Constructing positive maps from block matrices

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Abstract

Positive maps are useful for detecting entanglement in quantum information theory. Any entangled state can be detected by some positive map. In this paper, the relation between positive block matrices and completely positive trace-preserving maps is characterized. Consequently, a new method for constructing decomposable maps from positive block matrices is derived. In addition, a method for constructing positive but not completely positive maps from Hermitian block matrices is also obtained.

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1. Introduction

Positive maps play a crucial role in quantum information theory [1–3]. In the context of quantum physics, every quantum operation is characterized by a trace-preserving completely positive (CP) map transforming quantum states to quantum states, where a quantum state is described by a density matrix, i.e., a positive matrix with a unit trace. Positive but not completely positive (PNCP) maps are useful tools for detecting an entanglement of quantum states [2, 4].

Recall that a bipartite quantum state \(\rho \in \mathcal{M}_m \otimes \mathcal{M}_n\) is called separable if it can be written as a convex combination [5, 6]

\[
\rho = \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)}, \quad \sum_i p_i = 1, \quad p_i \geq 0,
\]

where \(\rho_i^{(1)}\) and \(\rho_i^{(2)}\) are the quantum states in \(\mathcal{M}_m\) (i.e., the algebra of all \(m \times m\) complex matrices) and \(\mathcal{M}_n\), respectively. Otherwise, \(\rho\) is called entangled. It is well known that a bipartite state \(\rho \in \mathcal{M}_m \otimes \mathcal{M}_n\) is separable iff [2]

\[
(id_{m} \otimes \Phi)\rho \geq 0
\]
holds for any PNCP map $\Phi : M_m \rightarrow M_m$. That is, if $(id_m \otimes \Phi)\rho$ is not positive, then $\rho$ is entangled; namely, $\rho$ is detected by $\Phi$.

A well-known example of a PNCP map is the transpose $\tau$; another is the so-called reduction map defined by $\Phi(A) = \text{tr}(A)I - A$ [7]. The trace map $\text{tr}(\cdot)$ is one of the most frequently used CP maps in the quantum literature. Every CP map $\Phi : M_m \rightarrow M_m$ admits the form of [8, 9]

$$\Phi(A) = \sum_i X_i A X_i^\dagger, \quad A \in M_m,$$

where $X_i$ are $n \times m$ matrices. In particular, $\Phi$ is trace-preserving (i.e. $\text{tr}(\Phi(A)) = \text{tr}(A)$ for any $A \in M_m$) iff $\sum_i X_i X_i^\dagger = I_m$, and in such a case, it is called a quantum channel in quantum physics.

The structure of positive maps has been studied extensively by researchers on both mathematics and quantum physics [8, 10–32]. Several methods of constructing PNCP maps are proposed (see [18, 24] and references therein for details). The transpose map and the reduction map are the most well-known decomposable maps; the first example of an indecomposable map was given by Choi [10]; some variants of Choi-type maps are proposed in [18, 24]; in [17, 18], a new type of indecomposable maps is deduced via the reduction map; the Robertson map [29] and the Breuer map [30, 31] are indecomposable maps as well; some other indecomposable maps are introduced in [21, 25, 32]; Ha [23] extended Choi’s another example in [11]; the reduction and Robertson maps were generalized by Chruściński et al [19], where the generalized reduction maps are decomposable maps and the generalized Robertson maps are indecomposable ones; in [27, 28], the elementary operators method was proposed; etc. The main purpose of this paper is to propose a new method for constructing PNCP maps, from which we can obtain some useful tools for detecting entanglement in quantum information theory since any PNCP map can lead to an entanglement witness via the Choi–Jamiołkowski isomorphism [8, 12, 13] and the entanglement witnesses are observables that allow us to detect entanglement experimentally. We stress here that the method we proposed is very different from the previous ones, since our method is based on the relation between the positive block matrix and the trace-preserving positive map. (Note that the relation here is different from the Choi–Jamiołkowski isomorphism; see the remark in section 2.)

The paper is organized as follows: in section 2, we characterize the relation between the positive block matrices and the trace-preserving PNCP maps (lemma 2.1), and reveal some relation between bipartite states and quantum channels (corollary 2.2). Several special bipartite states are considered, which correspond to special quantum channels. In addition, we obtain a sufficient condition for a bipartite state to be separable (corollary 2.3). In section 3, the correspondence between decomposable maps and non-positive partial transpose (NPPT) block matrices is derived (theorem 3.1). It is illustrated with several examples of new decomposable maps. Section 4 is devoted to another way of constructing PNCP maps from Hermitian block matrices (theorem 4.1). Finally, we draw some conclusions and point out some questions for further investigation in section 5.

