On the BLM scale–fixing procedure, its generalizations and the “genuine” higher order corrections

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Abstract

The question of the uniqueness of the Brodsky–Lepage–Mackenzie procedure for fixing the renormalization scale in perturbative QCD is discussed. It is shown that the resulting finite order approximants are as ambiguous as the original truncated perturbative expansions. This inherent ambiguity of the BLM procedure undermines the recent attempts to define “genuine” higher order perturbative corrections.

1 Introduction

Over the last 15 years the problem of the renormalization prescription dependence of finite order approximants to perturbation expansions in QCD \cite{1} has been a subject of lively and sometimes even heated debate. From time to time a “resolution” of this problem is announced, but invariably it turns out that these “solutions” contain the original ambiguity in some disguise. One of such methods has been suggested some years ago by Brodsky, Lepage and Mackenzie (BLM) \cite{1}. Soon after the appearance of the preprint version of \cite{1} Stevenson and Celmaster \cite{2} pointed out its inherent ambiguity, but their criticism has largely been ignored and the BLM procedure has since been used in many phenomenological analyses. In the journal version of their work the authors of \cite{1} have acknowledged the presence of the ambiguity pointed out in \cite{2}, but claimed that it can “be eliminated to a large extent by adopting some physical process as a theoretical standard for defining $\alpha_s(Q)$.” As the BLM procedure is based on the generalization (called “naive nonabelianization” \cite{3}) of the QED procedure of incorporating the effects of quark loops in the renormalized coupling constant (couplant for short), it has been conjectured to be closely related to the large order behaviour of the expansion coefficients in perturbation theory. This observation has recently been exploited in several attempts \cite{3,4,5} to help answer the question of the importance of higher order corrections in QCD perturbative expansions. The idea suggested in these papers is to distinguish the “genuine” higher order corrections from those governed by the choice of the renormalization scale and, supposedly, related to the large order behaviour of perturbative expansions. As this question is of principal as well as practical importance, it is certainly useful to reappraise the old criticism voiced in \cite{2} in order to see how much of it is relevant for these recent efforts. The main purpose of this paper is to carry out such an analysis.

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1In this article only the renormalization prescription dependence of physical quantities in QCD with $n_f$ massless quark flavours will be considered.
The paper is organized as follows. After introducing in Section 2 the notation and recalling some basic facts concerning the description of ambiguities of finite order perturbative approximants, I shall discuss in Section 3 the BLM procedure in QCD. This is followed in Section 4 by a short detour to QED, in order to see why there the BLM procedure does, indeed, lead to a unique result. In Section 5 the abovementioned suggestion [1] to avoid the “residual” RS dependence of the BLM results by selecting some physical process as a standard for the definition of \( \alpha_s \) is shown to be no real cure to the problem. The implications of the inherent ambiguity of the BLM procedure for the basic strategy of the papers [3, 4, 5] are discussed in Section 6, followed by a short summary and conclusions in Section 7.

## 2 Ambiguities of finite order approximants in perturbative QCD

Before coming to the essence of the BLM procedure, let me introduce notation and recall a few basic facts concerning the renormalization prescription ambiguity of finite order perturbative approximants. I shall make a clear difference between the concepts of renormalization prescription, renormalization scheme and renormalization convention, which all appear in the literature and which are not always used in the same sense. I shall emphasize the mathematically complete quantitative description of these concepts as this will be crucial for the later discussion. Let us start by considering perturbation expansion for the generic physical quantity \( r(Q) \), depending, for simplicity, on a single external momentum \( Q \),

\[
r(Q) = a(\mu, \text{RS}) \left[ 1 + r_1(Q/\mu, \text{RS})a(\mu, \text{RS}) + r_2(Q/\mu, \text{RS})a^2(\mu, \text{RS}) + \cdots \right].
\]

