Abstract

The \( k \)-Dirac operator is a first order differential operator which is natural to a particular class of parabolic geometries which include the Lie contact structures. A natural task is to understand the set of local null solutions of the operator at a given point. We will show that this set has a very nice and simple structure, namely we will show that there is a submanifold passing through the point such that any section defined on the submanifold extends locally to an unique null solution of the operator.

This result also indicates that these parabolic geometries are naturally associated to a certain constant coefficient operator which has been studied in Clifford analysis and this is the original motivation for this paper. In order to prove the claim about the set of initial conditions for the \( k \)-Dirac operator we will adapt some parts of the theory of exterior differential systems and the Cartan-Kähler theorem to the setting of differential operators which are natural to geometric structures that are equipped with a filtration of the tangent bundle.

Keywords: Cartan-Kähler theorem, exterior differential systems, weighted jets, initial value problem

2010 Mathematics Subject Classification. Primary 35F46, 58A15, 58A20. Secondary 58A30.

1 Introduction

We will first recall the constant coefficient differential operator from the abstract that has been intensively studied in Clifford analysis (see [3] or [14]). It will be then more natural to explain the motivation for this paper. This operator generalizes the \( k \)-Cauchy-Riemann operator just as the Dirac operator can be viewed as a generalization of the Cauchy-Riemann operator.

Let \( \{\varepsilon_1, \ldots, \varepsilon_{n+1}\} \) be the standard basis of \( \mathbb{R}^{n+1} \), \( g \) the standard inner product, \( S \) be the complex space of spinors (see Section 6 in [1]) of the complexified Clifford algebra of \( (\mathbb{R}^{n+1}, g) \) and \( M(n+1, k, \mathbb{R}) \) be the space of matrices of size \( (n+1) \times k \). We assume throughout this paper that \( k \geq 2 \) and \( n \geq 2 \). Let \( \mathcal{C}^\infty(M(n+1, k, \mathbb{R}), \mathbb{V}) \) be the space of all smooth functions on \( M(n+1, k, \mathbb{R}) \).
with values in some vector space $V$. We define a differential operator
\[ E : C^\infty(M(n+1,k,\mathbb{R}), \mathcal{S}) \to C^\infty(M(n+1,k,\mathbb{R}), \mathbb{R}^k \otimes \mathcal{S}) \] (1)
\[ E\psi := (E_1\psi, \ldots, E_k\psi) \text{ where } E_i\psi := \sum_{\alpha=1}^{n+1} \varepsilon_{\alpha} \partial_{x_{\alpha i}} \psi \]

and $x_{\alpha i}$ are the usual matrix coefficients on $M(n+1,k,\mathbb{R})$ and the dot denotes the Clifford multiplication. We will call this operator a $k$-Dirac operator in the flat setting and a solution of $E\psi = 0$ a monogenic function. It is not hard to see that any monogenic function $\psi$ is uniquely determined by its restriction $\psi|_{M(n,k,\mathbb{R})}$ to the affine subset $M(n,k,\mathbb{R}) = \{ x_{n+1,1} = \ldots = x_{n+1,k} = 0 \}$. Moreover, the restriction has to satisfy the following equations:
\[ [\hat{E}_i, \hat{E}_j](\psi)|_{M(n,k,\mathbb{R})} = 0; \ i, j = 1, \ldots, k \] (2)

where $\hat{E}_i := \sum_{\alpha=1}^{n} \varepsilon_{\alpha} \partial_{x_{\alpha i}}$. The equations (2) can be easily derived from the fact that the coordinate vector fields commute. It follows that not every $S$-valued function on $M(n,k,\mathbb{R})$ is the restriction of a (necessarily unique) monogenic function and it is easy to see that this implies (actually it is equivalent to the fact) that $E$ is not an involutive operator (see [12] that the first prolongation of the 2-Dirac operator is involutive and it is probably not known which prolongation of the $k$-Dirac operator with $k > 2$ is involutive) in the sense of the Cartan-Kähler theorem (see [1]).

Recall that the involutivity of the system of PDE’s can be most easily checked by the well known Cartan’s test (see [1]). Let us denote by $M_i$ the space of all monogenic functions whose components are homogeneous polynomials of degree $i$. Now as $E$ is a first order constant coefficient differential operator without zero order part, it follows that a real analytic function is monogenic if its $i$-th homogeneous component belongs to $M_i$. Moreover, as any coordinate vector field is an infinitesimal symmetry of $E$, differentiation by such a vector field $X$ induces a linear map $M_i \to M_{i-1}$ which we also denote by $X$. A choice of an ordered basis $\{ X_1, \ldots, X_{(n+1)k} \}$ of the vector space of coordinate vector fields induces filtration $\{ M^j_i : j = 1, \ldots, (n+1)k \}$ of $M_i$ where $M^j_i := \{ \psi \in M_i : X_j\psi = \ldots = X_{(n+1)k}\psi = 0 \}$. Notice that since coordinate vector fields commute, the map $X : M_i \to M_{i-1}$ is compatible with these filtrations. Now it is easy to show that $\dim M_{i+1} \leq \dim M_i + \sum_{j=1}^{k(n+1)} \dim M^j_i$. If the equality holds then we know by the Cartan’s test that the system is involutive and we get an explicit characterization of the set of initial conditions.

There is a generalization of the Cartan’s test for differential operators with real analytic coefficients (see Chapter 9.1 in [1]). Here one looks at the symbol of the operator at a fixed point and defines a filtration of the kernel of the symbol map. The filtration comes from a choice of an ordered basis of the cotangent space at the point. From this point of view, one simply regards the operator at the given point as a homogeneous constant coefficient differential operator and uses the Cartan’s test we explained above.

\[ ^1 \text{Note that this operator is called in [3] and [14] the } k\text{-Dirac operator but we choose this longer name in order to distinguish it from the } k\text{-Dirac operator which lives in the world of parabolic geometries.} \]

\[ ^2 \text{Actually any monogenic function has to be real analytic as components are harmonic functions.} \]

\[ ^3 \text{Meaning homogeneous in the degree of derivatives.} \]
In the paper we will consider the differential operator which is called in [12] and [13] the $k$-Dirac operator in the parabolic setting. We will denote this operator by $D$ and we will call it for simplicity the $k$-Dirac operator and a solution of $D\Psi = 0$ a monogenic spinor. The operator $D$ is a differential operator which is natural to a particular class of parabolic geometries (see Section 2). The geometry is $|2|$-graded which means that any manifold with this geometric structure has a bracket generating distribution such that the Levi form has a special algebraic type. If $k = 2$, then this is a contact structure known as the Lie contact structure, see Section 4.2.5 in [2]. As $D$ is a first order operator, it can be shown (see [15]) that $D$ is given by differentiating only in the directions which belong to the canonical distribution. This means that $D$ is not just a differential operator of order one but more is true, it is a differential operator of the weighted order one (see Section 3 for the definition).

We will consider $D$ on the homogeneous space $M$ of the parabolic geometry. Given any two points $x, x' \in M$ there is a symmetry $\Phi$ of the geometric structure which maps $x$ to $x'$. By definition, any natural operator commutes with the induced pullback map $\Phi^*$ and so $\Phi^*$ maps bijectively germs of monogenic sections at $x'$ to germs of monogenic sections at $x$. So it is enough to understand the germs of monogenic sections at a fixed point $x_0$ (which we will call the origin).

We will choose an open affine neighbourhood $A = A(k, \mathbb{R}) \times M(n+1, k, \mathbb{R})$ of $x_0$ where $A(k, \mathbb{R})$ is the space of skew-symmetric matrices of size $k \times k$. The set $A$ inherits in a non-canonical way the structure of a Lie group with a 2-graded nilpotent Lie algebra. The naturality of $D$ means that this operator is left invariant and so it commutes with right invariant vector fields which are the infinitesimal symmetries. Fixing trivializations of natural vector bundles over $A$, the $k$-Dirac operator can be viewed as a differential operator

$$D : C^\infty(A, S) \to C^\infty(A, \mathbb{R}^k \otimes S)$$

with polynomial coefficients (see the formula (44)). From the formula (44) also immediately follows that there is a commutative diagram

$$\begin{array}{ccc}
C^\infty(A, S) & \xrightarrow{D} & C^\infty(A, \mathbb{R}^k \otimes S) \\
\rho^* \downarrow & & \rho^* \\
C^\infty(A, S) & \xrightarrow{E} & C^\infty(A, \mathbb{R}^k \otimes S)
\end{array}$$

where we have for a moment denoted by $\rho$ the canonical projection $A \to M(n+1, k, \mathbb{R})$. So $D$ can be viewed as a non-trivial "extension" of $E$ to the larger affine set $A$ and this also justifies the name for $D$. As $\rho^*$ is injective, we know that $\psi$ is a monogenic function iff $D(\rho^*\psi) = 0$.

As $D$ is a differential operator with polynomial coefficients, it is no longer true that homogeneous components of a monogenic spinor are again monogenic spinors. However, this can be fixed by introducing (see Section 2.1) the weighted degree of polynomials. We will write $\operatorname{wd}(f) = i$ if $f$ is a vector valued on $A$ such that each component of $f$ is a polynomial which is homogeneous of the weighted degree $i$ and put $\mathcal{M}_i := \{ \Psi \in C^\infty(A, S) : D\Psi = 0, \operatorname{wd}(\Psi) = i \}$. Then $D$ is a homogeneous operator of degree $-1$, i.e. $\operatorname{wd}(D(\Psi)) = i - 1$ whenever $\operatorname{wd}(\Psi) = i$. It follows that $\Psi$ is a real analytic monogenic spinor iff $\Psi$ is a converging sum $\sum_{i \geq 0} \Psi_i$ with $\Psi_i \in \mathcal{M}_i$. 

3
If we fix the origin and apply the classical Cartan-Kähler theorem to $D$, then we would actually view $D$ as the constant coefficient differential operator $E$ with its trivial extension to the larger affine set $A$ and this turns out not to be a good move (see [12]). Rather than this, one could try to adapt the Cartan’s lemma to the concept of homogeneous functions with respect to the weighted degree. In particular, we need to define a filtration on the spaces $\mathfrak{M}$ (more precisely on spaces which are isomorphic to $\mathfrak{M}$, which have more invariant meaning). Taking into account that the right invariant vector fields on $A$ are infinitesimal symmetries of $D$, it is clear that the filtrations should be defined with respect to these vector fields. However, there is a problem that the right invariant vector fields do not commute and so in general differentiation by a right invariant vector field does not give map $\mathfrak{M}_i \to \mathfrak{M}_{i-1}$ that is compatible with the filtrations. Nevertheless, we will show that there is an ordered basis (see Lemma [5,2]) of the vector space of right invariant vector fields for which we get maps that are compatible with the filtrations. Here we will need some particular properties of the operator $D$ (this is used in the proof of Lemma [5,1]) and so at this point the machinery presented in this paper fails to work for general left invariant differential operators without further assumptions. With this at hand we can imitate the proof of Proposition 2.5 from [1] that a prolongation of an involutive system is again involutive and this is the hardest step in proving the main result of the paper.

**Theorem 1.1.** Let $i$ be a non-negative integer and $\psi$ be a $S$-valued function defined on the subset $M(n,k,\mathbb{R}) := \{x_{n+1,1} = \ldots = x_{n+1,k} = y_{12} \ldots = y_{k-1,k} = 0\}$ of $\mathcal{A}$ such that each component of $\psi$ is a homogeneous polynomial of the degree $i$. Then there is a unique monogenic spinor $\Psi \in \mathfrak{M}$ such that $\Psi|_{M(n,k,\mathbb{R})} = \psi$.

