On the difference between the volatility swap strike and the zero vanna implied volatility

Elisa Alòs† Frido Rolloos Kenichiro Shiraya‡

December 12, 2019

Abstract

In this paper, Malliavin calculus is applied to arrive at exact formulas for the difference between the volatility swap strike and the zero vanna implied volatility for volatilities driven by fractional noise. To the best of our knowledge, our estimate is the first to show the rigorous relationship between the zero vanna implied volatility and the volatility swap strike. In particular, we will see that the zero vanna implied volatility has a higher rate of convergence than the at-the-money (ATM) implied volatility for both zero and non-zero correlation and for all values of the Hurst parameter.

Keywords:
AMS subject classification: 91G99

1 Introduction

The pricing and hedging of volatility derivatives continue to be an active and fruitful area of research in quantitative finance. One of the first volatility derivatives to be traded in the over-the-counter market is the variance swap. Another instrument to trade volatility is the volatility swap, which unlike the variance swap has a payoff that is linear in volatility. However, volatility swaps are less liquid than variance swaps. The reason for this is because the price of a volatility swap was for a long time considered to be highly model-dependent.

It was Carr and Lee (2008) that first challenged the idea that volatility swaps are highly model-dependent. In the case where the correlation between the volatility and the underlying asset is zero, Carr and Lee proved in their seminal paper that the exact volatility swap strike is in fact model-free, and like the variance swap can be synthesised using a continuous strip of options. The difference is that in the volatility swap case the replicating strip of options has to be continuously rebalanced. An elegant derivation of the replicating portfolio for volatility swaps is given by Friz and Gatheral (2005). When correlation deviates from zero, there is indeed model dependence. But a substantial part of the exact volatility swap price, regardless of the model, is still model-independent.

†Dpt. d’Economia i Empresa, Universitat Pompeu Fabra.
‡Graduate School of Economics, The University of Tokyo. Kenichiro Shiraya is supported by CARF.
In recent years, the fractional volatility models introduced by Comte and Renault (1998) have led to several papers which explore the valuation of volatility derivatives under the models. For example, Bergomi and Guyon (2011) and El Euch, Fukawasa, Gatheral and Rosenbaum (2019) derive approximation formulas for the variance swap strike by using expansion techniques. Alós and Shiraya (2019) approximates the volatility swap strike by immunising correlation dependence to first order and also provides an estimation method for the Hurst parameter from ATM implied volatilities.

While the aforementioned papers establish relationships between volatility derivatives and the ATM implied volatility, a different approach to non-parametric pricing of volatility swaps has been put forth by Rolloos and Arslan (2017). Using only the generalised Hull-White formula and Taylor expansions, they show that the volatility swap strike is approximately equal to the implied volatility at the strike where the Black-Scholes vanna of a vanilla option is zero. Like the Carr-Lee approximation, the Rolloos-Arslan approximation is to a large extent immune to correlation to first order. Furthermore, although their two approximations are not equal, numerical tests thus far have shown that both are of comparable accuracy.

A pleasing feature of the zero vanna implied volatility approximation is that it is not only intuitive and easy to implement, but also lends itself to rigorous quantification of the error between the true volatility swap price and the zero vanna implied volatility. This paper extends the model of Rolloos and Arslan (2017) to general fractional volatility models and provides the rigorous relationship between the zero vanna implied volatility and the volatility swap strike. We show that in the uncorrelated case the zero vanna implied volatility does not coincide with the volatility swap strike even though the approximation is very accurate. Furthermore, in the correlated case we prove that the first order of $\rho$ is not immunised completely. However, numerical examples show that the zero vanna implied volatility is a better approximation for the volatility swap strike than both the ATM implied volatility and the approximation formula of Alós and Shiraya (2019) for the cases we consider in this paper.

The paper is organised as follows. In Section 2 we introduce the relevant concepts and establish notation. Section 3 is devoted to deriving exact expression for the difference between the volatility swap strike and the zero vanna implied volatility for the zero correlation case. This result is generalised in Section 4 to the case when correlation deviates from zero. In Section 5 numerical examples are presented for various values of the Hurst parameter. Section 6 contains concluding remarks.

2 The main problem and notations

Consider a stochastic volatility model for the log-price of a stock under a risk-neutral probability measure $P$:

$$X_t = X_0 - \frac{1}{2} \int_0^t \sigma_s^2 ds + \int_0^t \sigma_s \left( \rho dW_s + \sqrt{1 - \rho^2} dB_s \right), \quad t \in [0, T].$$  \hspace{1cm} (2.1)

Here, $X_0$ is the current log-price, $W$ and $B$ are standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{G}, P)$, and $\sigma$ is a square-integrable and right-continuous stochastic process adapted to the filtration generated by $W$. We denote by $\mathcal{F}^W$ and $\mathcal{F}^B$ the filtrations generated by
We make use of the following notation: where $E$, $W$, and $r$. The same arguments in this paper hold for $r 
eq 0$.

Under the above model, the price of a European call with strike price $K$ is given by the equality

$$V_t = E_t[(e^{X_t} - K)_+],$$

where $E_t$ is the $\mathcal{F}_t$-conditional expectation with respect to $P$ (i.e., $E_t[Z] = E[Z|\mathcal{F}_t]$). In the sequel, we make use of the following notation:

- $v_t = \sqrt{\frac{Y_t}{2\sigma^2 t}}$, where $Y_t = \int_t^T \sigma^2 du$. That is, $v$ represents the future average volatility, and it is not an adapted process. Notice that $E_t[v_t]$ is the fair strike of a volatility swap with maturity time $T$.

- $BS(t, T, x, k, \sigma)$ is the price of a European call option under the classical Black-Scholes model with constant volatility $\sigma$, stock price $e^x$, time to maturity $T - t$, and strike $K = \exp(k)$. Remember that (if $r = 0$)

$$BS(t, T, x, k, \sigma) = e^x N(d_1(k, \sigma)) - e^\lambda N(d_2(k, \sigma)),$$

where $N$ denotes the cumulative probability function of the standard normal law and

$$d_1(k, \sigma) := \frac{x - k}{\sigma \sqrt{T - t}} + \frac{\sigma}{2 \sqrt{T - t}}, \quad d_2(k, \sigma) := \frac{x - k}{\sigma \sqrt{T - t}} - \frac{\sigma}{2 \sqrt{T - t}}.$$

For the sake of simplicity, we make use of the notation $BS(k, \sigma) := BS(t, T, x, k, \sigma)$.

- The inverse function $BS^{-1}(t, T, x, k, \cdot)$ of the Black-Scholes formula with respect to the volatility parameter is defined as

$$BS(t, T, x, k, BS^{-1}(t, T, x, k, \lambda)) = \lambda,$$

for all $\lambda > 0$. For the sake of simplicity, we denote $BS^{-1}(k, \lambda) := BS^{-1}(t, T, x, k, \lambda)$.

- For any fixed $t, T, X_t, k$, we define the implied volatility $I(t, T, X_t, k)$ as the quantity such that

$$BS(t, T, X_t, k, I(t, T, X_t, k)) = V_t.$$  

Notice that $I(t, T, X_t, k) = BS^{-1}(t, T, X_t, k, V_t)$.

- $\hat{k}_t$ is the zero vanna implied volatility strike at time $t$. That is, the strike such that

$$d_2(\hat{k}_t, I(t, T, X_t, \hat{k}_t)) = 0.$$

Moreover, we will refer to $I(t, T, X_t, \hat{k}_t)$ as the zero vanna implied volatility.

- $\Lambda_t := E_r[BS(t, T, X_t, k, v_t)]$.

- $\Theta_t(k) := BS^{-1}(k, \Lambda_t)$. Notice that $\Theta_t(k) = I(t, X_t, k, V_t)$ and $\Theta_T(k) = v_t$. 

3
• \( G(t, T, x, k, \sigma) := \left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(t, T, x, k, \sigma) \).

• \( H(t, T, x, k, \sigma) := \left( \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) BS(t, T, x, k, \sigma) \).

In the remaining of this paper \( D_{W}^{1,2} \) denotes the domain of the Malliavin derivative operator \( D_{W} \) (see Appendix 1) with respect to the Brownian motion \( W \). We also consider the iterated derivatives \( D_{W}^{n} \), for \( n > 1 \), whose domains will be denoted by \( D_{W}^{n,2} \). We will use the notation \( L_{W}^{n,2} = L^{2}([0, T]; D_{W}^{n,2}) \).

