Study of the reactions $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0\pi^0\pi^0\pi^0\pi^0\pi^0\pi^0\eta$ at center-of-mass energies from threshold to 4.5 GeV using initial-state radiation

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We study the processes $e^+e^-\to\pi^+\pi^-\pi^0\pi^0\pi^0\pi^0\gamma$ and $\pi^+\pi^-\pi^0\pi^0\pi^0\pi^0\eta\gamma$ in which an energetic photon is radiated from the initial state. The data were collected with the BABAR detector at the SLAC National Accelerator Laboratory. About 7300 and 870 events, respectively, are selected from a data sample corresponding to an integrated luminosity of 469 $fb^{-1}$. The invariant mass of the hadronic final state defines the effective $e^+e^-$ center-of-mass energy. The center-of-mass energies range from threshold to 4.5 GeV. From the mass spectra, the first ever measurements of the $e^+e^-\to\pi^+\pi^-\pi^0\pi^0\pi^0\pi^0\eta\gamma$ and the $e^+e^-\to\pi^+\pi^-\pi^0\pi^0\pi^0\eta\gamma$ cross sections are performed. The contributions from $\omega\pi^0\pi^0\pi^0\pi^0\pi^0\pi^0\eta\gamma$, $\eta\pi^0\pi^0\pi^0\pi^0\pi^0\eta\gamma$, and other intermediate states are presented. We observe the $J/\psi$ and $\psi(2S)$ in most of these final states and measure the corresponding branching fractions, many of them for the first time.
I. INTRODUCTION

Many precision Standard Model (SM) predictions require the hadronic vacuum polarization (HVP) terms to be taken into account. At a relatively large momentum transfer, these terms are measured by studying the inclusive hadron production in $e^+e^-$ annihilation and are relatively well calculated by perturbative quantum chromodynamics. However, in the energy region from the hadronic threshold to about 2 GeV, the inclusive hadronic cross section cannot be measured or calculated reliably, and a sum of exclusive states must be used. It is particularly important for the calculation of the muon anomalous magnetic moment $(g^\mu - 2)$, which is most sensitive to the low-energy region. Despite large data sets of $e^+e^-$ cross sections, accumulated in the past years, and the studies performed [1, 2], there still is a discrepancy between the SM calculation and the experimental value. With the latest result of the $(g^\mu - 2)$ experiment at Fermilab [3], this discrepancy increased to 4.2 sigma.

Electron-positron annihilation events with initial-state radiation (ISR) can be used to study processes over a wide range of energies below the nominal $e^+e^-$ center-of-mass (c.m.) energy $(E_{\text{c.m.}})$, as proposed in Ref. [4]. The possibility of exploiting ISR to make precise measurements of low-energy cross sections at high-luminosity $\phi$ and $B$ factories is discussed in Refs. [5–7], and motivates the studies described in this paper. Not all accessible states have yet been measured; thus new measurements will improve the reliability of the HVP calculation. In addition, studies of ISR events at $B$ factories are interesting in their own right, because they provide information on resonance spectroscopy for masses up to the charmomium region.

Studies of hadron ($h$) production in the ISR processes $e^+e^-\rightarrow h\gamma$ using data from the $\bar{B}A\bar{B}ar$ experiment at SLAC have been previously reported [8–22]. Initial-state radiation events with detection of the ISR photon are characterized by good reconstruction efficiency and by well understood kinematics, demonstrated in the references given above. The $\bar{B}A\bar{B}ar$ detector performance (tracking, particle identification, $\pi^0$, $K^0_S$, and $K^0_L$ reconstruction) is well suited to the study of ISR processes. This paper reports on analyses of the $\pi^+\pi^-4\pi^0$ and $\pi^+\pi^-3\pi^0\eta$ final states produced in conjunction with an energetic photon, assumed to result from ISR. While $\bar{B}A\bar{B}ar$ data cover effective c.m. energies up to 10.58 GeV, this analysis is restricted to energies below 4.5 GeV because of backgrounds from $Y(4S)$ decays.

There are no previous measurements of the $e^+e^-\rightarrow \pi^+\pi^-4\pi^0$ and $e^+e^-\rightarrow \pi^+\pi^-3\pi^0\eta$ cross sections. The six-pion cross sections have a sizable value below 2 GeV [11] and the two-charged plus four-neutral pion processes are currently included in the HPV calculation by assuming isospin relations [1]. The direct measurement of this channel can reduce the calculation uncertainty. It is also important to extract the contribution of the intermediate resonances, because the total cross section calculation depends on their decay rate to the measured final states. Below, we present the measurements of $e^+e^-\rightarrow \omega\pi^0\pi^0\pi^0$, $e^+e^-\rightarrow \eta\pi^+\pi^-\pi^0$, and $e^+e^-\rightarrow \omega\eta$ cross sections, with $\eta \rightarrow \pi^0\pi^0\pi^0$, that contribute to the $e^+e^-\rightarrow \pi^+\pi^-4\pi^0$ final state.

A clear $J/\psi$ signal is observed for both the $\pi^+\pi^-4\pi^0$ and $\pi^+\pi^-3\pi^0\eta$ channels, and the corresponding $J/\psi$ branching fractions are measured.

II. THE $\bar{B}A\bar{B}ar$ DETECTOR AND DATA SET

The data used in this analysis were collected with the $\bar{B}A\bar{B}ar$ detector at the PEP-II asymmetric-energy $e^+e^-$ storage ring. The total integrated luminosity used is 468.6 fb$^{-1}$ [24], which includes data collected at the $Y(4S)$ resonance (424.7 fb$^{-1}$) and at a c.m. energy 40 MeV below this resonance (43.9 fb$^{-1}$).

The $\bar{B}A\bar{B}ar$ detector is described in detail elsewhere [23]. Charged particles are reconstructed using the $\bar{B}A\bar{B}ar$ tracking system, which is comprised of the silicon vertex tracker (SVT) and the drift chamber (DCH), both located inside a 1.5 T solenoid. Separation of pions and kaons is accomplished by means of the detector of internally reflected Cherenkov light (DIRC) and energy-loss measurements in the SVT and DCH. Photons and $K^0_L$ mesons are detected in the electromagnetic calorimeter (EMC). Muon identification is provided by the instrumented flux return (IFR).

To evaluate the detector acceptance and efficiency, we have developed a special package of Monte Carlo (MC) simulation programs for radiative processes based on the approach of Kühn and Czyż [25]. Multiple collinear soft-photon emission from the initial $e^+e^-$ state is implemented with the structure function technique [26, 27], while additional photon radiation from final-state particles is simulated using the PHOTOS package [28]. The precision of the radiative simulation is such that it contributes less than 1% to the uncertainty in the measured...
hadronic cross sections.

We simulate $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0\gamma$ events assuming production through the $\omega(782)\eta$ intermediate channel, with decay of the $\omega$ to three pions and decay of the $\eta$ to all its measured decay modes [29].

A sample of 460k simulated events is generated for the signal reaction and processed through the detector response simulation, based on the GEANT4 package [30]. These events are reconstructed using the same software chain as the data. Most of the experimental events contain additional soft photons due to machine background or interactions in the detector material. Variations in the detector and background conditions are included in the simulation.

For the purpose of background estimation, large samples of events from the main relevant ISR processes ($5\pi\gamma$, $\rho\gamma\gamma$, $\pi^+\pi^-\pi^0\pi^0\gamma$, etc.) are simulated. To evaluate the background from the relevant non-ISR processes, namely $e^+e^- \rightarrow q\bar{q}$ ($q = u,d,s$) and $e^+e^- \rightarrow \tau^+\tau^-$, simulated samples with integrated luminosities about that of the data are generated using the JETSET [31] and KORALB [32] programs, respectively. The cross sections for the above processes are known with an accuracy slightly better than 10%, which is sufficient for the present purposes.

FIG. 1: (a) The invariant mass $m(\gamma\gamma)$ of the fourth photon pair vs $\chi^2_{2x3+0+\gamma}$. (b) The $m(\gamma\gamma)$ distribution for $\chi^2_{2x3+0+\gamma} < 70$ with additional selection criteria applied as described in the text. The double-peak structure near the $\pi^0$ mass is produced by the reconstruction procedure, as explained in the text.

