Abstract—This paper investigates the enumeration of a family of perfect quaternary arrays (PQAs) from Zeng, et al.’s constructions. By deriving the conditions which result in distinct PQAs, the family size of Zeng, et al.’s constructions is determined, and distinct PQAs are deduced from the obtained conditions. Finally, two examples are given. The proposed distinct PQAs provide lots of candidates for applications to communications, radar, and so on.

Keywords—perfect quaternary array, periodic autocorrelation, distinct array, family size, n-dimensional array

I. INTRODUCTION
A perfect array means a n-dimensional function whose autocorrelation is impulse-like. Perfect arrays are widely applied to high-dimensional communications, time-frequency-coding, spatial correlation or map matching, built-in tests of VLSI-circuits, coded aperture imaging, phased array antennas, arrays of sound sources, radar, and so on [1]-[3]. In the existing literature, there are a large number of perfect binary arrays (PBAs) [3]-[9], perfect ternary arrays (PTAs) [10] [11], and perfect multi-phase arrays [12]. However, the constructions of perfect quaternary arrays (PQAs) only have a few. To the best of the author’s knowledge, Arasu and Launey gave a family of PQAs by making use of polynomial theory [13], and Zeng, et al proposed a method converting a PBA into a PQA [14]. With regard to the advances on perfect arrays, please refer to [15].

This paper follows [14] so as to solve the problem of the enumeration of Construction 2 in it. By deriving the existence conditions of distinct PQAs in Construction 2 in [14], this paper determines the family size of Construction 2, and distinct PQAs are deduced from the obtained conditions. More clearly, for given two n-dimensional PQAs with size \( N_1 \times \cdots \times N_{k-1} \times N_k \times N_{k+1} \times \cdots \times N_n \), where the positive integer \( N_k \) is odd, the family size of Construction 2 arrives at \( N_k \), in other words, Construction 2 can produce \( N_k \) distinct PQAs. Due to the fact that the users’ number in a system is decided by the number of perfect arrays employed, Construction 2 in [14] can provide lots of candidates for applications.

Incidentally, it should be noted that PQAs theory is fundamental, since constructions of perfect arrays over some constellations, such as 16-QAM constellation, depend on their results [16].

The remainder of this paper is organized as follows. In Sect. II, the necessary concepts are recalled. In Sect. III, Zeng, et al’s constructions are briefly given. In the following section, the main results are stated. Two examples will appear in Sect. V. Finally, we conclude this paper in Sect. VI.

II. BASIC CONCEPTS
A n-dimensional function is said to be an array, denoted traditionally by

\[
A = [a_{i_1, i_2, \ldots, i_n}] \quad 0 \leq i_k \leq N_k - 1, 1 \leq k \leq n,
\]

(1)

where \( N_1 \times \cdots \times N_{k-1} \times N_k \times N_{k+1} \times \cdots \times N_n \) is referred to as size.

For two arrays \( A \) and \( B \), we define their correlation function by

\[
R_{A, B}(\tau_1, \tau_2, \ldots, \tau_n)
= \sum_{i_1=0}^{N_1-1} \sum_{i_2=0}^{N_2-1} \ldots \sum_{i_n=0}^{N_n-1} a_{i_1, i_2, \ldots, i_n} b_{i_1, i_2, \ldots, i_n}^{*} e^{i \pi (\tau_1 i_1 + \tau_2 i_2 + \cdots + \tau_n i_n)},
\]

(2)

where the symbol “*” denotes the complex conjugate, and the addition “\( i_k + \tau_k \) (1 \leq k \leq n)” is performed modulo \( N_k \). If \( A = B \), we call \( R_{A, A}(\tau_1, \tau_2, \ldots, \tau_n) \) an periodic autocorrelation function, otherwise, a periodic cross-correlation function.

If the autocorrelation function of an array \( A \) satisfies
where the addition \( i_k + \mu_k \) \((1 \leq k \leq n)\) is performed modulo \( N_k \).

For two arrays \( A \) and \( B \), if there are \( n \) integers \( \mu_k \)'s \((1 \leq k \leq n)\) so as to satisfy
\[
T(\mu_1, \mu_2, \ldots, \mu_n) A = B,
\]
we say that these two arrays are cyclical shift equivalence, otherwise, distinct arrays.

Let \( A = [a_{i_1, i_2, \ldots, i_n}] \) and \( B = [b_{i_1, i_2, \ldots, i_n}] \) be two binary arrays with size \( N_1 \times \cdots \times N_k \times N_{k+1} \times \cdots \times N_n \). We construct two quaternary arrays by
\[
Q_l' = [q_{i_1, i_2, \ldots, i_n}^l] \quad (l = 1, 2),
\]
where \( x, y \in \{0, 1\} \), and \( \eta_{i_k} \) and \( \delta_{i_k} \) \((1 \leq k \leq n)\) are integers.