3 The uncorrelated case

Let us consider the following hypotheses:

(H1) There exist two positive constants \( a, b \) such that \( a \leq \sigma_t \leq b \), for all \( t \in [0, T] \).

(H2) \( \sigma^2 \in L^{2} \) and there exist two constants \( \nu > 0 \) and \( H \in (0, 1) \) such that, for all \( 0 < r, \theta < s < T \),

\[ |E_r[D_{r}\sigma^2_s]| \leq \nu(s-r)^{H-\frac{1}{2}}, \quad |E_r[D_{r}\sigma^2_s]| \leq \nu^2(s-r)^{H-1}(s-\theta)^{H-\frac{1}{2}}. \]

The key tool in our analysis will be the following relationship between the zero vanna implied volatility and the fair strike of a volatility swap.

Proposition 1. Consider the model (2.1) with \( \rho = 0 \) and assume that hypotheses (H1) and (H2) hold. Then the zero vanna implied volatility admits the representation

\[
I(t, T, X_t, \hat{k}_t) = E_t[v_t] + \frac{1}{2}E_t \left[ \int_t^T \left( \frac{\partial^3}{\partial x^3} \left( \frac{\partial}{\partial x} \right) BS(t, T, X_t, \hat{k}_t, v_t) \right) \right] + \frac{1}{4}E_t \left[ \int_t^T \left( \frac{\partial^2}{\partial x^2} \left( \frac{\partial}{\partial x} \right) \right) A_t U_r^2 dr \right],
\]

where

\[
A_r := \frac{1}{2} \int_r^T U_r^2 ds, \quad (D^{-}A)_r := \int_r^T D_r U_r^2 ds,
\]

and

\[
U_r := E_r[D_r(\sigma^2)] = E_r \left[ \frac{\partial BS(t, T, X_t, \hat{k}_t, v_t)}{\partial \sigma} \right] \frac{1}{2v_t(T-t)} \int_r^T D_r \sigma^2 ds.
\]

Proof. This proof is decomposed into several steps.
Step 1 First, we will show that

\[ I(t, T, X_t, \hat{k}_t) = E_t [v_t] + \frac{1}{2} E_t \left[ \int_t^T (BS^{-1})''(k, \Lambda_t) U_r^2 dr \right]. \] (3.3)

Observe that, as \( \rho = 0 \), the Hull and White formula gives is that \( V_t = \Lambda_t \). Then, as in the proof of Proposition 3.1 in Alòs and Shiraya (2019) we can write

\[ I(t, T, X_t, \hat{k}_t) = BS^{-1}(\Lambda_t) = E_t[BS^{-1}(\Lambda_t)]. \] (3.4)

Now, (H2) and the Clark-Ocone formula (see Appendix 1) give us that \( \Lambda_t \) admits the martingale representation given by

\[ d\Lambda_t = E_r [D_r(BS(t, T, X_t, \hat{k}_t, v_t))] \]
\[ = E_r \left[ \frac{dBS}{d\sigma}(t, T, X_t, \hat{k}_t, v_t) \frac{1}{2} \int_r^T D_r \sigma_s^2 ds \right] dW_r \]
\[ = U_r dW_r. \] (3.5)

Then, a direct application of the classical Itô’s formula gives us that, after taking expectations:

\[ E_t[BS^{-1}(\hat{k}_t, \Lambda_t)] = E_t[BS^{-1}(\hat{k}_t, \Lambda_T)] - \frac{1}{2} E_t \left[ \int_t^T (BS^{-1})''(\hat{k}_t, \Lambda_t) d\langle \Lambda, \Lambda \rangle_r \right]. \] (3.6)

Now, as \( \Lambda_T = BS(t, T, X_t, \hat{k}_t, v_t) \), (3.4) and (3.7) imply that

\[ I(t, T, X_t, \hat{k}_t) = E_t [v_t] - \frac{1}{2} E_t \left[ \int_t^T (BS^{-1})''(\hat{k}_t, \Lambda_t) U_r^2 dr \right]. \] (3.7)

That is,

\[ I(t, T, X_t, \hat{k}_t) = E_t [v_t] - \frac{1}{2} E_t \left[ \int_t^T (BS^{-1})''(\hat{k}_t, \Lambda_t) U_r^2 dr \right]. \]

Step 2 Next, let us see that

\[ E_t \left[ \int_t^T (BS^{-1})''(\hat{k}_t, \Lambda_r) U_r^2 dr \right] \]
\[ = E_t \left[ \int_t^T (BS^{-1})(\hat{k}_t, \Lambda_r))''(D^{-1}A)_r U_r^2 dr \right] \]
\[ + \frac{1}{2} E_t \left[ \int_t^T (BS^{-1})(\hat{k}_t, \Lambda_r))^{(iv)} A_r U_r^2 dr \right]. \] (3.8)

Towards this end, we apply the anticipating Itô’s formula (see Appendix 1) to the process

\[ (BS^{-1})''(\hat{k}_t, \Lambda_r) A_r, \]
and, taking into account that \( dA_t = -U_t^2 dr \), we get

\[
E_t \left[ (BS^{-1})'' \left( \hat{k}_t, \Lambda_T \right) A_T \right] = E_t \left[ (BS^{-1})'' \left( \hat{k}_t, \Lambda_t \right) A_t \right] - \frac{1}{2} E_t \left[ \int_t^T (BS^{-1})'' \left( \hat{k}_t, \Lambda_t \right) U_t^2 dr \right] + \frac{1}{2} E_t \left[ \int_t^T (BS^{-1} (\hat{k}_t, \Lambda_t))''' (D^- A)_t U_t dr \right] + \frac{1}{4} E_t \left[ \int_t^T (BS^{-1} (\hat{k}_t, \Lambda_t))^{(vi)} \right] A_t U_t^2 dr. \tag{3.9}
\]

Now, a direct computation gives us that

\[
(BS^{-1})'' (\hat{k}_t, \Lambda_t) = - \frac{\partial^2 BS}{\partial \sigma^2} (\hat{k}_t, \Theta_t(\hat{k}_t)) \left( \frac{\partial BS}{\partial \sigma} (k_t, \Theta_t(\hat{k}_t)) \right)^3 \quad \frac{(\Theta_t(\hat{k}_t))^4(T-t)^2 - 4(X_t - k_t)^2}{4 \left( \exp(X_t)N'(d_1 (k_t, \Theta_t(\hat{k}_t))(T-t)) \right)^2 (\Theta_t(\hat{k}_t))^3}.
\]

In particular, \((BS^{-1})'' (\hat{k}_t, \Lambda_t) = 0\) and

\[
(BS^{-1})' (\hat{k}_t, \Lambda_T) = \frac{(v_t^4 - (\Theta_t(\hat{k}_t))^4)}{4 \left( \exp(X_t)N'(d_1 (k_t, \Theta_t(\hat{k}_t))) \right)^2 v_t^4}, \tag{3.10}
\]

which implies that \((BS^{-1})'' (\hat{k}_t, \Lambda_T) A_T = 0\). Then

\[
\frac{1}{2} E_t \left[ \int_t^T (BS^{-1})'' (\hat{k}_t, \Lambda_t) U_t^2 dr \right] = \frac{1}{2} E_t \left[ \int_t^T (BS^{-1} (\hat{k}_t, \Lambda_t))''' (D^- A)_t U_t dr \right] + \frac{1}{4} E_t \left[ \int_t^T (BS^{-1} (\hat{k}_t, \Lambda_t))^{(vi)} \right] A_t U_t^2 dr, \tag{3.11}
\]

which completes the proof. \(\square\)

In order to prove our limit results, we will need the following hypothesis.

