Enhanced Euclidean supersymmetry, 11D supergravity and $SU(\infty)$ Toda equation

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ABSTRACT: We show how to lift solutions of Euclidean Einstein–Maxwell equations with non–zero cosmological constant to solutions of eleven–dimensional supergravity theory with non–zero fluxes. This yields a class of 11D metrics given in terms of solutions to $SU(\infty)$ Toda equation. We give one example of a regular solution and analyse its supersymmetry.

We also analyse the integrability conditions of the Killing spinor equations of $N = 2$ minimal gauged supergravity in four Euclidean dimensions. We obtain necessary conditions for the existence of additional Killing spinors, corresponding to enhancement of supersymmetry. If the Weyl tensor is anti-self-dual then the supersymmetric metrics satisfying these conditions are given by separable solutions to the $SU(\infty)$ Toda equation. Otherwise they are ambi–Kähler and are conformally equivalent to Kähler metrics of Calabi type or to product metrics on two Riemann surfaces.

KEYWORDS: Differential and Algebraic Geometry, Supergravity Models

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1 Introduction

Euclidean solutions to Einstein–Maxwell equations with non–zero cosmological constant (and their lifts to eleven–dimensional supergravity) have recently appeared in the context of AdS/CFT correspondence. In [18, 19] it was argued that a class of supersymmetric gauge theories on three–dimensional Berger spheres posses gravity duals given by Euclidean \( N = 2 \) minimal gauged supergravity solutions. See [17] for some earlier results on Euclidean supersymmetry.

The aim of this paper is two–fold. Firstly we shall show how to lift the Euclidean Einstein–Maxwell space times with \( \Lambda > 0 \) to solutions of \( D = 11 \) Lorentzian supergravity with non–vanishing fluxes. The Fubini–Study metric \( g_{\mathbb{C}P^2} \) on \( \mathbb{C}P^2 \) with the Maxwell field given by the Kähler form leads to an explicit regular eleven–dimensional solution which is a non–trivial bundle over a product manifold \( \mathbb{C}P^2 \times \tilde{\mathbb{C}P}^3 \), where \( \tilde{\mathbb{C}P}^3 \) is the non–compact dual to \( \mathbb{C}P^3 \) with the Bergmann metric. The metric and the four–form are given by

\[
\begin{align*}
ds^2 &= g_{\mathbb{C}P^2} + dt^2 + e^\tau g_{\mathbb{R}^4} - 2e^\tau(dt + A)(d\chi - \alpha + \frac{1}{2}e^{-\tau}(dt + A)) \\
G &= 3\text{vol}_{\mathbb{C}P^2} - J \wedge F,
\end{align*}
\]

where \( g_{\mathbb{R}^4} \) is the flat metric on \( \mathbb{R}^4 \) with the Kähler form \( da \), the Maxwell field in four dimensions is the Kähler form on \( \mathbb{C}P^2 \) given by \( F = dA \), and \( J = -d(e^\tau(d\chi - \alpha)) \). This solution admits a null non-hyper-surface-orthogonal isometry \( \frac{\partial}{\partial \chi} \). Our procedure is a modification of the ansatz made by Pope [23], adapted to the Euclidean signature, and anti–self–dual Maxwell fields. Moreover we show that SUSY solutions in four dimensions in general lift to non SUSY solutions in eleven dimensions.
Secondly we shall investigate solutions of $N = 2$ minimal gauged supergravity in four Euclidean dimensions (these are the same as Euclidean Einstein–Maxwell space–times with non–vanishing cosmological constant), where the Killing spinor equations admit more than one solution. In particular, we derive necessary conditions for enhanced supersymmetry by analysing the integrability conditions of the Killing spinor equations. This will select a subclass of solutions constructed in [7, 8] (see [6] and [16] for discussion of supersymmetric solutions to Euclidean $N = 2$ SUGRA with $\Lambda = 0$). In particular the underlying metric has to be ambi–Kähler in the sense of [2] w.r.t. both self–dual and anti–self–dual parts of the Maxwell fields. If the self–dual part of the Maxwell field vanishes, the integrability conditions impose additional constraints on solutions of the $SU(\infty)$ Toda equation [3]

$$U_{XX} + U_{YY} + (e^U)_{ZZ} = 0, \quad U = U(X, Y, Z) \quad (1.2)$$

which underlies some solutions of [7]. Here $U_X = \partial_X U$ etc. Imposing these constraints enables us to integrate the $SU(\infty)$ Toda equation completely to find that the corresponding solutions are separable, i.e. $U(X, Y, Z) = U_1(Z) + U_2(X, Y)$ and thus fall into the class studied by Tod [27]. This class of solutions includes the non–compact analogue of the Fubini–Study metric, $\tilde{CP}^2 = SU(2,1)/U(2)$ with cosmological constant $\Lambda < 0$. In this case the norm of the Maxwell is constant and the solution of (1.2) is

$$U = \ln \frac{4Z(Z + 4\Lambda)}{(1 + X^2 + Y^2)^2}. \quad (1.3)$$

If the Maxwell field is neither self–dual nor anti–self–dual, then the integrability conditions imply that the solutions are type $D$ Euclidean Einstein–Maxwell metrics. They are either conformal rescalings of a product metric on two Riemann surfaces, or conformal rescalings of

$$\tilde{g} = yh_\Sigma + yQ^{-1}dy^2 + y^{-1}Q(d\psi + \phi)^2 \quad (1.4)$$

where $Q(y)$ is a product of two quadratic polynomials in $y$ and $d\phi$ is a volume form of a metric $h_\Sigma$ on a 2D surface $\Sigma$. This form of the metric is a special case of the metric appearing in equation (10) of [2].

