First results for the Coulomb gauge integrals using NDIM

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The Coulomb gauge has at least two advantages over other gauge choices in that bound states between quarks and studies of confinement are easier to understand in this gauge. However, perturbative calculations, namely Feynman loop integrations are not well-defined (there are the so-called energy integrals) even within the context of dimensional regularization. Leibbrandt and Williams proposed a possible cure to such a problem by splitting the space-time dimension into $D = \omega + \rho$, i.e., introducing a specific one parameter $\rho$ to regulate the energy integrals. The aim of our work is to apply negative dimensional integration method (NDIM) to the Coulomb gauge integrals using the recipe of split-dimension parameters and present complete results – finite and divergent parts – to the one and two-loop level for arbitrary exponents of propagators and dimension.

Key-words: Coulomb gauge, negative dimensional integration method, Feynman loop integrals, non-covariant gauges.

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I. INTRODUCTION

Perturbative approach in quantum field theory (QFT) was responsible for several breakthrough ideas in Physics and Mathematics. One of them is dimensional regularization [4], i.e., analytic continuation of space-time dimension $D$ into an extended domain that allows for complex values. Feynman loop integrals gained a solid theoretical foundation and renormalization process became simpler than it was (where one had to use cut-offs and so on). Of course, this is only a partial picture of it all, say, the covariant side of the coin.

In algebraic non-covariant gauges [2], on the other hand, dimensional regularization was able to control divergences, e.g., in the light-cone gauge, but the results were not physically acceptable. In other words, double-poles did appear in one-loop integral calculations and Wilson loops did not have the correct behavior [3]. These problems were first overcome with the advent of what is known as the Mandelstam-Leibbrandt (ML) prescription [4]. More recently, we have shown that in the NDIM approach we do not need to invoke any kind of prescription to perform Feynman loop integrals in this gauge [5].

Among the non-covariant gauges we also have the Coulomb gauge (often referred to as the radiation gauge) where confinement and bound states are easier to deal with, ghost propagator has no pole and unitarity is manifest. However, in such a gauge, no further insight has been achieved with the standard dimensional regularization technique, because it presents a gauge boson propagator of the form

$$G_{\mu\nu}(q) = -\frac{i\delta_{ab}}{q^2} \left[ \eta_{\mu\nu} + \frac{n^2}{q^2} q_\mu q_\nu - \frac{q \cdot n}{q^2} (q_\mu n_\nu + n_\mu q_\nu) \right], \quad \text{with} \quad n_\mu = (1, 0, 0, 0),$$

(1)

which generates loop integrals like,

$$\int \frac{d^Dq}{q^2(q - p)^2}. \quad (2)$$

where bold face letters stand for three momentum vectors.

The integral over the fourth-component (in Euclidean space) or zeroth-component (in Minkowski’s), the so-called energy integral, is not defined even within the context of dimensional regularization. Doust and Taylor discussed Coulomb gauge loop integrals and presented a possible remedy for this problem in terms of an interpolating gauge (between Feynman and Coulomb, see also [4]). Leibbrandt, again, and Williams, presented another approach...
for this ill-defined integrals, a procedure called split dimensional regularization. Both parts, namely energy and 3-momentum sectors, need to be separately dimensionaly regularized, that is, one parameter only, \( D \), is not sufficient to render the integrals well-defined. To overcome this problem, they introduced another regulating parameter, i.e., split the dimensinality of space-time into two distinct sectors, namely, \( D = 4 - 2\epsilon = \omega + \rho \) and the divergences contained in energy integrals are expressed as poles in \( \rho \) besides the usual ones in terms of \( \omega \), that is to say, the integration measure is written down as \( d^D q = d^\omega q \ d^\rho q_4 \).

In a series of papers [11], Leibbrandt studied Coulomb gauge integrals to one and two-loop level (with Heinrich) and presented results for divergent parts of several of them. Our aim, in this work is two-fold: show that NDIM is the most versatile technique to carry out loop integrals, whether they come from covariant or non-covariant gauges; and to present complete results for the Coulomb gauge integrals to one and two-loop level for arbitrary exponents of propagators and dimension.

