Abstract

The relativistic two-body problem is considered for spinless particles subject to an external electromagnetic field. When this field is made of the monochromatic superposition of two counter-propagating plane waves (and provided the mutual interaction between particles is known), it is possible to write down explicitly a pair of coupled wave equations (corresponding to a pair of mass-shell constraints) which takes into account also the field contribution. These equations are manifestly covariant; constants of the motion are exhibited, so one ends up with a reduced problem involving five degrees of freedom.

1 Introduction

A large litterature is devoted to the effect of applying a laser field to various objects (electron beams, two-level atoms, etc). In early papers [1-3] the laser wave was modelled as purely electric, the effect of a magnetic contribution being introduced more recently [4-5]. Experimental devices are concerned with a wide variety of atoms, but theoretical developments so far have mainly considered an atom as a two-level system; however this simplification (which seems to be adequate in most realistic cases) does not fully take into account the few-body structure of atoms. So, at least in principle, a more accurate treatment is desirable; as a very first step, it is natural to consider a two-body problem involving two charges in interaction in the presence of a plane electromagnetic wave. In this spirit hydrogen-like bound states are the most simple targets offering the two-body structure, but for simplicity we focus here on spinless particles.

A relativistic two-particle system can be covariantly described by a pair of coupled Klein-Gordon equations referred to as mass-shell constraints; these constraints determine the evolution of a wave function which depends on a pair of

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four-dimensional arguments [6-16]. This approach is suited for situations where particle creation can be neglected whereas other relativistic effects must be taken into account. In this article we consider a pair of charged particles undergoing some mutual interaction whereas they are coupled with an external electromagnetic field. The following assumptions are made:
The external field is not affected by the motion of the system,
This external field does not create pairs.
We neglect the radiation emitted by the charges in their motion.
Thus the validity of this picture is implicitly limited by some conditions involving the strength and the analytic shape of the field, the Compton wavelength of the particles, etc. The conditions under which laser fields may create pairs have been widely discussed in the literature (see Ref. [4,17] for fermionic pairs), but here we are interested in the cases where pairs cannot be created.

In the absence of external field the contact of the constraint formalism with QED is well established, through the quasi-potential approach [18-21] or by reducing the Bethe-Salpeter (BS) equation [22,23], and realistic mass-shell constraints can be exhibited, see Crater et al. [24], see Sazdjian and Jallouli [25] when the mutual interaction also is of electromagnetic nature.

In the presence of external field, it seems natural to perform in the same manner a reduction of the BS equation and so derive the coupled wave equations. This task was carried out by Bijtebier and Broekaert [26] in the case of static external potentials, but the tractability of the method for time-dependent cases is problematic. Moreover it is sometimes interesting to consider also phenomenological models of (mutual) interaction, not provided with a given BS kernel.

So we prefer a direct coupling to the field, constructed with help of these reasonable prescriptions: correct limits when one of the interactions vanishes, and invariance under the spacetime symmetries that survive application of the field.

In the present paper the explicit form of the mass-shell constraints is supposed to be known for the isolated system. Naturally the mutual interaction between the charges may be simply electromagnetic, but in several models this electromagnetic term is neglected in front of other forces of phenomenological origin.

When an external field is applied, we face this old problem that relativistic interactions cannot be just linearly composed; this was soon understood by Sokolov in early attempts to construct n-body systems, in the context of a very different approach [27,28]. In our framework, the trouble is that the coupled wave equations must remain compatible one with the other, and this requirement is essentially nonlinear. As a result we generally do not know how to write down in closed form a new pair of compatible wave equations that takes into account also the coupling to the external field.

Nevertheless this problem becomes solvable when the external potential corresponding to the external field has a particular symmetry that we proposed to call strong translation invariance. The solution is furnished by an ansatz elaborated in such a way that wave equations have the correct limits when either the mutual interaction or the external potential is turned off; these equations also have the correct nonrelativistic limit. Moreover in this context, under reasonably general
assumptions, the principle of isometric invariance [29] is satisfied, say: the coupling
to the external field must respect the surviving isometries; in other words the space-
time isometries that would survive (as symmetries of the system) application of the
external field to mutually independent particles should remain symmetries of the
system when these particles also undergo the mutual interaction. This property was
not considered when the ansatz was proposed for the first time; having it finally
satisfied is a bonus.

Our method was initiated by J. Bijtebier [30] in the simplest case of a stationary
external interaction. Introducing the notion of strong translation invariance, we
carried out a systematic generalization [31] which can be applied to a lot of more
complicated situations.

Focusing on the case where external interaction is due to some electromagnetic
plane waves, we already started many years ago a discussion about the tractability
of the ansatz [32,33] and we pointed out the interest of considering a monochromatic
superposition of two waves; but the subject was not developed further. In the
present paper our purpose is to construct (for the first time explicitly) the mass-
shell constraints and to exhibit the first integrals of the motion, allowing for a
reduction of the number of degrees of freedom. By the same token several complicated
calculations and involved arguments given in our aforementioned articles will
be replaced by a more comprehensive exposition.

Section 2 is devoted to a survey of the general formalism we employ, in both the
one-body and two-body cases.

In Section 3 we return to one-body motion, focusing on the case where external
field is made of plane electromagnetic waves. Indeed the one-body motion must be
considered first, because strong translation invariance, which helps us to formulate
the two-body problem in closed form, is basically defined as a property of the one-
body motion in the field. We present and discuss two cases of external fields:
i) the single monochromatic plane wave,
ii) a monochromatic superposition of two such waves, which can be seen as made of
two counter-propagating waves.

Here we show how strong translation invariance arises and we discuss its usefulness
with respect to our purpose.

Then, in Section 4, the details of formulating the two-body wave equations in presence
of the monochromatic superposition are displayed. Section 5 is devoted to
concluding remarks.

Notation, terminology  
Signature $+ - - -$, units $c = \hbar = 1$. Isometry refers to
any transformation of the Poincaré group acting on spacetime. Let us call surviving
symmetries of a system those symmetries that are not destroyed by application of
the external field. Particle labels $a, b$ run from 1 to 2.

2 Basic formulas

This section is not limited to the case of electromagnetic plane waves; here we
intend to consider a more general case where the external interaction enjoys the
appropriate symmetry. In fact strong translation invariance which (when it arises) helps us to formulate the two-body problem, is primarily a property of the one-body motion in the field. Therefore in the next subsection we first consider a single test particle submitted to an external field.

