What is the dual of a dipole?

Luis F. Alday,\textsuperscript{1} Jan de Boer,\textsuperscript{2} Ilies Messamah\textsuperscript{2}

\textsuperscript{1}Institute for Theoretical Physics and Spinoza Institute, Utrecht University
3508 TD Utrecht, The Netherlands
\textsuperscript{2}Instituut voor Theoretische Fysica,
Valckenierstraat 65, 1018XE Amsterdam, The Netherlands.

l.f.alday@phys.uu.nl, jdeboer@science.uva.nl, imessama@science.uva.nl

Abstract

We study gravitational solutions that admit a dual CFT description and carry non zero dipole charge. We focus on the black ring solution in $\text{AdS}_3 \times S^3$ and extract from it the one-point functions of all CFT operators dual to scalar excitations of the six-dimensional metric. In the case of small black rings, characterized by the level $N$, angular momentum $J$ and dipole charge $q_3$, we show how the large $N$ and $J$ dependence of the one-point functions can be reproduced, under certain assumptions, directly from a suitable ensemble in the dual CFT. Finally we present a simple toy model that describes the thermodynamics of the small black ring for arbitrary values of the dipole charge.
1 Introduction

One of the surprising features of holography in general and of the AdS/CFT correspondence in particular is that a local field theory in $d$ dimensions can look like classical gravity in $d + 1$ dimensions. A proper understanding of this relation is crucial in order to make progress in our understanding of quantum gravity. In particular, we would like to understand the map between states/ensembles in the field theory, and smooth/singular gravitational solutions in the bulk. There are several cases where we know what this map is. For example, black holes are believed to be dual to thermal ensembles in the field theory \(^1\) as both have a finite periodic Euclidean time. Other examples include the map between half-BPS states and geometries in AdS\(_3\) (see [2–4] and references therein) and AdS\(_5\) [5–10]. However, in general very little is known. We do not even know what characterizes ensembles that describe general smooth geometries or geometries with horizons.

Gravitational solutions that are asymptotically AdS have conserved charges such as mass and angular momentum. These are relatively easily described in the dual field theory, as for each conserved charge in the bulk there is a corresponding conserved charge in the boundary field theory. This is only a very small subset of the data that characterize the gravitational solution. Since the latter is asymptotically AdS, it corresponds to a normalizable deformation of AdS, and this in turn is characterized in full generality by the one-point functions of all gauge invariant operators of the dual field theory. It is in principle straightforward to extract these one-point functions from the gravitational solution, but the reverse is difficult, as one needs to integrate the equations of motion of gravity subject to the boundary conditions specified by the one-point functions. Similarly, in the boundary theory the relation between one-point functions and ensembles is known (the one point functions are simply $\langle O \rangle = \text{tr}(\rho O)$ with $\rho$ the density matrix), but to convert one-point functions into ensembles and vice-versa is practically impossible. Given one-point functions $\langle O_1 \rangle$ of operators $O_1$, the density matrix that reproduces those expectation values and has maximal entropy is

$$\rho = \exp(c + \sum a_i O_i).$$  \hfill (1.1)

Here the coefficients $c$ and $a_i$ have to be adjusted in such a way that the one-point functions come out right and such that $\rho$ is properly normalized. If the set operators $O_i$ is a finite subset of all operators and all other operators have vanishing expectation value it is not true that (1.1) should only involve the operators whose vev is nonzero. This is because it is by no means obvious that in a density matrix of the form (1.1) those other operators will indeed have a

\(^1\)There are several subtleties regarding this statement. For example, thermal ensembles can be dual to sums over geometries instead of a single geometry [1]. Also, most pure states in the thermal ensemble will presumably have a geometrical description which is similar to that of the black hole in the regime in which the supergravity description is reliable. From this point of view the black hole spacetime arises by coarse graining the underlying microstate geometries. These issues have little bearing on the results in this paper but will be mentioned whenever relevant.
vanishing expectation value, and if this happens it is purely an accident. Conversely, it may happen that many one-point functions are nonzero, but all but a finite number of $a_i$ vanish, as for example in a thermal ensemble. Such ensembles are relatively simple and one may hope that they have a corresponding simple gravitational solution, but we have little evidence to support this beyond what we find in this paper.

In this paper we will try to shed some light on these issues by considering gravitational solutions that are not just characterized by conserved charges but also carry a nonzero dipole charge. This class of solutions typically include giant gravitons such as in the Myers-Tafjord superstar [11] in AdS$_5$ and black rings in AdS$_3$ [12]. Dipole charges are not conserved and they will therefore not correspond to a simple conserved charge in the dual field theory. On the other hand, they do enter in the generalizations of the first law for solutions with dipole moments [12, 13], and as such one would expect that dipole charges can somehow be incorporated in the dual field theory. However, the potential that multiplies the dipole charge in [12, 13] involves the difference of a field at infinity and at the horizon and this field cannot be put equal to zero at horizon without introducing a singularity at the rotation axis [13]. This suggests that dipole charges should correspond to complicated, probably nonlocal, operators.

In the case of dipole charges in the 1/2-BPS sector of $\mathcal{N} = 4$ super Yang-Mills theory this is indeed the case. States in the 1/2-BPS sector of $U(N)$ $\mathcal{N} = 4$ SYM can be represented, either via a free fermion or via a free boson representation, in terms of Young diagrams with $N$ rows [14, 15]. In [9] it was argued that the number of giant gravitons appearing in the supergravity solution dual the state is given by the length of the first row of the Young diagram. The various giant gravitons carry different dipole charges (depending on how one defines this notion), so with a slight abuse of language we will call the number of giant gravitons the dipole charge of the solution. Strictly speaking we do the same in the black ring case where the word dipole charge refers to the number of dipole branes. We hope this will not cause confusion and keep on calling this the dipole charge of the system. Thus for the Young diagram the dipole charge is the length of the first row. This can be reexpressed in terms of the fields of the free fermion/boson system, but it is easy to see that it will be rather strange looking non-local operator.

The case of black rings is even less well understood, and these will be the subject of this paper. Being the first example of black objects with non-spherical horizon topology, they are also of interest in their own right. One can take a decoupling limit of black rings so that they are embedded in AdS$_3 \times S^3$ [16, 17] and therefore should be describable in terms of the D1-D5 CFT. This is sometimes called the UV CFT, not to be confused with the near-horizon IR CFT, the entropy of the black ring is most easily understood in terms of the latter. A precise understanding of the black ring in terms of the UV CFT is more problematic, and a completely convincing picture has not yet been given. In [17] a phenomenological dual description of black rings is given in terms of a condensate of strings of a fixed length together with a thermal ensemble of the remaining strings, employing the familiar short/long string picture of the D1-
D5 CFT. Unfortunately, it is not clear whether this dual picture arises as a phase in a suitable thermodynamic system, and the size of the strings in the condensate is fixed by hand by requiring that the ensemble yields the right entropy and angular momentum.

In the 1/2-BPS limit the situation is somewhat better. 1/2-BPS black rings, sometimes called small black rings, are characterized by a central charge $c = 6N$ (which is the central charge of the UV CFT), an angular momentum $J$ and a KK dipole charge $q_3$. There is a fair amount of evidence that these are dual to ensembles in the CFT that consist of a Bose-Einstein condensate of $J$ short strings with length $q_3$ and one unit of angular momentum, plus a thermal distribution of strings that make up the remaining $N - q_3J$ units of string length $[2,17–19]$. They therefore have a residual entropy of order $S \sim \sqrt{N - q_3J}$. The Bose-Einstein condensate only forms for sufficiently large $J$, typically for $J \sim N$. For $q_3 = 1$, which corresponds to the case with no dipole charge, this can be studied explicitly using a partition function $Z = \text{Tr}(e^{-\beta(H + \mu J)})$ restricted to the 1/2-BPS sector. This correctly captures the physics of the solution and indeed yields the picture described above $[18–21]$. One should in principle also be able to further verify this by comparing the correlation functions of untwisted fields in the D1-D5 CFT of $[19]$ to computations in the small black ring geometry. The thermodynamic description confirms that for large $J$ the Bose-Einstein condensate picture is correct, but for small $J$ it is not a very accurate description of the system. The thermodynamic description therefore clearly shows what the generic states carrying fixed angular momentum look like for different values of the conserved charges.

In the presence of a dipole charge, $q_3 > 1$, less is known. For $N = q_3J$, the small black ring becomes a conical defect as studied in $[22–24]$. However, a thermodynamic description of such conical defects and small black rings with $q_3 > 1$, i.e. one of the form

$$Z = \text{Tr}e^{-\beta(H + \mu J + \nu D)}$$

with $D$ a “dipole operator” is unknown. One of the aims of this paper was to find such a description. Since this system should have Bose-Einstein condensation of an excited state (strings of length larger than one), and not of the ground state, it is clear that the operator $D$ will have to be rather peculiar to achieve this.

In this paper we will first study the general black ring in its decoupling limit in which it is embedded in $\text{AdS}_3 \times S^3$. Using standard AdS/CFT machinery, we will extract the one-point functions of operators in the dual CFT in the ensemble dual to the black ring. It turns out that virtually all operators have non-trivial one-point functions which are complicated functions of the charges and dipole moments of the black ring. This is described in section 2. Since it seems rather hopeless to use these complicated results to extract useful information about the CFT, we will study 1/2-BPS black rings in section 3. They can be obtained from the full black ring solution by taking some charges equal to zero. The one-point functions can be directly extracted from the results in section 2. These small black rings have vanishing classical horizon area, but do have a residual entropy, which might become visible after including higher order
curvature corrections in the supergravity equations of motion \[18,25\]. We notice that it is possible to choose the charges of the full black ring in such a way that the solution reproduces both the one-point functions as well as the entropy of the small black rings. These solutions are therefore candidates for what the full 1/2-BPS solution in the presence of higher order corrections could look like. This could be tested further by studying subleading corrections to the entropy or one-point functions. It would also be interesting to understand whether these small extra charges can be understood as arising from some polarization effect.

In section 4 we turn to the D1-D5 CFT, which we take to be at the orbifold point. We study one-point functions of operators in the type of ensembles that are believed to be dual to black rings. In general, the calculations are way too complicated to perform, as they involve arbitrarily many twist fields in the orbifold CFT. Once we restrict to the 1/2-BPS sector the situation improves somewhat. We argue that in the large \(N\) limit correlation functions simplify and receive only contributions from certain irreducible pieces. From this we can infer, modulo some assumptions, what the leading behavior at large \(N\) of the one-point functions is, and we find agreement with the supergravity analysis. We will also find that one-point functions in conical defects are generically non-vanishing but are all suppressed by a single power of \(1/N\). This indicates that conical defect solutions will also receive corrections once higher order terms in supergravity are included.

