Shortest closed curve to contain a sphere in its convex hull

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Abstract
We show that in Euclidean 3-space any closed curve which contains the unit sphere within its convex hull has length $L \geq 4\pi$, and characterize the case of equality. This result generalizes the authors’ recent solution to a conjecture of Zalgaller. Furthermore, for the analogous problem in $n$ dimensions, we include the estimate $L \geq Cn\sqrt{n}$ by Nazarov, which is sharp up to the constant $C$.

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1  |  INTRODUCTION

The convex hull of a set $X$ in Euclidean space $\mathbb{R}^3$ is the intersection of all convex sets that contain $X$. The inradius of $X$ is the supremum of the radii of spheres which are contained in $X$. Here, we show the following:

**Theorem 1.1.** Let $\gamma : [a, b] \rightarrow \mathbb{R}^3$ be a closed rectifiable curve of length $L$, and $r$ be the inradius of the convex hull of $\gamma$. Then,

$$L \geq 4\pi r.$$  

Equality holds only if, up to a reparameterization, $\gamma$ is simple, $C^{1,1}$, lies on a sphere of radius $\sqrt{2}r$, and traces consecutively 4 semicircles of length $\pi r$. 

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In 1996, V. A. Zalgaller [18, 22] conjectured that the above theorem holds subject to the additional assumption that \( \gamma \) lie outside a sphere \( S \) of radius \( r \) within its convex hull. The length minimizer, called the \textit{baseball curve}, together with \( S \), is shown in Figure 1.

Zalgaller’s conjecture was proved recently in [15] following earlier work in [13]. Here, we refine the methods introduced in those papers to establish the more general result above. Our approach will be similar to that in [15]. We start by setting \( r = 1 \) and assuming that \( \gamma \) has the smallest length among closed curves, which contain the unit sphere \( S^2 \) within their convex hull [15, section 2]. The horizon of \( \gamma \) is the measure in \( S^2 \) counted with multiplicity of the set of points \( p \in S^2 \), where the affine tangent plane \( T_p S^2 \) intersects \( \gamma \):

\[
H(\gamma) := \int_{p \in S^2} \#(T_p S^2) \, dp.
\]

Since \( \gamma \) is closed, one quickly sees that \( \#(T_p S^2) \geq 2 \) for almost every \( p \in S^2 \) [13, Lemma 7.1]. Hence, \( H(\gamma) \geq 8\pi \). The \textit{efficiency} of \( \gamma \) is given by

\[
E(\gamma) := \frac{H(\gamma)}{L(\gamma)}.
\]

So to establish (1), it suffices to show \( E(\gamma) \leq 2 \). To this end, we note that for any partition of \( \gamma \) into subcurves \( \gamma_i \),

\[
E(\gamma) = \sum_i \frac{H(\gamma_i)}{L(\gamma)} = \sum_i \frac{L(\gamma_i)}{L(\gamma)} E(\gamma_i).
\]

So it suffices to construct a partition with \( E(\gamma_i) \leq 2 \). Similar to [15], this is achieved by \textit{unfolding} \( \gamma \) into the plane (Section 3), and identifying a collection of subcurves of \( \gamma \) we call \textit{spirals} (Section 4); however, these operations need to be generalized here as they were defined only for curves with \( |\gamma| \geq 1 \) in [15]. Furthermore, we will show that if \( E(\gamma) = 2 \), then \( |\gamma| \geq 1 \). So the rigidity of (1) follows from Zalgaller’s conjecture established in [15], and completes the proof of Theorem 1.1 (Section 5).

For curves in \( \mathbb{R}^2 \), the isoperimetric inequality quickly yields \( L \geq 2\pi r \) as the analog of (1). We will include in the Appendix a version of (1) by F. Nazarov for curves in \( \mathbb{R}^n \), which is obtained by covering the unit sphere \( S^{n-1} \) with certain slabs, and applying the correlation inequality [16, 19] to their Gaussian volume. This approach has implications for covering problems for the sphere by congruent disks [5], and yields a new proof of a result of Tikhomirov [21] (Note A.4). There are
many natural optimization problems for a convex hull of space curves, which remain open, including other questions of Zalgaller [22], which are closely related to well-known problems of Bellman [2–4] in operations research and search theory [1, 12]; see also [13, 15, 17] and references therein.

### 2 MINIMAL INSPECTION CURVES

\( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space with inner product \( \langle \cdot, \cdot \rangle \), norm \( | \cdot | := \langle \cdot, \cdot \rangle^{1/2} \), and origin \( o \). A curve is a continuous rectifiable mapping \( \gamma : [a, b] \to \mathbb{R}^n \) with length \( L = L(\gamma) \).

