Sequential Posted Pricing
and Multi-parameter Mechanism Design

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Abstract

We consider the classical mathematical economics problem of Bayesian optimal mechanism design where a principal aims to optimize a given objective when allocating resources to self-interested agents. In single-parameter settings (where each agent preference is given by a private value for being allocated the resource and zero for not being allocated) this problem is solved [19]. While this economic solution is tractable whenever the non-economic optimization problem is tractable, it is complicated enough that it is rarely employed. Moreover, the techniques do not seem to generalize to multi-parameter settings. Our first result is that for general product distributions on agent preferences and resource allocation problems that satisfy matroid properties (e.g., multi-unit auctions, matchings, spanning trees), sequential posted price mechanisms, where agents are approached in-turn and offered a pre-computed take-it-or-leave-it offer, are at most a 4-approximation to the optimal single-round mechanism. Furthermore, a suitable sequence of prices can be effectively computed by sampling the agents’ distributional preferences. Notably, the analysis of this sequential posted price mechanism can be extended to give approximation mechanisms for the unsolved multi-parameter setting. In stark contrast to the single-parameter setting, in multi-parameter settings there is no general description or tractable implementation of optimal mechanisms. For decades, this unanswered issue has been widely considered one of the most important in the economic theory on mechanism design. We focus on the unit-demand special case, e.g., houses, where each agent has a different value for each resource or service but desires at most one. Our second result is that for general product distributions on unit-demand preferences and matroid problems, an easy-to-compute mechanism inspired by the class of Vickrey-Clarke-Groves (VCG) mechanisms is an 8-approximation to the optimal (deterministic) mechanism. Our exposition focuses on the objective of profit maximization, however, the approaches generalize to other objectives that are linear in valuations and payments.

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1 Introduction

Why do sellers prefer simple posted-pricing to more complex auction mechanisms? For example, a large fraction of eBay’s transactions are from “buy it now” (posted price) sales and not auctions [16]. Indeed, the auction that economic theory predicts a seller would prefer, i.e., the revenue-optimal auction, is rarely used. We propose two answers, (a) because despite its simplicity posted-pricing is approximately optimal in very general settings of single-dimensional preferences, and, (b) because it generalizes naturally to (more realistic) multi-dimensional preferences.

Consider the following example. The local organizers for a prominent symposium on discrete mathematics need to find suitable hotel accommodations in the Austin area for the attendees of the conference. There are a number of hotel rooms available and attendees have preferences over the rooms. The organizers need a mechanism for soliciting preferences, assigning rooms, and calculating payments. Fortunately, they have distributional knowledge over the participants’ preferences (e.g., from past conferences). How can the organizers do this to maximize their objective (revenue, of course)?

Our analysis begins with the economically well understood setting of single-dimensional preferences. In the single-dimensional version of the hotel room example, each attendee is assumed to consider a particular (known) subset of hotel rooms to be satisfactory and has a single private value for obtaining any of these rooms. This is naturally modeled as a bipartite graph with hotel rooms on the right, attendees on the left, and an edge between a room and an attendee if the room is satisfactory. An “allocation” of rooms to attendees is a matching in this graph. We will ignore the actual matching (since attendees are assumed to be indifferent between each satisfactory room) and focus on the sets of attendees that can be simultaneously served, a.k.a., the feasible sets. Such a set system is known as the transversal matroid.

Economic theory gives a succinct, albeit sometimes difficult to interpret, description of optimal mechanisms for general single-dimensional settings: apply a distribution dependent transformation to the agents’ values to get virtual values and then maximize the sum of the virtual values of the agents served, a.k.a., the virtual surplus. When the values of the agents are distributed i. i. d. and the set system forms a matroid, this optimal mechanism is the Vickrey-Clarke-Groves (VCG) mechanism with a uniform reservation price [19]. When the agents’ values are not identically distributed or the set system is non-matroid, the optimal mechanism is not as easily interpreted; however, VCG with non-uniform reservation prices is nearly optimal for many settings [15]. While the VCG mechanism has many nice theoretical properties, it is seldom used in practice. The essay “The Lovely but Lonely Vickrey Auction” by Ausubel and Milgrom [1] discusses why this is the case — among other reasons they cite VCG’s vulnerability to collusion, and the non-monotonicity of the seller’s revenue in the set of bidders and the amounts bid.

Just as eBay is switching from auctions to “buy it now”, we consider replacing the optimal mechanism with a sequential posted price mechanism. Intuitively sequential posted pricing looks at each agent in turn and makes a “take-it-or-leave-it while-supplies-last” offer. If the agent can be served feasibly alongside the other agents who have previously accepted their offers, then the agent is offered service at a pre-computed price. The only strategic action available to the agent is to accept or reject the offer, each with the natural consequence. Sequential posted pricing has a number of advantages over the VCG mechanism. Accepting or rejecting an offer is a simpler task for agents than bidding in an auction. Agents learn immediately whether they will be served or not. From the seller’s point of view, posted pricings are simpler to implement, satisfy revenue monotonicity, as well as strong notions of collusion resistance, e.g., group strategyproofness [12]. A natural question is how much the seller loses in switching to this simpler and collusion resistant class of mechanisms. We show that in many domains the answer is “not much”.

Our first main result is that for single-dimensional agent preferences distributed according to any product distribution and any matroid set system, sequential posted pricing gives a 4-approximation to the optimal mechanism. We believe this approximate optimality is one of the main justifications for the widespread use of these mechanisms in practice. Matroid properties are important for sequential posted pricing and we show the existence of downward-closed set systems where no such constant approximation is possible.

Single-dimensional preferences are rarely realistic and have limited applicability in practice. Our hotel example could perhaps have been better modeled by multi-dimensional agent preferences, i.e., agents have different values for different hotel rooms but desire at most one. Unfortunately, despite much effort in this direction economic theory has had very little to say in describing optimal mechanisms for these more general multi-dimensional mechanism design problems (see, for example, [18] [20] [21]). Many of the techniques and insights of the single-dimensional case (e.g. virtual values, and characterizing truthfulness via monotonicity) fail to carry over to the multi-dimensional setting, or apply only weakly. Indeed no concise descriptions are known for optimal mechanisms in multidimensional
settings [21]. Our second main result is the first general approximation to revenue in any multi-dimensional Bayesian mechanism design setting. We show that VCG with easy-to-compute reservation prices is a 8-approximation to the Bayesian optimal mechanism in the matching setting.

Beyond the multi-dimensional setting given by matchings, we consider the following abstract setting. Each agent is interested in any of a set of services. Without loss of generality, we assume that the subsets of services each agent is interested in partition the set of all offered services. We consider downward-closed feasibility constraints over the sets of services that can be simultaneously provided. Our third main result is that for matroid set systems, a VCG-type mechanism with easy-to-compute reservation prices gives an 8-approximation to optimal. Although this mechanism is not truthful, the revenue guarantee holds in every Nash equilibrium of the mechanism. Furthermore, this result follows from the techniques developed in our consideration of sequential posted pricing.

Other contributions. In single-dimensional settings we also study the performance of posted price mechanisms when agents arrive in an adversarial order. We show that in matroid settings with adversarial ordering posted prices are $O(\log k)$ approximate, where $k$ is the rank of the matroid. While our exposition focuses on revenue maximization, all of our techniques and results apply equally well to any one in a large class of objectives — those that are linear in agents’ valuations and payments. This class contains, for example, social welfare, and residual surplus with money-burning [14].

Related work. As mentioned earlier, ours is the first work to provide a general purpose approximation to revenue in multi-dimensional, Bayesian settings. See [21] and references therein for work in economics literature on optimal multi-dimensional mechanism design.

In single dimensional settings, Hartline and Roughgarden [15] recently considered the issue of whether nearly optimal revenue can be obtained through simple mechanisms. They focused on the class of VCG mechanisms with reserve prices and showed that in a number of settings—value distributions satisfying the so-called monotone hazard rate condition, and regular value distributions with a matroid set system—the gap is a small constant. Our work is similar in spirit to theirs, and as a simple consequence of the near-optimality of sequential posted prices, we answer one of their open questions in the positive, namely, that the gap between the revenue optimal mechanism and a VCG mechanism with appropriate reserve prices is a constant (4) in a matroid setting but with arbitrary valuation distributions.

Sequential posted price mechanisms have been studied previously but in the simple setting of a single-item auction with i. i. d. agents. Blumrosen and Holenstein [6] show how to compute the optimal posted prices in this setting, and also that the revenue of these mechanisms approaches the optimal revenue asymptotically. Several papers study revenue maximization through online posted pricings in the context of adversarial values, albeit in the simpler context of multi-unit auctions [5, 17, 4].

