A RANDOM VARIABLE RELATED TO THE HURWITZ ZETA-FUNCTION WITH ALGEBRAIC PARAMETER

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Abstract. In this paper, we introduce a certain random variable closely related to the value-distribution of the Hurwitz zeta-function with algebraic parameter. We prove a version of the limit theorem, where the limit measure is presented by the law of this random variable. Then we apply it to show that any complex number can be approximated by values of the Hurwitz zeta-function for quadratic irrational parameters but with finite exceptions.

1. Introduction

Let \( s = \sigma + it \) be a complex variable. For a real number \( \alpha \) satisfying \( 0 < \alpha \leq 1 \), the Hurwitz zeta-function \( \zeta(s, \alpha) \) is defined as

\[
\zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n + \alpha)^s}.
\]

This series is convergent absolutely for \( \sigma > 1 \). The Riemann zeta-function \( \zeta(s) \) is a special case of the Hurwitz zeta-function. Indeed, we have obviously \( \zeta(s, 1) = \zeta(s) \). Some properties on \( \zeta(s) \) are generalized to \( \zeta(s, \alpha) \) for every \( \alpha \). For example, \( \zeta(s, \alpha) \) can be continued to a holomorphic function on \( \mathbb{C} \) except only for a simple pole at \( s = 1 \). On the other hand, Davenport–Heilbronn [5] proved that \( \zeta(s, \alpha) \) has infinitely many zeros for \( \sigma > 1 \) if \( \alpha \neq 1/2 \) or 1 is rational or transcendental, while \( \zeta(s) \) has no zeros for \( \sigma > 1 \) due to the Euler product representation

\[
\zeta(s) = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1},
\]

where \( p \) runs through all prime numbers. Then, the absence of the Euler product for the Hurwitz zeta-function causes significant differences in methods for investigating the value-distributions of \( \zeta(s) \) and \( \zeta(s, \alpha) \).

Let \( X(p) \) be random variables indexed by prime numbers \( p \), which are independent and uniformly distributed on the unit circle of the complex plane. According to (1.2), we define the random variable \( \zeta(\sigma, X) \) as

\[
\zeta(\sigma, X) = \prod_p \left( 1 - \frac{X(p)}{p^\sigma} \right)^{-1}.
\]

This infinite product is convergent for \( \sigma > 1/2 \) almost surely. Due to the unique factorization of the integers, the complex numbers \( p^{-it} \) for \( t \in \mathbb{R} \) behave as if they are independent random variables. Therefore we expect that the value-distribution of \( \zeta(\sigma + it) \) in the \( t \)-aspect is approximated by using \( \zeta(\sigma, X) \). In fact, we have the

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following limit theorem proved essentially by Jessen–Wintner \cite{8}. Let $\sigma > 1/2$ be a fixed real number. Then the probability measure
\[
P_{\sigma,T}(A) = \frac{1}{T} \operatorname{meas} \{ t \in [0,T] \mid \zeta(\sigma + it) \in A \}, \quad A \in \mathcal{B}(\mathbb{C})
\]
converges weakly as $T \to \infty$ to the law of $\zeta(\sigma,X)$. Here, $\operatorname{meas} S$ stands for the one-dimensional Lebesgue measure of a set $S$, and $\mathcal{B}(\mathbb{C})$ is the algebra of Borel sets of $\mathbb{C}$. See Laurinčikas \cite{12} for more information about the limit theorem for $\zeta(s)$.

For the Hurwitz zeta-function, we consider another random variable to obtain the limit theorem. For $0 < \alpha \leq 1$, we define
\[
\zeta(\sigma,\mathcal{X}_\alpha) = \sum_{n=0}^{\infty} \frac{\mathcal{X}_\alpha(n)}{(n + \alpha)^\sigma},
\]
where $\mathcal{X}_\alpha(n)$ are some random variables indexed by integers $n \geq 0$. Our first purpose is to choose $\mathcal{X}_\alpha(n)$ suitably and to show that the probability measure
\[
P_{\sigma,\alpha,T}(A) = \frac{1}{T} \operatorname{meas} \{ t \in [0,T] \mid \zeta(\sigma + it, \alpha) \in A \}, \quad A \in \mathcal{B}(\mathbb{C})
\]
converges weakly as $T \to \infty$ to the law of the random variable $\zeta(\sigma,\mathcal{X}_\alpha)$ as in \cite{13}. This is already achieved for the case in which $\alpha$ is a transcendental number. In this case, we choose $\mathcal{X}_\alpha(n)$ as independent random variables uniformly distributed on the unit circle of $\mathbb{C}$. Then we obtain the desired limit theorem for $\zeta(s,\alpha)$ as a consequence of \cite{8} Theorem 29). Here, we remark that it is necessary for the proof to use the fact that the real numbers $\log(n + \alpha)$ are linearly independent over $\mathbb{Q}$ if $\alpha$ is transcendental. The linear independence of $\log(n + \alpha)$ is not necessary valid for the case where $\alpha$ is algebraic. For example, Dubickas \cite{6} proved that the equation
\[
(x_1 + \alpha)(x_2 + \alpha)(x_3 + \alpha) = (u + \alpha)(v + \alpha)
\]
has a solution in positive integers $x_1, x_2, x_3, u, v \geq n_0$ for any $n_0 \in \mathbb{Z}_{>0}$ if $\alpha$ is an algebraic integer of degree $2$, which shows the $\mathbb{Q}$-linear dependence of the real numbers $\log(n + \alpha)$. Therefore we must change the choice of $\mathcal{X}_\alpha(n)$ to obtain the result for an algebraic parameter $\alpha$. The first main result provides suitable random variables $\mathcal{X}_\alpha(n)$.

**Theorem 1.1.** Let $\alpha$ be an algebraic number satisfying $0 < \alpha \leq 1$. Then there exist random variables $\mathcal{X}_\alpha(n)$ for $n \geq 0$ which are distributed on the unit circle, and
\[
\mathbb{E} [\mathcal{X}_\alpha(n_1)^{e_1} \cdots \mathcal{X}_\alpha(n_k)^{e_k}] = \begin{cases} 1 & \text{if } (n_1 + \alpha)^{e_1} \cdots (n_k + \alpha)^{e_k} = 1, \\ 0 & \text{otherwise} \end{cases}
\]
is satisfied for any integers $n_1, \ldots, n_k \geq 0$ and $e_1, \ldots, e_k \in \mathbb{Z}$.

The precise construction of $\mathcal{X}_\alpha(n)$ is explained in Section \cite{2}. Note that we have $\mathcal{X}_\alpha(n)^{-1} = \mathcal{X}_\alpha(n)^{-1}$ almost surely since $\mathcal{X}_\alpha(n)$ are distributed on the unit circle. Therefore the random variables $\mathcal{X}_\alpha(n)$ of Theorem \cite{1} are orthonormal in the sense that the following condition is satisfied:
\[
\mathbb{E} [\mathcal{X}_\alpha(n) \overline{\mathcal{X}_\alpha(n)}] = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise}. \end{cases}
\]
Then we apply a version of the Menshov–Rademacher theorem [10, Theorem B.10.5] to see that (1.3) is convergent for $\sigma > 1/2$ almost surely due to
$$\sum_{n=0}^{\infty} \frac{(\log n)^2}{(n+\alpha)^{2\sigma}} < \infty.$$ 
With the above choice of $X_\alpha(n)$, we prove the following limit theorem for $\zeta(s, \alpha)$.

**Theorem 1.2.** Let $\alpha$ be an algebraic number satisfying $0 < \alpha \leq 1$, and let $X_\alpha(n)$ be random variables as in Theorem 1.1. Then, for any fixed real number $\sigma > 1/2$, the probability measure $P_{\sigma, \alpha, T}$ defined as (1.3) converges weakly as $T \to \infty$ to the law of $\zeta(\sigma, X_\alpha)$.

Let $\sigma$ be a fixed real number with $1/2 < \sigma \leq 1$. Bohr–Courant [2] proved that the set $\{\zeta(\sigma + it) \mid t \in \mathbb{R}\}$ is dense in $\mathbb{C}$. Furthermore, this denseness result was generalized to the Hurwitz zeta-function $\zeta(s, \alpha)$ by Gonek [7] if $\alpha$ is rational or transcendental. More precisely, it holds that

$$\lim \inf_{t \to \infty} \frac{1}{T} \text{meas}\{t \in [0, T] \mid |\zeta(\sigma + it, \alpha) - z_0| < \epsilon\} > 0 \tag{1.6}$$

for any $z_0 \in \mathbb{C}$ and $\epsilon > 0$ for such $\alpha$. Gonek derived his result as a consequence of the so-called universality theorem for $\zeta(s, \alpha)$. He further conjectured in [7] that the universality theorem remains true if $\alpha$ is an algebraic irrational number. Then we also believe that (1.6) is true for any algebraic irrational $\alpha$, but it is still open. Some evidence to this conjecture was recently presented by Lee–Mishou [14] and Sourmelidis–Steuding [16]. In this paper, we prove the following result which also supports the truth of (1.6) in the case of quadratic irrational parameters.

**Theorem 1.3.** Let $0 < c < 1$ be a real number, and let $d$ be a positive non-square integer. Denote by $\mathcal{A}_{c,d}$ the set of all quadratic irrational numbers $\alpha = (b \pm \sqrt{d})/a$ with $a, b \in \mathbb{Z}_{>0}$ such that $c < (b - \sqrt{d})/a < (b + \sqrt{d})/a < 1$ is satisfied. Let $\sigma$ be a fixed real number with $1/2 < \sigma < 1$. Then, for any $z_0 \in \mathbb{C}$ and $\epsilon > 0$, there exists a finite subset $\mathcal{E}_{c,d} = \mathcal{E}_{c,d}(\sigma, z_0, \epsilon) \subset \mathcal{A}_{c,d}$ such that (1.4) holds for any $\alpha \in \mathcal{A}_{c,d} \setminus \mathcal{E}_{c,d}$.

Note that $\mathcal{A}_{c,d}$ contains infinitely many quadratic irrationals. Unfortunately, the exceptional subset $\mathcal{E}_{c,d}$ depends on $z_0$ and $\epsilon$. This prevents us from proving completely that the set $\{\zeta(\sigma + it, \alpha) \mid t \in \mathbb{R}\}$ is dense in $\mathbb{C}$. One can just deduce from Theorem 1.3 that $\{\zeta(\sigma + it, \alpha) \mid t \in \mathbb{R}, \alpha \in \mathcal{A}_{c,d}\}$ is dense for any $c, d$. This is far from expected, but it makes the first progress toward Gonek’s conjecture by the probabilistic approach.

**Remark 1.4.** The first version of the limit theorem for $\zeta(s, \alpha)$ containing the case of algebraic irrational parameters was established by Laurinčikas [11]. Let $\sigma > 1/2$ be a fixed real number. Then he proved that the probability measure $P_{\sigma, \alpha, T}$ defined as (1.3) converges weakly to some probability measure for any $0 < \alpha \leq 1$. However, this result did not present the limit measure explicitly. Laurinčikas–Steuding [13] attempted to present the limit measure in terms of random variables for an algebraic irrational number $\alpha$. Let $L(\alpha) = \{\log(n + \alpha) \mid n \in \mathbb{Z}_{\geq 0}\}$. Then we fix a maximal $\mathbb{Q}$-linearly independent subset of $L(\alpha)$ and denote it by $I(\alpha)$. Put

$$\mathcal{M}(\alpha) = \{m \in \mathbb{Z}_{\geq 0} \mid \log(m + \alpha) \in I(\alpha)\},$$

$$\mathcal{N}(\alpha) = \{n \in \mathbb{Z}_{\geq 0} \mid \log(n + \alpha) \notin I(\alpha)\}.$$
Then we define random variables $X_\alpha(m)$ for $m \in \mathcal{M}(\alpha)$ and $X_\alpha(n)$ for $n \in \mathcal{N}(\alpha)$ as follows. Let $X_\alpha(m)$ be independent random variables for $m \in \mathcal{M}(\alpha)$ which are uniformly distributed on the unit circle. For $n \in \mathcal{N}(\alpha)$, there exist elements $m_j \in \mathcal{M}(\alpha)$ and $r_j \in \mathbb{Q}$ for $j = 1, \ldots, k$ such that

\begin{equation}
\log(n + \alpha) = r_1 \log(m_1 + \alpha) + \cdots + r_k \log(m_k + \alpha)
\end{equation}

since $\mathcal{M}(\alpha) \cup \{\log(n + \alpha)\}$ is linearly dependent over $\mathbb{Q}$. It implies that

$$n + \alpha = (m_1 + \alpha)^{r_1} \cdots (m_k + \alpha)^{r_k},$$

and according to this identity, we define $X_\alpha(n)$ for $n \in \mathcal{N}(\alpha)$ as

\begin{equation}
X_\alpha(n) = X_\alpha(m_1)^{r_1} \cdots X_\alpha(m_k)^{r_k}
\end{equation}

with the principal values of the roots. Laurinčikas–Steuding [13, Theorem 1] claimed that the probability measure $P_{\sigma,\alpha,T}$ converges weakly as $T \to \infty$ to the law of the random variable $\zeta(\sigma, X_\alpha)$ defined as ($\sigma,\alpha$). However, it appears that the proof has a serious gap. They actually used the formula

$$(n + \alpha)^it = ((m_1 + \alpha)^{r_1} \cdots (m_k + \alpha)^{r_k})^it = ((m_1 + \alpha)^{it})^{r_1} \cdots ((m_k + \alpha)^{it})^{r_k}$$

for all $t \in \mathbb{R}$; see [13, p. 423, 1.2]. Remark that the second equality fails when the rational numbers $r_j$ are not integers since they used the principal values of the roots, e.g. for any $\theta \in (\pi, 2\pi)$ we have $\theta - 2\pi \in (-\pi, \pi)$ and

$$(e^{i\theta})^{1/2} = (e^{i(\theta - 2\pi)})^{1/2} = e^{i(\theta - 2\pi)/2} = - (e^{1/2})^{i\theta} \neq (e^{1/2})^{i\theta}.$$ 

If one can confirm that all rational numbers $r_j$ in equation (1.7) are integers for any $n \in \mathcal{N}(\alpha)$, then such a trouble does not occur and the proof of [13] will make sense. However, it seems quite difficult to show the integrality for $r_j$. The present paper avoids using the integrality for $r_j$, and we present another construction of the random variables $X_\alpha(n)$. Then Theorem 1.2 is the first result on the limit theorem for $\zeta(s, \alpha)$ with algebraic parameter $\alpha$ whose limit measure is explicitly presented in terms of random variables.

Remark 1.5. Theorem 1.3 is similar to the result by Sourmelidis–Steuding [16]. The advantage of their result is that it is an effective result, which means that one can find effectively a real number $T > 0$ such that $|\zeta(\sigma + it, \alpha) - z_0| < \epsilon$ holds with some $t \in [T, 2T]$. Furthermore, they proved a weak form of the universality theorem for $\zeta(s, \alpha)$ for an algebraic irrational parameter $\alpha$ as a consequence of joint denseness results for $\zeta(s, \alpha)$ and its derivatives. As remarked in [16, Section 1], their results have meaning only when $\sigma > 1 - \xi$ with $\xi \approx 0.00186$. Theorem 1.3 of the present paper is not an effective result, but it has an advantage that we do not need such restriction for $\sigma$.

Remark 1.6. Another difference between Theorem 1.3 of this paper and the result of Sourmelidis–Steuding is that we consider only quadratic irrational parameters to show 1.6, while in 1.6 they dealt with algebraic irrational numbers of arbitrary degree. The reason why we restrict the parameter $\alpha$ in Theorem 1.3 to the form $\alpha = (b \pm \sqrt{d})/a$ is coming from Lemma 1.2 which is a variant of the following assertion of Cassels [4]. Let $\alpha$ be an algebraic irrational number, and denote by $\alpha$
the ideal denominator of \( \alpha \) in the algebraic field \( K = \mathbb{Q}(\alpha) \). Then \((n + \alpha)a\) is an integral ideal of \( K \) for any rational integer \( n \). We write its ideal decomposition as
\[
(n + \alpha)a = b \prod_p p^{a(p)},
\]
where \( p \) runs through all prime ideals of \( K \) with the following properties: (i) \( p \) is of the first degree and unambiguous, i.e. \( N(p) = p \) and \( p^2 \nmid (p) \) for a rational prime \( p \),
(ii) for any integer \( m \), if \( p \mid (m + \alpha)a \) then \( p' \nmid (m + \alpha)a \) for any prime ideal \( p' \neq p \) with \( N(p') = p \). Hence the ideal \( b \) contains all prime factors of \((n + \alpha)a\) that do not satisfy (i) or (ii). In [4, Section 2], Cassels claimed that the norm of \( b \) is bounded for \( n \in \mathbb{Z} \). However, we have a counterexample to this boundedness. Let \( \alpha = 7\sqrt{2} \). Then we have \( K = \mathbb{Q}(\sqrt{2}) \) and \( a = (1) \). Note that \( p_1 = (3 + \sqrt{2}) \) and \( p_2 = (3 - \sqrt{2}) \) are prime ideals of \( K \) satisfying \( N(p_1) = N(p_2) = 7 \) and \( (7) = p_1p_2 \). For any \( k \in \mathbb{Z} \), we have
\[
(7k + \alpha) = p_1p_2(k + \sqrt{2}).
\]
It implies that the above condition (ii) is not satisfied for \( p_1 \) and \( p_2 \). Therefore, they are prime factors of the ideal \( b \) of (1.9) for \( n = 7k \). The norm of \((k + \sqrt{2})\) is equal to \( k^2 - 2 \). Furthermore, a simple induction shows that, for any \( v \geq 0 \) there exists an integer \( k_v \) such that \( k_v^2 - 2 \) is divisible by \( 7^v \). Hence \((k_v + \sqrt{2})\) is divisible by either \( p_1^v \) or \( p_2^v \) for any \( v \geq 0 \). As a result, either one of
\[
p_1^v \mid b \quad \text{and} \quad p_2^v \mid b
\]
is valid for \( n = 7k_v \), and we have \( N(b) \geq 7^v \) in either case. Therefore \( N(b) \) is unbounded as \( n \) varies over rational integers. In the present paper, we slightly modify the way of decomposition (1.9) in the case where \( \alpha \) is a quadratic irrational number. Then we obtain a similar boundedness for an ideal in this decomposition; see Lemma 1.2. In this way, we can correct the proof of Cassels [4] in the quadratic case.

