Abstract

In this paper we introduce the distinguishing number of a poset $P$ as the minimum number of colors needed to color the points of $P$ so that any automorphism of $P$ preserves colors. We find the distinguishing number of any distributive lattice and certain classes of ranked planar posets by constructing appropriate colorings. In addition, we suggest two natural definitions for the distinguishing chromatic number of a poset. The first of these reduces to the width of the poset, but the second is more interesting and we prove an upper bound for distributive lattices.

Keywords: distributive lattice, distinguishing number, distinguishing chromatic number

1 Introduction

The distinguishing number of a graph, introduced by Albertson and Collins [1], is the smallest integer $k$ for which the vertices can be colored using $k$ colors so that the only automorphism of the graph that preserves colors is the identity. The distinguishing chromatic number, introduced by Collins and Trenk [8], has the additional requirement that the coloring of the vertices is proper, that is, adjacent vertices get different colors. The distinguishing number of graph $G$ is denoted by $D(G)$ and the distinguishing chromatic number by $\chi_D(G)$. These and related topics have received considerable

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attention by many authors in recent years; see, for example, [2, 4, 9, 11, 12, 13]. In
this paper, we introduce the distinguishing number of a poset and also two definitions
for the distinguishing chromatic number of a poset. The additional structure inherent
in posets makes these parameters qualitatively different from the graph versions.

In the remainder of this section, we provide some background definitions about
posets followed by an overview of the rest of the paper. The posets we consider are
finite and reflexive. If \( P \) is the poset \((X, \preceq)\), we call \( X \) the ground set
of \( P \) and refer to the elements of \( X \) as points. We write \( x \prec y \) if \( x \preceq y \) and
\( x \neq y \). If \( x \preceq y \) or \( y \preceq x \), we say points \( x \) and \( y \) are comparable,
and otherwise they are incomparable and we write \( x \parallel y \). The comparability graph
of poset \( P \) is the graph \( G_P = (V, E) \) where \( V \) is the ground set of \( P \) and
\( xy \in E \) if and only if \( x \) and \( y \) are comparable in \( P \). The dual of poset \( P \),
denoted by \( P^\dual \), is the poset with the same ground set as \( P \) and for which
\( x \preceq y \) in \( P \) if and only if \( y \preceq x \) in \( P^\dual \). By definition, a poset and its dual have
the same comparability graph. We say that \( y \) covers \( x \) if \( x \prec y \) and there is no other
point \( v \) with \( x \prec v \prec y \). Incomparable points \( x \) and \( y \) are twins if they have the same
relationship to all other points of the poset. A poset is twin-free if it has no twins.

For poset \( P = (X, \preceq) \), the downset of a point \( a \in X \) is defined as \( \downa = \{ x \in X : x \preceq a \} \)
and the downset of a subset \( A \subseteq X \) is defined as \( \downa = \{ x \in X : x \preceq a \) for some \( a \in A \} \). A set of pairwise comparable points in a poset is called a chain,
and if the points are pairwise incomparable they form an antichain. An \( r \)-chain is a
chain with \( r \) points, and such a chain has length \( r - 1 \). The height of a poset is the
size of a maximum chain and the width is the size of a maximum antichain.

Figure 1: Examples of posets with distinguishing labelings (\( p \), \( q \) and \( r \) are distinct
primes).

If a poset has a unique minimal element, we call this element \( \hat{0} \) and if it has a
unique maximal element we call it \( \hat{1} \). We say that a poset with a \( \hat{0} \) and \( \hat{1} \) is ranked
if every maximal chain from \( 0 \) to \( 1 \) has the same length. The rank of a point \( x \) in a
poset is the length of a longest chain that has \( x \) as its largest element.

A poset is planar if its Hasse diagram can be drawn in the plane with no edges
crossing and so that the edge from \( a \) to \( b \) has strictly increasing \( y \)-coordinate when
\( a \prec b \). An automorphism of poset \( P = (X, \preceq) \) is a bijection from \( X \) to \( X \) that
preserves the relation \( \preceq \).

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We illustrate some of these definitions using the posets $L_{pq^2}, L_{p^2q^2}, M, B, S_4$, whose Hasse diagrams are shown in Figure 1. Each of posets $L_{pq^2}, L_{p^2q^2}, M$ and $B$ has a 0 and a 1, while poset $S_4$ has neither. Posets $L_{pq^2}, L_{p^2q^2}$ and $B$ are ranked, while poset $M$ is not. Posets $L_{pq^2}, L_{p^2q^2}, M$ and $S_4$ are planar, even though the drawing of $S_4$ given in Figure 1 has edges crossing. We demonstrate later that poset $B$ is not planar (see Remark 20).

We end this section with an overview of the rest of the paper. In Section 2 we provide preliminary results about the distinguishing number of a poset, including showing that the distinguishing number of a poset is not bounded in Proposition 3. We also prove a lemma that is useful in induction proofs for the distinguishing number of ranked posets. In Section 3, we consider lattices. In Theorem 8 we provide an elementary proof that the distinguishing number of a divisibility lattice is 1 or 2. Using a classical result of Birkhoff, our proof of Theorem 15 constructs distinguishing 2-colorings of any distributive lattice. In Section 3.3 we provide a proof of the folklore result that any ranked, planar poset with a 0 and a 1 has a Hasse diagram in which the points of each rank are on a horizontal line and all edges are straight line segments. In Section 4 we introduce two versions of the distinguishing chromatic number for posets. The chain-proper version equals the width of the poset. We show that the antichain-proper version is unbounded in Proposition 26 and provide an upper bound for distributive lattices in Theorem 27. The proof of the latter result uses some of the ideas of the proof of Theorem 15. We conclude with several open questions.

