THE LEFSCHETZ-HOPF THEOREM AND AXIOMS FOR THE LEFSCHETZ NUMBER

MARTIN ARKOWITZ AND ROBERT F. BROWN

Dartmouth College, Hanover and University of California, Los Angeles

Abstract. The reduced Lefschetz number, that is, $L(·) - 1$ where $L(·)$ denotes the Lefschetz number, is proved to be the unique integer-valued function $\lambda$ on selfmaps of compact polyhedra which is constant on homotopy classes such that (1) $\lambda(fg) = \lambda(gf)$, for $f:X \to Y$ and $g:Y \to X$; (2) if $(f_1, f_2, f_3)$ is a map of a cofiber sequence into itself, then $\lambda(f_1) = \lambda(f_1) + \lambda(f_3)$; (3) $\lambda(f) = -(\deg(p_1 e_1) + \cdots + \deg(p_k e_k))$, where $f$ is a selfmap of a wedge of $k$ circles, $e_r$ is the inclusion of a circle into the $r$th summand and $p_r$ is the projection onto the $r$th summand. If $f:X \to X$ is a selfmap of a polyhedron and $I(·)$ is the fixed point index of $f$ on all of $X$, then we show that $I(·) - 1$ satisfies the above axioms. This gives a new proof of the Normalization Theorem: If $f:X \to X$ is a selfmap of a polyhedron, then $I(·)$ equals the Lefschetz number $L(f)$ of $f$. This result is equivalent to the Lefschetz-Hopf Theorem: If $f:X \to X$ is a selfmap of a finite simplicial complex with a finite number of fixed points, each lying in a maximal simplex, then the Lefschetz number of $f$ is the sum of the indices of all the fixed points of $f$.

1. Introduction.

Let $X$ be a finite polyhedron and denote by $\tilde{H}_*(X)$ its reduced homology with rational coefficients. Then the reduced Euler characteristic of $X$, denoted by $\tilde{\chi}(X)$, is defined by

$$\tilde{\chi}(X) = \sum_j (-1)^j \dim \tilde{H}_j(X).$$

Clearly, $\tilde{\chi}(X)$ is just the Euler characteristic minus one. In 1962, Watts [13] characterized the reduced Euler characteristic as follows: Let $\epsilon$ be a function from the set of finite polyhedra with base points to the integers such that (i) $\epsilon(S^0) = 1$, where $S^0$ is the 0-sphere, and (ii) $\epsilon(X) = \epsilon(A) + \epsilon(X/A)$, where $A$ a subpolyhedron of $X$. Then $\epsilon(X) = \tilde{\chi}(X)$.

Let $\mathcal{C}$ be the collection of spaces $X$ of the homotopy type of a finite, connected CW-complex. If $X \in \mathcal{C}$, we do not assume that $X$ has a base point except when $X$ is a sphere or a wedge of spheres. It is not assumed that maps between spaces with base points are based. A map $f:X \to X$, where $X \in \mathcal{C}$, induces trivial homomorphisms $f_j:H_j(X) \to H_j(X)$ of rational homology vector spaces for all $j > \dim X$. The Lefschetz number $L(f)$ of $f$ is defined by

$$L(f) = \sum_j (-1)^j Tr f_j,$$

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where $Tr$ denotes the trace. The reduced Lefschetz number $\tilde{L}$ is given by $\tilde{L}(f) = L(f) - 1$ or, equivalently, by considering the rational, reduced homology homomorphism induced by $f$.

Since $\tilde{L}(id) = \tilde{\chi}(X)$, where $id: X \to X$ is the identity map, Watts’s Theorem suggests an axiomatization for the reduced Lefschetz number which we state below as Theorem 1.1.