In other words, we will consider here only formal solutions of the $k$-Dirac operator and we will not go into analysis of real analytic or smooth solutions of this operator.

Replacing the degree of polynomials by the weighted degree corresponds in more invariant language to replacing usual jets by weighted jets. Similarly, one needs to replace the order of a differential operator by the weighted order. These concepts (see Section 3) were developed in the 90’s by Morimoto in [6, 7] and [8]. See also [5] and [11] for an application of this theory. Many parts of the classical theory of exterior differential systems and the Cartan-Kähler theorem was generalized to the setting of weighted differential operators in [8, 9] and [10]. Namely, we will need the Spencer complex for weighted differential operators which has been introduced already in [17].

I would like to thank Katharina Neusser and the unknown referee for useful comments and to Boris Krukligov and Peter Vassiliou for their kind support.

### 1.1 Notation

Let $p : V \to M$ be a fibre bundle. We denote the fibre $p^{-1}(x)$ over $x \in M$ by $V_x$. If $U \subset M$ is an open subset, we denote $p^{-1}(U)$ also by $V|_U$.

**Used symbols.**

- $I(n,k) = \{(\alpha, i) : \alpha = 1, \ldots, n, i = 1, \ldots, k\}$
- $I(k) = \{(r, s) : r, s \in \mathbb{Z}; 1 \leq r < s \leq k\}$
- $M(n,k,\mathbb{R})$ matrices of size $n \times k$ with real coefficients
$A(k, \mathbb{R})$ skew-symmetric matrices of size $k \times k$ with real coefficients
$1_k$ is the identity $k \times k$ matrix
$[v_1, \ldots, v_\ell]$ the linear span of vectors $v_1, \ldots, v_\ell$
$\mathbb{V}$, resp. $\mathbb{V}^*$ the dual of vector space $\mathbb{V}$, resp. of vector bundle $\mathbb{V}$

## 2 The parabolic geometry

In this section we will recall some basic knowledge from the theory of parabolic geometries and talk about the geometric structure for which the $k$-Dirac operator is natural. In the second part of the section we will go to an affine subset of the homogeneous model of the geometry, we will set some notation and definitions. Most importantly, we will introduce the weighted degree of polynomials on this affine set.

Let $\{e_1, \ldots, e_k, \varepsilon_1, \ldots, \varepsilon_{n+1}, e_1', \ldots, e_k'\}$ be the standard basis of $\mathbb{R}^{2k+n+1}$. Then the bilinear form $h$ on $\mathbb{R}^{2k+n+1}$ which satisfies

$$h(e_i, e_j) = \delta^i_j, \quad h(e_i, \varepsilon) = \delta_{i\alpha}, \quad h(e_i, \varepsilon_j) = h(e_j, e_i) = 0$$

where $\delta$ is the Kronecker delta and $i, j = 1, \ldots, k; \alpha, \beta = 1, \ldots, n + 1$, is non-degenerate and the associated quadratic form $H$ has signature $(k, k + n + 1)$. We will sometimes write $\mathbb{R}^{k,n+k+1}$ instead of $\mathbb{R}^{2k+n+1}$ to indicate that the vector space comes with the bilinear form $h$.

Let $\hat{G}$ be the associated special orthogonal group $\text{SO}(k, n + k + 1)$. The Lie algebra of $\hat{G}$ is the simple matrix algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & Z^T & W \\ X & B & -Z \\ Y & -X^T & -A^T \end{pmatrix} \bigg| \begin{array}{c} A \in M(k, \mathbb{R}), B \in A(n + 1, \mathbb{R}), \\ X, Z \in M(n + 1, k, \mathbb{R}), Y, W \in A(k, \mathbb{R}) \end{array} \right\}.$$  \tag{5}$$

The block decomposition determines the direct sum decomposition $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0$ of $\mathfrak{g}$ which is given by

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_1 & \mathfrak{g}_0 & \mathfrak{g}_2 \\ \mathfrak{g}_2 & \mathfrak{g}_0 & \mathfrak{g}_1 \end{pmatrix}.$$  \tag{6}$$

If $X \in \mathfrak{g}_1$, $Y \in \mathfrak{g}_2$, then $[X, Y] \in \mathfrak{g}_{1+2}$ where we agree that $\mathfrak{g}_s = \{0\}$ if $|s| \neq 0, 1, 2$. Moreover, $\mathfrak{g}_1$ generates $\mathfrak{g}_- := \mathfrak{g}_- \oplus \mathfrak{g}_- - 1$ as Lie algebra. This means that the decomposition is a $[2]$-gradation (see Definition 3.1.2 in [2]) on $\mathfrak{g}$. The associated filtration of $\mathfrak{g}$ is $\{ \mathfrak{g}^i := \oplus_{j \geq i} \mathfrak{g}_j ; i \in \mathbb{Z} \}$. Then $\mathfrak{p} := \mathfrak{g}^0$ is a subalgebra and each subspace $\mathfrak{g}^i$ is $\mathfrak{p}$-invariant. Moreover $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ where $\mathfrak{p}_+ := \mathfrak{g}^1$ is a nilradical (i.e. a maximal nilpotent subalgebra) and $\mathfrak{g}_0$ is a reductive Levi factor of $\mathfrak{p}$ (see Section 2.1.8 in [2]). Notice that $\mathfrak{g}_0$ is isomorphic to $\mathfrak{sl}(k, \mathbb{R}) \oplus \mathfrak{so}(n + 1)$ and that the subspaces $\mathbb{F} := [e_1, \ldots, e_k], \mathbb{F} := [e_1, \ldots, \varepsilon_{n+1}]$ are $\mathfrak{g}_0$-invariant. The space $\mathbb{F}$ has $\mathfrak{g}_0$-invariant inner product $h|_{\mathbb{F}}$ which yields isomorphism $\mathbb{F} \cong \mathbb{F}^*$ as $\mathfrak{g}_0$-module. We have the following list of isomorphisms of $\mathfrak{g}_0$-modules.

\footnote{4Here we mean the following: $\mathfrak{g}_0$ is the subspace of block diagonal matrices, $\mathfrak{g}_1$ lives in the blocks $(1, 2)$ and $(2, 1)$ etc.}
\[g_{-1} \cong \mathfrak{E}^* \otimes F, \ g_{-2} \cong \Lambda^2 \mathfrak{E}^* \otimes \mathbb{R}, \ g_1 \cong \mathfrak{E} \otimes F, \ g_2 \cong \Lambda^2 \mathfrak{E} \otimes \mathbb{R} \]

where \(\mathbb{R}\) is a trivial representation of \(\mathfrak{so}(n)\).

Let us now look for a homogeneous model of the geometry. The group \(\hat{G}\) has a canonical transitive action on the Grassmannian variety \(M\) of \(k\)-dimensional totally isotropic subspaces in \(\mathbb{R}^{k,n+k+1}\). We denote the action by dot, i.e. if \(x \in M, g \in \hat{G}\) then \(g.x \in M\). In particular, the subspace \(\mathbb{E}\) is totally isotropic and we will call this point the origin of \(M\) and denote it by \(x_0\). The stabilizer of \(x_0\) is a closed subgroup of \(\hat{G}\) with Lie algebra \(\mathfrak{p}\). This group, we denote it by \(\hat{P}\), is called a parabolic subgroup of \(G\) corresponding to the \([2]\)-gradation. The group \(\hat{P}\) is (see Proposition 3.1.3 in \([2]\)) isomorphic to a semidirect product \(\hat{G}_0 \ltimes \exp(\mathfrak{p}_+)\) with normal subgroup \(\exp(\mathfrak{p}_+)\) where \(\hat{G}_0\) is the subgroup of \(\hat{G}\) of the block diagonal matrices. Notice that \(\hat{G}_0\) is isomorphic to \(\text{GL}(k, \mathbb{R}) \times \text{SO}(n+1)\) and that its Lie algebra is obviously \(\mathfrak{g}_0\). So any \(\hat{G}_0\)-module is naturally also a \(\hat{P}\)-module with trivial action of \(\exp(\mathfrak{p}_+)\) and it is well known (see Proposition 3.1.12 from \([2]\)) that any irreducible \(\hat{P}\)-module is of this form. This in particular applies to the nilpotent graded Lie algebra \(\mathfrak{g}_-\) which becomes a (non-irreducible) \(\hat{P}\)-module. Then the canonical map \(\mathfrak{g}_- \rightarrow \text{gr}(\mathfrak{g}/\mathfrak{p})\) where \(\text{gr}(\mathfrak{g}/\mathfrak{p})\) is the graded vector space associated to the filtration \(\mathfrak{g}^{-1}/\mathfrak{p} \subset \mathfrak{g}/\mathfrak{p}\) is an isomorphism of \(\hat{P}\)-modules.

The canonical projection \(\hat{p} : \hat{G} \rightarrow \hat{M}, \hat{p}(g) := g.x_0\) is a principal \(\hat{P}\)-bundle over \(M\). The quotient \(\hat{G}_0\) of \(\hat{G}\) under the principal action of \(\exp(\mathfrak{p}_+)\) is (see Section 3.1.5 in \([2]\)) the total space of the principal bundle \(\hat{p}_0 : \hat{G}_0 \rightarrow \hat{M}\) with typical fibre \(\hat{G}_0 \cong \hat{P}/\exp(\mathfrak{p}_+)\) where \(\hat{p}_0(g, \exp(\mathfrak{p}_+)) := \hat{p}(g), g \in \hat{G}\). It is a well known fact that the tangent bundle \(TM\) is isomorphic (see Section 1.4.3 in \([2]\)) to the associated vector bundle \(\hat{G} \times_{\hat{p}} (\mathfrak{g}/\mathfrak{p})\). This bundle contains the subbundle \(\hat{G} \times_{\hat{p}} (\mathfrak{g}^1/\mathfrak{p})\) and so we see that \(TM\) contains a canonical subbundle which we denote by \(H\). Put \(Q := TM/H\) and let \(\text{gr}(TM)\) be the associated graded vector bundle. The bundle \(\text{gr}(TM)\) is isomorphic to \(\hat{G} \times_{\hat{p}} \text{gr}(\mathfrak{g}/\mathfrak{p})\). We will view \(\text{gr}(\mathfrak{g}/\mathfrak{p})\) rather as \(\mathfrak{g}_-\). As \(\exp(\mathfrak{p}_+)\) acts trivially on \(\mathfrak{g}_-\), it follows that \(\text{gr}(TM)\) is isomorphic to \(\hat{G}_0 \times_{\hat{p}} \mathfrak{g}_-\). We see that for each \(x \in M\) the graded vector space \(\text{gr}(T_xM)\) has the structure of a nilpotent graded Lie algebra. This structure has a geometric origin, i.e. it is well known that the structure coincides with the Levi bracket \(L_x : \Lambda^2 \text{gr}(T_xM) \rightarrow \text{gr}(T_xM)\) which is naturally induced by the Lie bracket of vector fields (see Section 3.1.7 in \([2]\)). In particular, we see that \(H\) is a bracket generating distribution.

