Precise deviations for disk counting statistics of invariant determinantal processes

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We consider two-dimensional determinantal processes which are rotation-invariant and study the fluctuations of the number of points in disks. Based on the theory of mod-phi convergence, we obtain Berry-Esseen as well as precise moderate to large deviation estimates for these statistics. These results are consistent with the Coulomb gas heuristic from the physics literature. We also obtain functional limit theorems for the stochastic process $(\# D_r)_{r>0}$ when the radius $r$ of the disk $D_r$ is growing in different regimes. We present several applications to invariant determinantal processes, including the polyanalytic Ginibre ensembles, Gaussian analytic function and other hyperbolic models. As a corollary, we compute the precise asymptotics for the entanglement entropy of (integer) Laughlin states for all Landau levels.

1. Introduction and results

1.1. Introduction

Determinantal point processes arise in several contexts in probability theory, such as random matrices, zeros of Gaussian analytic functions, domino tilings, etc. [Bor15]. In particular, the eigenvalues of a random matrix with i.i.d. standard complex Gaussian entries form a determinantal process in $\mathbb{C}$ which is known as the Ginibre ensemble [Gin65]. This process can also be viewed as a 2-dimensional Coulomb gas at inverse temperature $\beta = 2$ [Ser18]. Random systems such as determinantal processes or Coulomb gas, due to their built-in repulsion, exhibit remarkable hyperuniformity or rigidity properties, that is, the fluctuations of the number of points in a given region is smaller than compared to a Poisson process with the same intensity. In fact, a well known principle from electrostatics introduced in [MY80] states that the variance of the number of points in a box should grow like the surface area instead of the volume as in case of i.i.d. random points. Based on this physical reasoning, Jancovici, Lebowitz, Manificat [JLM93] also derived large deviation asymptotics for the number of points in a ball. We aim at rigorously verifying this conjecture for the Ginibre ensemble ($\beta = 2$), as well as to obtain precise deviations for counting statistics. Our method relies on the determinantal structure of the Ginibre ensemble combined with the theory of mod-phi convergence. In Section 1.4, we discuss other examples which include fermion point processes associated with higher Landau levels, the zero process of the hyperbolic Gaussian analytic function as well as other determinantal processes which are invariant (in law) under certain groups of Möbius transformations.

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It is easier to present our results in the infinite volume case. The (infinite) Ginibre point process is the microscopic scaling limit in the bulk of the Ginibre eigenvalue point process. It turns out that this limit is universal for many different ensembles of random matrices, including normal matrices [AHM11] and random matrices with general i.i.d. entries [TV15, CES19]. Let us denote by $\xi$ this point process (viewed as a discrete random measure on $\mathbb{C}$). It turns out that $\xi$ is a determinantal process with kernel $K_0(z, w) = e^{x \bar{y}}$ with respect to the standard Gaussian measure $\frac{d\mu}{2\pi}(z) = \frac{1}{2\pi} e^{-|z|^2}$, where $m$ denotes the Lebesgue measure. [RV07a]. It follows that the process $\xi$ is translation-invariant on $\mathbb{C}$ with intensity $1/\pi$. Since it is also rotation-invariant, it has the following remarkable property which was first obtained by Kostlan [Kos92], see [HKPV06] and [Dub18] for generalizations.

**Theorem 1.1 (Kos92).** The set of square-modulus of the points of $\xi$ has the same distribution as $\{\Gamma_1, \Gamma_2, \ldots\}$, where $\Gamma_k$ are independent gamma random variables with shape $k$ and rate 1.

We are interested in the fluctuations of the number of points in a disk $D_r = \{|z| < r\}$ for a large $r > 0$. Theorem 1.1 shows that this counting statistic is given by $\xi(D_r) = \sum_{k=1}^{\infty} \mathbb{I}_{\Gamma_k \leq r^2}$ in law, from which one can easily deduce a central limit theorem:

$$\pi \frac{\xi(D_r) - r^2}{r^\frac{1}{2}} \xrightarrow{d} \mathcal{N}_0, 1,$$

with $\mathcal{N}_0, 1$ being a standard Gaussian random variable, see [MY80, Shi06]. The CLT (1.1) can also be deduced from a general result from [Sos02] for the fluctuations of linear statistics of determinantal processes. In particular, the statistic $\xi(D_r)$ has variance

$$\text{Var}(\xi(D_r)) = \frac{r}{\sqrt{\pi}} (1 + o(1)),$$

which is indeed proportional to the surface area of $D_r$ for large $r$. As for large deviations of $\xi(D_r)$, it is argued in [JLM93] by using electrostatic arguments that for any $x \in \mathbb{R}_+$,

$$\log \mathbb{P}(\xi(D_r) - r^2 \geq x r^\gamma) = \begin{cases} \frac{x}{2\pi} r^{\gamma-1} \left( \sqrt{\pi} + o(1) \right), & \frac{1}{2} \leq \gamma < 1, \\ \frac{3}{8} x r^{\gamma-2} \left( 1 + o(1) \right), & 1 < \gamma < 2, \\ \frac{(\gamma-2)x^2}{2} r^{2\gamma} \log r \left( 1 + o(1) \right), & 2 < \gamma. \end{cases}$$

In this paper, we establish (1.2) and also investigate the precise deviations of the statistics $\xi(D_r)$ at different scales complementing the above-mentioned results. A similar study is also done for the eigenvalues of large Ginibre random matrices and our findings agree inside of the bulk. At the edge, there are some cut-off phenomena occurring (see Section 1.4).

Our analysis relies on the theory of mod-phi convergence, a concept which provides a unified framework to study deviations of certain random variables beyond the central limit theorem, see [FMN16]. In particular, this allows to makes precise the idea that $\xi(D_r) = \sum_{k=1}^{\infty} \mathbb{I}_{\Gamma_k \leq r^2}$ behaves like its mean $r^2$ plus a sum of $r$ independent, roughly identically distributed Bernoulli random variables. For a review of mod-phi convergence, we refer to [Section 1.2]. As a result, we obtain the following asymptotics for the counting statistics $\xi(D_r)$ of the infinite Ginibre process.

**Theorem 1.2 (Precise large deviations).** There exist two functions $I: \mathbb{R} \to \mathbb{R}_+$ with $I(0) = I'(0) = 0$ and $I'' > 0$ and $\psi: \mathbb{R} \to \mathbb{R}_+$ with $\psi(0) = 1$ such that locally uniformly for all $y \in \mathbb{R}_+$ it holds

$$\mathbb{P}(\xi(D_r) - r^2 \geq ry) = e^{-r I(y)} \sqrt{\frac{I''(y)}{2\pi r}} \frac{\psi_0(I'(y))}{1 - e^{-r I(y)}} \left( 1 + O(r^{-1}) \right).$$

The function $I$ is usually called the *rate function* and its convex conjugate $A_0$, as well as the correction terms. Moreover, our method allows us to obtain Berry–Essen estimates in the CLT (1.1) and an extension of the CLT up to the optimal regime in which a Gaussian
approximation is still valid (see Corollary 1.8). We also verified that our expression for $I$ matches with the prediction from [JLM93 (2.9a) and (2.9b)]. This rate function as well as the asymptotics \[I_k\] have also been obtained recently in the physics literature using different methods, \[LGC\].

In general, there is an extensive literature on large deviation estimates for hyperuniform two-dimensional point processes. Most relevant in our context are [BAZ08] on eigenvalues of the Ginibre ensemble, \[Kri06\] NSV08 for zeros of Gaussian analytic functions and \[ZZ10\] for zeros of random polynomials, as well as \[ST05\] [GN19] on the so-called hole probabilities of the Gaussian entire function. We also refer to [HKPV09 Chapter 7] or [GN18] for a review of these results and further references. For the Ginibre ensemble, it is also worth mentioning the rates of convergence to the circular law obtained recently in [GJ18].

For deviations at higher scale, i.e. $P(\xi(D_r) - r^2 \geq r^\gamma x)$ with $x_r \to \infty$, the approach of mod-phi convergence does not apply. Instead, inspired by \[Shi06\], we obtain the asymptotics directly from a moderate (in case $\gamma < 2$) or a superlarge deviation principle (in case $\gamma > 2$) for the random variables $\Gamma_k$ from \[Theorem 1.1\]. The proof is given in \[Section 4\].

**Proposition 1.3.** For any $x > 0$ it holds

$$\log \mathbb{P}(\xi(D_r) - r^2 \geq r^\gamma x) = \begin{cases} -\frac{\pi^2}{4} r^{2\gamma - 2} (1 + o(1)) & \text{for } 1 < \gamma < 2, \\ -J(x) r^\gamma (1 + o(1)) & \text{for } \gamma = 2, \\ -(\gamma - 2\frac{\pi^2}{2}) r^\gamma \log r (1 + o(1)) & \text{for } 2 < \gamma. \end{cases}$$

The rate function $J(x) = \frac{1+\pi^2}{2} \log(1+x) - \frac{2(3\pi^2 + 2)}{x^2}$ for $x \in \mathbb{R}_+$ has already been computed by \[Shi06\] Theorem 1.1. Our results show that as predicted by [LGC1], there are basically three regimes of large deviations which are separated by two critical points at $\gamma = 1$ (deviations of the order of the variance) and $\gamma = 2$ (deviations of the order of the mean). \[Theorem 1.2\] describes precisely the crossover behaviour for $\gamma = 1$.

**Remark 1.** An inspection of the proof of \[Proposition 1.3\] shows that it extends to any sequences $q_r \sim r^\gamma$ and that we can also cover larger deviation probabilities. Namely, if $q_r \geq r^\gamma$ for every constant $\gamma > 0$, then

$$\log \mathbb{P}(\xi(D_r) - r^2 \geq q_r) = q_r^2 \log q_r (1 + o(1)).$$

We can also investigates the joint law of the statistics $\xi(D_r)$ for different $r$. To do so, we use the notation

$$X_r(t) = \frac{r}{\pi^2} \xi(D_{r^2}) - \frac{(rt)^2}{r^2}, \quad t > 0.$$ 

The random variables $X_r(t)$ have mean 0 and its variance converges to $t$ as $r \to +\infty$. We obtain the following central limit theorem:

**Theorem 1.4** (Multidimensional CLT). The following convergence of the finite dimensional distributions holds:

$$(X_r(t))_{t \in \mathbb{R}_+} \overset{fdd}{\underset{r \to \infty}{\to}} \left( t^2 N_t \right)_{t \in \mathbb{R}_+},$$

where $(N_t)_{t \in \mathbb{R}_+}$ is a white noise, i.e. $N_t$ are i.i.d. standard Gaussian random variables.

This is in contrast with what happens if the points where independent. If $\mathcal{F}$ is a homogeneous Poisson point process with intensity $1/\pi$ on $\mathbb{C}$, then we verify that

$$\left( \frac{\mathcal{F}(D_{r^2}) - (rt)^2}{r} \right)_{t \in \mathbb{R}_+} \overset{\mathbb{S}(\mathbb{R}_+)}{\underset{r \to \infty}{\to}} \left( W_t \right)_{t \in \mathbb{R}_+},$$

where $(W_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion and the convergence is with respect to the Skorokhod topology. This difference lies in the rigidity of the Ginibre configurations. Namely, the Coulomb gas heuristic suggests that only the particles which lie near the boundary of $D_r$ contribute

\[1\] Recall that a sequence of càdlàg functions $X_n: \mathbb{R} \to \mathbb{R}$, is said to converge to a càdlàg function $X: \mathbb{R} \to \mathbb{R}$ in the Skorokhod topology if there exists a sequence of strictly increasing, continuous functions $(w_n)_{n \in \mathbb{N}}$ such that for all compact sets $B \subseteq \mathbb{R}$ it holds $\sup_{t \in B} |w_n(t) - t| \to 0$ and $\sup_{t \in B} |X_n(w_n(t)) - X(t)| \to 0$.
Remark 3. A classical result asserts that the (asymptotics) fluctuations of the eigenvalues of the Ginibre are described by an $H^1$-valued Gaussian process \[ \mathbb{R}^2 \to \mathbb{R}^2 \] This means that for any function $f \in H^1 \cap L^1(\mu)$,
\[
\xi(f_\rho) - \int_C f_\rho(z) \frac{dm(z)}{\pi} \xrightarrow{d_{\rho \to \infty}} \|f\|_{H^1(C)} N_{0,1}, \quad \text{where } f_\rho(z) = f\left(\frac{z}{\rho}\right).
\] (1.3)
and $\|f\|_{H^1(C)} = \frac{1}{2\pi} \int_C |\nabla f(z)|^2 \, dm(z)$. If one considers $f = \mathbb{1}_D \ast \phi_\rho$ where $\phi_\rho$ is an approximate $\delta$-function at microscopic scale $\rho^{-1/2}$, then one expects from Theorem 1.3 that $(\xi(D_\rho) - \rho^2) \sim \sqrt{\rho} N_{0,1}$ as $\rho \to \infty$. Similarly, by considering the test function $f = \sum_{j=1}^m \alpha_j \mathbb{1}_{D_{t_j}} \ast \phi_\rho$ with $0 < t_1 < \cdots < t_m$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ for $m \in \mathbb{N}$, one can informally recover the multi-dimensional central limit Theorem 1.3

1.2. Background and notation

Notation. Throughout this article, $\mathbb{1}_E$ denotes the indicator function of the event $E$. As in the introduction, $N_{0,1}$ stands for a standard Gaussian random variable and we let $\Phi_0(x) = \mathbb{P}(N_{0,1} \geq x)$ be its probability tail function. We denote the set of non-negative integers by $\mathbb{N} = \{0, 1, 2, \ldots\}$ and the disk of radius $r > 0$ in the complex plane by $D_r = \{z \in \mathbb{C} : |z| \leq r\}$. We denote the cumulant generating function of a centred Bernoulli distribution with parameter $p$ by
\[
\kappa_p(z) = \log \mathbb{E}[e^{z \cdot \text{Ber}_p}] = \log(1 + p(e^z - 1)) - pz
\]
for $p \in [0, 1]$ and $z \in \mathbb{C}$ and by $\kappa_p(z) = \frac{dp(z)}{dp}$. We also let $S = \{z \in \mathbb{C} : |3z| < \pi\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Determinantal point processes. Determinantal processes is a class of point processes which has been introduced by Macchi to describe ensembles of free fermions [Mac75]. Such processes are characterized by the property that all their correlation functions (with respect to a reference measure $\mu$) are of the form
\[
\det_{i,j \in \{1, \ldots, n\}} [K(x_i, x_j)], \quad n \in \mathbb{N}.
\]
The function $K$ is called the correlation kernel and we assume that it is continuous and Hermitian-symmetric. Moreover, if $K$ defines a locally trace-class (integral) operator $K$, then the kernel
characterizes the law of the process. We refer to [Sos02, ST03a, Joh06, HKPV06, Bor15] as introductions to determinantal processes and some examples. For instance, the eigenvalues of unitary invariant ensembles like the GUE and the CUE and the eigenvalues of normal random matrices are determinantal processes [AGZ10, AHM11]. In particular, the correlation kernel of the \( n \times n \) Ginibre ensemble is \( K_N(z, w) = \sum_{k=0}^{\infty} a_k z^k \) with respect to \( \frac{d\mu}{dm}(z) = \frac{1}{\pi} e^{-|z|^2} \) on \( \mathbb{C} \).

In the present article, we are interested in certain two-dimensional processes which are rotation-invariant such as the (polyanalytic) Ginibre ensembles whose kernels with respect to the Gaussian measure \( \frac{d\mu}{dm}(z) = \frac{1}{\pi} e^{-|z|^2} \) are given by

\[
K_\alpha(z, w) = L_\alpha(\{z - w\}) e^{z \bar{w}}, \quad z, w \in \mathbb{C},
\]

where \( L_\alpha \) denotes the (orthonormal) Laguerre polynomial of degree \( \alpha \in \mathbb{N}_0 \). These ensembles generalize the infinite Ginibre point process \( \alpha = 0 \) and they describe the thermodynamical limit of two-dimensional free fermions in a uniform magnetic field in the Landau level \( \alpha \), see [Zab06].

Another example that we consider for comparison reasons is the zeros point process of the hyperbolic Gaussian analytic function \( F(z) = \sum_{k=0}^{\infty} a_k z^k \) with \( (a_k)_{k \in \mathbb{N}_0} \) being a sequence of i.i.d. standard complex Gaussian random variables. In [PV05] it is shown that \( \{ z \in D : F(z) = 0 \} \) defines a simple point process which is determinantal with kernel \( K(z, w) = \frac{1}{\pi(1 - z \bar{w})} \) with respect to the Lebesgue measure \( m \) on the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \).

We focus on rotation-invariant determinantal processes because our analysis relies on the following property which is a consequence of [ST03b] Proposition 2.8 or [HKPV06, Proposition 9]. If \( \xi \) is a rotation-invariant determinantal process in \( \mathbb{C} \) associated with a locally trace-class operator \( K \), then for any \( r_1 > \cdots > r_\ell > 0 \), it holds

\[
\left( \xi(D_{r_\ell}) \right)_{\ell=1}^\ell \overset{d}{=} \left( \sum_{k=0}^{\infty} \mathbb{1}_{U_k \leq \lambda_k(r_\ell)} \right)_{\ell=1}^\ell, \tag{1.4}
\]

where \( \{U_k\}_{k=0}^{\infty} \) is a sequence of i.i.d. random variables which are uniform in \([0, 1]\) and \( \{\lambda_k(r)\}_{k=0}^{\infty} \) denotes the eigenvalues of the operator \( K|_{D_{r_\ell}} \). In particular, it follows from the general theory that \( \lambda_k(r) \in [0, 1] \) for all \( k \in \mathbb{N}_0 \) and \( r > 0 \), see [Sos02, Theorem 3] or [HKPV06, Theorem 22].

Mod-\( \phi \) convergence. The concept of mod-\( \phi \) convergence was first introduced in [KN10, JKN11, KN12] and recently further developed in the series of works [FMN16, FMN17, FMN19] with the goal of providing a unified framework to obtain precise information on a sequence satisfying a central limit theorem, such as Berry-Esseen estimates and precise moderate to large deviation estimates. By now, the theory of mod-\( \phi \) convergence has been successfully applied in different contexts such as asymptotic combinatorics [FMN18], in connection to the Ising model [MN15] and determinants of classical random matrix ensembles [BHR19]. Intuitively, the idea is to compare a sequence of random variables \( \{X_n\}_{n \in \mathbb{N}} \) with a sum of independent, identically distributed random variables at the level of cumulant generating functions to obtain precise tail estimates. As we are interested in counting statistics, we introduce the concept of mod-\( \phi \) convergence for discrete random variables following the approach of [FMN16].

Definition 1.6. The sequence of \( \mathbb{Z} \)-valued random variables \( \{X_n\}_{n \in \mathbb{N}} \) converges in the mod-\( \phi \) sense with parameters \( (t_n)_{n \in \mathbb{N}} \), normalizing distribution \( \nu \) on \( \mathbb{R} \) and limiting function \( \psi \) (analytic in \( S \)) if as \( t_n \to \infty \), it holds locally uniformly on \( S \),

\[
\frac{\mathbb{E}[e^{X_n t_n} \Lambda(z)]}{e^{t_n \Lambda(z)}} \xrightarrow{n \to \infty} \psi(z),
\]

where \( \Lambda(z) = \log \int_S e^{z x} d\nu(x) \) denotes the cumulant generating function of \( \nu \). We also assume that the following condition holds: there exists \( C, \varepsilon > 0 \) such that for all \( x \in [-\varepsilon, \varepsilon] \) and all \( \delta > 0 \),

\[
\max_{y \in (-\pi, \pi) \setminus (-\delta, \delta)} |\exp(\Lambda(x + iy) - \Lambda(x))| \leq 1 - C\delta^2. \tag{1.5}
\]
In contrast to the works mentioned above, the distributions $\nu$ observed in this article are discrete signed measures, which is a rather surprising fact. To fix the normalization, we assume that the total mass of $\nu$ is 1 in which case $\psi(0) = 1$. In Definition 1.6, we implicitly assume that $\int_{\mathbb{R}} e^{izx} \, d\nu(x)$ for all $z > 0$ where $|\nu|$ denotes the variation of $\nu$ and that $\log(\int_{\mathbb{R}} e^{izx} \, d\nu(x))$ is well-defined for the principle branch of log in the strip $\mathbb{S}$. Then $\Lambda$ is an analytic function in this strip and

$$\Lambda'(0) = \lim_{n \to +\infty} \frac{\mathbb{E}[X_n]}{t_n} \quad \text{and} \quad \Lambda''(0) = \lim_{n \to +\infty} \frac{\text{Var}(X_n)}{t_n}.$$ 

We always assume that $\Lambda''(0) > 0$, in which case a Taylor expansion shows that if a sequence $(X_n)_{n \in \mathbb{N}}$ converges in the above mod-phi sense, then it satisfies a standard central limit theorem:

$$\frac{X_n - t_n \Lambda'(0)}{\sqrt{t_n \Lambda''(0)}} \xrightarrow{d} N_{0,1}.$$ 

Moreover, we have the following precise large deviation estimates, [FMN16, Section 3].

**Theorem 1.7.** Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of centred, discrete, real-valued random variables converging in the mod-phi sense of Definition 1.6. Let $I : \mathbb{R} \to [0, +\infty]$ be the convex conjugate of $\Lambda$, then the following asymptotics hold locally uniformly for $y \in \mathbb{R}_+$ as $n \to \infty$,

$$\mathbb{P}(X_n \geq t_n y) = e^{-t_n I(y)} \sqrt{\frac{I''(y)}{2\pi t_n}} \frac{1}{1 - e^{-I'(y)}} (1 + O(t_n^{-1})). \quad (1.6)$$

$I$ is usually called the *rate function*. Recall that $I(y) = \sup_{x \in \mathbb{R}} \{xy - \Lambda(x)\}$ is convex on $\mathbb{R}$ and that $I \geq 0$ since $\Lambda(0) = 0$ because of our normalization. The RHS of (1.6) needs to be interpreted as 0 whenever $I(y) = +\infty$. The main differences with [FMN16, Theorem 3.2.2] is that we only require that $\Lambda(z)$ exists for $z \in \mathbb{S}$ and we assume the technical condition (1.5). Since we are restricted to $\mathbb{Z}$-valued random variables, the proof (which is based on Fourier’s inversion formula) goes through directly. In the formulation of [Theorem 1.7] we assume that $y > 0$, but one can also obtain the left tail by applying the results to the sequence $(-X_n)$ which has rate function $I(-y)$.

Note that compared to usual large deviation estimates, the asymptotics (1.5) hold for $\mathbb{P}(X_n \geq t)$, rather than its log, uniformly for all $t \in \mathbb{R}$ up to the order $t_n$. Thus, if we normalize the random variables $X_n$, we deduce the following results.

