UNIFORM STABILITY AND MEAN-FIELD LIMIT FOR THE AUGMENTED KURAMOTO MODEL

SEUNG-YEAL HA
Department of Mathematical Sciences and Research Institute of Mathematics
Seoul National University
Seoul 08826, Korea
Korea Institute for Advanced Study
Hoegiro 87
Seoul 02455, Korea

JEONGHO KIM
Department of Mathematical Sciences
Seoul National University
Seoul 08826, Korea

JINYEONG PARK*
Department of Mathematics and Research Institute of Natural Sciences
Hanyang University
222 Wangsimni-ro, Seongdong-gu, Seoul 04763, Korea

XIONGTAO ZHANG
Center for Mathematical sciences
Huazhong University of Science and Technology
Wuhan, China

Abstract. We present two uniform estimates on stability and mean-field limit for the “augmented Kuramoto model (AKM)” arising from the second-order lifting of the first-order Kuramoto model (KM) for synchronization. In particular, we address three issues such as synchronization estimate, uniform stability and mean-field limit which are valid uniformly in time for the AKM. The derived mean-field equation for the AKM corresponds to the dissipative Vlasov-McKean type equation. The kinetic Kuramoto equation for distributed natural frequencies is not compatible with the frequency variance functional approach for the complete synchronization. In contrast, the kinetic equation for the AKM has a similar structural similarity with the kinetic Cucker-Smale equation which admits the Lyapunov functional approach for the variance. We present sufficient frameworks leading to the uniform stability and mean-field limit for the AKM.

2010 Mathematics Subject Classification. 70F99, 92D25.
Key words and phrases. The Kuramoto model, mean-field limit, synchronization, uniform stability.

The works of S.-Y. Ha and X. Zhang are supported by the Samsung Science and Technology Foundation under Project Number SSTF-BA1401-03). The work of J. Kim is supported by the German Research Foundation (DFG), project number IRTG2235.
* Corresponding author: Jinyeong Park.
1. Introduction. Synchronization of weakly coupled oscillators is ubiquitous in our nature, e.g., rhythmic heart beatings of pacemaker cells, synchronous flashing of fireflies and collective hand clapping in a concert hall, etc [1, 6, 34, 35]. After Huygen’s observation on two pendulum clocks hanging on the same bar, collective behaviors of weakly coupled oscillators have been reported from time to time in scientific literature (see [34]). However, major scientific progress on the collective dynamics of complex systems was initiated by Winfree and Kuramoto about a half century ago in [26, 27, 41]. Recently, research on the collective dynamics of complex systems has received lots of attention due to engineering applications in sensor network, mobile network and control of unmanned aerial vehicles (UAV) etc. In [22], the authors observed a formal analogy between the Cucker-Smale flocking model and the Kuramoto model for synchronization, and provide a quantitative estimate for the synchronization based on Lyapunov functional approach. In the paper, we further investigate this formal analogy and study dynamic asymptotic properties of the AKM.

Consider an ensemble of Kuramoto oscillators lying on the nodes of the complete graph with \( N \)-nodes, and assume that the state of an oscillator is described by a real-valued function “phase”. Let \( \theta_i \) be the phase of the \( i \)-th oscillator whose dynamics is given by the Kuramoto model:

\[
\dot{\theta}_i = \nu_i + \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_i), \quad 1 \leq i \leq N, \tag{1}
\]

where \( \kappa \) and \( \nu_i \) denote the coupling strength, and the natural frequency of the \( i \)-th Kuramoto oscillator, respectively. On the other hand, it is well known in [28] that the dynamics of system (1) with \( N \gg 1 \) can be described by the corresponding mean-field equation, namely kinetic Kuramoto equation. To be specific, let \( f = f(\theta, \nu, t) \) be the one-particle distribution function at phase \( \theta \), natural frequency \( \nu \) at time \( t \). Then, the kinetic Kuramoto equation reads as follows.

\[
\partial_t f + \partial_\theta (L[f] f) = 0, \quad (\theta, \nu) \in \mathbb{T} \times \mathbb{R}, \quad t > 0,
\]

\[
L[f](\theta, \nu, t) := \nu + \kappa \int_{-\infty}^{2\pi} \int_0^{\infty} \sin(\theta_* - \theta) f(\theta_*, \nu_*, t) d\nu_* d\theta_*, \tag{2}
\]

where \( \mathbb{T} \) denotes the 1-dimensional torus. The Kuramoto model (1) and its corresponding mean-field kinetic equation (2) have been extensively studied in applied mathematics, control theory and statistical physics communities from various aspects, e.g., existence of partial and fully phase-locked states [2, 8, 29, 39], emergence of complete synchronization [4, 5, 7, 9, 10, 11, 14, 17, 18, 19, 22, 25, 37], stability of partial and fully phase-locked states and incoherent state [3, 30, 31, 32], slow-fast dynamics [23], existence of the critical coupling strength and its computing algorithm for phase-locked states [13, 38], phase-transition phenomena at critical coupling strength [1], relation with other models [22, 36] and rigorous mean-field limit [28] etc. For details, we refer to survey articles and a book [1, 12, 20, 34, 35]. Our main concern is to derive complete synchronization estimate for (2). For identical oscillators with \( g(\nu) = \delta_0 \), the kinetic equation (2) can be reduced to

\[
\partial_t f + \partial_\theta (L[f] f) = 0, \quad \theta \in \mathbb{T}, \quad t > 0,
\]

\[
L[f](\theta, t) := \kappa \int_0^{2\pi} \sin(\theta_* - \theta) f(\theta_*, t) d\theta_* . \tag{3}
\]
In this case, the variance function \( \Lambda[f] := \int_{T^2} |\theta - \theta^*|^2 f(\theta, t)f(\theta^*, t)d\theta d\theta \) can measure the emergence of complete (phase) synchronization. However, for distributed natural frequencies, the functional \( \Lambda[f] \) cannot be used for the complete synchronization. Then, it is not clear how to derive a complete synchronization estimate directly for the kinetic equation (2) without lifting corresponding particle results as in [7]. This motivates the works in this paper.

Next, we briefly explain how to bypass the aforementioned difficulty for the complete synchronization of the kinetic equation (2) for distributed natural frequencies. Our idea is to lift the first-order model (1) into a second-order model by introducing an auxiliary frequency variable \( \omega_i = \dot{\theta}_i \), i.e., we differentiate the equation (1) with respect to \( t \) and obtain an augmented Kuramoto model (AKM):

\[
\begin{align*}
\dot{\theta}_i &= \omega_i, \quad t > 0, \quad 1 \leq i \leq N, \\
\dot{\omega}_i &= \frac{\kappa}{N} \sum_{k=1}^{N} \cos(\theta_k - \theta_i)(\omega_k - \omega_i) .
\end{align*}
\] (4)

On the other hand, we consider a mean-field limit \( N \to \infty \). In this case, we set \( f = f(\theta, \omega, t) \) to be a probability density function corresponding to (4). Then, a formal BBGKY Hierarchy argument yields the kinetic equation:

\[
\begin{align*}
&f_t + \omega \partial_\theta f + \kappa \partial_\omega (L[f]f) = 0, \quad (\theta, \omega) \in T \times \mathbb{R}, \quad t > 0, \\
&L[f](\theta, \omega, t) := \int_{0}^{2\pi} \int_{-\infty}^{\infty} \cos(\theta^* - \theta)(\omega^* - \omega)f(\theta^*, \omega^*, t)d\theta^*d\omega^* .
\end{align*}
\] (5)

As an analogy with the kinetic Cucker-Smale model in [16], we can use the frequency variance functional

\[
\Lambda_1[f] := \int_{T^2 \times \mathbb{R}^2} |\omega - \omega^*|^2 f(\theta, \omega, t)f(\theta^*, \omega^*, t)d\omega d\omega d\theta d\theta
\]

to measure the emergence of complete synchronization for (5).

In this paper, we are interested in the following questions for (4) and (5):

- (Q1): Under what conditions, can system (4) exhibit the complete synchronization?
- (Q2): Is the system (4) uniformly \( \ell_p \)-stable with respect to initial data?
- (Q3): Can we derive the mean-field kinetic equation (5) from the particle model (4) as \( N \to \infty \) uniformly in time?

The first two questions might be generalized to the locally coupled Kuramoto model on a general symmetric and connected networks. However, the last question, i.e., uniform mean-field limit can be treated only for mean-field couplings (e.g., BBGKY hierarchy arguments break down for the locally coupled case). As aforementioned, since our main motivation is to study the complete synchronization of the kinetic level in a direct manner, we consider only the complete network case.

The main results of this paper are four-fold: First, we provide a sufficient framework for the complete synchronization estimate. Our sufficient conditions are expressed in terms of the coupling strength \( \kappa \), the diameter of the set of natural frequencies and initial data, and they are free of the number of oscillators (see Theorem 3.2). Second, we provide the uniform stability estimate of (4) with respect to initial data in a metric equivalent to the \( \ell_p \)-distance in phase space. Our uniform stability roughly says that the \( \ell_p \)-distance between two configurations at time \( t \) is
uniformly bounded by the constant multiple of initial $\ell_p$-distance between two initial data (see Theorem 4.3). Third, we present a uniform-in-time mean-field limit for (4) as a direct application of the exponential flocking estimate in Theorem 3.2 and uniform-in-time stability estimates in Theorem 4.3. Our last result is to derive the complete synchronization estimate for the mean-field kinetic equation (5) using a robust Lyapunov functional approach.

The rest of this paper is organized as follows. In Section 2, we briefly discuss the theoretical minimum for the Kuramoto model and its augmented model. In Section 3, we present a synchronization estimate for the AKM (4). In Section 4, we present a uniform $\ell_p$-stability estimate for the augmented model. In Section 5, we present a uniform-in-time mean-field limit from the particle model (4) to the corresponding kinetic equation uniformly in time, and we also study the complete frequency synchronization estimate for the kinetic equation. In Section 6, we study the uniform mean-field limit from the particle model (4) to the corresponding kinetic equation uniformly in time, and we also study the complete frequency synchronization estimate for the kinetic equation. Finally, Section 7 is devoted to a brief summary of our main results and discussion for future works.

Before we proceed to the next section, we introduce the notation which will be used in the rest of the paper.

**Notation.** When we discuss the distance in the spatial dimension $\mathbb{T}$, we use the orthodromic distance: let $\theta$ be a constant in $\mathbb{R}$, then we define

$$|\theta|_o := |\bar{\theta}|$$

where $\bar{\theta} \in (-\pi, \pi]$ and $\theta \equiv \bar{\theta}$ (mod $2\pi$).

In the following discussion, we only consider the case in which the oscillators are confined in a half circle. It is obvious that

$$|\theta|_o = |\theta| \quad \text{for} \quad \theta \in (-\pi, \pi).$$

For notational simplicity, we use $|\cdot|$ instead of $|\cdot|_o$ for the spatial distance. Throughout the paper, we use the following simplified notation: for $Z := (z_1, \cdots, z_N)$, we set

$$D(Z) := \max_{1 \leq i,j \leq N} |z_i - z_j|,$$

$$\|Z\|_p := \left( \sum_{i=1}^N |z_i|^p \right)^{\frac{1}{p}}, \quad p \in [1, \infty),$$

$$\|Z\|_\infty := \max_{1 \leq i \leq N} |z_i|,$$

and

$$\Theta := (\theta_1, \cdots, \theta_N), \quad \Omega := (\omega_1, \cdots, \omega_N), \quad \mathcal{V} := (\nu_1, \cdots, \nu_N).$$

2. Preliminaries. In this section, we briefly review a theoretical minimum for the Kuramoto model and the augmented Kuramoto model.