The bundle \(\hat{G}_0\) can be viewed also as the frame bundle of a tautological vector bundle over \(M\). Let \(E_x, \text{resp. } E_\perp\) be the total space of the vector bundle over \(M\) such that \(E_x = x, \text{resp. } E_\perp = x^\perp\). Then \(E \subset E_\perp \subset M \times \mathbb{R}^{2k+n+1}\) and the vector bundle \(F := E_\perp / E\) has a canonical inner product which comes from restricting \(h\) to its fibres. The quotient bundle \((M \times \mathbb{R}^{2k+n+1})/E_\perp\) is canonically isomorphic to \(E^\ast\) where the canonical pairing with \(E\) is again naturally induced by \(h\). As the canonical volume form on \(\mathbb{R}^{2k+n+1}\) is invariant under \(\hat{G}\) and the bundle \(E \oplus E^\ast\) has a canonical orientation, we can fix the orientation on the bundle \(F\) so that for each \(x \in M\) the induced orientation on \(E_x \oplus F_x \oplus E_x^\perp\) coincides with the canonical one on \(\mathbb{R}^{2k+n+1}\). Obviously \(E, \text{resp. } F\) is a \(\hat{G}\)-homogeneous vector bundle whose fibre over the origin \(x_0\) is isomorphic to the (irreducible) \(P\)-module \(\mathfrak{E}, \text{resp. } \mathfrak{F}\). It follows (see Section 1.4.3 in \([2]\)) that \(E, \text{resp. } F\) is isomorphic to the associated vector bundle \(\hat{G} \times_{\hat{p}} \mathfrak{E}, \text{resp. } \hat{G} \times_{\hat{p}} \mathfrak{F}\)
and since \( \exp(p+) \) acts trivially on \( \mathbb{E} \), resp. \( \mathbb{F} \), it follows that it is isomorphic to \( \hat{G}_0 \times \mathbb{C}_0 \), resp. \( \hat{G}_0 \times \mathbb{C}_0 \), \( \mathbb{F} \). Hence, we may view \( (\hat{G}_0)_x : x \in M \) as the set of pairs \((p,q)\) where \( p \) is a frame of \( E_x \) and \( q \) is an orthonormal frame of \( F_x \) which is compatible with the orientation on \( F_x \). Repeating this argument also to the bundles \( H, Q, H^*, Q^* \) respectively and using (7), we obtain isomorphisms

\[
H \cong E^* \otimes F, \quad Q \cong \Lambda^2 E^*, \quad H^* \cong E \otimes F, \quad Q^* \cong \Lambda^2 E.
\] (8)

The group \( \hat{G} \) is not simply connected. In order to invariantly define the \( k \)-Dirac operator we will choose a 4:1 covering \( \rho : G \to \hat{G} \). It is easy to check that the inclusion \( \hat{G}_0 \hookrightarrow G \) induces isomorphisms on \( \pi_0 \) and \( \pi_1 \). It follows that

\[
\pi_1(\hat{G}) = \begin{cases} Z \times Z_2, \ k = 2, \\ Z_2 \times Z_2, \ k > 2. \end{cases}
\] (9)

If \( k = 2 \), then \( G \) is determined by the subgroup \( 2Z \times \{1\} \). If \( k > 2 \), then \( G \) is a universal covering of \( \hat{G} \). We put \( P := \rho^{-1}(\hat{P}), G_0 := \rho^{-1}(\hat{G}_0) \). It follows that \( G_0 \) is isomorphic to \( GL(k,\mathbb{R}) \times Spin(n+1) \) where \( GL(k,\mathbb{R}) \) is isomorphic to a connected 2:1 covering of \( GL(k,\mathbb{R}) \) (which is unique as the fundamental group of \( GL(k,\mathbb{R}) \) contains a unique subgroup of index 2). We obtain a principal fibre bundle \( p : G \to M \) with typical fibre \( P \) where we put \( p := \hat{p} \circ p \). We define \( \mathcal{G}_0 := G/\exp(p+) \) so that the map \( \rho \) factorizes to a 4:1 covering \( \rho_0 : \mathcal{G}_0 \to \hat{G}_0 \).

We get another principal bundle \( p_0 : \mathcal{G}_0 \to M \) where \( p_0 := \hat{p}_0 \circ \rho_0 \) with typical fibre \( G_0 \).

Let \( \mathcal{S} \) be the space of spinors for the complexified Clifford algebra \((\mathbb{F},h|\mathbb{C})\). We may view the pair \((\mathbb{F},h|\mathbb{C})\) as well as the pair \((\mathbb{R}^{n+1},g)\) from the introduction and so \( \mathcal{S} \) is the same of spinors. Recall that if \( n+1 \) is odd, then \( \mathcal{S} \) is an irreducible representation of \( Spin(n+1) \) while if \( n+1 \) is even, then \( \mathcal{S} \) is the direct sum of two irreducible subspaces. Let \( \mathcal{C}_\lambda \) be an irreducible complex \( GL(k,\mathbb{C}) \)-module with highest weight \( \lambda := (\frac{\omega_1}{2},\frac{\omega_2}{2},\ldots,\frac{\omega_k}{2}) \) as in [13]. Then \( \mathcal{C}_\lambda \) is also a 1-dimensional \( GL(k,\mathbb{R}) \)-module by restriction on which only the center (the subspace of multiples of the identity matrix) acts non-trivially. We get two irreducible \( G_0 \)-modules \( \mathcal{S}_\lambda \) := \( \mathcal{C}_\lambda \otimes \mathcal{S} \) and \( \mathcal{S}_\lambda^k := (\mathbb{E} \otimes \mathcal{C}_\lambda) \otimes \mathcal{S} \) where we indicate by subscript over which field we take the tensor product. These two space are also irreducible \( P \)-modules as we explained above. If we forget the structure of \( G_0 \)-modules on both spaces, then there are canonical isomorphisms

\[
\mathcal{S}_\lambda \to \mathcal{S}, \text{ resp. } \mathcal{S}_\lambda^k \to \mathbb{E} \otimes \mathbb{R} \mathcal{S}.
\] (10)

of vector spaces that are given by \( z \otimes \psi \mapsto z.\psi \), resp. \((v \otimes z) \otimes \psi \mapsto v \otimes (z.\psi)\) where \( z \in \mathcal{C}_\lambda, v \in \mathbb{E}, \psi \in \mathcal{S} \). Now we can form two associated vector bundles:

\[
\mathcal{S}_\lambda := G \times_P \mathcal{S}_\lambda, \quad \mathcal{S}_\lambda^k := G \times_P \mathcal{S}_\lambda^k,
\] (11)

The \( k \)-Dirac operator then maps section of \( \mathcal{S}_\lambda \) to sections of \( \mathcal{S}_\lambda^k \). We will give an invariant definition of \( D \) in Section [3]. Later on, we will trivialize the bundles over the affine subset \( \mathcal{A} \) and we will give a formula \( D \) with respect to these trivializations.
2.1 The affine set $A$

Recall that

$$g_- = \begin{cases} 
0 & 0 & 0 \\
X & 0 & 0 \\
Y & -X^T & 0 
\end{cases} \quad | X =\in M(n+1,k,\mathbb{R}), Y = -Y^T \in A(k,\mathbb{R}) \end{cases} \quad (12)$$

and that $g_- = g_{-2} \oplus g_{-1}$ where $g_{-2}$, respectively $g_{-1}$ is the subspace of those matrices where $X = 0$, resp. $Y = 0$. The map $p \circ \exp : g_- \to M$ is

$$p \circ \exp \left( \begin{array}{ccc} 0 & 0 & 0 \\
X & 0 & 0 \\
Y & -X^T & 0 \end{array} \right) = \begin{bmatrix} 1_k \\
X \\
Y - \frac{1}{2} X^T X \end{bmatrix} \quad (13)$$

where $[a_{ij}]$ is the $k$-plane in $\mathbb{R}^{2k+n+1}$ spanned by the columns of a matrix $(a_{ij}) \in M(2k+n+1,k,\mathbb{R})$. We see that the map (13) is injective and that its image $A$ is an affine subset of $M$. It is also easy to see that $G_- := \exp(g_-)$ is a closed connected analytic subgroup of $G$ with Lie algebra $g_-$ such that the exponential map $g_- \to G_-$ is bijective. Let us write the matrices above as $X = (x_{\alpha i})_{\alpha=1,...,k, i=1,...,n+1}$, resp. $Y = (y_{rs})_{r=1,...,k}$. We will view $g_-$ also as an affine space with coordinates $x_{\alpha i}, y_{rs}$ where $\alpha, i \in I(n+1,k), (r, s) \in \hat{I}(k)$. (14)

**Remark 2.1.** We see that both maps in the composition

$$g_- \xrightarrow{\exp} G_- \xrightarrow{p|_{G_-}} A \quad (15)$$

are diffeomorphisms which allows us to push-forward and pullback geometric objects from one set to another set. We will do that without any further comment. In particular, we will use the matrix coefficients on $g_-$ also as coordinates on $A$ and we will view left and right invariant vector fields on $G_-$ as vector fields on $A$.

Let $X \in g_-$. We denote by $L_X$, resp. $R_X$ the corresponding left, resp. right invariant vector field on $G_-$. As the distribution $H$ is invariant with respect to the canonical left action of $G$ on $M$, it follows that $H$ is spanned over $A$ by the vector fields $L_X$ as $X$ ranges over the set $g_{-1}$. It is a straightforward calculation to show that

$$L_{\alpha i} := L_{e_\alpha \otimes x_\alpha} = \partial_{x_{\alpha i}} - \frac{1}{2} \sum_{j=1}^{k} x_{\alpha j} \partial_{y_{ij}}, \quad (16)$$

$$R_{\alpha i} := R_{e_\alpha \otimes x_\alpha} = \partial_{x_{\alpha i}} + \frac{1}{2} \sum_{j=1}^{k} x_{\alpha j} \partial_{y_{ij}}, \quad (17)$$

$$L_{e_\alpha \wedge e_j} = R_{e_\alpha \wedge e_j} = \partial_{y_{ij}} \quad (18)$$

where we use the isomorphisms from (14) and the bases of $E,F$ that are given above that formula. Moreover, we will use the convention $\partial_{y_{rs}} = -\partial_{y_{sr}}$. We have that

$$[L_{\alpha i}, L_{\beta j}] = -[R_{\alpha i}, R_{\beta j}] = \delta_{\alpha \beta} \partial_{y_{ij}} \quad (19)$$
while all other Lie brackets of the vector fields given above are zero.

The left invariant vector fields form a natural framing over \( A \) which is adapted to the filtration \( H \subset TM \).

**Example 2.1. (Left invariant framing and coframing over \( A \))**

We see that \( \{ L_{\alpha i}, \partial_{y_{ij}} : (\alpha, i) \in I(n+1,k), (r, s) \in \hat{I}(k) \} \) is a framing over \( A \) which is adapted to the filtration of \( TA \) as for each \((\alpha, i) \in I(n+1,k)\) the vector field \( L_{\alpha i} \) is a section of \( H|_A \). Let \( \{ \omega_{\alpha i}, \theta_{rs} \} \) be the dual coframing, i.e. \( \omega_{\alpha i}, \theta_{rs} \) are differential 1-forms on \( A \) such that

\[
\omega_{ji}(L_{\alpha i}) = \delta_{\alpha,j} \delta_{ij}, \quad \theta_{rs}(\partial_{y_{ij}}) = \delta_{s,ij} - \delta_{r,j} \delta_{i,s}, \quad \omega_{ai}(\partial_{y_{ij}}) = \theta_{rs}(L_{\alpha i}) = 0
\]

Then we have

\[
\omega_{ai} = dx_{ai}, \quad \theta_{rs} = dy_{rs} - \frac{1}{2} \sum_{\alpha=1}^{n+1} (x_{\alpha r} dx_{\alpha s} - x_{\alpha s} dx_{\alpha r}). \tag{20}
\]

In particular, the exterior derivative of \( \theta_{rs} \) is

\[
d\theta_{rs} = - \sum_{\alpha=1}^{n+1} \omega_{\alpha r} \wedge \omega_{\alpha s}. \tag{21}
\]

The framing over \( A \) determined by the right invariant vector fields is not adapted to the filtration of the tangent bundle of \( M \). On the other hand, the right invariant vector fields are infinitesimal symmetries of the parabolic structure and this is why we will use these vector fields later on.

