On Computing the Shadows and Slices of Polytopes

Hans Raj Tiwary*

hansraj@cs.uni-sb.de

We study the projection of polytopes along $k$ orthogonal vectors for various input and output forms. We show that if $k$ is part of the input and we are interested in output-sensitive algorithms, then in most forms the problem is equivalent to enumerating vertices of polytopes, except in two where it is NP-hard. In two other forms the problem is trivial. We also review the complexity of computing projections when the projection directions are picked at random. For full-dimensional polytopes containing origin in the interior, projection is an operation dual to intersecting the polytope with a suitable hyperplane and so the results in this paper can be dualized by interchanging vertices with facets and projection with intersection. We would like to remark that even though most of the results in this paper do not appear to have been published before, they follow from straightforward reductions to other known results. The purpose of this paper is to serve as a reference to these results about the computational complexity of projection of polytopes onto affine subspaces.

1 Introduction

A polytope in $\mathbb{R}^d$ is a closed convex body that can be represented as either the convex hull of a finite number of points or as the intersection of a finite number of halfspaces. We will call the former $V$-representation and the latter $H$-representation. Accordingly a polytope will be called a $V$-polytope, an $H$-polytope or an $HV$-polytope depending on whether the polytope is given by $V$, $H$ or both representations. For any polytope each of these representations is unique if no redundancies are allowed and any of these representations completely determines the others. We refer the reader to [6, 9] for a thorough treatment of the subject.

Suppose we are given an $H$-polytope in $\mathbb{R}^n$ and we want to compute the facets of its projection onto $\mathbb{R}^{n-1}$. One can use Fourier-Motzkin elimination (or any other algorithm of choice [7]) and compute the facets easily. The projection can have as many as $\frac{n^2+1}{4}$ facets if $P$ has $m$ facets and as few as $n$ facets. Now suppose we want to further project the shadow of $P$ onto $\mathbb{R}^{n-2}$. Of course one can remove redundancies from the projection obtained in $\mathbb{R}^{n-1}$ and apply Fourier-Motzkin elimination again but a careful reader might already have noticed a problem. If one wants to compute the projection of $P$ onto a $d$-dimensional subspace, then applying Fourier-Motzkin incrementally might result in an intermediate polytope with very large (exponential) number

*Universität des Saarlandes, 66123 Saarbrücken Germany
of facets even if we finally get a polytope with relatively small number of facets. Is there a way around this problem?

This question is rather ill-defined at this point. To be precise, we want to know if there is an algorithm that computes the desired description of the projection and takes total time polynomial in the input and output. We need to include the size of the output in the picture because otherwise the problem clearly has no polynomial algorithm, since the number of facets of the projection of a polytope can be exponential in the size of the input. An algorithm that runs in time polynomial in the size of the input and the output is called output-sensitive polynomial algorithm. So the previous question can be rephrased as: Given an $\mathcal{H}$-polytope $P$, is there an output-sensitive polynomial algorithm that enumerates all facets of the projection of $P$ onto some subspace?

In this paper we answer this question in the negative. We also study several other natural variants of this problem arising from the simple fact that a polytope has two representations. One can ask the same question but in a different flavor by requiring the input and output to be in various different representations. We prove that in most forms this problem is equivalent to the problem of enumerating vertices of a polytope. In two forms, including the one described in previous paragraph, the problem is NP-hard and in two other forms the problem has a trivial solution. We also review the complexity of this projection problem if the projection directions are picked at random and prove that in many cases one can enumerate the desired form of the output in polynomial time. We would like to note that even though the algorithm for computing random projections was conceived independently by the author, it appears to be almost similar to the one presented in [8]. We nevertheless include it for completeness of discussion.

The problem of enumerating the vertices of a polytope given by its facets has been studied for a long time by a number of researchers. Still the complexity status of Vertex Enumeration problem (VE), for general dimension and for polytopes that are neither simple nor simplicial, is unknown. It is neither known to be in P nor is it known to be NP-complete. The dual problem of computing $\mathcal{H}$-representation from $V$-representation is known as Convex Hull problem (CH). These two problems are equivalent modulo solving a Linear Program. Thus, for rational input these two problems are polynomial time equivalent and a polynomial output-sensitive algorithm for one can be used to solve the other in output-sensitive polynomial time. For more details about the problems with various Vertex Enumeration methods, we refer the reader to [1].

