Research Article

Numerical Simulation and Symmetry Reduction of a Two-Component Reaction-Diffusion System

Jina Li and Xuehui Ji

College of Science, Zhongyuan University of Technology, Zhengzhou 450007, China

Correspondence should be addressed to Jina Li; lijina@zut.edu.cn

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In this paper, the symmetry classification and symmetry reduction of a two-component reaction-diffusion system are investigated, the reaction-diffusion system can be reduced to system of ordinary differential equations, and the solutions and numerical simulation will be showed by examples.

1. Introduction

The system

\[ u_t = \left( d_1 + d_2 u + d_3 v \right) u_{xx} + u(a_{11} - b_{11} u - c_1 v), \] (1)

\[ v_t = \left( d_4 + d_5 u + d_6 v \right) v_{xx} + v(a_{21} - b_{12} u - c_2 v), \] (2)

where the parameters \( a_{11}, a_{21} > 0 \) are the intrinsic growth coefficients, \( b_{11}, c_1 \geq 0 \) are the coefficients of intraspecific competitions, \( d_i \) (\( i = 1, 2, \cdots, 6 \)) are the diffusion rate, and parameters \( c_1 \) and \( b_{12} \) determine the types of species interactions. When \( c_1 > 0 \) and \( b_{12} > 0 \), this model is competition interaction; when \( c_1 > 0 \) and \( b_{12} < 0 \), this model is mutualism interaction; and when \( c_1 < 0 \) and \( b_{12} < 0 \), this model is prey-predator interaction. Systems (1) and (2) are proposed by Shigesada et al. [1] and include the classical Lotka-Volterra system, diffusive Lotka-Volterra system, and the generalization form [2–4]. The symmetry methods are also known one of the effective methods for construction exact solutions of differential equations; the symmetry method was created by Sophus Lie [5] and was developed by Ovsiannikov [6], Bluman [7], Olver [8], Cherniha [9], and other researchers [10–17]. The authors mainly research the Lie symmetry, exact solution, conditional Lie-Bäcklund symmetry (CLBS) of reaction-diffusion system, or the relevant research work [18–27]. This paper mainly research the symmetry reduction, solutions, and numerical simulation of systems (1) and (2).

2. Symmetry Reduction

In this section, we will illustrate the main feature of the reduction procedure. The systems (1) and (2) admit the conditional Lie-Bäcklund symmetry (CLBS)

\[ \eta_1 = u_{xx} - b_1 u_x, \]

\[ \eta_2 = v_{xx} - b_1 v_x, \] (3)

when

\[ b_{11} = 4d_2 b_1^2, \quad c_1 = 4d_3 b_1^2, \]

\[ b_{12} = 4d_5 b_1^2, \quad c_2 = 4d_6 b_1^2. \] (4)

We mainly consider the following two cases.

Case 1. When \( b_1 = 0 \), then system can be derived to the following form

\[ u_t = \left( d_1 + d_2 u + d_3 v \right) u_{xx} + a_{11} u, \] (5)

\[ v_t = \left( d_4 + d_5 u + d_6 v \right) v_{xx} + a_{21} v, \] (6)

and admits the CLBS:

\[ \eta_1 = u_{xx}, \] (7)

where
\[ \eta_2 = v_{xx}. \] (8)

The systems (7) and (8) are a system of ordinary differential equations (ODEs) with respect to variable \( x \), so the following forms are the corresponding solutions:

\[ u = \phi_1(t)x + \phi_2(t), \] (9)
\[ v = \psi_1(t)x + \psi_2(t). \] (10)

In the following, inserting solutions (9) and (10) into (7) and (8) yields the following ODEs:

\[ \frac{d\phi_1}{dt} = a_1\phi_1(t), \] (11)
\[ \frac{d\phi_2}{dt} = 2d_2\phi_1(t)^2 + 2d_4\phi_1(t)\psi_1(t) + a_{11}\phi_2(t), \] (12)
\[ \frac{d\psi_1}{dt} = a_1\psi_1(t), \] (13)
\[ \frac{d\psi_2}{dt} = 2d_5\phi_1(t)\psi_1(t) + 2d_6\psi_1(t)^2 + a_{12}\psi_2(t). \] (14)

