No-arbitrage pricing under cross-ownership

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Abstract

We generalize Merton’s asset valuation approach to systems of multiple financial firms where cross-ownership of equities and liabilities is present. The liabilities, which may include debts and derivatives, can be of differing seniority. We derive equations for the prices of equities and recovery claims under no-arbitrage. An existence result and a uniqueness result are proven. Examples and an algorithm for the simultaneous calculation of all no-arbitrage prices are provided. A result on capital structure irrelevance for groups of firms regarding externally held claims is discussed, as well as financial leverage and systemic risk caused by cross-ownership.

Key words: Absolute priority rule, capital structure irrelevance, contingent claims analysis, counterparty risk, credit risk, cross-ownership, derivatives pricing, financial contagion, leverage, Merton model, multi-asset valuation, no-arbitrage pricing, ownership structure, priority of claims, reciprocal ownership, seniority of debt, structural models, systemic risk.

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1 Introduction

1.1 Preliminaries

The main purpose of this paper is to investigate and overcome the intricacies of determining no-arbitrage prices for the equities and the liabilities of a group of firms where the financial fates of these firms are intertwined through the cross-ownership of financial assets issued or guaranteed by the same firms. In this theory, the liabilities can include debt and derivatives, and the claims belonging to these liabilities are allowed to have differing priorities in a potential liquidation.

On the one hand, the problem of no-arbitrage pricing under cross-ownership is real, and very important to consider, since cross-ownership of financial assets is present in the world’s financial markets. For instance, McDonald (1989) writes that in 1987 double counting from cross-ownership of equity (also called “reciprocal ownership”, “corporate cross-holding”, or “intercorporate shareholding”) accounted for “at least 24% of Japan’s
reported market capitalization.” An article by Bøhren and Michalsen (1994) shows at the example of the Oslo stock exchange how equity cross-ownership can lead to double counting of assets and overstated market equity. Ritzberger and Shorish (2002) state “that under cross-ownership the book value of a firm will tend to be overestimated with respect to the underlying cash flows.” Bøhren and Michalsen (1994) also show that financial leverage can be understated due to cross-ownership. Furthermore, given the evidence from more recent events in global financial markets, not only the danger of overstated market aggregates, but also the very real risk of financial contagion seems to at least partially stem from cross-ownership. However, especially in this context of systemic risk, cross-ownership should not only be considered for equities, but also for liabilities like bonds and derivatives issued by the considered firms.

On the other hand, the topic of financial cross-ownership is essentially non-existent in the literature of financial mathematics as far as asset valuation models, credit risk models, and derivatives pricing are concerned. In financial economics, or finance in general, cross-ownership is considered, but mostly in conjunction with the separation of ownership from the control of firms (e.g. in Bebchuk, Kraakman and Triantis (2000), Ritzberger and Shorish (2002), Dorofeenko et al. (2008), and references therein), or with respect to the distortion of market aggregates (e.g. McDonald (1989), Bøhren and Michalsen (1994), and references therein). Because of this, the focus of cross-ownership considerations in financial economics primarily lies on equity. This, however, is insufficient from a mathematical finance point of view, because it neglects aspects of liabilities like debt or derivatives, where cross-ownership can play a substantial role in the context of asset valuation under counterparty risk.

It is indeed somewhat surprising that mathematical finance has neglected the area of cross-ownership, since, for instance when trying to assess the credit-worthiness of a financial firm, one of the natural questions to ask would be: what does it mean for the balance sheet of company A if company B defaults on its debt, given that A owns parts of B’s debt and maybe even some of its equity? Also, how severely are the financial promises of A (debt or derivatives issued by A) affected when B defaults? For a larger number of firms that are intertwined by cross-ownership, these kind of questions can quickly become very complicated. As Ritzberger and Shorish (2002) write in the context of separation of ownership from the control of firms:

[...] The precise quantitative effect of cross-ownership between firms is, however, difficult to capture, both at the theoretical and the empirical level. [...] [...] (“A owns part of B, B owns part of A, so A owns part of B’s ownership of A, which is also part of a part of A’s ownership of B, which is...”). This recursion must be addressed, [...] In our context of asset valuation, the “recursion” Ritzberger and Shorish write about could be thought of as a self-feeding financial feedback loop – potentially a financial vicious circle. For instance, in the earlier example, the deterioration of A, that was originally caused by B, could damage the financial status of B even further. It is clear that these kind of scenarios have to be considered when it comes to pricing financial assets. The questions asked above go straight to the core of a very important and very timely problem: financial contagion and, indeed, systemic risk.
1.2 Asset valuation models

To understand why cross-ownership has virtually never been considered in mathematical finance, one needs to have a closer look at modern credit risk models. The reason for this is that Merton’s asset valuation model, also called the Merton model (Black and Scholes (1973), Merton (1973, 1974)) and its offspring today are mostly used in the context of credit risk management.

Merton’s model, where equity and debt are considered as derivatives of an underlying value process of the firm’s assets, inspired many extensions and refinements. These models are usually summarily called structural models, also “firm-value models” or “threshold models”, since they attempt to value assets by modelling the financial structure of the considered companies. In these credit risk models, a credit event is usually triggered when the assets (the “firm value”) of a company fall below a certain threshold – in Merton’s case the nominal amount of outstanding debt. This fundamental idea is the basis of a plethora of models that aim to improve the original Black-Merton-Scholes approach. Already Merton (1974) includes coupon paying bonds and explains how to incorporate stochastic interest rates. Further models that extend the original idea to stochastic interest rates, differing liabilities, differing maturities, counterparty risk (“vulnerable options”), and jump diffusion exist - including multi-firm models. It is far beyond the scope of this paper to give a comprehensive overview of these models. For a summary of the literature we refer to standard textbooks like Bingham and Kiesel (2004), as well as the references therein.

In contrast to the structural approaches, so-called reduced-form models attempt to model credit risk and especially default rates (therefore also the name “default rate models”) in a more statistical way, usually by assuming independence of credit events conditional on certain underlying stochastic factors. For an overview over credit risk models and over structural versus reduced-form models see for instance Crouhy, Galai and Mark (2000) and Arora, Bohn and Zhu (2005), or textbooks like Bingham and Kiesel (2004). In general, it is clear that default rate models are less useful if asset valuation (rather than credit risk management) for a broad spectrum of assets and liabilities is the main goal.

While the Black-Merton-Scholes approach is the basis of modern structural credit risk models (the model of Moody’s KMV (cf. Arora, Bohn and Zhu (2005) and Kealhofer and Bohn (2001)) possibly being the commercially most successful one), a rupture in the rationale of structural approaches seems to appear as soon as multiple firms are considered. While at the level of the individual firm the blueprint of the Merton model is usually clearly visible, at the multi-firm or inter-firm level, structural approaches seem to not get any consideration at all. For instance, models like the one of Moody’s KMV seem to not go far beyond modelling correlations (cf. Kealhofer and Bohn (2001)) or, more generally, multivariate distributions for the firms’ underlying assets. Such dependencies between assets certainly lead to dependencies between the firms’ default events when the original Black-Merton-Scholes idea is then applied to the individual firm – which is that default is triggered when the assets of the company fall below a certain level. However, it is obvious that such approaches do not attempt to model the actual ownership or cross-ownership structures that possibly contribute to the observed credit event correlations. In this sense, it is possibly fair to say that structural models turn reduced-form at the
multi-firm or inter-firm level.

Before moving on to the next section, in the context of contagion at the two-firm level, i.e. regarding counterparty risk or “vulnerable options”, there are certainly many papers considering this problem. For instance, see Jarrow and Turnbull (1995) and the references therein. Regarding financial contagion at a larger scale, it should be mentioned that there are articles on credit risk that focus especially on modelling credit contagion (e.g. Horst (2007)). However, the question of (and the role of) multi-firm, multi-liability, and multi-priority cross-ownership is generally not addressed in these papers.

1.3 General liabilities under cross-ownership

While the focus of the Merton model and its (structural) successors lies on credit risk, this is not the main focus of this paper. Let us assume that we had a ‘correct’ model of all considered firms’ financial structures which was also correctly describing the inter-firm structures in the form of cross-ownership – cross-ownership not only of equity, but also of general liabilities that besides debt could be in the form of derivatives that had been issued by or that were guaranteed by these firms. One would expect that such a ‘correct’ model should not only make it possible to model credit events more precisely. Additionally, better pricing of equities and liabilities in general should be possible. Similarly, it should be possible to properly value an issued derivative including any counterparty risk present.

In reality, and in contrast to most existing models, liabilities can be of differing seniority. The seniority of a liability, or the priority of the corresponding financial claim, defines the order of repayment in a potential liquidation event. For instance, senior debt must be paid before subordinate debt, where subordinate debt itself can again be ranked by so-called tiers. In general, equity in the form of common stock is subordinate to other liabilities, i.e. equity is the “residual claim”. It is therefore clear that the proper incorporation of the seniority of liabilities is important when it comes to valuation.

The general liabilities mentioned above should not only include debts or simple financial derivatives. In general, a liability could be any properly defined financial commitment. For example, it could be the (naked) short-sale of another asset or financial claim, a liability or a guarantee related to a mortgage-backed security, as well as an insurance liability. For instance, Walsh (2009) writes in The New York Times:

[…]

A.I.G.’s individual insurance companies have been doing an unusual volume of business with each other for many years – investing in each other’s stocks; borrowing from each other’s investment portfolios; and guaranteeing each other’s insurance policies, even when they have lacked the means to make good. […]

Here would be a real life situation where cross-ownership was (or is) possibly not only present in equities, but also in insurance liabilities or derivatives thereof – additionally to any other financial liabilities present, e.g. in the form of derivatives contracts with firms outside A.I.G. As far as not obvious anyway, this case clearly shows that cross-ownership is not only an issue in national Asian or European markets (cf. McDonald (1989) for Japan, Bøhren and Michalsen (1994) for Norway; also compare the concept of the Deutschland AG, a historic network of cross-ownership or “capital entanglement”
among Germany’s blue chip companies (cf. Höpner and Krempel (2003))), but it can very much also be an issue within a conglomerate of firms or within a holding company. As such, cross-ownership issues should have a very high priority for any regulator or regulatory framework like Basel II or Solvency II.

The importance of cross-ownership goes beyond the examples mentioned so far. Cross-ownership is present in more markets and at more levels than first meets the eye. For instance, consider a houseowner who serves a mortgage on her house. We can consider the houseowner as a financial firm, where the house is possibly the largest part of the assets, and the mortgage possibly is the main liability. Like a firm, the houseowner has a limited liability (at least in the U.S.) since she could declare personal bankruptcy or default on her mortgage should her total assets become worth less than her total liabilities. Assume now that the mortgage holder also owns a portfolio of stocks, for instance through her pension fund. Furthermore assume that the mortgage was arranged by a major investment bank, was sliced and packed into a mortgage-backed security which was sold by the investment bank, possibly together with a guarantee against default. It is now not at all beyond imagination that the pension fund of our mortgage holder not only owns shares in the mentioned investment bank, but maybe it also holds mortgage-backed securities that might include our homeowner’s mortgage, plus guarantee. On a nationwide or even global scale, it is therefore entirely imaginable that homeowners who owe a mortgage might in turn indirectly own parts of their own mortgage and possibly parts of other people’s mortgages, including the corresponding guarantees. They also might own shares in the involved investment banks that possibly still hold parts of these mortgages too. The implications are clear: a downturn in the housing market (for whatever reason) might not only affect the homeowner’s equity directly through the house ownership, but it might also affect her by owning the investment banks or mortgage-backed securities involved. This would affect the value of the mortgage, which in turn influences the value of the investment bank (if it holds any) and the value of any mortgage-backed securities and guarantees that are involved, etc. There is a clear risk of contagion or even a systemic risk present in such a situation. A realistic model should try to measure these risks and take them into account when pricing equities and liabilities.

