On Asymptotically Optimal Solvability of Euclidean Max $m$-$k$-Cycles Cover Problem

Edward Gimadi$^{1,2}$ and Ivan Rykov$^{1,2}$

$^1$ Sobolev Institute of Mathematics, Koptuga 4, Novosibirsk, Russia
$^2$ Department of Mechanics and Mathematics, Novosibirsk State University, 1 Pirogova Street, 630090 Novosibirsk, Russia

gimadi@math.nsc.ru, rykovweb@gmail.com
http://math.nsc.ru

Abstract. We consider the problem of finding $m$ edge-disjoint $k$-cycles covers formulated in $d$-dimensional Euclidean space. We construct a polynomial-time approximation algorithm for this problem and derive conditions of its asymptotical optimality.

Keywords: Cycles covering · $m$-TSP · Asymptotically optimal

1 Introduction

We consider the following problem: given a complete undirected weighted graph $G = (V, E)$, where the set $V$ consists of $n$ vertices represented by points in $d$-dimensional Euclidean space $R^d$ and the weight of each edge equals to the Euclidean distance between corresponding points.

A cycle cover of a graph is a spanning subgraph whose connected components are simple cycles.

Find a union $C = \{C_1 \cup C_2 \cup \cdots \cup C_m\}$ of $m$ edge-disjoint cycle covers, such that each set $C_i$ is a spanning 2-factor in $G$, consisting of exactly $k$ cycles, so that the sum of weights of all the edges in the union $C$ is maximized.

The paper is organized as follows. Section 2 summarizes related work on the studied problem. Section 3 contains necessary preliminary definitions and theorems. Section 4 introduces the proposed algorithm. Section 5 presents the theoretical analysis of the algorithm. Section 6 concludes the paper.

2 Related Works

We are considering the combinatorial optimization problem, which is closely related to the well-known Traveling Salesman Problem (TSP) and $k$-Cycles
Cover Problem ($k$-CsCP). For a fixed natural number $k$ and a given complete weighted graph $G = (V, E)$, it is required to find an extremal (minimum, or maximum)-weight cover of the set $V$ by $k$ vertex-disjoint cycles.

TSP is a particular case of $k$-CsCP for $k = 1$.

It is known [14] that the TSP is NP-hard even in the Euclidean case, the optimal solution can not be found in polynomial time, unless $P = NP$. Although the TSP is hardly approximable [15] in the general case, for some special cases polynomial-time approximation algorithms are developed. For instance, the Metric TSP [3] can be approximated in polynomial time with a ratio $3/2$, and, for Euclidean TSP, a polynomial-time approximations scheme [1] and an asymptotically correct algorithm are developed [16] and [5].

Other well-known generalizations of the TSP is $m$-PSP. Instances of this problem are given by undirected complete graphs with positive edge weights.

In $m$-PSP, the goal is to nd $m$ edge-disjoint Hamiltonian cycles $H_1, \ldots, H_m$ of minimum or maximum total edge weight. The problem remains intractable in metric and Euclidean special cases. For Euclidean Max $m$-PSP [12], there exist the asymptotically optimal algorithm [2].

In [10] and [11] it is shown that the Min $k$-CsCP is NP-hard in the strong sense, both in general and in particular cases, Metric and Euclidean. For the case $k = 2$ efficient 2-approximation algorithm is proposed, and for the Euclidean problem on the plane a PTAS is suggested.

Asymptotically optimal algorithms with running time $O(n^3)$ were suggested:

1) for the Euclidean Max $k$-CsCP with given lengths cycles and $k = o(n)$ [6];
2) for the Random Min $k$-CsCP (on instances UNI(0,1)) with $k \leq n^{1/3} \ln n$ [7].

In [18] a problem of finding a maximum-cost cycle cover which satisfies an upper bound on the number of cycles and a lower bound on the number of edges in each cycle is considered. There is suggested a polynomial-time algorithm in the geometric case when the vertices of the graph are points in a multidimensional real space and the distances between them are induced by a positively homogeneous function whose unit ball is an arbitrary convex polytope with a fixed number of facets.

3 Preliminary Facts

The integer numbers $n, m, k$, and $d > 1$ are the input parameters of the problem Euclidean Max $m$-$k$-CsCP, together with points $v_1 = (v_{11}, \ldots v_{1d}), \ldots v_n = (v_{n1}, \ldots, v_{nd})$ defining the graph.

