A GENERAL METHOD FOR NON-PERTURBATIVE
RENORMALIZATION OF LATTICE OPERATORS

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Abstract

We propose a non-perturbative method for computing the renormalization constants of generic composite operators. This method is intended to reduce some systematic errors, which are present when one tries to obtain physical predictions from the matrix elements of lattice operators. We also present the results of a calculation of the renormalization constants of several two-fermion operators, obtained, with our method, by numerical simulation of QCD, on a $16^3 \times 32$ lattice, at $\beta = 6.0$. The results of this simulation are encouraging, and further applications to four-fermion operators and to the heavy quark effective theory are proposed.
1 Introduction

Renormalization of lattice operators is a necessary ingredient to obtain many physical results from numerical simulations, such as meson decay constants, form factors, structure functions, mixing amplitudes, etc. In this paper we study the renormalization of composite operators in lattice QCD. We propose a method which avoids completely the use of lattice perturbation theory and allows a non-perturbative determination of the renormalization constants of any composite operator. For illustrative purposes, we consider some specific applications to matrix elements of two-fermion operators. The approach is particularly useful in those cases, such as the scalar or pseudoscalar densities, where it is not possible to use the chiral Ward identities to determine the renormalization constants non-perturbatively\(^1\)\[1\]–\[3\]. Moreover, our proposal can be applied to many other cases, such as the renormalization of the four-fermion operators of the effective weak Hamiltonian, and to heavy-light currents in the Heavy Quark Effective Theory. Preliminary results have been presented in ref. \[4\]. A similar attempt, limited to the divergent part of the \(\Delta I = 1/2\) penguin-operator, can be found in ref. \[5\].

Renormalized lattice operators must correspond, in the limit of infinite lattice cut-off, to finite chirally covariant operators, which obey the same renormalization conditions as those in the continuum. With just a few exceptions, lattice perturbation theory is used to evaluate the renormalization constants of lattice operators. The problem of mixing with lower dimensional operators requires however, a non-perturbative subtraction of all power divergences \[1\],\[6\]–\[9\]. Apart from this special case, the use of perturbation theory, in the computation of multiplicative constants and dimensionless mixing coefficients, is well justified, provided that the lattice spacing \(a\) is sufficiently small, i.e. \(a^{-1} \gg \Lambda_{QCD}\). However, in those cases where the results from perturbation theory, obtained using the bare lattice coupling constant \(\alpha_{s}^{\text{LAT}} = g_{0}^{2}/4\pi\) as an expansion parameter, can be checked using non-perturbative methods, there is generally a significant discrepancy. Several solutions to this problem have been proposed so far:

- In order to improve the convergence of the perturbative series, the au-

\(^1\)In the following we will denote the method to determine the renormalization constants, using the Ward identities, as the Ward identity method.
thors of ref. [10] have proposed a “tadpole improved” perturbation theory. Nevertheless, ignorance of higher order contributions still represents an important source of uncertainty in the extraction of physical results.

- In some limited cases, corresponding to finite operators, namely the vector and axial vector currents, and the ratio of pseudoscalar and scalar densities, a fully non-perturbative determination of the renormalization constants can be obtained with the use of chiral Ward identities [1]–[3].

- One may fix non-perturbative renormalization conditions directly on hadronic matrix elements. This procedure was used in ref. [7], to subtract the divergent part of the $\Delta I = 1/2$ operator. The price is that one has to sacrifice a physical prediction, for any subtraction imposed on hadronic states. Thus, when several renormalization conditions are necessary, one loses much predictive power [1, 6].

Our proposal is to impose renormalization conditions non-perturbatively, directly on quark and gluon Green functions, in a fixed gauge, with given off-shell external states, with large virtualities.

The method consists in mimicking what is usually done in perturbation theory. One fixes the renormalization conditions of a certain operator by imposing that suitable Green functions, computed between external off-shell quark and gluon states, in a fixed gauge, coincide with their tree level value. For example, if we consider the generic two-quark operator $O_\Gamma = \bar{\psi} \Gamma \psi$, we may impose the condition

$$Z_\Gamma \langle p|O_\Gamma|p\rangle |_{p^2 = -\mu^2} = \langle p|O_\Gamma|p\rangle_0,$$

(1)

where $\Gamma$ is one of the Dirac matrices and $\langle p|O_\Gamma|p\rangle_0$ is the tree level matrix element. The extension to more complicated cases, including four-fermion operators and operator mixing, is straightforward. This procedure defines the same renormalized operators, i.e. the same Wilson coefficient functions, in all regularization schemes, provided they are expressed in terms of the same renormalized coupling constant. However, the coefficient functions now depend on the external states and on the gauge, which must be specified.
In principle this scheme completely solves the problem of large corrections in lattice perturbation theory, which are automatically included in the renormalization constants. Matching of lattice operators to the corresponding ones, renormalized in a continuum renormalization scheme (for definiteness, the \( \overline{MS} \) scheme), then requires continuum perturbation theory only. When next-to-leading corrections are known, the error on the physical matrix elements is of \( O((\alpha_s^{\overline{MS}})^2) \), in the continuum perturbative expansion, which is expected to have a better convergence. A continuum perturbative calculation of the matching conditions is a common step in all approaches, standard or “tadpole improved” perturbation theory and non-perturbative renormalization. Our proposal can be applied to any composite operator, unlike the Ward identity method, which is limited to a few, albeit very important, cases\(^2\).

The feasibility of using Green functions computed between quark states, in a fixed gauge, was already discussed in ref. \[3\], in the context of non-perturbative renormalization, using Ward identities. Indeed, in the case of the axial current, it was shown that the signal was rather good and the result was in agreement with the corresponding hadronic gauge invariant determination.

As far as the renormalization conditions are concerned, our proposal is expected to work, whenever it is possible to fix the virtuality of the external states \( \mu \) and to satisfy the condition \( \Lambda_{QCD} \ll \mu \ll 1/a \), in order to keep under control both non-perturbative and discretization effects. We stress that this requirement is common to all methods. The results of our numerical investigation are encouraging: they suggest that there does exist a “window” in \( \mu \), where the method can be applied, at values of the lattice coupling constant used in current lattice simulations.

The plan for the remainder of the paper is as follows. In the next section the basic formulae necessary to define our non-perturbative method are presented, using as an example logarithmically divergent two-quark operators, such as the scalar and pseudoscalar densities. The discussion is generalised to all composite operators in section 3, and in the following section the implementation of the method in numerical simulations for two-quark operators

\(^2\)It is still true that the statistical accuracy is in general slightly better with the Ward identities method, in those cases where it can be applied, see sec. \[3\].
is discussed. In section 5 we give some details about the perturbative evaluation of the renormalisation constants on lattices of finite volume. The values of the renormalisation constants obtained in a numerical simulation are presented in section 6, and are compared to the results from perturbation theory. The final section contains our conclusions, a summary of the numerical results and a discussion of possible further applications of the ideas presented in this paper.

2 Non-perturbative renormalization conditions

In this section, the non-perturbative renormalization scheme is presented in detail. We also discuss the conditions which must be satisfied for our method to be applicable.

To facilitate the discussion, it is convenient to classify composite operators into three main classes, according to their ultra-violet behaviour, as $a \to 0$:

1. **Finite operators**: these include the vector and axial vector currents. Another example is the ratio of the scalar and pseudoscalar renormalization constants, $Z_S/Z_P$. For these cases, the Ward identity method is also applicable.