The generalization of the following considerations for the case when the leading term behaves as \( a^P \) is trivial. In the above expression \( a(\mu, \text{RS}) \equiv \alpha_s/\pi \) is the renormalized couplant (the adjective “renormalized” will be dropped in the following), evaluated at the renormalization scale \( \mu \) in a given renormalization scheme (RS), to be fully specified below. While the whole sum in (1) is independent of both the scale \( \mu \) and the RS, any finite order approximant \( r^{(N)}(Q/\mu, \text{RS}) \equiv \sum_{k=0}^{N-1} r_k(Q/\mu, \text{RS})a^{k+1}(\mu, \text{RS}) \),

\[
(2)
\]
does depend on \( \mu \) as well as the RS. The dependence of the couplant on the scale \( \mu \) is governed by the equation

\[
\frac{da(\mu, \text{RS})}{d \ln \mu} \equiv \beta(a) = -b a^2(\mu, \text{RS}) \left( 1 + ca(\mu, \text{RS}) + c_2 a^2(\mu, \text{RS}) + \cdots \right).
\]

In massless QCD the first two coefficients on the r.h.s. of (3) are unique functions of \( n_f \), the number of massless quarks (\( b = (33 - 2n_f)/6, c = (153 - 19n_f)/(33 - 4n_f) \)), while all the higher order ones are completely arbitrary. Once they are chosen and some initial condition on \( a \) is specified, (3) can be solved. One way of specifying this boundary condition is via the scale parameter \( \Lambda_{\text{RS}} \), introduced in the following implicit equation for the solution of (3),

\[
\frac{b \ln \frac{\mu}{\Lambda_{\text{RS}}}}{\Lambda_{\text{RS}}} = \frac{1}{a} + c \ln \frac{ca}{1 + ca} + \int_0^a dx \left[ - \frac{1}{x^2 B^{(n)}(x)} + \frac{1}{x^2(1 + cx)} \right]
\]

\[
(4)
\]

These coefficients define the so called renormalization convention (RC), introduced in [6].

\( ^2 \)The parameter \( \Lambda \) introduced in (3) differs from \( \Lambda \) used in most phenomenological analyses by multiplicative \( n_f \)-dependent factor \((2c/b)^{c/b}\), which for realistic values of \( n_f \) is close to unity.
where \( B^{(n)}(x) \equiv (1 + cx + c_2 x^2 + \cdots + c_{n-1} x^{n-1}) \). The dependence of the couplant on the parameters \( c_i \) for \( i \geq 2 \) is determined by equations similar to (3)

\[
\frac{d a(\mu, c_i)}{d c_i} \equiv \beta_i = -\beta(a) \int_0^a \frac{bx^{i+2}}{(\beta(x))^2} dx,
\]

which are uniquely determined by the basic \( \beta \)-function in (3) and thus introduce no additional ambiguity. The renormalization scheme is defined by choosing both the coefficients \( c_i \) and the parameter \( \Lambda_{\text{RS}} \), i.e., by the full specification of the solution of (3), RS \( \equiv \{ c_i, \Lambda_{\text{RS}} \} \). Note that at the NLO only \( \Lambda_{\text{RS}} \) labels the RS. Finally the renormalization prescription is defined by the specification of both the renormalization scale \( \mu \) and renormalization scheme RS. Only the specification of the renormalization prescription thus leads to unique results of finite order perturbative approximants (2).

In connection with \( \Lambda_{\text{RS}} \) the following fact is worth mentioning. The BLM procedure, to be discussed in the next section, is based on the isolation of the \( n_f \)-dependence of the expansion coefficients \( r_k \). This brings up the subtle question of the \( n_f \)-dependence of the chosen RS, which up to the NLO means the \( n_f \)-dependence of \( \Lambda_{\text{RS}} \). As worlds with different number of massless quarks are not related by any theoretical arguments, such as the renormalization group (RG) considerations, there is, however, no meaningful way of introducing this dependence. What can, however, be done in a well-defined manner is to discuss the \( n_f \)-dependence of the ratio \( \Lambda_{\text{RS'}}/\Lambda_{\text{RS}} \) of the \( \Lambda \)-parameters, corresponding to two different RS, as this ratio is determined solely by the RG considerations.