**Organization of the paper.** This paper consists of six sections.

- **Theorem 1.1** is proved in Section 2. We construct certain random variables \( X_\alpha(n) \) such that condition (1.5) is satisfied. Then we prove several properties of \( X_\alpha(n) \), which are used in the proof of Theorems 1.2 and 1.3.
- **Theorem 1.2** is proved in Section 3. A key step in the proof is to associate the complex numbers \((n_1 + \alpha)^{-t}, \ldots, (n_k + \alpha)^{-t}\) for \( t \in \mathbb{R} \) with the random variables \( X_\alpha(n_1), \ldots, X_\alpha(n_k) \) by applying (1.5).
- The next three sections are devoted to the proof of Theorem 1.3. The purpose of Section 4 is to show a variant of the Cassels lemma in [4], which asserts that at least 51 percent of elements in \( L(\alpha) = \{ \log(n + \alpha) \mid n \in \mathbb{Z}_{\geq 0} \} \) are linearly independent over \( \mathbb{Q} \) in the sense of density.
- In Section 5, we collect preliminary results on the Beurling–Selberg function which presents a nice approximation of the signum function \( \text{sgn}(x) \). Then we apply it to estimate some indicator functions of sets used later.
- Finally, the proof of Theorem 1.3 is completed in Section 6. To show (1.6), we apply Theorem 1.2 and the Cassels lemma proved in Section 4. The method is partially similar to that of Sourmelidis–Steuding [10].
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2. Certain random variables

Throughout this paper, a random variable means a measurable map \( X : \Omega \rightarrow \mathbb{C} \) from a probability space \( (\Omega, \mathcal{F}, P) \) to the Borel measurable space \( (\mathbb{C}, \mathcal{B}(\mathbb{C})) \). Then we denote the expectation of a random variable \( X \) by

\[
E[X] = \int_{\Omega} X(\omega) \, dP(\omega)
\]

as usual. Furthermore, the law of a random variable \( X \) is the measure on \( (\mathbb{C}, \mathcal{B}(\mathbb{C})) \) defined as

\[
P(\{\omega \in \Omega \mid X(\omega) \in A\})
\]

for \( A \in \mathcal{B}(\mathbb{C}) \). Let \( \Lambda \) be a countable set. Then it is well-known that there exist independent random variables \( X(\lambda) \) indexed by \( \lambda \in \Lambda \) which are uniformly distributed on the unit circle on \( \mathbb{C} \), i.e. the law of every \( X(\lambda) \) satisfies

\[
P(\lambda(\lambda) \in A(s, t)) = \frac{t - s}{2\pi}
\]

for any \( 0 < t - s \leq 2\pi \), where \( A(s, t) \) is the arc of the unit circle such that

\[
(2.1)
\]

Let \( \alpha \) be an algebraic number satisfying \( 0 < \alpha \leq 1 \). Then we define random variables \( X_\alpha(n) \) indexed by integers \( n \geq 0 \) as follows. Let \( K = \mathbb{Q}(\alpha) \) and denote by \( \mathcal{O}_K \) the ring of integers of \( K \). For any \( y \in K \) with \( y > 0 \), the fractional principal ideal \( (y) \) has the decomposition

\[
(2.2)
\]

where \( p_j \) are prime ideals of \( K \), and \( a_j \) are rational integers uniquely determined by \( y \). Let \( h \) be the class number of \( K \). Since \( a_h \) is a principal ideal for any ideal \( \mathfrak{a} \) of \( K \), we know that the set

\[
S_p = \{ \varpi_p \in \mathcal{O}_K \mid \mathfrak{p}^{h} = (\varpi_p) \}
\]

is non-empty for any prime ideal \( \mathfrak{p} \). Then we take an element \( \varpi = (\varpi_p)_p \in \prod_p S_p \), where \( p \) runs through all prime ideals of \( K \). By (2.2), we obtain

\[
(2.3)
\]

which yields the formula

for some \( u \in \mathcal{O}_K^\times \). Here, the unit \( u \) is uniquely determined only by \( y \) if the choice of \( \varpi = (\varpi_p)_p \) is fixed. Furthermore, we can choose \( \varpi \) so that every \( \varpi_p \) is positive. Then we see that \( u \) is positive due to \( y^h > 0 \). Let \( \mathcal{U} = (u_1, \ldots, u_d) \) be a fundamental system of units of \( \mathcal{O}_K \). We can also choose \( \mathcal{U} \) so that every \( u_j \) is positive. With the above choices of \( \varpi \) and \( \mathcal{U} \), we obtain

\[
(2.3)
\]
where \( b_j \) are rational integers uniquely determined by \( y \). Let \( \Lambda \) denote
\[
\Lambda = \{ \varpi_p \mid p \text{ is a prime ideal of } K \} \cup \{ u_1, u_2, \ldots, u_d \}.
\]
(2.4)

Using the expression of \( y^h \) as in (2.3), we define \( \text{ord}(y, \lambda) \) for \( \lambda \in \Lambda \) as
\[
\text{ord}(y, \lambda) = \begin{cases} 
  a_j & \text{if } \lambda = \varpi_p^j \text{ for some } \varpi_p \text{ in (2.3)}, \\
  b_j & \text{if } \lambda = u_j \text{ for some } u_j \text{ in (2.3)}, \\
  0 & \text{otherwise}.
\end{cases}
\]
(2.5)

By the uniqueness of \( a_j \) and \( b_j \), we have the formula
\[
\text{ord}(y_1 y_2, \lambda) = \text{ord}(y_1, \lambda) + \text{ord}(y_2, \lambda)
\]
for any \( y_1, y_2 \in K \) with \( y_1, y_2 > 0 \) and \( \lambda \in \Lambda \). Let \( \mathcal{X}(\lambda) \) be independent random variables for \( \lambda \in \Lambda \) uniformly distributed on the unit circle. We finally define \( \mathcal{X}_\alpha(n) \) for \( n \geq 0 \) as
\[
\mathcal{X}_\alpha(n) = \prod_{\lambda \in \Lambda} \mathcal{X}(\lambda)^{\text{ord}(n+\alpha, \lambda)}.
\]
(2.6)

In the following, we always denote by \( \mathcal{X}_\alpha(n) \) the random variables defined in the above way. First, we show that they fulfill the desired condition of Theorem 1.1.

**Proof of Theorem 1.1.** By definition, the random variables \( \mathcal{X}_\alpha(n) \) are distributed on the unit circle. Thus we show below that condition (1.5) is satisfied for any integers \( n_1, \ldots, n_k \geq 0 \) and \( e_1, \ldots, e_k \in \mathbb{Z} \). Put
\[
y = (n_1 + \alpha)^{e_1} \cdots (n_k + \alpha)^{e_k}.
\]
Then we have \( y \in K \) and \( y > 0 \). Using (2.5), we calculate \( \text{ord}(y, \lambda) \) for \( \lambda \in \Lambda \) as
\[
\text{ord}(y, \lambda) = e_1 \text{ord}(n_1 + \alpha, \lambda) + \cdots + e_k \text{ord}(n_k + \alpha, \lambda).
\]
(2.7)

Therefore we obtain
\[
\mathcal{X}_\alpha(n_1)^{e_1} \cdots \mathcal{X}_\alpha(n_k)^{e_k} = \prod_{\lambda \in \Lambda} \mathcal{X}(\lambda)^{e_1 \text{ord}(n_1 + \alpha, \lambda) + \cdots + e_k \text{ord}(n_k + \alpha, \lambda)} = \prod_{\lambda \in \Lambda} \mathcal{X}(\lambda)^{\text{ord}(y, \lambda)}
\]
by the definition of \( \mathcal{X}_\alpha(n_j) \). Since the random variables \( \mathcal{X}(\lambda) \) are independent, the expectation of the above is
\[
\mathbb{E}[\mathcal{X}_\alpha(n_1)^{e_1} \cdots \mathcal{X}_\alpha(n_k)^{e_k}] = \prod_{\lambda \in \Lambda} \mathbb{E}[\mathcal{X}(\lambda)^{\text{ord}(y, \lambda)}].
\]
Note that \( \mathbb{E}[\mathcal{X}(\lambda)^{\text{ord}(y, \lambda)}] \) equals to 1 if \( \text{ord}(y, \lambda) = 0 \) and to 0 otherwise since \( \mathcal{X}(\lambda) \) is uniformly distributed on the unit circle. Hence we arrive at
\[
\mathbb{E}[\mathcal{X}_\alpha(n_1)^{e_1} \cdots \mathcal{X}_\alpha(n_k)^{e_k}] = \begin{cases} 
  1 & \text{if } \text{ord}(y, \lambda) = 0 \text{ for any } \lambda \in \Lambda, \\
  0 & \text{otherwise}.
\end{cases}
\]
(2.7)

The condition that \( \text{ord}(y, \lambda) = 0 \) for any \( \lambda \in \Lambda \) is equivalent to \( y^h = 1 \) by the expression of \( y^h \) as in (2.3). Furthermore, \( y^h = 1 \) if and only if \( y = 1 \) due to \( y > 0 \). Hence we see that (2.7) is nothing but (1.5). \( \square \)
Let $X$ be a random variable. The characteristic function of $X$ is defined as
$$g(w; X) = \mathbb{E}[\psi_w(X)]$$
for $w \in \mathbb{C}$, where $\psi_w(z) = \exp(i \text{Re}(z \overline{w}))$ is an additive character of $\mathbb{C} \simeq \mathbb{R}^2$. Then the law of $X$ is uniquely determined by $g(w; X)$; see [1, Section 29]. For example, a random variable $X$ is uniformly distributed on the unit circle if and only if the characteristic function is represented as
$$g(w; X) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{|w|}{2}\right)^{2m} = J_0(|w|),$$
where $J_0(z)$ is the Bessel function of the first kind of order zero. In the remainder of this section, we prove several properties of the random variables $X_\alpha(n)$.

**Lemma 2.1.** Let $\alpha$ be an algebraic number satisfying $0 < \alpha < 1$. Then the random variables $X_\alpha(n)$ are uniformly distributed on the unit circle.

*Proof.* Since the additive character $\psi_w(z)$ is represented as
$$\psi_w(z) = \exp \left( \frac{i}{2}(z \overline{w} + \overline{z} w) \right) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{(\frac{1}{2})^{\mu+\nu}}{\mu! \nu!} z^\mu \overline{w}^\nu \overline{w}^\nu$$
for any $z, w \in \mathbb{C}$, we calculate the characteristic function of $X_\alpha(n)$ as
$$g(w; X_\alpha(n)) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{(\frac{1}{2})^{\mu+\nu}}{\mu! \nu!} \mathbb{E}[(X_\alpha(n)^{\mu} \overline{X_\alpha(n)^{\nu}})] w^\mu \overline{w}^\nu.$$ 

Applying [1,5] with the identity $\overline{X_\alpha(n)} = X_\alpha(n)^{-1}$, we have
$$\mathbb{E}[(X_\alpha(n)^{\mu} \overline{X_\alpha(n)^{\nu}})] = \begin{cases} 1 & \text{if } (n + \alpha)^{\mu} (n + \alpha)^{-\nu} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $(n + \alpha)^{\mu} (n + \alpha)^{-\nu} = 1$ if and only if $\mu = \nu$ since we have $n + \alpha \neq 1$ for $0 < \alpha < 1$. Therefore off-diagonal terms of (2.9) disappear, and we obtain
$$g(w; X_\alpha(n)) = \sum_{\mu=0}^{\infty} \frac{(-1)^\mu}{(\mu!)^2} \left(\frac{|w|}{2}\right)^{2\mu} = J_0(|w|).$$

This shows that $X_\alpha(n)$ is uniformly distributed on the unit circle. \hfill \Box

**Remark 2.2.** In the case of $\alpha = 1$, we have $X_\alpha(0) \equiv 1$ by definition. It is obviously not uniformly distributed on the unit circle.

For random variables $X_1, \ldots, X_k$, the joint characteristic function is defined as
$$g(w; X_1, \ldots, X_k) = \mathbb{E}[\psi_{w_1}(X_1) \cdots \psi_{w_k}(X_k)]$$
for $w = (w_1, \ldots, w_k) \in \mathbb{C}^k$. By using the inversion formula [1, Section 29], we find that $X_1, \ldots, X_k$ are independent if and only if the equality
$$g(w; X_1, \ldots, X_k) = g(w_1; X_1) \cdots g(w_k; X_k)$$
holds for any $w = (w_1, \ldots, w_k) \in \mathbb{C}^k$. By this fact, we prove the following results.

**Lemma 2.3.** Let $\alpha$ be an algebraic number satisfying $0 < \alpha < 1$. For any integers $n_1, \ldots, n_k \geq 0$, the random variables $X_\alpha(n_1), \ldots, X_\alpha(n_k)$ are independent if and only if the numbers $\log(n_1 + \alpha), \ldots, \log(n_k + \alpha)$ are linearly independent over $\mathbb{Q}$. 


Proof. First, we suppose that \( \log(n_1 + \alpha), \ldots, \log(n_k + \alpha) \) are linearly independent over \( \mathbb{Q} \). Then we apply (2.8) to derive (2.11)
\[
g(w; \mathcal{X}_\alpha(n_1), \ldots, \mathcal{X}_\alpha(n_k)) = \sum_{\mu_k=0}^{\infty} \cdots \sum_{\mu_1=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_k-\nu_k}}{(m_1!)^2 \cdots (m_k!)^2} \left( \frac{|w_1|}{2} \right)^{2\mu_1} \cdots \left( \frac{|w_k|}{2} \right)^{2\mu_k},
\]
where \( G_{\mu_1,\nu_1,\ldots,\mu_k,\nu_k} \) is calculated by using (1.5) as
\[
G_{\mu_1,\nu_1,\ldots,\mu_k,\nu_k} = \mathbb{E}[\mathcal{X}_\alpha(n_1)^{\mu_1} \mathcal{X}_\alpha(n_1)^{\nu_1} \cdots \mathcal{X}_\alpha(n_k)^{\mu_k} \mathcal{X}_\alpha(n_k)^{\nu_k}]
= \begin{cases} 1 & \text{if } (n_1 + \alpha)^{m_1-\nu_1} \cdots (n_k + \alpha)^{m_k-\nu_k} = 1, \\ 0 & \text{otherwise}. \end{cases}
\]
By the linear independence, \( (n_1 + \alpha)^{m_1-\nu_1} \cdots (n_k + \alpha)^{m_k-\nu_k} = 1 \) is equivalent to that \( \mu_j = \nu_j \) for all \( 1 \leq j \leq k \). Therefore (2.11) yields
\[
g(w; \mathcal{X}_\alpha(n_1), \ldots, \mathcal{X}_\alpha(n_k)) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_k}}{(m_1!)^2 \cdots (m_k!)^2} \left( \frac{|w_1|}{2} \right)^{2m_1} \cdots \left( \frac{|w_k|}{2} \right)^{2m_k}
= g(w_1; \mathcal{X}_\alpha(n_1)) \cdots g(w_k; \mathcal{X}_\alpha(n_k)),
\]
since the characteristic function \( g(w; \mathcal{X}_\alpha(n)) \) is calculated as (2.10). This derives that \( \mathcal{X}_\alpha(n_1), \ldots, \mathcal{X}_\alpha(n_k) \) are independent.