We also use \( \gamma \) to refer to its image \( \gamma([a, b]) \). If \( \gamma(a) = \gamma(b) \), then we say that \( \gamma \) is closed and identify \([a, b]\) with the topological circle \( \mathbb{R}/(b-a) \). Rectifiable curves may be parameterized with constant speed [6], which we assume is the case throughout this work. In particular, all curves below are Lipschitz continuous, and thus differentiable almost everywhere, with \( |\gamma'| = L/(b-a) \); see [15, section 2] and references therein for basic facts on rectifiable curves. We say \( \gamma \) is a (generalized) inspection curve provided that \( \gamma \) is closed and its convex hull, \( \text{conv}(\gamma) \), contains the unit sphere \( S^2 \). It follows from Arzela–Ascoli theorem that there exists an inspection curve \( \gamma \) whose length achieves the minimum value among all inspection curves [15, section 2]. Then, \( \gamma \) will be called a minimal inspection curve. We let \( \text{int} \), \( \text{cl} \), and \( \partial \), stand, respectively, for interior, closure, and boundary.

**Lemma 2.1.** Let \( \gamma : \mathbb{R}/L \to \mathbb{R}^3 \) be a minimal inspection curve. Suppose \( \gamma(t) \in \text{int}(\text{conv}(\gamma)) \), for some \( t \in \mathbb{R}/L \). Then, there exists a connected open set \( U \subset \mathbb{R}/L \), with \( t \in U \), such that \( \gamma \) maps \( \text{cl}(U) \) injectively to a line segment with end points on \( \partial \text{conv}(\gamma) \). In particular, \( \gamma(t) = o \) for at most finitely many \( t \in \mathbb{R}/L \).

**Proof.** Let \( U \) be the component of \( \gamma^{-1}(\text{int}(\text{conv}(\gamma))) \), which contains \( t \). If \( \gamma|_{\text{cl}(U)} \) does not trace a line segment, we may shorten \( \gamma \) by replacing \( \gamma(\text{cl}(U)) \) with the line segment connecting the end points of \( \gamma(\text{cl}(U)) \). But this operation preserves \( \text{conv}(\gamma) \), as it preserves the points of \( \gamma \) on \( \partial \text{conv}(\gamma) \). Hence, we obtain an inspection curve shorter than \( \gamma \), which is impossible. If \( \gamma(t) = o \), then \( L(\gamma|_U) \geq 2 \), since \( \gamma(U) \) contains a diameter of \( S^2 \). So there can be only finitely many such points, since \( \gamma \) is rectifiable.

We say that \( t \) is a regular point of a curve \( \gamma \) provided that \( \gamma \) is differentiable at \( t \) and \( \gamma'(t) \neq 0 \). Then, the tangent line of \( \gamma \) at \( t \) is well defined. Since we assume that curves are parameterized with constant speed, they are regular almost everywhere. Furthermore, by Lemma 2.1, all points \( t \in \mathbb{R}/L \) with \( \gamma(t) \in \text{int}(\text{conv}(\gamma)) \) of a minimal inspection curve \( \gamma \) are regular.

**Lemma 2.2.** Let \( \gamma : \mathbb{R}/L \to \mathbb{R}^3 \) be a minimal inspection curve, \( t \in \mathbb{R}/L \) be a regular point of \( \gamma \), and \( \ell \) be the tangent line of \( \gamma \) at \( t \). Suppose that \( \ell \) intersects \( \text{int}(\text{conv}(\gamma)) \). Then, there exists an open interval \( U \subset \mathbb{R}/L \), with \( t \in U \), which is mapped injectively by \( \gamma \) into \( \ell \cap \text{int}(\text{conv}(\gamma)) \).

**Proof.** If \( \gamma(t) \in \partial \text{conv}(\gamma) \), then either \( \gamma'(t) \) or \( -\gamma'(t) \) points outside \( \text{conv}(\gamma) \). Hence, for some \( s \) close to \( t \), \( \gamma(s) \) lies outside \( \text{conv}(\gamma) \), which is impossible. So \( \gamma(t) \in \text{int}(\text{conv}(\gamma)) \), in which case, Lemma 2.1 completes the proof.

Combining the last two observations we obtain the following:
**Proposition 2.3.** Let \( \gamma : \mathbb{R} / L \to \mathbb{R}^3 \) be a minimal inspection curve. Then, there exists an open set \( U \subset \mathbb{R} / L \) such that tangent lines of \( \gamma \) on \( U \) do not pass through \( o \). Furthermore if \( U \neq \mathbb{R} / L \), then \( \mathbb{R} / L \setminus U \) is the disjoint union of a finite number of closed intervals each mapped by \( \gamma \) into a line segment, which passes through \( o \) and ends on \( \partial \text{conv}(\gamma) \).

