A REDUCTION THEOREM FOR NONSOLVABLE FINITE GROUPS

FRANCESCO FUMAGALLI, FELIX LEINEN, AND ORAZIO PUGLISI

Abstract. Every finite group $G$ has a normal series each of whose factors is either a solvable group or a direct product of nonabelian simple groups. The minimum number of nonsolvable factors attained on all possible such series is called the nonsolvable length of the group and denoted by $\lambda(G)$. For every integer $n$, we define a particular class of groups of nonsolvable length $n$, called $n$-rarefied, and we show that every finite group of nonsolvable length $n$ contains an $n$-rarefied subgroup. As applications of this result, we improve the known upper bounds on $\lambda(G)$ and determine the maximum possible nonsolvable length for permutation groups and linear groups of fixed degree resp. dimension.

1. Introduction

Every finite group $G$ has a normal series each of whose factors is either a solvable group or a direct product of nonabelian simple groups. The minimum number of nonsolvable factors attained on all possible such series is called the nonsolvable length of $G$ and denoted by $\lambda(G)$ (see [11]). In a series of recent papers (see also [13], [12], [7], [3], [4]) this parameter has been investigated, as some bounds on $\lambda(G)$ have been proved to be useful in several situations.

The aim of this paper is to provide a tool that might be useful when investigating properties related to the nonsolvable length. For every natural number $n$ denote $\Lambda_n$ the class of groups of nonsolvable length $n$. We define a particular kind of $\Lambda_n$-groups, called $n$-rarefied (see Definition 4.1) whose structure of normal subgroups is quite restricted. E.g. any chief series of an $n$-rarefied group has exactly $n$ nonabelian factors and the nonabelian composition factors are simple groups isomorphic to one of the following:

$$L_2(2^r), L_2(3^r), L_2(p^a), L_3(3), 2B_2(2^s),$$

where $p, r, s$ are primes, $p, s$ odd and $a \geq 0$.

The main result of this paper is

**Theorem 1.1.** Every finite group in $\Lambda_n$ contains an $n$-rarefied subgroup.

The proof relies on the Classification of Finite Simple Groups. Due to the peculiar structure of $n$-rarefied groups, Theorem 1.1. plays a central role whenever we deal with questions related to the nonsolvable length that can be reduced to subgroups. In the last section of the paper we provide some examples of this situation. In particular, we use the above ideas to improve the bound on $\lambda(G)$, obtained in [11, Theorem 1.1.(a)]

**Theorem 5.2** Let $G$ be any finite group. Then $\lambda(G) \leq L_2(G)$.
Here $L_2(G)$ denotes the maximum of the 2-lengths of all possible solvable subgroups of $G$. Also, along the lines of [20], where the authors study group properties that can be detected from the 2-generated subgroups, we prove

**Theorem 5.3** Let $G$ be any finite group. If $G$ is not solvable then there exists a 2-generator subgroup $H$ of $G$ such that $\lambda(H) = \lambda(G)$.

The last application of Theorem 1.1 concerns the nonsolvable length of subgroups of $H = \text{Sym}(m)$ resp. of $H = \text{GL}(m, F)$, for any fixed natural number $m$ and any field $F$. Denoting by $\lambda(m)$ resp. $\lambda_F(m)$ the maximum value of $\lambda(G)$ when $G$ is a subgroup of $H$, we prove the following

**Theorem 5.6** For every $m \geq 5$ we have $\lambda(m) = \lfloor \log_5(m) \rfloor$.

**Theorem 5.9** For every $m \geq 2$ and every field $F$ with at least four elements, we have $\lambda_F(m) = 1 + \lfloor \log_5(m/2) \rfloor$. When $|F| \leq 3$, then $1 + \lfloor \log_5(m/3) \rfloor \leq \lambda_F(m) \leq 1 + \lfloor \log_5(m/2) \rfloor$.

2. Preliminaries

In this section we set up the basic definitions and notation, and collect a series of technical facts to be used in the proof of the main result (Theorem 1.1).

**Definition 2.1.** Let $G$ be any finite group. Define

$$R(G) = \langle B \mid B \text{ is a normal solvable subgroup of } G \rangle.$$ 

The subgroup $R(G)$ is the solvable radical of $G$. It is clearly normal and solvable, and $G/R(G)$ does not have any nontrivial normal solvable subgroup.

**Definition 2.2.** Let $G$ be any finite group. Define

$$S(G) = \langle A \mid A \text{ is a minimal normal nonabelian subgroup of } G \rangle.$$ 

If $S(G) \neq 1$, then it is the direct product of nonabelian simple groups. All its factors are subnormal in $G$ and they are called the components of $G$. It is easy to check that $S(G)$ is the subgroup generated by all the nonabelian simple subnormal subgroups of $G$.

A series can then be defined as follows.

**Definition 2.3.** Let $G$ be any finite group. Set $R_1(G) = R(G)$ and define $S_1(G)$ by the equation

$$S_1(G)/R_1(G) = S(G/R_1(G)).$$

Once $R_i(G)$ and $S_i(G)$ have been defined for all $i < n$, the subgroups $R_n(G)$ and $S_n(G)$ are determined by the equations

$$R_n(G)/S_{n-1}(G) = R(G/S_{n-1}(G))$$

and

$$S_n(G)/R_n(G) = S(G/R_n(G)).$$

We call this series the $RS$-series of $G$. 

The series thus defined always reaches $G$. If $G = R_{n+1}(G) > R_n(G)$, we say that $G$ has nonsolvable length $\lambda(G) = n$ (according to the notation introduced in [11]). Also, for $n \geq 1$ we denote by $\Lambda_n$ the class of finite groups $G$ of nonsolvable length $n$.

Note that the class of groups having nonsolvable length zero coincides with the one of solvable groups. Also, simple/quasisimple/almost simple groups, as well as their direct products are all groups of nonsolvable length one. A typical example of a group of nonsolvable length $n$ is an $n$-fold wreath product of a fixed nonabelian simple group.

The next Lemma collects some easy but useful observations. The proof is left to the reader.

**Lemma 2.4.** Let $G$ be a finite group and assume that $\lambda(G) = n$. The following hold.

1. $\lambda(G/R_i(G)) = n - i + 1$, for all $i = 1, \ldots, n$.
2. $\lambda(G/S_i(G)) = n - i$, for all $i = 1, \ldots, n$.
3. If $N \trianglelefteq G$, then $\lambda(G/N) \leq \lambda(G) \leq \lambda(N) + \lambda(G/N)$.
4. $C_{G/R_i(G)}(S_i(G)/R_i(G)) = 1$, for all $i = 1, \ldots, n$.
5. If $H$ is a subnormal subgroup of $G$ then $R_i(H) \leq R_i(G)$ and $S_i(H) \leq S_i(G)$, for all $i = 1, \ldots, n$.
6. The RS-series of $G$ is a shortest series whose factors are either solvable or semisimple.

In the course of our analysis of $\Lambda_n$-groups acting on sets/vector spaces we will need the following result.

**Lemma 2.5.** Let $G$ be a finite group with normal subgroups $N_1, N_2, \ldots, N_r$ such that $\bigcap_{i=1}^r N_i = 1$. Then $\lambda(G) = \max\{\lambda(G/N_i)|i = 1, \ldots, r\}$.

**Proof.** For every $i = 1, \ldots, r$ call $G_i = G/N_i$ and set also $n$ the greatest of all $\lambda(G_i)$. Since $G$ has an homomorphic image of nonsolvable length $n$, we have by Lemma 2.4(3), $\lambda(G) \geq n$. Note that $\lambda(G) \leq n$ is of course true if $\lambda(G) = 0$ or 1, since if all the $G_i$ are solvable groups then $G$ itself is solvable. Assume that $G$ is a minimal counterexample and therefore $\lambda(G) > 1$. For each $i$ let $A_i$ be such that $A_i/N_i$ is the solvable radical of $G_i$, and let $A = \cap_{i=1}^r A_i$. Then, since $A$ embeds in the direct product $\prod_{i=1}^r A_i/N_i$, $A$ is solvable and, being normal, $A \trianglelefteq R_1(G)$. For every $i$, $R_1(G)N_i/N_i$ is a solvable normal subgroup of $G_i$, therefore $R_1(G) \leq A_i$ for every $i$, hence $R_1(G) = A$. Now, if $R_1(G) \neq 1$, the claim is proved by induction because $\lambda(G) = \lambda(G/R_1(G))$ and $\lambda(G_i) = \lambda(G/A_i)$ for all $i = 1, \ldots, r$. We assume therefore that $R_1(G) = 1$ and moreover, being $G$ a subdirect product of the groups $G/A_i$, we can assume that each $G/N_i$ has trivial solvable radical, i.e. $N_i = A_i$ for every $i$. For each $i$ let $K_i$ be the subgroup of $G$ defined by $K_i/N_i = S_i(G_i)$ and let $K = \bigcap_{i=1}^r K_i$. Of course, as $S_1(G_i)/N_i \leq K_i$ for each $i$, we have that $S_1(G) \leq K$. In particular, $G/K$ has order strictly smaller than the order of $G$ and it is a subdirect product of the groups $G/K_i \simeq G_i/S_i(G_i)$, for $i = 1, \ldots, r$. Hence by the inductive assumption we have that $\lambda(G/K) = \max\{\lambda(G_i/K_i)|i = 1, \ldots, r\} = n - 1$. Also $K$ is a subdirect product of the groups $K_i/N_i$, for $i = 1, \ldots, r$. Hence if $K \neq G$, by the inductive assumption we have that $\lambda(K) = \max\{\lambda(K_i/N_i)|i = 1, \ldots, r\} = 1$. By Lemma 2.4(3) it follows that $\lambda(G) \leq n$. Therefore we have that $G = K = K_i$ for every $i = 1, \ldots, r$. In particular, $G_i = S_1(G_i)$ for every $i$ and so $G$ is a subdirect
product of semisimple groups. By [8, Lemma 4.3A] $G$ is a direct product of simple groups, then $\lambda(G) = 1 = \max \{\lambda(G_i) | i = 1, \ldots, r\}$ which completes the proof. □

3. Some technical results on finite simple groups

Given two finite groups $X$ and $Y$ with $X \subseteq Y$, let $K$ be a subgroup of $X$. We say that $K$ extends from $X$ to $Y$ if $Y = XN_Y(K)$. This is equivalent to say that the $Y$-conjugacy class of $K$, which we denote by $[K]_Y = \{K^y | y \in Y\}$, coincides with the $X$-conjugacy class, $[K]_X$. If, for example, $X$ contains a unique conjugacy class of subgroups isomorphic to $K$, then $K$ extends from $X$ to $Y$ by a Frattini argument.

