Scalar modes and the linearized Schwarzschild solution on a quantized FLRW space-time in Yang–Mills matrix models

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Abstract
We study scalar perturbations of a recently found 3+1-dimensional FLRW quantum space-time solution in Yang–Mills matrix models. In particular, the linearized Schwarzschild metric is obtained as a solution. It arises from a quasi-static would-be massive graviton mode, and slowly decreases during the cosmic expansion. Along with the propagating graviton modes, this strongly suggests that 3+1 dimensional (quantum) gravity emerges from the IKKT matrix model on this background. For the dynamical scalar modes, non-linear effects must be taken into account. We argue that they lead to non-Ricci-flat metric perturbations with very long wavelengths, which would be perceived as dark matter from the GR point of view.

Keywords: emergent gravity, matrix models, quantum space-time, quantum gravity models

(Some figures may appear in colour only in the online journal)
1. Introduction

The starting point of this paper is a recent solution of the IKKT-type matrix models with mass term [1], which is naturally interpreted as 3+1-dimensional cosmological FLRW quantum space-time. It was shown that the fluctuation modes around this background include spin-2 metric fluctuations, as well as a truncated tower of higher-spin modes which are organized in a higher-spin gauge theory. The two standard Ricci-flat massless graviton modes were found, as well as some additional vector-like and scalar modes whose significance was not fully clarified.

The aim of the present paper is to study in more detail the metric perturbations, and in particular to see if and how the (linearized) Schwarzschild solution can be obtained. We will indeed find such a solution, which is realized in the scalar sector of the linearized perturbation modes exhibited in [1]. This means that the model has a good chance to satisfy the precision solar system tests of gravity. We will also elaborate and discuss in some detail the extra scalar mode, which is not present in GR. This seems to provide a natural candidate for apparent dark matter.

Since the notorious problems in attempts to quantize gravity arise primarily from the Einstein–Hilbert action, it is very desirable to find another framework for gravity, which
is more suitable for quantization. String theory provides such a framework, but the traditional approach using compactifications leads to a host of issues, notably lack of predictivity. This suggests to use matrix models as a starting point, and in particular the IKKT or IIB model [2], which was originally proposed as a constructive definition of string theory. Remarkably, numerical studies in this non-perturbative formulation provide evidence [3–5] that 3+1-dimensional configurations arise at the non-perturbative level, tentatively interpreted as expanding universe. However, this requires a new mechanism for gravity on 3+1-dimensional non-commutative backgrounds as in [1], which does not rely on compactification. The present paper provides further evidence and insights for this mechanism.

The (linearized) Schwarzschild metric is clearly the benchmark for any viable theory of gravity. There has been considerable effort to find noncommutative analogs of the Schwarzschild metric from various approaches, leading to a number of proposals [6–9] and references therein, see also [10]; however, none is truly satisfactory. The proposals are typically obtained by some ad-hoc modification of the classical solution, without any intrinsic role of noncommutativity, which is put in by hand. In contrast, the quantum structure (or its semiclassical limit) plays a central role in the present framework. Our solution is a deformation of the noncommutative background which respects an exact $SO(3)$ rotation symmetry, even though there are only finitely many d.o.f. per unit volume. The solution has a good asymptotics at large distances, allowing superpositions corresponding to arbitrary mass distributions. In fact we obtain generic quasi-static Ricci-flat linearized perturbations, which complement the Ricci-flat propagating gravitons found in [1].

This realization of the (linearized) Schwarzschild solution is remarkable and may seem surprising, because the action is of Yang–Mills type, and no Einstein–Hilbert-like action is required\(^1\). This means that the theory has a good chance to survive upon quantization, which is naturally defined via integration over the space of matrices. The IKKT model is indeed well suited for quantization, and quite clearly free of ghosts and other obvious pathologies. It is background-independent in the sense that it has a large class of solutions with different geometries, and defines a gauge theory for fluctuations on any background.

The price to pay is a considerable complexity of the resulting theory. As explained in [1, 15] the background leads to a higher-spin gauge theory, with a truncated tower of higher spin modes, and many similarities with (but also distinctions from) Vasiliev theory [16]. Since space-time itself is part of the background solution, it is not unreasonable to expect Ricci-flat deformations, see [17–19]. However, Lorentz invariance is very tricky on noncommutative backgrounds. In the present case the space-like isometries $SO(3,1)$ of the $k = −1$ FLRW space-time are manifest, but invariance under (local) boosts is not. Nevertheless, the propagation of all physical modes is governed by the same effective metric. In particular, the concept of spin has to be used with caution, and would-be spin $s$ modes decompose further into sectors governed by the space-like $SO(3,1)$ isometry. The tensor fields are accordingly characterized by the transformation under the local $SO(3)$ stabilizer group, and the term ‘scalar modes’ is understood in this sense throughout the paper. However this complication is in fact helpful to identify physical degrees of freedom in the physical sector, and to understand the absence of ghosts.

\(^1\) It is well-known that gravity can be obtained from a Yang–Mills-type action by imposing constraints, see [11–13]. However this essentially amounts to a reformulation of classical GR, and the usual problems are expected to arise upon quantization. In contrast, we do not impose any constraints on the Yang–Mills action. Nevertheless, quantum effects are expected to induce an Einstein–Hilbert-like term, as discussed in [1]. This may play an important role here as well, but we focus on the classical mechanism. Another interesting possibility was proposed in [14], which has some similarities to the present mechanism but leads to many additional fields, possibly including ghosts.
Let us describe the new results in some details. We focus on the scalar fluctuation mode which was found in [1], and elaborate the associated metric fluctuations. The main result is that there is a preferred ‘quasi-static’ vacuum solution which leads to the linearized Schwarzschild metric on the FLRW background. This strongly suggests that a near-realistic gravity emerges on the background, however only the vacuum solution is considered here. Quasi-static means that the solution is static on local scales at late times, but slowly decays on cosmic scales, in a specific way. This is a somewhat unexpected result, whose significance is not entirely clear. The quasi-static solution is singled out because all other solutions lead to a large diffeo term, which makes the linearized treatment problematic. Hence the Schwarzschild solution is the ‘cleanest’ case, while the generic dynamical scalar modes require non-linear considerations somewhat reminiscent of the Vainshtein mechanism [20]. We offer a heuristic way to understand them, which points to the intriguing—albeit quite speculative—possibility that these non-Ricci-flat scalar modes might provide a geometrical explanation for dark matter at galactic scales.

It may seem strange to start with a curved cosmological background rather than flat Minkowski space, since the Schwarzschild solution is basically a local structure. The reason is that no flat counterpart of the underlying quantum-spacetime with the required structure is known. We will thus largely neglect the contributions of the Schwarzschild solution at the cosmic scales.

Along the way, we also find the missing 4th off-shell scalar fluctuation mode, which was missing in [1]. Thus all ten off-shell metric fluctuation modes for the most general metric fluctuations are realized, and the model is certainly rich enough for a realistic theory of gravity. That theory would clearly deviate from GR at cosmic scale, since the FLRW background solution is not Ricci flat, but requires no stabilization by matter (or energy) and no fine-tuning.

Finally, it should be stressed that even though the model is intrinsically noncommutative, it should be viewed in the spirit of almost-local and almost-classical field theory. Space-time arises as a condensation of matrices rather than some non-local holographic image, with dynamical local fluctuations described by an effective field theory.

The paper is meant to be as self-contained and compact as possible. We start with a lightning introduction to the $M_{3,1}^n$ space-time under consideration, and elaborate only the specific modes and aspects needed to obtain the Schwarzschild solution. For some results we have to refer to [1], but the essential new computations are mostly spelled out. For the skeptical reader, some of the missing steps may be uncovered from the file in the arXive.

2. Quantum FLRW space-time $M_{3,1}^n$

The quantum space-time under consideration is based on a particular representation $\mathcal{H}_n$ of $SO(4,2)$, which is a lowest weight unitary irrep in the short discrete series known as minireps or doubletons [21, 22]. Those are the unique irreps which remain irreducible under the restriction to $SO(4,1) \subset SO(4,2)$. We denote the generators in this representation by $\mathcal{M}^{ab}$, which are Hermitian operators satisfying
\[
[\mathcal{M}_{ab}, \mathcal{M}_{cd}] = i (\eta_{ac} \mathcal{M}_{bd} - \eta_{ad} \mathcal{M}_{bc} - \eta_{bc} \mathcal{M}_{ad} + \eta_{bd} \mathcal{M}_{ac})
\]
where $\eta^{ab} = \text{diag}(-1, 1, 1, 1, -1)$ is the invariant metric of $SO(4,2)$. We then define
\[
X^\mu := r \mathcal{M}^\mu, \quad X^4 := r \mathcal{M}^{45} \quad T^\mu := R^{-1} \mathcal{M}^{\mu 4} \quad \mu, \nu = 0, \ldots, 3. \tag{2.2}
\]
Then the $X^\mu$ transform as vector operators under $SO(4, 1)$, while the $T^\mu$ are vector operators under $SO(3, 1) \subset SO(4, 1)$. The $SO(3, 1)$-invariant fuzzy or quantum space-time $M^{3,1}$ is then defined through the algebra of functions $\phi(X^\mu)$ generated by the $X^\mu$ for $\mu = 0, 1, 2, 3$. Here $r$ is a microscopic length scale related to the internal quantum structure, while $R$ is a macroscopic scale as specified in (2.4a). The commutation relations (2.1) imply

$$[X^\mu, X^\nu] = -i r^2 M^{\mu\nu} = :i \Theta^{\mu\nu}, \quad (2.3a)$$

$$[T^\mu, X^\nu] = i \frac{1}{R} \eta^{\mu\nu} X^4, \quad (2.3b)$$

$$[T^\mu, T^\nu] = -\frac{i}{r^2 R^2} \Theta^{\mu\nu}, \quad (2.3c)$$

$$[T^\mu, X^4] = -\frac{1}{R} X^\mu, \quad (2.3d)$$

$$[X^\mu, X^4] = -i r^2 R T^\mu, \quad (2.3e)$$

and the irreducibility of $\mathcal{H}_n$ under $SO(4, 1)$ implies the relations [1]

$$X_\mu X^\mu = -R^2 - X^4 X^4, \quad R^2 = \frac{r^2}{4} (n^2 - 4) \quad (2.4a)$$

$$T_\mu T^\mu = \frac{1}{p^2} + \frac{1}{r^2 R^2} X^4 X^4, \quad (2.4b)$$

$$X_\mu T^\mu + T^\mu X_\mu = 0. \quad (2.4c)$$

There are some extra constraints involving $\Theta^{\mu\nu}$, which will only be given in the semi-classical version below. Unless otherwise stated, indices will be raised and lowered with $\eta^{ab}$ or $\eta^{\mu\nu}$. Apart from the extra constraints, the construction is quite close to that of Snyder [23] and Yang [24].

The proper interpretation of this structure is not obvious a priori, due to the extra generators $T^\mu$ and $\Theta^{\mu\nu}$. These cannot be dropped, because the full algebra $\text{End}(\mathcal{H}_n)$ is generated by the $X^\mu$ alone. A proper geometrical understanding is obtained by considering all the generators $\mathcal{M}_{ab}$ of $so(4, 2)$. As explained in [25–27], these are naturally viewed as quantized embedding
functions of a coadjoint orbit \( m^h : \mathbb{C}P^{1,2} \hookrightarrow \mathfrak{so}(4, 2) \cong \mathbb{R}^{15} \). Here \( \mathbb{C}P^{1,2} \) is a 6-dimensional noncompact analog of \( \mathbb{C}P^3 \), which is singled out by the constraints satisfied by \( m^h \). Hence the full algebra \( \text{End}(\mathcal{H}_n) \) can be interpreted as a quantized algebra of functions on \( \mathbb{C}P^{1,2} \), dubbed fuzzy \( \mathbb{C}P^{1,2} \). Furthermore, \( \mathbb{C}P^{1,2} \) is naturally a \( S^2 \) bundle over \( H^4 \), which is defined by the \( X^a \) satisfying (2.4a). Hence the space-time \( \mathcal{M}^{3,1} \) generated by the \( X^a \sim x^a \), \( a = 0, \ldots, 3 \) can be viewed as projection of \( H^4 \subset \mathbb{R}^{4,1} \) to \( \mathbb{R}^{3,1} \) along \( X^4 \), as sketched in figure 1. This is the space-time of interest here, which is covariant under \( SO(3, 1) \). For similar covariant quantum spaces see e.g. [26, 28–33].

