LANCZOS’S FUNCTIONAL THEORY OF ELECTRODYNAMICS
A commentary on Lanczos’s Ph.D. dissertation

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Abstract

Lanczos’s idea of classical electrodynamics as a biquaternionic field theory in which point singularities are interpreted as electrons is reevaluated. Using covariant quaternionic integration techniques developed by Paul Weiss in 1941, we show that the Lagrangian suggested by Lanczos in his thesis of 1919 is equivalent to the standard Lagrangian of classical electrodynamics. On the mathematical side, Lanczos’s thesis contains the correct generalizations of the Cauchy-Riemann regularity conditions, and of Cauchy’s formula, from complex numbers to quaternions. Lanczos therefore anticipated Moisil-Fueter’s discovery of 1931 by more than 12 years.

Introductory remarks

This commentary was written in 1994 and published in 1998 in the Cornelius Lanczos Collected Published Papers With Commentaries [18]. Since then we have made substantial progress in our understanding of the relations of Lanczos’s Lagrangian to that of standard classical electrodynamics. This is the subject of two new papers, one in which we complete the proof that the usual action integral of classical electrodynamics can be derived from Lanczos’s more fundamental Lagrangian, and show that there is no divergence in that derivation so that the
mass is finite [19]; and a second one in which we show that Paul Weiss’s derivation of the Abraham-Lorentz-Dirac equation of motion is fully consistent and avoiding several problems which plague other derivations [20]. Moreover, we have made Lanczos’s handwritten dissertation available in typseted form [21], together with a preface which further complements our commentary of 1994. This commentary is therefore reprinted here with only a few minor modifications, and with an appendix consisting of an exchange of correspondence between the authors and Professor Kuni Imaeda on the relations he had with Cornelius Lanczos who was like him at the Dublin Institute of Advanced Studies at the time he wrote his paper on “A new formulation of classical electrodynamics” using biquaternion function theory [7].

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While Lanczos was an assistant in physics at the Budapest Institute of Technology between 1916 and 1921, he wrote a dissertation under Rudolph Ortvay, Professor of Theoretical Physics, University of Szeged. Today, we can recognize Lanczos’s dissertation as an important scientific work [1]. Handwritten in 1919, with copies made by an early photolithographic method, this dissertation was thought to have been lost due to the destruction caused by the intervening turbulent periods in Hungary’s history. Fortunately, a copy of Lanczos’s dissertation was found a few years ago.¹

With today’s hindsight, a quick glance through Lanczos’s dissertation immediately reveals several important mathematical and physical ideas, each of them years or decades in advance of their time. The originality of these ideas is such that Lanczos most certainly wrote his dissertation rather independently of his professors, a point that is confirmed by George Marx’s analysis [2]. In fact, Lanczos’s dissertation, *The Functional Theoretical Relationships of the Maxwell Aether Equations — A Contribution to the Theory of Relativity and Electrons*, is a vision of mathematics and physics, fundamentally interrelated in such a way to provide a self-contained description of the world.

In skillful ingenuity, Lanczos displays his genius for inventing new ideas. His intuition is such that he does not stop at technicalities: he puts down his ideas, recognizes possible difficulties, but goes ahead. His vision provides him with a thread, and he goes all the way to his conclusion, with an invitation to readers to send him their comments “in writing — if possible registered” [1, p.78] !

¹Lanczos’s dissertation was found by Dr. Iván Abonyi, Department of Theoretical Physics of the Roland Eötvös University, Budapest, Hungary.

²Note of translator: In contemporary language, the word ‘Aether’ used by Lanczos in his dissertation should be translated by ‘vacuum’.
There is no doubt that Lanczos did not receive many comments. Indeed, he probably did not even receive remarks about the problems or possible mistakes one can easily find in his dissertation, or about the many fundamental questions raised by his very ambitious theory. Let us try to respond to Lanczos’s early invitation for comments on his dissertation.

Like many students at the beginning of the XXth century or today, Lanczos must have been deeply impressed by the power and elegance of complex function analysis. He certainly was not the first to have contemplated the dream of finding a three- or four-dimensional generalization of the Cauchy-Riemann regularity conditions and Cauchy’s formula.

Quaternions $Q = [s; \vec{v}]$ were discovered by Hamilton in 1843, 50 years before Lanczos’s birth. Replacing $s$, the scalar part of $Q$, by $ix_0$, where $x_0$ is the time component of a four-vector, one obtains a very compact and explicit notation for spatial rotations, Lorentz transformations and special relativity formulas. This is what Lanczos shows in Chapters 1 and 2 of his dissertation. The same was done earlier by Conway (1911) and Silberstein [3]. Since there are no references in Lanczos’s dissertation, we do not know if he knew of these publications. For lack of evidence to the contrary, we may therefore assume that Lanczos had discovered everything independently.

