Defects in affine Toda field theories

C. Zambon \(^1\)

Laboratoire de Physique Théorique et Modélisation
Université de Cergy-Pontoise (CNRS UMR 8089), Saint-Martin 2
2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France

ABSTRACT

In this talk some classical and quantum aspects concerning a special kind of integrable defect - called a jump-defect - will be reviewed. In particular, recent results obtained in an attempt to incorporate this defect in the affine Toda field theories, in addition to the sine-Gordon model, will be presented.

\(^1\)E-mail: cristina.zambon@ptm.u-cergy.fr
1 Jump-defects

The jump-defect is a purely transmitting defect, which can be incorporated in certain integrable field theories in such a way as to allow the integrability of the system to be preserved. Its existence has been proved originally for the sine-Gordon model, both in the classical \cite{[1]} and in the quantum context \cite{[2]}, and subsequently extended to other integrable systems (see for instance \cite{[3]}). From the very start, the jump-defect has displayed interesting features, which appear to be quite different from the ones enjoyed by a typical $\delta$-type impurity.

Consider two free massive scalar fields $\phi(x,t), x < 0$ and $\psi(x,t), x > 0$, with Lagrangian density given by

$$ L = \theta(-x)L_\phi + \theta(x)L_\psi + \delta(x)D(\phi, \psi). \quad (1.1) $$

The terms $L_\phi$ and $L_\psi$ represent the bulk Lagrangian densities $(-\infty < x < \infty)$ for the fields $\phi$ and $\psi$ respectively, while $D$ defines the condition for the defect, which is located at $x = 0$. The function $D$ can be chosen in many ways. For instance, for a $\delta$-type impurity

$$ D = -\frac{1}{2}[\sigma \phi \psi - (\phi_x + \psi_x)(\phi - \psi)], \quad (1.2) $$

which leads to the following set of equations of motion and defect conditions

$$ \partial^2 \phi = -m^2 \quad x < 0, \quad \phi = \psi \quad x = 0, $$
$$ \partial^2 \psi = -m^2 \quad x > 0, \quad \psi_x - \phi_x = \sigma \phi \quad x = 0. \quad (1.3) $$

Because of the presence of a defect, which breaks the space translation invariance, it can be verified that momentum is not conserved. Moreover, the system described by the Lagrangian density (1.1) with defect term (1.2) allows both transmission and reflection and possesses a bound state, provided $\sigma < 0$.

On the other hand, a defect term, which defines the condition for a jump-defect reads

$$ D = \frac{1}{2} (\phi \psi_t - \psi \phi_t) + \frac{m \sigma}{4} (\phi + \psi)^2 + \frac{m}{4 \sigma} (\phi - \psi)^2. \quad (1.4) $$

As a consequence, the Lagrangian density (1.1) with $D$ given by (1.4) leads to the following set of equations

$$ \partial^2 \phi = -m^2 \quad x < 0, \quad \partial_x \phi - \partial_t \psi = -\sigma \left(\frac{\phi + \psi}{2}\right) - \frac{1}{\sigma} \left(\frac{\phi - \psi}{2}\right) \quad x = 0, $$
$$ \partial^2 \psi = -m^2 \quad x > 0, \quad \partial_x \psi - \partial_t \phi = \sigma \left(\frac{\phi + \psi}{2}\right) - \frac{1}{\sigma} \left(\frac{\phi - \psi}{2}\right) \quad x = 0. \quad (1.5) $$

This time, surprisingly, it was discovered that momentum is conserved \cite{[1]}, provided a suitable contribution from the defect is added. Besides, the system described by equations (1.5) is purely transmitting and does not have a bound state. Finally, it is worth pointing out that contrary to the $\delta$-type impurity situation (1.3), if there is a jump-defect, the two fields $\phi$ and $\psi$ do not match at the defect location.

This very simple example is useful to elucidate some of the more striking features of a jump-defect, which persist when it is incorporated in some integrable field theory such as the sine-Gordon model. In addition, contrary to a $\delta$-type impurity, the jump-defect is able to preserve integrability.
2 Jump-defects and affine Toda field theories

The aim of this talk is to summarize some recent results obtained in the context of the affine Toda field theories (A TFTs) [4] (see also the review [5] and references therein) with a jump-defect, namely the progress achieved in stretching the investigation beyond the sine-Gordon model. For this purpose, only the A TFTs associated with the root data of the Lie algebra $a_r$ will be taken into account. Classically, integrable jump-defects incorporated into these models have been described extensively in [6], where their integrability was established via a Lax pair argument. However, only recently, it has been possible to add a quantum description of the $a_2$ affine Toda model with a jump-defect [7].

