Modified Born-Jordan Method For Constructing The
Commutation Relation Of Coordinate and Momentum

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Abstract

The Born-Jordan method for constructing the quantum condition of the Matrix Mechanics is pointed out to be inappropriate in the present work. We modify this method and reconstruct the quantum condition by setting up a new expression for the Bohr quantum condition with the help of the \((n, n)\) elements of the matrix \(\oint \hat{p}(t) d\hat{x}(t)\).

I. Introduction

It is well known that the quantum condition expressed as the commutation relation of coordinate and momentum was first constructed in the Matrix Mechanics by M. Born and P. Jordan in paper [1]. On the other hand the Born-Jordan method is inappropriate and needs certain modifications. Owing to the fact that the Born-Jordan method is a generalization of Heisenberg’s method for constructing the Heisenberg Quantum Condition we seek to reveal some features of the drawback of the former method here by analyzing the latter. Consider the special case that was studied by W. Heisenberg in paper [2]. Denote by \(x(t)\) and \(m \dot{x}\) the coordinate and momentum of periodic motion of one-dimensional system. The Bohr ”classical” quantum condition employed by Heisenberg can be written as

\[
J_H = \int_0^{2\pi/\omega(n)} m (\dot{x}(n, t))^2 dt = n h \sum_{\alpha=-\infty}^{\alpha=\infty} (\alpha \omega(n))X_{-\alpha}(n)X_{\alpha}(n),
\]

with

\[
x(n, t) = \sum_{\alpha=-\infty}^{\infty} X_{\alpha}(n) e^{i \omega(n) \alpha t},
\]
where the integer \( n \) stands for the stationary states. According to Heisenberg’s rule \( X_\alpha(n) \) and \( \alpha \omega(n)/(2\pi) \) should be replaced by the transition amplitude \( X_{n,n-\alpha} \) between states \((n, n-\alpha)\) and the transition frequency \( \omega(n, n-\alpha)/(2\pi) \). Moreover \( X_{n,n+\alpha} \) is regarded as \( X_{n,n-\alpha}^* \) in paper [2] so that \( x(n,t) \) is a real function. The formula (1) thus becomes

\[
J_H = 2\pi m \sum_{\alpha=-\infty}^{\infty} \alpha \omega(n, n-\alpha) X_{n,n+\alpha} X_{n,n-\alpha},
\]

\[
= 2\pi m \sum_{\alpha=-\infty}^{\infty} \alpha \omega(n, n-\alpha) X_{n,n-\alpha}^* X_{n,n-\alpha}. \tag{3}
\]

Next Heisenberg replaced equation 1 \( \frac{\partial J_H}{\partial J_H} \) with the difference equation

\[
h = 2\pi m \sum_{\alpha=-\infty}^{\infty} \alpha \frac{\delta}{\delta n} \left( \omega(n, n-\alpha) X_{n,n-\alpha}^* X_{n,n-\alpha} \right),
\]

and obtained his Quantum Condition

\[
h = m \sum_{\alpha=-\infty}^{\infty} \{ X_{n,n+\alpha,n}^* X_{n,n+\alpha,n} \omega(n + \alpha, n) \\
- X_{n,n-\alpha,n}^* X_{n,n-\alpha,n} \omega(n, n-\alpha) \}, \tag{4}
\]

which can be written as

\[
i\hbar = (X^\dagger P - P X^\dagger)_{nn}, \tag{5}
\]

where the \((n, n-\alpha)\) elements of the matrices \( X \) and \( P \) are \( X_{n,n-\alpha} \) and \( P_{n,n-\alpha} \) respectively and

\[
P_{n,n-\alpha} = im \omega(n, n-\alpha) X_{n,n-\alpha}.
\]

However M.Born regarded \( X^\dagger \) in the formula (5) to be \( X \). Actually it is not allowed to do so because the condition \( X_{n,n+\alpha} = X_{n,n-\alpha}^* \) together with the hermitian property of \( X \) leads to

\[
X_{n,n+\alpha} = X_{n,n-\alpha}^* , \quad X_{n+\alpha,n} = X_{n,n-\alpha}.
\]

(6)

Such equations are incorrect and yield wrong results such as \( X_{n,n-1} = X_{1,0} \) and \( X_{n-1,n} = X_{0,1} \). It might be worthwhile to see what happens when the formula (4) together with \( X = X^\dagger \) is applied to the linear harmonic oscillator problem with the Hamiltonian

\[
H = \frac{1}{2m} P^2 + \frac{1}{2} m \omega^2 X^2.
\]

