The intersection of spheres in a sphere and a new geometric meaning of the Arf invariant

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Abstract
Let $S^3_i$ be a 3-sphere embedded in the 5-sphere $S^5$ ($i = 1, 2$). Let $S^3_1$ and $S^3_2$ intersect transversely. Then the intersection $C = S^3_1 \cap S^3_2$ is a disjoint collection of circles. Thus we obtain a pair of 1-links, $C$ in $S^3_i$ ($i = 1, 2$), and a pair of 3-knots, $S^3_i$ in $S^5$ ($i = 1, 2$). Conversely let $(L_1, L_2)$ be a pair of 1-links and $(X_1, X_2)$ be a pair of 3-knots. It is natural to ask whether the pair of 1-links $(L_1, L_2)$ is obtained as the intersection of the 3-knots $X_1$ and $X_2$ as above. We give a complete answer to this question. Our answer gives a new geometric meaning of the Arf invariant of 1-links.

Let $f : S^3 \to S^5$ be a smooth transverse immersion. Then the self-intersection $C$ consists of double points. Suppose that $C$ is a single circle in $S^5$. Then $f^{-1}(C)$ in $S^3$ is a 1-knot or a 2-component 1-link. There is a similar realization problem. We give a complete answer to this question.
1 Introduction and Main results

Let $S^3_i$ be a 3-sphere embedded in the 5-sphere $S^5$ $(i = 1, 2)$. Let $S^3_i$ and $S^3_j$ intersect transversely. Then the intersection $C = S^3_i \cap S^3_j$ is a disjoint collection of circles. Then $C$ in $S^3$ is a 1-link $(i = 1, 2)$. Note that the orientation of $C$ is induced by that of $S^3_i$, that of $S^3_j$ and that of $S^5$. Thus we obtain a pair of 1-links, $C$ in $S^3_i$ ($i = 1, 2$), and a pair of 3-knots, $S^3_i$ in $S^5$ $(i = 1, 2)$.

Conversely let $(L_1, L_2)$ be a pair of 1-links and $(X_1, X_2)$ be a pair of 3-knots. It is natural to ask whether the pair of 1-links $(L_1, L_2)$ is obtained as the intersection of the 3-knots $X_1$ and $X_2$ as above. We give a complete answer to this question. (Theorem 1.1.)

To state our results we need some definitions.

An (oriented) (ordered) $m$-component n-(dimensional) link is a smooth, oriented submanifold $L = \{K_1, \ldots, K_m\}$ of $S^{n+2}$, which is the ordered disjoint union of $m$ manifolds, each PL homeomorphic to the $n$-sphere. If $m = 1$, then $L$ is called a knot. (See [1], [2], [8], [9].)

We say that $n$-links $L_1$ and $L_2$ are equivalent if there exists an orientation preserving diffeomorphism $f : S^{n+2} \rightarrow S^{n+2}$ such that $f(L_1) = L_2$ and $f|_{L_1} : L_1 \rightarrow L_2$ is an orientation preserving diffeomorphism.

Definition $(L_1, L_2, X_1, X_2)$ is called a 4-tuple of links if the following conditions (1), (2) and (3) hold.

1. $L_i = (K_{i1}, \ldots, K_{im})$ is an oriented ordered $m_i$-component 1-dimensional link $(i = 1, 2)$.
2. $m_1 = m_2$.
3. $X_i$ is a 3-knot.

Definition A 4-tuple of links $(L_1, L_2, X_1, X_2)$ is said to be realizable if there exists a smooth transverse immersion $f : S^3_i \bigcup S^3_j \rightarrow S^5$ with the following properties. We assume that the orientations of $S^3_i, S^3_j$ and $S^5$ are given.

1. $f|_{S^3_i}$ is a smooth embedding. $f(S^3_i)$ in $S^5$ is equivalent to the 3-knot $X_i (i = 1, 2)$.
2. For $C = f(S^3_i) \cap f(S^3_j)$, the inverse image $f^{-1}(C)$ in $S^3_i$ is equivalent to the 1-link $L_i (i = 1, 2)$. Here, the orientation of $C$ is induced naturally from the preferred orientations of $S^3_i, S^3_j$, and $S^5$, and an arbitrary order is given to the components of $C$.

The following theorem characterizes the realizable 4-tuples of links.

Theorem 1.1 A 4-tuple of links $(L_1, L_2, X_1, X_2)$ is realizable if and only if $(L_1, L_2, X_1, X_2)$ satisfies one of the following conditions (1) and (2).

1. Both $L_1$ and $L_2$ are proper links, and

$$\text{Arf}(L_1) = \text{Arf}(L_2).$$

2. Neither $L_1$ nor $L_2$ is a proper link, and

$$\text{lk}(K_{ij}, L_i - K_{ij}) \equiv \text{lk}(K_{ij}, L_2 - K_{ij}) \mod 2 \text{ for all } j.$$

In the case where $L_i$ is a 1-component link, that is, $L_i$ is a knot $K_i$, we have the following corollary.

Corollary 1.2 For 1-knots $K_1$ and $K_2$, a 4-tuple of links $(K_1, K_2, X_1, X_2)$ is realizable if and only if

$$\text{Arf}(K_1) = \text{Arf}(K_2).$$
Note. Theorem 1.1 and Corollary 1.2 give a new geometric meaning of the Arf invariant.

Note 1.2.1. The problem (26) in [3] says: Investigate ordinary sense slice 1-links, where ordinary sense slice 1-links are 1-links which are obtained as follows: Let \( S^2 \) be in \( R^4 = R^3 \times R \). Then \( S^2 \cap [R^3 \times \{0\}] \) in \( R^3 \times \{0\} \) is a 1-link. By using Theorem 1.1, the author gives an answer to this problem. The answer is: for every ordinary sense slice 1-link we can define the Arf invariant and it is zero (see [18]).

