A 3D model of the oculomotor plant including the pulley system

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Abstract. Early models of the oculomotor plant only considered the eye globes and the muscles that move them. Recently, connective tissue structures have been found enveloping the extraocular muscles (EOMs) and firmly anchored to the orbital wall. These structures act as pulleys; they determine the functional origin of the EOMs and, in consequence, their effective pulling direction. A three dimensional model of the oculomotor plant, including pulleys, has been developed and simulations in Simulink were performed during saccadic eye movements. Listing’s law was implemented based on the supposition that there exists an eye orientation related signal. The inclusion of the pulleys in the model makes this assumption plausible and simplifies the problem of the plant noncommutativity.

1. Introduction

Each eye is moved by six muscles: the medial, lateral, superior and inferior rectus and the superior and inferior oblique.

There has been a significant change in the conception of the oculomotor plant over the past twenty years. Connective tissue structures have been found enveloping the EOMs in the region posterior to the globe equator [1]. These structures, composed of collagen, elastin and smooth muscle, are anchored to the orbit walls and function as pulleys for the EOMs [2]. Clark [3] showed with high resolution magnetic resonance studies that the positions of the pulleys change with the eye orientation. The pulley position determines the EOM path and, as a consequence, the pulling direction of the muscle.

There are some fundamental differences between three dimensional and two dimensional models of the oculomotor plant. These differences are principally consequences of the characteristics of three dimensional rotations which complicate the direct extension of a simplified model to a more realistic one.

The eye can, in a two dimensional model, be represented as a disk that rotates around a central axis. The position of the eye in this model is determined if we specify the angle of rotation from a reference position. The representation of the orientation of a body that rotates in 3D space is less intuitive and mathematically more complicated.

In a two dimensional model, angular velocity is the derivative of orientation. In a three dimensional model this doesn’t hold any more. This is a consequence of the noncommutativity of rotations in space about arbitrary axes.
The goal of this paper was to develop a 3D model of the oculomotor plant and to verify that the neural control of the system can be simplified if the pulleys are included.

2. Materials

2.1. Representation of orientation

Six independent coordinates are necessary to define the location of a rigid body in space. The position is specified by locating a coordinate system fixed in the body relative to a global set of axes (figure 1). Three coordinates are used to represent the origin of the local system relative to the global axes. The remaining three coordinates are used to define the orientation of the local system relative to a system parallel to the global axes, but with the same origin [4].

There are many ways of specifying the orientation of a Cartesian set of axes relative to another set if their origins coincide. This could be accomplished by defining a rotation matrix that transforms the coordinates of a vector relative to one set of axes into the coordinates of that vector relative to the other set. The position of a point fixed in the rigid body can then be described as:

$$ r(t) = r_c(t) + R(t) \cdot r_0 $$

where \( t \) indicates time, \( r_0 \) the position of the fixed point relative to the local system and \( r_c(t) \) the position of the local set of axes relative to the global set. \( R(t) \) is the rotation matrix that transforms the local coordinates of the body into the global coordinates.

The eye is modeled as a sphere capable of rotating around any axis through its center, which is fixed in space. It is assumed that the head remains still during the eye movements. With this simplifications and taking the same origin for the local and global set of axes, equation (1) can be expressed as:

$$ r(t) = R(t) \cdot r_0 $$

Another way of specifying the orientation of a Cartesian set of axes relative to another set relies on Euler’s Theorem. Euler’s Theorem states that the general displacement of a rigid body with one point fixed in space is a rotation about some axis [4]. If the fixed point is taken as the origin for the global and local set of axes, there will be no translation of the local axes. The position of the body at time \( t \) can in this way be described by specifying the orientation of the body fixed axes relative to the global set. The orientation is defined by the angle \( \theta \) and the axis of rotation, which is represented by a unitary vector \( \vec{n} \), about which the initial local set of axes (which coincides with the global axes) has to be rotated to get to the final orientation.

Both ways of representing the orientation are equivalent. The vector \( \vec{n} \) and the angle \( \theta \) can be obtained from the rotation matrix \( R(t) \):

$$ \vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \frac{r_{32} - r_{23}}{2 \sin \theta} \\ \frac{r_{13} - r_{31}}{2 \sin \theta} \\ \frac{r_{21} - r_{12}}{2 \sin \theta} \end{bmatrix} $$

$$ \theta = \cos^{-1} \frac{\text{Tr}(R) - 1}{2} $$

where \( \text{Tr} \) indicates the trace operator and \( r_{ij} \) the element \( ij \) of \( R(t) \).
The rotation matrix $R(t)$ can be obtained at time $t$ from:

$$
u_i = \tilde{e}_i \cos \theta + \tilde{n} \cdot \mathbf{n}_i [1 - \cos \theta] + (\tilde{n} \times \tilde{e}_i) \sin \theta$$

(5)

, where $\times$ indicates the cross product, $\mathbf{n}_i$ the $i$th component of the vector $\tilde{n}$, $\tilde{e}_i = (1\ 0\ 0)^T$, $\tilde{e}_z = (0\ 1\ 0)^T$ and $\tilde{e}_x = (0\ 0\ 1)^T$ [4],[5]. Counter clockwise rotations are defined positive.