For clarity, we list the notations and the terminologies in this paper. $A'$ denotes the transpose of $A \in M_m$, and $A^\dagger$ stands for the transpose of the complex conjugate of $A$, i.e. $A^\dagger = \tilde{A}^\dagger$. $A$ is positive (or positive semi-definite), denoted by $A \succeq 0$, if $A^\dagger = A$ and its eigenvalues are non-negative. Let $M_m^+$ be the positive part of $M_m$. $M_m \otimes M_n(= M_m(M_n))$ is the algebra of all $m \times n$ block matrices with $n \times n$ complex matrices as entries. A linear map $\Phi : M_m \rightarrow M_n$ is positive (resp. Hermitian) if $\Phi(M_m^+) \subseteq M_n^+$ (resp. $\Phi(A)$ is Hermitian for any Hermitian matrix $A \in M_m$). Let $E_{ij}$ be the matrix units of the associated matrix algebra. $\Phi$ is CP if $(id_k \otimes \Phi)(\sum_{i,j=1}^d E_{ij} \otimes A_{ij}) = \sum_{i,j=1}^d E_{ij} \otimes \Phi(A_{ij})$ is positive for any positive maps $A_{ij}$.
integer $k$ and every $A = \sum_{i,j=1}^{k} E_{ij} \otimes A_{ij} \in \mathcal{M}_k \otimes \mathcal{M}_m$. A positive map $\Phi$ is decomposable if there exist two CP maps $\Phi_1$ and $\Phi_2$ such that $\Phi = \Phi_1 + \Phi_2 \circ \tau$, where $\tau$ denotes the transpose map (i.e. $\tau(A) = A^t, A \in \mathcal{M}_m$). Otherwise, $\Phi$ is defined to be indecomposable. For $A = \sum_{i,j=1}^{m} E_{ij} \otimes A_{ij} \in \mathcal{M}_m \otimes \mathcal{M}_n$, $A^t = \sum_{i,j=1}^{m} E_{ij}^t \otimes A_{ij}$ and $A^{\circ} = \sum_{i,j=1}^{m} E_{ij} \otimes A_{ij}^t$ are called the partial transpose of $A$. It is clear that $(A^t)^t = A^t, (A^{\circ})^t = A^{\circ}$. We call $A$ a positive partial transpose (PPT) matrix if $A^{\circ} \geq 0$. If $A \geq 0$ with $A^{\circ} \not\geq 0$, then we call $A$ an NPPT matrix.

2. CP maps and positive block matrices

For $A = \sum_{i,j=1}^{m} E_{ij} \otimes A_{ij} \in \mathcal{M}_m \otimes \mathcal{M}_n$, the reduced matrix of $A$, denoted by $A_{1,2}$, is defined by

\[
A_1 = \text{tr}_2(A) := (\text{id}_m \otimes \text{tr}) \sum_{i,j=1}^{m} E_{ij} \otimes A_{ij} = \sum_{i,j=1}^{m} \text{tr}(A_{ij}) E_{ij},
\]

\[
A_2 = \text{tr}_1(A) := (\text{tr} \otimes \text{id}_n) \sum_{i,j=1}^{m} E_{ij} \otimes A_{ij} = \sum_{i,j=1}^{m} \text{tr}(E_{ij}) A_{ij} = \sum_{i} A_{ii},
\]

where $\text{tr}(\cdot)$ denotes the trace operation. It is obvious that $\text{tr}(A) = \text{tr}(A_1) = \text{tr}(A_2)$ and $A_{1,2} \geq 0$ whenever $A \geq 0$. We begin our discussion with the famous Schmidt decomposition theorem [33] which reads as follows. Let $H_1$ and $H_2$ be two complex Hilbert spaces with $\dim H_1 = m$ and $\dim H_2 = n$, and let $|x\rangle$ be a vector (not necessarily normalized) in $H_1 \otimes H_2$, then there exist the orthonormal sets $\{|e_i\rangle\}$ and $\{|f_i\rangle\}$ of $H_1$ and $H_2$, respectively, and positive numbers $\{\lambda_i\}$, such that

\[
|x\rangle = \sum_{i} \lambda_i |e_i\rangle |f_i\rangle,
\]

where $\sum \lambda_i^2 = |||x\rangle||$. $\{\lambda_i\}$ are called the Schmidt coefficients of $|x\rangle$ and $r(x) \leq \min\{m, n\}$ is the Schmidt number of $|x\rangle$.

The following lemma is necessary.