After the arbitrary coefficients \( c_i \) are specified there are thus two parameters to vary: \( \mu \) and \( \Lambda_{\text{RS}} \). As, however, \( \mu \) enters the solution of (3) always in the ratio with \( \Lambda_{\text{RS}} \), we can

- select one \( \Lambda_{\text{RS}} \) and thus one of the RS \( = \{ c_i, \Lambda_{\text{RS}} \} \) and vary \( \mu \) only,
- fix \( \mu \) by identifying it with some external momentum, such as \( Q \), and vary \( \Lambda_{\text{RS}} \) instead.

Both of these options are completely equivalent and it is merely a matter of taste which one to use. To vary both the scale \( \mu \) and the \( \Lambda_{\text{RS}} \) is legal, but obviously redundant. Fixing the scale \( \mu \) without also specifying the RS is, on the other hand, insufficient to specify the renormalization prescription and thus does not lead to a unique result for the finite order approximants (2).

I emphasize this point as the scale \( \mu \) is often fixed by identifying it with some natural physical scale of the process, such as \( Q \). Although such a natural scale can usually be found, its mere existence does not fix the RS and thus does not specify the couplant. In most of phenomenological analyses the \( \overline{\text{MS}} \) RS is tacitly adopted, but there is no theoretical argument for this choice, except that in this RS and for the conventional choices of \( \mu = Q \), the coefficients \( r_k \) are often, but not always, small. If, however, the magnitude of the expansion coefficients \( r_k \) should be the criterion for the “best” choice of the scale \( \mu \) and the RS, we would be naturally drawn to the effective charges approach (2), where all higher order coefficients \( r_k \) vanish by definition.

While the explicit dependence of the couplant on the renormalization scale \( \mu \) and the parameters \( \Lambda_{\text{RS}}, c_i, i \geq 2 \) is given by (3) and (4), the dependence of the coefficients \( r_k \) on them is determined by the relations

\[
\frac{d r^{(N)}(Q/\mu, \text{RS})}{d \ln \mu} = O(a^{N+1}), \quad \frac{d r^{(N)}(Q/\mu, \text{RS})}{d \ln \Lambda_{\text{RS}}} = O(a^{N+1}), \quad \frac{d r^{(N)}(Q/\mu, \text{RS})}{d c_i} = O(a^{N+1}),
\]

which express the internal consistency of perturbation theory. Iterating these equations we easily find

\[
r_1(Q/\mu, \text{RS}) = b \ln \frac{\mu}{\Lambda_{\text{RS}}} - \rho(Q/\Lambda_{\text{RS}}),
\]
\[ r_2(Q/\mu, \text{RS}) = \rho_2 - c_2 + r_1^2 + cr_1, \]  
(7)

\[ r_n(Q/\mu, \text{RS}) = \rho_n - c_n + f(r_i, c_i; \rho_i; i \leq n-1), \]  
(7)

where the quantities \( \rho, \rho_i \) are RG invariants. The dependence of the perturbative approximants (3) on \( Q \) comes entirely through the invariant \( \rho(Q/\Lambda) \), which can be written as

\[ \rho = b \ln(Q/\Lambda_{\text{RS}}) - r_1(1, \text{RS}), \]

while all the higher order invariants \( \rho_i \) are just pure numbers. A nontrivial part of any perturbative calculation beyond the LO boils down to the evaluation of the invariants \( \rho_i \), the rest being essentially a straightforward exploitation of the RG considerations based on (3,5,7).

In the following section the BLM procedure will be discussed in detail at the NLO, as this is where the dependence on the scale \( \mu \) comes in and where also its basic ambiguity becomes evident. At this order only the first two, unique, coefficients \( b \) and \( c \) in (3) are taken into account and the renormalization prescription dependence of (1) is therefore essentially a one–parameter ambiguity. Varying either \( \mu \) for fixed \( \Lambda_{\text{RS}} \), or vice versa, we find

\[ a(\mu', \text{RS}) = a(\mu, \text{RS}) \left( 1 - b \ln \left( \frac{\mu'}{\mu} \right) a(\mu, \text{RS}) + \cdots \right), \]

(9)

\[ a(\mu, \text{RS}') = a(\mu, \text{RS}) \left( 1 - b \ln \left( \frac{\Lambda_{\text{RS}}}{\Lambda_{\text{RS}'}} \right) a(\mu, \text{RS}) + \cdots \right) \]

