Derivations for the even parts of modular Lie superalgebras $W$ and $S$ of Cartan type

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Let $\mathbb{F}$ be the underlying base field of characteristic $p > 3$ and denote by $W$ and $S$ the even parts of the finite-dimensional generalized Witt Lie superalgebra $\mathcal{W}$ and the special Lie superalgebra $\mathcal{S}$, respectively. We first give the generator sets of the Lie algebras $\mathcal{W}$ and $\mathcal{S}$. Using certain properties of the canonical tori of $\mathcal{W}$ and $\mathcal{S}$, we then determine the derivation algebra of $\mathcal{W}$ and the derivation space of $\mathcal{S}$ to $\mathcal{W}$, where $\mathcal{W}$ is viewed as $\mathcal{S}$-module by means of the adjoint representation. As a result, we describe explicitly the derivation algebra of $\mathcal{S}$. Furthermore, we prove that the outer derivation algebras of $\mathcal{W}$ and $\mathcal{S}$ are abelian Lie algebras or metabelian Lie algebras with explicit structure. In particular, we give the dimension formulae of the derivation algebras and outer derivation algebras of $\mathcal{W}$ and $\mathcal{S}$. Thus we may make a comparison between the even parts of the (outer) superderivation algebras of $\mathcal{W}$ and $\mathcal{S}$ and the (outer) derivation algebras of the even parts of $\mathcal{W}$ and $\mathcal{S}$, respectively.

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0. Introduction

This paper considers finite-dimensional Lie algebras and Lie superalgebras over a field of prime characteristic. As is well known, the theory of Lie superalgebras over a field of characteristic zero has obtained plentiful fruits (see [4–5, 14]). But that is not the situation for modular Lie superalgebras, for example, the classification of finite-dimensional simple modular Lie superalgebras has not been completed. We sketch the recent development for modular Lie superalgebras. As far as we know, [6] may be the earliest paper on modular Lie superalgebras, in which the $p$-structure and $2p$-structure for modular Lie superalgebras (analogous to $p$-mappings for modular Lie algebras) are introduced. The (restricted) enveloping algebras for modular Lie superalgebras are studied in [12].
The reader is also referred to a paper on Frobenius extensions and restricted modular Lie superalgebras (see [3]). In [19] the four families of finite-dimensional Cartan-type modular Lie superalgebras $X(m,n;\mathfrak{L})$ are constructed and the simplicity and restrictiveness are studied, where $X = W,S,H,$ or $K$ (These notations and other concepts mentioned in this introduction will be further explained in Section 1). These modular Lie superalgebras are analogous to both finite-dimensional modular Lie algebras of Cartan type and finite-dimensional Lie superalgebras of Cartan type over a field of characteristic zero (see [4, 15]). In a recent paper [9] (see also [10]), we consider a new family of finite-dimensional simple modular Lie superalgebras of Cartan type, which are analogous to neither the finite-dimensional modular Lie algebras of Cartan type, nor the finite-dimensional Lie superalgebras of Cartan type over a field of characteristic zero. It is therefore conceivable that the classification of finite-dimensional simple modular Lie superalgebras should not be trivial. For more information on modular Lie superalgebras, the reader is referred to [7–12, 19–23].

Given any Lie superalgebra $\mathfrak{g} = \mathfrak{g}_\mathbb{Z} \oplus \mathfrak{g}_\mathbb{Z}$, then the superderivation algebra of $\mathfrak{g}$ is also a Lie superalgebra, denoted by $\text{Der}(\mathfrak{g}) = \text{Der}(\mathfrak{g}_\mathbb{Z}) \oplus \text{Der}(\mathfrak{g}_\mathbb{Z})$. Since $\mathfrak{g}_\mathbb{Z}$ is a Lie algebra, one can consider the derivation algebra $\text{Der}(\mathfrak{g}_\mathbb{Z})$. A question naturally arises: What are the relations between the derivation algebra of the Lie algebra $\mathfrak{g}_\mathbb{Z}$ and the superderivation algebra of $\mathfrak{g}$? More precisely, if we denote still by $\text{Der}(\mathfrak{g}_\mathbb{Z})$ the Lie algebra consisting of restrictions of the even superderivations to $\mathfrak{g}_\mathbb{Z}$, can we assert that $\text{Der}(\mathfrak{g}_\mathbb{Z}) = \text{Der}(\mathfrak{g})$? Equivalently, can we assert that every derivation of the Lie algebra $\mathfrak{g}_\mathbb{Z}$, the even part of $\mathfrak{g}$, may extend to a superderivation of the Lie superalgebra $\mathfrak{g}$? In this paper, as direct consequences of our results, these questions will be answered for the Lie superalgebras $W$ and $S$ of Cartan type over a field of prime characteristic.

Let $\mathbb{F}$ be the underlying base field of characteristic $p > 3$. Let $W$ and $S$ denote the even parts of the Lie superalgebras $W$ and $S$, respectively. In this paper we shall study the derivation algebras and the outer derivation algebras of these two Lie algebras in a systematic way and one of the main purposes of this paper is to lay the foundations for future studies on modular Lie superalgebras of Cartan type. Let $\mathcal{L}$ denote $W$ or $S$, and $\text{Der}(\mathcal{L}, W)$ the space of derivations of $\mathcal{L}$ to $W$, where $W$ is viewed as $\mathcal{L}$-module by means of adjoint representation. By our notation, $\text{Der}(W) = \text{Der}(W,W)$ is the derivation algebra of $W$. Let $\mathcal{L} = \oplus_{i \geq -1} \mathcal{L}_i$ be the natural $\mathbb{Z}$-gradation of $\mathcal{L}$. Just as in the case of Lie superalgebras $W$ and $S$ of Cartan type, one may prove that any derivation in $\text{Der}(\mathcal{L}, W)$ can be reduced (by modulo a suitable inner derivation) to be a derivation vanishing on $\mathcal{L}_{-1}$. However, in contrast to the case of Lie superalgebras $W$ and $S$ of Cartan type, one cannot obtain directly that a derivation vanishing on $\mathcal{L}_{-1}$ must be zero, since $\mathcal{L}$ is neither transitive nor admissibly graded in general. Thus it is conceivable that the top $\mathcal{L}_{-1} \oplus \mathcal{L}_0$ will play an important role in the studies on the derivations. Indeed, just as we shall see, any two derivations of nonnegative $\mathbb{Z}$-degree in $\text{Der}(\mathcal{L}, W)$ which coincide on the top $\mathcal{L}_{-1} \oplus \mathcal{L}_0$ differ with only an inner derivation (see Corollaries 3.1.5 and 4.1.4), where $\mathcal{L} = W$ or $S$. This motivates us to consider whether every derivation of nonnegative $\mathbb{Z}$-degree in $\text{Der}(\mathcal{L}, W)$ can be reduced to be a derivation vanishing on the top. For $\text{Der}(W) = \text{Der}(W,W)$, using a known result which tells us that any homogeneous derivation of nonnegative $\mathbb{Z}$-degree of a centerless $\mathbb{Z}$-graded Lie algebra may be reduced to be vanishing on the given torus contained in the component of degree zero (see [15, Proposition 8.4, p. 193]), we can obtain the desired
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result by a brief argument (see Section 3.2). But the same work on the derivation space $\text{Der}(S, W)$ is more difficult, since we cannot apply the known result mentioned above in this case. This observation leads to the study on the exterior algebra $\Lambda(n)$ and the canonical torus of $S$ (see Corollary 2.1.5), where $\Lambda(n)$ is viewed as a module of the canonical torus of $S$ in the obvious way. Then it is proved that any homogeneous derivation of nonnegative $\mathbb{Z}$-degree in $\text{Der}(S, W)$ can be reduced to be vanishing on the canonical torus of $S$. As a result, we can show that the homogeneous derivations of nonnegative $\mathbb{Z}$-degree in $\text{Der}(L, W)$ are all inner. By giving the generator set of $L$, we can compute the homogeneous derivations of negative $\mathbb{Z}$-degree. Finally, the derivation algebras and the outer derivation algebras of $W$ and $S$ are determined completely; in particular, we give the dimension formulae of the derivation algebras of $W$ and $S$.

The original motivation for this paper comes from the encouragement of the anonymous referee for the paper [8]. Our work is motivated by the results and methods on Lie algebras and Lie superalgebras [2, 15, 17, 23] and based on certain results in [15, 23] on modular Lie algebras and Lie superalgebras of Cartan type. Certain results of this paper are closely parallel to those obtained by Celousov [2]. In particular, we use many ideas from [2, 15] and benefit much from reading [13, 18]. For more information on derivations of modular Lie superalgebras of cartan type, the reader is referred to [8, 10–11, 17, 23].

The paper is organized as follows. In Section 1, we review the necessary notions concerning Lie algebras and Lie superalgebras and the notions of modular Lie superalgebras $W$ and $S$ of Cartan type. In Section 2, we study certain subalgebras of the even parts $W$ and $S$ and give the generator sets of $W$ and $S$. We establish also some technical lemmas concerning the canonical tori of $W$ and $S$, which will be used throughout this paper. In Section 3, we first study the derivations vanishing on the top $W_{-1} \oplus W_0$. Then we characterize the homogeneous derivations of nonnegative $\mathbb{Z}$-degree and negative $\mathbb{Z}$-degree, respectively. As a result, the derivation algebra of $W$ is determined. In Section 4, the derivation space $\text{Der}(S, W)$ and the derivation algebra $\text{Der}(S)$ are determined. In Section 5, by using the results obtained in Sections 3 and 4, the outer derivation algebras of $W$ and $S$ are described explicitly and the dimension formulae of derivation algebras are given.

1. Preliminary

1.1. Basic notion

Let $F$ be an arbitrary field in this subsection and $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$ be the field of two elements. In this paper, all vector spaces, linear mappings, tensor products are over the underlying base field $F$.

Recall that a vector superspace is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. We denote by $p(a) = \theta$ the parity of a homogeneous element $a \in V_\theta$, $\theta \in \mathbb{Z}_2$. A subspace $U$ of a vector superspace $V$ is by definition $\mathbb{Z}_2$-graded; that is, $U = U \cap V_0 \oplus U \cap V_1$.

We assume throughout that if $p(x)$ occurs in an expression, then $x$ is assumed to be $\mathbb{Z}_2$-homogeneous.

A superalgebra is a vector superspace $A = A_0 \oplus A_1$ endowed with an algebra structure such that $A_\theta A_\mu \subset A_{\theta+\mu}$ for all $\theta, \mu \in \mathbb{Z}_2$. 
A Lie superalgebra is a superalgebra satisfying the super-anticommutativity and super-Jacobi identity (see [4, 14]). Let \( g = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be a Lie superalgebra. Then the even part \( \mathfrak{g}_0 \) is a Lie algebra. Note that in the case \( \text{char} F = 2 \), a Lie superalgebra is a \( \mathbb{Z}_2 \)-graded Lie algebra. Thus one usually adopts the convention that \( \text{char} F = p > 2 \) in the modular case.

Let \( V = V_0 \oplus V_1 \) be a vector superspace. The algebra \( \text{End}_F(V) \) consisting of the \( F \)-linear mappings of \( V \) into itself becomes an associative superalgebra if one defines

\[
\text{End}_F(V)_\theta := \{ A \in \text{End}_F(V) \mid A(V_\mu) \subseteq V_{\theta+\mu}, \mu \in \mathbb{Z}_2 \}
\]

for \( \theta \in \mathbb{Z}_2 \). On the vector superspace \( \text{End}_F(V) = \text{End}_F(V)_\overline{0} \oplus \text{End}_F(V)_\overline{1} \) we define a new multiplication \([ , ]\) by

\[
[A, B] = AB - (-1)^{p(A)p(B)}BA \quad \text{for} \quad A, B \in \text{End}_F(V).
\]

This algebra endowed with the new multiplication will be denoted by \( \text{pl}(V) = \text{pl}_\overline{0}(V) \oplus \text{pl}_\overline{1}(V) \); it is a Lie superalgebra and is said to be the general linear Lie superalgebra.

Suppose that \( A = A_\overline{0} \oplus A_\overline{1} \) is superalgebra. Let \( \text{Der}_\theta(A) := \{ D \in \text{pl}_\theta(A) \mid D(xy) = D(x)y + (-1)^{\theta p(x)}xD(y) \text{ for } x, y \in A \} \) for all \( \theta \in \mathbb{Z}_2 \). Define

\[
\text{Der}(A) := \text{Der}_\overline{0}(A) \oplus \text{Der}_\overline{1}(A),
\]

then it is easy to see that \( \text{Der}(A) \) is a subalgebra of \( \text{pl}(A) \), which is called the superderivation algebra of \( A \). If \( \theta = \overline{0} \) (resp. \( \theta = \overline{1} \)), the elements in \( \text{Der}_\theta(A) \) are called even superderivations (resp. odd superderivation) of \( A \) and the elements in \( \text{Der}(A) \) are called superderivations of \( A \). For more details on superderivations for Lie superalgebra, the reader is referred to [14].

Let \( g \) be a Lie algebra and \( V \) an \( g \)-module. A linear mapping \( D : g \to V \) is called a derivation of \( g \) to \( V \) if \( D(xy) = x \cdot D(y) - y \cdot D(x) \) for all \( x, y \in g \). A derivation \( D : g \to V \) is called inner if there is \( v \in V \) such that \( D(x) = x \cdot v \) for all \( x \in g \). Following [15, p. 13], denote by \( \text{Der}(g, V) \) the space of derivations of \( g \) to \( V \). Then \( \text{Der}(g, V) \) is a \( g \)-submodule of \( \text{Hom}_F(g, V) \). Assume in addition that \( g \) and \( V \) are finite-dimensional and that \( g = \oplus_{r \in \mathbb{Z}} g_r \) is \( \mathbb{Z} \)-graded and \( V = \oplus_{r \in \mathbb{Z}} V_r \) is a \( \mathbb{Z} \)-graded \( g \)-module. Then \( \text{Der}(g, V) = \oplus_{r \in \mathbb{Z}} \text{Der}_r(g, V) \) is \( \mathbb{Z} \)-graded \( g \)-module by setting

\[
\text{Der}_r(g, V) := \{ D \in \text{Der}(g, V) \mid D(g_i) \subseteq V_{r+i} \text{ for all } i \in \mathbb{Z} \}.
\]

In the case of \( V = g \), the derivation algebra \( \text{Der}(g) \) coincides with \( \text{Der}(g, g) \) and \( \text{Der}(g) = \oplus_{r \in \mathbb{Z}} \text{Der}_r(g) \) is a \( \mathbb{Z} \)-graded Lie algebra.

For a Lie superalgebra \( g = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \), the restriction of an even superderivation \( D \in \text{Der}_\overline{0}(g) \) to \( \mathfrak{g}_0 \) is a derivation of the Lie algebra \( \mathfrak{g}_0 \). For convenience, we shall write still \( \text{Der}_\overline{0}(g) \) for the set \( \{ D | D \in \text{Der}_\overline{0}(g) \} \). By using this notation, it is easy to see that \( \text{Der}_\overline{0}(g) \subseteq \text{Der}(\mathfrak{g}_0) \) is a subalgebra \( \text{Der}(\mathfrak{g}_0) \). As mentioned in the introduction, as one of the main results, we shall prove in this paper that the converse inclusion is also valid for \( g = W, S \); the generalized Witt superalgebra and the special superalgebra of Cartan type over a field of finite characteristic (for a definition, see Section 2.2).

If \( g = \oplus_{-r \leq i \leq s} \mathfrak{g}_i \) is a \( \mathbb{Z} \)-graded Lie algebra, then \( \oplus_{-r \leq i \leq s} \mathfrak{g}_i \) is called the top of \( g \) (with respect to the gradation).
1.2. Modular Lie superalgebras $W$ and $S$

In this subsection we review the notions of modular Lie superalgebras $W$ and $S$ of Cartan type and their gradation structures.

In the following sections, $\mathbb{F}$ denotes a field of characteristic $p > 3$. In addition to the standard notation $\mathbb{Z}$, we write $\mathbb{N}$ for the set of positive integers, and $\mathbb{N}_0$ for the set of nonnegative integers. Henceforth, we will let $m, n$ denote fixed integers in $\mathbb{N} \setminus \{1, 2\}$ without notice. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$, we put $|\alpha| = \sum_{i=1}^{m} \alpha_i$. Let $\mathfrak{A}(m)$ denote the divided power algebra over $\mathbb{F}$ with an $\mathbb{F}$-basis $\{x^{(\alpha)} | \alpha \in \mathbb{N}_0^m\}$. For $\varepsilon_i = (\delta_{i1}, \ldots, \delta_{im})$, we abbreviate $x^{(\varepsilon_i)}$ to $x_i$, $i = 1, \ldots, m$. Let $\Lambda(n)$ be the exterior superalgebra over $\mathbb{F}$ in $n$ variables $x_{m+1}, \ldots, x_{m+n}$. Denote the tensor product by $\mathfrak{A}(m, n) = \mathfrak{A}(m) \otimes_{\mathbb{F}} \Lambda(n)$. Obviously, $\mathfrak{A}(m, n)$ is an associative superalgebra with a $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $\mathfrak{A}(m)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$. Moreover, $\mathfrak{A}(m, n)$ is super-commutative.

For $g \in \mathfrak{A}(m), f \in \Lambda(n)$, we write $gf$ for $g \otimes f$. The following formulas hold in $\mathfrak{A}(m, n)$:

$$x^{(\alpha)} x^{(\beta)} = \left(\frac{\alpha + \beta}{\alpha}\right) x^{(\alpha + \beta)} \quad \text{for } \alpha, \beta \in \mathbb{N}_0^m,$$

$$x_k x_l = -x_l x_k \quad \text{for } k, l = m + 1, \ldots, m + n;$$

$$x^{(\alpha)} x_k = x_k x^{(\alpha)} \quad \text{for } \alpha \in \mathbb{N}_0^m, k = m + 1, \ldots, m + n,$$

where $\left(\frac{\alpha + \beta}{\alpha}\right) := \prod_{i=1}^{m} \left(\frac{\alpha_i + \beta_i}{\alpha_i}\right)$.

Put $Y_0 := \{1, 2, \ldots, m\}, Y_1 := \{m + 1, \ldots, m + n\}$ and $Y := Y_0 \cup Y_1$. For convenience, we adopt the notation $r' := r + m$ for $r = 1, \ldots, n$. Thus, $Y_1 := \{1', 2', \ldots, n'\}$. Set

$$\mathbb{B}_k := \{(i_1, i_2, \ldots, i_k) | m + 1 \leq i_1 < i_2 < \cdots < i_k \leq m + n\}$$

and $\mathbb{B} := \mathbb{B}(n) = \bigcup_{k=0}^{n} \mathbb{B}_k$, where $\mathbb{B}_0 := \emptyset$. For $u = \langle i_1, i_2, \ldots, i_k \rangle \in \mathbb{B}_k$, set $|u| := k, |\emptyset| := 0, x^0 := 1$, and $x^u := x_{i_1} x_{i_2} \cdots x_{i_k}$; we use also $u$ to stand for the set $\{i_1, i_2, \ldots, i_k\}$. Clearly, $\{x^{(\alpha)} x^{u} | \alpha \in \mathbb{N}_0^m, u \in \mathbb{B}\}$ constitutes an $\mathbb{F}$-basis of $\mathfrak{A}(m, n)$.

Let $D_1, D_2, \ldots, D_{m+n}$ be the linear transformations of $\mathfrak{A}(m, n)$ such that

$$D_r(x^{(\alpha)} x^{u}) = \begin{cases} x^{(\alpha - \varepsilon_r)} x^{u}, & r \in Y_0 \\ x^{(\alpha)} \cdot \partial x^{u}/\partial x_r, & r \in Y_1. \end{cases}$$

Then $D_1, D_2, \ldots, D_{m+n}$ are superderivations of the superalgebra $\mathfrak{A}(m, n)$. Let

$$W(m, n) = \left\{ \sum_{r \in Y} f_r D_r \mid f_r \in \mathfrak{A}(m, n), r \in Y \right\}.$$ 

Then $W(m, n)$ is a Lie superalgebra, which is contained in $\text{Der}(\mathfrak{A}(m, n))$.

Obviously, $p(D_i) = \tau(i)$, where

$$\tau(i) := \begin{cases} 0, & i \in Y_0 \\ 1, & i \in Y_1. \end{cases}$$
One may verify that
\[
[fD,gE] = fD(g)E - (-1)^p(f)gE(f)D + (-1)^p(g)fg[D,E]
\] (1.2.1)
for \(f, g \in \mathfrak{A}(m,n)\), \(D, E \in \text{Der} \mathfrak{A}(m,n)\). In particular, the following formula holds in \(W(m,n)\):
\[
[fD_r,gD_s] = fD_r(g)D_s - (-1)^{p(f)p(g)}gD_s(f)D_r
\]
for \(f, g \in \mathfrak{A}(m,n), r, s \in Y\).

Let
\[
\mathfrak{A}(m,n;\underline{t}) := \text{span}_F \{ x^{(\alpha)}x^u \mid \alpha \in \mathfrak{A}, u \in \mathfrak{B} \}
\]
is a finite-dimensional subalgebra of \(\mathfrak{A}(m,n)\) with a natural \(\mathbb{Z}\)-gradation \(\mathfrak{A}(m,n;\underline{t}) = \bigoplus_{r=0}^{\xi} \mathfrak{A}(m,n;\underline{t})_r\) by putting
\[
\mathfrak{A}(m,n;\underline{t})_r := \text{span}_F \{ x^{(\alpha)}x^u \mid |\alpha| + |u| = r \}, \quad \xi := |\pi| + n.
\]
Set
\[
W(m,n;\underline{t}) := \left\{ \sum_{r \in Y} f_r D_r \mid f_r \in \mathfrak{A}(m,n;\underline{t}), r \in Y \right\}.
\]
Then \(W(m,n;\underline{t})\) is a subalgebra of \(W(m,n)\). In particular, it is a finite-dimensional simple Lie superalgebra (see [19]). Obviously, \(W(m,n;\underline{t})\) is a free \(\mathfrak{A}(m,n;\underline{t})\)-module with \(\mathfrak{A}(m,n;\underline{t})\)-basis \(\{ D_r \mid r \in Y \}\). We note that \(W(m,n;\underline{t})\) possesses a standard \(\mathbb{F}\)-basis \(\{ x^{(\alpha)}x^u D_r \mid \alpha \in \mathfrak{A}, u \in \mathfrak{B}, r \in Y \}\).

Let \(r, s \in Y\) and \(D_{rs} : \mathfrak{A}(m,n;\underline{t}) \to W(m,n;\underline{t})\) be the linear mapping such that
\[
D_{rs}(f) = (-1)^{\tau(r)\tau(s)}D_r(f)D_s - (-1)^{\tau(r)\tau(s)+p(f)}D_s(f)D_r \quad \text{for } f \in \mathfrak{A}(m,n;\underline{t}).
\]
Then the following equation holds:
\[
[D_k, D_{rs}(f)] = (-1)^{\tau(k)\tau(r)}D_{rs}(D_k(f)) \quad \text{for } k, r, s \in Y; \ f \in \mathfrak{A}(m,n;\underline{t}).
\] (1.2.2)

Put
\[
S(m,n;\underline{t}) := \text{span}_F \{ D_{rs}(f) \mid r, s \in Y; \ f \in \mathfrak{A}(m,n;\underline{t}) \}.
\]
Then \(S(m,n;\underline{t})\) is a finite-dimensional simple Lie superalgebra (see [19]).

Let \(\text{div} : W(m,n;\underline{t}) \to \mathfrak{A}(m,n;\underline{t})\) be the divergence such that
\[
\text{div} \left( \sum_{r \in Y} f_r D_r \right) = \sum_{r \in Y} (-1)^{\tau(r)p(f_r)}D_r(f_r).
\]
Direct computation shows that \(\text{div}\) is superderivation of \(W(m,n;\underline{t})\) to \(\mathfrak{A}(m,n;\underline{t})\) (see [5] or [19]); that is,

\[
\text{div}[D,E] = D\text{div}(E) - (-1)^p(D)p(E)E\text{div}(D) \quad \text{for all } D, E \in W(m,n;\underline{t}).
\]
Following [19], put
\[ \mathcal{S}(m, n; \mathfrak{t}) := \{ D \in W(m, n; \mathfrak{t}) \mid \text{div}(D) = 0 \}. \]

Then \( \mathcal{S}(m, n; \mathfrak{t}) \) is a subalgebra of \( W(m, n; \mathfrak{t}) \) and \( S(m, n; \mathfrak{t}) \) is a subalgebra of \( \mathcal{S}(m, n; \mathfrak{t}) \).

