Asymptotically quasiperiodic solutions for time-dependent Hamiltonians

Donato Scarcella

Departament de Matemàtiques, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain

E-mail: donato.scarcella@upc.edu

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Abstract

Dynamical systems subject to perturbations that decay over time are relevant in describing many physical models, e.g. when considering the effect of a laser pulse on a molecule, in epidemiological studies, and celestial mechanics. For this purpose, we consider time-dependent Hamiltonian vector fields that are the sum of two components. The first has an invariant torus supporting quasiperiodic solutions, and the second decays as time tends to infinity. The time decay is modelled by functions satisfying suitable conditions verified by a proper polynomial decay in time. We prove the existence of orbits converging as time tends to infinity to the quasiperiodic solutions associated with the unperturbed system. The proof of this result relies on a new strategy based on a refined analysis of the Banach spaces and the functionals involved in the resolution of suitable nonlinear invariant equations. This result is proved for finite differentiable and real-analytic Hamiltonians. Analogous statements for time-dependent vector fields on the torus are also obtained as a corollary. These results extend a previous work of Canadell and de la Llave, where only exponential decay in time is considered. The relaxation of the decay in time makes the results in the present paper suited for applications in many physical problems, such as celestial dynamics.

Keywords: dynamical systems, Hamiltonian systems, KAM tori, time-dependence

Mathematics Subject Classification numbers: 37J25, 37J40, 70H08
1. Introduction

Many physical phenomena can be described by dynamical systems subjected to time-dependent perturbations decaying over time. We refer to [1, 5] for the example of a molecule interacting with another molecule or with a laser pulse, and to [9] for the development of some epidemiological models. The paper [7] considers the planar three-body problem perturbed by a given comet, coming from and going back to infinity asymptotically along a hyperbolic Keplerian orbit, modelled as a time-dependent perturbation.

For this reason, in the present paper, we are interested in studying the asymptotic dynamics of time-dependent perturbations of Hamiltonian systems having an invariant torus supporting quasiperiodic solutions. Before the description of the main results of this work, let us introduce the definition of $C^\sigma$-asymptotic KAM torus. Let $B \subset \mathbb{R}^n$ be an open ball centred at the origin, and let $\mathcal{P}$ be equal either to $\mathbb{T}^n$ or to $\mathbb{T}^n \times B$ (this is because we prove results about time-dependent Hamiltonian vector fields or time-dependent vector fields on the torus). Given $v \geq 0$, we set $J_v = [v, +\infty) \subset \mathbb{R}$ and we denote the time dependence with an apex $t$. In the sequel, $C^\sigma$ indicates the class of Hölder functions and $| \cdot |_{C^\sigma}$ the associated Hölder norm (we refer to appendix A for a brief introduction). Now, given a pair of real numbers $\sigma \geq 0$, $v \geq 0$, a positive integer $k \geq 0$ and a vector $\omega \in \mathbb{R}^n$, we consider two time-dependent vector fields $X', X_0 \in C^{\sigma+k}(\mathcal{P})$ for all fixed $t \in J_v$, and an embedding $\varphi_0 : \mathbb{T}^n \rightarrow \mathcal{P}$ of class $C^\sigma$ such that

$$\lim_{t \rightarrow +\infty} |X' - X_0|_{C^{\sigma+k}} = 0,$$

(1.1)

$$X_0(\varphi_0(q), t) = \partial_q \varphi_0(q) \omega \quad \text{for all} \quad (q, t) \in \mathbb{T}^n \times J_v.$$  

(1.2)

In words, we are considering a time-dependent vector field $X'$ converging as time tends to infinity to a vector field $X_0'$ having an invariant torus $\varphi_0$ supporting quasiperiodic solutions of frequency vector $\omega$. The most natural situation is when $X_0'$ is autonomous, but the results we will prove consider the more general case when $X_0'$ depends on time. With the setting above, we can give the following.

**Definition 1.1 ($C^\sigma$-asymptotic KAM torus).** We assume that $(X', X_0, \varphi_0)$ satisfy (1.1) and (1.2). A family of $C^\sigma$ embeddings $\varphi : \mathbb{T}^n \rightarrow \mathcal{P}$ is a $C^\sigma$-asymptotic KAM torus associated to $(X', X_0, \varphi_0)$ if there exists $v' \geq v \geq 0$ such that

$$\lim_{t \rightarrow +\infty} |\varphi' - \varphi_0|_{C^\sigma} = 0,$$

(1.3)

$$X(\varphi(q, t), t) = \partial_q \varphi(q, t) \omega + \partial_t \varphi(q, t),$$

(1.4)

for all $(q, t) \in \mathbb{T}^n \times J_v$. When $\mathcal{P}$ is a symplectic manifold with dim$\mathcal{P} = 2n$, then we say that $\varphi'$ is Lagrangian if $\varphi'(\mathbb{T}^n)$ is Lagrangian for all $t \in J_v$.

Hence, a $C^\sigma$-asymptotic KAM torus is a family of embedded tori converging in time to the invariant torus $\varphi_0$. Moreover, the dynamics on this family of embeddings converge, as time tends to infinity, to the quasiperiodic solutions associated with $X_0'$ on the invariant torus $\varphi_0$. We refer to section 3 for a more detailed analysis. The previous definition is due to Canadell and de la Llave (see [2]).

In this work, we prove the existence of a $C^\alpha$-asymptotic KAM torus for finite differentiable time-dependent Hamiltonian vector fields and time-dependent vector fields on the torus (see theorem A and corollary A). In the Hamiltonian case, we consider finite differentiable time-dependent Hamiltonian vector fields on the form $X' = X_0' + F'$. We assume that $X_0'$ has a Lagrangian invariant torus $\varphi_0$ supporting quasiperiodic solutions of frequency vector $\omega$ and
Theorem A (which we do not assume to be a perturbation) decays as time tends to infinity. In the case of time-dependent vector fields on the torus $X = \omega + F^t$ where $\omega$ is constant, and $F^t$ decays in time. The perturbative terms are bounded by positive, decreasing, integrable functions over $J_0$. These functions satisfy suitable conditions verified by a proper polynomial decay in time (see (2.5) and (2.6) for the Hamiltonian case and (2.8) for vector fields on the torus). The analytical setting is also considered and analogous results for Hamiltonian vector fields and vector fields on the torus are proved. We refer to theorem B and corollary C for the results and to definition 2.4 for the definition of an asymptotic KAM torus which is the equivalent of definition 1.1 in the analytic case.

The proof for finite differentiable Hamiltonians (theorem A) is based on an innovative strategy divided into two parts (see section 5). In the first part, we introduce a suitable functional $F$ related to equation (1.4) on specific Banach spaces (see (5.12)), and in section 5.3, we verify that it satisfies the hypothesis of the implicit function theorem. The solution of the associated linearised problem relies on the analysis of the homological equation (see (5.20) below) studied in section 5.2. Equation (5.20) is solved by integration thanks to a suitable change of coordinates that rectifies the dynamic on the torus without assuming any arithmetic condition on the frequency vector. In the second part of the proof, we use the functional $F$ to define a suitable quasi-Newton operator $\mathcal{L}$ (see (5.18)), which appears naturally in the proof of the implicit function theorem. In section 5.4, we fix the parameter $\nu'$ to be sufficiently large to ensure that the perturbative terms, which decay in time, become suitably small, thereby obtaining that $\mathcal{L}$ is a contraction and applying the fixed point theorem. This proves the existence of a $C^n$-asymptotic KAM torus defined for all $t \geq \nu'$. In section 5.1, a more detailed overview of the proof is provided. Similarly, we prove corollary A (the above-mentioned result about finitely differentiable time-dependent vector field on the torus) and theorem B (the real-analytic version of theorem A). We refer to sections 6 and 7, respectively. The proof of corollary B (the real-analytic version of corollary A) is obtained as an application of theorem B (see section 2 for more detail).

Before giving the statement and the proof of the main results contained in the present paper, we want to provide an overview of the previous works on this subject to help the reader better situate this work in the literature. In fact, this paper is part of a wide series of works [2, 3, 6–8] aimed at developing a non-autonomous KAM theory. The main characteristic of the latter with respect to the classical KAM theory relies on the absence of non-degeneracy conditions on the unperturbed part and on the lack of any arithmetic assumptions on the frequency vector (no Diophantine condition is needed). This last aspect is due to the absence of small denominators when considering the homological equations appearing in time-dependence problems. The lack of small denominators allows us to prove the main results of this paper without requiring any super-linear convergent method or Nash–Moser implicit function theorem.

The paper of Canadell and de la Llave [2] considers time-dependent vector fields of the form $X = X_0 + F^t$, where $F^t$ is a time-dependent perturbation, and $X_0$ is an autonomous vector field having an invariant torus $\varphi_0$ supporting quasiperiodic solutions of frequency vector $\omega$. They assume that the perturbation $F^t$ decays exponentially fast as time tends to infinity and a certain control over the normal dynamics to the invariant torus $\varphi_0$. Then, they prove the existence of a $C^n$-asymptotic KAM torus associated to $(X, X_0, \varphi_0)$. This work generalises a previous paper of Fortunati and Wiggins [3] where the Hamiltonian case is considered, arithmetic conditions on the frequency vector $\omega$, and non-degenerate assumptions on $X_0$ are assumed. Firstly, Canadell and de la Llave prove their result for mappings as an application of the implicit function theorem. Secondly, they use this result to prove the case of vector fields, as usual in the invariant manifolds theory.
In this paper, we generalise the work of Canadell and de la Llave \cite{Canadell2002} in the particular case of time-dependent Hamiltonian vector fields or time-dependent vector fields on the torus. The decay in time is considerably improved, making these results interesting in the description of many important physical problems, such as celestial dynamics. Furthermore, no smallness assumption on the perturbative terms is required. Canadell and de la Llave need exponential decay in time in order to control possible hyperbolic dynamics normal to the invariant torus $\varphi_0$ associated with the unperturbed system. In the case of time-dependent Hamiltonian vector fields discussed in the present paper, the motions normal to $\varphi_0$ are not hyperbolic. For this reason, we are able to relax the decay in time. This improvement is based on a different abstract formulation of the dynamical problem and a very refined analysis of the Banach spaces and functionals involved in the solution of nonlinear invariant equations.

The analysis carried out in the present work is motivated by the example of the planar three-body problem perturbed by a given comet coming from and going back to infinity asymptotically along a hyperbolic Keplerian orbit, which is the content of another work \cite{Scardelletti2002}. In this application, the interaction between the comet and the planets is modelled by a time-dependent perturbation decaying polynomially fast as time tends to infinity. We also apply the results in the present work in order to prove the existence of orbits converging to quasiperiodic solutions in the future and in the past for time-dependent perturbations of integrable Hamiltonians or nearly integrable Hamiltonians \cite{Llibre2002}. Finally, as a complement to the present paper, in \cite{Llibre2003}, the case of arbitrary dynamics at infinity is considered. More specifically, the dynamics associated with the unperturbed system are not necessarily quasiperiodic, but exponential decay in time for the time-dependent perturbation is required.

2. Results

Let us be more precise and introduce the main results of the present paper. For this reason, this section is divided into two subsections because we prove the same results in the finitely differentiable and analytical case. Before the statements, we need to define some notations.

We recall that $B \subset \mathbb{R}^n$ is an open ball around the origin and, for a given parameter $\nu \geq 0$, we have the following interval $J_0 = [\nu, +\infty) \subset \mathbb{R}$. Given $f : \mathbb{T}^n \times B \times J_0 \rightarrow \mathbb{R}$, for fixed $t \in J_0$ and $p \in B$, respectively, we define the following functions

\[
\begin{align*}
    f' : \mathbb{T}^n \times B \rightarrow \mathbb{R}, & \quad f'(q, p) = f(q, p, t) \\
    f_p : \mathbb{T}^n \times J_0 \rightarrow \mathbb{R}, & \quad f_p(q, t) = f(q, p, t).
\end{align*}
\]

We will refer to this notation also for vector-valued functions or matrices. We will use it for the rest of this work.

As mentioned before, we are interested in time-dependent perturbations of Hamiltonians having an invariant torus supporting quasiperiodic solutions. To describe the unperturbed Hamiltonians, we introduce the following set.

**Definition 2.1.** Given $\omega \in \mathbb{R}^n$, let $\mathcal{K}_{\omega}$ be the set of the Hamiltonians $h : \mathbb{T}^n \times B \times J_0 \rightarrow \mathbb{R}$ such that, for some $c \in \mathbb{R}$

\[ h(q, 0, t) = c, \quad \partial_p h(q, 0, t) = \omega \]

for all $(q, t) \in \mathbb{T}^n \times J_0$.
In what follows, we will always consider \( c = 0 \). Now, let \( X_H \) be the Hamiltonian system associated with a given Hamiltonian \( H \). For all \( h \in \mathcal{K}_\omega \), it is straightforward to verify that the following trivial embedding

\[
\varphi_0 : \mathbb{T}^n \to \mathbb{T}^n \times B, \quad \varphi_0(q) = (q, 0)
\]

is an invariant torus for \( X_h \) supporting quasiperiodic solutions of frequency vector \( \omega \).

### 2.1. Finitely differentiable case

This section is dedicated to the statement of the main results of the present paper in the Hölder regularity. In order to quantify the regularity of smooth functions, we introduce the following space. We consider the positive parameters \( \sigma \geq 0 \) and \( \upsilon \geq 0 \), an integer \( k \geq 0 \) and \( i \in \mathbb{N}^{2n} \).

**Definition 2.2.** Let \( S_{(\upsilon, k)}^i \) be the space of functions \( f : \mathbb{T}^n \times B \to \mathbb{R} \) such that \( f \in C^\sigma (\mathbb{T}^n \times B) \) for all \( t \in J \) and \( \partial_{(q,p)}^i f \in C(\mathbb{T}^n \times B \times J_0) \) for all \( 0 \leq |i| \leq k \).

In the previous definition \( |i| = i_1 + \ldots + i_{2n} \) and \( \partial_{(q,p)}^i = \partial_{q_1}^{i_1} \cdots \partial_{q_n}^{i_n} \partial_{p_1}^{i_{n+1}} \cdots \partial_{p_{2n}}^{i_{2n}} \) stands for the partial derivatives of order \( |i| \) with respect to the variables \((q, p)\), conventionally \( \partial_{(q,p)}^0 f = f \). We will refer to the latter space also for maps defined on \( \mathbb{T}^n \times J_0 \), vector-valued functions or matrices. This will be specified in the context.

On the other hand, to measure the decay in time of the perturbations, we introduce positive, decreasing, integrable functions \( u : J_0 \to \mathbb{R}^+ \) and we denote

\[
\hat{u}(t) = \int_T^t u(\tau) \, d\tau
\]

for all \( t \in J_0 \).

Now, we have everything we need to state the following theorem. Given \( \omega \in \mathbb{R}^n \) and \( \sigma \geq 1 \), we consider a time-dependent Hamiltonian \( H \) of the form

\[
\begin{aligned}
H : \mathbb{T}^n \times B \times J_0 &\to \mathbb{R} \\
H(q, p, t) &= h(q, p, t) + f(q, p, t), \\
h &\in \mathcal{K}_\omega, \quad f_0, (\partial_p f_0), \partial_p^2 H \in S_{(\sigma, 2)}^0, \\
\sup_{t \in J_0} |f_0|_{C^{\sigma + 2}} < \infty, &\quad \sup_{t \in J_0} |\partial_p^2 H|_{C^{\sigma + 2}} < \infty, \\
|\partial_p^i f_0|_{C^{\sigma + 1}} \leq a(i), &\quad |(\partial_p^i f_0)|_{C^{\sigma + 2}} \leq b(i) \quad \text{for all } t \in J_0,
\end{aligned}
\]

where \( a : J_0 \to \mathbb{R}^+ \) and \( b : J_0 \to \mathbb{R}^+ \) are positive, decreasing, integrable functions on \( J_0 \). We point out that in the latter we used the notation introduced by (2.1) and (2.2). We also assume that there exists \( \upsilon \geq 0 \) such that \( a \) and \( b \) satisfy the following conditions

\[
\begin{aligned}
\bar{a}(t) &\leq \Lambda a(t) \\
\bar{b}(t) &\leq \Lambda b(t)
\end{aligned}
\]

for all \( t \in J_0 \) and a suitable constant \( \Lambda \). Finally, let \( \tilde{h} : \mathbb{T}^n \times B \times J_0 \to \mathbb{R} \) be the following Hamiltonian

\[
\tilde{h}(q, p, t) = h(q, p, t) + \int_0^1 (1 - \tau) \partial_p^2 f(q, \tau p, t) \, d\tau \cdot \tau^2.
\]

It is straightforward to verify that \( \tilde{h} \in \mathcal{K}_\omega \). Moreover, \( X_{\tilde{h}} \) and \( X_h \) verify (1.1).
Theorem A. Let $H$ be as in (2.5) with $a$ and $b$ satisfying (2.6). Then, there exists a Lagrangian $C^\sigma$-asymptotic KAM torus $\phi'$ associated to $(X_H, X_\phi, \varphi_0)$, where $\varphi_0$ is the trivial embedding defined by (2.3).