2.1. Semi-classical structure of \( \mathcal{M}^{3,1} \)

We will mostly restrict ourselves to the semi-classical limit \( n \to \infty \) of the above space, working with commutative functions of \( x^\mu \sim X^\mu \) and \( t^\mu \sim T^\mu \), but keeping the Poisson or symplectic structure \([\cdot, \cdot] \sim \{\cdot, \cdot\}\) encoded in \( \theta^{\mu\nu} \). The constraints (2.4) etc imply the following relations

\[
x_\mu x^\mu = -R^2 - \frac{x_4^2}{4} = -R^2 \cosh^2(\eta), \quad R \sim \frac{r}{2n} \tag{2.5a}
\]

\[
t_\mu t^\mu = r^{-2} \cosh^2(\eta), \tag{2.5b}
\]

\[
t_\mu x^\mu = 0, \tag{2.5c}
\]

\[
t_\mu \theta^{\mu\alpha} = -\sinh(\eta)x^\alpha, \tag{2.5d}
\]

\[
x_\mu \theta^{\mu\alpha} = -r^2 R^2 \sinh(\eta)t^\alpha, \tag{2.5e}
\]

\[
\eta^\mu_{\cdot\cdot} \theta^{\mu\alpha} \theta^{\alpha\beta} = R^2 r^2 \eta^{\mu\beta} = R^2 r^4 t^\alpha t^\beta - r^2 x^\alpha x^\beta \tag{2.5f}
\]

where \( \mu, \alpha = 0, \ldots, 3 \). Here \( \eta \) is a global time coordinate defined by

\[
x^4 = R \sinh(\eta), \tag{2.6}
\]

which will be related to the scale parameter of the universe (2.23). Clearly the \( x^\mu : \mathcal{M}^{3,1} \hookrightarrow \mathbb{R}^{3,1} \) can be viewed as Cartesian coordinate functions. Similarly, the \( t^\mu \) describe the \( S^2 \) fiber over \( \mathcal{M}^{3,1} \) as discussed above. On the other hand, the relation (2.3b) implies that the derivations

\[-i[T^\mu, \cdot] \sim \{t^\mu, \cdot\} = \sinh(\eta)\partial_\mu \tag{2.7}\]

act as momentum generators on \( \mathcal{M}^{3,1} \), leading to the useful relation

\[
\partial_\mu \phi = \beta \{t_\mu, \phi\}, \quad \beta = \frac{1}{\sinh(\eta)} \tag{2.8}
\]

for \( \phi = \phi(x) \). In particular, a \( SO(3, 1) \)-invariant matrix d’Alembertian can be defined as

\[
\Box := [T^\mu, [T_\mu, \cdot]] \sim -\{t^\mu, \{t_\mu, \cdot\}\}. \tag{2.9}
\]
It acts on any $\phi \in \text{End}(\mathcal{H})$, and will play a central role throughout this paper. We also define a globally defined time-like vector field

$$\tau := x^\mu \partial_\mu.$$  \hspace{1cm} (2.10)

To get some insight into the $\theta^{\mu \nu}$, fix some reference point $\xi$ on $\mathcal{M}^{3,1}$, which using $SO(3,1)$ invariance can be chosen as

$$\xi = (x^0, 0, 0, 0), \quad x^0 = R \cosh(\eta).$$ \hspace{1cm} (2.11)

Then (2.5e) provides a relation between the $t^\mu$ and the $\theta^{\mu \nu}$ generators,

$$t^\mu = -\frac{1}{R^2} \frac{1}{\cosh(\eta)} \theta^{\eta \mu}, \quad \rho^0 \xi = 0.$$  \hspace{1cm} (2.12)

Conversely, the self-duality relation on $H_4$ \cite{34}

$$\epsilon_{abcd} \theta^{ab} \theta^{cd} = 2R \delta_{de}$$  \hspace{1cm} (2.13)

relates the space-like and the time-like components of $\theta^{\mu \nu}$ on $\mathcal{M}^{3,1}$, and an explicit expression of $\theta^{\mu \nu}$ in terms of $t^\mu$ can be derived \cite{1}

$$\theta^{\mu \nu} = c(x^\mu t^\nu - x^\nu t^\mu) + b \epsilon^{\mu \nu \alpha \beta} x_\alpha t_\beta$$  \hspace{1cm} (2.14)

with

$$c = \frac{r^2 \sinh(\eta)}{\cosh(\eta)} \quad \text{and} \quad b = \frac{r^2}{\cosh^2(\eta)}.$$  \hspace{1cm} (2.15)

### 2.1. Hyperbolic coordinates.

Now consider the adapted hyperbolic coordinates

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = R \cosh(\eta) \begin{pmatrix} \cosh(\chi) \\ \sinh(\chi) \sin(\theta) \cos(\varphi) \\ \sinh(\chi) \sin(\theta) \sin(\varphi) \\ \sinh(\chi) \cos(\theta) \end{pmatrix}.$$ \hspace{1cm} (2.16)

We will see that $\eta$ measures the cosmic time, see (2.6), while the space-like distance from the origin on each time slice $H^3$ is measured by $\chi$. Noting that

$$\frac{\sqrt{x^\mu x_\mu}}{R^2 \cosh^2(\eta)} \, dx^\mu dx_\nu = R^2 \sinh^2(\eta) d\eta^2$$ \hspace{1cm} (2.17)

which follows from (2.5a), we obtain the induced (flat) metric of $\mathbb{R}^{3,1}$ in these coordinates

$$ds^2 = \eta_{\mu \nu} dx^\mu dx^\nu = R^2 \left( - \sinh^2(\eta) d\eta^2 + \cosh^2(\eta) d\Sigma^2 \right)$$ \hspace{1cm} (2.18)

where $d\Sigma^2$ is the metric on the unit hyperboloid $H^3$,

$$d\Sigma^2 = d\chi^2 + \sinh^2(\chi) d\Omega^2,$$ \hspace{1cm} (2.19)

However, the effective metric is a different one, which is also $SO(3,1)$ invariant but not flat.

### 2.2. Effective metric and d’Alembertian

In the matrix model framework considered below, the effective metric on the background $\mathcal{M}^{3,1}$ under consideration is given by \cite{1}
\[ G^{\mu\nu} = \alpha \gamma^{\mu\nu} = \sinh^{-1}(\eta) \eta^{\mu\nu} \quad \alpha = \sqrt{\frac{1}{\beta^2 |\gamma^{\mu\nu}|}} = \sinh^{-3}(\eta) \]
\[ \gamma^{\alpha\beta} = \eta^{\alpha\beta} \theta^{\mu\alpha} \theta^{\nu\beta} = \sinh^2(\eta) \eta^{\alpha\beta}. \] (2.20)

This is an \( SO(3, 1) \)-invariant FLRW metric with signature \((-+++ \ldots)\). Here \( \beta^2 \) is an irrelevant constant which adjusts the dimensions. There are several ways to obtain this metric. One is by rewriting the kinetic term in covariant form \([1, 19]\)
\[ S[\phi] = \text{Tr}[T^\mu, \phi][T_\mu, \phi] \sim \int d^4x \sqrt{|G|} G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \] (2.21)
and another way is given below by showing (2.26). Using (2.18), this metric can be written as
\[ d\mathbf{s}^2_{\text{G}} = G^{\mu\nu} d\mathbf{x}_\mu d\mathbf{x}_\nu = -R^2 \sinh^3(\eta) d\eta^2 + R^2 \sinh(\eta) \cosh^2(\eta) d\Sigma^2 \] (2.22)
and we can read off the cosmic scale parameter \( a(t) \)
\[ a(t)^2 = R^2 \sinh(\eta) \cosh^2(\eta) \sim R^2 \sinh^3(\eta), \] (2.23)
\[ dr = R \sinh(\eta)^2 d\eta. \] (2.24)
Hence \( a(t) \sim \frac{3}{2}t \) for late times, and the Hubble rate is decreasing as \( \dot{a} / a \sim a^{-5/3} \). This is related to the time-like vector field \( \tau \) (2.10) via
\[ \frac{\partial}{\partial \eta} = \tanh(\eta) \tau, \quad \frac{\partial}{\partial t} = \frac{1}{R} \sqrt{\sinh(\eta) \cosh(\eta)} \tau \sim \frac{1}{R} \beta \tau. \] (2.25)
As a consistency check, it is shown in appendix A.6 that the covariant d’Alembertian \( \Box_G \) of a scalar field is indeed given by \( \Box \) up to a factor [19],
\[ -\Box \phi = \eta^{\alpha\beta} \{ t_\alpha, \{ t_\beta, \phi \} \} = \eta^{\alpha\beta} \beta^{-1} (\partial_\alpha \beta^{-1} \partial_\beta \phi) \]
\[ = \beta^{-2} (\eta^{\alpha\beta} \partial_\alpha \partial_\beta - \frac{1}{x^2} \chi^{\beta} \partial_\beta \phi) \]
\[ = \beta^{-3} \nabla^\alpha \partial_\alpha \phi = \beta^{-3} \Box_G \] (2.26)
where \( \nabla \) is the covariant derivative w.r.t. \( G_{\mu\nu} \). In particular, we note the useful formula
\[ \partial^\alpha \partial_\alpha \phi = \beta^2 \left(-\Box + \frac{1}{R^2 \tau} \phi \right). \] (2.27)
We would like to decompose \( \Box \) into time derivatives \( \tau \) and the space-like Laplacian \( \Delta^{(3)} \) on \( H^3 \)
\[ -\Delta^{(3)} \phi = \nabla^{(3)}_\mu \nabla^{(3)}(3) \phi = \partial_\mu (P^{(3)\mu}_\tau \partial_\nu \phi) \] (2.28)
using the time-like and space-like projectors
\[ P^{(3)\mu}_\tau := \frac{1}{x^0 x^0} x^\mu x^\nu, \quad P^{(3)\mu}_\perp := \eta^{\mu\nu} - P^{(3)\mu\nu}. \] (2.29)
After some calculations using (2.8) and the formulas in appendix A.2, one obtains
\[
\square \phi = \left( \beta^{-2} \Delta \phi + \frac{1}{R^2} \sinh^2(\eta) \left( \frac{\sinh(2\eta)}{R^2 \cosh^2(\eta)} (2 + \tau) \phi \right) \right)
\]
(2.30)
for scalar fields \( \phi(x) \). This can be checked e.g. for \( \phi = x^\alpha \). On the other hand we can use the above hyperbolic coordinates (2.16), where
\[
G_{\mu\nu} = R^2 \sinh(\eta) \text{diag} \left( - \sinh^2(\eta), \cosh^2(\eta), \cosh^2(\eta) \sinh^2(\chi), \cosh^2(\eta) \sinh^2(\theta) \right)
\]
so that
\[
\square_G = - \frac{1}{\sqrt{|G_{\mu\nu}|}} \partial_\mu \left( \sqrt{|G_{\mu\nu}|} G^{\mu\nu} \partial_\nu \right)
\]
\[
= \frac{1}{R^2 \sinh^3(\eta) \cosh^3(\eta)} \partial_\eta \left( \cosh^3(\eta) \partial_\eta \phi \right) + \frac{1}{\sinh(\eta)} \Delta^{(3)} \phi .
\]
(2.32)
This reduces indeed to (2.30) using \( \square = \beta^{-3} \square_G \) and (2.25). The Laplacian \( \Delta^{(3)} \) (2.28) on the space-like \( H^3 \) reduces for rotationally invariant functions \( \phi(\chi) \) to
\[
\Delta^{(3)} \phi(\chi) = - \frac{1}{R^2 \cosh^2(\eta) \sinh^2(\chi)} \partial_\chi \left( \sinh^2(\chi) \partial_\chi \phi \right).
\]
(2.33)

2.3. Higher spin sectors and filtration

Due to the extra generators \( t^\mu \), the full algebra of functions decomposes into sectors \( C^s \) which correspond to spin \( s \) harmonics on the \( S^2 \) fiber:
\[
\text{End}(H^3) = C = C^0 \oplus C^1 \oplus \ldots \oplus C^n \quad \text{with} \quad S^2|_{C^s} = 2s(s + 1).
\]
(2.34)
Here \( S^2 = \frac{1}{2} \sum_{a,b,c,d} [M_{ab}, [M_{cd}, \ldots ]] + r^{-2} [X_a, [X_a, \ldots ]] \) can be viewed as a spin operator \( r \) on \( H^3 \) [34], which commutes with \( \square \). In the semi-classical limit, the \( C^s \) are modules over \( C^0 \), and can be realized explicitly in terms of totally symmetric traceless space-like rank \( s \) tensor fields on \( M^{3,1} \)
\[
\phi^{(s)} = \phi_{\mu_1 \ldots \mu_s} (x) t^{\mu_1} \ldots t^{\mu_s}, \quad \phi_{\mu_1 \ldots \mu_s} x^{\mu_i} = 0
\]
(2.35)
due to (2.5). The underlying \( so(4,2) \) structure provides an \( SO(3,1) \)-invariant derivation
\[
D \phi := \{ x^a, \phi \} = r^2 R^2 \frac{1}{x^a} t^{\mu_1} \{ t_{\mu_1}, \phi \} = - \frac{1}{x^a} \chi_{\mu_1} \{ x^{\mu_1}, \phi \}
\]
\[
= r^2 R t^{\mu_1} \ldots t^{\mu_s} \nabla^{(3)} \phi_{\mu_1 \ldots \mu_s} (x)
\]
(2.36)
where \( \nabla^{(3)} \) is the covariant derivative along the space-like \( H^3 \subset M^{3,1} \). Hence \( D \) relates the different spin sectors in (2.34):
\[
D = D^- + D^+ : C^s \rightarrow C^{s-1} \oplus C^{s+1}, \quad D^{\pm} \phi^{(s)} = [D \phi^{(s)}]_{s \pm 1}
\]
(2.37)

\[2\]Since local Lorentz invariance is not manifest, the usual notion of spin cannot be used, and \( S^2 \) is a substitute.
where $[.]_0$ denotes the projection to $C^r$ defined through (2.34). For example, $Dx^u = r^2 R r^u$ and $D\mu u = R^{-1} x^u$. This allows to define a further refinement $[1]$ $C^{(s)} := C^{(s)}/C^{(s-1)}, \quad K^{(s)} = \ker(D^-)^3 \subset C^r$. (2.38)

Then $D^\pm : C^{(s)} \to C^{(s-1)}$. (2.39)

In particular, $C^{(s,0)} \subset C^r$ is the space of divergence-free traceless space-like rank $s$ tensor fields on $\mathcal{M}^{3,1}$, while $D^+ D\phi^{(0)} = [\mu u \bar{\mu}\nu] \partial \nu \phi^{(0)} \in C^{(2,2)} \subset C^2$ encodes the traceless second derivatives of the scalar field $\phi^{(0)}$. These will play an important role below. Finally, $\tau$ is extended to $C^r$ via $[1]$ $\sinh(\eta)(\tau + s)\phi^{(s)} = x^u \{t^u, \phi^{(s)}\}$, which gives (A.13).

### 2.3.1. Averaging

We will need some explicit formulas for the projection $[.]_0$ to $C^0$:

$$[r^\mu t^\nu]_0 =: \frac{\cosh^2(\eta)}{3r^2} P^\mu_{\perp} \nu,$$

$$\text{in terms or the projector } P_{\perp} \text{ (2.29) on the time-slices } H^3.$$

This can be viewed as an averaging over $S^2$. Explicitly, one finds $[1]$

$$[r^\mu t^\nu]_0 = \frac{1}{3} \left( \sinh(\eta)(\eta^{\alpha\nu} x^\mu - \eta^{\alpha\mu} x^\nu) + x_\beta \epsilon^{\beta\gamma\mu\nu} \right),$$

(2.42a)

$$[r^\mu t^\nu]_0 = \frac{3}{5} \left( [r^\mu t^\nu][r^\nu t^\mu]_0 + [r^\mu t^\nu][r^\nu t^\mu]_0 + [r^\mu t^\nu][r^\nu t^\mu]_0 \right),$$

$$[r^a t^\beta t^\gamma]_0 = \frac{3}{5} \left( [r^a t^\beta t^\gamma]_0 + r^a [r^\beta t^\gamma]_0 + r^a [r^\beta t^\gamma]_0 \right).$$

(2.42b)

As an application, one can derive the following formula

$$\{x^u, (x_\mu, \phi)\}_0 = \frac{r^2 R^2}{3} (3 - \cosh^2(\eta)) \beta^2 (-\Box + \frac{1}{R^2} \tau) \phi + \frac{r^2}{3} (2\tau + 7) \tau \phi$$

(2.43)

for $\phi \in C^0$. This could be another natural d’Alembertian on $\mathcal{M}^{3,1}$ which exhibits a transition from a Euclidean to a Minkowski era, as discussed in [25]. However in this paper the effective d’Alembertian will be $\Box$, which respects the spin sectors $C^r$ (2.34).