Let \( f : S^3 \to S^5 \) be a smooth transverse immersion. Then the self-intersection \( C \) consists of double points. Suppose that \( C \) is a single circle in \( S^5 \). Then the \( f^{-1}(C) \) in \( S^3 \) is a 1-knot or a 2-component 1-link. There is a similar realization problem. We consider which 1-knots (resp. 2-component 1-links) we obtain as above. We give complete answers.

**Theorem 1.3** Let \( f : S^3 \to S^5 \) be a smooth transverse immersion. Then the self-intersection \( C \) consists of double points. Suppose that \( C \) is a single circle in \( S^5 \).

1. Any 2-component 1-link is realizable as \( f^{-1}(C) \) in \( S^3 \) for an immersion \( f \).

2. Any 1-knot is realizable as \( f^{-1}(C) \) in \( S^3 \) for an immersion \( f \).

Remark. Suppose that \( K_1 \) is the trivial 1-knot, \( K_2 \) is the trefoil 1-knot, and \( X_1 \) and \( X_2 \) are 3-knots. Suppose that \( L = (L_1, L_2) \) is a split 1-link such that \( L_1 \) is the trivial 1-knot and \( L_2 \) is the trefoil knot. Then, by Corollary 1.2, a 4-tuple of links \( (K_1, K_2, X_1, X_2) \) is not realizable. But, by Theorem 1.3 (1), the two component split link \( L \) is realizable as in Theorem 1.3 (1).

In [16],[17],[18],[19] the author discussed some topics which are related to this paper. In [19] he discussed the intersection of three 4-spheres in a 6-sphere. In [17] he discussed the intersection of two \((n+2)\)-spheres in an \((n+4)\)-sphere. In [16] he discussed the following: Let \( L = (K_1, K_2) \) be a 2-link in \( S^4 = \partial B^5 \). Take a slice disc \( D^3_1 \) in \( B^5 \) for each component \( K_i \). He discussed the intersection of two slice discs \( D^3_1 \) and \( D^3_2 \) in the 5-ball \( B^5 \). In [18] he applied Theorem 1.1 to Fox’s problem as we state in Note 1.2.1.

**Problem 1.4.** Suppose each of \( S^3_i, S^3_j, \) and \( S^3_k \) is a 3-sphere embedded in \( S^5 \). Suppose \( S^3_i \) and \( S^3_j \) intersect transversely. Suppose each of \( S^3_i \cap S^3_j, S^3_j \cap S^3_k, \) and \( S^3_k \cap S^3_i \) is a single circle. Then we have a triple of 1-links, \( L_i = (S^3_i \cap S^3_j, S^3_j \cap S^3_k, S^3_k \cap S^3_i) \) in \( S^5 \), where \( (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2) \).

Which triple of 1-links do we obtain like this?

Do we characterize such triple by the Arf invariants, the linking numbers, and the Saito-Sato-Levine invariants? (See [20] for the definition of the Saito-Sato-Levine invariant.)

In [19] the author discussed a higher dimensional version of Problem 1.4. This paper is organized as follows. In §2 we review spin cobordism and the Arf invariant. In §3 we discuss a necessary condition for the realization of 4-tuple of links. We find the obstruction for the realization in the spin cobordism group \( \Omega^{spin}_2 \). In §4 we discuss a sufficient condition for the realization of 4-tuple of links. We carry out surgeries of submanifolds to carry out an (un)knotting operation. Theorem 1.1 is deduced from §3 and §4. In §5 we prove Theorem 1.3. In §6 we give a problem.
2 Spin cobordism and the Arf invariant

In this section we review some results on the Arf invariant and spin cobordism. See [5] and [6] for the Arf invariant. See [10] for spin structures and spin cobordism.

We suppose that, when we say $M$ is a spin manifold, $M$ is oriented.

Recall that a proper link is an $m$-component 1-link $L = \{K_1, ..., K_m\}$ such that $\text{lk}(K_j, L - K_j) = \sum_{1 \leq i \leq m, i \neq j} \text{lk}(K_j, K_i)$ is an even number for each $K_j$.

Let $L = (K_1, ..., K_m)$ be an $m$-component 1-link. Let $F$ be a Seifert surface for $L$. We induce a spin structure $\sigma$ on $F$ from the unique one on $S^3$. We induce a spin structure $\sigma_i$ on $K_i$ from $\sigma$ on $F$. Then we have:

**Proposition 2.1** Under the above condition, for each $i$,

$$\text{mod } 2 \text{ } \text{lk}(K_i, L - K_i) = [(K_i, \sigma_i)] \in \Omega^{\text{spin}}_1.$$  

In particular, $L$ is a proper link if and only if each $[(K_i, \sigma_i)] = 0$.

Suppose that $L$ is a proper link. Take $(F, \sigma)$ as above. Let $\hat{F}$ be the closed surface obtained from $F$ by attaching disks to the boundaries. Let $\hat{\sigma}$ be the unique extension of $\sigma$ over $\hat{F}$. Then we have:

**Proposition 2.2** Under the above condition, $\text{Arf}(L) = [(\hat{F}, \hat{\sigma})] \in \Omega^{\text{spin}}_2$.

Although they may be folklore, the author gives a proof of Proposition 2.1 and that of Proposition 2.2 in the appendix.

3 A necessary condition for the realization of 4-tuple of links

In this section we discuss a necessary condition for the realization of a 4-tuple of links. That is, we prove the following two propositions.

**Proposition 3.1** If $(L_1, L_2, X_1, X_2)$ is realizable then

$$\text{lk}(K_{1j}, L_1 - K_{1j}) \equiv \text{lk}(K_{2j}, L_2 - K_{2j}) \text{ } \text{mod } 2 \text{ } \text{for all } j.$$  

In particular, $L_1$ is a proper link if and only if $L_2$ is a proper link.

