THE RANGE OF UNITED K-THEORY

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Abstract. We prove that the united K-theory functor is a surjective functor from the category of real simple purely infinite C*-algebras to the category of countable acyclic CRT-modules.

1. Introduction

In this paper, we further investigate the united K-theory functor for real C*-algebras, developed in [2]. We will show that united K-theory is a surjective function from the category of real simple purely infinite C*-algebras to the category of countable acyclic CRT-modules, providing one part of a classification-type theorem for real C*-algebra along the lines of that of Kirchberg [12] and of Phillips [14]. Of course the other part, injectivity, is the more difficult aspect to such a classification program, but the universal coefficient theorem of [3] described below and the surjectivity result of the present paper give us confidence that such a classification theorem for real C*-algebras is possible.

The immediate purpose of the development of united K-theory was to state and prove a Künneth-type formula for the tensor product to two real C*-algebra (in [2]). Later, in [3], we proved a universal coefficient theorem (UCT) for united K-theory (more on this later). Recall that for a real C*-algebra \( A \), united K-theory \( K^{\text{CRT}}(A) \) consists of three graded modules and the collection of natural transformations between them. The three objects are

1. real K-theory \( K^O(A) \) — defined to be the K-theory of the real C*-algebra \( A \) as discussed for example in [19].
2. complex K-theory \( KU(A) \) — defined to be the K-theory of the complexification \( A_c = \mathbb{C} \otimes A \).
3. self-conjugate K-theory \( KT(A) \) — defined to be the K-theory of \( T \otimes A = \{ f : [0,1] \to \mathbb{C} \otimes A \mid f(0) = \overline{f(1)} \} \).

These objects are taken not just as graded groups, but as graded modules over the graded unital rings \( K^O(\mathbb{R}) \), \( KU(\mathbb{R}) \), and \( KT(\mathbb{R}) \) respectively; which are displayed in degrees 0 through 8 by

\[
K^O(\mathbb{R}) = \mathbb{Z} \quad \mathbb{Z}_2 \quad \mathbb{Z}_2 \quad 0 \quad \mathbb{Z} \quad 0 \quad 0 \quad 0 \quad \mathbb{Z}
\]

\[
K^U(\mathbb{R}) = \mathbb{Z} \quad 0 \quad \mathbb{Z} \quad 0 \quad \mathbb{Z} \quad 0 \quad \mathbb{Z} \quad 0 \quad \mathbb{Z}
\]

\[
K^T(\mathbb{R}) = \mathbb{Z} \quad \mathbb{Z}_2 \quad 0 \quad \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z}_2 \quad 0 \quad \mathbb{Z} \quad \mathbb{Z}
\]

(see page 23 in [19] and Tables 1, 2, and 3 in [2]). The generators are the elements \( 1_o \in K^O_0(\mathbb{R}) \), \( \eta_o \in K^O_1(\mathbb{R}) \), \( \eta_0^2 \in K^O_2(\mathbb{R}) \), \( \xi \in K^U_4(\mathbb{R}) \), and the invertible element \( \beta_o \in K^T_8(\mathbb{R}) \). The ring \( K^O(\mathbb{C}) \) is the free unital polynomial ring generated by the invertible Bott element \( \beta^U \in K^T_2(\mathbb{C}) \). The ring \( K^O(T) \) has generators \( 1_r \) in degree 0, \( \eta_r \) in degree 1, \( \omega \) in degree 3, and the invertible element \( \beta_r \) is degree 4. Thus, \( K^O(A) \) has period 8, \( KU(A) \) has period 2, and \( KT(A) \) has period 4. United

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The other operations, see Sections 1.1 and 1.2 of [2].

In [5], An abstract CRT-module is a triple \( M = (M^O, M^U, M^T) \) consisting of graded modules over \( KO_*(\mathbb{R}), KU_*(\mathbb{R}), \) and \( KT_*(\mathbb{R}) \) respectively. Furthermore there must be \( KO_*(\mathbb{R}) \)-module homomorphisms \( r, c, \varepsilon, \zeta, \psi_U, \psi_T, \gamma \) and \( \tau \) which satisfy the relations

\[

cr = 2 \quad \psi_U \beta_U = -\beta_U \psi_U \quad \xi = r \beta_U^2 c \\
cr = 1 + \psi_U \quad \psi_T \beta_T = \beta_T \psi_T \quad \omega = \beta_T \gamma \\
r = \tau \gamma \quad \varepsilon \beta_O = \beta_T^2 \varepsilon \quad \beta_T \varepsilon \tau = \varepsilon \beta_T + \eta_T \beta_T \\
c = \zeta \varepsilon \quad \zeta \beta_T = \beta_T^2 \zeta \quad \varepsilon \tau^2 = 1 + \psi_T \\
(\psi_U)^2 = 1 \quad \gamma \beta_T^2 = \beta_T \gamma \quad \gamma \varepsilon \tau = 1 - \psi_T \\
(\psi_T)^2 = 1 \quad \tau \beta_T^2 = \beta_T \tau \quad \tau = -\tau \psi_T \\
\psi_T \varepsilon = \varepsilon \quad \gamma = \gamma \psi_U \quad \tau \beta_T \varepsilon = 0 \\
\zeta \gamma = 0 \quad \eta_O = \tau \varepsilon \quad \varepsilon \xi = 2 \beta_T \varepsilon \\
\zeta = \psi_U \zeta \quad \eta_T = \gamma \beta_T \zeta \quad \xi \tau = 2 \tau \beta_T
\]

as in Section 1.9 of [5]. These relations are satisfied by united \( K \)-theory by Proposition 1.7 of [2].

Not every abstract CRT-module \( M \) can be realized as the united \( K \)-theory of a real \( C^* \)-algebra. According to Theorem 1.18 of [2] a necessary condition is that \( M \) be acyclic, i.e. the following complexes must be exact:

\[
\cdots \to M^U_{n+1} \to M^T_n \to M^U_n \stackrel{1-\psi_T}{\to} M^U_n \to \cdots \\
\cdots \to M^O_n \to M^O_{n+1} \to M^U_{n+1} \stackrel{\gamma}{\to} M^T_{n+1} \to M^O_{n+1} \to \cdots \\
\cdots \to M^O_n \to M^O_{n+2} \to M^T_{n+2} \stackrel{\tau \beta_T}{\to} M^O_{n+2} \to \cdots
\]

Our main theorem is that every countable acyclic CRT-module can be realized as the united \( K \)-theory of a real \( C^* \)-algebra. Furthermore, the real \( C^* \)-algebra can be taken to be simple and purely infinite. Recall that a complex \( C^* \)-algebra is said to be a Kirchberg algebra if it is separable, nuclear, simple, and purely infinite. We say that a real \( C^* \)-algebra \( A \) is a Kirchberg algebra if the complexification \( A_c \) is a Kirchberg algebra.

