A map for finding hidden quantum Markovian models

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Many current problems of interest in quantum non-equilibrium are described by time-local master equations (TLMEs) for the density matrix that are not of the Lindblad form, that is, that are not strictly probability conserving and/or Markovian. Here we describe an approach by which the system of interest that obeys the TLME is coupled to an ancilla, such that the dynamics of the combined system-plus-ancilla is Markovian and thus described by a Lindblad equation. This in turn allows to recover the properties of the original TLME dynamics from a physical unravelling of this associated Lindblad dynamics. We discuss applications of this generic mapping in two areas of current interest. The first one is that of “thermodynamics of trajectories”, where non-Markovian master equations encode the large-deviation properties of the dynamics, and we show that the relevant large-deviation functions (i.e. dynamical free-energies) can be recovered from appropriate observables of the ancilla. The second one is that of quantum filters, where we show by mapping to an enlarged Markovian system that it is not possible to simulate efficiently a quantum filter with quantum hardware. This results in a “no tracking theorem”, which is to weak measurements what the no-cloning theorem is to strong measurements.

Introduction. A central result in the theory of open quantum system is due to Lindblad [1–4] who proved that the general form for the quantum master equation (QME) for the density matrix of a quantum Markovian system is,

\[ \dot{\rho} = -i[\mathcal{H}, \rho] + \sum_{i=1}^{N} \mathcal{J}_i \rho \mathcal{J}_i^\dagger - \frac{1}{2} \{ \mathcal{J}_i^\dagger \mathcal{J}_j, \rho \} \equiv \mathcal{L}(\rho), \tag{1} \]

where the self-adjoint Hamiltonian \( \mathcal{H} \) generates the coherent part of the dynamics, \( \mathcal{J}_i \) are a set of \( N \) (bounded) jump operators which encode incoherent transitions [1–4], and \( \{,\} \) indicates anticommutator. A large number of open quantum systems of experimental relevance have been described using this Markovian approximation [1–4], and Eq. (1) has become the starting point for analysing the dynamics of a quantum system that interacts with a thermal bath or other environments.

The form (1) guarantees positiveness of the density matrix, conservation of probability, and the quantum (i.e. no memory) Markov property. Useful theorems have been proved for it, including on the existence of stochastic unravellings [4–6] (and so of physical realisations of the dynamics), and on steady state behaviour [7–8]. No such framework exists in general for master equations which are not of the Lindblad kind. This paper aims to remedy this problem for general time local master equations (TLMEs) of the form,

\[ \dot{\rho} = \mathbb{L}_\alpha(\rho) + B\rho + \rho C + \sum_{j=1}^{M} D_j \rho E_j^\dagger = \mathbb{K}(\rho), \tag{2} \]

where \( \mathbb{L}_\alpha \) is a Lindbladian, as in [1,4], and \( B, C, D_j, E_j \) are arbitrary (bounded) operators [2]. The super-operator \( \mathbb{K} \) defined by (2) does not in general preserve positivity or probability of the density matrix \( \rho \).

TLMEs such as (2) appear in three important contexts in quantum non-equilibrium. One is explicitly non-Markovian systems [11–17], where the TLME formulation is equivalent to the time non-local formalism of Nakajima and Zwanzig [18–19]. Stochastic unravellings

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of these non-Markovian TLMEs have involved adding an ancillary system \[20,26\]. The second area is full-counting statistics in quantum optics \[4\] and in mesoscopics \[27\], and the related thermodynamics of trajectories, the generalization of Ruelle’s thermodynamic formalism \[28\] to many-body stochastic systems, both classical \[29,30\] and quantum \[31,32\]. Here TLMEs encode the large-deviation properties \[33\] of time-integrated observables, and the associated large-deviation functions play the role of dynamical free-energies for ensembles of trajectories \[29,31\]. The third area is quantum feedback and control \[4,6\], in particular quantum filters \[5,34,35\] whose dynamics is also described by TLMEs \[35\].

Here we present a general way to map a system whose dynamics is described by a TLME to one where the same system is coupled to an ancilla, such that the density matrix \(\rho\) of the combined system-plus-ancilla evolves according to a master equation of the Lindblad form. The evolution of the system under the TLME is recovered via:

\[
\rho = \text{Tr}_a[w_t \varrho],
\]

where \(w_t\) is an appropriate (possibly time-dependent) operator on the ancillary space and \(\text{Tr}_a\) indicates trace over the ancillary Hilbert space. We achieve this by noting that all the applications described above can be unified under the idea of quantum weighting, named due to its analogous behaviour to classical weighting \[36,38\]. The generality of our approach also allows us to classify which TLMEs can efficiently be implemented with Markovian quantum hardware and which cannot. The mapping is shown schematically in Fig. 1.

**General framework.** Our aim is to describe the evolution of \(\rho\) of the system under a TLME \[2\] in terms of the evolution of \(\rho\), for the same system coupled to an ancilla, under a QME of the form \[1\] where \(\rho\) is obtained from \(\varrho\) via the mapping \[3\]:

\[
\dot{\varrho} = \mathbb{L}_{tot}(\varrho) \rightarrow \dot{\rho} = \partial_t \text{Tr}_a[w_t \varrho] = \mathbb{K}(\text{Tr}_a[w_t \varrho]) = \mathbb{K}(\rho).
\]

We describe this procedure in a manner as general as possible (with details deferred to the Supplemental Material). Note that while the operators in \[2\] act on the system, the self-adjoint \(\mathcal{H}\) and jump operators \(\mathcal{J}_i\) \((i = 1, \ldots, N)\) that define the QME \[1\] will act on the combined system and ancillary Hilbert space.

We posit that the dynamics of the system under the TLME should be recovered by tracing out over the ancilla as in \[3\] with an appropriate choice of “weight” \(w_t\), an (as of yet unrestricted) operator on the ancillary space. Furthermore, we require that the RHS of master equation for \(\rho\) only depend on \(\rho\), meaning its dynamical equation is closed:

\[
\dot{\rho} = \partial_t \text{Tr}_a[w_t \varrho] = \text{Tr}_a[w_t \varrho] + \text{Tr}_a[w_t \mathbb{L}_{tot}(\varrho)] \equiv \mathbb{M}(\rho).
\]

A general \(w_t\) and \(\mathbb{L}_{tot}\) is not guaranteed to have the closure property described by \[5\], as we define \(\mathbb{M}(\rho)\) to be a linear superopertor on \(\rho\) only (not on \(\varrho\)). By enforcing closure we will restrict the form of \(w_t\) and \(\mathbb{L}_{tot}\) which will implicitly restrict the form of \(\mathbb{M}(\rho)\). The first step to enforce closure is to require \(\text{Tr}_a[w_t \varrho] = \alpha \text{Tr}_a[w_t \varrho]\) where \(\alpha \equiv \alpha(t)\) is some undefined time dependent complex number, for general \(\varrho\) this is equivalent to the differential equation \(\dot{w}_t = \alpha w\) whose solution is \(w_t = e^{i\int_0^t \alpha(t) dt} w\), where \(w\) is a bounded time-independent operator on the ancillary system. The condition on the second term of \[5\] is discussed in detail in the Supplemental Material. What is readily found is that the general form of \(\mathbb{M}(\rho)\) must be

\[
\mathbb{M}(\rho) = \mathbb{L}_{sys}(\rho) + \alpha \rho + \left( A_1 - S_i - \frac{1}{2} \sum_i L_i^\dagger L_i \right) \rho + \rho \left( A_2 - S_i - \frac{1}{2} \sum_i R_i^\dagger R_i + \sum_i L_i \rho R_i^\dagger \right),
\]

where \(A_1 \) are anti-Hermitian, \(S_{r,1}\) positive semidefinite, \(R_i, L_i\) general operators and \(I\) is a constant. We can now bring \(\mathbb{M}(\rho)\) to the form of \(\mathbb{K}(\rho)\). First, we identify \(L_i = D_i \) and \(R_i = E_i\). Second, we identify \(A_{r,1}\) with the anti-Hermitian parts of \(B, C\), that is, \(A_1 = B_r - A_r = C_r\) \([\text{where } X_{r,l} \equiv (X \pm X^\dagger)/2]\). Third, we note the Hermitian parts of \(B\) and \(C\) are not necessarily negative semidefinite \([\text{which the form of } \mathbb{M}, \text{ Eq. } (6), \text{ requires} ]\), but can always be made so by the subtraction of a sufficiently large constant, so we define

\[
S_{l} \equiv \alpha_1 - H_l; \quad \text{and } S_{r} \equiv \alpha_r - H_r.
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\[
S_{l} \equiv \alpha_1 - H_l; \quad \text{and } S_{r} \equiv \alpha_r - H_r.
\]
From (3) we know that $\alpha \geq 0$. When $\alpha = 0$, the exponential growth will not occur, and in principle calculating $\langle X \rangle$ can be achieved efficiently.

