Super Giant Quantum Vortex in a Pion Bose-Einstein Condensate

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Under the circumstance of parallel rotation and magnetic field that are expected in off-central heavy ion collisions, charged pions may form a Bose-Einstein condensate because of the lift of the Landau level degeneracy by rotation. We show that the ground state of such a pion superfluid is a super giant quantum vortex, i.e., a quantized vortex with a large circulation. The size of the vortex is as large as the system size, which may yield to observable effects in off-central heavy ion collisions.

Pions are the lightest hadrons of the strong interaction and are regarded as the pseudo-Goldstone bosons associated with the dynamical chiral symmetry breaking. As bosons, they may undergo Bose-Einstein condensation (BEC) in certain circumstances. The studies of quantum chromodynamics (QCD) at finite isospin chemical potential indicate that BEC of charged pions takes place when the isospin chemical potential exceeds the mass of charged pions [1–3]. It was proposed that BEC of pions may be formed in compact stars [4–8], in heavy ion collisions [9–11], and in the early Universe [12–14].

In relativistic heavy ion collisions, large vorticity and magnetic fields can be generated. Theoretical studies predicted that non-central collisions involve large angular momenta in the range $10^3 \sim 10^5$ [15–18]. The global polarization of $\Lambda$ hyperon observed in off-central Au-Au collisions reported by the STAR collaboration indicates a large vorticity with an angular velocity $\Omega \approx (9 \pm 1) \times 10^{21}$ Hz $\sim 0.05 m_\pi$ [19]. Meanwhile, we expect that a large magnetic field $B$, parallel to the vorticity, is formed at the early stage of the collision. Numerical simulations indicated that the strength of the magnetic field reaches $eB \sim m_\pi^2$ [20–22]. The state of QCD matter under the circumstance of parallel rotation and magnetic field (PRM) arises as an interesting theoretical issue.

It was argued that PRM can induce BEC of charged pions or a pion superfluid phase based on the solution of the Klein-Gordon equation for non-interacting pions in PRM [23]. The mechanism is simple but robust: The Landau level degeneracy of charged pions in a constant magnetic field is lifted by rotation, and the rotation then plays the role of a chemical potential. However, the true ground state of the pion superfluid with realistic pion-pion interaction is yet unknown. Under the circumstance of PRM, the ground state would be inhomogeneous, such as a quantized vortex or a vortex lattice.

In this Letter, we show that the ground state of the pion superfluid in PRM is a super giant quantum vortex. Quantum vortices are a type of topological defect exhibited in superfluids and superconductors. In a cylindrical vortex state, the macroscopic wave function of the condensate can be written as

$$\psi(r) = f(r)e^{i\omega \theta}, \quad \omega \in \mathbb{Z},$$

with the cylindrical coordinates $r = (\rho, \theta, z)$. A stable vortex state in a quantum fluid is usually the singly quantized with winding number $w = 1$ (e.g., superfluid at a given rotation rate). A vortex state with winding number $w \geq 2$ is called a giant vortex since the radial profile behaves as $f(\rho) \sim \rho^w$ near the core. The giant vortex is usually unstable owing to its large energy cost ($\sim w^2$). Instead, a lattice of singly quantized vortices will form. Searching for giant vortices is a longstanding topic in the research of quantum fluids [24–32]. The pion superfluid studied in this work provides an extreme example: The ground state is a super giant quantum vortex with an extremely large winding number $w \gg 1$.

Minimal Model:— The minimal model to describe the charged pions is a relativistic complex scalar field model with a quartic self-interaction [33]. The Lagrangian density is given by

$$\mathcal{L} = (\partial_\mu \Phi^*) (\partial^\mu \Phi) - m_\pi^2 |\Phi|^2 - \lambda |\Phi|^4.$$  \hspace{1cm} (2)

The coupling constant can be set as $\lambda = m_\pi^2/(2f_\pi^2)$ so that the s-wave pion-pion interaction length in the $I = 2$ channel at the tree level, $a_{\pi\pi} = m_\pi/(16\pi f_\pi^2)$ [34], is recovered. However, we leave $\lambda$ as a free parameter in this work.

