Optimal quantum source coding with quantum side information at the encoder and decoder

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Abstract—Consider many instances of an arbitrary quadripartite pure state of four quantum systems $ACBR$. Alice holds the $AC$ part of each state, Bob holds $B$, while $R$ represents all other parties correlated with $ACB$. Alice is required to redistribute the $C$ systems to Bob while asymptotically retaining the purity of the global states. We prove that this is possible using $Q$ qubits of communication and $E$ ebits of shared entanglement between Alice and Bob provided that $Q \geq \frac{1}{2} I(R;C|B)$ and $Q + E \geq H(C|B)$. This matches the outer bound for this problem given in [1]. The optimal qubit rate provides the first known operational interpretation of quantum conditional mutual information. We also show how our protocol leads to a fully operational proof of strong subadditivity and uncover a general organizing principle, in analogy to thermodynamics, which underlies the optimal rates.

I. INTRODUCTION

The most fundamental situation in communication theory consists of a two-terminal source coding problem. Here one user, say Alice, attempts to describe a source of information to another user, who we call Bob. If the information source is modeled by a sequence of independent and identically distributed (i.i.d.) random variables $X$, one can ask for the ultimate rate at which the source can be described, in units of bits per sample. It is required that Alice’s description allows Bob to perfectly recreate the source sequence with high probability, although decreasing the error probability generally requires block coding on longer source sequences. According to Shannon’s noiseless channel coding theorem [2], this ultimate rate is given by the Shannon entropy

$$H(X) = - \sum_x p(x) \log p(x).$$

Intuitively, Shannon entropy can be understood as a measure of the information contained in the random variable $X$. Because Shannon entropy answers the question regarding the optimal rate for data compression, one says that the corresponding protocol for data compression provides an operational interpretation of Shannon entropy.

Suppose now that Bob had some a priori information regarding the random variable $X$, in the form of a correlated random variable $Y$. In this case, Slepian and Wolf demonstrated [3] that Alice would only need to send to Bob at a rate given by the conditional entropy

$$H(X|Y) = H(XY) - H(Y)$$

and that surprisingly, Alice would not need to know Bob’s side information to accomplish this task. The so-called Slepian-Wolf protocol for data compression with side information provides an operational interpretation of conditional entropy. Intuitively, one thinks of $H(X|Y)$ as a measure of the information that is to be gained by learning $X$ for one who already knows $Y$. Note that there is no advantage if Alice has additional side information regarding $X$, and that shared common randomness between Alice and Bob is also of no help.

In this paper, we provide a complete solution to a general quantum counterpart of the above scenario. We find that, in contrast to the classical case, additional Alice side information changes the problem, while quantum mechanical entanglement between Alice and Bob, the quantum analog of shared common randomness, is a useful resource. Our problem is fully quantum in a sense introduced by Schumacher [4], where Alice is asked to transfer part of a pure quantum state to Bob, while preserving the purity of the global state. For this, we consider a pure state of four quantum systems $|\psi\rangle_{ACBR}$. Initially, the $A$ and $C$ systems are held by Alice, while $B$ is in the possession of Bob. We refer to $R$ as the reference system and suppose that it is held by a third party, say Robert, and is inaccessible both to Alice and Bob. We determine the cost for Alice and Bob to redistribute the state, so that it is Bob who holds $C$ instead of Alice, thereby transferring the quantum information in $C$ to Bob. More specifically, we analyze the corresponding asymptotic scenario, asking that many copies of the same state be redistributed as above, while requiring that the redistributed states have arbitrarily high fidelity with the originals in the asymptotic limit.

To achieve this task, we allow the use of two fundamental quantum mechanical resources. First, Alice may send qubits (two-level quantum systems) to Bob over a noiseless quantum channel. Second, we allow Alice and Bob to use pre-existing entanglement, shared between themselves in the form of Bell states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle).$$

In the sequel, we refer to such a state as an ebit (entangled bit). We therefore phrase the asymptotic cost of redistributing $C$ to Bob in terms of the number $Q$ of qubits sent and the number $E$ of ebits consumed, per copy of the state. We allow the entanglement cost $E$ to be negative, in which case the corresponding protocol generates entanglement rather than consuming it. Our main result (Theorem [1]) states that it is possible to redistribute the state $|\psi\rangle_{ACBR}$ as above if and
The optimal qubit cost gives the first known operational interpretation of quantum conditional mutual information. In Section IV, we show how our results yield a fully operational proof of strong subadditivity which, unlike previous operational proofs, is logically independent even from the subadditivity of entropy. We conclude with a discussion in Section V where we reflect on the main result and provide a novel thermodynamic interpretation of the optimal rates.

A. Notational conventions

Throughout this paper, we assume familiarity with standard background material in quantum information theory; for a general reference, the reader is referred to [13]. We use capital Roman letters such as $A, B, C$ to denote Hilbert spaces. We write $|A|$ for the dimension of $A$ and use a superscripted label to associate a state to a Hilbert space, by writing $\rho^A$ or $|\psi|^A$. Computational basis states of $A$ are denoted with lower case Roman letters as in $\{|i\rangle^A\}$. Tensor products of Hilbert spaces are written $AB = A \otimes B$. Given a pure state $|\varphi\rangle^{AB}$, we abbreviate $\varphi^{AB} = |\varphi\rangle\langle\varphi|^{AB}$, while writing its partial traces as $\varphi^{A} = Tr_B \varphi^{AB}$. We write $\pi^A$ for the maximally mixed state on $A$, and given two isomorphic Hilbert spaces $A$ and $A'$, we write

$$|\Phi\rangle^{AA'} = \frac{1}{\sqrt{|A|}} \sum_{i=1}^{|A|} |i\rangle^A \langle i|^{A'}$$

for the unique maximally entangled state associated with the isomorphism $|i\rangle^A \mapsto |i\rangle^{A'}$. A quantum channel is a completely positive, trace-preserving linear map $\mathcal{N}^{A \rightarrow B}$ from density matrices on $A$ to those on $B$. Given an isometry $\mathcal{V}^{A \rightarrow B}$, we will abbreviate its adjoint action on density matrices as $\mathcal{V}(\rho) = \mathcal{V}^\dagger \rho \mathcal{V}$. A partial isometry is an isometry when restricted to its support subspace.

For the von Neumann entropy of a density matrix $\varphi^A$ we write

$$H(A) = - Tr \varphi^A \log_2 \varphi^A.$$ 

When the underlying state could be ambiguous we write $H(A)_\rho$. Given a multipartite state $\varphi^{ABC}$, various entropic quantities can be defined in exact analogy to the classical case (see e.g. [14]). Quantum conditional entropy is defined [15] as

$$H(A|B) = H(AB) - H(B),$$ 

quantum mutual information [15] is

$$I(A; B) = H(A) + H(B) - H(AB)$$

and quantum conditional mutual information is given by

$$I(A; B|C) = H(A|C) + H(B|C) - H(AB|C).$$

Observe that the conditional quantities above cannot generally be interpreted as averages, unless the conditioning system is purely classical. Furthermore, notice that conditional entropy

$$Q + E = H(C|B)$$

only if

$$Q \geq \frac{1}{2} I(R; C|B), \quad Q + E \geq H(C|B). \quad (1)$$

This region is depicted in Figure I. The quantities in these bounds, conditional mutual information and conditional entropy, are defined in Section II-A. Simultaneously minimizing the qubit rate $Q$ and the total sum rate $Q + E$ gives the optimal cost pair

$$Q^* = \frac{1}{2} I(R; C|B), \quad E^* = \frac{1}{2} I(C; A) - \frac{1}{2} I(C; B). \quad (2)$$

The optimal qubit cost gives the first known operational interpretation of quantum conditional mutual information. In Section IV, we show that $Q^*$ cannot be negative, which leads to an operational proof of the celebrated strong subadditivity inequality [5]. This proof differs from other such operational proofs [6], [7] in that it follows solely from a direct coding theorem and not from a converse proof. In [8], where our main result was first announced, we showed that $Q^*$ is symmetric under time-reversal, where now Bob redistributes $C$ back to Alice, while $E^*$ is anti-symmetric. The former gives an intuitive understanding to the curious identity

$$I(C; R|A) = I(C; R|B),$$

which holds on any quadripartite pure state. We comment further on this feature in Section V. We also demonstrated there that the corresponding protocol is perfectly composable. This constitutes an exact solution to the quantum analog of result of Cover and Equitz [9] on the successive refinement of classical information, although the classical problem is only known to be exactly soluble in the presence of a Markov condition.