### III. ZENG, ET AL.'S CONSTRUCTIONS

Zeng, et al gave two constructions in [14] so as to convert a PBA into a PQA. For the sake of saving the reader’s trouble in referring to the relevant reference, we briefly state them below.

Construction 1 [14].

Consider a PBA \( A \), namely, \( A = B \), with size \( N_1 \times \cdots \times N_k \times N_{k+1} \times \cdots \times N_n \), where all integers \( N_k \)'s \((1 \leq k \leq n)\) are even. If we have
\[
\eta_{i_k} = \delta_{i_k} \quad (\text{mod} \frac{N_i}{2}) \quad (1 \leq k \leq n),
\]
the resultant quaternary arrays \( Q' \) \((l = 1, 2)\) in (6) is perfect.

Construction 2 [14].

Consider two PBAs \( A \) and \( B \), with size \( N_1 \times \cdots \times N_k \times N_{k+1} \times \cdots \times N_n \), where at least an integer in the integers \( N_k \)'s \((1 \leq k \leq n)\) is odd.

We construct a quaternary array \( P = [p_{i_1, i_2, \ldots, i_n}] \) by
\[
P_{i_1, i_2, \ldots, i_n} = \begin{cases} h_1^1, & i' = 2i, \\ h_2^1, & i' = 2i + 1, \end{cases},
\]
where
\[
h_1^1 = \phi(a_{i_1, i_2, \ldots, i_n}), \quad b_{i_1, i_2, \ldots, i_n},
\]
\[
h_2^1 = \phi_2(a_{i_1, i_2, \ldots, i_n}), \quad b_{i_1, i_2, \ldots, i_n},
\]
and \( 0 \leq i_k \leq N_k - 1 (1 \leq k \leq n) \) and \( i' = 2i + t \) \((t \in \{0, 1\})\), that is, \( 0 \leq i' \leq 2N_r - 1 \). If we have
\[
\delta_l - \eta_l = \lambda_l - \zeta_l \quad \text{mod} \ N_r \quad (1 \leq l \leq n)
\]
\[
\delta_l - \eta_l = \lambda_l - \zeta_l \quad \text{mod} \ N_r \quad (1 \leq l \leq n, l \neq r)
\]
\[
\delta_r - \eta_l = \lambda_r - \zeta_r + 1 \quad \text{mod} \ N_r,
\]
the quaternary array \( P \) with size \( N_1 \times \cdots \times N_{r-1} \times 2N_r \times N_{r+1} \times \cdots \times N_n \) is perfect.

Apparently, Theorem 2 does not give the family size of Construction 2.

### IV. FAMILY SIZE OF CONSTRUCTION 2

First of all, we need to investigate the conditions under which Construction 2 results in distinct arrays.

Let arrays \( A \) and \( B \) be two PBAs with each of size \( N_1 \times \cdots \times N_{r-1} \times N_r \times N_{r+1} \times \cdots \times N_n \), where the integers \( N_r \) is odd. Again let the array \( P \) be produced by Eqs. (8)-(10) from the arrays \( A \) and \( B \) with integers \( \eta_l \)'s, \( \delta_l \)'s, and \( \zeta_l \)'s \((1 \leq l \leq n)\), and so does the array \( P' \) but with integers \( \eta_l \)'s, \( \delta_l \)'s, and \( \zeta_l \)'s \((1 \leq l \leq n)\). If the following conditions:
\[
\delta_l = \delta_l' \equiv \lambda_l' \equiv \zeta_l' \quad \text{mod} \ N_r \quad (1 \leq l \leq n, l \neq r)
\]
\[
\delta_l = \delta_l' \equiv \lambda_l' \equiv \zeta_l' \quad \text{mod} \ N_r \quad (1 \leq l \leq n, l \neq r)
\]
\[
\delta_l = \delta_l' \equiv \lambda_l' \equiv \zeta_l' \quad \text{mod} \ N_r
\]
holds, the arrays \( P \) and \( P' \) are distinct from each other.

Provided that there exist the integers \( \mu_l \)'s \((1 \leq l \leq n)\) so that \( T(\mu_1, \mu_2, \ldots, \mu_n)P = P' \), which means that the arrays \( P \) and \( P' \) are cyclical shift equivalence. In
accordance with Eq. (8), the entries of the array $T(\mu_1, \mu_2, \cdots, \mu_n)P$ can be calculated by four cases as follows.