The proof of the following Lemma is left to the reader.

**Lemma 3.1.** Assume that $K, X$ and $Y_1$ are subgroups of a group $Y$ such that $K < X < Y_1 < Y$.

1. If $K$ extends from $X$ to $Y$, then $K$ extends from $X$ to $Y_1$.
2. If $K$ is maximal in $X$ and $K$ extends from $X$ to $Y$, then $N_Y(K)$ is maximal in $Y$ provided $K$ is not normal in $Y$.

Beside the standard notation generally used in literature, we shall also make use of the one established in the Atlas ([5]), e.g.:

- $n$ denotes both a natural number and the cyclic group of that order;
- $A.B$ denotes an extension of the group $A$ by the group $B$;
- $A : B$ denotes a split extension;
- $A \cdot B$ denotes a non-split extension.

The notation for sporadic simple groups and simple groups of Lie type follows the Atlas, while $\text{Alt}(m)$ and $\text{Sym}(m)$ denote respectively the alternating and symmetric group of degree $m$. We refer to [17] for the definition of the Aschbacher’s classes $C_i$ (for $i = 1, \ldots, 8$) of natural subgroups of classical groups.

From now on $\mathcal{L}$ will denote the class consisting of the following simple groups:

- $L_2(q)$ for $q \in \{2^r, 3^r, p^{2^a}\}$ with $r$ a prime, $p$ an odd prime and $a \geq 0$,
- $L_3(3)$,
- $2B_2(2^s)$ for $s$ an odd prime.

Note that $\mathcal{L}$ contains $\text{Alt}(5) \simeq L_2(4)$ and $\text{Alt}(6) \simeq L_2(9)$. Moreover, every minimal simple group (i.e. every simple group whose proper subgroups are all solvable) lies in $\mathcal{L}$ (see [22, Corollary 1]).

In the proof of Lemma 3.2 we need a technical result about the existence of certain subgroups in finite nonabelian simple groups not belonging to $\mathcal{L}$. The proof of this result relies on the classification of finite simple groups.

**Lemma 3.2.** Let $S$ be a nonabelian finite simple group. Assume that $S$ does not belong to $\mathcal{L}$. Then $S$ contains a proper self-normalizing subgroup $H$ which belongs to $\Lambda_1$ and such that it extends to $\text{Aut}(S)$.

**Proof.**

Case $S$ alternating.

Let $S = \text{Alt}(n)$ with $n \geq 7$. We may choose $H$ to be the stabilizer of a point. Then $H$ is a maximal subgroup of $S$ isomorphic to $\text{Alt}(n-1)$, hence $N_S(H) = H \in \Lambda_1$. Trivially $H$ extends to $\text{Aut}(S) = \text{Sym}(n)$. 

The notation for sporadic simple groups and simple groups of Lie type follows the Atlas, while $\text{Alt}(m)$ and $\text{Sym}(m)$ denote respectively the alternating and symmetric group of degree $m$. We refer to [17] for the definition of the Aschbacher’s classes $C_i$ (for $i = 1, \ldots, 8$) of natural subgroups of classical groups.
Case $S$ a simple group of Lie type in characteristic $p$.

Let $G$ be the extension of $S$ by the diagonal and field automorphisms. We consider separately the two cases:

1. $G = \text{Aut}(S)$,
2. $G < \text{Aut}(S)$.

Case 1. By general $BN$-pair theory [2 Proposition 8.2.1 and Theorem 13.5.4], the lattice of proper overgroups in $S$ of a Borel subgroup $B = N_S(U)$ (where $U$ is a fixed Sylow $p$-subgroup of $S$) consists of the parabolic subgroups of $S$. In particular the maximal parabolic subgroup, $P_1$, that is correlated to the first node of the Dynkin diagram (or with the orbit of nodes labelled 1 in the twisted case), is a maximal subgroup of $S$. Now the group $G$ admits a $BN$-pair too, whose Borel subgroup is $N_G(U)$, since in order to construct $G$ from $S$ we can choose diagonal and field automorphisms that normalize every root subgroup of $U$. In particular, $P_1$ extends to $G$. As long as $P_1 \in \Lambda_1$, we may therefore choose $H = P_1$ to prove the Lemma. Note that $P_1 = U_1L_1$ is the product of a $p$-subgroup, its unipotent radical $U_1$, by the Levi complement $L_1$, which is a central product of groups of Lie type corresponding to the subdiagram obtained by deleting the first node (the first orbit of nodes). Thus $P_1 \in \Lambda_1$ if and only if $L_1$ does and this almost always happens. The only exceptions are when the Dynkin diagram is a point, or an orbit of two points (in the twisted case), or a pair of points and $q \in \{2, 3\}$, because this are the cases in which the group $L_2(q)$ is solvable. Namely, $H$ cannot be chosen to be $P_1$ in the following situations:

1. $L_2(q)$ for all $q \geq 4$,
2. $U_3(q)$ for all $q \geq 3$,
3. $U_4(3)$, $U_5(2)$, $U_5(3)$,
4. $2B_2(2r)$ with $r$ odd and $2r \geq 8$,
5. $2G_2(3r)$ with $r$ odd and $3r \geq 27$,
6. $S_4(3)$, $3D_4(2)$, $3D_4(3)$,

since $S_4(2) \simeq \text{Sym}(6)$, $U_3(2) \simeq 3^2 : Q_8$, $U_4(2) \simeq S_4(3)$, $G_2(2) \simeq U_3(3).2$, $2B_2(2) \simeq 5 : 4$, $2G_2(3) \simeq L_2(8).3$ and $2F_4(2)$ has a simple normal subgroup of index 2.

i. Let $S = L_2(q)$ for $q = p^f \geq 4$. We may now consider maximal subgroups of $S$ belonging to Aschbacher’s class $C_5$. In particular, if $p = 2$ and $f$ is not a prime there exists a maximal subgroup $H$ isomorphic to $L_2(q_0)$, with $q = q_0^a$ for some prime $r$ dividing $f$, that extends to $\text{Aut}(S)$ (see for instance [1 Table 8.1]). Similarly, if $p$ is odd and there exists an odd prime $r$ dividing $f$, then there is a maximal subgroup $H$ isomorphic to $L_2(q_0).2$, with $q = q_0^a$, that extends to $\text{Aut}(S)$. Note that such a subgroup lies in $\Lambda_1$, provided that $q_0 \neq 3$. Thus being

$$\{L_2(2^r), L_2(3^r), L_2(p^{2a})\rvert r, \text{p primes, } p \text{ odd, } a \geq 0\} \subseteq L$$

the Lemma is proved in this case.

We treat together ii. and iii. (and in general every unitary case).

Let $S = U_n(q)$, with $n \geq 3$. Let $H$ be the stabilizer of a non-isotropic point (in the natural unitary $n$-dimensional space). Then $H$ is a maximal subgroup of $S$ and it extends to $\text{Aut}(S)$ (see [11 Table 3.5.B] and [11 Table 8.1]). Also, $H'$ is a cyclic extension of $U_{n-1}(q)$, therefore $H \in \Lambda_1$, whenever $(n, q) = (3, 3)$. For $S = U_3(3)$, a direct inspection in [5] suggests that we may take $H$ to be the maximal subgroup $L_2(7)$.
iv. Let $S = 2B_2(2^r)$ with $r \geq 3$ odd and composite. Then if $l$ is a prime divisor of $r$ the centralizer in $S$ of a field automorphism of order $l$ is a maximal subgroup of $S$, isomorphic to $2B_2(2^{r/l})$ that extends to $\text{Aut}(S)$ (see [21]).

v. Let $S = 2G_2(3^r)$ with $r$ odd and $r \geq 3$. Then $S$ has a unique class of involutions (see [14]). Let $i$ be one involution, then $C_S(i)$ is a maximal subgroup of $S$ isomorphic to $2 \times L_2(q)$ ([15] Theorem C]) and therefore it can be chosen as $H$.

Finally consider the three cases in vi.

Let $S = S_4(3)$. In this case we may take $H \simeq 2^4 : \text{Alt}(5)$ (see [5]).

Let $S = 3D_4(2)$. A direct inspection in [5] suggests that we may take $H \simeq 2^9 : L_2(8)$.

Let $S = 3D_4(3)$. By [10] we may take as $H$ a copy of the maximal subgroup $3^9 : \text{SL}_2(27).2$.

Case 2. Suppose now that $G < \text{Aut}(S)$, that is, there exist graph automorphisms of $S$. Note that this happens exactly when $S$ is one of the following groups (see [2] or [5]):

$L_n(q)$ for $n \geq 3$, $S_4(2^r)$, $O_{2n}^+(q)$ for $n \geq 4$, $G_2(3^r)$, $F_4(2^r)$, $E_6(q)$.

Let $S = L_n(q)$ with $n \geq 3$. Let $\iota$ be a graph automorphism acting as the inverse-transpose on the elements of $S$, and let $H = P_1 \cap P_1^\iota$. Note that $H$ is the stabilizer in $S$ of a direct decomposition of the natural module $V = U \oplus W$, where $U$ is 1-dimensional and $W$ ($n - 1$)-dimensional. In particular we have that $H = N_S(H)$, since $n - 1 \geq 2$. Moreover, $H$ extends to $\text{Aut}(S)$ and $H'$ is a central extension of $L_{n-1}(q)$ (see [14] Prop. 4.1.4 for details). Therefore, as long as $(n, q) \notin \{(3, 2), (3, 3)\}$ the subgroup $H$ lies in $\Lambda_1$. Finally note that $L_2(2) \simeq L_2(7)$ and $L_3(3)$ are minimal simple groups and so they belong to $\mathcal{L}$.

Let $S = S_4(2^r)$. Note that $r \geq 2$ as $S_4(2)$ is not simple. If $r = 2$ then a direct inspection in [5] shows that we may take $H \simeq \text{Sym}(6)$. Let $r \geq 3$. If $r$ is odd, then $S$ contains a unique conjugacy class of maximal subgroups isomorphic to $2B_2(2^{r'})$, so we are done. For $r$ even, one may take $H \simeq S_4(q^{1/2})$, which is always a maximal subgroup of $S$, and $[H]_S = [H]_{\text{Aut}(S)}$ (see [17] Table 3.5.C or [1] Table 8.14).

