Heavy-Quark Hybrid Mass Splittings: Hyperfine and “Ultrafine”

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I. INTRODUCTION

The term hyperfine splitting refers to the difference between two energy levels of a particle, atom, or molecule due to interactions involving a coupling to the magnetic dipole or higher electromagnetic moments of a nucleus. In atoms or molecules, hyperfine splittings are suppressed due to the large size of the nuclear mass $m_N$ compared to that of the electrons, either because $1/m_N$ appears in the nuclear magnetic dipole moment, or due to large gradients of internal atomic electromagnetic fields in the vicinity of the nucleus (within a Compton wavelength $1/m_N$) that can produce an observable coupling to the nucleus.

In the case of heavy-quark hadrons, the heavy quark $Q$ assumes the role of the nucleus, and being spin-1/2, its only higher electromagnetic moment is the magnetic dipole $\mu_Q$, which is proportional to its spin $S_Q$ and inversely proportional to its mass $m_Q$. Since $m_Q$ is large compared to the scale $\Lambda_{QCD}$ of the light degrees of freedom (gluons and sea quarks) in the hadron, hyperfine splittings in heavy-quark systems are small. States with approximately the same light-field content but differing only in the relative spin states of the heavy quarks should therefore lie close together in mass, an observation that lies at the crux of heavy-quark spin symmetry; any two such states related in this way are said to differ only by a hyperfine splitting.

In the context of quarkonium with a heavy quark-antiquark pair $Q\bar{Q}$ ($c\bar{c}$ or $b\bar{b}$), states related by hyperfine splittings have the same wave functions with respect to the light degrees of freedom, and differ in mass only due to the relative $Q$ or $\bar{Q}$ spin states. In charmonium, for example, the $1S$ states ($\eta_c$, $J/\psi$) form a hyperfine doublet, while the $1P$ states ($h_c$, $\chi_{c0}$, $\chi_{c1}$, $\chi_{c2}$) form a hyperfine quartet.

The actual operators in a Hamiltonian formalism responsible for hyperfine splittings are easy to identify by exploiting the similarities of QED and QCD. Essentially the same terms emerge from the Breit reduction of Dirac fermions interacting through photons and gluons, respectively. These terms are the well-known spin-spin, spin-orbit, and tensor operators, as discussed in detail for quarkonium in Ref. [3]. The spin-spin operator $S_Q \cdot S_{\bar{Q}}$ is especially interesting, because in the Breit reduction it is accompanied by the Laplacian of the $1/r$ gauge interaction, which produces the contact interaction $\delta^{(3)}(r)$, a fact first noted by Fermi [4]. Such a term contributes only for wave functions that are nonzero at zero $Q\bar{Q}$ separation. In the context of nonrelativistic quantum mechanics, only the $S$ waves satisfy this criterion, and all $L > 0$ states have a zero spin-spin hyperfine contribution.

A primary result of Ref. [3] is the observation that, for any given hyperfine multiplet, one unique linear combination of masses is sensitive solely to the spin-spin and not to the spin-orbit or tensor operators. In the case that all quarks are heavy, the relevant combination for $L > 0$ states should therefore have a very small mass splitting — much smaller than typical hyperfine splittings. It is a linear combination of pairwise hyperfine splittings and is therefore a hyperfine splitting in its own right, dubbed ultrafine in Ref. [3]. The ultrafine combination is simply the difference between the mass of the state with total quark spin $S_{Q\bar{Q}} = 0$ and the spin-averaged mass of states with total quark spin $S_{Q\bar{Q}} = 1$. This combination for, e.g., $P$-wave quarkonium is

$$\Delta \equiv M_h - \frac{1}{9} [1 \cdot M_{\chi_0} + 3 \cdot M_{\chi_1} + 5 \cdot M_{\chi_2}] \quad . \quad (1)$$

The ultrafine splittings were seen in all cases where all four states have been observed to be extremely small—indeed, experimentally consistent with zero—both in quarkonium [2] and positronium [2]. The theoretically expected splittings are so small that any measured deviation from zero can be identified as the presence in at least some of the quarkonium states of a substantial non-$Q\bar{Q}$ Fock state (“coupled-channel exoticity”). Such

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a component can be most easily identified with a heavy-light meson pair \([QQ\bar{q}](QQ\bar{q})\) contribution, but it might also be due to a tetraquark or some other exotic component reviewed in Ref. 2.

