A Quasiorder-Based Perspective on Residual Automata

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Abstract

In this work, we define a framework of automata constructions based on quasiorders over words to provide new insights on the class of residual automata. We present a new residualization operation and a generalized double-reversal method for building the canonical residual automaton for a given language. Finally, we use our framework to offer a quasiorder-based perspective on NL*, an online learning algorithm for residual automata. We conclude that quasiorders are fundamental to residual automata as congruences are to deterministic automata.

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1 Introduction

Residual automata (RFAs for short) are finite-state automata for which each state defines a residual of its language, where the residual of a language $L$ by a word $u$ is defined as the set of words $w$ such that $uw \in L$. The class of RFAs lies between deterministic (DFAs) and nondeterministic automata (NFAs). They share with DFAs a significant property: the existence of a canonical minimal form for any regular language. On the other hand, they share with NFAs the existence of automata that are exponentially smaller (in the number of states) than the corresponding minimal DFA for the language. These properties make RFAs specially appealing in certain areas of computer science such as Grammatical Inference [10, 14].

RFAs were first introduced by Denis et al. [8, 9]. They defined an algorithm for residualizing an automaton, which is a variation of the well-known subset construction used for determination, and showed that there exists a unique canonical RFA, which is minimal in
the number of states, for every regular language. Moreover, they showed that the residual-
equivalent of the double-reversal method [4] holds, i.e., residualizing an automaton $N$ whose
reverse is residual yields the canonical RFA for the language accepted by $N$.

Later, Tamm [16] generalized the double-reversal method for RFAs by giving a sufficient
and necessary condition that guarantees that the residualization operation defined by Denis
et al. [9] yields the canonical RFA. In fact, this generalization comes in the same lines as that
of Brzozowski and Tamm [5] for the double-reversal method for building the minimal DFA.

These results evidence the existence of a relationship between RFAs and DFAs. In fact,
a connection between these two classes of automata was already established by Myers et
al. [1, 15] from a category-theoretical point of view. Concretely, they [1] use this perspective
to address the residual-equivalent of the double-reversal method proposed by Denis et al. [9]
to obtain the canonical RFA.

In this work we evidence this connection between RFAs and DFAs from the point of view
of quasiorders over words. Specifically, we show that quasiorders are fundamental to RFAs
as congruences are for DFAs.

Previously, we studied the problem of building DFAs using congruences, i.e., equivalence
relations over words with good properties w.r.t. concatenation [11]. This way, we derived
several well-known results about minimization of DFAs, including the double-reversal method
and its generalization by Brzozowski and Tamm [5]. While the use of congruences over
words suited for the construction of a subclass of residual automata, namely, deterministic
automata, these are no longer useful to describe the more general class of nondeterministic
residual automata. By moving from congruences over words to quasiorders, we are able to
introduce nondeterminism in our automata constructions.

We consider quasiorders with good properties w.r.t. right and left concatenation. In
particular, we define the so-called right language-based quasiorder, whose definition relies on
a given regular language; and the right automata-based quasiorder, whose definition relies
on a finite representation of the language, i.e., an automaton. We also give counterpart
definitions for quasiorders that behave well with respect to left concatenation. Relying on
quasiorders that preserve a given regular language, i.e., the closure of the language w.r.t. the
quasiorder coincides with the language, we will provide a framework of finite-state automata
constructions for the language.

When instantiating our automata constructions using the right language-based quasiorder,
we obtain the canonical RFA for the given language; while using the right automata-based
quasiorder yields an RFA for the language accepted by the automaton that has, at most, as
many states as the RFA obtained by the residualization operation defined by Denis et al. [9].
Similarly, left automata-based and language-based quasiorders yield co-residual automata,
i.e., automata whose reverse is residual.

Our quasiorder-based framework allows us to give a simple correctness proof of the
double-reversal method for building the canonical RFA. Moreover, it allows us to generalize
this method in the same fashion as Brzozowski and Tamm [5] generalized the double-reversal
method for building the minimal DFA. Specifically, we give a characterization of the class of
automata for which our automata-based quasiorder construction yields the canonical RFA.

We compare our characterization with the class of automata, defined by Tamm [16], for
which the residualization operation of Denis et al. [9] yields the canonical RFA and show
that her class of automata is strictly contained in the class we define. Furthermore, we
highlight the connection between the generalization of Brzozowski and Tamm [5] and the
one of Tamm [16] for the double-reversal methods for DFAs and RFAs, respectively.
Finally, we revisit the problem of learning residual automata from a quasiorder-based perspective. Specifically, we observe that the NL* algorithm defined by Bollig et al. [3], inspired by the popular Angluin’s L* algorithm for learning DFAs [2], can be seen as an algorithm that starts from a quasiorder and refines it at each iteration. At the end of each iteration, the automaton built by NL* coincides with our quasiorder-based automata construction applied to the refined quasiorder.

Structure of the paper. After preliminaries in Section 2, we introduce in Section 3 automata constructions based on quasiorders and establish the duality between these constructions when using right and left quasiorders. We instantiate these constructions in Section 4 with the language-based and automata-based quasiorders and study the relations between the resulting automata. As a consequence, we derive in Section 5 a generalization of the double-reversal method for building the canonical RFA for a language. In addition, we show a novel quasiorder-based perspective on the NL* algorithm for learning residual automata in Section 6. For space reasons, a formal description of the NL* algorithm as well as supplementary results, including the pseudocode of our quasiorder-based version of NL*, and missing proofs are deferred to the extended version of this paper [12].

2 Preliminaries

Languages. Let Σ be a finite nonempty alphabet of symbols. Given a word w ∈ Σ*, we will use |w| to denote the length of w. We denote w^R the reverse of w. Given a language L ⊆ Σ*, L^R := {w^R | w ∈ L} denotes the reverse language of L and L^c, its complement language.

We denote the left (resp. right) quotient of L by a word u, also known as residual, as u^{-1}L := {w ∈ Σ* | uw ∈ L} (resp. Lu^{-1} := {w ∈ Σ* | wu ∈ L}). Denis et al. [9] defined the notion of composite and prime residuals that we extend to right quotients as follows. A left (resp. right) quotient u^{-1}L (resp. Lu^{-1}) is composite iff it is the union of all the left (resp. right) quotients that it strictly contains, i.e. u^{-1}L = \bigcup_{x ∈ Σ^*} x^{-1}L ⊊ x^{-1}Lu^{-1} (resp. Lu^{-1} = \bigcup_{x ∈ Σ^*} Lx^{-1} ⊊ Lx^{-1}). Otherwise, we say the quotient is prime.

Automata. A (nondeterministic) finite-state automaton (NFA for short), or simply automaton, is a 5-tuple N = (Q, Σ, δ, I, F), where Q is a finite set of states, Σ is an alphabet, I ⊆ Q are the initial states, F ⊆ Q are the final states, and δ : Q × Σ → φ(Q) is the transition function, where φ(Q) denotes the powerset w.r.t. Q. We denote the extended transition function from Σ to Σ^* by δ̂, defined in the usual way, and, given w ∈ Σ^* and S ∈ φ(Q), we define post^w_N(S) := {q ∈ Q | ∃ q’ ∈ S, q ∈ δ̂(q’, w)} and prec^w_N(S) := {q ∈ Q | ∃ q’ ∈ S, w ∈ δ̂(q, q’)}.