Our results about the hardness, and the equivalence of vertex enumeration and computing projection (in most forms) imply that in all forms where the projection can not be computed by a trivial algorithm, finding an output-sensitive polynomial algorithm will be a challenging task. Equivalently, an output-sensitive polynomial algorithm for vertex enumeration will have significant impact in many fields outside algorithmic polytope theory, like Control Theory, Constraint Logic Programming Languages, Constraint Query Languages etc, where one frequently needs to solve the projection problem ([7]).

In order to be able to talk about the equivalence of Vertex Enumeration and projection, we will define a complexity class based on Vertex Enumeration. Keeping in line with other notions of completeness, we call an enumeration problem $\Phi$ VE-complete if any output-sensitive polynomial algorithm for VE can be used to solve $\Phi$ in output-sensitive polynomial time and vice-versa. Similarly, we call a problem VE-easy if it can be solved in output-sensitive polynomial time using an oracle for VE and we call a problem VE-hard if an oracle for this problem can be used to solve VE in output-sensitive polynomial time.

The results in this paper are summarized in Table 1 and Table 2. Table 1 summarizes the complexity of computing projection along arbitrary directions while Table 2 summarizes the complexity of computing projection along randomly picked projection directions.
Table 1: Complexity of computing projection of a polytope onto an arbitrary subspace, for various input and output representations.

| Input Output | $\mathcal{V}$ | $\mathcal{H}$ | $\mathcal{HV}$ |
|--------------|---------------|---------------|----------------|
| $\mathcal{V}$ | poly          | VE-Complete   | VE-complete    |
| $\mathcal{H}$ | NP-hard       | NP-hard       | VE-complete    |
| $\mathcal{HV}$ | poly          | VE-Complete   | VE-complete    |

Table 2: Complexity of computing projection of a polytope onto a randomly picked subspace, for various input and output representations.

| Input Output | $\mathcal{V}$ | $\mathcal{H}$ | $\mathcal{HV}$ |
|--------------|---------------|---------------|----------------|
| $\mathcal{V}$ | poly          | VE-complete   | VE-complete    |
| $\mathcal{H}$ | VE-hard       | poly          | VE-complete    |
| $\mathcal{HV}$ | poly          | poly          | poly           |

Since for bounded polytopes containing the origin in relative interior, projection is an operation dual to intersection with a hyperplane passing through origin, all our results can be dualized by interchanging vertices and facets and replacing projection with intersection.

The rest of the paper is organized as follows. In the next section we briefly review some related work about computing projection of a polytope. Our result section is divided into two parts. In Subsection 3.1 we present the results about the complexity of computing the projection of a polytope along arbitrary directions and in Subsection 3.2 we describe the complexity results for the case when the (orthogonal) projection directions are picked at random.

2 Related Work

Perhaps the best known algorithm for computing the facets of the projection of an $\mathcal{H}$-polytope is the Fourier-Motzkin elimination discovered by Fourier in 1824 and then rediscovered by Motzkin in 1936. This method is analogous to the method of Gaussian elimination for equations and works by eliminating one variable at a time. Since eliminating one variable from a system of $m$ inequalities can result in $\frac{m^2}{4}$ facets, the algorithm can have a terrible running time in bad cases where the intermediate polytopes have very large (exponential) number of facets but the final output has only a small number of facets.

Many improvements have been done over the original algorithm (See [7] for a survey) but there is no algorithm that has an output-sensitive polynomial running time. The natural question then is whether one can find a shortcut around the intermediate projection steps in the Fourier-Motzkin elimination and obtain an output-sensitive polynomial algorithm. As we will see in Section 3 the answer is no if $P \neq NP$. Thus the lack of any output-sensitive algorithm, for computing the facets of the projection of an $\mathcal{H}$-polytope, is somewhat natural because the problem turns out be NP-hard.

Farka’s Lemma provides a way of generating a cone whose extreme rays correspond to the facets of the projection of an $\mathcal{H}$-polytope ([3]), but unfortunately it does not yield a bijective mapping and many extreme rays of the resulting cone may correspond to redundant inequalities in the projection. Balas [3] found a way to get rid of these redundancies and provided a way to construct, given an $\mathcal{H}$-polytope $P$, another polyhedral cone $W$ in polynomial time whose
projection \( W' \) yields a one-to-one correspondence between the extreme rays of \( W' \) and the facets of the projection of \( P \). Also, \( W \) has polynomially many facets compared to \( P \).