2.1 Classical setting

The bulk Lagrangian density for a complex A TFT with root data of the Lie algebra $a_r$ reads

$$\mathcal{L}_\phi = \frac{1}{2} (\partial_\mu \phi \cdot \partial^\mu \phi) + \frac{m^2}{\beta^2} \sum_{j=0}^{r} (e^{i\beta \alpha_j \cdot \phi} - 1), \quad |\alpha_j|^2 = 2$$

(2.1)

where $m$ and $\beta$ are constants, and $r$ is the rank of the algebra. The vectors $\alpha_j$ with $j = 1, \ldots, r$ are simple roots and $\alpha_0$ is the extended root, defined by

$$\alpha_0 = -\sum_{j=1}^{r} \alpha_j.$$  

The field $\phi = (\phi_1, \phi_2, \ldots, \phi_r)$ takes values in the $r$-dimensional Euclidean space spanned by the simple roots $\{\alpha_j\}$. The A TFTs described by the Lagrangian density (2.1) are massive integrable field theories. They possess infinitely many conserved charges, a Lax pair representation, and many other interesting properties, both in the classical and quantum domains. The simplest choice $r = 1$ coincides with the sine-Gordon model. Apart this model, all other A TFTs described by the Lagrangian density (2.1) are not unitary theories.

What is particular interesting in the context of the present investigation is the fact that these models possess soliton solutions [8], for which the explicit expression reads

$$\phi^a = \frac{m^2}{\beta} \sum_{j=0}^{r} \alpha_j \ln (1 + E_a \omega^{\alpha_j}) \quad a = 1, \ldots, r, \quad E_a = e^{a_0 x - b_a t + \xi_a}, \quad \omega = e^{2\pi i/h},$$

(2.2)

where $(a_a, b_a) = m_a (\cosh \theta, \sinh \theta)$, $h = (r + 1)$ is the Coxeter number of the algebra, $\xi_a$ is a complex parameter, and $\theta$ is the soliton rapidity. These soliton solutions are complex, with the exception for the sine-Gordon soliton. Nevertheless, they possess real energy and momentum, and their masses are given by [8]

$$M_a = \frac{4hm}{\beta^2} \sin \left( \frac{\pi a}{h} \right).$$

(2.3)

Each solution (2.2) is characterized by a topological charge, which is defined to be

$$Q^a = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \phi^a = \frac{\beta}{2\pi} \phi^a(\infty, t),$$

(2.4)
which lies in the weight lattice $\Lambda_W(a_r)$ of the Lie algebra $a_r$. In particular, it can be noticed that for each $a = 1, \ldots, r$ there are several solitons whose topological charges lie in the set of weights of the fundamental $a^{th}$ representation of $a_r$. Looking at the expression (2.2), it can be noticed that the value of the topological charge depends on the imaginary part of parameter $\xi_a$. Shifting $\xi_a$ by $2\pi i a / h$ changes the topological charge, since that amount sets the boundaries between different topological charge sectors.

The system with a single jump-defect located at $x = 0$, which links two $a_r$ fields $\phi(x, t)$, $x < 0$ and $\psi(x, t)$, $x > 0$, is described by the Lagrangian density (1.1) with defect term given by

$$D = \left( \frac{1}{2} \phi \cdot E \partial_t \phi + \phi \cdot D \partial_t \psi + \frac{1}{2} \psi \cdot E \partial_t \psi - B(\phi, \psi) \right). \quad (2.5)$$

The requirement that integrability must be preserved, forces the matrices $E$ to be antisymmetric with $D = 1 - E$ and fixes the form of the defect potential $B$ to be

$$B = -\frac{m}{\beta^2} \sum_{j=0}^{r} \left( \sigma e^{i\beta \alpha_j \cdot (D^T \phi + D \psi) / 2} + \frac{1}{\sigma} e^{i\beta \alpha_j \cdot (D \phi - \psi) / 2} \right), \quad (2.6)$$

where $\sigma$ represent the defect parameter. Moreover, the matrix $D$ satisfies the following constraints

$$\alpha_k \cdot D \alpha_j = \begin{cases} 2 & k = j, \\ -2 & k = \pi(j), \\ 0 & \text{otherwise,} \end{cases} \quad (D + D^T) = 2, \quad (2.7)$$

where $\pi(j)$ indicates a permutation of the simple roots. Note that for $r = 1$, and after setting $\alpha_1 = 1 / \beta = \sqrt{2}$, the linearized version of (2.5) and (2.6) reduce to (1.4). It should be mentioned that the jump-defect setting presented in this section is not unique, as was explained in [9]. However, the alternative case will not be considered here.