Essentially the same reasoning holds for the yet-unobserved heavy-quark hybrid mesons, for which the gluon field connecting the \(QQ\) pair occurs in a nontrivial configuration (For a review, see Ref. 2). Such states were discussed briefly in Ref. 3. While of course no hybrid meson has ever been experimentally confirmed as such, the existence of excited gluon fields is a widely expected feature of QCD, and moreover, a definite spectrum of heavy-quark hybrid mesons is a well-established feature of increasingly sophisticated lattice QCD simulations \([8–14]\). In this paper we investigate the hyperfine structure of heavy-quark hybrid mesons, showing that all of the lowest-lying multiplets possess a combination of the heavy-quark description of hadrons in Section III, and it is shown that matrix elements of the hyperfine potentials are labeled by the irreducible representations \(\Lambda = 0, \pm 1, \pm 2, \ldots\) of the axial angular momentum operator \(\hat{r} \cdot \hat{J}_{\text{light}}\) provide a good quantum number; since the system is symmetric with respect to reflection through any plane containing \(\hat{r}\) (which takes \(\lambda \rightarrow -\lambda\)), energy eigenvalues cannot depend upon the sign of \(\lambda\), so that one defines \(\Lambda \equiv |\lambda|\). Analogously to the labels \(S, P, D, \ldots\), for the usual angular momentum quantum numbers \(L = 0, 1, 2, \ldots\), the values \(\Lambda = 0, 1, 2, \ldots\) are then denoted by \(\Sigma, \Pi, \Delta, \ldots\). The light degrees of freedom also possess a reflection symmetry about an origin given by the midpoint between the two nuclei, so that the eigenvalue \(\eta\) of the corresponding parity operator \(P_{\text{light}}\) is a good quantum number, with \(+1\) (\(-1\)) denoted by \(g (u)\), respectively. Lastly, the \(\Lambda = 0 (\Sigma)\) representations can be distinguished by their behavior under a reflection \(R_{\text{light}}\) through a plane containing the nuclei, its \(\pm 1\) eigenvalue being denoted by \(\epsilon\). The \(\Lambda > 0\) configurations \((\lambda, \eta; r)\) can also be combined into eigenstates of \(R_{\text{light}}\) with eigenvalue \(\epsilon\) by noting that \(R_{\text{light}}(\lambda, \eta; r) = (-1)^{\delta} |\lambda, \eta; r\rangle\), where \(\delta\) is the intrinsic parity of the light degrees of freedom, and by defining the eigenstates

\[
|\Lambda, \eta, \epsilon; r\rangle \equiv \frac{1}{\sqrt{2}} \left( |\Lambda, \eta; r\rangle + \epsilon (-1)^{\lambda} |\Lambda, \eta; r\rangle \right) .
\]  

Full physical states for the system are then obtained by solving the Schrödinger equation of the nuclei interacting with the BO potential \(V_T(r)\), which produces eigenvalues and eigenfunctions labeled by a principal quantum number \(n\). When the complete state including both light and heavy components is considered, additional good quantum numbers arise. Components of \(J_{\text{light}}\) orthogonal to \(\hat{r}\) are not among these, because the nuclei can possess their own relative orbital angular momentum \(L_{\text{nuc}}\) (which satisfies \(\hat{r} \cdot L_{\text{nuc}} = 0\)) that cannot be distinguished from \(J_{\text{light}}\) in the complete state; the conserved quantity is the combined orbital angular momentum \(L\), where

\[
L \equiv L_{\text{nuc}} + J_{\text{light}} .
\]  

as is the total nuclear spin \(S\), and ultimately, the total angular momentum quantum numbers \(J, J_z\) of the molecule. It is also easily seen from contracting Eq. 3 with \(\hat{r}\) that \(L \geq |\hat{r} \cdot L| = |\hat{r} \cdot J_{\text{light}}| = \Lambda\). In summary, the physical states are completely specified by the kets

\[
|n, L, S, Jm_J; \Lambda, \eta, \epsilon\rangle ,
\]  

