Localization of fields on a brane in six dimensions

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(received 27 June 2002; accepted in final form 11 November 2002)

PACS. 11.10.Kk – Field theories in dimensions other than four.
PACS. 04.50.+h – Gravity in more than four dimensions, Kaluza-Klein theory, unified field theories; alternative theories of gravity.
PACS. 98.80.Cq – Particle-theory and field-theory models of the early Universe (including cosmic pancakes, cosmic strings, chaotic phenomena, inflationary universe, etc.).

Abstract. – Universe is considered as a brane in infinite \((2 + 4)\)-space. It is shown that zero modes of all kinds of matter fields and 4-gravity are localized on the brane by increasing the transversal gravitational potential.

The papers [1–3] had excited recent interest in brane models. In the present paper we want to concentrate on the localization problem in the model where our world is considered as a single shell expanding in multi-dimensions [4–8]. Two observed facts of modern cosmology, the isotropic runaway of galaxies and the existence of a preferred frame in the Universe, where the relict background radiation is isotropic, have an obvious explanation in this context.

We assume that trapping of physical fields on the brane has a gravitational nature, since in our world gravity is known to be the unique interaction which has universal coupling with all matter fields. To provide universal and stable trapping, we assume also that on the brane (where all gravitating matter can be resided) the gravitational potential should have a minimal value with respect to extra coordinates. Growing gravitational potential (warp factor) is a choice opposite to that of Randall-Sundrum with the maximum on the brane [2]. However, Newton’s law on the brane is the result of the cancellation mechanism introduced in [3, 6] which allows both types of gravitational potential.

To have localized multi-dimensional fields on a brane “coupling” constants appearing after integration of their Lagrangian over extra coordinates must be non-vanishing and finite. In \((1 + 4)\)-dimensional models the following facts were clarified: the spin-0 field is localized on the brane with decreasing warp factor and the spin-(1/2) field is localized on the brane with increasing warp factor [9]; the spin-1 field is not normalizable at all [10] and the spin-2 fields are localized on the brane with decreasing warp factor [2, 3]. For the case of \((1 + 5)\)-dimensions it was found that spin-0, -1 and -2 fields are localized on the brane with decreasing warp factor...
and the spin-(1/2) field is localized on the brane with increasing warp factor [11]. So, to fulfill the localization of Standard Model particles in (1+4)-, or (1+5)-spaces it is required to introduce no other interaction but gravity.

Here we want to show that zero modes of spin-0, -1/2, -1 and -2 fields can be all localized on the brane in the (2+4)-space by increasing warp factor. Our motivation for the choice of the signature of the bulk is as follows. In the massless-field case (weakest coupling with gravity), symmetries of a multi-dimensional manifold can be restored. It is well known that in the zero-mass limit the main equations of physics are invariant under the 15-parameter non-linear conformal transformations. A long time ago it was also discovered that the conformal group can be written as a linear Lorentz-type transformation in a (2+4)-space (for these subjects see, for example, [12]).

The action of the gravitating system in six dimensions can be written in the form

$$S = \int d^6x \sqrt{\hat{g}} \left[ -\frac{M^4}{2} (\hat{R} + 2\Lambda) + \hat{L} \right],$$  \hspace{1cm} (1)

where $\hat{g}$ is the determinant, $M$ is the fundamental scale, $\hat{R}$ is the scalar curvature, $\Lambda$ is the cosmological constant and $\hat{L}$ is the Lagrangian of matter fields; all these values refer to six dimensions. Einstein’s 6-dimensional equations can be written in the form

$$\hat{R}_{AB} = -\frac{1}{2} \Lambda \hat{g}_{AB} + \frac{1}{M^4} \left( T_{AB} - \frac{1}{4} \hat{g}_{AB} T \right).$$ \hspace{1cm} (2)

Capital Latin indices run over $A, B, \ldots = 0, 1, 2, 3, 5, 6$.