Given S, T ⊆ Q, W_N^S,T := {w ∈ Σ^* | ∃ q ∈ S, q’ ∈ T, w ∈ δ̂(q, q’)}. In particular, when S = {q} and T = {F}, we say that W_N^q,F is the right language of state q. Likewise, when S = I and T = {q}, we say that W_N^I,q is the left language of state q. In general, we omit the automaton N from the superscript when it is clear from the context. We say that a state q is unreachable iff W_N^I,q = ∅ and we say that q is empty if W_N^q,F = ∅. Finally, the language accepted by an automaton N is L(N) = \bigcup_{q ∈ I} W_N^q,F = \bigcup_{q ∈ F} W_N^I,q = W_N^I,F.

The NFA N’ = (Q’, Σ, δ’, I’, F’) is a sub-automaton of N iff Q’ ⊆ Q, I’ ⊆ I, F’ ⊆ F and q’ ∈ δ̂(q, a) ⇒ q’ ∈ δ̂(q, a) with q, q’ ∈ Q and a ∈ Σ. The reverse of N, denoted by N^R, is defined as N^R = (Q, Σ, δ, F, I) where q ∈ δ̂(q’, a) iff q’ ∈ δ̂(q, a). Clearly, L(N)^R = L(N^R).

Residual Automata. A residual finite-state automaton (RFA for short) is an NFA such that the right language of each state is a left quotient of the accepted language. We write RFA instead of RFSA [9] to be consistent with the abbreviations NFA and DFA. Formally, an RFA is an automaton N = (Q, Σ, δ, I, F) such that ∀q ∈ Q, ∃ u ∈ Σ*, W_N^q,F = u^{-1}L(N).
We say an automaton is \textit{co-residual} (co-RFA for short) if its reverse is an RFA, i.e., \( \forall q \in Q, \exists u \in \Sigma^* \), \( W_q^+ \subseteq \mathcal{L}(N) u^{-1} \). We say \( u \in \Sigma^* \) is a \textit{characterizing word} for \( q \in Q \) iff \( W_q^+ = u^{-1} \mathcal{L}(N) \) and we say \( N \) is \textit{consistent} iff every state \( q \) is reachable by a characterizing word for \( q \). Moreover, \( N \) is \textit{strongly consistent} iff every state \( q \) is reachable by every characterizing word of \( q \).

Denis et al. [9] define a \textit{residualization} operation that, given NFA \( N \), builds an RFA \( N^{\text{res}} \) such that \( \mathcal{L}(N^{\text{res}}) = \mathcal{L}(N) \). Let \( N = (Q, \Sigma, \delta, \tilde{I}, \tilde{F}) \) be an NFA and \( u \in \Sigma^* \), the set \( \text{post}^+_u(N) \) is \textit{coverable} iff \( \text{post}^+_u(N) = \bigcup_{v \in \Sigma^*} \text{post}^+_v(N) \subseteq \text{post}^+_u(N) \). Define \( N^{\text{res}} \overset{\text{def}}{=} (\tilde{Q}, \Sigma, \tilde{\delta}, \tilde{I}, \tilde{F}) \) as an RFA with \( \tilde{Q} = \{ \text{post}^+_u(N) \mid u \in \Sigma^* \land \text{post}^+_u(N) \text{ is not coverable} \} \), \( \tilde{I} = \{ S \in \tilde{Q} \mid S \subseteq I \} \), \( \tilde{F} = \{ S \in \tilde{Q} \mid S \subseteq \delta(S, a) \} \) for every \( S \in \tilde{Q} \) and \( a \in \Sigma \). As shown by Denis et al. [9], the canonical RFA is a strongly consistent RFA and it is the minimal (in number of states) RFA such that \( \mathcal{L}(N) = L \). Moreover, the canonical RFA is maximal in the number of transitions.

\section*{Quasiorders.} A \textit{quasiorder} over \( \Sigma^* \) (qo for short) \( \preceq \) is a reflexive and transitive binary relation over \( \Sigma^* \). A symmetric qo is called an \textit{equivalence relation}. A quasiorder \( \preceq \) is a right (resp. left) \textit{quasiorder} and we denote it \( \preceq^r \) (resp. \( \preceq^l \)) iff for all \( u, v \in \Sigma^* \), we have that \( u \preceq v \Rightarrow u a \preceq v a \) (resp. \( u \preceq v \Rightarrow a u \preceq a v \)), for all \( a \in \Sigma \). For example, the quasiorder defined by \( u \preceq_{\text{lex}} v \overset{\text{def}}{=} |u| \leq |v| \), is a left and right qo but not an equivalence relation.

Given two qo’s \( \preceq \) and \( \preceq' \), we say that \( \preceq \) is \textit{finer} than \( \preceq' \) (or \( \preceq' \) is \textit{coarser} than \( \preceq \)) iff \( \preceq \subseteq \preceq' \). For every qo \( \preceq \), we define its \textit{strict version} as: \( u \preceq' \overset{\text{def}}{=} u \preceq v \land v \not\preceq u \) and we define \( (\preceq)^{-1} \) as: \( u (\preceq)^{-1} v \overset{\text{def}}{=} u \preceq v \Leftrightarrow v \not\preceq u \). Note that every qo \( \preceq \) induces an equivalence relation defined as \( \sim \overset{\text{def}}{=} \preceq \cap (\preceq)^{-1} \).

We adopt the definition of \textit{closure} of a subset of \( S \subseteq \Sigma^* \) w.r.t. a qo \( \preceq \) introduced by de Luca and Varricchio [7]. Concretely, given a qo \( \preceq \) on \( \Sigma^* \) and a subset \( S \subseteq \Sigma^* \), we define the \textit{upper closure} (or simply \textit{closure}) of \( S \) w.r.t. \( \preceq \) as \( \text{cl}_\preceq(S) \overset{\text{def}}{=} \{ w \in \Sigma^* \mid \exists x \in S, x \preceq w \} \). We say that \( \text{cl}_\preceq(S) \) is a \textit{principal} iff \( \text{cl}_\preceq(S) = \text{cl}_\preceq(\{u\}) \), for some \( u \in \Sigma^* \). In that case, we write \( \text{cl}_\preceq(u) \) instead of \( \text{cl}_\preceq(\{u\}) \). Note that, \( \text{cl}_\preceq(u) = \text{cl}_\preceq(v) \), for all \( v \in \Sigma^* \) such that \( u \sim v \). Finally, given a language \( L \subseteq \Sigma^* \), we say that a qo \( \preceq \) is \textit{L-preserving} iff \( \text{cl}_\preceq(L) = L \).

\section*{3 Automata Constructions from Quasiorders}

We will consider right and left quasiorders on \( \Sigma^* \) (and their corresponding closures) and we will use them to define RFAs constructions for regular languages. The following lemma gives a characterization of right and left quasiorders.

\textbf{Lemma 1.} The following properties hold:
1. \( \preceq^r \) is a right quasiorder iff \( \text{cl}_{\preceq^r}(u)v \subseteq \text{cl}_{\preceq^r}(uv) \), for all \( u, v \in \Sigma^* \).
2. \( \preceq^l \) is a left quasiorder iff \( v \text{cl}_{\preceq^l}(u) \subseteq \text{cl}_{\preceq^l}(ev) \), for all \( u, v \in \Sigma^* \).

Given a regular language \( L \), we are interested in left and right quasiorders that are \( L \)-preserving. We will use the principals of these quasiorders as states of automata constructions that yield RFAs and co-RFAs accepting the language \( L \). Therefore, in the sequel, we will only consider quasiorders that induce a finite number of principals, i.e., quasiorders \( \preceq \) such that the induced equivalence \( \sim \overset{\text{def}}{=} \preceq \cap (\preceq)^{-1} \) has finite index.
Next, we introduce the notion of $L$-composite principals which, intuitively, correspond to states of our automata constructions that can be removed without altering the language accepted by the automata.