For $k \geq 1$, denote by $\bigvee^k S^n$ the wedge of $k$ copies of the $n$-sphere $S^n$, $n \geq 1$. If we write $\bigvee^k S^n$ as $S^n_1 \vee S^n_2 \vee \cdots \vee S^n_k$, where $S^n_j = S^n$, then we have inclusions $e_j: S^n_j \to \bigvee^k S^n$ into the $j$-th summand and projections $p_j: \bigvee^k S^n \to S^n_j$ onto the $j$-th summand, for $j = 1, \ldots, k$. If $f: \bigvee^k S^n \to \bigvee^k S^n$ is a map, then $f_j: S^n_j \to S^n_j$ denotes the composition $p_j fe_j$. The degree of a map $f: S^n \to S^n$ is denoted by $\deg(f)$.

We characterize the reduced Lefschetz number as follows.

**Theorem 1.1.** The reduced Lefschetz number $\tilde{L}$ is the unique function $\lambda$ from the set of self-maps of spaces in $C$ to the integers that satisfies the following conditions:

1. **(Homotopy Axiom)** If $f, g: X \to X$ are homotopic maps, then $\lambda(f) = \lambda(g)$.

2. **(Cofibration Axiom)** If $A$ is a subpolyhedron of $X$, $A \to X \to X/A$ is the resulting cofiber sequence and there exists a commutative diagram

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
A & \longrightarrow & X/A \\
\end{array}
\]

then $\lambda(f) = \lambda(f') + \lambda(\bar{f})$.

3. **(Commutativity Axiom)** If $f: X \to Y$ and $g: Y \to X$ are maps, then $\lambda(gf) = \lambda(fg)$.

4. **(Wedge of Circles Axiom)** If $f: \bigvee^k S^1 \to \bigvee^k S^1$ is a map, $k \geq 1$, then

\[
\lambda(f) = -(\deg(f_1) + \cdots + \deg(f_k)),
\]

where $f_j = p_j fe_j$.

In an unpublished dissertation [10], Hoang extended Watts’s axioms to characterize the reduced Lefschetz number for basepoint-preserving self-maps of finite polyhedra. His list of axioms is different from, but similar to, those in Theorem 1.1.

One of the classical results of fixed point theory is

**Theorem 1.2 (Lefschetz-Hopf).** If $f: X \to X$ is a map of a finite polyhedron with a finite set of fixed points, each of which lies in a maximal simplex of $X$, then $L(f)$ is the sum of the indices of all the fixed points of $f$.

The history of this result is described in [3], see also [8, p. 458]. A proof that depends on a delicate argument due to Dold [5] can be found in [2] and, in a more condensed form, in [4]. In an appendix to his dissertation [12], D. McCord outlined a possibly more direct argument, but no details were published. The book of Granas and Dugundji [8, pp. 441 - 450] presents an argument based on classical techniques of Hopf [11]. We use the characterization of the reduced Lefschetz number in Theorem 1.1 to prove the Lefschetz-Hopf theorem in a quite natural manner by showing that the fixed point index satisfies the axioms of Theorem 1.1. That is, we prove...
Theorem 1.3 (Normalization Property). If \( f: X \to X \) is any map of a finite polyhedron, then \( L(f) = i(X, f, X) \), the fixed point index of \( f \) on all of \( X \).

The Lefschetz-Hopf Theorem follows from the Normalization Property by the Additivity Property of the fixed point index. In fact these two statements are equivalent. The Hopf Construction [2, p. 117] implies that a map \( f \) from a finite polyhedron to itself is homotopic to a map that satisfies the hypotheses of the Lefschetz-Hopf theorem. Thus the Homotopy and Additivity Properties of the fixed point index imply that the Normalization Property follows from the Lefschetz-Hopf Theorem.

2. Lefschetz numbers and exact sequences.

In this section, all vector spaces are over a fixed field \( F \), which will not be mentioned, and are finite dimensional. A graded vector space \( V = \{V_n\} \) will always have the following properties: (1) each \( V_n \) is finite dimensional and (2) \( V_n = 0 \) for \( n < 0 \) and for \( n > N \), for some non-negative integer \( N \). A map \( f: V \to W \) of graded vector spaces \( V = \{V_n\} \) and \( W = \{W_n\} \) is a sequence of linear transformations \( f_n: V_n \to W_n \). For a map \( f: V \to V \), the Lefschetz number is defined by

\[
L(f) = \sum_n (-1)^n Tr f_n.
\]

The proof of the following lemma is straightforward, and hence omitted.