2.1. The Kuramoto model. In this subsection, we briefly discuss an associated conservation law, and review the state-of-the-art results on the complete synchronization for the Kuramoto model. First, we introduce a time-dependent quantity $\mathcal{C}(\Theta, \mathcal{V}, t)$:

$$\mathcal{C}(\Theta, \mathcal{V}, t) := \sum_{i=1}^N \theta_i - t \sum_{i=1}^N \nu_i.$$
Lemma 2.1. Let $\Theta = \Theta(t)$ be a phase vector whose dynamics is governed by (1). Then, the quantity $\mathcal{C}(\Theta, \mathcal{V}, t)$ is a constant of motion.

$$\frac{d}{dt}\mathcal{C}(\Theta(t), \mathcal{V}, t) = 0, \quad t > 0.$$ 

Proof. Let $\Theta = \Theta(t)$ be a Kuramoto flow. Then, we have

$$\frac{d}{dt}\mathcal{C}(\Theta(t), \mathcal{V}, t) = \frac{d}{dt}\left( \sum_{i=1}^{N} \theta_i - t \sum_{i=1}^{N} v_i \right) = \sum_{i=1}^{N} \dot{\theta}_i - \sum_{i=1}^{N} v_i = 0.$$ 

This yields the desired estimate. \(\square\)

Remark 1. Note that unless $\sum_{i=1}^{N} v_i$ is zero, the total phase $\sum_{i=1}^{N} \theta_i$ is not conserved.

Next, we discuss the equilibrium for the Kuramoto model (1). Note that the equilibrium solution $\Theta = (\theta_1, \cdots, \theta_N)$ is a solution to the following equilibrium system:

$$v_i + \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_i) = 0, \quad 1 \leq i \leq N. \quad (6)$$

We sum up the equation (6) with respect to $i$ to obtain

$$0 = \sum_{i=1}^{N} v_i + \frac{\kappa}{N} \sum_{i,k=1}^{N} \sin(\theta_k - \theta_i) = \sum_{i=1}^{N} v_i.$$ 

Thus, if the system (6) does have a solution, then the total sum of natural frequencies is zero. Hence if $\sum_{i=1}^{N} v_i \neq 0$, then system (6) does not have a solution. This leads to the need of the relaxed equilibria in the following definition. We first recall several definitions for a phase-locked state and asymptotic phase-locking.

Definition 2.2. [1, 12, 19] Let $\Theta(t) = (\theta_1(t), \cdots, \theta_N(t))$ be a time-dependent phase vector.

1. $\Theta$ is a phase-locked state if all relative phase differences are constants:

$$\theta_i(t) - \theta_j(t) = \theta_i(0) - \theta_j(0), \quad t \geq 0, \quad 1 \leq i, j \leq N.$$ 

2. $\Theta$ exhibits asymptotic phase-locking (complete synchronization) if the relative frequencies tend to zero asymptotically:

$$\lim_{t \to \infty} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| = 0, \quad 1 \leq i, j \leq N.$$ 

Remark 2. Note that solutions to the following equilibrium system:

$$v_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) = 0, \quad i = 1, \cdots, N, \quad \sum_{i=1}^{N} v_i = 0$$

correspond to phase-locked states of (1).

We next briefly review the state-of-the-art result for the Kuramoto model (1). It is well known in [11, 24, 36] that system (1) can be lifted as a dynamical system on $\mathbb{R}^N$ and can also be written as a gradient flow with an analytical potential in $\mathbb{R}^N$:

$$\dot{\Theta}(t) = -\nabla_{\Theta} V(\Theta), \quad \text{where} \quad V(\Theta) := -\sum_{k=1}^{N} v_k \dot{\theta}_k + \frac{\kappa}{2N} \sum_{1 \leq k,l \leq N} (1 - \cos(\theta_k - \theta_l)).$$
For a gradient flow system with analytical potential, uniform boundedness is equivalent to the convergence of solution toward the phase-locked state. As a dynamical system in $\mathbb{R}^N$, the uniform boundedness of (1) in $\mathbb{R}^N$ in a rotating frame moving with the speed of average of natural frequencies is not obvious since nonidentical oscillators can cross each other.

In the following theorem, we summarize the state-of-the-art results for the emergence of phase-locked states from generic initial data in a large coupling strength regime with $\kappa \gg 1$.

Theorem 2.3. [9, 11, 13, 19] The following assertions hold.

1. Suppose that the initial phase configuration $\Theta^0$ is confined in a half circle and the coupling strength $\kappa$ is positive such that

$$D(\Theta^0) < \pi, \quad \kappa > 0, \quad D(V) = 0.$$ 

Then, for any solution $\Theta = \Theta(t)$ to (1), we have

$$\lim_{t \to \infty} D(\dot{\Theta}(t)) = 0.$$ 

2. Suppose that the initial phase configuration $\Theta^0$ is confined in a half circle and the coupling strength $\kappa$ is sufficient large such that

$$D(\Theta^0) < \pi, \quad \kappa > D(V) > 0.$$ 

Then, for any solution $\Theta = \Theta(t)$ to (1), there exists a positive constant $\lambda$ such that

$$D(\dot{\Theta}(t)) \leq C e^{-\lambda t}, \quad \text{as } t \to \infty.$$ 

3. Suppose that natural frequencies are distributed and initial configurations satisfy

$$D(V) > 0, \quad R^0 = \frac{1}{N} \left| \sum_{k=1}^{N} e^{i \theta_0^0} \right| > 0, \quad \theta_0^0 \neq \theta_0^k, \quad 1 \leq j \neq k \leq N.$$ 

Then there exists a large coupling strength $\kappa_\infty > 0$ such that if $\kappa \geq \kappa_\infty$ there exists a phase-locked state $\Theta^\infty$ such that the solution with initial data $\Theta_0$ satisfies

$$\lim_{t \to \infty} \| \Theta(t) - \Theta^\infty \|_\infty = 0,$$

where the norm $\| \cdot \|_\infty$ is the standard $\ell^\infty$-norm in $\mathbb{R}^N$.

Remark 3. 1. In the course of the proof for the first statement, we can show that there exists a finite time $t_0$ and $D^\infty \in (0, \frac{\pi}{2})$ such that

$$D(\Theta(t)) \leq D^\infty, \quad \text{for } t \geq t_0.$$ 

2. The result of [11] does not yield detailed asymptotic dynamics of identical Kuramoto oscillators. However, when the diameter of the emergent phase-locked state is less than $\pi$ and the coupling strength is sufficiently large, then we can show that convergence speed is at least exponential. See [9, 12, 13] for detailed discussion.
2.2. The augmented Kuramoto model. In this subsection, we discuss basic structural properties of the AKM and relationship between the Kuramoto model and AKM. We set averaged quantities and fluctuations of phase and frequency around them:

$$\theta_c := \frac{1}{N} \sum_{k=1}^{N} \theta_k, \quad \omega_c := \frac{1}{N} \sum_{k=1}^{N} \omega_k,$$

$$\dot{\theta}_i := \theta_i - \theta_c, \quad \dot{\omega}_i := \omega_i - \omega_c.$$ 

Then, it is easy to see that the averaged quantities and fluctuations satisfy

$$\dot{\theta}_c = \omega_c,$$

$$\dot{\omega}_i = \frac{\kappa}{N} \sum_{k=1}^{N} \cos(\theta_k - \theta_i)(\omega_k - \omega_i). \quad (7)$$

**Lemma 2.4.** Let $$\{(\theta_i, \omega_i)\}_{i=1}^{N}$$ be a solution to (4). Then, the averaged quantities $$(\theta_c, \omega_c)$$ satisfy the following relations:

$$\omega_c(t) = \omega_c(0), \quad \theta_c(t) = \theta_c(0) + t\omega_c(0), \quad t \geq 0.$$ 

**Proof.** We sum (4) with respect to $$i$$ and use the skew symmetry of $$\cos(\theta_j - \theta_i)(\omega_j - \omega_i)$$ in the transformation of $$(i,j) \iff (j,i)$$ to get

$$\frac{d}{dt} \sum_{i=1}^{N} \omega_i = \frac{\kappa}{N} \sum_{i,k=1}^{N} \cos(\theta_k - \theta_i)(\omega_k - \omega_i) = -\frac{\kappa}{N} \sum_{i,k=1}^{N} \cos(\theta_k - \theta_i)(\omega_k - \omega_i) = 0.$$ 

The second relation follows from (7). $\square$

Next, we discuss the relation between the first-order model (1) and the second-order model (4) which is stated in the following theorem.

**Theorem 2.5.** The Kuramoto model (1) is equivalent to the augmented Kuramoto model (4) in the following sense.

1. If $$\{\theta_i\}$$ is a solution to (1) with initial data $$\{\theta^0_i\},$$ then $$\{(\theta_i, \omega_i) := (\dot{\theta}_i)\}$$ is a solution to (4) with well-prepared initial data $$\{(\theta^0_i, \omega^0_i)\}$$:

$$\omega^0_i := \nu_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta^0_j - \theta^0_i), \quad i = 1, \ldots, N.$$ 

2. If $$\{(\theta_i, \omega_i)\}$$ is a solution to (4) with initial data $$\{(\theta^0_i, \omega^0_i)\},$$ then $$\{\theta_i\}$$ is a solution to (1) with natural frequencies:

$$\nu_i := \omega^0_i - \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta^0_j - \theta^0_i), \quad i = 1, \ldots, N.$$ 

**Proof.** (i) Let $$\Theta = \Theta(t)$$ be a solution to (1) with initial data $$\Theta^0.$$ Then, it satisfies

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_i). \quad (8)$$

We set

$$\omega_i = \dot{\theta}_i \quad (9)$$

and differentiate the above equation to obtain

$$\dot{\omega}_i = \frac{\kappa}{N} \sum_{k=1}^{N} \cos(\theta_k - \theta_i)(\omega_k - \omega_i). \quad (10)$$
We use (8) to find
\[ \omega_i = \nu_i + \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_i). \] (11)

Letting \( t \to 0^+ \) in (11), we obtain
\[ \omega_i^0 = \nu_i + \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k^0 - \theta_i^0). \] (12)

Finally, we combine (9), (10) and (12) to see that \((\theta_i, \omega_i)\) is a solution to (4) with initial data \((\theta_i^0, \omega_i^0)\).

(ii) Let \( \{ (\theta_i, \omega_i) \} \) be a solution to (4) with initial data \( \{ (\theta_i^0, \omega_i^0) \} \), i.e., it satisfies
\[ \dot{\omega}_i = \kappa \sum_{k=1}^{N} \cos(\theta_k - \theta_i)(\omega_k - \omega_i). \]

Then, we use the relations:
\[ \omega_i = \dot{\theta}_i \quad \text{and} \quad \cos(\theta_k - \theta_i)(\omega_k - \omega_i) = \frac{d}{dt} \sin(\theta_k - \theta_i) \]

to integrate (10) to obtain
\[ \dot{\theta}_i = \omega_i^0 - \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k^0 - \theta_i^0) + \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_i). \] (13)

Then, we set
\[ \nu_i := \omega_i^0 - \frac{\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k^0 - \theta_i^0). \] (14)

Finally, we combine (13) and (14) to recover the Kuramoto model.

