Critical Current in the Strip Wires from the Josephson Medium

M V Belodedov¹, L P Ichkitidze²,³ and S V Selishchev²

¹ Bauman Moscow State Technical University, Moscow, 105005 RF
² National Research University of Electronic Technology, Zelenograd, Moscow, 124498 RF
³ I.M. Sechenov First Moscow State Medical University, Moscow, 119435 RF

E-mail: ichkitidze@bms.zone, leo852@inbox.ru

Abstract. On the basis of the previously proposed phenomenological material equation of a superconducting granular medium, numerical simulation of some products from granular superconductors which include, in particular, products from high-temperature superconducting ceramics (and electrodynamic processes in them) is carried out. An expression is obtained for the first critical field of the medium in question and its relation to the phenomenological parameters of the medium. A strip wire model from a granular superconducting medium is constructed, the maximum current of these wires and its dependence on the external magnetic field were obtained. An analogue of the SQUID based on the strip from a granular superconductor is proposed, it is shown that its field characteristic has similarities with the field characteristics of the classical single-contact SQUID.

1. Introduction

The practical application of high-temperature superconducting ceramics (HTSC) is constrained by the lack of an adequate HTSC model as a multiple Josephson medium. It is generally accepted to consider such a medium in the form of a stochastic set of superconducting granules, at the points of contact of which Josephson junctions are formed [1]. By averaging the Ginzburg-Landau equation over a small volume containing a large number of granules [2], it is possible to obtain a material equation relating in the stationary case the vector of the average current density \( \vec{j} \) in the medium and the vector potential of the magnetic field \( \vec{A} \) [3]:

\[
\vec{j} = -\vec{A} I_c \rho a \frac{\pi}{8} \left( \frac{2\pi}{\Phi_0} Aa \right) \exp \left[ -\frac{\pi}{16} \left( \frac{2\pi}{\Phi_0} Aa \right)^2 \right] - \frac{\rho}{Rc} \frac{\pi a^2}{8} \frac{d\vec{A}}{dt} - \frac{C \rho}{c} \frac{\pi a^2}{8} \frac{d^2\vec{A}}{dt^2},
\]

where \( \Phi_0 = \hbar c / 2e = 2.07 \times 10^{-7} \text{ G} \times \text{cm}^2 \) – magnetic flux quantum, \( \rho \) – concentration of Josephson junctions in a medium, \( I_c \), \( R \) and \( C \) – mean values of critical current, active resistance and capacity of a single Josephson junction, \( a \) – the average size of the granules forming the junction.
The material equation (1) is derived under the assumption of the Maxwellian law of distribution of the size of individual granules. At the same time, as the modeling of processes in multiple Josephson medium shows, the nature of the processes remains unchanged in the cases of granule sizes distribution according to other laws [3].

As has been shown in [3], the penetration of a weak magnetic field \( A \ll \Phi_0/(a\pi) \) into a semi-infinite medium should be described by equation:

\[
V^2 \vec{B} = \frac{2\pi I_p \rho \pi^2 \mu a^2}{\Phi_0} \vec{B},
\]

(2)

where the value \( \mu \) has the meaning of the magnetic permeability of the medium, due only to the Meissner currents of individual granules. Equation (2) gives the grounds for introducing the depth of penetration of a weak magnetic field:

\[
\lambda = \sqrt{\frac{\Phi_0}{2\pi \mu \rho \pi^2 a^2}}.
\]

(3)

With allowance for (3), the material equation (1) can be written in the stationary case as

\[
\vec{j} = -1 \frac{e}{4\pi \mu} \exp \left[ -\frac{\pi}{16} \left( \frac{2\pi \Phi_0}{A} \right)^2 \right].
\]

(4)

which gives grounds for describing the distribution of the magnetic field in the medium by the equation:

\[
V^2 \vec{A} = \frac{1}{\lambda^2} \vec{A} \exp \left[ -\frac{\pi}{16} \left( \frac{2\pi \Phi_0}{A} \right)^2 \right].
\]

(5)