2. **Logarithmically divergent operators**: this is a large family of operators, relevant to hadron phenomenology. It includes scalar and pseudoscalar densities, four-fermion $\Delta F = 2$ operators, some components of the energy-momentum tensor, etc.

3. **Power divergent operators**: these operators are present when mixing with lower dimensional operators is possible. This happens in regularizations which have an intrinsic mass scale, such as the lattice regularization. Examples are often encountered in phenomenological applications of lattice QCD, such as four-fermion operators relevant to $\Delta I = 1/2$ transitions or operators of order $1/m$ in the heavy quark effective theory (HQET) [11]–[15].

For simplicity, we first consider the two-fermion operators of class 2., and extend the discussion to other operators in this class, and also to those in
classes 1. and 3., in the next section. Thus, the formulae given below apply
directly to the pseudoscalar and scalar densities. Throughout our discussion,
we assume that discretization errors are negligible: in field theory language,
this is equivalent to the statement that the renormalized Green functions do
not differ appreciably from their values in the limit of an infinite ultra-violet
cut-off. The discussion is presented in the limit of small quark masses and in
Euclidean space-time. The discretisation of the quark action is assumed to
be performed à la Wilson, characterized by explicit chiral symmetry breaking
of the Lagrangian. The extension to staggered fermions is straightforward.

Let us consider the forward amputated Green function $\Gamma_{O}(pa)$, of a two-
fermion bare lattice operator $O(a)$, computed between off-shell quark states
with four-momentum $p$, with $p^2 = \mu^2$, and in a fixed gauge, for example the
Landau gauge. Without gauge-fixing all Green functions computed between
quark and gluon external states are zero. This is also true when the gauge is
not completely fixed, e.g. in the Coulomb gauge. We define the renormalized
operator $O(\mu)$, by introducing the renormalization constant $Z_{O}$

$$O(\mu) = Z_{O}(\mu a, g(a)) O(a).$$

$Z_{O}$ is found, by imposing the renormalization condition

$$Z_{O}(\mu a, g(a)) Z_{\psi}^{-1}(\mu a, g(a)) \Gamma_{O}(pa)|_{p^2=\mu^2} = 1,$$

where $Z_{\psi}$ is the field renormalization constant, to be defined below. This
procedure defines a renormalized operator $O(\mu)$ which is independent of the
regularization scheme [17]–[19]. It depends, however, on the external states
and on the gauge. This does not affect the final results, which, combined
with the continuum calculation of the renormalization conditions, at any
given order of perturbation theory, will be gauge invariant and independent
of the external states. Let us specify the different quantities entering eq. (3).
$\Gamma_{O}$ is defined in terms of the expectation value\(^{\dagger}\) of the non-amputated Green
function $G_{O}(pa)$, and of the quark propagator $S(pa)$

$$\Gamma_{O}(pa) = \frac{1}{12} \text{tr} \left( \Lambda_{O}(pa) \hat{P}_{O} \right),$$

\(^{3}\)This includes the SW-Clover action [16].

\(^{4}\)“Expectation value” means, as usual, that one averages the Green functions over the
gauge field configurations, generated by Monte Carlo simulation, see sec. 4.
where

$$\Lambda_O(pa) = S(pa)^{-1}G_O(pa)S(pa)^{-1}. \quad (5)$$

$\hat{P}_O$ is a suitable projector on the tree-level operator: $\hat{P}_O = \gamma_5$ for the scalar (pseudoscalar) density. The factor $1/12$ ensures the correct overall normalization of the trace (colour x spin = 12). Projectors are very convenient when defining Green functions, particularly in the non-perturbative case. They have been extensively used in refs. [18, 19]. Of course one can also use other definitions of $Z_O$.

$Z_{\psi}^{1/2}$ is the renormalization constant of the fermion field. It can be defined in different ways, some of which are equivalent perturbatively. Beyond perturbation theory, the most natural definition of $Z_\psi$ is obtained from the amputated Green function of the conserved vector current $V^C$. Indeed, one knows that for $V^C$ the renormalization constant is equal to one:

$$Z_{VC}^{-1} = 1 \quad (6)$$

which implies

$$Z_\psi = \frac{1}{48} \text{tr} \left( A_{VC}^\mu (pa) \gamma_\mu \right) |_{p^2=\mu^2}.$$  \quad (7)

Equations (3)–(7) completely define our method. In the remainder of this section, we discuss some important aspects concerning its applicability.

In the above formulae, for simplicity, we have always considered only forward matrix elements. In general, one has the freedom to define the renormalization conditions at different external momenta $p$ and $p'$ ($p \neq p'$). The virtualities of the quark states must be much larger than $\Lambda_{QCD}$. The reason is that, in order to obtain the physical result, we have to combine the matrix element of the renormalized operator $O(\mu)$, with a Wilson coefficient function. The latter is computed in continuum perturbation theory, by expanding in $\alpha_{MS}$ at a scale of order $\mu$. Thus, for the validity of this perturbative calculation, $\mu$ must be large. An important question in our program is whether it is possible to find, on the lattice, a scale $\mu$ which is sufficiently low, in order to have small $O(a)$ effects, and sufficiently large, in order to have small higher order corrections. The range of $\mu$, where the above conditions are satisfied, depends on the value of $g_\alpha^2$ of the numerical simulation.
We stress that, if such a window does not exist, in current lattice simulations, an accurate matching of lattice operators to continuum ones becomes impossible, not only in our approach, but also with any other method.

It might be expected that the condition $\mu \gg \Lambda_{\text{QCD}}$ ensures that perturbation theory is valid. When spontaneous symmetry breaking occurs however, as is the case in QCD, a large value of $\mu$ may not be enough, because of the presence of the Goldstone boson, the pion in our case. At low momentum transfers $q = p - p'$, Green functions can receive a non-perturbative contribution from the pion pole, which cannot be computed. The naive expectation is that this effect vanishes as $1/q^2$, at large $q^2$. However, with fermion fields, this contribution is proportional to $1/p^2 = 1/\mu^2$, even when $q^2 = 0$. The proof is given in the appendix. We conclude that a large value of $\mu$ solves the problem. At low values of $\mu^2$, however, for the pseudoscalar density and the axial current, we expect to find visible effects, see secs. 3 and 6.

A danger to the non-perturbative scheme may come from the presence of Gribov copies. We do not address this problem here; it has been investigated in refs. [20, 21]. The results of ref. [21] indicate that, at least for the quantities of interest in the present study, the “Gribov uncertainty”, in current lattice simulations, is at most of the same order as the statistical error.

One may imagine applying the renormalization conditions at very large values of $\beta$, on lattices which have small physical volumes. This might seem to overcome the problem of the existence of the window in $\mu$. For physical applications however, it is necessary to evolve the renormalization constants to smaller values of $\beta$. Non-perturbative contributions, which are not detected at large $\beta$’s, may then become important. This demonstrates the necessity, for any method, of the existence of a window $\Lambda_{\text{QCD}} \ll \mu \ll 1/a$, where $a$ is the lattice spacing used in numerical simulations of physical quantities. Only then are non-perturbative effects and lattice artefacts small simultaneously.