### 2 CP$^2$ and $\tilde{CP}^2$ as SUSY solutions to Euclidean gauged supergravity

Let $\sigma_j, j = 1, 2, 3$ be the left–invariant one forms on the group manifold $SU(2)$ such that $d\sigma_1 = \sigma_2 \wedge \sigma_3$, etc. The Fubini–Study metric on $CP^2$ is (see e.g. [22], [5])

$$g_4 = f^2 dr^2 + \frac{1}{4}(r^2 f (\sigma_1^2 + \sigma_2^2) + r^2 f^2 \sigma_3^2), \quad (2.1)$$

where $f = (1 + \Lambda r^2)^{-1}$ and $\Lambda > 0$. This metric is conformally anti–self–dual (ASD) and Einstein with the cosmological constant $\Lambda$, i.e. $R_{ab} = 6\Lambda g_{ab}$. Taking instead $\Lambda < 0$ in (2.1) gives the Bergmann metric on the non–compact manifold $\tilde{CP}^2 = SU(2,1)/U(2)$. The
metric (2.1) is also Kähler, albeit with the opposite orientation: the ASD Kähler form is given by

\[ F = f^2 r \, dr \wedge \sigma_3 + \frac{1}{2} r^2 f \, \sigma_1 \wedge \sigma_2 \] (2.2)

\[ = d((1/2)r^2 f) \].

In [7] it was shown that supersymmetric solutions to \( N = 2 \) minimal gauged Euclidean supergravity in four dimensions, with anti–self–dual Maxwell field, and such that the Killing spinor generates a Killing vector (which we call \( K = \frac{\partial}{\partial \psi} \)) are of the form

\[ g = \frac{1}{Z^2} \left( V(dZ^2 + e^U(dX^2 + dY^2)) + V^{-1}(d\psi + \phi)^2 \right), \] (2.3)

where \( U = U(X,Y,Z) \) is a solution of the \( SU(\infty) \) Toda equation (1.2), the function \( V \) is given by \( 4AV = ZUZ - 2 \), and \( \phi \) is a one–form such that

\[ d\phi = -V_X dY \wedge dX - V_Y dZ \wedge dX - (V e^U)_Z dX \wedge dY. \] (2.4)

This is also the most general class of ASD Einstein metrics with \( \Lambda \neq 0 \) and an isometry [25], [26].

We can now read off the solution of the \( SU(\infty) \) Toda equation from the metric (2.1). This can be done in more than one way, as the isometry group of (2.1) is \( SU(2,1) \). Thus we make a choice of a left–invariant Killing vector \( K \) such that \( i_K \sigma_3 = 1 \) (this would be given by \( \frac{\partial}{\partial \psi} \) in the usual coordinates on \( SU(2) \), but there is no need to introduce the coordinates at this stage). Comparing (2.3) with (2.1) we find \( Z = -4/(r^2 f) \). We now introduce the coordinates \((X,Y)\) on the two–sphere \( \mathbb{CP}^1 \) such that

\[ \sigma_1^2 + \sigma_2^2 = \frac{4(dX^2 + dY^2)}{(1 + X^2 + Y^2)^2}, \] (2.5)

which yields the expression (1.3) for \( U \) and \( V = -(Z + 4\Lambda)^{-1} \). In these coordinates the ASD Kähler form is

\[ F = \frac{2}{Z^2} dZ \wedge (d\psi + \phi) - \frac{8}{Z} \frac{dX \wedge dY}{(1 + X^2 + Y^2)^2}. \] (2.6)

This coincides, up to a constant overall factor, with the ASD Maxwell field constructed in [7, 8] for a general solution to \( SU(\infty) \) Toda equation. The formula for \( F \) was only implicit in these papers - the explicit expression is given by

\[ F = \frac{\ell^{-1}}{4\Lambda(ZUZ - 2)}(U_{ZY}dZ \wedge dX - U_{ZX}dZ \wedge dY) \]

\[ + \frac{\ell^{-1}}{(ZUZ - 2)^2}(U_{ZX}dX + U_{ZY}dY) \wedge (d\psi + \phi) \]

\[ + \frac{\ell^{-1}}{8\Lambda(ZUZ - 2)} e^U(2U_{ZZ} + U_Z^2) dX \wedge dY + \frac{\ell^{-1}}{2(ZUZ - 2)^2}(2U_{ZZ} + U_Z^2) dZ \wedge (d\psi + \phi), \] (2.7)

\[ ^1 \text{In this paper we revert to the standard sign convention for the cosmological constant. Thus the Fubini–Study metric and the round sphere have } \Lambda > 0. \text{ In our previous papers [7, 8] these metrics had } \Lambda < 0. \]
where $\Lambda = -1/(2\ell^2)$. Thus we need $\Lambda < 0$ for this to be real, and we have established that $\mathbb{C}P^2$ is super-symmetric. The Maxwell potential is given by

$$A = \frac{1}{\ell} \left( \frac{1}{2} U_Z - (d\psi + \phi) + \frac{1}{8\Lambda} (U_X dY - U_Y dX) \right)$$

(2.8)

and the two–form (2.6) is given by $-4\ell\, dA$. If instead $\Lambda = 1/(2\ell^2) > 0$ then (2.3) is still supersymmetric if we replace $F$ by $-F$.