### 3. Matrix model and higher-spin gauge theory

Now we return to the noncommutative setting, and define a dynamical model for the fuzzy $\mathcal{M}^{3,1}$ space-time under consideration. We consider a Yang–Mills matrix model with mass term, $S[Y] = \frac{1}{g^2} \operatorname{Tr} \left( [Y^\mu, Y^\nu][Y^\mu, Y^\nu] \eta_{\mu\nu\rho} \eta_{\rho\sigma} + \frac{6}{R^2} Y^\mu Y^\nu \eta_{\mu\nu} \right).$ (3.1)

This includes in particular the IKKT or IIB matrix model [2] with mass term, which is best suited for quantization because maximal supersymmetry protects from UV/IR mixing [35]. As observed in [1], $\mathcal{M}^{3,1}$ is indeed a solution of this model$^3$, through

$^3$ Any other positive mass parameter in (3.1) would of course just result in a trivial rescaling. For negative mass parameter, $Y^u \sim X^u$ would be a solution [25], but the fluctuations are more difficult to analyze.
\[ Y^\mu = T^\mu. \] (3.2)

Now consider tangential deformations of the above background solution, i.e.
\[ Y^\mu = T^\mu + A^\mu, \] (3.3)

where \( A^\mu \in \text{End}(H_n) \otimes \mathbb{C}^4 \) is an arbitrary (Hermitian) fluctuation. The Yang–Mills action (3.1) can be expanded as
\[ S[Y] = S[T] + S_2[A] + O(A^3), \] (3.4)

and the quadratic fluctuations are governed by
\[ S_2[A] = -\frac{2}{g^2} \text{Tr} \left( A_\mu \left( D^2 - \frac{3}{R^2} \right) A^\mu + G(A)^2 \right). \] (3.5)

Here
\[ D^2 A = (\Box - 2t) A \] (3.6)
is the vector d'Alembertian, which involves the scalar matrix d'Alembertian \( \Box \sim \alpha^{-1} \Box_G \) on the \( \mathcal{M}^3,1 \) background (2.9) and (2.26) as discussed before, and the intertwiner
\[ \mathcal{I}(A)^\mu = -i[Y^\mu, Y^\nu], A_\nu = \frac{i}{r^2 R^2} [\Theta^{\mu\nu}, A_\nu] = -\frac{1}{r^2 R^2} \mathcal{I}(A)^\mu \] (3.7)
using (2.3c). As usual in Yang–Mills theories, \( A \) transforms under gauge transformations as
\[ \delta \Lambda A = -i[T^\mu + A^\mu, \Lambda] \sim \{ t^\mu, \Lambda \} + \{ A^\mu, \Lambda \} \] (3.8)
for any \( \Lambda \in \mathbb{C} \), and the scalar ghost mode
\[ G(A) = -i[T^\mu, A_\mu] \sim \{ t^\mu, A_\mu \}, \] (3.9)
should be removed to get a meaningful theory. This can be achieved by adding a gauge-fixing term \( -G(A)^2 \) to the action as well as the corresponding Faddeev–Popov (or BRST) ghost.

Then the quadratic action becomes
\[ S_2[A] + S_{gf} + S_{ghost} = -\frac{2}{g^2} \text{Tr} \left( A_\mu \left( D^2 - \frac{3}{R^2} \right) A^\mu + 2r^2 c^2 \right) \] (3.10)
where \( c \) denotes the fermionic BRST ghost; see e.g. [36] for more details.

4. Fluctuation modes

All indices will be raised and lowered with \( \eta^{\mu\nu} \) in this section. We should expand the vector modes into higher spin modes according to (2.34) and (2.35)
\[ A^\mu = A^\mu_\alpha(x) x^\alpha + A^\mu_{\alpha\beta}(x) r^\alpha r^\beta + \ldots \in C^0 \oplus C^1 \oplus C^2 \oplus \ldots. \] (4.1)

However these are neither irreducible nor eigenmodes of \( D^2 \). In [1], three series of spin \( s \) eigenmodes \( A_\mu \) were found of the form

\[
\begin{align*}
\mathcal{A}^{(s)}_\mu[\phi^{(s)}] &= \{ t_\mu, \phi^{(s)} \} \in C^s, \\
\mathcal{A}^{(+)}_\mu[\phi^{(s)}] &= \{ x_\mu, \phi^{(s)} \}_{C^{s+1}} \equiv \{ x_\mu, \phi^{(s)} \} \in C^{s+1}, \\
\mathcal{A}^{(-)}_\mu[\phi^{(s)}] &= \{ x_\mu, \phi^{(s)} \}_{C^{s-1}} \equiv \{ x_\mu, \phi^{(s)} \} \in C^{s-1}
\end{align*}
\] (4.2)
for any $\phi^{(s)} \in \mathcal{C}^s$, which satisfy

$$D^2 A^{(s)}_{\mu} [\phi] = A^{(s)}_{\mu} \left[ \left( \Box + \frac{3}{R^2} \right) \phi \right], \quad (4.3)$$

$$D^2 A^{(+)}_{\mu} [\phi^{(s)}] = A^{(+)}_{\mu} \left[ \left( \Box + \frac{2s + 5}{R^2} \right) \phi^{(s)} \right], \quad (4.4)$$

$$D^2 A^{(-)}_{\mu} [\phi^{(s)}] = A^{(-)}_{\mu} \left[ \left( \Box + \frac{-2s + 3}{R^2} \right) \phi^{(s)} \right]. \quad (4.5)$$

We provide in appendix A.1 a simple new derivation for the last two relations. Hence diagonalizing $D^2$ is reduced to diagonalizing $\Box$ on $\mathcal{C}^s$, and we have the on-shell modes

$$D^2 A^{(+)} [\phi^{(s)}] = A^{(+)} [\phi^{(s)}]$$

$$D^2 A^{(-)} [\phi^{(s)}] = A^{(-)} [\phi^{(s)}]$$

$$D^2 A^{(g)} [\phi^{(s)}] = A^{(g)} [\phi^{(s)}]. \quad (4.6)$$

We will focus on the physical helicity 0 or scalar mode, with on-shell condition

$$D^2 A^{(-)} [\phi^{(s)}] = A^{(-)} [\phi^{(s)}]$$

for

$$\Box \phi^{(s)} = 0, \quad \Box = \Box^0.$$ 

In particular for $s = 2$, $A^{(-)}_{\mu} [\phi^{(2)}]$ is already gauge fixed. This will lead to the physical spin 2 metric fluctuations. According to the discussion in section 2.3, they decompose into the modes $A^{(-)}_{\mu} [\phi^{(2)}]$, $A^{(-)}_{\mu} [D\phi^{(1)}]$ and $A^{(-)}_{\mu} [D^+ D\phi^0]$, which we will denote — in slight abuse of language—as helicity 2, 1 and 0 sectors of the would-be massive spin 2 modes, respectively.

We will focus on the physical helicity 0 or scalar mode, with on-shell condition

$$A^{(-)}_{\mu} [D^+ D\phi], \quad (\Box + \frac{2}{R^2}) \phi = 0, \quad \phi \in \mathcal{C}^0.$$ 

(4.11)

due to (A.3). However, one series of spin $s$ (off-shell) eigenmodes $A^{(s)}_{\mu}$ of $D^2$ is still missing, and was not known up to now. We will find the missing scalar mode in section 4.2, in terms of

$$A^{(s)}_{\mu} [\phi^{(s)}] = x_{\mu} \phi^{(s)}.$$ 

(4.12)

As a check, consider e.g. $A^{(-)}_{\mu} [\phi^{(1)}]$. It satisfies $\{\mu^\nu, A^{(1)}_{\nu} [\phi^{(1)}]\} = 0 = x_{\mu} A^{(1)}_{\mu}$ due to (2.36), and (A.45) gives $\nabla_{\mu} A^{(1)}_{\nu} = 0$ and $\tilde{I}(A^{(1)}_{\mu}) = r^2 A^{(1)}_{\mu}$, consistent with (A.33) in [1].

For $s \neq 2$ some linear combinations of $A^{(+)}_{\mu}$ and $A^{(-)}_{\mu}$ must be taken to obtain a gauge-fixed physical solution. However, this is not our concern here.
That ansatz was also considered in [1], where it was shown to satisfy
\[ \mathcal{D}^2 A_{\mu}^{\tau} \{ \phi^{(s)} \} = A_{\mu}^{\tau} \left( \Box + \frac{7}{R^2} \right) \phi^{(s)} + 2 \delta_{\mu}^{\tau} \phi^{(s)} \] (4.13)

\[ \{ t^{\mu}, A_{\mu}^{\tau} \{ \phi^{(s)} \} \} = \sinh(\eta) \left( 4 + s + \tau \right) \phi^{(s)} \] (4.14)

Here \( \delta_{\mu} \) will be defined in (4.23). We will show in the following that \( A_{\mu}^{\tau} [D^+ D \phi] \) provides the on-shell mode leading to the linearized Schwarzschild metric. Moreover, an ansatz based on \( A_{\mu}^{\tau} \) will give solutions which are equivalent on-shell, but not off-shell.

4.1. Scalar \( A^{(-)}[D^+ D \phi] \) mode

We need the explicit form of \( A^{(-)}[D^+ D \phi] \). This is quite tedious to work out and delegated to the appendix A.5, where we provide an exact expression in (A.32). This simplifies considerably using the on-shell condition \( (\Box + \frac{2}{R^2}) \phi = 0 \) (4.11), leading to
\[ A_{\mu}^{(-)}[D^+ D \phi] = \frac{2 \alpha}{3} \left( \beta t^{\mu} + x^{\mu} r^{\tau} \partial_{\tau} \right) \phi \] (4.15)

with \( \Lambda \) given in (A.34). This is a reasonable perturbation of the background \( Y^\mu = t^\mu \), as long as \( \phi \) remains bounded. Remarkably, (4.15) can be rewritten via \( \theta^{\mu} \partial_\gamma \phi = A^{(+)\mu}[\phi] \) as
\[ A_{\mu}^{(-)}[D^+ D \phi] = \frac{2 \alpha}{3} \left( D(x^{\mu} \phi') - \frac{R}{3} A_{\mu}^{(+)\mu}(\tau + 4 + \beta^2)(\tau + 2) \right) + \{ t^{\mu}, \Lambda \} \] (4.16)

where \( A_{\mu}^{(+)\mu}[\phi'] \) is the new mode defined in (4.38), with
\[ \phi' = \beta(t + 2) \phi, \quad \Lambda' = \Lambda + \frac{2}{15} \beta^2 \mathcal{R}D(\tau + 4 + \beta^2)(\tau + 2) \phi \] (4.17)

To see this, the identities
\[ \beta(t^{\mu} + x^{\mu} r^{\tau} \partial_{\tau}) (\tau + 2) \phi = \frac{1}{r^2} D(x^{\mu} \phi') \] (4.18)

\[ (\tau + 4 + \beta^2)(\tau + 2) \phi = (\sinh(\eta)(\tau + 5) + 2 \beta) \phi' \]

and the on-shell equations
\[ (\Box + \frac{2}{R^2}(3 + \tau - \beta^2)) \phi' = 0 \] (4.19)

\[ \Box \Lambda' = 0 \] (4.20)

are needed, which can be checked using the results of appendix A.2. The last form implies that \( A_{\mu}^{(-)}[D^+ D \phi] \) differs from \( A_{\mu}^{(+)\mu}[\phi'] \) by an on-shell pure gauge mode. This means that even though these are distinct off-shell modes, they become degenerate on-shell, so that there is only one physical scalar graviton mode. This is essential for a ghost-free theory.

Strictly speaking, the form (4.15) collapses for \( \tau = -2 \). However, its expression in terms of \( \phi' \)—or alternatively the form (4.16)—makes sense also in the limit \( \tau \to -2 \). This is important,
because \( \tau = -2 \) gives precisely the Ricci-flat quasi-Schwarzschild solution, as discussed in section 5.4.

For completeness we also provide the explicit form of the pure gauge field \( \mathcal{A}^{(n)} \) corresponding to (A.34)

\[
\mathcal{A}^{(n)} [\Lambda] = \{ t_\mu, \Lambda \} = \frac{2}{5} r^2 \left( \theta^{\mu\nu} \partial_\nu - R \sinh(\eta) D \partial_\mu \right) (\tau + 3) \phi .
\] (4.21)

4.1.1. Gauge fixing. A non-trivial consistency check of (4.15) is obtained by verifying that it satisfies the gauge-fixing constraint. For the pure gauge contribution, this is

\[
\{ t^\mu, \{ t_\mu, \Lambda \} \} = \frac{2}{5} r^2 D (\tau + 3) \phi = \frac{4r^2}{5} D \beta^2 (2 + \tau) \phi
\] (4.22)

using (A.12). Together with the relations (A.19), one verifies indeed

\[
\{ t^\mu, \mathcal{A}^{(\nu)} [D^+ D^+ \phi] \} = 0 .
\]

4.2. Time-like scalar mode \( \tilde{A}^{(\tau)}_\mu \)

In this section we will show that a refined ansatz involving \( \mathcal{A}^{(\tau)} [\phi] \) provides a further scalar eigenmode of \( D^2 \). This will also provide the missing 10th degree of freedom for the off-shell metric fluctuations. While this is not essential to understand the Schwarzschild solution, it provides further insights.