3 Results

We will denote the projection of a \( n \)-dimensional polytope \( P \) onto a given \( d \)-dimensional subspace as \( \pi_d(P) \). We will mostly omit the subscript and simply refer to the projection as \( \pi(P) \) and the projection subspace will depend on the context.

Formally, we are interested in the following problem: Given a polytope \( P \in \mathbb{R}^n \) in \( \mathcal{H} \), \( \mathcal{V} \) or \( \mathcal{HV} \)-representation and a set \( \Gamma \) of \( k \) orthogonal projection directions defining the projection space, we want to compute the non-redundant \( \mathcal{H} \), \( \mathcal{V} \) or \( \mathcal{HV} \)-representation of \( \pi(P) \).

3.1 Projection onto arbitrary subspaces

Depending on the input and output form, we have nine variants of the problem. It is obvious that if the vertices are part of the input and one wants to compute the vertices of the projection, then each vertex can be projected trivially and the vertices that become redundant in the projection can be identified by solving one Linear Program per vertex. Hence, we have the following:

**Lemma 1.** Given a polytope \( P \subset \mathbb{R}^n \) in \( \mathcal{V} \)- or \( \mathcal{HV} \)-representation and an arbitrary projection subspace, non-redundant \( \mathcal{V} \)-representation of \( \pi(P) \) can be computed in polynomial time.

Also, it is easy to see that every polytope can be represented as the projection of a suitable simplex. Moreover, given a polytope \( P \) by its vertices one can compute in polynomial time the vertices of this simplex \( \Delta \) and the projection subspace such that \( P \) is the projection of \( \Delta \). Since it is trivial to compute the facets of a simplex given its vertices, Vertex Enumeration can be solved in output-sensitive polynomial time using any algorithm that computes the \( \mathcal{H} \)- or \( \mathcal{HV} \)-representation of projection from \( \mathcal{V} \)- or \( \mathcal{HV} \)-representation of a polytope.

Clearly, one can also use any polynomial algorithm for Vertex Enumeration to compute the \( \mathcal{H} \)- or \( \mathcal{HV} \)-representation of the projection of any polytope given in \( \mathcal{V} \)- or \( \mathcal{HV} \)-representation in polynomial time. Hence, we have the following easy lemma:

**Lemma 2.** Computing the \( \mathcal{H} \)- or \( \mathcal{HV} \)-representation of the projection of a polytope given in \( \mathcal{V} \)- or \( \mathcal{HV} \)-representation is \( \text{VE-complete} \).

In what follows, we assume the input polytope \( P \) is of the form

\[
P(x, y) = \{x, y | Ax + By \leq 1\}
\]

and we want to compute the projection

\[
\pi(P) = \{x \in \mathbb{R}^d | (x, y) \in P \text{ for some } y\},
\]

where \( A \in \mathbb{Q}^{m \times d}, B \in \mathbb{Q}^{m \times k}, y \in \mathbb{R}^k \). We also assume \( P \) to be full-dimensional and to contain the origin in its relative interior. For rational polytopes this assumption is justified because one can always find a point in the interior of the polytope via Linear Programming and move the origin to this point. We will sometimes omit details like \( A \in \mathbb{Q}^{m \times d}, B \in \mathbb{Q}^{m \times k}, x \in \mathbb{R}^d, y \in \mathbb{R}^k \) where it can be inferred from the context.

\footnote{Assuming that the vertices are numbered 0 through \( m - 1 \), simply append \( e_i \) to the \( i \)-th vertex, where \( e_0 \) is the zero vector and \( e_i \) is the \( i \)-th unit vector in \( \mathbb{R}^{m-1} \).}
We are now left with the three cases where the input polytope is given by $\mathcal{H}$-representation. As we will see now, computing either the facets or the vertices of the projection in this case in hard while computing both facets and vertices of the projection is equivalent to Vertex Enumeration. Consider the following decision version of the problem:

**Input:** Polytopes $P = \{x, y | Ax + By \leq 1\}$ and $Q = \{x | A'x \leq 1\}$

**Output:** YES if $Q = \pi(P)$, NO otherwise.

We will now prove that this decision problem is NP-complete thus proving the NP-hardness of the enumeration problem.

**Theorem 1.** Given a polytope $P = \{x, y | Ax + By \leq 1\}$ and $Q = \{x | A'x \leq 1\}$ it is NP-complete to decide if $Q \neq \pi(P)$.