---

**Lemma 2.6.** [21] Suppose that two nonnegative Lipschitz functions \( X \) and \( V \) satisfy the system of differential inequalities:
\[
\left| \frac{dX}{dt} \right| \leq V, \quad \frac{dV}{dt} \leq -\alpha V + \gamma e^{-\alpha t} X, \quad \text{a.e. } t > 0,
\]
where \( \alpha \) and \( \gamma \) are positive constants. Then, \( X \) and \( V \) satisfy the uniform bound and decay estimates:
\[
X(t) \leq \frac{2M}{\alpha} (X(0) + V(0)), \quad V(t) \leq M(X(0) + V(0))e^{-\frac{\alpha t}{2}}, \quad t \geq 0,
\]
where \( M \) is a positive constant defined by,
\[
M := \max \left\{ 1, \frac{2\gamma}{\alpha \epsilon} \right\} + \frac{8\gamma}{\alpha^3 \epsilon^2}.
\]

**3. Emergence of the complete synchronization.** In this section, we present complete synchronization estimates for the AKM (4) in \( \ell_\infty \) and \( \ell_p \), \( p \in [1, \infty) \) frameworks by deriving a system of Grönwall’s inequalities.
3.1. $\ell_\infty$-framework. In this subsection, we present a synchronization estimate in $\ell_\infty$-framework. In each case, the synchronization estimates are obtained in the following three steps:

- **Step A** (Existence of positively invariant set). We identify a positively invariant set which is translation invariant in phase space.
- **Step B** (Derivation of Grönwall’s inequality). We introduce a Lyapunov type functional and derive a Grönwall type differential inequality.
- **Step C** (Complete synchronization estimate). Once we derive a Grönwall type inequality for a suitable Lyapunov functional, suitable Grönwall’s lemma and continuity arguments yield the desired synchronization estimate.

As candidates for Lyapunov functionals and an invariant set, we introduce phase and frequency diameters:

$$D(\Theta) := \max_{1 \leq i, j \leq N} |\theta_i - \theta_j|, \quad D(\Omega) := \max_{1 \leq i, j \leq N} |\omega_i - \omega_j|,$$

and for $D_\infty < \frac{\pi}{2}$, we set

$$S(D_\infty) := \{ \Theta = (\theta_1, \cdots, \theta_N) : D(\Theta) < D_\infty \}.$$

**Lemma 3.1.** Suppose that initial data $(\Theta^0, \Omega^0)$ and the coupling strength $\kappa$ satisfy

$$\Theta^0 \in S(D_\infty), \quad \kappa > \frac{D(\Omega^0)}{\cos(D_\infty)(D_\infty - D(\Theta^0))}, \quad \text{where } D_\infty < \frac{\pi}{2}.$$

Then, the set $S$ is positively invariant under the flow (4), i.e., for any solution $\Theta = \Theta(t)$ with initial data $\Theta^0 \in S$, we have

$$\Theta(t) \in S(D_\infty), \quad t \geq 0.$$

**Proof.** Let $\Theta^0$ be a initial data with $D(\Theta^0) < D_\infty$. Suppose that $S$ is not positively invariant under flow. Then, there exists a finite time $t^*$ such that

$$t^* = \sup\{ t : D(\Theta(s)) < D_\infty, \quad 0 \leq s \leq t \}.$$

By the continuity, we have

$$D(\Theta(t^*)) = D_\infty.$$

On the other hand, for any indices $i$ and $j$, we integrate (4)$_2$ from 0 to $t$ for $t < t^*$ to obtain

$$\omega_i(t) - \omega_j(t) = \omega_i^0 - \omega_j^0 + \frac{\kappa}{N} \sum_{k=1}^{N} \int_0^t \cos(\theta_k - \theta_i)(\omega_k(s) - \omega_i(s)) \, ds$$

$$- \frac{\kappa}{N} \sum_{k=1}^{N} \int_0^t \cos(\theta_k - \theta_j)(\omega_k(s) - \omega_j(s)) \, ds.$$

Now we choose maximal indices $M$ and $m$ which might be dependent on $t$:

$$\omega_M(t) := \max_{1 \leq i \leq N} \omega_i(t), \quad \omega_m(t) := \min_{1 \leq i \leq N} \omega_i(t).$$
Then, for $t \in [0, t^*]$, we get

$$D(\dot{\Theta}(t)) = \omega_M(t) - \omega_m(t)$$

$$= \omega_M(0) - \omega_m(0) + \frac{\kappa}{N} \sum_{k=1}^{N} \int_0^t \cos(\theta_k(s) - \theta_M(s))(\omega_k(s) - \omega_M(s)) \, ds$$

$$- \frac{\kappa}{N} \sum_{k=1}^{N} \int_0^t \cos(\theta_k(s) - \theta_m(s))(\omega_k(s) - \omega_m(s)) \, ds.$$  \hfill (15)

For $0 \leq s \leq t^*$, we have

$$|\theta_k(s) - \theta_M(s)| \leq D^\infty, \quad |\theta_k(s) - \theta_m(s)| \leq D^\infty. \hfill (16)$$

Therefore, it follows from (15) and (16) that we have

$$D(\dot{\Theta}(t)) \leq D(\dot{\Theta}(0)) + \kappa \cos D^\infty \int_0^t (\omega_k(s) - \omega_M(s)) \, ds$$

$$- \frac{\kappa \cos D^\infty}{N} \sum_{k=1}^{N} \int_0^t \cos(\theta_k(s) - \theta_m(s))(\omega_k(s) - \omega_m(s)) \, ds$$

$$= D(\dot{\Theta}(0)) - \kappa \cos D^\infty \int_0^t D(\dot{\Theta}(s)) \, ds. \hfill (17)$$

This yields

$$D(\dot{\Theta}(t)) \leq D(\dot{\Theta}(0)) - \kappa \cos D^\infty \int_0^t D(\dot{\Theta}(s)) \, ds. \hfill (18)$$

We set

$$u(t) := \int_0^t D(\dot{\Theta}(s)) \, ds.$$  

Then, it is easy to see that

$$\dot{u}(t) = D(\dot{\Theta}(t)), \quad u(0) = 0, \quad \dot{u}(0) = D(\dot{\Theta}(0)). \hfill (19)$$

Then, the relation (18) is equivalent to

$$\dot{u}(t) + \kappa \cos D^\infty u(t) \leq \dot{u}(0). \hfill (20)$$

Then, (19) and (20) yield

$$u(t) \leq \frac{\dot{u}(0)}{\kappa \cos D^\infty} \left(1 - e^{-\kappa(\cos D^\infty)t}\right) \leq \frac{\dot{u}(0)}{\kappa \cos D^\infty}, \quad t \geq 0. \hfill (21)$$

On the other hand, since $D(\Theta(t^*)) = D^\infty$, there exist indices $i$ and $j$ such that

$$\theta_i(t^*) - \theta_j(t^*) = D^\infty.$$

Then, it follows from (4)$_1$ that we have

$$D^\infty = \theta_i(t^*) - \theta_j(t^*)$$

$$= \theta_i^0 - \theta_j^0 + \int_0^{t^*} (\omega_i(s) - \omega_j(s)) \, ds$$

$$\leq D(\Theta^0) + \int_0^{t^*} D(\dot{\Theta}(s)) \, ds \leq D(\Theta^0) + \frac{D(\dot{\Theta}(0))}{\kappa \cos(D^\infty)} < D^\infty,$$

where we used the hypothesis on $\kappa$ and (21), which yields contradiction.
Theorem 3.2. Suppose that initial data and coupling strength satisfy
\[ \Theta^0 \in S(D^\infty), \quad \sum_{i=1}^N \omega_i^0 = 0, \quad \kappa > \frac{D(\Theta^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \quad \text{where } D^\infty < \frac{\pi}{2}. \]
Then, we have an exponential synchronization:
\[ D(\Omega(t)) \leq D(\Omega^0)e^{-\kappa \cos(D^\infty)t}, \quad t \geq 0. \]

Proof. Due to the conservation law in Lemma 2.4, we have
\[ \sum_{i=1}^N \omega_i(t) = 0, \quad t \geq 0. \]
We set extremal indices \( M \) and \( m \) such that
\[ \omega_M := \max_{1 \leq i \leq N} \omega_i, \quad \omega_m := \min_{1 \leq i \leq N} \omega_i. \]
Then, it follows from (4) that we have
\[ \dot{\omega}_M = \frac{\kappa}{N} \sum_{k=1}^N \cos(\theta_k - \theta_i)(\omega_k - \omega_M) \leq -\kappa \cos D^\infty \omega_M. \tag{22} \]
Similarly, we have
\[ \dot{\omega}_m \geq -\kappa \cos D^\infty \omega_m. \tag{23} \]
Then, it follows from (22) and (23) that we have
\[ \frac{d}{dt} D(\dot{\Theta}(t)) \leq -\kappa \cos D^\infty D(\dot{\Theta}), \quad t > 0. \]
This yields the desired exponential decay estimate. \( \square \)

3.2. \( \ell_p \)-framework with \( p \in [1, \infty) \). In this subsection, we present \( \ell_p \)-estimate for (4) for later use. For phase and frequency vectors \( \Theta = (\theta_1, \ldots, \theta_N) \) and \( \Omega = (\omega_1, \ldots, \omega_N) \),
we set \( \|\Theta\|_p \) and \( \|\Omega\|_p \):
\[ \|\Theta\|_p := \left( \sum_{i=1}^N |\theta_i|^p \right)^{\frac{1}{p}}, \quad \|\Omega\|_p := \left( \sum_{i=1}^N |\omega_i|^p \right)^{\frac{1}{p}}, \quad p \in [1, \infty). \]

Proposition 1. Suppose that initial data and coupling strength satisfy
\[ \Theta^0 \in S(D^\infty), \quad \sum_{i=1}^N \omega_i^0 = 0, \quad \kappa > \frac{D(\Theta^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \quad \text{where } D^\infty < \frac{\pi}{2}. \]
Then for any solution \( \{ (\theta_i, \omega_i) \}_{i=1}^N \) to (4), we have
\[ \frac{d}{dt} \|\Theta\|_p \leq \|\Omega\|_p, \quad \frac{d}{dt} \|\Omega\|_p \leq -\kappa \cos(D^\infty) \|\Omega\|_p, \quad a.e. \ t > 0. \tag{24} \]

Proof. (i) Note that
\[ \frac{d}{dt} |\theta_i| \leq |\omega_i|. \]
We multiply by \( p|\theta_i|^{p-1} \) to the above relation, take a sum the resulting relation, and use Hölder’s inequality to get the following estimate:
\[ \frac{d}{dt} \sum_{i=1}^N |\theta_i|^p \leq p \sum_{i=1}^N |\theta_i|^{p-1} |\omega_i| \leq p \left( \sum_{i=1}^N |\theta_i|^p \right)^{\frac{p-1}{p}} \left( \sum_{i=1}^N |\omega_i|^p \right)^{\frac{1}{p}} \leq p \|\Theta\|_p^{p-1} \|\Omega\|_p. \]
This yields the desired first differential inequality. 
(ii) It follows from (4) that we have
\[
|\omega_1| \frac{d|\omega_1|}{dt} = \frac{1}{2} |\omega_1|^2 \frac{d\omega_1}{dt} = \frac{1}{2} \omega_1 \frac{d\omega_1}{dt} = \frac{\kappa}{N} \sum_{j=1}^{N} \cos(\theta_j - \theta_i)\omega_1(\omega_j - \omega_1). \tag{25}
\]
We use (25) to obtain
\[
\frac{d\|\Omega\|^p}{dt} = \sum_{i=1}^{N} \frac{d}{dt} |\omega_i|^p = \sum_{i=1}^{N} p|\omega_i|^{p-2} |\omega_i| \frac{d}{dt} |\omega_i|
\]
\[
= \sum_{i=1}^{N} p|\omega_i|^{p-2} \left[ \frac{\kappa}{N} \sum_{j=1}^{N} \cos(\theta_j - \theta_i)\omega_1(\omega_j - \omega_1) \right]
\]
\[
= \frac{\kappa p}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \cos(\theta_j - \theta_i)|\omega_i|^{p-2}\omega_1(\omega_j - \omega_i)
\]
\[
= \frac{\kappa p}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \cos(\theta_j - \theta_i)(\omega_j - \omega_i)\left(|\omega_i|^{p-2}\omega_i - |\omega_j|^{p-2}\omega_j\right). \tag{26}
\]
We use the monotonicity of \( f(x) = |x|^{p-2}x \) to see
\[
(\omega_j - \omega_i)(|\omega_i|^{p-2}\omega_i - |\omega_j|^{p-2}\omega_j) \leq 0. \tag{27}
\]
Then, we use (26), (27), \( \sum_{i=1}^{N} \omega_i = 0 \) and a priori condition:
\[
\cos(\theta_j - \theta_i) \geq \cos D^\infty
\]
to obtain
\[
\frac{d\|\Omega\|^p}{dt} \leq \frac{\kappa p \cos D^\infty}{2N} \sum_{i,j=1}^{N} (\omega_j - \omega_i)(|\omega_i|^{p-2}\omega_i - |\omega_j|^{p-2}\omega_j)
\]
\[
= -\kappa p \cos D^\infty \sum_{i=1}^{N} |\omega_i|^p = -\kappa p \cos D^\infty\|\Omega\|^p.
\]
This yields the desired second differential inequality. \( \square \)

Finally, we combine Proposition 1 and Lemma 3.1 to derive the exponential synchronization.

