ONE-RELATOR GROUPS WITH TORSION ARE CONJUGACY SEPARABLE

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Abstract. We prove that one-relator groups with torsion are hereditarily conjugacy separable. Our argument is based on a combination of recent results of Dani Wise and the first author. As a corollary we obtain that any quasiconvex subgroup of a one-relator group with torsion is also conjugacy separable.

1. Introduction

Recall that a group $G$ is said to be conjugacy separable if for any two non-conjugate elements $x, y \in G$ there is a homomorphism from $G$ to a finite group $M$ such that the images of $x$ and $y$ are not conjugate in $M$. Conjugacy separability can be restated by saying that each conjugacy class $x^G = \{ gxg^{-1} \mid g \in G \}$ is closed in the profinite topology on $G$. The group $G$ is said to be hereditarily conjugacy separable if every finite index subgroup of $G$ is conjugacy separable. Conjugacy separability is a natural algebraic analogue of solvability of the conjugacy problem in a group and has a number of applications (see, for example, [11]). Any conjugacy separable group is residually finite, but the converse is false. Generally, it may be quite hard to show that a residually finite group is conjugacy separable.

In the recent breakthrough work [16] Dani Wise proved that one-relator groups with torsion possess so-called quasiconvex hierarchy, and groups with such hierarchy are virtually compact special. The class of special (or A-special, in the terminology of [7]) cube complexes was originally introduced by Fredéric Haglund and Dani Wise in [7], as cube complexes in which hyperplanes enjoy certain combinatorial properties. They also showed that a cube complex is special if and only if it admits a combinatorial local isometry to the Salvetti cube complex (see [3]) of some right angled Artin group. It follows that the fundamental group of every special complex $\mathcal{X}$ embeds into some right angled Artin group.

A group $G$ is said to be virtually compact special if $G$ contains a finite index subgroup $P$ such that $P = \pi_1(\mathcal{X})$ for some compact special cube complex $\mathcal{X}$. Thus Wise’s result implies that any one-relator group $G$, with torsion, is (virtually) a subgroup of a right...
angled Artin group. In particular, $G$ is residually finite, which answers an old question of G. Baumslag.

An important fact, established by Haglund and Wise in [7], states that the fundamental group $P$ of a compact special complex is a virtual retract of some finitely generated right angled Artin group. From the work of the first author [11] it follows that $P$ is hereditarily conjugacy separable. This shows that any one-relator group with torsion possesses a hereditarily conjugacy separable subgroup of finite index. Unfortunately, in general conjugacy separability is not stable under passing to finite index overgroups (see [6]). The aim of this note is to prove the following:

**Theorem 1.1.** If $G$ is a one-relator group with torsion then $G$ is hereditarily conjugacy separable.

This theorem answers positively Question 8.69 in Kourovka Notebook [10], posed by C.Y. Tang. This question was also raised in [15] in 1982; its special cases have been considered in [15] and [1].

As a consequence of Theorem 1.1 we also derive

**Corollary 1.2.** If $G$ is a one-relator group with torsion then every quasiconvex subgroup of $G$ is conjugacy separable.

Our proof of Theorem 1.1 uses the above mentioned results of Wise, Haglund-Wise and the first author, and employs the quasiconvex hierarchy for one-relator groups with torsion, that was investigated by Wise in [16].

2. **Background on one-relator groups with torsion**

Let

\[(1) \quad G = \langle S \parallel W^n \rangle \]

be a one-relator group with torsion, where $S$ is a finite alphabet, $n \geq 2$ and $W$ is a cyclically reduced word, which is not a proper power in the free group $F(S)$.

Newman’s spelling theorem [12, Thm. 3] (see also [9, IV.5.5]) implies that every freely reduced word over $S^{\pm 1}$, representing the identity element of $G$, contains a subword of $W^n$ of length strictly greater than $(n - 1)/n$ times the length of $W^n$. Since $(n - 1)/n \geq 1/2$ it follows that the presentation (1) satisfies Dehn’s algorithm ([9, IV.4]); in particular $G$ has a linear Dehn function, and hence it is word hyperbolic. For the background on hyperbolic groups and quasiconvex subgroups the reader is referred to [2].