Next, we suppose that \( \log(n_1 + \alpha), \ldots, \log(n_k + \alpha) \) are linearly dependent over \( \mathbb{Q} \). Then the equation
\[
(n_1 + \alpha)^{m_1} \cdots (n_k + \alpha)^{m_k} = 1
\]
is satisfied with some integers \( m_1, \ldots, m_k \) such that at least one of them is nonzero. Since every \( \mathcal{X}_\alpha(n) \) is uniformly distributed on the unit circle by Lemma 2.1, we have \( \mathbb{E}[\mathcal{X}_\alpha(n)^m] = 0 \) for any nonzero integer \( m \). Thus we obtain
\[
\mathbb{E}[\mathcal{X}_\alpha(n_1)^{m_1}] \cdots \mathbb{E}[\mathcal{X}_\alpha(n_k)^{m_k}] = 0.
\]
On the other hand, we have
\[
\mathbb{E}[\mathcal{X}_\alpha(n_1)^{m_1} \cdots \mathcal{X}_\alpha(n_k)^{m_k}] = 1
\]
by (1.5) and (2.12). Comparing these, we conclude that \( \mathcal{X}_\alpha(n_1), \ldots, \mathcal{X}_\alpha(n_k) \) are not independent, and we obtain the desired equivalence. \( \square \)

Lemma 2.4. Let \( \alpha \) be an algebraic number satisfying \( 0 < \alpha < 1 \), and let
\[
\mathcal{L}(N) = \{ n \in \mathbb{Z} \mid N < n \leq N \log N \},
\]
\[
\mathcal{M}(N) = \{ n \in \mathbb{Z} \mid 0 \leq n \leq N \log N \}
\]
for \( N \geq 3 \). We define \( \mathcal{K}_\alpha(n) \) as the subset of \( \mathcal{L}(N) \) consisting of all integers \( n \) such that \( (n + \alpha)\mathfrak{a} \) is divisible by a prime ideal \( \mathfrak{p} \) not dividing \( (m + \alpha)\mathfrak{a} \) for any \( m \in \mathcal{M}(N) \) with \( m \neq n \), where \( \mathfrak{a} \) denotes the ideal denominator of \( \alpha \). Put
\[
\mathcal{K}_\alpha(N) = \{ n_1, \ldots, n_k \} \quad \text{and} \quad \mathcal{M}(N) \setminus \mathcal{K}_\alpha(N) = \{ n_{k+1}, \ldots, n_{\ell} \}.
\]
Then the random variables
\[
\mathcal{X}_\alpha(n_1), \ldots, \mathcal{X}_\alpha(n_k), c_{k+1}\mathcal{X}_\alpha(n_{k+1}) + \cdots + c_{\ell}\mathcal{X}_\alpha(n_{\ell})
\]
are independent for any complex numbers $c_{k+1}, \ldots, c_{\ell}$.

**Proof.** Define $\mathcal{Y} = c_{k+1} \mathcal{X}_\alpha(n_{k+1}) + \cdots + c_{\ell} \mathcal{X}_\alpha(n_{\ell})$. Then we have

$$g(w; \mathcal{X}_\alpha(n_1), \ldots, \mathcal{X}_\alpha(n_k), \mathcal{Y}) = \mathbf{E} \left[ \psi_{w_1} (\mathcal{X}_\alpha(n_1)) \cdots \psi_{w_k} (\mathcal{X}_\alpha(n_k)) \psi_{w_{k+1}} (\mathcal{Y}) \right]$$

$$= \mathbf{E} \left[ \psi_{w_1} (\mathcal{X}_\alpha(n_1)) \cdots \psi_{w_k} (\mathcal{X}_\alpha(n_k)) \psi_{w_{k+1}} (c_{k+1} \mathcal{X}_\alpha(n_{k+1}) + \cdots + c_{\ell} \mathcal{X}_\alpha(n_{\ell})) \right]$$

for $w = (w_1, \ldots, w_{k+1}) \in \mathbb{C}^{k+1}$. Expanding $\psi_{w}(z)$ by using (2.8), we have further

(2.13)

$$g(w; \mathcal{X}_\alpha(n_1), \ldots, \mathcal{X}_\alpha(n_k), \mathcal{Y}) = \sum_{\mu_1 = 0}^{c_{k+1}} \sum_{\mu_2 = 0}^{c_{k+2}} \cdots \sum_{\mu_{\ell} = 0}^{c_{\ell}} \left( \prod_{\ell = 1}^{k} \mu_1^{\mu_1} \cdots \mu_{\ell}^{\mu_{\ell}} \right) \mathbf{E} \left[ \mathcal{X}_\alpha(n_1)^{\mu_1} \mathcal{X}_\alpha(n_2)^{\mu_2} \cdots \mathcal{X}_\alpha(n_k)^{\mu_k} \mathcal{X}_\alpha(n_{k+1}) \cdots \mathcal{X}_\alpha(n_{\ell}) \mathcal{Y} \right]$$

where $G_{\mu_1, \mu_2, \ldots, \mu_{\ell}} = \mathbf{E} \left[ \mathcal{X}_\alpha(n_1)^{\mu_1} \mathcal{X}_\alpha(n_2)^{\mu_2} \cdots \mathcal{X}_\alpha(n_k)^{\mu_k} \mathcal{X}_\alpha(n_{k+1}) \cdots \mathcal{X}_\alpha(n_{\ell}) \mathcal{Y} \right]$. By the definition of $K_\alpha(N)$, there exists a prime ideal $p_j$ for $j \in \{1, \ldots, k\}$ such that it is a prime divisor of $(n_j + \alpha)\mathfrak{a}$ but not dividing $(m + \alpha)\mathfrak{a}$ for any integer $m \in M(N)$ with $m \neq n_j$. Hence, for $\lambda_j = \varpi_{p_j} \in \mathfrak{A}$ as in (2.4), we have $\text{ord}(n_j + \alpha, \lambda_j) > 0$ but $\text{ord}(m + \alpha, \lambda_j) = 0$ for any $m \in M(N)$ with $m \neq n_j$. Note that $\lambda_1, \ldots, \lambda_k$ are distinct. Since every $\mathcal{X}_\alpha(n_j)$ is represented as

$$\mathcal{X}_\alpha(n_j) = \mathcal{X}(\lambda_j)^{\text{ord}(n_j + \alpha, \lambda_j)} \prod_{\lambda \in \mathfrak{A}, \lambda \neq \lambda_j} \mathcal{X}(\lambda)^{\text{ord}(n_j + \alpha, \lambda)}$$

by (2.6), we obtain

$$\mathcal{X}_\alpha(n_1)^{\mu_1} \mathcal{X}_\alpha(n_2)^{\mu_2} \cdots \mathcal{X}_\alpha(n_k)^{\mu_k} \mathcal{X}_\alpha(n_{k+1}) \cdots \mathcal{X}_\alpha(n_{\ell})$$

$$= \prod_{j=1}^{k} \mathcal{X}(\lambda_j)^{\text{ord}(n_j + \alpha, \lambda_j)}$$

$$\times \prod_{\lambda \in \mathfrak{A}, \lambda \neq \lambda_1, \ldots, \lambda_k} \mathcal{X}(\lambda)^{\text{ord}(n_1 + \alpha, \lambda) + \cdots + \text{ord}(n_\ell + \alpha, \lambda)}.$$

By the independence of $\mathcal{X}(\lambda)$, it deduces

$$G_{\mu_1, \mu_2, \ldots, \mu_{\ell}} = \prod_{j=1}^{k} \mathbf{E} \left[ \mathcal{X}(\lambda_j)^{\text{ord}(n_j + \alpha, \lambda_j)} \right]$$

$$\times \prod_{\lambda \in \mathfrak{A}, \lambda \neq \lambda_1, \ldots, \lambda_k} \mathbf{E} \left[ \mathcal{X}(\lambda)^{\text{ord}(n_1 + \alpha, \lambda) + \cdots + \text{ord}(n_\ell + \alpha, \lambda)} \right].$$

As a result, we see that $G_{\mu_1, \mu_2, \ldots, \mu_{\ell}} = 0$ if $\mu_j \neq \nu_j$ for some $1 \leq j \leq k$. Hence the terms in (2.13) with $\mu_j \neq \nu_j$ for some $1 \leq j \leq k$ disappear. Furthermore, we have

$$G_{\mu_1, \mu_2, \ldots, \mu_{\ell}} = \mathbf{E} \left[ \mathcal{X}_\alpha(n_{k+1})^{\mu_{k+1}} \mathcal{X}_\alpha(n_{k+2})^{\mu_{k+2}} \cdots \mathcal{X}_\alpha(n_{\ell})^{\mu_{\ell}} \mathcal{Y} \right].$$
if \( \mu_j = \nu_j \) for all \( 1 \leq j \leq k \) by definition. Thus we deduce from (2.13) that

\[
g(w; X_\alpha(n_1), \ldots, X_\alpha(n_k), \mathcal{Y}) = \sum_{\mu_1 = 0}^{\infty} \cdots \sum_{\mu_k = 0}^{\infty} \sum_{\nu_1 = 0}^{\infty} \cdots \sum_{\nu_k = 0}^{\infty} \frac{(-1)^{\mu_1 + \cdots + \mu_k}}{\mu_1! \cdots \mu_k! \nu_1! \cdots \nu_k!} \\
\times \mathbb{E}[X_\alpha(n_{k+1})^{\mu_k+1}] \sum_{\nu_k = 0}^{\nu_k} \cdots \sum_{\nu_1 = 0}^{\nu_1} \frac{(-1)^{\nu_1 + \cdots + \nu_k}}{\nu_1! \cdots \nu_k!} \\
\times (w|2)^{2\mu_k} \cdots (w|n_{k+1})^{2\mu_k+1} \cdots (w|n_2) \cdots (w|n_1) \\
= g(w_1; X_\alpha(n_1)) \cdots g(w_k; X_\alpha(n_k)) g(w_{k+1})
\]

by using (2.10), where we put

\[
g(w_{k+1}) = \sum_{\mu_{k+1} = 0}^{\infty} \cdots \sum_{\mu_1 = 0}^{\infty} \frac{(-1)^{\mu_{k+1} + \cdots + \mu_1}}{\mu_{k+1}! \cdots \mu_1!} \\
\times \mathbb{E}[X_\alpha(n_{k+1})^{\mu_{k+1}}] \sum_{\nu_{k+1} = 0}^{\nu_{k+1}} \cdots \sum_{\nu_1 = 0}^{\nu_1} \frac{(-1)^{\nu_1 + \cdots + \nu_{k+1}}}{\nu_1! \cdots \nu_{k+1}!} \\
\times c_{k+1}^{\mu_{k+1}} \cdots c_{k+1}^{\mu_1} \cdots c_{k+1}^{\mu_1} = 1 \cdots c_{k+1}^{\mu_{k+1} + \cdots + \mu_1} w_{k+1}^{\nu_{k+1} + \cdots + \nu_1}.
\]

Here, we can confirm the identity \( g(w_{k+1}) = g(w_{k+1}; \mathcal{Y}) \) by using the expansion of \( \psi_w(z) \) as in (2.5). Therefore the equality

\[
g(w; X_\alpha(n_1), \ldots, X_\alpha(n_k), \mathcal{Y}) = g(w_1; X_\alpha(n_1)) \cdots g(w_k; X_\alpha(n_k)) g(w_{k+1}; \mathcal{Y})
\]

holds, and we obtain the desired result. \( \square \)

3. PROOF OF THE LIMIT THEOREM

Let \( \sigma > 1/2 \) be a fixed real number, and let \( \alpha \) be an algebraic number satisfying \( 0 < \alpha \leq 1 \). For \( T > 0 \), we define

\[
(3.1) \quad g_T(w; \sigma + it, \alpha) = \frac{1}{T} \int_0^T \psi_w(\zeta(\sigma + it, \alpha)) dt,
\]

where \( \psi_w(z) = \exp(i \text{Re}(z\overline{w})) \) for \( z, w \in \mathbb{C} \) as before. In this section, we show that the function \( g_T(w; \sigma + it, \alpha) \) converges to

\[
g(w; \sigma, X_\alpha) = \mathbb{E}[\psi_w(\zeta(\sigma, X_\alpha))]
\]

as \( T \to \infty \) uniformly in the region \( |w| \leq R \) for any \( R > 0 \), where \( \zeta(\sigma, X_\alpha) \) is the random variable of (1.3). For this, we truncate the series of \( \zeta(s, \alpha) \) and \( \zeta(\sigma, X_\alpha) \) as

\[
(3.2) \quad \zeta_N(s, \alpha) = \sum_{n=0}^{N} \frac{1}{(n + \alpha)^s} \quad \text{and} \quad \zeta_N(\sigma, X_\alpha) = \sum_{n=0}^{N} \frac{X_\alpha(n)}{(n + \alpha)^\sigma}
\]

with an integer \( N \geq 0 \), and define

\[
(3.3) \quad g_{T,N}(w; \sigma + it, \alpha) = \frac{1}{T} \int_0^T \psi_w(\zeta_N(\sigma + it, \alpha)) dt,
\]

\[
g_N(w; \sigma, X_\alpha) = \mathbb{E}[\psi_w(\zeta_N(\sigma, X_\alpha))]
\]

similarly to the above. Then we have the following propositions.
Proposition 3.1. Let $\sigma > 1/2$ be a fixed real number, and let $\alpha$ be an algebraic number satisfying $0 < \alpha \leq 1$. Take an integer $N \geq 2$ arbitrary. For any $R > 0$ and $\epsilon > 0$, there exists a positive real number $T_0 = T_0(\alpha, N, R, \epsilon)$ such that the inequality

$$|g_{T,N}(w; \sigma + it, \alpha) - g_N(w; \sigma, X_\alpha)| < \epsilon$$

holds for all $T \geq T_0$ in the region $|w| \leq R$.

Proof. Let $M$ be any positive integer. Recall that the asymptotic formula

$$e^{i\theta} = \sum_{m<M} \frac{i^m}{m!} \theta^m + O\left(\frac{1}{M!} |\theta|^M\right)$$

holds for all $\theta \in \mathbb{R}$ with an absolute implied constant. Then we obtain

$$\psi_w(z) = \sum_{\mu+\nu<M} \frac{(\frac{i}{2})^{\mu+\nu}}{\mu!\nu!} z^\mu \bar{w}^\nu w^\nu + O\left(\frac{1}{M!} |zw|^M\right)$$

for all $z, w \in \mathbb{C}$. Applying this formula, we calculate the right-hand side of (3.3) as

$$g_{T,N}(w; \sigma + it, \alpha) = \sum_{\mu+\nu<M} \frac{(\frac{i}{2})^{\mu+\nu}}{\mu!\nu!} \left(\frac{1}{T} \int_0^T \zeta_N(\sigma + it, \alpha)^{\mu} \overline{\zeta_N(\sigma + it, \alpha)^\nu} dt\right) \bar{w}^\nu w^\nu + E_1,$$

where the error term is evaluated as

$$E_1 \ll \frac{R^M}{M!} \frac{1}{T} \int_0^T |\zeta_N(\sigma + it, \alpha)|^M dt$$

for $|w| \leq R$ with an absolute implied constant. By definition, we have

$$|\zeta_N(\sigma + it, \alpha)| \leq \sum_{n=0}^N \frac{1}{(n+\alpha)^{1/2}} \ll \alpha \sqrt{N}$$

for any $\sigma > 1/2$. Hence we obtain $E_1 \ll (c(\alpha)R\sqrt{N})^M / M!$ with a positive constant $c(\alpha)$ depending only on $\alpha$. As a result, there exists an integer $M = M(\alpha, N, R, \epsilon)$ such that $|E_1| < \epsilon/3$ holds for each $\epsilon > 0$. We can choose $M$ so that $M \geq 2$. Then,
the main term of (3.4) is evaluated as follows. We have

\[
\frac{1}{T} \int_0^T \zeta_N(\sigma + it, \alpha)^{\nu} \xi_N(\sigma + it, \alpha)^{\nu} \, dt
\]

\[
= \sum_{m_1=0}^{N} \cdots \sum_{n_1=0}^{N} \sum_{m_\nu=0}^{N} \cdots \sum_{n_\nu=0}^{N} \frac{1}{(m_1 + \alpha)^{\sigma} \cdots (m_\mu + \alpha)^{\sigma}} \int_0^T \left( \frac{(m_1 + \alpha) \cdots (m_\mu + \alpha)}{(n_1 + \alpha) \cdots (n_\nu + \alpha)} \right)^{-it} \, dt
\]

\[
= \sum_{m_1=0}^{N} \cdots \sum_{n_1=0}^{N} \sum_{m_\nu=0}^{N} \cdots \sum_{n_\nu=0}^{N} \frac{1}{(m_1 + \alpha)^{\sigma} \cdots (m_\mu + \alpha)^{\sigma}} \frac{1}{(n_1 + \alpha)^{\sigma} \cdots (n_\nu + \alpha)^{\sigma}}
\]

\[
\times \left\{ 1 - \left( \frac{(m_1 + \alpha) \cdots (m_\mu + \alpha)}{(n_1 + \alpha) \cdots (n_\nu + \alpha)} \right) \right\}^{-1}
\]

\[
= S_1 + S_2,
\]

say. Recall that the random variables \(X_\alpha(n)\) satisfy (1.3). Then the first term \(S_1\) is (3.5)

\[
S_1 = \sum_{m_1=0}^{N} \cdots \sum_{n_1=0}^{N} \sum_{m_\nu=0}^{N} \cdots \sum_{n_\nu=0}^{N} \frac{1}{(m_1 + \alpha)^{\sigma} \cdots (m_\mu + \alpha)^{\sigma}}
\]

\[
\times \frac{1}{(n_1 + \alpha)^{\sigma} \cdots (n_\nu + \alpha)^{\sigma}} \mathbb{E} \left[ X_\alpha(m_1) \cdots X_\alpha(m_\mu) X_\alpha(n_1) \cdots X_\alpha(n_\nu) \right]
\]

\[
= \mathbb{E} \left[ \zeta_N(\sigma, X_\alpha)^{\mu} \xi_N(\sigma, X_\alpha)^{\nu} \right].
\]