When $\alpha > 0$ the error in each individual estimation of $\langle X \rangle$ will grow exponentially in time. This means an exponential number of measurements will be required to achieve the same precision for any observable of the system over time. Thus we state that if $\alpha > 0$ for a given TMLE there is no physical realisation from Markovian quantum hardware in which observables can be efficiently measured. Given the generality of our assumptions this is the case irrespective of: (i) how many additional ancillary systems are added (as we made no assumptions about the ancilla in the derivations above); and (ii) what operations are performed on the system conditioned on the ancillary results [as all possible operations are covered by the quantum weighting (3)]. The only way to escape these limitation in the case $\alpha > 0$ is to perform some nonlinear operation, for example renormalisation or quantum amplification (however, no nonlinear operation can be performed to a density matrix deterministically according to the postulates of quantum mechanics [32]).

**General mappings for two-level ancilla.** Beyond whether an (efficient) physical realisation exists or not, our general framework suggests it is always possible to find a Markovian system-plus-ancilla that will give a TMLE under the mapping (3). A basic question is then: how big must the ancillary system be? As it happens, the minimum necessary ancillary Hilbert space is that of a qubit. We show this by providing two explicit general mappings. We denote the basis states of the ancilla by $|0\rangle$ and $|1\rangle$, and consider two possible choices of the weight operator $w$ (and from which other choices can be deduced): (i) $w$ is diagonal, say $w = |0\rangle\langle 0|$, (ii) off-diagonal, say $w = |1\rangle\langle 0|$. (i) **Diagonal ancilla and Hermiticity-preserving TLMEs:** A mapping with $w = |0\rangle\langle 0|$ is appealing because in this case the weight operator is Hermitian at all times, $w_t = w_t^\dagger$, and is therefore an observable. This mapping is only possible for TLMEs which maintain the Hermiticity of the system density matrix $\rho$ (given that the system-plus-ancilla $\rho$ is Hermitian at all times). This means that we can simulate all TLMEs where $C \equiv B^\dagger$ and $D_t \equiv E_t$. In this case, the following QME for the system-plus-ancilla gives the TMLE evolution of the system under the mapping (3).

\[
\dot{\rho} = \mathbb{L}_{\text{sys}}(\rho) + [B_- \otimes 1, \rho] + \mathbb{D}[\sqrt{2S_1} \otimes \sigma](\rho) + \sum_{i=1}^{M} \mathbb{D}[D_i \otimes 1](\rho). \tag{9}
\]

Here $\mathbb{L}_{\text{sys}}$ acts only on the system and not on the ancilla, $\mathbb{D}[X](\cdot) \equiv X \cdot X^\dagger - \frac{1}{2}(X^\dagger X + X X^\dagger)$, $\rho_0 \equiv |0\rangle\langle 0|$, $\rho_1 \equiv |1\rangle\langle 1|$, and $\sigma \equiv |1\rangle\langle 0|$. The TLME can be recovered from (9) using the quantum weighting: $\rho = \text{Tr}_a[w_t \rho]$, Eq. (3).

(ii) **Off-diagonal ancilla and generic TLMEs:** If $w = |1\rangle\langle 0|$ then the Hermiticity-preserving restrictions above do not apply and a mapping for any TLME can be found,

\[
\dot{\rho} = \mathbb{L}_{\text{sys}}(\rho) + [B_- \otimes p_0 + C_- \otimes p_1, \rho] + \mathbb{D}[\sqrt{S_1} \otimes p_0](\rho) + \mathbb{D}[\sqrt{S_1} \otimes p_1](\rho) + \sum_{i=1}^{M} \mathbb{D}[D_i \otimes p_0 + E_i \otimes p_1](\rho). \tag{10}
\]

where $\mathbb{L}_{\text{sys}}$ only acts on the system, and the operators $S_i$ are defined in (7). Once again the TMLE for $\rho$, Eq. (2), can be recovered using the quantum weighting as previously defined.

**Multiplicity of mappings.** The mappings (9) and (10) are not unique. Given a Lindbladian $\mathbb{L}$ which maps to $K$ under $w$, Eq. (3), one can obtain a second $\mathbb{L}' = \mathbb{L} + \mathbb{G}_t$ which also obeys (3), as long as $\text{Tr}_a[w_i \mathbb{G}_t(\cdot)] = 0$. This is a “gauge invariance” of the mapping, since we are describing one system by means of a larger one, and so we can make (possibly time-dependent) transformations of the operators on the system-plus-ancilla while maintaining the same dynamics of the system. This freedom can be exploited to obtain the most convenient mapping for the problem at hand, as we discuss below.

**Application I: thermodynamics of trajectories.** Consider a system evolving according to a QME with Lindbladian $\mathbb{L}_{\text{sys}}$ for which we wish to compute the probability $P_t(K)$ of observing $K$ quantum jumps due to jump operator, say, $J_1$. Such time-integrated quantities are convenient order parameters for classifying the dynamical phase structure of open systems [30, 31]. Instead of the probability $P_t(K)$ we may consider the generating function $Z_t(s) \equiv \sum_K e^{-s K} P_t(K)$. At large times this acquires a large-deviation (LD) form [33], $Z_t(s) \sim e^{t \theta(s)}$, where the LD function (scaled cumulant generating function) $\theta(s)$ plays the role of a free-energy density for trajectories [30, 31], where the “counting” field $s$ is conjugate to the observable $K$. This leads to the definition of the deform (or “tilted”) operator $\mathbb{W}_s[31]$,

\[
\dot{\rho} = \mathbb{W}_s(\rho) \equiv \mathbb{L}_{\text{sys}}(\rho) + e^{-s J_1} \rho J_1^\dagger - \frac{1}{2} \{J_1^\dagger J_1, \rho\}, \tag{11}
\]

where $\mathbb{L}_{\text{sys}} = \mathbb{L}_{\text{sys}} - \mathbb{D}[J_1]$. The LD function $\theta(s)$ is given by the largest eigenvalue of $\mathbb{W}_s$, such that $\text{Tr}[\rho] \sim e^{t \theta(s)}$ at large times. The above is a TMLE for the evolution of $\rho$. The dynamics it generates is related to that of a subset of trajectories of the original dynamics, reweighed such that the average $K$ is given by $-\theta(s)$ [and not $-\theta(0)$ as in the original dynamics] (sometimes called the s-ensemble of the dynamics [30]).

The general mapping allows to access this s-ensemble through the actual dynamics of a system-plus-ancilla. Since $\mathbb{W}_s$ is Hermiticity-preserving, we can choose case (i) or (ii) for the ancilla. If we choose $w = |0\rangle\langle 0|$, then from (9) we get the QME of the system-plus-ancilla,

\[
\dot{\rho} = \mathbb{L}_{\text{sys}}(\rho) + \mathbb{D}[e^{-s/2 J_1} \otimes 1](\rho) + \mathbb{D}[\sqrt{2S_1} \otimes \sigma](\rho). \tag{12}
\]
For $s > 0$ we have $\alpha = 0$, the mapping is efficient, and $\sqrt{2s} = 1 - e^{-s}J_1$. For $s < 0$, in contrast, $\alpha \neq 0$ and the mapping is inefficient. Alternative mappings to (12) are obtained by exploiting the gauge invariance which may prove more convenient than (12) for efficient numerical simulation.

**Example: measuring the LD function in the micromaser.** As an example of a system whose LD function can be observed with a system-plus-ancilla which corresponds to actual physical hardware, we consider the micromaser [11], an optical cavity pumped by excited two-level atoms interacting with a thermal bath.

The micromaser has four distinct jump operators, $J_1 = \sqrt{r}a^\dagger \sin(\theta \sqrt{ra})/(\sqrt{ra})$ and $J_2 = \sqrt{r} \cos(\theta \sqrt{ra})/(\sqrt{r})$ corresponding to the observation of output atoms in the ground and excited states, respectively, and $J_3 = \sqrt{\nu + \pi a}$ and $J_4 = \sqrt{\phi a^\dagger}$ associated to emission and absorption of quanta from the thermal bath. The micromaser has a rich dynamical phase diagram [10], and in particular it displays multiple transitions in $\theta(s)$ as a function of $s$, when $s$ is the counting field conjugate to the number of jumps due to $J_1$, i.e. the number of outgoing atoms that have ceded a quantum to the cavity.

