Steering criteria from general entropic uncertainty relations

Ana C. S. Costa, Roope Uola, and Otfried Gühne
Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Straße 3, 57068 Siegen, Germany
(Dated: October 13, 2017)

The effect of steering describes a possible action at a distance via measurements but characterizing the quantum states that can be used for this task remains difficult. We provide a method to derive sufficient criteria for steering from entropic uncertainty relations using generalized entropies. We demonstrate that the resulting criteria outperform existing criteria in several scenarios; moreover, they allow to detect weakly steerable states.

PACS numbers: 03.65.Ud, 03.67.-a

Introduction.— Steering is a term coined by Schrödinger in 1935 in order to capture the essence of the Einstein-Podolsky-Rosen argument [1]. It describes Alice’s ability to affect Bob’s quantum state through her choice of a measurement basis, without allowing for instantaneous signaling. In the modern view, steering is based on a quantum correlation between entanglement and the violation of Bell inequalities, meaning that not every entangled state can be used for steering and not every steerable state violates a Bell inequality [2].

In the last years the theory of steering has evolved quickly. It has been shown that the concept of steering is closely related to fundamental problems and open questions in quantum physics. For instance, steering has been used to find counterexamples to so-called Peres conjecture, which was an open problem in entanglement theory for more than fifteen years [3-5]. In addition, steering was shown to be equivalent to the notion of joint measurability of generalized measurements [6-11] and results from one problem can be transferred to the other. Finally, steering has been shown to be useful for tasks in quantum information processing, such as one-sided device-independent quantum key distribution [12] and subchannel discrimination [13].

Despite of all these results, the simple question whether or not a given bipartite quantum state is useful for steering is not easy to answer. If the conditional states of Bob are known, the problem can be solved via semidefinite programming [13-15], but this approach requires complete positive maps provide a guiding line for developing separability criteria [21].

In this paper we identify entropic uncertainty relations as a fundamental tool to develop steering criteria. Uncertainty relations in terms of entropies have already become important in many areas of quantum information theory [22-23]. We show that various entropic uncertainty relations can be transformed into a steering criterion. As examples, we consider generalized entropies such as the so-called Tsallis entropy and demonstrate that the resulting criteria outperform known criteria in many cases. Our approach is motivated by previous works on entanglement criteria from entropic uncertainty relations [24] and generalizes recent entropic criteria for steering [23-26], which were, however, restricted to the special case of the Shannon entropy.

Steering and entropies.— In steering scenarios, one assumes that Alice and Bob share a quantum state ρAB. Then, Alice makes measurements on her system and claims that with these measurements she can steer the state inside Bob’s laboratory. Bob, of course, is not convinced of Alice’s abilities. In a more formal manner, we can assume that Alice performs a measurement A with outcome i on her part of the system, while Bob performs a measurement B with outcome j on his part. From that, they can obtain the joint probability distribution of the outcomes. If for all possible measurements A and B one can express the joint probabilities in the form

\[ p(i,j|A,B) = \sum_\lambda p(\lambda)p(i|A,\lambda)p(j|B,\lambda), \]

(1)

then the system is called unsteerable. Here, \( p(i|A,\lambda) \) is a general probability distribution, while \( p(j|B,\lambda) = \text{Tr}_B[B(j)|\sigma_\lambda] \) is a probability distribution originating from a quantum state \( \sigma_\lambda \) being the same for all measurements B on Bob’s side. Furthermore, \( B(j) \) denotes a measurement operator such that \( \sum_j B(j) = 1 \), and \( \sum_\lambda p(\lambda) = 1 \), where \( \lambda \) is a label for the hidden quantum state \( \sigma_\lambda \). A model as in Eq. (1) is called a local hidden state (LHS) model, and if it exists, Bob can explain all the results through a set of local states \( \{\sigma_\lambda\} \) which is not altered by Alice’s measurements. But if it is not possible to find states \( \sigma_\lambda \) that make this probability distribution feasible, Bob has to assume that Alice can steer the state.