**Theorem 1.1.**

1. Let \( M \) be any countable acyclic CRT-module. Then there exists a real stable Kirchberg algebra \( A \) such that \( K^{CRT}(A) \cong M \) and \( A_c \) satisfies the UCT.

2. Let \( M \) be any countable acyclic CRT-module and let \( m \) be any element of \( M^O_0 \) (that is, \( m \) is a degree zero element in the real part of \( M \)). Then there exists a real unital Kirchberg algebra \( A \) such that \( (K^{CRT}(A), [1_A]) \cong (M, m) \) and \( A_c \) satisfies the UCT.
In [3], we developed united KK-theory (generalizing united K-theory in the sense that \( KK^{CRT}(\mathbb{R}, A) \cong K^{CRT}(A) \) for a real \( \sigma \)-unital C*-algebra \( A \)) allowing us to state and prove a Universal Coefficient Theorem for real C*-algebras. One of the important corollaries highlighting the strength of this theory says that two separable C*-algebras \( A \) and \( B \) such that \( A_\mathbb{C} \) and \( B_\mathbb{C} \) are in the bootstrap category are KK-equivalent (in the real sense) if and only if \( K^{CRT}(A) \) and \( K^{CRT}(B) \) are isomorphic CRT-modules. With Theorem 1.1 this implies there is an equivalence between the category of countable acyclic CRT-modules and the category of KK-equivalence classes of real Kirchberg algebras whose complexification satisfies the UCT. We believe that this result is a strong indication that united \( K \)-theory should play the same role for real C*-algebras that complex \( K \)-theory plays for complex C*-algebras, especially for any classification theorems for real C*-algebras.

Neither real \( K \)-theory nor complex \( K \)-theory by itself can do the job of united \( K \)-theory. In fact, in [2], we showed that the two tensor products of real Cuntz algebras \( O_3^R \otimes O_3^R \) and \( O_3^R \otimes O_5^R \) are nonisomorphic although their complexifications are isomorphic. Many more such examples can be obtained by applying Theorem 1.1 (using, for example, the \( CRT \)-modules \( M_i = \Sigma^i K^{CRT}(\mathbb{R}) \) for \( i = 0, 2, 4, 6 \)). Hence complex \( K \)-theory by itself is not sufficient to classify real simple purely infinite C*-algebras. Neither is real \( K \)-theory by itself sufficient, as we will see from Theorem 5.1.

In her dissertation [11], Beatrice Hewitt showed that acyclic \( CRT \)-modules can be classified in terms of their cores, which contains only the complex part and the image of \( \eta_O \) in the real part (and some natural transformations). Thus the self-conjugate part of united \( K \)-theory is strictly unnecessary. However, we know of no way to express \( CRT \) tensor product or \( Hom \) functors in terms of just the cores. So for purposes of the Künneth formula and the universal coefficient theorem, it is still necessary to work with the full united \( K \)-theory.

In our main theorem, we specified the properties that \( A_\mathbb{C} \) must satisfy, rather than \( A \). This is partly because it is easier to verify properties in the more familiar setting of complex C*-algebras. Nothing is lost by this approach since it is usually the case that when \( A_\mathbb{C} \) satisfies a certain property, the corresponding property is satisfied by \( A \).

For example, any real C*-algebra is simple if its complexification is simple. Indeed, if \( I \) is a closed ideal in \( A \), then \( I_\mathbb{C} \) is a closed ideal in \( A_\mathbb{C} \). The converse is not true in general. For example, the algebra \( C \otimes C \) is a simple C*-algebra whose complexification \( C \otimes C \cong C \oplus C \) is not simple. In fact, the complexification \( A_\mathbb{C} \) is simple if and only if \( A \) is simple and \( A \) is not itself isomorphic to the complexification of a real C*-algebra.

Following the definition in [20], a real C*-algebra \( A \) is purely infinite if each hereditary subalgebra of the form \( xA_\mathbb{C}x \) for a nonzero positive element \( x \) contains an infinite projection. Theorem 3.3 of [20] states that \( A \) is purely infinite if \( A_\mathbb{C} \) is purely infinite. The converse is still an open question, although there is a partial result in Section 4 of [20].

We know of no investigations into an intrinsic notion of nuclearity for real C*-algebras. For our purposes in this paper, we define a real C*-algebra to be nuclear if and only if its complexification is nuclear. At any rate, this is enough to imply that tensor products of real C*-algebras are unique if one of the factors is nuclear.

A real C*-algebra \( A \) is said to satisfy the UCT if there is an exact sequence of \( CRT \)-modules

\[
0 \to \text{Ext}_{CRT}(K^{CRT}(A), K^{CRT}(B)) \xrightarrow{\sim} KK^{CRT}(A, B) \xrightarrow{\gamma} \text{Hom}_{CRT}(K^{CRT}(A), K^{CRT}(B)) \to 0
\]

for all real separable C*-algebras \( B \). According to Theorem 1.1 of [3], a real C*-algebra \( A \) satisfies the UCT for united \( K \)-theory if \( A_\mathbb{C} \) is in the bootstrap category \( \mathcal{N} \). More generally, the proof in that paper shows that if \( A \) is any real C*-algebra such that \( A_\mathbb{C} \) satisfies the complex UCT for any complex separable C*-algebra \( B \), then \( A \) satisfies the UCT.
A partial converse is true since the complex part of the UCT exact sequence for united $K$-theory above is the same as the complex UCT exact sequence for the complex C*-algebras $A_C$ and $B_C$. Thus if a real C*-algebra $A$ satisfies the UCT for all real separable C*-algebras $B$, then the complexification $A_C$ will satisfy the UCT for all complex separable C*-algebras of the form $B_C$ for some real C*-algebra $B$. However, as shown in [15], not every complex C*-algebra is the complexification of a real C*-algebra.

The proof of the main theorem takes place through a series of approximating steps. In Section 2, we first show how to obtain a real separable C*-algebra whose united $K$-theory is isomorphic to the prescribed CRT-module. In Section 3, we show how to modify this algebra to form a real unital C*-algebra with the same $K$-theory. Finally in Section 4, we use a real version of Kumjian’s construction to make our algebra simple and purely infinite, allowing us to complete the proof of the main theorem.