Lately, the topic of envy-free pricing has received much attention in the algorithmic game theory community (see, for example, [13, 3, 7]). Many of these pricing problems can be viewed as Bayesian mechanism design problems in the special case where there is only a single agent (or equivalently, unlimited supply). In this context, the multi-dimensional results in this paper are an extension of the unit-demand pricing design and analysis techniques of [9] to the case where multiple agents are vying for limited resources.

Finally, our setting of an online mechanism with a matroid constraint is very closely related to the so-called matroid secretary problem studied by Babaioff et al. [2] Their model differs from ours in two important ways—(a) they assume that agents’ values are adversarial, whereas in our setting they are drawn from known distributions, and, (b) in their setting agents arrive in random order, whereas we consider two variants on the ordering—determined by the mechanism, or adversarial. In the adversarial ordering setting, we obtain the same approximation of $O(\log k)$ as Babaioff et al. do in the adversarial values but random order setting. (Here $k$ is the rank of the matroid.) However, this is coincidental. We show that their approach performs poorly in our setting.

2 Problem set-up and preliminaries

2.1 Bayesian optimal single-parameter mechanism design

We consider the following Bayesian single-parameter mechanism design problem. There are $n$ single-parameter agents and a single seller providing a certain service. Agent $i$’s value $v_i$ for getting served is distributed independently
according to distribution function $F_i$ with density $f_i$. The seller faces a feasibility constraint specified by a set system $\mathcal{J} \subseteq 2^{[n]}$, and is allowed to serve any set of agents in $\mathcal{J}$. We assume that the set system $\mathcal{J}$ is downward closed. That is, for any $A \subseteq B \subseteq 2^{[n]}$, $B \in \mathcal{J}$ implies $A \in \mathcal{J}$.

**Myerson’s optimal mechanism.** Myerson’s seminal work describes the revenue maximizing mechanism for the Bayesian single-parameter mechanism design problem. In Myerson’s mechanism the seller first computes so-called virtual values for each agent, and then allocates to a feasible subset of agents that maximizes the “virtual surplus”—the sum of the virtual values of agents in the set minus the cost of serving that set of agents. These quantities are formally defined as follows.

**Definition 1** For a valuation $v_i$ drawn from $F_i$, the virtual valuation of agent $i$ is given by

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

The virtual surplus of a set $S$ of agents is defined as $\Phi(S, v) = \sum_{i \in S} \phi_i(v_i)$.

Myerson’s optimal mechanism is based on the following observation.

**Proposition 1** The expected revenue of any truthful single-parameter mechanism $M$ is equal to its expected virtual surplus.

A direct consequence of Proposition 1 is that the expected revenue maximizing mechanism would be one that maximizes expected virtual surplus. Given a vector $v$ of values, Myerson’s mechanism serves the set $\mathrm{argmax}_S \Phi(S, v)$. This mechanism is truthful when the virtual valuation function is monotone non-decreasing for every $i$, or in other words, the distribution $F_i$ is regular. Note that we do not explicitly specify the prices charged by the mechanism. These are uniquely determined by the allocation rule assuming that agents that are not served pay nothing.

**Definition 2** A one dimensional distribution distribution $F$ is regular, if $\phi(v)$ is monotone non-decreasing in $v$.

**Irregular distributions and ironed virtual values.** When the distributions $F_i$ are irregular, that is, Definition 2 does not hold, Myerson’s mechanism as described above will no longer be truthful. Myerson fixed this case by “ironing” the virtual valuation function and converting it into a monotone non-decreasing function. We skip the description of this procedure; the reader is referred to [8, 9] for details. Let us denote the ironed virtual value of an agent with value $v_i$ by $\overline{\phi}_i(v_i)$. We then note the following.

**Proposition 2** The expected revenue of any truthful single-parameter mechanism $M$ is no more than its expected ironed virtual surplus. If the probability with which the mechanism serves agent $i$, as a function of $v_i$, is constant over any valuation range in which the ironed virtual value of $i$ is constant, the expected revenue is equal to expected ironed virtual surplus.

Myerson’s mechanism serves a subset of agents that maximizes the ironed virtual surplus, breaking ties in an arbitrary but consistent manner. Denoting the revenue of a mechanism $A$ by $R^A$ and the revenue of Myerson’s mechanism $M$ by $R^M$ we get the following:

**Theorem 3** $R^M \geq R^A$ for every truthful mechanism $A$.

**Sequential posted price mechanisms.** A sequential posted-price mechanism (SPM), $S$, is defined by an ordering $\sigma$ over agents and a collection of prices $p_i$ for $i \in [n]$. The mechanism is run as follows:

1. Initialize $A \leftarrow \emptyset$.

2. For $i = 1$ through $n$, do:
   
   (a) If $A \cup \{\sigma(i)\} \in \mathcal{J}$, offer to serve agent $\sigma(i)$ at price $p_i$.

   (b) If the agent accepts, $A \leftarrow A \cup \{\sigma(i)\}$.

3. Serve the agents in $A$.

Let $c_i$ denote the probability taken over values of agents $\sigma(1), \cdots, \sigma(i - 1)$ that the mechanism offers to serve agent $i$, and let $q_i = 1 - F_i(p_i)$. Then the expected revenue of the sequential mechanism, $R^S$, is given by $\sum_i c_i q_i p_i$. 

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Ordering-oblivious posted prices. We also study the revenue that the seller obtains when he is forced to offer service to agents in an adversarial order. The “ordering-oblivious” revenue of a set of prices \( p \) is defined as \( R_{p}^{obl} = \min_{\sigma} \mathcal{R}(p, \sigma) \) where \( \mathcal{R}(p, \sigma) \) is the revenue of a sequential mechanism that uses prices \( p \) and an ordering \( \sigma \).

Bounding the revenue of the Bayesian optimal mechanism. We now develop some simple general bounds on the revenue obtained by any truthful mechanism that will be useful in proving performance guarantees for posted-price mechanisms.

Lemma 4 If \( F_i \) is regular for each \( i \), for any truthful mechanism \( M \) over the \( n \) agents, the revenue of \( M \) is bounded from above by \( \sum_i p_i^M q_i^M \) where \( q_i^M \) is the probability (over \( v_1, \ldots, v_n \)) with which \( M \) allocates to agent \( i \) and \( p_i^M = F_i^{-1}(1 - q_i^M) \).

Proof: Consider the revenue that \( M \) draws from serving agent \( i \). This is clearly bounded above by the optimal mechanism that sells to only \( i \), but with probability at most \( q_i^M \). By Proposition 1 such a mechanism should sell to agent \( i \) with probability 1 whenever the value of the agent is above \( F_i^{-1}(1 - q_i^M) \) and with probability 0 otherwise. The revenue of the optimal such mechanism is therefore \( p_i^M q_i^M \).

While the lemma does not extend to non-regular distributions, an approximate version of it holds in general. The following (and all other missing proofs) are proven in Appendix F.

Lemma 5 For any arbitrary collection of distributions \( F_i \), and any truthful mechanism \( M \) over the \( n \) agents, there exist probabilities \( \tilde{q}_i \leq q_i^M \) for all \( i \) with \( \tilde{p}_i = F_i^{-1}(1 - \tilde{q}_i) \), such that the revenue of \( M \) is bounded from above by \( 2 \sum_i \tilde{p}_i \tilde{q}_i \).

2.2 Bayesian optimal multi-parameter mechanism design

We study the following multi-parameter mechanism design problem (MPMD): there are \( n \) buyers and 1 seller. The seller offers a number of different services indexed by set \( J \). The set \( J \) is partitioned into groups \( J_i \), with the services in \( J_i \) being targeted at agent \( i \). Each agent \( i \) is interested in getting any one of the services in \( J_i \) (that is, consumers are unit-demand agents). Agent \( i \) has value \( v_j \) for service \( j \in J_i \). \( v_j \) is independent of all other values and is drawn from distribution \( F_i \). Once again the seller faces a feasibility constraint specified by a set system \( \mathcal{J} \subseteq 2^J \). Note that for every \( S \in \mathcal{J} \) and \( i \in [n] \), \( |S \cap J_i| \leq 1 \), that is each agent gets at most one service. As in the single-parameter case, a mechanism for this problem maps any set of bids \( v \) to an allocation \( M(v) \in \mathcal{J} \) and a pricing \( \pi(v) \) with a price \( \pi_i \) to be paid by agent \( i \).

Comparing the revenues of single-parameter and multi-parameter mechanisms. Understanding the properties of optimal mechanisms in multi-parameter settings is tricky. We use the approach of [9] and construct a “full-competition” version of the above multi-parameter problem with single-parameter agents. We then argue that the revenue of Myerson’s mechanism in the full-competition setting is an upper bound on the revenue of an optimal mechanism for the multi-parameter problem.

