Dynamical content of Chern-Simons Supergravity

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Abstract

The dynamical content of local $AdS$ supergravity in five dimensions is discussed. The bosonic sector of the theory contains the vielbein ($e^a$), the spin connection ($\omega^{ab}$) and internal $SU(N)$ and $U(1)$ gauge fields. The fermionic fields are complex Dirac spinors ($\psi^i$) in a vector representation of $SU(N)$. All fields together form a connection 1-form in the superalgebra $SU(2,2|N)$. For $N = 4$, the symplectic matrix has maximal rank in a locally AdS background in which the dynamical degrees of freedom can be identified. The resulting effective theory have different numbers of bosonic and fermionic degrees of freedom.

I. INTRODUCTION

Over twenty years ago, Cremmer, Julia and Scherk presented a beautiful $N = 1$ theory of supergravity in 11 dimensions [1], which, apart from the metric and a gravitino ($\psi$), included a three-form field ($A_{\mu\nu\lambda}$). This theory is quite unique: a larger $D$ or $N$ would give rise to inconsistencies upon compactification to 4 dimensions (i.e., fields with spin greater that 2

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or more than one graviton). A “dual” alternative possibility which uses a six form \( A_{\mu_1...\mu_6} \) instead of the three form also leads to inconsistencies \[2\]. Additionally, it has been also shown that regardless of the compactification arguments, the theory cannot accommodate a cosmological constant in eleven dimensions \[3\].

One of the puzzling aspects of the theory is the conjecture contained in the original paper by Cremmer, Julia and Scherk in the sense that this theory should be related to another one based on the \( osp(32|1) \) algebra, a problem that they promised to discuss in a forthcoming article that never appeared. In fact, it can be seen that in a gauge theory for an \( osp(32|1) \) connection, the anticommutator of the fermionic generators takes the form \[4\]

\[
\{Q,Q\} \approx P_a \Gamma^a + Z_{ab} \Gamma^{ab} + Z_{abcde} \Gamma^{abcde},
\]

where one could identify the different components of the connection that accompany the generators in the RHS with the fields in the CJS theory: \( A^a_{\mu} \sim e^a_{\mu} \), \( A^{ab}_{\mu} \sim A_{\mu\nu\lambda} \), \( A^{abcde}_{\mu} \sim A_{\mu_1...\mu_6} \). However, no supergravity theory was known to contain all these fields in a simple and natural way. It was therefore a surprise for us to find a family of Lagrangians (one in each odd dimension) which could prove the conjecture of Cremmer, Julia and Scherk to be true \[5\]. The key ingredients in this new family of supergravity theories is their Chern-Simons form and the fact that the spacetime symmetry is realized in the tangent space and not on the base manifold.

II. SUPERGRAVITY IN ODD DIMENSIONS

One of the unique features of gravity in 2+1 dimensions is that it is a genuine gauge theory in the sense of having a fiber bundle structure. This is because the standard Einstein-Hilbert action (both with and without cosmological constant) is a Chern-Simons (CS) form \( \int < AdA + \frac{2}{3} A^3 > \). As a consequence, the theory is an integrable system, unlike the case of four-dimensional gravity \[6\]. Furthermore, the simplest \( D = 3, N = 1 \) supergravity theory \[7\] also shares this feature, and as a bonus, the theory is \textit{locally} invariant under the anti-de
Sitter group. It is easy to see that these features can be generalized to all odd dimensions: provided one has identified the superalgebra that extends AdS in a given dimension $D$, it is just a matter of constructing the appropriate Chern-Simons $D$-form. The resulting theory would be invariant by construction under the right supergroup in which the fields of the theory transform as different pieces of a connection. Thus, the algebra of the supersymmetry transformations is guaranteed to close off shell without requiring auxiliary fields [8].

However, both identifying the algebra and the construction of the CS form involve some subtleties that it is instructive to analyze in detail. The simplest example that captures all the problems—and which yields a theory with propagating local degrees of freedom—occurs in five dimensions. In the following we study the five-dimensional case in detail and indicate where appropriate how the results generalize to other dimensions.