FIG. 2: (a) The fourth-photon-pair invariant mass $m(\gamma\gamma)$ vs $m(2\pi3\pi^0\gamma\gamma)$ for (a) $\chi^2_{2x3+0+\gamma} < 70$ and (b) $70 < \chi^2_{2x3+0+\gamma} < 140$.

III. EVENT SELECTION AND KINEMATIC FIT

A relatively clean sample of $\pi^+\pi^-4\pi^0\gamma$ and $\pi^+\pi^-3\pi^0\eta\gamma$ events is selected by requiring that there be two tracks reconstructed in the DCH, SVT, or both, and nine or more photons (sometimes up to 20), with an energy above 0.02 GeV in the EMC. We assume the photon with the highest energy to be the ISR photon, and we require its c.m. energy to be larger than 3 GeV.

We allow either two or three tracks in an event, with exactly one opposite-sign pair that extrapolates within 0.25 cm of the beam axis and 3.0 cm of the nominal collision point along that axis. The reason a third track is allowed is to capture a relatively small fraction of signal events that contain a background track. The two tracks that satisfy the extrapolation criteria are fit to a vertex, which is used as the point of origin in the calculation of the photon directions.

We subject each candidate event to a set of constrained kinematic fits and use the fit results, along with charged-particle identification, to select the final states of interest and evaluate backgrounds from other processes. The kinematic fits make use of the four-momenta and covariance matrices of the initial $e^+, e^-$, and the set of selected tracks and photons. The fitted three-momenta of each track and photon are then used in further calculations.

Excluding the photon with the highest c.m. energy,
which is assumed to arise from ISR, we consider all independent sets of eight other photons, and combine them into four pairs. For each set of eight photons, we test all possible independent combinations of four photon pairs. We consider those combinations in which the di-photon mass of at least three pairs lies within ±0.7 MeV/c² (±3σ of the resolution) of the π⁰ mass m_{π⁰} [29]. The selected combinations are subjected to a fit in which the di-photon masses of the three pairs with \( m(\gamma\gamma) - m_{π⁰} < 35 \) MeV/c² are constrained to m_{π⁰}. For the signal hypothesis \( e^+e^- \rightarrow π^+π^- 3π^0γγ_{ISR} \), the fits can be performed with the constraints due to four-momentum conservation, there are thus seven constraints (7C) in the fit. The photons in the remaining (“fourth”) pair are treated as being independent. If all four photon pairs in the combination satisfy \( m(\gamma\gamma) - m_{π⁰} < 35 \) MeV/c², we rotate the combinations, allowing each of the four di-photon pairs in turn to be the fourth pair, i.e., without the m_{π⁰} constraint. The combination with the smallest \( χ^2 \) is retained, along with the obtained \( χ^2_{2π3πγγ} \) (\( χ^2_{7C} \)) value and the fitted three-momenta of each track and photon.

The above procedure allows us not only to search for events with \( π^0 \rightarrow γγ \) in the fourth photon pair, but also for events with \( π^0 \rightarrow γγ \) in any other pair.

Each retained event is also subjected to a 7C fit under the \( e^+e^- \rightarrow π^+π^- 3π^0γγ_{ISR} \) background hypothesis, and the smallest \( χ^2_{2π3πγγ} \) value from all photon combinations is retained. The \( π^+π^- 3π^0 \) process has a comparable cross section to the \( π^+π^- 4π^0 \) signal process and can contribute to the background, so we use these events to evaluate background.

IV. ADDITIONAL SELECTION CRITERIA

The results of the 7C fit to events with two tracks and at least nine photon candidates are used to perform the final selection of the six-pion and the five-pion plus eta sample. We require the tracks to lie within the fiducial region of the DCH (0.45-2.40 radians) and to be inconsistent with being a kaon or muon. The photon candidates are required to lie within the fiducial region of the EMC (0.35-2.40 radians) and to have an energy larger than 0.035 GeV. A requirement that there be no charged tracks within 1 radian of the ISR photon reduces the \( π^+π^- \) background to a negligible level. A requirement that any extra photons in an event each have an energy below 0.7 GeV slightly reduces the multi-photon background.

Figure 1(a) shows the invariant mass \( m(\gamma\gamma) \) of the fourth photon pair vs \( χ^2_{2π3π^0γγ} \). Clear \( π^0 \) and \( η \) peaks are visible at small \( χ^2 \) values. We require \( χ^2_{2π3π^0γγ} < 70 \) to select the signal events, and apply \( χ^2_{2π3π^0} > 30 \) condition if these events also satisfy the \( 2π3π^0 \) background hypothesis. This requirement reduces the contamination due to \( 2π3π^0 \) events from 30% to about 1-2% while reducing the detection efficiency by only 5%.

Figure 1(b) shows the \( m(\gamma\gamma) \) distribution after the above requirements have been applied. The dip in this distribution at the \( π^0 \) mass value is a consequence of the kinematic fit constraint of the best three photon pairs to the \( π^0 \) mass. Also, because of this constraint, the fourth photon pair is sometimes formed from photon candidates that are less well measured.

Figure 2 shows the \( m(γγ) \) distribution vs the invariant mass \( m(2π3π^0γγ) \) for events (a) in the signal region \( χ^2_{2π3π^0γγ} < 70 \) and (b) in a control region defined by \( 70 < χ^2_{2π3π^0γγ} < 140 \). Events from the \( e^+e^- \rightarrow π^+π^- 4π^0 \) and \( π^+π^- 3π^0 \) processes are clearly seen in the signal region, as well as \( J/ψ \) decays to these final states. No significant structures are seen in the control region, and we use these events to evaluate background.

Our strategy to extract the signals for the \( e^+e^- \rightarrow π^+π^- π^0\bar{π}^0 \) and \( π^+π^- π^0\bar{π}^0 \) processes is to perform a fit to the \( π^0 \) and \( η \) yields in intervals of 0.05 GeV/c² in the distribution of the invariant mass \( m(π^+π^- 3π^0γγ) \).

![Figure 3](image-url)

**FIG. 3:** The MC-simulated distribution for \( e^+e^- \rightarrow ωη \) events of (a) the fourth-photon-pair invariant mass \( m(γγ) \), and (b) \( m(γγ) \) vs \( m(π^+π^- 3π^0γγ) \).

V. DETECTION EFFICIENCY

A. Number of signal events in simulation

As mentioned in Sec. IV the model used in the MC simulation assumes that the six-pion final state arises
primarily through $\omega\eta$ production, with $\omega$ decays to three pions and $\eta$ decays to $3\pi^0$. As shown below, events with $\eta$ and $\omega$ dominate in the observed cross sections.

The selection procedure applied to the data is also applied to the MC-simulated events. Figure 3 shows (a) the $m(\gamma\gamma)$ distribution for the $\chi^2$ signal region and (b) the distribution of $m(\gamma\gamma)$ vs $m(2\pi3\pi^0\gamma\gamma)$ for the simulated $\omega\eta$ events. The $\pi^0$ signal shape is not Gaussian due to the procedure explained in the previous section. It also includes a combinatoric background arising from the combination of background photons, included in the simulation, with the photons from the signal reactions.

This background is subtracted as illustrated in Fig. 4, which shows the simulated $m(\gamma\gamma)$ distribution from Fig. 3(a) with a bin width of 0.02 GeV/c$^2$. The solid histogram in Fig. 4(a) corresponds to the two-photon mass distribution obtained from the $\chi^2$ signal region. The dashed histogram is obtained instead from the control region and represents a combinatoric background distribution, which is subtracted assuming a scale factor that is varied to estimate the uncertainty in its contribution. The signal yield is then extracted by fitting the $\pi^0$ peak of this distribution with a sum of three Gaussian functions for the signal plus a second-order polynomial function to account for a residual combinatoric background. If a scale factor 1.5 is used, the background level becomes negligible, and we can determine and fix parameters for the signal function. If then we change the scale factor to 1.0 or to 0.0 in the fit, the obtained signal yield does not change by more than 3%. The result, for a scale factor of 1.0, is shown by the points in Fig. 4(b). The fit is shown by the smooth solid curve, while the dashed curve shows the contribution of the remaining combinatoric background. The fitted signal yields $2639 \pm 66$ events.

