The Pseudo-Character of the Weil Representation
and its Relation with the Conley–Zehnder Index

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Abstract

We calculate the character of the Weil representation using previous results which express the Weyl symbol of metaplectic operators in terms of the symplectic Cayley transform and the Conley–Zehnder index.

1 Introduction

Let $\text{Sp}(2n, \mathbb{R})$ be the standard symplectic group: it consists of all linear automorphisms of $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ preserving the standard symplectic form $\sigma = \sum_{j=1}^n dp_j \wedge dx_j$ (the generic element of $\mathbb{R}^{2n}$ is $z = (x, p)$). It is well-known that $\text{Sp}(2n, \mathbb{R})$ is a connected Lie group and $\pi_1[\text{Sp}(2n, \mathbb{R})]$ is isomorphic to the integer group $(\mathbb{Z}, +)$ hence $\text{Sp}(2n, \mathbb{R})$ has covering groups $\text{Sp}_q(2n, \mathbb{R})$ of all orders $q = 2, 3, ..., \infty$. It turns out that the double cover $\text{Sp}_2(2n, \mathbb{R})$ can be faithfully represented by a group of unitary operators on $L^2(\mathbb{R}^n)$. This group is denoted by $\text{Mp}(2n, \mathbb{R})$ and is called the Weil (or metaplectic) representation of $\text{Sp}(2n, \mathbb{R})$. Its elements are called metaplectic operators. The covering projection is denoted by $\Pi : \text{Mp}(2n, \mathbb{R}) \longrightarrow \text{Sp}(2n, \mathbb{R})$.

Trace formulas for diverse Weil representations have been recently obtained (see for instance [9, 16], also see [12]); such formulas are important in many contexts, for instance in the theory of theta functions. In this Note we study the analogue of trace formulas for the continuous case, that is, for the full metaplectic representation. Of course, for metaplectic operators the notion of trace does not make sense since such operators are not of trace

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class. It is however possible to define what we call a “pseudo character” by the formula
\[
\text{Tr}(S) = \int_{\mathbb{R}^n} K_S(x, x) dx
\]
provided that \( s = \Pi(S) \) has no eigenvalue equal to one; here \( K_S \) is the kernel of \( S \in \text{Mp}(2n, \mathbb{R}) \). We will see that the phase of the “pseudo-trace” in the right-hand side is obtained in terms of the Conley–Zehnder index of symplectic paths, familiar from the theory of periodic orbits of Hamiltonian systems (see [11, 6, 10, 13] and the references in these works). This index has been expressed in terms of the Leray–Maslov index (see [11, 2]) in de Gosson [5, 4, 7, 8] and in de Gosson et al [6]. The Conley–Zehnder index also plays a key role in the semiclassical quantization of chaotic Hamiltonian systems (the physicist’s “Gutzwiller formula”) as has been recognized by Meinrenken [14, 15].

2 Metaplectic operators as Weyl operators

Recall that if \( a \in S'((\mathbb{R}^{2n})) \) the Weyl operator with symbol \( a \) is the operator \( A : S(\mathbb{R}^n) \to S'(\mathbb{R}^n) \) defined by
\[
A = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} a_\sigma(z)T(z)dz
\]
where \( a_\sigma \) is the symplectic Fourier transform of \( a \),
\[
a_\sigma(z) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{-i\sigma(z,z')} a(z')dz'
\]
and \( T(z) \) is the Heisenberg operator:
\[
T(z)f(x') = e^{-i(px' - \frac{1}{2}p^2)} f(x' - x).
\]
The distribution \( a \) is the Weyl symbol of \( A \).

In de Gosson [3, 4, 5] metaplectic operators are studied from the point of view of Weyl pseudo-differential calculus. The main results are summarized in the following Theorem:

**Theorem 1** Let \( S \in \text{Mp}(2n, \mathbb{R}) \) have projection \( \Pi(S) = s \) on \( \text{Sp}(2n, \mathbb{R}) \) such that \( \det(s - I) \neq 0 \). Then the symplectic Fourier transform of the Weyl symbol \( a^S \) of \( S \) is given by the formula
\[
a^S_\sigma(z) = \left( \frac{1}{2\pi} \right)^n \frac{\nu(S)}{\sqrt{\det(s - I)}} e^{2Mz \cdot z}
\]
where
\[ M = \frac{1}{2} J (s + I)(s - I)^{-1}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \]  
(2)

The number \( \nu(S) \), defined modulo 4, is the Conley–Zehnder index of a path joining the identity to \( s \) in \( \text{Sp}(2n, \mathbb{R}) \) and whose homotopy class depends on the choice of \( S \).