We couple the micromaser to a two-level ancilla with weight operator $w = |1\rangle\langle 0|$, i.e. scheme (ii). The QME which maps under $w$ to the corresponding $s$-ensemble, $\mathbb{W}_s$ (at $s > 0$), follows from (10),

$$\dot{\rho} = \mathcal{L}_s(\rho) + \frac{1}{2} \mathcal{D}[J_1 \otimes U_+](\rho) + \frac{1}{2} \mathcal{D}[J_1 \otimes U_-](\rho),$$

where $U_{\pm} \equiv e^{i\phi_{\pm} p_0 + p_1}$, and $\phi_{\pm} \equiv \pm \cos^{-1}(e^{-s})$. To obtain (13) we have exploited the gauge invariance of (10). Note that the coupling to the ancilla is through the unitaries $U_\pm$, and so it can be achieved by using feedback, or simply scattering the outgoing quasi from the system off the ancilla. This is shown in Fig. 1 with parameters $\vartheta = 4\pi$, $\nu = 1$ and $r = 1000$.

From (3) we have that $\rho(t) = \text{Tr}_\omega[\rho_\omega(t)]$ by measuring the time dependence of $\rho_\omega$, i.e. the coherence of the ancilla, we obtain the LD function, $\langle \omega || \rho_\omega(t) \rangle \sim e^{i\theta(s)}$. This means that from the rate of relaxation of $\langle \omega ||$ in the system-plus-ancilla we obtain the LD function of the system at a value of $s$ determined by the coupling to the ancilla through $U_\pm$. Figure 1 shows what would result from a quantum jump Monte Carlo simulation of the micromaser coupled to the ancilla. From the rate of decay of $\langle \omega ||$ we obtain $\theta(s)$; the LD function estimated in this way displays a first-order singularity at $s_c > 0$, as expected for the parameters of the figure 10.

**Application II: quantum filters.** A quantum filter allows an experimentalist to make an optimal estimate of the quantum state given the record of some weak measurement. For example, consider a quantum system evolving under a known Hamiltonian $H$, coupled to the environment with some operator $J$. The environment is measured using a homodyne detector, which produces a stochastic continuous signal $y_t$. This signal contains information about the system observable $J + J^\dagger$ and can be used to estimate the current state through the equation:

$$d\pi = -i[H, \rho]dt - A \left[ \frac{1}{2}(J^2 + J^\dagger J)dt + J \circ dy_t \right](\pi),$$

where $A[\mathcal{X}](\cdot) = X(\cdot) + (\cdot) X^\dagger$, $\pi$ is the unnormalised conditional density matrix which provides the best estimate for the state of the system and $\circ dy_t$ indicates a Stratonovich integral [4]. As the system size gets larger, integrating (14) on classical hardware rapidly becomes impractical due to the exponential growth of the Hilbert space dimension. One might hope that this could be overcome by using instead quantum hardware to simulate such an equation. This requires mapping the TLME (14) to a QME for an appropriate system-plus-ancilla. This is achieved by applying prescription (10) to (14). From (8) we see that

$$\alpha = \lambda_{\text{max}}^{+ve} \left[ \frac{1}{2} |J^2 + (J^\dagger)^2| dt + (J + J^\dagger) \circ dy_t \right].$$

For an implementation with quantum hardware to be efficient we require $\alpha(t) = 0$. As $dy_t$ can fluctuate between negative and positive values, we can only guarantee that $\alpha(t) = 0 \forall \ t$ only if $J$ is anti-Hermitian, $J^\dagger = -J$. Unfortunately, the measurement signal from the system in such a case will simply be noise (as the system observable now is $J + J^\dagger = 0$), in which case the conditional density matrix $\rho$ will no longer strictly converge towards the true state of the system [6]. Thus, when the filter produces a meaningful estimate of the system we must have $\alpha(t) > 0$. This demonstrates that it is impossible to create an efficient implementation of (14) with Markovian quantum hardware. We can prove analogous results for all other quadrature measurements and photon counting using the same method: details in Supplemental Material.

We call this result a “no tracking theorem”, as it is equivalent to the statement: a quantum system being weakly measured cannot be tracked efficiently with another quantum system of equal size (even with addition of an ancilla). This is true except when the initial state of the system is perfectly known and the weak measurement only measures the noise applied to the system. This is akin to the no cloning theorem which states a projective measurement cannot be used to deterministically clone a quantum state, unless the quantum state is perfectly known [39, 49]. This result has implications for coherent control where attempts have been made to realise so-called quantum observers [50].

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[1] G. Lindblad, Comm. Math. Phys. 48, 119 (1976)
SUPPLEMENTAL MATERIAL FOR: A MAP FOR FINDING HIDDEN MARKOVIAN MODELS

I. FORM OF $\mathcal{M}(\rho)$

Our aim in this section is to show the closure property, defined in Eq. (5), restricts the form of the superoperator $\mathcal{M}(\rho)$ to be Eq. (6) (where both equations are from the main text). We prove this making only one assumption: that the evolution of the system-ancilla obeys a Lindblad equation. Otherwise, we allow both the system evolution and its coupling to the ancilla to be of any form. We complete this proof by first defining precisely what we mean by the ancilla, system and the quantum weighted average discussed in the main text. We then state explicitly what we aim to show as a theorem. Proving this theorem directly is challenging, so to simplify matter we break it down into a sequence of lemmas which we string together at the end. These lemmas may be of interest, in their own right, to readers attempting to create their own ancilla-system couplings.

Definitions: Consider a some system and ancilla which have Hilbert spaces $\mathcal{H}_s$ and $\mathcal{H}_a$ respectively. We define the evolution of the density matrix for the combined system, $\rho \in \mathcal{B}(\mathcal{H}_s \otimes \mathcal{H}_a)$, to be the following general Lindblad master equation:

$$\dot{\rho} = \mathbb{L}_{\text{tot}}(\rho) = \mathbb{L}_{\text{sys}}(\rho) - i \left[ \sum_j J_j \otimes f_j + M_j^\dagger \otimes f_j^\dagger, \rho \right]$$

$$+ \sum_k \mathbb{D} \left[ \sum_j N_{j,k} \otimes g_{j,k} \right], (\rho)$$

where $\mathbb{D}[X](\cdot) = X \cdot X^\dagger - \{X^\dagger X, \cdot \}/2$, $J_j$, $J_d$ and $K_d$ are all finite integers, $f_j, g_{j,k} \in \mathcal{B}(\mathcal{H}_a)$, $M_j, N_{j,k} \in \mathcal{B}(\mathcal{H}_s)$, $\mathbb{L}_{\text{sys}}(\cdot)$ is Lindblad superoperator which operates on the system only and the notation $\mathcal{B}(\mathcal{H})$ means the set of all bounded operators generated by the Hilbert space $\mathcal{H}$. We allow $f_j, g_{j,k}, M_j$ and $N_{j,k}$ to all possibly time dependent, by the most strict definition this makes the system time-dependent Markovian as discussed in [51]. The inclusion of $\mathbb{L}_{\text{sys}}(\cdot)$ does over specify the problem slightly, as it could be considered as a special case of the main form where $f_j/g_{j,k} = 1$. But we consider it to be some kind of intrinsic Markovian evolution of the system which cannot be modified, thus it is useful to consider it separately. We are primarily interested in engineering the evolution of the ancilla and its coupling to the system.

We introduce a mapping between the system-ancilla and a reduced representation of the state, $\rho$, by taking a trace over the ancilla weighted by a general ancillary operator:

$$\rho \equiv \text{Tr}_a[w_t \varrho],$$

where $w_t \in \mathcal{B}(\mathcal{H}_a)$ and we assume $w_t \neq 0 \forall t$. Note: this mapping is not necessarily positive, which allows us to generate non-Markovian behaviour in the reduced density matrix $\rho$. This process we term quantum weighting due to its similarity to weighted averages in classical probability [30] [32].

