Neyman-Pearson classification: parametrics and power enhancement

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Summary.
The Neyman-Pearson (NP) paradigm in binary classification seeks classifiers that achieve a minimal type II error while enforcing the prioritized type I error under some user-specified level. This paradigm serves naturally in applications such as severe disease diagnosis and spam detection, where people have clear priorities over the two error types. Despite recent advances in NP classification, the NP oracle inequalities, a core theoretical criterion to evaluate classifiers under the NP paradigm, were established only for classifiers based on nonparametric assumptions with bounded feature support. In this work, we conquer the challenges arisen from unbounded feature support in parametric settings and develop NP classification theory and methodology under these settings. Concretely, we propose a new parametric NP classifier NP-sLDA which satisfies the NP oracle inequalities. Furthermore, we construct an adaptive sample splitting scheme that can be applied universally to existing NP classifiers and this adaptive strategy greatly enhances the power of these classifiers. Through extensive numerical experiments and real data studies, we demonstrate the competence of NP-sLDA and the new sample splitting scheme.

Keywords: classification, asymmetric error, Neyman-Pearson (NP) paradigm, NP oracle inequalities, sparse linear discriminant analysis, NP umbrella algorithm, unbounded feature support, adaptive splitting

1. Introduction

Classification aims to predict discrete outcomes (i.e., class labels) for new observations, using algorithms trained on labeled data. It is one of the most studied machine learning problems with applications including automatic disease diagnosis, email spam filters, and image classification. Binary classification, where the outcomes belong to one of two classes and the class labels are usually coded as \{0, 1\} (or \{-1, 1\} or \{1, 2\}), is the most common type. Most binary classifiers are constructed to minimize the expected classification error (i.e., risk), which is a weighted sum of type I and type II errors. Here, type I error is defined as the conditional probability of misclassifying a class 0 observation as a class 1 observation, and type II error is the conditional probability of misclassifying a class 1 observation as a class 0 observation. In the following, we refer to this paradigm as the classical classification paradigm.

Along this line, numerous methods have been proposed, including linear discriminant analysis (LDA) in both low dimensions and high dimensions (Guo et al., 2005; Cai and Liu, 2011; Shao et al., 2011; Witten and Tibshirani, 2012; Fan et al., 2012; Mai et al., 2012), logistic regression, support vector machine (SVM) (Vapnik, 1999), random forest (Breiman, 2001), among others.
In contrast, the Neyman-Pearson (NP) classification paradigm (Cannon et al., 2002; Scott and Nowak, 2005; Rigollet and Tong, 2011; Tong, 2013; Zhao et al., 2016) was developed to seek a classifier that minimizes the type II error while maintaining the type I error below a user-specified level \( \alpha \), usually a small value (e.g., 5%). We call this target classifier the NP oracle classifier. The NP paradigm is appropriate in applications such as cancer diagnosis, where a type I error (i.e., misdiagnosing a cancer patient to be healthy) has more severe consequences than a type II error (i.e., misdiagnosing a healthy patient as with cancer). The latter incurs extra medical costs and patients’ anxiety but will not result in tragic loss of life, so it is appropriate to have type I error control as the priority. Previous NP classification literature use both empirical risk minimization (ERM) (Cannon et al., 2002; Casasent and Chen, 2003; Scott, 2005; Scott and Nowak, 2005; Han et al., 2008; Rigollet and Tong, 2011) and plug-in approaches (Tong, 2013; Zhao et al., 2016), and its genetic application is suggested in Li and Tong (2016). More recently, Tong et al. (2018) took a different route, and proposed an NP umbrella algorithm that adapts scoring-type classification algorithms (e.g., logistic regression, support vector machines, random forest, etc.) to the NP paradigm, by setting proper thresholds for the classification scores. As argued intensively in Tong et al. (2018), to construct a classifier with type I error bounded from above by \( \alpha \) with high probability, it is not correct to just tune the empirical type I error to (no more than) \( \alpha \); instead, careful application of order statistics is the key. Cost-sensitive learning, which assigns different costs as weights of type I and type II errors (Elkan, 2001; Zadrozny et al., 2003) is a popular paradigm to address asymmetric errors. This approach has merits and many practical values, but when there is no consensus to assign costs to errors, or in applications such as medical diagnosis, where it is morally unacceptable to do a cost and benefit analysis, the NP paradigm is a more natural choice.

In literature, NP oracle inequalities, a core theoretical criterion to evaluate classifiers under the NP paradigm, were only established for nonparametric classifiers (Tong, 2013; Zhao et al., 2016). This work is the first to establish NP oracle inequalities under parametric settings. Concretely, we construct an NP classifier \( \text{NP-sLDA} \) based on the canonical linear discriminant analysis (LDA) model and show that it satisfies the NP oracle inequalities. The second major contribution is that we design an adaptive sample splitting scheme that can be applied universally to existing NP classifiers. This adaptive strategy greatly enhances the power (i.e., reduces type II error) and therefore raises the practicality of the NP algorithms. Through extensive numerical experiments and real data studies, we demonstrate the competence of NP-sLDA and the new sample splitting scheme.

The rest of this paper is organized as follows. Section 2 introduces the notations and model setup. Section 3 constructs the NP-sLDA classifier. Section 4 formulates new theoretical conditions for parametric models with unbounded support, and derives oracle inequalities for NP-sLDA in particular. Section 5 describes the new data-adaptive sample splitting scheme. Numerical results are presented in Section 6, followed by a short discussion in Section 7. Main proofs are relegated to the Appendix.

2. Notations and model setup

A few common notations are introduced to facilitate our discussion. Let \((X, Y)\) be a random pair where \(X \in \mathcal{X} \subset \mathbb{R}^d\) is a \(d\)-dimensional vector of features and \(Y \in \{0, 1\}\) indicates \(X\)’s class label. Denote respectively by \(\mathbb{P}\) and \(\mathbb{E}\) generic probability distribution and expectation. A classifier \(\phi : \mathcal{X} \to \{0, 1\}\) is a data-dependent mapping from \(\mathcal{X}\) to \(\{0, 1\}\) that assigns \(X\) to one of the classes. The classification error of \(\phi\) is \(R(\phi) = \mathbb{E}\mathbb{I}\{\phi(X) \neq Y\} = \mathbb{P}\{\phi(X) \neq Y\}\), where \(\mathbb{I}(\cdot)\) denotes the indicator function. By the law of total probability, \(R(\phi)\) can be decomposed into a weighted average of type I error \(R_0(\phi) = \)
\( \mathbb{P}\{ \phi(X) \neq Y | Y = 0 \} \) and type II error \( R_1(\phi) = \mathbb{P}\{ \phi(X) \neq Y | Y = 1 \} \) as
\[
R(\phi) = \pi_0 R_0(\phi) + \cdots \text{general } m_1 \times m_2 \text{ matrix } M, \|M\|_{\infty} = \max_{i=1, \ldots, m_1} \sum_{j=1}^{m_2} |M_{ij}|, \text{ and } \|M\| \text{ denotes the operator norm. For a vector } b, \|b\|_{\infty} =
\]

where \( \pi_0 = \mathbb{P}(Y = 0) \) and \( \pi_1 = \mathbb{P}(Y = 1) \). While the classical paradigm minimizes \( R(\cdot) \), the Neyman-Pearson (NP) paradigm seeks to minimize \( R_1 \) while controlling \( R_0 \) under a user-specified level \( \alpha \). The NP oracle classifier is thus
\[
\phi^*_\alpha \in \arg \min_{R_0(\phi) \leq \alpha} R_1(\phi),
\]
where the significance level \( \alpha \) reflects the level of conservativeness towards type I error.

In this paper, we assume that \((X|Y = 0)\) and \((X|Y = 1)\) follow multivariate Gaussian distributions with a common covariance matrix. That is, their probability density functions are comparable to or larger than the sample size. Therefore, we choose the LDA model as our first attempt under parametric settings and demonstrate theoretical analysis via this simple but powerful model.

It is well known that the Bayes classifier (i.e., oracle classifier) of the classical paradigm is \( \phi^*(x) = 1 I(\eta(x) > 1/2) \), where \( \eta(x) = \mathbb{E}(Y|X = x) = \mathbb{P}(Y = 1|X = x) \) is the regression function. Since
\[
\eta(x) = \frac{\pi_1 \cdot f_1(x)/f_0(x)}{\pi_1 \cdot f_1(x)/f_0(x) + \pi_0},
\]
the oracle classifier can be written alternatively as \( 1 I(f_1(x)/f_0(x) > \pi_0/\pi_1) \). When \( f_1 \) and \( f_0 \) follow the LDA model, the oracle classifier of the classical paradigm is
\[
\phi^*(x) = 1 \{ (x - \mu_\alpha)^\top \Sigma_\alpha^{-1} \mu_\alpha + \log \frac{\pi_1}{\pi_0} > 0 \} = 1 \{ (\Sigma_\alpha^{-1} \mu_\alpha)^\top x > \mu_\alpha^\top \Sigma_\alpha^{-1} \mu_\alpha - \log \frac{\pi_1}{\pi_0} \},
\]
where \( \mu_\alpha = \frac{1}{2}(\mu_0 + \mu_1), \mu_\alpha = \mu_1 - \mu_0 \), and \((\cdot)^\top\) denotes the transpose of a vector. In contrast, motivated by the famous Neyman-Pearson Lemma in hypothesis testing (attached in the Appendix for readers’ convenience), the NP oracle classifier is
\[
\phi^*_\alpha(x) = 1 \left\{ \frac{f_1(x)}{f_0(x)} > C_\alpha \right\},
\]
for some threshold \( C_\alpha \) such that \( P_0\{ f_1(X)/f_0(X) > C_\alpha \} \leq \alpha \) and \( P_0\{ f_1(X)/f_0(X) \geq C_\alpha \} \geq \alpha \), where \( P_0 \) is the conditional probability distribution of \( X \) given \( Y = 0 \) (\( P_1 \) is defined similarly).

Under the LDA assumption, the NP oracle classifier is \( \phi^*_\alpha(x) = 1 ((\Sigma_\alpha^{-1} \mu_\alpha)^\top x > C_{\alpha^{**}}) \), where \( C_{\alpha^{**}} = \log C_\alpha + \mu_\alpha^\top \Sigma_\alpha^{-1} \mu_\alpha \). Denote by \( \beta^{Bayes} = \Sigma_\alpha^{-1} \mu_\alpha \) and \( s^*(x) = (\Sigma_\alpha^{-1} \mu_\alpha)^\top x = (\beta^{Bayes})^\top x \), then the NP oracle classifier (4) can be written as
\[
\phi^*_\alpha(x) = 1 (s^*(x) > C_{\alpha^{**}}).
\]
We will construct a plug-in version of \( \phi^*_\alpha \) in the next section.

Other mathematical notations we use are introduced as follows. For a general \( m_1 \times m_2 \) matrix \( M, \|M\|_{\infty} = \max_{i=1, \ldots, m_1} \sum_{j=1}^{m_2} |M_{ij}|, \) and \( \|M\| \) denotes the operator norm. For a vector \( b, \|b\|_{\infty} =
\]
max_j |b_j|, |b|_{\text{min}} = \min_j |b_j|, and |b| denotes the L_2 norm. Let A = \{j : (\Sigma^{-1}\mu_d)_j \neq 0\}, and \mu_A^j be a sub-vector of \mu^1 of length s := \text{cardinality}(A) that consists of the coordinates of \mu^1 in A (similarly for \mu_A^j). Up to permutation, the \Sigma matrix can be written as

\[\Sigma = \begin{bmatrix} \Sigma_{AA} & \Sigma_{AA^c} \\ \Sigma_{A^cA} & \Sigma_{A^cA^c} \end{bmatrix}.\]

3. NP-sLDA

We assume the following sampling scheme in the theoretical analysis. Let \(S_0 = \{x_0^0, \ldots, x_0^n\}\) be an i.i.d. class 0 sample of size \(n_0\), \(S_0' = \{x_0^{n_0+1}, \ldots, x_0^{n_0+n_0'}\}\) be an i.i.d. class 0 sample of size \(n_0'\) and \(S_1 = \{x_1^1, \ldots, x_1^{n_1}\}\) be an i.i.d. class 1 sample of size \(n_1\). Moreover, assume that the samples are independent of each other. To plug-in \(\phi\), we need estimates for \(\beta^{\text{Bayes}}\) in \(s^*(x) = (\beta^{\text{Bayes}})^\top x\) and for \(C_a^\star\). Although the decision thresholds are different, the NP oracle \(\phi_a^\star\) in (5) and the classical oracle \(\phi^\star\) in (3) both project an observation \(x\) to the \(\beta^{\text{Bayes}} = \Sigma^{-1}\mu_d\) direction. Hence one can consult existing works on (sparse) LDA under the classical paradigm to find a \(\beta^{\text{Bayes}}\) estimate, using samples \(S_0\) and \(S_1\). In particular, we adopt \(\beta^{\text{lasso}}\), the lassoed (sparse) discriminant analysis (sLDA) direction in Mai et al. (2012), which is computed by

\[(\beta^{\text{lasso}}, \beta_0^\lambda) = \arg\min_{(\beta, \theta)} \left\{ n^{-1} \sum_{i=1}^{n} (y_i - \beta_0^\top x_i^\top \beta)^2 + \lambda \sum_{j=1}^{d} |\beta_j| \right\}, \tag{6}\]

where \(n = n_0 + n_1\) and \(y_i = -n/n_0\) if the \(i\)th observation is from class 0, and \(y_i = n/n_1\) if the \(i\)th observation is from class 1. Note that although the optimization program (6) is the same as in Mai et al. (2012), our sampling scheme is different from that in Mai et al. (2012), where they assumed i.i.d. samples from the joint distribution of \((X, Y)\). As a consequence, it is necessary to establish theoretical results that are counterparts to those in Mai et al. (2012). To estimate the threshold \(C_a^\star\), we use the left-out class 0 sample \(S_0' = \{x_0^{n_0+1}, \ldots, x_0^{n_0+n_0'}\}\), leveraging the next proposition adapted from Tong et al. (2018).

Proposition 1. Suppose that we use \(S_0\) and \(S_1\) to train a base algorithm (e.g., sLDA), and obtain a scoring function \(f\) (e.g., an estimate of \(s^\star\)). Applying \(f\) to \(S_0'\), we denote the resulting classification scores as \(T_1, \ldots, T_{n_0'}\), which are real-valued random variables. Then, denote by \(T_{(k)}\) the \(k\)th order statistic (i.e., \(T_{(1)} \leq \cdots \leq T_{(n_0')}\)). For a new observation \(X\), if we denote its classification score \(f(X)\) as \(T\), we can construct classifiers \(\hat{\phi}_k(X) = \mathbb{I}(T > T_{(k)})\), \(k \in \{1, \ldots, n_0'\}\). Then, the population type I error of \(\hat{\phi}_k\), denoted by \(R_0(\hat{\phi}_k)\), is a function of \(T_{(k)}\) and hence a random variable, and it holds that

\[\mathbb{P} \left[ R_0(\hat{\phi}_k) > \alpha \right] \leq \sum_{j=k}^{n_0'} \binom{n_0'}{j} (1 - \alpha)^j \alpha^{n_0' - j}. \tag{7}\]

That is, the probability that the type I error of \(\hat{\phi}_k\) exceeds \(\alpha\) is under a constant that only depends on \(k, \alpha\) and \(n_0'\). We call this probability the violation rate of \(\hat{\phi}_k\) and denote its upper bound by \(v(k) = \sum_{j=k}^{n_0'} \binom{n_0'}{j} (1 - \alpha)^j \alpha^{n_0' - j}\). When \(T_{(k)}\)’s are continuous, this bound is tight.