Alternatively, for $\omega\eta$ events, the $\omega$ mass peak can be used. Figure 4(a) shows the $\pi^0\pi^0\pi^0$ invariant mass (four entries per event) for selected MC-simulated events. A Breit-Wigner (BW) function, convolved with a Gaussian distribution to account for the detector resolution, is used to describe the $\omega$ signal. A second-order polynomial is used to describe the background. We obtain $2699 \pm 75$ events in total. We also obtain the number of events by fitting $m(\pi^+\pi^-\pi^0)$ in 0.05 GeV/c$^2$ intervals of the $m(\pi^+\pi^-\pi^0\gamma\gamma)$ invariant mass.

Because in our simulation the $\omega$ and $\eta$ mesons are produced in correlation, we can significantly reduce combinatoric background by selecting only one (from four) combination, in which the $3\pi^0$ invariant mass is closest to the $\eta$ mass. The distribution of $m(3\pi^0)$ for this combination is shown in Fig. 4(b) by the histogram. In the remaining $\pi^+\pi^-\pi^0$ combination the $\omega$ signal, shown by dots in Fig. 4(b), has much lower background, and the fit yields $2796 \pm 76$ events in total. The $\pi^+\pi^-3\pi^0\gamma\gamma$ mass distribution is obtained by similar fitting in each 0.05 GeV/c$^2$ interval.

Similarly, as an alternative for the $\omega\eta$ events, we determine the number of events by fitting the $\eta$ signal from the $\eta \rightarrow \pi^0\pi^0\pi^0$ decay: the simulated distribution is shown in Fig. 5(c) (four entries per event). The fit functions are the sum of three Gaussian functions and a polynomial for the combinatoric background. This fit yields $2569 \pm 79$ events in total. The $\pi^+\pi^-3\pi^0\gamma\gamma$ mass distribution is also obtained in each 0.05 GeV/c$^2$ interval.

B. Efficiency evaluation

The mass-dependent detection efficiency is obtained by dividing the number of fitted MC events in each 0.05 GeV/c$^2$ mass interval by the number generated in the same interval. By comparing the results of the four different methods, we conclude that the total efficiency does not change by more than 5% because of variations of the functions used to extract the number of events or the use of different background subtraction procedures. This value is taken as an estimate of the systematic uncertainty in the efficiency associated with the simulation model used and with the fit procedure. We average the four efficiencies in each 0.05 GeV/c$^2$ mass interval and fit the result with a third-order polynomial function, shown in Fig. 6. Although the signal simulation accounts for all $\eta$ decay modes, the efficiency calculation considers only
the $\eta \rightarrow \pi^0\pi^0\pi^0$ decay mode. From Fig. 6, it is seen that the reconstruction efficiency is about 2%, roughly independent of mass. The result of this fit is used for the cross section calculation.

This efficiency estimate takes into account the geometrical acceptance of the detector for the final-state photons and the charged pions, the inefficiency of the detector subsystems, and the event loss due to additional soft-photon emission from the initial and final states. Corrections to the efficiency that account for data-MC differences are discussed below.

VI. THE $\pi^+\pi^-4\pi^0$ FINAL STATE

A. Number of $\pi^+\pi^-4\pi^0$ events

The solid histogram in Fig. 7(a) shows the $m(\gamma\gamma)$ data of Fig. 6(b) binned in mass intervals of 0.02 GeV/$c^2$. The dashed histogram shows the distribution of data from the $\chi^2$ control region. The dotted histogram is the estimated remaining background from the $e^+e^- \rightarrow \pi^+\pi^-3\pi^0$ process. No evidence for a peaking background is seen below 0.45 GeV/$c^2$ in either of the two background distributions. We subtract the background evaluated using the $\chi^2$ control region with the scale factor 1.0. The resulting $m(\gamma\gamma)$ distribution is shown in Fig. 7(b).

We fit the data of Fig. 7(b) with a combination of a
signal function, taken from a fit to simulated data, and a background function, taken to be a third-order polynomial. The fit is performed in the \( m(\gamma\gamma) \) mass range from 0.0 to 0.45 GeV/c\(^2\). The result of the fit is shown by the solid and dashed curves in Fig. 7(b). In total 7306 ± 164 events are obtained. Note that this number includes a relatively small peaking background component, due to \( q\bar{q} \) events, which is discussed in Sect. IV.B. The same fit is applied to the corresponding \( m(\gamma\gamma) \) distribution in each 0.05 GeV/c\(^2\) interval in the \( \pi^+\pi^-3\pi^0\gamma\gamma \) invariant mass. The resulting number of \( \pi^+\pi^-4\pi^0 \) event candidates as a function of \( m(\pi^+\pi^-4\pi^0) \), including the peaking \( q\bar{q} \) background, is shown by the data points in Fig. 8.

**B. Peaking background**

The major background producing a \( \pi^0 \) peak following application of the selection criteria of Sect. IV.A is from non-ISR \( q\bar{q} \) events, the most important channel being \( e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0\pi^0\pi^0 \) in which one of the neutral pions decays asymmetrically, yielding a high energy photon that mimics an ISR photon. We apply all our selection criteria and fit procedures to the non-ISR light quark \( q\bar{q} \) (\( uds \)) simulation. Figure 9 shows the fourth-photon-pair invariant mass for \( \chi^2_{2\pi3\pi0\gamma\gamma} < 70 \) and 70 < \( \chi^2_{2\pi3\pi0\gamma\gamma} < 140 \): clear signals from \( \pi^0 \) and \( \eta \) are seen.

To normalize the \( uds \) simulation, we form the diphoton invariant mass distribution of the ISR candidate with each of the other photons in the event. A \( \pi^0 \) peak is observed, with approximately the same number of events in data and simulation, leading to a normalization factor of 1.0 ± 0.1. The resulting \( uds \) background is shown in Fig. 8; the \( uds \) background is negligible below 2 GeV/c\(^2\), but accounts for more than half the total spectrum for around 4 GeV/c\(^2\) and above. We subtract this background for the cross section calculation.

**C. Cross section for \( e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0\pi^0\pi^0 \)**

The \( e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0\pi^0\pi^0 \) Born cross section is determined from

\[
\sigma(2\pi4\pi^0)(E_{\text{c.m.}}) = \frac{dN_{6\pi\gamma}(E_{\text{c.m.}})}{d\mathcal{L}(E_{\text{c.m.}})\epsilon_{6\pi}^{\text{MC}}(E_{\text{c.m.}})(1 + \delta_R)} \tag{1}
\]

where \( E_{\text{c.m.}} \) is the invariant mass of the six-pion system; \( dN_{6\pi\gamma} \) is the background-subtracted number of selected six-pion events in the interval \( dE_{\text{c.m.}} \), and \( \epsilon_{6\pi}^{\text{MC}}(E_{\text{c.m.}}) \) is the corresponding detection efficiency from simulation.
The factor $\epsilon_{\text{ISR}}$ accounts for the difference between data and simulation in the tracking (1.0±1.0% per track) and π reconstruction efficiencies. The ISR differential luminosity, $dL$, is calculated using the total integrated BABAR luminosity of 469 fb$^{-1}$. The initial- and final-state soft-photon emission is accounted for in the luminosity calculation.

Our results for the $e^+e^−\to\pi^+\pi^-\pi^0\pi^0\pi^0\pi^0$ cross section are shown in Fig. 10. The cross section exhibits a structure around 1.7 GeV with a peak value of about 2 nb, followed by a monotonic decrease toward higher energies. Because we present our data in bins of width 0.050 GeV/$c^2$, compatible with the experimental resolution, we do not apply an unfolding procedure to the data. Numerical values for the cross section are presented in Table I. The $J/\psi$ region is discussed later.