We solve the systems (11)–(14); the solutions are shown as below:

\[ \phi_1(t) = C_2e^{a_{11}t}, \]
\[ \phi_2(t) = e^{a_{11}t}\left[2C_2\left(\frac{d_4C_1}{a_{11}}e^{a_{11}t} + \frac{d_2C_2}{a_{11}}e^{a_{11}t}\right) + C_3\right], \]
\[ \psi_1(t) = C_4e^{a_{12}t}, \]
\[ \psi_2(t) = e^{a_{12}t}\left[2C_4\left(\frac{d_6C_1}{a_{12}}e^{a_{12}t} + \frac{d_2C_2}{a_{12}}e^{a_{12}t}\right) + C_4\right]. \] (15)

Then, the solutions of systems (5) and (6) can be shown by substituting the above functions \( \phi_1(t) \), \( \phi_2(t) \), \( \psi_1(t) \), and \( \psi_2(t) \) into Eqs. (9) and (10).

Case 2. When \( b_1 \neq 0 \), the system

\[ u_t = [(d_1 + d_2u + d_3v)u]_{xx} + u(a_{11} - 4d_2b_1^2u - 4d_4b_1^2v), \] (16)
\[ v_t = [(d_4 + d_5u + d_6v)v]_{xx} + v(a_{21} - 4d_2b_1^2u - 4d_6b_1^2v), \] (17)

admits the CLBS:

\[ \eta_1 = u_{xx} - b_1u_x, \] (18)
\[ \eta_2 = v_{xx} - b_1v_x. \] (19)

The system (19) is a system of ODEs with respect to variable \( x \), so the following forms are the corresponding solutions.

\[ u = \phi_1(t) + \phi_2(t)e^{b_1t}, \] (20)
\[ v = \psi_1(t) + \psi_2(t)e^{b_1t}. \] (21)

In the following, inserting solutions (21) into (16) yields the following ODEs:

\[ \frac{d\phi_1}{dt} = -4\phi_1(t)^2b_1^2d_2 + (-4b_1^2d_4\psi_1(t) + a_{11})\phi_1(t), \] (22)
\[ \frac{d\phi_2}{dt} = (-6b_1^2d_5\phi_2(t) - 3\psi_2(t)b_1^2d_1)\phi_1(t) + (-3b_1^2d_4\psi_1(t) + d_1b_1^2 + a_{11})\phi_2(t), \] (23)
\[ \frac{d\psi_1}{dt} = -4\psi_1(t)^2b_1^2d_6 + (-4b_1^2d_4\phi_1(t) + a_{21})\psi_1(t), \] (24)
\[ \frac{d\psi_2}{dt} = (-3b_1^2d_5\psi_2(t) - 6\psi_2(t)b_1^2d_4)\psi_1(t) + (-3b_1^2d_4\phi_1(t) + d_1b_1^2 + a_{21})\psi_2(t). \] (25)

3. Numerical Simulation

In the following, we research the numerical simulations of systems (22) and (25). Systems (22) and (25) have four equilibria \( E_1(0, 0, 0, 0) \), \( E_2((a_{11}/4b_1^2d_2), 0, 0, 0) \), \( E_3((0, 0, a_{21}/4b_1^2d_4), 0) \) and \( E_4((a_{11} - a_{21}d_1/4b_1^2(d_1d_6 - d_2d_4)), 0, (a_{21}d_2 - a_{21}d_1/4b_1^2(d_1d_6 - d_2d_4)), 0) \). The Jacobian matrix of systems (22) and (25) at \( E_i(i = 1, 2, 3, 4) \) takes the form of

\[ J_{E_i} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & d_1b_1^2 + a_{11} & 0 & 0 \\ 0 & 0 & d_1b_1^2 + a_{21} & 0 \\ 0 & 0 & 0 & d_1b_1^2 + a_{21} \end{pmatrix}, \]