We begin with two examples of functions $a$ and $b$ satisfying (2.6). First, we consider the case of exponential decay. Let $a$ and $b$ be the following functions

$$a(t) = e^{-\lambda_1 t}, \quad b(t) = e^{-\lambda_2 t},$$

for some positive parameters $\lambda_1 \geq \lambda_2 > 0$. It is straightforward to see that (2.6) is verified for all $t \in J_0$ and $\Lambda \geq \max\{\frac{\lambda_1}{\lambda_2}, \frac{1}{\lambda_1}\}$. The following example is more interesting than the previous one. It is about polynomial decay. We consider

$$a(t) = \frac{1}{t^{\tau}}, \quad b(t) = \frac{1}{t},$$

for a positive real parameter $\tau > 1$. This couple of functions satisfy (2.6) for all $t \in J_1$ with $\Lambda = 1$. Finally, we point out that hypothesis (2.6) consists of some technical conditions we need to prove the above theorem. Both are used to verify that the functional $F$ (see (5.12) below) is well-defined and in the proof of lemma 5.3. In addition, the first condition of (2.6) is also used for proving lemma 5.2.

The previous theorem shows the existence of a $C^\sigma$-asymptotic KAM torus $\phi'$ of the form

$$\phi': \mathbb{T}^n \to \mathbb{T}^n \times B, \quad \phi'(q) = (q + u'(q), v'(q))$$

for all $t$ sufficiently large, where $u': \mathbb{T}^n \to \mathbb{R}^n$, $v': \mathbb{T}^n \to \mathbb{R}^n$, and $\text{id} + u'$ is a diffeomorphism of the torus for all fixed $t$. Furthermore, we also obtain some information about the decay in time of $u$ and $v$. More specifically

$$|u'|_{C^\sigma} \leq Cb(t), \quad |v'|_{C^\sigma} \leq Ca(t),$$

for all $t$ large enough and for a suitable constant $C$.

Concerning time-dependent vector fields on the torus, given $\sigma \geq 1$ and $\omega \in \mathbb{R}^n$, we consider the following time-dependent vector field

$$\begin{aligned}
Z: \mathbb{T}^n \times J_0 &\longrightarrow \mathbb{R}^n \\
Z(q,t) &= \omega + P(q,t) \\
P \in C^{(\sigma,1)}_r, &\quad |P'|_{C^{\sigma+1}} \leq P(t) \quad \text{for all } t \in J_0,
\end{aligned}
$$

(2.8)

where $P: J_0 \to \mathbb{R}^+$ is a positive, decreasing, integrable function on $J_0$.

Corollary A. Let $Z$ be as in (2.8). Then, there exists a $C^s$-asymptotic KAM torus $\psi'$ associated to $(Z, \omega, \text{Id})$.

We point out that if $P$ is not integrable on $J_0$, then, in general, there does not exist a $C^s$-asymptotic KAM torus associated to $(Z, \omega, \text{Id})$. We refer to appendix C for an example.

2.2. Real analytic case

As mentioned above, this section contains the real analytic version of the previous results. To this end, for some $s > 0$, we define complex domains

$$T^s_x = \{ q \in C^s / \mathbb{Z}^n : |\text{Im}(q)| \leq s \}, \quad B_x = \{ p \in C^s : |p| \leq s \},$$

and, given $v \geq 0$, we introduce the following space of functions.
Definition 2.3. Let $A^0_v$ be the space of the functions $f: T^n_v \times B_v \times J_v \to \mathbb{R}$ such that $f \in C(T^n_v \times B_v \times J_v)$ and, for all $t \in J_v, f'$ is real analytic on $T^n_v \times B_v$.

We will refer to the same notation for maps defined on $T^n_v \times J_v$, vector-valued functions, and matrices. In addition, we need to introduce the definition of analytic asymptotic KAM torus, which is the real-analytic version of definition 2.1. For this purpose, we recall that $B \subset \mathbb{R}^n$ is a ball centred at the origin, and $P$ is equal either to $T^n \times B$ or to $T^n$. In what follows, $| \cdot |_0$ indicates the analytic norm (see appendix A). Given $\omega \in \mathbb{R}^n$, we consider time-dependent real analytic vector fields $X'$ and $X_0$ on $P$ for all fixed $t \in J_v$, and a real analytic embedding $\varphi_0 : T^n \to P$ such that

$$\lim_{t \to +\infty} |X' - X_0|_t = 0,$$

$$X_0(\varphi_0(q), t) = \partial_q \varphi_0(q) \omega \text{ for all } (q, t) \in T^n \times J_v. \quad (2.9)$$

Definition 2.4 (analytic asymptotic KAM torus). We assume that $(X, X_0, \varphi_0)$ satisfy (2.9) and (2.10). A family of real analytic embeddings $\varphi : T^n \to P$ is an analytic asymptotic KAM torus associated to $(X, X_0, \varphi_0)$ if there exist $0 < s' \leq s$ and $\nu' \geq \nu \geq 0$ such that

$$\lim_{t \to +\infty} |\varphi' - \varphi_0|_t = 0,$$

$$X(\varphi(q, t), t) = \partial_q \varphi(q, t) \omega + \partial_t \varphi(q, t),$$

for all $(q, t) \in T^n \times J_v$. When $P$ is a symplectic manifold with $\dim P = 2n$, then we say that $\varphi'$ is Lagrangian if $\varphi'(T^n)$ is a Lagrangian for all $t \in J_v'.

Finally, given $\omega \in \mathbb{R}^n$ and a positive real parameter $0 < s < 1$, we consider the following time-dependent Hamiltonian $H$

$$H : T^n \times B \times J_0 \to \mathbb{R},$$

$$H(q, p, t) = h(q, p, t) + f(q, p, t),$$

$$h \in K_\omega, \quad h, f \in A^0_v$$

$$\sup_{t \in J_0} |f'_0|_s < \infty, \quad \sup_{t \in J_0} |\partial^2_H f|_s < \infty$$

$$\sup_{t \in J_0} |(\partial_q f)'_0|_s \leq a(t), \quad (\partial_q f)'_0 | \leq b(t), \quad \text{for all } t \in J_0, \quad (2.11)$$

where $a : J_0 \to \mathbb{R}^+ \text{ and } b : J_0 \to \mathbb{R}^+$ are positive, decreasing, integrable functions on $J_0$.

**Theorem B.** Let $H$ be as in (2.11) with $a$ and $b$ satisfying (2.6). Then, there exists a Lagrangian analytic asymptotic KAM torus $\varphi'$ associated to $(X_H, X_0, \varphi_0)$, where $h$ and $\varphi_0$ are defined by (2.7) and (2.3), respectively.

Similarly to theorem A, we prove the existence of an analytic asymptotic KAM torus $\varphi'$ of the form $\varphi' : T^n \to T^n \times B$ such that $\varphi'(q) = (q + u'(q, t), v'(q))$ for all $t$ sufficiently large, where $u' : T^n \to \mathbb{R}^n$, $v' : T^n \to \mathbb{R}^n$, and $id + u'$ is a diffeomorphism of the torus for all fixed $t$. Furthermore, for a suitable constant $C$

$$|u'|_2^2 \leq Cb'(t), \quad |v'|_2^2 \leq Ca'(t),$$

for all $t$ large enough.
Also in this case, we prove an analogous result regarding real analytic time-dependent vector fields on the torus. Let $Z$ be a non-autonomous vector field on $\mathbb{T}^n \times J_0$ of the form

$$
\begin{cases}
Z : \mathbb{T}^n \times J_0 \to \mathbb{R}^n, \\
Z(q,t) = \omega + P(q,t) \\
P \in \mathcal{A}^s_0, \quad |P'| \leq P(t) \quad \text{for all } t \in J_0
\end{cases}
$$

(2.12)

where $\omega \in \mathbb{R}^n$ and $0 < s < 1$. We assume that $P : J_0 \to \mathbb{R}^+$ is a positive, decreasing, integrable function on $J_0$.

**Corollary B.** Let $Z$ be as in (2.12). Then, there exists an analytic asymptotic KAM torus $\psi^t$ associated to $(Z, \omega, \text{Id})$.

**Proof.** The proof is a straightforward application of theorem B. We consider the Hamiltonian $H : \mathbb{T}^n \times B \times J_0 \to \mathbb{R}$ of the form

$$
H(q,p,t) = \omega \cdot p + P(q,t) \cdot p.
$$

The latter satisfies the hypotheses of theorem B. Then, there exist an analytic asymptotic KAM torus $\varphi^t$ associated to $(X_{H}, X_{\psi}, \varphi_0)$, where $\varphi_0$ is the trivial embedding defined by (2.3) and $\hat{h}(q,p,t) = \omega \cdot p$. Moreover, $\varphi^t = (\text{id} + u', v')$ and, for all fixed $t$, $\text{id} + u'$ is a diffeomorphism of the torus. This concludes the proof of this corollary with $\psi^t = \text{id} + u'$.

We point out that we cannot use the previous argument to prove corollary A without asking $P \in \mathcal{S}_0^{(e,2)}$, hence more regularity for $P$. However, we aim to avoid this stronger assumption. Therefore, the proof of corollary A is provided in section 6.

### 3. Asymptotic KAM tori

This section investigates a set of properties of $C^\infty$-asymptotic KAM tori (see definition 1.1). The same properties are also verified in the case of analytic asymptotic KAM tori. We recall that $B \subset \mathbb{R}^n$ is an open ball centred at the origin, $\mathcal{P}$ is equal to $\mathbb{T}^n$ or $\mathbb{T}^n \times B$ and, for all $\nu \geq 0$, $J_{\nu} = [\nu, +\infty) \subset \mathbb{R}$.

We consider $X, X_0$ and $\varphi_0$ as in definition 1.1. Let $\varphi^t$ be the $C^\infty$-asymptotic KAM torus associated to $(X, X_0, \varphi_0)$. Then, it is possible to rewrite (1.4) in terms of the flow of $X$. For this purpose, let $\psi^t_{\nu, X}$ be the flows at time $t$ with initial time $t_0$ of $X$.

**Proposition 3.1.** If the flow $\psi^t_{\nu, X}$ is defined for all $t$, $t_0 \in J_{\nu}$, then (1.4) is equivalent to

$$
\psi^t_{\nu, X} \circ \varphi_0^\nu(q) = \varphi^t(q + \omega(t - t_0)),
$$

(3.1)

for all $t, t_0 \in J_{\nu}$, and $q \in \mathbb{T}^n$. We point out that $\nu'$ is the positive parameter of definition 1.1.

**Proof.** It suffices to verify that both sides of (3.1) verify the same initial value problem.

Using the previous proposition, one can see that (1.4) is trivial

**Proposition 3.2.** If $\psi^t_{\nu, X}$ is defined for all $t$, $t_0 \in J_{\nu}$, it is always possible to find a family of embeddings satisfying (1.4).

**Proof.** Let $\varphi : \mathbb{T}^n \to \mathcal{P}$ be an embedding then, for all $t, t_0 \in J_{\nu}$, and $q \in \mathbb{T}^n$, we consider

$$
\varphi^t(q) = \psi^t_{\nu, X} \circ \varphi(q - \omega(t - t_0)).
$$
The latter is a family of embeddings satisfying (3.1). Indeed, by the above definition of \( \varphi^t \) we have that \( \varphi^h(q) = \tilde{\varphi}(q) \) for all \( q \in \mathbb{T}^n \). Then, by construction, \( \varphi^t \) satisfies (3.1) and thus (1.4).

Another important consequence of proposition 3.1 is the following property. We will see that if a \( C^\sigma \)-asymptotic KAM torus is defined for all \( t \) large, then we can extend the set of definition for all \( t \in \mathbb{R} \).

**Proposition 3.3.** We assume that \( \psi^t_{h,X} \) is defined for all \( t, t_0 \in \mathbb{R} \). If there exists a \( C^\sigma \)-asymptotic KAM torus \( \varphi^t \) defined for all \( t \geq v' \), then we can extend the set of definition for all \( t \in \mathbb{R} \).

**Proof.** For all \( q \in \mathbb{T}^n \), we consider

\[
\psi^t(q) = \begin{cases} 
\varphi^t(q) & \text{for all } t \geq v' \\
\psi^t_{h,X} \circ \varphi^{v'}(q - \omega(t - v')) & \text{for all } t \leq v'.
\end{cases}
\]

This is a family of embeddings that verify (1.3) and (1.4).

Unfortunately, we cannot deduce any asymptotic information for the family of embeddings (3.2) when \( t \to -\infty \).

Concerning the dynamics associated with a \( C^\sigma \)-asymptotic KAM torus, we introduce the definition of asymptotically quasiperiodic solutions in the sequel and discuss some properties of these orbits.

**Definition 3.1 (asymptotically quasiperiodic solutions).** We assume that \((X, X_0, \varphi_0)\) satisfy (1.3) and (1.4). An integral curve \( g(t) \) of \( X \) is an asymptotically quasiperiodic solution associated to \((X, X_0, \varphi_0)\) if there exists \( q \in \mathbb{T}^n \) in such a way that

\[
\lim_{t \to +\infty} |g(t) - \varphi_0(q + \omega(t - t_0))| = 0.
\]

In words, an asymptotically quasiperiodic solution is an orbit converging in time to a quasiperiodic solution associated with \( X_0 \) on the invariant torus \( \varphi_0 \).

The following proposition proves that if \( \varphi^t \) is a \( C^\sigma \)-asymptotic KAM torus associated to \((X, X_0, \varphi_0)\), each initial point \( \varphi^h(q) \) gives rise to an asymptotically quasiperiodic solution associated to \((X, X_0, \varphi_0)\).

**Proposition 3.4.** Let \( \varphi^t \) be a \( C^\sigma \)-asymptotic KAM torus associated to \((X, X_0, \varphi_0)\). Then, for all \( q \in \mathbb{T}^n \) and \( t_0 \in J_{v'} \),

\[
g(t) = \psi^t_{h,X} \circ \varphi^h(q)
\]

is an asymptotically quasiperiodic solution associated to \((X, X_0, \varphi_0)\).

**Proof.** Thanks to (3.1)

\[
g(t) = \psi^t_{h,X} \circ \varphi^h(q) = \varphi^t(q + \omega(t - t_0))
\]

and hence, by (1.3), we have the claim.

We conclude this section with an important property concerning the case when both \( X \) and \( X_0 \) are Hamiltonian vector fields. Let \( \mathcal{P} = \mathbb{T}^n \times B \), and assume \( \varphi^t \) to be a \( C^\sigma \)-asymptotic KAM torus associated to \((X, X_0, \varphi_0)\). In the specific context of Hamiltonian systems, if the invariant torus \( \varphi_0 \) is Lagrangian, then \( \varphi^t \) is Lagrangian for all \( t \). This property was originally proved by Canadell and de la Llave in the discrete case, whereas here, we verify it in the continuous case.
Proposition 3.5. Let \( \varphi^t \) be a \( C^*-\)asymptotic KAM torus associated to \( (X, X_0, \varphi_0) \). If \( \varphi_0 \) is Lagrangian, then \( \varphi^t \) is Lagrangian for all \( t \in J_\varphi \).