**Example 2.2. (Right invariant framing and coframing over \( A \))**

Let \( \{ \varpi_{ai}, \vartheta_{rs} \} \) be the dual coframing to the framing \( \{ R_{\alpha i}, \partial_{y_{ij}} \} \) over \( A \) where \((\alpha, i), (r, s)\) ranges over the sets \( I(n+1,k), \hat{I}(k) \) respectively. We find that

\[
\varpi_{ai} = dx_{ai}, \quad \vartheta_{rs} = dy_{rs} + \frac{1}{2} \sum_{\alpha=1}^{n+1} (x_{\alpha r} dx_{\alpha s} - x_{\alpha s} dx_{\alpha r}) \tag{22}
\]

and in particular

\[
d\vartheta_{rs} = \sum_{\alpha=1}^{n+1} \omega_{\alpha r} \wedge \omega_{\alpha s} \tag{23}
\]

The \( k \)-Dirac operator \( D \) is not a constant coefficient operator but it is a differential operator with polynomial coefficients (see the formula (14) but have in mind that this formula depends on choices). Hence, it is not true that homogeneous components of a real analytic monogenic spinor are again monogenic spinors. However, we may fix this by introducing a weighted degree.

Let \( \mathbb{R}[x,y] \), resp. \( \mathbb{R}[y] \), resp. \( \mathbb{R}[x] \) be the ring of polynomials with real coefficients on the affine space \( g_- \), resp. \( g_{-2} \), resp. \( g_{-1} \). Then since \( g_{-2} \), resp. \( g_{-1} \) is an affine subspace of \( g_- \), we may view \( \mathbb{R}[y] \), resp. \( \mathbb{R}[x] \) as a subring of \( \mathbb{R}[x,y] \). If \( s \in \mathbb{R}[x,y] \) is a monomial, then we can certainly find monomials \( p' \in \mathbb{R}[y], p'' \in \mathbb{R}[x] \) such that \( p' p'' = s \). Then the number \( \text{wd}(s) := 2\text{deg}(p') + \text{deg}(p'') \) where \( \text{deg} \) is the usual degree of polynomials is independent of the choice of \( p', p'' \) and we call it the *weighted degree* of \( s \). More generally, we call a polynomial \( p \in \mathbb{R}[x,y] \) a homogeneous polynomials of the weighted degree \( i \)
homogeneous function of a weighted degree \( i \) and \( \partial \) homogeneous polynomial of the weighted degree \( j \).

Recall the definition of the tableau and its prolongations (see also material [8] and [11] where this can be found).

In the second part of the section we will give an invariant definition of the \( k \) and the graded space of weighted jets of functions.) In the second part of the section we will give an invariant definition of the \( k \) and the graded space of weighted jets of functions.)

We will see that this a natural concept for filtered manifolds which generalizes the notion of the usual degree of differential operators and the jets of the germs of sections of vector bundles. We will also see that the weighted jet of a function at a point is an invariant analogue of the weighted degree of functions that we defined in the previous section (see Example 3.2 below that the map which assigns to a function its weighted jet at the origin of \( M \) defined in the previous section (see Example 3.2 below that the map which assigns to a function its weighted jet at the origin of \( M \) gives an isomorphism between the space of polynomials graded with respect to the weighted degree and the graded space of weighted jets of functions.) In the second part of the section we will give an invariant definition of the \( k \)-Dirac operator and we will recall the definition of the tableau and its prolongations (see also material [8] and [11] where this can be found).

Recall that there is a natural bracket generating distribution \( H \) on \( M \) such that \( (gr(TM), \mathcal{L}) \) is a locally trivial bundle of graded nilpotent Lie algebras over \( M \) with a typical fibre \( g \). Put \( F_{-1} := H, F_{-2} := TM \). Let \( X \) be a vector field which is defined over an open subset \( U \) of \( M \), then the weighted order of \( X \) is defined as the smallest integer \( i \) such that \( X \in \Gamma(F_{-i}|U) \). We write \( ord(X) = i \).

**Example 3.1.** We have that \( ord(L_{\alpha i}) = 1, ord(R_{\alpha i}) = ord(\partial_{y^r x^s}) = 2 \) where the vector fields were defined in [10] and \( (\alpha, i) \in I(n + 1, k), (r, s) \in \hat{I}(k) \).

A differential operator \( D \) acting on the space of smooth functions on \( M \) is called a differential operator of the weighted order at most \( r \) if for each point \( x \in M \) there is an open neighbourhood \( U \) of \( x \) with local framing \( \{X_1, \ldots, X_p\} \) such that

\[
D|_U = \sum_{a \in \mathbb{N}_0^p} f_a X_1^{a_1} \cdots X_p^{a_p} \quad (24)
\]

where \( \mathbb{N}_0^p := \{a = [a_1, \ldots, a_p] : a_i \in \mathbb{Z}, a_i \geq 0, i = 1, \ldots, p\}, f_a \in C^\infty(U) \) and for all \( a \) in the sum with \( f_a \) non-zero: \( \sum_{i=1}^p a_i, ord(X_i) \leq r \). If \( D \) is a differential
operator of the order at most $r$ but not of the order at most $r-1$ then we say that $D$ is a differential operator of the weighted order $r$ (and we will write $\text{ord}(D) = r$).

Let $f,f'$ be two germs of smooth functions at $x \in M$. Then we say that $f,f'$ are $r$-equivalent (we write $f \sim_r f'$) if $Df(x) = Df'(x)$ for all differential operators of the weighted order at most $r$. Clearly, $\sim_r$ is an equivalence relation and we denote by $I^r_x$ the equivalence class of the germ $f$ and by $\mathfrak{g}^r_x$ the space of all such equivalence classes. The disjoint union $\mathfrak{g}^r := \bigcup_{x \in M} \mathfrak{g}^r_x$ is naturally a vector bundle over $M$. There is a canonical projection $\pi_r : \mathfrak{g}^r \to \mathfrak{g}^{r-1}$ whose kernel $\mathfrak{g}^r$ is again a vector bundle.

Let $V$ be a vector bundle over $M$ and $s,s'$ be two germs of smooth sections of $V$ at $x \in M$. We say that $s,s'$ are $r$-equivalent ($s \sim_r s'$) if

$$D(\lambda, s - s')(x) = 0$$

for all sections $\lambda$ of the dual bundle $V^*$ and all differential operators $D$ of the weighted order at most $r$. Here $\langle -,- \rangle$ denotes the canonical pairing. We again denote by $I^r_s$ the equivalence class of the germ $s$ and the space of all such equivalence classes by $\mathfrak{g}^r_s$. The disjoint union $\mathfrak{g}^r := \bigcup_{x \in M} \mathfrak{g}^r_x,M$ is a vector bundle over $M$ (see for example Theorem 2.10 in [11]) such that the canonical map $\pi_r : \mathfrak{g}^r \to \mathfrak{g}^{r-1}$ is clearly a surjective vector bundle map whose kernel $\mathfrak{g}^r$ is again a vector bundle over $M$. We denote its fibre over $x \in M$ by $\mathfrak{g}^r_x$.

Let us now assume that $V$ is a $G$-homogeneous vector bundle. Then $V$ is naturally isomorphic to $G \times_p V$ where $V$ is the $P$-module $V_{x_0}$ (see Section 1.4.3 in [2]). The bundles $\mathfrak{g}^r \otimes V$, $\mathfrak{g}^r V$ have also a canonical $G$-action and so it suffices to understand their fibres (see the following example) over the origin $x_0$. We will use the following notation:

$$\mathfrak{g}^r \otimes V := \mathfrak{g}^r_{x_0} \otimes V, \mathfrak{g}^r V := \mathfrak{g}^r_{x_0} V.$$

**Example 3.2.** Recall that $g_- = g_{-2} \oplus g_{-1}$ is a nilpotent graded Lie algebra, i.e. $[g_i,g_j] \subset g_{i+j}$ where $i,j = -1,-2$ and we put $g_{-\ell} = \{0\}$ whenever $\ell \neq -1,-2$. The universal enveloping algebra $U(g_-)$ of $g_-$ is defined as $T(g_-)/I$ where $T(g_-)$ is the tensor algebra of $g_-$ and $I$ is the both sided ideal which is generated by the elements of the form $X \otimes Y - Y \otimes X - [X,Y]; X,Y \in g_-$. As the Lie bracket on $g_-$ is compatible with the grading it follows that $I$ is spanned as a vector space by homogeneous elements. It follows that the enveloping algebra $U(g_-)$ is naturally graded $\oplus_{r \geq 0} U_{-r}(g_-)$ where $U_{-r}(g_-) := T_{-r}(g_-)/(T_{-r}(g_-) \cap I)$ and $T_{-r}(g_-)$ is the vector subspace of $T(g_-)$ which is spanned by the elements of the form $X_{i_1} \otimes \cdots \otimes X_{i_u}$ where $X_{i_j} \in g_{s_j}; s_j = -1,-2; j = 1, \ldots, u$ and $\sum_{j=1}^u s_j = -r$.

Recall that we view $g_-$ as the graded space $gr(g/p)$ which is canonically isomorphic to $gr(T_{x_0}M)$. We will denote the $j$-th graded component of $gr(T_{x_0}M)$ by $gr_j(T_{x_0}M)$ and so $gr_j(T_{x_0}M)$ is isomorphic to $g_{j'; j'} = -2,-1$. Given vectors $X_{i_1}, \ldots, X_{i_u}$ as above, there are vector fields $\hat{X}_{i_1}, \ldots, \hat{X}_{i_u}$ on $M$ such that $\text{ord}(\hat{X}_{i_j}) = -s_j$ and

$$X_{i_j} = \begin{cases} \hat{X}_{i_j}(x_0), & s_j = -1, \\ q(X_{i_j})(x_0), & s_j = -2 \end{cases}$$

where $q : TM \to Q$ is the canonical projection.

Let $\mathcal{V}$ be the $G$-homogeneous vector bundle $G \times P V$. Then $\mathcal{V}$ can be trivialized over the affine set $\mathcal{A}$ and so we can view the germ of a section $s$ of $\mathcal{V}$ at
\(x_0\) as a vector valued function which we may differentiate by the vector fields \(\dot{X}_{i_1}, \ldots, \dot{X}_{i_u}\). If \(\dot{X}_{i_0}^{-1}s = 0\), then the value

\[
(\dot{X}_{i_1} \ldots \dot{X}_{i_u} s)(x)
\]

depends only \(s\) and on \(X_{i_1}, \ldots, X_{i_u}\), (in particular it does not depend on the way we have extended the vector field \(X_{i_j}\) to the vector field \(\dot{X}_{i_j}\), \(j = 1, \ldots, u\) and the choice of trivialization of \(V\)). Hence, we have obtained a well defined map

\[
\text{gr} V \to \mathcal{U}_{-r}^+ (g_-) \otimes V.
\]

(29)

The construction above works for a general point \(x \in M\) and we would get a map \(\text{gr}_1 V \to \mathcal{U}_{-r}^+ (\text{gr}(T_x M)) \otimes V\) which is bijective by Proposition 2.2 from [11]. Using the duality \(g_{-1} \cong g_1\) then the isomorphism (29) becomes for small \(r\):

\[
\text{gr}^1 V \cong g_1 \otimes V, \text{ gr}^2 V \cong S^2 g_1 \otimes V \oplus g_2 \otimes V, \ldots
\]

and in general

\[
\text{gr}^r V \cong \bigoplus_{i=0}^{[a]} S^{r-2i} g_1 \otimes S^i g_2 \otimes V
\]

where \([a]\) is the integer part of \(a \in \mathbb{R}\).