Lemma 2.1. Given a map of short exact sequences of vector spaces

\[
0 \to U \to V \to W \to 0
\]

\[
f \downarrow \quad g \downarrow \quad h \downarrow
\]

\[
0 \to U \to V \to W \to 0,
\]

then \( Tr g = Tr f + Tr h \).

Theorem 2.2. Let \( A, B \) and \( C \) be graded vector spaces with maps \( \alpha: A \to B, \beta: B \to C \) and selfmaps \( f: A \to A, g: B \to B \) and \( h: C \to C \). If for every \( n \), there is a linear transformation \( \partial_n: C_n \to A_{n-1} \) such that the following diagram is commutative and has exact rows:

\[
0 \to A_N \to B_N \to C_N \to A_{N-1} \to \cdots
\]

\[
f_N \downarrow \quad g_N \downarrow \quad h_N \downarrow \quad f_{N-1} \downarrow
\]

\[
0 \to A_N \to B_N \to C_N \to A_{N-1} \to \cdots
\]

\[
\cdots \to A_0 \to B_0 \to C_0 \to 0
\]

\[
f_0 \downarrow \quad g_0 \downarrow \quad h_0 \downarrow
\]

\[
\cdots \to A_0 \to B_0 \to C_0 \to 0,
\]

then

\[
L(g) = L(f) + L(h).
\]
Proof. Let \( Im \) denote the image of a linear transformation and consider the commutative diagram

\[
\begin{array}{c}
0 \longrightarrow \text{Im } \beta_n \longrightarrow C_n \longrightarrow \text{Im } \partial_n \longrightarrow 0 \\
h_n|\text{Im } \beta_n \downarrow \quad h_n \downarrow \quad f_{n-1}|\text{Im } \partial_n \downarrow \\
0 \longrightarrow \text{Im } \beta_n \longrightarrow C_n \longrightarrow \text{Im } \partial_n \longrightarrow 0.
\end{array}
\]

By Lemma 2.1, \( \text{Tr}(h_n) = \text{Tr}(h_n|\text{Im } \beta_n) + \text{Tr}(f_{n-1}|\text{Im } \partial_n) \). Similarly, the commutative diagram

\[
\begin{array}{c}
0 \longrightarrow \text{Im } \partial_n \longrightarrow A_{n-1} \longrightarrow \text{Im } \alpha_{n-1} \longrightarrow 0 \\
f_{n-1}|\text{Im } \partial_n \downarrow \quad f_{n-1} \downarrow \quad g_{n-1}|\text{Im } \alpha_{n-1} \downarrow \\
0 \longrightarrow \text{Im } \partial_n \longrightarrow A_{n-1} \longrightarrow \text{Im } \alpha_{n-1} \longrightarrow 0.
\end{array}
\]

yields \( \text{Tr}(f_{n-1}|\text{Im } \partial_n) = \text{Tr}(f_{n-1}) - \text{Tr}(g_{n-1}|\text{Im } \alpha_{n-1}) \). Therefore

\[
\text{Tr}(h_n) = \text{Tr}(h_n|\text{Im } \beta_n) + \text{Tr}(f_{n-1}) - \text{Tr}(g_{n-1}|\text{Im } \alpha_{n-1}).
\]

Now consider

\[
\begin{array}{c}
0 \longrightarrow \text{Im } \alpha_{n-1} \longrightarrow B_{n-1} \longrightarrow \text{Im } \beta_{n-1} \longrightarrow 0 \\
g_{n-1}|\text{Im } \alpha_{n-1} \downarrow \quad g_{n-1} \downarrow \quad h_{n-1}|\text{Im } \beta_{n-1} \downarrow \\
0 \longrightarrow \text{Im } \alpha_{n-1} \longrightarrow B_{n-1} \longrightarrow \text{Im } \beta_{n-1} \longrightarrow 0,
\end{array}
\]

so \( \text{Tr}(g_{n-1}|\text{Im } \alpha_{n-1}) = \text{Tr}(g_{n-1}) - \text{Tr}(h_{n-1}|\text{Im } \beta_{n-1}) \). Putting this all together, we obtain

\[
\text{Tr}(h_n) = \text{Tr}(h_n|\text{Im } \beta_n) + \text{Tr}(f_{n-1}) - \text{Tr}(g_{n-1}) + \text{Tr}(h_{n-1}|\text{Im } \beta_{n-1}).
\]