In our protocol, the systems $A$ and $B$ which respectively remain with Alice and Bob can be regarded as quantum side information that is used to reduce the cost of redistribution. By disregarding that side information in various ways, our main result incorporates various quantum Shannon-theoretic protocols as special cases. As we discuss in Section V these include the Schumacher compression [4], fully quantum Slepian Wolf (FQRS) [10], [11], fully quantum Reverse Shannon (FQRS) [10], [11], and state merging [7], [12] protocols.

The paper is organized as follows. In the next subsection we fix our notational conventions. The following section gives an introduction to the resource calculus, and a formal statement of the main result. Theorem I which is proved in Section III. In Section IV we show how our results yield a fully operational proof of strong subadditivity which, unlike previous operational proofs, is logically independent even from the subadditivity of entropy. We conclude with a discussion in Section V where we reflect on the main result and provide a novel thermodynamic interpretation of the optimal rates.
can in fact be negative, as is evidenced by any pure entangled state on $AB$. On the other hand, $I(A; B|C)$ is never negative, a fact which is known as strong subadditivity [5]. In Section [11] we show how our main result leads to a self-contained proof of strong subadditivity.

II. RESOURCE INEQUALITIES

It will be convenient for us to use the high-level notation of resource inequalities [16], [17] to express our main result, as well as to describe various intermediate protocols introduced during the proof. We take a somewhat more elementary path than that given in [17] which is sufficient for our purposes.

A. Finite resource inequalities

A single ebit shared between Alice and Bob is denoted $[qq]$. The notation $[q → q]$ represents a noiseless qubit channel from Alice to Bob, while a noiseless classical bit channel is written $[c → c]$. A finite resource inequality is an expression such as

$$[q → q] ≥ [c → c], \quad [q → q] ≥ [qq]$$

meaning that the resource on the left can simulate the one on the right. The above two examples respectively signify that a qubit channel can be used to send classical bits (by signaling with orthogonal pure states), or otherwise can be used to distribute entanglement (by transmitting halves of locally prepared ebits). Addition of two resources may be used to distribute entanglement (by transmitting halves of

$$[q → q] + 2[c → c] ≥ [q → q], \quad [qq] + [q → q] ≥ 2[c → c].$$

B. Approximate resource inequalities

Given two quantum states $ρ$ and $σ$ of the same quantum system, we may judge their closeness using either the trace distance $||ρ − σ||_1$ or the fidelity $F(ρ, σ) = \sqrt{\text{Tr}(√ρσ)}^2$. Note that when one of the states is pure, $F(⟨φ|σ⟩) = ⟨φ|σ|φ⟩$. A useful characterization of fidelity – Uhlmann’s theorem – says that if $|ψ⟩$ is a purification of $ρ$, then $F(ρ, σ)$ is the maximum of $|⟨ψ|φ⟩|^2$ over all purifications $|φ⟩$ of $σ$. Fidelity and trace distance related by the inequalities

$$F(ρ, σ) ≥ 1 - ||ρ − σ||_1 \quad (4) \quad ||ρ − σ||_1 ≤ 2\sqrt{1 - F(ρ, σ)} \quad (5)$$

Therefore, fidelity and trace distance are equivalent distance measures when one is interested in arbitrarily good approximations of states are we are here. An approximate resource inequality

$$\sum_i a_i ≥ ϵ \sum_j b_j$$

is a finite resource inequality which holds with an error of $ϵ$ in the following sense. Consider acting on half of a maximally entangled state with each target resource $b_j$ that is a channel, and call the resulting global state $Ω$. Note that $Ω$ should also contain the $b_j$ that are quantum states. Now, let $Ω'$ be the simulated version of this state, obtained by using the resources $a$. We require that $Ω$ and $Ω'$ are $ϵ$-close in either trace distance or fidelity. The particular measure is not important, as we are ultimately concerned with asymptotics, where $ϵ$ can be arbitrarily small.

C. Asymptotic resource inequalities

The notion of finite resource inequalities can be generalized to that of an asymptotic resource inequality, which is a formal expression of the form

$$\sum_i R_{in}^{(i)} a_i ≥ \sum_j R_{out}^{(j)} b_j. \quad (6)$$

Here the $a_i$ and $b_j$ are resources and the rates $R_{in}^{(i)}$ and $R_{out}^{(j)}$ are nonnegative real numbers. We shall consider the inequality (6) to be shorthand for the following formal statement: for every $c > 0$, every set of rates $R_{in}^{(i)} > R_{in}^{(i')}$, $R_{out}^{(j)} < R_{out}^{(j')}$ and all sufficiently large $n$, the approximate resource inequality

$$\sum_i |nR_{in}^{(i)}| a_i ≥ ϵ \sum_j |nR_{out}^{(j)}| b_j$$

holds. Below, we use Greek letters to denote linear combinations of finite resources which appear in asymptotic resource inequalities. In some asymptotic resource inequalities, we may only require a sublinear amount $o(n)$ of a particular input resource. In such cases, we write $\alpha + β ≥ γ$ if we have $Ra + β ≥ γ$ for every $R > 0$.

It will also be convenient for us extend the definition of asymptotic resource inequalities to have negative rates on the left. Such rates are interpreted as meaning that the corresponding resources are generated rather than consumed. Formally, these resources should be negated and moved to the right. Let us introduce two powerful lemmas which are the raison d’être for the entire formalism of asymptotic resource inequalities and which play important roles in our proofs.

Lemma 1 (Composition lemma [17]):

$$α ≥ β \text{ and } β ≥ γ \Rightarrow α ≥ γ.$$  

Lemma 2 (Cancellation lemma [17]): Given rates which satisfy $R_{in} > R_{out} ≥ 0$,

$$R_{in}α + β ≥ R_{out} a + γ \Rightarrow (R_{in} - R_{out})a + β ≥ γ.$$  

Otherwise, if $R_{out} ≥ R_{in} ≥ 0$, then

$$R_{in}α + β ≥ R_{out} a + γ \Rightarrow oα + β ≥ (R_{out} - R_{in})a + γ$$

D. Distributed states

In Schumacher data compression, Alice wishes to transmit the $C$ parts of many instances of the state $ψ^{CR}$ to Bob while asymptotically preserving the entanglement with $R$. We introduce the following notation to describe the corresponding coding theorem:

$$ψ^{[θ]} + H(C)|q → q| ≥ ψ^{θ|C}.$$
The notation $\psi^{C|B}$ indicates that Alice holds the $C$ parts of many i.i.d. instances of some fixed purification $|\psi\rangle^{CR}$ of the density matrix $\psi^C$ while Bob holds nothing. On the right, the expression $\psi^{B|C}$ refers to the same purifications as on the left, only it is Bob who is holding the $C$ systems. In other words, Alice attempts to simulate identity channels from the systems $C$ in her lab to identical systems $C$ located in Bob’s lab. This channel is only required to work well when the input is equal to $\psi^C$. In [17], the formalism of relative resources was introduced for these purposes, though our alternate notation is sufficient for our needs. State redistribution involves a purification $|\psi\rangle^{ACBR}$ of a tripartite density matrix $\psi^{ACB}$. We denote the distributed states before and after the protocol as $\psi^{AC|B}$ and $\psi^A|CB$ since Alice begins by holding $AC$ and ends by only holding $A$. The rates in an asymptotic resource inequality involving such distributed states will in general be more tractable than the expression $\psi^{AC|B}$.