Finally, let \(p \in \mathbb{R}[x, y]_r\). Then (recall Example 2.3 and 3.1) we have that \(\dot{X}_{i_0}^{-1} p = 0\) and it is easy to verify that the map \(p \mapsto \dot{X}_{i_0}^{-1} p\) induces isomorphism \(\mathbb{R}[x, y]_r \to \text{gr}^r V\). From [20] we know that \(\text{gr}^r_{x_0}\) is isomorphic to \(\mathcal{U}_{-r}^+ (g_-)\). Altogether, we obtain isomorphism

\[
\mathbb{R}[x, y]_r \otimes V \to \text{gr}^r V.
\]

(30)

Now we can give an invariant definition of the \(k\)-Dirac operator using the language of weighted jets.

### 3.1 The \(k\)-Dirac operator and the tableau

There is (up to a constant) a unique non-zero \(P\)-equivariant map

\[
\phi : \mathcal{J}^1 S_{\lambda} \to S^k_{\lambda}
\]

(31)

where \(S^k_{\lambda}, S_{\lambda}\) where defined at the end of Section 2. The map \(\phi\) can be extended in a unique way to a \(G\)-equivariant vector bundle map

\[
\Phi : \mathcal{J}^1 S_{\lambda} \to S^k_{\lambda}.
\]

(32)

For a non-negative integer \(i\) we define the \(i\)-th prolongation of \(\Phi\) as the vector bundle map

\[
p^i \Phi : \mathcal{J}^{1+i} S_{\lambda} \to \mathcal{J}^i S^k_{\lambda}
\]

\[
p^i \Phi(j_{i+1}^s) := j_{i}^s(\Phi(j_1^s))
\]

where \(s\) is a germ of a section of \(S_{\lambda}\) at \(x\). As \(\Phi\) is linear in fibres, the right hand side depends only the weighted \((i + 1)\)-jet of \(s\) and so \(p^i \Phi\) is well defined (see
also Section 1.3.1 in [11]). We denote $p^i\phi$ the restriction of $p^i\Phi$ to the fibres over the origin and put

$$R_i := \begin{cases} S_\lambda, & i = 0 \\ \text{Ker}(p^{i-1}_i\phi), & i \geq 1. \end{cases} \quad (34)$$

Then for each $i \geq 0$ there is a commutative diagram of vector spaces with exact rows and columns

$$0 \rightarrow A^{(i)} \rightarrow \text{gr}^{i+1}S_\lambda \rightarrow \text{gr}^iS_\lambda \rightarrow 0 \quad (35)$$

$$0 \rightarrow R^{i+1} \rightarrow \mathfrak{g}^{i+1}S_\lambda \rightarrow \mathfrak{g}^iS_\lambda \rightarrow 0$$

$$0 \rightarrow R^i \rightarrow \mathfrak{g}^iS_\lambda \rightarrow \mathfrak{g}^{i-1}S_\lambda \rightarrow 0$$

So $A^{(i)}$ is at the same time the kernel of the canonical projection $R^{i+1} \rightarrow R^i$ and the kernel of the restriction of $p^i\phi$ to $\text{gr}^{i+1}S_\lambda$. We will call $A := A^{(0)}$ the \textit{tableau} (determined by $\phi$) and $A^{(i)}$ the $i$-th prolongation of the tableau (see also [8]). It is shown in [12] that

$$A = E_\lambda \otimes (F \otimes S), \quad (36)$$

$$A^{(i)} = (S^2E \otimes C_\lambda) \otimes (S^2F \otimes S) \oplus (\Lambda^2E \otimes C_\lambda) \otimes (\Lambda^2F \otimes S \oplus S) \quad (37)$$

where $\otimes$ is the highest weight component in the tensor product and the $S^2_0$ denotes the trace-free part of the second symmetric power. In particular, we easily compute that

$$\dim A = nk \dim S_\lambda, \quad \dim A^{(i)} = \left(\frac{nk + 1}{2}\right) \dim S_\lambda. \quad (38)$$

The $k$-Dirac operator can be then invariantly defined as the composition

$$\Gamma(S_\lambda) \rightarrow \Gamma(\mathfrak{g}^iS_\lambda) \xrightarrow{D_k} \Gamma(S^k_{\lambda}) \quad (39)$$

where the first map is the canonical inclusion which assigns to a section of $S_\lambda$ its weighted 1-jet at each point. As $\Phi$ is a $G$-equivariant vector bundle map, it follows that $D$ is a $G$-invariant linear differential operator. In particular, the operator $D$ commutes with any infinitesimal symmetry. Recall that infinitesimal symmetries are the right invariant vector fields. Also notice that $Ds(x) = Ds'(x)$ whenever $j^1_1s = j^1_1s'$, $x \in M$ and so we say that the weighted order of $D$ is at most 1.

\footnote{In these special cases the highest weight components has a more direct description. It coincides with the kernel of $S^2_0F \otimes S \rightarrow S$, resp. of $\Lambda^2F \otimes S \rightarrow S$.}
Remark 3.1. Let $X$ be a vector field of the weighted order $r$ which is defined on an open subset $U$ of $M$. Then the formula (29) and the remark below the formula imply that differentiation by $X$ induces a well defined map $\mathfrak{g}^1_x V \rightarrow \mathfrak{g}^{-1}_x V$ for each $x \in U, i \in \mathbb{N}$. In particular, for $X \in \mathfrak{g}_{-1}$ we know that $\text{ord}(L_X) = 1$ and so it follows that the differentiation by $L_X, X \in \mathfrak{g}_{-1}$ induces a linear map $\mathfrak{g}^1_x V \rightarrow \mathfrak{g}^{-1}_x V$ for each $x \in A$. However, it is not true that if we choose $x = x_0$, that the map restricts to a map $\mathbb{A}^i \rightarrow \mathbb{A}^{i-1}$. This follows from the fact that $L_X$ is not an infinitesimal symmetry of the structure and so the operator $D$ does not commute in general with $L_X$ (compare with the local formula (44) of the $k$-Dirac operator).

On the other hand, by Example 2 and 30 it follows that differentiation by the right invariant vector field $R_X$ induces a well defined map

$$R_X : \mathfrak{g}^{i+1}_x S \rightarrow \mathfrak{g}^i S_x. \quad (40)$$

As the right invariant vector fields are infinitesimal symmetries of the parabolic structure, the map $R_X$ commutes with the $k$-Dirac operator $D$ (and hence, with all prolongations $\mathfrak{p}^\phi$). It follows that the map (40) restricts to a well defined map

$$R_X : \mathbb{A}^i \rightarrow \mathbb{A}^{i-1}. \quad (41)$$

This observation will be crucial in the sequel. As the vector field $\partial_{\lambda_x}$ is both left and right invariant, it follows that differentiation by this vector fields induces a well defined linear map

$$\partial_{\lambda_x} : \mathbb{A}^i \rightarrow \mathbb{A}^{i-2}. \quad (42)$$

Finally, notice that $s \in \mathfrak{g}^{i+1}_x S$ is contained in the subspace $\mathbb{A}^{i}, i \geq 1$ iff $R_X s \in \mathbb{A}^{i-1}$ for each $X \in \mathfrak{g}_{-1}$.

4 The set of initial condition for the $k$-Dirac operator

Consider the map $\rho : \mathbb{A} \xrightarrow{\mu} G_- \hookrightarrow G$ where $\mu$ is the inverse of the map on the right hand side in (15) and the second map is the canonical inclusion. Then clearly $\rho$ is a smooth section of the canonical P-bundle over $\mathbb{A}$. Given a section $\psi$ of the vector bundle $S_x$ over $\mathbb{A}$, there is (see Proposition 1.2.7 in [2]) a unique $P$-equivariant function $f \in C^\infty(\rho^{-1}(A), S_x)$ such that $\Gamma(S_x | A) \rightarrow C^\infty(A, S_X)$ is a bijection. As the structure of $G_0$-modules is no longer visible here, we may view, using the isomorphism of the vector spaces from (10), the latter space as $C^\infty(A, S)$. We similarly identify $\Gamma(S_X \Lambda | A)$ with $C^\infty(A, \mathbb{R}^k \otimes S)$. Then there is a unique linear differential operator (which we also for simplicity also denote by $D$)

$$D : C^\infty(A, S) \rightarrow C^\infty(A, \mathbb{R}^k \otimes S) \quad (43)$$

such that $D \psi = D\bar{\psi}$ for each $\psi \in \Gamma(S_x | U)$. This is the $k$-Dirac operator in the trivialization induced by $\rho$. We will call $\Psi \in C^\infty(A, S)$ a spinor valued function or simply a spinor. Then (see [13])

$$D\Psi = (D_1 \Psi, \ldots, D_k \Psi) \text{ where } D_i \Psi = \sum_{\alpha=1}^{n+1} \varepsilon_\alpha L_\alpha \Psi \quad (44)$$
and the dot denotes the Clifford multiplication as in the introduction. A solution of \( D\Psi = 0 \) is called a monogenic spinor. See that the formula (1) differs from (44) only by replacing each coordinate vector field \( \partial_{x_i} \) by \( L_i \).

A real analytic \( \mathbb{S} \)-valued function \( \Psi \) on \( A \) can be written in a unique way as a converging sum \( \sum_{i \geq 0} \Psi_i \) where \( \text{wd}(\Psi_i) = i \).

As \( D \) is a linear combination of vector fields \( L_X \) with \( X \in \mathfrak{g}_{-1} \), we see that \( \text{wd}(D\Psi_i) = i - 1 \). It follows that \( \Psi \) is a monogenic spinor iff each \( \Psi_i \) is a monogenic spinor. We will denote the set of all real analytic monogenic spinors over \( A \) by \( M \) and by \( M_i \) the vector space all homogeneous monogenic spinors of the weighted degree \( i \). The map in (30) then restricts to isomorphism of vector spaces

\[
M_i \to \mathcal{A}^{(i-1)},
\]

\[
\Psi \mapsto \Psi_{x_0},
\]

In this paper we will be interested in the vector spaces \( M_i \). Nevertheless, as we have just pointed out, this provides a good deal of information also about \( M \). But we will not discuss this issue here (see [8] about the issue of the convergence of formal solutions). Let us recall the main result of this article which has been already stated in the introduction.

**Theorem 4.1.** Let \( i \) be a non-negative integer and \( \psi \) be a \( \mathbb{S} \)-valued function defined on the subset \( M(n,k,\mathbb{R}) := \{ x_{n+1,1} = \ldots = x_{n+1,k} = y_{12} \ldots = y_{k-1,k} = 0 \} \) of \( A \) such that each component of \( \psi \) is a homogeneous polynomial of the degree \( i \). Then there is a unique monogenic spinor \( \Psi \in M_i \) such that \( \Psi |_{M(n,k,\mathbb{R})} = \psi \).

