Solution to a Zero-Sum Differential Game with Fractional Dynamics via Approximations

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Abstract
The paper deals with a zero-sum differential game in which the dynamical system is described by a fractional differential equation with the Caputo derivative of an order \( \alpha \in (0, 1) \). The goal of the first (second) player is to minimize (maximize) a given quality index. The main contribution of the paper is the proof of the fact that this differential game has the value, i.e., the lower and upper game values coincide. The proof is based on the appropriate approximation of the game by a zero-sum differential game in which the dynamical system is described by a first-order functional differential equation of a retarded type. It is shown that the values of the approximating differential games have a limit, and this limit is the value of the original game. Moreover, the optimal players’ feedback control procedures are proposed that use the optimally controlled approximating system as a guide. An example is considered, and the results of computer simulations are presented.

Keywords
Differential game · Value of the game · Optimal strategies · Fractional derivative · Fractional differential equation · Approximation · Control with a guide

1 Introduction
The paper is devoted to zero-sum differential games for dynamical systems described by fractional differential equations. For the basics of fractional calculus, theory of fractional differential equations and their applications, the reader is referred to [8,25,32,42,46,48]. Despite the fact that a great number of various control problems for fractional-order systems are intensively studied nowadays, only a few works deal with differential games for such systems (see [1,7,41,45] and the references therein). Furthermore, in these works, only some special classes of linear pursuit-evasion differential games are investigated.

In the paper, we mainly follow the game-theoretical approach [28,29,31,35,44,50,51] and consider a quite general formulation of a zero-sum differential game (see also [3,4,6,9,...]
We suppose that a motion of a dynamical system is described by a nonlinear fractional differential equation with the Caputo derivative of an order $\alpha \in (0, 1)$. The game is considered on a finite time interval. The goal of the first (second) player is to minimize (maximize) a given quality index evaluating the system’s motion. The main contribution of the paper is the proof of the fact that the considered differential game has the value, i.e., the lower and upper values of the game coincide.

Due to non-local structure of fractional-order derivatives, fractional differential equations are used for describing dynamical systems with the memory effects of a special kind. It makes these equations close to functional differential equations (see, e.g., [5,19,27]). In particular, the Riemann–Liouville fractional integral of the order $(1-\alpha)$ of the solution to the considered fractional differential equation is, by the definition, the solution to the corresponding first-order functional differential equation of a neutral type. It allows us to introduce a differential game for this neutral-type system and study it instead of the original game. However, to the best of our knowledge, there are no results that can be applied for investigating the obtained differential game. Namely, in [2,15–17,36,37,40,43], only some special classes of neutral-type systems are considered, and, in [53], the game is considered in the classes of players’ program (open-loop) strategies.

Nevertheless, following [13], based on the finite-difference Grünwald–Letnikov formulas for calculation of fractional derivatives (see, e.g., [48, p. 386]), one can approximate the obtained differential game for the first-order neutral-type system by a differential game for a first-order retarded-type system. Let us note that differential games for dynamical systems described by functional differential equations of a retarded type are quite well studied (see, e.g., [31,33–35,44] and the references therein), especially in comparison with differential games for neutral-type systems. Thus, applying the results of [33–35], we derive that the approximating differential game has the value in the appropriate classes of players’ positional (closed-loop) strategies.

Further, based on the ideas from [30] (see also [37]), in order to establish a connection between the original and approximating differential games, we consider the players’ feedback control procedures that use the optimally controlled approximating system as a guide (see, e.g., [31, § 8.2]). It allows us to prove that the values of the approximating games have a limit, and this limit coincides with the value of the original game. The key point here is the mutual aiming procedure between the original and approximating systems [13] that provides the desired proximity between the systems’ motions. Moreover, in particular, we obtain that the proposed players’ control procedures with guides guarantee the game value with a given accuracy, and, in this sense, they can be called optimal.

Let us note that differential games give a natural formalization of control problems under conditions of unknown disturbances (see, e.g., [28,29,31,51]). For some examples of control problems for fractional-order systems arising in applications, the reader is referred to [10,18,23,24,52]. It seems that the results of the paper may be useful for studying these problems when the systems under consideration contain dynamical disturbances or uncertainties, which are not necessarily small. On the one hand, these factors essentially complicate the problems, but, on the other hand, they often have to be taken into account, since they can be caused, for example, by inaccuracies of modeling. Let us note also that in some other frameworks, control problems for fractional-order systems under conditions of disturbances are studied, e.g., in [22,49].

The rest of the paper is organized as follows. In Sect. 2, we introduce the notations, recall the definitions of fractional-order integrals and derivatives and give some of their properties. In Sect. 3, the considered differential game for a fractional-order system is described, and in particular, the notion of the game value is defined. The corresponding differential game for a
first-order neutral-type system is discussed in Sect. 4. In Sect. 5, we propose an approximation of this game by a differential game for a first-order retarded-type system. In Sect. 6, the mutual aiming procedure between the original and approximating systems and the optimal players’ control procedures with guides are described, and the limit of the values of the approximating differential game is introduced. In Sect. 7, we prove that the original differential game has the value. An example is considered in Sect. 8. Concluding remarks are given in Sect. 9.

2 Notations and Definitions

Let \( t_0, \vartheta \in \mathbb{R}, t_0 < \vartheta \), and \( n \in \mathbb{N} \) be fixed. Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space with the inner product \( \langle \cdot, \cdot \rangle \) and the norm \( \| \cdot \| \). For any \( t_s \in [t_0, \vartheta] \), we denote by \( L^\infty([t_0, t_s], \mathbb{R}^n) \) the set of essentially bounded (Lebesgue) measurable functions \( x : [t_0, t_s] \to \mathbb{R}^n \) such that

\[
\| x(\cdot) \|_\infty = \text{ess sup} \| x(t) \| < \infty.
\]

Let \( \mathcal{C}([t_0, t_s], \mathbb{R}^n) \) be the Banach space of continuous functions \( x : [t_0, t_s] \to \mathbb{R}^n \) with the standard uniform norm, which is also denoted by \( \| \cdot \|_\infty \). Let \( \text{Lip}^0([t_0, t_s], \mathbb{R}^n) \) be the set of functions \( x(\cdot) \in \mathcal{C}([t_0, t_s], \mathbb{R}^n) \) that are Lipschitz continuous and satisfy the equality \( x(t_0) = 0 \). For \( L \geq 0 \), we denote by \( \text{Lip}^0([t_0, t_s], \mathbb{R}^n) \) the set of functions \( x(\cdot) \in \text{Lip}^0([t_0, t_s], \mathbb{R}^n) \) that satisfy the Lipschitz condition with this constant \( L \).

Let us shortly describe some basic properties of the fractional-order integrals and derivatives that are used in the paper. For a more detailed discussion, the reader is referred to \([8,25,32,42,46,48]\) (see also \([12,13]\)).

Let \( \alpha \in (0, 1) \) be fixed. For a function \( \varphi(\cdot) \in L^\infty([t_0, t_s], \mathbb{R}^n) \), the Riemann–Liouville fractional integral of the order \( \alpha \) is defined by

\[
(I^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\varphi(\tau)}{(t - \tau)^{1-\alpha}} \mathrm{d}\tau, \quad t \in [t_0, t_s],
\]

where \( \Gamma \) is the gamma function. Let us denote by \( I^\alpha(L^\infty([t_0, t_s], \mathbb{R}^n)) \) the set of functions \( x : [t_0, t_s] \to \mathbb{R}^n \) that can be represented by the Riemann–Liouville fractional integral of the order \( \alpha \) of a function \( \varphi(\cdot) \in L^\infty([t_0, t_s], \mathbb{R}^n) \), i.e., \( x(t) = (I^\alpha \varphi)(t), t \in [t_0, t_s] \). Note that, in the case \( t_s = t_0 \), the set \( I^\alpha(L^\infty([t_0, t_0], \mathbb{R}^n)) \) can be considered as consisting of the single vector \( 0 \in \mathbb{R}^n \).

Let \( x(\cdot) \in I^\alpha(L^\infty([t_0, t_s], \mathbb{R}^n)) \), and let \( x(\cdot) \in L^\infty([t_0, t_s], \mathbb{R}^n) \) be such that \( x(t) = (I^\alpha \varphi)(t), t \in [t_0, t_s] \). From the definition of \( I^\alpha(L^\infty([t_0, t_s], \mathbb{R}^n)) \) it follows that, for any \( t' \in [t_0, t_s] \), the restriction \( x_{t'}(t) = x(t), t \in [t_0, t'] \), of \( x(\cdot) \) [this notation is introduced below in (8)] satisfies the inclusion \( x_{t'}(\cdot) \in I^\alpha(L^\infty([t_0, t'], \mathbb{R}^n)) \) and the equality \( x_{t'}(t) = (I^\alpha \varphi)(t), t \in [t_0, t'] \). Let us consider the function \( y(t) = (I^{1-\alpha} x)(t), t \in [t_0, t_s] \). Then, according to \([48, \text{Theorem 2.5}]\), we obtain

\[
y(t) = (I^{1-\alpha}(I^\alpha \varphi))(t) = \int_{t_0}^{t} x(\tau) \mathrm{d}\tau, \quad t \in [t_0, t_s].
\]

Therefore, the inclusion \( y(\cdot) \in \text{Lip}^0([t_0, t_s], \mathbb{R}^n) \) is valid with the constant \( L = \| \varphi(\cdot) \|_\infty \), and the Riemann–Liouville fractional derivative of \( x(\cdot) \) of the order \( \alpha \) defined by

\[
(D^\alpha x)(t) = \frac{\mathrm{d}}{\mathrm{d}t} (I^{1-\alpha} x)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_0}^{t} \frac{x(\tau)}{(t - \tau)^\alpha} \mathrm{d}\tau
\]
exists for almost every \( t \in [t_0, t_s] \). In particular, we have \((D^\alpha x)(t) = \dot{y}(t) = \varphi(t)\) for almost every \( t \in [t_0, t_s] \), where we denote \( \dot{y}(t) = dy(t)/dt \). Hence, we derive \((D^\alpha x)(\cdot) \in L^\infty([t_0, t_s], \mathbb{R}^n)\) and \( x(t) = (I^\alpha(D^\alpha x))(t), t \in [t_0, t_*] \) (see, e.g., [12, Proposition 2.2] and the references therein). In particular, we obtain (see, e.g., [48, Lemma 2.2 and Theorem 2.5])

\[
(x(t) = (D^{1-\alpha} y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\dot{y}(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad t \in [t_0, t_\ast]. \tag{1}
\]

Note also that, in accordance with [12, Proposition 2.1] (see also the references therein), one can choose \( H > 0 \) such that for any \( t_* \in [t_0, \vartheta] \) and any \( x(\cdot) \in I^\alpha(L^\infty([t_0, t_*], \mathbb{R}^n)) \), the inequality below holds:

\[
\|x(t) - x(t')\| \leq H \|D^\alpha x(\cdot)\|_{L^\infty} |t - t'|^\alpha, \quad t, t' \in [t_0, t_*]. \tag{2}
\]

For a vector \( w \in \mathbb{R}^n \), let us consider the set \( \{w\} + I^\alpha(L^\infty([t_0, t_*], \mathbb{R}^n)) \) consisting of functions \( x : [t_0, t_*] \to \mathbb{R}^n \) that can be represented in the following form: \( x(t) = w + x_+(t), t \in [t_0, t_*] \), where \( x_+(\cdot) \in I^\alpha(L^\infty([t_0, t_*], \mathbb{R}^n)) \). Note that \( x(t_0) = w \) since \( x_+(t_0) = 0 \). For a function \( x(\cdot) \in \{w\} + I^\alpha(L^\infty([t_0, t_*], \mathbb{R}^n)) \), the Caputo fractional derivative of the order \( \alpha \) defined by

\[
(CD^\alpha x)(t) = (D^\alpha(x(\cdot) - x(t_0)))(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(\tau) - x(t_0)}{(t-\tau)^\alpha} d\tau \tag{3}
\]

exists for almost every \( t \in [t_0, t_*] \). In the case when \( x(t_0) = 0 \), the Riemann–Liouville and Caputo fractional derivatives coincide.

### 3 Differential Game with Fractional Dynamics

#### 3.1 Fractional-Order System

We consider a dynamical system which motion is described by the following fractional differential equation with the Caputo derivative of the order \( \alpha \):

\[
(CD^\alpha x)(t) = f(t, x(t), u(t), v(t)), \quad t \in [t_0, \vartheta], \\
\quad x(t) \in \mathbb{R}^n, \quad u(t) \in U, \quad v(t) \in V. \tag{4}
\]

Here, \( t \) is the time; \( x(t) \) is the state vector at the time \( t \); \( u(t) \) and \( v(t) \) are, respectively, the control vectors of the first and second players at the time \( t \); \( t_0 \) and \( \vartheta \) are called the initial and terminal times; the sets \( U \subset \mathbb{R}^{n_u} \) and \( V \subset \mathbb{R}^{n_v} \) are compact, \( n_u, n_v \in \mathbb{N} \). We suppose that the function \( f : [t_0, \vartheta] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n \) satisfies the following conditions:

(A.1) The function \( f \) is continuous.

