A parameter estimation method based on random slow manifolds *

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Abstract

A parameter estimation method is devised for a slow-fast stochastic dynamical system, where only
the slow component is observable. By using available observations on the slow component, a system
parameter is estimated by studying the slow system on the random slow manifold. This offers a benefit
of dimension reduction in quantifying parameters in stochastic dynamical systems. An example is
presented to illustrate this method, and to verify that the parameter estimator based on the lower
dimensional, reduced slow system is a good approximation of the parameter estimator for the original
slow-fast stochastic dynamical system.

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Keywords Parameter estimation; Slow-fast system; Random slow manifold; Quantifying uncertainty;
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1 Introduction

Invariant manifolds provide geometric structures for understanding dynamical behavior of nonlinear systems under uncertainty. Some systems evolve on fast and slow time scales, and may be modeled by coupled singularly perturbed stochastic ordinary differential equations (SDEs). A slow-fast stochastic system may have a special invariant manifold called a random slow manifold that captures the slow dynamics.

We consider a stochastic slow-fast system

\[ \dot{x} = Ax + f(x, y), \quad x(0) = x_0 \in \mathbb{R}^n, \]  
\[ \dot{y} = \frac{1}{\varepsilon}By + \frac{1}{\varepsilon}g(x, y) + \frac{\sigma}{\sqrt{\varepsilon}}W_t, \quad y(0) = y_0 \in \mathbb{R}^m, \]  

where \( A \) and \( B \) are matrices, \( \varepsilon \) is a small positive parameter measuring slow and fast scale separation, \( f \) and \( g \) are nonlinear Lipschitz continuous functions with Lipschitz constant \( L_f \) and \( L_g \) respectively, \( \sigma \) is a noise intensity constant, and \( \{ W_t : t \in \mathbb{R} \} \) is a two-sided \( \mathbb{R}^m \)-valued Wiener process (i.e., Brownian motion) on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Under a gap condition and the dissipative condition (H1 and H2 in Section 2) for matrix \( B \), for \( \varepsilon \) sufficiently small, there exists a random slow manifold \( (\xi, h^\varepsilon(\xi, \omega)) \), with \( h^\varepsilon(\xi, \omega) : \mathbb{R}^n \rightarrow \mathbb{R}^m, \omega \in \Omega \), for slow-fast stochastic system (1.1)-(1.2). When the nonlinearities \( f, g \) are only locally Lipschitz continuous but the system has a random absorbing set (e.g., in mean-square norm), we conduct a cut-off of the original system.

The random slow manifold is the graph of a random nonlinear mapping \( h^\varepsilon(\xi, \omega) = \sigma \eta^\varepsilon(\omega) + \tilde{h}^\varepsilon(\xi, \omega) \), with \( \tilde{h}^\varepsilon(\xi, \omega) \) determined by a Lyapunov-Perron integral equation \[10\], \[ \tilde{h}^\varepsilon(\xi, \omega) = \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{-\frac{B}{\varepsilon} s} g(x(s, \omega, \xi), y(s, \omega, \xi) + \sigma \eta^\varepsilon(\theta s \omega)) ds, \xi \in \mathbb{R}^n, \]

here \( \eta^\varepsilon(\omega) = \frac{1}{\sqrt{\varepsilon}} \int_{-\infty}^{0} e^{-\frac{B}{\varepsilon} s} dW_s \) and \( \eta^\varepsilon(\theta t \omega) = \frac{1}{\sqrt{\varepsilon}} \int_{-\infty}^{t} e^{-\frac{B}{\varepsilon} (t-s)} dW_s \). The random slow manifold exponentially attracts other solution orbits. We will find an analytically approximated random slow manifold for sufficiently small \( \varepsilon \), in terms of an asymptotic expansion in \( \varepsilon \), as in \[20\] \[21\]. This slow manifold may also be numerically computed as in \[12\]. Roberts \[18\] introduced a normal form transform method for stochastic differential systems with both slow modes and quickly decaying modes, in order to find the approximate formula for a slow manifold. Related works on the dynamics of stochastic differential equation or stochastic center manifold include \[1\] \[2\] \[3\] \[7\]. By restricting to the slow manifold, we obtain a lower dimensional reduced system of the original slow-fast system (1.1)-(1.2), for \( \varepsilon \) sufficiently small

\[ \dot{x} = Ax + f(x, \tilde{h}^\varepsilon(x, \theta t \omega) + \sigma \eta(\theta t \psi \omega)), \quad x \in \mathbb{R}^n, \]  

where \( \theta t \) and \( \psi \varepsilon \) will be defined in the next section.

If the original slow-fast system (1.1)-(1.2) contains unknown system parameters, but only the slow component \( x \) is observable, we conduct parameter estimation using the slow system (1.3). Since the slow system is lower dimensional than the original system, this parameter estimator offers an advantage in computational cost, in addition to the benefit of using only observations on slow variables.
This paper is arranged as follows. In the next section, we obtain an approximated random slow manifold and thus the random slow system. Then in Section 3, we provide an error estimation for our parameter estimator, in terms of $O(\varepsilon)$ (due to random slow reduction) and the observation error. Finally, we present a simple example in Section 4 to illustrate our method.

2 Random slow manifold and its approximation

In order to use the reduced system to estimate a parameter, we firstly give some results on slow manifold and its approximation [10, 12, 17, 18]. The slow manifold is considered under a driving flow $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. Here $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. And $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ is a flow on $\Omega$ which is defined as a mapping

$$\theta : \mathbb{R} \times \Omega \mapsto \Omega$$

satisfying

- $\theta_0 = id_\Omega$ (identity mapping on $\Omega$),
- $\theta_s \theta_t = \theta_{s+t}$ for all $s, t \in \mathbb{R}$, and
- the mapping $(t, \omega) \mapsto \theta_t \omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$–measurable and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

By a random transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} := \mathcal{V}_\varepsilon(\omega, x, y) = \begin{pmatrix} x \\ y - \sigma \eta^\varepsilon(\omega) \end{pmatrix}, \quad (2.1)$$

we convert the SDE system (1.1)-(1.2) to the following system with random coefficients

$$\begin{align*}
\dot{X}(t) &= A X(t) + f(X(t), Y(t) + \sigma \eta^\varepsilon(\theta_t \omega)), \\
\dot{Y}(t) &= \frac{1}{\varepsilon} B Y(t) + \frac{1}{\varepsilon} g(X(t), Y(t) + \sigma \eta^\varepsilon(\theta_t \omega)),
\end{align*} \quad (2.2-2.3)$$

where $\eta^\varepsilon(\omega) = \frac{1}{\sqrt{\varepsilon}} \int_{-\infty}^0 e^{-\frac{B}{\varepsilon} s} dW_s$ is the stationary solution of linear system $d\eta^\varepsilon = \frac{B}{\varepsilon} \eta^\varepsilon dt + \frac{1}{\sqrt{\varepsilon}} dW_t$. And $\theta_t : \Omega \to \Omega$ is the Wiener shift implicitly defined by $W_s(\theta_t \omega) = W_{t+s}(\omega) - W_t(\omega)$. Note that $\eta^\varepsilon(\theta_t \omega) = \frac{1}{\sqrt{\varepsilon}} \int_{-\infty}^t e^{\frac{B}{\varepsilon} (t-s)} dW_s$.