These results all indicate that the picture of small black rings with dipole moments works quite well. Then finally in section 5 we present a simple toy model of the form \[1.2\] which correctly reproduces the physics of the small black ring. A string of length \(n\) has \(H = n, J = 1\) (if the string carries a unit of angular momentum), and our proposal is that it carries “dipole charge” \(D = 1/n\). This may seem rather awkward, but it has a couple of appealing features. First of all, it will not effect the thermal nature of the partition function at high energies. It therefore has the flavor of a non-local normalizable deformation, as expected for a dipole charge. Second, it can be easily generalized to incorporate more complicated configurations. If, for example, concentric black rings solutions exist that allow for a decoupling limit (the solutions of \[26,27\] apparently do not allow such a limit because the charges obey restrictions incompatible with the decoupling limit) one could imagine simply including further negative powers of \(n\) in the partition function. Finally, weights of the form \(D = 1/n\) appear in many places in integrable systems. Amazingly, exactly the same operator also appears \[28\] by considering the pp-wave limit of the first non-local conserved charge of string theory in an AdS background found in \[29\]. Clearly, it would be interesting to explore this connection in more detail. Some further discussion can be found in the conclusions.
2 One point functions in SUGRA

2.1 The solution

In [16,17] supergravity solutions corresponding to supersymmetric black rings with three charges and three dipoles were obtained. In ten dimensions the solution can be realized as D1-D5-P black supertubes, carrying the usual charges of the D1-D5-P system. In addition the solution carries dipole charges of D1 and D5 branes as well as KK-monopoles. The distribution of the various charges is shown in the following table

| Q_1 D5: | z | z^2 | z^3 | z^4 | - |
| Q_2 D1: | z | - | - | - | - |
| Q_3 P: | z | - | - | - | - |
| q_1 d1: | - | - | - | - | ψ |
| q_2 d5: | - | z | z^2 | z^3 | z^4 | ψ |
| q_3 kkm: | (z) | z | z^2 | z^3 | z^4 | ψ |

(2.1)

The D5-branes wrap a four torus parametrized by z^1, ..., z^4, ψ parametrizes a contractible circle (the direction of the ring) and the solution carries momentum in the z-direction, which is the coordinate that describes the U(1) fiber of the KK-monopoles. The string frame metric of the black supertube is given by

\[ ds^2 = -(X^3)^{1/2} ds_5^2 + (X^3)^{-3/2} (dz + A^3)^2 + X^1 (X^3)^{1/2} dz_4^2 \]

\[ = -\frac{1}{H_3 \sqrt{H_1 H_2}} (dt + \omega)^2 + \frac{H_3}{\sqrt{H_1 H_2}} (dz + A^3)^2 + \sqrt{H_1 H_2} dx_3^2 + \sqrt{H_1 H_2} dz_4^2 \]  

(2.2)

with \( X^i = H_i^{-1} (H_1 H_2 H_3)^{1/3} \), and the harmonic functions \( H_i \) are given by

\[ H_1 = 1 + \frac{Q_1}{\Sigma} - \frac{q_2 q_3 R^2 \cos 2\theta}{\Sigma^2}, \quad H_2 = 1 + \frac{Q_2}{\Sigma} - \frac{q_1 q_3 R^2 \cos 2\theta}{\Sigma^2}, \quad H_3 = 1 + \frac{Q_3}{\Sigma} - \frac{q_1 q_2 R^2 \cos 2\theta}{\Sigma^2} \]  

(2.3)

where \( \Sigma = r^2 + R^2 \cos^2 \theta \). If we further define

\[ Y = q_1 Q_1 + q_2 Q_2 + q_3 Q_3 - q_1 q_2 q_3 \left( 1 + \frac{2 R^2 \cos 2\theta}{\Sigma} \right) \]  

(2.4)

then the one form \( \omega = \omega_\phi d\phi + \omega_\psi d\psi \) and gauge potential \( A^i \) are given by

\[ \omega_\phi = -\frac{r^2 \cos^2 \theta}{2 \Sigma^2} Y \]  

(2.5)

\[ \omega_\psi = -(q_1 + q_2 + q_3) \frac{R^2 \sin^2 \theta}{\Sigma} - \frac{(r^2 + R^2) \sin^2 \theta}{2 \Sigma^2} Y \]  

(2.6)

\[ A^i = H_i^{-1} (dt + \omega) + \frac{q_i R^2}{\Sigma} (\sin^2 \theta d\psi - \cos^2 \theta d\phi) \]  

(2.7)
Finally, the base space $dx_4^2$ is flat space written in a peculiar coordinate system

$$dx_4^2 = \sum \left( \frac{dr^2}{r^2 + R^2} + d\theta^2 \right) + (r^2 + R^2) \sin^2 \theta d\psi^2 + r^2 \cos^2 \theta d\phi^2$$

(2.8)

and $dz_4^2$ is the metric of the four-torus and will play no role in the following discussion. The coordinates take value in the range $0 \leq r < \infty$, $0 \leq \theta \leq \pi/2$, $\phi$ and $\psi$ have period $2\pi$ and $z$ has period $2\pi R_z$. The solution also contains a non-vanishing dilaton and RR 3-form field strength

$$e^{2\Phi} = \frac{H_2}{H_1}, \quad F^{(3)} = (X^1)^{-2} *_5 F_1^1 + F^2 \wedge (dz + A^3)$$

(2.9)

where $F^i = dA^i$, and the Hodge dual $*_5$ is with respect to the metric $ds_5^2$ appearing in (2.2). The solution depends on 7 parameters, the radius $R$, the charges $Q_i$ and the dipole charges $q_i$; note that the $Q_i$ are conserved charges at infinity but the dipole charges $q_i$ are not. These charges are quantized and related to the integer number of D-branes, momentum units and dipole branes through

$$N_{D5} = \frac{1}{g_s \ell_s^2} Q_1, \quad N_{D1} = \frac{1}{g_s \ell_s^2} \left( \frac{\ell}{\ell_s} \right)^4 Q_2, \quad N_P = \frac{1}{g_s \ell_s^2} \left( \frac{R_z}{\ell_s} \right)^2 \left( \frac{\ell}{\ell_s} \right)^4 Q_3, \quad n_{D1} = \frac{1}{g_s \ell_s} \left( \frac{R_z}{\ell_s} \right) \left( \frac{\ell}{\ell_s} \right)^4 q_1, \quad n_{D5} = \frac{1}{g_s \ell_s} \left( \frac{R_z}{\ell_s} \right) q_2,$$

(2.10)

with $g_s$ the string coupling constant, $\ell_s$ the string length and $\ell$ the radius of the torus. Furthermore the KK-dipole is also quantized in units of $R_z$,

$$q_3 = n_{KK} R_z$$

(2.11)

for some integer $n_{KK}$. It is useful to define

$$Q_1 = Q_1 - q_2 q_3, \quad Q_2 = Q_2 - q_1 q_3, \quad Q_3 = Q_3 - q_1 q_2$$

(2.12)

so that $H_i \geq 0$ implies $Q_i \geq 0$. It was shown in [16] that for the solution to be free of closed causal curves there is also an upper bound for $R^2$ which must be satisfied

$$2 \sum_{i<j} Q_i q_i Q_j q_j - \sum_i Q_i^2 q_i^2 \geq 4R^2 q_1 q_2 q_3 \sum_i q_i$$

(2.13)

The solution possesses angular momenta in the $\psi$ and $\phi$ directions

$$J_\phi = \frac{1}{2} \frac{R_z \ell_4}{\ell_s^8 g_s^2} \sum_i q_i (Q_i - q_1 q_2 q_3)$$

(2.14)

$$J_\psi = R^2 \frac{R_z \ell_4}{\ell_s^8 g_s^2} (q_1 + q_2 + q_3) + J_\phi$$

(2.15)
which need to be integer (or half-integer). Black rings have a horizon and an associated Bekenstein-Hawking entropy, proportional to the horizon area. It is most naturally expressed in terms of the quantized charges; to do so we will denote the integer charges to which $Q_i$ and $q_i$ are proportional by $N_i$ and $n_i$ respectively. Furthermore, we define $N_i = N_1 - n_2 n_3$ etcetera, in analogy with our definition of $Q_i$. The entropy is then equal to

$$S_H = 2\pi \sqrt{n_1 n_2 (\bar{N}_1 \bar{N}_2 - n_3 J) - \frac{1}{4}(\bar{N}_1 n_1 + \bar{N}_2 n_2 - \bar{N}_3 n_3)^2}$$

where we have defined $J = J_\psi - J_\phi$.

In the decoupling limit the geometry near the core decouples from the asymptotically flat region. In order to achieve this $\alpha' = l_s^2$ must be sent to zero keeping the energies of excitations near the core as well as the number of branes fixed, so in this limit

$$r \sim \alpha', \quad Q_{1,2} \sim \alpha', \quad Q_3 \sim \alpha'^2, \quad R_z \sim 1$$

$$R \sim \alpha', \quad q_{1,2} \sim \alpha', \quad q_3 \sim 1, \quad \ell^2 \sim \alpha'$$

Note that the region of parameters $Q_1 = Q_2 = Q_3$ and $q_1 = q_2 = q_3$, corresponding to the first supersymmetric black ring found in [30], is not captured by the decoupling limit. \footnote{In particular, due to the same reason, we expect that the multi-centered black rings found in [26,27] do not allow for a decoupling limit.}

The decoupled solution has the same form \footnote{The full decoupled solution is given in eq. (2.18).} with

$$H_{1,2} = \frac{Q_{1,2}}{\Sigma} - \frac{q_{2,1} q_3 R^2 \cos 2\theta}{\Sigma^2},$$

$$H_3 = 1 + \frac{Q_3}{\Sigma} - \frac{q_1 q_2 R^2 \cos 2\theta}{\Sigma^2},$$

$$\omega_\psi = -q_3 \frac{R^2 \sin^2 \theta}{\Sigma} - \frac{(r^2 + R^2) \sin^2 \theta}{\Sigma^2} \left( J_\phi - q_3 \frac{R^2 \cos 2\theta}{\Sigma} \right).$$

while $\omega_\phi$ etc have the same form as in the full solution. To explore the asymptotic region of this metric we first make a change of variables $z \to z - t$, after which for large values of $r$ the metric and three form become (to be precise what we do is to re-scale $r \to r/\epsilon, z \to \epsilon z, t \to \epsilon t$ and then take $\epsilon \to 0$ )

$$ds^2 = \frac{r^2}{\sqrt{Q_1 Q_2}} (dz^2 - dt^2) + \frac{\sqrt{Q_1 Q_2}}{r^2} dr^2 + \sqrt{Q_1 Q_2} (d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\psi^2)$$

$$F = \frac{2r}{\sqrt{Q_1 Q_2}} dz \wedge dt \wedge dr - \sqrt{Q_1 Q_2} \sin 2\theta d\theta \wedge d\psi \wedge d\phi.$$

We see that the asymptotic geometry is identical to that of $M = 0$ BTZ black hole times $S^3$.\footnote{In particular, due to the same reason, we expect that the multi-centered black rings found in [26,27] do not allow for a decoupling limit.}
which at large $r$ is asymptotically $AdS_3 \times S^3$, with radius $(Q_1 Q_2)^{1/4}$. So the black ring\footnote{Actually, once we uplift to six dimensions the solution is more properly thought of as a black supertube instead of a black ring, but we will continue to call the solution a black ring anyway, hoping that this will not cause any confusion.} must admit a description in terms of the two dimensional CFT that is dual to the D1/D5 system.

For certain values of the parameters the decoupled metric is everywhere locally $AdS_3 \times S^3$, for instance for $R = 0$ the solution reduces to the BMPV black hole \cite{31}. In \cite{16} (see also \cite{32}) the near horizon limit of the solution was studied in detail, where it was shown that provided the following condition is satisfied

$$Q_1 q_1 + Q_2 q_2 - Q_3 q_3 = 2 q_1 q_2 m R_z$$

for some integer $m$, the space factorizes into the near horizon limit of the extremal BTZ black hole times the quotient space $S^3 / Z_{q_3}$. In this near-horizon limit the black ring is no longer visible, so we will restrict our attention to the first decoupling limit only.

\subsection*{2.2 Decomposition of the fluctuations and vev’s}

In the previous section we have seen that the decoupling limit of the three charges supertube yields a space that is asymptotically $AdS_3 \times S^3$. To extract the one-point functions of CFT operators, we need to decompose the full solution into those degrees of freedom that diagonalize the linearized equations of motion; the leading large $r$ behavior of each of those degrees of freedom then provides us with the one-point functions. To obtain the properly normalized one-point functions we also need to know the precise normalization with which each degree of freedom appears in the action, after we expand to action to second order. Luckily, this analysis was already done in \cite{33,34}.

In order to diagonalize the linearized equations of motion, and also in order to extract the quantum numbers of the dual CFT operators, it is useful to organize all fluctuations in terms of representations of the isometry group $SO(2,2) \times SO(4)$ of $AdS_3 \times S^3$. At the linearized level, different representations cannot mix with each other, and must therefore couple to different operators in the CFT. This works as long as we perturb around global $AdS_3 \times S^3$. However, in view of (2.20), it might be more natural to think of the black ring as “small” perturbation of the $M = 0$ BTZ solution rather than as a “large” perturbation around global $AdS_3 \times S^3$. Because the $M = 0$ BTZ solution is locally the same as $AdS_3$, it will have locally the same Killing vector fields as $AdS_3$, but these will not be globally well-defined; their explicit expressions are summarized in appendix A.1. Though it would be interesting to develop this point of view in more detail, we will in this paper view the black ring as a perturbation of global $AdS_3$, which is the same as the $M = -1$ BTZ black hole. The relevant $SO(2,2) \times SO(4)$ generators are given in appendix A.2 and B. The $SO(2,2) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ generators, when viewed as vector fields on the $M = 0$ BTZ describe globally well-defined vector fields, but
they generate only asymptotic isometries, not global ones. Eventually, all these differences are more or less irrelevant, since the one-point functions are obtained from the leading asymptotics of the solutions of the field equations only.

