Building handicrafts for the study of the infinite
dimension of linear spaces

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Abstract. One of the main responsibilities of the pedagogical practice is to guide the teaching process towards significant knowledge obtention. The design and implementation of learning strategies contributes to this task. The present paper shows how a teaching strategy based on the inflationary universe theory creates images of the concept of infinite dimensional linear spaces from the concrete reality of the student, thus solving one epistemological obstacle associated with the assimilation of this concept. This work is a descriptive qualitative research that follows the design research paradigm to develop a teaching experiment over a linear algebra course conformed by 18 students from the mathematics education undergraduate program of the “Universidad del Atlantico, Colombia”. The results of the present work suggest that the designed experiment is a suitable teaching tool that solves an epistemological obstacle associated with the significant learning of the infinite dimension concept of vector spaces.

1. Introduction

According to Friedberg [1], a vector space \( V \) is called finite-dimensional if it has a basis consisting of a finite number of elements; the unique number of elements in each basis for \( V \) is called the dimension of \( V \) and is denoted by \( \dim(V) \). If a vector space is not finite-dimensional, then it is called infinite-dimensional.

To start working into the research line of mathematical analysis or quantum groups theory, any student must acquire a significant domain of the infinite dimension concept of vector spaces since this is a fundamental topic of many important results in these lines [2–5]. This is a reason to implement modifications into the curricular plan of the linear algebra subject in the postgraduate and undergraduate levels for the purpose of giving a better mathematical research formation to the students.

Sierpinski [6] states that “the didactic treatment shortcomings in the formation of the vector space concept have implied a more widespread use of algorithmic procedures by the students, an increased difficulty with the comprehension of the concept meaning, and an increased difficulty with the argumentation and manipulation of the different semiotic representations, since they manipulate the symbolic representation of the concept mechanically like a formal object, without the assimilation of the meaning and without the perception of the relations among all its different representations”. On the other hand, Hernandez [7] affirms that “the vector space concept allows the development and the interconnection among others linear algebra objects like the linear dependency, span sets, bases, dimension, and linear transformations, thus its didactic treatment is an important issue in the interpretation and the distinction of conceptual relations done by
the students into the theoretical system of the linear algebra”. Now, Whitehead [8] suggests that “one of the worst problems in the development of mathematical skills is to be able to explain to students the purpose behind each technique”.

According to Duroux [9], “an epistemological obstacle is a knowledge, a conception, but it is not a difficulty or lack of knowledge. This knowledge produces right answers in certain context but generates wrong solutions outside the same context. This learning manifests resistance to the contradictions (which are produced by the academic confrontation) and to the systematization of a better knowledge. After the deep understanding of the lack of accuracy, this knowledge maintains its stubborn and untimely manifestation.

A recurrent patron in the learning process of linear algebra students is observed during our teaching experience. This patron can be described as a deceleration in the conceptual development of the courses associated with a low assimilation of the infinite dimensional concept of vector spaces. The same patron was identified in one postgraduate course of linear algebra at the master’s in mathematics program of The Universidad del Atlantico, Colombia. The precedent of this situation encouraged a study to determine the causes that originated the apparition of this patron. An exploratory questionnaire was implemented in this group. This implementation concluded that the students try to move the infinite dimensional problems towards the finite dimensional context in a stubborn way, creating interactions between definitions and images associated with the finite concept to solve problems of the infinite context. We concluded also that the students do not have an image associated with the infinite dimension concept of vector spaces and that they are not able to make relations between their concrete reality and the concept. All these reasons originated the construction of strategies that yielded mistakes in their solving procedures.

As far as we are aware, a didactic strategy for the study of infinite dimension concept has not appeared in print before. The present work shows the first of them. We design a learning model based on the development of a teaching experiment.

2. Theoretical framework

2.1. The Vinner’s theory

In his theory [10–12], Vinner describes the acquisition process of the mathematical concepts. This theory states that the formal definition of these concepts are inevitable in the academic formation of the student but these definitions are not necessarily employed by the students in the solving procedures of problems. The concept image is defined as the set of all mental images associated with the name of the concept, which may be a visual representation or a sequence of experiences or impressions. Thus, the concept image is a no verbal object that has been formed along the years through many kinds of experiences and it can be comprised by some parts in contradiction with the formal definition.

