ONE-COMPONENT INNER FUNCTIONS II

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Abstract. We continue our study of the set $I_c$ of inner functions $u$ in $H^\infty$ with the property that there is $\eta \in ]0,1[$ such that the level set $\Omega_u(\eta) := \{ z \in \mathbb{D} : |u(z)| < \eta \}$ is connected. These functions are called one-component inner functions. Here we show that the composition of two one-component inner functions is again in $I_c$. We also give conditions under which a factor of one-component inner function belongs to $I_c$.

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1. Introduction

One-component inner functions, the collection of which we denote by $I_c$, were first studied by B. Cohn [4] in connection with embedding theorems and Carleson-measures. Recall that an inner function $u$ in $H^\infty$ is said to be a one-component inner function if there is $\eta \in ]0,1[$ such that the level set (also called sublevel set or filled level set) $\Omega_u(\eta) := \{ z \in \mathbb{D} : |u(z)| < \eta \}$ is connected. Unimodular constants are considered to belong to $I_c$. It was shown in [4, p. 355] for instance, that arclength on $\{ z \in \mathbb{D} : |u(z)| = \varepsilon \}$ is a Carleson measure whenever

$$\Omega_u(\eta) = \{ z \in \mathbb{D} : |u(z)| < \eta \}$$

is connected and $\eta < \varepsilon < 1$. A detailed study of the class $I_c$ was undertaken by A.B. Aleksandrov [1]. Classes of explicit examples of one-component inner functions were given by the present authors in [3]. The most fundamental ones are finite Blaschke products and singular inner functions $S_\mu$ with finite singularity set (or spectrum), $\text{Sing } S_\mu$. Infinite interpolating Blaschke products with real zeros $(x_n)$ satisfying $0 < \eta_1 \leq \rho(x_n, x_{n+1}) \leq \eta_2 < 1$ (where $\rho$ is the pseudohyperbolic distance in $\mathbb{D}$) were also shown to belong to $I_c$. On the other hand, no finite product of thin interpolating Blaschke products (these are (infinite) Blaschke products $B$ whose zeros $(z_n)$ satisfy $\lim_n \prod_{k \neq n} \rho(z_n, z_k) = 1$), can be in $I_c$. It also turned out the class of one-component inner functions is invariant under taking finite products. In the present note, we are considering when a factor of a one-component inner function is in $I_c$ again. A sufficient criterion is provided. On the other hand, as it is shown, there exist two non one-component inner

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functions $u$ and $v$ such that $uv \in \mathcal{I}_c$. Our main result will show that the class of one-component inner functions is also invariant under taking compositions, generalizing special cases dealt with in [3]. The results of this note stem from December 2016. Meanwhile (May 2018) a manuscript by A. Reijonen [10] provides other classes of one-component inner functions.

2. Main tools

Our results will mainly be based on the following known results which we recall for citational reasons.

**Lemma 2.1.** Given a non-constant inner function $u$ in $H^\infty$ and $\eta \in ]0, 1[. let $\Omega := \Omega_u(\eta) = \{z \in \mathbb{D}: |u(z)| < \eta\}$ be a level set. Suppose that $\Omega_0$ is a component (=maximal connected subset) of $\Omega$. Then

1. $\Omega_0$ is a simply connected domain; that is, $\mathbb{C} \setminus \Omega_0$ has no bounded components.
2. $\inf_{\Omega_0} |u| = 0$.
3. Either $\overline{\Omega_0} \subseteq \mathbb{D}$ or $\overline{\Omega_0} \cap \mathbb{T}$ has measure zero.

A detailed proof of parts (1) and (2) is given in [3]; part (3) is in [2, p. 733].