The next step was to generalize the fundamental complex function analysis axioms and theorems to quaternions. This is superbly done in Chapter 3. With great simplicity, Lanczos goes through the arguments that will be rediscovered by Moisil and Fueter [4] in 1931. Hence, in the non-commutative field of real quaternions $\mathbb{H}$, the left-regular functions $F$ are those for which the Cauchy-Riemann-Lanczos-Fueter condition is satisfied:

$$\nabla F = 0 \quad (1)$$

and the generalization of Cauchy’s formula becomes the Cauchy-Lanczos-Fueter formula:

$$F(\mathcal{X}) = \frac{-1}{2\pi^2} \int \int \int \frac{\mathcal{R}}{|\mathcal{R}|^4} d^3 \Sigma F(\mathcal{Y}) \quad (2)$$

where $\mathcal{R} = \mathcal{Y} - \mathcal{X}$, $|\mathcal{R}|^2 = \mathcal{R} \overline{\mathcal{R}}$, and $\Sigma(\mathcal{Y})$ a hypersurface surrounding $\mathcal{X}$.\footnote{For an introduction to basic quaternion notations, definitions, and methods, see [5].}
In Chapter 4, Lanczos rewrites Maxwell’s equations in vacuum as

\[ \nabla B = 0 \]  

(3)

where \( B \) stands for the quaternion \([0; \vec{E} + i\vec{H}]\). However, to make this identification, one has to use ‘imaginary time’ as the scalar part of the fundamental space-time quaternion \( \mathcal{X} = [it; \vec{x}] \). And, for consistency, one has to differentiate with respect to \( it \), whenever differentiation is to be performed with respect to the physical time variable. For instance, \( \nabla = [\partial_{it}; \partial_{\vec{x}}] \). As a result, the functional theory to be associated with Maxwell’s equation is not that of real quaternions, but that of complex quaternions, i.e., the eight-dimensional algebra of biquaternions \( \mathbb{B} \).

Hence, while it is clear that the definition (1) and the theorem (2) can be formally carried from \( \mathbb{H} \) to \( \mathbb{B} \), it is by no means evident how (and under which conditions) to use the Cauchy-Lanczos-Fueter formula for \( \mathbb{B} \), and therefore how to calculate the electromagnetic field at a point in space-time, knowing its value on some non-trivial boundary hypersurface. Other subtle problems, of which Lanczos (as other early users of quaternions) was not aware, arise with the full Maxwell’s equations in quaternion form [5]. These equations can be written in the form [3]:

\[ \nabla \wedge A = B , \quad \nabla B = -4\pi J . \]  

(4)

Here, \( A \) is the electromagnetic four-potential, \( B \) the electromagnetic field strength, and \( J \) the source four-current density. The symbol \( \wedge \) means that we discard the scalar part of \( \nabla A \). \( B \) is thus a gauge-invariant ‘bivector’. Writing \( (\cdot)^* \) for imaginary conjugation, and \( (\cdot) \) for quaternion conjugation, we follow Hamilton and call ‘biconjugation’ the operation \( (\cdot)^+ = \overline{(\cdot)}^* \). Hence, physical four-vectors in momentum space are ‘bireal’, i.e., they satisfy the condition \( A = A^+ \), \( J = J^+ \), etc. There is, however, a fourth fundamental involution which we denote \( (\cdot)^\sim \). This is ‘order reversal’ (or ‘ordinal conjugation,’) which has the effect of reversing the order of the factors in a product. The origin of this involution is the residual arbitrariness in the definition of the quaternion product which comes from their non-commutativity. Obviously, physically meaningful results should not depend on this arbitrariness [5].

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4The quaternionic generalization of the complex regularity condition and the corresponding interpretation of Maxwell’s equation will be given by Lanczos two more times: first, in his 1929 series of papers on Dirac’s equation [17, p.453], and later in his lectures at Purdue University. C.f., C. Lanczos, “Wave Mechanics. Part II,” Lecture Notes (1931/1932), page 348. We thank Professor W.R. Davis for a copy of these notes. Between 1919 and 1931, several people will generalize the Cauchy-Riemann condition to four-dimensions, but nobody will generalize Cauchy’s formula accordingly. See, e.g., D. Iwanenko and K. Nikolsky, Z. f. Phys., 63 (1930) 129-137, and references cited.
The problem with (4) is that while it is fully gauge- and relativistic-invariant, it is not manifestly ordinal-invariant because $B \neq B^\sim$. However, since we have assumed $A$ and $J$ to be bireal, for the present discussion of Lanczos’s ideas, it is enough to further postulate that $A = A^\sim$ and $J = J^\sim$. Maxwell’s equations are then exactly equivalent to (4).

In Chapter 5 Lanczos explains his conception of electrically charged particles:

“Durch eine solche funktionentheoretische Deutung ... erhält das Problem der Materie, insbesondere ihre atomistische Structur, eine ausserordentlich harmonische Lösung. Die Materie represäntiert die singulären Stellen derjenigen Funktionen, welche durch die in Aether gültigen Differentialgleichungen bestimmt werden” [1, p.33].