It is convenient to introduce the new dimensionless coordinates $z, v$ of the extra (1+1)-space except for the Cartesian ones:

$$x^5 = \epsilon \sqrt{z} \cosh v, \quad x^6 = \epsilon \sqrt{z} \sinh v, \quad z = \frac{x^2_5 - x^2_6}{\epsilon^2}, \quad \tanh v = \frac{x^6}{x^5}.$$ \hspace{1cm} (3)

The constant $\epsilon$ which makes $z, v$ to be dimensionless corresponds to the width of the brane.

We are looking for the solution of (2) in the form

$$d\hat{s}^2 = \phi^2(z) \eta_{\alpha\beta}(x^\nu) dx^\alpha dx^\beta + g_{ij}(z) dx^i dx^j,$$ \hspace{1cm} (4)

where Greek indices $\alpha, \beta, \ldots = 0, 1, 2, 3$ numerate coordinates in 4-dimensions, while small Latin indices $i, j, \ldots = 5, 6$ numerate coordinates of the transversal space. It is assumed that in ansatz (4) the 4-dimensional conformal factor $\phi^2$ and the metric tensor of transversal $(1+1)$-space $g_{ij}$ depend on the extra coordinates $x^i$ only via the coordinate $z$.

Suppose also that the extra coordinates enter the stress energy $T_{AB}$ from the metric (4) only. This means that the strength of a gauge field $A_B$ towards the extra directions and the covariant derivatives of scalar $\Phi$ and spinor $\Psi$ fields with respect to the extra coordinates are zero [5]:

$$F_{iB} = 0, \quad D_i \Phi = 0, \quad D_i \Psi = 0.$$ \hspace{1cm} (5)

Then the ansatz for multi-dimensional matter energy momentum tensor can be written in the form

$$T_{\alpha\beta} = \frac{\tau_{\alpha\beta}(x^\nu)}{\epsilon^2 \phi^2(z)}, \quad T_{ij} = -g_{ij}(z) \frac{L(x^\nu)}{\epsilon^2 \phi^2(z)}.$$ \hspace{1cm} (6)
Because of conformal mapping in the space (4) the 4-dimensional Lagrangian of matter fields \( L(x^\nu) \) and the 4-dimensional stress-energy \( \tau_{\alpha\beta}(x^\nu) \) automatically appear to be independent of \( z \) (see, for example, [12]).

So we are looking for the solution of 6-dimensional Einstein’s and matter field equations for the case of brane Universe when the metric and matter energy momentum tensor have the general structures (4) and (6), respectively. As was shown in [5], this configuration corresponds to the solution with the minimal energy and thus is stable.

On the brane we require to have 4-dimensional Einstein’s equations without the cosmological term

\[
R_{\alpha\beta} = \frac{1}{\epsilon^2 M^4 \phi^2} \left( \tau_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} \tau \right). \tag{7}
\]

The Ricci tensor in four dimensions \( R_{\alpha\beta} \) is constructed from the 4-dimensional metric tensor \( \eta_{\alpha\beta}(x^\nu) \) in the standard way. As was shown in [8], the remaining in (2) equations reduce to

\[
g_{ij} = c \eta_{ij}, \quad z \phi^3 \phi' + A \phi^5 + B \phi + C = 0, \tag{8}
\]

where prime denotes a derivative with respect to \( z \). Here \( \eta_{ij} \) is the metric tensor of flat extra \((1+1)\)-space, \( c \) and \( C \) are the integration constants and

\[
A = -\frac{\Lambda \epsilon^2 c}{40}, \quad B = \frac{c(\tau + 2L)}{16M^4} \tag{9}
\]

are dimensionless parameters. In general \( B \) depends on the 4-coordinates \( x^\nu \).

To localize matter on the brane without extra sources, the factor \( 1/\phi^2(z) \) in (6) and (7) should have a \( \delta \)-like behavior. It means that \( \phi^2(z) \) (and the transversal gravitational potential) must be a growing function starting from the brane location. On the brane we assume \( \phi(0) = 1 \), any other constant will correspond to an overall re-scaling of the coordinates. To have convergent transversal volume when \( z \) runs from 0 to \( \infty \), the needed solution \( \phi(z) \) of (8) must approach some finite value \( a > 1 \) at infinity.