The full-competition version: There are \( |J| \) distinct agents interested in a single service; agent \( j \)’s value for getting served, \( v_j \), is distributed independently according to \( F_j \). The mechanism again faces a feasibility constraint given by the set system \( \mathcal{J} \). Given an instance \( I \) of the MPMD, we denote this full-competition instance by \( I^c \). The following lemma then relates the revenue of Myerson’s mechanism on \( I^c \) to the revenue of the optimal mechanism for \( I \).

Lemma 6 Let \( A \) be any individually rational and truthful deterministic mechanism for instance \( I \) of the MPMD. Then the expected revenue of \( A \), \( R_A^I \) is no more than the expected revenue of Myerson’s mechanism for the full-competition instance \( I^c \).

3 Sequential posted-price mechanisms

In this section we present approximations to optimal revenue via sequential posted price mechanisms. Towards the end of the section we show that our techniques extend to a large class of objective functions, namely those that are linear in valuations and payments.
3.1 A constant approximation for matroids

We first consider the setting where the set system $([n], J)$ is a matroid. Precisely, it satisfies the following conditions:

1. (heredity) For every $A \in J$, $B \subset A$ implies $B \in J$.

2. (augmentation) For every $A, B \in J$ with $|A| > |B|$, there exists $e \in A \setminus B$ such that $B \cup \{e\} \in J$.

Sets in $J$ are called independent, and maximal independent sets are called bases. A simple consequence of the above properties is that all bases are equal in size. The rank of a set $S \subseteq [n]$, $\text{rank}(S)$, is equal to the size of a maximal independent subset of $S$. The span of a set $S \subseteq [n]$, $\text{span}(S)$, is the maximal set $T \supseteq S$ with $\text{rank}(T) = \text{rank}(S)$.

Our main result of this section is that there exists an SPM that approximates the expected revenue of the optimal mechanism to within a constant factor. Towards the end of this section we show that the mechanism can be computed efficiently.

The mechanism is described as follows. Let $q_i^M$ denote the probability that Myerson’s mechanism serves agent $i$. If the distribution $F_i$ is regular, we set $q_i = q_i^M/2$, otherwise, we set $q_i = \tilde{q}_i/2$ where $\tilde{q}_i$ is the quantity defined in Lemma 5. Let $p_i = F_i^{-1}(1 - q_i)$ for all $i$. The SPM sets a price of $p_i$ for agent $i$ and offers to serve the agents in decreasing order of their prices. We denote this mechanism by $S$.

We first show that for the prices picked above, there is some ordering that gives the desired approximation, and then that the greedy ordering described above performs no worse. Note that $\sum_{i \in S} q_i^M \leq \text{rank}(S)$ for any set $S$ and $q_i \leq q_i^M/2$ for all $i$, so the probabilities $q_i$ defined above satisfy the premise of the lemma below.

Lemma 7 Let $q_i$’s be such that for every set $S$, $\sum_{i \in S} q_i \leq \text{rank}(S)/2$. Then, for the prices $p = F_i^{-1}(1 - q_i)$, there exists an ordering such that the probability of allocating to agent $i$ is at least $q_i/2$.

Proof: The lemma follows if we can demonstrate an ordering where the probability that we can offer to serve each agent when we get to it is at least 1/2. We proceed by finding an agent for which this property holds when it is placed last; we place this agent at the end and apply our argument inductively on the remaining agents.

For any agent $i$ consider placing it last. We are interested in the probability that we may offer to the agent when we reach it. Let $D$ be the set of agents that “desire” service, that is agents with $v_i \geq p_i$. Note that each time we reach an element of $D$, we offer to serve it (and hence sell to it) if and only if the agents served so far don’t span it. Thus, at each step, the served set has the same span as the portion of $D$ seen so far. We then have that

$$\Pr[D \text{ offers to serve } i \text{ if placed last}] = \Pr[\tilde{i} \notin \text{span}(D - \{i\})]$$
$$\geq \Pr[\tilde{i} \notin \text{span}(D)];$$

(1)

Let $c_i$ denote the probability $\tilde{i}$, which is over the desired set $D$. Fix any basis $B$. For any desired set $D$, the augmentation property for matroids implies that at least $(k - \text{rank}(D))$ elements of $B$ are not spanned by $D$. Thus, we have

$$\sum_{i \in B} c_i = \sum_{i \in B; D : i \notin \text{span}(D)} \Pr[D \text{ is desired}]$$
$$\geq \sum_{D} (k - \text{rank}(D)) \Pr[D \text{ is desired}]$$
$$= k - E_{D}[\text{rank}(D)] \geq k/2,$$

where the inequality follows from the fact that

$$E_{D}[\text{rank} D] \leq E_{\|D\|}[\|D\|] = \sum_{i \in [n]} q_i \leq k/2.$$

As a basis, $B$ has cardinality $k$, so $c_i$ must be at least 1/2 for at least one of its elements; this is the element we place last. Since the sum of the $q_i$ over any subset is no more than half the rank of that subset, we may repeat the above argument on the remainder of the elements. By our choice of last element at each step, the final order ensures that each agent is served with probability at least $q_i/2$. 

\[\square\]
Theorem 8  The above mechanism gives a 4-approximation to the expected revenue of Myerson’s mechanism in the regular case and an 8-approximation in general.

Proof: If the mechanism $S$ were to offer items in the order found in Lemma 7, then its revenue would be at least

$$\sum_i p_i \Pr [i \text{ is served}] \geq \sum_i p_i q_i / 2,$$

In the regular case, noting that $q_i = q_i^M / 2$ and so $p_i \geq p_i^M$, Lemma 4 implies that this ordering achieves the desired revenue. In the general case, the observation follows from Lemma 5 by noting that $q_i = \tilde{q}_i / 2 < q_i$ and therefore $p_i \geq \tilde{p}_i$. Now the key observation is that since our feasible sets are defined by a matroid, for any particular set $D$ of desired items a maximally independent subset of $D$ is sold; furthermore, the greedy strategy of offering items by decreasing $p_i$ ensures that among such sets the one maximizing the sum of its $p_i$ is sold. Thus, our specified mechanism also achieves the claimed revenue.

Computing the mechanism.  We show in Appendix D that the SPMs guaranteed in Theorem 8 can be computed efficiently while losing a further negligible factor in approximation.

A 4-approximation in the non-regular case

The mechanism in the previous section gives a 4-approximation to the optimal revenue when all the distributions $F_i$ are regular, but only an 8-approximation in general. In this section we show that in fact a 4-approximate SPM also exists in the general case. We describe a randomized SPM in which prices are picked randomly that achieves the desired approximation. It follows that for some deterministic setting of prices the mechanism achieves a 4-approximation.

As in the previous section let $q_i^M$ denote the probability with which Myerson’s mechanism serves agent $i$, and let $q_i = q_i^M / 2$ and $p_i = F_i^{-1}(1 - q_i)$. Note that the value $p_i$ may fall in a valuation range that has constant ironed virtual value. Let $p_i$ denote the infimum $\inf \{ v : \tilde{\phi}_i(v) = \phi_i(p_i) \}$ of this range and $\overline{p_i}$ denote the supremum $\sup \{ v : \tilde{\phi}_i(v) = \phi_i(p_i) \}$. Let $q_i = 1 - F_i(p_i)$ and $\overline{q}_i = 1 - F_i(\overline{p_i})$. Then, $\overline{q}_i \leq q_i \leq q_i^M$, and there exists an $x_i$ such that $x_i q_i + (1 - x_i) \overline{q}_i = q_i$.

Our randomized SPM first determines an ordering over agents based on the probabilities $q_i$ as prescribed in the proof of Lemma 7. Then for each agent $i$ it randomly picks a price of $p_i$ with probability $x_i$ and a price of $\overline{p_i}$ with probability $1 - x_i$. The probability (taken over $i$’s value distribution as well as the mechanisms coin flips) that $i$ accepts if it is offered a price is exactly $x_i q_i + (1 - x_i) \overline{q}_i = q_i$. Therefore, by the analysis of Lemma 2 we have for the chosen ordering $c_j \geq 1/2$ for each agent $j$, and the expected revenue of the randomized mechanism is at least $1/2 \sum_i (x_i q_i p_i + (1 - x_i) \overline{q}_i \overline{p_i})$. To complete the proof we use the following bound on the expected revenue of Myerson’s mechanism. (See Appendix F for a proof.)

Lemma 9  The expected revenue of Myerson’s mechanism is no more than twice $\sum_i (x_i q_i p_i + (1 - x_i) \overline{q}_i \overline{p_i})$, where $q_i = q_i^M / 2$ for every $i$ and $p_i$ is defined as above.

We get the following theorem.