We replace the system in a constant magnetic field along the $z$ direction, $B = B\hat{z}$. Furthermore, a global rigid rotation along the magnetic field is applied, with angular velocity $\Omega = \Omega\hat{z}$. We consider the case $eB > 0$ and $\Omega > 0$ without loss of generality. It is convenient to study the system in a rotating frame. The spacetime metric $g_{\mu\nu}$ of the rotating frame is given by

$$ds^2 = (1 - \Omega^2 \rho^2)dt^2 + 2\Omega \rho dx dt - 2\Omega \rho dy dt - dr^2,$$  \hspace{1cm} (3)

where $r = (x, y, z)$ and $\rho = \sqrt{x^2 + y^2}$. The cylindrical coordinates $r = (\rho, \theta, z)$ will also be used in the following. The action of the system is given by

$$S = \int d^4x \sqrt{-g} \left[ g^{\alpha\beta} (D_\alpha \Phi)^* (D_\beta \Phi) - m_\pi^2 |\Phi|^2 - \lambda |\Phi|^4 \right],$$  \hspace{1cm} (4)

where $|g| = \text{det}(g_{\mu\nu})$. The constant magnetic field enters the Lagrangian density through the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$, where $A_\mu$ is the vector potential in
the rotating frame. Since \( \sqrt{-|g|} = 1 \), it is convenient to rewrite the action as \( S = \int d^4xL \), with the Lagrangian density
\[
L = \left| (D_t + \Omega y D_x - \Omega x D_y) \Phi \right|^2 - |D_t \Phi|^2 - m^2_\pi \Phi^2 - \lambda |\Phi|^4.
\]
(5)

In the rest frame, it is convenient to use the symmetric gauge \( A^B_t = (0, B ey/2, -B ex/2, 0) \) so that the rotational symmetry along the z-axis is manifested. The vector potential in the rotating frame is then given by \( A_t = (-\Omega y e^z/2, By/2, -B x/2, 0) \) according to the coordinate transformation to the rotating frame \( t_R = t \), \( \rho_R = \rho \), \( \theta_R = \theta + \Omega t \). Note that an additional electric field \( E = B \Omega \rho \) is induced in the rotating frame. However, according to the identity \( D_t + \Omega y D_x - \Omega x D_y = \partial_t + \Omega y \partial_x - \Omega x \partial_y \), the induced electric field \( E \) cancels out automatically, indicating that the rotating frame corresponds only to a frame change with no new force [23].

Therefore, the Lagrangian density (5) reduces to
\[
L = \left| (\partial_t - i\Omega L_z) \Phi \right|^2 - |D_t \Phi|^2 - m^2_\pi |\Phi|^2 - \lambda |\Phi|^4,
\]
where \( L_z \equiv -i(x \partial_y - y \partial_x) \) is the angular momentum along the z direction.

Free-Particle Picture:— In the absence of interaction \( (\lambda = 0) \), the Klein-Gordon equation in PRM is given by
\[
[-(\partial_t - i\Omega L_z)^2 + K_{2D} + \partial^2_z - m^2_\pi] \Phi(t, r) = 0,
\]
(7)
where the operator \( K_{2D} \) is defined as
\[
K_{2D} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{L_z^2}{\rho^2} - \frac{1}{4} e^2 B^2 \rho^2 + e B L_z.
\]
(8)

Consider a cylindrical system with radius \( R \). The solution can be written as \( \Phi(t, r) = e^{-iEt + ip_z z + il \theta \partial_n l(\rho)} \), where \( p_z \) is the momentum along the z-direction and \( l \) is the angular momentum quantum number. The solution of the radial part can be given by [23]
\[
\varphi_{nl}(\rho) = N_{nl} \rho^{|l|} e^{-\frac{1}{2} e B \rho^2} I_{1F1} \left( -a_{nl}, |l| + 1, \frac{e B \rho^2}{2} \right),
\]
where \( N_{nl} \) is a normalization factor and \( I_{1F1} \) is a confluent hypergeometrical function with the parameter
\[
-a_{nl} = \frac{1}{2} (|l| - l + 1) - \frac{1}{2eB} [(E + i \lambda)^2 - p_z^2 - m^2_\pi].
\]
(9)

The energy levels \( E = E_{nl} \) can be obtained by imposing a zero boundary condition at \( \rho = R \), i.e.,
\[
1 F_1 \left( -a_{nl}, |l| + 1, \frac{e B R^2}{2} \right) = 0,
\]
(11)
where \( a_{nl} \) is defined as the \((n + 1)\)-th zero of left-hand side for a given \( l \).