**Definition 2 (L-Composite Principal).** Let $L$ be a regular language and let $\preceq^r$ (resp. $\preceq^l$) be a right (resp. left) quasiorder on $\Sigma^*$. Given $u \in \Sigma^*$, the principal $\text{cl}_{\preceq^r}(u)$ (resp. $\text{cl}_{\preceq^l}(u)$) is $L$-composite iff

$$u^{-1}L = \bigcup_{x \in \Sigma^*, \ x \preceq^r u} x^{-1}L \quad \text{(resp. } Lu^{-1} = \bigcup_{x \in \Sigma^*, \ x \preceq^l u} Lx^{-1})$$

If $\text{cl}_{\preceq^r}(u)$ (resp. $\text{cl}_{\preceq^l}(u)$) is not $L$-composite then it is $L$-prime.

We sometimes use the terms composite and prime principal when the language $L$ is clear from the context. Observe that, if $\text{cl}_{\preceq^r}(u)$ is $L$-composite, for some $u \in \Sigma^*$, then so is $\text{cl}_{\preceq^l}(v)$, for every $v \in \Sigma^*$ such that $u \sim^r v$. The same holds for a left quasiorder $\preceq^l$.

Given a regular language $L$ and a right quasiorder $\preceq^r$ that is $L$-preserving, the following automata construction yields an RFA that recognizes exactly the language $L$.

**Definition 3 (Automata construction $H^r(\preceq^r, L)$).** Let $\preceq^r$ be a right quasiorder and let $L \subseteq \Sigma^*$ be a language. Define the automaton $H^r(\preceq^r, L) \overset{\text{def}}{=} (Q, \Sigma, \delta, I, F)$ where $Q = \{\text{cl}_{\preceq^r}(u) \mid u \in \Sigma^*, \text{cl}_{\preceq^r}(u) \text{ is } L\text{-prime}\}$, $I = \{\text{cl}_{\preceq^r}(u) \in Q \mid \exists \in \text{cl}_{\preceq^r}(u)\}$, $F = \{\text{cl}_{\preceq^r}(u) \in Q \mid u \in L\}$ and $\delta(\text{cl}_{\preceq^r}(u), a) = \{\text{cl}_{\preceq^r}(v) \in Q \mid \text{cl}_{\preceq^r}(v)a \subseteq \text{cl}_{\preceq^r}(v)\}$ for all $\text{cl}_{\preceq^r}(u) \in Q, a \in \Sigma$.

**Lemma 4.** Let $L \subseteq \Sigma^*$ be a regular language and let $\preceq^r$ be a right $L$-preserving quasiorder. Then $H^r(\preceq^r, L)$ is an RFA such that $L(H^r(\preceq^r, L)) = L$.

Given a regular language $L$ and a left $L$-preserving quasiorder $\preceq^l$, we can give a similar automata construction of a co-RFA that recognizes exactly the language $L$.

**Definition 5 (Automata construction $H^l(\preceq^l, L)$).** Let $\preceq^l$ be a left quasiorder and let $L \subseteq \Sigma^*$ be a language. Define the automaton $H^l(\preceq^l, L) = (Q, \Sigma, \delta, I, F)$ where $Q = \{\text{cl}_{\preceq^l}(u) \mid u \in \Sigma^*, \text{cl}_{\preceq^l}(u) \text{ is } L\text{-prime}\}$, $I = \{\text{cl}_{\preceq^l}(u) \in Q \mid u \in L\}$, $F = \{\text{cl}_{\preceq^l}(u) \in Q \mid \exists \in \text{cl}_{\preceq^l}(u)\}$, and $\delta(\text{cl}_{\preceq^l}(u), a) = \{\text{cl}_{\preceq^l}(v) \in Q \mid a\text{cl}_{\preceq^l}(v) \subseteq \text{cl}_{\preceq^l}(v)\}$ for all $\text{cl}_{\preceq^l}(u) \in Q, a \in \Sigma$.

**Lemma 6.** Let $L \subseteq \Sigma^*$ be a language and let $\preceq^l$ be a left $L$-preserving quasiorder. Then $H^l(\preceq^l, L)$ is a co-RFA such that $L(H^l(\preceq^l, L)) = L$.

Observe that the automaton $H^r = H^r(\preceq^r, L)$ (resp. $H^l = H^l(\preceq^l, L)$) is finite, since we assume $\preceq^r$ (resp. $\preceq^l$) induces a finite number of principals. Note also that $H^r$ (resp. $H^l$) possibly contains empty (resp. unreachable) states but no state is unreachable (resp. empty).

Moreover, notice that by keeping all principals of $\preceq^r$ (resp. $\preceq^l$) as states, instead of only the prime ones as in Definition 3 (resp. Definition 5), we would obtain an RFA (resp. a co-RFA) with (possibly) more states that also recognizes $L$.

The following lemma shows that $H^r$ and $H^l$ inherit the left-right duality between $\preceq^r$ and $\preceq^l$ through the reverse operation.

**Lemma 7.** Let $\preceq^r$ and $\preceq^l$ be a right and a left quasiorder, respectively, and let $L \subseteq \Sigma^*$ be a language. If $u \preceq^r v \iff u^R \preceq^l v^R$ then $H^r(\preceq^r, L)$ is isomorphic to $(H^l(\preceq^l, L^R))^R$.

Finally, it follows from the next theorem that given two right $L$-preserving quasiorders, $\preceq_1^r$ and $\preceq_2^r$, if $\preceq_1^r \subseteq \preceq_2^r$ then the automaton $H^r(\preceq_1^r, L)$ has, at least, as many states as $H^r(\preceq_2^r, L)$. The same holds for left $L$-preserving quasiorders and $H^l$. Observe that this is not obvious since only the $L$-prime principals correspond to states of the automata construction.
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Lemma 12. Let \( L \subseteq \Sigma^* \) be a language and let \( \preceq_1 \) and \( \preceq_2 \) be two left or two right \( L \)-preserving quasiorders. If \( \preceq_1 \subseteq \preceq_2 \) then:

\[
|\{cl_{\preceq_1}(u) \mid u \in \Sigma^* \land cl_{\preceq_1}(u) \text{ is } L\text{-prime}\}| \geq |\{cl_{\preceq_2}(u) \mid u \in \Sigma^* \land cl_{\preceq_2}(u) \text{ is } L\text{-prime}\}|
\]

4 Language-based Quasiorders and their Approximation using NFAs

In this section we instantiate our automata constructions using two classes of quasiorders, namely, the so-called Nerode’s quasiorders [6], whose definition is based on a given regular language; and the automata-based quasiorders, whose definition relies on a given automaton.

Definition 9 (Language-based Quasiorders). Let \( u, v \in \Sigma^* \) and let \( L \subseteq \Sigma^* \) be a language. Define:

\[
\begin{align*}
  u \preceq^r_L v & \iff u^{-1}L \subseteq v^{-1}L & \text{Right-language-based Quasiorder} \\
  u \preceq^f_L v & \iff Lu^{-1} \subseteq Lv^{-1} & \text{Left-language-based Quasiorder}
\end{align*}
\]

It is well-known that for every regular language \( L \) there exists a finite number of quotients \( u^{-1}L \) [7]. Therefore, the language-based quasiorders defined above induce a finite number of principals since each principal set is determined by a quotient of \( L \).