Proof. Let \( \alpha = dp \wedge dq \) be the standard symplectic form on \( T^n \times B \), and \( \psi^{b+\mu}(q) = q + \omega t \) for all \( q \in T^n \), \( t_0 \in J_\varphi \), and \( t > 0 \). Taking the pull-back with respect to the standard form \( \alpha \) on both sides of (3.1) and using that \( \psi^{b+\mu} \) is a symplectomorphism for all fixed \( t, t_0 \in J_\varphi \), one has

\[
(\varphi^t)^* \alpha = (\psi^{b+\mu})^* (\varphi^{b+\mu})^* \alpha
\]

for all \( t_0 \in J_\varphi \) and \( t \geq 0 \). We want to prove that \( ((\varphi^t)^* \alpha) \big|_q = 0 \) for all \( q \in T^n \), where \( ((\varphi^t)^* \alpha) \big|_q \) stands for the symplectic form calculated on \( q \in T^n \). The idea consists of verifying that, for all fixed \( q \in T^n \), the limit when \( t \to +\infty \) on the right-hand side of the above equation converges to zero. Then, taking the limit for \( t \to +\infty \) on both sides of the latter, we have the claim. Now, let us introduce the following notation. We consider the following normed spaces \( (Y, \| \cdot \|_Y) \), \( (Z, \| \cdot \|_Z) \) and a bilinear map \( B : X \times Y \to Z \). We define the following norm

\[
\| B \| = \sup_{\| x \|_Y \leq 1, \| y \|_Z \leq 1} \| B(x,y) \|_Z.
\]

Now, we recall that \( \varphi_0 \) is Lagrangian. This means that for all \( q \in T^n \), and \( v_1, v_2 \in \mathbb{R}^n \)

\[
0 = ((\varphi_0)^* \alpha) \big|_{(v_1, v_2)} = \alpha_{\varphi_0} (D\varphi_0(q) v_1, D\varphi_0(q) v_2) = dp \wedge dq (D\varphi_0(q) v_1, D\varphi_0(q) v_2).
\]

Then, thanks to (3.3), for all \( t_0 \in J_\varphi \), \( q \in T^n \), \( t > 0 \), and \( v_1, v_2 \in \mathbb{R}^n \), we have

\[
\| ((\varphi^t)^* \alpha) \big|_{(v_1, v_2)} \| = \left\| (\psi^{b+\mu})^* (\varphi^{b+\mu})^* \alpha \right\|_{q, (v_1, v_2)}
\]

\[
= \| dp \wedge dq (D\varphi^{b+\mu}(q + \omega t) v_1, D\varphi^{b+\mu}(q + \omega t) v_2) \|
\]

\[
= \| dp \wedge dq (D\varphi^{b+\mu}(q + \omega t) v_1, D\varphi^{b+\mu}(q + \omega t) v_2) \|
\]

\[
= \| dp \wedge dq (D\varphi^{b+\mu}(q + \omega t) - D\varphi_0(q + \omega t) v_1, (D\varphi^{b+\mu}(q + \omega t) - D\varphi_0(q + \omega t)) v_2) \|
\]

\[
\leq \| dp \wedge dq \| \| (\psi^{b+\mu} - \varphi_0)^2 \|_{v_1} \| v_2 \|
\]

where, in the last line of the latter, \( \| \cdot \| \) is the norme introduced by (3.4) and \( \| \cdot \| \) is the Euclidean norm in \( \mathbb{R}^n \). This concludes the proof of this proposition because, by (1.3), the last line of the latter converges to 0 if \( t \to +\infty \). 

4. Functional setting

This section is divided into two parts. First, we introduce a series of notations and Banach spaces we will use in the proofs of the main results of this paper. In the second part, we properly rewrite the Hamiltonians (2.5) and (2.11).

For this purpose, consider a given function \( f : T^n \times B \to \mathbb{R} \). We define \( \tilde{f} \) as the following map

\[
\tilde{f} : T^n \times B \times J_\varphi \to \mathbb{R} \times J_\varphi, \quad \tilde{f}(q, p, t) = (f(q, p, t), t).
\]

We will also use this notation for functions defined on \( T^n \times B \times J_\varphi \), \( T^n \times J_\varphi \), \( T^n \times J_\varphi \), vector-valued functions or matrices. This will be specified by the context.

For a given \( \omega \in \mathbb{R}^n \), we define \( \Omega = (\omega, 1) \in \mathbb{R}^{n+1} \). Letting \( u : T^n \times J_\varphi \to \mathbb{R}^n \), one can see that for all \( (q, t) \in T^n \times J_\varphi \)

\[
Du(q, t) \Omega = \partial_q u(q, t) \omega + \partial_t u(q, t),
\]

(4.2)
where $Du$ stands for the differential of $u$. We will use the same notation also for functions defined on $T^n \times J_\nu$.

Now, we need to introduce some Banach spaces. Let $s > 0$, $\sigma \geq 1$, $v \geq 0$, an integer $k \geq 0$, and a positive decreasing integrable function $u : J_\nu \to \mathbb{R}^+$. For every $f \in S^v_{(\sigma,k)}$ and $g \in A^v_{\nu}$, we define the following norms

$$
|f|_{v,k,u}^v = \sup_{t \in J_\nu} |f|^v_{u}(t), \quad |g|_{1,u}^v = \sup_{t \in J_\nu} |g|^v_{1}(t),
$$

(4.3)

$$
|f|_{\sigma+k,1}^v = \sup_{t \in J_\nu} |f|^v_{\sigma+k,1}, \quad |g|_{1,1}^v = \sup_{t \in J_\nu} |g|^v_{1,1},
$$

(4.4)

where we refer to definitions 2.2 and 2.3 for the spaces $S^v_{(\sigma,k)}$ and $A^v_\nu$, respectively. We recall that $| \cdot |_{C^v}$ is the Hölder norm (see appendix A), whereas $| \cdot |$ is the analytic norm (see appendix B).

In order to quantify the regularity on the perturbative terms in the finite differentiable and the analytical case, we define the following Banach spaces $(S^v_{(\sigma,k)}, u \cdot | s + k, u )$, $(S^v_{(\sigma,k)}, (1,u))$, $(A^v_{\nu}, | s + k, u )$, $(A^v_{\nu}, (1,u))$ such that

$$
S^v_{(\sigma,k),u} = \left\{ f : T^n \times J_\nu \to \mathbb{R} \mid f \in S^v_{(\sigma,k)} \text{ and } |f|_{\sigma+k,u} < \infty \right\}
$$

(4.5)

$$
S^v_{(\sigma,k),(1,u)} = \left\{ f : T^n \times J_\nu \to \mathbb{R} \mid f \in S^v_{(\sigma,k)}, \text{ and } |f|_{\sigma+k,(1,u)} = |f|^v_{\sigma+k,1} + |\partial_g f|^v_{\sigma+k-1,1} < \infty \right\}
$$

(4.6)

$$
A^v_{\nu,u} = \left\{ g : T^n \times J_\nu \to \mathbb{R} \mid g \in A^v_\nu \text{ and } |g|_{1,u}^v < \infty \right\}
$$

(4.7)

$$
A^v_{\nu,(1,u)} = \left\{ g : T^n \times J_\nu \to \mathbb{R} \mid g \in A^v_\nu, \text{ and } |g|_{1,1}^v = |g|^v_{1,1} + |\partial_g g|^v_{1,1} < \infty \right\}.
$$

(4.8)

We will use the same notation for spaces of vector-valued functions. This will be clear by the context. Given $\omega \in \mathbb{R}^n$, to quantify the regularity of the $C^\omega$-asymptotic KAM torus (or the analytic asymptotic KAM torus) components, we will look for in the following sections, we define the following Banach spaces $(T^v_{(\sigma,\nu,u,w)}, | s, \nu, u, w )$, $(V^v_{\nu,\nu,u,w})$

$$
T^v_{(\sigma,\nu,u,w)} = \left\{ f : T^n \times J_\nu \to \mathbb{R}^\omega \mid f, Df\Omega \in S^v_{(\sigma,0)} \text{ and } |f|_{\sigma,u,w}^v = \max \{ |f|^v_{\sigma,u,w}, |Df\Omega|^v_{\sigma,u,w} \} < \infty \right\}
$$

(4.9)

$$
V^v_{\nu,\nu,u,w} = \left\{ g : T^n \times J_\nu \to \mathbb{R}^\omega \mid g, Dg\Omega \in A^v_{\nu} \text{ and } |g|_{\nu,u,w}^v = \max \{ |g|^v_{\nu,u,w}, |Dg\Omega|^v_{\nu,u,w} \} < \infty \right\}
$$

(4.10)

we refer to (2.4) for the definition of $\bar{u}$ and to (4.2) for the one of $Df\Omega$ or $Dg\Omega$. Moreover, in appendix D we prove that the previous spaces are Banach spaces. This concludes the first part of this section.

In the second part, we expand the Hamiltonian $H$ in (2.5) and (2.11)) in a small neighbourhood of $0 \in B$

$$
H(q,p,t) = \omega \cdot p + f(q,0,t) + \partial_q f(q,0,t) \cdot p + \int_0^1 (1 - \tau) \partial^2_{\tau} H(q,\tau p,t) d\tau \cdot p^2
$$
where we recall that \( h \in K_\omega \) implies \( \partial_t h(q,0,t) = \omega \) for all \((q,t) \in \mathbb{T}^n \times J_0\) and we assumed \( h(q,0,t) = 0 \) for all \((q,t) \in \mathbb{T}^n \times J_0\), we can do it without loss of generality (see definition 2.1).

Letting

\[
a(q,t) = f(q,0,t), \quad b(q,t) = \partial_p f(q,0,t), \quad m(q,p,t) = \int_0^1 (1 - \tau) \partial^2_\tau H(q,\tau p,t) \, d\tau,
\]

for a positive real parameter \( \Upsilon \geq 1 \), we can rewrite the Hamiltonian \( H \) in (2.5) (resp. (2.11)) in the following form

\[
\begin{cases}
  H : \mathbb{T}^n \times B \times J_0 \to \mathbb{R} \\
  H(q,p,t) = \omega \cdot p + a(q,t) \cdot p + m(q,p,t) \cdot p^2,
\end{cases}
\]

\[a \in S^0_{(\sigma,2),(1,a)} \quad b \in S^0_{(\sigma,2),b} \quad \partial^2_\tau H \in S^0_{(\sigma,2)}
\]

\[\text{resp. } a \in A^0_{(1,a)} \quad b \in A^0_{b} \quad \partial^2_\tau H \in A^0
\]

(4.11)

\[
\left| \partial_t a \right|_{\sigma+1,a}^0 \leq 1, \quad \left| b \right|_{\sigma+2,b}^0 \leq 1, \quad \sup_{t \in J_0} |\partial^2_\tau H|_{C^{\sigma+2}} \leq \Upsilon
\]

(4.11)

where \( a(t) \) and \( b(t) \) are the functions introduced in (2.5) satisfying (2.6). Concerning the proof of theorems A and B, this Hamiltonian is our new starting point.

5. Proof of theorem A

5.1. Outline of the proof of theorem A

This section aims to provide an overview of the proof of theorem A. The objective is to find a \( C^\alpha \)-asymptotic KAM torus \( \varphi' \) associated to \((X_H, X_{\tilde{h}}, \varphi_0)\), where \( H \) and \( \tilde{h} \) are the Hamiltonians defined by (4.11) and (2.7), respectively, and \( \varphi_0 \) is the trivial embedding in (2.3). More specifically, for given \( H \), we are looking for \( \nu' \geq 0 \) sufficiently large and suitable functions \( u, v : \mathbb{T}^n \times J_{\nu'} \to \mathbb{R}^n \) such that

\[
\varphi : \mathbb{T}^n \times J_{\nu'} \to \mathbb{T}^n \times B, \quad \varphi(q,t) = (q + u(q,t), v(q,t))
\]

(5.1)

is a family of \( C^\alpha \) embedded tori verifying the following conditions

\[
X_H(\varphi(q,t),t) - \partial_q \varphi(q,t) \omega - \partial_t \varphi(q,t) = 0,
\]

\[
\lim_{t \to +\infty} |u'|_{C^\alpha} = 0, \quad \lim_{t \to +\infty} |v'|_{C^\alpha} = 0,
\]

(5.2)

(5.3)

for all \((q,t) \in \mathbb{T}^n \times J_{\nu'}\). The parameter \( \nu' \) is free and will be chosen large enough in lemma 5.3 below.

5.1.1. Definition of the functional \( \mathcal{F} \). To formulate the dynamical problem we want to solve, we will introduce a suitable functional \( \mathcal{F} \) related by (5.2). First, we need the following notation. For all \((q,p,t) \in \mathbb{T}^n \times B \times J_{\nu'}\), we define

\[
\check{m}(q,p,t) = \int_0^1 \partial^2_\tau H(q,\tau p,t) \, d\tau
\]

(5.4)

where \( H \) is the Hamiltonian in (4.11). We claim that, for all \((q,p,t) \in \mathbb{T}^n \times B \times J_{\nu'}\),

\[
\check{m}(q,p,t)p = \partial_p (m(q,p,t) \cdot p^2)
\]
where $m$ is defined in \eqref{eq:m_def}. Indeed, using the Taylor formula, we observe that

\begin{equation}
H(q, p, t) = H(q, 0, t) + \partial_p H(q, 0, t) \cdot p + \int_0^1 (1 - \tau) \partial^2_p H(q, \tau p, t) \, d\tau \cdot p^2 \tag{5.5}
\end{equation}

\begin{equation}
\partial_p H(q, p, t) = \partial_p H(q, 0, t) + \int_0^1 \partial^2_p H(q, \tau p, t) \, d\tau \cdot p, \tag{5.6}
\end{equation}

whereas differentiating \eqref{eq:H_def} with respect to $p$

\begin{equation}
\partial_p H(q, p, t) = \partial_p H(q, 0, t) + \partial_p \left( \int_0^1 (1 - \tau) \partial^2_p H(q, \tau p, t) \, d\tau \cdot p^2 \right), \tag{5.7}
\end{equation}

Now, comparing \eqref{eq:H_def} with \eqref{eq:H_def_p} one has

\begin{equation}
\int_0^1 \partial^2_p H(q, \tau p, t) \, d\tau \cdot p = \partial_p \left( \int_0^1 (1 - \tau) \partial^2_p H(q, \tau p, t) \, d\tau \cdot p^2 \right),
\end{equation}

which proves the claim.

Now, we have everything we need to define the above-mentioned functional $F$. To this end, we observe that the Hamiltonian system associated with the Hamiltonian $H$ is equal to

\begin{equation}
X_H(q, p, t) = \begin{pmatrix}
\omega + b(q, t) + \tilde{m}(q, p, t) p \\
-\partial_q a(q, t) - \partial_q b(q, t) p - \partial_q m(q, p, t) p^2
\end{pmatrix},
\end{equation}

for all $(q, p, t) \in \mathbb{T}^n \times B \times J_{\nu}$. To achieve a more elegant form, we define the following map

\begin{equation}
U: \mathbb{T}^n \times J_{\nu} \to \mathbb{R}^n, \quad U(q, t) = q + u(q, t) \tag{5.8}
\end{equation}

where $u$ is defined by \eqref{eq:u_def}. Composing the Hamiltonian system $X_H$ with $\tilde{\varphi}$, we can write $X_{H \circ \tilde{\varphi}}$ in the following form

\begin{equation}
X_{H \circ \tilde{\varphi}}(q, t) = \begin{pmatrix}
\omega + b \circ \tilde{U}(q, t) + \tilde{m} \circ \tilde{\varphi}(q, t) \nu(q, t) \\
-\partial_q a \circ \tilde{U}(q, t) - \partial_q b \circ \tilde{U}(q, t) \nu(q, t) - \partial_q m \circ \tilde{\varphi}(q, t) \cdot \nu(q, t)^2
\end{pmatrix} \tag{5.9}
\end{equation}

for all $(q, t) \in \mathbb{T}^n \times J_{\nu}$. We want to point out that in the latter we used the notations defined by \eqref{eq:H_def}, \eqref{eq:nu_def} and \eqref{eq:5.8}. On the other hand, one can see that

\begin{equation}
\partial_q \varphi(q, t) \omega + \partial_t \varphi(q, t) = \begin{pmatrix}
\omega + \partial_q u(q, t) \omega + \partial_t u(q, t) \\
\partial_t \nu(q, t) \omega + \partial_t \nu(q, t)
\end{pmatrix} \tag{5.10}
\end{equation}

for all $(q, t) \in \mathbb{T}^n \times J_{\nu}$. Then, thanks to \eqref{eq:5.9} and \eqref{eq:5.10}, we can rewrite \eqref{eq:5.2} in the following form

\begin{equation}
\begin{pmatrix}
b \circ \tilde{U} + (\tilde{m} \circ \tilde{\varphi}) \nu - D\nu \Omega \\
-\partial_q a \circ \tilde{U} - (\partial_q b \circ \tilde{U}) \nu - (\partial_q m \circ \tilde{\varphi}) \cdot \nu^2 - D\nu \Omega
\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}. \tag{5.11}
\end{equation}

The latter consists of sums and products of functions defined on $(q, t) \in \mathbb{T}^n \times J_{\nu}$. We have omitted the arguments $(q, t)$ to attain a more elegant form. We keep this notation for the rest of the proof. Furthermore, it is important to observe that the notations used in the above equation are defined by \eqref{eq:H_def}, \eqref{eq:nu_def}, \eqref{eq:5.4} and \eqref{eq:5.8}.