Let $S = O_{2n}^+(q)$ with $n \geq 4$. If $n \geq 5$, we may take $H = P_1$. This coincides with the stabilizer of an isotropic point. $H$ is a maximal subgroup of $S$ that extends to $\text{Aut}(S)$ and such that $H'$ is a central extension of $O_{2(n-1)}^+(q)$ (see [17] Table 3.5.E or [1] Table 8), thus $H \in \Lambda_1$.

For $S = O_7^+(q)$ we refer to [13]. If $q > 2$ then maximal parabolic subgroups $P_2$ that correspond to the second node of the Dynking diagram form a unique $S$-class of maximal subgroups of $S$, which is $\text{Aut}(S)$-invariant. Such subgroups are stabilizers of totally singular 2-subspaces and are denoted by $R_{s2}$ in [14]. Their isomorphism type is $[q^n]_0.4\text{GL}_2(q) \times O_3^+(q),d$, where $d = (2q - 1)$ ([14] Prop. 2.2.2), thus as long as $q > 3$, $H = P_2$ satisfies our requirements.

If $S = O_8^+(3)$, we may take $H$ to be an $N_2$-group in the notation of [14] Section 3.2. Then by [14] Prop. 3.2.3, $H \simeq L_3(3) : 2$ and it extends from $S$ to $\text{Aut}(S)$. Moreover $H$ is self-normalizing in $S$, as it is a maximal member in the class $C_2$ ([14] Prop. 4.2.1 and Table III).

Finally consider the case $S = O_8^+(2)$. According to the [12] (or again [14] Section 3.2) we may take $H$ a subgroup isomorphic to $U_3(3) : 2$. Such subgroup extends
Case S sporadic or $^2F_4(2)'$. 
In Table I we collect our choice for the subgroup $H$ when S is one of the 26 sporadic simple groups or the Tits group $^2F_4(2)'$. In each case, $H$ is maximal in $S$. The validity of Table I can be checked in the Atlas ([5]) and by the use of [10] for what concerns the Monster group $M$.

Table 1. Case S is sporadic or $^2F_4(2)'$.

| $S$   | $H$       | $S$   | $H$       | $S$   | $H$       |
|-------|-----------|-------|-----------|-------|-----------|
| $M_{11}$ | $M_{10}$ | $M_{24}$ | $M_{23}$ | $HN$ | $Alt(12)$ |
| $M_{12}$ | $L_2(11)$ | $M \cdot L$ | $U_4(3)$ | $L_9$ | $G_2(5)$ |
| $J_1$   | $L_2(11)$ | $He$   | $3'Sym(7)$ | $Th$ | $M_{10}$ |
| $M_{22}$ | $L_2(11)$ | $Ru$   | $^2F_4(2)$ | $Fi_{23}$ | $2F_{i_{22}}$ |
| $J_2$   | $U_3(3)$  | $Suz$  | $G_2(4)$  | $Co_1$ | $Co_2$ |
| $M_{23}$ | $M_{22}$ | $O'N$  | $J_1$     | $J_4$ | $2^{11}:M_{24}$ |
| $^2F_4(2)'$ | $L_2(25)$ | $Co_3$ | $M \cdot L : 2$ | $F_{i_{24}}$ | $Fi_{23}$ |
| $HS$    | $M_{22}$ | $Co_2$ | $M \cdot L$ | $B$ | $Th$ |
| $J_3$   | $L_2(17)$ | $Fi_{22}$ | $M_{12}$ | $M$ | $L_2(29) : 2$ |

The proof is now complete. □

The following fact will be needed in the last section.

**Lemma 3.3.** Let $S$ be a simple group belonging to $\mathcal{L}$. Then there exists a 2-nilpotent subgroup $D$ of $S$, with $O_2(D) = 1$ and $N_S(D) = D$ that extends to Aut($S$).

**Proof.** We treat the different cases separately.

Case $S \simeq L_2(2^r)$, with $r$ a prime.

We may take $D$ to be a dihedral group of order $2(q + 1)$. Then $D$ is a maximal subgroup of $S$ and it extends to Aut($S$), being the normalizer of maximal torus of order $q + 1$ (see [9] Theorem 6.5.1]).

Case $S \simeq L_2(3^r)$, with $r$ a prime.

When $r$ is odd, $q = 3^r \equiv -1$ (mod 4) and so $\frac{q - 1}{2}$ is odd. Then take $D$ the normalizer of a split torus of order $\frac{q - 1}{2}$. This is a maximal subgroup of $S$ isomorphic to Aut($S$) and is self-normalizing, as the proper overgroups of these are isomorphic to $S_6(2)$.

Let $S = G_2(3')$. If $r$ is odd, let $H$ be the centralizer in $S$ of a graph involution, $\phi_2$. Then $H$ is a maximal subgroup of $S$, isomorphic to $^2G_2(q)$ and such that $[H, S] = [H]_{Aut(S)}$ (see [13] Lemma 1.5.5 and Theorem B). For $r$ even one can take the centralizer of a field automorphism of order two, then $H$ is still a maximal subgroup that extends to Aut($S$) and it is isomorphic to $G_2(q^{1/2})$.

Let $S = F_4(2')$. According to [18] Table 5.1, $S$ admits a unique conjugacy class of subgroups (of maximal rank in Aut($S$)) isomorphic to $Sp_4(q^2).2$. These subgroups are not maximal in $S$ but they are self-normalizing. If $H$ is one of these then $H = N_S(S_1(H))$ and we know that $M := N_{Aut(S)}(S_1(H))$ is maximal in Aut($S$). Then $H$ is normal in $M$ and so $M = M_{Aut(S)}(H)$ forcing $H = M \cap S = N_S(H)$.

Let $S = E_6(q)$. We can take as $H$ either a maximal parabolic subgroup that corresponds to the middle node in the Dynkin diagram or the one below it.
to a dihedral group, that extends to Aut(S). (see [9, Theorem 6.5.1]). When \( r = 2 \),
take \( D \) the normalizer of a Sylow 3-subgroup, then \( D \simeq 3^2 : 4 \) is a maximal
subgroup that extends to Aut(S) (see [5]).

Case \( S \simeq L_2(p^a) \), with \( p \) an odd prime and \( a \geq 0 \).

If \( q = p^a \neq 5, 7, 11 \) then take \( D \) a dihedral subgroup of order \( q + \epsilon \), where \( \epsilon = \pm 1 \)
and \( q \equiv \epsilon (\text{mod 4}) \). Then \( D \) is maximal in \( S \) and it extends to Aut(S) (see [1, Table 8.1]). If \( q = 5 \), then \( S \simeq \text{Alt}(5) \) and one may take as \( D \) a dihedral subgroup
of order 10. If \( q = 7 \) the group \( L_2(7) \simeq L_3(2) \). In this case, take \( D \) the stabilizer in
\( L_3(2) \) of a direct decomposition \( V = U \oplus W \) of the natural 3-dimensional module
\( V \) by a 1-dimensional \( U \) and 2-dimensional \( W \). Then \( D \) is isomorphic to a dihedral
group of order 6, it is self-normalizing in \( S \) and it extends to Aut(S). If \( q = 11 \) take
\( D \) the normalizer in \( S \) of Sylow 5-subgroup, this self-normalizing subgroup extends
to Aut(S) (see [5]).

Case \( S \simeq L_3(3) \).

Take \( D \) to be a Borel subgroup of \( S \). This is a self-normalizing subgroup of \( S \)
isomorphic to \( P : 2^2 \), where \( P \) is extraspecial of order 27 and exponent 3.

Case \( S \simeq 2B_2(2^r) \), where \( r \) is an odd prime.

We may take as \( D \) a dihedral subgroup of order \( 2(q-1) \). This is a maximal subgroup
of \( S \) and, since \( S \) has a unique conjugacy class of subgroups of order \( q-1 \) (see [21]),
it extends to Aut(S).

□

4. \( n \)-rarefied subgroups

We start by defining some particular members of the class \( \Lambda_n \).

**Definition 4.1.** A group \( G \) of nonsolvable length \( n \geq 1 \) will be said to be \( n \)-rarefied
if the following conditions hold:

1. \( S_i(G)/R_i(G) \) is the unique minimal normal subgroup of \( G/R_i(G) \) for all
   \( i = 1, 2, \ldots, n \);
2. the components of \( S_i(G)/R_i(G) \) are simple groups in \( L \) for all \( i = 1, 2, \ldots, n \);
3. \( R_1(G) = \Phi(G) \) and \( R_{i+1}(G)/S_i(G) = \Phi(G/S_i(G)) \) for all \( i = 1, \ldots, n - 1 \).

Note that every \( n \)-rarefied group \( G \) is perfect with last RS-factor, \( G/R_n(G) \), a
simple group.

The aim of this section is to show that each group in \( \Lambda_n \) contains an \( n \)-rarefied
subgroup (Theorem 1.1). The existence of \( n \)-rarefied subgroups can be used to
reduce, in certain cases, the study of a question about \( \Lambda_n \)-groups to the case of
\( n \)-rarefied groups. However, if the problem we are dealing with, is not about abstract
group but concerns groups acting on some structure, then this reduction Theorem
may not be sufficient. In the last part of this section, we derive a reduction argument
that covers the case of permutation and linear groups (Proposition 4.7).

The class of rarefied groups is closed by quotients.

**Proposition 4.2.** Let \( G \) be an \( n \)-rarefied group and \( N \) a normal subgroup of \( G \).
Then \( G/N \) is \( m \)-rarefied for some \( m \leq n \).

**Proof.** Clearly we may assume that \( N \) is not contained in \( \Phi(G) \), otherwise the
claim is plainly true. We make induction on the order of \( G \). Assume the claim
true for groups of order smaller than \( |G| \) and choose \( N \) not contained in \( \Phi(G) \). If
Lemma 4.4. Let $\Phi(G) = 1$, then $N$ must contain $S_1(G)$, which is the only minimal normal subgroup of $G$. Since $G/S_1(G)$ is $(n-1)$-rarefied, we apply induction to $\overline{N} = N/S_1(G)$ as a normal subgroup of $\overline{G} = G/S_1(G)$, getting that $\overline{G}/\overline{N} \simeq G/N$ is $m$-rarefied for some $m \leq n - 1 < n$.