Prove of the uniqueness of \( \Psi \): Suppose that the claim is not true. Then there is a non-constant monogenic spinor \( \Psi \) which depends only on the variables \( y_{rs}, x_{n+1,i} \) where \( i, r, s = 1, \ldots, k \). Then for any \( 1 \leq i < j \leq k \) we have that

\[
0 = \partial_{x_1}D_j\Psi = \partial_{x_1} \sum_{\beta=1}^{n+1} \varepsilon_\beta L_\beta j \Psi = -\frac{1}{2} \varepsilon_1 \partial_{y_{ij}} \Psi.
\]

We see that \( \Psi \) does not depend also on any of the variable \( y_{ij} \). So for any \( i = 1, \ldots, k : 0 = D_i \Psi = \varepsilon_{n+1,i} \partial_{x_{n+1,i}} \Psi \). We see that \( \Psi \) is constant. \( \square \)

As we now know that the restriction map \( \Psi \in M_i \mapsto \Psi |_{M(n,k,\mathbb{R})} \) is injective, to finish the proof it suffices to show that the dimension of \( M_i \) is equal to the space of homogeneous spinors on \( M(n,k,\mathbb{R}) \) of the weighted degree \( i \). This is obviously true if \( i = 0 \). From (35) follows that this is true also if \( i = 1, 2 \). It remains to show the claim for \( i = 3, 4, 5, \ldots. \) This will occupy the rest of the paper.

5 The Spencer complex for left invariant differential operators

The Spencer complex (as already introduced in [17] and used in [8], etc.) is a complex of vector bundles which is natural to filtered manifolds. We will consider this complex over the origin. We will use a slightly different definition of the complex. More precisely we will not change the spaces in the complex but we will only change the definition of the co-differentials. Here we will take the
co-differential with respect to the right invariant structure on $A$ rather than the natural left invariant structure. Then we get the Spencer complex associated to the $k$-Dirac operator which has the same form as the Spencer complex used in the theory of exterior differential systems (see [1] and [16]). In the second part of the section we define filtration on the tableau, on its prolongations and on all other spaces in the Spencer complex associated to the $k$-Dirac operator. As we mentioned and explained in the introduction, it is most natural to do this with respect to infinitesimal symmetries of the parabolic structure. Then the co-differential in the Spencer complex associated to the $k$-Dirac operator is nicely adapted to the filtration and we will be able in the proof of Lemma 5.3 to repeat the proof of Proposition 2.5 from [1] which is the key point in the proof of Theorem 1.1.

Let us now recall the construction of the Spencer complex on filtered manifolds. Let $i \in \mathbb{N}$. The exterior derivative of vector valued functions induces an injective linear map of vector spaces

$$\partial : \text{gr}^{i+1}S_\lambda \rightarrow g_1 \otimes \text{gr}^iS_\lambda \oplus g_2 \otimes \text{gr}^{i-1}S_\lambda$$

(47)

$$\partial(i^{i+1}_x f) = (X \mapsto i^{i+1}_{x_0}(L_X f), Y \mapsto i^{i-1}_{x_0}(L_Y f)).$$

where $X \in g_{-1}, Y \in g_{-2}$ and $f$ is the germ of $S_\lambda$-valued function at $x_0$ which satisfies $i^{i}_{x_0} f = 0$. We have used here the notation from (26), isomorphisms $g_i \cong g_{-i}, i = 1, 2$ and we agree that $\text{gr}^0 S_\lambda = 0$ if $i < 0$. Notice that the map (47) is well defined at any point $x \in A$ and that even the first component of $\partial$ is injective (as $H$ is bracket generating). More generally, the exterior derivative of $S_\lambda$-valued $r$-forms gives linear map

$$\partial : \Lambda^r \text{gr}^{i+1}S_\lambda \rightarrow \Lambda^{r+1} \text{gr}^{i-r}S_\lambda$$

(48)

where we put

$$\Lambda^* \text{gr}^rS_\lambda := \bigoplus_{\ell = 0}^{\lfloor \frac{r}{2} \rfloor} \Lambda^{* - \ell} g_1 \wedge \Lambda^{\ell} g_2 \otimes \text{gr}^{r-\ell}S_\lambda.$$  

(49)

We obtain a complex $(\Lambda^* \text{gr}^{i+1}S_\lambda, \partial)$ with trivial cohomology groups (this follows from the fact that the de Rham complex is locally exact). Well known properties of the exterior derivative imply that

$$\partial(\omega \wedge \omega') = \partial \omega \wedge \omega' + (-1)^* \omega \wedge \partial \omega',$$  

(50)

$$\partial(e^i \otimes e_{\alpha}) = 0, \partial(e^r \wedge e^s) = \sum_{\alpha = 1}^{n+1}(e^r \otimes e_{\alpha}) \wedge (e^s \otimes e_{\alpha}).$$  

(51)

where $\omega \in \Lambda^* \text{gr}^rS_\lambda, \omega' \in \Lambda^* \text{gr}^rS_\lambda$. The line (51) follows from (21). However, as we already mentioned in Remark 3.1, it is not true that if we restrict $\partial$ in (47) to the subspace $A^{(i)}$ that we get a map $A^{(i)} \rightarrow g_1 \otimes A^{(i-1)} \otimes g_2 \otimes A^{(i-2)}$. The problem is that the left invariant vector fields are not infinitesimal symmetries of the structure and so they do not commute with the $k$-Dirac operator $D$. To fix this problem we need to consider exterior derivative with respect to right invariant objects on $A$ rather than left invariant objects. This leads us to consider the map:

$$\delta : \text{gr}^{i+1}S_\lambda \rightarrow g_1 \otimes \text{gr}^iS_\lambda \oplus g_2 \otimes \text{gr}^{i-1}S_\lambda$$

(52)

$$\delta(i^{i+1}_x f) := (X \mapsto i^{i+1}_{x_0}(R_X f), Y \mapsto i^{i-1}_{x_0}(R_Y f)).$$  

(53)
where $X \in \mathfrak{g}_{-1}, Y \in \mathfrak{g}_{-2}, \partial f$ are as in (17) and more generally:

$$\delta : \Lambda^r \mathfrak{g}^{r+1} \to \Lambda^r \mathfrak{g}^{r+1} \mathfrak{g}^\Lambda$$

such that (50) is still true with $\partial$ being replaced by $\delta$, but from (23) follows that (51) has to be replaced by

$$\delta(e^r \otimes \varepsilon_\alpha) = 0, \delta(e^r \wedge e^s) = - \sum_{\alpha=1}^{n+1} (e^r \otimes \varepsilon_\alpha) \wedge (e^s \otimes \varepsilon_\alpha).$$

This uniquely pins down $\delta$. Notice that we have not changed the spaces in the complexes but we have only replaced the map $\partial$ by the map $\delta$. By the same reason as above, the cohomology of the complex $(\Lambda^r \mathfrak{g}^{r+1} \mathfrak{g}^\Lambda, \delta)$ is trivial. Now, we can restrict the map (52) to the subspace $\mathfrak{A}^{(i)}$ and we get an injective map

$$\mathfrak{A}^{(i)} \to \mathfrak{g}^1 \otimes \mathfrak{A}^{(i-1)} \oplus \mathfrak{g}^2 \otimes \mathfrak{A}^{(i-2)}$$

which we still denote by $\delta$. More generally, we can restrict (54) also to

$$\mathfrak{A}^{*,*} := \bigoplus_{\ell=0}^{\lfloor \frac{i}{2} \rfloor} \Lambda^{*+\ell} \mathfrak{g}^1 \wedge \Lambda^\ell \mathfrak{g}^2 \otimes \mathfrak{A}^{(i+\ell)}.$$  

It follows that the complex $(\Lambda^r \mathfrak{g}^{r+1} \mathfrak{g}^\Lambda, \delta)$ contains the subcomplex $(\mathfrak{A}^{*,*}, \delta)$ which is the Spencer complex associated to the $k$-Dirac operator.

5.1 Filtration of the tableau

The map (10) is a special case of the (linear map induced by the) Lie derivative $\mathcal{L}_{RX} : \Lambda^r \mathfrak{g}^{r+1} \mathfrak{g}^\Lambda \to \Lambda^r \mathfrak{g}^{r+1} \mathfrak{g}^\Lambda$. This map satisfies $\mathcal{L}_{RX} (\omega \wedge \omega') = (\mathcal{L}_{RX} \omega) \wedge \omega' + \omega \wedge (\mathcal{L}_{RX} \omega')$ where $\omega, \omega'$ are as in (50). The Cartan formula $\mathcal{L}_{RX} = iX \delta + \delta iX$ is still valid where $iX$ is the insertion of $X \in \mathfrak{g}$ into the first entry. Notice that from (55) follows

$$\mathcal{L}_{RX} (e^r \otimes \varepsilon_\beta) = 0, \mathcal{L}_{RX} (e^r \wedge e^s) = \delta^r_se^r \otimes \varepsilon_\alpha - \delta^r_se^s \otimes \varepsilon_\alpha, \mathcal{L}_{\partial_{\alpha}} (e^r \otimes \varepsilon_\alpha) = \mathcal{L}_{\partial_{\alpha}} (e^r \wedge e^s) = 0.$$  

The Lie derivative restricts to a map

$$\mathcal{L}_{RX} : \mathfrak{A}^{*,*} \to \mathfrak{A}^{*,*-1}.$$  

Now we will introduce a filtration on the spaces $\mathfrak{A}^{*,*}$. Let us fix a basis $\{X_1, \ldots, X_{k(n+1)}\}$ of $\mathfrak{g}_{-1}$. We will for simplicity write $R_p$ instead of $R_X$. Then we put for each $r, i$ and $j = 0, \ldots, k(n+1)$:

$$\mathfrak{A}^{*,*}_j := \{s \in \mathfrak{A}^{*,*} | \mathcal{L}_{RX_s} s = \cdots = \mathcal{L}_{RX_j} s = 0\}.$$  

We obtain a filtration

$$\{0\} = \mathfrak{A}^{*,*}_k \supset \mathfrak{A}^{*,*}_{k(n+1) - 1} \supset \cdots \supset \mathfrak{A}^{*,*}_1 \subset \mathfrak{A}^{*,*}_0 = \mathfrak{A}^{*,*}$$

of $\mathfrak{A}^{*,*}$. We will for simplicity write $A^{(i)}_j$ instead of $\mathfrak{A}^{*,*}_j$.
We have now given all necessary definitions. Now we can proceed with the proof of Theorem 1.1. Let us briefly go through the next steps. We will start with Lemmas 5.1 and 5.2 which are needed in order to make the machinery of the Cartan-Kähler theorem running also for the $k$-Dirac operator. Notice that in contrast to the case of the classical Cartan-Kähler theorem, it is not at all clear that Lemma 5.2 is true. The problem here is that the right invariant vector fields do not commute but rather:

$$[L_{R^\alpha}, L_{R^\beta}] = -\delta_{\alpha\beta} L_{\partial y^uv}. \quad (63)$$

However, we will show that if we choose the basis

$$\{e_1 \otimes \varepsilon_1, \ldots, e_1 \otimes \varepsilon_n, e_2 \otimes \varepsilon_1, \ldots, e_2 \otimes \varepsilon_n, \ldots, e_k \otimes \varepsilon_1, \ldots, e_k \otimes \varepsilon_n, e_1 \otimes \varepsilon_{n+1}, \ldots, e_k \otimes \varepsilon_{n+1}\}, \quad (64)$$

then everything works just as for the classical Cartan-Kähler theorem. Then it follows that for each $i, j$ there is a sequence

$$0 \to A^{(i+1)}_j \to A^{(i+1)}_{j-1} \xrightarrow{R^j_i} A^{(i)}_{j-1} \to 0. \quad (65)$$

Clearly, the sequence is a complex which is exact in the middle by the definition of $A^{(i+1)}_i$. As $A^{(i+1)}_{k(n+1)} = 0$, we obtain an upper bound on the dimension of $A^{(i+1)}_{i+1}$:

$$\dim A^{(i+1)}_j \leq \sum_{j=0}^{k(n+1)} \dim A^{(i)}_j. \quad (66)$$

Notice that there is equality in (66) iff $R^j_i$ is surjective for each $j = 1, 2, \ldots, (n+1)k$. We will show in Lemma 5.3 that the latter condition holds. In other words, we will prove that (65) is a short exact sequence (in the language of the classical Cartan-Kähler theorem this property is the definition of the involutivity of the system). The fact that (65) is a short exact sequence for each $i, j$ is all we need to finish the proof of Theorem 1.1 (see the end of the section). After this short summary, we can proceed by verifying all steps.

