Thermoelastic non-stationary fields in a rigidly fixed plate

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Abstract. In this article, a new closed solution of the axisymmetric dynamic problem of the theory of thermoelasticity is constructed for a rigidly fixed circular isotropic plate in the case of temperature changes on its face surfaces. The mathematical formulation of the problem includes linear equations of motion and thermoelasticity in the spatial formulation with respect to the components of the displacement vector, as well as the function of temperature change. The study of non-self-adjoint equations is carried out in an unrelated statement. Initially, we consider the initial boundary value problem of thermoelasticity without taking into account the deformation of the plate, and at the next stage, we study the problem of elasticity theory under the action of a given (defined) temperature change function. To solve the problems, we use a mathematical apparatus for separating variables in the form of finite integral transformations: Fourier, Hankel, and generalized integral transformation (CIP). In this case, at each stage of the study, the procedure is performed to bring the boundary conditions to the form that allows you to apply the appropriate transformation.

1. Introduction

Uneven non-stationary heating of structures for various purposes leads to the thermal deformations and stresses occurrence, which must be taken into account in the case of a comprehensive analysis of the strength characteristics of elastic systems of finite dimensions [1, 2]. At present, various theories of thermoelasticity have been developed (CTE, GHI – GHIll, LS) [3, 4], which solves this problem with varying accuracy degrees.

The mathematical formulation of the considered initial-boundary value problems in a linear three-dimensional formulation includes coupled non-self-adjoint differential equations of motion and heat conduction. As a rule, this system of differential equations is considered in an unrelated setting [5-10]. In this case, when an external non-stationary heat load acts on the elastic system, the effect of the rate of change in the volume of the body on the temperature field is not taken into account.

In a related formulation, closed solutions of dynamic problems of thermoelasticity are presented in a few works.

In particular, studies [11-13] were carried out for a finite isotropic cylinder with membrane fixing of its end surfaces. In [11], using the generalized method of finite integral transformations [14], and [12, 13] - the biorthogonal integral transform [15, 16]. Research [17, 18] was carried out using hyperbolic (GHIll, GHIll) theories of thermoelasticity and helps to analyze the frequency equations, as well as the forms of harmonic waves in an infinite cylindrical waveguide.

In this work, a rigidly fixed round isotropic plate is investigated. The case of the action on the upper and lower surfaces of an unsteady axisymmetric temperature load (boundary conditions of the
1st kind) is considered. The numerical results of calculating this problem in an unconnected formulation [19] allow us to conclude that the inertia forces of an elastic system affect its stress-strain state only in very thin structures \((h^*/b \leq 0.01, h^*, b - \text{thickness and radius of the plate})\) under the action of a high-frequency load. Taking into account these results, the inertia forces are not taken into account when solving the system of non-self-adjoint differential equations of the classical (CTE) theory of thermoelasticity, i.e. the constraint is used for the considered constructions \(h^*/b \leq 0.01\).

The constructed solution of the coupled problem in a three-dimensional formulation makes it possible to take into account the effect of the rate of change in its volume (rate of dilatation) on the nature of the distribution of the temperature field and the stress-strain.

2. Materials and methods

Let a round rigidly fixed plate occupy the region \(\Omega: \{0 \leq r_e \leq b, 0 \leq \theta \leq 2\pi, 0 \leq z_e \leq h^*\}\) in the cylindrical coordinate system \((r_e, \theta, z_e)\). On the upper and lower surfaces, the temperature is set, the value of which depends on the radial coordinate \(r_e\) and time \(t_e\): at \(z_e = 0\) \(\omega_1^* (r_e, t_e)\), at \(z_e = h^*\) \(\omega_2^* (r_e, t_e)\) (figure 1).

![Figure 1. Calculation scheme.](image)

The mathematical formulation of the initial boundary value problem under consideration in a dimensionless form includes a system of linear axisymmetric non-self-adjoint differential equations for the components of the displacement vector \(U(r, z, t), W(r, z, t)\) and temperature \(\Theta(r, z, t)\):

\[
\frac{\partial}{\partial r} \nabla U + \frac{\partial^2 U}{\partial z^2} + a_1 \frac{\partial^2 W}{\partial r \partial z} - \frac{\partial \Theta}{\partial r} = 0
\]

\[
a_1 \frac{\partial W}{\partial r} + \frac{\partial^2 W}{\partial z^2} + a_1 \frac{\partial U}{\partial r} - \frac{\partial \Theta}{\partial z} = 0
\]