We now turn to the analysis of the black ring solution. We write the full metric and three form in the following way

\[ g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}, \quad F = F^{(0)} + db \]  

(2.22)

Then, following [33,34] we parametrize the metric fluctuations around the \( AdS_3 \times S^3 \) background as (we restrict ourselves to the sector describing scalar fields on \( AdS_3 \))

\[ h_{ab} = h_{(ab)} + \frac{1}{3} g_{ab}^{(0)} h^c_{c}, \quad (g^{(0)})^{ab} h_{(ab)} = 0 \]  

(2.23)

and we write the two form potential (which only has a self dual part) in terms of a vector field \( U^c \) as:

\[ b_{ab} = 2 \sin \theta \cos \theta \epsilon_{abc} U^c \]  

(2.24)

with \( a, b = 1, 2, 3 \) \( SO(3) \) vector indices (i.e. \( S^3 \) tangent space indices) which are raised and lowered with the \( S^3 \) metric \( g^{(0)}_{ab} \).

We can then expand the linearized fluctuations in terms of harmonic functions on \( S^3 \) (see appendix B for a detailed discussion on harmonics on \( S^3 \) )

\[ h^a_{a}(r, \theta) = \sum_{k=0}^{\infty} \pi^k(r) Y^k_a(\theta) \]  

(2.25)

\[ h_{ab}(r, \theta) = \sum_{k=4}^{\infty} \zeta^\pm_k(r) (Y^k_{t\pm})_{ab} + \sum_{k=2}^{\infty} \zeta^k(r) \nabla_{[a} \nabla_{b]} Y^k_a(\theta) \]  

(2.26)

\[ U_{a}(r, \theta) = \sum_{k=2}^{\infty} U^k(r) \partial_a Y^k_a(\theta). \]  

(2.27)

By using the completeness relations presented in the appendix one can easily check that the component \( \zeta^k(r) \) is different from 0 for the solution under consideration, that means we are outside the de Donder gauge. This component can be easily removed (as we are interested only in the leading \( 1/r \) piece) by an appropriate gauge transformation

\[ \delta h_{ab} = \nabla_a \xi_b + \nabla_b \xi_a, \quad \delta U_{a} = \xi_{a} \]  

(2.28)

with

\[ \xi_{a} = \delta_{a9} \sum_{i=1}^{\infty} d_i R_{2i}^{2i} \theta^i (\partial_\theta Y^2 \theta) \]  

(2.29)

\[ d_i = (-1)^{i/2} \frac{\sqrt{i + 1} (i - 2)!}{(i + 2)!} \frac{1}{C_{i/2}} \left( 1 - \frac{q^2 R^2}{Q_1 Q_2} \right) \]  

(2.30)
where we have restricted ourselves to the regime of large charges, more precisely we rescale
\[ Q_1 \to Q_1/\epsilon, Q_2 \to Q_2/\epsilon, R \to R/\epsilon \]
and then take \( \epsilon \to 0 \) and keep the leading term in the \( \epsilon \)-expansion; \( C_k = \frac{2^k k!}{(k+1)!k!} \) denote the Catalan numbers. We now obtain
\[
\varrho^\pm(r) = (-1)^{k/2} \frac{k \sqrt{k - 2}}{(k(k - 1))^{3/2}} \frac{1}{C_{k/2 - 1}} \frac{R^k}{(k/2 - 1) r^k} \left(1 - \frac{q_3^2 R^2}{Q_1 Q_2}\right) + \ldots \tag{2.31}
\]
\[
\pi^k(r) = \frac{3(-1)^{(k/2)} k}{2(k - 1)^2 \sqrt{k + 1} C_{k/2 - 1}} \frac{1}{r^k} \left(1 - \frac{q_3^2 R^2}{Q_1 Q_2}\right) + \ldots \tag{2.32}
\]
\[
U^k(r) = -\frac{(-1)^{(k/2)}}{4(k - 1)^2 \sqrt{k + 1} C_{k/2 - 1}} \frac{1}{r^k} \left(1 - \frac{q_3^2 R^2}{Q_1 Q_2}\right) + \ldots \tag{2.33}
\]
where the dots denote terms that are subleading in \( 1/r \) or in the large charge limit explained above.\(^4\) Note that the value for \( \varrho^\pm(r) \) does not depend on the gauge choice. We have also rescaled the full six-dimensional metric by a factor of \( 1/\sqrt{Q_1 Q_2} \) so that the \( S^3 \) has unit radius (as done for example in [35]). This results in a prefactor in the six-dimensional supergravity actions which equals, up to factors of \( 2\pi \), the product of the inverse ten-dimensional Newton constant, the volume of the four-torus and \( Q_1 Q_2 \). We will denote this number by \( N \), since it is equal to
\[
N \equiv \frac{1}{g_5^2 f_s} \ell^4 Q_1 Q_2 = N_{D1} N_{D5}. \tag{2.34}
\]
In order to diagonalize the linearized equations of motion we need to make the following field redefinition
\[
\pi^k = -6k \sigma_k + 6(k + 2) \tau_k, \quad U^k = \sigma_k + \tau_k \tag{2.35}
\]
Then, at leading order in the \( 1/r \) expansion (as we will see the vanishing of \( \tau \) is consistent with its conformal dimension)
\[
\sigma_k = -\frac{(-1)^{(k/2)}}{4(k - 1)^2 \sqrt{k + 1} C_{k/2 - 1}} \frac{1}{r^k} \left(1 - \frac{q_3^2 R^2}{N}\right) + \ldots \tag{2.36}
\]
\[
\tau_k \approx 0 \tag{2.37}
\]
To extract the one-point functions from the large-\( r \) behavior we first consider the action for a general massive scalar field \( \phi \)
\[
S = \frac{\eta_\phi}{2} \int_{AdS_{d+1}} \sqrt{G} \left(-\nabla_\mu \phi \nabla^\mu \phi - m_\phi^2 \phi^2\right) \tag{2.38}
\]
The scalar field \( \phi \) will act as a source of a dual boundary operator \( O_\phi \) and the two point function of this operator can be computed to be [36]
\[
< O(x) O(y) > = \frac{\eta_\phi}{2\pi^{d/2} \Gamma[\Delta + 1]} \frac{2\Delta - d}{\Delta} \frac{1}{|x - y|^{2\Delta}} \tag{2.39}
\]
\(^4\)Using the results in appendix B it is straightforward to compute the full dependence on the charges (at leading order in the \( 1/r \) expansion) but we could not find a manageable closed expression for general level \( k \).
where $\Delta = \frac{1}{2}(d + \sqrt{d^2 + 4m^2})$, for our particular case we should obviously set $d = 2$. The decoupled black ring is asymptotically $\text{AdS}_3 \times S^3$ and therefore corresponds to a normalizable deformation, described by expectation values for all operators $O$. If the solution of (2.38) near the boundary behaves as

$$\phi \to 1/r^\Delta(A(x) + ...) \quad \text{for } r \to \infty$$

then the expectation value for the operator $O$ is [37]

$$< O(x) >= \eta_\phi (2\Delta - d)(Q_1Q_2)^{-\Delta/2}A(x).$$

The extra factor $\eta_\phi$ was put to one in the conventions of [37], but it is relevant for us since it contains a non trivial dependence on $N$. The extra factor $(Q_1Q_2)^{-\Delta/2}$ comes from the fact that [34] and [37] use a different coordinate system with a radial coordinate $\rho$ in which the boundary is at $\rho = 0$. The coordinate $\rho$ is related to our coordinate $r$ by $r = \sqrt{Q_1Q_2}\rho$, and the change of variables yields the extra factor in (2.41).

From [34] we can read off

$$\eta_\sigma^k = \frac{N}{(2\pi)^3}16(k-1), \quad \eta_\nu^k = \frac{N}{2(2\pi)^3} \quad (2.42)$$

$$m^2_\sigma^k = m^2_\nu^k = k(k-2) \Rightarrow \Delta_\sigma^k = \Delta_\nu^k = k; \quad (2.43)$$

$$m^2_\tau^k = (k+2)(k+4) \Rightarrow \Delta_\tau^k = k+4 \quad (2.44)$$

Notice that the values of $\Delta$ for each scalar field are in correspondence with the leading behavior of the field near the boundary, in particular they imply the vanishing of $\tau^k$ at order $1/r^k$, which is consistent with (2.37).

For the fields under consideration

$$< O^\sigma_k(x)O^\sigma_k(y)> = \frac{4N}{\pi^4}k(k-1)^3 \frac{1}{|x-y|^{2k}} \quad (2.45)$$

$$< O^\nu_k(x)O^\nu_k(y)> = \frac{N}{9\pi^4}(k-1)^2 \frac{1}{|x-y|^{2k}}. \quad (2.46)$$

If we normalize operators in such a way that the two point correlation function is $1/|x-y|^\Delta$ we finally obtain

$$< O^\sigma_k > = \frac{2(-1)^{k/2}}{\pi}N^{1/2} \frac{k}{(k+2)\sqrt{k(k-1)}} C_{k/2} \left( \frac{R}{\sqrt{Q_1Q_2}} \right)^k \left( 1 - \frac{q_3^2R^2}{N} \right) \quad (2.47)$$

$$< O^\nu_k > = \frac{3(-1)^{k/2}}{8\pi}N^{1/2} \frac{k\sqrt{k-2}}{(k(k-1))^{3/2} C_{k/2-1}} \left( \frac{R}{\sqrt{Q_1Q_2}} \right)^k \left( 1 - \frac{q_3^2R^2}{N} \right). \quad (2.48)$$

This can be rewritten in a straightforward way in terms of the integer charges, using (2.10).
3 1/2 BPS case

A particularly interesting case of the above solution is the so called ”small black ring” [18, 19] obtained from the general solution by setting  

\[ q_1 = q_2 = Q_3 = 0 \]  

(3.1)

In this case (2.13) is trivially saturated but the absence of closed causal curves still imposes the non-trivial constraint

\[ q_3 R^2 \leq \frac{Q_1 Q_2}{q_3}. \]  

(3.2)

From now on we will express everything in terms of the integer charges using (2.10). We will also put \( R_z = 1 \), as this is the radius of the circle of the cylinder on which the dual CFT lives, so the \( R_z \) dependence can always be restored by a conformal transformation. Then \( q_3 = n_{KK} \) and we will use \( q_3 \) instead of \( n_{KK} \) to denote this particular integer. The absence of closed causal curves (3.2) becomes simply

\[ q_3 J \leq N. \]  

(3.3)

One can easily check that the macroscopic entropy vanishes in the limit (3.1), however the system still has a finite microscopic entropy [18, 21, 38]

\[ S_{micro} = 4\pi \sqrt{N - q_3 J}. \]  

(3.4)

When considering the system compactified on K3 evidence was given in [18, 21] that the solution develops a non-vanishing horizon once stringy \( R^2 \) corrections to the supergravity action are considered, furthermore, the area of such a horizon gives rise to the entropy (3.4). The full geometry including \( R^2 \) corrections, however, has not been computed.

From (2.16) we see that a way to get the small black ring microscopic entropy is by setting\(^6\)

\[ n_{D1} = n_{D5} = 1, \quad Q_3 = \frac{q_1 Q_1 + q_2 Q_2 - q_1 q_2 q_3}{q_3}. \]  

(3.5)

Notice that the entropy can by written in the following alternative form

\[ S = 2\pi \sqrt{|N_1 N_2 - \bar{N}_1 \bar{N}_2| N_3 + |\bar{N}_1 \bar{N}_2 - n_3 J| n_1 n_2 - J^2_{\phi}}. \]  

(3.6)

\(^5\)Actually in this limit the solution is singular but it is believed to become a small black ring with a string scale horizon once higher order curvature corrections are taken into account.