First we recall the relation (4.13), which involves the derivation

\[
\partial^\mu \phi = - \frac{1}{r^2 R^2} \theta^{\mu b} \{ x_b, \phi \} = \{ t^\mu, \beta \phi \} + \frac{1}{R^2} \lambda^\mu \left( - \beta^2 + \tau \right) \phi ,
\] (4.23)

for \( b = 0, ..., 4 \) and \( \phi \in C^0 \). The second form is obtained noting that

\[
\partial^\mu \phi = \partial^\mu \phi + \frac{1}{R^2} x^\mu \tau \phi \quad \text{for} \quad \phi \in C^0 ,
\] (4.24)

and rewriting the first term using \( \partial^\mu \phi = \{ t^\mu, \beta \phi \} - \frac{1}{R} x^\mu \beta^2 \phi \). Hence (4.13) can be written as

\[
D^2 A^{(\tau)}_\mu [\phi] = A^{(\tau)}_\mu \left( \left[ \Box + \frac{1}{R^2} \left( - 2 \beta^2 + 2 \tau + 7 \right) \right] \phi \right) + 2 \{ t^\mu, \beta \phi \} .
\] (4.25)

Since the last term is a pure gauge mode, this provides a new eigenmode of \( D^2 \):

4.2.1. Scalar time-like \( C^0 \) mode. Combining the above with (4.3), the ansatz

\[
\tilde{A}^{(\tau)}_\mu [\phi] = A^{(\tau)}_\mu [\phi] + \{ t^\mu, \tilde{\phi} \}
\] (4.26)

leads to new scalar eigenmode of \( D^2 \)

\[
D^2 \tilde{A}^{(\tau)}_\mu [\phi] = \lambda \tilde{A}^{(\tau)}_\mu [\phi]
\] (4.27)

provided

\[
\left( \Box + \frac{1}{R^2} \left( - 2 \beta^2 + 2 \tau + 7 \right) \right) \phi = \lambda \phi
\]

\[
(\Box + \frac{3}{R^2}) \tilde{\phi} + 2 \beta \tilde{\phi} = \lambda \tilde{\phi} .
\] (4.28)
The first equation can be solved, and has propagating solutions \( \phi \). Then \( \tilde{\phi} \) is determined by the second equation, up to solutions of \((\Box + \frac{1}{R^2} - \lambda)\tilde{\phi} = 0\). This 4th eigenmode is needed e.g. for the off-shell propagator. In particular, \( \tilde{A}_\mu^{(\tau)}[\phi] \) is on-shell, \((D^2 - \frac{1}{R^2})\tilde{A}^{(\tau)} = 0\) for

\[
\left(\Box + \frac{2}{R^2}(2 + \tau - \beta^2)\right)\phi = 0
\]
\[
\Box\tilde{\phi} + 2\beta\phi = 0.
\]

However the gauge fixing condition for this mode is very restrictive on-shell,

\[
\{t_\mu, \tilde{A}_\nu^{(\tau)}[\phi]\} = \{t_\mu, \tilde{A}^{(\tau)}_\nu[\phi]\} - \Box\tilde{\phi} = \left(\sinh(\eta)(4 + \tau) + 2\beta\right)\phi
\]

or

\[
\tau\phi = -(4 + 2\beta^2)\phi,
\]

which means that \( \phi \) is decaying in time with a fixed rate. Hence these modes are ‘frozen’ rather than propagating, which is good because they would otherwise be ghosts. We will see that these \( A \in C_0 \) modes do not contribute to the linearized metric fluctuations.

4.2.2. Scalar time-like \( C^1 \) mode. Based on the above mode and using the ladder property (A.4), we can similarly find a new eigenmode \( A \in C^1 \) with the ansatz

\[
D\tilde{A}^{(\tau)}_\mu[\phi] = D(A^{(\tau)}_\mu[\phi] + \{t^\mu, \tilde{\phi}\}) = r^2 p^\mu\phi + x^\mu D\phi + \frac{1}{R}\{x^\mu, \tilde{\phi}\} + \{t^\mu, D\tilde{\phi}\}.
\]

This is an eigenmode of \( D^2 \) provided \( \tilde{A}^{(\tau)}_\mu[\phi] \) is an eigenmode, with shifted eigenvalue

\[
D^2(D\tilde{A}^{(\tau)}_\mu[\phi]) = D(D^2 + \frac{2}{R^2})\tilde{A}^{(\tau)}_\mu[\phi].
\]

In particular, \( D^+\tilde{A}^{(\tau)}_\mu[\phi] \) is on-shell if \((D^2 - \frac{1}{R^2})\tilde{A}^{(\tau)}_\mu[\phi] = 0\), which means by (4.28)

\[
\left(\Box + \frac{2}{R^2}(-\beta^2 + \tau + 3)\right)\phi = 0
\]

\[
(\Box + \frac{2}{R^2})\tilde{\phi} + 2\beta\phi = 0.
\]

This provides the missing 4th scalar eigenmode in \( C^1 \). The gauge-fixing condition is

\[
\{t^\mu, D\tilde{A}^{(\tau)}_\mu[\phi]\} = \{t^\mu, r^2 p_\mu\phi + x_\mu D\phi\} + \{t^\mu, D\tilde{\phi}\}
\]

\[
= r^2 p_\mu\{t^\mu, \phi\} + 4\sinh(\eta)D\phi + x_\mu\{t^\mu, D\phi\} + \frac{1}{R}\{x_\mu, \tilde{\phi}\} - \Box\tilde{\phi}
\]

\[
= \sinh(\eta)D\phi + 4\sinh(\eta)D\phi + \sinh(\eta)(\tau + 1)D\phi + \frac{3}{R^2}D\tilde{\phi} - D(\Box + \frac{2}{R^2})\tilde{\phi}
\]

\[
= D\left(\sinh(\eta)(\tau + 5) + 2\beta\right)\phi + \frac{3}{R^2}\tilde{\phi}
\]

using (4.30), (2.40), (A.13), (2.36) and the on-shell equation (4.34). This implies

\[
(\sinh(\eta)(\tau + 5) + 2\beta)\phi + \frac{3}{R^2}\tilde{\phi} = f(x^4).
\]
For now we set $f = 0$. Then
\[ \tilde{\phi} = -\frac{R^2}{3} \left( \sinh(\eta)(\tau + 5) + 2\beta \right) \phi, \]  
(4.37)
and one can verify that the equations of motion (4.34b) for $\tilde{\phi}$ indeed follow from those of $\phi$, using the relations in appendix A.2. This means that the gauge-fixing condition leading to (4.37) is consistent with the equations of motion, and we have found a physical propagating mode of the form
\[ A^{(S)}_\mu[\phi] := D \left( A^{(\tau)}_\mu[\phi] - \frac{R^2}{3} \{ t^\mu, \left( \sinh(\eta)(\tau + 5) + 2\beta \right) \phi \} \right) \]  
(4.38)
with $\phi$ satisfying (4.34a). On-shell, this coincides precisely with the on-shell eigenmode $A^{(-)}[D^+ D\phi]$ (4.16), although off-shell (hence in the propagator) they are distinct modes. In the quasi-static case $\tau = -2$, this will give the linearized Schwarzschild metric.

5. Scalar metric fluctuation modes

In this section, we elaborate the metric fluctuations arising from the above scalar modes. The effective metric for functions of $M^{1,1}$ on a perturbed background $Y = T + A$ can be extracted from the kinetic term in (2.21), which defines the bi-derivation
\[ \gamma : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \]
\[ (\phi, \phi') \mapsto \{ Y^\alpha, \phi \} \{ Y^\alpha, \phi' \} \]  
(5.1)
up to a conformal factor as discussed in section 2.2. Specializing to $\phi = x^\mu$, $\phi' = x'^\nu$ we obtain the coordinate form
\[ \gamma^\mu_\nu = \delta^\mu_\nu + \delta_A \gamma^\mu_\nu + \left[ [A^\alpha, x^\mu] \{ A_\alpha, x'^\nu \} \right]_0 \]  
(5.2)
where the linearized contribution is given by
\[ \delta_A \gamma^\mu_\nu := \left[ [A^\alpha, x^\mu] \{ A_\alpha, x'^\nu \} \right]_0 + (\mu \leftrightarrow \nu) = \sinh(\eta) [A_\mu, x'^\nu]_0 + (\mu \leftrightarrow \nu). \]  
(5.3)
The projection on $\mathcal{C}^0$ ensures that this is the metric for functions on $M^{1,1}$. We will focus on the linearized contribution in $\mathcal{A}$ in the following. To evaluate this explicitly, it is convenient to consider the following rescaled graviton mode:
\[ h^\mu_\nu[A] := \left\{ A^\mu, x'^\nu \right\}_0 + (\mu \leftrightarrow \nu), \]  
(5.4)
Clearly only $A \in \mathcal{C}^1$ can contribute to $h^\mu_\nu[A]$. Taking into account the conformal factor as identified in section 2.2, the effective metric $G^\mu_\nu$ (2.20) is
\[ G^\mu_\nu = \tilde{G}^\mu_\nu + \delta G^\mu_\nu \]
\[ = \alpha \left[ \gamma^\mu_\nu + \delta_A \gamma^\mu_\nu - \frac{1}{2}  \eta^\mu_\nu \left( \eta_{\alpha\beta} \delta_A \gamma^\alpha_\beta \right) \right] \]
\[ \delta G^\mu_\nu = \beta^2 \left( h^\mu_\nu - \frac{1}{2} \eta^\mu_\nu h \right). \]  
(5.5)
Here $\tilde{G}^\mu_\nu = \alpha \gamma^\mu_\nu = \beta \eta^\mu_\nu$ (2.20) is the effective background metric, $\alpha = \beta^3$ is the conformal factor arising from the fixed symplectic measure on $\mathbb{C}P^{1,1}$, and $\beta = \sinh(\eta)^{-1}$ (2.8). Equivalently,
where \( h_{\alpha\beta} = \eta_{\alpha\alpha'} \eta_{\beta\beta'} h^{\alpha'}{}^{\beta'} \). One has to be very careful in rising and lowering indices, because there are different metrics in the game. The indices of the effective metric \( G \) will always be raised and lowered with the effective background metric \( \bar{G}^{\mu\nu} \), while the indices of \( h^{\mu\nu} \) and most other tensorial objects will be raised and lowered with \( \eta^{\mu\nu} \). In case of ambiguity, we will typically spell this out. With this convention, we can write the fluctuations of the effective background effective metric (2.22) as

\[
(G_{\mu\nu} - \delta G_{\mu\nu}) \, dx^\mu \, dx^\nu = -dr^2 + a^2(t) d\Sigma^2 - (h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{h}) \, dx^\mu \, dx^\nu. \quad (5.7)
\]

5.1. Linearized Ricci tensor

To understand the significance of the metric modes, we consider the linearized Ricci tensor

\[
2\delta R^{\mu\nu}_{(\text{lin})}[G] = -\nabla^\alpha \nabla^\beta \delta g_{\alpha\beta} + \nabla^\mu \nabla^\rho \delta g_{\mu\rho} + \nabla^\nu \nabla^\rho \delta g_{\nu\rho} - \nabla^\mu \nabla^\nu \delta G \quad (5.8)
\]

for a metric fluctuation \( \delta G^{\mu\nu} = \beta^2 \bar{h}^{\mu\nu} \) with

\[
\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \bar{h}, \quad \bar{h} = -h
\]

around the background \( \bar{G}^{\mu\nu} = \beta \eta^{\mu\nu} \). For simplicity, we will neglect contributions of the order of the cosmic background curvature. Then we can replace \( \nabla \) by \( \partial \) in Cartesian coordinates, and

\[
2R^{\mu\nu}_{(\text{lin})}[G] \eta \rightarrow \infty \approx \beta^2 \left( -\partial^\alpha \partial_\alpha \bar{h}^{\mu\nu} + \partial^\mu \partial_\rho \bar{h}^{\rho\nu} + \partial^\nu \partial_\rho \bar{h}^{\mu\rho} - \partial^\mu \partial^\nu \bar{h} \right)
\]

\[
= \beta^2 \left( -\partial^\alpha \partial_\alpha (h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \bar{h}) + \partial^\mu \partial_\rho h^{\rho\nu} + \partial^\nu \partial_\rho h^{\mu\rho} \right) \quad (5.10)
\]

neglecting the \( \partial \beta \) terms at late times \( \eta \rightarrow \infty \), because (A.8)

\[
\beta^{-1} \partial^\alpha \partial_\alpha = \frac{\beta^2}{R} \frac{G_{\mu\nu} x^\nu}{x^4} = O(\beta^2). \quad (5.11)
\]

Now we can use the intertwiner relation (6.25) in [1]

\[
\left( \square + \frac{2}{R^2} \mathcal{I} \right) \bar{h}^{\mu\nu}[A] = h_{\mu\nu} [\mathcal{D}^2 A] + \frac{2}{R^2} \left( 3h^{\mu\nu}[A] - \eta^{\mu\nu} h[A] \right) \quad (5.12)
\]

and the on-shell relation \((\mathcal{D}^2 - \frac{3}{R^2}) A = 0\). We should also drop the contribution from \( \mathcal{I} \) in the same approximation, because

\[
\square \phi \sim -\sinh^2(\eta) \partial^\alpha \partial_\alpha \phi \approx \frac{1}{R^2} \mathcal{I} (h^{\mu\nu}) \sim \frac{x}{R^2} \partial h^{\mu\nu} \quad (5.13)
\]

for \( \partial \gg \frac{1}{x^4} \), using (2.27) and \( \{ \theta^{\mu\nu}, \phi \} = r^2(x^\mu \partial^\nu - x^\nu \partial^\mu) \phi [1] \). Therefore (5.12) reduces on-shell to

\[
\partial^\alpha \partial_\alpha h^{\mu\nu} \approx -\frac{1}{x^4} (9h^{\mu\nu} - 2\eta^{\mu\nu} h) \quad (5.14)
\]
which is negligible at late times compared to the terms involving second derivatives $\partial^2 h^{\mu\nu}$ in (5.10), and similarly for the trace. This means that the linearized Ricci tensor reduces on-shell to

$$2R_{(\text{lin})}^{\mu\nu}(G^{\alpha\beta}) = \beta^2 \left( \partial^\mu \partial_\rho h^{\rho\nu} + \partial^\nu \partial_\rho h^{\rho\mu} + O\left(\frac{\partial h^{\mu\nu}}{x^4}\right) \right)$$

(5.15)
on scales much shorter than the cosmic curvature scale, or for late times i.e. large $\eta$.