2. Penetration of a homogeneous magnetic field into a semi-infinite medium

Let us consider the penetration of a magnetic field into a semi-infinite medium in a general form, without satisfying the condition \( A \ll \Phi_0/(a\pi) \). Suppose, for definiteness, the semi-infinite medium occupies the space \( x \geq 0 \), and the magnetic field in the space \( x < 0 \) is directed along the \( y \) axis and has the value \( B_0 \). In this case, the vectors \( \vec{A} \) and \( \vec{j} \) will be directed along the \( z \) axis, and their quantities must obey the equations:

\[
\frac{d^2}{dx^2} A = \frac{A}{\lambda^2} \exp \left[ -\frac{\pi}{16} \left( \frac{2\pi \Phi_0}{A} \right)^2 \right];
\]

(6a)

\[
j = -A \frac{e}{4\pi \mu} \exp \left[ -\frac{\pi}{16} \left( \frac{2\pi \Phi_0}{A} \right)^2 \right],
\]

(6b)

that for the dimensionless quantities

\[
\tilde{x} = \frac{x}{\lambda_M}, \quad \tilde{A} = \frac{2\pi}{\Phi_0} A, \quad \tilde{B} = \frac{2\pi a \lambda_M}{\Phi_0} B = \frac{d\tilde{A}}{d\tilde{x}}, \quad \tilde{j} = -\frac{8}{I_p \rho \pi a} \tilde{j}.
\]

(7)

will have the following form:

\[
\frac{d^2}{d\tilde{x}^2} \tilde{A} = \tilde{A} \exp \left[ -\frac{\pi}{16} \tilde{A}^2 \right];
\]

(8a)

\[
\tilde{j} = -\tilde{A} \exp \left[ -\frac{\pi}{16} \tilde{A}^2 \right].
\]

(8b)

Irrotational penetration of the magnetic field requires that equation (8a) has a solution under boundary conditions:
\[
B_{\|} = \left. \frac{dA}{dx} \right|_{x=0} = \frac{2\pi \alpha \lambda_M}{\Phi_0} B_0 \quad \text{and} \quad \lim_{x \to \infty} \tilde{A}(\tilde{x}) = 0. \tag{9}
\]

Numerical simulation of equation (8a) with boundary conditions (9) allows us to conclude that the maximum value of the external magnetic induction, at which there are no "hypervortices" and \( \lim_{x \to \infty} \tilde{B}(\tilde{x}) = 0 \), is equal to

\[
\left( B_{\|} \right)_{\text{max}} = 2.256903..., \]

which corresponds to the first critical field of the medium under in question:

\[
B_{c1} \approx 2.256903 \frac{\Phi_0}{2\pi \alpha \lambda_M}. \tag{10}
\]

### 3. Simulation of current flow through strip wires

Of great practical interest is the problem of the flow of a superconducting current through a stripline. Consider such a line lying in the XY plane with a thickness \( D \) and a width \( b \), directed along the y-axis. Let the magnetic field be directed along the Z axis and has the value of \( B = dA/dx \). The current density in the strip (under the condition that there are no vortices), in accordance with the Maxwell equations, is determined by the vector potential of the magnetic field:

\[
j(x) = -\frac{c}{4\pi \mu} \frac{d^2 A}{dx^2},
\]

whose distribution of along the coordinate \( x \) obeys the equation (8a):

\[
\frac{d^2}{dx^2} \tilde{A} = \tilde{A} \exp \left[ -\frac{\pi}{16} \tilde{A}^2 \right].
\]

A definite solution of equation (8a) is possible only if there two boundary conditions. The first of them in the absence of an external magnetic field is obvious – the values of the magnetic field at the edges of the strip should be opposite in sign:

\[
B\left( \frac{b}{2} \right) = -B\left( \frac{-b}{2} \right),
\]

which corresponds to:

\[
\left. \frac{d\tilde{A}}{dx} \right|_{x=\frac{-b}{2}} = -\left. \frac{d\tilde{A}}{dx} \right|_{x=\frac{b}{2}}. \tag{11}
\]

The second boundary condition can be, for example, the condition:

\[
\tilde{A}(\tilde{x}) \bigg|_{\tilde{x}=0} = \tilde{A}_0. \tag{12}
\]

Equation (8a) with boundary conditions (11)-(12) can be solved by standard Runge-Kutta method.