Theorem: If the evolution of the combined system and ancilla density matrix, $\varrho$, satisfies the closure property:

$$\dot{\rho} = \frac{d}{dt} \text{Tr}_a[w_t \mathbb{L}_{\text{tot}}(\varrho)] = \mathbb{M}(\text{Tr}_a[w_t \varrho]) = \mathcal{M}(\rho),$$

where $\mathbb{L}_{\text{tot}}(\varrho)$ is the general Lindblad superoperator defined in Eq. (1) and $\mathcal{M}(\rho)$ is currently an undetermined linear superoperator. Then the weighting operator must have the form:

$$w_t = e^{\int_0^t dt' \alpha(t')w},$$

where $\alpha(t')$ is a function, $w \in \mathcal{B}(\mathcal{H}_a)$, and the linear superoperator $\mathcal{M}(\rho)$ must have the form:

$$\mathcal{M}(\rho) = \mathbb{L}_{\text{sys}}(\rho) + \alpha \rho + \left( A_l - S_l - \frac{1}{2} \sum_i (L_i)^\dagger L_i \right) \rho$$

$$+ \rho \left( A_r - S_r - \frac{1}{2} \sum_i (R_i)^\dagger R_i \right) + \sum_i R_i \rho L_i^\dagger,$$

where $I$ is a finite integer, $A_{r/l}$ are independent general anti-hermitian operators, $S_{r/l}$ are positive-semidefinite operators and $L_i^\dagger R_i$ are general independent operators.

To start we prove our first lemma which determines what restrictions the closure property, Eq. (3), puts on the operators in Eq. (1). We find these restrictions manifests naturally as a set of algebraic conditions.

Lemma 1: If we require the quantum weighting process [2] generates a master equation for $\rho$ which is closed according to Eq. (3) then the weighting operator must be of the form Eq. (4) and the coupling operators for the ancilla system (in Eq. (1)) must obey the following algebra:

$$w f_j = \gamma_j^f w; w f_j^\dagger = \mu_j^f w;$$

$$f_j w = \gamma_j^w w; f_j^\dagger w = \mu_j^w w;$$

$$g_{j,k}^\dagger w g_{j,k} = \kappa_{i,j,k} w,$$

$$w g_{j,k}^\dagger g_{j,k} = \kappa_{i,j,k}^w w,$$

$$g_{j,k}^\dagger g_{j,k} w = \kappa_{i,j,k}^w w.$$
for the Eq. (7) to be closed we require \( \dot{w}(t) = \alpha w_t \),
we can solve this differential equation which gives us 
\( w_t = \expf{\int_0^t dt' \alpha(t')}w \) where \( w \) is some constant operator in \( \mathcal{A} \) and \( \alpha \equiv \alpha(t') \) is some function. This proves Eq. (4). We can now replace this definition for \( w_t \) into Eq. (7) and replace in Eq. (1) to find

\[
\dot{\rho} = \mathbb{I}_{\text{sys}}(\rho) + \alpha(t)\rho + \epsilon \int_0^\infty ds \alpha(s) \left( \sum_j (-i M_j Tr_\alpha [w f_j \rho] \\
- i M_j^\dagger Tr_\alpha [w f_j^\dagger \rho] + i Tr_\alpha [f_j w \rho | M_j + i Tr_\alpha [f_j^\dagger w \rho | M_j^\dagger] \
+ \frac{1}{2} \sum_{k,j} \sum_k (2 N_i,k Tr_\sigma [g_{j,k}^\dagger w_{i,k} \sigma] N_{j,k}^\dagger \\
- N_{j,k}^\dagger N_{i,k} Tr_\sigma [w_{i,k} \sigma] N_{j,k}^\dagger \\
- Tr_\sigma [g_{j,k}^\dagger g_{i,k} w_{i,k} \sigma] N_{j,k}^\dagger, N_{i,k} \right),
\]

(8)

where we have suppressed the tensor product notation [2]. Again, we require Eq. (8) is closed as described by Eq. (3). We assume \( M_j \) and \( N_{i,k} \) are set independently (although the proof still holds when we loosen this assumption see appendix [A]). Thus each term in Eq. (8) must individually satisfy the closure property. For example the first term enforces \( w f_j \rho = \gamma_j^i w \), where \( \gamma_j^i \) is a complex constant. Applying this logic to each term in Eq. (8) gives us Eq. (9).

We now have an algebra, Eq. (6), that the coupling operators must obey. We have many complex constants which currently appear to be independent. However, we require that these equations hold simultaneously, this puts strong restrictions on the values these complex coefficients can hold. E.g., the constants \( \mu_j^{\dagger/r} \) turns out to the complex conjugate of \( \gamma_j^{\dagger/r} \). In the next lemma we determine what restrictions are on the constants and thereby simplify Eq. (9).

**Lemma 2:** If we require Eq. (6) all hold simultaneously then:

\[
\begin{align*}
\mu_j^{\dagger} &= (\gamma_j^i)^*; \\
\mu_j^r &= (\gamma_j^i)^*; \\
\kappa_{i,j,k}^m &= (\delta_{i,j}^k)^*(\delta_{i,j}^k); \\
\kappa_{i,j,k}^l &= (\delta_{i,j}^k)^*(\delta_{i,j}^k) + (\epsilon_{i,j,k}^l)(\epsilon_{i,j,k}^l); \\
\text{and } \kappa_{i,j,k}^m &= (\delta_{i,j}^k)^*(\delta_{i,j}^k) + (\epsilon_{i,j,k}^l)(\epsilon_{i,j,k}^l);
\end{align*}
\]

(9a, 9b, 9c, 9d, 9e)

Where \( \delta_{i,j}^k \) are matrices and \((\cdot, \cdot)\) is our notation for an inner product.

**Proof:** First consider the equations: \( w f_j = \gamma_j^i w \) and \( w f_j^\dagger = \mu_j^l w \) from Eq. (6a). We apply the Moore-Penrose pseudoinverse [54] of \( w, w^* \), to the LHS of both equations:

\[
\begin{align*}
p f_j &= \gamma_j^i p \\
p f_j^\dagger &= \mu_j^l p
\end{align*}
\]

(10a, 10b)

Where \( p_{1} = w^+ w \) is a projector with \( p_{1} = 0 \) only when \( w = 0 \) (which we have assumed is not the case). Using Eq. (10a) we can show \( p f_j p_{1} = \gamma_j^i p_{1} = \gamma_j^i p_{1} \) and using Eq. (10b) we can show \( p f_j^\dagger p_{1} = \mu_j^l p_{1} = \mu_j^l p_{1} \), as \( p f_j p_{1} = (p f_j^\dagger p_{1})^\dagger \) this implies:

\[
\gamma_j^i = (\mu_j^l)^*.
\]

(11)

Eq. (11) proves Eq. (9a).

Using equations: \( f_j w = \gamma_j^i w \) and \( f_j w = \mu_j^l w \) from Eq. (6b) it can be shown

\[
\gamma_j^i = (\mu_j^l)^*.
\]

(12)

with the same methodology used to prove Eq. (11). This proves Eq. (9b).

Proving Eq. (9c) can be achieved using singular value decomposition (SVD) [55]. We use SVD to factorise \( w \) into \( w = usv^\dagger \) where \( u \) and \( v \) are unitary matrices and \( s \) is a diagonal matrix containing the singular values of \( s \). The singular values are all strictly positive and the rank of \( s \) is guaranteed to be greater than 1 for \( w \neq 0 \). Replacing this decomposition into \( g_{j,k}^l w_{i,k} = \kappa_{i,j,k}^m w_{i,k} \), from Eq. (6c), gives \( g_{j,k}^l u s v^\dagger g_{i,k} = \kappa_{i,j,k}^m u s v^\dagger \), which we can rearrange into:

\[
p^u g_{j,k}^l u s v^\dagger g_{i,k} u s v^\dagger = \kappa_{i,j,k}^m p.
\]

(13)

Where \( p = ss^\dagger \) which is a projector. We can express Eq. (13) in terms of the elements of the matrix as follows

\[
\sum_l \{\sqrt{s_{i,m}^l} g_{j,l} u s v^\dagger g_{i,k} \} \{\sqrt{s_{j,m}^l} u s v^\dagger g_{i,k} \} = \kappa_{m,j,k}^m \delta_{m,n}.
\]

(14)

Where \( \{\cdot\}_{m,n} \) is our notation for taking the \( (m, n) \) element of a matrix and the matrix indexes go from 1 to the rank of \( p \) which we define as \( P \). Eq. (14) is a set of equations which must be satisfied for \( w \) and \( g_{i,k} \). We assume an appropriate set of matrices have already been found, we are only interested in putting restrictions on \( \kappa_{i,j,k}^m \). Towards this end we consider the equation in Eq. (14) corresponding to the maximum \( s \), i.e. we take \( m = n = M \) where \( \{\cdot\}_{M,M} = \max[s] \equiv s_{\max} \); this equation is equivalent to:

\[
\kappa_{i,j,k}^m = (\delta_{i,j}^k)^*(\delta_{i,j}^k),
\]

(15)

where \( \{\delta_{i,j}^k\}_l \equiv \{\sqrt{s_{\max}^l} u s v^\dagger g_{i,k} \}_{l,M} \) and \( \{\delta_{i,j}^k\}_l \equiv \{\sqrt{s_{\max}^l} u s v^\dagger g_{i,k} \}_{l,M} \) are vectors and \((\cdot, \cdot) = \sum_l (\{\cdot\}_l)(\{\cdot\}_l)^\dagger \) is our notation for an inner product. This proves Eq. (9c).