3 Applications of the method to generic composite operators

In the previous section, using the renormalization of the scalar and pseudoscalar densities as a prototype, the general strategy of the non-perturbative
scheme has been presented. In the present section the necessary modifications needed in other cases are briefly described. We will follow the classification of operators introduced in sec. 2.

### 3.1 Finite operators

In the case of vector and axial vector currents, we do not have the freedom of fixing the renormalization conditions in an arbitrary way: the renormalization conditions are acceptable only when they are compatible with the Ward identities. To be specific, one can apply eq. (3) to the local vector current in order to determine $Z_{V_L}$

$$Z_{V_L} = \frac{Z_\psi}{\Gamma_{V_L}}.$$  

We now show that this definition of $Z_{V_L}$, which uses the general prescription given in the previous section, coincides with the one derived from the vector current Ward identity on quark states [1, 22]

$$q_\mu [Z_{V_L} \Lambda_{V_L}(p + \frac{q}{2}, p - \frac{q}{2})] = -i \left( S^{-1}(p + \frac{q}{2}) - S^{-1}(p - \frac{q}{2}) \right).$$  

(9)

where $\Lambda_{V_L}(p+q/2, p-q/2)$ is the amputated vertex with momentum transfer $q$. By applying $\partial/\partial q_\rho$, at $q = 0$, on both sides of eq. (9), one gets

$$Z_{V_L} \left( \Lambda_{V_L}(p) + q_\mu \frac{\partial}{\partial q_\rho} \Lambda_{V_L}(p + \frac{q}{2}, p - \frac{q}{2}) |_{q=0} \right) = -i \frac{\partial}{\partial p_\rho} S^{-1}(p).$$  

(10)

The second term on the l.h.s. vanishes when $q \to 0$. By tracing eq. (10) with $\gamma_\rho$, we then recover the definition of $Z_{V_L}$ of eq. (8).

One is tempted to follow the same path for the local axial vector current, i.e. to define

$$Z_{A_L} = \frac{Z_\psi}{\Gamma_{A_L}}.$$  

(11)

However this definition of $Z_{A_L}$ is not equivalent to the one obtained by the Ward identity

$$q_\mu [Z_{A_L} \Lambda_{A_L}(p + \frac{q}{2}, p - \frac{q}{2})] = -i \left( \gamma_5 S^{-1}(p + \frac{q}{2}) + S^{-1}(p - \frac{q}{2}) \gamma_5 \right).$$  

(12)

\[5\text{For the remainder of this subsection we work in lattice units, setting } a = 1.\]
To see this we proceed as for the vector current, applying $\partial/\partial q_\mu$ to both sides of eq. (12).

\[
Z_{A_L^\mu} \left( \Lambda_{A_L^\mu}(p) + q_\mu \frac{\partial}{\partial q_\mu} \Lambda_{A_L^\mu}(p + \frac{q}{2}, p - \frac{q}{2}) \right) =
\]

\[
- \frac{i}{2} \left(\gamma_5 \frac{\partial}{\partial p_\mu} S^{-1}(p) - \frac{\partial}{\partial p_\mu} S^{-1}(p) \gamma_5 \right).
\]

(13)

In the axial case, however, the second term on the l.h.s. does not vanish, as $q \to 0$ and, indeed, this term is necessary to saturate the Ward identity, in presence of a massless Goldstone boson. As demonstrated in the appendix however, this term becomes negligible for high values of $p^2$. In this case, by tracing eq. (13), one recovers (11). In conclusion, we expect to find a value of $Z_{A_L^\mu}$ close to those obtained from the Ward identity on hadron or on quark states [3], by using eq. (11), at large $\mu^2$.

$Z_S/Z_P$ is another finite quantity, which is completely fixed by the Ward identities, even though the renormalization conditions on $Z_S$ and $Z_P$ are separately arbitrary. In other words one is free to decide the renormalization conditions of one of the two operators, the scalar density say, and the value of $Z_P$ will automatically follow. The procedure of sec. 2, respects the Ward identities and thus guarantees that the correct value of $Z_S/Z_P$ is obtained. This is clearly true only at large values of $\mu^2$, for the same reasons as for the axial vector current.

### 3.2 Logarithmically divergent operators

For this category of operator only the question of operator-mixing is left to be discussed. There are two kinds of mixing: the one which can occur also in the continuum and the one induced by the explicit lattice chiral symmetry breaking. We are only concerned here with the latter case, and hence restrict our discussion to operators which renormalize multiplicatively in the continuum. To illustrate our arguments, we consider the four-fermion operator

\[
O^{\Delta S=2} = \bar{s} \gamma^\mu (1 - \gamma_5) d \bar{s} \gamma_\mu (1 - \gamma_5) d,
\]

(14)
which is relevant to the calculation of the $K^0-\bar{K}^0$ mixing amplitude. Because of the Wilson term, this operator mixes with operators of different chiralities \([1, 6, 23]\).

\[ O^{\Delta S=2}_{\text{cont}} = Z_O \left( O^{\Delta S=2}_{\text{latt}} + \sum_i Z_i O^i \right), \]  

(15)

where the operators $O^i$ are given by $O^1 = (\bar{s}d)(\bar{s}d), O^2 = (\bar{s}\gamma_5d)(\bar{s}\gamma_5d)$, etc. \([23]\). The mixing coefficients $Z_i$ are finite functions of $g_0^2(a)$. The over-all renormalization $Z_O$ is necessary to eliminate the logarithmic divergence.

We now sketch the renormalization procedure for $O^{\Delta S=2}$. Let us start from the four-point amputated, Green function of $O^{\Delta S=2}_{\text{latt}}$, between quark states with momentum $p^2 = \mu^2$

\[ \Lambda^{ABCD}_{\alpha\beta\gamma\delta}(pa), \]  

(16)

where $\alpha, \beta \ldots$ and $A, B \ldots$ are spin and colour indices respectively. To find the mixing coefficients $Z_i$, one uses suitable projectors $\hat{P}$, as was done for two-quark operators\([6]\), for example

\[ \text{tr}(\Lambda \hat{P}) = \Lambda^{AABB}_{\alpha\beta\gamma\delta}(\gamma_5)_{\beta\alpha}(\gamma_5)_{\delta\gamma}. \]  

(17)

By projecting on all the possible colour-spin structures, corresponding to the operators $O^i$, one obtains a system of linear equations in the mixing coefficients $Z_i$. Using the $Z_i$, determined in this way, one can compute the Green function of the subtracted operator $O^s = O^{\Delta S=2}_{\text{latt}} + \sum_i Z_i O^i$. From the amputated vertex of $O^s$, one can finally remove the logarithmic divergence, by imposing the renormalization condition

\[ Z_O(\mu, g(a)) Z_{\psi}^{-2}(\mu, g(a)) \Gamma^s_O(pa)|_{p^2=\mu^2} = 1, \]  

(18)

where

\[ \Gamma^s_O \sim \text{tr}(\Lambda^s \hat{P}_{L\times L}), \]  

(19)

and $P_{L\times L}$ is the suitable projector on the $\gamma_\mu(1-\gamma_5)\otimes\gamma^\mu(1-\gamma_5)$ Dirac tensor.

\[ \text{In order to simplify the presentation, we do not discuss here the complications arising from the different possible colour structures.} \]
Figure 1: Diagram which mixes the four fermion operator in eq. (20) with the scalar and pseudoscalar densities. The mixing is induced by the chiral breaking term present in the Wilson and Clover actions.