**Corollary 1.8.** Let $\bar{X}_n = \frac{X_n}{\sqrt{t_n \Lambda''(0)}}$. Under the assumptions of [Theorem 1.7] it holds

- **Berry-Esseen estimate:** There exists a constant $C > 0$ such that

  $$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(\bar{X}_n \geq x) - \mathbb{P}(N_{0,1} \geq x) \right| \leq \frac{C}{\sqrt{t_n}}.$$ 

- **Extended central limit theorem**:\footnote{In general, the condition $x_n = o(t_n^{1/6})$ for the extended CLT is optimal.} It holds for any sequence $x_n = o(t_n^{1/4})$,

  $$\mathbb{P}(\bar{X}_n \geq x_n) = \mathbb{P}(N_{0,1} \geq x_n) (1 + o(1)).$$

- **Precise moderate deviations:** If $x_n \geq 0$ is any sequence with $x_n = o(\sqrt{t_n})$,

  $$\mathbb{P}(\bar{X}_n \geq x_n) = \exp \left( -t_n I \left( \frac{x_n}{\sqrt{\Lambda''(0)t_n}} \right) \right) \frac{1}{\sqrt{2\pi x_n}} (1 + o(1)).$$
1.3. Main results

We take a general viewpoint and consider a rotation-invariant determinantal process \( \xi \) on \( \mathbb{C} \) with a Hermitian-symmetric correlation kernel \( K \) (with respect to a Radon measure \( \mu \)). Let us assume that \( K(z, w) = \sum_{k \in \mathbb{Z}} \rho_k(|z|^k) \rho_k(|w|^k) \bar{w}^k \) for \( z, w \in \mathbb{C} \) with \( \rho_k : \mathbb{R}_+ \to \mathbb{R} \) and \( \int_{\mathbb{C}} \rho_k^2(|z|)|z|^{2k} \mu(dz) \in \{0, 1\} \) for all \( k \in \mathbb{Z} \). This last conditions imply that the operator \( K \) with (integral) kernel \( K \) is a projection and that for any \( r > 0 \), the operator \( K|_{D_r} \), with kernel \( K \) acting on \( L^2(D_r, \mu) \), has eigenvalues:

\[
\lambda_k(r) = \int_{|z|<r} \rho_k^2(|z|)|z|^{2k} \mu(dz) \in [0, 1].
\]

We also assume that \( \text{Tr} K|_{D_r} = \sum_{k \in \mathbb{Z}} \lambda_k(r) < +\infty \) for all \( r > 0 \) so that the operator \( K \) is locally trace-class. Then, we obtain the following Lemma which is modelled after Theorem 1.1 in [Kos92], see also [PV05, Section 4.1]. Note that it immediately implies the identity (1.4).

**Lemma 1.9.** Denote by \( \{Z_k\}_{k \in \mathbb{Z}} \) the atoms of \( \xi \) and suppose that \( \sum_{|k| \leq N} \rho_k^2(|z|)|z|^{2k} \mu(dz) \) converges locally uniformly to the intensity \( \nu \) of \( \xi \) as \( N \to +\infty \). Then, \( |Z_k| \) are independent and we can label the point \( \xi \) so that for all \( k \in \mathbb{Z} \) and \( r > 0 \),

\[
P(|Z_k| \leq r) = \lambda_k(r).
\]

The kernel \( K_N(z, w) = \sum_{|k| \leq N} \rho_k(|z|^k) \rho_k(|w|^k) \bar{w}^k \) provides a finite rank approximation of the operator \( K \) and the corresponding (finite) point process \( \xi_N \) converges in law to \( \xi \) as \( N \to +\infty \) (this is guaranteed by the technical condition in Lemma 1.9). The proof relies on this fact and is given in Appendix A for completeness.

We use Lemma 1.9 to study the law of the counting statistic \( (\xi(D_r))_{r>0} \) as a stochastic process. For simplicity, we assume that \( \rho_k = 0 \) for all \( k < -\alpha \) with \( \alpha \in \mathbb{R}_+ \) and denote \( T(r) = \mathbb{E}[\xi(D_r)] \). Then, we define the random variables for \( k \geq -\alpha \),

\[
\Gamma_{k+\alpha+1} = T(|Z_k|).
\]

This transformation corresponds to unfolding\(^4\) the point process \( \xi \) so that the random variables \( \Gamma_k \) is located around \( k \). As for the applications to the invariant processes discussed in Section 1.4, this naturally takes in account the geometry at hand. In the following, we study the process

\[
\Xi_R = \sum_{k \in \mathbb{N}} I_{\Gamma_k \leq R} - R \overset{d}{=} \xi(D_r) - \mathbb{E}[\xi(D_r)], \quad \text{with } r = T^{-1}(R).
\]

Our goal is to show that the random variable \( \Xi_R \) converges as \( R \to +\infty \) in the mod-phi sense under some weak conditions on the sequence \( \{\Gamma_k\}_{k \in \mathbb{N}} \). As explained in Section 1.2, this allows us to deduce precise asymptotic results like Berry-Esseen estimates as well as precise moderate and large deviations for counting statistics of these determinantal processes (see Theorem 1.7 and Corollary 1.8). We work under the following assumptions.

**Assumptions A.** Suppose there exists an interval \( \tilde{I}_R = [a^-_R, a^+_R] \) with \( a^+_R, a^-_R \in \mathbb{N} \) such that, as \( R \to \infty \),

\[
\text{Var}(\Xi_R) = \sum_{k \in \mathbb{N}} P(\Gamma_k \leq R) P(\Gamma_k > R) + o(1)
\]

and that there exists a non-decreasing function \( \Sigma : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \Sigma_R \to +\infty \) and it holds uniformly for \( k \in \tilde{I}_R \),

\[
P(\Gamma_k \leq R) = \Phi\left( \frac{k - \theta R}{\Sigma_R} \right) + \frac{1}{\Sigma_R} \Psi\left( \frac{k - \theta R}{\Sigma_R} \right) + o(1),
\]

\(^4\)If \( \lambda_k = 0 \) for some \( k \in \mathbb{Z} \) the statement should be interpreted as “The corresponding point \( Z_k \) is non-existent.”

\(^4\)Notice that by definition, \( \mathbb{E}[\sum_k 1_{\Gamma_k \leq R}] = R \) for all \( R > 0 \), so that the densities of the \( \Gamma_k \) form a partition of unity. In particular (\( \Gamma_k \)) cannot be any sequence of absolutely continuous random variables.
with \( \vartheta \in \{0, 1\} \), \( \Phi(x) = \mathbb{P}(Z > x) \) for an absolutely continuous random variable \( Z \) and \( \Psi \in L^1(\mathbb{R}) \) is absolutely continuous. As for the interval \( I_R \), we further assume that \( I_R = \hat{I}_R \cup \hat{I}_R^{-} \rightarrow I = [-a^-, a^+] \) as \( R \rightarrow +\infty \) with \(-\infty \leq a^- < a^+ \leq +\infty \) and that

\[
\int_{I \triangle I_R} \Phi(x)(1 - \Phi(x)) \, dx = o(\Sigma_R^{-1}),
\]

where \( I \triangle I_R = (I \setminus I_R) \cup (I_R \setminus I) \) denotes the symmetric difference. In case \( \vartheta = 1 \) we further assume \( |I_R| = o(R) \). In case \( \vartheta = 0 \) we have \( a^- = 0 \) and \( \Sigma_R \sim R \).

The conditions (1.10) and (1.11) from Assumptions \( \text{A} \) mean that only the random variables \( \Gamma_k \) for \( k \in \hat{I}_R \) are relevant to the asymptotics of \( \Xi_R \) and that \( (k, R) \rightarrow \mathbb{P}(\Gamma_k \leq R) \) has a smooth profile in this window. In fact, when suitably normalized, \( \Gamma_k \) can be statistically approximated by the random variable \( Z \). As for the parameter \( \vartheta \), it distinguishes whether the \( \Gamma_k \) are concentrated (i.e. \( \vartheta = 1 \) as for the planar models of Section 1.1.2) and not \( \vartheta = 0 \) as for the hyperbolic models of Section 1.4.3). These conditions are natural to control the asymptotics of the cumulant general function of \( \Xi_R \) and thus to apply the theory of mod-phi convergence.

**Theorem 1.10.** Under Assumptions \( \text{A} \) the random variable \( \Xi_R \) converges as \( R \rightarrow +\infty \) in the mod-phi sense of Definition 1.6 with speed \( \Sigma_R \) with respect to a cumulant generating function

\[
\Lambda(z) = \int_I \kappa_{\Phi(z)}(z) \, dx \tag{1.12}
\]

and with limiting function

\[
\psi(z) = \exp \left( \int_I \Psi(x) \kappa_{\Phi(x)}(z) \, dx + \frac{\kappa_{\Phi(0)}(z) + \kappa_{\Phi(a^-)}(z)}{2} \right). \tag{1.13}
\]

The functions \( \Lambda \) and \( \psi \) are analytic in the strip \( \mathcal{S} = \{ z \in \mathbb{C} : |\Im z| < \pi \} \).

The proof of Theorem 1.10 is given in Section 2.1 and we immediately conclude that the precise asymptotics from Corollary 1.8 hold for the random variable \( \Xi_R / \sqrt{\Sigma_R \Lambda''(0)} \). Note that under the condition of the above theorem, \( \Lambda \) is convex on \( \mathbb{R} \) with \( \Lambda''(0) = \int_I \Phi(x)(1 - \Phi(x)) \, dx < +\infty \).

**Remark 4.** The condition \( \int_{I \triangle I_R} \Phi(x)(1 - \Phi(x)) \, dx = o(\Sigma_R^{-1}) \) on the tails of \( Z \) holds if \( \mathbb{E}[|Z|^p] < \infty \) and \( \Sigma_R^p = o(|I_R|) \) for some \( p > 0 \). Moreover, it holds as \( R \rightarrow +\infty \),

\[
\text{Var}(\Xi_R) = \Sigma_R \Lambda''(0) + O(1) = \Sigma_R \int_I \text{Var}(1_{Z \leq x}) \, dx + O(1).
\]

Our setup also allows to study \( (\xi(D_t))_{t \geq 0} \) as a stochastic process which is where we need to distinguish between the case \( \vartheta = 0, 1 \). For instance, in the planar case, we observe non-trivial correlations only in a microscopic regime.

**Theorem 1.11.** Recall (1.9) and denote \( \hat{\Xi}_R(t) = \frac{\Xi_R}{\sqrt{\Sigma_R \Lambda''(0)}} \). Under Assumptions \( \text{A} \) with \( \vartheta = 1 \), if \( \sqrt{\Sigma_R / \hat{\Xi}_R} \rightarrow g_t \) pointwise as \( R \rightarrow \infty \), then

\[
(\hat{\Xi}_R(t))_{t \in \mathbb{R}_+} \xrightarrow{fdd \ R \rightarrow \infty} (g_t N_t)_{t \in \mathbb{R}_+},
\]

where \( (N_t)_{t \in \mathbb{R}_+} \) is a Gaussian white noise, and

\[
(\hat{\Xi}_R \left( 1 + \frac{t \Sigma_R}{R^2} \right))_{t \in \mathbb{R}} \xrightarrow{R \rightarrow \infty} \mathcal{S}(\mathbb{R}) (G_t)_{t \in \mathbb{R}},
\]

where \( (G_t)_{t \in \mathbb{R}} \) is a centred Gaussian process with kernel

\[
\text{Cov}(G_s, G_t) = \frac{1}{\Lambda''(0)} \int_{|s+\Delta t| > t} \Phi(s - \Phi(x - t)) \, dx, \quad \text{for } s \leq t.
\]

Absolutely continuous means that \( \Psi \) has a weak derivative \( \Psi' \in L^1(\mathbb{R}) \).
In the hyperbolic case, as the area of the disk \( D_r \) is comparable to the size of its boundary, we obtain a microscopic functional limit theorem.

**Theorem 1.12.** Recall \((1.9)\) and define \( \Xi_R(t) = \frac{\Xi(t)}{\sqrt{N(0)R}} \). Under Assumptions A with \( \vartheta = 0 \), it holds as \( R \to +\infty \),

\[
(\Xi_R(t))_{t \in \mathbb{R}^+} \xrightarrow{R \to +\infty} (G_t)_{t \in \mathbb{R}^+},
\]

where \((G_t)_{t \in \mathbb{R}^+}\) is a centered Gaussian process with kernel

\[
\text{Cov}(G_s, G_t) = \frac{1}{\sqrt{N}(0)} \int_{x \in \mathbb{R}^d} \Phi \left( \frac{x}{\gamma} \right) (1 - \Phi \left( \frac{x}{\gamma} \right)) \, dx, \quad \text{for } 0 < s \leq t.
\]

**Remark 5.** Actually, for Theorems 1.11 and 1.12 it suffices to assume an expansion \((1.11)\) of order \( o(1) \) as a covariance computation up to the leading order suffices to prove these results.

### 1.4. Applications

In this section we present different models that fall in the framework of Section 1.3.

#### 1.4.1. The Ginibre ensemble

The Ginibre ensemble introduced in [Gin65] is the prototypical example of a non-Hermitian random matrix. It consists of a \( N \times N \) matrix filed with i.i.d. standard complex Gaussian entries and its eigenvalue process, denoted by \( \xi^{(N)} \), is determinantal process with correlation kernel

\[
K_N(z, w) = \sum_{k=0}^{N-1} \frac{(zw)^k}{k!},
\]

with respect the Gaussian measure \( d\mu(z) = \frac{1}{A}(z) e^{-|z|^2} \) on \( \mathbb{C} \). It is well-known that the point process \( \xi^{(N)} \) is invariant by rotation and that it distributes uniformly on the disk \( D_{\sqrt{N}} \) for large \( N \), this is known as the circular law, see e.g. [Cha15]. It is plain that \( K_N(z, w) \to e^{|z||w|} \) as \( N \to +\infty \), so that the (infinite) Ginibre ensemble as presented in Section 1.3 is the local scaling limit of the Ginibre eigenvalue process. In fact, \((1.14)\) is a finite rank approximation of the Ginibre kernel \( K_0 \) and as such, by applying Lemma 1.9 we find that according to \((1.15)\) with \( T(r) = r^2 \), \( \Gamma_k \) are gamma-distributed random variables with shape \( k \) and rate \( 1 \) for \( k \in \{1, \ldots, n\} \) and “non-existent” otherwise. In this section, we study the counting statistics \( \xi^{(N)}(D_{\sqrt{N}})_{\gamma \in [0,1]} \) as \( N \to +\infty \) and observe that there will be an edge effect for \( \gamma = 1 \). We let for \( \gamma > 0 \),

\[
\Xi_N(\gamma) = \sum_{k=1}^{N} \mathbb{1}_{\gamma_k \leq \gamma} - \gamma N \xrightarrow{d} \xi^{(N)}(D_{\sqrt{N}}) - \gamma N.
\]

**Theorem 1.13.** In the bulk, let \( I_{\infty} = \mathbb{R} \) for \( \gamma \in (0,1) \). At the edge, let \( I_{0} = (-\infty, t] \) for \( \gamma = 1 - t/\sqrt{N} \) for a fixed \( t \in \mathbb{R} \). In both cases, the sequence \( \{\Xi_N(\gamma)\}_{N \in \mathbb{N}} \) converges in the mod-phi sense at speed \( \sqrt{N} \) with cumulative generating function of the form \((1.12)\) where \( I = I_\gamma \), \( \Phi = \Phi_0 \) being the tail distribution function of \( N_{0,1} \) and with limiting function of the form \((1.13)\) with \( I = I_\gamma \) and \( \Psi = \frac{1}{2} \Phi_0'' \).

---

\(^6\text{Note that } T(r) = \mathbb{E}[\xi^{(N)}(D_{r})] + o(1) \text{ uniformly for } r < \sqrt{N} \text{ with } \gamma < 1 \text{ as } N \to +\infty \text{. This guarantees that we can still apply the results from Section 1.3 for } \gamma \in [0,1], \text{ i.e. up to the edge of the circular law. Namely, the Assumptions A hold with } R = \gamma N \text{ choosing the parameters } a_R^+ = R - R^{1+\varepsilon}, a_R^- = \min\{R + R^{1+\varepsilon}, N\} \text{ for a small } \varepsilon > 0 \text{ and } \Sigma_R = \sqrt{\Pi}. \text{ Then, we have } \Phi = \Phi_0, \text{ i.e. } Z = N_{0,1}, \Psi = \frac{1}{2} \Phi_0'', I = R \text{ for } \gamma \in (0,1) \text{ and } I = (-\infty, t) \text{ for } \gamma = 1 - t/\sqrt{N} \text{ with } t \in \mathbb{R} \text{. These conditions are verified in the exact same way as in the proof of Theorem 1.16 below about the Ginibre-type ensembles.}
The proof of Theorem 1.13 is given in Section 3.1 and it relies on the fact that \( \Gamma \) are infinitely divisible together with an Edgeworth expansion to obtain the approximation (1.11). Note that we can also obtain the error function \( \Phi_0 \) and the correction \( \Psi_0(x) = \frac{1}{4}(1-x^2)\Phi_0(x) \) by applying the steepest descent method to compute the asymptotics of \( P(\Gamma_k \leq R) \) with \( k = R + Ru \) for large \( R \) and suitable \( u \in \mathbb{R} \). By Corollary 1.8 this mod-phi convergence implies Berry-Esseen estimates and precise moderate and large deviations for the normalized counting statistics \( (\frac{t}{\sqrt{N}})\xi_N(\gamma) \) including at the edge of the circular law.

We also obtain functional central limit theorems as in Theorem 1.11.

**Theorem 1.14.** Recall (1.15). The following convergence of the finite dimensional distributions holds:

\[
\left( \left( \frac{1}{N} \right)^{\frac{1}{2}} \xi_N(\gamma) \right)_{\gamma \in \mathbb{R}_+^d} \xrightarrow{fdd} (g_{\gamma \sqrt{N}}, \gamma)_{\gamma \in \mathbb{R}_+^d},
\]

where \( (N_{\gamma})_{\gamma \in \mathbb{R}_+^d} \) is a Gaussian white noise and \( g_{\gamma \sqrt{N}} \) if \( \gamma < 1 \), \( g_{1} = \frac{1}{\sqrt{2}} \) and \( g_{\gamma} = 0 \) if \( \gamma > 1 \).

**Theorem 1.15.** For any \( \gamma \in (0, 1] \), the following weak convergence holds true with respect to the Skorokhod topology:

\[
\left( \left( \frac{1}{N} \right)^{\frac{1}{2}} \xi_N \left( \gamma + \frac{t}{\sqrt{N}} \right) \right)_{\gamma \in \mathbb{R}_+^d} \xrightarrow{fdd} (g_{\gamma \sqrt{N}}, \gamma)_{\gamma \in \mathbb{R}_+^d},
\]

where \( (G_{\gamma \sqrt{N}})_{\gamma \in \mathbb{R}_+^d} \) is a centred Gaussian process with covariance kernel

\[
\text{Cov}(G_s, G_t) = \sqrt{\pi} \int_I \mathbb{P}(N_{0,1} > x - s)\mathbb{P}(N_{0,1} \leq x - t) \, dx
\]

for \( s \leq t \), where \( I = \mathbb{R} \) if \( \gamma \in (0, 1) \) (i.e. in the bulk) and \( I = (-\infty, 0] \) if \( \gamma = 1 \) (i.e. at the edge).

Note that as expected, these results are consistent with those presented in Section 1.1 for the (infinite) Ginibre ensemble, except for the edge effects. In particular, by Remark 2 the Gaussian process \( (G_{\gamma \sqrt{N}})_{\gamma \in \mathbb{R}_+^d} \) observed in the bulk is the same as in Theorem 1.5.

### 1.4.2. Planar models: Polyanalytic Ginibre or Ginibre-type ensembles

These point processes are generalizations of the (infinite) Ginibre ensemble which arise from the physics of two-dimensional quantum (spinless) electrons subject to a strong (perpendicular) magnetic field, [Eza13, Zab06]. This physical problem is for instance relevant to the description of the fractional quantum Hall effect, see [Ron19] and reference therein. As explained in [Sh11], this system is described by the Landau Hamiltonian whose spectrum consists of only eigenvalues with infinite multiplicity; the eigenspaces \( \mathcal{K}_{\alpha} \) for \( \alpha \in \mathbb{N}_0 \) are called Landau levels and they can be identified with \( \alpha \)-analytic Bargmann-Fock spaces [7]. In particular, the ground state \( \alpha = 0 \) of the Landau Hamiltonian corresponds to the Hilbert space \( \mathcal{K}_0 \) of all entire functions in \( L^2(\mu) \) whose reproducing kernel is the Ginibre kernel \( K_0(z, w) = e^{z \overline{w}} \) for \( z, w \in \mathbb{C} \). In general, the Hilbert space \( \mathcal{K}_\alpha \) has a reproducing kernel

\[
K_\alpha(z, w) = \sum_{k=-\alpha}^{\infty} \varphi_{\alpha, k}(z)\overline{\varphi_{\alpha, k}(w)} = L_\alpha(|z-w|^2)e^{z\overline{w}}, \quad \alpha \in \mathbb{N}_0,
\]

where \( L_\alpha \) denotes the orthonormal Laguerre polynomial of degree \( \alpha \). The Ginibre-type ensembles, denoted by \( \xi_\alpha \), refers to the determinantal point process on \( \mathbb{C} \) with correlation kernel \( K_\alpha \). They describe the thermodynamic limit of an infinite system of electrons which are confined in the \( \alpha \)-th Landau level. Since \( L_0(0) = 1 \) for all \( \alpha \in \mathbb{N}_0 \), these ensembles are homogeneous with intensity \( 1/\pi \). Moreover, it is easy to check from \( K_\alpha \) that they are also translation and rotation invariance.

---

7 That is, we have the spectral decomposition \( L^2(\mu) \cong \bigoplus_{\alpha=0}^{\infty} \mathcal{K}_\alpha \) with \( \frac{d\mu}{d\mu_{\alpha}}(z) = \frac{1}{\pi}e^{-|z|^2} \) on \( \mathbb{C} \), \( \mathcal{K}_0 = \text{span}\{z^j : j \in \mathbb{N}_0\} \) in \( L^2(\mu) \) and \( \mathcal{K}_{\alpha} = \{e^{z\overline{w}}\partial^\alpha_\gamma(\varphi_{\alpha, k})^2 : \varphi \in \mathcal{K}_{\alpha} \} \) for \( \alpha \geq 1 \). Hence, it holds \( \partial^\alpha_\gamma(\varphi) = 0 \) for any \( \varphi \in \mathcal{K}_{\alpha} \) which motivates the name Polyanalytic Ginibre ensemble from [HH13].
The modes in (1.16) are given by \( \varphi_{\alpha,k}(z) = L_{\alpha}^{(k)}(|z|^2)z^k \), where \( L_{\alpha}^{(k)} \) are the orthonormal generalized Laguerre polynomials\(^8\) of degree \( \alpha \in \mathbb{N}_0 \). This expansion for the correlation kernels are not entirely obvious and can be found in [HH13 Section 2] or [Sh15 Section 2].