In the following sections, \( W(m, n; \mathfrak{t}), S(m, n; \mathfrak{t}), \mathcal{S}(m, n; \mathfrak{t}), \) and \( \mathfrak{A}(m, n; \mathfrak{t}) \) will be denoted by \( W, S, \mathcal{S}, \) and \( \mathfrak{A} \), respectively. In addition, the the even parts of \( W, S, \) and \( \mathcal{S} \) will be denoted by \( \mathcal{W}, \mathcal{S}, \) and \( \mathcal{A} \), respectively; that is, \( W := W_{\mathcal{W}}; S := S_{\mathcal{W}}; \) and \( \mathcal{S} := \mathcal{S}_{\mathcal{W}} \).

We note also that for the sake simplicity, as mentioned above we assume throughout that \( \text{char}\mathbb{F} > 3 \) and that the parameters \( m, n > 2 \) although sometimes a weaker hypothesis is sufficient.

\section{Subalgebras and generator sets of \( \mathcal{W} \) and \( \mathcal{S} \)}

In this section, we present certain results on some subalgebras of \( \mathcal{W} \) and then give the generator sets of \( \mathcal{W} \) and \( \mathcal{S} \). These results will be frequently used in the sequel.

\subsection{Subalgebras}

In this subsection, we deal with certain subalgebras of \( \mathcal{W} \), which are important for future studies in this paper. In particular, we will study the property of the canonical torus subalgebras of \( \mathcal{W} \) and give a reduction proposition (Proposition 2.1.6) for derivations of \( \mathcal{L} \) to \( \mathcal{W} \), where \( \mathcal{L} \) is a \( \mathbb{Z} \)-graded subalgebra of \( \mathcal{W} \) satisfying \( \mathcal{L}_{-1} = \mathcal{W}_{-1} \).

Let
\[ \mathcal{G} := \text{span}_\mathbb{F}\{ x^u D_r \mid r \in Y, u \in \mathbb{B}, p(x^u D_r) = 0 \}. \]

Then \( \mathcal{G} = C_{\mathcal{W}}(\mathcal{W}_{-1}) \), the centralizer of \( \mathcal{W}_{-1} \) in \( \mathcal{W} \). Therefore, \( \mathcal{G} \) is a \( \mathbb{Z} \)-graded subalgebra of \( \mathcal{W} \). Put \( \mathcal{G}_i := \mathcal{G} \cap \mathcal{W}_i \) and
\[ E(\mathcal{G}) := \bigoplus_{r \in \mathbb{Z}} \mathcal{G}_{2r}, \quad O(\mathcal{G}) := \bigoplus_{r \in \mathbb{Z}} \mathcal{G}_{2r+1}. \]

Since \([O(\mathcal{G}), O(\mathcal{G})] = 0\), \( O(\mathcal{G}) \) is an ideal of \( \mathcal{G} \). It is easily seen that
\[ \mathcal{G}/O(\mathcal{G}) \cong E(\mathcal{G}) \cong W(n)_{\mathfrak{t}}, \]
where \( W(n) \) is the simple Lie superalgebra of Cartan type (see [4, p. 57]) and \( W(n)_{\mathfrak{t}} \) is the even part of \( W(n) \).

Let \( \mathfrak{g} = \bigoplus_{q=0}^m \mathfrak{g}_q \) be a \( \mathbb{Z} \)-graded Lie algebra. Recall that \( \mathfrak{g} \) is called transitive (with respect to the \( \mathbb{Z} \)-gradation) provided that \( \{ x \in \mathfrak{g}_q \mid [x, \mathfrak{g}_{-1}] = 0 \} = 0 \) for all \( q \in \mathbb{N}_0 \). We say that \( \mathfrak{g} \) is admissibly graded if \( C_{\mathfrak{g}}(\mathfrak{g}_{-1}) = \mathfrak{g}_{-r} \).

By the remarks above, \( \mathcal{W} \) is neither transitive nor admissibly graded. In particular, \( \mathcal{W} \) is not a simple Lie algebra. Noticing that \( \mathcal{S}_{-1} = \mathcal{W}_{-1} \) and \( \mathcal{S} \cap \mathcal{G}_0 = 0 \), we have the same conclusion for \( \mathcal{S} \).

We shall use frequently the following simple fact. The proof is straightforward and therefore is omitted.

\begin{lemma}
Let \( \mathcal{L} \) be a \( \mathbb{Z} \)-graded subalgebra of \( \mathcal{W} \) such that \( \mathcal{L}_{-1} = \mathcal{W}_{-1} \). Suppose that \( \phi \in \text{Der}(\mathcal{L}, \mathcal{W}) \) and \( \phi(\mathcal{L}_{-1}) = 0 \). If \( E \) is an element of \( \mathcal{L} \) such that \([E, \mathcal{W}_{-1}] \subset \ker(\phi)\), then \( \phi(E) \in \mathcal{G} \).
\end{lemma}

\[ \Box \]
Put $\Gamma_r := x_r D_r$ for $r \in Y$ and $\Gamma = \sum_{r \in Y} \Gamma_r$. Then $\Gamma$ is called the degree derivation of $\mathfrak{A}(m,n;\ell)$. Correspondingly, $\text{ad}\Gamma$ is called the degree derivation of $\mathcal{W}$. Put $\Gamma' := \sum_{k \in Y_1} \Gamma_k$ and $\Gamma'' := \sum_{i \in Y_0} \Gamma_i$.

In this paper we adopt the following notation. Let $P$ denote a proposition. Put $\delta_P := 1$ if $P$ is true and $\delta_P := 0$, otherwise. We need the following computational lemma.

**Lemma 2.1.2.** Let $i \in Y_0$, $k \in Y_1$, $\alpha \in \mathbb{A}$ and $u \in \mathbb{B}$. Then the following statements hold.

(i) $\Gamma_i(x^{(\alpha)} x^u) = \alpha_i x^{(\alpha)} x^u$.

(ii) $\Gamma_k(x^{(\alpha)} x^u) = \delta_k e_u x^{(\alpha)} x^u$; in particular, $\Gamma_k^2 = \Gamma_k$.

(iii) For $f \in \mathfrak{A}$, $r \in \mathbb{N}_0$, $\Gamma(f) = rf$.

(iv) Let $r \in \mathbb{Z}$ and $D \in \mathcal{W}_r$. Then $[\Gamma, D] = rD$.

(v) $\Gamma_i^2 = \Gamma_i$.

(vi) Let $D \in \mathcal{G}_r$. If $r$ is even, then $[\Gamma', D] = rD$; if $r$ is odd, then $[\Gamma', D] = (r+1)D$.

(vii) Every standard basis element of $W$ is an eigenvector of $\text{ad}\Gamma_r$ for all $r \in Y$.

(viii) $[\Gamma'', x^{(\alpha)} x^u D_i] = (|\alpha| - 1)x^{(\alpha)} x^u D_i$ for all $\alpha \in \mathbb{A}, u \in \mathbb{B}$ and $i \in Y_0$. \hfill \Box

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra. Following [16, p. 119] and [15, p. 79], define $x \in \mathfrak{g}$ to be $p$-semisimple if $x \in \sum_{r \in \mathbb{N}} \mathbb{F} x^{[p]_r}$. If $x^{[p]} = x$ (and hence is $p$-semisimple) we say that $x$ is toral. An abelian restricted subalgebra $\mathfrak{T}$ of $\mathfrak{g}$ is called a torus if every element in $\mathfrak{T}$ is $p$-semisimple.

If $\mathfrak{g}$ is a (not necessarily restricted) Lie algebra with trivial center, then $\mathfrak{g}$ may be identified with a subalgebra of the restricted Lie algebra $\text{Der}(\mathfrak{g})$. Following [1, p.97], we say that a subalgebra $\mathfrak{T}$ is a torus of $\mathfrak{g}$ if $\mathfrak{T}$ is a torus of $\text{Der}(\mathfrak{g})$.

Let us consider $\mathcal{W}$ and $\mathcal{S}$. Note that $\mathcal{W}$ and $\mathcal{S}$ are all centerless. Put $\mathcal{T} := \sum_{r \in Y} \mathbb{F} \cdot \Gamma_r$ and $\mathcal{T}_S := \mathcal{S} \cap \mathcal{T}$. Then by Lemma 2.1.2, $\mathcal{T}$ and $\mathcal{T}_S$ are tori of $\mathcal{W}$ and $\mathcal{S}$, which are called canonical tori.

The following lemma will be heavily used in this paper. It is essentially a generalization of [15, Proposition 8.2, p. 192].

**Lemma 2.1.3.** Let $V$ be a vector space over $\mathbb{F}$ and $v_1, v_2, \ldots, v_k \in V$. Suppose that $A_i \in \text{End}_\mathbb{F} V$ is generalized invertible; that is, there is $B_i \in \text{End}_\mathbb{F} V$ such that

$$A_i B_i A_i = A_i, \quad 1 \leq i \leq k. \quad (2.1.1)$$

If the following conditions are satisfied, then there exists $v \in V$ such that $A_i(v) = v_i$ for $1 \leq i \leq k$.

(i) $A_1, A_2, \ldots, A_k$ commute mutually;

(ii) $A_i(v_j) = A_j(v_i), \quad 1 \leq i, j \leq k$;

(iii) $A_i B_i(v_i) = v_i, \quad 1 \leq i \leq k, \quad (2.1.2)$

$$A_i B_j = B_j A_i \text{ whenever } i \neq j. \quad (2.1.3)$$

**Proof.** We use induction on $k$. When $k = 1$, putting $v := B_1(v_1)$, one obtains from (2.1.2) that $A_1(v) = A_1 B_1(v_1) = v_1$. 

Let $k \geq 2$ and assume that the conclusion holds for $k-1$. Then there exists $w \in V$ such that $A_i(w) = v_i$, $i = 1, \ldots, k-1$. Set $v := w + B_k(v_k - A_k(w))$. Then, for $i = 1, \ldots, k-1$, we obtain from (2.1.3), (i) and (ii) that

$$A_i(v) = A_i(w) + A_iB_k(v_k - A_k(w))$$

$$= v_i + B_kA_i(v_k - A_k(w))$$

$$= v_i + B_k(A_i(v_k) - A_kA_i(v_i))$$

$$= v_i.$$

Moreover, by (2.1.1) and (2.1.2) we have

$$A_k(v) = A_k \left( w + B_k(v_k - A_k(w)) \right)$$

$$= A_k(w) + A_kB_k(v_k - A_k(w))$$

$$= A_k(w) + A_kB_k(v_k) - A_kB_kA_k(w)$$

$$= A_k(w) + v_k - A_k(w)$$

$$= v_k.$$

The proof is complete. $\Box$

Using Lemma 2.1.3, we give a result analogous to [15, Proposition 8.2, p.192], which will be used to prove the main result in this subsection (Proposition 2.1.6).

**Corollary 2.1.4.** Let $r \leq m$ be an integer and $f_1, \ldots, f_r \in \mathfrak{A}$. Suppose that

(a) $D_i(f_j) = D_j(f_i)$, $1 \leq i, j \leq r$;

(b) $D_i^{j-1}(f_i) = 0$, $1 \leq i \leq r$.

Then there is $f \in \mathfrak{A}$ such that $D_i(f) = f_i$ for $1 \leq i \leq r$.

**Proof.** Define $B_i : \mathfrak{A} \to \mathfrak{A}$ to be a linear transformation such that

$$B_i(x^{(\alpha)}x^u) = x^{(\alpha+\varepsilon_i)}x^u \quad \text{for } i \in Y_0$$

where $x^{(\alpha+\varepsilon_i)} := 0$ for $\alpha + \varepsilon_i \not\in \mathfrak{A}(m; \mathfrak{L})$. We check the conditions of Lemma 2.1.3. Evidently, $D_1, \ldots, D_r$ satisfy the condition (i). (b) shows that $B_1, \ldots, B_r$ satisfy (2.1.1) and that (iii) holds. By (a), $f_1, \ldots, f_r$ satisfy the condition (ii) in Lemma 2.1.3. The proof is complete. $\Box$

Obviously, $\Lambda(n)$ is an invariant subspace of $\mathfrak{A}$ under $\Gamma_k$ for all $k \in Y_1$. Then $\Gamma_k \in \text{End}_\mathbb{F}(\Lambda(n))$. The following corollary will be used in Section 4 to consider how a derivation $\phi$ of $\mathfrak{S}$ to $\mathcal{W}$ is determined by the action of $\phi$ on $\mathfrak{S}_0$ (see Lemma 4.2.4).

**Corollary 2.1.5.** Let $r \leq m$ be a positive integer and $f_1, \ldots, f_r \in \Lambda(n)$. Suppose that $\Gamma_q_i = x_{q_i}D_{q_i}$, $q_i \in Y_1$, $1 \leq i \leq r$, satisfy the following conditions:

(a) $\Gamma_q_i(f_j) = \Gamma_q_j(f_i)$, $1 \leq i \leq r$;

(b) $\Gamma_{q_i}(f_i) = f_i$, $1 \leq i \leq r$.

Then there exists $f \in \Lambda(n)$ such that $\Gamma_q_i(f) = f_i$ for $i = 1, 2, \ldots, r$. 

Proof. Set $B_i = \text{id}_{A(n)}$. We check the conditions of Lemma 2.1.3. Since $\Gamma_{ii}^2 = \Gamma_{qi}$ by Lemma 2.1.2(ii), (2.1.1) holds. The remaining conditions may be easily checked.

Let $V = \oplus_{r \in \mathbb{Z}} V_r$ be a $\mathbb{Z}$-graded vector space. For a $\mathbb{Z}$-homogeneous element $v \in V$, we denote by $zd(v)$ the $\mathbb{Z}$-degree of $v$.

For the sake of convenience, in the sequel we usually write $i, j$ for the elements in $Y_0$, and write $k, l$ for the elements in $Y_1$. View $W$ as an $L$-module by the adjoint representation. Recall the definition of a derivation of a Lie algebra to its module (see Section 1.1). Let $L$ be a $\mathbb{Z}$-subalgebra of $W$ satisfying $L_{-1} = W_{-1}$. Just as [15, Proposition 8.3, p. 193], the following proposition reduces derivations of $\text{Der}(L, W)$ to the derivations vanishing on $L_{-1}$.

**Proposition 2.1.6.** Let $L$ be a $\mathbb{Z}$-subalgebra of $W$ such that $L_{-1} = W_{-1}$. If $\phi \in \text{Der}_t(L, W)$ where $t := zd(\phi) \geq 0$, then there exists $E \in \mathcal{W}_t$ such that

$$ (\phi - \text{ad}E)(L_{-1}) = 0. $$

**Proof.** Suppose that $\phi(D_i) = \sum_{r \in Y} f_{ri} D_r$ for $i \in Y_0$, where $f_{ri} \in \mathfrak{A}$. Applying $\phi$ to the identity $[D_i, D_j] = 0$ for $i, j \in Y_0$, we have $[\phi(D_i), D_j] + [D_i, \phi(D_j)] = 0$. Then

$$ \left[ \sum_{r \in Y} f_{ri} D_r, D_j \right] + \left[ D_i, \sum_{r \in Y} f_{rj} D_r \right] = 0. $$

Consequently,

$$ \sum_{r \in Y} D_i (f_{rj}) D_r - \sum_{r \in Y} D_j (f_{ri}) D_r = 0. $$

Since $\{D_r \mid r \in Y\}$ is a free basis of $\mathfrak{A}$-module $W$, we have

$$ D_i (f_{rj}) = D_j (f_{ri}) \quad \text{for all } i, j \in Y_0, \ r \in Y. $$

This implies that for fixed $r \in Y$, $\{f_{r1}, \ldots, f_{rm}\}$ satisfies the condition (a) in Corollary 2.1.4 (see also [15, Proposition 8.2, p. 192]). On the other hand, since $(\text{ad}D_i)^{\pi+1} = 0$ for $i \in Y_0$, it follows from [15, Lemma 8.1, p. 191] that $(\text{ad}D_i)^{\pi} (\phi(D_i)) = 0$. Consequently, $D_i^{\pi} (f_{ri}) = 0$ for $r \in Y$ and $i \in Y_0$; that is, $\{f_{r1}, \ldots, f_{rm}\}$ fulfills the condition (b) in Corollary 2.1.4. Therefore, there is $g_r \in \mathfrak{A}$ such that

$$ D_i (g_r) = f_{ri} \quad \text{for all } i \in Y_0. $$

Let $E'' = - \sum_{r \in Y} g_r D_r$. Then for $i \in Y_0$, we have

$$ [E'', D_i] = \sum_{r \in Y} D_i (g_r) D_r - \sum_{r \in Y} f_{ri} D_r = \phi(D_i). \quad (2.1.4) $$

Note that $zd(\phi) = t$. (2.1.4) shows that

$$ [E''_i, D_i] = \phi(D_i) \quad \text{for } i \in Y_0. \quad (2.1.5) $$

Let $E' = E''$. Then (2.1.5) implies that $(\phi - \text{ad}E') |_{L_{-1}} = 0$. Let $E = E''_0$ (the even part of $E'$). Then $E \in \mathcal{W}_t$ and $(\phi - \text{ad}E) |_{L_{-1}} = 0$. □
Remark 2.1.8. Since $\mathcal{L}_{-1}$ is an abelian Lie algebra, one may consider the derivation space $\text{Der}(\mathcal{L}_{-1}, \mathcal{W})$. Then, under the conditions of Proposition 2.1.6, $\phi|_{\mathcal{L}_{-1}} \in \text{Der}(\mathcal{L}_{-1}, \mathcal{W})$ is inner.

Remark 2.1.7. In Proposition 2.1.6 the element $E$ does not necessarily lie in $\text{Nor}_W(\mathcal{L})$. In addition, in contrast to the cases of Cartan type Lie algebras and Lie superalgebras (see [15, Proposition 8.3, p. 193] and [23, Lemma 5]), we cannot prove directly that $\phi = (\text{ad}E)\big|_{\mathcal{L}}$, since $\mathcal{L}$ is not necessarily admissibly graded, as noted at the beginning of this subsection.

2.2. Generator sets

As is well known, the generators play an important role in the study on derivations of an $F$-algebra. More precisely, a derivation is completely determined by its action on the generator set. When we consider the derivation algebra of a given algebra, it is natural to hope finding a ‘good’ generator set, which is convenient for computing derivations.

In this subsection, we study mainly the generator sets of $\mathcal{W}$ and $\mathcal{S}$ for the purpose mentioned above. We know that $W(m, n; \mathfrak{t})$ is a free $\mathbb{Z}$-graded module of the associative super-commutative algebra $\mathbb{A}(m, n; \mathfrak{t})$ with basis $\{D_r \mid r \in Y\}$ consisting of special derivations of $W(m, n; \mathfrak{t})$. The ‘coefficients’ in $\mathbb{A}(m, n; \mathfrak{t})$ are tensor products of divided power series and exterior products. Thus it is desirable to find a generator set which consists of certain elements of low $\mathbb{Z}$-degree and certain elements of high $\mathbb{Z}$-degree but of simple form, such as the elements in Lie algebras $W(m; \mathfrak{t})$ or $S(m; \mathfrak{t})$ of Cartan type.

The results obtained in this subsection will be used frequently in the sequel.

Recall that $W = W(m, n; \mathfrak{t})$ and $\mathcal{W}$ denotes the even part of $W$. Recall also our notations that $\mathfrak{t} := (t_1, \ldots, t_m) \in \mathbb{N}$ and $\pi_i := p^{t_i} - 1$ for $i \in Y_0$. Put

\[
\mathcal{M} := \{x^{(q\epsilon_i)}D_j \mid 0 \leq q \leq \pi_i, \ i, j \in Y_0\},
\]

\[
\mathcal{N} := \{x_i x_k D_l \mid i \in Y_0, \ k, l \in Y_1\},
\]

\[
\mathcal{P} := \{x_k x_l D_i \mid i \in Y_0, \ k, l \in Y_1\}.
\]

For $u, v \in \mathbb{B}$ with $u \cap v = \emptyset$, define $u + v$ to be $w \in \mathbb{B}$ such that and $w = u \cup v$. If in addition $\text{max} u < \text{min} v$, then denote $u + v = w$.

**Proposition 2.2.1.** $\mathcal{W}$ is generated by $\mathcal{M} \cup \mathcal{N} \cup \mathcal{P}$.

**Proof.** Denote by $\mathcal{X}$ the subalgebra of $\mathcal{W}$ generated by $\mathcal{M} \cup \mathcal{N} \cup \mathcal{P}$. We make the following preparatory remarks.

(i) Assert that

\[
x^{(\alpha)}D_i \in \mathcal{X} \quad \text{for all } \alpha \in \mathbb{A} \text{ and all } i \in Y_0. \tag{2.2.1}
\]

In fact, it can be easily proved by induction on $r$ that for $i \in Y_0$,

\[
x^{(\pi_{1} \epsilon_{1} + \cdots + \pi_{r} \epsilon_{r})}D_i \in \mathcal{X} \quad \text{for all } r \leq m.
\]

Therefore, $x^{(\pi)}D_i \in \mathcal{X}$. The assertion follows, since $D_j \in \mathcal{X}$ for all $j \in Y_0$.

(ii) Assert that

\[
x^u D_i \in \mathcal{X} \quad \text{for } u \in \mathbb{B}, \ |u| \text{ even}; \ i \in Y_0. \tag{2.2.2}
\]
We are going to prove the assertion by induction on \(|u|\). For \(|u| = 2\), it is clear that \(x^u D_i \in \mathcal{P} \subseteq \mathcal{X}\). Assume that \(|u| > 2\). Then write \(u = v + w\), where \(v, w \in \mathbb{B}\) such that \(|v| = 2, |v| = |u| - 2\).

Note that \(x^u D_i \in \mathcal{P} \subseteq \mathcal{X}\) and \(x^v D_i \in \mathcal{X}\) by the inductive hypothesis. We have

\[
x^u D_i = [x^v D_i, [x^w D_i, x^{(2\xi_1)} D_i]] \in \mathcal{X},
\]
as asserted.

(iii) Assert that

\[
x^u D_k \in \mathcal{X} \quad \text{for } u \in \mathbb{B}, \ |u| \text{ odd; } k \in Y_1. \tag{2.2.3}
\]

We may suppose that \(|u| \geq 3\). Write \(u = v + w\) such that \(k \notin v\) and \(|v| = |u| - 1, |w| = 1\).

From (2.2.2), we have \(x^v D_1 \in \mathcal{X}\). Noticing that \(x_1 x^w D_k \in \mathcal{N}\), we obtain that

\[
x^u D_k = \lambda [x^v D_1, x_1 x^w D_k] \in \mathcal{X} \quad \text{where } \lambda = 1 \text{ or } -1.
\]

Now we are ready to show that \(\mathcal{X} = \mathcal{W}\). First claim that

\[
x^{(\alpha)} x^u D_i \in \mathcal{X} \quad \text{for } u \in \mathbb{B}, \ |u| \text{ even; } i \in Y_0. \tag{2.2.4}
\]

Without loss of generality, one may assume that \(|u| \geq 2\). Take \(k \in u\). Using (2.2.2) we have \(x^u D_i \in \mathcal{X}\). Since \(x^{(\alpha)} x_k D_k = [x^{(\alpha)} D_1, x_1 x_k D_k] \in \mathcal{X}\), we have

\[
x^{(\alpha)} x^u D_i = [x^{(\alpha)} x_k D_k, x^u D_i] \in \mathcal{X},
\]
as desired.

It remains to prove that

\[
x^{(\alpha)} x^u D_k \in \mathcal{X} \quad \text{for } u \in \mathbb{B}, \ |u| \text{ odd; } k \in Y_1. \tag{2.2.5}
\]

If \(|u| = 1\), then \(x_1 x^u D_k \in \mathcal{N} \subseteq \mathcal{X}\). It follows from (2.2.1) that \(x^{(\alpha)} x^u D_k = [x^{(\alpha)} D_1, x_1 x^u D_k] \in \mathcal{X}\). Assume that \(|u| \geq 3\). Choose \(r \in u \backslash k\). Since \(x^{(\alpha)} x_r D_r = [x^{(\alpha)} D_1, x_1 x_r D_r] \in \mathcal{X}\), it follows from (2.2.3) that

\[
x^{(\alpha)} x^u D_k = [x^{(\alpha)} x_r D_r, x^u D_k] \in \mathcal{X}.
\]

By (2.2.4) and (2.2.5), we have \(\mathcal{X} = \mathcal{W}\), completing the proof.

We note that in the three sets above only \(\mathcal{M}\) contains elements of \(\mathbb{Z}\)-degree higher than 1. Clearly, \(\mathcal{M}\) is contained in \(W(m; \lambda)\), the generalized Witt algebra (see [18]). As we shall see in the following sections, this allows us to compute efficiently.

Recall the torus \(T_S\) of \(S\) (see Section 2.1). Clearly,

\[
\{x_r D_r - x_s D_s \mid \tau(r) = \tau(s); r, s \in Y\} \cup \{x_r D_r + x_s D_s \mid \tau(r) \neq \tau(s); r, s \in Y\}
\]
is an \(\mathbb{F}\)-basis of \(T_S\) consisting of toral elements.