3.3 Power divergent operators

Composite operators are divergent in $1/a$ when they can mix with lower dimensional operators. Examples are given by the operators relevant to deep inelastic scattering and by the penguin-operators appearing in the $\Delta S = 1/2$ Hamiltonian. Let us consider the $\Delta S = 1/2$ operator

$$O^{\Delta S=1/2} = (\bar{s}\gamma_\mu(1 - \gamma_5)d) \sum_i (\bar{q}_i \gamma^\mu(1 + \gamma_5)q_i).$$

(20)

$O_{}\Delta S=1/2^{\text{latt}}$ mixes with the scalar and pseudoscalar densities, even at zeroth order in $\alpha_s$, through the diagram in fig. [7]

$$O^{\Delta S=1/2}_{\text{cont}} = Z_O O_{\text{latt}}^{\Delta S=1/2} + Z_1 (\bar{s}d) + Z_2 (\bar{s}\gamma_5d) + \ldots.$$  

(21)

To find the coefficients $Z_{1,2}$, one imposes that the two-point Green function of the operator on the r.h.s. of eq. (21), on off-shell quark states, is zero [4]. Once all the divergent parts have been removed, one can impose the renormalization conditions, necessary to remove the over-all logarithmic divergence, as discussed above.

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7 $O^{\Delta S=1/2}_{\text{latt}}$ also mixes with the chromo-magnetic operator, e.g. $\bar{s}\sigma_{\mu\nu}G^{\mu\nu}d$, which is not being considered here.
4 Implementation of the method in numerical simulations

To implement the ideas of the previous section, let us consider the generic local two-fermion operator

\[ O_\Gamma(x) = \bar{\psi}(x)\Gamma\psi(x) \]  

(22)

where \( \Gamma \) is any Dirac matrix. The lattice, bare Green function of \( O_\Gamma \), between off-shell quark states, can be obtained from

\[ G_O(x, y) = \langle \psi(x)O_\Gamma(0)\bar{\psi}(y) \rangle = \frac{1}{N} \sum_{i=1}^{N} S_i(x|0)\Gamma S_i(0|y), \]  

(23)

where \( i = 1, N \) labels the gauge field configurations and \( S_i(x|0) \) is the quark propagator on the single configuration \( i \), obtained by inverting the discretized Dirac operator. We denote by \( S_i(x|y) \) the inverse Dirac operator, which is not translationally invariant, in contrast to \( S(x - y) \), which is the usual quark propagator. The gauge field configurations are generated with some standard, gauge-invariant algorithm; the gauge fields and the quark propagators are then rotated into the Landau gauge, which will be defined in sec. \[6\].

From eq. (23), using \( S_i(x|y) = \gamma_5S_i^\dagger(y|x)\gamma_5 \), one gets

\[ G_O(p, a) \equiv \int d^4xd^4ye^{-i(p \cdot x - p \cdot y)}G_O(x, y) \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \left( \int d^4x S_i(x|0)e^{-i(p \cdot x)} \right)\Gamma\left( \int d^4y S_i(0|y)e^{i(p \cdot y)} \right) \]

\[ = \frac{1}{N} \sum_{i=1}^{N} S_i(p|0)\Gamma\gamma_5\left( \int d^4y S_i^\dagger(y|0)e^{i(p \cdot y)} \right)\gamma_5 \]

\[ = \frac{1}{N} \sum_{i=1}^{N} S_i(p|0)\Gamma\left( \gamma_5S_i^\dagger(p|0)\gamma_5 \right), \]  

(24)

where we have defined

\[ S_i(p|y) = \int d^4x S_i(x|y)e^{-i(p \cdot x)}, \]  

(25)
because, on a single configuration, the quark propagator is not translationally invariant. Translational invariance is recovered after averaging over the configurations. We can then write

\[ S(pa) = \langle S(p|0) \rangle = \frac{1}{N} \sum_{i=1}^{N} S_i(p|0). \]  

Eqs. (24) and (26) give \( G_O(pa) \) and \( S(pa) \) in terms of quark propagators, computed by inverting the Dirac operator in the presence of a background gauge field configuration, generated by numerical simulation. With these quantities at hand, one can apply the strategy and the formulae, introduced in sec. 2. We add some information, that can help the reader in practical applications. The quark wave-function renormalization constant \( Z_\psi \) has been computed using

\[ Z_\psi = \Gamma_{V_L} \times Z_{V_L} = \frac{1}{48} \text{tr} \left( \Lambda_{V_L}(pa) \gamma_\mu \right) \big|_{p^2=\mu^2} \times Z_{V_L} \]  

(27)

where \( \Lambda_{V_L}(pa) \) is defined as in eq. (5); the trace is taken over colours and spins and \( Z_{V_L} \) is the renormalization constant of the local vector current \( V_\mu(x) = \bar{\psi}(x) \gamma_\mu \psi(x) \). Equations (7) and (27) are two equivalent definitions of \( Z_\psi \). We have used eq. (27) instead of eq. (7), because the former is more convenient when, as in our case, one has only computed quark propagators which stem from the origin. The three-point \((q - V - q)\) Green function is obtained by inserting the local vector current in the origin. \( Z_{V_L} \) can be obtained with very high accuracy from ratios of matrix elements of the conserved and local currents [2, 24]. From the amputated Green functions

\[ \Gamma_1(pa) = \frac{1}{12} \text{tr} \left( \Lambda_1(pa) \hat{1} \right), \]

\[ \Gamma_{\gamma_5}(pa) = \frac{1}{12} \text{tr} \left( \Lambda_{\gamma_5}(pa) \gamma_5 \right), \]

\[ \Gamma_{\gamma_\mu\gamma_5}(pa) = \frac{1}{48} \text{tr} \left( \Lambda_{\gamma_\mu\gamma_5}(pa) \gamma_5 \gamma_\mu \right). \]  

(28)

by imposing the renormalization conditions (3), at several values of \( p^2 = \mu^2 \), to be specified in sec. 3, we have determined \( Z_{S,P,A}(\mu, g(a)) \), as functions of \( \mu^2 \).
If we wish to compute $Z_{VL}$ using the procedure defined in sec. 3 we clearly cannot take $Z_\psi$ from eq. (27). From the Ward identity we have

$$Z_\psi = -i \frac{1}{12} \mathrm{tr} \left( \frac{\partial S(p)^{-1}}{\partial \phi} \right) |_{\mu^2}$$

where $\partial S(p)^{-1}/\partial \phi = 1/4 \times \gamma_\rho \partial S(p)^{-1}/\partial p_\rho$. To avoid derivatives with respect to a discrete variable, we have used

$$Z_\psi' (\mu, g(a)) = \frac{\mathrm{tr} \left( -i \sum_{\lambda=1,4} \gamma_\lambda \sin(p_\lambda a) S^{-1}(pa) \right) \gamma_\mu}{4 \sum_{\lambda=1,4} \sin^2 p_\lambda a} |_{\mu^2}.$$ 

(30)

In one loop perturbation theory, $Z_\psi' = Z_\psi$, in the Landau gauge\footnote{In general, they differ by a finite term of order $\alpha_s$.}. By using

$$\Gamma_{\gamma_\mu} (pa) = \frac{1}{48} \mathrm{tr} \left( \Lambda_{\gamma_\mu} (pa) \gamma_\mu \right),$$

(31)

and imposing the renormalization conditions \footnote{In general, they differ by a finite term of order $\alpha_s$.}, we have determined $Z_{VL}$ at several $\mu^2$. From the general discussion of the previous section, one expects that $Z_{VL}$, defined through the present procedure, is independent of $\mu^2$, up to terms of $O(a)$, as has been effectively found, see sec. 3.