This problem is a generalization of two problems, considered earlier, both of them being generalizations of Euclidean Max TSP:

- problem of finding $m$ edge-disjoint Hamiltonian cycles in complete Euclidean graph (Euclidean Max $m$-PSP): Max TSP is a particular case with $m = 1$;
- problem of finding a maximum-weight single covering of complete Euclidean graph with $k$ cycles (the Max $k$-CsCP): Max TSP is a particular case with $k = 1$. 
First of these problems was considered in [2], where an asymptotically optimal solvability for the problem Euclidean Max $m$-PSP was established.

**Theorem 1.** [2] Euclidean Max $m$-PSP problem is solved with relative error

$$\varepsilon_A(n) \leq \frac{1}{n} + \alpha_d \left( \frac{m}{n} \right)^{\frac{2}{d+1}},$$

$\alpha_d$ being a constant depending only on the dimension $d$.

In [6] an algorithm for the latter problem (of finding $k$-cycles coverage of maximal length) was introduced, with the following estimation for the relative error:

**Theorem 2.** [6] The Euclidean Max $k$-CsCP is solved with relative error

$$\varepsilon_A(n) \leq \frac{2k}{n} + \beta_d \left( \frac{1}{n} \right)^{\frac{2}{d+1}},$$

$\beta_d$ being a constant depending only on the dimension $d$.

Our goal is to explore a more general problem with both $m$ and $k$ being given as a part of input. Note that, using properties of the remote angle between two vectors in the normed space, introduced in [17], an asymptotically optimal approach was realized for the Normed Max $m$-PSP [8], and then for the Normed Max $m$-$k$-CsCP [9].

**Theorem 3.** [9] The Normed Max $m$-$k$-CsCP is solved with relative error

$$\varepsilon_A(n) \leq \frac{2k+1}{n} + 2(d+1) \left( \frac{m}{n} \right)^{1/(d+1)}.$$

In the current work we present $O(n^3)$-time Algorithm $A$ for the Euclidean Max $m$-$k$-CsCP. It turns out that in Euclidean space the problem is solved more accurately than in a normed space.

Let $w(u, v)$ be the weight of edge $e = (u, v) \in E$ and $W(G') = \sum_{e \in E'} w(e)$ be the total weight of a subgraph $G' = (V; E')$ of the initial graph $G = (V; E)$ with the set of edges $E' \subseteq E$. The goal of the algorithm $A$ for the maximum $m$-$k$-CsCP is to find a subset of edges $\tilde{C} \subseteq E$, consisting of $m$ edge-disjoint $k$-Cycles Coverings $C_1, \ldots, C_m$. At the beginning of the algorithm, $\tilde{C}$ is empty.

Let $M^* = \{I_1, \ldots, I_\mu\}$ be the set of edges (intervals in $R^d$) of a maximum weight matching in $G$; $\mu = \lfloor n/2 \rfloor$.

**Definition 1.** Two edges $e_1, e_2 \in E$ are linked (with respect to set $\tilde{C}$), if there exists an edge $e \in E$ ($e \in \tilde{C}$), that connects the end vertices of $e_1$ and $e_2$.

**Definition 2.** An $I$-chain is a sequence of edges, where each two neighboring edges are linked.

**Definition 3.** Two $I$-chains are linked (with respect to set $\tilde{C}$) if their end edges are linked.
We declare one of the end edges of an $I$-chain to be the master edge and the other one to be the inferior edge.

**Definition 4.** An $\alpha$-chain is an $I$-chain, where the remote angle between any two neighboring edges of the chain is at most $\alpha$.

For a description of the Algorithm it is convenient to use the spatial figure, which we will call a $\mu$-pseudo-prism $P$ (Fig. 4), $\mu = \lceil n/2 \rceil$.

![Fig. 1. The example of the $\mu$-pseudo-prism, $\mu = 14$.](image)

**Definition 5.** A subgraph of a graph $G$ forms the $\mu$-pseudo-prism if it consists of two edge-disjoint $\mu$-vertex circuits $(u_1, \ldots, u_\mu, u_1)$ and $(v_1, \ldots, v_\mu, v_1)$, connected between itself by $\mu$ edge jumpers $(u_j, v_j)$, $j = 1, \ldots, \mu$, where $\mu = \lfloor n/2 \rfloor$. Thus, the prism consists of $2\mu$ vertices and $3\mu$ edges.