Theorem 10  There exists an SPM with expected revenue at least a quarter of the expected revenue of Myerson’s mechanism.

An improved approximation for the 1-uniform matroid

We show in Appendix A that there exists an SPM that approximates the expected revenue of Myerson’s mechanism to within an improved factor of $e$ in the special case where the set of feasible agents are the independent sets of a 1-uniform matroid i.e. the seller is allowed to sell to exactly one of the agents.
3.2 The non-matroid case

We now show that Theorem 10 cannot extend to non-matroid set systems. In particular, the example below describes a family of instances with i.i.d. agents and a symmetric non-matroid constraint for which the ratio between the expected revenue of Myerson’s mechanism and that of the optimal SPM is $\Omega(\log n / \log \log n)$ where $n$ is the number of agents.

Example 1 For a given $m$ set $n = m^{m+1}$. Partition $[n]$ into $m^m$ groups $G_1, \ldots, G_m$ of size $m$ each, with $G_i \cap G_j = \emptyset$ for all $i \neq j$. The set system $\mathcal{F}$ contains all subsets of groups $G_i$, that is, $\mathcal{F} = \{ A : \exists i \text{ with } A \subseteq G_i \}$. Each agent has a value of 1 with probability $1 - 1/m$ and $m$ with probability $1/m$.

For any given valuation profile, let us call the agents with a value of $m$ to be good agents and the rest to be bad agents. The probability that a group contains $m$ good agents is $m^{-m}$. Therefore in expectation one group has $m$ good agents and Myerson’s mechanism can obtain revenue $m^2$ from such a group: $R^M = \Omega(m^2)$.

Next consider any SPM. The mechanism can serve at most $m$ agents. If all the served agents are bad, the mechanism obtains a revenue of at most $m$. On the other hand, once the mechanism commits to serving a good agent, it can only serve agents within the same group in the future. These have a total expected value less than $2m$. Therefore, the revenue of any SPM is at most $3m$, and we get a gap of $\Omega(n) = \Omega(\log n / \log \log n)$.

The above example also shows that while in many single-parameter pricing problems when the values are distributed in the range $[1, h]$ it is possible to obtain a $\log h$ approximation to social welfare, the same does not hold in our general setting. In the example we have $h = m$ and the gap between the expected revenue of the optimal SPM and that of Myerson’s mechanism is $\Omega(h)$. On the other hand, the gap is always bounded by $O(h)$ and is achieved by an SPM that charges each agent a uniform price of 1.

3.3 Approximating social welfare and other objectives

While the main focus of this work is on revenue maximization we note that sequential posted price mechanisms give good approximations in matroid settings to a large class of objectives, namely those that are linear in the valuations of the served agents and the payment received by the mechanism. In Appendix E we show that our approach from Section 3.1 extends to this general class of objectives implying the existence of an $8$-approximate SPM (or a $4$-approximate one when all value distributions are regular with respect to the objective).

4 Ordering-oblivious posted-prices

The approximations designed in Section 3 rely heavily on a specific ordering of the agents. A natural question is whether the seller can obtain good revenue when he has no control over the ordering. In such a case the seller picks a set of prices in advance, and then offers them to the agents on a first-come first-served basis. We show that in the matroid setting it is possible to determine a set of prices for which such “order-oblivious” mechanisms perform well.

We first note that for the case of uniform matroids (where every set of size at most $k$ is independent), the ordering oblivious revenue of the pricing implied by Theorem 8 is at least a quarter of the optimal revenue. In this setting, however, an improved approximation of $3$ can be obtained via ordering-oblivious prices by employing techniques developed by Chawla, Hartline and Kleinberg [9] for pricing problems in multi-parameter settings. We present this approximation in Appendix B.

For general matroids we give an $O(\log k)$ bound on the gap below, where $k$ is the rank of the matroid. We remark that a similar result was obtained by Babaioff et al. [2] for the related matroid secretary problem. In Babaioff et al.’s setting agents arrive in a random order but their values are adversarial. They present an $O(\log k)$ approximation by picking a price uniformly at random in the set $\{h/k, 2h/k, \ldots, h\}$ and charging it to every agent; Here $h$ is the largest among all values. Note that in our setting where values may arrive from an unbounded domain, such an approach does not work. In fact the example below shows that no uniform pricing can achieve an $o(\log h)$ approximation even when $k = 1$.

Example 2 Let $k = 1$ and consider a group of $h$ agents where agent $i$ has a value of $i$ with probability $1/2i^2$ and zero otherwise. Then an SPM that sets a price of $i$ for agent $i$ obtains an expected revenue of $\Omega(\log h)$. On the other hand, an SPM that uses a uniform price of $c$ only obtains expected revenue $\sum_{i \in [c,h]} c/2i^2 < c/2c = 1/2$. 


**Theorem 11** Given a matroid feasibility constraint with rank $k$, there exists a set of prices $p$ such that the ordering oblivious revenue of $p$ is at least an $O(\log k)$ fraction of the expected revenue of Myerson’s mechanism.

**Proof:** We present the proof for the regular case. For the non-regular case, modifications identical to those in the proof of Theorem 10 can be applied. The proof approach is similar to that of Lemma 7. Recall from that proof that for an element $i$, $c_i$ denotes the probability that $i$ does not belong to the span of the desired set $D$. Then for any basis $B$ of the matroid we have

$$\sum_{i \in B} c_i \geq k/2.$$  

Since we always have $c_i \leq 1$, we may conclude that at least one third of the agents in $B$ must have $c_i \geq 1/4$. Let

$$G = \{i| c_i \geq 1/4\};$$  

Intuitively, any one of the agents in $G$ can be placed last in the ordering, and therefore, if we ignore all agents outside $G$ (by offering them a price of $\infty$ instead of our originally chosen price) and order the agents arbitrarily, then our expected revenue on $G$ is at least

$$\sum_{i \in G} c_i p_i q_i \geq 1/8 \sum_{i \in G} p_i^M q_i^M.$$  

Consider iterating this process by recursively applying the above argument to the elements outside $G$. At step $j$, let $G_j$ be the set defined as in (2), and let $E_j$ be the set of agents still under consideration, defined as

$$E_j = [n], \text{ and,}$$

$$E_j = E_{j-1} - G_{j-1} \text{ for } j > 1.$$  

Now, at each stage $G_j$ contains at least one third of the elements of every basis; it follows that

$$\text{rank}(E_j) \leq 2 \text{rank}(E_{j-1})/3.$$  

Since any non-empty set has rank at least 1, we may conclude that this process can continue for at most $\log_{3/2} k$ steps. Note that the collection of $G_j$’s form a size $O(\log k)$ partition of $[n]$, and summing (3) over the collection gives a total expected revenue of $R^M/8$. We thus conclude that there is some $G_j$ which gives a $\Omega(1/\log k)$-fraction of $R^M$ regardless of ordering.

We remark that while the 4-approximate SPM in Section 3 can be computed efficiently, we do not know of an efficient algorithm for computing an $O(\log k)$-approximate ordering-oblivious pricing.

**The non-matroid case.** Example 1 in Section 3.2 already implies that ordering-oblivious pricings cannot obtain more than an $O(\log n/\log \log n)$ fraction of the revenue of Myerson’s mechanism in general in non-matroid settings. How do they compare to the optimal SPM? We show in Appendix C that the gap between the optimal ordering-oblivious pricing and the optimal SPM can be large — $\Omega(\log n/\log \log n)$ — in the non-matroid setting.

## 5 Revenue maximization through VCG mechanisms

A consequence of our constant-factor approximation to revenue through SPMs is that in matroid settings VCG mechanisms with appropriate reserve prices are near-optimal in terms of revenue. This follows from noting, as we show below, that VCG mechanisms perform no worse in terms of expected revenue than SPMs with the same reserve prices. Although VCG mechanisms aim to maximize the social welfare of the outcome, setting high enough reserve prices allows them to also obtain good revenue.

Formally, a Vickrey-Clarke-Groves (VCG) mechanism $\mathcal{V}p$ with reserve prices $p$ serves the set $S$ of agents, with $v_i \geq p_i$ for all $i \in S$, that maximizes $\sum_{i \in S} v_i$.

Hartline and Roughgarden [15] show that in several single-parameter settings the VCG mechanism with monopoly reserve prices gives a constant factor approximation to revenue. This result holds when all the value distributions
satisfy the so-called monotone hazard rate condition, or with a matroid feasibility constraint when all the value distributions are regular. Their result does not extend to the case of matroids with general (non-regular) value distributions. One of the main questions left open by their work is whether there is some set of reserve prices (not necessarily equal to the monopoly reserve prices) for which the VCG mechanism gives a constant factor approximation to revenue in the matroid setting with general value distributions. We answer this question in the positive. We use the following fact about matroids.