We next look at the left end of the original diagram and get

\[
0 = \text{Tr}(h_{N+1}) = \text{Tr}(f_N) - \text{Tr}(g_N) + \text{Tr}(h_N|\text{Im } \beta_N)
\]

and at the right end which gives

\[
\text{Tr}(h_1) = \text{Tr}(h_1|\text{Im } \beta_1) + \text{Tr}(f_0) - \text{Tr}(g_0) + \text{Tr}(h_0).
\]

A simple calculation now yields

\[
\sum_{n=0}^{N} (-1)^n \text{Tr}(h_n) = \sum_{n=0}^{N+1} (-1)^n \left( \text{Tr}(h_n|\text{Im } \beta_n) + \text{Tr}(f_{n-1}) - \text{Tr}(g_{n-1}) + \text{Tr}(h_{n-1}|\text{Im } \beta_{n-1}) \right)
\]

\[
= - \sum_{n=0}^{N} (-1)^n \text{Tr}(f_n) + \sum_{n=0}^{N} (-1)^n \text{Tr}(g_n).
\]

Therefore \( L(h) = -L(f) + L(g) \).  \( \Box \)

We next give some simple consequences of Theorem 2.2.

If \( f: (X, A) \to (X, A) \) is a selfmap of a pair, where \( X, A \in C \), then \( f \) determines \( f_X: X \to X \) and \( f_A: A \to A \). The map \( f \) induces homomorphisms \( f_j: H_j(X, A) \to H_j(X, A) \) of relative homology with coefficients in \( F \). The relative Lefschetz number \( L(f; X, A) \) is defined by

\[
L(f; X, A) = \sum_j (-1)^j \text{Tr} f_j.
\]

Applying Theorem 2.2 to the homology exact sequence of the pair \( (X, A) \), we obtain
Corollary 2.3. If \( f: (X, A) \to (X, A) \) is a map of pairs, where \( X, A \in \mathcal{C} \), then

\[
L(f; X, A) = L(f_X) - L(f_A).
\]

This result was obtained by Bowszyc [1].

Corollary 2.4. Suppose \( X = P \cup Q \) where \( X, P, Q \in \mathcal{C} \) and \( (X; P, Q) \) is an proper triad [6, p. 34]. If \( f: X \to X \) is a map such that \( f(P) \subseteq P \) and \( f(Q) \subseteq Q \) then, for \( f_P, f_Q \) and \( f_{P \cap Q} \) the restrictions of \( f \) to \( P, Q \) and \( P \cap Q \) respectively, we have

\[
L(f) = L(f_P) + L(f_Q) - L(f_{P \cap Q}).
\]

Proof. The map \( f \) and its restrictions induce a map of the Mayer-Vietoris homology sequence [6, p. 39] to itself so the result follows from Theorem 2.2. \( \square \)

A similar result was obtained by Ferrario [7, Theorem 3.2.1].

Our final consequence of Theorem 2.2 will be used in the characterization of the reduced Lefschetz number.

Corollary 2.5. If \( A \) is a subpolyhedron of \( X \), \( A \to X \to X/A \) is the resulting cofiber sequence of spaces in \( \mathcal{C} \) and there exists a commutative diagram

\[
\begin{array}{ccc}
A & \to & X & \to & X/A \\
\downarrow & & \downarrow & & \downarrow \\
A & \to & X & \to & X/A,
\end{array}
\]

then

\[
L(f) = L(f') + L(\bar{f}) - 1.
\]

Proof. We apply Theorem 2.2 to the homology cofiber sequence. The ‘minus one’ on the right hand side arises because that sequence ends with

\[
\to H_0(A) \to H_0(X) \to \check{H}_0(X/A) \to 0.
\]

\( \square \)

3. Characterization of the Lefschetz number.

Throughout this section, all spaces are assumed to lie in \( \mathcal{C} \).