In actual numerical simulations, the renormalization procedure could be obscured by lattice artefacts, i.e. terms of $O(\mu a)$, which are present when one imposes the renormalization conditions at large values of $\mu^2$, required to avoid large non-perturbative or higher order effects. As shown in ref.\footnote{In general, they differ by a finite term of order $\alpha_s$.} (we use here the same notation), in order to reduce discretization errors, due to the finite value of the lattice spacing, one can compute correlation functions using a nearest-neighbour improved fermion action \footnote{In general, they differ by a finite term of order $\alpha_s$.}

$$S_F^I = S_F^W + a^4 \sum_{x,\mu\nu} \left[ -ig_0 \frac{a r}{4} \tilde{\psi}(x) \sigma_{\mu\nu} F_{\mu\nu}(x) \psi(x) \right],$$

(32)

where $S_F^W$ is the usual Wilson action, $r$ is the Wilson parameter and $F_{\mu\nu}$ is the lattice field strength tensor. Moreover, it is also necessary to use improved operators $O^I_\Gamma$ defined as

$$O^I_\Gamma (x) = \tilde{\psi}(x) \overline{R}_x \Gamma \overline{R}_x \psi(x),$$

(33)
where the rotations of the fermion fields are defined as

$$\vec{R}_x = (1 - a\frac{r}{4}(\vec{D}_x - m_0)) \quad \text{and} \quad \vec{R}_y = (1 + a\frac{r}{4}(\vec{D}_y + m_0)). \quad (34)$$

By using the improved action and operators, one eliminates discretization errors of \(O(a)\), and only those of \(O(\alpha_s a)\) or \(O(a^2)\), or higher, remain. In eq. (33), \(\vec{R}_x\) and \(\vec{R}_y\) can be replaced by

$$\vec{R}'_x = 1 - a\frac{r}{2}\vec{D}_x \quad \text{and} \quad \vec{R}'_y = 1 + a\frac{r}{2}\vec{D}_y, \quad (35)$$

in computations of the hadron spectrum and of on-shell operator matrix elements. As discussed in refs. [26, 27], the improvement procedure is equivalent to expressing all correlation functions in terms of “effective” quark propagators

$$S_{i}^{\text{eff}}(x|0) = (1 - a\frac{r}{2}\vec{D}_x)S_{i}^{L}(x|0)(1 + a\frac{r}{2}\vec{D}_y=0) \quad (36)$$

where \(S_{i}^{L}(x|0)\) is the fermion propagator of the improved theory from the point \(x\) to the origin. In order to eliminate terms of \(O(a)\) in the off-shell correlation functions, used in this study, a further step is necessary [3]: we have to shift the quark propagator by a local term, which in Fourier space appears as a constant, to be added to its diagonal components

$$S_{\alpha\beta}^{AB}(pa) = S_{\alpha\beta}^{\text{eff},AB}(pa) + \frac{r}{2}\delta^{AB}\delta_{\alpha\beta}$$

In the following, the vector current \(V_{\mu}^L\), the axial vector current \(A_{\mu}^L\), the pseudoscalar density \(P\) and the scalar density \(S\) are improved operators, obtained by using eq. (36).

5 Lattice perturbation theory

In order to gain some insight into the non-perturbative results presented in the next section, we have performed the corresponding one-loop perturbative calculation both in the continuum and on the lattice in the Landau gauge. This calculation is an extension of the one presented in ref. [27].
The standard calculation of the one-loop corrections corresponds to a computation of the renormalization constants with a fixed ultra-violet cut-off on the loop momenta, $|p_i| \leq \pi/a$, neglecting terms of $O(a)$ in the final result. The renormalization constants obtained in this limit depend on the quark mass $m_q$ and on the external momenta only through the logarithmically divergent terms, which are related to the anomalous dimension of the operator. Since, at one loop, these terms are universal, i.e. they are the same on the lattice and in any continuum regularization, the renormalization constants, necessary to relate lattice to continuum renormalized operators, are independent of the quark mass and external momenta (they depend however on the lattice spacing and on the continuum renormalization scale $\mu$). We call this limit “Standard Perturbation Theory” (STP). The non-perturbative calculation however, is performed on a lattice of finite size, and at a fixed value of the lattice spacing and quark mass: thus we cannot make terms of $O(a)$ arbitrarily small. In particular, since we want to study the $\mu$-dependence of the renormalization constants, it is important to know at which values of $\mu$ (and $m_q$), the perturbative results, obtained on a finite lattice and with a fixed lattice spacing, differ from those obtained in SPT. In order to follow closely the non-perturbative computation, the perturbative results have also been obtained, on a lattice of size $16^4 \times 32$. In this way, we keep all terms of $O(\alpha_s a)$ or $O(a^2)$, or higher, in the final result (we call this procedure “Discrete Perturbation Theory”, DPT). The loop integrals were computed numerically, both in SPT and in DPT.

The non-perturbative results have been obtained by tracing suitable amputated Green functions, computed between off-shell quark states, in the Landau gauge, cf. eq. (4). In perturbation theory, this procedure differs from the standard one, where one identifies the correction from the term proportional to the tree-level matrix element of the operator under consideration \[18, 19\]. The difference is due to finite terms which, in the standard procedure, are considered as part of the matrix element of the operator, whereas with the projectors, are considered as part of the one-loop corrections. At one-loop order, these terms are the same in the continuum and on the lattice (in the limit $a \to 0$) and cancel when one computes the difference between the continuum and the lattice renormalization constants. To be more specific, let us consider the diagrams in fig. \[2\] computed in Na"ive Dimensional
where \( C_F = (N^2 - 1)/2N \), \( 1/\hat{\epsilon} = 1/\epsilon - \ln 4\pi - \gamma_E \) and \( p \) is the external momentum. Usually, one takes the term proportional to \( (1/\hat{\epsilon} + 1) \), i.e. the coefficient of \( \gamma_\mu \), as the one loop contribution to the bare operator. With the projector (see eq. (4)) the last term in eq. (37) also gives a contribution, so that the factor now becomes \( (1/\hat{\epsilon} + 1/2) \). In table [1], we give the one-loop corrections obtained, with the projectors and in the ’t Hooft and Veltman (HV) dimensional renormalization scheme [28], for the two-quark operators considered in this paper. The NDR regularization differs from the HV one by the definition of the anti-commuting properties of \( \gamma_5 \). For details see for example [19]. In NDR the corrections to the axial current and pseudoscalar density coincide with those to the vector and scalar density respectively. By writing the gluon propagator as \( G(k) = (-\delta_{\mu\nu}k^2 + (1 - \lambda)k_\mu k_\nu)/k^4 \), we denote as „Landau“ the contribution to the one-loop corrections proportional to \( (1 - \lambda) \).

