Flat symplectic Lie algebras

Mohamed Boucetta\(^a\), Hamza El Ouali\(^b\), and Hicham Lebzioui\(^b\)

\(^{a}\)Faculté des sciences et techniques, Université Cadi-Ayyad, Marrakech, Maroc; \(^{b}\)École Supérieure de Technologie-Khénifra, Université Sultan Moulay Slimane, Khénifra, Maroc

**Abstract**

Let \((G, \Omega)\) be a symplectic Lie group, i.e., a Lie group endowed with a left invariant symplectic form. If \(g\) is the Lie algebra of \(G\) then we call \((g, \omega = \Omega(e))\) a symplectic Lie algebra. The product \(\bullet\) on \(g\) defined by \(3\omega(x \bullet y, z) = \omega([x, y], z) + \omega([x, z], y)\) extends to a left invariant connection \(\nabla\) on \(G\) which is torsion free and symplectic (\(\nabla \Omega = 0\)). When \(\nabla\) has vanishing curvature, we call \((G, \Omega)\) a flat symplectic Lie group and \((g, \omega)\) a flat symplectic Lie algebra.

In this paper, we study flat symplectic Lie groups. We start by showing that the derived ideal of a flat symplectic Lie algebra is degenerate with respect to \(\omega\). We show that a flat symplectic Lie group must be nilpotent with degenerate center. This implies that the connection \(\nabla\) of a flat symplectic Lie group is always complete. We prove that the flat symplectic double extension process can be applied to characterize all flat symplectic Lie algebras. More precisely, we show that every flat symplectic Lie algebra is obtained by a sequence of flat symplectic double extension of flat symplectic Lie algebras starting from \(\{0\}\). As examples in low dimensions, we classify all flat symplectic Lie algebras of dimension \(\leq 6\).

**1. Introduction**

Let \((M, \Omega)\) be a symplectic manifold, i.e., a manifold endowed with a nondegenerate closed 2-form. A symplectic connection is a connection \(\nabla\) such that \(\nabla \Omega = 0\). Contrasting with pseudo-Riemannian manifolds where there exits a unique torsion free connection which preserves the metric, namely, the Levi-Civita connection, on a symplectic manifold, there are many torsion free symplectic connections (see [3]). However, in a symplectic Lie group \((G, \Omega)\), among all the many torsion free symplectic left invariant connections there is one which depends only on the structure of Lie group and \(\Omega\). This connection appeared first in [4]. Indeed, let \((G, \Omega)\) be a symplectic Lie group \((G, \Omega)\), i.e., a Lie group \(G\) endowed with a left invariant symplectic form \(\Omega\). Its Lie algebra \(g\) endowed by \(\omega = \Omega(e)\) is called a symplectic Lie algebra and denoted by \((g, \omega)\). The product \(\bullet\) on \(g\) defined by

\[
\omega(u \bullet v, w) = \frac{1}{3} (\omega([u, v], w) + \omega([u, w], v)),
\]

for any \(u, v, w \in g\), extends to a left invariant connection \(\nabla\) on \(G\) which is torsion free and symplectic. As one can see from (1.1), \(\nabla\) depends only on the structure of Lie group and \(\Omega\) and can be thought of as the Levi-Civita connection of a symplectic Lie group. In this paper, we study the case where this connection is of vanishing curvature. In this case, we call \((G, \Omega)\) a flat symplectic Lie group and \((g, \omega)\) a flat symplectic Lie algebra.

Recall that if \((G, \Omega)\) is a symplectic Lie group, then the left-invariant connection \(\nabla\) defined by

\[
\omega(\nabla_u v, w) = \omega(v, [w, u]),
\]

for any \(u, v, w \in g\), is of vanishing curvature. Note that \(\nabla\) is symplectic (\(\nabla \Omega = 0\)) if and only if \(G\) is abelian.
In this paper, we show that if \((g, \omega)\) is a flat symplectic Lie algebra then \(Z(g)\) and \([g, g]\) must be degenerate with respect to \(\omega\). We adapt the process of double extension developed in [1, 6, 9] to our context and we show that all flat symplectic Lie algebras can be obtained by this process. Using this method, we prove that a flat symplectic Lie group must be nilpotent. Thus, the connection \(\nabla\) in a flat symplectic Lie group is always geodesically complete. In particular, we deduce that flat symplectic Lie algebras are obtained by a sequence of flat symplectic double extension of flat symplectic Lie algebras starting from \(\{0\}\). Finally, as applications, we classify all flat symplectic Lie algebras of dimension \(\leq 6\).

The paper contains five sections. In Section 2, we give some general results on flat symplectic Lie algebras, and in particular, we prove that \([g, g]\) is degenerate in this case. We describe the method of flat symplectic double extension of flat symplectic Lie algebras in Section 3. In Section 4, we prove that every flat symplectic Lie algebra is nilpotent with degenerate center. We also show in this section that the flat symplectic double extension process characterize all flat symplectic Lie algebras. In Section 5, we show that, in dimension 4, there exists only one non-abelian flat symplectic Lie algebra, and we classify up to a Lie isomorphism, all 6-dimensional flat symplectic Lie algebras.

Throughout this paper, \(g\) is a finite dimensional Lie algebra over the real field \(\mathbb{R}\). A symplectic vector space \((V, \omega)\) is a real finite dimensional vector space \(V\) endowed with a non-degenerate bilinear skew-symmetric form \(\omega\). A symplectic basis of a symplectic vector space \((V, \omega)\) is a basis \(\{e_1, \ldots, e_{2n}\}\) of \(V\) such that \(\omega(e_i, e_{i+1}) = -\omega(e_{i+1}, e_i) = 1, i = 1, \ldots, 2n-1,\) and all others are zero. Let \(F\) be a subspace of \((V, \omega)\). \(F^\perp\) is the subspace defined by \(F^\perp = \{x \in g/\omega(x,y) = 0\text{ for any } y \in F\}\). \(F\) is said to be degenerate (resp. non-degenerate, totally isotropic, lagrangian) if \(F \cap F^\perp = \{0\}\) (resp. \(F \cap F^\perp = \{0\}, F \subset F^\perp, F = F^\perp\)). If \(f : V \rightarrow V\) then \(f^*\) is the endomorphism of \(V\) defined by \(\omega(f(x), y) = \omega(x, f^*(y))\) for any \(x, y \in V\).

2. Preliminaries

Let \((G, \Omega)\) be a symplectic Lie group, i.e., a Lie group endowed with a left invariant symplectic form. We consider the associated symplectic Lie algebra \((g, \omega)\). For any \(u \in g\), we denote by \(u^\dagger\) the left invariant vector field on \(G\) associated to \(u\). The condition \(\text{d}\Omega = 0\) is equivalent to,

\[
\omega([u, v], w) + \omega([v, w], u) + \omega([w, u], v) = 0,
\]

for any \(u, v, w \in g\). The product \(\bullet\) given by (1.1) defines a left invariant connection \(\nabla\) on \(G\) given by

\[
\nabla_{u^\dagger} v^\dagger = (u \bullet v)^\dagger, \quad u, v \in g.
\]

Note that (1.1) can be written

\[
L_u = \frac{1}{3} (\text{ad}_u - \text{ad}_u^*)
\]

where \(L_u v = u \bullet v\). This implies that \(L_u\) is skew-symmetric with respect to \(\omega\), i.e.,

\[
\omega(u \bullet v, w) + \omega(v, u \bullet w) = 0
\]

for any \(u, v, w \in g\). This is equivalent to \(\nabla \Omega = 0\). Moreover, by using (2.1), we can deduce that \(\bullet\) is Lie admissible, i.e., \([u, v] = u \bullet v - v \bullet u\), for any \(u, v \in g\). This can be written \(\text{ad}_u = L_u - R_u\) where \(\text{ad}_u v = [u, v]\) and \(R_u v = v \bullet u\). At the Lie group level this is equivalent to \(\nabla\) is torsion free.

Let \(Z(g)\) be the center of the Lie algebra \(g, [g, g]\) its derived ideal, \(g_0 = \text{span}\{u \bullet v/ u, v \in g\}\), \(N^e(g) = \{u \in g, L_u = 0\}\) and \(N^r(g) = \{u \in g, R_u = 0,\}\). It is obvious from (2.2) that \(N^e(g) = \{u \in g, \text{ad}_u^* = \text{ad}_u\}\).

**Proposition 2.1.** Let \((g, \omega)\) be a symplectic Lie algebra. Then

\[
[g, g]^\perp = \{u \in g/\text{ad}_u + \text{ad}_u^* = 0\}
\]

and

\[
Z(g) = (gg)^\perp \cap N^r(g) = N^e(g) \cap [g, g]^\perp.
\]

If \(g\) is not solvable, then the solvable radical \(R(g)\) is degenerate with respect to \(\omega\).
Proof. The first relation is an immediate consequence of (2.1). The equality \((gg)^\perp = \{ u \in g/R_u = 0 \}\) follows immediately from (2.3). Moreover, by virtue of (2.2) if \(u \in Z(g)\) then \(L_u = 0\) and hence \(R_u = 0\). Suppose now that \(R_u = 0\). Then \(\text{ad}_u = L_u\) and hence \(\text{ad}_u^* = -\text{ad}_u\). But \(L_u = \frac{1}{2}(\text{ad}_u - \text{ad}_u^*)\) and hence \(\text{ad}_u = -\frac{1}{2}\text{ad}_u^*\). It follows that \(\text{ad}_u = 0\). If \(g\) is not solvable and \(R(g)\) is nondegenerate then, according to (1.2), \(R(g)^\perp\) is stable by the left symmetric product induced by \(\nabla\). But \(R(g)^\perp\) is isomorphic to \(g/R(g)\) which is semi-simple. This is a contradiction, since the Lie algebra associated to a left symmetric product cannot be semi-simple [7].

Recall that \(g\) is unimodular (\(G\) is unimodular) if, for any \(u \in g\), \(\text{tr}(\text{ad}_u) = 0\). Let \(H\) be the vector defined by \(\omega(H, u) = \text{tr}(\text{ad}_u)\) for any \(u \in g\). Then \(g\) is unimodular if and only if \(H = 0\).

A vector subspace \(I\) is called a left ideal (resp. right ideal) if \(g \bullet I \subset I\) (resp. \(I \bullet g \subset I\)). It is called two-sided ideal if it is left and right ideal. We call \(I\) a Lie ideal if \([g, I] \subset I\). It is obvious that a two-sided ideal is a Lie ideal.