We let \( \lambda \) be a function from the set of self-maps of spaces in \( \mathcal{C} \) to the integers that satisfies the Homotopy Axiom, Cofibration Axiom, Commutativity Axiom and Wedge of Circles Axiom of Theorem 1.1 as stated in the Introduction.

We draw a few simple consequences of these axioms. From the Commutativity Axiom, we obtain

Lemma 3.1. If \( f: X \to X \) is a map and \( h: X \to Y \) is a homotopy equivalence with homotopy inverse \( k: Y \to X \), then \( \lambda(f) = \lambda(h fk) \). \( \square \)
Lemma 3.2. If \( f: X \to X \) is homotopic to a constant map, then \( \lambda(f) = 0 \).

Proof. Let \(*\) be a one-point space and \(*: * \to *\) the unique map. From the map of cofiber sequences

\[
\begin{array}{cccc}
* & \longrightarrow & * & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & * & \longrightarrow & * \\
\end{array}
\]

and the Cofibration Axiom, we have \( \lambda(*) = \lambda(*) + \lambda(*), \) and therefore \( \lambda(*) = 0 \).

Write any constant map \( c: X \to X \) as \( c(x) = * \) for some \(* \in X\), let \( e:* \to X \) be inclusion and \( p:X \to * \) projection. Then \( c = ep \) and \( pe = * \), and so \( \lambda(c) = 0 \) by the Commutativity Axiom. The lemma follows from the Homotopy Axiom. \( \square \)

If \( X \) is a based space with base point \(*\), i.e., a sphere or wedge of spheres, then the cone and suspension of \( X \) are defined by \( CX = X \times I/(X \times 1 \cup * \times I) \) and \( \Sigma X = CX/(X \times 0) \), respectively.

Lemma 3.3. If \( X \) is a based space, \( f: X \to X \) is a based map and \( \Sigma f: \Sigma X \to \Sigma X \) is the suspension of \( f \), then \( \lambda(\Sigma f) = -\lambda(f) \).

Proof. Consider the maps of cofiber sequences

\[
\begin{array}{cccc}
X & \longrightarrow & CX & \longrightarrow & \Sigma X \\
\downarrow f & & \downarrow caf & & \downarrow \Sigma f \\
X & \longrightarrow & CX & \longrightarrow & \Sigma X. \\
\end{array}
\]

Since \( CX \) is contractible, \( caf \) is homotopic to a constant map. Therefore, by Lemma 3.2 and the Cofibration Axiom,

\[
0 = \lambda(Cf) = \lambda(Sf) + \lambda(f) \quad \square
\]

Lemma 3.4. For any \( k \geq 1 \) and \( n \geq 1 \), if \( f: \bigvee^k S^n \to \bigvee^k S^n \) is a map, then

\[
\lambda(f) = (-1)^n(\deg(f_1) + \cdots + \deg(f_k)),
\]

where \( e_r: S^n \to \bigvee^k S^n \) and \( p_r: \bigvee^k S^n \to S^n \) for \( r = 1, \ldots, k \) are the inclusions and projections, respectively, and \( f_r = p_r f e_r \).

Proof. The proof is by induction on the dimension \( n \) of the spheres. The case \( n = 1 \) is the Wedge of Circles Axiom. If \( n \geq 2 \), then the map \( f: \bigvee^k S^n \to \bigvee^k S^n \) is homotopic to a based map \( f': \bigvee^k S^n \to \bigvee^k S^n \). Then \( f' \) is homotopic to \( \Sigma g \), for some map \( g: \bigvee^k S^{n-1} \to \bigvee^k S^{n-1} \). Note that if \( g_j: S_j^{n-1} \to S_j^{n-1} \), then \( \Sigma g_j \) is homotopic to \( f_j: S_j^n \to S_j^n \). Therefore by Lemma 3.3 and the induction hypothesis,

\[
\lambda(f) = \lambda(f') = -\lambda(g) = -(-1)^{n-1}(\deg(g_1) + \cdots + \deg(g_k)) \\
= (-1)^n(\deg(f_1) + \cdots + \deg(f_k)). \quad \square
\]
Proof of Theorem 1.1.