(10)

In both ways of labelling the ambiguity, we have

\[ a' = a(1 - \kappa a + \cdots), \quad r'_1 = r_1 + \kappa \]

(11)

At the NLO and for some “initial” \( a \equiv a(\mu, \text{RS}) \) \( \kappa \) can thus be used as yet another way of labelling this ambiguity. After selecting

a) the scale \( \mu \) in a fixed RS, or

b) the parameter \( \Lambda_{\text{RS}} \) for fixed \( \mu \), or

c) the parameter \( \kappa \) for a fixed initial \( a \) in (11)

we should, however, check that the resulting couplant and expansion coefficient \( r_1 \) do not depend on the RS (for a), \( \mu \) (for b) or initial \( a \) (for c). At the NLO and in all three ways of labelling the inherent one–parameter ambiguity, both the Principle of Minimal Sensitivity (PMS) \[6\] and Effective Charges (ECH) \[7\] approaches do, indeed, lead to a unique result for \( r_2(Q) \). On the other hand, as will be shown in detail in the next section, this basic requirement is not satisfied by the BLM procedure.

Although the scale \( \mu \) appears naturally and unavoidably in the process of renormalization, perturbation expansions for physical quantities can actually do without it. Combining eqs. (4) and (7) allows us to express all the expansion coefficients \( r_k \) as unique functions of the \( \beta \)–function coefficients \( c_k \) and the couplant \( a \). To specify a unique result for any finite order approximant (4) we can thus use the set \( \{a, c_i\} \) instead of \( \{\mu, \Lambda_{\text{RS}}, c_i\} \). At the NLO this means that instead of the pair \( \mu, \Lambda_{\text{RS}} \) the couplant itself can serve to label the one–parameter ambiguity! Note that neither the PMS nor the ECH approaches fix the scale \( \mu \) and the RS=\{\Lambda_{\text{RS}}\} separately, but merely their ratio, or, equivalently, just the couplant \( a \).

\[ ^4 \text{Despite the appearance of } \Lambda_{\text{RS}} \text{ in this expression, } \rho \text{ is actually RS–independent as the RS–dependences of the two terms on the r.h.s. of (8) mutually cancel.} \]

\[ ^5 \text{Starting at the NNLO, there are certain complications with the existence and/or uniqueness of the PMS and ECH “optimized” results, but this is irrelevant for the present discussion.} \]
3 The BLM procedure in QCD

This method borrows its basic idea from QED, where the renormalization of the electric charge is entirely due to the vacuum polarization by the charged fermion–antifermion pairs. In QCD the scale dependence of the couplant is due to other effects as well, but the authors of [1] suggest to fix the scale \( \mu \) by absorbing the effects of quark loops entirely in the definition of the renormalized couplant. Let us consider the quantity (1) up to the NLO and define the class of the so-called regular RS\( = \{ \Lambda_{RS} \} \) by the condition that the expansion coefficient \( r_1 \) can be written as a linear function of \( n_f \)

\[
\begin{equation}
(\mu/Q, \Lambda_{RS}) = a(\mu, RS) \left( 1 + r_{10} \left( \frac{\mu}{Q}, RS \right) + n_f r_{11} \left( \frac{\mu}{Q}, RS \right) \right) a(\mu, RS),
\end{equation}
\]

where \( r_{10}, r_{11} \) are \( n_f \)-independent coefficients. \( ^{6} \) The BLM procedure fixes \( \mu \) by the condition \( r_{11}(\mu/Q, RS) = 0 \). This implies [1]

\[
\begin{equation}
\begin{aligned}
\mu_{BLM}(\mu, RS) & \equiv \mu \exp \left[ 3r_{11}(\mu/Q, RS) \right], \\

r_{1}(\mu_{BLM}/Q, RS) & = r_{10}(\mu/Q, RS) + \frac{33}{2} r_{11}(\mu/Q, RS)
\end{aligned}
\end{equation}
\]

and consequently

\[
\begin{equation}
\begin{aligned}
r^{(2)}_{BLM}(Q/\mu, RS) & = a(\mu_{BLM}, RS) \left( 1 + r_{10} \left( \frac{\mu}{Q}, RS \right) + \frac{33}{2} r_{11} \left( \frac{\mu}{Q}, RS \right) \right) a(\mu_{BLM}, RS). \\
\end{aligned}
\end{equation}
\]