Proposition 1 is the key step towards the NP umbrella algorithm proposed in Tong et al. (2018), which applies to all scoring type classification methods (base algorithms), including logistic regression, support vector machines, random forest, etc. In the following, we always assume the continuity of scoring functions. Under this mild assumption, \(v(k)\) is the violation rate of type I error for \(\hat{\phi}_k\). It is obvious that
\(v(k)\) decreases as \(k\) increases. So to choose from \(\hat{\phi}_1, \ldots, \hat{\phi}_{n'_0}\) a classifier with minimal type II error whose type I error violation rate is less than or equal to a user’s specified \(\delta_0\), the right order is

\[
k^* = \min \{k \in \{1, \ldots, n'_0\} : v(k) \leq \delta_0\}.
\] (8)

Note that to construct an NP classifier, one not only needs to specify a type I error upper bound \(\alpha\), but also has to specify an upper bound \(\delta_0\) on type I error violation rate. To achieve \(\mathbb{P}[R_0(\hat{\phi}) > \alpha] \leq \delta_0\) for some \(\hat{\phi}_k\), we need to control the violation rate under \(\delta_0\) at least in the extreme case when \(k = n'_0\); that is, it is necessary to ensure \(v(n'_0) = (1 - \alpha)^{n'_0} \leq \delta_0\). Clearly if the \((n'_0)\)-th order statistic cannot guarantee the violation rate control, other order statistics certainly cannot. Therefore, for a given \(\alpha\) and \(\delta_0\), there exists a minimum left-out class 0 sample size requirement \(n'_0 \geq \log \delta_0 / \log(1 - \alpha)\) for type I error violation rate control. Note that the control on type I error violation rate does not demand any sample size requirements on \(S_0\) and \(S_1\). But these two parts will have an impact on estimation accuracy of the scoring functions, and on the type II error performance.

Having estimates for \(s^*\) and \(C_{\alpha}^*\), we propose the following NP classifier,

\[
\hat{\phi}_{k^*}(x) = \mathbb{I}(\hat{s}(x) > \hat{C}_\alpha),
\]

(9)

where \(\hat{s}(x) = (\hat{\beta}_{\text{lasso}})^\top x\) with \(\hat{\beta}_{\text{lasso}}\) determined in optimization program (6), and \(\hat{C}_\alpha\) is the \((k^*)\)-th smallest element in \(\{\hat{s}(x^0_{m+1}), \ldots, \hat{s}(x^0_{m+n'_0})\}\). Because the estimate \(\hat{s}\) is inspired from the sLDA classifier in Mai et al. (2012), we name the classifier \(\hat{\phi}_{k^*}\) NP-sLDA.

### 4. Theoretical analysis

In this section, we establish NP oracle inequalities for the NP-sLDA classifier \(\hat{\phi}_{k^*}\) specified in equation (9). The NP oracle inequalities were formulated for classifiers under the NP paradigm to reckon the spirit of oracle inequalities in the classical paradigm. They require two properties to hold simultaneously with high probability: i) type I error \(R_0(\hat{\phi}_{k^*})\) is bounded from above by \(\alpha\), and ii) excess type II error, that is \(R_1(\hat{\phi}_{k^*}) - R_1(\phi_{\alpha}^*)\), diminishes as sample sizes increase. By construction of the order \(k^*\) in the NP-sLDA classifier \(\hat{\phi}_{k^*}\), the first property is already fulfilled, so in the following we focus on bounding the excess type II error.

In the NP classification literature, nonparametric NP classifiers constructed in Tong (2013) and Zhao et al. (2016) were shown to satisfy the NP oracle inequalities. Both papers assume bounded feature support \([-1, 1]^d\). Under this assumption, uniform deviation bounds between \(\hat{f}_1 / \hat{f}_0\) and its nonparametric estimate \(\hat{f}_1 / \hat{f}_0\) were derived, and such uniform deviation bounds were crucial in bounding the excess type II error. However, as canonical parametric models in classification (such as LDA and QDA) have unbounded feature support, the development of NP theory under parametric settings cannot bypass the challenges arisen from the unboundedness of feature support. To address these challenges, we formulate conditional marginal assumption and conditional detection condition. For the LDA model, we elaborate these high level conditions in terms of specific parameters. Before presenting the new assumptions and main theorem, we need a few technical lemmas to make the “conditioning” work.

#### 4.1 A few technical lemmas

With kernel density estimates \(\hat{f}_1, \hat{f}_0\), and an estimate of the threshold level \(\hat{C}_\alpha\) based on VC inequality, Tong (2013) constructed a plug-in classifier \(\mathbb{I}(\hat{f}_1(x) / \hat{f}_0(x) \geq \hat{C}_\alpha)\) that is of limited practical value unless
the feature dimension is small and sample size is large. Zhao et al. (2016) analyzed high-dimensional Naive Bayes models under the NP paradigm, and innovated the threshold estimate by invoking order statistics with explicit analytic formula for the chosen order. We denote that order by $k^*$, and it will be introduced in the next proposition. The order $k^*$ derived in Tong et al. (2018) is a refinement of the order statistics approach to estimate the threshold. However, although the order $k^*$ is optimal, it does not take an explicit formula and thus is not helpful in bounding the excess type II error. Interestingly, efforts to approximate $k^*$ analytically for type II error control leads to $k'$, and so $k'$ will be employed as a bridge in establishing NP oracle inequalities for $\hat{\phi}_{k^*}$.

To derive an upper bound for excess type II error, it is essential to bound the deviation between type I error of $\hat{\phi}_{k^*}$ and that of the NP oracle $\hat{\phi}'_{k^*}$. To achieve this, we first quote the next proposition from Zhao et al. (2016) and derive from it a corollary.

**Proposition 2.** Given $\delta_0 \in (0, 1)$, suppose $n'_0 \geq 4/(\alpha \delta_0)$, let the order $k'$ be defined as follows

$$k' = \lceil (n'_0 + 1)A_{\alpha, \delta_0}(n'_0) \rceil,$$

where $\lceil z \rceil$ denotes the smallest integer larger than or equal to $z$, and

$$A_{\alpha, \delta_0}(n'_0) = \frac{1 + 2\delta_0(n'_0 + 1)(1 - \alpha) + \sqrt{1 + 4\delta_0(1 - \alpha)n'_0 + 2}}{2\{\delta_0(n'_0 + 1) + 1\}}.$$

Then we have

$$\mathbb{P} \left( R_0(\hat{\phi}_{k'}) > \alpha \right) \leq \delta_0.$$

In other words, the type I error of classifier $\hat{\phi}_{k'}$ ($\hat{\phi}_k$ was defined in Proposition 1) is bounded from above by $\alpha$ with probability at least $1 - \delta_0$.

**Corollary 1.** Under continuity assumption of the classification scores $T_i$'s (which we always assume in this paper), the order $k^*$ is smaller than or equal to the order $k'$.

**Proof.** Under the continuity assumption of $T_i$'s, $v(k)$ is the exact violation rate of classifier $\hat{\phi}_k$. By construction, both $v(k')$ and $v(k^*)$ are smaller than or equal to $\delta_0$. Since $k^*$ is the smallest $k$ that satisfies $v(k) \leq \delta_0$, we have $k^* \leq k'$.

**Lemma 1.** Let $\alpha, \delta_0 \in (0, 1)$ and $n'_0 \geq 4/(\alpha \delta_0)$. For any $\delta'_0 \in (0, 1)$, the distance between $R_0(\hat{\phi}_{k'})$ and $R_0(\hat{\phi}'_{k^*})$ can be bounded as

$$\mathbb{P} \left[ |R_0(\hat{\phi}_{k'}) - R_0(\hat{\phi}'_{k^*})| > \xi_{\alpha, \delta_0, n'_0}(\delta'_0) \right] \leq \delta'_0,$$

where

$$\xi_{\alpha, \delta_0, n'_0}(\delta'_0) = \sqrt{\frac{k'(n'_0 + 1 - k^*)}{(n'_0 + 1)(n'_0 + 1)^2(\delta'_0)^2} + A_{\alpha, \delta_0}(n'_0) - (1 - \alpha) + \frac{1}{n'_0 + 1}},$$

in which $k'$ and $A_{\alpha, \delta_0}(n'_0)$ are the same as in Proposition 2. Moreover, if $n'_0 \geq \max(\delta_0^{-2}, \xi_{\alpha, \delta_0, n'_0}^{-2})$, we have $\xi_{\alpha, \delta_0, n'_0}(\delta'_0) \leq (5/2)n'_0^{-1/4}$.

Lemma 1 is borrowed from Zhao et al. (2016), so its proof is omitted. Based on Lemma 1 and Corollary 1, we can derive the following result whose proof is in the Appendix.
Lemma 2. Under the same assumptions as in Lemma 1, the distance between \( R_0(\hat{\phi}_k^*) \) and \( R_0(\phi^*_n) \) can be bounded as

\[
P\{|R_0(\hat{\phi}_k^*) - R_0(\phi^*_n)| > \xi_{\alpha, \delta_0, \nu_0}(\delta'_0)\} \leq \delta_0 + \delta'_0.
\]

If the features have bounded support, Lemma 2 would be exactly the desired deviation bound on type I error. But as feature support is unbounded, it can only serve as an intermediate step towards the final “conditional” version to be elaborated in Lemma 4.

Moving towards Lemma 4, we construct a set \( C \in \mathbb{R}^d \), such that \( C^c \) is “small”. We also show that the uniform deviation between \( \hat{s} \) and \( s^* \) on \( C \) is controllable (Lemma 3). To achieve that, we digress to introduce some more notations. Suppose the lassoed linear discriminant analysis (sLDA) finds the set \( C \) for some positive constant, and in which \( \beta \) is only for theoretical analysis, as the definition assumes knowledge of the true support set \( A \). The next proposition is a counterpart of Theorem 1 in Mai et al. (2012), but due to different sampling schemes, it differs from that theorem and a proof is attached in the Appendix.

Proposition 3. Assume \( \kappa := \|\Sigma_{x^1,A}(\Sigma_{x^1A})^{-1}\|_\infty < 1 \) and choose \( \lambda \) in the optimization program (6) such that \( \lambda < \min\{|\beta^*|_{min}/(2\varphi), \Delta\} \), where \( \beta^* = (\Sigma_{x^1A})^{-1}(\mu_1^A - \mu_1^0) \), \( \varphi = \|((\Sigma_{x^1A})^{-1})\|_\infty \) and \( \Delta = \|\mu_1^A - \mu_1^0\|_\infty \), then it holds that

1. With probability at least \( 1 - \delta_1^* \), \( \hat{\beta}_{lasso} = \tilde{\beta}_A \) and \( \beta_{lasso} = 0 \), where

\[
\delta_1^* = \sum_{i=0}^{1} 2d \exp \left(-c_1r_1 \frac{\lambda^2(1 - 2\varphi^2)^2}{16(1 + \kappa)^2}\right) + f(d, s, n_0, n_1, (\kappa + 1)\varphi(1 - \varphi)^{-1}),
\]

in which \( \epsilon \) is any positive constant less than \( \min[\epsilon_0, \lambda(1 - \kappa)/(4\varphi)^{-1}(\lambda/2 + (1 + \kappa)\Delta^{-1})] \) and \( \epsilon_0 \) is some positive constant, and in which

\[
f(d, s, n_0, n_1, \epsilon) = (d + s)s \exp \left(-c_1\epsilon^2 n_1^2/4s^2n_1\right) + (d + s)s \exp \left(-c_1\epsilon^2 n_1^2/4s^2n_1\right),
\]

for some constants \( c_1 \) and \( c_2 \).

2. With probability at least \( 1 - \delta_2^* \), none of the elements of \( \tilde{\beta}_A \) is zero, where

\[
\delta_2^* = \sum_{i=0}^{1} 2s \exp(-n_1\epsilon^2c_2) + \sum_{i=0}^{1} 2s^2 \exp \left(-c_1\epsilon^2 n_1^2/4n_1s^2\right),
\]

in which \( \epsilon \) is any positive constant less than \( \min[\epsilon_0, \epsilon(3 + \xi)^{-1}/\varphi, \Delta\xi(6 + 2\xi)^{-1}] \), where \( \xi = |\beta^*|_{min}/(\Delta\varphi) \).

3. For any positive \( \epsilon \) satisfying \( \epsilon < \min\{\epsilon_0, \lambda(2\varphi\Delta)^{-1}, \lambda\} \), we have

\[
P\left(\|\tilde{\beta}_A - \beta^*\|_\infty \leq 4\varphi\lambda\right) \geq 1 - \delta_2^*.
\]

Aided by Proposition 3, the next lemma constructs \( C \), a high probability set under both \( P_0 \) and \( P_1 \). Moreover, a high probability bound is derived for the uniform deviation between \( \hat{s} \) and \( s^* \) on this set.
Lemma 3. Suppose \( \max\{\text{tr}(\Sigma AA), \text{tr}(\Sigma^2 A A), \|\Sigma AA\|, \|\mu_0 A\|^2, \|\mu_1 A\|^2\} \leq c_0 s \) for some constant \( c_0 \), where \( s = \text{cardinality}(A) \). For \( \delta_3 = \exp\{-(n_0 \land n_1)^{1/2}\} \), there exists some constant \( c_1 > 0 \), such that \( C = \{X \in \mathbb{R}^d : \|X\| \leq c_1 s^{1/2}(n_0 \land n_1)^{1/4}\} \) satisfies \( P_0(X \in C) \geq 1 - \delta_3 \) and \( P_1(X \in C) \geq 1 - \delta_3 \). Moreover, let \( \|s - s^\ast\|\infty,C := \max_{x \in C} |s(x) - s^\ast(x)| \). Then for \( \delta_1 \geq \delta_1^\ast \) and \( \delta_2 \geq \delta_2^\ast \), where \( \delta_1^\ast \) and \( \delta_2^\ast \) are defined as in Proposition 3, it holds that with probability at least \( 1 - \delta_1 - \delta_2 \),

\[
\|s - s^\ast\|\infty,C \leq 4c_1 \varphi \lambda \delta s(n_0 \land n_1)^{1/4}.
\]

The set \( C \) was constructed with two opposing missions in mind. On one hand, we want to restrict the feature space \( \mathbb{R}^d \) to \( C \) so that the restricted uniform deviation of \( \hat{s} \) from \( s^\ast \) is controlled. On the other hand, we also want \( C \) to be sufficiently large, so that \( P_0(X \in C^\ast) \) and \( P_1(X \in C^\ast) \) diminish as sample sizes increase. The next lemma is implied by Lemma 2 and Lemma 3.

Lemma 4. Let \( C \) be defined as in Lemma 3. Then under the same conditions as in Lemma 1, the distance between \( R_0(\hat{\phi}_{\delta_{\ast}}|C) := P_0(\hat{s}(X) \geq \tilde{C}_\alpha X \in C) \) and \( R_0(\phi_0^\ast|C) := P_0(s^\ast(X) \geq C_0^\ast X \in C) \) can be bounded as

\[
\mathbb{P}\{\|R_0(\hat{\phi}_{\delta_{\ast}}|C) - R_0(\phi_0^\ast|C)\| > 2[\xi_{\alpha,\delta_{\ast},n_0}(\delta_{\ast}^{\ast}) + \exp\{-(n_0 \land n_1)^{1/2}\}]} \leq \delta_0 + \delta_0^{\ast},
\]

where \( \xi_{\alpha,\delta_{\ast},n_0}(\delta_{\ast}^{\ast}) \) is defined in Lemma 1.

4.2. Margin assumption and detection condition

Margin assumption and detection condition are important theoretical assumptions in Tong (2013) and Zhao et al. (2016) for bounding excess type II error of the nonparametric NP classifiers constructed in those papers. Unlike Tong (2013) and Zhao et al. (2016) which assumed bounded feature support, parametric models (e.g., the LDA model) often have the entire \( \mathbb{R}^d \) as the support. To assist our proof strategy that divides the \( \mathbb{R}^d \) space into a high probability set (e.g., \( C \) defined in Lemma 3) and its complement, we introduce conditional versions of these assumptions.