The Maxwell two–form satisfies $F \wedge F = c.\, \text{vol}$ where $c = \text{const}$ if

$$e^U (U_Z^2 + 2U_ZZ)^2 + 4(U_ZX^2 + U_ZY^2) = \frac{1}{4} c e^U \left( \frac{ZU_Z - 2}{Z} \right)^4.$$  

(2.9)

We note that, up to a transformation of $(X,Y)$ coordinates, the function (1.3) is the most general solution to the above constraint which also satisfies the $SU(\infty)$ Toda equation, and is separable in the sense that $U(X,Y,Z) = U_1(Z) + U_2(X,Y)$.

To sum up, if $(M,g)$ is conformally ASD, $F_+ = 0$ and $F_-$ is covariantly constant then $(M,g)$ is a space of constant holomorphic sectional curvature, i.e. a complex space form $\mathbb{C}P^2$, its non–compact dual $\mathbb{C}P^2$, or $\mathbb{C}^2$.

3 Lift to eleven dimensions

Let $(M_4, g_4, F)$ be a Riemannian solution of Einstein–Maxwell equations in four dimensions with $\Lambda > 0$ and anti–self–dual Maxwell field $F = dA$, and let $(M_6, g_6, J)$ be a Kähler–Einstein manifold with the Ricci tensor $R_{\alpha\beta} = k_2 g_{\alpha\beta}$ and the Kähler form $J$. Let us consider a Lorentzian metric

$$ds^2 = g_4 + g_6 - (dt + k_1 A + B)^2,$$

(3.1)

together with the the four–form

$$G = \sigma_1 \text{vol}_4 + \sigma_2 J \wedge F + \sigma_3 J \wedge J,$$

(3.2)

where $dB = 2k_3 J$ and $k_1, k_2, k_3, \sigma_1, \sigma_2, \sigma_3$ are constants which will be fixed by the field equations of $D = 11$ supergravity

$$R_{AB} = \frac{1}{12} g_{AN_1N_2N_3} g_{B}^{N_1N_2N_3} - \frac{1}{144} g_{AB} g_{N_1N_2N_3N_4} g^{N_1N_2N_3N_4}$$

(3.3)

and

$$d \star_{11} G - \frac{1}{2} G \wedge G = 0.$$  

(3.4)

Here $R_{AB}$ is the Ricci tensor of $ds^2$, and the capital letter indices run from 0 to 10. The case $\sigma_3 = 0$ is a modification of the ansatz [23] adapted to the Euclidean signature. We chose the eleven–dimensional volume to be

$$\text{vol}_{11} = \frac{1}{6} (dt + k_1 A + B) \wedge \text{vol}_4 \wedge J \wedge J \wedge J$$

(3.5)
which yields
\[ *_{11} G = (dt + k_1 A + B) \wedge \left( \frac{1}{6} \sigma_1 J \wedge J \wedge J - \frac{1}{2} \sigma_2 J \wedge J \wedge F + 2 \sigma_3 J \wedge \text{vol}_4 \right). \tag{3.6} \]

We now substitute this to the gauge field equations (3.4) and get the following three conditions for the constants
\[ -k_3 \sigma_2 + \frac{1}{6} k_1 \sigma_1 = \sigma_2 \sigma_3, \quad \sigma_2 (k_1 + \sigma_2) = 0, \quad \sigma_3 (4 k_3 - \sigma_1) = 0. \tag{3.7} \]

The analysis of the Einstein equations is more complicated, and requires the computation of the spin connection coefficients. We skip the tedious calculations and only give the answer. Equations (3.3) hold if and only if
\[ 6k_3^2 = 2 \sigma_2^2 + \frac{1}{6} \sigma_1^2, \quad k_1^2 = \sigma_2^2, \quad k_2 + 2k_3^2 = 2 \sigma_3^2 - \frac{1}{6} \sigma_2^2, \quad 6 \Lambda = \frac{1}{3} \sigma_1^2 - 2 \sigma_3^2. \tag{3.8} \]

Consider the conditions (3.7) together with (3.8). It is straightforward to show that there are no solutions to these equations with both \( \sigma_2 \neq 0 \) and \( \sigma_3 \neq 0 \). There are three distinct classes of solutions:

1. Solutions with \( \sigma_2 = \sigma_3 = 0 \). These have \( k_1 = 0 \). Furthermore one must have \( \Lambda \geq 0 \), with
   \[ \sigma_1^2 = 18 \Lambda, \quad k_3^2 = \frac{1}{2} \Lambda, \quad k_2 = -4 \Lambda. \tag{3.9} \]
   Thus the Maxwell field decouples in this lift, and we end up with an analogue of the Freund–Rubin solution.

2. Solutions with \( \sigma_2 = 0, \sigma_3 \neq 0 \). These solutions also have \( k_1 = 0 \), and \( \Lambda > 0 \) with
   \[ k_3^2 = 2 \Lambda, \quad \sigma_3^2 = \frac{-10}{3} \Lambda, \quad \sigma_1 = \pm 4 \sqrt{2 \Lambda}, \quad k_2 = \frac{-8}{3} \Lambda. \tag{3.10} \]
   The Maxwell field also decouples in this case. This is an analogue of the Englert solution.