For a large system with radius \( R \to \infty \), \( a_{nl} \to n \), and the function \( I_{1F1} \) reduces to an associated Laguerre polynomial. At \( \Omega = 0 \), the energy spectrum recovers the Landau levels \( E_n = \left[ p_z^2 + m^2_\pi + eB(2n + 1) \right]^{1/2} \), with \(-n < l < N - n \). Here \( N \equiv eBR^2/2 \) is the degeneracy of the Landau levels. When a rotation is turned on, the degeneracy is lifted and the spectrum becomes \( E_{nl} = E_n \pm \Omega l \). The term \( \Omega l \) plays the role of a chemical potential. Bose-Einstein condensation takes place if the largest chemical potential \( \Omega N \) exceeds the effective mass \( \sqrt{m^2_\pi + eB} \) in the lowest Landau level [23].

Ground State with Interaction:— Now we turn on the interaction. It is useful to work in the imaginary-time formalism. The partition function of the system is given by
\[
Z = \int \mathcal{D}[\Phi^*] [\mathcal{D}[\Phi]] \exp(-\mathcal{S}_E),
\]
where the action reads
\[
\mathcal{S}_E = \int_X \left\{ \Phi^* \mathcal{G} \Phi + \lambda |\Phi|^4 \right\},
\]
(12)
with \( f_X \equiv \int_0^\beta d\tau \int d^3r \mathcal{G} = m^2_\pi - (\partial_t - \Omega L_z)^2 - K_{2D} - \partial^2_z \) and \( \mathcal{G} = m^2_\pi - (\partial_t - \Omega L_z)^2 - K_{2D} - \partial^2_z \). Here \( \tau \) is the imaginary time and \( \beta \) is the inverse of the temperature. We focus on the zero temperature limit \( (\beta \to \infty) \) in the following. If the ground state is a Bose-Einstein condensate, the complex scalar field \( \Phi(\tau, r) \) acquires a nonzero expectation value. Therefore, we decompose the quantum field \( \Phi(\tau, r) \) into its classical part \( \psi(r) \) and its quantum fluctuation \( \phi(\tau, r) \), i.e.,
\[
\Phi(\tau, r) = \psi(r) + \phi(\tau, r).
\]
(13)

While the condensate \( \psi(r) \) is static, it can be inhomogeneous. The profile of the condensate is determined by minimizing the effective potential, which is rather hard to evaluate beyond the tree level. At the tree level, the treatment is in analogy to the Gross-Pitaevskii (GP) theory of nonrelativistic Bose-Einstein condensates [35, 36].

The GP potential of the present system is given by
\[
U[\psi(r)] = \int d^3 r \left[ \psi^* (r) \left( m^2_\pi - \Omega^2 L_z^2 - K_{2D} - \partial^2_z \right) \psi(r) + \lambda |\psi(r)|^4 \right].
\]
(14)

Furthermore, we assume that the condensate is homogeneous along the z direction, \( \psi(r) = \psi(\rho, \theta) \). The GP potential per length along the z-direction reads
\[
U[\psi(\rho, \theta)] = \frac{U[\psi(r)]}{\int dz} = \int_{2D} \left[ \psi^* (\rho, \theta) \left( m^2_\pi - \Omega^2 L_z^2 - K_{2D} \right) \psi(\rho, \theta) + \lambda |\psi(\rho, \theta)|^4 \right],
\]
(15)
where \( \int_{2D} \equiv \int_0^{2\pi} d\theta \int_0^R \rho d\rho \). The minimization of \( U \) leads to a relativistic GP equation in PRM,
\[
\left[ m^2_\pi - \Omega^2 L_z^2 - K_{2D} + 2\lambda |\psi(\rho, \theta)|^2 \right] \psi(\rho, \theta) = 0.
\]
(16)
Here $\Theta$ with eigenvalues is given by of the operator in the quadratic term, $m^2 - \Omega^2 L^2 - K_{2D}$, is given by