(A.2) For any \( R \geq 0 \), there exists \( \lambda > 0 \) such that

\[
\|f(t, x, u, v) - f(t, x', u, v)\| \leq \lambda\|x - x'\|
\]

for any \( t \in [t_0, \vartheta], x, x' \in B(R) = \{y \in \mathbb{R}^n : \|y\| \leq R\}, u \in U, \) and \( v \in V \).

(A.3) There exists \( c > 0 \) such that

\[
\|f(t, x, u, v)\| \leq (1 + \|x\|)c
\]

for any \( t \in [t_0, \vartheta], x \in \mathbb{R}^n, u \in U, \) and \( v \in V \).
According to (3), we have
\[ \min_{u \in U} \max_{v \in V} \langle s, f(t, x, u, v) \rangle = \max_{v \in V} \min_{u \in U} \langle s, f(t, x, u, v) \rangle = H(t, x, s) \]
(5)
for any \( t \in [t_0, \vartheta] \) and \( x, s \in \mathbb{R}^n \).

Note that these conditions are quite typical for the differential games theory with first-order dynamics (see, e.g., [31, p. 7]).

3.2 Admissible Positions of the System

Let \( R_0 > 0 \) be fixed. By a position of system (4), we mean a pair \((t, w(\cdot))\) consisting of a time \( t \in [t_0, \vartheta] \) and a function \( w : [t_0, t] \to \mathbb{R}^n \), which is treated as a motion history on the interval \([t_0, t]\). A position \((t, w(\cdot))\) is called admissible if the relations below are valid:

\[
\begin{align*}
 w(t_0) & \in B(R_0), \\
 w(\cdot) & \in [w(t_0)] + I^\alpha(L^\infty([t_0, t], \mathbb{R}^n)), \\
 \| (C D^\alpha w)(\tau) \| & \leq (1 + \| w(\tau) \|)c \text{ for a.e. } \tau \in [t_0, t],
\end{align*}
\]
(6)
where \( c \) is the constant from condition (A.3). According to the definition given in Sect. 2, the second inclusion in (6) means that there exists a function \( \varphi(\cdot) \in L^\infty([t_0, t], \mathbb{R}^n) \) such that \( w(\tau) = w(t_0) + (I^\alpha \varphi)(\tau), \tau \in [t_0, t] \). The set of the admissible positions is denoted by \( G_* \).

**Proposition 1** The set \( G_* \) is not empty, and there exist \( R_1 > 0, M_1 > 0, \) and \( H_1 > 0 \) such that, for any \((t, w(\cdot)) \in G_* \), the inequalities below are valid:

\[
\begin{align*}
 \| w(\tau) \| & \leq R_1, \quad \tau \in [t_0, t], \\
 \| (C D^\alpha w)(\tau) \| & \leq M_1 \text{ for a.e. } \tau \in [t_0, t], \\
 \| w(\tau) - w(\tau') \| & \leq H_1 |\tau - \tau'|^\alpha, \quad \tau, \tau' \in [t_0, t].
\end{align*}
\]

**Proof** Let \( t \in [t_0, \vartheta] \) and \( w_0 \in B(R_0) \). Let us consider the function \( w(\tau) = w_0, \tau \in [t_0, t] \). According to (3), we have \((C D^\alpha w)(\tau) = 0, \tau \in [t_0, t] \). Hence, the inclusion \((t, w(\cdot)) \in G_* \) is valid, and, therefore, the set \( G_* \) is not empty.

Further, let us define
\[
R_1 = (1 + R_0)E_\alpha((\vartheta - t_0)^\alpha c) - 1, \quad M_1 = (1 + R_1)c, \quad H_1 = H M_1,
\]
where \( c \) is the constant from (A.3), \( E_\alpha = E_{\alpha, 1} \) is the Mittag–Leffler function (see, e.g., [48, (1.90)]), and \( H \) is the constant from (2). Let \((t, w(\cdot)) \in G_* \). Then, due to (6) and the results given in Sect. 2, for \( \tau \in [t_0, t] \), we have
\[
\| w(\tau) - w(t_0) \| = \left\| \frac{1}{\Gamma(\alpha)} \int_{t_0}^\tau \frac{(C D^\alpha w)(\xi)}{(\tau - \xi)^{1-\alpha}} d\xi \right\| \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^\tau (1 + \| w(\xi) \|)c (\tau - \xi)^{1-\alpha} d\xi,
\]
and, therefore,
\[
\| w(\tau) \| \leq R_0 + \frac{c}{\Gamma(\alpha)} \int_{t_0}^\tau \frac{1 + \| w(\xi) \|}{(\tau - \xi)^{1-\alpha}} d\xi.
\]
From this inequality, applying the fractional version of Bellman–Gronwall lemma (see, e.g., [8, Lemma 6.19] and also [12, Lemma 1.1]), we conclude \( \| w(\tau) \| \leq R_1, \tau \in [t_0, t] \). Hence, according to (6), we obtain
\[
\| (C D^\alpha w)(\tau) \| \leq (1 + \| w(\tau) \|)c \leq (1 + R_1)c = M_1 \text{ for a.e. } \tau \in [t_0, t].
\]
Finally, by the choice of $H$, we derive
\[
\|w(\tau) - w(\tau')\| \leq HM_1|\tau - \tau'|^\alpha = H_1|\tau - \tau'|^\alpha, \quad \tau, \tau' \in [t_0, t].
\]
The proposition is proved. \qed

Let $(t_*, w_*(\cdot)) \in G_*, \ t_* < \vartheta$, and $t^* \in (t_*, \vartheta)$. By admissible control realizations (controls) of the first and second players on the interval $[t_*, t^*)$, we can measure functions $u : [t_*, t^*) \rightarrow U$ and $v : [t_*, t^*) \rightarrow V$, respectively. The sets of the admissible control realizations of the players are denoted by $U(t_*, t^*)$ and $V(t_*, t^*)$. Following [20] (see also [12]), by a motion of system (4) generated from the initial position $(t_*, w_*(\cdot))$ by players’ control realizations $u(\cdot) \in U(t_*, t^*)$ and $v(\cdot) \in V(t_*, t^*)$, we mean a function $x(\cdot) \in \{w_*(t_0)\} + I^\alpha(L^\infty([t_0, t^*], \mathbb{R}^n))$ that satisfies the initial condition
\[
x(t) = w_*(t), \quad t \in [t_0, t_*],
\]
and, together with $u(\cdot)$ and $v(\cdot)$, satisfies the differential equation in (4) for almost every $t \in [t_*, t^*)$. For such a motion $x(\cdot)$ and a time $t \in [t_0, t^*)$, we denote by $(t, x_*(\cdot))$ the corresponding position of system (4), and here, $x_*(\cdot)$ is defined by
\[
x_*(\tau) = x(\tau), \quad \tau \in [t_0, t].
\]

**Proposition 2** Let $(t_*, w_*(\cdot)) \in G_*, \ t_* < \vartheta$, and $t^* \in (t_*, \vartheta)$. Then, any players’ control realizations $u(\cdot) \in U(t_*, t^*)$ and $v(\cdot) \in V(t_*, t^*)$ generate from the initial position $(t_*, w_*(\cdot))$ a unique motion $x(\cdot)$ of system (4). Moreover, for any $t \in [t_0, t^*)$, the inclusion $(t, x_*(\cdot)) \in G_*$ is valid.

**Proof** Let $(t_*, w_*(\cdot)) \in G_*, \ t_* < \vartheta$, $t^* \in (t_*, \vartheta)$, $u(\cdot) \in U(t_*, t^*)$, and $v(\cdot) \in V(t_*, t^*)$. The existence and uniqueness of the corresponding motion $x(\cdot)$ of system (4) can be proved by the standard scheme (see, e.g., [8, Theorem 6.1], [54, Theorem 3.1], and also [12, Theorem 3.1]), if we note that $x(\cdot)$ is the motion of system (4) if and only if $x(\cdot)$ satisfies the inclusion $x(\cdot) \in C([t_0, t^*], \mathbb{R}^n)$, initial condition (7), and the integral equation
\[
x(t) = w_*(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} \left( \frac{C}{D^\alpha} w_*(\tau) \right) \frac{\partial}{(t - \tau)^{1-\alpha}} \, d\tau
\]
\[
\quad + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} \left( \frac{f(\tau, x(\tau), u(\tau), v(\tau))}{(t - \tau)^{1-\alpha}} \right) d\tau, \quad t \in [t_*, t^*].
\]
Further, for $t \in [t_0, t_*]$, the inclusion $(t, x_*(\cdot)) \in G_*$ follows from initial condition (7) and the inclusion $(t_*, w_*(\cdot)) \in G_*$. For $t \in (t_*, t^*)$, the inclusion $(t, x_*(\cdot)) \in G_*$ is valid due to (A.3). The proposition is proved. \qed

Directly from Proposition 2 we derive the following property of motions of system (4). Let $(t_*, w_*(\cdot)) \in G_*, \ t_* < \vartheta$, $t^* \in (t_*, \vartheta)$, and let $x(\cdot)$ be the motion generated from $(t_*, w_*(\cdot))$ by $u(\cdot) \in U(t_*, t^*)$ and $v(\cdot) \in V(t_*, t^*)$. Further, let $t^{**} \in (t^*, \vartheta)$, and let $x^{**}(\cdot)$ be the motion generated from $(t^*, x_*(\cdot))$ by $u^{**}(\cdot) \in U(t^*, t^{**})$ and $v^{**}(\cdot) \in V(t^*, t^{**})$. Then, $x^{**}(\cdot)$ can be considered as the motion generated from $(t_*, w_*(\cdot))$ by the control realizations
\[
u^{**}(t) = \begin{cases} u(t), & t \in [t_*, t^*), \\ v(t), & t \in [t^*, t^{**}). \end{cases}
\]
In particular, this property allows us to consider step-by-step feedback control procedures for constructing players’ control realizations (see Sect. 6).
Let us note that, in the degenerate case of initial positions \((\vartheta, w_\ast)\) \(\in G_\ast\), the motion \(x(\cdot)\) of system (4) is completely determined by initial condition (7). Therefore, there is no need in considering players’ control realizations \(u(\cdot)\) and \(v(\cdot)\) and defining the sets \(\mathcal{U}(\vartheta, \vartheta)\) and \(\mathcal{V}(\vartheta, \vartheta)\). However, for convenience, we formally say that the motion \(x(\cdot)\) is generated from \((\vartheta, w_\ast(\cdot))\) by \(u(\cdot) \in \mathcal{U}(\vartheta, \vartheta)\) and \(v(\cdot) \in \mathcal{V}(\vartheta, \vartheta)\).

From Propositions 1 and 2 we derive the following result.

**Corollary 1** Let \(x(\cdot)\) be the motion of system (4) generated from an initial position \((t_\ast, w_\ast(\cdot))\) \(\in G_\ast\) by players’ control realizations \(u(\cdot) \in \mathcal{U}(t_\ast, \vartheta)\) and \(v(\cdot) \in \mathcal{V}(t_\ast, \vartheta)\). Then, the following inequalities hold:

\[
\|x(t)\| \leq R_1, \quad \|x(t) - x(t')\| \leq H_1|t - t'|^\alpha, \quad t, t' \in [t_0, \vartheta],
\]

where the constants \(R_1\) and \(H_1\) are taken from Proposition 1.

### 3.3 Quality Index

Let \(x(\cdot)\) be the motion of system (4) generated from an initial position \((t_\ast, w_\ast(\cdot))\) \(\in G_\ast\) by players’ control realizations \(u(\cdot) \in \mathcal{U}(t_\ast, \vartheta)\) and \(v(\cdot) \in \mathcal{V}(t_\ast, \vartheta)\). Let quality of this motion be evaluated by the index

\[
\gamma = \sigma(x(\cdot)).
\]

(9) We suppose that the function \(\sigma : C([t_0, \vartheta], \mathbb{R}^n) \to \mathbb{R}\) satisfies the following condition:

\[(A.5)\] The function \(\sigma\) is continuous.

For dynamical system (4) and quality index (9), we consider some zero-sum differential game in which the first player aims to minimize the quality index, and the second player aims to maximize it.

### 3.4 Non-anticipative Strategies and the Game Value

To define the value of the differential game (4), (9), we consider non-anticipative strategies of the players (see, e.g., [3, Ch. VIII] and the references therein) and introduce the lower and upper values of the game. Note that, in another terminology, such strategies are called quasi-strategies (see, e.g., [51, p. 24]) or progressive strategies (see, e.g., [9, § XI.4]).

Let \((t_\ast, w_\ast(\cdot))\) \(\in G_\ast\), \(t_\ast \leq \vartheta\), be an initial position. By a non-anticipative strategy of the first player, we mean a function \(\gamma : \mathcal{V}(t_\ast, \vartheta) \to \mathcal{U}(t_\ast, \vartheta)\) with the following property. For any \(t^* \in [t_\ast, \vartheta]\) and any second player’s control realizations \(v(\cdot), v'(\cdot) \in \mathcal{V}(t_\ast, \vartheta)\), if the equality \(v(t) = v'(t)\) is valid for almost every \(t \in [t_\ast, t^*]\), then the corresponding images \(u(\cdot) = \kappa(v(\cdot))\) and \(u'(\cdot) = \kappa(v'(\cdot))\) satisfy the equality \(u(t) = u'(t)\) for almost every \(t \in [t_\ast, t^*]\). The lower value of the differential game (4), (9) is defined by

\[
\rho_\ast(t_\ast, w_\ast(\cdot)) = \inf_k \sup_{v(\cdot) \in \mathcal{V}(t_\ast, \vartheta)} \gamma,
\]

where \(\gamma = \sigma(x(\cdot))\) is the quality index of the motion \(x(\cdot)\) generated from \((t_\ast, w_\ast(\cdot))\) \(\in G_\ast\) by the second player’s control realization \(v(\cdot)\) and the first player’s control realization \(u(\cdot) = \kappa(v(\cdot))\).