Let us now explain some basic facts about entropy. For a general probability distribution \( \mathcal{P} = (p_1, \ldots, p_N) \), the Shannon entropy is defined as [24]

\[ S(\mathcal{P}) = -\sum_i p_i \ln(p_i). \]

(2)

Entropic uncertainty relations can easily be explained with an example. Consider the Pauli measurements \( \sigma_z \)
and \( \sigma_z \) on a single qubit. For any quantum state these measurements give rise to a probability distribution of the outcomes \( \pm 1 \) and the corresponding entropy \( S(\sigma_z) \). The fact that \( \sigma_x \) and \( \sigma_z \) do not share a common eigenstate can be expressed by \( \ref{eq:5} \)

\[
S(\sigma_x) + S(\sigma_z) \geq \ln(2),
\]

where the lower bound does not depend on the state.

For our approach, we also need the relative entropy, also known as Kullback-Leibler divergence \( \ref{eq:27} \), between two probability distributions \( P \) and \( Q \),

\[
D(P\|Q) = \sum_i p_i \ln \left( \frac{p_i}{q_i} \right).
\]

Two properties are essential: First, the relative entropy is additive for independent distributions, that is if \( P, P_2 \) are independent distributions, with the joint probability distribution \( P(x, y) = P_1(x)P_2(y) \) and the same for \( Q_1, Q_2 \) then one has that

\[
D(P\|Q) = D(P_1\|Q_1) + D(P_2\|Q_2).
\]

Second, the relative entropy is jointly convex. This means that for two pairs of distributions \( P_1, Q_1 \) and \( P_2, Q_2 \) one has

\[
D[\lambda P_1 + (1 - \lambda)P_2\|\lambda Q_1 + (1 - \lambda)Q_2] \leq \lambda D(P_1\|Q_1) + (1 - \lambda)D(P_2\|Q_2).
\]

The main idea. — The starting point of our method is the relative entropy between two distributions, namely

\[
F(A, B) = -D(A \otimes B\|A \otimes \mathbb{I}).
\]

Here, \( A \otimes B \) denotes the joint probability distribution \( p(i, j|A, B) \), which we denote by \( p_{ij} \) for convenience, \( A \) is the marginal distribution \( p(i|A) \), which we denote by \( p_i \), and \( \mathbb{I} \) is a uniform distribution with \( q_j = 1/N \) for all \( j \). As the relative entropy is jointly convex, \( F(A, B) \) is concave in the probability distribution \( A \otimes B \). We can directly calculate that

\[
F(A, B) = -\sum_{ij} p_{ij} \ln \left( \frac{p_{ij}}{p_i/q_j} \right) = S(B|A) - \ln(N),
\]

where \( S(B|A) = S(A, B) - S(A) \) is the conditional entropy. On the other hand, considering a product distribution \( P(i|A, \lambda)p_q(j|B, \lambda) \) with a fixed \( \lambda \) and the property from Eq. \( \ref{eq:5} \), we have

\[
F(A, B) = -D[p(i|A, \lambda)|p(i|A, \lambda)] - D[p_q(j|B, \lambda)|\mathbb{I}] = S(B|\lambda) - \ln(N).
\]

Consequently, for a product distribution and a set of measurements \( A_k \otimes B_k \), we have

\[
\sum_k S(B_k|A_k) = \sum_k S(B_k|\lambda).
\]

The right-hand side of this equation depends on probability distributions taken from the quantum state \( \sigma_\lambda \). Such distributions typically obey an entropic uncertainty relation,

\[
\sum_k S(B_k|\lambda) \geq C_B.
\]

So, for product distributions we have

\[
\sum_k S(B_k|A_k) \geq C_B.
\]

Finally, since \( F \) is concave, the same bound holds for convex combinations of product distributions \( p(i|A, \lambda)p_q(j|B, \lambda) \) from Eq. \( \ref{eq:1} \), meaning that any non-steerable quantum system obeys this relation. In this way entropic uncertainty relations can be used to derive steering criteria. The intuition behind these criteria is based on the interpretation of Shannon conditional entropy. In Eq. \( \ref{eq:12} \), one can see that the knowledge that Alice has about Bob’s outcomes is bounded. If this inequality is violated, then the system is steerable, meaning that Alice can do better predictions than those allowed by an entropic uncertainty relation.