with \(J_{\text{light}}\) and \(L_{\text{nuc}}\) eigenvalues implicit.
The same BO approach can be applied to any other quantum-mechanical system that possesses both heavy and light degrees of freedom. In the context of heavy quarkonium $QQ$, all the nomenclature discussed thus far in this section applies verbatim with small substitutions \cite{17}: The nuclei become the $QQ$ pair, $\mathbf{r}$ is the unit vector pointing from $Q$ to $Q$, the light electronic degrees of freedom become the glue field, and the presence of the antiparticle $\bar{Q}$ means that $q$ is the eigenvalue of $\gamma_L$ [readily seen to equal $\epsilon(-1)^{A}$ in this case], but $(\gamma^i)_{\text{light}}$. A hybrid heavy quarkonium meson in this terminology then simply refers to an energy eigenstate containing $QQ$ for which at least some of the eigenvalues $\Lambda, \eta, \epsilon$ are nontrivial, i.e., every BO eigenstate except $\Sigma^+_r$. Taking into account both the light and heavy degrees of freedom, the overall discrete quantum numbers for the physical state are determined to be:

$$P = \epsilon(-1)^{A+L+1}, \quad C = \eta\epsilon(-1)^{A+L+S}. \quad \quad (5)$$

The first treatment of hybrids within the BO approximation (and moreover, in the context of a lattice QCD simulation) appeared several decades ago in Ref. \cite{8}. In Sec. \textbf{V} we touch upon important landmarks in the lattice simulations of hybrids and their connection to the BO approximation. Here however, we especially note two important recent papers in this regard: First, Ref. \cite{17} established the BO approximation as a formalism useful not only for the description of hybrid mesons, but the full collection of exotic XYZ states \cite{3} as well; as noted in \cite{17}, the light degrees of freedom can be generalized to carry nontrivial isospin quantum numbers and therefore can also be used to study tetraquarks.

Moreover, Ref. \cite{17} noted (an observation dating back as far as \cite{18}) that some of the hybrid BO light-field potentials become degenerate in the $r \to 0$ limit, in which case the light configuration (specifically, one in the color-octet representation) is called a glue lump. Given eigenvalues of $L$ and $\Lambda$ satisfying $L \geq 1, \Lambda > \Lambda$, as well as eigenvalues of $\eta, \epsilon$, and $n$, and using Eqs. (5–6), one finds that the quartet of states (one from $S = 0$ and three from $S = 1$) derived from the $\Lambda^+(nL)$ and $(\Lambda+1)^-(nL)$ BO potentials produce the same set of $J^{PC}$ values. The lowest BO potentials above the ground state $\Sigma^+_r$ are calculated on the lattice to be $\Sigma^+_u(1P), \Pi^+_u(1P)$, and $\Pi^+_u(1P)$ (see Sec. \textbf{V}). The potentials $\Pi^+_u(1P)$ form a parity pair expected to be degenerate in the $m_Q$ limit (i.e., leading-order BO) limit. However, the pair $\Sigma^+_u(1P)$ and $\Pi^+_u(1P)$, an example of the above rule, each produce a set of states with the same $J^{PC}$ values (see Table \textbf{I}), and indeed arise from the same glue lump, $J^\text{light}_{\text{glue}} = 1_+^+$. The second important paper \cite{19} built upon the BO heavy-quark hybrid studies of Ref. \cite{17} to develop an effective theory in the expansion parameter $1/m_Q$, in order to study such states. As shown in \cite{19}, the $\Sigma^+_u(1P)$ and $\Pi^+_u(1P)$ potentials explicitly produce coupled Schrödinger equations that, in particular, lift the $\Pi^+_u(1P)$ degeneracy, an effect known in BO studies as $\Lambda$-doubling. Reference \cite{19} then showed how to diagonalize and numerically solve the Schrödinger equations, thus obtaining a spectrum of heavy-quark hybrid meson masses that compare favorably with the results of lattice simulations.

The precise choice for an expansion parameter in the heavy-quarkonium hybrid sector to obtain a useful BO expansion corresponds to the nature of the optimal effective field theory used to describe the states. In a very recent paper \cite{20} by several of the authors of Ref. \cite{19}, it is argued that the hierarchy relevant to hybrids is

$$m_Qv \gg \Lambda_{QCD} \gg m_Ql^2, \quad \quad (7)$$

where $v$ is the typical heavy-quark velocity ($\sim a_s \approx 0.3$ for charm). This choice is motivated by requiring that the typical Bohr radius-like heavy-quark separation, $1/(m_Qv)$, is small compared to the size $1/\Lambda_{QCD}$ of the light hadronic cloud, and that the typical heavy-quark kinetic energies are small compared to those of the light cloud. With $m_c \approx 1.5$ GeV and $\Lambda_{QCD} \approx 300$ MeV, Eq. (7) becomes $450$ MeV $\gg 300$ MeV $\gg 130$ MeV. A hadronic BO expansion would nevertheless still have been possible even under the weaker condition that $\Lambda_{QCD}/m_Q$ ($\approx 0.2$ for charm) is a small parameter.