2.1 The one-body wave equation, symmetries

Let \( x, p \) be the canonical variables satisfying the standard commutation relations \([x^\alpha, p_\mu] = i\delta^\alpha_\mu\). The one-body wave function is a scalar on spacetime, say \( \psi(x) \). The Klein-Gordon equation \( 2K\psi = m^2\psi \) is written using the half-squared mass operator

\[
K = \frac{1}{2} p^2 + G, \tag{1}
\]

where the interaction term \( G \) will be called external interaction potential. It is a scalar, not to be confused with the electromagnetic vector potential \( A_\mu \).

This scheme may be obtained by quantization of a classical relativistic (and manifestly covariant) formalism where the equations of motion stem from a scalar generator \( K_{cl} = \frac{1}{2} p_{cl} \cdot p_{cl} + G_{cl} \), the eight canonical variables \( x_{cl}^\alpha, p_{cl\alpha} \) being conjugate in terms of their Poisson brackets; the generator is an obvious constant of the motion and is interpreted as half of the squared mass. Although this approach is not very popular among physicists, it is very simple and natural from a geometrical point of view, as it rests on the symplectic structure of the eight-dimensional phase space. In this formulation the evolution parameter is affine, that is proportional to the proper time; the role of the half-squared mass with respect to this parameter is analogous to the role of the energy with respect to the absolute time in Newtonian mechanics.

**Strong translation invariance.**

Only some fields give rise to strong translation invariance. The most simple example corresponds to an interaction potential of the form \( G = G(\vec{x}, \vec{p}) \), with obvious notation using the lab frame, supposed to be unique\(^{30}\).

**Definition** Any phase-space function is strongly translation invariant along direction \( w \) when it commutes with both \( w \cdot x \) and \( w \cdot p \). When it commutes with only \( w \cdot x \) it is said simply translation invariant.

We are interested in having the external potential strongly translation invariant. In the space of four-vectors, the directions leaving \( G \) strongly translation invariant, when they exist, span a linear space \( E_L \) called longitudinal space. For our purpose it is essential that \( E_L \) admit an orthonormal basis\(^1\) in this situation referred to as normal, the space of four-vectors is an orthogonal direct sum \( E = E_L \oplus E_T \) where \( E_T \) is the transverse space.

Any four-vector \( \xi \) can be written as

\[
\xi = \xi_L + \xi_T, \quad \xi_L = \tau \xi
\]

\(^1\) But some situations of physical interest are not included in this case: for instance when the external field is a single monochromatic plane wave, it turns out that the wave vector is a direction of strong translation invariance; but it is a null vector, so in this case \( E_L \) admits no orthonormal frame.
with $\tau$ projector onto $E_L$. The canonical variables are split accordingly, say $x = x_L + x_T$, $p = p_L + p_T$. Phase-space functions are called longitudinal (resp. transverse) when they depend only on the longitudinal (resp. transverse) canonical variables, equivalently they commute with all the transverse (resp. longitudinal) canonical variables. The case where $E_L$ fails to admit an orthonormal basis will be referred to as degenerate.

**Proposition 1**  The interaction potential is a transverse quantity.

Proof Let $w_A$ be an orthonormal basis of $E_L$ (the number of values taken by indices $\alpha, \beta$ is just the dimension of $E_L$), so $w_A \cdot w_A = \varepsilon$, with $\varepsilon = \pm 1$ depending on the signature of $E_L$. The projector onto $E_L$ is

$$\tau^\alpha_{\beta} = \sum_A \varepsilon w^\alpha_A w_{A\beta}$$

$$x^\alpha_L = \tau^\alpha_{\beta} x^\beta = \sum_A \varepsilon w^\alpha_A (w_A \cdot x)$$

$$[G, x^\alpha_L] = \sum_A \varepsilon w^\alpha_A [G, w_A \cdot x] = 0 \quad (2)$$

Similarly $p^\alpha_L = \tau^\alpha_{\beta} p^\beta$ and one finds

$$[G, p^\alpha_L] = 0 \quad (3)$$



2.2 Two-body wave equations

The wave function is $\Psi(q_1, q_2)$ where the points $q_1, q_2$ in spacetime are canonically conjugate to the momenta $p^\alpha_a = -i \frac{\partial}{\partial q^\alpha_a}$. Notice that the arguments of $\Psi$ are denoted as $q_1, q_2$ rather than $x_1, x_2$ because their classical (i.e. non-quantum) analogs may fail to coincide with the positions in the whole phase space of classical relativistic dynamics [34-36].

The coupled wave equations involve the half-squared-mass operators $H_a$ as follows

$$2H_a \Psi = \mu^2 \Psi, \quad a = 1, 2 \quad (4)$$

where the commutator $[H_1, H_2]$ must vanish. First integrals (also called constants of the motion) are characterized by commuting with both $H_1$ and $H_2$.

The wave equations (4) can obviously be replaced by their sum and difference. Moreover it is convenient to define

$$P = p_1 + p_2, \quad Q = \frac{1}{2}(q_1 + q_2), \quad y = \frac{1}{2}(p_1 - p_2), \quad z = q_1 - q_2 \quad (5)$$

and $j_a = q_a \wedge p_a$. The Lie algebra of the Poincaré group is spanned by its generators $P^\alpha$ and $J_{\mu\nu}$, where $J = Q \wedge P + z \wedge y = j_1 + j_2$. 


In the absence of mutual interaction we would simply have the following equations

\[ 2K_a \Psi = m_a^2 \Psi \]  
\[ K_a = \frac{1}{2} p_a^2 + G_a \]

where \( G_a(q_a, p_a) \) is the external interaction potential acting on particle \( a \); in an external electromagnetic field, \( K_a \) and \( G_a \) are obtained by replacing \( x, p \) and the charge \( e \) respectively by \( q_a, p_a \) and \( e_a \) in \( K \) and in \( G \); see \[29\].

**Definition**  
The couple of potentials \( G_1, G_2 \) is **strongly invariant along direction** \( w \) when each potential separately is strongly invariant along \( w \) in the one-body sector, say

\[ [G_1, w \cdot q_1] = [G_1, w \cdot p_1] = [G_2, w \cdot q_2] = [G_2, w \cdot p_2] = 0 \]

Again the **longitudinal space** \( E_L \) is defined as the span of strong translation invariance directions, and the question arises as to know whether \( E_L \) admits an orthonormal basis. In the **normal case** (that is when it does admit such a basis) Proposition 1 entails that \( G_1 \) and \( G_2 \) are **transverse**.