**Proof.** Let \( X \) be the union of all closed intervals \( I \subset \mathbb{R} / L \) such that \( \gamma(I) \) is a line segment, which passes through \( o \) and ends on \( \partial \text{conv}(\gamma) \). By Lemma 2.1, there are at most finitely many such intervals. Thus, \( X \) is closed. Let \( U := \mathbb{R} / L \setminus X \). By Lemma 2.2, no tangent line of \( \gamma \) at a regular point of \( U \) may pass through \( o \), which completes the proof. \( \Box \)

## 3 UNFOLDING

Let \( \gamma : \mathbb{R} / L \to \mathbb{R}^3 \) be a minimal inspection curve. We will always assume that 0 is a local minimum point of \( |\gamma| \). By Lemma 2.1, \( \gamma \) passes through \( o \) at most finitely many times, which, if they exist, will be denoted by \( 0 =: t_0, \ldots, t_m =: L \). Then, the projection \( \bar{\gamma} : \mathbb{R} / L \to \mathbb{S}^2 \), given by \( \bar{\gamma} := \gamma / |\gamma| \) is well defined on \( \mathbb{R} / L \setminus \{t_k\} \). Furthermore, since by Proposition 2.3, \( \gamma \) traces line segments near \( t_k \), \( \bar{\gamma} \) is Lipschitz on each interval \((t_{k-1}, t_k)\). Thus, \( \bar{\gamma} \) is differentiable almost everywhere on \( \mathbb{R} / L \). Consequently, the arclength function

\[
\theta(t) := \int_0^t |\bar{\gamma}'(s)| \, ds
\]

is well defined on \([0, L]\) (\( \theta \) measures the “cone angle” [7] or “vision angle” [8] of \( \gamma \) from the point of view of \( o \)). The *unfolding* of \( \gamma \) is the planar curve \( \tilde{\gamma} : [0, L] \to \mathbb{R}^2 \) defined as

\[
\tilde{\gamma}(t) := |\gamma(t)| e^{i(\theta(t) + (k-1)\pi)}, \quad \text{for } t \in [t_{k-1}, t_k].
\]

Note that \( |\gamma| = |\tilde{\gamma}| \), and whenever \( \gamma \) passes through \( o \), then \( \tilde{\gamma} \) will pass through \( o \) as well on a line segment. As in [15], we may also compute that

\[
|\tilde{\gamma}'| = |\gamma'| + i |\gamma| \theta', \quad \text{and} \quad \theta' = |\tilde{\gamma}'| = \frac{1}{|\gamma|} \sqrt{|\gamma|^2 |\gamma'|^2 - \langle \gamma, \gamma' \rangle^2},
\]

almost everywhere. It follows that, for almost all \( t \in [0, L] \), \( |\tilde{\gamma}'| = |\gamma'| = 1 \). So \( \tilde{\gamma} \) is parameterized by arclength, and \( L(\gamma) = L(\tilde{\gamma}) \). Hence, by [15, Cor. 3.2], \( E(\gamma) = E(\tilde{\gamma}) \) since points of \( \gamma \) with \( |\gamma| \leq 1 \) make no contribution to \( E(\gamma) \). Furthermore, the angles \( \alpha := \angle(\gamma, \gamma') \) and \( \tilde{\alpha} := \angle(\tilde{\gamma}, \tilde{\gamma}') \) are defined almost everywhere, and

\[
\alpha = \cos^{-1} (|\gamma'|) = \cos^{-1} (|\tilde{\gamma}'|) = \tilde{\alpha}.
\]

**Lemma 3.1.** Let \( \gamma : \mathbb{R} / L \to \mathbb{R}^3 \) be a minimal inspection curve. Then, \( \tilde{\gamma} \) is locally one-to-one.

**Proof.** Let \( U \) be as in Proposition 2.3. Then, \( \gamma \) and \( \gamma' \) are linearly independent at all regular points of \( U \). So (2) shows \( \theta' > 0 \) almost everywhere on \( U \), via Cauchy–Schwarz inequality. Hence, \( \theta \) is strictly increasing on \( U \), which yields that \( \tilde{\gamma} \) is star-shaped with respect to \( o \) in a neighborhood of each point of \( U \). Since, by Proposition 2.3, \( \gamma \) traces a line segment on each component of \( \mathbb{R} / L \setminus U \),
|\gamma| \text{ is strictly monotone on each of these components. Hence, } \tilde{\gamma} \text{ is one-to-one on each component of } \mathbb{R}/L \setminus U, \text{ since } |\tilde{\gamma}| = |\gamma|. \text{ Finally, } \tilde{\gamma} \text{ is one-to-one in a neighborhood of each point of } \partial U, \text{ since } \tilde{\gamma} \text{ is locally star-shaped on } U \text{ and it maps each component of } \mathbb{R}/L \setminus U \text{ to a line passing through } o.

