INTEGRABILITY THEOREMS AND CONFORMALLY CONSTANT CHERN SCALAR CURVATURE METRICS IN ALMOST HERMITIAN GEOMETRY

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Abstract. The various scalar curvatures on an almost Hermitian manifold are studied, in particular with respect to conformal variations. We show several integrability theorems, which state that two of these can only agree in the Kähler case. Our main question is the existence of almost Kähler metrics with conformally constant Chern scalar curvature. This problem is completely solved for ruled manifolds and in a complementary case where methods from the Chern–Yamabe problem are adapted to the non-integrable case. Also a moment map interpretation of the problem is given, leading to a Futaki invariant and the usual picture from geometric invariant theory.

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The present paper is devoted to the conformal geometry of almost Hermitian structures, in particular to aspects relating to their scalar curvature.

The necessary background is briefly reviewed in §1. In particular, we recall the Chern connection and its torsion (see [23]) on almost Hermitian manifolds, which reflects also the almost complex structure. From it one derives three Ricci forms and two scalar curvatures: the Hermitian (or Chern) scalar curvature $s^H = 2s^C$ and the third scalar curvature $s$. From the Levi-Civita connection we also have the Riemannian scalar curvature $s^g$ and all three evidently coincide in the Kähler case. In §2 their precise relationship in general is established by careful calculation in local coordinates (see Propositions 2.1, 2.2, 2.3). The formulas generalize those of Gauduchon in the integrable case [22].

These are applied in §3 to prove several new integrability theorems, which assert that when two scalar curvatures coincide we must already be in the Kähler case.
These holds in any dimension when we have a nearly Kähler structure (Corollary 3.5). We also have results in any dimension in the almost Kähler case (see Corollary 3.1 and also Apostolov–Drăghici [6]). The completely general almost Hermitian case is restricted to dimension 4 (Theorem 3.2). We also obtain an interesting result (Corollary 3.7) on 6-dimensional compact non-Kähler, nearly Kähler manifolds: they all have $s^H = 0$.

We then compute in §4 the behaviour of our Ricci forms and scalar curvatures under conformal variations (see Corollaries 4.4 and 4.5). This allows us to prove another integrability theorem (Theorem 4.8) for conformally almost Kähler structures, relating the Hermitian and Riemannian scalar curvature.

In §5 we state the basic problem that will concern us for the rest of the paper: an almost Hermitian structure is \textit{conformally constant} if some conformal variation has constant Hermitian scalar curvature. We first extend the results of Angella–Calamai–Spotti [1] on the Chern–Yamabe problem to the non-integrable case, and show some independent results of interest. In Corollary 5.10 we obtain that every almost Hermitian structure with non-positive fundamental constant (44) is conformally constant. The remaining case is much more difficult. It is not even known in general whether \textit{one can find any conformally constant almost Hermitian structure}. This is our Existence Problem 5.2, where we restrict to the symplectic case.

In §6 we solve this problem for ruled manifolds given by the generalized Calabi construction (Theorem 6.4). Drawing on the fundamental work by Apostolov–Calderbank–Gauduchon–Tønnesen-Friedman [2, 4], the proof is carried out in §6.3 and quickly reduces to an ODE for a metric on the moment polytope, an interval in our case. The main difficulty is to show positivity of the solution and this is done by a careful asymptotic analysis in §6.4. The manifolds thus constructed are new examples of non-Kähler structures of constant Hermitian scalar curvature with positive fundamental constant.

Finally in §7 we give an interpretation of our existence problem in the framework of moment maps (Theorem 7.9). Here we assume a symmetry on the manifold, namely a Hamiltonian vector field which is Killing for some metric.

This leads to the usual existence and uniqueness conjectures in terms of geometric invariant theory, and also to a Futaki invariant (see §7.4). We end in §7.5 with concrete calculations in the toric case.

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1. Preliminaries

Let $(M, J, g)$ be an almost Hermitian manifold of real dimension $m = 2n$. Thus $J: TM \to TM$ is an almost complex structure $J^2 = -1$ that is orthogonal for the Riemannian metric $g$. The associated fundamental form is $F = g(J\cdot, \cdot)$. We usually do not distinguish between the metric and the almost complex structure and write $g_J := F(\cdot, J\cdot)$ for the metric corresponding to $J$. The volume form is $\text{vol}_g = F^n$.

On the complexification $TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$ we consider the $\mathbb{C}$-bilinear extension of $g$, the Hermitian form $h(X \otimes z, Y \otimes w) = z\overline{w}g(X, Y)$, and the restriction of $h$ to $TM$, which we identify with $T^{1,0}$. 


1.1. Complex Coordinates. Let \( z_\alpha \) denote a complex basis of \( T^{1,0} \). Then \( \bar{z}_\alpha \) is the basis of \( T^{0,1} \) obtained by conjugation. The dual basis is denoted \( z^\alpha, \bar{z}^\beta \). The components of the Hermitian form are
\[
h_{\alpha\bar{\beta}} = h(z_\alpha, \bar{z}_\beta).
\]
The transposed inverse of \( h_{\alpha\bar{\beta}} \) is denoted \( h^{\alpha\bar{\beta}} \). Thus \( h_{\alpha\bar{\gamma}} h^{\beta\bar{\gamma}} = \delta^\beta_\alpha \) and \( h_{\alpha\bar{\beta}} = \bar{h}_{\beta\alpha} \). The fundamental form is then \( F = i h_{\alpha\bar{\beta}} z^\alpha \wedge \bar{z}^\beta \). We shall use the Hermitian form to raise and lower tensor indices.

We also get a \( J \)-adapted orthonormal frame \( e_1, e_2 = Je_1, \ldots, e_{m-1}, e_m = Je_{m-1} \) of the real tangent bundle \( TM \) by decomposing \( z_\alpha \) into the real and imaginary part:
\[
z_\alpha = \frac{1}{2}(e_{2\alpha - 1} - i e_{2\alpha})
\]
By convention \( \alpha, \beta, \gamma, \ldots \) range over \( 1, \ldots, n \) while \( i, j, k, \ldots \) range over \( 1, \ldots, m \).

The twisted exterior differential of a \( p \)-form \( \psi \) is defined as (see [19, (1.11.1)])
\[
d^\star \psi = J\psi J^{-1} \psi,
\]
where \( J \) acts on forms by \((J^{-1})^*\) (some authors have a different sign convention).

1.2. Type Decomposition. Let \( E \) be a vector bundle on \( M \) with complex structure \( J^E \). Unless \( E = \mathbb{C} \) the space \( \Omega^p(M; E) \) of \( E \)-valued differential \( p \)-forms has two different type decompositions.

Definition 1.1. A form \( \psi \in \Omega^p(M; E) \) has \( E \)-type \((r, s)\) when \( p = r + s \) and
\[
\sum_{k=1}^{p} \psi_{X_1 \ldots X_k \ldots X_p} = (r - s)J^E(\psi_{X_1 \ldots X_p}) \quad \forall X_i \in TM.
\]
The subspace of forms of \( E \)-type \((r, s)\) is denoted by \( \Omega^{r,s}(M; E) \). We write \( \psi^{r,s} \) for the projection with respect to this direct sum decomposition of \( \Omega^p(M; E) \).

Hence \( \psi \) behaves like an ordinary \((r, s)\)-form, except that it is vector-valued. For example, the Nijenhuis tensor \( N \in \Omega^2(M; TM) \) has \( TM \)-type \((0,2)\).

Lemma 1.2. For a connection with \( \nabla_X J = 0, \nabla_X \psi \) has the same \( E \)-type as \( \psi \).

To understand the \( E \)-type with respect to contractions, let us say that \( \psi \) has ordinary type \((r, s)\) when in a local frame as in Section 1.1 we may write
\[
\psi = \frac{1}{r!s!} \psi_{\alpha_1 \ldots \alpha_r, \beta_1 \ldots \beta_s} z^{\alpha_1} \ldots z^{\alpha_r} \wedge \bar{z}^{\beta_1} \ldots \bar{z}^{\beta_s},
\]
the coefficients being sections of \( E \), anti-symmetric for \( \alpha_1 \ldots \alpha_r \) and for \( \beta_1 \ldots \beta_s \). Thus, the ordinary type behaves as expected under contraction with \((1,0)\) and \((0,1)\)-vector fields. Concerning the \( E \)-type, we have the following observation:

Lemma 1.3. A form \( \psi \) has \( E \)-type \((r, s)\) precisely when it is the sum of an \( E^{1,0} \)-valued form of ordinary type \((r, s)\) and an \( E^{0,1} \)-valued form of ordinary type \((s, r)\).

Finally, in case \( E = TM \) we may use the metric \( g \) to identify \((p + 1)\)-forms with \( TM \)-valued \( p \)-forms. Using the musical isomorphism we get a map
\[
i: \Omega^{p+1}(M) \to \Omega^p(M; TM), \quad i(\phi)_{X_1 \ldots X_p} = \phi_{-X_1 \ldots X_p} \bar{z}^{s}.
\]
Note that \( \phi \in \Omega^{p+1}(M) \) are real forms. From [23] we get a third type decomposition. This has been used by Gauduchon [23] (1.3.2) in the case \( p = 2 \).
Definition 1.4. A \((r + s)\)-form \(\phi\) has real type \((r, s) + (s, r)\) when the complexification of \(\phi\) is a sum of a complex \((r, s)\)-form and (its conjugate) \((s, r)\)-form. We write \(\Omega^{(r, s) + (s, r)}(M)\) for the space of real forms of real type \((r, s) + (s, r)\).

Lemma 1.5. The map (4) identifies the real type decomposition
\[
  \Omega^{(r, s + 1) + (s + 1, r)}(M) = \left[\Omega^{r,s}(M; TM) \oplus \Omega^{s+1,r-1}(M; TM)\right] \cap \Omega^{p+1}(M),
\]
with the \(TM\)-grading. When \(p = n\), we get \(\Omega^{0,n}(M; TM) \cap \Omega^{n+1}(M) = \{0\}\).

Proof. Since both gradings decompose the entire space, it is enough to show an inclusion, which is a straightforward direct verification. \(\square\)

1.3. Traces. We identify \(A \in \text{End}_C(TM \otimes \mathbb{C})\) with the \((0, 2)\)-tensor
\[
  \phi_{X Y} = g(A(X), Y).
\]
Then \(A\) is skew-Hermitian when \(\phi\) is a 2-form and complex linear for \(J\) when \(\phi\) is \(J\)-invariant. The (Lefschetz) trace of \(\phi\) is
\[
  \Lambda(\phi) = \frac{1}{2} h(\phi, F) = \frac{i}{2} (\phi^\alpha_\alpha - \phi^\alpha_\bar{\alpha})
\]
using the inner product on \((0, 2)\)-tensors. This is just the Lefschetz trace of the anti-symmetrization of \(\phi\). When \(A\) is skew-Hermitian we thus have \(\Lambda(\phi) = -i \phi^\alpha_\alpha\).