**Proof.** It is easy to see that deciding whether a given set of hyperplanes completely define the projection of a higher dimensional polytope is in NP. So the only thing remaining is to show that it is NP-hard as well.

It is known (\cite{5}) that Given an $\mathcal{H}$-polytope $P = \{x | Ax \leq 1\}$ and a $\mathcal{V}$-polytope $Q = CH(V)$, it is NP-complete to decide whether $P \nsubseteq Q$. Clearly, $P \subseteq Q$ if and only if $P \cap Q = P$. Now, $P \cap Q$ has the following $\mathcal{H}$-representation

\[
\begin{align*}
Ax & \leq 1 \\
x - \sum_{v \in V} \lambda_v \cdot v & = 0 \\
\sum_{v \in V} \lambda_v & = 1 \\
\lambda_v & \geq 0, \forall v \in V
\end{align*}
\]

The variables $\lambda$ ensure that we consider only those points in $P$ that can be represented as a convex combination of vertices of $Q$. One can further, in polynomial time, get a full dimensional representation of $P \cap Q$ by eliminating the $d - 1$ equations. The resulting polytope is a full-dimensional polytope in $\mathbb{R}^{|V| - 1}$.

Since we are interested in only the (vector) variable $x$, the projection of this polytope along the axes corresponding to the variables $\lambda_v$ gives us the facets of $P \cap Q$ in the subspace of variables $x$. It follows that $P \cap Q = P$ if and only if the projection of the above defined polytope onto the subspace of variables $x$ has the $\mathcal{H}$-representation same as that of $P$ i.e. $Ax \leq 1$.

Thus any polynomial algorithm for deciding whether a given set of hyperplanes completely define the projection of some high dimensional $\mathcal{H}$-polytope, can be used to decide whether an $\mathcal{H}$-polytope is contained in a $\mathcal{V}$-polytope or not, which is a NP-complete problem. \hfill \Box

Balas (\cite{3}) has shown that for a given $\mathcal{H}$-polytope ($P$) and a set of projection axes, one can compute the facets of another pointed polyhedral cone $W$ and a set of projection axes such that the facets of $\pi(P)$ are in one-to-one correspondence with the extreme rays $\pi(W)$. The number of facets of $W$ is polynomial in the number of facets of $P$. It is not difficult to modify the construction in \cite{3} so that $W$ is bounded i.e. a polytope and the vertices in the projection of $W$ are in one-to-one correspondence with the facets of the projection of $P$. For completeness we state the result of Balas and describe the modification here.

\footnote{Note that the projections of $P$ and $W$ are defined in different spaces and should not be confused as the same projection map despite the abuse of notation here.}
Lemma 3. Given an $H$ polytope $P$ and a set of projection directions, there exists a polyhedral cone $W$ and another set of directions such that the facets of $\pi(P)$ are in one-to-one correspondence with the extreme rays of $\pi(W)$. Furthermore, $W$ has polynomially many facets compared to $P$ and the facets of $W$ can be computed in polynomial time.

We will use the notion of polar duality to prove that the cone $W$ obtained from the construction of Balas can be turned into a bounded polytope. For a polyhedral cone $W$ in $\mathbb{R}^n$ with facet inequalities $Ax \leq 0$ and extreme rays the row vectors of $V$, where $A$ and $V$ are matrices with each row a vector in $\mathbb{R}^n$. The polar dual $W^*$ has the roles of the extreme rays and facets reversed. In particular, the facet inequalities of $W^*$ are $Vx \leq 0$ and the extreme rays are the row vectors of $A$. We again refer the reader to [6, 9] for more details of the properties of this duality.

Lemma 4. Given a pointed polyhedral cone $W \in \mathbb{R}^n$ and a set of projection directions $\Gamma$ one can construct, in polynomial time, a polytope $Q \in \mathbb{R}^n$ such that the extreme rays of $\pi(W)$ are in one-to-one correspondence with the vertices of $\pi(Q)$, where both the projections are onto the same subspace.

Proof. Clearly, none of the projection directions lie in the relative interior of $W$, otherwise the projection spans the whole subspace. Let $W^*$ be the polar dual of $W$. For any vector $\alpha$ in the relative interior of $W^*$ the hyperplane $\alpha \cdot x = 0$ touches $W$ only at the origin and hence $W \cap \{x | \alpha \cdot x \leq 1\}$ is a bounded polytope. It is actually a pyramid with origin as the apex.