A planar curve } \gamma: [a, b] \to \mathbb{R}^2 \text{ is locally convex provided that it is locally one-to-one and each point } t \in [a, b] \text{ has a neighborhood } U \subset [a, b] \text{ such that } \gamma(U) \text{ lies on the boundary of a convex set. A side of a line } \ell \subset \mathbb{R}^2 \text{ is one of the two closed half spaces determined by } \ell. \text{ A local supporting line } \ell \text{ for } \gamma \text{ at } t \text{ is a line passing through } \gamma(t) \text{ with respect to which } \gamma(U) \text{ lies on one side. If } \gamma(U) \text{ lies on a side of } \ell, \text{ which contains } o, \text{ then we say that } \ell \text{ lies above } \gamma. \text{ Finally, if } \gamma \text{ is locally convex and through each point of it there passes a local support line, which lies above } \gamma, \text{ then we say that } \gamma \text{ is locally convex with respect to } o. \text{ Note that if } \gamma \text{ is locally convex with respect to } o \text{ and passes through } o, \text{ then } \gamma \text{ must trace a line segment near } o.

**Lemma 3.2.** Let } \gamma: \mathbb{R}/L \to \mathbb{R}^3 \text{ be a minimal inspection curve. Then, } \tilde{\gamma} \text{ is locally convex with respect to } o.

**Proof.** Let } U \text{ be as in Proposition 2.3, and } t \in U. \text{ By Lemma 3.1, there exists a neighborhood } V \text{ of } t \text{ in } U \text{ on which } \tilde{\gamma} \text{ is one-to-one. Furthermore, } \tilde{\gamma}(V) \text{ is star-shaped with respect to } o. \text{ So connecting the end points of } \tilde{\gamma}(V) \text{ to } o \text{ by line segments yields a simple closed curve. It is shown in the proof of [15, Proposition 4.3] that this curve bounds a convex set, due to minimality of } \gamma. \text{ Thus, } \tilde{\gamma} \text{ is locally convex with respect to } o \text{ on } U. \text{ Next, suppose } t \in \partial U, \text{ and let } V \text{ be a small neighborhood of } t \text{ in cl}(U). \text{ By Proposition 2.3, } \tilde{\gamma} \text{ connects one end point of } \tilde{\gamma}(V) \text{ to } o \text{ by tracing a line segment. Connect the other end point of } \tilde{\gamma}(V) \text{ to } o \text{ by another line segment. Then, the resulting simple closed curve again bounds a convex set by the argument in the proof of [15, Proposition 4.3]. So } \tilde{\gamma} \text{ is locally convex with respect to } o \text{ on cl}(U). \text{ Finally, } \tilde{\gamma} \text{ is locally convex with respect to } o \text{ on the complement of cl}(U), \text{ since these regions are mapped to line segments, by Proposition 2.3.} \square

## 4 SPIRAL DECOMPOSITION

If } \gamma: [a, b] \to \mathbb{R}^2 \text{ is a locally convex curve, parameterized with constant speed, then its one sided derivatives, } \gamma'_\pm, \text{ are well-defined everywhere and are nonvanishing [14, Lemma 5.1]. Set } \gamma'(a) := \gamma'_+(a). \text{ We say that } \gamma: [a, b] \to \mathbb{R}^2 \text{ is a (generalized) spiral provided that (i) } \gamma \text{ is locally convex with respect to } o, \text{ (ii) } |\gamma| \text{ is nondecreasing, and (iii) } \langle \gamma(a), \gamma'(a) \rangle = 0. \text{ A spiral is called strict if } |\gamma| \text{ is increasing. A spiral decomposition of a curve } \gamma: [a, b] \to \mathbb{R}^2 \text{ is a collection } U_i \text{ of mutually disjoint open subsets of } [a, b] \text{ such that (i) } \gamma|_{\text{cl}(U_i)} \text{ is a strict spiral, after switching the direction of } \gamma|_{\text{cl}(U_i)} \text{ if necessary, and (ii) } |\gamma'| = 0 \text{ almost everywhere on } [a, b] \setminus \cup_i \text{ cl}(U_i).

**Lemma 4.1.** Let } \gamma: \mathbb{R}/L \to \mathbb{R}^3 \text{ be a minimal inspection curve. Then, } \tilde{\gamma} \text{ admits a spiral decomposition.