Definition 10 (Automata-based Quasiorders). Let \( u, v \in \Sigma^* \) and let \( \mathcal{N} = (Q, \Sigma, \delta, I, F) \) be an NFA. Define:

\[
\begin{align*}
  u \preceq^r_{\mathcal{N}} v & \iff post^\mathcal{N}_u(I) \subseteq post^\mathcal{N}_v(I) & \text{Right-Automata-based Quasiorder} \\
  u \preceq^f_{\mathcal{N}} v & \iff prec^\mathcal{N}_u(F) \subseteq prec^\mathcal{N}_v(F) & \text{Left-Automata-based Quasiorder}
\end{align*}
\]

Clearly, the automata-based quasiorders induce a finite number of principals since each principal is represented by a subset of the states of \( \mathcal{N} \).

Remark 11. The pairs of quasiorders \( \preceq^r_L \prec \preceq^f_L \) and \( \preceq^r_{\mathcal{N}} \prec \preceq^f_{\mathcal{N}} \) from Definitions 9 and 10 are dual, i.e. \( u \preceq^r_L v \Leftrightarrow u^R \preceq^f_L v^R \) and \( u \preceq^f_{\mathcal{N}} v \Leftrightarrow u^R \preceq^r_{\mathcal{N}} v^R \).

The following result shows that the principals of \( \preceq^r_{\mathcal{N}} \) and \( \preceq^f_{\mathcal{N}} \) can be described, respectively, as intersections of left and right languages of the states of \( \mathcal{N} \) while the principals of \( \preceq^r_L \) and \( \preceq^f_L \) correspond to intersections of quotients of \( L \).

Lemma 12. Let \( \mathcal{N} = (Q, \Sigma, \delta, I, F) \) be an NFA with \( \mathcal{L}(\mathcal{N}) = L \). Then, for every \( u \in \Sigma^* \),

\[
\begin{align*}
  cl_{\preceq^r_{\mathcal{N}}}(u) & = \bigcap_{q \in post^\mathcal{N}_u(I)} W^\mathcal{N}_{L,q} \\
  cl_{\preceq^f_{\mathcal{N}}}(u) & = \bigcap_{q \in prec^\mathcal{N}_u(F)} W^\mathcal{N}_{q,F} \\
  cl_{\preceq^r_L}(u) & = \bigcap_{w \in \Sigma^*, w \in u^{-1}L} Lw^{-1} \\
  cl_{\preceq^f_L}(u) & = \bigcap_{w \in \Sigma^*, w \in Lv^{-1}} w^{-1}L
\end{align*}
\]

As shown by Ganty et al. [13], given an NFA \( \mathcal{N} \) with \( L = \mathcal{L}(\mathcal{N}) \), the quasiorders \( \preceq^r_L \) and \( \preceq^r_{\mathcal{N}} \) are right \( L \)-preserving quasiorders, while the quasiorders \( \preceq^f_L \) and \( \preceq^f_{\mathcal{N}} \) are left \( L \)-preserving quasiorders. Therefore, by Lemma 4 and 6, our automata constructions applied to these quasiorders yield automata for \( L \).

Finally, as shown by de Luca and Varricchio [6], we have that \( \preceq^r_{\mathcal{N}} \) is finer than \( \preceq^r_L \), i.e., \( \preceq^r_{\mathcal{N}} \subseteq \preceq^r_L \). In that sense we say that \( \preceq^r_{\mathcal{N}} \) approximates \( \preceq^r_L \). As the following lemma shows, the approximation is precise, i.e., \( \preceq^r_{\mathcal{N}} = \preceq^r_L \), whenever \( \mathcal{N} \) is a co-RFA with no empty states.

Lemma 13. Let \( \mathcal{N} \) be a co-RFA with no empty states such that \( L = \mathcal{L}(\mathcal{N}) \). Then \( \preceq^r_{\mathcal{N}} = \preceq^r_L \).

Similarly, if \( \mathcal{N} \) is an RFA with no unreachable states and \( L = \mathcal{L}(\mathcal{N}) \) then \( \preceq^f_{\mathcal{N}} = \preceq^f_L \).
The upper part of the diagram follows from Theorem 15 (f), the squares follow from Theorem 15 (c) and the bottom curved arc follows from Theorem 15 (b). Incidentally, the diagram shows a new relation which is a consequence of the left-right dualities between \( \preceq^L \) and \( \preceq^R \), and \( \preceq^L \) and \( \preceq^R \): \( \text{Can}^f(L(N^R)) \) is isomorphic to \( \text{Res}^r(\text{Res}^r(N^R)) \).

**Figure 1** Relations between the constructions \( \text{Res}^r, \text{Res}^s, \text{Can}^r \) and \( \text{Can}^f \). Note that constructions \( \text{Can}^r \) and \( \text{Can}^f \) are applied to the language accepted by the automaton in the origin of the labeled arrow while constructions \( \text{Res}^r \) and \( \text{Res}^s \) are applied directly to the automaton.

## 4.1 Automata Constructions

In what follows, we will use \( \text{Can}^r, \text{Can}^f \) and \( \text{Res}^s, \text{Res}^f \) to denote the constructions \( H^r, H^f \) when applied, respectively, to the language-based quasiorders induced by a regular language and the automata-based quasiorders induced by an NFA.

**Definition 14.** Let \( N \) be an NFA accepting the language \( L = \mathcal{L}(N) \). Define:

\[
\text{Can}^r(L) \overset{\text{def}}{=} H^r(\preceq^r_L, L) \quad \text{and} \quad \text{Res}^r(N) \overset{\text{def}}{=} H^r(\preceq^r_N, L).
\]

\[
\text{Can}^f(L) \overset{\text{def}}{=} H^f(\preceq^f_L, L) \quad \text{and} \quad \text{Res}^f(N) \overset{\text{def}}{=} H^f(\preceq^f_N, L).
\]

Given an NFA \( N \) accepting the language \( L = \mathcal{L}(N) \), all constructions in the above definition yield automata accepting \( L \). However, while the constructions using the right quasiorders result in RFAs, those using left quasiorders result in co-RFAs. Furthermore, it follows from Remark 11 and Lemma 7 that \( \text{Can}^f(L) \) is isomorphic to \( (\text{Can}^r(L^R))^R \) and \( \text{Res}^f(N) \) is isomorphic to \( (\text{Res}^r(N^R))^R \).

It follows from Theorem 8 that the automata \( \text{Res}^r(N) \) and \( \text{Res}^f(N) \) have more states than \( \text{Can}^r(L) \) and \( \text{Can}^f(L) \), respectively. Intuitively, \( \text{Can}^r(L) \) is the minimal RFA for \( L \), i.e. it is isomorphic to the canonical RFA for \( L \), since \( \preceq^r_L \) is the coarsest right \( L \)-preserving quasiorder [6]. On the other hand, as we evidenced in Example 17, \( \text{Res}^r(N) \) is a sub-automaton of \( N^{\text{res}} \) [9] for every NFA \( N \).

Finally, it follows from Lemma 13 that residualizing \( \text{Res}^r \) a co-RFA with no empty states \( (\text{Res}^f(N)) \) results in the canonical RFA for \( \mathcal{L}(N) \) \( \text{Can}^r(\mathcal{L}(N)) \).

We formalize all these notions in Theorem 15. Figure 1 summarizes all these connections between the automata constructions given in Definition 14.