Definition 1 (Conditional margin assumption). A function \( f(\cdot) \) is said to satisfy conditional margin assumption restricted to \( C^\ast \) of order \( \bar{\gamma} \) with respect to probability distribution \( P \) (i.e., \( X \sim P \)) at the level \( C^\ast \) if there exists a positive constant \( M_0 \), such that for any \( \delta \geq 0 \),

\[
P\{|f(X) - C^\ast| \leq \delta | X \in C^\ast \} \leq M_0 \delta^{\bar{\gamma}}.
\]

The unconditional version of such an assumption was first introduced in Polonik (1995). In the classical binary classification framework, Mammen and Tsybakov (1999) proposed a similar condition named “margin condition” by requiring most data to be away from the optimal decision boundary and this condition has become a common assumption in classification literature. In the classical classification paradigm, Definition 1 reduces to the margin condition by taking \( f = \eta \), \( C^\ast = \text{support}(X) \) and \( C^\ast = 1/2 \), with \( \{x : |f(x) - C^\ast| = 0\} = \{x : \eta(x) = 1/2\} \) giving the decision boundary of the classical Bayes classifier. For a \( C^\ast \) with nontrivial probability, the conditional margin assumption is actually weaker than the unconditional version. For example, suppose \( P(X \in C^\ast) \geq 1/2 \), then the condition \( P(|f(X) - C^\ast| \leq \delta) \leq 2M_0 \delta^{\bar{\gamma}} \) would imply the conditional margin assumption in view of the Bayes Theorem.

Definition 1 is a high level assumption. In view of explicit Gaussian modeling assumptions, it is preferable to derive it based on more elementary assumptions on \( \mu^0 \), \( \mu^1 \) and \( \Sigma \), for our choices of \( f \), \( P \), \( C^\ast \) and \( C^\ast \). Recall that the NP oracle classifier can be written as

\[
\phi_0^\ast(x) = \mathbb{I}\{(\Sigma^{-1}\mu_d)\top x > C_0^\ast\}.
\]
Here we take \( f(x) = s^*(x) = (\Sigma^{-1}\mu_d)^\top x \), \( C^* = C_{\alpha}^{**} \), \( P = P_0 \), and \( C^* = C^\star \) in Lemma 3. When \( X \sim N(\mu^0, \Sigma), (\Sigma^{-1}\mu_d)^\top X \sim N(\mu_d^0, \Sigma_{\Sigma}^{-1}\mu_d) \). Lemma 3 guarantees that for \( \delta_3 = \exp\{(n_0/n_1)^{1/2}\} \), \( P_0(X \in C) \geq 1 - \delta_3 \). Moreover,

\[
P_0(|s^*(X) - C_{\alpha}^{**}| \leq \delta |X \in C) \\
\leq P_0(C_{\alpha}^{**} - \delta \leq (\Sigma^{-1}\mu_d)^\top X \leq C_{\alpha}^{**} + \delta) / (1 - \delta_3) \\
= |\Phi(U) - \Phi(L)| / (1 - \delta_3),
\]

where \( \Phi \) is the cumulative distribution function of the standard normal distribution, \( U = (C_{\alpha}^{**} + \delta - \mu_d^\top \Sigma^{-1}\mu^0)/\sqrt{\mu_d^\top \Sigma^{-1}\mu_d} \), and \( L = (C_{\alpha}^{**} - \delta - \mu_d^\top \Sigma^{-1}\mu^0)/\sqrt{\mu_d^\top \Sigma^{-1}\mu_d} \). By the mean value theorem, we have

\[
\Phi(U) - \Phi(L) = \phi(z)(U - L) = \phi(z)\frac{2\delta}{\sqrt{\mu_d^\top \Sigma^{-1}\mu_d}},
\]

where \( \phi \) is the probability distribution function of the standard normal distribution, and \( z \) is in \([L, U]\). Clearly \( \phi \) is bounded from above by \( \phi(0) \). Hence, under the assumptions of Lemma 3, if we additionally assume that \( \mu_d^\top \Sigma^{-1}\mu_d \geq C \) for some universal positive constant \( C \), the conditional margin assumption is met with the restricted set \( C \), the constant \( M_0 = 2\phi(0)/\sqrt{C(1 - \delta_3)} \) and \( \gamma = 1 \). Since \( \delta_3 < 1/2 \), we can take \( M_0 = 4\phi(0)/\sqrt{C} \).

**Assumption 1.** i) \( \max\{tr(\Sigma_{AA}), tr(\Sigma_{AA}^2), \|\Sigma_{AA}\|, \|\mu_A^0\|^2, \|\mu_A^1\|^2\} \leq c_0 s \) for some constant \( c_0 \), where \( s = \text{cardinality}(A) \), and \( A = \{j : \{\Sigma^{-1}\mu_d\} \neq 0\} \); ii) \( \mu_d^\top \Sigma^{-1}\mu_d \geq C \) for some universal positive constant \( C \); iii) the set \( C \) is defined as in Lemma 3.

**Remark 1.** Under Assumption 1, the function \( s^*(\cdot) \) satisfies the conditional margin assumption restricted to \( C \) of order \( \gamma = 1 \) with respect to probability distribution \( P_0 \) at the level \( C_{\alpha}^{**} \). In addition, the constant \( M_0 \) can be taken as \( M_0 = 4\phi(0)/\sqrt{C} \).

Unlike the classical paradigm where the optimal threshold 1/2 on regression function is known, the optimal threshold level in the NP paradigm is unknown and needs to be estimated, suggesting the necessity of having sufficient data around the decision boundary to detect it. This concern motivated Tong (2013) to formulate a detection condition that works as an opposite force to the margin assumption, and Zhao et al. (2016) improved upon it and proved its necessity in bounding excess type II error of an NP classifier. However, formulating a transparent detection condition for feature spaces of unbounded support is subtle: to generalize the detection condition in the same way as we generalize the margin assumption to a conditional version, it is not obvious what elementary general assumptions one should impose on the \( \mu^0, \mu^1 \) and \( \Sigma \). The good side is that we are able to establish explicit conditions for \( s \leq 2 \), aided by the literature on truncated normal distribution. Also, we need a two-sided detection condition as in Tong (2013), because the technique in Zhao et al. (2016) to get rid of one side does not apply in the unbounded feature support situation.

**Definition 2 (Conditional Detection Condition).** A function \( f(\cdot) \) is said to satisfy conditional detection condition restricted to \( C^\star \) of order \( \gamma \) with respect to \( P \) (i.e., \( X \sim P \)) at level \((C^\star, \delta^\star)\) if there exists a positive constant \( M_1 \), such that for any \( \delta \in (0, \delta^\star) \),

\[
P(C^\star \leq f(X) \leq C^\star + \delta |X \in C^\star) \land P(C^\star - \delta \leq f(X) \leq C^\star |X \in C^\star) \geq M_1 \delta^2.
\]
Unlike the conditional margin assumption, the conditional detection condition is stronger than its unconditional counterpart, in view of the Bayes Theorem. Although we do not have a proof of the necessity for the conditional detection condition, much efforts to bound excess type II error without it failed.

**Assumption 2.** The function $s^*(\cdot)$ satisfies conditional detection condition restricted to $C$ (defined in Lemma 3) of order $\gamma \geq 1$ with respect to $P_0$ at the level $(C^*_\alpha, \delta^*)$.

Proposition 4 in the Appendix shows that under restrictive settings ($s \leq 2$), Assumption 2 can be implied by more elementary assumptions on the parameters of the LDA model.

### 4.3. NP oracle inequalities

Having introduced the technical assumptions and lemmas, we present the main theorem.

**Theorem 1.** Suppose Assumptions 1 and 2, and the assumptions for Lemmas 1-4 hold. Further suppose $n_0' \geq \max\{4/(\alpha \alpha_0), \delta_0 n_0^2, (\delta_0')^{-2}, (\frac{1}{16} M_1 \lambda s^2)^{-4}\}$, $n_0 \land n_1 \geq \lfloor -\log(M_1 \delta^2)/4 \rfloor^2$, and $C_\alpha$ and $\mu_0^{-1} \Sigma^{-1} \mu$ are bounded from above and below. For $\delta_0, \delta_0' > 0$, $\delta_1 \geq \delta_1^*$ and $\delta_2 \geq \delta_2^*$, there exist constants $c_1, c_2$ and $\bar{c}_3$ such that, with probability at least $1 - \delta_0 - \delta_0' - \delta_1 - \delta_2$, it holds that

$$R_0(\hat{\phi}_{K*}) \leq \alpha,$$

$$R_1(\hat{\phi}_{K*}) - R_1(\phi_{K*}) \leq \bar{c}_1(n_0)^{-\frac{1}{2}(\frac{1+\gamma}{\gamma} \land 1)} + \bar{c}_2(\lambda s)^{1+\gamma}(n_0 \land n_1)^{\frac{1+\gamma}{\gamma}} + \bar{c}_3 \exp \left\{ -(n_0 \land n_1)^{\frac{1}{2}(\frac{1+\gamma}{\gamma} \land 1)} \right\}. $$

Theorem 1 establishes the NP oracle inequalities for the NP-sLDA classifier $\hat{\phi}_{K*}$. By Assumption 1, $\gamma = 1$ and by Proposition 4 in the Appendix, $\gamma = 1$ for $s \leq 2$ under certain conditions. Substituting in these numbers will greatly simplify the upper bound for the excess type II error. But we choose to keep $\gamma$ so that the explicit dependency on this parameter is clear. Also note that the upper bound for excess type II error does not contain the overall feature dimensionality $d$ explicitly. However, the indirect dependency is two-fold: first, the choice of $\lambda$ might depend on $d$; second, the minimum requirements (i.e., lower bounds) for $\delta_1$ and $\delta_2$, which are $\delta_1^*$ and $\delta_2^*$ defined in Proposition 3, depend on $d$.

### 5. Data-adaptive sample splitting scheme

In practice, researchers and practitioners are not given data as separate sets $S_0, S_0'$ and $S_1$. Instead, they have a single dataset $S$ that consists of mixed class 0 and class 1 observations. More class 0 observations to better train the base algorithm and more class 0 observations to provide more candidates for threshold estimate each has their own merits. Hence how to split the class 0 observations into two parts, one to train the base algorithm and the other to estimate the score threshold, is far from intuitive. Tong et al. (2018) adopts a half-half split for class 0 and ignores this issue in the development of the NP umbrella algorithm, as that paper focuses on the type I error violation rate control.

Now switching the focus to type II error, we propose a data-adaptive splitting scheme that universally enhances the power (i.e., reduces type II error) of NP classifiers, as demonstrated in the subsequent numerical studies. The procedure is to choose a split proportion $\tau$ according to rankings of $K$-fold cross-validated type II errors. Concretely, for each split proportion candidate $\tau \in \{.1, .2, \ldots, .9\}$, the following steps are implemented.
(a) Randomly split class 1 observations into $K$-folds.

(b) Use all class 0 observations and $K - 1$ folds of class 1 observations to train an NP classifier. For class 0 observations, $\tau$ proportion is used to train the base algorithm, and $1 - \tau$ proportion for threshold estimate.

(c) For this classifier, calculate its classification error on the validation fold of the class 1 observations (type II error).

(d) Repeat steps (b) and (c) for $K$ times, with each of the $K$ folds used exactly once as the validation data. Compute the mean of type II errors in step (c), and denote it by $e(\tau)$.

Our choice of the split proportion is

$$\tau_{\text{min}} = \arg \min_{\tau \in \{1/9, \ldots, 9/9\}} e(\tau).$$

Note that $\tau_{\text{min}}$ not only depends on the dataset $S$, but also on the base algorithm one uses, as well as on the user-specified $\alpha$ and $\delta_0$. Merits of this adaptive splitting scheme will be revealed in the next simulation section. Here we elaborate how to reconcile this adaptive scheme with the violation rate control objective. The type I error violation rate control was proved based on a fixed split proportion of class 0 observations, so will the adaptive splitting scheme be overly aggressive on type II error such that we can no longer keep the type I error violation rate under control? If for each realization (among infinite realizations) of the mixed sample $S$, we do adaptive splitting on class 0 observations before implementing NP-sLDA $\hat{\phi}_k$ (or other NP classifiers), then the overall procedure indeed does not lead to a classifier with type I error violation rate controlled under $\delta_0$. However, this is not how we think about this process; instead, we only adaptively split for one realization of $S$, getting a split proportion $\hat{\tau}$, and then fix $\hat{\tau}$ in all rest realizations of $S$. This implementation of the overall procedure keeps the type I error violation rate under control.

6. Numerical analysis

Through extensive numerical experiments and real data studies, we demonstrate the competence of NP-sLDA and the new sample splitting scheme.

6.1. Simulation studies

In this subsection, $N_0$ denotes the total class 0 training sample size (we do not use $n_0$ and $n'_0$ here, as class 0 observations are not assumed to be pre-divided into two parts), and $n_1$ denotes the class 1 training sample size. In Examples 1 – 3, we conduct simulations to compare the empirical performance of the proposed NP-sLDA with other NP classifiers as well as the sLDA (Mai et al., 2012). In Examples 4-5, we study how the adaptive splitting scheme enhances power (i.e., reduces type II error) upon the default half-half choice. In every simulation setting, the experiments are repeated 1,000 times.

**Example 1.** The data are generated from an LDA model with common covariance matrix $\Sigma$, where $\Sigma$ is set to be an AR(1) covariance matrix with $\Sigma_{ij} = 0.5^{|i-j|}$ for all $i$ and $j$. The true $\beta^{\text{Bayes}} = \Sigma^{-1} \mu_d = 0.556 \times (3, 1.5, 0, 0, 2, 0, \ldots, 0)^\top$, $\mu^0 = 0^\top$, $d = 1,000$, and $N_0 = n_1 = 200$. Bayes error = 10% under $\pi_0 = \pi_1 = 0.5$.

**Example 2.** The data are generated from an LDA model with common covariance matrix $\Sigma$, where $\Sigma$ is set to be a compound symmetric covariance matrix with $\Sigma_{ij} = 0.5$ for all $i \neq j$ and $\Sigma_{ii} = 1$ for
Table 1. Violation rate and type II error for Examples 1, 2 and 3 over 1,000 repetitions.

|       | NP-sLDA | NP-penlog | NP-svm | sLDA |
|-------|---------|-----------|--------|------|
| Ex 1  | violation rate | .068     | .055   | .054 | .764 |
|       | type II error (mean) | .189     | .205   | .621 | .104 |
|       | type II error (sd) | .057     | .063   | .077 | .010 |
| Ex 2  | violation rate | .073     | .081   | .081 | 1.000 |
|       | type II error (mean) | .246     | .255   | .615 | .129 |
|       | type II error (sd) | .051     | .053   | .070 | .010 |
| Ex 3  | violation rate | .079     | .088   | .099 | .997 |
|       | type II error (mean) | .332     | .334   | .584 | .231 |
|       | type II error (sd) | .044     | .044   | .045 | .012 |

all i. The true \( \beta^{Bayes} = \Sigma^{-1} \mu_d = 0.551 \times (3, 1.7, -2.2, -2.1, 2.55, 0, \cdots, 0)^\top \). \( \mu^0 = 0^\top \), \( d = 2,000 \), and \( N_0 = n_1 = 300 \). Bayes error = 10% under \( \pi_0 = \pi_1 = 0.5 \).