III. HIGHER DIMENSIONAL CS THEORIES

In 2+1 dimensions, gravity—or any other Chern-Simons theory for that matter—has no dynamical degrees of freedom. The field equations are

$$F = 0, \quad (1)$$

where $F$ is the (anti-de Sitter or Poincaré) curvature, and it therefore means that all on-shell configurations are locally gauge-equivalent to a flat connection. However, the field equations of a CS theory in dimensions five and higher—for any gauge group—take the form

$$< F \wedge F \wedge \cdots \wedge F G_M > = 0, \quad (2)$$

where $G_M$ is a gauge generator. This equation in general does not imply a flat connection, allowing for the existence of propagating degrees of freedom.

For dimensions $D > 3$, the dynamical structure of a CS system becomes highly nontrivial. The root of the complexity lies in three independent features inherent of CS theories: i) they are gauge systems; ii) they have coordinate invariance built in; and iii) they are first order
systems. Although each of these items are neither exclusive of CS systems, nor particularly difficult to deal with, their conjunction requires special care. As discussed in [9], there are two problems: diffeomorphism and gauge invariances are not completely independent, and because of the first order nature, CS systems have first class constraints inextricably mixed with second class ones. This makes the Dirac matrix hard to express in a simple form and almost impossible to invert.

Moreover, for $D > 3$ the symplectic form $(\Omega)$ that multiplies the velocities in (2) is a function of the gauge field $\Omega = \Omega(A)$, and it can degenerate for certain field configurations. In particular, for $D = 5$,

$$\Omega^{ij}_{MN}(A) = \Delta_{MNP} \epsilon^{ijkl} F^{P}_{kl},$$

(3)

where $\Delta_{MNP} \equiv \langle G_{M} G_{N} G_{P} \rangle$ is an invariant tensor of the gauge group (see Appendix B). At the singular configurations (as $F^{P}_{kl} = 0$), the rank of the Dirac matrix is reduced and becomes noninvertible, so that some (or all) second class constraints could actually be viewed as first class, and this further complicates the identification of the propagating degrees of freedom. Outside those regions, however, other dynamical structure is well behaved and the symplectic form has maximal rank.

Here we will not discuss the problems arising from the presence of degenerate surfaces. This seems to be a reasonable point of view as these singularities constitute sets of measure zero in the configuration space of the theory.

The dynamical analysis of a higher-dimensional bosonic CS theory is discussed in [9] and we summarize it here: If the gauge algebra takes the form $G \times U(1)$, where $G$ is a semisimple algebra, the symplectic form can be computed in a background where $\Omega$ has maximal rank. The configurations that satisfy this requirement are called generic in the sense that under small deformations the rank of $\Omega$ remains maximal. Also, in these configurations, the first and second class constraints can be separated and the degrees of freedom computed. If the algebra has dimension $f > 1$ and the spacetime has dimension $D = 2n + 1 > 3$, it is shown that the number of propagating local degrees of freedom of the theory is
\[ g = nf - f - n \]

and, in five dimensions, the symplectic matrix has the form

\[ \Omega^i_{MN}(\bar{A}) = g_{MN} \epsilon^{ijkl} \bar{f}_{kl}, \quad (4) \]

where the bar stands for background fields, and \( \bar{f}_{kl} = \partial_k \bar{b}_l - \partial_l \bar{b}_k \) is the curvature of the \( U(1) \) field.

\section*{IV. LOCAL ADS\textsubscript{5} SUPERGRAVITY}

The supersymmetric extension of the anti-de Sitter algebra in five dimensions is \( su(2,2|N) \) \cite{10}, whose associated connection can be written as,

\[ A = e^a J_a + \frac{1}{2} \omega^{ab} J_{ab} + a^\Lambda T_\Lambda + (\bar{\psi}^r Q_r - \bar{Q}_r \psi_r) + bZ, \]

where the generators \( J_a, J_{ab} \), form an AdS algebra \( (so(4,2)) \), \( T_\Lambda (\Lambda = 1, \ldots N^2 - 1) \) are the generators of \( su(N) \), \( Z \) generates a \( U(1) \) subgroup and \( Q_r, \bar{Q}_r \) are the supersymmetry generators, which transform in a vector representation of \( SU(N) \). The Chern-Simons Lagrangian for this gauge algebra is defined by the relation \( dL = i \langle F_3 \rangle \), where \( F = dA + A^2 \) is the (antihermitean) curvature, and \( \langle \cdots \rangle \) stands for the supertrace in the representation described in the Appendix A. Using this definition, one obtains the Lagrangian originally discussed by Chamseddine in \cite{11}.