The examples above should be sufficient to understand why modelling cross-ownership relationships and properly taking them into account when pricing equities and liabilities should be a major priority of mathematical finance. A generalized multi-firm structural model that incorporates cross-ownership structures should have a profound impact not only on the theory, but also on the practice of asset valuation. It is clear that it can be difficult to obtain cross-ownership information and that optimally one would like to have perfect information (lack of information, especially of price information about the underlying assets, is a general problem of all structural models). It is also clear that in many cases influences other than cross-ownership might play large roles in (or even dominate) the pricing procedure as well. However, some observers, like regulators, might have more information than ordinary market participants, and it is clear that any properly used cross-ownership information at all can improve a model by making it more realistic than one where such information is ignored. In that sense, any cross-ownership information is valuable.
1.4 Valued added

The model of this paper is an extension of the Merton model in the sense that it has only one maturity date and in the sense that prices at maturity are determined assuming no-arbitrage, which includes the assumption of no-arbitrage in a potential liquidation event. However, the main focus of the paper is not the calculation of no-arbitrage prices prior to maturity, but the determination of no-arbitrage prices at maturity, which is trivial in the Merton model, but not in the cross-ownership case. It will become clear that the calculation of no-arbitrage prices before maturity is straightforward by risk-neutral pricing (cf. Sec. 6.1) once no-arbitrage prices at maturity for any given scenario of the underlying exogenous assets have been determined. The ways in which our model extends the theory of asset valuation in Merton’s model are the following:

1. It models the cross-ownership of assets and liabilities for groups of firms.
2. Multiple different classes of liabilities, e.g. debt and derivatives, are allowed.
3. Multiple different priorities of claims in a possible liquidation are allowed.
4. It directly incorporates counterparty risk at the multi-firm level into derivatives pricing.
5. Underlying exogenous assets can be stochastically dependent, a feature other existing multi-firm models have.
6. A principle of capital structure irrelevance for a system of firms in a (no-arbitrage) price equilibrium is derived (value of externally held claims = value of exogenous assets).
7. Situations where valuation is impossible (no no-arbitrage price equilibrium/multiple no-arbitrage price equilibria) can exist.
8. It should be possible to extend any model based on Merton’s original idea (for instance the Moody’s KMV model) to incorporate cross-ownership along the lines of the model of this paper.

The main challenge in this paper is to formulate and solve the (in Merton’s single firm model: trivial) no-arbitrage equations (cf. Sec. 4.2) that properly account for cross-ownership in terms of the balance sheet items of all firms at maturity, under any given possible economic scenario of the underlying exogenously priced assets. Why this is trivial in Merton’s model but much less so in the extended model will become clear in the simple example of Section 2. A further important insight will be that in the case of a unique no-arbitrage price equilibrium, similarly to the Merton model, all equities and liabilities are direct derivatives of the underlying exogenously priced assets. This makes no-arbitrage pricing at times before maturity possible by means of risk-neutral valuation techniques.

The paper is structured as follows. After the motivating example of Section 2 and the introduction of some notation in Section 3 Section 4 presents and interprets the main results. These results are essentially contained in Theorem 2 which is on the existence
and uniqueness of no-arbitrage price equilibria under the cross-ownership of equities and general liabilities of differing seniority. Section 4 also derives results for the accounting equations of the considered system of firms. Measures of the degree of cross-ownership with regards to financial leverage and financial contagion are defined and discussed. After the proof of Theorem 2 in Section 5, we consider several applications and examples in Section 6. A conclusion and a technical appendix follow.

2 Cross-ownership: a two-firm example

To illustrate and motivate the main results of this paper, we will have a look at an example with only two companies. Assume that company \( i \) \((i = 1, 2)\) has outstanding nominal debt of \( b_i \geq 0 \). There are no coupon or interest payments, and both loans have to be paid back at the same future point in time (maturity). Company \( i \) holds assets of market value \( a_i \geq 0 \) at maturity. We assume that these assets are exogenously priced in the sense that the capital structure of the companies \( i = 1, 2 \) has no influence on the value of these assets. In the original Merton model, the equity of company \( i \) at maturity, \( s_i \), is now given by

\[
s_i = (a_i - b_i)^+ = \max\{0, a_i - b_i\}.
\]  

The reason for this valuation formula is that, under the bankruptcy laws of many economies, equity in firm \( i \) essentially is a European call option on the assets of \( i \), \( a_i \), with a strike price of \( b_i \). Expressed in layman’s terms, the owners of \( i \) get what is left of the company’s assets after the debt \( b_i \) has been paid. However, because of limited liability, the owners will never encounter negative equity, i.e. the owners will never have to make up for losses of the creditors of firm \( i \) by paying them out of their own pockets after their (the shareholders’) stake in the equity of \( i \) has become worthless. We can therefore say that \( s_i \) is the liquidation value (in a perfectly liquid market with no frictions like transaction costs or taxes) of the equity (“common stock”) of company \( i \) at maturity. The no-arbitrage price of company \( i \) at maturity should therefore be identical to \( s_i \) as otherwise arbitrage opportunities would exist. The creditors who gave company \( i \) the loan with principal \( b_i \) at outset recover at maturity the amount

\[
r_i = \min\{b_i, a_i\}.
\]  

From the creditors’ point of view, \( r_i \) is recoverable part of the claim \( b_i \) (also called recovery claim). From company \( i \)’s point of view, \( r_i \) is the payable part of the liability \( b_i \). The reason for (2) is that the creditors cannot collect more than the market value of the company’s assets, \( a_i \), even if \( a_i \) is less than the outstanding principal \( b_i \). Therefore, \( r_i \) can be interpreted as the liquidation value of the nominal debt \( b_i \) at maturity, and hence it must be identical to the price of this debt at maturity under the assumption of no arbitrage. Actual market prices and no-arbitrage prices might converge in real markets (even in the absence of a realistic chance of immediate liquidation) for the reason of so-called capital-structure arbitrage where hedge funds “attempt to use these models to buy the underpriced part of a firm’s capital structure, be it debt or equity, and sell the overpriced part” (Robert C. Merton in Mitchell (2004)).
As an example, if \( b_i = 100 \), and at maturity \( a_i = 150 \), then equity would be \( s_i = 50 \) and the loan recovery \( r_i = 100 \), i.e. the full loan would be recovered. In contrary, was \( b_i = 100 \) as before, but at maturity \( a_i = 50 \), then equity would be \( s_i = 0 \) since the owners of the firm would not exercise their call option on the de-facto negative equity. The loan recovery would therefore only be \( r_i = a_i = 50 \).

In the classical Merton model, the (by \( a_i \) and \( b_i \)) uniquely determined no-arbitrage prices at maturity, (1) and (2), are used to obtain no-arbitrage prices for stocks and bonds at times before maturity by modelling the price process of the assets \( (a_i \) at maturity) before maturity by a geometric Brownian motion. The Black-Scholes model (Black and Scholes (1973), Merton (1973)) then provides a direct solution for the no-arbitrage prices of equities and debts (Merton (1974)). The crucial non-stochastic ingredients of this pricing approach are the no-arbitrage prices at maturity given by (1) and (2) which show that equity and debt are mere derivatives of the underlying exogenous assets.

Expressed at the level of our example with two firms, the question this paper will investigate is: what happens to the no-arbitrage prices of stocks and bonds at maturity, \( s_i \) and \( r_i \), if company \( i \) \((i = 1, 2)\) is allowed to own stock or bonds of company \( j \) \((j = 2, 1)\)? In other words, can (and if so, how) the Merton model be generalized to the case where cross-ownership is allowed?

This is a very important question for two reasons. First and foremost, as pointed out before, cross-ownership is present in real markets, therefore it has to be modelled appropriately. At the time of writing this paper, we are not aware of any extension or generalization of Merton’s corporate debt model in this direction. Second, if under cross-ownership there were unique no-arbitrage prices at maturity as in the case of the classical Merton model without cross-ownership, then, because of the existence of an implicit function that mapped exogenous asset prices on endogenous assets’ no-arbitrage prices, the calculation of no-arbitrage prices at times other than maturity would be straightforward from the completeness of the original Black-Scholes approach used in Merton (1974). The reason for this is that as in the case of the classical Merton model, the no-arbitrage prices of equities and debts at maturity would simply be derivatives of the underlying exogenous assets and they could therefore be priced by the risk-neutral pricing approach using an equivalent martingale measure. For this reason, the present paper will focus on the investigation of no-arbitrage prices at maturity, since at least from a theoretical perspective, no-arbitrage prices before maturity are straightforward to obtain if unique price equilibria at maturity exist and if a complete model for the price processes of the exogenous assets is chosen.

For an illustration of the changed situation under cross-ownership again consider the above two-firm example \((b_i = 100 \text{ for } i = 1, 2)\), but with the following changes to its set-up: assume that company \( i \) \((i = 1, 2)\) owns 50% of the equity of company \( j \) \((j = 2, 1)\). We assume here that ownership in any of the two companies is an homogeneous asset class where all owners have the same rights in proportion. Assume now that there were no-arbitrage prices for equities and debts, \( s_i \) and \( r_i \) \((i = 1, 2)\). As before in the case with no cross-ownership, we define no-arbitrage prices as no-arbitrage prices under liquidation. If this is the case, equation (1) turns into \((i = 1, 2; j = 2, 1)\)

\[
  s_i = (a_i + 0.5s_j - b_i)^+, \tag{3}
\]
since the total assets of company $i$ ($i = 1, 2$) are given by the exogenous assets $a_i$ plus the share in the equity of company $j$ ($j = 2, 1$), $0.5s_j$. The recoverable debt at maturity, $r_i$, is

$$r_i = \min\{b_i, a_i + 0.5s_j\}.$$  \hfill (4)

The fundamental difference to the set-up with no cross-ownership in equations (1) and (2) is that (3) is not a direct expression of $s_i$ in terms of the exogenous assets $a_i$ and the debt level $b_i$. Instead, (3) is a system of two equations ($i = 1, 2; j = 2, 1$) for the two unknowns $s_1$ and $s_2$. Having determined (if possible) $s_1$ and $s_2$, the recoveries $r_i$ follow immediately from (4). The fundamental question in this example is therefore: given $a_i \geq 0$ and $b_i \geq 0$ ($i = 1, 2$), has system (3) a solution $(s_1^*, s_2^*)$, and if so, is this solution unique? If both was the case and if $b_i$ ($i = 1, 2$) was assumed to be fixed, then

$$(s_1^*, s_2^*) = (s_1^*(a_1, a_2), s_2^*(a_1, a_2))$$ \hfill (5)

would be an implicit function (in finance terms: a derivative) of the $a_i$ ($i = 1, 2$). If this function was (Lebesgue-)measurable, then one would know how to price the equities and loans at times before maturity using risk-neutral valuation for a given complete model of the exogenous assets.

Similarly to the above case, we could consider a set-up with no equity cross-ownership, but cross-ownership of debt instead: assume that company $i$ ($i = 1, 2$) owns 50% of the debt of company $j$ ($j = 2, 1$). Again, assume that ownership of the debt of any of the two companies is an homogeneous asset class where all owners have the same rights in proportion. The new system for no-arbitrage equity prices would therefore be ($i = 1, 2; j = 2, 1$)

$$s_i = (a_i + 0.5r_j - b_i)^+,$$ \hfill (6)

and the recovery of debt at maturity would be ($i = 1, 2; j = 2, 1$)

$$r_i = \min\{b_i, (a_i + 0.5r_j)^+\}. \hfill (7)$$

The $(\cdot)^+$ in the last expression is there since debt recovery can not be negative – similar to the equity owned by the shareholder. While this time the equity system (6) is straightforward, the important question is whether the loan recovery system (7) has a unique solution $(r_1^*, r_2^*)$.

The answer to the questions asked in both of the above examples is yes: no-arbitrage prices exist and they are unique. This is a direct consequence of the main result of this paper in Section 4.5. Existence and uniqueness of no-arbitrage prices also apply in much more general set-ups than above, for instance, we can allow cross-ownership of equities and loans (and even derivatives that have been issued on these) at any percentage level at the same time and for any number of companies with differing debt levels. This is a major generalization of Merton’s original model and should have direct implications for many existing credit and derivatives pricing models in theory and practice that use the Merton model as a basis.

To give a simple numeric example, assume for each of the three set-ups above (no cross-ownership, 50% stock cross-ownership, 50% bond cross-ownership) that the companies 1 and 2 have exactly the same debt levels, $b_1 = b_2$, and that they own exactly the same
type of exogenous assets, i.e. $a_1 = a_2$. The no-arbitrage prices for their equities and debts will therefore be identical. Table 1 contains in each row for each of the three different set-ups a value (scenario) for $a_1 = a_2$ that implies the same no-arbitrage prices under all three set-ups. While large-scale numerical examples and more detailed examples lie outside the scope of this paper, a few remarks regarding Table 1 seem appropriate. First, the second row clearly marks the bankruptcy level for all three set-ups. Second, in both cases, 50% stock cross-ownership and 50% bond cross-ownership, starting with the first row, a change of -75 (or for both firms a total of -150) in the exogenous assets, down to the level in row three, is enough to wipe out all equity and 50% of the bond value. This is a loss of value of 100 per firm, or 200 for both firms together. In the case of no cross-ownership, a change of -100 (or for both firms a total of -200) is needed for the same result. So, as should be entirely expected, starting at the same level of equity and debt, the leverage caused by the cross-ownership structure in the market clearly creates a higher risk regarding moves in the prices of the exogenous assets. This is a feature classical corporate debt models or credit risk models can not properly reflect since they are not attempting to model cross-ownership structures that are present. Mere modelling of correlations or other dependency structures between exogenously priced assets (which is possible within the model of this paper as well) is not sufficient to reflect the actual interdependencies that stem from the cross-ownership of assets.

