The main purpose of this paper is mathematical analysis on time-periodic flows of electrons and holes in semiconductors. The flows appear in a situation that alternating-current voltages are applied to devices. In this paper, we study the drift-diffusion model for semiconductors in a three-dimensional bounded domain and investigate the existence and stability of time-periodic solutions. We first derive uniform-in-time estimates of time-global solutions and then prove by the relative entropy method that the difference of any two solutions decays exponentially fast as time tends to infinity. These facts enable us to show the unique existence and global stability of time-periodic solutions.

**KEYWORDS**

Drift-diffusion model, global stability, initial-boundary value problem, mixed-boundary condition, parabolic–elliptic system, time-periodic solution

**MSC CLASSIFICATION**

35B10; 35B35; 35B40; 35B45; 82D37

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**1 | DRIFT-DIFFUSION MODEL**

This paper is concerned with the asymptotic behavior of solutions to the drift-diffusion model for semiconductors. The model was proposed by Roosbroeck \(^1\) as a system of partial differential equations for the transport of electrons and holes in semiconductor devices. Let \( \Omega \subseteq \mathbb{R}^3 \) be a domain occupied by a semiconductor device. Then the model is written as the parabolic–elliptic system

\[
\begin{align*}
    n_t &= \nabla \cdot (\nabla n - n \nabla v) - R(n, p), \\
    p_t &= \nabla \cdot (\nabla p + p \nabla v) - R(n, p), \\
    \epsilon \Delta v &= n - p - D(x),
\end{align*}
\]

where \( I \subseteq \mathbb{R} \) is an open interval with \( \sup I = \infty \). The unknown functions \( n, p, \) and \( v \) stand for the electron density, the hole density, and the electrostatic potential, respectively. The recombination-generation term \( R \) accounts for instantaneous generation or annihilation of electron-hole pairs. The doping profile \( D \) denotes the density of ionized impurities in semiconductors and determines the performance of devices. The positive constant \( \epsilon \) is the scaled Debye length. For more details of this model, see the books.\(^2-5\)

We divide the boundary \( \partial \Omega \) into two parts \( \Gamma_D \) and \( \Gamma_N \) and impose a mixed boundary condition as follows.

\[
\begin{align*}
    n &= N_0(x), \quad p = P_0(x), \quad v = V_0(t, x) \\
    (\nabla n - n \nabla v) \cdot \mathbf{n} &= (\nabla p + p \nabla v) \cdot \mathbf{n} = 0, \quad \epsilon \nabla v \cdot \mathbf{n} + b(x)v = g(t, x),
\end{align*}
\]

\((t, x) \in I \times \Gamma_D, \quad (t, x) \in I \times \Gamma_N.
\]
Here, \( N_b, P_b, V_b, b, \) and \( g \) are given functions, and \( n \) denotes the outer unit normal vector to \( \partial \Omega \). From a physical point of view, this boundary condition corresponds to Ohmic, Schottky, or Metal-Oxide contact arising in widely used semiconductor devices such as MOSFETs, p-n diodes, and thyristors.

There have been many researches on the existence and asymptotic behavior of solutions to the initial-boundary value problem of (1.1) with time-independent boundary data. A pioneer work was made by Mock\(^6\)–\(^8\) for the simpler case \( \Gamma_D = \emptyset \) and \( b = g = 0 \). It was shown that a solution exists globally in time and converges to a stationary solution. Physically speaking, the boundary condition in this case does not allow any electron and hole to flow through the boundary. Gajewski and Gröger\(^9\) proved the time-globalsolvability for the more relevant case \( \Gamma_D \neq \emptyset, g \neq 0, \) and \( b \geq 0 \) (see also the study of Gajewski\(^10\)). In this case, electrons and holes can flow through the boundary. Furthermore, they investigated the asymptotic state of solutions for a special boundary data \( N_b, P_b, \) and \( V_b \) and then showed the global stability of a special stationary solution \( (N, P, V) \) which represents a thermal equilibrium, that is,

\[
NP = 1, \quad \nabla (\log N - V) = \nabla (\log P + V) = 0. \tag{1.2}
\]

The second and third equalities mean that the currents vanish, and therefore, their results do not cover physically important situations that semiconductor devices are used in integrated circuits.

For general time-independent boundary data, Gröger\(^11\) constructed a stationary solution in which the current is flowing (see the studies of Gajewski and Seidman\(^12,13\)). Of course, it is of great interest to study its stability. A difficulty in proving the global stability lies in the derivation of uniform-in-time estimates. The study of Gajewski and Gröger\(^9\) used (1.2) in an essential way to resolve this difficulty. For a simpler case \( b = g = 0 \), Fang and Ito\(^14\)–\(^16\) derived uniform-in-time estimates and then constructed a compact attractor. The relation between the attractor and stationary solutions was not clarified.

In this paper, we first extend their result on uniform-in-time estimates to the case \( b \neq 0 \) and \( g \neq 0 \) assuming that the area of \( \{ b \neq 0 \} \) is small. The set \( \{ b \neq 0 \} \) corresponds to interfaces between semiconductors and oxides. It is worth pointing out that the uniform-in-time estimates can be obtained even in the case when the current is large and/or the boundary data \( g \) and \( V_b \) are time-dependent.

Besides stationary flows, time-periodic flows are also physically important. Indeed, time-periodic flows appear when PN junction diodes act like a rectifier by converting alternating current into direct current. In a one-dimensional case, Kan and Suzuki\(^17\) studied the unique existence and global stability of time-periodic solutions in a situation that the applied voltage is periodic in time. This time-periodic solution has nonzero currents. Seidman\(^18\) also investigated time-periodic solutions for a generalized drift-diffusion model. In this paper, we show the global stability of time-periodic solutions for time-periodic boundary data. Our main theorem also ensures the stability of stationary solutions in which small currents are flowing.

## 2 | MAIN THEOREMS

We begin with introducing notation and making assumptions to be used throughout the paper. For \( 1 \leq q \leq \infty, \) \( 1 \leq q, \) and \( 1 \leq q, \) denote the norms of the Lebesgue spaces \( L^q(\Omega) \) and \( L^q(\Gamma_N) \), respectively. Furthermore, \( \| \cdot \| \) stands for the norm of the Sobolev space \( H^1(\Omega) \). We denote by \( H^1(\Omega) \) the subspace \( \{ f \in H^1(\Omega); f = 0 \text{ on } \Gamma_D \} \) and by \( H^1(\Omega)^* \) its dual space. The notation \( f' \) means the derivative of a function \( f \) with respect to \( t \). For \( a, \sigma \in \mathbb{R} \), we write \( a_+ := \max\{a, 0\}, \) \( a_- := \min\{a, 0\}, \) and \( a_\sigma := (a - \sigma)_+ + \sigma = \max\{a, \sigma\} \).

### Assumption 2.1

We assume conditions (H1)–(H8) below.

(H1) \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with Lipschitz boundary.

(H2) \( \partial \Omega \) consists of the disjoint union of \( \Gamma_D \) and \( \Gamma_N \), and the measure of \( \Gamma_D \) is nonzero.

(H3) The recombination-generation term \( R \) is given by the Shockley–Read–Hall form, that is,

\[
R(n, p) := c_R \frac{np - 1}{n + p + 2},
\]

where \( c_R \) is a positive constant.

(H4) \( N_b = N_b(x), P_b = P_b(x), V_b = V_b(t, x), D = D(x), b = b(x), \) and \( g = g(t, x) \) are given functions satisfying \( N_b, P_b \in H^1(\Omega) \cap L^{\infty}(\Omega), V_b \in W^{1,\infty}(\mathbb{R}; W^{1,\rho}(\Omega)) \cap L^{\infty}(\mathbb{R} \times \Omega), D \in L^{\infty}(\Omega), b \in L^{\infty}(\Gamma_N), g \in W^{1,\infty}(\mathbb{R}; L^{\sigma}(\Gamma_N)) \) for some \( \rho > 3 \) and \( r > 2 \).
Definition 2.1. We say that \((n, p, v)\) is a solution of (1.1) if it satisfies the following conditions.

(i) For any bounded interval \(J \subset I\),

\[
\begin{align*}
&n - N_b \in L^2(J; H^1_0(\Omega)) \cap L^\infty(J \times \Omega), \quad n' \in L^2(J; H^1(\Omega)^*), \\
p - P_b \in L^2(J; H^1_0(\Omega)) \cap L^\infty(J \times \Omega), \quad p' \in L^2(J; H^1(\Omega)^*), \\
v - V_b \in C(I; H^1_0(\Omega)).
\end{align*}
\]

(ii) \(n, p \geq 0\) a.e. in \(I \times \Omega\).

(iii) For any \(\phi_1, \phi_2, \phi_3 \in H^1(\Omega)\) and a.e. \(t \in I\),

\[
\begin{align*}
\langle n', \phi_1 \rangle + \int_\Omega \{(\nabla n - n\nabla v) \cdot \nabla \phi_1 + R(n, p)\phi_1 \} \, dx &= 0, \\
\langle p', \phi_2 \rangle + \int_\Omega \{(\nabla p + p\nabla v) \cdot \nabla \phi_2 + R(n, p)\phi_2 \} \, dx &= 0, \\
\varepsilon \int_\Omega \nabla v \cdot \nabla \phi_3 \, dx + \int_{\Gamma_N} (bv - g) \phi_3 \, dS &= - \int_\Omega (n - p - D)\phi_3 \, dx.
\end{align*}
\]

Furthermore, if \((n, p, v)\) is a solution of (1.1) with \(I = \mathbb{R}\) and additionally satisfies the condition (iv) below, we say that \((n, p, v)\) is a time-periodic solution of (1.1) with period \(T^* > 0\).

(iv) \((n, p, v)(t + T^*, x) = (n, p, v)(t, x)\) for some constant \(T^* > 0\).

We are now in a position to state our main theorems. As mentioned above, Gajewski and Gröger\(^9\) considered the problem (1.1) with the initial condition \((n, p)(0, \cdot) = (n_0, p_0)\) and showed the global existence of a solution \((n, p, v)\) for any nonnegative initial data \((n_0, p_0) \in L^\infty(\Omega) \times L^\infty(\Omega)\). The result on the time-global solvability will be introduced precisely in Section 3. Our first theorem deals with the uniform-in-time estimates of \((n, p)\). Here and subsequently, we fix \(q_0 > 2\) and set

\[
\theta := |b|q_0\Gamma_N.
\]

Theorem 2.1. There exist positive constants \(\hat{\theta}\) and \(\hat{C}\) depending only on \(\max\{\delta, 1\}, c_0, \ C_0, \ \Omega, \ \Gamma_D, \ \rho, \ r, \ \varepsilon, \) and \(c_R\) such that if \(\theta < \hat{\theta}\), then any solution \((n, p, v)\) of (1.1) satisfies

\[
\limsup_{t \to \infty} \left( \frac{1}{n(t)} + \frac{1}{p(t)} + |n(t)|_\infty + |p(t)|_\infty \right) \leq \hat{C}.
\]

We are interested in the asymptotic behavior of solutions in the case that the applied voltage is periodic in time as an AC voltage. We prove that if \(\delta\) is sufficiently small, then (1.1) has a unique time-periodic solution, and any solution converges to it as \(t \to \infty\). This result is summarized in the following theorem.

Theorem 2.2. Suppose that \(V_b\) and \(g\) are periodic in \(t\) with period \(T^* > 0\) and that \(\theta < \hat{\theta}\) holds for \(\hat{\theta}\) being in Theorem 2.1. Then there exists a constant \(\delta_0 > 0\) depending only on \(c_0, \ C_0, \ \Omega, \ \Gamma_D, \ \rho, \ r, \ \varepsilon, \) and \(c_R\) such that (1.1) has a unique time-periodic solution \((n^*, p^*, v^*)\) if \(\delta < \delta_0\). Furthermore, any solution \((n, p, v)\) of (1.1) converges to \((n^*, p^*, v^*)\) in \(L^2(\Omega) \times L^2(\Omega) \times H^1(\Omega)\) exponentially fast as \(t \to \infty\). Specifically,

\[
\limsup_{t \to \infty} e^{\theta t} \left( |n - n^*(t)|_2 + |p - p^*(t)|_2 + |(v - v^*)(t)|_1 \right) < +\infty.
\]
where \( c > 0 \) is a constant depending on \( c_0, C_0, \Omega, \Gamma_D, \rho, r, \epsilon, \) and \( c_R \).

Remark 2.1. Theorem 2.2 provides the global stability of asymptotic states which do not satisfy \((1.2)\), and therefore, it is an extension of the result\(^9\) for the case \( \theta \ll 1 \). The smallness assumption on \( \theta \) holds if the area \( S_0 \) of the set \( \{ b \neq 0 \} \) is small, because of the inequality \( |b|_{L^\infty} \leq |b|_{L^\infty} \Omega^{1/4} \). From a physical point of view, \( S_0 \) denotes the areas of interfaces between semiconductors and oxides. We also remark that stationary solutions may not be unique for large \( \delta \) (see the previous studies\(^{19–21}\)).

A main task in the proof of Theorem 2.3 is to establish the uniform-in-time estimate of \( L^1 \)-norms. Then a difficulty arises from the condition \( b \neq 0 \) in (2.2c). Specifically, when we rewrite the drift terms \( n V v \) and \( p V v \) in (2.2a) and (2.2b) by using (2.2c), we have boundary terms having a strong nonlinearity. To handle the difficulty, we decompose \( n \) and \( p \) into the parts of the lower and higher values. The lower part is not issue at all. If the \( L^1 \)-norm of the higher part is large, the dissipative effect is strong enough so that the rewrite mentioned above is not necessary. For the case that the \( L^1 \)-norm of the higher part is small, we use a new technique to show that the nonlinear term with \( b \) can be absorbed into the dissipative terms. The details will be given in Lemma 4.3.

A main difficulty of the proof of Theorem 2.4 arises from the low regularity of solutions. It is not expected that the solutions have a better regularity than (2.1) due to the mixed boundary condition (1.1b). In such a case, it is not straightforward to handle some nonlinear terms only by applying the well-known inequalities. To overcome this difficulty, we use the new estimate proved in Lemma 5.2.

This paper is organized as follows. In Section 3, we introduce basic inequalities and facts. Section 4 is devoted to the proof of the uniform-in-time estimates in Theorem 2.1. In Section 5, we study the estimate of the difference of two solutions. It enables us to prove Theorem 2.2. The proof of Theorem 2.2 will be discussed in Section 6.

### 3 | PRELIMINARIES

In this section, we introduce basic facts to be used in our arguments. First we mention the time-global solvability of the problem (1.1) supplemented with the initial condition \((n, p)(0, \cdot) = (n_0, p_0)\). It is proved by Gajewski and Gröger\(^9\), Theorem 1 in the case that the function \( V_b \) is time-independent. Even if \( V_b \) and \( g \) are time-dependent, their proof works under the condition \((H4)\) in Assumption 2.1.

**Proposition 3.1** (Gajewski and Gröger\(^9\)). Let \( I = (0, \infty) \). For any \((n_0, p_0) \in L^\infty(\Omega) \times L^\infty(\Omega) \) with \( n_0, p_0 \geq 0 \), the problem (1.1) has a unique solution \((n, p, v)\) satisfying \((n, p)(t, \cdot) \to (n_0, p_0)\) in \( L^2(\Omega) \times L^2(\Omega) \) as \( t \downarrow 0 \).