5.2. Pure gauge modes

Now consider the metric fluctuation corresponding to the pure gauge fields $A^{(g)}[\phi]$, where $\phi = \phi^{(1)}$ is a spin 1 field. This has the form (see [1])

$$h^{\mu\nu}(g) [\phi] := h^{\mu\nu}[A^{(g)}] = - \{ t^\mu, A^{(-)\nu} [\phi] \} + (\mu \leftrightarrow \nu) + \frac{1}{3} h^{\mu
u} \eta^{\mu
u}, \quad (5.16a)$$

$$h^{(g)}[\phi] := \eta^{\mu
u} h^{\mu\nu} = \frac{6}{R} D^- \phi = 6 \{ t^\mu, A^{(-)\nu} [\phi] \}. \quad (5.16b)$$

It is not hard to show the following formulas

$$\{ t^\mu, h^{\mu\nu}(g) [\phi] \} = - \{ \Box \phi, x^- \} - \frac{2}{R} D^- \{ t^\nu, \phi \}, \quad (5.17)$$

$$x^- h^{\mu\nu}(g) [\phi] = 2 R \sinh^2(\eta) D^- \tau \phi \quad (5.18)$$

using (2.36) see [1], and in particular

$$\{ t^\mu, h^{\mu\nu}(g) [\phi^{(1,0)}] \} = - \frac{2}{R} A^{(-)\nu} [\phi] \quad \text{for } \Box \phi^{(1,0)} = 0$$

$$x^- h^{\mu\nu}(g) [\phi^{(1,0)}] = - \sinh(\eta)(\tau - 1) A^{(-)\nu} [\phi]. \quad (5.19)$$

Taking into account the conformal factor (5.5), the pure gauge contribution to the effective metric is

$$\delta G^{\mu\nu}(g) = \beta^2 \left( h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h \right)$$

$$= \beta^2 \left( - \{ t^\mu, A^\nu \} - \{ t^\nu, A^\mu \} - \eta^{\mu\nu} \{ t_\alpha, A^\alpha \} \right)$$

$$= - \partial^\mu A^\nu - \partial^\nu A^\mu - G^{\mu\nu} (\partial_\alpha A^\alpha) \quad (5.20)$$

where $A^\alpha = A^{(-)\alpha} [\phi]$ and $\partial^\mu = G^{\mu\nu} \partial_\nu$. This formula is valid in Cartesian coordinates, and we must be very careful with using upper indices, e.g. $\{ t^\mu, \phi \} = \sinh(\eta) \eta^{\mu\nu} \partial_\nu \phi = \sinh^2(\eta) G^{\mu\nu} \partial_\nu \phi$.

5.2.1. Relation with diffeomorphisms. We can rewrite these pure gauge modes as diffeomorphism modes by comparing with (A.44) on the present FLRW background. This gives

$$\delta G^{\mu\nu}(g) = \partial^\mu \xi^\nu + \partial^\nu \xi^\mu - \frac{1}{x^4} G^{\mu\nu} x \cdot \xi$$

$$= \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu, \quad \xi^\mu = - A^\mu \quad (5.21)$$
using
\[ x_\alpha A^\alpha = \eta_{\alpha\beta} x^\alpha \{ x^\beta, \phi \} - = -x^4 D^- \phi \]
\[ \sinh \partial_\alpha A^\alpha = \{ t_\alpha, \{ x^\alpha, \phi \} \} - = \frac{1}{R} D^- \phi \]
\[ \partial_\alpha A^\alpha = -\frac{1}{x_4^2} x \cdot A \] (5.22)
where \( A^\alpha = A^{\alpha(-)}[\phi] \), using the notation \( x \cdot A \equiv \eta_{\alpha\beta} x^\alpha A^\beta \). Hence the pure gauge metric modes in the present framework can be identified with diffeomorphisms generated by \( \xi = -A \). This also provides a non-trivial consistency check for the correct identification of \( G \).

It is easy to check using (A.45) that these diffeomorphisms satisfy the constraint
\[ \nabla_\alpha \xi^\alpha = -\frac{3}{2} x^2 \frac{4}{x_4^2} x \cdot \xi \] (5.23)
or equivalently
\[ \nabla_\alpha (\beta^{3/2} \xi^\alpha) = 0 \] (5.24)
Hence they are essentially volume-preserving diffeos up to the factor \( \beta^{3/2} \), leaving only 3 rather than 4 diffeomorphism d.o.f., unlike in GR. This reflects the presence of a dynamical scalar metric degree of freedom, which we will study in detail below.

5.3. Generalities for the \( A^{(-)} \) metric modes

Among the \( A^{(-)}[\phi^{(s)}] \) modes, only the ones with spin \( s = 2 \) can contribute to the metric, and these are in fact physical degrees of freedom as shown in (4.8). The corresponding linearized metric fluctuation is [1]
\[ h^{\mu\nu}_{(-)}[\phi] := h^{\mu\nu}[A^{(-)}[\phi]] = -2 \{ x^\mu, \{ x^\nu, \phi \} \} - = -2 \{ x^\nu, \{ x^\mu, \phi \} \} - \]
\[ h_{(-)}[\phi] := \eta_{\mu\nu} h^{\mu\nu}_{(-)} = -2 \{ x^\nu, \{ x_\mu, \phi \} \} - = 2D_\nu D_\mu \phi \] (5.25a)
for \( \phi = \phi^{(2)} \). It is not hard to derive the following formulas
\[ \{ t_\mu, h^{\mu\nu}_{(-)} \} = -\frac{2}{R} \{ x^\nu, D^- \phi \} - \] (5.26a)
\[ \{ t_\mu, \{ \rho^{\nu}, h^{\rho\sigma}_{(-)} \} \} + (\mu \leftrightarrow \nu) = \frac{2}{R^2} \left( h_{\mu\nu}^{(\phi)} - \frac{1}{3} \eta_{\mu\nu} h^{(\phi)} \right) [D^- \phi] \] (5.26b)
\[ x_\mu h^{\mu\nu}_{(-)} = 2x_4 \{ x^\nu, D^- \phi \} - \] (5.26c)
since \( \{ \rho^{\nu}, \phi^{(2)} \}_0 = 0 \). Comparing (5.26c) and (5.26a), we obtain
\[ \partial_\mu h^{\mu\nu}_{(-)} = -\frac{1}{x_4^2} x_\mu h^{\mu\nu}_{(-)} \] (5.27)
or equivalently
\[ \partial_\mu (\beta h^{\mu\nu}_{(-)}) = 0 \] (5.28)
This looks like a gauge-fixing condition. We can write it in covariant form using the explicit form of the Christoffel symbols (A.42) and (A.43), which gives
\[ \nabla_\mu h^{\mu\nu} = \partial_\mu h^{\mu\nu} - \frac{3}{2\kappa_4^2} x_\mu h^{\mu\nu} + \frac{1}{2\kappa_4^2} x^\nu h. \] (5.29)

Since the \( \mathcal{A}(-)[\phi^{(2,0)}] \) and the \( \mathcal{A}(-)[\phi^{(2,1)}] \) modes satisfy \( h = 0 \), this can be written using (5.27) as
\[ \nabla_\mu (\beta^j h^{\mu\nu} [\phi^{(2,\beta)}]) = 0 \quad \text{for } j = 0, 1. \] (5.30)

Since this condition (5.27) is not quite the same as (5.19) for the on-shell pure gauge gravitons, it follows that the extra 2 on-shell metric fluctuations \( h^{\mu\nu}[\mathcal{A}(-)[\phi^{(2,1)}]] \) are not in fact physical.

5.3.1. Linearized Ricci tensor. Using the constraint (5.28) for \( h^{\mu\nu}[\mathcal{A}(-)[\phi^{(2)}]] \), it follows from (5.15) that all these on-shell (would-be massive) spin 2 modes are Ricci-flat up to cosmic scales,
\[ 2R^{\mu\nu}_{\text{lin}} = 0 + O(\frac{\partial G^{\mu\nu}}{x_4}). \] (5.31)

This seems to suggest that these modes are exactly massless with only 2 physical degrees of freedom, but this is not true, as pointed out above. The point is that the \( h^{\mu\nu} \) contributions from the would-be helicity 1 and 0 modes are typically dominated by diffeos, which are trivially flat. However, we will see in the next section that the linearized Schwarzschild solution which arises from \( h^{\mu\nu}[\mathcal{A}(-)[D^+ D\phi]] \) is not dominated by diffeos, but a genuine non-trivial Ricci-flat metric.

5.4. Scalar modes \( \mathcal{A}(-)[D^+ D\phi] \) and the Schwarzschild metric

Now we work out the explicit metric perturbation arising from the on-shell \( \mathcal{A}(-)[D^+ D\phi] \) mode, which is part of the would-be massless spin 2 multiplet \( \mathcal{A}(-)[\phi^{(2)}] \). We will see that this includes a quasi-static Schwarzschild metric, as well as other solutions which might be related to dark matter. We will use the on-shell condition \( \Box \phi = -\frac{\kappa_4}{2} \phi (4.11) \) throughout, and focus on the late-time limit \( \eta \to \infty \). Starting with the explicit form (4.15) for \( \mathcal{A}(-)[D^+ D^+ \phi] \), dropping the pure gauge contribution \( \{\mu, \Lambda\} \) and using the results of appendix A.3, we obtain
\[ \frac{5}{2R} h^{\mu\nu}[\mathcal{A}^{(2)}[D^+ D\phi]] = h^{\mu\nu}[\frac{1}{2} (r^2 \partial t^t + r^2 \partial \alpha^\alpha + \frac{1}{\kappa_4} \partial \phi^\alpha \partial \Lambda^\alpha (\tau + 4)) (\tau + 2) \phi] \]
\[ \eta \to \infty \]
\[ \frac{5}{2R} \left( \eta^{\mu\nu} (2 + \tau) (3 + \tau^2 + 4\tau) + \frac{1}{\kappa_4^2} x^\mu x^\nu (\tau^2 - 1) - (x^\tau \partial \tau + x^t \partial t^t) (\tau^2 + 3\tau + 2) - R^2 \partial^\tau \partial^t (\tau + 4) \right) (\tau + 2) \phi. \] (5.32)

Therefore
\[ h^{\mu\nu} = \frac{4\tau^t}{45} \left( (2 + \tau) (3 + \tau) \eta^{\mu\nu} + \frac{\beta^2}{\kappa_4^2} x^\mu x^\nu (\tau - 1) - (x^\tau \partial \tau + x^t \partial t^t) (\tau + 2) \right) (\tau + 1) (\tau + 2) \phi \]
\[ - \frac{4\tau^t}{45} R^2 \partial^\tau \partial^t (\tau + 4) (\tau + 2) \phi \] (5.33)
with trace
\[ h^{\eta} \equiv \frac{4r^4}{45} (\tau + 1)(2\tau + 5)(\tau + 5)\phi. \] (5.34)

Then the trace-reversed metric fluctuation \( \tilde{h}^{\mu\nu} \) is
\[
\tilde{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} h^{\mu\nu} - \frac{4r^4}{45} \left( - \frac{1}{2} (5\tau + 13) h^{\mu\nu} + \frac{\beta^2}{R^2} x^\nu x^\rho (\tau - 1) - (x^\nu \partial^\mu + x^\mu \partial^\nu) (\tau + 2) \right) (\tau + 1) (\tau + 2) \phi
\]
\[
- \frac{4r^4}{45} R^2 \partial^\mu (\tau + 4) (\tau + 2) \phi. \quad (5.35)
\]

Observe that for \( \tau \neq -2 \), the term \((x^\nu \partial^\mu + x^\mu \partial^\nu) \phi\) is dominant at late times, since \( x^0 \sim R \cosh(\eta) \). However this is essentially a large diffeomorphism contribution, which can be removed from the effective metric fluctuation using (A.48), with the result
\[
\tilde{h}^{\mu\nu} \sim \frac{4r^4}{45} \left( \frac{1}{2} (\tau - 1) h^{\mu\nu} + \frac{3\beta^2}{R^2} x^\nu x^\rho (\tau + 1) \right) (\tau + 1) (\tau + 2) \phi
\] (5.36)

for large \( \eta \). Hence
\[
\tilde{h}_{\mu\nu} \, dx^\mu dx^\nu = \frac{2r^4 R^2}{45} \sinh^2(\eta) (d\eta^2 (5\tau + 7) + d\Sigma^2 (\tau - 1)) (\tau + 1) (\tau + 2) \phi
\]
\[
\tau \overset{\approx -2}{\rightarrow} \frac{2r^4 R^2}{15} \sinh^2(\eta) (\tau + 2) \phi (d\eta^2 + d\Sigma^2)
\]
\[
= -4\phi' (dr^2 + a(t)^2 d\Sigma^2) \quad (5.37)
\]

using (2.17) where \( \tilde{h}_{\mu\nu} = \eta_{\mu\nu} \eta_{\rho\sigma} \tilde{h}^{\rho\sigma} \), and using the explicit form (2.22) of the scale parameter \( a(t) \) for large \( \eta \). Here we define
\[
\phi' := -\frac{r^4}{30} (\tau + 2) \phi \quad (5.38)
\]
as in (4.17) (up to rescaling), which allows to take \( \tau \rightarrow -2 \). We will see that this reduces to the linearized Schwarzschild metric for \( \tau \rightarrow -2 \), while for \( \tau \neq -2 \) it is a distinct metric which is not Ricci-flat. However for \( \tau \neq -2 \) the diffeo contribution in (5.35) grows very large at late times, which may invalidate the linearized approximation as discussed below. Therefore we focus on \( \tau \approx -2 \), which is the most interesting and most reliable case. Then the full perturbed metric can be written in the form (5.7)

\[
ds^2 = (G_{\mu\nu} - \delta G_{\mu\nu}) dx^\mu dx^\nu = (\sinh(\eta)\eta_{\mu\nu} - \tilde{h}_{\mu\nu}) \, dx^\mu dx^\nu
\]
\[
= -dr^2 + a(t)^2 d\Sigma^2 + 4\phi' (dr^2 + a(t)^2 d\Sigma^2). \quad (5.39)
\]

The on-shell condition reduces to \( \Delta(3) \phi = 0 \) for \( \tau = -2 \) due to (2.30), and in the spherically symmetric case the Newton potential on a \( k = -1 \) geometry is recovered (A.65), with
\[
\phi = \frac{e^{-\chi}}{\sinh(\chi) \cosh^2(\eta)} \sim \frac{1}{\rho} e^{-\chi - 2\eta}, \quad \rho = \sinh(\chi). \quad (5.40)
\]
Strictly speaking we should use $\phi'$ rather than $\phi$ in the $(\tau + 2)\phi = 0$ case. Then the quasi-static condition becomes $(\tau + 3 + \beta^2)\phi' = 0$, and the on-shell condition (4.19) is $(\Delta^{(3)} - 4\beta^2)\phi' = 0$. However the $\beta^2$ contributions can be dropped in the large $\eta$ limit giving again $\Delta^{(3)}\phi' = 0$, so that

$$
\phi' \sim \frac{1}{\rho} e^{-\chi - 3\eta} \sim \frac{e^{-\chi}}{\rho} \frac{1}{a(t)^2}
$$

(5.41)

for large $\eta$, using (2.25) and recalling $a(t) \sim e^{-\frac{1}{2}\eta}$ (2.23). This metric is very close to the Vittie solution [37] for the Schwarzschild metric for a point mass $M$ in a FRW spacetime, whose linearization for $k = -1$ is given by

$$
d\bar{s}^2 = -dt^2 + a(t)^2 d\Sigma^2 + 4\mu(dr^2 + a(t)^2 d\Sigma^2) + O(\mu^2). 
$$

(5.42)

Here

$$
\mu = \mu(t, \chi) = M \frac{1}{2\rho a(t)}
$$

is the mass parameter, which is not constant but decays during the cosmic expansion; this is as it should be, because local gravitational systems do not participate in the expansion of the universe. Comparing with (5.41) we have

$$
\phi' \sim \mu(t, \chi) \frac{e^{-\chi}}{a(t)}. 
$$

(5.44)

Since $\mu$ (5.43) looks like a constant mass for a comoving observer [37], the effective mass parameter in our solution effectively decreases like $a(t)^{-1}$ during the cosmic evolution. This might be interpreted in terms of a time-dependent Newton constant, although this a bit premature since we have not properly investigated the coupling to matter, and quantum effects may modify the result. Nevertheless, the result is suggestive. Also, while both metrics have the characteristic $\frac{1}{\rho}$ dependence of the Newton potential, our solution has an extra $e^{-\chi}$ factor, which reduces its range at space-like curvature scales. Both effects are irrelevant at solar system scales, but they will be important for cosmological considerations, reducing the gravitational attraction at long scales.