Note that the BLM–fixed scale \( \mu_{BLM} \) is a function of both the “initial” scale \( \mu \) and RS. This by itself is not surprising as the same happens in the PMS and ECH approaches as well. In these approaches the corresponding couplant \( a \), as well as the coefficients \( r_k \), are, however, independent of both \( \mu \) and the RS, while this is not the case in the BLM procedure. In the explicit calculation Celmaster and Stevenson have shown that for the \( \Upsilon \) hadronic decay width the coefficient \( r_1 \) has different values in two most frequently used RS’s, namely the \( \overline{\text{MS}} \) and symmetric MOM RS based on the 3–gluon vertex in the Landau gauge. Despite the fact that this simple example is sufficient to demonstrate the basic shortcoming of the BLM procedure, its message has largely been ignored, presumably because the dependence of BLM results on the RS has been considered as a kind of “residual” dependence, less important than the dependence on the scale \( \mu \). In the rest of this section I shall demonstrate that this “residual” RS dependence of the BLM procedure actually coincides with the original scale ambiguity in a fixed RS.

To see how the BLM results depend on \( \mu \) and the RS, we need to evaluate the ratio

\[
\begin{equation}
\frac{\mu_{BLM}(\mu, RS)}{\Lambda_{RS}} = \exp \left[ 3r_{11} \left( \frac{\mu}{Q}, RS \right) + \ln \left( \frac{\mu}{\Lambda_{RS}} \right) \right] = \exp[Z(\mu, RS)].
\end{equation}
\]

Going from RS to RS’ (for fixed \( \mu \)), or from \( \mu \) to \( \mu’ \) (for fixed RS), we get, exploiting (7),

\[
\begin{equation}
\begin{aligned}
3r_1(\mu/Q, RS') & = 3r_1(\mu/Q, RS) + \left( \frac{33}{2} - n_f \right) \ln \left( \frac{\Lambda_{RS}}{\Lambda_{RS'}} \right), \\
3r_1(\mu'/Q, RS) & = 3r_1(\mu/Q, RS) + \left( \frac{33}{2} - n_f \right) \ln \left( \frac{\mu'}{\mu} \right).
\end{aligned}
\end{equation}
\]

\( ^6 \)In this statement \( \mu \) as well as the RS (labelled by \( \Lambda_{RS} \)) are held fixed and only the explicit dependence of \( r_1 \) on \( n_f \) is taken into account.
Within the class of regular RS’s we have

\[ b \ln \left( \frac{\Lambda_{RS}}{\Lambda_{RS'}} \right) = A + B n_f, \]

\[ Z(\mu, RS') = Z(\mu, RS) + \frac{\kappa}{b}, \]

\[ r_1(\mu_{BLM}(\mu, RS')) = r_1(\mu_{BLM}(\mu, RS)) + \kappa, \quad \kappa \equiv \frac{33}{2}, \]

\[ a_{BLM}(RS') = a_{BLM}(RS) \left[ 1 - k_{BLM}(RS) \right]. \]

The last two equations coincide with the transformations [14]. We see that the BLM results are unique only provided \( \kappa = 33B/2 + A = 0 \), which is equivalent to the condition that the ratio \( \Lambda_{RS}/\Lambda_{RS'} \) is \( n_f \)-independent. This condition divides all regular RS into disjoint subclasses of RS’s, characterized by the property that for any pair of RS’s from the same subclass \( \kappa = 33B/2 + A = 0 \), while for RS, RS’ from different subclasses \( \kappa \neq 0 \). The BLM results are thus unique only within each of these subclasses, while different subclasses lead to different results. To show that the latter situation arises naturally for quite conventional RS’s, let me take RS = each of these subclasses, while different subclasses lead to different results. To show that the latter situation arises naturally for quite conventional RS’s, let me take RS =

\[ r_1(\mu_{BLM}(\mu, RS')) = r_1(\mu_{BLM}(\mu, RS)) + \kappa, \quad \kappa \equiv \frac{33}{2}B + A, \]

\[ a_{BLM}(RS') = a_{BLM}(RS) \left[ 1 - k_{BLM}(RS) \right]. \]

We conclude that not only do the BLM results depend on the chosen RS(\( \alpha_G \)), as claimed in [2], but quantitatively this ambiguity is completely equivalent to the scale ambiguity in any fixed RS.