Next we discuss estimates of a solution \( w \) of the boundary value problem

\[
\begin{cases}
\Delta w = h(x), & x \in \Omega, \\
w = W_b(x), & x \in \Gamma_D, \\
\nabla w \cdot n + b(x)w = \tilde{g}(x), & x \in \Gamma_N, 
\end{cases}
\]

(3.1)

where \( h \in L^{6/5}(\Omega) \), \( W_b \in H^1(\Omega) \), \( \tilde{b} \in L^2(\Gamma_N) \), \( \tilde{b} \geq 0 \), and \( \tilde{g} \in L^{4/3}(\Gamma_N) \). We say that \( w \in H^1(\Omega) \) is a solution of (3.1) if

\[
\int_\Omega \nabla w \cdot \nabla \phi dx + \int_{\Gamma_N} (\tilde{b}w - \tilde{g})\phi dS = -\int_\Omega h\phi dx
\]

for all \( \phi \in H^1_D(\Omega) \). This is written as

\[
\int_\Omega \nabla (w - W_b) \cdot \nabla \phi dx + \int_{\Gamma_N} (\tilde{b}w - \tilde{g})\phi dS = -\int_\Omega \nabla W_b \cdot \nabla \phi dx - \int_\Omega h\phi dx - \int_{\Gamma_N} (\tilde{b}W_b - \tilde{g})\phi dS. \tag{3.2}
\]

The Sobolev embedding theorem \( H^1(\Omega) \hookrightarrow L^6(\Omega) \) and the inequality (3.7) below imply that the mapping \( \phi \mapsto -\int_\Omega \nabla W_b \cdot \nabla \phi dx + \int_{\Gamma_N} (\tilde{b}W_b - \tilde{g})\phi dS \) defines a bounded linear functional on \( H^1_D(\Omega) \). Therefore, a standard theory of elliptic
partial differential equations shows that (3.1) has a unique solution $w$ satisfying

$$
\|w - W_b\|_1 \leq C (|h|_{6/5} + |\tilde{g}|_{4/3, \Gamma_N} + \|W_b\|_1 (1 + |\tilde{b}|_{2, \Gamma_N}))
$$  \hfill (3.3)

for some constant $C = C(\Omega, \Gamma_D) > 0$.

**Lemma 3.1.** Let $1 \leq \alpha < 3/2$ and $1 \leq \beta < 2$. Then the solution $w$ of (3.1) satisfies

$$
|\nabla w|_{\alpha} + |w|_{\beta, \Gamma_N} \leq C (|h|_1 + |\tilde{g}|_1, \Gamma_N + \|W_b\|_1 (1 + |\tilde{b}|_{4/3, \Gamma_N})),
$$  \hfill (3.4)

where $C = C(\Omega, \Gamma_D, \alpha, \beta) > 0$ is a constant. Suppose further that $h \in L^q(\Omega), W_b \in L^\infty(\Omega)$ and $\tilde{g} \in L^r(\Gamma_N)$ for some $q > 3/2$ and $r > 2$. Then there exists a constant $\tilde{C} = \tilde{C}(\Omega, \Gamma_D, q, r) > 0$ such that $w \in L^\infty(\Omega)$ and

$$
|w|_\infty \leq \tilde{C} (|h|_q + |\tilde{g}|_{r, \Gamma_N} + |W_b|_\infty).
$$  \hfill (3.5)

The following lemmas will also be utilized.

**Lemma 3.2.** The following hold.

(i) For $1 \leq q \leq 2$, there exists a constant $C = C(\Omega, \Gamma_D, q) > 0$ such that for all $f \in H^1_D(\Omega)$,

$$
|f|_{q, \Gamma_N} \leq C |\nabla f|_q.
$$  \hfill (3.6)

(ii) There exists a constant $C = C(\Omega) > 0$ such that for all $f \in H^1(\Omega)$,

$$
|f|_{1, \Gamma_N} \leq C |f|_1.
$$  \hfill (3.7)

(iii) For $1 \leq q < 4$, there exist constants $C = C(\Omega, \Gamma_D, q) > 0$ and $\alpha = \alpha(q) > 0$ such that for all $f \in H^1_D(\Omega)$ and $\mu > 0$,

$$
|f|_{q, \Gamma_N} \leq \mu |\nabla f|_2 + C \mu^{-\alpha} |f|_1.
$$  \hfill (3.8)

**Lemma 3.3.** Let $d \geq 1$. Then there is a constant $C = C(d) > 0$ such that

$$
\int_a^b \log \frac{y}{A} dy + \int_b^\infty \log \frac{y}{B} dy \leq C \left\{ (a - b)^2 + \frac{ab - 1}{a + b + 2} \log(ab) + 1 \right\}
$$  \hfill (3.9)

for all $a, b > 0$ and $1/d \leq A, B \leq d$.

**Lemma 3.4.** Let $\sigma > 0$. For $M > e$ and $s \in \mathbb{R}$, we set

$$
h(M) := \frac{1}{2} \log \log M, \quad H_M(s) := \begin{cases}
-h(M) & (s < -h(M)), \\
s & (|s| \leq h(M)), \\
h(M) & (s > h(M)).
\end{cases}
$$

Then there is a constant $M_* = M_*(\sigma) > e$ such that

$$
(a + b)\chi_{\{a + b \geq M\}} \leq \frac{2}{h(M)} \left\{ (a - b)H_M \left( \log \frac{a_s}{b_s} \right) + \frac{ab - 1}{a + b + 2} \log(ab) \right\}
$$  \hfill (3.10)

for all $a, b > 0$ and $M \geq M_*$.

**Lemma 3.5.** Let $E$ be a measurable set in a Euclidean space. Assume that sequences $\{f_k\} \subset L^2(E)$ and $\{g_k\} \subset L^2(E)$ are convergent in $L^2(E)$ and that $|f_k| \leq C$ in $E$ for some constant $C > 0$ independent of $k$. Then $\{f_k g_k\}$ is convergent in $L^2(E)$.
We will give the proofs of Lemmas 3.1–3.5 in Appendix A1.

# Upper and Lower Bounds of Solutions

Throughout this section, we assume that \((n, p, v)\) is a solution of (1.1). The goal of this section is to prove Theorem 2.1. We set

\[
E_1 := \int \log \frac{y}{N_b} dy + \int \log \frac{y}{P_b} dy, \quad E_2 := \frac{\varepsilon}{2} |\nabla (v - \nabla V_b)|^2, \quad E_3 := \frac{1}{2} b (v - \nabla V_b)^2.
\]

Note that these are all nonnegative. The upper and lower bounds of \(n\) and \(p\) follow from the following propositions which will be proved in Sections 4.1–4.3.

**Proposition 4.1.** There exist constants \(\theta_0 > 0\) and \(C > 0\) depending only on \(\max\{\delta, 1\}, c_0, C_0, \Omega, \Gamma_D, \rho, \varepsilon, \) and \(c_R\) such that if \(\theta < \theta_0\), then

\[
\limsup_{t \to \infty} \left\{ \int \left( E_1 + E_2 \right) dx + \int_{\Gamma_N} E_3 ds \right\} \leq C.
\]

**Proposition 4.2.** Suppose that

\[
L := \limsup_{t \to \infty} (|n(t)|_1 + |p(t)|_1 + ||v(t)||_1) < \infty.
\]

Then there exists a constant \(C = C(L, c_0, C_0, \Omega, \Gamma_D, r, \varepsilon, c_R) > 0\) such that

\[
\limsup_{t \to \infty} (|n(t)|_\infty + |p(t)|_\infty) \leq C.
\]

**Proposition 4.3.** Suppose that

\[
\bar{L} := \limsup_{t \to \infty} ||v(t)||_1 < \infty.
\]

Then there exists a constant \(c = c(\bar{L}, c_0, C_0, \Omega, \Gamma_D, r, \varepsilon, c_R) > 0\) such that

\[
\liminf_{t \to \infty} \inf_{\Omega} n(t, \cdot) \geq c, \quad \liminf_{t \to \infty} \inf_{\Omega} p(t, \cdot) \geq c.
\]

For a moment, we assume that Propositions 4.1–4.3 hold and then complete the proof of Theorem 2.1.

**Proof of Theorem 2.1.** It is elementary to show that \(E_1 \geq (n - eN_b)_+ + (p - eP_b)_+\). From this and the Poincaré inequality, we have

\[
|(n - eN_b)_+|_1 + |(p - eP_b)_+|_1 + ||v - \nabla V_b||_1 \leq \int \left( \int_{\Omega} E_1 dx + C \left( \int_{\Omega} E_2 dx \right)^{1/2} \right),
\]

where \(C = C(\Omega, \Gamma_D, \varepsilon) > 0\) is a constant. Combining this with Proposition 4.1 implies that

\[
\limsup_{t \to \infty} (|n|_1 + |p|_1 + ||v||_1) \leq C
\]

for some constant \(C = C(\max\{\delta, 1\}, c_0, C_0, \Omega, \Gamma_D, \rho, \varepsilon, c_R) > 0\). This together with Propositions 4.2 and 4.3 leads to (2.3). \(\square\)

## 4.1 Proof of Proposition 4.1

This subsection is devoted to the proof of Proposition 4.1. We begin by deriving an energy inequality. We put

\[
D_1 := n |\nabla \log n - \nabla v|^2 + p |\nabla \log p + \nabla v|^2, \quad D_2 := R(n, p) \log(np).
\]
Lemma 4.1. There holds that
\[ n, p > 0 \text{ a.e. in } I \times \Omega. \tag{4.1} \]
Furthermore, there is a constant \( C_1 = C_1(C_0, \Omega, \Gamma_D, \rho, \epsilon, c_R) > 0 \) such that
\[
\frac{d}{dt} \left\{ \int_\Omega (E_1 + E_2) dx + \int_{\Gamma_N} E_3 dS \right\} + \frac{3}{4} \int_\Omega (D_1 + D_2) dx \leq C_1 \delta \int_\Omega (n + p) dx + C_1 \delta. \tag{4.2} \]

Proof. We first show that for \( t, t_0 \in I, \)
\[
\left\{ \int_\Omega (E_1 + E_2) dx + \int_{\Gamma_N} E_3 dS \right\} \bigg|_{t_0}^{t} + \int_{t_0}^{t} \left( \int_\Omega (D_1 + D_2) dx + \int_{\Gamma_N} (I_1 + I_2 + I_3) dx dt + \int_{I_0} I_4 dS dt, \right. \tag{4.3} \]
where
\[
I_1 := (\nabla n - n \nabla v) \cdot (\nabla \log \rho - \nabla V_b) + (\nabla p + p \nabla v) \cdot (\nabla \log \rho_b + \nabla V_b),
\]
\[
I_2 := R(n, p) \log(\rho_b/p_b), \quad I_3 := -\epsilon \nabla V_b' \cdot \nabla (v - V_b), \quad I_4 := -(bV_b' - g')(v - V_b).
\]
To make the computation in this paragraph rigorous, we use a mollifier with respect to the time variable \( t \) due to the insufficiency of the regularity of solutions. We omit the argument since it is standard. We differentiate (2.2c) with respect to \( t \) to find that
\[
\epsilon \int_\Omega \nabla v' \cdot \nabla \phi_3 dx + \int_{\Gamma_N} (bv' - g') \phi_3 dS = -(n' - p', \phi_3).
\]
Combining (2.2a), (2.2b), and this equality gives
\[
\langle n', \phi_1 + \phi_3 \rangle + \langle p', \phi_2 - \phi_3 \rangle + \epsilon \int_\Omega \nabla (v - V_b)' \cdot \nabla \phi_3 dx + \int_{\Gamma_N} b(v - V_b)' \phi_3 dS + \int_\Omega n(\nabla \log n - \nabla v) \cdot \nabla \phi_3 dx + \int_{\Gamma_N} p(\nabla \log p + \nabla v) \cdot \nabla \phi_3 dx + \int_\Omega R(n, p)(\phi_1 + \phi_2) dx
\]
\[
= -\epsilon \int_\Omega \nabla V_b' \cdot \nabla \phi_3 dx - \int_{\Gamma_N} (bV_b' - g') \phi_3 dS.
\]
Let \( 0 < \sigma \leq c_0 \), where \( c_0 \) is the constant in the assumption (H6). Taking \( \phi_1 = \log(\sigma/n_b) - (v - V_b), \phi_2 = \log(p_a/P_b) + (v - V_b) \) and \( \phi_3 = v - V_b \) and integrating the result over \([t_0, t]\), we have
\[
\left\{ \int_\Omega \left( \int_{t_0}^t \log \frac{\rho_b}{P_b} dy + \int_{t_0}^t \log \frac{\rho}{n_b} dy + E_2 \right) dx + \int_{\Gamma_N} E_3 dS \right\} \bigg|_{t_0}^{t}
\]
\[
+ \int_{t_0}^{t} \left( \int_\Omega n|\nabla \log n_e|^2 + p|\nabla \log p_a|^2 - 2\nabla(n - p) \cdot \nabla v + (n + p)|\nabla v|^2 \right) dx dt + \int_{t_0}^{t} \int_\Omega c_R \frac{n_e P_a - 1}{n + p + 2} \log(n_e P_a) dx dt
\]
\[
= \int_{t_0}^{t} \left( I_1 + I_2 + I_3 \right) dx dt + \int_{t_0}^{t} I_4 dS dt + \int_{t_0}^{t} \left( J_1^\sigma + J_2^\sigma \right) dx dt, \tag{4.4}
\]
where \( a_\sigma = \max\{a, \sigma\} \) for \( a \in \mathbb{R}, \)
\[
J_1^\sigma := -(\nabla n - \nabla n_e) \cdot \nabla v + (\nabla p - \nabla p_a) \cdot \nabla v, \quad J_2^\sigma := c_R \frac{n_e P_a - np}{n + p + 2} \log(n_e P_a).
We now derive (4.3) by letting $\sigma \to 0$ in (4.4). The monotone convergence theorem shows that the left-hand side of (4.4) converges to that of (4.3). One sees that $J_1' \to 0$ a.e. in $\{(0, t) \times \Omega$ and $|J_1'| \leq (|\nabla n| + |\nabla p|) |\nabla v|$. Furthermore, $J_2'$ is estimated as

$$|J_2'| \leq c_r \frac{(\sigma - p)n}{n + p + 2} |\log(\sigma n)| \chi_{[\sigma, p \leq \sigma]} + c_r \frac{(\sigma - n)p}{n + p + 2} |\log(\sigma p)| \chi_{[\sigma, n \leq \sigma]} + c_r \frac{\sigma^2 - np}{n + p + 2} |\log(\sigma^2)| \chi_{[\sigma, n \leq \sigma]}$$

$$\leq \frac{c_r}{\sigma n} |\log(\sigma n)| + \frac{c_r}{\sigma p} |\log(\sigma p)| + c_r \sigma^2 |\log \sigma|.$$  

It follows from the dominated convergence theorem that the rightmost term of (4.4) converges to 0. Thus, (4.3) is verified. In particular, we have $\int D_2 dx < \infty$, and therefore, (4.1) holds.

To derive (4.2), we estimate the right-hand side of (4.3). The Schwarz inequality yields

$$I_1 = n(\nabla \log n - \nabla v) \cdot (\nabla \log n - \nabla v) + p(\nabla \log p + \nabla v) \cdot (\nabla \log p + \nabla v)$$

$$\leq n \left( \frac{1}{4} |\nabla \log n - \nabla v|^2 + |\nabla \log n - \nabla v|^2 \right) + p \left( \frac{1}{4} |\nabla \log p + \nabla v|^2 + |\nabla \log p + \nabla v|^2 \right)$$

$$\leq \frac{1}{4} D_1 + \delta(n + p).$$  

(4.5)

It is elementary to show that $|R(n, p)| \leq c_r(n + p + 1)$. From this, we see that

$$I_2 \leq c_r \delta(n + p + 1).$$  

(4.6)

Let $\alpha \in [1, 3/2]$ and $\beta \in [1, 2]$ be the Hölder conjugates of $\rho > 3$ and $r > 2$, respectively. By the Hölder inequality and (3.4), we have

$$\int \int I_3 dx \leq \epsilon |\nabla V'_{n,r}|_\sigma |\nabla (v - V_b)|_\alpha \leq C \delta(|n - p|_1 + 1).$$  

(4.7)

Using (3.4), (3.7), and the Hölder inequality, we deduce that

$$\int I_4 dS \leq |b V'_{g,1/4,\Gamma_n} |v - V_b|_{4/3, \Gamma_n} + |g'(r,\Gamma_n)| v - V_b|_{\beta, \Gamma_n}$$

$$\leq C(|V'_{g,1} + |g'(r,\Gamma_n)|(|n - p|_1 + 1)$$

$$\leq C \delta(|n - p|_1 + 1).$$  

(4.8)

Here, $C > 0$ is a constant depending only on $C_0, \Omega, \Gamma_D, \rho, \sigma, \sigma$, and $\epsilon$.

The equality (4.3) implies that $\int (\mathcal{E}_1 + \mathcal{E}_2) dx + \int E_3 dS$ is absolutely continuous in $t$. Differentiating (4.3) in $t$ and then plugging (4.5)–(4.8) into the result, we obtain the desired inequality.  