Another important fact, proved by Newman in [12, Thm. 2] (see also [8, p. 956]), states that centralizers of non-trivial elements in one-relator groups with torsion are cyclic.

Many results about one-relator groups are proved using induction on some complexity depending on the word $W$. In this paper we will use the repetition complexity $RC(W)$ of $W$ employed by Wise in [16]. This is defined as the difference between the length of $W$, and the number of distinct letters from $S$ that occur in $W$. For example, if $S = \{a, b, c\}$ then $RC(ab^2a^{-1}c^{-3}) = 7 - 3 = 4$. 
Start with a one relator-group $G$ given by presentation (1). Recall that a Magnus subgroup $M$ of $G$ is a subgroup generated by a subset $U \subset S$ such that $U$ omits at least one generator appearing in $W$. By the famous Magnus’s Freiheitssatz, $M$ is free and $U$ is its free generating set.

Observe that if $RC(W) = 0$ then every letter appears in $W$ exactly once. In this case, using Tietze transformations, it is easy to see that $G$ is isomorphic to the free product of a free group of rank $|S| - 1$ with the cyclic group of order $n$.

Assume, now, that $RC(W) > 0$. Then, following [16, 18.2], one can let $H = G \ast \langle t \rangle$, and represent $H$ as an HNN-extension of another one-relator group $K = \langle \langle S \parallel W^n \rangle \rangle$, where $|S| < \infty$, $W$ is some cyclically reduced word in the free group $F(S)$, and the associated subgroups are Magnus subgroups $M_1, M_2$ of $K$. In other words, there are subsets $U_1, U_2 \subset S$, each of which omits some letter of $W$, and a bijection $\alpha : U_1 \to U_2$ such that $M_i = \langle U_i \rangle$, $i = 1, 2$, and $H$ has the presentation

\[(2) \quad H = \langle S, t \parallel W^n, tut^{-1} = \alpha(u) \text{ for all } u \in U_1 \rangle.\]

Moreover, in [16, 18.3] Wise shows that one can do this in such a way that $RC(W) < RC(W)$.

**Lemma 2.1.** The group $H$ defined above contains a finite index normal subgroup $L \triangleleft H$ such that $L$ is hereditarily conjugacy separable.

**Proof.** In [16, Ch. 18] Wise shows that $H$ is virtually compact special. By the work of Haglund and Wise from [7, Ch. 6], $H$ contains a finite index subgroup $L$ such that $L$ is a virtual retract of some finitely generated right angled Artin group $A$. Now, a result of the first author [11, Cor. 2.1] implies that $L$ is hereditarily conjugacy separable. \[\square\]

The next statement follows from a combination of results of Wise [16] and Haglund-Wise [7]:

**Lemma 2.2.** Let $P$ be a finite index subgroup of $K$ or $M_1$, or $M_2$. Then $P$ is closed in the profinite topology of $H$.

**Proof.** The group $H$ is hyperbolic as a free product of two hyperbolic groups, and by [16, Lemma 18.8] $K$, $M_1$ and $M_2$ are all quasiconvex subgroups of $H$. Since a finite index subgroup of a quasiconvex subgroup is itself quasiconvex, it follows that $P$ is quasiconvex in $H$.

As we already mentioned above, [16, Cor. 18.3] states that $H$ is virtually compact special. Now we can use [7, Thm. 7.3, Lemma 7.5], which imply that any quasiconvex subgroup of $H$ is separable in $H$. Thus the lemma is proved. \[\square\]

### 3. Some auxiliary facts

First let us specify some notation. If $A$ is a group and $C, D \subseteq A$, then $C^D$ will denote the subset defined by $C^D = \{ dcd^{-1} \mid c \in C, d \in D \}$. If $x \in A$ and $E \subseteq A$ then $C_E(x) = \{ g \in E \mid gx = xg \}$ will denote the centralizer of $x$ in $E$. 
Recall that a subset $C$ of a group $A$ is said to be separable if $C$ is closed in the profinite topology of $A$. This is equivalent to the following property: for every $y \in A \setminus C$ there exist a finite group $Q$ and an epimorphism $\psi : H \to Q$ such that $\psi(y) \notin \psi(C)$ in $Q$.