The outline for our paper is as follows: in section II we consider scalar and tensorial Coulomb gauge integrals at one-loop level, while section III is devoted to two-loop integrals and in section IV we present our conclusions. In the appendix we discuss some technical issues.

II. ONE-LOOP COULOMB GAUGE INTEGRALS

To show how NDIM can handle Coulomb gauge integrals with ease we consider in this section one-loop integrals. Recall that negative dimensional integration is equivalent to positive dimensional integration over Grassmannian variables [12] — a property demonstrated by Dunne and Halliday — and for this very reason, propagators are raised to positive powers (they appear in the numerator of integrands) and usual variables become Grassmannian ones. Another important point is that in the NDIM context it is as simple to work with arbitrary exponents of propagators 

\[ \text{variables [12]} \] — a property demonstrated by Dunne and Halliday — and for this very reason, propagators are raised to positive powers (they appear in the numerator of integrands) and usual variables become Grassmannian ones. Another important point is that in the NDIM context it is as simple to work with arbitrary exponents of propagators as if we choose particular values for them. This is why we consider the general case. It is also worth remembering that for some types of diagrams, e.g., box integrals [13,14], there are divergences that are not related to space-time singularities expressed as, for instance \( \Gamma(i-j) \), see also [15]. So, within the NDIM approach we can also trace back the origin of divergences.

The first two integrals we choose to work with are scalar ones,

\[
g_1(i,j,k) = \int d^D q \ (q^2)^i (q + p)^{2j} (q^2)^k, \tag{3}
\]

\[
g_2(i,j,k) = \int d^D q \ (q^2)^i (q + p)^{2j} (q^2)^k, \tag{4}
\]

where generating functions for these referred Coulomb gauge integrals are,

\[
G_1 = \int d^D q \ \exp \left[ -\alpha q^2 - \beta (q + p)^2 - \gamma q^2 \right], \tag{5}
\]

\[
G_2 = \int d^D q \ \exp \left[ -\alpha q^2 - \beta (q + p)^2 - \gamma q^2 \right], \tag{6}
\]

with \( D = \omega + \rho = 4 - \epsilon \) and \( d^D q = d^\omega q \ d^\rho q_4 \), in Euclidean space, following the split dimension recipe of Leibbrandt et al.

Completing the square we can easily carry the integration out to get,

\[
G_1 = \left( \frac{\pi}{\alpha + \beta} \right)^{\rho/2} \left( \frac{\pi}{\lambda_1} \right)^{\omega/2} \exp \left[ -\frac{(\alpha + \gamma)\beta p^2}{\lambda_1} \right] \exp \left( -\frac{\alpha\beta p_4^2}{\alpha + \beta} \right), \tag{7}
\]

\[
G_2 = \left( \frac{\pi}{\alpha} \right)^{\rho/2} \left( \frac{\pi}{\lambda_1} \right)^{\omega/2} \exp \left[ -\frac{(\alpha + \gamma)\beta p^2}{\lambda_1} \right], \tag{8}
\]

where \( \lambda_1 = \alpha + \beta + \gamma \).

Taylor expanding both expressions in (7) and (8) we get the NDIM solutions for \( g_1 \) and \( g_2 \) by solving systems of linear algebraic equations.
The system of linear algebraic equations for the first integral is given by a $5 \times 8$ matrix,

\[
\begin{align*}
X_{13} + Y_{14} &= i \\
X_{123} + Y_{25} &= j \\
X_2 + Y_3 &= k, \\
Y_{123} &= -X_{12} - \omega/2 \\
Y_{45} &= -X_{3} - \rho/2
\end{align*}
\]

with five equations and eight “unknowns”, corresponding to the various summation indices coming from the Taylor and multinomial expansions. They are solvable only within the lower quadratic $5 \times 5$ dimension matrices. There are a grand total of 56 possible square matrices of this type (i.e., $5 \times 5$) from which 36 yield relevant non vanishing and workable solutions while the remaining 20 yield a set of trivial solutions (i.e., the related systems do not have a solution). We know from our previous works (see for instance [13]) that all those non-trivial solutions will generate power series of hypergeometric type, known as hypergeometric functions [16]. Moreover, all of them are related by analytic continuation, either directly or indirectly. In our present case, namely integral $g_1$, there are triple as well as double series, among which we choose to consider only the simplest ones,