- (H2') \( \sigma \in \mathbb{L}^{3.2} \) and there exists two constants \( \nu > 0 \) and \( H \in (0, 1) \) such that, for all \( 0 < r < u, s, \theta < T \)

\[
|E_t[D_o \sigma_s^2]| \leq \nu (s - r)^{H-\frac{1}{2}}, \quad |E_t[D_o D_r \sigma_s^2]| \leq \nu^2 (s - r)^{H-\frac{1}{2}} (s - \theta)^{H-\frac{1}{2}},
\]

and

\[
|E_t[D_o D_o D_r \sigma_s^2]| \leq \nu^3 (s - r)^{H-\frac{1}{2}} (s - \theta)^{H-\frac{1}{2}} (s - u)^{H-\frac{1}{2}}.
\]
\textbf{Theorem 2.} Consider the model (2.1) and assume that hypotheses (H1) and (H2') hold. Then, there exist a constant \( C_t \) such that
\[
\lim_{T \to t, \nu \to 0} \frac{I(t, T, X_t, \hat{k}_t) - E_t [v_t]}{\nu^4 (T - t)^{4H+1}} = C_t.
\]

\textit{Proof.} Again, the proof is decomposed into several steps.

\textbf{Step 1} We start by showing that
\[
I(t, T, X_t, \hat{k}_t) = E_t [v_t]
\]
\[
+ \frac{1}{2} \left( BS^{-1} (\hat{k}_t, \Lambda_t) \right)^{(iv)} \int_t^T (D^{-A})_r U_r dr
\]
\[
+ \frac{1}{4} \left( BS^{-1} (\hat{k}_t, \Lambda_t) \right)^{(v)} \int_t^T A_r U_r^2 dr
\]
\[
+ T_1 + T_2 + T_3 + T_4,
\]
where
\[
T_1 = E_t \left[ \int_t^T \left( BS^{-1} (\hat{k}_t, \Lambda_r) \right)^{(iv)} (D^{-A})_r U_r dr \right],
\]
\[
T_2 = \frac{1}{2} E_t \left[ \int_t^T \left( BS^{-1} (\hat{k}_t, \Lambda_r) \right)^{(v)} \Psi_r U_r^2 dr \right],
\]
\[
T_3 = E_t \left[ \int_t^T \left( BS^{-1} (\hat{k}_t, \Lambda_r) \right)^{(v)} (D^{-\Phi})_r U_r dr \right],
\]
and
\[
T_4 = \frac{1}{2} E_t \left[ \int_t^T \left( BS^{-1} (\hat{k}_t, \Lambda_r) \right)^{(v)} \Phi_r U_r^2 dr \right],
\]
with \( \Psi_t := \int_t^T (D^{-A})_r U_r dr \) and \( \Phi_t := \int_t^T A_r U_r^2 dr \). Towards this end we can apply the anticipating Itô’s formula to the processes
\[
\left( BS^{-1} (\hat{k}_t, \Lambda_t) \right)^{(iv)} \int_t^T (D^{-A})_r U_r dr = \left( BS^{-1} (\hat{k}_t, \Lambda_t) \right)^{(iv)} \Psi(t),
\]
and
\[
\frac{1}{4} \left( BS^{-1} (\hat{k}_t, \Lambda_t) \right)^{(v)} \int_t^T A_r U_r^2 dr = \frac{1}{4} \left( BS^{-1} (\hat{k}_t, \Lambda_t) \right)^{(v)} \Phi(t).
\]
Then, the same arguments as in the proof of Proposition \(\Box\) allow us to write

\[
E_t \left[ \int_t^T (BS^{-1}(\hat{k}_t, \Lambda_t))''''(D^{-}A)_tU_tdr \right]
\]

\[
= (BS^{-1}(\hat{k}_t, \Lambda_t))'''' E_t \left[ \int_t^T (D^{-}A)_tU_tdr \right]
\]

\[
+ E_t \left[ \int_t^T (BS^{-1}(\hat{k}_t, \Lambda_t))^{(iv)} (D^{-}\Psi)_tU_tdr \right]
\]

\[
+ \frac{1}{2} E_t \left[ \int_t^T (BS^{-1}(\hat{k}_t, \Lambda_t))^{(v)} \Psi_tU_t^2dr \right]
\]

\[
= (BS^{-1}(\hat{k}_t, \Lambda_t))'''' E_t \left[ \int_t^T (D^{-}A)_tU_tdr \right] + T_1 + T_2, \quad (3.13)
\]

and

\[
E_t \left[ \int_t^T (BS^{-1}(\hat{k}_t, \Lambda_t))^{(iv)} A_tU_t^2dr \right]
\]

\[
= (BS^{-1}(\hat{k}_t, \Lambda_t))^{(iv)} E_t \left[ \int_t^T A_tU_t^2dr \right]
\]

\[
+ E_t \left[ \int_t^T (BS^{-1}(\hat{k}_t, \Lambda_t))^{(iv)} (D^{-}\Phi)_tU_tdr \right]
\]

\[
+ \frac{1}{2} E_t \left[ \int_t^T (BS^{-1}(\hat{k}_t, \Lambda_t))^{(v)} \Phi_tU_t^2dr \right]
\]

\[
= (BS^{-1}(\hat{k}_t, \Lambda_t))^{(iv)} E_t \left[ \int_t^T A_tU_t^2dr \right] + T_3 + T_4. \quad (3.14)
\]

**Step 2** Now, let us study the term

\[
(BS^{-1}(\hat{k}_t, \Lambda_t))'''' E_t \left[ \int_t^T (D^{-}A)_tU_tdr \right].
\]

On one hand,

\[
(BS^{-1}(\hat{k}_t, \Lambda_t))'''' = \frac{-d^3BS}{d\sigma^3}(\hat{k}_t, \Theta_t(\hat{k}_t)) \left( \frac{d^2BS}{d\sigma^2}(\hat{k}_t, \Theta_t(\hat{k}_t)) \right)^3 + 3 \left( \frac{d^2BS}{d\sigma^2}(\hat{k}_t, \Theta_t(\hat{k}_t)) \right)^2 \left( \frac{dBS}{d\sigma}(\hat{k}_t, \Theta_t(\hat{k}_t)) \right)^2
\]

\[
= \frac{-d^3BS}{d\sigma^3}(\hat{k}_t, \Theta_t(\hat{k}_t)) \left( \frac{d^2BS}{d\sigma^2}(\hat{k}_t, \Theta_t(\hat{k}_t)) \right)^4 o \left( (T-t)^{-\frac{1}{2}} \right)
\]

\[
= (2\pi)^{\frac{1}{2}} \exp \left( -3X_t + \frac{3}{2} \Theta_t(\hat{k}_t) \right) (T-t)^{-\frac{1}{2}} + o \left( (T-t)^{-\frac{1}{2}} \right). \quad (3.15)
\]
On the other hand,

$$(D^- A)_r = \int_r^T D_t U_r^2 ds = 2 \int_r^T U_t D_t U_t ds. \quad (3.16)$$

The vega-delta-gamma relationship allows us to write

$$U_s = E_s \left[ \frac{dB S}{\partial \sigma} (t, T, X_t, \hat{k}, v_i) \frac{1}{2v_i(T-t)} \int_s^T D_s \sigma_u^2 du \right]$$

$$= \frac{1}{2} E_s \left[ G(t, T, X_t, \hat{k}, v_i) \int_s^T D_s \sigma_u^2 du \right], \quad (3.17)$$

and

$$D_t U_s = E_s \left[ \frac{1}{2} G(t, T, X_t, \hat{k}, v_i) \left( \frac{d_1(k, v_i)}{\sqrt{v_i T}} - \frac{1}{2v_i^2(T-t)} \left( \int_s^T D_s \sigma_u^2 du \right) \left( \int_s^T D_s \sigma_u^2 du \right) \right] + \frac{1}{2} G(t, T, X_t, \hat{k}, v_i) \left( \int_s^T D_s D_t \sigma_u^2 du \right) \right].$$