We first give the following simple fact:

**Lemma 2.2.2.** \(S_0\) is spanned by \(T_S \cup \{x_r D_s \mid \tau(r) = \tau(s), r \neq s; r, s \in Y\}\).
Proof. It is straightforward. □

Put
\[ Q := \{ D_{ij}(x^{(r \varepsilon_j)}) \mid i, j \in Y_0, r \in \mathbb{N}_0 \}; \]
\[ R := \{ D_{il}(x^{(2 \varepsilon_l)} x_k) \mid i \in Y_0, k, l \in Y_1 \} \cup \{ D_{ij}(x^{(r \varepsilon_j)}) \mid i, j \in Y_0, v \in \mathbb{B}_2 \}. \]

We now consider the generator set of \( S \).

Proposition 2.2.3. \( S \) is generated by \( Q \cup R \cup S_0 \).

Proof. Let \( \mathcal{X} \) be the subalgebra generated by \( Q \cup R \cup S_0 \). We proceed in several steps to show that \( S = \mathcal{X} \).

(i) Let \( i, j \in Y_0 \) with \( i \neq j \), and let \( k, l \in Y_1 \). Then
\[
D_{jl}(x^{(2 \varepsilon_l)} x_k) = [D_{ij}(x^{(3 \varepsilon_l)}), D_{il}(x^{(r \varepsilon_j)})] + 3 \delta_{kl} D_{ij}(x^{(3 \varepsilon_l)}) \in \mathcal{X} \tag{2.2.6}
\]
and
\[
D_{il}(x^{(2 \varepsilon_l)} x_k) = [D_{il}(x^{(2 \varepsilon_l)} x_k), D_{ij}(x^{(2 \varepsilon_j)})] \in \mathcal{X}. \tag{2.2.7}
\]
In addition, for any \( r \in Y_0 \setminus \{ i, j \} \), we have \( D_{rl}(x^{(2 \varepsilon_l)} x_k) = [D_{ij}(x^{(3 \varepsilon_l)}), D_{ji}(x^{(2 \varepsilon_i)})] \in \mathcal{X} \). Therefore,
\[
D_{rl}(x^{(2 \varepsilon_l)} x_k) = [D_{ir}(x^{(2 \varepsilon_i)}), D_{jl}(x^{(2 \varepsilon_j)} x_k)] - \delta_{kl} D_{rl}(x^{(2 \varepsilon_i + 2 \varepsilon_j)}) \in \mathcal{X}. \tag{2.2.8}
\]
Combining (2.2.6)–(2.2.8) and the assumption that \( D_{il}(x^{(2 \varepsilon_l)} x_k) \in \mathcal{R} \) for all \( i \in Y_0 \) and \( k, l \in Y_1 \), we obtain that
\[
D_{rl}(x^{(2 \varepsilon_l)} x_k) \in \mathcal{X} \quad \text{for all } i, j \in Y_0 \text{ and all } k, l \in Y_1. \tag{2.2.9}
\]

(ii) Let \( k, l \in Y_1 \) with \( k \neq l \). For all \( r \in Y_1 \) and all \( i \in Y_0 \), it follows that
\[
D_{kr}(x^{(2 \varepsilon_l)} x_k) = [D_{ik}(x^{(2 \varepsilon_l)} x_k), D_{ir}(x^{(r \varepsilon_l)} x_l)] - D_{ir}(x^{(2 \varepsilon_l)} x_l) + 2 \delta_{kl} D_{ik}(x^{(2 \varepsilon_l)} x_k) \in \mathcal{X}. \tag{2.2.10}
\]

(iii) Let \( u \in \mathbb{B}_3 \). Write \( u = v + w \) with \( |v| = 2, |w| = 1 \). Given \( i \in Y_0 \) and \( k \in Y_1 \), take \( j \in Y_0 \setminus i \) and \( l \in v \setminus k \). Remember the assumption \( D_{ij}(x^{(r \varepsilon_j)}) \in \mathcal{R} \). We have
\[
D_{ik}(x^u) = -[D_{ij}(x^{(r \varepsilon_j)}), D_{kl}(x^{(r \varepsilon_l)} x_l)] \in \mathcal{X}. \tag{2.2.11}
\]
Combining (2.2.9)–(2.2.11) and the assumption that \( D_{ij}(x^{(r \varepsilon_j)}) \in \mathcal{R} \) for all \( i, j \in Y_0 \) and \( v \in \mathbb{B}_2 \), we obtain that
\[
S_1 \subset \mathcal{X}. \tag{2.2.12}
\]

(iv) Assert that \( D_{ij}(x^{(\alpha)}) \in \mathcal{X} \) for all \( i, j \in Y_0 \) and \( \alpha \in \mathfrak{A} \). For the purpose, we first show that \( D_{12}(x^{(r \varepsilon_1 + 2 \varepsilon_2)}) \in \mathcal{X} \). It is easy to verify the following equations:
\[
D_{12}(x^{(2 \varepsilon_1 + 2 \varepsilon_2)}) = [D_{12}(x^{(3 \varepsilon_1)}), D_{12}(x^{(2 \varepsilon_2)})] \in \mathcal{X},
D_{12}(x^{(r \varepsilon_2)}) = [D_{12}(x^{(r \varepsilon_2)}), D_{12}(x^{(2 \varepsilon_1 + 2 \varepsilon_2)})] \in \mathcal{X},
\]
\[
D_{12}(x^{(r \varepsilon_1 + 2 \varepsilon_2)}) = [D_{12}(x^{(r \varepsilon_1 + 2 \varepsilon_2)}), D_{12}(x^{(r \varepsilon_1)})] \in \mathcal{X},
\]
and for $k \in Y_1$,
\[
D_{2k}(x^{(\pi_1 \xi_1 + (\pi_2 - 1) \xi_2)} x_k) = -[D_{12}(x^{(\pi_1 \xi_1 + (\pi_2 - 1) \xi_2)}), D_{1k}(x^{(2 \xi_1)} x_k)] \in \mathcal{X},
\]
\[
D_{2k}(x^{((\pi_1 - 1) \xi_1 + (\pi_2 - 1) \xi_2)} x_k) = [D_1, D_{2k}(x^{(\pi_1 \xi_1 + (\pi_2 - 1) \xi_2)} x_k)] \in \mathcal{X},
\]
\[
D_{2k}(x^{((\pi_1 - 1) \xi_1 + \pi_2 \xi_2)} x_k) = -\frac{1}{2}[D_{2k}(x^{((\pi_1 - 1) \xi_1 + (\pi_2 - 1) \xi_2)} x_k), D_{2k}(x^{(2 \xi_2)} x_k)] \in \mathcal{X}.
\]
Similarly, we also have $D_{ik}(x^{(\pi_1 \xi_1 + (\pi_2 - 1) \xi_2)} x_k) \in \mathcal{X}$. Therefore,
\[
D_{12}(x^{(\pi_1 \xi_1 + \pi_2 \xi_2)}) = [D_{1k}(x^{(2 \xi_1)} x_k), D_{12}(x^{((\pi_1 - 1) \xi_1 + \pi_2 \xi_2)})]
+ D_{1k}(x^{(\pi_1 \xi_1 + (\pi_2 - 1) \xi_2)} x_k) - 2D_{2k}(x^{((\pi_1 - 1) \xi_1 + \pi_2 \xi_2)} x_k) \in \mathcal{X}.
\]
Now, proceeding by induction on $q$ one may easily prove that
\[
D_{q, q+1}(x^{(\pi_1 \xi_1 + \cdots + \pi_{q+1} \xi_{q+1})}) \in \mathcal{X}
\]
(cf. [23, Theorem 1]). As a result, $D_{\text{m} \to \text{m}}(x^{(\pi)}) \in \mathcal{X}$. It follows that $D_{ij}(x^{(\pi)}) \in \mathcal{X}$ for all $i, j \in Y_0$. Hence the assertion holds.

(v) Suppose that $|u|$ is odd. We propose to prove that
\[
D_{ik}(x^u) \in \mathcal{X} \quad \text{for all } i \in Y_0, k \in Y_1. \tag{2.2.13}
\]
For $|u| = 3$, from (2.2.12) it is easily seen that (2.2.13) holds. Use induction on $|u| \geq 3$ to prove (2.2.13). Assume that $|u| \geq 5$ and write $u = v + w$ where $|v| = |u| - 2$ and $|w| = 2$. Let $l = \min v$. By inductive hypothesis, $D_{li}(x^u) \in \mathcal{X}$ for all $i \in Y_0$. Then for $i \in Y_0$ and $k \in Y_1$,
\[
D_{ik}(x^u) = [D_{li}(x^v), D_{ik}(x_i x_l x^w)] = [D_{li}(x^v), [D_{li}(x_l x^w), D_{ik}(x^{(2 \xi_1)} x_l)]] \in \mathcal{X}.
\]
This proves (2.2.13).

(vi) Suppose that $|u|$ is even. Assert that
\[
D_{kl}(x^u) \in \mathcal{X} \quad \text{for all } k, l \in Y_1. \tag{2.2.14}
\]
We first note that if $v \in \mathbb{B}$ and $|v|$ is odd, then
\[
D_{ik}(x_i x^v) \in \mathcal{X} \quad \text{for all } i \in Y_0, k \in Y_1. \tag{2.2.15}
\]
In fact, this follows from (2.2.13) and the following equation
\[
D_{ik}(x_i x^v) = [D_{qi}(x^v), D_{ik}(x^{(2 \xi_1)} x_q)] \in \mathcal{X}
\]
where $q = \min v$.

One may write $u = v + w$ where $|v| = |u| - 1, |w| = 1$. Using (2.2.15) we know that for $k, l \in Y_1$,
\[
D_{kl}(x^u) = [D_{1k}(x_1 x^v), D_{1l}(x_1 x^w)] - D_{li}(x_1 D_k(x^v) x^w) + D_{k1}(x_1 x^v D_l(x^w)) \in \mathcal{X}.
\]
Then (2.2.14) follows.
(vii) Let us complete the proof of this proposition. For the purpose, first, we show that if \(|u|\) is odd where \(u \in \mathbb{B}\), then

\[
D_{ik}(x^{(\pi)}x^u) \in \mathcal{X} \quad \text{for all } i \in Y_0, k \in Y_1. \tag{2.2.16}
\]

For \(|u| = 1\), let \(j \in Y_0 \setminus i\). Then

\[
D_{ik}(x^{(\pi)}x^u) = [D_{ij}(x^{(\pi)}), D_{kj}(x^{(2\varepsilon_j)}x^u)] \in \mathcal{X}.
\]

Suppose that \(|u| \geq 3\). To prove (2.2.16), we note that

\[
D_{jk}(x^{(3\varepsilon_j)}x_l) = -\frac{1}{3}[D_{jk}(x^{(2\varepsilon_j)}x_l), D_{jl}(x^{(2\varepsilon_j)}x_l)] \in \mathcal{X} \quad \text{for all } j \in Y_0, k, l \in Y_1 \text{ with } k \neq l. \tag{2.2.17}
\]

Given \(k \in Y_1\), since \(|u| \geq 3\), one may choose \(l \in u \setminus k\). Without loss of generality, assume that \(l = \min u\). Then we obtain from (2.2.13) and (2.2.17) that

\[
D_{kj}(x^{(2\varepsilon_j)}x^u) = [D_{ij}(x^u), D_{kl}(x^{(3\varepsilon_j)}x_l)] \in \mathcal{X} \quad \text{for all } j \in Y_0, k \in Y_1. \tag{2.2.18}
\]

Take \(j \in Y_0 \setminus i\). Then (iv) and (2.2.18) ensure that

\[
D_{ik}(x^{(\pi)}x^u) = -[D_{ij}(x^{(\pi)}), D_{kj}(x^{(2\varepsilon_j)}x_l)] \in \mathcal{X},
\]

proving (2.2.16).

Next, we show that for \(u \in \mathbb{B}\),

\[
D_{ij}(x^{(\pi)}x^u) \in \mathcal{X} \quad \text{for } i, j \in Y_0, \ |u| \text{ even}; \tag{2.2.19}
\]

\[
D_{kl}(x^{(\pi)}x^u) \in \mathcal{X} \quad \text{for } k, l \in Y_1, \ |u| \text{ even}. \tag{2.2.20}
\]

One may write \(u = v + w\) where \(|v| = |u| - 1, |w| = 1\). Let \(l := \min v\). Using (2.2.16), we have

\[
D_{ij}(x^{(\pi)}x^u) = -[D_{il}(x^{(\pi)}x^v), D_{ji}(x_lx_lx^w)] \in \mathcal{X}, \ |u| \text{ even};
\]

that is, (2.2.19) holds. Similarly, we have also

\[
D_{kl}(x^{(\pi)}x^u) = -[D_{lk}(x^{(\pi)}x^v), D_{kl}(x^{(2\varepsilon_l)}x^w)] \in \mathcal{X}, \ |u| \text{ even},
\]

proving (2.2.20).

By (2.2.16), (2.2.19) and (2.2.20), it is easily seen that \(\mathcal{X} = S\). The proof is complete.

\(\square\)

**Remark 2.2.4.** In contrast to the case of Lie superalgebra \(S\), the elements in \(R\) cannot be generated by \(Q\) and \(S_0\) (cf. [23, Theorem 1]). The reason is that there are not odd elements in \(S_0\). On the other hand, in contrast to the case \(W\), the elements of \(Z\)-degree zero of \(S_0\) cannot be generated by \(Q \cup R\).

### 3. Derivations of \(W\)

In this section, we shall describe explicitly the derivation algebra of \(W\). To do that, we proceed in two steps. First, we show in Section 3.1 that every derivation of nonnegative \(Z\)-degree vanishing on the top of \(W\) is necessarily inner. Next, we show in Section 3.2 that every derivation of nonnegative \(Z\)-degree can be reduced to being vanishing on the top and therefore, every derivation of nonnegative \(Z\)-degree is necessarily inner. Finally, using the generator set, we compute the derivations of negative \(Z\)-degree and then formulate the derivation algebra \(\text{Der}(W)\).
3.1. The top of $W$

In this subsection, we discuss the influence of the top $W_{-1} \oplus W_0$ over the derivations of $W$. We shall see that the homogeneous derivations of nonnegative $\mathbb{Z}$-degree of $W$ which vanish on the top $W_{-1} \oplus W_0$ turn out to be inner. In other words, if two homogenous derivations of nonnegative $\mathbb{Z}$-degree of $W$ coincide on the top $W_{-1} \oplus W_0$ then they are congruent modulo $adW$. This observation allows us to focus attention to $\text{Der}(W_{-1} \oplus W_0, W)$ in next subsection.

**Lemma 3.1.1.** Let $\phi \in \text{Der}(W, t \in \mathbb{Z}$). Suppose that $\phi(W_{-1} \oplus W_0) = 0$ and $t \neq 0 \pmod{p}$. Then $\phi = 0$.

**Proof.** Consider the degree derivation $\Gamma := \sum_{i \in Y} x_i D_i$. By Lemma 2.1.2(iv), we have

$$[\Gamma, D] = rD \quad \text{for all } D \in W_r, \ r \in \mathbb{Z}.$$ 

Then noting that $\Gamma \in W_0$. We have

$$[\Gamma, \phi(D)] = r\phi(D). \quad (3.1.1)$$

Since $zd(\phi(D)) = r + t$, again by Lemma 2.1.2(iv), we have

$$[\Gamma, \phi(D)] = (r + t)\phi(D). \quad (3.1.2)$$

It follows from (3.1.1) and (3.1.2) that $t\phi(D) = 0$. Since $t \neq 0 \pmod{p}$, we have $\phi(D) = 0$ for all $D \in W_r, \ r \in \mathbb{Z}$. This proves that $\phi = 0$.

Recall that $P = \{x_kx_lD_i \mid i \in Y_0, k, l \in Y_1\}$. We have the following

**Lemma 3.1.2.** Let $\phi \in \text{Der}(W)$ with $zd(\phi) = t \geq 0$. Suppose that $\phi(W_{-1} \oplus W_0) = 0$. Then $\phi$ is a scalar transformation of $\text{span}_\mathbb{F}P$.

**Proof.** Since $\phi(W_{-1}) = \phi(W_0) = 0$, by Lemma 2.1.1 we have

$$\phi(x_kx_lD_i) \in G_{t+1} \quad \text{for } i \in Y_0, k, l \in Y_1. \quad (3.1.3)$$

In view of Lemma 3.1.1, we may assume that $t \equiv 0 \pmod{p}$. Consider $\Gamma' = \sum_{r \in Y_1} x_r D_r \in W_0$. Clearly, $[\Gamma', x_kx_lD_i] = 2x_kx_lD_i$. Applying $\phi$ to this equation, we obtain that

$$[\Gamma', \phi(x_kx_lD_i)] = 2\phi(x_kx_lD_i). \quad (3.1.4)$$

Note that $zd(\phi(x_kx_lD_i)) = t + 1$. By (3.1.3) and Lemma 2.1.2(vi) we have

$$[\Gamma', \phi(x_kx_lD_i)] = \begin{cases} (t + 2)\phi(x_kx_lD_i), & \text{zd(\phi) even;} \\ (t + 1)\phi(x_kx_lD_i), & \text{zd(\phi) odd.} \end{cases}$$

If $zd(\phi)$ is odd, then we obtain from (3.1.4) and the above equation that $\phi(x_kx_lD_i) = 0$, since $t \equiv 0 \pmod{p}$. That is, $\phi(P) = 0$. Now consider the case that $zd(\phi)$ is even. In view of (3.1.3), we may assume that

$$\phi(x_kx_lD_i) = \sum_{r \in Y_0} c_{u,r}x^u D_r \quad \text{where } c_{u,r} \in \mathbb{F}. \quad (3.1.5)$$
Since \( n > 2 \), choose \( s \in Y_1 \setminus \{ k, l \} \). Then \([x_kx_lD_1, x_sD_s] = 0\). Noting that \( \phi(W_0) = 0 \), we have \([\phi(x_kx_lD_1), x_sD_s] = 0\). It follows that
\[
\left[ \sum_{r \in Y \atop |r| = t + 2} c_{u, r} x^u D_r, x_s D_s \right] = 0 \quad \text{for all } s \in Y_1 \setminus \{ k, l \}.
\] (3.1.6)

From (3.1.5) and (3.1.6), we have \( c = 0 \) for all \( s \in Y \). Therefore, we obtain from (3.1.7) that
\[
\phi(x_kx_lD_i) = \sum_{r \in Y_0} c_{< k, l >, r} x_kx_lD_r.
\] (3.1.7)

If \( zd(\phi) > 0 \), by (3.1.7) we have \( \phi(x_kx_lD_i) = 0 \) and therefore \( \phi(P) = 0 \). Thus, it remains only to consider the case \( zd(\phi) = 0 \). Applying \( \phi \) to the equation that \([x_kx_lD_1, x_iD_i] = x_kx_lD_i \), we obtain from (3.1.7) that
\[
\phi(x_kx_lD_i) = c_{< k, l >, i} x_kx_lD_i.
\] (3.1.8)

Let \( \lambda := c_{< k, l >, i} \). For arbitrary \( j \in Y_0 \), applying \( \phi \) to the equation that \([x_kx_lD_i, x_jD_j] = x_kx_lD_j \), we have
\[
\phi(x_kx_lD_j) = \lambda(x_kx_lD_j).
\] (3.1.9)

For arbitrary \( r \in Y_1 \), applying \( \phi \) to the equation that \([x_rD_k, x_kx_lD_j] = x_rx_lD_j \), we obtain from (3.1.9) that
\[
\phi(x_rx_lD_j) = \lambda x_rx_lD_j.
\] (3.1.10)

(3.1.8)–(3.1.10) show that \( \lambda = c_{< k, l >, i} \) is independent of the choice of \( k, l, i \). Therefore,
\[
\phi(x_kx_lD_i) = \lambda x_kx_lD_i \quad \text{for all } k, l \in Y_1, i \in Y_0.
\]

The proof is complete. \( \square \)

Recall that \( \mathcal{N} = \{ x_i x_kD_l \mid i \in Y_0, k, l \in Y_1 \} \). We have the following fact.

**Lemma 3.1.3.** Let \( \phi \in \text{Der}(W), zd(\phi) = t \geq 0 \). Suppose that \( \phi(W_{-1} \oplus W_0) = 0 \). Then \( \phi(\mathcal{N}) = 0 \).

**Proof.** By Lemma 2.1.1, \( \phi(x_i x_kD_l) \in G_{t+1} \) for \( i \in Y_0, k, l \in Y_1 \). In view of Lemma 3.1.1, it suffices to consider the case that \( t \equiv 0 \pmod{p} \). Recall \( \Gamma' := \sum_{r \in Y_1} x_rD_r \). Clearly,
\[
[\Gamma', x_ix_kD_l] = 0 \quad \text{for } i \in Y_0, k, l \in Y_1.
\] (3.1.11)

Therefore,
\[
[\Gamma', \phi(x_i x_kD_l)] = 0 \quad \text{for all } i \in Y_0, k, l \in Y_1.
\] (3.1.12)

Since \( \phi(x_i x_kD_l) \in G_{t+1} \), using Lemma 2.1.2 we obtain that
\[
[\Gamma', \phi(x_i x_kD_l)] = \begin{cases} 
(t + 2)\phi(x_i x_kD_l), & \text{zd}(\phi) \text{ even;} \\
(t + 1)\phi(x_i x_kD_l), & \text{zd}(\phi) \text{ odd.}
\end{cases}
\] (3.1.13)

Comparing (3.1.11) and (3.1.12) and noticing that \( t \equiv 0 \pmod{p} \), we have
\[
\phi(x_i x_kD_l) = 0 \quad \text{for all } i \in Y_0, k, l \in Y_1.
\]

The proof is complete. \( \square \)

Recall that \( \mathcal{M} = \{ x^{(q \pi_i)}D_j \mid i, j \in Y_0, 0 \leq q \leq \pi_i \} \). We have the following
Lemma 3.1.4. Let \( \phi \in \text{Der}(W) \), \( \text{zd}(\phi) = t \geq 0 \). Suppose that \( \phi(W_{-1} \oplus W_0) = 0 \). Then \( \phi(M) = 0 \).

**Proof.** We proceed by induction on \( q \geq 2 \) to prove that
\[
\phi(x^{(q\varepsilon_i)}D_j) = 0 \quad \text{for all } i, j \in Y_0, \ q \in \mathbb{N}_0.
\]
For \( q = 2 \), Lemma 2.1.1 shows that \( \phi(x^{(2\varepsilon_i)}D_j) \in G_{t+1} \). Thus we may assume that
\[
\phi(x^{(2\varepsilon_i)}D_j) = \sum_{r \in Y} c_{u,r}x^uD_r \quad \text{where } c_{u,r} \in \mathbb{F}.
\] (3.1.13)

Note that \( \text{zd}(\phi(x^{(2\varepsilon_i)}D_j)) \geq 1 \). For every pair \( (u,r) \), there is \( k \in u \setminus r \). Clearly, \([x^{(2\varepsilon_i)}D_j, x_kD_k] = 0\). Then
\[
[\phi(x^{(2\varepsilon_i)}D_j), x_kD_k] = 0.
\] (3.1.14)

(3.1.13) and (3.1.14) imply that \( c_{u,r} = 0 \). Since the pair \( (u,r) \) is arbitrary, we have \( \phi(x^{(2\varepsilon_i)}D_j) = 0 \).

By the inductive hypothesis and Lemma 2.1.1, it is easily seen that \( \phi(x^{(q\varepsilon_i)}D_j) \in G \). Just as in the case \( q = 2 \), one may check that \( \phi(x^{(q\varepsilon_i)}D_j) = 0 \).

Now we are able to prove the following fact.

**Corollary 3.1.5.** Let \( \phi \in \text{Der}_t(W), t \geq 0 \). Suppose that \( \phi(W_{-1} \oplus W_0) = 0 \). Then \( \phi \in \text{ad}W \).

**Proof.** By Lemma 3.1.2, there is \( \lambda \in \mathbb{F} \) such that
\[
\phi(x_kx_lD_i) = \lambda x_kx_lD_i \quad \text{for all } k, l \in Y_1, i \in Y_0.
\]
Put \( \psi := \phi - \frac{1}{2}\lambda \text{ad}\Gamma' \) where \( \Gamma' = \sum_{r \in Y_1} x_rD_r \). Then \( \psi(W_{-1} \oplus W_0) = 0 \) and \( \psi(P) = 0 \). By Lemma 3.1.3 and 3.1.4, we have \( \phi(N) = \phi(M) = 0 \). By Proposition 2.2.1, we obtain that \( \psi = 0 \); that is, \( \phi = \lambda \text{ad}\Gamma' \in \text{ad}W \). \( \square \)

**Remark 3.1.6.** In view of Corollary 3.1.5, for determining the homogeneous derivations of nonnegative \( Z \)-degree of \( W \), it suffices to reduce such a derivation to be vanishing on the top. Just as in Proposition 2.1.6 and Corollary 3.1.5, the idea of reduction will lead us throughout.