**Theorem 3.3.** Let \( \{(\theta_i, \omega_i)\} \) be a solution to (4) with initial data and coupling strength:
\[
\Theta^0 \in \mathcal{S}(D^\infty), \quad \sum_{i=1}^{N} \omega_i^0 = 0, \quad \kappa > \frac{D(\Theta^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \quad \text{where } D^\infty < \frac{\pi}{2}.
\]
Then, there exists a positive constant \( \theta^p_\infty \) such that for \( p \in [1, \infty) \),
\[
\|\Omega(t)\|^p \leq \|\Theta^0\|^p e^{-\kappa \cos(D^\infty)t}, \quad \|\Theta(t)\|^p \leq \theta^p_\infty, \quad t \geq 0.
\]

**Proof.** The exponential decay of \( \Omega \) follows from the second equation of (24). On the other hand, it follows from (24) that we have
\[
\left|\|\Theta(t)\|^p - \|\Theta(0)\|^p\right| \leq \int_{0}^{t} \|\Omega(s)\|^p ds \leq \frac{\|\Theta^0\|^p}{\kappa \cos D^\infty} \left(1 - e^{-\kappa \cos(D^\infty)t}\right) \leq \frac{\|\Theta^0\|^p}{\kappa \cos D^\infty}. \]
Thus, we have
\[
\|\Theta(t)\|_p \leq \|\Theta^0\|_p + \frac{\|\Omega^0\|_p}{\kappa \cos D^\infty} =: \theta^\infty_p(D^\infty, \kappa, \|\Theta^0\|_p, \|\Omega^0\|_p).
\]

Thanks to Theorem 3.3, we can conclude that there exists a unique phase-locked state $\Theta^\infty$. Moreover, $\Theta(t)$ will tend to $\Theta^\infty$ exponentially.

**Corollary 1.** Let $\{\theta_i, \omega_i\}$ be a solution to (4) with initial data $\{\theta_i^0, \omega_i^0\}$ and coupling strength $\kappa$:

\[
\Theta^0 \in S(D^\infty), \quad \sum_{i=1}^N \omega_i^0 = 0, \quad \kappa > \frac{D(\Theta^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \quad \text{where } D^\infty < \frac{\pi}{2}.
\]

Then, for any solution $\{(\theta_i, \omega_i)\}$, there exists a unique phase lock state $\Theta^\infty := (\theta^\infty_1, \cdots, \theta^\infty_N)$ such that

\[
|\theta_i(t) - \theta^\infty_i| \leq Ce^{-\kappa \cos D^\infty}t, \quad i = 1, \cdots, N.
\]

**Proof.** Let $\Theta = \Theta(t)$ be a solution to system (4). Then, since $\kappa$ is sufficiently large, we have

\[
\sup_{0 \leq t < \infty} D(\Theta(t)) \leq D^\infty.
\]

Then, we use Theorem 3.2 to obtain

\[
|\theta_i(t) - \theta_i(\tilde{t})| \leq \int_{t}^{\tilde{t}} |\theta_i'(s)| ds \leq \int_{t}^{\tilde{t}} \left( \sum_{i=1}^N |\theta_i'(s)|^p \right)^{1/p} ds \leq \int_{t}^{\tilde{t}} \left( \sum_{i=1}^N |\theta_i'(s)|^p \right)^{1/p} ds \leq \frac{\|\Omega^0\|_p}{\kappa \cos D^\infty} \left( e^{-\kappa \cos D^\infty} - e^{-\kappa \cos D^\infty} \right).
\]

Then for any $\varepsilon > 0$, we can find a positive number $T$ such that if $\tilde{t} \geq T$ and $t \geq T$, then

\[
|\theta_i(\tilde{t}) - \theta_i(t)| < \varepsilon.
\]

This immediately implies that there exists a unique asymptotic limit $\theta^\infty_i$. Moreover, we combine (28) to show that

\[
|\theta_i(t) - \theta^\infty_i| \leq Ce^{-\kappa \cos D^\infty}t.
\]

4. **Uniform $\ell_p$-stability estimate.** In this section, we study the uniform $\ell_p$-stability for the augmented system (4) with respect to initial data.

Let $Z := (\Theta, \Omega)$ and $\tilde{Z} := (\tilde{\Theta}, \tilde{\Omega})$ be two solutions to (4) corresponding to initial data $(\Theta^0, \Omega^0)$ and $(\tilde{\Theta}^0, \tilde{\Omega}^0)$, respectively. For the uniform stability estimate, we introduce a metric which is equivalent to $\ell_p$-distance: for $p \in [1, \infty)$ and two solutions $Z = (\Theta, \Omega)$ and $\tilde{Z} = (\tilde{\Theta}, \tilde{\Omega})$, we define the distance as

\[
d_p(Z(t), \tilde{Z}(t)) := \|\Theta(t) - \tilde{\Theta}(t)\|_p + \|\Omega(t) - \tilde{\Omega}(t)\|_p.
\]

Next, we present a uniform $\ell_p$-stability of system (4) with respect to initial data as follows.
Definition 4.1. The system (4) is uniformly $\ell_p$-stable with respect to initial data if the following relation holds: For two solutions $Z$ and $\tilde{Z}$ to (4) with initial data $Z^0$ and $\tilde{Z}^0$, respectively, there exists a positive constant $G$ independent of $t$ such that

$$d_p(Z(t), \tilde{Z}(t)) \leq G d_p(Z_0, \tilde{Z}_0), \quad t \geq 0.$$ 

In the following lemma, we will derive differential inequalities for two subfunctionals $\|\Theta(t) - \Theta(t)\|_p$ and $\|\Omega(t) - \Omega(t)\|_p$ for $p \in [1, \infty)$.

Lemma 4.2. Let $(\Theta, \Omega)$ and $(\tilde{\Theta}, \tilde{\Omega})$ be two solutions to (4) corresponding to initial data $(\Theta^0, \Omega^0)$ and $(\tilde{\Theta}^0, \tilde{\Omega}^0)$, respectively. Suppose that initial data and coupling strength satisfies following conditions:

$$\Theta^0 \in S(D^\infty), \quad \tilde{\Theta}^0 \in S(D^\infty), \quad \sum_{i=1}^N \omega_i^0 = 0 \quad \text{and} \quad \sum_{i=1}^N \tilde{\omega}_i^0 = 0,$$

$$\kappa > \max \left\{ \frac{D(\Theta^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \frac{D(\tilde{\Theta}^0)}{\cos(D^\infty)(D^\infty - D(\tilde{\Theta}^0))} \right\}.$$ 

Then, we have

$$\frac{d}{dt}\|\Theta - \tilde{\Theta}\|_p \leq \|\Omega - \tilde{\Omega}\|_p, \quad a.e., \quad t > 0,$$

$$\frac{d}{dt}\|\Omega - \tilde{\Omega}\|_p \leq -\kappa \cos(D^\infty)\|\Omega - \tilde{\Omega}\|_p + 2\kappa\|\Omega^0\|_p e^{-\kappa \cos(D^\infty)t} \|\Theta - \tilde{\Theta}\|_p. \quad (30)$$

Proof. • Case A (Derivation of the first inequality \((30)_1\)). Note that $\theta_i - \tilde{\theta}_i$ satisfies

$$\frac{d}{dt}(\theta_i - \tilde{\theta}_i) = \omega_i - \tilde{\omega}_i.$$ 

This yields

$$\frac{d}{dt}|\theta_i - \tilde{\theta}_i| \leq |\omega_i - \tilde{\omega}_i|.$$ 

We multiply by $p|\theta_i - \tilde{\theta}_i|^{p-1}$ on both sides, sum up the resulting relations with respect to $i$ and apply Hölder’s inequality to obtain

$$\frac{d}{dt} \sum_{i=1}^N |\theta_i - \tilde{\theta}_i|^p \leq p\|\Theta - \tilde{\Theta}\|_p^{p-1} \|\Omega - \tilde{\Omega}\|_p.$$ 

This implies the desired estimate.

• Case B (Derivation of the first inequality \((30)_2\)). Note that $\omega_i - \tilde{\omega}_i$ satisfies

$$\frac{d}{dt}(\omega_i - \tilde{\omega}_i) = \frac{\kappa}{N} \sum_{k=1}^N \left[ \cos(\theta_k - \theta_i)(\omega_k - \omega_i) - \cos(\tilde{\theta}_k - \tilde{\theta}_i)(\tilde{\omega}_k - \tilde{\omega}_i) \right]$$

$$= \frac{\kappa}{N} \sum_{k=1}^N \cos(\theta_k - \theta_i) \left[ (\omega_k - \omega_i) - (\tilde{\omega}_k - \tilde{\omega}_i) \right]$$

$$+ \frac{\kappa}{N} \sum_{k=1}^N \left[ \cos(\theta_k - \theta_i) - \cos(\tilde{\theta}_k - \tilde{\theta}_i) \right] (\tilde{\omega}_k - \tilde{\omega}_i)$$

$$= \frac{\kappa}{N} \sum_{k=1}^N \cos(\theta_k - \theta_i) \left[ (\omega_k - \tilde{\omega}_k) - (\omega_i - \tilde{\omega}_i) \right]$$

$$+ \frac{\kappa}{N} \sum_{k=1}^N \left[ \cos(\theta_k - \theta_i) - \cos(\tilde{\theta}_k - \tilde{\theta}_i) \right] (\tilde{\omega}_k - \tilde{\omega}_i).$$
\[- \frac{\kappa}{N} \sum_{k=1}^{N} \sin \theta_{ik}^* \left[ (\theta_k - \theta_i) - (\tilde{\theta}_k - \tilde{\theta}_i) \right] (\dot{\omega}_k - \dot{\omega}_i). \tag{31} \]

where \( \theta_{ik}^* \) is located between \((\theta_k - \theta_i)\) and \((\tilde{\theta}_k - \tilde{\theta}_i)\) by mean value theorem. By multiplying \((\omega_i - \hat{\omega}_i)\) on both sides of (31), we have

\[ |\omega_i - \hat{\omega}_i| \frac{d}{dt} |\omega_i - \hat{\omega}_i| = (\omega_i - \hat{\omega}_i) \frac{d}{dt} (\omega_i - \hat{\omega}_i) \]

\[ = \frac{\kappa}{N} \sum_{k=1}^{N} \cos(\theta_k - \theta_i)(\omega_i - \hat{\omega}_i) \left[ (\omega_k - \hat{\omega}_k) - (\omega_i - \hat{\omega}_i) \right] \]

\[ - \frac{\kappa}{N} \sum_{k=1}^{N} \sin \theta_{ik}^* \left[ (\theta_k - \theta_i) - (\tilde{\theta}_k - \tilde{\theta}_i) \right] (\omega_k - \hat{\omega}_k)(\omega_k - \hat{\omega}_i). \]

\[ \leq \frac{\kappa}{N} \sum_{k=1}^{N} \cos(\theta_k - \theta_i)(\omega_i - \hat{\omega}_i) \left[ (\omega_k - \hat{\omega}_k) - (\omega_i - \hat{\omega}_i) \right] + \frac{\kappa}{N} \sum_{k=1}^{N} \left[ |\theta_k - \hat{\theta}_k| + |\theta_i - \hat{\theta}_i| \right] |\omega_i - \hat{\omega}_i| |\omega_k - \hat{\omega}_i|. \tag{32} \]