\[
g_1^{A\{AC\}}(i,j,k) = f_1^{A\{AC\}} \sum_{n_1,n_2,n_3=0}^{\infty} \left( \frac{p_3^2}{p_4^2} \right)^{n_{123}} \frac{(-1)^{n_3}(-i|n_{123})(k + \omega/2|n_{12}) (D/2 + j|n_{23})(1 - i - \rho/2|n_{123})}{n_1!n_2!n_3!(1 + j + k + D/2|n_{23})(1 - i - \rho/2|n_{123})(\sigma + \omega/2 - i|n_{123})},
\]

where the superscript “$[AC]$” means analytic continuation (to positive dimensional region) and we define the shorthand notation $n_{AB} = n_A + n_B$, while $(x|y) \equiv (x)_y = \Gamma(x + y)/\Gamma(x)$ is the Pochhammer symbol and

\[
f_1^{A\{AC\}} = \pi^{D/2}(-p_4^2)^i (p^2)^{j+k+D/2}(-j|j + k + \omega/2)(-k|j + k + D/2)(-i + \sigma + \omega/2|i - \sigma - \omega/2 - D/2 - j - k),
\]

where $\sigma = i + j + k + D/2$. Observe that the above result is valid for negative $j, k$. Among the 36 possible series this is the only one that has the form $\Sigma(\cdots)^{a+b+c}$, where the $\cdots$ stands for the specific kinematical configuration.

For the case of double series we have, e.g.,

\[
g_1^{B\{AC\}}(i,j,k) = f_1^{B\{AC\}} \sum_{n_1,n_2=0}^{\infty} \left( \frac{p_3^2}{p_4^2} \right)^{n_{12}} \frac{(-\sigma|n_{12})(-k|n_2)(\omega/2 + k|n_1)(1 - k - \sigma - D/2|n_2)}{n_1!n_2!(\omega/2|n_{12})(1 - j - D/2|n_2)},
\]

where

\[
f_1^{B\{AC\}} = \pi^{D/2}(p_4^2)^\sigma (-i|\sigma)(-j|\sigma)(\omega/2|k)(k + \sigma + D/2) - 2\sigma - k - D/2),
\]

and the result is valid when $i, j$ are negative.

This hypergeometric series representation is 4-fold degenerate. Here we mention an important point in the process of analytic continuation to positive dimension and negative values of exponents of propagators referred to above. The result for the negative dimensional space region for $g_1^B$ is in fact given by a sum of two terms – the second one being also 4-fold degenerate – however, when we perform the analytic continuation this second term vanishes because it contains a factor of the form (forgetting about the minus sign)

\[
\frac{1}{(1 - \rho/2)} = \frac{\Gamma(1)}{\Gamma(1 - \rho/2)} \Delta \rightarrow \Gamma(0|\rho/2) = \frac{\Gamma(\rho/2)}{\Gamma(0)} = 0,
\]

which always vanishes (see also [18]) since $\rho \neq 0$ by definition and where, as usual in NDIM approach we make use of the Pochhammer’s symbol property $a|b = (-1)^{|b|} (1 - |a| - b)$.

To close this part of the computation, we just mention that of course there are other hypergeometric series which represent the same Feynman integral in other kinematical regions, e.g., there is a double series of the form,

\[
\sum_{a,b=0}^{\infty} \frac{\Gamma(\cdots)}{a!b!} \left( \frac{p_3^2}{p_4^2} \right)^{a},
\]

that is, one of the series (with summation index $b$) has unit argument and can be recast as a $3F_2(\cdots|1)$.

These are just the few different manners in which we may write down the result for the integral $g_1$. 