Now consider the projection $\pi(W)$, which is a pointed cone with origin as apex. This cone is a full-dimensional cone in the subspace containing it and we can consider its polar dual in that subspace. Let $\pi^*(W)$ be the polar dual of $\pi(W)$. For any vector $\alpha'$ in the relative interior of $\pi^*(W)$, $\pi(W) \cap \{x | \alpha' \cdot x \leq 1\}$ is a bounded polytope. Moreover, for such an $\alpha'$, $\gamma_i \cdot \alpha' = 0$ for all $\gamma_i \in \Gamma$. Since $\pi(W)$ is a pointed cone such an $\alpha'$ exists.

Since $\pi^*(W)$ can be obtained as the intersection of $W^*$ with $\bigcap_{\gamma_i \in \Gamma} \{\gamma_i \cdot x = 0\}$, a vector $\alpha'$ in the relative interior of $\pi^*(W)$ can be computed in polynomial time if one knows either the vertices or facets of $W$. Also, $\alpha'$ lies in the relative interior of $W^*$ as well. Define $Q = W \cap \{\alpha' \cdot x \leq 1\}$. Given the extreme rays (facets respectively) of $W$ and the projection directions $\Gamma$ one can compute the vertices (facets respectively) of $Q$ in polynomial time.

Since $\alpha'$ is orthogonal to each of the projection directions, the vertices and facets of $\pi(Q)$ are in one-to-one correspondence with the extreme rays and the facets of $\pi(W)$. 

Theorem 1 together with Lemma 3 and Lemma 4 gives the following:

Theorem 2. Given a polytope $P = \{x, y | Ax + By \leq 1\}$ and $Q = CH(V)$ it is NP-complete to decide if $Q \neq \pi(P)$.

Now we consider the last variant of the projection problem where we are given an $H$-polytope and we want to compute the $H\mathcal{V}$-representation of the projection. As it turns out, although computing either the vertices or facets of the projection is NP-hard, computing both vertices and facets is VE-complete.

Before we prove this, we would like to remark that the notion of output-sensitiveness can have various meanings. An output-sensitive polynomial algorithm for an enumeration problem (like VE) could enumerate vertices such that a new vertex is reported within incremental polynomial delay i.e. each new reporting takes time polynomial in the input and the output produced so far. It is equally conceivable that the algorithm takes total time polynomial in the input and output but there is no guarantee that successive reportings take only incremental polynomial delay.


delay. We will assume that if we have an output-sensitive algorithm of the latter kind, then we actually know the complexity of its running time. Under this assumption the two notions are same for VE.

To see why this is true, consider the following. Given an $H$-polytope $P$ and a $V$-polytope $Q$, determining whether $P = Q$ is polynomial time equivalent to VE (See [1]). Also, solving this problem gives an algorithm for VE that is not only output-sensitive polynomial but also has a polynomial delay guarantee. If we have an enumeration algorithm that has no guarantee of polynomial delay between successive outputs, but for which we know the running time, then we can use this procedure to create a polynomial algorithm for deciding the equivalence of $H$- and $V$-polytopes: Simply compute the time needed by the algorithm to enumerate all vertices of $P$ assuming $P = Q$ and run the enumeration algorithm for the time required to output $|V| + 1$ vertices. If the procedure stops then we can compare the list of vertices of $P$ with that of $Q$ in polynomial time. If, on the other hand, the procedure doesn’t finish within the given time then $P$ must have more vertices than $Q$ and hence, $P \neq Q$.

So for proving VE-completeness in the next theorem, when we assume the existence of an output-sensitive polynomial algorithm for VE, we also assume that this algorithm has a guarantee of polynomial delay between successive outputs. Although we will work with the Convex Hull problem which is the dual version of VE, with a slight abuse of language we will refer to this dual problem as VE as well.

**Theorem 3.** Given a polytope $P = \{x, y|Ax + By \leq 1\}$ it is VE-complete to compute the facets and vertices of $\pi(P)$.

**Proof.** Since every polytope $P \subseteq \mathbb{R}^n$ given by $m$ vertices can be converted to a $(m - 1)$-dimensional simplex $\Delta$ such that $P$ is a projection of $\Delta$ it is clear that computing $HV$-representation of the projection of an $H$-polytope is VE-hard.