On the other hand, the second term \(S_2\) is evaluated as follows. Put

\[
P(x) = (m_1 + x) \cdots (m_\mu + x) - (n_1 + x) \cdots (n_\nu + x),
\]

and denote its degree and height by \(\text{deg}(P)\) and \(\text{ht}(P)\), respectively. For \(\mu + \nu < M\), \(\text{deg}(P)\) is at most \(M\). For \(m_1, \ldots, m_\mu, n_1, \ldots, n_\nu \leq N\), we also find that \(\text{ht}(P)\) is less than the height of \((N + x)^M\), which is at most \((2N)^M\). We also denote by \(\text{deg}(\alpha)\) and \(\text{ht}(\alpha)\) the degree and height of \(\alpha\), respectively. Then we obtain

\[
|P(\alpha)| \geq (\text{deg}(P) + 1)^{1 - \text{deg}(\alpha)}(\text{deg}(\alpha) + 1)^{-\text{deg}(P)/2} \text{ht}(P)^{1 - \text{deg}(\alpha)} \text{ht}(\alpha)^{-\text{deg}(P)}
\]

\[
\geq (2N)^{-\omega(\alpha)M}
\]
by applying [3] Theorem A.1, where \( \omega(\alpha) \) is a positive constant that depends only on \( \alpha \). Hence we derive
\[
\left| \log \frac{(m_1 + \alpha) \cdots (m_\mu + \alpha)}{(n_1 + \alpha) \cdots (n_\nu + \alpha)} \right|
\geq \frac{|(m_1 + \alpha) \cdots (m_\mu + \alpha) - (n_1 + \alpha) \cdots (n_\nu + \alpha)|}{\max\{(m_1 + \alpha) \cdots (m_\mu + \alpha), (n_1 + \alpha) \cdots (n_\nu + \alpha)\}}
\geq (2N)^{-(\omega(\alpha)+1)M}.
\]
Therefore we arrive at
\[
S_2 \ll \left( \sum_{n=0}^N \frac{1}{(n + \alpha)\sigma} \right)^{\mu+\nu} \frac{1}{T} (2N)^{\omega(\alpha)+1)M}
\leq \frac{1}{T} (d(\alpha)N)^{\omega(\alpha)+1)M}
\]
with a positive constant \( d(\alpha) \) depending only on \( \alpha \). Combining (3.5) and (3.6), we deduce the formula
\[
\frac{1}{T} \int_0^T \zeta_N(\sigma + it, \alpha)^\mu \overline{\zeta_N(\sigma + it, \alpha)^\nu} \, dt
= E \left[ \zeta_N(\sigma, \mathcal{X}_\alpha)^\mu \overline{\zeta_N(\sigma, \mathcal{X}_\alpha)^\nu} \right] + O \left( \frac{1}{T} (d(\alpha)N)^{\omega(\alpha)+1)M} \right),
\]
where the implied constant is absolute. Furthermore, it yields
\[
\sum_{\mu+\nu < M} \frac{(\frac{1}{2})^{\mu+\nu}}{\mu!\nu!} \left( \frac{1}{T} \int_0^T \zeta_N(\sigma + it, \alpha)^\mu \overline{\zeta_N(\sigma + it, \alpha)^\nu} \, dt \right) \mathfrak{w}^\mu \mathfrak{w}^\nu
= \sum_{\mu+\nu < M} \frac{(\frac{1}{2})^{\mu+\nu}}{\mu!\nu!} E \left[ \zeta_N(\sigma, \mathcal{X}_\alpha)^\mu \overline{\zeta_N(\sigma, \mathcal{X}_\alpha)^\nu} \right] \mathfrak{w}^\mu \mathfrak{w}^\nu + E_2,
\]
where the error term is estimated as
\[
E_2 \ll \frac{1}{T} (d(\alpha)N)^{\omega(\alpha)+1)M} \sum_{\mu+\nu < M} \frac{1}{\mu!\nu!} \left( \frac{|w|}{2} \right)^{\mu+\nu} \ll \frac{1}{T} (d(\alpha)N)^{\omega(\alpha)+1)M} \exp(R)
\]
for \(|w| \leq R\). Thus, there exists a positive real number \( T_0 = T_0(\alpha, M, N, R, \epsilon) \) such that \(|E_2| < \epsilon/3\) holds for all \( T \geq T_0 \). Finally, we obtain
\[
g_N(w; \sigma, \mathcal{X}_\alpha)
= \sum_{\mu+\nu < M} \frac{(\frac{1}{2})^{\mu+\nu}}{\mu!\nu!} E \left[ \zeta_N(\sigma, \mathcal{X}_\alpha)^\mu \overline{\zeta_N(\sigma, \mathcal{X}_\alpha)^\nu} \right] \mathfrak{w}^\mu \mathfrak{w}^\nu + E_3
\]
similarly to (3.3), where
\[
E_3 \ll \frac{R^M}{M!} E \left[ \left| \zeta_N(\sigma, \mathcal{X}_\alpha)^{\left| M \right|} \right| \right] \ll \frac{(c(\alpha)R\sqrt{N})^M}{M!}
\]
with absolute implied constants. Then we again obtain \(|E_3| < \epsilon/3\) by recalling the choice of \( M \). The desired result follows from above formulas (3.4), (3.7), (3.8) with the above error estimates. \( \square \)
Proposition 3.2. Let \( \sigma > 1/2 \) be a fixed real number, and let \( \alpha \) be an algebraic number satisfying \( 0 < \alpha \leq 1 \). For any \( R > 0 \) and \( \epsilon > 0 \), there exists an integer \( N_0 = N_0(\sigma, R, \epsilon) \geq 0 \) such that the inequalities

\[
\limsup_{T \to \infty} |g_T(w; \sigma + it, \alpha) - g_{T, N}(w; \sigma + it, \alpha)| < \epsilon,
\]

\[
|g(w; \sigma, \mathcal{X}_\alpha) - g_N(w; \sigma, \mathcal{X}_\alpha)| < \epsilon
\]

hold for all \( N \geq N_0 \) in the region \( |w| \leq R \).

Proof. Since the inequality \( |e^{i\theta_1} - e^{i\theta_2}| \leq |\theta_2 - \theta_1| \) holds for all \( \theta_1, \theta_2 \in \mathbb{R} \), we have

\[
|\psi_w(z_1) - \psi_w(z_2)| \leq |z_1 - z_2| |w|
\]

for all \( z_1, z_2, w \in \mathbb{C} \). By (3.1), (3.3), and \( |\psi_w(z)| \leq 1 \), we obtain

\[
|g_T(w; \sigma + it, \alpha) - g_{T, N}(w; \sigma + it, \alpha)|
\]

\[
\leq \frac{R}{T} \int_{2\pi}^{T} |\zeta(\sigma + it, \alpha) - \zeta_N(\sigma + it, \alpha)| dt + \frac{4\pi}{T}
\]

for \( |w| \leq R \). We evaluate this integral as follows. If \( \sigma \geq 2 \), then we have \( \zeta(s, \alpha) = \zeta_N(s, \alpha) + O(N^{-1}) \) by applying (1.1). It yields the estimate

\[
\frac{1}{T} \int_{2\pi}^{T} |\zeta(\sigma + it, \alpha) - \zeta_N(\sigma + it, \alpha)| dt \ll N^{-1}.
\]

If \( 1/2 < \sigma < 2 \), then we apply the formula [11] Theorem 1 on p. 78] to deduce

\[
\zeta(s, \alpha) = \sum_{0 \leq n \leq x} \frac{1}{(n + \alpha)^s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma})
\]

for \( 2\pi \leq |t| \leq \pi x \). Taking \( x = T/\pi \), we obtain

\[
\frac{1}{T} \int_{2\pi}^{T} |\zeta(\sigma + it, \alpha) - \zeta_N(\sigma + it, \alpha)| dt
\]

\[
\ll \left( \frac{1}{T} \int_{0}^{T} \left| \sum_{N < n \leq x} \frac{1}{(n + \alpha)^{\sigma + it}} \right|^2 dt \right)^{1/2} + \frac{1}{T}(\log T)x^{1-\sigma} + x^{-\sigma}
\]

due to the Cauchy–Schwarz inequality. Furthermore, it holds that

\[
\int_{0}^{T} \left| \sum_{N < n \leq x} \frac{1}{(n + \alpha)^{\sigma + it}} \right|^2 dt \ll (T + x)N^{1-2\sigma}
\]

by [15] Corollary 2, where the implied constant depends only on \( \sigma \). Hence we have

\[
\frac{1}{T} \int_{2\pi}^{T} |\zeta(\sigma + it, \alpha) - \zeta_N(\sigma + it, \alpha)| dt \ll_{\sigma} N^{1/2-\sigma} + T^{-\sigma} \log T
\]

in this case. Using (3.11) for \( \sigma \geq 2 \) and (3.12) for \( 1/2 < \sigma < 2 \), we obtain

\[
\limsup_{T \to \infty} |g_T(w; \sigma + it, \alpha) - g_{T, N}(w; \sigma + it, \alpha)| \ll_{\sigma} RN^{\max(-1,1/2-\sigma)}
\]
Hence we conclude that the probability measure $\alpha \mapsto \infty P$. Thus the characteristic function of any rational integer $(3.9) \text{ and the Cauchy–Schwarz inequality, we have}$

$$|g(w; \sigma, X_\alpha) - g_N(w; \sigma, X_\alpha)| \leq R \mathbb{E} \left[ |\zeta(\sigma, X_\alpha) - \zeta_N(\sigma, X_\alpha)| \right]$$

$$\leq R \mathbb{E} \left[ |\zeta(\sigma, X_\alpha) - \zeta_N(\sigma, X_\alpha)|^2 \right]^{1/2}.$$ Here, we note that $(3.3)$ is convergent for $\sigma > 1/2$ almost surely. Thus we derive

$$\mathbb{E} \left[ |\zeta(\sigma, X_\alpha) - \zeta_N(\sigma, X_\alpha)|^2 \right] = \sum_{m,n>N} \frac{\mathbb{E}[X_\alpha(m)]^2}{(m + \alpha)^\sigma(n + \alpha)^\sigma} = \sum_{n>N} \frac{1}{(n + \alpha)^{2\sigma}}$$

by applying $(1.5)$. Therefore we obtain

$$|g(w; \sigma, X_\alpha) - g_N(w; \sigma, X_\alpha)| \ll R N^{1/2-\sigma}$$

with the implied constant depending only on $\sigma$, which completes the proof. □

Let $N_0 = N_0(\sigma, R, \epsilon)$ be the integer as in Proposition $3.2$. Then we choose a positive real number $T_0$ as $T_0 = T_0(\sigma, \alpha, N_0, R, \epsilon)$ of Proposition $5.1$. With these setting, we obtain the inequality

$$|g_T(w; \sigma + it, \alpha) - g(w; \sigma, X_\alpha)| \leq |g_T(w; \sigma + it, \alpha) - g_{T,N_0}(w; \sigma + it, \alpha)|$$

$$+ |g_{T,N_0}(w; \sigma + it, \alpha) - g_{N_0}(w; \sigma, X_\alpha)|$$

$$+ |g_{N_0}(w; \sigma, X_\alpha) - g(w; \sigma, X_\alpha)|$$

$$< 3\epsilon$$

for all $T \geq T_0$ in the region $|w| \leq R$. Thus the function $g_T(w; \zeta(\sigma + it, \alpha))$ converges to $g(w; \zeta(\sigma, X_\alpha))$ as $T \to \infty$ uniformly in the region $|w| \leq R$ for any $R > 0$. Finally, we prove Theorem $1.2$ as follows.

**Proof of Theorem 1.2.** Let $P_{\sigma,\alpha,T}$ be the probability measure defined as $(1.4)$, and denote by $P_{\sigma,\alpha}$ the law of the random variable $\zeta(\sigma, X_\alpha)$. The characteristic functions of these probability measures are represented as

$$\int_{\mathbb{C}} \psi_w(z) P_{\sigma,\alpha,T}(dz) = \frac{1}{T} \int_0^T \psi_w(\zeta_N(\sigma + it, \alpha)) dt = g_{T,N}(w; \sigma + it, \alpha),$$

$$\int_{\mathbb{C}} \psi_w(z) P_{\sigma,\alpha}(dz) = \mathbb{E} [\psi_w(\zeta_N(\sigma, X_\alpha))] = g_{N}(w; \sigma, X_\alpha).$$

Thus the characteristic function of $P_{\sigma,\alpha,T}$ converges to that of $P_{\sigma,\alpha}$ for any $w \in \mathbb{C}$. Hence we conclude that the probability measure $P_{\sigma,\alpha,T}$ converges weakly to $P_{\sigma,\alpha}$ as $T \to \infty$ by Lévy’s criterion. See [10, Theorem B.5.1]. □

4. A VARIANT OF THE CASSELS LEMMA

Let $\alpha$ be an algebraic irrational number, and denote by $a$ the ideal denominator of $\alpha$ in the algebraic field $K = \mathbb{Q}(\alpha)$. Then $(n + \alpha)a$ is an integral ideal of $K$ for any rational integer $n$. The following lemma by Cassels [4] is fundamental to study the Hurwitz zeta-function with algebraic irrational parameter.

**Cassels lemma.** Let $\alpha$ be an algebraic irrational number. Then there exists an integer $N_0 = N_0(\alpha) > 10^6$ depending on $\alpha$ with the following property. Suppose that $N \geq N_0$ and put $M = \lfloor 10^{-6}N \rfloor$. Then at least $51M/100$ integers in $N < n \leq N + M$
Let \( y \prod \) By definition, where \( \text{(4.2)} \) with showing preliminary lemmas toward the proof of Proposition 4.1.

In the following, the norm of a prime ideal \( \mathfrak{p} \) a prime ideal \( \mathfrak{p} \) not dividing \( (m + \alpha) \mathfrak{a} \) for any integer \( 0 \leq m \leq N \log N \) with \( m \neq n \).

As in Lemma 2.4, we put \( L(N) = \{ n \in \mathbb{Z} \mid N < n \leq N \log N \} \), and define \( K_{\alpha}(n) \) as the subset of \( L(N) \) consisting of all integers \( n \) such that \((n + \alpha) \mathfrak{a}\) is divisible by a prime ideal \( \mathfrak{p} \) and \((m + \alpha) \mathfrak{a}\) for any integer \( 0 \leq m \leq N \log N \) with \( m \neq n \).

The ultimate goal of this section is to prove the following result which is a weighted version of the Cassels lemma.

**Proposition 4.1.** Let \( 0 < c < 1 \) be a real number, and let \( d \) be a positive non-square integer. Denote by \( A_{c,d} \) the set of all quadratic irrational numbers \( \alpha = (b \pm \sqrt{d})/a \) with \( a, b \in \mathbb{Z}_{>0} \) such that \( c < (b - \sqrt{d})/a < (b + \sqrt{d})/a < 1 \) is satisfied. Let \( \sigma \) be a fixed real number with \( 1/2 < \sigma < 1 \). Then there exists an integer \( N_1 = N_1(c,d,\sigma) \) depending only on \( c,d,\sigma \) such that the inequality

\[
\sum_{n \in K_{\alpha}(N)} \frac{1}{(n + \alpha)^\sigma} > \frac{51}{100} \sum_{n \in L(N)} \frac{1}{(n + \alpha)^\sigma}
\]

holds for all \( \alpha \in A_{c,d} \) and \( N \geq N_1 \).

Remark that the integer \( N_1 \) of Proposition 4.1 is uniform for \( \alpha \) in the set \( A_{c,d} \). This fact is used essentially to prove Theorem 1.3 in Section 6. The following proof of Proposition 4.1 is largely based on the method of Cassels [11], Worley [18], and Lee–Mishou [14], but we make some modifications.