A typical form of \( j(x) \) distributions with boundary condition (11) for different values of the \( \tilde{A}_0 \), constant in strips of different widths is shown in Fig.1.

In narrow \( j \) stripes with a widths of the order of or less than \( \lambda \) a practically uniform distribution of the current is observed, in wide (several \( \lambda \)) – all the current is concentrated only on the edges of a strip 1-2 \( \lambda \) wide.

When modeling the flow of current through strips in an external magnetic field with induction \( \tilde{B}_0 \), perpendicular to the plane of the strip, the picture becomes more complicated. It is logical to assume that the current distribution along the width of the strip remains symmetrical. In this case, the changes in the magnetic field caused by the current of the strip at the opposite edges of the strip will be the same.
in magnitude, and opposite in sign, so the ratio must be respected:

\[
\left( B\bigl|_{x=\frac{b}{2}} + \tilde{B}\bigl|_{x=\frac{b}{2}} \right) / 2 = \left( \frac{d\tilde{A}}{dx}\bigl|_{x=\frac{b}{2}} + \frac{d\tilde{A}}{dx}\bigl|_{-\frac{b}{2}} \right) / 2 = \tilde{B}_0.
\]  

(13)

Fig. 1. Current distribution in strips of different widths. Dimensionless units (7) are used.

Thus, specifying two values \( \tilde{A}_0 \) and \( \tilde{B}_0 \) means setting two boundary conditions (12) and (11) and, thus, completely determines the solution of the equation (8a) and the current density distribution over the width of the strip \( \tilde{j}(x) \). The distribution of the current density over the width of the strip makes it possible to calculate the total current through the strips:

\[
\bar{I}_{\text{tot}} = D \int_{-b/2}^{b/2} \tilde{j}(\tilde{x})d\tilde{x}.
\]

The numerical solution of equation (8a) with boundary conditions (11) and (12) make it possible to obtain a tabular dependence \( \bar{I}_{\text{tot}}(\tilde{A}_0, \tilde{B}_0) \). If one poses the problem of determining the maximum current through strips for a given external magnetic field, it is necessary to find the maximum value of \( \bar{I}_{\text{tot}} \) in the set of all values of the constant \( \tilde{A}_0 \):
\[ \tilde{I}_{\text{max}}(\tilde{B}_0) = \max_{\tilde{A}_0} \left[ \tilde{I}_{\text{sub}}(\tilde{A}_0, \tilde{B}_0) \right] = \max_{\tilde{A}_0} \left\{ D \int_{-\alpha/2}^{\beta/2} \tilde{j}(\tilde{x}) \, d\tilde{x} \right\}. \] (14)

Obtained by the described method using the second-order Runge-Kutta method, the values of the maximum current through the strips for different values of the external magnetic field \( B_0 \) for strips of different widths are shown in the graphs of Fig.2.

The resulted results allow to formulate recommendations for the designing of power wires on the basis of granular superconductors (including those based on HTS-ceramics): to achieve the maximum possible current, it is advisable to make wires in the form of a kind of "litzwire", that is, a bundle of separate (without mutual superconducting contact) threads of width no more than \( \lambda \). At first glance, it seems that it is sufficient to "cut" a wide strip into several narrow strips with a width of no more than 1-2 \( \lambda \). This, however, is not so. Numerical calculations show that if the gaps between individual thin strips tend to zero width, the current distribution in the set of strips begins to seek to the current distribution in a wide strip of total thickness. So the gaps between the strips must be made sufficiently wide, much larger than the width of the individual thin strips.

Fig.2. The dependence of the maximum superconducting current through the Josephson medium strips on the external magnetic field for strips of different width (in units \( \lambda \)). \( \tilde{B}_0 \) is measured in units of \( \Phi_0/(2\pi a\lambda) \), \( \tilde{I}_{\text{max}} \) – in units of \( I_c \rho D a \lambda \pi/8 \).