$$F_{nl}(\rho, \theta) = \varphi_{nl}(\rho)\Theta_l(\theta),$$

with eigenvalues

$$K_{nl} = m^2 - \Omega^2 l^2 + eB(2a_{nl} + |l| - l + 1).$$

Here $\Theta_l(\theta) = e^{il\theta}/\sqrt{2\pi} \sqrt{e}$ and $\varphi_{nl}(\rho)$ is the solution (9) of the Klein-Gordon equation. Therefore, it is natural to expand $\psi(\rho, \theta)$ in terms of $F_{nl}(\rho, \theta)$, i.e.,

$$\psi(\rho, \theta) = \sum_{n=0}^{\infty} \sum_{l=-\infty}^{\infty} c_{nl} F_{nl}(\rho, \theta).$$

The GP potential can be expressed in terms of the variational parameters $c_{nl}$ as

$$U = \sum_{nl} K_{nl} |c_{nl}|^2 + \lambda \int_{2D} \left| \sum_{nl} c_{nl} F_{nl}(\rho, \theta) \right|^4.$$

The expression (20) has a transparent quadratic plus quartic structure. Nonzero value of $\psi$ develops when at least one of the eigenvalues $K_{nl}$ becomes negative. Therefore, the critical condition for BEC is given by

$$\min_{nl} K_{nl} = 0.$$

Noting that $a_{nl}$ is always positive, we see clearly that the BEC is driven by the rotation: The quantity $\Omega$ plays the role of an $l$-dependent chemical potential. Because $a_{nl}$ is an increasing function of $n$, the critical condition for BEC will be first fulfilled for $n = 0$. For negative $l$, $K_{nl}$ is an increasing function of $|l|$ and is hence always positive. Therefore, BEC occurs only for positive $l$. For positive $l$, we have $K_{nl} = m^2 + eB(2a_{nl} + 1) - \Omega^2 l^2$. In Fig. 1, we demonstrate the $l$-dependence of $K_{nl}$ at $l > 0$. At large $l$, $a_{nl}$ goes faster than $l^2$. The competition between the decreasing term $-\Omega^2 l^2$ and the increasing term $2eB a_{nl}$ results in a unique global minimum at a certain quantum number $l$. For sufficiently large $\Omega$, this minimum for $n = 0$ becomes negative and the BEC is induced. The critical angular velocity is given by

$$\Omega_c = \frac{\sqrt{m^2 + eB(2a_{nl} + 1)}}{l_*}.$$

Here $l_*$ is the location of the minimum that is exactly...
Taking $eB = m^2_\pi$, for $N = 25$ and $N = 100$, we have $l_v = 20$ and $l_v = 84$, respectively. As $N$ goes to infinity ($R \to \infty$), $l_v \to N$. The critical angular velocity approaches $\Omega_c R \simeq 2/\sqrt{N}$, which is vanishingly small. Thus the mechanism of rotation induced pion BEC in a magnetic field is quite robust. Note that the critical condition (21) is consistent with the analysis in the free-particle picture [23].

We then perform full variational calculations to determine the ground state in the BEC phase $\Omega > \Omega_c$. A typical numerical calculation has been performed for $eB = m^2_\pi$ and $N = 25$, corresponding to a system size $R \simeq 10$fm. In this case, $\Omega_c R \simeq 0.59$. The two-dimensional profile of the condensate is shown in Fig. 2. We find that it is always isotropic, indicating that the condensate wave function $\psi(\rho, \theta)$ is actually composed of a single $l$-component. This is supported by the results for the variational parameters: $c_{nl}$ is finite only for a certain quantum number $l = w$, while it is vanishingly small for all other values of $l$. Therefore, the condensate wave function can actually be expressed as $\psi(\rho, \theta) = f(\rho)e^{i\nu \theta}$ with $f(\rho) = \sum_n c_n \phi_n(\rho)$, corresponding to a vortex state. This can be understood from the $l$-dependence of the quantity $K_{\Omega l}$. For $\Omega > \Omega_c$, the location of the unique minimum, $l = l_0$, moves to larger values ($l_0 > l_*$) and the minimum becomes negative. As a result, the superposition (19) favors a single $l$-component with $w \simeq l_0$. Numerical results confirm this understanding. For example, the locations of the minima are $l_0 = 24, 30, 42$ for $\Omega R = 0.6, 0.7, 0.8$, respectively. For $\lambda = 1$, the winding numbers are respectively $w = 23, 29, 39$. The small discrepancy comes from the combined effects of the interaction and the higher energy levels ($n \geq 1$).