**Lemma 5.1.** Let $i \geq 0$ be an integer and \{\(A^{(i+1)}_j : j = 0, \ldots, k(n+1)\}\} be the filtration of $A^{(i+1)}$ with respect to the basis (63). If $f \in A^{(i+1)}_{nr}$ where $r = 1, \ldots, k-1$, then $\partial_{y^sf} = 0 \in A^{(i-1)}$ whenever $s \leq r$.

**Remark 5.1.** Before going through the proof, let us make few observations. We will view the element $f \in A^{(i+1)}_{nr}$ from the statement of Lemma 5.1 as a homogeneous monogenic spinor of the weighted degree $i + 2$ (here we use the isomorphism from (45)). If $f$ satisfies the hypothesis and the conclusion of the lemma, then $f$ depends only on the variables $x_\alpha, y_{cd}$ where $j > r, d > c > r$ and $\alpha = 1, \ldots, n + 1$. This easily follows from the following computation. Suppose that $s \leq r, \alpha = 1, \ldots, n$. Then we have

$$0 = R_{\alpha r} f = \partial_{x_\alpha}, f + \frac{1}{2} \sum_{t=1}^{k} x_\alpha t \partial_{y^sf} f = \partial_{x_\alpha}, f. \quad (67)$$

$$0 = D^r f = \sum_{\beta=1}^{n+1} \varepsilon^r_{\beta} L^r_{\beta r} f = \sum_{\beta=1}^{n+1} \varepsilon^r_{\beta} \partial_{x_\alpha}, f = \varepsilon^r_{n+1} \partial_{x_{n+1}}, f. \quad (68)$$
In particular, if \( f \in A_{n(k-1)}^j \), then \( f \) depends only on the variables \( x_{\alpha k}, \alpha = 1, \ldots, n+1 \). Then \( D_i f = 0, i = 1, \ldots, k-1 \) and \( D_k f \) is the usual Dirac operator (in one variable) which is an involutive system. It also follows from \( \text{(58)} \) that

\[
A_j^{r,i} \subset \bigoplus_{\ell=0}^{\frac{j}{2}} \Lambda^{r-\ell} g_1 \wedge \Lambda^\ell g_2 \otimes A_{n,j}^{i-\ell}
\]

(69)

where we write \( j = nJ + \rho, \rho = 1, \ldots, n-1 \).

Proof of Lemma 5.1 Let us suppose that the claim is not true. Then we may choose \( f \) (which we view as an element of \( M_{i+2} \) as in Remark 5.1) which satisfy the hypothesis but \( \partial_{y_{u}}, f \neq 0 \) for some \( s \leq r, t > s \). We will show that this leads to a contradiction. We may also assume that \( i, r \) are minimal, i.e. if \( g \in A_{n(i)}^{j} \) satisfies \( \partial_{y_{u}}, g \neq 0 \) for some \( s \leq j \), then \( t > i \) or \( t = i, j \geq r \). In particular, as \( f \in A_{n(i)}^{j} \subset A_{n+1}^{(i-1)} \), we have that \( s = r \) by the choice of \( f \). As we have showed in Remark 5.1 \( f \) may depend only on the variables \( x_{\alpha j}, y_{cd} \) where \( j \geq r, d > c \geq r, \alpha = 1, \ldots, n+1 \). We may also assume that \( f \) depends only on the variables \( x_{\alpha j}, y_{cd} \) where \( j > r, d > c > r, \alpha = 1, \ldots, n+1 \). This implies that

\[
f = \sum_{u > r} y_{ru} f_{ru} + \ldots \tag{70}
\]

where \( \ldots \) is a function which does not depend on any of the variable \( y_{cd}, c \leq r \). By the assumption above, \( f_{ru} \neq 0 \). From (19) follows that \( R_{n+1,j} f \in A_{n(i)}^{j} \) and hence as before, \( R_{n+1,j} f \) does not depend on any of the variable \( y_{cd}, c \leq r \). Using (70), we have

\[
R_{n+1,j} f = \sum_{u > r} y_{ru} \partial_{x_{n+1}, j} f_{ru} + \ldots \tag{71}
\]

where \( \ldots \) is a function which does not depend on any of the variable \( y_{ru}, u = 1, \ldots, k \). It follows that \( f_{ru} \) does not depend on \( x_{n+1,j} \) for \( j = 1, \ldots, k \). We have that

\[
0 = R_{n+1,j} f = \sum_{u > r} \frac{1}{2} x_{au} \partial_{y_{ru}} f + \partial_{x_{au}} f - \sum_{u > r} \frac{1}{2} x_{au} f_{ru} + \partial_{x_{au}} f; \alpha < n \tag{72}
\]

\[
0 = \sum_{\alpha=1}^{\alpha=n+1} \varepsilon_\alpha L_{n+1} f = \sum_{\alpha=1}^{\alpha=n} \varepsilon_\alpha (R_{n+1} f - \sum_{u > r} x_{au} \partial_{y_{ru}} f) + \varepsilon_{n+1} L_{n+1,j} f
\]

\[
= - \sum_{\alpha=1}^{\alpha=n+1} \varepsilon_\alpha (x_{n+1,j} f_{ru}) + \varepsilon_{n+1} L_{n+1,j} f. \tag{73}
\]

Now the first term in (73) depends only the variables \( x_{\alpha j}, y_{cd} \) where \( j > r, d > c > r, \alpha = 1, \ldots, n \). In particular, it does not depend on \( x_{n+1,j} \) for \( j = 1, \ldots, k \). This implies that the same is true for

\[
L_{n+1,j} f = \varepsilon_{n+1} (\partial_{x_{n+1,j} r} - \frac{1}{2} \sum_{u > r} x_{n+1,u} \partial_{y_{ru}}) f = \varepsilon_{n+1} (\partial_{x_{n+1,j} r} f - \frac{1}{2} \sum_{u > r} x_{n+1,j} f_{ru}).
\]

19
Combining this together with (72), we obtain that

\[
f = \sum_{u > r} (y_{ru} f_{ru} - \frac{1}{2} \sum_{a=1}^{n} x_{ar} x_{au} f_{ru} + \frac{1}{2} x_{r+1,ru} x_{n+1,u} f_{ru}) + \ldots
\]

where \ldots is a function which depends only on \( x_{ac}, y_{uv} \) where \( c, u, v > r \). Then

\[
0 = \sum_{\alpha=1}^{n+1} \varepsilon_{\alpha} L_{\alpha} f = \sum_{\alpha \leq n} \varepsilon_{\alpha} L_{\alpha} f + \varepsilon_{n+1} L_{n+1} f
\]

\[
= \sum_{\alpha \leq n} \varepsilon_{\alpha} (-\frac{1}{2} x_{ar} f_{rt} + \frac{1}{2} x_{ar} f_{rt}) + \frac{1}{2} \varepsilon_{n+1} (x_{n+1,r} f_{rt} + x_{n+1,r} f_{rt}) + \ldots
\]

where \ldots represents a function which depends on \( x_{ac}, y_{uv} \) where \( c, u, v > r, \alpha = 1, \ldots, n+1 \). We see that the equality holds iff \( f_{rt} = 0 \). The claim is proved. \( \square \)

We will formulate Lemma 5.3 not only for the basis (64) but for a particular set of bases of \( g \) although this is not necessary in order to show that (65) is a well defined sequence. We will need this more general statement in the proof of Lemma 5.3.

Lemma 5.2. Let \( \sigma \) be a permutation of \( \{1, 2, \ldots, n\} \). Let \( \{A_{j}^{r,i}[\sigma] : j = 0, \ldots, (n+1)k \} \) be the filtration of \( \mathcal{H}^{r,i} \) with respect to the basis

\[
\{e_{1} \otimes \varepsilon_{\sigma(1)}, \ldots, e_{1} \otimes \varepsilon_{\sigma(n)}, e_{2} \otimes \varepsilon_{\sigma(1)}, \ldots, e_{2} \otimes \varepsilon_{\sigma(n)}, \ldots, e_{k} \otimes \varepsilon_{\sigma(1)}, \ldots, e_{k} \otimes \varepsilon_{\sigma(n)}\},
\]

(74)

\[
\ldots, e_{k} \otimes \varepsilon_{\sigma(n)}, e_{1} \otimes \varepsilon_{n+1}, \ldots, e_{k} \otimes \varepsilon_{n+1}\}
\]

of \( g_{-1} \). Then for each \( i, r, j : L_{R_{i}} f \in \mathcal{K}_{j-1}^{r,i+1}[\sigma] \) whenever \( f \in \mathcal{K}_{j}^{r,i}[\sigma] \).

Proof: The claim follows from (63), the Leibniz property of Lie derivative, the formulas (59), (60) and Lemma 5.1. \( \square \)

We will use the following notation. Let us for simplicity denote the basis (64) by \( \{X_{1}, \ldots, X_{k(n+1)}\} \) and let \( \{\varpi_{1}, \ldots, \varpi_{k(n+1)}\} \) be the dual basis of \( g_{1} \) so that \( \varpi_{p}(X_{q}) = \delta_{pq} \) for each \( p, q \). We will put \( e^{r_{j}} := e^{r} \wedge e^{s} \) so we obtain a basis \( \{e^{12}, \ldots, e^{k-1,k}\} \) which is dual to the basis \( \{e_{1} \wedge e_{2}, \ldots, e_{k-1} \wedge e_{k}\} \) of \( g_{-2} \). Then we can write down the map (52) as

\[
\delta(f) = \sum_{l=1}^{k(n+1)} \varpi_{l} \otimes R_{l} f + \sum_{1 \leq r < s \leq k} e^{r} \otimes \partial_{y_{rs}} f
\]

(75)

where \( f \in \mathcal{K}^{i}(\sigma) \).

By Remark 5.1 we have that \( \dim(\mathcal{K}^{i}(\sigma)) = 0 \) for \( j \geq kn \). In Section 4.3 in 12 can be found that

\[
\dim(\mathcal{A}^{i}(0)) = \left\{ (nk - j) \dim S; \ j = 0, \ldots, nk, \ 0; \ j \geq nk. \right\}
\]

(76)

It can be easily seen that the formula holds also if we replace \( \mathcal{A}^{i}(0) \) by \( \mathcal{A}^{i}(0)[\sigma] := \mathcal{A}_{j}^{0,0}[\sigma] \) where \( \sigma \) is any permutation from Lemma 5.2. As \( \dim(\mathcal{A}^{i}(1)) = \left\lfloor \frac{(nk+1)^{2}}{4} \right\rfloor \dim S \)
(see formula (38)), we have that

$$\dim \mathcal{A}(l) = \sum_{j=0}^{k(n+1)} \dim(\mathcal{A}^{(0)}_{j} [\sigma])$$  \hspace{1cm} (77)$$

and so we have that for each \( j = 1, 2, \ldots, k(n + 1) \) the map

$$R_j : \mathcal{A}^{i+1}_{j-1} [\sigma] \to \mathcal{A}^{i}_{j-1} [\sigma]$$  \hspace{1cm} (78)$$
is surjective if \( i = 0 \).