\[ L = L_G(\omega^{ab}, e^a) + L_{su(N)}(a^r_s) + L_{u(1)}(\omega^{ab}, e^a, b) + L_F(\omega^{ab}, e^a, a^r_s, b, \psi_r), \quad (5) \]

with

\begin{align*}
L_G & = \frac{1}{8} \epsilon_{abcde} \left[ R^{ab} R^{cd} e^e / l + \frac{2}{3} R^{ab} e^c e^d e^e / l^3 + \frac{1}{3} e^a e^b e^c e^d e^e / l^5 \right] \\
L_{su(N)} & = - Tr \left[ a(da)^2 + \frac{3}{2} a^3 da + \frac{3}{8} a^5 \right] \\
L_{u(1)} & = \left( \frac{1}{4} - \frac{1}{N} \right) b(db)^2 + \frac{3}{4l^2} \left[ T^a T_a - R^{ab} e_a e_b - l^2 R^{ab} R_{ab} / 2 \right] b + \frac{3}{N} f^r_s f^s_r b \\
L_F & = \frac{3}{2l} \left[ \bar{\psi}^r \nabla \psi_r + \bar{\psi}^s \mathcal{F}_s^r \nabla \psi_r \right] + c.c.
\end{align*}
where $a_s^r \equiv a^L(\tau_\Lambda)_s^r$ is the $su(2, 2)$ connection, $f_s^r$ is its curvature, and the bosonic blocks of the supercurvature: $\mathcal{R} = \frac{1}{2} T^a \Gamma_a + \frac{1}{4} (R^{ab} + e^a e^b) \Gamma_{ab} + \frac{i}{4} db \delta_s^r - \frac{1}{2} \bar{\psi}_s \psi^s$, $\mathcal{F}_s^r = f_s^r + \frac{i}{N} db \delta_s^r - \frac{1}{2} \bar{\psi}_s \psi^s$. The cosmological constant is $- l^{-2}$, and the AdS covariant derivative $\nabla$ acting on $\psi_r$ is

$$\nabla \psi_r = D \psi_r + \frac{1}{2l} e^a \Gamma_a \psi_r - a_s^r \psi_s + i \left( \frac{1}{4} - \frac{1}{N} \right) b \psi_r. \tag{7}$$

where $D$ is the Lorentz covariant derivative.

The above relation implies that the fermions carry a $u(1)$ “electric” charge given by $e = (\frac{1}{4} - \frac{1}{N})$. The purely gravitational part, $L_G$ is equal to the standard Einstein-Hilbert action with cosmological constant, plus the dimensionally continued Euler density.

The action is by construction invariant –up to a surface term– under the local (gauge generated) supersymmetry transformations $\delta \lambda A = -(d \lambda + [A, \lambda])$ with $\lambda = \bar{\epsilon}^r Q_r - \bar{Q}_r \epsilon_r$, or

$$\delta e^a = \frac{1}{2} (\bar{\epsilon}^r \Gamma^a \psi_r - \bar{\psi}^r \Gamma^a \epsilon_r)$$

$$\delta \omega^{ab} = - \frac{1}{4} (\bar{\epsilon}^r \Gamma^{ab} \psi_r - \bar{\psi}^r \Gamma^{ab} \epsilon_r)$$

$$\delta a_s^r = - i (\bar{\epsilon}^r \psi_s - \bar{\psi}^r \epsilon_s)$$

$$\delta \psi_r = - \nabla \epsilon_r$$

$$\delta \bar{\psi}^r = - \nabla \bar{\epsilon}^r$$

$$\delta b = - i (\bar{\epsilon}^r \psi_r - \bar{\psi}^r \epsilon_r).$$

As can be seen from (6) and (7), for $N = 4$ the $b$ field looses its kinetic term and decouples from the fermions (the gravitino becomes uncharged with respect to the $U(1)$ field). The only remnant of the interaction with the $b$ field is a dilaton-like coupling with the Pontryagin four forms for the AdS and $SU(N)$ groups (in the bosonic sector). As it is also shown in