**Proposition 12** Let $B_1$ and $B_2$ be any two independent sets of equal size in a matroid set system $\mathcal{J}$. Then there is a bijective function $g : B_1 \setminus B_2 \to B_2 \setminus B_1$ such that for all $e \in B_1 \setminus B_2$, $B_1 \setminus \{e\} \cup \{g(e)\}$ is independent in $\mathcal{J}$.

**Theorem 13** For any instance of the single-parameter Bayesian mechanism design problem with a matroid feasibility constraint, there exists a set of reserve prices such that the expected revenue of the VCG mechanism with those reserve prices is at least a quarter of the expected revenue of Myerson’s mechanism.

**Proof:** We prove that when the set system $\mathcal{J}$ is a matroid, for any collection of prices $\mathbf{p}$, the revenue of the SPM $\mathcal{S}^\mathbf{p}$ is no more than the revenue of the VCG mechanism $\mathcal{V}^\mathbf{p}$. The result then follows from Theorem 10.

Fix a value vector $\mathbf{v}$ and let $A$ denote the set served by $\mathcal{S}^\mathbf{p}$ and $B$ denote the set served by $\mathcal{V}^\mathbf{p}$. Then, since both mechanisms serve a maximal independent set among the set of agents with $v_i \geq p_i$, we have $|A| = |B|$. Proposition 12 then implies the existence of a bijection $g$ such that for all $e \in B \setminus A$, $B \setminus \{e\} \cup \{g(e)\}$ is independent. This implies that $\mathcal{V}^\mathbf{p}$ charges $\mathbf{e}$ a price of at least the value of $g(e)$, which is at least the reserve price $p_{g(e)}$. On the other hand, by definition, the price charged to any $e \in B \cap A$ is at least $p_e$. Therefore, the revenue of $\mathcal{V}^\mathbf{p}$ in this case is at least $
sum_{e \in B \cap A} p_e + \nsum_{e \in B \setminus A} p_{g(e)} = \nsum_{e \in A} p_e$ which is equal to the revenue of $\mathcal{S}^\mathbf{p}$.

6 Multi-parameter optimal mechanism design

We now study revenue maximization in the multidimensional setting via VCG-type mechanisms. We first consider the special case of multi-unit auctions, and provide a truthful 8-approximate VCG mechanism. For general matroids, we present a VCG-type mechanism that is not truthful. Nevertheless we show that in any Nash equilibrium, the set of agents served by the optimal mechanism serves a maximal independent set among the set of agents with the highest social value.

6.1 The multi-unit multi-item case

We first consider a simpler version of the multi-parameter mechanism design problem, where the seller has $m$ different items, with $k_j$ identical copies of item $j$. We use slightly different notation in this section and denote by $v_{i,j}$ the value of agent $i$ for item $j$. The seller’s constraint is to allocate item $j$ to no more than $k_j$ agents, and to allocate at most one item to each agent. That is, the set system $\mathcal{J}$ is the collection of all (partial) matchings from the set of agents to the multiset of items.

To describe our approximately optimal mechanism we need some more notation. Recall that given an instance $\mathcal{I}$ of the MPMD, we use $\mathcal{I}^c$ to denote its full-competition version. Agents in $\mathcal{I}^c$ are indexed by pairs $(i,j)$. Let $q_{i,j}$ denote the probability with which Myerson’s mechanism serves agent $(i,j)$ in the $\mathcal{I}^c$. Let $q_{i,j} = q_{i,j}^m/2$, and $p_{i,j} = F_{i,j}^{-1}(1 - q_{i,j})$. Our mechanism is the VCG mechanism with reserve prices $\mathbf{p}$, defined as follows.

Let $\mathcal{J}$ denote the multiset of items and recall that the collection of feasible allocations in this setting can be represented by all (partial) matchings from $[n]$ to $J$. For a partial matching $\sigma : [n] \to J$, let $\hat{sv}(\sigma) = \nsum_{i \in [n]} (v(i,\sigma(i)) - p_{i,\sigma(i)})$ be the “pseudo social value” of $\sigma$ (this is not the actual social value because $\mathbf{p}$ are not true costs). Then VCG allocates according to a matching $\sigma \in \argmax_{\sigma : [n] \to J} \hat{sv}(\sigma)$. Payments are defined as follows. For an agent $i$, let $\hat{\sigma}_i = \argmax_{\sigma : [n] \setminus \{i\} \to J} \hat{sv}(\sigma)$. The payment of agent $i$ is $\hat{sv}(\hat{\sigma}_i) - \hat{sv}(\sigma) + v(i,\sigma(i))$. Note that since the mechanism falls in the class of VCG mechanisms, it is truthful. Also, $\hat{sv}(\hat{\sigma}_i) \geq \hat{sv}(\sigma) - (v(i,\hat{\sigma}_i) - p_{i,\hat{\sigma}_i})$, therefore, $i$’s payment is at least the reserve price for the pair $(i,\hat{\sigma}_i)$.

We now have the following theorem. (See Appendix A for a proof.)

**Theorem 14** The mechanism described above is a 8-approximation to the MPMD when the seller has identical copies of $m$ items on sale.
6.2 The general matroids case

Next we consider the case of unit-demand agents with a matroid set constraint. Specifically the set system $(J, J)$ forms a matroid in addition to the property that $|A \cap J_i| \leq 1$ for all $A \in J$ and $i \in [n]$.

Our mechanism for this general case mimics a VCG mechanism with reserve prices in the single-parameter setting. We again focus on the regular case, and can apply the modifications in the proof of Lemma 9 for the non-regular case. The mechanism is defined as follows. Once again let $q_j^M$ denote the probability with which Myerson’s mechanism serves agent $j$ in the $I^c$. Let $q_j = q_j^M / 4$, and $p_j = F_j^{-1}(1 - q_j)$. The mechanism first solicits bids from agents on each of the services. Let $b_i$ denote the bid of agent $i$ for service $j \in J_i$. Then the mechanism allocates to a set $S(b)$ in $\arg\max_{S \in J, \forall j \in S, b_j \geq p_j} \sum_{j \in S} b_j$. The payment of an agent $i$ with $j \in J_i \cap S(b)$ is the smallest value of $b_j$ for which $j \in S(b_{-i}, b_j)$. We denote this mechanism $V(p)$. We note that each agent pays at least the reserve price for the service it is allocated.

**Theorem 15** In the unit-demand matroid setting, in any Nash equilibrium, the mechanism $V(p)$ with reserve prices $b$ as defined above gives a 8-approximation to the optimal expected revenue.

Given the set of values $v$, for an agent $i$, let $R_i = \{j \in J_i : v_j \geq p_j\}$ denote the set of services “desired” by $i$ given the reserve prices $p_j$. We note that the mechanism $V(p)$ defined above is not truthful. However, we show below that it satisfies some weak truthfulness properties, and under those properties provides an 8-approximation to revenue in any Nash equilibrium. (See Appendix for a proof.)

**Lemma 16** For any vector of values $v$, in any Nash equilibrium of the mechanism, the following hold:

- For any $i$ with $|R_i| \leq 1$ and $j \in R_i$, $b_j \geq p_j$.
- For any $i$ and $j \in J_i \setminus R_i$ (i.e. $v_j < p_j$), $b_j < p_j$.

**Proof of Theorem 15** Fix a Nash equilibrium of the mechanism $V(p)$ and let $b$ denote the bids of the agents. The proof of the theorem has two steps. We start with an SPM for $I^c$ with reserve prices $p$ and the ordering described by Lemma 7 and analyze the performance of this SPM in the case where agents accept or decline offers based on $b$ instead of their true values. We show that the expected revenue of the SPM in this setting is still within a factor of 8 of that of Myerson’s mechanism. Then we show that the expected revenue of the mechanism $V(p)$ for $I$ described above is no less than that of the SPM with $b$ for $I^c$.

Following the argument in Lemma 7 and using the fact that $q_j = q_j^M / 4$ for all $j$, we get that in the SPM, for every $j$, the probability of offering to serve $j$ is at least $3/4$. Now consider what happens when the agents bid according to $b$. As in the proof of Lemma 7 let $D$ denote the set of agents desiring service. Then, the second claim in Lemma 16 implies that when the agents bid according to $b$, the set $D$ potentially shrinks but can be no larger. Therefore, the probability of offering to serve $j$ is still at least $3/4$ for every $j$. Now we bound from below the expected revenue of the SPM by accounting for the contribution by all $j \in J$ that satisfy the following:

1. $j$ is offered service in the SPM when agents play according to $b$.
2. $j \in J_i$ with $|R_i \setminus \{j\}| = 0$.
3. $v_j \geq p_j$.