**Case 1:** $i' = 2i$ and $\mu_r = 2\rho_r$.

\begin{equation}
P_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0} = h_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0}.
\end{equation}

**Case 2:** $i' = 2i + 1$ and $\mu_r = 2\rho_r$.

\begin{equation}
P_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0} = h_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0}.
\end{equation}

**Case 3:** $i' = 2i$ and $\mu_r = 2\rho_r + 1$.

\begin{equation}
P_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0} = h_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0}.
\end{equation}

**Case 4:** $i' = 2i + 1$ and $\mu_r = 2\rho_r + 1$.

\begin{equation}
P_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0} = h_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0}.
\end{equation}

According to $T(\mu_1, \mu_2, \cdots, \mu_n)P = P'$, for Cases 1-4 we have

\begin{equation}
\phi_i(a_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0}) = b_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0},
\end{equation}

\begin{equation}
\phi_i(a_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0}) = b_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0},
\end{equation}

\begin{equation}
\phi_i(a_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0}) = b_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0},
\end{equation}

and

\begin{equation}
\phi_i(a_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0}) = b_{h \cdot j + \delta_r - \delta_0, \cdots, j + \delta_r - \delta_0, j + \delta_r - \delta_0, j + \delta_r - \delta_0},
\end{equation}

respectively.

Consider $\mu_r$ even, which results in that Cases 1 and 2 appear in $T(\mu_1, \mu_2, \cdots, \mu_n)P = P'$ synchronously. For Case 1, from Eqs. (6) and (17) we have

\begin{equation}
\left[(-1)^{\alpha} - (-1)^{\gamma} \right] - j((-1)^{\beta} + (-1)^{\delta}) = 0,
\end{equation}

where

\begin{equation}
\alpha = a_{h \cdot j + \delta_0, \cdots, j + \delta_0, j + \delta_0, j + \delta_0},
\end{equation}

\begin{equation}
\beta = b_{h \cdot j + \delta_0, \cdots, j + \delta_0, j + \delta_0, j + \delta_0},
\end{equation}

\begin{equation}
\gamma = a_{h \cdot j + \delta_0, \cdots, j + \delta_0, j + \delta_0, j + \delta_0},
\end{equation}

\begin{equation}
\theta = b_{h \cdot j + \delta_0, \cdots, j + \delta_0, j + \delta_0, j + \delta_0}.
\end{equation}

Hence, from Eq. (21) we have

\begin{equation}
\begin{cases}
(-1)^{\alpha} = (-1)^{\gamma} \\
(-1)^{\beta} = (-1)^{\delta}.
\end{cases}
\end{equation}

Since the equation system in (23) holds for arbitrary integers $i'$'s $(1 \leq l \leq n)$, we must have

\begin{equation}
\begin{cases}
\eta_l + \mu_l = \eta'_{l'} \mod N_r, (1 \leq l \leq n, l \neq r) \\
\eta_r + \rho_r = \eta'_{l'} \mod N_r \\
\eta_l + \mu_l - \delta_l + \mu_r = \eta'_{l'} \mod N_r, (1 \leq l \leq n, l \neq r) \\
\delta_l + \mu_r = \delta'_l \mod N_r,
\end{cases}
\end{equation}

which results in $\rho_r \equiv 0$ (mod $N_r$) and $\eta_l \equiv \eta'_{l'}$ (mod $N_r$) under given conditions (e.g., $\delta_l = \delta'_l$ (mod $N_r$)) in Theorem 3. But, we have set $\eta_l \neq \eta'_{l'}$ (mod $N_r$) in the conditions of Theorem 3. Apparently, here is a contradiction.

Similarly, consider $\mu_r$ odd, which results in that Cases 3 and 4 appear in $T(\mu_1, \mu_2, \cdots, \mu_n)P = P'$ synchronously. For Case 3, from Eqs. (6) and (19) we have

\begin{equation}
\left[(-1)^{\alpha} - (-1)^{\gamma} \right] - j((-1)^{\beta} + (-1)^{\delta}) = 0,
\end{equation}

where

\begin{equation}
\alpha = a_{h \cdot j + \delta_0, \cdots, j + \delta_0, j + \delta_0, j + \delta_0},
\end{equation}

\begin{equation}
\beta = b_{h \cdot j + \delta_0, \cdots, j + \delta_0, j + \delta_0, j + \delta_0},
\end{equation}

\begin{equation}
\gamma = a_{h \cdot j + \delta_0, \cdots, j + \delta_0, j + \delta_0, j + \delta_0},
\end{equation}

\begin{equation}
\theta = b_{h \cdot j + \delta_0, \cdots, j + \delta_0, j + \delta_0, j + \delta_0},
\end{equation}