Recall that the spectrum $\text{Sing}(u)$ of an inner function $u$ is the set of all boundary points $\zeta$ for which $u$ does not admit a holomorphic extension; or equivalently, for which $\text{Cl}(u, \zeta) = \mathbb{D}$, where

$$\text{Cl}(u, \zeta) = \{w \in \mathbb{C} : \exists(z_n) \in \mathbb{D}^\mathbb{N}, \lim z_n = \zeta \text{ and } \lim u(z_n) = w\}$$

is the cluster set of $u$ at $\zeta$ (see [6, p. 80]).

The pseudohyperbolic disk of center $z_0 \in \mathbb{D}$ and radius $r$ is denoted by $D_r(z_0, r)$.

**Theorem 2.2** (Aleksandrov). [1, Theorem 1.11 and Remark 2, p. 2915] Let $u$ be an inner function. The following assertions are equivalent:

1. $u \in \mathcal{I}_c$.
2. There is a constant $C > 0$ such that for every $\zeta \in \mathbb{T} \setminus \text{Sing}(u)$ we have
   
   $$i) \ |u''(\zeta)| \leq C \ |u'(\zeta)|^2,$$
   
   and
   
   $$ii) \ \liminf_{r \to 1} |u(r\zeta)| < 1 \text{ for all } \zeta \in \text{Sing}(u).$$

Note that, due to this theorem, $u \in \mathcal{I}_c$ necessarily implies that $\text{Sing}(u)$ has measure zero.

3. Splitting off factors

In this section we give a condition under which a factor of a one-component inner function is in $\mathcal{I}_c$ again. Recall from [3] that for the atomic inner function
We claim that $\sum$ and $\phi$ are not in the class of one-component inner functions. Any inner function $u$ following result tells us that one can split off finitely many zeros without leaving the class of one-component inner functions. Any inner function $u$ has the form $u = BS\mu$, where $B$ is a Blaschke product and $S\mu$ a singular inner function

$$S\mu(z) := \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)\right)$$

associated with a positive Borel measure $\mu$ which is singular with respect to Lebesgue measure on $\mathbb{T}$.

**Proposition 3.1.** Let $\Theta \in \mathcal{I}_c$ and $a \in \mathbb{D}$. If $\Theta(a) = 0$, then $v := \Theta/\varphi_a \in \mathcal{I}_c$.

**Proof.** Note that $\Theta = \varphi_a v$. We may assume that $v$ is not constant, otherwise we are done. Choose $\eta \in [0, 1]$ so that $\Omega_{\Theta}(\eta)$ is connected. Let

$$\delta := \inf\{|\varphi_a(z)| : |\Theta(z)| = \eta\}.$$

We claim that $\eta < \delta < 1$. In fact, since the set $L := \{z \in \mathbb{D} : |\Theta(z)| = \eta\}$ is not empty, and $|\varphi| < 1$ in $\mathbb{D}$, we see that $\eta < 1$. Moreover, if $z_0 \in L$, then

$$L' := \{|\varphi_a(z)| : |\Theta(z)| = \eta, |\varphi_a(z)| \leq |\varphi_a(z_0)|\}$$

is a compact set in $[0, 1]$, and so

$$\inf\{|\varphi_a(z)| : |\Theta(z)| = \eta\} = \inf L' = \min L'.$$

Hence $\delta = |\varphi_a(z_1)|$ for some $z_1 \in L$. Since $v$ is not a unimodular constant, we deduce from $|\Theta(z_1)| = |\varphi_a(z_1)||v(z_1)|$ that $\eta < \delta$.

Consequently, if $|\Theta(z)| = \eta$,

$$(3.1) \quad |v(z)| = \frac{|\Theta(z)|}{|\varphi_a(z)|} \leq \frac{\eta}{\delta} := \eta' < 1.$$  

We claim that

$$\Omega_v(\eta) \subseteq \Omega_{\Theta}(\eta) \subseteq \Omega_v(\eta').$$

Notice that the first inclusion is obvious. To verify the second inclusion, let $z_0 \in \Omega_{\Theta}(\eta)$. We discuss three cases: $\rho(z_0, a) < \delta, \rho(z_0, a) = \delta$ and $\rho(z_0, a) > \delta$.