\(^6\)Notice that \( Q_3 \) is such that (2.21) is satisfied with \( m = 0 \), but in fact in the limit of large charges we will reproduce the entropy for any finite \( m \). With the values \( n_{D1} = n_{D5} = 1 \) the entropy will have the correct form but with a different proportionality factor, in general the black ring entropy reduces to \( S_{micro} = 2\pi \sqrt{n_{D1} n_{D5} \sqrt{N - q_3 J}} \), so we must set e.g. \( n_{D1} = n_{D5} = 2 \) in order to reproduce the microscopic entropy with the correct proportionality factor for the system on K3, and e.g. \( n_{D1} = 1 \) and \( n_{D5} = 2 \) for the system on \( T^4 \).
and as argued in [16, 17] this suggests the interpretation that the system decomposes into two sectors, the "BMPV" sector, with central charge $c' = 6[N_1N_2 - \bar{N}_1\bar{N}_2]$ and angular momentum $J_\phi$ and the "supertube" sector, with central charge $c'' = \bar{N}_1\bar{N}_2$ and angular momentum $J$. The condition (3.5) implies $c'N_3 = 6J_\phi^2$ so that the BMPV sector carries no entropy. It would be interesting to understand better whether there is a physical mechanism behind the decoupling of the BMPV sector. In any case we would like to conjecture that once we include $R^2$ corrections to the small black ring, the geometry we get is similar to the general black ring with the conditions (3.5).

The small black ring limit is particularly interesting since the bulk geometry is 1/2-BPS, this implies that typical states that describe it will also be 1/2 BPS and the corresponding density matrix in the dual CFT is of the form $\sum_{a,b} \rho_{ab} |a\rangle \langle b|$, with $|a\rangle$ and $|b\rangle$ chiral primaries, as we will see in the next section, this will simplify the computation of the one-point functions in the orbifold CFT.

It is easy to see that in the limit of large charges considered in the previous section the answer for the one point functions obtained from supergravity in the small black ring limit are independent on whether we choose $q_1 = q_2 = Q_3 = 0$ or (3.5). Expressed in terms of $N = N_{D1}N_{D5}$, $J$ and $q_3$ dependence, we obtain

$$< O > \sim N^{1/2} q_3^{-k/2} \left( \frac{J}{N} \right)^{k/2} (1 - q_3 J N).$$

Our aim will be to obtain the same result from the dual orbifold CFT.

In [17] a microscopic interpretation of the small black ring was given (see also [18] for the case $q_3 = 1$), in the regime $J \sim N$ the correct entropy and angular momentum are accounted for if we consider states of the form

$$\left( a_{n=-q_3}^+ \right)^J \times \prod_{n=1}^{\infty} \left( \prod_{i=\pm,3,\ldots,24} (a_{-n}^i)^{N_{n_i}} \right) |0\rangle \quad (3.8)$$

where the first factor represents a Bose-Einstein condensate and accounts for the angular momentum, and the second factor has no net angular momentum and accounts for the entropy. The notation will be further explained in the next section but is based on the identification of the space of chiral primaries with the states at level $N$ in a Fock space built out of 24 free bosons. In section 5 we present a thermodynamic toy model which will lead to the Bose condensate in (3.8) for general $q_3$. 

14
4 Orbifold computation

4.1 Preliminaries

In this section we would like to compare the one-point functions obtained from supergravity to those obtained from the dual conformal field theory. As is well-known, the dual conformal field theory is believed to be a deformation of the $N = (4, 4)$ orbifold SCFT which is a sigma model with target space $M^N/S_N$, with $M = T^4$ or $K3$ (see e.g. [39–46]. We will perform our calculations at the orbifold point. In the 1/2-BPS case one may hope that the results will be independent of the deformation, though we are not aware of any proof of this; in general the results will certainly depend on it.

Since we will in this section focus on the 1/2 BPS case, we will consider density matrices of the form $\sum_{a,b} \rho_{ab} |a\rangle \langle b|$ with $|a\rangle$ and $|b\rangle$ chiral primaries. The set of chiral primaries, as a vector space, is the same as the set of states at level $N$ in a Fock space built out of $b^{\text{even}} = b^0 + b^2 + b^4$ bosonic oscillators and $b^{\text{odd}} = b^1 + b^3$ fermionic oscillators, with $b^i = \text{dim } H^i(M)$ the Betti numbers of $M$. Of course the chiral primaries also form a ring, but this ring structure is much more difficult to obtain from the F1-P system [47, 48]. The ring structure is very interesting and has been the subject of various mathematics papers in the past few years (see e.g. [49–51]). It turns out to be highly non-trivial to express the ring structure in terms of the free bosons and fermions and we will not discuss this here. In any case, most one-point functions require information which is not contained in the chiral ring.

Because for $M = K3$ all elements of the chiral ring are bosonic, we will only discuss this case here. It also has the advantage that for $M = K3$ one can explicitly include $\alpha'$ corrections in the supergravity solutions and see a horizon form in case the classical horizon area vanishes [25]. Thus this is a natural case in which to study various aspects of black hole physics.

The free boson description of the chiral ring can also be extracted directly from the orbifold CFT. The latter has various twisted sectors labeled by conjugacy classes of $S_N$. Any group element of $S_N$ can be written in its cycle decomposition as $(c_1)(c_2)\ldots$ with $|c_i|$ the length of the cycles. Each cycle $(c_i)$ describes $i$ copies of $M$ that are being cyclically twisted as we move once around the string. In other words each cycle describes effectively a long string of length $|c_i|$. Each such long string gives rise to a set of chiral primaries, one for each element $\gamma \in H^*(M)$. In the twisted sector $(c_1)(c_2)\ldots$ we therefore find operators that we will denote as $\sigma_{c_1}(\gamma_1)\sigma_{c_2}(\gamma_2)\sigma_{c_3}(\gamma_3)\ldots$. However, to get an honest operator in the orbifold CFT we need to sum over the centralizer of the group element $(c_1)(c_2)\ldots$, and also over the entire conjugacy class this group element belongs to. Notice that $\sigma_c(\gamma)$ refers to a particular cyclic permutation $c$
and not yet to its conjugacy class. We will denote the operator obtained by averaging \( \sigma_c(\gamma) \) over its conjugacy class by \( \sigma_c|_{\gamma} \). This is an honest operator in the orbifold CFT, and corresponds to the bosonic creation operator \( \alpha_{\gamma} \) in the free boson description.

It is important to keep in mind that the free boson representation does not reflect the chiral ring structure. The product of the two operators \( \sigma_c(\gamma_1) \) and \( \sigma_c(\gamma_2) \) is quite complicated. Each operator involves a sum over conjugacy classes, and therefore in the product there will be terms where the two cycles are disjoint, but also where the two-cycles overlap. Taking the naive product of the boson creation operators would miss the second type of contributions.

Chiral primaries have (in the NS sector) conformal weights \((h_L, h_R)\), with \( J = h_L - h_R \) the angular momentum. These conformal weights are related to the Hodge decomposition of the complex four-manifold \( M \). A form \( \gamma \) of degree \( (p, q) \) yields a chiral primary with weights \((h_L, h_R) = (p/2, q/2)\), whereas \( \sigma_c(\gamma) \) yields a chiral primary with weights \((h_L, h_R) = ((p + |c| - 1)/2, (q + |c| - 1)/2)\) in the orbifold theory. The cohomology of K3 has unique elements of degrees \((0, 0)\), \((2, 0)\), \((0, 2)\) and \((2, 2)\). The corresponding operators \( \sigma_c(\gamma) \) are denoted by \( \sigma_n^-\), \( \sigma_n^+\), \( \sigma_n^=\) and \( \sigma_n^{++}\) with \( n = |c| \) in [53]. These operators exist for any hyperkahler \( M \) and can be constructed using just the \( N = 4 \) SCFT without explicit reference to \( M \). For \( M = K3 \) there are also 20 elements of degree \((1, 1)\). Therefore, the chiral primaries of \( K3^N/S_N \) can be represented in terms of the Fock space constructed out of 24 bosons. Of these, 22 carry no angular momentum, one carries angular momentum +1 and one −1.

### 4.2 The sum over conjugacy classes

The solutions of the field equations of 6d supergravity on \( \text{AdS}_3 \times \text{S}^3 \) (obtained by KK reduction of type II string theory on K3) can be put in one-to-one correspondence with the “single particle” chiral primaries \( \sigma_c(\gamma) \) and their superconformal descendants, at least as far as the quantum numbers go [40, 54]. Strictly speaking it has not been shown that the supergravity fields are dual to precisely these one-particle operators and that they are not modified by products of operators with smaller \( |c| \). We will nevertheless use this as our working assumption. A good test would for example be to compare orbifold three-point functions to the dual supergravity result. A detailed comparison has not been done but see [35, 53]. It is also plausible that at large \( N \), the leading contribution simply comes from the single particle operator \( \sigma_c(\gamma) \) and its superconformal descendants. A simple estimate shows that the admixture of operators based on multiple cycles is suppressed by factors of \( N^{-1/2} \). As we will only focus on the leading large \( N \) behavior of correlation functions, this puts further faith in our working assumption.

A further simplifying assumption that we will be making is that the density matrix in which we compute one-point functions is diagonal in the twist field basis. This seems to include all examples of interest, and once we drop this assumption the calculation become quickly untractable.

With these simplifications the correlations functions that we need to compute are sums of
correlators of the form $\langle A|O|A \rangle$ with

$$O = (J_-)^u (\bar{J}_-)^{\bar{u}} \sigma_{|c|}(\gamma) = (J_-)^u (\bar{J}_-)^{\bar{u}} \sum_{\text{cent conj}} \sigma_c(\gamma)$$

(4.1)

and $A$ a chiral primary of the form

$$A = \sum_{\text{cent conj}} \prod_i \sigma_{c_i}(\gamma_i)$$

(4.2)

where $c_i$ are disjoint cycles and $\sum_{\text{cent conj}}$ indicates a suitably normalized sum over the centralizer and conjugacy class of the corresponding group element. The supergravity operators that will have a nonzero vev should clearly have $U(1)_L \times U(1)_R$ charge equal to zero. Therefore $O$ cannot be a chiral primary, it has to be a descendant of a chiral primary in such a way that the $U(1)_L \times U(1)_R$ charge vanishes. To achieve this we lower the charges of $O$ using the $SU(2)_L \times SU(2)_R$ lowering operators. There are also other descendants that make use of the supersymmetry generators that could have a one-point vev. Our calculations and estimates below apply to such operators as well.

The sums over centralizers and conjugacy classes are important for the overall normalization and the large $N$ dependence of the various correlators. The centralizer of a group element $(c_1)(c_2)\ldots$ consists of those group elements that either cyclically permute the cycles, or that permute cycles of equal length. If we assume that there are $r_i$ cycles of length $i$, then the order of the centralizer is

$$z[\{c\}] = \prod_i r_i^i r_i!.$$  

(4.3)

Once we associate different classes $\gamma_i \in H^*(M)$ to the cycles $c_i$, the centralizer of the group element no longer acts trivially on the state. A sum over the centralizer effectively symmetrizes the classes $\gamma_i$ over the different cycles of equal length. Incidentally, this is also why the corresponding operators $\alpha_{-|c|}(\gamma)$ obey Bose statistics. It will be convenient in what follows to separate the centralizer in a piece which preserves the state, and a remainder. The piece that preserves the state consists again of the cyclic permutations of the cycles, plus permutations of those cycles of equal length that carry the same label $\gamma_i$. The remaining elements of the centralizer together with the sum over conjugacy classes form a group $G(A)$. We will explicitly keep track of this group in what follows. Altogether the precise definition of our operator $A$ now reads

$$A = \frac{1}{N(A)} \sum_{g \in G(A)} g \left( \prod_i \sigma_{c_i}(\gamma_i) \right),$$

(4.4)

where the normalization constant $N(A)$ will be chosen such that $\langle A|A \rangle = 1^8$. In addition, we will assume the operators $\sigma_c(\gamma)$ are normalized so that $\langle \sigma_c(\gamma) | \sigma_c(\gamma) \rangle = 1$. In general, if we have operators $A_i$ living in a sector twisted by $g_i \in \mathcal{S}_N$, the correlator of the $A_i$ on the sphere will

---

8As usual, we assume here that $|A\rangle = A(0)|0\rangle$ and that $\langle A|$ is the BPZ conjugate of $|A\rangle$. 