Similarly, a function \(\beta : \mathcal{U}(t_\ast, \vartheta) \to \mathcal{V}(t_\ast, \vartheta)\) is a non-anticipative strategy of the second player if, for any \(t^* \in [t_\ast, \vartheta]\) and any \(u(\cdot), u'(\cdot) \in \mathcal{U}(t_\ast, \vartheta)\) such that \(u(t) = u'(t)\) for almost
every \( t \in [t_*, t^*] \), we have \( v(t) = v'(t) \) for almost every \( t \in [t_*, t^*] \), where \( v(\cdot) = \beta(u(\cdot)) \) and \( v'(\cdot) = \beta(u'(\cdot)) \). The upper value of the game is defined by
\[
\rho^{(u)}(t_*, w_*(\cdot)) = \sup_{\beta} \inf_{u(\cdot) \in \mathcal{U}(t_*, \vartheta)} \gamma.
\]

Let us note that, according to the agreement made in Sect. 3.2, in the case of initial positions \((\vartheta, w_*(\cdot)) \in G_*\), we put
\[
\rho^{(u)}(\vartheta, w_*(\cdot)) = \rho^{(v)}(\vartheta, w_*(\cdot)) = \sigma(w_*(\cdot)).
\]

If the lower and upper game values coincide for any initial position \((t_*, w_*(\cdot)) \in G_*\), then we say that the game has the value
\[
\rho(t_*, w_*(\cdot)) = \rho^{(u)}(t_*, w_*(\cdot)) = \rho^{(v)}(t_*, w_*(\cdot)), \quad (t_*, w_*(\cdot)) \in G_*.
\]

The goal of the paper is to prove that the differential game (4), (9) has the value, and, for any initial position \((t_*, w_*(\cdot)) \in G_*\), construct the players’ feedback control procedures that guarantee the game value \(\rho(t_*, w_*(\cdot))\) with a given accuracy \(\zeta > 0\). These results are formulated in Theorem 1 (see Sect. 7). The proof of this theorem follows the scheme from [37, Theorem 2] and is based on the appropriate approximation of the differential game (4), (9). Before describing this approximation, in the next section, we rewrite the considered differential game in another form.

### 4 Differential Game for a Neutral-Type System

Let \( x(\cdot) \) be the motion of system (4) generated from an initial position \((t_*, w_*(\cdot)) \in G_*\), \( t_* < \vartheta \), by players’ control realizations \( u(\cdot) \in \mathcal{U}(t_*, \vartheta) \) and \( v(\cdot) \in \mathcal{V}(t_*, \vartheta) \). Let us consider the function
\[
y(t) = (I^{1-\alpha}(x(\cdot) - w_*(t_0)))(t), \quad t \in [t_0, \vartheta].
\]
Since \( x(\cdot) \in \{w_*(t_0)\} + I^\alpha(L^\infty([t_0, \vartheta], \mathbb{R}^n)) \), then, according to the results given in Sect. 2, we have
\[
y(\cdot) \in \text{Lip}^0([t_0, \vartheta], \mathbb{R}^n),
\]
\[
y(t) = (\xi D^\alpha x)(t) \text{ for a.e. } t \in [t_0, \vartheta],
\]
\[
x(t) = w_*(t_0) + (D^{1-\alpha}y)(t), \quad t \in [t_0, \vartheta].
\]

Substituting these equalities into (4), we obtain that, instead of the original differential game (4), (9), one can consider the differential game for the dynamical system
\[
\dot{y}(t) = f(t, w_*(t_0) + (D^{1-\alpha}y)(t), u(t), v(t)), \quad t \in [t_*, \vartheta],
\]
under the initial condition
\[
y(t) = (I^{1-\alpha}(w_*(\cdot) - w_*(t_0)))(t), \quad t \in [t_0, t_*],
\]
and the quality index
\[
\gamma = \sigma(w_*(t_0) + (D^{1-\alpha}y)(\cdot)).
\]

Furthermore, due to (1), one can rewrite (14) as follows:
\[
\dot{y}(t) = f\left(t, w_*(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{\dot{y}(\tau)}{(t - \tau)^{1-\alpha}} \, d\tau, u(t), v(t)\right), \quad t \in [t_*, \vartheta].
\]
Note that the right-hand side of the differential equation in (17) depends explicitly on the history of the derivative $\dot{y}(\tau)$ for $\tau \in [t_0, t]$. Therefore, in the terminology of the theory of functional differential equations (see, e.g., [5,19,27]), this equation is a functional differential equation of a neutral type. To the best of our knowledge, in the theory of differential games for neutral-type systems (see the references in Introduction), there are no results that can be directly applied for studying the game (14), (16) and, therefore, the original game (4), (9) too. However, as it is shown in the next section, the game (14), (16) can be approximated by a differential game for a retarded-type system.

5 Approximating Differential Game

Following [13, Sect. 6] and taking into account that $y(t_0) = 0$ [see (13)], let us approximate in relations (14), (16) the fractional derivative $(D^{1-\alpha} y)(t) = (\mathcal{C} D^{1-\alpha} y)(t)$ by the divided fractional difference $h^{1-\alpha} \left( \Delta^{1-\alpha}_h (y(\cdot) - y(t_0)) \right)(t)$ with a step size $h > 0$, where (see, e.g., [48, p. 385])

$$
\left( \Delta^{1-\alpha}_h (y(\cdot) - y(t_0)) \right)(t) = \sum_{i=0}^{\lfloor (t-t_0)/h \rfloor} (-1)^i \binom{1-\alpha}{i} (y(t-ih) - y(t_0)), \quad t \in [t_0, \vartheta],
$$

the symbol $[\tau]$ means the integer part of $\tau \geq 0$, and $\binom{1-\alpha}{i}$ are the binomial coefficients. In this section, we study the differential game obtained after this approximation.

Remark 1 In [13], the difference $(\Delta^{1-\alpha}_h y)(t)$ is used instead of $(\Delta^{1-\alpha}_h (y(\cdot) - y(t_0)))(t)$. However, in order to directly apply the results of [33–35] for studying the approximating differential game introduced below, it is convenient to consider the case when $y(t_0)$ does not necessarily equal zero.

5.1 Approximating Dynamical System

Let us fix a vector $w_0 \in B(R_0)$ and a sufficiently small value of the parameter $h > 0$. Note that, in what follows, the vector $w_0$ corresponds to an initial position $(t_*, w_*(\cdot)) \in G_*$ of system (4) such that $w_0 = w_*(t_0)$. The approximating differential game below is determined by these two parameters $w_0$ and $h$.

Let us denote by $G$ the set of pairs $(t, r(\cdot))$ such that $t \in [t_0, \vartheta]$ and $r(\cdot) \in C([t_0, t], \mathbb{R}^n)$. This set is considered with the following metric (see, e.g., [34] and also [35, p. 25]):

$$
\rho((t, r(\cdot)), (t', r'(\cdot))) = \max \left\{ \rho^*(t, r(\cdot), (t', r'(\cdot))), \rho^*(t', r'(\cdot)), (t, r(\cdot)) \right\},
$$

where $(t, r(\cdot)), (t', r'(\cdot)) \in G$, and

$$
\rho^*(t, r(\cdot), (t', r'(\cdot))) = \max_{\tau \in [t_0, t]} \min_{\tau' \in [t_0, t']} \left( (\tau - \tau')^2 + \|r(\tau) - r'(\tau')\|^2 \right)^{1/2}.
$$

Let us define the function

$$
f_{w_0,h}(t, r(\cdot), p, q) = f(t, w_0 + h^{1-\alpha} \left( \Delta^{1-\alpha}_h (r(\cdot) - r(t_0)) \right)(t), p, q),
$$

where $(t, r(\cdot)) \in G$, $p \in \mathbb{U}$, and $q \in \mathbb{V}$. From properties (A.1)–(A.4) of the function $f$ it follows that the function $f_{w_0,h}$ satisfies the following conditions:
The functions $f_{w_0, h}$ are equicontinuous in $w_0 \in B(R_0)$ for each $h > 0$.

For any $h > 0$ and any $R \geq 0$, there exists $\lambda_h > 0$ such that, for any $w_0 \in B(R_0)$, the inequality

$$\|f_{w_0, h}(t, r(\cdot), p, q) - f_{w_0, h}(t, r'(\cdot), p, q)\| \leq \lambda_h \max_{\tau \in [0, t]} \|r(\tau) - r'(\tau)\|$$

is valid for any $(t, r(\cdot), t, r'(\cdot)) \in G$ satisfying $\|r(\cdot)\| \leq R$, $\|r'(\cdot)\| \leq R$ and any $p \in \mathbb{U}$, $q \in \mathbb{V}$.

For any $h > 0$, there exists $c_h > 0$ such that, for any $w_0 \in B(R_0)$, the estimate

$$\|f_{w_0, h}(t, r(\cdot), p, q)\| \leq (1 + \max_{\tau \in [0, t]} \|r(\tau)\|)c_h$$

holds for any $(t, r(\cdot)) \in G$, $p \in \mathbb{U}$, and $q \in \mathbb{V}$.

For any $w_0 \in B(R_0)$ and any $h > 0$, the function $f_{w_0, h}$ satisfies the saddle point condition in a small game, i.e.,

$$\min_{p \in \mathbb{U}} \max_{q \in \mathbb{V}} (s, f_{w_0, h}(t, r(\cdot), p, q)) = \max_{q \in \mathbb{V}} \min_{p \in \mathbb{U}} (s, f_{w_0, h}(t, r(\cdot), p, q))$$

for any $(t, r(\cdot)) \in G$ and $s \in \mathbb{R}^n$.

Let us consider the following approximating dynamical system which motion is described by the differential equation

$$\dot{y}(t) = f_{w_0, h}(t, y_1(\cdot), p(t), q(t)), \quad t \in [t_0, \vartheta],$$

$$y(t) \in \mathbb{R}^n, \quad p(t) \in \mathbb{U}, \quad q(t) \in \mathbb{V}.$$  (2.1)

Here, $y(t)$ is the state vector; $y_1(\cdot)$ is a motion history on the interval $[t_0, t]$, which, in accordance with (8), is defined by $y_1(\tau) = y(\tau), \tau \in [t_0, t]$; $p(t)$ and $q(t)$ are respectively the control vectors of the first and second players. Note that, according to (18) and (20), at a time $t \in [t_0, \vartheta]$, the right-hand side of the differential equation in (2.1) depends on $y(t_0)$ and $y(t-ih)$ for $i \in 0,\ldots,\frac{(t-t_0)}{h}$ and, in contrast to (17), does not depend explicitly on the history of the derivative $\dot{y}(\tau), \tau \in [t_0, t]$. Thus, this equation is a functional differential equation of a retarded type.

By a position of approximating system (2.1), we mean a pair $(t, r(\cdot)) \in G$. Let $(t_0, r_0(\cdot)) \in G$, $t_0 < \vartheta$, and $t_0 \in (t_0, \vartheta)$. As in Sect. 3.2, by admissible control realizations of the players, we mean functions $p(\cdot) \in \mathcal{U}(t_0, t^*)$ and $q(\cdot) \in \mathcal{V}(t_0, t^*)$. Due to properties (B.1)–(B.3), from the initial position $(t_0, r_0(\cdot))$, such control realizations $p(\cdot)$ and $q(\cdot)$ uniquely generate (see, e.g., [34, 35, Theorem P1.1] and the references therein) the motion of approximating system (2.1), which is the function $y(\cdot) \in C([t_0, t^*], \mathbb{R}^n)$, such that it is absolutely continuous on $[t_0, t^*]$, satisfies the initial condition $y(t) = r_0(t), t \in [t_0, t_0]$, and, together with $p(\cdot)$ and $q(\cdot)$, satisfies the differential equation in (2.1) for almost every $t \in [t_0, t^*]$. As usual, in the case of initial positions $(\vartheta, r_0(\cdot)) \in G$, by the motion $y(\cdot)$ generated from $(\vartheta, r_0(\cdot))$ by $p(\cdot) \in \mathcal{U}(\vartheta, \vartheta)$ and $q(\cdot) \in \mathcal{V}(\vartheta, \vartheta)$, we mean the function $y(t) = r_0(t), t \in [t_0, \vartheta]$.

According to (15), if an initial position $(t_0, w_0(\cdot)) \in G_0$ of original system (4) is given, we define the corresponding initial position $(t_0, r_0(\cdot)) \in G$ of approximating system (2.1) as follows:

$$r_0(t) = (I^{1-\alpha}(w_0(\cdot) - w_0(t_0)))(t), \quad t \in [t_0, t_0].$$  (2.2)

Due to Proposition 1 and the results given in Sect. 2, the function $r_0(\cdot)$ satisfies the inclusion $r_0(\cdot) \in \text{Lip}_{M_0}^0([t_0, t_0], \mathbb{R}^n)$. Taking this into account, we consider the set $G_h^0$ of positions $(t, r(\cdot)) \in G$ of approximating system (2.1) such that
Hence, by (1), taking into account that
\[ r(\cdot) \in \text{Lip}_0([t_0, t], \mathbb{R}^n), \]
\[ \|\dot{r}(\tau)\| \leq (1 + \max_{\xi \in [t_0, t]}\|r(\xi)\|) \tilde{c}_h \text{ for a.e. } \tau \in [t_0, t], \]
where \( \tilde{c}_h = \max \{c_1, c_h\} \), and \( c_h \) is the constant from condition (B.3). Note that this set is independent on the parameter \( w_0 \).