**Lemma 2.1.** Let $A \in (\mathcal{M}_m \otimes \mathcal{M}_n)^+$ and $|x\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ be a vector (not necessarily normalized) with $\text{tr}_2(|x\rangle \langle x|) = A_1$, then there exists a trace-preserving CP map $\Lambda : \mathcal{M}_m \to \mathcal{M}_n$, such that

\[
A = (\text{id}_m \otimes \Lambda) |x\rangle \langle x|. 
\]

**Proof.** Let $A = \sum_{i,j=1}^{m} E_{ij} \otimes A_{ij}$, then $A_1 = \sum_{i,j=1}^{m} \text{tr}(A_{ij}) E_{ij}$. Let

\[
A_1 = \sum_{i} \lambda_i^2 |\psi_i\rangle \langle \psi_i|
\]

be the spectral decomposition of $A_1$. Suppose

\[
|x\rangle = \sum_{i=1}^{r(x)} \lambda_i |\psi_i\rangle |\phi_i\rangle
\]

for some orthonormal set $\{|\phi_i\rangle\}$ in $\mathbb{C}^n$ and let $\{|\psi_i\rangle\}_{i=1}^{m}$ be an orthonormal basis of $\mathbb{C}^m$ induced from the eigenvectors of $A_1 - ||\psi_i\rangle \langle \psi_i||_{i=1}^{m}$ (we assume with no loss of generality that the
rank\( (A_1) = r(x) = r, \ 1 \leq r < m; \) the case of \( r = m \) can be discussed similarly. We write \( E_{ij} = |\psi_i\rangle \langle \psi_j| \) and \( E_{ij} = \sum_{k,l=1}^{m} \omega_{kl}^{(ij)} E_{kl} \) for some complex numbers \( \omega_{kl}^{(ij)} \), then

\[
A = \sum_{i,j=1}^{m} \left( \sum_{k,l=1}^{m} \omega_{kl}^{(ij)} E_{kl} \right) \otimes A_{ij}
\]

\[
= \sum_{k,l=1}^{m} E_{kl} \otimes \left( \sum_{i,j=1}^{m} \omega_{kl}^{(ij)} A_{ij} \right) = \sum_{i,j=1}^{m} E_{ij} \otimes A_{ij},
\]

where \( A_{ij}' = \sum_{k,l=1}^{m} \omega_{kl}^{(ij)} A_{kl} \). It follows from

\[
A_1 = \sum_{i,j=1}^{m} \text{tr}(A_{ij}') E_{ij} = \sum_{i=1}^{r} \lambda_i^2 E_{ii}'
\]

that

\[
\text{tr}(A_{ij}') = \begin{cases} 
0 & \text{when } i \neq j, \\
\lambda_i^2 & \text{when } i = j,
\end{cases}
\]

and

\[
\text{tr}(A_{ij}') = 0 \quad \text{when } i > r \text{ or } j > r.
\]

Since \( \text{tr}(A_{ij}') = 0 \) implies \( A_{ij}' = 0 \) when \( i > r \), and thus \( A_{ij}' = 0 \) when \( i > r \) or \( j > r \),

\[
A = \sum_{i,j=1}^{m} E_{ij} \otimes A_{ij}'.
\]

Let \( \{|\phi_i\rangle\}_{i=1}^{n} \) be an orthonormal basis of \( \mathbb{C}^n \) extended from \( \{|\phi_i\rangle\}_{i=1}^{r} \). We now define a linear map \( \Lambda : \mathcal{M}_m \to \mathcal{M}_n \) by

\[
\Lambda (|\phi_i\rangle \langle \phi_j|) = A_{ij}'' = \begin{cases} 
\frac{1}{\lambda_i} A_{ij}' & \text{when } 1 \leq i, j \leq r, \\
0 & \text{when } r < i = j \leq n, \\
X_i & \text{when } r < i, j \leq n, i \neq j.
\end{cases}
\]  \hfill \text{(6)}

where \( X_i \) are positive matrices in \( \mathcal{M}_n \) with \( \text{tr}(X_i) = 1, \ r \leq i \leq n \). Then \( A = (\text{id}_m \otimes \Lambda) |x\rangle \langle x| \). It remains to show that \( \Lambda \) is a trace-preserving CP map. By the definition above, it is easy to see that \( \Lambda \) preserves the trace. We next show that it is CP. For clarity, we denote by \( A' \) and \( A'' \) the block matrices \( \sum_{i,j=1}^{r} E_{ij} \otimes A_{ij} + \sum_{i=r}^{m} E_{ij} \otimes X_i \) and \( \sum_{i,j=1}^{m} E_{ij} \otimes A_{ij} \), respectively, under the basis \( \{E_{ij}\} \) of \( \mathcal{M}_r \). Observe that

\[
A'' = \begin{bmatrix}
\frac{1}{\lambda_1} I \\
\vdots \\
\frac{1}{\lambda_r} I \\
I \\
\end{bmatrix} \begin{bmatrix}
A' \\
\frac{1}{\lambda_1} I \\
\vdots \\
\frac{1}{\lambda_r} I \\
I \\
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\lambda_1} I \\
\vdots \\
\frac{1}{\lambda_r} I \\
I \\
\end{bmatrix},
\]

where \( I \) is the \( n \times n \) unit matrix. Therefore, \( A'' \geq 0 \) iff \( A' \geq 0 \), and in turn, iff \( A \geq 0 \), which implies that \( \Lambda \) is a CP map, since the positive block matrix \( A'' \) is the Choi matrix of \( \Lambda \) [8]. \qed
The lemma above implies that any positive block matrix can induce trace-preserving CP maps.