Then, from the equation for $G$ and (H2') we can deduce that

$$\begin{align*}
(D^- A)_r &= \frac{1}{4} \int_t^T E_s \left[ \frac{e^{X_t N(d_2(k, v_i))}}{v_t \sqrt{T - t}} \int_s^T D_s \sigma_u^2 du \right] \\
&\quad \times E_s \left[ \frac{e^{X_t N(d_2(k, v_i))}}{v_t \sqrt{T - t}} \left( \int_s^T D_s \sigma_u^2 du \right) \left( \int_s^T D_s \sigma_u^2 du \right) + \left( \int_s^T D_s D_t \sigma_u^2 du \right) \right] ds \\
&\quad + o(v^3(T - t)^{3H}), 
\end{align*} \quad (3.18)$$

which implies that

$$\begin{align*}
E_t \left[ \int_t^T (D^- A)_t U_t dr \right] &= E_t \left[ \int_t^T \frac{e^{X_t N(d_2(k, v_i))}}{v_t \sqrt{T - t}} \left( \int_r^T D_r \sigma_u^2 du \right) \left( \int_r^T D_r \sigma_u^2 du \right) \right] \\
&\quad \times E_s \left[ \frac{e^{X_t N(d_2(k, v_i))}}{v_t \sqrt{T - t}} \left( \int_s^T D_s \sigma_u^2 du \right) \left( \int_s^T D_s \sigma_u^2 du \right) + \left( \int_s^T D_s D_t \sigma_u^2 du \right) \right] dr \\
&\quad + o(v^4(T - r)^{4H+1}). 
\end{align*} \quad (3.19)$$

This, jointly with (3.15) and (H2') allow us to see that there exists a constant $C_t$ such that

$$\lim_{T \to t, v \to 0} \frac{1}{v^4(T - t)^{4H+1}} (BS^{-1}(k, T, \Lambda))'''' E_t \left[ \int_t^T (D^- A)_t U_t dr \right] = C_t.$$
Step 3 In order to calculate the term

\[
\frac{1}{4} \left( BS^{-1} \left( \hat{k}_t, \Lambda_t \right) \right)^{(iv)} E_t \left[ \int_t^T A_r U_r^2 dr \right].
\]

Note that

\[
\left( BS^{-1} \left( \hat{k}_t, \Lambda_t \right) \right)^{(iv)} = -(2\pi)^2 \exp \left( -4X_t + 2(\Theta_t(\hat{k}_t))(T - t) \right) (T - t)^{-1}. \tag{3.20}
\]

On the other hand,

\[
E_t \left[ \int_t^T A_r U_r^2 dr \right] = E_t \left[ \int_t^T \left( \int_r^T U_r^2 ds \right) U_r^2 dr \right]
\]

\[
= \frac{1}{2} E_t \left[ \left( \int_t^T U_r^2 dr \right)^2 \right]
\]

\[
= \frac{1}{2} E_t \left[ \left( \int_t^T \left( E_r \left( \frac{\partial BS}{\partial \sigma} (t, T, X_t, \hat{k}_t, v_t) \frac{1}{2v_t(T-t)} \int_r^T D_s \sigma^2 ds \right) \right)^2 \right] ds \right)^2. \tag{3.21}
\]

Together with (3.20) this gives us

\[
\lim_{T \to t, \nu \to 0} \nu^{-4} \left( T - t \right)^{4H+1} \left( BS^{-1} \left( \hat{k}_t, \Lambda_t \right) \right)^{(iv)} E_t \left[ \int_t^T A_r U_r^2 dr \right] = c_t,
\]

for some positive constant \( c_t \).

Step 4 Next, let us prove that \( T_2 + T_4 = \nu^A(T - t)^{4H+1} \). The computations in Step 2 and Step 3 prove that \( \Psi_r = O(\nu^A(T - t)^{4H+2}) \) and \( \Phi_r = O(\nu^A(T - t)^{4H+2}) \). Moreover, \( U_r = O(\nu(T - t)^H) \) and direct computations give us that, for all \( i \geq 3 \)

\[
BS^{-1} \left( \hat{k}_t, \Lambda_t \right)^{(vi)} \leq C(T - r)^{-\frac{1}{2}},
\]

and

\[
BS^{-1} \left( \hat{k}_t, \Lambda_t \right)^{(vi)} \leq C(T - r)^{-2},
\]

for some positive constant \( C \). Then, straightforward computations allow us to check that \( T_2 + T_4 = o(\nu^A(T - t)^{4H+1}) \).

Step 5 The final step is to show that \( T_1 + T_3 = o(\nu^A(T - t)^{4H+1}) \). We have that

\[
D^- \Psi_t := \int_t^T D_t (D^- A_t) U_t dr
\]

\[
= \int_t^T (D_t (D^- A_t)) U_t dr + \int_t^T (D^- A_t) D_t U_t dr, \tag{3.22}
\]

10
and
\[
D^{-} \Phi_t := \int_t^T D_t(A_r U_r^2)dr
= \int_t^T (D_t A_r)U_r^2 dr + 2 \int_t^T U_r A_r D_t(U_r)dr.
\] (3.23)

The computations in Section 2 give us that
\[
U_r = O(\nu(T - t)^H),
D_t U_r = O(\nu^2(T - t)^{2H - \frac{1}{2}}),
D^{-} A_r = O(\nu^3(T - t)^{3H + \frac{1}{2}}),
\]
and with the same arguments we can easily see that, under (H2')
\[
D_t(D^{-} A)_r = \int_r^T D_t(U_s D_t U_s)ds
= 2 \int_r^T D_t U_s D_t U_s ds + \int_r^T U_s (D_t D_t U_s)ds
= O(\nu^4(T - r)^{4H - 1}).
\] (3.24)

Then we deduce that
\[
D^{-} \Psi_t = O(\nu^5(T - t)^{5H + 1}) \quad \text{and} \quad D^{-} \Phi_t = O(\nu^5(T - t)^{5H + 1}).
\]
Again, direct computations allow us to see that
\[
BS^{-1} \left( \hat{k}_t, \Lambda_r \right)^{(\omega)} \leq C(T - r)^{-1},
\]
and
\[
BS^{-1} \left( \hat{k}_t, \Lambda_r \right)^{(\omega)} \leq C(T - r)^{-\frac{3}{2}},
\]
which allows us to see that \( T_1 + T_3 = o(\nu^4(T - t)^{4H + 1}) \). Now the proof is complete. \( \Box \)

4 The correlated case

We will consider the following hypothesis.

\( \text{(H3)} \) Hypotheses (H1), (H2'), hold and terms
\[
\frac{1}{(T - t)^{3 + 2H}} E_t \left[ \left( \int_t^T \int_s^T D_s W_s \sigma_s^2 dr ds \right)^2 \right],
\]
\[
\frac{1}{(T - t)^{2 + 2H}} E_t \left[ \int_t^T \left( \int_s^T D_s W_s \sigma_s dr \right)^2 ds \right],
\]
and
\[
\frac{1}{(T - t)^{2 + 2H}} E_t \left[ \int_t^T \int_s^T \int_r^T D_s W_s D_r \sigma_s^2 dudr ds \right],
\]
have a finite limit as \( T \to t \).
The following result, that follows from the same arguments as Proposition 4.1 in Alòs and Shiraya (2019), gives us an exact decomposition for the zero vanna implied volatility that will be the main tool in this Section.

Proposition 3. Consider the model (2.1) and assume that hypotheses (H1), (H2') and hold for some \( H \in (0, 1) \). Then, for every \( k \in \mathbb{R} \)

\[
I(t, T, X_t, \tilde{k}_t) = I^0(t, T, X_t, \tilde{k}_t) + \frac{\rho}{2} E_t \left[ \int_t^T (BS^{-1})'(\tilde{k}_t, s) H(s, T, X_s, \tilde{k}_s, v_s) \zeta_s ds \right],
\]

(4.1)

where \( I^0(t, T, X_t, \tilde{k}_t) \) denotes the zero vanna implied volatility in the uncorrelated case \( \rho = 0 \),

\[
\Gamma_s := E_t[BS(t, T, X_t, \tilde{k}_t, v_t)] + \frac{\rho}{2} E_t \left[ \int_t^s H(r, T, X_r, \tilde{k}_r, v_r) \zeta_r dr \right],
\]

and \( \zeta_t := \sigma_t^T D^W \sigma_r^2 dr \).

Theorem 2 and Proposition 3 allow us to prove the following result.