**Proposition 2.2.** If \(I\) is a Lie ideal of \(g\) then

\[
I^\perp \bullet I \subset I, \quad I \bullet I^\perp \subset I \quad \text{and} \quad I^\perp \bullet I^\perp \subset I^\perp.
\]

In particular, \(I^\perp\) is a Lie subalgebra.

**Proof.** From (2.1), we have for any \(u, v \in I^\perp\) and \(w \in I\), we get that \(\omega([u, v], w) = 0\) and hence \(I^\perp\) is a Lie subalgebra. Moreover, for any \(u, v \in I^\perp\) and any \(w \in I\), by using (1.1), we get that \(\omega(u \bullet v, w) = 0\) and hence \(I^\perp \bullet I^\perp \subset I^\perp\). Since the \(L_u\) are skew-symmetric, we deduce that \(I^\perp \bullet I \subset I\). On the other hand, for any \(u \in I\) and \(v, w \in I^\perp\), by virtue of (1.1), \(\omega(u \bullet v, w) = 0\) and hence \(I \bullet I^\perp \subset I^\perp\).

The symplectic Lie group \((G, \Omega)\) is flat if the curvature \(K\) of \(\nabla\) vanishes identically. This is equivalent to

\[
L_{[u, v]} = [L_u, L_v]
\]

for any \(u, v \in g\). One can see easily that (2.4) is equivalent to

\[
R_{u \bullet v} - R_v \circ R_u = [L_u, R_v],
\]

(2.5)

for any \(u, v \in g\). In this case we call \((g, \omega)\) a flat symplectic Lie algebra.

We give here, a particular family of examples of flat symplectic Lie algebras.

**Proposition 2.3.** Let \((g, \omega)\) be a symplectic Lie algebra. If \(Z(g)^\perp \subset Z(g)\) (in particular if \(Z(g)\) is lagrangian), then \((g, \omega)\) is flat. In this case \(\bullet\) is associative and \((g, [\ , \ ])\) is a two-step nilpotent Lie algebra.

**Proof.** If \(Z(g)^\perp \subset Z(g)\), then from \(Z(g) = (gg)^\perp\), we deduce that \((gg) \subset Z(g)\). Since \(Z(g) \subset N^\ell(g)\), then \((u \bullet v) \bullet w = u \bullet (v \bullet w) = 0\) for any \(u, v, w \in g\). Thus \(\bullet\) is an associative product and \((g, \omega)\) is flat. Since \([u, v] = u \bullet v - v \bullet u\), then \([g, g] \subset Z(g)\) and \((g, [\ , \ ])\) is two-step nilpotent in this case.

**Example 2.1.** Let \(g_6^2\) the 6-dimensional two-step nilpotent Lie algebra defined by the only non-vanishing Lie brackets \([x_1, x_2] = x_3\) and \([x_1, x_3] = x_6\). It is proved in page 52 of [8] that \(g_6^2\) admits three non-isometric symplectic forms: \(\omega_1 = x_1^* \wedge x_6^* + x_2^* \wedge x_3^* + x_3^* \wedge x_4^*\), \(\omega_2 = x_1^* \wedge x_4^* + x_2^* \wedge x_5^* + x_3^* \wedge x_6^*\) and \(\omega_3 = x_1^* \wedge x_5^* + x_2^* \wedge x_4^* - x_3^* \wedge x_6^*\). Since \(Z(g)\) is lagrangian with respect to \(\omega_1, \omega_2\) and \(\omega_3\), then every symplectic form on \(g_6^2\) is flat.

**Proposition 2.4.** Let \((g, \omega)\) be a flat symplectic Lie algebra. Then

1. For any \(u, v \in [g, g]^\perp\), \(u \bullet v = 0\) and \(\text{ad}_u \circ \text{ad}_v = 0\). In particular, \([g, g]^\perp\) is abelian and, for any \(u \in [g, g]^\perp\), \(\text{ad}_u^2 = 0\).
2. \( H \in [\mathfrak{g}, \mathfrak{g}] \cap [\mathfrak{g}, \mathfrak{g}]^\perp. \)

**Proof.** 1. Let \( u, v \in [\mathfrak{g}, \mathfrak{g}]^\perp. \) By virtue of (1.1), one has for any \( w \in \mathfrak{g}, \)

\[
\omega(u \cdot v, w) = \frac{1}{3} \omega([u, v], w) + \frac{1}{3} \omega([u, w], v),
\]

\[
= \frac{1}{3} \omega([u, v], w),
\]

\[
= 0.
\]

Then \( u \cdot v = 0 \) and \( [u, v] = u \cdot v - v \cdot u = 0. \)

On the other hand, from (2.1) and the fact that \( \text{ad}_u = L_u - R_u, \) one can deduce easily that \( L_u = \frac{2}{3} \text{ad}_u \) and \( R_u = -\frac{1}{3} \text{ad}_u \) for any \( u \in [\mathfrak{g}, \mathfrak{g}]^\perp. \) Now, for \( u, v \in [\mathfrak{g}, \mathfrak{g}]^\perp, \) and since \( u \cdot v = 0, \) the relation (2.5) implies that \(-\frac{1}{3} \text{ad}_u \circ \text{ad}_v = -\frac{2}{3} \text{ad}_{[v, u]}\), which implies that \( \text{ad}_u \circ \text{ad}_v = 0 \) and in particular \( \text{ad}_u^2 = 0. \)

2. For any \( u, v \in \mathfrak{g}, \) one has \( \omega(H, [u, v]) = \text{tr}(\text{ad}_{[u, v]}) = 0 \) and hence \( H \in [\mathfrak{g}, \mathfrak{g}] \). Let \( u \in [\mathfrak{g}, \mathfrak{g}]^\perp. \) Since \( \text{ad}_u \) is nilpotent then \( \omega(H, u) = \text{tr}(\text{ad}_u) = 0. \) It follows that \( H \in [\mathfrak{g}, \mathfrak{g}] \cap [\mathfrak{g}, \mathfrak{g}]^\perp \) as desired.

\[
\square
\]

It is known that there is a correspondence between connected and simply connected affine Lie groups and left symmetric algebras. An affine Lie group is a Lie group endowed with a torsion free flat left invariant connection. A left symmetric algebra is an algebra \((A, \cdot)\) satisfying

\[
\text{ass}(u, v, w) = \text{ass}(v, u, w),
\]

where \( \text{ass}(u, v, w) = (u \cdot v)z - u \cdot (v \cdot w), \) for any \( u, v, w \in \mathfrak{g}. \) It is known that a left symmetric algebra is Lie-admissible. Let \( G(A) \) be a Lie group whose Lie algebra is \((A, [\cdot, \cdot])\) where \([u, v] = u \cdot v - v \cdot u\) and denote by \( u^+ \) the left invariant vector field on \( G(A) \) associated to \( u \in A. \) Then the left invariant connection \( \nabla \) on \( G(A) \) given by \( \nabla_{u^+} v^+ = (u \cdot v)^+ \) is torsion free and has vanishing curvature. Moreover, \( \nabla \) is geodesically complete if and only if \( R_u \) is nilpotent for any \( u \in A \) [11]. This is also equivalent to \( \text{tr}(R_u) = 0 \) for any \( u \in A. \) It is known that if \( \nabla \) is complete then \((A, [\cdot, \cdot])\) is solvable and hence \( G(A) \) is solvable [7].

Let \((G, \Omega)\) be a flat symplectic Lie group and \((\mathfrak{g}, \omega)\) its associated symplectic Lie algebra. Then the vanishing of the curvature of \( \Omega \) is equivalent to \((\mathfrak{g}, \cdot)\) is a left symmetric algebra. Since for any \( u \in \mathfrak{g}, \) \( L_u \) is skew-symmetric then \( \text{tr}(R_u) = -\text{tr}(\text{ad}_u) \) and hence \( \nabla \) is complete if and only if \( G \) is unimodular and we get the following result.

**Remark 2.1.** The symplectic connection \( \nabla \) of a flat symplectic Lie group is complete if and only if the group is unimodular and in this case it is solvable.

**Proposition 2.5.** Let \((\mathfrak{g}, \omega)\) be a flat symplectic Lie algebra. Then \([\mathfrak{g}, \mathfrak{g}] \) is degenerate.

**Proof.** Suppose in the contrary that \([\mathfrak{g}, \mathfrak{g}] \) is nondegenerate. Then \( \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus [\mathfrak{g}, \mathfrak{g}]^\perp. \) Since \( H \in [\mathfrak{g}, \mathfrak{g}] \cap [\mathfrak{g}, \mathfrak{g}]^\perp \) then \( \mathfrak{g} \) must be unimodular and hence \( G \) is solvable. Thus \([\mathfrak{g}, \mathfrak{g}] \) is nilpotent and hence its center \( Z([\mathfrak{g}, \mathfrak{g}]) \neq \{0\}. \) Let us show that \( Z([\mathfrak{g}, \mathfrak{g}]) \subset Z(\mathfrak{g}) \) which is a contradiction since, according to **Proposition 2.1**, \( Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]^\perp. \) Indeed, if \( u \in Z([\mathfrak{g}, \mathfrak{g}]) \) then for any \( v \in [\mathfrak{g}, \mathfrak{g}]^\perp, \) we have for any \( w \in [\mathfrak{g}, \mathfrak{g}], \)

\[
[[u, v], w] = [u, [v, w]] + [v, [w, u]] = 0.
\]

On the other hand, for any \( w \in [\mathfrak{g}, \mathfrak{g}]^\perp, \) according to **Proposition 2.4**, \( \text{ad}_v \circ \text{ad}_w = 0 \) and hence \([u, v, w] = 0. \) Thus \([u, v] \in Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]^\perp \) and hence \([u, v] = 0 \) which completes the proof. \( \square \)
Proposition 2.6. 1. Let \((A,\cdot)\) be a left-symmetric algebra. Then \(N^2(A) = \{u \in A/L_u = 0\}\) is a two-sided ideal and in particular, it is a Lie ideal.

2. Let \((\mathfrak{g},\omega)\) be a flat symplectic Lie algebra. Then \([\mathfrak{g},[\mathfrak{g},\mathfrak{g}]^\perp] \subset N^2(\mathfrak{g})\).

Proof. 1. Let \((A,\cdot)\) be a left-symmetric algebra. It is trivial that \(N^2(A)\) is a right ideal. Let us show that it is also a left ideal. Let \(u \in N^2(A)\) and \(v, w \in A\). From

\[(v.u).w - v.(u.w) = (u.v).w - u.(v.w),\]

we deduce that \((v.u).w = 0\) and hence \(v.u \in N^2(\mathfrak{g})\).