**Proof.** The argument follows the same outline as in [15, Proposition 5.2], with minor modifications. Recall that we assume 0 is a local minimum point of } |\gamma|. \text{ If } |\gamma(0)| > 0, \text{ then it follows that } \tilde{\alpha}(0) = \tilde{\alpha}(L) = \pi/2. \text{ Otherwise, } |\tilde{\gamma}(0)| = |\tilde{\gamma}(L)| = 0, \text{ since } |\gamma| = |\tilde{\gamma}|. \text{ Let } X \text{ be the set of points } t \in [0, L] \text{ such that } \tilde{\gamma} \text{ has a local support line at } \tilde{\gamma}(t), \text{ which is orthogonal to } \tilde{\gamma}(t), \text{ or } |\tilde{\gamma}(t)| = 0. \text{ Then, } 0, L \in X \text{ and } |\tilde{\gamma}'| = 0 \text{ almost everywhere on } X. \text{ Also note that } X \text{ is closed, since the limit of any
sequence of support lines of a convex body is a support line, and the set of points with $|\tilde{\gamma}(t)| = 0$ is compact. Consequently each component $U$ of $[0, L] \setminus X$ is an open subinterval of $[0, L]$. It remains to show that $\tilde{\gamma} |_{\text{cl}(U)}$ is a spiral. By Lemma 3.2, $\tilde{\gamma} |_{\text{cl}(U)}$ is locally convex with respect to $o$. Furthermore, as argued in the proof of [15, Proposition 5.2], $|\tilde{\gamma}|'$ is always positive or always negative at differentiable points of $|\tilde{\gamma}|$ on $U$. So we may suppose that $|\tilde{\gamma}|$ is increasing on $U$, after switching the direction of $\tilde{\gamma} |_{\text{cl}(U)}$ if necessary. Finally, let $x \in \partial U$ be the initial point of $\tilde{\gamma} |_{\text{cl}(U)}$. If $|\tilde{\gamma}(x)| = 0$, then $\tilde{\gamma} |_{\text{cl}(U)}$ is a spiral. If $|\tilde{\gamma}(x)| > 0$, it follows that $\tilde{\gamma}(x)$ is orthogonal to $\tilde{\gamma}' + (x)$, which again shows that $\tilde{\gamma} |_{\text{cl}(U)}$ is a spiral and completes the proof. □

Let $S^1$ denote the unit circle in $\mathbb{R}^2$. The last observation quickly yields the following:

**Lemma 4.2.** Let $\gamma, \tilde{\gamma}$ be as in Lemma 4.1 and $\sigma : [a, b] \to \mathbb{R}^2$ be a spiral in the unfolding of $\tilde{\gamma}$. Let $t \in [a, b]$ be a regular point of both $\sigma$ and $\gamma$, and $\ell$ be the tangent line of $\sigma$ at $t$. Suppose that $\ell$ crosses $S^1$. Then, $\sigma([a, t])$ lies on $\ell$.

**Proof.** Let $\bar{\ell}$ be the tangent line of $\gamma$ at $t$. If $\bar{\ell}$ crosses $S^1$, then $\bar{\ell}$ crosses $S^2$, by (3). In particular, $\bar{\ell}$ intersects the interior of $\text{conv}(\gamma)$. Then, Lemma 2.2 completes the proof. □

The key point in the proof of Theorem 1.1 is the following:

**Proposition 4.3.** Let $\sigma : [a, b] \to \mathbb{R}^2$ be a spiral in the unfolding of a minimal inspection curve. Then, $E(\sigma) \leq 2$. Furthermore, if $|\sigma(a)| < 1$, then $E(\sigma) < 2$.

**Proof.** If $|\sigma(a)| \geq 1$, then $E(\sigma) \leq 2$ by [15, Proposition 2.7]. So we assume $|\sigma(a)| < 1$. We may also assume $|\sigma(b)| > 1$ for otherwise $H(\sigma) = 0$, which yields $E(\sigma) = 0$. Let $b'$ be the supremum of $t \in [a, b]$ such that $\sigma([a, t])$ is a line segment. By Lemma 2.1, $|\sigma(b')| \geq 1$. We may assume that $\sigma(a)$ lies on the nonnegative portion of the $y$-axis, and $\sigma([a, b'])$ lies to the right of the $y$-axis, see Figure 2. If $b' < b$, then we may choose $b'' < b''' < b$ such that $\sigma([b', b'''])$ is convex, and lies to the right of the $y$-axis. Since $\sigma$ is locally convex with respect to $o$, $\sigma([b', b'''])$ lies below the line $\lambda$ spanned by $\sigma([a, b'])$, if $|\sigma(a)| > 0$. If $|\sigma(a)| = 0$, we may still assume that $\sigma([b', b'''])$ lies below $\lambda$ after a reflection. Consider the line that passes through $\sigma(b')$ and is tangent to the upper half of $S^1$, say at a point $x$. Let $\tau$ be the curve obtained by joining the line segment $x\sigma(b')$ to the beginning of $\sigma |_{[b', b]}$. We will show that (i) $\tau$ is a spiral and (ii) $E(\sigma) < E(\tau)$. Then, we are done, because $E(\tau) \leq 2$ since its initial height is $\geq 1$.