In such pseudo-prism, unlike a regular spatial prism, the opposite sides of the quadrangles can be non-parallel, and the vertices of the cycles do not have to be in the same plane.

We will describe an asymptotically optimal algorithm $A$, which

- builds $m$ maximum-total weight $\mu$-pseudo-prisms $P_1, \ldots, P_m$ in the Euclidean space, where the maximum matching edges of the original graph $G$ are used as the jumpers of the prisms, but all of other edges are non adjacent.
- constructs the maximum $m$-$k$-CsCP.

### 4 Description of the Algorithm $A$

**Preliminary Stage**

In the given graph $G$ find a matching $\mathcal{M}^* = \{I_1, \ldots, I_\mu\}$ of maximum weight, where $\mu = \lceil n/2 \rceil$ is the number of its edges (intervals).
Set $\tilde{C} = \emptyset$ and fix a parameter $t \leq \mu/2$. Sort the edges of $M^*$ in the non-increasing order. We will refer to the first $(\mu - t)$ edges of $M^*$ as heavy edges, and the last $t$ edges as light.

**Stage 1: Designing pseudo-prisms $P_i$, $i = 1, \ldots, m$.**

Define angle $\alpha_i$ according to the relation:

$$\sin^2 \frac{\alpha_i}{2} = \gamma_d t_i^{-2/(d-1)}.$$  

(1)

where $\gamma_d$ being a constant depending only on the dimension $d$;

$t_i$ is the number of admitible edges when building a prism $P_i$:

$$t_i = \begin{cases} t, & \text{if } i = 1; \\ t/(2i - 2) & \text{for } 1 < i \leq m. \end{cases}$$  

(2)

Set $i = 1$.

**Step 1.** Constructing a sequence $S = \{S_1, \ldots, S_t\}$ of $t$ $\alpha$-chains.

Build a set $I$ of $\alpha$-chains, which consist only of the heavy edges. (Note that an edge is a one-element $\alpha_i$-chain.)

We start with $I_t$ consisting of the first $t$ heaviest edges of $\tilde{M}^*$: $I_t = \{I_1, I_2, \ldots, I_t\}$.

Set $j = t$.

In the current $t$-chain $I_j$, find a pair of non-linked (with respect to the set $\tilde{C}$) $I$-chains such that the angle between their master edges is at most $\alpha_i$.

Join these chains into one $\alpha_i$-chain by setting their master edges to be neighbors and assign one of the end edges of the joined chain (one of the former inferior edges) to be the new master edge. Set $j := j + 1$. If $j < \mu - t$, then append one more heavy edge $I_j$ to the current set $I$ and repeat Stage 1.

Otherwise, we have obtained a sequence $S = \{S_1, \ldots, S_t\}$ of $t$ $\alpha$-chains such that each of them consists of a sequence of heavy edges with the angle between any consecutive (neighboring) pair of edges at most $\alpha_i = \alpha_d(t_i)$.

**Fig. 2.** The bold lines correspond to the edges of $M^*$, while the dashed lines indicates the $\alpha$-chains. The $t$ light edges of $M^*$ were placed to the positions between the $\alpha$-chains. The last light edge $I_{\nu_t}$ is placed between the first and the $t$-th $\alpha$-chains.
Step 2. Constructing a pseudo-prism $P_i$.

Let’s regard the sequence $S$ as a cycle, i.e. the $\alpha_i$-chain $S_t$ is followed by the $\alpha_i$-chain $S_1$. Let the edges of the $\alpha_i$-chains $S_1, \ldots, S_t$ be enumerated so that $S_r = \{I_{\nu_r-1+1}, \ldots, I_{\nu_r-1}\}$, $1 \leq r \leq t$, where $\nu_1 < \nu_2 < \ldots < \nu_t$ are the numbers reserved for the remaining light edges of $M^*$ ($\nu_0 = 0$, $\nu_t = \mu$).

Place $t$ light edges of $M^*$ to the positions $\nu_1, \nu_2, \ldots, \nu_t = \mu$, such that no light edge is linked to the neighboring end edges of the $\alpha$-chains.