3. Solutions with \( \sigma_2 \neq 0, \sigma_3 = 0 \). These solutions have \( \Lambda \geq 0 \), with
   \[ \sigma_1 = \pm 3 \sqrt{2 \Lambda}, \quad k_3 = \mp \frac{1}{\sqrt{2}} \sqrt{\Lambda}, \quad k_2 = -4 \Lambda, \tag{3.11} \]
   and \( k_1 \) and \( \sigma_2 \) satisfy \( k_1 = -\sigma_2 \) but are otherwise unconstrained. This is the most interesting class from our perspective, as the four–dimensional Maxwell field contributes non–trivially to the flux in eleven dimensions.

Note that in all cases, only solutions with \( \Lambda \geq 0 \) can be uplifted. This should be contrasted with the original Pope ansatz [23] where the Lorentzian Einstein–Maxwell space times with \( \Lambda < 0 \) have been uplifted to eleven dimensions.
3.1 Example

We shall consider the third possibility (3.11), and chose \( \sigma_2 = -1 \) so that the flux is given by

\[
G = 3\sqrt{2} \Lambda \text{vol}_4 - J \wedge F.
\]  

(3.12)

Let us take \( M_4 = \mathbb{C}P^2, g_4 = g_{\mathbb{C}P^2} \) with the ASD Fubini study metric (2.1) and the Maxwell field given by the ASD Kähler form (2.2). The six–dimensional Kähler–Einstein manifold is taken to be the non–compact version of the complex projective three space, \( \mathbb{C}P^3 \) known as the Bergmann space. The Ricci scalars of \( \mathbb{C}P^2 \) and \( \mathbb{C}P^3 \) have the same magnitude but opposite signs. We shall chose \( \Lambda = 1/2 \), so that the Ricci scalar of \( \mathbb{C}P^2 \) is 12.

To construct the metric on \( \mathbb{C}P^3 \) explicitly, consider the metric

\[
|dZ_1^2| + |dZ_2^2| + |dZ_3^2| - |dZ_4^2|
\]  

(3.13)

on \( \mathbb{C}^4 \). This is \( SU(3,1) \) invariant. Restricting this metric to the quadric \(|Z_1|^2 + |Z_2|^2 + |Z_3|^2 - |Z_4|^2 = -1\) reduces it to the constant curvature metric on \( AdS_7 \). Both the initial metric, and the quadric constraint are invariant under \( Z_\alpha \rightarrow \exp(i\theta)Z_\alpha \), and the Bergmann manifold is the space of orbits under this circle action. Thus we can express the \( AdS_7 \) as a non-trivial \( U(1) \) or \( \mathbb{R}^* \) bundle over \( \mathbb{C}P^3 \):

\[
g_{AdS_7} = -(dt + B)^2 + g_{\mathbb{C}P^3}, \quad \text{where} \quad J = -dB.
\]  

(3.14)

Finally, the lift (3.1) of the Fubini–Study metric on \( \mathbb{C}P^2 \) to the solution of \( D = 11 \) supergravity is given by a regular metric

\[
ds^2 = g_{\mathbb{C}P^2} + g_{\mathbb{C}P^3} - (dt + A + B)^2,
\]  

(3.15)

which is a non–trivial line bundle over the ten dimensional Riemannian manifold \( \mathbb{C}P^2 \times \mathbb{C}P^3 \) with its product metric. This could, if desired, be reduced along the time–like direction to a \( D = 10 \) Euclidean supergravity. Alternatively we can exhibit a reduction along a space–like Killing vector in \( \mathbb{C}P^3 \) which leads to a Lorentzian solution to the type IIA string theory.

It worth remarking that (3.15) belongs to the class of 11D SUGRA solutions with null isometry. To exhibit this isometry, use local coordinates \((\tau, \chi, p, q, r, s)\) on \( \mathbb{C}P^3 \) found in [24] so that the Bergmann metric takes the form

\[
g_{\mathbb{C}P^3} = d\tau^2 + e^\tau(dp^2 + dq^2 + dr^2 + ds^2) + e^{2\tau}(d\chi - pdr - qds)^2,
\]  

(3.16)

In these coordinates \( B = e^\tau(d\chi - pdr - qds) \), and (3.15) can be written as

\[
ds^2 = g_{\mathbb{C}P^2} + d\tau^2 + e^\tau g_{\mathbb{R}^4} - 2e^\tau(dt + A)(d\chi - \alpha + \frac{1}{2}e^{-\tau}(dt + A))
\]  

(3.17)

where \( g_{\mathbb{R}^4} \) is the flat metric on \( \mathbb{R}^4 \) with the Kähler form \( d\alpha \). The null isometry is generated by \( \frac{\partial}{\partial \chi} \). The corresponding one–form \( g(\frac{\partial}{\partial \chi}, \cdot) \) is not hyper–surface–orthogonal, and so (3.15) is not a plane wave solution.
3.2 Supersymmetry of Uplifted $\mathbb{C}P^2$ Solution

The Killing spinor equations of $D = 11$ supergravity are

$$\mathcal{D}_M \epsilon = 0$$

(3.18)

where

$$\mathcal{D}_M = \nabla_M - \frac{1}{288} \left( \Gamma_M N_1 N_2 N_3 N_4 G_{N_1 N_2 N_3 N_4} - 8 G_{M N_1 N_2 N_3} \Gamma^{N_1 N_2 N_3} \right).$$

(3.19)

Here $\nabla_M$ is the Levi–Civita connection of the 11D metric $ds^2$, $\Gamma^M$ are the Dirac matrices in eleven dimensions and $\Gamma^{MNP} = \Gamma[^{[M} \Gamma^{N]P]}$ etc. It follows from the work of [20] that the uplifted $\mathbb{C}P^2$ solution is a spin manifold. However, this does not imply that there exists a spinor satisfying (3.18). It is straightforward to show that the uplifted $\mathbb{C}P^2$ solution exhibits both a timelike and null isometry, corresponding to $\partial / \partial t$ and $\partial / \partial \chi$ respectively. One might therefore attempt to match the uplifted geometry to the conditions derived in [11] or [12], in which the conditions necessary for a solution of $D=11$ supergravity to preserve the minimal amount of supersymmetry were derived, by comparing the isometry of the uplifted solution to the isometry generated by the vector field dual to the 1-form Killing spinor bilinear associated with any supersymmetric solution. This naive matching fails, but this does not constitute a proof that the solution is not supersymmetric, as the apparent discrepancy in the geometric conditions may simply be an artefact of a poor choice of gauge.