### 3.2. Derivation algebra of \( W \)

In this subsection we shall determine the derivation algebra of \( W \), the even part of the generalized Witt Lie superalgebra \( W \). For the Lie superalgebra \( W \), since the natural \( Z \)-gradation is transitive, by a result analogous to Proposition 2.1.6 (see [23, Lemma 5]), one may easily prove that the homogeneous superderivations of nonnegative \( Z \)-degree of \( W \) are all inner. But, as mentioned above, the \( Z \)-gradation of \( W \) is not transitive. Thus we cannot obtain the corresponding conclusion for \( W \) by using Proposition 2.1.6 directly. This observation leads us naturally to devote our attention to the gradation component of \( Z \)-degree zero. To do that, we shall use a known result (see [15, Proposition 8.4, p. 193]).
Let $V$ be a finite-dimensional vector space over $F$. A linear transformation $\varphi : V \to V$ is called semisimple if the minimum polynomial of $\varphi$ has distinct roots in some base field extension.

**Lemma 3.2.1.** ([15, Proposition 8.4, p. 193]) Let $\mathfrak{g} = \bigoplus_{t=0}^{\infty} \mathfrak{g}_t$ be a $\mathbb{Z}$-graded centerless Lie algebra and $\mathfrak{T} \subset \mathfrak{g}_0$ a torus of $\mathfrak{g}$. If $\varphi \in \text{Der}(\mathfrak{g})$ is homogeneous of $\mathbb{Z}$-degree $t$, then there is $e \in \mathfrak{g}_t$ such that $(\varphi - \text{ad } e) |_{\mathfrak{T}} = 0$.

Recall that $\mathcal{T} := \text{span}_F \{ \Gamma_r \mid r \in Y \}$ is the canonical torus of $W$. Using the lemma above, we can prove the following

**Corollary 3.2.2.** Let $\phi \in \text{Der}_t(W), t \geq 0$. Suppose that $\phi(W_{-1}) = 0$. Then there exists $D \in \mathcal{G}_t$ such that $(\phi - \text{ad } D) |_{\mathcal{T}} = 0$.

**Proof.** By Lemma 2.1.2, all the standard basis elements of $W$ are eigenvectors of $\text{ad } \Gamma_r$ for each $r \in Y$. Note that every element of $\mathcal{T}$ is semisimple. By Lemma 3.2.1, there exists $E \in W_t$ such that $(\phi - \text{ad } E) |_{\mathcal{T}} = 0$. Note that $\phi(\mathcal{T}) \subset \mathcal{G}_t$. Then

$$[E, \Gamma_r] = \phi(\Gamma_r) \in \mathcal{G}_t \quad \text{for all } r \in Y.$$  

By Lemma 2.1.2(vii), there is $D \in \mathcal{G}_t$ such that $[D, \Gamma_r] = [E, \Gamma_r]$ for all $r \in Y$. Hence $(\phi - \text{ad } D) |_{\mathcal{T}} = 0$.

Now we can prove that every homogeneous derivation of nonnegative $\mathbb{Z}$-degree of $W$ is inner. We first prove the following

**Proposition 3.2.3.** Let $t > 0$. Then $\text{Der}_t W = \text{ad } W_t$.

**Proof.** It suffices to prove that $\text{Der}_t W \subset \text{ad } W_t$. Let $\phi \in \text{Der}_t W, t > 0$. By Proposition 2.1.6, we may assume that $\phi(W_{-1}) = 0$. Then Corollary 3.2.2 shows that there is $D \in \mathcal{G}_t$ such that $\phi(\mathcal{T}) = 0$, where $\psi := \phi - \text{ad } D$.

We treat separately the two cases $t \geq 3$ and $0 < t < 3$ in order to prove that

$$\psi(x_r D_s) = 0 \quad \text{for } r, s \in Y \text{ with } \tau(r) = \tau(s). \quad (3.2.1)$$

**Case (i):** $t \geq 3$. Pick $k \in Y_1 \setminus \{ r, s \}$. Then $[\Gamma_k, x_r D_s] = 0$. Consequently,

$$[\Gamma_k, \psi(x_r D_s)] = 0 \quad \text{for all } k \in Y_1 \setminus \{ r, s \}. \quad (3.2.2)$$

Note that $\psi(x_r D_s) \in \mathcal{G}$ by Lemma 2.1.1. Therefore, by Lemma 2.1.2(vii), we obtain from (3.2.2) that $\text{zd} (\psi(x_r D_s)) \leq 2$. Since $\text{zd}(\psi) = t \geq 3$ by our hypothesis, we obtain that $\psi(x_r D_s) = 0$; that is, (3.2.1) holds in this case.

**Case (ii):** $0 < t < 3$. Then $t = 1$ or $t = 2$. Consider $\Gamma' := \sum_{k \in Y_1} \Gamma_k$. Obviously, $[\Gamma', x_r D_s] = 0$. Then

$$[\Gamma', \psi(x_r D_s)] = 0. \quad (3.2.3)$$

On the other hand, noticing that $\psi(x_r D_s) \in \mathcal{G}$ (by Lemma 2.1.1), we obtain by Lemma 2.1.2(vi) that

$$[\Gamma', \psi(x_r D_s)] = 2 \psi(x_r D_s). \quad (3.2.4)$$
Since char\(\mathbb{F}\) \(\neq 2\), (3.2.3) and (3.2.4) imply that \(\psi(x_rD_s) = 0\).

So far, we have proved that \(\psi(\mathcal{W}_{-1} + \mathcal{W}_0) = 0\). Using Corollary 3.1.5, we have \(\psi \in \text{ad}\mathcal{W}_1\) and therefore, \(\phi \in \text{ad}\mathcal{W}_1\). \(\square\)

We have also the following

**Proposition 3.2.4.** \(\text{Der}_0\mathcal{W} = \text{ad}\mathcal{W}_0\).

**Proof.** Let \(\phi \in \text{Der}_0\mathcal{W}\). In view of Proposition 2.1.6 and Corollary 3.2.2, we may assume that \(\phi(\mathcal{W}_{-1}) = \phi(T) = 0\).

Let \(i, j \in \mathcal{Y}_0\) with \(i \neq j\). Noting that \(\phi(x_iD_j) \in \mathcal{G}_0\), we have

\[
\phi(x_iD_j) = \phi([x_iD_i, x_jD_j]) = [x_iD_i, \phi(x_jD_j)] = 0.
\] (3.2.5)

We want to prove that

\[
\phi(x_kD_l) = c_{kl}x_kD_l \quad \text{for all } k, l \in \mathcal{Y}_1 \text{ with } k \neq l,
\] (3.2.6)

where \(c_{kl} \in \mathbb{F}\). Obviously, we may assume for fixed \(k, l \in \mathcal{Y}_1\) with \(k \neq l\) that

\[
\phi(x_kD_l) = \sum_{r,s \in \mathcal{Y}_1} c_{rs}x_rD_s \quad \text{where } c_{rs} \in \mathbb{F}.
\] (3.2.7)

Then

\[
\phi(x_kD_l) = \phi([\Gamma_k, x_kD_l]) = [\Gamma_k, \phi(x_kD_l)].
\] (3.2.8)

By (3.2.7) and (3.2.8), we can easily compute that

\[
\sum_{r,s \in \mathcal{Y}_1} c_{rs}x_rD_s = \sum_{s \in \mathcal{Y}_1 \setminus k} c_{ks}x_kD_s + (c_{kk}x_k - \sum_{r \in \mathcal{Y}_1} c_{rk}x_r)D_k.
\] (3.2.9)

Comparing the coefficients of \(D_k\) in (3.2.9), we have

\[
\sum_{r \in \mathcal{Y}_1} c_{rk}x_r = c_{kk}x_k - \sum_{r \in \mathcal{Y}_1} c_{rk}x_r = - \sum_{r \in \mathcal{Y}_1 \setminus k} c_{rk}x_r.
\]

Consequently, \(c_{kk} = 0\). Comparing the coefficients of \(D_s\) for \(s \in \mathcal{Y}_1 \setminus k\), one may get \(c_{rr} = 0\) for all \(r \in \mathcal{Y}_1 \setminus k\). Thus we obtain from (3.2.7) that

\[
\phi(x_kD_l) = \sum_{r,s \in \mathcal{Y}_1, r \neq s} c_{rs}x_rD_s \quad \text{where } c_{rs} \in \mathbb{F}.
\] (3.2.10)

For any \(q \in \mathcal{Y}_1\), we have \([\Gamma_q, x_kD_l] = (\delta_{qk} - \delta_{ql})x_kD_l\). Then

\[
[\Gamma_q, \phi(x_kD_l)] = (\delta_{qk} - \delta_{ql})\phi(x_kD_l).
\] (3.2.11)

Using (3.2.10) we obtain from (3.2.11) that

\[
\sum_{r,s \in \mathcal{Y}_1, r \neq s} (\delta_{qr} - \delta_{qs})c_{rs}x_rD_s = (\delta_{qk} - \delta_{ql}) \sum_{r,s \in \mathcal{Y}_1, r \neq s} c_{rs}x_rD_s.
\]
It follows that
\[(\delta_{qr} - \delta_{qs})c_{rs} = (\delta_{qk} - \delta_{ql})c_{rs} \quad \text{for all } q \in Y_1.\]
This implies that \(c_{rs} = 0\) unless \((r, s) = (k, l)\). Hence (3.2.6) holds. Applying \(\phi\) to the following equation
\[[x_rD_k, x_kD_l] = x_rD_l \quad \text{for } r, k, l \in Y_1, l \neq r,
we obtain that
\[c_{rk} + c_{kl} = c_{rl}. \quad (3.2.12)\]
Similarly, we obtain from the equation \([x_lD_k, x_kD_l] = x_lD_l - x_kD_k\) that
\[c_{lk} + c_{kl} = 0. \quad (3.2.13)\]
Clearly, the following system of \(n - 1\) linear equations in \(n\) unknowns \(\lambda_1', \lambda_2', \ldots, \lambda_n'\) has solutions:
\[
\begin{align*}
\lambda_1' - \lambda_2' &= c_{1'2'} \\
\lambda_1' - \lambda_3' &= c_{1'3'} \\
& \quad \ldots \quad \ldots \quad \ldots \\
\lambda_1' - \lambda_n' &= c_{1'n'}.
\end{align*}
\]
Let \((\lambda_1', \lambda_2', \ldots, \lambda_n')\) be a solution. Then \(\lambda_k - \lambda_l = (\lambda_k - \lambda_1) + (\lambda_1 - \lambda_l) = c_{k1} + c_{1l} = c_{kl}\).
Set
\[D := \sum_{r \in Y_1} \lambda_r \Gamma_r, \quad \psi := \phi - \text{ad}D.\]
Then we have, for all \(k, l \in Y_1\) with \(k \neq l\),
\[
\psi(x_kD_l) = \phi(x_kD_l) - [D, x_kD_l] = c_{kl}x_kD_l - (\lambda_k - \lambda_l)x_kD_l = 0.
\]
Using (3.2.5) we have also \(\psi(x_iD_j) = 0\) for \(i, j \in Y_0\). Therefore, \(\psi(W_0) = 0\). Clearly, \(\psi(W_{-1}) = 0\). Corollary 3.1.5 ensures that \(\psi \in \text{ad}\mathcal{W}\) and therefore, \(\phi \in \text{ad}\mathcal{W}_0\). \(\square\)

Summarizing, we have the following

**Proposition 3.2.5.** The homogeneous derivations of nonnegative \(Z\)-degree of \(\mathcal{W}\) are all inner.

**Proof.** This is a direct consequence of Propositions 3.2.3 and 3.2.4. \(\square\)

In view of Proposition 3.2.5, it remains only to determine the homogeneous derivations of negative \(Z\)-degree of \(\mathcal{W}\). Our work is motivated by the corresponding results and methods in [2, 23] and will depend heavily on the generator set of \(\mathcal{W}\) (Proposition 2.2.1). We first determine \(\text{Der}_{-1}\mathcal{W}\). To do that, we need the following lemma, which asserts that the derivations of \(Z\)-degree \(-1\) are completely determined by \(\mathcal{W}_0\).

**Lemma 3.2.6.** Suppose that \(\varphi \in \text{Der}_{-1}(\mathcal{W})\) and \(\varphi(\mathcal{W}_0) = 0\). Then \(\varphi = 0\).
Proof. First, we prove that \( \varphi(N \cup P) = 0 \). Observe that

\[
[x_k x_l D_i, x_l D_i] = x_k x_l D_i \quad \text{for all } k, l \in Y_1, i \in Y_0.
\] (3.2.14)

Note that \( \varphi(W_0) = 0 \) and \( \varphi(x_k x_l D_i) \in \mathcal{G}_0 \). Applying \( \varphi \) to (3.2.14), one gets

\[
\varphi(x_k x_l D_i) = [\varphi(x_k x_l D_i), x_l D_i] = 0.
\]

Similarly, one may check that \( \varphi(x_i x_k D_l) = 0 \) by applying \( \varphi \) to the equation that

\[
[x_i x_k D_l, x_i x_k D_l] = x_i x_k D_l. \quad \text{Hence, } \varphi(N) = \varphi(P) = 0.
\]

Next, we prove that \( \varphi(M) = 0 \). To do that we assert that

\[
\varphi(x^{(r \epsilon_i)} D_i) = 0 \quad \text{for all } r \in \mathbb{N}, i \in Y_0. \quad (3.2.15)
\]

For \( r = 1 \), it is clear that (3.2.15) holds. For \( r = 2 \), noticing that \( \varphi(x^{(2 \epsilon_i)} D_i) \in \mathcal{G}_0 \subset E(\mathcal{G}) \), we obtain that

\[
\varphi(x^{(2 \epsilon_i)} D_i) = \varphi([x_i D_i, x^{(2 \epsilon_i)} D_i]) = [x_i D_i, \varphi(x^{(2 \epsilon_i)} D_i)] = 0.
\]

Now we proceed by induction on \( r > 2 \) to prove (3.2.15). By inductive hypothesis, \( \varphi(x^{(r \epsilon_i)} D_i) \in \mathcal{G}_{r-2} \). Thus, we may assume that \( \varphi(x^{(r \epsilon_i)} D_i) = \sum u_q c_{uq} x^u D_q \) where \( c_{uq} \in \mathbb{F} \). For any given pair \((u, q)\), find \( k \in u \setminus q \) since \( r > 2 \). Then we may get \( c_{uq} = 0 \) from the equation that \([x_k D_k, \varphi(x^{(r \epsilon_i)} D_i)] = 0 \). Thus, (3.2.15) holds. Therefore,

\[
\varphi(x^{(r \epsilon_i)} D_j) = \varphi([x^{(r \epsilon_i)} D_i, x_j D_j]) = 0 \quad \text{for all } j \in Y_0 \setminus i.
\]

Hence \( \varphi(M) = 0 \). By Proposition 2.2.1, we conclude that \( \varphi = 0 \). \qed

Using Lemma 3.2.6, we can prove the following

**Proposition 3.2.7.** Der\(_{-1}(W) = \text{ad}W_{-1} \).

Proof. Let \( \varphi \in \text{Der}_{-1}(W) \). For \( i \in Y_0, k, l \in Y_1 \), applying \( \varphi \) to the equation that \( [x_i D_i, x_k D_l] = 0 \), one gets

\[
[\varphi(x_i D_i), x_k D_l] + [x_i D_i, \varphi(x_k D_l)] = 0.
\]

Note that \([\varphi(x_i D_i), x_k D_l] = 0 \), since \( \varphi(x_i D_i) \in W_{-1} \). It follows that

\[
[x_i D_i, \varphi(x_k D_l)] = 0 \quad \text{for } i \in Y_0, k, l \in Y_1.
\]

Since \( \varphi(x_k D_l) \in W_{-1} \), this implies that \( \varphi(x_k D_l) = 0 \) for all \( k, l \in Y_1 \).

Given \( i \in Y_0 \), suppose that \( \varphi(x_i D_i) = \sum_{r \in Y_0} c_{ir} D_r \) where \( c_{ir} \in \mathbb{F} \). Applying \( \varphi \) to the equation that \([x_i D_i, x_j D_j] = 0 \) for \( i, j \in Y_0 \) with \( i \neq j \), one may get \( c_{ij} = 0 \) whenever \( j \neq i \). Therefore, \( \varphi(x_i D_i) = c_{ii} D_i \). Applying \( \varphi \) to the equation that \([x_i D_i, x_j D_j] = x_i D_j \), one gets

\[
c_{ii} D_j + [x_i D_i, \varphi(x_i D_j)] = \varphi(x_i D_j).
\]

This implies that \( \varphi(x_j D_j) = c_{ii} D_j \) for all \( j \in Y_0 \), since \( \varphi(x_i D_j) \in W_{-1} \). Put \( \psi := \varphi - \sum_{r \in Y_0} c_{rr} (\text{ad} D_r) \). Then \( \psi(W_0) = 0 \). By Lemma 3.2.6, we have \( \psi = 0 \); that is, \( \varphi = \sum_{r \in Y_0} c_{rr} (\text{ad} D_r) \in \text{ad}W_{-1} \). \qed

To determine the derivations of \( Z \)-degree less than \(-1 \), we first establish a technical lemma.
Lemma 3.2.8. Let $\phi \in \text{Der}_{-q}(W)$, $q > 1$. If $\phi(x^{(q)})D_i = 0$ for all $i \in Y_0$, then $\phi = 0$.

Proof. We first show that $\phi(M) = 0$. If $t \leq q$, then $\phi(x^{(t)})D_i = 0$ for all $i \in Y_0$. Suppose that $t > q$. We use induction on $t$ to prove that $\phi(x^{(t)})D_i = 0$. By inductive hypothesis and Lemma 2.1.1, $\phi(x^{(t)})D_i \in G_{t-q-1}$. Thus we may assume that

$$\phi(x^{(t)})D_i = \sum_{u \in B, r \in Y} c_{u,r}x^uD_r$$

where $c_{u,r} \in \mathbb{F}$. (3.2.16)

If $|u| \geq 2$, then for the pair $(u, r)$, there is $k \in u \setminus r$. Note that $[x_kD_k, x^{(t)}D_i] = 0$. Applying $\phi$ to this equation and using (3.2.16), one may get

$$\phi(x^{(t)})D_i = \sum_{w \in B_1, r \in Y_1} c_{w,r}x^wD_r = \sum_{l,r \in Y_1} c_{l,r}x_lD_r,$$

where $c_{l,r} := c_{u,r}$ if $u = (l)$. Applying $\phi$ to the equation that $[x^{(t)}D_i, x_lD_r] = 0$ for $l, r \in Y_1$, we obtain that $c_{l,r} = 0$ whenever $l \neq r$ and that $c_{l,l} = c_{r,r}$ for all $l, r \in Y_1$. Denote $\lambda := c_{r,r}$ for all $r \in Y_1$. It follows from (3.2.17) that

$$\phi(x^{(t)})D_i = \sum_{l \in Y_1} \lambda x_lD_i = \lambda \Gamma'.$$

Applying $\phi$ to the equation that $[x^{(t)}D_i, x_kx_lD_j] = 0$ for $i, j \in Y_0$ with $j \neq i$ and $k, l \in Y_1$, we have

$$2\lambda x_kx_lD_j + [x^{(t)}D_i, \phi(x_kx_lD_j)] = 0.$$

Since $zd(\phi(x_kx_lD_j)) \leq -1$, the equation above yields $\lambda = 0$. For $j \in Y \setminus i$, it follows that $\phi(x^{(t)}D_j) = \phi([x^{(t)}D_i, x_kx_lD_j]) = 0$. Therefore, $\phi(M) = 0$.

It remains to show that $\phi(N \cup P) = 0$. Since $N \cup P \subseteq W_1$ and $zd(\phi) = -q \leq -2$, it suffices to consider only the case $zd(\phi) = -2$. Let $k, l \in Y_1, i \in Y_0$. Note that $\phi(x_kx_lD_i) \in W_{-1}$. Applying $\phi$ to the equation $[\Gamma', x_kx_lD_i] = 2x_kx_lD_i$, we have

$$\phi(x_kx_lD_i) = \frac{1}{2}\phi([\Gamma', x_kx_lD_i]) = \frac{1}{2}[\Gamma', \phi(x_kx_lD_i)] \in [\Gamma', W_{-1}] = 0,$$

proving that $\phi(P) = 0$. Similarly, we have

$$\phi(x_kx_lD_i) = \phi([\Gamma''', x_kx_lD_i]) = [\Gamma''', \phi(x_kx_lD_i)] = -\phi(x_kx_lD_i).$$

Thus $\phi(x_kx_lD_i) = 0$, since $\text{char}\mathbb{F} \neq 2$.

So far, we have showed that $\phi(M) = \phi(N) = \phi(P) = 0$. By Proposition 2.2.1, $\phi = 0$. \(\square\)

We are in the position to compute the homogeneous derivations of negative $\mathbb{Z}$-degree. We treat two cases separately. First, we give the following

Proposition 3.2.9. Suppose that $q > 1$ is not any $p$-power. Then $\text{Der}_{-q}(W) = 0$. 

Proof. Let \( \phi \in \text{Der}_{-q}(W) \). Clearly, \( \phi(W_{-1} + W_0) = 0 \). If \( q \not\equiv 0 \pmod{p} \), then Lemma 3.1.1 shows that \( \phi = 0 \). Thus we assume that \( q \equiv 0 \pmod{p} \). Write \( q \) to be the \( p \)-adic expression: \( q = \sum_{k=1}^r c_k p^k \), where \( 0 \leq c_k < p \) and \( c_r \neq 0 \). We note that \( \binom{q}{p^r} \not\equiv 0 \pmod{p} \) and \( \binom{q}{p^r} \equiv 0 \pmod{p} \). According to Lemma 3.2.8, it suffices to show that

\[
\phi (x^{(q \xi_i)} D_i) = 0 \quad \text{for all } i \in Y_0.
\]

Direct computation shows that

\[
[x^{(q - p^r + 1 \xi_i)} D_i, x^{(p^r \xi_i)} D_i] = \left( \binom{q}{p^r} - 1 \right) x^{(q \xi_i)} D_i = -\binom{q}{p^r} x^{(q \xi_i)} D_i.
\]

Since \( q - p^r + 1 < q \) and \( p^r < q \), from the equation above one easily gets \( \phi (x^{(q \xi_i)} D_i) = 0 \). The proof is complete. \( \square \)

For the remaining case we have the following

**Proposition 3.2.10.** Let \( q \) be \( p \)-power \( p^r \) for some \( r \in \mathbb{N} \). Then

\[
\text{Der}_{-q}(W) = \text{span}_F \{ (\text{ad} D_i)^q | i \in Y_0 \}.
\]

*Proof.* By Leibniz rule, it is easily seen that \( (\text{ad} D_i)^q \) is a derivation of \( W \) for all \( i \in Y_0 \). It follows that one implication holds. To prove the converse implication, let \( \phi \in \text{Der}_{-q}(W) \). One may assume that

\[
\phi (x^{(q \xi_i)} D_i) = \sum_{r \in Y_0} a_{ir} D_r \quad \text{where } a_{ir} \in F.
\]

Applying \( \phi \) to the equation that \( [x^{(q \xi_i)} D_i, x_j D_j] = 0 \) for \( j \in Y_0 \setminus i \), yields that \( a_{ij} = 0 \). Therefore, \( \phi (x^{(q \xi_i)} D_i) = a_{ii} D_i \) where \( i \in Y_0 \). Set \( \psi := \phi - \sum_{r \in Y_0} a_{ir} (\text{ad} D_i)^q \). Then \( \psi (x^{(q \xi_i)} D_i) = 0 \) for all \( i \in Y_0 \). By Lemma 3.2.8 we have \( \psi = 0 \); that is, \( \phi = \sum_{r \in Y_0} a_{rr} (\text{ad} D_r)^q \). The proof is complete. \( \square \)

Assembling the main results obtained in this subsection we are able to describe the derivation algebra of \( W \). Recall that \( W \) stands for the even part of \( W = W(m, n; \underline{t}) \), where \( \underline{t} = (t_1, t_2, \ldots, t_m) \in \mathbb{N}^m \).

