NONSTABLE $K$–THEORY FOR EXTENSION ALGEBRAS OF THE SIMPLE PURELY INFINITE $C^*$–ALGEBRA BY CERTAIN $C^*$–ALGEBRAS

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Abstract. Let $0 \rightarrow B \stackrel{j}{\rightarrow} E \stackrel{\pi}{\rightarrow} A \rightarrow 0$ be an extension of $A$ by $B$, where $A$ is a unital simple purely infinite $C^*$–algebra. When $B$ is a simple separable essential ideal of the unital $C^*$–algebra $E$ with $RR(B) = 0$ and (PC), $K_0(E) = \{[p] \mid p$ is a projection in $E \setminus B\}$; When $B$ is a stable $C^*$–algebra, $\mathcal{U}(C(X, E))/\mathcal{U}_0(C(X, E)) \cong K_1(C(X, E))$ for any compact Hausdorff space $X$.

Keywords $K$–groups; simple purely infinite $C^*$–algebra; real rank zero.
2000 MR Subject Classification 46L05.

1. Introduction

Let $\mathcal{E}$ be a $C^*$–algebra. Denote by $M_n(\mathcal{E})$ the $C^*$–algebra of all $n \times n$ matrices over $\mathcal{E}$. If $\mathcal{E}$ is unital, write $\mathcal{U}(\mathcal{E})$ to denote the unitary group of $\mathcal{E}$ and $\mathcal{U}_0(\mathcal{E})$ to denote the connected component of the unit in $\mathcal{U}(\mathcal{E})$. Put $U(\mathcal{E}) = \mathcal{U}(\mathcal{E})/\mathcal{U}_0(\mathcal{E})$. If $\mathcal{E}$ has no unit, we set $U(\mathcal{E}) = \mathcal{U}(\mathcal{E}^+)/\mathcal{U}_0(\mathcal{E}^+)$, where $\mathcal{E}^+$ is the $C^*$–algebra obtained by adding a unit to $\mathcal{E}$. Two projections $p$, $q$ in $\mathcal{E}$ are equivalent, denoted $p \sim q$, if $p = v^*v, q = vv^*$ for some $v \in \mathcal{E}$. Let $[p]$ denote the equivalence of $p$ with respect to $\sim$. Let $p$, $r$ be projections in $\mathcal{E}$. $[p] \leq [r]$ (resp. $[p] < [r]$) means that there is projection $q \leq r$ (resp. $q < r$) such that $p \sim q$. A projection $p$ in $\mathcal{E}$ is called to be infinite, if $[p] < [p]$. The simple $C^*$–algebra $\mathcal{E}$ is called to be purely infinite if every nonzero hereditary subalgebra of $\mathcal{E}$ contains an infinite projection.

Let $K_0(\mathcal{E})$ and $K_1(\mathcal{E})$ be the $K$–groups of the $C^*$–algebra $\mathcal{E}$ and let $i_\mathcal{E}: U(\mathcal{E}) \rightarrow K_1(\mathcal{E})$ be the canonical homomorphism (cf. [1]).

The main tasks in non–stable $K$–theory are how to use the projection in $\mathcal{E}$ to represent $K_0(\mathcal{E})$ and how to show $i_\mathcal{E}$ is isomorphic. Cuntz showed in [2] that $K_0(\mathcal{E}) \cong \{[p] \mid p \in \mathcal{E}$ nonzero projection\} and $i_\mathcal{E}$ is isomorphic, when $\mathcal{E}$ is a simple unital purely infinite $C^*$–algebra. Rieffel and Xue proved
that under some restrictions of stable rank on the $C^\ast$–algebra $E$, $i_E$ may be injective, surjective or isomorphic (cf. [6, 7], [12]).

Let $B$ be a closed ideal of a unital $C^\ast$–algebra $E$. Let $\pi: E \rightarrow E/B = A$ be the quotient map. We will use these symbols $E$, $B$, $A$ and $\pi$ throughout the paper. Liu and Fang proved in [5] that

1. $K_0(E) = \{[p]| p \text{ is a projection in } E\B\}$ and

2. $i_E: U(E) \rightarrow K_1(E)$ is isomorphic.

when $B = K$ (the algebra of compact operators on some separable Hilbert space) and $A$ is a unital simple purely infinite $C^\ast$–algebra. Visinescu showed in [10] that the above results are also true when $B$ is purely infinite.