Theorem 1:

$$\psi^{AC|B} + Q[q \rightarrow q] + E[qq] \geq \psi^A|CB$$

if and only if $Q$ and $E$ satisfy (1), i.e. are contained in the region depicted in Figure 4. The converse part of the proof of Theorem 4 i.e. that $Q$ and $E$ must satisfy (1), is proved in [11]. We thus focus on proving a coding theorem showing that (7) is satisfied whenever $Q$ and $E$ satisfy (1). Because $[q \rightarrow q] \geq [qq]$, it suffices for us to demonstrate (7) for the corner point $(Q^*, E^*)$ defined in 2.

III. PROOF OF THEOREM 4

To prove Theorem 4 we will demonstrate the existence of the following auxiliary protocol which transfers $C^n$ to Bob, which has the desired net communication and entanglement cost.

Theorem 2:

$$\psi^{AC|B} + \frac{1}{2}I(C; RB)[q \rightarrow q] + \frac{1}{2}I(C; A)[qq] \geq \psi^{ACB} + \frac{1}{2}I(C; B)[q \rightarrow q] + \frac{1}{2}I(C; B)[qq].$$

Together with the cancellation lemma (Lemma 3), Theorem 2 yields a proof of Theorem 4. However, observe that if $I(B; C) \geq I(A; C)$, the cancellation lemma still requires a sublinear amount of entanglement on the left. Similarly, if we have $I(C; RB) = I(C; R)$ (i.e. if strong subadditivity is saturated), a sublinear amount of communication will also be required. However, because $[q \rightarrow q] \geq [qq]$, the additional entanglement cost can be absorbed into the communication rate and is therefore only relevant if the state $\psi^{C|RB}$ saturates strong subadditivity. We discuss this point further in Section V.

We prove Theorem 2 by means of another protocol which simulates coherent channels [18]. A coherent channel $[q \rightarrow qq]$ is a type of quantum feedback channel which is an isometry from Alice to Alice and Bob:

$$|0\rangle^A |0\rangle^B (|0\rangle^A + |1\rangle^A) |1\rangle^B (|1\rangle^A).$$

Using a coherent version of teleportation, where Alice and Bob apply only local unitaries, it is known that [18]

$$[qq] + 2[q \rightarrow qq] \geq 2[qq] + [q \rightarrow q].$$

Repeated concatenation yields the following asymptotic resource inequality [18]:

$$2[q \rightarrow qq] \geq [q \rightarrow q] + [qq].$$

In fact, the opposite direction holds as a finite resource inequality, but it will not be useful for us here. In this paper, we devote most of our efforts toward proving the following theorem which, when combined with (8) and the composition lemma (Lemma 1), provides a proof of Theorem 2.

**Theorem 3:**

$$\psi^{AC|B} + \frac{1}{2}I(C; RB)[q \rightarrow q] + \frac{1}{2}I(C; A)[qq] \geq \psi^{ACB} + I(C; B)[q \rightarrow qq].$$

A. Proof of Theorem 5

Our proof of Theorem 5 relies on the following robust one-shot version, whose proof we delay until Section III-B.

**Theorem 4 (Robust one-shot redistribution protocol):**

Let a pure state $|\psi\rangle^{ACBR}$ and a maximally entangled state $|\Phi\rangle^{AB}$ be given, where $|A\rangle = |B\rangle$ divides $|C\rangle$. Suppose that $|\varphi\rangle^{ACBR}$ and $|\phi\rangle^{ACBR}$ are states satisfying

$$\max \{ \| \psi^{ACBR} - \varphi^{ACBR} \|_1, \| \psi^{ACBR} - \phi^{ACBR} \|_1 \} \leq \epsilon.$$

Then there exist a quantum system $S$ with $|S| = |C|/|\tilde{A}|$, $\kappa$ encoding isometries $V_k^{AS}$ and a decoding isometry $W_{k^{SB}}$ under which

$$\sum_{k=1}^{K} \langle k | |v\rangle^{ACBR} W_{k} | \psi\rangle^{ACBR} |\Phi\rangle^{AB} \geq 1 - 2\eta$$

where $\eta$ is equal to

$$6\sqrt{7} + 4 \left( \frac{|C| \|v^{BR}\|_1 \|v^{CR}\|_2}{|S|^2} \right)^{1/4} + 4K \|d^{CB}\|_0 \|d^{B}\|_\infty. $$

Now we show how to apply Theorem 4 to pure states of the form $|\psi\rangle^{ACBR}$ to obtain a proof of Theorem 5. This is accomplished via the following theorem. The direct part is proved in [11], while the converse part follows from standard arguments in classical information theory (see e.g. [14]).

**Theorem 5 (Method of types):**

Let a tripartite state $|\psi\rangle^{ABC}$ be given. For every $\epsilon, \delta > 0$ and all sufficiently large $n$, there are projections $\Pi^{A^n}_{\epsilon}$, $\Pi^{B^n}_{\delta}$, and $\Pi^{C^n}_{\epsilon}$ such that for $T \in \{A, B, C\}$,

$$\Tr \Pi^{T^n}_{\epsilon} \psi^{T^n} \geq 1 - \epsilon.$$

Also, the normalized version $|\varphi\rangle^{A^n B^n C^n}$ of the subnormalized state

$$\Pi^{A^n}_{\epsilon} \otimes \Pi^{B^n}_{\delta} \otimes \Pi^{C^n}_{\epsilon} |\psi\rangle^{ACBR}.$$
satisfies
\[ \left\| \varphi^{A^nB^nC^n} - (\psi^{ABC})^{\otimes n} \right\|_1 \leq \epsilon \]
and for each \( T \in \{A, B, C, AB, BC, AC\} \),
\[ 2^{nH(T) - n\delta} \leq \left\| \varphi^{T^n} \right\|_0 \leq 2^{nH(T) + n\delta} \]
\[ 2^{-nH(T) - n\delta} \leq \left\| \varphi^{T^n} \right\|_2 \leq 2^{-nH(T) + n\delta} \]
\[ 2^{-nH(T) - n\delta} \leq \left\| \varphi^{T^n} \right\|_\infty \leq 2^{-nH(T) + n\delta}. \]
The entropies in these bounds are evaluated on \( |\psi^{ABC}\rangle \).
Additionally, the normalized version \( |\Psi^{ABC}\rangle^{\otimes n} \) of the sub-normalized state
\[ (1^A \otimes I^B \otimes \Pi_\delta^{CM}) (|\psi^{ABC}\rangle^{\otimes n}) \]
satisfies
\[ \left\| \Psi^{A^nB^nC^n} - (\psi^{ABC})^{\otimes n} \right\|_1 \leq \epsilon. \]
Finally, there is a constant \( c > 0 \) for which we may take \( \epsilon = 2^{-n\delta^2} \) in all of the above bounds.

**Proof of Theorem 3** We apply Theorem 5 two separate times to the state \( (|\psi^{ABC}\rangle)^{\otimes n} \), obtaining two auxiliary states which we will use to control the main quantities appearing in the error bound (10) of Theorem 4. For the first, we consider \( |\psi^{ABC}\rangle^{\otimes n} \) to be a tripartite state of the systems \( A, C, B, \). We thus obtain, for every \( \delta > 0 \) and all sufficiently large \( n \), a state \( |\varphi^{A^nC^nB^nR^n}\rangle \) which is \( \epsilon \)-close to \( |\Psi^{ABC}\rangle^{\otimes n} \) in trace distance for \( \epsilon = 2^{-n\delta^2} \), where all of the operator norms in the second term in (10) have the exponential properties.
Within respect to the partition \( AB, C, B, \) we similarly obtain another state \( |\phi^{A^nC^nB^nR^n}\rangle \) for which the operator norms in the last term of (10) are bounded accordingly. Alice initiates the protocol by Schumacher compressing the system \( C^n \). For this, she performs the projective measurement \( \{\Pi_\delta^{CM}, I^{CM} - \Pi_\delta^{CM}\} \) on \( C^n \). According to Theorem 5, the first outcome occurs with probability at least \( 1 - \epsilon \). In this case, the global state is replaced by the normalized version \( |\Psi^{A^nC^nB^nR^n}\rangle \) of the projected state \( \Pi_\delta^{CM} (|\psi^{ABC}\rangle^{\otimes n}) \). In the case the other outcome occurs, Alice declares an error and the protocol is aborted. We condition on the first case. In what follows, we identify \( |\Psi^{A^nC^nB^nR^n}\rangle \) with its restriction \( |\Psi^{A^nC^nB^nR^n}\rangle \) to the support \( C_\delta \) of the typical projection \( \Pi_\delta^{CM} \).