For completeness, we also recall the linearized Schwarzschild solution in isotropic coordinates

$$
d\bar{s}^2 = \frac{(1 - \frac{M}{2r})^2}{(1 + \frac{M}{2r})^2} dr^2 + (1 + \frac{M}{2r})^4(d\Omega^2 + d\Sigma^2 + dz^2) = \eta_{\mu\nu}dx^\mu dx^\nu + \frac{2M}{r}dx_3^2 + \frac{2M}{r} (d\Sigma^2 + dz^2) + O(\frac{1}{r^2}).
$$

(5.45)

(5.42) reduces to this metric for a local comoving observer for large $a(t)$, while we obtained an extra factor $\frac{1}{a(t)}$ in the effective mass.

Let us discuss the consistency and significance of these results. The most striking point is that even though the metric (5.36) is Ricci-flat for $\tau = -2$, for other values of $\tau$ it is not. This seems to contradict the general result (5.31) for the linearized Ricci tensor, which should always vanish at scales shorter than the background curvature i.e. for large $\eta$. The resolution of this puzzle lies in the diffeo contributions $(x\partial_x + x\partial) \bar{g}$ in (5.35), which were eliminated by a change of coordinates in (5.36). The point is that for $\tau \neq -2$, this term becomes very large
for large $\eta$ as $x^0 \sim R \cosh(\eta)$, and completely dominates the other, non-trivial contributions to $\tilde{h}^{\mu\nu}$. But the Ricci-tensor for a diffeo contribution vanishes trivially, leading to (5.31). In other words, if the first terms in (5.35) are non-trivial, they are dominated by the $(x\partial + x\bar{\partial})$ term, so that for large $\eta$ the linearized approximation becomes invalid, unless $\tau \approx -2$. Then the effective metric fluctuation (5.3) must be completed by the non-linear contribution, which will be discussed briefly below.

In contrast for $\tau \approx -2$, the pure gauge contribution in (5.35) vanishes, hence our Schwarzschild-like solution is fully justified.

A similar issue may arise for the would-be helicity 1 modes $A^{(-)}[\phi^{(2,1)}]$, but not for the helicity-2 gravitons $A^{(-)}[\phi^{(2,0)}]$, because there are no helicity 2 pure gauge contributions. Therefore these are indeed Ricci-flat and non-trivial, as stated in [1].

One might worry that the restriction to $\tau = -2$ of the Schwarzschild solution is too rigid for real physical systems such as the solar system. However, systems with non-uniform motion lead to dynamical metric perturbations corresponding to physical spin 2 gravitons, which are realized here by the $A^{(-)}[\phi^{(2,0)}]$ modes. Therefore there should not be an obstacle to obtain dynamical Ricci-flat metric perturbations as a combination of $A^{(-)}[D^+D\phi^{(0)}]$ and $A^{(-)}[\phi^{(2,0)}]$ modes.

5.4.1 Interpretation and physical significance. We found that the scalar on-shell modes provide a Ricci-flat metric perturbation only for the specific quasi-static time-dependence $\tau \approx -2$. Indeed it should be expected that a dynamical scalar metric mode, which does not exist in GR, is not Ricci-flat in general. From a GR point of view, such non-Ricci-flat perturbations would be interpreted as dark matter. Nevertheless, there better be a reason why in typical situations such as the solar system, such non-Ricci-flat deformations are suppressed. Strictly speaking this question can only be settled once the coupling of matter to the various modes is properly taken into account. Quantum effects may also be important here, because they typically lead to an induced Einstein–Hilbert term [1], which would distinguish Ricci-flat and -non-flat solutions.

However heuristically, we can give a classical mechanism which achieves that effect at the non-linear level as follows. For scalar modes with large diffeo contribution in (5.35), the linearized metric (5.3) must be replaced by the full non-linear expression (5.2). Then the large would-be diffeo contribution no longer decouples from a conserved $T^{\mu\nu}$, but strongly couples to matter. But if this large contribution governs the dynamics, the first two terms in (5.35) are effectively suppressed, and this suppression is stronger for shorter wavelengths due to the derivatives. On the other hand $A$ becomes small sufficiently far from matter, so that the linearized treatment will suffice. Then the large contribution is indeed a flat diffeo, while the sub-leading non-Ricci-flat contribution in (5.36) is strongly suppressed. This does not apply to the Ricci-flat $\tau = -2$ contribution since the diffeo vanishes, and we conclude that the non-Ricci-flat contributions are strongly suppressed, as desired.

For very long wavelengths, this suppression mechanism becomes weak, so that some non-Ricci-flat perturbations with very long—possibly galactic—wavelengths are expected. This would then be interpreted as dark matter from a GR point of view. Moreover, the suppression
mechanism is weaker in the earlier Universe, which might explain why dark matter seems to be more abundant in older galaxies [38].

This non-linear effect is somewhat reminiscent of the vDVZ discontinuity in massive gravity and its resolution through the Vainshtein mechanism [20]. Indeed, the present modes arise precisely from would-be massive spin 2 modes, albeit the details are different.

A time dependence \( \sim a(t)^{-1} \) of the Newton constant seems to be somewhat large in view of recent estimates [39, 40]. However, we have not properly taken into account the coupling to matter, and the underlying FLRW cosmology is non-standard. Including an induced Einstein–Hilbert action in the quantum effective action could also affect the result. These issues need to be understood before solid predictions can be made.

Finally, we note that the case \( \tau = -1 \) is also special. After suitable rescaling this leads to \( \tilde{h}^{\mu\nu} \sim \eta^{\mu\nu} \), hence to a modification of the cosmological evolution \( a(t) \) for \( \phi \sim e^{-\phi} \). A similar modification may arise from \( f \neq 0 \) in (4.36). This shows that modifications of the cosmic evolution are possible, but again this needs to be studied in more detail.

5.4.2. Pure gauge contribution and checks. An instructive check can be obtained by computing the metric fluctuation arising from the pure gauge term (4.21):

\[
h^{\mu\nu} = \frac{2}{5} r^2 h^{\mu\nu}[\theta^{\mu\alpha} \partial_\alpha (\tau + 3)\phi] - \frac{2}{5} r^4 R^2 h^{\mu\nu}[\sinh(\eta) \tau^\mu \partial_\nu (\tau + 3)\phi]
\]

\[
\eta \to \infty \quad \frac{4}{15} r^4 \left(-\tau \eta^{\mu\nu} + \frac{1}{R^2} \beta^2 x^\mu x^\nu - R^2 \sinh^2(\eta) \partial_\mu \partial_\nu + (x^\nu \partial^\mu + x^\mu \partial^\nu)\right)(\tau + 3)(\tau + 2)\phi
\]

using (A.18), with trace

\[
h = -\frac{12}{15} (\tau + 1)(\tau + 2)(\tau + 3) r^4 \phi
\]

consistent with (5.16b). Then the trace-reversed pure gauge metric fluctuation is

\[
\tilde{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} n^{\mu\nu}
\]

\[
= \frac{4}{15} r^4 \left(\frac{1}{2} (\tau + 3)\eta^{\mu\nu} + \frac{1}{R^2} \beta^2 x^\mu x^\nu - R^2 \sinh^2(\eta) \partial_\mu \partial_\nu + (x^\nu \partial^\mu + x^\mu \partial^\nu)\right)(\tau + 3)(\tau + 2)\phi
\]

and one can check using the results in appendix A.6 that this is indeed a diffeomorphism on the FLRW background.

Various other non-trivial checks were performed for the combined and the pure gauge contributions, comparing the trace \( h \) and the time component \( x_\mu x_\nu h^{\mu\nu} \) with the general formulas (5.18), (5.25a) and (A.27). All tests work out, so that we can be very confident that the above expressions for the metric fluctuations are correct.

5.5. Unphysical scalar \( \mathcal{A}^{(+)} \) modes

Among the \( \mathcal{A}^{(+)}[\phi^{(s)}] \) modes, only the scalar mode \( \mathcal{A}^{(+)}[\phi^{(0)}] \) contributes to the linearized metric. Even though it is unphysical because it does not satisfy the gauge-fixing constraint, we give its metric contribution for completeness:
This is part of the $D\tilde{A}^{(r)}$ (4.32) mode. As a check, we recover (2.43) by taking the trace.

To summarize, the $A^{(-)}[\phi^{(2)}], A^{(0)}[\phi^{(1)}], A^{(+)}[\phi^{(0)}]$ and $D\tilde{A}^{(r)}[\phi]$ modes provide all $5+3+1+1=10$ off-shell d.o.f. of the most general metric fluctuation. They lead to five physical on-shell modes comprising 2 graviton modes from $A^{(-)}[\phi^{(2)}]$, one scalar mode $A^{(-)}[D^+D\phi^{(0)}]$, and presumably 2 helicity 1 modes $A^{(-)}[\phi^{(2,1)}]$.

6. Summary and conclusions

We have studied in detail the scalar fluctuations of the FLRW quantum space-time solution $M^{3,1}$ of Yang–Mills matrix models, based on the general results in [1]. In particular, we recovered the quasi-static linearized Schwarzschild metric as a solution, which arises from the scalar sector of the physical would-be massive spin 2 modes. Quasi-static indicates that the corresponding effective mass is found to decrease slowly during the cosmic evolution.

It is very remarkable that the linearized Schwarzschild solution can be obtained within the framework of Yang–Mills matrix models, as we have shown. Along with the propagating spin 2 graviton modes found in [1], this strongly supports the claim that $3+1$-dimensional gravity can emerge from the matrix model framework without compactification, in particular for the IKKT or IIB model [2]. The mechanism is very simple in the spirit of noncommutative but almost-local field theory, by considering fluctuations around a background solution.

The present result is tied to the specific structure of the background solution, which is a twisted $S^2$ bundle over space-time, leading to a tower of higher-spin modes. It does not seem to work e.g. on simpler Moyal–Weyl type backgrounds, where the linearized modes only lead to restricted metric fluctuations, which includes some Ricci-flat metrics [17–19] but not enough.

An important issue is (local) Lorentz invariance, which is only partially manifest in the present framework. This leads to a different organization of modes in terms of the space-like $SO(3,1)$ isometry group. For example, the 5 modes of a generic spin 2 irrep decompose into $2+2+1$ modes of $\phi^{(2)}$ as in (2.38). This is best understood in space-like gauge, somewhat reminiscent of helicity modes. This structure is indicated by the name ‘would-be massive’ modes. Nevertheless, Lorentz-invariance appears to be largely respected, presumably due to the large underlying gauge invariance. In particular, the propagation of all physical modes is governed by the same effective metric.

Aside from the higher spin modes, the present model includes extra on-shell metric modes beyond those of GR. This is not surprising, since the gauge invariance of the metric sector is reduced to 3 rather than 4 diffeomorphism d.o.f. We studied in some detail the extra scalar modes, which arise from the would-be helicity zero sector of $A^{(-)}[\phi^{(2)}]$. Those are in general not Ricci-flat, but their proper treatment is quite subtle and require non-linear considerations, except (!) for the quasi-static Schwarzschild case. We propose a heuristic argument why the non-Ricci-flat modes should be suppressed at the non-linear level, somewhat reminiscent of the Vainshtein mechanism [20]. They may however play a role at very long wavelengths, in
the guise of dark matter. Similarly, there are presumably two more physical modes arising from the would-be helicity 1 gravitons, which are not studied here, and may also require the non-linear theory.

This leads us to the list of open issues and questions which need to be addressed in future work. One important step is the inclusion of matter, in order to clarify how matter acts as a source of metric deformations. This was briefly discussed in [1], but it needs to be studied in detail, and at the non-linear level in order to clarify the above mechanism. Only then a reliable assessment can be made whether a satisfactory behavior arises at the classical level, or if quantum effects such as an induced Einstein–Hilbert action are essential.

Another obvious task is to extend the present Schwarzschild solution, and more generally the full higher spin theory, to the non-linear regime as far as possible. Even though some computations in the present paper are quite involved, the basic structure of the underlying solution is very simple and based only on Lie-algebraic structures. This—along with black hole solutions in higher spin theories [41–43]—leads to the hope that an exact analytic solution can be found, not only at the semi-classical level, but also at the fully non-commutative level. These are only some of many open questions which can be studied using the tools provided here and in [1].

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Appendix

A.1. Ladder operators and eigenmodes

We provide a simpler and more conceptual derivation of the eigenmodes $A^{(±)}(4.4)$ and (4.5) found in [1]. Starting from the observation $[\Theta^{\mu\nu}, X^4] \sim i(\theta^{\mu\nu}, \tilde{x}^4) = 0$ we obtain

$$\tilde{I}(D^{±}(A_\mu)) = D^{±}(\tilde{I}(A_\mu)).$$

Together with the relations [1]

$$\Box D^+ \phi^{(i)} = D^+ \left( \Box + \frac{2s + 2}{R^2} \right) \phi^{(i)},$$

$$\Box D^- \phi^{(i)} = D^- \left( \Box - \frac{2s}{R^2} \right) \phi^{(i)}$$

we obtain

$$D^2 D^+ A^{(i)} = \left( \Box + \frac{2}{r^2 R^2} \tilde{I} \right) D^+ A^{(i)} = (D^+ (\Box + \frac{2s + 2}{R^2}) + \frac{2}{r^2 R^2} D^+ \tilde{I}) A^{(i)}$$

$$= D^+ (D^2 + \frac{2s + 2}{R^2}) A^{(i)}, \quad A^{(i)} \in C^i$$
and similarly for $D^-$. Therefore $D^\pm$ are intertwiners for $D^2$ which rise or lower the eigenvalues. Now observe

$$D^2 A(\pm) \phi(s) = D^\pm(\pm 2s + 2R^2) A(\pm) \phi(s) = D^\pm A(\pm) (\Box + \frac{2s + 2}{R^2}) \phi(s)$$

(A.5)

But this implies that $A(\pm) \phi(s) = A(\pm) (\Box + \frac{2s + 2}{R^2}) \phi(s)$ has the same intertwiner property,

$$D^2 A(\pm) \phi(s) = A(\pm) (\Box + \frac{2s + 2}{R^2}) \phi(s)$$

(A.5)

as desired. These properties originate from the underlying $so(4, 2)$ Lie algebra structure, and they should apply to the fully noncommutative case as well as the semi-classical Poisson limit. In particular, the solution $A(\pm) [D^\pm D\phi]$ underlying the Schwarzschild metric should easily generalize to the noncommutative setting.