Define a mapping (between random samples) $\psi_\varepsilon : \Omega \to \Omega$ implicitly by $W_t(\psi_\varepsilon \omega) = \frac{1}{\sqrt{\varepsilon}} W_{t \varepsilon}(\omega)$. Then $\frac{1}{\sqrt{\varepsilon}} W_{t \varepsilon}(\omega)$ is also a Wiener process with the same distribution as $W_t(\omega)$. Moreover, $\eta^\varepsilon(\theta_{t \varepsilon} \omega)$ and $\eta^\varepsilon(\omega)$ are identically distributed with $\eta(\theta_t \psi_\varepsilon \omega) = \int_{-\infty}^t e^{B(t-s)} dW_s(\psi_\varepsilon \omega)$ and $\eta(\psi_\varepsilon \omega) = \int_{-\infty}^0 e^{-Bs} dW_s(\psi_\varepsilon \omega)$, respectively.
By a time change $\tau = t/\varepsilon$ and using the fact that $\eta^\varepsilon(\theta_\tau \omega)$ and $\eta(\theta_\tau \psi \varepsilon \omega)$ are identically distributed, the system (2.2)-(2.3) is reformulated as

\begin{align}
X' &= \varepsilon [AX + f(X, Y + \sigma \eta(\theta_\tau \psi \varepsilon \omega))], \\
Y' &= BY + g(X, Y + \sigma \eta(\theta_\tau \psi \varepsilon \omega)),
\end{align}

where $' = \frac{d}{d\tau}$.

We make the following two hypotheses.

**H1**: There are positive constants $\alpha$, $\beta$ and $K$, such that for every $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, the following exponential estimates hold:

$$
|e^{At}x|_{\mathbb{R}^n} \leq Ke^{\alpha t}|x|_{\mathbb{R}^n}, \quad t \leq 0; \quad |e^{Bt}y|_{\mathbb{R}^m} \leq Ke^{\beta t}|y|_{\mathbb{R}^m}, \quad t \geq 0.
$$

**H2**: $\beta > KL_g$.

Then there exists a random slow manifold $\tilde{M}^\varepsilon(\omega) = \{(\xi, \tilde{h}^\varepsilon(\xi, \omega)) : \xi \in \mathbb{R}^n\}$ for the random system (2.2)-(2.3), with $\tilde{h}^\varepsilon$ being expressed as follows [10, 17],

$$
\tilde{h}^\varepsilon(\xi, \omega) = \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{-\frac{B}{\varepsilon} s} g(X(s, \omega, \xi), Y(s, \omega, \xi) + \sigma \eta^\varepsilon(\theta_s \omega)) \, ds, \quad \xi \in \mathbb{R}^n.
$$

We can get a small $\varepsilon$ approximation for $\tilde{h}^\varepsilon$. Start with the expansion $Y = Y_0 + \varepsilon Y_1 + O(\varepsilon^2)$ and the integral expression $X = X(0) + \varepsilon \int_{0}^{\tau} [AX + f(X, Y + \sigma \eta(\theta_\tau \psi \varepsilon \omega))] \, dr$ in the system (2.4)-(2.5). Using the Taylor expansions of $f(X, Y + \sigma \eta(\theta_\tau \psi \varepsilon \omega))$ and $g(X, Y + \sigma \eta(\theta_\tau \psi \varepsilon \omega))$ at point $(x_0, Y_0 + \sigma \eta(\theta_\tau \psi \varepsilon \omega))$ and denoting $\xi = X(0)$, we obtain

$$
\tilde{h}^\varepsilon(\xi, \omega) = \tilde{h}^\varepsilon_0(\xi, \omega) + \varepsilon \tilde{h}^\varepsilon_1(\xi, \omega) + O(\varepsilon^2),
$$

for $\varepsilon$ sufficiently small, with

$$
\tilde{h}^\varepsilon_0(\xi, \omega) = \int_{-\infty}^{0} e^{-B_s} g(\xi, Y_0(s) + \sigma \eta(\theta_s \psi \varepsilon \omega)) \, ds,
$$

and

$$
\tilde{h}^\varepsilon_1(\xi, \omega) = \int_{-\infty}^{0} e^{-B_s} \left\{ g_x(\xi, Y_0(s) + \sigma \eta(\theta_s \psi \varepsilon \omega)) [A_\xi + \int_{0}^{s} f(\xi, Y_0(r) + \sigma \eta(\theta_r \psi \varepsilon \omega)) \, dr] \\
+ g_y(\xi, Y_0(s) + \sigma \eta(\theta_s \psi \varepsilon \omega)) Y_1(s) \right\} \, ds,
$$

where $Y_0(t)$ and $Y_1(t)$ satisfy the following random differential equations, respectively

\begin{align}
\begin{cases}
Y_0'(\tau) = BY_0(\tau) + g(\xi, Y_0(\tau) + \sigma \eta(\theta_\tau \psi \varepsilon \omega)), \\
Y_0(0) = \tilde{h}^\varepsilon_0(0, \omega),
\end{cases}
\end{align}
and

\[
\begin{align*}
Y_1'(\tau) &= [B + g_y(\xi, Y_0(\tau) + \sigma \eta(\theta_\tau \psi_\varepsilon \omega)))]Y_1(\tau) \\
& \quad + g_z(\xi, Y_0(\tau) + \sigma \eta(\theta_\tau \psi_\varepsilon \omega))\{A\tau \xi + \int_0^\tau f(\xi, Y_0(s) + \sigma \eta(\theta_s \psi_\varepsilon \omega)) \, ds\}, \\
Y_1(0) &= \hat{h}_1^\varepsilon(0, \omega).
\end{align*}
\]

Noticing that \(\eta^\varepsilon(\omega)\) and \(\eta(\psi_\varepsilon \omega)\) have identical distribution, together with the fact that

\[
\eta^\varepsilon(\theta_\tau \omega) = \sqrt{\varepsilon} \int_{-\infty}^0 e^{-B_\tau S} \, dW_s,
\]

we see that \(\eta^\varepsilon(\theta_\tau \omega)\) and \(\eta(\psi_\varepsilon \omega)\) are identically distributed. Recall the transformation introduced in the beginning of this section \(X(t) = x(t)\) and \(Y(t) = y(t) - \sigma \eta^\varepsilon(\theta_\tau \omega)\). We thus obtain a lower dimensional, slow stochastic system on the slow manifold,

\[
\dot{x} = Ax + f(x, \hat{h}^\varepsilon(x, \theta_\tau \omega) + \sigma \eta(\theta_\tau \psi_\varepsilon \omega)),
\]