The set \( G^0_h \) has the following properties (see also [34, (2.9)]). Firstly, for any \((t_*, w_*(\cdot)) \in G_*\), the inclusion \((t_*, r_*(\cdot)) \in G^0_h\) is valid for the function \( r_*(\cdot) \) defined by (22). Secondly, for the motion \( y(\cdot) \) of the approximating system (21) generated from \((t_*, r_*(\cdot)) \in G^0_h\) by \( p(\cdot) \in \mathcal{U}(t_*, \vartheta) \) and \( q(\cdot) \in \mathcal{V}(t_*, \vartheta) \), the inclusion \((t, y_1(\cdot)) \in G^0_h\) holds for any \( t \in [t_0, \vartheta] \), and, therefore, in particular, \( y(\cdot) \in \text{Lip}_0^0([t_0, \vartheta], \mathbb{R}^n) \). Finally, the set \( G^0_h \) is a compact subset of \( G \), considered with the metric \( \varrho \) defined in (19).

Let us note that, in the case when an initial position \((t_*, r_*(\cdot)) \in G^0_h\) is defined by \((t_*, w_*(\cdot)) \in G_*\) according to (22), we have \( y(t_0) = 0 \), and, consequently, approximating system (21) is the same as proposed in [13].

**Proposition 3** There exists \( L_1 > 0 \) such that the following statement holds. Let \((t_*, w_*(\cdot)) \in G_*\) be an initial position of original system (4). Let us consider approximating system (21) for \( w_0 = w_*(t_0) \), any \( h > 0 \), and under the initial position \((t_*, r_*(\cdot)) \) defined by (22). Then, the inclusion \( y(\cdot) \in \text{Lip}_{L_1}^0([t_0, \vartheta], \mathbb{R}^n) \) is valid for any motion \( y(\cdot) \) of the approximating system generated from \((t_*, r_*(\cdot)) \) by \( p(\cdot) \in \mathcal{U}(t_*, \vartheta) \) and \( q(\cdot) \in \mathcal{V}(t_*, \vartheta) \).

**Proof** The proof essentially repeats that of [13, Lemma 2], and we only sketch it here. Let us define \( E_\alpha(\varrho - t_0)^\alpha a \), where \( a > 0 \) is chosen from the same conditions as in the proof of [13, Lemma 2], but the constant \( R_1 \) from Proposition 1 is used instead of \( R_0 \). Let \( y(\cdot) \in \text{Lip}_0^0([t_0, \vartheta], \mathbb{R}^n) \) be the motion of the approximating system described in the statement of the proposition. According to (21) and (22), we have
\[
\dot{y}(t) = (C D^\alpha w_*) (t) \text{ for a.e. } t \in [t_0, t_*],
\]
\[
\dot{y}(t) = f_{w_0, h}(t, y_1(\cdot), p(t), q(t)) \text{ for a.e. } t \in [t_*, \vartheta].
\]
Let \( \varphi(t) = (D^{1-\alpha} y)(t), t \in [t_0, \vartheta] \). One can show that
\[
\|f_{w_0, h}(t, y_1(\cdot), p(t), q(t))\| \leq (1 + \max_{\xi \in [t_0, t]}\|\varphi(\xi)\|)a, \quad t \in [t_*, \vartheta].
\]
Hence, by (1), taking into account that \( \|C D^\alpha w_*\| \leq a \) for almost every \( t \in [t_0, t_*] \) due to the inequality in (6), we obtain
\[
\|\varphi(t)\| \leq \frac{a}{\Gamma(\alpha)} \int_{t_0}^t \frac{1 + \max_{\xi \in [t_0, \tau]}\|\varphi(\xi)\|}{(\tau - \tau)^{1-\alpha}} d\tau, \quad t \in [t_0, \vartheta].
\]
From this inequality, we derive
\[
1 + \max_{\xi \in [t_0, t]}\|\varphi(\xi)\| \leq E_\alpha(\varrho - t_0)^\alpha a, \quad t \in [t_0, \vartheta].
\]
Thus, we conclude \( \|\dot{y}(t)\| \leq L_1 \) for almost every \( t \in [t_0, \vartheta] \). Therefore, the inclusion \( y(\cdot) \in \text{Lip}_{L_1}^0([t_0, \vartheta], \mathbb{R}^n) \) is proved.

**5.2 Approximating Quality Index**

Let us define the function
\[
\sigma_{w_0, h}(y(\cdot)) = \sigma\left(w_0 + h^{1-\alpha} (\Delta^1_h (y(\cdot) - y(t_0))) (\cdot)\right), \quad y(\cdot) \in C([t_0, \vartheta], \mathbb{R}^n).
\]
From property \((A.5)\) of the function \(\sigma\), we derive that the function \(\sigma_{w_0,h}\) satisfies the following condition:

\[ (B.5) \text{ The functions } \sigma_{w_0,h} \text{ are equicontinuous in } w_0 \in B(R_0) \text{ for each } h > 0. \]

Let us introduce the approximating quality index

\[ y_{w_0,h} = \sigma_{w_0,h}(y(\cdot)). \]  

(23)

**Remark 2** Let us note that, even in a simple case when original quality index \((9)\) is terminal, i.e., \(\gamma = \mu(x(\vartheta))\) for a function \(\mu : \mathbb{R}^n \to \mathbb{R}\), the corresponding approximating quality index \(y_{w_0,h} = \mu(w_0 + h^{\alpha - 1}(\Delta^1_h(y(\cdot) - y(t_0))(\vartheta)))\) is still non-terminal, since, according to \((18)\), it depends on \(y(t_0)\) and \(y(\vartheta - ih)\) for \(i \in \mathbb{Z}, \lceil(\vartheta - t_0)/h\rceil\).

The approximating differential game is considered for dynamical system \((21)\) and quality index \((23)\). The first player minimizes the quality index, and the second player maximizes it. Let us note that, due to conditions \((B.1)-(B.5)\), for studying the approximating game, the results of \([33-35]\) can be applied. In particular, following these works, we consider the approximating game in the classes of positional strategies of the players.

### 5.3 Positional Strategies and the Value of the Approximating Game

Let \(w_0 \in B(R_0)\) and \(h > 0\) be fixed. In the approximating differential game \((4), (9)\), by the positional strategies \(P = P_{w_0,h}\) and \(Q = Q_{w_0,h}\) of the players, we mean arbitrary functions

\[ G \times (0, \infty) \ni (t, r(\cdot), \varepsilon) \mapsto P(t, r(\cdot), \varepsilon) \in U, \]

\[ G \times (0, \infty) \ni (t, r(\cdot), \varepsilon) \mapsto Q(t, r(\cdot), \varepsilon) \in V, \]

where \(\varepsilon\) is the accuracy parameter.

Let \((t_0, r_0(\cdot)) \in G, t_0 < \vartheta, \varepsilon > 0, \) and let

\[ \Delta = \{\tau_j\}_{j=1}^{k+1}, \quad \tau_1 = t_0, \quad \tau_{j+1} > \tau_j, \quad j \in \mathbb{Z}, \quad \tau_{k+1} = \vartheta, \]  

(24)

be a partition of the interval \([t_0, \vartheta]\). The triple \(\{P, \varepsilon, \Delta\}\) is called a control law of the first player. This law forms in the approximating system a piecewise constant control realization \(p(\cdot) \in U(t_0, \vartheta)\) by the following step-by-step feedback rule:

\[ p(t) = P(\tau_{j}, y(\tau_{j} \cdot, \varepsilon)), \quad t \in [\tau_{j}, \tau_{j+1}), \quad j \in \mathbb{Z}, \]  

(25)

where \(y_{\tau_j(\cdot)} = r_0(\cdot)\). Thus, from the initial position \((t_0, r_0(\cdot))\), the control law of the first player \(\{P, \varepsilon, \Delta\}\) together with a control realization of the second player \(q(\cdot) \in V(t_0, \vartheta)\) uniquely generate the motion \(y(\cdot)\) of the approximating system and, therefore, determine its quality index \(y_{w_0,h}\) according to \((23)\). Let us consider the guaranteed result of the control law \(\{P, \varepsilon, \Delta\}\)

\[ \rho^{(p)}_{w_0,h}(t_0, r_0(\cdot); P, \varepsilon, \Delta) = \sup_{v(\cdot) \in V(t_0, \vartheta)} y_{w_0,h}. \]

The guaranteed result of the strategy \(P\) is defined by

\[ \rho^{(p)}_{w_0,h}(t_0, r_0(\cdot); P) = \limsup_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \sup_{\Delta: \text{diam}(\Delta) \leq \delta} \rho^{(p)}_{w_0,h}(t_0, r_0(\cdot); P, \varepsilon, \Delta), \]

where \(\text{diam}(\Delta) = \max_{j \in \mathbb{Z}} (\tau_{j+1} - \tau_j)\) is the diameter of the partition \(\Delta\).
In the case of initial positions \((\vartheta, r_\ast(\cdot)) \in G\), we formally consider the partition \(\Delta = \{\vartheta\}\) and say that the motion \(y(t) = r_\ast(t), \, t \in [0, \vartheta]\), is generated from \((\vartheta, r_\ast(\cdot))\) by \(\{P, \varepsilon, \Delta\}\) and \(q(\cdot) \in \mathcal{V}(\vartheta, \vartheta)\). Hence, in this case, we put
\[
\rho_\ast^{(p)}(\vartheta, r_\ast(\cdot); P) = \sigma_{w_0, h}(r_\ast(\cdot)).
\]

Further, we define the optimal guaranteed result of the first player by
\[
\rho_\ast^{(p)}(t_\ast, r_\ast(\cdot)) = \inf_P \rho_{w_0, h}(t_\ast, r_\ast(\cdot); P), \quad (t_\ast, r_\ast(\cdot)) \in G.
\]

A strategy of the first player \(P^0_{w_0, h}\) is optimal for a position \((t_\ast, r_\ast(\cdot)) \in G\) if
\[
\rho_{w_0, h}(t_\ast, r_\ast(\cdot); P^0_{w_0, h}) = \rho_{w_0, h}(t_\ast, r_\ast(\cdot)).
\]

Similarly, we consider control laws of the second player \(\{Q, \varepsilon, \Delta\}\) that form \(q_\ast(\cdot) \in \mathcal{V}(t_\ast, \vartheta)\) as follows:
\[
q(t) = Q(\tau_j, y_{\tau_j}(\cdot), \varepsilon), \quad t \in [\tau_j, \tau_{j+1}), \quad j \in \overline{1, k}.
\]

The optimal guaranteed result \(\rho_{w_0, h}(t_\ast, r_\ast(\cdot))\) and the optimal strategy \(Q^0_{w_0, h}\) of the second player are defined in a similar way as for the first player with clear changes (see, e.g., [33–35] for details).

According to [33, Theorem 1] (see also [35, Theorem 17.1]), due to conditions (B.1)–(B.5), the approximating differential game has the value
\[
\rho_{w_0, h}(t_\ast, r_\ast(\cdot)) = \rho_{w_0, h}(t_\ast, r_\ast(\cdot)) = \rho_{w_0, h}(t_\ast, r_\ast(\cdot)), \quad (t_\ast, r_\ast(\cdot)) \in G,
\]

and the value function \(G \ni (t_\ast, r_\ast(\cdot)) \mapsto \rho_{w_0, h}(t_\ast, r_\ast(\cdot)) \in \mathbb{R}\) is continuous [let us recall that the set \(G\) is endowed with the metric \(\bar{Q}\) defined in (19)]. Furthermore, for any initial position \((t_\ast, r_\ast(\cdot)) \in G\), the players’ optimal positional strategies \(P^0_{w_0, h}\) and \(Q^0_{w_0, h}\) exist, and one can choose them in such a way that they are optimal uniformly in \((t_\ast, r_\ast(\cdot)) \in G^0_h\) and \(w_0 \in B(R_0)\), the set \(G^0_h\) is introduced in Sect. 5.1. More precisely, the following result is valid (see also [16] for a related technique).

**Lemma 1** For any \(h > 0\) and any \(\zeta > 0\), one can choose \(\varepsilon^{(1)} = \varepsilon^{(1)}(h, \zeta) > 0\) and \(\delta^{(1)}(\varepsilon) = \delta^{(1)}(\varepsilon, h, \zeta) > 0\) for every \(\varepsilon \in (0, \varepsilon^{(1)})\) such that the following statement holds. Let \(w_0 \in B(R_0)\), \((t_\ast, r_\ast(\cdot)) \in G^0_h, \varepsilon \in (0, \varepsilon^{(1)})\), and let \(\Delta\) be a partition (24) with the diameter \(\text{diam}(\Delta) \leq \delta^{(1)}(\varepsilon)\). Then, the control law \(\{P^0_{w_0, h}, \varepsilon, \Delta\}\) of the first player guarantees the inequality
\[
\gamma_{w_0, h} \leq \rho_{w_0, h}(t_\ast, r_\ast(\cdot)) + \zeta
\]
for any control realization of the second player \(q(\cdot) \in \mathcal{V}(t_\ast, \vartheta)\), and the control law \(\{Q^0_{w_0, h}, \varepsilon, \Delta\}\) of the second player guarantees the inequality
\[
\gamma_{w_0, h} \geq \rho_{w_0, h}(t_\ast, r_\ast(\cdot)) - \zeta
\]
for any control realization of the first player \(p(\cdot) \in \mathcal{U}(t_\ast, \vartheta)\).