3
The second integral is easier than the first, and its result is also degenerate, i.e., there are a total of five $4 \times 4$ systems to be solved of which one has no solution and the remaining four, after properly summed give the same result, yielding

$$g_2 = (-\pi)^{D/2}(p^2)^{\sigma} \frac{\Gamma(1+i)\Gamma(1+j)\Gamma(1-\sigma-\omega/2)\Gamma(1+i+k+\rho/2)}{\Gamma(1+\sigma)\Gamma(1+i+\rho/2)\Gamma(1-i-k-D/2)\Gamma(1-j-\omega/2)}.$$  \hspace{1cm} (13)

which after analytically continuing to positive $D$ becomes,

$$g_2^{\{AC\}}(i,j,k) = \pi^{D/2}(p^2)^{\sigma}(\sigma+\omega/2)-2\sigma-\omega/2)(-i-\rho/2)(-j|\sigma)(-i-k-\rho/2)|\sigma).$$  \hspace{1cm} (14)

Next we consider Feynman integrals in the Coulomb gauge with tensorial structures. Again, as we treated in [17] the case of covariant gauge we show here how NDIM can handle these Coulomb gauge tensorial integrals in a similar manner. Let,

$$g_3(i,j,k) = \int d^Dq \ (q^2)^i(q+p)^2j(2q \cdot p)^k,$$

and

$$g_4(i,j,k,m) = \int d^Dq \ (q^2)^i(q+p)^2j(q^2)^k(2q \cdot p)^m,$$

so that, after some algebraic manipulations, we eventually get the result,

$$g_3^{\{AC\}}(i,j,k) = \pi^{D/2}(-2)^k(p^2)^{\sigma}(-i-j-D/2)(\sigma+\omega/2)(-\sigma)(-j|\sigma) \ _3F_2\{\{3\}|1,$$

where the set of parameters $\{3\} \equiv \{a_3, b_3, c_3; e_3, f_3\}$ for the hypergeometric function $\ _3F_2\{\{3\}|1\}$ is given in the table.

We must observe here that for (15) the exponent $k \geq 0$ always, and in (16) the exponent $m \geq 0$ always, and these must not be analytically continued into the region of negative values whereas the exponents $i, j$ do follow the usual analytic continuation process to get the final result for the integrals.

From our previous work [17] on NDIM approach to tensorial integrals, we know that the best solution for such kind of integrals is a truncated hypergeometric function because it contains all the cases of interest in the same formula: scalar, vector and arbitrary tensor rank. The hypergeometric function above is clearly truncated for positive integers $k$. This result, among the five possible hypergeometric series representations of such integral, is the only one that is a truncated series for even and odd values of propagator exponent $k$, since it assumes only positive values.

Finally, the result for the tensorial integral with three propagators,

$$g_4^{\{AC\}}(i,j,k,m) = \pi^{D/2}(-2)^m(p^2)^{\sigma}(-i-\rho/2)(\sigma^\prime+\omega/2)(j+\sigma^\prime)(-i-k-\rho/2)|j+\omega/2) \ _3F_2\{\{4\}|1,$$  \hspace{1cm} (16)

where $\sigma^\prime = \sigma + m = i + j + k + m + D/2$. Note that the result [18] contains the previous one, [17] in the particular case when $k = 0$; it is valid also when $i, j, k$ are negative and $m$ positive. The five parameters $\{4\} \equiv \{a_4, b_4, c_4; e_4, f_4\}$ are given in the table, and clearly the hypergeometric function is also truncated for even and odd positive integers $m$.

The well-known hypergeometric function $\ _3F_2$ is defined by the series,

$$\ _3F_2 \left[\begin{array}{c}a, b, c \\ e, f \end{array} \bigg| z \right] = \sum_{n=0}^{\infty} \frac{(a[n](b[n](c[n](z^n))}{(e[n](f[n])n)!} n!$$

so, when we refer to $\ _3F_2(\ldots|1)$ we are meaning the above series. For more details about hypergeometric functions the reader is referred to, e.g., [16].

| Parameters | $\ _3F_2\{\{3\}|1\}$ | $\ _3F_2\{\{4\}|1\}$ |
|-----------|----------------|----------------|
| a         | $-k/2$         | $-m/2$         |
| b         | $1/2 - k/2$    | $1/2 - m/2$    |
| c         | $j + \omega/2$ | $j + \omega/2$ |
| e         | $1 + i + j + D/2$ | $1 + i + j + k + D/2$ |
| f         | $1 - i - k - D/2$ | $1 - i - k - m - D/2$ |

Parameters for hypergeometric functions in equations (17) and (18).
III. TWO-LOOP COULOMB GAUGE INTEGRALS