\[
\frac{\partial \Theta}{\partial r} + \frac{\partial^2 \Theta}{\partial z^2} - \frac{\partial U}{\partial t} + \frac{\partial W}{\partial t} = 0
\]

boundary conditions:

\[
r = 0,1 \quad \{U(0,z,t),W(0,z,t),\Theta(0,z,t)\} < \infty, \quad \frac{\partial \Theta}{\partial r_{n-1}} = 0 \cdot \{U(1,z,t),W(1,z,t)\} = 0;
\]

\[
z = 0,h \quad \frac{\nu}{1-\nu} \nabla U + \frac{\partial W}{\partial z} = \{\omega_1, \omega_2\}, \quad \frac{\partial W}{\partial \xi} + \frac{\partial U}{\partial \xi} = 0, \quad \Theta(r, z, t)_{|z=0} = \{\omega_1, \omega_2\};
\]

and initial conditions:

\[
t = 0 \quad \Theta(r, z, 0) = 0.
\]

where \(\{U, W, r, z, h\} = \{U^*, W^*, r, z, h^*\}/b\), \(\{\Theta, \omega_1, \omega_2\} = \{\Theta^*, \omega_1^* - T_0, \omega_2^* - T_0\}\), \(a_1 = a_2 (1-2\nu)\), \(a_2 = 0.5(1-\nu)^{-1}\), \(a_1 = \frac{\nu^2 (1+\nu)(1-2\nu)}{E(1-\nu)\xi^3} T_0\), \(a_1 = \frac{\nu^2}{1-\nu} a_1\), \(t = t_e \frac{\Lambda}{b^2 \xi^2}\), \(U^*(r, z, t), W^*(r, z, t), \Theta^*(r, z, t) - 2\)
displacement vector components and temperature increment in dimensional form: \( \Theta^* = T - T_0 \), \( \Theta^*, T, T_0 \) — current temperature and temperature of the initial state of the body, in which there are no mechanical stresses; \( E, \nu \) — elastic modulus and Poisson's ratio of the material; \( \alpha, c, \Lambda \) — coefficients of linear thermal expansion, volumetric heat capacity and thermal conductivity of the material.

3. Results and discussion

The initial boundary value problem (1) – (3) is solved by the method of integral transformations, using successively the Hankel transform [20] with finite limits in the variable and the degenerate biorthogonal finite transformation [15] in the coordinate \( z \). At each stage of the solution, the procedure of standardization of the corresponding boundary conditions is carried out [21].

Transformants \( R(n, z, t) \), \( G(\lambda_n, n, t) \) and the inversion formulas of the corresponding transformations have the following form:

\[
R(n, z, t) = \int_0^1 N(r, z, t) P(n, r) dr, \quad N(r, z, t) = \sum_{n=0}^{\infty} \Omega_n^{-1} R_P;
\]

\[
G(\lambda_n, n, t) = \int_0^1 R(n, z, t) Y(\lambda_n, z) dz, \quad R = \sum_{i=1}^{\infty} G \|K_i\|^{-1};
\]

where \( N(r, z, t) = [U(r, z, t), W(r, z, t), \Theta(r, z, t)]^T \), \( P = [s_{np}] \) — 3-order diagonal matrix,
\[
(s_{11} = J_0(j_1 r), s_{22} = J_0(j_2 r), s_{12} = J_0(j_1 r)), \quad Y(\lambda_n, z) = [0,0,K_1(\lambda_n, z)]^T, \quad H(\lambda_n, z) = [N_1(\lambda_n, z), N_2(\lambda_n, z), N_3(\lambda_n, z)]^T;
\]

\( K_1, N_1, N_2, N_3 \) — components of the vector-function of biorthogonal transformations; \( \Omega_n, \|K_n\| \) — square of the norm of transformation kernels; \( j_n, \lambda_n \) — eigenvalues (\( n = 0, 1, 2, ..., i = 1, 2, 3, ... \)).

As a result, we obtain an expression for the functions \( U(r, z, t), W(r, z, t), \Theta(r, z, t) \) in the form of spectral expansions:

\[
N(r, z, t) = \sum_{n=0}^{\infty} \Omega_n^{-1} P(n, r) \left[ F_i + \sum_{i=1}^{\infty} G(\lambda_n, n, t) H(\lambda_n, z) \|K_i\|^{-1} \right],
\]

where \( F_i \) — matrix is a column of standardizing functions.

The algorithm for solving the initial – boundary value problem of thermoelasticity (1) - (3) is described in detail in [22].