**Theorem 15.** Let \( N \) be an NFA with \( L = \mathcal{L}(N) \). Then the following properties hold:

\begin{itemize}
  \item[(a)] \( \mathcal{L}(\text{Can}^r(L)) = \mathcal{L}(\text{Can}^f(L)) = \mathcal{L}(\text{Res}^s(N)) = \mathcal{L}(\text{Res}^f(N)) \).
  \item[(b)] \( \text{Can}^f(L) \) is isomorphic to \( (\text{Can}^r(L^R))^R \).
  \item[(c)] \( \text{Res}^r(N) \) is isomorphic to \( (\text{Res}^r(N^R))^R \).
  \item[(d)] \( \text{Can}^r(L) \) is isomorphic to the canonical RFA for \( L \).
  \item[(e)] \( \text{Res}^r(N) \) is isomorphic to a sub-automaton of \( N^{\text{res}} \) and \( \mathcal{L}(\text{Res}^r(N)) = \mathcal{L}(N^{\text{res}}) = L \).
  \item[(f)] \( \text{Res}^r(\text{Res}^r(N)) \) is isomorphic to \( \text{Can}^r(L) \).
\end{itemize}

Let \( N \) be an NFA with \( L = \mathcal{L}(N) \). If \( \preceq^r_L = \preceq^r_N \) then the automata \( \text{Can}^r(L) \) and \( \text{Res}^r(N) \) are isomorphic. The following result shows that the reverse implication also holds.

**Lemma 16.** Let \( N \) be an NFA with \( L = \mathcal{L}(N) \). Then \( \preceq^r_L = \preceq^r_N \) iff \( \text{Res}^r(N) \) is isomorphic to \( \text{Can}^r(\mathcal{L}(N)) \).
The following example illustrates the differences between our residualization operation, \( \text{Res}^r(\mathcal{N}) \), and the one defined by Denis et al. [9], \( \mathcal{N}^\text{res} \), on a given NFA \( \mathcal{N} \): the automaton \( \text{Res}^r(\mathcal{N}) \) has, at most, as many states as \( \mathcal{N}^\text{res} \). This follows from the fact that for every \( u \in \Sigma^* \), if \( \text{post}^N_u(I) \) is coverable then \( \text{cl}_{<^\mathcal{N}}(u) \) is composite but not vice-versa.

**Example 17.** Let \( \mathcal{N} = (Q, \Sigma, \delta, I, F) \) be the automaton on the left of Figure 2 and let \( L = \mathcal{L}(\mathcal{N}) \). To build \( \mathcal{N}^\text{res} \) we compute \( \text{post}^N_w(I) \), for all \( u \in \Sigma^* \). Let \( C \equiv \mathcal{L}^c \setminus \{ \varepsilon, a, b, c \} \).

\[
\begin{align*}
\text{post}^N_\varepsilon(I) &= \{0\} & \text{post}^N_\varepsilon(I) &= \{1, 2\} & \forall w \in L, \text{post}^N_w(I) &= \{5\} \\
\text{post}^N_I(I) &= \{1, 2, 3, 4\} & \text{post}^N_I(I) &= \{1, 3\} & \forall w \in C, \text{post}^N_w(I) &= \emptyset 
\end{align*}
\]

Since none of these sets is coverable by the others, they are all states of \( \mathcal{N}^\text{res} \). The resulting RFA \( \mathcal{N}^\text{res} \) is shown in the center of Figure 2. On the other hand, let us denote \( \text{cl}_{<^\mathcal{N}} \) simply by \( \text{cl} \). In order to build \( \text{Res}^r(\mathcal{N}) \) we need to compute the principals \( \text{cl}(u) \), for all \( u \in \Sigma^* \). By definition of \( <^\mathcal{N} \), we have that \( w \in \text{cl}(u) \iff \text{post}^N_w(I) \subseteq \text{post}^N_u(I) \). Therefore, we obtain:

\[
\text{cl}(c) = \{c\} \quad \text{cl}(a) = \{a, c\} \quad \text{cl}(b) = \{b, c\} \quad \text{cl}(c) = \{c\} \quad \forall w \in L, \text{cl}(w) = L \quad \forall w \in C, \text{cl}(w) = \Sigma^* 
\]

Since \( a <^\mathcal{N} c \), \( b <^\mathcal{N} c \) and \( \forall w \in \Sigma^* \), \( cw \subseteq L \iff (aw \subseteq L \lor bw \subseteq L) \), it follows that \( \text{cl}(c) \) is \( L \)-composite. The resulting RFA \( \text{Res}^r(\mathcal{N}) \) is shown on the right of Figure 2.

### 5 Double-Reversal Method for Building the Canonical RFA

Denis et al. [9] show that their residualization operation satisfies the residual-equivalent of the double-reversal method for building the minimal DFA. More specifically, they prove that if an NFA \( \mathcal{N} \) is a co-RFA with no empty states then their residualization operation applied to \( \mathcal{N} \) results in the canonical RFA for \( \mathcal{L}(\mathcal{N}) \). As a consequence, \( (\text{Res}^r(\mathcal{N}^R))^{\text{res}} \) is the canonical RFA for \( \mathcal{L}(\mathcal{N}) \).

In this section, we first show that the residual-equivalent of the double-reversal method holds within our framework, i.e. \( \text{Res}^r((\text{Res}^r(\mathcal{N}^R))^{\text{res}}) \) is isomorphic to \( \text{Can}^r(\mathcal{N}) \). Then, we generalize this method along the lines of the generalization of the double-reversal method for building the minimal DFA given by Brzozowski and Tamm [5]. To this end, we extend our previous work [11] in which we provided a congruence-based perspective on the generalized double-reversal method for DFAs. By moving from congruences to quasiorders, we find a necessary and sufficient condition on an NFA \( \mathcal{N} \) so that \( \text{Res}^r(\mathcal{N}) \) yields the canonical RFA for \( \mathcal{L}(\mathcal{N}) \). Finally, we compare our generalization with the one given by Tamm [16].

#### 5.1 Double-reversal Method

We give a simple proof of the double-reversal method for building the canonical RFA.
Theorem 18 (Double-Reversal). Let $\mathcal{N}$ be an NFA. Then $\text{Res}'((\text{Res}^r(\mathcal{N}^r))^{r})$ is isomorphic to the canonical RFA for $L(\mathcal{N})$.

Proof. It follows from Theorem 15 (c), (d) and (f).

Note that Theorem 18 can be inferred from Figure 1 by following the path starting at $\mathcal{N}$, labeled with $R - \text{Res}' - R - \text{Res}'$ and ending in $\text{Can}^r(L(\mathcal{N}))$.

5.2 Generalization of the Double-reversal Method

Next we show that residualizing an automaton yields the canonical RFA iff the left language of every state is closed w.r.t. the right Nerode quasiorder.

Theorem 19. Let $\mathcal{N} = (Q, \Sigma, \delta, I, F)$ be an NFA with $L = L(\mathcal{N})$. Then $\text{Res}'(\mathcal{N})$ is the canonical RFA for $L$ iff $\forall q \in Q$, $\text{cl}_{q}^L(W_{I,q}^N) = W_{I,q}^N$.

Proof. We first show that $\forall q \in Q$, $\text{cl}_{q}^L(W_{I,q}^N) = W_{I,q}^N$ is a necessary condition, i.e. if $\text{Res}'(\mathcal{N})$ is the canonical RFA for $L$ then $\forall q \in Q$, $\text{cl}_{\text{Res}^r}^L(W_{I,q}^N) = W_{I,q}^N$ holds. By Lemma 16 we have that if $\text{Res}'(\mathcal{N})$ is the canonical RFA then $\preceq_{L}^r = \preceq_{\mathcal{N}}^r$. Moreover,

$$\text{cl}_{q}^L(W_{I,q}^N) = \{w \in \Sigma^* | \exists u \in W_{I,q}^N, u^{-1}L \subseteq w^{-1}L\} = \{w \in \Sigma^* | \exists u \in W_{I,q}^N, \text{post}_{q}^N(I) \subseteq \text{post}_{q}^N(I)\} \subseteq \{w \in \Sigma^* | \exists u \in W_{I,q}^N, \text{post}_{q}^N(I) \subseteq \text{cl}_{q}^L(W_{I,q}^N)\}$$

By reflexivity of $\preceq_{L}^r$, we conclude that $\text{cl}_{q}^L(W_{I,q}^N) = W_{I,q}^N$.