As far as we know, NDIM is the only approach where Feynman integrals in different gauges, covariant and non-covariant alike, can be neatly performed, without reference to any special prescription to handle peculiar non-covariant singularities in the boson propagator. In the usual covariant gauges several calculations were carried out, e.g., one-loop $n$-point function \([3]\), scalar integrals for photon-photon scattering in QED \([3]\) and genuine \([13]\) two-loop three-point integrals. On the non-covariant side, we have gotten an important original result: Light-cone integrals in the NDIM context do not need the famous ML-prescription \([3]\) to circumvent the gauge dependent singularities \([3]\), as well as avoiding other features which turn the calculation cumbersome – such as using partial fractioning \([20]\) (a mandatory feature there) and integration over components.

Coulomb gauge two-loop integrals can be treated within NDIM methodology as well. Consider, for example,

\[ I_1(i, j, k, m) = \int d^Dq \, d^D r \, (q^2)^i (r^2)^j (p - r)^{2k} (r^2)^m, \]

\[ I_2(i, j, k, m) = \int d^Dq \, d^D r \, (q^2)^i (q - r)^{2j} (p - q)^2 m, \]

which can be generated by,

\[ I_1 = \int d^Dq \, d^Dr \, \exp \left[ -\alpha q^2 - \beta r^2 - \gamma (p - r)q^2 - \theta r^2 \right], \]

\[ I_2 = \int d^Dq \, d^Dr \, \exp \left[ -\alpha q^2 - \beta (q - r)^2 - \gamma r^2 - \theta (p - q)^2 \right]. \]

Following the usual steps of NDIM \([21]\), we get for the first integral a $6 \times 11$ matrix for the system of linear algebraic equations to be solved

\[ \begin{align*}
X_{123} + Y_{12467} & = i \\
X_{13} + Y_{2368} & = j \\
X_{123} + Y_{13578} & = k \\
X_2 + Y_{45} & = m , \\
X_{12} + Y_{12345} & = -\omega/2 \\
X_3 + Y_{678} & = -\rho/2
\end{align*} \]

which generates 462 $(6 \times 6)$ possible hypergeometric series representations for the integral in question. Of these, 216 have solutions in terms of hypergeometric series whose variable is either $z = p^2_1/p^2$ or $z^{-1}$. Among these power series the simplest ones are double series which we consider in more detail. Of course, there are also series of the following form,

\[ \sum_{a,b,c,e=0}^{\infty} \frac{z^{a+b} (z^{-1})^{c+e}}{a! b! c! e!} \Gamma(...), \]

\[ \sum_{a,b,c,e=0}^{\infty} \frac{z^{a} (z^{-1})^{b+c+e}}{b! c! e!} \Gamma(...), \]

which can only be convergent if $z = z^{-1} = 1$. Since this is a particular case, where $p^2 = p^2_1$, we will not study it.

Let us consider the solution written in terms of double hypergeometric series,

\[ I_1^{[AC]}(i, j, k, m) = \pi^D(p^2_1)^{\sigma''} P_1^{[AC]} \sum_{n_1, n_2=0}^{\infty} \frac{(-\sigma''|n_1|n_2)(-m|n_2|(m + \omega/2|n_1|(1 - m - \sigma'' - D/2|n_2)\left(\frac{p^2}{p^2_1}\right)^{n_1} \left(\frac{p^2}{p^2_1}\right)^{n_2}, \]

where $\sigma'' = \sigma' + D/2 = i + j + k + m + D$ and $P_1^{[AC]}$ is a product of Pochhammer symbols,

\[ P_1^{[AC]} = (-i + k + D/2)(-k + i + D/2)(\omega/2|m)(i + k + D|m)(-j - \sigma''|m + D/2|j - \sigma''), \]

where the exponents of propagators $i, j, k$ must assume negative values. There is another double series, in the other kinematical region, where $|p^2_1/p^2| < 1$, namely,

\[ I_1^{[AC]}(i, j, k, m) = \pi^D(p^2)^{\sigma''} P_2^{[AC]} \sum_{n_1=0}^{\infty} \frac{(-\sigma''|n_2|(-j - m - \omega/2|n_2 - n_1)(m + \omega/2|n_1)(j + m + D/2|n_1)\left(\frac{p^2}{p^2_1}\right)^{n_1} \left(\frac{p^2}{p^2_1}\right)^{n_2}, \]