When all of these events hold, Lemma 16 implies that $b_j \geq p_j$, and so $j$ generates a revenue of $p_j$. The first two events hold with probabilities $3/4$ and $3/4$ respectively, and the third holds with probability $q_j$ independent of the other two. Therefore, the total probability with which the events hold is at least $1/2q_j$. So we get that the expected revenue of the SPM with $b$ is at least $1/2 \sum_j q_j p_j$, which is at least an eighth of the expected revenue of Myerson’s mechanism, by the definition of $q_j$ and $p_j$.

Finally it remains to show that mechanism $V(p)$ obtains good revenue. Given a set of bids $b$, we compare the revenue of $V(p)$ against that of the SPM under the same set of bids. We note that the revenue of the mechanism is identical to the revenue of the corresponding VCG mechanism for $I^c$, given the same bids; then the claim follows from an argument analogous to the proof of Theorem 14.
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A An $\varepsilon$-approximation to revenue via SPMs for 1-uniform matroids

In this section we present an improvement over the 4-approximation of Section 3.1 in the special case where the set of feasible agents are the independent sets of a 1-uniform matroid i.e. the seller is allowed to sell to exactly one of the agents.

Let $q_i^M$ be the probability with which Myerson’s mechanism serves agent $i$ and let $p_i^M = 1 - F_i(q_i^M)$. The SPM $S$ is defined as follows. Let $\sigma$ be any ordering in which the last element $j$ has $q_j^M = \max_i q_i^M$, and let $p_i = p_i^M$ for all $i$. $S$ serves agents in the order prescribed by $\sigma$ at prices $p$. Without loss of generality, let $n$ denote the last element.

The Revenue $R^S$ of the SPM can be written as

$$R^S = \sum_{i=1}^{n-1} \prod_{j=1}^{i-1} (1 - q_j) p_i^M q_i^M \geq \left( \prod_{j \neq n} (1 - q_j) \right) \sum_{i=1}^{n} p_i^M q_i^M \geq \left( \prod_{j \neq n} (1 - q_j) \right) R^M$$

The factor $\varepsilon$ approximation follows from the claim that $\prod_{j \neq n} (1 - q_j) \geq 1/e$. To prove the claim, we first note that for $q_n \geq 1/2$,

$$\prod_{j \neq n} (1 - q_j) \geq 1 - \sum_{i \leq n-1} q_i \geq 1 - (1 - q_n) \geq 1/2$$

For $q_n < 1/2$ we have the following lemma.

**Lemma 17** $\sum_i q_i \leq 1$ and $q_n < 1/2$ imply $\prod_{i \neq n} (1 - q_i) \geq 1/e$.

**Proof:** Let $q_n = \alpha$. As before, we have that $\sum_{i \neq n} q_i \leq 1 - q_n \leq 1 - \alpha$. We wish to minimize the function $\prod_{i \neq n} (1 - q_i)$ subject to $q_i \in [0, \alpha]$ for all $i \neq n$ and $\sum_{i \neq n} q_i \leq 1 - \alpha$.

We first note that for $q_1, q_2 \in [0, \alpha]$ with $q_1 + q_2 = s$ for some fixed $s$, the minimum value of $(1 - q_1)(1 - q_2) = 1 - s + q_1 q_2$ is achieved by either setting one of the quantities to 0 or one of them to $\alpha$. Therefore, the minimum of the above expression is achieved by setting all but one of the probabilities $q_i$ to 0 or $\alpha$. Now, taking the worst case of $\sum_{i \neq n} q_i = 1 - \alpha$, we find that the minimum value is attained when $(1 - \alpha)/\alpha$ of the $q_i$ are set to $\alpha$ and the rest to 0. The product in this case becomes $(1 - \alpha)^{1/(1-\alpha)}$.

Consider the function $f(x) = (1 - x)^{1/(1-\alpha)}$. In the range $0 < x < \frac{1}{2}$, $f'(x) > 0$. Thus $f(x)$ attains its minimum when $x \to 0$. The lemma now follows by noting

$$(1 - \alpha)^{1/(1-\alpha)} \geq \lim_{x \to 0} (1 - x)^{1/(1-\alpha)} \geq \lim_{x \to 0} (1 - x)^{1/\alpha} = \frac{1}{e}$$

The claim immediately implies the $\varepsilon$-approximation.

**Theorem 18** Mechanism $S$ defined above gives an $\varepsilon$-approximation to the expected revenue of Myerson’s mechanism in the case of a 1-uniform matroid.

B Ordering oblivious pricings for uniform matroids

We consider the case of uniform matroids, that is matroids where the independent sets are exactly all sets of size at most $k$, for some threshold $k \leq n$. We present a $3$-approximation in the case of uniform matroids based on a $3$-approximation to a multi-parameter pricing problem developed by Chawla et al. [10]. Chawla et al. consider a setting where a single unit-demand agent is interested in buying one of $n$ items and the seller wants to maximize its revenue.

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from offering a price menu to the agent. Chawla et al. show that there exists a set of prices such that if the agent is allowed to pick any item of its choice at these prices (even adversarially), the seller obtains a revenue of no less than a third of the revenue of Myerson’s mechanism for the corresponding single-parameter single-item mechanism design problem. It is easy to see that this result implies that the same set of prices has an ordering-oblivious revenue in our setting in the 1-uniform matroid that is \( \frac{3}{4} \)-approximate.

We now show that Chawla et al.’s technique extends also to \( k \)-uniform matroids. In order to extend the result to the \( k \)-uniform case, we redefine one key quantity from [10], and then describe what changes this necessitates in the main theorems of that paper. Since the rest of the argument is nearly identical to that in [10], we omit the details. We also use notation from that paper without redefining it. The reader is encouraged to familiarize herself with the argument in [10] before reading the remainder of this section.

For any mechanism \( \mathcal{A} \), the quantity \( \chi (\cdot) \) is defined in [10] as

\[
\chi (\mathcal{A}) = \Pr[\mathcal{A} \text{ serves no agent}].
\]

We redefine this quantity as

\[
\chi (p) = k - \mathbb{E}[\text{number of agents served under prices } p].
\]

Thus, we can see that

\[
\mathcal{R} \geq \sum_i p_i q_i \Pr[\text{fewer than } k \text{ agents served}]
\]

\[
\geq \sum_i p_i q_i \Pr[\text{fewer than } k \text{ agents served}]
\]

\[
\geq \frac{\chi (p)}{k} \sum_i p_i q_i,
\]

where the last inequality follows from the noted bound on \( \chi (p) \).

\[ \blacksquare \]

The additional factor of \( \frac{1}{k} \) in Lemma 8 must be carried through to Corollaries 10 and 11 as well; similarly, the factor of \( (1 - \chi (p)) \) in the statement of Lemma 7 must be replaced with \( (k - \chi (p)) \). These changes, however, do not change the reasoning behind the associated proofs; furthermore, when they are carried through to Lemma 12, we achieve the same result as before, when \( p \) is chosen such that \( \chi (p) = k/2 \). Thus, the key theorem from that paper extends to the uniform matroid setting as well and we obtain the following.

**Theorem 20** In the uniform matroid setting there exists a collection of prices \( p \) such that \( \mathcal{R}^{\text{obl}} \geq \frac{1}{3} \mathcal{R}^{\mathcal{M}} \).

### C Ordering-oblivious pricings in the non-matroid setting

In this section we present an example with a non-matroid constraint for which the revenue obtained by ordering the agents in the optimal way is a factor of \( \Omega(\log n / \log \log n) \) larger than that obtained by ordering the agents in the least optimal way.

**Lemma 21** There exists an instance of the single-parameter mechanism design problem with a non-matroid feasibility constraint, along with two orderings \( \sigma_1 \) and \( \sigma_2 \) such that the revenue of the optimal SPM using ordering \( \sigma_1 \) is a factor of \( \Omega(\log n / \log \log n) \) larger than that of the optimal SPM using ordering \( \sigma_2 \).

**Proof:** Consider the following example. Construct a complete \( m \)-ary tree of height \( m + 1 \), and place a single agent at each node other than the root. The agents’ valuations are i. i. d., where any agent has a valuation of \( m \) with probability...
We now describe how to compute an approximately optimal SPM, achieving the bounds guaranteed in Theorem 8, given access to the following oracles and algorithms:

1. We assume that we are given an algorithm to compute the optimal price to charge to a single-parameter agent given the agent’s value distribution. Note that given such an algorithm and some value \( x \), we can modify it to return the optimal price in the range \([x, \infty)\) to charge the agent.

2. We also assume access to an oracle that given a value \( v \) and index \( i \) returns \( f_i(v) \) and \( F_i(v) \), as well as, given a probability \( \alpha \) returns \( F_i^{-1}(\alpha) \). Note that the oracle can be used to compute the virtual value \( \phi_i(v) \).