By the other hand, the acquisition concept means to achieve a mechanism to construct and identify all the examples of the concept such as it is conceived by the mathematical community.

The work exposed by Vinner [11, 12] shows that even though it is tended to think that the students base their reasoning in the formal definitions of the concepts to solve problems, letting the images of this concept in a second place, the most frequent student behaviour is too different. Vinner’s theory prove that the most used schemes to solving maths problems by the students have the image concept implementation as the only resource to construct solving problem procedures.

2.2. Description of the didactic tool

Let $B$ be a non-empty set with arbitrary cardinality. For each $b \in B$ the indicator function $\psi_b : B \rightarrow \mathbb{R}$ is defined by $\psi_b(x) = 1$ if $x = b$, and $\psi_b(x) = 0$ if $x \neq b$. Let consider $\text{In}(B)$ as
the set of all the indicator functions over the set $B$. The set $V_B$ comprised by the finite support functions $f: B \rightarrow \mathbb{R}$ is a vector space with the canonical operations of the function space.

Let us show that $\text{In}(B)$ is a base for the vector space $V_B$. Let consider $f \in V_B$, then the support of $f$ is finite, that is, $\text{supp}(f) = \{a_1, a_2, \ldots, a_n\}$ where $a_k \in \mathbb{R}$ for all $k = 1, 2, \ldots, n$. Now we take $c_k = f(a_k)$ for all $k = 1, 2, \ldots, n$. Therefore every $x \in B$ satisfies the Equation (1), that is:

$$f(x) = c_1 \psi_{a_1}(x) + c_2 \psi_{a_2}(x) + \cdots + c_n \psi_{a_n}(x). \quad (1)$$

Then we have that $\text{In}(B)$ generates to $V_B$. On the other hand, let $b_1, b_2, \ldots, b_m$ be a finite number of elements in $B$. Let suppose now that certain real numbers $r_1, r_2, \ldots, r_m$ satisfy $c_1 \psi_{b_1} + c_2 \psi_{b_2} + \cdots + c_m \psi_{b_m} = 0$. Thus for all $x \in B$ the Equation (2) is satisfied, that is:

$$c_1 \psi_{b_1}(x) + c_2 \psi_{b_2}(x) + \cdots + c_m \psi_{b_m}(x) = 0. \quad (2)$$

In particular, we can consider $x = b_1$ and then $c_1 = 0$. Analogously, it is proved that $c_1 = c_2 = \cdots = c_m = 0$ concluding that $\text{In}(B)$ is linearly independent. In this way, we have that $\text{In}(B)$ is a base of $V_B$. We observe now that the map $b \rightarrow \psi_b$ from the set $B$ over $\text{In}(B)$ is a representation of $B$ in $\text{In}(B)$. In this way we conclude that $B$ can be regarded as a base of $V_B$.

We note that the free construction of $V_B$ enable us to interpret the addition operation of this vector space as the juxtaposition of the elements. Since the vectors of this space are linear combinations of elements in $B$, then the juxtaposition of the elements of $B$ produces vectors in $V_B$.

2.3. Good choice of a vector base

The inflationary universe theory of Alan Guth [13–15], affirms that the universe experiences an eternal and chaotic expansion. This theory proves the existence of certain points where the universe locally stops its growth around these points, thus forming one kind of universe called the type one universes. It is demonstrated that there are an infinite number of type one universes. We choose a numerable collection $\{A_n: n \in \mathbb{N}\}$ of type one universes. Now we construct the numerable set $P_2 = \{p_1, p_2, p_3, \ldots\}$ where $p_n$ is chosen uniquely in $A_n$. Each $p_n$ is called particle of the universe $A_n$. It is clear that the family $P_2$ is infinitely countable. Let $P_1$ be the finite set of craft stones that will be given to the students for the handicrafts construction.

We define $B$ by the formula $B = P_1 \cup P_2$. Then the cardinality of $B$ is infinitely countable since the cardinality of $P_2$ is $\aleph_0$. Consider now the vector space $V_B$, in this infinite-dimensional linear space the vectors can be constructed by juxtaposition among the elements of $B$. That is, we can construct vectors by the juxtaposition of craft stones of the set $P_1$. Therefore, the handicrafts constructions (necklace or bracelets with craft stones) with elements of $P_1$ is equivalent to the construction of vectors in an infinite-dimensional linear space $V_B$. In this form, the student can to build a concept image of the infinite dimension concept very close to their concrete reality.