4. Conclusions

As an example, we consider a rigidly fixed round plate(\( b = 1 \)m) made of steel, which has the following physical and mechanical characteristics of the material: \( E = 2 \times 10^5 \) Pa, \( \Lambda = 50 \), W/(m\(^3\)K), \( v = 0.28 \), \( c_v = 3.8 \times 10^6 \) J/(m\(^3\)K), \( \alpha_v = 1.2 \times 10^{-5} \) 1/°K.

The case of the action of (\( z = 0 \)) a temperature load on the upper front surface in the form of:

\[
\omega_1'(r, t) = (1 - r) T_{\text{max}} \sin \left( \frac{\pi - r_{\min}}{2 T_{\text{max}}} \right) H(t - t_{\text{max}}) + H(t - t_{\text{max}}), \quad \omega_2'(r, t) = 0,
\]

where \( H(t) \) — the single function of Heaviside (\( H(\tilde{t}) = 1 \) at \( \tilde{t} \geq 0 \), \( H(\tilde{t}) = 0 \) at \( \tilde{t} < 0 \)), \( T_{\text{max}} - T_0 = T_{\text{max}}' - T_0 \), \( T_{\text{max}}', T_{\text{max}} \) — maximum value of the external temperature effect and the corresponding time in the dimensional form (\( T_{\text{max}}' = 373 \)°K (100°C), \( T_0 = 293 \)°K, (20°C)).

Figures 2-4 show graphs of temperature changes \( \Phi'(0, h/2, t) \), components of the vector of displacements \( U'(0.5, z, t), W'(0, z, t) \) in time and axial coordinate with allowance for (solid line) and
also without (dashed line) for the coupling of thermoelastic fields \( t_{\text{max}} = 10 \), \( t_{\text{max}} = \frac{\Lambda}{b^2 c^2} t_{\text{max}} = 1.3 \times 10^{-4} \).

The temperature field and stress-strain state are analyzed for plates with a thickness \( h = 0.1, 0.2 \).

**Figure 2.** Graphs of temperature \( \Theta^t(0, h) \) changes over time \( t \).

**Figure 3.** Graphs of the \( W^t(0, z, t) \) change in the height of the plate \( 1-t = t_{\text{max}}, 2-t = 5t_{\text{max}} \).

**Figure 4.** Graphs of change \( U^t(0.5, z, t) \) along the height of the plate \( 1-t = t_{\text{max}}, 3-t = 100t_{\text{max}} \).

Analysis of the calculation results allows us to draw the following conclusions:

- the connectivity of thermoelastic fields at a given temperature load (7) leads to a slower heating of the plate over time (figure 2). In this case, the rate of change in the volume of the body, which is taken into account in the heat conduction equation (1), has a significant effect at the first stage of the study of the temperature regime, when \( t_{\text{max}} < t < 10t_{\text{max}} \). In the future, this effect is not observed;
• a decrease in the rate of temperature change inside the plate due to its dilation gives an increase in the gradients \( \frac{\partial \sigma}{\partial r}, \frac{\partial \sigma}{\partial z} \) that are used in the initial differential equations of thermoelasticity (1). As a result, there is an increase in the numerical values of the axial component of the displacement vector (figure 3, graphs 1, 2);
• at a given temperature load, the coupling of thermoelastic fields decreases over time (figure 3). In addition, as a result of warming up the structure, an increase in displacements is observed (figure 3, 4), and with a steady temperature regime on the lower surface there are no radial displacements (figure 4, graph 3);
• the linear nature of the change in the radial component of the displacement vector along the height of the plate, allows us to conclude that when solving thermoelasticity problems for homogeneous elastic systems with the help of applied theories, it is possible to use the kinematic hypothesis of plane sections;
• the numerical values of radial displacements \( U^{r}(0.5, z, t) \) at a steady temperature regime do not depend on the thickness of the plate (figure 4, graph 3).

Figure 5 shows graphs of changes in normal mechanical stresses along \( \sigma_{r}(r, 0, t) \) the radial coordinate at different points in time taking into account (solid line), as well as disregarding (dashed line) the connectivity of thermoelastic fields.

![Graphs of changes in normal mechanical stresses](image)

Figure 5. Graphs of changes \( \sigma_{r}(r, 0, t) \) in the radial coordinate \((h=0.1)\).

Here you can draw a conclusion:

• the greatest influence of the field coupling on the stress tensor component \( \sigma_{r}(r, 0, t) \) is observed at \( t=t_{\text{max}} \). Subsequently, this effect sharply decreases (figure 5);
• in the process of warming up \( t>t_{\text{max}} \) the structure, when there is a decrease in normal stresses \( \sigma_{r}(r, 0, t) \). At \( t=t_{\text{max}} \) this component of the stress tensor is higher at a faster temperature loading of the plate;
• it should be noted that when the condition is met \( \omega_{r}(r, t)=0 \) the lower face plane \((z=h)\) is a neutral surface \( \sigma_{r}(r, h, t)=0 \), since \( \frac{\partial W}{\partial z}=0 \) (figure 3) and (figure 4).