[“By such a functional theory interpretation ... one obtains for the problem of matter, and more particularly for its atomistic structure, an extraordinary harmonious solution. Matter represents the singular points of the corresponding functions which are determined by the vacuum differential equations.”]5

Of course, the idea was not entirely new. In fact, even before the theory of relativity, Conway (1903) and Herglotz had used Cauchy’s theorem to calculate the retarded potential of an electron interpreted as a moving pole in the complex plane [6]. In other words, the concepts of ‘analyticity’ and ‘poles’ that in the 1950’s and 1960’s will become standard tools in elementary particle theory, have a long history. However, Lanczos was certainly the first one to have explicitly formulated the idea that electrons could be poles in a biquaternionic space-time. This very idea will be forgotten until 1976, when Imaeda [7] will eventually formulate a similar theory and show how to calculate the retarded potential and electromagnetic field of a relativistic electron using a generalization of Fueter’s function theory.

In his later years, between 1941 and 1949, Fueter started extending his methods to complex quaternions, Dirac fields and Clifford algebras. The problem is that, in comparison with real quaternions (which correspond to an Euclidian four-dimensional space), the singularities and the parametrization of hypersurfaces are much more complicated. Imaeda, who apparently was not aware of Fueter’s later work, achieved therefore a kind of a ‘tour de force’. Fortunately, today, and especially in about the last twenty years, there is a growing number of mathematical publications on the subject [8].

In Chapter 6 Lanczos introduces Hamilton’s principle and further discusses the far-reaching implications of his conception of matter. For example: “An electron

5Lanczos used underlining for emphasis in his dissertation.
can be seen as a structure with an infinite number of degrees of freedom” [1, p.36]. This is a premonition of the concept of the quantized field.

In order to define the Lagrangian density, Lanczos soon discovers that the only possibility is to write the action $S$ as

$$iS = \iiint d^4\!\!X \left[ B\!\!B + (B\!\!B)^\sim \right]$$

where $B$ is the total electromagnetic field of all particles [1, p.39–40].

The simplicity and the universal character of this Lagrangian, both from the mathematical and physical points of view, is recognized by Lanczos and emphasized in a short addendum [1, p.79–80]. In effect, most remarkably, contrary to the usual Lagrangian of classical electrodynamics, there is no ‘kinetic’ and no ‘interaction’ terms explicit in (5), only a ‘field’ term! The reason why Lanczos’s Lagrangian (5) is sufficient is that $B$ is assumed to be a biquaternion vector function, everywhere differentiable and continuous, except at the poles which are the electrons. Under these conditions, for instance, there is always a potential $A$ such that $B = \nabla \wedge A$. Moreover, minimizing the action, and assuming as usual $B$ to be zero at infinity, one recovers (3), i.e., Maxwell’s equations in vacuum. In other words, as emphasized in the addendum, the variation principle which in biquaternionic functional theory leads to the regularity condition (1), is simply Hamilton’s principle in the physical interpretation of the theory. But this is not all.

To find the full implications of (5), Lanczos takes the case of a particle in an external field. Let the self-potential and field of the electron be $A_1$, $B_1$, and the external potential and field $A_2$, $B_2$. Lanczos’s Lagrangian (5) takes then the form

$$iS = \iiint d^4\!\!X \left[ (B_1 B_1 + 2B_1 \cdot B_2 + B_2 B_2) + (\cdots)^\sim \right]$$

Using (3) and Gauss’s theorem, the first and the second terms can be rewritten as integrals over a hypersurface surrounding the electron. Hence, with $\tau$ the proper time

$$iS = \int i d\tau \left[ \iiint A_1 \cdot (d^2\!\!X B_1) + 2 \iiint A_2 \cdot (d^2\!\!X B_1) \right]$$

$$+ \iiint d^4\!\!X \left[ B_2 B_2 + (\cdots)^\sim \right]$$

In fact, Lanczos postulated just the first term because he did not realize the necessity of ordinal invariance. As a result, while equation (5) insures that $S$ is real, the action in Lanczos’s dissertation is a complex number of which the imaginary part has to be discarded.
Anticipating the result, provided the integrations over the self-field $A_1$, $B_1$, are feasible and finite, one immediately sees that the first term could correspond to the ‘kinetic’ term, and the second one to the ‘interaction’ term. The problem is that if these integrals are calculated ‘naively,’ for example in the rest frame of the electron, one meets with the well known problem of the infinite self-energy of a point charge.

Lanczos realizes this problem very clearly. He also understands that calculating integrals in a hyperbolic space like the biquaternion algebra is not a trivial thing at all. For this reason, he simply assumes that in the real world these integrals may well be finite, in which case (7) is equivalent to the standard text-book Lagrangian of classical electrodynamics for a point charge in an external field. To make this assertion plausible, Lanczos gives in Chapter 7 a model of the electron: the “Kreiselektron,” i.e., the ‘circle electron.’ In effect, if one assumes that the poles corresponding to electrons are in fact not in real three-space, but slightly offset by a small imaginary amount, the singularity is no more a point, but a circle in complex three-space. In this case the action integral is finite.7

In Chapter 8 Lanczos derives Lorentz’s equation of motion. But, first, he attempts to include gravitation in his theory. For this purpose, he tries to make use of the boundary conditions at infinity. Indeed, by some kind of a Mach principle, he gets a relativistic form of Newton’s equation. He then fixes the problem that equal sign charges must attract in gravitation, and repel in electrodynamics, by inserting an $i$ at the right place. But, as Lanczos already knows, and very explicitly states in his first letter to Einstein [2], his theory, entirely confined to the limits of special relativity, may possibly already have been made obsolete by the general theory of relativity.