2 Distinguishing number of a poset

2.1 Preliminaries

We begin with the definition of the distinguishing number of a poset.

\begin{definition}
Let $P$ be a poset. A coloring of the points of $P$ is \textit{distinguishing} if the only automorphism of $P$ that preserves colors is the identity. The \textit{distinguishing number} of $P$, denoted $D(P)$ is the least integer $k$ so that $P$ has a distinguishing coloring using $k$ colors.
\end{definition}

Distinguishing colorings are shown in Figure 1 and the distinguishing numbers are the following: $D(L_{pq^2}) = 1$, $D(L_{p^2q^2}) = 2$, $D(M) = 1$, $D(B) = 2$, $D(S_4) = 2$. Observe that antichains are the only posets for which each point must receive a different color in a distinguishing coloring.

Any automorphism of a poset $P$ is also an automorphism of its dual $P^d$, its comparability graph $G_P$, and its incomparability graph $\overline{G_P}$. This justifies the following remark.

\begin{remark}
$D(P) = D(P^d)$, $D(P) \leq D(G_P)$, and $D(P) \leq D(\overline{G_P})$.
\end{remark}
Some automorphisms of the graph $G_P$ are not automorphisms of the poset $P$ because they do not preserve the ordering of points in $P$. The following example shows that $D(P)$ can differ significantly from $D(G_P)$ and $D(G_P^c)$. If $P$ is an $n$-chain then $D(P) = 1$. However, the comparability graph of $P$ is the complete graph $K_n$, and the incomparability graph of $P$ is its complement $\overline{K_n}$, and each of these has distinguishing number $n$.

In the next two results, we find the distinguishing number for posets consisting of the sum of chains.

**Proposition 3.** Let $P$ be the poset consisting of the sum of $t$ chains, each consisting of $r$ points and let $k$ be the positive integer for which $(k-1)r < t \leq kr$. Then $D(P) = k$.

**Proof.** First we find a distinguishing coloring of $P$ using $k$ colors. There are $k^r$ different ways to color the elements of an $r$-chain when $k$ colors are available. Coloring the elements of each $r$-chain differently is a distinguishing coloring since any automorphism of $P$ maps an $r$-chain to an $r$-chain. Thus $D(P) \leq k$. We next show $D(P) > k - 1$. For a contradiction, suppose there is a distinguishing coloring of $P$ using $k - 1$ colors. There are $(k-1)^r$ ways to color each chain and since $t > (k-1)^r$, two chains have the same coloring. The automorphism that swaps those two chains is non-trivial, a contradiction. \(\Box\)

Combining Proposition 3 with the following proposition, allows us to compute the distinguishing number of any poset that consists of the sum of chains.

**Proposition 4.** Let $P$ be the sum of chains and partition $P$ as $P_1 + P_2 + \cdots + P_m$ where $P_i$ consists of $t_i$ chains, each consisting of $r_i$ points, where $r_1, r_2, \cdots, r_m$ are distinct. Then $D(P) = \max\{D(P_i) : 1 \leq i \leq m\}$.

**Proof.** The result follows immediately from the fact that any automorphism of $P$ will map $P_i$ to itself for each $i$. \(\Box\)

In showing that a coloring is distinguishing it can be helpful to analyze the points individually or in groups. If $P$ is a poset with a color assigned to each point, we say that a point $x$ is pinned if every automorphism of $P$ that preserves colors maps $x$ to itself. Thus a coloring of the ground set of a poset $P$ is distinguishing precisely when every point is pinned.

For ranked posets, it can be useful to prove that points are pinned by using induction on their rank. The next lemma facilitates such arguments.

**Lemma 5** (Pinning Lemma for Ranked Posets). Let $P$ be a ranked poset with a color assigned to each point. If every element at rank at most $t$ is pinned and the downsets of the elements at rank $t + 1$ are distinct, then every element at rank $t + 1$ is pinned.
Proof. Let \( P \) be a ranked poset with a color assigned to each point. Furthermore assume every element at rank at most \( t \) is pinned and for which the downsets of the elements with rank \( t+1 \) are distinct. Let \( x \) be a point with rank \( t+1 \) and \( f \) an automorphism of \( P \). If \( y = f(x) \) then \( f(down(x)) = down(y) \). The points in \( down(x) \) have rank at most \( t \), hence are pinned, so \( down(x) = f(down(x)) = down(y) \). Since rank is preserved by automorphisms of \( P \), point \( y \) has rank \( t+1 \). By assumption, downsets of distinct elements of rank \( t+1 \) are unequal. Since \( down(x) = down(y) \) we must have \( x = y \), and thus \( x \) is pinned. \( \square \)

3 Lattices

We next focus on lattices. In section 3.1 we consider divisibility lattices and in section 3.2 we consider the more general class of distributive lattices.

A point \( z \) in poset \( P \) is called the meet of \( x \) and \( y \) in \( P \), and denoted by \( x \wedge y \), if it is the unique largest element in \( P \) such that \( z \preceq x \) and \( z \preceq y \). Thus, if \( x \wedge y \) exists and \( a \preceq x \) and \( a \preceq y \), then \( a \preceq x \wedge y \). Similarly, a point \( w \in P \) is called the join of \( x \) and \( y \) in \( P \), and denoted by \( x \vee y \), if it is the unique smallest element \( w \in P \) such that \( w \succeq x \) and \( w \succeq y \). Thus, if \( x \vee y \) exists and \( a \succeq x \) and \( a \succeq y \), then \( a \succeq x \vee y \).

A poset \( L \) is a lattice if \( x \wedge y \) and \( x \vee y \) both exist for all points \( x \) and \( y \) in \( L \). Furthermore, \( L \) is a distributive lattice if \( \wedge \) and \( \vee \) satisfy the distributive laws

\[
(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z) \\
(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)
\]

for all \( x, y, z \in L \). For example, all the posets in Figure 1 are lattices except for \( S_4 \), and all the lattices are distributive except for \( M \). A lattice need not be ranked (\( M \) is not ranked), but a distributive lattice is ranked, as a consequence of Birkhoff’s Theorem (Theorem 12 in Section 3.2).