Choosing a particular cyclic permutation, namely

$$\alpha_{\pi(j)} = \alpha_{j-1} \quad j = 1, \ldots, r, \quad \alpha_{\pi(0)} = \alpha_r,$$

it is possible to write explicitly the matrix $D$ as follows

$$D = 2 \sum_{j=1}^{r} w_j (w_j - w_{j+1})^T, \quad w_0 \equiv w_{r+1} = 0, \quad (2.8)$$

where $w_j$ with $j = 1, \ldots, r$ are the fundamental highest weights of the Lie algebra $a_r$ ($\alpha_i \cdot w_j = \delta_{ij}$). The Lagrangian density (1.1) with defect term (2.5) leads to the following equations of motion

$$\partial^2 \phi = \frac{m^2 i}{\beta} \sum_{j=0}^{r} \alpha_j e^{i\beta \alpha_j \cdot \phi} \quad x < 0, \quad \partial^2 \psi = \frac{m^2 i}{\beta} \sum_{j=0}^{r} \alpha_j e^{i\beta \alpha_j \cdot \psi} \quad x > 0, \quad (2.9)$$

and defect conditions

$$\partial_x \phi - E \partial_t \phi - D \partial_t \psi = 0 - \partial_x B \quad x = 0, \quad \partial_x \psi - D^T \partial_t \phi + E \partial_t \psi = 0 \partial_x B \quad x = 0. \quad (2.10)$$

As already pointed out in the case of the free massive field in section (1), a generalized momentum is conserved. Again, the system allows only transmission, and it is instructive to look at what
When a soliton solution $\phi^a (x < 0)$ (2.2) travels across the jump-defect from the left to the right ($\theta > 0$). As expected, the emerging soliton $\psi^a (x > 0)$ will experience a delay since its form will be

$$\psi^a = \frac{m^2 i}{\beta} \sum_{j=0}^{r} \alpha_j \ln(1 + z_a E_a \omega^a_j),$$

where the explicit expression for the delay $z_a$ is provided by the defect conditions (2.10), namely

$$z_a = \left( \frac{i e^{-(\theta-\eta)} + i e^{-i\gamma_a}}{e^{-(\theta-\eta)} + i e^{i\gamma_a}} \right), \quad \gamma_a = \frac{\pi a}{h}, \quad \sigma = e^{-\eta}. \quad (2.12)$$

This expression is in general complex and diverges when

$$\theta = \eta + \frac{i\pi}{2} \left( 1 - \frac{2a}{h} \right). \quad (2.13)$$

However, for the self-conjugate soliton $a = h/2$ (provided $r$ is odd), the delay becomes real and coincides with the delay for the sine-Gordon model [1]. When this happens the soliton can be absorbed by the defect since the pole (2.13) appears for a real value of the rapidity, namely $\theta = \eta$. Finally, in [6] it was also pointed out that a soliton might be turned into one and only one of the adjacent solitons by the jump-defect, provided the argument of the delay (2.12) is sufficiently large. In fact, the argument is given by

$$\tan(\text{arg } z_a) = -\left( \frac{\sin 2\gamma_a}{e^{-2(\theta-\eta)} + \cos 2\gamma_a} \right), \quad (2.14)$$

and therefore the phase shift produced by the defect can vary between zero (as $\theta \to -\infty$) and $-2\gamma_a$ (as $\theta \to \infty$) allowing a change in the topological charge of the incoming soliton, since, as pointed out before, the topological charge sectors are separated exactly by $2\gamma_a$.