In the concluding chapter, Lanczos starts by spelling out his dream: as a result of some variation, a good theory should not only predict the behaviour of an electron in the electromagnetic field, but also its mechanical mass. He confesses that his theory does not give any hint of the reason why electrons have an universal and constant charge and mass, why there is positive and negative electricity, or where the quantum-like character of radiation comes from. Nevertheless, in a footnote [1, p.75], remarking that the number $2\hbar c/e^2 \approx 1720$ is empirically almost equal to the proton to electron mass ratio, he dares suggesting that the problems of ‘positive electricity’ and ‘quanta’ might be related.

7The concept of displacing the singularity off the world-line is a standard “regularization” technique used since many years in quantum electrodynamics to obtain finite results. However, its use similar to Lanczos in the context of Maxwell’s equations expressed in complex space-time is more recent, and due to E.T. Newman. See, “Maxwell’s equations in complex Minkowski space,” J. Math. Phys., 14 (1973) 102-103; see also reference [22].
In the last few lines of his conclusion, Lanczos gives a very lucid and personal assessment of his dissertation. It is therefore most appropriate to quote these lines in extenso:

“Die hier skizzierte Theorie möchte einen Beitrag zum konstruktiven Aufbau der modernen theoretischen Physik liefern, wie er insbesondere durch die Arbeiten Einsteins eingeleitet worden ist. Ihr Wert oder Unwert will eben deshalb nicht nach praktischen positivistisch-ökonomischen Prinzipien beurteilt sein — da sie keine blosse ‘Arbeitshypothese’ sein will. Ihre Überzeugungskraft — wenn es sich nicht bloss um meine subjektive Täuschung handelt — liegt nicht in ‘schlagenden Beweisen,’ sondern in der Folgerichtigkeit und Unwillkürlichkeit ihrer Konstruktion, durch welche sie, die eigentliche Seele der Maxwellsschen Gleichungen erfassend, die Maxwellsche Theorie verschmolzen mit der Relativitätstheorie naturgemäß zu den Elektronen hinführt. In dieser systematischen Einfachheit und Notwendigkeit liegt meiner Ansicht nach ihre Überlegenheit gegenüber der gewöhnlichen Elektronentheorie. Es war mir hier nicht um die ins Einzelne gehende Ausarbeitung, sondern mit um die grosse Umrisse zu tun. Es war mir darum zu tun, mehr deutend, als direktführend einen Weg anzubahnen, dessen Spuren verfolgend vielleicht neue Perspektiven nach den unergründlichen Tiefen der Natur sich eröffnen werden” [1, p.76–78].

[“The theory which is here sketched is meant to be a contribution to the constructive formulation of modern physical theory, in the sense that has particularly been introduced by the works of Einstein. Its value, or lack thereof, should therefore not be judged according to practical positivist-economic principles — because it does not pretend to provide any simple ‘working hypothesis.’ Its convincing power — when I am not missled by my subjectivity — does not lie in ‘striking proofs,’ but in the consistency and non-arbitrariness of its construction, by which, in capturing the proper soul of Maxwell’s equations, the theory of Maxwell fused with the theory of relativity, it leads to electrons in a natural way. This systematic simplicity and necessity provides the basis for my view of its superiority over the usual theory of the electron. I have not gone here into the details, but just into the outlines. More precisely, I have been concerned with merely preparing a direct way, the path of which when followed may possibly open new perspectives into the inscrutable depths of Nature.”]

Lanczos’s Lagrangian (5) and his conception of classical electrodynamics as
a biquaternionic field theory are such beautiful and simple ideas, that we would like to retain them. Instead of imagining some kind of a model for the electron, with today’s hindsight, we may simply recognize that there is no explanation for its properties, such as mass or charge, at the classical level. In this perspective, Lanczos’s suggestion of a ‘circle’ electron is not acceptable because we would like to keep the singularities entirely in Minkowski’s 4-dimensional space-time. We also would like to keep the concept of a strictly point-like and structureless classical electron.

In order to calculate the integrals in (7), one possible choice for an appropriate hypersurface is the proper tube of constant retarded-distance introduced by Weiss [9] in 1941.

Paul Weiss [10] was certainly the most brilliant student of Max Born and Paul Dirac. His doctoral thesis and his publications on quantum mechanics are at the foundation of contemporary quantum field theory [11]. Even though he was a Jew and a refugee from Hitler’s fascist policies, he was interned as an ‘enemy alien’ and sent to Canada for six months in 1940. This probably gave him the time to very carefully correct and polish a masterpiece: “On Some Applications of Quaternions to Restricted Relativity and Classical Radiation Theory” [9]. In effect, by introducing the spinor decomposition of 4-vectors, a concept to be rediscovered by Proca [12] in 1946, he shows how to make complicated calculations of radiation theory in a direct and exact way, because the quaternion formalism provides explicit formulas which are difficult to obtain by the ordinary methods of analysis.