In this paper, we will make frequent use of the following important theorem, which is implicit in [2] and we state here for convenience. It is an immediate consequence of the results of Section 2.3 of [5] (restated as Propositions 1.14 and 1.15 of [2]) and Theorem 1.12 in [2].

**Theorem 1.2.**

1. Let $A$ be a real C*-algebra. If one of the three graded modules $KO_*(A)$, $KU_*(A)$, and $KT_*(A)$ is trivial, then all three are trivial.

2. Let $f: A \to B$ be a homomorphism of real C*-algebras. If one of the three graded homomorphisms $f_*: KO_*(A) \to KO_*(B)$, $f_*: KU_*(A) \to KU_*(B)$, and $f_*: KT_*(A) \to KT_*(B)$ is an isomorphism, then all three are isomorphisms.

2. The First Construction

For any acyclic CRT-module $M$, there is according to Theorem 2.9 of [8] a topological spectrum $E$ such that $K^{CRT}(E) \cong M$. It isn’t known in general whether $E$ can be taken to be a actual topological space; however, by Theorem 11.1 of [5], it is possible to find a CW-complex $X$ such that $K^{CRT}(X) \cong M$ if $M$ is finitely generated. In this section, we prove the following theorem which only requires that $M$ be countable, but leaves the commutative setting far behind.

**Theorem 2.1.** Let $M$ be a countable acyclic CRT-module. Then there is a real separable nuclear C*-algebra $A$ satisfying the UCT such that $K^{CRT}(A) \cong M$.

First we establish some preliminary notation. Given a real C*-algebra $A$ we define the suspension by $SA = C_0(\mathbb{R}, A)$ and the desuspension by

$$S^{-1}A = \{ f \in C_0(\mathbb{R}, \mathbb{C} \otimes A) \mid f(-x) = \overline{f(x)} \}.$$  

This nomenclature is justified by the result that $SS^{-1}\mathbb{R}$ (and $S^{-1}S\mathbb{R}$) is KK-equivalent to $\mathbb{R}$ (Proposition 1.20 of [2]). More generally, we define

$$S^nA = \begin{cases} SS\ldots S A & \text{if } n \geq 0 \\ S^{-1}S^{-1}\ldots S^{-1}A & \text{if } n < 0 \end{cases}.$$  

Let $\iota$ represent the orientation-reversing involution of $SA$, which induces multiplication by $-1$ on $K$-theory.

Recall from Section 2.1 of [2] that $K^{CRT}(\mathbb{R})$, $K^{CRT}(\mathbb{C})$, and $K^{CRT}(T)$ are free CRT-modules. The CRT-module $K^{CRT}(\mathbb{R})$ is generated by $1_D$, the class of the identity in $KO_0(\mathbb{R})$; the element

$$1_D = \begin{cases} SS\ldots S A & \text{if } n \geq 0 \\ S^{-1}S^{-1}\ldots S^{-1}A & \text{if } n < 0 \end{cases}.$$  

Let $\iota$ represent the orientation-reversing involution of $SA$, which induces multiplication by $-1$ on $K$-theory.
\(\kappa_1 \in KU_0(\mathbb{C})\) generates \(K^{\text{CRT}}(\mathbb{C})\) as a CRT-module and satisfies \(r(\kappa_1) = 1_U \in KO_0(\mathbb{C})\); and the element \(\chi \in KT_{-1}(T)\) generates \(K^{\text{CRT}}(T)\) and satisfies \(\tau(\chi) = 1_T \in KO_0(T)\).

**Lemma 2.2.** Let \(B\) be any real unital C*-algebra and let \(x \in KO_0(B)\). Then there is a positive integer \(n\) and a C*-algebra homomorphism \(\alpha : \mathbb{R} \to M_n SB\) such that \(\alpha_s(1_O) = x\).

Note that we are making use of the identifications \(KO_0(\mathbb{R}) = KO_{-1}(\mathbb{R})\) and \(KO_0(B) = KO_{-1}(SB)\), claiming that \(\alpha_\ast : KO_{-1}(\mathbb{R}) \to KO_{-1}(SB)\) sends \(1_O\) to \(x\).

**Proof.** Let \(x = [p_1] - [p_2]\) where \(p_i\) is a projection in \(M_{n_i}(B)\) for \(i = 1, 2\). First define \(\alpha_i : \mathbb{R} \to M_{n_i} B\) by \(\alpha_i(t) = tp_i\). Then let \(n = n_1 + n_2\) and define \(\alpha = S\alpha_1 \oplus (S\alpha_2 \circ \iota)\). Then \(\alpha_s(1_O) = [p_1] + t_s[p_2] = [p_1] - [p_2] = x\). \(\square\)

**Lemma 2.3.** Let \(B\) be any real unital C*-algebra and let \(y \in KU_0(B)\). Then there is a positive integer \(n\) and a C*-algebra homomorphism \(\alpha : SC \to M_n S^{-1} B\) such that \(\alpha_s(\kappa_1) = \beta y^{-1} y\).

**Proof.** Consider the unital inclusion \(c : \mathbb{R} \to C\). We apply the mapping cone construction as in the proof of Theorem 1.18 of [2] to obtain a C*-algebra homomorphism \(\nu : SC \to C\). In that proof, we found that the mapping cone \(C\) is homotopy equivalent to \(S^{-1}\) and we proved that the element of \(KK_{-2}(C, \mathbb{R})\) represented by \(\nu\) is \(\pm r\beta^{-1}_{y}\). If the sign is negative, replace \(\nu\) by \(\nu \circ \iota\) to make it positive.

Let \(y = [p_1] - [p_2]\) where each \(p_i\) is a projection in \(M_{n_i} C \otimes B\). Define a C*-algebra homomorphism \(\rho_i : C \to M_{n_i} C \otimes B\) by \(\rho_i(t) = tp_i\) for all \(t \in C\). The composition \(h_i = \nu \circ S\rho_i\) defines a homomorphism from \(SC\) to \(M_{n_i} S^{-1} B\) such that \((h_i)_*(1_U) = r\beta^{-1}_{y}[p_i]\). Let \(n = n_1 + n_2\) and define \(h = h_1 \oplus (h_2 \circ \iota)\) from \(SC\) to \(M_{n} S^{-1} B\), so that \(h_s(1_U) = r\beta^{-1}_{y}(y)\). Then \(rh_s(\kappa_1) = h_s r(\kappa_1) = h_s(1_U) = r\beta^{-1}_{y}(y)\). Since \(proof = \text{image} \beta^{-1}_{y} c(\text{Theorem 1.18 of [2]}),\) there is an element \(x \in KO_1(S^{-1} B)\) such that \(h_s(\kappa_1) = \beta^{-1}_{y}(y) + \beta^{-1}_{y} c(x)\).