### 2.2 Quantum domain

In this section, recent developments concerning the quantization of the $a_r$ ATFTs will be presented. In particular, the example elucidated in this talk concerns the $a_2$ affine Toda model, for which a complete analysis has been carried out in [7]. The purpose of that investigation was to find the transmission matrices, describing the interaction amongst a jump-defect and the soliton and antisoliton solutions of the model. Two different approaches were used for this purpose and they will be sketched in the next section. Both methods make use of the assumption that the topological charge of the system containing two $a_2$ fields $\phi (x < 0)$, $\psi (x > 0)$ and the jump-defect located in $x = 0$ is conserved. This fact relies on the classical investigation, briefly presented in section 2.1 which suggests that both solitons and defects carry a topological charge that can be exchanged due to their mutual interaction.

Both procedures allow to determine the transmission matrices up to an overall function of the rapidity. The first method consists of a functional integral approach, which makes use of the Lagrangian density (1.1) with defect term (2.5), together with a bootstrap procedure. The second method consists in solving directly the triangular equations, which represent a set of consistency conditions among the bulk scattering $S$-matrices and the unknown transmission

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matrices. Primary ingredients needed for both investigations are the $S$-matrices for the $a_2$ ATFT. Together with the $S$-matrices for the other $a_r$ models (with the exception of the sine-Gordon model), they have been conjectured by Hollowood [10], who made use of the trigonometric solutions of the Yang-Baxter equation found originally by Jimbo [11] (see also references therein).

The Lie algebra $a_2$ has two fundamental representations, and the weights belonging to the first representation can be written in terms of simple roots as follows

$$l_1 = \frac{1}{3}(2\alpha_1 + \alpha_2), \quad l_2 = -\frac{1}{3}(\alpha_1 - \alpha_2), \quad l_3 = -\frac{1}{3}(\alpha_1 + 2\alpha_2). \quad (2.15)$$

As a consequence this representation contains three solitons, while the corresponding antisolitons have weights which are the negative of these and lie in the second representation. The knowledge of the soliton-soliton $S$-matrix suffices since it can be used to derive the other $S$-matrices, which describe the interactions soliton-antisoliton and antisoliton-antisoliton, by means of a bootstrap procedure. Having said that, the soliton-soliton $S$-matrix for the $a_2$ model can be written in the following explicit form

$$S_{kl}^{mn}(\theta_{12}) = R_{kl}^{mn}(x_{12}) \rho(\theta_{12}), \quad \theta_{12} = (\theta_1 - \theta_2), \quad x_{12} = \frac{x_1}{x_2}, \quad (2.16)$$

where $k, l$ label the incoming particles and $m, n$ label the outgoing particles in a two-body scattering process, with the particle $k, n$ having rapidity $\theta_1$, and the particle $l, m$ having rapidity $\theta_2$. The explicit form for the $R$-matrix is

$$R(x_{12}) = \begin{pmatrix}
a(x_{12}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_{12}^{1/3} c & 0 & b(x_{12}) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_{12}^{-1/3} c & 0 & 0 & 0 & b(x_{12}) & 0 & 0 \\
0 & b(x_{12}) & 0 & x_{12}^{-1/3} c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a(x_{12}) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x_{12}^{1/3} c & 0 & b(x_{12}) & 0 \\
0 & 0 & b(x_{12}) & 0 & 0 & 0 & x_{12}^{1/3} c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b(x_{12}) & 0 & x_{12}^{-1/3} c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(x_{12})
\end{pmatrix}$$

with

$$a(x_{12}) = (q x_{12} - q^{-1} x_{12}^{-1}), \quad b(x_{12}) = (x_{12} - x_{12}^{-1}), \quad c = (q - q^{-1}), \quad (2.17)$$

and

$$x_k = e^{k\gamma\theta_k/2} \quad k = 1, 2, \quad q = -e^{-i\pi\gamma}, \quad \gamma = \frac{4\pi}{\beta^2} - 1.$$ 

Finally, $\rho$ is a scalar function constrained by consistency relations such as bootstrap constraints, and requirements such as crossing, which a scattering matrix must satisfy. Its expression can be found in [10].