1The first term in $L_G$ is the dimensional continuation of the Euler (or Gauss-Bonnet) density from two and four dimensions, exactly as the three-dimensional Einstein-Hilbert Lagrangian is the continuation of the the two dimensional Euler density. This is the leading term in the limit of vanishing cosmological constant ($l \to \infty$), whose local supersymmetric extension yields a nontrivial extension of the Poincaré group [12].
the Appendix A, the case $N = 4$ is also special at the level of the algebra, which becomes a superalgebra with a $u(1)$ central extension.

In the bosonic sector, for $N = 4$, the field equation obtained from the variation with respect to $b$ states that the Pontryagin four form of AdS and $SU(N)$ groups are proportional. Consequently, if the curvatures approach zero sufficiently fast at spatial infinity, there is a conserved topological current which states that, for the spatial section, the second Chern characters of AdS and $SU(4)$ are proportional. Consequently, if the spatial section has no boundary, the corresponding Chern numbers are related. Using the fact that $\Pi_4(SU(4)) = 0$, the above implies that the Hirzebruch signature plus the Nieh-Yan number of the spatial section cannot change in time.

V. SYMPLECTIC FORM

We will consider a background that is a solution for the field equations, for which $\Omega$ has maximum rank. Realizing this is in general a difficult task. However, an amazing simplification happen when $N = 4$, which has its root in the form of the invariant tensor $\Delta_{MNP}$. As stated in the Appendix, considering the splitting: $M = \{M', Z\}$, when $N = 4$ we have “the accident”: $\Delta_{ZZZ} = 0$, and $\Delta_{ZM'N'} = -\frac{i}{2}g_{M'N'}$, where $g_{M'N'}$ is the Killing metric of $PSU(2,2|4)$.

This fact will help us find an adequate background. Consider any locally AdS spacetime with pure gauge matter fields (Bosons and fermions), with the exception of the $b$ field. That is, a background $\bar{A}$ such that $F^{AB} = F_A = \psi_s = \bar{\psi}^r = 0 \neq F^Z$.

It is easy to see also that for $N = 4$ the conditions that make the separation between first and second class constraints possible in the construction of $[9]$ can also be applied, even if in this case the algebra is bigraded and not a direct sum of a semisimple one and $u(1)$, but a central extension.

Indeed, it can be directly checked that the symplectic form takes the form
\[
\Omega_{MN}^{ij} [\tilde{A}] = \begin{bmatrix}
0_{4 \times 4} & 0 \\
0 & \tilde{\Omega}_{M^{'N^{'}}\!}^{ij}
\end{bmatrix}
\]  \tag{8}

where \(\tilde{\Omega}_{M^{'N^{'}}\!}^{ij}\) is generically an invertible matrix. In fact, for a flat AdS curvature (e.g., a spacetime of constant negative curvature and vanishing torsion) the symplectic matrix takes the form

\[
\Omega_{MN}^{ij} [\tilde{A}] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & \eta_{[AB][CD]} & 0 & 0 & 0 \\
0 & 0 & g_{\alpha_1\alpha_2} & 0 & 0 \\
0 & 0 & 0 & 2\delta^\gamma_\alpha \delta^\beta_\delta & 0 \\
0 & 0 & 0 & -2\delta^\alpha_\delta \delta^\beta_\gamma & 0
\end{bmatrix} \otimes -\frac{i}{4} \epsilon^{ijkl} (\partial_k \bar{b}_l - \partial_l \bar{b}_k). \tag{9}
\]

The nonvanishing block in the RHS can be recognized as the Killing metric for \(PSU(2, 2|4)\), while the factor on the right is the space-dual of the \(b\) field, \(*db\).

This shows that the requirement for the algebra to be of the form \(G \times U(1)\) in order to decouple first and second class constraints is sufficient but not necessary. In general, any semisimple (super) group with a \(U(1)\) central extension [as in case of \(N = 4\) super AdS discussed above] will be sufficient too.