In general, let \( A \) be a Hermitian matrix in \( \mathcal{M}_m \otimes \mathcal{M}_n \) and \( |x\rangle \in \mathbb{C}^m \otimes \mathbb{C}^m \) be a vector (not necessarily normalized) with \( A_1 = \text{tr}_2 (|x\rangle \langle x|) \), then there exists a trace-preserving Hermitian map \( \Lambda : \mathcal{M}_m \rightarrow \mathcal{M}_n \), such that \( \Lambda = (\text{id}_m \otimes \Lambda) |x\rangle \langle x| \). (Note that \( \Lambda \) is a Hermitian map iff the Choi matrix of \( \Lambda \) is Hermitian [8].)

**Remark.** (i) The trace-preserving CP map satisfying equation (5) is not unique. If \( |x\rangle = \sum_{r=1}^m A_i |\psi_r\rangle \langle \psi_r| \in \mathbb{C}^m \otimes \mathbb{C}^m \) is its Schmidt decomposition such that \( A_1 = \text{tr}_2 (|x\rangle \langle x|) \) and \( r(x) = m \), then the trace-preserving CP map \( \Lambda \) is unique. (ii) Equation (5) is different from the Choi–Jamiolkowski isomorphism between block matrices and positive maps. Recall that the Choi–Jamiolkowski isomorphism reads as \([8, 12, 13]\)

\[
W = (\text{id}_m \otimes \Phi) \left( \sum_{i,j=1}^m E_{ij} \otimes E_{ij} \right) = \sum_{i,j=1}^m E_{ij} \otimes \Phi(E_{ij}),
\]

where \( W \in \mathcal{M}_m \otimes \mathcal{M}_n \) and \( \Phi \) is a positive map from \( \mathcal{M}_m \) to \( \mathcal{M}_n \). That is, the Choi–Jamiolkowski isomorphism is established via a positive map acting on a maximally pure entangled state (i.e. \[ E_{ij} \sum_{i,j=1}^m E_{ij} \otimes E_{ij} \]), while lemma 2.1 (resp. corollary 2.2 below) is a relation between any positive block matrix and the corresponding trace-preserving CP map (i.e. quantum channel) associated with the rank-1 block matrix (resp. pure state) which is the purification of the reduced matrix. In addition, the Choi matrix \( W \) is uniquely determined by \( \Phi \) and \( \Phi \) is unique when \( W \) is fixed. Also, note that map \( \Phi \) in equation (7) is not necessarily trace-preserving. (iii) If \( \text{rank}(A_1) = r < m \), then \( A \) can be viewed as a matrix in \( \mathcal{M}_r \otimes \mathcal{M}_n \).

In the quantum literature the term pure state is sometimes used for both the rank-1 density matrix \( |\psi\rangle \langle \psi| \) and the ket \( |\psi\rangle \). A pure state \( |x\rangle \) is called a purification of a density matrix \( \rho \) if \( \rho_1 = \text{tr}_2 (|x\rangle \langle x|) \).

The following are some consequences of lemma 2.1.

**Corollary 2.2.** Let \( \rho \in \mathcal{M}_m \otimes \mathcal{M}_n \) be a bipartite density matrix and \( |x\rangle \in \mathbb{C}^m \otimes \mathbb{C}^m \) be a purification of the reduced density matrix \( \rho_1 \). Then there exists a quantum channel \( \Lambda \) from \( \mathcal{M}_m \) to \( \mathcal{M}_n \), such that

\[
\rho = (\text{id}_m \otimes \Lambda) |x\rangle \langle x|.
\]

That is, any bipartite state can arise from a quantum channel acting on the purification of the reduced state.

Let \( \rho \in \mathcal{M}_m \otimes \mathcal{M}_n \) be a bipartite density matrix and \( \rho_1 \) be a pure state. By corollary 2.2, \( \rho \) is separable; in particular, \( \rho \) is a product state, i.e. \( \rho = \rho_1 \otimes \rho_2 \).

**Corollary 2.3.** Let \( \rho \in \mathcal{M}_m \otimes \mathcal{M}_n \) be a bipartite density matrix. If either \( m = 2 \) and \( \text{rank}(\rho_1) \leq 3 \) or \( m = 3 \) and \( \text{rank}(\rho_1) \leq 2 \), then \( \rho \) is separable iff it is PPT.