A point \( x \) in a lattice is called join-irreducible if in the Hasse diagram of the lattice \( x \) has exactly one downward edge. For example, in Figure 1 the join-irreducible points of \( L_{pq^2} \) are \( p, q, \) and \( q^2 \), while there are no join-irreducible points in poset \( S_4 \). As we will see in Birkhoff’s Theorem, the join-irreducible points of a distributive lattice generate all the elements in the lattice by the join operation. In this way, they act like the prime numbers in the prime factorization of an integer.

3.1 Divisibility Lattices

For positive integer \( n \), the divisibility lattice is the poset \( L_n \) consisting of the positive integer divisors of \( n \) ordered by divisibility. Figure 1 shows the poset \( L_n \) for \( n = pq^2 \) and \( n = p^2q^2 \) when \( p \) and \( q \) are distinct primes. As illustrated by this figure, the structure of \( L_n \) is determined by the prime factorization of \( n \).
Lemma 6. Let \( n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \) where the \( p_i \) are distinct primes and each \( a_i \) is a positive integer. If \( f \) is an automorphism of divisibility lattice \( L_n \) and \( f(p_i) = p_j \) then \( a_i = a_j \).

Proof. Suppose \( f \) is an automorphism of \( L_n \) for which \( f(p_i) = p_j \). The join-irreducible points of \( L_n \) are those of the form \( p_i^{b_i} \) for \( 1 \leq b_i \leq a_i \). Each of these points belongs to exactly one maximal chain of join-irreducible points, namely \( C_i : 1 < p_i < p_i^2 < \cdots < p_i^{b_i} \). Chains are preserved under automorphisms of \( L_n \), thus \( f \) maps the chain \( C_i \) to the chain \( C_j \). Therefore, these chains have the same length and \( a_i = a_j \). \( \square \)

The next lemma shows that if a coloring pins the rank 1 points of a divisibility lattice, then it is a distinguishing coloring.

Lemma 7. If the points of a divisibility lattice are colored so that the rank 1 points are pinned, then all points are pinned.

Proof. We prove by induction that all points of \( P \) are pinned. By hypothesis, the rank one points are pinned. Assume all points of rank at most \( t \) are pinned and let \( m \) be a point of rank \( t + 1 \). The points at rank \( t + 1 \) have distinct prime factorizations and hence they have distinct downsets at rank \( t \). Thus by Lemma 5 all elements at rank \( t + 1 \) are pinned, and by induction, all elements of \( P \) are pinned. \( \square \)

The next theorem allows us to determine the distinguishing number of any divisibility lattice and the proof produces a distinguishing coloring. This proof is self-contained and uses only elementary arguments. Theorems 14 and 15 prove a more general result, but rely on a result of Birkhoff given in Theorem 12. Figure 1 illustrates the distinguishing coloring used in the proof of Theorem 8 for the lattice \( L_{pq^2} \) and \( L_{p^2q^2} \), and similarly, Figure 2 illustrates this coloring for \( L_{150} \).

Theorem 8. Let \( n \) be an integer greater than 1 and write \( n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \) where \( p_i \) are distinct primes and \( a_i \geq 1 \) for each \( i \). The divisibility lattice \( L_n \) has \( D(L_n) = 1 \) if the \( a_i \) are distinct and \( D(L_n) = 2 \) otherwise.

Proof. First consider the case in which the \( a_i \) are distinct and color all points with the same color. The rank 1 points of \( P \) are the join-irreducibles \( p_1, p_2, \ldots, p_k \), and since the \( a_i \) are distinct, Lemma 6 implies that they are pinned. Now by Lemma 7 all points are pinned, so this coloring is distinguishing and \( D(L_n) = 1 \).

Next we show that \( D(L_n) > 1 \) if the \( a_i \) are not distinct. Without loss of generality we may assume \( k \geq 2 \) and \( a_1 = a_2 \). If all points receive the same color then the non-trivial automorphism \( f \) defined by \( f(p_1^{b_1} p_2^{b_2} p_3^{b_3} \cdots p_k^{b_k}) = p_2^{b_1} p_2^{b_2} p_3^{b_3} \cdots p_k^{b_k} \), that is swapping the roles of \( p_1 \) and \( p_2 \), is color-preserving. Hence \( D(L_n) > 1 \).

Finally we show \( D(L_n) \leq 2 \). Consider the set of rank 1 elements, \( A = \{ p_1, p_2, \ldots, p_k \} \), and partition it into parts so that \( p_i \) and \( p_j \) are in the same part if and only if \( a_i = a_j \). (For the divisibility lattice \( L_{150} \) shown in Figure 2 \( A = \{ 2, 3, 5 \} \) and the partition of \( A \) is \( A_1 = \{ 2, 3 \}, A_2 = \{ 5 \} \).) Consider one part of this partition, without loss of
generality we may assume this part is $A_1 = \{p_1, p_2, \ldots, p_r\}$. Color the elements of the chain $p_1 \prec p_1 p_2 \prec p_1 p_2 p_3 \prec \cdots \prec p_1 p_2 \cdots p_r$ using the color red. For each $p_i \in A_1$ there is a unique lowest ranked point of this chain in the upset of $p_i$, namely the point $p_1 p_2 \cdots p_i$. The unique chain in $L_n$ from $p_i$ to $p_1 p_2 \cdots p_i$ has length $i - 1$ and thus no automorphism of $P$ can map an element of $A_1$ to a different element of $A_1$. Repeat this process for each part of the partition of $A$, again using the color red. Color the remaining points blue. Thus, no automorphism of $L_n$ can map a rank 1 element to another rank 1 element in its part of the partition. By Lemma 6, no automorphism of $L_n$ can map a rank 1 point to a rank 1 point in another part of the partition. Thus all points of $L_n$ at rank 1 are pinned. Now by Lemma 7, all elements of $L_n$ are pinned, so $D(L_n) \leq 2$.