**Theorem 3.2.11.** \( \text{Der}(W) = \text{ad} W \oplus \{ (\text{ad} D_i)^{p^r} | i \in Y_0, 1 \leq r_i < t_i \} \).

*Proof.* By Leibniz rule, it is sufficient to show the inclusion \( \subset \). Note that when \( r_i \geq t_i \), the derivation \( (\text{ad} D_i)^{p^r} \) vanishes for \( i \in Y_0 \). Then the theorem follows from Propositions 3.2.5, 3.2.7, 3.2.9 and 3.2.10. \( \square \)

**Remark 3.2.12.** Now we can answer the question for \( W \) mentioned in the introduction. By Theorem 3.2.11 and [23, Theorem 1], it is easily seen that \( \text{Der}(W) = \text{Der}(W) \), where \( W \) stands for the Lie superalgebra \( W(m, n; \underline{t}) \). As mentioned in Introduction, this implies that every derivation of the Lie algebra \( W \) can extend to be a superderivation of the Lie superalgebra \( W \).
4. The derivation algebra of \(S\)

In this section we shall determine the derivation algebra of \(S\) (\(S\) is the even part of the Lie superalgebra of \(S(m,n;\mathfrak{L})\)). In contrast to the case of \(W\), we do not find a way to reduce directly a homogeneous derivation of nonnegative \(\mathbb{Z}\)-degree of \(S\) to be a derivation vanishing on \(S_{-1} \oplus S_0\). Thus we encounter the difficulty that Lemma 3.2.1 is by no means applicable for \(S\). This observation forces us to establish a proposition similar to Proposition 2.1.6, but, where \(W_{-1}\) is replaced with the canonical torus \(T_S\) contained in \(S_0\). This is why we generalize Proposition 8.2 in [15, p. 192] to be Lemma 2.1.3 and then give Corollary 2.1.5. As a result, we establish such a proposition indeed, which is written to be Lemmas 4.2.4 and 4.2.5 in Section 4.2.

Recall our convention that \(m, n > 2\).

4.1. The top of \(S\)

Just as in the case of \(W\), in this subsection we shall study the derivations of nonnegative \(\mathbb{Z}\)-degree of \(S\) to \(W\), which vanish on the top of \(S\). As the final result in this subsection, it is proved that such a derivation must be inner and determined by a scalar multiple of \(\Gamma' = \sum_{r \in Y_1} x_r D_r\). The arguments will be based on the generator set of \(S\) (see Proposition 2.2.3).

First, we consider the generators of the form \(D_{il}(x^{(2\varepsilon_i)}x_k)\) in \(R\), where \(i \in Y_0\) and \(k, l \in Y_1\).

**Lemma 4.1.1.** Suppose that \(\phi \in \text{Der}(S, W)\) with \(zd(\phi) \geq 0\) and that \(\phi(S_{-1} + S_0) = 0\). Then
\[
\phi(D_{il}(x^{(2\varepsilon_i)}x_k)) = 0 \quad \text{for all } i \in Y_0, k, l \in Y_1.
\] (4.1.1)

**Proof.** By the definition of \(D_{il}\),
\[
D_{il}(x^{(2\varepsilon_i)}x_k) = x_i x_k D_{il} + \delta_{kl} x^{(2\varepsilon_i)} D_l \quad \text{for all } i \in Y_0, k, l \in Y_1.
\] (4.1.2)

By our assumption and Lemma 2.1.1, one may assume that
\[
\phi(D_{il}(x^{(2\varepsilon_i)}x_k)) = \sum_{r \in Y, u \in B} c_{u,r,x^u} D_r \quad \text{where } c_{u,r} \in \mathbb{F}.
\] (4.1.3)

We proceed in two steps to prove the equation (4.1.1).

*Case (i): \(zd(\phi)\) is even.* Then it follows from (4.1.3) that \(c_{u,r} = 0\) for all \(r \in Y_1\). Thus
\[
\phi(D_{il}(x^{(2\varepsilon_i)}x_k)) = \sum_{r \in Y_0, u \in B} c_{u,r,x^u} D_r.
\] (4.1.4)

First, consider the case \(k \neq l\). Then (4.1.2) yields
\[
D_{il}(x^{(2\varepsilon_i)}x_k) = x_i x_k D_{il} \quad \text{for } k \neq l.
\] (4.1.5)

Clearly,
\[
[\Gamma_k - \Gamma_l, D_{il}(x^{(2\varepsilon_i)}x_k)] = 2D_{il}(x^{(2\varepsilon_i)}x_k) \quad \text{for } k \neq l.
\] (4.1.6)
Consequently,

$$[\Gamma_k - \Gamma_l, \phi(D_{il}(x^{(2\epsilon_i)}x_k))] = 2\phi(D_{il}(x^{(2\epsilon_i)}x_k)) \quad \text{for} \quad k \neq l. \quad (4.1.7)$$

Then we obtain from (4.1.4) and (4.1.7) that

$$\sum_{r \in Y_0, u \in \mathbb{B}} (\delta_{k \in u} - \delta_{l \in u})c_{u,r} x^u D_r = 2 \sum_{r \in Y_0, u \in \mathbb{B}} c_{u,r} x^u D_r. \quad (4.1.8)$$

It follows immediately from (4.1.8) that

$$(\delta_{k \in u} - \delta_{l \in u})c_{u,r} = 2c_{u,r} \quad \text{for} \quad r \in Y_0, u \in \mathbb{B}. \quad (4.1.9)$$

This implies that $c_{u,r} = 0$ for all $r \in Y_0, u \in \mathbb{B}$. Therefore, (4.1.1) holds for the case $k \neq l$.

Second, we consider the case $k = l$. By (4.1.2),

$$D_{ik}(x^{(2\epsilon_i)}x_k) = x_i x_k D_k + x^{(2\epsilon_i)} D_i. \quad (4.1.10)$$

For $q_1, q_2 \in Y_1$, it is easy to see that $[\Gamma_{q_1} - \Gamma_{q_2}, D_{ik}(x^{(2\epsilon_i)}x_k)] = 0$. Applying $\phi$, one gets

$$[\Gamma_{q_1} - \Gamma_{q_2}, \phi(D_{ik}(x^{(2\epsilon_i)}x_k))] = 0. \quad (4.1.11)$$

We then obtain from (4.1.4) and (4.1.10) that

$$\sum_{r \in Y_0, u \in \mathbb{B}} (\delta_{q_1 \in u} - \delta_{q_2 \in u})c_{u,r} x^u D_r = 0 \quad \text{for all} \quad q_1, q_2 \in Y_1, r \in Y_0. \quad (4.1.12)$$

Consequently,

$$(\delta_{q_1 \in u} - \delta_{q_2 \in u})c_{u,r} = 0 \quad \text{for all} \quad q_1, q_2 \in Y_1, r \in Y_0. \quad (4.1.13)$$

Note that (4.1.11) implies that, if $u \neq \omega$ then $c_{u,r} = 0$ for all $r \in Y_0$. Therefore, it follows from (4.1.4) that

$$\phi(D_{ik}(x^{(2\epsilon_i)}x_k)) = \sum_{r \in Y_0} c_{\omega,r} x^\omega D_r. \quad (4.1.14)$$

Find $j \in Y_0 \setminus i$, since $|Y_0| \geq 3$. Note that (Notice (4.1.9))

$$[\Gamma_i - \Gamma_j, D_{ik}(x^{(2\epsilon_i)}x_k)] = D_{ik}(x^{(2\epsilon_i)}x_k).$$

Applying $\phi$ and substituting (4.1.12), we obtain that

$$[\Gamma_i - \Gamma_j, \sum_{r \in Y_0} c_{\omega,r} x^\omega D_r] = \sum_{r \in Y_0} c_{\omega,r} x^\omega D_r.$$

It follows immediately that

$$-c_{\omega,i} x^\omega D_i + c_{\omega,j} x^\omega D_j = \sum_{r \in Y_0} c_{\omega,r} x^\omega D_r.$$
This implies that 
\[ c_{\omega, r} = 0, \quad \text{whenever } r \neq j. \]

Then (4.1.12) yields that 
\[ \phi(D_{ik}(x^{(2\varepsilon_i)}x_k)) = c_{\omega, j}x^\omega D_j \quad \text{whenever } j \in Y_0 \setminus i. \] (4.1.13)

The general assumption \( m \geq 3 \) and (4.1.13) show that (4.1.1) holds in the case \( k = l \).

Case (ii): zd(\( \phi \)) is odd. Note that in this case it is easily seen in (4.1.3) that \( c_{u, r} = 0 \) for all \( r \in Y_0 \). Thus 
\[ \phi(D_{il}(x^{(2\varepsilon_i)}x_k)) = \sum_{r \in Y_1, u \in \mathbb{B}} c_{u, r}x^n D_r. \] (4.1.14)

Recall \( \Gamma' := \sum_{r \in Y_1} x_r D_r \). Using the formulas (4.1.5) and (4.1.9), one obtains by direct computation that 
\[ [\Gamma' + n\Gamma_i, D_{il}(x^{(2\varepsilon_i)}x_k)] = nD_{il}(x^{(2\varepsilon_i)}x_k), \] (4.1.15)

\[ [\Gamma' + (n - 1)\Gamma_i + \Gamma_j, D_{il}(x^{(2\varepsilon_i)}x_k)] = (n - 1)D_{il}(x^{(2\varepsilon_i)}x_k) \quad \text{for } j \in Y_0 \setminus i. \] (4.1.16)

Applying \( \phi \) to (4.1.15) and (4.1.16), respectively, one gets from (4.1.14) that 
\[ [\Gamma', \phi(D_{il}(x^{(2\varepsilon_i)}x_k))] = [\Gamma' + n\Gamma_i, \phi(D_{il}(x^{(2\varepsilon_i)}x_k))] = n\phi(D_{il}(x^{(2\varepsilon_i)}x_k)), \] (4.1.17)

and 
\[ [\Gamma', \phi(D_{il}(x^{(2\varepsilon_i)}x_k))] = [\Gamma' + (n - 1)\Gamma_i + \Gamma_j, \phi(D_{il}(x^{(2\varepsilon_i)}x_k))] = (n - 1)\phi(D_{il}(x^{(2\varepsilon_i)}x_k)). \] (4.1.18)

Comparing (4.1.17) and (4.1.18), we have \( \phi(D_{il}(x^{(2\varepsilon_i)}x_k)) = 0 \). Summarizing, (4.1.1) holds.

Next, we consider the generators of the form \( D_{ij}(x_i x^v) \) in \( \mathcal{R} \), where \( i, j \in Y_0 \) and \( v \in \mathbb{B}_2 \).

Lemma 4.1.2. Let \( \phi \in \text{Der}_t(S, \mathcal{W}) \) with \( t \geq 0 \). Suppose that \( \phi(S_{-1} + S_0) = 0 \). Then there is \( \lambda \in \mathbb{F} \) such that \( \phi(D_{ij}(x_i x^v)) = \lambda D_{ij}(x_i x^v) \) for all \( i, j \in Y_0 \) and \( v \in \mathbb{B}_2 \).

Proof. Note that 
\[ \phi(D_{ij}(x_i x^v)) = \phi([\Gamma_q - \Gamma_j, D_{ij}(x_i x^v)]) = [\Gamma_q - \Gamma_j, \phi(D_{ij}(x_i x^v))] \quad \text{for all } q \in Y_0 \setminus j. \] (4.1.19)

Assume that 
\[ \phi(D_{ij}(x_i x^v)) = \sum_{u \in \mathbb{B}, r \in Y} c_{u, r}x^u D_r \quad \text{where } c_{u, r} \in \mathbb{F}. \] (4.1.20)

Case (i): zd(\( \phi \)) is even. Then 
\[ \phi(D_{ij}(x_i x^v)) = \sum_{u \in \mathbb{B}, r \in Y_0} c_{u, r}x^u D_r. \] (4.1.21)
If follows from (4.1.19) and (4.1.21) that
\[ \phi(D_{ij}(x_i x^v)) = \sum_u c_{u,j} x^u D_j - \sum_u c_{u,q} x^u D_q \quad \text{for all } q \in Y_0 \setminus j. \] (4.1.22)

This implies that \( c_{u,q} = 0 \) whenever \( q \in Y_0 \setminus j \) and therefore,
\[ \phi(D_{ij}(x_i x^v)) = \sum_u c_{u,j} x^u D_j. \] (4.1.23)

Find \( k \in Y_1 \setminus v \), since \( n \geq 3 \). Then
\[ \left[ \Gamma_q + \Gamma_k, D_{ij}(x_i x^v) \right] = 0 \quad \text{for } q \in Y_0 \setminus j. \]
Applying \( \phi \) to this equation and then substituting (4.1.23), one gets
\[ \sum_{u \in \mathbb{B}} \delta_{k\in u} c_{u,j} x^u D_j = 0. \]

For any fixed \( u \neq v \) with \(|u| \geq 2\), if \( k \in u \setminus v \), then the equation above forces \( c_{u,j} = 0 \). It follows from (4.1.23) that
\[ \phi(D_{ij}(x_i x^v)) = c_{v,j} x^v D_j. \]

Note that \( D_{ij}(x_i x^v) = x^v D_j \). Just as in the proof of Lemma 3.1.2, one may easily check that \( \lambda := c_{v,j} \) is independent of the choice of \( v \) and \( j \). Therefore, the lemma holds in this case.

**Case (ii):** \( zd(\phi) \) is odd. Then
\[ \phi(D_{ij}(x_i x^v)) = \sum_{u \in \mathbb{B}, r \in Y_1} c_{u,r} x^u D_r. \]

Find \( q \in Y_0 \setminus j \). Then
\[ \left[ \Gamma_q - \Gamma_j, D_{ij}(x_i x^v) \right] = D_{ij}(x_i x^v). \]

Hence,
\[ \phi(D_{ij}(x_i x^v)) = \left[ \Gamma_q - \Gamma_j, \phi(x^v D_j) \right] = \left[ \Gamma_q - \Gamma_j, \sum_{u \in \mathbb{B}, r \in Y_1} c_{u,r} x^u D_r \right] = 0. \]

The proof is complete. \( \square \)

Finally, we consider the generators in \( Q \).

**Lemma 4.1.3.** Let \( \phi \in \text{Der}(\mathcal{S}, W), zd(\phi) \geq 0 \). Suppose that \( \phi(S_{-1} + S_0) = 0 \). Then
\[ \phi(D_{ij}(x^{(\alpha_i^a)})) = 0 \quad \text{for all } i, j \in Y_0 \text{ and all } a \in \mathbb{N}. \] (4.1.24)

**Proof.** We proceed by induction on \( a \geq 2 \) to prove (4.1.24). Assume that (4.1.24) holds for \( a - 1 \). Then Lemma 2.1.1 ensures that
\[ \phi(D_{ij}(x^{(\alpha_i^a)})) = \sum_{u \in \mathbb{B}, r \in Y} c_{u,r} x^u D_r \quad \text{where } c_{u,r} \in \mathbb{F}. \] (4.1.25)
Note that $D_{ij}(x^{(a\varepsilon_i)}) = x^{((a-1)\varepsilon_i)}D_j$. Direct computation shows that

$$[x_kD_k + x_jD_j, D_{ij}(x^{(a\varepsilon_i)})] = -D_{ij}(x^{(a\varepsilon_i)})$$

for all $k \in Y_1$.

Applying $\phi$ to the equation and using (4.1.25), one obtains that

$$\sum_{u,r} (\delta_{k \in u} - \delta_{kr})c_{u,r}x^uD_r - \sum_{u} c_{u,j}x^uD_j = - \sum_{u,r} c_{u,r}x^uD_r. \tag{4.1.26}$$

Comparing coefficients in (4.1.26), we have

$$(\delta_{k \in u} - \delta_{kr})c_{u,r} = -c_{u,r} \quad \text{for } r \neq j; \tag{4.1.27}$$

$$(\delta_{k \in u} - 1)c_{u,j} = -c_{u,j}. \tag{4.1.28}$$

For arbitrary $u \in \mathbb{B}$ with $|u| \geq 2$, find $k \in u$. Then (4.1.27) and (4.1.28) imply that $c_{ur} = 0$ for $r \in Y$, since char$\mathbb{F} \neq 2$. This proves (4.1.24).

Now we can conclude this subsection by the following main result.

**Corollary 4.1.4.** Suppose that $\phi \in \text{Der}(S, W)$ with $zd(\phi) \geq 0$ and that $\phi(S_{-1} + S_0) = 0$. Then $\phi \in \mathbb{F} \cdot \text{ad}\Gamma'$. In particular, $\phi$ is inner.

**Proof.** According to Lemma 4.1.2, there is $\lambda \in \mathbb{F}$ such that

$$\phi(D_{ij}(x_i x^v)) = \lambda D_{ij}(x_i x^v) \quad \text{for all } i, j \in Y_0 \text{ and all } v \in \mathbb{B}_2.$$ 

Set $\psi := \phi - \frac{1}{2}\lambda \text{ad}\Gamma'$. We then use Lemmas 4.1.1 and 4.1.3 in order to see that $\psi(Q \cup R) = 0$. Lemma 2.2.2 applies and $\psi = 0$. Therefore, $\phi = \frac{1}{2}\lambda \text{ad}\Gamma'$, completing the proof.

**Remark 4.1.5.** As noted in Remark 3.1.6, the central work in the sequel is to reduce the homogeneous derivation of nonnegative $Z$-degree to be vanishing on the top of $S$. In contrast to the case $W$, just as remarked in the introduction of this paper, we shall encounter the phenomenon that the reduction proposition [15, Proposition 8.4. p. 193] is not applicable in this case.

### 4.2. Derivations of nonnegative $Z$-degree

In this subsection, we shall determine $\text{Der}_t(S, W)$ and $\text{Der}_t(S)$ for $t \geq 0$ (Propositions 4.2.9 and 4.2.10). For derivations of positive odd $Z$-degree in $\text{Der}(S, W)$, we establish a lemma (Lemma 4.2.4) analogous to Proposition 2.1.6, which contends that such a derivation vanishing on $S_{-1}$ can be reduced to be vanishing on the toral elements $\Gamma_k - \Gamma'_{1'}$, $k \in Y_1 \setminus 1'$ (Lemma 4.2.4). For a derivation of nonnegative even $Z$-degree in $\text{Der}(S, W)$, we establish a corresponding lemma (Lemma 4.2.5). The reason that we treat those two cases separately is that the images of elements in the canonical torus under a derivation are of different forms with respect to the standard $\mathbb{F}$-basis of $W$. In view of the results mentioned above, the remaining works in this subsection will be
devoted to reducing the derivations of nonnegative Z-degree to be vanishing on the top \(S_{-1} \oplus S_0\). Thus we may conclude this subsection by using the result obtained in Section 4.1 (Corollary 4.1.4).

We first give two technical lemmas which will simplify our discussion. Recall the notation \(n = |\omega| = |Y_1|\).

**Lemma 4.2.1.** Suppose that \(\phi \in \text{Der}_t(S, \mathcal{W})\) and \(\phi(\mathcal{W}_{-1}) = 0\).

(i) If \(t = n - 1\) is odd, then

\[
\phi(\Gamma_{1'} - \Gamma_k) = 0 \quad \text{for all } k \in Y_1 \setminus 1'.
\]

(ii) If \(t = n - 1\) is even, then there is \(\lambda \in \mathbb{F}\) such that

\[
(\phi - \lambda \text{ad}(x^\omega D_{t'}))(\Gamma_{1'} - \Gamma_k) = 0 \quad \text{for all } k \in Y_1 \setminus 1'.
\]

(iii) If \(t > n - 1\), then \(\phi = 0\).

**Proof.** (i) We may assume that

\[
\phi(\Gamma_{1'} - \Gamma_k) = \sum_{r \in Y_0} c_r x^\omega D_r \quad \text{where } c_r \in \mathbb{F}. \tag{4.2.1}
\]

Applying \(\phi\) to \([\Gamma_1 - \Gamma_i, \Gamma_{1'} - \Gamma_k] = 0\) for \(i \in Y_0, k \in Y_1\), we have

\[
[\Gamma_1 - \Gamma_i, \phi(\Gamma_{1'} - \Gamma_k)] = [\Gamma_{1'} - \Gamma_k, \phi(\Gamma_1 - \Gamma_i)] = 0, \tag{4.2.2}
\]

since \(\phi(\Gamma_1 - \Gamma_i) \in \text{span}\{x^\omega D_r \mid r \in Y_0\}\). (4.2.1) and (4.2.2) then yield

\[-c_1 x^\omega D_1 + c_i x^\omega D_i = 0, \quad i \in Y_0 \setminus 1.
\]

Therefore \(c_r = 0\) for all \(r \in Y_0\) and (i) holds.

(ii) Applying \(\phi\) to the equation that \([\Gamma_{1'} - \Gamma_k, \Gamma_{1'} - \Gamma_l] = 0\), one gets

\[
[\Gamma_{1'} - \Gamma_k, \phi(\Gamma_{1'} - \Gamma_l)] = [\Gamma_{1'} - \Gamma_l, \phi(\Gamma_{1'} - \Gamma_k)] \quad \text{for all } k, l \in Y_1 \setminus 1'. \tag{4.2.3}
\]

Since \(t\) is even and \(\phi(\mathcal{W}_{-1}) = 0\), by virtue of Lemma 2.1.1, we may assume that

\[
\phi(\Gamma_{1'} - \Gamma_k) = \sum_{r \in Y_1} c_r^{(k)} x^\omega D_r \quad \text{where } c_r^{(k)} \in \mathbb{F}. \tag{4.2.4}
\]

Combining (4.2.3) and (4.2.4), we have

\[
-c_1^{(l)} x^\omega D_1 + c_k^{(l)} x^\omega D_k = -c_1^{(k)} x^\omega D_1 + c_l^{(k)} x^\omega D_l.
\]

Consequently, \(c_k^{(l)} = c_l^{(k)} = 0\). Let \(\lambda := c_1^{(2')} = c_1^{(3')} = \cdots = c_1^{(n')}\). Then

(4.2.4) shows that \(\phi(\Gamma_{1'} - \Gamma_k) = \lambda x^\omega D_{1'}\) for all \(k \in Y_1 \setminus 1'\). Direct calculation shows that \(\lambda \in \mathbb{F}\) is the desired scalar.

(iii) By Lemma 2.1.1, \(\phi(S_0) \subseteq \mathcal{G}\). Note that \(\mathcal{G} \subseteq \bigoplus_{i=1}^{n-1} \mathcal{W}_i\). Since \(t > n - 1\),

\[
\phi(S_0) \subseteq \mathcal{W}_t \cap \mathcal{G} = 0.
\]

Now, one may easily show by induction on \(r\) that \(\phi(S_r) = 0\) for all \(r \in \mathbb{N}\), proving \(\phi = 0\). \(\Box\)
Lemma 4.2.2. Suppose that $\phi \in \text{Der}(\mathcal{S}, \mathcal{W})$ and $zd(\phi) > 0$ is odd.

(i) If $zd(\phi) < n - 1$ and
\[
\phi(\Gamma_{1'} - \Gamma_{2'}) = \phi(\Gamma_{1'} - \Gamma_{3'}) = \cdots = \phi(\Gamma_{1'} - \Gamma_{n'}) = 0
\]
then
\[
\phi(\Gamma_1 - \Gamma_2) = \phi(\Gamma_1 - \Gamma_3) = \cdots = \phi(\Gamma_1 - \Gamma_m) = 0; \quad \phi(\Gamma_1 + \Gamma_{1'}) = 0.
\]

(ii) If $zd(\phi) = n - 1$ then there are $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ such that
\[
\left( \phi - \text{ad} \left( \sum_{i \in Y_0} \lambda_i x^{\omega_i} D_i \right) \right)(\Gamma_1 - \Gamma_j) = 0 \quad \text{for all } j \in Y_0 \setminus 1;
\]
\[
\left( \phi - \text{ad} \left( \sum_{i \in Y_0} \lambda_i x^{\omega_i} D_i \right) \right)(\Gamma_1 + \Gamma_{1'}) = 0.
\]

Proof. Note that $zd(\phi)$ is odd and $\phi(S_0) \subseteq \mathcal{G}$, by Lemma 2.1.1. We may assume that
\[
\phi(\Gamma_1 - \Gamma_{i}) = \sum_{r \in Y_0, u \in \mathbb{B}} c_{u,r}^{(i)} x^{u} D_r \quad \text{where } c_{u,r}^{(i)} \in \mathbb{F}. \quad (4.2.5)
\]

(i) Since $zd(\phi) < n - 1$, the coefficients $c_{u,r}^{(i)}$ in (4.2.5) vanish for all $r \in Y_0$. For any fixed $v \neq \omega$, find $k \in v$ and $l \in Y_1 \setminus v$. Applying $\phi$ to the equation $[\Gamma_k - \Gamma_l, \Gamma_1 - \Gamma_i] = 0$, one may obtain by the hypothesis that
\[
[\Gamma_k - \Gamma_l, \phi(\Gamma_1 - \Gamma_i)] = 0 \quad \text{for } i \in Y_0 \setminus 1.
\]
Combining (4.2.5) with the equation above, we have
\[
\sum_{r \in Y_0, u \in \mathbb{B}} (\delta_{k \in u} - \delta_{l \in u}) c_{u,r}^{(i)} x^{u} D_r = 0. \quad (4.2.6)
\]
Since $\delta_{k \in v} - \delta_{l \in v} = 1$, (4.2.6) implies that
\[
c_{u,r}^{(i)} = 0 \quad \text{for all } r \in Y_0.
\]
Then by (4.2.5), $\phi(\Gamma_1 - \Gamma_{i}) = 0$ for $i \in Y_0 \setminus 1$. One may show in the same way that $\phi(\Gamma_1 + \Gamma_{1'}) = 0$.