Following the convention from Section 1.3 we let \( R = T(r) = r^2 \) and observe that the square-modulus of the points have law given for \( k \in \mathbb{N} \)

\[
P \left( \Gamma_k^{(\alpha)} \leq R \right) = \int_0^R L_{\alpha}^{(k-\alpha-1)}(x^2)x^{k-\alpha-1}e^{-x} \, dx
\]

In the case of the Ginibre ensemble, \( \alpha = 0 \), note that \( \Gamma_k^{(0)} \) are gamma-distributed random variables as in Theorem 1.1. In Lemma 3.7, we show that these random variables have the following asymptotic property: if we let \( \tilde{\Gamma}_k^{(\alpha)} = \frac{\Gamma_k^{(\alpha)} - k}{\sqrt{k}} \) so that \( E[\tilde{\Gamma}_k^{(\alpha)}] = 0 \) and it holds as \( k \to +\infty \),

\[
E \left[ e^{it\tilde{\Gamma}_k^{(\alpha)}} \right] = E \left[ e^{i\alpha Z_k} \right] \left( 1 - \frac{e^{4x^2}}{3\sqrt{k}} \right) + O \left( \frac{1}{k(1 + x^2)} \right), \quad (1.17)
\]

uniformly for all \( x \in \mathbb{R} \). Let \( h_\alpha(x) = H_\alpha(x)e^{-x^2/4}, \) \( x \in \mathbb{R} \) and \( \alpha \in \mathbb{N}_0 \) be the Hermite functions – that is \( H_\alpha \) are Hermite polynomials of degree \( \alpha \) which are orthonormal with respect to the weight \( e^{-x^2/2} \) on \( \mathbb{R} \). Then, the random variables \( Z_\alpha \) which appear on the RHS of (1.17) have probability density function \( h_\alpha^2 \):

\[
\Phi_\alpha(x) = P(Z_\alpha > x) = \int_{x}^{\infty} h_\alpha^2(t) \, dt. \quad (1.18)
\]

Note that \( \Phi_0(x) = P(N_{0,1} > x) \) corresponds to the standard error function. For \( \alpha \geq 1 \), unlike gamma random variables, \( \Gamma_k^{(\alpha)} \) are not infinitely divisible, since otherwise the random variable \( Z_\alpha \) on the RHS of (1.17) would be a Gaussian. For \( \alpha \geq 0 \), \( Z_\alpha \) can be interpreted as the position of a quantum particle confined by a harmonic trap in the \( \alpha^{th} \) excited state. This connection between the point processes associated with different Landau levels and the harmonic oscillator is rather surprising to us.

Let for \( \alpha \in \mathbb{N}_0, R > 0 \)

\[
\Xi_R^{(\alpha)} = \sum_{k=\alpha+1}^{\infty} \mathbb{I}_{\Gamma_k^{(\alpha)} \leq R} - R \stackrel{d}{=} \xi_\alpha(D_r) - E[\xi_\alpha(D_r)] \quad \text{with } r = \sqrt{R}.
\]

From Theorem 1.10 and the asymptotics (1.17), we can infer that the random variables \( \Xi_R^{(\alpha)} \) converge as \( R \to +\infty \) in the mod-phi sense with speed \( \sqrt{R} \) and cumulant generating functions of the form (1.12) which are associated to the tail functions \( \Phi_\alpha \) given by (1.18). Namely, we prove the following result in Section 3.2

**Theorem 1.16.** For any \( \alpha \in \mathbb{N}_0 \), the random variable \( \Xi_R^{(\alpha)} \) converges in the mod-phi sense at speed \( \sqrt{R} \) with the cumulant generating function

\[
\Lambda_\alpha(z) = \int_{\mathbb{R}} \log \left( 1 + \Phi_\alpha(x)(e^z - 1) \right) - \Phi_\alpha(x)z \, dx,
\]

and limiting function \( \psi(z) = \exp \left( \int_{\mathbb{R}} \left( \frac{e^z - 1}{1 + \Phi_\alpha(x)(e^z - 1)} - z \right) \Phi_\alpha(x) \, dx \right) \) where \( \Phi_\alpha = \frac{1}{\beta} \Phi_\alpha'' \).

By Corollary 1.8 this implies an (extended) central limit theorem together with a Berry-Esseen bound and precise moderate deviations for the counting statistics \( \xi_\alpha(D_r) \) (after appropriate normalization). In view of giving functional central limit theorems for counting statistics, let us define

\(^8\) It is straightforward to check that \( \partial_x^2(z^\beta e^{-x^2/2}) = L_\beta^{(\alpha)}(|z|^2)z^{\beta - \alpha}e^{-x^2/2} \) for any \( j, \alpha \in \mathbb{N}_0 \) where \( \tilde{L}_\beta^{(\alpha)}(x) = x^{-\beta}e^{-x^2/2} \sum_{\kappa=0}^{\alpha} \binom{\beta}{\alpha} \binom{\alpha}{\kappa} x^{\kappa}e^{-x^2} \) is a polynomial of degree \( \alpha \) for any \( \beta \in \mathbb{R} \). Hence, with \( L_{\alpha}^{(\beta)}(x) = \Gamma(\alpha + \beta + 1)^{-1/2}L_\beta^{(\alpha)}(x) \) it holds \( \int_{\mathbb{R}} L_{\alpha}^{(\beta)}(x)^2x^\beta e^{-x} \, dx = 1 \) for \( \beta \geq -\alpha \) and we obtain that for any \( \alpha \in \mathbb{N}_0, \varphi_{\alpha,k}^{(n)}(z) \) is an orthonormal basis of the Hilbert space \( \mathcal{H}_\alpha \). Moreover, it turns out that for \( \beta = 0 \), \( L_{\alpha}^{(0)} = L_\alpha \) for all \( \alpha \in \mathbb{N}_0 \).
the processes for $R > 0$ and $t > 0$,

$$
\tilde{X}^{(a)}_R(t) = \frac{\widetilde{z}^{(a)}_R(t)}{\sqrt{A''_0(0)}\sqrt{R}}.
$$

Then by applying Theorem 1.1 we obtain the following results which describe the global, respectively microscopic, asymptotic fluctuations of the processes $\tilde{X}^{(a)}_R$. These results generalize Theorem 1.4 and Theorem 1.5 given in the introduction in the context of the Ginibre ensemble ($\alpha = 0$) to Ginibre-type ensembles associated with the higher $\alpha$ Landau levels. In particular, the large deviations rate function in (1.6) as well as the correlation structure of the underlying Gaussian process (at the microscopic level) depend non-trivially on the index $\alpha$.

**Theorem 1.17.** For any fixed $\alpha \in \mathbb{N}_0$, the following convergences hold

$$
\big(\tilde{X}^{(a)}_R(t)\big)_{t \in \mathbb{R}_+} \xrightarrow{\text{fdd}} \big(\hat{t}^\alpha N_t\big)_{t \in \mathbb{R}_+}
$$

where $(N_t)_{t \in \mathbb{R}_+}$ is a Gaussian white noise and

$$
\bigg(\tilde{X}^{(a)}_R\left(1 + \frac{t}{R}\right)\bigg)_{t \in \mathbb{R}_+} \xrightarrow{\text{fdd}} \big(\hat{G}^{(\alpha)}_t\big)_{t \in \mathbb{R}_+},
$$

where, for $\Phi_\alpha$ as in (1.18), $(\hat{G}^{(\alpha)}_t)_{t \in \mathbb{R}}$ is a centred Gaussian process with kernel

$$
\text{Cov}(G^{(\alpha)}_{s,t}) = \frac{1}{\Lambda''_0(0)} \int_{\mathbb{R}} \Phi_\alpha(x-s)(1-\Phi_\alpha(x-t)) \, dx, \quad \text{for } s \leq t.
$$

**Remark 6.** Counting statistics for fermions in the lowest Landau level ($\alpha = 0$) of a magnetic Laplacian have been recently considered in [CE19] in a more general geometric setting. These ensembles correspond to determinantal processes on a Kähler manifold $M$ and the law of the statistic $N_A$ for the number of points in a subset $A \subset M$ can be described in terms of the eigenvalues of a Toeplitz operator $T_A$. By semiclassical methods, if $A$ has a smooth boundary, the authors obtain two-term Weyl asymptotics for the operator $T_A$ and discuss applications to computing entanglement entropy. They show that the corrections are universal and described in terms of the error function $\Phi_\alpha$ from (1.18). This also allows to obtain asymptotic expansions of the cumulants $\chi^{(q)}(N_A)$ for a general set $A$, see [CE19] Theorem 1.6. In particular, there results also apply to the Ginibre ensemble in which case, with our conventions, one has for $A \subset \mathbb{D}$, for any $q \in \mathbb{N}$, as $N \to +\infty$,

$$
\begin{align*}
\mathbb{E}[\xi^{(N)}(\sqrt{N}A)] &= N \frac{m(A)}{\pi} + O(1), \\
\chi^{(2q)}(\xi^{(N)}(\sqrt{N}A)) &= \sqrt{N} \text{Vol}(\partial A) \int_{-\infty}^{\infty} \chi^{(2q)}\left(1_{U \leq \Phi_0(t)}\right) \, dt + O(N^{-\frac{1}{2}}), \\
\chi^{(2q+1)}(\xi^{(N)}(\sqrt{N}A)) &= O(N^{-1}),
\end{align*}
$$

(1.19)

where $U$ is a random variable uniform in $[0,1]$. If $A$ is a disk, these results are consistent with the mod-phi convergence from [Theorem 1.13]. However, since the eigenvalues of the operator are explicit in the case of a disk, our results are more precise and we also obtain for any $\gamma \in (0,1)$,

$$
\chi^{(2q+1)}(\xi(D_{\gamma}N)) = -\frac{2}{3} \int_{-\infty}^{\infty} t \chi^{(2q+1)}(1_{\xi^{(N)}(\sqrt{N}A)}(t)) \, dt + O(N^{-\frac{1}{2}}).
$$

In fact, our method gives in principle all order asymptotic expansions of the cumulants in terms of $\Phi_0$, see the beginning of Section 3.1. Let us also point out that using the asymptotics (1.19) to deduce mod-phi convergence for counting statistics in a general set $A \subset \mathbb{D}$ seems out of reach as the estimates from [CE19] lack control on the growth of the cumulants.

\footnote{Except that it seems there is a typo in the asymptotics of [CE19] Theorem 1.6 for odd cumulants and we have shown instead that $\chi^{(2q+1)}(\xi(\sqrt{N}A))$ is of order 1 as $N \to +\infty$ for any $q \in \mathbb{N}$.}
The problem of obtaining asymptotics of the entanglement entropy for counting statistics have been considered for different ensembles of free fermions. Using our asymptotics (see e.g. [1.4.1]), we can compute the entanglement entropy, denoted $S_a(D_r)$, between a disk $D_r$ and its complement for (integer) Laughlin states for all Landau levels. We obtain the following precise result which is consistent with the well-known area law for fermionic ensembles. This is to be compared with [CE19, Theorem 1.8] in the context of Remark 6 and we note that the entanglement entropy depends non-trivially on the index $\alpha$ of the Landau levels.

**Corollary 1.18.** For any $\alpha \in \mathbb{N}_0$, it holds as $r \to \infty$,

$$S_a(D_r) = \text{Tr}[f(K_\alpha|_{D_r})] = \sqrt{r} \int_{\mathbb{R}} f(\Phi_\alpha(x)) \, dx + o(1),$$

where $f(x) = -x \log(x) - (1-x) \log(1-x)$ for $x \in [0,1]$.

### 1.4.3. Hyperbolic models and Zero set of the hyperbolic Gaussian analytic function

The behaviour of the Ginibre ensemble is often compared to the one of the zero set of the hyperbolic Gaussian analytic function $F(z) = \sum_{k=0}^{\infty} a_k z^k$ with $(a_k)_{n\in\mathbb{N}_0}$ being a sequence of i.i.d. standard complex Gaussian random variables. $K_{\rho}$ corresponds to the Bergman kernel for the unit disk $\mathbb{D}$ and the intensity function of $\zeta_1$ is the invariant measure $d\nu(z) = \frac{\rho dm(z)}{\pi(1-|z|^2)^{\rho+1}}$ of the Poincaré disk $\mathbb{D}$. In particular, the point process $\zeta_1$ is invariant (in law) under all linear fractional transformations of $\mathbb{D}$ which preserve $\nu$, i.e. the group $SU(1,1)$.

It turns out that there is a 1-parameter family of determinantal process on $\mathbb{D}$ which generalizes the zero set of the hyperbolic GAF that we call hyperbolic models after [Kri06b, Chapter 3]. These point processes are also invariant under the action of SU(1,1) and they are associated with Bargmann-Fock spaces of analytic functions. Namely for any $\rho > 0$, the reproducing kernel of the space $\mathcal{H}_0 \cap L^2(\mathbb{D}, \frac{\rho dm(z)}{\pi(1-|z|^2)^{\rho+1}})$ is

$$K_{\rho}(z, w) = \frac{1}{(1-z\overline{w})^{\rho+1}} = \sum_{k \in \mathbb{N}_0} \binom{k + \rho}{\rho} z^k \overline{w}^k$$

and we let $\zeta_{\rho}$ be the determinantal process on $\mathbb{D}$ with kernel $K_{\rho}$ with respect to $\frac{\rho dm(z)}{\pi(1-|z|^2)^{\rho+1}}$. Moreover, by [Kri06b, Theorem 4], when $\rho \in \mathbb{N}$, $\zeta_{\rho}\rho$ corresponds to the zero set of matrix-valued GAF with i.i.d. coefficients from the $\rho \times \rho$ Ginibre ensembles. These processes have been studied in the thesis [Kri06b] and the central limit theorem for smooth linear statistics has been established in [RV07a]. Since they are rotation-invariant, we can study the distributions of their counting statistics in (hyperbolic) disks using the formalism from Section 1.3. Since the hyperbolic models have intensity $\rho \, d\nu(z) = \frac{\rho dm(z)}{\pi(1-|z|^2)^{\rho+1}}$, we let $T(r) = \frac{\rho^2}{2\pi r^2}$ for $r \in [0,1)$ and after unfolding according to (1.8), the modulus of the points have laws given by for $\rho > 0$ and $k \in \mathbb{N}$

$$\mathbb{P}(\Gamma_k^{(\rho)} \leq R) = \rho \left(\binom{k + \rho - 1}{k - 1}\right) \int_0^R x^{k-1}(1+x)^{-k-\rho} \, dx,$$

that is $\Gamma_k^{(\rho)}$ are $\rho \text{Beta}(k, \rho)$-distributed\(^{10}\). Then, according to (1.9), let for $\rho > 0$

$$Z_k^{(\rho)} = \sum_{k \in \mathbb{N}} 1_{\Gamma_k^{(\rho)} \leq R} - R \overset{d}{=} \zeta_{\rho}(D_r) - \mathbb{E}[(\zeta_{\rho}(D_r)], \quad \text{with } r = \sqrt{\frac{R}{\rho+1}}.$$

In Section 3.3 we verify that the random variables $(\Gamma_k^{(\rho)})_{k \in \mathbb{N}}$ satisfy the Assumptions A with $\theta = 0$, $\Sigma_R = R$ and $\Phi_\rho(x) = \mathbb{P}((\text{Gamma}(\rho, \rho) \leq x)$. Hence, by applying Theorem 1.10 we obtain the following mod-phi convergence result and therefore by Theorem 1.1 precise moderate and large deviations.

---

\(^{10}\)Since this is a model of 2-dimensional hyperbolic geometry, this motivates the name hyperbolic Gaussian analytic function.

\(^{11}\)Recall that a random variable is $\rho \, \text{Beta}(k, \rho)$-distributed if it is the image by $T$ of a $\text{Beta}(k, \rho)$ random variable with parameters $k, \rho > 0$. 

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Theorem 1.19. For any $\rho \geq 1$, the random variables $\Xi_R^{(\rho)}$ converges in the mod-phi sense at speed $R$ with cumulant generating function of the form \( (1.12) \) with $I = \mathbb{R}_+$, $\Phi = \mathbb{P}(Z_\rho > x)$ for $x > 0$ where $Z_\rho$ is a gamma random variable with shape $\rho$ and rate $\rho$. The limiting function is of the form \( (1.13) \) with $\Psi = \Psi_\rho$.

Note that in \([FMN16]\) the authors show that $\Xi_R^{(1)}/R^{1/3}$ converges in the mod-Gaussian sense with speed $R^{1/3}$ and limiting function $\varphi(z) = \exp(\frac{z^2}{\Phi})$ (with our slightly different scaling). It turns out that this mod-Gaussian convergence is merely an artefact of our Theorem 1.19.

By applying Theorem 1.12 we also obtain a functional central limit theorem for counting statistics in hyperbolic disks. We define the processes $\Xi_R^{(\rho)}(t) = \frac{\Xi_R^{(\rho)}}{\sqrt{\Lambda^R(0)R}}$ for $t > 0$, $R > 0$.

Theorem 1.20. For $\rho \geq 1$, it holds
\[
\Xi_R^{(\rho)}(t) \xrightarrow{t \to \infty} \mathbb{E}^{(G)}(t) \xrightarrow{t \to \infty} G_t^{(\rho)},
\]
where $G_t^{(\rho)}$ is a centred Gaussian process with covariance kernel given by
\[
\text{Cov}(G_s^{(\rho)},G_t^{(\rho)}) = \rho^\alpha I^{(\rho)}(s)\mathbb{P}\left(\text{Gamma}(\rho,\rho) > \frac{X}{s}\right)\mathbb{P}\left(\text{Gamma}(\rho,\rho) \leq \frac{x}{t}\right)dx,
\]
for $0 < s \leq t$.

1.5. Summary

For the convenience of the reader let us summarize our main results for the different models considered within this article: the Ginibre ensemble together with its generalizations to higher Landau levels and the invariant hyperbolic ensembles which are a generalization of the zero set of the hyperbolic Gaussian analytic function.

| Kernel |
|---|
| Ginibre ensemble |
| $\frac{1}{\pi} e^{-\frac{1}{2}|z|^2}$ |
| $\frac{1}{\pi}$ |
| Gamma($k,1$) |
| $\sqrt{R}$ |
| $\mathbb{N}(0,1)$ |
| Ginibre $\alpha$-type |
| $\frac{1}{\pi} e^{-\frac{|1-|z|^2|}{2}}$ |
| $\frac{1}{\pi}$ |
| Gamma$_\alpha$($k,1$) |
| $\sqrt{R}$ |
| $\mathbb{N}_\alpha(0,1)$ |
| Hyperbolic GAF |
| $\frac{1}{\pi(1-|z|^2)} e^{-\frac{|1-|z|^2|}{2}}$ |
| $\frac{1}{\pi(1-|z|^2)}$ |
| Beta($k,1$) |
| $R$ |
| $\mathbb{E}(1)$ |
| Hyperbolic model |
| $\frac{1}{\pi(1-|z|^2)} e^{-\frac{|1-|z|^2|}{2}}$ |
| $\frac{1}{\pi(1-|z|^2)}$ |
| Beta($k,\rho$) |
| $R$ |
| Gamma($\rho,\rho$) |

The first two columns depict the kernel of the respective model with respect to the Lebesgue reference measure as well as the first intensity measure. Notice that the first intensity measure in all cases is (proportional to) the invariant measure on the respective space, i.e. on the complex plane and the hyperbolic disk.

For all these models we show that the point count statistic of growing disks converges in the mod-phi sense (see Theorem 1.10). This convergence is obtained from the fact that the modulus of the points of the point process, after the natural transformation $R = T(r) = \nu(D_r)$, are independent random variables $\Gamma_k$ with explicit distribution as given in the third column of the table. In particular, the general theory of mod-phi convergence allows us to conclude a central limit theorem at scaling $\sqrt{\Sigma_R}$ together with a Berry-Esseen bound (Corollary 1.8) as well as precise large deviations (Theorem 1.7).

\[
\mathbb{P}\left(\xi(D_{T^{-1}(R)}) - R \geq y \Sigma_R \right) = e^{-\Sigma_R^\lambda(y)} \frac{\psi'(\bar{y})}{2\pi \Sigma_R} \frac{\psi'(\bar{y})}{1 - e^{-\psi(y)}} \left(1 + O\left(\Sigma_R^{-1}\right)\right)
\]
for $y > 0$ with rate function $I$ being the convex conjugate of
\[
\Lambda(z) = \int \log(1 + \Phi(x)(e^z - 1)) - \Phi(x)z \, dx.
\]
where $\Phi$ is the cumulative distribution function of the distribution presented in the last column of the table, the integral is over the support of the respective distribution (or a smaller set in case of finite versions of the models) and $\psi$ is given in (1.13). Since the above asymptotic is even locally uniform in $y$, we also conclude precise moderate deviations (Corollary 1.8). In addition, our analysis also allows us to obtain multidimensional limit theorems for the point count statistic (Theorems 1.11 and 1.12). It turns out that at the microscopic scaling $T^{-1}(R + t\Sigma_R)$ the process converges in the Skorokhod topology to a centred Gaussian process with covariance function explicitly given via an integral over $\Phi$.

2. Proof of main results

2.1. Proof of mod-phi convergence (Theorem 1.10)

Let us quickly describe the main idea for proving mod-phi convergence. First, we show that the main contribution to the cumulant generating function arises from the concentrated region $\hat{I}_R$. Then, we use the expansion for $\mathbb{P}(\Gamma_k \leq R)$ to rewrite the parameters in terms of its asymptotic expansion and finally apply a Riemann sum argument in form of the Euler-Maclaurin formula to approximate the remaining sum with the integral given in $\Lambda$. Throughout this section we will always work under the setting of Theorem 1.10.

Before starting to prove Theorem 1.10 let us recall some notation. We denote the cumulant generating function of a centred Bernoulli distribution $\kappa_p$ and its first and second derivative in $p$ by $\bar{\kappa}_p$ and $\ddot{\kappa}_p$, respectively. The horizontal strip will be denoted by $S = \{z \in \mathbb{C} : |3z| < \pi\}$. Moreover, we will use the notation $\lambda_k = \mathbb{P}(\Gamma_k \leq R)$ within this section.

First, let us establish some general bounds on $\kappa_p$.