Example 3. Same as in Example 2, except \( d = 3,000 \), \( N_0 = n_1 = 400 \), and the true \( \beta^{Bayes} = \Sigma^{-1} \mu_d = 0.362 \times (3, 1.7, -2.2, -2.1, 2.55, 0, \cdots, 0)^\top \). Bayes error = 20% under \( \pi_0 = \pi_1 = 0.5 \).

Examples 1-3 compare the empirical type I/II error performance of NP-sLDA, NP-penlog (penlog stands for penalized logistic regression), NP-svm and sLDA on a test data set of size 20,000 that consist of 10,000 observations from each class. In all NP methods, \( \tau \), the class 0 split proportion, is fixed at 0.5 and \( \delta_0 \), the upper bound on the type I error violation rate, is set at 0.1. For Examples 1 and 2, we set the type I error upper bound \( \alpha = 0.1 \). For Example 3, we set \( \alpha = 0.2 \). These choices for \( \alpha \) match the corresponding Bayes errors, so that comparison between NP methods and classical methods does not obviously favor the former.

Table 1 indicates that while the sLDA method cannot control the type I error violation rate under \( \delta_0 \), all the NP classifiers are able to do so. In addition, among the three NP classifiers, NP-sLDA gives the smallest mean type II error. Strictly speaking, the observed type I error violation rate is only an approximation to the real violation rate. The approximation is two-fold: i). in each repetition of an experiment, the population type I error is approximated by empirical type I error on a large test set; ii). the violation rate should be calculated based on infinite repetitions of the experiment, but we only calculate it based on 1,000 repetitions. However, such approximation is unavoidable in numerical studies.

By explanations in the last paragraph of Section 5, type I error violation rate is under control by \( \delta_0 \) for the adaptive splitting scheme. The next two examples investigate the power enhancement as a result of the adaptive splitting scheme. They include an array of situations, including low and high dimensional settings \( (d = 20 \text{ and } 1,000) \), balanced and imbalanced classes \( (N_0 : n_1 = 1 : 1 \text{ to } 1 : 256) \), and small to medium sample sizes \( (N_0 = 100 \text{ to } 500) \).

Example 4. Same as in Example 1, except taking the following sample sizes.

(4a). \( N_0 = 100 \) and varying \( n_1/N_0 \in \{1, 2, 4, 8, 16, 32, 64, 128, 256\} \).
(4b). Varying \( n_1 = N_0 \in \{100, 150, 200, 250, 300, 350, 400, 450, 500\} \).

Example 5. Same as in Example 1, except that \( d = 20 \), \( N_0 = 100 \) and varying \( n_1/N_0 \in \{1, 2, 4, 8, 16\} \).

Note that Examples 4a and 4b each includes 9 different simulation settings, and Example 5 includes 5. For each simulation setting, we generate 1,000 (training) datasets and a common test set of size...
100,000 from class 1. Only class 1 test data are needed because only type II error is investigated in these examples. In each simulation setting, we train 10 NP classifiers of the same base algorithm using each of the 1,000 datasets. Nine of these 10 NP classifiers use fixed split proportions in \{.1, \cdots, .9\}, and the last one uses adaptive split proportion using 5-fold cross-validation. Overall in Examples 4 and 5, we set \( \alpha = \delta_0 = 0.1 \), and train an enormous number of NP classifiers. For instance, in Example 4a, we train \( 9 \times 1,000 \times 10 = 90,000 \) NP-sLDA classifiers, and the same number of NP classifiers for any other base algorithm under investigation.

For each simulation setting, denote by \( \tilde{R}_1(\cdot) \) the empirical type II error on the test set. We fix a simulation setting so that we do not need to have overly complex sub or sup indexes in the following discussion. Denote by \( \hat{h}_{i,b,\tau} \) an NP classifier with base algorithm \( b \), trained on the \( i \)-th dataset (\( i \in \{1, \cdots, 1000\} \)) using split proportion \( \tau \). This classifier also depends on users’ choices of \( \alpha \) and \( \delta_0 \), but we suppress these dependencies here to highlight our focus. In fixed proportion scenarios, \( \tau \in \{.1, \cdots, .9\} \).

Let \( \tau^{ada}(j,b) \) represent the adaptive split proportion trained on the \( j \)-th dataset with base algorithm \( b \) using adaptive splitting scheme described in Section 5. Therefore, \( \hat{h}_{i,b,\tau^{ada}(j,b)} \) refers to the NP classifier with base algorithm \( b \), trained on the \( i \)-th dataset using the split proportion \( \tau^{ada}(j,b) \) pre-determined in the \( j \)-th dataset, where \( i,j \in \{1, \cdots, 1000\} \). Let \( \text{Ave}_{b,\tau} \) and \( \text{Ave}_{b,\tau^{ada}} \) be our performance measures for fix proportion and adaptive proportion respectively, which are defined by,

\[
\text{Ave}_{b,\tau} = \frac{1}{1000} \sum_{i=1}^{1000} \tilde{R}_1(\hat{h}_{i,b,\tau}), \quad \text{Ave}_{b,\tau^{ada}} = \text{median}_{j=1,\cdots,1000} \left( \frac{1}{1000} \sum_{i=1}^{1000} \tilde{R}_1(\hat{h}_{i,b,\tau^{ada}(j,b)}) \right).
\]

While the meaning of the measure \( \text{Ave}_{b,\tau^{ada}} \) is almost self-evident, \( \text{Ave}_{b,\tau} \) deserves some elaboration. As we explained in the last paragraph of Section 5, the adaptive splitting scheme returns a proportion based on one realization of \( S \), and then we adopt it in all subsequent realizations. Let \( w_b(j) = \frac{1}{1000} \sum_{i=1}^{1000} \tilde{R}_1(\hat{h}_{i,b,\tau^{ada}(j,b)}) \), then \( w_b(j) \) is a performance measure of the adaptive scheme if the proportion is returned from training on the \( j \)-th dataset. To account for the variation among \( w_b(j) \)'s for different choices of \( j \), we take the median over \( w_b(j) \)'s as our final measure. Also, denote the average of adaptively selected proportions by

\[
\tau_{b,ada} = \frac{1}{1000} \sum_{j=1}^{1000} \tau^{ada}(j,b), \quad \tau_{b,\text{opt}} = \frac{1}{1000} \sum_{j=1}^{1000} \arg \min_{\tau \in \{1,\cdots,9\}} \tilde{R}_1(\hat{h}_{i,b,\tau}).
\]

With Example 4, we investigate i). the effectiveness (in terms of type II error) of the adaptive splitting strategy compared to a fixed half-half split, illustrated by the left panels of Figures 1 and 2; ii). how close is \( \tau_{b,\text{opt}} \) compared to \( \tau_{b,\text{opt}} \), illustrated by the right panels of Figures 1 and 2; iii). how the class imbalance affects NP-sLDA and NP-penlog, illustrated by both panels of Figure 1; and iv). how the absolute class 0 sample size affects NP-sLDA and NP-penlog, illustrated by both panels of Figure 2.

In Figure 1 (Example 4a), the left panel presents the trend of type II errors (\( \text{Ave}_{b,\tau} \) and \( \text{Ave}_{b,\tau^{ada}} \)) as the sample size ratio \( n_1/N_0 \) increases from 1 to 256 for fixed \( N_0 = 100 \). For both NP-penlog and NP-sLDA, type II error decreases as \( n_1/N_0 \) increases from 1 to 16 and gradually stabilizes afterwards. Neither NP-penlog nor NP-sLDA suffers from training on imbalanced classes. In terms of type II error performance, the adaptive splitting strategy significantly improves over the fixed split proportion 0.5. The right panel of Figure 1 shows that, on average the adaptive split proportion is very close to the optimal one throughout all sample size ratios.
Fig. 1. Example 4a. Left panel: type II error (\(\text{Ave}_{b,.5}\) and \(\text{Ave}_{b,\hat{\tau}}\)) of NP-sLDA and NP-penlog vs. \(n_1/N_0\); Right panel: average split proportion (\(\tau_{b,\text{ada}}\) and \(\tau_{b,\text{opt}}\)) vs. \(n_1/N_0\). \(N_0\) is fixed to be 100 for both panels.

![Left panel: type II error of NP-sLDA and NP-penlog](image1)

![Right panel: average split proportion](image2)

Fig. 2. Example 4b. Left panel: type II error (\(\text{Ave}_{b,.5}\) and \(\text{Ave}_{b,\hat{\tau}}\)) of NP-sLDA and NP-penlog vs. \(N_0\); Right panel: average split proportion (\(\tau_{b,\text{ada}}\) and \(\tau_{b,\text{opt}}\)) vs. \(n_1 = N_0\) for both panels.

![Left panel: type II error of NP-sLDA and NP-penlog](image3)

![Right panel: average split proportion](image4)
In Figure 2 (Example 4b), the left panel presents the trend of type II errors (\(Ave_{b, 0.5}\) and \(Ave_{b, 0.7}\)) as the class 0 sample size \(N_0 (n_1 = N_0)\) increases from 100 to 500, indicating that type II error clearly benefits from increasing training sample sizes of both classes. For the same base algorithm, the adaptive splitting strategy significantly improves over the fixed split proportion 0.5 for \(N_0\) and \(n_1\) small, although the improvement diminishes as both sample sizes become large. The right panel of Figure 2 shows that, on average the adaptive split proportion is very close to the optimal one throughout all sample sizes. Furthermore, the average optimal split proportion seems to increase as \(N_0\) increases in general. An intuition is that when \(N_0\) is smaller, a higher proportion of class 0 observations is needed for threshold estimate, in order to guarantee the type I error violation rate control.

With Example 5, we investigate the interaction between adaptive splitting strategy and multiple random splits on different NP classifiers. Multiple random splits of class 0 observations were proposed in the NP umbrella algorithm in Tong et al. (2018) to increase the stability of the type II error performance. When an NP classifier uses \(M > 1\) multiple splits, each split will result in a classifier, and the final prediction rule is a majority vote of these classifiers. Figure 3 shows the trend of type II error of NP-sLDA, NP-penlog, NP-randomforest, and NP-svm, as the sample size ratio \(n_1/N_0\) increases from 1 to 16 while keeping \(N_0 = 100\). For each base algorithm, four scenarios are considered: (fixed 0.5 split proportion, single split), (adaptive split proportion, single split), (fixed 0.5 split proportion, multiple splits), and (adaptive split proportion, multiple splits). Figure 3 suggests the following interesting findings: i). type II error decreases for NP-sLDA and NP-penlog but increases for NP-randomforest and NP-svm, as a function of \(n_1/N_0\) while keeping \(N_0\) constant; ii). with both fixed 0.5 split proportion and adaptive splitting strategy, performing multiple splits leads to a smaller type II error compared with their single split counterparts; iii). for both single split and multiple splits, the adaptive split always improves upon the fixed 0.5 split proportion; iv). NP-svm and NP-randomforest are affected by the imbalance scenario, and one might consider downsampling or upsampling methods before applying an NP algorithm; and v). adding multiple splits to the adaptive splitting strategy leads to a further reduction on the type II error. Nevertheless, the reduction in type II error from adaptive splitting scheme alone is much larger than the marginal gain from adding multiple splits on top of it. Therefore, when computation power is limited, one should implement the adaptive splitting scheme before considering multiple splits.

### 6.2. Real data analysis

We now evaluate NP-sLDA on a neuroblastoma dataset containing \(d = 43,827\) gene expression measurements from \(n = 498\) neuroblastoma samples generated by the Sequencing Quality Control (SEQC) consortium (Wang et al., 2014). The samples fall into two classes: 176 high-risk (HR) samples and 322 non-HR samples. It is usually understood that misclassifying an HR sample as non-HR will have more severe consequences than the other way around. Formulating this problem under the NP classification framework, we label the HR samples as class 0 observations and the non-HR samples as class 1 observations and, use all gene expression measurements as features to perform classification. We set \(\alpha = \delta_0 = 0.1\), and compare NP-sLDA with NP-penlog, NP-randomforest and NP-svm. We randomly split the dataset 1,000 times into a training set (70%) and a test set (30%), and then train the NP classifiers on each training data and compute their empirical type I and type II errors over the corresponding test data. We consider each fixed split proportion in \(\{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}\) as well as the adaptive splitting strategy. Here, the split proportion 0.9 is not considered since it leads to a left-out sample size which is too small to control the type I error at the given \(\alpha\) and \(\delta_0\) values. Figure 4 indicates that the average type I error
is less than $\alpha$ across different split proportions for all four methods considered. Regarding the average type II error, NP-sLDA has the smallest values for a wide range of split proportions. In particular, the smallest average type II error for NP-sLDA corresponds to split proportion 0.4. The average location of the split proportion chosen by the adaptive splitting scheme would lead to a type II error close to the minimum. This demonstrates that the adaptive splitting scheme works well for different NP classifiers.

7. Discussion

This work develops NP classification theory and methodology under parametric assumptions. Concretely, we propose NP-sLDA, an NP version of the sparse linear discriminant analysis classifier (sLDA) and show that NP-sLDA achieves NP oracle inequalities under certain conditions, including the newly minted conditional margin assumption and conditional detection condition. A second major contribution of this work is a new adaptive sample splitting scheme, which pairs well with any base classification algorithm in the NP umbrella algorithm in Tong et al. (2018). Our numerical analysis shows that power increases (i.e., type II error drops) tremendously once we adopt the adaptive splitting scheme. The NP-sLDA algorithm and data-adaptive sample splitting scheme have been incorporated into the R package nproc, available on CRAN. For future work, it would be interesting to investigate NP classifiers under other parametric settings, such as quadratic discriminant analysis (QDA) model and heavy-tailed distributions which are appropriate to model financial data. The theoretical assumptions set up in this work, although minted for the LDA model, shed light on the development of NP theory for other parametric models.
Fig. 4. The average type I and type II errors vs. splitting proportion on the neuroblastoma data set for NP-sLDA, NP-penlog, NP-randomforest and NP-svm over 1,000 random splits. The *" point on each line represents the average split proportion chosen by adapting splitting.

A. Appendix

The appendix contains additional technical lemmas, a proposition, and proofs.

A.1. Neyman-Pearson Lemma

The oracle classifier under the NP paradigm arises from its close connection to the Neyman-Pearson Lemma in statistical hypothesis testing. Hypothesis testing bears strong resemblance to binary classification if we assume the following model. Let \( P_1 \) and \( P_0 \) be two known probability distributions on \( X \subset \mathbb{R}^d \). Assume that \( Y \sim \text{Bern}(\zeta) \) for some \( \zeta \in (0, 1) \), and the conditional distribution of \( X \) given \( Y \) is \( P_Y \). Given such a model, the goal of statistical hypothesis testing is to determine if we should reject the null hypothesis that \( X \) was generated from \( P_0 \). To this end, we construct a randomized test \( \phi : \mathcal{X} \to [0, 1] \) that rejects the null with probability \( \phi(X) \). Two types of errors arise: type I error occurs when \( P_0 \) is rejected yet \( X \sim P_0 \), and type II error occurs when \( P_0 \) is not rejected yet \( X \sim P_1 \). The Neyman-Pearson paradigm in hypothesis testing amounts to choosing \( \phi \) that solves the following constrained optimization problem

\[
\max \mathbb{E}[\phi(X)|Y = 1], \quad \text{subject to } \mathbb{E}[\phi(X)|Y = 0] \leq \alpha,
\]

where \( \alpha \in (0, 1) \) is the significance level of the test. A solution to this constrained optimization problem is called a most powerful test of level \( \alpha \). The Neyman-Pearson Lemma gives mild sufficient conditions for the existence of such a test.