We show the following two lemmas to be used in the proof of Proposition 4.1. A key of the proof is to decompose $n$ and $p$ into the parts of the lower and higher values. To do so, we introduce a function

$$I^M := (n + p) \chi_{(n+p \geq M)},$$

where $M \geq 0$ and $\chi_A$ denotes the indicator function of a set $A$. 
**Lemma 4.2.** There exist positive constants \( \theta_1 = \theta_1(C_0, \Omega, \Gamma_D, \varepsilon, c_R) \), \( \tilde{C}_1 = \tilde{C}_1(C_0, \Omega, \Gamma_D, \varepsilon, c_R) \), and \( M_1 = M_1(C_0) \) such that if \( \theta < \theta_1 \) and \( M \geq M_1 \), then

\[
\int \Omega I^M dx \leq \tilde{C}_1 \left( \theta + \frac{1}{h(M)} \right) \left\{ \int \Omega (D_1 + D_2) dx + 1 \right\}. \tag{4.9}
\]

Here, \( h(M) \) is defined in Lemma 3.4.

**Proof.** In the proof, \( C \) denotes a positive constant depending only on \( C_0, \Omega, \Gamma_D, \varepsilon, \) and \( c_R \). Set \( \sigma := \max\{|N_b|_{\infty}, |P_b|_{\infty}\} \) and let \( M_1 := M_1(\sigma) \), where \( M(\sigma) \) is the constant being in Lemma 3.4. We first claim that it suffices to show the inequality

\[
\int \Omega (n - p)H_M \left( \log \frac{n_{\sigma}}{p_{\sigma}} \right) dx \leq C \left\{ \int \Omega D_1 dx + 1 \right\} + C\theta h(M) \left\{ \int \Omega I^0 dx + 1 \right\}. \tag{4.10}
\]

where \( H_M \) is the function defined in Lemma 3.4. This claim is verified as follows. We see from (3.10) and (4.10) that

\[
\int \Omega I^M dx \leq \frac{2}{h(M)} \left\{ \int \Omega (n - p)H_M \left( \log \frac{n_{\sigma}}{p_{\sigma}} \right) dx \right\} + \frac{n \sigma - 1}{n + p + 2} \log(n p) \left\{ \int \Omega I^0 dx + 1 \right\}. \tag{4.11}
\]

Choosing \( M = M_1 \) in (4.11) yields

\[
\int \Omega I^0 dx \leq \int \Omega I^{M_1} dx + M_1 |\Omega| \leq C \left\{ \int \Omega (D_1 + D_2) dx + 1 \right\} + C\theta \left\{ \int \Omega I^0 dx + 1 \right\}.
\]

Hence, we have

\[
\int \Omega I^0 dx \leq C \left\{ \int \Omega (D_1 + D_2) dx + 1 \right\},
\]

provided that \( \theta \) is less than some constant \( \theta_1 = \theta_1(C_0, \Omega, \Gamma_D, \varepsilon, c_R) > 0 \). Plugging this into (4.11), we obtain (4.9). Thus, the claim is proved.

Let us complete the proof by showing (4.10). We take \( \phi_3 = H_M(\log(n_{\sigma}/p_{\sigma})) \in H^1_D(\Omega) \) in (2.2c) to obtain

\[
\int (n - p)H_M \left( \log \frac{n_{\sigma}}{p_{\sigma}} \right) dx = \varepsilon \int_\Omega I_1 dx + \int_\Gamma_N I_2 ds + \int_\Gamma_N I_3 dS + \int_\Omega I_4 dx,
\]

\[
I_1 := -\nabla \cdot \nabla H_M \left( \log \frac{n_{\sigma}}{p_{\sigma}} \right), \quad I_2 := -b H_M \left( \log \frac{n_{\sigma}}{p_{\sigma}} \right), \quad I_3 := g H_M \left( \log \frac{n_{\sigma}}{p_{\sigma}} \right), \quad I_4 := D H_M \left( \log \frac{n_{\sigma}}{p_{\sigma}} \right). \tag{4.12}
\]

Notice that \( I_1 \) is written as

\[
I_1 = \frac{1}{2} \left( |\nabla \log n - \nabla |^2 \chi_{(|n| > \sigma)} + |\nabla \log p + \nabla |^2 \chi_{(|p| > \sigma)} \right) \frac{dH_M}{ds} \left( \log \frac{n_{\sigma}}{p_{\sigma}} \right)
- \frac{1}{2} \left( |\nabla \log n_{\sigma}|^2 + |\nabla \log p_{\sigma}|^2 + |\nabla |^2 \left( \chi_{(|n| > \sigma)} + \chi_{(|p| > \sigma)} \right) \right) \frac{dH_M}{ds} \left( \log \frac{n_{\sigma}}{p_{\sigma}} \right). \]
From this and the fact that \( 0 \leq dH_M/ds \leq 1 \), we have
\[
\int_{\Omega} I_1 dx \leq \frac{1}{2} \int_{\Omega} \left( \frac{n}{\sigma} |\nabla \log n - \nabla v|^2 \chi_{(n>\sigma)} + \frac{p}{\sigma} |\nabla \log p + \nabla v|^2 \chi_{(p>\sigma)} \right) dx \\
- \frac{1}{2} \int_{\Omega} (|\nabla \log n_\sigma|^2 + |\nabla \log p_\sigma|^2) \frac{dH_M}{ds} \left( \log \frac{n_\sigma}{p_\sigma} \right) dx \\
\leq \frac{1}{2\sigma} \int_{\Omega} D_1 dx - \frac{1}{2} \int_{\Omega} (|\nabla \log n_\sigma|^2 + |\nabla \log p_\sigma|^2) \frac{dH_M}{ds} \left( \log \frac{n_\sigma}{p_\sigma} \right) dx. \tag{4.13}
\]

By the fact that \(|H_M| \leq h(M)\), the Hölder inequality and (3.4), the integral of \( I_2 \) is estimated as
\[
\int_{\Gamma_N} I_2 dS \leq h(M) \theta |\nabla|_{\beta} \chi_{\Gamma_N} \leq C \theta h(M) \left( \int_{\Omega} I_0 dx + 1 \right), \tag{4.14}
\]
where \( \beta \in [1, 2) \) is the Hölder conjugate of \( q_0 > 2 \). For the third and fourth terms of the right-hand side of (4.12), we utilize the Schwarz and Poincaré inequalities and (3.6) to obtain
\[
\int_{\Gamma_N} I_3 dS + \int_{\Omega} I_4 dx \leq \int_{\Gamma_N} \left( \mu H_M \left( \log \frac{n_\sigma}{p_\sigma} \right)^2 + \frac{1}{4\mu} g^2 \right) dS + \int_{\Omega} \left( \mu H_M \left( \log \frac{n_\sigma}{p_\sigma} \right)^2 + \frac{1}{4\mu} D^2 \right) dx \\
\leq C \mu \int_{\Omega} \left| \nabla H_M \left( \log \frac{n_\sigma}{p_\sigma} \right) \right|^2 dx + \frac{C}{\mu} \\
\leq C \mu \int_{\Omega} (|\nabla \log n_\sigma|^2 + |\nabla \log p_\sigma|^2) \frac{dH_M}{ds} \left( \log \frac{n_\sigma}{p_\sigma} \right) dx + \frac{C}{\mu}, \tag{4.15}
\]
where \( \mu > 0 \) is an arbitrary number. Plugging (4.13)–(4.15) into (4.12) and then taking \( \mu \) appropriately, we obtain (4.10). Thus the lemma follows. \( \square \)

**Lemma 4.3.** There exist constants \( c_1 > 0 \) and \( C > 0 \) depending only on \( c_0, C_0, \Omega, \Gamma, r, \epsilon, \) and \( c_R \) such that if
\[
M \geq \max(\|N_0\|_\infty, \|P_0\|_\infty) \quad \text{and} \quad \int_{\Omega} I^M dx \leq c_1,
\]
then
\[
\int_{\Omega} (E_1 + E_2) dx + \int_{\Gamma_N} E_3 dS \leq C \left\{ \int_{\Omega} (D_1 + D_2) dx + M^2 + 1 \right\}.
\]

**Proof.** Let \( M \geq \max(\|N_0\|_\infty, \|P_0\|_\infty) \). In the proof, \( C \) denotes a generic positive constant depending only on \( c_0, C_0, \Omega, \Gamma, r, \epsilon, \) and \( c_R \). We set
\[
J := \int \frac{|\nabla n|^2}{n} dx + \int \frac{|\nabla p|^2}{p} dx + \int (n - p)^2 dx.
\]

The assertion follows if we show the inequality
\[
J \leq C \left( \int_{\Omega} I^M dx \right)^{1/2} \int_{\Omega} D_1 dx + \int \frac{dx}{\int_{\Omega} I^M dx + M^2 + 1}. \tag{4.16}
\]
Indeed, this inequality gives
\[ J \leq C \left( \int_{\Omega} D_1 \, dx + M^2 + 1 \right), \]
provided that \( \int_{\Omega} I^M \, dx \) is sufficiently small. Combining this with (3.3), (3.6), and (3.9), we see that
\[
\int_{\Omega} \mathcal{E}_1 \, dx \leq C \left[ \int_{\Omega} (n - p)^2 + D_2 \, dx + 1 \right] \leq C \left( \int_{\Omega} (D_1 + D_2) \, dx + M^2 + 1 \right),
\]
\[
\int_{\Omega} \mathcal{E}_2 \, dx + \int_{\Gamma_N} \mathcal{E}_3 \, dS \leq C \left[ \int_{\Omega} (n - p)^2 + 1 \right] \leq C \left( \int_{\Omega} D_1 \, dx + M^2 + 1 \right),
\]
which lead to the desired inequality.

Let us verify (4.16). It is seen that
\[
\frac{|\nabla n|^2}{n} = n |\nabla \log n - \nabla v|^2 + 2 \nabla n \cdot \nabla v - n |\nabla v|^2,
\]
\[
\frac{|\nabla p|^2}{p} = p |\nabla \log p + \nabla v|^2 - 2 \nabla p \cdot \nabla v - p |\nabla v|^2.
\]
These inequalities give
\[
\int_{n \geq M} \frac{|\nabla n|^2}{n} \, dx + \int_{p \geq M} \frac{|\nabla p|^2}{p} \, dx \leq \int_{\Omega} D_1 \, dx + 2 \int_{n \geq M} \nabla n \cdot \nabla v \, dx - 2 \int_{p \geq M} \nabla p \cdot \nabla v \, dx.
\] (4.17)

Owing to the condition \( M \geq |N_0| \), we can substitute \( \phi_3 = (n - M)_+ \) into (2.2c) to obtain
\[
\int_{n \geq M} \nabla n \cdot \nabla v \, dx = \int_{\Omega} \nabla (n - M)_+ \cdot \nabla v \, dx
\]
\[
= -\frac{1}{\varepsilon} \int_{\Gamma_N} (bv - g)(n - M)_+ \, dS - \frac{1}{\varepsilon} \int_{\Omega} (n - p - D)(n - M)_+ \, dx.
\]
Similarly, putting \( \phi_3 = (p - M)_+ \) yields
\[
\int_{p \geq M} \nabla p \cdot \nabla v \, dx = -\frac{1}{\varepsilon} \int_{\Gamma_N} (bv - g)(p - M)_+ \, dS - \frac{1}{\varepsilon} \int_{\Omega} (n - p - D)(p - M)_+ \, dx.
\]
By substituting these equalities into (4.17), we have
\[
\int_{n \geq M} \frac{|\nabla n|^2}{n} \, dx + \int_{p \geq M} \frac{|\nabla p|^2}{p} \, dx \leq \int_{\Omega} D_1 \, dx + \frac{2}{\varepsilon} \left( \int_{\Gamma_N} I_1 \, dS + \int_{\Gamma_N} I_2 \, dS + \int_{\Omega} I_3 \, dx \right),
\] (4.18)
where
\[
I_1 := -bv((n - M)_+ - (p - M)_+), \quad I_2 := g((n - M)_+ - (p - M)_+), \quad I_3 := -(n - p - D)((n - M)_+ - (p - M)_+).
\]
Let us first estimate the integral of \( I_1 \). From (3.5), we see that \( |v|_\infty \leq C(|n - p|_2 + 1) \), and hence,

\[
|v|_\infty \leq C(J^{1/2} + 1). \tag{4.19}
\]

By (3.6) and the Hölder inequality, we have

\[
|(n - M)_+|_{1, \Gamma_N} \leq C|\nabla (n - M)_+|_1 = C \int_{n \geq M} |\nabla n| \, dx \\
\leq C \left( \int_{n \geq M} n \, dx \right)^{1/2} \left( \int_{n \geq M} |\nabla n|^2 \, dx \right)^{1/2} \\
\leq C \left( \int_{\Omega} I^M \, dx \right)^{1/2} \cdot J^{1/2}.
\]

Since we also have the inequality with \( n \) replaced by \( p \), we deduce that

\[
|(n - M)_+|_{1, \Gamma_N} + |(p - M)_+|_{1, \Gamma_N} \leq C \left( \int_{\Omega} I^M \, dx \right)^{1/2} \cdot J^{1/2}. \tag{4.20}
\]

It follows from (4.19), (4.20), and the Schwarz inequality that

\[
\int_{\Gamma_N} I_1 \, dS \leq |b|_{\infty, \Gamma_N} |v|_\infty (n - M)_+ - (p - M)_+|_{1, \Gamma_N} \\
\leq C \left( \int_{\Omega} I^M \, dx \right)^{1/2} (J + J^{1/2}) \\
\leq C \left( \int_{\Omega} I^M \, dx \right)^{1/2} J + \mu J + \frac{C}{\mu} \int_{\Omega} I^M \, dx, \tag{4.21}
\]

where \( \mu \) is an arbitrary positive number.

Next we deal with \( I_2 \). Put \( n_M = (n - M)_+ + M \) and \( p_M = (p - M)_+ + M \), and let \( r' < 2 \) be the Hölder conjugate of \( r \). Then

\[
|n_M|_{r', \Gamma_N} = \sqrt{\frac{n_M}{2r', \Gamma_N}} \leq 2\sqrt{n_M} - \sqrt{M} \left( \frac{2r'}{2r', \Gamma_N} \right) + 2\sqrt{M} \left( \frac{2r'}{2r', \Gamma_N} \right) \\
\leq \mu |\nabla (\sqrt{n_M} - \sqrt{M})|^2 + C\mu^{r'} |\sqrt{n_M} - \sqrt{M}|^2 + CM,
\]

where we have used (3.8) with \( q = 2r' \) in deriving the last inequality. Notice that

\[
|\nabla (\sqrt{n_M} - \sqrt{M})|^2 = |\nabla \sqrt{n_M}|^2 = \frac{1}{4} \int_{\Omega} \frac{|\nabla n_M|^2}{n_M} \, dx = \frac{1}{4} \int_{n \geq M} |\nabla n|^2 \, dx \leq \frac{1}{4} J, \\
|\sqrt{n_M} - \sqrt{M}|^2 = \left( \int_{n \geq M} (\sqrt{n} - \sqrt{M}) \, dx \right)^2 \leq \left( \int_{n \geq M} \sqrt{n} \, dx \right)^2 \leq |\Omega| \int_{n \geq M} n \, dx \leq |\Omega| \int_{\Omega} I^M \, dx.
\]
Thus, we arrive at
\[ |n_M|_{ \Gamma_N} \leq \mu J + C \mu^{-a} \int_\Omega I^M dx + CM. \]

Since we also have the inequality with \( n \) replaced by \( p \), we see that
\[ |(n - M_+ - (p - M)_+|_{ \Gamma_N} = |n_M - p_M|_{ \Gamma_N} \leq \mu J + C \mu^{-a} \int_\Omega I^M dx + CM. \]

This together with the Hölder inequality gives
\[ \int_{\Gamma_N} I_3 dS \leq |g|_{ \Gamma_N} |(n - M)_+ - (p - M)_+|_{ \Gamma_N} \leq C \mu J + C \mu^{-a} \int_\Omega I^M dx + CM. \quad (4.22) \]

To estimate \( I_3 \), we use the following:
\[ (n - M)_+ - (p - M)_+ = n - p - (n - M)_- + (p - M)_-. \]
\[ |(n - M)_- - (p - M)_-| \leq M, \quad |(n - M)_+ - (p - M)_+| \leq |n - p|. \]

From these and the Schwarz inequality, we deduce that
\[ I_3 = - (n - p)^2 + (n - p)| (n - M)_- - (p - M)_-| + D((n - M)_+ - (p - M)_+) \]
\[ \leq - (n - p)^2 + M |n - p| + |D||n - p| \]
\[ \leq - \frac{1}{2} (n - p)^2 + M^2 + |D|^2. \quad (4.23) \]

Plugging (4.21)–(4.23) into (4.18) and choosing \( \mu \) appropriately small, we obtain (4.16). Thus, the proof is complete.