If the external field were zero, we would have (the superscript \( (0) \) refers to the absence of external field)

\[ 2H^{(0)}_a \Psi = m_a^2 \Psi \]  
\[ H^{(0)}_a = \frac{1}{2} p_a^2 + V^{(0)}_a \]

where we could have \( H^{(0)}_a = \frac{1}{2} p_a^2 + V^{(0)}_a \) but in this work we assume a **unipotential model**, say \( V^{(0)}_1 = V^{(0)}_2 = V^{(0)} \), hence

\[ H^{(0)}_a = \frac{1}{2} p_a^2 + V^{(0)} \]  
\[ \text{and we suppose that mutual interaction is explicitly known when the external field vanishes, in other words} \; V^{(0)} \text{ is given.} \]

For simplicity let us assume that the mutual interaction takes on the form

\[ V^{(0)} = f(Z, P^2, y \cdot P) \]  
\[ Z = z \cdot z P^2 - (z \cdot P)^2 \]

is the main ingredient; so the three dynamical variables in \( V^{(0)} \) are mutually commuting.

Let us stress that the simplification which results from assuming \( (8)(9) \) leaves still enough generality to encompass a wide class of mutual interactions; as an example see the model of electromagnetic interaction derived from Feynman’s diagrams by Jallouli and Sazdjian \[25\].

When all interactions are present, eqs.\[11\] hold with

\[ H_a = K_a + V \]
where \( V \) is a suitable modification of \( V^{(0)} \), to be constructed in order to satisfy the vanishing of \([H_1, H_2]\) and to reproduce the correct limits when either of the interactions is absent.

In general the compatibility condition

\[
[K_1 - K_2, V] = 0
\]

cannot be solved for \( V \) in closed form. But when the external potentials \( G_a \) enjoy some special symmetry, this equation can be transformed to a tractable problem, owing to a change of representation, as follows

Let \( \mathcal{O} \) be any operator; the \textit{external-field representation} is formally defined by

\[
\Psi' = e^{iB} \Psi, \quad \mathcal{O}' = e^{iB} \mathcal{O} e^{-iB}
\]

where \( B \) suitably depends on the external potentials. Of course we are left with the task of solving for \( V' \)

\[
[K_1' - K_2', V'] = 0
\]

which will be possible provided \( B \) is chosen such that \( K_1' - K_2' \) is computable and takes on a simple form. Moreover we have to write down explicitly the transformed of system (1), which requires that both \( K_1' \) and \( K_2' \) be computable by (12).

The concept of \textit{strong translation invariance, naturally extended to the two-body sector}, offers a possibility to carry out this program, choosing \( B \) such that

\[
K_1' - K_2' = y_L \cdot P_L
\]

Before going further a few remarks are in order.

In many cases of interest, the presence of external potentials does not kill all the isometries of spacetime.

Example We shall see later on in Section 4 that, in an electromagnetic field satisfying the equations (35)(36) below, the survivors of the Poincaré algebra are, in an adapted frame

\[
P_1 = p_{(1)1} + p_{(2)1}, \quad P_2 = p_{(1)2} + p_{(2)2}
\]

(in these formulas and whenever necessary in the sequel, we put \textit{parenthesis around the particle indices}).

In the above example \( P_{L1} = P_1 \) is the only nonvanishing component of \( P_L \). In contrast \( P_2 \) is just another conserved quantity.

\textit{Notation} In order to avoid confusing the square of a vector with its second contravariant component, we make the following convention:

use covariant components for the momenta \( P_\alpha, y_\beta \), and contravariant components for the coordinates \( Q^\mu, z^\nu \). So \( P^2 \) stands for \( P \cdot P \), but \( z^2 \) is the second component of \( z \), whereas the square of \( z \) is explicitly noted as \( z \cdot z \). This convention also holds with longitudinal and transverse parts, for example \( P_L^2 = P_L \cdot P_L \), etc, but we write the square of \( z_T \) as \( z_T \cdot z_T \), etc.
In the remaining part of this section we are concerned with strong invariance in the normal case where (by definition) longitudinal and transverse parts are well defined, which corresponds to the existence of an orthonormal basis in $E_L$. Under this assumption it was proved that the longitudinal piece of linear momentum, say $P_{L\alpha}$, is conserved (see Section 3.2 of [31]).

Formula (14) is ensured by taking $B = TL$, where $T$ and $L$ are respectively transverse and longitudinal operators suitably chosen, namely

$$T = y_T \cdot P_T + G_1 - G_2$$  \hspace{1cm} (16)  
$$L = \frac{P_L \cdot z_L}{P_L^2}$$  \hspace{1cm} (17)  

Note that only $T$ depends on the external field. But the denominator in $L$ requires some caution in order to make sense; so we are led to cut off the space of states a sector corresponding to the vanishing eigenvalues of $P_L^2$, as follows. The wave function can be considered as a function of $Q$ and $z$, which can be written as a Fourier expansion with respect to $Q$,

$$\Psi = \frac{1}{(2\pi)^2} \int e^{iK \cdot Q} \Upsilon(K, z) d^4K$$  \hspace{1cm} (18)  

Introducing an arbitrarily small but positive constant $\epsilon$, we shall restrict the support of $\Upsilon$ to the values of the vector $K_\alpha$ satisfying

$$K_L^2 \geq \epsilon$$

Using once for all the Fourier development (18) it is easy to check that the operators $z, y, Q, P$ and $1/P_L^2$ respect the cut-off; in other words, any of these operators maps into itself the space of wave functions whose Fourier transform satisfies the support condition above.

Naturally $P_\alpha = -i \frac{\partial}{\partial Q_\alpha}$ and $y_\alpha = -i \frac{\partial}{\partial z_\alpha}$ thus for instance we have that

$$\frac{1}{P_L^2} \Psi = \frac{1}{(2\pi)^2} \int e^{iK \cdot Q} \frac{1}{K_L^2} \Upsilon(K, z) d^4K$$

$P_L$ is a constant of the motion and we shall eventually focus on eigenstates of it; in that case the support condition will get trivially simplified.

Note that $T$ is manifestly transverse in (16); however we can write equivalently

$$T = K_1 - K_2 - y_L \cdot P_L$$  \hspace{1cm} (19)  

It is obvious in (17) that $P_L$ commutes with $L$, thus also with $B$, hence $P_L' = P_L$. Similarly transformation (12) brings no change in $L$, nor in $T$, say $L' = L$, $T' = T$.

The explicit form of $K'_1$ and $K'_2$ was derived from (6) in [31]. In order to get rid of a clumsy notation ($L^2 \neq L^\alpha L_\alpha$) we replace here the four-vector $L^\alpha$ proposed in [31] by its definition, say $L^\alpha = P_L^\alpha / P_L \cdot P_L$, hence for the sum

$$K'_1 + K'_2 = K_1 + K_2 - 2T \frac{y_L \cdot P_L}{P_L^2} + \frac{T^2}{P_L^2}$$  \hspace{1cm} (20)
and (14) for the difference; in fact the external-field representation was tailored for having that (14) holds true.