Therefore we suppose $\Phi(G) \neq 1$. If $A = N \cap \Phi(G) \neq 1$, we apply the inductive hypothesis to $N/A$ as a normal subgroup of $G/A$, and conclude as in the above paragraph. We are then left with the case $\Phi(G) \cap N = 1$. In this situation $N$ has trivial solvable radical, hence $B = S_1(N)$ is semisimple and from this it follows that $S_1(G) = B\Phi(G)$. If we can prove that $G/B$ is $(n-1)$-rarefied, the claim will follow as before. The first remark we should make is that $\lambda$ is trivial solvable radical, hence

$$H/R = 1, \text{ a case that we do not need to consider.}$$

Since $N$ is still $n$-rarefied. Thus we assume $\Phi(G) \neq 1$, and pick $M/B$ a maximal subgroup of $G/B$. Since $M$ is maximal in $G$ it must contain $\Phi(G)$. Thus $M$ contains $B\Phi(G) = S_1(G)$ and we have that

$$A = \{M \mid B \leq M \text{ and } M/B \text{ is maximal in } G/B\}$$

coincides with

$$B = \{M \mid S_1(G) \leq M \text{ and } M/S_1(G) \text{ is maximal in } G/S_1(G)\}.$$

Hence

$$F = \cap\{M \mid M \in A\} = \cap\{M \mid M \in B\} = R_2(G)$$

and the proof is completed. \qed

The next Lemma says that for the class of $n$-rarefied groups the nonsolvable length behaves well.

**Lemma 4.3.** Let $G$ be an $n$-rarefied group and $N \leq G$. Then $\lambda(G) = \lambda(N) + \lambda(G/N)$.

**Proof.** Since $\lambda(N) = \lambda(N\Phi(G))$, we assume $\Phi(G) \leq N$. When $\Phi(G) \neq 1$, an obvious inductive argument gives the claim, since $\lambda(G) = \lambda(G/\Phi(G))$ and $G/\Phi(G)$ is still $n$-rarefied. Thus we assume $\Phi(G) = 1$ which implies $S_1(G) \leq N$, unless $N = 1$, a case that we do not need to consider. Since $S_1(G) = S_1(N)$, we apply induction on $N/S_1(G)$ as a subgroup of $G/S_1(G)$, getting

$$\lambda(G/S_1(G)) = \lambda(N/S_1(G)) + \lambda(G/N)$$

and the claim follows because $\lambda(G/S_1(G)) = \lambda(G) - 1$ and $\lambda(N/S_1(G)) = \lambda(N) - 1$. \qed

Our first step towards Theorem 4.4 is to show that every group in $\Lambda_n$ has a subgroup whose RS-series satisfies some restrictions.

**Lemma 4.4.** Let $G$ be a group in $\Lambda_n$. Then there exists a subgroup $H$ of $G$ such that $H \in \Lambda_n$ and $S_i(H)/R_i(H)$ is the unique minimal normal subgroup of $H/R_i(H)$, for all $i = 1, \ldots, n$. 

Proof. We prove the claim by induction on \( n \). If \( n = 1 \) the section \( S_1(G)/R_1(G) \) is a direct product of simple groups. Choose \( H \) such that \( R_1(G) < H \) and \( H/R_1(G) \) is one of the simple direct factors of \( S_1(G)/R_1(G) \). Thus \( R_1(H) = R_1(G) \) and the claim holds.

Assume \( n > 1 \) and the claim true for groups in \( \Lambda_{n-1} \). If the claim does not hold in \( \Lambda_n \), choose a counterexample \( G \in \Lambda_n \) of minimal order. If \( R_1(G) \neq 1 \), the group \( G/R_1(G) \) is still in \( \Lambda_n \) (by Lemma 2.4) but its order is strictly smaller than \( |G| \). There is therefore a subgroup \( H \) of \( G \) such that \( H/R_1(G) \) belongs to \( \Lambda_n \) and has the required property. Clearly \( R_1(G) \leq R_1(H) \), so that the preimages of the terms of the \( RS \)-series of \( H/R_1(G) \) are the terms of the \( RS \)-series of \( H \), contradicting the fact that \( G \) was a counterexample. Hence \( R_1(G) = 1 \).

For every \( i \) for all \( S \) is a minimal normal subgroup of \( RS \). Thus \( S \) is a minimal normal subgroup of \( G \). Since \( S \) is a minimal normal subgroup of \( G \), there is therefore a subgroup \( K \) of \( G \) such that \( K \) is the preimage of the required property. Clearly \( R_1(G) \leq R_1(H) \), so that the preimages of the terms of the \( RS \)-series of \( H/R_1(G) \) are the terms of the \( RS \)-series of \( H \), contradicting the fact that \( G \) was a counterexample. Hence \( R_1(G) = 1 \).

The group \( G/S_1(G) \) belongs to \( \Lambda_{n-1} \), so that there exists \( K = K/S_1(G) \leq G/S_1(G) \) such that \( K \in \Lambda_{n-1} \) and \( S_1(K)/R_i(K) \) is the unique minimal normal subgroup of \( K/R_i(K) \) for all \( i = 1, \ldots, n-1 \). Since \( [R_1(K), S_1(G)] \leq R_1(K) \cap S_1(G) = 1 \), the subgroup \( R_1(K) \) lies in \( C_G(S_1(G)) \) which is trivial by Lemma 2.4; therefore the \( RS \)-series of \( K \) starts with \( S_1(K) \). Of course \( S_1(G) \leq S_1(K) \) and, being \( S_1(K) \) semisimple, if \( S_1(G) \neq S_1(K) \) we have a direct decomposition \( S_1(K) = S_1(G) \times C \) with \( C \neq 1 \). But then \( C \) is contained in \( C_G(S_1(G)) \) = 1, a contradiction. As a consequence we have that \( K \) belongs to \( \Lambda_n \) and, by minimality of \( G, G = K \).

It is then possible to decompose \( S_1(G) \) as \( S_1(G) = T_1 \times \cdots T_r \), where each \( T_i \) is a minimal normal subgroup of \( G \). Since \( G \) is a counterexample, \( r \) is at least 2. For every \( i = 1, 2, \ldots, r \), set \( C_i = C_G(T_i) \). If none of the \( C_i \) is contained in \( R_2(G) \), for \( i = 1, \ldots, r \), then each subgroup \( C_i R_2(G) \) must contain \( S_2(G) \), because \( S_2(G)/R_2(G) \) is the only minimal normal subgroup of \( G/R_2(G) \). Therefore, writing \( S \) for \( S_2(G) \), we have

\[
S = S \cap C_i R_2(G) = (S \cap C_i) R_2(G),
\]

for all \( i = 1, 2, \ldots, r \). Thus

\[
S' = [S, S] = [(S \cap C_1) R_2(G), (S \cap C_2) R_2(G)] \leq R_2(G) (C_1 \cap C_2).
\]

In this way we prove that

\[
\gamma_r(S) \leq R_2(G) (\cap_{i=1}^r C_i).
\]

However \( \cap_{i=1}^r C_i = 1 \), showing that \( S_2(G)/R_2(G) \) is nilpotent of class at most \( r - 1 \). This can happen only if \( S_2(G)/R_2(G) = 1 \) which, in turn, implies \( n = 1 \), a contradiction. We can therefore assume, without loss of generality, that \( C = C_G(T_1) \leq R_2(G) \). If \( T = \prod_{i \neq 1} T_i \), set \( \overline{G} = G/T \). We claim that \( \overline{G} \in \Lambda_n \).

First of all we prove that \( R_1(\overline{G}) = C/T \). Write \( R_1(\overline{G}) = U/T \). Being \( T_1 \) a minimal normal subgroup of \( G \) and \( U/T \) solvable, we have \( T_1 \cap U = 1 \). Therefore \( [T_1, U] = 1 \), thus \( U \leq C \). On the other hand, we know that \( C \leq R_2(G) \), hence \( C S_1(G) \) is solvable and, for some \( d \),

\[
C^{(d)} \leq S_1(G) \cap C = T.
\]

This shows that \( C/T \) is solvable or, in other words, that \( C \leq U \) thus proving that \( R_1(\overline{G}) = C/T \).

In order to prove our claim, we shall show that \( S_1(\overline{G}) = S_1(G) C/T \). What we need is to identify the socle of \( \overline{G}/R_1(\overline{G}) \) and, since this group is isomorphic to \( G/C \), we prefer to work in this quotient of \( G \). Of course \( S_1(G)/C \) is contained in the socle of \( G/C \). The subgroup \( S_1(G) C/C \) is normal so, in particular, it is normal in \( S_1(G)/C \), which is semisimple as \( G/C \approx \overline{G}/R_1(\overline{G}) \). This means that we can write \( S_1(G)/C = S_1(G) C/C \times L/C \), where \( L/C \) is normal and, if non trivial, it
is the direct product of nonabelian simple groups. Taking commutators we get 
\([T_1, L] \leq T_1 \cap C = 1\) thus showing that \(L \leq C\). Hence \(S_1(G/C) = S_1(G)C/C\) and 
from this it follows that

\[
\frac{\overline{G}}{S_1(G)} \cong \frac{\overline{G}/R_1(\overline{G})}{\overline{S_1(G)/R_1(\overline{G})}} \cong \frac{G/C}{S(G/C)} \cong \frac{G/C}{S_1(G)C/C} \cong \frac{G}{S_1(G)C}.
\]

Now, the subgroup \(S_1(G)C\) is contained in \(R_2(G)\) so that \(G/S_1(G)C\) has an image 
iso morphic to \(G/R_2(G) \in \Lambda_{n-1}\). This proves that \(\overline{G}\) belongs to \(\Lambda_n\).