We move our attention to Eq. (9d). We start by replacing the SVD of \( w = usv^\dagger \) into \( wg_{j,k}^l g_{i,k} = \kappa_{i,j,k}^m u s v^\dagger \) from Eq. (6d), after rearranging we find:

\[
p^u g_{j,k}^l v \left( \frac{s}{s_{\max}} + p - \frac{s}{s_{\max}} + q \right) u s v^\dagger g_{i,k} v p = \kappa_{i,j,k}^m p.
\]

(16)
Where we defined $q = 1 - p$ which is another projector, and used the identity $1 = s/s_{\text{max}} + p - s/s_{\text{max}} + q$. Again if we looked at the matrix elements of Eq. (16), we would have an set of equations that have to be satisfied. If we take the same approach used to create Eq. (15), and take the equation corresponding to matrix element $m = n = M$, we find:

$$\kappa^l_{i,j,k} = (\delta^l_{i,j,k} | \delta^l_{i,k}) + (\epsilon^l_{i,j,k} | \epsilon^l_{i,k})$$

(17)

Where $\delta^l_{i,j,k}$ was previously defined and $\{\epsilon^l_{i,j,k}\}_l = \{(\sqrt{p - s/s_{\text{max}} + q})u^l g_{i,j,k} u\}_{1,M}$ and $l$ in this case actually goes from 1 to the rank of $p + q$ (instead of only up to the rank of $p$). Note that $\sqrt{p - s/s_{\text{max}} \geq 0}$ by definition, this ensures we can express the term $(\epsilon^l_{i,j,k} | \epsilon^l_{i,k})$ as an inner product. This proves Eq. (17).

There is currently a complicated interdependence between the vectors $\epsilon^l_{i,j,k}$ and $\delta^l_{i,k}$. Fortunately this will not have any impact for the rest of this proof. Also, if one was engineering a particular ancilla-system coupling and found this interdependence a problem. One can always simply select a $w$ which has equal singular values, e.g. $s = p$, in which case $\delta^l_{i,k}$ becomes independent from $\delta^l_{i,k}$.

By starting with $g^j_{i,k} g_{i,j,k} = \kappa^l_{i,j,k} w$ from Eq. (16d) and applying the same procedure used to derive ?? it can be shown

$$\kappa^l_{i,j,k} = (\delta^l_{i,j,k} | \delta^l_{i,k}) + (\epsilon^l_{i,j,k} | \epsilon^l_{i,k})$$

(18)

where $\delta^l_{i,k}$ was previously defined and $\{\epsilon^l_{i,k}\}_l = \{(\sqrt{p - s/s_{\text{max}} + q})u^l g_{i,k} u\}_{1,M}$. This proves Eq. (16c), which completes the proof of Eq. (9).

We are finally in a position to prove the main theorem. We first combine the lemmas we have proven. In Lemma 1 we showed that the coupling operators between the system and ancilla defined in Eq. (1) must obey Eq. (9), in order to ensure the evolution of $\rho$ obeys the closure property Eq. (3). In Lemma 2 we showed the constants in Eq. (6) are not independent and must obey Eq. (9). We combine these lemmas, by replacing Eq. (9) into Eq. (6), to get the final quantum weight algebra the ancillary coupling operators and weighting operator must obey:

$$w f_j = \gamma^j_w f_j; w f^\dagger_j = (\gamma^j_w)^* w;$$

(19a)

$$f_j w = \gamma^j_w f_j; f^\dagger_j w = (\gamma^j_w)^* w;$$

(19b)

$$g^j_{i,k} g_{i,j,k} = \delta^l_{i,j,k} | \delta^l_{i,k}) w;$$

(19c)

$$w g^j_{i,k} g_{i,j,k} = \{(\delta^l_{i,j,k} | \delta^l_{i,k}) + (\epsilon^l_{i,j,k} | \epsilon^l_{i,k})\} w;$$

(19d)

and $g^j_{i,k} g_{i,j,k} = \{(\delta^l_{i,j,k} | \delta^l_{i,k}) + (\epsilon^l_{i,j,k} | \epsilon^l_{i,k})\} w$. (19e)

Where the constants were previously defined.

We now assume that some set of operators for $w$, $f_j$ and $g_{i,k}$ have been found that obey Eq. (19). Using these operators we can find the form of $M(\rho)$ and determine if the closure property has resulted in it being restricted.

We achieve this goal by replacing Eq. (19) into Eq. (8):

$$\dot{\rho} = M(\rho) \equiv L_{\text{sys}}(\rho) + \alpha(t)\rho$$

$$+ \sum_j \left( -i(\gamma_j^w M_j + (\gamma^j_w)^* M_j^\dagger) \rho + i\rho(\gamma_j^w M_k + (\gamma^j_w)^* M_k^\dagger) \right)$$

$$+ \sum_{i,j,k} \sum_{l,m} \left( N_{i,j,k} \rho N_{i,j,k}^\dagger (\delta_{i,j,k} | \delta_{i,k}) \right)$$

$$- \frac{1}{2} \left( (\delta_{i,j,k} | \delta_{i,k}) + (\epsilon_{i,j,k} | \epsilon_{i,k})\right) N_{i,j,k} N_{i,j,k}^\dagger \rho$$

$$- \frac{1}{2} \left( (\delta_{i,j,k} | \delta_{i,k}) + (\epsilon_{i,j,k} | \epsilon_{i,k})\right) \rho N_{i,j,k} N_{i,j,k}^\dagger,$$

(20)

We can simply the form of this equation by taking advantage of our inner product notation. If we define $| \cdot \rangle = | \cdot \rangle$$ and use the convention $(b/a) = (a^* | b^*) = |a\rangle |b\rangle$ Eq. (20) reduces to

$$\dot{\rho} = M(\rho) \equiv L_{\text{sys}}(\rho) + \alpha(t)\rho$$

$$+ \sum_j \left( -i(\gamma_j^w M_j + (\gamma^j_w)^* M_j^\dagger) \rho + i\rho(\gamma_j^w M_k + (\gamma^j_w)^* M_k^\dagger) \right)$$

$$+ \sum_{i,j} \left( N_{i,j} \rho N_{i,j}^\dagger \right) \frac{1}{2} \left( (N_{i,j}^\dagger)^t N_{i,j}^\dagger \rho + \rho (N_{i,j}^\dagger)^t N_{i,j}^\dagger \right)$$

$$- \frac{1}{2} \left( (N_{i,j}^\dagger)^t N_{i,j}^\dagger \rho + \rho (N_{i,j}^\dagger)^t N_{i,j}^\dagger \right),$$

(21)

where $N_{i,j}^\dagger$ for a general ancilla-system coupling have a complicated interdependence. However, a particular ancilla-system coupling can always be found that allows us to treat them all as independent operators which act on only on the system space this is discussed in more detail in appendix 3. Whether or not such an ancilla-system coupling is being used does not affect the rest of this proof, so we continue without making any such assumptions.

We have technically achieved our primary goal of finding the form of $M(\rho)$ through Eq. (21). However, if one was given a superoperator it would be very difficult to state if it is in the form of Eq. (21) or not. We have not provided a clear understanding on what restrictions the closure property puts on $M(\rho)$. In order to make this more transparent we identify specific features the super-operators must possess to be in the form of Eq. (21).