Moreover, \(\hat{h}^\varepsilon(\xi, \omega) = \tilde{h}_\varepsilon^\omega(\xi, \omega) + \varepsilon \hat{h}_1^\varepsilon(\xi, \omega)\) is an approximation or a first order truncation of \(\hat{h}^\varepsilon(\xi, \omega)\), and hence we have an approximate slow system

\[
\dot{x} = Ax + f(x, \hat{h}^\varepsilon(x, \theta_\tau \omega) + \sigma \eta(\theta_\tau \psi_\varepsilon \omega)).
\]

This is the slow system we will work on for parameter estimation in the next section.

3 Parameter estimation on a random slow manifold

Suppose that the original slow equation (1.1) contains an unknown system parameter \(a \in \mathbb{R}\), and we know or we can determine its range \(\Lambda\), which is a closed interval. That is, the slow-fast system (1.1)-(1.2) becomes

\[
\begin{align*}
\dot{x} &= Ax + f(x, y, a), \quad x(0) = x_0 \in \mathbb{R}^n, \\
\dot{y} &= \frac{1}{\varepsilon} By + \frac{1}{\varepsilon} g(x, y) + \frac{\sigma}{\sqrt{\varepsilon}} \dot{W}_t, \quad y(0) = y_0 \in \mathbb{R}^m.
\end{align*}
\]

When both slow component and fast component are observable, it is possible to make a good estimation of \(a\), as in [6, 11, 22]. If only slow variable is observable, can we get a good estimator based on the reduced slow system? In this section, we will devise a parameter estimation method using only the slow system, together with its error estimation.

3.1 Parameter estimation based on the reduced slow system
Let \( x_{ob}(t), t \in [0, T] \), be the observation of \( x(t) \), corresponding to a true system parameter value \( a \). Although the fast component is not observable, we still temporarily denote it by \( y_{ob}(t) \) (in fact, it will not appear in the error estimation below). We attempt to estimate the system parameter \( a \) in (3.1)-(3.2), using the random slow system (2.11), which is now in the form

\[
\dot{x} = Ax + f(x, \tilde{h}^\varepsilon(x, \theta_t \omega) + \sigma \eta(\theta_t \psi_\varepsilon \omega), a), \quad x \in \mathbb{R}^n,
\]

for \( \varepsilon \) sufficiently small. Assume that \( f(x, y, a) \) is Lipschitz continuous with respect to \( x, y, a \) with Lipschitz constant \( L_f \). For \( x_S(t) \) satisfying the reduced slow system (3.3) with parameter \( a \), denoted as \( a^S_E \) to distinguish from \( a \) in (3.1), and initial value \( x_0 \), define an objective function \( F^S(a^S_E) = \mathbb{E} \int_0^T |x_S(t) - x_{ob}(t)|^2 \|a\|^2 dt \). We take the corresponding minimizer \( a^S_E \) as the estimation of the true parameter \( a \).

Now we provide an error estimation for this parameter estimation method. By the mean value theorem, there is a \( t^* \in (0, T) \), such that

\[
\int_0^T |x_{ob}(t) - x_S(t)| \|a\|^2 dt = T|x_{ob}(t^*) - x_S(t^*)| \|a\|^2,
\]

and then by Cauchy-Schwarz inequality

\[
|x_{ob}(t^*) - x_S(t^*)| \|a\|^2 = \frac{1}{T} \int_0^T |x_{ob}(t) - x_S(t)| \|a\|^2 dt \leq \frac{1}{T} \left( T \int_0^T |x_{ob}(t) - x_S(t)|^2 \|a\|^2 dt \right)^{\frac{1}{2}}.
\]

We calculate the difference between \( x_{ob}(t) \) and \( x_S(t) \), which satisfies

\[
\dot{x}_{ob}(t) - \dot{x}_S(t) = A(x_{ob} - x_S) + [f(x_{ob}, y_{ob}, a) - f(x_S, \tilde{h}^\varepsilon(x_S, \theta_t \omega) + \sigma \eta(\theta_t \psi_\varepsilon \omega), a^S_E)], \quad x_{ob}(0) - x_S(0) = 0.
\]

By the variation of constants formula, we get

\[
e^{-At^*} [x_{ob}(t^*) - x_S(t^*)]
\]

\[
= \int_0^{t^*} e^{-At} [f(x_{ob}(t), y_{ob}(t), a) - f(x_S(t), \tilde{h}^\varepsilon(x_S(t), \theta_t \omega) + \sigma \eta(\theta_t \psi_\varepsilon \omega), a)] dt
\]

\[+ \int_0^{t^*} e^{-At} [f(x_S(t), \tilde{h}^\varepsilon(x_S(t), \theta_t \omega) + \sigma \eta(\theta_t \psi_\varepsilon \omega), a) - f(x_S(t), \tilde{h}^\varepsilon(x_S(t), \theta_t \omega) + \sigma \eta(\theta_t \psi_\varepsilon \omega), a^S_E)] dt.
\]

and then

\[
\int_0^{t^*} e^{-At} [f(x_S(t), \tilde{h}^\varepsilon(x_S(t), \theta_t \omega) + \sigma \eta(\theta_t \psi_\varepsilon \omega), a) - f(x_S(t), \tilde{h}^\varepsilon(x_S(t), \theta_t \omega) + \sigma \eta(\theta_t \psi_\varepsilon \omega), a^S_E)] dt
\]

\[= - \int_0^{t^*} e^{-At} [f(x_{ob}(t), y_{ob}(t), a) - f(x_S(t), \tilde{h}^\varepsilon(x_S(t), \theta_t \omega) + \sigma \eta(\theta_t \psi_\varepsilon \omega), a)] dt + e^{-At^*} [x_{ob}(t^*) - x_S(t^*)].
\]