The trace of \(A\) is given by
\[
  \text{tr}_C A = A^\alpha_\alpha = \phi^\alpha_\alpha + \phi^{\bar{\alpha}}_{\bar{\alpha}}.
\]
Then for \(\tilde{A} = -J \circ A\) we have
\[
  \phi^{\alpha}_{\bar{\alpha}} = \frac{1}{2} \text{tr}_C A + i \Lambda(\phi) = \frac{1}{2} \text{tr}_C A + \frac{i}{2} \text{tr}_C(\tilde{A})
\]
Suppose \(A\) is the complexification of \(a \in \text{End}_R(TM)\). Then \(\phi^{\alpha}_{\bar{\alpha}} = \phi^\alpha_{\bar{\alpha}}\) and the general fact \(\phi^{\alpha}_{\bar{\alpha}} = \phi^\alpha_{\bar{\alpha}}\) imply that (6), (7) are real. Moreover, \(\text{tr}_C A = \text{tr}_R a = a_{+}\).

If \(a\) is complex linear for \(J\) then \(\text{tr}_R a = 2 \text{tr}_C a\).

1.4. Norms. Let \(E\) be a complex vector bundle on \(M\) with Hermitian form \(\langle \cdot, \cdot \rangle\). Generalizing the case \(E = \mathbb{C}\), the norm of an \(E\)-valued differential \(p\)-form is
\[
  |\psi|^2_{\Omega^p(M; E)} = \frac{1}{p!} \langle \psi_{i_1 \ldots i_p}, \psi_{i_1 \ldots i_p} \rangle = \frac{1}{p!} g^{i_1j_1} \cdots g^{i_pj_p} \langle \psi_{i_1 \ldots i_p}, \psi_{j_1 \ldots j_p} \rangle.
\]

Unless \(p = 1\) we shall not follow [23] in identifying \(TM\)-valued \(p\)-forms with \((0, p + 1)\)-tensors, since this leads to different conventions for the norm. We will only need (9) in the cases \(E = \mathbb{C}\) and \(E = TM\). When an \(E\)-valued \(p\)-form \(\psi\) is decomposed as a sum of elements (3) then
\[
  |\psi|^2_{\Omega^p(M; E)} = \frac{1}{p! |g|} \langle \psi_{\alpha_1 \ldots \alpha_r, \beta_1 \ldots \beta_s}, \psi^{\bar{\alpha}_1 \ldots \bar{\alpha}_r, \bar{\beta}_1 \ldots \bar{\beta}_s} \rangle.
\]
In particular, the decomposition into \(E\)-type is orthogonal.

Lemma 1.6. Let \(\phi \in \Omega^{(r, r+1) + (r+1, r)}(M)\) be a real form with associated \(TM\)-valued form \(\psi = i(\phi)\) using (4). For the \(\Omega^{2r}(M; TM)\)-norm of the projections we have
\[
  \frac{1}{r! r!} |\psi^{r+1, r-1}|^2 = \frac{1}{(r-1)!(r+1)!} |\psi^{r, r}|^2.
\]
When \(n = 2\), we have \(2|\psi^{2, 0}| = |\psi^{1, 1}|^2\) for every 3-form \(\psi\) (see also [23] (1.3.9)).
1.5. Chern Connection. The almost complex structure is parallel for the Levi-Civita connection $D^\theta$ precisely when $M$ is K"{a}hler. Therefore one considers other metric connections that make $J$ parallel.

**Definition 1.7.** The Chern connection $\nabla$ is the unique Hermitian connection on $TM$ whose $(0,1)$-part is the canonical Cauchy–Riemann operator

$$\bar{\partial}_X Z = [X^{0,1}, Z]^{1,0}, \quad X \in TM, Z \in \mathcal{C}^\infty(M, T^{1,0}).$$

(recall that a Hermitian connection is required to satisfy $\nabla g = 0, \nabla J = 0$.)

Equivalently, the Chern connection is the unique Hermitian connection whose torsion tensor $T_{XY} = \nabla_X Y - \nabla_Y X - [X, Y]$ is $J$-anti-invariant. The decomposition of $T \in \Omega^2(M; TM)$ into $TM$-type is then given by (see [22] p. 272)

$$T^{0,2} = N, \quad T^{1,1} = 0, \quad T^{2,0} = (dF)^{2,0}.$$

Here $dF$ is a $TM$-valued 2-form via $\bar{\partial}$ and we take the $(2,0)$-part of its $TM$-type.

**Remark 1.8.** If $T = 0$ for the torsion of the Chern connection of an almost Hermitian manifold, then $\nabla = D^\theta$. Hence $J$ is parallel for $D^\theta$ and the structure is K"{a}hler.

1.6. Torsion 1-Form. Besides not being integrable, the difficulty in dealing with almost Hermitian manifolds is that the fundamental form is not closed ($dF = 0$ holds when $M$ is almost K"{a}hler). We thus consider the torsion 1-form

$$\theta = \Lambda(dF) = J\delta F.$$

The almost Hermitian structure is Gauduchon if $\delta^\theta \theta = 0$, and is balanced if $\theta = 0$.

It is easy to check that the torsion 1-form $\theta_X = \text{tr}(Z \mapsto T_{XZ})$ is the trace of the torsion tensor of the Chern connection. Thus

$$\theta = T_{\alpha\beta}^\gamma z^\alpha + T_{\alpha\bar{\beta}}^\bar{\gamma} \bar{z}^{\bar{\alpha}}.$$

An equivalent definition is $dF = (dF)_0 + \frac{1}{n} \theta \wedge F$, for the trace-free part $(dF)_0$.

1.7. Ricci Forms. Let $R$ be a 2-form with values in skew-Hermitian endomorphisms of $TM$, for example the curvature tensor $R^\nabla_{XY} = [\nabla_X, \nabla_Y] - [\nabla_X, \nabla_Y]$ of the Chern connection. In the integrable case, the 2-form $R^\nabla$ is $J$-invariant. In general, the complexification of $R$ is not of type $(1,1)$ and has more components

$$R = \left( \frac{1}{2} R_{\alpha\beta\gamma}^\delta z^\alpha \wedge z^\beta + R_{\alpha\beta\gamma}^\delta \bar{z}^\alpha \wedge \bar{z}^\beta + \frac{1}{2} R_{\alpha\bar{\beta}\bar{\gamma}}^\delta \bar{z}^{\bar{\alpha}} \wedge \bar{z}^{\bar{\beta}} \right) \otimes z^\gamma \otimes \bar{z}^\delta.$$

Following [22], we consider three ways to contract the tensor $R$:

**Definition 1.9.** The first (or Hermitian) Ricci form $\rho$ or $R$ is the trace

$$\rho_{XY} = \text{tr}_C(J \circ R_{XY}) = -\Lambda(R_{XY}).$$

The complexification of $\rho$ has components

$$\rho = \frac{i}{2} R_{\alpha\beta\gamma}^\delta z^\alpha \wedge z^\beta + i R_{\alpha\beta\gamma}^\delta \bar{z}^\alpha \wedge \bar{z}^{\bar{\beta}} + \frac{i}{2} R_{\alpha\bar{\beta}\bar{\gamma}}^\delta \bar{z}^{\bar{\alpha}} \wedge \bar{z}^{\bar{\beta}}.$$

The first Ricci form is always closed and, when $J$ is integrable, is of type $(1,1)$. Its cohomology class $2\pi c_1(TM, J)$ is the first Chern class of $M$. 

Definition 1.10. The second Ricci form of $R$ is $r_{XY} = -\Lambda(R_{\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 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Proposition 2.1. We have \( s^C - s = \frac{1}{2}|\theta|^2 + \frac{1}{2}\delta \phi \theta - \frac{9}{2}|t|^2 + \frac{1}{2}|T|^2 \).

Here, \( |T|^2 \) is given by (1) as a \( TM \)-valued 2-form and \( |\theta|^2 \) and \(|t|^2 \) are the usual norms for differential 3-forms. The codifferential \( \delta \phi \) is taken with respect to \( g \).

Proof. Since \( \nabla(J) = 0 \), (22) reduces in complex coordinates to
\[(26) \quad R^\gamma_{\alpha\delta} \alpha \gamma + R^\gamma_{\beta\gamma} \delta = (\nabla_{\alpha} T)_{\gamma} \delta - T_{\beta \gamma} T(\gamma, \alpha) \delta.
\]
(see that \( \nabla \) preserves the type decomposition of \( TM \)-valued forms.)

Choose an orthonormal frame \( e_{2\alpha - 1}, e_{2\alpha} = Je_{2\alpha - 1} \) near \( p \in M \) with \( \nabla_X e_i = 0 \) for all \( X \in T_p M \). Such a frame may be constructed by extending parallelly an adapted orthonormal basis in \( p \) along geodesic rays. We work in the basis \( z_{\alpha} = \frac{1}{4}(e_{2\alpha - 1} - ie_{2\alpha}) \) of \( T^{1,0} \). Evaluating (23) at \( p \) gives
\[D_{\alpha}^g(z_{\alpha}) = D_{\alpha}^g(z_{\alpha}) = -\frac{1}{4}\theta^\beta z_{\beta} - \frac{1}{4}\theta^\beta z_{\beta}, \]
so at \( p \) the codifferential reduces to
\[\frac{1}{2}\delta \phi \theta = -\bar{z}_{\alpha} \theta_{\alpha} - z_{\alpha} \theta_{\alpha} - \frac{1}{2}|\theta|^2. \]
Taking the double trace over (26) we obtain
\[s^C - s = R^\gamma_{\alpha\delta} \alpha \gamma + R^\gamma_{\beta\gamma} \delta = (\nabla_{\alpha} T)_{\gamma} \delta - T_{\beta \gamma} T(\gamma, \alpha) \delta.
\]
(the last equation holds since \( s^C - s \) is real.) In our frame
\[\frac{1}{2}(\nabla_{\alpha} T)_{\gamma} \delta - (\nabla_{\alpha} T)_{\gamma} \delta = -\bar{z}_{\alpha} \theta_{\alpha} - z_{\alpha} \theta_{\alpha} \]
at the point \( p \). Thus
\[s^C - s = \frac{1}{2}(|\theta|^2 + \delta \phi \theta - \frac{1}{2}\left(T^\alpha T(\gamma, \alpha) \gamma + T_{\beta \gamma} T^\beta T(\gamma, \alpha) \delta\right). \]
Now apply the easy identities \( T^\alpha T(\gamma, \alpha) \gamma = T_{\beta \gamma} T^\beta \gamma \) and
\[(27) \quad 9|t|^2 - |T|^2 = 2T_{\alpha \beta \gamma} T^\beta \gamma \gamma = T_{i \alpha} T_{j \alpha} \gamma \gamma. \]