3. Methodology

Our work is a descriptive qualitative research that follows the design research paradigm to develop a teaching experiment over a linear algebra undergraduate course conformed by 18 students from the mathematics education program of the “Universidad del Atlántico, Colombia”. According to Cobb [16], a teaching experiment is constituted by three phases. Down below we describe the phases of our experiment.

3.1. Preparation of the experiment

Our approach is based on the teaching model of the linear algebra proposed by Sierpinska [6] which establishes a distinction between two kinds of thinking of the students. The theoretical
thinking is a mental activity specialized on the itself which is manifested by the writing word and the creation of texts. By other hand, the practical thinking is an auxiliary activity that accompanies other activities and it takes the guide role in some contexts. The practical thinking is expressed through the direct action over the environment. We propose a teaching process where both kind of thinking interact each other. We design a sequence of pedagogical habits that recognize and solve the difficulties related to the teaching-learning process of this subject proposed by Kaput [17], who affirms that such difficulties are consequence of the variety of languages and symbolic representations of this subject. Additionally, Kaput asserts that the immersed languages are the geometrical language which illustrates the properties of the vectors in the plane; the arithmetical language which gives the necessary tools for the good operations of vectors, matrices, equation systems, etc; and the algebraic-abstract language which enables the generalization, formalization and characterization of concepts, with or without geometric representation. The objectives of our experiment are:

- To identify and enrich the strategies implemented by the students in the problem resolutions related with the vector dimension concept of spaces.
- To construct a concept image of the infinite dimension of linear spaces from a strong relationship with the concrete reality of the student.

It was decided to incorporate an auxiliary section where topics related to set theory and pre-calculus were treated. This was designed to contextualize the student into the academical framework of the experiment. The topics constituted this auxiliary section were algebra of sets, relations, functions, domain, range. In addition, it was proposed to develop exercises about function supports. Thus, the necessary conceptual baggage was given to the participants. We implemented the team work and the oral discussions as support of this section. Teacher observations, group interviews and resolution of questionnaires were implemented for the data recollection. Videos exposed the philosophical treatment of the inflationary universe theory of Alan Guth. The interdisciplinary development of the curriculum was done by these kind of activities for the knowledge enrichment of the students. The auxiliary section was designed to be extended by the requirement of the participants. The arbitrary dimension of linear spaces was planned to be the last topic of the course aiming the recollection significant information for the results of the experience.

The distribution of the topics of the work plan of the subject was made in three periods and the inclusion of the auxiliary section was made.

The data recollection was planned in three sessions coinciding with the ending of each period, then the temporalization of the recollection process was clearly deliberate. The first session was made to identify the previous knowledge of the group. The second one had the aim of measure the impact of the developing of the experiment and the last one desired the significant results of all the experience.

From the acquired experience in the previous linear algebra courses and the observations made into the master course, we were able to predict many behaviors in our experimental group. An initial conjecture was done about the learning process that would be experienced by the group, this conjecture is described bellow. It was conjectured that the group would experience a deceleration in the conceptual development of the course associated with too low assimilation of the infinite dimension concept. We conjectured that the students would find difficulties in the interpretation of the problems and the construction of solving strategies. These problems are not attributed to the lack of preparation but the low capacity to construct images related to their concrete reality.

However, we estimate that the different solving problems strategies and the practicing developed in the experiment were going to enable the bigger part of the group to construct a suitable comprehension of the finite dimension concept. It was conjectured that the students
would create strategies based on the computation of finite dimensions to address the infinite dimension issue. We planned to make explicit this mistake for the deep understanding of the epistemological obstacle that was being tested by group, thus fixing a starting point to the resolution of the problematic.

3.2. Experimentation

After commencement of the experimentation, we realized that some adjustments had to be implemented into the design, the temporalization and into the subject work plan, due to the reading of the data that was being recollected. The interventions at the classroom was given according the plan, emphasising on the construction of images of the concepts translated to the reality of the students. Discussions spaces were opened for the interpretation of each exposed example. No participate observations were done by some last semester students during the development of some activities.

By the application of personal and group interviews, questionnaires and class videos a huge amount of information was recollected about the academic behavior of the students while they were elaborating the proposed class activities. It was observed a increased motivation during the progress of the didactic of handicraft constructions. A huge enthusiasm was manifested by the group, and they also expressed the team work desire to do the activity. The concepts associated with the teaching experiment had a easy assimilation.