The following notion is helpful for proving hereditary conjugacy separability of groups. It is similar to [11, Def. 3.1].

**Definition 3.1.** Let $H$ be a group and $x \in H$. We will say that the element $x$ satisfies the Centralizer Condition in $H$ (briefly, $CC_H$), if for every finite index normal subgroup $P \triangleleft H$ there is a finite index normal subgroup $N \triangleleft H$ such that $N \leq P$ and $C_{H/N}(\psi(x)) \subseteq \psi(C_H(x)P)$ in $H/N$, where $\psi : H \to H/N$ is the natural homomorphism.

The condition $CC_H$ defined above is actually quite natural from the viewpoint of the profinite completion $\widehat{H}$ of $H$. Indeed, in [11, Prop. 12.1] it is shown that if $H$ is residually finite then $x \in H$ has $CC_H$ if and only if $C_{\widehat{H}}(x) = \overline{C_H(x)}$, where the right-hand side is the closure of $C_H(x)$ in the profinite completion $\widehat{H}$.

The next two lemmas were proved by the first author in [11, Lemmas 3.4 and 3.7]. The first one shows why the Centralizer Condition is useful, and the second lemma provides a partial converse to the first one.

**Lemma 3.2.** Suppose that $H$ is a group, $H_1 \leq H$ and $x \in H$. Assume that the element $x$ satisfies $CC_H$ and the conjugacy class $x^H$ is separable in $H$. If the double coset $C_H(x)H_1$ is separable in $H$, then the $H_1$-conjugacy class $x^{H_1}$ is also separable in $H$.

**Lemma 3.3.** Let $H$ be a group. Suppose that $x \in H$, $P \triangleleft H$ and $|H : P| < \infty$. If the subset $x^P$ is separable in $H$, then there is a finite index normal subgroup $N \triangleleft H$ such that $N \leq P$ and $C_{H/N}(\psi(x)) \subseteq \psi(C_H(x)P)$ in $H/N$ (where $\psi : H \to H/N$ denotes the natural homomorphism).

The proof of Theorem 1.1 will also use the following two auxiliary statements.

**Lemma 3.4.** Let $A$ be a group and let $C_1, C_2 \leq A$ be isomorphic subgroups with a fixed isomorphism $\varphi : C_1 \to C_2$. Let $B = \langle A, t \mid t^g = \varphi(g) \text{ for all } g \in C_1 \rangle$ be the corresponding HNN-extension of $A$. Suppose that $x, y \in A$ are elements such that $y \notin x^A$ and $x \notin C_i^A$ for $i = 1, 2$. Then $y \notin x^B$ and $C_B(x) = C_A(x)$ in $B$.

**Proof.** Let $\mathcal{T}$ be the Bass-Serre tree associated to the splitting of $B$ as an HNN-extension of $A$. Then $x$ fixes a particular vertex $v$ of $\mathcal{T}$, where the stabilizer $St_B(v)$ of $v$ in $B$ is equal to $A$. The stabilizer of any edge $e$, adjacent to $v$, is $C_i^a$ for some $i \in \{1, 2\}$ and some $a \in A$ (see [14]). Therefore, the assumptions imply that $x$ does not fix any edge of $\mathcal{T}$ adjacent to $v$. Since the fixed point set of an isometry of a tree is connected, it follows that $v$ is the only vertex of $\mathcal{T}$ fixed by $x$.

Arguing by contradiction, suppose that $y \in x^B$, thus there is $b \in B$ such that $y = bxb^{-1}$ in $B$. Then $b \circ v$ is the only vertex of $\mathcal{T}$ fixed by $y$. Since $A = St_B(v)$ and $y \in A$ the latter implies that $b \circ v = v$. Hence $b \in St_B(v) = A$, i.e., $y \in x^A$, contradicting one of the assumptions. Thus $y \notin x^B$, as claimed.
For the final assertion, suppose that \( b \in C_B(x) \), i.e., \( x = bxb^{-1} \). The same argument as above shows that \( b \in A \), hence \( b \in C_A(x) \).