Construct a pseudo-prism $P_i$ such that $P_1 \setminus M^*, P_2 \setminus M^*, \ldots, P_{t-1} \setminus M^*$ in the following way. We assume that the sequence of edges $\{I_1, I_2, \ldots, I_\mu\}$ is given according to their order in the sequence $S$, $I_j = (x_j, y_j)$, $j = 1, \ldots, \mu$. Now we are going to construct a pseudo-prism $P_i$.

For $j = \nu_1, \ldots, \nu_t$ execute the following operator

\[
\text{if } w(u_{j-1}, x_j, u_{j+1}) + w(v_{j-1}, y_j, v_{j+1}) \geq w(u_{j-1}, y_j, u_{j+1}) + w(v_{j-1}, x_j, v_{j+1}),
\]

then set $u_j = x_j$; $v_j = y_j$; otherwise, set $u_j = y_j$ and $v_j = x_j$.

As a result of the Step 2 we have obtained a pseudo-prism $P_i$, consisting of the two non-intersecting circuits $(u_1, u_2, \ldots, u_\mu, u_1)$ and $(v_1, v_2, \ldots, v_\mu, v_1)$, and of the maximum matching $M^* = \{I_1, I_2, \ldots, I_\mu\}$ where $I_j = (u_j, v_j)$, $j = 1, \ldots, \mu$.

Fig. 3. The $\mu$-pseudo-prism: the dashed lines form two edge-disjoint $\mu$-vertex circuits; $\mu$ bold lines are edge jumpers of $M^*$.

Stage 3. Constructing $m$-$k$-CsC.

Common Step: Constructing $k$-Cycles Covering $C_i$, $i = 1, \ldots, m$.

In order to obtain coverage with $k$ cycles, it (arbitrary) chooses $k > 1$ pairs of adjacent edges in the corresponding ordering of matching $M^*$ and performs “reverse operation”, i.e. returns edges of matching into the solution $C_i$ and removes pair of edges that connected endpoints of this pair of adjacent edges.

Stage 4. Cases of evenness and oddness of $n$.

In the case of even $n$ we have $k$-CsC $C_i$ of graph $G$ and go to Stage 5.

If $n$ is odd, there exists a vertex $x_0$ that is not in $M^*$. In this case replace one of the matching edge $(u, v)$ of the constructed covering by the pair of edges $(u, x_0)$ and $(v, x_0)$ so that none of these edges intersects the set $\tilde{C}$. The triangle inequality guarantees that the weight of the cycle will not decrease.
Stage 5. Append the edges of the obtained $k$-covering $C_i$ to the set $\tilde{C}$. Further these $2ik$ edges of matching $M^*$ added to $C_1, \ldots, C_i$ are marked as forbidden.

The description of the algorithm $A$ is complete.

5 Algorithm Analysis

Lemma 1. The admissible number of cycles in the Max $m$-$k$-CsCP satisfies the inequality

$$k \leq \frac{n}{8m}.$$  

Proof. We search for another $k$ pairs of adjacent edges. In the worst case $2ik$ edges are forbidden and $2ik - 1$ edges are located between them, one between each two forbidden and thus can’t be used. So, in order to be able to finish this operation on $m$-th coverage, we need that $2k + 4(m - 1)k - 1 \leq n/2$. From this it follows restriction on parameters of the problem: $k \leq \frac{n}{8m}$.

Lemma 2. [2] For $d \geq 2$ the following inequality holds:

$$\sum_{i=1}^{m} t_i^{-2/(d-1)} \leq \left( \frac{m}{t} \right)^{-2/(d-1)}.$$  \hspace{1cm} (3)

Theorem 4. The Max $m$-$k$-CsCP is approximately solved by the algorithm $A$ in running time $O(n^3)$ with relative error estimated by the inequality

$$\varepsilon_A(n) \leq \frac{2(t + k) + 1}{n} + \gamma_d \left( \frac{m}{t} \right)^{-2/(d-1)}.$$  \hspace{1cm} (4)

Proof. The running-time of the algorithm is determined by the time one needs to construct a maximum weight matching, which is $O(n^3)$ [4].