\[ J_{E_i} = \begin{pmatrix} -a_{11} & 0 & -d_1a_{11}/d_2 & 0 \\ 0 & -\frac{1}{2}a_{11} + d_1b_1^2 & 0 & -\frac{3d_1a_{11}}{4d_2} \\ 0 & 0 & -d_1b_1^2 + a_{21} & 0 \\ 0 & 0 & 0 & -d_1b_1^2 + a_{21} \end{pmatrix}, \]

\[ J_{E_i} = \begin{pmatrix} \frac{d_1a_{11} + a_{21}}{d_1} & 0 & 0 & 0 \\ 0 & 0 & -\frac{3d_1a_{21}}{4d_1} + d_1b_1^2 & a_{21} & 0 \\ 0 & 0 & 0 & -\frac{3d_1a_{21}}{4d_1} & -\frac{3}{2}a_{21} + d_1b_1^2 \end{pmatrix}. \] (26)
Figure 1: The equilibrium $E_2$ of system is locally asymptotically stable in the case of $d_3 > 0$ and $d_5 > 0$. Here, $b_1 = 1$, $a_{11} = 2$, $a_{32} = 1$, $d_1 = 0.1$, $d_2 = 0.1$, $d_3 = 0.1$, $d_4 = 0.1$, $d_5 = 0.15$, and $d_6 = 0.1$.

Figure 2: In the case of $d_3 > 0$ and $d_5 > 0$, the asymptotical behavior of system with equilibrium $E_2$ of the corresponding ODE system is stable. Here, $b_1 = 1$, $a_{11} = 2$, $a_{32} = 1$, $d_1 = 0.1$, $d_2 = 0.1$, $d_3 = 0.1$, $d_4 = 0.1$, $d_5 = 0.15$, and $d_6 = 0.1$.

Figure 3: The equilibrium $E_4$ of system is locally asymptotically stable in the case of $d_3 > 0$ and $d_6 > 0$. Here, $b_1 = 1$, $a_{11} = 10$, $a_{32} = 1$, $d_1 = 0.1$, $d_2 = 0.1$, $d_3 = 0.1$, $d_4 = 0.1$, $d_5 = 0.001$, and $d_6 = 0.1$.

Figure 4: In the case of $d_3 > 0$ and $d_6 > 0$, the asymptotical behavior of system with equilibrium $E_4$ of the corresponding ODE system is stable. Here, $b_1 = 1$, $a_{11} = 10$, $a_{32} = 1$, $d_1 = 0.1$, $d_2 = 0.1$, $d_3 = 0.1$, $d_4 = 0.1$, $d_5 = 0.001$, and $d_6 = 0.1$. 

Figure 5: The equilibrium $E_5$ of system is locally asymptotically stable in the case of $d_3 > 0$ and $d_6 > 0$. Here, $b_1 = 1$, $a_{11} = 10$, $a_{32} = 1$, $d_1 = 0.1$, $d_2 = 0.1$, $d_3 = 0.1$, $d_4 = 0.1$, $d_5 = 0.001$, and $d_6 = 0.1$.
respectively, where

\[
\begin{align*}
J_{11} &= -4b_1^3 (2d_1 \phi_1^3 + d_3 \psi_1^3) + a_{11}, \\
J_{22} &= -3b_2^3 (2d_2 \phi_1^3 + d_1 \phi_2^3) + d_1 b_2^3 + a_{11}, \\
J_{33} &= -4b_3^3 (2d_3 \psi_1^3 + d_2 \phi_2^3) + a_{21}, \\
J_{44} &= -3b_4^3 (2d_4 \psi_1^3 + d_3 \phi_2^3) + d_4 b_2^3 + a_{21}, \\
\phi_1^3 &= \frac{a_1 d_6 - a_{21} d_3}{4b_1^3 (d_6 - d_3)}, \\
\psi_1^3 &= \frac{a_{21} d_2 - a_{11} d_5}{4b_1^3 (d_6 - d_3)}.
\end{align*}
\]