Since \( \tilde{L}(f) = L(f) - 1 \), Corollary 2.5 implies that \( \tilde{L} \) satisfies the Cofibration Axiom. We next show that \( \tilde{L} \) satisfies the Wedge of Circles Axiom. There is an isomorphism \( \theta: \bigoplus^k H_1(S^1) \to H_1(\vee^k S^1) \) defined by \( \theta(x_1, \ldots, x_k) = e_1(x_1) + \cdots + e_k(x_k) \), where \( x_i \in H_1(S^1) \). The inverse \( \theta^{-1}: H_1(\vee^k S^1) \to \bigoplus^k H_1(S^1) \) is given by \( \theta^{-1}(y) = (p_1(y), \ldots, p_k(y)) \). If \( u \in H_1(S^1) \) is a generator, then a basis for \( H_1(\vee^k S^1) \) is \( e_1(u), \ldots, e_k(u) \). By calculating the trace of \( f: H_1(\vee^k S^1) \to H_1(\vee^k S^1) \) with respect to this basis, we obtain \( \tilde{L}(f) = -(\deg(f_1) + \cdots + \deg(f_k)) \). The remaining axioms are obviously satisfied by \( \tilde{L} \). Thus \( \tilde{L} \) satisfies the axioms of Theorem 1.1.

Now suppose \( \lambda \) is a function from the self-maps of spaces in \( C \) to the integers that satisfies the axioms. We regard \( X \) as a connected, finite CW-complex and proceed by induction on the dimension of \( X \). If \( X \) is 1-dimensional, then it is the homotopy type of a wedge of circles. By Lemma 3.1, we can regard \( f \) as a self-map of \( \vee^k S^1 \), and so the Wedge of Circles Axiom gives

\[
\lambda(f) = -(\deg(f_1) + \cdots + \deg(f_k)) = \tilde{L}(f).
\]

Now suppose that \( X \) is \( n \)-dimensional and let \( X^{n-1} \) denote the \((n-1)\)-skeleton of \( X \). Then \( f \) is homotopic to a cellular map \( g: X \to X \) by the Cellular Approximation Theorem [9, Theorem 4.8, p. 349]. Thus \( g(X^{n-1}) \subseteq X^{n-1} \), and so we have a commutative diagram

\[
\begin{array}{ccc}
X^{n-1} & \longrightarrow & X \\
\downarrow g' & & \downarrow g \\
X^{n-1} & \longrightarrow & X/X^{n-1} = \vee^k S^n
\end{array}
\]

Then, by the Cofibration Axiom, \( \lambda(g) = \lambda(g') + \lambda(\tilde{g}) \). Lemma 3.4 implies that \( \lambda(\tilde{g}) = \tilde{L}(\tilde{g}) \) so, applying the induction hypothesis to \( g' \), we have \( \lambda(g) = \tilde{L}(g') + \tilde{L}(\tilde{g}) \). Since we have seen that the reduced Lefschetz number satisfies the Cofibration Axiom, we conclude that \( \lambda(g) = \tilde{L}(g) \). By the Homotopy Axiom, \( \lambda(f) = \tilde{L}(f) \).

4. The Normalization Property.

Let \( X \) be a finite polyhedron and \( f: X \to X \) a map. Denote by \( I(f) \) the fixed point index of \( f \) on all of \( X \), that is, \( I(f) = i(X, f, X) \) in the notation of [2] and let \( \tilde{I}(f) = I(f) - 1 \).

In this section we prove Theorem 1.3 by showing that, with rational coefficients, \( I(f) = L(f) \).

Proof of Theorem 1.3.

