Characterization of the energy level-structure of a trapped dipolar Bose gas via mean-field parametric resonances

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Abstract
Analyzing the energy levels of a trapped Bose–Einstein condensate (BEC) can be difficult when dipole–dipole interactions (DDIs) are present. To address this issue, this study focuses on the parametric resonances (PRs) in the mean-field dynamics of a one-dimensional dipolar BEC (DBEC) over widely varying trapping geometries, with the primary objective of characterizing the energy levels of this system via analytical methods. This is achieved by matching the PR energies to the energy levels of the confining trap using perturbative methods. Further, this research reveals the role of the interplay between DDIs and the trapping geometry in defining the energies and amplitudes of the PRs. The PRs are induced by a negative Gaussian potential with a depth that oscillates with respect to time; DDIs also play a role in this induction. The dynamics of this system are modeled using the time-dependent Gross–Pitaevskii equation (TDGPE), which is numerically solved via the Crank–Nicolson method.

The PRs are discussed based on analytical methods. First, we show that PRs similar to the ones obtained from the TDGPE can be reproduced via the Lagrangian variational method. Second, the energies at which the PRs occur are closely matched with the energy levels of the corresponding trap, calculated using the time-independent perturbation theory. Third, the most probable transitions between the trap energy levels yielding PRs are determined based on the time-dependent perturbation theory. The primary contribution of this research is that the energy levels of a DBEC within a complex trapping potential could be characterized.

1. Introduction
Parametric resonance (PR), which is a ubiquitous phenomenon in nature and a widely examined fundamental physical property, is one of the outstanding features observed in Bose–Einstein condensates (BECs) [1–8], where resonances are usually the result of modulating a system parameter, such as the scattering length [7, 9–11]. These resonances are also generated by external factors such as the time-dependent trapping geometry [7], laser stirring [2], and laser-intensity modulation [1]. PRs have also been shown to occur in classical systems [12, 13] and quantum devices such as birefringent optical fibers [14], magnetometers [15], superconducting wave guides [16], and quantum dots [17, 18]. The importance of PRs in transatlantic telecommunication fiber optics with regard to modulational instability has been demonstrated [19].

In this study, we explore the PRs in a dipolar BEC (DBEC) within large traps, which is expected to enable an examination of the effects of long-range interactions. Dipole–dipole interactions (DDIs) are capable of shifting the frequencies of PRs [1, 20, 21], and such shifts are sensitive to the trapping geometry [20]. As a result, the interplay between the trapping geometry and DDIs is inferred to determine the energies at which PRs occur. These energies are equivalent to the energies of the trap levels, i.e. the trap in which the DBEC is confined. The main objective is to characterize the energy-level structure (ELS) of a DBEC in a large trap by matching the PR energies to the energy levels of the trap using perturbative methods. Thus, the importance of the PRs in characterizing the shape of the BEC trap and its energy levels is elucidated [1]. However, in the presence of
interactions, analytical calculations of the energy levels of a trapped BEC can be rather difficult and this is the reason why this study was conducted.

To the best of our knowledge, the present investigation has not been undertaken before. The novelty of this study lies in the observation of mean-field PRs in a DBEC that is driven by a negative Gaussian potential (NGP) with oscillating depth. This is in contrast to PRs associated with the dynamical fragmentation of a rotationally stirred BEC [22], where in the present mean-field treatment this fragmentation is absent. The properties of the PRs observed here have not been demonstrated before. Further, the new feature of the present treatment is that the dynamics of the BEC and its properties are scanned along a broad range of DDI strength that is performed via a large number of Crank-Nicholson (CN) runs. Moreover, this research is conducted to facilitate evaluations of the ELS under complex potentials. This work is expected to inspire future research on determining the ELSs of more complicated traps that are yet to be discovered and engineered. This study also aims to reveal the role of the interplay between complex trapping geometry and DDIs in defining the frequencies at which PRs occur. Although the effects of the trapping geometry, such as the influence of the trap aspect ratio on the oscillation frequencies [26], stability [27, 28], and the density profiles as well as breathing mode oscillations of DBECs [29] have been investigated in previous studies, the relationship between complex geometry and the structure of trap energy levels in the presence of interactions has not been discovered.

In this study, we consider a one-dimensional (1D) trapped DBEC that is driven by an NGP, the depth of which is periodically modulated with respect to time. This system is simulated numerically under different traps using the mean-field time-dependent Gross–Pitaevskii equation (TDGPE). A 1D Bose gas is an important and well-known system in that it can represent many physical systems such as optical fibers [14, 30–32]. It is studied to reveal the physics in 1D, which is considerably different from that in higher dimensions. The addition of an NGP is intended to cause the particles to condense into lower energy levels as ‘catch particles’ and then throw (excite) them to higher energy levels based on the oscillating NGP depth. The oscillatory NGP has been shown to function similar to, and is therefore regarded as analogous to, a modulated contact interaction [6]. In another study, an NGP was used to induce a BEC and study its growth dynamics [33]. Using the aforementioned NGP, a laser-light source was modeled with an intensity that oscillated with respect to time [1, 34] and afforded a stirring of the BEC in the energy space, which was softer than the stirring by time-dependent spatial modulations. In this study, this method, which has rarely been used or mentioned in BEC-related literature, is demonstrated to be able to afford significant excitations.

PRs are measured using a quantity that resembles the time average of the square of the kinetic energy, referred to as the ‘signal energy’ [35]. We have previously reported that signal energy, which is a term used in engineering in the processing of an oscillating electrical signal, is considerably effective for revealing PRs [2].

Our present investigation was inspired by the work of Balik et al [1], who applied a CO2-laser generated optical-dipole trap to confine a sample of 87Rb atoms. By modulating the laser intensity, the authors were able to excite PRs whose frequencies were observed to shift with a change in the laser modulation depth. Analogously, in this study, we explain the frequency shift resulting from a change in the depths of the effective time-averaged potential of a trapped DBEC. This potential behaves like a single-particle trap. One could ascribe to it energy levels analogous to those of the particle in a quantum well that can be solved by the Schrödinger equation. A change in the depth of this potential, therefore, leads to a change in its energy levels which, in turn, causes the PR frequency shift. If the DBEC is scanned over a broad range of these depths, which are controlled by DDIs, it is possible to observe PRs at the DDI values corresponding to the resonance frequencies. This project involves heavy computations because it requires a large number of runs to locate the DDI regimes where PRs occur.

The most significant contribution of this research is that we were able to characterize the energy levels of a DBEC within a complex trapping potential, i.e. a trap to which an NGP is added. The key results can be summarized as follows:

(i) By varying the DDI strength, we detect mean-field PRs in the dynamics of a DBEC induced by an NGP with a depth that is periodically modulated with respect to time. The PRs obtained are an inherent feature deeply embedded in the physics of the DBEC and could also be generated by other approaches such as the Lagrangian variational method (LVM) [6, 36–39].

(ii) The positions of the PRs are dictated by the ELS of the trapping geometry. The energies at which PRs occur are shown to be characterizeable by the second-order perturbation theory, enabling us to determine the ELS.

(iii) The depth of the time-averaged effective potential is found to reduce with an increase in the repulsive DDI strength, which results in the PR frequencies shifting toward larger values in the case of a harmonic oscillator (HO) + NGP. However, in the case of a BOX + NGP, the PR frequencies decrease with increasing DDI strength. Thus, the interplay between trapping geometry and DDIs influences the DDI strength at
which PRs arise in the signal energy [35]. Moreover, the amplitudes of the PRs decline with increasing DDIs owing to the simultaneous reduction in the depth of the time-averaged mean-field effective potential.