(3.5)
Furthermore, taking norm on two sides, and using the mean value theorem, there is a \( \kappa \in (0,1) \), and \( a' = a + \kappa(a'_E - a) \) such that the left hand side of (3.5) can be rewrite as the left hand side of (3.6). And the exponential estimation property \( H1 \), Lipschitz continuity of \( f \), together with Cauchy-Schwarz inequality lead to the estimation for the right hand side,

\[
|a - a'_E| \cdot |\int_0^t e^{-\lambda t} \nabla_a f(x_s(t), \tilde{h}^\varepsilon(x_s(t), \theta_i\omega) + \sigma\eta(\theta_i\psi_\varepsilon\omega), a') dt |_{\mathbb{R}^n}
\]

\[
\leq \int_0^T K e^{-\alpha t} L_f |x_{ob}(t) - x_s(t)|_{\mathbb{R}^n} + |y_{ob}(t) - \tilde{h}^\varepsilon(x_s(t), \theta_i\omega) - \sigma\eta(\theta_i\psi_\varepsilon\omega)|_{\mathbb{R}^m} dt + K e^{-\alpha t} |x_{ob}(t^*) - x_s(t^*)|_{\mathbb{R}^n}
\]

\[
< KL_f \left( T \int_0^T |x_{ob}(t) - x_s(t)|_{\mathbb{R}^n}^2 dt \right)^{\frac{1}{2}} + KL_f \int_0^T |y_{ob}(t) - \tilde{h}^\varepsilon(x_s(t), \theta_i\omega) - \sigma\eta(\theta_i\psi_\varepsilon\omega)|_{\mathbb{R}^m} dt
\]

\[
+ K |x_{ob}(t^*) - x_s(t^*)|_{\mathbb{R}^n}.
\]

As stated in Section 2, the random transformation (2.1) converts the SDE system to a random system. Thus for \( (x_{ob}(t), y_{ob}(t)) \) satisfying stochastic system (3.1)-(3.2), the corresponding \( (x_{ob}(t), y_{ob}(t) - \sigma\eta(\theta_i\psi_\varepsilon\omega)) \) satisfies a random system (2.2)-(2.3) with an unknown true parameter value \( a \) and initial value \( (x_0, y_0 - \sigma\eta(\psi_\varepsilon\omega)) \). We know that the slow manifold is composed of special paths of the corresponding system. Thus \( (x_s(t), \tilde{h}^\varepsilon(x_S(t), \theta_i\omega)) \) also satisfies the random system (2.2)-(2.3) with a parameter denoted as \( a'_E \) and initial value \( (x_0, \tilde{h}^\varepsilon(x_0, \omega)) \). From random system (2.3), the difference between \( \tilde{h}^\varepsilon(x_S(t), \theta_i\omega) \) and \( y_{ob}(t) - \sigma\eta(\theta_i\psi_\varepsilon\omega) \) satisfies,

\[
\frac{d}{dt} [y_{ob}(t) - \sigma\eta(\theta_i\psi_\varepsilon\omega) - \tilde{h}^\varepsilon(x_S(t), \theta_i\omega)]
\]

\[
= \frac{B}{\varepsilon} [y_{ob}(t) - \sigma\eta(\theta_i\psi_\varepsilon\omega) - \tilde{h}^\varepsilon(x_S(t), \theta_i\omega)] + \frac{1}{\varepsilon} [g(x_{ob}(t), y_{ob}(t)) - g(x_S(t), \tilde{h}^\varepsilon(x_S(t), \theta_i\omega) + \sigma\eta(\theta_i\psi_\varepsilon\omega))].
\]

By the help of the variation of constants formula, we get

\[
|y_{ob}(t) - \sigma\eta(\theta_i\psi_\varepsilon\omega) - \tilde{h}^\varepsilon(x_S(t), \theta_i\omega)|_{\mathbb{R}^m}
\]

\[
= |e^\frac{B}{\varepsilon} t [y_0 - \sigma\eta(\psi_\varepsilon\omega) - \tilde{h}^\varepsilon(x_0, \omega)] + \frac{1}{\varepsilon} \int_0^t e^\frac{B}{\varepsilon} (t-s) [g(x_{ob}(s), y_{ob}(s)) - g(x_S(s), \tilde{h}^\varepsilon(x_S(s), \theta_s\omega) + \sigma\eta(\theta_s\psi_\varepsilon\omega))] ds |_{\mathbb{R}^m}
\]

\[
\leq K e^{-\frac{B}{\varepsilon} t} |y_0 - \sigma\eta(\psi_\varepsilon\omega) - \tilde{h}^\varepsilon(x_0, \omega)|_{\mathbb{R}^m}
\]

\[
+ \frac{KL_q}{\varepsilon} \int_0^t e^\frac{B}{\varepsilon} (s-t) |x_{ob}(s) - x_S(s)|_{\mathbb{R}^n} + |y_{ob}(s) - \sigma\eta(\theta_s\psi_\varepsilon\omega) - \tilde{h}^\varepsilon(x_S(s), \theta_s\omega)|_{\mathbb{R}^m} ds.
\]

We rewrite this as

\[
e^\frac{B}{\varepsilon} t |y_{ob}(t) - \sigma\eta(\theta_i\psi_\varepsilon\omega) - \tilde{h}^\varepsilon(x_S(t), \theta_i\omega)|_{\mathbb{R}^m} \leq K |y_0 - \sigma\eta(\psi_\varepsilon\omega) - \tilde{h}^\varepsilon(x_0, \omega)|_{\mathbb{R}^m} + \frac{KL_q}{\varepsilon} \int_0^t e^\frac{B}{\varepsilon} s |x_{ob}(s) - x_S(s)|_{\mathbb{R}^n} ds
\]

\[
+ \frac{KL_q}{\varepsilon} \int_0^t e^\frac{B}{\varepsilon} s |y_{ob}(s) - \sigma\eta(\theta_s\psi_\varepsilon\omega) - \tilde{h}^\varepsilon(x_S(s), \theta_s\omega)|_{\mathbb{R}^m} ds.
\]
By Gronwall’s inequality \cite{9}, we have
\[
e^{\hat{\beta} t}|y_{ob}(t) - \sigma \eta(\theta_t \psi \omega) - \tilde{h}^\varepsilon(x_S(t), \theta_t \omega)|_{\mathbb{R}^m} \\
\leq K |y_0 - \sigma \eta(\psi \omega) - \tilde{h}^\varepsilon(x_0, \omega)|_{\mathbb{R}^m} e^{\frac{KLg}{\varepsilon} t} + \frac{KLg}{\varepsilon} \int_0^t e^{\hat{\beta} s}|x_{ob}(s) - x_S(s)|_{\mathbb{R}^n} e^{\frac{KLg}{\varepsilon} (t-s)} ds,
\]
and thus
\[
|y_{ob}(t) - \sigma \eta(\theta_t \psi \omega) - \tilde{h}^\varepsilon(x_S(t), \theta_t \omega)|_{\mathbb{R}^m} \\
\leq K |y_0 - \sigma \eta(\psi \omega) - \tilde{h}^\varepsilon(x_0, \omega)|_{\mathbb{R}^m} e^{\frac{\beta - KLg}{\varepsilon} t} + \frac{KLg}{\varepsilon} \int_0^t |x_{ob}(s) - x_S(s)|_{\mathbb{R}^n} e^{\frac{\beta - KLg}{\varepsilon} (t-s)} ds.
\]