**Proposition 3.2** Let $L_1$ and $L_2$ be proper links. If $(L_1, L_2, X_1, X_2)$ is realizable then

$$\text{Arf}(L_1) = \text{Arf}(L_2).$$

In order to prove them, we prepare a lemma.

Let $M_1$ and $M_2$ be codimension one submanifolds of an $n$-dimensional compact spin manifold $N$. Suppose that $M_1$ and $M_2$ are compact oriented manifolds. Suppose that $M_1$, $M_2$, and $N$ may have the boundary and the corner. Let $M_1$ and $M_2$ intersect transversely. Suppose $M_i$ may be embedded in the boundary (resp. the corner) of $N$.

We induce a spin structure $\sigma_i$ on $M_i$ from $N$. We induce a spin structure $\xi_i$ on $M_1 \cap M_2$ from $\sigma_i$ on $M_i$. ($i = 1, 2$). Then it is easy to prove:

**Lemma** $\xi_1$ and $\xi_2$ are same.

The spin structure $\xi_1 = \xi_2$ on $M_1 \cap M_2$ is called the unique spin structure induced by $M_1$, $M_2$ and $N$. 

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Proof of Proposition 3.1. Let \( f : S^3_1 \amalg S^3_2 \to S^5 \) be an immersion to realize the 4-tuple of links \((L_1, L_2, X_1, X_2)\). Let \( C = \text{II}C_i \) denote \( f(S^3_1) \cap f(S^3_2) \).

We abbreviate \( f(S^3_i) \) to \( S^3_i \). Let \( V_i \) be a Seifert hypersurface for \( X_i \), i.e.,

\[
S^3_i = \partial V_i \subset V_i \subset S^5.
\]

We make \( V_1 \) and \( V_2 \) intersect transversely. We induce a spin structure \( \nu \) on \( V_i \) from the unique one on \( S^5 \).

Put \( W = V_1 \cap V_2 \). We have:

\[
\begin{array}{ccc}
V_1 & \subset & \subset \phantom{1} S^5 \\
W & \subset & \subset V_2
\end{array}
\]

We give \( W \) the unique spin structure \( w \) induced by \( V_1, V_2 \) and \( W \).

Here, we see that \( \partial W = (\partial V_1 \cap V_2) \cup (V_1 \cap \partial V_2) \). Put \( F_1 = \partial V_1 \cap V_2 \). Put \( F_2 = V_1 \cap \partial V_2 \). Then \( F_1 \) (resp. \( F_2 \)) is a Seifert surface for \( L_1 \) (resp. \( L_2 \)).

We induce a spin structure on \( \partial V_i = S^3_i \) from \( \nu \) on \( V_i \). Note that it is the unique one on \( S^3_i \).

We have:

\[
\begin{array}{ccc}
S^3_1 & \subset & \subset V_1 \\
F_1 & \subset & \subset W
\end{array}
\]

We give \( F_1 \) the unique spin structure \( \sigma_1 \) induced by \( S^3_1, W \) and \( V_1 \).

We have:

\[
\begin{array}{ccc}
W & \subset & \subset V_2 \\
F_2 & \subset & \subset S^3_2
\end{array}
\]
We give $F_2$ the unique spin structure $\sigma_2$ induced by $S_2^3$, $W$ and $V_2$. We have:

\[ F_1 \subset \subset C \subset \subset W \subset \subset F_2 \]

We give $C_j$ the unique spin structure $\tau_j$ induced by $F_1$, $F_2$ and $W$. Then mod2 $\text{lk}(K_{1j}, L_1 - K_{1j}) = \text{mod2 } \text{lk}(K_{2j}, L_2 - K_{2j}) = [(C_j, \tau_j)] \in \Omega^{\text{spin}}_1$ for all $j$. This completes the proof of Proposition 3.1.

We confirm that we have:

\[ S_1^3 \subset \subset F_1 \subset \subset V_1 \subset \subset F_2 \subset \subset S_2^3 \]

\[ C \subset \subset W \subset \subset S_3^5 \]

**Proposition 4.2** Let $X_1$ and $X_2$ be the trivial 3-knots. Let $L_1$ and $L_2$ be 1-links. Suppose that $L_1$ and $L_2$ satisfies one of the conditions (1) and (2) of Theorem 1.1. Then the 4-tuple of links $(L_1, L_2, X_1, X_2)$ is realizable.

We prove
Lemma 4.3  Let $X_1$ and $X_2$ be the trivial 3-knots. Let $L$ be a 1-link. Then the 4-tuple of links $(L, L, X_1, X_2)$ is realizeable.

Proof of Lemma 4.3. Let $f : S^1_1 \coprod S^3_2 \rightarrow S^5$ be an embedding such that $f(S^1_1 \coprod S^3_2)$ in $S^5$ is equivalent to the trivial 3-link. We take a chart $(U, \phi)$ of $S^5$ with the following properties (1) and (2).

1. $\phi : U \cong R^5 = \{(x, y, z, u, v)|x, y, z, u, v \in R\} = R^3 \times R_u \times R_v$.
2. $U \cap f(S^1_1) = \{(x, y, z, u, v)|u = 0, v = 0\} = R^3_1$.

Figure 1

Obviously Lemma 4.3 follows from Lemma 4.4.

Lemma 4.4  There exists an immersion $g : S^1_1 \coprod S^3_2 \rightarrow S^5$ with the following conditions.

1. $g|_{S^1_1} = f|_{S^1_1}$.
2. $g|_{S^3_2}$ is isometric to $f|_{S^3_2}$.
3. $g|_{S^1_1} - g^{-1}(U) = f|_{S^3_2} - g^{-1}(U)$. Hence $g(S^1_1) \cap g(S^3_2) \subset U$.
4. $(g(S^1_1) \cap g(S^3_2))$ and that in $g(S^3_2)$ are both equivalent to the 1-link $L$.