(iv) The occurrence of a PR requires two conditions: (1) its energy should closely match one of the trap levels, and (2) the probability of transition to this level should be significantly high.

1.1. Wider context of research
This investigation considers trapped DBECs in light of their interesting physics [40] resulting from the anisotropy of their DDI, which depends on the direction of polarization of a pair of atoms with respect to their relative positions. The DDI are long–ranged and can thus introduce more effects than the widely used, short-ranged, s–wave interactions. Further effects can be introduced by exciting these systems via external driving agents such as the oscillating NGP of this work. The NGP is introduced by subjecting the DBEC to a red-detuned focusing laser beam, which polarizes the cold atoms and induces a potential \( U = -p \cdot E \) arising from the interaction between the electric dipoles \( p \) and the electric field \( E \) of the incident laser light. The gradient arising from \( U \) generates a confining force on the atoms, pushing them towards the laser-beam axis. Therefore, the NGP can be considered to act like an attractive interparticle interaction potential [41] and has been found very efficient in realizing traps for matter-waves [42] and in generating BECs [33] without cooling. The NGP is proportional to the power of the laser beam [43], due to which its depth can be easily controlled by varying the laser intensity—for instance, a periodic modulation of this intensity causes the NGP depth to oscillate and excite the DBEC. Because the latter modulation introduces an effect that is analogous to that of varying the s–wave scattering length \( a_{s} \) [37], the present work can be placed in the context of investigations that modulate interatomic interactions [44].

Parametric resonance (PR) is a phenomenon manifested in the maximum response of a system, via the amplitude of its oscillations, to a time-dependent variation in at least one of its parameters. A PR occurs when the natural frequency of a physical system coincides with the driving frequency of an external periodic force [13, 45]. As noted before, PRs also arise in many other areas of science, such as in plasmons, solid–state devices, and even astrophysics [46]. A more thorough understanding of PRs can enable us to improve quantum devices such as PR entangling gates [47], delve deeper into the secrets of the early universe at the time of its formation [48] and expansion [49], and gain further access to the physics of non-linear, non-equilibrium dynamics—a topic which, although it has been dealt with earlier [50], is not so well-understood [45]. In the light of these applications, the quest to develop a further understanding of PRs is, therefore, very important.

The remainder of this paper is organized as follows. Section 2 presents the study methodology, while section 3 discusses the results. In section 4, the origins of the PRs are explained. Lastly, section 5 provides the conclusions drawn from this research.

2. Method
This section presents a rudimentary outline of the methodology. A more detailed description can be found in [51].

2.1. Basic units
In this study, the lengths and energies are expressed in units of the trap \( a_{ho} = \sqrt{\hbar/(m\bar{\omega})} \) and \( \hbar \bar{\omega} \), respectively, where \( \bar{\omega} = (\omega_{x} \omega_{y} \omega_{z})^{1/3} \) is the geometric average of the trapping frequencies along the coordinate axes, and \( m \) is the mass of the atom. Time, on the other hand, is unitless; \( t = \bar{\omega} \tau \), and \( \tau \) is the time in seconds.

2.2. Systems
The systems investigated in this study are 1D strongly repulsive DBECs that are confined within a different power-law traps that have the following form:

\[
V_{D}(x) = \frac{1}{2} \frac{x^{p}}{L_{x}},
\]

where \( p \) is the trapping exponent, and \( L_{x} \) is a length scale that provides a certain degree of flatness for \( V_{D}(x) \). The latter is expressed in units of \( \hbar \bar{\omega} \), whereas \( x \) and \( L_{x} \) are in \( a_{ho} \).

The DBECs are excited by an NGP modelled by

\[
V_{NGP}(x, t) = [A + \delta A \cos(\Omega t)]\exp(-\beta x^{2}),
\]

where \( \Omega = 2\pi f \) is the driving frequency, \( A \) is the principal depth, \( \delta A \) is the modulation amplitude, and \( 1/\sqrt{3} \) is a measure of the NGP width. \( A \) and \( \delta A \) are expressed in units of \( \hbar \bar{\omega} \), \( \beta \) in \( a_{ho}^{-1} \), and \( \Omega \) is unitless. It should be noted
that there exists an oscillating NGP force acting on the BEC (see appendix A) responsible for inducing excitations. Experimentally, the NGP is generated via the application of a focusing red laser beam [33, 52, 53] with a depth that oscillates with respect to time. A red-detuned laser beam interacts with the BEC such that it introduces an NGP into it via phase-imprinting [54].

Appendix B presents a review of the methods for manipulating DDIs. Additionally, the DDIs in the present study are repulsive.

The present system consists of \( N = 2000 \) Rb\(^{87} \) atoms in a 1D trap, which can be experimentally generated—for instance, on an atom chip [55] with an axial trapping frequency \( \omega_{a} = 2\pi \times 16 \) Hz and a radial one, \( \omega_{r} = 2\pi \times 1100 \) Hz, leading to a geometric average of \( \bar{\omega} = 2\pi \times 132.7 \) Hz. As such, the corresponding 3D trap length would be \( a_{ho} = 0.936 \) \( \mu m \), with \( a_{a} = 0.325 \) \( \mu m \) and \( a_{r} = 2.695 \) \( \mu m \) being the radial and longitudinal trap lengths, respectively. For \( G_{ID} \) values of 50, 100, and 150, the s-wave scattering lengths read \( a_{s} = 221 \) \( d_{0} \), 442 \( d_{0} \), and 663 \( d_{0} \), respectively, where \( d_{0} \) is the Bohr radius. Using these values for \( a_{s} \) would satisfy the condition for the present setup to be 1D with the transverse modes completely frozen, i.e. \( a_{s} \ll a_{a} \ll a_{r} \) [56]. On the other hand, the range of dipolar scattering lengths, \( d_{dd} \), along which the DBEC is scanned, ranges from 0 to 7408 \( d_{0} \). This change is achieved by applying an external electric field, \( E \), from 0 up to \( 6.019 \times 10^{6} \) V/cm, which induces an electric dipole moment, \( p \), of \( 3.174 \times 10^{-30} \) A \( \cdot \) m\(^{2} \), given that the electric polarizability of Rb\(^{87} \) is \( \alpha = 320 \) [57]. Electric fields of such magnitudes (or higher) have been adapted for the same purpose [58]. On the other hand, the change in the s-wave scattering length was brought about by applying an external magnetic field through the Feshbach resonance method [59]. Furthermore, the principle depth of the NGP was \( U = 387 \) nK, which brings it in line with the value of 210 nK of [43].