With the \( \overline{\text{MS}} \) procedure, after the subtraction of the pole, the irreducible
Table 1: The results of the calculation of the one-loop diagram of fig. 4 for different two-quark operators, in the HV scheme, are given. We also give the result of the calculation of the self-energy diagram of fig. 3. The mass of the quark has been taken to be zero, the inverse quark propagator is written as $i\not{p}\Sigma(p^2)$ and a factor $(\alpha_s/4\pi)C_F\left(p^2/\mu^2\right)^{-\epsilon}$ is omitted.

| Operator | Feynman | Landau |
|----------|---------|---------|
| $\gamma_\mu$ | $1/\hat{\epsilon} + 1/2$ | $-1/\hat{\epsilon} - 1/2$ |
| $\gamma_\mu\gamma_5$ | $1/\hat{\epsilon} + 5/2$ | $-1/\hat{\epsilon} - 1/2$ |
| $1$ | $4/\hat{\epsilon} + 6$ | $-1/\hat{\epsilon} - 2$ |
| $\gamma_5$ | $4/\hat{\epsilon} + 10$ | $-1/\hat{\epsilon} - 2$ |
| $\Sigma$ | $-1/\hat{\epsilon} - 1$ | $1/\hat{\epsilon} + 1$ |

From table 1, one finds $C_{\gamma_\mu}^{\overline{MS}} = 1/2$, $C_{11}^{\overline{MS}} = 6$, etc., in the Feynman gauge and $C_{\gamma_\mu}^{\gamma_5} = -1/2$, $C_{11}^{\gamma_5} = -2$, etc., for the Landau (longitudinal) terms.

On the lattice, one gets a similar expression

$$\Gamma_O(p/\mu) = 1 + \frac{\alpha_s}{4\pi}C_F\left(-\gamma^{(O)}\ln(p^2/\mu^2) + C_{\gamma_\mu}^{\overline{MS}}\right). \quad (38)$$

The values of $C_O^{\text{Latt}}$, for the Feynman and Landau corrections are given in table 2. In table 2, we have denoted by “Rotated Feynman” the contribution coming from the rotated part of the two-fermion operators, see eqs. (33) and (34). The Landau contribution to the rotated part of the operators vanishes. This is required by the fact that the renormalization constants which relate lattice to continuum operators are gauge invariant.

Following refs. [27], [28], we write

$$O(\mu) = \left(1 + \frac{\alpha_s}{4\pi}C_F\left(-\gamma^{(0)}\ln(\mu a)^2 + \Delta_O\right)\right)O(a), \quad (40)$$

$^9\Gamma_O(p/\mu)$ is obtained from $\Lambda O(p/\mu)$ using a projector, as described in eq. (4).
\( \Delta O \) can be computed from tables 1 and 2

\[
\Delta O = C_O^{\text{MS}} \bar{O} - C_O^{\text{Latt}} + C_O^{\text{MS}} \bar{O} - C_O^{\text{Latt}}. \tag{41}
\]

In the difference between continuum and lattice one-loop corrections, all Landau pieces cancel, as expected. The results for \( \Delta_{\gamma_\mu}, \Delta_{\gamma_\mu\gamma_5}, \cdots \) all agree with those of ref. [27].

On a lattice, for a given quark mass and external momenta, there are terms of order \( a \) which can become important at large values of \( m_q \) and \( p \). Since with our method we have to impose the renormalization conditions at values of the momenta which are large, it is important to monitor, at least in perturbation theory, the values of masses and momenta at which \( O(a) \) effects become important. The Green function in DPT has a form similar to (39)

\[
\Gamma_O(pa) = 1 + \frac{\alpha_s}{4\pi} C_F \left( -\gamma^{(O)} \ln(p^2 a^2) + C_O^{\text{Latt}}(pa, m_q a) \right), \tag{42}
\]

where, however, the “constant” \( C_O^{\text{Latt}} \) now depends on the terms of \( O(a) \). Thus, in the calculation of the one-loop corrections in DPT, all the cut-off dependent terms have been included. On a finite volume, the propagators of massless particles at zero momentum must be regulated. The results of Feynman diagrams depend on the regulator, however this dependence vanishes as the volume is increased. In the results presented below, we have set the zero-momentum terms in both the quark and gluon propagators to zero. We observe a small difference in the results from SPT and DPT, at low

| Operator | Feynman | Rotated Feynman | Landau |
|----------|---------|-----------------|--------|
| \( \gamma_\mu \) | 6.62 | -3.52 | -4.29 |
| \( \gamma_\mu\gamma_5 \) | 5.09 | -11.68 | -4.29 |
| 1 | 16.11 | -12.00 | -5.79 |
| \( \gamma_5 \) | 19.18 | 4.31 | -5.79 |
| \( \Sigma \) | 8.21 | -- | 4.79 |

Table 2: The values of \( C_O^{\text{Latt}} \) for different two-quark operators and for the self-energy (\( C_\Sigma \)), with the Clover action. The mass of the quark has been taken to be zero.
values of $\mu^2$, which is due to this choice. At the value of the quark mass at which we have performed the simulation, mass corrections are negligible.

The coupling constant appearing in eqs. (39) and (42) is the bare lattice coupling. In ref. [10] it was argued that it is possible to optimize the convergence of the perturbative series by reorganizing the expansion in terms of a “boosted” coupling $\alpha_s^V$, defined in terms of a physical quantity. In the comparison with the non-perturbative results below we follow the suggestion of ref. [10], and compute the perturbative corrections using an effective coupling

$$\alpha_s^V = \frac{1}{\langle \frac{1}{3} \text{Tr} U_{\text{plaq}} \rangle} \alpha_s^{\text{LATT}} \simeq 1.68 \alpha_s^{\text{LATT}},$$

(43)

where $\alpha_s^{\text{LATT}} = g_0^2/4\pi$ and $\langle \frac{1}{3} \text{Tr} U_{\text{plaq}} \rangle$ is the expectation value of the plaquette. We call this expansion “Boosted Discrete Perturbation Theory” (BDPT). Analogously we denote the “Boosted Standard Perturbation Theory” by BSPT.

The results of the calculation of the renormalization constants in BDPT will be compared to the non-perturbative results in the next section.

6 Non-perturbative results

In this section, we give the main results of our numerical study of the non-perturbative renormalization procedure proposed in this paper and a comparison of these results with perturbation theory.

We have performed a simulation, by generating 36 independent gluon field configurations, on a $16^3 \times 32$ lattice, at $\beta = 6.0$. The quark propagators have been computed for a single value of the quark mass ($am_q \sim 0.07$), corresponding to a value of the hopping parameter $K = 0.1425$. All the Green functions have been computed in the lattice Landau gauge, which is obtained by minimizing the functional

$$\text{tr} \left[ \sum_{\mu=1}^{4} \left( U_\mu(x) + U_\mu^\dagger(x) \right) \right].$$

(44)

Possible effects from Gribov copies or spurious solutions have not been considered. As shown below, for those quantities which can be determined
also in a gauge invariant way, the non-perturbative results obtained on quark state in a fixed gauge are in good agreement with those obtained using the Ward identity method [3].