Let \( \alpha = (b \pm \sqrt{d})/a \in A_{c,d} \) with \( a, b \in \mathbb{Z}_{>0} \). If we put \( d = (d_1)^2d_2 \) with positive integers \( d_1, d_2 \) such that \( d_2 \) is square-free, then we have \( K = \mathbb{Q}(\sqrt{d_2}) \). Furthermore, an integral basis of \( K \) is obtained as \( (1, 1 + \sqrt{d_2}) \) if \( d_2 \equiv 1 \pmod{4} \), and \( (1, \sqrt{d_2}) \) if \( d_2 \equiv 2,3 \pmod{4} \). Then the ideal denominator \( \mathfrak{a} \) is an integral ideal of \( K \) which divides the principal ideal \((a)\) in the former case, and \((2a)\) in the latter case. For any rational integer \( n \geq 0 \), we have

\[
(n + \alpha) \mathfrak{a} = \prod_{\mathfrak{p} \in J_\alpha} \mathfrak{p}^u_n(\mathfrak{p}),
\]

where \( J_\alpha \) denotes the set of all prime ideals of \( K \), and \( u_n(\mathfrak{p}) \) are non-negative integers. The set \( J_\alpha \) can be divided into the following three subsets:

- \( I_\alpha = \{ \mathfrak{p} \in J_\alpha \mid (\mathfrak{p}) = \mathfrak{p} \text{ for a rational prime } \mathfrak{p} \} \),
- \( R_\alpha = \{ \mathfrak{p} \in J_\alpha \mid (\mathfrak{p}) = \mathfrak{p}^2 \text{ for a rational prime } \mathfrak{p} \} \),
- \( S_\alpha = \{ \mathfrak{p}_1, \mathfrak{p}_2 \in J_\alpha \mid (\mathfrak{p}) = \mathfrak{p}_1\mathfrak{p}_2, \mathfrak{p}_1 \neq \mathfrak{p}_2 \text{ for a rational prime } \mathfrak{p} \} \).

By definition, \( \prod_{\mathfrak{p} \in I_\alpha} \mathfrak{p}^u_n(\mathfrak{p}) \) is a principal ideal \((x_n)\) for some rational integer \( x_n > 0 \). Let \( y_n > 0 \) be the maximum rational integer satisfying \( (y_n) \mid \prod_{\mathfrak{p} \in R_\alpha} \mathfrak{p}^u_n(\mathfrak{p}) \), and put

\[
\prod_{\mathfrak{p} \in R_\alpha} \mathfrak{p}^u_n(\mathfrak{p}) = (y_n)b_n,
\]

where \( b_n \) is an integral ideal of \( K \). Then (4.1) derives

\[
(n + \alpha) \mathfrak{a} = (x_n y_n)b_n \prod_{\mathfrak{p} \in S_\alpha} \mathfrak{p}^u_n(\mathfrak{p}).
\]

In the following, the norm of a prime ideal \( \mathfrak{p} \in S_\alpha \) is always denoted by \( \mathfrak{p} \). We begin with showing preliminary lemmas toward the proof of Proposition 4.1.
4.1. Preliminary lemmas.

**Lemma 4.2.** Let \( \mathcal{A}_{c,d} \) denote the set of Proposition 4.1. Then, with the notation as in (4.3), we have \( N((x_n y_n) b_n) \leq 16d \) for any \( \alpha \in \mathcal{A}_{c,d} \) and \( n \in \mathbb{Z}_{\geq 0} \).

**Proof.** For \( \alpha = (b + \sqrt{d})/a \in \mathcal{A}_{c,d} \) with \( d = (d_1)^2 d_2 \), we have

\[
\frac{an + b + d_1 \sqrt{d_2}}{2a} = \frac{2an + 2b + d_1 + \sqrt{d_2}}{2}
\]

for any \( n \in \mathbb{Z}_{\geq 0} \). By (4.3), we see that \( x_n y_n \) divides \( 2an + 2b + d_1 \) if \( d_2 \equiv 1 \pmod{4} \), and \( 2an + 2b + d_1 \sqrt{d_2} \) if \( d_2 \equiv 3 \pmod{4} \). Since \( x_n y_n \) is a rational integer, and \( (1, \frac{1+\sqrt{d}}{2}) \) or \( (1, \sqrt{d_2}) \) is an integral basis of \( K \), we find that \( x_n y_n \mid 2d_1 \) in either case. This yields (4.4)

\[
N((x_n y_n)) = (x_n y_n)^2 \leq 4(d_1)^2.
\]

Then, we consider the ideal \( b_n \) in (4.3). Recall that any prime factor \( p \mid b_n \) satisfies \( p^2 = (p) \) with a rational prime \( p \). Hence we see that \( u_n(p) = 1 \) for \( p \mid b_n \) since \( y_n \) is the maximum rational integer such that (4.2) holds. Thus the norm of \( b_n \) is

\[
N(b_n) = \prod_{p \mid b_n} N(p).
\]

For any prime ideal \( p \in R_{\alpha} \), the norm \( N(p) \) divides the discriminant of \( \mathbb{Q}(\sqrt{d_2}) \), namely, \( d_2 \) if \( d_2 \equiv 1 \pmod{4} \), and \( 4d_2 \) if \( d_2 \equiv 2, 3 \pmod{4} \). Therefore we obtain (4.5)

\[
N(b_n) \leq 4d_2
\]

in either case. The desired result follows from (4.4) and (4.5) by \( d = (d_1)^2 d_2 \).

**Lemma 4.3.** Let \( \mathcal{A}_{c,d} \) denote the set of Proposition 4.1. Take an element \( \alpha \in \mathcal{A}_{c,d} \). Assume that a prime ideal \( p \in S_\alpha \) satisfies \( \mathfrak{p}^v \mid (m + \alpha)\mathfrak{a} \) and \( \mathfrak{p}^v \mid (n + \alpha)\mathfrak{a} \) for integers \( m, n \geq 0 \) and \( v \geq 1 \). Then we have \( m \equiv n \pmod{p^v} \).

**Proof.** Since \( (m + \alpha)\mathfrak{a} \subset \mathfrak{p}^v \) and \( (n + \alpha)\mathfrak{a} \subset \mathfrak{p}^v \) by the assumption, we derive

\[
(m - n)\mathfrak{a} \subset (m + \alpha)\mathfrak{a} + (n + \alpha)\mathfrak{a} \subset \mathfrak{p}^v.
\]

Note that we have \( p \nmid \mathfrak{a} \). Thus \( \mathfrak{p}^v \mid (m - n) \) follows, and we deduce \( \mathfrak{p}^v \mid (m - n)^2 \) by taking norms. Since \( p \) is a prime number, it yields at least that

\[
p \mid (m - n).
\]

Using this, we can put \( m - n = p^w N \) with \( w, N \geq 1 \) such that \( (p, N) = 1 \). Then we see that \( p \nmid (N) \) by taking norms. Hence \( \mathfrak{p}^v \mid (p)^w \) follows. Note that we have \( \mathfrak{p}^2 \nmid (p) \) for \( p \in S_\alpha \). Then \( v \leq w \) must be satisfied. Therefore \( \mathfrak{p}^v \mid (m - n) \) holds, and we arrive at the conclusion.

**Lemma 4.4.** Let \( 1/2 < \sigma < 1 \) be a fixed real number. For any \( 0 < \alpha \leq 1 \), we have

\[
\sum_{\substack{n \in \mathbb{L}(N) \\ n \equiv a \pmod{q}}} \frac{1}{(n + \alpha)^\sigma} = \frac{1}{a} \sum_{\substack{n \in \mathbb{L}(N) \\ n \equiv a \pmod{q}}} \frac{1}{(n + \alpha)^\sigma} + O(N^{-\sigma})
\]

with arbitrary integers \( q \geq 1 \) and \( 0 \leq a < q \), where the implied constant is absolute.
Proof. Let \( f(x) = (x + \alpha)^{-\sigma} \) and \( c_n = 1 \) if \( n \equiv a \pmod{q} \) and \( c_n = 0 \) otherwise. Then we have

\[
\mathcal{C}(x) := \sum_{N < n \leq x} c_n = \frac{1}{q}(x - N) + O(1).
\]

By the partial summation formula \([9, \text{Theorem 1 on p. 326}]\), we obtain

\[
\sum_{n \in L(N)} \frac{1}{(n + \alpha)^\sigma} = -\int_N^{N \log N} \mathcal{C}(x) f'(x) \, dx + \mathcal{C}(N \log N) f(N \log N)
= \frac{1}{q} \left\{ \sigma \int_N^{N \log N} \frac{x - N}{(x + \alpha)^{\sigma + 1}} \, dx + \frac{N \log N - N}{(N \log N + \alpha)^\sigma} \right\} + O\left( \frac{1}{(N + \alpha)^\sigma} \right).
\]

Letting \( q = 1 \), we also obtain

\[
\sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma}
= \sigma \int_N^{N \log N} \frac{x - N}{(x + \alpha)^{\sigma + 1}} \, dx + \frac{N \log N - N}{(N \log N + \alpha)^\sigma} + O\left( \frac{1}{(N + \alpha)^\sigma} \right).
\]

Comparing these formulas, we obtain the desired result. \( \square \)

4.2. Cassels method. Let \( 1/2 < \sigma < 1 \) be a fixed real number. Define

\[
\mathcal{S}_\alpha(N) = \left\{ n \in \mathcal{L}(N) \mid p^{\nu_n(p)} \leq N \log N \text{ for all } p \in S_\alpha \right\}.
\]

Then we put

\[
(4.6) \quad \sum_{n \in \mathcal{S}_\alpha(N)} \frac{1}{(n + \alpha)^\sigma} = \rho \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma},
\]

where \( \rho = \rho(\sigma, \alpha, N) \) is a real number such that \( 0 < \rho \leq 1 \). In this subsection, we prove that \( \rho \leq 0.48 + o(1) \) holds as \( N \to \infty \) uniformly for \( \alpha \in \mathbb{A}_{c,d} \) according to the method of Cassels \([4]\). For \( n \in \mathcal{L}(N) \), we define

\[
(4.7) \quad \sigma(n) = \sum_{p \in S_\alpha} \sum_{v=1}^{\infty} \phi(p^v, n),
\]

where \( \phi(p^v, n) = \log p \) if \( p^v \mid (n + \alpha)a \), and \( \phi(p^v, n) = 0 \) otherwise. Then we have the following result.

**Proposition 4.5.** Let \( 1/2 < \sigma < 1 \) be a fixed real number. Then we have

\[
(4.8) \quad \sum_{n \in \mathcal{S}_\alpha(N)} \frac{\sigma(n)}{(n + \alpha)^\sigma} \geq (2\rho + o(1)) \log(N \log N) \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma}
\]

as \( N \to \infty \) uniformly for \( \alpha \in \mathbb{A}_{c,d} \), where \( \rho = \rho(\sigma, \alpha, N) \) satisfies \((4.6)\).

**Proof.** For \( \alpha \in \mathbb{A}_{c,d} \) and \( n \in \mathbb{Z}_{\geq 0} \), we have

\[
N((n + \alpha)a) = N_{K/Q}(n + \alpha) N(a) \geq (n + c)^2 \gg n^2.
\]
since \((b \pm \sqrt{d})/a > c\) is satisfied for any \(\alpha = (b \pm \sqrt{d})/a \in \mathbb{A}_{c,d}\). Applying this lower bound and Lemma 4.3, we deduce from (4.3) the inequality
\[
\sum_{\alpha \in \mathbb{A}_{c,d}} u_n(\alpha) \log p \geq 2 \log n + O(1),
\]
where the implied constant depends only on \(d\). For \(n \in \mathcal{S}_n(N)\), it further yields that \(\sigma(n) \geq 2 \log n + O(1)\) by the definition of \(\phi(p^v, n)\). Therefore,
\[
\sum_{\alpha \in \mathbb{A}_{c,d}} \frac{\sigma(n)}{(n + \alpha)^\sigma} \geq \sum_{\alpha \in \mathbb{A}_{c,d}} \frac{2 \log n + O(1)}{(n + \alpha)^\sigma} \geq (2 \rho + o(1)) \log N \sum_{\alpha \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma}.
\]
Using that \(\log N = \log(N \log N)(1 + o(1))\) as \(N \to \infty\), we obtain the conclusion.

Furthermore, we divide \(\sigma(n)\) of (4.7) into the following three parts:
\[
\begin{align*}
\sigma_1(n) &= \sum_{p^v \leq N \log N} \sum_{p \leq \sqrt{N \log N}} \phi(p^v, n), \\
\sigma_2(n) &= \sum_{p^v \leq N \log N} \sum_{\sqrt{N \log N} < p \leq N \log N} \phi(p, n), \\
\sigma_3(n) &= \sum_{p^v \leq N \log N} \phi(p, n).
\end{align*}
\]

**Proposition 4.6.** Let \(1/2 < \sigma < 1\) be a fixed real number. Then we have
\[
\begin{align*}
(4.9) \quad &\sum_{n \in \mathcal{S}_n(N)} \frac{\sigma_1(n)}{(n + \alpha)^\sigma} = o(\log(N \log N)) \sum_{\alpha \in \mathbb{A}_{c,d}} \frac{1}{(n + \alpha)^\sigma}, \\
(4.10) \quad &\sum_{n \in \mathcal{S}_n(N)} \frac{\sigma_2(n)}{(n + \alpha)^\sigma} \leq \left(\frac{1}{2} + o(1)\right) \log(N \log N) \sum_{\alpha \in \mathbb{A}_{c,d}} \frac{1}{(n + \alpha)^\sigma}, \\
(4.11) \quad &\sum_{n \in \mathcal{S}_n(N)} \frac{\sigma_3(n)^2}{(n + \alpha)^\sigma} \leq \left(\frac{3}{8} + o(1)\right) \log^2(N \log N) \sum_{\alpha \in \mathbb{A}_{c,d}} \frac{1}{(n + \alpha)^\sigma}
\end{align*}
\]
as \(N \to \infty\) uniformly for \(\alpha \in \mathbb{A}_{c,d}\).

**Proof.** We have
\[
\sum_{n \in \mathcal{S}_n(N)} \frac{\sigma_1(n)}{(n + \alpha)^\sigma} \leq \sum_{n \in \mathcal{L}(N)} \frac{\sigma_1(n)}{(n + \alpha)^\sigma} = \sum_{p \in \mathcal{S}_n} \sum_{\substack{v = 2 \to \infty \leq N \log N}} \phi(p^v, n). \]

By Lemma 4.3, the inner sum is evaluated as
\[
\sum_{n \in \mathcal{L}(N)} \phi(p^v, n) \leq \log p \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} \]

for some integer $a$ with $0 \leq a < p^n$. Then we apply Lemma 4.4 to obtain
\[
\sum_{n \in \mathfrak{O}_a(N)} \frac{\sigma_1(n)}{(n + \alpha)^\sigma} \leq \sum_{p \in S_\alpha} \sum_{v=2}^{\infty} \frac{\log p}{p^v} \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + O(N^{-\sigma})
\]
\[
\leq \sum_{p \in S_\alpha} \sum_{v=2}^{\infty} \frac{\log p}{p^v} \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + N^{-\sigma} \sum_{p \in S_\alpha} \log p \sum_{p \leq \sqrt{N \log N}} 1
\]
\[
\leq \sum_{p \in S_\alpha} \frac{\log p}{p^2} \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + N^{-\sigma} \log N \sum_{p \leq \sqrt{N \log N}} 1,
\]
where the implied constants are absolute. The prime ideal theorem yields
\[
\sum_{p \leq \sqrt{N \log N}} \frac{\log p}{p^2} \ll 1 \quad \text{and} \quad \sum_{p \leq \sqrt{N \log N}} 1 \ll N^{1/2} (\log N)^{-1/2}.
\]
Hence we have
\[
\sum_{n \in \mathfrak{O}_a(N)} \frac{\sigma_1(n)}{(n + \alpha)^\sigma} \ll \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + N^{1/2-\sigma} (\log N)^{1/2}
\]
with an absolute implied constant. Note that
\[
(4.12) \quad \sum_{n \in \mathcal{L}(N)} (n + \alpha)^{-\sigma} \asymp_{\sigma} (N \log N)^{1-\sigma}
\]
holds uniformly for $\alpha \in \mathbb{A}_{c,d}$. Hence we obtain (4.9). Furthermore, we can prove (4.10) in the same line. Indeed, it holds that
\[
\sum_{n \in \mathfrak{O}_a(N)} \frac{\sigma_2(n)}{(n + \alpha)^\sigma} \leq \sum_{p \in S_\alpha} \log p \left( \frac{1}{p} \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + O(N^{-\sigma}) \right)
\]
\[
= \sum_{p \in S_\alpha} \log p \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + O \left( N^{-\sigma} \log N \sum_{p \in S_\alpha} \frac{1}{p \leq N \log N} \right)
\]
by Lemmas 4.3 and 4.4. Then we apply the prime ideal theorem, and derive
\[
\sum_{n \in \mathfrak{O}_a(N)} \frac{\sigma_2(n)}{(n + \alpha)^\sigma} \leq \left( \frac{1}{2} \log(N \log N) + O(1) \right) \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + O \left( N^{1-\sigma} \log N \right)
\]
\[
= \frac{1}{2} \log(N \log N) \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + O \left( N^{1-\sigma} \log N \right).
Therefore we obtain (4.10) by using (4.12). The proof of (4.11) requires more careful treatments. By the definition of $\sigma_3(n)$, we have

$$\sum_{n \in S_\alpha(N)} \frac{\sigma_3(n)^2}{(n + \alpha)^\sigma} \leq \sum_{p_1, p_2 \in S_\alpha} \sum_{n \in \mathcal{L}(N)} \frac{\phi(p_1, n)\phi(p_2, n)}{(n + \alpha)^\sigma},$$