Let \(H\) be a supplement to \(T\) in \(G\) of minimal order. Of course, \(H\) is a proper 
subgroup of \(G\) (by an easy application of Frattini argument for instance). The 
subgroup \(D = H \cap T\) is then nilpotent. For, if \(Q\) is a nontrivial Sylow \(p\)-subgroup 
of \(D\), the Frattini argument gives \(H = DN_H(Q)\), so that \(N_H(Q)\) is a supplement 
to \(T\) in \(G\). By the minimal choice of \(H\), we have \(N_H(Q) = H\) and, since all the 
Sylow subgroups of \(D\) are normal, \(D\) is nilpotent. Clearly \(D \leq R_1(H)\) and 
\[
\frac{H}{D} = \frac{H}{H \cap T} \cong \frac{HT}{T} = \frac{G}{T} = \overline{G}.
\]

Therefore we have 
\[
\frac{H}{R_1(H)} \cong \frac{H/D}{R_1(H)/D} \cong \frac{\overline{G}}{R_1(\overline{G})},
\]
showing that \(H\) is in \(\Lambda_n\) and has the desired property. This contradiction com-
pletes the proof of the Lemma. \(\square\)

In the course of our analysis we have to treat the following situation. A finite 
group \(G\) has a unique minimal normal subgroup \(T\) which is nonabelian, and so a 
direct product of simple groups \(S_i\), for \(i = 1, \ldots, r\) all isomorphic say to \(S\). Since \(G\) 
acts transitively on the set \(\{S_i | i = 1, \ldots, r\}\), for each \(i = 1, \ldots, r\) we fix \(g_i \in G\) in 
such a way that \(S_i = S_1^{g_i}\), and choose \(g_1 = 1\). We also fix an isomorphism \(\alpha\) from 
\(S\) to \(S_1\) and, given any subgroup \(H\) of \(S\), we let 
\[
H^* = H_1 \times H_2 \times \ldots \times H_r
\]
the subgroup of \(T\) such that for every \(i = 1, \ldots, r\), \(H_i = (H^*)^{g_i}\). In particular, 
\(T = S^*\).

**Lemma 4.5.** Let \(G\) be a finite group and, with the above notation, suppose that \(S^*\) 
is the unique minimal normal subgroup of \(G\) which is nonabelian. If \(H\) is a proper 
subgroup of \(S\) that extends from \(S\) to \(\text{Aut}(S)\), then \(H^*\) extends from \(S^*\) to \(G\).

**Proof.** For every \(x \in G\) denote by \(\sigma_x \in \text{Sym}(r)\) the permutation induced by \(x\) 
on the set \(\{S_i | i \leq r\}\), so that, in our notation, for every \(i = 1, \ldots, r\):
\[
S_1^{\sigma_x} = S_i^x = S_{\sigma_x(i)} = S_1^{g_{\sigma_x(i)}}.
\]

Now for every \(i\), the component \(S_i^x\) contains both the subgroups \(H_i^x\) and \(H_{\sigma_x(i)}\), and 
we claim that these subgroups are \(S_i^x\)-conjugate. Note that \(g_i x (g_{\sigma_x(i)})^{-1} \in N_G(S_1)\) 
and since \(H_1\) extends from \(S_1\) to \(\text{Aut}(S_1)\), there exists an element \(s_i \in S_1\) such that 
\[
H_1^{g_1 x (g_{\sigma_x(i)})^{-1}} = H_1^{s_i},
\]
equivalently:
\[
H_i^x = H_1^{s_i, g_{\sigma_x(i)}} = (H_{\sigma_x(i)})^{(g_{\sigma_x(i)})^{-1} s_i g_{\sigma_x(i)}},
\]
which proves our claim since 
\[(g_{\sigma_i(i)})^{-1} s_i g_{\sigma_i(i)} \in S_i \times S_i \times S_i \times \cdots \times S_i \text{ if we set}
\]
\[t = \prod_{i=1}^{r} ((g_{\sigma_i(i)})^{-1} s_i g_{\sigma_i(i)})
\]
then \(t \in S^*\) and we have that
\[(H^*)^x = (H^*)^t,
\]
therefore \(x \in S^* N_G(H^*)\), which proves the Lemma. \(\square\)

**Lemma 4.6.** Let \(G\) be a group in \(\Lambda_n\). Then \(G\) contains a subgroup \(M\) belonging to \(\Lambda_n\), such that the components of \(S_i(M)/R_i(M)\) are in \(\mathcal{L}\).

**Proof.** Let \(G\) be a minimal counterexample, which, by Lemma 4.4, we assume having \(S_i(G)/R_i(G)\) as unique minimal normal subgroup of \(G/R_i(G)\), for all \(i = 1, \ldots, n\). In particular, write
\[S_1(G) = S^* = S_1 \times S_2 \times \cdots \times S_r,
\]
where each \(S_i \simeq S\) is simple and \(r \geq 1\).

Note that \(S \not\in \mathcal{L}\), otherwise we may take \(M = G\) and \(G\) is no more a counterexample. By Lemma 4.2 we choose a proper self-normalizing subgroup \(H\) of \(S\) which belongs to \(\Lambda_1\) and such that it extends from \(S\) to \(\text{Aut}(S)\). By Lemma 4.3 the \(\Lambda_1\)-subgroup \(H^*\) of \(S^*\) may be extended to an \(\mathcal{G}\), i.e. \(G = S_1(G)M\), where \(M = N_G(H^*)\) is a proper subgroup of \(G\). We show that \(M \in \Lambda_n\), then the contradiction will follow by the minimal choice of \(G\).

We set \(A = R_1(H^*) = (R_1(H))^*, B = S_1(H^*) = (S_1(H))^*\) and \(C = C_M(B/A)\) and we proceed by steps.

1) \(A = H^* \cap R_1(M)\) and \(B = H^* \cap S_1(M).
\)

Since \(H^* \not\subset M\), by Lemma 4.4 we have that \(A \leq H^* \cap R_1(M)\) and \(B \leq H^* \cap S_1(M)\).

Conversely, both \(H^* \cap R_1(M)\) and \(H^* \cap S_1(M)\) are normal in \(H^*\), thus by Lemma 2.4 again, we obtain
\[H^* \cap R_1(M) = R_1(H^* \cap R_1(M)) \leq R_1(H^*) = A
\]
and
\[H^* \cap S_1(M) = S_1(H^* \cap S_1(M)) \leq S_1(H^*) = B.
\]

2) \(C\) is solvable.

Note that \(C\) normalizes every component \(S_i\) of \(G\). This is trivial when \(r = 1\), while if \(r > 1\) let \(x \in C\) and assume that \(x\) moves \(S_i\) to \(S_j\) (with \(j \neq i\)), then \(x\) moves \(H_iA/A\) onto \(H_jA/A\) and then \(B/A\) is no more centralized by \(x\), which is a contradiction.

Hence every element of \(C = C_M(B/A)\) induces an automorphism of each \(S_i\) and in particular the group \(CS^*/S^* \simeq C/C \cap S^*\) is solvable (being a subdirect product of outer automorphism group of a simple group). Also, \(C \cap S^* \leq N_S(H^*) = H^*,\) thus \((C \cap S^*)A/A \leq C_{H^*}(B/A),\) which is trivial by Lemma 2.4(4). Thus \(C \cap S^* \leq A\) and \(C\) is solvable.

3) \(BR_1(M) = S_1(M)\).

As \(H^* \not\subset M,\) by Lemma 2.4 we have that \(BR_1(M) \leq S_1(M)\). Assume by contradiction that the inclusion is proper and choose a direct complement \(D/R_1(M)\) of \(BR_1(M)/R_1(M)\) in \(S_1(M)/R_1(M)\). Then
\[[B, D] \leq R_1(M) \cap B \leq R_1(M) \cap H^* = A,
\]
by step 1). This means that $D \leq C$ and so by step 2), $D$ is solvable, which is a contradiction.

4) $M \in \Lambda_n$.

Note first that being $H$ self-normalizing in $S$, the subgroup $H^*$ is self-normalizing in $S^*$. Hence

$$\frac{M}{H^*} = \frac{N_G(H^*)}{N_{S^*}(H^*)} = \frac{N_G(H^*)}{S^* \cap N_G(H^*)} \cong \frac{G}{S^*}$$

and therefore it belongs to $\Lambda_{n-1}$. Moreover, as $H^* = R_2(H^*)$ and $H^* \leq M$, we have $H^* \leq R_2(M)$. Also by step 3) that $H^*S_1(M) = H^*R_1(M) \leq R_2(M)$ and hence $H^*S_1(M)/H^*$, being solvable and normal in $M/H^*$, lies in the solvable radical of $M/H^*$, which is $R_2(M)/H^*$. This with $M/H^* \in \Lambda_{n-1}$ implies that $M/R_2(M) \in \Lambda_{n-1}$ and therefore $M \in \Lambda_n$. □

We are now in a position to prove our main result.

Proof of Theorem 1.1. Let $G \in \Lambda_n$ be a counterexample such that $|G| + n$ is minimal. By Lemma 4.6 we may assume that $S_1(G)/R_1(G)$ is the unique minimal normal subgroup of $G/R_1(G)$, for every $i = 1, 2, \ldots, n$.

Assume that $R_1(G) \neq 1$. Then by the minimal choice, the group $\overline{G} = G/R_1(G)$ contains $n$-rarefied subgroup $\overline{K} = K/R_1(G)$. Note that being $R_1(G) \leq R_1(K)$ we have that $R_i(G) \leq R_i(K)$ and $S_i(G) \leq S_i(K)$ for each $i = 1, \ldots, n$ and therefore

$$\frac{R_{i+1}(K)}{S_i(K)} \simeq \frac{R_{i+1}(K)}{S_i(K)}$$

and

$$\frac{S_i(K)}{R_i(K)} \simeq \frac{S_i(K)}{R_i(K)}$$

In particular, if $R_1(K) \leq \Phi(K)$, then $K$ is an $n$-rarefied subgroup of $G$, which is a contradiction. Hence we have that $R_1(K) \not\leq \Phi(K)$. Let $M$ be a maximal subgroup of $K$ such that $K = R_1(K)M$. Since

$$\frac{R_1(K)}{R_1(G)} = R_1(K) = \Phi(K),$$

we have that $R_1(G) \not\leq M$ and therefore $K = R_1(G)M$. Call $H$ a minimal supplement of $R_1(G)$ in $K$. Note that $R_1(H) \not\leq \Phi(H)$, otherwise taken $L$ a maximal subgroup of $H$ that supplements $R_1(H)$ in $H$, since $R_1(H)/H \cap R_1(G) = \Phi(H/H \cap R_1(G))$, $L$ does not contain $H \cap R_1(G)$, forcing $H = (H \cap R_1(G))L$ and therefore $K = R_1(G)L$, which contradicts the minimal choice of $H$. Therefore $R_1(H) \not\leq \Phi(H)$ and, since $H \cap R_1(G) \leq R_1(H)$ and $H/H \cap R_1(G) \simeq \overline{K}$, it easily follows that $H$ is an $n$-rarefied subgroup of $G$, which is a contradiction.