By the triangle inequality, each of \( |\varphi^{A^nC^nB^nR^n}\rangle \) and \( |\phi^{A^nC^nB^nR^n}\rangle \) is \( 2\epsilon \)-close to \( |\Psi^{A^nC^nB^nR^n}\rangle \) in trace distance because all three states are \( \epsilon \)-close to \( (|\psi^{ABC}\rangle)^{\otimes n} \). Therefore, the one-shot theorem (Theorem 2) implies that there exist a quantum system \( S \) and a maximally entangled state \( |\Phi^{AB}_{\delta}\rangle \) with \( |A| \cdot |S| = |C_\delta| \), together with \( \kappa \) encoding isometries \( \gamma_k^{A^nC^nS^n} \to A^nS^n \) and a decoding isometry \( \gamma_k^{S^nB^nC^nK_{out}} \) satisfying (9) and (10) with \( \epsilon \) replaced by \( 2\epsilon \). If Alice applies one of the isometries \( \gamma_k \) uniformly at random and sends \( S \) to Bob, after which he applies \( \gamma \), the system \( C_\delta \) will be transferred with high global fidelity. By measuring \( K_{out} \), Bob can, on the average, identify Alice’s encoding. Rather than send Bob classical information, Alice can instead simulate a coherent channel from a system \( K_{in} \) to \( K_{in}K_{out} \) by applying a controlled isometry
\[ \mathcal{V} = \sum_k |k\rangle\langle k|_{K_{in}} \otimes \tilde{\Lambda}^{A^nC^nS^n} \to A^nS^n. \]
If she tries to send half of a maximally entangled state \( |\Phi^{AB}_{\delta}^{K_{in}}\rangle \), the global pure state \( |\Omega\rangle \) on \( A^nC^nB^nR^nR'K_{in}K_{out} \) that results from the protocol is
\[ |\Omega\rangle = \mathcal{V} \circ |\Phi^{A^nC^nB^nR^n}|_{K_{in}} \otimes |\Phi^{AB}_{\delta}^{K_{in}}\rangle \tilde{\Lambda}^B. \]
It is then immediate from (9) that
\[ (\mathcal{I} |R'K_{in}K_{out} \otimes |\Phi^{A^nC^nB^nR^n}|_{K_{in}} \otimes |\Phi^{AB}_{\delta}^{K_{in}}\rangle \tilde{\Lambda}^B) \]
is a GHZ state. Squaring this estimate and using monotonicity of fidelity, we obtain
\[ F((|\Phi^{AB}_{\delta}^{K_{in}}\rangle)^{\otimes n}) \geq 1 - 4\eta. \]
By similar reasoning, together with (5), we find that
\[ F((\mathcal{I} |R'K_{in}K_{out}) \otimes \mathcal{V} |\Phi^{A^nC^nB^nR^n}|_{K_{in}}) \geq 1 - 4\eta. \]
Because \( |\Psi^{A^nC^nB^nR^n}\rangle \) is \( \epsilon \)-close to \( (|\psi^{ABC}\rangle)^{\otimes n} \) in trace distance, the triangle inequality implies that
\[ F((\mathcal{I} |R'K_{in}K_{out}) \otimes |\Phi^{AB}_{\delta}^{K_{in}}\rangle^{\otimes n}) \geq 1 - 4\eta. \]
We may combine the estimates (11) and (12) using a lemma from [19], yielding
\[ F((|\Phi^{AB}_{\delta}^{K_{in}}\rangle)^{\otimes n}) \geq 1 - 4\eta. \]
provided \( \eta \) is sufficiently small. At this point, all that remains is to bound the two main terms in the expression (10) for \( \eta \). Taking \( |S| = 2^mQ \) and \( \kappa = 2^nR \), the first main quantity in (10) satisfies
\[ |C_\delta| \left\| \varphi^{B^nR^n} \right\|_0 \left\| \varphi^{C^nR^n} \right\|_2 \leq 2^{n[H(C) + H(B)R - H(CBR) - 2Q] + 3n\delta} \]
\[ = 2^{n[I(C; B)R - 2Q] + 3n\delta}. \]
Therefore, this term goes to zero exponentially fast provided that
\[ Q \geq \frac{1}{2} I(C; B) + 2\delta. \]
For the second term,
\[ \frac{K_{out}}{C_\delta} \left\| \varphi^{C^nB^nR^n} \right\|_\infty \leq 2^n[R + H(C) - H(C\dot{B}) + 3n\delta] \]
\[ = 2^n[R - I(C; B)] + 3n\delta \]
so that if \( R \leq I(C;B) - 4\delta \), this term also goes to zero exponentially with \( n \). Since each of these terms will eventually be less than \( \epsilon = 2^{-n c \delta^2} \), we find that

\[
\eta \leq 6\sqrt{2}e + 4e^{1/4} + 4\epsilon \leq 5e^{1/4}
\]

if \( \epsilon \) is sufficiently small. Therefore, the final bound on the fidelity in (13) can be expressed as \( 1 - 4e^{1/8} \) for sufficiently small \( \epsilon \).

Combining (15) with the bound on \(|C_0| = \text{Tr} \Pi_C^n \) obtained in Theorem 5, we find that the rate at which this protocol uses entanglement satisfies

\[
E_{\text{in}} = \frac{1}{n} \log \left( \frac{|C_0|}{|S|} \right) \leq \frac{1}{2} I(C;A) - \delta + \frac{1}{n}.
\]

Because \( \delta > 0 \) can be taken arbitrarily small, it follows that whenever

\[
Q > \frac{1}{2} I(C;RB), \quad E_{\text{in}} > \frac{1}{2} I(A;C), \quad \text{and} \quad R < I(B;C),
\]

we have, for all sufficiently large \( n \),

\[
\psi^{AC|B} + [nQ][q \rightarrow q] + [nE_{\text{in}}][qq] \geq_{5e^{1/8}} \psi^{A|CB} + [nR][q \rightarrow qq].
\]

Since this holds for arbitrarily small \( \epsilon > 0 \), the asymptotic resource inequality of Theorem 5 follows.

### B. Proof of Theorem 4

Our proof of Theorem 4 makes essential use of the following robust one-shot decoupling lemma, which is proved in the appendix. After stating the lemma, we briefly recall how it is used in two previously studied special cases of our redistribution result, to help the reader understand the context into which it fits with our proof.