### 2.3. Mean-field Gross–Pitaevskii equation

The trapped DBEC is described using the time-dependent Gross–Pitaevskii equation (TDGPE), which is reduced from the 3D to the 1D form when the contributions in the transverse direction are integrated out (refer to, for example, [51, 60]). We thus consider a cigar-shaped DBEC that is elongated along the \( x \)-axis, with strong radial confinement in the transverse directions. The dynamics in the transverse directions is frozen in the radial ground state:

\[
\psi(x, t) = \frac{1}{d_{p} \sqrt{\pi}} e^{-\rho^{2}/(2d_{p}^{2})},
\]

where \( d_{p} \) is the width of the Gaussian. As in [51], the reduced 1D TDGPE can be expressed as

\[
\frac{\partial \psi(x, t)}{\partial t} = \left\{ -\frac{1}{2} \frac{\partial^{2}}{\partial x^{2}} + V_{DB}(x) + V_{DT}(x, t) + G_{ID} |\psi(x, t)|^{2} + \frac{4\pi}{3} G_{dd} \int_{-\infty}^{+\infty} \frac{dk_{x}}{2\pi} e^{-i\kappa_{x}x} |\tilde{\psi}(k_{x}, \tau)|^{2} j_{ID}(\tau) \right\} \psi(x, t),
\]

where \( \tilde{\psi}(k_{x}, \tau) \) is the Fourier transform of \( \psi(x, t) \). The latter \( \psi(x, t) \) is the longitudinal wave function, which is normalized to one vis-à-vis \( \int_{-\infty}^{+\infty} |\psi(x, t)|^{2} dx = 1 \) with \( \psi(x, t) \) in units of \( a_{ho}^{-1/2} \). On the other hand, \( j_{ID}(\tau) \) is a function expressed as

\[
j_{ID}(\tau) = \frac{\sqrt{\sigma}}{2\pi d_{p}} \int_{-\infty}^{+\infty} dt \tau e^{-\tau^{2}} h_{2D}(\tau),
\]

where \( \tau_{p} = d_{p} k_{x} \sqrt{\tau_{x}^{2} + \tau_{y}^{2}} \), and

\[
h_{2D}(\tau) = \frac{1}{\sqrt{\pi} d_{p}} \left[ 2 - 3\sqrt{\pi} e^{-\tau^{2}} \{ 1 - erf(\tau) \} \right].
\]

\( G_{ID} \) and \( G_{dd} \) are the 1D s-wave and dipolar interaction parameters, respectively (see appendix C). The fourth term on the right-hand side of (4) introduces the usual mean-field s-wave interaction nonlinearity. The last term introduces the mean-field DDI nonlinearity.

Equation (4) is solved numerically by utilizing the widely applied split-step Crank–Nicolson (CN) method [51, 60] in real time. In the first step it is solved in imaginary time to initialize the BEC in the trapping geometry (1), over which the NGP (2) is superimposed without the modulated portion, i.e.

\[
V_{DT}(x) = A \exp(-\beta x^{2}).
\]

In the second step, the DBEC is driven by the NGP (2) in real time, enabling an examination of the ensuing dynamics. Further technical details can be found in appendix E. Appendix C provides also values for the parameters used in this study. The codes used for solving the TDGPE were written by the group of Antun Balaz in Belgrade; recent versions treating DBECs are fully explained in [51], while earlier versions on ordinary BECs...
are explained in [60]. Numerous other TDGPE codes have been made available by this group of researchers [61–67] and have also been used extensively.

2.4. Gross–Pitaevskii energies
The Gross–Pitaevskii (GP) energies are evaluated as

\[
E_{\text{GP}}(t) = \int_{-\infty}^{+\infty} dx \left[ \frac{\partial \psi(x, t)}{\partial x} \right]^2 + [V_N(x) + V_{\text{D}}(x, t)]|\psi(x, t)|^2 + \frac{1}{2} G_{\text{DD}}|\psi(x, t)|^4
+ \frac{2\pi}{3} G_{\text{DD}} \int_{-\infty}^{+\infty} dk x e^{-i k x} \bar{\psi}(x, t) |\psi(x, t)|^2 \psi(\tau x) |\psi(x, t)|^2.
\]  

(8)

The time average is then computed using

\[
\langle E_{\text{GP}} \rangle_t = \frac{1}{T} \int_0^T E_{\text{GP}}(t) dt,
\]  

(9)

and the associated errors by

\[
\delta E_{\text{GP}} = \sqrt{(\langle E_{\text{GP}}^2 \rangle_t - \langle E_{\text{GP}} \rangle_t^2) / T},
\]  

(10)

where \(T\) is the total simulation time after initializing the DBEC. \(E_{\text{GP}}(t)\) is in units of \(\hbar \omega\), and \(T\) is unitless.

2.5. Signal energy
For a time-dependent physical observable \(f(t)\), the signal energy \(\mathcal{D}\) is formally defined by the integral

\[
\mathcal{D} = \int_{-\infty}^{+\infty} |f(t)|^2 dt,
\]  

(11)

where \(f(t)\) is the mean-field kinetic energy,

\[
E_{\text{kin}}(t) = \frac{1}{2} \int_{-\infty}^{+\infty} \left| \frac{\partial \psi(x, t)}{\partial x} \right|^2 dx,
\]  

(12)

and \(\psi(x, t)\) is the time-dependent wavefunction describing the DBEC. We refer to the signal energy using \(\mathcal{D}_{\text{kin}}\) because it is derived from \(E_{\text{kin}}\). With regard to the units, \(E_{\text{kin}}\) is expressed in \(\hbar \omega\) and \(\mathcal{D}_{\text{kin}}\) is expressed in \((\hbar \omega)^2 / \hbar \omega\).

However, because we cannot numerically integrate to infinity neither temporally nor spatially, we limit the integral in equation (11) from \(t = 0\) to some time \(T\) that is sufficiently long to adequately reveal the dynamical properties, and in equation (12) to the length of the simulation grid from \(x = -L_x\) to \(L_x\). (see appendix \(D\) for a discussion on the applicability of the signal energy to the present study.)

3. Results

3.1. Resonances in box potential
Figure 1 displays the plot of \(\mathcal{D}_{\text{kin}}/\mathcal{D}_{\text{kin}}^{(0)}\) versus \(G_{\text{DD}}\) for a few different values of \(G_{\text{ID}}\) and \(\Delta \Delta A\) in a BOX + NGP trap. \(\mathcal{D}_{\text{kin}}^{(0)}\) is the maximum signal energy for \(G_{\text{ID}} = 50\) at \(\Delta A = 20\) in the \(G_{\text{DD}}\) range considered. It is used for the normalization of signals in all frames. Several PRs are discovered with amplitudes that notably decline with increasing values of \(G_{\text{DD}}\). This is in line with [68, 69], where DDI is reported to reduce the amplitude of DBEC dynamics. An increase in the DDI, therefore, causes a weaker response to the oscillating NGP; the repulsive DDIs are observed to reduce the depth of the effective mean-field potential (see appendix \(F\)) and, with it, the occupancy of the NGP, such that fewer particles are excited.