One can now count the degrees of freedom for the effective theory in this background. There are 15 generators of \(AdS_5\), 15 for \(SU(2, 2)\), and 1 for \(U(1)\), plus \(4 \times 4 Q^i_\alpha\)'s and an equal number of \(\bar{Q}_i^\alpha\)'s. According to the argument outlined in [9], this makes \(f = 63, n = 2\), and a total of 61 independently propagating degrees of freedom.

The previous result is puzzling. There can be no matching of fermionic and bosonic degrees of freedom in this case. In fact, there seems to be a hidden subtlety in this counting because, at least in the perturbative sense, the number of degrees of freedom around this background is different. Consider a fluctuation in the connection around a fixed background \(\tilde{A}\),

\[
A^M = \tilde{A}^M + u^M, \tag{10}
\]
where the dynamical fields will be the spatial components of $A = A^M G_M$, with $M$ ranging
over the whole supergroup indices. Then, for small $u$, the effective action in the linearized
approximation, is given by

$$I_{E_{\text{f}}} (u^M_i) \sim \int < uF \nabla u >$$

$$= \int d^5 x [u^M_i \Omega_{MN}^{ij} (\bar{A}) \nabla_0 u^N_j + 2u^M_0 \Omega_{MN}^{ij} (\bar{A}) \nabla_i u^N_j - 2\varepsilon^{ijkl} u^A_i \Delta_{ABC} \bar{F}_B^D \nabla_k u^C_l].$$ (11)

If we consider the AdS background where $\Omega_{MN}^{ij} (\bar{A})$ takes the form \([8]\), then the counting
comes to 58.

VI. DISCUSSION AND OUTLOOK

1. From the previous discussion it is clear that for $N = 4$ the theory is extremely
simple and the $b$ field almost completely decouples from the rest of the fields. In fact, the
$b$ field is analogous to a Lagrange multiplier, and this analogy is completely accurate in
the effective action for the perturbations, where the perturbation associated with this field
doesn’t appear at all in the effective action. Actually, around the same AdS background
one cannot distinguish between the effective linearized theory described above from that
containing only the Kähler-CS like action \([13]\) described by the second term of $L_{u(1)}$. We
will discuss this issue in detail elsewhere.

2. Another important problem is to show that the algebra generated by the conserved
charges reproduces an algebra which is not isomorphic to the gauge algebra $su(2,2|4)$, but
is a non-trivial central extension of $psu(2,2|4)$. Here the result is that this is indeed the case
and that the algebra sets bounds to the values of the charges and conditions for the existence
of Killing spinors that ensure that the background saturates the Bogomolnyi bound. This
issue, in turn, raises the question of identifying the nontrivial BPS states. These states must
keep a fraction of supersymmetry; in fact a solution of the field equations with these features
is some sort of “topological black hole” \([14]\) and it is reassuring to verify that the extreme
case saturates the bound. These problems are going to be discussed in a forthcoming article.
3. The five dimensional theory generates a new four-dimensional superconformal theory at the boundary. This theory is constructed on the generalization of the Kać-Moody extension of the superconformal algebra without central charge, that is, the WZW$_4$ algebra for PSU$(2,2|1)$. This theory could be a rich test ground in the context of AdS/CFT duality conjecture [15].

4. Another interesting problem is to try to generalize what is known about $D = 5$ to higher dimensions. We have shown [5] that the $D = 11$, $N = 32$ theory admits a nontrivial extension of the AdS superalgebra with one abelian generator for which anti-de Sitter space without matter fields is a background of maximal rank, and the gauge superalgebra is realized in the Dirac brackets. On a background like the used in the five dimensional theory, the $D = 11$ theory has $2^{12}$ fermionic and $2^{12} - 1$ bosonic degrees of freedom, and the (super) charges obey a non-trivial central extension of the $OSP(32|32)$ algebra.