We are now in a position to prove Proposition 4.1.

**Proof of Proposition 4.1.** Put
\[ E = E(t) := \int_\Omega (\varepsilon_1 + \varepsilon_2) dx + \int_{\Gamma_N} \varepsilon_3 dS \]
and define
\[ \theta_0 := \min \left\{ \theta_1, \frac{1}{4C_1\hat{C}_1 \max\{\delta, 1\}}, \frac{c_1}{4\hat{C}_1(5C_1 \max\{\delta, 1\} + 1)} \right\}, \]
\[ M_2 := \max \left\{ h^{-1} \left( \frac{1}{\theta_0} \right), M_1, \max\{|N_0|_{\infty}, |P_0|_{\infty}\} \right\}, \]
where \( \theta_1, C_1, \hat{C}_1, c_1, \) and \( M_1 \) are the constants given in Lemmas 4.1–4.3.

In what follows, we assume that \( \theta < \theta_0 \). The proof is completed by showing that
\[ E' + \tilde{c} E \leq \tilde{C} \]
(4.24)
in some interval \((T_0, \infty)\), where \( \tilde{c} \) and \( \tilde{C} \) are positive constants depending only on \( \max\{\delta, 1\}, c_0, C_0, \Omega, \Gamma_D, \rho, \gamma, \) and \( c_R \). Indeed, applying the Gronwall inequality gives
\[ \limsup_{t \to \infty} E(t) \leq \limsup_{t \to \infty} \left\{ E(T_0) e^{-\tilde{c}(t - T_0)} + \frac{\tilde{C}}{\tilde{c}} \left( 1 - e^{-\tilde{c}(t - T_0)} \right) \right\} = \frac{\tilde{C}}{\tilde{c}}. \]
First let us determine $T_0$. By the definitions of $\theta_0$ and $M_2$, we have

$$\tilde{C}_1 \left( \theta_0 + \frac{1}{h(M_2)} \right) \leq 2\tilde{C}_1 \theta_0 \leq \frac{1}{2C_1 \max\{\delta, 1\}}.$$ 

Hence, we see from (4.9) that

$$\int (n + p) dx \leq \int I M_1 dx + M_2 |\Omega| \leq \frac{1}{2C_1 \max\{\delta, 1\}} \left\{ \int (D_1 + D_2) dx + 1 \right\} + M_2 |\Omega|.$$ 

This together with (4.2) gives

$$E' + \frac{1}{4} \int \frac{D_1 + D_2}{\Omega} dx \leq C_2 \delta.$$ (4.25)

where $C_2 = C_2(\max\{\delta, 1\}, c_0, C_0, \Omega, \Gamma_D, \rho, \epsilon, c_R) > 0$ is a constant. Integrating this over $[t_0, t] \subset I$ leads to

$$\lim_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} (D_1 + D_2) dx dt \leq \lim_{t \to \infty} \left( \frac{4E(t_0)}{t - t_0} + 4C_2 \delta \right) = 4C_2 \delta.$$ 

Therefore, we can take $T_0 > 0$ such that

$$\int_{\Omega} (D_1 + D_2) dx \bigg|_{t = T_0} \leq 5C_2 \delta.$$ (4.26)

The constants $\tilde{c}$ and $\tilde{C}$ are chosen as follows. The definitions of $\theta_0$ and $M_2$ give

$$\tilde{C}_1 \left( \theta_0 + \frac{1}{h(M_2)} \right) \leq \frac{c_1}{2(5C_1 \max\{\delta, 1\} + 1)}.$$ (4.27)

This together with (4.9) and (4.26) shows that

$$\int_{\Omega} I M_1 dx \bigg|_{t = T_0} \leq \frac{c_1}{2(5C_1 \max\{\delta, 1\} + 1)} \left\{ \int_{\Omega} (D_1 + D_2) dx \bigg|_{t = T_0} + 1 \right\} \leq \frac{c_1}{2}.$$ 

This gives $\int_{\Omega} I M_1 dx \leq c_1$ in some open interval $J_0 \ni T_0$. Applying Lemma 4.3, we deduce that

$$E \leq C \left\{ \int_{\Omega} (D_1 + D_2) dx + 1 \right\} \text{in} J_0.$$ (4.28)

where $C = C(\max\{\delta, 1\}, c_0, C_0, \Omega, \Gamma_D, \rho, \epsilon, c_R) > 0$ is a constant. This together with (4.25) yields

$$E' + c_1 E \leq C_3 \text{ in } J_0.$$ (4.29)

Furthermore, (4.9), (4.26), and (4.28) imply that

$$E(T_0) \leq C_4.$$ (4.30)
Here, $c_3$, $C_3$, and $C_4$ are positive constants depending only on $\max\{\delta, 1\}$, $c_0$, $C_0$, $\Omega, \Gamma_D, p, r, \epsilon$, and $c_R$. Now we choose $\tilde{c}$ and $\tilde{C}$ as

$$
\tilde{c} := \min \left\{ c_3, \frac{C_2 \max\{\delta, 1\}}{4C_4} \right\}, \quad \tilde{C} := C_3.
$$

We complete the proof by showing (4.24). From (4.29), we see that the set

$$
\mathcal{T} := \{ \tau \in (T_0, \infty) \mid (4.24) \text{ holds in } (T_0, \tau) \}
$$

is nonempty, and therefore, $T_1 := \sup \mathcal{T} \in (T_0, \infty]$. What is left is to show that $T_1 = \infty$. On the contrary, suppose that $T_1 < \infty$. Then we have either

$$
\int_\Omega \frac{M_2}{c_1} \leq c_1 \quad \text{or} \quad \int_\Omega M_2 \geq c_1
$$

We first consider the former case. In the same way as the derivation of (4.29), one can show that (4.24) holds in some open interval $J_1 \ni T_1$. From this, we find that $J_1 \subset \mathcal{T}$, which contradicts the fact that $T_1$ is the supremum of $\mathcal{T}$. Next let us consider the latter case. We take $T_2 > T_1$ such that $\int_\Omega M_2 \geq c_1/2$ on $[T_1, T_2]$. Then, from (4.9) and (4.27), we have

$$
\int_\Omega (D_1 + D_2) dx \geq \frac{2(5C_2 \max\{\delta, 1\} + 1)}{c_1} \int_\Omega M_2 dx - 1 \geq 5C_2 \max\{\delta, 1\}
$$

on $[T_1, T_2]$. Plugging this into (4.25) leads to

$$
E' \leq -\frac{1}{4}C_2 \max\{\delta, 1\} \text{ on } [T_1, T_2].
$$

which particularly yields

$$
E \leq E(T_1) \text{ on } [T_1, T_2].
$$

Since (4.24) holds on $[T_0, T_1]$, we see from the Gronwall inequality that

$$
E(T_1) \leq E(T_0) e^{-\tilde{c}(T_1 - T_0)} + \frac{\tilde{C}}{\tilde{c}} \left( 1 - e^{-\tilde{c}(T_1 - T_0)} \right) \leq E(T_0) + \frac{\tilde{C}}{\tilde{c}}.
$$

This together with (4.30) and (4.32) shows that

$$
E \leq C_4 + \frac{\tilde{C}}{\tilde{c}} \text{ on } [T_1, T_2].
$$

By (4.31), (4.33), and the definition of $\tilde{c}$, we obtain

$$
E' + \tilde{c}E \leq -\frac{1}{4}C_2 \max\{\delta, 1\} + \tilde{c}C_4 + \tilde{C} \leq \tilde{C} \text{ on } [T_1, T_2].
$$

This gives $T_2 \in \mathcal{T}$, a contradiction. We thus conclude that $T_1 = \infty$, and the proof is complete.

4.2 | Proof of Proposition 4.2

We prove Proposition 4.2 by the iteration argument of Moser. To this end, we put

$$
J_\gamma = J_{\gamma}(t) := \int_\Omega \{ (n - M_0)_+^\gamma + (p - M_0)_+^\gamma \} dx,
$$

$$
K_\gamma = K_{\gamma}(t) := \int_\Omega \left\{ |\nabla(n - M_0)_+^\gamma|^2 + |\nabla(p - M_0)_+^\gamma|^2 \right\} dx,
$$

where $M_0 := \max\{|N_b|_\infty, |P_b|_\infty\}$ and $\gamma \geq 1$. 
Lemma 4.4. There exist constants $C = C(C_0, \Omega, \Gamma_D, r, \varepsilon, c_R) > 0$ and $\beta = \beta(r) > 0$ such that

$$J'_{2\gamma} + K_{2\gamma} \leq C(\|v\|_1 + 1)^\beta \gamma \beta (J'_{2\gamma} + 1). \quad (4.34)$$

Proof. We put $\zeta := (n - M_0)^2_+ + \xi := (p - M_0)^2_+$. Take $\phi_1 = (n - M_0)^2_+ + M_0$ in (2.2a) to obtain

$$\langle n', (n - M_0)^2_+ \rangle + (2\gamma - 1) \int_{\Omega} (n - M_0)^{2\gamma}_+ \nabla n \cdot \nabla (n - M_0)_+ dx$$

$$- \int_{\Omega} n \nabla v \cdot \nabla (n - M_0)^{2\gamma}_+ dx + \int_{\Omega} R(n, p)(n - M_0)^{2\gamma}_+ dx = 0. \quad (4.35)$$

The first two terms of the left-hand side of this equality are written as

$$\langle n', (n - M_0)^2_+ \rangle = \frac{1}{2\gamma} \frac{d}{dt} \int_{\Omega} (n - M_0)^{2\gamma}_+ dx = \frac{1}{2\gamma} \frac{d}{dt} \|\zeta\|^2_2. \quad (4.36)$$

$$\int_{\Omega} (n - M_0)^{2\gamma}_+ \nabla n \cdot \nabla (n - M_0)_+ dx = \frac{1}{2\gamma} \int_{\Omega} |\nabla (n - M_0)^{2\gamma}_+|^2 dx = \frac{1}{\gamma^2} \|\nabla \zeta\|^2_2. \quad (4.37)$$

Here, (4.36) is validated by (2.1). Note that $n \nabla (n - M_0)^{2\gamma}_+ = \nabla F(n)$, where

$$F(n) := \frac{2\gamma - 1}{2\gamma} (n - M_0)^{2\gamma}_+ + M_0 (n - M_0)^{2\gamma}_-. $$

Hence, using (2.2c) with $\phi_3 = F(n)$, we have

$$\int_{\Omega} n \nabla v \cdot \nabla (n - M_0)^{2\gamma}_+ dx = \int_{\Omega} \nabla v \cdot \nabla F(n) dx$$

$$= -\frac{1}{\varepsilon} \int_{\Gamma_N} (bv - g) F(n) dS - \frac{1}{\varepsilon} \int_{\Omega} (n - p - D) F(n) dx. \quad (4.38)$$

Substituting (4.36)–(4.38) into (4.35) yields

$$\frac{1}{2\gamma} \frac{d}{dt} \|\zeta\|^2_2 + \frac{2\gamma - 1}{2\gamma^2} \|\nabla \zeta\|^2_2 + \frac{1}{\varepsilon} \int_{\Omega} (n - p) F(n) dx = \frac{1}{\varepsilon} \int_{\Gamma_N} I_1 dS + \int_{\Omega} I_2 dx,$$

$$I_1 := -(bv - g) F(n), \quad I_2 := \frac{1}{\varepsilon} DF(n) - R(n, p) \zeta^{2\gamma - 1} / \gamma. \quad (4.39)$$

Let us estimate the right-hand side of (4.39). From now on, let $C$ denote a positive constant depending only on $C_0, \Omega, \Gamma_D, r, \varepsilon, \text{and} \ c_R$. By the Young inequality, we have

$$F(n) = \frac{2\gamma - 1}{2\gamma} \zeta^{2\gamma} + M_0 \zeta^{2\gamma - 1}/\gamma \leq \frac{2\gamma - 1}{2\gamma} \zeta^2 + M_0 \left( \frac{2\gamma - 1}{2\gamma} \zeta^{2\gamma} + \frac{1}{2\gamma} \right) \leq C(\zeta^2 + 1). \quad (4.40)$$
From this, the Hölder inequality, (3.7) and (3.8), we see that

\[
\int_{\Gamma_N} I_1 dS \leq C \int_{\Gamma_N} (|b| + |g|)(\zeta^2 + 1) dS \\
\leq C |b|_{L^4} |\zeta_{1/3}^N| \zeta^2 + 1 |4/3|_{\Gamma_N} + C |g|_{L^8} |\zeta_{1/3}^N| \zeta^2 + 1 |4/3|_{\Gamma_N} \\
\leq C (|\zeta_{1/3}^N| + 1) \left( |\zeta_{2/3}^N| + |\zeta_{1/3}^N| \right) \\
\leq C (\|v\|_1 + 1) \left( \mu |\nabla \zeta|_2^2 + \mu^{-5} |\zeta|_1^2 + 1 \right). \quad (4.41)
\]

where \( r < 2 \) is the Hölder conjugate of \( r \), \( \tilde{a} = \tilde{a}(r) > 0 \) is a constant, and \( \mu > 0 \) is an arbitrary number. Furthermore, (4.40) and \(|R(n, p)| \leq c_R (n + p + 1)\) together with the Young inequality yield

\[
I_2 \leq C (\zeta^2 + \zeta^2 + 1).
\]

By the Gagliardo–Nirenberg, Poincaré, and Young inequalities, we have

\[
\int_{\Omega} I_2 dx \leq C (|\nabla \zeta|_2^{5/2} |\zeta|_1^{4/5} + |\nabla \zeta|_2^{5/2} |\zeta|_1^{4/5}) + C \\
\leq C \mu (|\nabla \zeta|_2^2 + |\nabla \zeta|_2^2) + C \mu^{-3/2} (|\zeta|_1 + |\zeta|_1^2 + 1). \quad (4.42)
\]

Plugging (4.41) and (4.42) into (4.39), we deduce that

\[
\frac{d}{dt} |\zeta|_2^2 + \frac{2(2\gamma - 1)}{\gamma} |\nabla \zeta|_2^2 + \frac{2\gamma}{\epsilon} \int_{\Omega} (n - p) F(n) dx \\
\leq C (\|v\|_1 + 1) \gamma \left\{ \mu (|\nabla \zeta|_2^2 + |\nabla \zeta|_2^2) + \max \{ \mu^{-5}, \mu^{-3/2} \} (|\zeta|_1 + |\zeta|_1^2 + 1) \right\}.
\]

Performing the same computation for \( \xi \) and adding the result to the above inequality, we obtain

\[
\frac{d}{dt} I_{2\gamma} + \frac{2(2\gamma - 1)}{\gamma} K_{\gamma} + \frac{2\gamma}{\epsilon} \int_{\Omega} (n - p)(F(n) - F(p)) dx \\
\leq C (\|v\|_1 + 1) \gamma \left( \mu K_{\gamma} + \max \{ \mu^{-5}, \mu^{-3/2} \} J_{2\gamma} + 1 \right).
\]

Note that \((n - p)(F(n) - F(p)) \geq 0\), since \(F(z)\) is nondecreasing in \(z\). Therefore, by choosing \( \mu \) as \( \mu = c (\|v\|_1 + 1)^{-1/\gamma} / c \) with a suitable constant \( c = c(C_0, \Omega, \Gamma_D, r, \epsilon, c_R) > 0 \), we conclude that (4.34) holds with \( \beta := \max \{ \tilde{a} + 1, 5/2 \} \). \( \square \)

Let us prove Proposition 4.2.