5. $g(S^1_1) \cap \{(x, y, z, u, v)|u = 0, v \in R\} = g(S^3_1) \cap \{(x, y, z, u, v)|u = 0, v = 0\}$.

Proof of Lemma 4.4. We modify the embedding $f$ to construct an immersion $g$.

In $R^3_2$ we take the 1-link $L$ and a Seifert surface $F$ for $L$. Let $N(F)$ = $F \times \{t - 1 \leq t \leq 1\}$ be a tubular neighborhood of $F$ in $R^3_2$.

We define a subset $E$ of $N(F) \times R_u \times R_v = \{(p, t, u, v)|p \in F; -1 \leq t \leq 1, u \in R, v \in R\}$ so that

$E = \{(p, t, u, v)|p \in F, 0 \leq u \leq \frac{\pi}{2}, t = k \cdot \cos u, \ v = k \cdot \sin u, -1 \leq k \leq 1\}$.

Put $P = \partial E$ and $\partial E \cap f(S^1_1)$. Put $Q = f(S^1_1) \cap \partial E \cap f(S^1_1)$. Note that $\partial P = \partial Q = \partial N(F)$.

Put $\Sigma = P \cup Q$. Then, by the construction, $\Sigma$ is a 3-sphere embedded in $S^5$ and is the trivial 3-knot.

Note. In $U$ the following hold.

1. $g(S^1_1) \cap \{(x, y, z, u, v)|u = 0, v \in R\} = g(S^3_1) \cap \{(x, y, z, u, v)|u = 0, v = 0\}$

2. $g(S^1_1) \cap \{(x, y, z, u, v)|0 < u \leq \frac{\pi}{2}, v \in R\}$ is $\text{Int } N(F)$.

3. Let $0 < u' < \frac{\pi}{2}$.

4. Let $0 < u' < \frac{\pi}{2}$.

5. Let $0 < u' < \frac{\pi}{2}$.

6. Let $0 < u' < \frac{\pi}{2}$.

7. $g(S^3_1) \cap \{(x, y, z, u, v)|u = u', v \in R\}$ is diffeomorphic to $\text{Int } F$.

8. $g(S^1_1) \cap \{(x, y, z, u, v)|u = u', v = 0\}$ is diffeomorphic to $\text{Int } F$.

9. $g(S^3_1) \cap \{(x, y, z, u, v)|u = u', v = 0\}$ is diffeomorphic to $\partial(N(F))$.

Let $F_i$ be diffeomorphic to $F$ $(i = 1, 2)$. Recall $F$ is a compact oriented surface with boundary. We identify $\partial F_1$ with $\partial F_2$ to obtain $F_0 = F_1 \cup F_2$. Note $\partial F_0$ is diffeomorphic to $\partial(N(F))$.

10. $g(S^1_1) \cap \{(x, y, z, u, v)|u = \frac{\pi}{2}, v \in R\}$ is diffeomorphic to $N(F)$.

11. $g(S^1_1) \cap \{(x, y, z, u, v)|u = \frac{\pi}{2}, v = 0\}$ is diffeomorphic to $F$. 

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(9) $g(S^3_1) \cap \{(x, y, z, u, v)|0 < u \leq \frac{\pi}{2}, v = 0\}$ is diffeomorphic to Int $F$.

In Figure 2, 3, 4, $\Sigma \cap U$ and $g(S^3_2) \cap U$ are drawn. There, we replace $R^3 \times R_u \times R_v$ with $R^2 \times R_u \times R_v$.

By the construction, $\Sigma \cap g(S^3_1)$ in $\Sigma$ and that in $g(S^3_2)$ are both equivalent to $L$. Define $g|_{S^3_1}$ so that $g(S^3_1) = \Sigma$.

This completes the proof of Lemma 4.4 and therefore Lemma 4.3.

Note. Lemma 4.3 gives an alternative proof of the results of [4] and Lemma 1 of [15].

In order to prove Proposition 4.2, we review the pass-moves. See [5] and [6] for detail.

Definition (See [5][6].) Two 1-links are pass-move equivalent if one is obtained from the other by a sequence of pass-moves. See Figure 5 for an illustration of the pass-move.

In $B^3_3$ there are four arcs. Each of four arcs may belong to different components of the 1-link. We do not assume the four arcs belong to one component of the 1-link.

The following propositions are essentially proved in [5]. A proof is written in the appendix of [14].

**Proposition 4.5** (See [5][14].) Let $L_1$ and $L_2$ be 1-links. Then $L_1$ and $L_2$ are pass-move equivalent if and only if $L_1$ and $L_2$ satisfy one of the conditions (1) and (2) of Theorem 1.1.

**Proposition 4.6** (See [5][14].) Let $L_1$ and $L_2$ be 1-links. Suppose that $L_1$ is pass-move equivalent to $L_2$. Let $F$ be an oriented Seifert surface for $L_2$ such that the genus of $F$ is not less than the genus of $L_1$. Then there exists a disjoint union of 3-balls $B^3$ such that $B^3 \cap F$ is as in Figure 6 and the pass-moves in all $B^3$ change $L_2$ to $L_1$.

Let $L_1$ and $L_2$ be 1-links. Suppose $L_1$ and $L_2$ satisfy one of the conditions (1) and (2) of Theorem 1.1. Then by Proposition 4.5 $L_1$ is obtained from $L_2$ by a sequence of pass-moves. We choose a Seifert surface $F$ for $L_2$ and the disjoint union of 3-balls $B^3$ in $S^3$ as in Proposition 4.6.

Take a 1-link $(Y_1, Y_2)$ in each $B^3$ as in Figure 7. By considering the Kirby moves of framed links (in [10]), it is easy to prove:

**Lemma 4.7** We carry out 0-framed surgeries along all $Y_1$ and all $Y_2$. After these surgeries, we have the following. (1) $S^3$ becomes a 3-sphere again. (2)
each $B^3$ becomes a 3-ball again, (3) $L_2$ in the old sphere $S^3$ changes to $L_1$ in the new 3-sphere.