To prove that this problem is also VE-easy, we give an algorithm that uses a routine for VE to enumerate the facets and vertices of $\pi(P)$. The algorithm proceeds as follows: At any point we have a list of vertices $V$ of $\pi(P)$ and we want to verify that $V$ indeed contains all vertices of $\pi(P)$. If the list is not complete, we want to find another vertex of $\pi(P)$ that is not already in $V$. To do this, we start enumerating facets of $CH(V)$ and we verify that each generated facet is indeed a facet of $\pi(P)$. This is easy to check because of the following:

Suppose $h = \{x|a \cdot x = 1\}$ be a hyperplane in the projection space. We say that $h$ intersects $P$ properly if the intersection $P \cap \{(a, 0, \cdots, 0) \cdot (x, y) = 1\}$ has some point in the relative interior of $P$. We will call such an intersection a proper intersection.

We claim that the defining hyperplane of every facet $f$ of $CH(V)$, that is not a facet of $\pi(P)$, intersects $P$ properly. To see this, pick a point $x_1$ in the relative interior of $f$. Such a point exists because $CH(V) \subset \pi(P)$ if some facet $f$ of $CH(V)$ is not a facet of $\pi(P)$. This point also lies in the relative interior of $\pi(P)$. Also, there is a point $(x_1, y_1)$ that lies in the relative interior of $P$ that projects to $x_1$. Clearly the hyperplane $\{(a, 0, \cdots, 0) \cdot (x, y) = 1\}$ contains $(x_1, y_1)$ and hence the hyperplane defining $f$ intersects $P$ properly.

It follows that, if $V$ does not contain all vertices of $\pi(P)$ then there exists a facet $f \in CH(V)$ intersecting $P$ properly. So if the enumeration procedure for facets of $CH(V)$ stops and none of the facets intersect $P$ properly then $V$ contains all the vertices of $\pi(P)$. If some intermediate facet $\{a \cdot x = 1\}$ of $CH(V)$ does intersect $P$ properly then maximizing the objective function $(a, 0, \cdots, 0) \cdot (x, y)$ over $P$ produces a vertex of $P$ that also gives a vertex $v$ of $\pi(P)$ upon projection. Moreover this vertex is not in the list $V$. Thus, if $V$ is not a complete
vertex description of $\pi(P)$ we can find another vertex of $\pi(P)$ in polynomial time. This gives an output-sensitive polynomial algorithm for enumerating all facets and vertices of $\pi(P)$. Hence, computing all vertices and facets of the projection $\pi(P)$ of an $\mathcal{H}$-polytope $P$ is VE-easy as well.

Given that it is widely believed that $P \neq \text{NP}$ and VE has been studied quite closely by a number of researchers, one can infer that computing the projection is going to be a challenging problem for arbitrary projection directions. Also, one should note that if the projection is known to be a simple of simplicial polytope then computing both facets and vertices of the projection can be done in output-sensitive polynomial time because VE for simple or simplicial polytopes can be done in output-sensitive polynomial time ([2, 4]). But if the projection yields a degenerate polytope then we do not have such an algorithm.

### 3.2 Projection onto random subspaces

As stated before, the result that the facets of a random projection of a polytope can be computed in output sensitive polynomial time was independently established earlier by Jones et. al. (See [8]). We nevertheless discuss random projections for completeness of the discussion and to highlight how the computational complexity of the projection problem changes when one is interested only in random projection directions.

Now we describe a polynomial time algorithm for a more general case where the projection directions are, in some sense, non-degenerate. To make this notion precise, note that if $P$ is the input polytope then every face of projection $\pi(P)$ is the shadow of some proper face of $P$. Call the maximal dimensional face $f'$ of $P$ a pre-image of the face $f$ of $\pi(P)$ if $f$ is obtained by projecting all vertices defining $f'$ and taking their convex hull. In general, the dimensions of $f$ and $f'$ are not the same. This can happen if some projection directions lie in the affine hull of $f'$. We call a set of projection directions non-degenerate with respect to $P$ if no directions lie in the affine hull of $f'$. We call a set of projection directions non-degenerate with respect to $P$ if no directions lie in the affine hull of any face of $P$.

**Fact 1:** If the projection directions are picked randomly, then they are non-degenerate with respect to any polytope $P$ with probability 1.