Finally, after defining
\[
\mu = \frac{1}{2}(m_1^2 + m_2^2), \quad \nu = \frac{1}{2}(m_1^2 - m_2^2)
\]
the coupled wave equations \( H_a' \Psi' = \frac{1}{2} m_a^2 \Psi' \) obtained from (11) take on this form
\[
(K_1' + K_2' + 2V')\Psi' = \mu \Psi' \tag{21}
\]
\[
y_L \cdot P_L \Psi' = \nu \Psi' \tag{22}
\]
(note that the latter equation does not depend on the mutual interaction).

The explicit form of \( V' \) is constructed from that of \( V(0) \) as follows: according to (14) we have to ensure that
\[
[y_L \cdot P_L, V'] = 0 \tag{23}
\]
A solution is easy to find, introducing the no-field ”limit” of \( B \), obtained by canceling \( G_1 \) and \( G_2 \) in \( T \), say
\[
B^{(0)} = y_T \cdot P_T L
\]
indeed we observe that
\[
e^{iB^{(0)} y \cdot P} e^{-iB^{(0)}} = e^{iB} y \cdot P e^{-iB} = y_L \cdot P_L
\]
and we know that \( Z \) commutes with \( y \cdot P \), therefore defining \( \hat{Z} \) as the no-external-field limit of \( Z' \), namely
\[
\hat{Z} = e^{iB^{(0)} Z} e^{-iB^{(0)}} \tag{24}
\]
it turns out that \( \hat{Z} \) commutes with \( y_L \cdot P_L \). Moreover it is obvious that \( P^2 \) commutes with \( y_L \cdot P_L \), thus finally any function of \( \hat{Z}, P^2, y_L \cdot P_L \) is expected to solve (23).

The ansatz consists in constructing \( V' \) from \( V^{(0)} \) as follows
\[
V' = f(\hat{Z}, P^2, y_L \cdot P_L) \tag{25}
\]
where \( f \) is the function given in (9). It is not difficult to check that this formula yields the correct limits when either \( f \) or both \( G_1 \) and \( G_2 \) vanish.

Fortunately \( \hat{Z} \) is explicitly computed; formula (24) yields
\[
\hat{Z} = Z + 2(P_L^2 z \cdot P - P^2 z_L \cdot P_L) L + P_L^2 P^2 L^2 \tag{26}
\]
In [37] this formula was cast into this equivalent but more compact form
\[
\hat{Z} = z_T \cdot z_T P^2 - (z_T \cdot P)^2 + P^2(z_L \cdot z_L - \frac{(z_L \cdot P_L)^2}{P_L^2}) \tag{27}
\]
We emphasize that in Ref. [37] all numbered formulas from (1) up to (7) included are general and by no means limited to the case of constant electromagnetic field [3]. Indeed inserting (17) into the middle term of (26) we obtain
\[
2(P_L^2 z \cdot P - P^2 z_L \cdot P_L) L = 2(z \cdot P)(z_L \cdot P) - 2 \frac{P^2}{P_L^2} (z_L \cdot P)^2
\]
\[\text{In contrast eq. (8) of that reference holds only if } P_L \text{ is timelike, which is not the case eventually considered in the present paper.}\]
\[ 2(P_L^2 z \cdot P - P^2 z_L \cdot P_L)L = 2(z \cdot P)(z_L \cdot P) - 2(z_L \cdot P)^2 (1 + P_T^2 / P_L^2), \]

splitting \( z \cdot P \) yields
\[ 2(P_L^2 z \cdot P - P^2 z_L \cdot P_L)L = 2(z_L \cdot P)^2 + 2(z_T \cdot P)(z_L \cdot P) - 2(z_L \cdot P)^2 (1 + P_T^2 / P_L^2) \]

Now add \( Z \), taking (10) into account; we find a cancellation of \((z_T \cdot P)(z_L \cdot P)\) and remain with
\[ Z + 2(P_L^2 z \cdot P - P^2 z_L \cdot P_L)L = z \cdot z P^2 - (z_T \cdot P)^2 - (z_L \cdot P)^2 (1 + 2P_T^2 / P_L^2) \]

Owing to (26), \( \hat{Z} \) is given by adding \( P_T^2 P_L^2 L^2 \) to this quantity. But (17) implies that \( P_T^2 P_L^2 L^2 = (z_L \cdot P)^2 P_T^2 / P_L^2 \), thus
\[ \hat{Z} = z \cdot z P^2 - (z_T \cdot P)^2 - (z_L \cdot P)^2 (1 + P_T^2 / P_L^2) \]

after splitting \( z \cdot z P^2 \) we are left with (27), that is formula (7) of [37].

In order to develop (21) we need to compute the r.h.s. of (20). Eq. (6) implies \( K_1 + K_2 = P^2 / 4 + y^2 + G_1 + G_2 \) that we insert into (20), hence
\[ K_1' + K_2' = P^2 / 4 + y^2 + G_1 + G_2 - 2T \frac{y_L \cdot P_L}{P_L^2} L + \frac{T^2}{P_L^2} \] (28)

In the following sections we shall specify the external field.

3 One-body motion in electromagnetic waves

Consider first the motion of a single charged pointlike and spinless body (treated as a test particle) subject to any electromagnetic field \( F = \partial \wedge A \). Let angular momentum be noted as \( j = x \wedge p \). In the Klein-Gordon equation we have
\[ K = \frac{1}{2}(p - eA)^2, \quad G = -\frac{1}{2}(eA \cdot p + ep \cdot A - e^2 A^2) \] (29)

Note that
\[ [A^\mu, p_\nu] = i \partial_\nu A^\mu \] (30)

3.1 Single electromagnetic plane wave

The behavior of a relativistic charged particle in an electromagnetic plane wave has been studied a long time ago, even including spin [38-40].

Although the structure of a single electromagnetic plane wave is well known, here we insist on its manifestly covariant formulation as follows.