Qualitatively, the same features are observed for all values of \(G_{\text{ID}}\) as shown in figure 1. Notably, an increase in \(G_{\text{ID}}\) shifts the entire spectrum backwards (i.e. toward the left), maintaining the same distance \(\Delta G_{\text{DD}}\) between each pair of peaks. This demonstrates that \(\Delta G_{\text{DD}}\) is not influenced by \(G_{\text{ID}}\). For example, the third peak from the left in frame (A) at \(G_{\text{DD}} \approx 200\) is shifted backwards by 50 in frame (B) from its position in (A) and by 100 in frame (C). In other words, an increase in \(G_{\text{DD}}\) by 50 causes all the resonance positions to shift backwards by the same value, i.e. 50 (and similarly, for an increase by 100), which is a rather unprecedented and remarkable feature. Thus, there appears to be no effect for the interplay between s-wave interactions and DDIs on the principal features of the spectrum of \(\mathcal{D}_{\text{kin}}\) under a box potential. This demonstrates that s-wave interactions and DDI function similarly in determining the positions of PRs. The aforementioned results reveal information regarding the ELS of the DBEC and the shape of the trapping potential. The equidistant peaks demonstrate confinement homogeneity, i.e. the flatness of the box. This is further proven by a comparison with the HO potential, as discussed in the next section.
3.2. Resonances in harmonic trap

As shown in figure 2, PRs are also encountered in an HO + NGP trap. However, the separation $D_{1dd}$ is not uniform as in the box; instead, it increases with $\Gamma_{1dd}$, demonstrating confinement inhomogeneity. Further, this result reveals the role of the interplay between trapping geometry and the DDIs in determining PR energies. A uniform trap leads to equidistant PRs, whereas a nonuniform one leads to non-equidistant PRs along the $\Gamma_{1dd}$ axis. Nonetheless, the uniform shift of the entire spectrum as a result of changing $\Gamma_{1D}$ is also observed, similar to that in figure 1. In other words, increasing $\Gamma_{1D}$ by 50 causes the PR peaks to shift backwards by 50 along the $\Gamma_{1dd}$ axis.

3.3. Resonances in quartic trap

As shown in figure 3, in a QT + NGP, there is only one well-defined PR at $\delta A = 5$ within the same $\Gamma_{1dd}$ range examined for the other traps. At the larger $\delta A$, a disordered excitation pattern arises. Compared to those in figures 1 and 2, this result is unexpected because a pattern similar to that for the HO + NGP trap was anticipated. Thus, in a QT at stronger driving, a larger number of modes are excited than in the HO trap and box. As such, there exist trapping geometries that, under certain conditions, do not support ordered PR patterns. Furthermore, the geometry of the trap is observed to strongly influence the PR phenomenon and its pattern, and it can, therefore, be used to control the PRs. As in the previous figures, an increase in $\Gamma_{1D}$ by 50 causes the entire PR spectrum to shift by 50 along the $\Gamma_{1dd}$ axis.

Figure 1. Signal energy $D_{1uw}$ of driven one-dimensional DBEC in the box as a function of the dipolar interaction parameter, $\Gamma_{1dd}$ [equation (C1)]. $D_{1uw}$ is normalized using $D_{1uw}^{(0)}$, the maximum of $D_{1uw}$ at $\delta A = 20$ in $\Gamma_{1dd}$-range. $\Gamma_{1D}$ is the s-wave interaction parameter [equation (C1)]. The box is generated from equation (1) using $L_x = 51.2$ and $p = 100$. DBEC is excited by the NGP with the principal depth $A$ being periodically modulated by amplitude $\delta A$ according to equation (3), with $A = -30$, $f = 10$, and $\beta = 4$. Various values are used for $\delta A$ and $\Gamma_{1D}$. Frame (A): thick-solid line: $D_{1uw}$ as a function of $\Gamma_{1dd}$ using $\Gamma_{1D} = 50$ and NGP modulation $\delta A = 5$. The signal is magnified (mag.) 60 $\times$ for comparison with the other cases; thin-dashed line: as for the prior label; but for $\delta A = 10.0$, thin-dashed line: as for the prior label; but for $\delta A = 10.0$, with magnification 10 $\times$; and dashed-dotted line: similarly; but for $\delta A = 20.0$ and no magnification. Frames (B, C): as in (A), but for $\Gamma_{1D} = 100$ and 150, respectively. $A$ and $\delta A$ are in units of $\bar{\omega} \bar{w}$; $L_x$ is in $\bar{a}_{\bar{w}}$; $\Gamma_{1D}$ and $\Gamma_{1dd}$ in $\bar{a}_{\bar{w}}^2$; whereas $\beta$ in $\bar{a}_{\bar{w}}^2$. Meanwhile, $D_{1uw}$ and $D_{1uw}^{(0)}$ are in $(\bar{\omega} \bar{w})^2$.
The origin of the PRs and the shifts in their positions with $\Gamma_D$ are discussed in section 4, based on well-known theoretical methods, namely LVM, time-independent and time-dependent perturbation theories, and Wenzel–Kramer–Brillouin (WKB) approximation for one case. Most importantly, PRs occur whenever the time-averaged energy of the DBEC closely matches one of the energy levels of the trap + NGP.

### 4. Origins of parametric resonances

The appearances of the PRs at certain values of $\mathcal{G}_{1dd}$ are based on numerical solutions of the TDGPE; however, they can be difficult to decipher. The only approach to circumvent this issue is to seek qualitative explanations based on other models and methods, such as LVM [6, 36–39], WKB approximation, and time-dependent and time-independent perturbation theories [70]. Although LVM has previously been applied to investigate the same HO + NGP system as in this study, the DDIs were not considered [6]. This previous work also demonstrated the presence of PRs, albeit in a different framework.

In the present study, by using LVM, it was possible to generate a few resonances in a manner similar to those in figures 1–3. LVM analyses of these PRs can be used in a qualitative manner to better understand them and their origins.

The time-averaged GP energies of some of our systems at PR matched the energy levels calculated using the perturbation theory corresponding to either the HO + NGP, BOX + NGP, or QT + NGP traps. This enables the characterization of their level structure. Through the application of the time-dependent perturbation theory, each PR is demonstrated to correspond to a certain transition between an energy level $m$ of the trap and the energy level $n$ with which the PR is matched, which is identified by the maximum transitional probability (among a set of other probabilities) to the state $n$. The applied LVM theory is explained in SM section 2, and

![Figure 2](image-url)

**Figure 2.** As in figure 1; but for the HO trap with $p = 2$ and $L_x = 1$. Solid line: $D_{\text{kin}}$ as a function of $\mathcal{G}_{1dd}$ for modulation $\delta A = 5$. The signal is magnified $30 \times$; dashed line: $\delta A = 10$ and magnification of $4 \times$; dashed-dotted line: $\delta A = 20$ and no magnification. $D_{\text{kin}}$ is the maximum of $D_{\text{kin}}$ at $\delta A = 20$ in frame (A). $A$ and $\delta A$ are in units of $\hbar / 2$; $L_x$ is in $\alpha_{ho}$; $\mathcal{G}_{1dd}$ and $\mathcal{G}_{1id}$ in $\alpha_{ho}^{-1}$; whereas $\beta$ in $\alpha_{ho}^{-2}$. Meanwhile, $D_{\text{kin}}$ and $D_{\text{kin}}^{(0)}$ are in $(\hbar / 2)^2$. The origin of the PRs and the shifts in their positions with $\mathcal{G}_{1D}$ are discussed in section 4, based on well-known theoretical methods, namely LVM, time-independent and time-dependent perturbation theories, and Wenzel–Kramer–Brillouin (WKB) approximation for one case. Most importantly, PRs occur whenever the time-averaged energy of the DBEC closely matches one of the energy levels of the trap + NGP.
evaluations of the energy levels and transitional probabilities via the perturbation theory are explained in SM section 3.

4.1. LVM resonances

The LVM equation, SM equation (2), is solved numerically with Mathematica over the same \( \Gamma_{dd} \) range as in section 3, using similar parameters for \( \Gamma_D \) and \( \Lambda_0 \), except for \( \Omega \), which is set to different values for the best response of the system to changes in \( \Gamma_{dd} \). For example, an integer value for \( \Omega \) yields resonant behavior in \( \Delta \kappa_n \) and \( \kappa_n \), unlike the real value [6]. For this portion of the study, we principally aimed at a qualitative comparison with the TDGPE that would help explain the PRs reported in section 3.

