Envelopes are Solving Machines for Quadratics and Cubics and Certain Polynomials of Arbitrary Degree

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Everybody knows from school how to graphically solve a quadratic equation

\[ x^2 - px + q = 0, \]

if \( p, q \in \mathbb{R} \) are given. Simply plot the graph of \( f(x) = x^2 - px + q \) and find the point(s) of intersection with the \( x \)-axis. If \( p \) and \( q \) are modified, you have to start over and draw a new parabola to find the solutions. Can’t there be one single curve that simultaneously solves all quadratic equations?

Stunningly, there actually is such a magic parabola that serves as a solving machine for any quadratic equation, namely \( f(x) = \frac{1}{4}x^2 \). Of course, the solutions are no longer given by points of intersection with the \( x \)-axis, but are obtained by drawing tangent lines to \( f \) through a given point \((p, q)\). Moreover, the technique can be generalized to equations of the form \( x^n - px + q = 0 \), and the number of real solutions of such an equation can be seen immediately.

In this article, which is strongly inspired by Lecture 8 from the wonderful book [1], we derive the above mentioned methods in an elementary way and conclude by pointing out relations to the duality of points and lines in the plane and the concept of Legendre transformation.

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Solving quadratic equations

We consider an ordinary quadratic equation \( x^2 - px + q = 0 \) with \( p, q \in \mathbb{R} \). As we stated at the outset, everybody knows how to solve it, but we want to look at it from an unusual perspective. We solve for \( q \) obtaining \( q = xp - x^2 \) and interpret \( q \) as a function of \( p \), depending on the parameter \( x \). This gives a family of linear functions

\[
Q_x(p) = xp - x^2,
\]

where \( Q_x \) has slope \( x \) and vertical intercept \(-x^2\). Figure 1 shows \( Q_1 \) and \( Q_2 \).

The graph of \( Q_1 \) describes all quadratic equations having \( x = 1 \) as a solution, i.e., \( x = 1 \) is a solution of \( x^2 - px + q = 0 \) if and only if \((p, q)\) lies on \( Q_1 \). Therefore, the unique pair \((p, q)\) of parameters of a quadratic equation with solution set \( \{1, 2\} \) should be given by the point of intersection of \( Q_1 \) and \( Q_2 \). In fact, the point of intersection is \((3, 2)\) and the corresponding equation is \( x^2 - 3x + 2 = 0 \).

As a quadratic equation has at most two solutions, not more than two of the lines \( Q_x \) can go through one given point \((p, q)\). Nevertheless, we want to derive this fact from the mere form of the lines \( Q_x \). Therefore, we consider \( x \neq y \) such that \( Q_x \) and \( Q_y \) go through \((p, q)\). We show that if \((p, q)\) lies also on \( Q_z \), then it follows that \( z = x \) or \( z = y \).

The point \((p, q)\) lies on \( Q_x \) if and only if \( q = xp - x^2 \). Thus, that \( Q_x \) and \( Q_y \) intersect in \((p, q)\) implies \( xp - x^2 = yp - y^2 \), from which we obtain \( p(x - y) = x^2 - y^2 \). Division by \( x - y \) \((\neq 0)\) yields \( x + y = p \).\(^2\) Now, if \( Q_z \) goes also through \((p, q)\) and \( x \neq z \), it follows analogously that \( x + z = p \). Subtracting these equations gives \( y = z \).

Considering Figure 2, which shows a few more of the lines \( Q_x \), leads to the conjecture that they have a quadratic envelope,\(^3\) and we are going to determine it.

To this end, we find the point of intersection \( p_\varepsilon \) of two lines \( Q_x \) and \( Q_{x+\varepsilon} \) for some small \( \varepsilon \neq 0 \) and determine the limit as \( \varepsilon \to 0 \). We have

\[
Q_{x+\varepsilon}(p_\varepsilon) = Q_x(p_\varepsilon) \iff (x + \varepsilon)p_\varepsilon - (x + \varepsilon)^2 = xp_\varepsilon - x^2,
\]

\(^1\)We use the form \( x^2 - px + q = 0 \) instead of \( x^2 + px + q = 0 \) only for convenience; this decision simplifies most calculations in what follows.

\(^2\)Note that we hereby obtained one part of Vieta’s formula for a quadratic. We will investigate this later in more detail.