**Lemma 2.1.** The functions $(p, z) \mapsto \kappa_p(z)$, $(p, z) \mapsto \bar{\kappa}_p(z)$ and $(p, z) \mapsto \ddot{\kappa}_p(z)$ are well-defined for all $z \in S$ and $p \in [0, 1]$. Moreover, they are bounded uniformly in $p \in [0, 1]$ and locally uniformly in $z \in S$, i.e. for all compact subsets $K \subseteq S$ there exists a constant $C > 0$ such that

$$\sup_{z \in K} \sup_{p \in [0, 1]} |\kappa_p(z)| \leq C,$$

$$\sup_{z \in K} \sup_{p \in [0, 1]} |\bar{\kappa}_p(z)| \leq C$$

and

$$\sup_{z \in K} \sup_{p \in [0, 1]} |\ddot{\kappa}_p(z)| \leq C.$$

In particular, $\kappa_p$ satisfies the Lipschitz bound $\sup_{z \in K} |\kappa_p(z)| \leq 2Cp(1 - p)$.

**Proof.** The claims follow immediately by calculating the respective derivative. \qed

To prove mod-phi convergence, we want to investigate

$$\log \mathbb{E}[e^{z\Xi_k}] = \sum_{k \in \mathbb{N}} \kappa_{\lambda_k}(z).$$

We now show that the sum can be truncated, i.e. that the main contribution comes indeed from $k \in I_R$.

**Lemma 2.2.** Locally uniformly on $S$, as $R \to \infty$, it holds

$$\log \mathbb{E}[e^{z\Xi_k}] = \sum_{k \in I_R} \kappa_{\lambda_k}(z) + o(1).$$

**Proof.** Indeed, by Lemma 2.1 and (1.10), as $R \to \infty$, locally uniformly on $S$ we have

$$\left| \sum_{k \in I_R} \log \left( 1 + \lambda_k(e^z - 1) \right) - \lambda_kz \right| \leq 2C \sum_{k \in I_R} \lambda_k(1 - \lambda_k) = o(1).$$

\qed

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We now apply the expansion for $\lambda_k(r)$ to obtain the following result.

**Lemma 2.3.** Locally uniformly on $S$, as $R \to \infty$, it holds

$$\sum_{k \in I_R} \kappa_{\lambda_k}(z) = \sum_{k \in I_R} \kappa_\Phi(\frac{\kappa}{\kappa}) (z) + \frac{1}{\Sigma R} \sum_{k \in I_R} \Psi \left( \frac{k - \Phi R}{\Sigma R} \right) \kappa_\Phi(\frac{\kappa}{\kappa}) (z) + o(1).$$

**Proof.** For notational reasons, denote $x = \frac{k - \Phi R}{\Sigma R}$. The statement now follows from a Taylor expansion of $p \to \kappa_p(z)$ together with the expansion (1.11) of $\lambda_k$. Indeed,

$$\sum_{k \in I_R} \kappa_{\lambda_k}(z) = \sum_{k \in I_R} \kappa_\Phi(x) (z) + \sum_{k \in I_R} (\lambda_k - \Phi(x)) \kappa_\Phi(x) (z)$$

$$+ \sum_{k \in I_R} \frac{(\lambda_k - \Phi(x))^2}{2} \kappa_{p_0}(z)$$

for some $p_0 \in [\lambda_k, \Phi(x)]$. By Lemma 2.1, $\kappa_p(z)$ is bounded locally uniformly in $z \in S$ and uniformly in $p \in [0,1]$ so that the uniform expansion of $\lambda_k$ implies

$$\sum_{k \in I_R} (\lambda_k - \Phi(x)) \kappa_\Phi(x) (z) = \frac{1}{\Sigma R} \sum_{k \in I_R} \Psi(x) \kappa_\Phi(x) (z) + o(1)$$

locally uniformly in $z \in S$. For the other term notice that any integrable, absolutely continuous function is bounded. Since $\Psi$ in integrable and absolutely continuous it is also bounded. This together with the (locally) uniform bound for $\kappa$ yields, as $R \to \infty$, that

$$\sum_{k \in I_R} \frac{(\lambda_k - \Phi(x))^2}{2} \kappa_{p_0}(z) = o(1)$$

locally uniformly in $z \in S$. This proves the claim. \qed

Recall the Euler-MacLaurin formula: For integers $K_0 < K_1$ and any absolutely continuous function $f$ it holds

$$\sum_{k = K_0}^{K_1} f(k) = \int_{K_0}^{K_1} f(u) \, du + \frac{f(K_1) + f(K_0)}{2} + \int_{K_0}^{K_1} \left( u - \frac{1}{2} \right) f'(u) \, du. \quad (2.1)$$

Indeed, the full Euler-MacLaurin formula provides an expansion in even higher derivatives of $f$ if they exist. In this simple form, it basically corresponds to an integration by parts.

**Lemma 2.4.** Locally uniformly on $S$, as $R \to \infty$, it holds

$$\sum_{k \in I_R} \kappa_\Psi(\frac{\kappa - \Phi R}{\Sigma R}) (z) = \Sigma R \int_I \kappa_\Phi(x) (z) \, dx + \frac{\kappa_\Phi(b) (z) + \kappa_\Phi(a) (z)}{2} + o(1).$$

**Proof.** Apply Euler-MacLaurin’s formula (2.1) to $f(u) = \kappa_\Psi(\frac{\kappa - \Phi R}{\Sigma R}) (z)$, $K_0 = a^-_R$ and $K_1 = a^+_R$.

For the first term we perform the change of variable $x = \frac{u - \Phi R}{\Sigma R}$ which yields

$$\int_{I_R} \kappa_\Phi(\frac{u - \Phi R}{\Sigma R}) (z) \, du = \Sigma R \int_{I_R} \kappa_\Phi(x) (z) \, dx.$$

Now, Lemma 2.1 together with the tail assumption on $\Phi$ yields that

$$\left| \Sigma R \int_{I_R} \kappa_\Phi(x) (z) \, dx - \Sigma R \int_I \kappa_\Phi(x) (z) \, dx \right| \leq 2C\Sigma R \int_{I \Delta I_R} \Phi(x) (1 - \Phi(x)) \, dx = o(1)$$

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This proves the local uniform convergence to zero for the last term in the Euler-Maclaurin formula \( z \) which clearly converges to zero locally uniformly for \( \Psi \).

By a generalized Riemann-Lebesgue lemma, the right-hand side converges to 0 locally uniformly in \( z \in S \). We need locally uniform convergence for \( \Psi \) as \( \Phi \) is continuous and \( p \mapsto \kappa_p(z) \) locally uniformly Lipschitz continuous.

Proof. The proof follows in exactly the same way as the one for Lemma 2.4. Apply the Euler-Maclaurin formula (2.1) with \( f \) and hence concludes the proof.

Because the first derivative of \( \kappa \) is (locally uniformly) bounded. Finally, the last term converges to zero locally uniformly as \( \Psi \) and \( \tilde{\kappa} \) are (locally uniformly) bounded and \( \Phi' \) and \( \Psi' \) are integrable.

This proves the local uniform convergence to zero for the last term in the Euler-Maclaurin formula and hence concludes the proof.

Lemma 2.5. Locally uniformly on \( S \), as \( R \to \infty \), it holds

\[
\frac{1}{\Sigma_R} \sum_{k \in I_R} \Psi \left( \frac{\kappa - \partial R}{\Sigma_R} \right) \hat{\kappa} \left( \frac{\kappa_0}{\Sigma_R} \right) (z) = \int_I \Psi(x) \hat{\kappa}(\frac{z}{\Sigma_R})(x) dx + o(1).
\]

Proof. The proof follows in exactly the same way as the one for Lemma 2.4. Apply the Euler-Maclaurin formula (2.1) with \( f(u) = \Psi \left( \frac{\kappa - \partial R}{\Sigma_R} \right) \hat{\kappa} \left( \frac{\kappa_0}{\Sigma_R} \right) (z) \), \( K_0 = a_R \) and \( K_1 = a_R \).

For the first term, the change of variables \( x = \frac{u}{\Sigma_R}c_n \) implies

\[
\frac{1}{\Sigma_R} \int_{I_R} \Psi \left( \frac{u - \partial R}{\Sigma_R} \right) \hat{\kappa} \left( \frac{\kappa_0}{\Sigma_R} \right) (z) du = \int_{I_R} \Psi(x) \hat{\kappa}(\frac{z}{\Sigma_R})(x) dx.
\]

Because the first derivative of \( \kappa \) is (locally uniformly) bounded and \( \Psi \) is integrable we conclude that

\[
\int_{I_R} \Psi(x) \hat{\kappa}(\frac{z}{\Sigma_R})(x) dx = \int_I \Psi(x) \hat{\kappa}(\frac{z}{\Sigma_R})(x) dx + o(1)
\]

locally uniformly for \( z \in S \).

The term corresponding to \( f \) at the endpoints of the summation vanishes locally uniformly for \( z \in S \) as \( \Psi \) and \( \hat{\kappa} \) are (locally uniformly) bounded. Finally, the last term

\[
\frac{1}{\Sigma_R} \int_{I_R} \left( \sum_{R} x + \partial R \right) \left( \Psi'(x) \hat{\kappa}(\frac{z}{\Sigma_R})(x) + \Psi(x) \Phi'(x) \hat{\kappa}(\frac{z}{\Sigma_R})(x) \right) dx
\]

also converges to zero locally uniformly as \( \Psi \), \( \hat{\kappa} \) and \( \Phi' \) are (locally uniformly) bounded and \( \Phi' \) and \( \Psi' \) are integrable.
Proof of Theorem 1.10. The theorem immediately follows by combining Lemmas 2.2 to 2.5. Analyticity of $\Lambda$ and $\psi$ follow immediately from the analyticity of $z \mapsto \kappa_p(z)$ for all $p \in [0, 1]$ together with the locally uniform bounds on $\kappa$ and $\dot{\kappa}$ from Lemma 2.1 (e.g. by Morera’s theorem). It only remains to show that the function $\Lambda$ indeed satisfy (1.5). A direct calculation shows

$$\left| \exp(\Lambda(x + iy) - \Lambda(x)) \right| = \exp\left( \int_I \log|1 + \Phi(u)(e^{x+iy} - 1)| - \log|1 + \Phi(u)(e^x - 1)|\, du \right)$$

$$= \exp\left( \frac{1}{2} \int_I \log\left( 1 - \frac{2\Phi(u)(1 - \Phi(u)e^x(1 - \cos y))}{(1 + \Phi(u)(e^x - 1))^2} \right)\, du \right)$$

$$\leq \exp\left( -(1 - \cos y) \min\{e^{-x}, e^x\} \int_I \Phi(u)(1 - \Phi(u))\, du \right)$$

$$\leq 1 - \min\{e^{-x}, e^x\} \int_{\mathbb{R}} \Phi(u)(1 - \Phi(u))\, du \frac{y^2}{12}.$$  

This proves the condition (1.5). □

2.2. Proof of functional central limit theorems (Theorems 1.11 and 1.12)

The convergence of $(\xi(D_r))_{r \in \mathbb{R}^+}$ as a stochastic process relies on a general limit theorem for sums of independent Bernoulli random variables. Basically, we show that convergence as a stochastic process follows from the convergence of covariances. The results presented in Theorems 1.11 and 1.12 then follow immediately as under Assumptions A one can show that the covariances indeed converge.

Fix some (possibly unbounded) interval $J \subseteq \mathbb{R}$. Let $f_R(t)$ be a non-negative, increasing function for $R \in \mathbb{R}^+$ and $t \in J$ such that $\lim_{R \to \infty} f_R(t) = \infty$ for all $t \in J$. Let us denote

$$X_R(t) = \sum_{k=1}^{\infty} \frac{\mathbb{I}_{k \leq f_R(t)} - f_R(t)}{\sigma_R}$$

for some function $\sigma_R$ with $\sigma_R \to \infty$ as $R \to \infty$. Within this section we denote $\lambda_{k,1} = \mathbb{P}(\Gamma_k \leq f_R(t))$.

Let us first state a multivariate central limit theorem for the stochastic processes $X_R$.

Proposition 2.6. Assume that there exists a function $c : J \times J \to \mathbb{R}$ such that for all $s, t \in J$

$$\text{Cov}(X_R(s), X_R(t)) \xrightarrow{R \to \infty} c(s, t).$$

Then,

$$\left( X_R(t) \right)_{t \in J} \xrightarrow{\text{fdd}} \left( G_t \right)_{t \in J},$$

where $(G_t)_{t \in J}$ denotes a centred Gaussian process with covariance kernel $\text{Cov}(G_s, G_t) = c(s, t)$ for all $s, t \in J$.

Proof. Fix $n \in \mathbb{N}$. We show that, for any sequence $t_1 \leq \cdots \leq t_n$, $t_i \in J$, and for any choice $a_1, \ldots, a_n \in \mathbb{R}$, it holds

$$a_1 X_R(t_1) + \cdots + a_n X_R(t_n) \xrightarrow{d} a_1 G_{t_1} + \cdots + a_n G_{t_n}.$$  

Then, the result follows by the Cramér-Wold device. Clearly, the mean of both sides is equal to zero. For the variance, observe that the convergence of the covariance function immediately implies

$$\lim_{R \to \infty} \text{Var}(a_1 X_R(t_1) + \cdots + a_n X_R(t_n))$$

$$= \sum_{i=1}^{n} a_i^2 \lim_{R \to \infty} \text{Var}(X_R(t_i)) + \sum_{1 \leq i < j \leq n} 2a_ia_j \lim_{R \to \infty} \text{Cov}(X_R(t_i), X_R(t_j))$$

$$= \sum_{i=1}^{n} a_i^2 \text{Var}(G_{t_i}) + \sum_{1 \leq i < j \leq n} 2a_ia_j \text{Cov}(G_{t_i}, G_{t_j})$$

$$= \text{Var}(a_1 G_{t_1} + \cdots + a_n G_{t_n}).$$
Finally, we show that the higher order cumulants converge to zero. Firstly, notice for \( q \in \mathbb{N}, q \geq 3 \), that
\[
\zeta(q) \left( a_1 X_R(t_1) + \cdots + a_n X_R(t_n) \right) = \sum_{k=1}^{\infty} \zeta(q) \left( \sum_{i=1}^{n} \frac{a_i}{\sigma_R} (I_{\Gamma_k \leq f_R(t_i)} - \lambda_{k,t_i}) \right).
\]
Since \( \sum_{i=1}^{n} \frac{a_i}{\sigma_R} (I_{\Gamma_k \leq f_R(t_i)} - \lambda_{k,t_i}) \) is centred and bounded by \( \sum_{i=1}^{n} \frac{|a_i|}{\sigma_R} \), the moment-cumulant formula implies
\[
\zeta(q) \left( \sum_{i=1}^{n} \frac{a_i}{\sigma_R} (I_{\Gamma_k \leq f_R(t_i)} - \lambda_{k,t_i}) \right)
\leq \sum_{\Pi \in \mathcal{Q}_q} (|\Pi| - 1)! \sum_{\pi \in \Pi} \mathbb{E} \left[ \left( \sum_{i=1}^{n} \frac{|a_i|}{\sigma_R} (I_{\Gamma_k \leq f_R(t_i)} - \lambda_{k,t_i}) \right)^{q} \right]
\leq 2^q \left( \sum_{i=1}^{n} \frac{|a_i|}{\sigma_R} \right)^{q-2} \sum_{i=1}^{n} \frac{a_i^2}{\sigma_R^2} \lambda_{k,t_i} (1 - \lambda_{k,t_i}),
\]
where \( \mathcal{Q}_q \) denotes the set of all set partitions of \( \{1, \ldots, q\} \). Therefore, we conclude
\[
\zeta(q) \left( a_1 X_R(t_1) + \cdots + a_n X_R(t_n) \right) \leq 2^q \left( \sum_{i=1}^{n} \frac{|a_i|}{\sigma_R} \right)^{q-2} n \sum_{i=1}^{n} \frac{a_i^2}{\sigma_R^2} \text{Var}(X_R(t_i)) \xrightarrow{R \to \infty} 0. \tag{2.2}
\]
This proves the convergence in finite dimensional distributions. \( \square \)

**Remark 7.** Notice that in (2.2) we actually provide an explicit bound on the cumulants of \( X_R(t) \). The theory of \[ \text{SS91} \] and \[ \text{DE13} \] on bounds on cumulants thus immediately allows to deduce moderate deviations and Berry-Esseen estimates for \( X_R(t_0) \). For a summary of the precise statement we refer the reader to \[ \text{Ven19} \] Proposition 3.18.

The convergence in finite dimensional distributions can be upgraded to convergence in the Skorokhod topology under a mild growth condition on the expectation \( t \to f_R(t) \).

**Proposition 2.7.** Under the assumptions of Proposition 2.6 and further assume that there exist constants \( \varepsilon > 0, C > 0 \) and \( R_0 \in \mathbb{R}_+ \) such that for all \( R \geq R_0 \) and \( s, t \in J \) with \( s \leq t \leq s + 1 \),
\[
f_R(t) - f_R(s) \leq C \sigma_R^2 (t - s)^{1/2 + \varepsilon}, \tag{2.3}
\]
then the convergence
\[
(X_R(t))_{t \in J} \xrightarrow{\xi \left( D_{\tilde{T}^{-1}(f_R(t))} \right) \overset{d}{=} \sum_{k=1}^{\infty} \mathbb{1}_{\Gamma_k \leq f_R(t)} }{R \to \infty} (G_t)_{t \in J}
\]
holds true with respect to the Skorokhod topology.

**Remark 8.** Probabilistically, the condition (2.3) implies that the increments of the counting statistics \( \xi(D_{\tilde{T}^{-1}(f_R(t))}) \overset{d}{=} \sum_{k=1}^{\infty} \mathbb{1}_{\Gamma_k \leq f_R(t)} \) are Hölder continuous and of the order of the variance. This condition is used to obtain tightness of the process \( (X_R(t))_{t \in J} \). From an analytic viewpoint, the condition (2.3) means that the operators \( K|_{D_{\tilde{T}^{-1}(f_R(t))}} \) are Hölder continuous in the Schatten 1-norm with a constant which is controlled by \( \sigma_R^2 \).

**Proof.** Notice that by Proposition 2.6 \( (X_R(t))_{t \in J} \) converges in finite dimensional distributions to \( (G(t))_{t \in J} \). Thus, it remains to show tightness. This will be proven by using the following Kolmogorov criterion: Fix \( t_0 \in J \).

(i) The collection of random variables \( (X_R(t_0))_{R \in \mathbb{R}_+} \) is tight.
(ii) \( \lim_{t \to t_0} \limsup_{R \to \infty} \mathbb{P}( |X_R(t) - X_R(t_0)| > \varepsilon ) = 0 \) for all \( \varepsilon > 0 \).
(iii) There exist constants $C, R_0 \in \mathbb{R}_+$ and $\beta > 1$ such that
\[
\mathbb{E}[(X_R(t) - X_R(s))^2 | X_R(u) - X_R(t)]^2 \leq C(u - s)^{\beta}
\]
for all $s \leq t \leq u$, $t, u \in J$ and all $R \geq R_0$.

Concerning (i), the claim follows immediately from the convergence in distribution of $X_R(t_0)$ for $R \to \infty$. For (ii) notice the mean condition implies
\[
\text{Var}(X_R(t) - X_R(t_0)) = \frac{1}{\sigma_R^2} \sum_{k=1}^{\infty} (\lambda_{k,t} - \lambda_{k,s}) \left(1 - (\lambda_{k,t} - \lambda_{k,s})\right)
\]
\[
\leq \frac{1}{\sigma_R^2} \sum_{k=1}^{\infty} (\lambda_{k,t} - \lambda_{k,s})
\]
\[
\leq C(t - s).
\]

Next, apply Chebyshev’s inequality and the above variance bound to obtain
\[
\limsup_{R \to \infty} \mathbb{P}(|X_R(t) - X_R(t_0)| > \varepsilon) \leq \limsup_{R \to \infty} \frac{\text{Var}(X_R(t) - X_R(t_0))}{\varepsilon^2} \leq \frac{C(t - t_0)}{\varepsilon^2}.
\]

The last term clearly converges to zero as $t \to t_0$.

Lastly, we show that (iii) holds. Notice that
\[
\mathbb{E}[(X_R(t) - X_R(s))^2 | X_R(u) - X_R(t)]^2 = \text{Var}(X_R(t) - X_R(s)) \text{Var}(X_R(u) - X_R(t)) + 2 \text{Cov}(X_R(t) - X_R(s), X_R(u) - X_R(t))^2
\]
\[
+ \mathbb{E}(X_R(t) - X_R(s), X_R(t) - X_R(s), X_R(u) - X_R(t), X_R(u) - X_R(t)),
\]

where the last term denotes the joint cumulant. By the above variance bound, the first two summands are bounded by
\[
\text{Var}(X_R(t) - X_R(s)) \text{Var}(X_R(u) - X_R(t)) \leq C^2(t - s)(u - t)
\]
and
\[
2 \text{Cov}(X_R(t) - X_R(s), X_R(u) - X_R(t))^2 \leq 2C^2(t - s)(u - t).
\]

As $(u - t)(t - s) \leq \frac{(u - s)^2}{4}$, it only remains to bound the joint cumulant term. For simplicity, denote $\lambda_{k,s,t} = \lambda_{k,t} - \lambda_{k,s}$ and similarly $\lambda_{k,u} = \lambda_{k,u} - \lambda_{k,t}$. Then,
\[
\varepsilon(X_R(t) - X_R(s), X_R(t) - X_R(s), X_R(u) - X_R(t), X_R(u) - X_R(t))
\]
\[
= \frac{1}{\sigma_R^2} \sum_{k=0}^{\infty} \varepsilon(\mathbb{1}_{f_R(s) \leq f_R(t)} \mathbb{1}_{f_R(s) < f_R(u)} \mathbb{1}_{f_R(t) < f_R(u)} \mathbb{1}_{f_R(t) < f_R(u)} \mathbb{1}_{f_R(u) \geq f_R(t)} - \varepsilon)
\]
\[
= \frac{1}{\sigma_R^2} \sum_{k=0}^{\infty} (-6(\lambda_{k,u})^2(\lambda_{k,t,u})^2 + 2(\lambda_{k,u})^2(\lambda_{k,t,u})^2 + 2\lambda_{k,s,t}(\lambda_{k,t,u})^2 - \lambda_{k,s,t}^2).
\]

Let us now distinguish two cases.

Case 1: $\frac{2C(u - s)}{\sigma_R^2} \leq u - s$. Then, the above bound on the variance yields
\[
\varepsilon(X_R(t) - X_R(s), X_R(t) - X_R(s), X_R(u) - X_R(t), X_R(u) - X_R(t)) \leq \frac{2C(u - s)}{\sigma_R^2} \sum_{k=1}^{\infty} \lambda_{k,s,t} \lambda_{k,t,u}
\]
\[
= -2C(u - s) \text{Cov}(X_R(t) - X_R(s), X_R(u) - X_R(t))
\]
\[
\leq 2C(u - s) \sqrt{\text{Var}(X_R(t) - X_R(s)) \text{Var}(X_R(u) - X_R(t))}
\]
\[
\leq C^2(u - s)^2.
\]
This proves the claim in the first case. 