Proposition 2.2. For the Riemannian scalar curvature \( s^g \) we have
\[(28) \quad 2s - s^g = |T|^2 - \frac{9}{2}|t|^2 - 2\delta \phi - |\theta|^2 \]

Proof. This is a similar computation, using normal coordinates \( e_i \) at \( p \in M \). Thus \( [e_i, e_j] = 0 \) and \( D_{e_i}^g(e_j)\big|_p = 0 \). The Riemannian scalar curvature at \( p \) is then
\[s^g(p) = e_i g(D_{e_i}^g e_j, e_i) - e_j g(D_{e_j}^g e_i, e_i) \]
(we omit all summation signs) which, using
\[(29) \quad \nabla_{e_i} e_j = D_{e_i}^g(e_j) + \left(\frac{3}{2} T^k_{i j} - T^k_{j i}\right) e_k \]
from (23), becomes
\[e_i g(\nabla_{e_j} e_j, e_i) + e_j T_{ij} - e_j g(\nabla_{e_i} e_j, e_i) - e_i T_{ij} = 2s(p) + g(\nabla_{e_i} e_i, \nabla_{e_j} e_j) - g(\nabla_{e_j} e_j, \nabla_{e_i} e_i) - 2e_j T_{ij}. \]
Now $\delta \theta(p) = -c_jT_{jii}$ and inserting (29) gives
\[
s^g(p) - 2s(p) = T_{iki}T_{jkj} + \frac{9}{4}t_{jik} + 3t_{ij}T_{ij} - T_{jki}T_{ikj} + 2\delta \theta
\]
Now apply (27) and $|\theta|^2 = T_{iki}T_{jkj}$ to get (28). \hfill \Box

Combining Propositions 2.1 and 2.2 gives (recall $s_H = 2s_C$):

**Corollary 2.3.** $s_H - s^g = -\delta \theta - \frac{27}{2}|t|^2 + 2|T|^2$.

**Remark 2.4.** When $J$ is integrable, we have $9|t|^2 = |T|^2$. In this case (28) reduces to [22, (32)] and Proposition 2.1 specializes to [22, Corollaire 2]:
\[
2s - s^g = \frac{1}{2}|dF|^2 - 2\delta \theta - |\theta|^2
\]
\[
s_H - 2s = |\theta|^2 + \delta \theta
\]
\[
s_H - s^g = \frac{1}{2}|dF|^2 - \delta \theta
\]
(in this case $dF$ is of type $(2,1)+(1,2)$ and so $|(d^{c}F)^{2,0}|_{\Omega^2}^2 = |dF|_{\Omega^2}^2$ by Lemma 1.6)

### 3. Integrability Theorems

In the almost Kähler case, $dF = 0$, Propositions 2.1 and 2.2 immediately imply vanishing theorems. For in this case, $T = N$, $\theta = 0$, and $t = 0$ from (13), (14), and (25), respectively. The formulas then reduce to
\[
2s - s^g = |N|^2, \quad s_H - 2s = |N|^2.
\]

**Corollary 3.1.** [6] On an almost Kähler manifold we have $s^g \leq 2s \leq s_H$ with either equality precisely when $(J,g,F)$ is Kähler.

With some care in dimension four, these conclusions can be extended. Thus, assuming equality of various scalar curvatures on an almost Hermitian manifold will guarantee both the integrability of $J$ and the Kähler condition $dF = 0$.

**Theorem 3.2.** Let $(M, J, g, F)$ be a closed almost Hermitian $4 = 2n$-manifold.

i) $\int_M (2s - s^g + |\theta|^2) \frac{F^n}{n!} \geq 0$.

ii) $\int_M (s_H - s^g) \frac{F^n}{n!} \geq 0$.

iii) $\int_M (s_C - s) \frac{F^n}{n!} \geq 0$.

In any case, equality holds if and only if the structure is Kähler.

**Proof.** Recall from (25) and (13) that
\[
T = N + (d^{c}F)^{2,0}, \quad t = \frac{1}{3}d^{c}F.
\]
Since we are in dimension four, $t = t^{2,0} + t^{1,1}$ by Lemma 1.3. Also for the $\Omega^2(M;TM)$-norm defined in [19], Lemma 1.3 gives
\[
|t|^2_{\Omega^2} = |t^{2,0}|^2 + |t^{1,1}|^2 = 3|t^{2,0}|^2 = \frac{1}{3}|(d^{c}F)^{2,0}|^2.
\]
Combined with \(|t|_{\Omega^3}^2 = \frac{1}{3}|t|_{\Omega^2(M,T,M)}^2\) we get

\[(32)\quad |T|_{\Omega^2}^2 - \frac{9}{2}|t|_{\Omega^3}^2 = |N|^2 + \frac{1}{2}|(d^c F)^{2,0}|^2.\]

Putting this into Proposition \(2.2\) and integrating gives

\[
\int_M (2s - s^g + |\theta|^2) = \int_M \left(|N|^2 + \frac{1}{2}(|d^c F|^{2,0})^2\right) \frac{F^n}{n!} \geq 0
\]

where we use that the integral over \(\delta \theta\) vanishes (since \(\frac{F^n}{n!}\) is the Riemannian volume form). Of course, the left hand side can only vanish when \(N = 0\) and \((d^c F)^{2,0} = 0\).

Then \(T = N + (d^c F)^{2,0} = 0\) so by Remark \(1.8\) we are in the Kähler case. Part ii) is a similar application of Corollary \(2.3\) while iii) uses Proposition \(2.1\).

**Remark 3.3.** When \(M\) is a closed Hermitian manifold (the integrable case), one can deduce Theorem \(3.2\) in any dimension (see \([22, 35]\) or apply the technique above to Remark \(2.4\)). On the other hand, Dabkowski–Lock \([16]\) have examples of non-compact Hermitian with \(s^H = s^g\) which are not Kähler. Do higher-dimensional closed almost Hermitian non-Kähler manifolds with \(s^H = s^g\) exist?

In the conformally almost Kähler case, we will extend ii) to higher dimensions in Theorem \(4.8\) below. We now proceed by proving an ‘opposite’ of Corollary \(3.1\) for nearly Kähler structures.

**Definition 3.4.** An almost Hermitian manifold \((J,g,F)\) is nearly Kähler if

\[
(D_X g) J + (D_Y g) J = 0
\]

where \(D\) is the Levi-Civita connection \([25, 26]\).

It follows from the definition that \(D^g F = \frac{1}{3}dF\). Moreover, \(dF\) is of type \((3,0) + (0,3)\) and \(N = \frac{1}{4}d^c F\) is totally anti-symmetric. In particular, a nearly Kähler manifold is balanced. Furthermore, a nearly Kähler manifold of dimension \(2n = 4\) is Kähler. Also, if the nearly Kähler manifold is Hermitian then it is Kähler. Examples of nearly Kähler manifolds are \(S^6\) with its standard almost-complex structure and metric, \(S^3 \times S^3\) equipped with the bi-invariant almost complex structure and its 3-symmetric almost Hermitian structure and the twistor spaces over Einstein self-dual 4-manifolds, endowed with the anti-tautological almost complex structure (for more details about nearly Kähler manifolds see \([38, 12, 37, 43, 14]\)). We deduce from Propositions \(2.1\) and \(2.2\) that for any nearly Kähler manifold

\[
2s - s^g = -\frac{1}{6}|d^c F|_{\Omega^2}^2, \quad s^H - 2s = -\frac{2}{3}|d^c F|_{\Omega^3}^2.
\]

**Corollary 3.5.** On a nearly Kähler manifold we have \(s^H \leq 2s \leq s^g\) with either equality precisely when \((J,g,F)\) is Kähler.

**Remark 3.6.** This does not contradict Theorem \(3.2\) because in dimension 4 the notions Kähler and nearly Kähler agree.

A feature of nearly Kähler manifolds is that \(dF\) is of constant norm \([30, 13]\). Moreover, Gray proved \([27, \text{Theorem 5.2}]\) that any non-Kähler nearly Kähler manifold of dimension 6 is Einstein of positive (constant) \(s^g\). Furthermore, their first Chern class vanishes. Hence, the Hermitian scalar curvature \(s^H\) of any closed non-Kähler nearly Kähler manifold of dimension 6 vanishes.
Corollary 3.7. On a closed non-Kähler, nearly Kähler manifold \((M, J, g)\) of dimension 6, we have \(s^H = 0\).

4. Conformal Variations

Let \(\tilde{g} = e^{2f}g\) be a conformal variation of the metric along a smooth real-valued function \(f\). Then \((M, J, \tilde{g}, \tilde{F})\) is again an almost Hermitian manifold and we shall be interested in how the associated Chern connection, Ricci forms, and scalar curvatures behave under this variation.

We begin by deriving an alternative expression for the Chern connection:

Lemma 4.1. The Chern connection is given by
\[
h(W, \nabla_X Z) = X^0_1 h(W, Z) + h(W, [X^0_1, Z]) + h([W, X^0_1], Z),
\]
where \(X \in TM\) and \(W, Z \in C^\infty(M, T^{1,0})\).

Proof. \([33]\) is easily seen to define a Hermitian connection whose \((0, 1)\)-part is given by \([12]\). The result then follows from the uniqueness of such a connection. □

Lemma 4.2. \(\tilde{\nabla}_X Z = \nabla_X Z + 2 X^1_0(f) \cdot Z\) for all \(X \in TM, Z \in T^{1,0}\).