17
vanish unless $\prod_i g_i = 1$. If $|A|$ is twisted by $g \in S_N$, then $\langle A |$ is twisted by $g^{-1}$, and when we compute $\langle A |A \rangle$ only terms with $g = g'$ contribute to $N(A)^{-2}\sum_{g\in G(A)} \sum_{g'\in G(A)}$. Therefore,

$$\langle A |A \rangle = \frac{|G(A)|}{N(A)^2}. \quad (4.5)$$

The order of the group $G(A)$ can be expressed as follows. Suppose that there are $r_i(\gamma_j)$ occurrences of operators $\sigma_{c_l}(\gamma_j)$ with $|c_l| = i$ in $A$. Then by a simple generalization of (4.3) we find that

$$|G(A)| = \frac{N!}{\prod_{i,j} i^{r_i(\gamma_j)} r_i(\alpha_j)!} \quad (4.6)$$

and it is clear that the normalization factor will be

$$N(A) = \sqrt{|G(A)|}. \quad (4.7)$$

The “single-particle” operator $O$ is part of the $g$-twisted sector of the theory where $g$ consists of just one cycle $c$ with length $|c|$, which we will denote by $k$ from now on. An analysis similar to the one we just did shows that in order to properly normalize $O$ we need an extra factor of

$$\frac{1}{N(O)} = \frac{1}{\sqrt{|G(O)|}} = \left( \frac{(N-k)!k}{N!} \right)^{\frac{1}{2}}. \quad (4.8)$$

The final correlator therefore involves a sum

$$\sum_{g\in G(A)} \sum_{g'\in G(A)} \sum_{g''\in G(O)} \frac{1}{N(A)^2 N(O)} \quad (4.9)$$

and there can only be nonzero contributions if the three group elements are suitably aligned. More precisely, denote by $g_A$ and $g_O$ the twists in $A$ and $O$ before doing the sum. Then the only non-vanishing contributions can arise when

$$(gg_A g^{-1})(g'' g_O (g'')^{-1}) = g' g_A (g')^{-1}. \quad (4.10)$$

To make a precise counting of the number of sets of three group elements for which this holds is a daunting task. Since $g_A$ is a relatively “long” group element, and $g_O$ is a relatively “short” group element, one expects that the single cycle of length $k$ in $g_O$ will only interact with a few of the cycles in $g_A$. Geometrically, this means the following. The three group elements $g_1 = gg_A g^{-1}$, $g_2 = g'' g_O (g'')^{-1}$ and $g_3 = g' g_A (g')^{-1}$ obeying (4.10) define an $N$-fold cover $\Sigma \subset M^N$ of the string world-sheet $\mathbb{P}^1$, such that the map $\Sigma \to \mathbb{P}^1$ has three branch points with monodromies $g_1$, $g_2$ and $g_3$ at $0, 1, \infty$. In general, $\Sigma$ will be disconnected; Since $g_2$ consists of a single cycle, there must be a single connected component $\Sigma_0$ of $\Sigma$ that contains a branch point with monodromy $g_2$. This distinguished connected component is also a branched cover of $\mathbb{P}^1$ and its degree will be denoted by $s$. As $O$ does not contribute to the remaining components
of $\Sigma$, the calculations involving these other components reduce to that of two-point functions, and they can be done exactly as we did above with the same combinatorics. Suppose that of all the cycles in $A$, $d_i(\gamma_j)$ cycles of length $i$ with operator $\alpha_j$ inserted “live” on $\Sigma_0$, by which we mean that the monodromies of $\Sigma_0 \to \mathbb{P}^1$ at $0, \infty$ are group elements consisting of $d_i(\gamma_j)$ cycles of length $i$. These cycles somehow interact with the single cycle of length $k$ over in total $s$ elements in an irreducible way. Clearly, $\sum_{i,j} id_i(\gamma_j) = s$. There can in general be different topological ways to do this, but one would expect the number of topologically inequivalent contractions (i.e. ones that are not related by overall conjugation) to be very small. Call this number $T$. In addition, there could be a few overall conjugations that leave the configuration precisely invariant. Again, this will be a small number, perhaps one can even prove it will be always one. Denote this number by $R$.

The additional combinatoric factors that appear are the number of conjugacy classes of $A$ for those cycles which does not belong to $\Sigma_0$; this equals

$$L = \frac{(N-s)!}{\prod_{i,j} i^{r_i(\gamma_j)-d_i(\gamma_j)} (r_i(\gamma_j) - d_i(\gamma_j))!}.$$  

(4.11)

Next, there are $s!$ different ways to do an overall conjugation of the three monodromies of $\Sigma_0$. Accidental symmetries and different topological contraction possibilities are taken into account by the factor $T/R$. Finally, there are $\left( \begin{array}{c} N \\ s \end{array} \right)$ different possible ways to choose the irreducible component $\Sigma_0$ in $M^N$. The final total combinatorial prefactor is therefore

$$K = \frac{1}{N(A)^2 N(O)} \left( \begin{array}{c} N \\ s \end{array} \right) s! L \frac{T}{R}.$$  

(4.12)

Massaging this a bit we obtain

$$K = \left( \frac{(N-k)!k}{N!} \right)^{1/2} T \frac{R}{R} \prod_{i,j} i^{d_i(\gamma_j)} (r_i(\gamma_j) - d_i(\gamma_j))!.$$  

(4.13)

4.3 Large $N$ analysis

The combinatorial factor $K$ controls most of the $N$ dependence of the correlation function, since all that remains is a relatively small correlator times this combinatorial factor. To study large $N$, it is useful to use the following observation. In the irreducible component $\Sigma_0$ of degree $s$, there are three branch points with monodromy $h_1$, $h_2$ and $h_3$ whose product equals one. Suppose that the number of cycles of each of these group elements, viewed as elements of $S_s$, is $y_i$. Then the genus of $\Sigma_0$ is

$$g = \frac{1}{2} (s + 2 - y_1 - y_2 - y_3).$$  

(4.14)
which clearly has to be a nonnegative integer. This is called the graph defect in [49], since we can associate a simple graph to the three group elements and use the combinatorial genus of the graph to obtain (4.14). The number of orbits in the piece from $A$ that lives on $\Sigma_0$ is

$$y_1 = y_3 = \sum_{i,j} d_i(\gamma_j). \tag{4.15}$$

The number of orbits from $O$ is $s - k + 1$. Thus

$$g = \frac{1}{2}(k + 1 - 2 \sum_{i,j} d_i(\gamma_j)). \tag{4.16}$$

For large $N$ and finite $k$, $N!/(N-k)! \sim N^k$. Thus the combinatorial factor behaves for large $N$ as

$$N^{\frac{1}{2} - g - \sum_{i,j} d_i(\gamma_j)} \prod_{i,j} \frac{\text{i}^{d_i(\gamma_j)} r_i(\gamma_j)!}{(r_i(\gamma_j) - d_i(\gamma_j))!}. \tag{4.17}$$

As a check, if $A$ consists of a single cycle, we find that the correlation function scales as $N^{-1/2}$, which is the correct answer. Similarly, if $A$ is a relatively simple operator, it is easy to extract the large $N$ dependence. However, we are mainly interested in the case where $A$ is very complicated and consists of many cycles which are typically randomly distributed. In such a situation, there are many different contractions which contribute to the correlation function. The combinatorial factor involves (for $r_i(\gamma_j) \gg d_i(\gamma_j)$) a factor

$$\prod_{i,j} (\text{i} r_i(\gamma_j))^{d_i(\gamma_j)}. \tag{4.18}$$

This still has to be multiplied by the appropriate correlation function, summed over all possible sets of choices of the numbers $d_i(\gamma_j)$, and averaged over the ensemble to which the state $|A\rangle$ belongs. If we ignore the contribution from the correlation function, and assume that the total length of the set of cycles from which we randomly select $d_i(\gamma_j)$ of type $\sigma_i(\gamma_j)$ is $P$, then the factor (4.18) appears to scale like $P^{\sum_{i,j} d_i(\alpha_j)}$. For example, if $P = N$, then (4.18) appears to scale like $N^{\sum_{i,j} d_i(\alpha_j)}$ which would cancel against a similar factor in (4.17). In other words, it would appear that arbitrarily complicated contractions with genus $g = 0$ would contribute equally at large $N$. This is a very peculiar conclusion and there are several reasons why we believe it is incorrect. First of all, we ignored the contributions of the actual correlation functions. These are notoriously difficult to compute, but will certainly be nontrivial functions of $i$ and $j$ (see e.g. [53] for a three-point function calculation) and putting these in may well change this naive conclusion. Furthermore, it is quite possible that averaging over an ensemble will involve various signs that suppress the above naive estimate. Finally, if the above estimate were correct, it would predict that in the case of the $M = 0$ BTZ black hole all one-point functions would be turned on with equal strength. This is certainly not the case. To leading order in the $\alpha'$ expansion, only untwisted operators in the $M = 0$ BTZ have a nonzero one-point
function. This situation will probably change once higher order corrections are included (after all the $M = 0$ BTZ black hole has a residual entropy of order $\sqrt{N}$), but those higher order corrections will be suppressed by higher orders of $\alpha'$, i.e. $N^{-1/2}$. It would be interesting to work this out in more detail (see also [25]), but for us it motivates us to conjecture that after including correlation functions, and after averaging over a random ensemble of cycles of total length $P$, the factor (4.18) scales as $P^{(\sum_{i,j} d_i(\gamma_j)+1)/2}$:

$$\prod_{i,j}(ir_i(\gamma_j))^{d_i(\gamma_j)} \sim P^{(\sum_{i,j} d_i(\gamma_j)+1)/2}.$$  

(4.19)

Clearly, it would be nice to study this conjecture further. Here we will see that it provides a self-consistent picture of one-point functions in various situations, one that moreover agrees with supergravity calculations in cases where these are available.

### 4.4 One-point functions

We will now use the above results to find the large $N$ behavior of correlation functions in various ensembles. Notice that most results given here carry over to typical states in the ensembles as well, provided (4.19) still holds for the typical state.

#### $M = 0$ BTZ

We already have all the ingredients in place to do this calculation. The ensemble consists of random cycles of total length $N$. This is a microcanonical point of view, from a canonical point of view it is perhaps better to think of it as a system of 24 free bosons at a finite temperature proportional to $N^{1/2}$. Either way, using (4.17) and (4.19) we find that

$$\langle O_k \rangle \sim N^{1-g-\frac{1}{2}\sum_{i,j} d_i(\gamma_j)}.$$  

(4.20)

The equation for the genus (4.14) shows that $k$ has to be odd, otherwise the one-point function is identically zero. It also shows that the leading contribution comes from $g = 0$ and we finally get

$$\langle O_k \rangle \sim N^{\frac{k-1}{4}}.$$  

(4.21)

In other words, to leading order all one-point functions of twist fields vanish, but at subleading order in the $\alpha' \sim 1/\sqrt{N}$ expansion they are potentially turned on. It would be interesting to verify this more explicitly using higher order curvature corrections to supergravity.

#### Small black ring, $q_3 = 1$

For the small black ring, we separate the state into a condensate $(\alpha_{-1}^\perp)^J$ and a randomly distributed piece of total length $P = N - J$. We denote by $\hat{d}$ the number of elements of the condensate that live on the irreducible component $\Sigma_0$. Also, by $\sum'_{i,j}$ we denote a sum over all cycles except the condensate. Then the combinatorical factor works out to be

$$N^{\frac{1}{2}-g-\sum'_{i,j} d_i(\gamma_j)} \left(\frac{J}{N}\right)^{\hat{d}} \frac{T}{R} (N - J)^{(1+\sum'_{i,j} d_i(\gamma_j))/2}.$$  

(4.22)
The leading contribution appears when \( g = 0, \sum'_{i,j} d_i(\gamma_j) = 1 \) and \( \hat{d} = (k-1)/2 \). In this case \( \Sigma_0 \) contains many states from the condensate and only three cycles of length larger than one. One can also check (this requires a bit of work) that for these configurations \( T/R = 1 \). Therefore, we finally obtain that for odd \( k \)

\[
\langle \mathcal{O}_k \rangle \sim N^{-1/2} \left( \frac{J}{N} \right)^{\frac{k-1}{2}} (N-J).
\]

(4.23)

When we compare this to (3.7) we need to take into account that the \( k \) used here is not the same as the \( k \) used in (3.7). The supergravity fields used in the calculations in sections 2,3 couple to chiral primary fields with conformal weights \((\frac{k}{2}, \frac{k}{2})\). One can check that these are twisted fields that arise from the identity element of \( H^*(K3) \) in the \( \mathbb{Z}_{k+1} \)-twisted sector; therefore, in order to compare (4.23) to (3.7) we should replace \( k \) by \( k + 1 \) in (4.23). After this substitution it indeed agrees perfectly with (3.7).