### 2.3 Transmission matrices: two different approaches

Consider the following static field configurations

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \frac{2\pi}{\beta}(r, s), \quad (2.18)$$

Finally, $\rho$ is a scalar function constrained by consistency relations such as bootstrap constraints, and requirements such as crossing, which a scattering matrix must satisfy. Its expression can be found in [10].
where \( r, s \) are any two elements of the root lattice. It is not difficult to check, looking at (1.1) and (2.5) that, despite having a discontinuity at the location of the defect, the constant configurations (2.18) all have the same energy and momentum, namely

\[
(\mathcal{E}_0, \mathcal{P}_0) = -\frac{2hm}{\beta^2}(\cosh \eta, -\sinh \eta),
\]

and they are the vacuum configurations of the system. Suppose that a jump-defect is labelled by these vacuum configurations, in the sense that when the fields \( \phi, \psi \) have the constant values (2.18), the label \((r, s)\) is ascribed to the defect. The idea, first presented in [2], is to compare the transmission matrix elements describing the evolution of the field configurations in the presence of two different defects: one labelled \((r, s)\) and the other \((0, 0)\). For doing so, the fields \( \phi, \psi \) are shifted as follows

\[
\phi \to \phi - \frac{2\pi r}{\beta}, \quad \psi \to \psi - \frac{2\pi s}{\beta}.
\]

Note that the bulk and the defect potential \( B \) (2.6) do not change under this shift, but the part linear in time derivatives appearing in the defect term (2.5) does. As a consequence, the functional integrals, which represent the transmission factors related to the two differently labelled defects, will differ by a constant amount, namely

\[
T(r, s) = e^{i\tau(r,s)} T(0,0),
\]

where

\[
\tau(r, s) = \frac{\pi}{\beta} (-\delta \phi \cdot (Er + Ds) + (rD + sE) \cdot \delta \psi),
\]

and \( \delta \phi, \delta \psi \) are the changes in the field configurations from initial to final states. To obtain explicit expressions for the elements of the soliton transmission matrix for the \( a_2 \) model, consider that a soliton passing the defect will either retain its topological charge or change it to one of the other weights \( l_k \) listed in (2.15). Therefore, the effect of a soliton passing the defect must be to change the defect labels by

\[
r \to r - l_i, \quad s \to s - l_j,
\]

which implies

\[
\delta \phi = -\frac{2\pi l_i}{\beta}, \quad \delta \psi = -\frac{2\pi l_j}{\beta} \quad i, j = 1, 2, 3.
\]

Consequently, expression (2.19) becomes

\[
T(r, s, l_i, l_j) = e^{i\tau(r,s,l_i,l_j)} T(0,0,l_i,l_j)
\]

where

\[
\tau(r, s, l_i, l_j) = \frac{2\pi^2}{\beta^2} \left( l_i \cdot (Er + Ds) - (rD + sE) \cdot l_j \right).
\]

In the end, the functional integral approach suggests the following form for the elements of the transmission matrix (see [7] for details)

\[
T_{ij}^{\alpha \beta}(\theta) = Q^ {\alpha, [E(l_i - l_j) + l_i + l_j]/2} T_{ij}^{\alpha} (\theta) \delta_{\alpha - l_i + l_j}.
\]
where \( \alpha = s - r, \quad Q = -e^{i\pi \gamma} \).

Note that the matrix (2.22) is infinite dimensional with roman and greek labels denoting soliton states and defect charges, respectively. Naturally, this kind of argument does not provide any information concerning the rapidity dependent part of the transmission matrix (2.22). However, important information concerning this unknown quantity can be collected making use of a bootstrap procedure.

Consider \( D_\alpha \) to be the defect operator. Then, it is formally possible to describe the interaction between a defect and a soliton or antisoliton as follows (\( \theta > 0 \)),

\[
A_i(\theta) D_\alpha = T_{ij}^{\beta}(\theta) D_\beta A_j(\theta), \quad \bar{A}_i(\theta) D_\alpha = \bar{T}_{ij}^{\beta}(\theta) D_\beta \bar{A}_j(\theta) \quad i = 1, 2, 3, \quad (2.23)
\]

where \( A_i, \bar{A}_i \) are operators representing the soliton and antisoliton states, respectively. Since the antisoliton states \( \bar{A}_i \) can be built making use of the soliton states \( A_i \), the two expressions in (2.23) can be combined together to provide a link amongst the elements of \( T \) and \( \bar{T} \). The constraints obtained allow to fix, up to an overall scalar function of the rapidity, both the matrices \( T, \bar{T} \) and, surprisingly, to determine the constraints (2.7) that the classical quantity \( D = 1 - E \) has to satisfy. A complete discussion and explicit calculations are reported in [7].