Next, we show that $\forall q \in Q$, $\text{cl}_{q}^L(W_{I,q}^N) = W_{I,q}^N$ is also a sufficient condition. By Lemma 12 and condition $\forall q \in Q$, $\text{cl}_{q}^L(W_{I,q}^N) = W_{I,q}^N$, we have that

$$\text{cl}_{q}^N(u) = \bigcup_{q \in \text{post}_{u}^N(I)} W_{I,q}^N = \bigcup_{q \in \text{post}_{u}^N(I)} \text{cl}_{q}^L(W_{I,q}^N).$$

Since $u \in \text{cl}_{q}^L(W_{I,q}^N)$ for all $q \in \text{post}_{u}^N(I)$, it follows that $\text{cl}_{q}^L(u) \subseteq \text{cl}_{q}^L(W_{I,q}^N)$ for all $q \in \text{post}_{u}^N(I)$ and, since $\text{cl}_{q}^N(u) = \bigcup_{q \in \text{post}_{u}^N(I)} \text{cl}_{q}^L(W_{I,q}^N)$, we have that $\text{cl}_{q}^L(u) \subseteq \text{cl}_{q}^N(u)$ for every $u \in \Sigma^*$, i.e., $\preceq_{L}^r \subseteq \preceq_{\mathcal{N}}^r$.

On the other hand, as shown by de Luca and Varricchio [6], we have that $\preceq_{\mathcal{N}}^r \subseteq \preceq_{L}^r$. We conclude that $\preceq_{\mathcal{N}}^r = \preceq_{L}^r$, hence $\text{Res}'(\mathcal{N}) = \text{Can}^r(L)$.

It is worth to remark that Theorem 19 does not hold when considering the residualization operation $N_{\mathcal{N}}$ of Denis et al. [9] instead of $\text{Res}'(\mathcal{N})$. As a counterexample we have the automata $\mathcal{N}$ in Figure 2 where $\text{Res}'(\mathcal{N})$ is the canonical RFA for $L(\mathcal{N})$, hence $\mathcal{N}$ satisfies the condition of Theorem 19, while $N_{\mathcal{N}}$ is not canonical.

Co-atoms and co-rests

The condition of Theorem 19 is analogue to the one we gave for building the minimal DFA [11], except that the later is formulated in terms of congruences instead of quasiorders. In that case we proved that determinizing a given NFA $\mathcal{N}$ yields the minimal DFA iff $\text{cl}_{q}^L(W_{I,q}^N) = W_{I,q}^N$ for every state $q$ of $\mathcal{N}$, where $\sim_{L}^r = \preceq_{L}^r \cap (\preceq_{L}^r)^{-1}$ is the right Nerode’s congruence [7].

Moreover, we showed that the principals of $\sim_{L}^r$ coincide with the so-called co-atoms [11], which are non-empty intersections of complemented and uncomplemented right quotients of the language. This allowed us to connect our result for DFAs [11] with the generalization...
of the double-reversal method for building the minimal DFA proposed by Brzozowski and Tamm [5], who establish that determinizing an NFA $N$ yields the minimal DFA for $L(N)$ if and only if the left languages of the states of $N$ are unions of co-atoms of $L(N)$.

Next, we give a formulation of the condition from Theorem 19 along the lines of the one given by Brzozowski and Tamm [5] for their generalization of the double-reversal method for building the minimal DFA.

To do that, let us call the intersections used in Lemma 12 to describe the principals of $\lesssim_L$ and $\lesssim_L'$ as rests and co-rests of $L$, respectively. As shown by Theorem 19, residualizing an NFA $N$ yields the canonical RFA for $L(N)$ if and only if the left language of every state of $N$ satisfies $\text{cl}_{\lesssim_L}(W^N_{I,q}) = W^N_{I,q}$. By definition, $\text{cl}_{\lesssim_L}(S) = S$ if the left languages of the states of $N$ are union of principals of $\lesssim_L$ which, by Lemma 12 are the co-rests of $L$.

Therefore we derive the following statement, equivalent to Theorem 19, that we consider as the residual-equivalent of the generalization of the double-reversal method for building the minimal DFA proposed by Brzozowski and Tamm [5].

> Corollary 20. Let $N = (Q, \Sigma, \delta, I, F)$ be an NFA with $L = L(N)$. Then $\text{Res}^r(N)$ is the canonical RFA for $L$, if and only if the left languages of $N$ are union of co-rests.

**Tamm’s Generalization of the Double-reversal Method for RFAs**

Tamm [16] generalized the double-reversal method of Denis et al. [9] by showing that $N^{\text{res}}$ is the canonical RFA for $L(N)$ iff the left languages of $N$ are union of the left languages of the canonical RFA for $L(N)$.

In this section, we compare the generalization of Tamm [16] with ours. The two approaches differ in the definition of the residualization operation they consider and, as the following lemma shows, the sufficient and necessary condition from Theorem 19 is more general than that of Tamm [16, Theorem 4].

> Lemma 21. Let $N = (Q, \Sigma, \delta, I, F)$ be an NFA and let $C = \text{Can}^r(\lesssim_L, L) = (\tilde{Q}, \Sigma, \tilde{\delta}, \tilde{I}, \tilde{F})$ be the canonical RFA for $L = L(N)$. If $W^N_{I,q} = \bigcup_{q \in \tilde{Q}} W^C_{I,q}$ then $\text{cl}_{\lesssim_L}(W^N_{I,q}) = W^N_{I,q}$.

**Proof.** Since the canonical RFA, $C$, is strongly consistent then $\lesssim_C = \lesssim_L$ [12], hence $\text{Res}^r(C)$ is isomorphic to $\text{Can}^r(L)$. It follows from Theorem 19 that $\text{cl}_{\lesssim_L}(W^C_{I,q}) = W^C_{I,q}$ for every $q \in \tilde{Q}$. Therefore,

$$\text{cl}_{\lesssim_L}(W^N_{I,q}) = \text{cl}_{\lesssim_L}(W^C_{I,q}) = W^C_{I,q}$$

Observe that, since the canonical RFA $C = (\tilde{Q}, \Sigma, \tilde{\delta}, \tilde{I}, \tilde{F})$ for a language $L$ is strongly consistent, the left language of each state is a principal of $\text{cl}_{\lesssim_L}$. In particular, if the right language of a state is $u^{-1}L$ then its left language is the principal $\text{cl}_{\lesssim_L}(u)$. Therefore, if $W^N_{I,q} = \bigcup_{q \in \tilde{Q}} W^C_{I,q}$ then $W^N_{I,q}$ is a closed set in $\text{cl}_{\lesssim_L}$. However, the reverse implication does not hold since only the $L$-prime principals are left languages of states of $C$.

On the other hand, $L$-composite principals for $\lesssim_L$ can be described as intersections of $L$-prime principals [12]. As a consequence, $\text{Res}^r(N)$ is isomorphic to $C$ iff the left languages of states of $N$ are union of non-empty intersections of left languages of $C$, while, as shown by Tamm [16], $N^{\text{res}}$ is isomorphic to $C$ iff the left languages of the states of $N$ are union of left languages of $C$. 