In this short note, we show that (1) is true when $B$ is a separable simple $C^\ast$–algebra with $\text{RR}(B) = 0$ and (PC) (see §2 below) and $A$ is unital simple purely infinite; We also prove that $i_{C(X,E)}$ is isomorphic for any compact Hausdorff space $X$ when $B$ is stable and $A$ is unital simple purely infinite.

2. $K_0$–GROUP OF THE EXTENSION ALGEBRA

Let $E$ be a $C^\ast$–algebra. $E$ is of real rank zero, denoted by $\text{RR}(E) = 0$, if every self–adjoint element in $E$ can be approximated by a self–adjoint element in $E$ with finite spectra (cf. [3]). A non–unital, $\sigma$–unital $C^\ast$–algebra $E$ with $\text{RR}(E) = 0$ is said to have property (PC) if it has finitely many (densely defined) traces, say $\{\tau_1, \ldots, \tau_k\}$ such that following conditions are satisfied:

1. there is an approximate unit $\{e_n\}$ of $E$ consisting of projections such that $\lim_{n \to \infty} \tau_i(e_n) = \infty$, $i = 1, \ldots, k$;

2. for two projections $p, q \in E$, if $\tau_i(p) < \tau_i(q)$, $i = 1, \ldots, k$, then $[p] \leq [q]$.

Obviously, stable simple AF–algebras with only finitely many extremal traces have (PC) and $A_\theta \otimes K$ also has (PC), where $A_\theta$ is the irrational rotation algebra and $K$ is the algebra of compact operators on some complex separable Hilbert space.

**Remark 2.1.** Let $E$ be a non–unital, $\sigma$–unital $C^\ast$–algebra with $\text{RR}(E) = 0$ and (PC). Let $\{f_n\}$ be an approximate unit of $E$ consisting of increased projections. Suppose $\lim_{n \to \infty} \tau_i(e_n) = \infty$, $i = 1, \ldots, k$, for some approximate unit $\{e_n\}$ of $E$ consisting of projections. Then there $\{e_{n_j}\} \subset \{e_n\}$ such that $\tau_i(e_{n_j}) > j$, $j \geq 1$, $i = 1, \ldots, k$. Since $\lim_{s \to \infty} \|fs_{e_{n_j}} f_s - e_{n_j}\| = 0$, $j \geq 1$, we can find projections $f_{s_j} \leq f_s$ for $s$ large enough such that $f_{s_j} \sim e_{n_j}$, $j \geq 1$. Then

$$\tau_i(f_s) \geq \tau_i(f_{s_j}) = \tau_i(e_{n_j}) > j, \quad i = 1, \ldots, k,$$
so that \( \lim_{n \to \infty} \tau_i(f_n) = \infty, \ i = 1, \cdots, k. \)

With symbols as above, we can extend \( \tau_i \) to \( M(\mathcal{E}) \) by \( \tau_i(x) = \sup_{n \geq 1} \tau_i(f_n x f_n) \)

for positive element \( x \in M(\mathcal{E}) \) (cf. \[11, \text{P324}] \), \( i = 1, \cdots, k \), where \( M(\mathcal{E}) \) is the multiplier algebra of \( \mathcal{E} \).

**Lemma 2.2.** Suppose that \( \mathcal{B} \) is an essential ideal of \( E \) and \( A, \mathcal{B} \) are simple. Then every positive element in \( E \setminus \mathcal{B} \) is full.

**Proof.** Let \( a \in E \setminus \mathcal{B} \) with \( a \geq 0 \) and let \( I(a) \) be closed ideal generated by \( a \) in \( E \). Since \( \pi(I(a)) \) is a nonzero closed ideal in \( A \) and \( A \) is simple, we get that \( 1_A \in \pi(I(a)) \) and hence there is \( x \in \mathcal{B} \) such that \( 1_E + x \in I(a) \). Since \( \mathcal{B} \) is an essential ideal, it follows that \( aB a \neq \{0\} \). Choose a nonzero element \( b \in aB a \subset I(a) \). Since \( \mathcal{B} \) is simple, \( x \) is in the closed ideal of \( \mathcal{B} \) generated by \( b \). Thus, \( x \in I(a) \) and consequently, \( 1_E \in I(a) \). \( \Box \)

The following lemma slightly improves Lemma 2.1 of \[10\], whose proof is essentially same as it in \[11\] Lemma 3.2 and \[10\] Lemma 2.1.