Thus

\[
x_n(t) = X_{n,n-1} e^{i\omega t} + X_{n,n+1} e^{-i\omega t},
\]

\[
\omega(n, n \pm 1) = \mp \omega, \quad \omega(n \pm 1, n) = \pm \omega.
\]

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One therefore has from (4) that

$$\hbar = 2m \omega \left( X_{n+1,n}^* X_{n+1,n} - X_{n-1,n}^* X_{n-1,n} \right).$$

If the matrix $X$ is hermitian then $X_{n,n+1}$ is independent of $n$ and the right hand side of this formula is equal to zero.

What can be learned is that the formula (1) and (3) for the Bohr quantum condition is incompatible with the hermitian property of the matrices $X$ and $P$ and should not be employed to construct the quantum condition of the Matrix Mechanics.

II. The Drawback of the Born-Jordan Method

For carefully examining the drawback of the Born-Jordan method we now analyze the steps of constructing the quantum condition in paper [1]. Let $x(t)$ and $p(t)$ be the canonical coordinate and momentum of periodic motion of one-dimensional system. In the first step, M.Born and P.Jordan followed Heisenberg’s paper [2] to express Bohr’s ”classical” quantum condition. Their formula can be written as:

$$J_{BJ} = \oint p(n,t)dx(n,t) = \int_0^{2\pi/\omega(n)} p(n,t)\dot{x}(n,t)dt = n\hbar + J_0$$

$$= 2\pi m \sum_{\alpha=-\infty}^{\infty} \alpha P_{-\alpha}(n) X_\alpha(n),$$

with

$$x(n,t) = \sum_{\alpha=-\infty}^{\infty} X_\alpha(n)e^{i\omega(n)\alpha t},$$

$$p(n,t) = \sum_{\alpha=-\infty}^{\infty} P_\alpha(n)e^{i\omega(n)\alpha t},$$

Now the underline assumption here is that the coordinate and momentum and their real functions are represented by hermitian matrices, the $(n,n')$ elements of $\dot{x}(t)$ and $\dot{p}(t)$ are as follows

$$x_{nn'}(t) = X_{n,n'} e^{i\omega(n,n')t}, \quad p_{nn'}(t) = P_{n,n'} e^{i\omega(n,n')t}.$$  

After replacing $X_\alpha(n)$ and $P_\alpha(n)$ by $X_{n,n-\alpha}$ and $X_{n,n-\alpha}$ the formula (7) becomes

$$J_{BJ} = 2\pi i \sum_{\alpha=-\infty}^{\infty} \alpha P_{n,n+\alpha} X_{n,n-\alpha} = -2\pi i \sum_{\alpha=-\infty}^{\infty} \alpha P_{n,n-\alpha} X_{n,n+\alpha}. $$
Step (2), The equation \( 1 = \frac{\partial J}{\partial J_{BJ}} \) was replaced with the difference equation

\[
h = 2\pi i \sum_{\alpha = -\infty}^{\infty} \alpha \frac{\delta}{\delta n} \left( P_{n,n+\alpha} X_{n,n-\alpha} \right),
\]

which was interpreted as

\[
i \hbar = \sum_{\alpha = -\infty}^{\infty} \left( P_{n-\alpha,n} X_{n,n-\alpha} - P_{n,n+\alpha} X_{n+\alpha,n} \right),
\]

or

\[
i \hbar = (X P - P X)_{nn}.
\]

For the case \( p(t) = m \dot{x}(t) \), substituting \( P_{n-\alpha,n} = i m \omega(n-\alpha,n) X_{n-\alpha,n} \) and \( P_{n,n+\alpha} = P_{n+\alpha,n}^* \) in equation (12) leads to

\[
\hbar = m \sum_{\alpha = -\infty}^{\infty} \left\{ X_{n,n+\alpha} X_{n+\alpha,n} \omega(n+\alpha,n) - X_{n,n-\alpha} X_{n-\alpha,n} \omega(n,n-\alpha) \right\},
\]

which was regarded as Heisenberg’s quantum condition or Thomas-Kuhn equation\(^{[3,4]}\).

Step (3), With the help of the equation of motion M.Born and P.Jordan came to the conclusion that the matrix \( \dot{x}(t) \dot{p}(t) - \dot{p}(t) \dot{x}(t) \) is independent of time and is diagonal.

The first two steps sought to carry through and extend Heisenberg’s method of paper [2] under the condition that matrices \( X \) and \( P \) are hermitian. The matter is not so simple However. Obviously, further assumptions should be introduced for getting the formula (12) from (11). It seems that the assumption \( X_{n,n+\alpha} = X_{n-\alpha,n} \) or \( P_{n,n+\alpha} = P_{n-\alpha,n} \) is useful for going from equation (11) to (12) but they are unreasonable themselves. As stated in Sec.I \( X_{n,n+\alpha} = X_{n-\alpha,n} \), which together with \( X = X^\dagger \) means \( x(n,t) = x^*(n,t) \), leads to wrong results and therefore makes the formula (12) invalid. Similarly, \( P_{n,n+\alpha} = P_{n-\alpha,n} \), which together with \( P = P^\dagger \) means \( P(n,t) = P^*(n,t) \), also yield \( P_{n,n+1} = P_{0,1} \) and other wrong results and makes the formula (12) invalid. In particular, if both assumptions \( x_{n,n+\alpha} = X_{n-\alpha,n} \) and \( P_{n,n+\alpha} = P_{n-\alpha,n} \) are employed then the right hand side of the formula (12) equates to zero.

Actually, it can be verified that the Born-Jordan method is not available for constructing the quantum condition of the Matrix Mechanics. For the special case \( p(t) = m \dot{x}(t) \), the formula (10) becomes

\[
J_{BJ} = 2\pi m \sum_{\alpha = -\infty}^{\infty} \alpha \omega(n,n-\alpha) X_{n,n+\alpha} X_{n,n-\alpha},
\]

which should not be regarded as Heisenberg’s formula (3) because \( X \) is hermitian. Let us now suppose the formula (14) can be obtained reasonably from (15) and consider again the linear harmonic oscillator
problem one has

\[ \hbar = 2 m \omega \left\{ X_{n+1,n} X_{n,n+1} - X_{n,n-1} X_{n-1,n} \right\}. \quad (16) \]

From this formula and other principles of the Matrix Mechanics the coordinate momentum commutation relation can be established (see the end of Sect.3) and the elements of the Matrix \( X \) can also be found. On the other hand the simplified form of the formula (15) is

\[ J_{BJ} = 2 \pi m \sum_{\alpha = \pm 1} \alpha \omega(n,n) X_{n,n-\alpha} X_{n,n+\alpha}. \]

Consequently the Quantum Condition can also be written as

\[ \hbar = m \omega \sum_{\alpha = \pm 1} \alpha \left\{ X_{n+\alpha,n} X_{n+\alpha,n+2\alpha} - X_{n,n-\alpha} X_{n,n+\alpha} \right\} = m \omega \left\{ X_{n+1,n} X_{n+1,n+2} - X_{n-1,n} X_{n-1,n-2} \right\}. \]

It can easily be checked that this formula is invalid.

To sum up, with the Born-Jordan method it is not possible to construct the quantum condition reasonably even for linear harmonic oscillator problem.

### III. Modification of the Born-Jordan Method and the Reconstruction of the Quantum Condition

In order to modify the Born-Jordan Method and reconstruct the quantum condition, it is natural to express the Bohr quantum condition with reference to the \((n, n)\) element of the matrix \( \oint p(t) d\hat{x}(t) \). Since

\[ \left( \hat{p}(t) d\hat{x}(t) \right)(n, n) = - \sum_{\alpha} P_{n,-\alpha,n} X_{n-\alpha,n} i \omega(n,n-\alpha) dt, \quad (17) \]

\[ \left( \hat{p}(t) d\hat{x}(t) \right)^*(n, n) = \sum_{\alpha} P_{n,-\alpha,n} X_{n,n-\alpha} i \omega(n,n-\alpha) dt, \quad (18) \]

\[ \int_{0}^{2\pi/\omega(n)} p(n,t) \hat{x}^*(n,t) dt = -2\pi i \sum_{\alpha} \alpha P_{\alpha}(n) X_{\alpha}^{*}(n), \]

\[ = -2\pi i \sum_{\alpha} \alpha P_{n,-\alpha} X_{n,n-\alpha}^{*}, \quad (19) \]

and \( \omega(n,n-\alpha) \) corresponds to the classical quantities \( \alpha \omega(n) \), one sees that \( \oint (\hat{p}(t) d\hat{x}(t))(n, n) \) corresponds to \( \oint p(n,t) dx^*(n,t) \). It is thus reasonable to write the Bohr quantum condition as:

\[ J = \frac{1}{2} \left\{ \oint p^*(n,t) dx(n,t) + \oint p(n,t) dx^*(n,t) \right\} = n \hbar + J_0, \quad (20) \]
It is obvious from (17) that \( \frac{\delta}{\delta n} \oint p(n, t) dx^*(nt) \) is real, namely

\[
\frac{\delta}{\delta n} \oint p(n, t) dx^*(n, t) = \oint p^*(n, t) dx(n, t) = -2\pi i \sum_{\alpha} P_{n+\alpha,n} X_{n,n+\alpha} + 2\pi i \sum_{\alpha} P_{n,n-\alpha} X_{n-\alpha,n} .
\]  

Therefore \( 1 = \frac{\delta J}{\delta J} \) is equivalent to \( h = \frac{\delta}{\delta n} \oint p(n, t) dx^*(n, t) \) and the Quantum Condition is obtained from the difference equation

\[
h = \frac{\delta}{\delta n} \oint p(n, t) dx^*(n, t) ,
\]

or

\[
i\hbar = \sum_{\alpha=-\infty}^{\infty} (P_{n+\alpha,n} X_{n,n+\alpha} - P_{n,n+\alpha} X_{n+\alpha,n}) = (XP - PX)_{nn} .
\]

For the case \( p(t) = m \dot{x}(t) \), the above new expression for the Bohr quantum condition becomes

\[
J = n h + J_0 = \oint p^*(n, t) dx(n, t) = \oint p(n, t) dx^*(n, t) = 2m\pi \sum_{\alpha=-\infty}^{\infty} \alpha \omega(n, n - \alpha) X_{n,n-\alpha} X_{n-\alpha,n} .
\]

It follows from the formula (22) or (23) that

\[
h = m \sum_{\alpha=-\infty}^{\infty} \{X_{n,n+\alpha} X_{n+\alpha,n} \omega(n + \alpha, n) - X_{n,n-\alpha} X_{n-\alpha,n} \omega(n, n - \alpha)\} .
\]

It should be remembered that owing to \( X_{n\pm\alpha,n} = X_{n,n\pm\alpha}^* \neq X_{n,n\mp\alpha} \) this formula is not equivalent to Heisenberg’s quantum condition.

Finally it is an easy task to establish the commutation relation \( (XP - PX) = i\hbar \) from it’s diagonal part by improving Born-Jordan’s argument (see also the footnote (58) of reference [5]). In their paper [1] M.Born and P.Jordan judged the conservation matrix \( \dot{x}(t)\dot{p}(t) - \dot{p}(t)\dot{x}(t) \) to be diagonal by employing an assumption that \( \omega(n, n') \neq 0 \) for \( n \neq n' \). However B.L.Van Der Waerden pointed out in reference [6] that since Born-Jordan’s argument relied on this assumption they did not give an exact mathematical proof of the formula for \( (XP - PX) \). Actually, the whole independent stationary states \( \{n\} \) can be defined to form a representation so that both of the conservation quantity \( (\dot{x}\dot{p} - \dot{p}\dot{x})/(i\hbar) \) and the Hamiltonian are diagonal. Consequently one sees from the formula (23) that \( (XP - PX)/(i\hbar) \) is an unit matrix.
IV. Concluding Remarks

Historically (see Waerden’s paper of reference [6]), M.Born rewrote Heisenberg’s Quantum Condition as $i\hbar = (X P - PX)_{n,n}$ and guessed the matrix $(X P - PX)$ to be diagonal. One of the main tasks of Born-Jordan’s paper [1] is to verify M.Born’s guess. Correspondingly, in the present work we first verify that Heisenberg’s Quantum Condition can not be rewritten in Born’s way and therefore the formula (3) should not be applied to the Matrix Mechanics. We then show that Born-Jordan method, which is based on the formulas (7), can not be employed to construct the quantum condition of the Matrix Mechanics. It is also clear from our argument that even if Heisenberg’s assumption $X_{n,n-\alpha} = X^*_{n,n+\alpha}$ is excluded, his formula (3) could not be applied to the Matrix Mechanics.

Furthermore a new formula is setting up to express the Bohr quantum condition with the help of the $(n, n)$ elements of the matrix $\oint \hat{p}(t)d\hat{x}(t)$. This new formula is employed to modify the Born-Jordan method and reconstruct the quantum condition.

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