### Table I: Summary of the $e^+e^−\to\pi^+\pi^-\pi^0\pi^0\pi^0\pi^0$ cross section measurement. The uncertainties are statistical only.

| $E_{\text{cm}}$ (GeV) | $\sigma$ (nb) | $E_{\text{cm}}$ (GeV) | $\sigma$ (nb) | $E_{\text{cm}}$ (GeV) | $\sigma$ (nb) | $E_{\text{cm}}$ (GeV) | $\sigma$ (nb) |
|------------------------|-------------|------------------------|-------------|------------------------|-------------|------------------------|-------------|
| 1.425                  | 0.03 ± 0.05 | 2.075                  | 1.69 ± 0.21 | 2.075                  | 0.94 ± 0.14 | 3.375                  | 3.60 ± 0.10 |
| 1.475                  | 0.17 ± 0.06 | 2.125                  | 1.46 ± 0.20 | 2.775                  | 1.12 ± 0.14 | 3.425                  | 4.055 ± 0.09 |
| 1.525                  | 0.47 ± 0.08 | 2.175                  | 1.96 ± 0.19 | 2.825                  | 0.78 ± 0.13 | 3.475                  | 0.71 ± 0.10 |
| 1.575                  | 0.92 ± 0.13 | 2.225                  | 1.46 ± 0.19 | 2.875                  | 0.99 ± 0.14 | 3.525                  | 0.40 ± 0.08 |
| 1.625                  | 1.92 ± 0.18 | 2.275                  | 1.44 ± 0.18 | 2.925                  | 1.16 ± 0.14 | 3.575                  | 0.41 ± 0.08 |
| 1.675                  | 2.13 ± 0.21 | 2.325                  | 1.11 ± 0.15 | 2.975                  | 0.92 ± 0.14 | 3.625                  | 0.48 ± 0.09 |
| 1.725                  | 1.99 ± 0.20 | 2.375                  | 1.45 ± 0.18 | 3.025                  | 0.68 ± 0.15 | 3.675                  | 0.46 ± 0.09 |
| 1.775                  | 1.88 ± 0.20 | 2.425                  | 1.60 ± 0.17 | 3.075                  | 1.75 ± 0.17 | 3.725                  | 0.32 ± 0.10 |
| 1.825                  | 1.83 ± 0.20 | 2.475                  | 1.15 ± 0.15 | 3.125                  | 1.61 ± 0.16 | 3.775                  | 0.24 ± 0.08 |
| 1.875                  | 1.48 ± 0.18 | 2.525                  | 1.33 ± 0.16 | 3.175                  | 0.75 ± 0.13 | 3.825                  | 0.10 ± 0.09 |
| 1.925                  | 1.96 ± 0.21 | 2.575                  | 1.26 ± 0.16 | 3.225                  | 0.72 ± 0.10 | 3.875                  | 0.18 ± 0.07 |
| 1.975                  | 1.49 ± 0.20 | 2.625                  | 1.30 ± 0.15 | 3.275                  | 0.75 ± 0.10 | 3.925                  | 0.13 ± 0.06 |
| 2.025                  | 1.76 ± 0.21 | 2.675                  | 1.07 ± 0.14 | 3.325                  | 0.85 ± 0.11 | 3.975                  | 0.10 ± 0.06 |

The distribution of the $\omega\eta$ invariant mass can be taken to represent $m(\pi^+\pi^-3\pi^0\gamma\gamma)$. Figure 11(a) shows the distribution of the $\pi^0\pi^0\pi^0\pi^0\gamma\gamma$ invariant mass (four entries per event). The distribution is seen to exhibit a prominent $\eta$ peak, which is due to the $e^+e^-\to\eta\pi^+\pi^-\pi^0$ reaction. Figure 11(b) presents a scatter plot of the $\pi^0\pi^0\pi^0\pi^0\pi^0\pi^0$ invariant mass. From this plot, the $\omega\eta$ intermediate state is seen. Figure 11(c) presents a scatter plot of the $3\pi^0$ invariant mass versus $m(\pi^+\pi^-3\pi^0\gamma\gamma)$. The distribution of the $\pi^+\pi^-\pi^0$ invariant mass (four entries per event) is shown in Fig. 12(a). A prominent $\omega$ peak from $e^+e^-\to\omega3\pi^0$ is seen. The scatter plot in Fig. 12(b) shows $\pi^0\pi^0\pi^0\pi^0\pi^0\pi^0$ vs the $\pi^+\pi^-\pi^0$ invariant mass for events from Fig. 11(b) when only the $3\pi^0$ combination with the invariant mass closest to the nominal $\gamma$ mass is kept. Correlated $\eta$ and $\omega$ production is seen. A scatter plot of the $\pi^+\pi^-\pi^0$ vs the $\pi^+\pi^-\pi^0\gamma\gamma$ mass is shown in Fig. 12(c). A clear signal for a $J/\psi$ peak is also observed. Figure 13(a) shows the $\pi^+\pi^-\pi^0\eta$ (solid) and $\pi^+\pi^-\pi^0$ (points) invariant masses (four entries per event). Prominent $\rho(770)^\pm$ peaks, corresponding to $e^+e^-\to\rho^+\pi^+3\pi^0$ (or $\rho^\mp\rho^\pm2\pi^0$), are visible. The shaded histogram shows the presence of the $\rho^0$ signal. The scatter plot in Fig. 13(b)
shows the $\pi^+\pi^-\pi^0$ vs the $3\pi^0$ invariant mass. An indication of the $\rho^+\pi^-\eta$ (or $\rho^0\pi^-\pi^0$ - not shown) intermediate state is visible. Figure 13(c) shows the $\pi^+\pi^-\pi^0$ invariant mass vs the six-pion invariant mass: a clear signal for the $J/\psi$ and an indication for the $\psi(2S)$ are seen.
TABLE III: Summary of the $e^+e^- \rightarrow \eta\pi^+\pi^-\pi^0$ cross section measurement. The uncertainties are statistical only.

| $E_{c.m.}$ (GeV) | $\sigma$ (nb) | $E_{c.m.}$ (GeV) | $\sigma$ (nb) | $E_{c.m.}$ (GeV) | $\sigma$ (nb) | $E_{c.m.}$ (GeV) | $\sigma$ (nb) |
|------------------|--------------|------------------|--------------|------------------|--------------|------------------|--------------|
| 1.425            | 0.08 ± 0.46  | 2.075            | 0.96 ± 0.53  | 2.725            | 0.21 ± 0.23  | 3.375            | 0.14 ± 0.12  |
| 1.475            | 0.90 ± 0.39  | 2.125            | 0.71 ± 0.54  | 2.775            | 0.11 ± 0.19  | 3.425            | 0.06 ± 0.12  |
| 1.525            | 0.45 ± 0.61  | 2.175            | 1.33 ± 0.48  | 2.825            | 0.16 ± 0.23  | 3.475            | 0.47 ± 0.13  |
| 1.575            | 1.57 ± 0.75  | 2.225            | 0.11 ± 0.42  | 2.875            | 0.47 ± 0.24  | 3.525            | 0.16 ± 0.12  |
| 1.625            | 4.80 ± 0.99  | 2.275            | 0.77 ± 0.42  | 2.925            | 0.60 ± 0.22  | 3.575            | 0.08 ± 0.09  |
| 1.675            | 5.09 ± 1.01  | 2.325            | 0.39 ± 0.37  | 2.975            | 0.50 ± 0.23  | 3.625            | 0.15 ± 0.11  |
| 1.725            | 4.07 ± 0.95  | 2.375            | 0.88 ± 0.33  | 3.025            | 0.41 ± 0.23  | 3.675            | 0.10 ± 0.10  |
| 1.775            | 2.35 ± 0.82  | 2.425            | 1.03 ± 0.37  | 3.075            | 0.53 ± 0.32  | 3.725            | 0.29 ± 0.14  |
| 1.825            | 3.05 ± 0.76  | 2.475            | 0.26 ± 0.33  | 3.125            | 1.83 ± 0.28  | 3.775            | 0.36 ± 0.12  |
| 1.875            | 0.31 ± 0.66  | 2.525            | 0.65 ± 0.25  | 3.175            | 0.20 ± 0.21  | 3.825            | 0.05 ± 0.06  |
| 1.925            | 2.13 ± 0.75  | 2.575            | 0.08 ± 0.26  | 3.225            | 0.32 ± 0.18  | 3.875            | 0.07 ± 0.08  |
| 1.975            | 1.04 ± 0.65  | 2.625            | 1.04 ± 0.31  | 3.275            | 0.13 ± 0.14  | 3.925            | 0.00 ± 0.14  |
| 2.025            | 0.65 ± 0.60  | 2.675            | 0.53 ± 0.28  | 3.325            | 0.17 ± 0.15  | 3.975            | 0.18 ± 0.08  |

FIG. 14: The $3\pi^0$ invariant mass for data. The curves show the fit functions. The solid curve shows the $\eta$ peak (based on MC simulation) plus the non-$\eta$ continuum background (dashed).