4. Magnetization of a cylinder from a granular superconductor
Consider a solid cylinder of a Josephson medium in an external magnetic field directed along the axis of the cylinder. Let the length of the cylinder \( L \) substantially exceeds its radius \( R \). If the magnetic field has a sufficiently small induction \( B \ll B_j \) (10), there should be an vortex-free penetration of the magnetic field described by equation (5):

\[ \nabla^2 \tilde{A} = \frac{1}{\lambda^2} \tilde{A} \exp \left[ -\frac{\pi}{16} \left( \frac{2\pi \rho A a}{\Phi_0} \right)^2 \right] \]

If the consider structure is sufficiently solitary, the magnetic field and current density distributions have a radial symmetry and it will be logical to go over to the polar coordinates and assume that the vector potential of the magnetic field has only a component directed along the direction of the magnetic field \( (z) \) and is described by the equation:
\[ \nabla^2 A = \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (rA) \right) = A \exp \left[ -\frac{\pi}{16} \left( \frac{2\pi A_0}{A} \right)^2 \right]. \] (15)

Equation (15) is an ordinary differential equation of the second order, the unique solution of which is possible when two boundary conditions are given. The first of them is obvious—the magnetic field on the surface of the cylinder must coincide with the external magnetic field:

\[ \left. \frac{1}{r} \frac{d}{dr} (rA) \right|_{r=R} = B_0,\quad \Rightarrow \quad \left. \frac{dA(r)}{dr} \right|_{r=R} = B_0 - \frac{A(R)}{R}, \] (16)

the second is less obvious. In order to determine it, we integrate the function \( A(r) \) along a circular contour of radius \( r \) with the center at point \( r = 0 \), which will give the magnetic flux, covered by this contour:

\[ \oint A(r) dl = A(r) \times 2\pi r = \Phi(r). \] (17)

If there are no vortices, the aspiration \( r \) to zero ensures that the right-hand side of (17) is tending to zero, which implies the second boundary condition:

\[ \lim_{r \to 0} \left[ rA(r) \right] = 0. \] (18)

Equation (15) with boundary conditions (16), (17) is easily solved by numerical methods. The results of the simulation are shown in Fig.3, where dependences of the magnetic flux, captured by the cylinder, normalized to \( 2R^2 \), on the external magnetic field \( B_0 \) are given for cylinders with radii of \( R = 1, 3, 10, 30 \). If instead of a cylinder from a granular superconductor an empty space would be used, the ratio \( \Phi/R^2 \) would be reduced exactly to \( \pi B_0 \), so in Fig.3 it is also depicted by a dotted line with a slope \( \pi \). Fig.3 uses dimensionless units (7) too.

![Fig.3](image)

**Fig.3.** Magnetization curve of a long cylinder from granular superconductor for different values of cylinder radius

Sections with a steep slope of the magnetization curve of an HTSC cylinder with a sufficiently large radius (\( R \geq 30 \)) can be very useful in the construction of sensors of a weak magnetic field. Indeed, almost immediately after the discovery of HTSC materials, cylindrical rods of them are used in magnetically modulated magnetometers (MMM) as magnetically sensitive elements [4–9]. The MMM achieved a resolution in the magnetic field (10–1 pT), which is comparable to the sensitivity level in HTSC SQUIDs [4,5,10].

5. **The possibility to building a SQUID based on a granular superconducting medium**

We consider strips of a Josephson medium, coiled into a closed thin-walled cylinder with a diameter much larger than the wall thickness and height, not larger than \( \lambda \). We assume that a circular current flows through the wall of cylinder with a density \( j \), uniformly distributed along the length of the cylinder, as shown on Fig.4. This current has a surface density of \( jd \), where \( d \) is the thickness of the cylinder walls. We find the connection between the magnetic flux \( \Phi \), captured by the cylinder and the external...
magnetic field $B$. The magnetic induction inside the cylinder $B_0$ can be written in the first approximation as $B_0 = \Phi_0 / (\pi R^2)$, where $R$ is the radius of the cylinder. The external magnetic field, according to the circulation theorem, must have an induction.