**Proof.** If \( m = 2 \) (or \( m = 3 \)) and \( \text{rank}(\rho_1) = r \leq 3 \) (or \( r \leq 2 \)), then \( \rho = \sum_{i,j=1}^r E'_{ij} \otimes A'_{ij} \) is in fact a state in the \( r \otimes 2 \) (or \( r \otimes 3 \)) bipartite system. The theorem is now clear from the fact that a state in the \( m \otimes n \) system with \( mn \leq 6 \) is separable iff it is PPT [2].

At the end of this section, we list some special bipartite states which lead to special quantum channels (see table 1). A quantum channel \( \Lambda : \mathcal{M}_m \rightarrow \mathcal{M}_n \) is entanglement breaking if \( (\text{id}_m \otimes \Lambda) \rho \) is separable for any state \( \rho \in \mathcal{M}_k \otimes \mathcal{M}_m \) [34]. It is shown in [34] that \( \Lambda \) is entanglement breaking iff \( \Lambda(A) = \sum_k \text{tr}(WkA)Qk \) for any \( A \in \mathcal{M}_m \), where each \( Qk \) is a density matrix in \( \mathcal{M}_n \), \( W_k \geq 0 \) and \( \sum_k W_k = I_m \). \( \Lambda \) is called a completely contractive channel if...
\( \Lambda(A) = \text{tr}(A) \sigma \) holds for any \( A \in \mathcal{M}_m \) for some fixed state \( \sigma \in \mathcal{M}_n \) \[35\]. Let \( \rho \) be a bipartite density matrix in \( \mathcal{M}_m \otimes \mathcal{M}_n \). If \( \rho \) is a classical quantum state (i.e. \( \rho = \sum p_i |i\rangle \otimes \sigma_i \) with \( \{|i\rangle\} \) the orthonormal set of \( \mathbb{C}^m \) and \( \sigma_i \) are density matrices in \( \mathcal{M}_n \) and \( \text{rank}(\rho_1) = m \), then one can check that the channel \( \Lambda \) in equation \( (8) \) is entanglement breaking. In particular, if \( \rho \) is a product state with \( \text{rank}(\rho_1) = m \), then the channel \( \Lambda \) in equation \( (8) \) is completely contractive. If \( \rho = |y\rangle \langle y| \) is a pure state in \( \mathcal{M}_m \otimes \mathcal{M}_n \) with \( r(y) = m \), then \( \Lambda \) in equation \( (8) \) is a unitary operation (channel), i.e. \( \Lambda(A) = UAU^\dagger \) for some unitary matrix in \( \mathcal{M}_m \).

### 3. Decomposable maps derived from NPPT positive block matrices

For simplicity, we fix some notations. Let \( A \) be a positive matrix in \( \mathcal{M}_m \otimes \mathcal{M}_n \). We write

\[
A = \sum_{i,j=1}^n A_{ij} \otimes E_{ij},
\]

where \( E_{ij} \) are matrix units in \( \mathcal{M}_m \) and \( E_{kl} \) are matrix units in \( \mathcal{M}_n \). That is, \( A \) can be denoted by \( [A_{ij}] \) or \( (A_{kl}) \). It is straightforward that

\[
\tilde{A}_{kl} = \begin{bmatrix} a_{kl}^{(i)} \end{bmatrix} \text{ iff } A_{ij} = \begin{bmatrix} a_{ij}^{(i)} \end{bmatrix}.
\]

Let \( A_1 = \text{tr}_2(A) = \sum_{i,j=1}^n \lambda_i |\psi_i\rangle \langle \psi_i| \) be its spectral decomposition and \( E'_{ij} = |\psi_i\rangle \langle \psi_j| \). Then \( A \) can be represented as \( A = \sum_{i,j=1}^r E'_{ij} \otimes A'_{ij} \), where \( A'_{ij} \) are \( n \times n \) matrices. Let \( A''_{ij} \) be the same as defined in equation \( (6) \); then it is clear that \( A'' \geq 0 \) iff \( (A'')^{\dagger} \geq 0 \), \( A'' = \{ A''_{ij} \} = \sum_{i,j=1}^m E'_{ij} \otimes A''_{ij} \).

The following is the main result of this paper.

**Theorem 3.1.** Let \( A \) be an NPPT matrix in \( \mathcal{M}_m \otimes \mathcal{M}_n \). If \( \Phi : \mathcal{M}_m \rightarrow \mathcal{M}_n \) is a linear map satisfying \( \Phi(E_{ij}') = (A'_{ij})' \), where \( E_{ij}' \) and \( A''_{ij} \) are defined as above, then \( \Phi \) is a PNCP map; moreover, it is trace-preserving and decomposable.