**Lemma 3.5.** Let \( A \) be a group with a free subgroup \( F \leq A \) and let \( g \in A \setminus \{1\} \) be an element of finite order. Suppose that every finite index subgroup of \( F \) is separable in \( A \). Then there exists a finite index normal subgroup \( N \triangleleft A \) such that \( \psi(g) \notin \psi(F)^{A/N} \), where \( \psi : A \to A/N \) denotes the natural epimorphism.

**Proof.** Since every finite index subgroup of \( F \) is separable in \( A \) and \( F \) is residually finite, the assumptions imply that \( A \) is residually finite and the profinite topology of \( A \) induces the full profinite topology on \( F \). Therefore by Lemma 3.2.6 in [13] the closure \( \hat{F} \), of \( F \), in the profinite completion \( \hat{A} \) of \( A \), is naturally isomorphic to the profinite completion \( \hat{F} \) of \( F \). Then in the profinite completion \( \hat{A} \), of \( A \), the claim of the lemma reads as follows: \( g \) is not conjugate to \( \hat{F} \cong \hat{F} \) in \( \hat{A} \). Indeed, \( \hat{F} = \lim \psi_N(F)^{A/N} \), where \( \psi_N : F \to F/N \) denotes the natural epimorphism and the inverse limit is taken over the directed set of all finite index normal subgroups \( N \triangleleft_f A \). Therefore \( \psi_N(g) \notin \psi_N(F)^{A/N} \) for some \( N \triangleleft_f A \) if and only if \( g \notin \hat{F} \). But \( \hat{F} \cong \hat{F} \) is torsion-free by Proposition 22.4.7 in [3], hence the result follows.

### 4. Proofs

**Proof of Theorem 1.1.** Let \( G \) be a one-relator group given by the presentation \( \langle S | \mathcal{W} \rangle \). The result will be proved by induction on \( RC(W) \). If \( RC(W) = 0 \) then \( G \) is isomorphic to the free product \( F_m \ast \mathbb{Z}/n\mathbb{Z} \), where \( m = |S| - 1 \) and \( F_m \) is the free group of rank \( m \). Therefore \( G \) is virtually free and so it is hereditarily conjugacy separable by Dyer’s theorem [4].

Thus we can further assume that \( RC(W) > 0 \). Let \( H \cong G \ast \mathbb{Z} = K, M_1, M_2, U_1, U_2 \) and \( \alpha : M_1 \to M_2 \) be as described in Section 2. Then \( K = \langle S \parallel \mathcal{W} \rangle \), where \( RC(\mathcal{W}) < RC(W) \), and so \( K \) is hereditarily conjugacy separable by induction. Since \( G \) is a retract of \( H \), to prove the theorem it is enough to show that \( H \) is hereditarily conjugacy separable (cf. [11] Lemma 9.5).

Observe that \( H \) is itself a one-relator group with torsion. Therefore, by Newman’s theorem [12] Thm. 2], centralizers of non-trivial elements in \( H \) are cyclic. We also recall that, according to Lemma 2.1, \( H \) contains a finite index normal subgroup \( L \) which is hereditarily conjugacy separable.

Let \( H_1 \leq H \) be an arbitrary finite index subgroup and let \( x \in H \) be an arbitrary element. We will show that the subset \( x^{H_1} \) is separable in \( H \) by considering two different cases.

**Case 1:** \( x \) has infinite order in \( H \). Since \( L \) is hereditarily conjugacy separable, \( L_1 = H_1 \cap L \) is a normal conjugacy separable subgroup of finite index in \( H \). Set \( l = |H : L_1| \). Then \( x^l \in L_1 \setminus \{1\} \) and \( C_H(x^l) \) is infinite cyclic. It follows that for any \( y \in H \setminus x^{H_1} \), \( y^l \notin (x^l)^{H_1} \).

Indeed, if \( x^l = hyh^{-1} \) for some \( h \in H_1 \), then both \( x \) and \( hyh^{-1} \) belong to the infinite
cyclic subgroup $C_H(x^l)$. But in the infinite cyclic group any element can have at most one $l$-th root, thus $x = h y h^{-1}$, contradicting the assumption that $y \notin x^{H_1}$.