We use (32) and similar argument used in Proposition 1 to obtain

\[ \frac{d}{dt} \| \Omega - \hat{\Omega} \|_p^p = \sum_{i=1}^{N} \frac{d}{dt} |\omega_i - \hat{\omega}_i|^p \]

\[ = \sum_{i=1}^{N} p |\omega_i - \hat{\omega}_i|^{p-2} |\omega_i - \hat{\omega}_i| \frac{d}{dt} |\omega_i - \hat{\omega}_i| \]

\[ \leq \kappa p \sum_{i,k} \cos(\theta_k - \theta_i) |\omega_i - \hat{\omega}_i|^{p-2} (\omega_i - \hat{\omega}_i) \left[ (\omega_k - \hat{\omega}_k) - (\omega_i - \hat{\omega}_i) \right] \]

\[ + \kappa p \sum_{i,k} \left( |\theta_k - \hat{\theta}_k| + |\theta_i - \hat{\theta}_i| \right) |\omega_i - \hat{\omega}_i|^{p-1} |\omega_k - \hat{\omega}_i| \]

\[ \leq N D(\Omega) \| \Theta - \hat{\Theta} \|_p \| \Omega - \hat{\Omega} \|_p^{p-1}. \tag{33} \]

By Hölder’s inequality, we have

\[ \sum_{i,k} |\theta_k - \hat{\theta}_k| |\omega_i - \hat{\omega}_i|^{p-1} |\omega_k - \hat{\omega}_i| \leq \left( \sum_{i,k} |\omega_i - \hat{\omega}_i|^p \right)^{\frac{p-1}{p}} \left( \sum_{i,k} |\theta_k - \hat{\theta}_k|^p |\omega_k - \hat{\omega}_i|^p \right)^{\frac{1}{p}} \]

\[ \leq N D(\Omega) \| \Theta - \hat{\Theta} \|_p \| \Omega - \hat{\Omega} \|_p^{p-1} \]

and similarly

\[ \sum_{i,k} |\omega_i - \hat{\omega}_i|^{p-1} |\theta_i - \hat{\theta}_i| |\omega_k - \hat{\omega}_i| \leq N D(\Omega) \| \Omega - \hat{\Omega} \|_p^{p-1} \| \Theta - \hat{\Theta} \|_p. \]

Then, by using these estimation and relation (33), we obtain

\[ \frac{d}{dt} \| \Omega - \hat{\Omega} \|_p^p \leq -\kappa p \cos(D^\infty) \| \Omega - \hat{\Omega} \|_p^p + 2\kappa p D(\hat{\Omega}) \| \Theta - \hat{\Theta} \|_p \| \Omega - \hat{\Omega} \|_p^{p-1}. \]

By applying the relation \( D(\hat{\Omega}) \leq \| \hat{\Omega} \|_p \) and Theorem 3.3, we attain the desired result. \( \square \)
We combine Lemma 2.6 and Lemma 4.2 to obtain the uniform $\ell_p$-stability.

**Theorem 4.3.** Suppose that initial data and coupling strength satisfy the following relations:

\[
\Theta^0 \in S(D^\infty), \quad \tilde{\Theta}^0 \in S(D^\infty), \quad \sum_{i=1}^{N} \omega_i^0 = 0 \quad \text{and} \quad \sum_{i=1}^{N} \tilde{\omega}_i^0 = 0,
\]

\[
\kappa > \max \left\{ \frac{D(\Theta^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \frac{D(\tilde{\Theta}^0)}{\cos(D^\infty)(D^\infty - D(\tilde{\Theta}^0))} \right\}.
\]

Then, for any two solutions $(\Theta, \Omega)$ and $(\tilde{\Theta}, \tilde{\Omega})$, we have uniform $\ell_p$-stability estimate (29).

As a direct application of Theorem 4.3, we have the following corollary for the first-order Kuramoto model (1).

**Corollary 2.** Suppose that initial data and coupling strength $\kappa$ satisfy the following relations:

\[
\Theta^0 \in S(D^\infty), \quad \tilde{\Theta}^0 \in S(D^\infty), \quad \sum_{i=1}^{N} \nu_i = 0, \quad \sum_{i=1}^{N} \tilde{\nu}_i = 0,
\]

\[
\kappa > \max \left\{ \frac{D(\Theta^0)}{\cos(D^\infty)(D^\infty - D(\Theta^0))}, \frac{D(\tilde{\Theta}^0)}{\cos(D^\infty)(D^\infty - D(\tilde{\Theta}^0))} \right\}.
\]

Then, for any two solutions $\Theta$ and $\tilde{\Theta}$ to (1) with natural frequency $\nu := (\nu_i)$ and $\tilde{\nu} := (\tilde{\nu}_i)$ respectively, there exists a positive constant $C$ independent of $t$ such that

\[
\|\Theta(t) - \tilde{\Theta}(t)\|_p \leq C \left[ \|\Theta^0 - \tilde{\Theta}^0\|_p + \|\nu - \tilde{\nu}\|_p \right], \quad t \geq 0.
\]

**Proof.** Let $\Theta$ and $\tilde{\Theta}$ be phase processes for (1) corresponding to the following initial data and natural frequencies, respectively:

\[
(\theta^0_0, \ldots, \theta^0_N), \quad (\nu_1, \ldots, \nu_N); \quad (\tilde{\theta}^0_0, \ldots, \tilde{\theta}^0_N), \quad (\tilde{\nu}_1, \ldots, \tilde{\nu}_N).
\]

On the other hand, we also set initial frequencies:

\[
\omega_i^0 := \nu_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta_j^0 - \theta_i^0),
\]

\[
\tilde{\omega}_i^0 := \tilde{\nu}_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\tilde{\theta}_j^0 - \tilde{\theta}_i^0).
\]

Then, we solve the second-order system (4) with initial data (34) and (35). It follows from the equivalence relation between KM (1) and AKM (4) in Theorem 2.5 and Theorem 4.3 that we have

\[
\|\Theta(t) - \tilde{\Theta}(t)\|_p \leq C \left[ \|\Theta^0 - \tilde{\Theta}^0\|_p + \|\Omega^0 - \tilde{\Omega}^0\|_p \right],
\]

where $\Omega^0 = (\omega_0^0, \ldots, \omega_N^0)$. We again use the relations (35) to find

\[
|\omega_i^0 - \tilde{\omega}_i^0| \leq |\nu_i - \tilde{\nu}_i| + \frac{\kappa}{N} \sum_{j=1}^{N} |\sin(\theta_j^0 - \theta_i^0) - \sin(\tilde{\theta}_j^0 - \tilde{\theta}_i^0)|
\]

\[
\leq |\nu_i - \tilde{\nu}_i| + \frac{\kappa}{N} \sum_{j=1}^{N} \left( |\theta_j^0 - \tilde{\theta}_j^0| + |\theta_i^0 - \tilde{\theta}_i^0| \right).
\]
This yields
\[ \| \Omega^0 - \tilde{\Omega}^0 \|_p \leq C \left[ \| \Theta^0 - \tilde{\Theta}^0 \|_p + \| \mathcal{V} - \tilde{\mathcal{V}} \|_p \right]. \] (37)

Finally, we combine (36) and (37) to obtain the desired stability estimate.

5. Uniform mean-field limit from the AKM to kinetic equation. In this section, we present the uniform mean-field limit for the AKM in a measure theoretic framework. The limiting mean-field kinetic equation can be formally derived from the particle model (4) via the formal procedure of BBGKY hierarchy, and it can be rigorously justified using the standard empirical measure approximations and local-in-time stability estimates in Monge-Kantorovich distance which is equivalent to Wasserstein-1 distance in any finite time.

The formal BBGKY hierarchy procedure yields a formal mean-field limit of system (4) toward the mean-field kinetic equation as \( N \to \infty \). More precisely, let \( f = f(\theta, \omega, t) \) be the one-particle distribution function. Then, the kinetic equation reads as follows.

\[
\begin{aligned}
f_t + \omega \partial_\theta f + \partial_\omega (L[f]f) &= 0, \quad (\theta, \omega) \in \mathbb{T} \times \mathbb{R}, \ t > 0, \\
L[f](\theta, \omega, t) := \kappa \int_0^{2\pi} \int_{-\infty}^{\infty} \cos(\theta_s - \theta) (\omega_s - \omega) f(\theta_s, \omega_s, t) \, d\theta_s \, d\omega_s.
\end{aligned}
\] (38)

Recall that our main purpose of this section is to justify the rigorous transition from (4) to (38) in the mean-field limit \( N \to \infty \).

5.1. A measure theoretic framework. In this subsection, we briefly discuss some framework which embodies (4) and (38) in a common framework. For this, we first review concept of measure-valued solutions to (38).

Let \( \mathcal{P}(\mathbb{T} \times \mathbb{R}) \) be the set of all Radon probability measures with compact support on the phase space \( \mathbb{T} \times \mathbb{R} \), which can be understood as normalized nonnegative bounded linear functionals on \( C_0(\mathbb{T} \times \mathbb{R}) \). For a probability measure \( \mu \in \mathcal{P}(\mathbb{T} \times \mathbb{R}) \), we use a standard duality relation:

\[ \langle \mu, f \rangle = \int_{\mathbb{T} \times \mathbb{R}} f(\theta, \omega) \, d\mu(\theta, \omega), \quad f \in C_0(\mathbb{T} \times \mathbb{R}). \]

Next, we recall several definitions to be used later.

**Definition 5.1.** [7] For \( T \in [0, \infty) \), let \( \mu_t \in L^\infty([0, T) ; \mathcal{P}(\mathbb{T} \times \mathbb{R})) \) be a measure-valued solution to (38) with initial data \( \mu_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R}) \) if the following three assertions hold:

1. Total mass is normalized: \( \langle \mu_t, 1 \rangle = 1 \).
2. \( \mu_t \) is weakly continuous in \( t \):

\[ \langle \mu_t, f \rangle \text{ is continuous in } t, \quad \forall f \in C_0^1(\mathbb{T} \times \mathbb{R} \times [0, T]). \]

3. \( \mu \) satisfies the equation (38) in a weak sense: for \( \forall \varphi \in C_0^1(\mathbb{T} \times \mathbb{R} \times [0, T]), \)

\[ \langle \mu_t, \varphi(\cdot, \cdot, t) \rangle - \langle \mu_0, \varphi(\cdot, \cdot, 0) \rangle = \int_0^t \langle \mu_s, \partial_s \varphi + \omega \partial_\theta \varphi + L[\mu_s] \partial_\omega \varphi \rangle \, ds, \]

**Remark 4.** Note that for a solution \( \{ (\theta_i, \omega_i) \} \) to (4), the empirical measure

\[ \mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{\theta_i} \otimes \delta_{\omega_i}, \]
We now discuss how to measure the distance between the solutions of (4) and (38) by equipping a metric to the probability measure space \( \mathcal{P}(\mathbb{T} \times \mathbb{R}) \), and the concept of local-in-time mean-field limit. In fact, we can endow Wasserstein-\( p \)-distance \( W_p \) in the probability space \( \mathcal{P}(\mathbb{T} \times \mathbb{R}) \).