3. In order to compute the approximately optimal SPM for non-regular distributions we require access to an oracle for computing ironed virtual values.

4. Finally, we assume that we can maximize social welfare over the given feasibility constraint in order to be able to compute the outcome of Myerson’s mechanism.

Our algorithm runs as follows:

1. Let \( \epsilon = 1/3n \). Sample \( N = 4n^4 \log n/\epsilon^2 \) value profiles from \( F_1 \times F_2 \times \cdots \times F_n \). For each sample, compute the (ironed) virtual value for each agent, and use these to compute the outcome of Myerson’s mechanism for that value profile.

2. Estimate the probabilities \( q_i^M \) using the samples. Call the estimates \( \hat{q}_i^M \).

3. Let \( \hat{q}_i = 1/2(\hat{q}_i^M/(1 - \epsilon) + 1/n^2) \). Compute for each \( i \) the value \( \hat{p}_i = F_i^{-1}(1 - \hat{q}_i) \).

4. Find the optimal price in the range \([\hat{p}_i, \infty)\) to charge to agent \( i \). Call it \( p_i \).

5. Output the prices computed in the last step and order the agents in order of decreasing prices.

**Theorem 22** The above algorithm gives a \( 4 + o(1) \)-approximate SPM in the regular case with a matroid constraint and an \( 8 + o(1) \)-approximate SPM in the non-regular case with a matroid constraint.
To prove the theorem we first show that with a high probability we obtain a good approximation to the probabilities $q_i^M$.

**Lemma 23** With probability at least $1 - 2/n$, we have $\hat{q}_i \in 1/2[q_i^M, (1 + 3\epsilon)q_i^M + 2/n^2]$.

**Proof:** First, for any $i$ with $q_i^M \geq 1/n^4$, using Chernoff bounds we get that
$$q[|\hat{q}_i^M - q_i^M| \geq \epsilon q_i^M] \leq 2e^{-\epsilon^2 q_i^M N/2} \leq 2/n^2$$

Therefore we have $\hat{q}_i \in 1/2[q_i^M, (1 + 3\epsilon)q_i^M + 1/n^2]$. On the other hand, for $q_i^M < 1/n^4$, by Markov’s inequality, with probability $1 - 1/n^2$, $q_i^M < 1/n^2$, and so $\hat{q}_i \in [q_i^M, 1/n^2]$. The lemma now follows by employing the union bound.

This implies that the prices $p_i$ give a good estimate on the revenue of Myerson’s mechanism.

**Corollary 24** When all the distributions $F_i$ are regular, with probability at least $1 - 2/n$, $\sum_i q_i p_i \geq 1/2 R^M$. In the general case, with probability at least $1 - 2/n$, $\sum_i q_i p_i \geq 1/4 R^M$.

**Proof:** From the previous lemma we have that with probability $1 - 2/n$, for every $i$, $\hat{q}_i \geq q_i^M/2$. Then, since we pick the optimal price in the range $[\hat{q}_i, \infty)$, the result follows from Lemmas 4 and 5.

Finally, we claim that the SPM serves each agent with reasonable probability.

**Lemma 25** For the SPM returned by the above algorithm, with probability at least $1 - 2/n$, $R^S \geq 1/2(1 - 2/n) \sum_i q_i p_i$.

**Proof:** Lemma 23 implies that with probability at least $1 - 2/n$, for all $i$, $q_i \leq 1/2(1 + 3\epsilon)q_i^M + 1/n^2$. We claim that under these new probabilities, there still exists an element in the matroid with $c_e \geq 1/2(1 - 1/n)$. The lemma then follows. To prove the claim, we note that the expected rank of the desired set $D$ is at most $k/2(1 + 3\epsilon)q_i^M + 1/n^2 \leq k/2(1 + 3\epsilon)q_i^M + 1/n \leq k/2(1 + 2/n)$ where the last inequality follows from noting $\epsilon = 1/3n$. Then, as in the proof of Lemma 7 we have $\sum c_e B \geq k - k/2(1 + 2/n) \geq k/2(1 - 2/n)$, implying that there exists an element $e$ with $c_e \geq 1/2(1 - 2/n)$.

The theorem follows immediately from Corollary 24 and Lemma 25.

**E Approximating social welfare and other objectives via SPMs**

We now show that our approach from Section 3.1 in fact extends to the problem of maximizing any objective that is linear in social value and revenue via SPMs.

We start with some definitions. For all $i \in [n]$ let $g^i(v, p) = \alpha_i v + \beta_i p$ denote an arbitrary linear function of $v$ and $p$. For a mechanism $A$ with payment rule $p$, let $G(A, p)$ be the expected value of $g$ over the outcome of the mechanism, that is, $G(A, p) = E_v[\sum_{i \in A(v)} g^i(v, p)]$. Define the virtual value of $i$ with respect to $g^i$ to be
$$\phi^G_i(v) = (\alpha_i + \beta_i) v - \beta_i \frac{1 - F_i(v)}{f_i(v)}$$

and the virtual surplus with respect to $G$ of a set $S$ of agents to be $\Phi^G(S) = \sum_{i \in S} \phi^G_i(v_i)$. Then, the lemma below follows from standard techniques, and allows us to ignore the payment function in trying to maximize $G$.

**Lemma 26** For any truthful mechanism $A$ with payment rule $p$, the expected virtual surplus with respect to $G$ of $A$ is equal to the expected value of $G$ for $A$’s outcome. That is,
$$G(A, p) = E_v[\Phi^G(A(v))]$$
The lemma suggests that a mechanism $M^G$ with allocation rule $M^G(v) = \arg \max_S G^G(S)$ maximizes $G$ over the class of all truthful mechanisms. However, as for revenue-maximizing mechanisms, in order for this mechanism to be truthful, the distributions $F_i$ must satisfy a certain regularity condition.

**Definition 3** A one dimensional distribution distribution $F$ is regular with respect to function $G$, if $\phi^G(v)$ is monotone non-decreasing in $v$.

The following theorem is straightforward:

**Theorem 27** If for all $i$, $F_i$ is regular with respect to $G$, the mechanism $M^G$ defined above is truthful and obtains the maximum value of $G$ over the class of all truthful mechanisms.

In order to optimize $G$ over the class of SPMs in the matroid setting, we follow an approach similar to the one in Section 3. We focus on the regular setting. Our approximately optimal mechanism is defined as follows. Let $\gamma_i$ denote the optimal mechanism in Theorem 27 above. Let $q_i^G$ denote the probability that $M^G$ serves agent $i$. Define for all $i$

$$q_i = q_i^G / 2, \quad p_i = F_i^{-1}(1 - q_i), \quad \gamma_i = \left( \int_{p_i}^{\infty} \phi_i^G(v_i)f_i(v_i)dv_i \right) / q_i$$

The SPM sets a price of $p_i$ for agent $i$ and offers to serve the agents in decreasing order of their corresponding $\gamma_i$’s. The $\gamma_i$ reflects the expected virtual value we get from agent $i$ upon serving the agent. We denote this mechanism by $G^{\gamma_i}$.

We first note that the performance of $G^{\gamma_i}$ can be bounded in terms of the $\gamma_i$’s. In particular, for a certain (implicit) ordering over the agents, as before let $c_i$ denote the probability that agent $i$ is offered the price $p_i$. Then, Lemma 26 and the definition of $\gamma_i$ imply that

$$G(S^G) = \sum_i c_i q_i \gamma_i$$

Lemma 7 shows that there exists an ordering over agents for which $c_i \geq 1/2$ for every $i$, implying that for this ordering $G(S^G) \geq 1/2 \sum_i q_i \gamma_i$. Since we use a greedy ordering based on the $\gamma_i$’s, this bound holds for that ordering as well.

In order to complete our argument, we bound the performance of $M^G$ in terms of the $\gamma_i$’s.

**Lemma 28** If for all $i$, $F_i$ is regular with respect to $G$, then $G(M^G) \leq 2 \sum_i \gamma_i q_i$.

**Proof:** Let us consider the contribution of agent $i$ to the objective function value for $M^G$. This is no more than the objective function value achieved by an optimal mechanism that sells only to $i$ and with probability at most $q_i^G$. By the definition of $\Phi^G$ and using regularity, this is exactly $\int_{p_i}^{\infty} \phi_i^G(v_i)f_i(v_i)dv_i$ where $p_i = F_i^{-1}(1 - q_i^G)$. Again using the fact that $\phi_i^G$ is non-decreasing, this is no more than $2 \int_{p_i}^{\infty} \phi_i^G(v_i)f_i(v_i)dv_i$ where $p_i = F_i^{-1}(1 - q_i^G / 2)$. Finally, the integral is exactly equal to $\gamma_i q_i$ by the definition of $\gamma_i$.