**Proof of Proposition 4.2.** In the proof, \( c \) and \( C \) denote positive constants depending only on \( L \), \( c_0 \), \( C_0 \), \( \Omega \), \( \Gamma_D \), \( r \), \( \epsilon \), and \( c_R \). From the definition of \( L \), we can take \( t_0 \in I \) such that for all \( t \geq t_0 \),

\[
J_1(t) + \|v(t)\|_1 \leq 2L. \quad (4.43)
\]

We first take \( \gamma = 1 \) in (4.34). Since the Poincaré inequality gives \( K_1 \geq c J_2 \), we see from the Gronwall inequality that

\[
J_2(t) \leq J_2(t_0) e^{-\alpha(t-t_0)} + C \int_{t_0}^t e^{-\alpha(t-r)} (\|v(r)\|_1 + 1) \beta (J_1(r)^2 + 1) dr.
\]
for all $t \geq \tau_0$. From this and (4.43), we have

$$\limsup_{t \to \infty} J_2(t) \leq \limsup_{t \to \infty} \left\{ J_2(\tau_0) e^{-c(t-\tau_0)} + C \int_{\tau_0}^{t} e^{-c(t-\tau)} d\tau \right\} = \frac{C}{c}.$$  

This particularly gives

$$\limsup_{T \to \infty} \int_{T-1}^{T+1} (n^2 + p^2) dx dt \leq C \limsup_{T \to \infty} \left( J_2(t) + 1 \right) dt \leq C. \quad (4.44)$$  

By the iteration argument of Moser,\(^{22}\) one can show that for all $T \geq \tau_0 + 1$,

$$\sup_{[T, T+1] \times \Omega} (n + p) \leq C \left\{ \int_{T-1}^{T+1} (n^2 + p^2) dx dt \right\}^{1/2} + C. \quad (4.45)$$  

The proposition immediately follows by combining (4.44) and (4.45). To complete the proof, we briefly derive (4.45). Let $0 < \kappa \leq 1/2$ and let $\rho \in C^\infty(\mathbb{R})$ satisfy

$$\rho(t) = \begin{cases} 0 & \text{for } t \leq T - 2\kappa, \\ 1 & \text{for } t \geq T - \kappa, \end{cases} \quad 0 \leq \rho'(t) \leq \frac{2}{\kappa}. \quad (4.46)$$  

Multiplying (4.34) by $\rho$ and integrating it, we see that for all $T - 1 \leq t_1 \leq T + 1$,

$$J_{2\gamma}(t_1) \rho(t_1) + \int_{T-1}^{t_1} K_\gamma \rho dt \leq \int_{T-1}^{t_1} J_{2\gamma} \rho' dt + C \gamma^\beta \int_{T-1}^{t_1} (J_{2\gamma}^2 + 1) \rho dt$$

$$\leq C \left( \frac{1}{\kappa} + \gamma^\beta \right) \int_{T-2\kappa}^{T+1} J_{2\gamma} dt + 1 \right\}. \quad (4.47)$$  

where we have used (4.43) and (4.46) and the fact that $J_{2\gamma}^2 \leq CJ_{2\gamma}$ in deriving the last inequality. We take $t_1 = T + 1$ in (4.47) to obtain

$$\int_{T-\kappa}^{T+1} K_\gamma dt \leq C \left( \frac{1}{\kappa} + \gamma^\beta \right) \int_{T-2\kappa}^{T+1} J_{2\gamma} dt + 1 \right\}. \quad (4.48)$$  

We choose $t_1 \in [T - \kappa, T + 1]$ such that

$$J_{2\gamma}(t_1) = \max_{t \in [T - \kappa, T + 1]} J_{2\gamma}(t).$$  

Then (4.47) also gives

$$\max_{t \in [T - \kappa, T + 1]} J_{2\gamma}(t) \leq C \left( \frac{1}{\kappa} + \gamma^\beta \right) \int_{T-2\kappa}^{T+1} J_{2\gamma} dt + 1 \right\}. \quad (4.49)$$  

We know from Moser\(^{22}\), Lemma 2 that for $\lambda = 5/3$,

$$\int_{T-\kappa}^{T+1} J_{2\gamma} dt \leq C \left( \max_{t \in [T - \kappa, T + 1]} J_{2\gamma}(t) \right)^{2/3} \int_{T-\kappa}^{T+1} K_\gamma dt.$$
This together with (4.48) and (4.49) leads to
\[
\left( \int_{T-\kappa}^{T+1} J_{2\kappa} dt \right)^{1/(2\gamma)} + 1 \leq \tilde{C}(1/(2\gamma)) \left( \frac{1}{\kappa} + \gamma \right)^{1/(2\gamma)} \left( \int_{T-2\kappa}^{T+1} J_{2\kappa} dt \right)^{1/(2\gamma)} + 1,
\]
where \( \tilde{C} = \tilde{C}(L, C_0, \Omega, \Gamma_D, r, \varepsilon, c_R) > 0. \) Substituting \( \gamma = \gamma_n := \lambda^n \) and \( \kappa = \kappa_n := 2^{-n-1} \) into this inequality, we have \( I_{n+1} \leq f_j n^{1/(2\gamma_n)} I_n, \) where
\[
I_n := \left( \int_{T-2\kappa_n}^{T+1} J_{2\kappa_n} dt \right)^{1/(2\gamma_n)} + 1, \quad f_j n := \tilde{C}(2^{n+1} + \lambda^{\beta n}), \quad n = 0, 1, \ldots .
\]
Hence, we see that
\[
I_n \leq \prod_{k=0}^{n} f_k n^{1/(2\kappa_k)} I_0 \leq C I_0.
\]
Letting \( n \to \infty \) gives (4.45), and the proof is complete.

**4.3 Proof of Proposition 4.3**

To obtain the lower bounds of \( n \) and \( p \), we derive an inequality similar to (4.34). Let \( m_0 \) be a constant defined by
\[
m_0 := \min \left\{ c_0, \frac{\varepsilon c_R}{2c_0 + 2 + |D|_\infty + \varepsilon c_R} \right\}.
\]
For \( \gamma > 0, \gamma \neq 1, 0 < \sigma \leq m_0/2, \) and \( z \geq 0, \) we write
\[
G_0(z) := \left\{ (z_\sigma)^{-\gamma} - m_0^{-\gamma} \right\}^+ = (z_\sigma)^{-\gamma} - m_0^{-\gamma}, \quad G_1(z) := \int_z^{m_0} G_0(y) dy,
\]
\[
G_2(z) := \int_z^{m_0} \sqrt{zG_0(y)} dy = \frac{2\sqrt{\gamma}}{\gamma - 1} \left\{ (z_\sigma)^{-\gamma+1} - m_0^{-\gamma+1} \right\},
\]
\[
G_3(z) := \int_z^{m_0} z \frac{dG_0(y)}{dz} dy = \frac{\gamma}{\gamma - 1} \left\{ (z_\sigma)^{-\gamma+1} - m_0^{-\gamma+1} \right\},
\]
where
\[
z_\sigma^{m_0} := (z_\sigma - m_0)^+ + m_0 = \left\{ \begin{array}{ll}
\sigma & (z \leq \sigma), \\
z & (\sigma < z < m_0), \\
m_0 & (z \geq m_0).
\end{array} \right.
\]

**Lemma 4.5.** There exist constants \( C = C(C_0, \Omega, \Gamma_D, r, \varepsilon) > 0 \) and \( \beta = \beta(r) > 0 \) such that
\[
\frac{d}{dt} \int_{\Omega} G_1(n) dx + \frac{1}{2} \int_{\Omega} |\nabla G_2(n)|^2 dx + \sqrt{\frac{c_R}{\varepsilon}} \int_{\Omega} \sqrt{G_3(n)G_0(n)} dx
\]
\[
\leq C(||v||_1 + 1)^\beta |\gamma - 1|^\beta \int_{\Omega} G_2(n)^2 dx + \frac{C(||v||_1 + 1)\gamma}{|\gamma - 1|} m_0^{-2\gamma+2}.
\]
(4.50)
Proof. We can choose \( \phi_1 = G_0(n) \in H_D^1(\Omega) \) as a test function in (2.2a). Then we have

\[
\frac{d}{dt} \int_{\Omega} G_1(n) dx + \int_{\Omega} |\nabla G_2(n)|^2 dx + \int_{\Omega} n \nabla v \cdot \nabla G_0(n) dx - \int_{\Omega} R(n, p) G_0(n) dx = 0,
\]

where we have used the fact that

\[
\langle n', G_0(n) \rangle = -\frac{d}{dt} \int_{\Omega} G_1(n) dx, \quad \int_{\Omega} \nabla n \cdot \nabla G_0(n) dx = -\int_{\Omega} |\nabla G_2(n)|^2 dx.
\]

By using (2.2c) with \( \phi_3 = G_3(n) \in H_D^1(\Omega) \), the third term of the left-hand side is computed as

\[
\int_{\Omega} n \nabla v \cdot \nabla G_0(n) dx = \int_{\Omega} \nabla v \cdot \nabla G_3(n) dx = -\frac{1}{\varepsilon} \int_{\Gamma_1} (bv - g) G_3(n) dS - \frac{1}{\varepsilon} \int_{\Omega} (n - p - D) G_3(n) dx.
\]

Therefore, we arrive at

\[
\frac{d}{dt} \int_{\Omega} G_1(n) dx + \int_{\Omega} |\nabla G_2(n)|^2 dx + \int_{\Omega} I_2 dx = \frac{1}{\varepsilon} \int_{\Gamma_1} I_1 dS,
\]

where

\[
I_1 := (bv - g) G_3(n), \quad I_2 := -\frac{1}{\varepsilon} (n - p - D) G_3(n) - R(n, p) G_0(n).
\]

Notice that \( G_3(z) = (\gamma - 1) G_2(z)^2 / 4 + \sqrt{\gamma \rho_{m_0} r} + G_2(z) \). From this, we have

\[
G_3(z) \leq |\gamma - 1| G_2(z)^2 + \frac{\gamma m_0^{-2r+2}}{|\gamma - 1|}.
\]

Therefore, in the same way as the derivation of (4.41), the right-hand side of (4.51) is estimated as

\[
\int_{\Gamma_1} I_1 dS \leq \int_{\Gamma_1} (|v| + |g|) \left( |\gamma - 1| G_2(n)^2 + \frac{\gamma}{|\gamma - 1|} m_0^{-2r+2} \right) dS
\]

\[
\leq \frac{\varepsilon}{2} |\nabla G_3(n)|^2 + C(||v||_1 + 1)^{\beta} |\gamma - 1| G_2(n)^2 + \frac{C_0}{|\gamma - 1|} (||v||_1 + 1) m_0^{-2r+2}.
\]

Here, \( \beta = \beta(r) > 0 \) and \( C = C(C_0, \Omega, \Gamma_D, r, \varepsilon) > 0 \) are constants. To estimate \( I_2 \), we note that

\[
zG_0(z) = \int_{-z}^{m_0} \frac{dG_0}{dz}(y) dy \leq G_3(z) \leq \int_{z}^{m_0} \frac{dG_0}{dz}(y) dy = m_0 G_0(z).
\]

Hence, we see that

\[
I_2 = \left\{ \frac{1}{\varepsilon} (n + p + 2) G_3(n) + \frac{c_0}{n + p + 2} G_0(n) \right\} - \left\{ \frac{1}{\varepsilon} (2n + 2 - D) G_3(n) + \frac{c_0}{n + p + 2} m_0 G_0(n) \right\}
\]

\[
\geq 2 \frac{c_0}{\varepsilon} G_3(n) G_0(n) - \left\{ \frac{1}{\varepsilon} (2m_0 + 2 + |D|) G_3(n) + c_0 G_3(n) \right\}
\]

\[
\geq 2 \frac{c_0}{\varepsilon} G_3(n) G_0(n) - \left\{ \frac{1}{\varepsilon} (2m_0 + 2 + |D|) + c_0 \right\} \sqrt{m_0 G_3(n) G_0(n)}.
\]
Proof of Proposition 4.3. In the proof, we denote by $C$ a positive constant depending only on $\bar{L}$, $c_0$, $C_0$, $\Omega$, $\Gamma_D$, $r$, $\varepsilon$, and $c_R$. By the assumption, we can choose $r_0 > 0$ such that $\|v(t)\|_1 \leq 2\bar{L}$ for all $t \geq r_0$.

In what follows, we fix $T \geq r_0 + 1$. Let us show that

$$\int_{\Omega} \left( \sum_{i=1}^{T+1} \int_{t=1}^{t+1} (n^{m_0})^{-\gamma} \right) dt \leq C.$$  \hfill (4.55)

For this purpose, we integrate both sides of (4.50) over $[T - 1, T + 1]$ to obtain

$$\int_{\Omega} \left( \int_{T-1}^{T+1} G_3(n)G_0(n) \right) dx dt \leq \int_{\Omega} G_1(n) dx + C \left( \int_{T-1}^{T+1} \int_{\Omega} G_2(n) G_0(n) dx dt + \frac{\gamma}{\left| \gamma - 1 \right|} m_0^{-2\gamma + 2} \right).$$  \hfill (4.56)

It is elementary to show that if $\gamma < 1$, then

$$G_1(z) \leq \frac{1}{1 - \gamma} m_0^{-\gamma}, \quad G_2(z) \leq \frac{2\sqrt{\gamma}}{1 - \gamma} m_0^{-\gamma}, \quad G_3(z) \geq \frac{\gamma}{1 - \gamma} \left( 1 - \frac{1}{2\gamma} \right) m_0^{-\gamma} \mathbb{1}_{z \leq m_0/2}.$$

Hence, by choosing $\gamma = 1/2$ in (4.56), we obtain (4.55).

From now on, we suppose that $\gamma > 1$. Let $t_1 \in [T - 1, T + 1]$ and let $\rho \in C^\infty(\mathbb{R})$ be a nonnegative function satisfying $\rho(T - 1) = 0$. Then, multiplying (4.50) by $\rho$ and integrating over $[T - 1, t_1]$ yield

$$\rho \int_{\Omega} G_1(n) dx + \int_{t=1}^{t_1} \int_{\Omega} |\nabla G_2(n)|^2 \rho dx dt \leq \int_{t=1}^{t_1} \int_{\Omega} G_1(n) \rho' dx dt + \frac{\gamma}{\gamma - 1} \int_{t=1}^{t_1} \int_{\Omega} G_2(n)^2 \rho dx dt + \frac{\gamma}{\gamma - 1} m_0^{-2\gamma + 2} \int_{T-1}^{T+1} \rho dt.$$

Note that

$$\frac{1}{\gamma - 1} (c_\sigma^{m_0})^{1 - \gamma} - \frac{\gamma}{\gamma - 1} m_0^{-\gamma + 1} \leq G_1(z) \leq \frac{\gamma}{\gamma - 1} (c_\sigma^{m_0})^{-1 + 1}, \quad G_2(z) \leq \frac{2\sqrt{\gamma}}{\gamma - 1} (c_\sigma^{m_0})^{-\gamma + 1}.$$

From these, we have

$$\rho \int_{\Omega} \left( \sum_{i=1}^{T+1} \int_{t=1}^{t+1} (n^{m_0})^{-(\gamma - 1)} \right) dx dt + \frac{\gamma}{\gamma - 1} \int_{t=1}^{t_1} \int_{\Omega} |\nabla \left( \left( n^{m_0} \right)^{\frac{\gamma - 1}{\gamma}} \right) |^2 \rho dx dt \leq \frac{\gamma}{\gamma - 1} \int_{\Omega} \left( \sum_{i=1}^{T+1} \int_{t=1}^{t+1} \left( \gamma |\rho'| + \gamma \rho \right) dx dt \right) + C \int_{T-1}^{T+1} \rho dt + C \gamma m_0^{-\gamma + 1}.$$
Thus, by the same argument as in the derivation of (4.45), we deduce that

\[
\sup_{[T,T+1] \times \Omega} \left( n_\sigma^{m_0} \right)^{-1} \leq C \left\{ \int_{T-1}^{T+1} \int_{\Omega} \left( \frac{n_\sigma^{m_0}}{\sigma} \right)^{-\frac{1}{2}} \, dx dt \right\}^4 + C.
\]

Plugging (4.55) into this inequality and letting \( \sigma \to 0 \), we obtain \( |n(T)^{-1}|_\infty \leq C \). The inequality \( |p(T)^{-1}|_\infty \leq C \) can be shown in the same way, and therefore, the proof is complete. \( \square \)

## 5 Estimates of the Difference of Solutions

In this section, we estimate the relative entropy of any two solutions \((n_1, p_1, v_1)\) and \((n_2, p_2, v_2)\) of (1.1). Theorem 2.1 ensures that if \( \theta < \theta_0 \), then \((n_1, p_1, v_1)\) and \((n_2, p_2, v_2)\) satisfy

\[
(2\hat{C})^{-1} \leq n_1, n_2, p_1, p_2 \leq 2\hat{C} \quad \text{in} \quad (\bar{t}, \infty) \times \Omega
\]

for some \( \bar{t} \in \mathbb{R} \), where \( \hat{C} = \hat{C}(c_0, C_0, \Omega, \Gamma_D, \rho, r, \epsilon, c_R) \) is the constant being in (2.3). Throughout this section, we suppose that \( \theta < \theta_0 \) and \( t \geq \bar{t} \). We set

\[
\varphi := \frac{n_1}{n_2} - 1, \quad \psi := \frac{p_1}{p_2} - 1, \quad \eta := v_1 - v_2.
\]

The goal of this section is to prove the following proposition.