To correct the error, let \(x = [q_1] - [q_2]\) where \(q_i\) is a projection in \(M_{m_i} B\) for \(i = 1, 2\). Define \(\mu_i : \mathbb{R} \to M_{m_i} B\) by \(\mu_i(t) = t_q\) and then define \(j_i = S^{-1}_i \mu_i \circ \iota\). We patch together these two homomorphisms by letting \(l = m + n\) and defining \(\alpha = h \oplus j\) from \(SC\) to \(M_l S^{-1} B\) so that \(\alpha_s(\kappa_1) = \beta^{-1}_{y} y\). \(\square\)

**Lemma 2.4.** Let \(B\) be any real unital C*-algebra and let \(z \in KU_0(B)\). Then there is a positive integer \(n\) and a C*-algebra homomorphism \(\alpha : ST \to M_n S^{-2} B\) such that \(\alpha_s(\chi) = \beta^{-1}_{z} z\).

**Proof.** The mapping cone of the unital inclusion \(\varepsilon : \mathbb{R} \to T\) is homotopy equivalent to \(S^{-2}\) (as in the proof of Theorem 1.18 of [2]). Thus we obtain a C*-algebra homomorphism \(\sigma : ST \to S^{-2}\). Also in the proof of Theorem 1.18, we proved that the element of \(KK_{-3}(T, \mathbb{R})\) represented by \(\sigma\) is \(\pm r\beta^{-1}_{z}\). If the sign is negative, replace \(\sigma\) by \(\sigma \circ \iota\) to make it positive.

Let \(z = [p_1] - [p_2]\) where \(p_i\) is a projection in \(M_{n_i} T \otimes B\) for \(i = 1, 2\). Since \(T\) is commutative, there is a C*-algebra homomorphism \(\rho_i : T \to M_{n_i} T \otimes B\) defined by \(\rho_i(t) = tp_i\) for all \(t \in T\). The composition \(h_i = \sigma \circ S\rho_i\) defines a homomorphism from \(ST\) to \(M_{n_i} S^{-2} B\) such that \((h_i)_*(1_T) = r\beta^{-1}_{z}[p_i]\). Let \(n = n_1 + n_2\) and define \(h = h_1 \oplus (h_2 \circ \iota)\) from \(ST\) to \(M_{n} S^{-2} B\) so that \(h_s(1_T) = r\beta^{-1}_{z}(z)\). Then \(rh_s(\chi) = h_s r(\chi) = h_s(1_T) = r\beta^{-1}_{z}(z)\). Since \(\ker\tau = \text{image} \beta^{-1}_{z} \varepsilon\) (Theorem 1.18 of [2]), there is an element \(x \in KO_2(S^{-2} B) = KO_0(B)\) such that \(h_s(\chi) = \beta^{-1}_{z}(z) + \beta^{-1}_{z} \varepsilon(x)\).

To correct the error, let \(x = [q_1] - [q_2]\) where \(q_i\) is a projection in \(M_{m_i} B\) for \(i = 1, 2\). Define \(\mu_i : \mathbb{R} \to M_{m_i} B\) by \(\mu_i(t) = t_q\) and then define \(j_i = S^{-2}_i \mu_i \circ \sigma\). Let \(m = m_1 + m_2\) and define
Let $j: ST \to M_nS^{-2}B$ by $j = (j_1 \circ i) \oplus j_2$. Since $\beta_I \sigma_*(\chi) = \sigma_*(\beta_I(\chi)) = \tau(\chi) = 1_T$, we have $\sigma_*(\chi) = \beta^{-1}_T(1_T)$ in $KT^{-4}(\mathbb{R})$. Thus
\[
j_*(\chi) = ((\mu_2)_* - (\mu_1)_*)\beta^{-1}_T\varepsilon(1_0) = -\beta^{-1}_T \varepsilon((\mu_1)_* - (\mu_2)_*)(1_0) = -\beta^{-1}_T \varepsilon(x).
\]
We patch these two homomorphisms together by letting $l = m + n$ and defining $\alpha = h \oplus j$ from $ST$ to $M_lS^{-2}B$ so that $\alpha_*(\chi) = \beta^{-1}_T(\varepsilon)$.

**Proof of Theorem 2.7.** If $M$ is a free CRT-module, then it can be written as a direct sum of monogenic free CRT-modules, and each monogenic CRT-module can be realized as the united $K$-theory of $\mathbb{R}$, $\mathbb{C}$, $T$, or a suspension thereof. Therefore, $M$ can be realized as the united $K$-theory of a direct sum of countably many such $C^*$-algebras.

Now, let $M$ be an arbitrary countable acyclic CRT-module. By Theorems 3.2 and 3.4 in [5], we can find a resolution
\[
0 \to F_1 \xrightarrow{\mu_1} F_0 \xrightarrow{\mu_0} M \to 0
\]
where $F_0$ and $F_1$ are countable and free CRT-modules.

As in the first paragraph, find real separable $C^*$-algebras $B$ and $C$ such that $F_0 = K_{CRT}(B)$ and $F_1 = K_{CRT}(C)$. In particular, we set
\[
B = \bigoplus_{i \in I_O} S^{k_i} \mathbb{R} \oplus \bigoplus_{i \in I_U} S^{k_i} \mathbb{C} \oplus \bigoplus_{i \in I_T} S^{k_i} T
\]
where $I_O$, $I_U$, and $I_T$ are disjoint countable index sets and where $k_i \in \{0, 1, \ldots, 7\}$ for each $i$. Our strategy is to realize $\mu_1$ geometrically. That is, we wish to produce a $C^*$-algebra homomorphism $\beta: B \to C$ whose induced homomorphism on united $K$-theory is $\mu_1$. Actually, we will replace $B$ and $C$ with algebras $B'$ and $C'$ and the induced homomorphism $\beta_*: K_{CRT}(B') \to K_{CRT}(C')$ will not be identical to $\mu_1$ but will be injective and will have the same cokernel as $\mu_1$.