Since $L_1$ is conjugacy separable, $(x^l)^{L_1}$ is closed in the profinite topology of $L_1$, and since $|H : L_1| < \infty$ this implies that $(x^l)^{L_1}$ is separable in $H$. Moreover, we can also deduce that the subset $(x^l)^{H_1}$ is separable in $H$, because it equals to a finite union of conjugates of $(x^l)^{L_1}$, as $L_1$ has finite index in $H_1$. Since $y^l \notin (x^l)^{H_1}$, there are a finite group $Q$ and an epimorphism $\psi : H \to Q$ such that $\psi(y^l) \notin \psi((x^l)^{H_1}) = (\psi(x)^l)^{\psi(H_1)}$. Therefore $\psi(y) \notin \psi(x^{H_1})$ in $Q$, as required. Thus $x^{H_1}$ is separable in $H$.

Case 2: $x$ has finite order in $H$. Note that we can assume that $x \neq 1$ in $H$ because otherwise $x^{H_1} = \{1\}$ is separable in $H$ as $H$ is residually finite (by Wise’s work [16] $H$ possesses a finite index subgroup that embeds into a right angled Artin group, and right angled Artin groups are well-known to be residually finite). Now we are going to verify that all the assumptions of Lemma [5, 2] are satisfied.

Claim 1: the conjugacy class $x^H$ is separable in $H$.

By the torsion theorem for HNN-extensions ([9, IV.2.4]), $x \in K^H$. Thus, without loss of generality, we can assume that $x \in K$.

Consider any element $y \in H \setminus x^H$. If $y$ has infinite order then, since $H$ is residually finite, there is a finite group $Q$ and an epimorphism $\psi : H \to Q$, such that the order of $\psi(y)$ in $Q$ is greater than the order of $x$ in $H$ (and, hence, of $\psi(x)$ in $Q$). It follows that $\psi(x)$ is not conjugate to $\psi(y)$ in $Q$.

Thus we can further suppose that $y$ also has finite order in $H$; as before this allows us to assume that $y \in K$. Consequently $y \in K \setminus x^K$, and by conjugacy separability of $K$, we can find a finite index normal subgroup $K_0 \triangleleft K$ such that the images of $x$ and $y$, under the natural epimorphism $K \to K/K_0$, are not conjugate in $K/K_0$.

According to Lemmas [2, 2] and [5, 3] $H$ contains finite index normal subgroups $N_1, N_2 \triangleleft H$ such that the image of $x$ in $H/N_i$ is not conjugate to the image of $M_i$ for $i = 1, 2$. By Lemma [2, 2] $K_0$ is separable in $H$, hence there exists a finite index normal subgroup $N_0 \triangleleft H$ such that $N_0 \cap K \subseteq K_0$. Let $N' \triangleleft H$ and $K_1 \triangleleft K$ denote the finite index normal subgroups of $H$ and $K$ respectively, defined by $N' = N_0 \cap N_1 \cap N_2$ and $K_1 = K \cap N'$.

Let $\xi : K \to K/K_1$ denote the natural epimorphism. Note that the isomorphism $\alpha : M_1 \to M_2$ gives rise to the isomorphism $\tilde{\alpha} : \xi(M_1) \to \xi(M_2)$, defined by $\tilde{\alpha}(\xi(g)) = \xi(\alpha(g))$ for all $g \in M_1$. Indeed, the fact that $\tilde{\alpha}$ is well-defined is essentially due to the construction of $K_1$ as the intersection of $K$ with a normal subgroup $N'$ of $H$, and so $\xi$ is a restriction to $K$ of $\tilde{\xi} : H \to H/N'$. Thus for any $g, h \in M_1$ with $\xi(g) = \xi(h)$ we have

$$\tilde{\alpha}(\xi(g)) = \tilde{\xi}(\alpha(g)) = \tilde{\xi}(t g t^{-1}) = \tilde{\xi}(t) \tilde{\xi}(g) \tilde{\xi}(t^{-1}) = \tilde{\xi}(t h t^{-1}) = \tilde{\xi}(\alpha(h)) = \tilde{\alpha}(\xi(h)).$$