\section{III. \textsc{Heavy-quark QCD}}

The splittings that occur in heavy-quark hybrid multiplets are determined by the spin-dependent structure of QCD, which can be determined by constructing the heavy-quark expansion of the Hamiltonian of QCD in Coulomb gauge (a convenient choice because all degrees of freedom are physical) \cite{21}. A gauge-invariant approach based upon the Wilson loop yields the same results \cite{22}. The first spin-dependent interaction arises at order $1/m_Q$ and is generated by the Foldy-Wouthuysen term

$$\mathcal{H}_1 = \frac{1}{2m_Q} \int d^3x \ h^\dagger(x) \left( D^2 - g \sigma \cdot \mathbf{B} \right) h(x)$$

$$- (h \to \chi; m_Q \to m_{\bar{Q}}). \quad \quad (8)$$

Here, $h(\chi)$ annihilates a heavy (anti)quark, $\mathbf{B}$ is the chromomagnetic field, and $D$ is the covariant derivative. Spin-dependence is carried by the matrix element of $\sigma \cdot \mathbf{B}$ and is zero in conventional mesons because the only available vector is $\mathbf{r} = \mathbf{r}_Q \to \mathbf{r}_Q$, which has the wrong parity to yield a nonzero result \cite{22}. However, this conclusion need not follow in the case of hybrids where additional vectors, such as $J_{\text{light}}$, can contribute (see Ref. \cite{23}; Eq. (18) and subsequent discussion). The effect this interaction has on the ultrafine splitting will be discussed below.

The sole spin-dependent term at order $1/m_Q^2$ is given
The energy denominator is expressed in terms of adiabatic potentials of the color source and sink and of the quantum numbers \( \Gamma \) of the spin-orbit interaction \([21]\):

\[
\begin{align*}
V^{\text{LS}}_t(r = r_Q - r_\bar{Q}) &= \left( \frac{\sigma_Q \cdot L_Q}{4m_Q^2} - \frac{\sigma_\bar{Q} \cdot L_\bar{Q}}{4m_\bar{Q}^2} \right) \frac{1}{r} \frac{dV_f}{dr}, \quad (10)
\end{align*}
\]

where \( V \) is the static (Wilson-loop) interquark potential. Dependence upon the gluonic adiabatic quantum numbers \( \Gamma = \Lambda^L_n \) has been made explicit here, and reveals that the classical spin-orbit interaction is fixed by the relevant adiabatic potential. Since hybrid potentials are relatively flat at distance scales around 1 fm (corresponding to their expected equilibrium size), one concludes that the classical spin-orbit contribution to hybrid mass splittings is smaller than for conventional mesons.

Additional spin splittings arise at order \( 1/m_Q^2 \) by iterating the first-order terms in the Foldy-Wouthuysen expansion \([\text{Eq. } 3]\). In particular, the expression involving two powers of \( \sigma \cdot B \) gives rise to the (generalized) hyperfine interaction:

\[
V^{\text{hyp}}_t(r = r_Q - r_\bar{Q}) = \alpha_s \frac{4\pi}{3m_Q m_\bar{Q}} S_f \cdot S_{\bar{f}}
\]

\[
\times \sum_{r' \neq \Gamma} \frac{1}{E_{\Gamma}(r) - E_{\Gamma'}(r)} \langle \Gamma' ; r_Q, r_\bar{Q} \mid \frac{d^3x h^3(x) B(x) h(x)}{r} \mid \Gamma ; r_Q, r_\bar{Q} \rangle \times \langle \Gamma' ; r_Q, r_\bar{Q} \mid \frac{d^3y \chi^3(y) B(y) \chi(y)}{r} \mid \Gamma ; r_Q, r_\bar{Q} \rangle + \langle h \leftrightarrow \chi \rangle. \quad (11)
\]

The energy denominator is expressed in terms of adiabatic energies as a function of the distance \( r \) between the color source and sink and of the quantum numbers \( \Gamma \) of the relevant adiabatic surface. Notice that the intermediate state is summed over all adiabatic surfaces that are coupled to the initial surface by a chromomagnetic field.