**Lemma 3 (Robust one-shot decoupling):** Let a density matrix \( \psi^{CE} \) be given, fix \( \epsilon > 0 \) and let \( \varphi^{CE} \) be any state satisfying \( \|\psi^{CE} - \varphi^{CE}\|_1 \leq \epsilon \). Fix a unitary decomposition \( WC \rightarrow S \hat{B} \) of \( C \) into subsystems and define, for each \( UC \rightarrow C' \),

\[
\psi^{SBE}_U = WU\psi^{CE}U^\dagger W^\dagger.
\]

Then the average state

\[
\psi^{\hat{B}E} = \int_{U(C)} \psi^{SBE}_U dU.
\]

satisfies

\[
\|\psi^{\hat{B}E} - \pi^{\hat{B}} \otimes \varphi^E\|_1 \leq 2\epsilon + \sqrt{\|C\|_0 \|\varphi^{BR}\|_1^2 \|\varphi^{C|BR}\|_2^2 / |S|^2}.
\]

The fully quantum Slepian-Wolf theorem says that, given a tripartite pure state \( |\psi^{CBR}\rangle \),

\[
|\psi^{C|B} + \frac{1}{2} I(C;R)[q \rightarrow q] - |\psi^{BR}\rangle_{10} - |\varphi^{C|BR}\rangle_{12}| \\
\geq |\psi^{AC|B} + \frac{1}{2} I(A;C)[qq].
\]

Together with the method of types (Theorem 5), the above decoupling lemma provides an immediate proof of this resource inequality. Indeed, if Alice encodes with a random unitary, Lemma 3 ensures that a system \( A \) (identified with \( \hat{B} \) in the lemma), which will hold Alice’s half of the generated entanglement, is approximately maximally mixed and decoupled from \( R \). This can easily be shown to imply that \( \hat{A} \) is maximally entangled with \( BS \) (see [11], or compare with the proof of Theorem 4 below). Because all transformations are unitary and the global state is pure, this ensures that Bob can apply a local isometry to reconstruct \( C \), while at the same time obtaining the other half of the generated entanglement. This scenario is illustrated on the left of Figure 2.

![Fig. 2. Circuits for fully quantum Slepian-Wolf (left) and fully quantum reverse Shannon (right), related by time-reversal and swapping \( A \leftrightarrow B \). We have included the rates one gets by applying the method of types (Theorem 5) to the one-shot decoupling lemma (Lemma 3). Note that for fully quantum Slepian-Wolf, the random encoding determines the decoding, while for fully quantum reverse Shannon, the random decoding determines the encoding.](image-url)

By simply reversing time (and also the labels \( A \leftrightarrow B \)), a circuit for fully quantum Slepian-Wolf provides one for the fully quantum reverse Shannon theorem, which given a tripartite state \( |\psi^{ACR}\rangle \), states that

\[
|\psi^{AC|\emptyset} + \frac{1}{2} I(C;R)[q \rightarrow q] + \frac{1}{2} I(C;A)[qq] + \geq |\psi^{A|C}.n\rangle.
\]

A circuit for this is pictured on the right of Figure 2.

We prove Theorem 4 as follows. If Bob’s side information is considered as part of the reference (i.e. it is disregarded as side information), the fully quantum reverse Shannon protocol can be used to transfer \( C \) from Alice to Bob, at least making use of Alice’s side information. By a modification of that protocol provided below, Bob’s side information can be utilized to simulate the required coherent channels \( [q \rightarrow qq] \) as follows. Rather than choosing a single random unitary for the decoding, we choose exponentially many (roughly \( 2^{nI(C;B)} \) when we apply the method of types to the one-shot result). We further guarantee that if Alice chooses one of the corresponding encodings uniformly at random, Bob can, on average, correctly distinguish that encoding in order to apply the correct decoding. Thus, it is possible for Alice to “piggyback” classical information on the transmitted qubits, which Bob can access by means of his side information (cf. [20], [21]). We further ensure that this can all be done coherently, where Alice instead applies a superposition of encodings by using a controlled isometry which is controlled by an arbitrary quantum state. The circuit we construct for performing this task noncoherently is illustrated in Figure 3.

Our proof of Theorem 4 relies on two other lemmas. First, we require the operator inequality [22]:
We define the decoupling fidelity for $\psi_k^{BBR}$ as
$$F_k = F(\psi_k^{BBR}, \pi^B \otimes \psi_k^{BR}).$$
Since $|\Phi\rangle\hat{\Lambda}_B|\psi\rangle^{ACBR}$ is a purification of $\pi^B \otimes \psi^{BBR}$, Uhlmann’s theorem implies that there is an isometry $V_k^{BBR} \rightarrow \hat{\Lambda}_ACBR$ under which
$$F_k = |\langle \psi_k^{ACBR} | \psi_k^{ACBR} \rangle|^2.$$
To send the message $k$, Alice will apply the isometry $V_k = V_k^\dagger$.
We now define
$$|\psi'_k\rangle^{ACBR} = W^\dagger V_k^\dagger |\Phi\rangle \hat{\Lambda}_B |\psi\rangle^{ACBR},$$
which is the state that is created after Alice performs $V_k$ and gives $S$ to Bob, who then applies $W^{\dagger S B \rightarrow C}$. We may therefore equivalently write
$$F_k = |\langle \psi_k^{ACBR} | \psi'_k \rangle^{ACBR} |^2.$$
The average decoupling fidelity is a random variable
$$F_{ave} = \frac{1}{K} \sum_{k=1}^K F_k$$
which depends on the random choice of unitaries. We lower bound its expectation as follows. Define the average states with respect to Haar measure $dU$ as
$$\overline{\psi}_U^{ACBR} = \int_{U(C)} \psi_U^{ACBR} dU$$
$$\overline{\psi}_{ASBBR} = W \overline{\psi}_U^{ACBR} W^\dagger.$$
We now use the robust one-shot decoupling lemma (Lemma 3) to bound the expectation of $F_{ave}$ over the random choice of unitaries:
$$1 - \mathbb{E} F_{ave} = 1 - \frac{1}{K} \sum_{k=1}^K \mathbb{E} F_k$$
$$= 1 - F(\overline{\psi}_{ASBBR}, \pi^B \otimes \psi^{BBR})$$
$$\leq 2\epsilon + \frac{\mathbb{E} \left[ \| \psi_{SBR} \|_2 \| \psi^{CBR} \|_2 \right]}{|S|^2}.$$
A related estimate to be used later is
$$\mathbb{E} \left[ \sqrt{1 - F_{ave}} \right] \leq \sqrt{1 - \mathbb{E} F_{ave}}$$
$$\leq \sqrt{2\epsilon + \frac{\mathbb{E} \left[ \| \psi_{SBR} \|_2 \| \psi^{CBR} \|_2 \right]}{|S|^2}}^{1/4},$$
which follows by concavity and the inequality $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$, valid for $x, y \geq 0$.
Next, we consider Bob’s ability to distinguish the states $\psi^{CB}_k$. For this, we design a measurement which distinguishes the nearby states $\phi^{CB}_k = U_k \phi^{CB}_k U_k^\dagger$. Let $\Pi$ be the projection onto the support of $\phi^{CB}$ and define
$$\Pi_k = U_k \Pi U_k^\dagger,$$
while defining the “pretty good measurement”
$$\Lambda = \sum_{k=1}^K \Pi_k, \quad \Lambda_k = \Lambda^{-1/2} \Pi_k \Lambda^{-1/2}.$$

**Lemma 4:** If $0 \leq \Pi \leq 1$ and $\Pi \leq A$, then
$$\mathbb{I} - \Lambda^{-1/2} \Pi \Lambda^{-1/2} \leq 2(\mathbb{I} - \Pi) + 4(\Lambda - \Pi).$$
We also will use the following coherification lemma, which allows us to convert protocols which transmit classical information to ones which simulate coherent channels. We give a short proof in the appendix.