The wave vector is a four-vector \( k \), null and oriented toward the future; the electromagnetic field is a tensor
\[ F = a(k \wedge u) \sin(k \cdot x + \alpha), \] (31)
u is a constant spacelike unit vector \((u \cdot u = -1)\), \(a\) and \(\alpha\) scalar constants. Although \(k\) is given, the factor \(u\) in the bi-vector \(k \wedge u\) is not unique since we can add to \(au\alpha\) an arbitrary null vector proportional to \(k\). A vector-potential for this field is

\[
A^\mu = au^\mu \cos(k \cdot x + \alpha), \quad k \cdot k = 0
\]

The Lorenz gauge condition requires \(u \cdot k = 0\)
The linear space of four-vectors can be written as

\[
E = \mathcal{E}_{03} \oplus \mathcal{E}_{12},
\]

where \(k \in \mathcal{E}_{03}\) and \(u \in \mathcal{E}_{12}\). Note however that this direct-sum decomposition is not intrinsically defined.

We can use an orthonormal basis \((E_\alpha)\) defined such that \(E_0\) points toward the future, \(k = \omega(E_0 + E_3)\) and \(E_2 = u\).

So we have

\[
k^\mu = (\omega, 0, 0, \omega), \quad \omega > 0 \quad (33)
\]
equivalently \(k_\mu = (\omega, 0, 0, -\omega)\). Note that \(k \cdot x = \omega(x_0 + x_3) = \omega(x^0 - x^3)\).

Moreover \(u^\mu = (0, 0, 1, 0)\).

**First integrals**
The constants of the motion are induced by the symmetries of \(K\) in \((29)\). Since \(k \cdot x\) does not depend on \(x^1, x^2\) it follows that \(A^\mu\) is invariant under translation and rotation in \(\mathcal{E}_{12}\), in other words \(A^\mu\) commutes with \(p_1, p_2\), and \(j_{12}\). Further observe that \(u\), thus also \(A\), lies in \(\mathcal{E}_{12}\) thus \(A \cdot p + p \cdot A\) depends only on \(x^0, x^3, p_1, p_2\), which entails \([G, p_1] = [G, p_2] = 0\), therefore \([K, p_1] = [K, p_2] = 0\), so both \(G\) and \(K\) are (at least simply) invariant by translation in \(\mathcal{E}_{12}\).

Hence \(p_1\) and \(p_2\) as first integrals.

Moreover \([k \cdot x, k \cdot p] = 0\), thus \([A, k \cdot p]\) vanishes as well, implying that \(A \cdot p, \quad p \cdot A\) and \(A \cdot A\) separately all commute with \(k \cdot p\). Hence \([G, k \cdot p] = [K, k \cdot p] = 0\), so finally \(k \cdot p\) is another constant of the motion.

In any adapted frame, \(k \cdot p = \omega(p_0 + p_3)\) so conservation of \(k \cdot p\) amounts to having \(p_0 + p_3 = \text{const.}\) (note that in contrast \((p_0 - p_3)\) is not a constant of the motion).

To summarize: the system enjoys translation invariance along \(E_1, E_2\) and \(k\), more precisely \(K\) as well as \(G\) are invariant under these translations. We could state equivalently:

**Proposition 2** \(K\) and \(G\) are at least simply invariant by translation along direction \(w\) iff \(w \cdot k = 0\).

This condition characterizes the 3-plane tangent to the light cone along \(k\), say \(\Pi_3\).

But we are interested in the possibility of having \(G\) strongly translation invariant; the question is as to know whether \(G\) admits directions of strong translation invariance. Translation invariance along \(w\) will be strong iff additionally \([G, w \cdot x]\)
vanishes. One finds that (irrespective of the analytical shape of $A$) $[w \cdot x, G]$ vanishes iff $w \cdot u = 0$. Finally

$G$ is strongly translation invariant along $w$ provided

$$w \cdot k = w \cdot u = 0$$

In other words $w$ must belong to the two-dimensional vector space $\Pi_2$ spanned by $k$ and $u$. Unfortunately it is trivial to check that in $\Pi_2$ any direction orthogonal to $u$ is necessarily colinear with $k$. Thus this vector space fails to admit any orthonormal frame; it does not provide a unique and straightforward definition of longitudinal and transverse directions. This drawback with the single plane wave was pointed out in a previous work; in the same paper we already advocated the interest of considering rather a superposition of two plane waves [32].

### 3.2 Superposition of two plane waves

Now let us consider the case where the electromagnetic field is a superposition of two plane waves travelling along the same right line, with respective wave vectors $k$, $l$, non-colinear, both null and future oriented, so that their scalar product is strictly positive, say

$$k \cdot l = 2\omega^2$$

With an obvious notation the field is $F = F_I + F_{II}$, namely

$$F = a(k \wedge u)\sin(k \cdot x + \alpha) + b(l \wedge u)\sin(l \cdot x + \beta)$$

(34)

where $k$, $l$ are constant null vectors, $a, b, \alpha, \beta$ constant scalars; $u$ is a constant spacelike unit vector ($u \cdot u = -1$) orthogonal to both $k$ and $l$.

**Proposition 3** Since $k,l$ are given null and non-colinear, the factor $u$ in the bivectors $k \wedge u$, $l \wedge u$ is unique.

**Proof** Looking for a possible $u'$ such that $k \wedge u' = k \wedge u$ and $l \wedge u' = l \wedge u$ entails $u' - u = \lambda k$, $u' - u = \mu l$, for some $\lambda, \mu \in \mathbb{R}$, thus $\lambda k = \mu l$ which is impossible unless $\lambda = 0 = \mu$, therefore $u' = u$. []

The most simple vector-potential, such that $F = \partial \wedge A$, can be written as

$$A^\mu = A_1^\mu + A_{II}^\mu = Wu^\mu$$

(35)

with

$$W = W_I + W_{II} = a \cos(k \cdot x + \alpha) + b \cos(l \cdot x + \beta)$$

(36)

The Lorenz gauge condition is ensured by requiring that $u \cdot k = u \cdot l = 0$.

**Proposition 4** Given two non-colinear null vectors, $k$, $l$ future oriented, it is always possible to find orthonormal basis where $k^0 = l^0$. 

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Proof Define 
\[ E_0 = \frac{(k + l)}{2\omega}, \quad E_3 = \frac{(k - l)}{2\omega} \]

We get 
\[ E_0^2 = 1, \quad E_3^2 = -1, \quad E_0 \cdot E_3 = 0 \]

and \((k + l)(k - l) = 0\), moreover
\[ k \cdot E_0 = \frac{k \cdot l}{2\omega} = \omega, \quad k \cdot E_3 = -\frac{k \cdot l}{2\omega} = -\omega \]
\[ l \cdot E_0 = \omega, \quad l \cdot E_3 = \omega \]

and \((k + l)^2 = 4\omega^2, \quad (k - l)^2 = -4\omega^2\).