**Case 2:** $u - s < \frac{1}{C\sigma_R}$. Then,

$$
\lambda_{k,s,u} = \lambda_{k,u} - \lambda_{k,s} \leq \sum_{k=0}^{\infty} (\lambda_{k,u} - \lambda_{k,s}) = f_R(u) - f_R(s) \leq C\sigma_R^2(u - s) \leq \frac{1}{2}.
$$

In particular,

$$
-6(\lambda_{k,s,u})^2(\lambda_{k,t,u})^2 + 2(\lambda_{k,s,t})^2 \lambda_{k,t,u} + 2\lambda_{k,s,t}(\lambda_{k,t,u})^2 - \lambda_{k,s,t}\lambda_{k,t,u} \\
\leq -\lambda_{k,s,t}(1 - 2\lambda_{k,s,t})(1 - 2\lambda_{k,t,u}) \leq 0.
$$

Hence, $\mathcal{X}(X_R(t) - X_R(s), X_R(t) - X_R(s), X_R(u) - X_R(t), X_R(u) - X_R(t)) \leq 0$ in this case and the claim follows. 

**Proof of Theorem 1.11** Since convergence in the Skorokhod topology and in finite dimensional distributions is local it suffices to prove the respective convergence on any fixed compact subset $J$. Let $s, t \in J$ with $s \leq t$. Then, (1.10) together with Cauchy-Schwartz’ inequality implies that

$$
\text{Cov}(\mathcal{X}_{f_R(s)}, \mathcal{X}_{f_R(t)}) = \sum_{k=1}^{\infty} \lambda_{k,s} (1 - \lambda_{k,t}) = \sum_{k \in \tilde{I}_{f_R(s)} \cap \tilde{I}_{f_R(t)}} \lambda_{k,s} (1 - \lambda_{k,t}) + o(1). \quad (2.4)
$$

By combining this with the asymptotic expansion (1.11) we further get

$$
\text{Cov}(\mathcal{X}_{f_R(s)}, \mathcal{X}_{f_R(t)}) = \sum_{k \in \tilde{I}_{f_R(s)} \cap \tilde{I}_{f_R(t)}} \Phi \left( \frac{k - \vartheta f_R(s)}{\sum_{f_R(s)}} \right) + \frac{1}{\sum_{f_R(s)}} \Psi \left( \frac{k - \vartheta f_R(s)}{\sum_{f_R(s)}} \right) + o\left( \frac{1}{\tilde{I}_{f_R(t)}} \right)
\quad (1 - \Phi \left( \frac{k - \vartheta f_R(t)}{\sum_{f_R(t)}} \right) + \frac{1}{\sum_{f_R(t)}} \Psi \left( \frac{k - \vartheta f_R(t)}{\sum_{f_R(t)}} \right) + o\left( \frac{1}{\tilde{I}_{f_R(t)}} \right) + o(1)
$$

with $o$-terms being uniform in $k \in \tilde{I}_{f_R(s)} \cap \tilde{I}_{f_R(t)}$. Since $\Phi$ is bounded and $\Psi$ is integrable and absolutely continuous (and hence also bounded), we further get

$$
\text{Cov}(\mathcal{X}_{f_R(s)}, \mathcal{X}_{f_R(t)}) = \sum_{k \in \tilde{I}_{f_R(s)} \cap \tilde{I}_{f_R(t)}} \Phi \left( \frac{k - \vartheta f_R(s)}{\sum_{f_R(s)}} \right) \left( 1 - \Phi \left( \frac{k - \vartheta f_R(t)}{\sum_{f_R(t)}} \right) \right) + O(1) \quad (2.5)
$$

with an error term depending only on $J$.

Let us consider first the case $f_R(t) = tR$. Fix a constant $c \in I$. As $\vartheta = 1$ we have $\tilde{I}_{f_R(t)} \rightarrow I$ and thus in particular $\liminf_{R \rightarrow \infty} \frac{a_n - R}{\Sigma_R} < c$ and $\limsup_{R \rightarrow \infty} \frac{a_n - R}{\Sigma_R} > c$. This shows that for all $R$ large enough we have $R + c\Sigma_R \in \tilde{I}_R$. As $\Sigma_R \leq |\tilde{I}_R| = o(R)$ by assumption, we further conclude that $(s + \varepsilon)R \notin \tilde{I}_R$. In particular, this proves that $\tilde{I}_R \cap \tilde{I}_R = \emptyset$ for all $R$ large enough. From (2.4) we therefore conclude that

$$
\text{Cov}(\mathcal{X}_{sR}, \mathcal{X}_{tR}) = o(1)
$$

In case $s = t$, notice that Assumption A implies mod-phi convergence and hence an asymptotic expansion of the variance

$$
\text{Var}(\mathcal{X}_{tR}) = \Sigma_{tR}\Lambda''(0) + O(1)
$$

as in Remark 4. Since $\sum_{\Sigma_R} \rightarrow g_2$, this shows that

$$
\text{Cov}\left( \frac{\mathcal{X}_{sR}}{\sqrt{\Lambda'(0)}} \Sigma_R, \frac{\mathcal{X}_{tR}}{\sqrt{\Lambda'(0)}} \Sigma_R \right) \rightarrow g_2 \|_{s=t}.
$$

and by Proposition 2.6 we conclude that $\left( \frac{\mathcal{X}_{sR}}{\sqrt{\Lambda'(0)}} \Sigma_R \right)_{t \in \mathbb{R}^+}$ converges in finite dimensional distribution to $(\xi_t N_t)_{t \in \mathbb{R}^+}$. 

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Let us now consider the case \( f_R(t) = R + t\Sigma_R \). Then, (2.3) and a Riemann sum argument imply that

\[
\text{Cov}(\Xi_{R+s\Sigma_R}, \Xi_{R+t\Sigma_R}) = \sum_{k \in I_{R}(s) \cap I_{R}(t)} \Phi\left( \frac{k - R - s\Sigma_R}{\Sigma_R + s\Sigma_R} \right) \left( 1 - \Phi\left( \frac{k - R - t\Sigma_R}{\Sigma_R + t\Sigma_R} \right) \right) + O(1)
\]

\[
= \int_{I_{R}(s) \cap I_{R}(t)} \Phi\left( \frac{x - R - s\Sigma_R}{\Sigma_R + s\Sigma_R} \right) \left( 1 - \Phi\left( \frac{x - R - t\Sigma_R}{\Sigma_R + t\Sigma_R} \right) \right) dx + O(1).
\]

Notice that by assumption \( \Sigma_{R+s\Sigma_R} \rightarrow R \rightarrow \infty \) so that \( \hat{I}_{R+s\Sigma_R} - R \rightarrow I + t \). Hence, if we were allowed to interchange limit and integral, a change of variables would yield

\[
\text{Cov}(\Xi_{R+s\Sigma_R}, \Xi_{R+t\Sigma_R}) = \Sigma_R \int_{I + s \cap I + t} \Phi(x - s)(1 - \Phi(x - t)) + o(\Sigma_R).
\]

Since the growth condition (2.3) is trivially satisfied, Proposition 2.7 can be applied to yield the convergence of \( \frac{\Xi_{R+s\Sigma_R}}{\Lambda^{1/2}(\Sigma_R)} \) \( \rightarrow \Xi \). We are thus left to show that the limit \( R \rightarrow \infty \) can be interchanged with the integral. Recall that we denote \( I_R = \frac{\hat{I}_R - R}{\Sigma_R} \). First notice that

\[
\frac{1}{\Sigma_R} \int_{I_R + s \cap I_R + t} \Phi\left( \frac{x - R - s\Sigma_R}{\Sigma_R + s\Sigma_R} \right) \left( 1 - \Phi\left( \frac{x - R - t\Sigma_R}{\Sigma_R + t\Sigma_R} \right) \right) dx
\]

\[
= \int_{I_R + s \cap I_R + t} \Phi\left( \frac{\hat{x} - R - s\Sigma_R}{\hat{\Sigma}_R} \right) \left( 1 - \Phi\left( \frac{\hat{x} - R - t\Sigma_R}{\hat{\Sigma}_R} \right) \right) dx + O(1)
\]

since the integrand is bounded by \( (1 + \Phi(x)\mathbb{1}_{x > 0})(1 - \Phi(x)) \) and

\[
\int_{I_R} (1 + \Phi(x)\mathbb{1}_{x > 0})(1 - \Phi(x)) dx = o(1).
\]

Further, the same uniform bound implies that

\[
\lim_{R \rightarrow \infty} \int_{I_R} \Phi\left( \frac{\hat{x} - R - s\Sigma_R - s\Sigma_R}{\hat{\Sigma}_R} \right) \left( 1 - \Phi\left( \frac{\hat{x} - R - t\Sigma_R}{\hat{\Sigma}_R} \right) \right) dx
\]

\[
= \int_{I_R} \lim_{R \rightarrow \infty} \Phi\left( \frac{\hat{x} - R - s\Sigma_R}{\hat{\Sigma}_R} \right) \left( 1 - \Phi\left( \frac{\hat{x} - R - t\Sigma_R}{\hat{\Sigma}_R} \right) \right) dx
\]

\[
= \int_{I_R} \mathbb{1}_{x \in I} (1 - \Phi(x)) dx.
\]

This proves the claim. \( \square \)

**Proof of Theorem 1.12** Since convergence in the Skorokhod topology is defined locally, it suffices to prove the claim for any compact subset \( J \subseteq \mathbb{R} \). Let \( s, t \in J \) with \( s \leq t \). Analogue to (2.5) in the proof of Theorem 1.11 we have

\[
\text{Cov}(\Xi_{R}, \Xi_{s\Sigma_R}) = \sum_{k \in I_{R} \cap \hat{I}_{R}} \Phi\left( \frac{k}{\Sigma_R} \right) \left( 1 - \Phi\left( \frac{k}{\Sigma_{s\Sigma_R}} \right) \right) + O(1).
\]

By a Riemann sum argument we further get

\[
\text{Cov}(\Xi_{R}, \Xi_{s\Sigma_R}) = \int_{\hat{I}_{R} \cap \hat{I}_{R}} \Phi\left( \frac{x}{\Sigma_R} \right) \left( 1 - \Phi\left( \frac{x}{\Sigma_{s\Sigma_R}} \right) \right) dx + o(\Sigma_R).
\]
The same argument as in the proof of Theorem 1.11 shows again that one can interchange the limit \( R \to \infty \) with the integral (after a change of variables). From this, together with the assumption that \( \Sigma_R \sim R \) and thus \( \Sigma_R \to t \), one concludes

\[
\text{Cov}(\Xi_{sR}, \Xi_{tR}) = \Sigma_R \int_{sI \cap tI} \Phi\left(\frac{x}{s}\right) \left(1 - \Phi\left(\frac{x}{t}\right)\right) \, dx + o(\Sigma_R).
\]

This proves the needed covariance asymptotic. Since also the mean growth condition (2.3) is trivially satisfied, Proposition 2.7 implies the claim. \( \Box \)

3. Applications to different models

We now show that all models presented in Section 1.4 indeed fall into the framework of Section 1.3 and satisfy Assumptions A.

3.1. The Ginibre ensemble (using Edgeworth expansion)

Recall that in the Ginibre ensemble the random variables \( \Gamma_k \) are gamma-distributed with shape parameter \( k \) and rate parameter 1. The proof of Assumptions A will consist of the following two parts: First we establish bounds on the decay of \( \Gamma_k \) which allows to deduce (1.10). The expansion in (1.11) follows from the observation that \( \Gamma_k \) is a sum of independent and identically distributed random variables and an Edgeworth expansion argument which will be made precise by the help of a Berry-Esseen bound. Before going into the detail, let us provide the basic idea for the Edgeworth expansion. Consider a sequence \( (X_n)_{n \in \mathbb{N}} \) of random variables together with a reference random variables \( X \) and denote by \( \kappa_k^{(n)} \) and \( \kappa_k \) the respective \( k \)-th cumulant. Then, the expansion

\[
\mathbb{E}[e^{izX_n}] = \mathbb{E}[e^{izX}] \exp\left(\sum_{k=1}^{\infty} \left(\kappa_k^{(n)} - \kappa_k \right) \frac{(iz)^k}{k!}\right)
\]

can (formally) be transferred by Fourier inversion into the expansion

\[
F_{X_n}(x) = F_X(x) + \sum_{k=1}^{\infty} B_k \left(\kappa_1^{(n)} - \kappa_1, \ldots, \kappa_k^{(n)} - \kappa_k \right) \frac{(-1)^k}{k!} F_X(k)
\]

for the distribution functions, where \( B_k \) denotes the \( k \)-th Bell polynomial. A priori, this expansion does not yield any information as it is not clear if it converges. In the case when \( X_n \) can be written as a sum of \( n \) independent and identically distributed random variables and \( X \) is chosen according to the central limit theorem, then the sum can be controlled and yields an expansion in terms of negative powers of \( \sqrt{n} \). This is made formal under the so-called Berry-Esseen expansion. In principle the Edgeworth expansion allows to even obtain higher orders in (1.11) and thus one could obtain even higher order cumulant expansion and higher order corrections in the deviation probabilities in Theorem 1.7. In the following, we first present the proof for the infinite Ginibre ensemble. The necessary changes to adapt it to the finite Ginibre ensemble are discussed below.

Lemma 3.1. For all \( k \geq R \) it holds

\[
e^{-R \frac{R^k}{k!}} \leq \mathbb{P}(\Gamma_k \leq R) \leq e^{k-R} \left(\frac{R}{k}\right)^k \leq 3\sqrt{k} e^{-R \frac{R^k}{k!}}.
\]

Similarly, for all \( 1 \leq k \leq R \) it holds

\[
e^{-R \frac{R^k}{k!}} \leq \mathbb{P}(\Gamma_k > R) \leq e^{k-R} \left(\frac{R}{k}\right)^k \leq 3\sqrt{k} e^{-R \frac{R^k}{k!}}.
\]

Proof. We only prove the result for \( k \geq R \). The other result follows in exactly the same way. For the lower bound, apply the relationship between the Poisson and the gamma distribution to obtain

\[
\mathbb{P}(\Gamma_k \leq R) = \mathbb{P}(P_R \geq k) \geq \mathbb{P}(P_R = k) = e^{-R \frac{R^k}{k!}}.
\]
where \( P_R \) denotes a Poisson-distributed random variable with parameter \( R \). For the upper bound, apply Chernoff’s inequality to conclude for any \( x > 0 \) that
\[
\mathbb{P} (\Gamma_k \leq R) \leq e^{-xR (1 - x)^{-k}}.
\]
Minimizing the right-hand side over \( x \) and applying Stirling’s formula yields the claim. \( \square \)

**Lemma 3.2.** The infinite Ginibre ensemble satisfies Assumptions A with \( \vartheta = 1 \), \( a_R = [R - R^{\frac{1}{2} + \varepsilon}] \), \( a_R^+ = [R + R^{\frac{1}{2} + \varepsilon}] \) for some \( \varepsilon > 0 \), \( \Sigma_R = \sqrt{R} \),
\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}} \, dy \quad \text{and} \quad \Psi(x) = \frac{1}{3\sqrt{2\pi}} (1 - x^2) e^{-\frac{x^2}{2}}.
\]

**Proof.** We first show that the variance is concentrating as in (1.10). Indeed, by applying the bound from Lemma 3.1 twice we obtain
\[
\sum_{k=a_R^+}^\infty \mathbb{P} (\Gamma_k \leq R) \mathbb{P} (\Gamma_k > R) \leq \sum_{k=a_R^+}^\infty \mathbb{P} (\Gamma_k \leq R) \leq \sum_{k=a_R^+}^\infty 3\sqrt{ke^{-\frac{Rk}{k!}}} \leq 3\sqrt{R} \mathbb{P} (P_R \geq a_R^+ - 1)
\]
\[
= 3\sqrt{R} \mathbb{P} (\Gamma_{a_R^+-1} \leq R) \leq 3\sqrt{R} e^{R^{\frac{1}{2} + \varepsilon}} \left( \frac{R}{R + R^{\frac{1}{2} + \varepsilon} - 1} \right)^{R + R^{\frac{1}{2} + \varepsilon} - 1} \leq 3\sqrt{R} e^{R^{\frac{1}{2} + \varepsilon}},
\]
where \( P_R \) denotes a Poisson-distributed random variable with parameter \( R \). The estimate for
\[
\sum_{k=a_R^+}^\infty \mathbb{P} (\Gamma_k \leq R) \mathbb{P} (\Gamma_k > R)
\]
follows in exactly the same way.

Let us now show that \( \mathbb{P} (\Gamma_k \leq R) \) indeed admits a uniform expansion within \( I_R \). This follows from an application of the Berry-Esseen expansions \([Fe71, \text{Theorem XVI.4.1}].\) Denote by \((Z_n)_{n \in \mathbb{N}}\) a sequence of independent random variables with exponential distribution of parameter 1. Then, the equality in distribution \( \Gamma_k \overset{d}{=} \sum_{n=1}^k Z_n \) implies
\[
\mathbb{P} (\Gamma_k \leq R) = \mathbb{P} \left( \sum_{n=1}^k \frac{Z_n - k}{\sqrt{k}} \leq \frac{R - k}{\sqrt{k}} \right) = \Phi \left( \frac{k - R}{\sqrt{k}} \right) - \frac{1}{3\sqrt{k}} \left( 1 - \frac{(R - k)^2}{k} \right) \Phi^\prime \left( \frac{k - R}{\sqrt{k}} \right) + O \left( \frac{1}{k} \right)
\]
with \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}} \, dy \) being the tail distribution of the Gaussian distribution. Denote
\[
\Psi(x) = \frac{1}{3\sqrt{2\pi}} (1 - x^2) e^{-\frac{x^2}{2}}. \quad \text{Notice that } O \left( \frac{1}{k} \right) = O \left( \frac{1}{\sqrt{k}} \right) \text{ uniformly in } k \in I_R. \quad \text{To show that}
\]
\[
\Phi \left( \frac{k - R}{\sqrt{k}} \right) = \Phi \left( \frac{k - R}{\sqrt{k}} \right) + O \left( \frac{1}{\sqrt{k}} \right) \text{ apply a Taylor expansion argument. By the fact that } |\Phi^\prime (t)| \leq \frac{1}{|t|} \text{ we obtain for some } x \in \left[ \frac{1}{\sqrt{k}}, \frac{1}{\sqrt{k}} \right] \text{ or } x \in \left[ \frac{1}{\sqrt{k}}, \frac{1}{\sqrt{k}} \right],
\]
that
\[
|\Phi \left( \frac{k - R}{\sqrt{k}} \right) - \Phi \left( \frac{k - R}{\sqrt{k}} \right)| = |\Phi^\prime ((R - k)x)| \left| \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k}} \right| \leq \frac{\sqrt{2R}}{R - k} \left( \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k}} \right).
\]
As the last term is decreasing in \( k \), we can further bound it uniformly for \( k \in I_R \) by
\[
|\Phi \left( \frac{k - R}{\sqrt{k}} \right) - \Phi \left( \frac{k - R}{\sqrt{k}} \right)| \leq \frac{\sqrt{2R}}{R^{\frac{1}{2} + \varepsilon}} \left( \frac{1}{\sqrt{R - R^{\frac{1}{2} + \varepsilon}}} - \frac{1}{\sqrt{R}} \right) \leq \frac{C_\varepsilon}{R}
\]
for some constant \( C_\varepsilon > 0 \). This shows that \( \Phi \left( \frac{k - R}{\sqrt{k}} \right) = \Phi \left( \frac{k - R}{\sqrt{k}} \right) + O \left( \frac{1}{\sqrt{k}} \right) \text{ uniformly for } k \in I_R. \)

Finally, let us consider the term corresponding to \( \Psi \). Notice first that \( |\Psi(t)| \leq \frac{1}{|t|} \) so that as before
\[
\left| \frac{1}{\sqrt{k}} \Psi \left( \frac{k - R}{\sqrt{k}} \right) - \frac{1}{\sqrt{k}} \Psi \left( \frac{k - R}{\sqrt{k}} \right) \right| \leq \left( \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k}} \right) \sqrt{k} \leq \left( \frac{1}{\sqrt{R - R^{\frac{1}{2} + \varepsilon}}} - \frac{1}{\sqrt{R}} \right) \sqrt{R - R^{\frac{1}{2} + \varepsilon}} \leq \frac{C_\varepsilon}{R}
\]
uniformly for \( k \in I_R \) with some constant \( C_\varepsilon > 0 \) depending only on \( \varepsilon \). Further, since \( \Psi' \) is bounded by a constant \( C \) say, we conclude

\[
\left| \frac{1}{\sqrt{R}} \Psi\left( \frac{k-R}{\sqrt{k}} \right) - \frac{1}{R} \Psi\left( \frac{k-R}{\sqrt{k}} \right) \right| \leq \frac{C}{\sqrt{R}} \left( \frac{1}{\sqrt{R}} - \frac{1}{R} \right) \leq \frac{C_\varepsilon C}{R}
\]

uniformly for \( k \in I_R \) for some constant \( C_\varepsilon \) depending only on \( \varepsilon \). Therefore, we proved that

\[
\frac{1}{\sqrt{k}} \Psi\left( \frac{k-R}{\sqrt{k}} \right) = \frac{1}{R} \Psi\left( \frac{k-R}{R} \sqrt{R} \right) + O\left( \frac{1}{R} \right)
\]

uniformly for \( k \in I_R \). Combining the above three parts proves that \( \mathbb{P}(\Gamma_k \leq R) \) indeed satisfies the uniform expansion

\[
\mathbb{P}(\Gamma_k \leq R) = \Phi\left( \frac{k-R}{\sqrt{R}} \right) + \frac{1}{\sqrt{R}} \Psi\left( \frac{k-R}{\sqrt{R}} \right) + O\left( \frac{1}{R} \right).
\]

The tail bounds for \( \Phi \) follow from the tail bound for a Gaussian distribution as discussed in Remark 4. The integrability conditions for \( \Psi \) follow immediately from the explicit form of the function.

By our main results, Lemma 3.2 in particular implies that the point count for the infinite Ginibre ensemble satisfies a mod-\( \phi \)-phi result and therefore the precise deviation result in Theorem 1.2 follows from Theorem 1.7. Moreover, the functional central limit theorems presented in Theorems 1.4 and 1.5 hold as well. Let us now show that Assumptions A also is satisfied for the finite Ginibre ensemble.

**Lemma 3.3.** The finite Ginibre ensemble \( (Z_N(\gamma))_{N \in \mathbb{N}} \) satisfies Assumptions A with \( \vartheta = 1, \sum_N = \sqrt{\gamma}N, \Phi \) and \( \Psi \) as in the infinite Ginibre ensemble, \( a_N^\gamma = \lfloor \gamma N - (\gamma N)^{\frac{1}{2} + \varepsilon} \rfloor \) and \( a_N^+ = \lceil \gamma N + (\gamma N)^{\frac{1}{2} + \varepsilon} \rceil \) in the bulk \((0 < \gamma < 1)\) or \( a_N^+ = N \) at the edge \((\gamma = 1)\).