**Lemma 5 (Neyman-Pearson Lemma).** Let \( P_1 \) and \( P_0 \) be two probability measures with densities \( f_1 \) and \( f_0 \) respectively, and denote the density ratio as \( r(x) = f_1(x)/f_0(x) \). For a given significance level \( \alpha \), let \( C_\alpha \) be such that \( P_0\{r(X) > C_\alpha\} \leq \alpha \) and \( P_0\{r(X) \geq C_\alpha\} \geq \alpha \). Then, the most powerful test of level \( \alpha \) is

\[
\phi^*_\alpha(X) = \begin{cases} 
1 & \text{if } r(X) > C_\alpha, \\
0 & \text{if } r(X) < C_\alpha, \\
\frac{\alpha - P_0\{r(X) > C_\alpha\}}{P_0\{r(X) = C_\alpha\}} & \text{if } r(X) = C_\alpha.
\end{cases}
\]
Under mild continuity assumption, we take the NP oracle classifier
\[
\phi_\alpha^*(x) = \mathbb{I}\{f_1(x)/f_0(x) > C_\alpha\} = \mathbb{I}\{r(x) > C_\alpha\},
\]
as our plug-in target for NP classification.

A.2. Additional Lemmas and Propositions

**Lemma 6 (Hsu et al. (2012)).** Let \(A \in \mathbb{R}^{m \times n}\) be a matrix, and let \(\Sigma := A^T A\). Let \(x = (x_1, \cdots, x_n)^T\) be an isotropic multivariate Gaussian random vector with mean zero. For all \(t > 0\),
\[
P\left(\|Ax\|^2 > \text{tr}(\Sigma) + 2\sqrt{\text{tr}(\Sigma^2)t} + 2\|\Sigma\|t\right) \leq e^{-t}.
\]

**Lemma 7.** Recall that \(\beta_{\text{Bayes}} = \Sigma^{-1}\mu_d = \Sigma^{-1}(\mu^1 - \mu^0)\) and \(A = \{j : \{\Sigma^{-1}\mu_d\}_j \neq 0\}\). Denote by \(\beta^* = (\Sigma AA^{-1}(\mu_A^1 - \mu_A^0)\) and \(\beta_{\text{Bayes}}\) by letting \(\beta_{\text{Bayes}}^* = \beta^* = \beta_{\text{Bayes}} = 0\). Then \(\beta_{\text{Bayes}} = \beta_{\text{Bayes}}^*\).

Recall these notations for the following lemma: let \(S_0 = \{x_0^1, \cdots, x_0^m\}\) be an i.i.d. sample of class 0 and \(S_1 = \{x_1^1, \cdots, x_1^m\}\) be an i.i.d. sample of class 1 of size \(n_1\), and \(n = n_0 + n_1\). We use \(S_0\) and \(S_1\) to find an estimate of \(\beta_{\text{Bayes}}\). Let \(\hat{X}\) be the \((n \times d)\) centered predictor matrix, whose column-wise mean is zero, which can be decomposed into \(\hat{X}^0\), the \((n_0 \times d)\) centered predictor matrix based on class 0 observations and \(\hat{X}^1\), the \((n_1 \times d)\) centered predictor matrix based on class 1 observations. Let \(C^{(n)} = (\hat{X})^T \hat{X} / n\), then
\[
C^{(n)} = \frac{n_0}{n} \hat{S}_0 + \frac{n_1}{n} \hat{S}_1,
\]
where \(\hat{S}_0 = (\hat{X}^0)^T \hat{X}^0 / n_0\), and \(\hat{S}_1 = (\hat{X}^1)^T \hat{X}^1 / n_1\).

**Lemma 8.** Suppose there exists \(c > 0\) such that \(\Sigma_{jj} \leq c\) for all \(j = 1, \cdots, d\). There exist constants \(c_0\) and \(c_1, c_2\) such that for any \(\varepsilon \leq \varepsilon_0\) we have,
\[
P(\|\hat{\mu}_j^1 - \hat{\mu}_0^1\| - (\mu_j^1 - \mu_0^1) \geq \varepsilon) \leq 2\exp(\varepsilon_0 c_2 ) + 2\exp(\varepsilon_0 c_2 ), \text{ for } j = 1, \cdots, d.
\]

\[
P(\|\hat{S}_{ij}^l - \Sigma_{ij}\| \geq \varepsilon) \leq 2\exp(\varepsilon_0 c_2), \text{ for } l = 0, 1, i, j = 1, \cdots, d.
\]

\[
P(\|\hat{S}_{ij}^l - \Sigma_{ij}\| \geq \varepsilon) \leq 2\exp(\varepsilon_0 c_2), \text{ for } l = 0, 1, i, j = 1, \cdots, d.
\]

\[
P(\|\hat{S}_{ij}^l - \Sigma_{ij}\| \geq \varepsilon) \leq 2\exp(\varepsilon_0 c_2), \text{ for } l = 0, 1, i, j = 1, \cdots, d.
\]

\[
P(\|\hat{S}_{ij}^l - \Sigma_{ij}\| \geq \varepsilon) \leq 2\exp(\varepsilon_0 c_2), \text{ for } l = 0, 1, i, j = 1, \cdots, d.
\]

\[
P(\|\hat{S}_{ij}^l - \Sigma_{ij}\| \geq \varepsilon) \leq 2\exp(\varepsilon_0 c_2), \text{ for } l = 0, 1, i, j = 1, \cdots, d.
\]

\[
P(\|\hat{S}_{ij}^l - \Sigma_{ij}\| \geq \varepsilon) \leq 2\exp(\varepsilon_0 c_2), \text{ for } l = 0, 1, i, j = 1, \cdots, d.
\]

\[
P(\|\hat{S}_{ij}^l - \Sigma_{ij}\| \geq \varepsilon) \leq 2\exp(\varepsilon_0 c_2), \text{ for } l = 0, 1, i, j = 1, \cdots, d.
\]
Lemma 9. Recall that $\kappa = \|\Sigma_{A^c}^{-1}(\Sigma_{AA})^{-1}\|_\infty$, $\varphi = \|(\Sigma_{AA})^{-1}\|_\infty$ and $\Delta = \|\mu_1 - \mu_0\|_\infty$. Let $C_{A^cA}^{(n)} = \frac{n}{\bar{w}}(\tilde{X}_n^0)^\top \tilde{X}_n^0 + \frac{n}{\bar{w}}(\tilde{X}_n^1)^\top \tilde{X}_n^1 = \frac{n}{\bar{w}}\hat{S}_0^{00} + \frac{n}{\bar{w}}\hat{S}_0^{01}$, and $C_{AA}^{(n)} = \frac{n}{\bar{w}}\hat{S}_0^{10} + \frac{n}{\bar{w}}\hat{S}_0^{11}$. There exist constants $c_1$ and $\varepsilon_0$ such that for any $\varepsilon \leq \min(\varepsilon_0, 1/\varphi)$, we have

$$P \left( \|C_{A^cA}^{(n)}(\Sigma_{AA})^{-1} - \Sigma_{A^cA}(\Sigma_{AA})^{-1}\|_\infty \geq (\kappa + 1)\varepsilon \varphi (1 - \varphi \varepsilon)^{-1} \right) \leq f(d, s, n_0, n_1, \varepsilon),$$

where $f(d, s, n_0, n_1, \varepsilon) = (d + s)\exp \left(-\frac{c_1^2\varepsilon^2}{4s(n_0)}\right) + (d + s)\exp \left(-\frac{c_1^2\varepsilon^2}{4s(n_1)}\right)$, and $n = n_0 + n_1$.

Lemma 10. Let $\tilde{C}$ be in Equation (28), $C$ as in Lemma 3, and $\tilde{X}_A = \Sigma_{AA}^{-1/2}X_A$. Assume $\lambda_m = \lambda_{\min}(\Sigma_{AA}^{-1/2})$ is bounded from below, then we have

$$P_0(C_\alpha^{**} \leq s^*(X) \leq C_\alpha^{**} + \delta | X \in C) \geq (1 - \delta_3)P_0(C_\alpha^{**} \leq (\mu_0^1 - \mu_0^0)^\top \Sigma_{AA}^{-1/2} \tilde{X}_A \leq C_\alpha^{**} + \delta|\tilde{C}_0),$$

where $\delta_3 = \exp\{-n_0 \wedge n_1\}^{1/2}$.

Lemma 11. Let us denote $a = \Sigma_{AA}^{-1/2}(\mu_0^1 - \mu_0^0)$. Assume there exist $M > 0$ such that the following conditions hold:

i) $C_\alpha^{**} - (\mu_0^1 - \mu_0^0)^\top \Sigma_{AA}^{-1} \mu_0^1, \mu_0^0 \in (C^1, C^2)$ for some constants $C^1, C^2$.

ii) When $s = 1$, $a$ is a scalar. $f_{\tilde{N}(0, |a|)}$ is bounded below on interval $(C^1, C^2 + \delta^*)$ by $M$.

iii) When $s = 2$, $a = (a_1, a_2)^\top$ is a vector.

$$\left(\frac{1}{\sqrt{2\pi|a_1|}}\left(\frac{a_2^2}{a_1^2} + 1\right)\exp\left(-\frac{a_2^2 + a_2^2 - a_2^2 a_2^2}{a_1^2(a_1^2 + a_2^2)}\right)\right)$$

is bounded below on interval $t \in (C^1, C^2 + \delta^*)$ by $M$.

Then, for $s \leq 2$, for any $\delta \in (0, \delta^*)$, there exists $M_1$ which is a constant depending on $M$, such that the following inequality holds

$$P_0(C_\alpha^{**} \leq (\mu_0^1 - \mu_0^0)^\top \Sigma_{AA}^{-1/2} \tilde{X}_A \leq C_\alpha^{**} + \delta|\tilde{C}_0) \geq M_1.$$

Proposition 4. Suppose that $\lambda_{\min}(\Sigma_{AA}^{-1/2})$, the minimum eigenvalue of $\Sigma_{AA}^{-1/2}$, is bounded from below. Let us denote $a = \Sigma_{AA}^{-1/2}(\mu_0^1 - \mu_0^0)$. Let us also assume that there exists $M > 0$ such that the following conditions hold:

i) $C_\alpha^{**} - (\mu_0^1 - \mu_0^0)^\top \Sigma_{AA}^{-1} \mu_0^1, \mu_0^0 \in (C^1, C^2)$ for some constants $C^1, C^2$.

ii) When $s = 1$, $a$ is a scalar. $f_{\tilde{N}(0, |a|)}$ is bounded below on interval $(C^1 - \delta^*, C^2 + \delta^*)$ by $M$.

iii) Let $\tilde{L} = \lambda_{\min}(\Sigma_{AA}^{-1/2})c_1 s^{1/2}(n_0 \wedge n_1)^{1/4}$. When $s = 2$, $a = (a_1, a_2)$ is a vector.

$$\left(\frac{1}{\sqrt{2\pi|a_1|}}\left(\frac{a_2^2}{a_1^2} + 1\right)\exp\left(-\frac{a_2^2 + a_2^2 - a_2^2 a_2^2}{a_1^2(a_1^2 + a_2^2)}\right)\right)$$

is bounded below on interval $t \in (C^1 - \delta^*, C^2 + \delta^*)$ by $M$.

Then for $s \leq 2$, the function $s^*(\cdot)$ satisfies conditional detection condition restricted to $C$ of order $\gamma = 1$ with respect to $P_0$ at the level $(C_\alpha^{**}, \delta^*)$. In other words, Assumption 2 is satisfied.
A.3. Proofs

Proof (of Lemma 2). By Corollary 1, $k^* \leq k'$. This implies that $R_{0}(\hat{\phi}_{k^*}) \geq R_{0}(\hat{\phi}_{k})$. Moreover, by Lemma 1, for any $\delta_0 \in (0, 1)$ and $n' \geq 4/(\alpha \delta_0)$,
\[
P\left(\left|R_{0}(\hat{\phi}_{k}) - R_{0}(\phi_{0}^*)\right| > \xi_{\alpha, \delta_{0}, n'_{0}}(\delta_{0}'_0)\right) \leq \delta_{0}'.
\]
Let $\mathcal{E}_0 = \{R_{0}(\hat{\phi}_{k^*}) \leq \alpha\}$ and $\mathcal{E}_1 = \{|R_{0}(\hat{\phi}_{k}) - R_{0}(\phi_{0}^*)| \leq \xi_{\alpha, \delta_{0}, n'_{0}}(\delta_{0}')\}$. On the event $\mathcal{E}_0 \cap \mathcal{E}_1$, we have
\[
\alpha = R_{0}(\phi_{0}^*) \geq R_{0}(\hat{\phi}_{k^*}) \geq R_{0}(\hat{\phi}_{k}) \geq R_{0}(\phi_{0}^*) - \xi_{\alpha, \delta_{0}, n'_{0}}(\delta_{0}'),
\]
This implies that
\[
|R_{0}(\hat{\phi}_{k}) - R_{0}(\phi_{0}^*)| \leq \xi_{\alpha, \delta_{0}, n'_{0}}(\delta_{0}').
\]

Proof (of Lemma 3). Note that $\Sigma_{AA}^{-1/2}(X_{A} - \mu_{0}^{A}) \sim \mathcal{N}(0, I_{s})$. By Lemma 6, for all $t > 0$,
\[
P_{0}\left(\|X_{A} - \mu_{0}^{A}\|^2 > tr(\Sigma_{AA}) + 2\sqrt{tr(\Sigma_{AA}^2)t} + 2\|\Sigma_{AA}\|t\right) \leq e^{-t}.
\]
For $t = (n_0 \land n_1)^{1/2} > 1$, the above inequality implies there exists some $c'_t > 0$ such that
\[
P_{0}(\|X_{A} - \mu_{0}^{A}\|^2 > c'_t st) \leq e^{-t}.
\]
Similarly, $P_{1}(\|X_{A} - \mu_{1}^{A}\|^2 > c'_t st) \leq e^{-t}$. Let $C^0 = \{X : \|X_{A} - \mu_{0}^{A}\|^2 \leq c'_t st\}$ and $C^1 = \{X : \|X_{A} - \mu_{1}^{A}\|^2 \leq c'_t st\}$. There exists some $c'_t > 0$, such that both $C^0$ and $C^1$ are subsets of $C = \{X : \|X_{A}\| \leq c'_t s^{1/2}t^{1/2}\}$. Then $P_{0}(X \in C) \geq 1 - \delta_3$ and $P_{1}(X \in C) \geq 1 - \delta_3$, for $\delta_3 = \exp\{-(n_0 \land n_1)^{-1/2}\}$.

By Proposition 3, for $\delta_1 \geq \delta_3^1$ and $\delta_2 \geq \delta_3^2$, we have with probability at least $1 - \delta_1 - \delta_2$, $\beta_{\text{lasso}}^{\text{Bayes}} = \beta_{\text{Bayes}}^{\text{Bayes}} = 0$. Moreover,
\[
\|\hat{s} - s^{*}\|_{\infty, C} \leq \max_{x \in C} |x^{T}_{A} \hat{\beta}_{\text{lasso}}^{A} - x^{T}_{A} \hat{\beta}_{\text{Bayes}}^{A}| + \max_{x \in C} |x^{T}_{A} \hat{\beta}_{\text{lasso}}^{A} - x^{T}_{A} \hat{\beta}_{\text{Bayes}}^{A}|
\]
\[
\leq \max_{x \in C} \|x^{T}_{A} \hat{\beta}_{\text{lasso}}^{A} - x^{T}_{A} \hat{\beta}_{\text{Bayes}}^{A}\|_{\infty, C} \max_{x \in C} \|X_{A}\|_{1}
\]
\[
\leq \|\hat{\beta}_{\text{lasso}}^{A} - \hat{\beta}_{\text{Bayes}}^{A}\|_{\infty, C} \cdot \sqrt{s} \max_{x \in C} \|X_{A}\|_{2}
\]
\[
\leq 4\varphi \cdot c'_t s^{1/4} (n_0 \land n_1)^{1/4}.
\]
where the last inequality uses a relation $\beta^{*} = \beta_{\text{Bayes}}^{\text{Bayes}}$, which is derived in Lemma 7.