Therefore, in the presence of interaction, the ground state of the pion BEC is a super giant vortex with winding number $w \gg 1$. The condensate wave function takes the form (1). The radial profile $f(\rho)$ is shown in Fig. 3. Because of the large winding number, the size of the vortex is always as large as the system size. We have also performed calculations for a larger system size, $N = 100$, corresponding to $R \simeq 20$fm [37]. The conclusion remains.

**Excitation Spectra:** The elementary excitations in the pion BEC can also be calculated. They are quanta of the quantum fluctuation $\phi(\tau, r)$. Taking the quantum fluctuation into account, the action becomes

$$S_E = \beta U[\psi(r)] + S_B(\phi^*, \phi),$$

where the fluctuation contribution $S_B$ can be obtained by substituting (13) into (12). To calculate the excitation spectra, only the quadratic terms are relevant, which can be evaluated to be

$$S_2(\phi^*, \phi) = \int_X \left[ \phi^* \left( G + 4\lambda|\psi|^2 \right) \phi + \lambda (\psi^2 \phi^2 + \psi^2 \phi^2) \right].$$

(23)

Considering the fact that the condensate takes the form $\psi(r) = f(\rho)e^{i\nu \theta}$, we expand the fluctuation as $\phi(\tau, r) = \sum_{n, Q} \phi_n(Q)e^{-i\nu \tau + ip_z z} f_n^*(\rho, \theta)$, where $Q = (i\nu, p_z, l)$ and $\omega_v = 2\pi\sqrt{\nu / \beta}$ ($\nu \in \mathbb{Z}$) is the boson Matsubara frequency. Then we can write

$$S_2 = \frac{1}{2} \sum_Q \sum_{n_n'} \Lambda_n^*(Q) M_{n_n' n} \Lambda_{n'}(Q),$$

(24)

where $\Lambda_n(Q) = (\tilde{\phi}_n(Q), \tilde{\phi}_n^*(-Q))^T$. The explicit form of the matrix $M$ is given in the Supplemental Material [37]. The excitation spectra can be determined by $\det M = 0$ with the analytical continuation $i\omega_v \to E + i\epsilon$.

Typical behavior of the excitation spectra is shown in Fig. 4 for $\Omega R = 0.6$ and $\lambda = 1$. The lowest level of the spectra fulfills the Goldstone’s theorem. Interestingly, the Goldstone mode in this system is an anisotropic generalization of the type-II Goldstone boson [38–43]: In the low energy limit, the excitation energy $E$ shows a quadratic rather than linear dispersion. Especially, for $l = 0$, the dispersion relation of the lowest excitation can be well approximated as $E(p_z) \simeq \sqrt{p_z^2 + (w\Omega)^2} - w\Omega$, where $w$ plays the role of a chemical potential. Thus the excitation energy exhibits a quadratic dispersion.

**FIG. 3.** Radial profile $f(\rho)$ of the condensate wave function for various values of interaction strength and rotation rate. In this calculation we take $eB = m^2_\pi$ and $N = 25$.

**FIG. 4.** Energy of the elementary excitations as a function of $l$ for $p_z = 0$ (a) and as a function of $p_z$ for $l = 0$ (b) at $\Omega R = 0.6$ for $\lambda = 1$. The lowest three levels are shown in this plot. In this calculation we take $eB = m^2_\pi$ and $N = 25$. 

\[ E(p_z) \sim p_z^2 \] for \( p_z \to 0 \). This quadratic behavior is similar to the previously discussed type-II Goldstone boson at finite density \([40]\). On the other hand, if the dynamical electromagnetic field is taken into account, the system becomes a superconductor and it is interesting to study how this gapless mode will be modified.