We go back to the proof of Proposition 4.2.

As in Lemma 4.4, take an immersion $g : S^3_1 \coprod S^3_2 \to S^5$ to realize $(L_2, L_2, X_1, X_2)$, where $X_1$ is the trivial 3-knot.

Obviously Proposition 4.2 follows from Lemma 4.8.

**Lemma 4.8** There exists an immersion $h : S^3_1 \coprod S^3_2 \to S^5$ with the following conditions.

1. $h|_{S^3_2} = g|_{S^3_2}$.
2. $h(S^3_1)$ in $S^5$ is equivalent to the trivial 3-knot.
3. $h|_{S^3_1-g^{-1}(U)} = g|_{S^3_1-g^{-1}(U)}$. Hence $h(S^3_1) \cap h(S^3_2) \subset U$.
4. $h(S^3_1) \cap h(S^3_2)$ in $h(S^3_1)$ is equivalent to $L_1$. $h(S^3_1) \cap h(S^3_2)$ in $h(S^3_2)$ is equivalent to $L_2$.
5. $g(S^3_1) \cap \{ u \neq 0, v \in R \} = h(S^3_1) \cap \{ u \neq 0, v \in R \}$

**Proof of Lemma 4.8.** We modify the immersion $g$ to define an immersion $h$ as follows.

As in the proof of Lemma 4.4, we take $L_2$ and a Seifert surface $F$ for $L_2$ in $g(S^3_1)$. Suppose $L_3 \subset R^3_1$ and $F \subset U$. Suppose the genus of $F$ satisfies the condition in Proposition 4.6. Take 3-balls $B^3$ in $g(S^3_1)$ as in Proposition 4.6.

Take $(Y_1, Y_2)$ in $B^3$ as in Lemma 4.7. See Int $N(F)$ in $g(S^3_1)$. Int $N(F)$ is not in $R^3_1$ and is in $\{ u > 0 \}$. But by Proposition 4.6, we can suppose $Y_1$ and $Y_2$ are in $R^3_1$.

Take an embedded 2-disc $h_1^2$ in $\{(x, y, z, u, v) \mid u = 0, v \geq 0 \}$ so that $h_1^2$ meets $\{(x, y, z, u, v) \mid u = 0, v = 0 \}$ at $Y_1$ transversely. Suppose that $h_1^2$ is embedded trivially.

Take an embedded 2-disc $h_2^2$ in $\{(x, y, z, u, v) \mid u = 0, v \leq 0 \}$ so that $h_2^2$ meets $\{(x, y, z, u, v) \mid u = 0, v = 0 \}$ at $Y_2$ transversely. Suppose that $h_2^2$ is embedded trivially.

See Figure 8.

**Figure 8**

Let $h_1^2$ be a tubular neighborhood of $h_1^2$ in $\{(x, y, z, u, v) \mid u = 0, v \geq 0 \}$.
Let $h_2^2$ be a tubular neighborhood of $h_2^2$ in $\{(x, y, z, u, v) \mid u = 0, v \leq 0 \}$.

Then $h_1^2$ and $h_2^2$ are diffeomorphic to the 4-ball. Furthermore we can regard $h_1^2$ as a 4-dimensional 2-handle attached to $g(S^3_1)$ along $Y_1$ with 0-framing.

Put $R = g(S^3_1) - \{ g(S^3_1) \cap \partial h_1^2 \} - \{ g(S^3_1) \cap \partial h_2^2 \}$. Put $S_1 = \partial h_1^2 - \{ g(S^3_1) \cap \partial h_1^2 \}$. Put $S_2 = \partial h_2^2 - \{ g(S^3_1) \cap \partial h_2^2 \}$. Here, we consider all $h_i^2$. Put $\Lambda = R \cup S_1 \cup S_2$.

Then we can regard $\Lambda$ as the result of 0-framed surgeries on $g(S^3_1)$ along all $Y_1$ and $Y_2$. Then the pass-moves are carried out in all $B^3$.

Therefore $\Lambda$ is an embedded 3-sphere in $S^5$. Furthermore $\Lambda \cap g(S^3_2)$ in $\Lambda$ is equal to the 1-link $L_1$ and $\Lambda \cap g(S^3_2)$ in $g(S^3_2)$ is equal to the 1-link $L_2$.

Put $h|_{S^3_2}$ so that $h(S^3_1) = \Lambda$. By the above construction, $h$ satisfies the conditions (1)(3)(4)(5) in Lemma 4.8.

We prove:

**Lemma 4.9** $h|_{S^3_1}$ is equal to the trivial 3-knot.

**Proof.** By the construction, $h(S^3_1)$ bounds a 4-manifold represented by the disjoint union of some copies of the framed link in Figure 9.

**Figure 9**
See [10] for dot-circles. Hence this 4-manifold is diffeomorphic to the 4-ball. Therefore \( h(S^3) \) bounds a 4-ball. This completes the proof of Lemma 4.9 and therefore Lemma 4.8. This completes the proof of Proposition 4.2 and therefore Proposition 4.1.

5 The proof of Theorem 1.3

**Lemma 5.1.1** Let \( L \) be the trivial 2-component 1-link. There exists a self-transverse immersion \( f : S^3 \rightarrow B^5 \) with the following properties.

1. The singular point set (in \( B^5 \)) is a single circle \( C \).
2. \( f^{-1}(C) \) in \( S^3 \) is equivalent to \( L \).
3. \( f(S^3 - N(L)) \subseteq \partial B^5 \) and \( f(\text{Int}(N(L))) \subseteq \text{Int} B^5 \), where \( N(L) \) is a tubular neighborhood of \( L \) in \( S^3 \).