**Lemma 5.3.** Let \( \sigma \) be a permutation of \( \{1, \ldots, n\} \), \( i \geq 0 \) and \( j = 1, \ldots, k(n+1) \). Then the map \( R_j \) in (78) is surjective.

Proof: As we have just seen above, the claim is true if \( i = 0 \). Let \( r \geq 0, p = 1, \ldots, k(n + 1) \) and assume that (78) is surjective whenever \( \sigma \) is a permutation of \( \{1, \ldots, n\} \) and \( i < r, j = 1, \ldots, k(n+1) \) or \( i = r, j = p + 1, \ldots, k(n+1) \). We will show that then (78) is surjective also when \( \sigma \) is the identity permutation and \( i = r, p = j \). The proof for arbitrary permutation is similar. Let us consider the following commutative diagram

$$
\begin{array}{ccccccccc}
\mathcal{A}^{(r+1)}_{p-1} & \xrightarrow{\delta} & \mathcal{A}^{1,r}_{p-1} & \xrightarrow{\delta} & \mathcal{A}^{2,r-1}_{p-1} & \xrightarrow{\delta} & \cdots \\
\downarrow{R_p} & & \downarrow{L_R_p} & & \downarrow{L_R_p} & & \\
\mathcal{A}^{(r)}_{p-1} & \xrightarrow{\delta} & \mathcal{A}^{1,r-1}_{p-1} & \xrightarrow{\delta} & \mathcal{A}^{2,r-2}_{p-1} & \xrightarrow{\delta} & \cdots \\
\end{array}
\hspace{1cm} (79)$$

Let \( Q \in \mathcal{A}^{(r)}_{p-1} \). Suppose that there exists \( T \in \mathcal{A}^{1,r}_{p-1} \) such that the following is true:

1. \( R_p T = \delta Q \).
2. \( \delta T = 0 \).
3. \( i_{R_{p}} T = \cdots = i_{R_{p-1}} T = 0 \).

Then by the exactness of the complex (54) there is \( P \in \gr^{r+2} S \) such that \( \delta P = T \). As \( R_s P = i_{R_s} \delta P = i_{R_s} T = s = 1, \ldots, k(n+1) \) we have that \( R_s T = \cdots = R_{p-1} T = 0 \). Since \( R_s P \in \mathcal{A}^{(r)} \) for each \( s = 1, \ldots, k(n+1) \) it follows that \( P \in \mathcal{A}^{(r+1)}_{p-1} \). Then \( \delta R_p (P) = L_R_p \delta (P) = L_R_p (T) = \delta Q \). By the injectivity of \( \delta \) at the beginning, it follows that \( Q = R_p (P) \). Hence, it is enough to find \( T \).

If \( p > nk \), then by Remark 5.1 we have \( \mathcal{A}^{(r+1)}_{p-1} = 0 \) and so there is nothing to prove. If \( nk \geq p > k(n-1) \), then by the Remark 5.1 we are dealing with the Dirac operator, this is an involutive system and so the claim follows also in this case. So we may actually assume that \( p \leq n(k-1) \). Write \( p = nJ + \rho \) where \( 0 < \rho \leq n \). Then \( J = 0, 1, \ldots, k-2 \) and \( X_p = e_{J+1} \otimes e_{\rho} \). By Lemma 5.1 we have that \( \delta Q = \sum_{l \geq p} \varpi \otimes q_l + \sum_{1 \leq c < d \leq n} c d \otimes q_{cd} \) where \( q_l = R_x Q \), \( q_{rs} = \delta q_{rd} \).

We will first show that there is \( U \in \mathcal{A}^{1,r}_{p-1} \) such that \( L_R_p U = \delta Q \). If \( q_{cd} = 0 \) for all \( c, d \), then \( q \in \mathcal{A}^{(r-1)}_{p-1} \) for each \( l \geq p \) and so by the induction hypothesis there are \( u \in \mathcal{A}^{(r)}_{p-1}, l \geq p \) such that \( R_p u \delta q_l \) and so we may put \( U = \sum_{l \geq p} \varpi \otimes u_l \).

In general, each \( q_{cd} \in \mathcal{A}^{(r-2)}_{p-1} \) and so by the induction hypothesis there are
follows by (69) that $L_{\sigma_t} = \delta_{i}^{\prime}$. Let us fix $t = 1, \ldots, \rho - 1$ and let $\sigma_t$ be the transposition $(t, \rho - 1)$. We have that $L_{\rho-1}^{(r-1)[[\sigma_t]} \subset L_{\rho-2}^{(r-1)[[\sigma_t]}$. So by the induction hypothesis: for each $d > J + 1, t = 1, \ldots, \rho - 1$ there is $u'_{dt} \in \mathbb{H}^{(r)}_p[\sigma_t]$ such that $u_{dt} = u_{J+1,t}$. Notice that $X_nJ+t = e_{J+1} \otimes e_t$ and so from (58) follows that $\mathcal{L}_{R_nJ+t,\varepsilon} = -\delta_{i+1}^{\prime} + (d-1)n+t$ whenever $J < c < d$.

Consider

$$U'' := \sum_{J+1 < t < J+\rho-1} \mathscr{w}_{n(d-1)-t} \otimes u_{dt} + \sum_{J < c < d} e^{cd} \otimes u_{cd}.$$ 

Then we clearly have that $U'' \in \mathbb{H}^{(r)}_{nJ}$. For each $t = 1, \ldots, \rho - 1$ we have

$$\mathcal{L}_{R_nJ+t,\varepsilon} = \sum_{J < c < d} \mathscr{w}_{n(d-1)-t} \otimes u_{J+1,t} - \mathscr{w}_{(d-1)n+t} \otimes u_{J+1,t} = 0$$

and so we see actually see that $U'' \in \mathbb{H}^{(r)}_{nJ}$. Then we have already proved that there is $U'' = \mathcal{L}_{R_nJ''} = \delta Q - \mathcal{L}_{R_nJ}U''$ and so we can put $U := U' + U''$.

We have that $V := \delta U \in \mathbb{H}^{(r)}_{nJ-1}$ and for each $j < p : i_{J,n} \delta U = R_{i_{J,n}}U = 0$. Since $\mathcal{L}_{R_nJ} \delta U = \delta \mathcal{L}_{R_nJ}U = \delta^2 Q = 0$ we find that $V \in \mathbb{H}^{(r)}_{nJ}$.

Now we can complete the proof of the main theorem.

Proof of Theorem 1.14 As we have already proved injectivity of the restriction map $\Psi \in \mathfrak{M}^{1,1}_+ \rightarrow \Psi_{[M(n,k,R)]}$, we know that $\dim \mathfrak{M}^{1,1}_+ \leq \binom{n+k+1}{k+1} \dim \mathbb{S}$. Hence, $\bigstar$
it is enough to show that there is equality. We will prove by induction on \( p, r \) that
\[
\dim A_r^{(p)} = \left( \frac{nk - r + p}{p + 1} \right) \dim S. \tag{80}
\]
Check that the right hand side is equal to the dimension of the space of homogeneous polynomials of the degree \( p + 1 \) in \( nk - r \) variables multiplied by \( \dim S \) (where we agree that the binomial coefficient is zero if \( nk - r + p < p + 1 \)). By (76) the claim is true for \( p = 0, r = 0, 1, \ldots, (n + 1)k \). Let us fix non-negative integers \( i, j \). We suppose that (80) is true if \( p < i \) or \( p = i, k(n + 1) \geq r > j \). We want to prove the claim also \( i, j \). As we already know that (65) is short exact for each \( i, j \), we have that
\[
\dim A_j^{(i)} = \dim A_{j+1}^{(i)} + \dim A_{j-1}^{(i)}
= \left( \left( nk - j - 1 + i \right) + \left( nk - j + i - 1 \right) \right) \dim S
= \left( \frac{nk - j + i}{i + 1} \right) \dim S.
\]
The equality we wanted to prove is the particular case \( j = 0 \) as \( \dim M_{i+1} = \dim A^{(i)} \).

\[ \square \]

References

[1] Bryant R. L., S. S. Chern, R. B. Gardner, H. L. Goldschmidt and P. A. Griffiths. Exterior differential systems, Mathematical Sciences Research Institute Publications, vol. 18, Springer-Verlag, New York, 1991.

[2] Čap, Andreas, Jan Slovák. Parabolic Geometries I, Background and General Theory. American Mathematical Society, Providence, 2009. ISBN 978-0-8218-2681-2.

[3] Colombo, Fabrizio, Irene Sabadini, Franciscus Sommen, Daniele C. Struppa. Analysis of Dirac Systems and Computational Algebra. Birkhauser, Boston, 2004. ISBN 0-8176-4255-2.

[4] Goodman Roe, Nolan R. Wallach: Representations and Invariants of the Classical Groups. Springer, New York, 2009. ISBN 978-0-387-79581-6

[5] Kruglikov Boris. Symmetries of filtered structures via filtered Lie equations. Journal of Geometry and Physics, vol. 85, 2014, p. 164-170.

[6] Morimoto, Tohru. Théorème de Cartan-Kähler dans une classe de fonctions formelles Gevrey. C. R. Acad. Sci. Paris. 311, série A. 1990, p. 443-436.

[7] Morimoto, Tohru. Théorème d’existence de solutions analytiques pour des systèmes d’équations aux dérivées partielles non-linéaires avec singularités. C.R. Acad. Sci. Paris. 321, série I. 1995. p. 1491-1496.
[8] Morimoto, Tohru. Lie algebras, geometric structures and differential equations on filtered manifolds. In Lie Groups Geometric Structures and Differential Equations - One Hundred Years after Sophus Lie, Adv. Stud. Pure Math., Math. Soc. of Japan, Tokyo. 2002, p. 205-252.

[9] Morimoto, Tohru: Differential Equations Associated to a Representation of a Lie algebra from the Viewpoint of Nilpotent Analysis. RIMS Kokyuroku 1502, Kyoto University. 2006/07, p. 238-250.

[10] Morimoto, Tohru. Generalized Spencer Cohomology Groups and Quasi-Regular Bases. Tokyo J. Math. vol. 14, no. 1, 1991, p. 165-179.

[11] Neusser, Katharina. Prolongation on regular infinitesimal flag manifolds. International Journal of Mathematics. vol. 23, no. 4, 2012, p. 1-41.

[12] Salač, Tomáš. k-Dirac operator and the Cartan-Kähler theorem. Archivum mathematicum. vol. 49, no. 5, 2013, p. 333-346.

[13] Salač, Tomáš. k-Dirac operator and parabolic geometry. J. comp. anal. and oper. theo. vol. 8, no. 2, 2014, p. 383-408.

[14] Sabadini, Irene, Franciscus Sommen, Daniell C. Struppa, Peter van Lancker. Complexes of Dirac operators in Clifford algebras. Mathematische Zeitschrift. vol. 239, no. 2, 2002, p. 293-320.

[15] Slovák, Jan, Vladimír Souček. Invariant Operators of the First Order on Manifolds with a Given Parabolic Structure. Societé Mathématique de France. vol. 4, 2001, p. 251-276.

[16] Spencer, D.C. Overdetermined systems of linear partial differential equations. Bull. Amer. Math. Soc., vol. 75, no. 2, 1969, p. 179-239.

[17] Tanaka, Noboru. On the equivalence problems associated with simple graded Lie algebras. Hokkaido Math. J., vol. 8, no. 1, 1979, p. 23-84.