Proof. This is an immediate consequence of Lemma 4.1. □

Proposition 4.3. For the curvature tensors of the Chern connection we have
\[
\tilde{R}^\nabla(Z) = R^\nabla(Z) + idf \cdot Z.
\]

Proof. Beginning with Lemma 4.2 a straightforward calculation gives
\[
\tilde{R}^\nabla_{XY} Z = R^\nabla_{XY} Z + 2 (X(Y^{1,0}f) - Y(X^{1,0}f) - [X, Y]^{1,0} f) \cdot Z.
\] □

Corollary 4.4. The conformal variations of the three Ricci forms are given by
\[
\tilde{\rho} = \rho - n \Lambda(dd^c f),
\]
\[
\tilde{r} = r - \Lambda(dd^c f) \cdot F,
\]
\[
\tilde{\sigma} = \sigma - dd^c f.
\]

Proof. Compute the three Ricci forms of the tensor \(dd^c f \otimes F\). □

Corollary 4.5. The conformal variations of the scalar curvatures are
\[
e^{2f}s^C = s^C - n\Lambda(dd^c f),
\]
\[
e^{2f}\tilde{s} = s - \Lambda(dd^c f).
\]

Lemma 4.6. For \(f \in C^\infty(M)\) real we have
\[
-\Lambda(dd^c f) = \Delta^g(f) + g(\theta, df).
\]
Here, \(\Delta^g\) denotes the Hodge–de Rham operator \(\Delta^g(f) = \delta^g df\).
Proof. Let \((e_i)_{i=1,…,m}\) be a \(J\)-adapted orthonormal frame as in (1). Then
\[
\Lambda(\dd c f) = \frac{1}{2} \sum_{i=1}^{m} (\dd c f)(e_i, Je_i)
\]
\[
= \sum_{i=1}^{m} e_i (d^c f (Je_i)) - d^c f (JD_{e_i}^g Je_i) - (d^c f) ((D_{e_i}^g J)e_i)
\]
\[
= - \Delta^g (f) - g(\theta, df).
\]
In the last equality we have used \(\theta = J\delta F = - \sum_{i=1}^{m} g(J(D_{e_i}^g J)e_i, \cdot)\), which is a straightforward consequence of \(D^g g = 0\) and \(F = g(J, \cdot, \cdot)\). □

Corollary 4.7. For the Hermitian scalar curvature of \(\tilde{g} = e^{2f} g = u^{-2} g\) we have
\[
e^{2f} \tilde{s}^H = s^H + m\Delta^g (f) + mg(\theta, df)
\](41)
\[
\tilde{s}^H = u^2 s^H - mu \Delta^g (u) + g(\theta, du) - m|du|^2
\]
(42)
For (42) use \(-e^f \Delta e^{-f} = \Delta f + |df|^2\). When \(J\) is integrable we recover \(22\) (23)).

Theorem 4.8. Let \((M, J, g, F)\) be a closed almost Hermitian manifold of real dimension \(m = 2n\). Assume that \(g\) is conformally almost Kähler. Then
\[
\int_M (s^H - s^g) \frac{F^n}{n!} \geq 0.
\]
Equality holds precisely when \(J\) is integrable and \((J, g, F)\) is Kähler.

Proof. Suppose \((J, \tilde{g} = e^{2f} g)\) is almost Kähler. Then by (30) we have
\[
\tilde{s}^H - \tilde{s}^g = 2|N|_g^2 = 2e^{-2f}|N|_g^2
\]
Since \(\tilde{F} = e^{2f} F\) is closed, we have
\[
0 = d(e^{2f} F) = e^{2f} \left( (dF)_0 + \left(2 df + \frac{\theta}{n-1}\right) \wedge F \right)
\]
for the trace-free part \((dF)_0\). From this we read off the torsion 1-form
\[
\theta = (2 - m) df.
\]
Putting this into (11) and combining with the formula for the conformal variation of the Riemannian scalar curvature (see Besse [11, Theorem 1.159]) we get
\[
e^{2f} \left( \tilde{s}^H - \tilde{s}^g \right) = (s^H - s^g) + (2 - m) \Delta f - (m - 2) |df|^2
\]
(43)
Hence
\[
\int_M 2|N|_g^2 \frac{F^n}{n!} = \int_M (s^H - s^g) \frac{F^n}{n!} - (m - 2) \int_M |df|^2 \frac{F^n}{n!}.
\]
□
5. Conformally Constant Metrics

We shall be concerned with the existence of the following type of metrics:

**Definition 5.1.** An almost Hermitian metric \((J, g, F)\) is **conformally constant** if for some \(f \in C^\infty(M)\) the structure \((\tilde{J}, \tilde{g}, \tilde{F}) := (J, e^{2f}g, e^{2f}F)\) has \(\tilde{s}^H = \text{const.}\)

In Corollary 5.10 we prove a sufficient criterion for \((J, g, F)\) to be conformally constant (non-positive fundamental constant). This can be regarded as a generalization of the Chern–Yamabe problem \([1]\) to the non-integrable case. Thus the problem is divided into the cases \(C(J, [g]) \leq 0\) and \(C(J, [g]) > 0\) according to the fundamental constant. The positive case in the Chern–Yamabe problem is difficult because (41) loses its nice analytic properties stemming from the maximum principle. The question remains open in this case. Restricting to symplectic manifolds, we shall consider instead the following more basic existence problem:

**Existence Problem 5.2.** Let \((M, \omega)\) be a closed symplectic manifold. Does \(M\) admit any almost complex structure \(J\) such that \((J, g, \omega)\) is conformally constant?

**Remark 5.3.** We allow conformal variations because if we fix \(\omega\) the existence of compatible metrics with \(s^H = \text{const}\) is not guaranteed: sometimes one cannot find an extremal Kähler metric \([15]\) and for instance on toric manifolds the existence of extremal Kähler metrics is conjecturally equivalent to the existence of extremal almost-Kähler metrics (see \([18]\) and also for example \([3, 4, 7, 29, 34, 41, 39]\)).

**Chern–Yamabe Problem 5.4.** Given a closed almost Hermitian manifold \(M\), find a conformal structure \((J, e^{2f}g, e^{2f}F)\) of constant Hermitian scalar curvature. In other words, is every \((J, g, F)\) conformally constant?

As shown in (41), the Hermitian scalar curvature transforms by the same formula as in the integrable case. Here we show how to extend the main results of \([1]\) to the non-integrable case, as well as some results of independent interest. We mention also the work \([17]\) where a similar problem for the \(J\)-scalar curvature is studied, which is derived from the Riemannian curvature.

Recall that Gauduchon showed in \([21]\) that every conformal class \([g]\) has a natural base-point \(g_0 = e^{-2f_0}g\). It is characterized by having a co-closed torsion 1-form \(\theta_0\), once we normalize \(g_0\) to unit volume. In terms of the complex Laplacian

\[
L^2(f) := \Delta^g f + g(\theta, df),
\]

this is equivalent to \((L^2)^* e^{(m-2)f_0} = 0\) and \(\int_M e^{-m f_0} \frac{F^n}{n!} = 1\).

**Definition 5.5.** \((J, g_0 := e^{-2f_0}g, F_0 := e^{-2f_0}F)\) is the **Gauduchon metric** in the conformal class \([g]\). The fundamental constant is (see \([11, 15, 9, 20]\))

\[
C(M, J, [g]) := \int_M e^{(2-m)f_0} s^H F_0^n \frac{F^n}{n!} \int_M s^H_0 \frac{F_0^n}{n!}.
\]

In the Hermitian setting, the fundamental constant plays a central role in the Plurigenera Theorem \([20]\) and is closely related to the Kodaira dimension. The different cases in the Chern–Yamabe problem are \(C < 0\), \(C = 0\), and \(C > 0\).

Recall that the **Yamabe constant** is defined in terms of the Riemannian structure

\[
Y[g] := \inf \left\{ \int_M \tilde{s} \text{vol}_{\tilde{g}} \middle| \tilde{g} = e^{2f}g, \int_M \text{vol}_{\tilde{g}} = 1 \right\}.
\]
We remark that Yamabe, Trudinger, Aubin, and Schoen have shown that $[g]$ contains metrics of constant Riemannian scalar curvature $Y[g]$ (see [31] for a full account). From Theorem 3.2 we immediately get (see [5] in the integrable case):

**Proposition 5.6.** In dimension $2n = 4$ we have the estimate

$$Y[g] \leq C(M^4, J, [g])$$

with equality if and only if the Gauduchon metric $(J, g_0, F_0)$ is Kähler of constant scalar curvature.

**Proposition 5.7.** Let $(M^m, J, g, F)$ be a closed almost Hermitian manifold. Then there exists a conformal metric $\tilde{g} \in [g]$ whose Hermitian scalar curvature has the same sign as $C$ at every point (meaning zero when $C = 0$).

**Proof.** The adjoint of the complex Laplacian $L_{g_0}$ of the Gauduchon metric $g_0$ is $\Delta_{g_0} f - g_0(\theta_0, df)$, where we use $\delta_{g_0} \theta_0 = 0$. By the maximum principle $\ker(L_{g_0})^*$ are the constant functions (for more details, see [21]). Hence the equation

$$L_{g_0} f = C(J, [g]) - s_{H_{g_0}}$$

is solvable for $f$, since the right hand side is orthogonal to the constants. Defining $\tilde{g} = e^{2f} g_0$, equation (41) shows $s_{H_{g_0}} = e^{-2f} C(J, [g])$. \qed

**Remark 5.8.** This generalizes [1, Theorem 3.1] to the non-integrable case. It follows that the Chern–Yamabe problem is solvable when $C = 0$. The same conclusion (and same proof) holds for the third scalar curvature, where $C$ is replaced by the integral of the third scalar curvature of the Gauduchon metric $g_0$.

A careful review of the analytic content of the argument given by Angella–Calamai–Spotti for [1, Theorem 4.1] reveals the following statement:

**Theorem 5.9 ([1]).** Let $(M^m, J, g, F)$ be a closed almost Hermitian manifold, and let $S: M \to \mathbb{R}$ be any strictly negative smooth function (not necessarily the scalar curvature). Then the PDE

$$mL^g(f) + S = \lambda e^{2f}$$

has a solution $(\lambda, f) \in \mathbb{R} \times C^\infty(M)$; in fact we must have $\lambda < 0$. The solution is unique up to replacing $(\lambda, f)$ by $(\lambda e^{-2c}, f + c)$ for a constant $c$. Thus by scaling we may solve (45) for any given negative $\lambda$.

**Proof.** This is proven in [1, p. 11] by the continuity method. For their argument it is only important to see that for any solution $f$ of (45) we have $\lambda < 0$. In our general setting, this follows since putting the formula $L^g(f) = e^{-2f_0}L_{g_0}(f)$ into (45) and integrating gives

$$m \int_M (\Delta_{g_0} f + g_0(\theta_0, df)) \frac{F^a_n}{n!} \bigg|_{=0} + \int_M Se^{2f_0} \frac{F^a_n}{n!} \bigg|_{<0} = \lambda \int_M e^{2(f + f_0)} \frac{F^a_n}{n!} \bigg|_{>0}. \quad \square$$

Combining this with Proposition 5.7 and (41) we thus obtain the following generalization of [1, Theorem 4.1]:

**Corollary 5.10.** Every closed almost Hermitian manifold with $C(J, [g]) \leq 0$ is conformally constant (see also Remark 5.8).
Remark 5.11. We refer also to Berger [10] for a related question. When \((J,g)\) is Kähler (or more generally when \([g]\) is a balanced conformal class) he essentially constructs solutions of (45) where \(S\) is the Hermitian scalar curvature of \((J,g)\) and \(\lambda\) is a given non-positive function.