First, we check that $\tau$ is a spiral. This is obvious if $b' = b$. So assume $b' < b$, and let $b'' < b''' < b$ be as defined above. It suffices to check that $\tau$ is locally convex at $\sigma(b')$. Connect the end points of
the portion $x\sigma(b'')$ of $\tau$ to $\sigma(a)$ to obtain a closed curve $\Gamma$. Note that $\Gamma$ is simple since $x\sigma(b')$ lies above $\lambda$ while $\sigma([b', b''])$ lies below it. Let $\theta$ be the interior angle of $\Gamma$ at $\sigma(b')$. We need to show $\theta \leq \pi$. To this end, let $t_i \in (b', b'')$ be a sequence of regular points of $\sigma$ converging to $b'$, and $\ell_i$ be tangent lines of $\sigma$ at $t_i$. Then, $\ell_i$ converge to a support line of $\sigma([b', b''])$ at $\sigma(b')$, which we call $\ell$. By Lemma 4.2, $\ell_i$ do not cross $S^1$. Consequently, $\ell$ does not cross $S^1$ either. So $\ell$ also supports $x\sigma(b')$. Hence, $\ell$ is a support line of $\Gamma$ at $\sigma(b')$, which yields that $\theta \leq \pi$ as desired.

It remains to check that $E(\sigma) < E(\tau)$. To see this, consider the triangle $\sigma(a)x\sigma(b')$. The interior angle of this triangle at $x$ is $\geq \pi/2$, since $\sigma(a)$ lies on the nonnegative portion of the $y$-axis. Hence, $|x\sigma(b')| < |\sigma(a)x\sigma(b')|$, which yields $L(\tau) < L(\sigma)$. On the other hand, tangent planes of $S^2$ intersect $\mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ in lines that do not cross $S^1$, and any such line has exactly the same number of transverse intersections with $\sigma$ as it does with $\tau$. Hence, $H(\tau) = H(\sigma)$ by definition of horizon. So $E(\sigma) < E(\tau)$ as desired.

\[ \square \]

5 \hspace{1cm} PROOF OF THEOREM 1.1

Set $r = 1$ and let $\gamma : \mathbb{R}/L \to \mathbb{R}^3$ be a minimal inspection curve, as discussed in Section 2. To establish (1), it suffices to show then that $E(\gamma) \leq 2$, as outlined in Section 1. In Section 3, we established that $E(\gamma) = E(\bar{\gamma})$ where $\bar{\gamma} : [0, L] \to \mathbb{R}^2$ is the unfolding of $\gamma$. By Lemma 4.1, $\bar{\gamma}$ admits a spiral decomposition, generated by a collection of mutually disjoint open sets $U_i \subset [0, L)$, $i \in I$. Set $U_0 := [0, L] \setminus \bigcup_i \text{cl}(U_i)$, and let $\bar{\gamma}_i := \big|\bar{\gamma}|_{\text{cl}(U_i)}, \bar{\gamma}_0 := \bar{\gamma}|_{U_0}$. As in the proof of Zalgaller’s conjecture in [15, section 10], we have

\[ E(\gamma) = \frac{H(\bar{\gamma})}{L(\bar{\gamma})} \leq \frac{1}{L(\bar{\gamma})} \sum_i H(\bar{\gamma}_i) = \frac{1}{L(\bar{\gamma})} \left( L(\bar{\gamma}_0)E(\bar{\gamma}_0) + \sum_i L(\bar{\gamma}_i)E(\bar{\gamma}_i) \right). \tag{4} \]

By Lemma 2.2, every point $t \in [0, L]$ with $|\gamma(t)| < 1$ lies on a line segment in $\gamma$ with end points on $S^2$, and thus $\bar{\gamma}(t)$ belongs to a strict spiral (with origin of the spiral corresponding to the midpoint of that line segment). So $|\bar{\gamma}_0| \geq 1$. Then, as described in [15, section 10], $E(\bar{\gamma}_0) \leq 2$. Furthermore $E(\bar{\gamma}_i) \leq 2$ for all $i$ by Proposition 4.3. So $E(\bar{\gamma}) \leq 2$ by (4), as desired. To characterize the case of equality in (1), note that by (4), if $E(\bar{\gamma}) = 2$, then $E(\bar{\gamma}_i) = 2$. Consequently, by Proposition 4.3, $|\bar{\gamma}_i| \geq 1$. So $|\bar{\gamma}| \geq 1$, which yields $|\gamma| \geq 1$. Hence, by the proof of Zalgaller’s conjecture [15, Theorem 1.1], $\gamma$ is the baseball curve.