In analogous way it can be shown that for \( n_f \)-independent scale \( \mu \) the BLM results are, indeed, \( \mu \)-independent. Formally we could introduce \( n_f \)-dependent scale \( \mu \) as well and thus destroy the scale independence of the BLM results, but as for each fixed \( n_f \) the scale \( \mu \) is completely arbitrary there is no meaning in introducing this dependence. The situation looks differently as far as the \( n_f \)-dependence of the chosen RS’s is concerned, primarily because in the MOM–based RS’s this dependence arises quite naturally.

It is also simple to see why, contrary to the BLM approach, the ECH one [7] leads to unique results. Taking into account [7] the condition \( r_1 = 0 \) implies

\[ \frac{\mu_{ECH}(RS)}{\Lambda_{RS}} = \exp \left( \frac{\rho(Q)}{b} \right). \]

\footnote{Or for the Landau gauge \( \alpha_G = 0 \).}
The scale µ does not appear in (25) at all and as the r.h.s. of (25) contains only RG invariants ρ(Q) and b, it is manifestly RS–independent. Similar considerations hold in the PMS approach. It is thus the very essence of the BLM procedure, i.e. the attempt to separate the coefficient r₁ into the n_f–dependent and n_f–independent parts, that causes its inherent ambiguity.

4 The case of QED

The results of the BLM procedure in QED can be obtained from (20)–(21) simply by taking into account that in QED the corresponding b_{QED} = −2n_f/3 and contains thus no term analogous to 33/2 = 11N_c/2, coming in QCD from gluon selfinteraction. Consequently the BLM results are unique provided κ = A = 0, which is again equivalent to the condition that Λ_{RS} is n_f–independent. In QED, however, the scale as well as RS–dependence of the corresponding couplant come entirely from the renormalization of fermion loops and it is therefore natural to define the regular RS’s as those satisfying this condition. In QED and within any class of regular RS’s the BLM results are, indeed, unique.

5 Fixing the RS by means of the “standard” physical process

The idea, suggested in [1], is to fix the RS of the couplant a(µ, RS) with the help of some “standard” physical quantity, such as the familiar ratio

\[ R(s) \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to µ^+µ^-)} = 3 \sum_i e_i^2 (1 + r(s)), \]  

by demanding \( r(s) = a(\sqrt{s}, RS) \). Note that in this way chosen couplant is just the effective charge corresponding to the physical quantity \( r(s) \).

The main problem with this strategy is that there is no theoretical reason to prefer one particular physical quantity to another to serve as the “standard”. By appealing to some “standard” physical process to fix the RS this ambiguity is transformed into the “initial condition” ambiguity [10, 11]. Moreover, if the ECH approach is used for the “standard” physical quantity, why not to use it for the one under study?

Secondly, in writing \( r(s) = a(\sqrt{s}, RS) \) we have already set the scale of the couplant a(µ, RS) to \( µ = \sqrt{s} \). Recall that, as emphasized in Section 3, the ECH approach does not actually fix the scale and the RS, but determines directly the couplant \( a_{ECH} \), or equivalently, the ratio \( µ/Λ_{RS} \)! Without specifying the scale µ in the equation \( r(s) = a_{ECH}(µ, RS) \) the RS is not fixed and the reference to the “standard” physical quantity thus of no help. However, to set \( µ = \sqrt{s} \) in (26) has no theoretical justification, in particular taking into account that the basic aim of the BLM procedure is just to find some plausible scale fixing method!

6 Generalization of the BLM procedure and “genuine” higher order corrections

Let me now turn to the implications of the inherent ambiguity of the BLM procedure for some of the essential ingredients of the recent papers [3, 4, 5], where higher order perturbative corrections are separated into two parts:

- the so called “genuine” higher order corrections, which are “hard to anticipate” [3], and
those related to the renormalization scale dependence of the couplant, which by an “improper” choice of \( \mu \) may become artificially large.