(27)
where,

\[ P_B^{[AC]} = (-i| - \sigma'')( -j| - m - \omega/2)( -k| j + k + m + D/2)(\omega/2|m) (i + k + D| - k - D/2)(\rho/2|k + \omega/2). \]  \hspace{1cm} (28)

The second integral is much simpler than the former, in that the hypergeometric series representations involved are all summable. Summing them is an easy task and the result can be written in terms of gamma functions,

\[ I_2^{[AC]}(i, j, k, m) = \pi^D(p^2)^{\sigma''}(-i| - \rho/2)(\sigma'' + \omega/2) - 2\sigma'' - \omega/2)( -k|2k + \omega/2)( -m|2m + \omega/2)( -j|i + 2j + k + D) \times(-i - j - k - D/2 - \rho/2)(j + k + D - \rho/2| - k - \omega/2). \]  \hspace{1cm} (29)

This is a 36-fold degenerate result, i.e., of the overall 56 \((5 \times 5)\) systems, 36 of them have non-trivial solutions, which after summed (see e.g. [21]) and analytically continued give (29). It is important to note that this result allows negative values for \((i, j, k, m)\) and positive ones for \(D\). For negative \(m\) see appendix.

IV. CONCLUSION

Splitting the space-time dimension \(D\) in the dimensional regularization context into energy-sector and momentum sector, each with a specific regularizing parameter, it was possible to Leibbrandt et al to perform perturbative calculations in the Coulomb gauge at one and two-loop level. However, the calculations are very involved and they were able to present explicit results only for the divergent parts of the integrals. On the other hand, using NDIM we showed here that we can calculate complete results for the same integrals, and not only that, they did not have to be carried out separately. In our approach we can consider several of them at the same time, because we leave the exponents of propagators arbitrary, the integrals being either scalar or tensorial.

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APPENDIX A: SPECIAL CASES: EXTRACTING POLES

To make things a little more illuminating we consider in this appendix some technical issues relevant to the results for sample special cases.

1. One-loop.

Let us consider for instance the particular case where the exponents of propagators in the integral (16) are \( i = j = -1, k = -2, m = 2 \). The result for this integral is obtained from eq. (18),

\[
g_4^{\text{AC}}(-1, -1, -2, 2) = 4\pi^{D/2}(p^2)^{D/2 - 2} \frac{\Gamma(1 - \rho/2)\Gamma(\omega/2 - 1)\Gamma(3 - D/2)\Gamma(D/2 - 1)}{\Gamma(3 - \rho/2)\Gamma(D/2 - 2 + \omega/2)} {}_3F_2 \left( -1, -1/2, \omega/2 - 1 \left| \frac{D/2 - 3, 2 - D/2}{1} \right. \right),
\]

(A1)

observing that now \( \rho' = D/2 - 2 \). We proceed as usual in dimensional regularization, taking \( D = 4 - 2\epsilon, \omega = 3 - \epsilon, \rho = 1 - \epsilon \) and Taylor expanding around \( \epsilon = 0 \), to get

\[
g_4^{\text{AC}}(-1, -1, -2, 2) = 4\pi^{2-\epsilon}(p^2)^{-\epsilon} \left[ \frac{8}{3} + \left( -\frac{8\gamma_E}{3} + \frac{64}{9} - \frac{16}{3}\ln 2 \right) \epsilon + O(\epsilon^2) \right] {}_3F_2 \left( -1, -1/2, \omega/2 - 1 \left| \frac{D/2 - 3, 2 - D/2}{1} \right. \right), \quad (A2)
\]

where \( \gamma_E \) is the Euler’s constant.