FIG. 15: The $m(\pi^+\pi^-4\pi^0)$ invariant mass dependence of the selected data events for $e^+e^- \rightarrow \eta\pi^+\pi^-\pi^0$, $\eta \rightarrow 3\pi^0$ (triangles) in comparison with all six-pion events (dots).

FIG. 16: Comparison of the current results (dots) for the $e^+e^- \rightarrow \pi^+\pi^-\pi^0\eta$ cross section with those from the SND experiment with $\eta \rightarrow \gamma\gamma$, shown by squares [33] and with those from the CMD-3 experiment, also based on $\eta \rightarrow \gamma\gamma$, shown by triangles [33]. The insert shows an expanded view of the resonant region.

F. The $\eta\pi^+\pi^-\pi^0$ intermediate state

To determine the contribution of the $\eta\pi^+\pi^-\pi^0$ intermediate state, we fit the events of Fig. 11(a) using a triple-Gaussian function to describe the signal peak, as in Fig. 15(c), and a polynomial to describe the background. The result of the fit is shown in Fig. 14. We obtain 1539 ± 89 $\eta\pi^+\pi^-\pi^0$ events. The number of $\eta\pi^+\pi^-\pi^0$ events as a function of the six-pion invariant mass is determined by performing an analogous fit to the events in Fig. 11(c) in each 0.05 GeV/$c^2$ interval of $m(\pi^+\pi^-4\pi^0)$. The resulting distribution is shown in Fig. 15 by triangles in comparison with all $\pi^+\pi^-4\pi^0$ events (dots).

Using Eq. (1), we determine the cross section for the $e^+e^- \rightarrow \eta\pi^+\pi^-\pi^0$ process. The results, which account for the $\eta \rightarrow 3\pi^0$ branching fractions of 0.327, are reported in Fig. 16 and Table III. Systematic uncertainties in this measurement are the same as those listed in Table II. Figure 16 shows our measurement in comparison to the SND result [33] and to those from the CMD-3 experiment [35]. These previous results are based on a different $\eta$ decay mode from that considered here. The insert shows an expanded view for the c.m. energies below 2 GeV, where the resonance, interpreted as the $\omega(1650)$,
TABLE IV: Summary of the $e^+ e^- \rightarrow \eta \omega$ cross section measurement. The uncertainties are statistical only.

| $E_{c.m.}$ (GeV) | $\sigma$ (nb) | $E_{c.m.}$ (GeV) | $\sigma$ (nb) | $E_{c.m.}$ (GeV) | $\sigma$ (nb) | $E_{c.m.}$ (GeV) | $\sigma$ (nb) |
|------------------|--------------|------------------|--------------|------------------|--------------|------------------|--------------|
| 1.425            | 0.11 ± 0.22  | 2.075            | 0.14 ± 0.15  | 2.725            | 0.05 ± 0.04  | 3.375            | 0.08 ± 0.08  |
| 1.475            | 0.27 ± 0.22  | 2.125            | 0.12 ± 0.11  | 2.775            | 0.09 ± 0.05  | 3.425            | 0.03 ± 0.05  |
| 1.525            | 0.42 ± 0.31  | 2.175            | 0.18 ± 0.12  | 2.825            | 0.07 ± 0.05  | 3.475            | 0.04 ± 0.03  |
| 1.575            | 1.24 ± 0.36  | 2.225            | 0.00 ± 0.08  | 2.875            | 0.07 ± 0.05  | 3.525            | 0.08 ± 0.05  |
| 1.625            | 2.16 ± 0.40  | 2.275            | 0.12 ± 0.09  | 2.925            | 0.09 ± 0.06  | 3.575            | 0.00 ± 0.03  |
| 1.675            | 1.98 ± 0.40  | 2.325            | 0.06 ± 0.07  | 2.975            | 0.09 ± 0.06  | 3.625            | 0.04 ± 0.03  |
| 1.725            | 1.06 ± 0.34  | 2.375            | 0.13 ± 0.09  | 3.025            | 0.05 ± 0.06  | 3.675            | 0.04 ± 0.03  |
| 1.775            | 0.33 ± 0.28  | 2.425            | 0.12 ± 0.07  | 3.075            | 0.38 ± 0.10  | 3.725            | 0.00 ± 0.04  |
| 1.825            | 0.62 ± 0.28  | 2.475            | 0.14 ± 0.08  | 3.125            | 0.21 ± 0.08  | 3.775            | 0.04 ± 0.03  |
| 1.875            | 0.28 ± 0.22  | 2.525            | 0.13 ± 0.08  | 3.175            | 0.07 ± 0.06  | 3.825            | 0.01 ± 0.02  |
| 1.925            | 0.10 ± 0.24  | 2.575            | 0.10 ± 0.05  | 3.225            | 0.11 ± 0.04  | 3.875            | 0.03 ± 0.03  |
| 1.975            | 0.31 ± 0.19  | 2.625            | 0.15 ± 0.07  | 3.275            | 0.04 ± 0.04  | 3.925            | 0.03 ± 0.02  |
| 2.025            | 0.46 ± 0.19  | 2.675            | 0.08 ± 0.05  | 3.325            | 0.05 ± 0.08  | 3.975            | 0.03 ± 0.02  |

To agree within the uncertainties.

FIG. 17: (a) The $3\pi^0$ invariant mass closest to the $\eta$ mass. (b) The $\pi^+ \pi^- \pi^0$ invariant mass for events with $m_{\min}(3\pi^0) < 0.7$ GeV/$c^2$. The curves show the fit functions. The solid curve is the fit (based on MC simulation fit) plus the continuum background (dashed).

FIG. 18: $m(\pi^+ \pi^- 3\pi^0)$ invariant mass dependence of the selected data events for $e^+ e^- \rightarrow \eta \omega, \eta \rightarrow 3\pi^0$ (triangles) and $e^+ e^- \rightarrow \eta \phi, \eta \rightarrow 3\pi^0$ (open circles) in comparison with all six-pion events (dots).

G. The $\eta \omega$ intermediate state

To determine the contribution of the $\eta \omega$ intermediate state to the $\eta \pi^+ \pi^- \pi^0$ events, we select the $3\pi^0$ combination with the invariant mass closest to the nominal $\eta$ mass, $m_{\min}(3\pi^0)$, and search for the $\omega$ signal in the remaining $\pi^+ \pi^- \pi^0$ combination. Figure 17(a) shows the $m_{\min}(3\pi^0)$ distribution; Figure 17(b) is the distribution for the corresponding $\pi^+ \pi^- \pi^0$ invariant mass in the event. An additional requirement $m_{\min}(3\pi^0) < 0.7$ GeV/$c^2$ is applied. Prominent $\omega$ and $\phi$ peaks are seen. The latter arises from the $e^+ e^- \rightarrow \eta \phi, \phi \rightarrow \pi^+ \pi^- \pi^0$ reaction.

We fit the events of Fig. 17(b) using a double-Gaussian function to describe the signal from the $\omega$ and $\phi$ peaks, and a polynomial to describe the background. We obtain $351 \pm 43$ and $100 \pm 32 \eta \omega$ and $\eta \phi$ events, respectively. The number of $\eta \omega$ and $\eta \phi$ events as a function of the six-pion invariant mass is determined by performing an
analogous fit to the events in each 0.05 GeV/c² interval of $m(\pi^+\pi^-4\pi^0)$. The resulting distributions are shown in Fig. 18.

Using Eq. (1), we determine the cross section for the $e^+e^-\rightarrow \eta\omega$ process. The results, accounting for the $\eta$ branching fractions, are reported in Fig. 19 and listed in Table IV. Systematic uncertainties in this measurement are the same as those listed in Table I. Figure 19 shows our measurement in comparison to the Babar result [11] (open circles), the SND result [33] (squares), and the CMD-3 result [35] (triangles). The insert shows an expanded view of the resonant region, where signal from the $\omega(1650)$ dominates. These previous results are based on different $\eta$ decay modes ($\eta \rightarrow \pi^+\pi^-\pi^0$ for Babar, $\eta \rightarrow \gamma\gamma$ for SND and CMD-3) from that considered here. The results from the different experiments are seen to agree within the uncertainties. Including the results of the present study, we have thus now measured the $e^+e^-\rightarrow \eta\omega$ cross section in two different $\eta$ decay modes.