Also, for non-degenerate projection directions a face $f$ of $\pi(P)$ and its pre-image $f'$ have the same affine dimension. This is easy to see because a projection reduces the dimension of some face $f'$ of $P$, that does not disappear in the projection, if and only if the projection direction lies in the affine hull of $f'$ which is not possible for non-degenerate projection directions. Thus,

**Fact 2:** For non-degenerate projection directions and a face $f$ of $\pi(P)$, if $f'$ is the pre-image of $f$ then $\text{dim}(f) = \text{dim}(f')$.

Now, given a polytope $P$ in $\mathcal{H}$-representation and a set of non-degenerate projection directions $\Gamma$ we want to compute the facets of the projection $\pi(P)$. Again, we assume that the facets of $P$ are presented as inequalities of the form $Ax \leq 1$, where $A$ is a matrix of size $m \times (d + k)$ and the projection has dimension $d$. Since, we will need to solve Linear Programs we also assume that the polytope is rational i.e. the entries in $A$ are rational numbers. We will omit $\Gamma$ from the discussion below and assume that the projection directions are aligned along a subset of co-ordinate axes. If not, we can apply a suitable affine transform to $P$ depending on the orthogonal projection directions. Thus the reader should bear in mind that the polytope in what follows is a result of an affine transformation determined by a set of randomly picked orthogonal projection directions.
Our algorithm for enumerating the facets of $\pi(P)$ proceeds as follows: Given a partial list of facets of $\pi(P)$, for each facet $f$ we identify its pre-image $f'$ in $P$. For each of these faces of $P$ we identify their $(d-2)$-dimensional faces and among all such $(d-2)$-faces of $f'$ some give rise to ridges in $\pi(P)$. We identify which faces form the pre-image of some ridge of $\pi(P)$ and from the corresponding ridge, we identify the two facets defining this ridge, thus finding a new facet of $\pi(P)$ if the current list of facets is not complete.

**Lemma 5.** Given an $\mathcal{H}$-polytope $P$ and a facet $f$ of its projection $\pi(P)$, one can find the facets of $P$ defining the pre-image of $f$ in polynomial time.

**Proof.** Let $\{x \in \mathbb{R}^d | a \cdot x \leq 1, a \in \mathbb{Q}^d\}$ be the halfspace defining the facet $f$ of $\pi(P)$. Clearly, the hyperplane $h$ in $\mathbb{R}^{d+k}$ with normal $a' = (a, 0, \cdots, 0)$ defines the supporting hyperplane $\{x \in \mathbb{R}^{d+k} | a' \cdot x = 1\}$. Also, $P \cap h$ is a face of $P$ and is exactly the pre-image of $f$. A facet $F$ of $P$ contains this face iff $P \cap h \cap F$ has the same dimensions as $P \cap h$. Thus, one can find all the facets of $P$ containing the pre-image of $f$ in time polynomial in the size of $P$. \hfill \square

The next lemma follows immediately from the non-degeneracy of the projection directions, so we mention it without the proof (See Fact 2).

**Lemma 6.** Given $P$ and a facet $f$ of its projection $\pi(P)$, if $g$ is another facet of $\pi(P)$ sharing a ridge with $f$ then the pre-images $f'$ and $g'$ share a face in $P$. Furthermore,

$$\dim(f' \cap g') = \dim(f \cap g) = \dim(f') - 1 = \dim(g') - 1 = d - 2$$

Since the facets of $P$ are known, we can identify all $(d-2)$-faces of $f'$. The number of these faces is at most $m$ for each pre-image $f'$ and since $f'$ is itself a polytope of dimension $d-1$, we can compute the non-redundant inequalities defining the facets $(d-2)$-dimensional faces) of $f'$.

At this point, what remains is to identify these ridges and the facets defining these ridges. The following two lemmas achieve this.

**Lemma 7.** Let $P = \{(x, y) | A \cdot (x, y) \leq 1\}$ be a polytope in $\mathbb{R}^d \times \mathbb{R}^k$, where $A \in \mathbb{Q}^{m \times (d+k)}$, $x \in \mathbb{R}^d, y \in \mathbb{R}^k$. Also, let $f$ be a $(d-2)$-face of $P$ defined as $f = \{(x, y) | A' \cdot (x, y) = 1, (x, y) \in P\}$, where $A' \subset A$. Then, $f$ defines a ridge in the projection $Q(x)$ if and only if

- there exists $\alpha \in \mathbb{R}^d$ such that $(\alpha, 0, \cdots, 0) \in \text{conv}(A')$, where each row of $A'$ is treated as a point in $\mathbb{R}^{d+k}$. And,

- The feasible region of all such $\alpha$ is a line segment.