In order to determine if the uplifted solution is actually supersymmetric, a necessary condition is that the integrability conditions of the $D=11$ Killing spinor equations (3.18) should admit a non-zero solution $\epsilon$. In particular, consider the integrability conditions

$$\mathcal{R}_{MN} \epsilon = 0,$$

(3.20)

where

$$\mathcal{R}_{MN} = [\mathcal{D}_M, \mathcal{D}_N].$$

(3.21)

The actual form of the supercovariant curvature $\mathcal{R}_{MN}$ is set out in [10]. Due to the rather complicated structure of these integrability conditions, the analysis of (3.20) has been performed using a computer. First, it is straightforward to check that

$$\Gamma^M \mathcal{R}_{MN} \epsilon = 0$$

(3.22)

for any spinor $\epsilon$; this also follows as a consequence of the bosonic field equations [15]. We choose a basis $e^0, e^\mu, e^a$, where

$$e^0 = dt + A + B$$

(3.23)

and let $e^a$ be an appropriately chosen basis for $\mathbb{C}P^2$, and $e^a$ be a basis for $\mathbb{C}
\tilde{P}^3$. Using some further computer analysis, one finds that the integrability condition (3.20) is equivalent to

$$F_{\mu \nu} \Gamma^{\mu \nu} \epsilon = 0, \quad \Gamma_0 J_{ab} \Gamma^{ab} \epsilon = 6 \epsilon.$$  

(3.24)
These conditions reduce the number of real degrees of freedom in $\epsilon$ from 32 down to 4, so the solution can preserve at most $N = 4$ supersymmetry. We remark that these conditions also imply that

$$G_{N_1N_2N_3N_4}\Gamma^{N_1N_2N_3N_4}\epsilon = -72\epsilon$$

(3.25)

and using this identity the Killing spinor equation (3.18) simplifies to

$$\left(\nabla_M + \frac{1}{24}G_{MN_1N_2N_3}\Gamma^{N_1N_2N_3} + \frac{1}{4}\Gamma_M\right)\epsilon = 0.$$  

(3.26)

It remains to analyse (3.26), making use of the conditions (3.24). The non-zero components of the $D = 11$ spin connection are

$$\Omega_{0,\mu\nu} = \Omega_{\mu,0\nu} = \frac{1}{2}F_{\mu\nu}, \quad \Omega_{0,ab} = \Omega_{a,0b} = -\frac{1}{2}J_{ab}, \quad \Omega_{\mu,\nu\rho} = \tilde{\omega}_{\mu,\nu\rho}, \quad \Omega_{a,bc} = \tilde{\omega}_{a,bc},$$

(3.27)

where $\tilde{\omega}_{\mu,\nu\rho}$ is the spin connection of $\mathbb{C}P^2$, and $\tilde{\omega}_{a,bc}$ is the spin connection of $\tilde{\mathbb{C}P}^3$.

Then the $M = 0$ component of (3.26) implies that

$$\partial_t \epsilon = -\Gamma_0 \epsilon$$

(3.28)

so

$$\epsilon = \left(\cos t \mathbf{1} - \sin t \Gamma_0\right)\dot{\epsilon}$$

(3.29)

where $\partial_t \dot{\epsilon} = 0$. The conditions (3.24) are equivalent to

$$F_{\mu\nu}\Gamma^{\mu\nu}\dot{\epsilon} = 0, \quad \Gamma_0 J_{ab}\Gamma^{ab}\dot{\epsilon} = 6\dot{\epsilon}.$$  

(3.30)

Next, consider the $M = \mu$ component of (3.26). This is equivalent to

$$\left(\partial_\mu + \frac{1}{4}\omega_{\mu,\nu_1\nu_2}\Gamma^{\nu_1\nu_2}\right)\epsilon - \frac{1}{2}\Gamma_\mu\epsilon - F_{\mu\lambda}\Gamma_0\Gamma^\lambda\epsilon = 0.$$  

(3.31)

On substituting (3.29) into this expression, and evaluating the terms dependent on $\sin t$ and $\cos t$ independently, one obtains the condition

$$\frac{1}{2}\Gamma_\mu\dot{\epsilon} + F_{\mu\lambda}\Gamma_0\Gamma^\lambda\dot{\epsilon} = 0.$$  

(3.32)

On contracting this expression with $\Gamma^\mu$, and using (3.30), one finds $\dot{\epsilon} = 0$. It follows that the uplifted solution (3.17, 3.12) is not supersymmetric.
4 Solutions with Enhanced Supersymmetry

In this section, we examine solutions of minimal Euclidean gauged supergravity with enhanced supersymmetry. We shall consider solutions which admit a spinor $\epsilon$ satisfying the following Killing spinor equation:

$$ \left( \partial_\mu + \frac{1}{4} \Omega_{\mu \nu_1 \nu_2} \Gamma^{\nu_1 \nu_2} + \frac{i}{4} F_{\nu_1 \nu_2} \Gamma^{\nu_1 \nu_2} \Gamma_\mu + \frac{1}{2\ell} \Gamma_\mu - \frac{i}{\ell} A_\mu \right) \epsilon = 0. \quad (4.1) $$

In this case, the cosmological constant is given by $\Lambda = -\frac{1}{2\ell^2}$, and we do not assume that the Maxwell field strength $F = dA$ is either self or anti-self-dual.