Let $\tilde{H}$ be the HNN-extension of $K/K_1$ with associated subgroups $\tilde{\xi}(M_1)$ and $\tilde{\xi}(M_2)$, defined by

$$\tilde{H} = \langle K/K_1, \tilde{t} \parallel \tilde{t} \tilde{\xi}(u) \tilde{t}^{-1} = \tilde{\alpha}(\xi(u)) \text{ for all } u \in U_1 \rangle.$$
Note that $\bar{H}$ is virtually free since $|K/K_1| < \infty$ (see, for example, [13 II.2.6, Prop. 11]). Clearly $\xi$ extends to a homomorphism $\eta : H \rightarrow \bar{H}$, given by $\eta(t) = t$ and $\eta(g) = \xi(g)$ for all $g \in K$.

Let us show that $\eta(x) = \xi(x)$ is not conjugate to $\eta(y) = \xi(y)$ in $\bar{H}$. Indeed, $\xi(y) \notin \xi(K^{K/K_1})$ because the homomorphism $K \rightarrow K/K_0$ factors through $\xi$ by construction (as $K_1 = K \cap N' \subseteq K \cap N_0 \subseteq K_0$) and the images of $x$ and $y$ are not conjugate in $K/K_0$. On the other hand, since $K_1 \subseteq N_1 \cap N_2$, we have $\xi(x) \notin \xi(M_i^{K/K_1})$ for $i = 1, 2$. Therefore, $\xi(y) \notin \xi(x)^{\bar{H}}$ by Lemma [3.4].

It remains to recall that $\bar{H}$ is conjugacy separable by Dyer’s theorem [3], and so there exist a finite group $Q$ and a homomorphism $\zeta : \bar{H} \rightarrow Q$ such that $\zeta(\eta(y)) \notin \zeta(\eta(x))^Q$ in $Q$. Hence the homomorphism $\psi = \zeta \circ \eta : H \rightarrow Q$ distinguishes the conjugacy classes of $x$ and $y$, as required. Thus we have shown that $x^R$ is separable in $H$.

**Claim II:** $x$ satisfies the Centralizer Condition $CC_H$ from Definition [3.1].

This will be proved similarly to Claim I. As above, without loss of generality, we can assume that $x \in K$. Consider any finite index normal subgroup $P < H$ and let $R = K \cap P$.

Since $K$ is hereditarily conjugacy separable by induction, the finite index subgroup $E = R \langle x \rangle \leq K$ is conjugacy separable. Hence the subset $x^E = x^R$ is separable in $E$. And since $|K : E| < \infty$ we see that $x^R$ is separable in $K$. Therefore we can apply Lemma [3.3] to find a finite index normal subgroup $K_0 < K$ such that $K_0 \leq R$ and the centralizer of the image of $x$ in $K/K_0$ is contained in the image of $C_K(x)R$ in $K/K_0$.

Arguing as in Claim I, we can choose finite index normal subgroups $N_0, N_1, N_2 < H$ such that $K \cap N_0 \subseteq K_0$, and the image of $x$ is not conjugate to the image of $M_i$ in $H/N_i$ for $i = 1, 2$. Set $N' = N_0 \cap N_1 \cap N_2$ and $K_1 = K \cap N'$. Similarly to Claim I, the homomorphism $\xi : K \rightarrow K/K_1$ extends to a homomorphism $\eta : H \rightarrow \bar{H}$, where $\bar{H}$ is an HNN-extension of $K$ with associated subgroups $\xi(M_1)$ and $\xi(M_2)$.

Denote $\bar{x} = \eta(x) = \xi(x) \in K/K_1 \leq \bar{H}$. As before, since $K_1 \leq N_i$, we have that $\bar{x} \notin \xi(M_i^{K/K_1})$, $i = 1, 2$, and so we can use Lemma [3.4] to conclude that $C_{\bar{H}}(\bar{x}) = C_{K/K_1}(\bar{x})$. Recall that $K_1 \leq K_0$, hence the epimorphism from $K$ to $K/K_0$ factors through $\xi$. Therefore in $\bar{H}$ we have

$$C_{\bar{H}}(\bar{x}) = C_{K/K_1}(\bar{x}) \subseteq \xi(C_K(x)R_{K_0}) = \xi(C_K(x)R) \subseteq \eta(C_H(x)P),$$

because $K_0 \leq R \leq P$ by construction.