We therefore have the following theorem:

**Theorem 29** The mechanism $G^\gamma$ defined above obtains a 4-approximation to the objective $G$ in the matroid case when all the input distributions are regular with respect to $G$.

Finally, we note that if the distributions are not regular as defined in Definition 3, we can apply an ironing procedure to the virtual values in much the same way as in Myerson’s approach. This implies a general 8-approximation to the objective $G$ via SPMs. We leave the details to the reader.
F Missing proofs

Proof of Lemma 5 Once again, $M$’s expected revenue from agent $i$ is bounded from above by the revenue of a mechanism that sells to only agent $i$ but with probability at most $q_i^M$. Let $p_i^M = F_i^{-1}(1 - q_i^M)$.

Let $p_i$ denote the infimum of values with ironed virtual value equal to $p_i^M$, $p_i = \inf \{ v : \tilde{\phi}_i(v) = \tilde{\phi}_i(p_i^M) \}$ and $\overline{p_i}$ denote the supremum $\sup \{ v : \tilde{\phi}_i(v) = \tilde{\phi}_i(p_i^M) \}$ over the same range. Also let $\overline{p_i} = 1 - F_i(\overline{p_i})$ and $\underbar{q_i} = 1 - F_i(p_i)$. Proposition 2 implies that the optimal mechanism $A$ with selling probability $q_i^M$ sells to the agent with probability $x_i$ if the agent’s value is between $p_i$ and $\overline{p_i}$, and with probability 1 if the value is above $\overline{p_i}$, where $x_i$ is defined such that $q_i^M = x_i(\overline{q_i} - \underbar{q_i}) + \underbar{q_i} = x_i\overline{q_i} + (1 - x_i)\underbar{q_i}$.

From Proposition 2 the revenue $R_i^A$ of $A$ is equal to its expected ironed virtual surplus. So,

$$R_i^A = x_i \int_{p_i}^{\overline{p_i}} \tilde{\phi}_i(v_i)f_i(v_i) dv_i + \int_{\overline{p_i}}^{\infty} \tilde{\phi}_i(v_i)f_i(v_i) dv_i$$

$$= x_i \int_{p_i}^{\overline{p_i}} \tilde{\phi}_i(v_i)f_i(v_i) dv_i + (1 - x_i) \int_{\overline{p_i}}^{\infty} \tilde{\phi}_i(v_i)f_i(v_i) dv_i = x_i p_i \overline{q_i} + (1 - x_i) \underbar{q_i} \overline{p_i}$$

Therefore at least one of $x_i p_i \overline{q_i}$ and $(1 - x_i) \underbar{q_i} \overline{p_i}$ should be at least $R_i^A / 2$. In the latter case, $\underbar{q_i} \overline{p_i} \geq R_i^A / 2$ and we set $\overline{q_i} = \underbar{q_i}$. In the former case, $p_i q_i^M \geq p_i \overline{q_i} \geq x_i p_i \overline{q_i} \geq R_i^A / 2$ and we set $\overline{q_i} = q_i^M$.

Proof of Lemma 6 We first note that a mechanism is individually rational if we have $\pi_j \leq v_j$ for $j \in S \cap J_i$, and $\pi_i = 0$ if $S \cap J_i = \emptyset$. Truthful mechanisms in multi-parameter settings satisfy the weak monotonicity condition defined below.

Definition 4 A mechanism $(M, \pi)$ satisfies weak monotonicity if for any agent $i$ and any two types (value vectors) $v^1$ and $v^2$ with $v^1_j = v^2_j$ for all $j \in J \setminus J_i$, the following holds:

$$v^1_{M_i(v^1)} + v^2_{M_i(v^2)} \geq v^1_{M_i(v^2)} + v^2_{M_i(v^1)}$$

Here $M_i(v)$ denotes the unique index in $M(v) \cap J_i$.

We show that we can construct a truthful mechanism $A^{fc}$ for the $I^{fc}$ with revenue no less than that of $A$. The lemma then follows from the optimality of Myerson’s mechanism. Given a vector of values $v$, the mechanism $A^{fc}$ allocates to the set that $A$ allocates to in $I$ given the same value vector. We first claim that the allocation rule of $A^{fc}$ is monotone non-decreasing in any $v_j$, implying that there exists a payment rule that makes the mechanism truthful. To prove the claim, fix any agent $i$ and $j \in J_i$, and consider two value vectors $v^1$ and $v^2$ with $v^1_j = x$, $v^2_j = y$, and $v^1_j = v^2_j$ for $j' \neq j$. Let $\alpha_x$ and $\alpha_y$ denote the probabilities of serving agent $i$ with service $j$ under the two value vectors respectively, and let $\beta_x$ and $\beta_y$ denote the total value that agent $i$ obtains from other services $j' \in J_i$, $j' \neq j$, in the two cases respectively. Then the weak-monotonicity (Definition 4) of $A$ implies that

$$(x \alpha_x + \beta_x) + (y \alpha_y + \beta_y) \geq (x \alpha_y + \beta_y) + (y \alpha_x + \beta_x)$$

or,

$$(x - y)(\alpha_x - \alpha_y) \geq 0$$

Therefore the claim holds.

It remains to prove that the expected revenue of $A^{fc}$ given $I^{fc}$ is no less than the expected revenue of $A$ given $I$. Note that any deterministic multi-parameter mechanism can be interpreted as offering a price menu with one price for each item or service to each agent as a function of other agents’ bids [22]. The agent then chooses the item or service that brings her the most utility. Given this characterization, suppose that for a fixed set $v$ of values, mechanism $A$ offers a price menu with prices $p$ to agent $i$. Then, it draws a revenue of $p_i$ from $i$ whenever service $j$ is offered. On the other hand, mechanism $A^{fc}$ charges the agent $j$ the minimum amount it needs to bid to be served, which is no less than $p_j$, as $A$ is individually rational.
Proof of Lemma 9. Once again, consider the revenue that Myerson’s mechanism obtains from agent $i$. This is no more than the revenue obtained by an optimal mechanism that sells to only agent $i$ but with probability no more than $q_i^M$. The latter in turn is no more than twice the revenue obtained by an optimal mechanism that sells to agent $i$ with probability no more than $q_i^M/2$. This is because one way of selling to agent $i$ with probability $q_i^M/2$ is to reject the agent with probability half, and with probability a half use the optimal mechanism with selling probability $q_i^M$. Finally, an easy consequence of Proposition 2 is that the optimal mechanism with selling probability $q_i$ sells to the agent with probability $x_i$ if the agent’s value is between $p_i$ and $p_i$, and with probability 1 if the value is above $p_i$, where $x_i$ is defined such that $x_i q_i + (1 - x_i) q_i p_i = q_i$. The revenue of this mechanism is exactly $x_i q_i p_i + (1 - x_i) q_i p_i$.

Proof of Theorem 14. We prove the theorem for the case of regular distributions. In the non-regular setting, modifications similar to those in the proof of Lemma 9 can be applied. We first claim that the mechanism’s revenue is at least $1/4 \sum_{(i,j)} q(i,j) p(i,j)$. To see this, note that the probability that agent $i$ gets item $j$ is at least $q(i,j)$ times the probability that for all $j' \neq j$, $v(i,j') < p(i,j')$, times the probability that for at most $k_j - 1$ of the agents $i' \neq i$, $v(i',j) > p(i',j)$. The claim follows by noting that:

- $\sum_j q(i,j)^M \leq 1$ and therefore $\sum_j q(i,j) \leq 1/2$, and,
- $\sum_i q(i,j)^M \leq k_j$, so $\sum_i q(i,j) \leq k_j/2$ and therefore $\Pr[\{|i: v(i',j) > p(i',j)\}| \geq k_j] \leq 1/2$.

Recall from Lemma 3 that the expected revenue of the optimal mechanism is no more than the expected revenue of Myerson’s mechanism for $T^c$, which in turn is no more than $\sum_{(i,j)} q(i,j)^M p(i,j)^M \leq 2 \sum_{(i,j)} q(i,j)p(i,j)$ using Lemma 4 and noting that $q(i,j) = 1/2q(i,j)^M$, where $p(i,j) \geq p(i,j)^M$. Therefore the theorem follows.

Proof of Lemma 16. For the first part of the lemma note that $v_j \geq p_j$; if the agent bids $b_j < p_j$, then the agent wins no service and has a utility of 0, whereas if he bids $b_j \geq p_j$, his utility is non-negative. For the second part, note that if the agent bids $b_j \geq p_j$ and wins service $j$, he obtains strictly negative utility; otherwise the outcome of the mechanism is unaffected.