In figure 4, the signal energy \( D_{kin} \) obtained from \( e_{kin}(t) \), where the latter is given by SM equation (10), is plotted as a function of \( \Gamma_{dd} \). It indicates that LVM generates PRs along the same lines as those displayed in section 3. Importantly, the positions of these resonances are shifted along the \( \Gamma_D \) axis when \( \Gamma_D \) is changed. This is similar to the result observed in figures 1–3 of the TDGPE results. However, in LVM, they shift in the opposite direction, and their amplitudes and shapes, shown in figure 4, are altered slightly because of this shift. This opposite behavior is an artefact of the model arising from the reliance of LVM on the variational Gaussian Ansatz [36, 37, 71]

\[
\psi(x) = \frac{1}{\pi^{1/4} \omega_0} e^{-\frac{x^2}{2\omega_0^2} + i \alpha^2}
\]  

for evaluating the mean-field Lagrangian [6, 36–39]. This Ansatz is significantly less flexible than the numerical solution to the TDGPE, resulting in a different behavior. It is mostly suitable for BECs in an HO trap and thus needs to be further developed for other types of traps. Nevertheless, the discovery of the LVM PRs, as presented
Figure 4. Parametric resonances generated by the Euler–Lagrange differential equation [SM equation (2)] for 1D DBEC in the same HO+NGP trap as figure 2. The graph displays the signal energy $D_{\text{in}}$ of LVM kinetic energy $a_{\text{in}}(t)$ [SM equation (10)] with respect to the DDI parameter $G_{\text{dd}}$. The value of $D_{\text{in}}$ is normalized by $D_{\text{in}}^0 = 1 \times 10^6$. Different values for $G_{\text{dd}}$ and $\Omega$ are considered: 10 and 2, respectively (solid line); 10 and 4 (dashed line); 50 and 2 (dotted line); and 50 and 4 (dashed-dotted line). Initial conditions used in solving the Euler–Lagrange equation at time $t = 0$ are initial speed $v = w(0) = 0$ and the values of equilibrium $w_0 = w(0)$, satisfying SM equation (3) at each $G_{\text{dd}}$ with the value of $d_\omega$ in $\kappa = d_\omega/w$ set to 0.6. All the parameters in this figure are considered unitless to be in line with the LVM formulation of Ref. [36]. Meanwhile, $D_{\text{in}}$ and $D_{\text{in}}^0$ are in units of $(\hbar c)^2/2$.

in figure 4, strongly substantiates the results regarding the TDGPE PRs and supports our claims that they are an inherent feature in the DBECs and not just a result of other influences such as noise or numerical chaos.

Figure 5 displays the time-average of the breathing frequency $\Omega_B(t)$ [SM equation (7)], i.e. $\langle \Omega_B(t) \rangle$, plotted against $G_{\text{dd}}$ for various $G_{1D}$. For a larger $G_{1D}$, the height of the curve decreases. This can be related to the PR shifting under larger resonant values of $G_{\text{dd}}$ in figure 4. In SM equation (7), the term proportional to $G_{1D}$ is linearly added to that proportional to $G_{\text{dd}}$. Thus, when $\langle \Omega_B(t) \rangle$ changes as a result of varying $G_{1D}$, the position of the corresponding PR is updated.

4.2. Perturbative analysis of HO+NGP trap

This section demonstrates that the time-averaged total GP energies $\langle E_{\text{GP}}(t) \rangle$ [equation (9)], at which the driven DBEC resonances, can be generated using the first- and second-order perturbation theory, thereby revealing a certain ELS via the quantum numbers $n$ corresponding to $\langle E_{\text{GP}}(t) \rangle$. First, the HO+NGP is considered. Within this context, the NGP is considered to be a small perturbation within a large HO trap, given that the size of the simulated system is 60 ($a_0, a_0)$, which induces a significantly high potential energy at the trap edges in the order of $\sim 450(\hbar c/2)$. Therefore, time-independent perturbation theory is applied to calculate the energy levels of the DBEC.

Table 1 lists, for example, the $\langle E_{\text{GP}}(t) \rangle$ of the DBEC at resonant values of $G_{\text{dd}}$ for the system in figure 2(A) at $\delta A = 10$, with the corresponding values of $E_n$ obtained from SM equations (11)–(15). The values of $\langle E_{\text{GP}}(t) \rangle$ and $E_n$ were found to be in good agreement after matching and, consequently, an analytical ELS could be deduced.

When $\langle E_{\text{GP}}(t) \rangle$ matches with an $E_n$, a PR occurs. Therefore, a Green’s function of the form

$$G(E) \sim \frac{1}{E - E_n + i\Gamma}$$

that accounts for these PRs could be proposed, where $\Gamma$ is the width, and $E = \langle E_{\text{GP}}(t) \rangle$. This $\langle E_{\text{GP}}(t) \rangle$ increases with $G_{\text{dd}}$, whereas $\langle |V_{\text{GP}}(t)| \rangle$ [see equation (F1) and figure F1 in appendix F] decreases. The quantum number $n$ also indicates that the PRs shift to higher frequencies. In fact equation (F1) is the effective potential already mentioned in the introduction in connection to the results of the experiment of Balik et al [1]. For this case, our findings presented above are analogous to theirs. It should further be added that in a different system than ours, such as one described by mean–field methods with condensate depletion, the Fourier transform of the DDI [SM equation (20)] in a homogeneous system, along with the condensate fraction, contribute to the self–energy of the interactions. The latter plays an important role in defining the poles of the Green’s function, and as a result a change in the strength of the DDI shifts the positions of the resonances described by this function.

Next, the transition probabilities at which PRs occur in an HO+NGP trap are evaluated. Based on the assumption that the NGP and DDI are perturbations inducing transitions between different states in the traps, the probabilities for these can be computed according to the time–dependent perturbation theory, as outlined in SM section 3.

Table 2 displays $\mathcal{P}(n, m, t)$ [SM equation (25)] (in units of arbitrary $\lambda^3$) for the most probable transitions to the states $n$ matched in table 1 for a length of one driving cycle $t = 0.1$. The assumption is $\lambda \ll 1$. A significant number of $m \leftrightarrow n$ transitions are inferred to be prohibited with identically zero values for $\mathcal{P}(n, m, t)$. This
explains the absence of PRs for the values of $\dot{G}_{1,dd}$ (not shown) with a time-averaged GP that can nevertheless be matched to one of the energy levels obtained via the perturbation theory.

### 4.3. Perturbative analysis of BOX$^+$NGP trap

This section is similar to the previous one, except that it focuses on the box potential. Table 3 lists the same quantities as table 1, but for the BOX$^+$NGP. We demonstrate again that perturbation theory reproduces the GP energies at which PRs occur. Furthermore, when $\langle E_{GP}\rangle$ matches $E_n$, a PR arises. Notably, the PRs correspond to significantly high energy levels because the size of the system $2L_0$ is considerably large ($L_0 = 51.2$), yielding small differences in $E_n$ and consequently a large density of states. However, the behavior of $\langle E_{GP}\rangle$ with respect to $G_{1,dd}$ is opposite to that observed for HO$^+$NGP. The reduction in the depth of the effective potential with respect to $G_{1,dd}$ causes the PR energies to decrease, shifting them toward lower frequencies. Thus, the DDI for BOX$^+$NGP cannot prevent the bosons from falling deeper into the NGP. This phenomenon is observed only for the box and not for the HO because of the following two reasons: (1) the density in the BOX$^+$NGP is lower than that in the HO$^+$NGP, resulting in a weaker dipolar nonlinearity in the box; and (2) the quantum pressure in the HO trap is larger than that in the box. This finding, in conjunction with that presented in the previous section, demonstrates the role of the interplay between the trapping geometry and the interactions in defining the positions of the PRs in the spectrum of $\Delta_k$.