In summary, we have shown that under the circumstance of parallel magnetic field and rotation, the charged pions get Bose condensed and the ground state is a super giant quantum vortex with a winding number \( w \gg 1 \). The condensate is almost located at the edge of the system, indicating that the size of the vortex is as large as the system size. The formation of super giant vortex may yield to observable effects in off-central heavy ion collisions. In the future, it is interesting to investigate the multi-pion Bose-Einstein correlations \([44–47]\) and see useful discussions. This work was supported by the National Key R&D Program (Grant No. 2018YFA0306503) and the National Natural Science Foundation of China (Grant No. 11775123 and 11890712).

Finally, the mechanism of forming super giant quantum vortices studied in this work is robust and general. Since (pseudo) relativistic particles can be engineered in condensed matter and cold atom systems \([48–50]\), relativistic bosons under parallel magnetic field and rotation may also be engineered to explore the super giant quantum vortices.

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Supplemental Material for “Formation of Super Giant Vortices in a Pion Bose-Einstein Condensate Driven by Parallel Magnetic Field and Rotation”

In this Supplemental Material, we provide the explicit form of the matrix $M$ in Eq. (24) and the results for a larger system with $N = 100$.

### A. Explicit form of the matrix $M$

The matrix $M(Q)$ can be expressed as

$$M_{nn'}(Q) = \begin{pmatrix} \mathcal{K}_n(Q)\delta_{nn'} + A_{nn'}(l) & B_{nn'}(l) \\ B_{nn'}(-l) & \mathcal{K}_n(-Q)\delta_{nn'} + A_{nn'}(-l) \end{pmatrix},$$

(25)

where $Q = (i\omega, p_z, l)$. The elements are given by

$$\mathcal{K}_n(Q) = -[i\omega + (l + w)\Omega]^2 + p_z^2 + m_z^2 + eB[2a_{n,l+w} + |l + w| - (l + w) + 1],$$

$$A_{nn'}(l) = \frac{2\lambda}{\pi} \int_0^R \rho d\rho f^2(\rho)\varphi_{n,l,w}(\rho)\varphi_{n',l+w}(\rho),$$

$$B_{nn'}(l) = \frac{\lambda}{\pi} \int_0^R \rho d\rho f^2(\rho)\varphi_{n,l,w}(\rho)\varphi_{n',w-l}(\rho).$$

(26)

### B. Results for $N = 100$

We have performed calculations for $N = 100$ with the same magnetic field $eB = m_z^2$, indicating a larger system size, $R \approx 20$ fm. In this case, $\Omega R \approx 0.25$. In Fig. 5, we show the $l$-dependence of the quantity $K_{nl}$ for different values of $\Omega$. The qualitative behavior for the lowest level $n = 0$ is the same as the case $N = 25$, leading to similar results for the condensate profile (Fig. 6). The radial profile of the condensate is shown in Fig. 7. We see that the size of the vortex is still as large as the system size. The locations of the minima of $K_{nl}$ are $l_0 = 84, 88, 98$ for $\Omega R = 0.26, 0.3, 0.4$, respectively. For $\lambda = 1$, the winding numbers are respectively $w = 84, 86, 93$. 
FIG. 5. $l$-dependence of the quantity $K_{nl}$ for different values of $\Omega$. Here we show the result for $n = 0$ and $n = 1$. In this plot we take $eB = m^2$ and $N = 100$.

FIG. 6. Profile of the condensate $|\psi(\rho, \theta)|$ in the $x - y$ plane for various values of the interaction strength: (a)(b)(c)$\lambda = 0.1$, (d)(e)(f)$\lambda = 0.5$, (g)(h)(i)$\lambda = 1$. In this plot we take $eB = m^2$ and $N = 100$. 
FIG. 7. Radial profile $f(\rho)$ of the condensate wave function for various values of interaction strength and rotation rate. In this plot we take $eB = m^*_{\pi}$ and $N = 100$. 

(a) $\lambda = 0.10$

(b) $\lambda = 0.25$

(c) $\lambda = 0.50$

(d) $\lambda = 1.00$