To this end, we first note that $D_\rho(a, \delta) \subseteq \Omega_{\Theta}(\eta)$. In fact, if $\rho(a, z) = |\varphi_a(z)| < \delta$, then $|\Theta(z)| < \eta$, since otherwise $\Theta(a) = 0$ implies the existence of $z_0 \in D_\rho(a, \delta)$ with $|\Theta(z_0)| = \eta$ and so, by the definition of $\delta$, $|\varphi_a(z)| \geq \delta$. An obvious contradiction.
Hence $|\Theta(z)| \leq \eta$ for $\rho(z,a) = \delta$. Thus (3.1) holds true for $z \in \partial D_\rho(a,\delta)$. By the maximum principle, $|v(z)| < \eta'$ on $D_\rho(a,\delta)$. If $\rho(z,a) \geq \delta$ and $|\Theta(z)| < \eta$, then, as in (3.1), $|v(z)| < \eta'$, too. We deduce that $\Omega_\Theta(\eta) \subseteq \Omega_v(\eta')$.

Now we are able to prove that $\Omega_v(\eta')$ is connected. Assuming the contrary, there would exist a component $\Omega_v$ of $\Omega_v(\eta')$ distinct (and so disjoint) from that containing the connected set $\Omega_\Theta(\eta)$. In particular, $|v| \geq |\Theta| \geq \eta$ on $\Omega_1$. By Lemma 2.1, $\inf_{\Omega_1} |v| = 0$; an obvious contradiction. 

The preceding result admits the following generalization.

**Proposition 3.2.** Let $u, v$ be two non-constant inner functions and put $\Theta = uv$. Suppose that

(i) $\Theta \in J_c$ and that $\eta \in ]0,1[$ is chosen so that $\Omega_\Theta(\eta)$ is connected.

(ii) $\sigma := \sup_{|\Theta|=\eta} |v| \in [\eta, 1[$ (or equivalently, $\delta := \inf_{|\Theta|=\eta} |u| \in ]\eta, 1[$).

Then $v \in J_c$. The assertion does not necessarily hold if $\sigma = 1$ (or, equivalently, if $\delta = \eta$).

**Proof.** Due to hypothesis (ii), we have the following estimate on $|\Theta| = \eta$:

(3.2)  
$$|u| = \frac{|\Theta|}{|v|} \geq \frac{\eta}{\sigma} = \delta.$$  

Note that $\delta \in ]\eta, 1[$. We claim that

(3.3)  
$$\Omega_u(\delta) \subseteq \Omega_\Theta(\eta) \cap \Omega_v(\sigma).$$

To this end, we first show that $|\Theta| < \eta$ on $\Omega_u(\delta)$. In fact, assuming the contrary, there exist $z_0 \in \Omega_u(\delta)$ such that $|\Theta(z_0)| \geq \eta$. Let $\Omega_0$ be that component of $\Omega_u(\delta)$ containing $z_0$. By Lemma 2.1 (2), $\inf_{\Omega_0} |u| = 0$. Since $u$ is a factor of $\Theta$, we conclude that there exists $z_1 \in \Omega_0 \subseteq \Omega_u(\delta)$ such that $|\Theta(z_1)| < \eta$. Thus, the connected set $\Omega_0$ meets $\{|\Theta| < \eta\}$ as well as its complement. Hence $\Omega_0$ meets the topological boundary of $\Omega_\Theta(\eta)$. Because $\Omega_0 \subseteq \mathbb{D}$, we obtain $z_2 \in \Omega_0$ such that $|\Theta(z_2)| = \eta$. Hence, by (ii), $|v(z_2)| \leq \sigma$ and so $|u(z_2)| \geq \delta$ by (3.2). Both assertions $|u(z_2)| \geq \delta$ and $z_2 \in \Omega_0 \subseteq \Omega_u(\delta)$ cannot hold. Thus our assumption right at the beginning of this paragraph was wrong. We deduce that