In conclusion, we can conclude that when calculating structures of finite dimensions in the case of a high-speed thermal load, the coupling of temperature and elastic fields has a significant effect on its stress-strain state. Moreover, this feature is more pronounced in thin plates.
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References
[1] Podstrigach Y S 1984 Thermal elasticity of bodies of non-uniform structure (M.: Science)
[2] Novatsky V 1970 Dynamic problems of thermoelasticity (M.: World)
[3] Radaev Y N and Taranova M V 2011 Wave numbers of thermoelastic waves in a waveguide with heat exchange on the side wall Bulletin of the Samara state University Ser. Phys.-Math. Sci. 2(23) 53-61
[4] Shashkov A G, Bubnov V A and Yanovsky S Y 2004 Wave phenomena of thermal conductivity System-structural approach 2 (M.: Unitorialal URSS)
[5] Kovalenko A D 1965 Introduction to thermoelasticity (Kiev: Science Duma)
[6] Sargsyan S H 2013 Mathematical Model of Micropolar Thermo-Elasticity of Thin Shells J. of Thermal Stresses 36(11) 1200-16
[7] Verma K L 2008 Thermoelastic waves in anisotropic plates using normal mode expansion method World Academy of Science, Engineering and Technology (37) 573-80
[8] Zhornik A I, Zhornik V A and Savochka PA 2012 On a thermoelasticity problem for a solid cylinder News of the Southern Federal University. Techn. Sci. 9(1) 63-9
[9] Makarova I S 2012 Solution of an unrelated thermoelasticity problem with boundary conditions of the first kind Bulletin of the Samara state technical university. Ser. Phys.-Math. Sci. 28(3) 191-5
[10] Harmatij H, Król M and Popovycz V 2013 Quasi-Static Problem of Thermoelasticity for Thermosensitive Infinite Circular Cylinder of Complex Heat Exchange Advances in Pure Mathematics (3) 430-7
[11] Senitsky Yu E 1982 To the solution of the coupled dynamic thermoelasticity problem for an infinite cylinder and sphere Applied Mechanics of the AS USSR 18(6) 34-41
[12] Lychev S A 2003 The related dynamical problem of thermoelasticity for a finite cylinder Bulletin of the Samara State University 4(30) 112-24
[13] Lychev S A, Manzhirov A V and Uber S V 2010 Closed solutions of boundary value problems of coupled thermoelasticity J. of Russian Academy of Sciences 4 138-54
[14] Senitsky Yu E 1991 Multicomponent generalized finite integral transformation and its application to nonstationary problems of mechanics Proceedings of high schools 4 57-63
[15] Senitskiy Y E 1996 Biorthogonal multi-component finite integral transform and its application to boundary value problems of mechanics Proceedings of universities Mathematika 8 71-81
[16] Lychev S A and Senitsky Yu E 2002 Nonsymmetric integral transformations and their applications to problems of viscoelasticity Bulletin of Samara un-ta. Natur. Sci. Series 16-38
[17] Kovalev V A, Radaev Yu N and Semenov D A 2009 Related dynamic problems of hyperbolic thermoelasticity Bulletin of the Saratov Uni. Ser. Math-s. Mechan. Inform. 4(2) 94-127
[18] Kovalev V A, Radaev Yu N and Revinsky R A 2011 Passage of a generalized GHIII thermoelastic wave through a waveguide with a heat-permeable wall Bulletin of the Saratov University, New.ser. Ser. Math. Mechan. Inform. 11(1) 59-70
[19] Shlyakhin D A and Dauletmuratova Zh M 2018 Nonstationary axisymmetric problem of thermo-elasticity for a rigidly fixed circular plate Eng. J.: Science and Innovation 5(77) doi:10.1086/1757-899X/1181/1/012026
[20] Sneddon I N 1955 Fourier Transforms (Moscow: Foreign Literature Publishing House)
[21] Butkovskiy A G 1979 Characteristics of systems with distributed parameters (M.: Science)
[22] Shlyakhin D A and Dauletmuratova Zh M 2019 Nonstationary coupled axisymmetric thermoelasticity problem for a rigidly fixed round plate Bulletin of the PNRPU. Mechan. 4 191-200