So the two-dimensional vector space spanned by \(k, l\) has signature \(+−\) and admits \(E_0, E_3\) as orthonormal basis.

Its orthocomplement in the space \(E\) of four-vectors, say \(E_{12}\), has elliptic signature; the couple \(E_0, E_3\) can be completed by \(E_1, E_2\) as to form an orthonormal basis.

In contrast to the case of a single plane wave, in the present case the splitting \(E = E_{03} \oplus E_{12}\) is intrinsically defined; note that \(u \in E_{12}\). \[\]

Any basis constructed that way (which gives to both waves the same frequency) will be called a \textit{monochromatic basis}. In such a basis we can write
\[ k^\mu = (\omega, 0, 0, \omega) \quad l^\mu = (\omega, 0, 0, -\omega) \quad (37) \]

3.2.1 One-body motion, first integrals

\(A\) depends only on \(x^0, x^3\), thus \([A, p_1] = [A, p_2] = 0\). Moreover \((35)\) implies that \(A\) lies in \(E_{12}\). It follows that \(A \cdot p + p \cdot A\) depends only on the mutually commuting arguments \(x^0, x^3, p_1, p_2\). Finally \(G\) and also \(K\) commutes with both \(p_1\) and \(p_2\), say
\[ [p_1, G] = [p_1, K] = [p_2, G] = [p_2, K] = 0, \quad (38) \]

this property of simple translation invariance in the plane \(E_{12}\) entails that \(p_1\) and \(p_2\) are constants of the motion.

In contrast \(A_1^\mu\) commutes with \(p_0 + p_3\) whereas \(A_{11}^\mu\) commutes with \(p_0 - p_3\), moreover \([A_1^\mu, p_0 - p_3]\) is a function of \(x^0 - x^3\) only whereas \([A_{11}^\mu, p_0 + p_3]\) is a function of \(x^0 + x^3\) only. The most general direction of \(E_{03}\) can be written as \(w = w_+ (E_0 + E_3) + w_- (E_0 - E_3)\) (with \(w_\pm\) constant scalars). One finds that \([A^\mu, w \cdot p]\) is a sum of two independent functions; it cannot be identically zero, thus no direction of \(E_{03}\) could generate a translation leaving \(A^\mu\) invariant (in fact \(A_1^\mu + A_{11}^\mu\) exhibits no invariance at all in the plane \(E_{03}\)). Similar argument holds for \([A \cdot p + p \cdot A, w \cdot p]\) and \([A \cdot A, w \cdot p]\), and finally \(G\) cannot be translation invariant along a direction of \(E_{03}\).
3.2.2 Strong translation invariance

Let us prove more briefly this statement announced several years ago with a complicated justification [33].

**Proposition 5** The interaction term \( G \) corresponding to (35) is strongly translation invariant along a unique direction \( w \), defined as orthogonal to \( k, l \) and \( u \).

**Proof** in the previous subsection we saw that \( G \) is (at least simply) translation invariant along any direction of the plane \( E_{12} \), and by no direction of the plane \( E_{03} \). Thus any possible direction of strong invariance of \( G \) must be searched only within the plane \( E_{12} \). Remind that \( u \) being orthogonal to \( k \) and \( l \), it belongs to the plane \( E_{12} \).

Hereafter we shall specify further the monochromatic basis by taking \( E_2 = u \), so we have \( u^\mu = (0, 0, 1, 0) \). This choice determines the adapted monochromatic basis; as a result we can write the second formula (29) on this form

\[
G = -\frac{1}{2}e(Wp_2 + p_2W) - \frac{1}{2}e^2W^2
\]

where the only momentum involved is the component \( p_2 \).

Let us now look for a direction \( w \) of strong translation invariance. In addition to ordinary invariance just characterized above, we must have that \([G, w \cdot x] \) vanishes. Since the quadratic piece (with respect to \( eA \)) of \( G \) trivially commutes with \( x \), we are left with

\[
2[G, w \cdot x] = -eA^\alpha[p_\alpha, w \cdot x] - e[p_\alpha, w \cdot x]A^\alpha
\]

but \( [p_\alpha, w \cdot x] = -iw_\alpha \), therefore \([G, w \cdot x] \) vanishes iff \( w \) is orthogonal to \( A \), which means orthogonal to \( u \). In the \( E_{12} \) plane the only direction orthogonal to \( u \) is that of \( E_1 \). []

We shall normalize \( w \) by choosing \( w = E_1 \), say

\[
w^\mu = (0, 1, 0, 0)
\]

This spacelike direction determines a \( 1 \oplus 3 \) decomposition longitudinal/transverse in the linear space of four-vectors \(^4\).

In agreement with the convention made in 2.2, the ray spanned by \( w \) will be called longitudinal, whereas the span of \( k, l, u \) will be called the transverse 3-plane.

To summarize: in the field which corresponds to (35) the wave equation is \((p - eA)^2\psi = m^2\psi\), neither \( k \cdot x \) nor \( l \cdot x \) commutes with the squared-mass operator; in contrast the quantities \( p_1, p_2 \) are constants of the motion; they can be diagonalized, say \( p_1\psi = \rho_1\psi, \quad p_2\psi = \rho_2\psi \), with \( \rho_1, \rho_2 \) numerical constants. So we can write, up to a normalization factor

\[
\psi = e^{i(p_1x^1 + p_2x^2)} \gamma(x^0, x^3)
\]

In the sequel we shall tackle the two-body problem, in order to construct a pair of compatible mass-shell constraints.

\(^4\) space \( \oplus \) three-dimensional hyperbolic, not to be confused with the usual time \( \oplus \) space decomposition.
4 Two-body system in a monochromatic superposition

Let us resume our analysis of the two-body problem initiated in subsection 2.2. As we saw up there, in the absence of mutual interaction the motion of each particle would be ruled by the Hamiltonian generator $K_a = \frac{1}{2}p_a^2 + G_a$. Now we focus on the situation characterized by the form (35) of electromagnetic vector-potential. We use the adapted frame described in the previous section and extend formula (39) to the two-body system by replacing $x, p, e$ as indicated in subsection 2.2; we find

$$2G_a = -e_a(W_a p_{(a)2} + p_{(a)2} W_a) - (e_a W_a)^2$$

where

$$W_a = a \cos(k \cdot q_a + \alpha) + b \cos(l \cdot q_a + \beta)$$

and with the following

**Notation**  When there is a risk that particle label be confused with coordinate label, the former is put between parenthesis; no parenthesis otherwise: for instance $p_{(1)2}$ is the second component of the momentum of particle 1.