Boundary conditions are taken in the form

\[
\phi(z \rightarrow 0) \approx 1 + z/|c|, \quad \phi(z \rightarrow \infty) \approx a - 1/b|c|z^b, \tag{10}
\]

where \( b > 0 \) is some parameter. This choice corresponds to the following geometries on the brane and in the transversal infinity:

\[
ds^2(z \rightarrow 0) \approx \eta_{\alpha\beta}(x^\nu) dx^\alpha dx^\beta + \eta_{ij} dx^i dx^j, \\
ds^2(z \rightarrow \infty) \approx a^2 \eta_{\alpha\beta}(x^\nu) dx^\alpha dx^\beta + \frac{1}{z^{b+1}} \eta_{ij} dx^i dx^j. \tag{11}
\]

Substitution of the conditions (10) to (8) impose the following relations:

\[
Aa^5 + Ba + C \simeq 0, \quad A + B + C \simeq 0, \\
ba^3 - 5Aa^4 - B \simeq 0, \quad 1 + 5A + B \simeq 0. \tag{12}
\]

From these relations one can find [8]

\[
b \approx \frac{4a^3 + 3a^2 + 2a + 1}{a^3(a^3 + 2a^2 + 3a + 4)}, \\
A = -\frac{\Lambda \epsilon^2 c}{40} \approx \frac{1}{a^4 + a^3 + a^2 + a - 4}, \\
B = \frac{c(\tau + 2L)}{16M^4} \approx \frac{a^4 + a^3 + a^2 + a + 1}{a^4 + a^3 + a^2 + a - 4}. \tag{13}
\]
Using the relations (12), it can be shown also that the solution of (8) has an inflection point on the brane $z = 0$ (at the inflection point the second derivative of a function is zero, while the first is not). This means that on the brane, at the minimum of the transversal gravitational potential and of the total energy of the gravitating system, the transversal curvature $R_{zz}$ is zero. The function $\phi$ has no other inflection point outside the brane and smoothly grows from 1 to its maximal value $a$.

Now it is easy to show that 4-dimensional gravity is localized on the brane, in spite of the growing character of the transversal potential. Using formulae for the decomposition of the scalar curvature and determinant

$$6R = \frac{R}{\phi^2} - 3\Lambda + \frac{\tau + 2L}{2\epsilon^2 M^4 \phi^4},$$
$$\sqrt{\frac{6}{g}} = |c\phi'|\sqrt{-\eta},$$

the integral of the gravitational part of the action (1) can be written in the form

$$S_g = -\int d^6x \sqrt{\frac{6}{g}} \frac{M^4}{2} (6R + 2\Lambda) =$$
$$= -|c|\epsilon^2 \int_{-1}^{1} dv \int d^4x \int_0^\infty dz \phi^4 \phi' \sqrt{-\eta} \frac{M^4}{2} \left( \frac{R}{\phi^2} - \Lambda + \frac{\tau + 2L}{2\epsilon^2 M^4 \phi^4} \right).$$

Putting the minimum of the warp factor at $z = 0$ means that in the frame of the center of the expanding shell Universe ($x^5, x^6 = 0$) its walls move towards the transversal (1+1)-space with a velocity close to the speed of light. In the considered space (4), the physical fields are independent of $v$. Integration of the action over $v$ gives a large but finite universal factor for all kinds of fields (corresponding to the transversal velocity of the brane) and can be ignored in the calculations. So we must show that physical fields are localized on the brane only with respect to the coordinate $z$.