\(^3\)By envelope we mean a differentiable function \( e \) such that for every \( p \) there is one unique \( x \) for which \( Q_x \) is a tangent to \( e \) at the point \((p, e(p))\).
Figure 2. Some lines from the family \( \{ Q_x \mid x \in \mathbb{R} \} \).

Figure 3. Solving \( x^2 - x - 2 = 0 \).

and the latter is equivalent to \( p_\varepsilon = 2x + \varepsilon \). Thus, we obtain \( p_\varepsilon \to 2x \) as \( \varepsilon \to 0 \), let \( p = 2x \) be the “point of intersection of two infinitesimal distinct lines” from our family, and denote the envelope function by \( e \). It follows that \( x = \frac{p}{2} \), and

\[
e(p) = Q_{\frac{p}{2}}(p) = \frac{p}{2} \cdot p - \left( \frac{p}{2} \right)^2 = \frac{p^2}{4}.
\]

Now that we know \( e(p) \) it can be easily used to graphically solve any quadratic equation \( x^2 - px + q = 0 \). Given \( p, q \) we just have to find the line(s) \( Q_x \) on which the point \((p, q)\) lies. Therefore we construct the tangent line(s) to \( e \) through \((p, q)\). Figure 3 illustrates the method by the example of solving \( x^2 - x - 2 = 0 \). For convenience we marked the rescaling \( x = \frac{p}{2} \) on the graph of \( e \), so that one can simply read off the solutions from the figure.

Isn’t it wonderful? We can solve any quadratic equation with one picture. After discovering this “magic parabola” with a class, one could draw a large version of it on a whiteboard and construct the tangent lines with ropes and magnets. Of course, also a GeoGebra applet is easily constructed and fun.

Moreover, Figure 3 tells us the number of solutions of any given quadratic equation. By convexity there are two tangent lines (i.e., two solutions of the equation) to the graph of \( e \) through a given point \((p, q)\) if and only if \((p, q)\) lies “below” \( e \). There is exactly one tangent line if and only if \((p, q)\) lies on the graph of \( e \), and there is no
tangent line if and only if \((p, q)\) lies “above” \(e\). This can be expressed algebraically by noting that \((p, q)\) lies on [below, above] \(e\) if and only if \(q = e(p)\) \([q < e(p), q > e(p)]\). As

\[ q = e(p) \iff \frac{p^2}{4} - q = 0, \]

we have rediscovered the well-known discriminant of a quadratic, which indicates the number of solutions.

**Solving cubic equations**

As a next step we want to extend our method to cubic equations of the form

\[
x^3 - px + q = 0. \quad (1)
\]

Later, we will deal with such equations for arbitrary \(n\) instead of 3, and of course, the results of this and the previous section follow from the general considerations. Nevertheless, we think it is worth doing these special cases first, because they are illustrative for the case study (\(n\) even or odd) that is needed later.

Following the same idea as above we rearrange to \(q = xp - x^3\) and interpret \(q\) as a function of \(p\) with parameter \(x\). That is, we consider the family of linear functions

\[ Q_x(p) = xp - x^3, \]

where \(Q_x\) has slope \(x\) and axis intercept \(-x^3\).

Figure 4 shows some of these lines, and we get the impression that there exists an envelope with two branches, which we like to determine.

As before, we find the point \(p_\varepsilon\) of intersection of two lines \(Q_x\) and \(Q_{x+\varepsilon}\) for some small \(\varepsilon \neq 0\) and let \(\varepsilon \to 0\). We have

\[ Q_{x+\varepsilon}(p_\varepsilon) = Q_x(p_\varepsilon) \iff (x + \varepsilon)p_\varepsilon - (x + \varepsilon)^3 = xp_\varepsilon - x^3. \]

The last equation is equivalent to

\[ \varepsilon p_\varepsilon = (x + \varepsilon)^3 - x^3. \]
Expanding on the right hand side and dividing by $\varepsilon$ on both sides yields

$$p_\varepsilon = 3x^2 + 3x\varepsilon + \varepsilon^2,$$

and we see that $p_\varepsilon \to 3x^2$ as $\varepsilon \to 0$. Again, we interpret $p = 3x^2$ as ‘infinitesimal point of intersection’ and obtain that $p \geq 0$. Solving for $x$ gives $x = \pm \sqrt{p/3}$, and we see that we will actually get two envelope branches. For the first branch we obtain

$$e(p) = Q \sqrt[p]{p} = \sqrt[p]{p} \cdot p - \left(\sqrt[p]{p}\right)^3 = 2\left(\frac{p}{3}\right)^{3/2},$$

and the second branch equals $-e(p)$. Figure 5 shows both branches of the envelope.