2. Let \((\mathfrak{g},\omega)\) be a flat symplectic Lie algebra and \(u, v \in \mathfrak{g}\). From (1.1) one has, \(3L_u = \text{ad}_u - \text{ad}_u^\ast\). Thus

\[0 = L_{[u,v]} - [L_u,L_v]\]

\[= \frac{1}{3} (\text{ad}_{[u,v]} - \text{ad}_{[u,v]}^\ast) - \frac{1}{9} \left[\text{ad}_u - \text{ad}_u^\ast, \text{ad}_v - \text{ad}_v^\ast\right]\]

\[= \frac{2}{9} \text{ad}_{[u,v]} - \frac{2}{9} \text{ad}_{[u,v]}^\ast + \frac{1}{9} \left[\text{ad}_u^\ast, \text{ad}_v\right] + \frac{1}{9} \left[\text{ad}_u, \text{ad}_v^\ast\right]. \hspace{1cm} (2.6)\]

Now for any \(u \in \mathfrak{g}\) and \(v \in [\mathfrak{g},\mathfrak{g}]^\perp\), we have \(\text{ad}_u^\ast = -\text{ad}_v\) and if we replace in the relation above, we get

\[0 = \frac{2}{9} \text{ad}_{[u,v]} - \frac{2}{9} \text{ad}_{[u,v]}^\ast + \frac{1}{9} \left[\text{ad}_u^\ast, \text{ad}_v\right] + \frac{1}{9} \left[\text{ad}_u, \text{ad}_v^\ast\right]\]

\[\Rightarrow \text{ad}_{[u,v]} = \text{ad}_{[u,v]}^\ast\] and hence \([u,v] \in N^2(\mathfrak{g})\). \hfill \square

The symplectic reduction is an important tool in the study of symplectic Lie algebras (see [2]). If \((\mathfrak{g},\omega)\) is a symplectic Lie algebra and \(I \subset \mathfrak{g}\) is a totally isotropic Lie ideal, then \(I^\perp/I\) carries a natural symplectic structure called the symplectic reduction of \((\mathfrak{g},\omega)\) by \(I\). The following proposition shows that the symplectic reduction of a flat symplectic Lie algebra is also flat. A symplectic Lie algebra is called completely reducible if it admits a sequence of symplectic reduction to the trivial symplectic Lie algebra. Otherwise, it is called irreducible.

Proposition 2.7. Let \((\mathfrak{g},\omega)\) be a flat symplectic Lie algebra and \(I\) is a totally isotropic Lie ideal. Then the symplectic Lie algebra structure on \(I^\perp/I\) is flat.

Proof. According to Proposition 2.2, \(I^\perp\) is stable by \(\bullet\) and \(I\) is a two-sided ideal of \(I^\perp\). Hence \(\bullet\) defines a product \(\bullet\) which is the product associated to the symplectic Lie algebra structure on \(I^\perp/I\). \hfill \square

Example 2.2. It is known that the Lie algebra \(\text{aff}(n,\mathbb{R})\) of the affine group has a symplectic Lie algebra structure (see [5]). Moreover, Baues and Cortés in [2] showed that \(\text{aff}(n,\mathbb{R})\) has a symplectic reduction to \(\text{aff}(n-1,\mathbb{R})\). It is easy to check that the symplectic structure of \(\text{aff}(1,\mathbb{R})\) is not flat (see Example 4.1). Then the symplectic structure of \(\text{aff}(n,\mathbb{R})\) is not flat.

Proposition 2.8. A flat symplectic Lie algebra is completely reducible.

3. Flat symplectic double extension of flat symplectic Lie algebras

The double extension method constitute a powerful tool to study Lie groups endowed with left-invariant structures (see for instance [1, 6, 9]). In [1], the authors developed the double extension method to study
Lie groups endowed with flat left-invariant pseudo-Riemannian metrics. In [12], the author adapted this process to study Lie groups endowed by flat symplectic connection. In this section, we adapt this method to our connection given by (1.1), and we will show in the next section, that all flat symplectic Lie algebras are obtained by this method.

Let \((B, \omega_B)\) be a flat symplectic Lie algebra. We denote by \(ab\) the product defined by (1.1) for any \(a, b \in B\). Let \(\xi \in \text{End}(B)\) and \(b_0 \in B\) such that for any \(a, b \in B\)

\[
\begin{align*}
\left[\xi^*, \xi^*\right] &= \xi^2 - \frac{1}{3} R_{b_0}, \\
\xi^* &\in \ker \left(\xi^* - \xi\right), \\
\xi^* \circ \xi &= \frac{1}{3} \left( R_{b_0} + R_{b_0}^* \right), \\
\xi (\{a, b\}) &= a\xi(b) - b\xi(a), \\
\xi^*(ab) - \xi^*(a)b - a\xi^*(b) &= \xi(ab) - a\xi(b) - 2\xi(a)b.
\end{align*}
\]

A couple \((\xi, b_0) \in \text{End}(B) \times B\) which satisfies the equations (3.1)–(3.5) is called \textit{admissible}. Let \(g\) be the vector space defined by \(g = \mathbb{R}^e \oplus B \oplus \mathbb{R}\xi\) endowed with the non-degenerate skew-symmetric form \(\omega\) defined by \(\omega_{/B \times B} = \omega_B, \omega(e, e) = \omega(\xi, \xi) = 0, \omega(e, B) = 0\) and \(\omega(e, \xi) = 1\). We define also in \(g\) the Lie brackets

\[
[\xi, a] = (\xi^* - 2\xi)(a) + \omega_B(b_0, a)e \quad \text{and} \quad [a, b] = [a, b]_B + \omega_B \left((\xi + \xi^*)(a), b\right) e.
\]

Let us show that \((g, \omega)\) is also a flat symplectic Lie algebra. First, one can show easily that \(\omega\) satisfies (2.1). Thus, the product defined by (1.1) satisfies \([u, v] = u \circ v - v \circ u\) for any \(u, v \in g\). Let us show that this product is left symmetric. From (1.1), one has, for any \(a, b \in B\),

\[
\begin{align*}
L_e &= R_e = 0, \\
\xi^* - \xi &= (\xi^* - 2\xi)(a) + \frac{1}{3} \omega_B(b_0, a)e, \\
\left(\xi^* - \xi\right)(a) &= \frac{2}{3} \omega_B(b_0, a)e, \\
\xi^* - \xi &= \frac{1}{3} b_0,
\end{align*}
\]

where the product of \(a, b \in B\) is denoted in \(B\) by \(ab\) and in \(g\) by \(a \circ b\). Since \(L_e = R_e = 0\) then \(\text{ass}(e, u, v) = \text{ass}(u, e, v) = \text{ass}(u, v, e) = 0\) for any \(u, v \in g\). Let \(a, b, c \in B\). The equation \(\xi^* - \xi\) is equivalent to (3.1) and (3.2). The equation \(\text{ass}(\xi, a, \xi) = \text{ass}(a, \xi, \xi)\) is equivalent to (3.1) and (3.5). The equation \(\text{ass}(a, b, \xi) = \text{ass}(b, a, \xi)\) is equivalent to (3.3) and (3.4). Finally, The equation \(\xi^* - \xi\) is equivalent to (3.4) and to the fact that \((B, \omega_B)\) is a flat symplectic Lie algebra. In summary, if \((B, \omega_B)\) is a flat symplectic Lie algebra of dimension \(2n\) and \((\xi, b_0) \in \text{End}(B) \times B\) such that (3.1)–(3.5) are satisfied, then \((g, \omega)\) is a flat symplectic Lie algebra of dimension \(2n + 2\).

**Definition 3.1.** The flat symplectic Lie algebra \((g, \omega)\) constructed as above is called the flat symplectic double extension of \((B, \omega_B)\) by means of \(\xi\) and \(b_0\).

Conversely, we have the following result.

**Proposition 3.1.** Let \((g, \omega)\) be a flat symplectic Lie algebra which admits a one-dimensional two-sided ideal \(I\) (with respect to the product given by (1.1)) such that \(I^1\) is also a two-sided ideal. Then \((g, \omega)\) is a flat symplectic double extension of another flat symplectic Lie algebra of dimension \(\dim g - 2\) by means of \(\xi\) and \(b_0\).
Remark 3.1. • As we have shown in Proposition 3.1, if $I = \mathbb{R}e$ is a two-sided ideal of a flat symplectic Lie algebra such that $I^\perp$ is also a two-sided ideal, then $e \in Z(\mathfrak{g})$. 

Proof. Let $(\mathfrak{g}, \omega)$ be a flat symplectic Lie algebra which admits a one-dimensional two-sided ideal $I$. Since $I^\perp$ is also a two-sided ideal, then $I^\perp/I$ can be endowed by a flat symplectic structure. In fact, for any $\tilde{x}, \tilde{y} \in I^\perp/I$, we put $\tilde{\omega}(\tilde{x}, \tilde{y}) = \omega(\tilde{x}, \tilde{y})$. Then $\tilde{\omega}$ is a symplectic structure on $I^\perp/I$. Furthermore, the product defined on $I^\perp/I$ by

$$\tilde{\omega}(\tilde{x} \tilde{y}, \tilde{z}) = \frac{1}{3} \tilde{\omega}([\tilde{x}, \tilde{y}], \tilde{z}) + \tilde{\omega}([\tilde{x}, \tilde{z}], \tilde{y})$$

is left symmetric. Thus $(I^\perp/I, \tilde{\omega})$ is a flat symplectic Lie algebra. We put $I = \mathbb{R}e$ and let $\tilde{e} \in \mathfrak{g}$ such that $\omega(e, \tilde{e}) = 1$. Let $B' = (e, \tilde{e})^\perp$. Then $\mathfrak{g} = \mathbb{R}e \oplus B' \oplus \mathbb{R}\tilde{e}$ where $I^\perp = \mathbb{R}e \oplus B'$. Since $I^\perp$ is a two-sided ideal, then for any $a, b \in B'$ one has

$$(\alpha e + a) \bullet (\beta e + b) = f(a, b)e + a \bullet b,$$

where $a \bullet b$ is the component of $a \bullet b$ over $B'$. The product on $I^\perp$ is left symmetric is equivalent to the fact that $(B', \bullet)$ is a left symmetric algebra and

$$f([a, b]_{B'}, c) = f(a, b * c) - f(b, a * c), \text{ for any } a, b, c \in B'. \quad (3.7)$$