The starting point of these attempts is the claim \( [3] \) that “some prescriptions may be closer to general expectations.” Without specifying the meaning of the term “general expectation”, all the mentioned papers single out the BLM procedure as the best way of fixing the scale \( \mu \). One of the arguments in favour of this conjecture is the widely held view that because the BLM approach fixes the scale \( \mu \) via the “naive nonabelianization” of the original QED procedure of incorporating the effects of quark loops in the renormalized couplant, it is related to the large order behaviour of expansion coefficients \( r_k \) in (1) \( [3] \). By generalizing the BLM procedure to higher orders the papers \( [3, 4] \) attempt to incorporate the effects of the leading IR renormalon in the BLM couplant, thereby isolating the remaining “genuine” higher order perturbative corrections.

The generalization of the BLM scale–fixing procedure to arbitrary an order \( N \) suggested in \( [3, 4] \) is based on the generalization of the relation \( (12) \)

\[
8 \quad r_n = r_{n0} + r_{n1} n_f + \cdots + r_{nn} n_f^n. \tag{27}
\]

Writing \( r_n \) as

\[
8 \quad r_n = \delta_n + (b/2)^n d_n, \tag{28}
\]

where \( d_n \) are \( n_f \)–independent, absorbs the leading \( n_f \)–dependence in \( d_n \) and allows the authors of \( [3, 4] \) to define the generalized BLM scale \( Q_{BLM}^{(N)} \) at order \( N \) via the relation \( 7 \)

\[
8 \quad a(Q_{BLM}^{(N)}) \equiv a(Q) M_N, \quad M_N \equiv 1 + \sum_{n=1}^{N} d_n (b/2)^n a^n. \tag{29}
\]

With these definitions eq. (1) can be written as

\[
8 \quad r(Q) = a(Q_{BLM}^{(N)}) + \sum_{n=1}^{N} \delta_n a(Q)^{n+1} \tag{30}
\]

Except for the argument of the couplant in the second term of (30), which, however, does not influence the coefficient \( \delta_1 = r_1(\mu_{BLM}/Q, RS) \), (30) reduces for \( N = 1 \) to (15). According to \( [3, 4] \) the coefficients \( \delta_n \) represent the “genuine” higher order corrections, in contrast to those included in \( M_N \), which are incorporated, via the BLM procedure, in the leading–order couplant \( a(Q_{BLM}) \).

Moreover, in the limit \( N \to \infty \), \( Q_{BLM} \equiv \lim_{N \to \infty} Q_{BLM}^{(N)} \) is claimed to be independent of the finite renormalization of the fermion loop.

However, in QCD the renormalization of the colour charge is not given by the gluon polarization due to fermion loops only. Therefore the fact that each of the terms on the r.h.s. of (30) is independent of the finite renormalization of the basic fermion loop does not imply that it is also RS–invariant and thus unique. In Section 3 I have shown explicitly that within the class of MOM–based RS’s both the BLM couplant \( a(Q_{BLM}^{(2)}) \) and the leading coefficient \( \delta_1 \) of the second term in (30) are RS–dependent. The value of the “genuine” NLO correction coefficient \( \delta_1 \) is a function of the the gauge–fixing parameter \( \alpha_G \), and can take any prescribed value. The RS–dependence of \( \delta_1 \) automatically implies that the separation of \( r(Q) \) into the two terms in (30) is ambiguous even if the BLM procedure is generalized to an arbitrary order \( N \). Recall that in perturbative expansions

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8 As the leading order coefficient \( r_0 \) of \( [3, 4] \) is unique, it can be set to unity, as done in (1), without losing any generality.

9 Following \( [3, 4] \) I set in the rest of this section \( \mu = Q \).
of RS–independent quantities $\sum_{n=p} r_n a^n$ the leading–order coefficient $r_p$ has to be unique as there is no way how its RS–dependence could be compensated by the RS–dependence of the expansion parameter $a$! The only way to secure the RS–invariance of the full sum in (30) is therefore the mutual cancellation of this dependence between the BLM couplant $a(Q_{BLM})$ and the expansion describing the supposed “genuine” higher order perturbative corrections.