To use the envelope for solving cubic equations we have to find all tangents to these branches through a given point $(p, q)$. Of course, the number of tangents (or solutions) depends on the position of the point.

Both branches are continuously differentiable on $\mathbb{R}_{>0}$, with $e'(p) = \sqrt[p]{p}$ and differentiable from the right at the origin with $e'(0) = 0$. Furthermore, we have $\lim_{p \to \infty} e'(p) = \infty$, and therefore $e$ takes on every slope in $[0, \infty)$, whereas $-e(p)$ takes on every nonpositive slope. $e$ is strictly convex and $-e$ is strictly concave. From these considerations we can conclude the following.

• There are exactly three different tangents through $(p, q) \neq (0, 0)$ to the envelope if and only if $(p, q)$ lies in the region strictly “between the branches” (see Figure 6), because in this case there are two tangents to one branch and one tangent to the other branch. A special case is when $(p, q)$ lies on the $x$-axis. Then there are two tangents to both branches, but the axis itself is an identical tangent.

• There are exactly two different tangents through $(p, q) \neq (0, 0)$ to the envelope if and only if $(p, q)$ lies on one of the branches, because in this case there is a unique tangent to both branches.

• There is one unique tangent through $(p, q) \neq (0, 0)$ to the envelope if and only if $(p, q)$ lies in the region strictly “not between the branches” (see Figure 6), because in this case there is no tangent to one of the branches and one unique tangent to the
other one. A special case is when \((p, q)\) lies on the \(x\)-axis. Then the axis itself is the only mutual tangent.

The discriminant of a cubic equation. The point \((p, q) \neq (0, 0)\) lies on the envelope if and only if \(|q| = e(p)\). It lies strictly between the branches if the analogous condition holds with < instead of =, and strictly not between the branches if we have the inequality with >. Therefore, we obtain that equation (1) has exactly two solutions if and only if \(q^2 = 4(p/3)^3\) and this equivalent to

\[
\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3 = 0.
\]

For one and three solutions we have analogous inequalities, and hereby rediscovered the well-known discriminant of a cubic equation without quadratic term.\(^4\)

Solving equations of arbitrary degree

We want to generalize our ideas to equations of the form

\[
x^n - px + q = 0, \quad (2)
\]

where \(n \geq 2\) is an arbitrary integer. As before, the equation defines a family of linear functions with parameter \(x \in \mathbb{R}\):

\[
Q_x(p) = xp - x^n
\]

To determine the envelope we use the same approach as above, namely

\[
Q_{x+\varepsilon}(p_x) = Q_x(p_x) \Leftrightarrow (x + \varepsilon)p_x - (x + \varepsilon)^n = xp_x - x^n
\]

\(^4\)By a simple substitution of the form \(x \to x + c\) every cubic equation can be transformed into one without quadratic term. For more details on this and various considerations on cubic equations from a mathematics-educational perspective see [2].
Letting \( \varepsilon \to 0 \) each term including a factor \( \varepsilon \) with a positive exponent vanishes. As the only summand with a nonpositive \( \varepsilon \)-exponent is the \( (n - 1) \)th, it follows that \( p_\varepsilon \to (n-1)x^{n-1} = nx^{n-1} \) for \( \varepsilon \to 0 \). Thus, we obtain

\[
p = nx^{n-1}
\]
as “infinitesimal point of intersection” and distinguish two cases to proceed.

**Case 1: \( n \) is even.** Then \( n - 1 \) is odd, we obtain \( x = \left(\frac{p}{n}\right)^{\frac{1}{n-1}} \), and therefore

\[
e(p) = Q\left(\frac{p}{n}\right)^{\frac{1}{n-1}}(p) = \left(\frac{p}{n}\right)^{\frac{1}{n-1}} \cdot p - \left(\frac{p}{n}\right)^{\frac{n}{n-1}} = (n-1) \left(\frac{p}{n}\right)^{\frac{n}{n-1}}.
\]

The following figure shows the envelopes for \( n = 2, 4, 6 \) (dotted, dashed, solid).