### 3 Notation and mathematical preliminaries

We write $M > N$ if for two matrices $M, N \in \mathbb{R}^{n \times n}$ with $M = (M_{ij})_{i,j=1,...,n}$, $N = (N_{ij})_{i,j=1,...,n}$ we have $M_{ij} \geq N_{ij}$ for $i, j \in \{1, \ldots, n\}$, and $M_{ij} > N_{ij}$ for at least one pair $(i, j)$. A matrix $M \in \mathbb{R}^{n \times n}$ is called left substochastic if $M$ is non-negative, i.e. $M_{ij} \geq 0$ for $i, j \in \{1, \ldots, n\}$, and for any $j$ one has $\sum_{i=1}^{n} M_{ij} \leq 1$, i.e. the sums of columns are less than or equal to 1. We will call a column $j$ of $M$ strictly substochastic if $\sum_{i=1}^{n} M_{ij} < 1$. We will call $M$ strictly left substochastic if all columns of $M$ are strictly substochastic. A right substochastic matrix is a transposed left substochastic matrix. A left substochastic matrix can be interpreted as an ownership matrix (see also Ritzberger and Shorish (2002)). The meaning and usage of an ownership matrix will be explained in Section 4.1. We will distinguish between column vectors $a = (a_1, \ldots, a_n)^t \in \mathbb{R}^{n \times 1}$ and row vectors $a^t = (a_1, \ldots, a_n) \in \mathbb{R}^{1 \times n}$. However, for column vectors we will often conveniently write $a \in \mathbb{R}^n$ and $a = (a_1, \ldots, a_n)$. From the context, no confusion should arise. The meaning

| no XOS | 50% stock XOS | 50% bond XOS | no-arbitrage prices |
|--------|----------------|--------------|---------------------|
| $a_1 = a_2$ | $a_1 = a_2$ | $a_1 = a_2$ | $s_1 = s_2$ | $r_1 = r_2$ |
| 150    | 125            | 100          | 50                  | 100                |
| 100    | 100            | 50           | 0                   | 100                |
| 50     | 50             | 25           | 0                   | 50                 |

Table 1: Example for $b_1 = b_2 = 100$
of \( a > b \) and \( a' > b' \) for \( a, b \in \mathbb{R}^n \) is analogue to the conventions for matrices. We use the symbols \( 0 = (0, \ldots, 0) \) (where \( 0 \) might also be used for a zero matrix), \( 1 = (1, \ldots, 1) \), and \( I \) for the identity matrix. In all cases where these symbols are used, the dimension should be clear from the context. The following operations will apply element-wise to matrices and vectors: the positive part, \((\cdot)^+\); the negative part, \((\cdot)^-\); the maximum, \(\max\{\cdot, \cdot\}\); the minimum, \(\min\{\cdot, \cdot\}\). We will make use of the \(\ell^1\)-norm on \(\mathbb{R}^n\), for \(x \in \mathbb{R}^n\) given by

\[
||x||_1 = \sum_{i=1}^{n} |x_i|.
\]  

(8)

For the \(\ell^1\)-norm of any strictly left substochastic matrix \(M\) we have \(x \in \mathbb{R}^n\)

\[
||M||_1 = \max_{||x||_1 = 1} ||M \cdot x||_1 = \max_j \sum_{i=1}^{n} M_{ij} < 1,
\]

(9)

that means \(||M||_1\) is the maximum of the column sums. As expected, it follows that

\[
||M \cdot x||_1 \leq ||M||_1 \cdot ||x||_1,
\]

since

\[
||M \cdot x||_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} x_j \leq \sum_{j=1}^{n} |x_j| \cdot \sum_{i=1}^{n} M_{ij}
\]

\[
\leq \left( \max_j \sum_{i=1}^{n} M_{ij} \right) \cdot ||x||_1 = ||M||_1 \cdot ||x||_1.
\]

4 No-arbitrage pricing of general liabilities

4.1 General liabilities of differing seniority

In Section 2 we considered a situation where only stocks and bonds could be cross-owned and where the priorities of claims in a possible liquidation were clear by default (bonds more senior than stocks). In this section, where we consider the case with \(n\) companies \((n \in \{1, 2, \ldots\})\), we allow the cross-ownership of equities and more general liabilities, including debts and derivatives. For these liabilities we allow differing priorities of the corresponding recovery claims under liquidation.

Suppose that the vector \(a = (a_i)_{i=1,\ldots,n} \geq 0\) summarily denotes the exogenous assets of market value \(a_i \geq 0\) held by company \(i\) \((i = 1, \ldots, n)\) at maturity. These assets are exogenously priced in the sense that it is assumed that the capital structure of the companies \(1, \ldots, n\) has no influence on the price of these assets. The exogenous assets \(a\) could include physical assets like commodities or property, the work force of a company, intellectual property, but also cash, future cash flows or claims on (equities or liabilities of) external firms as long as these are not affected by the considered \(n\) firms' capital structures. We assume that the exogenous assets or parts thereof can be sold at any time at the given price \(a\). A sale of only a part of the assets would not affect the price of the remaining parts. We will see in Section 6.2 that the dimension of the exogenous assets' price vector \((n\) above, and hence identical to the number of firms\) is irrelevant to our
considerations, and any number of exogenous assets could be considered. We choose the dimension \( n \) for convenience only.

As mentioned earlier, a left substochastic matrix can be interpreted as an ownership matrix. Let \( s \in \mathbb{R}^n \) denote the prices of the equities of the companies \( 1, \ldots, n \). Furthermore, suppose that the equities of the companies \( 1, \ldots, n \) with prices \( s \) are at least partially owned by the companies themselves. In particular, we assume that company \( i \) owns a proportion \( 0 \leq M^s_{ij} \leq 1 \) of the equity of company \( j \) (called the “cross-owned fraction” in Bøhren and Michalsen (1994)). This (partial) ownership is worth \( M^s_{ij} s_j \). The ownership structure of the equities can therefore be described by the left substochastic matrix \( M^s = (M^s_{ij})_{i,j=1,\ldots,n} \) (cf. Ritzberger and Shorish (2002)). The total value of any equity that company \( i \) owns is given by the \( i \)-th entry of the vector \( M^s \cdot s \), i.e. by

\[
\sum_{j=1}^{n} M^s_{ij} s_j. \tag{11}
\]

Similarly, we can describe the cross-ownership structure of any outstanding liabilities by means of a left substochastic matrix.

In order to properly reflect cross-ownership structures that are present in a group of firms, any firm that owns equities or liabilities of firms in this group, where at the same time the group owns part of the equity or liabilities of the considered firm, should be included in the model. Firms which only own equities or liabilities of other firms, but who’s own equity or liabilities are not at least partially owned by other firms, do not have to be considered since they are price takers in this market in the sense that their own assets and liabilities do not influence other firms’ balance sheets.

**ASSUMPTION 1.** Let \( M^a \) and \( M^{d,i} \) \( (i = 1, \ldots, m) \) be strictly left substochastic matrices. Let \( a \in (\mathbb{R}_0^+)^n \). For \( i = 1, \ldots, m \), let \( d^i \) be a function

\[
\mathbb{R}^{n(m+1)} \rightarrow (\mathbb{R}_0^+)^n \tag{12}
\]

\[
\begin{pmatrix}
  r^1 \\
  \vdots \\
  r^m \\
  s
\end{pmatrix} \mapsto d^i_{r^1, \ldots, r^m, s}. \tag{13}
\]

Suppose now that the \( n \) elements of the vector \( d^i \) define the liabilities of the \( n \) considered firms, where \( i = 1, \ldots, m \) are the priorities of the corresponding claims in a possible liquidation of any firm (e.g. by Chapter 7), such that 1 is the highest priority (paid first) and \( m \) is the lowest (paid last). We assume that these liabilities are payable in cash. The functions \( d^i \) are non-negative since negative liabilities are assets (another firm’s liability) and hence they will be modelled as such. Suppose that the \( r^i \in \mathbb{R}^n \) are the vectors of recovery claims belonging to \( d^i \). So, while the \( d^i \) describe what is supposed to be paid at maturity, the \( r^i \) stand for the actual payoff, which could be after a liquidation due to the default of the writer of the liability. In general, we should therefore have

\[
0 \leq r^i \leq d^i. \tag{14}
\]
For simplicity, we assume that there is only one liability per firm per level of seniority. However, as long as it was clear how to split the corresponding recovery claim in any economic scenario of a liquidation situation in which the claim would be recovered at less than par, it would be no problem to assume that any of these liabilities was a sum of several liabilities of the same seniority. If such an agreement (of a split) did not exist, one could possibly assume a repayment pari passu.

A very simple version of a liabilities function \( d_i \) would be where \( d_{r_1, \ldots, r_m, s} \equiv b_i \in (\mathbb{R}_+^\infty)^n \). In this case, the liabilities would be simple loans, rather than general liabilities, like derivatives that could depend on other assets like \( s \). However, since \( d_{r_1, \ldots, r_m, s} \) has the \( r_i \) as arguments, the liability functions can even depend on their own (!) eventual payoff (see an example in Sec. 6.4). Assumption 1 is also very general in the sense that no restrictions on derivatives regarding exogenous assets have been made. This is because exogenous assets are treated as constants in our framework since all considerations are conditional on given prices for exogenous assets. So, to be very clear about this and although we will not use this notation later, the \( d_i \) may also depend on the exogenously priced assets \( a \), i.e. one generally has

\[
d_{r_1, \ldots, r_m, s} = d_{r_1, \ldots, r_m, s, a}.
\]  

For an example see Section 6.3.

Regarding the issue of strictly substochastic matrices in Assumption 1 (rather than just substochastic ones), an example in Sec. 6.5 will illustrate the kind of problems that can arise if ownership matrices are not strictly substochastic. A more thorough discussion of the possible effects of non-strictly substochastic matrices is beyond the scope of this paper.

### 4.2 The liquidation value equations

Suppose now that the matrices \( M^s \) and \( M^{d,i} \ (i = 1, \ldots, m) \) are ownership matrices as in Section 4.1. Assume that \( M^s \) describes the equity (common stock) cross-ownership in the system of \( n \) firms, while the \( M^{d,i} \ (i = 1, \ldots, m) \) describe the cross-ownership of general liabilities, with \( M^{d,i} \) belonging to \( d^i \). Under this set-up and under Assumption 1 we demand the following at maturity:

**ASSUMPTION 2** (Absolute Priority Rule). *The priority of claims is honored. Equity is the residual claim.*

The Absolute Priority Rule, which requests strict adherence to the seniority of liabilities in a liquidation in the sense that any higher rank claim has to be fully paid off before any lower rank claim can be paid, is not always honored in real life. Longhofer and Carlstrom (1995) write:

While this rule would seem quite simple to implement, it is routinely circumvented in practice. In fact, bankruptcy courts themselves play a major role in abrogating this feature of debt contracts.
However, the circumstances (liquidation vs. reorganization, or political intervention) and also the extent to which the rule is disregarded vary (see e.g., Eberhart, Moore and Roenfeldt (1990)) and can not be discussed in this paper. For the purpose of our theory, Assumption 2 is reasonably close to practice.