**Proof.** Since the Choi matrix of \( \Phi \), i.e. \( (\text{id}_m \otimes \Phi) \left( \sum_{i,j=1}^m E_{ij} \otimes E_{ij}' \right) = (A'')^{\dagger} \), is not positive, by theorem 2 in \[8\], \( \Phi \) is not CP. In order to show the positivity of \( \Phi \), it suffices to prove that \( \Phi(|w\rangle \langle w|) \geq 0 \)

holds for any rank-1 projection \( |w\rangle \langle w| \in \mathcal{M}_m \). Let \( |w\rangle \langle w| = \sum_{i,j=1}^m t_{ij} E_{ij}' \). We claim that \( T = [t_{ij}] \) is a rank-1 projection. In fact, there exists a unitary matrix \( U \) such that \( U|\psi_i\rangle = |e_i\rangle \), \( i = 1, 2, \ldots, m \), where \( \{|\psi_i\rangle\} \) is the orthonormal basis of \( \mathbb{C}^n \) derived from the eigenvectors of \( A_1 \), and \( \{|e_i\rangle\} \) is the standard orthonormal basis of \( \mathbb{C}^m \). It turns out that

\[
U|w\rangle \langle w|U^\dagger = \sum_{i,j=1}^m t_{ij} U E'_{ij} U^\dagger = \sum_{i,j=1}^m t_{ij} E_{ij} = [t_{ij}],
\]

which implies that \( T \) is a rank-1 projection. We define \( \Lambda : \mathcal{M}_m \rightarrow \mathcal{M}_n \) by \( \Lambda(E_{ij}') = A''_{ij} \). By lemma 2.1, it is CP, and thus positive. Hence,

\[
\Lambda(|w\rangle \langle w|) = \sum_{i,j=1}^m t_{ij} \Lambda(E_{ij}') = \sum_{i,j=1}^m t_{ij} A''_{ij} = [\text{tr}(T'\tilde{A}'_{ij})] \geq 0.
\]

### Table 1. The correspondence between the state \( \rho \) and the channel \( \Lambda \).

| \( \rho \)                | System    | \( \Lambda \)                        |
|--------------------------|-----------|--------------------------------------|
| \( \rho = \sum p_i |i\rangle \otimes \sigma_i \) with \( \text{rank}(\rho_1) = m \) | \( \mathcal{M}_m \otimes \mathcal{M}_n \) | \( \Lambda(A) \) is entanglement breaking |
| \( \rho = \rho_1 \otimes \rho_2 \) with \( \text{rank}(\rho_1) = m \) | \( \mathcal{M}_m \otimes \mathcal{M}_n \) | \( \Lambda(A) \) is completely contractive  |
| \( \rho = |y\rangle \langle y| \) with \( r(y) = m \) | \( \mathcal{M}_m \otimes \mathcal{M}_n \) | \( \Lambda(A) \) is a unitary channel    |
It follows that
\[
\Phi(|w\rangle\langle w|) = \sum_{i,j=1}^{m} t_{ij} \Phi(E_{ij}') = \sum_{i,j=1}^{m} t_{ij} (A_{ij}')'
\]
\[
= [\text{tr}(T'(\tilde{A}_{ij}'))] = [\text{tr}(T\tilde{A}_{ij})] = \Lambda(|\tilde{w}\rangle\langle \tilde{w}|) \geq 0,
\]
that is, $\Phi$ is positive. We define $\Lambda'$ : $M_{m} \rightarrow M_{n}$ by $\Lambda' = \Lambda \circ \tau$, $\Phi$ is decomposable. It is easy to see that $\Phi$ is trace-preserving from the definition. □

That is, any NPPT block matrix can deduce some decomposable map(s). We thus derive a broad class of decomposable maps. This result provides a new method of constructing decomposable positive maps and it can thus provide new tools for detecting entanglement since decomposable maps can detect NPPT entangled states [3].

We illustrate our results with some well-known NPPT bipartite density matrices.

Example 3.2. We consider a $3 \otimes 3$ density matrix,
\[
\rho(a) = \frac{1}{21}
\begin{bmatrix}
2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5-a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5-a & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5-a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5-a & 0 \\
2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\
\end{bmatrix}
\]
where $2 \leq a \leq 5$. It is proved in [36] that $\rho(a)$ is separable iff $2 \leq a \leq 3$, is PPT entangled iff $3 < a \leq 4$, is NPPT entangled iff $4 < a \leq 5$. One can check that $\rho(a)$ is NPPT when $0 \leq a < 1$. By theorem 3.1,
\[
\Phi_{1}([a_{ij}]) = 2
\begin{bmatrix}
a_{11} & a_{21} & a_{31} \\
a_{12} & a_{22} & a_{32} \\
a_{13} & a_{23} & a_{33} \\
\end{bmatrix}
\begin{bmatrix}
(5-a)a_{22} + aa_{33} & 0 & 0 \\
0 & a_{11} + (5-a)a_{33} & 0 \\
0 & 0 & (5-a)a_{11} + aa_{22} \\
\end{bmatrix}
\]
and
\[
\Phi_{2}([a_{ij}]) = 2
\begin{bmatrix}
a_{11} & a_{21} & a_{31} \\
a_{12} & a_{22} & a_{32} \\
a_{13} & a_{23} & a_{33} \\
\end{bmatrix}
\begin{bmatrix}
aa_{22} + (5-a)a_{33} & 0 & 0 \\
0 & (5-a)a_{11} + aa_{33}+ & 0 \\
0 & 0 & aa_{11} + (5-a)a_{22} \\
\end{bmatrix}
\]
are decomposable when $4 \leq a \leq 5$ or $0 \leq a < 1$. In fact, $\Phi_{1,a}^{2}$ is positive iff $0 \leq a \leq 5$ and is CP iff $2 \leq a \leq 4$.