The observed contribution from the $e^+e^-\rightarrow \eta\phi$ reaction is small and we do not calculate its cross section.

![Figure 20](image1.png)

**FIG. 20:** (a) The $\pi^+\pi^-\pi^0$ invariant mass for data. The solid curve shows the fit function for signal (based on a fit to MC simulation) plus the polynomial for the combinatorial background (dashed curve). (b) The mass distribution of the $\pi^+\pi^-4\pi^0$ events in the $\omega$ peak (open squares) and the estimated contribution for $\omega\eta$ (triangles), $\omega2\pi^0$ (circles), and $uds$ (filled squares).

**FIG. 21:** The energy dependent $e^+e^-\rightarrow 3\pi^0$ cross section in the $\pi^+\pi^-4\pi^0$ mode.

**H. The $\omega3\pi^0$ intermediate state**

To determine the contribution of the $\omega3\pi^0$ intermediate state, we fit the events of Fig. 12(a) using a BW function to model the signal and a polynomial to model the background. The BW function is convolved with a Gaussian distribution that accounts for the detector resolution. The result of the fit is shown in Fig. 20(a). We obtain 2808 ± 180 $\omega3\pi^0$ events. The number of $\omega3\pi^0$ events as a function of the six-pion invariant mass is determined in each 0.05 GeV/c² interval of $m(\pi^+\pi^-4\pi^0)$. The resulting distribution is shown in Fig. 20(b).

![Figure 21](image2.png)

For the $e^+e^-\rightarrow \omega3\pi^0$ channel, there can be a peaking background from $e^+e^-\rightarrow \omega2\pi^0$ when the fourth $\pi^0$ is formed from background photons. A simulation of this reaction with proper normalization leads to the peaking-background estimation, which is found to be small as shown in Fig. 20(b). There is also a small peaking background from the generic $uds$ reaction. Finally, we need to remove events with correlated $\omega$ and $\eta$ production in the $\omega\eta$ final state, described in Sec. VIG and shown in Fig. 20(b). These contributions are subtracted from the $\omega3\pi^0$ signal candidate distribution.

The $e^+e^-\rightarrow \omega3\pi^0$ cross section, not associated with $\eta\omega$ and corrected for the $\omega \rightarrow \pi^+\pi^-\pi^0$ branching fraction, is shown in Fig. 21 and summarized in Table V. The uncertainties are statistical only. The systematic uncertainties are about 12%. No previous measurement exists for this process. The cross section exhibits a rise at threshold, a decrease at large $E_{c.m.}$ with a signal from $J/\psi$, and a possible resonance activity around 1.7-2.0 GeV.

**I. The $\rho(770)^\pm \pi^+3\pi^0$ intermediate state**

A similar approach is followed to study events with a $\rho^\pm$ meson in the intermediate state. Because the $\rho$ meson is broad, a BW function is used to describe the signal shape. There are eight $\rho^\pm$ candidates per event, leading
TABLE V: Summary of the $e^+e^- \to \omega\pi^0\pi^0$ cross section measurement. The uncertainties are statistical only.

| $E_{c.m.}$ (GeV) | $\sigma$ (nb) | $E_{c.m.}$ (GeV) | $\sigma$ (nb) | $E_{c.m.}$ (GeV) | $\sigma$ (nb) | $E_{c.m.}$ (GeV) | $\sigma$ (nb) |
|------------------|--------------|------------------|--------------|------------------|--------------|------------------|--------------|
| 1.425            | 0.13 ± 0.15  | 2.075            | 0.98 ± 0.22  | 2.725            | 0.28 ± 0.11  | 3.375            | 0.05 ± 0.05  |
| 1.475            | -0.03 ± 0.12 | 2.125            | 0.81 ± 0.21  | 2.775            | 0.29 ± 0.10  | 3.425            | 0.24 ± 0.06  |
| 1.525            | 0.22 ± 0.19  | 2.175            | 0.47 ± 0.19  | 2.825            | 0.32 ± 0.10  | 3.475            | 0.16 ± 0.06  |
| 1.575            | -0.02 ± 0.22 | 2.225            | 0.92 ± 0.18  | 2.875            | 0.36 ± 0.10  | 3.525            | 0.05 ± 0.05  |
| 1.625            | 0.25 ± 0.30  | 2.275            | 0.68 ± 0.17  | 2.925            | 0.22 ± 0.09  | 3.575            | 0.14 ± 0.05  |
| 1.675            | 0.57 ± 0.31  | 2.325            | 0.84 ± 0.17  | 2.975            | 0.28 ± 0.09  | 3.625            | 0.17 ± 0.05  |
| 1.725            | 0.61 ± 0.29  | 2.375            | 0.69 ± 0.16  | 3.025            | 0.51 ± 0.10  | 3.675            | 0.13 ± 0.05  |
| 1.775            | 0.11 ± 0.28  | 2.425            | 0.39 ± 0.14  | 3.075            | 0.36 ± 0.10  | 3.725            | 0.06 ± 0.04  |
| 1.825            | 1.44 ± 0.30  | 2.475            | 0.44 ± 0.13  | 3.125            | 0.34 ± 0.09  | 3.775            | 0.08 ± 0.04  |
| 1.875            | 0.77 ± 0.25  | 2.525            | 0.55 ± 0.15  | 3.175            | 0.23 ± 0.07  | 3.825            | 0.06 ± 0.04  |
| 1.925            | 1.01 ± 0.25  | 2.575            | 0.36 ± 0.12  | 3.225            | 0.17 ± 0.07  | 3.875            | 0.13 ± 0.04  |
| 1.975            | 0.85 ± 0.24  | 2.625            | 0.47 ± 0.12  | 3.275            | 0.28 ± 0.07  | 3.925            | 0.10 ± 0.04  |
| 2.025            | 1.09 ± 0.24  | 2.675            | 0.29 ± 0.10  | 3.325            | 0.20 ± 0.06  | 3.975            | 0.06 ± 0.03  |

FIG. 22: (a) The $\pi^\pm\pi^0$ invariant mass for data. The dashed curve shows the fit to the combinatorial background. The solid curve is the sum of the background curve and the BW function for the $\rho^\pm$. (b) The result of the $\rho$ fit in bins of 0.05 GeV/$c^2$ in the $\pi^0\pi^-3\pi^0\gamma\gamma$ mass. The squares show the contribution from $uds$ background.

FIG. 23: The circles show the number of events determined from the $\pi^0$ fit. The squares show the sum of the number of events with an $\eta$, $\omega$, $\rho$, or $uds$ contribution.

5965 ± 667 combinations with $\rho^\pm$ signals. The distribution of these events vs the six-pion invariant mass is shown by the triangle symbols in Fig. 22(b), while a similar fit for the $uds$ simulation is shown by squares. The $uds$ background dominates at higher energies.

We expect more than one $\rho^\pm$ per event, namely that there is a significant production of $e^+e^- \to \rho^+\rho^-2\pi^0$. Because of the large combinatoric background, we do not perform a study of correlated $\rho^+\rho^-$ production.

A similar study of the $\rho^04\pi^0$ final state yields 407 ± 45 events in total, but obtained mass dependence is not reliable for this contribution.

J. The sum of intermediate states

The circle symbols in Fig. 23 show the total number of $\pi^+\pi^-4\pi^0$ events, already shown in Fig. 8. We perform a sum of the number of $\eta\pi^+\pi^-\pi^0$, $\omega\pi^0$, $uds$ and $\rho^+\pi^-3\pi^0$ intermediate state candidates, found as described in the previous sections, and we show this sum by the square symbols in Fig. 23. This summed curve is seen to be in agreement with the total number of $\pi^+\pi^-4\pi^0$ events except in the region above 2.5 GeV, where the sum is dominated by the $\rho^\pm$ signal extraction. The observed overcount indicates a possible contribution from correlated $\rho^+\rho^-$ production.