So far, this criterion is the same as the one in Ref. \( \ref{eq:26} \), but our proof highlights the three central ingredients:

First, we needed an additivity relation for independent distributions in Eq. \( \ref{eq:5} \), second we needed the state independent entropic uncertainty relation in Eq. \( \ref{eq:11} \), and finally we needed the joint convexity of the relative entropy in Eq. \( \ref{eq:6} \). These properties are not at all specific for the Shannon entropy, so our strategy works also for generalized entropies.

Steering criteria for generalized entropies. — As a possible generalized entropy, we consider the so-called Tsallis entropy \( \ref{eq:28} \) which depends on a parameter \( q > 1 \). It is given by

\[
S_q(P) = -\sum_i p_i^q \ln_q (p_i),
\]

where the \( q \)-logarithm is defined as \( \ln_q(x) = (x^{1-q} - 1)/(1-q) \). Note that in the limit \( q \to 1 \) this entropy converges to the Shannon entropy. The generalized relative entropy can be defined as \( \ref{eq:30} \)

\[
D_q(P\|Q) = -\sum_i p_i \ln_q \left( \frac{q_i}{p_i} \right),
\]

it is jointly convex and obeys the following relation for product distributions:

\[
D_q(P\|Q) = D_q(P_1\|Q_1) + D_q(P_2\|Q_2) + (q - 1)D_q(P_1\|Q_1)D_q(P_2\|Q_2).
\]

The additional term is due to non-additivity of the generalized entropy.

Now we can apply the machinery derived above and consider the quantity \( F(A, B) = -D_q(A \otimes B\|A \otimes \mathbb{I}) \). It
follows by direct calculation that if the measurements $B_k$ obey the entropic uncertainty relation
\[ \sum_k S_q(B_k) \geq C_B^{(q)} \]
then one has the steering criterion
\[ \sum_k \left[ S_q(B_k|A_k) + (1 - q)C(A_k; B_k) \right] \geq C_B^{(q)} \]
and violation of it implies steerability of the state. Here, $S_q(B|A) = S_q(A, B) - S_q(A)$ is the conditional entropy \[31\] and the additional term is given by
\[ C(A, B) = \sum_i p_i^A \ln_q(p_i) - \sum_{i,j} p_{ij}^B \ln_q(p_{ij}) \ln_q(p_{ij}). \]
From Eq. \[16\] it is easy to see that if we consider $q \to 1$, we arrive at Eq. \[12\]. Note that we can also rewrite
\[ \ln \] in terms of probabilities as
\[ \frac{1}{1-q} \sum_k \left[ 1 - \sum_{i,j} \left( \frac{p_{ij}^{(k)}}{p_i^{(k)}} \right)^q \right] \geq C_B^{(q)} \]
Here, $p_{ij}^{(k)}$ is the probability of Alice and Bob for outcome $(i,j)$ when measuring $A_k \otimes B_k$, and $p_i^{(k)}$ are the marginal outcome probabilities of Alice’s measurement $A_k$. This form of the criterion is very easy to evaluate.

**Application I: Isotropic states.**— To test the strength of our steering criteria, we consider $d$-dimensional isotropic states \[32\]
\[ \rho_{\text{iso}} = \alpha|\phi_+^d\rangle\langle\phi_+^d| + \frac{1 - \alpha}{d^2} \mathbb{1}, \]
where $|\phi_+^d\rangle = (1/\sqrt{d}) \sum_{i=1}^{d-1} |i\rangle|i\rangle$ is a maximally entangled state. These states are known to be entangled for $\alpha > 1/(d+1)$ and separable otherwise. As observables, we consider a set of mutually unbiased bases (MUBs) in dimension $d$. One can directly check that the marginal probabilities for this class of states are $p_i = 1/d$ for all $i$ and the joint probabilities are $p_{ij} = (1 + (d-1)\alpha)/d^2$ (occurring $d$ times), and $p_{ij} = (1 - \alpha)/d^2$ [for $i \neq j$ and occurring $d(d-1)$ times]. These probabilities are the same for all measurements. Inserting them in Eq. \[18\], the condition for non-steerability reads
\[ \frac{m}{q-1} \left[ 1 - \frac{1}{d^q} (1 + (d-1)\alpha)^q + (d-1)(1-\alpha)^q \right] \geq C_B^{(q)} , \]
which depends on the parameter $q$ and the number of MUBs $m$. For certain values of $q$ and $m$, the bounds of the entropic uncertainty relations $C_B^{(q)}$ are known (see Appendix A). For other cases they can be approximated numerically.