Figure 1. Location of a rigid body in space. The axes $x',y'z'$ define the global coordinate system. The axes $x,y,z$ define the local coordinate system.

Figure 2. Listing’s law. The axes $x',y'z'$ define the global coordinate system. The $y'z'$ plane coincides with Listing’s plane. The axis $\mathbf{r}$ lies on Listing’s plane and defines the axis of rotation.

2.2. Listing’s law
Listing’s law establishes a restriction about the orientations the eye can take. Listing’s law states that all the axes of rotation in Euler’s representation of orientation, lie closely scattered along a plane. The orientation of this plane, called Listing’s plane, depends on the reference set of axes used to describe the eye positions. We will assume that the $x’z’$ plane of the global set of axes and Listing’s plane coincide (figure 2). The $x’$ axis is taken along the line of sight when the eye is at reference position. This reference position is called primary position. The torsional component is defined as the first element of the vector $\theta \cdot \tilde{n}$, where $\tilde{n}$ is the unitary vector in the direction of the axis of rotation and $\theta$ is the angle of rotation. Listing’s law can then be formulated very simply: all rotation vectors $\theta \cdot \tilde{n}$ characterizing 3D eye position have zero torsional component [6]. It is important not to confuse the term torsional with the concept of deformation. The term “torsional component” refers to the first element of the vector $\theta \cdot \tilde{n}$.

2.3. Saccadic eye movement characteristics
Saccadic movements are rapid eye movements from one point of fixation to another and have the objective to project a target of interest onto the fovea.

Saccades obey Listing’s law, during and after the eye movements. Straumann [7] registered the torsional component of the eye orientation during horizontal and vertical saccadic movements and noticed transient torsional deviations of the rotation vector $\theta \cdot \tilde{n}$. The amplitude of this torsional deviations ranges between $1^\circ$ and $2^\circ$ and decays exponentially towards zero.

Recordings in motoneurons of the extraocular muscles show that the innervation signal is composed of a tonic (step) and a phasic (pulse) component. The pulse brings the eye rapidly to the final orientation and the step keeps the eye in that position.
3. Methods

Due to the surrounding tissue of the eye, viscous and elastic counter torques act over the eye when it is moving or not at primary position. In one dimension, the elastic torque can be represented as:

\[ T = -k \cdot \alpha \]  

(6)

where \( T \) is the elastic torque, \( k \) is the elastic constant and \( \alpha \) is the angle the eye moved from its reference position \([5]\).

An important aspect of Euler’s axis-angle representation will be used to define an orientation dependant elastic torque in three dimensions. The vector \( \vec{n} \) and the angle \( \theta \) represent the shortest path the eye takes from the primary position to a given orientation. Thus, it is reasonable to expect that the restoring torque should be along the axis defined by Euler’s theorem because it is the optimal path for returning to the primary position \([5][6]\). Therefore, the elastic restitution torque is defined as:

\[ T(R) = -K \cdot \theta \cdot (R) \cdot \vec{n}(R) \]  

(7)

If we consider the viscous counter torque and the torque applied by the EOMs,

\[ J \cdot \frac{d \vec{w}}{dt} = -B \cdot \vec{w} - K \cdot \theta \cdot \vec{n} + m \]  

(8)

where \( J \) represents the moment of inertia, \( \vec{w} \) the angular velocity of the eyes relative to the head, \( B \cdot \vec{w} \) and \( K \cdot \theta \cdot \vec{n} \) the viscous and elastic torques and \( m \) the torque applied by the muscles.

The change in time of the rotation matrix \( R(t) \) is \([5]\):

\[ \frac{dR(t)}{dt} = -\vec{w} \times R(t) \]  

(9)

where \( \vec{w} \times R(t) \) indicates the cross product of \( \vec{w} \) with each column of \( R(t) \).

The relation between the muscle torque \( m \) and the motoneurons activity \( m_n \) is assumed to be linear. That is,

\[ m = M \cdot m_n \]  

(10)

Pulleys are included in the model by considering the matrix \( M \) as the product of a rotation matrix \( R_m \) and a matrix \( M_p \) related to the primary position \([8]\),

\[ M = R_m \cdot M_p \]  

(11)

The matrix \( M_p \) represents the relation between motoneuron activity and the muscle torque when the eye is at primary position. The rotation matrix \( R_m \) describes the action of the pulleys by changing the muscles pulling directions. The matrix \( R_m \) is defined as:

\[ R_m(\vec{n},\alpha) = R(\vec{n},k_\theta \cdot \theta) \]  

(12)

The EOMs’ torque vectors and the eye are rotated about the same axis \( \vec{n} \), but the torque vectors are rotated by a fraction \( k_\theta \) of the angle of rotation of the eye; \( k_\theta \) is the pulley coefficient.