Hence, from Eq. (25) we have

\begin{equation}
(-1)^{\beta} = (-1)^{\delta}.
\end{equation}

In accordance with Eq. (12), from Eq. (27) we have

\begin{equation}
\begin{cases}
\beta_r = b_{h \cdot j + \delta_0, \cdots, j + \delta_0, j + \delta_0, j + \delta_0} \\
\theta_r = b_{h \cdot j + \delta_0, \cdots, j + \delta_0, j + \delta_0, j + \delta_0}.
\end{cases}
\end{equation}

Notice the fact that for given integers $\delta_l$'s $(1 \leq l \leq n)$, the array $B' = [b_{h \cdot j + \delta_0, \cdots, j + \delta_0, j + \delta_0}]$ is perfect due to the perfect array $B$. On the other hand, we count the autocorrelation of the array $B'$ as follows.
Theorem 3 is true.

The next theorem will answer the family size of Construction 2.

For a given PBA. Construction 2 yields \( N_r \) distinct PQAs.

**Proof:** From Theorem 3, when \( \eta_t \) ranges the range from 0 to \( N_r - 1 \) with other parameters unaltered in Eq. (12), the obtained arrays resulting from Construction 2 are distinct from one another. We complete the proof of this theorem.

V. EXAMPLES

In order to help the reader understand our results, here are two examples.

**Example 1:**

Consider the arrays A and B be an identical PBA with size \( 3 \times 12 \) [5] as follows.

\[
A = \begin{bmatrix}
- & + & + & - & + & + & + & + & - & + & - & - \\
+ & + & + & + & - & + & - & - & - & - & - & -
\end{bmatrix}
\]

Consider the odd integer \( N_1 = 3 \). According to Eq. (12) we set

\[
\begin{align*}
(\eta_1, \eta_2) &= (\Gamma, 0) \\
(\delta_1, \delta_2) &= (0, 0) \\
(\sigma_1, \sigma_2) &= (\Gamma + 2, 0) \\
(\lambda_1, \lambda_2) &= (0, 0),
\end{align*}
\]

Where \( \Gamma = 0, 1, 2 \).

From Construction 2 the resultant distinct PQAs \( P_t \)'s \( (\Gamma = 0,1,2) \) with size \( 6 \times 12 \), depending on the choice of \( \Gamma \), are given as follows, respectively.

\[
P_0 = \begin{bmatrix}
3 & 0 & 0 & 3 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 2 \\
0 & 1 & 2 & 3 & 1 & 1 & 2 & 2 & 2 & 3 & 2 & 0 \\
0 & 0 & 1 & 1 & 3 & 0 & 1 & 2 & 1 & 2 & 1 & 3 \\
2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 1 & 3 & 1 & 3 \\
1 & 0 & 3 & 2 & 0 & 0 & 3 & 3 & 3 & 2 & 3 & 1 \\
1 & 1 & 0 & 0 & 2 & 1 & 0 & 3 & 0 & 3 & 0 & 2
\end{bmatrix},
\]

\[
P_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 2 \\
3 & 1 & 1 & 3 & 1 & 1 & 1 & 1 & 3 & 1 & 3 & 1 \\
0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 0 \\
1 & 1 & 1 & 1 & 3 & 1 & 1 & 3 & 1 & 3 & 1 & 3 \\
2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\
1 & 1 & 3 & 3 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 1
\end{bmatrix},
\]

and

\[
P_2 = \begin{bmatrix}
0 & 0 & 3 & 3 & 1 & 0 & 3 & 2 & 3 & 2 & 3 & 1 \\
0 & 1 & 1 & 0 & 2 & 1 & 1 & 2 & 1 & 3 & 1 & 3 \\
3 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 3 \\
1 & 1 & 2 & 2 & 0 & 1 & 2 & 3 & 2 & 3 & 2 & 0 \\
1 & 0 & 0 & 1 & 3 & 0 & 0 & 3 & 0 & 2 & 0 & 2 \\
2 & 1 & 0 & 3 & 1 & 1 & 0 & 0 & 0 & 3 & 0 & 2
\end{bmatrix}.
\]

The periodic autocorrelation function of whose each is depicted by Figure 1. Apparently, this function is impulse-like.

\[\text{Figure 1. Autocorrelation function of the array in Example 1.}\]

**Example 2:**

Consider the arrays A and B be an identical PBA with size \( 4 \times 3 \times 3 \) [5] as follows.