Example 3.3. We consider the $m \otimes m$ Werner state
\[
\omega = \frac{m-x}{m^3 - m} I_A \otimes I_B + \frac{mx - 1}{m^3 - m} F, \quad x \in [-1, 1].
\]
with $F = \sum_{i,j=1}^{m} E_{ij} \otimes E_{ji}$ being the flip operator. It is known that [5] $\omega$ is separable iff it is PPT and in turn, iff $0 \leq x \leq 1$. Take $m = 3$, that is,

$$
\omega = \begin{bmatrix}
1 + x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 3 - x & 0 & 0 & 3x - 1 & 0 & 0 & 0 \\
0 & 24 & 3 - x & 0 & 24 & 3x - 1 & 0 & 0 \\
0 & 0 & 3 - x & 0 & 0 & 24 & 3x - 1 & 0 \\
0 & 3x - 1 & 0 & 0 & 24 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3x - 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 24 & 3x - 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 + x & 0 \\
\end{bmatrix}.
$$

It turns out that

$$
\Phi_{3}^{J}(A) = (3x - 1)A + (3 - x)\text{tr}(A)I, \quad A \in \mathcal{M}_3, \quad \text{is CP when } 0 \leq x \leq 1, \quad \text{and is decomposable when } -1 \leq x < 0. \quad \text{In general,}
$$

$$
\Phi_{m}^{mx}(A) = (mx - 1)A + (m - x)\text{tr}(A)I, \quad A \in \mathcal{M}_m, \quad \text{(14)}
$$

is decomposable when $-1 \leq x < 0$. Interestingly, for the case of $x = -1$, it is the well-known reduction map. Furthermore, one can check that $\Phi_{3}^{mx}$ is CP iff $0 \leq x \leq m$ and positive iff $-1 \leq x \leq m$.

**Example 3.4.** For the $m \otimes m$ isotropic state

$$
\zeta = \frac{1 - y}{m^2} I_m \otimes I_m + yP^+, \quad -\frac{1}{m^2 - 1} \leq y \leq 1, \quad \text{(15)}
$$

with $P^+ = \frac{1}{m} \sum_{i,j=1}^{m} E_{ij} \otimes E_{ji}$ is the so-called maximally entangled state. It is known that $\zeta$ is separable iff $y \leq \frac{1}{m^2 + 1}$, and in turn, iff it is PPT [7]. For the case of $m = 3$,

$$
\zeta = \frac{1}{9} \begin{bmatrix}
2y + 1 & 0 & 0 & 0 & 3y & 0 & 0 & 0 & 3y \\
0 & 1 - y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 - y & 0 & 0 & 0 & 0 & 0 & 0 \\
3y & 0 & 0 & 0 & 2y + 1 & 0 & 0 & 0 & 3y \\
0 & 0 & 0 & 0 & 0 & 1 - y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 - y & 0 & 0 \\
3y & 0 & 0 & 0 & 3y & 0 & 0 & 0 & 2y + 1 \\
\end{bmatrix}.
$$

It turns out that

$$
\Phi_{4}^{J}(A) = 3yA' + (1 - y)\text{tr}(A)I, \quad A \in \mathcal{M}_3, \quad \text{is decomposable when } \frac{1}{2} < y \leq 1. \quad \text{In general, we have}
$$

$$
\Phi_{4}^{mx}(A) = myA' + (1 - y)\text{tr}(A)I, \quad A \in \mathcal{M}_m, \quad \text{(16)}
$$

which is decomposable when $\frac{1}{m + 1} < y \leq 1$. In particular, if $y = 1$, it reduces to the transpose map; if $y = 0$, it reduces to $\Phi(A) = \text{tr}(A)I$; if $y = \frac{1}{m}$, it reduces to $\Phi(A) = \text{tr}(A)I - A'$. One can check that $\Phi_{4}^{mx}$ is CP iff $-\frac{1}{m - 1} \leq y \leq \frac{1}{m + 1}$ and is positive iff $-\frac{1}{m - 1} \leq y \leq 1$. 