The algorithm $A$ produces $m$ $k$-coverings $C_1, C_2, \ldots, C_m$, and since each time we arrange the edges in $S$ so that they are not linked with respect to the edges of $\tilde{E}$, the obtained coverings are edge-disjoint.
In [16] it is shown, that

\[ W(C_i) \geq 2W(\tilde{M}^*) \left(1 - \frac{1}{n}\right) \cos \frac{\alpha_i}{2}, \]

where for the total weight \(W(\tilde{M}^*)\) of the first \((\mu - t)\) heaviest edges of \(M^*\) the following inequality satisfies:

\[ W(\tilde{M}^*) \geq W(M^*) \left(1 - \frac{\mu - t}{\mu}\right). \quad (5) \]

It follows from this

\[ W(C_i) \geq 2W(M^*) \left(1 - \frac{1}{n}\right) \left(1 - \frac{t}{\mu}\right) \cos \frac{\alpha_i}{2} \geq 2W(M^*) \left(1 - \frac{2t + 1}{n} - \frac{\sin^2 \alpha_i/2}{2}\right) \quad (6) \]

where the angle \(\alpha_i\) is defined by (1).

On the other hand, the upper bound of optimum for the coverage problem is estimated as

\[ 2W(M^*) \geq \left(1 - \frac{k}{n}\right)W(C_i^*), \quad (7) \]

(the worst case of all odd cycles), used in [6].

where \(C_i^*\) is the solution of the Max \(k\)-CsCP in the given graph.

Using (4), (6) and (7), for the relative error \(\varepsilon_A(n)\) of algorithm \(A\) we have

\[ \varepsilon_A(n) = 1 - \frac{W_A}{OPT} \leq 1 - \frac{W_A}{mW(C_i^*)} = 1 - \frac{W(C_1) + \ldots + W(C_m)}{mW(C_i^*)} \]

\[ \leq \frac{2(t + k) + 1}{n} + \frac{\gamma d}{2} \sum_{i=1}^{m} t_i^{-2/(d-1)} \leq \frac{2(t + k) + 1}{n} + \frac{\gamma d}{2} \left(\frac{m}{t}\right)^{-2/(d-1)}. \]

Theorem 4 is proved.

Now we can formulate the main result of the article.

**Theorem 5.** A polynomial-time approximation algorithm \(A\) with parameters \(t^* = m\left(\frac{n}{m}\right)^{\frac{d+1}{d}}\) and \(k \leq \min\left(\frac{n}{8m}, o(n)\right)\) gives asymptotically optimal solutions for the Max \(m-k\)-CsCP in the Euclidean space with a fixed dimension \(d\) and \(m = o(n)\).

**Proof.** Setting the given value of the parameter \(t = t^*\) in the condition of Theorem, we obtain

\[ \varepsilon_A(n) \leq \frac{2k + 1}{n} + \left(2 + \frac{\gamma d}{2}\right) \left(\frac{m}{n}\right)^{2/(d+1)} \rightarrow 0 \]

as \(n \rightarrow \infty\).
Remark. The specificity of the Euclidean space made it possible to obtain a solution of the Max $m$-$k$-CsCP with better accuracy in comparison with the solution in a Normed space in [9].

Indeed, it follows directly from the estimations of the relative error $\varepsilon_A(n)$ in Theorems 3 and 5:

\[
\left(\frac{m}{n}\right)^{2/(d+1)} \leq \left(\frac{m}{n}\right)^{1/(d+1)}.
\]

6 Conclusion

Using angle estimations obtained in [16], we construct an algorithm for the Euclidean Max $m$-$k$-CsCP, which gives asymptotically optimal solution for the problem in an Euclidean space of fixed dimension, given that $m = o(n)$ and $k \leq \min\left(\frac{n}{8m}, o(n)\right)$.

As a topic for further research it is interesting to extend this approach to different modifications of the considered problem. For example, it is of natural interest to construct an asymptotically optimal algorithm for the $m$-$k$-CsCP that would essentially rely on the specifics of problem statement and would have a better relative error or less tight condition for the numbers $m$ and $k$ of coverings.