**Lemma 5:** Given a pure state $|\psi\rangle^{DE}$ and $\kappa$ unitaries $U_k^{D\rightarrow D}$, let $|\psi_k\rangle^{DE} = U_k |\psi\rangle^{DE}$ Given any other set of pure states $|\psi'_k\rangle^{DE}$ and a POVM $\{P_k\}$ on $D$, there are complex phases $\alpha_k$ such that the isometry
$$L^{D\rightarrow DK} = \sum_k (\alpha_k U_k^{\dagger} \sqrt{\Lambda_k}) \otimes |k\rangle^K$$
satisfies
$$\frac{1}{K} \sum_{k=1}^K \langle k | \langle \psi | L | \psi'_k \rangle \geq 1 - 2(1 - F),$$
where
$$P = 1 - \frac{1}{K} \sum_k \mathrm{Tr} \psi_k \Lambda_k, \quad F = \frac{1}{K} \sum_k |\langle \psi_k | \psi'_k \rangle|^2.$$

**Proof of Theorem 4** As in the statement of the theorem, we fix nearby states $|\phi\rangle$ and $|\phi\rangle$ and let $W^{C\rightarrow S B}$ be any unitary decomposition of $C$ into subsystems. Independently choose $\kappa$ unitaries $\{U_1, \ldots, U_\kappa\}$ according to the Haar measure on $U(C)$. For each $k$, define the states
$$|\psi_k\rangle^{ACBR} = U_k |\psi\rangle^{ACBR}$$
$$|\psi_k\rangle^{ASBBR} = W U_k |\psi\rangle^{ACBR} = W |\psi_k\rangle^{ACBR}.$$
The probability that this measurement fails to identify the state $\psi^C_k$ is
\[
P_k = \text{Tr}(\mathbb{I} - \Lambda_k)\psi^C_k.
\]

Observe that
\[
|P_k - \text{Tr}(\mathbb{I} - \Lambda_k)\phi_k^C| \leq \|\psi^C_k - \phi_k^C\|_1 \\
\leq \|\psi^C_k - \psi^C_k'\| + \|\psi^C_k' - \phi_k^C\|_1 \\
\leq 2\sqrt{1 - F_k} + \epsilon \\
\equiv D_k.
\]

Because $x \to \sqrt{x}$ is concave, we have
\[
D_{\text{ave}} \equiv \frac{1}{K} \sum_{k=1}^K D_k \leq \epsilon + 2\sqrt{1 - F_{\text{ave}}}.
\]

Therefore, the average of the $P_k$ can be bounded using Lemma 4 obtaining a random variable satisfying
\[
P_{\text{ave}} \equiv \frac{1}{K} \sum_{k=1}^K P_k \\
\leq D_{\text{ave}} + \frac{1}{K} \sum_{k=1}^K \text{Tr}(\mathbb{I} - \Lambda_k)\phi_k^C \\
\leq D_{\text{ave}} + \frac{1}{K} \sum_{k=1}^K \left(2(1 - \text{Tr}\Pi_k\phi_k^C) + 4 \sum_{k' \neq k} \text{Tr}\Pi_{k'}\phi_{k'}^C\right) \\
= D_{\text{ave}} + 4 \frac{1}{K} \sum_{k=1}^K \sum_{k' \neq k} \text{Tr}\Pi_{k'}\phi_{k'}^C
\]
The last line holds because for each $k$, $\Pi_k$ projects onto the support of $\phi_k^C$. By taking the expectation over the random choice of unitaries, this yields
\[
E[P_{\text{ave}}] \leq E[D_{\text{ave}}] + 4K E[\text{Tr}\Pi_k\phi_k^C] \\
= E[D_{\text{ave}}] + 4K E[\text{Tr}(E[I]\Pi E[k\phi^C])] \\
= E[D_{\text{ave}}] + 4K E[\text{Tr}(E[I]\Pi(I\otimes k\phi^C))] \\
\leq E[D_{\text{ave}}] + 4K E[\phi_k^C]\phi_k^C \|\|_\infty \\
\leq 2E \sqrt{1 - F_{\text{ave}}} + \epsilon + \frac{4K E[\phi_k^C]\phi_k^C \|\|_\infty}{|C|}.
\]

We now apply Lemma 5 identifying $AR \to E$ and $CB \to D$. We can thus conclude that there is an isometry $\mathcal{L}^{CB \to CBK}$ under which
\[
\frac{1}{K} \sum_{k=1}^K \langle k | \psi | L \rangle |\psi^L_k^I\rangle \geq 1 - 2(P_{\text{ave}} + \sqrt{1 - P_{\text{ave}}}).
\]

Taking expectations, we find that
\[
1 - \frac{1}{K} \sum_{k=1}^K \langle k | \psi | L \rangle |\psi^L_k^I\rangle \\
\leq 2E \sqrt{1 - F_{\text{ave}}} + 2E P_{\text{ave}} \\
\leq 4E \sqrt{1 - F_{\text{ave}}} + \epsilon + \frac{4K E[\phi_k^C]\phi_k^C \|\|_\infty}{|C|} \\
\leq 6\sqrt{\epsilon} + 4 \left(\frac{|C|\phi BR \|\|_0}{|S|^2} \phi_{BR}^\| \|_2 \right)^{1/4} + \frac{4K E[\phi_k^C]\phi_k^C \|\|_\infty}{|C|}.
\]
The second inequality is by (18), while the third is due to (17) and holds for sufficiently small $\epsilon$, i.e. sufficiently large $n$. We may then conclude that for a particular value of the randomness, the same bound holds without the expectations. Finally, we define Bob’s decoding isometry to be $W = LW^\dagger$, completing the proof.

IV. AN OPERATIONAL PROOF OF STRONG SUBADDITIVITY

Let $|\psi^ACBR\rangle$ be an arbitrary pure state. In this section, we show how our results lead to an operational proof of strong subadditivity, i.e. that $I(C; RB) \geq 0$. By discarding some resources on the right in Theorem 2, we obtain:
\[
|\psi^AC|B + \frac{1}{2}I(C; RB)[q \rightarrow q] + \frac{1}{2}I(C; A)[qq] \geq \frac{1}{2}I(C; B)[q \rightarrow q].
\]

Intuitively, it makes sense that we should have
\[
I(C; RB) - I(C; B) = I(C; R|B) \geq 0
\]
since otherwise, a noiseless qubit channel could be used to faithfully transmit more than one qubit in the presence of entanglement between the sender and receiver. Of course this inequality is guaranteed by strong subadditivity. However, our aim is to provide an alternative proof of this fundamental inequality. The above asymptotic resource inequality implies that for every $\epsilon, \delta > 0$ and all sufficiently large $n$, we have
\[
\psi^L|L'\rangle \geq 2^{\frac{1}{2}I(C; RB)} + 2\delta |\psi^AC|B + \frac{1}{2}I(C; RB)[q \rightarrow q] + 2\delta I(C; B)[q \rightarrow q].
\]

$|\psi^L|L'\rangle$ represents the prior entanglement between Alice and Bob, although its precise form is irrelevant for our argument. Now consider the following lemma, whose proof we delay until the end of this section.

Lemma 6: Let $K$ and $Q$ be quantum systems and let $|\psi^L|L'\rangle$ be arbitrary. Consider any potential simulation $N^{K\rightarrow K}(\rho^K) = D^{QL\rightarrow Q} o (C^{KL\rightarrow P} \otimes I^{L'}) (\rho^K \otimes |\psi^L|L')$ of the identity quantum channel $id^{K\rightarrow K}$ by the possibly smaller one $id^{Q\rightarrow Q}$, assisted by the bipartite state $|\psi^L|L'\rangle$. If $|\Phi^K|K'K$ is maximally entangled, then the entanglement fidelity [23] satisfies
\[
F(|\Phi^K|K'K, |I^{K'} \otimes N^Q|(|\Phi^K|K')) \leq \frac{|Q|}{|K|}.
\]

Plugging in to the right side of (20), we find that the entanglement fidelity is upper bounded by $2^{\frac{1}{2}I(C; RB)} + \frac{4}{2} \delta I(C; B)[q \rightarrow q]$. Suppose now that strong subadditivity was not satisfied. Then, for some sufficiently small $\delta > 0$ the entanglement fidelity would tend to zero exponentially fast. However, (19) implies that the entanglement fidelity can be made arbitrarily close to 1. Therefore, we must have the inequality $I(C; R|B) \geq 0$.