6 Learning Residual Automata

Bollig et al. [3] devised the NL* algorithm for learning the canonical RFA for a given regular language. The algorithm describes the behavior of a Learner that infers a language L by performing membership queries on L (which are answered by a Teacher) and equivalence queries between the language accepted by a candidate automaton and L (which are answered by an Oracle). The algorithm terminates when the Learner builds an RFA accepting the language L.

In this section we present a quasiorder-based perspective on the NL* algorithm in which the Learner iteratively refines a quasiorder \( \preceq \) on \( \Sigma^* \) by querying the Teacher and uses an adaption of the automata construction \( H^r(\preceq, L) \) from Definition 3 to build an automaton that is used to query the Oracle. We capture this approach in the so-called NL签字alm algorithm whose pseudocode we defer to the extended version of this paper [12]. Here we give the definitions and general steps of the NL签字alm algorithm.

The Learner maintains a prefix-closed finite set \( \mathcal{P} \subseteq \Sigma^* \) and a suffix-closed finite set \( \mathcal{S} \subseteq \Sigma^* \). The set \( \mathcal{S} \) is used to approximate the principals in \( \preceq_L^r \) for the words in \( \mathcal{P} \). In order to manipulate these approximations, we define the following two operators.

**Definition 22.** Let \( L \) be a language, \( \mathcal{S} \subseteq \Sigma^* \) and \( u,v \in \Sigma^* \). Then \( u^{-1}L =_{\mathcal{S}} v^{-1}L \iff (u^{-1}L \cap \mathcal{S}) = (v^{-1}L \cap \mathcal{S}) \). Similarly, \( u^{-1}L \subseteq_{\mathcal{S}} v^{-1}L \iff (u^{-1}L \cap \mathcal{S}) \subseteq (v^{-1}L \cap \mathcal{S}) \).

These operators allow us to define a version of Nerode’s quasiorder restricted to \( \mathcal{S} \).

**Definition 23 (Right-language-based quasiorder w.r.t. \( \mathcal{S} \)).** Let \( L \) be a language, \( \mathcal{S} \subseteq \Sigma^* \) and \( u,v \in \Sigma^* \). Define \( u \preceq_{L_S}^r v \iff u^{-1}L \subseteq_{\mathcal{S}} v^{-1}L \).

Recall that the Learner only manipulates the principals for the words in \( \mathcal{P} \). Therefore, we need to adapt the notion of composite principal for \( \preceq_{L_S}^r \).

**Definition 24 (\( L_S \)-Composite Principal w.r.t. \( \mathcal{P} \)).** Let \( \mathcal{P}, \mathcal{S} \subseteq \Sigma^* \) with \( u \in \mathcal{P} \) and let \( L \subseteq \Sigma^* \) be a language. We say that the principal \( \text{cl}_{\preceq_{L_S}^r}(u) \) is \( L_S \)-composite w.r.t. \( \mathcal{P} \) iff \( u^{-1}L =_{\mathcal{S}} \bigcup_{x \in \mathcal{P}, x \prec_{L_S} L} x^{-1}L \). Otherwise, we say it is \( L_S \)-prime w.r.t. \( \mathcal{P} \).

The Learner uses the quasiorder \( \preceq_{L_S}^r \) to build an automaton by adapting the construction from Definition 3 in order to use only the information that is available by means of the sets \( \mathcal{S} \) and \( \mathcal{P} \). Building such an automaton requires the quasiorder to satisfy two conditions: it must be closed and consistent w.r.t. \( \mathcal{P} \).

**Definition 25 (Closedness and Consistency of \( \preceq_{L_S}^r \) w.r.t. \( \mathcal{P} \)).**

(a) \( \preceq_{L_S}^r \) is closed w.r.t. \( \mathcal{P} \) iff \( \forall u \in \mathcal{P}, a \in \Sigma, \text{cl}_{\preceq_{L_S}^r}(ua) \) is \( L_S \)-prime w.r.t. \( \mathcal{P} \) \( \Rightarrow \exists v \in \mathcal{P}, \text{cl}_{\preceq_{L_S}^r}(v) \).

(b) \( \preceq_{L_S}^r \) is consistent w.r.t. \( \mathcal{P} \) iff \( \forall u,v \in \mathcal{P}, a \in \Sigma : u \preceq_{L_S}^r v \Rightarrow ua \preceq_{L_S}^r va \).

At each iteration, the Learner checks whether the quasiorder \( \preceq_{L_S}^r \) is closed and consistent w.r.t. \( \mathcal{P} \). If \( \preceq_{L_S}^r \) is not closed w.r.t. \( \mathcal{P} \), then it finds \( \text{cl}_{\preceq_{L_S}^r}(ua) \) with \( u \in \mathcal{P}, a \in \Sigma \) such that \( \text{cl}_{\preceq_{L_S}^r}(ua) \) is \( L_S \)-prime w.r.t. \( \mathcal{P} \) and it is not equal to some \( \text{cl}_{\preceq_{L_S}^r}(v) \) with \( v \in \mathcal{P} \). Then the Learner adds \( ua \) to \( \mathcal{P} \).

Similarly, if \( \preceq_{L_S}^r \) is not consistent w.r.t. \( \mathcal{P} \), the Learner finds \( u,v \in \mathcal{P}, a \in \Sigma, x \in \mathcal{S} \) such that \( u \preceq_{L_S}^r v \) but \( uax \in L \land vax \notin L \). Then the Learner adds \( ax \) to \( \mathcal{S} \). When the quasiorder \( \preceq_{L_S}^r \) is closed and consistent w.r.t. \( \mathcal{P} \), the Learner builds the automaton \( R(\preceq_{L_S}^r, \mathcal{P}) \).

Definition 26 is an adaptation of the automata construction \( H^r \) from Definition 3. Instead of considering all principals, it considers only those that correspond to words in \( \mathcal{P} \). Moreover,
the notion of L-prrimality is replaced by $L_\mathcal{S}$-primality w.r.t. $\mathcal{P}$, since the algorithm does not manipulate quotients of $L$ by words in $\Sigma^*$ but the approximation through $\mathcal{S}$ of the quotients of $L$ by words in $\mathcal{P}$ (see Definition 22). Note that, if $\mathcal{S} = \mathcal{P} = \Sigma^*$ then $\text{Can}^*(L) = R(\equiv_{L_\mathcal{S}}, \mathcal{P})$.

**Definition 26** (Automata construction $R(\equiv_{L_\mathcal{S}}, \mathcal{P})$). Let $L \subseteq \Sigma^*$ be a language and let $\mathcal{P}, \mathcal{S} \subseteq \Sigma^*$. Define the automaton $R(\equiv_{L_\mathcal{S}}, \mathcal{P}) = (Q, \Sigma, I, F)$ with $Q = \{ \text{cl}_{\equiv_{L_\mathcal{S}}} (u) \mid u \in \mathcal{P}, \text{cl}_{\equiv_{L_\mathcal{S}}} (u) \in L_\mathcal{S} \}$, $I = \{ \text{cl}_{\equiv_{L_\mathcal{S}}} (u) \in Q \mid \varepsilon \in \text{cl}_{\equiv_{L_\mathcal{S}}} (u) \}$, $F = \{ \text{cl}_{\equiv_{L_\mathcal{S}}} (u) \in Q \mid u \in L \}$ and $\delta(\text{cl}_{\equiv_{L_\mathcal{S}}} (u), a) = \{ \text{cl}_{\equiv_{L_\mathcal{S}}} (v) \in Q \mid \text{cl}_{\equiv_{L_\mathcal{S}}} (u)a \subseteq \text{cl}_{\equiv_{L_\mathcal{S}}} (v) \}$ for all $\text{cl}_{\equiv_{L_\mathcal{S}}} (u) \in Q$ and $a \in \Sigma$.