**Lemma 2.3.** Suppose that \( \text{RR}(\mathcal{B}) = 0 \). Let \( p, q \) be projections in \( E \) and assume that there is \( v \in A \) such that \( \pi(p) = v^*v \) and \( vv^* \leq \pi(q) \) in \( A \). Then there is a projection \( e \in pBp \) and a partial isometry \( u \in E \) such that \( p - e = u^*u \), \( uu^* \leq q \) and \( \pi(u) = v \).

**Proof.** Let \( v \in A \) such that \( \pi(p) = v^*v \), \( vv^* \leq \pi(q) \). Choose \( u_0 \in E \) such that \( \pi(u_0) = v \) and set \( w = qu_0p \). Then \( \pi(w^*w) = \pi(p) \), \( \pi(w) = v \). Thus, \( p - w^*w \in pBp \). Since \( \text{RR}(\mathcal{B}) = 0 \), \( pBp \) has an approximate unit consisting of projections. So there is a projection \( e \in pBp \) such that

\[
\|(p - e)(p - w^*w)(p - e)\| = \|(p - e) - (p - e)w^*w(p - e)\| < 1.
\]

Then \( z = (p - e)w^*w(p - e) \) is invertible in \( (p - e)E(p - e) \) and \( \pi(z) = \pi(p) \). Let \( s = ((p - e)w^*w(p - e))^{-1} \), i.e., \( zs = sz = p - e \). Then \( \pi(s) = \pi(p) \). Put \( u = ws^{1/2} \). Then \( uu^* = wsw^* \leq q \), \( \pi(u) = v \) and

\[
\begin{align*}
    u^*u &= s^{1/2}w^*ws^{1/2} = s^{1/2}(p - e)w^*w(p - e)s^{1/2} \\
    &= (p - e)w^*w(p - e)s = p - e.
\end{align*}
\]

\( \Box \)

**Lemma 2.4.** Suppose that \( A \) is unital simple purely infinite and \( \mathcal{B} \) is an essential ideal of a unital \( C^* \)-algebra \( E \), moreover \( \mathcal{B} \) is separable simple with \( \text{RR}(\mathcal{B}) = 0 \) and \( \text{PC} \). Let \( p, q \) be projections in \( E \setminus \mathcal{B} \) and let \( r \) be a nonzero projection in \( pBp \). Then there is a projection \( r' \) in \( qBq \) such that \( |r| \leq |r'| \).
Proof. Since $\mathcal{B}$ has (PC), there are densely defined traces $\tau_1, \cdots, \tau_k$ on $\mathcal{B}$ and an approximate unit $\{f_n\}$ of $\mathcal{B}$ consisting of increased projections such that $\lim_{n \to \infty} \tau_i(f_n) = \infty$, $i = 1, \cdots, k$ and $\tau_i(e) < \tau_i(f)$, $i = 1, \cdots, k$ implies that $[e] \leq [f]$ for any two projections $e, f$ in $\mathcal{B}$.

By Lemma $2.2$, there are $x_1, \cdots, x_m \in \mathcal{B}$ such that $\sum_{i=1}^{m} x_i^* q x_i = 1_E$. We regard $E$ as a $C^*$-subalgebra of $M(\mathcal{B})$ for $\mathcal{B}$ is essential. Thus,

$$\infty = \tau_i(1_E) = \sum_{j=1}^{m} \tau_i(x_j^* q x_j) \leq \sum_{j=1}^{m} \tau_i(\|x_j\|^2 q),$$

i.e., $\tau_i(q) = \infty$, $i = 1, \cdots, k$. Let $r$ be a nonzero projection in $p\mathcal{B}p$. Let $\{g_n\}$ be an approximate unit for $q\mathcal{B}q$ consisting of increased projections. Since $\sup_{n \geq 1} \tau_i(g_n) = \tau_i(q) = \infty$, $i = 1, \cdots, k$, it follows that there is $n_0$ such that $\tau_i(g_{n_0}) > \tau_i(r)$, $i = 1, \cdots, k$. Put $r' = g_{n_0}$. Then we get $[r] \leq [r']$. \hfill $\Box$

Now we can prove the main result of the section as follows:

**Theorem 2.5.** Suppose that $\mathcal{A}$ is unital simple purely infinite and $\mathcal{B}$ is an essential ideal of $E$, moreover $\mathcal{B}$ is separable simple with $\text{RR}(\mathcal{B}) = 0$ and (PC). Then

$$K_0(E) = \{ [p] \mid p \text{ is a projection in } E \setminus \mathcal{B} \}.$$  