Now we turn to the theory of hypergeometric functions (16). When a numerator parameter is a negative integer the series is a truncated one. This is exactly our case, and the hypergeometric function above has only two terms,

\[
{}_3F_2 \left( -1, -1/2, \omega/2 - 1 \left| \frac{D/2 - 3, 2 - D/2}{1} \right. \right) = 1 + \frac{(-1)[(-1)\omega/2 - 1]}{1!(D/2 - 3)[(2 - D/2)]} = 1 - \frac{(1 - \epsilon)}{4\epsilon(1 + \epsilon)}, \quad (A3)
\]

which substituted into (A2) yields,

\[
g_4^{\text{AC}}(-1, -1, -2, 2) = 4\pi^{2-\epsilon}(p^2)^{-\epsilon} \left( -\frac{2}{3\epsilon} + \frac{20}{9} + \frac{2\gamma_E}{3} + \frac{4\ln 2}{3} \right). \quad (A4)
\]

We remember that the original integral was contracted with \( 4p_ip_j \), so rewriting \( 4(p^2)^{-\epsilon} = 4(p^2)^{2-\epsilon - 2} = 4(p_ip_j)(p^2)^{-2-\epsilon} \), we obtain the final result,

\[
g_4^{\text{AC}}(-1, -1, -2, 2) = \pi^{2-\epsilon}p_ip_j(p^2)^{-2-\epsilon} \left( -\frac{2}{3\epsilon} + \frac{20}{9} + \frac{2\gamma_E}{3} + \frac{4\ln 2}{3} \right). \quad (A5)
\]

2. Two-loops.

Let us consider two particular cases for integral (21): The first one where \( i = k = -2, j = -1, m = 1 \) and a second one where \( i = k = -2, j = m = -1 \).

Observe that in the first case there is one exponent, \( m \), which is positive, so we must not analytic continue it (see for instance our previous papers [17]). The related Pochhammer symbol \( (-m)\Gamma(m + \omega/2) \) was generated by,

\[
\frac{\Gamma(1 + m)}{\Gamma(1 - m - \omega/2)} = \frac{1}{(1 + m - 2m - \omega/2)} \frac{\text{AC}}{\text{AC}} \frac{\Gamma(1 + m)}{(1 - m - \omega/2)} = (-1)^{2m + \omega/2}(-m)\Gamma(m + \omega/2).
\]

However it must not be analytically continued since we are interested in the special case where \( m \) is positive \( (m = 1) \). So the result for (21), which allows one to take \( m \) positive, reads now,

\[
I_2^{AC}(i, j, k, m) = \pi^D(p^2)^{\sigma''}(-i - \rho/2)(\sigma'' + \omega/2 - 2\sigma'' - \omega/2)(-k + 2\omega/2)(-j + 2j + k + D)
\times(-i - j - k - D/2 - \rho/2)[i + \rho/2(j + k + D - \rho/2) - k - \omega/2] \frac{\Gamma(1 + m)}{\Gamma(1 - m - \omega/2)}
\times(-1)^{2m + \omega/2}. \quad (A6)
\]
Now it is easy to expand (we use the MAPLE V software) around $D = 4 - 2\epsilon$, with $\sigma'' = D - 4$, to obtain the poles, double and simple ones, plus finite part

$$I_2^{[AC]}(-2, -1, -2, 1) = -i^{1-\epsilon}\pi^{4-2\epsilon}(p^2)^{2-2\epsilon} \left[ -\frac{1}{2\epsilon^2} + \frac{7 + 12 \ln 2 + 6\gamma_E}{6\pi\epsilon} + \frac{-520 - 112\gamma_E - 192\gamma_E \ln 2 + 43\pi^2 - 224\ln 2}{48\pi} \right. \right.
\left. \left. - \frac{48\gamma_E + 192\ln^2 2}{48\pi} + O(\epsilon) \right] . \right) . \quad (A7)

The second special case can be studied using (29) directly since in this case all exponents ($i = k = -2$, $j = m = -1$) are negative, and $\sigma'' = D - 6$). Using again MAPLE V software to expand around $\epsilon = 0$, we get a simple pole plus finite part,

$$I_2^{[AC]}(-2, -1, -2, -1) = \pi^{4-2\epsilon}(p^2)^{2-2\epsilon} \left[ \frac{1}{3\epsilon} + \frac{1}{3} - \frac{2\gamma_E}{3} - \frac{4\ln 2}{3} \right]. \quad (A8)