**Definition 5.2.** [33, 40]

1. For \( p \in \mathbb{Z}_+ \), let \( \mathcal{P}_p(\mathbb{T} \times \mathbb{R}) \) be a collection of all probability measures with finite \( p \)th moment: for some \( z_0 \in \mathbb{T} \times \mathbb{R} \)
\[
\langle \mu, \|z - z_0\|_p^p \rangle < +\infty.
\]
Then, Wasserstein-\( p \)-distance \( W_p(\mu, \nu) \) is defined for any \( \mu, \nu \in \mathcal{P}_p(\mathbb{T} \times \mathbb{R}) \) as
\[
W_p(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu, \nu)} \left( \int_{\mathbb{T} \times \mathbb{R}^2} \|z - z^*\|_p^p d\gamma(z, z^*) \right)^{\frac{1}{p}},
\]
where \( \Gamma(\mu, \nu) \) denotes the collection of all probability measures on \( \mathbb{T} \times \mathbb{R}^2 \) with marginals \( \mu \) and \( \nu \).

2. If \( \lim_{p \to \infty} W_p \) exists, then we define \( W_\infty \) metric as the limit.

3. For any \( T \in (0, \infty] \), the kinetic equation (38) is derivable from the particle model (4) in \([0, T)\), or equivalent to say the mean-field limit from the particle system (4) to the kinetic equation (38), which is valid in \([0, T)\), if for every solution \( \mu_t \) of the kinetic equation (38) with initial data \( \mu_0 \), the following condition holds: for some \( p \in [1, \infty) \) and \( t \in [0, T) \),
\[
\lim_{N \to +\infty} W_p(\mu_0^N, \mu_0) = 0 \iff \lim_{N \to +\infty} W_p(\mu_t^N, \mu_t) = 0,
\]
where \( \mu_t^N \) is a measure valued solution of the particle system (4) with initial data \( \mu_0^N \).

For later use, we quote two results on the approximation of a measure by empirical measures and mean-field limit in any finite time interval without proofs.

**Proposition 2.** [40] For any given \( p \in [1, \infty) \) and \( \mu \in \mathcal{P}_p(\mathbb{T} \times \mathbb{R}) \) with compact support, there exists a sequence of empirical measures \( \mu^N \in \mathcal{P}_p(\mathbb{T} \times \mathbb{R}) \) such that \( \mu^N \) has a common compact support with \( \mu \) and \( \lim_{N \to +\infty} W_p(\mu^N, \mu) = 0 \).

**Remark 5.** The construction of the approximation can be followed by the method of Theorem 6.18 in the book [40] by finding a sequence of atomic measures \( \sum_{j=1}^{N} a_j \delta_{j} \) with rational numbers \( a_j \) such that \( \sum_{j=1}^{N} a_j = 1 \).

### 5.2. A uniform mean-field limit

In this subsection, we present a uniform mean-field limit to the kinetic equation (38). We basically follow the approach given in [21], Corollary 1 and Lemma 3.2.
Theorem 5.3. Suppose that the initial probability measure $\mu_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R})$ and coupling strength satisfy

$$D_\Theta^{\mu_0} \leq D_\infty^\mu < \frac{\pi}{2}, \quad \int_{\mathbb{T} \times \mathbb{R}} \omega \mu_0(d\theta, d\omega) = 0, \quad \int_{\mathbb{T} \times \mathbb{R}} \mu_0(d\theta, d\omega) \leq m_0,$$

where $D_\Theta^{\mu_0}$ and $D_\infty^\mu$ are diameters of the projected supports of $\mu_0$ in $\theta$ and $\omega$-spaces.

Then, the following assertions hold: for $p \in [1, \infty)$,

1. There exists a unique measure-valued solution $\mu_t \in L^\infty([0, \infty); \mathcal{P}(\mathbb{T} \times \mathbb{R}))$ to (38) with initial data $\mu_0$ such that $\mu_t$ is approximated by empirical measure $\mu_t^n$ in Wasserstein-$p$ distance uniformly in time:

$$\lim_{N \to +\infty} \sup_{t \in [0, +\infty)} W_p(\mu_t^n, \mu_t) = 0.$$

2. Suppose that $\nu_t$ is the measure-valued solution to (38) with initial measure $\nu_0$ which has the same property in (39). Then there exists nonnegative constant $G$ independent of $t$ such that

$$W_p(\mu_t, \nu_t) \leq GW_p(\mu_0, \nu_0), \quad t \in [0, \infty).$$

Proof. Since the overall proof of Theorem 5.3 is almost the same as that of Corollary 1.1 in [21], we will provide only sketch of the proof.

- Step A (Extraction of Cauchy approximation for $\mu_0$ in $W_p$). We take a sequence of empirical measures $\mu_0^n$ that approximate $\mu_0$ satisfying

$$\lim_{N \to +\infty} W_p(\mu_0^n, \mu_0) = 0. \quad (40)$$

The existence of such approximation is guaranteed by [40]. Then, owing to (40), for any $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that

$$W_p(\mu^n_0, \mu_0^m) < \varepsilon, \quad \text{for } n, m > N(\varepsilon).$$

- Step B (Approximation of $W_p(\mu_0^n, \mu_0^m)$). Using the argument used in the proof of the Corollary 1.1 in [21], we can find a natural number $M_{mn}$ such that

$$\left| W_p(\mu_0^n, \mu_0^m) - \frac{1}{M_{mn}} \sum_{k=1}^{M_{mn}} \| z_{k0} - \bar{z}_{k0} \|_p \right| \leq \varepsilon^p, \quad (41)$$

where, $z_{k0} := (\theta_{k0}, \omega_{k0})$ and $\bar{z}_{k0} := (\bar{\theta}_{k0}, \bar{\omega}_{k0})$ are support of initial approximated empirical measures $\mu_0^n$ and $\mu_0^m$ respectively.

- Step C (Lifting the information at time $s = 0$ to $s = t > 0$). Now, using (41) and the previous $\ell_p$-stability in particle level, Theorem 4.3, we can directly estimate $W_p(\mu_t^n, \mu_t^m)$ as

$$W_p(\mu_t^n, \mu_t^m) \leq 2^{p-1} G^p (W_p(\mu_0^n, \mu_0^m) + \varepsilon^p) \leq 2^{p} G^p \varepsilon^p. \quad (42)$$

which implies that the sequence $\mu_t^n$ is Cauchy in $W_p$-metric. Thus, we can find a limit measure $\mu_t$. We next apply similar arguments in [15] and show that the limit measure $\mu_t$ is the unique measure-valued solution of the kinetic equation (38) with
initial data \( \mu_0 \). Moreover, because of the estimate (42), we can conclude that for any \( \varepsilon \), there exists a positive constant \( L \), such that
\[
\sup_{t \in [0,+\infty)} W_p(\mu^n_t, \mu_t) \leq 4G\varepsilon, \quad \text{for} \quad n > L.
\]
This yields
\[
\lim_{N \to +\infty} \sup_{t \in [0,+\infty)} W_p(\mu^N_{t}, \mu_{t}) = 0. \quad (43)
\]
The uniform compact support of \( \mu_t \) follows this uniform convergence.

- Step D (Uniform stability of kinetic equation). For measures \( \mu_0 \) and \( \nu_0 \) in \( \mathcal{P}(\mathbb{T} \times \mathbb{R}) \), let \( \mu \) and \( \nu \) be measure-valued solutions to (38). Then, it follows from (43) that for any \( \varepsilon \ll 1 \), there exists \( N_0(\varepsilon) \in \mathbb{N} \) such that
\[
W_p(\mu^n_t, \nu^n_t) < \frac{\varepsilon}{2}, \quad W_p(\nu^n_t, \nu) < \frac{\varepsilon}{2} \quad \text{and} \quad n \geq N_0(\varepsilon).
\]
Then, we use the above estimates and (42) to obtain
\[
W_p(\mu_t, \nu_t) \leq \left( W_p(\mu_t, \mu^n_t) + W_p(\mu^n_t, \nu^n_t) + W_p(\nu^n_t, \nu_t) \right)^p \leq \left( \varepsilon + W_p(\mu^n_t, \nu^n_t) \right)^p \leq 2^{p-1} \left( \varepsilon^p + W_p(\mu^n_t, \nu^n_t) \right) \leq 2^{p-1} \left( 2\varepsilon^p + GpW_p(\mu^n_0, \nu^n_0) \right).
\]
Letting \( n \to \infty \), we have
\[
W_p(\mu_t, \nu_t) \leq 2^p \varepsilon^p + 2^{p-1} GpW_p(\mu_0, \nu_0).
\]
Since \( \varepsilon \) was arbitrary, we have the uniform \( W_p \)-stability:
\[
W_p(\mu_t, \nu_t) \leq 2^{p-1} GW_p(\mu_0, \nu_0), \quad t \geq 0.
\]

**Remark 6.** The same arguments can be applied for the mean-field limit for the Kuramoto model to the corresponding kinetic equation uniform in time in the class of synchronizing solutions in the next section.

As a direct application of Theorem 5.3, we have the following synchronization estimate for the measure-valued solutions to (38).

**Corollary 3.** Suppose that the assumptions (39) hold, and let \( \mu_t \) be a measure-valued solution to (38) whose existence is guaranteed by Theorem 5.3. Then, we have the complete frequency synchronization:
\[
\left( \int_{\mathbb{T} \times \mathbb{R}} |\omega|^p \, d\mu_t \right)^\frac{1}{p} \leq Ce^{-\kappa(cos D^\infty)t} \left( \int_{\mathbb{T} \times \mathbb{R}} |\omega|^p \, d\mu_0 \right)^\frac{1}{p}.
\]

**Proof.** Let \( \mu^n_t \) be a sequence of empirical measures appearing in the course of the proof of Theorem 3.2. Then, it follows from Theorem 3.2 that we have
\[
\left( \int_{\mathbb{R}^{2d}} |\omega|^p \, d\mu^n_t \right)^\frac{1}{p} \leq e^{-\kappa(cos D^\infty)t} \left( \int_{\mathbb{R}^{2d}} |\omega|^p \, d\mu_0^n \right)^\frac{1}{p}, \quad (44)
\]
On the other hand, since $\mu_N^t$ has a common compact support, we can view $||V||^p_p$ as a test function. Then, due to Theorem 5.3, we have,
\[
\lim_{N \to 0} W_p(\mu_N^t, \mu_t) = 0.
\]
This implies the weak convergence of $\mu_N^t$ to $\mu_t$. Thus, we can pass to the limit $N \to \infty$ to (44) to obtain
\[
\left( \int_{\mathbb{R}^{2d}} |\omega|^p \, d\mu_t \right)^{\frac{1}{p}} \leq e^{-\kappa(\cos D^\infty)} t \left( \int_{\mathbb{R}^{2d}} |\omega|^p \, d\mu_0 \right)^{\frac{1}{p}}.
\]

5.3. Complete synchronization estimate. In this subsection, we present an alternative approach for the complete synchronization estimate for (38) introduced in previous subsection. As mentioned in abstract, the kinetic equation (38) for the AKM is more suitable for the Lyapunov functional approach, compared to the kinetic Kuramoto equation for the KM with distributed natural frequencies. For simplicity of presentation, we suppress $t$-dependence in $f$:
\[
f(\theta, \omega) := f(\theta, \omega, t), \quad \theta \in [0, 2\pi], \quad \omega \in \mathbb{R}.
\]

Lemma 5.4. Let $f$ be a classical solution of (38) whose support is compact. Then, we have
\[
\frac{d}{dt} \int_0^{2\pi} \int_{-\infty}^{\infty} f \, d\omega \, d\theta = 0, \quad \frac{d}{dt} \int_0^{2\pi} \int_{-\infty}^{\infty} \omega f \, d\omega \, d\theta = 0, \quad t > 0.
\]

Proof. It directly comes from multiplying by 1 and $\omega$ to (38) and integrating the resulting relation over the phase and frequency space, hence we omit the detailed calculation.