**Lemma 5.1.2** Let \( L \) be the Hopf link. There exists a self-transverse immersion \( f : S^3 \rightarrow B^5 \) satisfying with the properties (1), (2) and (3) in Lemma 5.1.1.

**Lemma 5.1.3** Let \( L \) be the trivial 1-knot. There exists a self-transverse immersion \( f : S^3 \rightarrow B^5 \) satisfying with the properties (1), (2) and (3) in Lemma 5.1.1.

**Proof of Lemma 5.1.1** Take a chart of \((U, \phi)\) of \( B^5 \) such that \( \phi(U) = \{(x, y, z, v, t) \mid x, y, z, v \in \mathbb{R}, t \leq 0\} \).

Put \( F = \{(x, y, z, v, t) \mid x, y \in \mathbb{R}, z \geq 0, v = 0, t \leq 0\} \).

Put \( A = \{(x, y, z, v, t) \mid x, y \in \mathbb{R}, z = 0, v = 0, t \leq 0\} \).

We can regard \( U \) as the result of rotating \( F \) around the axis \( A \).

Take an immersed 2-disc \( D \) in \( F \) as in Figure 10, 11.

### Note

1. \( \{(x, y, z, v, t) \mid x, y \in \mathbb{R}, z \geq 0, v = 0, t = 0\} \cap D \) is diffeomorphic to \( D - \text{(two 2-discs)} \).
2. \( \{(x, y, z, v, t) \mid x, y \in \mathbb{R}, z \geq 0, v = 0, t < 0\} \cap D \) is a union of the interior of two 2-discs, where the intersection of the two 2-discs is one point. The point is \( p = (1, 0, 0, 0, -1) \).
3. Let \(-1 < t' < 0\).
   \( \{(x, y, z, v, t) \mid x, y \in \mathbb{R}, z \geq 0, v = 0, t = t'\} \cap D \) is the Hopf link in
   \( \{(x, y, z, v, t) \mid x, y \in \mathbb{R}, z \geq 0, v = 0, t = t'\} \).
4. \( \{(x, y, z, v, t) \mid x, y \in \mathbb{R}, z \geq 0, v = 0, t = -1\} \cap D \) is a union of two circles, where the intersection of the two circles is one point. The point is \( p = (1, 0, 0, 0, -1) \).
5. Let \(-2 < t' < -1\).
   \( \{(x, y, z, v, t) \mid x, y \in \mathbb{R}, z \geq 0, v = 0, t = t'\} \cap D \) is the trivial link in
   \( \{(x, y, z, v, t) \mid x, y \in \mathbb{R}, z \geq 0, v = 0, t = t'\} \).
6. \( \{(x, y, z, v, t) \mid x, y \in \mathbb{R}, z \geq 0, v = 0, t = -2\} \cap D \)
is a disjoint union of two 2-discs.

As we rotate $F$ as above, we rotate $D$ as well. We obtain an immersed 3-sphere $X$.

Take $f$ so that $f(S^3)=X$.

This completes the proof of Lemma 5.1.1.

**Proof of Lemma 5.1.2** Take $B^5$, $F$, $A$, and $D$ as above.

Put $G=\{(x, y, z, v, t) | x = 0, y = 0, z \geq 0, v = 0, t \leq 0\}$.

In Figure 10, 11 we suppose: If, in $F$, we rotate $D$ around $G$ by any angle, then $D \cap A$ does not change.

As we rotate $F$ as above, we rotate $D$ around $A$ so that in $F$ we rotate $D$ around $G$ one time. We obtain an immersed 3-sphere $X$.

Take $f$ so that $f(S^3)=X$.

This completes the proof of Lemma 5.1.2.

**Proof of Lemma 5.1.3** Take $B^5$, $F$, $A$, $D$, and $G$ as above.

In Figure 10, 11 we suppose: If we rotate $D$ around $G$ half time, then the resultant $D$ coincides with the original $D$.

As we rotate $F$ as above, we rotate $D$ around $A$ so that in $F$ we rotate $D$ around $G$ half time. We obtain an immersed 3-sphere $X$.

Take $f$ so that $f(S^3)=X$.

This completes the proof of the proof of Lemma 5.1.3.

**Lemma 5.2.1** Let $L$ be a 2-component 1-link whose linking number is even.

There exists a self-transverse immersion $g : S^3 \rightarrow \sharp lS^2 \times B^3$, for a non-negative integer $l$, with the following properties.

1. The singular point set (in $\sharp lS^2 \times B^3$) is a single circle $C$.
2. $g^{-1}(C)$ in $S^3$ is equivalent to $L$.
3. $g(S^3 - N(L)) \subset \partial(\sharp lS^2 \times B^3) = \sharp lS^2 \times S^2$ and $g(\text{Int}(N(L))) \subset \text{Int}(\sharp lS^2 \times B^3)$, where $N(L)$ is a tubular neighborhood of $L$ in $S^3$.

Here, we suppose that $\sharp 0S^2 \times B^3$ means $B^5$.

**Lemma 5.2.2** Let $L$ be a 2-component 1-link whose linking number is odd.

There exists a self-transverse immersion $g : S^3 \rightarrow \sharp lS^2 \times B^3$, for a non-negative integer $l$, satisfying with the properties (1), (2) and (3) in Lemma 5.2.1.

**Lemma 5.2.3** Let $L$ be a 1-knot. There exists a self-transverse immersion $g : S^3 \rightarrow \sharp lS^2 \times B^3$, for a non-negative integer $l$, satisfying with the properties (1), (2) and (3) in Lemma 5.2.1.

We prove:

**Claim 5.2.4** Lemma 5.2.1, 5.2.2, and 5.2.3 imply Theorem 1.3.

There is an embedding $h : \sharp lS^2 \times B^3 \hookrightarrow S^5$. Then $h \circ g$ is an immersion in Theorem 1.3. This completes the proof.