When $D(L_n) = 2$, the coloring in the proof of Theorem 8 does not always use a minimum number of red vertices, as seen in the following example.

**Example 9.** Consider the divisibility lattice $L_n$ where $n = 2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. In the proof of Theorem 8, the set $A = \{2, 3, 5, 7, 11\}$ is partitioned into one part and the five points in the chain $2 \prec 2 \cdot 3 \prec 2 \cdot 3 \cdot 5 \prec 2 \cdot 3 \cdot 5 \cdot 7 \prec 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ are colored red while the remaining points are colored blue. Instead, we could color the three points $5 \cdot 7, 7 \cdot 11, 3 \cdot 5 \cdot 7$ red and the remaining points blue. Each of the rank 1 points is pinned as follows: 2 is pinned since it is the only rank one point not below any red point, 3 is pinned since it below a rank 3 red point and no others, 5 is pinned since it is below one rank 2 red point and one rank 3 red point, 7 is pinned since it is below all three red points, and 11 is pinned since it is below one rank 2 red point and no others.

### 3.2 Distributive Lattices

The definition of a distributive lattice appears just before section 3.1. We next define the downset lattice of a poset and illustrate how distributive lattices are generated by their join-irreducible elements.

**Definition 10.** Let $P = (X, \preceq)$ be a poset. The downset lattice $J(P)$ has ground set $\{\text{down}(S) : S \subseteq X\}$ and the relation is $\subseteq$.

Observe that if $P$ is a poset, then $J(P)$ is a distributive lattice, where the meet of elements $S$ and $T$ is $S \cap T$ and the join of these elements is $S \cup T$.

**Example 11.** In Figure 2, the join-irreducible elements of $L_{150}$ are $a = 2$, $b = 3$, $c = 5$, and $d = 25$. When these are ordered using the ordering induced by $L_{150}$ they produce the poset labeled $P$ also shown in Figure 2. There are 16 subsets of elements of $P$, producing 12 distinct downsets, which are given in the following table.

| $S$ | $\emptyset$ | $a$ | $b$ | $c$ | $d$ | $ab$ | $ac$ | $ad$ | $bc$ | $bd$ | $cd$ | $abc$ | $abd$ | $acd$ | $bcd$ | $abcd$ |
|-----|-------------|-----|-----|-----|-----|------|------|------|------|------|------|-------|------|------|------|-------|
| $\text{down}(S)$ | $\emptyset$ | $a$ | $b$ | $c$ | $cd$ | $ab$ | $ac$ | $acd$ | $bc$ | $bcd$ | $cd$ | $abc$ | $abcd$ | $acd$ | $bcd$ | $abcd$ |
When these 12 downsets are ordered by set inclusion, we obtain the downset lattice $J(P)$ which is isomorphic to the original lattice $L_{150}$.

Example 11 illustrates the following theorem of Birkhoff [5], which is called the Fundamental Theorem of Finite Distributive Lattices in [15].

**Theorem 12** (Birkhoff [5]). If $L$ is a distributive lattice and $P$ is the poset induced by the join-irreducible points of $L$, then $J(P)$ is isomorphic to $L$. Indeed, if $Y$ denotes the set of join-irreducible points of $L$, the function $f : L \rightarrow J(P)$ defined by $f(w) = \{y \in Y : y \preceq w\}$ is an isomorphism.

A proof of Theorem 12 appears in [15]. This theorem is fundamental in several ways. First it provides a method for checking whether a poset is a distributive lattice without having to verify that every pair of points has a meet and a join. Simply construct the induced poset $P$ of join-irreducibles, and check whether the mapping $f$ is an isomorphism. Additionally, any distributive lattice can be generated by starting with a poset $P$ and constructing $J(P)$. We utilize Theorem 12 in proving that all distributive lattices have distinguishing number at most two (Theorem 15) and in characterizing which have distinguishing number one (Theorem 14).

Observe that in lattice $L_{150}$ shown in Figure 2 the point 75 can be written as the join of all the join-irreducibles less or equal to it, namely $75 = 3 \vee 5 \vee 25$. Equivalently, in $J(P)$, $\{b,c,d\} = \{b\} \cup \{c\} \cup \{c,d\}$. The next lemma shows this holds in general, that is, every point $w \in L$ can be written as the join of a unique subset of join-irreducibles of $L$. It is a well-known consequence of Birkoff’s Theorem (Theorem 12).

**Lemma 13.** Let $L = (X, \preceq)$ be a distributive lattice and $f : L \rightarrow J(P)$ be the isomorphism from Theorem 12 defined by $f(w) = \{y \in Y : y \preceq w\}$ where $Y$ is the set
of join-irreducible points of \( L \). If \( w \in X \), then \( w = \bigvee_{z \in f(w)} z \) and this representation is unique.

**Proof.** First we show

\[
\text{f}(w) = \bigcup_{z \in f(w)} \text{f}(z)
\]

(1)

Let \( t \preceq w \) in \( L \). Then \( \text{f}(t) \preceq \text{f}(w) \) in \( J(P) \) because \( f \) is an isomorphism. Hence \( \text{f}(t) \subseteq \text{f}(w) \) by the definition of \( J(P) \). This is true for every \( t \), hence \( \text{f}(w) \supseteq \bigcup_{z \in f(w)} \text{f}(z) \). To show the reverse containment, observe that if \( t \) is a join-irreducible, then \( t \in \text{f}(t) \) by definition. Thus, \( \text{f}(w) \subseteq \bigcup_{z \in f(w)} \text{f}(z) \).

In \( J(P) \), the join of two sets is their union, so (1) becomes \( \text{f}(w) = \bigvee_{z \in f(w)} \text{f}(z) \) in \( J(P) \). Since \( f \) is a isomorphism, \( f^{-1} \) is an isomorphism and hence \( f^{-1} \circ f(w) = \bigvee_{z \in f(w)} f^{-1} \circ f(z) \). Thus, \( w = \bigvee_{z \in f(w)} z \) as desired. Since the right hand side is completely determined by \( w \), the representation is unique.