All supersymmetric solutions preserving one quarter of the supersymmetry were classified in [8], and one finds that the metric and gauge potential are given by

$$ g = 2\lambda^2 \sigma^2 (d\psi + \phi)^2 + \frac{1}{\lambda^2 \sigma^2} \left( \frac{1}{2} dx^2 + 2 e^{2u} dz d\bar{z} \right) $$

$$ A = \frac{1}{\sqrt{2}} \left( \lambda^2 - \sigma^2 \right) (d\psi + \phi) - \frac{i\ell}{2} \partial_z u dz + \frac{i\ell}{2} \partial_{\bar{z}} u d\bar{z} \quad (4.2) $$

where $\lambda, \sigma, u$ are functions, and $\phi = \phi_x dx + \phi_z dz + \phi_{\bar{z}} d\bar{z}$ is a 1-form. All components of the metric and gauge potential are independent of the co-ordinate $\psi$; and $u, \lambda, \sigma, \phi$ must satisfy

$$ \partial_x u = -\frac{1}{\sqrt{2\ell}} (\lambda^{-2} + \sigma^{-2}) \quad (4.3) $$

and

$$ \partial_z \partial_{\bar{z}} (\lambda^{-2} - \sigma^{-2}) + e^{2u} \left( \partial_x^2 (\lambda^{-2} - \sigma^{-2}) + 3(\lambda^{-2} - \sigma^{-2}) \partial_x^2 u \right. $$

$$ + 3(\lambda^{-2} - \sigma^{-2})(\partial_x u)^2 + 3\partial_x u \partial_x (\lambda^{-2} - \sigma^{-2}) + \frac{1}{2} \ell^{-2} (\lambda^{-2} - \sigma^{-2})^3 \right) = 0 \quad (4.4) $$

and

$$ \partial_z \partial_{\bar{z}} u + e^{2u} \left( \partial_x^2 u + \frac{1}{2} (\partial_x u)^2 + \frac{3}{4} \ell^{-2} (\lambda^{-2} - \sigma^{-2})^2 \right) = 0 \quad (4.5) $$

and

$$ d\phi = -\frac{i}{(\lambda \sigma)^2} (\partial_z \log \frac{\lambda}{\sigma}) dx \wedge dz + \frac{i}{(\lambda \sigma)^2} (\partial_{\bar{z}} \log \frac{\lambda}{\sigma}) dx \wedge d\bar{z} $$

$$ + \frac{i e^{2u}}{(\lambda \sigma)^2} \left( 2\partial_x \log \frac{\sigma}{\lambda} + \sqrt{2\ell^{-1}} \left( \frac{\lambda^2 - \sigma^2}{(\lambda \sigma)^2} \right) \right) dz \wedge d\bar{z}. \quad (4.6) $$

We remark that the self-dual and anti-self-dual parts of $F$ are given by:

$$ F^- = -2(d\psi + \phi) \wedge (\sqrt{2}\lambda d\lambda + \frac{1}{2\ell} dx) - \sqrt{2} i e^{2u} (2\lambda \partial_x \lambda + \frac{1}{\sqrt{2\ell}}) dz \wedge d\bar{z} $$

$$ - \sqrt{2} i \frac{1}{\lambda \sigma^2} dx \wedge (\partial_z \lambda dz - \partial_{\bar{z}} \lambda d\bar{z}) \quad (4.7) $$
and
\[ F^+ = 2(d\psi + \phi) \wedge (\sqrt{2} \sigma d\sigma + \frac{1}{2\ell} dx) - \sqrt{2} i e^{2u} \lambda^2 \sigma^2 (2\sigma \partial_x \sigma + \frac{1}{\sqrt{2}\ell}) dz \wedge d\bar{z} \]
\[ - \frac{\sqrt{2}}{\sigma \lambda^2} dx \wedge (\partial_x \sigma dz - \partial_x \sigma d\bar{z}). \tag{4.8} \]

On setting \((F^\pm)^2 = F^\pm_{\mu\nu} F^{\pm\mu\nu}\), one obtains
\[ (F^+)^2 = \frac{4}{\ell^2} \left( 8\ell^2 e^{-2u} \sigma^2 \partial_x \sigma + (1 + 2\sqrt{2}\ell \sigma \partial_x \sigma)^2 \right) \]
\[ (F^-)^2 = \frac{4}{\ell^2} \left( 8\ell^2 e^{-2u} \lambda^2 \partial_x \lambda + (1 + 2\sqrt{2}\ell \lambda \partial_x \lambda)^2 \right). \tag{4.9} \]

In addition, the conditions (4.3), (4.4) and (4.5) imply
\[ e^{-2u} \partial_x \partial_x \sigma^2 + \partial_x \partial_x \sigma^2 - \frac{3\sqrt{2}}{\ell} \sigma^{-2} \partial_x \sigma^{-2} + 2\ell^{-2} \sigma^{-6} = 0 \]
\[ e^{-2u} \partial_x \partial_x \lambda^2 + \partial_x \partial_x \lambda^2 - \frac{3\sqrt{2}}{\ell} \lambda^{-2} \partial_x \lambda^{-2} + 2\ell^{-2} \lambda^{-6} = 0. \tag{4.10} \]