Theorem 4. Consider the model (2.1) and assume that hypotheses (H1), (H2') and (H3) hold for some \( H \in (0, 1) \). Then

\[
\lim_{T \to t} \frac{I(t, T, X_t, \tilde{k}_t) - E_t[v_t]}{(T - t)^{2H}} = + \lim_{T \to t} \frac{3\rho^2}{8\sigma_t^2(T - t)^{2+2H}} E_t \left[ \left( \int_t^T \int_s^T D^W_s \sigma_r^2 dr ds \right)^2 \right] - \lim_{T \to t} \frac{\rho^2}{2\sigma_t^2(T - t)^{2+2H}} E_t \left[ \int_t^T \left( \int_s^T D^W_s \sigma_r dr \right)^2 ds \right] - \lim_{T \to t} \frac{\rho^2}{2\sigma_t^2(T - t)^{2+2H}} E_t \left[ \int_t^T \int_s^T D^W_t D^W_r \sigma_u^2 du dr ds \right].
\]

(4.2)

Proof. The proof of this result follows similar ideas as the proof Theorem 2 in Alòs and Shiraya (2019). Notice that Proposition 3 gives us that

\[
I(t, T, X_t, \tilde{k}_t) - E_t[v_t] = T_1 + T_2,
\]

where

\[
T_1 = I^0(t, T, X_t, \tilde{k}_t) - E_t[v_t],
\]

\[
T_2 = \frac{\rho}{2} E_t \left[ \int_t^T (BS^{-1})'(\tilde{k}_t, s) H(s, T, X_s, \tilde{k}_s, v_s) \zeta_s ds \right].
\]

12
Let us first see that $T_1 = O((T - t)^{2H+1})$. Notice that

$$T_1 = \frac{1}{2} E_t \int_t^T (BS^{-1})''(k_t, \Lambda_r) U_r^2 dr.$$

Now, as

$$(BS^{-1})''(k_t, \Lambda_r) = \frac{(\Theta_t(\hat{k}_t))^4 (T - t)^2 - 4(X_t - k_t)^2}{4 (\exp(X_t) N'(d_1(k_t, \Theta_t(\hat{k}_t)))) (\Theta_t(\hat{k}_t))^3}$$

and $U_r = O((T - r)^H)$ it follows directly that $T_1 = O((T - t)^{2H+1})$.

Now, let us study $T_2$. Towards this end, we apply the anticipating Itô’s formula (9) to the process

$$H(s, T, X_s, \hat{k}_t, v_s) J_s,$$

where $J_s = \int_s^T (BS^{-1})'(\hat{k}_t, \Gamma_u) \zeta_u du$. Then, taking conditional expectations we get

$$0 = E_t \left[ H(t, T, X_t, \hat{k}_t, v_t) J_t ight.$$ 

$$+ \int_t^T H(s, T, X_s, \hat{k}_t, v_s) dJ_s$$

$$+ \int_t^T \frac{\partial^2}{\partial x \partial \sigma} H(s, T, X_s, \hat{k}_t, v_s) J_t \frac{\partial}{\partial y} (D^W Y_s) \sigma_s ds$$

$$+ \int_t^T \frac{\partial}{\partial \sigma} H(s, T, X_s, \hat{k}_t, v_s) (D^W J_s) \sigma_s ds$$

$$+ \int_t^T \frac{\partial}{\partial t} H(s, T, X_s, \hat{k}_t, v_s) J_s ds$$

$$+ \int_t^T \frac{\partial}{\partial \sigma} H(s, T, X_s, \hat{k}_t, v_s) \frac{\partial}{\partial t} J_s ds$$

$$+ \int_t^T \frac{\partial}{\partial \sigma} H(s, T, X_s, \hat{k}_t, v_s) \frac{\partial}{\partial y} J_s dY_s$$

$$+ \int_t^T \frac{\partial}{\partial x} H(s, T, X_s, \hat{k}_t, v_s) J_s dX_s$$

$$+ \frac{1}{2} \int_t^T \frac{\partial^2}{\partial x^2} H(s, T, X_s, \hat{k}_t, v_s) J_s d\langle X \rangle_s \right].$$

13
Now, using the relationships

\[
\frac{1}{\sigma(T-t)} \frac{\partial}{\partial \sigma} BS(t, T, x, k, \sigma) = \left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) BS(t, T, x, k, \sigma),
\]

\[
\left( \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} - \frac{1}{2} \sigma^2 \frac{\partial}{\partial x} \right) BS(t, T, x, k, \sigma) = 0,
\]

\[
D_s^W J_s = \rho \int_s^T (BS^{-1})' (\hat{k}_t, \Gamma_r) D_s^W \zeta_dr,
\]

\[
D_s^W Y_s = \rho \int_s^T D_s^W \sigma^2_r dr,
\]

we obtain

\[
0 = E_t \left[ H(t, T, X_t, \hat{k}_t, v_t) J_{lt} 
- \int_t^T H(s, T, X_s, \hat{k}_t, v_s)(BS^{-1})'(X_s, \Gamma_s) \zeta_s ds 
+ \frac{\rho}{2} \int_t^T \left( \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H(s, T, X_s, \hat{k}_t, v_s) J_s \zeta_s ds 
+ \rho \int_t^T \frac{\partial}{\partial x} H(s, T, X_s, \hat{k}_t, v_s) \left( \int_s^T (BS^{-1})'(\hat{k}_t, \Gamma_r)(D_s^W \zeta_r)dr \right) \sigma_s ds \right],
\]

which implies that

\[
T_2 = E_t \left[ \frac{\rho}{2} H(t, T, X_t, \hat{k}_t, v_t) J_{lt} 
+ \frac{\rho^2}{4} \int_t^T \left( \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H(s, T, X_s, \hat{k}_t, v_s) J_s \zeta_s ds 
+ \frac{\rho^2}{2} \int_t^T \frac{\partial}{\partial x} H(s, T, X_s, \hat{k}_t, v_s) \left( \int_s^T (BS^{-1})'(\hat{k}_t, \Gamma_r)(D_s^W \zeta_r)dr \right) \sigma_s ds \right] 
= T_2^1 + T_2^2 + T_2^3.
\]

Now, the study of \( T_2 \) is decomposed into two steps.

**Step 1** Notice that

\[
H(t, T, X_t, \hat{k}_t, v_t) = \frac{e^{X_t N'(d_1(\hat{k}_t, v_t))}}{v_t \sqrt{T-t}} \left( 1 - \frac{d_1(\hat{k}_t, v_t)}{v_t \sqrt{T-t}} \right)
= \frac{e^{X_t N'(d_1(\hat{k}_t, v_t))}}{2v_t^3} \left( (l_t(t, T, X_t, \hat{k}_t))^2 - v_t^2 \right).
\]
Then

\[
\lim_{t \to t} \frac{T^1_2}{(T - t)^{2H}} \leq \lim_{t \to t} \frac{\rho}{2(T - t)^{2H}} E_t \left[ \frac{e^{X_t} N'(d_1(\hat{k}_t, v_t))}{2 \sigma_t^3 \sqrt{T - t}} \left( (I_1(t, T, X_t, \hat{k}_t))^2 - \sigma_t^2 \right) \right]
\]

\[
\times \int_t^T \frac{1}{e^{X_s} N'(d_1(\hat{k}_s, BS^{-1}(\hat{k}_s, \Gamma_s))) \sqrt{T - s}} \zeta ds \right].
\]

(4.3)

and the norm of this is of the order \(O(\nu(T - t)^{H + \frac{1}{2}})\). Then, as

\[
(I_1(t, T, X_t, \hat{k}_t))^2 - \sigma_t^2 = (I(I(t, T, X_t, \hat{k}_t) + v_t)(I(t, T, X_t, \hat{k}_t) - v_t)
\]

we get

\[
\lim_{t \to t} T^1_2 = \lim_{t \to t} \frac{\rho}{4\sigma_t^2(T - t)}
\]

\[
\times E_t \left[ (I_1(t, T, X_t, \hat{k}_t) + v_t) \left( (I(t, T, X_t, \hat{k}_t) - E_{I_1}[v_t]) + (E_{I_1}[v_t] - v_t) \right) \int_t^T \int_s^T D_s W_r \sigma_r^2 dr ds \right]
\]

\[
= \lim_{t \to t} \frac{\rho}{4\sigma_t^2(T - t)} E_t \left[ I_1(t, T, X_t, \hat{k}_t) + v_t) (I(t, T, X_t, \hat{k}_t) - E_{I_1}[v_t]) \int_t^T \int_s^T D_s W_r \sigma_r^2 dr ds \right]
\]

\[
+ \lim_{t \to t} \frac{\rho}{4\sigma_t^2(T - t)} E_t \left[ (I_1(t, T, X_t, \hat{k}_t) + v_t)(E_{I_1}[v_t] - v_t) \int_t^T \int_s^T D_s W_r \sigma_r^2 dr ds \right]
\]

\[
=: \lim_{t \to t} T^{1,1}_2 + \lim_{t \to t} T^{1,2}_2.
\]