Under Assumption 2 the equations

\[
\begin{align*}
    r^1 &= \min \left\{ d^{1,*}, \ldots, d^{m,*} \cdot s, \ a + M^s \cdot s + \sum_{i=1}^{m} M_{d,i} \cdot r^i \right\} \\
    s &= \left( a + M^s \cdot s + \sum_{i=1}^{m} M_{d,i} \cdot r^i \right)^+ 
\end{align*}
\]

follow for the liquidation values of \(r^1, \ldots, r^m\) and \(s\). Liquidation means here that at maturity all assets are converted into cash and subsequently all equity is paid out in cash as well. Because of Assumption 2 each claim on a liability will only pay the minimum of the promised payoff and the value of the remaining assets of the firm after liabilities of higher seniority have been paid. In this sense, the equations (16) – (18) are the generalization of the Merton equations (1) and (2) for the multi-firm case with general liabilities under cross-ownership. However, while in Merton’s case the liquidation of equity is trivial since any equity is identical to the remaining exogenous assets after paying the debt, it is much less obvious how such a liquidation would work in our case due to the equity entanglement caused by cross-ownership. For a further and more thorough discussion of the issue of liquidation see therefore Section 4.6. In that section we will also show under fairly reasonable assumptions that the equations (16) – (18) must hold for the no-arbitrage prices of equities and liabilities at maturity. Hence, no-arbitrage prices and liquidation values are identical.

Note that \(r^1 \) are the recovery claims with the highest priority in a liquidation, and \(r^m \) are the recoveries of the lowest priority claims. Equity \(s \) is, of course, the first asset to be wiped out, then, in this order, the recoveries \(r^m \) to \(r^1 \) get wiped out (component-wise). The model is flexible enough to not only incorporate bonds and derivatives of differing seniority, but also some derivatives could rank higher than some bonds. We call a non-negative solution \((r^{1*}, \ldots, r^{m*}, s*)\) of the system (16) – (18) a no-arbitrage price equilibrium. In Assumption 2 we defined equity as the “residual claim”. In practice, the asset class closest to this would be “common stock”. The asset class of “preferred stock”, which usually has more rights and a higher seniority in a default event, would in our set-up be modelled as one of the liabilities ranking higher than equity.

In a first attempt, one would possibly formulate the liquidation value equations (16) – (18) such that the first one would read

\[
\begin{align*}
    r^1 &= \min \left\{ d^{1,*}, \ldots, d^{m,*} \cdot s, \ a + M^s \cdot s + \sum_{i=1}^{m} M_{d,i} \cdot r^i \right\} \\
    s &= \left( a + M^s \cdot s + \sum_{i=1}^{m} M_{d,i} \cdot r^i \right)^+ 
\end{align*}
\]
LEMMA 1. Under Assumption [4] any solution \((r_1^*, \ldots, r^*_m, s^*)\) of the system \((16) - (18)\) is non-negative. Hence, \((16) - (18)\) and \((19), (17)\) and \((18)\) have identical solutions.

For the proof of the lemma we will use the following notation. Let \(\pi\) be a permutation (i.e. a bijection) on \(\{1, \ldots, n\}\) and \(\mathbf{a} \in \mathbb{R}^n\) and \(\mathbf{M} \in \mathbb{R}^{n \times n}\). We denote \(\pi \mathbf{a} = (a_{\pi(1)}, \ldots, a_{\pi(n)})\), and we denote \(\pi \mathbf{M}\) for the matrix obtained from \(\mathbf{M}\) by permuting elements such that \(\pi M_{ij} = M_{\pi(i)\pi(j)}\) for \(i, j \in \{1, \ldots, n\}\). The latter is a simultaneous permutation of rows and columns.

**Proof.** Let \((r_1^*, \ldots, r^*_m, s^*)\) be a solution of \((16) - (18)\). Hence, \(r_2^*, \ldots, r^*_m, s^* \in (\mathbb{R}_0^+)^n\). Consider now Eq. \((16)\), and substitute with a solution \((r_1^*, \ldots, r^*_m, s^*)\) which contains at least one \(r_i^* < 0\) \((i \in \{1, \ldots, n\}\)). By the help of a permutation \(\pi\) on \(\{1, \ldots, n\}\), we can now re-arrange rows and columns of this new system such that it has the equivalent form

\[
\pi r^*_i = \min \left\{ \pi (d_{1i}^*, \ldots, r^*_m, s^*), \pi \mathbf{a} + \pi \mathbf{M} s^* \cdot \pi s^* + \sum_{i=1}^{m} \pi \mathbf{M} d_{i} \cdot \pi r^*_i \right\},
\]

where rows 1 to \(j\) \((j \in \{1, \ldots, n\})\) are non-negative, and rows \(j + 1\) to \(n\) are negative. The matrices \(\pi \mathbf{M}^s\) and \(\pi \mathbf{M}^d\) are again strictly left substochastic matrices since sums of columns are still less than 1. Consider now the subsystem that consists of the negative rows from \(j + 1\) to \(n\). Row \(k\) \((j + 1 \leq k \leq n)\) has the form

\[
\pi r^*_k = \pi a_k + \sum_{i=1}^{n} \pi \mathbf{M}^d_{kl} \cdot \pi s^*_l + \sum_{i=1}^{m} \sum_{i=1}^{n} \pi \mathbf{M}^d_{kl} \cdot \pi r^*_i.
\]

However, since \(\pi r^*_2, \ldots, \pi r^*_m, s^* \in (\mathbb{R}_0^+)^n\) and \(\pi r^*_l \geq 0\) for \(1 \leq l \leq j\), we can write

\[
\pi r^*_k = c_k + \sum_{l=j+1}^{n} \pi \mathbf{M}^d_{kl} \cdot \pi r^*_l
\]

for some \(c_k \geq 0\). Define now

\[
\mathbf{r}' = (r_{j+1}^1 \cdot \ldots, r^*_n)^t,
\]

\[
\mathbf{c} = (c_{j+1}, \ldots, c_n)^t,
\]

\[
\mathbf{M}' = (\pi \mathbf{M}^d_{kl})_{k,l=j+1, \ldots, n}.
\]

The system \((22)\) (for \(j + 1 \leq k \leq n\)) can now be written as

\[
\mathbf{r}' = \mathbf{c} + \mathbf{M}' \cdot \mathbf{r}.
\]
where \( r' < 0 \) and \( c \geq 0 \). Since the matrix \( M' \) as the lower right \((n-j) \times (n-j)\)-submatrix of \( \pi \pi M^{d,1} \) is again a strictly left substochastic matrix, we can apply Lemma 3 of Section A.1 to the system

\[
r' = (I - M')^{-1} \cdot c,
\]

which is equivalent to (26). Since \((I - M')^{-1}\) is non-negative according to Lemma 3, Eq. (27) is a contradiction. \( \square \)

### 4.3 Accounting equations

Before we turn to results about the existence of no-arbitrage price equilibria in Section 4.5, it is useful to first consider the balance sheet equations under no-arbitrage, as well as measures of leverage and cross-ownership (in Sec. 4.4).

Under no-arbitrage, the accounting equations (or balance sheet equations),

\[
a + M^s \cdot s + \sum_{i=1}^{m} M^{d,i} \cdot r^i = s + \sum_{i=1}^{m} r^i
\]

\( \text{assets + receivables} = \text{equity + payable liabilities} \),

follow directly from applying Eq. (84) in Lemma 4 in Section A.3 to the right side of the sum of the system (16) – (18). Under no-arbitrage, Eq. (28) holds componentwise (per firm) with all components being non-negative. It therefore also holds in absolute terms (\( \ell_1 \)-terms) for the total capital in the considered system:

\[
||a||_1 + ||M^s \cdot s||_1 + \sum_{i=1}^{m} ||M^{d,i} \cdot r^i||_1 = ||s||_1 + \sum_{i=1}^{m} ||r^i||_1
\]

\( \text{total assets + total receivables} = \text{total equity + total payable liabilities} \).

The right side of (29) represents all claims on equity and liabilities of the financial firms \( 1, \ldots, n \). The left side, apart from \( ||a||_1 \), represents all claims on equity and liabilities that are claimed within the system of the firms \( 1, \ldots, n \) itself. Expressed differently, the following mathematically trivial (from (29)) but from an economic perspective certainly important equation,

\[
||s||_1 + \sum_{i=1}^{m} ||r^i||_1 - \left( ||M^s \cdot s||_1 + \sum_{i=1}^{m} ||M^{d,i} \cdot r^i||_1 \right) = ||a||_1
\]

\( \text{all claims} - \text{internally held claims} = \text{exogenous assets} \),

means that under no-arbitrage the sum of the values of all externally held claims regarding the financial firms \( 1, \ldots, n \) is identical to the value of all exogenous assets owned by these firms. It is remarkable that ownership structures and the amount of financial leverage therefore have no influence on the aggregate value of externally held assets regarding these companies. In other terms, all asset value that stems from internal leverage (internal leverage will be properly defined in Eq. (32) below) is contained within the system and
hence somewhat irrelevant to outsiders who hold claims of this system but who are not themselves partially owned or financially liable to this system. Although being relevant for a system of firms only, and not for individual firms, this seems to be an interesting principle of capital structure irrelevance. We therefore state it as a separate theorem.

**THEOREM 1.** Under the Assumptions 1 and 2 the no-arbitrage value of all externally held claims (liabilities and equity) belonging to a group of firms is identical to the no-arbitrage value of the exogenous assets owned by the group:

\[ \text{value of externally held claims} = \text{value of exogenous assets}. \quad (31) \]

The aggregate value of the externally held claims is therefore independent of any other aspects of the financial structure of this group.

### 4.4 Measures of leverage and cross-ownership

From Eq. (30) in the previous section it is fairly obvious that a high degree of cross-ownership within a group of financial firms (i.e. when the matrices \( M \) have large entries; for instance, all column sums could be close to 1) leads to artificially large balance sheets when compared with the underlying exogenous asset base. In this section we therefore look at measures of financial leverage and cross-ownership under no-arbitrage.

It is clear that the standard measure of financial leverage in the case of a single firm, the debt-to-equity ratio (it should be liabilities-to-equity anyway), is not very useful if we consider a whole group of firms and if we want to assess the degree of their (internal) leverage as a group. In fact, Eq. (30) shows that even in the case of no liabilities at all, the cross-ownership of equity alone could cause balances to be a large multiple of the value of the underlying exogenous assets. We therefore need to find a meaningful measure of the internal leverage for a group of financial firms. Assume now \( ||a||_1 > 0 \), and define

\[
L = \frac{||M^s \cdot s||_1 + \sum_{i=1}^{m} ||M^{d,i} \cdot r^i||_1}{||a||_1}, \quad (32)
\]

\[= \frac{\text{internally held claims}}{\text{exogenous assets}} \text{ internally held claims} \quad \text{(31)} \frac{\text{externally held claims}}{\text{exoterically held claims}}. \quad (33)\]

This value can be interpreted as the level of internal financial leverage in the considered system. Note that \( L = L(a) \) since not only \( ||a||_1 \), but also \( s \) and the \( r^i \) depend on \( a \). Furthermore, by (29),

\[
L + 1 = \frac{||s||_1 + \sum_{i=1}^{m} ||r^i||_1}{||a||_1}, \quad (34)
\]

\[= \frac{\text{total claims}}{\text{exoterically held claims}} \text{ total assets} \frac{\text{exogenous assets}}{\text{exoterically held claims}}. \quad (35)\]

Related to (32) is for \( ||s||_1 + \sum_{i=1}^{m} ||r^i||_1 > 0 \) the value defined by (cf. (30))

\[
I = \frac{||M^s \cdot s||_1 + \sum_{i=1}^{m} ||M^{d,i} \cdot r^i||_1}{||s||_1 + \sum_{i=1}^{m} ||r^i||_1} = \frac{L}{L + 1}, \quad (36)
\]

\[= \frac{\text{internally held claims}}{\text{total claims}}. \quad (37)\]
which could be seen as a measure of the degree of financial cross-ownership. We could also consider this to be a measure of financial inbreeding or self-excitement. Because of \( ||M^s||_1, ||M^{d,i}||_1 \in [0, 1) \) for \( i = 1, \ldots, m \) (cf. Eq. (27)), it follows from (11) that for

\[ I^{\text{max}} = \max\{ ||M^s||_1, ||M^{d,1}||_1, \ldots, ||M^{d,m}||_1 \} \quad (38) \]

one has

\[ 0 \leq I \leq I^{\text{max}} < 1. \quad (39) \]

Since \( L = \frac{I}{1-I} \) and since \( L \) is a strictly monotonically increasing function of \( I \), it follows immediately that for

\[ L^{\text{max}} = \frac{I^{\text{max}}}{1 - I^{\text{max}}} \quad (40) \]

one has

\[ 0 \leq L \leq L^{\text{max}} < +\infty \quad (41) \]

and

\[ ||s||_1 + \sum_{i=1}^m ||r_i||_1 \overset{34}{=} (L+1)||a||_1 \leq (L^{\text{max}} + 1)||a||_1. \quad (42) \]

The upper boundary \( L^{\text{max}} \) is sharp, as an example in Section 6.6 demonstrates. Another straightforward conclusion from the Equations (39) and (29) is that \( ||a||_1 = 0 \) is equivalent to \( ||s||_1 = ||r^1||_1 = \ldots = ||r^m||_1 = 0 \).