Now, let $\sigma$, $\Upsilon$, $\omega$, $a$, and $b$ be as defined in \eqref{eq:5.11}. Furthermore, we fix $m$ and $\tilde{m}$ as specified in \eqref{eq:5.11}, with reference to \eqref{eq:5.4} for the definition of $\tilde{m}$. For $\nu' \geq 0$ sufficiently large, to be specified later, we introduce the following functional
\[ F : S_{(\sigma,2),(1,a)}^{
u'} \times \mathcal{S}_{(\sigma,2),b}^\nu \times \mathcal{T}_{\sigma,b,\omega}^\nu \times \mathcal{T}_{\sigma,a,\omega}^\nu \rightarrow \mathcal{S}_{(\sigma,0),b}^\nu \times \mathcal{S}_{(\sigma,0),a}^\nu \]

\[ F(a,b,u,v) = (F_1(b,u,v), F_2(a,b,u,v)) \quad (5.12) \]

such that

\[ F_1(b,u,v) = b \circ \tilde{U} + (\tilde{m} \circ \hat{\phi}) v - D\Omega \]
\[ F_2(a,b,u,v) = \partial_q a \circ \tilde{U} + (\partial_q b \circ \tilde{U}) v + (\partial_q \tilde{m} \circ \hat{\phi}) \cdot v^2 + D\Omega \cdot v. \]

We refer to section 4 for the Banach spaces involved in this definition. The latter is obtained by (5.11) and we observe that

\[ F(0,0,0,0) = 0. \]

Hence, we can reformulate our problem in the following form. For \((a,b) \in S_{(\sigma,2),(1,a)}^\nu \times \mathcal{S}_{(\sigma,2),b}^\nu\) sufficiently close to \((0,0)\), we are looking for some functions \((u,v) \in \mathcal{T}_{\sigma,b,\omega}^\nu \times \mathcal{T}_{\sigma,a,\omega}^\nu\) in such a way that \(F(a,b,u,v) = 0\).

In section 5.3, we will verify that \(F\) satisfies the hypotheses of the implicit function theorem. Always in section 5.3 we will see that the differential of \(F\) with respect to the variables \((u,v)\) evaluated at \((0,0)\) equals

\[ D_{(u,v)}F(0,0,0,0) : \mathcal{T}_{\sigma,b,\omega}^\nu \times \mathcal{T}_{\sigma,a,\omega}^\nu \rightarrow \mathcal{S}_{(\sigma,0),b}^\nu \times \mathcal{S}_{(\sigma,0),a}^\nu \]

\[ D_{(u,v)}F(0,0,0,0)(\hat{u},\hat{v}) = (m_0 \hat{v} - D\Omega \cdot \hat{v}, D\Omega \cdot \hat{v}) \quad (5.13) \]

where, according to (2.2), for all \((q,t) \in \mathbb{T}^n \times J_{\nu'}\) we let \(\tilde{m}_0(q,t) = \tilde{m}(q,0,t)\). We will see in section 5.3 that (5.13) is invertible (see lemma 5.2), the proof of which is based on the solution of the homological equation (5.20) analysed and solved in section 5.2 (see lemma 5.1).

Now, if we include the condition that the following norms

\[ |a|_{\sigma+2,(1,a)}^{\nu'}, |b|_{\sigma+2,b}^{\nu'} \]

must be sufficiently small among the hypotheses of theorem A, then its proof would straightforwardly follow from applying the implicit function theorem. However, we choose not to impose this condition. In order to avoid the requirement of smallness, we take a different perspective.

In the second part of this section, we will approach this problem differently by looking for a fixed point of the quasi-Newton operator \(L\) (see (5.18) below). We observe that by choosing \(\nu'\) sufficiently large, we have the following norms

\[ |\partial_q a'|_{\sigma+1}, |\partial_q b'|_{\sigma+1} \]

as small as we need for all \(t \in J_{\nu'}\). This allows us to establish that \(L\) is a contraction and conclude the proof of the theorem.

5.1.2. Definition of the operator \(L\). Let us be more precise by introducing the operator \(L\) and explaining the strategy to conclude the proof of theorem A.

We fix \(x = (a,b)\), where \(a\) and \(b\) are those defined by (4.11). Obviously \((a,b) \in \mathcal{S}_{(\sigma,2),(1,a)}^\nu \times \mathcal{S}_{(\sigma,2),b}^\nu\). Furthermore, we introduce the Banach space \(\mathcal{Y}_{\sigma,a,b,\omega}^{\nu'}\) such that

\[ \mathcal{Y}_{\sigma,a,b,\omega}^{\nu'} = \mathcal{T}_{\sigma,b,\omega}^{\nu'} \times \mathcal{T}_{\sigma,a,\omega}^{\nu'} \quad (5.16) \]

and for all \(y = (u,v) \in \mathcal{Y}_{\sigma,a,b,\omega}^{\nu'}\)

\[ \|y\|_{\sigma,a,b,\omega}^{\nu'} = \max \left\{ |u|_{\sigma,a,b,\omega}^{\nu'}, |v|_{\sigma,a,\omega}^{\nu'} \right\}. \quad (5.17) \]
We rewrite $F$ in the following form

$$F(x, y) = D_{(u, v)} F(0, 0, 0, 0) y + R(x, y).$$

The aim is always to find $y \in \mathcal{Y}^{u' \omega}_{\sigma, a, b, \omega}$ such that $F(x, y) = 0$, which is equivalent to find $y \in \mathcal{Y}^{u' \omega}_{\sigma, a, b, \omega}$. In such a way that

$$y = -D_{(u, v)} F(0, 0, 0, 0)^{-1} R(x, y)$$

This is well defined because $D_{(u, v)} F(0, 0, 0, 0)$ is invertible (see lemma 5.2). For this reason, we introduce the following functional

$$\mathcal{L}(x, \cdot) : \mathcal{Y}^{u' \omega}_{\sigma, a, b, \omega} \to \mathcal{Y}^{u' \omega}_{\sigma, a, b, \omega}$$

such that

$$\mathcal{L}(x, y) = y - D_{(u, v)} F(0, 0, 0, 0)^{-1} F(x, y).$$

(5.18)

Due to the regularity properties of $F$, in section 5.4 we will see that $\mathcal{L}$ is continuous, differentiable with respect to $y = (u, v)$ with differential $D_y \mathcal{L}$ is continuous. Now, the proof of theorem A is reduced to find a fixed point of the latter. For this reason, for some $c \in \mathbb{R}^+$ we define the following closed set

$$\mathcal{D}^{u' \omega}_{\sigma, a, b, \omega} = \left\{ y \in \mathcal{Y}^{u' \omega}_{\sigma, a, b, \omega} \ | \ \| y \|^{u' \omega}_{\sigma, a, b, \omega} \leq c \right\}.$$  

(5.19)

In section 5.4, specifically in lemma 5.3, we choose $u'$ sufficiently large to make the norms (5.15) small enough to establish the existence of a constant $C_u$ in such a way that $\mathcal{L}(\mathcal{D}^{u' \omega}_{\sigma, a, b, \omega, C_u}) \subset \mathcal{D}^{u' \omega}_{\sigma, a, b, \omega, C_u}$, $\mathcal{L}$ restricted to $\mathcal{D}^{u' \omega}_{\sigma, a, b, \omega, C_u}$ is a contraction and applying the fixed point theorem.

### 5.2. Homological equation

This section is dedicated to solving the homological equation (see (5.20) below). As mentioned before, the following lemma (see lemma 5.1) will be the main ingredient to prove that the differential (5.13) is invertible. Given $\sigma \geq 1$, $\nu \geq 0$ and $\omega \in \mathbb{R}^n$, we want to solve the following equation for the unknown $\varkappa : \mathbb{T}^n \times J_u \to \mathbb{R}$

$$\begin{cases}
\varkappa \cdot \partial_q \varkappa(q, t) + \partial_t \varkappa(q, t) = g(q, t), \\
g \in S^u_{(\sigma, 0), \nu}.
\end{cases}$$

(5.20)

where $g : J_u \to \mathbb{R}^+$ is a positive, decreasing, integrable function on $J_u$ and $g : \mathbb{T}^n \times J_u \to \mathbb{R}$ is given. We refer to (4.5) for the definition of $S^u_{(\sigma, 0), \nu}$.

**Lemma 5.1 (homological equation).** There exists a unique solution $\varkappa \in S^u_{(\sigma, 0)}$ of (5.20) such that

$$\lim_{t \to +\infty} |\varkappa|_{C^0} = 0.$$  

(5.21)

Moreover,

$$|\varkappa|^{u'}_{(\sigma, \nu, g} \leq |g|^{u'}_{(\sigma, \nu}).$$

(5.22)

We refer to definition 2.2 for the definition of the space of functions $S^u_{(\sigma, 0)}$ and to (4.3) for the norm $| \cdot |_{(\sigma, \nu, g}$ Whereas, $\bar{g}$ is defined by (2.4).
**Proof.** Existence: let us define the following transformation
\[ \phi : T^n \times J_\nu \rightarrow T^n \times J_\nu, \quad \phi(q,t) = (q-\omega t, t). \]

We claim that it is enough to prove the first part of this lemma for the following much simpler equation in the unknown \( \kappa : T^n \times J_\nu \rightarrow \mathbb{R} \)
\[ \partial_t \kappa = g(q + \omega t, t). \tag{5.23} \]

As a matter of fact, if \( \kappa \) is a solution of (5.23) satisfying the asymptotic condition (5.21), then \( \chi = \kappa \circ \phi \) is a solution of (5.20) satisfying the same asymptotic condition and vice versa. For the sake of clarity, we prove this claim. Let \( \kappa \) be a solution of (5.20) verifying the asymptotic condition (5.21), then
\[ \partial_t (\kappa \circ \phi^{-1}) = \partial_q \kappa \circ \phi^{-1} \cdot \omega + \partial_t \kappa \circ \phi^{-1} = g \circ \phi^{-1}, \]
where the last equality is due to (5.20). This implies that \( \kappa = \kappa \circ \phi^{-1} \) is a solution of (5.23) and by
\[ |\kappa'|_{C^0} = |(\kappa \circ \phi^{-1})'|_{C^0} \leq |\kappa'|_{C^0} \]
\( \kappa = \kappa \circ \phi^{-1} \) satisfies the asymptotic condition because \( \chi \) does. Vice versa, let \( \kappa \) be a solution of (5.23) satisfying the asymptotic condition (5.21), then
\[ \partial_q (\kappa \circ \phi) \cdot \omega + \partial_t (\kappa \circ \phi) = \partial_q \kappa \circ \phi \cdot \omega - \partial_t \kappa \circ \phi = g. \]

By (5.23), we have the last equality of the latter. Hence, \( \kappa \circ \phi \) is a solution of (5.20). Moreover, thanks to
\[ |\chi'|_{C^0} = |(\kappa \circ \phi)'|_{C^0} \leq |\kappa'|_{C^0} \]
\( \chi = \kappa \circ h \) satisfies the asymptotic condition (5.21). This proves the claim.

Now, we can solve equation (5.23) by integration. Indeed, for all \((q,t) \in T^n \times J_\nu\) a solution of (5.23) exists and it is given by
\[ \kappa(q,t) = e(q) + \int_{\nu}^t g(q + \omega \tau, \tau) d\tau \]
where \( e : T^n \rightarrow \mathbb{R} \) is free. To ensure that \( \kappa \) satisfies the following asymptotic condition for all fixed \( q \in T^n \)
\[ 0 = \lim_{t \to +\infty} \kappa(q,t) = e(q) + \int_{\nu}^{+\infty} g(q + \omega \tau, \tau) d\tau, \]
there is only one possible choice for \( e \), which is
\[ e(q) = -\int_{\nu}^{+\infty} g(q + \omega \tau, \tau) d\tau. \]
This implies that
\[ \kappa(q,t) = -\int_{\nu}^{+\infty} g(q + \omega \tau, \tau) d\tau \tag{5.24} \]
is the solution of (5.23) we are looking for. Therefore, \( e \) is well defined. Indeed,
\[ \left| \int_{\nu}^{+\infty} g(q + \omega \tau, \tau) d\tau \right| \leq |g|^v_{\nu, g} \int_{\nu}^{+\infty} g(\tau) d\tau = |g|^v_{\nu, g} R(v) < \infty \]
where in the latter we used that \( g \in S^{1}_{\psi(\sigma,0)} \). Moreover, thanks to (5.24) and \( g \in S^{1}_{\psi(\sigma,0)} \),
\[
|\kappa^t|_{C^0} \leq \int_{t}^{+\infty} |g^\tau|_{C^0} \mathrm{d}\tau \leq |g|_{\psi}^{\infty} \int_{t}^{+\infty} g(\tau) \mathrm{d}\tau = |g|_{\psi}^{\infty} \bar{g}(t),
\]

since \( \bar{g}(t) \) converges to 0 when \( t \to +\infty \), taking the limit for \( t \to +\infty \) on both sides of the latter, we have that \( \lim_{t \to +\infty} |\kappa^t|_{C^0} = 0 \). This concludes the first part of the proof because
\[
\varpi(q,t) = \kappa \circ h(q,t) = -\int_{t}^{+\infty} g(q + \omega(\tau - t), \tau) \mathrm{d}\tau
\]
is the unique solution of (5.20) verifying (5.21) that we are looking for.

**Regularity and estimates**: we observe that \( g \in S^{1}_{\psi(\sigma,0)} \) implies \( \kappa \in S^{1}_{\psi(\sigma,0)} \) and hence \( \varpi = \kappa \circ h \in S^{1}_{\psi(\sigma,0)} \). Moreover, for all fixed \( t \in J_{\psi} \)
\[
|\varpi|_{C^0} \leq |g|_{\psi}^{\infty} \bar{g}(t).
\]

Multiplying both sides of the latter by \( \frac{1}{\bar{g}(t)} \) and taking the sup for all \( t \in J_{\psi} \), we prove the second part of this lemma.

We point out that an alternative proof of the first part of the previous lemma can be provided by using the characteristics method for first-order partial differential equations. In this case, one obtains that for any \( (q,t,s) \in \mathbb{T}^n \times J_{\psi} \times J_{\psi} \)
\[
\varpi(q,t) = \varpi(q + (s-t)\omega,s) + \int_{t}^{s} g(q + (\tau-t)\omega,\tau) \mathrm{d}\tau
\]
is a formal solution of (5.20). Taking the limit for \( s \to +\infty \) on both sides of the latter, one obtains (5.25).

We want to emphasise that, unlike classical KAM theory, in this paper, we solve the homological equation without any arithmetic conditions on the frequency vector \( \omega \). As one can see from the proof of the previous lemma, it suffices to introduce a suitable change of coordinates \( \phi \) that rectifies the dynamics on the torus. Then, the homological equation can be solved by integration. We observe that, in general, the solution has a loss of regularity in terms of decay in time (see (5.22)). To illustrate this point, we encourage the reader to consider the case when \( \psi > 0 \) and \( g(t) = \frac{1}{t} \) with \( t > 1 \). This issue does not arise in the case of exponential decay (that is, \( g(t) = e^{-\lambda t} \) with \( \lambda > 0 \).

In the present paper, this loss of regularity is solved in the associated nonlinear problem through a careful analysis of the Banach spaces used to define the functional \( F \).

### 5.3. Regularity of \( F \)

In this section, we verify that the functional \( F \) given by (5.12) satisfies the hypotheses of the implicit function theorem. To this end, using the properties in proposition A.2 (see appendix A) and (2.6), one can prove that \( F \) is well-defined, continuous, and differentiable with respect to the variables \((u,v)\). Let \( D_{(u,v)} F \) be the differential of \( F \) with respect to \((u,v)\), we have that
\[
D_{(u,v)} F (b, u, v) (\hat{u}, \hat{v}) = D_u F_1 (b, u, v) \hat{u} + D_v F_1 (b, u, v) \hat{v}
\]
\[
= (\partial_q b \circ \hat{U}) \hat{u} + v^T (\partial_q m \circ \hat{\varphi}) \hat{u} + v^T (\partial_p m \circ \hat{\varphi}) \hat{v}
\]
\[
+ (m \circ \hat{\varphi}) \hat{v} - D\hat{\Omega}
\]

(5.26)
\[D_{(u,v)}F_2(a,b,u,v)\left(\hat{u},\hat{v}\right) = D_{u}F_2(a,b,u,v)\hat{u} + D_{v}F_2(a,b,u,v)\hat{v}
\]
\[= (\partial_{q}^{2}a \circ \hat{U}) \hat{u} + v^{T}(\partial_{q}^{2}b \circ \hat{U}) \hat{u} + (v^{T})^{2}(\partial_{q}^{2}m \circ \varphi) \hat{u}
\]
\[+ (\partial_{q}b \circ \hat{U}) \hat{v} + (v^{T})^{2}(\partial_{q}^{2}m \circ \varphi) \hat{v} + 2v^{T}(\partial_{q}b \circ \varphi) \hat{v} + D\hat{u} \Omega,
\]
(5.27)

where $T$ stands for the transpose of a vector and $D_{u}$ and $D_{v}$ are, respectively, the differentials with respect to $u$ and $v$. Furthermore, one has that $D_{(u,v)}F$ is continuous. Now, we observe that $D_{(u,v)}F$ evaluated at $(0,0,0,0)$ is equal to

\[D_{(u,v)}F(0,0,0,0) : \mathcal{T}_{σ,b,ω}^{υ′} \times \mathcal{T}_{σ,a,ω}^{υ′} \rightarrow \mathcal{S}_{σ(0),b}^{υ′} \times \mathcal{S}_{σ(0),a}^{υ′},
\]
\[D_{(u,v)}F(0,0,0,0)(\hat{u},\hat{v}) = (m_0\hat{v} - D\hat{u}Ω, D\hat{v}Ω)
\]
(5.28)

where in agreement with the notation (2.2), we denote $m(q,t) = m(q,0,t)$ for all $(q,t) \in \mathbb{T}^{n} \times J_{υ′}$. We refer to (4.5) and (4.9) for the definition of the Banach spaces $\mathcal{S}_{σ(0),b}^{υ′}$ (or $\mathcal{S}_{σ(0),a}^{υ′}$) and $\mathcal{T}_{σ,b,ω}^{υ′}$ (or $\mathcal{T}_{σ,a,ω}^{υ′}$), respectively.