For \( J \sim \sqrt{N} \) there is no longer a condensate, and we expect the CFT result to scale in the same way as the \( M = 0 \) BTZ. This is indeed the case, which shows that our assumptions, in particular (4.19), are at least self-consistent.

**Conical defect**

The conical defect has been conjectured \([2, 24]\) to be dual to a pure state \( |A⟩ = (\alpha - p)^{N/p} \). The irreducible component will therefore be of size \( s = up \) for some \( u \), and contain precisely \( u \) cycles of length \( p \) from \( |A⟩ \). We again expect the leading contribution to come from genus zero surfaces, which implies \( k = 2u - 1 \) (see (4.14)). One may check however that there are no such genus zero configurations. The cycle of length \( k \) would have exactly one element in common with at least one of the cycles of length \( p \) coming from \( A \). The product of the two will then involve a cycle of length larger than \( p \), which is inconsistent with the form of the state \( |A⟩ \).

It therefore appears that the leading contributing is from genus one, with \( k = 2u + 1 \), and one can explicitly find corresponding explicit group elements that obey (4.10). With some more work we found that \( T/R = u \) for this genus one case.

The precise combinatorial factor is then

\[
K = \left( \frac{(N-k)!k}{N!} \right)^{1/2} \frac{p^u(N/p)!}{(N/p - u)!}.
\]

(4.24)

Amazingly, this implies that

\[
\langle \mathcal{O}_k \rangle \sim \left( \frac{k-1}{2} \right) N^{-\frac{1}{2}}
\]

(4.25)

whose \( N \)-dependence is universal and independent of \( k \). It would be interesting to understand how this could arise from a supergravity solution in the presence of higher order \( \alpha' \) corrections.

**Small black ring, \( q_3 > 1 \)**

This case is somewhat similar to the small black ring with \( q_3 = 1 \). The main difference is that now the condensate is made up out of \((\alpha_{-q_3})^J \) instead of \((\alpha_{-1})^J \), and that the thermal
distribution has length $P = N - q_3 J$. The leading contribution is from the same configuration as for the case $q_3 = 1$, namely $\hat{d} = (k - 1)/2$, $g = 0$ and $\sum_{i,j} d_i(\gamma_j) = 1$. There could in this case also have been a contribution with $\sum_{i,j} d_i(\gamma_j) = 0$ which cannot happen for $q_3 = 1$, but in view of our conical defect discussion this one would have to have at least $g = 1$ and is therefore subleading. Putting in all the factors we get for odd $k$ that

$$
\langle O_k \rangle \sim N^{-1/2} \left( \frac{q_3 J}{N} \right)^{k-1} (N - q_3 J)
$$

which agrees (after shifting $k \to k + 1$ as explained above) with the supergravity result (3.7), up to a factor of $q_3^k$. Such a factor could well arise from the CFT calculation once actual correlation functions are included, but this is beyond the scope of the present calculation. It is already quite nontrivial that the large $N,J$ dependence of the CFT and supergravity results is identical.

5 A toy model for higher condensation

The statistical mechanics of the ensemble corresponding to small black rings, in absence of dipole charges, i.e. $q_3 = 1$, can be studied by considering a partition function of the form $Z = Tr(e^{-\beta(H+\mu J)})$, where for the oscillators $\alpha_{-n}^\pm, H = n$ and $J = \pm 1$, while for $\alpha_{-n}^i$ with $i = 3\ldots24$ $H = n$ and $J = 0$. It was shown in [18,19,38] that in the regime $J \sim N$ a Bose-Einstein condensate forms and the ensemble consists of $J$ strings of length 1 plus a thermal distribution giving rise to the entropy $S \sim \sqrt{N - J}$.

In this section we study the partition function in presence of a "dipole" charge $D$, i.e. $Z = Tr(e^{-\beta(H+\mu J+\nu D)})$. We show that if we assign to a given oscillator $\alpha_{-n}^\pm$ a dipole charge $D = 1/n$ then in a suitable regime the expectation values of $J,D$ will be of order $N$ and a Bose-Einstein condensation takes place but now of string components of length $q_3 \geq 1$, furthermore we exactly reproduce the small black ring entropy in the presence of dipoles charges $S \sim \sqrt{N - q_3 J}$.

The partition function we want to study is given by

$$
Z = Tr_{\mathcal{H}}(e^{-\beta(H+\mu J+\nu D)})
$$

The Hilbert space $\mathcal{H}$ consists of a Fock space built out of 24 free oscillators $\alpha_{-n}^\pm$ and $\alpha_{-n}^i$, $i = 3,\ldots,24$, carrying the following charges

$$
\quad
$$

$$
[\hat{J},\alpha_{-n}^\pm] = \pm \alpha_{-n}^\pm, \quad [\hat{J},\alpha_{-n}^i] = 0, \quad [D,\alpha_{-n}^\pm] = \frac{1}{n} \alpha_{-n}^\pm
$$

$^9$Note that this assignment for the dipole charge $D$ exactly coincides with (4.18) of [28] which gives the action of the first non-local charge of the infinite tower found in [29] when acting on a BMN state.
The charge of the other oscillators with respect to \( D \) will not be relevant for the discussion below, but will be relevant for the subleading behavior of the entropy. This is discussed further in the conclusions.

Let us focus on the \( \alpha^+ \) oscillator, its contribution to the partition function is

\[
\log Z = - \sum_{n=1}^{\infty} \log (1 - e^{\beta(-n+\mu+\nu/n)}) = \sum_{n=1}^{\infty} C_n
\]  

and \( C_n \) can be written as

\[
C_n = \sum_{l=1}^{\infty} e^{-ln\beta} \left( \sum_{j,k=0}^{\infty} \frac{(\mu\beta l)^j(l\beta\nu/n)^k}{j!k!} \right) = \sum_{j,k=0}^{\infty} \frac{\mu^j \nu^k \beta^{k+j}}{j!k!n^k} Li_{1-j-k}(e^{-\beta n}).
\]  

After changing variables \( k + j = s \) and summing over \( 0 \leq j \leq s \) we get

\[
C_n = \sum_{s=0}^{\infty} \beta^s Li_{1-s}(e^{-\beta n}) \frac{(\nu + n\mu)^s}{n^s\Gamma(1+s)}
\]  

Up to this point the above computation is exact, in order to proceed we approximate in the limit \( \beta \ll 1 \) the polylogarithm \( Li_{1-s} \) for \( s \geq 1 \) by

\[
Li_{1-s}(e^{-\beta n}) \approx \frac{(s-1)!}{\beta^s n^s}.
\]  

Then

\[
\tilde{C}_n = \sum_{s=1}^{\infty} \beta^s Li_{1-s}(e^{-\beta n}) \frac{(\nu + n\mu)^s}{n^s\Gamma(1+s)} \approx -\log (1 - \frac{\mu}{n} - \frac{\nu}{n^2}).
\]  

The contribution from \( s = 0 \) can be taken care of separately and the sum over \( n \) can easily be performed and gives the usual term depending only on \( \beta \). Taking into account all the oscillators we get

\[
\log Z \approx \frac{4\pi^2}{\beta} - \sum_{n=1}^{\infty} \log (1 - \frac{\mu}{n} - \frac{\nu}{n^2}).
\]  

The first term here is obtained by summing over all 24 oscillators, but the second term is due only to \( \alpha^+ \). There are similar \( \mu, \nu \)-dependent terms for the other oscillators as well, but the reason for not including their contribution will become clear momentarily. Computing the level \( N \), the average angular momenta \( J \) and the average dipole charge \( D \) from (5.9) we get

\[
N = - \left( \frac{d \log Z}{d \beta} \right)_{\beta_\mu, \beta_\nu} = \frac{4\pi^2}{\beta^2} + \mu J + \nu D
\]  

\[
J = \left( \frac{d \log Z}{d \beta_\mu} \right)_{\beta_\beta, \nu} = \sum_{n=1}^{\infty} \frac{n}{n^2\beta - n\beta\mu - \beta\nu}
\]  

\[
D = \left( \frac{d \log Z}{d \beta_\nu} \right)_{\beta_\mu, \beta} = \sum_{n=1}^{\infty} \frac{1}{n^2\beta - n\beta\mu - \beta\nu}.
\]
The expression for $J$ appears to diverge, but that is due to the approximation that we made. If we include the contribution from $\alpha^{-}$, which is similar to that of $\alpha^{+}$ in (5.9) except that $\mu$ is replaced by $-\mu$, the expression for $J$ will be convergent. This $\alpha^{-}$ contribution will not be relevant for most of what follows though. The expressions for $J, D$ are at first sight of order $\sqrt{N} \sim \beta^{-1}$. To see this we need to include the contribution from $\alpha^{-}$ in $J$. In order for $J, D$ to be of order $N$, one term in the sum must be very large; if this happens for the term with $n = q_{3}$ then in order to have $J, D \sim N$ we need that

$$q_{3}^{2} - q_{3}\mu - \nu \sim \beta \ll 1 \quad (5.13)$$

Notice that this will imply condensation of modes with $n = q_{3}$, indeed

$$< 0|\alpha_{n}^{+}\alpha_{n}^{+}|0> = \frac{e^{\beta(-n+\mu+\nu/n)}}{1-e^{\beta(-n+\mu+\nu/n)}}, \quad (5.14)$$

which has a pole at $n = q_{3}$ for $q_{3}^{2} - q_{3}\mu - \nu = 0$. Obviously, the combination $n^{2} - n\mu - \nu$ has to be greater than 0 for all $n$ otherwise the thermodynamic system is ill-defined. If we also require that this quantity has a minimum obeying (5.13) at $n = q_{3}$, we find

$$\mu \approx 2q_{3}, \quad \nu \approx -q_{3}^{2}. \quad (5.15)$$

With these values of $\mu, \nu$ the term with $n = q_{3}$ will dominate the sum that appears in the partition function in (5.9). If we keep only that term together with the other contribution $4\pi^{2}/\beta$ we can compute the entropy and we find

$$S = \beta(N - \mu J - \nu D) + \log Z \approx \frac{8\pi^{2}}{\beta} = 4\pi\sqrt{N - \mu J - \nu D} = 4\pi\sqrt{N - q_{3}J}. \quad (5.16)$$

That agrees exactly with the (small) black ring entropy for a general dipole charge $q_{3}$!

### 6 Conclusions

Black rings have many features that make them interesting objects to study, in particular they provide examples of gravitational solutions that on the one hand carry a non trivial dipole charge and on the other hand admit a description in terms of a dual CFT. In this paper we have studied several aspects of this system, in particular we tried to understand the nature of the dipole charge from the dual CFT perspective.

Using standard AdS/CFT one can extract the one point functions of CFT operators in the ensemble dual to the black ring. In this paper we have focussed on scalar operators, to complete the analysis one should consider vector and tensor operators as well, which is in principle straightforward but rather tedious. The one point functions are complicated expressions of the seven parameters of the solution. Once we restrict to the case of 1/2-BPS or small black rings.
they simplify considerably (as many of the parameters are set to zero). The small black ring has vanishing macroscopic entropy, however we have shown that choosing appropriately the value of some parameters in the full seven-parameter solution both the microscopic entropy and one point functions of the small black ring can be reproduced. This would suggest these solutions have similarities with the small black ring once stringy corrections are included. It would be interesting to explore this further. As we have worked in the limit of large charges, such a conjecture could be further tested/refined by studying subleading corrections to the entropy and one point functions. Another interesting question would be to understand whether these small extra charges, which are introduced by hand but strictly speaking absent in the small black ring, can be understood as arising from some polarization effect.