Note that the uniformness in the parameter \(w_0 \in B(R_0)\) is provided by the corresponding uniformness in properties (B.1)–(B.3) and (B.5).

Let us describe shortly one of the ways of constructing such optimal strategies \(P^0_{w_0, h}\) and \(Q^0_{w_0, h}\). We apply the method of extremal shift to accompanying points (see, e.g., [28, 29, 33, 35]). For simplicity of the notation below, it is convenient to consider the so-called
pre-strategies of the players in the approximating game (21), (23). Namely, by pre-strategies \( p_{w_0,h} \) and \( q_{w_0,h} \) of the first and second players, we mean functions
\[
G \times \mathbb{R}^n \ni (t, r(\cdot), s) \mapsto p_{w_0,h}(t, r(\cdot), s) \in U,
\]
\[
G \times \mathbb{R}^n \ni (t, r(\cdot), s) \mapsto q_{w_0,h}(t, r(\cdot), s) \in V
\]
that, for any \((t, r(\cdot)) \in G\) and any \(s \in \mathbb{R}^n\), satisfy the inclusions
\[
p_{w_0,h}(t, r(\cdot), s) \in \arg\min_{p \in U} \max_{q \in V} (s, f_{w_0,h}(t, r(\cdot), p, q)),
\]
\[
q_{w_0,h}(t, r(\cdot), s) \in \arg\max_{q \in V} \min_{p \in U} (s, f_{w_0,h}(t, r(\cdot), p, q)).
\]

Let \((t, r(\cdot)) \in G_h^0\) and \(\varepsilon > 0\). For the first and second players, we choose the accompanying points \(r_{\varepsilon}^{(p)}(\cdot)\) and \(r_{\varepsilon}^{(q)}(\cdot)\) from the conditions
\[
r_{\varepsilon}^{(p)}(\cdot) \in \arg\min \rho_{w_0,h}(t, r_{\varepsilon}(\cdot)), \quad r_{\varepsilon}^{(q)}(\cdot) \in \arg\max \rho_{w_0,h}(t, r_{\varepsilon}(\cdot)),
\]
where the minimum and maximum are calculated over the functions \(r_{\varepsilon}(\cdot)\) such that
\[(t, r_{\varepsilon}(\cdot)) \in G_h^0, \quad e^{-2(t-t_0)\lambda_h} \max_{\tau \in [t_0, t]} \|r(\tau) - r_{\varepsilon}(\tau)\|^2 \leq (t - t_0) \varepsilon,
\]
and the constant \(\lambda_h\) is chosen by the set \(G_h^0\) in accordance with property (B.2). After that, we define
\[
P_{w_0,h}^0(t, r(\cdot), \varepsilon) = p_{w_0,h}(t, r(\cdot), r(t) - r_{\varepsilon}^{(p)}(t)),
\]
\[
Q_{w_0,h}^0(t, r(\cdot), \varepsilon) = q_{w_0,h}(t, r(\cdot), r_{\varepsilon}^{(q)}(t) - r(t)).
\]

For \((t, r(\cdot)) \in G \setminus G_h^0\), the strategies \(P_{w_0,h}^0\) and \(Q_{w_0,h}^0\) are defined arbitrarily.

**Remark 3** There are another methods for constructing the optimal positional strategies \(P_{w_0,h}^0\) and \(Q_{w_0,h}^0\) (see, e.g., [31, 33–35]). For example, if the value function \(\rho_{w_0,h}\) is coinvariantly (ci-) smooth, then the method of extremal shift in the direction of the ci-gradient of \(\rho_{w_0,h}\) can be applied (see also the example in Sect. 8). In the general non-smooth case, such strategies can be constructed by the extremal shift in the direction of the ci-gradient of a suitable ci-smooth auxiliary function. Also, one can use the methods based on the notions of maximal \(u\)- and \(v\)-stable bridges. Furthermore, there are some specific methods for constructing the optimal strategies in the linear case (see, e.g., [14, 39]).

**Remark 4** Similarly to Sect. 3.4, in the approximating differential game (21), (23), one can consider non-anticipative strategies of the players and introduce the corresponding lower \(\tilde{\rho}_{w_0,h}^{(p)}(t_*, r_*(\cdot))\) and upper \(\tilde{\rho}_{w_0,h}^{(q)}(t_*, r_*(\cdot))\) game values. However, following, e.g., [29, § 29] and [16] (see also the proof of Theorem 1 in Sect. 7), one can show that, under conditions (B.1)–(B.5), the equalities below are valid:
\[
\rho_{w_0,h}(t_*, r_*(\cdot)) = \tilde{\rho}_{w_0,h}^{(p)}(t_*, r_*(\cdot)) = \tilde{\rho}_{w_0,h}^{(q)}(t_*, r_*(\cdot)), \quad (t_*, r_*(\cdot)) \in G.
\]

### 6 Players’ Control Procedures with a Guide

In this section, we propose the players’ feedback control procedures that use the optimally controlled approximating system (21) as a guide. It allows us to show that the values of the approximating differential games (21), (23) have the limit when \(h \downarrow 0\). This fact constitutes the basis of the proof of the main result of the paper formulated in Theorem 1 (see Sect. 7).
6.1 Mutual Aiming Procedures Between the Systems

According to [13, Sect. 7], let us consider the following mutual aiming procedure between original (4) and approximating (21) systems. First of all, let us introduce pre-strategies of the players in the original game (4), (9). By pre-strategies \( u \) and \( v \) of the first and second players, we mean functions

\[
[t_0, \vartheta] \times \mathbb{R}^n \times \mathbb{R}^n \ni (t, x, s) \mapsto u(t, x, s) \in U,
\]

\[
[t_0, \vartheta] \times \mathbb{R}^n \times \mathbb{R}^n \ni (t, x, s) \mapsto v(t, x, s) \in V.
\]

that, for any \( t \in [t_0, \vartheta] \) and any \( x, s \in \mathbb{R}^n \), satisfy the inclusions

\[
u(t, x, s) \in \arg\min_{u \in U} \max_{v \in V} (s, f(t, x, u, v)),
\]

\[
v(t, x, s) \in \arg\max_{v \in V} \min_{u \in U} (s, f(t, x, u, v)).
\]

Further, for \( (t, w(\cdot)) \in G_1 \) and \( (t, r(\cdot)) \in G \), let us denote

\[
s(t, w(\cdot), r(\cdot)) = w(t) - w(t_0) - h^{a-1} (\Delta_h^{1-a}(r(\cdot) - r(t_0)))(t).
\]

Let \( (t_s, w_s(\cdot)) \in G_1 \) be an initial position of original system (4). Let us fix \( h > 0 \), put \( w_0 = w_s(t_0) \) and consider the corresponding approximating system (21) under the initial position \( (t_s, r_s(\cdot)) \) defined by (22). Let us fix also a partition \( \Delta (24) \). Let a first player’s control realization \( u(\cdot) \in U(t_s, \vartheta) \) in the original system and a second player’s control realization \( q(\cdot) \in V(t_s, \vartheta) \) in the approximating system be formed simultaneously according to the following step-by-step feedback rule:

\[
u(t) = u(\tau_j, x(\tau_j), s_j), \quad q(t) = q_{\alpha_0,h}(\tau_j, y_{\tau_j}(\cdot), s_j), \quad t \in [\tau_j, \tau_{j+1}), \quad j \in \overline{1, k},
\]

(27)

where

\[
s_j = s(\tau_j, x(\tau_j)(\cdot), y(\tau_j)(\cdot)),
\]

(28)

and \( q_{\alpha_0,h} \) is a pre-strategy of the second player in the approximating game.

**Lemma 2** For any \( \xi > 0 \), there exist \( h^{(2)} = h^{(2)}(\xi) > 0 \) and \( \delta^{(2)} = \delta^{(2)}(\xi) > 0 \) such that, for any initial position \( (t_s, w_s(\cdot)) \in G_1 \) of original system (4) and any partition \( \Delta (24) \) with the diameter \( \text{diam}(\Delta) \leq \delta^{(2)} \), the following statement is valid. Let us consider approximating system (21) for \( w_0 = w_s(t_0) \) and any \( h \in (0, h^{(2)}] \) under the initial position \( (t_s, r_s(\cdot)) \) defined by (22). Then, for any control realizations \( v(\cdot) \in V(t_s, \vartheta) \) and \( p(\cdot) \in U(t_s, \vartheta) \), if control realizations \( u(\cdot) \in U(t_s, \vartheta) \) and \( q(\cdot) \in V(t_s, \vartheta) \) are formed according to the mutual aiming procedure (27), (28), then the corresponding motions \( x(\cdot) \) and \( y(\cdot) \) of the original and approximating systems satisfy the inequality

\[
x(t) - w_0 - h^{a-1}(\Delta_h^{1-a} y(t)) \leq \xi, \quad t \in [t_0, \vartheta].
\]

(29)

**Proof** The proof follows the scheme from [13, Theorem 3]. Applying [13, Propositions 5 and 7], by the constant \( L_1 \) from Proposition 3, one can choose \( R_2 > 0 \) and \( H_2 > 0 \) such that the inequalities

\[
\|(D_1^{1-a} y(t))\| \leq R_2,
\]

\[
\|h^{a-1}(\Delta_h^{1-a} y(t))\| \leq R_2,
\]

\[
\|h^{a-1}(\Delta_h^{1-a} y(t)) - h^{a-1}(\Delta_h^{1-a} y(t'))\| \leq H_2 |t - t'|^a, \quad t, t' \in [t_0, \vartheta],
\]

(30)
are valid for any \( y(\cdot) \in \text{Lip}^0_{L_1}([t_0, \vartheta], \mathbb{R}^n) \) and any \( h > 0 \). Taking the constants \( R_1 \) and \( H_1 \) from Corollary 1, we define \( R_3 = R_0 + R_1 + R_2, H_3 = H_1 + H_2 \). By the constant \( R_3 \), let us choose \( \lambda > 0 \) according to (A.2). Let \( \xi > 0 \) be fixed. Let us denote

\[
\eta = \frac{\Gamma(\alpha + 1)\xi^2}{4(\vartheta - t_0)^aE_\alpha(2(\vartheta - t_0)^a\lambda)}, \quad \varkappa = \min\left\{ \frac{\eta}{32(1 + R_3)c}, \frac{\eta}{32\lambda R_3} \right\},
\]

where \( c \) is the constant from (A.3). By [13, Lemma 1], there exists \( h^{(2)} > 0 \) such that the inequality

\[
\| (D^{1-\alpha}y)(t) - h^{a-1}(\Delta_h^{1-\alpha}y)(t) \| \leq \min\{\varkappa, \xi/2\}, \; t \in [t_0, \vartheta]. \tag{31}
\]

holds for any \( y(\cdot) \in \text{Lip}^0_{L_1}([t_0, \vartheta], \mathbb{R}^n) \) and any \( h \in (0, h^{(2)}) \). Due to (A.1), one can choose \( \delta^{(2)} > 0 \) such that \( (\delta^{(2)})^a \leq \varkappa/H_3 \) and, for any \( t, t' \in [t_0, \vartheta] \), \( x, x' \in B(R_3), u \in U \), and \( v \in V \), from the inequalities \( |t - t'| \leq \delta^{(2)} \) and \( \|x - x'\| \leq H_3(\delta^{(2)})^a \) it follows that \( \|f(t, x, u, v) - f(t', x', u, v)\| \leq \eta/(16R_3) \). Let us show that the statement of the lemma is valid for the chosen parameters.