Finally, the Learner asks the Oracle whether $L(R(\equiv_{L_\mathcal{S}}, \mathcal{P})) = L$. If the Oracle answers yes then the algorithm terminates. Otherwise, the Oracle returns a counterexample $w$ for the language equivalence. Then, the Learner adds every suffix of $w$ to $\mathcal{S}$ and repeats the process.

Theorem 27 shows that the NL$\leq$ algorithm exactly coincides with NL$^*$. 

**Theorem 27.** NL$\leq$ builds the same sets $\mathcal{P}$ and $\mathcal{S}$, performs the same queries to the Oracle and the Teacher and returns the same RFA as NL$^*$, provided that both algorithms resolve nondeterminism the same way.

It is worth to remark that, by replacing the right quasiorder $\equiv_{L_\mathcal{S}}$ by the right congruence $\sim_{L_\mathcal{S}} \overset{\text{def}}{=} \equiv_{L_\mathcal{S}} \cap (\equiv_{L_\mathcal{S}})^{-1}$ in the above algorithm (precisely, in Definitions 25 and 26), the resulting algorithm corresponds to Angluin’s L$^*$ algorithm [2]. Note that, in that case, all principals $\text{cl}_{\sim_{L_\mathcal{S}}} (u)$, with $u \in \Sigma^*$, are $L_\mathcal{S}$-prime w.r.t. $\mathcal{P}$.

### 7 Related Work and Conclusions

Denis et al. [9] introduced the notion of RFA and canonical RFA for a language and devised a procedure, similar to the subset construction for DFAs, to build the RFA $\mathcal{N}^{\text{res}}$ from a given automaton $\mathcal{N}$. Furthermore, they showed that $\mathcal{N}^{\text{res}}$ is isomorphic to the canonical RFA $\mathcal{C}$ for $L(\mathcal{N})$ when $\mathcal{N}$ is a co-RFA with no empty states. Later, Tamm [16] showed that $\mathcal{N}^{\text{res}}$ is isomorphic to $\mathcal{C}$ iff the left language of every state of $\mathcal{N}$ is a union of left languages of states of $\mathcal{C}$. This result generalizes the double-reversal method for building the canonical RFA along the lines of the generalization by Brzozowski and Tamm [5] of the double-reversal method for DFAs, which claims that determinizing an automaton $\mathcal{N}$ yields the minimal DFA iff the left language of each state of $\mathcal{N}$ is a union of co-atoms of $L(\mathcal{N})$. Although the two generalizations have a common foundation, the connection between the two results is not immediate.

Recently [11], we offered a congruence-based perspective of the generalized double-reversal method for DFAs and showed that determining an NFA, $\mathcal{N}$, yields the minimal DFA for $L(\mathcal{N})$ iff $\text{cl}_{\sim_{L}} (W_{\mathcal{N}}^N I_q) = W_{\mathcal{N}}^N I_q$. In this paper we extend our previous work and devise quasiorder-based automata constructions that result in RFAs. One of these constructions, when instantiated with the automata-based quasiorder from Definition 10, defines a residualization operation that, given an NFA $\mathcal{N}$, produces the RFA $\text{Res}^*(\mathcal{N})$ with, at most, as many states as $\mathcal{N}^{\text{res}}$, the residualization operation defined by Denis et al. [9]. Observe that if $\mathcal{N}$ is a co-RFA with no empty states then both $\mathcal{N}^{\text{res}}$ and $\text{Res}^*(\mathcal{N})$ are isomorphic to $\mathcal{C}$.

On the other hand, Theorem 19 shows that $\text{Res}^*(\mathcal{N})$ is isomorphic to $\mathcal{C}$ iff $\text{cl}_{\equiv_{L}} (W_{\mathcal{N}}^N I_q) = W_{\mathcal{N}}^N I_q$. We believe that the similarity between the generalizations of the double-reversal methods for DFAs $(\text{cl}_{\sim_{L}} (W_{\mathcal{N}}^N I_q) = W_{\mathcal{N}}^N I_q)$ and for RFAs $(\text{cl}_{\equiv_{L}} (W_{\mathcal{N}}^N I_q) = W_{\mathcal{N}}^N I_q)$ evidences that quasiorders are for RFAs as congruences are for DFAs. Indeed, determining an NFA $\mathcal{N}$ with $L = L(\mathcal{N})$ yields the minimal DFA for $L$ iff $\sim_{\mathcal{N}} = \equiv_{\mathcal{N}}$ [11] and, similarly, when residualizing $\mathcal{N}$ with our residualization operation we obtain the canonical RFA for $L$ iff $\equiv_{\mathcal{N}} = \equiv_{L}$, as shown by Lemma 16.
| Brzozowski and Tamm [5]                  | Ganty et al. [11]                        |
|------------------------------------------|------------------------------------------|
| $N^D \equiv M$                           | $N^D \equiv M$                           |
| iff                                      | iff                                      |
| $\forall q, W_{I,q}^N$ is a union of co-atoms | $\forall q, \text{cl}_{\sim_L}^c(W_{I,q}^N) = W_{I,q}^N$ |

| Tamm [16]                                | Theorem 19                               |
|------------------------------------------|------------------------------------------|
| $N^\text{res} \equiv C$                  | Res$^\text{r}(N)$ $\equiv C$             |
| iff                                      | iff                                      |
| $\forall q, W_{I,q}^N$ is a union of $W_{I,q}^C$ | $\forall q, \text{cl}_{\sim_L}^c(W_{I,q}^N) = W_{I,q}^N$ |

In the diagram: $N$ is an NFA with $L = \mathcal{L}(N)$; $N^D$ is the result of determinizing $N$ with the standard subset construction; $M$ is the minimal DFA for $L$; $C = \text{Can}^r(L)$ is the canonical RFA for $L$ and $N_1 \equiv N_2$ denotes that automaton $N_1$ is isomorphic to $N_2$.

**Figure 3** Summary of the existing results about the generalized double-reversal method for building the minimal DFA (first row) and the canonical RFA (second row) for a given language. The results on the first column are based on the notion of *atoms* of a language while the results on the second column are based on quasiorders.

It is worth to remark that the left languages of the minimal DFA for $L$ are principals of $\sim_L^{\text{r}}$ [11]. Therefore, the condition \( \text{cl}_{\sim_L^{\text{r}}} (W_{I,q}^N) = W_{I,q}^N \) which guarantees that determining $N$ yields the minimal DFA, can be stated as: the left language of each state of $N$ is a union of left languages of states of the minimal DFA. Thus, this characterization is the DFA-equivalent of Tamm’s condition [16] for RFAs.

Figure 3 summarizes the existing results about these double-reversal methods.

Moreover, we support the idea that quasiorders are natural to residual automata by observing that the NL$^*$ algorithm can be interpreted as an algorithm that, at each iteration, refines an approximation of the Nerode’s quasiorder and builds an RFA using our automata construction.

Finally, it is worth to mention that Myers et al. [15] describe different canonical non-determinism automata constructions for a given regular language and show how to obtain the canonical RFA. They do it by first constructing the minimal DFA for the language interpreted in a variety of join-semilattices and then applying a dual equivalence between this variety and the category of closure spaces. In some sense, this already establishes a connection between the class of DFAs and RFAs. Indeed, the same authors [1] use this category-theoretical perspective to address the residual-equivalent of the double-reversal method proposed by Denis et al. [9]. In contrast, this work revisit different methods to construct the canonical RFA relying on the simple notion of quasiorders on words, as a natural extension of our work on congruences for the study of minimization techniques for DFAs.

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