VII. THE $\pi^+\pi^-3\pi^0\eta$ FINAL STATE

A. Determination of the number of events

An analogous approach to that described above for $e^+e^- \to \pi^+\pi^-4\pi^0$ events is used to study $e^+e^- \to \pi^+\pi^-3\pi^0\eta$ events. We fit the $\eta$ signal in the fourth-photon-pair invariant mass distribution (cf., Fig. 7(b)) to the sum of two Gaussians with a common mean, while the relatively smooth background is described by a second-order polynomial function, as shown in Fig. 24(a). We
TABLE VI: Summary of the $e^+ e^- \rightarrow \pi^+ \pi^- \pi^0 \pi^0 \eta$ cross section measurement. The uncertainties are statistical only.

| $E_{c.m.}$ (GeV) | $\sigma$ (nb) | $E_{c.m.}$ (GeV) | $\sigma$ (nb) | $E_{c.m.}$ (GeV) | $\sigma$ (nb) | $E_{c.m.}$ (GeV) | $\sigma$ (nb) |
|----------------|-------------|----------------|-------------|----------------|-------------|----------------|-------------|
| 1.925          | 0.05 ± 0.05 | 2.475          | 0.34 ± 0.08 | 3.025          | 0.10 ± 0.07 | 3.575          | 0.04 ± 0.05 |
| 1.975          | 0.04 ± 0.05 | 2.525          | 0.13 ± 0.08 | 3.075          | 1.00 ± 0.13 | 3.625          | 0.04 ± 0.05 |
| 2.025          | 0.01 ± 0.06 | 2.575          | 0.09 ± 0.04 | 3.125          | 0.70 ± 0.11 | 3.675          | 0.19 ± 0.06 |
| 2.075          | 0.08 ± 0.08 | 2.625          | 0.18 ± 0.07 | 3.175          | 0.18 ± 0.07 | 3.725          | 0.04 ± 0.05 |
| 2.125          | 0.04 ± 0.06 | 2.675          | 0.10 ± 0.06 | 3.225          | 0.08 ± 0.07 | 3.775          | 0.13 ± 0.05 |
| 2.175          | 0.16 ± 0.08 | 2.725          | 0.13 ± 0.07 | 3.275          | 0.13 ± 0.07 | 3.825          | 0.11 ± 0.05 |
| 2.225          | 0.07 ± 0.07 | 2.775          | 0.02 ± 0.06 | 3.325          | 0.18 ± 0.06 | 3.875          | 0.03 ± 0.04 |
| 2.275          | 0.20 ± 0.10 | 2.825          | 0.15 ± 0.08 | 3.375          | 0.08 ± 0.05 | 3.925          | 0.03 ± 0.04 |
| 2.325          | 0.17 ± 0.08 | 2.875          | 0.12 ± 0.06 | 3.425          | 0.20 ± 0.06 | 3.975          | 0.05 ± 0.04 |
| 2.375          | 0.18 ± 0.08 | 2.925          | 0.20 ± 0.09 | 3.475          | 0.11 ± 0.07 | 4.025          | 0.06 ± 0.10 |
| 2.425          | 0.05 ± 0.11 | 2.975          | 0.25 ± 0.08 | 3.525          | 0.20 ± 0.06 | 4.075          | 0.05 ± 0.04 |

obtain 870 ± 52 events. Figure 24(b) shows the mass distribution of these events.

![Figure 24](image)

(a) The expanded view of Fig. 7(b). The solid curve shows the sum of background and the two-Gaussian fit function used to obtain the number of events with an $\eta$. (b) The invariant mass distribution for the $\pi^+ \pi^- 3\pi^0 \eta$ events obtained from the $\eta$ signal fit. The contribution of the $uds$ background events is shown by the squares.

B. Peaking background

The major background producing an $\eta$ peak is the non-ISL background, in particular $e^+ e^- \rightarrow \pi^+ \pi^- 4\pi^0 \eta$ in which one of the neutral pions decays asymmetrically, producing a photon interpreted as ISR. The $\eta$ peak from the $uds$ simulation is visible in Fig. 9. We fit the $\eta$ peak in the $uds$ simulation in intervals of 0.05 GeV/c² in $m(\pi^+ \pi^- 3\pi^0 \gamma \gamma)$.

To normalize the $uds$ simulation, we form the diphoton invariant mass distribution of the ISR candidate with all the remaining photons in the event. Comparing the number of events in the $\pi^0$ peaks in data and $uds$ simulation, we assign a scale factor of 1.5 ± 0.2 to the simulation. The results are shown by the squares in Fig. 24(b). We subtract these events from the data distribution.

![Figure 25](image)

FIG. 25: The energy-dependent cross section for $e^+ e^- \rightarrow \pi^+ \pi^- 3\pi^0 \eta$. The uncertainties are statistical only.

C. Cross section for $e^+ e^- \rightarrow \pi^+ \pi^- 3\pi^0 \eta$

The cross section for $e^+ e^- \rightarrow \pi^+ \pi^- 3\pi^0 \eta$ is determined using Eq. (1). We assume the same detection efficiency for the photons from the $\pi^0 \rightarrow \gamma \gamma$ and $\eta \rightarrow \gamma \gamma$ decays. The results are shown in Fig. 25 and listed in Table VI. These are the first results for this process. The systematic uncertainties and corrections are the same as those presented in Table IV except that the uncertainty in the detection efficiency increases to 15%.

The cross section is sizable above 2 GeV. Since this is
above the energy range where the final states are summed for the \((\eta\pi - 2)\) value calculation, we do not attempt to study the intermediate channels.

VIII. THE J/ψ REGION

A. The \(\pi^+\pi^- 4\pi^0\) final state

Figure 26(a) shows an expanded view of the J/ψ mass region from Fig. 8 for the six-pion data sample. Signals from \(J/ψ \rightarrow \pi^+\pi^- 4\pi^0\) and \(\psi(2S) \rightarrow \pi^+\pi^- 4\pi^0\) are seen. The non-resonant background distribution is well described by the second-order polynomial function in this region.

![FIG. 26: (a) The \(\pi^+\pi^- 4\pi^0\) mass distribution for ISR-produced \(e^+e^- \rightarrow \pi^+\pi^- 4\pi^0\) events in the \(J/ψ\)-\(\psi(2S)\) region. (b) The MC-simulated signals. The \(J/ψ\)-\(\psi(2S)\) signals ratio is arbitrary. The curves show the fit functions described in the text.](image)

The observed peak shapes are not purely Gaussian because of radiation effects and resolution, as seen in the simulated signal distributions shown in Fig. 26(b). The sum of two Gaussians is used in the fit. We obtain 340 ± 42 \(J/ψ\) events and 28 ± 19 \(\psi(2S)\) events. Using the results for the number of events, the detection efficiency, and the ISR luminosity, we determine the product:

\[
B_{J/ψ \rightarrow 6\pi} \cdot \Gamma_{J/ψ} = \frac{N(J/ψ \rightarrow \pi^+\pi^- 4\pi^0) \cdot m^2_{J/ψ}}{6\pi^2 \cdot dL/dE \cdot \epsilon^{MC} \cdot \epsilon^{corr} \cdot C} \tag{2}
\]

where \(\Gamma_{J/ψ}^{el}\) is the electronic width, \(dL/dE = 180 \text{ nb}^{-1}/\text{MeV}\) is the ISR luminosity at the \(J/ψ\) mass \(m_{J/ψ}^{MC} = 0.018 ± 0.002\) is the detection efficiency from simulation with the corrections \(\epsilon^{corr} = 0.85\), discussed in Sec. V.ID and \(C = 3.894 \times 10^{11} \text{ nb MeV}^2\) is a conversion constant [29]. We estimate the systematic uncertainty for this region to be 15%. The subscript “6\(\pi\)” for the branching fraction refers to the \(\pi^+\pi^- 4\pi^0\) final state exclusively.

Using \(\Gamma_{J/ψ}^{el} = 5.55 ± 0.14 \text{ keV}\) [29], we obtain \(B_{J/ψ \rightarrow 6\pi} = (6.5 ± 0.8 ± 1.0) \times 10^{-3}\); no other measurements for this channel exist.

Using Eq. (2) and the result \(dL/dE = 228 \text{ nb}^{-1}/\text{MeV}\) at the \(\psi(2S)\) mass, we obtain:

\[
B_{\psi(2S) \rightarrow 6\pi} \cdot \Gamma_{ee}^{(2S)} = (3.3 ± 2.3 ± 0.5) \text{ eV}.
\]

With \(\Gamma_{ee}^{(2S)} = 2.34 ± 0.06 \text{ keV}\) [29] we find \(B_{\psi(2S) \rightarrow 6\pi} = (1.4 ± 1.0 ± 0.2) \times 10^{-3}\). For this channel also, no previous result exists.