Let us discuss the strength of this criterion. First, numerical investigations suggest that the criterion is strongest for $q = 2$. For this value of $q$ the violation of Eq. \[20\] occurs for $\alpha > 1/\sqrt{d}$. Considering a complete set of MUBs $(m = d + 1)$ (this exists for $d$ being a power of a prime) the violation happens for $\alpha > 1/\sqrt{d + 1}$.

For qubits ($d = 2$) isotropic states are equivalent to Werner states \[33\]. Then, with a complete set of MUBs the violation occurs for $\alpha > 1/\sqrt{3} \approx 0.577$, which is known to be the optimal threshold \[34\]. More generally, in Ref. \[35\], a steering inequality for MUBs and isotropic states has been presented which is violated for $\alpha > (d^{3/2} - 1)/(d^2 - 1)$. It is straightforward to show that our inequality is stronger. Recently, the same problem has been investigated using semi-definite programming \[36\]. For $3 \leq d \leq 5$ a better threshold than ours was obtained, but it is worth to mention that our criteria directly use probability distributions from few measurements, without the need of performing full tomography on Bob’s conditional state. In addition, numerical approaches are naturally limited to small dimensions.

In Fig. \[1\] we compare our criterion with the ones mentioned above. We concentrate in the values of $q \to 1$ and $q = 2$, since the former is related to the usual entropic steering criteria and the latter is the optimal value of $q$ for the detection of steerable states.

**Connection to existing entanglement criteria.**— At this point, it is interesting to compare our approach with entanglement criteria derived from entropic uncertainty relations \[24\]. The mathematical formulation goes as follows. Let $A_1$ and $A_2$ ($B_1$ and $B_2$) be observables on Alice’s (Bob’s) laboratory. Assume that Bob’s observables obey an entropic uncertainty relation $S(B_1) + S(B_2) \geq C_B$, where $S(B_i)$ is a generalized entropy, such as the Shannon or Tsallis entropy. Then it can be shown that...
for separable states

\[ S(A_1 \otimes B_1) + S(A_2 \otimes B_2) \geq CB \]  

(21)

holds. Here, \( S(A_k \otimes B_k) \) is the entropy of the probability distribution of the outcomes of the global observable \( A_k \otimes B_k \). Note that this implies that for a degenerate \( A_k \otimes B_k \) the probability distribution differs from the local ones. For instance, measuring \( \sigma_z \otimes \sigma_z \) gives four possible local probabilities \( p_{++}, p_{+-}, p_{-+}, p_{--} \), but for the evaluation of \( S(A_k \otimes B_k) \) one combines them according to \( q_+ = p_{++} + p_{--} \) and \( q_- = p_{-+} + p_{+-} \), as these correspond to the global outcomes.

Some connections to our derivation of steering inequalities are interesting. First, if one reconsiders the proof in Ref. [24] one realizes that Eq. (21) is indeed a steering criterion and not a criterion for entanglement. That is, all probability distributions of the form in Eq. (11) fulfill it. Second, also in Ref. [24] it was observed that the criterion is strongest for values \( 2 \leq q \leq 3 \). Finally, if one asks for a direct comparison between Eq. (21) and Eqs. (10, 12) one finds that Eq. (21) is of the same strength for special scenarios (e.g. Bell-diagonal two-qubit states and Pauli measurements), while it seems weaker in the general case (see below).

**Application II: General two-qubit states.** Let us now consider the application of our methods to general two-qubit states. Any two-qubit state can, after application of local unitaries, be written as

\[ e_{AB} = \frac{1}{4} (1 \otimes 1 + (\bar{a}\vec{\sigma}) \otimes 1 + 1 \otimes (\bar{b}\vec{\sigma}) + \sum_{i=1}^3 c_i \sigma_i \otimes \sigma_i) \]  