The rank of \( w \) in \( L \) is the same as the rank of \( f(w) \) in \( J(P) \), which is \( |f(w)| \). □

We now have the tools to determine the distinguishing number of any distributive lattice.

**Theorem 14.** If \( L \) is a distributive lattice and \( P \) is the poset induced by the join-irreducible elements of \( L \), then \( D(L) = 1 \) if and only if \( D(P) = 1 \).

**Proof.** Let \( Y \) be the set of join-irreducible elements of \( L \) and \( P \) the poset induced in \( L \) by \( Y \). Since the property of being a join-irreducible is preserved by isomorphism, any automorphism of \( L \) must fix \( Y \). By Lemma 13, every element of \( L \) is the join of a unique set of elements of \( Y \). Since joins are preserved by isomorphism, for any automorphism \( \sigma \) of \( L \), it follows that \( \sigma \) is determined by its action on \( Y \). Thus, the set of automorphisms of \( L \) are in 1-1 correspondence with the set of automorphisms of \( P \).

For any poset \( Q \), \( D(Q) = 1 \) if and only if \( Q \) has no non-trivial automorphisms. Since \( L \) has no non-trivial automorphisms if and only if \( P \) has no non-trivial automorphisms, it follows that \( D(L) = 1 \) if and only if \( D(P) = 1 \). □

Examples 17 and 18 illustrate the proof of Theorem 15 for the lattices given in Figures 2 and 3.

**Theorem 15.** If \( L = (X, \preceq) \) is a distributive lattice and \( P \) is the poset induced by the join-irreducible elements of \( L \), then \( D(L) = 2 \) if and only if \( D(P) > 1 \).

**Proof.** Let \( L = (X, \preceq) \) be a distributive lattice, \( Y = \{y_1, y_2, \ldots, y_t\} \) be the set of join-irreducible points of \( L \), and \( P = (Y, \preceq) \) be the poset induced in \( L \) by the set \( Y \). By Theorem 12, \( L \) is isomorphic to \( J(P) \). We will provide a distinguishing coloring of \( J(P) \) using two colors.
Figure 3: A lattice $L$, its poset $P$ of join-irreducibles, and the downset lattice $J(P)$, together with a distinguishing coloring of $J(P)$.

Let $f : L \to J(P)$ be the isomorphism defined in Theorem 12, let $f(Y) = \{f(y_1), f(y_2), \ldots, f(y_t)\}$. The property of being join-irreducible is preserved under isomorphism, thus $f(Y)$ is the set of join-irreducible points of $J(P)$.

Let $E : y_1 \prec y_2 \prec y_3 \prec \cdots \prec y_t$ be a linear extension of $P$. Color the following chain of elements of $J(P)$ using the color red:

\[
\emptyset, \{f(y_1)\}, \{f(y_1), f(y_2)\}, \{f(y_1), f(y_2), f(y_3)\}, \cdots, \{f(y_1), f(y_2), f(y_3), \ldots, f(y_t)\}.
\]

Color the remaining elements green. We show this is a distinguishing coloring of $J(P)$ by showing that every nontrivial automorphism of $J(P)$ preserves colors.

Since poset automorphisms preserve rank and there is exactly one red vertex at each rank of $J(P)$, we know the red vertices are pinned. Next we show all green points in $f(Y)$ are pinned. Each $f(y_i) \in f(Y)$ is less than a unique lowest red point in the chain of red vertices of $J(P)$. In particular, the point $f(y_i)$ is less than $\{f(y_1), f(y_2), f(y_3), \ldots, f(y_i)\}$ but incomparable to all lower ranked red points. Thus the green points in $f(Y)$ are pinned. By Lemma 13 every point of $J(P)$ that is not in $f(Y)$ is the join of a unique set of elements of $f(Y)$, and hence is pinned. Thus all points are pinned and the coloring is distinguishing.

Our proof of Theorem 15 provides a distinguishing coloring of $L$ for each linear extension of $P$. We record this in Corollary 16.

Corollary 16. For any distributive lattice $L$ whose poset of join-irreducibles is $P$, each linear extension of $P$ leads to a distinguishing labeling of $L$ using two colors, one of which appears on exactly $|P| + 1$ points.

Example 17. For the distributive lattice $L_{150}$ in Figure 2, the set of join-irreducible points is $Y = \{a, b, c, d\}$, where $a = 2$, $b = 3$, $c = 5$ and $d = 25$. $E : a \prec c \prec d \prec b$
of \( P \), the chain of points in \( J(P) \) colored red is \( \emptyset \prec a \prec ac \prec acd \prec abcd \) and the remaining vertices are green. Observe that each join-irreducible point in \( J(P) \) is indeed less than or equal to a unique lowest red point: \( a \preceq a \) (rank 1), \( b \preceq abcd \) (rank 4), \( c \preceq ac \) (rank 2), \( cd \preceq acd \) (rank 3).

**Example 18.** For the distributive lattice in Figure 3, the set of join-irreducible points is \( Y = \{a, b, c, d\} \), where \( a = w_1 \), \( b = w_4 \), \( c = w_5 \), and \( d = w_2 \). For the linear extension \( E : d \prec a \prec b \prec c \) of \( P \), the chain of points in \( J(P) \) colored red is \( \emptyset \prec d \prec ad \prec abd \prec abcd \) and the remaining vertices are green. Observe that each join-irreducible point of \( J(P) \) is indeed less than or equal to a unique lowest red point: \( a \preceq ad \) (rank 2), \( ab \preceq abd \) (rank 3), \( ac \preceq abcd \) (rank 4), \( d \preceq d \) (rank 1). Each point of \( J(P) \) is the join of a unique set of join-irreducible points of \( J(P) \), for example, point \( acd \) is the join of \( a, d, ac \).