We now argue that the spin-spin hyperfine interaction should be short-ranged \( (r \ll 1/\Lambda_{QCD}) \). First, recall that in Coulomb gauge the vector potential propagates over short distances, in distinction to the instantaneous interaction that gives rise to confinement. Perturbatively, the matrix element leading order is proportional to two derivatives acting upon \( 1/r \), which yields a \( \delta^{(3)}(r) \); its leading perturbative corrections are presented in \([24]\). Nonperturbatively, one expects this behavior to be replaced by two derivatives acting on a Yukawa-like interaction (its mass scale appearing because the primary nonperturbative effect in QCD is the generation of a mass gap), although the long-range string potential produces an additional term proportional to \( 1/r^2 \). Each also yields a short-range matrix element.

This expectation is supported by lattice measurements of the ground-state hyperfine interaction in quenched-lattice QCD, which indicate that the potential is zero within statistical errors for \( r > 0.2 \) fm \([26]\). This result implies that the hybrid hyperfine interaction is also short-ranged, by the following argument. The lattice computation of the hybrid static potential is made by placing the gluonic source and sink into nontrivial configurations. These gluonic configurations differ from the ground-state ones only up to distance scales of some fraction of a fm; thus, one expects the ground-state and excited-state static potentials to differ by no more than a constant at large distances. In fact, the slopes of all measured static potentials are the same past approximately 2 fm \([3]\), while their differences at large distances are known from the strict QCD string picture to scale as \( 1/r \). Measurements of the spin-spin interaction are obtained by inserting operators on the temporal legs of the relevant Wilson loops \([26]\). Because these are short-distance operators, and because their matrix elements are observed to decorrelate in the ground-state Wilson loop at large distance, they are also expected to decorrelate in the excited-state Wilson loop at large distance. We shall see shortly that this expectation is confirmed in heavy-quark lattice data.

### IV. ULTRAFINE HYBRID SPLITTINGS

The general definition of the ultrafine mass splitting for a set of states in the same multiplet (hence with the same \( \Lambda^L_n \), principal quantum number \( n \), and orbital quantum number \( L > 0 \)) is simply the spin-averaged difference between the states with total heavy-quark spin \( S = 0 \) and \( S = 1 \), and is easily seen to be \([3]\):

\[
\Delta_{n,L} = M(n^3L_{J=L}) - \frac{2L+1}{3(2L+1)} M(n^3L_{J=L-1}) - \frac{2L+1}{3(2L+1)} M(n^3L_{J=L+1}), \quad (12)
\]

of which Eq. (11) is merely the \( L = 1 \) case for ordinary quarkonium. The expression for \( L = 0 \) (S waves, hence consisting solely of a \( \Sigma \) BO potential) is even simpler:

\[
\Delta_{n,0} = M(n^3S_0) - M(n^3S_1).
\]

---

1 The expectation of a short-ranged hyperfine interaction for the lowest bound-state configurations is supported by observed ultrafine splittings in heavy quarkonia and in positronium \([3]\).
We now turn to the issue of whether the ultrafine combination is truly unique in a given hybrid hyperfine multiplet. After all, hybrid quarkonia are more complicated states than conventional quarkonia; in the latter case, the only available operators to form Hamiltonian terms that can split the hyperfine multiplet are the heavy-quark spins $S_Q, S_{\bar{Q}}$, which are coupled to each other and to the sole orbital angular momentum operator $L$ and the $Q\bar{Q}$ direction $\hat{r}$ to form the well-known operators [2]:

\[ S_Q \cdot S_{\bar{Q}} \text{ (hyperfine), } (13) \]

\[ S \cdot L \text{ (spin-orbit), } (14) \]

\[ \bar{T} \equiv (S_Q \cdot \hat{r})(S_{\bar{Q}} \cdot \hat{r}) - \frac{1}{3} S_Q \cdot S_{\bar{Q}} \text{ (tensor), } (15) \]

where $S \equiv S_Q + S_{\bar{Q}}$. As noted in [3], the contribution of the latter two operators to the ultrafine combination Eq. (12) vanishes for group-theoretical reasons: They have rank $> 0$ in $L$ space, while Eq. (12) includes multiplets in $L$ space only in the rank-0 (symmetric) combination.