In the following lemma, we verify that the latter is invertible. First, to avoid a flow of constants, let $C(\cdot)$ be constants depending on $n$ and the other parameters in brackets. On the other hand, $C$ stands for constants depending only on $n$. We will use this notation for the rest of the proof of theorem A.

**Lemma 5.2.** For all $(z,g) \in \mathcal{S}_{σ(0),b}^{υ′} \times \mathcal{S}_{σ(0),a}^{υ′}$ there exists a unique $(\hat{u},\hat{v}) \in \mathcal{T}_{σ,b,ω}^{υ′} \times \mathcal{T}_{σ,a,ω}^{υ′}$ such that

\[D_{(u,v)}F(0,0,0,0)(\hat{u},\hat{v}) = (z,g).
\]

Moreover, there exists a positive constant $\bar{C} \geq 1$ depending on $n$, $σ$, $T$, and $Λ$ such that

\[|\hat{u}|_{σ,b,ω}^{υ′} \leq \bar{C} \left( |g|_{σ,b}^{υ′} + |z|_{σ,b}^{υ′} \right), \quad |\hat{v}|_{σ,a,ω}^{υ′} \leq |g|_{σ,a}^{υ′},
\]
(5.29)

where we refer to (4.9) and (4.3) for the definition of the norms $|\cdot|_{σ,b,ω}^{υ′}$ (or $|\cdot|_{σ,a,ω}^{υ′}$) and $|\cdot|_{σ,a}^{υ′}$ (or $|\cdot|_{σ,b}^{υ′}$), respectively. Whereas, $Λ$ is the constant in (2.6) and $T$ is defined in (4.11).

**Proof.** The proof of this lemma relies on lemma 5.1. Indeed, thanks to (5.28), we can reformulate the problem in the following form. Given $(z,g) \in \mathcal{S}_{σ(0),b}^{υ′} \times \mathcal{S}_{σ(0),a}^{υ′}$, we are looking for the unique solution $(\hat{u},\hat{v}) \in \mathcal{T}_{σ,b,ω}^{υ′} \times \mathcal{T}_{σ,a,ω}^{υ′}$ of the following system

\[\begin{align*}
\text{D}\hat{u}Ω & = m_0\hat{v} - z \\
\text{D}\hat{v}Ω & = g.
\end{align*}
\]
(5.30)

By lemma 5.1, the unique solution $\hat{v}$ of the last equation of the latter system exists and satisfies

\[|\hat{v}|_{σ,a}^{υ′} \leq |g|_{σ,a}^{υ′}.
\]

Moreover, by $|\text{D}\hat{v}Ω|_{σ,a}^{υ′} = |g|_{σ,a}^{υ′}$, we have the second estimate in (5.29)

\[|\hat{v}|_{σ,a,ω}^{υ′} = \max \left\{ |\hat{v}|_{σ,a}^{υ′}, |\text{D}\hat{v}Ω|_{σ,a}^{υ′} \right\} \leq |g|_{σ,a}^{υ′}.
\]
(5.31)

Now, it remains to solve the first equation of (5.30) where $\hat{v}$ is known. For all fixed $t \in J_{υ′}$ and thanks to property (2) of proposition A.2, the first condition of (2.6) and (5.31)
Taking the sup for all \( t \in J_{\sigma,\vartheta} \) on the left-hand side of the latter, we obtain
\[
|\bar{m}_{\sigma,\vartheta} - z|_{\sigma,\vartheta,\omega}^{|\vartheta|} \leq C(\sigma, \Upsilon, \Lambda) \left( |g|_{\sigma,\vartheta,\omega}^{|\vartheta|} + |z|_{\sigma,\vartheta,\omega}^{|\vartheta|} \right)
\]
and hence
\[
|D\bar{u} \Omega|_{\sigma,\vartheta,\omega}^{|\vartheta|} = |\bar{m}_{\sigma,\vartheta} - z|_{\sigma,\vartheta,\omega}^{|\vartheta|} \leq C(\sigma, \Upsilon, \Lambda) \left( |g|_{\sigma,\vartheta,\omega}^{|\vartheta|} + |z|_{\sigma,\vartheta,\omega}^{|\vartheta|} \right).
\]
Thanks to lemma 5.1 the unique solution \( \bar{u} \) of the first equation of (5.30) exists verifying
\[
|\bar{u}|_{\sigma,\vartheta,\omega}^{|\vartheta|} = |\bar{m}_{\sigma,\vartheta} - z|_{\sigma,\vartheta,\omega}^{|\vartheta|} \leq C(\sigma, \Upsilon, \Lambda) \left( |g|_{\sigma,\vartheta,\omega}^{|\vartheta|} + |z|_{\sigma,\vartheta,\omega}^{|\vartheta|} \right).
\]
This concludes the proof of this lemma because
\[
|\bar{u}|_{\sigma,\vartheta,\omega}^{|\vartheta|} = \max \left\{ |\bar{u}|_{\sigma,\vartheta,\omega}^{|\vartheta|}, |D\bar{u} \Omega|_{\sigma,\vartheta,\omega}^{|\vartheta|} \right\} \leq C(\sigma, \Upsilon, \Lambda) \left( |g|_{\sigma,\vartheta,\omega}^{|\vartheta|} + |z|_{\sigma,\vartheta,\omega}^{|\vartheta|} \right).
\]
\[
\square
\]
We have proved that the functional \( \mathcal{F} \) satisfies the hypotheses of the implicit function theorem. In section 5.1, it was mentioned that by introducing an additional smallness condition on the perturbative terms (5.14) among the hypotheses of theorem A, we can prove it as an application of the implicit function theorem. As explained in section 5.1, we avoid this unnecessary hypothesis, reformulating our problem by finding a fixed point of a suitable operator \( \mathcal{L} \) (see (5.18)). This will be the subject of the following section.

5.4. The operator \( \mathcal{L} \)

In this section, we conclude the proof of theorem A by proving that the functional \( \mathcal{L} \) (see (5.18)) is a contraction. For the sake of clarity, let us recall its definition. Let \( x = (a, b) \), where \( a \) and \( b \) are the perturbative terms in (4.11). We recall that the operator \( \mathcal{L} \) is given by
\[
\mathcal{L}(x, \cdot) : \mathcal{Y}^{|\vartheta|}_{\sigma, a, b, \omega} \longrightarrow \mathcal{Y}^{|\vartheta|}_{\sigma, a, b, \omega},
\]
such that
\[
\mathcal{L}(x, y) = y - D(x, v) \mathcal{F}(y),
\]
where we refer to (5.16) for the definition of the Banach space \( \mathcal{Y}^{|\vartheta|}_{\sigma, a, b, \omega} \). The latter is well-defined. Moreover, thanks to the regularity of \( \mathcal{F} \), one can see that \( \mathcal{L} \) is continuous, differentiable with respect to \( y = (u, v) \) with differential \( D_x \mathcal{L} \) continuous. As mentioned before, the proof of theorem A is reduced to find a fixed point of the above operator. For this purpose, we have the following

**Lemma 5.3.** Let \( C \) be the constant introduced by lemma 5.2. Then,
\[
\| \mathcal{L}(x, 0) \|_{\sigma, a, b, \omega} \leq 2C,
\]
where we refer to (5.17) for the definition of the norm $\| \cdot \|_{\sigma,\omega}^{\omega'}$. Moreover, there exists $v'$ large enough with respect to $\sigma$, $\Upsilon$, $\Lambda$ and $b$, such that, for all $y_*, y \in \mathcal{Y}_{\sigma,\omega}^{\omega'}$ with $\| y_* \|_{\sigma,\omega}^{\omega'} \leq 4C$

$$\| D_t \mathcal{L} (x, y_*) y \|_{\sigma,\omega}^{\omega'} \leq \frac{1}{2} \| y \|_{\sigma,\omega}^{\omega'}.$$  \hfill (5.33)

**Proof.** The proof relies on lemma 5.2. First, we verify (5.32). For this purpose, we observe that

$$\mathcal{L} (x, \mathcal{L}) = -D(\omega, \omega)\mathcal{F} (0, 0, 0, 0)^{-1} \mathcal{F} (x, 0).$$

Remembering the definition of $\mathcal{F}$ (see (5.12)), one can see that $\mathcal{F} (x, 0) = (b, \partial_q a)$. Then, we can reformulate this problem in terms of estimating the unique solution $\tilde{y}$ of the following system

$$D(\omega, \omega)\mathcal{F} (0, 0, 0, 0) \tilde{y} = - (b, \partial_q a).$$  \hfill (5.34)

Thanks to lemma 5.2, a unique solution of the above equation exists and

$$\| \tilde{y} \|_{\sigma,\omega}^{\omega'} \leq C \left( |\partial_q a|_{\sigma,\omega}^{\omega'} + |b|_{\sigma,\omega}^{\omega'} \right) \leq 2C$$

where the last inequality of the latter is a consequence of (4.11). This proves (5.32). It remains to prove (5.33). To this end, we observe that for all $y_*, y \in \mathcal{Y}_{\sigma,\omega}^{\omega'}$ with $\| y_* \|_{\sigma,\omega}^{\omega'} \leq 4C$

$$D_t \mathcal{L} (x, y_*) y = D(\omega, \omega)\mathcal{F} (0, 0, 0, 0)^{-1} \left( D(\omega, \omega)\mathcal{F} (0, 0, 0, 0) - D(\omega, \omega)\mathcal{F} (x, y_*) \right) y.$$

Similarly to the previous case, we can reformulate this problem in terms of estimating the unique solution $\tilde{y} = (\tilde{u}, \tilde{v}) \in \mathcal{Y}_{\sigma,\omega}^{\omega'}$ of the following system

$$D(\omega, \omega)\mathcal{F} (0, 0, 0, 0) \tilde{y} = \left( D(\omega, \omega)\mathcal{F} (0, 0, 0, 0) - D(\omega, \omega)\mathcal{F} (x, y_*) \right) y.$$  \hfill (5.35)

Therefore, in order to prove (5.33), it suffices to estimate the right-hand side of the latter and apply lemma 5.2. First, let us introduce the following notation. We denote $y_* = (u_*, v_*) \in \mathcal{Y}_{\sigma,\omega}^{\omega'}$ and we define

$$U_* (q, t) = q + u_* (q, t), \quad \varphi_* (q, t) = (q + u_* (q, t), v_* (q, t))$$

for all $(q, t) \in \mathbb{T}^n \times \mathbb{T}^{\omega'}$. Now, thanks to (5.13), the right-hand side of (5.35) is equal to

$$\left( D(\omega, \omega)\mathcal{F} (0, 0, 0, 0) - D(\omega, \omega)\mathcal{F} (x, y_*) \right) y = \left( \frac{\bar{m}_0 v - D(\omega) - D(\omega, \omega) F_1 (b, y_*)}{D(\omega) - D(\omega, \omega) F_2 (x, y_*)} \right) y,$$

where, by (5.26) and (5.27),

$$\bar{m}_0 v - D(\omega) - D(\omega, \omega) F_1 (b, y_*) y = \left( \bar{m}_0 - \bar{m} \circ \varphi_* \right) v - \left( \partial_q b \circ \bar{U}_* \right) u$$

$$- \nu_*^T \left( \partial_q \bar{m} \circ \varphi_* \right) u - \nu_*^T \left( \partial_q \bar{m} \circ \varphi_* \right) v$$

$$D(\omega) - D(\omega, \omega) F_2 (x, y_*) y = \left( \partial^2_q \bar{m} \circ \varphi_* \right) u - \left( \partial_q b \circ \bar{U}_* \right) v$$

$$- \left( \nu_*^T \right)^2 \left( \partial^2_q \bar{m} \circ \varphi_* \right) u - \left( \partial_q b \circ \bar{U}_* \right) v$$

$$- \left( \nu_*^T \right)^2 \left( \partial^2_q \bar{m} \circ \varphi_* \right) v - 2\nu_*^T \left( \partial_q \bar{m} \circ \varphi_* \right) v.$$
Thus, by (4.11), we use the first condition in (2) of proposition A.2, we can estimate the first member on the right-hand side of the latter as follows
\[
\left| \left( \tilde{m}_0v - Du \Omega - D_{(u,v)}F_1(b,y_*)y \right) \right|_{C^1} \leq C(\sigma) \left( \left| \tilde{m}_0' - \tilde{m} \circ \tilde{\varphi}_* \right|_{C^1} \right|v'|_{C^0} \\
+ \left| \left( \partial_{\nu}b \circ \tilde{U}_* \right)' \right|_{C^0} \left| u' \right|_{C^0} \\
+ \left| \left( \partial_{\nu} \tilde{m} \circ \tilde{\varphi}_* \right)' \right|_{C^0} \left| v' \right|_{C^0} \\
+ \left| \left( \partial_{\nu} \tilde{m} \circ \tilde{\varphi}_* \right)' \right|_{C^0} \left| v' \right|_{C^0} \right)
\]
for all \( t \in J_{\nu} \). Now, we have to estimate each member on the right-hand side of the latter. For this reason, for all \( t \in J_{\nu} \),
\[
\left| \tilde{m}_0' - \tilde{m} \circ \tilde{\varphi}_* \right|_{C^0} \leq C(\sigma) \left( \left| \tilde{m}_0' - \tilde{m} \circ \tilde{\varphi}_* \right|_{C^0} \right) |v'|_{C^0} \\
+ \left| \left( \partial_{\nu} \tilde{m} \circ \tilde{\varphi}_* \right)' \right|_{C^0} \left| u' \right|_{C^0} \\
\leq C(\sigma, C) \left( |u'_0|_{C^0} + |v'_0|_{C^0} \right) |v'|_{C^0} \\
\leq C(\sigma, C) \left( \| \tilde{b} \|_{\sigma, b, \omega} \right) |v'|_{\sigma, b, \omega} \\
+ C(\sigma, C) \left( \| \tilde{b} \|_{\sigma, b, \omega} \right) |v'|_{\sigma, b, \omega}
\]
The first inequality (5.36) of the latter is a consequence of the mean value theorem for a suitable \( \tau \in [0, 1] \). Concerning the second inequality (5.37), it is due to properties (2) and (5) of proposition A.2 and \( \| u'_0 \|_{\sigma, a, b, \omega, \omega} \leq 4C \). More specifically, we may assume \( \tilde{b}(t) \leq \tilde{b}(\nu') \leq 1 \) and \( \tilde{a}(t) \leq \Lambda b(t) \leq \Lambda \tilde{b}(\nu') \leq 1 \) for all \( t \in J_{\nu} \). Thanks to (2.6) and at the cost of taking \( v' \) large enough. Thus, by (5) of proposition A.2 and \( \| u'_0 \|_{\sigma, a, b, \omega} \leq 4C \) we can estimate \( |\partial_{\nu} \tilde{m}(id + \tau u_*, \tau v_*)|_{C^0} \) by \( \Upsilon C(\sigma, C) \) where, in agreement with the notation made above, \( C(\sigma, C) \) is a suitable constant depending on \( \sigma \) and \( C \). In (5.38), we apply the following estimate \( \| y'_0 \|_{\sigma, a, b, \omega} \leq 4C \) and the definition of the norm \( | \cdot |_{\sigma, a, \omega} \). For the last inequality (5.39), we use the first condition in (2.6), the definition of the norm \( \| \cdot \|_{\sigma, a, b, \omega} \) (see (5.17)) and \( \| y'_0 \|_{\sigma, a, b, \omega} \leq 4C \).