To start we recognise that $-i(\gamma_j^w M_j + (\gamma^j_w)^* M_j^\dagger)\rho$, the second term in Eq. (20), can only generate terms which operate on the LHS of $\rho$ and are anti hermitian, while $-\frac{1}{2} \rho (N_{i,j}^\dagger)^t N_{i,j}^\dagger$ can only generate terms which operate on the RHS of $\rho$ and are negative semidefinite. Systematically applying this logic get the following form:

$$L_{\text{tot}}(\rho) = L_{\text{sys}}(\rho) + \alpha(t)\rho + (A_t - S_t)\rho + \rho(A_t - S_t)$$

$$+ \sum_i L_i \rho R_i^\dagger - \frac{1}{2} (L_i)^t L_i \rho - \frac{1}{2} \rho (R_i)^t R_i,$$

(22)
where $A_\ell$ and $A_r$ are arbitrary anti-hermitian operators, $S_\ell$ and $S_r$ are positive-semidefinite operators and $L_\ell$ and $L_r$ are arbitrary operators.

To make it a bit clearer how we reached this form we give the explicit relationship between (21) and (22). Before stating, we need to take into account the special cases for the $N_{k,l/r}^{\ell/r}$ operators. We group the $K_d$ operators as follows: we assign cases where $N_{k,l/r}^{\ell/r} = 0$ to $k \in K_d' = (1, K_1)$, cases where $N_{k,l/r}^{\ell/r} = 0$ to $k \in K_d' = (1 + K_1, K_r)$ and cases where $N_{k,l/r}^{\ell/r} \neq 0$ to $k \in K_d' = (K_r + 1, K_d)$ (Note by this definition $K_d - K_r = I$). Taking this into account we find:

$$A_{l/r} = \sum_j (-i (\gamma_j^{l/r} M_j + (\gamma_j^{l/r})^* M^\dagger_j));$$

$$S_{l/r} = \frac{1}{2} \sum_{k \in K_d'} (N_{k,l/r}^{\ell/r})^\dagger N_{k,l/r}^{\ell/r} + \frac{1}{2} K_d \sum_{k \in K_d'} (N_{k,l/r}^{\ell/r})^\dagger N_{k,l/r}^{\ell/r};$$

$$L_i = N_i^\dagger \forall i \in K_d^\ell;$$

$$R_i = N_i^\dagger \forall i \in K_d^r;$$

From the form of $A_{l/r}$ and $S_{l/r}$ it should be immediately clear they are anti-Hermitian and positive-semidefinite respectively. This completes the theorem as Eq. (22) is precisely in the form of Eq. (2) as required.

For a general ancilla-system coupling there may be a complicated interdependence between $L_i/R_i$ and $S_{l/r}$ given in Eq (22). However, it should be clear that such interdependence will not change the form of $\mathcal{M}(\cdot)$. Also, given the discussion in appendix B if the reader requires it: a particular ancilla-system coupling can always be picked such that $L_i/R_i$ and $S_{l/r}$ are all independent operators. Indeed the explicit mapping given in Eq. (10) (from the main text) is an ancilla-system coupling where this is the case.

II. NO TRACKING THEOREM

We prove that a quantum filter can not be efficiently implemented with quantum hardware. This results in a so-called “no tracking” theorem which we describe. In the main text we argued why this is the case for a system being measured with a homodyne detector in the amplitude quadrature. Here we explicitly consider all other possible filters, namely, homodyne detection in a different quadrature and monitoring of quantum jumps. We find that these extra cases work out to be essentially identical to the amplitude homodyne detection case, hence we include them in the supplementary material.

For complete clarity we now restate precisely what we mean by a quantum filter. Consider some physical system with known Hamiltonian $H_p$ coupled to an environment with operator $L_p$. The master equation for this system would be

$$\dot{\rho}_p = -i[H_p, \rho_p] + \mathbb{D}[L_p]\rho_p,$$

where $\rho_p$ is a density matrix that describes the systems average evolution. The derivation of the master equation assumes the environment coupled to the system is averaged. However in principal the environment can be projectively measured using single photon detection or homodyne detection. It can be shown that the signal from this measurement can contain information about a particular observable from the system related to $L_p$. A better estimate for the state system other than $\rho_p$ can be determined conditioned on this information.

The optimal estimate for the state of the system conditioned on the measurement signal of the environment is given by a quantum filter. For a homodyne detection measuring at an angle $\theta$ the quantum filter is

$$d\pi_{\theta} = (-i[H_p, \pi_{\theta}] + \mathbb{D}[L_p](\pi_{\theta}))dt + (L_{\pi_{\theta}} \pi_{\theta} e^{i\theta} + \pi_{\theta} L_p e^{-i\theta})dy_{\theta},$$

where $\pi_{\theta}$ is an unnormalised density matrix which encodes the optimal estimate for the state of the system conditioned on the measurement signal $y_{\theta}(t)$ and $dy_{\theta}$ is the change in the measurement signal over a time $dt$. While for photon detection

$$d\pi_{j} = \left(-i[H_p, \pi_j] - \frac{1}{2} \{L_p^{\dagger} L_p, \pi_j\}\right)dt + (L_p^{\dagger} L_p - \pi_j)dy_{j},$$

where $\pi_j$ is the optimal estimate for the state conditioned on the signal $y_j(t)$ and $dy_j$ is the change in the measurement signal. In both cases optimal estimates for observables can be calculated using $\langle X \rangle = \text{Tr}[X \pi_{\theta/j}] / \text{Tr}[\pi_{\theta/j}]$.

We prove making an efficient physical realisation of $\pi_{\theta}$ or $\pi_j$ which provides a meaningful estimate of state of the system being monitored is impossible using markovian quantum hardware. We achieve this by simply applying the physical realisability result derived in the main text to Eq. (25) and Eq. (26). We repeat the physical realisability result for clarity: Given an TLME in the form:

$$\dot{\rho} = I_{\text{sys}}(\rho) + B \rho + \rho C + \sum_{j=1}^M D_j \rho E_j^\dagger,$$

(repeat of Eq. (2) from the main text), an efficient physical realisation exists of the system as long as $\alpha \equiv \alpha(t)$, defined as:

$$\alpha = \lambda_{\text{max}}^+ \left[ \frac{1}{2} (B + B^\dagger) + \frac{1}{2} \sum_i D_i^\dagger D_i \right] + \lambda_{\text{max}}^+ \left[ \frac{1}{2} (C + C^\dagger) + \frac{1}{2} \sum_i E_i^\dagger E_i \right],$$
is equal to zero, $\alpha(t) = 0$, for all time (repeat of Eq. (8) from the main text). Where $\lambda^+_{\text{max}}[X]$ returns the largest positive eigenvalue of matrix $X$ or return 0 is $X$ has no positive eigenvalues.

Before we begin there is something the reader should take note of. Both Eq. (25) and Eq. (26) are not in the form more commonly seen in the literature. Typically filters are presented in normalised form in terms of some underlying stochastic process $\pi \bar{\theta}$ [4] [6]. Specifically Eq. (25) is mathematically equivalent to

$$d\bar{\theta} = (\theta[I][\bar{\theta}] + d[L_p](\bar{\theta}))/dt + \mathbb{E}[L_p e^{i\theta}](\bar{\theta})dW,$$

(30)

where $\mathbb{E}[X](\cdot) = X \cdot \cdot X \cdot - (X \cdot X \cdot)$, $\bar{\theta} = \pi_\theta /\mathbb{E}[\pi_\theta]$ and $dW = dy_\theta - (L_p e^{i\theta} + L_p e^{-i\theta})dt$ which is a Wiener process. While Eq. (26) is equivalent to

$$d\bar{\pi}_j = \left(-i[H_p, \bar{\pi}_j] - \frac{1}{2} \mathbb{E}[L_p L_p](\bar{\pi}_j)\right)dt + \left(L_p e^{i\theta} + \pi_\theta \right) e^{i\theta} \circ dN,$$

(31)

where $\bar{\pi}_j = \pi_j /\mathbb{E}[\pi_j]$ and $dN = dy_j$ which is a Poissonian process with rate $(L_p L_p)$. The normalised and un-normalised forms are mathematically equivalent. However, when the question arises which of these equations is easier to realise with markovian quantum hardware there is a significant difference. Both Eq. (30) and Eq. (31) are nonlinear with regard to the quantum state. It is impossible to deterministically realise nonlinear evolution of the quantum state with markovian quantum hardware [39]. The unnormalised equation Eq. (25) and Eq. (26) act linearly on the state, indeed they are both TLME, thus it is at least possible to realise them in terms of quantum hardware. However, we show such an implementation is not efficient.