Proof (of Lemma 4). Note that by Lemma 3, $P_{0}(X \in C) \geq 1 - \exp\{-(n_0 \land n_1)^{-1/2}\}$, so we have
\[
|R_{0}(\hat{\phi}_{k}) - R_{0}(\phi_{0}^{*})|
\]
\[
\geq |R_{0}(\hat{\phi}_{k^*}) - R_{0}(\phi_{0}^*)|P_{0}(X \in C) + |R_{0}(\hat{\phi}_{k^*} - C^0) - R_{0}(\phi_{0}^{*} - C^0)|P_{0}(X \in C^0)\n\]
\[
\geq |R_{0}(\hat{\phi}_{k^*} - C^0) - R_{0}(\phi_{0}^{*})|P_{0}(X \in C) - |R_{0}(\hat{\phi}_{k^*} - C^0) - R_{0}(\phi_{0}^{*} - C^0)|P_{0}(X \in C^0)\n\]
\[
\geq |R_{0}(\hat{\phi}_{k^*} - C^0) - R_{0}(\phi_{0}^{*})|(1 - \exp\{-(n_0 \land n_1)^{-1/2}\}) - 1 \cdot \exp\{-(n_0 \land n_1)^{-1/2}\}.
\]
Lemma 2 says that
\[
P\{|R_{0}(\hat{\phi}_{k}) - R_{0}(\phi_{0}^{*})| > \xi_{\alpha, \delta_{0}, n'_{0}}(\delta_{0}')\} \leq \delta_{0} + \delta_{0}'.
\]
This combined with the above inequality chain implies
\[
P\{|R_{0}(\hat{\phi}_{k} - C) - R_{0}(\phi_{0}^{*} - C)| > \frac{\xi_{\alpha, \delta_{0}, n'_{0}}(\delta_{0}') + \exp\{-(n_0 \land n_1)^{-1/2}\}}{1 - \exp\{-(n_0 \land n_1)^{-1/2}\}}\} \leq \delta_{0} + \delta_{0}'.
\]
Since $\exp\{-(n_0 \land n_1)^{-1/2}\} \leq 1/2$, the conclusion follows.
Proof (of Lemma 7). Note that $\mu^1 - \mu^0 = \Sigma \beta_{\text{Bayes}}$. After shuffling the A coordinates to the front if necessary, we have

$$\mu^1 - \mu^0 = \begin{bmatrix} \Sigma_{AA} & \Sigma_{A^cA} \\ \Sigma_{A^cA} & \Sigma_{A^cA^c} \end{bmatrix} \begin{bmatrix} \beta_{\text{Bayes}} \\ \beta_{A^c} \end{bmatrix}. $$

Then, $\mu^1_A - \mu^0_A = (\Sigma_{AA})^{-1} \beta_{\text{Bayes}}$ as $\beta_{A^c} = 0 \in \mathbb{R}^{|A^c|}$ by definition. Therefore we have,

$$\beta^* = \Sigma_{AA}^{-1}(\mu^1_A - \mu^0_A) = \beta_{\text{Bayes}},$$

which combined with $\beta_{A^c} = 0$ leads to $\beta_{A^c} = \beta_{\text{Bayes}}$.

Proof (of Lemma 8). Inequalities (12)-(17) can be proved similarly as in Mai et al. (2012), so proof is omitted.

Inequalities (18)-(20) can be proved by applying (13)-(15) respectively and observe that $A + B \geq \varepsilon$ implies $A \geq \varepsilon/2$ or $B \geq \varepsilon/2$. More concretely, they are proven by the following arguments:

$$\mathbb{P}\left( |C_{ij}^{(n)}| - \Sigma_{ij} \geq \varepsilon \right) = \mathbb{P}\left( \left| \frac{n_0}{n} \bar{S}_{ij}^0 + \frac{n_1}{n} \bar{S}_{ij}^1 - \Sigma_{ij} \right| \geq \varepsilon \right) \leq \mathbb{P}\left( \left| \frac{n_0}{n} \bar{S}_{ij}^0 - \Sigma_{ij} \right| \geq \varepsilon/2 \right) + \mathbb{P}\left( \left| \frac{n_1}{n} \bar{S}_{ij}^1 - \Sigma_{ij} \right| \geq \varepsilon/2 \right) \leq 2\exp\left( -c_1 \varepsilon^2 n^2 \right) + 2\exp\left( -c_1 \varepsilon^2 n^2 \right).
$$

$$\mathbb{P}\left( |C_{AA}^{(n)}| - \Sigma_{AA} \geq \varepsilon \right) = \mathbb{P}\left( \left| \frac{n_0}{n} \bar{S}_{AA}^0 + \frac{n_1}{n} \bar{S}_{AA}^1 - \Sigma_{AA} \right| \geq \varepsilon \right) \leq \mathbb{P}\left( \left| \frac{n_0}{n} \bar{S}_{AA}^0 - \Sigma_{AA} \right| \geq \varepsilon/2 \right) + \mathbb{P}\left( \left| \frac{n_1}{n} \bar{S}_{AA}^1 - \Sigma_{AA} \right| \geq \varepsilon/2 \right) \leq 2s^2 \exp\left( -c_1 \varepsilon^2 n^2 \right) + 2s^2 \exp\left( -c_1 \varepsilon^2 n^2 \right).
$$

$$\mathbb{P}\left( |C_{A^cA}^{(n)}| - \Sigma_{A^cA} \geq \varepsilon \right) = \mathbb{P}\left( \left| \frac{n_0}{n} \bar{S}_{A^cA}^0 + \frac{n_1}{n} \bar{S}_{A^cA}^1 - \Sigma_{A^cA} \right| \geq \varepsilon \right) \leq \mathbb{P}\left( \left| \frac{n_0}{n} \bar{S}_{A^cA}^0 - \Sigma_{A^cA} \right| \geq \varepsilon/2 \right) + \mathbb{P}\left( \left| \frac{n_1}{n} \bar{S}_{A^cA}^1 - \Sigma_{A^cA} \right| \geq \varepsilon/2 \right) \leq (d - s)s \exp\left( -c_1 \varepsilon^2 n^2 \right) + (d - s)s \exp\left( -c_1 \varepsilon^2 n^2 \right).
$$

Proof (of Lemma 9). Let $\eta_1 = \|\Sigma_{AA} - C_{AA}^{(n)}\|_\infty$, $\eta_2 = \|\Sigma_{A^cA} - C_{A^cA}^{(n)}\|_\infty$, and $\eta_3 = \|(C_{AA}^{(n)})^{-1} - (\Sigma_{AA})^{-1}\|_\infty$.

$$\|C_{A^cA}^{(n)}(C_{AA}^{(n)})^{-1} - \Sigma_{A^cA}(\Sigma_{AA})^{-1}\|_\infty \leq \|C_{A^cA}^{(n)} - \Sigma_{A^cA}\|_\infty \times \|(C_{AA}^{(n)})^{-1} - (\Sigma_{AA})^{-1}\|_\infty + \|C_{A^cA}^{(n)} - \Sigma_{A^cA}\|_\infty \times \|\Sigma_{AA} - C_{AA}^{(n)}\|_\infty \times \|(\Sigma_{AA})^{-1}\|_\infty + \|\Sigma_{A^cA}^{(n)} - (\Sigma_{AA})^{-1}\|_\infty \times \|\Sigma_{AA} - C_{AA}^{(n)}\|_\infty \times \|(C_{AA}^{(n)})^{-1} - (\Sigma_{AA})^{-1}\|_\infty \leq (\kappa \eta_1 + \eta_2)(\varphi + \eta_3).$$
Moreover, \( \eta_1 \leq \| (C_{AA}^{(n)})^{-1} \|_\infty \times \| C_{AA}^{(n)} - \Sigma_{AA} \|_\infty \times \| (\Sigma_{AA})^{-1} \|_\infty \leq (\varphi + \eta_3) \varphi \eta_1 \). Hence, if \( \varphi \eta_1 < 1 \), we have \( \eta_3 \leq \varphi^2 \varphi_1 (1 - \varphi \eta_1)^{-1} \). Hence we have,

\[
\| C_{A'A}^{(n)} (C_{AA}^{(n)})^{-1} - \Sigma_{A'A} (\Sigma_{AA})^{-1} \|_\infty \leq (\kappa \eta_1 + \eta_2) \varphi (1 - \varphi \eta_1)^{-1}.
\]

Then we consider the event \( \max(\eta_1, \eta_2) \leq \varepsilon \). Note that \( \varepsilon < 1/\varphi \) ensures that \( \varphi \eta_1 < 1 \) on this event. The conclusion follows from inequalities (19) and (20).

**Proof (of Lemma 10).** Since \((\Sigma^{-1} \mu_0)_A = \Sigma_{AA}^{-1}(\mu^1_A - \mu^0_A)\) (by Lemma 7) and \( \tilde{C}^0 \subset \tilde{C}^0 \subset \tilde{C} \), we have

\[
P_0(C_{\alpha}^{**} \leq \alpha^*(X) \leq C_{\alpha}^{**} + \delta | X \in \mathcal{C})
\]

\[
= P_0(C_{\alpha}^{**} \leq \alpha^*(X) \leq C_{\alpha}^{**} + \delta, \tilde{C}^0)
\]

\[
\geq (1 - \delta) P_0(C_{\alpha}^{**} \leq \alpha^*(X) \leq C_{\alpha}^{**} + \delta),
\]

where the last inequality uses \( P_0(\tilde{C}^0) \geq 1 - \delta \). To derive this inequality, let \( V^0 \) (defined in the proof of Proposition 4) play the role of \( x \) and take \( A = I_s \) in Lemma 6, then we have

\[
P \left( \| V^0 \|^2 \geq s + 2\sqrt{s} + t \right) \leq e^{-t}, \quad \text{for all } t > 0.
\]

For \( s, t \in \mathbb{N} \), the above inequality clearly implies \( P(\| V^0 \|^2 \geq 4st) \leq \exp(-t) \). Take \( t = (n_0 \wedge n_1)^{1/2} \), then as long as \( c_1 \geq 2/\lambda_m \),

\[
\{ x : \| V^0 \|^2 \leq 4st \} \subset \{ x : \| V^0 \|^2 \leq \lambda^2_m (c_1^2) \}
\]

Since \( \lambda_m \) is bounded from below, we can certainly take \( c_1^2 \geq 2/\lambda_m \) is the proof of Lemma 3 in constructing \( \tilde{C}^0 \). Therefore, \( P(\| V^0 \|^2 \leq s + 2\sqrt{s} + t) \geq 1 - \exp(-t) \) implies that \( P(\tilde{C}^0) \geq 1 - \exp(-t) \) for \( t = (n_0 \wedge n_1)^{1/2} \).

**Proof (of Lemma 11).** Since \( \tilde{V}^0 = \tilde{X}_A - \Sigma_{AA}^{-1/2} \mu_A \), it follows that,

\[
P_0(C_{\alpha}^{**} \leq (\mu_A^1 - \mu_A^0)^\top \Sigma_{AA}^{-1/2} \tilde{X}_A \leq C_{\alpha}^{**} + \delta),
\]

\[
\quad = P_0(C_{\alpha}^{**} - (\mu_A^1 - \mu_A^0)^\top \Sigma_{AA}^{-1/2} \tilde{X}_A \leq a^\top V^0 \leq C_{\alpha}^{**} + \delta, \tilde{C}^0)
\]

By Mukerjee and Ong (2015), the probability density function of \( V^0 | \tilde{C}^0 \) is given by

\[
f_{V^0 | \tilde{C}^0}(v) = \begin{cases} k_{L,A} \Pi_{i=1}^s \phi(v_i) & \text{if } \| v \| \leq \tilde{L} \\ 0, & \text{otherwise,} \end{cases}
\]

(21)

where \( \phi \) is the pdf for the standard normal random variable, \( \tilde{L} \) is defined in equation (28), and \( k_{L,A} \) is a normalizing constant. Note that \( k_{L,s} \) is a monotone decreasing function of \( \tilde{L} \) for each \( s \), and when \( \tilde{L} \) goes to infinity, \( k_{L,s} = k_0 \) is a positive constant. Therefore, \( k_{L,s} \) is bounded below by \( k_0 \). Since we only consider \( s \in \{1, 2\} \), we can take \( k_0 \) as a universal constant independent of \( s \), and \( k_{L,s} \) is bounded below by \( k_0 \) universally.

Let \( f_{a^\top V^0 | \tilde{C}^0}(z) \) be the density of \( a^\top V^0 | \tilde{C}^0 \). Thus, we want to lower bound

\[
P_0(C_{\alpha}^{**} \leq (\mu_A^1 - \mu_A^0)^\top \Sigma_{AA}^{-1/2} \tilde{X}_A \leq a^\top V^0 \leq C_{\alpha}^{**} + \delta, \tilde{C}^0)
\]

\[
= \int_{C_{\alpha}^{**} - (\mu_A^1 - \mu_A^0)^\top \Sigma_{AA}^{-1/2} \tilde{X}_A \leq a^\top V^0 \leq C_{\alpha}^{**} + \delta, \tilde{C}^0} f_{a^\top V^0 | \tilde{C}^0}(z) dz.
\]
Let us analyze \( f_{a|V_0|C_0}(z) \) when \( s = 1 \) and \( s = 2 \).

**Case 1** \((s = 1)\): \( a \) is a scalar. Hence

\[
f_{a|V_0|C_0}(z) = \begin{cases} \frac{k_{L,2}}{|a|} \phi\left(\frac{z}{a}\right), & \text{for } |z| \leq |a|L \\ 0, & \text{otherwise}, \end{cases}
\]

which is the density function of a truncated Normal random variable with parent distribution \( \mathcal{N}(0,|a|) \) symmetrically truncated to \(-|a|L\) and \(|a|L\), i.e. \( TN(0,|a|, -|a|L, |a|L) \). Here \(|a|\) is the standard deviation of the parent Normal distribution. Therefore,

\[
f_{a|V_0|C_0}(z) \geq f_{N(0,|a|)}(z), \text{ for } |z| \leq |a|\tilde{L}.
\]

This implies

\[
\int_{C_{\alpha}^{\ast}} f_{a|V_0|C_0}(z) dz 
\geq \delta \min \left\{ f_{N(0,|a|)}(C_{\alpha}^{\ast} - \mu_1^0 \Sigma_{AA}^{-1} \mu_2^0), f_{N(0,|a|)}(C_{\alpha}^{\ast} - \mu_1^0 \Sigma_{AA}^{-1} \mu_2^0 + \delta^*) \right\} 
\geq \delta M.
\]

where the inequality follows from the mean-value theorem and our assumption (ii).

**Case 2** \((s = 2)\): \( a = (a_1, a_2) \) is a vector. Now let us do the following change of variable from \((V_1, V_2) = V^0|\tilde{C}\) to \((Z_1, Z_2) = (a^\top V^0|\tilde{C}, V_2)\).

\[
\begin{cases}
Z_1 = a_1 V_1 + a_2 V_2 \\
Z_2 = V_2
\end{cases}
\]

and thus \((V_1 = Z_1 - a_2 Z_2)\), \( V_2 = Z_2 \).