The value \( L + 1 \) is the value of the sum of the balance sheets of all firms expressed in terms of the sum of all exogenous assets (cf. Eq. (35)). It seems therefore plausible that the higher \( L \), or, equivalently, the lower the percentage of exogenous assets in the balance sheets, the higher the risk for the balance sheets that stems from these exogenous assets in absolute terms. Such risk could for instance materialize in the form of an instantaneous shock in the prices of the exogenous assets just before (at) maturity. The values \( L \) and \( I \) could therefore also be seen as measures of systemic risk or financial contagion in the absence of liquidity risk (see also Section 5.6). As we pointed out earlier, \( L = L(a) \) and \( I = I(a) \) (see also the example in Table 2). Therefore, the considered measures are calculated for one particular scenario for the exogenous assets. For a physical probability measure, \( \mathbb{P} \), or for an equivalent martingale measure, \( \mathbb{Q} \), for instance taken from a stochastic model for \( a \) (cf. Sec. 6.1), one method to obtain such measures for all scenarios at the same time could be to calculate the expectations

\[ \mathbb{E}_\mathbb{P}[L], \mathbb{E}_\mathbb{Q}[L] \leq L^{\text{max}}, \quad \text{or} \quad \mathbb{E}_\mathbb{P}[I], \mathbb{E}_\mathbb{Q}[I], \leq I^{\text{max}}. \quad (43) \]

It is clear that no-arbitrage prices before maturity, obtained for instance by risk-neutral pricing (cf. Sec. 6.1), would already reflect any risk from systemic cross-ownership and leverage. See Sec. 6.1 also for a further idea how measures of cross-ownership could be obtained at times before maturity.

In summary it can be said that under no-arbitrage cross-ownership does not overstate equity – the equity is there, according to no-arbitrage prices. However, cross-ownership can somewhat artificially inflate balance sheets (and potentially share prices) in comparison to the underlying exogenous assets. Cross-ownership should therefore be an integral part...
of any measure of market leverage. For instance, Table 2 shows the values of $L$ for the example in Section 2. The first row of values demonstrates from left to right how decreasing exogenous assets prices but increasing leverage lead to the same no-arbitrage prices of equities and liabilities, i.e. the total assets on the balance sheets in the system are identical in all three cases. In Section 1.1 we quoted Ritzberger and Shorish (2002).

Table 2: Example of Sec. 2 for $b_1 = b_2 = 100$

| no XOS | $50\%$ stock XOS | $50\%$ bond XOS | no-arbitrage prices |
|--------|-----------------|----------------|-------------------|
| $a_1 = a_2$ | $L$ | $a_1 = a_2$ | $L$ | $a_1 = a_2$ | $L$ | $s_1 = s_2$ | $r_1 = r_2$ |
| 150 0 | 125 0.2 | 100 0.5 | 50 | 100 |
| 100 0 | 100 0 | 50 1 | 0 | 100 |
| 50 0 | 50 0 | 25 1 | 0 | 50 |

with the statement “that under cross-ownership the book value of a firm will tend to be overestimated with respect to the underlying cash flows.” This would be exactly the case here if we considered discounted cashflows for the exogenous assets. In that sense, cross-ownership can cause leverage that not only increases the risk of contagion, but there also exists a moral issue as far as some investors might not be aware of the kind of hidden leverage cross-ownership of equity can cause. However, as we have seen in Theorem 1, external owners of claims and equities are in aggregate not affected by cross-ownership leverage.

In contrast to our measure of internal leverage, for $||s||_1 > 0$, a measure for the external financial leverage in the system could be defined as the ratio of payable externally held liabilities to externally held equity (cf. debt-to-equity),

$$L_{\text{ex}} = \frac{\sum_{i=1}^{m} ||r^i||_1 - \sum_{i=1}^{m} ||M^{d,i} \cdot r^i||_1}{||s||_1 - ||M^e \cdot s||_1} = \frac{\text{externally held liabilities}}{\text{externally held equity}}.$$  

However, in the light of Theorem 1, for external claimholders, external leverage is essentially as irrelevant as internal one since there is no influence on the value of all externally held assets, which is constant $||a||_1$.

While we are not considering measures for the leverage of individual companies here, they certainly serve an important purpose. However, on a market-wide scope, internal leverage as defined in (32) is possibly the most relevant measure of leverage, even though it matters only within a group of firms, and not outside.

4.5 Existence and uniqueness results

ASSUMPTION 3. The functions $d^i$ of Assumption 1 are continuous for $i = 1, \ldots, m.$
ASSUMPTION 4. For \( i = 1, \ldots, m \) and \( j = 1, \ldots, n, \)

\[
d_{r_1, \ldots, r_m, s} = \left( \psi_j^i \left( \sum_{k=1}^n M_{j+k}^s + \sum_{l=1}^m \sum_{k=1}^n M_{j+l}^s r_k \right) \right)_{j=1, \ldots, n} \tag{46}
\]

where \( \psi_j^i : \mathbb{R} \to \mathbb{R}_0^+ \) are monotonically increasing functions such that for any \( \mathbf{y}^1, \mathbf{y}^2 \in \mathbb{R}^n \) with \( \mathbf{y}^1 \geq \mathbf{y}^2 \)

\[
\mathbf{y}^1 - \mathbf{y}^2 \geq \sum_{i=1}^m \left( \psi_j^i(\mathbf{y}^1_j) - \psi_j^i(\mathbf{y}^2_j) \right)_{j=1, \ldots, n}. \tag{47}
\]

Assumption 4 is obviously much stronger than Assumption 3. Condition (47) alone is a stronger condition than Lipschitz continuity. Note that while Assumption 4 restricts allowed liabilities (derivatives) in the sense that they can only be written honoring restriction (46) with regards to the sum of the endogenous assets owned by the underwriter itself, the assumption is still liberal in the sense that no restrictions on derivatives regarding exogenous assets have been made (see also Eq. (15) and the remarks there). In that sense, we could as well write \( \psi_j^i = a \psi_j^i \) in Assumption 4. For example, liabilities, where \( d_{r_1, \ldots, r_m, s} = b(a) \in (\mathbb{R}_0^+)^n \), fulfill Assumption 4 (see also Sec. 6.3 for an example).

The following theorem is this paper’s main result on the existence and uniqueness of no-arbitrage prices in the presence of cross-ownership of equities and general liabilities of differing seniority.

**THEOREM 2.** Under Assumption 4, the following hold:

1. The system (16) – (18) can only have non-negative solutions.
2. For any solution of (16) – (18), the size of the sum of all balance sheets is less than or equal to \( (L_{\text{max}} + 1)||\mathbf{a}||_1 \), where \( L_{\text{max}} \) is as in (40).
3. Under the additional Assumption 5, the system (16) – (18) has at least one solution.
4. Under the additional Assumption 4, the solution of (16) – (18) is unique, i.e. all endogenous assets are derivatives of the exogenous assets. The implicit function

\[
\Psi : \mathbf{a} \mapsto \begin{pmatrix} r_1^s(a) \\ \vdots \\ r_m^s(a) \\ s^*(a) \end{pmatrix} \tag{48}
\]

(the ‘derivative’) that maps the exogenous assets \( \mathbf{a} \) on the solution of (16) – (18) is Lebesgue-measurable.

The first part of Theorem 2 is Lemma 1. The second part follows directly from Eq. (12). Proofs for the other parts and an algorithm for the solution in part four can be found in Section 5. The fourth part means that under Assumption 4 similarly as in the original Merton model, all (recovery) claims on equities and liabilities are derivatives of the underlying exogenously priced assets. The payout of a derivative is, as always, understood to be a function of the underlying asset(s), i.e. the payout is uniquely determined by the
value of the underlying. The existence of a unique solution for the system \((16) - (18)\) therefore implies the function \((48)\). For instance, if \(a\) was replaced by a vector of random variables (i.e. a stochastic model for the endogenously priced assets), then this derivative would be a random variable too due to Lebesgue-measurability. Obviously, it is here assumed that the ownership matrices \(M^s\) and \(M^{d,1}, \ldots, M^{d,m}\) and other features of \((16) - (18)\) are fixed – as they would be in a real life situation. Since \(\Psi\) is a Lebesgue-measurable derivative of \(a\) that uniquely determines the no-arbitrage prices of equity and recovery claims in terms of the exogenous assets, this means that, as in the original Merton valuation model for one firm, the theory of risk-neutral pricing can be applied for the valuation of equities and liabilities in the multi-firm case with cross-ownership. In particular, this means that no-arbitrage prices before maturity can be determined (see Sec. 6.1).

Clearly, Assumption 4 is a restriction that one would want to weaken as much as possible. However, for uniqueness, Assumption 3 alone is not enough, as an example with more than one no-arbitrage equilibrium in Section 6.4 will show (see also Sec. 4.7).

4.6 A comment on liquidation and no-arbitrage

Equation \((28)\) implies that

\[
a + \sum_{i=1}^{m} M^{d,i} \cdot r^i - \sum_{i=1}^{m} r^i = s - M^s \cdot s. \tag{49}
\]

Netting or paying all recovery claims (payable liabilities) at maturity is no problem if \(M^s = 0\), even if some (or all) firms are in default. The reason is that in this case the right side of \((49)\) is non-negative, which means that the left side implies that, for any of the \(n\) companies, exogenous assets plus receivable recovery claims are sufficient to pay the recovery claims on all liabilities they have written. Hence, no sale or liquidation of equity is needed to pay for the cash liabilities.

If for any company the corresponding row of \((49)\) was identical zero, then exogenous assets and receivables would exactly cover the payable liabilities – a sign that the firm’s equity was zero and that it might be in default.

Problems arise if some components of the right side of \((49)\) are negative, since in this case selling the exogenous assets (which can be treated like cash for this purpose) and receiving cash claims will not be enough to pay the recovery claims belonging to the liabilities of these firms. This means that assets in the form of equity, i.e. components of \(M^s \cdot s\), have to be liquidated in order to pay for liabilities. This is only possible if there either is a buyer (inside or outside the group of firms) willing to buy at the determined no-arbitrage prices, or if equity owners (shareholders) are able to cash in on equity by liquidating their share of the firm’s assets. In Section 4.2 this problem was avoided by considering directly liquidation values, i.e. implicitly it was assumed that all \(n\) firms got liquidated at maturity. Such a liquidation could happen as follows. A bank, e.g. a central bank, lends each firm an amount of the size of the firm’s balance sheet according to no-arbitrage prices. Each company then pays out liabilities and equity as summarized by the right side of the balance sheet \((28)\). Hence, each company receives equities and claims as
listed on the left side of the balance sheet, and it can also sell the exogenous assets. Each bank loan is then paid back using the proceeds (left side equals right side). It is assumed that this happens instantaneously and no interest is paid on the bank loans. The result in the end is that all values in liabilities and equities as determined by the liquidation value equations (16) – (18) have been paid off. Note that in this case of a bank-buffered liquidation there is no need to sell any assets other than the exogenous ones outside the group of considered firms.

We now turn to an alternative approach of justifying the liquidation value equations (16) – (18) as equations for prices under no-arbitrage by a no-arbitrage argument in which a complete liquidation is not necessarily required. However, at least the threat of (partial) liquidation is necessary to make the argument work.

**ASSUMPTION 5.** As long as there are outstanding liabilities of a firm \( i \) \((i = 1, \ldots, m)\), any remaining assets of firm \( i \) have to be liquidated in order to pay these.

**ASSUMPTION 6.** Because \( M^s \) and \( M^{d,i} \) \((i = 1, \ldots, m)\) are strictly left substochastic matrices, we assume that there is a market outside the group of \( n \) firms that instantaneously trades fractions of the liabilities and equities that are not owned within the group of \( n \) firms at maturity. In this market, any amount of liabilities or equities of the group of \( n \) firms, or perfect replications thereof, can be sold at prevailing prices. Assume that this market also trades the exogenous assets and that shortselling is allowed. Any amount of cash can be borrowed instantaneously at no interest. Further, assume that this market is free of (instantaneous) arbitrage.

**ASSUMPTION 7.** Any participant in the market of Assumption 6 who owns equity of any of the \( n \) firms is allowed to liquidate her part of the equity.