Therefore $R_1(G) = 1$. If $n = 1$, our minimal choice implies that $S_1(G) = G$ is a simple group not belonging to $\mathcal{L}$. Take $H$ a proper subgroup $G$ satisfying the properties of Lemma 4.2. As $H \in \Lambda_1$ and $|H| < |G|$, then $H$ (and therefore $G$ too) contains an 1-rarefied subgroup, which is a contradiction. Therefore $n \geq 1$. By inductive assumption, $G/S_1(G)$ contains a $(n-1)$-rarefied subgroup $H/S_1(G)$. Of course, $S_1(G) \leq S_1(H)$. If $S_1(G) < S_1(H)$ let $B/R_1(H)$ be a complement of $S_1(G)R_1(H)/R_1(H)$ in $S_1(H)/R_1(H)$. Then

$$[S_1(G), B] \leq S_1(G) \cap R_1(H) \leq R_1(G) = 1,$$

which implies that $C_G(S_1(G)) \neq 1$ and this contradicts Lemma 4.2 (4). It follows that $S_1(G) = S_1(H)$ and, since $H/S_1(H) \in \Lambda_{n-1}$, $H$ belongs to $\Lambda_n$. Again the minimality of $G$ implies that $G = H$. Remember that $S_1(G)$ is the unique minimal normal subgroup of $G$, therefore if its components lie in $\mathcal{L}$, then $G$ is $n$-rarefied, which is a contradiction. We may therefore apply Lemma 4.6 to find a subgroup $M$
of $G$ having nonsolvable length $n$ and such that the components of $S_1(M)/R_1(M)$ are in $\mathcal{L}$. This last condition implies that $M < G$. By induction, $M$, and therefore $G$, contains an $n$-rarefied subgroup, which is the last contradiction.

The next result is useful when dealing with permutation or linear groups.

**Proposition 4.7.** Let $G$ be a group in $\Lambda_n$ and $H$ an $n$-rarefied subgroup of $G$. Then

1. if $G$ acts faithfully on the set $\Omega$, there is an $H$-orbit $\Delta$, such that $H/C_H(\Delta)$ is in $\Lambda_n$;
2. if $G$ acts faithfully of the finite dimensional $\mathbb{F}$-vector space $V$, there exist $H$-invariant irreducible section $W$ of $V$, such that $H/C_H(W)$ is in $\Lambda_n$.

**Proof.** Let $\{N_i \mid i = 1, \ldots, l\}$ be the set of kernels of the action of $H$ on its orbits, we have $\bigcap_{i=1}^l N_i = 1$. By Lemma 2.5 we must have $H/N_i \in \Lambda_n$ for at least one index $i \in \{1, \ldots, l\}$, proving the claim. When $G \leq \text{GL}(V)$, the same argument works, if the $N_i$ are the kernels of the action of $H$ on the factors of an $H$-composition series in $V$. □

5. SOME APPLICATIONS

In this section we will apply the result obtained to the study of some questions concerning the nonsolvable length.

Our first aim is to improve the bound obtained in [11, Theorem 1.1.(a)], where the authors showed that $\lambda(G) \leq 2L_2(G) + 1$, being

$$L_2(G) = \max\{l_2(H) \mid H \text{ is a solvable subgroup of } G\}$$

and $l_2(H)$ the minimal number of 2-factors in a $2^l$-series of the solvable group $H$.

**Lemma 5.1.** Let $G$ be an $n$-rarefied group with trivial Frattini subgroup. Then $G$ contains a solvable subgroup $H$ such that $l_2(H/O_2(H)) \geq n$.

**Proof.** We prove the Lemma by induction on $n$.

If $n = 1$ then $G$ is a simple group in $\mathcal{L}$ and an inspection of the subgroups of $G$ reveals that the claim holds (see e.g. [9, Section 6.5]).

Assume the claim true for $n - 1$ and choose $G \in \Lambda_n$ with $\Phi(G) = 1$. Write

$$S_1(G) = S^* = \prod_i S_i$$

each $S_i \simeq S$ a simple group in $\mathcal{L}$. The group $G$ permutes the components and $G = S^* \cdot X$, where $X$ acts on the sets of components and, according to Lemma 3.3 it is chosen such that $X \leq N_G(D^*)$, where $D^*$ is a 2-nilpotent subgroup of $S^*$ with $O_2(D^*) = 1$ and $N_{S^*}(D^*) = D^*$ (we used Lemma 4.5).

Note that $X/X \cap S^*$ is a $(n-1)$-rarefied group, therefore if we set $F/X \cap S^* = R_1(X/X \cap S^*) = \Phi(X/X \cap S^*)$, by inductive hypothesis we have that there exists a solvable subgroup $L/F \leq X/F$ such that, if $A/F = O_2(L/F)$,

$$l_2\left(\frac{L/F}{A/F}\right) = l_2(L/A) \geq n - 1.$$

Let also $C$ be the kernel of the permutation action of $X$ on $\{S_i\}$, $C = \bigcap_i N_X(S_i)$. Then $C/C \cap S^*$ embeds, as a subdirect product, into $\prod_i \text{Out}(S_i)$, which is solvable.
Therefore $C \leq F$. We set $H = D^*L$ and we claim that $l_2(H/O_2(H)) \geq n$. Call $B/O_2(H) = O_{2'}(H/O_2(H))$. Then

\[ l_2(H/O_2(H)) \geq l_2(H/B) + 1. \]

Note that

\[ \frac{H}{D^*A} = \frac{(D^*A)L}{D^*A} \simeq \frac{L}{D^*A \cap L} = \frac{L}{A(D^* \cap L)} = \frac{L}{A}. \]

since $D^* \cap L \leq S^* \cap X \leq F \leq A$. It follows that if we show

\[ B \leq D^*A \]

then

\[ l_2(H/B) \geq l_2(H/D^*A) = l_2(L/F) \geq n - 1 \]

and so, by \[\text{II}\],

\[ l_2(H/O_2(H)) \geq n. \]

Now, $O_2(H)$ centralizes every $U_i \leq U^*$, thus $O_2(H) \leq D^*C \leq D^*A$. Let $R = O_{2'}(H) = O_2(H) \rtimes T$, where $T$ exists by Shur-Zassenhaus and has odd order, so that $B = O_{2'}(H) = RP$ for any Sylow 2-subgroup $P$ of $B$. Let $t$ be any element of $T$. If $S_i^t = S_j$ for $j \neq i$, then chosen a 2-element $v_i \in D_i$, we have that

\[ [t, v_i] = (v_i^{-1})^t v_i \]

is a nontrivial 2-element of $S^* \cap R$, thus a nontrivial 2-element of $S^* \cap O_2(H)$. But $S^* \cap O_2(H) \leq O_2(D^*) = 1$. We have therefore that $T$ normalizes every component, and thus $T \leq D^*C \leq D^*A$, forcing $R$ itself to be contained in $D^*A$.

Finally note that $B = RP \leq (D^*A)P$ and so $(D^*A)P = (D^*A)B$ is a normal subgroup of $H$ such that

\[ \frac{(D^*A)B}{D^*A} \simeq \frac{P}{P \cap D^*A} \]

is a 2-group. Therefore

\[ \frac{(D^*A)B}{D^*A} \leq O_2 \left( \frac{H}{D^*A} \right) = O_2 \left( \frac{L}{A} \right) = 1, \]

proving that $B \leq D^*A$. \hfill \qed

The improvement of the bound obtained in \[\text{II}\] Theorem 1.1] follows easily.

**Theorem 5.2.** Let $G$ be any finite group. Then $\lambda(G) \leq L_2(G)$.

**Proof.** The group $G$ contains an $n$-rarefied subgroup $G_0$. By Lemma \[\text{II}\] $G_0/\Phi(G_0)$ contains a solvable subgroup $H/\Phi(G_0)$ with $l_2(H/\Phi(G_0)) \geq n$ and $O_2(H/\Phi(G_0)) = 1$. As $\Phi(G_0)$ is nilpotent, $l_2(H) \geq l_2(H/\Phi(G_0)) \geq n$. Therefore $n \leq L_2(G_0)$ and, since we clearly have $L_2(G_0) \leq L_2(G)$, we obtain $n \leq L_2(G)$, as claimed. \hfill \qed

Our next application of Theorem \[\text{II}\] is related to a general problem studied in \[\text{II0}\]. In that paper the authors address the following question.

Let $\mathcal{P}$ be a group theoretical property, and $G$ a finite group possessing $\mathcal{P}$. What is the minimal number $d$ such that $G$ has a $d$-generated subgroup possessing $\mathcal{P}$?
The following result answers the aforementioned question for the property $\mathcal{P}$ consisting of having nonsolvable length $n$. At the same time it is an improvement of [7, Theorem 1.1].

**Theorem 5.3.** Let $G$ be any finite group. If $G$ is not solvable then there exists a 2-generator subgroup $H$ of $G$ such that $\lambda(H) = \lambda(G)$.

*Proof.* Let $n \geq 1$ and $H$ an $n$-rarefied group. We prove that $H$ is 2-generated. It is known, as a byproduct of the classification of finite simple groups, that every simple group can be generated by two elements. So, if $H$ is 1-rarefied, then $H/\Phi(H)$ is 2-generated. Clearly $H$ is 2-generated as well. Assume our claim true for groups of nonsolvable length at most $n-1$ and choose $H$ any $n$-rarefied group. It is harmless to assume that $\Phi(H)$ is trivial, since any set of elements generating $H$ modulo $\Phi(H)$ also generates $H$. Thus $S_1(H)$ is the unique minimal normal subgroup of $H$, and we can use the main result of [19] to see that the minimal number of generators of $H/S_1(H)$ is the same as the minimal number of generators of $H$. The inductive hypothesis yields that $H$ is 2-generated and Theorem 1.1 completes the proof. □

We consider now a problem of different nature.

For any natural number $m$ and any field $\mathbb{F}$, define

$$
\lambda(m) = \max \{ \lambda(G) \mid G \leq \text{Sym}(m) \}
$$

$$
\lambda_\mathbb{F}(m) = \max \{ \lambda(G) \mid G \leq \text{GL}(m, \mathbb{F}) \}
$$

We shall use Theorem 1.1 to find $\lambda(m)$ and $\lambda_\mathbb{F}(m)$ for all $m$ and $\mathbb{F}$. We start by establishing lower bounds for the degree of permutation and linear representations of $n$-rarefied groups.