A.2. Useful relations

From the basic commutation relations (2.3) it is easy to obtain

$$\beta^{-1} \{ x^\mu, \beta \} = -\beta \{ x^\mu, \beta^{-1} \} = r^2 \beta t^\mu$$

$$\beta^{-1} \{ t^\mu, \beta \} = -\beta \{ t^\mu, \beta^{-1} \} = \frac{1}{R^2} \beta x^\mu$$

(A.8)

$$\beta^{-1} \tau \beta = -\beta \tau \beta^{-1} = - (\beta^2 + 1) .$$

(A.9)

Furthermore, it is not hard to derive

$$\Box x^\alpha = \frac{1}{R^2} x^\alpha$$

$$\Box x^i = \frac{4}{R^2} x^i$$

(A.10)

and

$$\Box(\sinh(\eta) \phi) = \sinh(\eta) (\Box + \frac{2}{R^2} (\tau + 2)) \phi$$

$$\Box \beta \phi = \beta \Box \phi - \frac{2}{R^2} (\tau + 2) \beta \phi$$

(A.11)

$$\Box \tau \phi = \tau \Box \phi + 2 \beta (\Box + \frac{1}{R^2} \tau) \phi$$

(A.12)

for scalar functions $\phi \in C^0$. Finally, we note that (2.40) gives
\[ D(\tau + s)\phi = (\tau + s)D\phi, \]
\[ D^+ D^- \tau = \tau D^+ D^- . \]  
(A.13)

### A.3. Metric fluctuations from $A$ contributions

In this section we obtain the metric fluctuations $h^{\mu\nu}[A]$ (5.4) arising from the various terms in the tangential perturbations $A$. We will use the averaging formulas (2.42) and the on-shell relation $\Box \phi = -\frac{2}{R^2} \phi$ throughout, as well as

\[ \partial^\sigma \partial_\sigma \phi = \beta^2 (-\Box + \frac{1}{R^2}) \phi = \frac{\beta^2}{R^2} (2 + \tau) \phi \]  
(A.14)

using (2.27).

Consider first $A^\mu = \theta^{\mu\nu} \partial_\nu \phi$. Then

\[ h^{\mu\nu} = -\{x^\mu, A^\nu\}_0 + (\mu \leftrightarrow \nu) = -\{x^\mu, \theta^{\mu\nu} \partial_\nu \phi\} + (\mu \leftrightarrow \nu) \]
\[ = -\theta^{\mu\nu}[\partial_\alpha \beta_\nu] \partial_\alpha \phi - \{x^\mu, \theta^{\mu\nu}\} \partial_\alpha \phi + (\mu \leftrightarrow \nu) \]
\[ = \frac{r^2}{3} \left(-2(\beta^2 + \tau)(2 + \tau) + \frac{2\beta^2}{R^2} x^\mu x^\nu (2 + \tau) + (x^\mu \partial^\mu + x^\nu \partial^\nu) (2\tau + 1) + 2R^2 \partial^\sigma \partial^\nu \right) \phi \]  
(A.15)

using (2.27) and the on-shell condition. Next consider $A^\mu = \beta \theta^\mu \phi$. Then

\[ h^{\mu\nu} = -\{x^\mu, A^\nu\}_0 + (\mu \leftrightarrow \nu) = -\{x^\mu, \beta \theta^\nu \phi\}_0 + (\mu \leftrightarrow \nu) \]
\[ = \sinh(\eta) \theta^{\mu\nu} \beta \phi - \theta^{\mu\nu} \beta \phi + (\mu \leftrightarrow \nu) \]
\[ = \frac{2}{3} \sinh(\eta) \theta^{\mu\nu} (2 + \tau - \beta^2) \phi + \frac{2\beta^2}{3R^2} x^\mu x^\nu (2 + \tau - \beta^2) \phi . \]  
(A.16)

For $A^\mu = \beta x^\mu \theta^\nu \partial_\nu \phi = \frac{\beta}{R^2} x^\mu D\phi$, we obtain

\[ h^{\mu\nu} = -\{x^\mu, A^\nu\}_0 + (\mu \leftrightarrow \nu) = -\{x^\mu, \beta x^\nu \theta^\sigma \partial_\sigma \phi\}_0 + (\mu \leftrightarrow \nu) \]
\[ = x^\nu \partial_\nu \phi - x^\nu \{\theta^\sigma \theta^\nu \} \partial_\sigma \phi - \beta x^\nu \{\theta^\sigma \theta^\nu \} \partial_\sigma \phi + (\mu \leftrightarrow \nu) \]
\[ = -\frac{4}{3} \sinh(\eta) \theta^{\mu\nu} (1 + \tau) \phi + \frac{1}{3} (1 + \tau - \beta^2) (x^\mu \partial^\mu + x^\nu \partial^\nu) \]  
(A.17)

using (A.8). Again the trace provides some check. Finally, for $A^\mu = \sinh(\eta) x^\mu \partial_\alpha \partial_\mu \phi$ we obtain

\[ h^{\mu\nu} = -\{x^\mu, A^\nu\}_0 + (\mu \leftrightarrow \nu) = -\{x^\mu, \sinh(\eta) x^\nu \partial_\nu \phi\}_0 + (\mu \leftrightarrow \nu) \]
\[ = r^2 \{\theta^\nu \nu\}_0 \partial_\nu \phi + \sinh(\eta) \partial_\nu \phi - \sinh(\eta) \{\theta^\nu \nu\} \partial_\nu \partial_\nu \phi + (\mu \leftrightarrow \nu) \]
\[ = \frac{2}{3} \sinh(\eta) \partial_\nu \partial_\nu (2 + \tau) \phi - \frac{1}{R^2} (x^\nu \partial_\nu + x^\nu \partial_\mu) \phi - \frac{4}{3R^2} \beta^2 x^\nu x^\nu (2 + \tau) \phi \]  
(A.18)

using the on-shell relation. We also note the relations

\[ \{t^\mu, \beta \theta^\nu \partial_\nu \phi\} = t^\mu \partial_\mu (\tau + 2) \phi \]
\[ \{t^\nu, \beta x^\nu \partial_\nu \phi\} = t^\nu \partial_\nu (\tau + 3 - \beta^2) (\tau + 2) \phi \]
\[ \{t^\nu, \theta^\nu \partial_\nu \phi\} = 3 \tau t^\nu \partial_\nu (\tau + 2) \phi \]  
(A.19)

due to $t^\nu \partial_\nu \beta = 0$, which are used to check gauge invariance.
A.4. DD operator on scalar fields

Let $\phi \in \mathcal{C}^0$. The explicit formula (2.36) for $D$ gives

$$DD\phi = r^2 R^4\phi^{\mu \nu} \nabla^\mu \nabla^\nu \phi$$

$$DDDD\phi = r^2 R^4\phi^{\mu \nu} \nabla^\mu \nabla^\nu \nabla^\rho \nabla^\sigma \phi$$

where $\nabla^\mu$ is the covariant derivative along the space-like $H^3$. In particular,

$$D^- D^+ \phi = r^2 R^2 \phi^{\mu \nu} \nabla^\mu \nabla^\nu \phi = \frac{r^2 R^2}{3} \cosh^2(\eta) P_{\perp \perp} \nabla^\mu \nabla^\nu \phi$$

where $\Delta^{(3)} = -\nabla^\mu \nabla^\nu \phi^\mu \phi^\nu$ is the covariant Laplacian on $H^3$. Note that both expressions are $SO(3,1)$-invariant second order differential operators. The averaging is given in terms of the projector $P_{\perp}$ on $H^3$ in (2.41). Now we compute

$$[DDDD]_0 = r^2 R^4 \phi^{\mu \nu} \nabla^\mu \nabla^\nu \nabla^\rho \nabla^\sigma \phi$$

$$=\frac{3}{5} r^2 R^4 \left[ [\phi^{\mu \nu}] [\phi^{\rho \sigma}] | + [\phi^{\mu \rho}] [\phi^{\nu \sigma}] | + [\phi^{\mu \sigma}] [\phi^{\nu \rho}] \right] \nabla^\mu \nabla^\nu \nabla^\rho \nabla^\sigma \phi$$

$$= \cosh^4(\eta) \frac{R^4}{15} (P^{\mu \nu} P^{\rho \sigma} + P^{\mu \rho} P^{\nu \sigma} + P^{\mu \sigma} P^{\nu \rho}) \nabla^\mu \nabla^\nu \nabla^\rho \nabla^\sigma \phi$$

$$= R^4 \phi^{\mu \nu} \nabla^\mu \nabla^\nu \nabla^\rho \nabla^\sigma \phi$$

where $P^{\mu \nu} = g^{\mu \nu \alpha \beta} P_{\perp \perp}^{\alpha \beta}$ is the tangential induced metric on $H^3$ which satisfies $\nabla^\mu P^{\mu \nu} = 0$. The individual terms are given by

$$P^{\mu \nu} P^{\rho \sigma} \nabla^\mu \nabla^\nu \nabla^\rho \nabla^\sigma \phi = \Delta^{(3)} \Delta^{(3)} \phi$$

$$P^{\mu \rho} P^{\nu \sigma} \nabla^\mu \nabla^\nu \nabla^\rho \nabla^\sigma \phi = \Delta^{(3)} \Delta^{(3)} \phi + \nabla^\mu (R^{(3)}_{\mu \alpha \beta \gamma} \phi)$$

$$P^{\mu \sigma} P^{\nu \rho} \nabla^\mu \nabla^\nu \nabla^\rho \nabla^\sigma \phi$$

where

$$R^{(3)}_{\mu \nu} = \frac{1}{3} P^{\mu \nu} R^{(3)}, \quad R^{(3)} = - \frac{6}{R^2 \cosh^2(\eta)} \quad (A.24)$$

are the Ricci tensor and scalar on $H^3$. Combining these, we obtain

$$D^- D^+ D^+ D^+ \phi = [DDDD]_0 - D^- D^+ D^- D^+ \phi$$

$$= \frac{4}{15} R^4 \phi^{\mu \nu} \nabla^\mu \nabla^\nu \nabla^\rho \nabla^\sigma \phi$$

Note that this vanishes for $x^4$, consistent with $D^- D^+ D^+ D^+ x^4 = 0$. Now we apply this to on-shell solution with $(\Box + \frac{\phi}{R^2}) \phi = 0$. Then (2.30) gives

$$\cosh^2(\eta) \Delta^{(3)} \phi = - \frac{1}{R^2} (1 + \tau + \beta^2) (2 + \tau) \phi$$

(A.25)
so that
\[
\frac{1}{2} h = D^- D^- D^+ D^+ \phi \eta \sim \infty - \frac{4}{45} r^4 (1 - 3\tau - \tau^2)(\tau + 1)(\tau + 2) \phi .
\]
(A.27)
This can be used as a consistency check for the computation of the trace \( h \) in section 5.4.

**A.5. Evaluation of \( A^-(\cdot)(D^+ D\phi) \)**

To find the corresponding metric fluctuation mode, we need to elaborate the fluctuation mode \( A^-(\cdot) \) explicitly. For \( \phi \in C^0 \), we have
\[
DD\phi = r^4 R^2 \rho^3 \partial_\alpha \partial_\beta \phi - r^2 R \frac{1}{x_4} \theta^{\alpha \beta} \partial_\beta \phi
\]
(A.28)

\[
= r^4 R^2 \rho^3 \partial_\alpha \partial_\beta \phi + r^2 \tau \phi
\]

hence
\[
\{x^\mu, DD\phi\}_1 = r^4 R^2 \{x^\mu, r^4 \partial_\alpha \partial_\beta \phi\}_1 + r^2 \{x^\mu, \tau \phi\}
\]

\[
= 2 r^4 R^2 \{x^\mu, r^4 \partial_\alpha \partial_\beta \phi\}_1 + r^2 R^2 [r^4 \theta^{\alpha \beta} \partial_\mu (\partial_\nu \partial_\alpha \phi) + r^2 \theta^{\alpha \beta} \partial_\mu \tau \phi]
\]

\[
= -2 r^4 R^2 \{\mu \nu, r^4 \partial_\alpha \partial_\beta \phi\} - 2 r^4 \theta^{\alpha \beta} \partial_\mu \phi + \frac{1}{5} r^4 R^2 \cosh^2(\eta) \theta^{\alpha \beta} \partial_\mu (\partial_\nu \partial_\alpha \phi) + \frac{1}{5} r^2 \theta^{\alpha \beta} x^\mu \partial_\nu \partial_\alpha \phi
\]

\[
+ r^2 \theta^{\alpha \beta} \partial_\mu \tau \phi + \frac{2}{5} r^4 R^2 x^\nu \sinh(\eta) \left( x^\mu \partial_\nu (\partial_\alpha \partial_\beta \phi) \right)
\]
(A.29)
using the averaging formulas (2.42), (2.42b) and (2.27). The first term is pure gauge, and the last term can be rewritten as
\[
t^\mu \sinh(\eta) x^\nu \partial_\mu \partial_\nu \partial_\phi = \{\mu \nu, t^\alpha \partial_\nu (\tau - 2) \phi\} + \frac{1}{r^2 R^2} \theta^{\mu \alpha} \partial_\nu ((\tau - 2) \phi)
\]
(A.30)

using \( \tau \partial = \partial (\tau - 1) \). Further,
\[
x^\mu x^\nu \partial_\mu (\partial_\nu \partial_\beta \phi) = \partial_\beta ((\tau - 1) (\tau - 2) \phi) .
\]
(A.31)