Before revealing the explicit expressions of the two transmission matrices, a few words must be said on the alternative approach mentioned in section (2.2). It consists in solving directly the triangular equations, which relate, for instance, the elements of the soliton transmission matrix \( T \) to the scattering soliton-soliton \( S \)-matrix elements. Adopting the same conventions as before for the roman and greek labels, and considering solitons travelling along the positive \( x \)-axis (\( \theta_1 > \theta_2 \)), the triangular equations read

\[
S_{\kappa \lambda}^{m n}(\theta_{12}) T_{i \alpha}^{t \beta}(\theta_{1}) T_{m \gamma}^{s \delta}(\theta_{2}) = T_{i \alpha}^{n \beta}(\theta_{2}) T_{m \gamma}^{s \delta}(\theta_{1}) S_{\kappa \lambda}^{t \delta}(\theta_{12}). \quad (2.24)
\]

These equations have been discussed first in the context of purely transmitting defects by Delfino, Mussardo and Simonetti in [12]. Making use of the \( S \)-matrix (2.16) and of the following ansatz for the transmission matrix elements

\[
T_{i \alpha}^{n \beta}(\theta) = t_{i \alpha}^{n}(\theta) \delta_{\beta}^{\alpha - l_i + l_n} \quad i, n = 1, 2, 3, \quad (2.25)
\]

it is possible to classify the solutions of (2.24). This much has been done in [7], where it was found that one of the solutions obtained coincides exactly with the soliton transmission matrix \( T \) conjectured by the functional integral approach. Some of the other solutions may be related to an alternative setting for the jump-defect with respect to the one presented in section (2.1), while others do not seem to be relevant for the jump-defect problem. Details are available in [7].

To summarize, the transmission matrices for solitons and antisolitons related to the jump-defect presented in section (2.1) are, respectively,

\[
T_{i \alpha}^{n \beta}(\theta) = g(\theta) \left( \begin{array}{cccc}
Q^{-a_{1} \beta} \delta_{\alpha}^{\beta} & \hat{x} Q^{-a_{1} \beta} \delta_{\alpha}^{a_{1}} & \hat{x} Q^{-a_{1} \beta} \delta_{\alpha}^{a_{0}} \\
\hat{x}^{2} \delta_{\alpha}^{a_{0}} & Q^{a_{2} \beta} \delta_{\alpha}^{a} & \hat{x}^{2} \delta_{\alpha}^{a} \\
\hat{x} Q^{-a_{1} \beta} \delta_{\alpha}^{a_{2}} & \hat{x} Q^{-a_{1} \beta} \delta_{\alpha}^{a_{2}} & Q^{a_{3} \beta} \delta_{\alpha}^{a}
\end{array} \right), \quad (2.26)
\]

and

\[
\bar{T}_{i \alpha}^{n \beta}(\theta) = \bar{g}(\theta) \left( \begin{array}{cccc}
Q^{-a_{1} \beta} \delta_{\alpha}^{\beta} & \hat{x} \delta_{\alpha}^{a_{1}} & 0 \\
0 & Q^{-a_{2} \beta} \delta_{\alpha}^{a} & \hat{x} \delta_{\alpha}^{a} \\
\hat{x} \delta_{\alpha}^{a_{2}} & 0 & Q^{-a_{3} \beta} \delta_{\alpha}^{a}
\end{array} \right), \quad (2.27)
\]
where
\[ \tilde{g}(\theta) = g(\theta - i\pi/3) g(\theta + i\pi/3) (1 + \hat{x}^3), \quad \hat{x} = e^{\gamma(\theta-\Delta)}. \]

Eventually, the constant $\Delta$ will be related to the Lagrangian defect parameter $\sigma$ introduced in (2.6), but, first, a few comments are in order. First of all, note the striking asymmetry of $T$ and $\tilde{T}$. Classically, there is little difference in behaviour between solitons and antisolitons, and in section (2.1) it was pointed out that in either case the jump-defect causes a phase shift. Depending on the size of this shift, the topological charge of a soliton or antisoliton passing through the defect could be converted to just one of the adjacent topological charges. Comparing expression (2.26) and (2.27) with the argument of the classical delay (2.14), it can be seen that $\tilde{T}$ provides a good match to the classical situation because of the presence of zeros in expected positions, while $T$ does not possess the expected zeros corresponding to the classical selection rule. It appears that in the quantum context a soliton passing through the defect may change into either of the solitons adjacent to it, though the classically allowed transition remains the most probable.