(4.4)

Notice that

\[
T^{1,1}_2 = \lim_{t \to t} (I(t, T, X_t, \hat{k}_t) - E_{I_1}[v_t]) \frac{\rho}{4\sigma_t^2(T - t)} E_t \left[ I_1(\hat{k}_t) + v_t \right] \int_t^T \int_s^T D_s W_r \sigma_r^2 dr ds
\]

\[
= (I(t, T, X_t, \hat{k}_t) - E_{I_1}[v_t]) \times O((T - t)^{H + \frac{1}{2}}).
\]

(4.5)

On the other hand,

\[
T^{1,2}_2 \leq \lim_{t \to t} \frac{\rho}{4\sigma_t^2(T - t)} \left( E_t \left[ (I_1(t, T, X_t, \hat{k}_t) + v_t) \int_t^T \int_s^T D_s W_r \sigma_r^2 dr ds \right]^2 \right)^{1/2}
\]

\[
\times \left( E_t \left[ (E_{I_1}[v_t] - v_t)^2 \right] \right)^{1/2}
\]

(4.6)
Then, as

$$E_t[v_t] - v_t = -\int_t^T E_t[D_rv_t]dW_r,$$

and then, since $I(t, T, X_t, \hat{k}_t) + E_t[v_t] < 2b$ (see (H1)),

$$\lim_{T \to -t} T^\frac{1}{2}^{1,2}
\leq \lim_{T \to -t} b\rho \left( E_t \left[ \left( \int_t^T \int_s^T D_s^W \sigma_s^2 dr ds \right) \right] \right)^{1/2} \left( \left( \int_t^T \int_s^T E_t \left[ \left( \frac{1}{2} \sqrt{T - t} \int_t^T E_t \left[ \left( \frac{1}{2} \sqrt{T - t} \int_t^T E_t \left[ \left( \int_t^T D_r\sigma_r^2 dr \right) \right] dr \right) \right] \right] \right)^{1/2} \left( \left( \int_t^T \int_s^T E_t \left[ \left( \int_t^T D_r\sigma_r^2 dr \right) \right] dr \right) \right)^{2/3} \right)^{1/2}

= O(T - t)^{\frac{1}{2}+2H}. \tag{4.7}

**Step 2.** In order to see that $T_2^2$ and $T_2^3$ are $O(T - t)^{2H}$ we apply again the anticipating Itô's formula to the processes

$$\left( \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H(s, T, X_s, \hat{k}_s, v_s)Z_s,$$

and

$$\frac{\partial H}{\partial x}(s, T, X_s, \hat{k}_s, v_s)R_s,$$

where

$$Z_s := \int_s^T \zeta_u J_u du,$$

$$R_s := \int_s^T \left( \int_u^T (BS^{-1})'(\hat{k}_r, \Gamma_r)(D_s^W \zeta_r) dr \right) \sigma_u du.$$

Then we get

$$T_2^2 = \frac{\rho^2}{4} E_t \left[ \left( \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H(t, T, X_t, \hat{k}_t, v_t)Z_t \right]$$

$$+ \frac{\rho}{2} \int_t^T \left( \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right)^2 H(s, T, X_s, \hat{k}_s, v_s)Z_s \zeta_s ds$$

$$+ \rho \int_t^T \frac{\partial}{\partial x} \left( \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H(s, T, X_s, \hat{k}_s, v_s)(D_s^W Z_s)\sigma_s ds \right], \tag{4.8}$$
Lemma 4.1 in Alòs, León and Vives (2007) gives us that the last two terms in (4.8) and (4.9) are $O(\nu^3(t-T)^3)$. Now, as

$$T^2 = \frac{\rho^2}{2} E_t \left[ \frac{\partial H}{\partial x}(t, T, X_t, \hat{k}_t, v_t) R_t \right]$$

$$\frac{\rho}{2} \int_t^T \left( [\frac{\partial^3}{[\partial x]^3} - \frac{\partial^2}{[\partial x]^2}] \frac{\partial H}{\partial x}(s, T, X_s, \hat{k}_t, v_s) R_s \right) ds$$

$$+ \rho \int_t^T \frac{\partial^2 H}{\partial x^2}(s, T, X_s, \hat{k}_t, v_s)$$

$$\times \left( \int_s^T \left( \frac{1}{(s-T)^{\gamma}} \right) \right) (\hat{k}_t, \Gamma_u)(D_t^w D_s^\nu \zeta_u) du dr \sigma_s ds \right] \right].$$

(4.9)

and

$$\left| \left( \frac{\partial^3}{[\partial x]^3} - \frac{\partial^2}{[\partial x]^2} \right) H(t, T, X_t, \hat{k}_t, v_t) \right|$$

$$= \frac{1 - \frac{d_1(\hat{k}_t, v_t)}{v_t \sqrt{T-t}}}{v_t \sqrt{T-t}} \left( 1 - \frac{d_1(\hat{k}_t, v_t)}{v_t \sqrt{T-t}} \right)^3$$

$$- \frac{1}{(v_t \sqrt{T-t})^3} \left( 1 - \frac{d_1(\hat{k}_t, v_t)}{v_t \sqrt{T-t}} \right) + \frac{3eX_i' \nu'(d_1(\hat{k}_t, v_t))}{(v_t \sqrt{T-t})^5}$$

$$= \frac{3eX_i' \nu'(d_1(\hat{k}_t, v_t))}{v_t^5} (T-t)^{-\frac{3}{2}} + O((T-t)^{-\frac{5}{2}}),$$

and

$$\left| \frac{\partial H}{\partial x}(t, T, X_t, \hat{k}_t, v_t) \right|$$

$$= \frac{eX_i' \nu'(d_1(\hat{k}_t, v_t))}{v_t \sqrt{T-t}} \left( 1 - \frac{d_1(\hat{k}_t, v_t)}{v_t \sqrt{T-t}} \right)^2$$

$$- \frac{eX_i' \nu'(d_1(\hat{k}_t, v_t))}{(v_t \sqrt{T-t})^3}$$

$$= \frac{eX_i' \nu'(d_1(\hat{k}_t, v_t))}{v_t^3} (T-t)^{-\frac{3}{2}} + O((T-t)^{-\frac{5}{2}}).$$
\[
\lim_{t \to t} \frac{T_2^2}{(T-t)^{2H}} = \frac{\rho^2}{4(T-t)^{2H}} E_t \left[ \left( \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H(t, T, X_t, \hat{k}_t, v_t) Z_t \right]
\]
\[
= \frac{\rho^2}{4(T-t)^{2H}} E_t \left[ 3 \frac{\tilde{\rho}}{t^5} \right]
\times \int_t^T \sigma_s \left( \int_s^T D_s^W \sigma_r^2 dr \right) \left( \int_s^T \frac{\zeta_r}{e^{\lambda_1 N'(d_1(\hat{k}_t, v_t))}} dr \right) ds
\]
\[
= \lim_{t \to t} \frac{3 \rho^2}{4 \sigma_t^5 (T-t)^{3+2H}} E_t \left[ \int_t^T \left( \int_s^T D_s^W \sigma_r^2 dr \right) \left( \int_s^T \zeta_r dr \right) \sigma_s ds \right]
\]
\[
= \lim_{t \to t} \frac{3 \rho^2}{4 \sigma_t^5 (T-t)^{3+2H}} E_t \left[ \int_t^T \left( \int_s^T D_s^W \sigma_r^2 dr \right) \left( \int_s^T \sigma_r \int_r^T D_r^W \sigma_s^2 d\theta dr \right) \sigma_s ds \right]
\]
\[
= \lim_{t \to t} \frac{3 \rho^2}{8 \sigma_t^3 (T-t)^{3+2H}} E_t \left[ \left( \int_t^T \int_s^T D_s^W \sigma_r^2 dr ds \right)^2 \right], \tag{4.10}
\]