For any unital $C^*$-algebra $D$, let $S^{\sim-n}D = (S^{-1}D)^\sim$ denote the united desuspension of $D$ and let $S^{\sim-n}D$ denote the $n$-fold united desuspension. Let $C'' = S^{10}S^{-2}S^{\sim-8}(C^\sim)$ and let $C' = K \otimes C''$ where $K$ is an algebra of compact operators on a separable Hilbert space. Note that $K_{CRT}(C') = K_{CRT}(C) \oplus K_{CRT}(S^{10}S^{-2}S^{\sim-8} \mathbb{R})$ because of the split exact sequence
\[
0 \to S^{10}S^{-2}S^{\sim-8} \mathbb{R} \to S^{10}S^{-2}S^{\sim-8}C^\sim \to S^{10}S^{-10}C \to 0.
\]

For each $i \in I_O$, we construct a geometric realization of the restricted homomorphism
\[
K_{CRT}(S^{k_i} \mathbb{R}) \to K_{CRT}(C)
\]
as follows. Let $x \in KO_{-k_i}(C) = KO_0(S^{-k_i}C)$ be the image of $1_O \in KO_{-k_i}(S^{k_i} \mathbb{R}) = KO_0(\mathbb{R})$. By Lemma 2.2, there is a homomorphism $\alpha_i: S \mathbb{R} \to M_{n_i}S(S^{-k_i}C)^\sim$ such that $(\alpha_i)_*(1_O) = x$. Then apply the suspension and desuspension operations to $\alpha_i$ and follow it by the inclusion into $C''$ to form the homomorphism
\[
\beta_i: S^{10}S^{k_i-10} \mathbb{R} \to M_{n_i}S^{10}S^{k_i-10}(S^{-k_i}C)^\sim \hookrightarrow M_{n_i}C''
\]
which agrees on united $K$-theory with the restriction of $\mu_1$ to $K_{CRT}(S^{k_i} \mathbb{R})$.

Similarly, for each $i \in I_U$, consider the restriction of $\mu_1$
\[
K_{CRT}(S^{k_i} \mathbb{C}) \to K_{CRT}(C),
\]
and let $y_i \in KU_{-k_i}(C)$ be the image of $\kappa_1 \in KU_{-k_i}(S^{k_i} \mathbb{C})$. Using Lemma 2.3, let $\alpha_i: SC \to M_{n_i}S^{-1}(S^{-k_i}C)^\sim$ be given satisfying $(\alpha_i)_*(\kappa_1) = \beta^{-1}_U y$. Again suspend and desuspend to form the composition
\[
\beta_i: S^{11}S^{k_i-9} \mathbb{C} \to M_{n_i}S^{10}S^{k_i-10}(S^{-k_i}C)^\sim \hookrightarrow M_{n_i}C''.
\]
The induced homomorphism \((\beta_i)_\ast\) on united \(K\)-theory agrees with the restriction of \(\mu_1\) to \(K^{\text{CRT}}(S^{k_i}C)\) up to multiplication by \(\beta_i^{\text{CRT}}\). This is not a problem for us; since \(\beta_i^{-1}\) is an isomorphism on united \(K\)-theory, the homomorphism \((\beta_i)_\ast\) is still injective and its image is the same as that of \(\mu_1\).

Thirdly, for each \(i \in I_U\), consider the restriction of \(\mu_1\)

\[K^{\text{CRT}}(S^{k_i}T) \to K^{\text{CRT}}(C)\]

and let \(z_i \in KT_{-k_i-1}(C)\) be the image of \(\chi \in KT_{-k_i-1}(S^{k_i}C)\). By Lemma 2.4, let \(\alpha_i: ST \to M_n S^{-2}(S^{-k_i-1}C)\) be given satisfying \((\alpha_i)_\ast(\chi) = \beta_i^{-1} z_i\). Again suspend and desuspend to form

\[\beta_i: S^{11}S^{k_i-7}T \to M_n S^{10}S^{k_i-9}(S^{-k_i-1}C) \to M_n C''\]

a map which on united \(K\)-theory agrees with the restriction of \(\mu_1\) to \(K^{\text{CRT}}(S^{k_i}C)\) up to multiplication by \(\beta_i^{-1}\).

We need one more homomorphism,

\[\beta_0: S^{10}S^{-2}S^{-8} \to S^{10}S^{-2}S^{-8}(C'')\]

based on the unital inclusion \(\mathbb{R} \to C''\).

We assemble the homomorphisms using a big Hilbert space. Let \(K\) be the algebra of compact operators on a separable Hilbert space and let \(\phi_i\) be a collection of mutually orthogonal inclusions from \(M_n\) to \(K\) for \(i \in I_O \cup I_U \cup I_T \cup \{0\}\). Let

\[B' = S^{10}S^{-2}S^{-8} \oplus \bigoplus_{i \in I_O} S^{10}S^{k_i-10} \oplus \bigoplus_{i \in I_U} S^{11}S^{k_i-9}C \oplus \bigoplus_{i \in I_T} S^{11}S^{k_i-7}T\]

and we define \(\beta: B' \to C' = K \otimes C''\) by setting it to be \(\phi_i \circ \beta_i\) on each summand.

Therefore, we have a geometric realization of \(\mu_1\) in the sense that \(\beta\) is injective and has the same cokernel as \(\mu_1\). Let \(A'\) be the mapping cone of \(\beta\). Then we have a short exact sequence

\[0 \to SC' \to A' \to B' \to 0.\]

In the resulting long exact sequence, the homomorphism \(K^{\text{CRT}}(B') \to K^{\text{CRT}}(SC')\) of degree \(-1\) is the same as \(\beta_\ast\) (see Proposition 2.5 of [18] or Theorem 1.1 of [9]). Since \(\beta_\ast\) is injective, the long exact sequence collapses to the short exact sequence

\[0 \to K^{\text{CRT}}(B') \xrightarrow{\beta_\ast} K^{\text{CRT}}(C') \xrightarrow{i_*} K^{\text{CRT}}(A') \to 0\]

where \(i_*\) has degree \(-1\). The united \(K\)-theory of \(A'\) is thus a shift of the \(\text{CRT}\)-module \(M\) so the algebra \(A = S^{-1}A'\) finishes the job.