The main result obtained by Weiss using quaternion methods is that the Lorentz-Dirac equation can be derived from a variation principle. As noted by Weiss [9, p.162], the existence of a variation principle is of some importance.

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8The tube of constant retarded-distance was previously used by Homi Bhabha in the context of the Lorentz-Dirac equation. See, H.J. Bhabha, Proc. Roy. Soc., A172 (1939) 384. However, Weiss was the first to use the spinor decomposition provided by the biquaternion formalism to parametrize this hypersurface in a way that greatly facilitates explicit calculations involving the retarded potentials and fields of a relativistic electron.

9In this article Weiss explicitly shows that in order to derive the Lorentz-Dirac equation of motion it is not necessary to symmetrize between the retarded and advanced fields of a the point charge. As previously shown by H. Bhabha, Proc. Roy. Soc. A172 (1939) 384, and later emphasized by P. Havas, Phys. Rev. 74 (1948) 456, the retarded field is indeed sufficient for this purpose. Also, C. Teitelboim, Phys. Rev. D1 (1970) 1572; D2 (1970) 1763; D3 (1971) 297, has confirmed that the Lorentz-Dirac equation can be obtained employing only the retarded potential and rather simple physical arguments involving the conservation laws. This work was discussed in the survey article of F. Rohrlich on classical theories of the electron in J. Mehra (ed.), The Physicist’s Conception of Nature (D. Reidel Pub. Co., Dordrecht, 1973) 331-369.

10As shown in reference [20] Weiss’s derivation is fully consistent and avoiding several problems which plague other derivations.
since it leads to the definition of canonically conjugate variables, a notion which usually shows the way towards the quantization of the corresponding classical theory. \(^{11}\) In the present case, the canonical momentum \(P\) conjugated of \(X\) is

\[ P = kU - \frac{2}{3}ie^2\dot{U} \]  

where \(k\) is a constant. Hence, the canonical momentum depends on the velocity \(U\) as well as on the acceleration \(\dot{U}\) of the particle. Using this momentum to naively construct a Dirac-like equation, one finds that there are two solutions for the mass, one of the order of \(k\) and the other \(\frac{3}{2}\hbar c/e^2 \approx 205\) times larger. In other words, the two solutions are in the same ratio as the electron to muon mass ratio. This can be seen as an improved theoretical justification of Rosen’s ‘radiation reaction algorithm’ \(^{13}\); and as a hint that, beyond classical electrodynamics, there is some underlying physics which explains the elementary particle mass spectrum in such a way that Schrödinger’s constant \(e^2/\hbar c\) is somehow the universal fundamental interaction constant.

By using Weiss’s quaternionic expressions for the electromagnetic potential and field, and taking as hypersurface the proper tube, the two-dimensional integrations in (7) are straight-forward. The result, after dividing by \(16\pi\), is

\[ iS = \frac{e^2}{2\xi} \int \int \int \int \mathcal{V} \cdot (U + i\xi \dot{U}) + \frac{1}{16\pi} \int \int \int \int \mathcal{V} \cdot \mathcal{F} + (\cdots)^\sim \]  

where the electric charge \(e\), the invariant retarded distance \(\xi\), and the four-velocity \(U\) refer to the singularity, while \(A\) and \(B\) refer to the external field. As expected, if we take the limit \(\xi \to 0\), the first term becomes infinite. However, in the limit \(\xi \to \xi_0\) with the identification

\[ -\frac{e^2}{2\xi_0} = mc^2 \]  

we obtain the standard classical electrodynamics Lagrangian from Lanczos’s Lagrangian (5), with an additional interaction term which depends on the four-acceleration \(\dot{U}\). The minus sign means that the retarded distance is on the light cone stretching into the past.\(^{12}\)

\(^{11}\)In addition to Weiss, Höhn and Papapetrou where among other early researchers to stress the importance of variation principles in the context of classical theories of the electron. References to these and other related works can be found in H. Höhn’s survey paper in Ergebnisse der Exakten Naturwissenschaften, 26 (1952) 291.

\(^{12}\)As mentioned in the introductory remarks, we have made substantial progress in our understanding of the relations of Lanczos’s Lagrangian (5 – 7) to that of standard classical electrodynamics. We therefore refer the reader to reference [19] which supersedes most of the discussion in this and the two next paragraphs. In particular some of the signs in equations (9) and (10), and their interpretation, are wrong.
The additional term corresponds to some correction related to radiation of electromagnetic energy, a process that is usually (and for good reasons) not explicitly included in the standard Lagrangian. In effect, a local Lagrangian like (9) is not sufficient to account for radiation reaction [14]. But, for all phenomena in which the energy lost in radiation is negligible, the Lagrangian (9) is a very good approximation. The condition for this approximation to be valid is usually expressed in terms of the energies or times which correspond to distances of the order of the ‘classical electron radius’ $e^2/mc^2$. This is exactly the condition for neglecting the $\dot{U}$ term relative to $U$ in (9).