References

1. Arora, S.: Polynomial-programming: methods and applications. In: Reeves, C.R. (ed.) NATO Advanced Study Institute Series, Series C: Mathematics and Physics Science, vol. 19, pp. 1730–178. Reidel, Dordrecht (1975)
2. Baburin, A.E., Gimadi, E.Kh.: On the asymptotic optimality of an algorithm for solving the maximum m-PSP in a multidimensional Euclidean space. Proc. Steklov Inst. Math. 272(Suppl. 1), 1–13 (2011)
3. Christodes, N.: Worst-case analysis of a new heuristic for the traveling salesman problem. In: Symposium on New Directions and Recent Results in Algorithms and Complexity, p. 441. Academic Press, New York (1976)
4. Gabow, H.N.: An efficient reduction technique for degree-restricted subgraph and bidirected network flow problems. In: Proceedings of the 15th Annual ACM Symposium on Theory of Computing 1983, Boston, USA, pp. 448–456. ACM, New York (1983)
5. Gimadi, E.Kh.: A new version of the asymptotically optimal algorithm for solving the Euclidean maximum traveling salesman problem. In: Proceeding of the 12th Baykal International Conference 2001, Irkutsk, vol. 1, pp. 117–123 (2009). (in Russian)
6. Gimadi, E.Kh., Rykov, I.A.: Asymptotically optimal approach to the approximate solution of several problems of covering a graph by nonadjacent cycles. Proc. Steklov Inst. Math. 295, 57–67 (2016). https://doi.org/10.1134/S0081543816090078
7. Gimadi, E.Kh., Rykov, I.A.: On asymptotical optimality of solving maximum-weight Euclidean m-cycles covering problem. Doklady Math. 93(1/2), 117–120 (2016)
8. Gimadi, E.Kh., Tsidulko, O.Yu.: Asymptotically optimal algorithm for the maximum m-peripatetic salesman problem in a normed space. In: Battiti, R., Brunato, M., Kotsireas, I., Pardalos, P.M. (eds.) LION 12 2018. LNCS, vol. 11353, pp. 402–410. Springer, Cham (2019). https://doi.org/10.1007/978-3-030-05348-2_33

9. Gimadi, E.Kh., Rykov, I.A.: On asymptotically optimal solvability of max m-k-cycles cover problem in a normed space. In: Kononov, A., Khachay, M., Kalyagin, V.A., Pardalos, P. (eds.) MOTOR 2020. LNCS, vol. 12095, pp. 85–97. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-49988-4_6

10. Khachay, M., Neznakhina, E.: Approximation of Euclidean k-size cycle cover problem. Croatian Oper. Res. Rev. (CRORR) 5, 177–188 (2014)

11. Khachay, M., Neznakhina, E.: Polynomial-time approximations scheme for a Euclidean problem on a cycle covering of a graph. Trudy instituta matematiki i mehaniki UrORAN 20(4), 297–311 (2014). (in Russian)

12. Krarup, J.: The peripatetic salesman and some related unsolved problems. In: Roy, B. (eds.) Combinatorial Programming: Methods and Applications. NATO Advanced Study Institutes Series (Series C – Mathematical and Physical Sciences), vol. 19, pp. 173–178. Springer, Dordrecht (1975). https://doi.org/10.1007/978-94-011-7557-9_8

13. Manthey, B.: Minimum-weight cycle covers and their approximability. In: Brandstätter, A., Kratsch, D., Müller, H. (eds.) WG 2007. LNCS, vol. 4769, pp. 178–189. Springer, Heidelberg (2007). https://doi.org/10.1007/978-3-540-74839-7_18

14. Papadimitriou, C.: Euclidean TSP is NP-complete. Theoret. Comput. Sci. 4, 237–244 (1977)

15. Sahni, S., Gonzales, T.: P-complete approximation problems. J. ACM 23(3), 555–565 (1976)

16. Serdyukov, A.I.: An asymptotically optimal algorithm for the maximum traveling salesman problem in Euclidean space. Upravlyaemye sistemy, Novosibirsk 27, 79–87 (1987). (in Russian)

17. Shenmaier, V.V.: Asymptotically optimal algorithms for geometric Max TSP and Max m-PSP. Discret. Appl. Math. 163(2), 214–219 (2014)

18. Shenmaier, V.V.: An algorithm for the polyhedral cyclic covering problem with restrictions on the number and length of cycles. Tr. IMM UrORAN. 24(3), 272–280 (2018). https://doi.org/10.21538/0134-4889-2018-24-3-272-280