6. Ruled Manifolds

We begin our study of the Existence Problem 5.2 for positive fundamental constant with ruled manifolds. On complex manifolds, \(C(J,[g]) > 0\) implies Kodaira dimension \(-\infty\), by the Gauduchon Plurigenera Theorem [20]. The Kodaira dimension of ruled manifolds is \(-\infty\) (conversely, this however does not imply \(C(J,[g]) > 0\)).

Angella–Calamai–Spotti [1] Section 5 have given first simple examples of Hermitian non-Kähler manifolds of positive constant Hermitian scalar curvature (for instance on the Hopf surface or abstractly by deformations using the implicit function theorem). In this section, we demonstrate the existence of almost Hermitian non-Kähler metrics of positive constant Hermitian scalar curvature (see [12] in the case of extremal Kähler metrics). We mention also Hong’s work [28], which is different in that only the Kähler class is fixed.

6.1. The generalized Calabi construction. Let us briefly review the construction. The reader may consult [2 3 4] for more details and greater generality.

Let \((S,\omega_S)\) be a symplectic manifold. For a torus \(T\) with Lie algebra \(t\) let \(\pi: P \to S\) be a principal \(T\)-bundle with connection \(\theta \in \Omega^1(P; t)\). Assume

\[
d\theta = p \cdot \pi^* \omega_S
\]

for some fixed \(p \in t\). Let \((V,g_V,\omega_V)\) be a toric almost Kähler manifold for the same torus and moment map \(\mu: V \to \Delta \subset t^*\). Pick \(c \in \mathbb{R}\) with (here \(\langle \cdot, \cdot \rangle\) is evaluation)

\[
P(v) := \langle v, p \rangle + c > 0 \quad \forall v \in \Delta.
\]

Given this data, the generalized Calabi construction determines a symplectic structure \(\omega_M\) on the total space of the associated bundle

\[
M := P \times_T V \rightarrow S.
\]

On the free stratum \(V^0 = \mu^{-1}(\Delta^0)\) over the interior of the Delzant polytope let \(\alpha: TV^0 \to t\) be the \(g\)-orthogonal projection onto the orbits. The linear map

\[
T_vP \times T_vV^0 \to t, \quad (X,Y) \mapsto \theta(X) + \alpha(Y)
\]

is invariant under the action of the tangent group and thus induces a 1-form \(\theta^0\) on \(M^0 := P \times_T V^0\). The moment map factors over the projection to \(\mu: M \to \Delta\). Set

\[
\omega_M = P(\mu) \pi^* \omega_S + \langle d\mu \wedge \theta^0 \rangle.
\]

Generally, the \(g\)-orthogonal projection \(\alpha\) is a map \(T_vV \to t/t_v\) up to the isotropy Lie algebra \(t_v\). Since \(d\mu^\xi\) vanishes for \(\xi \in t_v\) the definition of \(\langle d\mu \wedge \theta^0 \rangle\) naturally extends so that \(\omega_M\) is also defined over all of \(M\). [45] implies that \(\omega_M\) is closed.

When \(S\) has an almost Kähler metric \((J_S,g_S,\omega_S)\) we get an almost Kähler metric on \(M\) as follows. Let \(G\) be the metric on \(\Delta^0 \subset t^*\) that turns \(\mu\) into a Riemannian submersion, let \(H\) be the dual metric on the cotangent bundle of \(\Delta^0\). Precomposing with \(\mu\) we obtain pairings \(G_p: t^* \otimes t^* \to \mathbb{R}, H_p: t \otimes t \to \mathbb{R}\) at each \(p \in M\). Then

\[
g_M := P(\mu) \pi^* g_S + G(d\mu \otimes d\mu) + H(\theta^0 \otimes \theta^0).
\]
6.2. Ruled manifolds. We now restrict to $T = S^1$. We shall say that a Hermitian line bundle $L$ with connection has degree $p \in \mathbb{R}$ if $R^L = p\omega$ for the curvature.

Remark 6.2. Modifying $\omega_S$ slightly, such line bundles always exist for closed $S$. Indeed, an arbitrary small perturbation of $\omega_S$ is a symplectic form that represents a rational cohomology class, so some $q\omega_S$ with $q \in \mathbb{Q}$ represents an integer cohomology class (see [24, Observation 4.3]). The corresponding Konstant–Souriau line bundle has the required properties, with $p = 1/q$. Another important class of examples is when $S$ is a Riemann surface. Here, holomorphic line bundles are determined by their degree $p \in \mathbb{Z}$ with $c_1(L) = p[\omega_S]$. Using the $\partial\bar{\partial}$-Lemma we find a Hermitian connection whose curvature 2-form is precisely $p\omega_S$.

Let $V = \mathbb{C}P^1$ with Fubini–Study symplectic form $\omega_{FS}$ and Delzant polytope $\Delta = [0, 1]$. As in (24) choose $c$ with $P(x) := px + c$ positive on $[0, 1]$.

Definition 6.3. The ruled manifold belonging to $(L \to S, c)$ is $M := \mathbb{P}(L \oplus \mathbb{C})$ equipped with the symplectic form $\omega_{M,c}$ from (19).

Hence $M$ is obtained by compactifying each fiber of $L$ to a sphere. Following [4] we assume also that the base $S$ has constant Hermitian scalar curvature.

6.3. Existence problem. Since the scalar curvature is $S^1$-invariant, it is our strategy to consider only conformal variations $u = \varphi \circ \mu$ for $\varphi : [0,1] \to \mathbb{R}^+$. 

Theorem 6.4. Let $(M^m = \mathbb{P}(L \oplus \mathbb{C}), \omega_{M,c})$ be a ruled manifold over a closed Kähler manifold $(S^{m-2}, g_S, \omega_S)$ of constant positive scalar curvature. Choose $u = a\mu + b$ with $a, b > 0$. Rescaling the volume of $S$ if necessary, there exists for $c$ sufficiently large a compatible $S^1$-invariant Kähler metric $g$ on $M$ so that $\tilde{g} := u^{-2}g$ has constant Hermitian scalar curvature (for $\tilde{\omega} = u^{-2}\omega_{M,c}$).

This solves the Existence Problem [5,2] on ruled manifolds. They are examples of almost Hermitian manifolds of positive fundamental constant (see Proposition [5,0] which are not covered by the results of the previous section. Instead of rescaling the base one may also change the Fubini–Study form on the fibers by a fixed factor.

Remark 6.5. By Apostolov–Calderbank–Gauduchon–Tønnesen-Friedman, ruled manifolds with $c$ sufficiently large also admit an extremal metric [4, Theorem 4].

As recalled above in Remark 6.4, compatible $S^1$-invariant metrics on $(\mathbb{C}P^1, \omega_{FS})$ correspond to smooth functions $H : [0,1] \to \mathbb{R}$ satisfying the boundary conditions

\begin{align}
H(0) &= H(1) = 0, \\
H'(0) &= 2 = -H'(1), \\
H(x) &> 0 \quad (0 < x < 1).
\end{align}

(51) (52)

Then $H$ and $g_S$ determine an $\omega_M$-compatible almost Kähler metric $g_M$ via (50). The metric $g$ we seek will be of the form (50) and is hence determined by a function $G^{-1} := H$ satisfying (51), (52). Let $t : V^0 \to S^1$ be the angle coordinate on the
round sphere \( \mathbb{C}P^1 \setminus \{N,S\} \). Let \((x^i)\) be local coordinates on \(S\) over which \(L\) is trivialized. The connection then corresponds to a local 1-form \(A = A_i dx^i\) on the base. We have local coordinates \((x^i, \mu, t)\) on \(M^0\) in which the induced 1-form can be written \(\theta^0 = A + dt\). From (49) we then get the volume form

\[
\text{vol}_M = \frac{\omega_{M,c}^n}{n!} = \frac{1}{n} P(\mu)^{n-1} \text{vol}_S \wedge d\mu \wedge dt.
\]

According to (50), the local expression for the metric \(g\) is:

\[
[P(\mu)g^0_{ij} + H(\mu)A_i(\mu)A_j(\mu)] dx^i dx^j + G(\mu) d\mu dt + 2H(\mu) A_i(x) dx^i dt
\]

From this we see \(d\mu^2 = H(\mu) \frac{dt}{\mu}\). Putting this into the formula \(d(i_{\text{grad}_s} f \text{vol}_M) = -\Delta^g (f) \text{vol}_M\) we get for the Hodge–de Rham Laplacian of the moment map

\[
\Delta^g(\mu) = - \frac{(P^{n-1}H)'(\mu)}{P(\mu)^{n-1}}.
\]

Note also the general formula \(\Delta^g(\varphi \circ \mu) = \varphi'(\mu) \Delta^g \mu - \varphi''(\mu) |d\mu|^2_g\).

We shall refer the analytic part of the proof to the next subsection.

**Proof of Theorem 6.4.** According to Proposition 6.8 below with \(\lambda := b/a\) we find unique \(A, B > 0\) and \(f \in C^\infty([\lambda, 1 + \lambda])\) strictly positive on the interior satisfying (59), (60). Rescaling the volume, we assume that \(B\) is the scalar curvature of \(S\).

By [4, Lemma 9] the Hermitian scalar curvature of \((M, g, \omega_{M,c})\) is (omitting the argument \(\mu\) and where the derivatives are taken as functions of \(x \in [0, 1]\))

\[
s^H = B \frac{P}{P} - \frac{(P^{n-1}H)''}{P^{n-1}}.
\]

Combining (12), (54), and (55) we get for \(\tilde{g} = u^{-2} g\), where \(u = \varphi \circ \mu\):

\[
\tilde{s}^H = \varphi^2 \frac{B}{P} - \varphi^2 \frac{(P^{n-1}H)''}{P^{n-1}} + m \varphi \varphi' \frac{(P^{n-1}H)'}{P^{n-1}} + m \left(\varphi \varphi'' - (\varphi')^2\right) H
\]

If \(\varphi(x) = ax + b\) and defining \(f(x + \lambda) = P(x)^{n-1}H(x)\) we see from (59) that we have found a solution \(H\) to this equation where \(s^H = a^2 A\). Condition (51) is (60) and \(\tilde{s}^H\) is just the positivity of \(f\).