Since \(A\) is constructed from the commutative algebras \(\mathbb{R}, \mathbb{C}\) and \(T\) using the operations of countable direct sum, suspensions, desuspensions, unitization, forming matrix algebras, stabilization, and forming mapping cones we know that \(A\) is separable, nuclear, and in the category of real \(C^\ast\)-algebras that satisfy the Universal Coefficient Theorem. \(\square\)

3. Unital

The goal of this section is to show that given a real \(C^\ast\)-algebra \(A\), we can obtain a unital algebra with the same united \(K\)-theory. For this, we will use the real analog of the construction of Proposition 4.1 in [1].

We begin by recording some results regarding real simple purely infinite \(C^\ast\)-algebras and their \(K\)-theory. These results are analogs of well-known results in the theory of complex simple purely infinite \(C^\ast\)-algebras. In each case, the proof follows directly from the corresponding result in the complex case, or can be proven in the same way as the complex version.
It is well-known that the inclusion of a full corner in a complex C*-algebra induces an isomorphism on $K$-theory. It is an easy consequence of Theorem 1.2 that the same is true for real C*-algebras. For completeness, we record the proofs of both statements below.

**Proposition 3.1.**

1. Let $p$ be a full projection in a complex C*-algebra $A$. Then the inclusion $i: pAp \to A$ induces an isomorphism on $K$-theory.

2. Let $p$ be a full projection in a real C*-algebra $A$. Then the inclusion $i: pAp \to A$ induces an isomorphism on united $K$-theory.

**Proof.** Let $A$ be a complex C*-algebra and let $p$ be a full projection. By Lemma 2.5 of [7], there is a partial isometry $v \in M(pAp \otimes K)$ such that $v^*v = 1$ and $vv^* = p \otimes 1$. Replacing $v$ by $(p \otimes 1)v$, we may assume that $v \in (pAp \otimes K)^+$. Then there is an isomorphism $\alpha: pAp \otimes K \to A \otimes K$ defined by $x \mapsto v^* xv$.

Now, if $q$ is any projection in $(pAp \otimes K)^+$, then $(qv)^*(qv) = v^*qv$ and $(qv)(qv)^* = q(p \otimes 1)q^* = q$. Thus in $K_0(A^+)$ we have $[i(q)] = [q] = [v^*qv] = [\alpha(q)]$. Similarly, if $u$ is any unitary in $(pAp \otimes K)^+$, then in $K_1(A^+)$ we have $[i(u)] = [u] = [v^*uv] = [\alpha(u)]$.

Therefore, $i_*$ and $\alpha_*$ agree as homomorphisms from $K_*(pAp)$ to $K_*(A)$. Since $\alpha_*$ is an isomorphism, so is $i_*$. This proves part (1). To prove part (2), let $p$ be a full projection in a real C*-algebra $A$. By part (1) the inclusion $i_*$ induces an isomorphism on complex $K$-theory $KU_*(pAp) \to KU_*(A)$. Therefore, $i_*$ is an isomorphism on united $K$-theory by Theorem 1.2. □

**Lemma 3.2.** Let $p$ and $q$ be non-trivial projections in a simple purely infinite C*-algebra. Then there is a projection $p'$ such that $p' \sim p$ and $p' < q$.

The complex version of Lemma 3.2 can be found as Proposition 1.5 in [8] or Lemma V.5.4 in [10]. The proof of Lemma 3.2 follows exactly the proof of Lemma V.5.4 in [10]. (This was also observed by Stacey in the proof of Proposition 4.1 in [20]. Once this lemma is established, the proof of Proposition 3.3 below follows exactly the proof of Theorem 1.4 of [8].

**Proposition 3.3.** Let $A$ be a real simple purely infinite C*-algebra. Then

$$KO_0(A) \cong \{ [p] \mid p \text{ is a non-zero projection in } A \}$$

where $[p]$ represents the Murray-von Neumann equivalence class of a projection $p$ in $A$.

**Proposition 3.4.** There is a functor $F$ from the category of all real C*-algebras (and real C*-algebra homomorphisms) to the category of all real unital C*-algebras (and real unital C*-algebra homomorphisms) and a natural transformation $\eta: A \to F(A)$ which induces an isomorphism on united $K$-theory. Furthermore,

1. If $A$ is nuclear, then $F(A)$ is nuclear.

2. If $A$ is separable, then $F(A)$ is separable and $\eta$ is a KK-equivalence.

3. If $A$ is separable and satisfies the UCT, then $F(A)$ satisfies the UCT.

**Proof.** Let $O_\infty^\mathbb{R}$ be the real Cuntz algebra generated by a sequence of mutually orthogonal isometries. By Theorem 1.2 the unital inclusion $\mathbb{R} \to O_\infty^\mathbb{R}$ induces an isomorphism on united $K$-theory since the complexification $C \to O_\infty^\mathbb{R}$ induces an isomorphism on $K$-theory. By Proposition 3.3 there is a non-zero projection $e \in O_\infty^\mathbb{R}$ and a projection $q < e$ such that $[e] = 0$ and $[q] = [1_{O_\infty^\mathbb{R}}]$.

Since $e$ is infinite, there exists a proper subprojection $p_1$ such that $p_1 \sim e$. Let $p_2 = e - p_1$. Then $[p_1] = [p_2] = [e] = 0$. Therefore (again by Proposition 3.3) there are partial isometries $s_1$ and $s_2$ in $eO_\infty^\mathbb{R}e$ such that $s_1^*s_1 = e$ and $s_2s_2^* = p_1$. Let $D = C^*(s_1, s_2)$. Then the algebra $D = C^*(s_1, s_2)$ is a unital subalgebra of $eO_\infty^\mathbb{R}e$ which is isomorphic to $O_2^\mathbb{R}$.
Now, for any real C*-algebra $A$, let $A^+$ be the unitization of $A$ and let $\pi_A: A^+ \to \mathbb{R}$ be the usual projection with kernel $A$. We define

$$F(A) = \{ b \in e\mathcal{O}_\infty^R e \otimes A^+ \mid (1 \otimes \pi_A)(b) \in D \}.$$ 

The element $e \otimes 1$ is a unit for $F(A)$. The natural transformation $\eta: A \to F(A)$ is defined by $a \mapsto q \otimes a$.