From the figure one might get the impression that \( e \) is no longer differentiable at the origin for larger \( n \), but this is not the case. \( e \) is continuously differentiable on the whole domain \( \mathbb{R} \) with \( e'(p) = \left(\frac{p}{n}\right)^{\frac{1}{n-1}} \), and in particular \( e'(0) = 0 \). Furthermore, we see \( e'(p) \to \infty \) for \( p \to \pm \infty \). Therefore \( e \) takes on every slope in \([0, \infty)\), and, analogously to the quadratic case on page 3, we can conclude that equation (2) has exactly one [two, no] solution(s) if and only if \( q = e(p) \) [\( q < e(p) \), \( q > e(p) \)].

**Case 2: \( n \) is odd.** Then \( n - 1 \) is even, and since \( p = nx^{n-1} \) we obtain that \( p \geq 0 \).

We have \( x = \pm \left(\frac{p}{n}\right)^{\frac{1}{n-1}} \), so that the envelope will have two branches. Similar to case 1 we obtain the first branch by

\[
e(p) = Q\left(\frac{p}{n}\right)^{\frac{1}{n-1}}(p) = \left(\frac{p}{n}\right)^{\frac{1}{n-1}} \cdot p - \left(\frac{p}{n}\right)^{\frac{n}{n-1}} = (n-1) \left(\frac{p}{n}\right)^{\frac{n}{n-1}},
\]

and the second branch equals \(-e(p)\). Figure 8 shows both envelope branches for \( n = 3, 5, 7 \) (dotted, dashed, solid).
Figure 8. $e(p)$ for $n = 3, 5, 7$.

The branch $e$ (and of course also $-e$) is continuously differentiable on $\mathbb{R}_{>0}$, and we have $e'(p) = \left(\frac{p}{n}\right)^{\frac{1}{n-1}}$. At the origin $e$ and $-e$ are differentiable from the right and we have $e'(0) = -e'(0) = 0$. Furthermore, it holds true that $e'(p) > 0$ for $p > 0$, $e'(p) \to \infty$ for $p \to \infty$, and $-e'(p) \to -\infty$ for $p \to \infty$. Therefore, $e$ takes on every slope in $[0, \infty)$ and $-e$ takes on every slope in $(-\infty, 0]$. Thus, analogously to the cubic case on page 5, we obtain that equation (2) has exactly two [three, one] solution(s) if and only if $|q| = e(p) [|q| < e(p), |q| > e(p)]$.

**Summary of both cases and the discriminant.** For all $n$ it holds true that $e(p) = (n - 1) \left(\frac{p}{n}\right)^{\frac{n}{n-1}}$. If $n$ is even, the envelope has one branch and is defined on the entire real line. If $n$ is odd, the envelope has the two branches $e$ and $-e$, and both of them are defined on $\mathbb{R}_{\geq 0}$. For both, odd and even $n$, the relation ($<$, $>$ or $=$) between $q$ (or $|q|$) and $e(p)$ indicates the number of solution of equation (2). From this we can derive the determinant of (2). For even $n$ it holds true that

$$q = e(p) \iff \frac{q}{n-1} = \left(\frac{p}{n}\right)^{\frac{n}{n-1}} \iff \left(\frac{p}{n}\right)^n - \left(\frac{q}{n-1}\right)^{n-1} = 0.$$  

The calculation for odd $n$ is analogous, so the determinant of equation (2) is given by

$$\left(\frac{p}{n}\right)^n - \left(\frac{q}{n-1}\right)^{n-1} .$$  

Isn’t it beautiful?

**Final remarks**

**Duality.** It is worth mentioning that we used a more general concept here, namely the duality between straight lines and points in the plane. On the one hand, a linear equation of the form $q = mp + n$ represents a line in the $pq$-plane, which is uniquely determined by the pair $(m, n)$ of slope and $y$-intercept. On the other hand, this pair represents a point in the $mn$-plane. Thus, we have a one-to-one correspondence between the (non-vertical) straight lines in the $pq$-plane and the points in the $mn$-plane.

If we consider a linear function $Q(p) = mp + n$ with slope $m$ and $y$-intercept $n$, a point $(p, q)$ in the $pq$-plane lies on the graph of $Q$ if and only if $q = mp + n$, which is
equivalent to \( n = -pm + q \). This means that in the \( mn \)-plane the point \((m, n)\), which is corresponding to \( Q \), lies on the graph of \( N(m) = -pm + q \) with slope \(-p\) and \( y\)-intercept \( q \).

To illustrate this, Figure 9 shows the lines \( a \), given by \( q = -p + 3 \) and \( b \), given by \( q = p - 1 \), which intersect in the point \( S = (2, 1) \). The corresponding points \( A \) and \( B \) in the \( mn \)-plane lie on the line \( s \), given by \( n = -2m + 1 \).