### 3.3 Rank-Connected Planar Posets

In Section 1 we defined planar posets and ranked posets. A ranked poset is rank-connected if every pair of consecutive ranks, considered as a vertex-induced subgraph is connected. We say that a Hasse diagram for ranked planar poset is a standard diagram if it is planar, all points at a given rank have the same \( y \)-coordinate, and all edges are straight line segments. The following result appears to be part of the folklore of the field. We include a proof for completeness.

**Proposition 19.** If \( P \) is a ranked planar poset with \( \hat{0} \) and \( \hat{1} \) then \( P \) has a standard diagram.

**Proof.** Partition the points of \( P \) by rank so that \( R_i \) is the set of points of rank \( i \) for \( 0 \leq i \leq n \). Since \( P \) is ranked, each covering edge in \( P \) is between points at consecutive ranks. Suppose we have a planar Hasse diagram for \( P \) in which the points of each rank have the same \( y \)-coordinate for ranks \( k \) and lower and every edge between points of rank at most \( k \) is a straight line segment. If \( k = n \) we are done, so assume \( k < n \).
Let \( w_1, w_2, w_3, \ldots, w_r \) be the elements of \( R_k \) listed from left to right in the Hasse diagram. If \( n = k + 1 \) then \( R_{k+1} = \{ \hat{1} \} \) and we can draw a straight line segment from each element of \( R_k \) to \( \hat{1} \), and this completes the proof. Otherwise, \( n \geq k + 2 \).

Let \( L_k \) be the horizontal line containing the points of \( R_k \) and \( L_{k+1} \) the horizontal line containing the point(s) of \( R_{k+1} \) with lowest \( y \)-coordinate. In Figure 4 the only point of \( R_{k+1} \) with lowest \( y \)-coordinate is \( z_2 \). Order the edges between \( R_k \) and \( R_{k+1} \) from left to right as \( e_1, e_2, \ldots, e_t \) by their order in the strip between lines \( L_k \) and \( L_{k+1} \). This order is uniquely determined because the diagram is planar. This ordering of edges induces an ordering of the points in \( R_{k+1} \) as follows: for any \( u, v \in R_{k+1} \), we order \( u \) before \( v \) if the leftmost edge \( e_i \) incident to \( u \) is to the left of the leftmost edge \( e_j \) incident to \( v \). The resulting order is \( z_1, z_2, z_3, \ldots, z_s \) and this is illustrated by the example in Figure 4.

For each \( z_i \) that is above line \( L_{k+1} \), choose an edge \( e_j \) incident to \( z_i \) and relocate \( z_i \) to the point \( z_i' \) where edge \( e_j \) meets line \( L_{k+1} \). In Figure 4 in each case the first such edge was selected. For each \( z_i \in R_{k+1} \), let \( N(z_i) \) be the set of points in \( R_k \) covered by \( z_i \). The points in \( N(z_i) \) have consecutive indices, for otherwise, there would be a point in \( R_k \) with no upward route to \( \hat{1} \). Thus we can think of \( z_i \) and its edges to \( N(z_i) \) as forming a cone. If \( N(z_i) = \{ w_r, w_{r+1}, \ldots, w_t \} \), then \( z_i \) is the only member of \( R_{k+1} \) that covers any of the internal points \( w_{r+1}, w_{j+2}, \ldots, w_{t-1} \), for any other such element in \( R_{k+1} \) would have no upward path to \( \hat{1} \). Thus cones intersect only at their outermost points. Indeed, if \( w_r \neq w_t \), then no other \( z_j \) of \( R_{k+1} \) can cover both \( w_r \) and \( w_t \), because this would imply that one of \( z_i, z_j \) would have no upward path to \( \hat{1} \). Thus for all \( z_i, z_j \in R_{k+1} \) with \( i \neq j \) we have, \( |N(z_i) \cap N(z_j)| \leq 1 \). Furthermore, if \( i < j \) and \( N(z_i) \cap N(z_j) \neq \emptyset \), then by the planarity assumption and the way we indexed the \( z_i \)’s, the unique point in \( N(z_i) \cap N(z_j) \) is the rightmost element of \( R_k \) incident to \( z_i \) and also is the leftmost element of \( R_k \) incident to \( z_j \). Starting with edges incident to \( z_1 \) and continuing rightward, we can draw the edges between \( R_{k+1} \) and \( R_k \) as straight line segments from the points \( z_1', z_2', \ldots, z_s' \) on \( L_{k+1} \) to the points \( w_1, w_2, \ldots, w_r \) on \( L_k \) and by the properties above we know that none of these segments cross. This is illustrated in the right portion of Figure 4.

It remains to reroute the edges between \( R_{k+1} \) and \( R_{k+2} \). For each \( z'_i \in R_{k+1} \), form a narrow band from \( z'_i \) to \( z_i \) along the original edge \( e_j \) that is incident to \( z'_i \) (see Figure 4). Each edge between \( z_i \) and \( R_{k+2} \) will now start at \( z'_i \), travel through this band to \( z_i \) and continue on its original path to its destination in \( R_{k+2} \). The result is a planar Hasse diagram for \( P \) in which points at each rank are each located on a horizontal line for ranks \( k+1 \) and lower, and edges between points of ranks at most \( k+1 \) are straight line segments. The result follows by induction.

**Remark 20.** Using Proposition 19 it is not hard to show that the poset \( B \) in Figure 7 is not planar.

The next lemma appears in [6] and is helpful to us in the induction proof of Theorem 22.

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Lemma 21. Given a standard diagram of a planar rank-connected poset that has a \( \hat{0} \) and a \( \hat{1} \), there exists a rank in which the leftmost element \( x \) covers exactly one element, \( a \), and is covered by exactly one element, \( b \). Furthermore, if there is an element immediately to the right of \( x \), it also covers \( a \) and is covered by \( b \).