Using the relations (12) and

$$\int \sqrt{\frac{6}{g}} dz = |c| \int_0^\infty \phi^4 \phi' \sqrt{-\eta} = |c| \int_1^a \phi^4 \sqrt{-\eta} d\phi,$$

one can find that after integration over $z$ the last two terms in (15) exactly cancel each other. Also, we can see that the integral over $z$ of the remaining term is finite in spite of the growing character of $\phi(z)$, since $\phi(z)$ varies in the finite range (1–$a$). So 4-dimensional gravity is localized on the brane and the total action (1) reduces to

$$S = \int d^6x \sqrt{\frac{6}{g}} \left[ -\frac{M^4}{2} (6R + 2\Lambda) + 6L \right] \simeq$$
$$\simeq \int d^4x \int_0^\infty dz \phi^4 \phi' \sqrt{-\eta} \left( -\frac{M^4}{2} \frac{R}{\phi^2} + \frac{L}{\epsilon^2 \phi^4} \right) \simeq \int d^4x \sqrt{-\eta} \left( -\frac{m_P^2}{2} R + L \right).$$

Appearing in (17), the effective 4-dimensional scale (Planck’s scale)

$$m_P^2 \sim M^4 \epsilon^2 a^2$$

is constructed from the fundamental scale $M$, the width of our world $\epsilon$ and the value of the transversal gravitational potential at infinity $a$.

For the realistic values (similar to [1]) of our physical parameters

$$m_P^2 \gg M^4 \epsilon^2, \quad (\tau + 2L) \sim M^4 > 0,$$
from the relations (13) there follows:
\[ a \gg 1, \quad c \sim -10, \quad \Lambda > 0, \quad b \sim 1/a^3, \quad c^2 \sim 1/\Lambda a^4. \] (20)

The smallness of Newton’s constant \( \sim 1/m_P^2 \) and of the width of our world \( \sim \epsilon \) can be a result of the large values of the transversal gravitational potential \( a \) and of the bulk cosmological constant \( \Lambda \).

It must be noted that, since \( c \) is negative and \( \phi' \) is positive, as can be seen from the first of equations (8), a suitable solution of our model does not exist in the case of space-like transversal 2-space of the (1 + 5)-models studied in [13–19].

Now we want to check that in (2 + 4)-space zero-modes of matter fields too are localized on the brane with increasing warp factor. To have a self-consistent theory, we must follow the assumptions (5) we had used to show the localization of 4-dimensional gravity on the brane.

The equation of a massless scalar field in six dimensions coupled to gravity has the form
\[ \partial_A (\sqrt{6}g g^{AB} \partial_B \Phi) = 0. \]
If we assume that \( \Phi \) is independent of the extra coordinates, we shall obtain an ordinary 4-dimensional Klein-Gordon equation and the action of spin-0 field can be cast in the form
\[ S_\Phi = -\frac{1}{2} \int d^6 x \sqrt{\det g} \partial_A \Phi \partial_B \Phi \sim -\frac{1}{2} \epsilon^2 \int d^4 x \phi^2 \int d^1 \nu \sqrt{-\eta} \epsilon \partial_\mu \Phi \partial_\nu \Phi. \] (21)

The localization condition requires the integral over \( \phi \) in (21) to be finite, as it actually is. The equation and the action of the \( U(1) \) vector field in the case of constant extra components \( A_i = \text{const} \) also reduce to the 4-dimensional Maxwell equations and to the action which is multiplied by a finite integral over extra coordinates:
\[ S_A = -\frac{1}{4} \int d^6 x \sqrt{\det g} g^{MN} F_{AM} F_{BN} \sim -\frac{1}{4} \epsilon^2 \int d^4 x \phi \int d^1 \nu \sqrt{-\eta} \epsilon \partial_\mu F_{\mu\nu}. \] (22)

From the convergent character of the volume element (16) and formulae (21) and (22) it is easy to see that localization of the Abelian-Higgs model (investigated in the papers [18,19] for the signature \( (1 + 5) \)) is a particular example of our model. In addition, here we have localization of the zero modes of spinor fields too.