Proof of Lemma 6: Let $\{E_i\}$ and $\{D_j\}$ be Kraus matrices for the encoding $C^{KL\rightarrow Q}$ and decoding $D^{QL\rightarrow K}$. Fixing orthonormal bases of $L$ and $L'$ which Schmidt-decompose the assistance state as
\[
|\psi^L|L'\rangle = \sum_{\ell} \sqrt{\lambda_\ell} |\ell\rangle^{L'} |\ell\rangle^L.
\]
the above Kraus matrices can be written in block form

\[ E_i = \begin{bmatrix} E_{i1} & \cdots & E_{i|L_i|} \end{bmatrix}, \quad D_j = \begin{bmatrix} D_{j1} & \cdots & D_{j|L_j|} \end{bmatrix}. \]

Because these maps are trace-preserving, we have

\[ \sum_i E_i^\dagger E_i = 1^K_L, \quad \sum_j D_j^\dagger D_j = 1^Q_L \]

which in turn implies that

\[ \sum_i E_i^\dagger E_i = \delta_{i'i'}, 1^K, \quad \sum_j D_j^\dagger D_j = \delta_{j'j}, 1^Q. \]

The overall map \( N^{K \rightarrow K} \) has Kraus matrices \( \{N_{ij}\} \) given by

\[ N_{ij} = \sum_{\ell} \sqrt{\lambda_{i\ell}} D_{j\ell} E_{i\ell}. \]

The entanglement fidelity (20) can be written in the following form [23]:

\[ F(\{\Phi_k\}^{K'},(\{\Phi_k^{K'}\} \otimes N)(\Phi_k^{K'})) = \sum_{ij} \left| \text{Tr} N_{ij} \pi^K \right|^2 \]

\[ = \frac{1}{|K|^2} \sum_{ij} \left| \text{Tr} N_{ij} \right|^2. \]

On the other hand,

\[ \sum_{ij} \left| \text{Tr} N_{ij} \right|^2 = \sum_{i\ell j} \lambda_{i\ell} \left| \text{Tr} D_{j\ell} E_{i\ell} \right|^2 \]

\[ \leq \sum_{i\ell j} \lambda_{i\ell} |Q| \text{Tr} E_{i\ell}^\dagger D_{j\ell}^\dagger D_{j\ell} E_{i\ell} \]

\[ = |Q| \sum_{i\ell j} \lambda_{i\ell} \text{Tr} \left( \sum_i E_{i\ell}^\dagger \left( \sum_j D_{j\ell}^\dagger D_{j\ell} \right) E_{i\ell} \right) \]

\[ = |Q| \sum_{i\ell j} \lambda_{i\ell} \text{Tr} 1^K \]

\[ = |Q| \cdot |K|. \]

Above, (22) is by the Cauchy-Schwarz inequality because \( \text{rank}(D_{j\ell} E_{i\ell}) \leq |Q| \), while (23) follows from the identities (21). The last line holds because the squares of the Schmidt coefficients sum to unity. Combining this estimate with the previous expression for the entanglement fidelity proves the lemma. 

V. Discussion

We have identified the cost, in terms of entanglement and transmitted qubits, of moving a subsystem of a multipartite quantum state between two spatially separated parties when the sender and receiver each have quantum side information. Our strategy was to prove a new resource inequality which, when combined with other known results, implies the existence of the optimal protocol. The minimal rate at which qubits must be sent provides the first known operational interpretation of quantum conditional mutual information. While operational interpretations of quantum mutual information are known [6], [24], these do not simply lead to one for the conditional quantity by naively subtracting mutual informations. An operational interpretation for a quantity is a proof that it is an optimal rate for performing some information processing task. Such a proof must consist of a protocol which achieves that rate, together with a converse demonstrating optimality, as was given in [1] for our situation.

Our interpretation provides an explanation of the quadrupartite pure state identity \( \frac{1}{2} I(C; R|A) = \frac{1}{2} I(C; R|B) \) because the inherent reversibility of our protocol implies that the communication cost is the same in both directions. Indeed, with the exception of the Schumacher compression step, which is essentially reversible because it succeeds with high probability, the protocol constructed to prove Theorem 3 consists entirely of isometries. Moreover, the additional steps required to prove Theorem 1 do not introduce further nonunitarity.

Throughout this paper, we have adhered to the convention of always conditioning on Bob’s side information, although this was an arbitrary notational choice. We thus interpret quantum conditional mutual information as a measure of the quantum correlations between \( C \) and \( R \), from the perspective of either party. Technically, our result provides an interpretation for one half of the conditional mutual information; nonetheless, we observed in [8] that by teleportation, we obtain a bona fide interpretation of conditional mutual information (i.e. without the 1/2) as the optimal classical communication rate for state merging [7].

Our protocol includes various others as special cases. When neither Alice nor Bob has any side information, or if they simply disregard it so that is considered part of the reference, our protocol reduces to Schumacher compression — the technical components for this are contained in Theorem 5. When it is only Alice who has side information, our protocol reduces to the fully quantum reverse Shannon protocol, which itself can be proved by trivially combining Theorem 5 and the robust decoupling lemma (Lemma 3). The same theorems can be shown to imply the fully quantum Slepian-Wolf theorem, in which it is only Bob who uses side information. As pointed out in [8], the formal time-reversal duality between fully quantum Slepian-Wolf and fully quantum reverse Shannon observed in [10] is embodied in a more natural way by our new protocol, which is in fact self-dual with respect to time reversal. In [8], we also observed the intuitively satisfying — but nonetheless surprising — fact that successive redistribution can be performed optimally using the optimal redistribution protocol.

State redistribution solves the most general two-terminal source coding problem that can be considered for the class of “fully quantum” coding problems for i.i.d. quantum states. Because the main technical part of our proof is proved in a one-shot fashion, it is likely that it could be applied to more general quantum sources that do not satisfy the i.i.d. property but which do have some internal regularity structure; for instance, to ground states of many-body Hamiltonians in statistical physics. However, it would perhaps be most useful for such applications to have a more direct proof of Theorem 1 which does not use coherent channels or the cancellation lemma. While it would be most desirable to have a one-shot version of Theorem 1 it would perhaps be most natural to
find a one-shot version of the related resource inequality
\[\psi^{AC|B} + \frac{1}{2} I(C; R|B)_{[q \rightarrow q]} + \frac{1}{2} I(C; A)_{[qq]} \geq \psi^{AB|C} + \frac{1}{2} I(C; B)_{[qq]}\].

The corresponding circuit for this case makes the time-reversal symmetry most apparent, as illustrated in Figure 4.

We expect state redistribution to be a useful primitive for studying more complicated state transfer problems. Most generally, one can imagine \(n\) spatially separated parties all holding various parts of a global multipartite state, wishing to shuffle their subsystems around in some arbitrary but predetermined way. There is a multitude of ways in which redistribution could be applied to give achievable rate regions for such problems, where each round of communication would fit our general setting, although they would most likely be suboptimal in general. A simple example along these lines, for which the optimal solution is not yet known, was considered in [25], where Alice and Bob wish to swap two systems. We expect that judicious use of state redistribution should lead to new achievable rates for this problem, by optimizing over ways of splitting the systems to be swapped into subsystems.

During the proof of our main result, we found that in some cases where strong subadditivity is satisfied \(I(C; R|B) = 0\), our protocol nonetheless requires a nonzero amount of quantum communication. One cannot hope for much better, as the following example illustrates. Suppose that Alice wishes to redistribute to Bob the \(C\) parts of many copies of an arbitrary product state \(|\psi\rangle^{AC} |\psi\rangle^{BR}\). Alice can simply throw away the states \(|\psi\rangle^{AC}\) and then dilute [26] her entanglement with Bob to create those states nonlocally. Despite the fact that that strong subadditivity is saturated on the global state, it is nonetheless known [27], [28] that in general, entanglement dilution requires a sublinear amount of communication.