As in the previous section, the next objective is to determine the most probable transitions at which the PRs arise in the BOX$^+$NGP trap. Table 4 is similar to table 2, but it provides the results for the box. The most probable transitions occur from relatively high states $m$ toward $n$ matched in table 3.

### 4.4. Energy levels in QT$^+$NGP via perturbation theory

Table 5 shows that $\langle E_{GP}\rangle$ at the PRs in the QT can be closely matched with the energies from the second-order perturbation theory via SM equations (34) to (40). We intend to emphasize that well-behaved, exact analytic solutions to the Schrödinger equation with a quartic oscillator are hitherto unknown. Although the Heun function [72] may provide a solution within a restricted range in the neighborhood of the trap center, outside of this center, it diverges to significantly large values. We observed that even numerical solutions yielded by Mathematica diverged away from the center (not shown here).

Finally, because of the divergent wavefunction of the QT$^+$NGP, analytical evaluations of the $P_{DIT}$ and $P_{DDII}$, as explained for the HO$^+$NGP and BOX$^+$NGP are currently extremely difficult, if not impossible.

To this end, the validity of perturbation theory applied in generating the above matches of $E_n$ to $\langle E_{GP}\rangle$ in the aforementioned tables 1, 3, and 5 has been checked and verified. For more details on this issue refer to appendix G.

### 5. Summary and conclusions

In summary then, we reported mean-field parametric resonances (PRs) in a one-dimensional (1D) dipolar Bose–Einstein condensate (DBEC) excited by a negative Gaussian potential (NGP) with a periodically oscillating depth. PRs were detected using the signal energy [35] with unprecedented features. The PR energies matched...
the elucidation of the effects of condensate depletion

avenue.

dynamical fragmentation of a condensate, such as it appears in the case of a rotationally stirred BEC

address the role of condensate depletion in the properties of PRs. It cannot account for PRs associated

DDI nonlinearity would be interesting.

the form of a nonlinearity, and thus, comparisons between the many-body effects of the DDIs and those of the

the same systems using the many-body Schrödinger equation with DDI. In this case, the DDIs do not occur in

interplay between the con

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inherent feature present in the DBECs awakened by external driving agents.

However, it was not possible to use LVM for the box and quartic traps because the Gaussian Ansatz was

most importantly, the DBEC PRs could be produced via the Lagrangian variational method (LVM).

energy levels computed via time-independent perturbation theory, allowing us to characterize the energy-level structure (ELS) of a DBEC in a complex trap. A few different traps were then applied to test the effect of the interplay between the confining geometry and the dipole–dipole interactions (DDI) on PRs. DDI was inferred to be influential in defining the amplitudes of PRs and their energies by its control over the depth of the effective mean-field potential.

The trapping geometry in conjunction with the NGP and DDIs was shown to govern the transition probabilities between different quantum states of confinement. It was rather surprising that these PRs corresponded only to specific values of the principal quantum numbers of high transitional probabilities. The presence of magic quantum numbers may even be assumed, although this hypothesis needs to be proven.

Most importantly, the DBEC PRs could be produced via the Lagrangian variational method (LVM).

Table 1. Matching of time-averaged GP energies $\langle E_{GP} \rangle$ [equation (9)] with energy levels $E_n$ obtained from the second-order perturbation theory. The system considered is the HO + NGP trap in figure 2(A) for $\gamma_{NGP} = 50$ and $\delta A = 10$. $n$ is a quantum number for the HO states. $E_n$ values obtained from SM equations (11)–(15). From left to right, the table lists $\bar{G}_{id}$ at PR, $\langle E_{GP} \rangle$, $E_n$ at quantum state $n$, number of states $M$ determined for second-order correction SM equation (13), difference between matched energies, and the error in the GP energy $\delta E_{GP}$ [equation (10)]. $\langle E_{GP} \rangle$, $\delta E_{GP}$, and $E_n$ are expressed in units of $\mathcal{Z}_C^n$ and $\mathcal{G}_{id}$ is in $\tilde{a}_0^{-1}$.

| $\bar{G}_{id}$ | $\langle E_{GP} \rangle$ | $E_n$ | $n$ | $M$ | $\langle E_{GP} \rangle - E_n$ | $\delta E_{GP}$ |
|---------------|----------------|------|----|-----|-----------------|----------|
| 29.9297       | 2.67631        | 2.66726 | 5  | 23  | 0.00905         | ±2.63090 |
| 67.8894       | 5.79083        | 5.77971 | 7  | 21  | 0.01112         | ±2.11159 |
| 120.449       | 9.62129        | 9.62126 | 9  | 6   | 0.00003         | ±1.70173 |
| 191.988       | 15.1178        | 15.1395 | 13 | 12  | 0.02172         | ±1.76877 |
| 292.727       | 18.3912        | 18.4233 | 19 | 8   | 0.03410         | ±1.32672 |

Table 2. Transition probability $P(m, n; t)$ from states $m$ to $n$ at the resonant values of $\bar{G}_{id}$ listed in table 1. $P(m, n; t)$ is computed using SM equations (18), (24), and SM (25). The values of $n$ are HO quantum numbers at which GP energies are matched in table 1 for HO+NGP trap at time $t = 0.1$. The values of $m$ correspond to the largest $P(m, n; t)$. From left to right, the table lists $\bar{G}_{id}$, $m$, $n$, $P(m, n; t)$, and the error $\delta P(m, n; t)$ obtained from SM equation (41). $\bar{G}_{id}$ is expressed in units of $\tilde{a}_0^{-1}$.

| $\bar{G}_{id}$ | $m$ | $n$ | $P(m, n; t) / \mathcal{X}$ | $\delta P(m, n; t) / \mathcal{X}$ |
|---------------|----|----|---------------------------|-------------------------------|
| 29.9297       | 3  | 5  | 0.0118                     | 3.9152 $\times 10^{-3}$      |
| 67.8894       | 5  | 7  | 0.0511                     | 1.6969 $\times 10^{-4}$      |
| 120.449       | 7  | 9  | 0.1298                     | 4.3119 $\times 10^{-4}$      |
| 191.988       | 11 | 13 | 0.2210                     | 7.3413 $\times 10^{-4}$      |
| 292.727       | 17 | 19 | 0.3184                     | 1.0576 $\times 10^{-3}$      |

The results presented in sections 3.1–3 were also a manifestation of the role of the mean-field dipolar nonlinearity in defining the properties of the PRs and, through its interplay with the trapping geometry, their spacings $\Delta \bar{G}_{id}$. However, a question that arises is whether similar results could be obtained by simulations of the same systems using the many-body Schrödinger equation with DDI. In this case, the DDIs do not occur in the form of a nonlinearity, and thus, comparisons between the many-body effects of the DDIs and those of the DDI nonlinearity would be interesting.

Despite the power of the applied mean-field method, it has its limitations. For example, GP theory cannot address the role of condensate depletion in the properties of PRs. It cannot account for PRs associated with dynamical fragmentation of a condensate, such as it appears in the case of a rotationally stirred BEC [22]. Thus, the elucidation of the effects of condensate depletion/fragmentation on PRs is an interesting future research avenue.