We have also computed the one point functions directly in the orbifold CFT in the type of ensembles that are believed to be dual to the 1/2 − BPS black ring. The computation is very complicated to perform and we need to make a number of assumptions (in particular eq. (4.19)) in order to proceed; it would be nice to test further the validity of our assumptions. We have found that the leading $N$ contribution to the one point function for the small black ring agrees with those computed from supergravity. As a by-product we have found that one point functions in conical defects are subleading in $1/N$ but there is no obvious reason why they should vanish. It would be interesting to see whether this can be reproduced from supergravity once higher $\alpha'$ corrections are included.

In section 5 we have presented a simple toy model that correctly seems to reproduce the physics of the small black ring. To a string of length $n$ we associate a "dipole charge" $D = 1/n$, which indicates that the dipole charge corresponds to some non-local operator in the CFT; this is perhaps exactly what one expects for a dipole charge. Our proposal can easily be generalized (by including further negative powers of $n$) to more general solutions, for instance concentric black rings, and it would be interesting to study this in more detail. This same kind of expressions appear in many places in integrable systems. For instance, when considering the tower of non-local charges for strings on $AdS_5 \times S^5$ acting on BMN states the first non-local charge has this same expression. This suggests that it may be worthwhile to study a thermodynamic system of strings in AdS which includes a potential for each of the non-local charges, perhaps using the integrable model approach to strings in AdS.

Finally it remains an open problem to study the phase diagram of the thermodynamic toy model, and the corresponding supergravity solutions. For example, the toy model includes the description of conical defects. There is no reason, in the toy model, why $q_3$ should be an integer, and allowing arbitrary real $q_3$ could perhaps lead to families of supergravity solutions that interpolate between different conical defects. We should also point out that in general it is not clear exactly what type of ensemble one should use in general to describe gravitational solutions. Different ensembles usually yield the same leading answer for the entropy, but different subleading pieces. In [21] it was argued that small black rings should not be described as an ensemble with fixed angular momentum and energy, but rather as a system with a fixed
condensate and energy. Both points of view yield different subleading terms in the expansion of the entropy in large charges. It would be interesting to understand the connection between this suggestion and our toy model.

**Acknowledgments**

We would like to thank Iosif Bena, Justin David, Per Kraus, Finn Larsen, Donald Marolf and Pedro Silva for useful discussions, and Vijay Balasubramanian and Roberto Emparan for useful discussions and insightful comments on a draft of this paper. This work was supported in part by the stichting FOM.

**A**  

$SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ **generators**

**A.1**  

$M = 0$ **BTZ generators**

The isometry group of $AdS_3$, $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ is generated by the Virasoro generators $L_0, L_{\pm 1}$ and $\bar{L}_0, \bar{L}_{\pm 1}$. In cylindrical coordinates the $AdS_3$ metric reads

$$\frac{ds^2}{Q} = -\cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2 + d\rho^2$$  \hspace{1cm} (A.1)

and the Virasoro generators are given by [39] [40]

$$L_0 = i \partial_u,$$  \hspace{1cm} (A.2)

$$L_{-1} = i e^{-iu} \left( \frac{\cosh 2\rho}{\sinh 2\rho} \partial_u - \frac{1}{\sinh 2\rho} \partial_v + \frac{i}{2} \partial_\rho \right),$$  \hspace{1cm} (A.3)

$$L_1 = i e^{iu} \left( \frac{\cosh 2\rho}{\sinh 2\rho} \partial_u - \frac{1}{\sinh 2\rho} \partial_v - \frac{i}{2} \partial_\rho \right)$$  \hspace{1cm} (A.4)

with $u = \tau + \phi$, $v = \tau - \phi$. They satisfy the following commutation relations

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \quad [L_1, L_{-1}] = 2L_0.$$  \hspace{1cm} (A.5)

The right moving generators $\bar{L}_0, \bar{L}_{\pm 1}$ are given by similar expressions with $u \leftrightarrow v$.

The change of coordinates from (A.1) to the $M = 0$ BTZ metric is given by

$$r = Q (\cosh \rho \cos \tau + \sinh \rho \cos \phi),$$  \hspace{1cm} (A.6)

$$t = \frac{Q}{r} \cosh \rho \sin \tau,$$  \hspace{1cm} (A.7)

$$z = \frac{Q}{r} \sinh \rho \sin \phi.$$  \hspace{1cm} (A.8)
From this one obtains that the local $SL(2, \mathbb{R})_L$ generators of the $M=0$ BTZ metric are given by the vector fields
\begin{align*}
L_0 &= -\frac{1}{2}ir(t+z)\partial_r + \frac{i(-Q^2 + r^2(1 + (t+z)^2))}{4r^2}\partial_z + \frac{i(Q^2 + r^2(1 + (t+z)^2))}{4r^2}\partial_t \\
L_1 &= \frac{1}{2}ir(-i + t + z)\partial_r + \frac{i(Q^2 - r^2(-i + t + z)^2)}{4r^2}\partial_z - \frac{i(Q^2 + r^2(-i + t + z)^2)}{4r^2}\partial_t \\
L_{-1} &= \frac{1}{2}ir(i + t + z)\partial_r + \frac{i(Q^2 - r^2(i + t + z)^2)}{4r^2}\partial_z - \frac{i(Q^2 + r^2(i + t + z)^2)}{4r^2}\partial_t.
\end{align*}
(A.9)

The $SL(2, R)_R$ generators are obtained simply by taking $z \to -z$. The corresponding quadratic Casimirs are given by
\begin{align*}
L^2 &= \frac{1}{2}(L_{-1}L_{-1} + L_{+1}L_{-1}) - L_0^2 \\
\bar{L}^2 &= \frac{1}{2}(\bar{L}_{-1}\bar{L}_{-1} + \bar{L}_{+1}\bar{L}_{-1}) - \bar{L}_0^2.
\end{align*}
(A.10, A.11)

Of course, the $M=0$ BTZ metric does not have a global $SL(2, \mathbb{R})$ isometry group because it is a quotient of global AdS$_3$. This is reflected in (A.9) by the fact that these generators are not globally well-defined, due to the periodicity of the coordinate $z$. Still, the quadratic Casimirs are well-defined, and they provide the kinetic terms for the various fields that propagate in the $M=0$ BTZ background.

### A.2 Asymptotic Virasoro generators

As discussed in section 2, when we compute the one-point functions we should really view the large black ring as a “large” perturbation of global AdS$_3$, not as a “small” perturbation of the $M=0$ BTZ black hole. Thus, the right Virasoro generators we should use are those of AdS$_3$ written in coordinates in which the asymptotic behavior is identical to that of the black ring solution, which in turn is identical to asymptotic behavior of the $M=0$ BTZ solution. The right coordinates are those in which we view global AdS$_3$ as the $M=-1$ BTZ solution. They can be obtained from (A.12) through the change of coordinates
\[
\sinh \rho = r/Q
\]
(A.12)

and the Virasoro generators then become (in $r, t = \tau, z = \phi$ coordinates)
\begin{align*}
L_0 &= \frac{i}{2}(\partial_t + \partial_z) \\
L_+ &= \frac{i}{2}e^{i(t+z)}(-i\sqrt{Q^2 + r^2}\partial_r + \sqrt{1 + Q^2/r^2}\partial_z + \frac{1}{\sqrt{1 + Q^2/r^2}}\partial_t) \\
L_- &= \frac{i}{2}e^{-i(t+z)}(i\sqrt{Q^2 + r^2}\partial_r + \sqrt{1 + Q^2/r^2}\partial_z + \frac{1}{\sqrt{1 + Q^2/r^2}}\partial_t).
\end{align*}
(A.13, A.14, A.15)
with analogous expressions for the right-moving generators, with \( z \to -z \). Notice that these generators are globally well defined. On the other hand, they are exact isometries of the \( M = -1 \) BTZ metric (i.e. global AdS\(_3\)), but only asymptotic (at large \( r \)) isometries of the \( M = 0 \) BTZ.

**B  \( SO(4) \) harmonics**

In terms of embedding coordinates \( X^0, ..., X^3 \) the \( S^3 \) metric is written as

\[
 ds^2 = (dX^0)^2 + ... + (dX^3)^2.
\] (B.1)

The isometry group of \( S^3 \) is \( SO(4) \), whose generators in terms of the embedding coordinates \( X^0, ..., X^3 \) are given by

\[
 L_{ab} = X^a \partial_b - X^b \partial_a
\] (B.2)

subject to the constraint \((X^0)^2 + ... + (X^3)^2 = 1\). The appropriate change of coordinates to the metric given in (2.2) is

\[
 X^0 = \cos \theta \cos \phi, \quad X^1 = \cos \theta \sin \phi,
 X^2 = \sin \theta \cos \psi, \quad X^3 = \sin \theta \sin \psi.
\] (B.3)

The group \( SO(4) \) is isomorphic to two copies of \( SU(2) \). One of these is generated by

\[
 G_1 = \frac{L_{23} - L_{01}}{2} = \frac{1}{2}(\partial_\psi - \partial_\phi)
\] (B.5)

\[
 G_2 = \frac{L_{12} - L_{03}}{2} = \frac{1}{2}(-\cot \theta \cos (\phi - \psi) \partial_\psi - \tan \theta \cos(\phi - \psi) \partial_\phi + \sin(\phi - \psi) \partial_\theta)
\] (B.6)

\[
 G_3 = \frac{L_{02} - L_{13}}{2} = \frac{1}{2}(\cot \theta \sin (\phi - \psi) \partial_\psi + \tan \theta \sin(\phi - \psi) \partial_\phi + \cos(\phi - \psi) \partial_\theta)
\] (B.7)

and the other one with generators \( \tilde{G}_i \) is generated by the same vector fields but with \( \phi \) replaced by \(-\phi\). They satisfy the following commutation relations

\[
 [G_1, G_2] = -G_3, \quad [G_2, G_3] = -G_1, \quad [G_3, G_1] = -G_2
\] (B.8)

together with \([G_i, \tilde{G}_j] = 0\). Notice that the perturbation is trivially invariant under the action of \( G_1 \) and \( \tilde{G}_1 \) that we will take as the Cartan generators. The quadratic Casimirs are given by

\[
 G^2 = -4 \left( G_1^2 + G_2^2 + G_3^2 \right)
\] (B.9)

\[
 \tilde{G}^2 = -4 \left( \tilde{G}_1^2 + \tilde{G}_2^2 + \tilde{G}_3^2 \right).
\] (B.10)

The scalar spherical harmonics are given by

\[
 Y^k_s(\theta) = \sqrt{1 + k} \; F\left(1 + \frac{k}{2}, -\frac{k}{2}; 1, \sin^2 \theta\right)
\] (B.11)
for \( k = 0, 2, 4, 6, \ldots \). The action of the quadratic Casimirs on the spherical harmonics is as follows

\[
G^2 Y^k_s(\theta) = G^2 Y^k_s(\theta) = k(k + 2)Y^k_s(\theta), \quad (B.12)
\]

where the normalization constant has been chosen so that

\[
\int_{S^3} Y^k_s Y^{k'}_s = \int_0^{\pi/2} 2\sin \theta \cos \theta Y^k_s(\theta) Y^{k'}_s(\theta) d\theta = \delta^{kk'}. \quad (B.13)
\]

Notice that with this normalization

\[
\int_{S^3} \nabla^a Y^k_s \nabla_a Y^{k'}_s = k(k + 2)\delta^{kk'}, \quad (B.14)
\]

\[
\int_{S^3} \nabla^a(\nabla^b) Y^k_s \nabla_a Y^{k'}_s = \frac{2}{3} k(k + 2)(k(k + 2) - 3)\delta^{kk'}, \quad (B.15)
\]

where we use \( (a, b) \) to denote the symmetric traceless combination and indices are raised and lowered with the \( S^3 \) metric \( G^0_{ab} = \text{Diag}(\sin^2 \theta, \cos^2 \theta, 1) \).