1. The \(\eta\pi^+\pi^-\pi^0\), \(\eta\omega\) intermediate states

Figure 27(a) shows an expanded view of Fig. 15 with the \(\pi^+\pi^- 4\pi^0\) mass distribution for events obtained by a fit to the \(3\pi^0\) mass distribution to select events with an \(\eta\). The two-Gaussian fit, implemented as described above, yields 200 ± 16 and < 20 events at 90% C.L. for the \(J/ψ\) and \(\psi(2S)\), respectively. Using Eq. (2) we obtain:

\[
B_{J/ψ \rightarrow \eta\pi^+\pi^-\pi^0} \cdot B_{\eta \rightarrow 3\pi^0} \cdot \Gamma_{J/ψ}^{el} = (21.1 ± 17.3 ± 3.2) \text{ eV},
\]

\[
B_{\psi(2S) \rightarrow \eta\pi^+\pi^-\pi^0} \cdot B_{\eta \rightarrow 3\pi^0} \cdot \Gamma_{ee}^{(2S)} < 3 \text{ eV}.
\]

Using \(B_{\eta \rightarrow 3\pi^0} = 0.3268\) and the value of \(\Gamma_{ee}\) from Ref. [29], we obtain \(B_{J/ψ \rightarrow \eta\pi^+\pi^-\pi^0} = (11.9 ± 0.9 ± 2.3) \times 10^{-3}\) and \(B_{\psi(2S) \rightarrow \eta\pi^+\pi^-\pi^0} < 3.5 \times 10^{-3}\) at 90% C.L. There are no other measurements of these decays.

Similarly, the expanded view of Fig. 15 in Fig. 27(b) for the subsample of \(\pi^+\pi^- 4\pi^0\) events with \(\eta \rightarrow 3\pi^0\) and an additional signal from \(\omega \rightarrow \pi^+\pi^-\pi^0\). The fit yields 47 ± 20 events corresponding to

\[
B_{J/ψ \rightarrow \eta\omega} \cdot B_{\eta \rightarrow 3\pi^0} \cdot B_{\omega \rightarrow \pi^+\pi^-\pi^0} \cdot \Gamma_{ee}^{J/ψ} = (4.9 ± 2.1 ± 0.7) \text{ eV},
\]

which yields \(B_{J/ψ \rightarrow \eta\omega} = (3.0 ± 1.3 ± 0.5) \times 10^{-3}\), compatible with the current world average result \(B_{J/ψ \rightarrow \eta\omega} = (1.74 ± 0.20) \times 10^{-3}\) [29].

We can set only an upper limit for the \(\psi(2S) \rightarrow \eta\omega\) decay: we observe < 20 events corresponding to \(B_{\psi(2S) \rightarrow \eta\omega} < 14 \times 10^{-4}\) at 90% C.L., consistent with the world average value < 1.1 × 10^{-5} [29].

2. The \(\omega 3\pi^0\) intermediate state

The expanded view of Fig. 27(b) is shown in Fig. 27(c). The fit yields 89 ± 22 for the \(J/ψ \rightarrow \omega 3\pi^0\) events corresponding to

\[
B_{J/ψ \rightarrow \omega 3\pi^0} \cdot B_{\omega \rightarrow \pi^+\pi^-\pi^0} \cdot \Gamma_{J/ψ}^{el} = (9.4 ± 2.3 ± 1.5) \times 10^{-3},
\]

\[
B_{J/ψ \rightarrow \omega 3\pi^0} = (1.9 ± 0.5 ± 0.3) \times 10^{-3}.
\]

We use \(B_{\omega \rightarrow \pi^+\pi^-\pi^0} = 0.892\) from Ref. [29]. No other measurements are available for this decay mode.

We can set only an upper limit for the \(\psi(2S) \rightarrow \omega 3\pi^0\) decay: we observe < 14 events corresponding to \(B_{\psi(2S) \rightarrow \omega 3\pi^0} < 8 \times 10^{-4}\) at 90% C.L. which is the only measured limit for this decay.
B. The $\pi^+\pi^-3\pi^0\eta$ final state

The expanded view of Fig. 27(b) is shown in Fig. 27(d). The fit yields 101 ± 16 for the $J/\psi \rightarrow \pi^+\pi^-3\pi^0\eta$ events corresponding to

$$B_{J/\psi \rightarrow \pi^+\pi^-3\pi^0\eta} \cdot B_{\eta \rightarrow \gamma\gamma} \cdot \Gamma_{\pi^+\pi^-3\pi^0\eta} = (10.6 \pm 1.6 \pm 1.6) \text{ eV},$$

$$B_{J/\psi \rightarrow \pi^+\pi^-3\pi^0\eta} = (4.9 \pm 0.8 \pm 0.8) \times 10^{-3}.$$ 

We set an upper limit for the $\psi(2S) \rightarrow \pi^+\pi^-3\pi^0\eta$ decay: we observe < 16 events at 90% C.L. corresponding to $B_{\psi(2S) \rightarrow \pi^+\pi^-3\pi^0\eta} < 2.0 \times 10^{-3}$. There are no previous results for this final state.

C. Summary of the charmonium region study

The rates of $J/\psi$ and $\psi(2S)$ decays to $\pi^+\pi^-4\pi^0$, $\pi^+\pi^-3\pi^0\eta$, and several intermediate final states have been measured. The measured products and calculated branching fractions are summarized in Table VII together with the available PDG values for comparison. Most of the measurements are performed for the first time.

| Quantity                                      | Measured Value (eV) | $J/\psi$ or $\psi(2S)$ Branching Fraction (10^{-3}) |
|-----------------------------------------------|---------------------|--------------------------------------------------|
| $\Gamma_{\pi^+\pi^-3\pi^0\eta}$             | 35.8 ± 4.4 ± 5.4    | 6.5 ± 0.8 ± 1.0                                  |
| $\Gamma_{\pi^+\pi^-3\pi^0\eta}$             | 21.1 ± 1.7 ± 3.2    | 11.9 ± 0.9 ± 2.3                                 |
| $\Gamma_{\pi^+\pi^-3\pi^0\eta}$             | 4.9 ± 2.1 ± 0.7     | 3.0 ± 1.3 ± 0.5                                 |
| $\Gamma_{\pi^+\pi^-3\pi^0\eta}$             | 9.4 ± 2.3 ± 1.5     | 1.9 ± 0.5 ± 0.3                                 |
| $\Gamma_{\pi^+\pi^-3\pi^0\eta}$             | 10.6 ± 1.6 ± 1.6    | 4.9 ± 0.8 ± 0.8                                 |
| $\Gamma_{\pi^+\pi^-3\pi^0\eta}$             | 3.3 ± 2.3 ± 0.5     | 1.4 ± 1.0 ± 0.2                                 |

D. Summary of the charmonium region study

The rates of $J/\psi$ and $\psi(2S)$ decays to $\pi^+\pi^-4\pi^0$, $\pi^+\pi^-3\pi^0\eta$, and several intermediate final states have been measured. The measured products and calculated branching fractions are summarized in Table VII together with the available PDG values for comparison. Most of the measurements are performed for the first time.

The excellent photon-energy and charged-particle momentum resolutions, as well as the particle identification capabilities of the BABAR detector, allow the reconstruction of the $\pi^+\pi^-4\pi^0$ and $\pi^+\pi^-3\pi^0\eta$ final states produced at center-of-mass energies below 4.5 GeV via initial-state radiation in data collected at the $\Upsilon(4S)$ mass region.

The cross sections for the $e^+e^- \rightarrow \pi^+\pi^-4\pi^0$ and the $e^+e^- \rightarrow \pi^+\pi^-3\pi^0\eta$ reactions have been measured for the first time. The accuracies are 12% and 15%, respectively.

The selected multi-hadronic final states in the broad range of accessible energies provide new information on hadron spectroscopy. The observed $e^+e^- \rightarrow \eta\eta\gamma\gamma$, $e^+e^- \rightarrow \eta\pi^+\pi^-\pi^0\eta$, and $e^+e^- \rightarrow \eta\omega$ cross sections provide additional information for the hadronic contribution calculation of the muon g-2.
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