and

\[
\lim_{t \to t} \frac{T_3^3}{(T-t)^{2H}} = \lim_{t \to t} \frac{\rho^2}{2(T-t)^{2H}} E_t \left[ \frac{\partial H}{\partial x} (t, T, X_t, \hat{k}_t, v_t) R_t \right]
\]
\[
= \lim_{t \to t} \frac{\rho^2}{2(T-t)^{2H}} E_t \left[ \frac{1}{4 \sqrt{T-t}^3} \frac{\tilde{\rho}}{(v_t \sqrt{T-t})^3} (\sigma_t^3 (T-t) - 4) \left( \int_t^T \int_s^T e^{\lambda_1 N'(d_1(\hat{k}_t, v_t))) \sqrt{T-t} \right) \left( D_s^W \left( \sigma_s \int_s^T D_s^W \sigma_s^2 du \right) \right) dr ds \right]
\]
\[
= -\lim_{t \to t} \frac{\rho^2}{2 \sigma_t^3 (T-t)^{2+2H}} E_t \left[ \int_t^T \int_s^T D_s^W \sigma_r \int_r^T D_r^W \sigma_s^2 d\theta dr \right]
\]
\[
+ \int_t^T \int_s^T \sigma_s \int_r^T D_s^W D_r^W \sigma_s^2 d\theta dr ds \right]
\]
\[
= -\lim_{t \to t} \frac{\rho^2}{2 \sigma_t^3 (T-t)^{2+2H}} E_t \left[ \int_t^T \left( \int_s^T D_s^W \sigma_r dr \right)^2 ds \right]
\]
\[
- \lim_{t \to t} \frac{\rho^2}{2 \sigma_t^3 (T-t)^{2+2H}} E_t \left[ \int_t^T \int_s^T \int_r^T D_s^W D_r^W \sigma_s^2 d\theta dr ds \right], \tag{4.11}
\]
Let us now summarize the previous computations. We have seen that
\[
I(t, T, X_t, \hat{k}_t) - E_t[v_t] = T_1 + T_2
\]
\[
= T_1 + T_2^{1,1} + T_2^2 + T_2^3
\]
where
\[
T_1 + T_2^{1,2} = o(T - t)^{2H},
\]
\[
T_2^{1,1} = (I(t, T, X_t, \hat{k}_t) - E_t[v_t]) \frac{\rho}{4\sigma_t^2(T - t)} E_t \left[ I(t, T, X_t, \hat{k}_t) + v_t \right] \int_t^T \int_s^T D_s^W \sigma_r^2 dr ds
\]
\[
T_2^2 = \frac{3\rho^2}{8\sigma_t^3(T - t)^3} E_t \left( \left( \int_t^T \int_s^T D_s^W \sigma_r^2 dr ds \right)^2 + o(T - t)^{2H} \right)
\]
and
\[
T_2^3 = -\lim_{T \to t} \frac{\rho^2}{2\sigma_t^2(T - t)^{2+2H}} E_t \left[ \int_t^T \left( \int_s^T D_s^W \sigma_r^2 dr \right)^2 ds \right]
\]
\[
-\lim_{T \to t} \frac{\rho^2}{2\sigma_t^2(T - t)^{2+2H}} E_t \left[ \int_t^T \int_s^T \int_r^T D_s^W D_r^W \sigma_r^2 dr ds dr ds \right]
\]
\[
+ o(T - t)^{2H}.
\]
Then, as there is some \( \epsilon \) such that, if \( T - t < \epsilon \)
\[
\left| \frac{\rho}{4\sigma_t^2(T - t)} E_t \left( I(t, T, X_t, \hat{k}_t) + v_t \right) \int_t^T \int_s^T D_s^W \sigma_r^2 dr ds \right| < 1
\]
we can write
\[
\lim_{T \to t} \frac{I(t, T, X_t, \hat{k}_t) - E_t[v_t]}{(T - t)^{2H}} = \lim_{T \to t} \frac{1}{(T - t)^{2H}} \frac{T_1 + T_2^{1,2} + T_2^2 + T_2^3}{1 - \frac{\rho}{4\sigma_t^2(T - t)} E_t \left( I(t, T, X_t, \hat{k}_t) + v_t \right) \int_t^T \int_s^T D_s^W \sigma_r^2 dr ds}
\]
\[
= \lim_{T \to t} \frac{3\rho^2}{8\sigma_t^3(T - t)^{3+2H}} E_t \left( \left( \int_t^T \int_s^T D_s^W \sigma_r^2 dr ds \right)^2 \right)
\]
\[
-\lim_{T \to t} \frac{\rho^2}{2\sigma_t^2(T - t)^{2+2H}} E_t \left[ \int_t^T \int_s^T \int_r^T D_s^W D_r^W \sigma_r^2 dr ds dr ds \right]
\]
\[
-\lim_{T \to t} \frac{\rho^2}{2\sigma_t^2(T - t)^{2+2H}} E_t \left[ \int_t^T \int_s^T \int_r^T D_s^W D_r^W \sigma_r^2 dr ds dr ds \right],
\]
(4.14)
as we wanted to prove.

\[ \square \]

**Corollary 5.** Assume that \( \sigma_t = f(B_t^H) \), where \( f \in C^1 \) with a range in a compact set of \( \mathbb{R}^+ \) and \( B_t^H \) is a fBm with Hurst parameter \( H \). Then the above result proves that, in the correlated case
\[
I(t, T, X_t, \hat{k}_t) - E_t[v_t] = O((T - t)^{2H}).
\]
Remark 6. Notice that the term $T_{1,2}^{1,2}$ is of the order $(\rho(T-t)^{\frac{1}{2}+H})$. When $T-t$ does not tend to zero, this term can not be neglected.

Remark 7. Hypotheses (H1)-(H3) have been chosen for the sake of simplicity. The same results can be extended to other stochastic volatility models (see e.g., Section 5 in Alòs and Shiraya (2019)).

5 Numerical examples

In this section, we confirm the validity of our estimates by using numerical examples. We assume the following stochastic volatility model,

\begin{align*}
\sigma_s &= \sigma_0 \exp \left( \nu W^H_s - \frac{\nu^2 s^{2H}}{4H} \right), \quad s \in [0, T], \\
W^H_s &:= \int_0^s \frac{dW_r}{(s-r)^{\frac{1}{2}-H}}.
\end{align*}

(5.1) (5.2)

with positive constants $\nu$, $\sigma_0$ and Hurst parameter $H \in (0, 1)$. We set the parameters $\sigma_0 = 20\%$, $\nu = 0.4$, the correlation between the asset price and its volatility $\rho = 0$ or $-0.8$, and the Hurst parameters $H = 0.1, 0.3, 0.5, 0.6, 0.9$.

In order to calculate the implied volatilities and volatility swap prices, we use Monte Carlo simulation with 500 time steps for one year and twenty million trials. To increase accuracy, the Black-Scholes model has been used as the control variate for the Monte Carlo simulations to obtain the option premiums. Once the exact volatility swap strikes and options prices have been calculated, the bisection method is used to infer implied volatilities, including zero vanna implied volatilities. To compare our new results to the approximation formula (4.8) of Alòs-Shiraya (2019), we also calculate the ATM skew ($\frac{\partial I}{\partial k}$) using the difference method on the implied volatilities.