$$B = B_0 - \frac{4\pi}{c} j d. \quad (19)$$

The uniform circular current flowing along the wall of cylinder is determined by the vector potential $A$, codirected with this current:

$$j = -I_A \rho \frac{\pi}{8} \left[ \frac{2\pi a}{\Phi_0 A} \right] \exp \left[ -\frac{\pi}{16} \left( \frac{2\pi a}{\Phi_0 A} \right)^2 \right]. \quad (20)$$

we obtain $A_0 = RB_0/2$. Since the magnetic field inside the cylinder is connected to the magnetic field in the thickness of the strip by the coefficient $\mu$, the corresponding values of the vector potentials must be connected in the same way: $A = \mu A_0$. Thus, we obtain:

$$A = \mu RB_0/2. \quad (21)$$

Passing to dimensionless quantities (7)

$$B = \tilde{B} \frac{\Phi_0}{2\pi a R}; \quad \Phi = \tilde{\Phi} \frac{\phi_0}{2\pi a}; \quad R = \tilde{R} \lambda; \quad d = \tilde{d} \lambda,$$

and combining the ratios (19)-(21), we obtain the connection between the magnetic flux $\tilde{\Phi}$, captured by the cylinder, and the external magnetic field $\tilde{B}$:

$$\tilde{B} = \frac{\Phi}{\pi R^2} + \left( \frac{\Phi}{2\pi R} \right) \exp \left[ -\frac{\pi}{16} \left( \frac{\Phi \lambda}{2\pi R} \right)^2 \right] \tilde{d}. \quad (22)$$

The calculated dependence of the magnetic flux captured by the structure of Fig.5, on the induction of an external magnetic field can easily be calculated, a typical form of the dependence $\Phi(B)$ for $\tilde{R} = 20$, $\mu = 0.1$, and different values $\tilde{d}$ is shown in Fig.5.

![Thin-walled cylinder of Josephson medium](image)

![Magnetization curve of a granular SQUID](image)

Fig.4. Thin-walled cylinder of Josephson medium

Fig.5. Magnetization curve of a granular SQUID for $\tilde{R} = 20$, $\mu = 0.1$ and different values $\tilde{d}$.

The resulting magnetization curves, as well as the magnetization curves of classical SQUIDS [4,10], have sections with a negative slope. Therefore, the element shown in Figure 4 exhibits hysteresis magnetic properties, and based on it you can create magnetically sensitive sensors.
6. Conclusion
On the basis the previously obtained material equation for the Josephson medium, the following problems are investigated:
- penetration of a homogeneous magnetic field into a semi-infinite medium;
- current flow through the strip wires;
- magnetization of a solid cylinder and the possibility of constructing a magnetically sensitive sensor having characteristics similar to those of a SQID.

Materials with properties of the Josephson medium can include such real materials as granular superconductors, HTSC ceramics.

The main conclusions are:
1. The first critical magnetic field of a Josephson medium does not depend on the shape of the granules and their size distribution, but is determined only by the effective depth of penetration of the weak magnetic field $\lambda$ and the average size of the granules $a$.
2. In fields smaller than the first critical field $B_{1c}$, the critical current density decreases, and the slope of the critical current dependence on the magnetic field increases with increasing band width ("size effect"); this behavior is similar to that observed in films of the traditional superconductor [11,12].
3. For solid cylindrical samples, the dependences of the captured magnetic flux on the external magnetic field are obtained, it is shown that for thick cylinders ($R \gg 10\lambda$) this dependence has areas with a sharp slope, which can be used to create magnetically sensitive elements.
4. The magnetization curves of thin-walled cylinders are obtained, the conditions for their similarity to the magnetization curves of classical SQUIDs and the prospects of creating on their basis magnetically sensitive elements are noted.
5. It can be assumed that when improving the procedure for the preparation of Josephson medium (high concentration of Josephson junctions and the values of their critical currents) various elements based on it can be used as sensors of weak magnetic fields. It is expected that their sensitivity will be at the sensitivity level of classical SQUIDs, surpassing them in terms of simplicity of manufacture, reliability and cheapness. Such sensors of a weak magnetic field are promising in many areas of science and technology, including medical applications. In particular, for non-invasive monitoring: an auxiliary pump of the left ventricle of the heart [13], a system for wireless transmission of electricity in the human body [14], or transportation of magnetic particles intended for various functions (diagnosis, drug delivery, treatment) [15,16].

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