**Proposition 6.6.** For \(c\) sufficiently large, the metric \(\tilde{g}\) on \(M = P(L \oplus \mathbb{C})\) in Theorem 6.4 has positive constant Hermitian scalar curvature. When \(n = \dim_{\mathbb{C}} M = 2\)

\[
C(M, J, [g]) = \frac{2s^H + 8c + 4p}{2c + p}.
\]

**Proof.** The coarea formula applied to the submersion \(\mu : M \to [0, 1]\) and volume form \(\omega^\mu_M/n!\) shows that for measurable \(f : [0, 1] \to \mathbb{R}\) we have (see also [4, p. 17])

\[
\int_M (f \mu) \frac{\omega^\mu_M}{n!} = \int_0^1 f(x) \text{vol}(\mu^{-1} x) dx.
\]

On \(\mu^{-1}(x)\) the form (19) restricts to \(P(x)^{-1} \omega_S\), hence \(\text{vol}(\mu^{-1} x) = P(x)^{n-1} \text{vol}(B)\). Without loss we may suppose \(\text{vol}(B) = 1\). The Kähler metric \((\omega_M, g)\) is Gauduchon and normalizing (11) to unit volume gives

\[
C(M, J, [g]) = \frac{1}{\text{vol}(M)} \int_M s^H \frac{\omega^\mu_M}{n!}.
\]
Recall $P(x) = px + c$. Evaluating using (67) gives
\[ \text{vol}(M) = \int_M \frac{\omega^n}{n!} = \text{vol}(B) \int_0^1 P(x)^{n-1} dx = \frac{(p + c)^n - c^n}{pn} \]
\[ \int_M s^H \frac{\omega^n}{n!} = \int_0^1 \frac{s^H - (P\varphi)''(x)}{P(x)} \text{vol}(\mu^{-1} x) dx \]
\[ = \text{vol}(B) \int_0^1 \left[ s^H - (P\varphi)''(x) \right] P(x)^{n-2} dx \]

Integrating by parts and inserting the boundary condition (51) gives
\[ \int_M s^H \frac{\omega^n}{n!} = s^H \frac{(p + c)^{n-1} - c^{n-1}}{p(n - 1)} - \left[ (P\varphi)'(x)P(x)^{n-2} \right]_0^1 + \int_0^1 (P\varphi)'(x)P(x)^{n-2} dx \]
\[ = s^H \frac{(p + c)^{n-1} - c^{n-1}}{p(n - 1)} + 2c^{n-1} + 2(p + c)^{n-1} + p \int_0^1 \varphi'(x)P(x)^{n-2} dx + \int_0^1 \varphi(x)P(x)^{n-1} dx \]

The last integral is $\geq 0$ since $\varphi$ and $P$ are non-negative on $[0,1]$ by (52), while the second to last integral is of order $O(c^{n-2})$. The result follows. If $n = 2$ then $\int_0^1 (P\varphi)'(x)P(x)^{n-2} dx = 0$ by the boundary conditions, and we obtain (50). \[ \square \]

6.4. **Analytic part.** We will use Landau notation in a narrower sense than usual.

For us $O(c^n)$ stands for an arbitrary Laurent polynomial in $c$ of degree less than $n$ with smooth coefficients (constants unless we are working with functions of $x$).

The reason for this restriction is that we need to ensure the rules
\[ (58) \int O(c^n)dx = O(c^n), \quad \frac{d}{dx}O(c^n) = O(c^n), \quad x^kO(c^n) = O(c^n). \]

Elementary properties of the determinant (Cramer’s rule) show:

**Lemma 6.7.** Let $M_c = (M_c^1, \cdots, M_c^N) \in \mathbb{R}^{N \times N}$ be a matrix whose columns are functions of $c$ satisfying $M_c^j = M_{c,\infty}^j c^{n_j} + O(c^{n_j-1})$. Then for $M_{\infty} = (M_{\infty}^1, \cdots, M_{\infty}^N)$
\[ \det(M_c) = \det(M_{\infty}) + O(c^{n_1 + \cdots + n_N - 1}). \]

In particular, when $M_{\infty}$ is invertible it follows that $M_c$ is invertible for $c$ sufficiently large. Assume $b_c = b_\infty = c^n + O(c^{n-1}) \in \mathbb{R}^N$ and let $x_c$ and $x_\infty$ be the respective solutions to $M_c x = b_c$ and $M_{\infty} x = b_\infty$. Then for the $j$-th entry
\[ x_c^j = x_\infty^j c^{n_j} + O(c^{n_j-1}). \]

To complete the proof of Theorem 6.4 it remains to prove the following:

**Proposition 6.8.** For any $m = 2n \in \mathbb{N}$, $\lambda > 0$, and $P(x) = px + c$ sufficiently large, there exists a unique solution $(A, B, f) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times C^\infty([\lambda, 1 + \lambda])$ of
\[ (59) \quad x^2 f''(x) - mx f'(x) + mf(x) = -A \cdot P(x - \lambda)^{n-1} + B \cdot x^2 \cdot P(x - \lambda)^{n-2} \]
with initial values
\[ (60) \quad f(\lambda) = f(1 + \lambda) = 0 \quad f'(\lambda) = 2c^{n-1} \quad f'(1 + \lambda) = -2(p + c)^{n-1}. \]

Moreover, $f$ is strictly positive on $[\lambda, 1 + \lambda]$.

Care must be taken to prove the positivity since near $\lambda$ the solution functions $f_c$ for $c \to \infty$ could oscillate around zero into the negative, even if the limiting function $f_\infty$ is strictly positive on the interior.
Proof. The corresponding homogeneous equation has solutions \(x, x^m\). Applying ‘reduction of order’ [40, p. 242] we obtain all solutions of (69) in the form
\[
(61) \quad f(x) = (u_1(x) + C)x + (u_2(x) + D)x^m
\]
for arbitrary constants \(C, D\) and where
\[
(62) \quad u_1(x) = \frac{-1}{m-1} \int Q(x)x^{-2}dx \quad u_2(x) = \frac{1}{m-1} \int Q(x)x^{-m-1}dx
\]
are chosen primitives. Since the integrands are Laurent polynomials in \(x\), the singularities are at zero and we conclude that all solutions \(f(x)\) are smooth on \([\lambda, 1 + \lambda]\).

For each parameter \(c\) we consider the initial value problem (60). From (61) we see that these are linear equations for \(\ddot{x} = (A, B, C, D)\). Let \(M_c\) denote the corresponding 4 \(\times\) 4 matrix, so that we are looking for solutions of
\[
(63) \quad M_c \cdot \ddot{x}_c = (0, 0, 2c^{n-1}, -2(p + c)^{n-1})^T.
\]
Once we show that \(M_c\) is invertible, this singles out a unique solution \(f_c\). By Lemma 6.7 to show that \(M_c\) is invertible and to understand the asymptotics of the solution \(\ddot{x}_c\) we need only keep track of the dominant powers of \(c\) in front of the variables. Putting
\[
Q(x) = Bc^{n-2}x^2 - Ac^{n-1} + A \cdot O(c^{n-2}) + O(c^{n-3})
\]
into (62) and remembering (68) gives
\[
(64) \quad f(x) = A \left( \frac{-c^{n-1}}{m} + O(c^{n-2}) \right) + B \left( \frac{-c^{n-2}}{m-2} + O(c^{n-3}) \right) x^2 + Cx + Dx^m.
\]
Inserting this into (60) leads to the matrix
\[
M_c = \begin{pmatrix}
\frac{-c^{n-1}}{m} + O(c^{n-2}) & \frac{-c^{n-2}}{m-2} + O(c^{n-3}) & \lambda^m \\
\frac{-c^{n-2}}{m-2} + O(c^{n-3}) & \frac{-c^n}{n-1} + (1 + \lambda)^2 + O(c^{n-3}) & 1 + \lambda & (1 + \lambda)^m \\
\frac{-c^n}{n-1} + (1 + \lambda) + O(c^{n-3}) & 1 & m\lambda m^{-1}
\end{pmatrix}.
\]
Also \(b_c = (0, 0, 2, -2)c^{n-1} + O(c^{n-2})\). The matrix
\[
M_\infty = \begin{pmatrix}
\frac{-1}{m} & \frac{-1}{m} & \lambda^m \\
\frac{-1}{m} & \frac{-1}{m} & 1 + \lambda & (1 + \lambda)^m \\
0 & 0 & m\lambda m^{-1} \\
0 & 0 & 1 & m(1 + \lambda)^{m^{-1}}
\end{pmatrix}
\]
is invertible. The equation \(M_\infty \cdot x_\infty = (0, 0, 2, -2)^T\) has the solution
\[
x_\infty = (2m\lambda(1 + \lambda), 2(m - 2), 2(1 + 2\lambda), 0)
\]
which by Lemma 6.7 gives us

\[ A_c = 2m\lambda(1 + \lambda) + O(e^{-1}), \quad B_c = 2(m - 2)c + O(e^0), \]
\[ C_c = 2(1 + 2\lambda)e^{n-1} + O(e^{n-2}), \quad D_c = O(e^{n-2}). \]

Putting this into (64) shows

\[ f_c(x + \lambda) = 2e^{n-1}x(1 - x) + O(e^{-2}). \]

Hence

\[ \frac{f'_c(x + \lambda)}{2e^{n-1}} = 1 - 2x + O(e^{-1}) \]

uniformly in \( x \). It follows that for \( c \) sufficiently large \( f_c \) has precisely one extreme point on \([\lambda, 1+\lambda]\) (close to \( 1/2+\lambda \)). From (60) we see that this must be a maximum and hence \( f_c \) is positive on \([\lambda, 1+\lambda]\). \( \square \)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Plot of the solution \( f_c \) for \( p = 3, \lambda = \frac{1}{2}, m = 4, c = 3 \) and of the ‘ideal solution’ at infinity}
\end{figure}

7. Moment Map Setup

In this section we give a moment map interpretation of the Existence Problem 5.2 inspired by the work of Apostolov and Maschler [8]. This leads to the familiar existence and uniqueness conjectures formulated in terms of geometric invariant theory, as well as to a version of the Futaki invariant. Our method applies to closed symplectic manifolds \((M^m, \omega)\) admitting a symmetry given by a Hamiltonian vector field, meaning we assume \( \mathcal{AC}^f(\omega) \neq \emptyset \) below. An example is a manifold with Hamiltonian circle action, as above.