The equation (3.7) is equivalent to the fact that $f$ is a scalar 2-cocycle for the left symmetric algebra $(B', \bullet)$ (see [1, 10]). Let $H^2_{SG}(B', \mathbb{R})$ be the second space of scalar cohomology of the left symmetric algebra $B'$ and $H^1_{L}(B', B')$ the first space of cohomology of the Lie algebra $B'$ with respect to the representation $L$. It is proved in [1] that $H^2_{SG}(B', \mathbb{R})$ is isomorphic to $H^1_{L}(B', B')$ via the equation

$$f(a, b) = \omega(\xi(a), b), \text{ for any } a, b \in B',$$

where $\xi : B' \rightarrow B'$ such that $\xi ([a, b]_{B'}) = a \star \xi (b) - b \star \xi (a)$. On the other hand, the application $\phi : B' \rightarrow I^\perp/I$ defined by $\phi(a) = \tilde{a}$ for any $a \in B'$ is an isomorphism of left symmetric algebra. Thus, we can identify $B'$ with $B = I^\perp/I$ and hence $\mathfrak{g} = \mathbb{R}e \oplus B \oplus \mathbb{R}\tilde{e}$ where $(B, \omega_B) = (I^\perp/I, \tilde{\omega})$ is a flat symplectic Lie algebra. The product on $I^\perp = \mathbb{R}e \oplus B$ is given by

$$(\alpha e + a) \bullet (\beta e + b) = \omega_B(\xi(a), b) e + ab.$$

Thus, the Lie brackets on $\mathfrak{g}$ are given by, for any $a, b \in B$

$$[\tilde{e}, e] = \lambda e, \quad [\tilde{e}, a] = D(a) + \omega_B(b_0, a)e \text{ and } [a, b] = [a, b]_B + \omega_B(\xi + \xi^*)(a), b) e,$$

where $D \in \text{End}(B), \lambda \in \mathbb{R} \text{ and } b_0 \in B$. We have $e \bullet \tilde{e} = -\frac{2}{3} \lambda e, \tilde{e} \bullet e = \frac{\lambda}{3} e \text{ and } \tilde{e} \bullet \tilde{e} = \frac{1}{3} b_0 + \tilde{e}$. Since $(\mathfrak{g}, \omega)$ is flat then

$$(e \bullet \tilde{e}) \bullet \tilde{e} e = (\tilde{e} \bullet e) \bullet \tilde{e} e = (e \bullet \tilde{e}) \bullet \tilde{e} - (\tilde{e} \bullet e) \bullet (e \bullet \tilde{e}),$$

which implies that $\lambda = 0$. From,

$$\omega(a \bullet b, \tilde{e}) = \frac{1}{3} \omega_B(\xi + \xi^* - D(a), b),$$

and $\omega(a \bullet b, \tilde{e}) = \omega_B(\xi(a), b), \text{ one can deduce that } D = \xi^* - 2\xi$. Therefore, the Lie brackets in $\mathfrak{g} = \mathbb{R}e \oplus B \oplus \mathbb{R}\tilde{e}$ are reduced to

$$[\tilde{e}, a] = (\xi^* - 2\xi)(a) + \omega_B(b_0, a)e \text{ and } [a, b] = [a, b]_B + \omega_B((\xi + \xi^*)(a), b)e.$$

As in the previous paragraph, one can show that $(\mathfrak{g}, \omega)$ is a flat symplectic Lie algebra if and only if (3.1)–(3.5) hold. Thus $(\mathfrak{g}, \omega)$ is a flat symplectic double extension of a flat symplectic Lie algebra $(B, \omega_B)$ by means of $\xi$ and $b_0$. 

□
If \((g, \omega) \) is a flat symplectic double extension of \((B, \omega_B)\) by means of \(\xi\) and \(b_0\), then \(D = \xi^* - 2\xi\) is a derivation of the Lie algebra \(B\) and \(\omega_B ((D^* (D + D^*)) + (D + D^*)D),..) is a scalar coboundary (see [6]). Indeed, according to (3.5), one can check that for any \(a, b \in B\),

\[
\xi^* - 2\xi (\langle a, b \rangle) = \left[ \xi^* - 2\xi (a), b \right] + \left[ a, \xi^* - 2\xi (b) \right],
\]

and according to (3.1), one can deduce that \(\xi^2 - \frac{1}{3} R_{b_0}\) is \(\omega_B\)-symmetric which is equivalent to the coboundary condition.

**Proposition 3.2.** Let \((B, \omega_B)\) be a flat symplectic Lie algebra and \((g, \omega)\) the flat symplectic double extension of \((B, \omega_B)\) by means of \((\xi, b_0)\). If \(B\) is nilpotent then \(g\) is nilpotent.

**Proof.** Let \((B, \omega_B)\) be a flat symplectic Lie algebra and let \((\xi, b_0)\) be an admissible couple of \(\text{End}(B) \times B\). Let us show that, for any \(k \in \mathbb{N}^*\),

\[
\text{tr} \left( \xi^k \circ R_a \right) = \text{tr} \left( R_{\xi^k(a)} \right).
\]

Indeed, for any \(a, b \in B\), \(\xi ([b, a]) = b.\xi (a) - a.\xi (b)\), thus \(\xi \circ R_a - \xi \circ L_a = R_{\xi(a)} - L_a \circ \xi\), and hence \(\text{tr} (\xi \circ R_a) = \text{tr} (R_{\xi(a)})\). Then the equation is true for \(k = 1\). We suppose the property is true for \(k\) and let us show that it is also correct for \(k + 1\). We have

\[
\xi^{k+1} ([a, b]) = \xi^k \circ \xi ([a, b]) = \xi^k (a.\xi (b) - b.\xi (a)) = \xi^k \circ L_a \circ \xi (b) - \xi^k \circ R_{\xi(a)} (b).
\]

It follows that \(\text{tr} (\xi^{k+1} \circ R_a) = \text{tr} (\xi^k \circ R_{\xi(a)}) = \text{tr} (R_{\xi^{k+1}(a)})\).

We put \(D = \xi^* - 2\xi\). From (3.1), one has \([\xi, D] = \xi^2 - \frac{1}{3} R_{b_0}\). By induction, one can show that for any \(k \in \mathbb{N}^*\),

\[
\left[ \xi^k, D \right] = k \xi^k + 1 - \frac{1}{3} \sum_{p=0}^{k-1} \xi^p \circ R_{b_0} \circ \xi^{k-1-p}.
\]

Thus \(\text{tr} (\xi^{k+1}) = \frac{1}{3} \text{tr} (R_{b_0} \circ \xi^{k-1})\), and from (3.8), one has \(\text{tr} (\xi^{k+1}) = \frac{1}{3} \text{tr} (R_{\xi^{k-1}(b_0)})\), for any \(k \in \mathbb{N}^*\).

If \(B\) is nilpotent then \(B\) is a complete left symmetric algebra which implies that \(\text{tr} (R_a) = 0\) for any \(a \in B\). It follows that \(\text{tr} (\xi^{k+1}) = 0\) for any \(k \in \mathbb{N}^*\) and hence \(\xi\) is nilpotent. From (3.1) and (3.3), one can deduce that \(\xi \circ \xi^* = \xi^2 + \frac{1}{3} R_{b_0}\) and \((\xi^*)^2 = [\xi, \xi^*] + \frac{1}{3} R_{b_0}^*\). We replace in the equation

\[
D^2 = 4\xi^2 + (\xi^*)^2 - 2\xi \circ \xi^* - 2\xi^* \circ \xi\quad \text{and we get}\quad D^2 = 3\xi^2 - 3\xi^* \circ \xi = -3D \circ \xi - 3\xi^2.
\]

By induction, one can deduce that for any \(k \geq 2\),

\[
D^k = a_k D \circ \xi^{k-1} + b_k \xi^k,
\]

where \(a_k, b_k \in \mathbb{R}\). Since \(\xi\) is nilpotent, then \(D\) is nilpotent. Now, if \((g, \omega)\) is a flat symplectic double extension of \((B, \omega_B)\), then the Lie brackets in \(g\) is given by

\[
[\xi, a] = D(a) + \omega_B (b_0, a) B e, \quad [a, b] = [a, b]_B + \omega_B (\xi + \xi^*(a), b) e.
\]

Since \(D\) is nilpotent then \(\text{ad}_\xi\) is nilpotent, and according to the nilpotency of \(B\) we deduce that \(\text{ad}_\xi^*\) is nilpotent and hence \(\text{ad}_\xi\) is also nilpotent, which implies that \(g\) is a nilpotent Lie algebra. This completes the proof of the proposition.

\(\Box\)

**4. Flat symplectic Lie groups are nilpotent**

In this section, we show that any flat symplectic Lie group is nilpotent. We show also that its Lie algebra is obtained by a sequence of flat symplectic double extension of flat symplectic Lie algebras starting from \(\{0\}\).
Proposition 4.1. Let \((\mathfrak{g}, \omega)\) be a flat symplectic Lie algebra. If \(\mathfrak{g}\) is solvable, then its center is not trivial.

Proof. We consider the sequence of vector subspace given by

\[ C_0 = [\mathfrak{g}, \mathfrak{g}] \cap [\mathfrak{g}, \mathfrak{g}]^\perp \quad \text{and} \quad C_k = ([\mathfrak{g}, \mathfrak{g}], C_{k-1}) \quad k \geq 1. \]

Since \(\mathfrak{g}\) is solvable, then \([\mathfrak{g}, \mathfrak{g}]\) is nilpotent and hence there exists \(k_0 \in \mathbb{N}\) such that \(C_{k_0} = \{0\}\). Suppose that \(Z(\mathfrak{g}) = \{0\}\). Let us show that, for any \(k \geq 1\), if \(C_k = \{0\}\) then \(C_{k-1} = \{0\}\). Suppose that \(C_k = \{0\}\). Then for any \(u \in C_{k-1}\), for any \(v \in [\mathfrak{g}, \mathfrak{g}]\) and for any \(w \in \mathfrak{g}\),

\[ 0 = \omega([u, v], w) = \pm \omega(v, [u, w]) \]

since \(\text{ad}_u = -\text{ad}_u\) if \(k = 1\) and \(\text{ad}_u^* = \text{ad}_u\) if \(k \geq 2\). So \([u, w] \in [\mathfrak{g}, \mathfrak{g}]^\perp \cap N^*(\mathfrak{g}) = Z(\mathfrak{g}) = \{0\}\) and hence \(u = 0\). It follows that \(C_0 = \{0\}\) which contradicts the fact that \([\mathfrak{g}, \mathfrak{g}]\) is degenerate by virtue of Proposition 2.5.