(22)

where \( \bar{a}, \bar{b}, \bar{c} \in \mathbb{R}^3 \) are vectors with norm less than one, \( \vec{\sigma} \) is a vector composed of the Pauli matrices and \( (\bar{a}\vec{\sigma}) = \sum_i a_i \sigma_i \). Let us assume that Alice performs projective measurements with effects \( P^{3}_{A} = [1 + \mu_{k} (\bar{u}_k \vec{\sigma})]/2 \) and Bob with the effects \( P^{3}_{B} = [1 + \nu_{k} (\bar{v}_k \vec{\sigma})]/2 \) with \( \mu_{k}, \nu_{k} = \pm 1 \) and \( \{\bar{u}, \bar{v}\} \in \mathbb{R}^3 \). Then, Eq. (15) can be written as

\[
\sum_{k} \left[ 1 - \sum_{\mu_{k}, \nu_{k}} \frac{1 + \mu_{k} (\bar{u}_k \vec{\sigma}) + \nu_{k} (\bar{v}_k \vec{\sigma}) + \mu_{k} \nu_{k} T_k}{2^{q+1} [1 + \mu_{k} (\bar{u}_k \vec{\sigma})]^{q-1}} \right] \\
\geq (q-1) c_i^{(c)} \beta, 
\]

(23)

where \( T_k = \sum_{i=1}^3 c_i u_{ik} v_{ik} \). The optimization over measurements of this criterion for general two-qubit states is involving. We will focus on the simple case of Pauli measurements, meaning that \( \bar{u}_k = \bar{v}_k = \{1, 0, 0\}^T, \{0, 1, 0\}^T, \{0, 0, 1\}^T \) and \( q = 2 \). Then we have that

\[
\sum_{i=1}^3 \left[ 1 - a_i^2 - b_i^2 + c_i^2 \frac{1}{2} (1 - a_i^2) \right] \geq 1. 
\]

(24)

If this inequality is violated, then the system is steerable.

Now, we can compare our criteria with other proposals for the detection of steerable states using three measurements, see Appendix B for detailed calculations. The criteria from Eq. (21) prove steerability if \( \sum_{i=1}^3 c_i^2 > 1 \), and from the linear criteria [2, 37] steerability follows if \( \sum_{i=1}^3 c_i^2/2 > 1 \), which is equivalent. Not surprisingly, Eq. (24) is stronger, since it uses more information about the state. This statement can be made hard by analyzing \( 10^6 \) two-qubit states randomly generated from a process based on Hilbert-Schmidt ensemble [38]. 94.34% of the states do not violate any of the criteria, 3.81% are steerable according to all criteria, 1.85% violate only criterion (24), and no state violates only the linear criteria.

A special case of two-qubit states are Bell diagonal states, which can be obtained if we set \( \bar{a} = \bar{b} = 0 \) in Eq. (22). For this class of states it is easy to see that the three criteria are equivalent. Note, however, that a necessary and sufficient condition for steerability of this class for projective measurements has recently been found [13].

**Application III: One-way steerable states.** As an example of weakly steerable states that can be detected with our methods we consider one-way steerable states, i.e., states that are steerable from Alice to Bob and not the other way around. We consider the state

\[ e_{AB} = (\beta |\psi(\theta)\rangle \langle \psi(\theta)| + (1 - \beta) \frac{1}{2} \otimes e^0_B, \]

(25)

where \( |\psi(\theta)\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle \) and \( e^0_B = \text{Tr}_A[|\psi(\theta\rangle \langle \psi(\theta)|] \). It has been shown that for \( \theta \in [0, \pi/4] \) and \( \cos^2(2\theta) \geq (2\beta - 1)((2 - \beta)^3) \) this state is not steerable from Bob to Alice considering an infinite number of projective measurements [10], while Alice can steer Bob for \( \beta > 1/2 \).

Considering three measurement settings, this state is one-way-steerable for \( 1/\sqrt{3} < \beta \leq \beta_{\text{max}} = 1 + 2 \sin^2(2\theta) \) [39]. For our entropic steering criteria we consider three Pauli measurements and \( q = 2 \) and we find that this state is one-way steerable for

\[
\frac{1}{2 \cos(2\theta)} \sqrt{3 - \sqrt{1 + 8 \sin^2(2\theta)}} < \beta \leq \beta_{\text{max}}. 
\]

(26)

For any \( \theta \) this gives a non-empty interval of \( \beta \) for which our criterion detects these weakly steerable states. An attempt of optimizing over the set of measurements will be addressed in a future work.