**ASSUMPTION 8.** The group of \( n \) firms participates in the market of Assumption 6 and sells any amount of equity at prevailing prices if asked to do so. Furthermore, any participant in the market who owns the entire equity of any of the \( n \) firms can liquidate this firm.

We now check that the Assumptions 2, 5, 6, and 7 (first set of assumptions), alternatively the Assumptions 2, 5, 6, and 8 (second set of assumptions), lead to the equations (16) – (18) for the no-arbitrage prices of equities and liabilities in the market of Assumption 6.

Under both sets of assumptions, the right hand sides of the no-arbitrage equations (16) – (18) correctly describe the payoffs of any liabilities, respectively the liquidation value of any equity. Therefore, no-arbitrage prices of any recovery claims, which are paid in cash, have to equal the amounts paid. Hence, equations (16) and (17) apply to these prices. For equities, the case where equity costs less than the liquidation value directly leads to arbitrage by liquidation due to Assumption 7 or Assumption 8. However, in the case of Assumption 8, the market agent would have to borrow enough cash to buy all the equity of the corresponding firm, while under Assumption 7 any fractional amount would suffice. If equity is overvalued compared to the liquidation value (right hand side of Eq. (18) for the row corresponding to that firm), then a participant can (proportionally) replicate the right hand side of Eq. (18) and sell this product in the market at higher
than purchasing price, creating instant arbitrage. For this argument we have to assume that market participants are rational in the sense that they do not discriminate between actual equity and perfectly replicated equity (cf. Assumption 6).

In summary, it seems appropriate to say that the equations (16) – (18) describe the prices of equities and liabilities under the assumption of no instantaneous arbitrage given the possibility of (partial) liquidation. The no-arbitrage arguments that were used obviously relate to Merton’s comment on capital-structure arbitrage in Mitchell (2004) (cf. Section 2).

It is clear that a violation of either the no-arbitrage equations (16) – (18), or the balance sheet equations (28) would mean either the violation of contractual law (the priority of claims), or the manipulation of balance sheets against better knowledge, for instance by ignoring current market prices of the assets on that balance sheet, creating an opportunity of instantaneous arbitrage. In this sense, a no-arbitrage equilibrium determined by (16) – (18) is indisputable from a theoretical point of view. However, a main ingredient of our argument is the possibility of a liquidation, potentially a liquidation of all firms involved. This is of course also true for the original Merton approach, but seems less rigorous there since only one firm with a very simple capital structure is affected. Nonetheless, we would argue that any pricing approach conflicting with the one outlined in this paper could lead to significant problems at some stage since inconsistent pricing at one instance could lead to even bigger inconsistencies over longer periods of time.

4.7 A comment on uniqueness

We will see in Section 6.4 that there are examples when Assumption 3 holds but no unique price equilibrium exists. In such cases, endogenous asset values are no derivatives of the underlying exogenous assets in the sense of an implicit function $\Psi$ as in (18), since the considered relation is not uniquely determined on the right hand side. In such situations, the market can not price the assets in an indisputable and rational manner. Prices at maturity but also before maturity are indeterminable. This is a very unhealthy market situation because of the uncertainty it creates. It is our opinion that financial contracts or derivatives that cause such situations should be illegal. In this sense, this theory could be useful to regulators in order to outrule such or similar situations and in order to assess these kind of situations before they become a real world problem.

Note that this problem is different from the problem of the incompleteness of a market. In the case of incompleteness, payoffs, that can not be replicated, can have a range of prices within which no arbitrage can occur, i.e. the no-arbitrage price is not unique, and usually the choice of one equivalent martingale measure (EMM) out of a range of possible EMMs leads to one specific no-arbitrage price system. In the case of the example above, even the choice of one specific EMM is of no use since there exists no uniquely determined expectation of the discounted payoffs, because the payoffs themselves are not uniquely determined given any state of the world. However, the situation is comparable with incompleteness to some extent since, in the case of non-unique no-arbitrage prices, the counterparties could agree on one specific no-arbitrage price set for each possible scenario of the exogenous assets, similar to the choice of one equivalent martingale measure in the incomplete market case. It is clear that the task of determining the set of all priceable...
or allowed liabilities (derivatives) in a given system is equivalent to the question of the weakest conditions for the liabilities functions \( d^i \) that still allow for a unique solution of the no-arbitrage equations \((16) – (18)\).

### 5 Proof and algorithm

#### 5.1 Third part of Theorem 2

Important ingredients for the proof of the third part of Theorem 2 are the Brouwer–Schauder Fixed Point Theorem (Theorem 3 in Section A.2) and Lemma 4 of Section A.3.

**Proof.** Any solution of the system \((16) – (18)\) is a fixed point of the mapping \( \Phi : \)

\[
\begin{pmatrix}
    r^1 \\
    r^2 \\
    \vdots \\
    r^m \\
    s
\end{pmatrix}
\right)

\[
\begin{pmatrix}
    \min\{d^1_{r^1,\ldots,r^m,s}, (a + M^s \cdot s + \sum_{i=1}^m M^{d,i} \cdot r^i)\} \\
    \min\{d^2_{r^1,\ldots,r^m,s}, (a + M^s \cdot s + \sum_{i=1}^m M^{d,i} \cdot r^i - d^1_{r^1,\ldots,r^m,s})\} \\
    \vdots \\
    \min\{d^m_{r^1,\ldots,r^m,s}, (a + M^s \cdot s + \sum_{i=1}^m M^{d,i} \cdot r^i - \sum_{i=1}^{m-1} d^i_{r^1,\ldots,r^m,s})\} \\
    (a + M^s \cdot s + \sum_{i=1}^m M^{d,i} \cdot r^i - \sum_{i=1}^{m-1} d^i_{r^1,\ldots,r^m,s})
\end{pmatrix}
\]

and vice versa. Because of Lemma 4, we only need to consider \( \Phi \) on \((\mathbb{R}^+_0)^{n(m+1)}\). Because of Eq. (84) in Lemma 4 (cf. Section A.3), one has with \( f^{\max} \) as in (38) and for \( r^1, r^2, \ldots, r^m, s \in (\mathbb{R}^+_0)^n \) that

\[
\| \Phi \left( \begin{pmatrix} r^1 \\ \vdots \\ r^m \\ s \end{pmatrix} \right) \|_1 = \| a + M^s \cdot s + \sum_{i=1}^m M^{d,i} \cdot r^i \|_1
\]

\[
\leq \| a \|_1 + \| M^s \|_1 \cdot \| s \|_1 + \sum_{i=1}^m \| M^{d,i} \|_1 \cdot \| r^i \|_1
\]

\[
\leq \| a \|_1 + f^{\max} \cdot \left( \| s \|_1 + \sum_{i=1}^m \| r^i \|_1 \right)
\]

\[
= \| a \|_1 + f^{\max} \cdot \left( \begin{pmatrix} r^1 \\ \vdots \\ r^m \\ s \end{pmatrix} \right) \|_1
\]

25
From (51), and since \( L^{\text{max}} + 1 = \frac{1}{1 + 1^{\text{max}}} \), we furthermore obtain that if

\[
0 \leq \left\| \begin{pmatrix} r^1 \\ \vdots \\ r^m \\ s \end{pmatrix} \right\|_1 \leq (L^{\text{max}} + 1) \| a \|_1
\]  

(52)

then

\[
0 \leq \left\| \Phi \left( \begin{pmatrix} r^1 \\ \vdots \\ r^m \\ s \end{pmatrix} \right) \right\|_1 \leq (L^{\text{max}} + 1) \| a \|_1.
\]  

(53)

Under Assumption 3, \( \Phi \) (cf. (50)) is continuous and (52) and (53) mean that we can apply the Brouwer–Schauder Fixed Point Theorem by considering \( \Phi \) on the compact subset of \( (\mathbb{R}_0^+)^{n(m+1)} \) defined by (52). The system (16) – (18) has therefore at least one non-negative solution with a balance sheet size of less than or equal to \( (L^{\text{max}} + 1) \| a \|_1 \).

\[ \square \]

5.2 Fourth part of Theorem 2 and algorithm

The main ingredients for the proof of the fourth part of Theorem 2 are Banach’s Contraction Mapping Theorem, Lemma 2 below, and Lemma 5 of Section A.3. Readers who are less familiar with the Contraction Mapping Theorem and the notion of a contraction in general might want to have a look at Section A.2 (Definition 1 and Theorem 1) before reading on.

**Lemma 2.** Under Assumptions 3 and 4, the mapping \( \Phi \) in (50) is a strict contraction on \( \mathbb{R}^{n(m+1)} \) under the metric implied by the \( \ell^1 \)-norm on \( \mathbb{R}^{n(m+1)} \).

**Proof.** Let \( r^{1,1}, \ldots, r_{m,1}, s^1, r^{1,2}, \ldots, r_{m,2}, s^2 \in \mathbb{R}^n \). Define \( g \in \mathbb{R}^{n(m+1)} \) as

\[
g = \Phi \left( \begin{pmatrix} r^{1,1} \\ \vdots \\ r_{m,1} \\ s^1 \end{pmatrix} \right) - \Phi \left( \begin{pmatrix} r^{1,2} \\ \vdots \\ r_{m,2} \\ s^2 \end{pmatrix} \right),
\]

(54)

and \( h \in \mathbb{R}^n \) as

\[
h = M^s \cdot s^1 + \sum_{i=1}^m M^{d,i} \cdot r^{i,1} - \left( M^s \cdot s^2 + \sum_{i=1}^m M^{d,i} \cdot r^{i,2} \right)
\]

\[
= M^s \cdot (s^1 - s^2) + \sum_{i=1}^m M^{d,i} \cdot (r^{i,1} - r^{i,2}).
\]

For \( k \in \{1, \ldots, n\} \), define \( x = a_k \), and for \( l = 1, 2 \) define

\[
y^l = \sum_{j=1}^n M^s_{kj} s^l_j + \sum_{i=1}^m \sum_{j=1}^n M^{d,i}_{kj} r_{i,j}^l.
\]

(56)

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This means that \( h_k = y^1 - y^2 \) and that
\[
g_k = \min \{ \psi_k^1(y^1), \, x + y^1 \} - \min \{ \psi_k^1(y^2), \, x + y^2 \}
\] (57)

For \( 0 < j < m \):
\[
g_{k+nj} = \min \left\{ \psi_k^{j+1}(y^1), \left( x + y^1 - \sum_{i=1}^{j} \psi_k^i(y^1) \right)^+ \right\} - \min \left\{ \psi_k^{j+1}(y^2), \left( x + y^2 - \sum_{i=1}^{j} \psi_k^i(y^2) \right)^+ \right\}
\] (58)
\[
g_{k+nm} = \left( x + y^1 - \sum_{i=1}^{m} \psi_k^i(y^1) \right)^+ - \left( x + y^2 - \sum_{i=1}^{m} \psi_k^i(y^2) \right)^+
\] (59)

Lemma 5 in Section A.3, in conjunction with Assumption 4, now implies that
\[
\sum_{j=0}^{m} |g_{k+nj}| = |h_k|.
\] (60)

Therefore \( ||g||_1 = ||h||_1 \). With \( I_{\text{max}} \) as in (38) and (39),
\[
\begin{align*}
||g||_1 & = \left| \Phi \left( \begin{array}{c} r_{1,1}^1 \\ \vdots \\ r_{m,1}^1 \\ s_1^1 \end{array} \right) - \Phi \left( \begin{array}{c} r_{1,2}^1 \\ \vdots \\ r_{m,2}^1 \\ s_2^1 \end{array} \right) \right|_1 \\
& = \left| M^s \cdot (s^1 - s^2) + \sum_{i=1}^{m} M^{d,i} \cdot (r_i^{1,1} - r_i^{1,2}) \right|_1 \\
& \leq \left| M^s \cdot (s^1 - s^2) \right|_1 + \sum_{i=1}^{m} \left| M^{d,i} \cdot (r_i^{1,1} - r_i^{1,2}) \right|_1 \\
& \leq \left| M^s \right|_1 \cdot \left| (s^1 - s^2) \right|_1 + \sum_{i=1}^{m} \left| M^{d,i} \right|_1 \cdot \left| (r_i^{1,1} - r_i^{1,2}) \right|_1 \\
& \leq I_{\text{max}} \cdot \left( \left| (s^1 - s^2) \right|_1 + \sum_{i=1}^{m} \left| (r_i^{1,1} - r_i^{1,2}) \right|_1 \right) \\
& = I_{\text{max}} \cdot \left| \Phi \left( \begin{array}{c} r_{1,1}^1 \\ \vdots \\ r_{m,1}^1 \\ s_1^1 \end{array} \right) - \Phi \left( \begin{array}{c} r_{1,2}^1 \\ \vdots \\ r_{m,2}^1 \\ s_2^1 \end{array} \right) \right|_1.
\end{align*}
\]