Theorem 22. If \( P \) is a twin-free, ranked, planar poset with a \( \hat{0} \) and a \( \hat{1} \) and for which the points of consecutive ranks induce a connected graph then \( D(P) \leq 2 \).

Proof. Let \( r \) be the maximum rank of points in \( P \). In any automorphism of \( P \), rank is preserved. If there exists a point \( x \) in \( P \) (other than 0 and 1) for which \( x \) is the only element of its rank, then \( x \) is pinned. Thus we need only pin the elements below \( x \) and separately the elements above \( x \). So without loss of generality, we may assume \( P \) has at least two points at each rank other the lowest and highest ranks.

By Proposition 19 we may fix a standard diagram of \( P \). At each rank, there is a leftmost point in the diagram, and because the diagram is planar and \( P \) is rank-connected, the union of these points forms maximal chain \( C_0 \) from 0 to 1. Color the points on chain \( C_0 \) red. Since there is a unique red point at each rank, these points are pinned. Color the remaining points blue. We show this coloring is distinguishing.

We apply Lemma 21 repeatedly to obtain a sequence of chains \( C_0, C_1, \ldots, C_n \), each from 0 to 1, so that \( C_i \) and \( C_{i+1} \) are identical except for two points \( x_i \in C_i \) and \( x_{i+1} \in C_{i+1} \) where \( x_i \) and \( x_{i+1} \) have the same rank in \( P \) and \( x_{i+1} \) is immediately to the right of \( x_i \) in the diagram. We know that \( x_0 \) is pinned since it is red and we proceed by induction.

Assume the points \( x_0, x_1, \ldots, x_{j-1} \) are pinned, thus the points in \( C_{j-1} \) are pinned. If there are no points at \( x_j \)'s rank that lie to the right of \( x_j \) then \( x_j \) is pinned since all remaining points at that rank are already pinned. Otherwise, there exists one or more points at \( x_j \)'s rank that lie to the right of \( x_j \).

Let \( a \) be the point immediately below \( x_j \) on chain \( C_j \) and \( b \) the point immediately above \( x_j \) on \( C_j \). Suppose \( x_j \) is not pinned, so thus there exists a nontrivial automorphism \( \phi \) of \( P \) with \( \phi(x_j) = w_j \) for some \( w_j \neq x_j \). We know \( w_j \) has the same rank as \( x_j \) and is located to the right of \( x_j \) since the points to the left of \( x_j \) are already pinned. Since \( \phi \) is an automorphism, \( \phi(a) \prec \phi(x_j) \prec \phi(b) \) and since \( a \) and \( b \) are pinned we have \( a \prec w_j \prec b \). Partition the set of points with \( b \)'s rank as \( B_1 \cup B_2 \cup \{ b \} \) where the points in \( B_1 \) lie to the left of \( b \) and the points in \( B_2 \) lie to the right of \( b \). By planarity, \( w_j \) is not adjacent to any point in \( B_1 \). However, the points in \( B_1 \cup \{ b \} \) are pinned by our induction hypothesis, and \( \phi(x_j) = w_j \), so \( x_j \) cannot be adjacent to any point of \( B_1 \) either. Also by planarity, \( x_j \) is not adjacent to any points in \( B_2 \), thus the only point at \( b \)'s rank that is adjacent to \( x_j \) is the point \( b \). Similarly, the only point at \( a \)'s rank adjacent to \( x_j \) is \( a \). So \( x_j \) is adjacent only to \( a \) and \( b \). The same must be true of \( w_j \) since \( \phi(x_j) = w_j \) and \( a \) and \( b \) are pinned. This means \( x_j \) and \( w_j \) are twins in \( P \), a contradiction. Thus \( x_j \) is pinned and this completes the induction. \( \Box \)
4 Distinguishing Chromatic Number

We will define two analogs of a proper coloring, each of which leads to a distinguishing chromatic number. The analogous parameter for graphs, \( \chi_D(G) \), is introduced in [8] and studied further by other authors, see for example [3, 4, 7, 10].

Definition 23. A coloring of the points of poset \( P \) is chain-proper if each color class induces a chain (or equivalently, if incomparable points are assigned different colors). The chain distinguishing chromatic number of poset \( P \), denoted \( \chi^c_D(P) \), is the least integer \( k \) for which there is a coloring of \( P \) that is both chain-proper and distinguishing.

For example, for the poset \( M \) in Figure 1, \( \chi^c_D(M) = 2 \) and one chain-proper distinguishing coloring is \( x, y, z \) are colored 1 and \( w, v \) are colored 2.

The definition of \( \chi^c_D(P) \) is related to the following problem of assigning rooms to a set of scheduled events. Represent the set of events as a poset \( P \) in which the events are the points of \( P \) and \( x \prec y \) if event \( x \) ends before event \( y \) begins. In a chain-proper coloring, each color class is a set of events that can be assigned to the same room, and thus the minimum number of color classes is the number of rooms needed to schedule all of the events. If the coloring is distinguishing as well as proper, then the poset together with its coloring will uniquely identify the events as well as specifying which room each would occupy.

Proposition 24. For any poset \( P \) we have \( \chi^c_D(P) = \text{width}(P) \).

Proof. Let \( k \) be the width of \( P \) and let \( A \) be an antichain of \( P \) with \( |A| = k \). Coloring the points of \( A \) properly requires \( k \) colors, hence \( \chi_D(P) \geq k \). For the reverse inequality, use Dilworth’s theorem to partition the points of \( P \) into \( k \) sets, each of which induces a chain in \( P \). Color all points on chain \( i \) using color \( i \) for \( i = 1, 2, 3, \ldots, k \). By definition, this coloring is proper. It is also distinguishing because each point on chain \( i \) has a unique height on that chain, and height is preserved by automorphisms. This shows \( \chi_D(P) \leq k \) and completes the proof.