APPENDIX: HIGHER DIMENSIONS

Here, we establish a higher dimensional version of (1) due to Fedor Nazarov:

**Theorem A.1** (Nazarov). Let $\gamma : [a, b] \to \mathbb{R}^n$ be a curve of length $L$, and $r$ be the inradius of the convex hull of $\gamma$. Then,

\[ L \geq Cn \sqrt{n} r, \tag{A.1} \]

where $C > 0$ is an absolute constant.

By absolute constant, here we mean that $C$ does not depend on $n$ or $\gamma$. A Hamiltonian path in the edge graph of the cross polytope, that is, the unit ball with respect to the $L^1$-norm in $\mathbb{R}^n$, gives
an example of a curve with $L \leq 2n\sqrt{2\pi}r$ [2]. Thus, (A.1) is sharp up to the constant $C$. To establish (A.1), we may set $r = 1$. Furthermore, we may assume that $n$ is even. Indeed suppose that (A.1) holds for even $n$. If $n$ is odd and bigger than 1, then we may project $\gamma$ into $\mathbb{R}^{n-1}$ to obtain $L \geq C(n-1)^{3/2} \geq (C/2)n^{3/2}$. Finally, it is enough to show that if $L \leq Cn\sqrt{n}$, for some absolute constant $C$, then the inradius of $\text{conv}(\gamma)$ is less than 1, which means that there exists $u \in S^{n-1}$ such that $\langle \gamma(t), u \rangle \leq 1$ for all $t \in [a, b]$. Equivalently, if $L \leq 2n\sqrt{n}$, then $\langle \gamma(t), u \rangle \leq C/2$. In summary, it suffices to show the following:

**Proposition A.2.** Let $\gamma : [a, b] \to \mathbb{R}^{2n}$ be a curve of length $\leq 2n\sqrt{n}$. Then, there exists $u \in S^{2n-1}$ such that $\langle \gamma(t), u \rangle \leq C$ for all $t \in [a, b]$.

To prove the above proposition, we again assume that $\gamma$ has constant speed. Let $t_i \in [a, b]$, $i = 0, \ldots, n$, be equidistant points with $t_0 := a$, $t_n := b$, and set $s_i := (t_{i-1} + t_i)/2$ for $i = 1, \ldots, n$. Let $H$ be an $n$-dimensional subspace of $\mathbb{R}^{2n}$, which is orthogonal to each $\gamma(s_i)$, and $\bar{\gamma}$ be the projection of $\gamma$ into $H$. Then, $\bar{\gamma}|_{[s_{i-1}, s_i]}$, $\bar{\gamma}|_{[s_i, t_i]}$ are curves of length $\leq \sqrt{n}$ with one end at $o$, since $\gamma$ has constant speed. So, identifying $H$ with $\mathbb{R}^n$, we have reduced Proposition A.2 to the following:

**Proposition A.3.** Let $\gamma_i : [a, b] \to \mathbb{R}^n$, $i = 1, \ldots, 2n$, be curves of length $\leq \sqrt{n}$ with $\gamma_i(a) = o$. Then, there exists $u \in S^{n-1}$ such that $\langle \gamma_i(t), u \rangle \leq C$ for all $t \in [a, b]$.

To prove the last proposition, we employ the standard Gaussian measure, which is defined for Borel sets $A \subset \mathbb{R}^n$ as

$$
\mu(A) := \frac{1}{(\sqrt{2\pi})^n} \int_A e^{-|x|^2/2} d\lambda(x),
$$

where $\lambda$ is the $n$-dimensional Lebesgue measure. We also record that if $K_i$ are a family of convex sets that are symmetric with respect to $o$, then

$$
\mu \left( \bigcap_i K_i \right) \geq \prod_i \mu(K_i)
$$

(A.2)

by the Gaussian correlation inequality [16, 19]. Here, we need this fact only for slabs, which had been established in [20].

**Proof of Proposition A.3.** We set $[a, b] = [0, 1]$ and assume that $\gamma_i$ have constant speed. For every $t \in [0, 1]$ and $i$ there exist vectors $v_{ik}(t) \in \mathbb{R}^n$, such that

$$
\gamma_i(t) := \sum_{k=1}^{\infty} v_{ik}(t), \quad \text{and} \quad |v_{ik}(t)| \leq \frac{\sqrt{n}}{2^k}.
$$