**Conclusions.** In this work we have proposed a straightforward technique for the construction of strong steering criteria from entropic uncertainty relations. These criteria are easy to implement using a finite set of measurement settings only, and do not need the use of semi-definite programming and full tomography on Bob’s conditional states.

For future work, several directions seem promising. First, besides the usual entropic uncertainty relations, such as entropic uncertainty relations in the presence of...
quantum memory [40] or relative entropy formulations of the uncertainty principle [41] are promising starting points for other criteria. Second, one can try to make quantitative statements on steerability from steering criteria. Recently, some attempts in this direction have been pursued [12]. Finally, it would be highly desirable to embed our approach in a general theory of multiparticle steering.

We thank Marcus Huber and Renato M. Angelo for discussions. This work was supported by the DFG, the ERC (Consolidator Grant No. 683107/TempoQ) and the Finnish Cultural Foundation.

APPENDIX

A: Known entropic uncertainty relations

In this Appendix we will present different entropic uncertainty relations that were used in this work and known from literature. For the Shannon entropy ($q \to 1$) and a complete set of MUBs, entropic uncertainty relations were analytically derived in Ref. [43] and are given by

$$C_B = \begin{cases} (d + 1) \log \left( \frac{d+1}{2} \right), & d \text{ odd} \\ \frac{4}{q} \log \left( \frac{4}{q} \right) + \frac{4}{q} + 1 \log \left( \frac{d+1}{2} \right), & d \text{ even}. \end{cases}$$ (27)

For the Tsallis entropy and $m$ MUBs it has been shown in Ref. [44] that, for $q \in (0; 2]$, the bounds are given by

$$C_B^{(q)} = m \ln_q \left( \frac{md}{d+m-1} \right).$$ (28)

If we consider the case $q \to 1$, this bound is not optimal for even dimensions, so in this case it is more appropriate to consider the bounds given in Eq. (27).

B: Calculations for two-qubit states

First, consider the steering criterion in Eq. (21), developed in Ref. [24]. For three Pauli measurements and the Tsallis entropy, we have the following relation

$$\sum_{k=1}^{3} S_q(A_k \otimes B_k) \geq C_B^{(q)}.$$ (29)

where $A_k = (\vec{u}_k \vec{\sigma})$ and $B_k = (\vec{v}_k \vec{\sigma})$. In terms of probabilities this criterion can be rewritten as

$$\frac{1}{q-1} \sum_{k=1}^{3} \left\{ 1 - \left[ p_{\vec{u}_k, \vec{v}_k}(+1,+1) + p_{\vec{u}_k, \vec{v}_k}(-1,-1) \right]^q 
- \left[ p_{\vec{u}_k, \vec{v}_k}(+1,-1) + p_{\vec{u}_k, \vec{v}_k}(-1,+1) \right]^q \right\} \geq C_B^{(q)}.$$ (30)

Inserting the probabilities for general two-qubit systems, we have that

$$\frac{1}{q-1} \sum_{k=1}^{3} \left\{ 1 - 2^{-q} \left[ (1+T_k)^q + (1-T_k)^q \right] \right\} \geq C_B^{(q)}.$$ (31)

If we fix the measurements and the value of $q$ in the same way as in Eq. (23), this criterion gives $\sum_{i=1}^{3} c_i^2 \leq 1$. Then, if this inequality is violated, the system is steerable.