Therefore
\[
\{x^\mu, DD\phi\}_1 = -2 r^2 \theta^{\mu \alpha} \partial_\nu \partial_\phi + \frac{1}{5} r^4 R^2 \cosh^2(\eta) \theta^{\mu \alpha} \partial_\nu (\partial_\nu \partial_\phi) + \frac{1}{5} r^2 \theta^{\mu \alpha} x^\nu \partial_\nu (\partial_\alpha \partial_\beta \phi)
\]

\[
+ \frac{2}{5} r^4 R^2 x^\nu \sinh(\eta) t^\nu (\partial_\alpha (\partial_\beta \phi) - \phi)
\]

\[
+ \frac{1}{5} r^4 R^2 \frac{1}{x_4} \theta^{\mu \alpha} (\partial_\nu (\partial_\beta \phi) - 1 \partial_\nu \phi)
\]

\[
= \frac{2}{5} r^4 R^2 \beta^2 \frac{1}{x_4} \sinh(\eta) t^\nu (\partial_\nu (\partial_\beta \phi) - \phi)
\]

\[
= \frac{2}{5} r^4 R^2 x^\nu \frac{1}{x_4} t^\nu (\partial_\nu (\partial_\beta \phi) - 1 \partial_\nu \phi)
\]

\[
+ \frac{1}{5} r^4 R^2 \frac{1}{x_4} \theta^{\mu \alpha} (\partial_\nu (\partial_\beta \phi) - \phi)
\]
(A.32)

using
\[
\theta^{\mu \alpha} \partial_\nu (\partial_\beta \phi) = 2 \beta^3 r^4 R^2
\]
(A.33)
where
\[ \Lambda = -\frac{2}{5} R^2 D ((\tau + 3) \phi). \]  

(A.34)

**A.5.1. Degenerate case.** In the special case \((\Box - \frac{1}{R^2} \tau) \phi = 0\), we obtain
\[
\{ x^\mu, D D \phi \}_1 = \frac{r^2}{5} \theta^{\mu \gamma} \partial_\gamma (\tau^2 - 4) \phi + \{ t_\mu, \Lambda \} = \frac{r^2}{5} \{ x^\mu, (\tau^2 - 4) \phi \} + \{ t_\mu, \Lambda \}
\]  

(A.35)
i.e. there is a linear dependence between the \( A^{(\pm)} \) modes. Imposing also the on-shell condition would imply \((2 + \tau) \phi = 0\). We will see that then \( A^{(\pm)} | D^+ D \phi | \) vanishes, but a non-trivial mode can be extracted by taking a suitable limit, which corresponds precisely to the Schwarzschild solution.

**A.5.2. On-shell condition.** Now consider on-shell solutions, so that \( \Box \phi = -\frac{2}{R^2} \phi \). Then (A.32) becomes
\[
\{ x^\mu, D D \phi \}_1 = \frac{2}{5} r^4 \beta (1 + \beta^2) t^\mu (2 + \tau) \phi + \frac{1}{5} r^2 (1 + \beta^2) \theta^{\mu \gamma} \partial_\gamma (2 + \tau) \phi
\]  
\[
+ \frac{2}{5} r^4 \beta x^\mu t^\alpha \partial_\alpha (2 + \tau) \phi + \frac{r^2}{5} \theta^{\mu \gamma} \partial_\gamma (\tau^2 - 4) \phi + \{ t_\mu, \Lambda \}.
\]  

(A.36)

But in fact we need
\[
A^{(\pm)} [ D^+ D^+ \phi ] = \{ x_\mu, D^+ D^+ \phi \}_1 = A^{(\pm)} [ D^- D^- \phi ]
\]  

(A.37)

where
\[
A^{(\pm)} [ D^- D^- \phi ] = \theta^{\mu \nu} \partial_\nu (D^- D^+ \phi) = \frac{r^2}{3} \theta^{\mu \nu} \partial_\nu (\beta^2 + \tau + 1) (2 + \tau) \phi
\]  

(A.38)
on-shell, using (A.21) and (A.26). Combining with the above and using
\[
-\frac{r^2}{5} \theta^{\mu \gamma} (\partial_\gamma \beta^2) (\tau + 2) \phi = -\frac{2}{5} r^4 \beta^3 (\tau + 2) \phi
\]  

(A.39)
one finds the on-shell form
\[
A^{(\pm)} [ D^+ D^+ \phi ] = \frac{2 r^4}{3} \left( \beta (t^\mu + x^\mu t^\rho \partial_\rho) - \frac{1}{5 r^2} \theta^{\mu \gamma} \partial_\gamma (\tau + 4 + \beta^2) (\tau + 2) \phi + \{ t_\mu, \Lambda \}. 
\]  

(A.40)

**A.6. Background FLRW geometry and covariant derivatives**

The effective FLWR metric (2.20) is conformally flat,
\[
G^{\mu \nu} = \beta \eta^{\mu \nu}, \quad \beta = \frac{1}{\sinh(\eta)}.
\]  

(A.41)

Then the Christoffel symbols in the Cartesian coordinates \( x^\mu \) are
\[
\Gamma^\rho_{\mu \nu} = -\frac{1}{2 \chi^4} (\delta^\rho_{\mu \alpha} x^\alpha + \delta^\rho_{\nu \alpha} x^\alpha - \eta_{\mu \nu} x^\rho)
\]  
\[
= -\frac{1}{2 \chi^4} \beta^3 (\delta^\rho_{\mu \alpha} G^{\alpha \rho} + \delta^\rho_{\nu \alpha} x^\alpha - G^{\mu \nu} \Lambda^\rho)
\]  

(A.42)
so that
\[ \Gamma^\rho = G^{\mu\nu}\Gamma^\rho_{\mu\nu} = \frac{R}{x_4} \delta^\rho, \quad \Gamma^\mu_{\mu\nu} = -\frac{1}{2x_4^2} \eta_{\mu\alpha} \chi^\alpha \] (A.43)

using (A.9). Note that $\Gamma^\mu_{\mu\nu}$ is suppressed by the cosmic curvature scale. For example, the pure gauge metric perturbations arising from diffeomorphisms generated by $\xi^\mu$ are given by
\[ \delta_\xi G^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu = \partial^\mu \xi^\nu + \partial^\nu \xi^\mu = -\frac{1}{x_4^2} G^{\mu\nu} x \cdot \xi . \] (A.44)

As an application, the divergence of a vector field can be expressed as follows
\[ \nabla_\mu A^\mu = \partial_\mu A^\mu + \Gamma^\mu_{\mu\nu} A^\nu = \partial_\mu A^\mu - \frac{1}{2x_4^2} \chi^\alpha \eta_{\mu\nu} A^\nu . \] (A.45)

### A.6.1 Diffeomorphisms and standard form on the FRW background.

The terms $(x^\mu \partial^\nu + x^\nu \partial^\mu) \phi$ and $\partial^\mu \partial^\nu \phi$ in the expression (5.35) for $\tilde h^{\mu\nu}$ can be eliminated by a suitable diffeomorphism. Since $(x^\mu \partial^\nu + x^\nu \partial^\mu) \phi$ becomes large at late times, one must be careful to use the proper covariant derivatives. For example, consider the following vector fields on the FRW background
\[ \xi^\mu = x^\mu \beta \phi . \] (A.46)

Then
\[ \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu = G^{\mu\nu} \partial_\mu (x^\nu \beta \phi) + (\mu \leftrightarrow \nu) - \frac{1}{x_4^2} \beta G^{\mu\nu} x \cdot x \phi \]
\[ = \left( 2 + \frac{\cosh^2}{\sinh^2} \right) \beta \phi G^{\mu\nu} + (x^\nu G^{\mu\nu} \partial_\mu \beta + ...) \phi + \beta (x^\nu G^{\mu\nu} \partial_\mu + ...) \phi \]
\[ \eta \to \infty = \beta^2 \left( 3\phi^\mu + 2x^\mu x^2 \beta^2 \phi + (x^\nu \eta^\mu \partial_\alpha x^\mu \eta^\nu \partial_\alpha \phi) \right) . \] (A.47)

Hence
\[ \beta^2 (x^\nu \eta^\mu \partial_\alpha x^\mu \eta^\nu \partial_\alpha \phi) \sim -\beta^2 \left( 3\phi^\mu + 2x^\mu x^2 \beta^2 \phi \right) . \] (A.48)

where $\sim$ indicates equivalence up to diffeos.

Next, consider the following vector fields
\[ \xi^\mu = x^\mu \phi . \] (A.49)

Then
\[ \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu = \partial^\mu (x^\nu \phi) + (\mu \leftrightarrow \nu) - \frac{1}{x_4^2} G^{\mu\nu} x \cdot x \phi \]
\[ \eta \to \infty = \beta \left( 3\phi^\mu + (x^\nu \eta^\mu \partial_\alpha x^\mu \eta^\nu \partial_\alpha \phi) \right) \] (A.50)

hence
\[ \beta (x^\mu \partial^\nu + x^\nu \partial^\mu) \phi \sim -3\beta \phi . \] (A.51)
Finally, consider
\[ \xi^\mu = \beta^{-1} \eta^{\mu \nu} \partial_\nu \phi . \]  
(A.52)

Then
\[ \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu = \partial^\mu (\beta^{-1} \eta^{\nu \alpha} \partial_\alpha \phi) + (\mu \leftrightarrow \nu) - \frac{1}{x_4^2} \beta^{-1} G^{\mu \nu} \tau \phi \]
\[ = 2 \eta^{\mu \nu} \eta^{\nu \rho} \partial_\rho \phi - \frac{1}{R^2} \beta^2 (x^\mu \eta^{\nu \rho} \partial_\rho \phi + x^\nu \eta^{\mu \rho} \partial_\rho \phi) \phi - \frac{1}{x_4^2} \eta^{\mu \nu} \tau \phi . \]  
(A.53)

The second term can be rewritten using (A.48), and therefore
\[ R^2 \eta^{\mu \nu} \eta^{\nu \rho} \partial_\rho \partial_\mu \phi \sim - \beta^2 \left( \frac{1}{2} (3 - \tau) \eta^{\mu \nu} + x^\alpha \lambda^\mu \frac{\beta^2}{R^2} \right) \phi . \]  
(A.54)

One can check with these results that the pure gauge contribution (5.47) is indeed a diffeomorphism.

**A.7. Massless scalar fields** \((\Box + \frac{2}{R^2}) \phi = 0\)

Using (2.32), the on-shell relation can be written for rotationally invariant \(\phi(\eta, \chi)\) in the form
\[ 0 = (\Box + \frac{2}{R^2}) \phi = \sinh(\eta)^3 \Box \phi + \frac{2}{R^2} \phi \]
\[ = \frac{\tanh^2(\eta)}{R^2} \left( \frac{1}{\sinh^2(\eta) \cosh(\eta)} \partial_\eta \left( \cosh^3(\eta) \partial_\eta \phi \right) + 2 \frac{\cosh^2(\eta)}{\sinh^2(\eta)} \right) + R^2 \cosh^2(\eta) \Delta^{(3)} \phi . \]  
(A.55)

We make a separation ansatz
\[ \phi(\eta, \chi) = f(\eta) g(\chi) . \]  
(A.56)

Then the eom becomes
\[ \frac{1}{\sinh(\eta)^3 \cosh(\eta)} \frac{1}{f} \partial_\eta \left( \cosh^3(\eta) \partial_\eta f \right) + 2 \frac{\cosh^2(\eta)}{\sinh^2(\eta)} = - R^2 \cosh^2(\eta) \Delta^{(3)} g . \]  
(A.57)

The factor \( \cosh^2(\eta) \) in front of \( \Delta^{(3)} \) drops out, see (2.33), which leads to two equations
\[ - R^2 \cosh^2(\eta) \Delta^{(3)} g = c g \]  
(A.58)

\[ \frac{1}{\sinh(\eta)^3 \cosh(\eta)} \partial_\eta \left( \cosh^3(\eta) \partial_\eta f \right) + 2 \frac{\cosh^2(\eta)}{\sinh^2(\eta)} = c f \]  
(A.59)

where \( c = \text{const} \).

**A.7.1. Space-like harmonics.** Consider first the space-like equation (A.58). For rotationally invariant functions \(\phi(\chi)\), this reduces using (2.33) to
\[ \frac{1}{\sinh^2(\chi)} \partial_\chi \left( \sinh^2(\chi) \partial_\chi g \right) = c g . \]  
(A.60)
The general solution is
\[ g(\chi) = \frac{c_1 e^{-\sqrt{1+c}\chi} + c_2 e^{\sqrt{1+c}\chi}}{\sinh(\chi)}. \]  
(A.61)

For \((1 + c) > 0\), there is at least one solution which is decreasing for \(\chi \to \infty\). For \((1 + c) < 0\), the solutions are oscillating in radial direction.

### A.7.2. Time dependence.

The second equation (A.59) is
\[ \frac{\cosh^2(\eta)}{\sinh(\eta)} f'' + \frac{3}{\sinh(\eta)} f' + \frac{2 \cosh^2(\eta)}{\sinh^2(\eta)} f - cf = 0. \]  
(A.62)

Asymptotically, this is
\[ e^{-3\eta} \partial_\eta (e^{3\eta} \partial_\eta f) = (-2 + c)f \]
\( (\partial^2_\eta + 3 \partial_\eta + 2 - c)f = 0 \)  
(A.63)

which is solved by \( f = e^{\lambda \eta} \) with
\[ \lambda^2 + 3\lambda + 2 - c = 0 \]
\[ \lambda_{1,2} = \frac{1}{2} (-3 \pm \sqrt{1 + 4c}). \]  
(A.64)

The most interesting quasi-static Schwarzschild solution arises for \( \tau \phi = -2\phi \), which corresponds to \( \Delta^{(3)} \phi = 0 \) via (2.30) hence to \( c = 0 \). Then (A.61) and (A.62) have the exact solutions
\[ g(\chi) = \frac{e^{-\chi}}{\sinh(\chi)}, \quad f(\eta) = \frac{1}{\cosh^2(\eta)} \sim e^{-2\eta}, \]  
(A.65)

where \( \rho = \sinh(\chi) \) (2.19) is the appropriate distance variable on \( H^3 \). Thus \( \phi = f(\eta)g(\chi) \) exhibits the typical \( \frac{1}{\rho} \) behavior of the harmonic Newton potential in three dimensions, with time dependence given by \( f(\eta) \sim e^{-2\eta} \). Note that \( \phi(\eta) \) remains finite for \( \eta \to 0 \), so that the Schwarzschild solution does not blow up at any time. For \( c < -\frac{1}{4} \), this will lead to propagating scalar modes.

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