**Proof and algorithm**

*Proof.* Given Lemma 2 Banach’s Contraction Mapping Theorem (Section A.2, Theorem 4) immediately implies that the system (16) – (18) has a unique solution. To obtain
the unique solution of system (16) – (18) under Assumption 4, the recursion (83) of the Contraction Mapping Theorem (also sometimes called Picard iteration) can be used with \( \Phi \) as in (50). To better account for the dependency of (50) on \( a \), we denote (50) now by \( \Phi_a \). Hence,

\[
\Psi(a) \equiv \lim_{m \to \infty} \Phi_a^m(\cdot) = \lim_{m \to \infty} \Phi_a \circ \ldots \circ \Phi_a(\cdot).
\]

Regarding the proof of Lebesgue-measurability, it is straightforward to show that the function \( \Phi(\cdot) \) is a continuous map \( \mathbb{R}^n \times \mathbb{R}^{n(m+1)} \to \mathbb{R}^{n(m+1)} \). Similarly, it is straightforward that the function (note the somewhat sloppy notation) \( \Phi_a^m(\cdot) : \mathbb{R}^n \times \mathbb{R}^{n(m+1)} \to \mathbb{R}^{n(m+1)} \) \((m = 1, 2, \ldots)\), which is obtained from \( \Phi_a^m(\cdot) = \Phi_a \circ \ldots \circ \Phi_a(\cdot) \)

by replacing \( a \) by \( \cdot \), is continuous too. Continuity implies Lebesgue-measurability. However, we know that \( \Phi_a^m(\cdot) \) has the non-functional limit (62), hence \( \lim_{m \to \infty} \Phi_a^m(\cdot) = \Psi(\cdot) \). A point-wise limit of measurable functions is measurable, therefore \( \Psi(\cdot) \) is Lebesgue-measurable on \( \mathbb{R}^n \times \mathbb{R}^{n(m+1)} \), with the second argument being irrelevant. It follows now from basic measure theory that \( \Psi(\cdot) \) is Lebesgue-measurable on \( \mathbb{R}^n \).

6 Applications and examples

6.1 No-arbitrage prices before maturity

So far, only the existence of no-arbitrage prices at maturity has been considered. Assume now that the exogenous assets are given by an \( n \)-dimensional price process \( a(t) \) where \( t \in \mathbb{T} \), with \( \min \mathbb{T} = 0 \) being the present time, and where \( \max \mathbb{T} = T \) is the time of maturity. Irrespective of the particular structure of \( \mathbb{T} \), we assume that \( a(\cdot) \) is adapted to a filtration \( (\mathcal{F}_t)_{t \in \mathbb{T}} \) that lives on a probability space \( (\Omega, \mathcal{F}_T, \mathbb{P}) \). For convenience, assume that \( a_1(\cdot) \) is a numéraire, i.e. non-dividend paying and almost surely strictly positive. Assume now that \( \mathbb{Q} \) is an equivalent martingale measure (EMM) for the price process \( a(\cdot) \), i.e. by the Fundamental Theorem of Asset Pricing, we assume that the market spanned by \( a(\cdot) \) is arbitrage-free. Furthermore, consider a payoff \( X \) at maturity \( T \), given by a random variable on \( (\Omega, \mathcal{F}_T, \mathbb{P}) \). Assuming sufficient integrability, or even boundedness where necessary, the Fundamental Theorem of Asset Pricing implies that for \( t \in \mathbb{T} \) the risk-neutral valuation formula

\[
X(t) = a_1(t) \mathbb{E}_\mathbb{Q} \left[ \frac{X}{a_1(T)} \bigg| \mathcal{F}_t \right]
\]

defines an arbitrage-free price process \( X(\cdot) \) for the payoff \( X = X(T) \) at \( T \). Because of Lebesgue-measurability, the implicit function \( \Psi \) in (13) is a vector of payoffs like \( X \) when evaluated at maturity using \( a = a(T) \). Therefore, if uniqueness of the solution of (16) – (18) is given for all possible outcomes of \( a(T) \), e.g. by part four of Theorem 2, then the risk-neutral valuation formula

\[
a_1(t) \mathbb{E}_\mathbb{Q} \left[ \frac{\Psi(a(T))}{a_1(T)} \bigg| \mathcal{F}_t \right]
\]
provides no-arbitrage prices of the equities and the liabilities of the considered system at any time \( t \in T \). While completeness of the market given by the underlying exogenous assets’ price processes plays no role in this consideration, uniqueness of the no-arbitrage prices at maturity (given any scenario for the underlying assets) is crucial. However, uniqueness of equities’ and liabilities’ no-arbitrage prices at any time before maturity naturally depends on the replicability of \( \Psi(a(T)) \) at maturity in terms of the exogenous assets, which would be given in a generally complete market.

Regarding the balance sheet considerations of Sec. 4.3, it is clear that the accounting equation (28) also holds when \( a, s \), and the \( d_i \) are replaced by corresponding no-arbitrage prices before maturity. This follows directly from the application of (64) to both sides of (28).

Regarding the measures of leverage and cross-ownership in Sec. 4.4, similarly to the balance sheet equation case, in (32) – (37) one could substitute \( a, s \), and the \( d_i \) by corresponding no-arbitrage prices before maturity. In general, this would lead to results different from (43).

### 6.2 More exogenous assets than firms

So far we have assumed the exogenous assets to be given by a vector \( a \in (\mathbb{R}^+)^n \), where the dimension \( n \) is the number of firms considered in the system. This rather artificial assumption, which was entirely sufficient for our considerations at maturity conditional on one price scenario of the exogenous assets, can naturally be extended in the following way. We assume that the exogenous assets are given by a vector \( a \in (\mathbb{R}^0^+)^q \) where \( q \in \{1, 2, \ldots\} \). To describe the ownership of these assets we need an ownership matrix \( M^a \in \mathbb{R}^{n \times q} \) with column sums in \([0, 1]\). Similar to the obvious change in the liquidation value (no-arbitrage price) equations (16) – (18), which become

\[
\begin{align*}
  r_1 &= \min \left\{ \sum_{i=1}^{m} M_{d, i} \cdot r_i, M^a \cdot a + M^s \cdot s + \sum_{i=1}^{m} M_{d, i} \cdot r_i \right\} \\
  For \ 0 < j < m: \\
  r_{j+1} &= \min \left\{ d_{r_1, \ldots, r_m, s, a}^{j+1}, \left( M^a \cdot a + M^s \cdot s + \sum_{i=1}^{m} M_{d, i} \cdot r_i - \sum_{i=1}^{j} d_{r_1, \ldots, r_m, s, a}^{j} \right)^{+} \right\} \\
  s &= \left( M^a \cdot a + M^s \cdot s + \sum_{i=1}^{m} M_{d, i} \cdot r_i - \sum_{i=1}^{m} d_{r_1, \ldots, r_m, s, a}^{j} \right)^{+} 
\end{align*}
\] (66)

(67)

(68)

the theory as described so far stays exactly the same, or can be adapted in the entirely obvious way by replacing \( a \) with \( M^a \cdot a \), including Theorem 2 and Section 6.1 on prices before maturity. However, using the extended set-up allows for even more general liabilities (derivatives) due to \( d_{r_1, \ldots, r_m, s}^{j} = d_{r_1, \ldots, r_m, s, a}^{j} \) (cf. Eq. (15)). See Sec. 6.3 for an example.
6.3 An example with stocks, bonds, and derivatives

Consider a system of two firms, \( n = 2 \), and three exogenous assets, \( a \in (\mathbb{R}_0^+)^3 \) and \( \mathbf{M}^a \in \mathbb{R}^{2 \times 3} \) (cf. Sec. 6.2). Suppose now for the liabilities \( (m = 3) \)

\[
d^1 = \begin{pmatrix} b_1 \\ b_3 \end{pmatrix}, \quad d^2_a = \begin{pmatrix} b_2 \\ c_2(0.5a_1 + a_3 - k_2)^- \end{pmatrix}, \quad d^3_a = \begin{pmatrix} c_1(a_2 - k_1)^+ \\ 0 \end{pmatrix},
\]

where \( b_1, b_2, b_3, c_1, c_2, k_1, k_2 > 0 \). As in Assumption 1, let \( \mathbf{M}^s, \mathbf{M}^{d,i} \in \mathbb{R}^{2 \times 2} \) \( (i = 1, 2, 3) \) be strictly left substochastic matrices. The described system is one of two firms where the first one has issued two bonds of differing seniority (that of nominal \( b_1 \) higher than that of \( b_2 \)), and one derivative – a European Call on exogenous asset \( a_2 \) with a strike price of \( k_1 \) and a ‘size’ of \( c_1 \). The second firm has issued one bond (nominal \( b_3 \)) and one derivative – a European Put on a basket (a mix) of exogenous assets \( a_1 \) and \( a_3 \) with a strike price of \( k_2 \) and a ‘size’ of \( c_2 \). Additionally, and not specified in more detail, any level of cross-ownership of any of these five liabilities and the two equities could be present. It is clear in this case that part four of Theorem 2 applies, with the unique no-arbitrage price equilibrium given by Eq. (66) – (68). These no-arbitrage prices could be calculated using the algorithm (62) of Sec. 5.2. Using risk-neutral pricing techniques under a stochastic model for the exogenous assets (cf. Sec. 6.1), one could therefore simultaneously calculate no-arbitrage prices of all claims (equities, loans, derivatives) in this system, while fully accounting for the priority of claims, as well as for leverage and counterparty risk caused by cross-ownership.

As a remark, in (69) the second entries of \( d^2_a \) and \( d^3_a \) could be swapped without any consequences for pricing. Furthermore, it is clear in what way the above example would simplify if one was only interested in the valuation of equity, bonds and derivatives of differing seniority issued by one single firm, free of any cross-ownership entanglements.

6.4 No unique prices under Assumption 3

Suppose \( n = 2 \) and \( m = 1 \), \( \mathbf{M}^s = 0 \) and

\[
\mathbf{M}^d = \begin{pmatrix} 0 & 0.8 \\ 0.8 & 0 \end{pmatrix}, \quad \mathbf{d}_{r,s} = \begin{pmatrix} (r_2 - 2)^2 \\ (r_1 - 2)^2 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

It can now easily be checked that the no-arbitrage equations

\[
\begin{align*}
    r_1 & = \min\{(r_2 - 2)^2, 1 + 0.8r_2\} \\
    r_2 & = \min\{(r_1 - 2)^2, 1 + 0.8r_1\} \\
    s_1 & = (1 + 0.8r_2 - (r_2 - 2)^2)^+ \\
    s_2 & = (1 + 0.8r_1 - (r_1 - 2)^2)^+.
\end{align*}
\]

are solved by

\[
\mathbf{r}^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{s}^* = \begin{pmatrix} 0.8 \\ 0.8 \end{pmatrix},
\]

as well as by

\[
\mathbf{r}^* = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{s}^* = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix}.
\]
Note that in this example (47) does not hold. For the theoretical meaning of this example see also Sec. 4.7.

6.5 No price equilibrium under maximum cross-ownership

Let $M^s$ be a maximum ownership matrix in the sense that the matrix is left stochastic, that is each column adds up to 1. Let $||a||_1 > 0$, $d^i \equiv 0$ and $M^{d,i} = 0$ ($i = 1, \ldots, m$). Suppose now that a no-arbitrage equilibrium for this set-up exists. Because of the assumptions, $||s||_1 = ||M^s \cdot s||_1$. Under no-arbitrage, (29) holds. Therefore,

$$||s||_1 = ||a||_1 + ||M^s \cdot s||_1 = ||a||_1 + ||s||_1,$$

which is a contradiction since $||a||_1 > 0$. Also, since $I_{\text{max}} = 1$, one has $L_{\text{max}} = \frac{I_{\text{max}}}{1-I_{\text{max}}} = +\infty$.