**Proof.** The proof of Assumptions A follows as the proof of Lemma 3.2 with the following changes:

In the bulk, choose \( R = \gamma N \) and notice that we have \( a_N^\gamma = \lfloor \gamma N + (\gamma N)^{\frac{1}{2} + \varepsilon} \rfloor < N \) for \( N \) large enough. Therefore, the cutoff of \( \sum_{k=1}^N \mathbb{1}_{k \leq R} \) at \( N \) is not active after the truncation \((1.10)\). Hence, the expansion \((1.11)\) as in the case of the infinite Ginibre ensemble applies and the claim follows without any changes as in the infinite model.

At the edge, choose \( R = \gamma N \) with \( \gamma = 1 - t/\sqrt{N} \) so that the right boundary \( a_R^\gamma = N < \lfloor \gamma N + (\gamma N)^{\frac{1}{2} + \varepsilon} \rfloor \) for \( N \) large enough. Therefore, the cutoff at \( N \) becomes active and the concentration of variance \((1.10)\) occurs on a reduced interval compared to the infinite case. The reduced interval can also be seen if one thinks of the functions \( \Phi \) and \( \Psi \) as being multiplied by \( \mathbb{1}_{x \leq N} \). Hence, the expansion \((1.11)\) as in the case of the infinite Ginibre ensemble applies and the claim follows with the only modification that limiting interval is given by \( I = (-\infty, t] \).

Our main results in Section 1.3 immediately allow to conclude Theorems 1.13 to 1.15 from Lemma 3.3.

Interestingly, there exists an explicit formula for the covariance of the infinite Ginibre ensemble.

**Proposition 3.4.** Consider the infinite Ginibre ensemble \( \xi \). For \( 0 \leq s \leq t \), it holds

\[
\text{Cov}(\xi(D_s), \xi(D_t)) = e^{-s^2-t^2} \left( s^2 I_0(2st) + st I_1(2st) \right) - (t^2 - s^2) K(s^2, t^2),
\]

where \( K(s, t) = \int_0^s e^{-t-x} I_0(2\sqrt{tx}) \, dx \) and \( I_\nu \) denotes the modified Bessel function of kind \( \nu \in \mathbb{N}_0 \).

In particular, as \( r \to \infty \),

\[
\text{Var}(\xi(D_r)) = r^2 e^{-2r^2} \left( I_0(2r^2) + I_1(2r^2) \right) = \frac{r}{\sqrt{\pi}} + O\left( \frac{1}{r} \right).
\]

**Proof.** The proof basically follows from the identity

\[
\mathbb{P}(\Gamma_k \leq R) = \int_0^R e^{-x} \frac{x^{k-1}}{(k-1)!} \, dx = \sum_{n=k}^{\infty} \frac{e^{-R} R^n}{n!}
\]

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and a careful treatment of sums. Denote $S = s^2$ and $T = t^2$. By reordering the sum one obtains

$$\text{Cov}(\xi(D_s), \xi(D_t)) = \sum_{k=1}^{\infty} \mathbb{P}(\Gamma_k \leq S) \mathbb{P}(\Gamma_k > T)$$

$$= \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-S} S^i e^{-T} T^j}{i! j!}$$

$$= \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} (i-j) e^{-S} S^i e^{-T} T^j$$

$$= \sum_{j=0}^{\infty} S e^{-S} S^i e^{-T} T^j j! + \sum_{j=0}^{\infty} T e^{-S} S^j+1 e^{-T} T^j j! - (T - S) \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} e^{-S} S^i e^{-T} T^j j!.$$

Next, recall the definition of the modified Bessel function of first kind:

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{(k+\nu)!} \left( \frac{x}{2} \right)^{2k+\nu}$$

for $\nu \in \mathbb{N}_0$. Hence we showed that the first two summands are given by $S e^{-S} T I_0(2\sqrt{ST})$ and $\sqrt{ST} e^{-S} T I_1(2\sqrt{ST})$, respectively. For the last summand, notice that

$$(T - S) \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} \frac{e^{-S} S^i e^{-T} T^j j!}{i!} = (T - S) \sum_{j=0}^{\infty} \frac{e^{-T} T^j}{j!} \int_0^S e^{-x} x^{j-1} dx$$

$$= (T - S) \int_0^S e^{-T-x} I_0(2\sqrt{Tx}) dx = (T - S) K(S,T).$$

Putting all this together the claimed formula for the covariance follows. To obtain the expansion for the variance use the known expansion of the Bessel functions

$$I_\nu(r) = \frac{e^r}{2\pi r^\nu} \left( 1 + O\left( \frac{1}{r} \right) \right),$$

as $r \to \infty$ for $\nu \in \mathbb{N}_0$. \hfill \qed

The explicit formula of the covariance allows to present an alternative proof for the functional central limit theorems [Theorems 1.4 and 1.5] by calculating the covariance asymptotic by hand and applying [Proposition 2.6] and [Proposition 2.7] directly.

**Lemma 3.5.** For $0 < s \leq t$, as $r \to \infty$, it holds that

$$\text{Cov}(\xi(D_{sr}), \xi(D_{tr})) = \frac{tr}{\sqrt{\pi}} 1_{s=1} + O(1).$$

For $-\infty < s < t < \infty$, as $r \to \infty$, it holds that

$$\text{Cov}(\xi(D_{r+s}), \xi(D_{r+t})) = \left( \frac{r}{\sqrt{\pi}} e^{-(t-s)^2} - 2r(t-s) \mathbb{P}(N_{0,1} > \sqrt{2}(t-s)) \right) (1 + o(1)).$$

**Proof.** Consider first the macroscopic regime $(\xi(D_{tr}))_{t \in \mathbb{R}_+}$. If $s = t$ we have seen already that

$$\text{Var}(\xi(D_{tr})) = \frac{tr}{\sqrt{\pi}} (1 + o(1)).$$

In case $s < t$, the asymptotic of the Bessel functions $I_\nu(r) = \frac{e^r}{2\pi r^{\nu}} (1 + O_{\nu}(\frac{1}{r}))$ shows that

$$e^{-r^2(s^2+t^2)} (r^2 s^2 I_0(2r^2 st) + r^2 st I_1(2r^2 st)) = e^{-r^2(t-s)^2} \left( \frac{s^2}{\sqrt{4\pi st}} + \frac{1}{\sqrt{4\pi st}} \right) (1 + o(1)).$$

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For the other summand notice $K(x, y) \in [0, 1]$ for all $x, y \in \mathbb{R}_+$ and apply $I_0(x) \leq \frac{e^x}{\sqrt{x}}$ for all $x \in \mathbb{R}_+$ so that

$$K(r^2 s^2, r^2 t^2) = \int_0^{r^2 s^2} e^{-x} x I_0(2rt\sqrt{x}) dx \leq \int_0^{r^2 s^2} \frac{1}{2rt\sqrt{x}} e^{-x(1-\sqrt{x})^2} dx \leq e^{-r^2(t-s)^2} \int_0^{r^2 t^2} \frac{1}{2rt\sqrt{x}} dx = e^{-r^2(t-s)^2} \frac{8}{t}.$$ 

Hence, for $s < t$ the covariance converges to zero even exponentially fast and Proposition 2.6 applies as claimed.

Now, consider the microscopic regime $\langle \xi(D_{r+t}) \rangle_{t \in \mathbb{R}}$. Fix $s \leq t$ and assume that $r$ is large enough such that $r + s > 0$ and $r + t > 0$. For the first summand we apply again the asymptotic $I_0(r) = \frac{e^r}{2\pi r} (1 + O(\frac{1}{r}))$ to obtain

$$e^{-(r+s)^2} e^{-(r+t)^2} \left((r+s)^2 I_0(2r(r+s) + (r+s)(r+t))\right) = \frac{r}{\sqrt{\pi}} e^{-((r-s)^2} (1 + o(1)).$$

For the second summand, the asymptotic of $K$ can be found in [Luk62] 12.1.5 formula 8] and is given by

$$K(x^2, y^2) = \frac{1}{2} e^{-(y-x)^2} e^{-2xy} I_0(2xy) - 2e^{(y-x)^2} P(N_{0,1} \geq \sqrt{2}(y-x)) + O(\frac{1}{xy}),$$

as $xy \to \infty$ with $\frac{y}{x} \geq 1$ and $\frac{(y-x)^2}{xy} = o(1)$. Applying this here yields

$$((r+t)^2 - (r+s)^2) K((r+s)^2, (r+t)^2) = 2(r-t) P(N_{0,1} \geq \sqrt{2}(t-s)) (1 + o(1)).$$

The claimed asymptotic for the covariance of $(\xi(D_{r+\sqrt{m}}))_{t \in \mathbb{R}}$ can be concluded by combining the above asymptotic results.

### 3.2. The Ginibre-type ensemble

Recall that for the Ginibre-type ensemble the variables $\Gamma_k^{(\alpha)}$ are given through the density

$$L_{\alpha}^{(k-\alpha-1)}(x^2) x^{k-\alpha-1} e^{-x} \mathbb{1}_{x \geq 0}$$

for $\alpha \in \mathbb{N}_0$ and $k \geq 1$. As $E[\Gamma_k^{(\alpha)}] = k + \alpha$ we shift the index accordingly to $k - \alpha$ as already discussed in the presentation of the model. To prove Assumptions A we unfortunately cannot apply the theory of Edgeworth expansion as we did for the Ginibre ensemble since the random variables $\Gamma_k^{(\alpha)}$ are not infinitely divisible. Nevertheless, by carefully expanding the moment generating function of $\Gamma_k^{(\alpha)}$ and an Fourier inversion argument we are still able to find an asymptotic expansion [Luk62]. This basically mimics the proof of the Berry-Esseen expansion for sums of i.i.d. random variables.

Recall from [Shi15] Proposition 4.1 that the moment generating function of $\Gamma_k^{(\alpha)}_{k-\alpha}$ is given by

$$E[e^{x \Gamma_k^{(\alpha)}_{k-\alpha}}} = (1 - x)^{-k} \sum_{l=0}^{\alpha \wedge (k-\alpha-1)} \binom{\alpha}{l} \binom{k - \alpha - 1}{l} x^l$$

for all $x < 1$. By expanding the right-hand side in powers of $x$ one obtains in particular that $E[\Gamma_k^{(\alpha)}_{k-\alpha}] = k$ and $\text{Var}(\Gamma_k^{(\alpha)}_{k-\alpha}) = 2a(k - \alpha - 1) + k$.

#### Lemma 3.6. For all $k \geq R$ and $R$ large enough, it holds

$$\mathbb{P}(\Gamma_k^{(\alpha)}_{k-\alpha} \leq R) \leq 3\sqrt{k - 3} e^{3\alpha - R} \frac{R^{k-2\alpha}}{\alpha!(k-3\alpha)!}.$$
Similarly, for all $\alpha + 1 \leq k \leq R$, it holds
\[
\mathbb{P}(\Gamma_{k-\alpha}^{(\alpha)} > R) \leq 3\sqrt{k - \alpha e^{3\alpha - R} R^{k} a!(k - \alpha)!}.
\]

**Proof.** Let us consider first the case $k \geq R$. By Chernoff’s inequality we have
\[
\mathbb{P}(\Gamma_{k-\alpha}^{(\alpha)} \leq R) \leq e^{xR} \mathbb{E}[e^{-x\Gamma_{k-\alpha}^{(\alpha)}}] = e^{xR} (1 + x)^{-k} \sum_{l=0}^{\alpha} \binom{\alpha}{l} \binom{k - \alpha - 1}{l} x^l.
\]
for any $x > 0$. By choosing $R > 3\alpha$ we find that $k > 3\alpha$ so that
\[
\mathbb{P}(\Gamma_{k-\alpha}^{(\alpha)} \leq R) \leq e^{xR} (1 + x)^{-k} \left(\frac{k - \alpha - 1}{\alpha}\right) (1 + x^2)^{\alpha} \leq e^{xR} (1 + x)^{-k + 2\alpha} \left(\frac{k - \alpha - 1}{\alpha}\right).
\]
Choose $x = \frac{k - R}{R} > 0$ and let $R > 4\alpha$ so that $k > 4\alpha$. Then Stirling’s formula implies that
\[
\mathbb{P}(\Gamma_{k-\alpha}^{(\alpha)} \leq R) \leq e^{kR} \left(\frac{R}{k}\right)^{k - 2\alpha} \left(\frac{k - \alpha - 1}{\alpha}\right) \leq 3\sqrt{k - 4\alpha e^{3\alpha - R} R^{k - 2\alpha} a!(k - 3\alpha)!}.
\]
For the other inequality, we apply again Chernoff’s inequality to obtain
\[
\mathbb{P}(\Gamma_{k-\alpha}^{(\alpha)} > R) \leq e^{-xR} \mathbb{E}[e^{-x\Gamma_{k-\alpha}^{(\alpha)}}] = e^{-xR} (1 - x)^{-k} \sum_{l=0}^{\alpha} \binom{\alpha}{l} \binom{k - \alpha - 1}{l} x^l
\]
for any $0 < x < 1$. We now have to distinguish two cases: If $k \leq 2\alpha$ then
\[
\sum_{l=0}^{\alpha} \binom{\alpha}{l} \binom{k - \alpha - 1}{l} x^l \leq \left(\frac{\alpha}{k - \alpha - 1}\right) \sum_{l=0}^{\alpha - 1} \binom{k - \alpha - 1}{l} \binom{k - \alpha - 1}{l} \leq 4^{\alpha - 1} \left(\frac{\alpha}{k - \alpha - 1}\right).
\]
Thus, by choosing $x = \frac{R - k}{R}$ we conclude by Stirling’s formula
\[
\mathbb{P}(\Gamma_{k-\alpha}^{(\alpha)} > R) \leq e^{kR} \left(\frac{R}{k}\right)^{k - \alpha} \left(\frac{\alpha}{k - \alpha - 1}\right) \leq 3\alpha e^{3\alpha - R} R^{k} a! \leq 3\sqrt{k - 4\alpha e^{3\alpha - R} R^{k} a!}.
\]
Similarly, for $k > 2\alpha$ we obtain
\[
\sum_{l=0}^{\alpha} \binom{\alpha}{l} \binom{k - \alpha - 1}{l} x^l \leq \left(\frac{\alpha}{k - \alpha - 1}\right) \sum_{l=0}^{\alpha - 1} \binom{\alpha}{l} \binom{\alpha}{l} \leq 4^{\alpha} \left(\frac{\alpha}{k - \alpha - 1}\right).
\]
Thus, by choosing $x = \frac{R - k}{R}$ we conclude by Stirling’s formula we conclude the same bound
\[
\mathbb{P}(\Gamma_{k-\alpha}^{(\alpha)} > R) \leq e^{kR} \left(\frac{R}{k}\right)^{k - \alpha} \left(\frac{\alpha}{k - \alpha - 1}\right) \leq 3\alpha e^{3\alpha - R} R^{k} a! \leq 3\sqrt{k - 4\alpha e^{3\alpha - R} R^{k} a!}.
\]
Recall that we denote $Z_{\alpha}$ a random variable with density given by the Hermite function, i.e. $\Phi_{\alpha}(x) = \mathbb{P}(Z_{\alpha} > x) = \int_{x}^{\infty} h_{\alpha}^{2}(t) \, dt$.

**Lemma 3.7.** Uniformly over $x \in \mathbb{R}$, it holds
\[
\mathbb{P}\left(\frac{\Gamma_{k-\alpha}^{(\alpha)} - k}{\sqrt{k}} \leq x\right) = \mathbb{P}(Z_{\alpha} \leq x) - \frac{1}{\sqrt{3k}} \left(h_{\alpha}^{2}(x)\right)^{\prime\prime} + O_{\alpha}\left(\frac{1}{k}\right).
\]

**Proof.** We start by expanding the characteristic function of $\Gamma_{k-\alpha}^{(\alpha)}$ and then use a Fourier inversion argument to obtain the claimed expansion for its tail function. Recall that for any $\alpha \in \mathbb{N}_{0}$, $k \geq \alpha + 1$ and $x \in \mathbb{R}$ the characteristic function of a Ginibre-type random variable is given by
\[
\mathbb{E}\left[e^{ix \Gamma_{k-\alpha}^{(\alpha)} / \sqrt{k}}\right] = \left(1 - \frac{ix}{\sqrt{k}}\right)^{-k} e^{-ix \sqrt{k} \sum_{l=0}^{\alpha} \binom{\alpha}{l} \binom{k - \alpha - 1}{l} \left(\frac{-x^{2}}{k}\right)^{l}}.
\]
Now, notice that for $k \to \infty$ and $l \leq \alpha$ it holds

\[ \left( k - \frac{\alpha - 1}{l} \right) = \frac{k^l}{l} \left( 1 + O_\alpha \left( \frac{1}{k} \right) \right) \]

with a constant in the error term depending only on $\alpha$. Thus, uniformly in $x \in \mathbb{R}$,

\[ \sum_{l=0}^{\alpha} \left( \frac{\alpha}{l} \right) \left( k - \frac{\alpha - 1}{l} \right) \left( \frac{\sqrt{x}}{k} \right)^l = \text{L}_\alpha(x^2) \left( 1 + O_\alpha \left( \frac{1}{k} \right) \right). \]

Moreover, by using $|1 - e^z| \leq |ze^z|$ for all $z \in \mathbb{C}$ with $\Re(z) \geq 0$ we get $|e^z - e^w| \leq |z - w|e^{|\max\{\Re(z), \Re(w)\}|}$ for all $z, w \in \mathbb{C}$. Apply this to obtain

\[
\left| 1 - \frac{ix}{\sqrt{k}} \right|^{-k} e^{-ix\sqrt{k}} - e^{-\sqrt{k}} \left( 1 - \frac{ix^3}{3\sqrt{k}} \right) \leq \log \left( 1 - \frac{ix^3}{3\sqrt{k}} \right) + \frac{ix^3}{3\sqrt{k}} \left( 1 + \frac{x^6}{9k} \right) e^{-\frac{x^2}{2}} \leq \frac{x^6}{18k} \left( 1 + \frac{x^6}{9k} \right) e^{-\frac{x^2}{2}}.
\]

Therefore we proved that for $k \geq \alpha + 1$, uniformly in $x \in \mathbb{R}$,

\[
\left( 1 - \frac{ix}{\sqrt{k}} \right)^{-k} e^{-ix\sqrt{k}} = e^{-\frac{x^2}{2}} \left( 1 - \frac{ix^3}{3\sqrt{k}} \right) + \frac{1}{(1 + x^2)^{\alpha + 1}} O_\alpha \left( \frac{1}{k} \right).
\]

Putting everything together we conclude that, uniformly over $x \in \mathbb{R}$, as $k \to \infty$,

\[
\mathbb{E} \left[ e^{ix\frac{\text{L}_\alpha(x^2)}{\sqrt{k}}} \right] = e^{-\frac{x^2}{2}} \text{L}_\alpha(x^2) \left( 1 - \frac{ix^3}{3\sqrt{k}} \right) + \frac{1}{1 + x^2} O_\alpha \left( \frac{1}{k} \right).
\]

This proves the expansion for the characteristic function. Notice that the expansion we proved is uniform in $x$, so that we can apply a Fourier inversion method. To do so, recall from [Sh15 (4.9)] that the characteristic function of $Z_\alpha$ is given by

\[
\mathbb{E}[e^{ixZ_\alpha}] = \text{L}_\alpha(x^2) e^{-\frac{x^2}{2}}.
\]

Since the Fourier transform of a derivative of some function $f$ is given by

\[
\mathcal{F}(f')(x) = -ix\mathcal{F}(f)(x)
\]

we obtain for the density $f^{(\alpha)}_k$ of $\frac{\text{L}_\alpha(x^2)}{\sqrt{k}}$ as $k \to \infty$, uniformly in $x \in \mathbb{R}$, that

\[
f^{(\alpha)}_k(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iyx} \left( e^{-\frac{x^2}{2}} \text{L}_\alpha(y^2) \left( 1 - \frac{iy^3}{3\sqrt{k}} \right) + \frac{1}{1 + y^2} O_\alpha \left( \frac{1}{k} \right) \right) dy
\]

\[
= h_\alpha(x)^2 - \frac{1}{3\sqrt{k}} (h_\alpha''(x))'' + O_\alpha \left( \frac{1}{k} \right) e^{-|x|}.
\]

The claim now follows by integrating this identity:

\[
\mathbb{P} \left( \frac{\text{L}_\alpha(x^2)}{\sqrt{k}} \leq x \right) = \mathbb{P}(Z_\alpha \leq x) - \frac{1}{3\sqrt{k}} (h_\alpha''(x))'' + O_\alpha \left( \frac{1}{k} \right),
\]

where we used that $\int_{\mathbb{R}} e^{-|x|} \, dx < \infty$. \[\square\]
we obtain the following relation between the Hermite polynomials and the Laguerre polynomials:

\[ \Phi_n(x) = \int_x^\infty h_n(t) \, dt \quad \text{and} \quad \Psi_n(x) = -\frac{1}{3}(h_n''(x))'' = \frac{1}{3} \Phi_n''(x). \]

**Proof.** We first show that the variance is concentrating as in (1.10). We will only show the bound for \( k \geq a_R^+ \). The other one follows in the same way. Denote by \( P_n \) a Poisson-distributed random variable with parameter \( n \). Then, by Lemma 3.6 for \( R \) large enough it holds that

\[
\sum_{k=a_R^+}^{\infty} P(\Gamma_{k-\alpha}^{(a)} \leq R) P(\Gamma_{k-\alpha}^{(a)} > R) \leq \sum_{k=a_R^+}^{\infty} P(\Gamma_{k-\alpha}^{(a)} \leq R) \leq 3\sqrt{k - 3\alpha e^{-R} - R^2 e^{-3\alpha/k}}
\]

which clearly converges to zero. As already claimed before the estimate for \( \sum_{k=a_R^+}^{\infty} P(\Gamma_{k-\alpha}^{(a)} \leq R) P(\Gamma_{k-\alpha}^{(a)} > R) \) follows in exactly the same way. This proves the concentration of the variance.