Proof. Set $\mathcal{P}(E) = \{ p \text{ is a projection in } E \setminus \mathcal{B} \}$. By [2, Theorem 1.4], when $\mathcal{P}(E)$ satisfies following conditions:

$$(\Pi_1)\text{ If } p, q \in \mathcal{P}(E) \text{ and } pq = 0, \text{ then } p + q \in \mathcal{P}(E);$$

$$(\Pi_2)\text{ If } p \in \mathcal{P}(E) \text{ and } p' \text{ is a projection in } E \text{ such that } p \sim p', \text{ then } p' \in \mathcal{P}(E);$$

$$(\Pi_3)\text{ For any } p, q \in \mathcal{P}(E), \text{ there is } p' \text{ such that } p' \sim p, p' < q \text{ and } q - p' \in \mathcal{P}(E);$$

$$(\Pi_4)\text{ If } q \text{ is a projection in } E \text{ and there is } p \in \mathcal{P}(E) \text{ such that } p \leq q, \text{ then } p \in \mathcal{P}(E),$$

then $K_0(E) = \{ [p] \mid p \in \mathcal{P}(E) \}$. Therefore, we need only check that $\mathcal{P}(E)$ satisfies above conditions.

Let $\mathcal{P}(A)$ be the set of all nonzero projections in $\mathcal{A}$. By [2, Proposition 1.5], $\mathcal{P}(A)$ satisfies $(\Pi_1) \sim (\Pi_4)$. Clearly, $\mathcal{P}(E)$ satisfies $(\Pi_1)$, $(\Pi_2)$ and $(\Pi_4)$. We now show that $\mathcal{P}(E)$ satisfies $(\Pi_3)$.

Let $p, q \in \mathcal{P}(E)$. Then there exists a projection $f \in \mathcal{P}(A)$, such that $f \sim \pi(p)$, $f < \pi(q)$ and $\pi(q) - f \in \mathcal{P}(A)$, that is, there is a partial isometry $v \in A$ such that $f = vv^* < \pi(q)$ and $\pi(p) = v^* v$. Thus, there are $u \in E$ and a projection $r \in p\mathcal{B}p$ such that $p - r = u^* u$, $uu^* \leq q$ and $\pi(u) = v$ by Lemma $2.3$. Note that $q - uu^* \notin \mathcal{B}$ and $(q - uu^*)\mathcal{B}(q - uu^*) \neq \{0\}$ (B is an
we have following exact sequence of groups:

\[ j \quad \text{(cf. [12, lemma 2.2]), where} \ E \quad \text{we say} \ \partial \ \text{(for} \ \pi \ \text{in} \ M_{\pi} \text{w essential ideal). Then by Lemma 2.4, there is}\]

\[ p \quad \text{jection}\]

\[ \text{24.4.3 of [1] that the sequence of groups}\]

\[ \text{functor, it follows from Proposition 21.4.1, Corollary 21.4.2 and Theorem}\]

\[ i\]

\[ \text{is isomorphic and}\]

\[ \partial \quad \text{We also have the exact sequence}\]

\[ \text{Suppose that}\]

\[ \text{Combining (3.1), (3.2) with (3.3), we have following diagram}\]

\[ \text{Proof. Combining (3.1), (3.2) with (3.3), we have following diagram}\]

\[ \eta = \partial_0 \circ i_A, \ \pi_* \circ i_E = i_A \circ \pi_*, \ j_* \circ i_B = i_E \circ j_* .\]
Since \( e_\ast \) is isomorphic, it follows from the commutative diagram
\[
\begin{array}{ccc}
U(SA) & \xrightarrow{i_\ast} & U(C_\pi) \\ \downarrow i_{SA} & & \downarrow i_{C_\pi} \\ K_1(SA) & \xrightarrow{i_\ast} & K_1(C_\pi)
\end{array}
\]
that \( \partial \circ i_{SA} = i_B \circ \partial \). Thus, (3.4) is a commutative diagram. Using the Five–Lemma to (3.4), we can obtain the assertion. \( \square \)

For a \( C^\ast \)–algebra \( \mathcal{E} \), let \( csr(\mathcal{E}) \) and \( gsr(\mathcal{E}) \) be the connected stable rank and general stable rank of \( \mathcal{E} \), respectively, defined in [6]. We summarize some properties of these stable ranks as follows:

**Lemma 3.2.** Let \( \mathcal{E} \) be a \( C^\ast \)–algebra. Then

1. \( gsr(\mathcal{E}) \leq csr(\mathcal{E}) \) (cf. [6]);
2. \( csr(\mathcal{E}) \leq 2 \) when \( \mathcal{E} \) is a stable \( C^\ast \)–algebra (cf. [9, Theorem 3.12]);
3. \( \mathcal{E} \) has 1–cancellation if \( gsr(\mathcal{E}) \leq 2 \) (cf. [12]);
4. if \( csr(\mathcal{E}) \leq 2 \) and \( gsr(C(S^1, \mathcal{E})) \leq 2 \), then \( i_\mathcal{E} \) is isomorphic (cf. [7, Theorem 2.9] or [12, Corollary 2.2]).