(3.4)  
$$\Omega_u(\delta) \subseteq \Omega_\Theta(\eta).$$

By continuity, this inclusion implies that $|\Theta| \leq \eta$ on $\{|u| = \delta\}$. Hence, for $|u(z)| = \delta$,

(3.5)  
$$|v(z)| = \frac{|\Theta(z)|}{|u(z)|} \leq \frac{\eta}{\delta} \overset{(3.2)}{=} \sigma.$$  

Now $\partial \Omega_u(\delta) \cap \mathbb{D} = \{|u| = \delta\}$. If $\Omega$ is a component of $\Omega_u(\delta)$ whose closure belongs to $\mathbb{D}$, then by the maximum principle and (3.5), $|v| < \sigma$ on $\Omega$. If $E := \overline{\Omega} \cap \mathbb{T} \neq \emptyset$, then $E$ has measure zero by Lemma 2.1 (3). The maximum principle
with exceptional points (see [2, p. 729] or [5]) now implies that $|v| < \sigma$ on $\Omega$. Consequently,

\begin{equation}
\Omega_u(\delta) \subseteq \Omega_v(\sigma).
\end{equation}

Thus (3.3) holds. Next we will deduce that

\begin{equation}
\Omega_v(\eta) \subseteq \Omega_\Theta(\eta) \subseteq \Omega_v(\sigma).
\end{equation}

To see this, observe that the first inclusion is obvious because $v$ is a factor of $\Theta$. To prove the second inclusion, we write the $\eta$-level set of $\Theta$ as

\begin{equation}
\Omega_\Theta(\eta) = \left( \Omega_\Theta(\eta) \cap \Omega_u(\delta) \right) \cup \left( \Omega_\Theta(\eta) \setminus \Omega_u(\delta) \right).
\end{equation}

By (3.6), the first set in this union is contained in $\Omega_v(\sigma)$. The second set is also contained in $\Omega_v(\sigma)$, because if $|u(z)| \geq \delta$ and $z \in \Omega_\Theta(\eta)$, then

\begin{equation}
|v(z)| = \frac{|\Theta(z)|}{|u(z)|} < \frac{\eta}{\delta} = \sigma.
\end{equation}

To sum up, we have shown that for every $z \in \Omega_\Theta(\eta)$ we have $|v(z)| < \sigma$ both in the case where $|u(z)| \leq \delta$ and $|u(z)| \geq \delta$. Thus

\begin{equation}
\Omega_\Theta(\eta) \subseteq \Omega_v(\sigma),
\end{equation}

and so, (3.7) holds. Using these inclusions (3.7), we are now able to prove that $\Omega_v(\sigma)$ is connected. Assuming the contrary, there would exist a component $\Omega_1$ of $\Omega_v(\sigma)$, distinct (and so disjoint) from that containing the connected set $\Omega_\Theta(\eta)$. In particular, $|v| \geq |\Theta| \geq \eta$ on $\Omega_1$. By Lemma 2.1 (2), $\inf_{\Omega_1} |v| = 0$; an obvious contradiction.

Finally we construct an example showing that in (ii) the parameter $\sigma$ cannot be taken to be 1. In fact, let $v$ be a thin interpolating Blaschke product with positive zeros clustering at 1, for example

$$v(z) = \prod_{n=1}^{\infty} \frac{1 - 1/n! - z}{1 - (1 - 1/n!)z},$$

and let $u(z) = S(z) := \exp[-(1+z)/(1-z)]$ be the atomic inner function. Then, by [3, Proposition 2.8], $\Theta = uv \in \mathcal{I}_c$. However, $v \notin \mathcal{I}_c$, [3, Corollary 21]. Thus, by the main assertion of this Proposition, $\sigma = \sup_{|\Theta| = \eta} |v| = 1$. (A direct proof of the assertion $\sigma$ can also be given using [7, p. 55]), by noticing that the boundary of the component $\Omega_\Theta(\eta)$ is a closed curve in $\mathbb{D} \cup \{1\}$.)