By exchanging the order of integrals, we conclude that
\[
\int_0^T \int_0^t |x_{ob}(s) - x_S(s)|_{\mathbb{R}^n} e^{\frac{\beta - KLg}{\varepsilon} (t-s)} ds \; dt = \int_0^T \int_s^T |x_{ob}(s) - x_S(s)|_{\mathbb{R}^n} e^{\frac{\beta - KLg}{\varepsilon} (t-s)} dt \; ds \\
< \frac{\varepsilon}{\beta - KLg} \int_0^T |x_{ob}(s) - x_S(s)|_{\mathbb{R}^n} \; ds.
\]

Thus, by Cauchy-Schwarz inequality
\[
\int_0^T |y_{ob}(t) - \sigma \eta(\theta_t \psi \omega) - \tilde{h}^\varepsilon(x_S(t), \theta_t \omega)|_{\mathbb{R}^m} \; dt \\
< K |y_0 - \sigma \eta(\psi \omega) - \tilde{h}^\varepsilon(x_0, \psi \omega)|_{\mathbb{R}^m} \frac{\varepsilon (1 - e^{\frac{\beta - KLg}{\varepsilon} T})}{\beta - KLg} + \frac{KLg}{\beta - KLg} \int_0^T |x_{ob}(s) - x_S(s)|_{\mathbb{R}^n} \; ds \\
< K |y_0 - \sigma \eta(\psi \omega) - \tilde{h}^\varepsilon(x_0, \psi \omega)|_{\mathbb{R}^m} \frac{\varepsilon}{\beta - KLg} + \frac{KLg}{\beta - KLg} (T \int_0^T |x_{ob}(t) - x_S(t)|_{\mathbb{R}^n}^2 \; dt)^{\frac{1}{2}}. \quad (3.7)
\]

Inserting (3.3) and (3.7) into (3.6), and taking expectation on two sides, and using Cauchy-Schwarz inequality \(E(|X|^2) \leq E(|X|^2)^{\frac{3}{2}} E(|Y|^2)^{\frac{1}{2}}\), we get
\[
|a - a^S_E| \cdot E \left[ \int_0^t e^{-At} \nabla a f(x_S(t), \tilde{h}^\varepsilon(x_S(t), \theta_t \omega)) + \sigma \eta(\theta_t \psi \omega), a') \; dt \right]_{\mathbb{R}^n} \\
< KL \left( T \mathbb{E} \int_0^T |x_{ob}(t) - x_S(t)|_{\mathbb{R}^n}^2 \; dt \right)^{\frac{1}{2}} + \frac{1}{T} \mathbb{E} \int_0^T |x_{ob}(t) - x_S(t)|_{\mathbb{R}^n}^2 \; dt \right)^{\frac{1}{2}} \\
+ K \mathbb{E} |y_0 - \sigma \eta(\psi \omega) - \tilde{h}^\varepsilon(x_0, \psi \omega)|_{\mathbb{R}^m} \frac{\varepsilon}{\beta - KLg} + \frac{KLg}{\beta - KLg} (T \mathbb{E} \int_0^T |x_{ob}(t) - x_S(t)|_{\mathbb{R}^n}^2 \; dt)^{\frac{1}{2}}. \quad (3.8)
\]
We then consider the integral term on the left hand side of (3.8), and set

\[ G(a, a_E^S) := \mathbb{E} \int_0^{t^*} e^{-At} \nabla_a f(x_S(t), \tilde{h}^e(x_S(t), \theta_I \omega) + \sigma \eta(\theta_I \psi_e \omega), a') \, dt |_{\mathbb{R}^n}, \]

which is a nonnegative function. Assume \( \nabla_a f(x, y, a) \) is continuous with respect to \( x, y \) and \( a \). Then \( G(a, a_E^S) \) is a continuous function of \( a_E^S \).

If \( G(a, a_E^S) > 0 \) for \( a_E^S \in \Lambda \), there is a \( \tilde{a} \in \Lambda \), such that \( G(a, \tilde{a}) = \min_{a_E^S \in \Lambda} G(a, a_E^S) \), which is a positive number. Then we obtain an error estimation for \( a_E^S \):

\[
|a - a_E^S| < \frac{1}{G(a, \tilde{a})} \left\{ K \mathbb{E} |y_0 - \sigma \eta(\omega) - \tilde{h}^e(x_0, \psi_e \omega)|_{\mathbb{R}^m} + \frac{\varepsilon}{\beta - KL_g} + \frac{KL_g}{\beta - KL_g} (T \mathbb{E} \int_0^T |x_{ob}(t) - x_S(t)|_{\mathbb{R}^n}^2 \, dt)^{\frac{1}{2}} \right. \\
+ KL_f \left. (T \mathbb{E} \int_0^T |x_{ob}(t) - x_S(t)|_{\mathbb{R}^n}^2 \, dt)^{\frac{1}{2}} + K \left( \frac{1}{T} \mathbb{E} \int_0^T |x_{ob}(t) - x_S(t)|_{\mathbb{R}^n}^2 \, dt \right)^{\frac{1}{2}} \right\},
\]

(3.9)

The estimation indicates that \(|a - a_E^S|\) can be controlled by objective function (observation error) and the error due to slow reduction, i.e. by \( O((F^S(a_E^S))^\frac{1}{2}) \) and \( O(\varepsilon) \).

If there is a \( \tilde{a} \), such that \( G(a, \tilde{a}) = 0 \), we further examine (3.9). It means that

\[ \mathbb{E} \int_0^{t^*} e^{-At} [f(x_S(t), \tilde{h}^e(x_S(t), \theta_I \omega) + \sigma \eta(\theta_I \psi_e \omega), a) - f(x_S(t), \tilde{h}^e(x_S(t), \theta_I \omega) + \sigma \eta(\theta_I \psi_e \omega), \tilde{a})] \, dt |_{\mathbb{R}^n} = 0. \]

By the mean value theorem, there is a \( t \in (0, t^*) \), such that

\[ \mathbb{E} |t^* e^{-At} [f(x_S(t), \tilde{h}^e(x_S(t), \theta_I \omega) + \sigma \eta(\theta_I \psi_e \omega), a) - f(x_S(t), \tilde{h}^e(x_S(t), \theta_I \omega) + \sigma \eta(\theta_I \psi_e \omega), \tilde{a})]|_{\mathbb{R}^n} = 0, \]

i.e.

\[ f(x_S(t), \tilde{h}^e(x_S(t), \theta_I \omega) + \sigma \eta(\theta_I \psi_e \omega), a) = f(x_S(t), \tilde{h}^e(x_S(t), \theta_I \omega) + \sigma \eta(\theta_I \psi_e \omega), \tilde{a}), \text{ a.s..} \]

Denote \( H(a) = \mathbb{E} f(x_S(t), \tilde{h}^e(x_S(t), \theta_I \omega) + \sigma \eta(\theta_I \psi_e \omega), a) \). Thus

\[ \tilde{a} = H(a). \]

Assume that for \((x, y)\) confined to slow manifold \( \{(x, \tilde{h}^e(x, \omega) + \sigma \eta(\psi_e \omega)) : x \in \mathbb{R}^n\} \), one component of \( \mathbb{E} f(x, y, a) \) is strictly monotonic with respect to \( a \in \Lambda \). Then \( H(a) = H(\tilde{a}) \) leads to \( a = \tilde{a} \). This means \( \tilde{a} \) is actually the true parameter value and we take \( a_E^S = \tilde{a} \).