Let us now show that \( P(\Gamma_{k-\alpha}^{(a)} \leq R) \) indeed admits a uniform expansion (1.11) within \( I_R \). This follows from the asymptotic expansion in Lemma 3.7,

\[
P(\Gamma_{k-\alpha}^{(a)} \leq R) = 1 - \Phi_n \left( \frac{R - k}{\sqrt{k}} \right) + \frac{1}{\sqrt{k}} \Psi_n \left( \frac{R - k}{\sqrt{k}} \right) + O_n \left( \frac{1}{k} \right)
\]

where we used that the density \( h_n^2 \) is symmetric, i.e. \( h_n^2(-x) = h_n^2(x) \). To conclude, we reuse the argument presented in the proof of Lemma 3.2. We only need to show for some large constant \( C_\alpha > 0 \) that \( |\Phi_n(x)| \leq \frac{C_\alpha}{|x|} \) and \( |\Psi_n(x)| \leq \frac{C_\alpha}{|x|} \) for all \( x \in \mathbb{R} \) and that \( \Phi_n' \) is bounded. All three claims follow immediately from the exponential decay of \( h_n(x)^2 \) for \( x \) near \( \pm \infty \). Hence, we can conclude the uniform expansion

\[
P(\Gamma_{k-\alpha}^{(a)} \leq R) = \Phi_n \left( \frac{k - R}{\sqrt{k}} \right) + \frac{1}{\sqrt{k}} \Psi_n \left( \frac{k - R}{\sqrt{k}} \right) + O_n \left( \frac{1}{k} \right).
\]

The proof of the remaining assumptions, in particular the tail bounds for \( \Phi_n \), follow immediately from the explicit form of the functions \( \Phi_n \) and \( \Psi_n \).

**Lemma 3.8** allows us to immediately apply our main results Theorems 1.10 and 1.11 to conclude Theorems 1.16 and 1.17.

By comparing the leading order term for the variance with the one found in [Shi15] Theorem 1.1, we obtain the following relation between the Hermite polynomials and the Laguerre polynomials:

**Corollary 3.9.** It holds

\[
\int_{\mathbb{R}} \left( \int_{-\infty}^{x} h_\alpha(u)^2 \, du \right) \left( \int_{x}^{\infty} h_\alpha(v)^2 \, dv \right) \, dx = \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}^+} \sqrt{x} L_\alpha(x)^2 e^{-x} \, dx.
\]

### 3.3. The hyperbolic Gaussian analytic ensemble

Recall that for the hyperbolic models we have that \( \Gamma_k^{(\rho)} \) is \( \rho \text{Beta}'(k, \rho) \)-distributed for \( k \in \mathbb{N} \) and \( \rho > 0 \), i.e.

\[
P(\Gamma_k^{(\rho)} \leq R) = \frac{\Gamma(k + \rho)}{\Gamma(k) \Gamma(\rho)} \int_0^R x^{k-1} (1 + x)^{-k-\rho} \, dx.
\]
Unlike for the Ginibre model and the Ginibre-type models for the hyperbolic ensembles the expansion (3.11) is not due to a central limit theorem-like result. Instead, the expansion is derived by a direct expansion of the distribution function.

**Lemma 3.10.** The hyperbolic model satisfies the concentration of variance (3.11) with \( a_R^- = [R^{1-\varepsilon}] \) and \( a_R^+ = [R^{1+\varepsilon}] \) for any \( \varepsilon > 0 \).

**Proof.** First notice that

\[
\sum_{k \in I_R^-} \mathbb{P}(\Gamma_k^{(\rho)} \leq R) \mathbb{P}(\Gamma_k^{(O)} > R) \leq \sum_{k=1}^{a_R^-} \mathbb{P}(\Gamma_k^{(\rho)} > R) + \sum_{k=a_R^+}^{\infty} \mathbb{P}(\Gamma_k^{(\rho)} \leq R)
\]

and we bound both terms separately. For the first one notice that

\[
\mathbb{P}(\Gamma_k^{(\rho)} > R) \leq \frac{\Gamma(k + \rho)}{\Gamma(k) \Gamma(\rho)} \int_R^\infty \frac{1}{(1 + x)^{\rho-1}} \, dx = \frac{\Gamma(k + \rho)}{\Gamma(k) \Gamma(\rho) (\rho + R)^\rho}.
\]

By the hockey stick identity we thus obtain

\[
\sum_{k=1}^{a_R^-} \mathbb{P}(\Gamma_k^{(\rho)} > R) \leq \frac{\rho^\rho}{(\rho + R)^\rho} \sum_{k=1}^{a_R^-} \binom{k + \rho - 1}{\rho} = \frac{\rho^\rho}{(\rho + R)^\rho} \binom{a_R^- + \rho}{\rho + 1} \leq \frac{\rho^\rho}{(\rho + R)^\rho} \binom{R + a_R^-}{\rho + 1}
\]

which clearly converges to zero for our choice of \( a_R^- \).

For the other term notice that

\[
\mathbb{P}(\Gamma_k^{(\rho)} \leq R) \leq \frac{\Gamma(k + \rho)}{\Gamma(k) \Gamma(\rho)} \int_0^R \left( \frac{x}{1 + x} \right)^{k-1} \frac{\rho + R}{\rho (1 + x)^2} \, dx \leq \frac{(k + \rho - 1)}{\rho} \left( \frac{R}{\rho + R} \right)^{k-1} \leq \frac{(2k)^\rho}{\Gamma(\rho + 1)} \left( \frac{R}{\rho + R} \right)^{k-1}
\]

for \( k \geq R \geq \rho \). Notice that \( t \mapsto t^\rho (\frac{R}{\rho + R})^t \) is decreasing for \( t \geq R + \rho \geq \frac{\rho}{\log(1 + R)} \) so that an integral comparison yields

\[
\sum_{k=a_R^+}^{\infty} \mathbb{P}(\Gamma_k^{(\rho)} \leq R) \leq \frac{2 \rho^\rho (R + R)}{(\rho + R)^{k+1}} \left( \frac{R}{\rho + R} \right)^{a_R^+} + \int_{a_R^+}^{\infty} t^\rho e^{-t \log(1 + \frac{R}{\rho + R})} \, dt
\]

To bound the integral, apply a Chernoff bound for a gamma-distributed random variable as in Lemma 3.1 to obtain

\[
\int_{a_R^+}^{\infty} t^\rho e^{-t \log(1 + \frac{R}{\rho + R})} \, dt = \frac{\Gamma(\rho + 1)}{(\log(1 + \frac{R}{\rho + R}))^{\rho+1}} \mathbb{P}(\Gamma(\rho + 1, \log(1 + \frac{R}{\rho + R})) > a_R^+)
\]

\[
= \frac{\Gamma(\rho + 1)}{(\log(1 + \frac{R}{\rho + R}))^{\rho+1}} \mathbb{P}(\Gamma(\rho + 1, 1) > \log(1 + \frac{R}{\rho + R}) a_R^+)
\]

\[
\leq \frac{3}{\sqrt{\rho + 1}} \left( \frac{R}{\rho + R} \right)^{a_R^+} (a_R^+)^{\rho+1}.
\]

Combine the above bounds to conclude that

\[
\sum_{k=a_R^+}^{\infty} \mathbb{P}(\Gamma_k^{(\rho)} \leq R) \leq C_\rho (a_R^+)^{\rho+1} \left( \frac{R}{\rho + R} \right)^{a_R^+}
\]

for some large constant \( C_\rho \). The right-hand side clearly converges to zero for our choice of \( a_R^+ \) so that the claim follows.
**Lemma 3.11.** Let $\hat{I}_R = [a^-_R, a^+_R]$ with $a^-_R$ and $a^+_R$ given as in Lemma 3.10. Uniformly for $k \in \hat{I}_R$, as $R \to \infty$, it holds
\[
P(\Gamma^{(\rho)}_k \leq R) = \Phi_{\rho}(\frac{k}{R}) + \frac{\rho}{R} \Psi_{\rho}(\frac{\rho k}{R}) + \varepsilon_R,
\]
where
\[
\Phi_{\rho}(x) = P(\text{Gamma}(\rho, \rho) > x), \quad \Psi_{\rho}(x) = \frac{\rho^{\rho+1}}{2!} x^n e^{-\rho x} - \frac{\rho}{2} x^{\rho+1} e^{-\rho x}
\]
and $\varepsilon_R = O(R^{-\frac{3}{2} + \varepsilon})$ for an arbitrary small $\varepsilon > 0$ in case $1 < \rho < 2$ and $\varepsilon_R = O(R^{-2})$ in case $\rho = 1$ or $\rho \geq 2$.

**Remark 9.** An expansion of $P(\Gamma^{(\rho)}_k \leq R)$ as in Lemma 3.11 still holds if $\rho < 1$. In this case we would obtain an additional contribution to $\Psi_{\rho}$, which is not well-behaved as needed to apply our results from Section 1.3.

**Proof.** By doing the change of variables $u = \frac{R}{x}$ we obtain
\[
P(\Gamma^{(\rho)}_k \leq \rho R) = P(\text{Beta}'(k, \rho) \leq R)
\]
\[
= 1 - \frac{\Gamma(k + \rho)}{\Gamma(k) \Gamma(\rho)} \int_{R}^{\infty} x^{k-1} (1 + x)^{-(k+\rho)} \, dx
\]
\[
= 1 - \frac{\Gamma(k + \rho)}{\Gamma(k) \Gamma(\rho)} \int_0^1 \frac{1}{u} \left(1 + \frac{R}{u}\right)^{-(k+\rho)} \, du.
\]
Next, notice that $0 \leq \frac{1}{1 + 2aR} = \frac{1}{1 + \frac{a}{R}} \leq a(a+1)$ for all $a > 0$ and all $x > 0$ which implies that
\[
(1 + \frac{R}{u})^{-\rho} = (\frac{u}{R})^{-\rho} - \rho (\frac{u}{R})^{-\rho+1} + O\left(\frac{u^2}{R^{\rho+2}}\right)
\]
with the error term being uniform in $u \in [0, 1]$. Similarly,
\[
\frac{1}{u} \left(1 + \frac{R}{u}\right)^{-k} = \exp\left(-k \log \left(1 + \frac{u}{R}\right)\right) = \exp\left(-\frac{uk}{R} + \frac{ku^2}{2R^2} + O\left(\frac{k^3 u^3}{R^4}\right)\right)
\]
with the error term being uniform in $u \in [0, 1]$ and uniform in $k \in [1, R^{2-\varepsilon}]$ for some $\varepsilon > 0$. Combining the above estimates we obtain
\[
P(\text{Beta}'(k, \rho) \leq R)
\]
\[
= 1 - \frac{\Gamma(k + \rho)}{\Gamma(k) \Gamma(\rho)} \int_0^1 \left(\frac{u}{R}\right)^{\rho} - \rho \left(\frac{u}{R}\right)^{\rho+1} + \frac{k}{2} \left(\frac{u}{R}\right)^{\rho+2}
\]
\[
+ O\left(\frac{u^2}{R^{\rho+2}} + \frac{k^2}{R^2} \left(\frac{u}{R}\right)^{\rho+2} + \frac{k}{2} \left(\frac{u}{R}\right)^{\rho+3}\right) e^{-\frac{uk}{R}} \, du
\]
\[
= 1 - \frac{\Gamma(k + \rho)}{k^\rho \Gamma(k) \Gamma(\rho)} \int_0^\infty \left(x^{\rho-1} - \frac{px^\rho}{k} + x^{\rho+1} + O\left(\frac{x^{\rho+1} + x^{\rho+2}}{k^2 + \frac{x^{\rho-1} + x^{\rho+1}}{R^2}}\right)\right) e^{-x} \, dx
\]
\[
= 1 - \frac{\Gamma(k + \rho)}{k^\rho \Gamma(k) \Gamma(\rho)} \int_0^\infty \left(x^{\rho-1} - \frac{px^\rho}{k} + x^{\rho+1} + O\left(\frac{x^{\rho+1}}{R^2}\right)\right) e^{-x} \, dx.
\]
In the last estimate we used that $x \leq \frac{k}{R}$ to obtain the uniform error $O\left(\frac{k^3}{R^4}\right)$.

Let’s for now assume $\rho = 1$. The case $\rho > 1$ we will discuss afterwards. For $\rho = 1$ it holds $\frac{\Gamma(k + \rho)}{k^\rho \Gamma(k)} = 1$ and thus we obtain
\[
P(\text{Beta}'(k, \rho) \leq R) = 1 - \int_0^\infty \left(1 - \frac{x}{k} + \frac{x^2}{2k} + O\left(\frac{1 + x^2}{R^2}\right)\right) e^{-x} \, dx
\]
\[
= \Phi_1\left(\frac{k}{R}\right) + \frac{1}{R} \Psi_1\left(\frac{k}{R}\right) + O\left(\frac{1}{R^2}\right)
\]
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with \[ \Phi_1(x) = \mathbb{P}\left( \text{Exp}(1) > \frac{k}{R} \right) \quad \text{and} \quad \Psi_1(x) = \frac{1}{x} \int_0^x (t - \frac{t^2}{2}) e^{-t} dt = \frac{x e^{-x}}{2}. \]

For the error term in the integral we used that the associated constant is uniform in \( x \) and that \( \int_0^\infty (1 + x^2) e^{-x} dx < \infty \). Now, consider the case \( \rho > 1 \). Then

\[
\frac{\Gamma(k + \rho)}{k \rho \Gamma(k) \Gamma(\rho)} = \frac{1}{\Gamma(\rho)} \left( 1 + \frac{\rho (\rho - 1)}{2k} + O\left( \frac{1}{k^2} \right) \right)
\]

for \( k \to \infty \). As already observed in the case \( \rho = 1 \), the distribution function of \( \Gamma_k^{(\rho)} \) is approximated by a sum of functions of the form \( f_{ab}(x) \frac{1}{\pi} \int_0^\pi \beta^{-1} e^{-t} dt \) for \( a \in \mathbb{R}_+ \) and \( b > 0 \). In general, such a function is bounded if \( b \geq a \). Thus, we get as \( R \to \infty \) uniformly for \( k \in \mathcal{I}_R \) that

\[
\mathbb{P}(\text{Beta}'(k, \rho) \leq R) = 1 - \frac{1}{\Gamma(\rho)} \int_0^\pi \left( x^{\rho-1} + \frac{\rho (\rho - 1)}{2k} x^{\rho-1} - \frac{\rho x^\rho}{k} + \frac{x^{\rho+1}}{2k} \right)
\]

\[
+ O\left( \frac{x^{\rho-1} + x^\rho + x^{\rho+1}}{k^2} + \frac{x^{\rho-1} + x^{\rho+1}}{R^2} \right) e^{-x} dx
\]

\[
= \tilde{\Phi}_\rho\left( \frac{k}{R} \right) + \frac{1}{R} \tilde{\Psi}_\rho\left( \frac{k}{R} \right) + \varepsilon_R
\]

with

\[
\tilde{\Phi}_\rho(x) = \mathbb{P}(\text{Gamma}(\rho, 1) > x),
\]

\[
\tilde{\Psi}_\rho(x) = -\frac{\rho (\rho - 1)}{2 \Gamma(\rho)} f_{1, \rho}(x) + \frac{\rho}{\Gamma(\rho)} f_{1, \rho+1}(x) - \frac{1}{2 \Gamma(\rho)} f_{1, \rho+2}(x) = \frac{e^{-x}}{2 \Gamma(\rho)} (x^{\rho} - (\rho - 1)x^{\rho-1}),
\]

\[
\varepsilon_R = \int_0^\pi O\left( \frac{x^{\rho-1} + x^\rho + x^{\rho+1}}{k^2} + \frac{x^{\rho-1} + x^{\rho+1}}{R^2} \right) dx,
\]

where we used integration by parts to obtain the second expression for \( \Psi \). Let us treat these terms one by one. The leading order term is given by

\[
1 - \frac{1}{\Gamma(\rho)} \int_0^\pi x^{\rho-1} e^{-x} dx = \mathbb{P}(\text{Gamma}(\rho, 1) > \frac{k}{R}).
\]

The error term \( \varepsilon_R \) can be obtained by integrating the error terms in the integrand as the associated constant is uniform in \( x \). We have

\[
\int_0^\pi x^{\rho-1} + x^{\rho+1} e^{-x} dx = O\left( \frac{1}{R^2} \right)
\]

and

\[
\int_0^\pi \frac{x^\rho + x^{\rho+1}}{k^2} e^{-x} dx = \frac{1}{R^2} \left( f_{2, \rho+1}\left( \frac{k}{R} \right) + f_{2, \rho+2}\left( \frac{k}{R} \right) \right) = O\left( \frac{1}{R} \right)
\]

where we used the boundedness of \( f_{ab} \) for \( b \geq a \). Notice that the last term \( \frac{1}{R^2} \int_0^\pi x^{\rho-1} e^{-x} dx \) can only be controlled like this if \( \rho \geq 2 \). Otherwise we this is not true as \( f_{2, \rho} \) is not bounded. Instead, we use for \( 1 < \rho < 2 \) that \( k \in \mathcal{I}_R \) implies

\[
\frac{1}{R^2} \int_0^\pi e^{\rho-1} e^{-x} dx = \frac{1}{R^2 k^{\rho-1} } f_{\rho, \rho}\left( \frac{k}{R} \right) = O\left( R^{-\rho - \frac{\rho (\rho - 1)}{2k} + \varepsilon} \right) = O\left( R^{-\frac{3}{2} + 2\varepsilon} \right).
\]

Combining all error terms yields the asymptotic expansion for \( \mathbb{P}(\text{Beta}'(k, \rho) \leq R) \). The claimed expansion for \( \mathbb{P}(\Gamma_k^{(\rho)} \leq R) \) follows immediately by replacing \( R \) by \( \frac{k}{R} \) in the former one, i.e. \( \Phi_\rho(x) = \tilde{\Phi}_\rho(\rho x) \) and \( \Psi_\rho(x) = \rho \tilde{\Psi}_\rho(\rho x) \). \( \square \)
Lemma 3.12. The hyperbolic ensemble with $\rho \geq 1$ satisfies Assumptions A with $\vartheta = 0$, $a_R^\pm = [R^{1+\varepsilon}]$, $a_R^- = [R^{1+\varepsilon}]$ for some small $\varepsilon > 0$, $\Sigma_R = R$,

$$\Phi(x) = \frac{\rho^\vartheta}{\Gamma(\vartheta)} \int_x^\infty t^{\vartheta-1} e^{-\rho t} \mathrm{d}t \quad \text{and} \quad \Psi(x) = \frac{\rho^{\vartheta+1}}{2\Gamma(\vartheta+1)} x^{\vartheta+1} e^{-\rho x} - \frac{\rho^\vartheta}{2\Gamma(\vartheta)} x^{\vartheta-1} e^{-\rho x}.$$

Proof. The claim follows from combining Lemmas 3.10 and 3.11. Only notice that $\Psi_\rho$ is only absolutely continuous if $\rho > 1$. □

Lemma 3.12 allows us to immediately apply our main results Theorems 1.10 and 1.11 to conclude Theorems 1.19 and 1.20. It turns out that in the case $\rho = 1$ one can calculate the covariance in Theorem 1.20.

Lemma 3.13. It holds

$$\text{Cov}(G_k^{(1)}, G_t^{(1)}) = \frac{2s^2}{(s+t)}$$

Proof. Indeed, we have $\Phi_1(x) = e^{-x}$ so that

$$\int_{\mathbb{R}_+} \Phi_1 \left( \frac{x}{s} \right) \left( 1 - \Phi_1 \left( \frac{x}{t} \right) \right) \mathrm{d}x = \frac{s^2}{s+t}.$$ □

4. Proof of the JLM-result (Proposition 1.3)

In this part, we prove the superlarge deviation principles Proposition 1.3 for the Ginibre point process for the cases $1 < \gamma < 2$ and $2 < \gamma$. The proofs are motivated by the similar result in [Shi06] where the author proves the deviations for $\gamma = 2$. Basically, the proof consists of two parts: Firstly, we apply the standard principle that among all the unlikely events it is the most likely one that contributes solely to the rate function. Secondly, we show that the rate function in the different regimes can be computed from a moderate (MDP), large (LDP) or superlarge deviation principle (SLDP) for the $\Gamma_k^\alpha$. Let us start with the latter part first. Within this section we denote $q_R = R^\alpha$. Since our argument also covers the case $\alpha = 2$ we reprove it here as well. Also notice that we use the notation as presented in Section 1.4.1. The result as presented in Section 1.1 follows by replacing $R$ by $r^2$.

Lemma 4.1. Let $k = R + xq_R$.

- MDP: For $1 < \gamma < 2$ and all $x > 0$ it holds

$$\frac{R}{q_R} \log \mathbb{P}(\Gamma_k \leq R) \xrightarrow{R \to \infty} - \frac{x^2}{2}.$$

- LDP: For $\gamma = 2$ and all $x > 0$ it holds

$$\frac{1}{R} \log \mathbb{P}(\Gamma_k \leq R) \xrightarrow{R \to \infty} x - (1 + x) \log(1 + x).$$

- SLDP: For $2 < \gamma$ and all $x > 0$ it holds

$$\frac{1}{q_R \log q_R} \log \mathbb{P}(\Gamma_k \leq R) \xrightarrow{R \to \infty} \frac{x(2 - \gamma)}{\gamma}.$$  

Proof. Notice that $\Gamma_k \overset{d}{=} \sum_{i=1}^k Z_i$ with $Z_i$ being independent exponentially distributed random variables with parameter 1. Thus, the results follow immediately from a moderate, large or superlarge deviation principle (see e.g. [Roz08, Roz15] for the not so classical theory of superlarge deviations). As superlarge deviations are not standard we provide here also an alternative analytic proof.
Let us start by proving first an upper bound. As the function \( x \mapsto x^{k-1}e^{-x} \) is unimodal and attains its maximum at \( x = k - 1 \geq R \) for \( R \) large enough, we can bound
\[
\sup_{x \in [0,R]} |x^{k-1}e^{-x}| \leq R^{k-1}e^{-R}.
\]
and therefore obtain
\[
\limsup_{R \to \infty} \frac{1}{q_R \log q_R} \log \mathbb{P}(\Gamma_k^{(\alpha)} \leq R) \leq \limsup_{R \to \infty} \frac{1}{q_R \log q_R} \log \left( \frac{R^k e^{-R}}{(k-1)!} \right) = \frac{x(2 - \gamma)}{\gamma}.
\]
Similarly, for the lower bound we use again the monotonicity to get
\[
\mathbb{P}(\Gamma_k \leq R) \geq \frac{(1 - \delta)R(\delta R)^{k-1}e^{-\delta R}}{(k-1)!}
\]
for an arbitrary small \( \delta > 0 \). Thus,
\[
\liminf_{R \to \infty} \frac{1}{q_R \log q_R} \mathbb{P}(\Gamma_k \leq R) \geq \frac{x(2 - \gamma)}{\gamma}.
\]
This finishes the proof of the superlarge deviation regime.

We now translate the above deviation results for \( \Gamma_k \) into results for \( \Xi_R \). To do so, notice that due to the Bernoulli structure we get
\[
\mathbb{P}(\Xi_R \geq xq_R) = \sum_{J \subseteq I} \prod_{k \in J} \mathbb{P}(\Gamma_k \leq R) \prod_{k \in I \setminus J} \mathbb{P}(\Gamma_k > R).
\]
The following lemma derives the size of the main term in the above sum.