Next, we discuss the derivation of the complete (frequency) synchronization estimate. For this, we use the Lyapunov functional defined as follows.
\[
\Lambda[f(t)] := \int_0^{2\pi} \int_{-\infty}^{\infty} |\omega - \omega_c|^2 f(\theta, \omega) \, d\omega \, d\theta, \quad \omega_c := \frac{\int_0^{2\pi} \int_{-\infty}^{\infty} \omega f \, d\omega \, d\theta}{\int_0^{2\pi} \int_{-\infty}^{\infty} f \, d\omega \, d\theta},
\]
where $\omega_c$ is the mean frequency which is constant due to Lemma 5.4.

If complete synchronization occurs, it is natural to expect that frequency will converge to $\omega_c$, i.e., the Lyapunov functional $\Lambda[f]$ converges to 0. To show this, we will use the standard Lyapunov functional estimate on $\Lambda(f)$.

Theorem 5.5. Let $f$ be a classical solution of (38) whose support is compact and initial datum $f^0$ satisfying
\[
D^\Theta_0 \leq D^\infty < \frac{\pi}{2},
\]
where $D^\Theta_0$ is the diameter of support of $f_0$ projected to $\theta$-space. Then, if the coupling strength $\kappa$ is large enough, the Lyapunov functional $\Lambda[f]$ decays exponentially:
\[
\Lambda[f(t)] \leq \Lambda[f^0] e^{-2\kappa(\cos D^\infty) ||f_0||_{L^1} t}, \quad as \ t \to \infty.
\]

Proof. It follows from Lemma 3.1 and condition for support of initial data that we have
\[
D^\Theta(0) \leq D^\infty < \frac{\pi}{2}.
\]
where $D_\theta(t)$ is the diameter of support of $f(\cdot, \cdot, t)$. Now we use (45) and use the periodicity of $f$ in $\theta$-variable to obtain

$$\frac{d}{dt} \Lambda[f] = \int_0^{2\pi} \int_\mathbb{R} (\omega - \omega_c)^2 \partial_\omega f d\omega d\theta$$

$$= - \int_0^{2\pi} \int_\mathbb{R} (\omega - \omega_c)^2 \omega \partial_\theta f d\omega d\theta - \int_0^{2\pi} \int_\mathbb{R} (\omega - \omega_c)^2 \partial_\omega (\Lambda[f] f) d\omega d\theta$$

$$= \int_0^{2\pi} \int_\mathbb{R} 2(\omega - \omega_c) (\Lambda[f] f) d\omega d\theta$$

$$= 2\kappa \int_{[0,2\pi]^2 \times \mathbb{R}^2} \cos(\theta_* - \theta)(\omega_* - \omega_c)(\omega_* - \omega) f(\theta, \omega) f(\theta_*, \omega_*) d\theta d\omega d\omega_*$$

$$+ 2\kappa \int_{[0,2\pi]^2 \times \mathbb{R}^2} \cos(\theta_* - \theta_\omega)(\omega_* - \omega) f(\theta, \omega) f(\theta_*, \omega_*) d\theta d\omega d\omega_*$$

$$- 2\kappa \int_{[0,2\pi]^2 \times \mathbb{R}^2} \cos(\theta_* - \theta)(\omega_* - \omega_\omega) f(\theta, \omega) f(\theta_*, \omega_*) d\theta d\omega d\omega_*$$

$$= I_1 + I_2.$$ 

Next, we estimate the terms $I_i$ separately.

- (estimate of $I_1$). By interchanging $\omega$ and $\omega_*$, we obtain

$$I_1 = 2\kappa \int_{[0,2\pi]^2 \times \mathbb{R}^2} \cos(\theta_* - \theta)(\omega_* - \omega) f(\theta, \omega) f(\theta_*, \omega_*) d\theta d\omega d\omega_*$$

$$= - \kappa \int_{[0,2\pi]^2 \times \mathbb{R}^2} \cos(\theta_* - \theta)(\omega_* - \omega)^2 f(\theta, \omega) f(\theta_*, \omega_*) d\theta d\omega d\omega_*$$

$$\leq - \kappa \cos D^\infty \int_{[0,2\pi]^2 \times \mathbb{R}^2} ((\omega_* - \omega_c) - (\omega - \omega_c))^2 f(\theta, \omega) f(\theta_*, \omega_*) d\theta d\omega d\omega_*$$

$$= - \kappa \cos D^\infty \int_{[0,2\pi]^2 \times \mathbb{R}^2} (\omega_* - \omega_c)^2 f(\theta, \omega) f(\theta_*, \omega_*) d\theta d\omega d\omega_*$$

$$+ 2\kappa \cos D^\infty \int_{[0,2\pi]^2 \times \mathbb{R}^2} (\omega_* - \omega_c)(\omega - \omega_c) f(\theta, \omega) f(\theta_*, \omega_*) d\theta d\omega d\omega_*$$

$$- \kappa \cos D^\infty \int_{[0,2\pi]^2 \times \mathbb{R}^2} (\omega - \omega_c)^2 f(\theta, \omega) f(\theta_*, \omega_*) d\theta d\omega d\omega_*$$

$$= -2\kappa \cos D^\infty \|f_0\|_{L^1} \Lambda[f],$$

where we use the condition $|\theta_* - \theta| \leq D^\infty$ when $\theta$ and $\theta_*$ are contained in support of $f(\cdot, \cdot, t)$, which is guaranteed by condition on support of initial data.

- (estimate of $I_2$). From the anti-symmetry of integrand, it is easy to see

$$I_2 = -2\kappa \omega_c \int_{[0,2\pi]^2 \times \mathbb{R}^2} \cos(\theta_* - \theta)(\omega_* - \omega) f(\theta, \omega) f(\theta_*, \omega_*) d\theta d\omega d\omega_* = 0.$$ 

From the estimation of $I_i$, $i = 1, 2$, we derive following differential inequality

$$\frac{d}{dt} \Lambda[f] \leq -2\kappa \cos D^\infty \|f_0\|_{L^1} \Lambda[f].$$

By using Grönwall’s lemma, we can obtain the desired exponential decay. \qed
6. Applications to the kinetic Kuramoto Model. In this section, we study the uniform mean-field limit of the Kuramoto model and as a direct application of previous results, we also show the existence of phase-locked states for the kinetic Kuramoto equation via uniform mean-field limit by lifting particle results to the kinetic regime. For the local-in-time stability and mean-field limit of the kinetic Kuramoto equation, we refer to [28].

Let \( f = f(\theta, \nu, t) \) be a one-oscillator probability density function for the ensemble of Kuramoto oscillators. Then, the dynamics of \( f \) is governed by the kinetic Kuramoto equation:

\[
\begin{aligned}
\partial_t f + \partial_\theta (v[f]f) &= 0, \quad (\theta, \nu) \in \mathbb{T} \times \mathbb{R}, \ t > 0, \\
v[f](\theta, \nu, t) &= \nu + \kappa \int_0^{2\pi} \sin(\theta_* - \theta) f(\theta_*, \nu_*, t) \, d\nu_* d\theta_*.
\end{aligned}
\]

(46)

Note that the probability density function \( g = g(\nu) \) for natural frequencies appears as a \( \nu \)-marginal density function of \( f \):

\[
\int_0^{2\pi} f(\theta, \nu, t) \, d\theta = g(\nu).
\]

Unlike to \( (38) \), it is not clear how to show the emergence of the complete frequency synchronization for \( (46) \) using the nonlinear functional approach as in Section 5.3. This is why we introduce a second order model \( (4) \) and its mean-field limit \( (38) \). Similar to Definition 5.2, we can define the measure valued solution of the kinetic Kuramoto equation \( (46) \).

**Definition 6.1.** [7] For \( T \in [0, \infty) \), \( \mu_t \in L^\infty([0, T); \mathcal{P}(\mathbb{T} \times \mathbb{R})) \) is a measure valued solution to \( (46) \) with initial data \( \mu_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R}) \) if the following three assertions hold:

1. Total mass is normalized: \( \langle \mu_t, 1 \rangle = 1 \).
2. \( \mu \) is weakly continuous in \( t \):

\[
\langle \mu_t, f \rangle \text{ is continuous in } t \quad \forall \ f(\theta, \nu) \in C^1_0(\mathbb{T} \times \mathbb{R} \times [0, T]).
\]
3. \( \mu \) satisfies the equation \( (46) \) in a weak sense: for \( \forall \ \varphi \in C^1_0(\mathbb{T} \times \mathbb{R} \times [0, T]) \),

\[
\langle \mu_t, \varphi(\cdot, \cdot, t) \rangle - \langle \mu_0, \varphi(\cdot, \cdot, 0) \rangle = \int_0^t \langle \mu_s, \partial_s \varphi + v(\mu) \partial_\theta \varphi \rangle \, ds.
\]

(47)

**Remark 7.** As mentioned in Remark 4, both the solution of the original Kuramoto model \( (1) \) and the solution of the kinetic equation \( (46) \) can be viewed as a measure valued solution in the sense of Definition 6.1. Thus, we can apply the Wasserstein metric in Definition 5.2 to measure the distance between two measure valued solutions.

According to Proposition 2 and Remark 5, we have the following result.

**Theorem 6.2.** Suppose that initial probability measure \( \mu_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R}) \) and coupling strength satisfy

\[
D^\infty_\Theta \leq D^\infty < \frac{\pi}{2}, \quad \int_0^{2\pi} \nu \mu_0(d\theta, d\nu) = 0, \quad \int_{\mathbb{T} \times \mathbb{R}} \mu_0(d\theta, d\nu) \leq m_0,
\]

\[
\int_{\mathbb{T} \times \mathbb{R}} (|\theta|^p + |\nu|^p) \mu_0(d\theta, d\nu) \leq m_2, \quad \kappa > \frac{D^\infty_\Theta}{\cos(D^\infty)(D^\infty - D^\infty_\Theta)}.
\]

(48)

Then, the following assertions hold. For \( p \in [1, \infty) \),
1. There exists a unique measure-valued solution $\mu_t \in L^\infty([0, \infty); \mathcal{P}(T \times \mathbb{R}))$ to (46) with initial data $\mu_0$ such that $\mu_t$ is approximated by empirical measure $\mu_t^N$ in Wasserstein-$p$ distance uniformly in time:

$$\lim_{N \to +\infty} \sup_{t \in [0, +\infty)} W_p(\mu_t^N, \mu_t) = 0.$$ 

2. Moreover, if $\bar{\mu}_t$ is the measure-valued solution to (46) with initial measure $\bar{\mu}_0$ with compact support and finite moments (48), then there exists nonnegative constant $G$ independent of $t$ such that

$$W_p(\mu_t, \bar{\mu}_t) \leq GW_p(\mu_0, \bar{\mu}_0), \quad t \in [0, \infty).$$

**Proof.** The construction of the proof is similar to Theorem 5.3. In fact, as the distribution of natural frequency $\nu$ does not have its own dynamics, i.e., it does not change in time, thus the variance of $\nu$ between $\mu$ and $\bar{\mu}$ will be a constant, i.e.,

$$\inf_{\gamma \in \Gamma(\mu_0, \bar{\mu}_0)} \int_{T^2 \times \mathbb{R}^2} |\nu - \bar{\nu}|^p \gamma(\nu, \bar{\nu}) = \inf_{\gamma \in \Gamma(\mu_0, \bar{\mu}_0)} \int_{T^2 \times \mathbb{R}^2} |\nu - \bar{\nu}|^p \gamma(\nu, \bar{\nu}).$$

Therefore, we only need to control the variance on $\theta$. Applying the uniform stability in Corollary 2, we can construct the uniform mean-field limit and stability for kinetic Kuramoto model (46) as same as Theorem 5.3.