Example 4.1. The non abelian symplectic Lie algebra of dimension 2 is solvable and has a vanishing center so it is not flat.

Proposition 4.2. Let \((\mathfrak{g}, \omega)\) be a flat symplectic Lie algebra. Then its center is not trivial.

Proof. Let \((\mathfrak{g}, \omega)\) be a flat symplectic Lie algebra. If \(\mathfrak{g}\) is unimodular, then it is solvable, and the result follows from Lemma 4.1. Suppose \(\mathfrak{g}\) is non-unimodular which implies that \(H \neq 0\). We will show that \(H \in Z(\mathfrak{g})\). Let \(u \in \mathfrak{g}\), we put \(x_0 = u\) and \(x_{n+1} = L_{x_n}(u)\) for any \(n \geq 0\). Let us show by induction that, for any \(n \geq 1\)

\[ R_{x_n} = R_u^{n+1} + R_u^{n-1} \circ [L_u, R_u] + R_u^{n-2} \circ [L_{x_1}, R_u] + \cdots + [L_{x_{n-1}}, R_u]. \]

From (2.5), the relation is true for \(n = 1\). We assume the property established at the rank \(n \geq 1\). According to the relation (2.5) we have

\[ R_{x_{n+1}} = R_{x_n - u} = R_u \circ R_{x_n} + [L_{x_n}, R_u] \]

\[ = R_u \circ (R_u^{n+1} + R_u^{n-1} \circ [L_u, R_u] + R_u^{n-2} \circ [L_{x_1}, R_u] + \cdots + [L_{x_{n-1}}, R_u]) + [L_{x_n}, R_u] \]

\[ = R_u^{n+2} + R_u^n \circ [L_u, R_u] + R_u^{n-1} \circ [L_{x_1}, R_u] + \cdots + [L_{x_n}, R_u] \]

Now, according to the Proposition 2.6, we have \([H, x] \in N_\ell(\mathfrak{g})\), for any \(x \in \mathfrak{g}\). Moreover, let \(u \in N_\ell^\perp(\mathfrak{g})\) we put, for any \(n \geq 0\) \(x_{n+1} = L_{x_n}(u)\) where \(x_0 = u\), according to the relation (1.1), we have

\[ \omega(H, x_{n+1}) = \omega(H, L_{x_n}(u)) = \frac{1}{3} \omega([H, x_n], u) = 0. \]

On the other hand, we have

\[ \omega(H, x_{n+1}) = \text{tr}(ad_{x_{n+1}}) = -\text{tr}(R_{x_{n+1}}) \]

and according to the relation (4.1) and to the fact that \(\text{tr}\left( R_u^0 \circ [L_u, R_u] \right) = 0\), we have, for any \(n \geq 0\)

\[ \text{tr}(R_{x_{n+1}}) = \text{tr}(R_u^{n+2}) = 0 \]

which implies that, for all \(u \in N_\ell^\perp(\mathfrak{g})\), \(R_u\) is nilpotent. It follows that \(H \in N_\ell(\mathfrak{g})\) and since \(H \in [\mathfrak{g}, \mathfrak{g}]^\perp\)

one has \(H \in Z(\mathfrak{g})\) as desired.

\[ \square \]

Theorem 4.1. Let \((\mathfrak{g}, \omega)\) be a flat symplectic Lie algebra. Then \((\mathfrak{g}, \omega)\) is obtained by a sequence of flat symplectic double extension starting from \(\{0\}\). Furthermore, \(\mathfrak{g}\) must be nilpotent with degenerate center.
Proof. Let \((g, \omega)\) be a flat symplectic Lie algebra. According to Proposition 4.2, \(Z(g) \neq \{0\}\). Let \(e\) be a non-null vector in \(Z(g)\). Thus \(L_e = R_e = 0\) and hence \(I = Re\) is a two-sided ideal. Furthermore, for any \(a \in I^\perp\) and \(x \in g\),

\[
\omega(ax, e) = -\omega(x, ae) = 0 \quad \text{and} \quad \omega(xa, e) = -\omega(a, xe) = 0.
\]

Then \(I^\perp\) is also a two-sided ideal. Thus, according to Proposition 3.1, \((g, \omega)\) is a flat symplectic double extension of a flat symplectic Lie algebra \((B_1, \omega_{B_1})\). Since \((B_1, \omega_{B_1})\) is a flat symplectic Lie algebra, then \(Z(B_1) \neq \{0\}\) and hence \((B_1, \omega_{B_1})\) is also a flat symplectic double extension of another flat symplectic Lie algebra \((B_2, \omega_{B_2})\). Thus, \((g, \omega)\) is obtained by a sequence of flat symplectic double extension starting from \(\{0\}\). According to Proposition 3.2, the flat symplectic double extension preserves the nilpotency of the Lie algebra. Since the trivial Lie algebra \(\{0\}\) is nilpotent and \((g, \omega)\) is obtained by a sequence of flat symplectic double extension starting from \(\{0\}\) thus \(g\) must be nilpotent. From \(Z(g) \subset [g, g]^+\), we deduce \(Z(g) \cap [g, g] \subset Z(g) \cap Z(g)^\perp\) and hence \(Z(g)\) is degenerate. \(\square\)

5. Examples of flat symplectic Lie algebras of low dimensions

In this section, we give all flat symplectic Lie algebras of dimension \(\leq 6\).

5.1. Four dimensional flat symplectic Lie algebras

Four dimensional flat symplectic Lie algebras are flat symplectic double extension of the 2-dimensional symplectic abelian Lie algebra by means of an admissible couple \((\xi, b_0)\). Note that if \(B\) is abelian then \((\xi, b_0) \in \text{End}(B) \times B\) is admissible if and only if \(\xi \xi^* = \xi^2\), \(\xi^* \xi = 0\) and \(b_0 \in \ker(\xi^* - \xi)\).

Proposition 5.1. Let \((B, \omega_B)\) be the symplectic abelian Lie algebra of dimension 2, then \((\xi, b_0)\) is admissible if and only if \((\xi = 0 \text{ and } b_0 \in B)\) or there exists a symplectic basis \(\{e_1, e_2\}\) of \((B, \omega_B)\) such that

\[
\xi = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b_0 = \alpha e_1,
\]

where \(\alpha \in \mathbb{R}\) and \(a \neq 0\).

Proof. Since \(\xi \xi^* = \xi^2\) and \(\xi^* \xi = 0\) we deduce that \(\xi\) is nilpotent, then \(\xi = 0\) and \(b_0 \in B\), or \(\xi^2 = 0\) and \(\xi \neq 0\). In the last case, there exists a symplectic basis \(\{e_1, e_2\}\) of \(B\) such that

\[
\xi = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b_0 = \alpha e_1 + \beta e_2
\]

where \(\alpha, \beta \in \mathbb{R}\) and \(a \neq 0\).

Form \(\xi(b_0) = \xi^*(b_0)\) we deduce that \(\beta = 0\). \(\square\)

Proposition 5.2. The only non-abelian flat symplectic Lie algebra of dimension 4 is \((\mathbb{R} \times h_3, \omega)\) where

\[
\mathbb{R} \times h_3 : [x_1, x_2] = x_3 \quad \text{and} \quad \omega_0 = x_1^* \wedge x_4^* + x_2^* \wedge x_3^*.
\]

Proof. Let \((g, \omega)\) be a flat symplectic non-abelian Lie algebra of dimension 4. According to Theorem 4.1, \(g\) is a flat symplectic double extension of the symplectic abelian Lie algebra of dimension 2. According to Proposition 5.1 we have two cases:

- If \(\xi = 0\) and \(b_0 = \alpha e_1 + \beta e_2\) where \(\alpha, \beta \in \mathbb{R}\), then according to (3.6), there exists a symplectic basis \(\{e, \tilde{e}, e_1, e_2\}\) of \(g\) such that the Lie brackets are given by

\[
[\tilde{e}, e_1] = -\beta e, \quad [\tilde{e}, e_2] = \alpha e.
\]
Since $\mathfrak{g}$ is non-abelian, then $(\alpha, \beta) \neq (0, 0)$. If for example $\beta \neq 0$, then we put
\[(x_1, x_2, x_3, x_4) = \left(-\frac{1}{\beta}e_1, \bar{e}, -e, -\beta e_2 - \alpha e_1\right).
\]
Thus $(\mathfrak{g}, \omega)$ is symplecto-isomorphic to $(\mathbb{R} \times \mathfrak{h}_3, \omega_0)$.

- If $\xi \neq 0$ and $b_0 = \alpha e_1$ where $\alpha \in \mathbb{R}$, then there exists a symplectic basis $\{e, \bar{e}, e_1, e_2\}$ of $\mathfrak{g}$ such that the Lie bracket is given by
\[\{\bar{e}, e_2\} = \alpha e - 3ae_1, \ a \neq 0.\]

Put
\[(x_1, x_2, x_3, x_4) = \left(3ae + \alpha e_2, \bar{e} + \frac{\alpha + 1}{3a}e_2, \alpha e - 3ae_1, -\frac{\alpha + 1}{3a}e + e_1\right).
\]
Thus, in this new basis, the Lie brackets and the flat symplectic form are reduced to
\[\{x_1, x_2\} = x_3 \text{ and } \omega = x_1^* \wedge x_4^* + x_2^* \wedge x_3^*,\]
which implies that $(\mathfrak{g}, \omega)$ is symplecto-isomorphic to $(\mathbb{R} \times \mathfrak{h}_3, \omega_0)$ as desired.

\[\Box\]

### 5.2. Flat symplectic Lie algebras of dimension 6

According to Theorem 4.1, flat symplectic Lie algebras of dimension 6 are obtained from 4-dimensional flat symplectic Lie algebras by using the flat symplectic double extension process. From Proposition 5.2, any flat symplectic Lie algebra of dimension 4 is abelian or isomorphic to $\mathbb{R} \times \mathfrak{h}_3$. Thus, we must determine all admissible couples $(\xi, b_0)$ in a flat symplectic Lie algebra $(B, \omega_B)$ where $B$ is abelian or $B$ is isomorphic to $\mathbb{R} \times \mathfrak{h}_3$.