Let \((t_*, w_*(\cdot)) \in G_*, h \in (0, h^{(2)})\), and let \( \Delta \) be a partition (24) with the diameter \( \text{diam}(\Delta) \leq \delta^{(2)} \). Let us consider original system (4) and approximating system (21) for \( w_0 = w_*(t_0) \), the fixed \( h \), and with the initial position \((t_*, r_*(\cdot))\) defined by (22). Let \( x(\cdot) \) and \( y(\cdot) \) be the motions of these systems generated by control realizations \( u(\cdot) \in \mathcal{U}(t_*, \vartheta), v(\cdot) \in \mathcal{V}(t_*, \vartheta) \) and \( p(\cdot) \in \mathcal{U}(t_*, \vartheta), q(\cdot) \in \mathcal{V}(t_*, \vartheta) \), when \( u(\cdot) \) and \( q(\cdot) \) are formed according to (27), (28). Let us note that, by the choice of \( L_1 \), we have \( y(\cdot) \in \text{Lip}^0_{L_1}([t_0, \vartheta], \mathbb{R}^n) \), and, therefore, the inequalities in (30) hold. Moreover, due to the choice of \( h \), the inequality in (31) is also valid. Let us introduce the Lyapunov function \( V(t) = \|x(t) - w_0 - (D^{1-\alpha}y)(t)\|^2, t \in [t_0, \vartheta] \). Then, taking into account (22) and the results given in Sect. 2, we obtain \( V(t) = 0, t \in [t_0, t_*] \), and, as in the proof of [13, Theorem 2]), we derive \( V(\cdot) \in I^a(L^\infty([t_0, \vartheta], \mathbb{R})) \) and

\[
(D^aV)(t) \leq 2(x(t) - w_0 - (D^{1-\alpha}y)(t), (C^aD^a)x(t) - \dot{y}(t))
\]

\[
= 2(x(t) - w_0 - (D^{1-\alpha}y)(t), f(t, x(t), u(t), v(t)))
\]

\[
- 2(x(t) - w_0 - (D^{1-\alpha}y)(t), f(t, x'(t), p(t), q(t))) \quad \text{for a.e. } t \in [t_*, \vartheta],
\]

where we denote

\[
x'(t) = w_0 + h^{a-1}(\Delta_h^{1-\alpha}y)(t), \; t \in [t_0, \vartheta]. \tag{32}
\]

Let \( j \in 1, k \) and \( t \in [t_j, t_{j+1}] \). We have

\[
\|x'\| \leq R_3, \quad \|x' - x'(\tau_j)\| \leq H_3(\delta^{(2)})^a,
\]

\[
\|x(t) - w_0 - (D^{1-\alpha}y)(t)\| \leq R_3,
\]

\[
\|s_j\| \leq R_3,
\]

where \( s_j \) is defined by (28). Due to the choice of \( h \) and \( \Delta \), we obtain

\[
\|x(t) - w_0 - (D^{1-\alpha}y)(t) - s_j\| \leq 2\varkappa, \quad \|s_j\|^2 \leq V(t) + \eta/(4\lambda).
\]

Therefore, according to the choice of \( \Delta \) and (27), we deduce

\[
\langle x(t) - w_0 - (D^{1-\alpha}y)(t), f(t, x(t), u(t), v(t)) \rangle
\]

\[
\leq \langle s_j, f(t_j, x(t_j), u(t), v(t)) \rangle + \eta/8
\]

\[
\leq \max_{v \in V} \langle s_j, f(t_j, x(t_j), u(t), v) \rangle + \eta/8 = \mathcal{H}(t_j, x(t_j), s_j) + \eta/8,
\]
where \( \mathcal{H} \) is the function from (5). In a similar way, we have
\[
\begin{align*}
(x(t) - w_0 - (D^{1-\alpha}y)(t), f(t, x'(t), p(t), q(t))) \\
\geq (s_j, f(\tau_j, x'(\tau_j), p(t), q(t))) - \eta/8 \\
\geq \min_{p \in \mathcal{U}} (s_j, f(\tau_j, x'(\tau_j), p(t), q(t))) - \eta/8 = \mathcal{H}(\tau_j, x'(\tau_j), s_j) - \eta/8.
\end{align*}
\]
Moreover, by the choice of \( \lambda \),
\[
\mathcal{H}(\tau_j, x(\tau_j), s_j) - \mathcal{H}(\tau_j, x'(\tau_j), s_j) \leq \lambda \|s_j\|^2 \leq \lambda V(t) + \eta/4.
\]
Hence, we conclude
\[
(D^\alpha V)(t) \leq 2\lambda V(t) + \eta \text{ for a.e. } t \in [t_0, \vartheta].
\]
From this inequality, due to the results given in Sect. 2, we derive
\[
V(t) \leq \frac{(\vartheta - t_0)^\alpha \eta}{\Gamma(\alpha + 1)} + \frac{2\lambda}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{V(\tau)}{(t - \tau)^{1-\alpha}} d\tau, \quad t \in [t_0, \vartheta],
\]
wherefrom, by the fractional version of Bellman–Gronwall lemma (see, e.g., [8, Lemma 6.19] and also [12, Lemma 1.1]), we deduce \( V(t) \leq \xi^2/4, \quad t \in [t_0, \vartheta] \). According to the choice of \( h \), we obtain
\[
\|x(t) - x'(t)\| \leq \sqrt{V(t)} + \|D^{1-\alpha}y(t) - h^{\alpha-1}(\Delta_t^{1-\alpha}y)(t)\| \leq \xi, \quad t \in [t_0, \vartheta].
\]
The lemma is proved. \( \square \)

Similarly, one can consider another mutual aiming procedure between the original and approximating systems. Namely, let \( v(\cdot) \in \mathcal{V}(t_0, \vartheta) \) and \( p(\cdot) \in \mathcal{U}(t_0, \vartheta) \) be formed on the basis of the partition \( \Delta \) as follows:
\[
v(t) = v(\tau_j, x(\tau_j), s_j), \quad p(t) = p_{w_0,h}(\tau_j, x(\tau_j), s_j), \quad t \in [\tau_j, \tau_{j+1}), \quad j \in \overline{1, k}, \quad (33)
\]
where
\[
s_j = -s(\tau_j, x(\tau_j), y(\cdot)). \quad (34)
\]
By analogy with Lemma 2, we obtain the following result.

**Lemma 3** For any \( \xi > 0 \), there exist \( h^{(3)} = h^{(3)}(\xi) > 0 \) and \( \delta^{(3)} = \delta^{(3)}(\xi) > 0 \) such that, for any initial position \( (t_0, w_0(\cdot)) \in G_a \) of original system (4) and any partition \( \Delta \) (24) with the diameter \( \text{diam}(\Delta) \leq \delta^{(3)} \), the following statement is valid. Let us consider approximating system (21) for \( w_0 = w_a(t_0) \) and any \( h \in (0, h^{(3)}] \) under the initial position \( (t_0, r_a(\cdot)) \) defined by (22). Then, for any control realizations \( u(\cdot) \in \mathcal{U}(t_0, \vartheta) \) and \( q(\cdot) \in \mathcal{V}(t_0, \vartheta) \), if control realizations \( v(\cdot) \in \mathcal{V}(t_0, \vartheta) \) and \( p(\cdot) \in \mathcal{U}(t_0, \vartheta) \) are formed according to the mutual aiming procedure (33), (34), then the corresponding motions \( x(\cdot) \) and \( y(\cdot) \) of the original and approximating systems satisfy the inequality in (29).

### 6.2 First Player’s Control Procedure with a Guide

Let \( (t_0, w_a(\cdot)) \in G_a, \quad h > 0, \quad \varepsilon > 0, \quad \text{and a partition } \Delta \) (24) be fixed. We propose the following control procedure of the first player in the original differential game \( (4), (9) \). Let us consider the approximating differential game \( (21), (23) \) for \( w_0 = w_a(t_0) \) and the fixed \( h \), with the initial position \( (t_0, r_a(\cdot)) \) defined by (22). By the steps of the partition \( \Delta \), the first player forms a control realization \( u(\cdot) \in \mathcal{U}(t_0, \vartheta) \) in the original system and, at the same
time, control realizations \( p(\cdot) \in U(t_*, \theta) \) and \( q(\cdot) \in V(t_*, \theta) \) in the approximating system as follows: \( u(\cdot) \) and \( q(\cdot) \) are formed according to the mutual aiming procedure (27), (28), and \( p(\cdot) \) is formed by the control law \( P_{w_0,h}^0(\theta, \varepsilon, \Delta) \) [see (25)] on the basis of the optimal strategy \( P_{w_0,h}^0 \) taken from Lemma 1. Note that, from the initial position \((t_*, w_*(\cdot))\), the described control procedure together with \( v(\cdot) \in V(t_*, \theta) \) uniquely generate the motion \( x(\cdot) \) of the original system and, therefore, determine its quality index \( \gamma \). Moreover, during this control procedure, the first player generates the auxiliary motion \( y(\cdot) \) of the approximating system, which can be considered as a guide (see, e.g., [31, § 8.2]). For convenience, in what follows, the described control procedure is referred as \( U(t_*, w_*(\cdot), h, \varepsilon, \Delta) \).

For any \( h > 0 \), let us introduce the function

\[
\hat{\rho}_h(t_*, w_*(\cdot)) = \rho_{w_0,h}(t_*, r_*(\cdot)), \quad (t_*, w_*(\cdot)) \in G_*, \tag{35}
\]

where \( r_*(\cdot) \) is defined by \( w_*(\cdot) \) according to (22), and \( \rho_{w_0,h}(t_*, r_*(\cdot)) \) is the value of the approximating differential game (21), (23) for \( w_0 = w_0(t_0) \) and the fixed \( h \).

**Lemma 4** For any \( \xi > 0 \), there exist

\[
\begin{align*}
\hat{h}^{(4)}(\xi) & = h^{(4)}(\xi) > 0, \\
\hat{\varepsilon}^{(4)}(h) & = \varepsilon^{(4)}(h, \xi) > 0, \quad h \in (0, h^{(4)}], \\
\hat{\delta}^{(4)}(\varepsilon, h, \xi) & = \delta^{(4)}(\varepsilon, h, \xi) > 0, \quad \varepsilon \in (0, \varepsilon^{(4)}(h)], \quad h \in (0, h^{(4)}],
\end{align*}
\]

such that, for any \((t_*, w_*(\cdot)) \in G_*, \ v(\cdot) \in V(t_*, \theta)\), \( h \in (0, h^{(4)}]\), \( \varepsilon \in (0, \varepsilon^{(4)}(h)] \), and any partition \( \hat{\Delta} \) (24) with the diameter \( \text{diam}(\hat{\Delta}) \leq \delta^{(4)}(\varepsilon, h) \), the first player’s control procedure with a guide \( U(t_*, w_*(\cdot), h, \varepsilon, \hat{\Delta}) \) guarantees the inequality

\[
\gamma \leq \hat{\rho}_h(t_*, w_*(\cdot)) + \xi
\]

for any control realization of the second player \( v(\cdot) \in V(t_*, \theta) \).

**Proof** Let the constants \( R_3 \) and \( H_3 \) be chosen as in the proof of Lemma 2. Let us consider the compact set \( K \subset C([t_0, \theta], \mathbb{R}^n) \) consisting of the functions \( x(\cdot) \) such that

\[
\|x(t)\| \leq R_3, \quad \|x(t) - x(t')\| \leq H_3|t - t'|^\alpha, \quad t, t' \in [t_0, \theta].
\]

Let \( \zeta > 0 \) be fixed. Due to (A.5), there exists \( \hat{\xi} = \xi(\zeta) > 0 \) such that, for any \( x(\cdot), x'(\cdot) \in K \), from the inequality \( \|x(\cdot) - x'(\cdot)\|_{\infty} \leq \zeta \), it follows that \( |\sigma(x(\cdot)) - \sigma(x'(\cdot))| \leq \zeta/2 \). Let us choose \( h^{(2)}(\hat{\xi}) > 0 \) and \( \delta^{(2)}(\hat{\xi}) > 0 \) by Lemma 2, and put \( h^{(4)} = h^{(2)}(\hat{\xi}) \). Finally, for any \( h \in (0, h^{(4)}] \), we take \( \varepsilon^{(1)}(h, \zeta/2) > 0 \) and \( \delta^{(1)}(\varepsilon, h, \zeta/2) > 0 \) for any \( \varepsilon \in (0, \varepsilon^{(1)}(h, \zeta/2)] \) from Lemma 1, and define

\[
\begin{align*}
\hat{\varepsilon}^{(4)}(h) & = \varepsilon^{(1)}(h, \zeta/2), \\
\hat{\delta}^{(4)}(\varepsilon, h, \zeta/2) & = \min \{\delta^{(1)}(\varepsilon, h, \zeta/2), \delta^{(2)}(\hat{\xi})\}, \quad \varepsilon \in (0, \varepsilon^{(4)}(h)].
\end{align*}
\]

Let us show that the statement of the lemma is valid for the chosen parameters.