The same argument applies equally for the corresponding operators in the case of hybrid states: The matrix elements of Eqs. (14) and (15) vanish when the full orbital angular momentum $L$ is used, leaving only the spin-spin operator of Eq. (13). However, now one must consider additional possible operators. One begins by noting that the analogue of $L$ in the conventional case (the operator dictating the short-distance behavior of the wave function) is $L_{Q\bar{Q}}$ in the hybrid case which, like $J_{\text{light}}$, does not provide good quantum numbers. Nevertheless, we have noted above that all the lightest hybrid multiplets $H_{1,2,3,4}$ arise from the same $J_{\text{light}} = 1$ glueump, while $H_1$ has $L_{Q\bar{Q}} = 0$ and $H_{2,3,4}$ have $L_{Q\bar{Q}} = 1$. Inasmuch as each $H_i$ corresponds to a unique BO potential with a good $L$ quantum number, states of the same $J^{PC}$ in different $H_i$ (specifically, $1^{-+}$ and $2^{+-}$) do not mix; however, the presence of an operator like $S \cdot L_{Q\bar{Q}}$ could accomplish this mixing. Within a given BO potential, however, one can check explicitly (by expanding all states in terms of eigenstates of the operator $J_{Q\bar{Q}} \equiv S + L_{Q\bar{Q}}$) that contributions to Eq. (12) by spin-orbit or tensor operators containing $L_{Q\bar{Q}}$ or $J_{\text{light}}$ cancel.

As discussed in Sec. [11], a spin-dependent interaction can occur in hybrids at order $1/m_Q$ due to the availability of additional spin vectors. Such contributions appear in the mixing of hybrids with quarkonium, as well as among hybrid states (see Eq. (14) in Ref. [28]). Even so, the ultrafine splittings stand out as the special case of hyperfine splittings that receive a nonzero contribution only from the heavy-quark spin-spin coupling; indeed, using the results from analyzing Eq. (14)–(17) of Ref. [28], one can show explicitly that our ultrafine combination Eq. (12) vanishes.

On the other hand, the mixing of the BO potentials $\Sigma_{\alpha}(1P)$ and $\Pi_{\alpha}^+(1P)$ in $H_1$ leaves $L (= 1)$ invariant, and indeed, the mixing between all states in these configurations derived in [19] depends solely on $L$ (and similarly for $L = 2$ in $H_4$). Therefore, the mixing in this case does not spoil the cancellation in Eq. (12) of matrix elements of Eqs. (13)–(15) for the states in $H_1$.

The question of whether the combination Eq. (12) is ultrafine for a given $H_i$ thus comes down to whether the spatial wave function associated with the BO potential vanishes as $r \to 0$. One of course anticipates the usual $r^{-L}$ behavior so that only $S$-wave multiplets fail to have an ultrafine splitting, but the BO potentials exhibit some interesting quirks.

First, the multiplets $H_2$ and $H_4$ are $P$-wave and $D$-wave, respectively, so that one expects each multiplet to have an ultrafine splitting $\Delta$ defined by Eq. (12). Indeed, using the lattice values in Table I, one finds $\Delta$ to be consistent with zero for $H_2$ and to differ from zero by only $1.3\sigma$ for $H_4$. Note also that the largest hyperfine splitting, $D$, within these multiplets is only about a factor two larger in $H_2$ and $H_4$. The pattern of splittings in these multiplets is quite peculiar: in all the known quarkonium and positronium cases [2], the spin-triplet states have masses that increase monotonically with $J$, and the spin-singlet state lies between the lowest and highest triplet state. However, in $H_2$ the $1^{-+}$ triplet state lies below the $0^{-+}$ triplet state, and the $1^{+-}$ singlet state lies above all the triplet states. In $H_4$, the $2^{+-}$ singlet state lies below all the triplet states. One may attribute these peculiarities to the size of lattice uncertainties, neglect of mixing between $1^{-+}$ or $2^{+-}$ states between BO multiplets, or misidentification of lattice states with the correct BO multiplets. The smallness of the ultrafine splitting in both cases, however, argues that none of these possibilities need be true, and that the calculated hybrid spectrum ordering is indeed correct; refined lattice simulations of the splittings would certainly serve to clarify the situation further.

Second, $H_3$ is an $S$-wave multiplet, suggesting a nonzero wave function as $r \to 0$, but its $(CP)_{\text{light}}$ and $R_{\text{light}}$ quantum numbers are both $-1$; and being a $\Lambda = 0$ ($\Sigma$) state, it has $P_{\text{light}} = -1$ as well, suggesting a wave function that vanishes at the origin. The only way to reconcile these facts is to allow the wave function to be odd, changing sign discontinuously when passing through the origin. In that case, integrating over the symmetric $\delta^{(3)}(\vec{r})$ distribution gives a vanishing result. Indeed, the splitting Eq. (12) for $H_3$ from Table I is extremely small, suggesting an ultrafine splitting.