First we recall the general definition (\( g^* \) gets the coadjoint action):

**Definition 7.1.** A symplectic action of a Lie group on a symplectic manifold \((S, \omega)\) is *Hamiltonian* if there exists a \( G \)-equivariant moment map \( \mu: X \to g^* \) with

\[ d\mu^\xi(X) = \omega(X, \xi^*), \quad \forall \xi \in g, X \in TS. \]  

(write \( \mu^\xi = \mu(-)(\xi) \in C^\infty(S) \) and let \( \xi^* \in \mathcal{X}(S) \) be the infinitesimal action.)
7.1. The action. For fixed \( f \in C^\infty(M) \) with \( \int_M f \operatorname{vol} = 0 \) let
\[ u := e^{-nf}, \quad K := \operatorname{grad}_\omega u. \]

**Definition 7.2.** \( \mathcal{AC}^f(\omega) \) is the Fréchet manifold of \( \omega \)-compatible almost complex structures \( J \) satisfying \( \Sigma_K J = 0 \).

We equip \( \mathcal{AC}^f(\omega) \) with the symplectic form
\[ \Omega_J(A, B) = \frac{1}{2} \int_M \operatorname{tr}(J \circ A \circ B)e^{nf} \operatorname{vol}, \quad A, B \in T_J \mathcal{AC}^f(\omega). \]

**Proposition 7.3.** When non-empty, \( \mathcal{AC}^f(\omega) \) is contractible. This is the case precisely when \( K \) is the a Killing vector field for some metric \( g \) on \( M \).

**Proof.** The usual map restricts to a retraction
\[ \mathfrak{Met}^f(M) \to \mathcal{AC}^f(\omega), \quad g \mapsto J_g, \quad \text{where} \quad J_g := A_g |A_g|^{-1}, \quad g(A_g -, -) := \omega \]
on the convex space of metrics \( g \) satisfying \( \Sigma_K g = 0 \). \( \square \)

**Remark 7.4.** As mentioned above, we make the assumption that \( \mathcal{AC}^f(\omega) \) is non-empty. According to Bochner, there are no non-trivial Killing fields in the case of strictly negative Ricci curvature. We have \( s^H = 2|N|^2 + s^g \) and so the results of this section mainly concern the case of positive fundamental constant.

**Definition 7.5.** \( \operatorname{Ham}^f(\omega) \subset \operatorname{Symp}(\omega) \) is the subgroup of Hamiltonian symplectomorphisms \( \phi \) satisfying \( f \circ \phi = f \) (equivalently \( \phi_* \) preserves \( \operatorname{grad}_\omega f \)).

We recall that by definition the Lie algebra of the Hamiltonian symplectomorphisms are the Hamiltonian vector fields \( d\phi = -\iota_X \omega \), where \( \phi \in C^\infty(M) \). The Lie algebra \( \mathfrak{ham}^f(\omega) \) consists of Hamiltonian vector fields with \( [\operatorname{grad}_f, f, X] = 0 \). Let \( C_f^\infty(M) \) be the space of \( \phi \in C^\infty(M) \) with constant Poisson bracket \( \{f, \phi\} \). Then \( \mathfrak{ham}^f(\omega) \) is canonically identified with \( C_f^\infty(M)/\mathbb{R} \). The adjoint action of \( \phi \in \operatorname{Ham}^f(\omega) \) on \( \varphi \in C_f^\infty(M)/\mathbb{R} \) can be written \( (\phi^{-1})^* \varphi \). On \( C_f^\infty(M) \) consider
\[ \langle h_1, h_2 \rangle_{e^{(2+n)f}} = \int_M h_1 h_2 e^{(2+n)f} \operatorname{vol}. \]

It places \( C_f^\infty(M)/\mathbb{R} \) in duality with \( C_{0,f}^\infty(M) := \{ \varphi \in C_f^\infty(M) \mid \langle \varphi, 1 \rangle_{e^{(2+n)f}} = 0 \} \). We have an isomorphism to \( C_{0,f}^\infty(M) \to C_f^\infty(M)/\mathbb{R} \) with inverse
\[ C_f^\infty(M)/\mathbb{R} \to C_{0,f}^\infty(M), \quad \varphi \mapsto \hat{\varphi} := \varphi - \frac{\langle \varphi, 1 \rangle_{e^{(2+n)f}}}{\langle 1, 1 \rangle_{e^{(2+n)f}}} = \varphi - \frac{\int_M \varphi e^{(2+n)f} \operatorname{vol}}{\int_M e^{(2+n)f} \operatorname{vol}}. \]

For \( \varphi, \psi \in C_f^\infty(M) \) note the formula
\[ \langle \hat{\varphi}, \psi \rangle_{e^{(2+n)f}} = \langle \varphi, \hat{\psi} \rangle_{e^{(2+n)f}}. \]

Since \( e^{nf} \operatorname{vol} \) is preserved by \( \phi \), the action of \( \operatorname{Ham}^f(\omega) \) on \( \mathcal{AC}^f(\omega) \) by \( \phi_* \circ J \circ \phi_*^{-1} \) preserves the symplectic form \( \Omega_J \). We will show that it is Hamiltonian.
7.2. Technical preparations.

Lemma 7.6 (see [33 Lemma 1.3]). Let \((J, g, \omega)\) be almost Kähler. Suppose the symplectic gradient \(K = \text{grad}_g u\) is a \(g\)-Killing field. Then the \(J\)-anti-invariant part \((D^gJdu)^J\) is anti-symmetric.

**Proof.** Because \(\mathcal{L}_K\omega = 0\) is automatic, the field \(K = Jdu^J\) is Killing precisely when it is holomorphic. Therefore, combined with the fact that \(D^g\) is torsion-free,
\[
0 = \mathcal{L}_K J = D^g_KJ - [D^gK, J].
\]
So \((D^gK)^J\) is anti-symmetric. \(\square\)

Consider a path \(J_t \in \mathcal{AC}(\omega)\) representing \(\dot{J} = \frac{d}{dt}\big|_0 J_t\). Write \(g_t = \omega(\cdot, J_t)\). The variation of the scalar curvature is given by the Mohsen formula:

**Proposition 7.7** (see [36]), \[\frac{d}{dt}\big|_0 s^H_{g_t} = -\delta J(\delta \dot{J})^b.\]

In this formula the codifferential of an endomorphism \(A\) is defined by

\[
g(\delta A, X) = \delta \langle A, X \rangle + g(A, D^g X), \quad X \in \mathfrak{X}(M)
\]
using the evaluation pairing \(\langle \cdot, \cdot \rangle\). For 1-forms \(\alpha, \beta\) we note also the simple formulas
\[
g(\alpha^2, A(X)) = g(A^\flat, \alpha \otimes X),
\]
\[
\frac{d}{dt}\big|_0 g_t(\alpha, \beta) = -g(\alpha, \dot{J}J\beta),
\]
which are used in the proof of our main technical lemma:

**Lemma 7.8.** For the metrics \(g_t = e^{2f} \omega(\cdot, J_t)\) and any \(h \in C^\infty(M)\) we have
\[
\frac{d}{dt}\big|_0 \int_M s^H_{g_t} he^{(2+n)f} \text{vol} = \int_M g(\dot{J}, D^gJdh^J)e^{nf} \text{vol}.
\]

**Proof.** Recall \(u := e^{-nf}\). By (11) the scalar curvature of the conformal variation is
\[
s^H_{g_t} = e^{-2f} \left(s^H_{g_t} + m\Delta g_t(f)\right) = u^2 s^H_{g_t} - 2u^{2/n} \Delta g_t(u) + 2u^{2/n - 2} |du|^2_{g_t}.
\]
Putting this and \(e^{(2+n)f} = u^{-1-2/n}\) into the left hand side of (73) gives
\[
\frac{d}{dt}\big|_0 \int_M s^H_{g_t} hu^{-1} \text{vol} - 2 \int_M \Delta g_t(u) hu^{-2} \text{vol} - 2 \int_M g_t(du, du) hu^{-3} \text{vol}.
\]
Applying Proposition 7.7 and (72) to the second and third summand we get
\[
\int_M \delta J(\delta \dot{J})^b hu^{-1} \text{vol} + 2 \int_M g(du, \dot{J}Jd(hu^{-2})) \text{vol} + 2 \int_M g(du, \dot{J}Jdu) hu^{-3} \text{vol}
\]
which, in view of (70) and (74) becomes
\[
\int_M g(\dot{J}, D^gJdh^J) + 2du \otimes Jd(hu^{-2}) + 2hu^{-3} du \otimes Jdu^J \text{vol}.
\]

Now expand using the Leibniz rule:
\[
D^gJd(hu^{-1})^J = 2u^{-3} hdu \otimes Jdu^J - u^{-2} du \otimes Jdh^J - u^{-2} dh \otimes Jdu^J + u^{-1} D^g(Jdh^J) - u^{-2} hD^gJdu^J
\]
\[
du \otimes Jd(hu^{-2})^J = u^{-2} du \otimes Jdh^J - 2u^{-3} hdu \otimes Jdu^J
\]
From (71) we see $g(J, du \otimes J dh^J) = g(J, dh \otimes J du^J)$. Moreover, Lemma 7.4 implies $g(J, D^g J du^J) = 0$ since $J$ is symmetric and $J$-anti-invariant, while the $J$-anti-invariant part of $D^g J du^J$ is anti-symmetric. Inserting (75) into (74) and applying these facts then gives the right hand side of (73). \hfill \Box

7.3. Proof of main theorem. We write $g_{f, J} := e^{2f} \omega(\cdot, J \cdot)$ and $g_J := g_{0, J}$. 

**Theorem 7.9.** Let $(M, \omega)$ be a closed symplectic manifold with Hamiltonian vector field $K = \text{grad}_\omega u$, $u = e^{-nf}$, $\int_M f \text{ vol} = 0$, and $AC^f(\omega) \neq \emptyset$. The action of $\text{Ham}^f(\omega)$ on $AC^f(\omega)$ with symplectic form (67) is Hamiltonian with moment map

\begin{equation}
\mu : AC^f(\omega) \times C_{0, f}^\infty(M) \to \mathbb{R}, \quad \mu^h(J) = \int_M s^H_{g_{f, J}} h e^{(2+n)f} \text{ vol}.
\end{equation}

Here the Hermitian scalar curvature of $g_{f, J}$ is viewed as a functional using (68).