Note that

\[
|y_{ob}(t) - \sigma \eta(\theta_I \psi_e \omega) - \tilde{h}^e(x_S(t), \theta_I \omega)|_{\mathbb{R}^m} \leq |y_{ob}(t) - \sigma \eta(\theta_I \psi_e \omega) - \tilde{h}^e(x_S(t), \theta_I \omega)|_{\mathbb{R}^m} + |\tilde{h}^e(x_S(t), \theta_I \omega) - \tilde{h}^e(x_S(t), \theta_I \omega)|_{\mathbb{R}^m},
\]

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we can use the approximation \( \hat{h}^\varepsilon(\xi, \omega) \) instead of \( \tilde{h}^\varepsilon(\xi, \omega) \) to get an estimator with error which is also controlled by \( O((F^S(a_E^S))^\frac{1}{2}) \) and \( O(\varepsilon) \).

Since this slow system is lower dimensional than the original system, our method offers an advantage in computational cost, in addition to a benefit of using only observations on slow variables. It is often more feasible to observe slow variables than fast variables [6].

### 3.2 Parameter estimation in the simulation

In the simulation, when observations \( x_{ob}^i = x_{ob}(t_i), y_{ob}^i = y_{ob}(t_i) \), with \( i = 0, 1, 2, \cdots, K \) and \( 0 = t_0 < t_1 < t_2 < \cdots < t_K \leq T \), are available for both components \( x \) and \( y \) from the original slow-fast system \((3.1)-(3.2)\) with true parameter value \( a \). Denote \( x^i(a_E) = x(t_i, a_E), y^i(a_E) = y(t_i, a_E) \) generated from system \((3.1)-(3.2)\) corresponding to parameter \( a = a_E \), an estimator \( a_E \) for \( a \) may be obtained with existing techniques as reviewed in our earlier work [22] or [6, 11] and references therein. We can get an estimator \( a_E \) by minimizing the objective function (or give a terminal value for objective function in the simulation) \( F(a_E) \triangleq \mathbb{E} \sum_{i=0}^{K} (|x_{ob}^i - x^i(a_E)|^2_{\mathbb{R}^n} + |y_{ob}^i - y^i(a_E)|^2_{\mathbb{R}^m}) \).

Assume that we only have observation on the slow component, \( x_{ob}^i = x_{ob}(t_i) \), from the initial system \((3.1)-(3.2)\), and we want to estimate the system parameter \( a \). We tried to give an estimator \( a_S^E \) by using the observation from system \((3.1)\) and conducted on slow system \((3.3)\). For example, \( a_S^E \) may be obtained by minimizing the objective function \( F^S(a_S^E) \triangleq \mathbb{E} \sum_{i=0}^{K} |x_{ob}^i - x^i_S(a_S^E)|^2_{\mathbb{R}^n} \).

We use a stochastic Nelder-Mead method to get an estimator. Given a terminal value, by compare the objective function values with different \( a \), we can get a better and better estimator, until get a \( a_E \) (or \( a_S^E \)) such that the objective function value is less than the terminal value. In the next section, we demonstrate this method with an example.

### 4 An example

In this section, we demonstrate our parameter estimation method based on random slow manifolds by a simple example.

**Example 1.** Consider a slow-fast stochastic system

\[
\dot{x} = 0.001x - axy, \quad x(0) = x_0 \in \mathbb{R},
\]

\[
\dot{y} = \frac{1}{\varepsilon} (-y + \frac{1}{600}x^2) + \frac{\sigma}{\sqrt{\varepsilon}} \dot{W}_t, \quad y(0) = y_0 \in \mathbb{R},
\]

where \( a \) is a real unknown positive parameter, \( \varepsilon \) is a small positive scale separation constant, \( \sigma \) is a constant noise intensity, and \( W_t \) is a scalar Wiener process.
See Figure 1 for a phase portrait of the corresponding deterministic system ($\sigma = 0$).

Figure 1: Deterministic dynamics – Phase portrait for $\dot{x} = 0.001x - xy$, $\dot{y} = \frac{1}{\varepsilon}(-y + \frac{1}{500}x^2)$ with $\varepsilon = \frac{1}{6}$: The global attractor is clearly seen (Red or thick curve within $-7 < x < 7$ and near the $x$–axis).

In this system, the nonlinear terms are not global Lipschitz. But if additionally it has an absorbing set, we can cut-off the nonlinearities without affecting the long time, slow dynamics (almost surely). Indeed, for arbitrary constants $M$ and $K$, we have

$$d(x^2) = 2x\, dx = (0.002x^2 - 2ax^2y)\, dt,$$

$$d((My - K)^2) = 2(My - K)\, M\, dy + M^2(\frac{\sigma}{\sqrt{\varepsilon}})^2\, dt$$

$$= (-\frac{2M^2}{\varepsilon}y^2 + \frac{M^2}{300\varepsilon}x^2y + \frac{2MK}{\varepsilon}y - \frac{MK}{300\varepsilon}x^2 + \frac{M^2\sigma^2}{\varepsilon})\, dt + 2M(My - K)(\frac{\sigma}{\sqrt{\varepsilon}})\, dW_t.$$  \hspace{1cm} (4.3)

Therefore,

$$d\left(\frac{M^2}{300\varepsilon}x^2 + 2a(My - K)^2\right)$$

$$= -\frac{1}{\varepsilon} \left(\frac{M^2}{300\varepsilon}x^2 + 2a(My - K)^2\right)\, dt - \frac{2aM^2}{\varepsilon}y^2\, dt + \frac{0.002M^2 - 2aMK + M^2/\varepsilon}{300\varepsilon}x^2\, dt$$

$$+ \frac{2aK^2 + 2aM^2\sigma^2}{\varepsilon}\, dt + 4aM(My - K)(\frac{\sigma}{\sqrt{\varepsilon}})\, dW_t. \hspace{1cm} (4.3)$$