\[
P_0 = \begin{bmatrix}
3 & 0 & 0 & 3 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 2 \\
0 & 1 & 2 & 3 & 1 & 1 & 2 & 2 & 2 & 3 & 2 & 0 \\
0 & 0 & 1 & 1 & 3 & 0 & 1 & 2 & 1 & 2 & 1 & 3 \\
2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 1 & 3 & 1 & 3 \\
1 & 0 & 3 & 2 & 0 & 0 & 3 & 3 & 3 & 2 & 3 & 1 \\
1 & 1 & 0 & 0 & 2 & 1 & 0 & 3 & 0 & 3 & 0 & 2
\end{bmatrix},
\]

\[
P_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 2 \\
3 & 1 & 1 & 3 & 1 & 1 & 1 & 1 & 3 & 1 & 3 & 1 \\
0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 0 \\
1 & 1 & 1 & 1 & 3 & 1 & 1 & 3 & 1 & 3 & 1 & 3 \\
2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\
1 & 1 & 3 & 3 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 1
\end{bmatrix},
\]

and

\[
P_2 = \begin{bmatrix}
0 & 0 & 3 & 3 & 1 & 0 & 3 & 2 & 3 & 2 & 3 & 1 \\
0 & 1 & 1 & 0 & 2 & 1 & 1 & 2 & 1 & 3 & 1 & 3 \\
3 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 3 \\
1 & 1 & 2 & 2 & 0 & 1 & 2 & 3 & 2 & 3 & 2 & 0 \\
1 & 0 & 0 & 1 & 3 & 0 & 0 & 3 & 0 & 2 & 0 & 2 \\
2 & 1 & 0 & 3 & 1 & 1 & 0 & 0 & 0 & 3 & 0 & 2
\end{bmatrix}.
\]
Consider the odd integer \( N_3 = 3 \). According to Eq. (12) we set
\[
(\eta_1, \eta_2, \eta_3) = (0, 0, \Gamma)
\]
\[
(\delta_1, \delta_2, \delta_3) = (0, 0, 1)
\]
\[
(\zeta_1, \zeta_2, \zeta_3) = (0, 0, \Gamma + 2)
\]
\[
(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0),
\]
where \( \Gamma = 0, 1, 2 \).

In accordance with Construction 2, the resultant distinct PQAs \( P_i \)'s (\( \Gamma = 0, 1, 2 \)) with size \( 4 \times 3 \times 6 \), depending on the choice of \( \Gamma \), are given as follows, respectively.

\[
P_0 = \begin{bmatrix}
3 & 0 & 2 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 3 & 0 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 2 & 1 & 1 & 3 & 0 & 2 & 1 & 3 & 2 & 3 & 2 \\
0 & 2 & 1 & 1 & 3 & 0 & 2 & 1 & 3 & 2 & 3 & 2 \\
0 & 2 & 1 & 1 & 3 & 0 & 2 & 1 & 3 & 2 & 3 & 2 \\
\end{bmatrix}
\]

\[
P_1 = \begin{bmatrix}
0 & 3 & 0 & 1 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
2 & 1 & 0 & 3 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 2 & 1 & 0 & 3 & 2 & 3 & 2 & 3 & 2 & 3 \\
0 & 1 & 2 & 1 & 0 & 3 & 2 & 3 & 2 & 3 & 2 & 3 \\
0 & 1 & 2 & 1 & 0 & 3 & 2 & 3 & 2 & 3 & 2 & 3 \\
\end{bmatrix}
\]

and

\[
P_2 = \begin{bmatrix}
0 & 0 & 3 & 1 & 1 & 2 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 2 & 0 & 0 & 3 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
3 & 1 & 1 & 2 & 0 & 0 & 2 & 3 & 2 & 3 & 2 & 3 \\
3 & 1 & 1 & 2 & 0 & 0 & 2 & 3 & 2 & 3 & 2 & 3 \\
3 & 1 & 1 & 2 & 0 & 0 & 2 & 3 & 2 & 3 & 2 & 3 \\
\end{bmatrix}
\]

the periodic autocorrelation function of whose each is
\[
R(\tau_1, \tau_2, \tau_3) =
\begin{bmatrix}
72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]
where \( 0 \leq \tau_1 \leq 3, 0 \leq \tau_2 \leq 2, \) and \( 0 \leq \tau_3 \leq 5 \).

Apparently, as predicted, Eq. (36) shows that \( P_0, P_1, \) and \( P_2 \) are perfect. In addition, it is not difficult for the reader to check up that the arrays are distinct from one another.

VI. CONCLUSION

This paper discusses the enumeration of Construction 2 in Zeng, et al’s constructions, and gives the conditions that the resultant arrays are distinct. By the obtained conditions, more PQAs than the original theorem are produced. These proposed PQAs provide lots of candidates for applications.

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