where $p_1 = N(p_1)$ and $p_2 = N(p_2)$. The diagonal contribution is estimated as

$$\sum_{p \in S_\alpha} \frac{\log^2 p}{p \leq \sqrt{N \log N}} \leq \sum_{p \in S_\alpha} \frac{\log^2 p}{p \leq \sqrt{N \log N}} \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + O \left( N^{-\sigma} \log^2 N \sum_{p \in S_\alpha} \frac{1}{p \leq \sqrt{N \log N}} \right),$$

$$= \left( \frac{1}{8} \log^2 (N \log N) + O(1) \right) \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + O \left( N^{1/2 - \sigma} (\log N)^{3/2} \right),$$

$$= \frac{1}{8} \log^2 (N \log N) \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + O \left( N^{1-\sigma} \log N \right),$$

similarly to the calculations for (4.9) and (4.10). Assume $p_1 \neq p_2$ for $p_1, p_2 \in S_\alpha$. If $p_1 \neq p_2$, then we have

$$\sum_{n \in \mathcal{L}(N)} \frac{\phi(p_1, n)\phi(p_2, n)}{(n + \alpha)^\sigma} \leq (\log p_1)(\log p_2) \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma}$$

by Lemma 4.3 and the Chinese remainder theorem. Thus Lemma 4.4 yields

$$\sum_{n \in \mathcal{L}(N)} \frac{\phi(p_1, n)\phi(p_2, n)}{(n + \alpha)^\sigma} \leq (\log p_1)(\log p_2) \left( \frac{1}{p_1 p_2} \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + O(N^{-\sigma}) \right)$$

for such $p_1$ and $p_2$. On the other hand, if $p_1 = p_2 = p$, then we have $p_1 p_2 = (p)$. Hence the condition $\phi(p_1, n)\phi(p_2, n) \neq 0$ implies $(p) \mid (n + \alpha)a$. By the same argument as in the proof of Lemma 4.2, the prime number $p$ must divide $2d_1$, where we write $d = (d_1)^2d_2$. This deduces that we have at most finite number of $p_1, p_2$ with
Let $p_1 = p_2$ such that $\phi(p_1, n)\phi(p_2, n) \neq 0$. As a result, the off-diagonal contribution is

$$\sum_{p_1, p_2 \in S_n, p_1 \neq p_2, p_1, p_2 \leq \sqrt{N\log N}} \frac{\phi(p_1, n)\phi(p_2, n)}{(n + \alpha)^\sigma} \leq \sum_{p_1, p_2 \in S_n, p_1 \neq p_2, p_1, p_2 \leq \sqrt{N\log N}} \frac{\log p_1\log p_2}{p_1p_2} \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + O(N^{-\sigma}) + O_d\left(\sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma}\right)$$

$$= \sum_{p_1, p_2 \in S_n, p_1 \neq p_2, p_1, p_2 \leq \sqrt{N\log N}} \frac{\log p_1\log p_2}{p_1p_2} \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + O_d\left(N^{1-\sigma}\log N\right)$$

$$\leq \frac{1}{4}\log^2(N\log N) + O(\log N) \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma} + O_d\left(N^{1-\sigma}\log N\right)$$

$$= \frac{1}{4}\log(N\log N)^2 \sum_{n \in \mathcal{L}_N} (n + \alpha)^{-\sigma} + O_d\left(N^{1-\sigma}(\log N)^{3/2}\right).$$

By (4.13) and (4.14), the sum under consideration is

$$\sum_{n \in \mathcal{L}_N} \frac{\sigma_3(n)}{(n + \alpha)^\sigma} \leq \frac{3}{8}\log(N\log N)^2 \sum_{n \in \mathcal{L}_N} (n + \alpha)^{-\sigma} + O\left(N^{1-\sigma}(\log N)^{3/2}\right)$$

with the implied constant depending only on $d$. Hence we obtain (4.11) by (4.12). □

**Remark 4.7.** As mentioned in Remark 1.6, the original method by Cassels required the condition for prime ideals $p$ such that for any integer $m$, if $p \mid (m + \alpha)a$ then $p' \mid (m + \alpha)a$ for any $p' \neq p$ with $N(p') = p$. Thus the terms in (4.13) with $p_1 = p_2$ did not appear in [3]. However, as seen above, the contribution of the terms with $p_1 = p_2$ can be negligible at least in the case $\alpha \in \mathbb{A}_{c,d}$. The author hopes that the method of this paper can be extended for algebraic numbers of higher degree.

Now we are ready to show $\rho \leq 0.48 + o(1)$ as $N \to \infty$ uniformly for $\alpha \in \mathbb{A}_{c,d}$. By (4.8), (4.9), (4.10), we have the lower bound

$$\sum_{n \in \mathcal{L}_N} \frac{\sigma_3(n)}{(n + \alpha)^\sigma} \geq \left(2\rho - \frac{1}{2} + o(1)\right) \log(N\log N) \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma}.$$
by the Cauchy–Schwarz inequality. Thus it is deduced from (4.11) that
\[
\sum_{n \in \mathcal{G}_\alpha(N)} \frac{\sigma_3(n)}{(n + \alpha)^\sigma} \leq \left( \sqrt{\frac{3\rho}{8}} + o(1) \right) \log(N \log N) \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma}.
\]
Comparing the lower and upper bounds, we obtain the inequality
\[
2\rho - \frac{1}{2} \leq \sqrt{\frac{3\rho}{8}} + o(1).
\]
If \(2\rho - 1/2 \leq 0\), then \(\rho \leq 1/4\) follows. Otherwise, the above inequality yields
\[
\rho \leq \frac{19 + \sqrt{105}}{64} + o(1) = 0.4569 \ldots + o(1).
\]
Therefore we obtain \(\rho \leq 0.48 + o(1)\) in either case.

4.3. **Completion of the proof.** Finally, we complete the proof of Proposition 4.1.

**Proof of Proposition 4.1.** With the notation above, we define
\[
\mathfrak{T}_\alpha(N) = \mathcal{L}(N) \setminus \mathcal{G}_\alpha(N)
\]
\[
= \left\{ n \in \mathcal{L}(N) \mid p^{\alpha_n(p)} > N \log N \text{ for some } p \in S_\alpha \right\}.
\]
Hence, for any \(n \in \mathfrak{T}_\alpha(N)\), there exists a prime ideal \(p \in S_\alpha\) such that
\[
(4.15) \quad p^v \mid (n + \alpha)a \quad \text{and} \quad p^v > N \log N
\]
for some integer \(v \geq 1\). When \(p \in S_\alpha\) and \(v \geq 1\) are given, there exists at most one element \(n \in \mathcal{L}(N)\) satisfying (4.15). Indeed, if \(p^v \mid (m + \alpha)a\) and \(p^v \mid (n + \alpha)a\), then \(m \equiv n \pmod{p^v}\) holds by Lemma 4.3. For \(m, n \in \mathcal{L}(N)\), we have \(m, n \leq N \log N\). Thus the condition \(m \equiv n \pmod{p^v}\) with \(p^v > N \log N\) implies \(m = n\). As a result, we have
\[
\sum_{n \in \mathfrak{T}_\alpha(N)} \frac{1}{(n + \alpha)^\sigma} \leq N^{-\sigma} \sum_{p \in S_\alpha} \frac{1}{p \leq N \log N} \ll N^{1-\sigma}
\]
by the prime ideal theorem. If \(p\) in (4.15) satisfies \(p > N \log N\), then the condition \(p \mid (m + \alpha)a\) for some \(0 \leq m \leq N \log N\) implies \(m = n\) since we have \(m \equiv n \pmod{p}\) by Lemma 4.3. In other words, such a prime ideal \(p\) does not divide \((m + \alpha)a\) for any integer \(0 \leq m \leq N \log N\) with \(m \neq n\). Therefore, we have
\[
\sum_{n \in \mathfrak{K}_\alpha(N)} \frac{1}{(n + \alpha)^\sigma} \geq \sum_{n \in \mathfrak{T}_\alpha(N)} \frac{1}{(n + \alpha)^\sigma} + O\left(N^{1-\sigma}\right)
\]
\[
= (1 - \rho + o(1)) \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)^\sigma}
\]
by using (4.12). Thus we derive the desired result by \(\rho \leq 0.48 + o(1)\). \(\square\)
5. Beurling–Selberg function

For $x \in \mathbb{R}$, the signum function $\text{sgn}(x)$ is defined as

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

In 1930s, Beurling noticed nice relations between $\text{sgn}(x)$ and the entire function

$$B(z) = \left( \frac{\sin \pi z}{\pi} \right)^2 \left\{ \sum_{m=0}^{\infty} \frac{1}{(z-m)^2} - \sum_{m=-\infty}^{-1} \frac{1}{(z-m)^2} + \frac{2}{z} \right\},$$

which is of exponential type $2\pi$. Indeed, he showed that $\text{sgn}(x) \leq B(x)$ is satisfied for any $x \in \mathbb{R}$, and we have

$$\int_{\mathbb{R}} (B(x) - \text{sgn}(x)) \, dx = 1.$$

Furthermore, if $F(z)$ is an entire function of exponential type $2\pi$ which satisfies that $\text{sgn}(x) \leq F(x)$ for any $x \in \mathbb{R}$, then the inequality

$$(5.1) \quad \int_{\mathbb{R}} (F(x) - \text{sgn}(x)) \, dx \geq 1$$

holds. Beurling further proved that the equality in $(5.1)$ is valid if and only if $F(z)$ equals to $B(z)$. Although Beurling’s results were not published, the entire function $B(z)$ played an important role in the work of Selberg on the large sieve. Many of the properties on the Beurling–Selberg function $B(z)$ and related topics are collected in the survey of Vaaler [17].

Define the entire functions $H(z)$ and $K(z)$ as

$$H(z) = \left( \frac{\sin \pi z}{\pi} \right)^2 \left\{ \sum_{m=\infty}^{\infty} \frac{\text{sgn}(m)}{(z-m)^2} + \frac{2}{z} \right\} \quad \text{and} \quad K(z) = \left( \frac{\sin \pi z}{\pi z} \right)^2.$$

Then they are of exponential type $2\pi$ and satisfy $B(z) = H(z) + K(z)$. It was proved in [17] Lemma 5] that

$$(5.2) \quad |\text{sgn}(x) - H(x)| \leq K(x) \quad \text{and} \quad |H(x)| \leq 1$$

hold for any $x \in \mathbb{R}$. Let $s, t, \Delta \in \mathbb{R}$ with $s < t$ and $\Delta > 1$. Then we define

$$U_{s,t}(z, \Delta) = \frac{1}{2} \left\{ H \left( \frac{\Delta}{2\pi} (z-s) \right) + H \left( \frac{\Delta}{2\pi} (t-z) \right) \right\},$$

$$K_{s,t}(z, \Delta) = \frac{1}{2} \left\{ K \left( \frac{\Delta}{2\pi} (z-s) \right) + K \left( \frac{\Delta}{2\pi} (t-z) \right) \right\},$$

which are entire functions of exponential type $\Delta$.

**Lemma 5.1.** Let $s, t \in \mathbb{R}$ with $s < t$. Denote by $1_{(s,t)}(x)$ the indicator function of the interval $(s, t) \subset \mathbb{R}$. Then, for any $\Delta > 1$, we have

$$(5.3) \quad |1_{(s,t)}(x) - U_{s,t}(x, \Delta)| \leq K_{s,t}(x, \Delta)$$

for any $x \in \mathbb{R}$. 
Proof. Note that \( 1_{(s,t)}(x) \) is represented as
\[
1_{(s,t)}(x) = \frac{1}{2} \left\{ \text{sgn} \left( \frac{\Delta}{2\pi} (x - s) \right) + \text{sgn} \left( \frac{\Delta}{2\pi} (t - x) \right) \right\}
\]
for any \( x \in \mathbb{R} \) with \( x \neq s, t \). Therefore, we deduce from the first inequality in (5.2) that (5.3) is valid for \( x \in \mathbb{R} \) with \( x \neq s, t \). Since \( U_{s,t}(x, \Delta) \) and \( K_{s,t}(x, \Delta) \) are continuous in \( x \), we obtain that it remains true at \( x = s, t \) by taking the limits \( x \to s - 0 \) and \( x \to t + 0 \).

\[ \square \]

Lemma 5.2. Let \( s, t \in \mathbb{R} \) with \( 0 < t - s \leq 2\pi \). Denote by \( 1_{A(s,t)}(z) \) the indicator function of the arc \( A(s,t) \) as in (2.1). Then, for any \( \Delta > 1 \), we have
\[ |1_{A(s,t)}(z) - \Upsilon_{s,t}(z, \Delta)| \leq \mathcal{K}_{s,t}(z, \Delta) \]
for any \( z \in \mathbb{C} \) with \( |z| = 1 \), where \( \Upsilon_{s,t}(z, \Delta) \) and \( \mathcal{K}_{s,t}(z, \Delta) \) are real valued functions represented as
\[ \Upsilon_{s,t}(z, \Delta) = \frac{1}{2\pi} \sum_{|m| \leq \Delta} \tilde{U}_{s,t}(m, \Delta) z^m, \quad \mathcal{K}_{s,t}(z, \Delta) = \frac{1}{2\pi} \sum_{|m| \leq \Delta} \tilde{K}_{s,t}(m, \Delta) z^m. \]

Here, \( \tilde{U}_{s,t} \) and \( \tilde{K}_{s,t} \) denote the Fourier transforms of \( U_{s,t} \) and \( K_{s,t} \), respectively.

Proof. Fix a branch of \( \text{arg}(z) \) for \( z \in \mathbb{C} \) with \( |z| = 1 \) arbitrarily. Then we have
\[ 1_{A(s,t)}(z) = \sum_{n \in \mathbb{Z}} 1_{(s,t)}(\text{arg}(z) + 2n\pi). \]
Applying (5.4), we obtain the inequality
\[ |1_{A(s,t)}(z) - \sum_{n \in \mathbb{Z}} U_{s,t}(\text{arg}(z) + 2n\pi, \Delta)| \leq \sum_{n \in \mathbb{Z}} K_{s,t}(\text{arg}(z) + 2n\pi, \Delta). \]
Note that the right-hand side is finite due to \( K_{s,t}(x, \Delta) \ll x^{-2} \) by definition. Since \( \sum_{n \in \mathbb{Z}} U_{s,t}(x + 2n\pi, \Delta) \) is periodic, it is represented as the Fourier series
\[ \sum_{n \in \mathbb{Z}} U_{s,t}(x + 2n\pi, \Delta) = \sum_{m \in \mathbb{Z}} c_{s,t}(m, \Delta) e^{imx} \]
for \( x \in \mathbb{R} \). Here, the Fourier coefficients are calculated as
\[ c_{s,t}(m, \Delta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} U_{s,t}(x + 2n\pi, \Delta) e^{-imx} \, dx \]
\[ = \frac{1}{2\pi} \int_{\mathbb{R}} U_{s,t}(x, \Delta) e^{-imx} \, dx = \frac{1}{2\pi} \tilde{U}_{s,t}(m, \Delta). \]
Furthermore, by the Paley–Wiener theorem, we see that \( c_{s,t}(m, \Delta) = 0 \) for \( |m| > \Delta \).

From the above, we obtain
\[ \sum_{n \in \mathbb{Z}} U_{s,t}(\text{arg}(z) + 2n\pi, \Delta) = \frac{1}{2\pi} \sum_{|m| \leq \Delta} \tilde{U}_{s,t}(m, \Delta) z^m = \Upsilon_{s,t}(z, \Delta) \]
for \( z \in \mathbb{C} \) with \( |z| = 1 \). Along the same line, we also obtain
\[ \sum_{n \in \mathbb{Z}} K_{s,t}(\text{arg}(z) + 2n\pi, \Delta) = \frac{1}{2\pi} \sum_{|m| \leq \Delta} \tilde{K}_{s,t}(m, \Delta) z^m = \mathcal{K}_{s,t}(z, \Delta). \]
Therefore (5.4) follows from (5.6). \[ \square \]
Lemma 5.3. Let \( s_n, t_n \in \mathbb{R} \) with \( 0 < t_n - s_n \leq 2\pi \) for \( 0 \leq n \leq N \). Then, for any \( \Delta \geq 3 \), we have

\[
\left| \prod_{n=0}^{N} 1_{A(s_n, t_n)}(z_n) - \prod_{n=0}^{N} \mathcal{U}_{s_n, t_n}(z_n, \Delta) \right| \ll (\log \Delta)^{N+1} \sum_{n=0}^{N} \mathcal{X}_{s_n, t_n}(z_n, \Delta)
\]

for \( (z_0, \ldots, z_N) \in \mathbb{C}^{N+1} \) with \( |z_0| = \cdots = |z_N| = 1 \), where \( \mathcal{U}_{s, t}(z, \Delta) \) and \( \mathcal{X}_{s, t}(z, \Delta) \) are as in Lemma 5.2 and the implied constant is absolute.