**Vector current:** In fig. 4, the renormalization constant of the local vector current $Z_{V_L}$, obtained using eqs. (30) and (31), is given. As expected, $Z_{V_L}$ is independent of the scale, within statistical errors, up to large values of $\mu^2$, where distortions due to lattice discretization become important. This is a consequence of the equivalence, which can be established up to terms of $O(\alpha_s a)$, of the method used here to determine $Z_{V_L}$, and of the Ward identity for the local vector current, see sec. 3. By using the points at $\mu^2 a^2 \sim 1$, corresponding to $\mu \sim 2$ GeV, we get $Z_{NP}^{V_L} = 0.84(1)$, to be compared with $Z_{WI}^{V_L} = 0.824(2)$ from the Ward identity method, and $Z_{BPT}^{V_L} = 0.83 (0.86 - 0.87$ on a $16^3 \times 32$ lattice, at $\mu^2 a^2 \sim 1$) from BSPT. The three methods are in good agreement.

**Scalar density:** $Z_S$ is an ideal quantity with which to check the validity of our method for a number of reasons: it cannot be determined using the Ward identity method, it is logarithmically divergent and gauge dependent, and it is not affected by the pole of the Goldstone boson (in contrast to the pseudoscalar density and the axial current), sec. 2.

In fig. 5, $Z_S$ is shown as a function of the renormalization scale. We report separately the results, obtained by using $Z_\psi$, eq.(27), and $Z'_\psi$, eq.(30). Notice that the points, corresponding to different definitions of $Z_\psi$, are in very good agreement where $\mu^2 a^2$ is not too large, and discretization errors are small. This can also be seen by comparing the results in BSPT (dashed curve) to those in BDPT (continuous curve). We do not expect agreement with perturbation theory, either at low $\mu^2$, where higher order effects become very large, or at high $\mu^2$, where lattice distortions are important, as shown by the continuous curve. The numerical results follow the theoretical expectations and we find good agreement between the non-perturbative determination of $Z_S$ and the predictions from boosted perturbation theory for $0.3 \leq \mu^2 a^2 \leq 1$ ($1.1 \text{ GeV} \leq \mu \leq 2 \text{ GeV}$). Notice the $Z_S (Z_P)$ is defined here from the off-shell Green function and depends on the anomalous dimension of the scalar (pseudoscalar) density, and hence on $\log(\mu^2 a^2)$. Moreover, the result is gauge dependent, even in the continuum, as can be shown by an explicit calculation in one-loop perturbation theory.
Axial vector current: The local axial-vector current shares some of the features of the vector current. Its renormalization constant is finite at all orders in perturbation theory \[31\] and can be determined from Ward identities. However, the axial-vector current is coupled to the would-be Goldstone boson of QCD, which can give an important contribution at low \(\mu^2\). \(Z_{A_L}\), which like \(Z_{V_L}\), should be independent of \(\mu\), is shown as a function of \(\mu^2\) in fig. 6.

We interpret the strong \(\mu\)-dependence of \(Z_{A_L}\), at low \(\mu^2\), as the non-perturbative effect of the pseudoscalar state. Unfortunately, there is no clear sign of the existence of a plateau between the non-perturbative regime and the large \(\mu\) region, where lattice artefacts become important\[10\]. Without the information coming from the Ward identity method \[3\], it would be difficult to determine \(Z_{A_L}\) confidently. Nevertheless, we do observe that, at \(\mu^2a^2 \sim 1\), just before lattice artefacts become large in DPT, the results are close to the value determined with the Ward identity, \(Z_{A_L}^{WI} = 1.06(2)\ [3, 21]\), see fig. 6. As observed in ref. \[3\], perturbation theory gives values of \(Z_{A_L}\) smaller than one, while the Ward identity method gives values larger than one, for \(\beta = 6.0–6.2\). A value larger than one is also suggested by our non-perturbative results.

Pseudoscalar density: The pseudoscalar density shares the main features of the scalar density, with two important differences. Firstly it is coupled to the would-be Goldstone boson, and secondly the one-loop perturbative corrections, with the Clover action, are quite large. It is then not surprising that the non-perturbative value of the corresponding renormalization constant \(Z_P\) lie well below the perturbative result, as shown in fig. 6. Had we used standard perturbation theory in fig. 6, instead of boosted perturbation theory, the discrepancy would have been even worse. The discrepancy could have been anticipated from the results of ref. \[3\], where it was shown that the ratio \(Z_S/Z_P\), determined by using the Ward identity method, was significantly larger than the result obtained in boosted perturbation theory. Combining this information with the agreement between the non-perturbative determination of \(Z_S\) and the result from BPT, one would conclude that the difference is due to \(Z_P\). This may also be due to the fact that \(Z_P\) has larger one-loop finite corrections, \(\sim 35\%\), and that perhaps the higher order terms, which are not accounted for by one loop boosted

\[10\] The plateau could eventually appear more clearly at larger values of \(\beta\).
perturbation theory, are important.

We also present, in fig. 8, $Z_S/Z_P$ as a function of $\mu^2$. The ratio $Z_S/Z_P$ has a behaviour similar to $Z_A$. Some discretization effects appear to be smaller than those observed in $Z_S$ and $Z_P$ separately. The numerical results support, qualitatively, the theoretical interpretation. In the intermediate range of $\mu$, this ratio stays almost constant. At small or large values of $\mu$ the results are unstable, due to non-perturbative or discretization effects respectively. At $\mu^2 a^2 \sim 1$, the value of this ratio obtained from our non-perturbative method is in reasonable agreement with that determined with the Ward identity method, $Z_S/Z_P = 1.64(5)$ \textsuperscript{[3]}.}

\section{Conclusion}

We have proposed a new non-perturbative method to renormalize lattice composite operators. It can be used in all cases and is particularly useful when the Ward identity method is not applicable. This method avoids the need to perform any calculations using lattice perturbation theory in the computation of physical quantities from lattice simulations. The success of our proposal is subject to the existence, at current values of $\beta$, of a window between the non-perturbative region at low momenta and the region of large momenta, where discretization errors become important. This limitation is however, common to all methods (with the exception of the Ward identity method, which can only be applied to a few cases, corresponding to finite operators). The window, necessary to implement the renormalization programme described here, seems to exists already at $\beta = 6.0$ and we expect that the range of momenta, useful for non-perturbative renormalization, will become larger as $\beta$ increases. We are planning to apply the approach presented in this paper to the renormalization of the $\Delta S = 2$ four fermion operator \textsuperscript{[14]} and of the heavy-light axial vector current in the static theory.

\footnote{The value published in ref. \textsuperscript{[3]}, using half of the present statistics, was $Z_P/Z_S = 0.64(2)$, corresponding to $Z_S/Z_P = 1.56(5)$.}
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Appendix

In this appendix we demonstrate that the non-perturbative contributions to forward two-quark matrix elements are suppressed as inverse powers of \( p^2 \), where \( p \) is the external momentum. Let us consider the non-amputated, two-fermion Green function

\[
F_O(p) = \int d^4x d^4y e^{-ip \cdot x} \langle \bar{q}(0) \Gamma q(x) O \Gamma(y) \rangle,
\]

where \( O \Gamma(y) = \bar{q}(y) \Gamma q(y) \) and \( \Gamma \) is one of the Dirac matrices. It is natural to worry about the infrared behaviour of \( F_O(p) \), because it contains the insertion of a zero-momentum operator. On general grounds, we expect infrared divergences to appear in Green functions computed at exceptional momenta. In physical terms, this is due to the fact that the matrix elements of \( O \Gamma \), at zero momentum, carry information about the physical mass spectrum, which is unaccessible to perturbation theory. In particular, if the operator \( O \Gamma \) is the pseudoscalar density (\( \gamma_5 \)) and we work in the chiral limit, a contribution from a Goldstone pole, corresponding to the pion, should appear in \( F_O(p) \).