Now we present the main result of this section as follows:

**Theorem 3.3.** Assume that \( \mathcal{A} \) is a unital simple purely infinite \( C^\ast \)–algebra and \( \mathcal{B} \) is a stable \( C^\ast \)–algebra. Let \( X \) be a compact Hausdorff space. Then \( i_{C(X, \mathcal{E})} \) is an isomorphism.

**Proof.** If \( \mathcal{B} \) is stable, then so is \( C(Y, \mathcal{B}) \) for any compact Hausdorff space \( Y \). Thus, \( gsr(C(S^1, C(X, \mathcal{B}))) \leq 2 \) and \( csr(C(X, \mathcal{B})) \leq 2 \) by Lemma 3.2 (1) and (2). So we get that \( i_{C(X, \mathcal{B})} \) is isomorphic by Lemma 3.2 (4). Since \( \mathcal{A} \) is unital simple purely infinite, it follows from [12, Corollary 3.1] that \( i_{C(X, \mathcal{A})} \) and \( i_{SC(X, \mathcal{A})} \) are all surjective. Now we prove \( i_{C(X, \mathcal{A})} \) is injective by using some methods appeared in [8].

Let \( f \in \mathcal{U}(C(X, \mathcal{A})) \) with \( i_{C(X, \mathcal{A})}(\langle f \rangle) = 0 \) in \( K_1(C(X, \mathcal{A})) \). Let \( p \) be a non–trivial projection in \( \mathcal{A} \). Then there exists \( g \in \mathcal{U}(C(X, p\mathcal{A})) \) such that \( f \) is homotopic to \( g + 1 - p \) by [13, Lemma 2.7]. Thus, there is a continuous path \( f_t : [0, 1] \to \mathcal{U}(M_{n+1}(C(X, \mathcal{A}))) \) such that \( f_0 = 1_{n+1} \) and \( f_1 = \text{diag}(g + 1 - p, 1_n) \) for some \( n \geq 2 \). Since \( M_{n+1}(\mathcal{A}) \) is purely infinite, we can find a partial isometry \( v = (v_{ij}) \in M_{n+1}(\mathcal{A}) \) such that \( \text{diag}(1 - p, 1_n) = v^*v, vv^* \leq \text{diag}(1 - p, 0) \). Consequently, we get that

\[
v_{11}'v_{11} = 1 - p, \ v_{1j}'v_{1j} = 1, \ v_{1i}'v_{1i} = 0, \ i \neq j, \sum_{i=1}^{n+1} v_{ii}v_{ii}' \leq 1 - p.
\]
Set \( v_1 = p + v_{11}, \) \( v_i = v_{1i}, \) \( i = 2, \cdots, n + 2. \) Then \( v_1, \cdots, v_{n+1} \) are isometries in \( \mathcal{A} \) and \( v_i^* v_j = 0, \) \( i \neq j, \) \( s = \sum_{i=1}^{n+1} v_i v_i^* \) is a projection. Put

\[
w_t(x) = (v_1, \cdots, v_{n+1}) f_t(x) \left( \begin{array}{c} v_1^* \\ v_2^* \\ \vdots \\ v_{n+1}^* \end{array} \right) + 1 - s, \quad t \in [0, 1], \ x \in X.\]

It is easy to check that \( w_t \) is a continuous path in \( \mathfrak{U}(M_n(\mathbb{C}(X, \mathcal{A}))) \) with \( w_0 = 1 \) and \( w_1 = g + 1 - p. \) Thus, \( i_{\mathcal{A}(X, \mathcal{A})} \) is injective.

The final result follows from Proposition [3.1].

Combining Theorem 3.3 with standard argument in Algebraic Topology, we can get

**Corollary 3.4.** Let \( \mathcal{A}, \mathcal{B} \) and \( E \) be as in Theorem 3.3. Then

\[
\pi_n(\mathfrak{U}(E)) = \begin{cases} 
K_0(E) & n \text{ odd} \\
K_1(E) & n \text{ even} 
\end{cases}
\]

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