**Proposition 5.1.** There exist positive constants \( \delta_0, c, \) and \( C \) depending only on \( c_0, C_0, \Omega, \Gamma_D, \rho, r, \epsilon, \) and \( c_R \) such that if \( \delta < \delta_0 \), then the following inequalities hold:

\[
|n_2\varphi(t)|_2 + |p_2\psi(t)|_2 + \|\eta(t)\|_1 \leq Ce^{-\alpha(t-\bar{t})},
\]

where \( \varphi, \psi, \) and \( \eta \) by (2.2a), we have

\[
\langle (n_1' - n_2', \phi_1) + \int_{\Omega} \{ (\nabla n_1 - n_1 \nabla v_1 - n_2 \nabla v_2) \cdot \nabla \phi_1 + (R(n_1, p_1) - R(n_2, p_2))\phi_1 \} \, dx \rangle = 0,
\]

where \( \phi_1 \in H^1_0(\Omega) \). From the following two equalities

\[
n_1 - n_2 = n_2 \varphi, \quad \nabla n_1 - n_1 \nabla v_1 - n_2 \nabla v_2 = n_1 (\nabla \log(1 + \varphi) - \nabla \eta) + \varphi (\nabla n_2 - n_2 \nabla v_2),
\]

we see that \( \varphi \) satisfies

\[
\langle (n_2 \varphi'), \phi_1 \rangle + \int_{\Omega} \{ n_1 (\nabla \log(1 + \varphi) - \nabla \eta) + \varphi (\nabla n_2 - n_2 \nabla v_2) \} \cdot \nabla \phi_1 \, dx + \int_{\Omega} (R(n_1, p_1) - R(n_2, p_2))\phi_1 \, dx = 0.
\]

Similarly, \( \psi \) solves

\[
\langle (p_2 \psi'), \phi_2 \rangle + \int_{\Omega} \{ p_1 (\nabla \log(1 + \psi) + \nabla \eta) + \psi (\nabla p_2 + p_2 \nabla v_2) \} \cdot \nabla \phi_2 \, dx + \int_{\Omega} (R(n_1, p_1) - R(n_2, p_2))\phi_2 \, dx = 0.
\]
where \( \phi_2 \in H^1_0(\Omega) \). We see from (2.2c) that \( \eta \) satisfies

\[
\varepsilon \int_{\Omega} \nabla \eta \cdot \nabla \phi_3 \, dx + \int_{\Gamma_N} b \eta \phi_3 \, dS = - \int_{\Omega} (n_2 \varphi - p_2 \psi) \phi_3 \, dx
\]

(5.5)

for all \( \phi_3 \in H^1_0(\Omega) \).

Now we derive an equality on the relative entropy of solutions.

**Lemma 5.1.** There holds that

\[
\frac{d}{dt} \left( \int_{\Omega} \mathcal{E} \, dx + \int_{\Gamma_N} \tilde{\mathcal{E}} \, dS \right) + \int_{\Omega} D \, dx = \int_{\Omega} (\mathcal{K} + \mathcal{L} + \mathcal{M}) \, dx,
\]

(5.6)

where

\[
\mathcal{E} := n_2 \int_0^\varphi \log(1 + y) \, dy + p_2 \int_0^\psi \log(1 + y) \, dy + \frac{\varepsilon}{2} |\nabla \eta|^2,
\]

\[
\tilde{\mathcal{E}} := \frac{b}{2} \eta^2,
\]

\[
D := n_1 |\nabla \log(1 + \varphi) - \nabla \eta|^2 + p_1 |\nabla \log(1 + \psi) + \nabla \eta|^2.
\]

\[
\mathcal{K} := (R(n_2, p_2) - R(n_1, p_1))(\log(1 + \varphi) + \log(1 + \psi)),
\]

\[
\mathcal{L} := R(n_2, p_2) \left( \varphi - \log(1 + \varphi) + \psi - \log(1 + \psi) \right),
\]

\[
\mathcal{M} := \varphi(n_2 - n_2 \nabla v_2) \cdot \nabla \eta - \psi(n_2 \nabla v_2 + p_2 \nabla v_2) \cdot \nabla \eta.
\]

**Proof.** It suffices to show

\[
\left( \int_{\Omega} \mathcal{E} \, dx + \int_{\Gamma_N} \tilde{\mathcal{E}} \, dS \right) \bigg|_t^i + \int_t^i \int_{\Omega} D \, dx \, dt = \int_t^i \int_{\Omega} (\mathcal{K} + \mathcal{L} + \mathcal{M}) \, dx \, dt,
\]

since this gives the absolute continuity of \( \int_0^t \mathcal{E} \, dx + \int_{\Gamma_N} \tilde{\mathcal{E}} \, dS \).

To make the following computation rigorous, we use a mollifier with respect to the time variable \( t \) due to the insufficiency of the regularity of solutions. We omit the argument since it is standard.

Choose \( \phi_1 = \log(1 + \varphi) - \eta \) in (5.4) to obtain

\[
\langle n_1' - n_2', \log(1 + \varphi) \rangle - \langle n_1' - n_2', \eta \rangle + \int_{\Omega} (n_1(\nabla \log(1 + \varphi) - \nabla \eta) + \varphi(\nabla n_2 - n_2 \nabla v_2)) \cdot \nabla (\log(1 + \varphi) - \eta) \, dx
\]

\[+ \int_{\Omega} (R(n_1, p_1) - R(n_2, p_2))(\log(1 + \varphi) - \eta) \, dx = 0.
\]

(5.7)
Let us rewrite the first and third terms of the left-hand side. Noting \( n_1 - n_2 = n_2 \varphi \), we have

\[
(n'_1 - n'_2) \log(1 + \varphi) = (n_2 \varphi)' \log(1 + \varphi)
\]

\[
= \left[ \int_0^{n_2 \varphi} \log \left( 1 + \frac{y}{n_2} \right) \, dy \right]' - \left[ \int_0^{n_2 \varphi} \log \left( 1 + \frac{y}{n_2} \right) \, dy \right]
\]

\[
= \left[ n_2 \int_0^\varphi \log(1 + y) \, dy \right]' + n'_2 (\varphi - \log(1 + \varphi)).
\]

Then, using (2.2a), we arrive at

\[
\langle n'_1 - n'_2, \log(1 + \varphi) \rangle
\]

\[
= \left\{ \int \limits_\Omega n_2 \left( \int \limits_0^\varphi \log(1 + y) \, dy \right) \, dx \right\}' - \int \limits_\Omega (\nabla n_2 - n_2 \nabla \varphi_2) \cdot \nabla (\varphi - \log(1 + \varphi)) \, dx - \int \limits_\Omega R(n_2, p_2)(\varphi - \log(1 + \varphi)) \, dx.
\]

One can rewrite the integrand of the third term on the left-hand side of (5.7) as

\[
\{ n_1(\nabla \log(1 + \varphi) - \nabla \eta) + \varphi(\nabla n_2 - n_2 \nabla \varphi_2) \} \cdot \nabla (\log(1 + \varphi) - \eta)
\]

\[
= n_1|\nabla \log(1 + \varphi) - \nabla \eta|^2 + (\nabla n_2 - n_2 \nabla \varphi_2) \cdot (\nabla (\varphi - \log(1 + \varphi)) + (\nabla n_2 - n_2 \nabla \varphi_2) \cdot \varphi \nabla \eta.
\]

From these, we obtain

\[
\left\{ \int \limits_\Omega n_2 \left( \int \limits_0^\varphi \log(1 + y) \, dy \right) \, dx \right\}' - \langle n'_1 - n'_2, \eta \rangle + \int \limits_\Omega n_1 |\nabla \log(1 + \varphi) - \nabla \eta|^2 \, dx
\]

\[
= \int \limits_\Omega \{ R(n_2, p_2) \cdot \nabla (\varphi - \log(1 + \varphi)) + (\nabla n_2 - n_2 \nabla \varphi_2) \cdot \varphi \nabla \eta - (R(n_1, p_1) - R(n_2, p_2))(\log(1 + \varphi) - \eta) \} \, dx. \quad \text{(5.8)}
\]

Similarly,

\[
\left\{ \int \limits_\Omega p_2 \left( \int \limits_0^\varphi \log(1 + y) \, dy \right) \, dx \right\}' + \langle p'_1 - p'_2, \eta \rangle + \int \limits_\Omega p_1 |\nabla \log(1 + \psi) + \nabla \eta|^2 \, dx
\]

\[
= \int \limits_\Omega \{ R(n_2, p_2) \cdot \nabla (\psi - \log(1 + \psi)) - (\nabla p_2 + p_2 \nabla \varphi_2) \cdot \psi \nabla \eta - (R(n_1, p_1) - R(n_2, p_2))(\log(1 + \psi) + \eta) \} \, dx. \quad \text{(5.9)}
\]

Note that (5.5) yields

\[
\langle n'_1 - n'_2, \eta \rangle - \langle p'_1 - p'_2, \eta \rangle = \langle (n_2 \varphi - p_2 \psi)', \eta \rangle = -\left\{ \int \limits_\Omega \frac{\varepsilon}{2} |\nabla \eta|^2 \, dx + \int \limits_{\Gamma_N} \frac{b}{2} \eta^2 \, dS \right\}'. \quad \text{(5.10)}
\]

Summing up (5.8)–(5.10) and integrating over \([\bar{t}, t]\) complete the proof.
We remark that in the case that \((n_2, p_2, v_2)\) is a stationary solution \((N, P, V)\) satisfying (1.2), the term \(\mathcal{K}\) is nonpositive, and the terms \(\mathcal{L}\) and \(\mathcal{M}\) are zero. Therefore, it is easier to show its global stability. Among these terms, \(\mathcal{M}\) is problematic to handle if \((n_2, p_2, v_2)\) does not satisfy (1.2). For this reason, we establish a new inequality in the following lemma.

**Lemma 5.2.** There is a constant \(C = C(\Omega, \Gamma_D, r, \varepsilon) > 0\) such that

\[
|\varphi \nabla \eta|_2 + |\psi \nabla \eta|_2 \leq C(|n_2 \varphi|_2 + |p_2 \psi|_2)(|\nabla \varphi|_2 + |\nabla \psi|_2). \tag{5.11}
\]

**Proof.** We note that by (5.5),

\[
|\eta|_\infty \leq C|n_2 \varphi - p_2 \psi|_2. \tag{5.12}
\]

This follows from (3.5) with \(h = (n_2 \varphi - p_2 \psi)/\varepsilon\), \(W_0 = 0\), \(\bar{b} = b/\varepsilon\), and \(\bar{g} = 0\).

Let us show (5.11). Taking \(\phi_3 = (\varphi^2 + \psi^2)\eta\) in (5.5) yields

\[
\varepsilon \left( |\varphi \nabla \eta|_2^2 + |\psi \nabla \eta|_2^2 \right) + \int_{\Gamma_N} b(\varphi^2 + \psi^2)\eta^2 dS = -2\varepsilon \int_{\Omega} \eta(\varphi \nabla \varphi + \psi \nabla \psi) \cdot \nabla \eta dx - \int_{\Omega} (n_2 \varphi - p_2 \psi)(\varphi^2 + \psi^2)\eta dx.
\]

By the Schwarz inequality and (5.12), the first term of the right-hand side of this equality is estimated as

\[
-2\varepsilon \int_{\Omega} \eta(\varphi \nabla \varphi + \psi \nabla \psi) \cdot \nabla \eta \leq 2\varepsilon \left( |\nabla \varphi|_2 + |\nabla \psi|_2 \right) \left( |\varphi \nabla \eta|_2^2 + |\psi \nabla \eta|_2^2 \right) 
\leq 2\varepsilon |\eta|_\infty^2 \left( |\nabla \varphi|_2^2 + |\nabla \psi|_2^2 \right) + \frac{\varepsilon}{2} \left( |\varphi \nabla \eta|_2^2 + |\psi \nabla \eta|_2^2 \right) 
\leq C \left( |n_2 \varphi|_2^2 + |p_2 \psi|_2^2 \right) \left( |\nabla \varphi|_2^2 + |\nabla \psi|_2^2 \right) + \frac{\varepsilon}{2} \left( |\varphi \nabla \eta|_2^2 + |\psi \nabla \eta|_2^2 \right).
\]

The second term is handled as

\[
- \int_{\Omega} (n_2 \varphi - p_2 \psi)(\varphi^2 + \psi^2)\eta dx \leq |\eta|_\infty |n_2 \varphi - p_2 \psi|_2 \left( |\varphi|_2^4 + |\psi|_2^4 \right) 
\leq C \left( |n_2 \varphi|_2^2 + |p_2 \psi|_2^2 \right) \left( |\nabla \varphi|_2^2 + |\nabla \psi|_2^2 \right),
\]

where we have used (5.12) and the Sobolev and Poincaré inequalities in deriving the last inequality. Thus we obtain (5.11).

We are now in a position to prove Proposition 5.1.