The original event \(S_{V_1, V_2} = \{V_1^2 + V_2^2 \leq \tilde{L}^2\}\) is equivalent to

\[
S_{Z_1, Z_2} = \left( \frac{Z_1 - a_2 Z_2}{a_1} \right)^2 + Z_2^2 \leq \tilde{L}^2
\]

\[
\iff \left( a_2^2 + 1 \right) \left( Z_2 - \left( \frac{a_1 a_2}{a_1^2 + a_2^2} \right) Z_1 \right)^2 \geq a_2^2 + a_1^2 \left( \frac{a_1 a_2}{a_1^2 + a_2^2} \right)^2 \leq \tilde{L}^2.
\]

Now for any \(z_1\), the marginal density of \(a^\top V^0\) can be carried out as

\[
f_{Z_1}(z_1) 
= \int_{S_{z_1, z_2}} k_{L,2} \frac{\phi\left( \frac{z_1 - a_2 z_2}{a_1} \right)}{|a_1|} \phi(z_2) dz_2
\]

\[
= \int_{S_{z_1, z_2}} k_{L,2} \exp\left\{-\frac{1}{2} \left( \frac{z_1 - a_2 z_2}{a_1} \right)^2 - \frac{z_2^2}{2} \right\} dz_2
\]

\[
= \left( k_{L,2} \exp\left\{-\frac{a_1}{2} \left( a_2^2 + a_1^2 - a_2^2 \right) \right\} \right) \int_{S_{z_1, z_2}} \exp\left\{-\frac{z_2 - \left( \frac{a_1 a_2}{a_1^2 + a_2^2} \right) z_1}{\frac{a_2^2}{a_1^2 + a_2^2}} \right\} dz_2
\]

\[
= \left( k_{L,2} \exp\left\{-\frac{a_1}{2} \left( a_2^2 + a_1^2 - a_2^2 \right) \right\} \right) \int_{S_{z_1, z_2}} \phi_N\left( \frac{z_2}{\sqrt{a_1^2 + a_2^2}} \right) dz_2
\]
This implies
\[
\int C_\alpha^{\ast}\sigma - (\mu_\alpha - \mu_0)\Sigma_{\alpha\alpha}^{-1}\mu_\alpha
f_{\alpha,\nu}(z)dz
\geq \delta \min\{f_z((C_\alpha^{\ast} - (\mu_\alpha - \mu_0)\Sigma_{\alpha\alpha}^{-1}\mu_\alpha), f_z((C_\alpha^{\ast} - (\mu_\alpha - \mu_0)\Sigma_{\alpha\alpha}^{-1}\mu_\alpha + \delta^*)
\geq \delta M k_0.
\]
where the inequality follows from the mean-value theorem and our assumption (iii).

We can safely conclude our proof by combining cases \(s = 1\) and \(s = 2\), and taking \(M_1 = \min\{M, M k_0\}\).

**Proof (of Proposition 3).** The proof is largely identical to that of Theorem 1 in Mai et al. (2012), except the differences due to a different sampling scheme.

Similarly to Mai et al. (2012), by the definition of \(\hat{\beta}_A\), we can write \(\hat{\beta}_A = (n^{-1}X_A^T \tilde{X}_A)^{-1}\{(\hat{\mu}_A - \mu_0) - \lambda t_A/2\}\), where \(t_A\) represents the subgradient such that \(l_j = \text{sign}(\beta_j)\) if \(\beta_j \neq 0\) and \(-1 < t_j < 1\) if \(\beta_j = 0\). To show that \(\hat{\beta}_\text{lasso} = (\hat{\beta}_A, 0)\), it suffices to verify that
\[
\|n^{-1}X_A^T \tilde{X}_A\hat{\beta}_A - (\hat{\mu}_A - \mu_0)\|_\infty \leq \lambda/2.
\tag{23}
\]
The left-hand side of (23) is equal to
\[
\|C_A^{(n)}(C_A^{(n)})^{-1}(\hat{\mu}_A - \mu_0) - C_A^{(n)}(C_A^{(n)})^{-1}\lambda t_A/2 - (\hat{\mu}_A - \mu_0)\|_\infty.
\tag{24}
\]
Using \(\Sigma_{\alpha\alpha}^{-1}(\mu_\alpha - \mu_0) = (\mu_\alpha - \mu_0)\), (24) is bounded from above by
\[
U_1 = \|C_A^{(n)}(C_A^{(n)})^{-1} - \Sigma_{\alpha\alpha}^{-1}\|_\infty \Delta + \|\hat{\mu}_A - \mu_0\|_\infty
+ \|C_A^{(n)}(C_A^{(n)})^{-1} - \Sigma_{\alpha\alpha}^{-1}\|_\infty + \|\hat{\mu}_A - \mu_0\|_\infty
+ \|C_A^{(n)}(C_A^{(n)})^{-1} - \Sigma_{\alpha\alpha}^{-1}\|_\infty + \|\hat{\mu}_A - \mu_0\|_\infty/\lambda/2.
\]
If \(\|C_A^{(n)}(C_A^{(n)})^{-1} - \Sigma_{\alpha\alpha}^{-1}\|_\infty \leq \Delta(1 + \kappa)\varphi(1 - \varphi)^{-1}\) (invoke Lemma 9), and \(\|\hat{\mu}_A - \mu_0\|_\infty \leq 4^{-1}(1 - \kappa - 2\varphi)/(1 + \kappa)\), and given \(\varepsilon \leq \min[e_0, \lambda(1 - \kappa)(4\varphi^{-1}(\lambda/2 + (1 + \kappa)\Delta)^{-1}]\), then \(U_1 \leq \lambda/2\).

Therefore, by Lemmas 8 and 9, we have
\[
\mathbb{P}((\|n^{-1}X_A^T \tilde{X}_A\hat{\beta}_A - (\hat{\mu}_A - \mu_0)\|_\infty \leq \lambda/2)
\geq 1 - 2d \exp(-ne^{-2\varepsilon^2}c_2) - 2d \exp(-n_1 \varepsilon^2 c_2) - f(d, s, n_0, n_1, (\kappa + 1)\varepsilon \varphi/(1 - \varphi)^{-1})
\]
where \(\varepsilon^* = 4^{-1}(1 - \kappa - 2\varphi)/(1 + \kappa)\), and \(f\) is the same as in Lemma 9. Tidy up the algebra a bit, we can write
\[
\delta^*_\varepsilon = \sum_{i=0}^1 2d \exp\left(-c_2n_0 \frac{\lambda^2(1 - \kappa - 2\varepsilon^2)^2}{16(1 + \kappa)^2}\right) + f(d, s, n_0, n_1, (\kappa + 1)\varepsilon \varphi/(1 - \varphi)^{-1})
\]
To prove the 2nd conclusion, note that
\[
\hat{\beta}_A = (\Sigma_{\alpha\alpha})^{-1}(\mu_\alpha - \mu_0) + (C_A^{(n)})^{-1}\{(\hat{\mu}_A - \mu_0) - (\mu_\alpha - \mu_0)\}
+ \{(C_A^{(n)})^{-1} - (\Sigma_{\alpha\alpha})^{-1}\}(\mu_\alpha - \mu_0) - \lambda(C_A^{(n)})^{-1}t_A/2.
\tag{25}
\]
\[
\beta_A = (\Sigma_{\alpha\alpha})^{-1}(\mu_\alpha - \mu_0) + (C_A^{(n)})^{-1}\{(\hat{\mu}_A - \mu_0) - (\mu_\alpha - \mu_0)\}
+ \{(C_A^{(n)})^{-1} - (\Sigma_{\alpha\alpha})^{-1}\}(\mu_\alpha - \mu_0) - \lambda(C_A^{(n)})^{-1}t_A/2.
\tag{26}
\]
Let $\xi = |\beta^*|_{\min}/(\Delta \varphi)$. Write $\eta_1 = \|\Sigma_{AA} - C(n)_{AA}\|_{\infty}$ and $\eta_3 = \|(C(n)_{AA})^{-1} - \Sigma^{-1}_{AA}\|_{\infty}$. Then for any $j \in A$,

$$|\hat{\beta}_j| \geq \xi \Delta \varphi - (\eta_3 + \varphi)\{\lambda/2 + \|(\hat{\mu}_A - \mu_A^0) - (\mu_A - \mu_A^0)\|_{\infty}\} - \eta_3 \Delta.$$ 

When $\eta_1 \varphi < 1$, we have shown that $\eta_3 < \varphi^2 \eta_1 (1 - \eta_1 \varphi)^{-1}$ in Lemma 9. Therefore,

$$|\hat{\beta}_j| \geq \xi \Delta \varphi - (1 - \eta_1 \varphi)^{-1}\{\lambda \varphi/2 + \|(\hat{\mu}_A - \mu_A^0) - (\mu_A - \mu_A^0)\|_{\infty}\varphi + \varphi^2 \eta_1 \Delta\}.$$ 

Because $\|\beta^*\|_{\infty} \leq \Delta \varphi$, $\xi \leq 1$. Hence $\lambda \leq |\beta^*|_{\min}/(2\varphi) \leq 2|\beta^*|_{\min}/[(3 + \xi)\varphi]$. The events $\eta_1 \leq \varepsilon$ and $\|(\hat{\mu}_A - \mu_A^0) - (\mu_A - \mu_A^0)\|_{\infty} \leq \varepsilon$, together with restriction on $\varepsilon$ and $\lambda$ in the assumption, we have $\|\hat{\beta}_A - \beta^*\|_{\infty} \leq 4\varphi \lambda$.

Hence,

$$\mathbb{P}(\|\hat{\beta}_A - \beta^*\|_{\infty} \leq 4\varphi \lambda) \geq 1 - \sum_{l=0}^1 2s \exp(-nl^2c_2) - \sum_{l=0}^1 2s^2 \exp\left(-\frac{c_1l^2n^2}{4sl^2}\right).$$

To prove the 3rd conclusion, equation (25) and $\eta_1 \varphi < 1$ imply that

$$\|\hat{\beta}_A - \beta^*\|_{\infty} \leq (1 - \eta_1 \varphi)^{-1}\{\lambda \varphi/2 + \|(\hat{\mu}_A - \mu_A^0) - (\mu_A - \mu_A^0)\|_{\infty}\varphi + \varphi^2 \eta_1 \Delta\}.$$ 

On the events $\{\eta_1 < \varepsilon\}$ and $\{\|(\hat{\mu}_A - \mu_A^0) - (\mu_A - \mu_A^0)\|_{\infty} \leq \varepsilon\}$, and under restrictions for $\varepsilon$ and $\lambda$ in the assumption, we have $\|\hat{\beta}_A - \beta^*\|_{\infty} \leq 4\varphi \lambda$.

Hence,

$$\mathbb{P}(\|\hat{\beta}_A - \beta^*\|_{\infty} \leq 4\varphi \lambda) \geq 1 - \sum_{l=0}^1 2s \exp(-nl^2c_2) - \sum_{l=0}^1 2s^2 \exp\left(-\frac{c_1l^2n^2}{4sl^2}\right).$$

**Proof (of Proposition 4).** For simplicity, we will derive the lower bound for one of the two probabilities in the definition:

$$P_0[C^*_a \leq s^*(X) \leq C^*_a + \delta|X \in C] \geq (1 - \delta)\mu_1 \delta, \text{ for } \delta \in (0, \delta^*) \tag{27}$$

The lower bound for the other probability can be derived similarly.

Recall that $C^0 = \{X \in \mathbb{R}^d : \|X - \mu^0_A\| \leq c'_1s^{1/2}(n_0 \wedge n_1)^{1/4} \leq L\}$ (in the proof of Lemma 3). Let $V^0 = \Sigma^{-1/2}_{AA} (X_A - \mu^0_A) = \tilde{X}_A - \Sigma^{-1/2}_{AA} \mu^0_A$, where $\tilde{X}_A = \Sigma^{-1/2}_{AA} X_A$, then $V^0 \sim \mathcal{N}(0, I_s)$ under $P_0$. Define an event

$$\tilde{C}^0 = \{X \in \mathbb{R}^d : \|V^0\| \leq \lambda_m L \leq \hat{L}\}, \tag{28}$$

where $\lambda_m = \lambda_{\min}(\Sigma^{-1/2}_{AA})$ and $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a matrix. Since $\|V^0\| \geq \lambda_{\min}(\Sigma^{-1/2}_{AA})\|X_A - \mu^0_A\| = \lambda_m\|X_A - \mu^0_A\|$, we have $\tilde{C}^0 \subset C^0$. Then inequality (27) holds by invoking Lemma 10 and Lemma 11.

**Remark 2.** Proof of Proposition 4 indicates that the same conclusion would hold for a general $s \in \mathbb{N}$, if the density of $(\mu^0_A - \mu^0_A)^{\top} \Sigma^{-1/2}_{AA} \tilde{X}_A|C^0$ is bounded below on $(C^1 - \delta^*, C^2 + \delta^*)$ by some constant.

**Proof (of Theorem 1).** The first inequality follows from Proposition 1 and the choice of $k^*$ in (8) of the main paper. In the following, we prove the second inequality.

Let $G^* = \{s^* \leq C^*_a\}$ and $\hat{G} = \{\hat{s} \leq \hat{C}_a\}$. The excess type II error can be decomposed as

$$P_1(\hat{G}) - P_1(G^*) = \int_{\hat{G} \setminus G^*} |r - C_a|dP_0 + \int_{G^* \setminus \hat{G}} |r - C_a|dP_0 + C_a\{R_0(\phi_a) - R_0(\hat{\phi}_k^*)\} \tag{29}.$$ 

In the above decomposition, the third part can be bounded via Lemma 2. For the first two parts, let

$$T = \|\hat{s} - s^*\|_{\infty, C} := \max_{x \in C} |\hat{s}(x) - s^*(x)|,$$ and
\[ \Delta R_{0,C} := |R_0(\phi^* \mid C) - R_0(\hat{\phi}_k \mid C)| = |P_0(s^*(X) > C^{**} \mid X \in C) - P_0(\hat{s}(X) > \hat{C}_\alpha \mid X \in C)|, \]

where \( C \) is defined in Lemma 3. A high probability bound for \( \Delta R_{0,C} \) was derived in Lemma 4.

It follows from Lemma 1 that if \( n'_0 \geq \max\{4/(\alpha \sigma_0), \delta_0^{-2}, (\frac{1}{M_1} M_1 \delta^{**})^{-4}\} \),

\[ \xi_{\alpha, \delta_0, n'_0}(\delta_0') \leq \frac{5}{2}(n'_0)^{-1/4} \leq \frac{1}{4} M_1(\delta^*)^2. \]

Because the lower bound in the detection condition should be smaller than 1 to make sense, \( M_1 \delta^{**} < 1 \).

This together with \( n_0 \land n_1 \geq [-\log(M_1 \delta^{**}/4)]^2 \) implies that \( \exp\{-(n_0 \land n_1)^{1/2}\} \leq M_1 \delta^{**}/4 \).