To generate these vectors, set $t_0 := 0$, and let $t_k := t_{k-1} - 1/2^k$, if $t < t_{k-1}$, and $t_k := t_{k-1} + 1/2^k$ otherwise. Then, we set $v_{ik}(t) := \gamma_i(t_k) - \gamma_i(t_{k-1})$. Note that each $v_{ik}(t)$ is chosen from a set $V_{ik}$.
of cardinality \(2^{k-1}\), which is independent of \(t\). Now consider the slabs

\[
S(v) := \left\{ x \in \mathbb{R}^n \left| \langle x, v \rangle \leq \frac{\sqrt{n}}{k^2} \right. \right\}, \quad v \in V_i^k,
\]

which have width \(2(\sqrt{n}/k^2)/|v| \geq 2k/k^2\), and set

\[
A := \bigcap_{i=1}^{2n} \bigcap_{k=1}^{\infty} \bigcap_{v \in V_i^k} S(v).
\]

By Fubini’s theorem, and a standard estimate for the Gaussian integral,

\[
\mu(S(v)) \geq \frac{1}{\sqrt{2\pi}} \int_{-a_k}^{a_k} e^{-t^2/2} \, dt \geq 1 - e^{-a_k^2/2},
\]

where \(a_k := 2k/k^2\). So by (A.2),

\[
\mu(A) \geq \prod_{i=1}^{2n} \prod_{k=1}^{\infty} \prod_{v \in V_i^k} \mu(S(v)) \geq \left( \prod_{k=1}^{\infty} \left( 1 - e^{-a_k^2/2} \right)^{2k-1} \right)^{2n}.
\]

Since \(\ln(1 - e^{-x}) \geq -2e^{-x}\) for \(x \geq 32/81\), which is the smallest value of \(a_k^2/2\) (achieved for \(k = 3\)), we have

\[
\prod_{k=1}^{\infty} \left( 1 - e^{-a_k^2/2} \right)^{2k-1} = \exp \left( \sum_{k=1}^{\infty} 2^{k-1} \ln \left( 1 - e^{-a_k^2/2} \right) \right) \geq \exp \left( - \sum_{k=1}^{\infty} 2^k e^{-a_k^2/2} \right) =: \sqrt{\delta} > 0.
\]

So we conclude that \(\mu(A) \geq \delta^n\) where \(\delta > 0\) is an absolute constant. Next note that, if \(B^n_r\) is the ball of radius \(r\) centered at \(o\) in \(\mathbb{R}^n\), with volume \(|B^n_r|\), then

\[
\mu(B^n_r) \leq \frac{|B^n_r|}{(\sqrt{2\pi})^n} = \frac{\sqrt{e}r^n}{\sqrt{n}} \left( \frac{|B^n_r|}{(\sqrt{2\pi})^n} \right)^n \leq \left( \frac{\sqrt{e}r^n}{\sqrt{n}} \right)^n \mu\left(B^n_{\sqrt{r}n}\right) \leq \left( \frac{\sqrt{e}r^n}{\sqrt{n}} \right)^n.
\]

So if \(r := \delta \sqrt{n}/\sqrt{e}\), then \(\mu(B^n_r) \leq \delta^n \leq \mu(A)\). Consequently, \(A \not\subset \text{int}(B^n_r)\), which means that there exists \(u_0 \in A\) with \(|u_0| \geq r\). Now setting \(u := u_0/|u_0|\), we have

\[
\langle y_i(t), u \rangle = \sum_{k=1}^{\infty} \langle v_{ik}, u \rangle \leq \frac{1}{r} \sum_{k=1}^{\infty} \langle v_{ik}, u_0 \rangle \leq \frac{\sqrt{e}}{\delta \sqrt{n}} \sum_{k=1}^{\infty} \frac{\sqrt{n}}{k^2} \leq \frac{2\sqrt{e}}{\delta} =: C,
\]

as desired. \(\square\)
Note A.4. When $\gamma_i$ in Proposition A.3 trace lines segments, we obtain the following result in discrete geometry: If $N \leq 2n$ points in $\mathbb{R}^n$ contain $S^{n-1}$ within their convex hull, then at least one of them has distance $\geq \sqrt{n}/C$ from o. Equivalently, if $N \leq 2n$ disks of geodesic radius $\rho$ cover $S^{n-1}$, then $\cos(\rho) \leq C/\sqrt{n}$, which had been observed earlier by Tikhomirov [21]. Furthermore, proof of Proposition A.3 allows an estimate for $C$ as follows. If $\gamma_i$ trace line segments, we may set $k = 1$. Then, $\mu(S(v)) \geq \int_{-2}^{2} e^{-t^2/2} dt/\sqrt{2\pi} \geq 0.95$. So $\delta = (0.95)^2$, which yields $C = \delta/(2\sqrt{e}) \approx 3.65$. It has been conjectured that the optimal value of $C$ is 1, which would correspond to the case where the points form the vertices of a cross polytope [5, Conjecture 1.3]. This has been shown only for $n = 3$ [10], see [11, p. 34], and $n = 4$ [9].

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