Identifying $\text{ham}^f(\omega) = C_g^\infty(M)/\mathbb{R}$ and using (69) we may rewrite

\begin{equation}
\mu : AC^f(\omega) \to (C^\infty(M)/\mathbb{R})^*, \quad \mu(J) = \int_M s^H_{g_{f, J}} h e^{(2+n)f} \text{ vol}.
\end{equation}

**Proof.** We must check that for any tangent vector $\dot{J} \in T_J AC^f(\omega)$ and $h \in C^\infty_{0, f}(M)$

$$
\Omega_J(h_{f, J}^*, \dot{J}) = \partial h^\mu(\dot{J}).
$$

Here $h_{J}^* = -\Sigma_J J \in T_J AC^f(\omega)$ for $Z = \text{grad}_\omega h$ denotes the infinitesimal action of $h$ at the point $J$. In terms of the adjoint of $D^g Z \in \text{End}(TM)$ with respect to $g = g_J$, the infinitesimal action can be rewritten as $h^* J = -J \circ (D^g Z + (D^g Z)^* \circ J)$ and so

$$
\Omega_J(h_{f, J}^*, \dot{J}) = \frac{1}{2} \int_M \left( \text{tr}(D^g Z \circ \dot{J}) + \text{tr}((D^g Z)^* \circ J) \right) e^{nf} \text{ vol} = \int_M \text{tr}(D^g Z \circ \dot{J}) e^{nf} \text{ vol}.
$$

This is the right hand side of (73), as $Z = J dh^J$, while the left hand side of (73) is simply $d\mu^h(\dot{J})$. From $\phi^* g_{f, J} = g_{f, \phi \cdot J}$ we get $\phi^* s_{g_{f, J}}^H = s_{g_{f, \phi \cdot J}}^H$. Now $f \circ \phi = f$ by Definition 7.5 and so $\phi^* \mu(\phi \cdot J) = \mu(J)$, proving that (76) is also equivariant. \hfill \Box

The zeros of the moment map $\mu$ are $J \in AC^f(\omega)$ such that the metric $g_{f, J}$ is of constant Hermitian scalar curvature. The geometric invariant theory formal picture suggests then the existence of a unique almost-Kähler metric in $AC^f(\omega)$ conformal to a constant Hermitian scalar curvature metric, modulo the action of $\text{Ham}^f(\omega)$, in every “stable” “complexified” orbit of the action of $\text{Ham}^f(\omega)$.

**Remark 7.10.** In [8 Remark 1], the zeros of the moment map are metrics $g_{f, J}$ with $s_{g_{f, J}}^H + |N|_{g_{f, J}}^2$ is constant, where $s_{g_{f, J}}^H$ is the Riemannian scalar curvature of $g_{f, J}$.

**Corollary 7.11.** Minima of $\|\mu\|^2$ on $AC^f(\omega)$ are conformally constant metrics.

It may also be of interest to consider critical points of $\|\mu\|^2$. 

From (71) we see $g(J, du \otimes J dh^J) = g(J, dh \otimes J du^J)$. Moreover, Lemma 7.4 implies $g(J, D^g J du^J) = 0$ since $J$ is symmetric and $J$-anti-invariant, while the $J$-anti-invariant part of $D^g J du^J$ is anti-symmetric. Inserting (75) into (74) and applying these facts then gives the right hand side of (73). \hfill \Box
7.4. Futaki invariant. Moment maps lead very generally to a Futaki invariant. In the context of Definition 7.1, this invariant is associated to any Lie subalgebra \( \mathfrak{h} \subset \mathfrak{g} \). Letting \( S^h = \{ p \in S \mid \xi_p^\ast = 0 \ \forall \xi \in \mathfrak{h} \} \) be the \( \mathfrak{h} \)-fixed points, the restriction \( \mu : S^h \to \mathfrak{h}^\ast \) is locally constant. Assuming \( S^h \) is connected, the common value of \( \mu \) is called the Futaki invariant \( F^h \subset \mathfrak{h}^\ast \).

Applied to our situation \( \mathfrak{h} = \mathbb{R} \cdot K \subset \text{ham}^I(\omega) \) corresponding to \( u \in C^\infty(M)/\mathbb{R} \). The \( \mathfrak{h} \)-fixed points are all of \( AC^I(\omega) \). Define \( \mathfrak{h}^\ast = \mathbb{R} \) by evaluating at \( u \).

**Definition 7.12.** For any \( J \in AC^I(\omega) \) the Futaki invariant is given by

\[
F^I(\omega) = \mu^u(J) = \langle \tilde{s}_g^H, u \rangle_{c(z+n)f} = \int_M \tilde{s}_g^H \cdot e^{2f} \text{vol}
\]

The main point of the Futaki invariant, that it is independent of \( J \), is a consequence of the general moment map setup and Proposition 7.3.

**Corollary 7.13.** If \( AC^I(\omega) \neq \emptyset \) then there exists \( J \in AC^I(\omega) \) such that \( g_{f,J} = e^{2f} \omega(\cdot,J\cdot) \) has constant Hermitian scalar curvature if and only if \( F^I(\omega) = 0 \).

Thus if for some \( f \) the Futaki invariant \( F^I(\omega) \) vanishes, we have an affirmative solution to the Existence Problem 5.2.

7.5. The toric case. Let \((M^{2n},\omega)\) be a closed symplectic manifold equipped with an effective Hamiltonian action of a \( n \)-dimensional torus \( T \). Let \( z : M \to \Delta \subset \mathbb{R}^n \) be the moment map, where \( \Delta \) is the Delzant polytope in \( \mathbb{R}^n \) the dual of \( t = \text{Lie}(T) \). Denote by \( \{ u_1, \cdots, u_d \} \) the normals to the polytope \( \Delta \). The action of the torus \( T \) is generated by a family of Hamiltonian vector fields \( \{ K_1, \cdots, K_n \} \) linearly independent on an open set of the \( 2n \)-dimensional symplectic manifold \((M,\omega)\) with \( \omega(K_i, K_j) = 0 \). The symplectic form \( \omega \) and an \( \omega \)-compatible \( T \)-invariant almost Kähler metric \( g \) are given on \( z^{-1} \) of the interior of \( \Delta \) by

\[
\omega = \sum_{i=1}^n dz_i \wedge dt_i,
\]

\[
g = \sum_{i,j=1}^n G_{ij}(z)dz_i \otimes dz_j + H_{ij}(z)dt_i \otimes dt_j + P_{ij}(z)dz_i \otimes dt_j,
\]

where \( G, H \) are symmetric positive definite matrix-valued functions satisfying the compatibility conditions \( GH - P^2 = Id \) and \( HP = P^t H \) (\( P^t \) is the transpose of \( P \)). The coordinates \( z_i \) are the moment coordinates and \( t_i \) are the angle coordinates.

Denote by \( H_{i,j,k} = \frac{\partial^2 H_{ij}}{\partial z_k \partial z_l} \) etc. It is shown \([18] \) and \([32] \) \((4.6)\) that the Hermitian scalar curvature is given by

\[
s^H = -\sum_{i,j=1}^n H_{i,j,i}.
\]
Let \( u = a_1 z_1 + a_2 z_2 + \cdots + a_n z_n + a_{n+1} \) be a Hamiltonian Killing potential \((a_i \text{ are real numbers})\). Then,

\[
J du = \sum_{i,l=1}^{n} a_i P_{i,l} dz_i + a_i H_{i,l} dt_l.
\]

Hence,

\[
dJ du = \sum_{i,l=1}^{n} a_i P_{i,l,j} dz_j \wedge dz_i + a_i H_{i,l,j} dz_j \wedge dt_l.
\]

Recall that \( \Delta g^u = -g(dJ du, \omega) \). We obtain

\[
\Delta g^u = -\sum_{i,j=1}^{n} a_i H_{ij,j}, |du|^2 = \sum_{i,j=1}^{n} a_i a_j H_{ij}.
\]

Hence, the conformal change equation (41) becomes

\[
(78) \quad s^H_{gf,J} = -u^2 n \sum_{i,j=1}^{n} H_{ij,ij} + 2u^{n-2} \sum_{i,j=1}^{n} a_i a_j H_{ij},
\]

where \( s^H_{gf,J} \) is the Hermitian scalar curvature of \( g_{f,J} = e^{2f} \omega(\cdot, J\cdot) \) with \( e^{-nf} = u \).

It is easy to check since \( H \) is symmetric that

\[
\sum_{i,j=1}^{n} (e^{nf} H_{ij})_{,ij} = \sum_{i,j=1}^{n} e^{nf} H_{ij,ij} - 2e^{2nf} a_i a_j H_{ij} + 2e^{3nf} a_i a_j H_{ij}.
\]

We conclude that (78) is equivalent to

\[
(79) \quad e^{(n+2)f} s^H_{gf,J} = -\sum_{i,j=1}^{n} (e^{nf} H_{ij})_{,ij}.
\]

Now, when the \( g \)-orthogonal distribution to the \( T \)-orbits is involutive (this is the case when \( P = 0 \)), \( H \) has to satisfy the boundary conditions in [2, Proposition 1] and hence we can apply [3, Lemma 2] to get

**Proposition 7.14.** For any \( H \) satisfying the boundary conditions [2, Proposition 1] and any affine function \( \xi = \xi(z_1, \ldots, z_n) \),

\[
-\int_{\Delta} \left( \sum_{i,j=1}^{n} (e^{nf} H_{ij})_{,ij} \right) \xi dv = 2 \int_{\partial \Delta} e^{nf} \xi d\mu,
\]

where \( \Delta \) is the polytope and \( \partial \Delta \) its boundary, \( dv = dz_1 \wedge \cdots \wedge dz_n \) and \( d\mu \) is defined by \( u_j \wedge d\mu = -dv \) for any codimension one face with inward normal \( u_j \).

If we suppose that \( s^H_{gf,J} \) is a constant, then (79) becomes using Proposition 7.14

\[
(80) \quad 2e^{(n+2)f} \frac{\int_{\partial \Delta} e^{nf} d\mu}{\int_{\Delta} e^{(n+2)f} dv} = -\sum_{i,j=1}^{n} (e^{nf} H_{ij})_{,ij}
\]

Define the Donaldson–Futaki invariant [18] for any smooth function \( \xi \) to be

\[
\mathcal{F}_{\Delta,f}(\xi) = 2 \int_{\partial \Delta} e^{nf} \xi d\mu - 2\frac{\int_{\partial \Delta} e^{nf} d\mu}{\int_{\Delta} e^{(n+2)f} dv} \int_{\Delta} \xi e^{(n+2)f} dv.
\]
It is straightforward from Proposition 7.14 to conclude that if there exists a solution $H$ (satisfying the boundary conditions) of (80), then $\mathcal{F}_{\Delta, \rho}(\xi) = 0$, for any affine function $\xi = \xi(z_1, \ldots, z_n)$. In fact, the existence of $(J, g, \omega)$ such that $g_{\rho, J}$ is of (positive) constant Hermitian scalar curvature can be related then to a notion of “stability” (see for instance [8, 18]).

**Remark 7.15.** Proposition 7.14 implies that for any toric almost Kähler manifold $(M, J, g, \omega)$ with $P = 0$,

$$C(M, J, [g]) = 2 \frac{\int_{\Delta} C^{nf} d\mu}{\int_{\Delta} e^{(n+2)/4} dv} > 0.$$ 

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