Tables 1 and 2 below show the results of the uncorrelated case and correlated case, respectively. In the tables, “vol swap” is the simulated volatility swap value, “IV (\(\hat{k}\))” and “ATMI” are the implied volatility at respectively the zero vanna strike and ATM strike, and “AS(4.8)” is the value of the formula (4.8) in Alòs and Shiraya (2019). We note that in the uncorrelated case AS(4.8) and ATMI are equal because the ATM skew in the uncorrelated case is 0.
| H index | T   | 0.25  | 0.5   | 1     | 2     | 3     |
|---------|-----|-------|-------|-------|-------|-------|
| 0.1     | vol swap | 20.48% | 20.98% | 21.58% | 22.28% | 22.76% |
|         | IV (k)    | 20.48% | 20.97% | 21.56% | 22.25% | 22.68% |
|         | ATMI       | 20.48% | 20.96% | 21.54% | 22.18% | 22.56% |
| 0.3     | vol swap | 20.28% | 20.44% | 20.67% | 21.03% | 21.32% |
|         | IV (k)    | 20.28% | 20.43% | 20.67% | 21.02% | 21.28% |
|         | ATMI       | 20.28% | 20.43% | 20.66% | 20.98% | 21.21% |
| 0.5     | vol swap | 20.07% | 20.13% | 20.26% | 20.52% | 20.77% |
|         | IV (k)    | 20.07% | 20.13% | 20.26% | 20.51% | 20.74% |
|         | ATMI       | 20.07% | 20.13% | 20.26% | 20.49% | 20.69% |
| 0.7     | vol swap | 20.02% | 20.06% | 20.15% | 20.38% | 20.66% |
|         | IV (k)    | 20.02% | 20.06% | 20.15% | 20.38% | 20.63% |
|         | ATMI       | 20.02% | 20.05% | 20.14% | 20.36% | 20.58% |
| 0.9     | vol swap | 20.01% | 20.03% | 20.10% | 20.35% | 20.69% |
|         | IV (k)    | 20.01% | 20.03% | 20.10% | 20.34% | 20.65% |
|         | ATMI       | 20.01% | 20.03% | 20.10% | 20.32% | 20.60% |

Table 1: Volatility swaps, and implied volatilities ($\rho = 0$)

| H index | T   | 0.25  | 0.5   | 1     | 2     | 3     |
|---------|-----|-------|-------|-------|-------|-------|
| 0.1     | vol swap | 20.48% | 20.98% | 21.58% | 22.28% | 22.76% |
|         | IV (k)    | 19.72% | 20.08% | 20.49% | 20.96% | 21.26% |
|         | ATMI       | 19.47% | 19.67% | 19.87% | 19.99% | 20.02% |
|         | AS(4.8)   | 19.72% | 20.06% | 20.46% | 20.87% | 21.12% |
| 0.3     | vol swap | 20.28% | 20.44% | 20.67% | 21.03% | 21.32% |
|         | IV (k)    | 20.07% | 20.10% | 20.16% | 20.21% | 20.24% |
|         | ATMI       | 19.92% | 19.85% | 19.73% | 19.48% | 19.25% |
|         | AS(4.8)   | 20.06% | 20.10% | 20.14% | 20.16% | 20.14% |
| 0.5     | vol swap | 20.07% | 20.13% | 20.26% | 20.52% | 20.77% |
|         | IV (k)    | 20.00% | 20.00% | 19.99% | 19.96% | 19.89% |
|         | ATMI       | 19.92% | 19.85% | 19.68% | 19.36% | 19.02% |
|         | AS(4.8)   | 20.00% | 20.00% | 19.98% | 19.92% | 19.82% |
| 0.7     | vol swap | 20.02% | 20.06% | 20.15% | 20.38% | 20.66% |
|         | IV (k)    | 20.00% | 20.00% | 19.99% | 19.95% | 19.86% |
|         | ATMI       | 19.96% | 19.90% | 19.76% | 19.43% | 19.04% |
|         | AS(4.8)   | 20.00% | 20.00% | 19.99% | 19.92% | 19.79% |
| 0.9     | vol swap | 20.01% | 20.03% | 20.10% | 20.35% | 20.69% |
|         | IV (k)    | 20.00% | 20.00% | 20.00% | 19.99% | 19.90% |
|         | ATMI       | 19.97% | 19.93% | 19.82% | 19.51% | 19.11% |
|         | AS(4.8)   | 20.00% | 20.00% | 20.00% | 19.96% | 19.83% |

Table 2: Volatility swaps, implied volatilities, and approximated volatility swaps ($\rho = -0.8$)

21
Since the error order on $T$ of IV $(\hat{k})$ is higher than that of ATMI, IV $(\hat{k})$ approximates the volatility swap better than ATMI in all cases. Also, since the error order in the uncorrelated case is higher than that of the correlated case, and the error is always multiplied by the correlation, IV $(\hat{k})$ is a more accurate approximation of the volatility swap when correlation is small. While the values of AS(4.8) are close to those of IV$(\hat{k})$, IV$(\hat{k})$ is better in our settings. Regarding the Hurst parameter, as the parameter increases, the order on $T$ increases, and the approximation errors in short terms becomes smaller as shown in Theorems\textsuperscript{[2]} and\textsuperscript{[4]}

6 Conclusion

By using techniques from Malliavin calculus we have extended the validity of the zero vanna implied volatility as an approximation for pricing volatility swaps to fractional stochastic volatility models. Furthermore, we have proved that even though the zero vanna approximation for the volatility swap strike is extremely accurate for zero correlation and for all values of the Hurst parameter, it is not exact. Thus, indirectly it is proved that the Rolloos-Arslan approximation is not equivalent to the Carr-Lee approximation for volatility swaps as the latter is exact for zero correlation. However, in the uncorrelated case and for most practical purposes it can be treated as exact. It has also been shown that for the cases we have considered the zero vanna approximation has a higher rate of convergence than the Alòs and Shiraya (2019) model-free result.

References

[1] Alòs, E., León, J. A., and Vives, J. “On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility,” Finance and Stochastics 11 (4), (2007): 571-589.

[2] Alòs, E., and Shiraya, K. “Estimating the Hurst parameter from short term volatility swaps: a Malliavin calculus approach.” Finance and Stochastics 23.2 (2019): 423-447.

[3] Bergomi, L. and Guyon, J.. The Smile in Stochastic Volatility Models (December 2, 2011). Available at SSRN: https://ssrn.com/abstract=1967470

[4] Carr, P. and Lee, R. “Robust replication of volatility derivatives,” PRMIA award for Best Paper in Derivatives, MFA 2008 Annual Meeting, (2008).

[5] Comte, F. and Renault, E.: “Long memory in continuous-time stochastic volatility models,” Math. Financ. 8, (1998): 291-323.

[6] El Euch, O. Fukasawa, M., Gatheral, J. and Rosenbaum, M. “Short-Term At-the-Money Asymptotics Under Stochastic Volatility Models,” SIAM Journal on Financial Mathematics 10.2 (2019): 491-511.

[7] Friz, P., and Gatheral, J. “Valuation of volatility derivatives as an inverse problem,” Quantitative Finance 5 (6), (2005): 531-542.
A Malliavin calculus

In this appendix, we present the basic Malliavin calculus results we use in this paper. The first one is the Clark-Ocone formula, that allows us to compute explicitly the martingale representation of a random variable \( F \in \mathcal{D}^{1,2}_W \).

Theorem 8. Clark-Ocone formula Consider a Brownian motion \( W = \{W_t, t \in [0,T]\} \) and a random variable \( F \in \mathcal{D}^{1,2}_W \). Then

\[
F = E[F] + \int_0^T E_r[D_rF]dW_r. 
\]

We will also make use of the following anticipating Itô’s formula (see for example, Nualart (2006)), that allows us to work with non-adapted processes.

Proposition 9. Assume model (2.1) and \( \sigma^2 \in \mathcal{L}^{1,2}_W \). Let \( F : [0,T] \times \mathbb{R}^2 \rightarrow \mathbb{R} \) be a function in \( C^{1,2}([0,T] \times \mathbb{R}^2) \) such that there exists a positive constant \( C \) such that, for all \( t \in [0,T] \), \( F \) and its partial derivatives evaluated in \( (t, X_t, Y_t) \) are bounded by \( C \). Then it follows that

\[
F(t, X_t, Y_t) = F(0, X_0, Y_0) + \int_0^t \partial_x F(s, X_s, Y_s) ds \\
- \int_0^t \partial_x F(s, X_s, Y_s) \sigma_s^2 ds \\
+ \int_0^t \partial_x F(s, X_s, Y_s) \sigma_s \sigma_s \rho dW_s + \sqrt{1 - \rho^2} dB_s \\
- \int_0^t \partial_y F(s, X_s, Y_s) \sigma_s^2 ds + \rho \int_0^t \partial^2_{xy} F(s, X_s, Y_s) \xi_s ds \\
+ \frac{1}{2} \int_0^t \partial^2_{xx} F(s, X_s, Y_s) \sigma_s^2 ds, \tag{A.1}
\]

where \( \xi_s := (\int_0^t D_s W_s \sigma_s^2 dr) \sigma_s \).