**Proof of Proposition 5.1.** In the proof, \(c\) and \(C\) stand for generic positive constants depending only on \(c_0\), \(C_0\), \(\Omega\), \(\Gamma_D\), \(\rho\), \(r\), \(\varepsilon\), and \(c_R\). Define

\[
a = a(t) := |(n_2 p_2 - 1)(t)|_2^2 + |(v n_2 - n_2 v_2)(t)|_2^2 + |(v p_2 + p_2 v_2)(t)|_2^2.
\]

We claim that the desired inequalities are derived from the inequalities

\[
\int_{\Omega} Ddx \geq c \left( |\nabla \varphi|_2^2 + |\nabla \psi|_2^2 \right). \tag{5.13}
\]

\[
\int_{\Omega} (\mathcal{K} + \mathcal{L})dx \leq \mu \left( |\nabla \varphi|_2^2 + |\nabla \psi|_2^2 \right) + \frac{Ca}{\mu^3} \left( |\varphi|_2^2 + |\psi|_2^2 \right), \tag{5.14}
\]

\[
\int_{\Omega} \mathcal{M}dx \leq \mu \left( |\nabla \varphi|_2^2 + |\nabla \psi|_2^2 \right) + \frac{Ca}{\mu} \left( |\varphi|_2^2 + |\psi|_2^2 \right). \tag{5.15}
\]
\[ c \left( |\varphi|^2 + |\psi|^2 + ||\eta||^2 \right) \leq \int_{\Omega} E dx + \int_{\Gamma_N} \tilde{E} dS \leq C \left( |\nabla \varphi|^2 + |\nabla \psi|^2 \right), \quad (5.16) \]

where \( \mu > 0 \) is an arbitrary number. Let us verify this claim. Substituting (5.13)–(5.15) into (5.6) and taking \( \mu \) small enough, we deduce that

\[ \frac{d}{dt} \left( \int_{\Omega} E dx + \int_{\Gamma_N} \tilde{E} dS \right) + c \left( |\nabla \varphi|^2 + |\nabla \psi|^2 \right) \leq C a \left( |\varphi|^2 + |\psi|^2 \right). \quad (5.17) \]

Applying (5.16) to this inequality, we have

\[ \frac{d}{dt} \left( \int_{\Omega} E dx + \int_{\Gamma_N} \tilde{E} dS \right) + \delta \left( |\nabla \varphi|^2 + |\nabla \psi|^2 \right) \leq 0. \quad (5.18) \]

We now use (4.25) with \((n, p, v) = (n_2, p_2, v_2)\). Integrating (4.25) and applying (5.1) give

\[ \int_{s}^{t} a(\tau) d\tau \leq C + \delta(t - s) \quad (5.19) \]

for all \( t \geq s \geq \tilde{t} \). Multiply (5.17) by \( \exp \left( \int_{s}^{t} c - C a(\tau) d\tau \right) \), integrate the result and then use (5.18) to obtain

\[ e^{(t-\tilde{t})} \left( \int_{\Omega} E(t) dx + \int_{\Gamma_N} \tilde{E}(t) dS \right) + \int_{\tilde{t}}^{t} e^{(t-\tilde{t})} \left( |\nabla \varphi(s)|^2 + |\nabla \psi(s)|^2 \right) ds \leq C \left( \int_{\Omega} E(\tilde{t}) dx + \int_{\Gamma_N} \tilde{E}(\tilde{t}) dS \right) \]

provided that \( \delta \) is smaller than some number \( \delta_0 = \delta_0(c_0, C_0, \Omega, \Gamma_D, \rho, r, \epsilon, c_R) > 0 \). By (5.1), we have

\[ \int_{\Omega} E(\tilde{t}) dx + \int_{\Gamma_N} \tilde{E}(\tilde{t}) dS \leq C, \quad (5.20) \]
|∇φ|^2 + |∇ψ|^2 \geq c \left( |\nabla \log(1 + \varphi)|^2 + |\nabla \log(1 + \psi)|^2 \right). \tag{5.21}

Plugging (5.16), (5.20), and (5.21) into (5.19) yields (5.2) and (5.3) as claimed.
We complete the proof by showing (5.13)–(5.16). First let us show (5.13). From the inequality |a - bi|^2 \geq |a|^2/2 - |b|^2 (a, b \in \mathbb{R}^3) and (5.1), we have

\int_\Omega |\nabla \eta|^2 dx \geq \frac{1}{2} \int_\Omega \left( \frac{n_1^2}{n_1} |\nabla \varphi|^2 + \frac{p_2^2}{p_1^2} |\nabla \psi|^2 \right) dx - \int_\Omega (n_1 + p_1) |\nabla \eta|^2 dx

\geq c \int_\Omega (|\nabla \varphi|^2 + |\nabla \psi|^2) dx - C \int_\Omega |\nabla \eta|^2 dx. \tag{5.22}

To estimate |\nabla \eta|^2, we take \phi_3 = \eta in (5.5). Then

\epsilon \int_\Omega |\nabla \eta|^2 dx + \int_{\Gamma_N} b n \eta^2 dS = \int_\Omega (-n_2 \varphi + p_2 \psi) \eta dx. \tag{5.23}

By the fact that a \log(1 + a) \geq 0 (a > -1), the Schwarz inequality and (5.1), the integrand of the right-hand side of this equality is estimated as

\begin{align*}
(-n_2 \varphi + p_2 \psi) \eta &\leq n_2 \varphi (\log(1 + \varphi) - \eta) + p_2 \psi (\log(1 + \psi) + \eta) \\
&\leq \bar{\mu} (\varphi^2 + \psi^2) + \frac{C}{\bar{\mu}} \left\{ (\log(1 + \varphi) - \eta)^2 + (\log(1 + \psi) + \eta)^2 \right\},
\end{align*}

where \bar{\mu} > 0 is an arbitrary constant. Plugging this into (5.23) and then using the Poincaré inequality and (5.1), we deduce that

\begin{align*}
\int_\Omega |\nabla \eta|^2 dx &\leq C \bar{\mu} \int_\Omega (|\nabla \varphi|^2 + |\nabla \psi|^2) dx + \frac{C}{\bar{\mu}} \int_\Omega \left\{ |\nabla \log(1 + \varphi) - \nabla \eta|^2 + |\nabla \log(1 + \psi) + \nabla \eta|^2 \right\} dx \\
&\leq C \bar{\mu} \int_\Omega (|\nabla \varphi|^2 + |\nabla \psi|^2) dx + \frac{C}{\bar{\mu}} \int_\Omega |\nabla \varphi|^2 + |\nabla \psi|^2 dx.
\end{align*}

Substituting this into (5.22) and choosing \bar{\mu} appropriately small give (5.13).
Next we derive (5.14) and (5.15). Note that

\begin{align*}
\mathcal{K} &= -c_R \frac{n_1 p_1 - n_2 p_2}{n_1 + p_1 + 2} \log \frac{n_1 p_1}{n_2 p_2} + c_R \frac{(n_2 p_2 - 1)(n_2 \varphi + p_2 \psi)}{(n_1 + p_1 + 2)(n_2 + p_2 + 2)} (\log(1 + \varphi) + \log(1 + \psi)).
\end{align*}

Since the first term of the right-hand side of this equality is nonpositive, we have

\begin{align*}
\mathcal{K} &\leq c_R \frac{(n_2 p_2 - 1)(n_2 \varphi + p_2 \psi)}{(n_1 + p_1 + 2)(n_2 + p_2 + 2)} (\log(1 + \varphi) + \log(1 + \psi)) \leq C |n_2 p_2 - 1| (\varphi^2 + \psi^2).
\end{align*}
It is elementary to show that $\mathcal{L} \leq C|n_2p_2-1|((\varphi^2+\psi^2)$. Hence, by the Hölder, Sobolev, Poincaré, and Young inequalities, we have

$$
\int \left(\mathcal{K} + \mathcal{L}\right)dx \leq |n_2p_2 - 1|_2 \left( |\varphi|^{1/2} |\varphi|\_6^{1/2} + |\psi|^{1/2} |\psi|\_6^{1/2} \right)
\leq C|n_2p_2 - 1|_2 \left( |\varphi|^{1/2} |\varphi|\_2^{3/2} + |\psi|^{1/2} |\psi|\_2^{3/2} \right)
\leq \mu \left( |\nabla \varphi|^2 + |\nabla \psi|^2 \right) + \frac{C}{\mu} |n_2p_2 - 1|_2^3 \left( |\varphi|^2 + |\psi|^2 \right).
$$

Here, $\mu > 0$ is an arbitrary number. Owing to (5.1), the last term can be estimated as

$$
|n_2p_2 - 1|_2^3 \left( |\varphi|^2 + |\psi|^2 \right) \leq C |n_2p_2 - 1|_2^3 \left( |n_2\varphi|^2 + |p_2\psi|^2 \right).
$$

Therefore, (5.14) is proved.

The inequality (5.15) is verified by applying the Hölder and Schwarz inequalities together with (5.11) as

$$
\int \mathcal{M}dx \leq |\nabla n_2 - n_2\nabla \eta|_2 |\varphi| |\nabla \eta|_2 + |\nabla p_2 + p_2\nabla \eta|_2 |\psi| |\nabla \eta|_2
\leq Ca^{1/2} \left( |n_2\varphi|_2 + |p_2|_2 \right) (|\nabla \varphi|_2 + |\nabla \psi|_2)
\leq \mu \left( |\nabla \varphi|^2 + |\nabla \psi|^2 \right) + \frac{Ca}{\mu} \left( |n_2\varphi|^2 + |p_2\psi|^2 \right).
$$

Finally, we prove (5.16). It is easily seen from (5.1) and (5.23) that

$$
c(\varphi^2 + \psi^2) \leq n_2 \int_0^\varphi \log(1 + y)dy + p_2 \int_0^\psi \log(1 + y)dy \leq C(\varphi^2 + \psi^2), \quad \varepsilon \int |\nabla \eta|^2 dx + \int_{\Gamma_N} b\eta^2 dS \leq C \left( |\varphi|^2 + |\psi|^2 \right).
$$

Hence, we have

$$
c \left( |\varphi|^2 + |\psi|^2 + |\nabla \eta|^2 \right) \leq \int_{\Omega} \varepsilon dx + \int_{\Gamma_N} \tilde{\varepsilon} dS \leq C \left( |\varphi|^2 + |\psi|^2 \right).
$$

We thus obtain (5.16) by applying the Poincaré inequality to the right-hand side of this inequality. The proof is complete.  

\hfill \Box

6 | TIME-PERIODIC SOLUTIONS

This section is devoted to the proof of Theorem 2.2 stating the unique existence and global stability of time-periodic solutions.

Proof of Theorem 2.2. Throughout the proof, $c$ and $C$ denote generic positive constants depending only on $c_0$, $C_0$, $\Omega$, $\Gamma_D$, $\rho$, $r$, $\varepsilon$, and $c_R$. Furthermore, we assume that $\theta < \theta_0$ and $\delta < \min\{1, \delta_0\}$, where $\theta_0$ and $\delta_0$ are given in Theorem 2.1 and Proposition 5.1, respectively.

First we show the uniqueness of time-periodic solutions. Suppose that $(n_1, p_1, \nu_1)$ and $(n_2, p_2, \nu_2)$ are time-periodic solutions of (1.1). Then Theorem 2.1 ensures that

$$
\hat{C}^{-1} \leq n_1, p_{1*}, n_{2*} \leq \hat{C} \quad \text{in} \quad \mathbb{R} \times \Omega,
$$

where $\hat{C} = \hat{C}(c_0, C_0, \Omega, \Gamma_D, \rho, r, \varepsilon, c_R)$ is the constant being in (2.3). Hence, we see from Proposition 5.1 that

$$
|(n_1 - n_2)(t)|_2 + |(p_1 - p_2)(t)|_2 + ||(\nu_1 - \nu_2)(t)||_1 \leq Ce^{-\alpha(t-t_0)}
$$
for all \( t \geq \tilde{t} \), where \( \tilde{t} \in \mathbb{R} \) can be chosen arbitrarily. By letting \( \tilde{t} \to -\infty \), we obtain \((n_{s1}, p_{s1}, v_{s1}) = (n_{s2}, p_{s2}, v_{s2})\), which establishes the uniqueness.

Next we investigate the existence of time-periodic solutions. To this end, we fix a solution \((n, p, v)\) of (1.1). From Theorem 2.1, we can choose \( t_0 \) such that

\[
(2\hat{C})^{-1} \leq n, p \leq 2\hat{C} \quad \text{in} \quad (t_0, \infty) \times \Omega.
\] (6.1)

Take an integer \( k_0 \) satisfying \( k_0 T_* > t_0 \) and define a sequence \( \{(n_k, p_k, v_k)\}_{k=0}^{\infty} \) by

\[
(n_k, p_k, v_k)(t, x) := (n, p, v)(t + (k_0 + k)T_*, x), \quad (t, x) \in [-kT_*, \infty) \times \tilde{\Omega}.
\]

Owing to (6.1) and the fact that \( g \) and \( V_b \) are periodic with period \( T_* \), we see that \((n_k, p_k, v_k)\) solves (1.1) for \( I = (-kT_*, \infty) \) and satisfies

\[
(2\hat{C})^{-1} \leq n_k, p_k \leq 2\hat{C} \quad \text{in} \quad (-kT_*, \infty) \times \Omega.
\] (6.2)

We can apply Proposition 5.1 with \((n_1, p_1, v_1) = (n_2, p_2, v_2) = (n_k, p_k, v_k)\), and \( \tilde{t} = -kT_* \) to obtain

\[
\int_{-kT_*}^{t} e^{C(t + kT_*+)} \left( \left| \nabla \log \frac{n_k}{n} \right|_2^2 + \left| \nabla \log \frac{p_k}{p} \right|_2^2 \right) ds \leq C,
\] (6.4)

where \( l > k \). In particular, from (6.3), there exists \((n_*, p_*, v_*) \in C(\mathbb{R}; L^2(\Omega)) \times C(\mathbb{R}; L^2(\Omega)) \times C(\mathbb{R}; H^1_D(\Omega)) \) such that

\[
n_k \to n_*, p_k \to p_* \quad \text{in} \quad C_0(\mathbb{R}; L^2(\Omega)), \quad v_k \to v_* \quad \text{in} \quad C_0(\mathbb{R}; H^1(\Omega))
\] (6.5)
as \( k \to \infty \). Note that the limit \((n_*, p_*, v_*) \) is independent of the choice of \((n, p, v)\) used to define the sequence \((n_k, p_k, v_k)\), since we have shown the uniqueness of time-periodic solutions.

Let us prove that \((n_*, p_*, v_*) \) is a time-periodic solution of (1.1) by checking the conditions (i)–(iv) in Definition 2.1. We see from (6.2) and (6.5) that

\[
(2\hat{C})^{-1} \leq n_*, p_* \leq 2\hat{C} \quad \text{in} \quad \mathbb{R} \times \Omega,
\]

which particularly gives the condition (ii). By the definition of \((n_k, p_k, v_k)\), we have

\[
(n_k, p_k, v_k)(t + T_*, x) = (n_{k+1}, p_{k+1}, v_{k+1})(t, x).
\]

Hence letting \( k \to \infty \) yields the condition (iv). To check the conditions (i) and (iii), we show that

\[
n_k \nabla v_k \to n_* \nabla v, \quad p_k \nabla v_k \to p_* \nabla v, \quad R(n_k, p_k) \to R(n_*, p_*) \quad \text{in} \quad L^2(\mathbb{R}; L^2(\Omega))
\] (6.6)
as \( k \to \infty \). The convergence of \( \{n_k \nabla v_k\} \) follows from Lemma 3.5 with \( f_k = n_k \) and \( g_k = V v_k \). In the same way, we have \( p_k \nabla v_k \to p_* \nabla v \). By a simple calculation, one can check that \( |\partial R/\partial n(n, p)|, |\partial R/\partial p(n, p)| \leq c_R \) and hence, \( |R(n_k, p_k) - R(n_*, p_*)| \leq c_R (|n_k - n_*| + |p_k - p_*|) \). This inequality and (6.5) imply that \( R(n_k, p_k) \to R(n_*, p_*) \) in \( L^2(\mathbb{R}; L^2(\Omega)) \). Thus, (6.6) is verified. By (6.4), we deduce that \( \nabla \log n_k \) and \( \nabla \log p_k \) are Cauchy sequences in \( L^2(J; L^2(\Omega)) \) for any bounded interval \( J \subset \mathbb{R} \). From this, (6.2) and (6.5), we can apply Lemma 3.5 to conclude that

\[
\{\nabla n_k\} = \{n_k \nabla \log n_k\} \quad \text{and} \quad \{\nabla p_k\} = \{p_k \nabla \log p_k\}
\]

are convergent in \( L^2(\mathbb{R}; L^2(\Omega)) \). (6.7)

Note that \( n_l - n_k \) satisfies

\[
\langle n_l' - n_k', \phi_1 \rangle = - \int_\Omega \left\{ (\nabla n_l - \nabla n_k) \cdot \nabla \phi_1 - (n_l \nabla v_l - n_k \nabla v_k) \cdot \nabla \phi_1 + (R(n_1, p_1) - R(n_2, p_2)) \phi_1 \right\} dx
\]
for all $\phi_1$. This together with (6.6) and (6.7) gives

$$\|n'_l - n'_k\|_{H^1_0(\Omega)} \leq \|\nabla n_l - \nabla n_k\|_2 + |n_l \nabla v_l - n_k \nabla v_k|_2 + |R(n_l, p_l) - R(n_k, p_k)|_2 \to 0 \ (k, l \to \infty),$$

and therefore,

$$\{n'_k\} \text{ and } \{p'_k\} \text{ are convergent in } L^2_{\text{loc}}(\mathbb{R}; H^1_D(\Omega)^*). \tag{6.8}$$

From (6.5), (6.7), and (6.8), we see that derivatives $\nabla n_*, \nabla p_* \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$ and $n'_*, p'_* \in L^2_{\text{loc}}(\mathbb{R}; H^1_D(\Omega)^*)$ exist, and

$$\nabla n_k \to \nabla n_*, \nabla p_k \to \nabla p_* \text{ in } L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)), \ n'_k \to n'_*, p'_k \to p'_* \text{ in } L^2_{\text{loc}}(\mathbb{R}; H^1_D(\Omega)^*) \tag{6.9}$$

as $k \to \infty$. The condition (i) is therefore verified. Furthermore, from (6.5), (6.6), (6.9), and the fact that $(n_k, p_k, v_k)$ satisfies the condition (iii), we see that $(n_*, p_*, v_*)$ also satisfies the condition (iii). Consequently, we have proved the existence of time-periodic solutions.