**Lemma 5.4.** Let $n, k$ be natural numbers with $1 \leq k < n$. Then $(n-k)5^k \geq n$.

*Proof.* It is easy to check that the function $f(x) = (n-x)5^x$ is strictly increasing in the interval $[1, n-1/log(5)]$ so that, when restricted to $[1, n-1] \cap \mathbb{N}$, it attains its minimum at $x = 1$. Since $f(1) = 5(n-1) > n$, the Lemma is proved. □

**Lemma 5.5.** Let $G$ be any finite group of nonsolvable length $n$ acting faithfully on the set $\Omega$. Then $|\Omega| \geq 5^n$.

*Proof.* Of course we may assume that $G$ acts transitively on $\Omega$. Moreover by Theorem 1.1 and Proposition 4.7, $G$ can be considered to be $n$-rarefied.

We prove the claim by induction on $n$.

Assume $n = 1$. If $\Phi(G) = 1$ then $G$ is a simple group in $\mathcal{L}$. Being the minimal degree of a faithful representation 5, the claim holds. If $\Phi(G) \neq 1$ then $G$ can not be primitive because, in this case, $\Phi(G)$ would be transitive and then, for every $\omega \in \Omega$, we would have $G = \Phi(G)G_\omega$, which is clearly impossible. Thus the orbits of $\Phi(G)$ form a system of non-trivial blocks for $G$. Also, $G/\Phi(G)$ acts faithfully and transitively on the set of $\Phi(G)$-orbits and we may therefore conclude as above.

Assume $n > 1$ and suppose the claim is true for $\Lambda_k$-groups with $1 \leq k < n$. Consider first the case $G$ primitive. As we have pointed out before, in this situation the Frattini subgroup must be trivial. Thus $S_1(G)$ is the socle of $G$ and its components, all isomorphic to a fixed simple group $S \in \mathcal{L}$, are permuted by $G/S_1(G)$. 
If $K/S_1(G)$ is the kernel of this action, then $K/S_1(G)$ is solvable and, therefore $K \leq R_2(G)$ and $G/K$ is a $(n - 1)$-rarefied group. The inductive hypothesis shows that $S_1(G)$ has at least $5^{n-1}$ components. Clearly $S_1(G)$ acts transitively on $\Omega$. If $S_1(G)$ acts regularly, then

$$|\Omega| = |S_1(G)| \geq |\text{Alt}(5)|^{5^{n-1}} \geq 5^{n}.$$ 

When the socle is not regular we can use [8, Theorem 4.6A]. The only possibilities, in our situations, are the following:

1. $S_1(G)$ acts in diagonal action, so that $|\Omega| = |S_1(G)|/|S|$, or
2. $G$ is a transitive subgroup of $W = U \wr_{\Gamma} \text{Sym}(\Gamma)$ where $U$ is primitive nonregular, $\Gamma$ has at least two elements, and $W$ acts in product action.

If we are in case 1, then the claim holds, because $|S_1(G)|/|S|$ is at least $60^{5^{n-2}}$ which is bigger than $5^n$.

Assume then we are in case 2, so that the set $\Omega$ can be identified with $\Delta^\Gamma$, where $\Delta$ is a primitive $U$-set. If $B$ indicates the base subgroup of $W$, the group $G/G \cap B$ is isomorphic to a transitive subgroup of $\text{Sym}(\Gamma)$. Moreover by Proposition 4.2 $G/G \cap B$ is a $k$-rarefied group for some $1 \leq k < n$. Fix $j \in \Gamma$ and let $N_j$ be the kernel of the projection from $M = G \cap B$ onto the $j$-th component of $B$. The group $M$ embeds, as a subdirect product, into $\prod_{i \in \Gamma} M_i$ where all factors $M_i = M/N_i$ are isomorphic, since $G$ acts transitively on $\Gamma$. If $M_j$ belongs to $\Lambda_k$, then $M$ itself is in $\Lambda_k$, because of Lemma 2.3. From Lemma 4.3 it follows that $n = k + l$. Therefore, using Lemma 4.7 and the induction, we have that $|\Delta| \geq 5^l$. Thus

$$|\Omega| = |\Delta|^{|\Gamma|} \geq (5^{n-k})^{5^k} \geq 5^n$$

the last inequality holds by Lemma 5.3.

It remains to handle the case when $G$ is imprimitive. Let $\Sigma$ be the system of imprimitivity consisting of the $S_1(G)$-orbits and let $N$ be its stabilizer. Then $G/N$ acts transitively on $\Sigma$ and, if $G/N$ belongs to $\Lambda_k$, it is a $k$-rarefied group. Notice that, since $S_1(G) \leq N$, $k < n$. The inductive assumption gives $|\Sigma| \geq 5^k$. Using an argument similar to the one of the above paragraph, it is readily seen that each block in $\Sigma$ has size at least $5^{n-k}$. The claim follows easily.

**Theorem 5.6.** For every $m \geq 5$ we have $\lambda(m) = \lfloor \log_5(m) \rfloor$.

**Proof.** Let $n$ be such that $5^n \leq m < 5^{n+1}$, so that $n = \lfloor \log_5(m) \rfloor$. The symmetric group $\text{Sym}(m)$ contains a subgroup $G$ isomorphic to the $n$-fold wreath product of $\text{Alt}(5)$ in its natural action, acting on a set of $5^n$ points. Since $G$ is in $\Lambda_n$ (actually it is an $n$-rarefied group), it follows that $\lambda(m) \geq n = \lfloor \log_5(m) \rfloor$.

Now let $G$ be a subgroup of $\text{Sym}(m)$, whose nonsolvable length is $\lambda(m)$. By Theorem 1.1 there exists $H \leq G$ which is a $\lambda(m)$-rarefied group and, using Lemma 4.7 and Lemma 5.5, we infer that $m \geq 5^\lambda$. From this we get $\lambda(m) \leq \log_5(m)$. Therefore

$$\lfloor \log_5(m) \rfloor \leq \lambda(m) \leq \log_5(m)$$

and the equality $\lambda(m) = \lfloor \log_5(m) \rfloor$ follows.

**Lemma 5.7.** Let $G$ be a group with $G/\zeta(G) = \prod_{i=1}^l G_i$ where each $G_i$ is a finite simple nonabelian group, and $\mathbb{K}$ be any algebraically closed field. If the $\mathbb{K}$-vector space $V$ affords an irreducible projective representation $\rho$ with $\ker(\rho) = \zeta(G)$, then $\dim_{\mathbb{K}}(V) \geq 2^l$. 


Proof. The claim is true when \( l = 1 \), so assume \( l \geq 2 \) and that the result holds when the number of factors is smaller than \( l \). Let \( H \leq \text{GL}(V) \) be such that \( H = G^l \), where \( Z \) is the center of \( \text{GL}(V) \). It is then possible to apply Lemma 11.20 to \( H \).

Let \( K \) be such that \( K/Z = G_1 \), the space \( V \) can be decomposed as \( U \otimes W \) and the action of \( H \) is the tensor product of two projective representations \( \sigma : K \to \text{GL}(U) \) and \( \tau : H/K \to \text{GL}(W) \). It is easy to see that \( \tau \) has trivial kernel, because \( H/K \) is the direct product of simple groups and the kernel of \( \rho \) is \( \zeta(G) \). Since the number of simple factors of \( H/K \) is \( l - 1 \), induction yields \( \dim(W) \geq 2^{l-1} \), while \( U \) has dimension at least 2. Thus \( \dim(V) = \dim(U) \cdot \dim(W) \geq 2 \cdot 2^{l-1} = 2^l \), as claimed.

Lemma 5.8. Let \( G \) be any \( \Lambda_n \)-group acting faithfully and irreducibly on the \( F \)-vector space \( V \), where \( F \) is any field. Then \( \dim(V) \geq 2 \cdot 5^{n-1} \).

Proof. If \( K \) is any algebraically closed field containing \( F \), the group \( G \) acts on \( W = V \otimes_F K \). By Proposition 4.4 we know that \( G \) has a \( \Lambda_n \)-subgroup acting faithfully on a \( G \)-composition factor of \( W \). A lower bound on the \( K \)-dimension of such factor would entail a lower bound for the \( F \)-dimension of \( V \). There is therefore no loss of generality, if we assume that \( F \) is algebraically closed.

If \( n = 1 \) the result is clear since \( G \) can not have representations of degree 1.

Assume the statement true up to \( n - 1 \). It is harmless to assume that \( G \) is \( n \)-rarefied, because of Theorem 1.1 and Proposition 4.7.

When \( G \) is imprimitive, let \( B = \{ W_1, \ldots, W_l \} \) be a system of imprimitivity, and \( N \) the normalizer of \( B \) in \( G \). If \( k = \lambda(N) \) we have, by Proposition 4.2 and Lemma 4.3, that \( G/N \) is an \((n-k)\)-rarefied group. If \( k = 0 \) then \( G/N \) is in \( \Lambda_n \) and therefore \( B \) must contain at least \( 5^n \) elements, by Lemma 5.5. In this case \( V \) has dimension at least \( 5^n > 2 \cdot 5^{n-1} \). Assume then that \( k \) is at least 1. The group \( N \) is a subdirect product of the groups \( N/C_N(W_i) \) (which are all isomorphic) hence, by Lemma 2.6, each \( N/C_N(W_i) \) is in \( \Lambda_k \). The inductive hypothesis yields \( \dim(W_i) \geq 2 \cdot 5^{k-1} \) and Lemma 5.5 tells us that \( B \) contains at least \( 5^{n-k} \) elements. These two facts give

\[
\dim(V) \geq 2 \cdot 5^{n-k-1} \cdot 5^k = 2 \cdot 5^{n-1}.
\]

It remains to consider the case \( G \) primitive.

Assume first that \( \Phi(G) \) is abelian. When this happens \( \Phi(G) \) is the center of \( G \) and \( S_1(G)/\Phi(G) \cong \prod_{i=1}^l S_i \), where each \( S_i \) is isomorphic to a simple group \( S \) in \( \mathcal{L} \). Moreover the module \( V \) is the direct sum of copies of a single irreducible \( S_1(G) \)-module. It is therefore enough to prove that the claimed bound holds, when \( V \) is already irreducible as a module for \( S_1(G) \). By Lemma 5.7 we have that \( \dim(V) \geq 2^l \). On the other hand, we can use Lemma 5.5 applied to the group \( G/S_1(G) \) in its action on the components of \( S_1(G)/\Phi(G) \), to see that \( l \geq 5^{n-1} \).