**Observation 3.3.** We know from [3, Proposition 2.9] that $u, v \in \mathcal{I}_c$ implies $uv \in \mathcal{I}_c$. Here is an example showing that neither $u$ nor $v$ must belong to $\mathcal{I}_c$ for $uv$ to be in $\mathcal{I}_c$. In fact, let $b$ be a thin Blaschke product with real zeros clustering at 1 and $-1$ (just consider $b(z) = v(z)v(-z)$, $v$ as above). Let $\tilde{u} := Sb$ and $\tilde{v}(z) := S(-z)b(z)$. Then $\Theta := \tilde{u}\tilde{v} \in \mathcal{I}_c$, because $\Theta(z) = (S(z)v^2(z))(S(-z)v^2(-z))$ is the product of two functions in $\mathcal{I}_c$ (same proof as in [3, Proposition 11]), but
neither $\tilde{u}$ nor $\tilde{v}$ belong to $\mathcal{I}_c$. This can be seen as follows: since $S(-1) = 1$, $\tilde{u} = Sb$ behaves as $b$ close to $-1$. Thus, for $\eta$ arbitrarily close to $1$, the level set $\Omega_\eta(\eta)$ is contained in a union of pairwise disjoint pseudohyperbolic disks $D_\rho(x_n, \eta^*)$, $n = 0, 1, 2, \ldots$, together with some tangential disk $D$ at 1, where $x_0 = 0$ and $x_n$ is the $n$-th negative zero of $b$ (this works similarly as in [3, Corollary 21] and [3, Proposition 11]).

4. Composition of one-component inner functions

In [3] we showed that for every finite Blaschke product $B$ and $\Theta \in \mathcal{I}_c$, the compositions $S \circ B \in \mathcal{I}_c$ and $B \circ \Theta \in \mathcal{I}_c$. Using the following standard Lemma 4.1, we will extend this to arbitrary one-component inner functions.

Lemma 4.1. 1) Let $B$ be a Blaschke product with zero sequence $(a_n)_{n \in \mathbb{N}}$. Then the following inequalities hold for $\xi \in \mathbb{T} \setminus \text{Sing}(B)$:

$$|B'(\xi)| = \sum_{n \in \mathbb{N}} \frac{1 - |a_n|^2}{|a_n - \xi|^2} \geq \frac{1 - |a_n_0|}{1 + |a_n_0|} > 0, \ \forall n_0 \in \mathbb{N}.$$

2) If $u$ is an inner function for which $\text{Sing}(u) \neq \mathbb{T}$, then

$$\delta_u := \inf\{|u'(\xi)| : \xi \in \mathbb{T} \setminus \text{Sing}(u)\} > 0.$$

Proof. 1) Just compute the logarithmic derivative $B'/B$ and note that on $\mathbb{T} \setminus \text{Sing}(B)$ the Blaschke product $B$ converges.

2) Let $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$. By Frostman’s theorem (see [6, p. 79]) there is $a \in \mathbb{D}$ such that $B := \varphi_a \circ u$ is a Blaschke product. Of course, $\text{Sing}(u) = \text{Sing}(B)$, $u = \varphi_a \circ B$ and $\varphi_a'(z) = -(1 - |a|^2)/(1 - \bar{a}z)^2$. Hence, for $\xi \in \mathbb{T} \setminus \text{Sing}(u)$,

$$|u'(\xi)| = |\varphi_a'(B(\xi))| |B'(\xi)| \geq \frac{1 - |a|^2}{|1 - \bar{a}B(\xi)|^2} \delta_B \geq \frac{1 - |a|}{1 + |a|} \delta_B > 0.$$

□

Theorem 4.2. If $u$ and $v$ are two non-constant inner functions in $\mathcal{I}_c$, then $u \circ v \in \mathcal{I}_c$.