We first consider the homodyne detection at an angle $\theta$. Our aim is to apply the efficiency test developed in the main text to Eq. (25). We can not apply it directly to (25) as we have assumed normal calculus applies when we derived the efficiency test. To fix this inconsistency we change the stochastic integrals in Eq. (25) from Ito form to Stratonovich, giving us

$$d\pi_\theta = (-i[H_p, \pi_\theta] - \frac{1}{2} (L_p L_p + L_p e^{i\theta}) \pi_\theta - \frac{1}{2} \pi_\theta (L_p L_p + (L_p)^\dagger e^{-i\theta}) \pi_\theta$$

$$+ (L_p \pi_\theta e^{i\theta} + \pi_\theta L_p e^{-i\theta}) \circ dy_\theta.$$

(32)

Stratonovich integrals are compatible with regular calculus (but sacrifice the averaging property Ito integrals have), thus the efficiency test can be directly applied to Eq. (32). To begin we take $B = (-iH_p - L_p L_p/2 + L_p e^{i\theta})/dt + L_p e^{i\theta} \circ dy_\theta$, $C = (iH_p - L_p L_p/2 + (L_p)^\dagger e^{-i\theta}/2)dt + L_p e^{i\theta} \circ dy_\theta$, $D = 0$ and $E = 0$. Replacing these matrices into Eq. (29) we find

$$\alpha(t) = \lambda^+_{\text{max}} \left[ -(L_p e^{i\theta} + (L_p)^\dagger e^{-i\theta})dt + 2(L_p e^{i\theta} + L_p e^{-i\theta}) \circ dy_\theta \right].$$

(33)

For an efficient physical realisation to exist we require $\alpha(t) = 0 \forall t$. This is clearly not guaranteed to be the case for a general $L_p$. Furthermore, it should be clear from the form of Eq. (33) that we can only guarantee an efficient physical realisation if $L_p$ has the property $L_p = -L_p e^{-i\theta}$. This actually corresponds to a special case for the quantum filter.

When $L_p = -L_p e^{-i\theta}$ the evolution of the filter becomes independent of the evolution of the system that is being monitored. This corresponds to a special case where the measurement of the environment gives no information about the state of the system and only returns noise. This can be understood as follows. It can be shown that the homodyne measurement signal coming out of the system equals

$$dy(t) = (L_p e^{i\theta} + L_p e^{-i\theta})dt + dW,$$

(34)

where here $\langle \cdot \rangle$ refers to an expectation taken over the system (not our estimate) [20]. We can see in the case when $L_p = -L_p e^{-i\theta}$ the signal reduces to:

$$dy(t) = dW,$$

(35)

which is purely noise. Thus the filter becomes effectively decoupled from the evolution of the system. This means that if the filter and the system are started with precisely the same initial condition, the filter will follow the evolution of the system, but only because the measurement gives us the noise that the system is experiencing. However, if the filter is started with a different initial condition to the system, it will not strictly converge to the systems state. Thus in this case the filter is not truly tracking the evolution of the system.

We have proved it is impossible to implement a quantum filter that tracks the evolution of another quantum system with quantum hardware, for homodyne detection. We move our attention to photon detection.

The easiest way to study implementing equation Eq. (26) is to split it into two parts. The “no-jump” evolution when $dN = 0$ and the “jumps” when $dN = 1$. We consider the “no-jump” evolution first, setting $dN$ equal to zero we get:

$$d\pi_j = \left(-i[H_p, \pi_j] - \frac{1}{2} (L_p L_p, \pi_j)\right)dt.$$

(36)

We can apply the efficiency test directly to this equation. To begin we take $B = -iH_p - L_p L_p/2$, $C = (-iH_p - L_p L_p/2 + (L_p)^\dagger e^{-i\theta}/2)dt + L_p e^{i\theta} \circ dy_\theta$, $D = 0$ and $E = 0$. Replacing these matrices into Eq. (29) we find

$$\alpha(t) = \lambda^+_{\text{max}} \left[ - (L_p e^{i\theta} + (L_p)^\dagger e^{-i\theta})dt + 2(L_p e^{i\theta} + L_p e^{-i\theta}) \circ dy_\theta \right].$$

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(36)

We can apply the efficiency test directly to this equation. To begin we take $B = -iH_p - L_p L_p/2$, $C =
\[ iH_p - L_p^\dagger L_p/2, \quad D = 0 \text{ and } E = 0. \]
Replacing this into Eq. (29) we get
\[ \alpha = 0. \quad (37) \]
Thus we can realise the “no-jump” evolution of the system. This result on its own may be of interest to theoretical proposals where the “no-jump” evolution of a system is the most important. However, if the quantum filter, Eq. (26), is to track the evolution of a quantum system it must be capable of both the “no-jump” and “jump” evolution.

We now look at the jump evolution. We set \( dN = 1 \), we can assume while this is the case all the terms in the system are negligible, thus the system evolution is
\[ \pi_j(t + dt) = L_p \pi_j L_p^\dagger, \quad (38) \]
where we used the identity \( d\pi = \pi(t + dt) - \pi(t) \). It is difficult to apply the efficiency test directly to this equation as it implies there is a sudden non-continuous change to the system, and we derived our test assuming continuous time and evolution. We can approximate this instantaneous change by a very very quick one, specifically we take
\[ \Delta \pi_j = \frac{L_p \pi_j L_p^\dagger - \pi_j}{\Delta t}. \quad (39) \]
Applying the efficiency test to this equation we find \( B = 0, \quad C = 0, \quad D_1 = L/\sqrt{\Delta t}, \quad D_2 = 1/\sqrt{\Delta t}, \quad E_1 = L/\sqrt{\Delta t} \) and \( E_2 = 1/\sqrt{\Delta t} \) replacing these identities into Eq. (12) we find
\[ \alpha = \frac{\lambda^{\text{max}} [L_p^\dagger L_p + 1]}{\Delta t}. \quad (40) \]

\( L_p^\dagger L_p \) is a positive-semidefinite matrix, hence the only time when \( \alpha = 0 \) is when \( L_p = 0 \) in which case we are clearly measuring nothing. Thus there exists no efficient physical implementation for performing the “jump” part of the evolution.

We note that the approximation of treating Eq. (38) as a continuous time equation, may appear to be weakness in the proof. We can also consider the case where we are allowed to perform a general deterministic instantaneous quantum operation on the system. Although this goes against the spirit of the main text which is about mapping continuous time TLME to Markov systems, it is worth considering discrete time operations for quantum jumps as it is an intrinsic feature of their evolution. A general deterministic quantum operation to a state \( \rho \) can always be represented as an operator sum, i.e. \( \sum_m E_k \rho E_k^\dagger \) \[39\]. Thus we can efficiently simulate Eq. (38) for \( L \) operators which can be represented as \( L \pi_j L^\dagger = \sum_m E_k \pi_j E_k^\dagger \). However, quantum operations have the property \( \sum_m E_k^\dagger E_k = 1 \) \[39\]. Thus the observable that we are actually measuring for quantum jumps which have this representation is \( \langle L_p^\dagger L_p \rangle = (\sum_m E_k^\dagger E_k) = 1 \). Consequently the measurement signal, the photon jump rate, has no information about the system and can not be used to make a filter which will converge to state of the physical system. Thus even if we take into account discrete operations on quantum systems, it is still impossible to create a filter from quantum hardware that will efficiently track the original quantum system.

We have shown that either there exists no efficient physical implementation for filters based on homodyne detection or photon jumps. We only considered the case where we are performing only one weak measurement on the system. But, it is easy to see that the logic is extendable to multiple weak measurements on a single system.

This result can be thought of as a “no tracking theorem”, as we have shown given some quantum system under weak measurement it is impossible to create another quantum system which will efficiently track the evolution of the original. This is true even allowing for the possibility of attaching large ancillary system to the tracking quantum system.

This result is analogous to the no cloning theorem. The no cloning theorem shows a state can not be perfectly copied after a projective measurement \[39\]. However, there is a caveat, if the state of the system is perfectly known, then a measurement can always be picked such that the projective measurement produces a deterministic result, not destroying the state allowing us to clone it. Much the same holds here, we have proven you can not track a quantum system with another quantum system and can not be used to make a filter which will efficiently track the original quantum system. However, there is a caveat, that if you have perfect knowledge of the system and you choose a weak measurement which does not return any information of the system, then it is possible to “track”. Although to call this caveat “tracking” goes against the spirit of the theorem, much like how cloning a perfectly known state goes against the spirit of the no cloning theorem.