Let \( \mathcal{E}_2 = \{R_{0,C} \leq 2[\xi_{\alpha, \delta_0, n'_0}(\delta'_0) + \exp\{-(n_0 \land n_1)^{1/2}\}]\} \). On the event \( \mathcal{E}_2 \) we have

\[ \left\{ \frac{R_{0,C}}{M_1} \right\}^{1/2} \leq \left\{ \frac{2[\xi_{\alpha, \delta_0, n'_0}(\delta_0') + \exp\{-(n_0 \land n_1)^{1/2}\}]}{M_1} \right\}^{1/2} \leq \delta^*. \]

To find the relation between \( C^{**} \) and \( \hat{C}_\alpha \), we invoke the detection condition as follows:

\[ P_0\left(s^*(X) \geq C^{**} + (\Delta R_{0,C}/M_1)^{1/2} \mid X \in C\right) \]

\[ = R_0(\phi^* \mid C) - R_0(C^{**} < s^*(X) < C^{**} + (\Delta R_{0,C}/M_1)^{1/2} \mid X \in C) \]

\[ \leq R_0(\phi^* \mid C) - \Delta R_{0,C} \quad \text{(by detection condition)} \]

\[ \leq R_0(\hat{\phi}_k \mid C) = P_0(\hat{s}(X) > \hat{C}_\alpha \mid X \in C) \]

\[ \leq P_0(s^*(X) > \hat{C}_\alpha - T \mid X \in C). \]

This implies that \( C^{**} + (\Delta R_{0,C}/M_1)^{1/2} \geq \hat{C}_\alpha - T \), which further implies that

\[ \hat{C}_\alpha \leq C^{**} + (\Delta R_{0,C}/M_1)^{1/2} + T. \]

Note that

\[ \mathcal{C} \cap (\hat{G} \setminus G^*) \]

\[ = \mathcal{C} \cap \{s^* > C^{**} : \hat{s} \leq \hat{C}_\alpha\} \]

\[ = \mathcal{C} \cap \{s^* > C^{**} : \hat{s} \leq C^{**} + (\Delta R_{0,C}/M_1)^{1/2} + T\} \]

\[ \subset \mathcal{C} \cap \{C^{**} + (\Delta R_{0,C}/M_1)^{1/2} + 2T \geq s^* \geq C^{**} + (\Delta R_{0,C}/M_1)^{1/2} + T \} \]

\[ \subset \mathcal{C} \cap \{C^{**} + (\Delta R_{0,C}/M_1)^{1/2} + 2T \geq s^* \geq C^{**}\}. \]

We decompose as follows

\[ \int_{(\hat{G} \setminus G^*_{\tau})} |r - C_\alpha|dP_0 = \int_{(\hat{G} \setminus G^*_{\tau}) \cap \mathcal{C}} |r - C_\alpha|dP_0 + \int_{(\hat{G} \setminus G^*_{\tau}) \cap \mathcal{C}^c} |r - C_\alpha|dP_0 =: (I) + (II). \]

To bound (I), recall that

\[ r(x) = \frac{f_{\delta}(x)}{f_{\beta}(x)} = \exp\left(s^*(x) - \mu^\top \Sigma^{-1} \mu_d\right), \]

and that, \( r(x) > C_\alpha \) is equivalent to \( s^*(x) > C^{**} = \log C_\alpha + \mu^\top \Sigma^{-1} \mu_d \). By the mean value theorem, we have

\[ |r(x) - C_\alpha| = e^{-n^\top \Sigma^{-1} \mu_d} |e^{s^*(x)} - e^{C^{**}}| = e^{-n^\top \Sigma^{-1} \mu_d} e^{z'} |s^*(x) - C^{**}|, \]

where \( z' \) is some quantity between \( s^*(x) \) and \( C^{**} \). Denote by \( \mathcal{C}_1 = \{x : C^{**} + (\Delta R_{0,C}/M_1)^{1/2} + 2T \geq s^*(x) \geq C^{**}\} \). Restricting to \( \mathcal{C} \cap \mathcal{C}_1 \), we have

\[ z' \leq C^{**} + (\Delta R_{0,C}/M_1)^{1/2} + 2T. \]
This together with $C \cap (\hat{G} \setminus G^*) \subset C \cap C_1$ implies that

\[
(I) \leq \int_{C \cap C_1} |r - C_\alpha| dP_0 = \int_{C \cap C_1} \exp\{s - \mu_a^T \Sigma^{-1}_a \mu_d | dP_0 \}
\leq \int_{C \cap C_1} \exp\left\{C_{\alpha}^{**} + (\Delta R_{0,C}/M_1)^{1/2} + 2T - \mu_a^T \Sigma^{-1}_a \mu_d \right\} |s(x) - C_{\alpha}^{**}| dP_0.
\]

Since $C_{\alpha}$ and $\mu_a^T \Sigma^{-1}_a \mu_d$ are assumed to be bounded, $C_{\alpha}^{**}$ is also bounded.

Let $\mathcal{E}_2 = \{R_{0,C} \leq 2[\xi_{\alpha, \delta, \gamma}^0(\delta_0) + \exp\{-(n_0 \wedge n_1)^{1/2}\}] \}$. By Lemma 4, $\mathbb{P}(\mathcal{E}_2) \geq 1 - \delta_0 - \delta_0'$. Let $\mathcal{E}_3 = \{T \leq 4c_1 \varphi \lambda s(n_0 \wedge n_1)^{1/4}\}$. By Lemma 3, $\mathbb{P}(\mathcal{E}_3) \geq 1 - \delta_1 - \delta_2$. Restricting to the event $\mathcal{E}_2 \cap \mathcal{E}_3$, $R_{0,C}$ and $T$ are bounded. Therefore on the event $\mathcal{E}_2 \cap \mathcal{E}_3$, there exists a positive constant $c'$ such that

\[
(I) \leq c' \int_{C \cap C_1} |s(x) - C_{\alpha}^{**}| dP_0 \leq c' \left( (\Delta R_{0,C}/M_1)^{1/2} + 2T \right) P_0(C \cap C_1).
\]

Note that by the margin assumption (we know $\gamma = 1$, but we choose to reserve the explicit dependency of $\gamma$ by not substituting the numerical value),

\[
P_0(C \cap C_1) = P_0(C_{\alpha}^{**} + (\Delta R_{0,C}/M_1)^{1/2} + 2T \geq s^* \geq C_{\alpha}^{**}, C) \leq P_0(C_{\alpha}^{**} + (\Delta R_{0,C}/M_1)^{1/2} + 2T \geq s^* \geq C_{\alpha}^{**}|C) \leq M_0 \left( (\Delta R_{0,C}/M_1)^{1/2} + 2T \right)^\gamma.
\]

Therefore,

\[
(I) \leq c' M_0 \left( (\Delta R_{0,C}/M_1)^{1/2} + 2T \right)^{1+\gamma}.
\]

Regarding (II), by Lemma 3 we have

\[
(II) \leq \int_{C^c} |r - C_\alpha| dP_0 \leq \int_{C^c} r dP_0 + C_{\alpha} \int_{C^c} dP_0 = P_1(C^c) + C_{\alpha} P_0(C^c) \leq (1 + C_{\alpha}) \exp\{-(n_0 \wedge n_1)^{1/2}\}.
\]

Therefore,

\[
\int_{(G^c \setminus G)} |r - C_\alpha| dP_0 \leq c' M_0 \left( (\Delta R_{0,C}/M_1)^{1/2} + 2T \right)^{1+\gamma} + (1 + C_{\alpha}) \exp\{-(n_0 \wedge n_1)^{1/2}\}.
\]

To bound $\int_{(G^c \setminus G)} |r - C_\alpha| dP_0$, we decompose

\[
\int_{(G^c \setminus G)} |r - C_\alpha| dP_0 = \int_{(G^c \setminus G) \cap C} |r - C_\alpha| dP_0 + \int_{(G^c \setminus G) \cap C^c} |r - C_\alpha| dP_0 =: (I') + (II').
\]

To bound (I'), we invoke both the margin assumption and the detection condition, and we need to define a new a new quantity $\hat{\Delta} R_{0,C} := P_0(s^*(X) > C_{\alpha}^{**}|X \in C) - P_0(\hat{s}(X) > \hat{C}_\alpha|X \in C)$. When $\hat{\Delta} R_{0,C} \geq 0$, we
have
\[ P_0 \left( s^*(X) \geq C_{\alpha}^{**} + (\Delta R_{0,C}/M_0)^{1/\gamma} | X \in C \right) \]
\[ = P_0 \left( s^*(X) > C_{\alpha}^{**} | X \in C \right) - P_0 \left( C_{\alpha}^{**} < s^*(X) < C_{\alpha}^{**} + (\Delta R_{0,C}/M_0)^{1/\gamma} | X \in C \right) \]
\[ = P_0 \left( \delta(X) > \hat{C}_{\alpha} | X \in C \right) \]
\[ \geq P_0 \left( s^*(X) > \hat{C}_{\alpha} | X \in C \right) \]
\[ \geq P_0 \left( s^*(X) > \hat{C}_{\alpha} + T | X \in C \right). \]

So when $\Delta R_{0,C} \geq 0$, $\hat{C}_{\alpha} \geq C_{\alpha}^{**} - (\Delta R_{0,C}/M_0)^{1/\gamma} - T$. On the other hand, when $\Delta R_{0,C} < 0$,
\[ P_0 \left( s^*(X) \geq C_{\alpha}^{**} - (\Delta R_{0,C}/M_1)^{1/\gamma} | X \in C \right) \]
\[ = P_0 \left( s^*(X) > C_{\alpha}^{**} | X \in C \right) + P_0 \left( C_{\alpha}^{**} \geq s^*(X) > C_{\alpha}^{**} - (\Delta R_{0,C}/M_1)^{1/\gamma} | X \in C \right) \]
\[ \geq P_0 \left( s^*(X) > \hat{C}_{\alpha} | X \in C \right) \]
\[ \geq P_0 \left( s^*(X) > \hat{C}_{\alpha} + T | X \in C \right) \]

So when $\Delta R_{0,C} < 0$, $\hat{C}_{\alpha} \geq C_{\alpha}^{**} - (\Delta R_{0,C}/M_1)^{1/\gamma} - T$. Note that $\Delta R_{0,C} = |\Delta R_{0,C}|$. Therefore we have in both cases,
\[ \hat{C}_{\alpha} \geq C_{\alpha}^{**} - (\Delta R_{0,C}/M_0)^{1/\gamma} \land (\Delta R_{0,C}/M_1)^{1/\gamma} - T. \]

Using the above inequality, we have
\[ C \cap (G^* \backslash \hat{G}) \]
\[ = C \cap \{ s^* \leq C_{\alpha}^{**}, \delta > \hat{C}_{\alpha} \} \]
\[ = C \cap \{ s^* \leq C_{\alpha}^{**}, \delta \geq C_{\alpha}^{**} - (\Delta R_{0,C}/M_0)^{1/\gamma} \land (\Delta R_{0,C}/M_1)^{1/\gamma} - T \} \cap \{ \delta > \hat{C}_{\alpha} \} \]
\[ \subset C \cap \{ C_{\alpha}^{**} - (\Delta R_{0,C}/M_0)^{1/\gamma} \land (\Delta R_{0,C}/M_1)^{1/\gamma} - T \leq s^* \leq C_{\alpha}^{**} \} \]
\[ \subset C \cap \{ C_{\alpha}^{**} - (\Delta R_{0,C}/M_0)^{1/\gamma} \land (\Delta R_{0,C}/M_1)^{1/\gamma} - 2T \leq s^* \leq C_{\alpha}^{**} \}. \]

Denote by $C_2 = \{ x : C_{\alpha}^{**} - (\Delta R_{0,C}/M_0)^{1/\gamma} \land (\Delta R_{0,C}/M_1)^{1/\gamma} - 2T \leq s^*(x) \leq C_{\alpha}^{**} \}$. Then we just showed that $C \cap (G^* \backslash \hat{G}) \subset C \cap C_2$. Recall that
\[ |r(x) - C_{\alpha}| = e^{-\mu^T \Sigma^{-1} \mu} |s^*(x) - C_{\alpha}^{**}| = e^{-\mu^T \Sigma^{-1} \mu} \cdot e^{z} |s^*(x) - C_{\alpha}^{**}|, \]
where $z'$ is some quantity between $s^*(x)$ and $C_{\alpha}^{**}$. Restricting to $C \cap C_2$, we have
\[ z' \leq C_{\alpha}^{**}. \]

This together with $C \cap (G^* \backslash \hat{G}) \subset C \cap C_2$ implies that
\[ (I') \leq \int_{C \cap C_2} |r - C_{\alpha}| dP_0 \]
\[ = \int_{C \cap C_2} \exp(z' - \mu^T \Sigma^{-1} \mu) |s^*(x) - C_{\alpha}^{**}| dP_0 \]
\[ \leq \int_{C \cap C_2} c'' |s^*(x) - C_{\alpha}^{**}| dP_0 \]
\[ \leq c'' \left( (\Delta R_{0,C}/M_0)^{1/\gamma} \land (\Delta R_{0,C}/M_1)^{1/\gamma} + 2T \right) P_0(C \cap C_2). \]
Note that by the margin assumption,
\[ P_0(C \cap C_0) \leq \alpha - (\Delta R_0, C/M_0)^{1/\gamma} \leq 2T \leq s^*(X) \leq C_{\alpha}^{**} \]
\[ \leq M_0 \left( (\Delta R_0, C/M_0)^{1/\gamma} \wedge (\Delta R_0, C/M_1)^{1/\gamma} + 2T \right)^{1+\gamma}. \]

Therefore,
\[ (I') \leq c'' M_0 \left( (\Delta R_0, C/M_0)^{1/\gamma} \wedge (\Delta R_0, C/M_1)^{1/\gamma} + 2T \right)^{1+\gamma}. \]

Regarding (II'), by Lemma 3 we have
\[ (II') \leq \int_{C^c} r dP_0 + C_\alpha \int_{C^c} dP_0 = P_1(C^c) + C_\alpha P_0(C^c) \leq (1 + C_\alpha) \exp\{-(n_0 \wedge n_1)^{1/2}\}. \]

Therefore, by the excess type II error decomposition equation (29),
\[ P_1(\hat{G}) - P_1(G) = (I) + (II) + (I') + (II') + C_\alpha \{ R_0(\phi_{n_0}^* - R_0(\phi_k^*) \}. \]

Using the upper bounds for (I), (II), (I') and (II') and Lemma 2, With probability at least 1 - \delta_0 - \delta_0 - \delta_1 - \delta_2, we have
\[ P_1(\hat{G}) - P_1(G) \leq c' M_0 \left( (\Delta R_0, C/M_1)^{1/2} + 2T \right)^{1+\gamma} \]
\[ + c'' M_0 \left( (\Delta R_0, C/M_0)^{1/\gamma} \wedge (\Delta R_0, C/M_1)^{1/\gamma} + 2T \right)^{1+\gamma} \]
\[ + 2(1 + C_\alpha) \exp\{-(n_0 \wedge n_1)^{1/2}\} + C_\alpha \cdot \xi_{\alpha, \delta_0, n_0} (\delta_0') \]
\[ \leq c_1' \Delta R_0, C^{1+\gamma/2} + c_2' T^{1+\gamma} + c_3' \exp\{-(n_0 \wedge n_1)^{1/2}\} + C_\alpha \cdot \xi_{\alpha, \delta_0, n_0} (\delta_0'), \]

for some positive constants c_1', c_2' and c_3'. In the last inequality of the above chain, we used \gamma \geq \hat{\gamma}. Note that on the event \mathcal{E}_2 \cap \mathcal{E}_3, Lemma 4 guarantees \Delta R_0, C \leq 2[\xi_{\alpha, \delta_0, n_0} (\delta_0') + \exp\{-(n_0 \wedge n_1)^{1/2}\}]. Lemma 3 guarantees that T \leq 4c_1' \varphi \lambda (n_0 \wedge n_1)^{1/4}. Therefore,
\[ P_1(\hat{G}) - P_0(G) \leq c_1'' [\xi_{\alpha, \delta_0, n_0} (\delta_0')^{(1+\gamma)/2} + \xi_{\alpha, \delta_0, n_0} (\delta_0')^{1+\gamma}]^{1+\gamma} \]
\[ + c_3'' [\exp\{-(n_0 \wedge n_1)^{1/2}\}]^{1+\gamma/2}. \]

Lemma 1 guarantees that \xi_{\alpha, \delta_0, n_0} (\delta_0') \leq (5/2)(n_0')^{-1/4}. Then the excess type II error is bounded by
\[ P_1(\hat{G}) - P_0(G) \leq c_1 (n_0')^{-\gamma/2} + c_2 (\lambda s)^{1+\gamma}(n_0 \wedge n_1)^{1+\gamma} \]
\[ + c_3 \exp\left\{-(n_0 \wedge n_1)^{1/2} \left( \frac{1 + \gamma}{2} \wedge 1 \right) \right\}. \]

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