(ii) As $zd(\phi) = n - 1$, we know from (4.2.5) that
\[
\phi(\Gamma_1 - \Gamma_{i}) = \sum_{r \in Y_0} c_{\omega,r}^{(i)} x^{\omega} D_r. \quad (4.2.7)
\]
Applying $\phi$ to the equation $[\Gamma_1 - \Gamma_{i}, \Gamma_1 - \Gamma_{j}] = 0$ for $i, j \in Y_0 \setminus 1$ with $i \neq j$, and then combining with (4.2.7), we have
\[
\left[ \sum_{r \in Y_0} c_{\omega,r}^{(i)} x^{\omega} D_r, \Gamma_1 - \Gamma_{j} \right] = \left[ \sum_{r \in Y_0} c_{\omega,r}^{(j)} x^{\omega} D_r, \Gamma_1 - \Gamma_{i} \right].
Consequently,
\[ c_{\omega,1}^{(i)}x^\omega D_1 - c_{\omega,j}^{(i)}x^\omega D_j = c_{\omega,1}^{(j)}x^\omega D_1 - c_{\omega,i}^{(j)}x^\omega D_i. \]

This implies that
\[ c_{\omega,1}^{(i)} = c_{\omega,1}^{(j)}, \quad c_{\omega,j}^{(i)} = c_{\omega,i}^{(j)} = 0 \quad \text{for all } i, j \in Y_0 \setminus 1 \text{ with } i \neq j. \] (4.2.8)

By (4.2.8), let \( \lambda_i := c_{\omega,1}^{(i)} \) for \( i \in Y_0 \setminus 1 \). Then (4.2.7) and (4.2.9) show that
\[ \phi(\Gamma_1 - \Gamma_i) = \lambda_i x^\omega D_1 + c_{\omega,i}^{(i)}x^\omega D_i. \] (4.2.10)

Let \( \lambda_i := -c_{\omega,i}^{(i)} \) for \( i = 2, \ldots, m \) and \( \varphi := \phi - \sum_{i \in Y_0} \lambda_i \text{ad}(x^\omega D_i) \). Then (4.2.10) shows that
\[ \varphi(\Gamma_1 - \Gamma_i) = 0 \quad \text{for all } i \in Y_0 \setminus 1; \] (4.2.11)
that is, the first equation in (ii) holds. To show that \( \varphi(\Gamma_1 + \Gamma_{1'}) = 0 \), assume that
\[ \varphi(\Gamma_1 + \Gamma_{1'}) = \sum_{r \in Y_0} d_{\omega,r} x^\omega D_r \quad \text{where } d_{\omega,r} \in \mathbb{F}. \] (4.2.12)

For \( i \in Y_0 \setminus 1 \), applying \( \varphi \) to the equation \( [\Gamma_1 - \Gamma_i, \Gamma_1 + \Gamma_{1'}] = 0 \), and using (4.2.11) and (4.2.12), one may get
\[ d_{\omega,i} x^\omega D_i - d_{\omega,1} x^\omega D_1 = [\Gamma_1 - \Gamma_i, \sum_{r \in Y_0} d_{\omega,r} x^\omega D_r] = 0. \]

Consequently, \( d_{\omega,i} = d_{\omega,1} = 0 \) for \( i \in Y_0 \setminus 1 \). This proves that \( \varphi(\Gamma_1 + \Gamma_{1'}) = 0 \). Hence, the second equation in (ii) holds. The proof is complete. \( \square \)

Recall the canonical torus of \( S \)
\[ T_S := \text{span}_F \{ \Gamma_1 - \Gamma_2, \ldots, \Gamma_1 - \Gamma_m, \Gamma_1 + \Gamma_{1'}, \Gamma_{1'} - \Gamma_2, \ldots, \Gamma_{1'} - \Gamma_{n'} \}. \]

Summarizing, we have the following fact:

**Corollary 4.2.3.** Suppose that \( \phi \in \text{Der}(S, W) \) is homogeneous derivation of positive odd \( \mathbb{Z} \)-degree such that \( \phi(S_{-1}) = 0 \) and \( \phi(\Gamma_k - \Gamma_{1'}) = 0 \) for all \( k \in Y_1 \setminus 1' \). Then there is \( E \in \mathcal{G} \) such that \( (\phi - \text{ad}E) \) vanishes on the canonical torus \( T_S \).

**Proof.** This is a direct consequence of Lemma 4.2.1(iii) and Lemma 4.2.2. \( \square \)

We now prove two key lemmas in this subsection. First, consider the derivations of odd \( \mathbb{Z} \)-degree. We shall use Lemma 2.1.3.

**Lemma 4.2.4.** Suppose that \( \phi \in \text{Der}_t(S, W) \), where \( t > 0 \) is odd. If \( \phi(S_{-1}) = 0 \), then there is \( D \in \mathcal{G}_t \) such that
\[ (\phi - \text{ad}D)(\Gamma_k - \Gamma_{1'}) = 0 \quad \text{for all } k \in Y_1 \setminus 1'. \]
Proof. If \( t \geq n - 1 \), then Lemma 4.2.1(i) and (iii) show that
\[
\phi(\Gamma_k - \Gamma_{l'}) = 0 \quad \text{for all } k \in Y_1 \setminus Y_1'.
\]
We therefore assume that \( t < n - 1 \).

By Lemma 2.1.1, one may assume that
\[
\phi(\Gamma_k - \Gamma_{l'}) = \sum_{r \in Y_0} f_{rk} D_r \quad \text{where } k \in Y_1 \setminus Y_1'; \; f_{rk} \in \Lambda(n). \tag{4.2.13}
\]
Applying \( \phi \) to the equation that \( [\Gamma_k - \Gamma_{l'}, \Gamma_l - \Gamma_{l'}] = 0 \) for \( k, l \in Y_1 \setminus Y_1' \), and using (4.2.13), one gets
\[
[\sum_{r \in Y_0} f_{rk} D_r, \Gamma_l - \Gamma_{l'}] + [\Gamma_k - \Gamma_{l'}, \sum_{r \in Y_0} f_{rl} D_r] = 0.
\]
Consequently,
\[
\sum_{r \in Y_0} (\Gamma_k - \Gamma_{l'})(f_{rl}) D_r = \sum_{r \in Y_0} (\Gamma_l - \Gamma_{l'})(f_{rk}) D_r.
\]
Since \( \{D_r \mid r \in Y\} \) is a free basis of \( \mathfrak{A}\)-module \( W \), we have
\[
(\Gamma_k - \Gamma_{l'})(f_{rl}) = (\Gamma_l - \Gamma_{l'})(f_{rk}) \quad \text{for all } r \in Y_0; \; k, l \in Y_1 \setminus Y_1'. \tag{4.2.14}
\]
Note that \( f_{rk} \in \Lambda(n) \). Let
\[
f_{rk} = \sum_{|u| = t+1} c_{u,r,k} x^u \quad \text{where } c_{u,r,k} \in \mathbb{F}. \tag{4.2.15}
\]
Combining (4.2.15) and (4.2.14), we have
\[
\sum_{|u| = t+1} (\delta_{k \in u} - \delta_{l' \in u}) c_{u,r,k} x^u = \sum_{|u| = t+1} (\delta_{l \in u} - \delta_{l' \in u}) c_{u,r,k} x^u.
\]
Since \( \{x^u \mid u \in \mathbb{B}\} \) is an \( \mathbb{F}\)-basis of \( \Lambda(n) \), it follows that
\[
(\delta_{k \in u} - \delta_{l' \in u}) c_{u,r,k} = (\delta_{l \in u} - \delta_{l' \in u}) c_{u,r,k} \quad \text{for } r \in Y_0; \; k, l \in Y_1 \setminus Y_1'. \tag{4.2.16}
\]
Assume that \( c_{u,r,k} \) is an arbitrary fixed nonzero coefficient in (4.2.15), where \( |u| = t + 1 < n, r \in Y_0 \) and \( k \in Y_1 \setminus Y_1' \). If \( l' \notin u \), noting that \( 2 \leq |u| \), one may find \( l \in u \setminus k \).
Then (4.2.16) shows that \( \delta_{k \in u} = 1 \); that is, \( k \in u \). If \( l' \in u \), noting that \( |u| \leq n - 1 \), one may find \( l \in Y_1 \setminus u \). Then (4.2.16) shows that \( \delta_{k \in u} = 0 \); that is, \( k \notin u \). Summarizing, for any nonzero coefficient \( c_{u,r,k} \) in (4.2.15), we have \( \delta_{k \in u} + \delta_{l' \in u} = 1 \). Then, we can rewrite (4.2.15) as follows
\[
f_{rk} = \sum_{l' \in u, k \notin u} c_{u,r,k} x^u + \sum_{l' \notin u, k \in u} c_{u,r,k} x^u.
\]
Direct computation shows that
\[
(\Gamma_k - \Gamma_{l'})(f_{rk}) = - \sum_{l' \in u, k \notin u} c_{u,r,k} x^u + \sum_{l' \notin u, k \in u} c_{u,r,k} x^u.
\]
Therefore,
\[(\Gamma_k - \Gamma_{1'})^2(f_{rk}) = f_{rk} \quad \text{for all } r \in Y_0 \text{ and } k \in Y_1 \setminus 1'.\] (4.2.17)

Put \(V := \Lambda(n), v_k := f_{rk}\) and \(A_k = B_k := \Gamma_k - \Gamma_{1'}\) for \(k \in Y_1 \setminus 1'.\) Now, we are going to show that the conditions of Lemma 2.1.3 are fulfilled. (i) is automatic. (4.2.14) ensures that (ii) holds. (4.2.17) shows that (2.1.2) holds. (2.1.3) is clear. We check (2.1.1); that is,
\[(\Gamma_k - \Gamma_{1'})^3 = \Gamma_k - \Gamma_{1'} \quad \text{for } k \in Y_1 \setminus 1'.\] (4.2.18)

For every basis element \(x^n\) of \(V = \Lambda(n),\) noticing that \(\delta_{k \in u} - \delta_{1' \in u} = 0, 1\) or \(-1,\) we obtain that
\[(\Gamma_k - \Gamma_{1'})^3(x^n) = (\delta_{k \in u} - \delta_{1' \in u})^3 x^n = (\delta_{k \in u} - \delta_{1' \in u}) x^n = (\Gamma_k - \Gamma_{1'})(x^n).\]

Thus (4.2.18) holds. By Lemma 2.1.3, there is \(f_r \in \Lambda(n)\) for any fixed \(r \in Y_0,\) such that
\[(\Gamma_k - \Gamma_{1'})(f_r) = f_{rk} \quad \text{for all } k \in Y_1 \setminus 1'.\] (4.2.19)

Set \(D' = -\sum_{r \in Y_0} f_r D_r.\) We obtain by using (4.2.13) and (4.2.19) that, for \(k \in Y_1 \setminus 1',\)
\[\[D', \Gamma_k - \Gamma_{1'}\] = \sum_{r \in Y_0} (\Gamma_k - \Gamma_{1'})(f_r) D_r = \sum_{r \in Y_0} f_{rk} D_r = \phi(\Gamma_k - \Gamma_{1'}).\] (4.2.20)

Let \(D := D'.\) Then \(D \in \mathcal{G}_t\) and (4.2.20) shows that \([D, \Gamma_k - \Gamma_{1'}] = \phi(\Gamma_k - \Gamma_{1'})\) for all \(k \in Y_1 \setminus 1',\) since \(zd(\phi) = t.\) Hence
\[(\phi - \text{ad}D)(\Gamma_k - \Gamma_{1'}) = 0 \quad \text{for all } k \in Y_1 \setminus 1'.\]

\[\Box\]

Let us consider the case of even \(Z\)-degree. Notice that, in contrast to the case of odd degree, we consider different elements in the canonical torus. This will simplify the computation.

**Lemma 4.2.5.** Let \(\phi \in \text{Der}_t(S, W)\) where \(t \geq 0\) is even. If \(\phi(W_{-1}) = 0,\) then there is \(D \in \mathcal{G}_t\) such that
\[(\phi - \text{ad}D)(\Gamma_1 + \Gamma_k) = 0 \quad \text{for all } k \in Y_1.\] (4.2.21)

**Proof.** Since \(\phi\) is of even \(Z\)-degree, by Lemma 2.1.1, we may assume that for \(k \in Y_1,\)
\[(\Gamma_1 + \Gamma_k) = \sum_{r \in Y_1} f_{rk} D_r \quad \text{where } f_{rk} \in \Lambda(n).\] (4.2.22)

It is easily seen that
\[[\Gamma_1 + \Gamma_k, \phi(\Gamma_1 + \Gamma_l)] = [\Gamma_1 + \Gamma_l, \phi(\Gamma_1 + \Gamma_k)] \quad \text{for all } k, l \in Y_1.\] (4.2.23)

Thus we obtain from (4.2.22) and (4.2.23) that
\[\sum_{r \in Y_1} (\Gamma_k(f_{rl}) - \Gamma_l(f_{rk})) D_r + f_{rk} D_l - f_{kl} D_k = 0.\] (4.2.24)
Comparing coefficients we have
\[
\Gamma_k(f_{rl}) = \Gamma_l(f_{rk}) \quad \text{whenever} \quad r \neq k, l; \tag{4.2.25}
\]
\[
\Gamma_l(f_{kl}) = \Gamma_k(f_{kl}) - f_{kl} \quad \text{whenever} \quad k \neq l. \tag{4.2.26}
\]
For \(r, k \in Y_1\), one may assume that
\[
f_{rk} = \sum_{|u|=t+1} c_{urk}x^u \quad \text{where} \quad c_{urk} \in \mathbb{F}. \tag{4.2.27}
\]
We then obtain from (4.2.25) and (4.2.27) that
\[
\delta_{l \in u} c_{urk} = \delta_{k \in u} c_{url} \quad \text{whenever} \quad r \neq k, l. \tag{4.2.28}
\]
(4.2.28) implies that for \(r, k \in Y_1\) with \(r \neq k\), if \(c_{urk} \neq 0\), then (4.2.29) ensures that \(k \in u\). Thus
\[
\Gamma_k(f_{rk}) = f_{rk} \quad \text{whenever} \quad r \neq k. \tag{4.2.30}
\]
For any fixed \(r \in Y_1\), by Corollary 2.1.5, there is \(f_{rk} \in \Lambda(n)\) such that
\[
\Gamma_k(f_{rk}) = f_{rk} \quad \text{for all} \quad k \in Y_1 \setminus r. \tag{4.2.31}
\]
Assert that
\[
\Gamma_k(f_{kk}) = 0 \quad \text{for all} \quad k \in Y_1. \tag{4.2.32}
\]
We treat two cases separately.

**Case (i):** \(t \geq 2\). Note that \(\Gamma_k^2 = \Gamma_k\) for \(k \in Y_1\), by Lemma 2.1.2(ii). We obtain from (4.2.26) that
\[
\Gamma_l \Gamma_k(f_{kk}) = \Gamma_k \Gamma_l(f_{kk}) = \Gamma_k^2(f_{kl}) - \Gamma_k(f_{kl}) = 0 \quad \text{for all} \quad l \in Y_1 \setminus k. \tag{4.2.33}
\]
Since \(zd(f_{kk}) = t + 1 \geq 3\) and \(\Gamma_k(x^u) = \delta_{k \in u} x^u\), one may easily deduce (4.2.32) from (4.2.33).

**Case (ii):** \(t = 0\). Applying \(\phi\) to the equation that \([x_lD_k, \Gamma_1 + \Gamma_k] = x_lD_k\) for \(k, l \in Y_1\) with \(k \neq l\), we have
\[
\phi(x_lD_k) + [\Gamma_1 + \Gamma_k, \phi(x_lD_k)] = [x_lD_k, \phi(\Gamma_1 + \Gamma_k)]. \tag{4.2.34}
\]
On the other hand, noticing that \(zd(f_{rk}) = 1\), it is easily seen from (4.2.30) that
\[
f_{rk} = c_{rk}x_k \quad \text{whenever} \quad r \neq k, \tag{4.2.35}
\]
where \(c_{rk} \in \mathbb{F}\). Then we obtain from (4.2.22) and (4.2.35) that
\[
\phi(\Gamma_1 + \Gamma_k) = f_{kk}D_k + \sum_{r \in Y_1 \setminus k} c_{rk}x_kD_r. \tag{4.2.36}
\]
Combining (4.2.34) and (4.2.36) and noticing that $D_k(f_{kk}) \in \mathbb{F}$, we have
\[
\phi(x_lD_k) + [\Gamma_1 + \Gamma_k, \phi(x_lD_k)] = (D_k(f_{kk})x_l - c_{lk}x_k)D_k + \sum_{r \in Y_1 \setminus k} c_{rk}x_lD_r. \tag{4.2.37}
\]

Since $zd(\phi) = 0$, one may assume that for $k, l \in Y_1$ with $k \neq l$,
\[
\phi(x_lD_k) = \sum_{s, r \in Y_1} \mu_{s, r}^{(l, k)} x_sD_r \quad \text{where } \mu_{s, r}^{(l, k)} \in \mathbb{F}. \tag{4.2.38}
\]

Clearly,
\[
[\Gamma_1 + \Gamma_k, x_sD_r] = (\delta_{ks} - \delta_{kr})x_sD_r \quad \text{for } k, l \in Y_1. \tag{4.2.39}
\]

It follows from (4.2.38) and (4.2.39) that
\[
\phi(x_lD_k) + [\Gamma_1 + \Gamma_k, \phi(x_lD_k)] = \sum_{s, r \in Y_1} \mu_{s, r}^{(l, k)} x_sD_r + \sum_{s, r \in Y_1} (\delta_{ks} - \delta_{kr})\mu_{s, r}^{(l, k)} x_sD_r. \tag{4.2.40}
\]

The coefficient of $D_k$ in the right hand side of (4.2.40) is
\[
\sum_{s \in Y_1} \mu_{s, k}^{(l, k)} x_s + \sum_{s \in Y_1} (\delta_{ks} - \delta_{kk})\mu_{s, k}^{(l, k)} x_s = \sum_{s \in Y_1} \delta_{ks}\mu_{s, k}^{(l, k)} x_s = \mu_{k, k}^{(l, k)} x_k.
\]

We then obtain from (4.2.37) that
\[
D_k(f_{kk})x_l - c_{lk}x_k = \mu_{k, k}^{(l, k)} x_k.
\]

It follows that $D_k(f_{kk}) = 0$, proving (4.2.32).

For $r \in Y_1$, put
\[
f_r := -f_{rr} + (\Gamma_1 + \Gamma_r)(\vec{f}_r).
\]

Clearly, $f_r \in \Lambda(n)$, since $\vec{f}_r, f_{rr} \in \Lambda(n)$. Using (4.2.32), we have
\[
(\Gamma_1 + \Gamma_r)(f_r) - f_r = f_{rr}. \tag{4.2.41}
\]

For $k \in Y_1 \setminus r$, using (4.2.26) and (4.2.31), we obtain that
\[
(\Gamma_1 + \Gamma_k)(f_r) = -((\Gamma_1 + \Gamma_k)(f_{rr}) + (\Gamma_1 + \Gamma_k)(\Gamma_1 + \Gamma_r)(\vec{f}_r))
\]

\[
= -\Gamma_k(f_{rr}) + \Gamma_r\Gamma_k(\vec{f}_r)
\]

\[
= -(\Gamma_r(f_{rk}) - f_{rk}) + \Gamma_r(f_{rk})
\]

\[
= f_{rk}.
\]

Putting $D' = -\sum_{r \in Y_1} f_rD_r$, we then obtain by using (4.2.41) that
\[
[D', \Gamma_1 + \Gamma_k] = -\sum_{r \in Y_1} [f_rD_r, \Gamma_1 + \Gamma_k]
\]

\[
= \sum_{r \in Y_1} (\Gamma_1 + \Gamma_k)(f_r)D_r - f_kD_k
\]

\[
= \sum_{r \in Y_1 \setminus k} (\Gamma_1 + \Gamma_k)(f_r)D_r + ((\Gamma_1 + \Gamma_k)(f_k) - f_k)D_k
\]

\[
= \sum_{r \in Y_1 \setminus k} f_{rk}D_r + f_{kk}D_k
\]

\[
= \phi(\Gamma_1 + \Gamma_k).
\]
Let $D := D'_t$. Then $D \in G_t$. Since $zd(\phi) = t$, we have

$$[D, \Gamma_1 + \Gamma_k] = \phi(\Gamma_1 + \Gamma_k) \quad \text{for all } k \in Y_1.$$ 

Now it is easy to see that (4.2.21) holds. $\square$

For our purpose, we need still the following three reduction lemmas.

**Lemma 4.2.6.** Suppose that $\phi \in \text{Der}_t(S, W)$, and $\phi(S_{-1}) = 0$ where $t > 0$ is even. If $\phi(\Gamma_1 + \Gamma_k) = 0$ for all $k \in Y_1$, then $\phi(S_0) = 0$.

**Proof.** (i) First, we show that $\phi(\Gamma_1 - \Gamma_i) = 0$ for all $i \in Y_0 \setminus 1$. Since $zd(\phi)$ is even, by Lemma 2.1.1 we may assume that

$$\phi(\Gamma_1 - \Gamma_i) = \sum_{r \in Y_1, u \in B} c_{u, r}^{(i)} x^u D_r \quad \text{where } c_{u, r}^{(i)} \in \mathbb{F}. \quad (4.2.42)$$

For any fixed coefficient $c_{v, s}^{(i)}$, the assumption that $zd(\phi) = t > 0$ ensures that there is $k \in v \setminus s$. Applying $\phi$ to the equation that $[\Gamma_1 + \Gamma_k, \Gamma_1 - \Gamma_i] = 0$ and using (4.2.42), one gets

$$\sum_{r \in Y_1, u \in B} \delta_{k,u} c_{u, r}^{(i)} x^u D_r - \sum_u c_{u, k}^{(i)} x^u D_k = 0.$$

The choice of $k$ and the equation above imply that $c_{v, s}^{(i)} = 0$. Thus $\phi(\Gamma_1 - \Gamma_i) = 0$ for all $i \in Y_0 \setminus 1$.

(ii) Second, we show that $\phi(x_i D_j) = 0$ for all $i, j \in Y_0$ with $i \neq j$. By our general assumption that $m \geq 3$, one may find $r \in Y_0 \setminus \{i, j\}$. Then $[x_i D_j, \Gamma_r + \Gamma_k] = 0$ for all $k \in Y_1$. Notice the assumption that $\phi(\Gamma_1 + \Gamma_k) = 0$ for all $k \in Y_1$. By (i), it is easily seen that $\phi(\Gamma_r + \Gamma_k) = 0$. Then $[\Gamma_r + \Gamma_k, \phi(x_i D_j)] = 0$. Arguing as in (i), one may see that the assertion holds.

(iii) Finally, we assert that $\phi(x_k D_l) = 0$ for $k, l \in Y_1$ with $k \neq l$. Just as in (i), one may assume that

$$\phi(x_k D_l) = \sum_{r \in Y_1, u \in B} c_{u, r} x^u D_r \quad \text{where } c_{u, r} \in \mathbb{F}. \quad (4.2.43)$$

**Case 1:** $zd(\phi) = 2$. Note that $n\Gamma_1 + \Gamma' \in S$ and $\phi(n\Gamma_1 + \Gamma') = 0$ by the assumption. Noticing (4.2.43), we have

$$2\phi(x_k D_l) = [\Gamma', \phi(x_k D_l)] = [n\Gamma_1 + \Gamma', \phi(x_k D_l)] = 0.$$

Therefore, $\phi(x_k D_l) = 0$.

**Case 2:** $zd(\phi) \geq 4$. For any given coefficient $c_{v, s}$ in (4.2.43), one may take $q \in v \setminus \{k, l, s\}$, since $|v| = zd(\phi) + 1 \geq 5$. Then $[\Gamma_1 + \Gamma_q, x_k D_l] = 0$ and therefore,

$$[\Gamma_1 + \Gamma_q, \sum_{r \in Y_1, u \in B} c_{u, r} x^u D_r] = 0.$$
Furthermore,
\[ \sum_{r \in Y_1, u \in B} \delta_{q \in u} c_{u,r} x^u D_r - \sum_u c_{u,q} x^u D_q = 0. \]

Consequently, \( c_{v,s} = 0 \) and then \( \phi(x_k D_l) = 0 \) for all \( k, l \in Y_1 \) with \( k \neq l \). Summarizing, we have \( \phi(S_0) = 0 \). □

**Lemma 4.2.7.** Suppose that \( \phi \in \text{Der}(S, W) \) is a homogeneous derivation of positive odd \( \mathbb{Z} \)-degree and \( \phi(S_{-1} + T_S) = 0 \). Then \( \phi(S_0) = 0 \).