It is assumed that the oculomotor system senses the velocity of each pair of EOMs and forms the vector:

\[ \vec{w}_s = [w_{s_x}, w_{s_y}, w_{s_z}] \]  

(13)
This implies that the neuronal component of the oculomotor control system does not consider the changing direction of the torque vectors of each pair of muscles. $\bar{\omega}_s$ does not represent the angular velocity of the eye because the EOMs’ torque vectors actually rotate as a function of eye orientation. The vector $\bar{\omega}_s$ has to be rotated in the same way the muscle torque vectors have been rotated to obtain $\bar{\omega}$. That is,

$$\bar{\omega} = R_m \cdot \bar{\omega}_s \Leftrightarrow \bar{\omega}_i = R_m^{-1} \cdot \bar{\omega} \quad (14)$$

Because rotations about arbitrary axes in three dimensions are not commutative, angular velocity is not the time derivative of the orientation. However, Quaian and Optican [9] showed that the parameter $\Delta$ defined as:

$$\Delta = \left\| R_m^{-1} \cdot \bar{\omega}(t) - \frac{d}{dt} \left[ \theta(t) \cdot \bar{n}(t) \right] \right\|$$

is minimum when $k_\theta = 0.5$. In this case $\Delta = 0.25\%$. This result implies that if the EOMs’ torque vectors rotate about the same axis as the eye does but by half of its angle of rotation, then the integral of $\bar{\omega}_i$ is a good estimation of the eye orientation,

$$\int \bar{\omega}_i dt \cong \theta \cdot \bar{n} \quad (16)$$

Figure 3 shows a block diagram of the oculomotor system model described in this paper. We assume that the neuronal system obtains, from visual information of the retina, the final orientation where the eye is going to be brought to. According to Listing’s law, the torsional component of this orientation will be zero. We use this fact to evaluate the motoneuron activity signal necessary to keep the eye in the steady final orientation. In this situation $\bar{\omega} = 0$. Then, using equations (8) and (10),

$$m_n = M^{-1} K \theta \cdot \left[ 0, n_{z, \text{final}}, n_{\theta, \text{final}} \right]^T$$

(17)

The signal $m_n$ is multiplied by a factor $G$ until the error between the desired final orientation and the estimation of the orientation is less than a predefined threshold. The estimated orientation signal is also used to correct the motoneuron signal $m_n$, if the former deviates out of Listing’s plane.

![Figure 3. Block diagram of the oculomotor system.](image)

4. Simulations and Results
The simulations reported in this paper were performed using Simulink. The parameters used are the same as those used by Schnabolk and Raphan [5].
Figure 4 shows the eye orientation as a function of time during a simulation with $k_\theta = 0.5$. The eye performs a saccadic movement at $t=0$ towards the orientation $(\hat{n}, \theta) = ([0,0,1], 25^\circ)$ and at $t=1$ towards $(\hat{n}, \theta) = ([0,1,0], 30^\circ)$. Note that the eye remains in Listing’s plane during the saccadic movements.

Figure 5 shows the neurological signal of the motoneurons that innervate the extraocular muscles. The pulse durations at $t=0$ and $t=1$ coincide approximately with the saccadic eye movement durations.

Figure 6 shows the eye orientation as a function of time during the same simulation with $k_\theta = 0.8$. Note that the eye orientation deviates in this case out of Listing’s plane. This is due to a coarser estimation of the eye orientation.

5. Conclusions
The inclusion of pulleys in the model simplifies the oculomotor control system.

The positions of the pulleys change with the eye orientations; they determine the pulling direction of the EOMs and, as a consequence, their torque vectors. If the EOMs’ torque vectors rotate about the same axis as the eye does but by half of its angle of rotation, then a good estimation of the eye orientation can be obtained. Thus, the accuracy of the estimation of the eye orientation depends on where the pulleys are located.

This estimation signal is used in a closed loop control system to implement Listing’s law. Not optimally located pulleys produce a coarser estimation signal and deviations of the eye rotation vector $\theta \cdot \hat{n}$ out of Listing’s plane. This argument can be used to explain the transient deviations registered.
by Straumann [7]. The estimated orientation is also used to determine the duration of the phasic component of the motoneurons’ signals sent to the EOMs.

It was assumed that the oculomotor system senses the velocity of each pair of EOMs. This assumption is plausible if we consider that afferent proprioceptive signals from the EOMs influence the eye movement control. There is increasing evidence from experiments in both animals and humans that EOMs afferent signals are important in the oculomotor control and spatial localization [10].

The relation between the neuronal signal and the muscle torque was assumed to be linear and the head, during the eye movements, was considered fixed in space. The model could be improved by including a more realistic muscle model and by considering the head movements.

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