The integrability conditions of (4.1) can be decomposed into positive and negative chirality parts
\[ \left( \frac{1}{4} W^\pm_{\mu\nu\lambda_1\lambda_2} \Gamma^{\lambda_1\lambda_2} + \frac{i}{2\ell} F^\pm_{\lambda_1[g\nu]\lambda_2} \Gamma^{\lambda_1\lambda_2} - \frac{i}{\ell} F^\pm_{\mu\nu} \right) \epsilon_\pm + \frac{i}{2} \nabla_{\sigma} F^\pm_{\mu\nu} \Gamma^\sigma \epsilon_\pm = 0 \tag{4.11} \]
where \(W^\pm\) are the self-dual and anti-self-dual parts of the Weyl tensor \(W\), with \(W = W^+ + W^-\), and
\[ \gamma_5 \epsilon_\pm = \pm \epsilon_\pm. \tag{4.12} \]

We assume that there exists a Killing spinor \(\epsilon^1\) satisfying (4.1) and its associated integrability conditions (4.11). The components of \(\epsilon^1\) are the functions \(\lambda, \sigma\) in an appropriately chosen gauge, using spinorial geometry techniques as described in [8].

To begin, we consider solutions preserving half of the supersymmetry. We denote the additional spinor by \(\epsilon^2\), and it is particularly convenient to write the components of \(\epsilon^2\) as
\[ \epsilon_2^+ = \alpha \epsilon_1^+ + \beta C * \epsilon_1^+ \]
\[ \epsilon_2^- = \theta \epsilon_1^- + \rho C * \epsilon_1^- \tag{4.13} \]
where \(\alpha, \beta, \theta, \rho\) are complex functions.

On evaluating (4.11) acting on \(\epsilon_2^\pm\) we eliminate the Weyl tensor terms using the conditions on \(\epsilon_1^\pm\), to obtain
\[ - \frac{1}{2} (\alpha - \theta) \nabla_{\sigma} F_{\mu\nu} \Gamma^\sigma \epsilon_1^+ + \frac{1}{2} (\beta + \rho) \nabla_{\sigma} F_{\mu\nu} \Gamma^\sigma C * \epsilon_1^- \]
\[ + \ell^{-1} \beta (F^+_{\lambda_1[g\nu]\lambda_2} \Gamma^{\lambda_1\lambda_2} - 2F^-_{\mu\nu}) C * \epsilon_1^- = 0 \tag{4.14} \]
\[
\frac{1}{2}(\alpha-\theta)\nabla_\sigma F^+_{\mu\nu}\Gamma^\sigma_+ + \frac{1}{2}(\beta+\rho)\nabla_\sigma F^-_{\mu\nu}\Gamma^\sigma_- C \ast \epsilon^1_+ \\
+ \epsilon^{-1}\rho(F^+_{\lambda_1[\mu} g_{\nu]\lambda_2} \Gamma^{\lambda_1\lambda_2} - 2F^+_{\mu\nu}) C \ast \epsilon^1_- = 0.
\] (4.15)

Note that contracting (4.14) with \( F^-_{\mu\nu} \) and contracting (4.15) with \( F^+_{\mu\nu} \) one finds

\[
-\frac{1}{4}(\alpha-\theta)\nabla_\sigma (F^-)^2 \Gamma^\sigma \epsilon^1_- + \frac{1}{4}(\beta+\rho)\nabla_\sigma (F^-)^2 \Gamma^\sigma C \ast \epsilon^1_- - 2\epsilon^{-1}\beta(F^-)^2 C \ast \epsilon^1_- = 0
\] (4.16)

and

\[
\frac{1}{4}(\alpha-\theta)\nabla_\sigma (F^+)^2 \Gamma^\sigma \epsilon^1_+ + \frac{1}{4}(\beta+\rho)\nabla_\sigma (F^+)^2 \Gamma^\sigma C \ast \epsilon^1_+ - 2\epsilon^{-1}\rho(F^+)^2 C \ast \epsilon^1_+ = 0
\] (4.17)

respectively.

There are a number of possible cases. First, if both \( F^+ = 0 \) and \( F^- = 0 \); then \( W^+ = 0 \) and \( W^- = 0 \) as a consequence of (4.11). The condition \( W = 0 \) together with the requirement that the metric be Einstein imply in this case that the manifold must be locally isometric to \( H^4 \). Henceforth, we shall assume that \( F \neq 0 \).

Suppose also that \( \alpha - \theta = 0 \) and \( \beta + \rho = 0 \). Then (4.16) and (4.17) imply that either \( F = 0 \) or \( \beta = \rho = 0 \). Discarding the case \( F = 0 \), if \( \beta = \rho = 0 \) and \( \theta = \alpha \) then

\[ \epsilon^2 = \alpha \epsilon^1. \] (4.18)

However, as both \( \epsilon^2 \) and \( \epsilon^1 \) satisfy (4.1), this implies that \( \alpha \) must be constant. In this case, there is no supersymmetry enhancement.