Similarly to the previous case, thanks to property (5) of proposition A.2, the first condition in (2.6), (4.11), \( \| y'_0 \|_{\sigma, a, b, \omega} \leq 4C \), and \( \tilde{b}(\nu') \leq 1, \tilde{a}(\nu') \leq 1 \), we obtain
\[
\left| \left( \partial_{\nu} b \circ \tilde{U}_* \right)' \right|_{C^0} \left| u' \right|_{C^0} \leq C(\sigma, C) \left( |b|_{\sigma, \nu', b, \omega} \right) \left| u'_0 \right|_{\sigma, b, \omega} \tilde{b}(t) \\
\leq C(\sigma, C) \left( \| y'_0 \|_{\sigma, a, b, \omega} \right) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \\
\left| \left( \partial_{\nu} \tilde{m} \circ \tilde{\varphi}_* \right)' \right|_{C^0} \left| v' \right|_{C^0} \leq C(\sigma, C) \left( |v'|_{\sigma, a, \omega} \right) \Upsilon |v'|_{\sigma, a, \omega} \tilde{b}(t) \\
\leq C(\sigma, C) \left( \| y'_0 \|_{\sigma, a, b, \omega} \right) \left| v' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \\
\left| \left( \partial_{\nu} \tilde{m} \circ \tilde{\varphi}_* \right)' \right|_{C^0} \left| u' \right|_{C^0} \leq C(\sigma, C) \left( |u'|_{\sigma, a, \omega} \right) \Upsilon |u'|_{\sigma, a, \omega} \tilde{a}(t) \\
\leq C(\sigma, C) \left( \| y'_0 \|_{\sigma, a, b, \omega} \right) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \\
\left| \left( \partial_{\nu} \tilde{m} \circ \tilde{\varphi}_* \right)' \right|_{C^0} \left| v' \right|_{C^0} \leq C(\sigma, C) \left( |v'|_{\sigma, a, \omega} \right) \Upsilon |v'|_{\sigma, a, \omega} \tilde{a}(t) \\
\leq C(\sigma, C) \left( \| y'_0 \|_{\sigma, a, b, \omega} \right) \left| v' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \right)
\]
for all \( t \in J_{\nu} \). Now, thanks to the above estimates, taking \( v' \) large enough with respect to \( \sigma, \Upsilon, \Lambda \) and \( b \), one has that.
\[ \left\| (\tilde{m}_t v - D\tilde{\Omega} - D_{(u,v)} F_1 (b, y_s) y) \right\|_{C^r} \leq \frac{1}{4} \| y \|_{\sigma, a, b, \omega}^{\nu'} \]

for all \( t \in J_{v'} \). Multiplying both sides of the latter by \( \frac{1}{\| y \|_{\sigma, a, b, \omega}} \) and taking the sup for all \( t \in J_{v'} \), we obtain
\[ \left\| m_0 v - D\tilde{\Omega} - D_{(u,v)} F_1 (b, y_s) y \right\|_{C^r} \leq \frac{1}{4} \| y \|_{\sigma, a, b, \omega}^{\nu'} . \tag{5.40} \]

Similarly to the previous case,
\[ \left\| (D\tilde{\Omega} - D_{(u,v)} F_2 (x, y_s) y) \right\|_{C^r} \leq C(\sigma) \left( \left\| (\partial_{\alpha} b \circ \tilde{U}_s) \right\|_{C^r} |u'|_{C^r} \right. \]
\[ + \left| v_s \right|_{C^r} \left( \partial_{\alpha} b \circ \tilde{U}_s \right) \left\| u' \right\|_{C^r} \]
\[ + \left| v_s \right|_{C^r} \left( \partial_{\alpha} m \circ \tilde{\varphi}_s \right) \left\| u' \right\|_{C^r} \]
\[ + \left| (\partial_b b \circ \tilde{U}_s) \right|_{C^r} \left| v' \right|_{C^r} \]
\[ + \left| v_s \right|_{C^r} \left( \partial_{\alpha} m \circ \tilde{\varphi}_s \right) \left| v' \right|_{C^r} \), \]

for all \( t \in J_{v'} \). Therefore, we have to estimate each member on the right-hand side of the latter. We begin with the element in the second line. For all \( t \in J_{v'} \),
\[ \left| v_s \right|_{C^r} \left( \partial_{\alpha} b \circ \tilde{U}_s \right) \left\| u' \right\|_{C^r} \leq C(\sigma, \tilde{C}) \left| v_s \right|_{C^r} |b|_{\sigma, a, b, \omega}^{\nu'} b(t) \left| u' \right|_{C^r} \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \left| a(t) \right| \left| b \right|_{\sigma, a, b, \omega}^{\nu'} \left| b(t) \right| \left| u' \right|_{C^r} \]
\[ \leq C(\sigma, \tilde{C}) \Lambda \tilde{b}(v') \left| u' \right|_{\sigma, a, b, \omega} \tilde{a}(t) . \]

The first line of the above estimate is due to property (5) of proposition A.2. \( \tilde{b}(v') \leq 1 \) and \( \| y \|_{\sigma, a, b, \omega} \leq 4\tilde{C} \). In the second line we use \( |v_s|_{C^r} \leq \tilde{C} |a(t)| \) and \( |u'|_{C^r} \leq \| u' \|_{\sigma, a, b, \omega} b(t) \) for all \( t \in J_{v'} \). The last inequality is a consequence of the second condition of \( (2.6) \).

Thanks to property (5) of proposition A.2, (2.6), (4.11) and \( \tilde{b}(v') \leq 1 \), \( \tilde{a}(v') \leq 1 \), in the same way we have
\[ \left| (\partial_{\alpha} b \circ \tilde{U}_s) \right|_{C^r} \left\| u' \right\|_{C^r} \leq C(\sigma, \tilde{C}) \left| \partial_{\alpha} a \right|_{\sigma, a, b, \omega}^{\nu'} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{b}(v') \left| u' \right|_{\sigma, a, b, \omega} \tilde{a}(t) , \]
\[ \left| v_s \right|_{C^r} \left( \partial_{\alpha} m \circ \tilde{\varphi}_s \right) \left\| u' \right\|_{C^r} \leq C(\sigma, \tilde{C}) \left| v_s \right|_{\sigma, a, b, \omega}^{\nu'} \tilde{a}(t)^2 \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \left( \left| v_s \right|_{\sigma, a, b, \omega} \right)^{\nu'} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
\[ \leq C(\sigma, \tilde{C}) \tilde{C} \tilde{a}(t) \left| u' \right|_{\sigma, a, b, \omega} \tilde{b}(t) \]
for all \( t \in J_{\psi^r} \). Then, for \( \psi^r \) large enough

\[
\left| \left( Dv \Omega - D_{(u,y)} F_2 (x,y) \right) y \right|_{C^r} \leq \frac{1}{4C} \| y \|_{\sigma,a,b,\omega} (t)
\]

for all \( t \in J_{\psi^r} \). We recall that \( \tilde{C} \geq 1 \) is the constant introduced in lemma 5.2 depending on \( n, \sigma, Y \) and \( \Lambda \). Multiplying both sides of the latter by \( \frac{1}{a(t)} \) and taking the sup for all \( t \in J_{\psi^r} \), we obtain

\[
\left| Dv \Omega - D_{(u,y)} F_2 (x,y) \right|_{\sigma,a} \leq \frac{1}{4C} \| y \|_{\sigma,a,b,\omega}.
\]

This concludes the proof of this lemma because, thanks to lemma 5.2, the unique solution of (5.35) exists and by (5.40), (5.41)

\[
|\tilde{\psi}_{\sigma,b,\omega}^r | \leq \tilde{C} \| m_0 v - Dv \Omega - D_{(u,y)} F_1 (b,y) \|_{\sigma,b} \leq \frac{1}{2} | y \|_{\sigma,a,b,\omega} + \tilde{C} \| Dv \Omega - D_{(u,y)} F_2 (x,y) \|_{\sigma,a} \leq \frac{1}{2} | y \|_{\sigma,a,b,\omega} \leq \frac{1}{4C} \| y \|_{\sigma,a,b,\omega}.
\]

Thanks to the above lemma, one can verify that \( \mathcal{L}(D^r_{\sigma,a,b,\omega} \mathcal{A}) \subset D^r_{\sigma,a,b,\omega,4\tilde{C}} \) and \( \mathcal{L} \) restricted to \( D^r_{\sigma,a,b,\omega,4\tilde{C}} \) is a contraction. We refer to (5.19) for the definition of \( D^r_{\sigma,a,b,\omega,4\tilde{C}} \). This proves the existence of a \( C^r \)-asymptotic KAM torus \( \psi^r \) associated to \((X_H, X_{\psi^r}, \psi_0)\). Moreover, by proposition 3.5, the found \( C^r \)-asymptotic KAM torus \( \psi^r \) is Lagrangian. This concludes the proof of theorem A.

### 6. Proof of corollary A

The proof is essentially the same as that of theorem A. Because of that, we will not give all the details. However, we will provide the necessary elements to reconstruct the proof.

We are looking for a \( C^r \)-asymptotic KAM torus \( \psi^r \) associated to \((Z, \omega, \text{Id})\), where \( Z \) is the vector field defined by (2.8). This means that, for given \( Z \), we are searching for \( \psi^r \geq 0 \) sufficiently large and a suitable function \( u : \mathbb{T}^n \times J_{\psi^r} \rightarrow \mathbb{R}^n \) such that

\[
\psi : \mathbb{T}^n \times J_{\psi^r} \rightarrow \mathbb{T}^n, \quad \psi(q,t) = q + u(q,t)
\]

is a family of diffeomorphisms of the torus satisfying

\[
Z(\psi(q,t), t) - \partial_q \psi(q,t) \omega - \partial_t \psi(q,t) = 0,
\]

\[
\lim_{t \to +\infty} |u(t)| = 0
\]
for all \((q, t) \in \mathbb{T}^n \times J_{\nu'}\). We will choose \(\nu'\) sufficiently large in lemma 6.1. Similarly to the proof of theorem A, we introduce a suitable functional \(\mathcal{F}\) given by (6.1). To this end, one can see that the composition of \(Z\) with \(\tilde{\psi}\) equals
\[
Z \circ \tilde{\psi} (q, t) = \omega + P \circ \tilde{\psi} (q, t)
\] (6.3)
and, on the other hand,
\[
\partial_q \tilde{\psi} (q, t) \omega + \partial_t \tilde{\psi} (q, t) = \omega + \partial_q u (q, t) \omega + \partial_t u (q, t)
\] (6.4)
for all \((q, t) \in \mathbb{T}^n \times J_{\nu'}\). We refer to (4.1) for the notation \(\tilde{\psi}\). Thanks to (6.3) and (6.4), we can rewrite (6.1) in the following form
\[
P \circ \tilde{\psi} - D_u \Omega = 0,
\] (6.5)
where we have omitted the arguments \((q, t)\). Furthermore, we refer to (4.2) for the notation \(D_u \Omega\). Given \(\sigma, \omega\) and \(P\) as in (2.8), and for \(\nu'\) large enough, we define the following functional obtained by (6.5)
\[
\mathcal{F} : S_0^{\nu'} P \times T_0^{\nu'} P, \omega \to S_0^{\nu'} P, \quad \mathcal{F} (P, u) = P \circ \tilde{\psi} - D_u \Omega.
\]
We observe that \(\mathcal{F} (0, 0) = 0\), and we reformulate the dynamical problem in the following form. For \(P \in S_0^{\nu'} P\) sufficiently close to 0, we are looking for \(u \in T_0^{\nu'} P, \omega\) in such a way that \(\mathcal{F} (P, u) = 0\).

One can see that \(\mathcal{F}\) is well defined, continuous, and differentiable with respect to \(u\) with \(D_u \mathcal{F} (P, u)\) continuous. Furthermore, \(D_u \mathcal{F} (P, u)\) evaluated at \((0, 0)\) is equal to

\[
D_u \mathcal{F} (0, 0) : T_0^{\nu'} P, \omega \to S_0^{\nu'} P, \quad D_u \mathcal{F} (0, 0) \hat{u} = - D_u \Omega.
\] (6.6)

As a straightforward application of lemma 5.1, one can see that (6.6) is invertible. Then, \(\mathcal{F}\) satisfies the hypotheses of the implicit function theorem.

Now, we fix \(P\) as in (2.8), and we introduce the following quasi-Newton operator in order to avoid a smallness assumption on the perturbation \(P\)
\[
\mathcal{L} (P, \cdot) : T_0^{\nu'} P, \omega \to T_0^{\nu'} P, \quad \mathcal{L} (P, u) = u - D_u \mathcal{F} (0, 0) \mathcal{F} (P, u).
\]

Now, the proof of corollary A is reduced to find a fixed point of the latter. To this end, we have the following lemma, whose proof is omitted because similar to lemma 5.3.

**Lemma 6.1.** We have the following estimate
\[
|\mathcal{L} (P, 0)|_{\nu'} P, \omega \leq 1.
\]
Moreover, there exists \(\nu'\) large enough with respect to \(n, \sigma\) and \(P\), such that, for all \(u, u \in T_0^{\nu'} P, \omega\) with \(|u|_{\nu'} P, \omega \leq 2\),
\[
|D_u \mathcal{L} (P, u)|_{\nu'} P, \omega \leq \frac{1}{2} |u|_{\nu'} P, \omega.
\]

The previous lemma proves the existence of a unique fixed point for the operator \(\mathcal{L}\) restricted to the elements \(u \in T_0^{\nu'} P, \omega\) such that \(|u|_{\nu'} P, \omega \leq 2\).

**7. Proof of theorem B**

The proof of this theorem is similar to that of theorem A. We are looking for an analytic asymptotic KAM torus \(\varphi'\) associated to \((X_H, X_\tilde{h}, \varphi_0)\), where \(H\) is the Hamiltonian in (4.11), \(\tilde{h}\) is defined by (2.7), and \(\varphi_0\) the trivial embedding \(\varphi_0 : \mathbb{T}^n \to \mathbb{T}^n \times B, \varphi_0 (q) = (q, 0)\). More
specifically, for given $H$, we are searching for $\nu' \geq 0$ sufficiently large and suitable functions $u, v : \mathbb{T}^n \times J_{\nu'} \to \mathbb{R}^n$ such that
\[
\varphi : \mathbb{T}^n \times J_{\nu'} \to \mathbb{T}^n \times B, \quad \varphi(q, t) = (q + u(q, t), v(q, t))
\]
is a family of real-analytic embedded tori verifying the following conditions for all $(q, t) \in \mathbb{T}^n \times J_{\nu'}$,
\[
X_H(\varphi(q, t), t) - \partial_q \varphi(q, t) \omega - \partial_t \varphi(q, t) = 0,
\]
\[
\lim_{t \to +\infty} |u'|_2 = 0, \quad \lim_{t \to +\infty} |v'|_2 = 0,
\]
where $s$ and $\omega$ are defined in (4.11), and we recall that $|\cdot|_s$ is the analytic norm (see appendix B). We will choose $\nu'$ large enough in lemma 7.4 (contrary to the proof of theorem A, $\nu'$ will already be required to be large in lemma 7.1).

Let $s, \gamma, \Lambda, \omega, a, b, m, \text{and } \bar{m}$ be as in (4.11). We refer to (5.4) for the definition of $\bar{m}$. Let $\bar{C}$ be the constant depending on $s, \gamma$ and $\Lambda$ introduced by lemma 7.3 below. We consider $\nu'$ sufficiently large and we define the following subspace $\mathcal{X}_{\nu', a, b, \bar{C}, \bar{m}}$ such that
\[
\mathcal{X}_{\nu', a, b, \bar{C}, \bar{m}} = \left\{ (a, b, u, v) \in \mathcal{A}_{\nu', a, b} \times \mathcal{A}_{\nu', b} \times \mathcal{W}_{\nu', \bar{C}, \bar{m}} : \right. \\
\left. |\partial_q a|_{\nu', a} \leq 1, \quad |b|_{\nu', \bar{C}, \bar{m}} \leq 1, \quad |u|_{\nu', \bar{C}, \bar{m}} \leq 4\bar{C}, \quad |v|_{\nu', \bar{C}, \bar{m}} \leq 4\bar{C} \right\}.
\]
We refer to section 4 for the Banach spaces employed in the previous definition, especially (4.7), (4.8), and (4.10). Similarly to the proof of theorem A, let $\mathcal{F}$ be the following functional
\[
\mathcal{F} : \mathcal{X}_{\nu', a, b, \bar{C}, \bar{m}} \longrightarrow \mathcal{A}_{\nu', \bar{C}, \bar{m}}, \quad \mathcal{F}(a, b, u, v) = (F_1(b, u, v), F_2(a, b, u, v))
\]
with
\[
F_1(b, u, v) = b \circ \bar{U} + (\bar{m} \circ \bar{\varphi}) v - \text{D}u \Omega, \\
F_2(a, b, u, v) = \partial_q a \circ \bar{U} + (\partial_q b \circ \bar{U}) v + (\partial_q m \circ \bar{\varphi}) \cdot v^2 + \text{D}v \Omega.
\]
We refer to (5.8) for the definition of $U$, to (4.1) for the notation $\bar{U}$ (or $\bar{\varphi}$), and to (4.2) for the definition of $\text{D}u \Omega$ (or $\text{D}v \Omega$). We point out that the previous functional is the same as that used in the proof of theorem A. It is obtained by (7.1). The only difference lies in the choice of the Banach spaces. Contrary to theorem A, we have to define $\mathcal{F}$ on a suitable subspace $\mathcal{X}_{\nu', a, b, \bar{C}, \bar{m}}$ because we have to control the domain of analyticity of the components of $\mathcal{F}$.

In order to prove that $\mathcal{F}$ is well defined, we need the following lemma which imposes the first restriction on $\nu'$. We will take a stronger one after.