In order to prove Lemma 5.2.1, 5.2.2, and 5.2.3, we review the $\sharp$-moves. See [14] and [7] for detail.

**Definition** ([14]) Two 1-links are $\sharp$-move equivalent if one is obtained from the other by a sequence of $\sharp$-moves. See Figure 12 for an illustrations of the $\sharp$-move. The $\sharp$-move is different from the pass-move by the orientation.

In each 3-ball in Figure 12 there are four arcs. Each of four arcs may belong to different components of the 1-link. We do not assume the four arcs belong to one component of the 1-link.
The following proposition 5.3 is proved in the appendix of [14].

Proposition 5.3 ([14]) Let $L$ be a 2-component link. Then $L$ is $\sharp$-move equivalent to the trivial link if and only if the linking number is even.

(2) Let $L$ be a 2-component link. Then $L$ is $\sharp$-move equivalent to the Hopf link if and only if the linking number is odd.

(3) Any 1-knot is $\sharp$-move equivalent to the trivial knot.

We have:

Proposition 5.4 ([14]) Let $L_1$ and $L_2$ be 1-links. Suppose that $L_1$ is $\sharp$-move equivalent to $L_2$. Take $L_1$ in $S^3$. Then there exists a disjoint union of 3-balls $B^3$ in $S^3$ such that $L_1 \cap B^3$ is as in Figure 12 and that the $\sharp$-moves in all $B^3$ change $L_1$ to $L_2$.

Let $L_1$ and $L_2$ be 1-links. Suppose $L_1$ is $\sharp$-move equivalent to $L_2$. Take $L_1$ in $S^3$. Take a disjoint union of 3-balls $B^3$ in $S^3$ as in Proposition 5.3. Take a 1-link $(Y_1, Y_2)$ in each $B^3$ as in Figure 13. Suppose that $Y_i$ bounds a 2-disc $B^2_i$ as in Figure 13.

By considering the Kirby moves of framed links (in [10]), it is easy to prove:

Lemma 5.5 We carry out 0-framed surgeries on all $Y_1$ and all $Y_2$. After these surgeries, we have the following. (1) each $B^3$ becomes a 3-ball again, (2) $S^3$ becomes a 3-sphere again. (3) $L_1$ in the old sphere $S^3$ changes to $L_2$ in the new 3-sphere.

We go back to the proof of Lemma 5.2.1, 5.2.2, 5.2.3.

Proof of Lemma 5.2.1 (resp. 5.2.2, 5.2.3)

By Proposition 5.3, a sequence of $\sharp$-moves changes $L$ into the trivial 2-component 1-link (resp. the Hopf link, the trivial 1-knot).

As in Lemma 5.1.1 (resp. 5.1.2, 5.1.3), take $f : S^3 \rightarrow B^5$. See the 1-link $f^{-1}(C)$ in $S^3$. It is equivalent to the 1-link $L$. As in Proposition 5.4, take 3-balls $B^3$. As in Lemma 5.5, take $Y_1$ and $Y_2$ in $B^3$. Suppose that $f(Y_1)$ and $f(Y_2)$ are in $\partial B^5$.

Attach 4-dimensional 2-handles $h^2$ to $f(S^3 - N(L))$ in $\partial B^5$ along $f(Y_i)$ with 0-framing. Here, 0-framing means the following. When attaching the 4-dimensional 2-handles $h^2$ to $f(S^3 - N(L))$, we can attach 4-dimensional 2-handles to $S^3$ naturally. These attaching maps are 0-framing.

When attaching the 4-dimensional 2-handles $h^2$ to $f(S^3 - N(L))$, we can attach 5-dimensional 2-handles $h^2 \times [-1, 1]$ to $B^5$ along $f(Y_i)$ naturally. Of course the attached parts are in $\partial B^5$. We obtained a 5-manifold. Call it $M$.

Put $P = f(S^3) - (f(S^3) \cap \partial h^2)$. Put $Q = \partial h^2 - (f(S^3) \cap \partial h^2)$. Put $\Sigma = P \cup Q$. Then $\Sigma$ is an immersed 3-sphere in $M$.

We prove Lemma 5.6. Before the proof of Lemma 5.6, we prove Lemma 5.7.

Lemma 5.6 Let $M$ be as above. $M = \sharp [S^2 \times B^3]$, for a non-negative integer $l$.

Lemma 5.7 Lemma 5.6 implies Lemma 5.2.1, 5.2.2 and 5.2.3.

Proof of Lemma 5.7. Take a self-transverse immersion $g : S^3 \rightarrow M = \sharp [S^2 \times B^3]$ so that $g(S^3) = \Sigma$.

Proof of Lemma 5.6. Recall: $J = (one ~ 5$-dimensional 0-handle)$\cup$(one 5-dimensional 2-handle) is $S^2 \times B^3$ or $S^2 \times B^3$. $J$ is $S^2 \times B^3$ if and only if the attaching map of the 2-handle is spin preserving diffeomorphism map.
It suffices to prove that the attaching maps of the 5-dimensional 2-handles are spin-preserving diffeomorphism maps.

We give a spin structure on \( f(S^3 - N(L)) \) from the unique one on \( \partial B^5 \).

We give a spin structure on \( f(B_i^2) \cap \partial B^5 \) from the spin structure on \( f(S^3 - N(L)) \).

We give a spin structure on \( f(B_i^2) \cap \partial B^5 \) from the spin structure on \( f(S^3 - N(L)) \).

We give a spin structure \( \xi \) on \( f(Y_i) \) from the spin structure on \( f(B_i^2) \cap \partial B^5 \).

Put \( S^1_a \sqcup S^1_b = f(B_i^2) \cap \partial N(L) \). We give a spin structure \( \alpha \) (resp. \( \beta \)) on \( S^1_a \) (resp. \( S^1_b \)) from the spin structure on \( f(B_i^2) \cap \partial B^5 \). By the construction, \( \alpha \) and \( \beta \) are the \( S^1_{Lie} \) spin structure.