Now, for large time behavior, we can apply Corollary 1 to the approximate solution $\mu_t^N$. Notice that the decay rate in Corollary 1 is independent of $N$. Hence, the mean-field limit preserves the decay rate, when $N$ tends to infinity.

**Corollary 4.** (Emergence of a phase-locked state) Suppose that the initial data $\mu_0$ and coupling strength satisfy the assumptions (39). Then, there exists a phase-locked state $\mu_\infty$ such that

$$W_p(\mu_t, \mu_\infty) \leq Ce^{-\kappa D^\infty t}, \quad t \to \infty.$$ 

**Proof.** It follows from Corollary 1 that for each $\mu_t^N$, we have a unique asymptotic equilibrium $\mu_\infty^N$. Then from the uniform stability in Corollary 2, we can obtain the sequence $\{\mu_\infty^N\}$ is Cauchy, and thus generates a unique limit measure $\mu_\infty$. Moreover, it follows from Corollary 1 that we have

$$W_p(\mu_t^N, \mu_\infty^N) \leq Ce^{-\kappa D^\infty t}.$$

Notice here the $p$-th moment of $\nu$ would be cancelled because $\mu_t^N$ and $\mu_\infty^N$ has the same natural frequency distribution $\int \nu \mu_\infty^N(d\theta, d\Omega)$. Now for any $\varepsilon > 0$, we can find $N_0$ large enough such that, for $N \geq N_0$ we have

$$W_p(\mu_t, \mu_\infty) \leq W_p(\mu_t, \mu_t^N) + W_p(\mu_t^N, \mu_\infty^N) + W_p(\mu_\infty^N, \mu_\infty) \leq 2\varepsilon + Ce^{-\kappa D^\infty t}.$$

Thus, we have

$$W_p(\mu_t, \mu_\infty) \leq Ce^{-\kappa D^\infty t}.$$

**7. Conclusion.** We presented the dynamic properties of the augmented Kuramoto model which is a second-order lifting of the Kuramoto model for synchronization. For the particle Kuramoto model with distributed natural frequencies, the complete (frequency) synchronization can be studied by analyzing the temporal evolution of $D(\Theta)$. However, it is not clear how to verify the complete synchronization for the corresponding mean-field kinetic equation directly. This is why we introduced a
second-order lifting of the Kuramoto model. For the corresponding kinetic equation, the complete frequency synchronization can be obtained via the Lyapunov functional measuring the dispersion of the frequency variations. Our proposed second-order model has a formal similarity to the Cucker-Smale flocking model. As long as the phase diameter is confined in a quarter arc, the flocking estimate, uniform ℓ∞-stability and mean-field limit for the extended Kuramoto model can be analyzed using the similar techniques done for the Cucker-Smale model. As aforementioned in Introduction, the reason that we focus on the mean-field case (all-to-all couplings) is that we are interested in the complete synchronization estimate for the kinetic Kuramoto equation at the kinetic level without lifting particle results to the kinetic level. Other than this, some of the estimates studied in this paper can be extended to a more general setting. For example, we considered synchronization dynamics of Kuramoto oscillators on the complete network, but some estimates such as the uniform stability and synchronization estimates at the particle level can be certainly extended to the locally coupled Kuramoto oscillators over the general networks, say symmetric and connected networks. Moreover, nonlinear stability and instability of the incoherent state can be studied for the proposed kinetic equation. We leave these interesting issues as a future work.

REFERENCES

[1] J. A. Acebron, L. L. Bonilla, C. J. P. Pérez Vicente, F. Ritort and R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, Rev. Mod. Phys., 77 (2005), 137–185.
[2] D. Aeyels and J. Rogge, Existence of partial entrainment and stability of phase-locking behavior of coupled oscillators, Prog. Theor. Phys., 112 (2004), 921–941.
[3] D. Benedetto, E. Caglioti and U. Montemagno, Exponential dephasing of oscillators in the kinetic Kuramoto model. J. Stat. Phys., 162 (2016), 813–823.
[4] D. Benedetto, E. Caglioti and U. Montemagno, On the complete phase synchronization for the Kuramoto model in the mean-field limit, Commun. Math. Sci., 13 (2015), 1775–1786.
[5] J. Bronski, L. Deville and M. J. Park, Fully synchronous solutions and the synchronization phase transition for the finite-N Kuramoto model, Chaos, 22 (2012), 033133, 17pp.
[6] J. Buck and E. Buck, Biology of synchronous flashing of fireflies, Nature, 211 (1966), 562–564.
[7] J. A. Carrillo, Y.-P. Choi, S.-Y. Ha, M.-J. Kang and Y. Kim, Contractivity of transport distances for the kinetic Kuramoto equation, J. Stat. Phys., 156 (2014), 395–415.
[8] L. Casetti, M. Pettini and E. G. D. Cohen, Phase transitions and topology changes in conﬁguration space, J. Statist. Phys., 111 (2003), 1091–1123.
[9] Y.-P. Choi, S.-Y. Ha, S. Jung and Y. Kim, Asymptotic formation and orbital stability of phase-locked states for the Kuramoto model, Physica D, 241 (2012), 735–754.
[10] N. Chopra and M. W. Spong, On exponential synchronization of Kuramoto oscillators, IEEE Trans. Automatic Control, 54 (2009), 353–357.
[11] J.-G. Dong and X. Xue, Synchronization analysis of Kuramoto oscillators, Commun. Math. Sci., 11 (2013), 465–480.
[12] F. Dörfler and F. Bullo, Synchronization in complex networks of phase oscillators: A survey, Automatica, 50 (2014), 1539–1564.
[13] F. Dörfler and F. Bullo, On the critical coupling for Kuramoto oscillators, SIAM. J. Appl. Dyn. Syst., 10 (2011), 1070–1099.
[14] G. B. Ermentrout, Synchronization in a pool of mutually coupled oscillators with random frequencies, J. Math. Biol., 22 (1985), 1–9.
[15] S.-Y. Ha and J.-G. Liu, A simple proof of Cucker-Smale flocking dynamics and mean field limit, Commun. Math. Sci., 7 (2009), 297–325.
[16] S.-Y. Ha and E. Tadmor, From particle to kinetic and hydrodynamic description of flocking, Kinet. and Relat. Model., 1 (2008), 415–435.
[17] S.-Y. Ha, T. Y. Ha and J.-H. Kim, On the complete synchronization for the globally coupled Kuramoto model, Physica D, 239 (2010), 1692–1700.
[18] S.-Y. Ha, H. K. Kim and J.-Y. Park, Remarks on the complete synchronization of Kuramoto oscillators, *Nonlinearity*, 28 (2015), 1441–1462.
[19] S.-Y. Ha, H. K. Kim and S. W. Ryoo, Emergence of phase-locked states for the Kuramoto model in a large coupling regime, *Commun. Math. Sci.*, 4 (2016), 1073–1091.
[20] S.-Y. Ha, D. Ko, J. Park and X. Zhang, Collective synchronization of classical and quantum oscillators, *EMS Surveys in Mathematical Sciences*, 3 (2016), 209–267.
[21] S.-Y. Ha, J. Kim and X. Zhang, Uniform stability of the Cucker-Smale model and its application to the mean-field limit, To appear in Kinet. and Relat. Model.
[22] S.-Y. Ha, C. Lattanzio, B. Rubino and M. Slemrod, Flocking and synchronization of particle models, *Quart. Appl. Math.*, 69 (2011), 91–103.
[23] S.-Y. Ha and M. Slemrod, A fast-slow dynamical systems theory for the Kuramoto type phase model, *J. Differential Equations*, 251 (2011), 2685–2695.
[24] S.-Y. Ha, Z. Li and X. Xue, Formation of phase-locked states in a population of locally interacting Kuramoto oscillators, *J. Differential Equations*, 255 (2013), 3053–3070.
[25] A. Jadbabaie, N. Motee and M. Barahona, On the stability of the Kuramoto model of coupled nonlinear oscillators, *Proceedings of the American Control Conference*, (2014), 4296–4301.
[26] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence*, Springer-Verlag, Berlin, 1984.
[27] Y. Kuramoto, International symposium on mathematical problems in mathematical physics, *Lecture Notes in Theoretical Physics*, 30 (1975), 420.
[28] C. Lancellotti, On the vlasov limit for systems of nonlinearly coupled oscillators without noise, *Transport Theory and Statistical Physics*, 34 (2005), 523–535.
[29] D. Mehta, N. S. Daleo, F. Dörfler and J. D. Hauenstein, Algebraic geometrization of the Kuramoto model: Equilibria and stability analysis, *Chaos*, 25 (2015), 053103, 7pp.
[30] R. Miroslav and S. H. Strogatz, The spectrum of the partially locked state for the Kuramoto model, *J. Nonlinear Science*, 17 (2007), 309–347.
[31] R. Miroslav and S. H. Strogatz, The spectrum of the locked state for the Kuramoto model of coupled oscillators, *Physica D*, 205 (2005), 249–266.
[32] R. Miroslav and S. H. Strogatz, Stability of incoherence in a population of coupled oscillators, *J. Stat. Phys.*, 63 (1991), 613–635.
[33] H. Neunzert, An introduction to the nonlinear boltzmann-vlasov equation, In *Kinetic Theories and the Boltzmann Equation (Montecatini, 1981)*, 60–110, Lecture Notes in Math., 1048, Springer, Berlin, 1984.
[34] A. Pikovsky, M. Rosenblum and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*, Cambridge University Press, Cambridge, 2001.
[35] S. H. Strogatz, From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators, *Physica D*, 143 (2000), 1–20.
[36] J. L. van Hemmen and W. F. Wreszinski, Lyapunov function for the Kuramoto model of nonlinearly coupled oscillators, *J. Stat. Phys.*, 72 (1993), 145–166.
[37] M. Verwoerd and O. Mason, A convergence result for the Kurmoto model with all-to-all couplings, *SIAM J. Appl. Dyn. Syst.*, 10 (2011), 906–920.
[38] M. Verwoerd and O. Mason, On computing the critical coupling coefficient for the Kuramoto model on a complete bipartite graph, *SIAM J. Appl. Dyn. Syst.*, 8 (2009), 417–453.
[39] M. Verwoerd and O. Mason, Global phase-locking in finite populations of phase-coupled oscillators, *SIAM J. Appl. Dyn. Syst.*, 7 (2008), 134–160.
[40] C. Villani, *Optimal Transport, Old and New*, Springer-Verlag, Berlin, 2009.
[41] A. T. Winfree, Biological rhythms and the behavior of populations of coupled oscillators, *J. Theor. Biol.*, 16 (1967), 15–42.

Received July 2017; revised March 2018.

*E-mail address: syha@snu.ac.kr*
*E-mail address: jhkim206@snu.ac.kr*
*E-mail address: jinyeongpark@hanyang.ac.kr*
*E-mail address: xtzhang@hust.edu.cn*