Let \((t_*, w_*(\cdot)) \in G_*, \ v(\cdot) \in V(t_*, \theta)\), \( h \in (0, h^{(4)}] \), \( \varepsilon \in (0, \varepsilon^{(4)}(h)] \), and let \( \Delta \) be a partition (24) with the diameter \( \text{diam}(\Delta) \leq \delta^{(4)}(\varepsilon, h) \). Let us consider the motion \( x(\cdot) \) of system (4) generated from the initial position \((t_*, w_*(\cdot))\) by the first player’s control procedure with a guide \( U = U(t_*, w_*(\cdot), h, \varepsilon, \Delta) \) and a second player’s control realization \( v(\cdot) \in V(t_*, \theta) \). Let us consider the corresponding first player’s control realization \( u(\cdot) \in U(t_*, \theta) \) in the original system and players’ control realizations \( p(\cdot) \in U(t_*, \theta) \) and \( q(\cdot) \in V(t_*, \theta) \) in approximating system (21) for \( w_0 = w_0(t_0) \), the fixed \( h \), and with the initial position \((t_*, r_*(\cdot))\) defined by (22). Let \( y(\cdot) \) be the corresponding motion of the approximating system. By the definition
of $U$, the motion $y(\cdot)$ is generated by the control law \( \{P_{w_0,h}^0, \varepsilon, \Delta\} \) on the basis of the first player’s optimal positional strategy $P_{w_0,h}^0$. Hence, for the auxiliary function $x'(\cdot)$ defined by (32), due to the choice of $\varepsilon$ and $\Delta$, we obtain

$$ \sigma(x'(\cdot)) = \sigma_{w_0,h}(y(\cdot)) = \gamma_{w_0,h} \leq \rho_{w_0,h}(t_*, r_*(\cdot)) + \zeta/2 = \widehat{\rho}_h(t_*, w_*(\cdot)) + \zeta/2. $$

Moreover, the control realizations $u(\cdot)$ and $q(\cdot)$ are formed according to the mutual aiming procedure (27), (28). Therefore, according to the choice of $h$ and $\Delta$, we derive $\|x(\cdot) - x'(\cdot)\|_\infty \leq \xi$. Thus, taking into account the inclusions $x(\cdot), x'(\cdot) \in K$, by the choice of $\xi$, we have

$$ \gamma = \sigma(x(\cdot)) \leq \sigma(x'(\cdot)) + \zeta/2 \leq \widehat{\rho}_h(t_*, w_*(\cdot)) + \zeta. $$

The lemma is proved. \hfill \Box

6.3 Second Player’s Control Procedure with a Guide

Similarly to Sect. 6.2, we propose the following second player’s control procedure with a guide in the original differential game (4), (9). Let $(t_*, w_*(\cdot)) \in G_*, \; h > 0, \; \varepsilon > 0$, and a partition $\Delta$ (24) be fixed. Let us consider the approximating differential game (21), (23) for $w_0 = w_*(l_0)$, the fixed $h$, and with the initial position $(t_*, r_*(\cdot))$ defined by (22). By the steps of the partition $\Delta$, the second player forms a control realization $v(\cdot) \in V(t_*, \vartheta)$ in the original system and, at the same time, control realizations $p(\cdot) \in U(t_*, \vartheta)$ and $q(\cdot) \in V(t_*, \vartheta)$ in the approximating system as follows: $v(\cdot)$ and $p(\cdot)$ are formed according to the mutual aiming procedure (33), (34), and $q(\cdot)$ is formed by the control law $\{Q_{w_0,h}^0, \varepsilon, \Delta\}$ [see (26)] on the basis of the optimal strategy $Q_{w_0,h}^0$ taken from Lemma 1. From the initial position $(t_*, w_*(\cdot))$, the described control procedure together with $u(\cdot) \in U(t_*, \vartheta)$ uniquely generate the motion $x(\cdot)$ of the original system and determine its quality index $\gamma$. In what follows, this control procedure with a guide is referred as $V(t_*, w_*(\cdot), h, \varepsilon, \Delta)$.

By analogy with Lemma 4, on the basis of Lemma 3, the following result can be proved.

Lemma 5 For any $\zeta > 0$, there exist

$$ h^{(5)} = h^{(5)}(\zeta) > 0, $$
$$ \varepsilon^{(5)}(h) = \varepsilon^{(5)}(h, \zeta) > 0, \quad h \in (0, h^{(5)}], $$
$$ \delta^{(5)}(\varepsilon, h) = \delta^{(5)}(\varepsilon, h, \zeta) > 0, \quad \varepsilon \in (0, \varepsilon^{(5)}(h)], \quad h \in (0, h^{(5)}], $$

such that, for any $(t_*, w_*(\cdot)) \in G_*, \; h \in (0, h^{(5)}], \; \varepsilon \in (0, \varepsilon^{(5)}(h)],$ and any partition $\Delta$ (24) with the diameter $\mathrm{diam}(\Delta) \leq \delta^{(5)}(\varepsilon, h)$, the second player’s control procedure with a guide $V(t_*, w_*(\cdot), h, \varepsilon, \Delta)$ guarantees the inequality

$$ \gamma \geq \widehat{\rho}_h(t_*, w_*(\cdot)) - \zeta $$

for any control realization of the first player $u(\cdot) \in U(t_*, \vartheta)$.

6.4 Limit of the Values of the Approximating Games

Considering in the original differential game (4), (9) the case when both players use the described in Sect. 6.2 and 6.3 control procedures with guides, we obtain the result below.

Lemma 6 For any initial position $(t_*, w_*(\cdot)) \in G_*$, the following limit exists:

$$ \lim_{h \downarrow 0} \widehat{\rho}_h(t_*, w_*(\cdot)) = \widehat{\rho}(t_*, w_*(\cdot)), $$

(36)
where \( \hat{\rho}_h(t_*, w_*(\cdot)) \) is defined by (35). Moreover, the convergence is uniform in \((t_*, w_*(\cdot)) \in G_*\).

**Proof** By the Cauchy criterion, to prove the lemma, it is sufficient to show that, for any \( h > 0 \), there exists \( \delta(h) > 0 \) such that, for any \( h_1, h_2 \in (0, h] \) and any \((t_*, w_*(\cdot)) \in G_*\), the inequality below is valid:

\[
\hat{\rho}_{h_2}(t_*, w_*(\cdot)) \leq \hat{\rho}_{h_1}(t_*, w_*(\cdot)) + \xi. \tag{37}
\]

Let \( \xi > 0 \) be fixed. By Lemmas 4 and 5, for \( i \in \{4, 5\} \), let us choose

\[
\begin{align*}
h^{(i)} &= h^{(i)}(\xi/2) > 0, \\
\varepsilon^{(i)}(h) &= \varepsilon^{(i)}(h, \xi/2) > 0, \quad h \in (0, h^{(i)}], \\
\delta^{(i)}(\varepsilon, h) &= \delta^{(i)}(\varepsilon, h, \xi/2) > 0, \quad \varepsilon \in (0, \varepsilon^{(i)}(h)], \quad h \in (0, h^{(i)}],
\end{align*}
\]

and put \( h = \min\{h^{(4)}, h^{(5)}\} \). Let \( h_1, h_2 \in (0, h] \). We define

\[
\varepsilon = \min\{\varepsilon^{(4)}(h_1), \varepsilon^{(5)}(h_2)\}, \quad \delta = \min\{\delta^{(4)}(\varepsilon, h_1), \delta^{(5)}(\varepsilon, h_2)\}.
\]

Let \((t_*, w_*(\cdot)) \in G_*\), and \( \Delta \) be a partition (24) with the diameter \( \text{diam}(\Delta) \leq \delta \). Let us consider the motion \( x(\cdot) \) of system (4) generated by the players’ control procedures with guides \( U(t_*, w_*(\cdot), h_1, \varepsilon, \Delta) \) and \( V(t_*, w_*(\cdot), h_2, \varepsilon, \Delta) \). Then, for the quality index \( \gamma = \sigma(x(\cdot)) \), due to the choice of \( h_1, h_2, \varepsilon \) and \( \Delta \), we have

\[
\hat{\rho}_{h_2}(t_*, w_*(\cdot)) - \xi/2 \leq \gamma \leq \hat{\rho}_{h_1}(t_*, w_*(\cdot)) + \xi/2,
\]

wherefrom we derive (37). The lemma is proved. \( \Box \)

### 7 Value of the Game

The main result of the paper is the following.

**Theorem 1** Let conditions (A.1)–(A.5) be satisfied. Then:

1. The differential game (4), (7) has the value \( \rho(t_*, w_*(\cdot)) \) for all \((t_*, w_*(\cdot)) \in G_*\).
2. This value coincides with the limit \( \hat{\rho}(t_*, w_*(\cdot)) \) [see (36)] of the values of the approximating differential games (21), (23).
3. For any \( \xi > 0 \), there exist

\[
\begin{align*}
h_* &= h_*(\xi) > 0, \\
\varepsilon_* &= \varepsilon_*(h, \xi) > 0, \quad h \in (0, h_*], \\
\delta_* &= \delta_*(\varepsilon, h, \xi) > 0, \quad \varepsilon \in (0, \varepsilon_*(h)], \quad h \in (0, h_*],
\end{align*}
\]

such that the following statement holds. Let \((t_*, w_*(\cdot)) \in G_*\), \( h \in (0, h_*], \varepsilon \in (0, \varepsilon_*(h)]\), and let \( \Delta \) be a partition (24) with the diameter \( \text{diam}(\Delta) \leq \delta_*(\varepsilon, h) \). Then, the control procedure with a guide of the first player \( U(t_*, w_*(\cdot), h, \varepsilon, \Delta) \) guarantees the inequality

\[
\gamma \leq \rho(t_*, w_*(\cdot)) + \xi. \tag{39}
\]

for any control realization of the second player \( v(\cdot) \in V(t_*, \vartheta) \), and the control procedure with a guide of the second player \( V(t_*, w_*(\cdot), h, \varepsilon, \Delta) \) guarantees the inequality

\[
\gamma \geq \rho(t_*, w_*(\cdot)) - \xi. \tag{40}
\]

for any control realization of the first player \( u(\cdot) \in U(t_*, \vartheta) \).
Proof Let \( \zeta > 0 \) be fixed. Let us define

\[
    h_* = \min \{ h^{(4)}, h^{(5)}, h^{(6)} \}, \\
    \varepsilon_* (h) = \min \{ \varepsilon^{(4)} (h), \varepsilon^{(5)} (h) \}, \quad h \in (0, h_*], \\
    \delta_* (\varepsilon, h) = \min \{ \delta^{(4)} (\varepsilon, h), \delta^{(5)} (\varepsilon, h) \}, \quad \varepsilon \in (0, \varepsilon_*], \quad h \in (0, h_*], \\
\]

where \( h^{(4)} > 0 \), \( \varepsilon^{(i)} (h) > 0 \), and \( \delta^{(i)} (\varepsilon, h) > 0 \) for \( i \in \{4, 5\} \) are chosen as in (38), and \( h^{(6)} = h^{(6)} (\zeta/2) > 0 \) is chosen according to Lemma 6 such that, for any \( h \in (0, h^{(6)}] \) and any \( (t_*, w_*(\cdot)) \in G_* \), the inequality below is valid:

\[
    | \hat{\rho} (t_*, w_*(\cdot)) - \hat{\rho}_h (t_*, w_*(\cdot)) | \leq \zeta/2. 
\]

Let \((t_*, w_*(\cdot)) \in G_*\), \( h \in (0, h_*] \), \( \varepsilon \in (0, \varepsilon_* (h)] \), and let \( \Delta \) be a partition (24) with the diameter \( \text{diam}(\Delta) \leq \delta_* (\varepsilon, h) \). Let us consider the first player’s control procedure with a guide \( U = U (t_*, w_*(\cdot), h, \varepsilon, \Delta) \). By the choice of \( h, \varepsilon \) and \( \Delta \), for any \( v(*) \in \mathcal{V}(t_*, \vartheta) \), the motion \( x(*) \) of system (4) generated by \( U \) and \( v(*) \) satisfies the inequality

\[
    \gamma = \sigma (x(*)) \leq \hat{\rho}_h (t_*, w_*(\cdot)) + \zeta/2 \leq \hat{\rho} (t_*, w_*(\cdot)) + \zeta. \tag{41} \]

Further, in the case \( t_* < \vartheta \), on the basis of the control procedure \( U \), we define a first player’s non-anticipative strategy \( \kappa \) (see Sect. 3.4) as follows. For any \( v(*) \in \mathcal{V}(t_*, \vartheta) \), we consider the unique motion \( x(*) \) of system (4) and the control realization \( u(*) \) that are formed by \( U \) and \( v(*) \), and put \( \kappa (v(*)) = u(*) \). Then, by definition (10) of the lower game value \( \rho^{(u)} (t_*, w_*(\cdot)) \) and due to (41), we obtain

\[
    \rho^{(u)} (t_*, w_*(\cdot)) \leq \hat{\rho} (t_*, w_*(\cdot)) + \zeta. \tag{42} \]

In the degenerate case \( t_* = \vartheta \), from (12) and (41), we derive

\[
    \rho^{(u)} (\vartheta, w_*(\cdot)) = \sigma (w_*(\cdot)) \leq \hat{\rho} (\vartheta, w_*(\cdot)) + \zeta. \tag{43} \]

Since relations (42) and (43) are valid for any \( \zeta > 0 \), we conclude

\[
    \rho^{(u)} (t_*, w_*(\cdot)) \leq \hat{\rho} (t_*, w_*(\cdot)). \tag{44} \]

Now, arguing by contradiction, let us suppose that

\[
    \rho^{(u)} (t_*, w_*(\cdot)) + \zeta^* = \hat{\rho} (t_*, w_*(\cdot)) \tag{45} \]

for a number \( \zeta^* > 0 \). Similarly to above, based on Lemmas 5 and 6, by the number \( \zeta^*/3 \), one can choose \( h^* > 0 \), \( \varepsilon^* > 0 \), and a partition \( \Delta^* (24) \) such that the motion \( x(*) \) of system (4) generated from \((t_*, w_*(\cdot))\) by the second player’s control procedure with a guide \( V = V (t_*, w_*(\cdot), h^*, \varepsilon^*, \Delta^*) \) and a first player’s control realization \( u(*) \in \mathcal{U}(t_*, \vartheta) \) satisfies the inequality

\[
    \gamma = \sigma (x(*) - \hat{\rho} (t_*, w_*(\cdot)) - \zeta^*/3. \tag{45} \]

Further, in the case \( t_* < \vartheta \), let us consider a first player’s non-anticipative strategy \( \kappa^* \) such that, for the motion \( x(*) \) of system (4) generated by \( v(*) \in \mathcal{V}(t_*, \vartheta) \) and \( u(*) = \kappa^* (v(*) \) ), the inequality below is valid:

\[
    \gamma = \sigma (x(*) \leq \rho^{(u)} (t_*, w_*(\cdot)) + \zeta^*/3. \tag{45} \]