It is easily verified that \((s + I)(s - I)^{-1} \in \mathfrak{sp}(2n, \mathbb{R})\) (the Lie algebra of \( \text{Sp}(2n, \mathbb{R}) \)), hence \( M = M^T \) (the mapping \( s \mapsto (s + I)(s - I)^{-1} \) is sometimes called the symplectic Cayley transform). The index \( \nu(S) \) corresponds to a choice of the argument of \( \det(s - I) \):

\[ \arg \det(s - I) \equiv (\nu(S) - n)\pi \mod 2\pi. \]  
(3)

For a detailed study of this relationship see de Gosson [8], where the Conley–Zehnder index is expressed in terms of the Leray–Maslov index [2] on the symplectic space \( (\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \sigma \oplus (-\sigma)) \).

Recall that the symbol \( a \) of a Weyl operator \( A \) is related to the kernel \( K_A \) of \( A \) by the formula

\[ K_A(x, y) = \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{R}^n} e^{ip \cdot (x-y)} a\left( \frac{1}{2}(x+y), p \right) dp \]  
(4)

(interpreted in the sense of distributions).

### 3 Pseudo-Trace Formulas

An immediate consequence of Theorem 1 is the following formula:

**Corollary 2** Let \( S \in \text{Mp}(2n, \mathbb{R}) \) be as above. We have

\[ \text{Tr}(S) = \left( \frac{1}{2\pi} \right)^n \frac{i^{\nu(S)}}{\sqrt{|\det(s - I)|}}. \]  
(5)

**Proof.** In view of formula (4) we can write

\[ \text{Tr}(S) = \int_{\mathbb{R}^n} K_S(x, x) dx = \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{R}^{2n}} a(z) dz = a^S_{\sigma}(0) \]
hence (5) in view of (1). Assume now that $s \ell_\rho \cap \ell_P = \{0\}$ where $\ell_P = \{0\} \times \mathbb{R}^n$; in the canonical symplectic basis of $(\mathbb{R}^{2n}, \sigma)$ we may identify $s$ with a block matrix

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

with $\det B \neq 0$, and $S \in \text{Mp}(2n, \mathbb{R})$ has projection $\Pi(S) = s$ if and only if

\[
S f(x) = \left(\frac{1}{2\pi i}\right)^n i^m \sqrt{\det B^{-1}} \int_{\mathbb{R}^n} e^{iW(x, x')} f(x') dx'
\]

for $f \in \mathcal{S}(\mathbb{R}^n)$ (Leray [11], de Gosson [4]); here

\[
W(x, x') = \frac{1}{2} DB^{-1} x \cdot x - B^{-1} x \cdot x' + \frac{1}{2} B^{-1} A x' \cdot x'
\]

is the generating function of $s$ and $m$ is the Maslov index:

\[
\arg \det B^{-1} = m\pi \mod 2\pi.
\]

We will write from now on $s = s_W$ and $S = S_{W,m}$. It is proven in de Gosson ... that if $\det(s_W - I) \neq 0$ then

\[
\nu(S_{W,m}) = m - \text{Inert } W''_{xx}
\]

where $\text{Inert } W''_{xx}$ (the "Morse index") is the signature of the Hessian matrix of the mapping $x \mapsto W(x, x)$. Thus:

**Corollary 3** When $s = s_W$ and $\det(s_W - I) \neq 0$ then

\[
\text{Tr}(S_{W,m}) = \frac{i^m - \text{Inert } W''_{xx}}{\sqrt{\det(s_W - I)}}.
\]

Note that we have, explicitly,

\[
\det(s_W - I) = (-1)^n \det B \det(B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1})
\]

(see de Gosson [3], Lemma 4).

It turns out that we have the following factorization result (de Gosson [3]):

**Proposition 4** Every $S \in \text{Mp}(2n, \mathbb{R})$ can be written (in infinitely many ways) as a product $S = S_{W,m}S_{W',m'}$ such that $\det(s_W - I) \neq 0$ and $\det(s_W' - I) \neq 0$. 

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Using formula (6) together with the product formula

\[ \nu(SS') = \nu(S) + \nu(S') + \frac{1}{2} \text{sign}(M + M') \]

proven in [6,8] the result above allows the calculation of the Conley–Zehnder index in the general case. The constructions in [6] are certainly useful in this context.

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