Since \( (f(Y_i), \xi) \) is spin cobordant to

\( (S^1_a, \text{the } S^1_{Lie} \text{ spin structure}) \sqcup (S^1_b, \text{the } S^1_{Lie} \text{ spin structure}) \),

\( \xi \) is the \( S^1_{bd} \) spin structure.

See [10] for the \( S^1_{bd} \) spin structure and the \( S^1_{Lie} \) spin structure.

This completes the proof of Lemma 5.6.

This completes the proof of Lemma 5.2.1 (resp. 5.2.2, 5.2.3). By Claim 5.2.4, Theorem 1.3 holds.
Appendix. The proof of Proposition 2.1 and 2.2

Firstly we prove Proposition 2.1.

Let \( L = (K_1, ..., K_m) \) be a 1-link. Let \( F \) be a Seifert surface. Let \( K_i \times [0,1] \subset F \) be a collar neighborhood of \( K_i \) in \( F \). Let \( K_i \times \{0\} = K_i \). Then \( \text{lk}(K_i, L - K_i) = \text{lk}(K_i \times \{0\}, K_i \times \{1\}) \).

See [10] for spin structures.

Let \( \varepsilon^2 \) be the trivial bundle over \( S^1 \). Let \( (e_p^1, e_p^2) \) (for each \( p \in S^1 \)) denote the trivialization. Let \( e_p^0 \) (for each \( p \in S^1 \)) denote the trivialization on \( TS^1 \).

Spin structures on \( S^1 \) are defined by spin structures on \( TS^1 \oplus \varepsilon^2 \). [10] defines that the spin structure defined by \( (e_p^0, e_p^1, e_p^2) \) is the \( S^1 \) spin structure. The other is the \( S^1_{id} \) spin structure.

Let \( (M, \sigma) \) and \( (N, \tau) \) be spin manifolds. Let \( f : M \to N \) be an orientation preserving diffeomorphism. \( f \) is called a spin preserving diffeomorphism if \( df \oplus id : TM \oplus \varepsilon^p \to TN \oplus \varepsilon^p \) carries \( \sigma \) to \( \tau \), \( id \) carries the trivialization to the trivialization.

We give \( F \) a spin structure \( \alpha \) induced from the unique spin structure \( \beta \) on \( S^3 \). We give \( K_i \) a spin structure \( \gamma \) induced from \( \alpha \). Let \( K_i \times D^2 \) be the tubular neighborhood of \( K_i \) in \( S^3 \). Regard \( K_i \times D^2 \) as the product \( D^2 \) bundle over \( K_i \).

We give \( K_i \times D^2 \) a trivialization such that \( e_p^1 \) is in \( K_i \times [0,1] \) and that \( e_p^2 \) is perpendicular to \( K_i \times [0,1] \) for each \( p \in K_i \). Then we can regard \( \alpha \) as a spin structure on the trivialized bundle \( TK_i \oplus (K_i \times [0,1]) \).

Let \( G \) be any Seifert surface of \( K_i \). Let \( g \) be a spin structure on \( G \) induced from the unique one \( S^3 \). Let \( \mu \) be a spin structure on \( K_i \) induced from \( (G, g) \).

Since \( \partial(G, g) = (K_i, \mu), [(K_i, \mu)] = 0 \). We give \( K_i \times D^2 \) a trivialization such that \( g_p^1 \) is in \( G \) and \( g_p^2 \) is perpendicular to \( G \). Then the spin structure on \( K_i \), induced by the framing \( (e_p^0, g_p^1, g_p^2) \) is \( \mu \).

The homotopy class of the framing \( (e_p^0, e_p^1, e_p^2) \) coincides with the homotopy class of the framing \( (e_p^0, g_p^1, g_p^2) \) if and only if \( \text{lk}(K_i \times \{0\}, K_i \times \{1\}) \) is even.

Therefore Proposition 2.1 holds.

Next we prove Proposition 2.2.

Under the above condition, suppose \( L \) is a proper 1-link.

Let \( \hat{F} \) be the closed oriented surface \( F \cup (\cup_{i=1}^m D^2) \). Since \( \gamma \) is the \( S^3_{ike} \) spin structure, \( \alpha \) extends to \( \hat{F} \), say \( \hat{\alpha} \). Let \( a_1, ..., a_g, b_1, ..., b_g \) be circles in \( F \) representing symplectic basis of \( H_1(\hat{F}; Z) \). Let \( a_i \times [-1,1], b_i \times [-1,1] \) be the tubular neighborhood in \( F \). Put mod 2 \( \text{lk} (a_i \times \{-1\}, a_i \times \{1\}) = x_i \) and mod 2 \( \text{lk} (b_i \times \{-1\}, b_i \times \{1\}) = y_i \). Recall \( \text{Arf} L = \sum x_i \cdot y_i \).

We give a spin structure \( \sigma_i \) (resp. \( \tau_i \)) on \( a_i \) (resp. \( b_i \)). Then \( [(a_i, \sigma_i)] \in \Omega_1^{spin} = x_i \) and \( [(b_i, \tau_i)] \in \Omega_1^{spin} = y_i \). By P.36 of [10], \( [(\hat{F}, \hat{\alpha})] \in \Omega_2^{spin} \) is \( \Sigma x_i \cdot y_i \).

Hence Proposition 2.2 holds.
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Figure 1
\[ u = 0 \]

\[ g(S^1) \]

\[ 0 < u < 1 \]
$g(S^2_2)$
$v = \pi$

$1 < u < \frac{\pi}{2}$

Figure 4
Figure 5

pass-move
Figure 6

The shaded part is $F \cap B^3$. 
Figure 8
\[ t = 0 \]

\[ v = 0 \]

\[ -1 < t < 0 \]

Figure 10
Figure 11
Figure 12
Figure 13