In this regard, it is worthwhile to comment on the expected effects of condensate depletion on the PR properties, in connection to an important examination within the framework of the Hartree–Fock-Bogoliubov theory [73], where the depleted and anomalous densities—$\bar{n}$ and $\bar{m}$, respectively—are coupled with the order parameter, $\Phi(r)$. If we were to conduct this study using the Hartree–Fock-Bogoliubov theory, an increase in
driving strength would be expected to excite more particles out of the BEC, amplifying the PRs and changing their positions. On the one hand, such amplification occurs because of the enhancement of $\bar{n}$ and $\bar{n}'$ [73], accompanied by a drop in the condensate fraction, $n_0$. A peculiar fact is then that $\bar{n}'$ would contribute next to $\bar{n}$ in amplifying the PRs. On the other hand, the anticipated change in PR positions arises from the fact that $n_0$ in the beyond-mean-field methods determines the weight of the self-energy $\Sigma(k, \omega)$ in the Green’s function:

$$G(k, \omega) = \frac{1}{\omega - \epsilon_k + i0 - \Sigma(k, \omega)},$$

which, in turn, controls the excitation frequency at which a PR arises, particularly in homogeneous systems. Here $\omega$, $k$, and $\epsilon_k = \hbar^2 k^2/(2m)$ are the excitation frequency, momentum wave vector, and the free-particle kinetic energy, respectively.

Further, it would be interesting to apply the generalized Gross–Pitaevskii equation (GGPE)—which includes the Lee–Huang–Yang (LHY) correction—to study the role of quantum depletion in the PRs. In the GGPE, the LHY correction can be computed from equation (5) in Ref. [73] that reads (although for three dimensions):

$$\delta \mu_{\text{LHY}}(r) = \int d r' V(r - r') [\bar{n}(r, r') \Phi(r') + \bar{n}(r, r') \Phi^*(r')]$$

and includes the coupling of $\bar{n}$ and $\bar{n}'$ with $\Phi$. It is this coupling that yields a measure for condensate depletion in the beyond-mean-field method. This correction modifies the GGPE energy functional, leading to results that are different from the typical GP energy.

Moreover, since the pair-correlation function also depends on $\bar{n}$, the interplay between DDI and quantum correlations could further modify the properties of the PRs. Thus, it is anticipated that richer PR physics will be obtained in the case of beyond-mean-field methods.
Acknowledgments

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. NGP force

The time modulation of the NGP depth in (2), i.e. \( \delta V_{DT}(x,t) = \delta A \cos(\Omega t)e^{-\beta x^2} \), generates an oscillating force along the length of the DBEC, which is given by the potential gradient
\[
\delta F_x = -\frac{\partial \delta V_{DT}(x,t)}{\partial x} = 2 \beta \delta A \cos(\Omega t)e^{-\beta x^2}.
\]
This force transfers momentum \( \Delta p_x \) from the oscillating NGP to the BEC that is symmetric about \( x = 0 \), resulting in
\[
\Delta p_x = \int_0^\infty \delta F_x \, dt = 2 \beta \delta A \sin(\Omega t) \frac{\Delta x}{\Omega},
\]
The latter is maximum at \( x = \pm 1/\sqrt{2\beta} \) and minimum at \( x = 0 \) and the edges of the trap. \( \delta F_x \) and \( \Delta p_x \) are in units of \( \hbar \omega / a_{ho} \).

Appendix B. Dipolar interactions and their control

This section explains the manipulation of the DDI. Consider the DDI potential given by [40, 51]:
\[
U_{dd}(R) = C_{dd} \left(1 - 3 \cos^2 \theta \right) \frac{1}{|R|^3},
\]
where \( R = r - \hat{r}' \) is the relative position vector of two dipoles at \( r \) and \( \hat{r}' \), and \( \theta \) is the angle between \( R \) and the orientation of the dipoles; \( C_{dd} = E^2 \alpha^2 / (\hbar \epsilon_0 c) \) [40, 74] is the DDI strength in trap units, where \( E \) is an external electric field, \( \epsilon_0 \) is the permittivity of free space, and \( \alpha \) is the static polarizability. Thus, the DDI can be induced and tuned by an external field \( E \) [58] to various orders of magnitude [75]. Moreover, in polar molecules, the dipole moment can be set up to \( 10^6 \) times larger than that in atomic systems. The DDIs also occur naturally [68, 76, 77] if the atoms possess a magnetic dipole moment \( \mu_B \); in such a case, \( C_{dd} = \mu_B \beta^2 / (4 \pi \hbar \omega) \) [40, 51] (in trap units), where \( \mu_0 \) is the permittivity of free space. The control of DDIs has been further explained by Lahaye et al [40]. Moreover, by applying the linear Stark effect, Rydberg-dressed atoms can be excited to significantly high principal quantum numbers to achieve large dipole moments such as \( \rho \sim 1450D \), where \( D \) is the ratio between the dipolar and s-wave scattering length [78]. Based on these principles, we justify the use of large values for DDI parameters \( G_{idd} \) in the present study.

Appendix C. Parameters

The 3D s-wave and DDI parameters are defined as \( G = 4 \pi Na \) and \( G_{idd} = 3Na_{add} \), respectively, where \( a \) and \( a_{dd} \) are the s-wave and dipolar scattering lengths, respectively, and \( N \) is the number of particles. The interaction parameters acting in one dimension, i.e. \( G_{1db} \) and \( G_{idd} \), are obtained after reducing the 3D TDGPE to its 1D form (4). As a result, \( G \) and \( G_{idd} \) are divided by a factor \( 2\pi d_p^2 \), resulting in
\[
G_{1db} = \frac{G}{2\pi d_p^2} \quad \text{and} \quad G_{idd} = \frac{G_{idd}}{2\pi d_p^2},
\]
where \( d_p \) is the width of the integrated-out wave function in the transverse direction. \( G_{1db} \) and \( G_{idd} \) are input directly into the code without an explicit evaluation via \( N, a, a_{dd} \) and \( d_p \). These parameters define the strength of the s-wave and dipolar-interaction nonlinearities of TDGPE. The systems are simulated from \( x = -L_0 \) to \( x = L_0 \).
Appendix D. Applicability of signal energy

The applicability of using signal energy to elucidate important properties of the excitations in a driven BEC has been demonstrated in our previous work [2], where the signal energy was argued to be tantamount to the time-average of the squared amplitude of an oscillating signal that describes a dynamic variable, i.e. it can be likened to the root-mean-squared value of the alternating voltage or current. Here, because $E_{\text{kin}}(t)$ is observed to be oscillating with respect to time, applying this concept for its measurement is convenient. In addition, kinetic energy is used because it is an important property of the condensate [79]. Other quantities such as the potential energy, zero-point energy, and radial size could also have been used because they reveal the same PRs with the same properties as obtained for $E_{\text{kin}}$. Further, $E_{\text{kin}}$ has been used in a number of previous studies [6, 37, 50, 80–82], which further demonstrates its importance. The experimental measurement of $D_{\text{kin}}$ can be performed as described in the supplementary material (SM) section 1.