It should be pointed out that solutions (2.26) and (2.27) are related by a bootstrap procedure, in the sense that starting with a $T$ matrix (2.26) for the solitons, the bootstrap leads to the $\tilde{T}$ matrix (2.27) for the antisolitons. Similarly, starting with the antisoliton matrix (2.27), the bootstrap leads to the soliton matrix (2.26). A different setting for the jump-defect would present a situation in which the asymmetry of $T$ and $\tilde{T}$ is maintained but the role of solitons and antisolitons is interchanged.

2.4 The overall scalar function: additional constraints

Some additional requirements are needed to be able to fix the overall functions of the transmission matrices. They are provided by crossing
\[ \tilde{T}_{n\alpha}^{\beta\beta}(\theta) = T_{\alpha\alpha}^\dagger(i\pi - \theta), \quad (2.28) \]
which allows to relate the transmission matrix for antisolitons $T$ to the transmission matrix $\tilde{T}$, which represents a process in which the incoming particles meet the defect from the right. In the jump-defect problem, parity is explicitly violated and therefore the matrix $\tilde{T}$ is expected to differ from the matrix $T$. Nevertheless, the two matrices $T$ and $\tilde{T}$ are expected to be related by
\[ T_{\alpha\alpha}^{\beta\beta}(\theta) \tilde{T}^{\gamma\gamma}_{b\beta}(-\theta) = \delta^\gamma_c \delta^\alpha_b. \quad (2.29) \]
This constraint replaces the usual unitarity condition, which does not hold here due to the fact that the model investigated is not unitary. Making use of solutions (2.26) and (2.27) in (2.28) and (2.29) leads to a relationship between the functions $g$ and $\tilde{g}$, from which the following minimal solution for $g$ is derived
\[ g(\theta) = \frac{f(\theta)}{(2\pi)^{2/3} \hat{x}} \quad (2.30) \]
with
\[ f(\theta) = \Gamma[(1 + \gamma)/2 - z] \prod_{k=1}^{\infty} \frac{\Gamma[(1 + \gamma)/2 + 3k\gamma - z] \Gamma[(1 - \gamma)/2 + (3k - 2)\gamma + z]}{\Gamma[(1 - \gamma)/2 + 3k\gamma + z] \Gamma[(1 + \gamma)/2 + (3k - 1)\gamma - z]}^{1}. \quad (2.31) \]
where \( z = i3\gamma(\theta - \Delta)/2\pi \). Note the presence of a pole in (2.30) at
\[
\theta_P = \Delta - \frac{i\pi}{3} - \frac{i\pi}{3\gamma}.
\] (2.32)

Comparing this in the classical limit, \( 1/\gamma \to 0 \) (\( \beta \to 0 \)), with the pole (2.13) appearing in the classical delay allows a determination of the relationship between the parameter \( \Delta \) appearing in the transmission matrix and the defect parameter \( \sigma \) appearing in the Lagrangian density. This relationship reads
\[
\Delta = \eta + \frac{i\pi}{2}, \quad \sigma = e^{-\eta}.
\] (2.33)

The identification (2.33) is also supported by the results found during the calculation of the transmission factors for the lightest breathers, as explained in [7]. Besides, the computation of the energy of the state associated with the pole (2.32) reveals that it corresponds to an unstable bound state, provided \( \frac{1}{2} < \gamma < 2 \). Consequently, in the classical limit, this unstable state disappears completely. This fact agrees nicely with the classical finding that a soliton with real rapidity cannot be absorbed by the defect. It is worth pointing out that the latter phenomenon differs from the sine-Gordon case in which a soliton can be absorbed by the defect and consequently a quantum unstable bound state is always present, independently of the range of the coupling constant.

3 Conclusion

Recent results in the context of the \( a_2 \) affine Toda field theory concerning the existence of a special integrable defect - called a jump-defect - have been presented. For this model, it was possible to provide a complete and consistent description both in the classical and quantum domains. In particular, the interaction between the soliton solutions of the \( a_2 \) affine Toda model and a jump-defect was found to be described, in the quantum context, by infinite dimensional matrices that are solutions of the triangular equations. Unfortunately, there was no room to discuss here further interesting issues, such as the connection with Bäcklund transformations or the scattering of defects in motion.

The jump-defect problem can be extended to all the \( a_r \) affine Toda models. On the other hand, the existence of integrable, purely transmitting defects in the other ATFTs appears to be more difficult to prove. In principle, infinite dimensional solutions of the triangular equations can be found for some other Toda models, but it remains to be seen if these solutions can be regarded as transmission matrices.

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