**First case: $B$ is abelian.**

In the abelian case, $(\xi, b_0)$ is admissible if and only if
\[\xi\xi^* = \xi^2, \ \xi^*\xi = 0 \text{ and } b_0 \in \ker(\xi^* - \xi).
\]

**Lemma 5.1.** Let $(B, \omega_B)$ be the abelian symplectic Lie algebra of dimension 4. If $(\xi, b_0)$ is admissible then $\text{Im}(\xi)$ is totally isotropic and $\xi^2 = 0$.

**Proof.** From $\omega(\xi(u), \xi(v)) = \omega(u, \xi^*\xi(v)) = 0$, for any $u, v \in B$, we deduce that $\text{Im}(\xi)$ is totally isotropic. Let us show that $\xi^2 = 0$. First, we have $\xi^3 = \xi^*\xi^2 = 0$. Assume that $\xi^2 \neq 0$. Then $\ker\xi \subset \ker\xi^2 \subset \ker\xi^* \subset B$. Since $\text{Im}(\xi)$ is totally isotropic then $\dim \text{Im}(\xi) \leq 2$. If $\dim \text{Im}(\xi) = 1$ then $\dim \ker\xi = 3$ and hence $\xi^2 = 0$, contradiction. Thus, $\dim \text{Im}(\xi) = \dim \ker\xi = 2$ and $\dim \ker\xi^2 = 3$. We can choose a basis $\{e_1, e_2, e_3, e_4\}$ of $(B, \omega_B)$ such that $\{e_1, e_2\}$ is a basis of $\ker\xi$, $\{e_1, e_2, e_3\}$ is a basis of $\ker\xi^2$. In this basis, $\xi$ has the form
\[
\xi = \begin{pmatrix}
0 & a & 0 & c \\
0 & b & d & e \\
0 & 0 & e & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix}
\text{ and } \xi^2 = \begin{pmatrix}
0 & 0 & 0 & ae \\
0 & 0 & 0 & be \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix}
\]

If $\ker\xi$ is symplectic, then we can choose the basis $\{e_1, e_2, e_3, e_4\}$ to be a symplectic basis. Thus
\[
\xi^* = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
d & -c & 0 & -e \\
-b & a & 0 & 0 
\end{pmatrix}.
\]
From $\xi \xi^* = \xi^2$, we deduce that $ae = be = 0$ which contradicts the fact that $\xi^2 \neq 0$. If ker $\xi$ is totally isotropic then we can choose the basis such that $\{e_1, e_4, e_2, e_3\}$ is symplectic. In this case $\xi^*$ has the form

$$
\xi^* = \begin{pmatrix}
0 & e & -d & -c \\
0 & 0 & -b & -a \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

One can check in this case that $\xi \xi^* = 0$, but this contradicts the fact that $\xi \xi^* = \xi^2$ and $\xi^2 \neq 0$. This completes the proof of the lemma.

**Proposition 5.3.** Let $(B, \omega_B)$ be the abelian symplectic Lie algebra of dimension 4. If $(\xi, b_0)$ is admissible then there exists a basis $B = \{e_1, e_2, e_3, e_4\}$ of $B$ such that the matrix of $\xi$ with respect to such basis, $b_0$ and a symplectic basis $S$ have one of the following forms:

1. $\xi = 0$, $b_0 \in B$ and $S = B$.

2. $S = \{e_1, e_4, e_2, e_3\}$ and

$$
\xi = \begin{pmatrix}
0 & 0 & a & b \\
0 & 0 & c & d \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad b_0 = \alpha e_1 + \beta e_2, \text{ where } a, b, c, d, \alpha, \beta \in \mathbb{R},
$$

with $ad - bc \neq 0$ and $(a + d)^2 \neq 4bc$.

3. $S = \{e_1, e_4, e_2, e_3\}$ and

$$
\xi = \begin{pmatrix}
0 & 0 & a & 0 \\
0 & 0 & 0 & -a \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad b_0 \in B, \text{ where } a \neq 0.
$$

4. $S = \{e_1, e_4, e_2, e_3\}$ and

$$
\xi = \begin{pmatrix}
0 & 0 & a & b \\
0 & 0 & 0 & -a \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad b_0 = \alpha e_1 + \beta e_2 + \gamma e_3, \text{ where } ab \neq 0 \text{ and } \alpha, \beta, \gamma \in \mathbb{R}.
$$

5. $S = \{e_1, e_4, e_2, e_3\}$ and

$$
\xi = \begin{pmatrix}
0 & 0 & a & 0 \\
0 & 0 & b & -a \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad b_0 = \alpha e_1 + \beta e_2 + \gamma e_4, \text{ where } ab \neq 0 \text{ and } \alpha, \beta, \gamma \in \mathbb{R}.
$$

6. $S = \{e_1, e_4, e_2, e_3\}$ and

$$
\xi = \begin{pmatrix}
0 & 0 & a & b \\
0 & 0 & c & d \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad b_0 = \alpha e_1 + \beta e_2 + \gamma e_3 - \frac{a + d}{2b} \gamma e_4, \text{ where } ad - bc \neq 0, \text{ } bc \neq 0,
$$

$(a + d)^2 = 4bc$ and $\alpha, \beta, \gamma \in \mathbb{R}$.
7. $\mathcal{S} = \{e_1, e_4, e_2, e_3\}$ and
$$\xi = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ and $b_0 = \alpha e_1 + \beta e_2 + \gamma e_3$, where $\alpha, \beta, \gamma \in \mathbb{R}$ with $a \neq 0$.

8. $\mathcal{S} = \mathbb{B}$
$$\xi = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ and $b_0 = \alpha e_1 + \beta e_3$, where $\alpha, \beta \in \mathbb{R}$ with $a \neq 0$.

**Proof.** According to Lemma 5.1, $\xi^2 = 0$ and $\text{Im}(\xi)$ is totally isotropic. Thus, $\text{Im}(\xi) \subset \ker \xi$ and $\dim \text{Im}(\xi) \leq 2$.

**Case 1:** $\text{Im}(\xi) = \mathbb{R}e_1$, then we can choose a basis $\{e_1, e_2, e_3, e_4\}$ of $(\mathcal{B}, \omega_B)$ such that $\{e_1, e_2, e_3\}$ is a basis of $\ker \xi$. Thus, $\xi$ has the form
$$\xi = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- If $\text{Im}(\xi) = (\ker \xi)^\perp$, then we can choose the basis $\{e_1, e_4, e_2, e_3\}$ to be symplectic. In this case, $\xi^*$ has the form
$$\xi^* = \begin{pmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ with $a \neq 0$.

Since $b_0 \in \ker(\xi^* - \xi)$ then $b_0 = \alpha e_1 + \beta e_2 + \gamma e_3$ with $\alpha, \beta, \gamma \in \mathbb{R}$.

- If there exists $x \in \ker \xi$ such that $\omega_B(e_1, x) \neq 0$, then we can choose the basis $\{e_1, e_2, e_3, e_4\}$ to be symplectic. In this case, $\xi^*$ has the form
$$\xi^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \\ 0 & 0 & -a & 0 \end{pmatrix}$$ with $a \neq 0$.

Since $b_0 \in \ker(\xi^* - \xi)$ then $b_0 = \alpha e_1 + \beta e_3$ with $\alpha, \beta \in \mathbb{R}$.

**Case 2:** $\text{Im}(\xi) = \mathbb{R}e_1 + \mathbb{R}e_2$, then $\text{Im}(\xi) = \ker \xi$ and there exists a symplectic basis $\{e_1, e_4, e_2, e_3\}$ such that
$$\xi = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ with $ad - bc \neq 0$.

In this case, $\xi^*$ has the form
$$\xi^* = \begin{pmatrix} 0 & 0 & -d & -b \\ 0 & 0 & -c & -a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The fact that $b_0 \in \ker(\xi - \xi^*)$ is equivalent to $(a + d)\gamma + 2b\delta = 0$ and $2c\gamma + (a + d)\delta = 0$. If $(a + d)^2 \neq 4bc$ then $\gamma = \delta = 0$ and $b_0 = \alpha e_1 + \beta e_2$ ($2^{nd}$ case). Suppose $(a + d)^2 = 4bc$. If $b = c = 0$
then \( a = -d \) with \( a \neq 0 \) since \( ad \neq bc \). Thus, \( b_0 \) is any vector in \( B \) (3rd case). If \( b \neq 0 \) and \( c = 0 \) then \( a = -d \) with \( a \neq 0 \). In this case \( \delta = 0 \) and \( b_0 = \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4 \) (4th case). If \( b = 0 \) and \( c \neq 0 \) (resp. \( bc \neq 0 \)) then we obtain the 5th (resp. 6th) case.

\[
\text{Second case: } B \text{ is not abelian.}
\]

**Proposition 5.4.** Let \((B, \omega_B)\) be a non abelian flat symplectic Lie algebra of dimension 4. Then \((\xi, b_0)\) is admissible if and only if there exists a basis \( S = \{e_1, e_2, e_3\} \) of \( B \) such that the matrix of \( \xi \) with respect to such basis, \( b_0 \) and a symplectic basis \( \mathcal{S} \) have one of the following forms:

1. \( \mathcal{S} = \{e_1, e_4, e_2, e_3\} \) and

\[
\xi = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
a & b & 0 & 0 \\
c & d & 0 & 0
\end{pmatrix}
\]

and \( b_0 = \alpha e_3 + \beta e_4 \), with \( a, b, c, d, \alpha, \beta \in \mathbb{R} \).

2. \( \mathcal{S} = \{e_1, e_4, e_2, e_3\} \) and

\[
\xi = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
a & b & 0 & c \\
0 & d & 0 & 0
\end{pmatrix}
\]

and \( b_0 = -9 cde_1 + x e_3 + 9 d(a + d) e_4 \), with \( a, b, d, x \in \mathbb{R} \) and \( c \in \mathbb{R}^* \).