3.3. Data analysis

We made a qualitative interpretation of the recollected data. From the work done into the classroom, we could identify the procedures designed by the students to solving problems. We characterized the interaction between the theoretical and practical thinking by the group. We could analyses the ability of the students to create relationships among the concepts studied at the course and their concrete reality. The information showed the great potential of our teaching experiment. The most of the students evidenced a good interaction between the practical and theoretical thinking while they tried to resolve problems. At the beginning of the experiment, it was noted an almost empty ability to construct relation among concepts and reality. As a evidence of the last questionnaire, the group manifested concept images appropriation.

4. Results

Now we analyses the impact of the teaching experiment implementation on the improvement of the student’s ability to build solving strategies for the infinite dimension vector space problems. For this purpose a statistical hypothesis testing is implemented. We consider the linear algebra course conformed by 18 students where the experiment was developed as our test group. A specific knowledge test based on mathematical proof exercises is developed and the average score obtained by the test group ($\mu_1$) is compared with average score ($\mu_2$) obtained in the same test by the control group conformed by 15 students of the mathematics education program of the “Universidad del Atlantico” taking the same course in the traditional way. In this sense, the hypothesis that we need to prove is: The teaching experiment implementation into a linear algebra course produces a significant improvement on the score of infinite dimensional vector space exams. The acceptance of the null hypothesis $H_0: \mu_1 - \mu_2 \leq 0$ implies that the implementation of our teaching experiment does not induce a significant improvement on the average score of the linear algebra students. Therefore our working hypothesis coincides with the alternative hypotheses $H_0: \mu_1 > \mu_2$. Here, we are assuming that the level of confidence is $\alpha = 0.1$. Now we remember that the universitary educational system of Colombia states that the minimum value for exam qualifications is 0 and the maximum is 5.

The data associated with the test group is the arithmetic mean of the scores $\bar{x}_1 = 3.6$, the standard deviation $s_1 = 0.7348$ and the sample size $n_1 = 18$ corresponding with the number
of students in the test group. Analogously, the data associated with the control group is \( \bar{x}_1 = 2.6266, s_1 = 0.877, \) and \( n_1 = 15, \) respectively. Given that the both sample sizes of the study are lesser than 30, we apply the student’s t-distribution. The Equation (3) shows the calculation of the t-value associated with the problem.

\[
t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = 3.4144. \tag{3}
\]

By the other hand, the Equation (4) states the calculation of the degrees of freedom that we have to consider to determine the critical value \( t_c \) of the study. This value is compared then with the confidence level.

\[
df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1 - 1} \left(\frac{s_1^2}{n_1}\right) + \frac{1}{n_2 - 1} \left(\frac{s_2^2}{n_2}\right)} = 1.2277 \tag{4}
\]

Verifying the critical value \( t_c \) associated with the t-value and the degrees of freedom of the problem, we conclude that \( t_c < \alpha. \) Then the null hypothesis must be rejected, thus concluding the validation of the alternative hypothesis.

5. Conclusions
Epistemological obstacles associated with the study of the infinite dimension vector spaces can be identified in the learning process of students belonging to linear algebra courses into the graduate and undergraduate levels. So far, the mathematics education has no developed a result that generates a solution to this kind of situation. The present work states a first mathematical didactic for the study of the infinite dimension vector spaces at the university level. A teaching experiment that has as the principal objective the solution of this kind of obstacle is described in depth. Through a statistical hypothesis testing we conclude that the didactic tool induces a significant improvement of the student’s ability to construct solving strategies for the infinite dimension vector space problems, this due to the average score of evaluations based on proof problems presents a significant improvement in courses that implement the proposed didactic strategy. Through the didactic experience the students can acquire a better academic preparation for graduate mathematical studies focus on functional analysis research.

To lend continuity to this research, we suggest to apply the teaching experiment in the mathematics graduate framework and compare the results obtained with the students of superior levels with the results exposed in this paper. For this, it is necessary to consider that the infinite dimension vector spaces topic belongs to the last part of a undergraduate linear algebra course, but it belongs to the first part of a graduate course instead. In this sense, the conclusions of this document could have a feedback and they could be refuted or supported by the comparison with other works.

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