**Proof.** We first assert that
\[ \phi(x_i D_j) = 0 \quad \text{for all } i, j \in Y_0 \text{ with } i \neq j. \]

Since \( \text{zd}(\phi) \) is odd, by Lemma 2.1.1 one may assume that
\[
\phi(x_i D_j) = \sum_{r \in Y_0, u} c_{u,r} x^u D_r \quad \text{where } c_{u,r} \in \mathbb{F}.
\]

Since \( m \geq 3 \), one may take \( s \in Y_0 \setminus \{i, j\} \). Clearly,
\[ [\Gamma_s + \Gamma_k, x_i D_j] = 0 \quad \text{for all } k \in Y_1. \]

Applying \( \phi \) to the equation above, we have
\[
\sum_{r \in Y_0, u \in B} \delta_{k \in u} c_{u,r} x^u D_r - \sum_u c_{u,s} x^u D_s = 0.
\]

Consequently,
\[ \delta_{k \in u} c_{u,r} = 0 \quad \text{for } r \in Y_0 \setminus s; \ k \in Y_1. \quad (4.2.44) \]

Take \( k \in u \). We obtain from (4.2.44) that \( c_{u,r} = 0 \) for \( r \in Y_0 \setminus s \). Thus
\[ \phi(x_i D_j) = \sum_{u \in B} c_{u,s} x^u D_s. \quad (4.2.45) \]

Applying \( \phi \) to the following equation
\[ [x_j D_j + \Gamma_k, x_i D_j] = -x_i D_j \quad \text{for all } k \in Y_1, \]
we obtain by using (4.2.45) that
\[ [x_j D_j + \Gamma_k, \sum_u c_{u,s} x^u D_s] = -\sum_u c_{u,s} x^u D_s. \]

Then
\[ \sum_u \delta_{k \in u} c_{u,s} x^u D_s = -\sum_u c_{u,s} x^u D_s. \]

A comparison of coefficients shows that
\[ \delta_{k \in u} c_{u,s} = -c_{u,s} \quad \text{for all } k \in Y_1. \quad (4.2.46) \]
Take $k \in u$. Then (4.2.46) yields $c_{u,s} = 0$ for all $u \in \mathbb{B}$. Then we obtain from (4.2.45) that 
\[
\phi(x_iD_j) = 0 \quad \text{for } i, j \in Y_0 \text{ with } i \neq j.
\]
The assertion holds.

It remains only to show that 
\[
\phi(x_kD_l) = 0 \quad \text{for } k, l \in Y_1 \text{ with } k \neq l.
\]
As in the above, we may assume that 
\[
\phi(x_kD_l) = \sum_{r \in Y_0, u \in \mathbb{B}} c_{u,r}x^nD_r \quad \text{where } c_{u,r} \in \mathbb{F}.
\]
Clearly, 
\[
[x_kD_l, \Gamma_i - \Gamma_j] = 0 \quad \text{for } i, j \in Y_0 \text{ with } i \neq j.
\]
Applying $\phi$ to this equation, we obtain that 
\[
\sum_u c_{u,i}x^nD_i - \sum_u c_{u,j}x^nD_j = 0
\]
and therefore, $c_{u,i} = c_{u,j} = 0$. Hence $c_{u,r} = 0$ for all $r$ and all $u$. The proof is complete. \(\square\)

**Lemma 4.2.8.** Suppose that $\phi \in \text{Der}_0(S, W)$ and $\phi(W_{-1}) = 0$. If 
\[
\phi(\Gamma_1 + \Gamma_k) = 0 \quad \text{for all } k \in Y_1,
\]
then there is $D \in G_0$ such that $(\phi - \text{ad}D)(S_0) = 0$.

**Proof.** By Lemma 2.1.1, $\phi(x_iD_j) \in E(G)$ for all $i, j \in Y_0$ with $i \neq j$. Then 
\[
\phi(\Gamma_1 - \Gamma_i) = [\phi(x_1D_i), x_iD_1] + [x_1D_i, \phi(x_1D_1)] = 0 \quad \text{for } i \in Y_0 \setminus 1.
\]
We next show that $\phi(x_iD_j) = 0$ for $i, j \in Y_0$ with $i \neq j$. Find $r \in Y_0 \setminus \{i, j\}$, since $m \geq 3$. Then 
\[
\phi(x_iD_j) = \phi([\Gamma_i - \Gamma_r, x_iD_j]) = 0,
\]
since $\phi(\Gamma_i - \Gamma_r), \phi(x_iD_j) \in E(G)$. Finally, consider $\phi(x_kD_l)$ for $k, l \in Y_1$ with $k \neq l$.

We may assume that 
\[
\phi(x_kD_l) = \sum_{r \in Y_1} f_rD_r \quad \text{where } f_r \in \Lambda(n)_1.
\]
For $q \in Y_1 \setminus \{k, l\}$, we have $[\Gamma_1 + \Gamma_q, x_kD_l] = 0$ and therefore, 
\[
\sum_r (\Gamma_1 + \Gamma_q)(f_r)D_r - f_qD_q = 0.
\]
Consequently, \((\Gamma_1 + \Gamma_q)(f_q) = f_q\). This implies that \(f_q = c_{k,l}^{(q)} x_q\) for \(q \in Y_1 \setminus \{k,l\}\), since \(f_q \in \Lambda(n)_1\). Therefore,

\[
\phi(x_k D_l) = \sum_{r \in Y_1 \setminus \{k,l\}} c_r x_r D_r + f_k D_k + f_l D_l.
\]

Note that \([\Gamma_1 + \Gamma_k, \phi(x_k D_l)] = \phi(x_k D_l)\). Then

\[
(\Gamma_1 + \Gamma_k)(f_k)D_k - f_k D_k + \Gamma_k(f_l)D_l = \sum_{r \in Y_1 \setminus \{k,l\}} c_r x_r D_r + f_k D_k + f_l D_l.
\]

Comparing coefficients we have

\[
c_r = 0 \quad \text{for} \quad r \in Y_1 \setminus \{k,l\};
\]

\[
(\Gamma_1 + \Gamma_k)(f_k) = 2f_k;
\]

\[
\Gamma_k(f_l) = f_l.
\]

The last two equations imply that \(f_k = 0\) and there is \(\lambda_{k,l} \in F\) such that \(f_l = \lambda_{k,l} x_k\). Thus

\[
\phi(x_k D_l) = \lambda_{k,l} x_k D_l.
\]

Just as in the proof of Proposition 3.2.4, one may show from the equation above that there is \(D \in G_0\) such that

\[
(\phi - \text{ad}D)(S_0) = 0.
\]

Now we are able to characterize the homogeneous derivation space of nonnegative \(Z\)-degree.

**Proposition 4.2.9.** \(\text{Der}_t(S, W) = \text{ad}W_t\) for \(t \geq 0\).

**Proof.** Clearly, \(\text{ad}W_t \subset \text{Der}_t(S, W)\). To prove the converse inclusion, let \(\phi \in \text{Der}_t(S, W)\). In the light of Proposition 2.1.6, we may assume that \(\phi(S_{-1}) = 0\). We treat two cases separately.

(i) Suppose that \(zd(\phi)\) is odd. By Lemma 4.2.4 and Corollary 4.2.3, there is \(E \in G\) such that \(\phi - \text{ad}E\) vanishes on \(T_S\). Then Lemma 4.2.7 shows that \(\phi - \text{ad}E\) vanishes on \(S_0\). Clearly, \((\phi - \text{ad}E)(S_{-1}) = 0\). Now Corollary 4.1.4 ensures that \(\phi \in \text{ad}W_t\) and therefore, \(\phi \in \text{ad}W_t\).

(ii) Suppose that \(zd(\phi)\) is even. By Lemma 4.2.5, there is \(D \in G_t\) such that \((\phi - \text{ad}D)(\Gamma_1 + \Gamma_k) = 0\) for all \(k \in Y_1\). If \(t > 0\), then Lemma 4.2.6 implies that \((\phi - \text{ad}D)(S_0) = 0\). If \(t = 0\), then Lemma 4.2.8 shows that there is \(E \in G_0\) such that \((\phi - \text{ad}D - \text{ad}E)(S_0) = 0\). Now, Corollary 4.1.4 ensures that \(\phi \in \text{Der}_t(S, W)\) is inner. The proof is complete.

As an application of Proposition 4.2.9, we have:

**Proposition 4.2.10.** \(\text{Der}_t(S) = \text{ad}(S + T)_t\) for \(t \geq 0\).
Proof. Just as in the case of Lie algebras, one may prove that $S$ is an ideal of $\overline{S}$ and $\mathcal{T} \subset \text{Nor}_W(S)$ (cf. [19, 23]). It follows that $\text{ad}(\overline{S} + \mathcal{T}) \subset \text{Der}(S)$. To prove the converse inclusion, let $\phi \in \text{Der}(S)$. One may identify $\phi$ with a derivation of $\text{Der}(S, W)$. Then by Proposition 4.2.9, there is $D \in W_t$ such that $\phi = \text{ad}D$ as elements of $\text{Der}(S, W)$ and therefore, $\phi = \text{ad}D$ as elements of $\text{Der}(S)$. Clearly, $D \in \text{Nor}_W(S)_t$. Therefore, it is enough to show that $\text{Nor}_W(S)_t \subset (\overline{S} + \mathcal{T})_t$. Let $E$ be an arbitrary element of $\text{Nor}_W(S)_t$. Then $\text{div}([D_i, E]) = 0$ for all $i \in Y_0$. Since $\text{div}$ is a derivation of $\mathfrak{A}$ and $\text{div}(D_i) = 0$, it follows that $D_i(\text{div}(E)) = 0$ for all $i \in Y_0$. This implies that $\text{div}(E) \in \Lambda(n)_t$.

If $t = 0$ then $\text{div}(E) \in \mathbb{F}$. It follows that $\text{div}(E - \text{div}(E)\Gamma_1) = 0$. Hence $E \in (\overline{S} + \mathcal{T})_0 = \overline{S}_0 + \mathcal{T}$.

Suppose that $t > 0$. We contend that $\text{Nor}_W(S)_t \subset \overline{S}_t$. Assume that on the contrary that $\text{Nor}_W(S)_t \not\subset \overline{S}_t$. Then there is $E \in \text{Nor}_W(S)_t \setminus \overline{S}_t$. Since $\text{div}(E) \in \Lambda(n)_t$, one may write $E$ to be the following form

$$E = \sum_{i \in X_0, u \in \Omega_0} c_{ui} x_i u D_i + \sum_{k \in v, v \in \Omega_1} c_{vk} x^v D_k + G$$

(4.2.47)

such that

$$\text{div}(E) = \sum_{i \in X_0, u \in \Omega_0} c_{ui} x^u - \sum_{k \in v, v \in \Omega_1} c_{vk} D_k(x^v)$$

(4.2.48)

and that no any cancellation occurs in the right hand side of (4.2.48), where $X_0 \subset Y_0, X_1 \subset Y_1$; $\Omega_0 \subset \cup_{l \leq r \leq \frac{r}{2}} \mathbb{B}_2, \Omega_1 \subset \cup_{l \leq r \leq \frac{r}{2}} \mathbb{B}_{2r + 1}^2$; $c_{ui}, c_{vk} \in \mathbb{F}$; $G \in \overline{S}$. Evidently, the assumption that $\text{Nor}_W(S)_t \not\subset \overline{S}_t$ secures that $X_0 \neq \emptyset$ and $\Omega_0 \neq \emptyset$, or, $X_1 \neq \emptyset$ and $\Omega_1 \neq \emptyset$. Assume that $X_0 \neq \emptyset$ and $\Omega_0 \neq \emptyset$. Given $w \in \Omega_0$ and $j \in X_0$, choose $r \in w$. It follows from (4.2.47) that

$$\sum_{i \in X_0, u \in \Omega_0} \delta_{r \in u} c_{ui} x^u x_i D_i + \sum_{k \in v, v \in \Omega_1} (\delta_{r \in v} - \delta_{r \in k}) c_{vk} x^v D_k = [\Gamma_r + \Gamma_1, E] - [\Gamma_r + \Gamma_1, G] \in \mathcal{S},$$

since $\Gamma_r + \Gamma_1 \in \mathcal{S}$. Applying the divergence, we obtain from the equation above that

$$\sum_{i \in X_0, u \in \Omega_0} \delta_{r \in u} c_{ui} x^u - \sum_{k \in v, v \in \Omega_1} (\delta_{r \in v} - \delta_{r \in k}) c_{vk} D_k(x^v) = 0.$$  

(4.2.49)

Note that $\delta_{r \in v} - \delta_{r \in k} = 0$ or $1$. One should bear in mind, as remarked above, that no cancellation occurs in the right hand side of (4.2.48). Thus the nonzero summand $c_{wj} x^w$ in the first sum in the left hand side of (4.2.49) cannot be cancelled, contradicting to that the right hand side is zero.

It remains to discuss the case that $X_1 \neq \emptyset$ and $\Omega_1 \neq \emptyset$. Given $w \in \Omega_1$ and $l \in X_1$, one can choose $r \in w \setminus l$ since $|w| \geq 3$. Arguing as above one may find that $c_{wl} D_l(x^w)$ is a nonzero summand in the second sum in the left hand side of (4.2.49), which cannot be cancelled by the same token, contradicting that the right hand side is zero.

So far, we have proved that $\text{Nor}_W(S)_t \subset \overline{S}_t$ for $t > 0$. Summarizing, the proof is complete. \qed

Remark 4.2.11. Our original idea is to study the derivation algebra $\text{Der}(S)$ rather than the derivation space $\text{Der}(S, W)$, which contains the former in the obvious sense.
But, in practice, it is convenient and effective first to study the derivation space \( \text{Der}(S, W) \) rather than the algebra \( \text{Der}(S) \). Our work had ever stopped for a time, since we observed that the natural \( \mathbb{Z} \)-gradation is not admissibly graded (see Remark 2.1.8). However, when we considered the derivation space \( \text{Der}(S, W) \) and determined it at last, almost all problems were solved at the last moment.

### 4.3. Derivation algebra of \( S \)

In this subsection we first determine the homogeneous derivations of negative \( \mathbb{Z} \)-degree of \( S \) to \( W \). This combining with the results obtained in Section 4.2 will give the structure of the derivation space \( \text{Der}(S, W) \). Using these results we are able to characterize the derivation algebra \( \text{Der}(S) \).

To compute the derivations of negative \( \mathbb{Z} \)-degree, recall the generator set of \( S \) (see Proposition 2.2.3). We still adopt the notations

\[
R := \{ D_{il}(x^{(2\varepsilon_i)}x_k) \mid i \in Y_0, k, l \in Y_1 \} \cup \{ D_{ij}(x_ix_l x_k) \mid i, j \in Y_0, k, l \in Y_1 \}
\]

and

\[
Q = \{ D_{ij}(x^{(a\varepsilon_j)}) \mid i, j \in Y_0, i \neq j, a \in \mathbb{N} \}.
\]

The following lemma tells us that a derivation of \( \mathbb{Z} \)-degree \(-1\) of \( S \) to \( W \) is completely determined by its action on \( S_0 \).

**Lemma 4.3.1.** Suppose that \( \varphi \in \text{Der}_{-1}(S, W) \) and \( \varphi(S_0) = 0 \). Then \( \varphi = 0 \).

**Proof.** First of all, we show that \( \varphi(R) = 0 \). We shall use the following simple fact (by Lemma 2.1.1):

\[
\varphi(S_1) \subseteq G_0 \subseteq E(G).
\] (4.3.1)

Given \( i \in Y_0, k, l \in Y_1 \), take \( j \in Y_0 \setminus i \). Then

\[
[\Gamma_i - \Gamma_j, D_{il}(x^{(2\varepsilon_i)}x_k)] = D_{il}(x^{(2\varepsilon_i)}x_k).
\] (4.3.2)

From (4.3.1) and (4.3.2), we obtain that

\[
\varphi(D_{il}(x^{(2\varepsilon_i)}x_k)) = 0 \quad \text{for} \quad i \in Y_0, k, l \in Y_1.
\] (4.3.3)

By a same argument, we can also obtain that

\[
\varphi(D_{il}(x_ix_l x_k)) = 0 \quad \text{for all} \quad i \in Y_0, k, l \in Y_1.
\] (4.3.4)

It follows from (4.3.3) and (4.3.4) that \( \varphi(R) = 0 \).

We next show that \( \varphi(Q) = 0 \). If \( a = 3 \), it is easily showed as above that \( \varphi(D_{ij}(x^{(a\varepsilon_j)})) = 0 \). Now suppose that \( a \geq 4 \). We proceed by induction on \( a \) to show that

\[
\varphi(D_{ij}(x^{(a\varepsilon_j)})) = 0, \quad a \geq 4.
\] (4.3.5)

By inductive hypothesis,

\[
\varphi(D_{ij}(x^{(a\varepsilon_j)})) \in G_{a-3}.
\] (4.3.6)
If $a$ is odd, then (4.3.6) ensures that
\[
\varphi\left(D_{ij}(x^{(a\varepsilon_j)})\right) = \sum_{r \in Y_1, u \in \mathbb{B}_{a-2}} c_{u,r} x^u D_r \quad \text{where } c_{u,r} \in \mathbb{F}. \tag{4.3.7}
\]
Given any coefficient $c_{v,s}$ in (4.3.7), one may take $k \in v \setminus r$, since $|u| = a - 2 \geq 2$. One may also take $q \in Y_0 \setminus \{i, j\}$, since $m \geq 3$. Then
\[
[\Gamma_k + \Gamma_q, D_{ij}(x^{(a\varepsilon_j)})] = 0. \tag{4.3.8}
\]
Applying $\varphi$ to (4.3.8) and using (4.3.7), one gets
\[
c_{v,s} = 0. \quad \text{Thus (4.3.5) holds.} \tag{4.3.9}
\]
If $a$ is even, by (4.3.6),
\[
\varphi\left(D_{ij}(x^{(a\varepsilon_j)})\right) = \sum_{r \in Y_0, u \in \mathbb{B}_{a-2}} c_{u,r} x^u D_r. \tag{4.3.10}
\]
Given any coefficient $c_{v,s}$ in (4.3.9), take $k \in v, q \in Y_0 \setminus \{s, j\}$. Then
\[
[\Gamma_k + \Gamma_q, D_{ij}(x^{(a\varepsilon_j)})] = -\delta_{qj} D_{ij}(x^{(a\varepsilon_j)}). \tag{4.3.11}
\]
From (4.3.9) and (4.3.10), one may compute that
\[
c_{v,s} = -\delta_{qj} c_{v,s}. \quad \text{Therefore, } c_{v,s} = 0 \quad \text{and (4.3.5) holds. This proves } \varphi(\mathcal{Q}) = 0. \quad \Box
\]

Using Lemma 4.3.1 we can determine the derivations of $\mathbb{Z}$-degree $-1$; in particular, they are all inner.

**Proposition 4.3.2.** $\text{Der}_{-1}(S, W) = \text{ad}W_{-1}$. In particular, $\text{Der}_{-1}(S) = \text{ad}S_{-1}$.

**Proof.** Let $\phi \in \text{Der}_{-1}(S, W)$. For $i \in Y_0$, suppose that
\[
\phi(\Gamma_{1'} + \Gamma_i) = \sum_{r \in Y_0} c_{i,r} D_r \quad \text{where } c_{i,r} \in \mathbb{F}. \tag{4.3.12}
\]
Let $j \in Y_0 \setminus i$. Then $[\Gamma_{1'} + \Gamma_i, \Gamma_{1'} + \Gamma_j] = 0$ and therefore,
\[
[\Gamma_{1'} + \Gamma_i, \phi(\Gamma_{1'} + \Gamma_j)] = [\Gamma_{1'} + \Gamma_j, \phi(\Gamma_{1'} + \Gamma_i)].
\]
Then by (4.3.12),
\[
c_{i,j} = 0 \quad \text{whenever } i, j \in Y_0 \text{ with } i \neq j.
\]
Thus, by (4.3.12), one gets
\[
\phi(\Gamma_{1'} + \Gamma_i) = c_{i,i} D_i \quad \text{where } c_{i,i} \in \mathbb{F}. \tag{4.3.13}
\]
Obviously,
\[
[\Gamma_{1'} + \Gamma_i, x_i D_j] = x_i D_j \quad \text{for } i, j \in Y_0 \text{ with } i \neq j.
\]
Using (4.3.13), we then have
\[
c_{i,i} D_j + [\Gamma_{1'} + \Gamma_i, \phi(x_i D_j)] = \phi(x_i D_j).
\]
Noticing that $\phi(x_i D_j) \in S_{-1} = W_{-1}$, we obtain from the equation above that $\phi(x_i D_j) = c_{i, i} D_j$. Now let $\psi := \phi - \sum_{r \in Y_0} c_{r, r} \text{ad} D_r$. Then

$$\psi(\Gamma_i + \Gamma_j) = \psi(x_i D_j) = 0 \quad \text{for } i, j \in Y_0, \ i \neq j. \quad (4.3.13)$$

We want to prove that

$$\psi(x_k D_l) = 0 \quad \text{for } k, l \in Y_1 \text{ with } k \neq l. \quad (4.3.14)$$

To do that, we choose $q \in Y_1 \setminus \{k, l\}$. Then

$$[\Gamma_i + \Gamma_q, x_k D_l] = 0 \quad \text{for all } i \in Y_0.$$  

It follows that

$$[\psi(\Gamma_i + \Gamma_q), x_k D_l] + [\Gamma_i + \Gamma_q, \psi(x_k D_l)] = 0. \quad (4.3.15)$$

Since $\psi(\Gamma_i + \Gamma_q) \in W_{-1}$, we have $[\psi(\Gamma_i + \Gamma_q), x_k D_l] = 0$. Then (4.3.15) implies that $[\Gamma_i, \psi(x_k D_l)] = 0$ for all $i \in Y_0$. Hence $\psi(x_k D_l) = 0$, since $\psi(x_k D_l) \in W_{-1}$.

Similarly, we may check that

$$\psi(\Gamma_i - \Gamma_k) = 0 \quad \text{for all } k \in Y_1 \setminus Y'. \quad (4.3.16)$$

By (4.3.13), (4.3.14) and (4.3.16), $\psi(S_0) = 0$. By Lemma 4.3.1, we obtain that $\psi = 0$ and $\phi \in \text{ad} W_{-1}$.

To compute the derivations of $\mathbb{Z}$-degree less than $-1$ of $S$ to $W$, we establish the following lemma.

**Lemma 4.3.3.** Let $\phi \in \text{Der}_{-t}(S, W)$, $t > 1$. Suppose that $\phi(D_{ij}(x^{((t+1)\varepsilon_i)})) = 0$ for all $i, j \in Y_0$. Then $\phi = 0$.

**Proof.** First claim that $\phi(Q) = 0$. To that aim, we proceed by induction on $q$ to show that

$$\phi(D_{ij}(x^{(q\varepsilon_i)})) = 0 \quad \text{for all } i, j \in Y_0. \quad (4.3.17)$$

If $q \leq t+1$, then (4.3.17) holds. Suppose that $q > t+1$ in the following. By inductive hypothesis and Lemma 2.1.1, $\phi(D_{ij}(x^{(q\varepsilon_i)})) \in G_{q-t-2}$. Then one may assume that

$$\phi(D_{ij}(x^{(q\varepsilon_i)})) = \sum_{r \in Y, |u|=q-t-1} c_{u,r} x^u D_r \quad \text{where } c_{u,r} \in \mathbb{F}. \quad (4.3.18)$$

We treat two cases separately.

**Case (i):** $q - t \geq 3$. For any fixed coefficient $c_{u_0,r_0}$ in (4.3.18), choose $k \in u_0 \setminus r_0$ since $|u_0| \geq 2$. Choose also $s \in Y_0 \setminus \{r_0, i\}$. Then

$$[\Gamma_s + \Gamma_k, D_{ij}(x^{(q\varepsilon_i)})] = -\delta_{i,j}(D_{ij}(x^{(q\varepsilon_i)})).$$

Applying $\phi$ to the equation above and then combining that with (4.3.18), one may obtain by a comparison of the coefficients of $x^{u_0} D_{r_0}$ that

$$c_{u_0,r_0} = -\delta_{i,j} c_{u_0,r_0}. \quad \text{Case (i)}$$
Consequently, \( c_{u_0, r_0} = 0 \) and \( \phi(D_{ij}(x^{(q_{\epsilon_i})})) = 0 \).