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VII. APPENDICES
APPENDIX A: SUPERSYMMETRIC EXTENSION OF ADS$_5$ ALGEBRA

As discussed in [10], [8], the supersymmetric extension of the anti-de Sitter algebra in five dimensions is $su(2,2|N)$. This is the Lie algebra associated with the invariance group of the quadratic form $q = \theta^\alpha g_{\alpha\beta} \theta^\beta + z^r u_{rs} z^s$, with $\alpha, \beta = 1, ..., 4$ and $r, s = 1, ..., N$. Here $\theta^\alpha$ are complex Grassmann numbers and $g_{\alpha\beta}$ and $u_{pq}$ are sesquilinear metrics, which will be chosen as $g_{\alpha\beta} = i(\gamma_0)_{\alpha\beta}$ and $u_{rs} = \delta_{rs}$. The supersymmetric algebra contains $su(2,2) \oplus su(N) \oplus u(1)$ as the bosonic subalgebra. Using the isomorphism: $su(2,2) \simeq so(4,2)$, the generators are composed by the AdS generators ($J_{AB}$), with $A, B = 0, ..., 5$, the $2 \times 4N$ (complex) supersymmetry generators ($Q^\alpha, \bar{Q}^\alpha$), and the rest of the algebra is composed by the generators of internal (Lorentz scalar) $su(N) \otimes u(1)$ ($T_\Lambda, Z$), with $\Lambda = 1, ..., N^2 - 1$.

A natural representation acting in the vector superspace $(\theta^\beta, z^q)$ is given by $(4 + N) \times (4 + N)$ matrices as follows. Defining $I_{4 \times 4} = \begin{bmatrix} \delta_\beta^\alpha & 0 \\ 0 & 0 \end{bmatrix}$, $I_{N \times N} = \begin{bmatrix} 0 & 0 \\ 0 & \delta^r_s \end{bmatrix}$, the generators are:

- **AdS generators**

  \[
  J_{AB} = \frac{1}{2} \Gamma_{AB} \otimes I_{4 \times 4},
  \]

  \[
  = \begin{bmatrix} \frac{1}{2} (\Gamma_{AB})^\beta_\alpha & 0 \\ 0 & 0 \end{bmatrix},
  \]

- **$su(N)$ generators**

  \[
  T_\Lambda = \tau_\Lambda \otimes I_{N \times N}
  \]

  \[
  = \begin{bmatrix} 0 & 0 \\ 0 & (\tau_\Lambda)^r_s \end{bmatrix}
  \]

  where $(\tau_\Lambda)^r_s$ are the antihermitean generators of $su(n)$.

- **Supersymmetry generators**

  \[
  Q^\alpha_s = \begin{bmatrix} 0 & 0 \\ -\delta^\alpha_\beta \delta^r_s & 0 \end{bmatrix},
  \]
\[
\bar{Q}_\alpha^r = \begin{bmatrix}
0 & \delta^r \delta_\alpha^\beta \\
0 & 0 
\end{bmatrix}.
\]

\textbullet u(1) charge

\[
Z = \frac{i}{4} I_B + \frac{i}{N} I_F,
\]

\[
= \begin{bmatrix}
\frac{i}{4} \delta_\beta^\alpha & 0 \\
0 & \frac{i}{N} \delta^r 
\end{bmatrix}.
\]

The commutators of the bosonic generators are those for the algebra \(so(4,2) \oplus su(N) \oplus u(1)\): \([J,J] \sim J\), \([T,T] \sim T\), \([Z,Z] = 0\), \([J,T] = 0\), \([Z,J] = 0\), \([Z,T] = 0\). The supersymmetry generators transform as spinors under AdS (then also under Lorentz), as “vectors” under \(su(N)\), and carry \(u(1)\) charge,

\[
[J_{AB}, Q_\alpha^a] = -\frac{1}{2}(\Gamma_{AB})_\beta^a Q_\alpha^\beta
\]

\[
[J_{AB}, \bar{Q}_\beta^\alpha] = \frac{1}{2}(\Gamma_{AB})_\alpha^\beta \bar{Q}_\beta
\]

\[
[T_\Lambda, Q_\alpha^a] = (\tau_\Lambda)_s^r Q_\alpha^r
\]

\[
[T_\Lambda, \bar{Q}_\beta^\alpha] = -(\tau_\Lambda)_s^r \bar{Q}_\beta^s
\]

\[
[Z, Q_\alpha^a] = -i(\frac{1}{4} - \frac{1}{N}) Q_\alpha^a
\]

\[
[Z, \bar{Q}_\beta^\alpha] = i(\frac{1}{4} - \frac{1}{N}) \bar{Q}_\beta^\alpha.
\]