The vector spherical harmonics are given by

\[
Y^k_{v\pm}(\theta) = \sqrt{\frac{k}{2}} \left[ -\frac{k}{2} \sin^2 \theta F(1 - \frac{k}{2}, 1 + \frac{k}{2}; 2, \sin^2 \theta) \right]_{\pm} \pm F(-\frac{k}{2}, \frac{k}{2}, 1, \sin^2 \theta), \quad 0 \right) \quad (B.16)
\]

for \( k = 2, 4, 6, \ldots \). The quadratic Casimirs act as follows

\[
G^2 Y^k_{v\pm} = (k - 1 \pm 1)(k + 1 \pm 1)Y^k_{v\pm}, \quad G^2 Y^k_{v\pm} = (k - 1 \mp 1)(k + 1 \mp 1)Y^k_{v\pm} \quad (B.17)
\]

The normalization factor has been chosen so that

\[
\int_{S^3} (G^0)^{ab}(Y^k_{v\pm})_a (Y^k_{v\pm})_b = \delta^{kk'}, \quad (B.18)
\]

Finally, the tensor spherical harmonics are given by

\[
Y^k_{t\pm} = \frac{1}{4} \sqrt{k(k - 1)(k - 2)} \left( \begin{array}{ccc} f_1(\theta) & \pm g(\theta) & 0 \\ \pm g(\theta) & f_2(\theta) & 0 \\ 0 & 0 & f_3(\theta) \end{array} \right) \quad (B.19)
\]

with

\[
f_1(\theta) = \frac{1}{4(k - 1)} \left( -4 \sin^2 \theta F(1 - \frac{k}{2}, \frac{k}{2}; 2, \sin^2 \theta) + k(k - 2) \sin^4 \theta F(2 - \frac{k}{2}, 1 + \frac{k}{2}; 3, \sin^2 \theta) \right)
\]

\[
f_2(\theta) = \frac{F(1 - \frac{k}{2}, \frac{k}{2}; 2, \sin^2 \theta)}{k - 1} - \cot^2 \theta f_1(\theta)
\]

\[
f_3(\theta) = \frac{F(1 - \frac{k}{2}, \frac{k}{2}; 2, \sin^2 \theta)}{(1 - k) \cos^2 \theta}
\]

\[
g(\theta) = \sin^2 \theta F(1 - \frac{k}{2}, \frac{k}{2}; 2, \sin^2 \theta).
\]

\[30\]
The action of the quadratic casimirs is given by

\[ G_2 Y_{t\pm}^k = (k - 2 \pm 2)(k \pm 2) Y_{t\pm}^k, \quad \bar{G}_2 Y_{t\pm}^k = (k - 2 \mp 2)(k \mp 2) Y_{t\pm}^k. \tag{B.21} \]

The normalization factor has been chosen so that

\[ \int_{S^3} (G^0)^{ac}(G^0)^{bd}(Y_{t\pm}^k)^{ab}(Y_{t\pm}^{k'})_{cd} = \delta^{kk'}. \tag{B.22} \]

References

[1] J. M. Maldacena, JHEP **0304**, 021 (2003) [arXiv:hep-th/0106112].

[2] O. Lunin and S. D. Mathur, “AdS/CFT duality and the black hole information paradox,” Nucl. Phys. B **623** (2002) 342 [arXiv:hep-th/0109154].

[3] O. Lunin, S. D. Mathur and A. Saxena, “What is the gravity dual of a chiral primary?,” Nucl. Phys. B **655**, 185 (2003) [arXiv:hep-th/0211292].

[4] S. D. Mathur, “The fuzzball proposal for black holes: An elementary review,” Fortsch. Phys. **53** (2005) 793 [arXiv:hep-th/0502050].

[5] H. Lin, O. Lunin and J. Maldacena, “Bubbling AdS space and 1/2 BPS geometries,” JHEP **0410** (2004) 025 [arXiv:hep-th/0409174].

[6] A. Buchel, “Coarse-graining 1/2 BPS geometries of type IIB supergravity,” [arXiv:hep-th/0409271]

[7] N. V. Suryanarayana, “Half-BPS giants, free fermions and microstates of superstars,” [arXiv:hep-th/0411145]

[8] P. G. Shepard, “Black hole statistics from holography,” JHEP **0510**, 072 (2005) [arXiv:hep-th/0507260].

[9] V. Balasubramanian, J. de Boer, V. Jejjala and J. Simon, “The library of Babel: On the origin of gravitational thermodynamics,” [arXiv:hep-th/0508023]

[10] P. J. Silva, “Rational foundation of GR in terms of statistical mechanic in the AdS/CFT framework,” [arXiv:hep-th/0508081]

[11] R. C. Myers and O. Tafjord, “Superstars and giant gravitons,” JHEP **0111** (2001) 009 [arXiv:hep-th/0109127].

[12] R. Emparan, “Rotating circular strings, and infinite non-uniqueness of black rings,” JHEP **0403** (2004) 064 [arXiv:hep-th/0402149].
[13] K. Copsey and G. T. Horowitz, “The role of dipole charges in black hole thermodynamics,” arXiv:hep-th/0505278.

[14] S. Corley, A. Jevicki and S. Ramgoolam, “Exact correlators of giant gravitons from dual N = 4 SYM theory,” Adv. Theor. Math. Phys. 5 (2002) 809 arXiv:hep-th/0111222.

[15] D. Berenstein, “A toy model for the AdS/CFT correspondence,” JHEP 0407, 018 (2004) arXiv:hep-th/0403110.

[16] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, “Supersymmetric black rings and three-charge supertubes,” Phys. Rev. D 71 (2005) 024033 arXiv:hep-th/0408120.

[17] I. Bena and P. Kraus, “Microscopic description of black rings in AdS/CFT,” JHEP 0412 (2004) 070 arXiv:hep-th/0408186.

[18] N. Iizuka and M. Shigemori, “A note on D1-D5-J system and 5D small black ring,” JHEP 0508 (2005) 100 arXiv:hep-th/0506215.

[19] V. Balasubramanian, P. Kraus and M. Shigemori, “Massless black holes and black rings as effective geometries of the D1-D5 system,” arXiv:hep-th/0508110.

[20] B. C. Palmer and D. Marolf, “Counting supertubes,” JHEP 0406, 028 (2004) arXiv:hep-th/0403025.

[21] A. Dabholkar, N. Iizuka, A. Iqubal and M. Shigemori, “Precision microstate counting of small black rings,” arXiv:hep-th/0511120.

[22] V. Balasubramanian, J. de Boer, E. Keski-Vakkuri and S. F. Ross, “Supersymmetric conical defects: Towards a string theoretic description of black hole formation,” Phys. Rev. D 64 (2001) 064011 arXiv:hep-th/0011217.

[23] J. M. Maldacena and L. Maoz, “De-singularization by rotation,” JHEP 0212 (2002) 055 arXiv:hep-th/0012025.

[24] O. Lunin, J. Maldacena and L. Maoz, “Gravity solutions for the D1-D5 system with angular momentum,” arXiv:hep-th/0212210.

[25] A. Dabholkar, “Exact counting of black hole microstates,” Phys. Rev. Lett. 94 (2005) 241301 arXiv:hep-th/0409148.

[26] J. P. Gauntlett and J. B. Gutowski, “Concentric black rings,” Phys. Rev. D 71 (2005) 025013 arXiv:hep-th/0408010.

[27] J. P. Gauntlett and J. B. Gutowski, “General concentric black rings,” Phys. Rev. D 71 (2005) 045002 arXiv:hep-th/0408122.
[28] L. F. Alday, “Non-local charges on AdS(5) x S**5 and pp-waves,” JHEP 0312 (2003) 033 [arXiv:hep-th/0310146].

[29] I. Bena, J. Polchinski and R. Roiban, “Hidden symmetries of the AdS(5) x S**5 superstring,” Phys. Rev. D 69 (2004) 046002 [arXiv:hep-th/0305116].

[30] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, “A supersymmetric black ring,” Phys. Rev. Lett. 93 (2004) 211302 [arXiv:hep-th/0407065].

[31] J. C. Breckenridge, R. C. Myers, A. W. Peet and C. Vafa, “D-branes and spinning black holes,” Phys. Lett. B 391 (1997) 93 [arXiv:hep-th/9602065].

[32] R. Emparan and D. Mateos, “Oscillator level for black holes and black rings,” Class. Quant. Grav. 22 (2005) 3575 [arXiv:hep-th/0506110].

[33] S. Deger, A. Kaya, E. Sezgin and P. Sundell, “Spectrum of D = 6, N = 4b supergravity on AdS(3) x S(3),” Nucl. Phys. B 536 (1998) 110 [arXiv:hep-th/9804166].

[34] G. Arutyunov, A. Pankiewicz and S. Theisen, “Cubic couplings in D = 6 N = 4b supergravity on AdS(3) x S(3),” Phys. Rev. D 63 (2001) 044024 [arXiv:hep-th/0007061].

[35] M. Mihailescu, “Correlation functions for chiral primaries in D = 6 supergravity on AdS(3) x S(3),” JHEP 0002 (2000) 007 [arXiv:hep-th/9910111].

[36] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, “Correlation functions in the CFT(d)/AdS(d+1) correspondence,” Nucl. Phys. B 546 (1999) 96 [arXiv:hep-th/9804058].

[37] I. R. Klebanov and E. Witten, “AdS/CFT correspondence and symmetry breaking,” Nucl. Phys. B 556 (1999) 89 [arXiv:hep-th/9905104].

[38] J. G. Russo and L. Susskind, “Asymptotic level density in heterotic string theory and rotating black holes,” Nucl. Phys. B 437, 611 (1995) [arXiv:hep-th/9405117].

[39] J. M. Maldacena and A. Strominger, “AdS(3) black holes and a stringy exclusion principle,” JHEP 9812 (1998) 005 [arXiv:hep-th/9804085].

[40] J. de Boer, “Six-dimensional supergravity on S**3 x AdS(3) and 2d conformal field theory,” Nucl. Phys. B 548 (1999) 139 [arXiv:hep-th/9806104].

[41] A. Strominger and C. Vafa, “Microscopic Origin of the Bekenstein-Hawking Entropy,” Phys. Lett. B 379, 99 (1996) [arXiv:hep-th/9601029].

[42] N. Seiberg and E. Witten, “The D1/D5 system and singular CFT,” JHEP 9904 (1999) 017 [arXiv:hep-th/9903224].
[43] J. de Boer, “Large N Elliptic Genus and AdS/CFT Correspondence,” JHEP **9905** (1999) 017 [arXiv:hep-th/9812240].

[44] R. Dijkgraaf, “Instanton strings and hyperKaehler geometry,” Nucl. Phys. B **543** (1999) 545 [arXiv:hep-th/9810210].

[45] F. Larsen and E. J. Martinec, “U(1) charges and moduli in the D1-D5 system,” JHEP **9906** (1999) 019 [arXiv:hep-th/9905064].

[46] J. R. David, G. Mandal and S. R. Wadia, “D1/D5 moduli in SCFT and gauge theory, and Hawking radiation,” Nucl. Phys. B **564** (2000) 103 [arXiv:hep-th/9907075]; “Microscopic formulation of black holes in string theory,” Phys. Rept. **369**, 549 (2002) [arXiv:hep-th/0203048].

[47] J. A. Harvey and G. W. Moore, “Algebras, BPS States, and Strings,” Nucl. Phys. B **463** (1996) 315 [arXiv:hep-th/9510182].

[48] J. A. Harvey and G. W. Moore, “On the algebras of BPS states,” Commun. Math. Phys. **197** (1998) 489 [arXiv:hep-th/9609017].

[49] M. Lehn, C. Sorger, “The cup product of the Hilbert scheme for K3 surface” [arXiv:math.AG/0012166].

[50] Y. Ruan, “Stringy orbifolds” [arXiv:math.AG/0201123].

[51] W. Wang, “Universal rings arising in geometry and group theory” [arXiv:math.QA/0211093].

[52] A. Jevicki, M. Mihaiescu and S. Ramgoolam, “Gravity from CFT on S**N(X): Symmetries and interactions,” Nucl. Phys. B **577** (2000) 47 [arXiv:hep-th/9907144].

[53] O. Lunin and S. D. Mathur, “Three-point functions for M(N)/S(N) orbifolds with N = 4 supersymmetry,” Commun. Math. Phys. **227** (2002) 385 [arXiv:hep-th/0103169].

[54] F. Larsen, “The perturbation spectrum of black holes in N = 8 supergravity,” Nucl. Phys. B **536**, 258 (1998) [arXiv:hep-th/9805208].