Thus \( \dim(V) \geq 2 \cdot 5^{n-1} \geq 2 \cdot 5^{n-1} \).

It remains to treat the case \( \Phi(G) \) not abelian. In this case \( \zeta(\Phi(G)) \) and \( \zeta(G) \) have a subgroup \( A \) such that \( \zeta(\Phi(G)) \leq A \leq \zeta(G) \) and \( |A/\zeta(\Phi(G))| = r^2 \) for some \( r \) dividing \( \dim(V) \), see [23, Theorem 3.3]. The group

\[
B = A/\zeta(G) \cap \zeta(\Phi(G))/\zeta(G)
\]

is not trivial, so there is at least one prime \( p \) such that \( \overline{P} = P/\zeta(G) \), the \( p \)-Sylow of \( B \), is not 1. The group \( \overline{P} \) is elementary abelian (see [23, Theorem 3.3]). Since \([P, C_G(P)], C_G(P) = 1 \), the group \( C_G(\overline{P}) \) is nilpotent and therefore contained in \( \Phi(G) \). On the other hand \( \Phi(G) \leq C_G(\overline{P}) \), hence the group \( \overline{G} = G/C_G(\overline{P}) = \overline{G}/\zeta(G) \leq \Phi(G) \).
$G/\Phi(G)$ is an $n$-rarefied group contained in $\text{GL}(\mathcal{P})$. Since $G$ has trivial Frattini group, we deduce, using the first part of this proof (the imprimitive case and the primitive case for groups with abelian Frattini group), that the dimension of $\mathcal{P}$ over the field with $p$ elements is at least $2 \cdot 5^{n-1}$. Therefore

$$(\dim(V))^2 \geq |A| \geq p^{2 \cdot 5^{n-1}} \geq 2^{2 \cdot 5^{n-1}}.$$ 

It follows that $\dim(V) \geq 2^{5^{n-1}} \geq 2 \cdot 5^{n-1}$. The proof is then concluded. \hfill \qed

We can now prove the analogous of Theorem 5.9 for linear groups.

**Theorem 5.9.** For every $m \geq 2$ and every field $\mathbb{F}$ with at least four elements, we have $\lambda_\mathbb{F}(m) = 1 + \lfloor \log_5(m/2) \rfloor$. When $|\mathbb{F}| \leq 3$, then $1 + \lfloor \log_5(m/3) \rfloor \leq \lambda_\mathbb{F}(m) \leq 1 + \lfloor \log_5(m/2) \rfloor$.

**Proof.** If $G \leq \text{GL}(m, \mathbb{F})$ has $\lambda(G) = \lambda_\mathbb{F}(m)$, then $m \geq 2 \cdot 5^{\lambda_\mathbb{F}(m)-1}$, by Lemma 5.8. From this inequality we get $\lambda_\mathbb{F}(m) \leq 1 + \lfloor \log_5(m/2) \rfloor$.

Conversely, if $\mathbb{F}$ has order at least 4 let $n = \lfloor \log_5(m/2) \rfloor$ and consider $W$ an $n$-fold wreath product of copies of $\text{Alt}(5)$ in natural action, acting on a set $\Omega$ of cardinality $5^n$. The group $G = \text{GL}(2, \mathbb{F}) \wr_\Omega W$ can be embedded into $\text{GL}(m, \mathbb{F})$. Since $\mathbb{F}$ has at least four elements, $\lambda(\text{GL}(2, \mathbb{F})) = 1$ and therefore $\lambda(G) = n + 1$. Hence $\lambda_\mathbb{F}(m) \geq 1 + \lfloor \log_5(m/2) \rfloor$.

If $|\mathbb{F}| \leq 3$ we define $n = \lfloor \log_5(m/3) \rfloor$ and let $G$ be $\text{GL}(3, \mathbb{F}) \wr_\Delta X$, where $X$ is the $n$-fold wreath product of alternating groups of degree 5, acting naturally on the set $\Delta$ or order $5^n$. Clearly $\lambda(G) = 1 + \lfloor \log_5(m/3) \rfloor$. Hence in this case we have

$$1 + \lfloor \log_5(m/2) \rfloor \leq \lambda_\mathbb{F}(m) \leq 1 + \lfloor \log_5(m/2) \rfloor$$

which completes the proof of the Theorem. \hfill \qed

We remark that $k = \lfloor \log_5(m/2) \rfloor > \lfloor \log_5(m/3) \rfloor$ if and only if $2 \cdot 5^k \leq m < 3 \cdot 5^k$.

We finish this section establishing a lower bound for the exponent of a group in $\Lambda_n$.

**Proposition 5.10.** Let $G$ be a group in $\Lambda_n$. Then $G$ contains elements of order $2^n$ and therefore $\exp(G) \geq 2^n$.

**Proof.** Since the claim clearly holds for $n = 0, 1$, we argue by induction on $n$, assuming the result true for groups in $\Lambda_{n-1}$. Let $G$ be any group in $\Lambda_n$. In order to prove the claim it is enough to find an element of order $2^n$ in a suitable section of $G$. Therefore we can assume that $G$ is $n$-rarefied and that $\Phi(G) = 1$. Let $gR_2(G)$ be an element of order $2^{n-1}$ in $G/R_2(G) \in \Lambda_{n-1}$. If the 2-part of $|g|$ is greater that $2^{n-1}$ there is nothing to prove, therefore we may assume without loss of generality that $g$ can be chosen of order exactly $2^{n-1}$, so that $\langle g \rangle \cap R_2(G) = 1$. Consider the action of $\langle g \rangle$ on the components of $S_1(G) = \prod_{i=1}^n S_i$. Note that there is of course at least one orbit of length $2^{n-1}$, otherwise the nontrivial subgroup $\langle g^{2^{n-2}} \rangle$ will lie in $\bigcap_{i=1}^n N_G(S_i) \leq R_2(G)$, a contradiction. Assume that $\Delta = \{ S_i^x | x \in \langle g \rangle \}$ is an orbit of length $2^{n-1}$. Then if $a$ is any element of $S_1(G)$ whose only nontrivial entry is in position $i$, we have

$$(ag^{-1})^m = aa^g a^{g^2} \ldots a^{g^{m-1}} g^{-m}.$$ 

In particular for $m = 2^{n-1}$ this is an element of the same order of $a$. Choosing $a$ of order 2 we get the claim. \hfill \qed
References

[1] J. N. Bray, D. Holt, C. M. Roney-Dougal, The maximal subgroups of the low-dimensional finite classical groups, London Mathematical Society Lecture Note Series, 407. Cambridge University Press, Cambridge, 2013.

[2] R. W. Carter, Simple Groups of Lie Type, Pure and Applied Mathematics, vol. 28, John Wiley & Sons, London-New York-Sydney, 1972.

[3] Y. Contreras-Rojas, P. Shumyatsky, Pavel Nonsoluble length of finite groups with commutators of small order, Math. Proc. Cambridge Philos. Soc. 158 (2015), no. 3, 487–492.

[4] Y. Contreras-Rojas, P. Shumyatsky, Nonsoluble length of finite groups with restrictions on Sylow subgroups, Comm. Algebra 45 (2017), no. 8, 3606–3609.

[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups, Oxford University Press, Eynsham, 1985.

[6] C. W. Curtis, I. Reiner Methods of representation theory Vol. I. With applications to finite groups and orders, Pure and Applied Mathematics, Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1981.

[7] E. Detomi, P. Shumyatsky, On the length of a finite group and of its 2-generator subgroups, Bull. Braz. Math. Soc. (N.S.) 47 (2016), no. 3, 845–852.

[8] J. D. Dixon, B. Mortimer, Permutation groups, GTM 163, Springer-Verlag, New York - Berlin - Heidelberg 1996.

[9] D. Gorenstein, R. Lyons, R. Solomon, The classification of the finite simple groups. Number 3. Part I. Chapter A. Almost simple $K$-groups, Mathematical Surveys and Monographs, 40.3, American Mathematical Society, Providence, RI, 1998.

[10] P. E. Holmes, R. A. Wilson, A new maximal subgroup of the Monster, J. Algebra 251 (2002), no. 1, 435–447.

[11] E. I. Khukhro, P. Shumyatsky, Nonsoluble and non-$p$-soluble length of finite groups, Israel J. Math. 207 (2015), no. 2, 507–525.

[12] E. I. Khukhro, P. Shumyatsky, On the length of finite factorized groups, Ann. Mat. Pura Appl. (4) 194 (2015), no. 6, 1775–1780.

[13] E. I. Khukhro, P. Shumyatsky, Words and pronilpotent subgroups in profinite groups, J. Aust. Math. Soc. 97 (2014), no. 3, 343–364.

[14] P. B. Kleidman, The maximal subgroups of the finite 8-dimensional orthogonal groups $PQ_4(q)$ and of their automorphism groups, J. Algebra 110 (1987), no. 1, 173–242.

[15] P. B. Kleidman, The maximal subgroups of the Chevalley groups $G_2(q)$ with $q$ odd, the Ree groups $2G_2(q)$, and their automorphism groups, J. Algebra 117 (1988), no. 1, 30–71.

[16] P. B. Kleidman, The maximal subgroups of the Steinberg triality groups $3D_4(q)$ and of their automorphism groups, J. Algebra 115 (1988), no. 1, 182–199.

[17] P. B. Kleidman, M. Liebeck, The subgroup structure of the finite classical groups, London Mathematical Society Lecture Note Series, 129. Cambridge University Press, Cambridge, 1990.

[18] M. W. Liebeck, J. Saxl, G. M. Seitz, Subgroups of maximal rank in finite exceptional groups of Lie type, Proc. London Math. Soc. (3) 65 (1992), no. 2, 297–325.

[19] A. Lucchini, F. Menegazzo, Generators for finite groups with a unique minimal normal subgroup, Forum Math. 24 (2012) no. 4, 875–887.

[20] A. Lucchini, M. Moret,P. Shumyatsky, Boundedly generated subgroups of finite groups, Rend. Semin. Mat. Univ. Padova 98 (1997), 173–191.

[21] M. Suzuki, On a class of doubly transitive groups, Ann. of Math. 75 (1962), 105–145.

[22] J. G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable, Bull. Amer. Math. Soc. 74 (1968), 383–437.

[23] B. A. F. Wehrfritz, Infinite linear groups, Ergebnisse der Matematik und ihrer Grenzgebiete, Band 76. Springer-Verlag, New York-Heidelberg, 1973.