**Proof.** Let \((B, \omega_B)\) be a non abelian flat symplectic Lie algebra of dimension 4. Then \( B \) is isomorphic to \( \mathbb{R} \times \mathfrak{h}_3 \): \( \{e_1, e_2\} = e_3 \) and \( \{e_1, e_4, e_2, e_3\} \) is a symplectic basis. According to (1.1), one has

\[
e_1 e_2 = 2 e_3, \quad e_2 e_1 = -\frac{1}{3} e_3, \quad e_2 e_2 = -\frac{1}{3} e_4.
\]

Recall that \((\xi, b_0)\) is admissible if and only if (3.1)–(3.5) hold. From \( \xi = (a_{ij})_{1 \leq i,j \leq 4} \) we deduce that \( a_{13} = a_{23} = 0, 3 a_{33} = a_{11} + 2 a_{22} \) and \( 3 a_{43} = a_{21} \). In the same way, by replacing \([a, b]\) with \([e_2, e_4]\) in (3.4) we get \( a_{14} = a_{24} = 0 \). Thus

\[
\xi = \begin{pmatrix}
a_{11} & a_{12} & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 \\
a_{31} & a_{32} & \frac{a_{31} + 2 a_{22}}{3} & a_{34} \\
a_{41} & a_{42} & \frac{a_{41}}{3} & a_{44}
\end{pmatrix}
\]

Since \( \{e_1, e_4, e_2, e_3\} \) is a symplectic basis then

\[
\xi^\ast = \begin{pmatrix}
a_{44} & a_{34} & 0 & 0 \\
\frac{a_{41}}{3} & \frac{a_{31} + 2 a_{22}}{3} & 0 & 0 \\
- a_{42} & - a_{32} & a_{22} & a_{12} \\
- a_{41} & - a_{31} & a_{21} & a_{11}
\end{pmatrix}
\]

In (3.5), if we take \( a = b = e_1 \), we get \( a_{21} = 0 \). For \( a = e_1 \) and \( b = e_2 \), we get

\[
3 a_{44} = 4 a_{11} + 2 a_{22}.
\]

(5.1)

For \( a = e_2 \) and \( b = e_1 \), we get

\[
3 a_{44} = a_{11} + 5 a_{22}.
\]

(5.2)

Finally, for \( a = e_2 \) and \( b = e_2 \), we get \( a_{12} = 0 \). From (5.1)–(5.2) we deduce that \( a_{22} = a_{11} \) and \( a_{44} = 2 a_{11} \). Let \( b_0 = \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4 \). We have

\[
R_{b_0} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{2 \beta}{3} & - \frac{\alpha}{3} & 0 & 0 \\
0 & - \frac{\alpha}{3} & 0 & 0
\end{pmatrix}
\]
According to (3.1), we deduce that $a_{11} = 0$, $\alpha = -9a_{34}a_{42}$ and $\beta = 9a_{34}a_{41}$. From (3.3), we deduce that $\beta = 0$.

- If $a_{34} = 0$, then $\alpha = 0$ and hence $(\xi, b_0)$ is admissible in this case and has the form
  
  $$
  \xi = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  a & b & 0 & 0 \\
  c & d & 0 & 0
  \end{pmatrix}
  $$
  
  and $b_0 = xe_3 + ye_4$, with $a, b, c, d, x, y \in \mathbb{R}$.

- If $a_{34} \neq 0$ then $a_{41} = 0$ and $\alpha = -9a_{34}a_{42}$. The equation (3.2) is equivalent to $\delta = 9a_{42}(a_{31} + a_{42})$. In this case, $(\xi, b_0)$ is admissible and has the form
  
  $$
  \xi = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  a & b & 0 & c \\
  0 & d & 0 & 0
  \end{pmatrix}
  $$
  
  and $b_0 = -9cde_1 + xe_3 + 9d(a + d)e_4$, with $a, b, d, x \in \mathbb{R}$ and $c \in \mathbb{R}^*$.

Using Propositions 5.3 and 5.4, we can determine all flat symplectic Lie algebras of dimension 6.

**Theorem 5.1.** A 6-dimensional Lie algebra $\mathfrak{g}$ admits a flat symplectic form if and only if $\mathfrak{g}$ is isomorphic to one and only one of the following Lie algebras:

- $\mathbb{R}^6$: The 6-dimensional abelian Lie algebra.
- $\mathbb{R}^3 \times \mathfrak{h}_3$: $[x_1, x_2] = x_6$.
- $\mathfrak{g}_1^6$: $[x_1, x_2] = x_4$, $[x_1, x_3] = x_5$, $[x_2, x_3] = x_6$.
- $\mathfrak{g}_2^6$: $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$.
- $\mathfrak{g}_3^6$: $[x_1, x_2] = x_4$, $[x_1, x_3] = x_5$, $[x_1, x_4] = x_6$, $[x_2, x_3] = x_6$.

**Proof.** Any 6-dimensional flat symplectic Lie algebra $(\mathfrak{g}, \omega)$ is a flat symplectic double extension of a 4-dimensional flat symplectic Lie algebra $(\mathfrak{b}, \omega_\mathfrak{b})$ by means of an admissible couple $(\xi, b_0)$.

**Case 1:** If $\mathfrak{b}$ is abelian, then $(\xi, b_0)$ has one of the forms given in Proposition 5.3.

- If $\xi = 0$ and $b_0 = \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4 \in \mathfrak{b}$, then according to (3.6), one has
  
  $$
  [\bar{e}, e_1] = -\delta e, [\bar{e}, e_2] = -\gamma e, [\bar{e}, e_3] = \beta e, [\bar{e}, e_4] = \alpha e.
  $$

  If $\alpha = \beta = \gamma = \delta = 0$, then $\mathfrak{g}$ is the abelian Lie algebra $\mathbb{R}^6$. If, for example, $\alpha \neq 0$, then we put $x_1 = \bar{e}, x_2 = e_4, x_6 = \alpha e, x_3 = \alpha e_3 - \beta e_4, x_4 = \alpha e_2 + \gamma e_4$ and $x_5 = \alpha e_1 + \delta e_4$. Thus, the only non vanishing Lie bracket in this basis is $[x_1, x_2] = x_6$. It follows that $\mathfrak{g}$ is isomorphic in this case to $\mathbb{R}^3 \times \mathfrak{h}_3$.

- If
  
  $$
  \xi = \begin{pmatrix}
  0 & 0 & a & b \\
  0 & 0 & c & d \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
  \end{pmatrix}
  $$
  
  and $b_0 = \alpha e_1 + \beta e_2$,

  where $\{e_1, e_4, e_2, e_3\}$ is symplectic and $ad - bc \neq 0$ and $(a + d)^2 \neq 4bc$. Then

  $$
  \xi^* = \begin{pmatrix}
  0 & 0 & -d & -b \\
  0 & 0 & -c & -a \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
  \end{pmatrix},
  $$

  and according to (3.6), one has

  $$
  [\bar{e}, e_3] = -(2a + d)e_1 - 3ce_2 + \beta e, [\bar{e}, e_4] = -3be_1 - (a + 2d)e_2 + \alpha e,
  $$

  $$
  [e_3, e_4] = (a - d)e.
  $$
If the vectors $e_1' = -(2a + d)e_1 - 3ce_2 + \beta e$, $e_2' = -3be_1 - (a + 2d)e_2 + \alpha e$ and $e' = (a - d)e$ are linearly independent then $[\bar{e}, e_2] = e_1'$, $[\bar{e}, e_4] = e_2'$ and $[\bar{e}, e_3] = e'$ and hence $g$ is isomorphic in this case to $\mathfrak{g}_6^1$. If $\text{rank}(e_1', e_2', e') = 2$, for example, $(e_1', e_2')$ are linearly independent and $e' = \lambda_1 e_1' + \lambda_2 e_2'$, then $[\bar{e}, e_3] = e_1'$, $[\bar{e}, e_4] = e_2'$ and $[\bar{e}, e_1] = \lambda_1 e_1' + \lambda_2 e_2'$. We put $e_3' = e_3 - \lambda_2 \bar{e}$ and $e_4' = e_4 + \lambda_1 \bar{e}$. In this basis, the only non vanishing Lie brackets are $[\bar{e}, e_3'] = e_1'$, $[\bar{e}, e_4'] = e_2'$, and hence $g$ is isomorphic in this case to $\mathfrak{g}_6^2$. If $\text{rank}(e_1', e_2', e') = 1$, then it is easy to check that $g$ is isomorphic to $\mathbb{R}^3 \times \mathfrak{h}_3$.

If
\[
\xi = \begin{pmatrix}
0 & 0 & a & 0 \\
0 & 0 & 0 & -a \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
and $b_0 \in B$, where $a \neq 0$ and $\{e_1, e_4, e_2, e_3\}$ is symplectic.

In this case the Lie brackets are given by
\[
[\bar{e}, e_1] = -d e, \quad [\bar{e}, e_2] = -\gamma e, \quad [\bar{e}, e_3] = -ae_1 + \beta e, \\
[\bar{e}, e_4] = ae_2 + \alpha e, \quad [e_3, e_4] = 2ae.
\]

If $\delta = \gamma = 0$ then $g$ is isomorphic to $\mathfrak{g}_6^1$. If $(\delta, \gamma) \neq (0, 0)$ (for example $\delta \neq 0$), then by taking $e_2' = \delta e_2 - \gamma e_1$, the Lie brackets are reduced to
\[
[\bar{e}, e_1] = -d e, \quad [\bar{e}, e_3] = -ae_1 + \beta e, \\
[\bar{e}, e_4] = \frac{a}{\delta} e_2 + \frac{\gamma}{\delta} e_1 + \alpha e, \quad [e_3, e_4] = 2ae.
\]

We put $e' = 2ae$, $e_1' = -ae_1 + \beta e$ and $e'' = \frac{a}{\delta} e_2 + \frac{\alpha}{\delta} e_1 + \alpha e$. Then the Lie brackets in this basis are
\[
[\bar{e}, e_1'] = \frac{\delta}{2} e', \quad [\bar{e}, e_3] = e_1', \quad [\bar{e}, e_4] = e_2', \quad [e_3, e_4] = e'.
\]

Finally, by taking $x_1 = \bar{e}$, $x_4 = \frac{2}{\delta} e_1'$, $x_6 = e'$, $x_2 = \frac{2}{\delta} e_3$, $x_3 = \frac{\delta}{2} e_4$ and $x_5 = \frac{\delta}{2} e_2''$, one can deduce that $g$ is isomorphic to $\mathfrak{g}_6^3$. The $4^{th}$, $5^{th}$ and $6^{th}$ cases are similar to this case. Note that in the $6^{th}$ case, $[e_3, e_4] = (a - d)e \neq 0$ since $(a + d)^2 = 4bc$ and $ad \neq bc$.

If
\[
\xi = \begin{pmatrix}
0 & 0 & 0 & a \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
and $b_0 = \alpha e_1 + \beta e_2 + \gamma e_3$,

where $\{e_1, e_4, e_2, e_3\}$ is symplectic and $a \neq 0$. Then $\xi^* = -\xi$ and $g$ is isomorphic in this case to $\mathbb{R}^3 \times \mathfrak{h}_3$ if $\beta = \gamma = 0$, or to $\mathfrak{g}_6^2$ if $(\beta, \gamma) \neq (0, 0)$.