Proposition 24 shows that the parameter \( \chi^c_D(P) \) does not yield a new avenue for research. However, the parameter \( \chi^a_D(P) \) is more interesting.

Definition 25. A coloring of the points of poset \( P \) is antichain-proper if each color class induces an antichain (or equivalently, if comparable points are assigned different colors). The antichain distinguishing chromatic number of poset \( P \), denoted \( \chi^a_D(P) \), is the least integer \( k \) for which there is a coloring of \( P \) that is both antichain-proper and distinguishing.

For example, for the poset \( M \) in Figure 1, \( \chi^a_D(M) = 4 \) and one antichain-proper distinguishing coloring is \( z \) is colored 1, \( y \) and \( w \) are colored 2, \( x \) is colored 3 and \( v \) is colored 4.
The definition of $\chi^a_D(P)$ is related to the following problem of designing a student’s course schedule. Form a poset $P$ in which the points of $P$ are the courses a student plans to take to complete a major, and $x \prec y$ if course $x$ is a prerequisite for course $y$. In an antichain-proper coloring, if two courses receive the same color, neither is a prerequisite of the other and they can be taken in the same semester. The minimum number of colors needed for an antichain-proper coloring of $P$ is the minimum number of semesters needed to complete the major. If the coloring is also distinguishing, then $P$ together with its coloring will uniquely identify the courses as well as specify which ones are taken in which semester.

The next result is an analog of Proposition 3 and Proposition 4 for the antichain distinguishing chromatic number.

We denote the falling factorial as $(k)_r = k(k-1)(k-2)\cdots(k-r+1)$.

**Proposition 26.** (i) If $P$ is the poset consisting of the sum of $t$ chains in which each chain contains $r$ elements, and $k$ is the positive integer for which $(k-1)_r < t \leq (k)_r$, then $\chi^a_D(P) = k$.

(ii) Let $P$ be the sum of chains and partition $P$ as $P_1 + P_2 + \cdots + P_m$ where $P_i$ consists of $t_i$ chains, each consisting of $r_i$ points, where $r_1, r_2, \ldots, r_m$ are distinct. Then $\chi^a_D(P) = \max\{\chi^a_D(P_i) : 1 \leq i \leq m\}$.

**Proof.** Use the arguments given in the proofs of Propositions 3 and 4 except here the vertices of a chain must get different colors, so there are $(k)_r$ ways to color a chain of $r$ points if there are $k$ colors available.

In our final result, we again use Birkhoff’s Theorem, this time to relate the antichain distinguishing number of a distributive lattice to the antichain distinguishing number of its poset of join-irreducibles.

**Theorem 27.** If $L$ is a distributive lattice and $P$ is the poset induced by the join-irreducible points of $L$, then $\chi^a_D(L) \leq \chi^a_D(P) + |P|$.

**Proof.** Let $Y$ be the set of join-irreducible elements of $L$ and $P$ the poset induced in $L$ by the points in $Y$. By Theorem 12, $J(P)$ is isomorphic to $L$, and we will produce a coloring of the vertices of $J(P)$ which is antichain-proper and distinguishing. Fix an isomorphism from $L$ to $J(P)$ and let $f(Y)$ be the image of the elements of $Y$. Color the elements of $f(Y)$ using $\chi^a_D(P)$ colors so that the coloring is both distinguishing and antichain-proper. Since any automorphism of $J(P)$ maps $f(Y)$ to itself, the elements of $f(Y)$ are pinned.

There is only one minimal element of $J(P)$, so it is pinned, and every other element of $J(P)$ is the join of a unique set of points in $f(Y)$, so they are also pinned. It remains to give an antichain-proper coloring of the remaining elements of $J(P)$ using $|P|$ colors. The rank 1 points are already colored since they are join-irreducible. The remaining (uncolored) points of $J(P)$ have ranks 0, 2, 3, 4, \ldots, $|P|$. Color all uncolored points at rank $i$ using color $i$. Each of these color classes is an antichain, so it is an antichain-proper coloring as desired and uses $\chi^a_D(P) + |P|$ colors.
Note that Theorem 27 is tight for $L_{pq}$ when $p$ and $q$ are distinct primes.

5 Open Questions

We conclude with some open questions.

**Question 5.1.** Let $G$ be a graph, and $\text{Aut}(G)$ its automorphism group and $\sigma \in \text{Aut}(G)$. Define the motion of $\sigma$ to be $m(\sigma) = |\{u \in V(G) \mid \sigma(u) \neq u\}|$, that is, the number of vertices moved by $\sigma$. Russell and Sundaram prove the Motion Lemma in [14]: If $\min_{\sigma \in \text{Aut}(G)} \{m(\sigma) \mid \sigma \neq \text{identity}\} > 2 \log_2(|\text{Aut}(G)|)$, then $D(G) \leq 2$. Many theorems about 2-distinguishability can be proven using the Motion Lemma. Is there a proof of Theorem 15 using the Motion Lemma?

**Question 5.2.** Given a distributive lattice $L$, what is the total number of distinguishing 2-colorings of $L$? By Theorem 15, this number is at least the number of linear extensions of the poset of join-irreducibles of $L$.

**Question 5.3.** Let $L$ be a distributive lattice and $P$ be its poset of join-irreducibles. The distinguishing 2-colorings of distributive lattices in Theorem 15 use one color $|P| + 1$ times. Because the maximum and minimum elements are unique, this can be reduced to $|P| - 1$ uses of one color and $|L| - |P| + 1$ uses of the other. Given a distinguishing coloring of a poset $P$ with the colors red and blue, and assuming there are at least as many blue points as red points, what is minimum number of vertices colored red? For example in $L_{p^2 q^2}$ from Figure 1 coloring point $p$ red and the remaining points blue is a distinguishing coloring and $1 < 4 - 1 = |P| - 1$. In contrast, it is straightforward to check that $B$ from Figure 1 requires 2 red points and $2 = 3 - 1 = |P| - 1$ in this case.

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