It remains to show (2.4). By taking $k = 0$ and letting $l \to \infty$ in (6.3), we have

$$|n(t + k_0 T_*) - n_*(t)|_2 + |p(t + k_0 T_*) - p_*(t)|_2 + ||v(t + k_0 T_*) - v_*(t)||_1 \leq Ce^{-ct}.$$

This together with the fact that $(n_*, p_*, v_*)$ is periodic with period $T_*$ gives

$$|(n - n_*)(t)|_2 + |(p - p_*)(t)|_2 + ||v - v_*)(t)||_1 \leq Ce^{-ct}.$$

We thus obtain (2.4), and the proof is complete.

\[\square\]

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**APPENDIX A: PROOFS OF LEMMAS 3.1–3.5**

This section provides the proofs of Lemmas 3.1–3.5.

**Proof of Lemma 3.1.** We can verify (3.5) applying a similar argument given by Gajewski and Görgen, or Daners, Theorem 2.5. We omit its proof.

It remains to show (3.4). In what follows, *C* denotes a generic positive constant depending only on *Ω*, *Γ*, *α*, and *β*. We set *W* := *w* − *W* and *κ* := *|h|* + *|γ*1 + *|γ|* *|W*| *|W*| ≤ *Ck*. (A1)

By the condition 1 ≤ *α* < 3/2, we can take *γ* > 0 such that

*κ* := (1 + *γ*3/2)/2 < 3.

Let *κ*′ > 3/2 and *β*′ > 2 denote the Hölder conjugates of *κ* and *β*, respectively. We first verify that for all *F* ∈ *L*κ′(*Ω*) and

\[ \left| \int_{Ω} FWdx + \int_{Γ} GWdS \right| \leq Ck(|F|κ′ + |G|β′,Γ). \]  (A2)

To this end, we employ a solution *φ* ∈ *H*1p(Ω) of the problem

\[ \int_{Ω} \nabla φ \cdot \nabla φdx + \int_{Γ} bφφdS = \int_{Ω} Fφdx + \int_{Γ} GφdS \text{ for } φ ∈ H^1_p(Ω). \]  (A3)

Owing to (3.3) and (3.5), we see that *φ* satisfies

\[ \|φ\|_1 + |φ|_∞ \leq C(|F|κ′ + |G|β′,Γ). \]  (A4)
Taking $\phi = \varphi$ in (3.2) and $\tilde{\phi} = \mathcal{W}$ in (A3), and then combining the resulting equalities, we deduce that

$$\left| \int_{\Omega} F \mathcal{W} \, dx + \int_{\Gamma_N} G \mathcal{W}_b \, dS \right| = \left| - \int_{\Omega} \nabla W_b \cdot \nabla \varphi \, dx - \int_{\Omega} h \varphi \, dx - \int_{\Gamma_N} (\bar{b} W_b - \bar{g}) \varphi \, dS \right|$$

$$\leq \| W_b \|_1 \| \varphi \|_1 + |h|_1 \| \varphi \|_{\infty} + C(\| \bar{b} \|_{4/3, \Gamma_N} \| W_b \|_1 + \| \bar{g} \|_{1, \Gamma_N}) \| \varphi \|_{\infty}$$

$$\leq C k (\| F \|_{\varepsilon'} + |G|_{\beta', \Gamma_N})$$

where we have used (3.7) and (A4) in deriving the first and second inequalities, respectively.

From (A2) and the duality of $L^p$ spaces, we see that

$$|\mathcal{W}|_k \leq C k, \quad |\mathcal{W}|_{\beta', \Gamma_N} \leq C k.$$  \hspace{1cm} (A5)

In particular, the latter inequality of (A1) holds. What is left is to prove the former one. The Hölder inequality gives

$$\int_{\Omega} |\nabla \mathcal{W}|^a \, dx \leq \left( \int_{\Omega} |\nabla \mathcal{W}|^2 \frac{dF}{ds}(\mathcal{W}) \, dx \right)^{a/2} \left( \int_{\Omega} \left( \frac{dF}{ds}(\mathcal{W}) \right)^{-a/(2-a)} \, dx \right)^{1-\frac{a}{2}}, \hspace{1cm} (A6)$$

where

$$F(s) := \int_0^s \frac{k \gamma}{k \gamma + |\gamma|} \, ds.$$  

It is easily seen that

$$|F(s)| \leq \left( \int_0^\infty \frac{1}{1 + |\gamma|} \, d\gamma \right) k, \quad 0 \leq \frac{dF}{ds}(s) \leq 1.$$

Taking $\phi = F(\mathcal{W}) \in H^1_0(\Omega)$ in (3.2) and using the above inequalities, we have

$$\int_{\Omega} |\nabla \mathcal{W}|^2 \frac{dF}{ds}(\mathcal{W}) \, dx + \int_{\Gamma_N} \bar{b} \mathcal{W} F(\mathcal{W}) \, dS$$

$$= - \int_{\Omega} (\nabla W_b \cdot \nabla \mathcal{W}) \frac{dF}{ds}(\mathcal{W}) \, dx - \int_{\Omega} h F(\mathcal{W}) \, dx - \int_{\Gamma_N} (\bar{b} W_b - \bar{g}) F(\mathcal{W}) \, dS$$

$$\leq \frac{1}{2} \int_{\Omega} |\nabla \mathcal{W}|^2 \frac{dF}{ds}(\mathcal{W})^2 \, dx + \frac{1}{2} \int_{\Omega} |\nabla W_b|^2 \, dx + C K |h|_1 + C k (\| \bar{b} \|_{4/3, \Gamma_N} \| W_b \|_1 + \| \bar{g} \|_{1, \Gamma_N})$$

$$\leq \frac{1}{2} \int_{\Omega} |\nabla \mathcal{W}|^2 \frac{dF}{ds}(\mathcal{W}) \, dx + C k^2.$$

Since the second term of the left-hand side is nonnegative, we arrive at

$$\int_{\Omega} |\nabla \mathcal{W}|^2 \frac{dF}{ds}(\mathcal{W}) \, dx \leq C K^2.$$

It follows from (A5) that

$$\int_{\Omega} \left( \frac{dF}{ds}(\mathcal{W}) \right)^{-a/(2-a)} \, dx = \int_{\Omega} \left\{ 1 + \left( \frac{|\mathcal{W}|}{k} \right)^{\gamma+1} \right\}^{a/(2-a)} \, dx \leq C + C \int_{\Omega} \left( \frac{|\mathcal{W}|}{k} \right)^K \, dx \leq C.$$
Substituting these inequalities into (A6), we obtain (A1). Thus, the lemma follows.

**Proof of Lemma 3.3.** The assertion (i) immediately follows from the boundedness of the trace operator from $W^{1,q}(\Omega)$ to $L^q(\partial \Omega)$ and the Poincaré inequality $|f|_q \leq C|\nabla f|_q$.

One can show (ii) by combining the boundedness of the trace operator from $W^{1,1}(\Omega)$ to $L^1(\partial \Omega)$, the Hölder inequality, and the Sobolev embedding theorem $H^1(\Omega) \hookrightarrow L^6(\Omega)$. Indeed, for $f \in H^1(\Omega)$, we have

$$|f|_{4, \Gamma_N} = \|f\|^{1/4}_{1, \Gamma_N} \leq C \left( \|\nabla f\|^{3/4}_{1} + |f|^{1/4}_{1} \right) \leq C \left( \|\nabla f\|^{3/4}_{2} + |f|_6 \right) \leq C\|f\|_1.$$  

Let us show (iii). We need only to consider the case $q \geq 3/2$ owing to the fact that $L^q(\Gamma_N) \hookrightarrow L^{q^*}(\Gamma_N)$ for $q_1 \geq q_2$. Using (3.6) and the Hölder inequality, we have

$$|f|_{q, \Gamma_N} = \|f\|_{q, \Gamma_N} \leq C \|\nabla f\|_{q^{*},1} |f|^{1/q}_{1} \leq C \|\nabla f\|_{2} |f|^{\theta_1}_{1},$$

(A7)

where

$$\theta_1 = \frac{1}{2}, \quad \theta_2 = \frac{3}{4} \left( 2 - \frac{3}{q} \right), \quad \theta_3 = \frac{3}{4} \left( \frac{4}{q} - 1 \right).$$

We note that $\theta_1 + \theta_2 + \theta_3 = 1$ and that the condition $3/2 \leq q < 4$ gives $0 \leq \theta_1 + \theta_2 < 1$ and $0 < \theta_3 \leq 1$. The Sobolev embedding theorem $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the Poincaré inequality yield $|f|_6 \leq C|\nabla f|_2$, and therefore, we see from (A7) that $|f|_{q, \Gamma_N} \leq C|\nabla f|^{\theta_1+\theta_2}_{2} |f|^{\theta_1}_{1}$. We thus obtain (3.8) by applying the Young inequality to the right-hand side of this inequality.

**Proof of Lemma 3.3.** It suffices to show that

$$a \log(da) + b \log(db) \leq C \left\{ (a - b)^2 + \frac{ab - 1}{a + b + 2} \log(ab) + 1 \right\}, \quad a, b > 0$$

(A8)

for some constant $C = C(d) > 0$, since

$$\int_a^b \log \frac{y}{A} dy + \int_b^A \log \frac{y}{B} dy = a \log \frac{A}{B} + b \log \frac{B}{A} - (a + b) + A + B \leq a \log(da) + b \log(db) + 2d.$$

Suppose that (A8) fails. Then there exist sequences $\{a_j\} \subset (0, \infty)$ and $\{b_j\} \subset (0, \infty)$ such that $F(a_j, b_j) \to \infty$ as $j \to \infty$, where

$$F(a, b) := (a \log(da) + b \log(db)) \left\{ (a - b)^2 + \frac{ab - 1}{a + b + 2} \log(ab) + 1 \right\}^{-1}.$$

It is easily seen that either $\{a_j\}$ or $\{b_j\}$ is unbounded. Therefore, by taking a subsequence if necessary, we may assume that $a_j + b_j \to \infty$ and $a_j/b_j \to l \in [0, \infty]$ as $j \to \infty$. We first consider the case $l \in (1, \infty)$. Then, in particular, $b_j < a_j$ holds for large $j$. Hence, $a_j \to \infty$ as $j \to \infty$ and $b_j \log(db_j) \leq a_j \log(da_j)$ for large $j$. From these, we have

$$F(a_j, b_j) \leq 2a_j \log(da_j) \cdot (a_j - b_j)^{-2} = \frac{2 \log(da_j)}{a_j} \cdot \left( 1 - \frac{b_j}{a_j} \right)^{-2} \to 0 \quad (j \to \infty),$$

which contradicts $\lim_{j \to \infty} F(a_j, b_j) = \infty$. By a similar argument, we have a contradiction for $l \in [0, 1)$. Next we assume that $l = 1$. In this case, we have $a_j, b_j \to \infty$ as $j \to \infty$ and $\log(da_j), \log(db_j) \leq \log(a_j, b_j)$ for large $j$. It follows that

$$F(a_j, b_j) \leq (a_j + b_j) \log(a_j b_j) \cdot \left( \frac{a_j b_j - 1}{a_j + b_j + 2 \log(a_j b_j)} \right)^{-1}$$

$$= \frac{(a_j/b_j + 1)(a_j/b_j + 1 + 2/b_j)}{a_j/b_j - 1/b_j^2} \to 4 \quad (j \to \infty),$$

a contradiction. Thus, we obtain (A8), and the proof is complete.
Proof of Lemma 3.4. We take \( M^* > \max\{2\sigma, e\} \) such that the following hold for \( M \geq M^* \):

\[
\frac{\log M}{(\sqrt{\log M + 1})^2} \geq 1, \quad \frac{(\log M)^{3/2}}{(\sqrt{\log M + 1})^2} \geq 2h(M), \quad \frac{\log M}{2\sigma} \geq h(M), \quad \frac{\sqrt{\log M - 1}}{\sqrt{\log M + 1}} \geq \frac{1}{2}.
\]

(A9)

It is sufficient to show that

\[
\frac{1}{a+b} \left\{ (a-b)H_M \left( \log \frac{a}{b\sigma} \right) + \log(a+b) \right\} \geq \frac{h(M)}{2}
\]

for all \( a, b > 0 \) and \( M \geq M^* \) with \( a + b \geq M \). We divide the proof into the following three cases:

(a) \( \frac{1}{\sqrt{\log M}} \leq \frac{a}{b} \leq \sqrt{\log M} \),
(b) \( \frac{a}{b} > \sqrt{\log M} \),
(c) \( \frac{a}{b} < \frac{1}{\sqrt{\log M}} \).

We consider the case (a). It is easily seen that \( z/(z+1)^2 \geq z_0/(z_0 + 1)^2 \) if \( z \geq 1 \) and \( 1/z_0 \leq z \leq z_0 \). Hence, by the first inequality of (A9), we have

\[
ab = \frac{a/b}{(a/b + 1)^2} (a+b)^2 \geq \frac{\sqrt{\log M}}{(\sqrt{\log M + 1})^2} M^2 \geq M.
\]

In particular, we have \( ab \geq 2 \), and hence, \( ab - 1 \geq ab/2 \). This together with the fact that \( a + b \geq M \geq 2 \) shows that

\[
\frac{ab - 1}{(a+b)(a+b+2)} \geq \frac{ab}{4(a+b)^2} = \frac{a/b}{4(a/b + 1)^2} \geq \frac{\sqrt{\log M}}{4(\sqrt{\log M + 1})^2}.
\]

Thus,

\[
\frac{ab - 1}{(a+b)(a+b+2)} \log(ab) \geq \frac{\sqrt{\log M}}{4(\sqrt{\log M + 1})^2} \log M \geq \frac{h(M)}{2},
\]

where we have used the second inequality of (A9). This shows that (A10) holds in this case.

It remains to examine the cases (b) and (c). We only consider the case (b), since the case (c) can be dealt with in the same way. It is seen that

\[
a = \frac{1}{1 + b/a}(a+b) > \frac{1}{1 + 1/\sqrt{\log M}} M \geq \frac{M}{2}.
\]

(A11)

Note that this particularly gives \( a \geq \sigma \). Hence, we have

\[
\log \frac{a}{b\sigma} = \log \frac{a}{b\sigma} \geq \min \left\{ \log \frac{a}{b}, \log \frac{a}{\sigma} \right\} \geq \min \left\{ h(M), \log \frac{M}{2\sigma} \right\} = h(M),
\]

where the second inequality follows from the conditions (b) and (A11) and the last equality follows from the third inequality of (A9). This together with the last inequality of (A9) gives

\[
\frac{a-b}{a+b} H_M \left( \log \frac{a}{b\sigma} \right) = \frac{a/b - 1}{a/b + 1} h(M) \geq \frac{\sqrt{\log M - 1}}{\sqrt{\log M + 1}} h(M) \geq \frac{h(M)}{2}.
\]

Thus, (A10) is verified, and the proof is complete.
Proof of Lemma 3.5. Let $f \in L^2(E)$ and $g \in L^2(E)$ be the limits of $\{f_k\}$ and $\{g_k\}$, respectively. Then the assumption $|f_k| \leq C$ implies that $|f| \leq C$. For $M > 0$, we have

$$
\int_{E} (f_k g_k - f g)^2 \, dx \leq 2 \int_{E} f_k^2 (g_k - g)^2 \, dx + 2 \int_{E} (f_k - f)^2 g^2 \, dx
$$

$$
= 2 \int_{E} f_k^2 (g_k - g)^2 \, dx + 2 \int_{|g| < M} (f_k - f)^2 g^2 \, dx + 2 \int_{|g| \geq M} (f_k - f)^2 g^2 \, dx
$$

$$
\leq 2C^2 \int_{E} (g_k - g)^2 \, dx + 2M^2 \int_{E} (f_k - f)^2 \, dx + 8C^2 \int_{|g| \geq M} g^2 \, dx,
$$

and hence,

$$
\limsup_{k \to \infty} \int_{E} (f_k g_k - f g)^2 \, dx \leq 8C^2 \int_{|g| \geq M} g^2 \, dx.
$$

Thus, the lemma follows by letting $M \to \infty$. 