We will prove that \( \tilde{I} \) satisfies the axioms and therefore, by Theorem 1.1, \( \tilde{I}(f) = \tilde{L}(f) \). The Homotopy and Commutativity Axioms are well-known properties of the fixed point index (see [2, pp. 59 and 62]).

To show that \( \tilde{I} \) satisfies the Cofibration Axiom, it suffices to consider \( A \) a subpolyhedron of \( X \) and \( f(A) \subseteq A \). Let \( f': A \to A \) denote the restriction of \( f \) and \( \tilde{f}: X/A \to X/A \) the map induced on quotient spaces. Let \( r: U \to A \) be a deformation retraction of a neighborhood of \( A \) in \( X \) onto \( A \) and let \( L \) be a subpolyhedron
of a barycentric subdivision of $X$ such that $A \subseteq \text{int} L \subseteq L \subseteq U$. By the Homotopy Extension Theorem there is a homotopy $H: X \times I \to X$ such that $H(x,0) = f(x)$ for all $x \in X, H(a,t) = f(a)$ for all $a \in A$ and $H(x,1) = f_r(x)$ for all $x \in L$. If we set $g(x) = H(x,1)$ then, since there are no fixed points of $g$ on $L - A$, the Additivity Property implies that

$$I(g) = i(X,g,\text{int} L) + i(X,g,X-L).$$

We discuss each summand of (4.1) separately. We begin with $i(X,g,\text{int} L)$. Since $g(L) \subseteq A \subseteq L$, it follows from the definition of the index ([2, p. 56]) that $i(X,g,\text{int} L) = i(L,g,\text{int} L)$. Moreover, $i(L,g,\text{int} L) = i(L,g,L)$ since there are no fixed points on $L - \text{int} L$ (the Excision Property of the index). Let $e : A \to L$ be inclusion then, by the Commutativity Property [2, p. 62] we have

$$i(L,g,L) = i(L,\text{eg},L) = i(A,\text{ge},A) = I(f^{'})$$

because $f(a) = g(a)$ for all $a \in A$.

Next we consider the summand $i(X,g,X-L)$ of (4.1). Let $\pi: X \to X/A$ be the quotient map, set $\pi(A) = *$ and note that $\pi^{-1}( *) = A$. If $\bar{g}: X/A \to X/A$ is induced by $g$, the restriction of $\bar{g}$ to the neighborhood $\pi(\text{int} L)$ of $*$ in $X/A$ is constant, so $i(X/A,\bar{g},\pi(\text{int} L)) = 1$. If we denote the set of fixed points of $\bar{g}$ with $*$ deleted by $\text{Fix}_* \bar{g}$, then $\text{Fix}_* \bar{g}$ is in the open subset $X/A - \pi(L)$ of $X/A$. Let $W$ be an open subset of $X/A$ such that $\text{Fix}_* \bar{g} \subseteq W \subseteq X/A - \pi(L)$ with the property $\bar{g}(W) \cap \pi(L) = \emptyset$. By the Additivity Property we have

$$I(\bar{g}) = i(X/A,\bar{g},\pi(\text{int} L)) + i(X/A,\bar{g},W) = 1 + i(X/A,\bar{g},W).$$

Now, identifying $X - L$ with the corresponding subset $\pi(X - L)$ of $X/A$ and identifying the restrictions of $\bar{g}$ and $g$ to those subsets, we have $i(X/A,\bar{g},W) = i(X,g,\pi^{-1}(W)))$. The Excision Property of the index implies that $i(X,g,\pi^{-1}(W))) = i(X,g,X-L)$. Thus we have determined the second summand of (4.1): $i(X,g,X-L) = I(\bar{g}) - 1$.

Therefore from (4.1) we obtain $I(g) = I(f^{'}) + I(\bar{g}) - 1$. The Homotopy Property then tells us that

$$I(f) = I(f^{'}) + I(\bar{f}) - 1$$

since $f$ is homotopic to $g$ and $\bar{f}$ is homotopic to $\bar{g}$. We conclude that $\bar{I}$ satisfies the Cofibration Axiom.