**Lemma 7.1.** For $\nu'$ large enough with respect to $n, s, \gamma, \Lambda$ and $b$, if $(u, v) \in \mathcal{W}_{\nu', \bar{C}, \bar{m}} \times \mathcal{W}_{\nu', \bar{C}, \bar{m}}$ with $|u|_{\nu', \bar{C}, \bar{m}} \leq 4\bar{C}$ and $|v|_{\nu', \bar{C}, \bar{m}} \leq 4\bar{C}$, then
\[
\sup_{t \in J_{\nu'}} |u'|_2 \leq \frac{s}{8}, \quad \sup_{t \in J_{\nu'}} |v'|_2 \leq \frac{s}{8}.
\]

**Proof.** If $|u|_{\nu', \bar{C}, \bar{m}} \leq 4\bar{C}$ and $|v|_{\nu', \bar{C}, \bar{m}} \leq 4\bar{C}$, then by (2.6)
\[
|u'|_2 \leq 4\bar{C} \bar{b} (t) \leq 4\bar{C} \bar{b} (\nu'), \quad |v'|_2 \leq 4\bar{C} \bar{a} (t) \leq 4\bar{C} \bar{a} (\nu'),
\]
for all $t \in J_{\nu'}$. Now, for $\nu'$ large enough, we have the claim. \(\square\)
Thanks to the previous lemma, the properties in appendix B and (2.6), one can prove that \( \mathcal{F} \) is well defined. Moreover, it is continuous, differentiable with respect to the variables \((u, v)\) and \(D_{(u,v)}\mathcal{F}\) is continuous. We recall that \(D_{(u,v)}\mathcal{F}\) stands for the differential with respect to \((u, v)\).

Furthermore, \(D_{(u,v)}\mathcal{F}\) evaluated at \((0, 0, 0, 0)\) equals

\[
\begin{align*}
D_{(u,v)}\mathcal{F}(0, 0, 0, 0) &= \mathcal{W}^\nu_{a,b,\omega} \times \mathcal{W}^\nu_{a,b,\omega} \\
D_{(u,v)}\mathcal{F}(0, 0, 0, 0) &= (\hat{u}, \hat{v}) = (\hat{m} \hat{v} - \hat{D} \hat{\omega}, \hat{D} \hat{\omega}) \quad \text{in (7.4)} .
\end{align*}
\]

In order to prove that \(D_{(u,v)}\mathcal{F}(0, 0, 0, 0)\) is invertible, we need to solve the following equation.

Given \(s_0 > 0, v_0 > 0\) and \(\omega_0 \in \mathbb{R}^t\), we are looking for a solution of the following equation for the unknown \(\lambda: T^n_{s_0} \times J_{v_0} \rightarrow \mathbb{R}^t\)

\[
\left\{ \begin{array}{l}
\omega_0 \cdot \partial_t \lambda(q, t) + \partial_q \lambda(q, t) = g(q, t) , \\
g \in \mathcal{A}^{\nu}_{s_0} , \mathcal{F}^t ,
\end{array} \right.
\]

where \(g: J_{v_0} \rightarrow \mathbb{R}^t\) is a positive, decreasing, integrable function on \(J_{v_0}\) and \(g: T^n_{s_0} \times J_{v_0} \rightarrow \mathbb{R}^t\) is given.

**Lemma 7.2 (homological equation).** There exists a unique solution \(\lambda \in \mathcal{A}^{\nu}_{s_0}\) of (7.5) such that

\[
\lim_{t \rightarrow +\infty} |\lambda|_{C^0} = 0 ,
\]

we refer to definition 2.3 for the introduction of \(\mathcal{A}^{\nu}_{s_0}\). Moreover,

\[
|\lambda|^{\nu}_{s_0, \mathcal{F}} \leq |g|^{\nu}_{s_0, \mathcal{F}} .
\]

**Proof.** The proof of this lemma is essentially the same as that of lemma 5.1.

Now, we have everything we need to prove that the differential \(D_{(u,v)}\mathcal{F}(0, 0, 0, 0)\) in (7.4) is invertible. It is the subject of the following lemma, whose proof is omitted because similar to that of lemma 5.2

**Lemma 7.3.** For all \((z, g) \in \mathcal{A}^{\nu}_{s_0} \times \mathcal{A}^{\nu}_{s_0}\) there exists a unique \((\hat{u}, \hat{v}) \in \mathcal{W}^{\nu}_{s_0} \times \mathcal{W}^{\nu}_{s_0}\) such that

\[
D_{(u,v)}\mathcal{F}(0, 0, 0, 0) (\hat{u}, \hat{v}) = (z, g) .
\]

Moreover, there exists a suitable constant \(C \geq 1\) depending on \(n, \nu, \mathcal{F}, \) and \(\Lambda\) such that

\[
|\hat{u}|^{\nu}_{s_0, \mathcal{F}} \leq C \left( |u|^{\nu}_{s_0, \mathcal{F}} + |z|^{\nu}_{s_0, \mathcal{F}} \right) ,
\]

\[
|\hat{v}|^{\nu}_{s_0, \mathcal{F}} \leq C \left( |v|^{\nu}_{s_0, \mathcal{F}} + |g|^{\nu}_{s_0, \mathcal{F}} \right) ,
\]

where \(\Lambda\) is the constant in (2.6) and \(\mathcal{F}\) is defined by (4.11).

**Proof.** The proof relies on lemma 7.2 and the properties in appendix B.

The functional \(\mathcal{F}\) satisfies the hypotheses of the implicit function theorem. Now, we define the following Banach space \((\mathcal{Y}^{\nu}_{s_0} \times a, b, \omega, ||\cdot||^{\nu}_{s_0} a, b, \omega}) such that

\[
\mathcal{Y}^{\nu}_{s_0} a, b, \omega} = \left\{ y = (u, v) \in \mathcal{W}^{\nu}_{s_0} \times \mathcal{W}^{\nu}_{s_0, \mathcal{F}} \mid |u|^{\nu}_{s_0, \mathcal{F}} \leq 4C , \quad |v|^{\nu}_{s_0, \mathcal{F}} \leq 4C \right\}
\]

with the norm \(||y||^{\nu}_{s_0, a, b, \omega} = \max\{ |u|^{\nu}_{s_0, a, \omega}, |v|^{\nu}_{s_0, a, \omega} \} \) for all \((u, v) \in \mathcal{Y}^{\nu}_{s_0} a, b, \omega)\). We fix \(x = (a, b) \in \mathcal{A}^{\nu}_{s_0} \times \mathcal{A}^{\nu}_{s_0}\), where \(a\) and \(b\) are those in (4.11). Furthermore, we introduce the following quasi-Newton operator

\[
\mathcal{L}_0 (x, y) = y - D_{(u,v)}\mathcal{F}(0, 0, 0, 0)^{-1} \mathcal{F}(x, y) ,
\]

(7.6)
Similarly to the proof of theorem A, the proof of theorem B is reduced to find a fixed point of (7.6). To this end, we have the following lemma, whose proof is essentially the same of that of lemma 5.3.

**Lemma 7.4.** Let \( C \) be the constant introduced by lemma 7.3. Then,

\[
\|\mathcal{L}(x,0)\|_{\frac{1}{2},a,b,\omega} \leq 2C. \tag{7.7}
\]

Moreover, there exists \( \nu' \) large enough with respect to \( s, \mathcal{Y}, \Lambda \) and \( b \), such that, for all \( y, y' \in \mathcal{Y}_{\frac{1}{2},a,b,\omega} \) with \( \|y\|_{\frac{1}{2},a,b,\omega} \leq 4C \),

\[
\|D_1\mathcal{L}(x,y)\|_{\frac{1}{2},a,b,\omega} \leq \frac{1}{2} \|y\|_{\frac{1}{2},a,b,\omega}. \tag{7.8}
\]

Thanks to the previous lemma, (7.6) is well-defined, and it is a contraction. This concludes the proof of theorem B.

**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

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**Appendix A. Hölder classes of functions**

This part is dedicated to recalling the definition of Hölder classes of functions and some well-known properties. To this end, let \( E \) be an open subset of \( \mathbb{R}^n \) and \( k \geq 0 \) a positive integer. We denote by \( C^k(E) \) the spaces of functions \( f : E \rightarrow \mathbb{R} \) with continuous partial derivatives \( \partial^\alpha f \in C^k(E) \) for all \( \alpha \in \mathbb{N}^n \) with \( |\alpha| = \alpha_1 + \cdots + \alpha_n \leq k \). Moreover, we define the following norm for all \( f \in C^k(E) \)

\[
|f|_{C^k} = \sup_{|\alpha| \leq k} |\partial^\alpha f|_{C^0},
\]

where \( |\partial^\alpha f|_{C^0} = \sup_{x \in E} |\partial^\alpha f(x)| \). Given \( \sigma = k + \mu \), with \( k \in \mathbb{Z} \) and \( 0 < \mu < 1 \), we define the Hölder spaces \( C^\sigma(E) \) as the spaces of functions \( f \in C^k(E) \) verifying

\[
|f|_{C^\sigma} = \sup_{|\alpha| \leq k} |\partial^\alpha f|_{C^0} \sup_{x,y \in E, x \neq y} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^\mu} < \infty. \tag{A.1}
\]

We will use the same notations for vector-valued functions or matrices. More specifically, in the case of functions \( f = (f_1, \ldots, f_n) \) with values in \( \mathbb{R}^n \), we set \( |f|_{C^\sigma} = \max_{1 \leq i \leq n} |f_i|_{C^\sigma} \). Moreover, in agreement with the convention made above, if \( M = \{m_{ij}\}_{1 \leq i,j \leq n} \) is a \( n \times n \) matrix, we set \( |M|_{C^\sigma} = \max_{1 \leq i,j \leq n} |m_{ij}|_{C^\sigma} \).
In what follows, we present a series of properties widely used in this paper. First, we recall that $C(\cdot)$ stands for constants depending on $n$ and other parameters in brackets. The following proposition provides a convexity property of the above-mentioned norms.

**Proposition A.1.** For all $f \in C^\sigma(E)$, then

$$ |f|_{C^{\sigma_0}}^{\sigma_0} \leq C(\sigma_1) |f|_{C^{\sigma_0}}^{\sigma_1} |f|_{C^{\sigma_0}}^{\sigma_0} \quad \text{for all } 0 \leq \sigma_0 \leq \sigma \leq \sigma_1. $$

**Proof.** We refer to [4] for the proof. $\square$

The next proposition deals with the composition and product of Hölder functions.

**Proposition A.2.** We consider $f, g \in C^\sigma(E)$ and $\sigma \geq 0$.

1. For all $\beta \in \mathbb{N}^n$ and $s \geq 0$, if $|\beta| + s \leq \sigma$ then $\left| \frac{\partial^{|\beta|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} f \right|_{C^s} \leq C(f) |f|_{C^\sigma}$.
2. $|fg|_{C^\sigma} \leq C(\sigma) (|f|_{C^\sigma} |g|_{C^\sigma} + |f|_{C^\sigma} |g|_{C^0})$.

Now, we consider composite functions. Let $E_1$ be an open subset of $\mathbb{R}^n$ and $z : E_1 \to E$ a function taking values in the domain of $f$. In what follows $\partial z$ stands for the partial derivatives of $z$.

3. If $\sigma < 1$, $f \in C^1(E)$, $z \in C^\sigma(E_1)$ then $f \circ z \in C^\sigma(E_1)$ and

$$ |f \circ z|_{C^\sigma} \leq C(|f|_{C^1} |z|_{C^\sigma} + |f|_{C^0}). $$

4. If $\sigma < 1$, $f \in C^\sigma(E)$, $z \in C^1(E_1)$ then $f \circ z \in C^\sigma(E_1)$

$$ |f \circ z|_{C^\sigma} \leq C(|f|_{C^\sigma} |\partial z|_{C^0} + |f|_{C^0}). $$

5. If $\sigma \geq 1$ and $f \in C^\sigma(E)$, $z \in C^\sigma(E_1)$ then $f \circ z \in C^\sigma(E_1)$

$$ |f \circ z|_{C^\sigma} \leq C(\sigma) (|f|_{C^\sigma} |\partial z|_{C^\sigma} + |f|_{C^0} |\partial z|_{C^\sigma-1} + |f|_{C^0}). $$

**Proof.** We refer to [4] for the proof of (1), (2), and (3). The property (4) is straightforward. We prove (5). By (A.1),

$$ |f \circ z|_{C^\sigma} \leq |f|_{C^\sigma} + |(\partial f \circ z)^T \partial z|_{C^{\sigma-1}}, $$

where $\partial f$ stands for the partial derivatives of $f$, and $T$ for the transpose. Thanks to property (2)

$$ |f \circ z|_{C^\sigma} \leq |f|_{C^\sigma} + |(\partial f \circ z)^T \partial z|_{C^{\sigma-1}} \leq |f|_{C^\sigma} + C(\sigma) |\partial f \circ z|_{C^{\sigma-1}} |\partial z|_{C^\sigma} + C(\sigma) |\partial f \circ z|_{C^\sigma} |\partial z|_{C^{\sigma-1}}. $$

We observe that $|\partial f \circ z|_{C^\sigma} |\partial z|_{C^{\sigma-1}} \leq |f|_{C^\sigma} |\partial z|_{C^{\sigma-1}}$, it remains to estimate $|\partial f \circ z|_{C^{\sigma-1}} |\partial z|_{C^\sigma}$. If $\sigma \leq 2$, $|\partial f \circ z|_{C^{\sigma-1}} \leq |f|_{C^\sigma} |\partial z|_{C^{\sigma-1}} + |f|_{C^0}$, thanks to (4). Then

$$ |\partial f \circ z|_{C^{\sigma-1}} |\partial z|_{C^\sigma} \leq C(\sigma) (|f|_{C^\sigma} |\partial z|_{C^\sigma} + |f|_{C^0} |\partial z|_{C^\sigma}) \leq C(\sigma) (|f|_{C^\sigma} |\partial z|_{C^\sigma} + |f|_{C^0} |\partial z|_{C^{\sigma-1}}), $$

whence the property holds in this case. If $\sigma > 2$, assuming that (5) is already proven for $\sigma - 1$, we find

$$ |\partial f \circ z|_{C^{\sigma-1}} |\partial z|_{C^\sigma} \leq C(\sigma) (|\partial f|_{C^{\sigma-1}} |\partial z|_{C^\sigma} + |f|_{C^\sigma} |\partial z|_{C^{\sigma-1}} |\partial z|_{C^\sigma} + |f|_{C^0} |\partial z|_{C^\sigma}) \leq C(\sigma) (|f|_{C^\sigma} |\partial z|_{C^\sigma} + |f|_{C^0} |\partial z|_{C^{\sigma-1}}). $$
It remains to find a good estimate for the central term of the last line of the latter. By proposition A.1
\[ |f_c| |\partial z| |\partial c| \leq C(\sigma) \left( |f|_{c}^{|\frac{n}{2}|} |\partial z|_{c}^{|\frac{n}{2}|} \right) \left( |\partial z|_{c}^{|\frac{n}{2}|} |\partial z|_{c^{-1}}^{|\frac{n}{2}|} \right) |\partial z|_{c}^{|\frac{n}{2}|} \leq C(\sigma) (|f|_{c} |\partial z|_{c^{-1}})^{\frac{n}{2}} \left( |f|_{c}^{\sigma} |\partial z|_{c}^{\sigma} \right)^{\frac{1}{2}} , \]

since \(a^{\lambda} b^{1-\lambda} \leq C(a + b)\) for \(0 < \lambda < 1\), we have the claim. \(\square\)

Appendix B. Real analytic classes of functions

This section will collect some well-known facts about real analytic functions. For some \(s > 0\), we begin with the introduction of complex domains
\[ T_s^n := \{ q \in C^n/\mathbb{Z}^n : |\text{Im}(q)| \leq s \}, \quad B_s := \{ p \in C^n : |p| \leq s \}, \]
with \(T^n = \mathbb{R}^n/\mathbb{Z}^n\) and \(B \subset \mathbb{R}^n\) a sufficiently large neighbourhood of the origin. Let \(D\) be equal to \(T^n \times B\) or \(T^n\) and we consider a real analytic function in a neighbourhood of \(D\)
\[ f : D \rightarrow \mathbb{R}. \]
Let \(D_s\) be equal to \(T^n \times B_s\) or \(T^n_s\), for a suitable small \(s\). It is known that \(f\) extends to a function
\[ f : D_s \rightarrow \mathbb{C} \]
that is real, holomorphic and bounded. We define the following norm
\[ |f|_s = \sup_{z \in B_s} |f(z)|. \]
In the case of vector-valued functions \(f = (f_1, \ldots, f_n)\) with values in \(C^n\), we set \(|f|_s = \max_i |f_i|_s\).
Moreover, if \(C = \{ C_{ij} \}_{1 \leq i, j \leq n}\) is an \(n \times n\) matrix, we let \(|C|_s = \max_i |C_{ij}|_s\). We define \(A_s\) as the space of such functions. The rest of this section is devoted to a series of general well-known properties.