**Remark**  Strong translation invariance of $G_a$ is a consequence of that of $G$, which stems from the shape of the electromagnetic field as described in section 3.2 and stated in Proposition 5, without any further condition.

Note that each $W_a$ depends only on $Q^0, Q^3, z^0, z^3$ while $G_a$ additionally depends on $P_2$ and $y_2$, through the identities

$$p_{(1)2} = \frac{1}{2}P_2 + y_2, \quad p_{(2)2} = \frac{1}{2}P_2 - y_2$$

It stems from (35) that

$$[p_{(1)1}, K_1] = [p_{(1)2}, K_1] = 0 \quad (43)$$

$$[p_{(2)1}, K_2] = [p_{(2)2}, K_2] = 0 \quad (44)$$

and it is trivial that

$$[p_{(1)\alpha}, K_2] = [p_{(2)\alpha}, K_1] = 0 \quad (45)$$

whence we deduce

$$[P_1, K_a] = [P_2, K_a] = 0 \quad (46)$$

The unique longitudinal direction is $E_1$ with contravariant components $(0, 1, 0, 0, 0)$ and now we have for any four-vector

$$\xi_L = -(\xi \cdot w) w, \quad \xi_T = \xi + (\xi \cdot w) w$$

but $\xi \cdot w = \xi \cdot E_1 = \xi_1 = -\xi_1$.

Here $w$ is spacelike; in any adapted frame we can write for any couple $\xi, \eta$

$$\xi_L \cdot \eta_L = -(\xi \cdot w)(\eta \cdot w) = -\xi_1 \eta_1$$

(48)
thus we have

$$z_L \cdot P_L = -z_1 P_1, \quad P^2_L = -P^2_1, \quad z^2_L = -z^2_1$$

(49)

Owing to this last formula, the third term in the r.h.s. of (27) vanishes and we remain with

$$\hat{Z} = z_T \cdot z_T \ P^2 - (z_T \cdot P_T)^2$$

(50)

analogous with Bijtebier’s formula (see (4.3)(4.4) in [30]) concerning the case where the external field was stationary; but here the external potential is strongly invariant along a *spacelike* direction.

On the other hand, equation (28) gets simplified as follows

$$K_1' + K_2' = P^2/4 + y^2 + G_1 + G_2 - 2T \frac{y_1}{P_1} - \frac{T^2}{P^2_1}$$

(51)

Linear momentum is $P_\alpha = p_{(1)\alpha} + p_{(2)\alpha}$. As we mentioned in subsection 2.2 its longitudinal piece

$$P_L \alpha = -P_1 \ w_\alpha$$

is conserved, therefore we can diagonalize $P_1$, and fix its eigenvalue say $\lambda_1 \neq 0$ (according to the restriction made in subsection 2.2). So we get rid of one spacelike degree of freedom, namely $Q_1$.

Moreover, in view of theorem 2 of [20], we expect that $P_2$ also is to be conserved. Let us directly check this point. In view of (46) all we have to prove is that $P_2$ also commutes with $V'$, or equivalently that $P'_2$ commutes with $V'$. But we first observe that

**Proposition 6** $P_2$ is not affected by transformation (12), say $P'_2 = P_2$

Proof $P_2$ is a purely transverse quantity, thus it commutes with $L$. Then a glance at (19) and (46) ensures that $P_2$ commutes also with $T$, so finally with $LT$ which is the generator of the transformation.[1]

Then looking at (25) the question is whether $P_2$ commutes with $\hat{Z}$, which is obvious in (27), so

$P_2$ is a constant of the motion, as expected; we assign to it a sharp value, say $\lambda_2$.

We can summarize: the survivors of the Poincaré Lie algebra are $P_1, P_2$, they define a conserved vector, let it be noted as

$$P_{\perp \alpha} = (0, P_1, P_2, 0)$$

(52)

and the principle of isometric invariance is satisfied. Notice that $P'_\perp$ is not affected by transformation (12), say $P'_{\perp \alpha} = P_{\perp \alpha}$.

In contrast to the case without external field, $P^2$ is not anymore a first integral whereas $P^\perp_2$ remains conserved.
At this stage it is convenient to remind that in the absence of external field the coupled wave equations are usually reduced to a spectral problem for the quantity
\[ N = H_1 + H_2 - \frac{(H_1 - H_2)^2}{P_1^2} - \frac{1}{4}P_1^2 \]
which is intimately related to the properties of relative motion.

In the present case \( N \) is not anymore a first integral, but now it is natural to consider instead of it this invariant combination
\[ N' = H_1 + H_2 - \frac{(H_1 - H_2)^2}{P_1^2} - \frac{1}{4}P_1^2 \]
\[ \text{(53)} \]

The system \( (21)(22) \) is to be solved in the external-field representation; we can impose that \( \Psi' \) be an eigenfunction of \( P_{\perp \alpha} \), say
\[ P_{\perp \alpha} \Psi' = I_{\alpha} \Psi', \quad I_{\alpha} = (0, \lambda_1, \lambda_2, 0), \]
this choice renders \( N' \) diagonal. The cut-off introduced in Section 2.2 is simply expressed as \( \lambda_1^2 \geq \varepsilon \), hence \( I^2 = -(\lambda_1^2 + \lambda_2^2) < 0 \).

We have
\[ P^2\Psi' = (P^2_0 - P^2_3)\Psi' - (\lambda_1^2 + \lambda_2^2)\Psi' \]
note that \( P^2 \neq P'^2 \).

### 4.1 Reducing the wave equations.

Here we aim at solving the coupled wave equations \( (21)(22) \) by an eigenstate of \( P_1, P_2 \), say
\[ \Psi' = \exp(i(\lambda_1 Q^1 + \lambda_2 Q^2)) \phi(Q^0, Q^3, z^\alpha) \]
\[ \text{(54)} \]
Let us consider first \( (22) \) and remember that \( y_1 = -i\partial/\partial z^1 \). Since \( y_L \cdot P_L = -y_1 P_1 \) we must have \( y_1 P_1 \Psi' = -\nu \Psi' \), but \( P_1 \Psi' = \lambda_1 \Psi' \), so \( y_1 \) is constant of the motion with eigenvalue \(-\nu/\lambda_1\), and \( (22) \) is to be solved by writing
\[ \Psi' = \exp(k_1 Q^1 + k_2 Q^2 - \frac{\nu}{\lambda_1}z^1) \chi(Q^0, Q^3, z_T) \]
\[ \text{(55)} \]
In order to determine \( \chi \) we now develop equation \( (21) \).