Proof. For \( z \in \mathbb{C} \) with \( |z| = 1 \), we have

\[
|\mathcal{U}_{s, t}(z, \Delta)| \leq \frac{1}{2\pi} \sum_{|m| \leq \Delta} |\tilde{U}_{s, t}(m, \Delta)|
\]

by (5.5). For \( m = 0 \), we deduce from (5.3) the formula

\[
\tilde{U}_{s, t}(0, \Delta) = \int_{\mathbb{R}} U_{s, t}(x, \Delta) \, dx = (s - t) + O\left(\frac{1}{\Delta}\right)
\]

since the integral of \( K_{s, t}(x, \Delta) \) over \( \mathbb{R} \) is calculated as

\[
\int_{\mathbb{R}} K_{s, t}(x, \Delta) \, dx = \frac{1}{2} \int_{\mathbb{R}} \left\{ K\left(\frac{\Delta}{2\pi}(x - s)\right) + K\left(\frac{\Delta}{2\pi}(t - x)\right) \right\} \, dx
= \frac{2\pi}{\Delta} \int_{\mathbb{R}} \left(\frac{\sin \pi x}{\pi x}\right)^2 \, dx \ll \frac{1}{\Delta}.
\]

It was proved in [17] Theorem 6] that \( J'(x) := \frac{1}{2} H'(x) \ll (1 + |x|)^{-3} \) holds for \( x \in \mathbb{R} \). This yields

\[
\frac{\partial}{\partial x} U_{s, t}(x, \Delta) \ll \frac{\Delta}{(1 + \Delta |x - s|)^3} + \frac{\Delta}{(1 + \Delta |x - t|)^3}.
\]

Therefore, we have

\[
\tilde{U}_{s, t}(m, \Delta) = -\frac{1}{2\pi m} \int_{\mathbb{R}} \left(\frac{\partial}{\partial x} U_{s, t}(x, \Delta) \right) e^{-imx} \, dx \ll \frac{1}{|m|}
\]

for \( m \neq 0 \) by integrating by parts, where the implied constant is absolute. Combining (5.8) and (5.10), we deduce the estimate

\[
|\mathcal{U}_{s, t}(z, \Delta)| \ll 1 + \sum_{0 < m \leq \Delta} \frac{1}{m} \ll \log \Delta.
\]

Then we prove (5.7) by induction on \( N \). Firstly, Lemma 5.2 yields the result for \( N = 0 \). Furthermore, we have

\[
\left| \prod_{n=0}^{N} 1_{A(s_n, t_n)}(z_n) - \prod_{n=0}^{N} \mathcal{U}_{s_n, t_n}(z_n, \Delta) \right|
\leq \prod_{n=0}^{N} 1_{A(s_n, t_n)}(z_n) \cdot \left| 1_{A(s_{N+1}, t_{N+1})}(z_{N+1}) - \mathcal{U}_{s_{N+1}, t_{N+1}}(z_{N+1}, \Delta) \right|
+ \prod_{n=0}^{N} 1_{A(s_n, t_n)}(z_n) \cdot \left| \mathcal{U}_{s_{N+1}, t_{N+1}}(z_{N+1}, \Delta) \right|.
\]

Therefore, by the inductive assumption and (5.11), we obtain the result for \( N+1 \). \( \square \)
6. Proof of Theorem 1.3

6.1. Support of random variables. Let $\mu$ be a probability measure on $(\mathbb{C}, B(\mathbb{C}))$. The support of $\mu$ is defined as

$$\text{supp}(\mu) = \{ z \in \mathbb{C} \mid \mu(A) > 0 \text{ holds for any set } A \text{ with } z \in A^i \},$$

where $A^i$ denotes the interior of $A$. For a random variable $X$, we define $\text{supp}(X)$ as the support of the law of $X$. Then $\text{supp}(X)$ is a non-empty closed set of $\mathbb{C}$. For closed subsets $A_j \subset \mathbb{C}$ for $1 \leq j \leq n$, we denote by $A_1 + \cdots + A_n$ the set of all points $a_1 + \cdots + a_n$ with $a_j \in A_j$ for $1 \leq j \leq n$. If every $A_j$ is bounded, then we see that $A_1 + \cdots + A_n$ is again closed. Therefore, if $X_1, \ldots, X_n$ are independent random variables such that every $\text{supp}(X_j)$ is bounded, then

$$\text{(6.1)} \quad \text{supp}(S_n) = \text{supp}(X_1) + \cdots + \text{supp}(X_n) = \text{supp}(X_1) + \cdots + \text{supp}(X_n)$$

holds for $S_n = X_1 + \cdots + X_n$; see the proof of [10, Proposition B.10.8]. The purpose of this subsection is to study the support of $\zeta_N(\sigma, X_\alpha)$ defined as [13,2].

**Proposition 6.1.** Let $\mathcal{A}_{c,d}$ denote the set of Proposition 4.1. Let $1/2 < \sigma < 1$ be a fixed real number. For any $z_0 \in \mathbb{C}$, there exists an integer $N_2 = N_2(c, d, \sigma, z_0)$ depending only on $c, d, \sigma$, and $z_0$ such that

$$\text{(6.2)} \quad z_0 \in \text{supp} (\zeta_N(\sigma, X_\alpha))$$

holds for all $\alpha \in \mathcal{A}_{c,d}$ and $N \geq N_2$.

**Proof.** Let $\mathcal{L}(N)$ and $\mathcal{M}(N)$ be the sets $\mathcal{L}(N) = \{ n \in \mathbb{Z} \mid N < n \leq N \log N \}$ and $\mathcal{M}(N) = \{ n \in \mathbb{Z} \mid 0 \leq n \leq N \log N \}$. For $\alpha \in \mathcal{A}_{c,d}$, we denote by $\mathcal{K}_\alpha(N)$ the subset of $\mathcal{L}(N)$ as in Lemma 2.3. Then we put

$$\mathcal{K}_\alpha(N) = \{ n_1, \ldots, n_k \} \quad \text{and} \quad \mathcal{M}(N) \setminus \mathcal{K}_\alpha(N) = \{ n_{k+1}, \ldots, n_\ell \}.$$

By Lemma 2.3, the random variables

$$\text{(6.3)} \quad \frac{X_\alpha(n_1)}{(n_1 + \alpha)^\sigma} \cdots \frac{X_\alpha(n_k)}{(n_k + \alpha)^\sigma} \cdot \sum_{j=k+1}^\ell \frac{X_\alpha(n_j)}{(n_j + \alpha)^\sigma}$$

are independent. Note that the support of the random variable $X_\alpha(n)/(n + \alpha)^\sigma$ is equal to the circle $\{ z : |z| = (n + \alpha)^{-\sigma} \}$. Thus we apply (6.1) to deduce

$$\text{supp} \left( \sum_{n \in \mathcal{M}(N)} \frac{X_\alpha(n)}{(n + \alpha)^\sigma} \right)$$

$$= \text{supp} \left( \frac{X_\alpha(n_1)}{(n_1 + \alpha)^\sigma} \right) + \cdots + \text{supp} \left( \frac{X_\alpha(n_k)}{(n_k + \alpha)^\sigma} \right) + \text{supp} \left( \sum_{j=k+1}^\ell \frac{X_\alpha(n_j)}{(n_j + \alpha)^\sigma} \right).$$

Furthermore, we deduce from [8, Theorem 9] that

$$\text{supp} \left( \frac{X_\alpha(n_1)}{(n_1 + \alpha)^\sigma} \right) + \cdots + \text{supp} \left( \frac{X_\alpha(n_k)}{(n_k + \alpha)^\sigma} \right) = \{ z : a_N \leq |z| \leq b_N \},$$
where \( a_N \) and \( b_N \) are non-negative real numbers determined as follows. Assume that there exists an element \( n_{j_0} \in K_\alpha(N) \) such that the inequality
\[
\frac{1}{(n_{j_0} + \alpha)^\sigma} > \sum_{1 \leq j \leq k, j \neq j_0} \frac{1}{(n_j + \alpha)^\sigma}
\]
is satisfied. In this case we put
\[
a_N = \frac{1}{(n_{j_0} + \alpha)^\sigma} - \sum_{1 \leq j \leq k, j \neq j_0} \frac{1}{(n_j + \alpha)^\sigma}.
\]
Otherwise, we put \( a_N = 0 \). Also, the non-negative real number \( b_N \) is determined as
\[
b_N = \frac{1}{(n_1 + \alpha)^\sigma} + \cdots + \frac{1}{(n_k + \alpha)^\sigma} = \sum_{n \in K_\alpha(N)} \frac{1}{(n + \alpha)^\sigma}.
\]
Note that (6.4) is equivalent to
\[
\frac{2}{(n_{j_0} + \alpha)^\sigma} > \sum_{n \in K_\alpha(N)} \frac{1}{(n + \alpha)^\sigma}.
\]
However, we have \( 2/(n_{j_0} + \alpha)^\sigma \ll N^{-\sigma} \) due to \( n_{j_0} \in (N, N \log N] \), while the right-hand side has the lower bound
\[
\sum_{n \in K_\alpha(N)} \frac{1}{(n + \alpha)^\sigma} \gg (N \log N)^{1-\sigma}
\]
for \( \alpha \in \mathbb{A}_{c,d} \) and \( N \geq N_1 = N_1(c,d,\sigma) \) by Proposition 4.1. Therefore the case in which (6.4) is valid does not occur for \( N \geq N_1 \). Hence (6.3) yields
\[
supp \left( \sum_{n \in M(N)} \frac{\mathcal{X}_\alpha(n)}{(n + \alpha)^\sigma} \right) = \{ z : |z| \leq b_N \} + supp \left( \sum_{j=k+1}^\ell \frac{\mathcal{X}_\alpha(n_j)}{(n_j + \alpha)^\sigma} \right)
\]
for \( N \geq N_1 \). Then we prove (6.2) by using this formula. Let \( z_0 \in \mathbb{C} \). We take an arbitrarily complex number \( w_0 \) such that
\[
w_0 \in supp \left( \sum_{j=k+1}^\ell \frac{\mathcal{X}_\alpha(n_j)}{(n_j + \alpha)^\sigma} \right).
\]
Then the absolute value is bounded as
\[
|w_0| \leq \sum_{j=k+1}^\ell \frac{1}{(n_j + \alpha)^\sigma} = \sum_{n=0}^N \frac{1}{(n + \alpha)^\sigma} + \sum_{n \in \mathbb{L}(N) \setminus K_\alpha(N)} \frac{1}{(n + \alpha)^\sigma}.
\]
Note that we have
\[
\sum_{n=0}^N \frac{1}{(n + \alpha)^\sigma} \ll_{c,\sigma} N^{1-\sigma} \quad \text{and} \quad \sum_{n \in \mathbb{L}(N)} \frac{1}{(n + \alpha)^\sigma} \ll_{\sigma} (N \log N)^{1-\sigma}
\]
for \( \alpha \in \mathbb{A}_{c,d} \). Thus we see that
\[
\sum_{n=0}^N \frac{1}{(n + \alpha)^\sigma} \leq \frac{1}{200} \sum_{n \in \mathbb{L}(N)} \frac{1}{(n + \alpha)^\sigma}
\]
holds for any $N \geq N_{2,1}$, where $N_{2,1} = N_{2,1}(c, \sigma)$ is an integer depending only on $c$ and $\sigma$. By Proposition 4.1, we also obtain
\[
\sum_{n \in \mathcal{L}(N) \setminus \mathcal{K}_\alpha(N)} \frac{1}{(n + \alpha)\sigma} < \frac{49}{100} \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)\sigma}
\]
for $N \geq N_1$. Combining these inequalities, we deduce from (6.6) that
\[
|w_0| < \frac{99}{200} \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)\sigma}
\]
is satisfied for any $N \geq \max\{N_1, N_{2,1}\}$. On the other hand, there exists an integer $N_{2,2} = N_{2,2}(\sigma, z_0)$ depending only on $\sigma$ and $z_0$ such that
\[
|z_0| \leq \frac{1}{200} \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)\sigma}
\]
for any $N \geq N_{2,2}$. From these, we obtain
\[
|z_0 - w_0| \leq \frac{50}{100} \sum_{n \in \mathcal{L}(N)} \frac{1}{(n + \alpha)\sigma} < \sum_{n \in \mathcal{K}_\alpha(N)} \frac{1}{(n + \alpha)\sigma} = b_N
\]
for $N \geq \max\{N_1, N_{2,1}, N_{2,2}\}$ by applying Proposition 4.1 again. This implies that $z_0 - w_0 \in \{z : |z| \leq b_N\}$, and furthermore, we deduce from (6.5) that
\[
z_0 = (z_0 - w_0) + w_0 \in \text{supp} \left( \sum_{n \in \mathcal{M}(N)} \frac{X_\alpha(n)}{(n + \alpha)\sigma} \right).
\]
Therefore we obtain (6.2) by changing variables. □

6.2. A restricted expectation. Let $\mathcal{X}$ be a random variable defined on the space $(\Omega, \mathcal{F}, P)$. For an event $\Omega_0 \in \mathcal{F}$, we define
\[
\mathbb{E} \left[ \mathcal{X} : \Omega_0 \right] = \int_{\Omega_0} \mathcal{X}(\omega) \, P(d\omega).
\]
In this subsection, we evaluate $\mathbb{E} \left[ |\zeta_N^\sigma(X_\alpha) - \zeta_N^\sigma(X_\alpha)|^2 : \Omega_0(N) \right]$ with an event $\Omega_0(N)$ defined as follows. By the definition of the support, (6.2) implies that
\[
\mathbb{P} \left( |\zeta_N^\sigma(X_\alpha) - z_0| < \epsilon \right) > 0
\]
for any $\epsilon > 0$. Hence, for $N \geq N_2$ with the integer $N_2$ as in Proposition 6.1 we can take at least one element $\omega_0 \in \Omega$ such that $|\zeta_N^\sigma(X_\alpha)(\omega_0) - z_0| < \epsilon$ is satisfied. Put $\theta_n = \arg X_\alpha(n)(\omega_0)$ for $0 \leq n \leq N$. Then we define
\[
\Omega_0(N) = \Omega_0(N; \alpha, \delta)
\]
\[
= \{ \omega \in \Omega \mid X_\alpha(n)(\omega) \in A(\theta_n - 2\pi\delta, \theta_n + 2\pi\delta) \text{ for all } 0 \leq n \leq N \}.
\]
for $0 < \delta < 1/2$, where $A(s,t)$ denotes the arc as in (2.1). For $\omega \in \Omega_0(N)$, the difference between $\zeta_N(\sigma, \mathcal{X}_\alpha)(\omega)$ and $\zeta_N(\sigma, \mathcal{X}_\alpha)(\omega_0)$ is estimated as

$$|\zeta_N(\sigma, \mathcal{X}_\alpha)(\omega) - \zeta_N(\sigma, \mathcal{X}_\alpha)(\omega_0)| \leq \sum_{n=0}^{N} \left| \zeta_N(n)(\omega) - \zeta_N(n)(\omega_0) \right| \frac{1}{(n+\alpha)^\sigma}$$

$$\ll \delta \sum_{n=0}^{N} \frac{1}{(n+\alpha)^\sigma} \ll \epsilon \delta N^{1/2}$$

for any $\alpha \in \mathbb{A}_{c,d}$, where the implied constants depend at most on $c$. Therefore, the condition $\omega \in \Omega_0(N)$ implies

$$|\zeta_N(\sigma, \mathcal{X}_\alpha)(\omega) - z_0| < 2\epsilon$$

for any $\alpha \in \mathbb{A}_{c,d}$ if we suppose $\delta \leq A(c)\epsilon N^{-1/2}$, where $A(c)$ is a suitable positive constant.

**Proposition 6.2.** Let $N_2$ be the integer as in Proposition 6.1. For any integer $N \geq N_2$ and $\Delta \geq \Delta_0$ with some absolute constant $\Delta_0$, there exists a finite subset $E_{c,d} = E_{c,d}(N, \Delta) \subset \mathbb{A}_{c,d}$ such that

$$P(\Omega_0(N)) = (2\delta)^N + O\left( \frac{N(\log \Delta)^N}{\Delta} \right)$$

holds for all $\alpha \in \mathbb{A}_{c,d} \setminus E_{c,d}$, where the implied constant is absolute.