6.6 $L_{\text{max}}$ is sharp

Consider a system with $n = 2$, $m = 1$, $M^d = 0$, $d = 0$,

$$M^s = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}, \quad \text{and} \quad a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The no-arbitrage equations are

$$r = 0 \quad (78)$$

$$s = (a + M^s \cdot s)^+. \quad (79)$$

We therefore have $L_{\text{max}} = 1$, and the upper boundary of the sum of all balance sheets is $(L_{\text{max}} + 1)||a||_1 = 4$. It is clear from Theorem 2 that we have unique no-arbitrage prices in this set-up. It can easily be checked against (78) and (79) that this price equilibrium is given by $r_1 = r_2 = 0$ and $s_1 = s_2 = 2$, which means that the sum of all balance sheets equals 4, which is the value of $(L_{\text{max}} + 1)||a||_1$. Hence, $L_{\text{max}}$ is sharp.

7 Conclusion

This paper has presented a model for the no-arbitrage valuation of equities and general liabilities in a system of firms where cross-ownership is present and where liabilities, that can include debts and derivatives, can be of differing seniority in a liquidation. Cross-ownership is a widespread financial phenomenon (cf. Sec. 1.3), and as the presented theory directly accounts for counterparty risk and, in a way, systemic risk, its valuation procedure should be relevant for the theory and practice of general asset valuation, derivatives pricing, and credit risk management. For the application of the ideas of this paper, it might be helpful that our theory is a direct extension of the Merton (1974) model (see also Sec. 1.4), which is the basis of modern structural credit risk models. For instance, the theory presented in this paper is general enough to be applied in stochastic interest rates settings, in settings where underlying exogenous assets are modelled with copula
approaches, and for credit risk modelling, where specifically defined default barriers could be applied for the calculation of default probabilities. Future directions of research to extend this theory should include investigations on the range of liabilities that allow for unique no-arbitrage price equilibria (weaker forms of Assumption 4) and, of course, the question of no-arbitrage pricing in the multi-period case.

A Appendix

A.1 A result for substochastic matrices

**Lemma 3.** If $M \in \mathbb{R}^{n \times n}$ is a strictly (left or right) substochastic matrix, then $(I - M)^{-1}$ exists and is non-negative. The diagonal elements of $(I - M)^{-1}$ are greater than or equal to 1.

**Proof.** For $||M||_1 < 1$, it follows from standard results of functional analysis that $(I - M)^{-1}$ exists and

$$(I - M)^{-1} = \sum_{n=0}^{\infty} M^n. \tag{80}$$

The lemma then follows from $M^n \geq 0$ for $n = 0, 1, \ldots$, and $M^0 = I$. \hfill \Box

A.2 Two fixed point theorems

**Theorem 3** (Brouwer–Schauder Fixed Point Theorem). Every continuous function from a convex compact subset $K$ of a Banach space to $K$ itself has a fixed point.

**Definition 1.** Let $(\mathbb{X}, d)$ be a metric space. A map $\Phi : \mathbb{X} \to \mathbb{X}$ is called a strict contraction on $\mathbb{X}$ if there exists a number $0 \leq c < 1$ such that

$$d(\Phi(x), \Phi(y)) \leq c \cdot d(x, y) \quad \text{for} \quad x, y \in \mathbb{X}. \tag{81}$$

The map $\Phi$ is called a weak contraction if

$$d(\Phi(x), \Phi(y)) \leq d(x, y) \quad \text{for} \quad x, y \in \mathbb{X}. \tag{82}$$

**Theorem 4** (Banach Contraction Mapping Theorem). Let $\mathbb{X}$ be a complete metric space and $f$ be a strict contraction on $\mathbb{X}$. Then $\Phi$ has a unique fixed point $x^* \in \mathbb{X}$. For any $x \in \mathbb{X}$, one has

$$\lim_{n \to \infty} \Phi^n(x) = \lim_{n \to \infty} \underbrace{\Phi \circ \ldots \circ \Phi}_n(x) = x^*. \tag{83}$$

A.3 Two lemmas

**Lemma 4.** For $x \in \mathbb{R}$, $m \in \{1, 2, \ldots\}$, and $y^1, \ldots, y^m \in \mathbb{R}_0^+$,

$$x = \min \{y^1, x\} + \sum_{j=1}^{m-1} \min \left\{ y^{j+1}, \left( x - \sum_{i=1}^{j} y^i \right)^+ \right\} + \left( x - \sum_{i=1}^{m} y^i \right)^+. \tag{84}$$
Proof. This is easy to check. \hfill \Box

**Lemma 5.** For \( x, y^1, y^2 \in \mathbb{R} \) and for monotonically increasing functions \( \psi^i : \mathbb{R} \to \mathbb{R}_0^+ \) \((i = 1, \ldots, m)\) such that for any \( z^1, z^2 \in \mathbb{R} \) with \( z^1 \geq z^2 \)

\[
z^1 - z^2 \geq \sum_{i=1}^{m} (\psi^i(z^1) - \psi^i(z^2)) \tag{85}
\]

the following equation holds:

\[
|y^1 - y^2| = \left| \min \{ \psi^1(y^1), x + y^1 \} - \min \{ \psi^1(y^2), x + y^2 \} \right| + \sum_{j=1}^{m-1} \left| \min \left\{ \psi^{j+1}(y^1), \left( x + y^1 - \sum_{i=1}^{j} \psi^i(y^1) \right)^+ \right\} - \min \left\{ \psi^{j+1}(y^2), \left( x + y^2 - \sum_{i=1}^{j} \psi^i(y^2) \right)^+ \right\} \right| + \left| \left( x + y^1 - \sum_{i=1}^{m} \psi^i(y^1) \right)^+ - \left( x + y^2 - \sum_{i=1}^{m} \psi^i(y^2) \right)^+ \right|. \tag{86}
\]

Obviously, any of the functions \( \psi^i \) in Lemma 5 could be a constant.

Proof. We will prove the equation considering six cases (with sub-cases) for which we will derive simplified expressions for the right hand side of (86). It will be fairly straightforward to check that these expressions are correct. Without loss of generality, assume \( y^1 \geq y^2 \), and hence \( \psi^i(y^1) \geq \psi^i(y^2) \) \((i = 1, \ldots, m)\). Because of this, all absolute expressions \( | \cdot | \) below will be positive anyway. We keep the \( | \cdot | \) for convenience. Regarding the aforementioned cases, note that for \( y^1 \geq y^2 \) and for \( j \leq k, j, k \in \{1, \ldots, m\} \), it is impossible to have a situation where simultaneously

\[
\sum_{i=1}^{j} \psi^i(y^1) \geq x + y^1 \tag{87}
\]

\[
\sum_{i=1}^{k} \psi^i(y^2) \leq x + y^2. \tag{88}
\]

This becomes immediately clear from a subtraction of the two inequalities, (87)-(88), which leads to a contradiction of (86):

\[
y^1 - y^2 \leq \sum_{i=1}^{j} \psi^i(y^1) - \sum_{i=1}^{k} \psi^i(y^2) \leq \sum_{i=1}^{k} (\psi^i(y^1) - \psi^i(y^2)). \tag{89}
\]

We therefore only have to consider the following cases.

Case 1: \( \sum_{i=1}^{m} \psi^i(y^1) \leq x + y^1 \) and \( \sum_{i=1}^{m} \psi^i(y^2) \leq x + y^2 \). In this case, using (85), the
right side of (86) is identical to
\[
\sum_{i=1}^{m} |\psi^i(y^1) - \psi^i(y^2)| + \left| x + y^1 - \sum_{i=1}^{m} \psi^i(y^1) - \left( x + y^2 - \sum_{i=1}^{m} \psi^i(y^2) \right) \right| = |y^1 - y^2| \tag{90}
\]

Case 2: Assume that \( m > k > 0 \) and
\[
\sum_{i=1}^{m} \psi^i(y^1) \leq x + y^1 \tag{91}
\]
and
\[
\sum_{i=1}^{k} \psi^i(y^2) \leq x + y^2 \leq \sum_{i=1}^{k+1} \psi^i(y^2), \tag{92}
\]
which implies
\[
0 \leq x + y^2 - \sum_{i=1}^{k} \psi^i(y^2) \leq \psi^{k+1}(y^2) \leq \psi^{k+1}(y^1). \tag{93}
\]

Case 2.1: \( k = m - 1 \). The right hand side of (86) becomes
\[
\sum_{i=1}^{m-1} |\psi^i(y^1) - \psi^i(y^2)| + \left| \psi^m(y^1) - \left( x + y^2 - \sum_{i=1}^{m-1} \psi^i(y^2) \right) \right| \tag{94}
\]
\[
+ \left| x + y^1 - \sum_{i=1}^{m} \psi^i(y^1) \right| = |y^1 - y^2|.
\]

Case 2.2: \( k \leq m - 2 \). The right hand side of (86) turns into
\[
\sum_{i=1}^{k} |\psi^i(y^1) - \psi^i(y^2)| + \left| \psi^{k+1}(y^1) - \left( x + y^2 - \sum_{i=1}^{k} \psi^i(y^2) \right) \right| \tag{95}
\]
\[
+ |\psi^{k+2}(y^1)| + \ldots + |\psi^{m}(y^1)| + \left| x + y^1 - \sum_{i=1}^{m} \psi^i(y^1) \right| = |y^1 - y^2|.
\]

Case 3: Assume that \( m > j \geq k > 0 \) and
\[
\sum_{i=1}^{j} \psi^i(y^1) \leq x + y^1 \leq \sum_{i=1}^{j+1} \psi^i(y^1) \tag{96}
\]
as well as
\[
\sum_{i=1}^{k} \psi^i(y^2) \leq x + y^2 \leq \sum_{i=1}^{k+1} \psi^i(y^2), \tag{97}
\]
which again implies
\[
0 \leq x + y^2 - \sum_{i=1}^{k} \psi^i(y^2) \leq \psi^{k+1}(y^2) \leq \psi^{k+1}(y^1). \tag{98}
\]
Case 3.1: \( j = k \). The right hand side of (86) becomes
\[
\sum_{i=1}^{k} |\psi^i(y^1) - \psi^i(y^2)| + |x + y^1 - \sum_{i=1}^{k} \psi^i(y^1) - (x + y^2 - \sum_{i=1}^{k} \psi^i(y^2))| \quad (99)
\]
\[
\leq |y^1 - y^2|. \]

Case 3.2: \( j = k + 1 \). The right hand side of (86) turns into
\[
\sum_{i=1}^{k} |\psi^i(y^1) - \psi^i(y^2)| + |\psi^{k+1}(y^1) - (x + y^2 - \sum_{i=1}^{k} \psi^i(y^2))| + |x + y^1 - \sum_{i=1}^{k+1} \psi^i(y^1)| \quad (100)
\]
\[
\leq |y^1 - y^2|. \]

Case 3.3: \( j \geq k + 2 \). The right hand side of (86) becomes
\[
\sum_{i=1}^{k} |\psi^i(y^1) - \psi^i(y^2)| + |\psi^{k+1}(y^1) - (x + y^2 - \sum_{i=1}^{k} \psi^i(y^2))| + |\psi^{k+2}(y^1)| + \ldots + |\psi^j(y^1)| + |x + y^1 - \sum_{i=1}^{j} \psi^i(y^1)| \quad (101)
\]
\[
\leq |y^1 - y^2|. \]

Case 4: Assume \( m > j > 0 \) and
\[
\sum_{i=1}^{j} \psi^i(y^1) \leq x + y^1 \leq \sum_{i=1}^{j+1} \psi^i(y^1) \quad (102)
\]
as well as
\[
x + y^2 \leq \psi^1(y^2). \quad (103)
\]
The right hand side of (86) turns into
\[
|\psi^1(y^1) - (x + y^2)| + |\psi^2(y^1)| + \ldots + |\psi^j(y^1)| + |x + y^1 - \sum_{i=1}^{j} \psi^i(y^1)| = |y^1 - y^2|. \quad (104)
\]

Case 5: Assume \( \sum_{i=1}^{m} \psi^i(y^1) \leq x + y^1 \) and \( x + y^2 \leq \psi^1(y^2) \). The right hand side of (86) turns into
\[
|\psi^1(y^1) - (x + y^2)| + |\psi^2(y^1)| + \ldots + |\psi^m(y^1)| + |x + y^1 - \sum_{i=1}^{m} \psi^i(y^1)| = |y^1 - y^2|. \quad (105)
\]

Case 6: Assume \( x + y^1 \leq \psi^1(y^1) \) and \( x + y^2 \leq \psi^1(y^2) \). The right hand side of (86) becomes
\[
|(x + y^1) - (x + y^2)| = |y^1 - y^2|. \quad (106)
\]
The proof for \( y^1 \leq y^2 \) follows immediately from swapping \( y^1 \) and \( y^2 \) with each other in (86). □
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