Lastly, let us turn to the lowest multiplet, $H_1$. In this case, the ordering of the masses in Table I seems completely conventional, and both BO potentials are $P$-wave, suggesting a noncontroversial ultrafine splitting with a wave function $\sim r^1$ as $r \to 0$. Indeed, the value for $\Delta$ (especially compared to the largest intermultiplet splitting $D$) appears to confirm this suspicion. However, in this case the mixing of BO potentials discussed in [19] generates normalizable wave functions with asymptotic behavior $r^{-L-1}$ and $r^{L+1}$. In particular, since $L = 1$ in this case, one finds a wave function component that survives as $r \to 0!$ However, the angular part of the wave function...
function is still one that corresponds to $L = 1$. The general expression for these angular wave functions is given in [19], and indeed, in textbooks as well [16]; in the case $\Lambda = 0$ they reduce to the usual spherical harmonics. The important point, however, is that only the $L = \Lambda = 0$ angular wave function trivial angular dependence and hence is well defined at the origin, meaning that again, the full wave function changes sign at the origin and offers zero support to the symmetric $\delta^{(3)}(r)$ distribution. The splitting of Eq. (12) for $H_1$ is therefore indeed ultrafine.

V. HYBRIDS IN LATTICE QCD SIMULATIONS

In the absence of confirmed experimental evidence for hybrid mesons, one may rely upon the direct results of numerical simulations of QCD on a discretized lattice. As noted above, the first lattice simulation to make use of the BO approximation for hybrids appeared almost 35 years ago [8]. The first high-quality determinations of the BO potentials relevant to the heavy-quark hybrids were performed in Ref. [9], with computations on larger lattice volumes presented in Ref. [10]. Some details of the hybrid spectrum were also discussed in Ref. [9], with further improvements in Ref. [11]. These calculations were carried out in the quenched limit, the first unquenched simulations [12] (with an equivalent pion mass of 650 MeV) giving very similar results. Improvements in Ref. [11] using the numerical results of [14] showed that one can identify a genuine organization of the states by mass values into these multiplets, although the mass splitting between the highest state of one multiplet and the lowest state of the next can sometimes be small, as one can see in Table I. It is important to note that the method for distinguishing $H_{1,2,3,4}$ multiplets uses a somewhat less general theoretical approach than that for distinguishing between $c\bar{c}$ and hybrid states or between states of different $J^{PC}$—it depends upon identifying specific chromoelectric and chromomagnetic operators in the multipole expansion of the gluon field with specific BO potentials [31]—but the expected patterns definitely hold. Note also that the spectrum of $L_{QQ} = 0$ states matches that of $H_1$, while the spectrum of $L_{QQ} = 1$ states matches that of $H_2 \cup H_3 \cup H_4$, that the two sets have no states of the same $J^{PC}$ in common, and that $H_{2,3,4}$ all have distinct $L$ values.

Indeed, the statistical uncertainties are sufficiently small that one may use them to explore the spin structure of hybrid multiplets, as is done here. One must take caution to note that the uncertainties do not take into account the extrapolation to physically small light-quark masses, nor to the continuum limit of the lattice. However, one may argue that the differences of masses should be less sensitive to these effects than their absolute values (indeed, one may suspect the same argument to hold for some portion of the statistical uncertainties). In any case, we assume that lattice simulations of the hybrids are now sufficiently mature that one can at last make definitive statements about their mass splittings.

These arguments lead us to expect, on general and essentially model-independent grounds, that the ultrafine splitting in hybrid heavy-quark multiplets should be dominated by the matrix element of $V_{\Gamma}^\text{hyp}$, which in turn should be very small in the case of hybrids, provided there is little wave-function support at the origin. We expect splittings that are much smaller than $A_{QCD}$, and
in the multiplet, just as for conventional quarkonium \cite{3}. Indication of the presence of "coupled-channel exoticity" measured ultrafine splitting would thus be a very strong degree of freedom are negligible for hybrids. A large addition). Also presented are the maximum mass difference in the case of positronium, $1 \Delta_P$ which is added. Present are the maximum mass difference $D$ and the ultrafine combination $\Delta$ within each multiplet.