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Table 2. The correspondence between the block matrix A and the map \( \Lambda \).

| Condition | Map Correspondence |
|-----------|--------------------|
| \( A \geq 0 \) | \( A \) is CP |
| \( A \not\geq 0 \), \( A \) is PPT | \( A \) is decomposable |
| \( A \not\geq 0 \), \( A^\dagger = A \), \( A_{ii} \geq 0 \) and \( A_{11}^U \geq 0 \), \( \forall U \) | \( A \) is decomposable or indecomposable |

**Remark.** The maps \( \Phi_1^a \), \( \Phi_2^x \), \( \Phi_3^{m,x} \) and \( \Phi_4^{m,y} \) range from positive but not CP ones to CP ones continuously when the parameters \( a, x \) and \( y \) vary continuously. Equations (11), (12), (14) and (16) propose new types of decomposable maps.

### 4. PNCP maps derived from Hermitian block matrices

We now consider the relation between the PNCP maps and the Hermitian block matrices. Let \( A \in \mathcal{M}_m \otimes \mathcal{M}_n \), then \( A \) can be denoted by both \( [A_{ij}] \) and \( \tilde{A}_{kl} \) as in equation (9). For any unitary matrix \( U \in \mathcal{M}_m \), let \( \tilde{A}_{ij}^U = U \tilde{A}_{ij} U^\dagger \). In the following, we write

\[
A^U = \sum_{i,j=1}^m E_{ij} \otimes A_{ij}^U = \sum_{k,l=1}^n \tilde{A}_{ij}^U \otimes E_{kl};
\]

that is, \( A^U = \left( U \otimes I_n \right) A \left( U^\dagger \otimes I_n \right) = [\tilde{A}_{ij}^U] = [A_{ij}] \).

**Theorem 4.1.** Let \( A \) be a Hermitian matrix in \( \mathcal{M}_m \otimes \mathcal{M}_n \) and \( A \not\geq 0 \). If \( A_{ii} \geq 0 \) and \( A_{11}^U \geq 0 \) for any unitary matrix \( U \), then \( \Psi(E_{ij}) = A_{ij} \) is a PNCP map.

**Proof.** If \( A_{ii} \geq 0 \) and \( A_{11}^U \geq 0 \) for any unitary matrix \( U \), then for any rank-1 projection \(|w\rangle\langle w| \in \mathcal{M}_m\), writing

\[
|w\rangle\langle w| = \sum_{i,j=1}^m t_{ij} E_{ij},
\]

we have

\[
\Psi(|w\rangle\langle w|) = \sum_{i,j=1}^m t_{ij} \Psi(E_{ij}) = \sum_{i,j=1}^m t_{ij} A_{ij} = [\text{tr}(T^\dagger \tilde{A}_{ij})] \geq 0.
\]

The inequality above holds since

\[
[\text{tr}(T^\dagger \tilde{A}_{ij})] = A_{11}^U
\]

for some unitary matrix \( U \). That is, \( \Psi \) is positive. On the other hand, \( A = (\text{id}_m \otimes \Psi) \left( \sum_{i,j=1}^m E_{ij} \otimes E_{ij} \right) \not\geq 0 \); thus, it is not CP. \( \square \)

**Theorem 4.1** implies that we can derive PNCP map(s) from any non-positive Hermitian matrix \( A \) with \( A_{ii} \geq 0 \) and \( A_{11}^U \geq 0 \) for any unitary matrix \( U \). That is, we can also turn the task of constructing PNCP maps into structuring these special classes of block matrices. This method may lead to indecomposable maps. Note that indecomposable maps can detect the PPT entangled states \([3, 37]\) (PPT entangled states contain weaker entanglement than NPPT entangled states, so the indecomposable map is stronger than the decomposable one). From **Theorem 4.1**, one may obtain new indecomposable maps which are useful tools for detecting the PPT entangled states.
5. Conclusions

We obtained a correspondence between the positive block matrix and the trace-preserving CP map, which is not the case of the Choi–Jamiołkowski isomorphism. In particular, we showed that any bipartite state can be produced from a quantum channel acting on the purification of the reduced state. Consequently, we derive new methods of constructing PNCP maps from the block matrices by exploring the relation between the structure of the block matrix and the trace-preserving map. The correspondence between the block matrix $A$ and the map $\Lambda$ is characterized (see table 2). In particular, examples 3.2–3.3 propose new concrete types of PNCP maps. Then, by the Choi–Jamiołkowski isomorphism, we can obtain entanglement witnesses which can detect entanglement physically. Therefore, our results may shed new light on both the quantum entanglement theory and the positive map theory in operator theory.

Theorem 4.1 leads to interesting questions for further study, such as whether or not there exists a matrix $A$ that satisfies the condition in theorem 4.1? It is a difficult issue due to the arbitrariness of the unitary matrix $U$. We conjecture that such a matrix exists. In addition, can the map be indecomposable? If this were possible, when would it be indecomposable? We will conduct further research into this in the future.

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