**Proposition B.1.** Let \(f, g \in A_s\), then the product \(fg \in A_s\) and
\[ |fg|_s \leq |f|_s |g|_s. \]
Let \(f \in A_s\) and \(0 \leq \sigma \leq s\). Then \(\partial^\sigma f \in A_s\) and we have
\[ |\partial^\sigma f|_s \leq \frac{1}{\sigma} |f|_s. \]
Let \(f \in A_s\), \(0 \leq \sigma \leq s\) and \(\phi \in A_{s-\sigma}\) such that \(\phi : D_{s-\sigma} \rightarrow D_s\). Then \(f \circ \phi \in A_{s-\sigma}\) and
\[ |f \circ \phi|_{s-\sigma} \leq |f|_s. \]

We conclude this part with a very important property concerning the function \(f \in A^n_s\). We recall that the space of function \(A^n_s\) is introduced by definition 2.3.
For all \(k \in \mathbb{N}^n\) with \(|k| \geq 1\), we recall that \(\partial^k_{(q,p)} = \partial^{k_1}_{q_1} \cdots \partial^{k_n}_{p_n} \partial^k_{q_1} \cdots \partial^k_{p_n}\), where \(|k| = |k_1| + \cdots + |k_n|\).

**Proposition B.2.** For fixed \(s > 0\) and \(v \geq 0\), let \(f \in A^n_s\). Then, for all \(k \in \mathbb{N}^n\) with \(|k| \geq 1\), \(\partial^k_{(q,p)} f \in A^n_s\), for all \(0 < s' < s\).

**Proof.** The proof follows from Cauchy’s estimates. \(\square\)
Appendix C. Example of non-existence of a $C^\omega$-asymptotic KAM torus

In this section, we provide an example of a time-dependent vector field on the torus that does not admit a $C^\omega$-asymptotic KAM torus. To this end, let $Z$ be a time-dependent vector field on $T^1 \times J_0$ of the form

$$\hat{Z}(q,t) = \omega + \hat{P}(t),$$

where $\omega \in \mathbb{R}$ and $\hat{P}(t)$ is a continuous function such that $\hat{P}(t) > 0$ for all $t > 0$. We assume that

$$\int_{t_0}^{+\infty} \hat{P}(\tau) \, d\tau = +\infty,$$  \hspace{1cm} (C.1)

for all $t_0 \geq 0$. More specifically, the integrability assumption on the perturbative term in (2.8) fails.

Let $\psi^t_{b,Z}$ be the flow at time $t$ with initial time $t_0$ of $\hat{Z}$. Then, by the fundamental theorem of calculus, one has that for all $q \in \mathbb{T}^n$ and $t > 0$

$$\psi^{b+t}_{b,Z}(q) = q + \omega t + \int_{t_0}^{b+t} \hat{P}(\tau) \, d\tau.$$  \hspace{1cm} (C.2)

Now, we assume the existence of $\nu \geq 0$ and a $C^\omega$-asymptotic KAM torus $\varphi^t$ associated to $(\hat{Z}, \text{Id}, \omega)$ defined for all $t \in J_\nu$. Then, for all fixed $q \in \mathbb{T}^n$ and $t_0 \in J_0$, by proposition 3.4 one has that $\psi_{b,Z}^{b+t} \circ \varphi^t(q)$ is an asymptotically quasiperiodic solution associated to $(\hat{Z}, \text{Id}, \omega)$. Therefore, thanks to (C.2)

$$|\psi_{b,Z}^{b+t} \circ \varphi^t(q) - q - \omega t| = \left| \int_{t_0}^{b+t} \hat{P}(\tau) \, d\tau - (q - \varphi^t(q)) \right| \geq \int_{t_0}^{b+t} \hat{P}(\tau) \, d\tau - |q - \varphi^t(q)|$$  \hspace{1cm} (C.3)

for all $(q, t_0) \in \mathbb{T}^n \times J_\nu$ and $t > 0$. Thanks to (C.1), the last line of the latter diverges to $+\infty$ if $t \to +\infty$. This contradicts the definition of asymptotically quasiperiodic solutions (see definition 3.1). Hence, we have an absurd.

Appendix D. Banach spaces

Here, we prove that the normed spaces $(S_{(\sigma,q),u}^v, | \cdot |_{v+1,k,u})$ and $(T_{(\sigma,q),u}^v, | \cdot |_{v,u})$, introduced in section 4 (see (4.5) and (4.9), respectively) are Banach spaces. For the sake of clarity, let us recall their definition. Given $\sigma \geq 1$, $v > 0$, $\omega \in \mathbb{R}^n$, a positive integer $k \geq 0$, and a positive decreasing integrable function $u : J_0 \to \mathbb{R}^+$

$$S_{(\sigma,q),u}^v = \left\{ f : \mathbb{T}^n \times J_\nu \to \mathbb{R} \mid f \in S_{(\sigma,q)}^v \text{ and } |f|_{\sigma+1,k,u}^v < \infty \right\}$$

and

$$T_{(\sigma,q),u}^v = \left\{ f : \mathbb{T}^n \times J_\nu \to \mathbb{R}^n \mid f, D\Omega \in S_{(\sigma,q)}^v \text{ and } |f|_{\sigma,u}^v \leq \max \{ |f|_{\sigma,u}^v, |D\Omega|_{\sigma,u}^v \} < \infty \right\}$$

we refer to definition 2.2 for the space $S_{(\sigma,q)}^v$, to (2.4) for the definition of $\bar{u}$, to (4.3) for the one of the norm $| \cdot |_{v+1,k,u}$, and to (4.2) for the notation $D\Omega$. We prove that the latter are complete. Similarly, one can see that the other normed spaces in section 4 are Banach spaces.
Now, let us consider \((S^v_{\sigma,k})_u \setminus |v|_{\sigma+k,u}\). In the first part of this section we want to prove that it is complete. For this purpose, let \(\{f_d\}_{d \in \mathbb{N}} \subseteq S^v_{\sigma,k,u}\) be a Cauchy sequence. This means that, for all \(\varepsilon > 0\) there exists \(D \in \mathbb{N}\) such that for all \(d, m \geq D\), \(|f_d - f_m|^v_{\sigma+k,u} < \varepsilon\). For all fixed \(t \in J_v\), we claim that the sequence \(\{f_d^{(t)}\}_{d \in \mathbb{N}}\) contained in the Banach space \((C^{\sigma+k}(T^n), |\cdot|_{C^{\sigma+k}})\) is a Cauchy sequence. In fact, for all fixed \(t \in J_v\)

\[
|f_d^{(t)} - f_m^{(t)}|_{C^{\sigma+k}} \leq u(t) \frac{|f_d - f_m|_{C^{\sigma+k}}}{u(t)} \leq u(t) |f_d - f_m|^v_{\sigma+k,u}.
\]

This proves the claim and hence, for all fixed \(t \in J_v\), the existence of \(f^* \in C^{\sigma+k}\) such that \(\lim_{d \to +\infty} |f_d^{(t)} - f^*|^v_{\sigma+k,u} = 0\). Now, the proof of the completeness of \((S^v_{\sigma,k,u})_u \setminus |v|_{\sigma+k,u}\) is reduced to verify that the found \(f^*\) satisfies \(f^* \in S^v_{\sigma,k,u}\) and \(\lim_{d \to +\infty} |f_d - f^*|^v_{\sigma+k,u} = 0\). The proof is divided into the following three parts. First, we prove that \(f \in S^v_{\sigma,k}\), then \(\lim_{d \to +\infty} |f_d - f|^v_{\sigma+k,u} = 0\) and finally we show that \(|f|^v_{\sigma+k,u} < \infty\).

Proof of \(f \in S^v_{\sigma,k}\). Obviously, for all fixed \(t \in J_v\), \(f^* \in C^{\sigma+k}(T^n)\). It remains to verify that \(\partial_\tau f^* \in C(T^n \times J_v)\) for all \(0 \leq |\tau| \leq k\) with \(i \in \mathbb{N}^n\). To this end, for all \((q_1,t_1), (q_2,t_2) \in \mathbb{T}^n \times J_v\), and \(0 \leq |\tau| \leq k\) with \(i \in \mathbb{N}^n\)

\[
|\partial_\tau f^*(q_1) - \partial_\tau f^*(q_2)| \leq |\partial_\tau f_{\sigma,k}(q_1) - \partial_\tau f_{\sigma,k}(q_2)| + |\partial_\tau f_{\sigma,k}(q_1) - \partial_\tau f_{\sigma,k}(q_2)|
\]

Now, we observe that, for all \(\varepsilon > 0\) there exists \(D \in \mathbb{N}\) such that, for all \(d \geq D\), the first and the last term on the right-hand side of the latter are smaller than \(\frac{\varepsilon}{2}\). This is because, for all fixed \(t \in J_v\), \(f_{\sigma,k}\) converges to \(f^*\) in the norm \(C^{\sigma+k}\). Concerning the second term, we know that \(f_{\sigma,k} \in S^v_{\sigma,k}\). Hence, \(\partial_\tau f_{\sigma,k} \in C(T^n \times J_v)\) for all \(0 \leq |\tau| \leq k\) with \(i \in \mathbb{N}^n\). Then, there exists \(\delta > 0\) such that if \(|(q_1,t_1) - (q_2,t_2)| < \delta\) the second term on the right-hand side of the latter is smaller than \(\frac{\varepsilon}{2}\). This proves this first part.

Proof of \(\lim_{d \to +\infty} |f_d - f|^v_{\sigma+k,u} = 0\). Let \(f_{\sigma,k}\) be a subsequence of \(f_d\) such that

\[
|f_{\sigma,k+1} - f_{\sigma,k}|^v_{\sigma+k,u} \leq \left(\frac{1}{2}\right)^d
\]

for all \(d \in \mathbb{N}\). We claim that it suffices to prove \(\lim_{d \to +\infty} |f_{\sigma,k} - f|^v_{\sigma+k,u} = 0\). Indeed, if we assume that \(\lim_{d \to +\infty} |f_{\sigma,k} - f|^v_{\sigma+k,u} = 0\), then, for all \(d \in \mathbb{N}\), one has

\[
|f - f_{\sigma+k,u}^v| + |f_{\sigma,k} - f|^v_{\sigma+k,u}.
\]

We observe that, for all \(\varepsilon > 0\), there exists \(D \in \mathbb{N}\) such that \(|f_d - f_{\sigma,k}|^v_{\sigma+k,u} < \frac{\varepsilon}{2}\) and \(|f_{\sigma,k} - f|^v_{\sigma+k,u} < \frac{\varepsilon}{2}\) for all \(d \geq D\). The first estimate is a consequence of the fact that \(f_d\) is a Cauchy sequence. On the other hand, \(f_{\sigma,k}\) converges to \(f\) in the norm \(|\cdot|^v_{\sigma+k,u}\), that implies the second estimate. This proves \(\lim_{d \to +\infty} |f_d - f|^v_{\sigma+k,u} = 0\) and hence the claim. Now, for all fixed \(t \in J_v\),

\[
|f_{\sigma,k+1} - f_{\sigma,k}|^v_{\sigma+k,u} \leq \frac{1}{2}
\]

and hence, taking the sup for all \(t \in J_v\), we obtain

\[
|f_{\sigma,k} - f|^v_{\sigma+k,u} \leq 2 \left(\frac{1}{2}\right)^d.
\]
This conclude this second part of the proof because, for every \( \varepsilon > 0 \) there exists \( D \in \mathbb{N} \) such that \( |f_{d_k} - f_{d_k}^v| < \varepsilon \) for all \( d \geq D \).

Proof of \( |f|^v_{\sigma + k, u} < \infty \). For all \( d \in \mathbb{N} \), we can estimate \( |f|^v_{\sigma + k, u} \) as follows

\[
|f|^v_{\sigma + k, u} \leq |f_d - f|^v_{\sigma + k, u} + |f_d|^v_{\sigma + k, u}.
\]

For \( d \) sufficiently large, \( |f_d - f|^v_{\sigma + k, u} < \infty \) because \( \lim_{d \to +\infty} |f_d - f|^v_{\sigma + k, u} = 0 \). Moreover, \( |f_d|^v_{\sigma + k, u} < \infty \) because \( f_d \in S^v_{(\sigma, 0)} \) for all \( d \geq 0 \). This concludes this final part of the proof.

In the second part of this section we prove that \( (T^v_{\lambda, \omega}, \cdot |_{S^v_{(\sigma, 0)}}) \) is a Banach space. Let \( \{g_d\}_{d \in \mathbb{N}} \subset T^v_{\lambda, \omega} \) be a Cauchy sequence. Similarly to the previous case, there exist \( g, f \in S^v_{(\sigma, 0)} \) such that

\[
\lim_{d \to \infty} |g_d - g|^v_{\sigma, u} = 0, \quad \lim_{d \to +\infty} |Dg_d\Omega - f|^v_{\sigma, u} = 0.
\]  

(D.1)

The proof is reduced to verify that \( Dg(q, t)\Omega = f(q, t) \) for all \( (q, t) \in \mathbb{T}^n \times J^v \). Let us denote \( z = (q, t) \) and we recall that \( \Omega = (\omega, 1) \). We will prove that, for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\left| \frac{g(z + \tau\Omega) - g(z)}{\tau} - f(z) \right| < \varepsilon
\]

for all \( |\tau| < \delta \). Thanks to the triangle inequality

\[
\left| \frac{g(z + \tau\Omega) - g(z)}{\tau} - f(z) \right| < \left| \frac{g_d(z + \tau\Omega) - g_d(z)}{\tau} - \frac{g(z + \tau\Omega) - g(z)}{\tau} \right| + \left| \frac{g_d(z + \tau\Omega) - g_d(z)}{\tau} - Dg_d(z)\Omega \right| + \left| Dg_d(z)\Omega - f(z) \right|
\]

By (D.1) there exists \( D > 0 \), depending on \( \varepsilon \) and \( \tau \), such that the first and the third terms on the right-hand side of the latter are smaller than \( \frac{\varepsilon}{2} \) for all \( d \geq D \). Now, thanks to Taylor’s formula, we can rewrite the second term on the right-hand side of the latter as follows

\[
\left| \frac{g_d(z + \tau\Omega) - g_d(z)}{\tau} - Dg_d(z)\Omega \right| = \left| \int_0^1 Dg_d(z + \tau s\Omega) - Dg_d(z)\Omega ds \right|
\]

and using the triangle inequality

\[
\left| \int_0^1 Dg_d(z + \tau s\Omega) - Dg_d(z)\Omega ds \right| \leq \int_0^1 |Dg_d(z + \tau s\Omega) - f(z + \tau s\Omega)| ds + \int_0^1 |f(z + \tau s\Omega) - f(z)| ds + \int_0^1 |f(z) - Dg_d(z)\Omega| ds.
\]

We know that \( f \) is continuous, then there exists \( \delta \) such that for all \( |\tau| < \delta \) the second term on the right-hand side of the latter is smaller than \( \frac{\varepsilon}{2} \). Since the uniform convergence of \( Dw_d\Omega \) there exists \( D > 0 \), depending on \( \varepsilon \) and \( \tau \), such that the first and the third terms on the right-hand side of the latter are smaller than \( \frac{\varepsilon}{2} \). This concludes the proof.
ORCID ID

Donato Scarcella  https://orcid.org/0000-0003-1894-3325

References

[1] Blazevski D and de la Llave R 2011 Time-dependent scattering theory for ODEs and applications to reaction dynamics J. Phys. A: Math. Theor. 44 195101
[2] Canadell M and de la Llave R 2015 KAM tori and whiskered invariant tori for non-autonomous systems Physica D 310 104–13
[3] Fortunati A and Wiggins S 2014 Persistence of Diophantine flows for quadratic nearly integrable Hamiltonians under slowly decaying aperiodic time dependence Regul. Chaotic Dyn. 19 586–600
[4] Hörmander L 1976 The boundary problems of physical geodesy Arch. Ration. Mech. Anal. 62 1–52
[5] Kawai S, Bandrauk A D, Jaffé C, Bartsch T, Palacian J and Uzer T 2007 Transition state theory for laser-driven reactions J. Chem. Phys. 126 164306
[6] Scarcella D 2022 Biassymptotically quasiperiodic solutions for time-dependent Hamiltonians (arXiv:2211.11135)
[7] Scarcella D 2022 Weakly asymptotically quasiperiodic solutions for time-dependent Hamiltonians with a view to celestial mechanics (arXiv:2211.06768)
[8] Scarcella D 2024 Asymptotic motions converging to arbitrary dynamics for time-dependent Hamiltonians Nonlinear Anal. 243 113528
[9] Thieme H and Castillo-Chavez C 1995 Asymptotically Autonomous Epidemic Models. Mathematical Population Dynamics: Analysis of Heterogeneity vol I (Wuerz Publishing)