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Taking \( M = \varepsilon \), \( K = \frac{1}{a\varepsilon} \), then we see that

\[
\frac{0.002M^2 - 2aMK + M^2/\varepsilon}{300\varepsilon} = \frac{0.002\varepsilon^2 - 2 + \varepsilon}{300\varepsilon} < 0, \text{ for small } \varepsilon;
\]

and

\[
\frac{2aK^2 + 2aM^2\sigma^2}{\varepsilon} = \frac{2}{a\varepsilon^3} + 2a\varepsilon^2.
\]

Thus

\[
\frac{d}{dt} \mathbb{E}(\frac{M^2}{300\varepsilon}x^2 + 2aM^2(y - K/M)^2) \leq - \frac{1}{\varepsilon} \mathbb{E}(\frac{M^2}{300\varepsilon}x^2 + 2aM^2(y - K/M)^2) + \frac{2}{a\varepsilon^3} + 2a\varepsilon^2. \tag{4.4}
\]

By the Gronwall inequality, we conclude that

\[
\mathbb{E}(\frac{\varepsilon}{300}x^2 + 2a\varepsilon^2(y - \frac{1}{a\varepsilon^2})^2) \leq \mathbb{E}(\frac{\varepsilon}{300}x_0^2 + 2a\varepsilon^2(y_0 - \frac{1}{a\varepsilon^2})^2)e^{-\frac{t}{\varepsilon}} + \left( \frac{2}{a\varepsilon^3} + 2a\varepsilon^2 \right)\varepsilon(1 - e^{-\frac{t}{\varepsilon}})
\]

\[
\leq \left( \frac{\varepsilon}{300}x_0^2 + 2a\varepsilon^2(y_0 - \frac{1}{a\varepsilon^2})^2 \right)e^{-\frac{t}{\varepsilon}} + \left( \frac{2}{a\varepsilon^3} + 2a\varepsilon^2 \right). \tag{4.5}
\]

This means, for fixed \( \varepsilon \) and \( a \in \Lambda = [L, R] \) with \( L > 0 \), the dynamics of the system \((4.1) - (4.2)\) will eventually stay in an ellipse (almost surely), i.e. there is a random absorbing set. In fact, the numerical experiments below shows that we can determine an interval \( \Lambda = [L, R] \), with \( L > 0 \) at first. We can then cut-off the nonlinearities outside this absorbing set to obtain a modified system which has, almost surely, the same long time, slow dynamics as the original system \([12]\). In the following calculations, we actually have omitted this cut-off procedure for simplicity.

By the random transformation \((2.1)\), the SDE system \((4.1) - (4.2)\) are converted into the following system

\[
\dot{X} = 0.001X - aX(Y + \sigma\eta^\varepsilon(\theta_t\omega)), \tag{4.6}
\]

\[
\dot{Y} = -\frac{1}{\varepsilon}Y + \frac{1}{\varepsilon} \frac{1}{600}X^2. \tag{4.7}
\]

Therefore, there exists an \( \tilde{h}^\varepsilon \) satisfying

\[
\tilde{h}^\varepsilon(\xi, \omega) = \frac{1}{\varepsilon} \int_{-\infty}^{0} e^{\varepsilon s} \frac{[X(s, \omega, \xi)]^2}{600} ds, \quad \xi \in \mathbb{R}, \tag{4.8}
\]

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whose graph is a random slow manifold for the random system (4.6)-(4.7). In fact, \( \tilde{h}_\varepsilon(\xi, \omega) \) has an approximation \( \hat{h}_\varepsilon(\xi, \omega) \) (with error \( O(\varepsilon^2) \)),

\[
\hat{h}_\varepsilon(\xi, \omega) = \frac{\xi^2}{600} + \varepsilon \left( -\frac{\xi^2}{300} - 0.001 + \frac{\xi^4}{180000} a - \frac{\xi^2}{300} a \sigma \int_{-\infty}^{0} s e^s dW_s(\omega) \right). \tag{4.9}
\]

So the approximated slow system is

\[
\dot{x} = 0.001x - ax(\sigma \eta(\theta_t \psi \varepsilon \omega) + \hat{h}_\varepsilon(x, \theta_t \omega)). \tag{4.10}
\]

with \( \eta(\theta_t \psi \varepsilon \omega) = \int_{-\infty}^{t} e^{(s-t)} dW_s(\psi \varepsilon \omega) \).

We can test that, for a fixed \( x \) and for \( \varepsilon \) sufficiently small,

\[
\mathbb{E}[-ax(\sigma \eta(\theta_t \psi \varepsilon \omega) + \hat{h}_\varepsilon(x, \theta_t \omega))] = -a \left( \frac{x^5 \varepsilon}{180000} a + \frac{x^3}{600} - \frac{0.002x^3 \varepsilon}{600} \right)
\]

is strictly decreasing for \( a \) in our working range. We will illustrate that the parameter estimator for a based on this low dimensional random slow system (4.10) is a good approximation of the parameter estimator based on the original system (4.1)-(4.2).

**Slow-fast system and random slow manifold** The random slow manifold is the graph of \( h_\varepsilon(\xi, \omega) = \sigma \eta(\psi \varepsilon \omega) + \tilde{h}_\varepsilon(\xi, \omega) \), where \( \tilde{h}_\varepsilon \) is as in (4.8). It is a curve (depending on random samples). The random orbits of the system (4.1)-(4.2) approaches to this curve exponentially fast. Figures 2–3 show some orbits of the slow-fast system (4.1)-(4.2) started away from the random slow manifold, and the approximate random slow manifold \( h_\varepsilon(\xi, \omega) = \sigma \eta(\psi \varepsilon \omega) + \hat{h}_\varepsilon(\xi, \omega) \), where \( \hat{h}_\varepsilon \) is as in (4.9), with different \( \sigma \) and \( \varepsilon \) values.

The orbits of the dynamical system (4.1)-(4.2) decay quickly to the random slow manifold. Figure 4 shows several samples or realizations of the random slow manifold with different \( a \) values.

**Stochastic Nelder-Mead method** The deterministic Nelder-Mead method (NM) is a geometric search method to find a minimizer of an objective function \( F(a) \). Starting from an initial guess point, it generates a new point (reflection point, expand point, inside/outside contraction point or shrink point) by comparing function values, and thus get a better and better estimator until the smallest objective function value in this iteration reaches the termination value (prescribed error tolerance). The algorithm of the NM method and its improvements have been widely studied and utilized \( [4, 14, 15] \). The method has its advantage that the objective function need not to be differentiable, and it can thus be used in various applications. As noted in \( [8, 16] \), the Nelder-Mead method is a widely used heuristic algorithm. Only very limited convergence results exist for a class of low dimensional (one or two dimensions) problems, such as in the case when the objective function \( F \) is strictly convex \( [13] \). For the numerical simulation, with analytical objective functions, Matlab function fminsearch can be used to find a minimizer.