(i) In the case of \( c_1 > 0 \) and \( b_{12} > 0 \), that is \( d_3 > 0 \) and \( d_4 > 0 \), obviously, the eigenvalues of the matrix \( J_{E_1} \) are not all negative. So, equilibrium \( E_1 \) of system is not stable. The eigenvalues of the matrix \( J_{E_2} \) are \( \lambda_{21} = -a_{11} < 0 \), \( \lambda_{22} = -1/2a_{11} + d_1 b_1^3 \), \( \lambda_{23} = -(d_1 a_{11}/d_2) + a_{21} \), and \( \lambda_{24} = -(3d_1 a_{11}/4d_2) + b_1^3 + a_{21} \). If \( a_{11} > \max \{ 2 d_1 b_1^3, (d_1 a_{11}/d_2), (4d_1/3d_2)(d_1 b_1^3 + a_{21}) \} \), that is \( \lambda_{2i} < 0 \), \( i = (1, 2, 4) \), then equilibrium \( E_2 \) is locally asymptotically stable (please see Figures 1 and 2). The eigenvalues of the matrix \( J_{E_3} \) are \( \lambda_{31} = -(d_1 a_{11}/d_2) + a_{11}, \)
\( \lambda_{32} = -(3d_1 a_{21}/4d_2) + d_1 b_1^3 + a_{11}, \)
\( \lambda_{33} = -a_{21} < 0 \), and \( \lambda_{34} = -1/2a_{21} + d_2 b_1^3 \). If \( a_{21} > \max \{ (d_2 a_{21}/a_{11}), (4d_2/3d_1)(d_2 b_1^3 + a_{11}), 2d_2 b_1^3 \} \), that is \( \lambda_{3i} < 0 \), \( i = (1, 2, 4) \); then, equilibrium \( E_3 \) is locally asymptotically stable. Due to the complexity of the eigenvalues of the matrix \( J_{E_3} \), we do not give a theoretical result for the stability of equilibrium \( E_3 \) here, and we shall investigate it through numerical simulations (please see Figures 3 and 4).

(ii) In the case of \( c_1 < 0 \) and \( c_2 < 0 \), that is \( d_4 < 0 \) and \( d_5 < 0 \), similar to the analysis as in case (i), we can see that equilibrium \( E_1 \) is unstable, \( E_2 \) is locally asymptotically stable under the condition \( a_{11} > \max \{ 2 d_1 b_1^3, (d_1 a_{11}/d_2) a_{21}, (4d_1/3d_2)(d_1 b_1^3 + a_{21}) \} \), \( E_3 \) is locally asymptotically stable under the condition \( a_{21} > \max \{ (d_1 a_{21}/a_{11}), (4d_1/3d_2)(d_2 b_1^3 + a_{11}), 2d_2 b_1^3 \} \), and \( E_4 \) exists under conditions \( (a_{11} d_6 - a_{21} d_3)(d_2 d_6 - d_3 d_5) > 0, (a_{21} d_2 - a_{11} d_5)(d_2 d_6 - d_3 d_5) > 0 \).

(iii) In the case of \( c_1 > 0 \) and \( c_2 < 0 \), that is \( d_3 > 0 \) and \( d_5 < 0 \), similar to the analysis as in case (ii), we can see that equilibrium \( E_1 \) is unstable, \( E_2 \) is locally asymptotically stable under the condition \( a_{11} > \max \{ 2 d_1 b_1^3, (d_1 a_{11}/d_2) a_{21}, (4d_1/3d_2)(d_1 b_1^3 + a_{21}) \} \), \( E_3 \) is locally asymptotically stable under the condition \( a_{21} > \max \{ (d_1 a_{21}/a_{11}), (4d_1/3d_2)(d_2 b_1^3 + a_{11}), 2d_2 b_1^3 \} \), and \( E_4 \) exists under conditions \( d_4 d_6 - d_5 d_5 < 0, a_{21} d_2 - a_{11} d_5 < 0 \).

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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