It remains to verify the Wedge of Circles Axiom. Let $X = \bigvee_k S^1 = S^1 \lor \cdots \lor S^1$ be a wedge of circles with basepoint $*$ and $f : X \to X$ a map. We first verify the axiom in the case $k = 1$. We have $f : S^1 \to S^1$ and we denote its degree by $\text{deg}(f) = d$. We regard $S^1 \subseteq \mathbb{C}$, the complex numbers. Then $f$ is homotopic to $g_d$, where $g_d(z) = z^d$ has $|d| - 1$ fixed points for $d \neq 1$. The fixed point index of $g_d$ in a neighborhood of a fixed point that contains no other fixed point of $g_d$ is $-1$ if $d \geq 2$ and is $1$ if $d \leq 0$. Since $g_1$ is homotopic to a map without fixed points, we see that $I(g_d) = -d + 1$ for all integers $d$. We have shown that $I(f) = -\text{deg}(f) + 1$.

Now suppose $k \geq 2$. If $f(*) = *$ then, by the Homotopy Extension Theorem, $f$ is homotopic to a map which does not fix $*$. Thus we may assume, without loss of generality, that $f(*) \in S_1^1 - \{\ast\}$. Let $V$ be a neighborhood of $f(*)$ in $S_1^1 - \{\ast\}$ such that there exists a neighborhood $U$ of $*$ in $X$ disjoint from $V$ with $f(U) \subseteq V$. 
Since $\bar{U}$ contains no fixed point of $f$ and the open subsets $S_j^1 - \bar{U}$ of $X$ are disjoint, the Additivity Property implies

\begin{equation}
I(f) = i(X, f, S_j^1) + \sum_{j=2}^{k} i(X, f, S_j^1 - \bar{U}).
\end{equation}

The Additivity Property also implies that

\begin{equation}
I(f_j) = i(S_j^1, f_s, S_j^1 - \bar{U}) + i(S_j^1, f_s, S_j^1 \cap \bar{U}).
\end{equation}

There is a neighborhood $W_j$ of $(\text{Fix } f) \cap S_j^1$ in $S_j^1$ such that $f(W_j) \subseteq S_j^1$. Thus $f_j(x) = f(x)$ for $x \in W_j$ and therefore, by the Excision Property,

\begin{equation}
i(S_j^1, f_j, S_j^1 - \bar{U}) = i(S_j^1, f_j, W_j) = i(X, f, W_j) = i(X, f, S_j^1 - \bar{U}).
\end{equation}

Since $f(\bar{U}) \subseteq S_j^1$, then $f_1(x) = f(x)$ for all $x \in \bar{U} \cap S_j^1$. There are no fixed points of $f$ in $\bar{U}$, so $i(S_j^1, f_1, S_j^1 \cap \bar{U}) = 0$ and thus $I(f_1) = i(X, f, S_j^1 - \bar{U})$ by (4.3) and (4.4).

For $j \geq 2$, the fact that $f_j(U) = \star$ gives us $i(S_j^1, f_j, S_j^1 \cap U) = 0$ so $I(f_j) = i(X, f, S_j^1 - \bar{U}) + 1$ by (4.3) and (4.4). Since $f_j: S_j^1 \to S_j^1$, the $k = 1$ case of the argument tells us that $I(f_j) = -\deg(f_j) + 1$ for $j = 1, 2, \ldots, k$. In particular, $i(X, f, S_j^1 - \bar{U}) = -\deg(f_1) + 1$ whereas, for $j \geq 2$, we have $i(X, f, S_j^1 - \bar{U}) = -\deg(f_j)$. Therefore, by (4.2),

\begin{equation}
I(f) = i(X, f, S_1^1 - \bar{U}) + \sum_{j=2}^{k} i(X, f, S_j^1 - \bar{U}) = -\sum_{j=1}^{k} \deg(f_j) + 1.
\end{equation}

This completes the proof of Theorem 1.3. \qed

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Hanover, NH 03755-1890, USA
E-mail address: Martin.A.Arkowitz@Dartmouth.edu

Los Angeles, CA 90095-1555, USA
E-mail address: rfb@math.ucla.edu