VI. CONCLUSIONS

The QCD Hamiltonian specifies a limited number of spin-dependent interactions that can serve to split heavy-quark multiplets of mesons, either conventional or with excited gluonic degrees of freedom. In the absence of substantial light valence-quark degrees of freedom, the ultrafine splitting defined in Eq. (12) is expected to be small on quite general grounds. This observation holds in the case of positronium, $1P$ and $2P$ bottomonium, and $1P$ charmonium \cite{3}. We anticipate that the ultrafine splitting will not be small for $2P$ charmonium because of the widely accepted notion that the $X(3872)$ is not a pure $c\bar{c}$ state.

The expected small hyperfine splittings in hybrid multiplets will assist in interpreting future spectroscopic data concerning hybrid mesons, such as at PANDA. As with conventional mesons, significant ultrafine splittings will constitute essentially model-independent evidence for coupled-channel exoticity in the relevant hybrid multiplet. The magnitude of an ultrafine splitting that qualifies as "significant" can be estimated from the charmium $2P$ multiplet, assuming that the $X(3872)$ is purely an interloper state. Indeed, the $1P$ charmium ultrafine splitting is measured to be $\Delta_{1,2P} = 80 \pm 130$ keV, while typical quark models (e.g., Ref. \cite{32}) indicate that the $X$ is approximately $100$ MeV lighter than expected, hence $\Delta_{1,2P} = 20 \sim 30$ MeV. Thus, as a rough guide, ultrafine splittings that are larger than $1 \sim 10$ MeV are indicative of non-conventional valence content in at least one state of a heavy-quark multiplet.

A likely minimal condition for the presence of coupled-channel exoticity is the existence of nearby $S$-wave continuum thresholds. We therefore examine the possibility of large ultrafine splittings in the hybrids of Table I by mixing with open-charm $S$- and $P$-wave meson pairs. Positive-parity states can be made from $SS$ or $PP$ combinations. The former start at $3740$ MeV for $DD$ \cite{33}, and run to $4220$ MeV for $D^*_sD^*_s$, all of which are lighter than the "bare" positive-parity hybrids of $H_2 \cup H_3 \cup H_4$ in Table I. Alternatively, the $PP$ combinations start at $4640$ MeV ($D^*_sD^*_s$), and are therefore too heavy to create substantial light-quark valence degrees of freedom in the positive-parity hybrids. Thus we (rather naively) expect the multiplets $H_2$, $H_3$, and $H_4$ to have very small ultrafine splittings. Negative-parity channels can be constructed from $SP$ meson combinations. Of these, $DD^*$ lies close to the $J^{PC} = 2^{+}H_1$ state, while $DD$ lies close to the $1^{--}H_1$ state. Thus, one sees an intriguing possibility of a large ultrafine splitting in the lightest ($H_1$) multiplet.

\footnotesize

\begin{table}[h]
\centering
\begin{tabular}{lcccc}
\hline
\text{Multiplet} & \text{$J^{PC}$} & \text{$m$ (MeV)} & \text{$D$ (MeV)} & \text{$\Delta$ (MeV)} \\
\hline
$H_1$ $[\Sigma^+_0(1P), \Pi^+_0(1P)]$ & $1^{--}$ & 4285(14) & 139(21) & 5.4(17.8) \\
& $0^+$ & 4195(13) & & \\
& $0^+$ & 4217(16) & & \\
& $2^+$ & 4334(17) & & \\
$H_2$ $[\Pi^+_0(1P)]$ & $1^{++}$ & 4399(14) & 55(40) & 22(29) \\
& $0^+$ & 4386(09) & & \\
& $1^+$ & 4344(38) & & \\
& $2^+$ & 4395(40) & & \\
$H_3$ $[\Sigma^+_0(1S)]$ & $0^{++}$ & 4472(30) & 5(36) & $-5(36)$ \\
& $1^{--}$ & 4477(19) & & \\
$H_4$ $[\Sigma^+_0(1D), \Pi^+_0(1D)]$ & $2^{++}$ & 4492(21) & 56(30) & $-33(25)$ \\
& $1^-$ & 4497(39) & & \\
& $2^-$ & 4509(18) & & \\
& $3^-$ & 4548(22) & & \\
\hline
\end{tabular}
\caption{Charmonium hybrid masses from lattice QCD simulations by the Hadron Spectrum Collaboration \cite{14}, as adapted from Ref. \cite{19} (where the experimental value of $m_{cc}$ is added). Also presented are the maximum mass difference $D$ and the ultrafine combination $\Delta$ within each multiplet.}
\end{table}

\normalsize

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