Duality can sometimes be helpful, which we want to illustrate by deriving Vieta’s formula for quadratic equations\(^5\) in a nice and uncommon way. We assume that \( x^2 - px + q = 0 \) has the solutions \( u \) and \( v \). Then \((p, q)\) is the point of intersection of the lines \( Q_u(p) = up - u^2 \) and \( Q_v(p) = vp - v^2 \). By duality the corresponding points \((u, -u^2)\) and \((v, -v^2)\) lie on the line with slope \(-p\) and \( y\)-intercept \( q \).

As the line through these two points has slope \( \frac{v^2 - u^2}{v - u} = -(u + v) \), we obtain \( p = u + v \). Moreover, by plugging in \((u, -u^2)\), the \( y\)-intercept \( q \) is obtained:

\[
q = pu - u^2 = (u + v)u - u^2 = uv.
\]

**Relation to Legendre transforms.** Readers that are familiar with Legendre transforms might have noticed that our magic envelope matches with the Legendre transform of \( f(x) = x^n \) (at least for even \( n \), so that \( f \) is convex). In concluding this article we want to point out that this is not a coincidence.

If we consider any smooth and strictly convex function \( f : I \to \mathbb{R} \), where \( I \) is an interval of reals, we can represent \( f \) in terms of its first derivative in the following way: Because \( f \) is strictly convex, we have \( f''(x) > 0 \) for all \( x \in I \). Therefore, \( f' \) is strictly increasing on \( I \) and hence a one-to-one-function. Thus, for every \( p \in I^* := \{ f'(x) \mid x \in I \} \) there is exactly one \( x \) with \( p = f'(x) \), or – geometrically expressed – no two tangent lines to \( f \) have the same slope.

Now, instead of \( I \) we can use the set \( I^* \) of slopes \( p \) as the domain of a new function that contains all information about \( f \). The tangent line \( t \) to \( f \) at a given point \((x_0, f(x_0))\) has slope \( p := f'(x_0) \) and determines its \( y\)-intercept uniquely. The Legendre transform \( f^* \) of \( f \) maps \( p \) to the negative\(^6\) of the \( y\)-intercept of \( t \). A formula for

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\(^5\)Recall Vieta’s formula for quadratic equations: \( u \) and \( v \) are the solutions of the quadratic equation \( x^2 + ax + b = 0 \) if and only if \( a = -(u + v) \) and \( b = uv \).

\(^6\)It is a convention to take the negative.
$f^*(p)$ is easily derived. We have

$$t(x) = f'(x_0)(x - x_0) + f(x_0) = px - (px_0 - f(x_0)),$$

and therefore the $y$-intercept of $t$ equals $px_0 - f(x_0)$. For a given function $f$ we can express this solely in terms of $p$ using the one-to-one relation between $p = f'(x_0)$ and $x_0$.

For instance, let us consider $f(x) = x^2$. Let $p = f'(x) = 2x$ be the slope of a tangent line to $f$ at some point $(x, f(x))$.\(^7\) Using $x = \frac{p}{2}$ we obtain

$$f^*(p) = px - f(x) = p \cdot \frac{p}{2} - f(p) = \frac{p^2}{4}.$$

It is not surprising that this coincides with the envelope $e(p)$ considered above, because there we started with the equation $x^2 - px + q = 0$ and regarded $q = px - x^2$ as a function of $p$. That is precisely what Legendre transformation for $f(x) = x^2$ does. The concept of Legendre transformation is important in physics; for further explanation and a more detailed introduction see [3].

**Summary.** Everybody knows from school how to solve a quadratic equation of the form $x^2 - px + q = 0$ graphically. To solve more than one equation this method can become tedious, as for each pair $(p, q)$ a new parabola has to be drawn. Stunningly, there is one single curve that can be used to solve every quadratic equation by drawing tangent lines through a given point $(p, q)$ to this curve.

In this article we derive this method in an elementary way and generalize it to equations of the form $x^n - px + q = 0$ for arbitrary $n \geq 2$. Moreover, the number of solutions of a specific equation of this form can be seen immediately with this technique. In concluding the article, we point out connections to the duality of points and lines in the plane and to the concept of Legendre transformation.

**References**

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\(^7\) We only needed to write $x_0$ instead of $x$ to derive the formula for $f^*$. For simplicity we omit the index now.