We separate the coordinates \( Q^1, Q^2, z^1 \) from \( Q^0, Q^3, z_T \). It is clear from \( (55) \) that \( \Psi' \) is eigenstate not only of \( P_1, P_2 \), but also of \( y_1, \) with respective eigenvalues \( \lambda_1, \lambda_2 \) and \(-\nu/\lambda_1\). In view of this remark it is convenient to introduce the following

**Notation:** to any dynamical variable \( F(Q, P, z, y) \) we associate the substitution
\[ \mathcal{F} = \text{subs.}(F \mid P_1 = \lambda_1, P_2 = \lambda_2, y_1 = -\nu/\lambda_1) \]
\[ \text{(56)} \]
so \( \mathcal{F} \) and \( F \) yield the same result when applied to \( \Psi' \). For instance
\[ P^2 = P^2_0 - P^2_3 - \lambda_1^2 - \lambda_2^2, \quad \frac{P^2}{4} = -\lambda_1^2 \]
\[ \text{(57)} \]

---

5 The eigenvalue of \(-N\) appears denoted as \( b^2 \) in the work of Todorov [18-21]; divided by the reduced mass it is proportional to the leading term in the development of the mass defect \( M - (m_1 + m_2) \), insofar as an isolated system is concerned.
From (40) we obtain
\[ 2G_a = -e_a(W_a p_{(a)2} + p_{(a)2} W_a) - (e_a W_a)^2 \] (58)
where
\[ p_{(1)2} = \frac{1}{2} \lambda_2 + y_2, \quad p_{(2)2} = \frac{1}{2} \lambda_2 - y_2 \] (59)
Formula (51) yields
\[ K_1' + K_2' = \frac{1}{4} P^2 - (\nu \lambda_1)^2 + y_2^2 + G_1 + G_2 + \frac{1}{\lambda_1}(2\nu T - T^2) \] (60)
where \( G_a \) is given by (58) above, while the expression for \( T \) results from (16), that is
\[ T = y_0 P_0 - \lambda_2 y_2 - P_3 y_3 + G_1 - G_2 \] (61)
But in order to achieve writing the explicit form of (21) we still have to evaluate \( V'\Psi' \).
In view of (43) (11) (25) we consider the action of \( \hat{Z}, P^2 \) and \( y \cdot P \) on the wave function.
In the present case \( \hat{Z} \) is given by (50), hence \( \hat{Z} \Psi' = \hat{Z} \Psi' \) where of course we define
\[ \hat{Z} = z_r \cdot z_r P^2 - (z_0 P_0 + z_3 P_3 + \lambda_2 z_2)^2 \] (62)
Note that, as differential operators, \( P^2 \) and \( \hat{Z} \) act only on the variables \( Q^0, Q^3 \).
Finally \( V'\Psi' = V'\Psi' \), defining
\[ V' = f(\hat{Z}, P^2, \nu), \] (63)
in this expression \( f \) encodes all information about mutual interaction; it is a priori given and the details about its arguments are formulas (57) and (62).
The reduced wave equation thus takes on this form
\[ (K_1' + K_2' + 2V') \chi = \mu \chi \] (64)
wherein (60) and (63) are to be inserted. In spite of its formal aspect, (64) cannot be considered as an eigenvalue equation for \( \mu \), since \( m_1, m_2 \) are parameters fixed from the outset. In fact this equation is, in a trivial manner, equivalent to an eigenvalue problem for the quantity defined in (53), say
\[ N_1' \Psi' = (H_1' + H_2' - \frac{\nu^2}{T^2} - \frac{T^2}{4}) \Psi' \]
Indeed according to (11) we recall that \( H_1' + H_2' = K_1' + K_2' + 2V' \) therefore the number \( \sigma = \mu - (\frac{\nu^2}{T^2} + \frac{T^2}{4}) \) is the eigenvalue of \( N_1' \).
5 Conclusion and outlook

From the start we have discarded the apparently simpler model involving a single plane electromagnetic wave, which actually is problematic for our purpose, since it leads to a degenerate case of strong translation invariance.

In this work we obtained a pair of compatible mass-shell constraints describing the motion of two charged spinless particles subject to a laser made of two counter-propagating plane waves. The form of these equations is given explicitly, in the external-field representation, through formulas (21) and (22) with help of (16), (40), (51) and (25), assuming that the term of mutual interaction was known in closed form in the absence of external field.

The monochromatic superposition of two plane waves (although it preserves less symmetries than a single plane wave) provides a normal case of strong translation invariance. Moreover (in contrast to the single wave) this superposition allows us to distinguish, in an intrinsic way, a preferred frame of reference (the adapted basis) which could be viewed, rather naturally, as the laboratory frame.

Enough symmetry of translation is preserved anyway, as to furnish two constants of the motion, \( P_1, P_2 \), the former associated with strong translation invariance, and both of them in agreement with the principle of isometric invariance.

On the one hand these first integrals permit to factorize out two degrees of freedom, namely \( Q_1, Q_2 \). On the other hand (22) leads to the elimination of \( z^1 \) so we remain with a unique reduced wave equation to be solved for \( \chi(Q_0, Q_3, z_T) \). We are left with a problem of five degrees of freedom.

Note that, the longitudinal direction being spacelike, the spacelike relative coordinate \( z^1 \) is eliminated instead of the so-called "relative time"; a similar situation also occurs in the simple case where the external field is a constant homogeneous electric field [41].

At the present stage we are at least provided with a manifestly covariant formalism which has the correct limits when either of the interactions vanishes and which satisfies the principle of isometric invariance. Naturally it would be interesting to renew the contact with BS equation in the spirit of [26], and to compare the result with the present approach; but now the variation of external field in time might be a serious complication for this program. In the meanwhile it is encouraging that isometric invariance which was neither explicitly required nor even invoked in the early foundations of our method [30,31], turns out to be satisfied after all.

Some attention is still required in order to clarify the physical meaning of \( N_\perp \), but this issue is already transparent in the equal-mass case (\( \nu = 0 \)), where \( N_\perp \) is just the conserved piece of \( N \).

Further work could be devoted to solve the reduced wave equation for a relativistic harmonic oscillator as a toy model, choosing \( f = \text{const.} \ (P^2)^{-1/2} Z \) in formula (9).

Another open problem in the hope of realistic applications is of course the introduction of spin.
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misprints in this paper: in p.1107 one should read condition (2.15) instead of
condition (3.15). In Appendices B and C: Z was skipped from the r.h.s. of
formula B.5. One should read $K = \overline{H} + G$ and $\overline{H}_1 + \overline{H}_2 = \frac{P^2}{4} + y^2$.
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