In the case of spinor fields we shall introduce the vierbein \( h_M^i \), where \( M, i, \ldots \) denote local Lorentz indices. The relation between the curved gamma matrices \( \gamma^M \) and the flat gamma ones \( \gamma^M \) is given by the formula \( \Gamma^M = h_M^i \gamma^i \), so that
\[ \Gamma_\mu = \phi \gamma_\mu, \quad \Gamma_i = \sqrt{|\phi|} \gamma_i. \] (23)

The spin connection is defined as
\[ \omega_M^N = \frac{1}{2} h_N^M (\partial_M h_N^\nu - \partial_N h_M^\nu) - \frac{1}{2} h_N^M (\partial_M h^\nu_N - \partial_N h^\nu_M) - h^P_N h^Q_M (\partial_P h^R_{QR} - \partial_Q h^R_{PR} h^R_M). \] (24)

The non-vanishing components of the spin connection for the background metric (4) are
\[ \omega_{\nu}^{\tilde{N}} = (\delta^i_{\tilde{N}} \delta^\nu_{\tilde{M}} - \delta^i_{\tilde{M}} \delta^\nu_{\tilde{N}}) \partial_i \phi/\sqrt{|\phi|}, \quad \omega_{\tilde{M}}^{\tilde{N}} = (\delta^i_{\tilde{N}} \delta^\nu_{\tilde{M}} - \delta^i_{\tilde{M}} \delta^\nu_{\tilde{N}}) \partial_i \sqrt{|\phi|}/\sqrt{|\phi|}. \] (25)

Therefore, the covariant derivatives have the form
\[ D_\mu \Psi = (\partial_\mu + \Gamma^\nu \partial_\nu \partial_\phi/2 \phi) \Psi, \quad D_i \Psi = (\partial_i + \Gamma^\nu \partial_\nu \sqrt{\phi}/2 \sqrt{|\phi|}) \Psi. \] (26)

We are looking for a solution in the form \( \Psi(x^A) = \psi(x^\nu) H(x^i) \), where \( \psi \) satisfies the massless 4-dimensional Dirac equation \( \gamma^\nu \partial_\nu \psi = 0 \). Then the 6-dimensional Dirac equation reduces to
\[ (\partial_\nu - 2 \partial_\nu \phi/\phi - \partial_\nu \sqrt{\phi}/2 \sqrt{|\phi|}) H(x^i) = 0. \] (27)
The solution of this equation with unit integration constant is

\[ H(x^i) = \phi^2(\phi')^{1/4}. \tag{28} \]

Then the action of the spin-(1/2) field takes the form

\[ S_{\Psi} = \int d^6x\sqrt{-g}\bar{\Psi}i\Gamma^AD_A\Psi \simeq \epsilon^2\int_0^\infty dz\phi^7(\phi')^{3/2}\int d^4x\sqrt{-\eta}\bar{\psi}i\gamma^\nu\partial_\nu\psi. \tag{29} \]

Since \( \phi \) and \( \phi' \) are monotone and finite functions of \( z \), the integral over \( z \) in (29) is finite. So massless Dirac fermions are also localized on the brane.

When we consider the interaction of scalars or fermions with the electromagnetic field we must make the usual replacements

\[ \partial_i \rightarrow \partial_i - iA_i, \quad \Psi \rightarrow e^{iA_i x^i}\Psi \tag{30} \]

in the above formulae for localization. Here \( x^i \) are the coordinates of the transversal \((1+1)\)-space and \( A_i \) are the constant extra components of the electromagnetic field.

To summarize, in this paper it is shown that for the realistic values of the fundamental scale and the brane stress energy, there exists a non-singular static solution of \((2+4)\)-dimensional Einstein equations. This solution provides gravitational trapping of the 4-dimensional gravity and the matter on the brane without extra \( \delta \)-like sources. In contrast to Randall-Sundrum’s case, the factor responsible for this trapping is the growing away from the brane gravitational potential, but has a convergent volume integral, although the transversal 2-space is infinite. The study of the fluctuation of the metric, which is crucial for the stability of the model, will be the subject of future investigations.

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