Unfortunately, by either letting $\xi_0 \to 0$ and renormalizing the infinite mass term, or by assuming a finite cut-off such as (10) and neglecting the $\xi_0 \dot{U}$ term, one does not solve the problem that the Lagrangian (5) is infinite. Hence, the question that is raised by the simple reasoning which leads from Lanczos’s Lagrangian (5) to the usual Lagrangian of classical electrodynamics (9) is whether or not the explicit use of biquaternionic function theory would lead to further insight into classical electrodynamics problems or not. This question is of course unanswered by Lanczos’s dissertation and, despite the work of Imaeda [7], it will certainly require more research before it is settled. It is therefore very stimulating to take notice of the contemporary mathematical interest in the development of hypercomplex analysis [8], something that is essential in order to pursue Lanczos’s project.

To conclude, now that we understand the remarkable potential of Lanczos’s idea of elementary particles as singularities in a pure field theory of matter, it is important to mention two sequels of his work of 1919.

First, right after the discovery of ‘matrix mechanics’ by Heisenberg, Born and Jordan in 1925, and before the discovery of ‘wave mechanics’ by Schrödinger in 1926, Lanczos published [15] an interpretation of the ‘new mechanics’ involving integral equations. Although there are no direct references to his dissertation of 1919, there are several obvious links to this earlier work. The value of Lanczos’s field representation of quantum theory was not recognized until 1973, when B.L. van der Waerden [16] discussed the fate of this important paper.

Second, soon after the discovery of Dirac’s relativistic wave-equation for an electron with spin, Lanczos showed how to obtain a natural formulation of Dirac’s equation in terms of quaternions [17]. In a series of three articles published in 1929, he expounds the similarities and differences between Maxwell’s and Dirac’s theories, explains the concept of mass as the result of some feedback mechanism which stabilizes the particle, and discovers a ‘doubling’ which today we interpret as isospin [5]. For a third time since 1919, Lanczos was far too much in advance
of his time to be understood by most of his contemporaries.\textsuperscript{13}

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\textsuperscript{13}Finally, it is worth mentioning that Lanczos’s focus on singularities in his Ph.D. dissertation
made him aware of these problems in general relativity, where he was the first to cast doubt on the
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[7] K. Imaeda, “A new formulation of classical electrodynamics;” *Nuovo Cim.*, 32B (1976) 138–162.

[8] V. Souček, “Holomorphycity in quaternionic analysis. Complex quaternionic analysis, connections to mathematical physics,” in *Seminari di Geometria 1982-1983* (Universita di Bologna, 1983) 147-171.
F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis* (Pitman Books, London, 1982) 307 pp.
V. Souček, *Complex Variables*, 1 (1983) 327–346. In this article, Souček shows the link between Fueter type and Penrose type integral formulas.
J. Ryan, *J. of Diff. Equ.*, 67 (1987) 295–329.
J. E. Gilbert and M.A.M. Murray, *Clifford algebras and Dirac operators in harmonic analysis* (Cambridge University Press, 1991) 334 pp.
J. Ryan, *Proc. London Math. Soc.*, 64 (1992) 70–94.

[9] P. Weiss, “On some applications of quaternions to restricted relativity and classical radiation theory,” *Proc. Roy. Irish. Acad.*, 46 (1941) 129–168.
The only sequel of Weiss’s research on quaternions and Maxwell’s theory seems to be his article entitled “An extension of Cauchy’s integral formula by means of Maxwell’s stress tensor,” *J. London Math. Soc.*, 21 (1946) 210–218. In this article, Weiss gives a n-dimensional generalization of Cauchy-Lanczos-Fueter’s formula.

[10] For an obituary of Paul Weiss (1911-1991) see *SIAM News*, 24, Nb 3, (May 1991) pages 2 and 6.

[11] P. Weiss, *Proc. Roy. Soc.*, A156 (1936) 192–220; A169 (1938) 102–119; A169 (1938) 119–133.
A footnote in J.M. Jauch and F. Rohrlich, *The theory of photons and electrons* (Addison-Wesley, 1955) p.9, reads: “A theory involving more general space-like surfaces was developed for the first time by P. Weiss (1938). Later work especially by S. Tomonaga (1946), J. Schwinger (1948), K.V. Roberts (1950), P.A.M. Dirac (1950), T.S. Chang (1950).”
E.T. Corson’s commentary in *Introduction to tensors, spinors and relativistic wave-equations* (Blackie and Son Ltd, London, 1953, 1954) p.68, is as follows: “This was first emphasized by Weiss (1936,1938) and is essentially the starting point of the more recent considerations of Tomonaga (1946), Schwinger (1948, 1949), et al.”

[12] A. Proca, *Proceedings of the Conference on Fundamental Particles and Low-Temperature Physics*, Cambridge, 1946, *Phys. Soc. Camb. Conf.* (1947) 180-181.

[13] G. Rosen, *Nuovo Cim.*, **32** (1964) 1037-1045. See also, A.O. Barut, *Phys. Lett.*, **73B** (1978) 310–312.