[1] C.G.Bollini, J.J.Giambiagi, Nuovo Cim.B12 (1972) 20. G. ‘t Hooft, M.Veltman, Nucl.Phys. B44 (1972) 189.
[2] G.Leibbrandt, Rev.Mod.Phys. 59 (1987) 1067. G.Leibbrandt, *Non-covariant gauges: Quantization of Yang-Mills and Chern-Simons theory in axial type gauges*, World Scientific (1994). A.Bassetto, G.Nardelli, R.Soldati, *Yang-Mills theories in algebraic non-covariant gauges*, World Scientific (1991). S.Leupold, H.Weigert, Phys.Rev.D54 (1996) 7695.
[3] S.Caracciolo, G.Curci, P.Menotti, Phys.Lett.B113 (1982) 311.
[4] S.Mandelstam, Nucl.Phys.B 213 (1983) 149. G.Leibbrandt, Phys.Rev.D 29 (1984) 1699.
[5] A.T.Suzuki, A.G.M.Schmidt, Prog.Theor.Phys.103 (2000) 1011; Eur.Phys.J.C12 (2000) 361. A.T.Suzuki, A.G.M.Schmidt, R.Bentín, Nucl.Phys.B537 (1999) 549.
[6] L.Baulieu, D.Zwanziger, Nucl.Phys.B548 (1999) 527. D.Zwanziger, Nucl.Phys.B518 (1998) 237; Prog.Theor.Phys.Suppl.131 (1998) 233.
[7] W.Kummer, W.Mödrich, A.Vairo, Z.Phys.C72 (1996) 653; Z.Phys.C66 (1995) 225. G.S.Adkins, P.M.Mitrikov, R.N.Fell, Phys.Rev.Lett.78 (1997) 9. E.Gubanova, C-R.Ji, S.R.Cotanch, hep-ph/0003286. F.Lenz, E.J.Moniz, M.Thies, Ann.Phys. (N.Y.) 242 (1995) 429.
[8] P.Doust, Ann.Phys. 177 (1987) 169. P.J.Doust, J.C.Taylor, Phys.Lett.B197 (1987) 232. J.C.Taylor, in *Lecture notes in Physics*, 137, P.Gaigg, W.Kummer, M.Schweda (Eds.), Springer-Verlag (1989).
[9] H.S.Chan, M.B.Halpern, Phys.Rev.D33 (1985) 540.
[10] G.Leibbrandt, J.Williams, Nucl.Phys.B475 (1996) 469.
[11] G.Leibbrandt, Nucl.Phys.B521 (1998) 383; Nucl.Phys.Proc.Suppl.64 (1998) 101. G.Heinrich, G.Leibbrandt, hep-th/9911211.
[12] I.G.Halliday, R.M.Ricotta, Phys.Lett.B193 (1987) 241. G.V.Dunne, I.G.Halliday, Phys.Lett.B193 247. D.J.Broadhurst, Phys.Lett.B197 (1987) 179.
[13] A.T.Suzuki, A.G.M.Schmidt, J.Phys.A31 (1998) 8023.
[14] A.I.Davydychev, *Proc. International Conference "Quarks-92"* (1992) 260. hep-ph/9307323.
[15] N.I.Upšukina, A.I.Davydychev, Phys.Lett.B332 (1994) 159.
[16] Y.L.Luke, *The Special Functions and their Approximations*, Vol.I, (Academic Press, 1969). L.J.Slater, *Generalized Hypergeometric Functions*, (Cambridge Univ.Press, 1966).
[17] A.T.Suzuki, A.G.M.Schmidt, Eur.Phys.J.C10 (1999) 357.
[18] C.Anastasiou, E.W.N.Glover, C.Oleari, Nucl.Phys. B565 (2000) 445; Nucl.Phys. B572 (2000) 307. A.T.Suzuki, A.G.M.Schmidt, J.Phys.A33 (2000) 3713.
[19] A.T.Suzuki, A.G.M.Schmidt, to be published in Can.J.Phys.
[20] G.Leibbrandt, S.-L.Nyeo, J.Math.Phys. 27 (1986) 627; Z.Phys.C30 (1986) 501.
[21] A.T.Suzuki, A.G.M.Schmidt, Eur.Phys.J.C5 (1998) 175.