However, when dealing with problems with noise, NM method has the disadvantage \( [4, 8] \) that it lacks an effective sample size scheme for controlling noise, as shrinking steps are sensitive to the noise in the
objective function values and then may lead to the search in a wrong direction. In fact an analytical and empirical evidence is known for the false convergence [5] on stochastic function. So we use the stochastic Nelder-Mead simplex method (SNM) [8] to mitigate the possible mistakes in the stochastic setting. The newly developed Adaptive Random Search in [8] consists of a local search and a global search. It generates a new point and new objective function  \( \hat{F} = \frac{1}{N(k)} \sum_{i=1}^{N(k)} F_i \) in the k-th iteration with increasing number of the sample size scheme \( N(k) \). A proper choice for \( N(k) \) is \([\sqrt{k}]\), with \([c]\) the largest integer not bigger than \( c \). Here \( \sum_{i=1}^{N(k)} F_i \) is the sum of \( N(k) \) objective function values for \( F \). SNM leads to the convergence of \( F \) at search points to \( \min_a F(a) \) (and thus we obtain a minimizer \( a^* \)), with probability one.

**Parameter estimation** To illustrate our method for parameter estimation on the random slow manifold, we fix \( \varepsilon = 0.01 \) and \( \sigma = 0.01 \) in the following numerical experiments.

We want to estimate the parameter \( a \) by using both the original slow-fast system and the slow system, in order to demonstrate that the slow system is appropriate for parameter estimation, when \( \varepsilon \) is sufficiently small.

**Step 1: Generate observations**

Take the true value \( a = 0.1 \) (say) and numerically solve [4.1]-[4.2] with an initial condition \((x_0, y_0)\) to get \( J \) samples of observational data \((x_{ij}^{ob}, y_{ij}^{ob})\), \( j = 1, \ldots, J \) at time instants \( t_i, i = 1, \ldots, I \) (save these data).

**Step 2: Estimator \( a_E \) for the original slow-fast system**

Take two initial guesses for the unknown system parameter \( a = a_0 \) and \( a = a_1 \) randomly and solve the original slow-fast system [4.1]-[4.2], with the same initial condition \((x_0, y_0)\) and time points \( t_i, i = 1, \ldots, I \). Thus we obtain \( x^i \) and \( y^i \) values which depend on \( a \).

Using the stochastic Nelder-Mead Algorithm, we find a parameter estimator \( a_E \) such that the objective function \( F(a) \equiv \mathbb{E} \sum_{i=1}^{I} \sum_{j=1}^{J} ((x^i - x_{ij}^{ob})^2 + (y^i - y_{ij}^{ob})^2) \) is minimized.

**Step 3: Estimator \( a_E^S \) for the slow system**

Take two initial guesses \( a_0 \) and \( a_1 \) randomly and solve the slow system [4.10] with the same initial condition \( x_0 \) at the same time instants \( t_i, i = 1, \ldots, I \). We thus obtain \( x_S^i \) which depends on \( a_0 \) or \( a_1 \).

By the stochastic Nelder-Mead method as in Step 2, we find the parameter estimator \( a_E^S \) by minimizing the objective function \( F^S(a) \equiv \mathbb{E} \sum_{i=1}^{I} \sum_{j=1}^{J} (x_S^i - x_{ij}^{ob})^2 \).
Numerical experiments  Figure 5 shows the objective functions $F(a)$ and $F^S(a)$ with true $a = 0.1$ (left) and $a = 1$ (right). Here we used 30 paths to get the expectation in the objective function. From figure 5 we can get a range $[L, R]$ with $L > 0$, to estimate $a$ in it.

For $\varepsilon = 0.01$ and $\sigma = 0.01$, Figures 6 and 7 are objective function values and estimators of each iteration for slow-fast system (4.1)-(4.2) (top) and reduced system (4.10) (bottom) with $a = 0.1$ and $a = 1$, respectively.

We observe that, as proved in [8], the objective function value $F(a)$ tends to $\min_a F(a) (=0)$, while the minimizer or the parameter estimator $a_E$ provides an accurate estimation of the system parameter $a$. For sufficiently small $\varepsilon > 0$, the objective function value $F^S(a)$ for the slow system will also get closer and closer to 0, and the minimizer or the parameter estimator $a^S_E$ is a good approximation of $a_E$. 
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Figure 2: Orbits of slow-fast system (4.1)-(4.2) (blue curves) and its slow manifold expansion $h^\varepsilon(\xi, \omega) = \sigma \eta(\psi_\varepsilon \omega) + \hat{h}^\varepsilon(\xi, \omega)$ (red curve), where $\hat{h}^\varepsilon$ is as in (4.9), with $\sigma = 0.01$ and $\varepsilon = 0.01$: $a = 0.1$ (left) and $a = 1$ (right).

Figure 3: Orbits of slow-fast system (4.1)-(4.2) (blue curves) and its slow manifold expansion $h^\varepsilon(\xi, \omega) = \sigma \eta(\psi_\varepsilon \omega) + \hat{h}^\varepsilon(\xi, \omega)$ (red curve), where $\hat{h}^\varepsilon$ is as in (4.9), with $\sigma = 0.05$ and $\varepsilon = 0.01$: $a = 0.1$ (left) and $a = 1$ (right).
Figure 4: Random slow manifold expansion $h^\varepsilon(\xi, \omega) = \sigma \eta(\psi ; \omega) + \hat{h}^\varepsilon(\xi, \omega)$, where $\hat{h}^\varepsilon$ is as in (4.9) with $\varepsilon = 0.01$: $\sigma = 0.01$ (left) and $\sigma = 0.05$ (right).

Figure 5: Minimizers of objective function $F(a)$ for slow-fast system (4.1)-(4.2) and slow manifold reduced system (4.10), with $\sigma = 0.01$ and $\varepsilon = 0.01$: True minimizers $a = 0.1$ (left) and $a = 1$ (right).
Figure 6: Objective function $F(a)$ and estimator $a_E$ (top) for slow-fast system (4.1)-(4.2); and objective function $F^S(a)$ and estimator $a^S_E$ (bottom) for slow manifold reduced system (4.10): $\sigma = 0.01$, $\varepsilon = 0.01$ and true value $a = 0.1$. 


Figure 7: Objective function $F(a)$ and estimator $a_E$ (top) for slow-fast system (4.1)-(4.2), and objective function $F^S(a)$ and estimator $a^S_E$ (bottom) for slow manifold reduced system (4.10): $\sigma = 0.01$, $\varepsilon = 0.01$ and true value $a = 1$. 