[14] See especially, F. Rohrlich in C. Enz and J. Mehra (ed.), *Physical Reality and Mathematical Description* (Reidel Pub. Co., Dordrecht, 1974) 387–402.

[15] C. Lanczos, *Z. f. Phys.*, **35** (1926) 812–830.

[16] B.L. van der Waerden, in J. Mehra (ed.), *The Physicist’s Conception of Nature*, (Reidel Pub. Co, Dordrecht, 1973) 276–293.

[17] C. Lanczos, *Z. f. Phys.*, **57** (1929) 447–473, 474–483, 484–493.

[18] A. Gsponer and J.-P. Hurni, *Lanczos's functional theory of electrodynamics — A commentary on Lanczos’s PhD dissertation*, in W.R. Davis et al., eds., Cornelius Lanczos Collected Published Papers With Commentaries, I (North Carolina State University, Raleigh, 1998) 2-15 to 2-23.

[19] A. Gsponer and J.-P. Hurni, *Cornelius Lanczos’s derivation of the usual action integral of classical electrodynamics*; e-print arXiv:math-ph/0408028 available at http://arXiv.org/abs/math-ph/0408028.

[20] A. Gsponer and J.-P. Hurni, *Paul Weiss’s derivation of the Abraham-Lorentz-Dirac equation of motion*; e-print arXiv:math-ph/040xxxx soon available at http://arXiv.org/abs/math-ph/040xxxx.

[21] Cornelius Lanczos, *The relations of the homogeneous Maxwell’s equations to the theory of functions — A contribution to the theory of relativity and electrons* (1919, Typesetted by Jean-Pierre Hurni with a preface by Andre Gsponer, 2004) 59 pp; e-print arXiv:math-ph/0408027 available at http://arXiv.org/abs/quant-ph/0408027.

[22] A. Gsponer, *On the physical interpretation of singularities in Lanczos-Newman electrodynamics*; e-print arXiv:gr-qc/0405046 available at http://arXiv.org/abs/gr-qc/0405046.
Correspondence with Professor Kuni Imaeda

May 28, 2001

Dear Professor Kuni Imaeda,

A common friend of us, Professor James Edmonds, has given me your address in Japan.

First of all, I very much hope that everything is going well for you and that you will be pleased by receiving this letter and its enclosures.

The purpose of my letter is related to your paper entitled “A new formulation of classical electrodynamics” that was published in *Nuovo Cim.*, **32 B** (1976) 138–162. As you certainly know, this paper has become a “classic” for all those using biquaternions or Clifford numbers to formulate classical electrodynamics.

The first purpose of my letter is to send you a copy of two short papers: one published in 1998 on Cornelius Lanczos’s PhD dissertation of 1919 in which he started what you finally achieved in 1976; and a second one dedicated to Freeman Dyson showing that quaternions have indeed much to contribute to the formulation of physics.\(^\text{14}\) The first paper is copied from the *Lanczos Collection* of which I am including an announcement.

The second purpose of my letter is to ask you whether you had any discussion with Cornelius Lanczos on quaternions and physics when you were in Dublin. You were at the Dublin Institute of Advanced Studies (DIAS) from 1958 until 1981, while Lanczos was at DIAS from 1954 to 1974, so it would have been quite natural that you have met with him.

I would therefore greatly appreciate if you could tell me if you had any discussion with Lanczos on the use of quaternions in physics, and in particular if Lanczos had told you that the subject of his PhD dissertation of 1919 was precisely an attempt to a quaternionic formulation of classical electrodynamics of the kind that you have published in 1976.

With many thanks and my best wishes,

Yours sincerely,

Andre Gsponer

*Associate editor of the Lanczos Collection*

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\(^{14}\)A. Gsponer and J.-P. Hurni, “Comment on formulating and generalizing Dirac’s, Proca’s, and Maxwell’s equations with biquaternions or Clifford numbers,” *Found. Phys. Lett.*, **14** (2001) 77–85. Available at [http://arXiv.org/abs/math-ph/0201049](http://arXiv.org/abs/math-ph/0201049).
Dear Professor Andre Gsponer,

Thank you for sending me a letter, the copies of your two papers, and an information concerning the “Comments on Professor Lanczos’s Dissertation: Functional Theory of Electrodynamics” and another information on the “Collected papers of Lanczos.”

Sometimes ago Professor Edmonds sent me a letter telling me that he gave my address in Japan to you. It is a great surprise to learn that Professor Lanczos had done a work on the generalization of the Cauchy-Riemann equations of functions of a complex variable to those of a biquaternion variable to be used in electrodynamics as early as 1919 in his “Dissertation,” and I didn’t know it until I got your letter a few days ago. Also, he anticipated Moisil-Fueter’s type regularity conditions.

I began the study of quaternions for use in electrodynamics around 1942 and read many papers on quaternions as mentioned in your commentary: Conway, Silberstein, and so on including Lanczos but not his dissertation thesis for PhD. I was pursuing the same idea as Lanczos and some professor at the university told me in 1943 about the papers of Fueter in Comm. Math. Helvetici in which Fueter developed the theory of functions of a quaternion variable. But I did not develop the theory to extend it to electrodynamics. I graduated from the university in 1943 and immediately went to military service, and when I got a university position in 1950 I published a paper to qualify the position at the university in the journal Progress of Theoretical Physics.