We will show that $\eta$ induces an isomorphism on united $K$-theory. Note that $\eta$ is a composition of the homomorphism $A \to e\mathcal{O}_\infty^R e \otimes A$ defined by $a \mapsto q \otimes a$ and the inclusion $e\mathcal{O}_\infty^R e \otimes A \hookrightarrow F(A)$. The homomorphism $A \to e\mathcal{O}_\infty^R e \otimes A$ induces an isomorphism on united $K$-theory because the map $\mathbb{R} \to e\mathcal{O}_\infty^R e$ defined by $t \mapsto tq$ does using the Künneth formula for united $K$-theory (2). Secondly, the inclusion $e\mathcal{O}_\infty^R e \otimes A \hookrightarrow F(A)$ induces an isomorphism on united $K$-theory because of the short exact sequence

$$0 \to e\mathcal{O}_\infty^R e \otimes A \hookrightarrow F(A) \xrightarrow{1 \otimes \pi_A} D \to 0$$

and the fact that $KCRT(D) = KCRT(\mathcal{O}_2) = 0$. It follows that $\eta$ induces an isomorphism on united $K$-theory.

It is clear from the short exact sequence above that if $A$ is separable or nuclear, then the same is true of $F(A)$. Furthermore, the argument of the previous paragraph also works for $KK$-theory, showing that $\eta$ induces isomorphisms

$$KK^{CRT}(B, A) \to KK^{CRT}(B, F(A))$$

and

$$KK^{CRT}(F(A), B) \to KK^{CRT}(A, B)$$

for any real separable C*-algebra $B$. If $A$ is separable, then so is $F(A)$ and by the Yoneda Lemma, $\eta \eta$ induces a $KK$-equivalence. In particular, if $A$ is separable and satisfies the UCT, so does $F(A)$. \hfill \Box

4. Simple and Purely Infinite

In [13] Alex Kumjian presents a construction (based on a special case of Michael Pimsner’s construction in [16]) which turns any complex separable unital C*-algebra $A$ into a complex C*-algebra $\mathcal{O}_E$ which is simple and purely infinite such that there is an inclusion $A \hookrightarrow \mathcal{O}_E$ which is a (complex) $KK$-equivalence. In this section, we show that this construction can be applied to the real case. Combined with the results from Sections 2 and 3, this will complete the proof of Theorem 1.1.

Proposition 4.1. Let $A$ be a real separable unital C*-algebra. Then there is a real separable simple purely infinite C*-algebra $\mathcal{O}_E^R$ and a unital inclusion $A \hookrightarrow \mathcal{O}_E^R$ which induces an isomorphism on united $K$-theory. Furthermore, if $A$ is nuclear and satisfies the UCT, then the same is true of $\mathcal{O}_E^R$ and $\iota$ is a (real) $KK$-equivalence.

Recall that a complex C*-algebra $A$ is said to have a real structure if there is a conjugate linear involution $x \mapsto \bar{x}$. In that case, the set $A^R$ of fixed points is a real C*-algebra. Conversely, given a real C*-algebra $A$, the complexification $A_C$ has a real structure given by $a_1 + ia_2 \mapsto a_1 - ia_2$. These functors are inverse to each other so there is a bijection between complex C*-algebras with real structure and real C*-algebras. To prove Proposition 4.1 we will retrace Kumjian’s construction, showing that the real structure of $A_C$ passes to $\mathcal{O}_E$.

Definition 4.2. Let $A$ be a complex C*-algebra with a real structure.
(1) A Hilbert $A$-module $E$ is said to have a real structure if $E$ has a conjugate linear involution $e \mapsto \overline{e}$ that satisfies $\langle e, f \rangle = \overline{\langle f, e \rangle}$ and $\overline{e \cdot a} = \overline{e} \cdot \overline{a}$ for all $a \in A$ and $e, f \in E$.

(2) A Hilbert $A$-bimodule $(E, \phi)$ is said to have a real structure if the Hilbert $A$-module $E$ has a real structure as in part (1) and the homomorphism $\phi : A \to \mathcal{L}(E)$ satisfies $\phi(a)e = \phi(\overline{a})\overline{e}$ for all $a \in A$ and $e \in E$.

If $E$ is a Hilbert $A$-module with a real structure, then the $C^*$-algebra $\mathcal{L}(E)$ has a real structure defined by $\overline{T(e)} = \overline{T(e)}$ for all $T \in \mathcal{L}(E)$ and $e \in E$. With this language, the Hilbert bimodule condition above can be restated as $\phi(a)e = \phi(\overline{a})\overline{e}$, saying that the *-homomorphism $\phi : A \to \mathcal{L}(E)$ respects the real structures.

Kumjian’s construction begins with a faithful representation $\pi : A \to \mathcal{L}(H)$ where $H$ is a separable complex Hilbert space such that $\pi(A) \cap \mathcal{K}(H) = \{0\}$. If $A$ has a real structure, we can start with a representation of $A_\mathbb{R}$ on a real Hilbert space $H_\mathbb{R}$ and then complexify. Thus we can assume that $\pi$ respects the real structures of $A$ and $\mathcal{L}(H)$. Following Kumjian, we define a Hilbert $A$-bimodule $(E, \phi)$ by

$$ E = H \otimes_\mathbb{C} A $$

with bimodule structure given by $(\xi \otimes a) \cdot b = \xi \otimes (a \cdot b)$ and $\phi(b)(\xi \otimes a) = \pi(b)\xi \otimes a$ for all $a, b \in A$ and $\xi \in H$. We give $(E, \phi)$ a real structure by $\overline{\xi \otimes a} = \overline{\xi} \otimes \overline{a}$.

Similarly, the Fock space

$$ \mathcal{E}_+ = \bigoplus_{n=0}^{\infty} E \otimes e_n $$

is also a Hilbert $A$-bimodule with a real structure. The involution is defined on pure tensors by

$$ \overline{e_1 \otimes e_2 \otimes \cdots \otimes e_n} = \overline{e_1} \otimes \overline{e_2} \otimes \cdots \otimes \overline{e_n}. $$

For any element $e \in E$, we define the operator $T_e \in \mathcal{L}(\mathcal{E}_+)$ on pure tensors by $T_e(e_1 \otimes \cdots \otimes e_n) = e \otimes e_1 \otimes \cdots \otimes e_n$. Since $\overline{T_e} = T_{\overline{e}}$, the involution of $\mathcal{L}(E)$ restricts to an involution of the algebra $\mathcal{T}_E$ generated by $\{T_e\}_{e \in E}$.