It has been apparent that one half of the mutual information plays a central role in characterizing the optimal rates in this paper. In the following, somewhat mysterious fashion, this quantity can be considered as a “measure” of the correlations between two subsystems. By analogy with thermodynamics, it is possible to identify an underlying heuristic organizing principle governing our optimal rates which perhaps could lend itself to further generalizations of redistribution. The main task of state redistribution is to transform between two configurations of the subsystems as follows:

\[AC|B|R \rightarrow A|CB|R\].

Let \(A_{\text{initial/final}}\) (resp. \(B\)) denote the systems Alice (resp. Bob) holds at the beginning/end of the protocol. Consider the following “dynamic potentials” relative to Alice\(\rightarrow\)Bob communication:

\[D_{\text{initial}}^A \rightarrow B = \frac{1}{2} I(R; A_{\text{initial}}) = \frac{1}{2} I(R; AC)
\]

\[D_{\text{final}}^A \rightarrow B = \frac{1}{2} I(R; A_{\text{final}}) = \frac{1}{2} I(R; A).
\]

We interpret these as indicating the correlations between Alice’s systems and the reference, both before and after redistribution. The optimal qubit rate for redistribution is easily shown to equal the difference between the dynamic potentials

\[D_{\text{final}}^A \rightarrow B - D_{\text{initial}}^A \rightarrow B = \frac{1}{2} I(C; R|A) = \frac{1}{2} I(C; R|B).
\]

We are therefore operationally justified in interpreting this difference as measuring the correlations with the reference that Alice must transfer to Bob to redistribute the state. Analogously, we may also define “static potentials”

\[S_{\text{initial}}^A \rightarrow B = \frac{1}{2} I(A_{\text{initial}}; B_{\text{initial}}) = \frac{1}{2} I(AC; B)
\]

\[S_{\text{final}}^A \rightarrow B = \frac{1}{2} I(A_{\text{final}}; B_{\text{final}}) = \frac{1}{2} I(A; CB)
\]

which indicate the correlations between Alice’s and Bob’s systems at each state of redistribution. Similarly, the optimal qubit rate can be shown to equal the difference of the static potentials

\[S_{\text{final}}^A \rightarrow B - S_{\text{initial}}^A \rightarrow B = \frac{1}{2} I(A; C) - \frac{1}{2} I(B; C).
\]

It is operationally justifiable to consider this difference as the amount of excess correlation between Alice and Bob that is involved in going between the two configurations.

Relative to the Bob\(\rightarrow\)Alice direction, the dynamic potentials are subtracted from a constant

\[D_{\text{initial/final}}^B \rightarrow A = H(R) - D_{\text{initial/final}}^A \rightarrow B
\]

while the static potentials obey

\[S_{\text{initial/final}}^A \rightarrow B = S_{\text{final}}^B \rightarrow A.
\]

Subtracting these potentials as above, we find that

\[D_{\text{final}}^B \rightarrow A - D_{\text{initial}}^B \rightarrow A = D_{\text{final}}^A \rightarrow B - D_{\text{initial}}^A \rightarrow B
\]

while

\[S_{\text{final}}^A \rightarrow B - S_{\text{initial}}^A \rightarrow B = -(S_{\text{final}}^B \rightarrow A - S_{\text{initial}}^B \rightarrow A),
\]

providing another explanation of the symmetry properties of the optimal rates. One could easily imagine generalizations of the above in which more complicated potentials are defined for redistribution problems involving many more parties. However, we expect it would quite challenging to find operational justifications for such a theory.
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APPENDIX

Here we collect the proofs of some auxiliary results used in the proof of Theorem 3. Our proof of the robust decoupling lemma (Lemma 3) relies on the following non-robust version.

Lemma 7 (One-shot decoupling): Let a density matrix \( \varphi \) be given and fix a unitary decomposition \( W^C \rightarrow S^B \) of \( C \) into subsystems. For each unitary \( U^C \), define

\[
\varphi^{SBE} = WU \varphi \otimes W^\dagger U^C.
\]

Then

\[
\int_{U(C)} \left\| \varphi^{BE} - \pi_B \otimes \varphi^E \right\|_1^2 dU \leq \frac{|C| \| \varphi^E \|_1^2 \varphi^{CE}^2}{|S|^2}. \tag{24}
\]

Proof of Lemma 7: By convexity of the trace norm

\[
\left\| \varphi^{BE} - \pi_B \otimes \varphi^E \right\|_1 \leq \int_{U(C)} \left\| \varphi^{BE} - \pi_B \otimes \varphi^E \right\|_1 dU,
\]

where \( dU \) is Haar measure on \( U(C) \). We use the triangle inequality to bound the integrand:

\[
\left\| \varphi^{BE} - \pi_B \otimes \varphi^E \right\|_1 \leq \left\| \varphi^{BE} - \pi_B \otimes \varphi^E \right\|_1 + \left\| \pi_B \otimes \varphi^E - \pi_B \otimes \varphi^E \right\|_1 \leq \epsilon. \tag{25}
\]

Similarly, the last term satisfies

\[
\left\| \pi_B \otimes \varphi^E - \pi_B \otimes \varphi^E \right\|_1 \leq \left\| \varphi^E - \varphi^E \right\|_1 \leq \epsilon. \tag{28}
\]

Because \( x \mapsto x^2 \) is convex, the integral of the first term satisfies

\[
\left( \int_{U(C)} \left\| \varphi^{BE} - \pi_B \otimes \varphi^E \right\|_1 dU \right)^2 \leq \int_{U(C)} \left\| \varphi^{BE} - \pi_B \otimes \varphi^E \right\|_1^2 dU.
\]

The theorem follows by applying Lemma 7 to this integral.

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[17] ——, “A resource framework for quantum Shannon theory,” 2005, arXiv:quant-ph/0512015.

[18] A. W. Harrow, “Coherent communication of classical messages,” Phys. Rev. Lett., vol. 92, p. 097902, 2004, arXiv:quant-ph/0307091.

[19] J. Yard, I. Devetak, and P. Hayden, “Capacity theorems for quantum multiple access channels – Classical-quantum and quantum-quantum capacity regions,” 2005, arXiv:quant-ph/0501045.

[20] M. Horodecki, P. Horodecki, R. Horodecki, D. Leung, and B. Terhal, “Classical capacity of a noiseless quantum channel assisted by noisy entanglement,” Quantum Information and Computation, vol. 1, no. 3, pp. 70–78, 2001.

[21] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, “Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem,” IEEE Trans. Inform. Theory, vol. 48, no. 10, p. 2637, 2002, arXiv:quant-ph/0106052.

[22] M. Hayashi and H. Nagoaka, “General formulas for capacity of classical-quantum channels,” IEEE Trans. Inform. Theory, vol. 49, pp. 1753–1768, 2003.

[23] B. Schumacher, “Sending entanglement through noisy quantum channels,” Phys. Rev. A, vol. 55, no. 1, pp. 2614–2628, 1996. [Online]. Available: arXiv.org:quant-ph/9604023

[24] B. Schumacher and M. Westmoreland, “Quantum mutual information and the one-time pad,” arXiv.org:quant-ph/0604207.

[25] J. Oppenheim and A. Winter, “Uncommon information,” arXiv:quant-ph/0511082.

[26] H.-K. Lo and S. Popescu, “Classical communication cost of entanglement manipulation: Is entanglement an interconvertible resource?” Phys. Rev. Lett., vol. 83, no. 7, pp. 1459–1462, Aug 1999.

[27] P. Hayden and A. Winter, “Communication cost of entanglement transformations,” Phys. Rev. A, vol. 67, no. 1, p. 012326, Jan 2003. [Online]. Available: arXiv.org:quant-ph/0204092

[28] A. Harrow and H.-K. Lo, “A tight lower bound on the classical communication cost of entanglement dilution,” IEEE Trans. Inform. Theory, vol. 50, no. 2, pp. 319–327, Feb. 2004. [Online]. Available: arXiv.org:quant-ph/0204096