If
\[
\xi = \begin{pmatrix}
0 & 0 & 0 & a \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
and $b_0 = \alpha e_1 + \beta e_3$,

where $\{e_1, e_2, e_3, e_4\}$ is symplectic and $a \neq 0$. Then
\[
\xi^* = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -a & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
and the Lie brackets are given by
\[
[\bar{e}, e_2] = -ae_3 - \beta e, \quad [\bar{e}, e_4] = -2ae_1 + \alpha e, \quad [e_2, e_4] = -ae.
\]

Since $a \neq 0$, then by putting $e_3' = -ae_3 - \beta e$, $e_4' = -2ae_1 + \alpha e$ and $e' = -ae$ one can check that $g$ is isomorphic in this case to $\mathfrak{g}_6^1$. 

COMMUNICATIONS IN ALGEBRA®

4397
Case 2: If $B$ is not abelian, then $(\xi, b_0)$ has one of the two forms given in Proposition 5.4.

- If
  \[
  \xi = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  a & b & 0 & 0 \\
  c & d & 0 & 0
  \end{pmatrix}
  \quad \text{and} \quad
  b_0 = \alpha e_3 + \beta e_4,
  \]
  where $\{e_1, e_4, e_2, e_3\}$ is symplectic. Then
  \[
  \xi^* = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  -d & -b & 0 & 0 \\
  -c & -a & 0 & 0
  \end{pmatrix}
  \]
  and the Lie brackets are given by
  \[
  [\tilde{e}, e_1] = -(2a + d)e_3 - 3ce_4 - \beta e, \quad [\tilde{e}, e_2] = -3be_3 - (a + 2d)e_4 - \alpha e, \quad [e_1, e_2] = e_3 + (d - a)e.
  \]
  As we showed in the abelian case, one can check that $g$ is isomorphic to $\mathbb{R}^3 \times \mathfrak{h}_3$, $\mathfrak{g}_6^1$ or $\mathfrak{g}_6^2$.

- If
  \[
  \xi = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  a & b & 0 & c \\
  0 & d & 0 & 0
  \end{pmatrix}
  \quad \text{and} \quad
  b_0 = -9cde_1 + xe_3 + 9d(a + d)e_4,
  \]
  where $\{e_1, e_4, e_2, e_3\}$ is symplectic and $c \neq 0$. Then
  \[
  \xi^* = \begin{pmatrix}
  0 & c & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  -d & -b & 0 & 0 \\
  0 & -a & 0 & 0
  \end{pmatrix}
  \]
  and the Lie brackets are given by
  \[
  [\tilde{e}, e_1] = -(2a + d)e_3 - 9d(a + d)e, \quad [\tilde{e}, e_2] = ce_1 - 3be_3 - (a + 2d)e_4 - xe, \\
  [\tilde{e}, e_4] = -2ce_3 - 9cde, \quad [e_1, e_2] = e_3 + (d - a)e, \quad [e_2, e_4] = ce.
  \]
  By taking $\tilde{f} = \tilde{\xi}$, $f_1 = e_1 - \frac{3b}{c}e_3 - \frac{a + 2d}{c}e_4 - \frac{c}{e}f$, $f_2 = e_2, f_3 = e_3, f_4 = -\frac{c}{e}f$ and $f = -\frac{c}{e}e$. We reduce the Lie brackets to
  \[
  [\tilde{f}, f_1] = \frac{3d}{c}f_3 - \frac{18d^2}{c^2}f, \quad [\tilde{f}, f_2] = f_1, \\
  [\tilde{f}, f_4] = f_3 - \frac{9d}{c}f, \quad [f_1, f_2] = f_3 - \frac{6d}{c}f, \quad [f_2, f_4] = f.
  \]
  We replace $f_3$ by $f_3' = f_3 - \frac{6d}{c}f$ and $\tilde{f}$ by $\tilde{f}' = \tilde{f} + \frac{3d}{c}f_2$. Thus, the Lie brackets will be
  \[
  [\tilde{f}', f_2] = f_1, \quad [\tilde{f}', f_4] = f_3', \quad [f_1, f_2] = f_3', \quad [f_2, f_4] = f.
  \]
  By putting $x_1 = f_2, x_4 = -f_1, x_6 = f_3', x_5 = f, x_3 = f_4, x_2 = \tilde{f}'$, one can see that $g$ is isomorphic to $\mathfrak{g}_6^3$. This completes the proof of the theorem.

\[\square\]

Remark 5.1. • In Theorem 5.1, the Lie algebras $\mathbb{R}^3 \times \mathfrak{h}_3$, $\mathfrak{g}_6^1$ or $\mathfrak{g}_6^2$ are 2-step nilpotent and $\mathfrak{g}_6^3$ is 3-step nilpotent.

- We can also determine the flat symplectic form on the Lie algebras given in Theorem 5.1 by making the same changes we made for the basis $\{e, e_1, e_2, e_3, e_4, \tilde{e}\}$ to the initial flat symplectic form $e^* \wedge \tilde{e}^* + \omega_B$.  

---

4398 M. BOUSETA ETAL.
• We have preferred to describe the Lie algebras in Theorem 5.1 by giving the same Lie brackets in the classification of symplectic nilpotent Lie algebras given in page 51 of [8]. Note that the Lie algebras $\mathbb{R}^3 \times h_3$, $\mathfrak{g}_6^1$, $\mathfrak{g}_6^2$, and $\mathfrak{g}_6^3$ are respectively denoted in [8] by classes 25, 18, 23, and 13.

We give here examples of flat symplectic forms on the Lie algebras mentioned in Theorem 5.1 and the corresponding product $\bullet$. We use the same notations in [8].

| Lie algebras | Non-vanishing Lie brackets | Flat symplectic forms | Non-vanishing products $\bullet$ |
|--------------|---------------------------|----------------------|---------------------------------|
| $\mathbb{R}^6$ | $x_1^3 \wedge x_6^3 + x_2^3 \wedge x_5^3 + x_3^3 \wedge x_4^3$ | $x_1 \bullet x_1 = \frac{1}{3} x_3, x_1 \bullet x_2 = \frac{1}{3} x_6,$ | $x_1 \bullet x_1 = \frac{1}{3} x_3, x_1 \bullet x_2 = \frac{1}{3} x_6,$ |
| $\mathbb{R}^3 \times h_3$ | $[x_1, x_2] = x_6$ | $x_1^3 \wedge x_6^3 + x_2^3 \wedge x_5^3 + x_3^3 \wedge x_4^3$ | $x_1 \bullet x_1 = \frac{1}{3} x_3, x_1 \bullet x_2 = \frac{1}{3} x_6,$ |
| $\mathfrak{g}_6^1$ | $[x_1, x_2] = x_4, [x_1, x_3] = x_5,$ | $x_1^3 \wedge x_6^3 + \lambda x_2^3 \wedge x_5^3 + (\lambda - 1)$ | $x_1 \bullet x_1 = \frac{1}{3} x_3, x_1 \bullet x_2 = \frac{2}{3} x_6,$ |
| $[x_2, x_3] = x_6$ | $x_3^3 \wedge x_4^3, \lambda \in \mathbb{R} - \{0, 1\}.$ | $x_2 \bullet x_1 = \frac{2}{3} x_6, x_2 \bullet x_2 = \frac{2}{3} x_6,$ | $x_1 \bullet x_1 = \frac{1}{3} x_3, x_1 \bullet x_2 = \frac{2}{3} x_6,$ |
| $\mathfrak{g}_6^2$ | $[x_1, x_2] = x_5, [x_1, x_3] = x_6$ | $x_1^3 \wedge x_6^3 + x_2^3 \wedge x_5^3 + x_3^3 \wedge x_4^3$ | $x_1 \bullet x_1 = \frac{1}{3} x_3, x_1 \bullet x_2 = \frac{1}{3} x_6,$ |
| $\mathfrak{g}_6^3$ | $[x_1, x_2] = x_4, [x_1, x_3] = x_5,$ | $x_1^3 \wedge x_6^3 + \frac{1}{2} x_2^3 \wedge x_5^3$ | $x_2 \bullet x_1 = -x_4, x_2 \bullet x_3 = \frac{1}{3} x_6,$ |
| $[x_1, x_4] = x_6, [x_2, x_3] = x_6$ | $-\frac{1}{2} x_2^3 \wedge x_4^3$ | $x_3 \bullet x_1 = -x_5, x_3 \bullet x_2 = \frac{1}{3} x_6,$ | $x_4 \bullet x_1 = -\frac{2}{3} x_6.$ |

References

[1] Aubert, A., Medina, A. (2003). Groupes de Lie pseudo-riemanniens plats. Tohoku Math. J. 55:487–506.
[2] Baues, O., Cortés, V. (2016). Symplectic Lie Groups, Société Mathématique de France 2016, ASTERISQUE 379.
[3] Bieliavsky, P., Cahen, M., Gutt, S., Rawnsley, J., Schwachhöfer, L. (2006). Symplectic connections. Int. J. Geom. Methods Mod. Phys. 3(3):375–420.
[4] Benayadi, S., Boucetta, M. (2014). Special bi-invariant linear connections on Lie groups and finite dimensional Poisson structures. Differ. Geom. Appl. 36:66–89.
[5] Bordeman, M., Medina, A., Ouadfel, A. (1993). Le groupe affine comme variété symplectique. Tohoku Math. J. 45(3):423–436.
[6] Dardié, J. M., Medina, A. (1996). Double extension symplectique d’un groupe de Lie symplectique. Adv. Math. 117:208–227.
[7] Helmstetter, J. (1979). Radical d’une algèbre symétrique à gauche. Annales de l’institut Fourier 29(4):17–35.
[8] Khakimdjanov, Yu., Goze, M., Medina, A. (2004). Symplectic or contact structures on Lie groups. Differ. Geom. Appl. 21:41–54.
[9] Medina, A., Revoy, P. (1985). Algèbres de Lie et produit scalaire invariant. Annales scientifiques de l’École Normale Supérieure, Série 4 18(3):553–561.
[10] Nijenhuis, A. (1970). Sur une classe de propriétés communes à quelques types différents d’algèbres. L’enseignement Mathématique 14(34):225–275.
[11] Segal, D. (1992). The structure of complete left symmetric algebras. Math. Ann. 293(3):569–578.
[12] Valencia, F. (2021). Flat affine symplectic Lie groups. J. Lie Theory 31(1):063–092.