Appendix A: System co-dependence

When deriving Eq. (6) we assumed that the system operators in equation Eq. (8), were chosen independently
presented in Sec. (I) it is straightforward to show that

\[ \dot{\rho} = \mathbb{I}_{\text{sys}} (\rho) + \alpha (t) \rho + \epsilon e^{J_0} ds a (s) \left( \sum_j (-i M_j \text{Tr}_a[w f_j f_j^\dagger \rho] \right) 

- i M_j^\dagger \text{Tr}_a[w f_j f_j^\dagger \rho] + i \text{Tr}_a[w f_j \rho M_j + i \text{Tr}_a[ f_j^\dagger w \rho M_j^\dagger ] \right) + \frac{1}{2} \sum_i \sum_j (2N_{i,k} \text{Tr}_a[ g_j^\dagger w g_j^\dagger \rho ] N_{j,k}^\dagger 

- N_{j,k}^\dagger \text{Tr}_a[w g_j^\dagger g_j^\dagger \rho ] 

- \text{Tr}_a[ g_j^\dagger g_j^\dagger \rho ] N_{j,k}^\dagger N_{j,k} ) . \] (A1)

Consequently, when we enforced closure we had to do so to each individual term in Eq. (A1). In this paper we are considering engineering both the ancilla and the system, so it is reasonable to ask what if we choose specific operators for the system such that the terms in Eq. (A1) are no longer independent? This would raise the possibility that some terms could be factored into a different form. In these cases we would get a modified quantum weight algebra Eq. (9) which could lead to different conclusions. However, we now briefly justify why restricting \( N_{j,k} \) and \( M_j \) or making them dependent on one another does not change the central result of this paper: the form of the superoperator \( \mathcal{M}(\rho) \), given in Eq. (5).

To start we note that the term on third line of (A1), where \( N_{j,k} \) operates on both the left and right side of \( \rho \), will always produce an independent operator. We can not modify the dependence \( N_{j,k} \) or \( M_j \) to change this as the other terms in the equations all operate purely on the left or right of \( \rho \) and thus can not be added to this term and factorized. Thus we only need to consider the cases when \( M_j \) and \( N_{j,k} \) are modified such that terms acting purely on the left or right can be factorized which results in a different quantum weight algebra.

Consider the case where we set \( M_j^\dagger = M_j \) in this case Eq. (A1) becomes

\[ \dot{\rho} = e^{J_0} ds a (s) \left( \sum_j (-i M_j \text{Tr}_a[w f_j f_j^\dagger \rho] + i \text{Tr}_a[( f_j f_j^\dagger + f_j^\dagger f_j ) w \rho M_j^\dagger ] \right) 

+ i \text{Tr}_a[( f_j f_j^\dagger + f_j^\dagger f_j ) w \rho M_j^\dagger ] + \cdots . \] (A2)

The first and second terms produce the requirement \( w ( f_j + f_j^\dagger ) = \gamma_j \) and \( ( f_j + f_j^\dagger ) w = \gamma_j^\dagger \). Applying the Moore-Penrose inverse in a method identical to what was presented in Sec. (I) it is straightforward to show that \( \gamma_j^\dagger = (\gamma_j^\dagger)^\ast \) meaning both these constants are real. Replacing these identities into Eq. (A2) we find

\[ \dot{\rho} = e^{J_0} ds a (s) \left( \sum_j (-i \gamma_j^\dagger M_j \rho + i \gamma_j^\dagger \rho M_j + \cdots . \right) \] (A3)

Both \(-i \gamma_j^\dagger M_j \) and \( i \gamma_j^\dagger \rho M_j \) are anti-hermitian operators. Thus the form for the superoperator \( \mathcal{M}(\rho) \), as given in Eq. (5), is not modified by placing restrictions on the system operator \( M_j \).

Another option is to set \( M_j^\dagger = i N_{j,k}^\dagger N_{j,k} \), where we set \( J_c = J_d^2 / K_d \) and \( J_d = iJ_d(j - 1) + J_d^2(k - 1) \). Replacing into Eq. (A1) gives

\[ \dot{\rho} = e^{J_0} ds a (s) \left( \sum_j \sum_k ( -i M_j \text{Tr}_a[w f_j f_j^\dagger \rho] + i \text{Tr}_a[w f_j + g_j^\dagger g_j^\dagger \rho M_j^\dagger ] \right) 

- N_{j,k}^\dagger N_{j,k} \text{Tr}_a[w f_j f_j^\dagger \rho] - \text{Tr}_a[( f_j f_j^\dagger + g_j^\dagger g_j^\dagger ) w \rho N_{j,k} N_{j,k}] 

+ \text{Tr}_a[w f_j f_j^\dagger \rho] N_{j,k}^\dagger N_{j,k} + \cdots . \] (A4)

These terms produce the following modified quantum weight algebra: when \( i \neq j \) \( w ( f_j + f_j^\dagger ) = \kappa_{i,j,k}^1 \), \( ( f_j + f_j^\dagger ) w = \kappa_{i,j,k}^1 \ast \), \( w f_j = \gamma_j^\dagger \), \( w f_j^\dagger = (\gamma_j^\dagger)^\ast \), \( f_j^\dagger w = \gamma_j \), \( f_j w = (\gamma_j)^\dagger \). Where \( \kappa_{i,j,k}^1 = \kappa_{i,j,k}^1 \ast - \gamma_j \), \( \kappa_{i,j,k}^1 \ast = \kappa_{i,j,k}^1 + \gamma_j^\dagger \) and \( \mathbb{R}[\gamma_j^\dagger] = 0 \) when \( i = j \). We have successfully split the modified quantum algebra, into a form which is the same as Eq. (5), thus the proof continues in an almost identical manner to what is presented in Sec. (I).

The only caveat being the is \( i = j \) in which case the true imaginary property of \( \gamma_j^\dagger \) produces anti-Hermitian terms in an identical manner to the case previous case (Eq. (A2)). Thus the form for the superoperator \( \mathcal{M}(\rho) \), as given in Eq. (5), is not modified by setting \( M_j^\dagger = i N_{j,k}^\dagger N_{j,k} \).

We could consider other minor variants of changing the dependence between \( M_j \) and \( N_{j,k} \) however it should be clear that they would all reduce to the two extreme cases already considered. Thus even if we weaken the assumption that the system operators are defined independently the central result of the paper still holds.

**Appendix B: Operator independence**

In this appendix we discuss how an ancilla-coupling can be chosen such that the operators:

\[ N_k^{\delta_{i,l/r}} = \sum_i N_{i,k} |\delta_{i,l/r}/\epsilon_{i,k}^r|, \] (B1)

are all independent and operate on the system space. The first choice we make is that the vectors \( \delta_{i,l/r}^1 \) and \( \epsilon_{i,k}^r \) are both of dimension 1. Furthermore we assume that the singular values of \( w \) are all the same, in this case the constants \( \delta_{i,k}^1 \) and \( \epsilon_{i,k}^r \) become independent (as explained
after Eq. (17)). Ancilla-system coupling exist where this is the case, indeed Eq. (9) and (10) from the main text are both examples of this.

Next we split the index over $i$ in four parts and assign the following values for $\delta^{l/r}_{i,k}$ and $\epsilon^{l/r}_{i,k}$: when $i \in J^{0}_{d} = (1, J_{\delta,l})$ we set $\delta^{l}_{i,k} = 1$ and $(\delta^{r}/\epsilon^{l/r})_{i,k} = 0$; when $i \in J^{1}_{d} = (J_{\delta,l} + 1, J_{\delta,r})$ we set $\delta^{r}_{i,k} = 1$, $(\delta^{l}/\epsilon^{l/r})_{i,k} = 0$; when $i \in J^{2}_{d} = (J_{\delta,r} + 1, J_{\epsilon,l})$ we set $\epsilon^{l}_{i,k} = 1$, $(\delta^{l}/\epsilon^{l/r})_{i,k} = 0$; and $i \in J^{3}_{d} = (J_{\epsilon,l} + 1, J_{\epsilon,r})$ we set $\epsilon^{r}_{i,k} = 1$, $(\epsilon^{l}/\epsilon^{l/r})_{i,k} = 0$. Replacing these definitions into Eq. (B1) we find get

$$N^{\delta/\epsilon,l/r} = \sum_{i \in J^{0}_{d}} N_{i,k}, \quad \text{(B2)}$$

where we dropped the inner product notation, as the inner product reduces to the regular product when the dimension of the vector is 1. The sets $J^{0}_{d}$ are all disjoint by construction. Thus $N^{\delta/\epsilon,l/r}$ are all independent operators. Furthermore, $N^{\delta/\epsilon,l/r}_{k}$ are operators that act directly on the Hilbert space of the system, as the dimension of the vectors $(\delta/\epsilon)^{l/r}_{i,j}$ was reduced to 1.