Beyond the NLO the couplant $a$ as well as the expansion coefficients $r_k$ become functions of the additional free parameters $c_i; i \geq 2$, specifying the RC. This freedom implies that not only $\delta_1$, but in fact all the coefficients $\delta_k$ are completely arbitrary, exactly in the same way as the coefficients $r_k$ in the general RS! In the ECH approach, for instance, they are set to zero. The large order behaviour of the coefficients $r_k$ crucially depends on the choice of the RS and so do also all the higher order coefficients $\delta_k$. The reason is clear: the RG transformations mix the two terms on the r.h.s. of (30) so that we cannot attach an unambiguous meaning to each of them separately.

The fact that by an appropriate choice of the RS we can get rid of the divergent behaviour of $r_k$ does not, of course, mean that we can in this way solve the problem of the divergence of perturbation expansions. The standard derivation of the asymptotic behaviour of the expansion coefficients $r_k$ is carried out in the ’t Hooft RS, in which all higher order $\beta$–function coefficients $c_i; i \geq 2$ are set to zero by definition. The resulting growth behaviour of the coefficients $r_k$ at large $k$ implies via (7) factorial behaviour of the invariants $\rho_k \propto k!$. This, in turn, implies that $r_k$ are factorially divergent also in all those RS=$\{\Lambda_{RS}, c_i\}$, where the coefficients $c_k$ define a convergent series. In the RS’s where $r_k$ define convergent series, the factorial behaviour of the invariants $\rho_k$ reappears as the divergence of the perturbation expansion of the corresponding $\beta$–function. Note, however, that factorial behaviour of the coefficients $c_k$ in (3) cannot be influenced by the choice of the scale $\mu$ and thus has nothing to do with it. The fact that we can freely shuffle part or the whole factorial divergence of the RG invariants $\rho_k$ between the coefficients $r_k$ and $c_k$ clearly signals that it is impossible to relate the effects of the factorial divergence of the invariants $\rho_k$ to the choice of the scale, as suggested in [3, 4, 5]. In other words, there is no well–defined relation between the large order behaviour of the coefficients $r_k$ and the choice of the renormalization scale $\mu$. The former depends not only on $\mu$ but also on all the parameters $c_k$, specifying the RS.

It is also fair to say that we actually do not even know how the coefficients $c_k$ behave in the most popular RS, the MS. The usual expectation is that they define a divergent series, but there are no convincing arguments behind this conjecture. And without this knowledge the behaviour of the expansion coefficients $r_k$ in this RS is also an open question.

## 7 Summary and conclusions

The preceding sections demonstrate two closely related facts:

a) the inherent ambiguity of the BLM scale fixing procedure, and

b) the impossibility to define in a reasonably unambiguous (i.e. RS–independent) way the “genuine” higher order perturbative corrections to physical quantities.

The RG transformations bind inextricably the two terms in the sum of (30). The “genuine” higher order corrections, defined in [1, 2, 3] by incorporating the supposed large order behaviour of the expansion coefficients in the BLM scale, are in fact as ambiguous as the original perturbation expansions before the BLM scale–fixing.

The divergence of perturbative expansions can be expressed in an unambiguous way as the statement about the factorial divergence of the RG invariants $\rho_i$. Anything else, for instance the divergence of the expansions in (31) and/or (32), depends on the chosen RS and has thus no direct
physical meaning. In my view the best strategy how to proceed in such circumstances is to follow
the suggestion of [12], i.e. to choose at each finite order \( N \) of perturbation theory a definite
renormalization prescription and investigate the limit of finite order approximants (2), i.e.
\[
\lim_{N \to \infty} r^{(N)}(\mu(N), RS(N)).
\] (31)

There is no problem to choose the dependences \( \mu(N) \) and RS\( (N) \) in such a way that this limit is
finite, even for factorially divergent series with asymptotically constant sign [13]. The question is
how to do it in a reasonably unique manner. The nonperturbative power corrections should provide
a crucial piece of information in this respect.

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