Appendix E. Numerics

The TDGPE [equation (4)] is solved numerically via the split-step Crank–Nicolson (CN) method [51, 60] in real time for the HO ($p = 2$), QT ($p = 4$), and BOX ($p = 100$) potential traps. For the HO and QT, propagation occurs along a grid of $N_x = 6000$ pixels of size $\Delta x = 0.01$ with $L_x = 1$ in equation (1). The time step chosen is $\Delta t = 1 \times 10^{-4}$, and the number of time steps in the transient run is set to $N_{\text{pas}} = 10^3$; the same values are used for the final run $N_{\text{run}} = 10^6$. For the box potential $N_x = 2048$, $\Delta x = 0.05$, $L_x = 51.2$, $\Delta t = 0.00025$, $N_{\text{pas}} = 200000$, and $N_{\text{run}} = 200000$. In all cases, the system is initialized with a stationary NGP of depth $A = -30$ via the imaginary-time CN method for a number of $N_{\text{imp}} = 2 \times 10^3$ time steps and then taken through $N_{\text{pas}} = 20000$ steps in the transient and ending with $N_{\text{run}} = 20000$ steps in the final run. The time step $\Delta t$ is the same as that in the corresponding real-time simulations. Among the CN codes that are used, the final run (NRUN) complements the transient run (NPAS), which could be the stage during which the BEC is allowed to evolve and relax to a stable state. The final run could then be the stage where the BEC is optionally excited by an external agent, enabling an examination of its ensuing dynamics. The authors of the code originally separated the transient and final runs for organizational purposes, such that data files in the relaxation stage of the BEC toward a stable state are separate from those for where this stable BEC is suddenly excited by an external driving force. By contrast, the present study considers a BEC that is continuously excited under both transient and final runs, and thus, there is no difference between them.

Sets of runs were performed, each at a fixed value of $G_{1D}$. For each $G_{1D}$, the system was scanned over a range of $G_{1dd}$ from 0 to 400 in steps of 2, and for each $G_{1dd}$ one run was performed. Thus, a large number of runs were conducted in the form of parallel array jobs on the high-performance computational cluster of the Scientific Computing Laboratory of the Institute of Physics in Belgrade, Serbia. Each simulation required approximately five days for completion.

Appendix F. Effective potential

This section explains the reason for the decline in the amplitude of a PR with $G_{1dd}$, as observed in figures 1 and 2. This decline is attributed to the change in the depth of the time-averaged effective potential, which is expressed as

$$
\langle V_{\text{eff}}(x) \rangle_t = \langle \langle V_{\text{DT}}(x, t) + G_{1D} | \psi(x, t) |^2 \rangle_t \rangle + \frac{4\pi}{3} G_{1dd} \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} e^{-\frac{1}{2} \alpha_x x_{1D}(\tau_0)} |\tilde{\psi}(k_x, \tau_0)|^2 \rangle_t \rangle + V_0(x),
$$

where

$$
\langle \cdots \rangle = \frac{1}{T} \int_0^T (\cdots) dt,
$$

and

\[ x = + L_0, \text{where } L_0 = 30 \text{ for a harmonic oscillator (HO) trap and a quartic trap (QT), and } L_0 = 51.2 \text{ for a box potential. The parameters of the oscillating NGP are } A = -30, \beta = 4, \text{ and } \Omega = 2\pi f, \text{ where } f = 10. \text{ The applied values of } \delta A \text{ are 5, 10, and 20. The DBECs are scanned over a } G_{1dd} \text{ range from 0 to 400 in steps of 2, and for each } G_{1dd} \text{ one run is performed. The values of } G_{1D} = 50, 100, \text{ and 150 are used for each set of runs. } G_{1D} \text{ and } G_{1dd} \text{ are expressed in units of } \alpha^{-1}, \text{ where } a_{\text{ho}}, a_{\text{dd}}, \text{ and } d_j \text{ are all in units of } a_{\text{ho}}. \]
and $T$ is the total simulation time. As an example, figure F1 shows $\langle V_{\text{eff}}(x) \rangle_t$ for the HO+NGP of figure 2(A) at $\delta A = 10$ and indicated resonant values of $G_{1dd}$: Solid line: 29.9297; dotted line: 67.8894; triple-dotted line: 120.449; dashed-dotted line: 191.988; and dashed-double-dotted line: 292.727. Frame (B): Minima $V_{\text{eff,min}} = \langle V_{\text{eff}}(0) \rangle_t$, at PR values of $G_{1dd}$ for some of the systems presented in figures 1 and 2 at $\delta A = 10$. Solid circles: BOX+NGP at $G_{1D} = 100$; open circles: same at 50; open squares: HO+NGP at $G_{1D} = 50$; and solid squares: same at 100. $\langle V_{\text{eff}}(x) \rangle_t$, $V_{\text{eff,min}}$, and $\delta A$ are expressed in units of $\bar{w}$; $L_x$ and $x$ are in $\alpha_{ho}^{-1}$ in $a_{ho}^{-1}$; whereas $G_{1D}$ and $G_{1dd}$ are in units of $a_{ho}^{-1}$.

Figure F1. Frame (A): Time-averaged effective potential $\langle V_{\text{eff}}(x) \rangle_t$ [equation (F1)] for HO+NGP in figure 2(A) at $\delta A = 10$ and indicated resonant values of $G_{1dd}$: Solid line: 29.9297; dotted line: 67.8894; triple-dotted line: 120.449; dashed-dotted line: 191.988; and dashed-double-dotted line: 292.727. Frame (B): Minima $V_{\text{eff,min}} = \langle V_{\text{eff}}(0) \rangle_t$, at PR values of $G_{1dd}$ for some of the systems presented in figures 1 and 2 at $\delta A = 10$. Solid circles: BOX+NGP at $G_{1D} = 100$; open circles: same at 50; open squares: HO+NGP at $G_{1D} = 50$; and solid squares: same at 100. $\langle V_{\text{eff}}(x) \rangle_t$, $V_{\text{eff,min}}$, and $\delta A$ are expressed in units of $\bar{w}$; $L_x$ and $x$ are in $\alpha_{ho}^{-1}$ in $a_{ho}^{-1}$; whereas $G_{1D}$ and $G_{1dd}$ are in units of $a_{ho}^{-1}$.

Appendix G. Range of validity of perturbation theory

The range of validity of the perturbation theory, applied to the computation of PR energies in tables 1, 3, and 5 emerges from the conditions:

$$ \frac{E^{(1)}_n}{E^{(0)}_n} \ll 1 \quad (G1) $$

and

$$ \frac{E^{(2)}_n}{E^{(0)}_n} \ll 1. \quad (G2) $$

It is the range of quantum numbers $n$, for which the corrections $E^{(1)}_n$ and $E^{(2)}_n$ satisfy the conditions in equations (G1) and (G2) for a given NGP depth $A$ and number of states $M$. For the purpose at hand and for $n$ values in the aforementioned tables, the perturbation theory was found to be valid at an NGP depth of $A = -30$. For example, figure G1 displays the ratios in equations (G1) and (G2) for the BOX+NGP system. Choosing some of the values of $M$ listed in table 1 as an example, it was found that for $M = 63$ the ratio $|E^{(1)}_n / E^{(0)}_n| < 1$ when $n > 100$, whereas for $M = 17$ and 27, this occurred when $n > 75$. On the other hand, the ratio $|E^{(2)}_n / E^{(0)}_n| < 1$
when $n > 60$. This affirms that the values of $n = 131, 109, 79$ corresponding to the values of $M = 63, 17, 27$, respectively, yield corrections that lie within the validity regime of the perturbation theory.

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