[1] E. Schrödinger in a letter to A. Einstein, reprinted in K. v. Meyenn (ed.), Eine Entdeckung von ganz außerordentlicher Tragweite, Springer (2011), p. 551.
[2] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. 98, 140402 (2007).
[3] T. Moroder, O. Gittsovich, M. Huber, and O. Gühne, Phys. Rev. Lett. 113, 050404 (2014).
[4] T. Vertesi and N. Brunner, Nat. Commun. 5, 5297 (2014).
[5] S. Yu and C. H. Oh, Phys. Rev. A 95, 032111 (2017).
[6] M. T. Quintino, T. Vértesi, and N. Brunner, Phys. Rev. Lett. 113, 160402 (2014).
[7] R. Uola, T. Vértesi, and O. Gühne, Phys. Rev. Lett. 113, 160403 (2014).
[8] R. Uola, C. Budroni, O. Gühne, and J.-P. Pellonpää, Phys. Rev. Lett. 115, 230402 (2015).
[9] R. Uola, F. Lever, O. Gühne, and J.-P. Pellonpää, arXiv:1707.09237.
[10] J. Kiukas, C. Budroni, R. Uola, and J.-P. Pellonpää, arXiv:1704.05734.
[11] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, Phys. Rev. A 85, 010301(R) (2012).
[12] M. Piani and J. Watrous, Phys. Rev. Lett. 114, 060404 (2015).
[13] M. F. Pusey, Phys. Rev. A 88, 032313 (2013).
[14] D. Cavalcanti and P. Skrzypczyk, Rep. Prog. Phys. 80, 024001 (2017).
[15] I. Kogias, P. Skrzypczyk, D. Cavalcanti, A. Acín, and G. Adesso, Phys. Rev. Lett. 115, 210401 (2015).
[16] E. G. Cavalcanti, C. J. Foster, M. Fiwa, and H. M. Wiseman, J. Opt. Soc. Am. B 32, A74-A81 (2015).
[17] S. Jevtic, M. Pusey, D. Jennings, and T. Rudolph, Phys. Rev. Lett. 113, 020402 (2014).
[18] H. C. Nguyen and T. Yu, Europhys. Lett. 115, 10003 (2016).
[19] J. Bowles, F. Hirsch, M. T. Quintino, and N. Brunner, Phys. Rev. A 93, 022121 (2016).
[20] T. Moroder, O. Gittsovich, M. Huber, R. Uola, and O. Gühne, Phys. Rev. Lett. 116, 090403 (2016).
[21] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[22] H. Maassen and J. B. M. Uffink, Phys. Rev. Lett. 60, 1103 (1988).
[23] S. Wehner and A. Winter, New J. Phys. 12, 025009 (2010).
[24] O. Gühne and M. Lewenstein, Phys. Rev. A 70, 022316 (2004).
[25] S. P. Walborn, A. Salles, R. M. Gomes, F. Toscano, and P. H. Souto Ribeiro, Phys. Rev. Lett. 106, 130402 (2011).
[26] J. Schneeloch, C. J. Broadbent, S. P. Walborn, E. G. Cavalcanti, and J. C. Howell, Phys. Rev. A 87, 062103 (2013).
[27] T. M. Cover and J. A. Thomas, “Elements of Information Theory”, Second edition, John Wiley & Sons, 2006.
[28] J. Havrda and F. Charvat, Kybernetika 3, 30 (1967).
[29] C. Tsallis, J. Stat. Phys. 52, 479 (1988).
[30] S. Furuichi, K. Yanagi, and K. Kuriyama, J. Math. Phys. 45, 4868 (2004).
[31] S. Furuichi, J. Math. Phys. 47, 023302 (2006).
[32] M. Horodecki, and P. Horodecki, Phys. Rev. A 59, 4206 (1999).
[33] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[34] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, Phys. Rev. A 80, 032112 (2009).
[35] P. Skrzypczyk and D. Cavalcanti, Phys. Rev. A 92, 022354 (2015).
[36] J. Bavaresco, M. T. Quintino, L. Guerini, T. O. Maciel, D. Cavalcanti, and M. T. Cunha, arXiv: 1704.02994v2 (2017).
[37] A. C. S. Costa and R. M. Angelo, Phys. Rev. A 93, 010203(R) (2016).
[38] K. Życzkowski, K. A. Penson, I. Nechita, and B. Collins, J. Math. Phys. 52, 062201 (2011).
[39] Y. Xiao, X.-J. Ye, K. Sun, J.-S. Xu, C.-F. Li, and G.-C. Guo, Phys. Rev. Lett. 118, 140404 (2017).
[40] M. Berta, M. Christandl, R. Colbeck, J. M. Renes, and R. Renner, Nat. Phys. 6, 659 (2010).
[41] A. Barchielli, M. Gregoratti, and A. Toigo, Entropy 19, 301 (2017).
[42] J. Schneeloch and G. A. Howland, arXiv: 1709.0362v1 (2017).
[43] J. Sanchez-Ruiz, Phys. Lett. 201, 125 (1995).
[44] A. E. Rastegin, Eur. Phys. J. D 67, 269 (2013).