It is not difficult to see that this lemma is just a rephrasing of the basic properties of supporting hyperplanes of a polytope. In other words, any hyperplane whose normal is a convex combinations of the normals of facets defining the face $f$, is a supporting hyperplane of $P$ and vice-versa. Furthermore, if the normal lies in the subspace where the projection $\pi(P)$ lives, then it is also a supporting hyperplane of $\pi(P)$. Also, the normals of all hyperplanes that support a polytope at some ridge, when treated as points, form a 1-dimensional polytope i.e. a line segment. This formulation allows us to check in polynomial time whether a $(d-2)$-face of $P$ forms a pre-image of some ridge of $Q(x)$.

\[\text{Removing redundancies can be achieved via Linear Programming.}\]
Lemma 8. The end points of the feasible region of $\alpha$ in lemma 7 are the normals of the facets of $\pi(P)$ defining the ridge corresponding to face $f$.

As noted before, the normals of the hyperplanes supporting a polytope at a ridge $r$ form a line segment when viewed as points. The end points of the segment represent the normals of the two facets defining the ridge $r$. This lemma ensures that given a pre-image of some ridge of $\pi(P)$, one can compute the normals of the two facets of $\pi(P)$ defining the ridge $r$ by solving a polynomial number of linear programs each of size polynomial in the size of input.

Putting everything together we get the following theorem:

Theorem 4. Given a polytope $P$ defined by facets, and a set of non-degenerate orthogonal projection directions $\Gamma$ one can enumerate all facets of $\pi(P)$ in output-sensitive polynomial time.

Since randomly picked projection directions are non-degenerate, one can also enumerate the facets of the projection of an $H$-polytope for such directions. Note, that this also gives an output-sensitive polynomial algorithm for the case when the input is an $HV$-polytope irrespective of the output form. Also, if the vertices of $P$ are given then some tests like those in Lemma 7 and 8 become easier. We leave the proof of this to the reader since they do not affect our main argument about an output-sensitive polynomial algorithm.

Corollary 1. Given an $HV$-polytope $P$ and a set of projection directions $\Gamma$ that are non-degenerate with respect to $P$ there is an algorithm that can enumerate the vertices and/or facets of $\pi(P)$ in output-sensitive polynomial time.

It is easy to see that computing the vertices of the projection of an $H$-polytope along non-degenerate directions has VE as a special case\footnote{Just pick the set of projection directions to be the empty set.}. Hence, enumerating vertices of the projection of an $H$-polytope along non-degenerate projection directions remains VE-hard even though it is not clear if it remains NP-hard. Similarly one can argue that the complexity status of computing the projection of a $V$-polytope along non-degenerate projection directions remains the same as that of computing the projection along arbitrary directions.

References

[1] D. Avis, D. Bremner, and R. Seidel. How good are convex hull algorithms? *Comput. Geom.*, 7:265–301, 1997.

[2] D. Avis and K. Fukuda. A pivoting algorithm for convex hulls and vertex enumeration of arrangements and polyhedra. *Discrete & Computational Geometry*, 8:295–313, 1992.

[3] E. Balas. Projection with a minimal system of inequalities. *Comput. Optim. Appl.*, 10(2):189–193, 1998.

[4] D. Bremner, K. Fukuda, and A. Marzetta. Primal - dual methods for vertex and facet enumeration. *Discrete & Computational Geometry*, 20(3):333–357, 1998.

[5] R. M. Freund and J. B. Orlin. On the complexity of four polyhedral set containment problems. *Mathematical Programming*, 33(2):139–145, 1985.
[6] B. Grünbaum. *Convex Polytopes Second Edition prepared by V. Kaibel, V. L. Klee and G. M. Ziegler*, volume 221 of *Graduate Texts in Mathematics*. Springer, 2003.

[7] J.-L. Imbert. Fourier’s elimination: Which to choose? In *PPCP*, pages 117–129, 1993.

[8] C. Jones, E. Kerrigan, and J. Maciejowski. Equality set projection: A new algorithm for the projection of polytopes in halfspace representation. Technical Report CUED/F-INFENG/TR.463, ETH Zurich, 2004.

[9] G. M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag.