On the asymptotic densities of certain subsets of \( N^k \)

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Abstract

We determine the asymptotic density \( \delta_k \) of the set of ordered \( k \)-tuples \((n_1, \ldots, n_k) \in N^k, k \geq 2, \) such that there exists no prime power \( p^a, a \geq 1, \) appearing in the canonical factorization of each \( n_i, 1 \leq i \leq k, \) and deduce asymptotic formulae with error terms regarding this problem and analogous ones. We give numerical approximations of the constants \( \delta_k \) and improve the error term of formula (1.2) due to E. Cohen.

We point out that our treatment, based on certain inversion functions, is applicable also in case \( k = 1 \) in order to establish asymptotic formulae with error terms regarding the densities of subsets of \( N \) with additional multiplicative properties. These generalize an often cited result of G. J. Rieger.

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1 Introduction

Let \( k \geq 2 \) be a fixed integer. What is the asymptotic density \( \delta_k \) of the set of ordered \( k \)-tuples \((n_1, \ldots, n_k) \in N^k, \) such that there exists no prime power \( p^a, a \geq 1, \) appearing in the canonical factorization of each \( n_i, 1 \leq i \leq k, \) ?

This problem is analogous to the following one: What is the asymptotic density \( d_k \) of the set of \( k \)-tuples which are relatively prime, i.e. \( k \)-tuples \((n_1, \ldots, n_k) \in N^k \) such that there exists no prime \( p, \) appearing in the canonical factorization of each \( n_i, 1 \leq i \leq k, \) ?

It is known that \( d_k = 1/\zeta(k), \) where \( \zeta \) is the Riemann zeta function, and this value can be considered as the probability that \( k \) integers \((k \geq 2)\) chosen at random are relatively prime. More precisely,

\[
N_k(x) := \#\{(n_1, \ldots, n_k) \in (N \cap [1, x])^k : \gcd(n_1, \ldots, n_k) = 1\} = \frac{1}{\zeta(k)}x^k + R_k(x),
\]

where \( R_k(x) = O(x^{k-1}) \) for \( k > 2, \) \( R_2(x) = O(x \log x) \) for \( k = 2, \) and \( d_k = \lim_{x \to \infty} N_k(x)/x^k = 1/\zeta(k). \) This result goes back to the work of J. J. Sylvester [9] and D. N. Lehmer [3], see also J. E. Nymann [5].
There are several generalizations of (1.1) in the literature. For example, let $S$ be an arbitrary subset of $N$. Then

\[(1.2) \; N_k(x, S) := \#\{(n_1, \ldots, n_k) \in (N \cap [1, x])^k : \gcd(n_1, \ldots, n_k) \in S\} = \frac{\zeta_S(k)}{\zeta(k)} x^k + T_k(x),\]

where

\[\zeta_S(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}\]

and $T_k(x) = O(x^{k-1})$ for $k > 2$, $T_2(x) = O(x \log^2 x)$ for $k = 2$, for every $S \subseteq N$, due to E. Cohen [1]. Therefore the asymptotic density of the set of ordered $k$-tuples $(n_1, \ldots, n_k)$ for which $\gcd(n_1, \ldots, n_k)$ belongs to $S$ is $\lim_{x \to \infty} N_k(x, S)/x^k = \frac{\zeta_S(k)}{\zeta(k)}$.

J. E. Nymann [6] shows that if the characteristic function $\rho_S$ of $\emptyset \neq S \subseteq N$ is completely multiplicative and if $\#\{n : n \in S \cap [1, x]\} = Ax + O(1)$, where $A$ is the asymptotic density of $S$, then

\[(1.3) \; \#\{(n_1, \ldots, n_k) \in (S \cap [1, x])^k : \gcd(n_1, \ldots, n_k) = 1\} = A^k \prod_{p \in S} \left(1 - \frac{1}{p^k}\right) x^k + R_k(x),\]

where $R_k(x)$ is the same as above. Therefore, if $P_S^k(n)$ denotes the probability that $k$ integers ($k \geq 2$) chosen at random from $S \cap [1, n]$ are relatively prime, then

\[\lim_{n \to \infty} P_k^S(n) = \prod_{p \in S} \left(1 - \frac{1}{p^k}\right).\]

This result can be applied for $S = \{n : \gcd(n, p_1 \cdots p_r) = 1\}$, where $\{p_1, \ldots, p_r\}$ is a given finite set of distinct primes.

Now return to the problem at the beginning. It is obvious that $\delta_k \geq d_k = 1/\zeta(k)$ for every $k > 2$ and thus $\lim_{k \to \infty} \delta_k = 1$. Which is the exact value of $\delta_k$?

In order to solve this problem we use the concept of the unitary divisor. For $d, n \in N$, $d$ is called a unitary divisor (or block divisor) of $n$ if $d|n$ and $\gcd(d, n/d) = 1$, notation $d\|n$. Various other problems concerning unitary divisors, including properties of arithmetical functions and arithmetical convolutions defined by unitary divisors, have been studied extensively in the literature, see for example [4] and its bibliography. Denote the greatest common unitary divisor of $n_1, \ldots, n_k$ by $\text{gcud}(n_1, \ldots, n_k)$.

Our question can be reformulated in this way: What is the asymptotic density $\delta_k$ of the set of ordered $k$-tuples $(n_1, \ldots, n_k)$ such that $\text{gcud}(n_1, \ldots, n_k) = 1$, or more generally, $\text{gcud}(n_1, \ldots, n_k) \in S$ ?

Furthermore, what is the probability that for $k$ integers $n_1, \ldots, n_k$ chosen at random from $S \cap [1, n]$ one has $\text{gcud}(n_1, \ldots, n_k) = 1$ ?

In this paper we determine the value $\delta_k$ and deduce asymptotic formulae with error terms analogous to (1.1) - (1.3), regarding these problems. We give numerical approximations of the constants $\delta_k$ and also improve the error term of (1.2) of E. Cohen.

The treatment we use is based on the inversion functions $\mu_S^k$ and $\mu_S$ attached to the subset $S$. We point out that this is applicable also in case $k = 1$ in order to establish asymptotics regarding the densities of certain subsets $S$ of $N$, generalizing in this way an often cited result of G. J. Rieger [7].
Note that the value $\delta_2$ is given by D. Suryanarayana and M. V. Subbarao [8], Corollary 3.6.3, applying other arguments as those of the present paper.

Using the concept of regular cross-convolution, see [11], [12], it is possible to deduce more general results, including (1.1) - (1.3) and (2.1) and (2.4) of this paper. We do not go into details.

2 Results

Let $S \subseteq \mathbb{N}$. We say that $S$ is (completely) multiplicative if $1 \in S$ and its characteristic function $\rho_S(n)$ is (completely) multiplicative. Define the function $\mu^*_S(n)$ by

$$\sum_{d|n} \mu^*_S(d) = \rho_S(n), \quad n \in \mathbb{N},$$

that is

$$\mu^*_S(n) = \sum_{d|n} \rho_S(d) \mu^*(n/d), \quad n \in \mathbb{N},$$

where the sums are extended over the unitary divisors of $n$ and $\mu^*(n) := \mu^*_1(n) = (-1)^{\omega(n)}$, $\omega(n)$ denoting the number of distinct prime factors of $n$.

Furthermore, let $\phi(n)$ and $\theta(n)$ denote Euler’s function and the number of squarefree divisors of $n$, respectively.

Theorem 2.1 If $k \geq 2$ and $S$ is an arbitrary subset of $\mathbb{N}$, then

$$(2.1) \quad \#\{(n_1, ..., n_k) \in (\mathbb{N} \cap [1, x])^k : \gcd(n_1, ..., n_k) \in S\} = \delta_k(S)x^k + V_k(x, S),$$

where

$$\delta_k(S) = \sum_{n=1}^{\infty} \frac{\mu^*_S(n)\phi^k(n)}{n^{2k}}$$

and the remainder term can be evaluated as follows:

1. $V_k(x, S) = O(x^{k-1})$ for $k > 2$ and for an arbitrary $S$,
2. $V_2(x, S) = O(x \log^4 x)$ for an arbitrary $S$,
3. $V_2(x, S) = O(x \log^2 x)$ for an $S$ such that $\sum_{n \in S} \frac{\theta(n)}{n} < \infty$ (in particular for every finite $S$) and for every multiplicative $S$,
4. $V_2(x, S) = O(x)$ for every multiplicative $S$ such that $\sum_{p \notin S} \frac{1}{p} < \infty$ (in particular if the set $\{p : p \notin S\}$ is finite).

If $S$ is multiplicative, then

$$\delta_k(S) = \prod_p \left(1 - (1 - \frac{1}{p})^k \sum_{a=1}^{\infty} \frac{1}{p^{ak}}\right).$$

If $S = \{1\}$, then

$$\delta_k := \delta_k(\{1\}) = \prod_p \left(1 - \frac{(p-1)^k}{p^k(p^k-1)}\right).$$
Theorem 2.2 If $k \geq 2$ and $S$ is an arbitrary subset of $\mathbb{N}$, then the asymptotic densities of the sets of ordered $k$-tuples $(n_1, ..., n_k)$ such that $\gcd(n_1, ..., n_k) \in S$ and $\gcd(n_1, ..., n_k) = 1$ are $\delta_k(S)$ and $\delta_k$, respectively, given in Theorem 2.1.

Theorem 2.3 Let $p_n$ denote the $n$-th prime and for $r \in \mathbb{N}$ let $N = 10^r/2$. Then

$$\delta_k \approx \prod_{n=1}^{N} \left(1 - \frac{(p_n - 1)^k}{p_n^k (p_n^k - 1)}\right)$$

is an approximation of $\delta_k$ with $r$ exact decimals.

In particular, $\delta_2 \approx 0.8073, \delta_3 \approx 0.9637, \delta_4 \approx 0.9924, \delta_5 \approx 0.9963, \delta_6 \approx 0.9996, \delta_7 \approx 0.9999$, with $r = 4$ exact decimals.

Theorem 2.4 For $k = 2$ the error term $R_2(x)$ of (1.2) can be improved into $R(x, S)$, where

(i) $R(x, S) = O(x \log x)$ for an $S$ such that $\sum_{n \in S} \frac{1}{n} < \infty$ (in particular for every finite $S$) and for every multiplicative $S$,

(ii) $R(x, S) = O(x)$ for every multiplicative $S$ such that $\sum_{p \notin S} \frac{1}{p} < \infty$ (in particular if the set $\{p : p \notin S\}$ is finite).

Remark. It is noted in [1] that if $k = 2$ and if the function $\mu_S(n)$ is bounded, cf. proof of Theorem 2.4 of the present paper, then the error term is $R_2(x) = O(x \log x)$.

Theorem 2.5 Suppose that $S \subseteq \mathbb{N}$ is multiplicative and $\min\{a : p^a \notin S\} \geq r \geq 2$ for every prime $p$. Then

$$\sum_{n \leq x} \rho_S(n) = d(S)x + O(\sqrt{x}),$$

where

$$d(S) = \prod_{p} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{a=1}^{\infty} \frac{1}{p^a}\right)$$

is the asymptotic density of $S$.

Remark. In the special case $S$ = the set of $K$-void integers we reobtain from (2.2) the result of G. J. Rieger [7]. The $K$-void integers are defined as follows. Let $K$ be a nonempty subset of $\mathbb{N} \setminus \{1\}$. The number $n$ is called $K$-void if $n = 1$ or $n > 1$ and there is no prime power $p^a$, with $a \in K$, appearing in the canonical factorization of $n$.

Does the density exist for an arbitrary multiplicative subset $S$? Yes, and it is $d(S)$ given by (2.3), where the infinite product is considered to be 0 when it diverges (if and only if $\sum_{p \notin S} \frac{1}{p} = \infty$). This follows from a well-known result of E. Wirsing [13] concerning the mean-values of certain multiplicative functions $f$. A short direct proof for the case $f$ multiplicative and $0 \leq f(n) \leq 1$ for $n \geq 1$, hence applicable for the characteristic function of an arbitrary multiplicative $S$, is given in the book of G. Tenenbaum, [10], p. 48.

Theorem 2.6 Let $k \geq 2$ and suppose that $S$ is a completely multiplicative subset of $\mathbb{N}$ such that $\#\{n : n \in S \cap [1, x]\} = Ax + O(1)$. Then

$$\#\{(n_1, ..., n_k) \in (S \cap [1, x])^k : \gcd(n_1, ..., n_k) = 1\} = A^k \beta_k(S) x^k + T_k(x),$$
\[ \beta_k(S) = \prod_{p \in S} \left(1 - \frac{(p-1)^k}{p^k(p^k-1)}\right), \]

and \( T_k(x) = O(x^{k-1}) \) for \( k > 2 \), \( T_2(x) = O(x \log^2 x) \) for \( k = 2 \).

If \( Q_k^S(n) \) denotes the probability that for \( k \) integers \( n_1, \ldots, n_k \) chosen at random from \( S \cap [1, n] \) one has \( \gcd(n_1, \ldots, n_k) = 1 \), then

\[ \lim_{n \to \infty} Q_k^S(n) = \beta_k(S). \]

3 Proofs

Proof of Theorem 2.1 Using the definition of \( \mu_S^* \), the fact that \( d \| \gcd(n_1, \ldots, n_k) \) if and only if \( d \| n_i \) for every \( 1 \leq i \leq k \), which can be checked easily, and the well-known estimate

\[ \sum_{n \leq x} \frac{1}{\gcd(n, m)} = \frac{\varphi(m)x}{m} + O(\theta(m)) \]

which holds uniformly for \( x \geq 1 \) and \( m \in \mathbb{N} \), we obtain

\[ \#\{n_1, \ldots, n_k \in (\mathbb{N} \cap [1, x])^k : \gcd(n_1, \ldots, n_k) \in S\} = \sum_{n_1, \ldots, n_k \leq x} \rho_S(\gcd(n_1, \ldots, n_k)) = \]

\[ = \sum_{n_1, \ldots, n_k \leq x} \sum_{d \| (n_1, \ldots, n_k)} \mu_S^*(d) = \sum_{n_1, \ldots, n_k \leq x} \sum_{d \| n_1, \ldots, d \| n_k} \mu_S^*(d) = \]

\[ = \sum_{d \leq x} \mu_S^*(d) \sum_{1 \leq e \leq x/d \leq x} \varphi(e/d) = \sum_{d \leq x} \mu_S^*(d) \left( \sum_{1 \leq e \leq x/d} \frac{\varphi(e/d)}{d} \right)^k = \]

\[ = x^k \sum_{d \leq x} \mu_S^*(d) \left( \frac{x \varphi(d)}{d^2} + O(\theta(d)) \right)^k = \sum_{d \leq x} \mu_S^*(d) \left( \frac{x^k \varphi^k(d)}{d^{2k}} + O(\frac{x^{k-1} \varphi(d)}{d^{k-1}}) \right) = \]

\[ = \delta_k(S)x^k + O \left( x^k \sum_{d > x} \frac{\mu_S^*(d)}{d^k} \right) + O \left( x^{k-1} \sum_{d > x} \frac{\mu_S^*(d)}{d^{k-1}} \right). \]

The given error term yields now from the next statements:

(a)

\[ \sum_{n \leq x} \frac{\theta(n)}{n^s} = \begin{cases} O(\log^2 x), & s = 1, \\ O(1), & s > 1. \end{cases} \]

\[ \sum_{n \leq x} \frac{\theta^2(n)}{n^s} = \begin{cases} O(\log^4 x), & s = 1, \\ O(1), & s > 1, \end{cases} \]

\[ \sum_{n > x} \frac{1}{n^s} = O(\frac{1}{x^{s-1}}), \quad \sum_{n > x} \frac{\theta(n)}{n^s} = O(\frac{\log x}{x^{s-1}}), \quad s > 1. \]
(b) For an arbitrary $S \subseteq \mathbb{N}$ and for every $n \in \mathbb{N}$, $|\mu_S^*(n)| \leq \sum_{d|n} \rho_S(d) \leq \theta(n)$, $|\mu_S^*(n)|\theta(n) \leq \sum_{d|n} \rho_S(d)\theta(d)\theta(n/d)$ and
\[
\sum_{n \leq x} \frac{|\mu_S^*(n)|\theta(n)}{n} \leq \sum_{d \leq x} \frac{\rho_S(d)\theta(d)}{d} \sum_{e \leq x/d} \frac{\theta(e)}{e} =
\]
\[
= O \left( \log^2 x \sum_{d \leq x} \frac{\rho_S(d)\theta(d)}{d} \right) = \left\{ O(\log^2 x), \quad \text{if} \quad \sum_{n=1}^{\infty} \frac{\rho_S(n)\theta(n)}{n} < \infty, \right. \\
\left. O(\log^4 x), \quad \text{otherwise}. \right\}
\]

(c) If $S$ is multiplicative, then $\mu_S^* = \mu_S^*$ is multiplicative too, $\mu_S^*(p^a) = \rho_S(p) - 1$ for every prime power $p^a$ ($a \geq 1$) and $\mu_S^*(n) \in \{-1, 0, 1\}$ for each $n \in \mathbb{N}$.

(d) Suppose $S$ is multiplicative. Then
\[
\sum_{p} \sum_{a=1}^{\infty} \frac{|\mu_S^*(p^a)|\theta(p^a)}{p^a} = 2 \sum_{p} \sum_{a=1}^{\infty} \frac{1 - \rho_S(p^a)}{p^a} \leq
\]
\[
\leq 2 \sum_{p} \left( \frac{1 - \rho_S(p)}{p} + \sum_{a=2}^{\infty} \frac{1}{p^a} \right) = 2 \sum_{p \in S} \frac{1}{p(p-1)} + 2 \sum_{p \not\in S} \frac{1}{p-1} \leq
\]
\[
\leq 4 \left( \sum_{p \in S} \frac{1}{p^2} + \sum_{p \not\in S} \frac{1}{p} \right) < \infty \quad \text{if} \quad \sum_{p \not\in S} \frac{1}{p} < \infty.
\]
It follows that in this case the series $\sum_{n=1}^{\infty} \frac{|\mu_S^*(n)|\theta(n)}{n}$ is convergent.

If $S$ is multiplicative, then the series giving $\delta_k(S)$ can be expanded into an infinite product of Euler-type.

**Proof of Theorem 2.2** This is a direct consequence of Theorem 2.1.

**Proof of Theorem 2.3** Consider the series of positive terms
\[
\sum_{p} \log \left( 1 - \frac{(p-1)^k}{p^k(p^k-1)} \right)^{-1} =
\]
\[
= \sum_{n=1}^{\infty} \log \left( 1 + \frac{(p_n-1)^k}{p_n^k(p_n^k-1) - (p_n-1)^k} \right) = -\log \delta_k,
\]
where $p_n$ denotes the $n$-th prime.

The $N$-th order error $R_N$ of this series can be evaluated as follows:
\[
R_N := \sum_{n=N+1}^{\infty} \log \left( 1 + \frac{(p_n-1)^k}{p_n^k(p_n^k-1) - (p_n-1)^k} \right) < \sum_{n=N+1}^{\infty} \frac{(p_n-1)^k}{p_n^k(p_n^k-1) - (p_n-1)^k} <
\]
\[
< \sum_{n=N+1}^{\infty} \frac{1}{p_n^k - 1} \leq \sum_{n=N+1}^{\infty} \frac{1}{p_n^k} = \sum_{n=N+1}^{\infty} \frac{1}{p_n^k - 1}.
\]
Now using that $p_n > 2n$, valid for $n \geq 5$, we have
\[
R_N < \sum_{n=N+1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2} \sum_{n=N+1}^{\infty} \left( \frac{1}{2n-1} - \frac{1}{2n+1} \right) = \frac{1}{2(2N + 1)}.
\]
In order to obtain an approximation with \( r \) exact decimals we use the condition

\[
\frac{1}{22(2N + 1)} \leq \frac{1}{2} \cdot 10^{-r}
\]

and obtain \( N \geq \frac{1}{2}(10^r - 1) \).

The numerical values were obtained using the software package MAPLE.

**Proof of Theorem 2.4** Define the function \( \mu_S(n) \) by

\[
\sum_{d|n} \mu_S(d) = \rho_S(n), \quad n \in \mathbb{N},
\]

that is

\[
\mu_S(n) = \sum_{d|n} \rho_S(d)\mu(n/d), \quad n \in \mathbb{N},
\]

where \( \mu(n) := \mu\{1\}(n) \) is the Möbius function, see [1]. We have

\[
N_k(x, S) := \#\{(n_1, \ldots, n_k) \in (\mathbb{N} \cap [1, x])^k : \gcd(n_1, \ldots, n_k) \in S\} = \\
= \sum_{n_1, \ldots, n_k \leq x} \rho_S(\gcd(n_1, \ldots, n_k)) = \sum_{n_1, \ldots, n_k \leq x} \sum_{d|n} \mu_S(d)
\]

and the proof runs parallel to the proof of Theorem 2.1.

**Proof of Theorem 2.5**

\[
N_1(x, S) = \sum_{n \leq x} \rho_S(n) = \sum_{n \leq x} \sum_{d|n} \mu_S(d) = x \sum_{d \leq x} \frac{\mu_S(d)}{d} + O(\sum_{d \leq x} |\mu_S(d)|).
\]

Here \( \mu_S \) is multiplicative, \( \mu_S(p^a) = \rho_S(p^a) - \rho_S(p^{a-1}) \), \( a \geq 1 \) and since \( p, p^2, \ldots, p^{r-1} \in S \) we have \( \mu_S(p) = \mu_S(p^2) = \ldots = \mu_S(p^{r-1}) = 0 \) for every prime \( p \). Hence for each \( n \in \mathbb{N} \), \( |\mu_S(n)| \leq \rho_{L_r}(n) \), where \( L_r \) is the set of \( r \)-full numbers, i.e. \( L_r = \{1\} \cup \{n > 1 : p|n \Rightarrow p^r|n\} \).

We get

\[
N_1(x, S) = d(S)x + O(x \sum_{d > x} \frac{\rho_{L_r}(d)}{d}) + O(\sum_{d \leq x} \rho_{L_r}(d)),
\]

and using the elementary estimate

\[
\sum_{n \leq x} \rho_{L_r}(n) = C \sqrt{x} + O(r^{1/2} \sqrt{x}),
\]

where \( C \) is a positive constant, due to P. Erdős and G. Szekeres [2], obtain the given result.

**Proof of Theorem 2.6**

\[
\#\{(n_1, \ldots, n_k) \in (S \cap [1, x])^k : \gcd(n_1, \ldots, n_k) = 1\} = \\
= \sum_{n_1 \leq x} \rho_S(n_1) \sum_{n_k \leq x} \rho_S(n_k) \sum_{d|\gcd(n_1, \ldots, n_k)} \mu^*(d) = \sum_{n_1 \leq x} \rho_S(n_1) \sum_{n_k \leq x} \rho_S(n_k) \sum_{d|\gcd(n_1, \ldots, n_k)} \mu^*(d) = \\
= \sum_{d \leq x} \mu^*(d) \sum_{a_1 \leq x/d} \rho_S(da_1) \ldots \sum_{a_k \leq x/d} \rho_S(da_k)
\]
\[
\sum_{d \leq x} \rho_S(d) \mu^*(d) \left( \sum_{\substack{a \leq x/d \, \text{gcd}(a,d)=1}} \rho_S(a) \right)^k.
\]

Here we use the estimate, valid for every \(\ell \in \mathbb{N}\),
\[
\sum_{\substack{n \leq x \, \text{gcd}(n,\ell)=1}} \rho_S(n) = \sum_{n \leq x} \rho_S(n) \sum_{d \mid \text{gcd}(n,\ell)} \mu(d) =
\sum_{d \mid \ell} \mu(d) \sum_{e \leq x/d} \rho_S(e) = \sum_{d \mid \ell} \mu(d) \rho_S(d) \left( A \frac{x}{d} + O(1) \right) =
Ax \prod_{p \mid \ell} \left( 1 - \frac{1}{p} \right) + O(\theta(\ell))
\]
and obtain the desired result, see the proof of Theorem 2.1.

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