**Case (ii):** \( q - t < 3 \). Note that \( q > t + 1 \) and then \( q - t - 1 = 1 \). Then rewrite (4.3.18) as
\[
\phi(D_{ij}(x^{(q_{\epsilon_i})})) = \sum_{s \in Y_1, r \in Y_1} c_{s, r} x_s D_r \quad \text{where} \quad c_{s, r} \in \mathbb{F}.
\]

Arguing as in Case (i), one may easily obtain that \( c_{s, r} = 0 \) whenever \( s \neq r \). Then
\[
\phi(D_{ij}(x^{(q_{\epsilon_i})})) = \sum_{r \in Y_1} c_{r, r} x_r D_r.
\]

Applying \( \phi \) to the identity \([x_s D_l, D_{ij}(x^{(q_{\epsilon_i})})] = 0\) for \( s, l \in Y_1 \), we have \( c_{s, s} = c_{l, l} \). Let \( \lambda := c_{r, r}, \ r \in Y_1 \).

Then \( \phi(D_{ij}(x^{(q_{\epsilon_i})})) = \lambda \Gamma' \). Clearly, \( x_s x_s D_r \in S \) for \( s \in Y_1, r \in Y_0 \setminus i \). Note that \([x_s x_s D_r, D_{ij}(x^{(q_{\epsilon_i})})] = 0\). Applying \( \phi \) to this equation, one has
\[
[x_s x_s D_r, \lambda \Gamma'] + \phi(x_s x_s D_r, D_{ij}(x^{(q_{\epsilon_i})})) = 0. \tag{4.3.19}
\]

Note that \([x_s x_s D_r, \lambda \Gamma'] = -2 \lambda x_s x_s D_r \) and \( \phi(x_s x_s D_r) \in S_{-1} \). Then by (4.3.19), \( \lambda = 0 \). Thus, (4.3.17) holds for all \( q \) and therefore, \( \phi(Q) = 0 \).

We next prove that \( \phi(R) = 0 \). Since \( R \subseteq S_1 \), \( zd(\phi) \leq -2 \), it suffices to consider the case that \( zd(\phi) = -2 \). Note that \( \phi(S_1) \subseteq S_{-1} \).

For \( i \in Y_0, k, l \in Y_1 \), choose \( q \in Y_1 \setminus \{k, l\} \). Then \( m \Gamma_q + \Gamma'' \in S_0 \) and
\[
[m \Gamma_q + \Gamma'', D_{il}(x^{(2\epsilon_i)}x_k)] = D_{il}(x^{(2\epsilon_i)}x_k).
\]

Applying \( \phi \), one gets
\[
\phi(D_{il}(x^{(2\epsilon_i)}x_k)) = [m \Gamma_q + \Gamma'', \phi(D_{il}(x^{(2\epsilon_i)}x_k))] = -\phi(D_{il}(x^{(2\epsilon_i)}x_k)).
\]

Since \( \phi(D_{il}(x^{(2\epsilon_i)}x_k)) \in W_{-1} \),
\[
\phi(D_{il}(x^{(2\epsilon_i)}x_k)) = 0 \quad \text{for all} \quad i \in Y_0, k, l \in Y_1.
\]

Note that \( n \Gamma_q + \Gamma' \in S_0 \) for \( q \in Y_0 \) and that
\[
[n \Gamma_q + \Gamma', D_{ji}(x_j x_k x_l)] = (2 - n \delta_{qj}) D_{ji}(x_j x_k D_l) \quad \text{for} \quad i, j \in Y_0, k, l \in Y_1. \tag{4.3.20}
\]

Assume that
\[
\phi(D_{ji}(x_j x_k x_l)) = \sum_{r \in Y_0} a_r D_r \quad \text{where} \quad a_r \in \mathbb{F}.
\]

Then we obtain from (4.3.20) that
\[
-a_d D_q = (2 - n \delta_{qj}) \sum_{r \in Y_0} a_r D_r.
\]

Since \( p \neq 3 \), it follows that \( a_r = 0 \) for all \( r \in Y_0 \).

This proves that
\[
\phi(D_{ji}(x_j x_k x_l)) = 0 \quad \text{for all} \quad i, j \in Y_0, k, l \in Y_1.
\]
By Proposition 2.2.3, $\phi = 0$. 

We are in the position to determine the homogeneous derivations of $\mathbb{Z}$-degree $< -1$ of $S$ to $W$. We first give the following fact.

**Proposition 4.3.4.** Suppose that $t > 1$ is not any $p$-power. Then $\text{Der}_{-t}(S, W) = 0$. In particular, $\text{Der}_{-t}(S) = 0$.

**Proof.** Let $\phi \in \text{Der}_{-t}(S, W)$. In view of Lemma 4.3.3, it is sufficient to show that

$$
\phi(D_{ij}(x^{(t+1)\varepsilon_i})) = 0 \quad \text{for all } i, j \in Y_0. \tag{4.3.21}
$$

We treat two cases separately.

**Case (i):** $t \not\equiv 0 \pmod{p}$. Recall $\Gamma'' = \sum_{r \in Y_0} \Gamma_r$. Clearly, $m\Gamma_k + \Gamma'' \in S_0$ for any $k \in Y_1$.

By Lemma 2.1.2(viii),

$$
[m\Gamma_k + \Gamma'', D_{ij}(x^{(t+1)\varepsilon_i})] = (t - 1)D_{ij}(x^{(t+1)\varepsilon_i}).
$$

Applying $\phi$ to the equation above, we have

$$
[m\Gamma_k + \Gamma'', \phi(D_{ij}(x^{(t+1)\varepsilon_i}))] = (t - 1)\phi(D_{ij}(x^{(t+1)\varepsilon_i})). \tag{4.3.22}
$$

On the other hand, since $\phi(D_{ij}(x^{(t+1)\varepsilon_i})) \in S_{-1}$,

$$
[m\Gamma_k + \Gamma'', \phi(D_{ij}(x^{(t+1)\varepsilon_i}))] = -\phi(D_{ij}(x^{(t+1)\varepsilon_i})). \tag{4.3.23}
$$

A comparison of (4.3.22) and (4.3.23) shows that (4.3.11) holds, since $t \not\equiv 0 \pmod{p}$.

**Case (ii):** $t \equiv 0 \pmod{p}$. Write $t$ to be the $p$-adic expression

$$
t = \sum_{s=1}^{r} a_s p^s \quad \text{where } 0 \leq a_s < p \text{ and } a_r \neq 0.
$$

Note that

$$
zd(D_{ij}(x^{((t-p^r+2)\varepsilon_j)})) = t - p^r < t - 2;
$$

$$
zd(D_{ij}(x^{(p^r \varepsilon_j)} x_i)) = p^r - 1 < t - 2,
$$

since $t$ is not any $p$-power. Then

$$
\phi(D_{ij}(x^{((t-p^r+2)\varepsilon_j)})) = \phi(D_{ij}(x^{(p^r \varepsilon_j)} x_i)) = 0. \tag{4.3.24}
$$

Direct computation shows that (by using the fact that $\binom{t}{p^r-1} \equiv 0 \pmod{p}$)

$$
[D_{ij}(x^{((t-p^r+2)\varepsilon_j)}), D_{ij}(x^{(p^r \varepsilon_j)} x_i)] = -\binom{t+1}{p^r} D_{ij}(x^{((t+1)\varepsilon_j)}). \tag{4.3.25}
$$

Note that $\binom{t+1}{p^r} \not\equiv 0 \pmod{p}$. Applying $\phi$ to (4.3.25), we obtain (4.3.21) by using (4.3.24).

Now we give the following
Proposition 4.3.5. Let $t = p^r$, $r > 0$. Then $\text{Der}_{-t}(\mathcal{S}, W) = \text{span}_\mathbb{F}\{(\text{ad}D_i)^t | i \in Y_0\}$. In particular, $\text{Der}_{-t}(\mathcal{S}) = \text{span}_\mathbb{F}\{(\text{ad}D_i)^t | i \in Y_0\}$.

Proof. Clearly, $(\text{ad}D_i)^p$ is a derivation of $\mathbb{Z}$-degree $-p^r$ for any $i \in Y_0$ and $r \in \mathbb{N}$. Let $\phi \in \text{Der}_{-t}(\mathcal{S}, W)$. Consider the action of $\phi$ on the element $D_{ij}(x^{((t+1)\epsilon_i)})$ for $i, j \in Y_0$. Note that $zd(\phi(D_{ij}(x^{((t+1)\epsilon_i)}))) = -1$. Suppose that

$$\phi(D_{ij}(x^{((t+1)\epsilon_i)})) = \sum_{r \in Y_0} a_{ijr} D_r \quad \text{where } a_{ijr} \in \mathbb{F}. \quad (4.3.26)$$

For any $s \in Y_0 \setminus j$, by Lemma 2.2.2, $\Gamma_k + \Gamma_s \in \mathcal{S}_0$ for $k \in Y_1$. Moreover,

$$[\Gamma_k + \Gamma_s, D_{ij}(x^{((t+1)\epsilon_i)})] = \delta_{si} \binom{t}{1} x^{(t \epsilon_i)} D_j = 0,$$

since $\binom{t}{1} \equiv 0 \pmod{p}$. Applying $\phi$ to the equation above, we have

$$a_{ijr} = 0 \quad \text{for } s \in Y_0 \setminus j.$$

Therefore, we obtain from that (4.3.26) that

$$\phi(D_{ij}(x^{((t+1)\epsilon_i)})) = a_{ijj} D_j. \quad (4.3.27)$$

Observe that

$$[D_{ij}(x^{((t+1)\epsilon_i)}), x_j D_r] = D_{ir}(x^{((t+1)\epsilon_i)}) \quad \text{for } i, j, r \in Y_0 \text{ with } r \neq i, j.$$

Applying $\phi$, we obtain from (4.3.27) that $[a_{ijj} D_j, x_j D_r] = a_{irr} D_r$. Consequently,

$$a_{ijj} = a_{irr}, \quad i \neq j, i \neq r.$$

Write $a_i := a_{irr}$ for $r \in Y_0 \setminus i$. Put

$$\psi := \phi - \sum_{r \in Y_0} a_r (\text{ad}D_r)^t.$$

Then $\psi \in \text{Der}_{-t}(\mathcal{S}, W)$ and for all $i, j \in Y_0$ with $i \neq j$, we have

$$\psi(D_{ij}(x^{((t+1)\epsilon_i)})) = \phi(D_{ij}(x^{((t+1)\epsilon_i)})) - \sum_{r \in Y_0} a_r (\text{ad}D_r)^t(D_{ij}(x^{((t+1)\epsilon_i)}))$$

$$= a_{ijj} D_j - a_i D_j$$

$$= a_i D_j - a_i D_j$$

$$= 0.$$

Lemma 4.3.3 ensures that $\psi = 0$; that is, $\phi = \sum_{r \in Y_0} a_r (\text{ad}D_r)^t$. The proof is complete. \qed

Now we can describe the derivation space $\text{Der}(\mathcal{S}, W)$ and the derivation algebra $\text{Der}(\mathcal{S})$. 
Theorem 4.3.6. \( \text{Der}(S, W) = \text{ad}(W) \oplus \text{span}_F \{(\text{ad}D_i)^{p_{ e_i}} \mid i \in Y_0, 1 \leq r_i < t_i \} \).

Proof. This is a direct consequence of Propositions 4.2.9, 4.3.2, 4.3.4 and 4.3.5. The proof is complete. \( \Box \)

Theorem 4.3.7. \( \text{Der}(S) = \text{ad}(\mathfrak{S} + \mathcal{T}) \oplus \text{span}_F \{(\text{ad}D_i)^{p_{ e_i}} \mid i \in Y_0, 1 \leq r_i < t_i \} \).

Proof. This is a direct consequence of Propositions 4.2.10, 4.3.2, 4.3.4 and 4.3.5. \( \Box \)

5. Outer derivation algebras

Let \( g \) be a Lie algebra. Denote by \( \text{Der}_{\text{out}}(g) := \text{Der}(g)/\text{ad}g \) the outer derivation algebra of \( g \). Using the results obtained in Sections 3 and 4, we shall determine the outer derivation algebras of \( W \) and \( S \).

Recall the our notations \( \mathfrak{A} := \mathfrak{A}(m, n; \mathbb{L}) \), \( W := W(m, n; \mathbb{L}) \), and \( S := S(m, n; \mathbb{L}) \), where \( m, n \) are integers at least 3 and \( \mathbb{L} := (t_1, t_2, \ldots, t_m) \in \mathbb{N}^m \) is an \( m \)-tuple of positive integers. We use still the symbols \( p_i := p_{ e_i} - 1 \), \( \pi := (\pi_1, \pi_2, \ldots, \pi_m) \in \mathbb{N}^m \).

Recall the canonical torus of \( W \), denoted by \( \mathcal{T} := \text{span}_F \{ \Gamma_r \mid r \in Y \} \), where \( \Gamma_r := x_r D_r \).

Since the full superderivation algebra \( \text{Der}(\mathfrak{A}) \) is a restricted Lie superalgebra (see [6] and [12]), one see easily that \( D_i^{p_{ e_i}} \in \text{Der}(\mathfrak{A}) \) and \( \text{ad}D_i^{p_{ e_i}} = (\text{ad}D_i)^{p_{ e_i}} \) for any \( i \in Y_0 \) and \( r \in \mathbb{N}_0 \), where \( \text{ad} \) is the adjoint representation of the Lie superalgebra \( \text{Der}(\mathfrak{A}) \). Put \( \mathcal{J} := \text{span}_F \{ D_i^{p_{ e_i}} \mid i \in Y_0, 1 \leq r_i < t_i \} \). Note that both \( \mathcal{T} \) and \( \mathcal{J} \) are contained in the even part of the Lie superalgebra \( \text{Der}(\mathfrak{A}) \).

5.1. The outer derivation algebra of \( W \)

Lemma 5.1.1. \( \mathcal{J} \) is a \( (\sum_{i \in Y_0} t_i - m) \)-dimensional abelian Lie-subalgebra of \( \text{Der}(\mathfrak{A}) \) and \( [\mathcal{J}, \mathcal{T}] = 0 \).

Proof. Since \( [D_i, D_j] = 0 \), \( \mathcal{J} \) is abelian. By the formula (1.2.1), \( [\mathcal{J}, \mathcal{T}] = 0 \). Applying \( D_i^{p_{ e_i}} \) to all basis elements of \( \mathfrak{A} \) of the form \( x^{(\pi_j, \xi_j)} \), one may show that \( \{ D_i^{p_{ e_i}} \mid i \in Y_0, 1 \leq r_i < t_i \} \) is \( F \)-linear independent. The proof is complete. \( \Box \)

Now we can characterize the outer derivation algebra of \( W \).

Theorem 5.1.2. \( \text{Der}_{\text{out}}(W) \) is an abelian Lie algebra of dimension \( \sum_{i \in Y_0} t_i - m \).

Proof. This is a direct consequence of Theorem 3.2.11 and Lemma 5.1.1. \( \Box \)

As a direct consequence of Theorem 5.1.2, we give the dimension formula of the derivation algebra of \( W \).

Corollary 5.1.3. \( \dim_F(\text{Der}(W)) = (m + n)2^{n-1}p^{\sum_{i \in Y_0} t_i} + \sum_{i \in Y_0} t_i - m \).

Proof. Note that \( \dim_F W = (m + n)2^{n-1}p^{\sum_{i \in Y_0} t_i} \). Since \( W \) is centerless, the dimension formula is a direct consequence of Theorem 5.1.2. \( \Box \)
5.2. The outer derivation algebra of $S$

To study the outer derivation algebra of $S$, we first establish the following lemma.

**Lemma 5.2.1.** (i) $\mathcal{T} \subset \text{Nor}_{W}(\overline{S})$; (ii) $\mathcal{J} \subset \text{Nor}_{\text{Der}(\mathfrak{A})}(\overline{S})$; (iii) $\mathcal{T} \subset \text{Nor}_{W}(S)$; (iv) $\mathcal{J} \subset \text{Nor}_{\text{Der}(\mathfrak{A})}(S)$.

**Proof.** (i) Since the divergence is a derivation of $W$ to $\mathfrak{A}$, we have

$$\text{div}(\{\Gamma_r, E\}) = \Gamma_r(\text{div}(E)) \quad \text{for all } r \in Y, E \in W.$$ 

The assertion follows.

(ii) Note that $D_i^{p,i}$ and $D_s$ commute. Using the formula (1.2.1), we have

$$[D_i^{p,i}, fD_s] = D_i^{p,i}(f)D_s \quad \text{where } i \in Y_0, s \in Y, f \in \mathfrak{A}.$$ 

Furthermore, $\text{div}([D_i^{p,i}, fD_s]) = D_i^{p,i}(\text{div}(fD_s))$. Since $\text{div}$ is linear, (ii) holds.

(iv) Just as in (ii), one may obtain that $[D_i^{p,i}, D_{rs}(f)] = D_{rs}(D_i^{p,i}(f))$ for $r, s \in Y, f \in \mathfrak{A}$. Therefore, (iv) holds.

(iii) The proof is analogous to the one of (iv). \hfill \square

Let $\tilde{S} := \overline{S} + \mathcal{T} + \mathcal{J}$. By Lemma 5.1.1 and 5.2.1(i), (ii), $\tilde{S}$ is a subalgebra of $\text{Der}(\mathfrak{A})$. We need the following lemma.

**Lemma 5.2.2.** $S$ is an ideal of $\overline{S}$; in particular, $S$ is an ideal of $\tilde{S}$.

**Proof.** The first assertion is direct, since $S$ is an ideal of $\overline{S}$. Then the second follows from Lemma 5.2.1(iii) and (iv). \hfill \square

**Proposition 5.2.3.** $\text{Der}_{\text{out}}(S) \cong \tilde{S}/S$.

**Proof.** By Lemma 5.2.2, the mapping $\tilde{S} \rightarrow \text{Der}(S)$, $E \mapsto (\text{ad}E)|_S$ is well defined. By Theorem 4.3.7, it is an epimorphism of Lie algebras. Assert that it is also injective. It suffices to show that the centralizer $C_{\tilde{S}}(S)$ of $S$ in $S$ is trivial.

Clearly, $C_{W}(S) \subset \mathcal{G}$, since $S_{-1} = W_{-1}$. Then $C_{W}(S) \subset C_{\mathcal{G}}(S) \subset C_{\mathcal{G}}(S_0)$. Using Lemma 2.1.2, one may verify by an elementary computation that $C_{\mathcal{G}}(S_0) = 0$. Consequently, $C_{W}(S) = 0$ and therefore, $C_{\overline{S} + \mathcal{T}}(S) = 0$. On the other hand, using the identity that $[D_i^{p,i}, D_{rs}(f)] = D_{rs}(D_i^{p,i}(f))$ for $r, s \in Y, f \in \mathfrak{A}(m)$, one may easily show that $C_{\mathcal{J}}(S) = 0$.

Combining $C_{\overline{S} + \mathcal{T}}(S) = 0$ with $C_{\mathcal{J}}(S) = 0$, one may obtain that $C_{\tilde{S}}(S) = 0$, since $\overline{S}$ is spanned by certain homogeneous elements of $\mathbb{Z}$-degree at least $-1$, but, $\mathcal{J}$ is spanned by certain elements of $\mathbb{Z}$-degree less than $-1$. The proof is complete. \hfill \square

We have the following dimension formula for the derivation algebra of $S$.

**Corollary 5.2.4.**

$$\dim_{F}(\text{Der}(S)) = \begin{cases} (m+n-1)2^{n-1}p^{\sum_{i \in Y_0} t_i} + \sum_{i \in Y_0} t_i - m + 2 & \text{if } n \text{ even;} \\ (m+n-1)2^{n-1}p^{\sum_{i \in Y_0} t_i} + \sum_{i \in Y_0} t_i - m + 1 & \text{if } n \text{ odd.} \end{cases}$$
Proof. By the proof of Proposition 5.2.3, Der($S$) $\cong \tilde{S}$. Note that $\tilde{S} = S \oplus F \cdot \Gamma_{1'} \oplus J$. By Lemma 5.1.1, it suffices to determine the dimension of $\tilde{S}$. By the definition of the divergence, it is easily seen that
\[
\text{div}(\mathcal{W}) = \text{span}_F \{ x^\alpha x^u \mid \alpha \in \Lambda, u \in \mathbb{B}; |\alpha| + |u| < |\pi| + n \}.
\]
Therefore, $\dim_F(\text{div}(\mathcal{W})) = 2^{n-1}p^{\sum_{i \in Y_0} t_i} - 1$, if $n$ is even; $\dim_F(\text{div}(\mathcal{W})) = 2^{n-1}p^{\sum_{i \in Y_0} t_i}$, if $n$ is odd. Note that $\tilde{S} = \ker(\text{div } \mathcal{W})$ and that $\dim_F(\mathcal{W}) = (m+n)2^{n-1}p^{\sum_{i \in Y_0} t_i}$. Then
\[
\dim_F(\tilde{S}) = \dim_F(\mathcal{W}) - \dim_F(\text{div}(\mathcal{W})) = \left\{ \begin{array}{ll}
(m+n-1)2^{n-1}p^{\sum_{i \in Y_0} t_i} + 1 & \text{if } n \text{ is even;} \\
(m+n-1)2^{n-1}p^{\sum_{i \in Y_0} t_i} & \text{if } n \text{ is odd.}
\end{array} \right.
\]
The proof is complete. \qed

We shall describe explicitly the structure of the outer derivation algebra of $S$. If $n(= |\omega|)$ is even, put $\mathfrak{P} := \text{span}\{ x^{(\pi-\pi \vdash)} x^\omega D_i \mid i \in Y_0 \}$. Then $\mathfrak{P}$ is an abelian subalgebra of $\mathcal{W}$.

Let $V$ be an abelian Lie algebra of dimension $\sum_{i \in Y_0} t_i$. Let $A \in \text{End}_F(V)$ be semisimple with eigenvalues 0 and 1 such that $\dim_F V_0(A) = \sum_{i \in Y_0} t_i - m$ and $\dim_F V_1(A) = m$. Denote the semidirect product by $\mathcal{L}_S := F \cdot A \ltimes V$. Then $\mathcal{L}_S$ is a metabelian Lie algebra of dimension $1 + \sum_{i \in Y_0} t_i$.

**Theorem 5.2.5.** (i) If $n$ is odd then Der$_{\text{out}}(S)$ is an abelian Lie algebra of dimension $1 + \sum_{i \in Y_0} t_i - m$.

(ii) If $n$ is even then Der$_{\text{out}}(S)$ is isomorphic to the metabelian Lie algebra $\mathcal{L}_S$ of dimension $1 + \sum_{i \in Y_0} t_i$.

**Proof.** According to [8, Proposition 2.8], we know that $\tilde{S} = S \oplus \mathfrak{P}$ if $n$ is even; $\tilde{S} = S$ if $n$ is odd. Consequently, $\dim_F(S) = \dim_F(\tilde{S}) - m$, if $n$ is even; $\dim_F(S) = \dim_F(\tilde{S})$, if $n$ is odd. Then by Proposition 5.2.3, we have
\[
\dim_F(\text{Der}_{\text{out}}(S)) = \left\{ \begin{array}{ll}
1 + \sum_{i \in Y_0} t_i & \text{if } n \text{ is even;} \\
1 + \sum_{i \in Y_0} t_i - m & \text{if } n \text{ is odd.}
\end{array} \right.
\]

Again by Proposition 5.2.3, if $n$ is odd then Der$_{\text{out}}(S) \cong F \cdot \Gamma_{1'} \oplus \mathfrak{J}$ is an abelian Lie algebra, since we have noted that $\tilde{S} = S \oplus F \cdot \Gamma_{1'} \oplus \mathfrak{J}$.

It remains to consider the case that $n$ is even. By Proposition 5.2.3, we obtain in this case that
\[
\text{Der}_{\text{out}}(S) \cong (S \oplus F \cdot \Gamma_{1'} \oplus \mathfrak{J} \oplus \mathfrak{P})/S = (S \oplus F \cdot \Gamma_{1'})/(S \oplus (S + J))/S \oplus (S + \mathfrak{P})/S.
\]

Note that $[\Gamma_{1'}, x^{(\pi-\pi \vdash)} x^\omega D_i] = x^{(\pi-\pi \vdash)} x^\omega D_i, [\Gamma_{1'}, \mathfrak{J}] = 0, [\mathfrak{J}, \mathfrak{P}] \subset S$. Now, a straightforward verification shows that Der$_{\text{out}}(S) \cong \mathcal{L}_S$. The proof is complete. \qed

**Remark 5.2.6.** According to the known results on the outer superderivation algebra of Lie superalgebra $W$ and $S$ (see [8, Theorems 2.4 and 2.12]), we conclude from Theorem 5.1.2 that, for the generalized Witt Lie superalgebra $W(m,n;\mathfrak{L})$, the outer superderivation algebra coincides with the outer derivation algebra of the even part $W_{\mathfrak{P}}$. We can also conclude from Theorem 5.2.5 that, for the special Lie superalgebra $S(m,n;\mathfrak{L})$, the same conclusion holds (that is, Der$_{\text{out}}(S) \cong \text{Der}_{\text{out}}(S_{\mathfrak{P}})$) if and only if $n$ is even.
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