Once again, $\bar{H}$ is virtually free and so is any subgroup of it. Therefore $P(\bar{x}) \leq \bar{H}$ is conjugacy separable by Dyer’s theorem [3], where $\bar{P} = \eta(P)$ is a finite index normal subgroup of $\bar{H}$. As above this yields that the subset $\bar{x}^P(\bar{x}) = \bar{x}^P$ is separable in $\bar{H}$. By Lemma [3.3] there exists a finite index normal subgroup $\bar{N} < \bar{H}$ such that $\bar{N} \leq \bar{P}$ and

$$C_{\bar{H}/\bar{N}}(\zeta(\bar{x})) \subseteq \zeta(C_{\bar{H}}(\bar{x})\bar{P}),$$

where $\zeta : \bar{H} \rightarrow \bar{H}/\bar{N}$ is the natural epimorphism.

Let $N = \eta^{-1}(\bar{N})$ be the full preimage of $\bar{N}$ in $H$, and let $\psi : H \rightarrow H/N$ be the natural homomorphism. Then $\psi = \zeta \circ \eta$ and $\bar{H}/\bar{N}$ can be identified with $H/N$. A combination
of (1) and (3) gives the following inclusion in $H/N$:

\[ C_{H/N}(\psi(x)) \subseteq \zeta(C_H(x)P) \subseteq \zeta(\eta(C_H(x)P)\overline{P}) = \psi(C_H(x)P). \]

To finish the proof of Claim II it remains to show that $N \leq P$. Since $\eta(N) = \overline{N} \leq \overline{P} = \eta(P)$, it is enough to prove that $\ker \eta \leq P$. To this end, observe that $\ker \eta$ is the normal closure of $K_1 = \ker \xi$ in $H$ (this easily follows from the universal property of HNN-extensions and is left as an exercise for the reader). Since $K_1 \leq K_0 \leq R \leq P$ and $P \triangleleft H$, we see that the normal closure of $K_1$ in $H$ must also be contained in $P$. Thus $\ker \eta \leq P$, implying that $N \leq P$, which finishes the proof of Claim II.

In order to apply Lemma 3.2 we should also note that the subset $C_H(x)H_1$ splits in a finite union of left cosets modulo $H_1$ in $H$ because $[H : H_1] < \infty$, and hence this subset is separable in $H$. In view of Claims I, II we see that all of the assumptions of Lemma 3.2 are satisfied. Therefore $x^{H_1}$ is separable in $H$, and the consideration of Case 2 is finished.

Thus we have shown that $x^{H_1}$ is separable in $H$ for all $x \in H$ and any finite index subgroup $H_1 \leq H$. Since the profinite topology of a subgroup is finer than the topology induced from the ambient group, we can conclude that $x^{H_1}$ is separable in $H_1$ whenever $x \in H_1$. Consequently $H_1$ is conjugacy separable. Since $H_1$ was chosen as an arbitrary finite index subgroup of $H$, we see that $H$ is hereditarily conjugacy separable.

\[ \square \]

**Proof of Corollary 4.2.** Let $H \leq G$ be a quasiconvex subgroup. By Newman’s theorem [12] Thm. 2, for any $x \in H \setminus \{1\}$ there is $g \in G$ such that $C_G(x) = \langle g \rangle$. Hence $x = g^k \in H$ for some $k \in \mathbb{N}$ and so the subset $C_G(x)H$ splits in a finite union of left cosets modulo $H$. Now, since $G$ is virtually compact special by [10] Cor. 18.3, quasiconvex subgroups are separable in $G$ by [7] Thm. 7.3, Lemma 7.5. It follows that $H$ and, hence, $C_G(x)H$ are separable in $G$, for an arbitrary $x \in H$ (if $x = 1$ then $C_G(x)H = G$).

By Theorem 14.1 $G$ is hereditarily conjugacy separable and so every element $x \in G$ satisfies $CC_G$ (see [11] Prop. 3.2]). Therefore we can apply Lemma 5.2 to conclude that $x^H$ is separable in $G$ (and, hence, in $H$). Thus $H$ is conjugacy separable, as claimed. \[ \square \]

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