Proof. As in [3], we shall use Aleksandrov’s criterion 2.2.

(1) Let $\Theta := u \circ v$. It is well known that $\Theta$ is an inner function again (see e.g. [9, p.83]). Now

$$\text{Sing}(\Theta) = \text{Sing}(v) \cup \{\xi \in \mathbb{T} \setminus \text{Sing}(v) : v(\xi) \in \text{Sing}(u)\}.$$ 

Since $v \in \mathcal{I}_c$, $\liminf_{r \to 1} |v(r\zeta)| < 1$ for every $\zeta \in \text{Sing}(v)$ (Theorem 2.2). Say $v(r_n \zeta) \to w_0 \in \mathbb{D}$. Then

$$\Theta(r_n \zeta) = u(v(r_n \zeta)) \to u(w_0) \in \mathbb{D}.$$
If $\xi \in \text{Sing}(\Theta) \setminus \text{Sing}(v)$, then $v(r\xi) \to v(\xi) = e^{i\theta} \in \text{Sing}(u)$ for some $\theta \in \mathbb{R}$. By Lemma 4.1, $v'(\xi) \neq 0$; hence $v$ is a conformal map in small neighborhoods of $\xi$; in particular, due to the angle conservation law, the curve $\gamma : r \mapsto v(r\xi)$ stays in a cone with aperture $0 < 2\sigma < \pi$ and cusp at $e^{i\theta} \in \text{Sing} u$. Since $u \in \mathcal{I}_c$, $\lim \inf |u(re^{i\theta})| < 1$. We claim that $\lim \inf |u(v(r\xi))| < 1$, too. To see this, choose a pseudohyperbolic radius $\rho_0$ so big that for some $r_0 \in ]0,1[$ the cone

$$C := \{ z \in \mathbb{D} : |z| \geq r_0, |\arg z - \theta| < \sigma \}$$

is entirely contained in the domain

$$V := \bigcup_{-1 < x < 1} D_\rho(x, \rho_0).$$

Note that by [8], the boundary of $V$ is the union of two arcs of circles cutting the line $\{ se^{i\theta} : s \in \mathbb{R} \}$ at $e^{i\theta}$ under an angle $\alpha$ with $\sigma < \alpha < \pi/2$ (see the figure, where we sketched the situation for $\theta = 0$).

Choose $r_n$ so that $\lim u(r_ne^{i\theta}) = a \in \mathbb{D}$. Then the curve $\gamma$ cuts the boundary of infinitely many disks $D(r_ne^{i\theta}, \rho_0)$ twice. But for $z \in D(r_ne^{i\theta}, \rho_0)$ we have

$$\frac{|u(z)| - |u(r_ne^{i\theta})|}{1 - |u(r_ne^{i\theta})||u(z)|} \leq \rho(u(z), u(r_ne^{i\theta})) \leq \rho(z, r_ne^{i\theta}) \leq \rho_0,$$

and so

$$|u(z)| \leq \frac{\rho_0 + |u(r_ne^{i\theta})|}{1 + |u(r_ne^{i\theta})|\rho_0}.$$ 

This clearly implies that

$$\lim \inf |u(v(r\xi))| < 1.$$
Consequently, \( \liminf |\Theta(r\xi)| < 1 \) for every \( \xi \in \text{Sing}(\Theta) \). Next we verify the first condition in Aleksandrov’s theorem.

\[
A := \frac{(u \circ v)''}{[(u \circ v)']^2} = \frac{u'' \circ v}{(u' \circ v)^2} \frac{v''}{(u' \circ v)^2 v''} = \frac{u'' \circ v}{(u' \circ v)^2} + \frac{1}{(u' \circ v)^2 v''}.
\]