Finally, the anticommutator reads

\[
\{Q_\alpha^a, \bar{Q}_\beta^\rho\} = -\frac{1}{2} \delta^\rho_\beta (\Gamma^a)_\alpha^\beta J_\alpha + \frac{1}{4} \delta^\rho_\beta (\Gamma^{ab})_\alpha^\beta J_{ab} - \delta^\alpha_\beta (\tau^\Lambda)_s^r T_\Lambda + i \delta^\alpha_\beta \delta^r \bar{Z}.
\]

where \(J_a := J_{a5}\).

It is clear from the algebra that the case \(N = 4\) is a special one: the generator \(Z\) commutes with the rest of the algebra and it is just a central extension, as can be read from the right hand of (A8).

It is important to point out that, if \(Z\) were omitted, the new algebra, \(psu(2,2|4)\), does not admit the representation described above, but still satisfies the Jacobi identity. Because of this last feature, the full resulting algebra for \(N = 4\), is \textbf{not} \(psu(2,2|4) \oplus u(1)\).
APPENDIX B: THIRD RANK INVARIANT TENSOR FOR N EXTENDED
SUPER ADS$_5$

From the matrix representation described above, it is straightforward to obtain a third
rank invariant tensor for the AdS$_5$ supergroup, which is needed to analyze the dynamics of
local AdS$_5$ supergravity.

Let $G_M$ the generators of $su(2,2|N)$, where the index $M$ ranks from the whole superalgebra: $M = \{[AB], \Lambda, Z, (\frac{\alpha}{6}), (\frac{\beta}{6})\}$.

The required tensor is defined through $\Delta_{MNP} = \langle G_M, G_N, G_P \rangle$, where $\langle \ldots \rangle$ stands
for the supertrace, which ensure the invariance of the tensor under the action of the group.
Because the supertrace is the difference between the trace of the upper and lower diagonal
bosonic blocks, the invariant tensor has the form:

$$
\Delta_{ZZZ} = -i \left( \frac{1}{4} - \frac{1}{N^2} \right)
$$

$$
\Delta_{[AB][CD][EF]} = \frac{i}{2} \epsilon_{ABCDEF}
$$

$$
\Delta_{(\Lambda_1)(\Lambda_2)(\Lambda_3)} = -Tr[(\tau_{\Lambda_1})(\tau_{\Lambda_2})(\tau_{\Lambda_3})]
$$

$$
\Delta_{Z[AB][CD]} = -\frac{1}{4} \eta_{[AB][CD]}
$$

$$
\Delta_{Z(\Lambda_1)(\Lambda_2)} = -\frac{i}{N} g_{\Lambda_1\Lambda_2}
$$

$$
\Delta_{Z(\frac{\alpha}{6})_{(\frac{\alpha}{6})}} = i \left( \frac{1}{4} + \frac{1}{N} \right) \delta_{\alpha}\delta_{\beta}
$$

$$
\Delta_{[AB](\frac{\beta}{6})_{(\frac{\beta}{6})}} = \frac{1}{2}(\Gamma_{AB})_{\alpha}\delta_{\beta}^{\alpha}
$$

$$
\Delta_{(\Lambda)(\frac{\beta}{6})_{(\frac{\beta}{6})}} = -\delta_{\Lambda}^{\beta}(\tau_{\Lambda})_{s}^{r}
$$

where $\eta_{[AB][CD]} := \eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}$, and $g_{\Lambda_1\Lambda_2} = Tr[(\tau_{\Lambda_1})(\tau_{\Lambda_2})]$ is the Killing metric of $su(N)$.

Note that, again in the special case $N = 4$, and considering the splitting $M = \{M', Z\}$,
then $\Delta_{ZZZ} = 0$, and $\Delta_{ZM'N'} = -\frac{i}{2} g_{M'N'}$, where $g_{M'N'}$ is the (invertible) Killing metric of $PSU(2,2|4)$. 

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