The presence of non-perturbative contributions to \( F_O(p) \) is signaled by the appearance of infrared divergences in the perturbative expansion. This is what happens in general. For quark degrees of freedom in the limit of large \( p^2 \) however, the Operator Product Expansion (OPE) guarantees that \( F_O(p) \) can be reliably computed at all orders in perturbation theory. The argument goes as follows. In the limit of large \( p^2 \), the dominant contribution to \( F_O(p) \) comes from regions of integration (in \( x \) and \( y \)) where the integrand is singular. This implies that, in this limit, we have to study the behaviour of

\[
F_O(x, y) = \langle \bar{q}(0) \Gamma q(x) O \Gamma(y) \rangle,
\]

when \( x \sim 0 \) and \( x \sim y \). In both cases of course, we can use the OPE in the following form\(^\text{12}\)

\[
q_\alpha(x) O \Gamma(y) \rightarrow A_{\alpha \beta}^\Gamma(x - y) \Gamma_{\beta \delta} q_\delta(y) + \ldots,
\]

\(^{12}\)From a technical point of view, OPE is a weak operator statement and, when \( O \Gamma \) is inserted in a Green function containing other operators, a slightly modified form would be appropriate. It is easy to convince oneself, however, that the additional terms, required in the general case, are not relevant to the present discussion.
for $x \sim y$, and

$$\bar{q}(x)\Gamma q(0) \to B_\Gamma(x^2)\bar{q}(0)\Gamma q(0) + \ldots,$$

for $x \sim 0$.

Since QCD is asymptotically free, the behaviour of $A_{\alpha\beta}(x-y)$ and $B_\Gamma(x)$ can be computed in perturbation theory. The interaction gives rise to logarithmic corrections to the free field behaviour

$$A_\Gamma(x) = \not{x}C_\Gamma(x^2) \equiv \not{x}a \Gamma \ln^\alpha (\mu^2 x^2) + \ldots$$

$$B_\Gamma(x^2) = b \Gamma \ln^\beta (\mu^2 x^2) + \ldots$$

Using the above equations, in the limit $p^2 \to \infty$, we find

$$\int d^4x d^4y e^{-ip \cdot x} \left( \langle \bar{q}(0) \Gamma A_\Gamma(x - y) \Gamma q(y) \rangle + B_\Gamma(x^2) \langle \bar{q}(0) \Gamma q(0) \rangle \right) =$$

$$- i p_\mu \bar{C}_\Gamma(p^2) \bar{S}_\Gamma \mu(p) + B_\Gamma(p^2) \bar{\Delta}_\Gamma(0),$$

where

$$\bar{S}_\Gamma \mu(p) = \int d^4y e^{-ip \cdot y} \langle \bar{q}(0) \Gamma \gamma^\mu \Gamma q(y) \rangle \quad \bar{\Delta}_\Gamma(0) = \int d^4y \langle \bar{q}(0) \Gamma q(0) \rangle \Gamma q(0) \rangle.$$
This is the reason why, even at an exceptional external momentum, \( F_\phi(p) \) is infrared safe in perturbation theory, when \( p^2 \) is large. It is straightforward to trace the suppression of the infrared divergence, by analyzing the corresponding Feynman diagrams. For example, the infrared divergent part of the diagram in fig. 9-a is suppressed by one power of \( p^2 \), due to the momentum flow through the lower gluon line.

It is interesting to see what would happen in a (fictitious) theory, with quarks represented by scalar fields. In this case the infrared divergence is not suppressed, as can be seen from the diagram in fig. 9-b. The general argument goes as follows

\[
F(p^2) = \int d^4xd^4ye^{-ip\cdot x} \langle \bar{\phi}(0)\phi(x)J(y) \rangle,
\]

with

\[
J(y) = \bar{\phi}(y)\phi(y).
\]

When \( x \sim 0 \) and \( x \sim y \), the OPE gives

\[
\phi(x)J(y) \rightarrow A((x-y)^2)\phi(y) + \ldots \quad \bar{\phi}(x)\phi(0) \rightarrow B(x^2)\bar{\phi}(0)\phi(0) + \ldots,
\]

with

\[
A(x^2) = \frac{a \ln^\alpha(\mu^2x^2) + \ldots}{x^2} \quad B(x^2) = b \ln^\beta(\mu^2x^2) + \ldots,
\]

and we would get

\[
F(p^2) \rightarrow \bar{A}(p^2)\bar{S}(p^2) + \bar{B}(p^2)\bar{\Delta}(0)
\]

with

\[
\bar{S}(p^2) = \int d^4ye^{-ip\cdot y} \langle \bar{\phi}(0)\phi(y) \rangle \quad \bar{\Delta}(0) = \int d^4y(\bar{\phi}(0)\phi(0)J(y)).
\]

One now obtains

\[
\bar{A}(p^2)\bar{S}(p^2) \rightarrow c \frac{\ln^\gamma(p^2/\mu^2)}{p^4}
\]

and

\[
\bar{B}(p^2) \rightarrow d \frac{\ln^\delta(p^2/\mu^2)}{p^4}.
\]

The latter equations imply that non-perturbative and perturbative contributions have the same \( p^2 \) dependence. Thus, we expect the appearance of infrared divergences in \( F(p^2) \).
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Figure 4: $Z_{V_L}$ as a function of $\mu^2a^2$. Since the lattice is not symmetric in space and time, the results obtained by using the time-component of the local current $(V_0)$ and the space components, averaged in the three directions, $(V_k)$ are shown separately. The dashed line is $Z_{V_L}^{BSPT}$ from BSPT, and the curve is from BDPT, on a volume of size $16^3 \times 32$, see sec. 5, see sec. 5. The straight (continuous) line is the result obtained using the Ward identities method [3, 21].
Figure 5: $Z_S$ as a function of $\mu^2 a^2$. We give the renormalization constant obtained by using the two possible definitions of the quark wave function renormalization, denoted by $Z_\psi$, eq. (27), and $Z'_\psi$, eq. (30). The dashed curve is $Z_S^{BSPT}$, from boosted standard perturbation theory in the infinite volume limit, and the full curve is the result obtained in BDPT.
Figure 6: $Z_{A^L}$ as a function of $\mu^2 a^2$. We give the renormalization constant, obtained by using the time component $A_0$ and the space components $A_k$, separately. The dashed line is $Z_{A}^{BSPT}$, from boosted standard perturbation theory, and the full curve is from BDPT. The straight (continous) line is the result obtained using the Ward identities method [3, 24].
Figure 7: $Z_P$ as a function of $\mu^2a^2$. We give the renormalization constant obtained by using the two possible definitions of the quark wave function renormalization, denoted by $Z_\psi$, eq. (27), and $Z'_\psi$, eq. (30). The dashed curve is $Z_P^{\text{BSPT}}$, from boosted standard perturbation theory in the infinite volume limit, and the full curve is the result obtained in BDPT.
Figure 8: $Z_S/Z_P$ as a function of $\mu^2 a^2$. The dashed line corresponds to BSPT, the full line comes from the determination of this ratio, using the Ward identity method [3].

Figure 9: Diagrams with standard, a), and scalar quarks, b).