In the general case, $\mathcal{O}_E$ is the quotient of $\mathcal{T}_E$ by the $C^*$-algebra generated in $\mathcal{L}(\mathcal{E}_+)$ by $\mathcal{L}\left( \bigoplus_{n=0}^{N} E \otimes e_n \right)$ for all positive integers $N$. But under the assumption $\pi(A) \cap \mathcal{K}(H) = \{0\}$, we have $\mathcal{O}_E \cong \mathcal{T}_E$ (see [16], Corollary 3.14). In either case, the involution of $\mathcal{L}(\mathcal{E}_+)$ induces one on $\mathcal{O}_E$. Furthermore, the inclusion $\iota : A \hookrightarrow \mathcal{O}_E$ given by $\iota(a)(e_1 \otimes e_2 \otimes \cdots \otimes e_n) = \phi(a)(e_1) \otimes e_2 \otimes \cdots \otimes e_n$, respects the real structures of $A$ and $\mathcal{O}_E$.

If we begin with a real separable unital $C^*$-algebra $A$, then the complexification $A_\mathbb{C}$ has a real structure and the construction above yields an inclusion $\iota : A \to \mathcal{O}_E^\mathbb{R}$ where $\mathcal{O}_E^\mathbb{R}$ is the fixed point set of $\mathcal{O}_E$.

**Proof of Proposition 4.1.** Let $A$ be a real separable unital $C^*$-algebra. Applying the construction above, we obtain an inclusion $\iota : A \to \mathcal{O}_E^\mathbb{R}$. By Theorem 2.8 of [13], $\mathcal{O}_E$ is simple and purely infinite. Thus $\mathcal{O}_E^\mathbb{R}$ is simple and purely infinite by Theorem 3.3 of [20]. By Corollary 4.5 of [16], the inclusion $\iota : A_\mathbb{C} \to \mathcal{O}_E$ is a $KK$-equivalence. In particular, it induces an isomorphism on $K$-theory, so by Theorem 1.2 $\iota : A \to \mathcal{O}_E^\mathbb{R}$ induces an isomorphism on united $K$-theory.

If $A$ is nuclear and satisfies the UCT, then by Theorem 3.1 of [13], the same is true of $\mathcal{O}_E$. Thus $\mathcal{O}_E^\mathbb{R}$ is nuclear and satisfies the UCT. In particular, since $A$ and $\mathcal{O}_E^\mathbb{R}$ have isomorphic united $K$-theory and both satisfy the UCT, they are $KK$-equivalent. \[\square\]
The proof of Section 4 of [16] will probably carry over to show that $\iota$ is a (real) $KK$-equivalence in general, giving a stronger statement than our Theorem 1.1, but we don’t need this for our present purposes.

Note that absent a full classification theorem for real simple purely infinite $C^*$-algebras, there is no guarantee that $O_E^K$ is independent of the choice of $\pi$ (as $O_E$ is when $A_c$ is nuclear and satisfies the UCT).

**Proof of Theorem 1.1.** Let $M$ be a countable acyclic CRT-module. By Theorem 2.1, there is a real separable nuclear $C^*$-algebra $A_1$ satisfying the UCT such that $K^\text{CRT}(A_1) \cong M$. Applying the functor $F$ of Proposition 3.4, there is a real separable nuclear unital $C^*$-algebra $A_2$ satisfying the UCT such that $K^\text{CRT}(A_2) \cong M$. Then applying the real Kumjian construction (Proposition 4.1), there is a real separable nuclear unital simple purely infinite $A_3$ satisfying the UCT such that $K^\text{CRT}(A_3) \cong M$. Finally, let $A_4 = K(H) \otimes_r A_3$ where $H$ is a real separable Hilbert space. By Lemma 4.3 below, $A_4$ is purely infinite and is the real $C^*$-algebra needed to prove part (1).

Now, let $m$ be any element in $M^G_0$. By Proposition 3.3, there is a projection $p \in A_4$ such that $[p] = m$. Let $A_5$ be the corner algebra $pA_4p$. By Proposition 4.1, $K^\text{CRT}(A_5) \cong M$. This proves part (2). □

**Lemma 4.3.** If $A$ is a real purely infinite simple $C^*$-algebra, then the stabilization $K(H) \otimes_r A$ is also purely infinite and simple.

**Proof.** By Lemma 4.2 of [20], the matrix algebras $M_n(A)$ are purely infinite. The proof of Proposition 4.1.8 of [17] carries over immediately to the real case to show that the inductive limit of simple purely infinite $C^*$-algebras is again simple and purely infinite. □

### 5. The inadequacy of real $K$-theory

In this section, we will draw one small application from our main theorem, creating an example which shows that $K$-theory by itself cannot classify isomorphism classes or even $KK$-equivalence of real simple purely infinite $C^*$-algebras. Further applications of Theorem 1.1 will appear in [31].

**Theorem 5.1.** There exist two real $C^*$-algebras $A$ and $B$ such that $KO_\ast(A) \cong KO_\ast(B)$, but $K^\text{CRT}(A) \ncong K^\text{CRT}(B)$.

**Proof.** By Theorem 1.1 it suffices to find two distinct countable acyclic CRT-modules whose real parts are isomorphic. I am indebted to A.K. Bousfield for sharing with me the example of such CRT-modules. We will employ a CRT-module construction found in Chapter 8 of [11].

Let $(G, \alpha)$ be a group with involution satisfying $\ker(1 + \alpha) = \text{image}(1 - \alpha)$ and $\ker(1 - \alpha) = \text{image}(1 + \alpha)$. Let $G^+ = \{ g \in G \mid \alpha(g) = g \}$ and $G^- = \{ g \in G \mid \alpha(g) = -g \}$. Then there are exact sequences

$$0 \to G^+ \xrightarrow{i^+} G \xrightarrow{\pi^+} G^- \to 0$$

and

$$0 \to G^- \xrightarrow{i^-} G \xrightarrow{\pi^-} G^+ \to 0$$

where $\pi^+ = 1 + \alpha$, $\pi^- = 1 - \alpha$, and $i^+$ and $i^-$ are the inclusion homomorphisms. Then Table 1 displays the groups and natural transformations of an acyclic CRT-module $P(G, \alpha)$. It is easy, if tedious, to verify that the CRT relations hold and that the sequences are exact making it acyclic.
Let $G = \mathbb{Z}_2^4$ and $H = \mathbb{Z}_4 \oplus \mathbb{Z}_2^2$ with involutions

$$
\alpha = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

and

$$
\beta = \begin{pmatrix}
1 & 0 & 2 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

respectively. Then $G^+ \cong H^+ \cong \mathbb{Z}_2^2$ and $G^- \cong H^- \cong \mathbb{Z}_2^2$. Thus the real parts of $P(G, \alpha)$ and $P(H, \beta)$ agree while the complex parts do not. \(\Box\)

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