If \( \zeta \in \mathbb{T} \setminus \text{Sing}(u \circ v) \), then \( |v(\zeta)| = 1 \) and \( \xi := v(\zeta) \not\in \text{Sing}(u) \). Since \( u, v \in \mathcal{I}_c \), we deduce from Lemma 4.1 and Aleksandrov’s theorem 2.2 that

\[
|A(\zeta)| \leq \sup_{\beta \notin \text{Sing} u} \frac{|u''(\beta)|}{|u'(\beta)|^2} + \frac{1}{\delta_u} \sup_{\alpha \notin \text{Sing} v} \frac{|v''(\alpha)|}{|v'(\alpha)|^2} =: C < \infty,
\]

where

\[
\delta_u := \inf \{|u'(\xi)| : \xi \in \mathbb{T} \setminus \text{Sing}(u)\}.
\]

Hence \( \Theta \in \mathcal{I}_c \). \( \square \)

**Theorem 4.3.** 1) Let \( E \subseteq \mathbb{T} \) be a closed finite set. Then there exists a one-component inner function \( u \) such that for some \( \eta_0 \in ]0, 1[ \) (and hence for all \( \eta \in ]\eta_0, 1[ \)) the associated level set \( \Omega_u(\eta) \) is connected and has the property that

\[
\overline{\Omega_u(\eta)} \cap \mathbb{T} = \text{Sing}(u) = E.
\]

2) There exists \( u \in \mathcal{I}_c \) such that \( \overline{\Omega_u(\eta)} \cap \mathbb{T} = \text{Sing}(u) \) is an infinite set.

**Proof.** 1) Let \( E = \{\lambda_1, \ldots, \lambda_N\} \) be finite. Then the function \( S_\mu \) given by

\[
S_\mu(z) = \prod_{j=1}^N \exp \left(-\frac{\lambda_j + z}{\lambda_j - z}\right)
\]

belongs to \( \mathcal{I}_c \) (by [3, Corollary 17]) and satisfies (4.3).

2) Let \( E = S^{-1}(1) \) be the countably infinite set of points where the atomic inner function \( S(z) = \exp(-(1 + z)/(1 - z)) \) takes the value 1, and let \( b \) be the interpolating Blaschke product with zeros \( 1 - 2^{-n} \). Then \( b \) and \( S \) belong to \( \mathcal{I}_c \) (see [3, Theorem 6]). By Theorem 4.2, \( u := b \circ S \in \mathcal{I}_c \). It is easy to see that \( \overline{\Omega_u(\eta)} \cap \mathbb{T} = \text{Sing}(u) = E \). The same holds true for \( S \circ b \) as well; just note that the argument function of \( b \) on \( \mathbb{T} \setminus \{1\} \) is unbounded when approaching 1 from both sides on the circle (see [6, p. 92]), so that \( b^{-1}(\{1\}) \) is infinite. Thus we have a singular inner function in \( \mathcal{I}_c \) with infinitely many singularities. \( \square \)

Finally, let us mention that for inner functions \( u \), we always have

\[
\text{Sing } u = \overline{\Omega_u(\eta)} \cap \mathbb{T},
\]

\( 0 < \eta < 1 \).
Questions 4.4. i) Given any countable closed subset $E$ of $\mathbb{T}$, does there exist $u \in \mathcal{J}_c$ such that $\Omega_u(\eta) \cap \mathbb{T} = E$?

ii) Is the set $\Omega_u(\eta) \cap \mathbb{T}$ necessarily countable whenever $u \in \mathcal{J}_c$? We don’t think so. As indicated by Carl Sundberg [11], the usual Cantor ternary set may be the support of some singular measure $\mu$ whose associated singular inner function $S_\mu$ belongs to $\mathcal{J}_c$.

iii) Give a description of those closed subsets $E$ of $\mathbb{T}$ such that for some singular inner function $S_\mu$ with $\text{Sing} \ S_\mu = E$ every inner factor of $S_\mu$ belongs to $\mathcal{J}_c$.

For example, finite subsets of $\mathbb{T}$ have this property [3, Corollary 17] .

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