According to the definition (see Sect 6.3), the control procedure \( V \) forms \( v(*) \) by the steps of the partition \( \Delta^* \) on the basis of the information about the realized values of the state vectors of the original and approximating systems. Therefore, since \( \kappa^* \) is non-anticipative,
one can consider the motion $x^*(\cdot)$ generated by $u(\cdot) \in U(t_\ast, \vartheta)$ and $v(\cdot) \in V(t_\ast, \vartheta)$ such that $u(\cdot) = \kappa^*(v(\cdot))$, and, at the same time, $v(\cdot)$ is formed by $V$. For this motion $x^*(\cdot)$, we have

$$\hat{\rho}(t_\ast, w_\ast(\cdot)) - \zeta^*/3 \leq \sigma(x^*(\cdot)) \leq \rho^{(u)}(t_\ast, w_\ast(\cdot)) + \zeta^*/3,$$

wherefrom we obtain

$$\hat{\rho}(t_\ast, w_\ast(\cdot)) \leq \rho^{(u)}(t_\ast, w_\ast(\cdot)) + 2\zeta^*/3. \quad (46)$$

In the case $t_\ast = \vartheta$, due to (12) and (45), we derive

$$\hat{\rho}(\vartheta, w_\ast(\cdot)) \leq \sigma(w_\ast(\cdot)) + \zeta^*/3 = \rho^{(u)}(\vartheta, w_\ast(\cdot)) + \zeta^*/3. \quad (47)$$

Relations (46) and (47) contradict (44) since $\zeta^* > 0$. Hence, we conclude

$$\rho^{(u)}(t_\ast, w_\ast(\cdot)) = \hat{\rho}(t_\ast, w_\ast(\cdot)). \quad (48)$$

The validity of the equality $\rho^{(v)}(t_\ast, w_\ast(\cdot)) = \hat{\rho}(t_\ast, w_\ast(\cdot))$ can be established in a similar way with clear changes. Thus, the first and second parts of the theorem are proved. Inequality (39) in the third part of the theorem follows directly from (41) and (48). The validity of the inequality in (40) can be shown similarly. The theorem is proved. \(\square\)

**Remark 5** Let us note that, following [31, § 8.2] (see also [37] for details), one can consider another formalization of the differential game (4), (9). Namely, one can formally describe a sufficiently wide classes of players’ strategies with a guide and introduce the corresponding values of the players’ optimal guaranteed results. One can show that, from Theorem 1, it follows that these optimal guaranteed results coincide, i.e., the differential game has the value in the classes of strategies with a guide, and this value is equal to $\hat{\rho}(t_\ast, w_\ast(\cdot))$. Moreover, the players’ strategies with a guide that guarantee inequalities (39) and (40) can be constructed on the basis of the proposed in Sects. 6.2 and 6.3 control procedures. In this sense, these control procedures with guides can be called optimal.

**Remark 6** In addition to Remark 3, another possible way of solving the approximating differential game (21), (23) is to approximate functional differential equation of a retarded type (21) by a high-dimensional system of ordinary differential equations (see, e.g., [38] and the references therein). Note that this approach can also be used for proving the existence of the game value and constructing the players’ optimal control procedures with guides in the original differential game (4), (9).

**8 Example**

Let us illustrate the results presented in the paper by an example. Let us consider the differential game for the dynamical system

$$\begin{align*}
(CD^{\alpha} x)(t) &= F(t, x(t))(u(t) - v(t)), \quad t \in [t_0, \vartheta], \\
x(t) &\in \mathbb{R}^n, \quad u(t) \in U, \quad v(t) \in V = U, \quad (49)
\end{align*}$$

and the quality index

$$\gamma = \|x(\vartheta)\|^2. \quad (50)$$

Here, the set $U \subset \mathbb{R}^n$ is compact, and the matrix-function $F$ is such that $f(t, x, u, v) = F(t, x)(u - v)$ satisfies conditions (A.1)–(A.3). Note that, condition (A.4) is fulfilled due to the special form of the right-hand side of the differential equation in (49).
Let \((t_0, w_0(\cdot)) \in G_0\), and \(x^0(\cdot)\) be the solution to \((C D^a x^0)(t) = 0, t \in [t_0, \vartheta]\), under the initial condition \(x^0(t) = w_0(t), t \in [t_0, t_*]\). In the case \(t_* < \vartheta\), considering the players’ non-anticipative strategies \(\kappa(v(\cdot)) = v(\cdot)\) and \(\beta(u(\cdot)) = u(\cdot)\), where \(u(\cdot) \in \mathcal{U}(t_*, \vartheta)\) and \(v(\cdot) \in \mathcal{V}(t_*, \vartheta)\), according to (10) and (11), we derive

\[
\rho(\vartheta)(t_0, w_0(\cdot)) \leq \|x^0(\vartheta)\|^2 \leq \rho(v)(t_0, w_0(\cdot)),
\]

wherefrom, applying Theorem 1, we obtain

\[
\rho(t_0, w_0(\cdot)) = \|x^0(\vartheta)\|^2. \tag{51}
\]

In the case \(t_* = \vartheta\), this equality is also valid since, due to (12), we have

\[
\rho(\vartheta, w_0(\cdot)) = \|w_0(\vartheta)\|^2 = \|x^0(\vartheta)\|^2.
\]

Let us fix \(w_0 \in B(R_0)\) and \(h > 0\). For simplicity, we assume that \(\vartheta - t_0 = Mh\) for some \(M \in \mathbb{N}\). In accordance with Sect. 5, let us define the function

\[
F_{w_0, h}(t, r(\cdot)) = F(t, w_0 + h^{\alpha - 1}(\Delta_h^{1 - \alpha}(r(\cdot) - r(t_0)))(t)), \quad (t, r(\cdot)) \in G,
\]

and consider the approximating differential game for the dynamical system

\[
\dot{y}(t) = F_{w_0, h}(t, y_\vartheta)(p(t) - q(t)), \quad t \in [t_0, \vartheta],
\]

\[
y(t) \in \mathbb{R}^n, \quad p(t) \in \mathcal{U}, \quad q(t) \in \mathcal{V} = \mathbb{U},
\]

and the quality index

\[
\gamma_{w_0, h} = \|w_0 + h^{\alpha - 1}(\Delta_h^{1 - \alpha}(y(\cdot) - y(t_0)))(\vartheta)\|^2.
\]

In a similar way as above, taking into account Remark 4, for the value of the approximating differential game, we obtain

\[
\rho_{w_0, h}(t_0, r_\vartheta(\cdot)) = \|w_0 + h^{\alpha - 1}(\Delta_h^{1 - \alpha}(y_0(\cdot) - y_0(t_0)))(\vartheta)\|^2, \quad (t_0, r_\vartheta(\cdot)) \in G, \tag{52}
\]

where \(y_\vartheta(\cdot)\) is the solution to \(y_\vartheta(t) = 0, t \in [t_0, \vartheta]\), under the initial condition \(y_\vartheta(t) = r_\vartheta(t), t \in [t_0, t_*]\). For calculating the players’ optimal strategies \(F_{w_0, h}^0\) and \(G_{w_0, h}^0\) (see Lemma 1), let us apply the method of extremal shift in the direction of the ci-gradient of the value function \(\rho_{w_0, h}\) (see, e.g., [34, Theorem 3.1], [35, Theorem 3.1], and also [17, Theorem 1] for details).

For \(m \in \mathbb{N}, 0 < M - 1\), we denote \(G^{(m)} = \{(t, r(\cdot)) \in G : t \in [\vartheta - (m + 1)h, \vartheta - mh]\}\). Then, due to (52), for the ci-derivative in \(t\) and ci-gradient of \(\rho_{w_0, h}\) at \((t, r(\cdot)) \in G^{(m)}\), we obtain \(\partial_t \rho_{w_0, h}(t, r(\cdot)) = 0\) and

\[
\nabla \rho_{w_0, h}(t, r(\cdot)) = 2\left(w_0 + (t - t_0)h^{\alpha - 1}\sum_{i=0}^{m}(-1)^i(1 - \alpha)^i\right)
\]

\[
+ h^{\alpha - 1}\sum_{i=m+1}^{M}(-1)^i(1 - \alpha)^i(r(\vartheta - ih) - r(t_0))h^{\alpha - 1}\sum_{i=0}^{m}(-1)^i(1 - \alpha)^i.
\]

Since, for any \(m \in \mathbb{N}, 0 < M - 1\), the ci-gradient \(\nabla \rho_{w_0, h}\) is continuous on \(G^{(m)}\), the optimal strategies can be defined for \((t, r(\cdot)) \in G, t < \vartheta\), by

\[
P_{w_0, h}^0(t, r(\cdot)) \in \arg\min_{p \in \mathcal{U}} \langle F_{w_0, h}(t, r(\cdot))p, \nabla \rho_{w_0, h}(t, r(\cdot)) \rangle,
\]

\[
Q_{w_0, h}^0(t, r(\cdot)) \in \arg\max_{q \in \mathcal{V}} \langle -F_{w_0, h}(t, r(\cdot))q, \nabla \rho_{w_0, h}(t, r(\cdot)) \rangle.
\]
Let us note that these strategies do not depend on the accuracy parameter \( \varepsilon \).

Further, after the strategies \( P^0_{w_0,h} \) and \( Q^0_{w_0,h} \) are found, in the original differential game \((49), (50)\), one can consider the corresponding optimal players’ control procedures with guides \( U = U(t_*, w_*(\cdot), h, \Delta) \) and \( V = V(t_*, w_*(\cdot), h, \Delta) \) described in Sects. 6.2 and 6.3.

Let us present the results of computer simulations for

\[
n = 2, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad \alpha = 0.5, \quad t_0 = 0, \quad \vartheta = 5,
\]

\[
F(t, x) = \begin{pmatrix} \cos(x_2) \cos(x_1) \\ \sin(2x_1) 0.5t x_2 \end{pmatrix}, \quad \mathbb{U} = \{ u = (u_1, u_2) \in \mathbb{R}^2 : u_1^2 + u_2^2 \leq 1 \},
\]

\[
t_0 = 1, \quad w_*(t) = (t^{0.8} - 0.5, -t^{0.8}) \in \mathbb{R}^2, \quad t \in [0, 1], \quad h = 0.01,
\]

and the partition \( \Delta(24) \) of the interval \([1, 5] \) with the constant step 0.005. For numerical solving the original and approximating differential equations, we use the corresponding forward Euler methods (see, e.g., [32, p. 101] and [26, p. 115]) with the constant step 0.001.

The value of the original game calculated according to \((51)\) is

\[
\rho = \rho(t_*, w_*(\cdot)) \approx (-0.286)^2 + (-0.214)^2 \approx 0.128.
\]

We consider the following three cases:

1. Both players use the optimal control procedures \( U \) and \( V \). Quality index \((50)\) of the realized motion \( x(\cdot) \) of system \((49)\) is

\[
\gamma \approx (-0.28)^2 + (-0.229)^2 \approx 0.131 \approx \rho.
\]

In this case, the motion \( x(\cdot) \) is almost exactly coincides with the optimal motion \( x^0(\cdot) \), and, therefore, the illustration is omitted.

2. The first player uses the optimal control procedure \( U \), while the second player’s control realization \( v(\cdot) \in \mathcal{V}(t_0, \vartheta) \) is formed by the following non-optimal in general but reasonable in some sense positional control law

\[
v(t) = v_j = \arg\max_{v \in \mathcal{V}} (-F(j, x_j))v, x(x_j), \quad t \in [t_j, t_{j+1}), \quad j \in \overline{1, k}.
\]

The corresponding result is

\[
\gamma \approx 0.043^2 + (-0.021)^2 \approx 0.002 < \rho.
\]

The realized motion \( x(\cdot) \), its approximation \( x'(t) = v_*(t_0) + h^{a-1}(\Delta_h^{1-a} y)(t) \), \( t \in [t_0, \vartheta] \), calculated by the realized motion \( y(\cdot) \) of the first player’s guide, and also the optimal motion \( x^0(\cdot) \) are shown in Fig. 1. Note that bold lines depict the first coordinates of the functions, and thin lines depict their second coordinates.

3. The first player still uses the optimal control procedure \( U \), while the second player’s control realization \( v(\cdot) \) is defined by

\[
v_1(t) = \sin(t), \quad v_2(t) = \cos(t), \quad t \in [1, 5].
\]

In this case, the quality index of the corresponding motion \( x(\cdot) \) is

\[
\gamma \approx 0.082^2 + (-0.053)^2 \approx 0.01 < \rho.
\]

This motion \( x(\cdot) \) and its approximation \( x'(\cdot) \) are shown in Fig. 2.

Thus, the results of computer simulations are consistent with Theorem 1.
9 Conclusion

In the paper, we have considered a zero-sum differential game for a dynamical system which motion is described by a fractional differential equation. We have proved that the lower and upper game values coincide, i.e., the differential game has the value. The proof is based on the appropriate approximation of the game by a differential game for a dynamical system which motion is described by a first-order functional differential equation of a retarded type. This approach has also allowed us to propose the optimal players’ feedback control procedures with guides, which can be effectively applied if the optimal players’ positional strategies are found in the approximating game. The obtained results have been illustrated by an example.
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