In this paper, a brief mention of a function of a pseudo-quaternion variable is made and the method to obtain regular functions of a pseudo-quaternion variable as given by Fueter, though his paper is not cited. But soon after I published a paper which used the regularity condition and biquaternion functions in the theory of electromagnetic fields.

Soon after, in 1958, I got a position at the School of Cosmic Physics, Dublin Institute, and Professor Lanczos and I became a good relation. Even though he was a senior professor at the School of Theoretical Physics, and the two schools were about one hundred meters apart, I attended the conferences and seminars held

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15The English of Professor Imaeda’s handwritten letter is slightly edited in this transcription.

16“Linearization of Minkowski space and five-Dimensional space,” Prog. Theor. Phys., 5 (1950) 133

17In this paper Imaeda uses the word ‘pseudo-quaternion’ instead of Hamilton’s term ‘biquaternion’ to mean ‘complexified quaternion.’

18“A study of field equations and spaces by means of hypercomplex numbers,” Memoirs of the Faculty of Liberal Arts and Education, 2 (Yamanashi University, Kofu, Japan, 1951) 111–118.
at the School of Theoretical Physics often so that I met Professor Lanczos quite often. He used to invite some of his students at the School of Theoretical Physics with me at his house, and I invited Professor and Mrs. to my home. Mrs. Lanczos gave me advice on the house to live and on the school that my daughters should attend, and she introduced me to a Jewish friend who was interested in Japanese art.

Even in such an intimate relation, I could not ask a question on quaternions. Once in a private party, I asked a question on the utility of quaternions in physics in view of his quaternion study in his earlier days. He told me that Irish are not enthusiastic about quaternions now and that he was disappointed by the reaction of the Irish to quaternions. Afterwards, I regretted that shouldn’t have discussed a serious matter at a private party. I should have gone to his office with my published papers with me. This was probably around the years 1960–1964. I have not published any paper on quaternions in the Institute since I was fully engaged in the study of Cosmic Physics and the study of biquaternions was postponed for a while. I could not determine to discuss fully the biquaternion electrodynamics with Professor Lanczos. I didn’t know that he had done a work with nearly the same idea quite a long time ago, so that after he left the Institute I couldn’t show him the work that I later published.

Around 1974, the time that Professor Lanczos left the Institute, I had some experience which led me to send a paper on biquaternion formulation of classical electrodynamics to some journal. But it is not relevant here and I do not included it here.

A word about my paper “A new Formulation of Classical Electrodynamics.” This paper was refused publication by a journal and was sent afterwards to Nuovo Cimento which published it in 1976.19

On your question whether I had a discussion on “Biquaternion Formulation of Electrodynamics” with Professor Lanczos: I did not have such a discussion and I now regret that, and your letter reminds me of happy days with Professor and Mrs. Lanczos who were so kind with us.

With best wishes.

Yours sincerely,

K. Imaeda

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19Nuov. Cim. 32 B (1976) 138–162.
Dear Professor Kuni Imaeda,

Thank you very much for your letter of 18 June in which you kindly answer my letter of 28 May 2001. The answers and information your are giving will certainly be useful, as much for future *Lanczos studies*, then for confirming your own priority in the use of biquaternions in classical electrodynamics.

I am especially grateful to you for having included copies of your early papers on the subject, especially the Japanese original and the translation of *A study of field equations and spaces by means of hypercomplex numbers*, Memoirs of the Faculty of Liberal Arts and Education 2 (Yamanashi University, Kofu, Japan, 1951) 111–118, which I had never seen before.

Coming back to Lanczos and your own work with quaternions, your recollections confirm that Lanczos was always modest and apparently avoided using opportunities such as your questions to push forward his own papers and ideas. Your recollection that Lanczos was “disappointed at the reaction of Irish to quaternions” is also important in explaining why Lanczos did not return to quaternions when he was in Dublin (except in “The splitting of the Riemann tensor,” *Rev. Mod. Phys.* 34 (1962) 379–389, and “William Rowan Hamilton, an appreciation,” *Amer. Sci.* 2 (1967) 129–143).

This information is also useful to better understand why quaternions have never made it as a tool for teaching and working in physics, which is puzzling because it is so despite of very positive appreciations by well known physicists such as Freeman Dyson and Richard Feynman.21

With many thanks and my best wishes,

Yours sincerely,

Andre Gsponer

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20The same year Lanczos added a new chapter on relativistic mechanics to the second edition of his textbook, *The Variational Principles of Mechanics* (Dover, New York, 1949, Second edition 1962, Fourth edition, 1970) 418 pp., with the remark in the preface: “The general theory of the Lorentz transformations is developed on the basis of Hamilton’s quaternions, which are so eminently suited to this task that one could hardly find any other mathematical tool of equal simplicity and conciseness.”

21See paper mentioned in footnote number 14.