**Proof.** Put $s_n = \theta_n - \delta$ and $t_n = \theta_n + \delta$. Then we have

$$P(\Omega_0(N)) = E \left[ \prod_{n=0}^{N} 1_{A(s_n,t_n)}(\mathcal{X}_\alpha(n)) \right]$$

$$= E \left[ \prod_{n=0}^{N} \mathcal{U}_{s_n,t_n}(\mathcal{X}_\alpha(n), \Delta) \right] + O\left( (\log \Delta)^N \sum_{n=0}^{N} E[\mathcal{X}_{s_n,t_n}(\mathcal{X}_\alpha(n), \Delta)] \right)$$

by Lemma 5.3. Inserting (5.5), we calculate the main term as

$$E \left[ \prod_{n=0}^{N} \mathcal{U}_{s_n,t_n}(\mathcal{X}_\alpha(n), \Delta) \right]$$

$$= \left( \frac{1}{2\pi} \right)^{N+1} \sum_{\left| m_0 \right| \leq \Delta} \cdots \sum_{\left| m_N \right| \leq \Delta} \prod_{n=0}^{N} \mathcal{U}_{s_n,t_n}(m_n, \Delta) E \left[ \prod_{n=0}^{N} \mathcal{X}_\alpha(n)^{m_n} \right].$$

Here, we have

$$E \left[ \prod_{n=0}^{N} \mathcal{X}_\alpha(n)^{m_n} \right] = \begin{cases} 1 & \text{if } \prod_{n=0}^{N} (n+\alpha)^{m_n} = 1, \\ 0 & \text{otherwise} \end{cases}$$

by condition (1.5). Then we define $E_{c,d} = E_{c,d}(N, \Delta)$ as the set of all $\alpha \in \mathbb{A}_{c,d}$ such that $\prod_{n=0}^{N} (n+\alpha)^{m_n} = 1$ for some $(m_0, \ldots, m_N) \in \mathbb{Z}^{N+1} \setminus \{0\}$ with $|m_n| \leq \Delta$ for any $0 \leq n \leq N$. Note that $E_{c,d}$ is a finite set when $N$ and $\Delta$ are given. Furthermore,
we obtain
\[ (6.10) \]
by using \((5.8)\). Then we evaluate the error term in \((6.7)\). We have

\[ \mathbf{E}[\mathcal{X}_{n,t}(\mathbf{x},\mathbf{y})] = \frac{1}{2\pi} |m| \mathbf{K}_{n,t}(m) \mathbf{E}[\mathcal{X}_n(m)] = \frac{1}{2\pi} \mathbf{K}_{n,t}(0) \]

since \(\mathbf{E}[\mathcal{X}_n(m)] = 0\) for \(m \neq 0\) by \((1.5)\). Due to the definition of \(\mathbf{K}_{n,t}(x,\mathbf{y})\), we further obtain

\[ \mathbf{K}_{n,t}(0) = \int_{\mathbb{R}} \mathbf{K}_{n,t}(x,\mathbf{y}) \, dx \ll \frac{1}{\Delta} \]

by using \((5.9)\). Therefore, we see that

\[ (6.10) \]

\[ (\log \Delta)^{N+1} \sum_{n=0}^{N} \mathbf{E}[\mathcal{X}_{n,t}(\mathbf{x},\mathbf{y})] \ll \frac{N(\log \Delta)^{N+1}}{\Delta} \]

Combining \((6.9)\) and \((6.10)\), we obtain the desired result.

**Proposition 6.3.** Let \(N_2\) be the integer as in Proposition \(6.1\). Let \(1/2 < \sigma < 1\) be a fixed real number. For any integers \(N, L\) satisfying \(L > N \geq N_2\) and \(\Delta \geq \Delta_0\) with some absolute constant \(\Delta_0\), there exists a finite subset \(c^{(1)} = c^{(1)}(N, L, \Delta) \subset \mathbb{A}^{(1)}\) such that

\[ \mathbf{E}[|\zeta(\sigma, \mathbf{x}) - \zeta_N(\sigma, \mathbf{x})|^2 : \Omega_0(N)] \]

\[ \ll \mathbf{P}(\Omega_0(N))N^{1-2\sigma} + L^{1-2\sigma} + \frac{NL(\log \Delta)^{N+1}}{\Delta} \]

holds for all \(\alpha \in \mathbb{A}_{c,d} \setminus c^{(2)}\), where the implied constant depends only on \(\sigma\).

**Proof.** By the Cauchy–Schwarz inequality, we have

\[ (6.11) \]

\[ \mathbf{E}[|\zeta(\sigma, \mathbf{x}) - \zeta_N(\sigma, \mathbf{x})|^2 : \Omega_0(N)] \]

\[ \ll \mathbf{E}[|\zeta_L(\sigma, \mathbf{x}) - \zeta_N(\sigma, \mathbf{x})|^2 : \Omega_0(N)] + \mathbf{E}[|\zeta(\sigma, \mathbf{x}) - \zeta_L(\sigma, \mathbf{x})|^2 : \Omega_0(N)] \]

The first term is calculated as

\[ \mathbf{E}[|\zeta_L(\sigma, \mathbf{x}) - \zeta_N(\sigma, \mathbf{x})|^2 : \Omega_0(N)] \]

\[ = \mathbf{P}(\Omega_0(N)) \sum_{N < n \leq L} \frac{1}{(n+\alpha)^{2\sigma}} + \sum_{N < n_1, n_2 \leq L \atop n_1 \neq n_2} \mathbf{E}[\mathcal{X}_n(n_1)\mathcal{X}_n(n_2) : \Omega_0(N)] \]

\[ \frac{(n_1 + \alpha)^{\sigma}(n_2 + \alpha)^{\sigma}}{(n_1 + \alpha)^{2\sigma}} \]

since \(\mathbf{E}[1 : \Omega_0(N)] = \mathbf{P}(\Omega_0(N))\) by definition. We have

\[ (6.12) \]

\[ \mathbf{P}(\Omega_0(N)) \sum_{N < n \leq L} \frac{1}{(n+\alpha)^{2\sigma}} \ll \mathbf{P}(\Omega_0(N))N^{1-2\sigma}. \]
Furthermore, we apply Lemma 5.3 to obtain

\[
\sum_{N < n_1, n_2 \leq L \atop n_1 \neq n_2} \mathbb{E}[\mathcal{X}_\alpha(n_1)\mathcal{X}_\alpha(n_2) : \Omega_0(N)] \frac{(n_1 + \alpha)\sigma(n_2 + \alpha)\sigma}{(n_1 + \alpha)^\sigma(n_2 + \alpha)^\sigma} = S_1 + S_2,
\]

where

\[
S_1 = \sum_{N < n_1, n_2 \leq L \atop n_1 \neq n_2} \frac{1}{(n_1 + \alpha)^\sigma(n_2 + \alpha)^\sigma} \mathbb{E} \left[ \mathcal{X}_\alpha(n_1)\mathcal{X}_\alpha(n_2) \prod_{n=0}^{N} \mathcal{Y}_{s_n, t_n}(\mathcal{X}_\alpha(n), \Delta) \right],
\]

\[
S_2 \ll \sum_{N < n_1, n_2 \leq L \atop n_1 \neq n_2} \frac{1}{(n_1 + \alpha)^\sigma(n_2 + \alpha)^\sigma} (\log \Delta)^{N+1} \sum_{n=0}^{N} \mathbb{E}[\mathcal{X}_{s_n, t_n}(\mathcal{X}_\alpha(n), \Delta)].
\]

Similarly to (6.8), we have

\[
\mathbb{E} \left[ \mathcal{X}_\alpha(n_1)\mathcal{X}_\alpha(n_2) \prod_{n=0}^{N} \mathcal{Y}_{s_n, t_n}(\mathcal{X}_\alpha(n), \Delta) \right] = \left( \frac{1}{2\pi} \right)^{N+1} \sum_{|m_0| \leq \Delta} \cdots \sum_{|m_N| \leq \Delta} \prod_{n=0}^{N} \hat{U}_{s_n, t_n}(m_n, \Delta) \mathbb{E} \left[ \mathcal{X}_\alpha(n_1)\mathcal{X}_\alpha(n_2) \prod_{n=0}^{N} \mathcal{X}_\alpha(n)^{m_n} \right].
\]

Then we define \( \mathcal{E}_{c,d}^{(2)} = \mathcal{E}_{c,d}^{(2)}(N, L, \Delta) \) as the set of all \( \alpha \in \mathbb{A}_{c,d} \) such that

\[
n_2 + \alpha = \prod_{n=0}^{N} (n + \alpha)^{m_n}
\]

for some \( N < n_1, n_2 \leq L \) with \( n_1 \neq n_2 \) and \( (m_0, \ldots, m_N) \in \mathbb{Z}^{N+1} \) with \( |m_n| \leq \Delta \) for any \( 0 \leq n \leq N \). We see that \( \mathcal{E}_{c,d}^{(2)} \) is a finite set when \( N, L, \) and \( \Delta \) are given. By the definition, if \( \alpha \in \mathbb{A}_{c,d} \setminus \mathcal{E}_{c,d}^{(2)} \), then we have

\[
\mathbb{E} \left[ \mathcal{X}_\alpha(n_1)\mathcal{X}_\alpha(n_2) \prod_{n=0}^{N} \mathcal{X}_\alpha(n)^{m_n} \right] = 0
\]

for any \( N < n_1, n_2 \leq L \) with \( n_1 \neq n_2 \) and \( (m_0, \ldots, m_N) \in \mathbb{Z}^{N+1} \) with \( |m_n| \leq \Delta \). Hence we conclude that \( S_1 = 0 \) for \( \alpha \in \mathbb{A}_{c,d} \setminus \mathcal{E}_{c,d}^{(2)} \). Next, we have

\[
\sum_{N < n_1, n_2 \leq L \atop n_1 \neq n_2} \frac{1}{(n_1 + \alpha)^\sigma(n_2 + \alpha)^\sigma} \leq \left( \sum_{N < n \leq L} \frac{1}{(n + \alpha)^{1/2}} \right)^2 \ll L.
\]

As checked before, we also obtain \( \mathbb{E}[\mathcal{X}_{s_n, t_n}(\mathcal{X}_\alpha(n), \Delta)] \ll \Delta^{-1} \). Therefore, \( S_2 \) is estimated as

\[
S_2 \ll \frac{NL(\log \Delta)^{N+1}}{\Delta}.
\]
Thus (6.13) yields
\[ \sum_{N<n_1,n_2\leq L, n_1\neq n_2} \mathbb{E}[X_{\alpha}(n_1)X_{\alpha}(n_2) : \Omega_0(N)] = \frac{NL(\log \Delta)^{N+1}}{\Delta} \]
for \( \alpha \in \mathbb{A}_{c,d} \setminus \mathcal{E}_{c,d}^{(2)} \).

Then, together with (6.12), we obtain the upper bound
\[ \mathbb{E} \left[ |\zeta_L(\sigma,X_{\alpha}) - \zeta_N(\sigma,X_{\alpha})|^2 : \Omega_0(N) \right] \leq P(\Omega_0(N))N^{1-2\sigma} + \frac{NL(\log \Delta)^{N+1}}{\Delta}. \]

The estimate for the second term in (6.11) remains. We have
\[
\begin{align*}
\mathbb{E} \left[ |\zeta(\sigma,X_{\alpha}) - \zeta_L(\sigma,X_{\alpha})|^2 : \Omega_0(N) \right] \\
\leq \mathbb{E} \left[ |\zeta(\sigma,X_{\alpha}) - \zeta_L(\sigma,X_{\alpha})|^2 \right] = \sum_{n>L} \frac{1}{(n+\alpha)^{2\sigma}} \ll L^{1-2\sigma}.
\end{align*}
\]

From these, we obtain the conclusion. \( \square \)

### 6.3. Completion of the proof

By Propositions 6.1, 6.2, and 6.3, we obtain the following result.

**Corollary 6.4.** Let \( 1/2 < \sigma < 1 \) be a fixed real number. Then, for any \( z_0 \in \mathbb{C} \) and \( \epsilon > 0 \), there exists a finite subset \( \mathcal{E}_{c,d} = \mathcal{E}_{c,d}(\sigma, \epsilon) \subset \mathbb{A}_{c,d} \) such that
\[ P\left( |\zeta(\sigma,X_{\alpha}) - z_0| < \epsilon \right) > 0 \]

holds for any \( \alpha \in \mathbb{A}_{c,d} \setminus \mathcal{E}_{c,d} \).

**Proof.** Let \( N \geq N_2 \) with the integer \( N_2 = N_2(c,d,\sigma,z_0) \) as in Proposition 6.1. Recall that \( \omega \in \Omega_0(N) = \Omega_0(N;\alpha,\delta) \) implies \( |\zeta_N(\sigma,X_{\alpha})(\omega) - z_0| < 2\epsilon \) for any \( \alpha \in \mathbb{A}_{c,d} \) if we suppose \( \delta \leq A(c)\epsilon N^{-1/2} \), where \( A(c) \) is a positive real number. Hence we have
\[
\begin{align*}
P &\left( |\zeta(\sigma,X_{\alpha}) - z_0| < 3\epsilon \right) \\
&\geq P(\Omega_0(N) \cap \{ |\zeta(\sigma,X_{\alpha}) - \zeta_N(\sigma,X_{\alpha})| < \epsilon \}) \\
&= P(\Omega_0(N)) - P(\Omega_0(N) \cap \{ |\zeta(\sigma,X_{\alpha}) - \zeta_N(\sigma,X_{\alpha})| \geq \epsilon \})
\end{align*}
\]

if \( \delta \leq A(c)\epsilon N^{-1/2} \) is satisfied. Furthermore, Proposition 6.3 deduces
\[
\begin{align*}
P &\left( \Omega_0(N) \cap \{ |\zeta(\sigma,X_{\alpha}) - \zeta_N(\sigma,X_{\alpha})| \geq \epsilon \} \right) \\
&\leq \frac{1}{\epsilon^2} \mathbb{E} \left[ |\zeta(\sigma,X_{\alpha}) - \zeta_N(\sigma,X_{\alpha})|^2 : \Omega_0(N) \right] \\
&\leq \frac{B(\sigma)}{\epsilon^2} \left\{ P(\Omega_0(N))N^{1-2\sigma} + L^{1-2\sigma} + \frac{NL(\log \Delta)^{N+1}}{\Delta} \right\}
\end{align*}
\]

for any \( \alpha \in \mathbb{A}_{c,d} \setminus \mathcal{E}_{c,d}^{(2)} \), where \( B(\sigma) \) is a positive constant depending only on \( \sigma \). Here, the set \( \mathcal{E}_{c,d}^{(2)} = \mathcal{E}_{c,d}^{(2)}(N,L,\Delta) \) is as in Proposition 6.3. We can take a positive integer \( N_3 = N_3(\sigma,\epsilon,N_2) \geq N_2 \) such that
\[ \frac{B(\sigma)}{\epsilon^2} P(\Omega_0(N_3))N_3^{1-2\sigma} < \frac{1}{2} P(\Omega_0(N_3)) \]
is satisfied. Then, putting $N = N_3$ and $\delta = A(c)\epsilon N_3^{-1/2}$, we derive the inequality

$$P \left( | \zeta(\sigma, X_{\alpha}) - z_0 | < 3\epsilon \right) > \frac{1}{2} P \left( \Omega_0(N_3) \right) - \frac{B(\sigma)}{\epsilon^2} \left\{ L^{1-2\sigma} + \frac{N_3 L(\log \Delta) N_3^{1+}}{\Delta} \right\}$$

for $\alpha \in A_{c,d} \setminus E_{c,d}^{(2)}$. Furthermore, we apply Proposition 6.2 to deduce

$$P \left( | \zeta(\sigma, X_{\alpha}) - z_0 | < 3\epsilon \right) > \frac{1}{2} \left( \frac{2\delta}{N_3^{1+}} \right)^{N_3^{1+}} - \frac{B'(\sigma)}{\epsilon^2} \left\{ L^{1-2\sigma} + \frac{N_3 L(\log \Delta) N_3^{1+}}{\Delta} \right\}$$

for $\alpha \in A_{c,d} \setminus (E_{c,d}^{(1)} \cup E_{c,d}^{(2)})$, where $B'(\sigma)$ is some positive constant, and the set $E_{c,d}^{(1)} = E_{c,d}^{(1)}(N_3, \Delta)$ is as in Proposition 6.2. Then, we see that there exists an integer $L = L(\sigma, \epsilon, \delta, N_3)$ satisfying

$$\frac{B'(\sigma)}{\epsilon^2} L^{1-2\sigma} < \frac{1}{4} (2\delta)^{N_3^{1+}}.$$ 

Finally, we take a real number $\Delta = \Delta(\sigma, \epsilon, \delta, N_3) \geq 3$ so that

$$\frac{B'(\sigma)}{\epsilon^2} \frac{N_3 L(\log \Delta) N_3^{1+}}{\Delta} < \frac{1}{4} (2\delta)^{N_3^{1+}}$$

is satisfied. Then we obtain

$$P \left( | \zeta(\sigma, X_{\alpha}) - z_0 | < 3\epsilon \right) > \frac{1}{2} (2\delta)^{N_3^{1+}}$$

for $\alpha \in A_{c,d} \setminus E_{c,d}$, with $E_{c,d} = E_{c,d}^{(1)} \cup E_{c,d}^{(2)}$, which yields the desired conclusion. □

Proof of Theorem 1.3. By the Portmanteau theorem [1, Theorem 29.1], we deduce from Theorem 1.2 the inequality

$$\liminf_{T \to \infty} \frac{1}{T} \text{ meas } \left\{ t \in [0, T] \mid | \zeta(\sigma + it, \alpha) - z_0 | < \epsilon \right\} \geq P \left( | \zeta(\sigma, X_{\alpha}) - z_0 | < \epsilon \right).$$

Hence Corollary 6.4 yields Theorem 1.3. □

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