**Lemma 4.2.** Recall that \( q_R = R^\pm \). For any \( x > 0 \) and \( 1 < \gamma < 2 \), it holds
\[
\lim_{R \to \infty} \frac{R}{q_R^2} \log \prod_{k=1}^{R + xq_R} \mathbb{P}(\Gamma_k \leq R) \prod_{k=R+xq_R+1}^{\infty} \mathbb{P}(\Gamma_k > R) = -\frac{x^3}{6}.
\]
For any \( x > 0 \) and \( \gamma = 2 \), it holds
\[
\lim_{R \to \infty} \frac{R}{q_R^2} \log \prod_{k=1}^{R + xq_R} \mathbb{P}(\Gamma_k \leq R) \prod_{k=R+xq_R+1}^{\infty} \mathbb{P}(\Gamma_k > R) = \frac{x(3x + 2)}{4} - \frac{(1 + x)^2}{2} \log(1 + x).
\]
For any \( x \in \mathbb{R}_+ \) and \( q_R = R^\pm \) for \( 2 < \gamma \), it holds
\[
\lim_{R \to \infty} \frac{1}{q_R^2} \log \prod_{k=1}^{R + xq_R} \mathbb{P}(\Gamma_k \leq R) \prod_{k=R+xq_R+1}^{\infty} \mathbb{P}(\Gamma_k > R) = \frac{x^2(2 - \gamma)}{2\gamma}.
\]

**Proof.** We prove all three cases simultaneously. To do so, denote by \( v_R^\gamma \) and \( J^\gamma \) the respective speed and rate function for different choices of \( \gamma \) as shown in Lemma 4.1 i.e.

\[
v_R^\gamma = \begin{cases} \frac{R}{q_R} & \text{for } 1 < \gamma < 2, \\ \frac{1}{q_R \log q_R} & \text{for } \gamma = 2, \\ \end{cases} \quad \text{and} \quad J^\gamma(x) = \begin{cases} -\frac{x^2}{2} & \text{for } 1 < \gamma < 2, \\ x - (1 + x) \log(1 + x) & \text{for } \gamma = 2, \\ \frac{x(2 - \gamma)}{2} & \text{for } 2 < \gamma. \end{cases}
\]

Notice that \( \mathbb{P}(\Gamma_k \leq R) \leq 0.7 \) for all \( k \geq R \). Thus, we can apply the bound \(|\log(1 - x)| \leq 2x|\) for \( x \leq 0.7 \) to \( \mathbb{P}(\Gamma_k \leq R) \). Recall the bounds from Lemma 3.1 With this we get that
\[
\left| \log \prod_{k=R+xq_R}^{\infty} \mathbb{P}(\Gamma_k > R) \right| \leq 2 \sum_{k=R}^{\infty} \mathbb{P}(\Gamma_k \leq R) \leq 6 \sum_{k=R}^{\infty} \sqrt{ke^{-R}} \frac{R^k}{k!} \leq 6 \sqrt{ek} e^{-R} \sum_{k=R+1}^{\infty} \frac{R^k}{k!} \leq 6 \sqrt{ek}.
\]
(4.1)
Thus,
\[
\lim_{R \to \infty} v_R^\gamma \log \prod_{k=R+xq_R}^{\infty} \mathbb{P}(\Gamma_k > R) = 0.
\]
Similarly, one shows that also
\[
\left| \log \prod_{k=1}^{n} \mathbb{P}(\Gamma_k \leq R) \right| \leq 6 \sqrt{R}
\]
and hence
\[
\lim_{R \to \infty} v_R^\gamma \log \prod_{k=1}^{R} \mathbb{P}(\Gamma_k \leq R) = 0.
\]
It thus remains to calculate \(\lim_{R \to \infty} v_R^\gamma \log \prod_{k=R+xq_R}^{R+1} \mathbb{P}(\Gamma_k \leq R)\). By Lemma 4.1 and the monotonicity \(\mathbb{P}(\Gamma_k \leq R) \geq \mathbb{P}(\Gamma_{k+1} \leq R)\) we now get
\[
\limsup_{R \to \infty} \frac{v_R^\gamma}{q_R} \sum_{k=R}^{R+bq_R} \log \mathbb{P}(\Gamma_k \leq R) = \limsup_{R \to \infty} v_R^\gamma (b-a) \log \mathbb{P}(\Gamma_{R+bq_R} \leq R) = (b-a)J^\gamma(a).
\]
Similarly,
\[
\liminf_{R \to \infty} \frac{v_R^\gamma}{q_R} \sum_{k=R}^{R+bq_R} \log \mathbb{P}(\Gamma_k \leq R) \geq \liminf_{R \to \infty} v_R^\gamma (b-a) \log \mathbb{P}(\Gamma_{R+bq_R-1} \leq R) = (b-a)J^\gamma(b).
\]
Finally apply a Riemann sum argument to obtain
\[
\frac{v_R^\gamma}{q_R} \sum_{k=R}^{R+xq_R} \log \mathbb{P}(\Gamma_k \leq R) \xrightarrow{R \to \infty} \int_0^x J^\gamma(y) \, dy.
\]
By calculating the integral on the right-hand side we obtain the claim. \(\square\)

**Proof of Proposition 1.3.** As before we treat the different cases for \(\gamma\) simultaneously. Denote by \(v_R^\gamma\) and \(J^\gamma\) the respective speed and rate function for the different choices of \(\gamma\) as shown in Lemma 4.2, i.e.
\[
v_R^\gamma = \begin{cases} \frac{R}{q_R} & \text{for } 1 < \gamma < 2, \\ \frac{1}{q_R \log q_R} & \text{for } 2 < \gamma, \end{cases}
\]
and
\[
J^\gamma(x) = \begin{cases} \frac{-x^3}{2(3x+2)} & \text{for } 1 < \gamma < 2, \\ \frac{1}{2}\log(1 + x) & \text{for } 2 < \gamma. \end{cases}
\]
By using that
\[
\mathbb{P}(\Xi_R \geq xq_R) = \sum_{|I| \geq R+xq_R} \prod_{k \in I} \mathbb{P}(\Gamma_k \leq R) \prod_{k \not\in I} \mathbb{P}(\Gamma_k > R)
\]
and together with Lemma 4.2 we immediately obtain
\[
\liminf_{R \to \infty} v_R^\gamma \log \mathbb{P}(\Xi_R \geq xq_R) \geq \liminf_{R \to \infty} v_R^\gamma \log \prod_{k=1}^{R+xq_R} \mathbb{P}(\Gamma_k \leq R) \prod_{k=R+xq_R+1}^{\infty} \mathbb{P}(\Gamma_k > R) = J^\gamma(x).
\]
Thus, it only remains to prove the upper bound. By Eqs. (4.1) and (4.2) it suffices further to prove
\[
\limsup_{R \to \infty} v_R^\gamma \log \sum_{|I| \geq R+xq_R} \prod_{k \in I} \mathbb{P}(\Gamma_k \leq R) \leq \limsup_{R \to \infty} v_R^\gamma \log \prod_{k=R}^{R+xq_R} \mathbb{P}(\Gamma_k \leq R).
\]
Let us start with the case that $R^2 = o(q_R)$, i.e. $\frac{4}{3} < \gamma$, as for the other case a more careful bound is needed. By splitting the sum according to the largest value $k$ appearing, we get the worst case bound

$$\sum_{|I| \geq R + xq_R} \prod_{k \in I} \mathbb{P}(\Gamma_k \leq R) \leq \prod_{k = R}^{R + xq_R} \mathbb{P}(\Gamma_k \leq R) \ast \sum_{l = R + xq_R}^{\infty} \mathbb{P}(\Gamma_{R + xq_R} \leq R)^{(l - R - xq_R)} \sum_{k = l}^{\infty} \left(\frac{k - 1}{l - 1}\right) \mathbb{P}(\Gamma_k \leq R) \mathbb{P}(\Gamma_l \leq R).$$

Now, applying the bounds in Lemma 3.1, we get further

$$\sum_{k = l}^{\infty} \left(\frac{k - 1}{l - 1}\right) \mathbb{P}(\Gamma_k \leq R) \mathbb{P}(\Gamma_l \leq R) \leq 3\sqrt{\frac{l}{k!}} = 3\sqrt{l} e^l R.$$

We thus obtain

$$\sum_{|I| \geq R + xq_R} \prod_{k \in I} \mathbb{P}(\Gamma_k \leq R) \leq \prod_{k = R}^{R + xq_R} \mathbb{P}(\Gamma_k \leq R) \sum_{l = R + xq_R}^{\infty} 3\sqrt{l} e^l \mathbb{P}(\Gamma_{R + xq_R} \leq R)^{(l - R - xq_R)}.$$

Finally, use that $\sqrt{l + R + xq_R} \leq l + \sqrt{q}$ together with a geometric sum to conclude that

$$\sum_{l = R + xq_R}^{\infty} 3\sqrt{l} e^l \mathbb{P}(\Gamma_{R + xq_R} \leq R)^{(l - R - xq_R)} \leq \frac{3\mathbb{P}(\Gamma_{R + xq_R} \leq R)}{(1 - \mathbb{P}(\Gamma_{R + xq_R} \leq R))^2} + \frac{3\sqrt{R + xq_R}}{1 - \mathbb{P}(\Gamma_{R + xq_R} \leq R)}.$$

Since $\mathbb{P}(\Gamma_{R + xq_R} \leq R) \to 0$ as $R \to \infty$, we can therefore bound the right-hand side by $C\sqrt{R + xq_R}$ for some constant $C > 0$ independent $R$. Hence, we showed that

$$\sum_{|I| \geq R + xq_R} \prod_{k \in I} \mathbb{P}(\Gamma_k \leq R) \leq C\sqrt{R + xq_R} e^R \prod_{k = R}^{R + xq_R} \mathbb{P}(\Gamma_k \leq R).$$

As we assumed $\frac{4}{3} < \gamma$, we obtain that $\lim_{R \to \infty} v_R^{\gamma} \log(C\sqrt{R + xq_R} e^R) = 0$. This proves the claim in case $\frac{4}{3} < \gamma$.

Let us now consider the remaining cases $\sqrt{R} = o(q_R)$ with $q_R = O(R^{\frac{2}{3}})$, i.e. $1 < \gamma < \frac{4}{3}$. To treat those, we need to bound the terms more carefully. Split the sum into the following parts:

$$S_1 = \sum_{I \subseteq \mathbb{N}, |I| \geq R + xq_R} \prod_{k \in I} \mathbb{P}(\Gamma_k \leq R),$$

$$S_2 = \sum_{I \subseteq \mathbb{N}, |I| \geq R + xq_R} \prod_{k \in I} \mathbb{P}(\Gamma_k \leq R),$$

$$S_3 = \sum_{I \subseteq \mathbb{N}, |I| \geq R + xq_R} \prod_{k \in I} \mathbb{P}(\Gamma_k \leq R),$$

where $0 \leq C_1 \leq C_2$ may depend on $R$ and $q_R$ and will be determined later in the proof.

Let us start by bounding $S_1$ and $S_3$. A worst case bound implies

$$S_1 \leq \prod_{k = 1}^{R + xq_R} \mathbb{P}(\Gamma_k \leq R) \sum_{l = R + xq_R}^{\infty} \mathbb{P}(\Gamma_{R + xq_R} \leq R)^{(l - R - xq_R)} \left(1 + C_1\right).$$

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We claim that we can choose
\[ C_1 = \left\lfloor \frac{1}{v_R^2 (\log R)^2} \right\rfloor \]
and still obtain the needed bound on the right-hand side. Indeed, notice that by Stirling’s formula
\[ \left( \frac{l + C_1}{l} \right) \leq \left( \frac{l + C_1}{C_1} \right)^{C_1} \leq 2 \left( \frac{l + C_1}{C_1} \right)^{C_1} \]
as \((\frac{x+y}{y})^y \geq (\frac{x+y}{y})^y\) for all 0 < x ≤ y and l ≥ R + xqR while C_1 = o(R) due to the choice of qR. Bound further
\[ \left( \frac{l + C_1}{C_1} \right)^{C_1} \leq (2l)^{C_1} \leq (2(R + xqR))^{C_1} e^{l-R-xqR} \]
to conclude
\[ S_1 \leq (2(R + xqR))^{C_1} \prod_{k=R}^{R+xqR} \mathbb{P}(\Gamma_k \leq R) \sum_{l=R+xqR}^{\infty} (e \mathbb{P}(\Gamma_{R+xqR} \leq R))^{l-R-xqR}. \]

By the choice of C_1,
\[ \limsup_{R \to \infty} v_R^2 C_1 \log(4(R + xqR)C_1) \leq 0 \]
and the sum \(\sum_{l=R+xqR}^{\infty} (e \mathbb{P}(\Gamma_{R+xqR} \leq R))^{l-R-xqR}\) converges to zero. Therefore, we proved that
\[ \limsup_{R \to \infty} v_R^2 \log S_1 \leq \limsup_{R \to \infty} v_R^2 \log \prod_{k=R}^{R+xqR} \mathbb{P}(\Gamma_k \leq R). \]

Let us next consider S_3. We show that we can choose \(C_2 = [eR]\) to conclude
\[ \limsup_{R \to \infty} v_R^2 S_2 \leq \limsup_{R \to \infty} v_R^2 \log \prod_{k=R}^{R+xqR} \mathbb{P}(\Gamma_k \leq R). \]

Indeed, following the same idea as in the case \(R^2 = o(q_R)\) we get
\[ S_3 \leq \prod_{k=R}^{R+xqR} \mathbb{P}(\Gamma_k \leq R) \sum_{l=R+xqR}^{\infty} \mathbb{P}(\Gamma_{R+xqR} \leq R)^{l-R-xqR} \sum_{k=l+C_2}^{\infty} \left( \frac{k-1}{l-1} \right) \frac{\mathbb{P}(\Gamma_k \leq R)}{\mathbb{P}(\Gamma_l \leq R)} \]
\[ \leq 3 \prod_{k=R}^{R+xqR} \mathbb{P}(\Gamma_k \leq R) \sum_{l=R+xqR}^{\infty} \sqrt{l} \frac{\mathbb{P}(\Gamma_{R+xqR} \leq R)}{\sqrt{l}} C_2 \sum_{k=C_2}^{\infty} \frac{R^k}{k!}. \]

Now, by using again the bounds in Lemma 3.1 we obtain
\[ \sum_{k=C_2}^{\infty} \frac{R^k}{k!} = eR \mathbb{P}(P_R \geq C_2) = e\mathbb{E} \mathbb{P}(\Gamma_{C_2} \leq R) \leq e C_2 \left( \frac{R}{C_2} \right)^{C_2} \leq 1 \]
for our choice of C_2, where \(P_R\) denotes a Poisson-distributed random variable with parameter \(R\). Therefore, we conclude as before that
\[ \limsup_{R \to \infty} v_R^2 S_3 \leq \limsup_{R \to \infty} v_R^2 \log \prod_{k=R}^{R+xqR} \mathbb{P}(\Gamma_k \leq R). \]

For the remaining term notice that
\[ S_2 \leq \prod_{k=R}^{R+xqR} \mathbb{P}(\Gamma_k \leq R) \]
\[ \sum_{l=R+xqR}^{\infty} \mathbb{P}(\Gamma_{R+xqR} \leq R)^{l-R-xqR} \sum_{k=1}^{l} \frac{(l + C_1)}{(l-k)} \left( \frac{C_2 - C_1}{k} \right) \left( \frac{\mathbb{P}(\Gamma_{R+xqR} \leq R)}{\mathbb{P}(\Gamma_{R+xqR} \leq R)} \right)^{k}. \]
Next, use that
\[
\binom{l+C_1}{l-k}\binom{C_2-C_1}{k} \leq C_2^k \binom{l+C_1}{l}
\]
and that
\[
\frac{\mathbb{P}(\Gamma_{R+2R} \leq R)}{\mathbb{P}(\Gamma_{R+2R} \leq R)} \leq \frac{3\sqrt{R+xQR + C_1}}{\mathbb{P}(\Gamma_{R+2R} \leq R)}
\]
which leads to the weak convergence of the point processes.

Proof. Let \( \xi \) be a point process. Set
\[
S_1 = \prod_{R \rightarrow \infty} \mathbb{P}(\Gamma_{R} \leq R) \sum_{l=R-xQR}^{R+2R} \binom{l+C_1}{l} \binom{l}{k} \left( \frac{C_2 \mathbb{P}(\Gamma_{R+2R} \leq R)}{\mathbb{P}(\Gamma_{R+2R} \leq R)} \right)^k R^{+xQR}
\]
and
\[
S_2 = \prod_{R \rightarrow \infty} \mathbb{P}(\Gamma_{R} \leq R) \sum_{l=R-xQR}^{R+2R} \binom{l+C_1}{l} \binom{l}{k} \left( \frac{C_2 \mathbb{P}(\Gamma_{R+2R} \leq R)}{\mathbb{P}(\Gamma_{R+2R} \leq R)} \right)^k R^{+xQR}
\]
By the choice of \( C_1 \), the same argument as for \( S_1 \) shows that
\[
\lim_{R \rightarrow \infty} \sup_{l=R-xQR} \mathbb{P}(\Gamma_{R} \leq R) \sum_{l=R-xQR}^{R+2R} \binom{l+C_1}{l} \left( 2\frac{\mathbb{P}(\Gamma_{R+2R} \leq R)}{\mathbb{P}(\Gamma_{R+2R} \leq R)} \right)^{R-xQR} \leq 0.
\]
For the other term notice that
\[
\lim_{R \rightarrow \infty} \left( 1 + \frac{C_2 \mathbb{P}(\Gamma_{R+2R} \leq R)}{\mathbb{P}(\Gamma_{R+2R} \leq R)} \right) R^{+xQR} = 0
\]
as \( \frac{C_2 \mathbb{P}(\Gamma_{R+2R} \leq R)}{\mathbb{P}(\Gamma_{R+2R} \leq R)} = o\left( \frac{1}{R^{+xQR}} \right) \). This proves that
\[
\lim_{R \rightarrow \infty} \sup_{l=R-xQR} \mathbb{P}(\Gamma_{R} \leq R) \prod_{R \rightarrow \infty} \mathbb{P}(\Gamma_{R} \leq R).
\]
Combining this with the lower bound for the limes inferior yields the claim.

\[\square\]

A. Proof of Lemma 1.9

**Lemma A.1.** Let \( \xi \) and \( \xi \) be point processes associated to the correlation kernels \( K_n(z,w) = \sum_{k=-n}^{n} \rho_k(|z|) z^k \rho_k(|w|) w^k \) and \( K_n(z,w) = \sum_{k=-n}^{\infty} \rho_k(|z|) z^k \rho_k(|w|) w^k \), respectively, with respect to the same measure \( \mu \). If the 1-point correlation functions \( K_n(z,z) \) converge to the 1-point correlation function \( K(z,z) \) locally uniformly, then the point processes \( \xi_n \) converge weakly to the point process \( \xi \).

**Proof.** By Cauchy-Schwartz' inequality, the locally uniform convergence of the 1-point correlation functions implies the locally uniform convergence of the kernels itself. This immediately implies the weak convergence of the point processes.

**Proof of Lemma 1.9** Decompose the rotation symmetric Radon measure \( \mu \) into its radial part \( \mu^r \) and its angular part \( d\theta \). Let
\[
K_n(z,w) = \sum_{k=-n}^{n} \rho_k(|z|) z^k \rho_k(|w|) w^k
\]
for $z,w \in \mathbb{C}$. By Lemma A.1 we conclude that the determinantal point process associated with the kernel $K_n$ converge weakly to the one associated to $K$. Denote the set of points corresponding to the kernel $K_n$ by $(Z_k^{(n)})_{k=-n,\ldots,n}$. Recall that if $X_k$ are independent random variables with density $f_X$ for $k=-n,\ldots,n$ then the joint intensity of $\{X_k: k=-n,\ldots,n\}$ at $x_{-n},\ldots,x_n$ is given by

$$\sum_{\tau \in S_{2n+1}} \prod_{k=-n}^n f_{\tau(k)}(x_k),$$

where $S_{2n+1}$ denotes the symmetric group of $2n+1$ elements. By the above weak convergence it thus suffices to prove that the points $|Z_k^{(n)}|$ for $k=-n,\ldots,n$ have joint intensity

$$(2\pi)^{2n+1} \sum_{\tau \in S_{2n+1}} \prod_{k=-n}^n \rho_{\tau(k)}(r_k)^2 r_k^{2\tau(k)+1}$$

with respect to the radial measure $\bar{\mu}$. Notice that

$$\left( K_n(z_{-n},z_{-n}) \cdots K_n(z_{-n},z_n) \\
\vdots \quad \vdots \\
K_n(z_n,z_{-n}) \cdots K_n(z_n,z_n) \right)$$

$$= \left( \begin{array}{ccc} \rho_{-n}(|z_{-n}|)z_{-n}^{-n} & \cdots & \rho_{-n}(|z_{-n}|)z_{-n}^n \\
\vdots & \ddots & \vdots \\
\rho_{-n}(|z_n|)z_n^{-n} & \cdots & \rho_{-n}(|z_n|)z_n^n \end{array} \right).$$

Hence, the joint intensity of $|Z_k^{(n)}|$ for $k=-n,\ldots,n$ at $r_{-n},\ldots,r_n$ is given by

$$\int_{[0,2\pi]^{2n+1}} \det \left( K_n(r_{-j}e^{i\theta_j},r_{-j}e^{i\theta_j}) \right) r_{-n} \cdots r_n d\theta_{-n} \cdots d\theta_n$$

$$= \int_{[0,2\pi]^{2n+1}} \left( \sum_{\tau \in S_{2n+1}} \sgn(\tau) \prod_{j=-n}^n \rho_{\tau(j)}(r_j)^2 e^{i\theta_j} \right)$$

$$\left( \sum_{\tau \in S_{2n+1}} \sgn(\tau) \prod_{k=-n}^n \rho_{\tau(k)}(r_k)^2 e^{-i\theta_k} \right) r_{-n} \cdots r_n d\theta_{-n} \cdots d\theta_n$$

$$= \sum_{\tau,\tau' \in S_{2n+1}} \prod_{k=-n}^n \sgn(\tau) \sgn(\tau') \rho_{\tau(k)}(r_k)^2 r_k^{\tau(k)+\tau'(k)+1} \int_{[0,2\pi]} e^{i\theta_k} \left( \tau(k)-\tau'(k) \right) d\theta_k.$$

Notice that the integral in the last formula equals zero as soon as $\tau \neq \hat{\tau}$. Hence, we conclude that the joint intensity indeed equals

$$\int_{[0,2\pi]^{2n+1}} \det \left( K_n(r_{-j}e^{i\theta_j},r_{-j}e^{i\theta_j}) \right) r_{-n} \cdots r_n d\theta_{-n} \cdots d\theta_n$$

$$= (2\pi)^{2n+1} \sum_{\tau \in S_{2n+1}} \prod_{k=-n}^n \rho_{\tau(k)}(r_k)^2 r_k^{2\tau(k)+1}.$$

This concludes the proof of the theorem. \qed

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