To proceed, suppose that \( F^- \neq 0 \). Then on using (4.16) to solve for \( C \ast \epsilon^1_+ \) in terms of \( \epsilon^1_- \) and \( C \ast \epsilon^-_+ \), and substituting the resulting expression back into (4.14), one can rewrite (4.14) as

\[
\left( (F^-)^2 \nabla_\sigma F^-_{\mu\nu} - \frac{1}{2}F^-_{\mu\nu}\nabla_\sigma (F^-)^2 + \frac{1}{2}F^-_{\sigma[\mu} \nabla_{\nu]}(F^-)^2 - \frac{1}{2}g_{\sigma[\mu} F^-_{\nu]\lambda} \nabla^\lambda (F^-)^2 \right) \Gamma^\sigma \hat{\epsilon} = 0
\] (4.19)

where

\[ \hat{\epsilon} = (\alpha-\theta)\epsilon^1_- - (\beta+\rho)C \ast \epsilon^-_+. \] (4.20)

The spinor \( \hat{\epsilon} \) is non-zero, as \( \alpha - \theta \) and \( \beta + \rho \) cannot both vanish. The condition (4.19) then implies that

\[ \nabla^- F^- = 0 \] (4.21)

where \( \nabla^- \) is the Levi-Civita connection of the conformally rescaled metric \( g^- = |F^-| g \).

Similarly, if \( F^+ \neq 0 \), one finds that

\[ \nabla^+ F^+ = 0 \] (4.22)
where $\nabla^+$ is the Levi-Civita connection of $g^+ = |F^+|g$. It follows that if both $F^+ \neq 0$ and $F^- \neq 0$ then the manifold is ambi-Kähler. Such geometries have been classified in [2]. However, the integrability conditions of the Killing spinor equations (4.11) impose a number of additional conditions.

In particular, suppose that $F^- \neq 0$, and evaluate the condition $\nabla^- F^- = 0$ explicitly, substituting in the expression for $|F^-|$ given in (4.9). One obtains a set of PDEs for $\lambda$, and after some rather involved manipulation, one finds that if $\partial_z \lambda \neq 0$ then $\partial_x \lambda = 0$. However, (4.10) then implies that $\partial_x u = 0$, which is inconsistent with (4.3). Similarly, if $F^+ \neq 0$ then one finds that $\partial_z \sigma = 0$. Conversely, $F^- = 0$ implies that $\partial_x \lambda = 0$, and $F^+ = 0$ implies that $\partial_z \sigma = 0$ as a consequence of (4.9).

Hence the conditions

$$\partial_z \lambda = \partial_x \sigma = 0$$  \hspace{1cm} (4.23)

are necessary for the enhancement of supersymmetry. However, they are not sufficient, and an example of a quarter-supersymmetric solution found in [20] for which (4.23) holds is constructed in the following section.

Using (4.23), the equations (4.10) can then be solved for $\sigma, \lambda$ to give

$$\sigma^{-2} = -\frac{\ell}{\sqrt{2}} \left( \frac{k_1x + k_2}{\frac{1}{2}k_1x^2 + k_2x + k_3} \right), \quad \lambda^{-2} = -\frac{\ell}{\sqrt{2}} \left( \frac{n_1x + n_2}{\frac{1}{2}n_1x^2 + n_2x + n_3} \right)$$  \hspace{1cm} (4.24)

for constants $k_1, k_2, k_3, n_1, n_2, n_3$. This satisfies (4.4). The condition (4.3) implies that

$$u = \frac{1}{2} \log \left( \left( \frac{1}{2}k_1x^2 + k_2x + k_3 \right) \left( \frac{1}{2}n_1x^2 + n_2x + n_3 \right) \right) + G(z, \bar{z})$$  \hspace{1cm} (4.25)

where $G$ is a real function of $z, \bar{z}$. The equation (4.5) is equivalent to

$$\partial_z \partial_{\bar{z}} G + \frac{1}{2} (k_1n_3 + n_1k_3 - k_2n_2)e^{2G} = 0$$  \hspace{1cm} (4.26)

and the metric is

$$g = \frac{4}{\ell^2} \left( (k_1x + k_2)(n_1x + n_2) \right)^{-1} W^{-1} (d\psi + \phi)^2$$

$$+ \left( (k_1x + k_2)(n_1x + n_2) \right) \left( \frac{\ell^2}{4} W dx^2 + \frac{1}{2} \ell^2 ds_{M_2}^2 \right)$$  \hspace{1cm} (4.27)

where

$$W = \frac{1}{\left( \frac{1}{2}k_1x^2 + k_2x + k_3 \right) \left( \frac{1}{2}n_1x^2 + n_2x + n_3 \right)}$$  \hspace{1cm} (4.28)

and

$$ds_{M_2}^2 = 2e^{2G} dz d\bar{z}$$  \hspace{1cm} (4.29)

$M_2$ is either $S^2, \mathbb{R}^2$ or $H^2$, according as to whether $k_1n_3 + n_1k_3 - k_2n_2$ is positive, zero or negative respectively. Also,

$$d\phi = \frac{i}{2} \ell^2 (n_1k_2 - k_1n_2) e^{2G} dz \wedge d\bar{z}.$$  \hspace{1cm} (4.30)

We remark that the case for which $M_2 = S^2$ has been obtained in [20]. The relationship between the solutions found here and those in [20] will be investigated in greater detail in the following section.
**ASD case and separable solutions to SU(∞) Toda equation.**

The special cases $F^- = 0$ and $F^+ = 0$ correspond to $n_2^2 - 2n_1n_3 = 0$ and $k_2^2 - 2k_1k_3 = 0$ respectively. In the ASD case, the metric $g$ is Einstein, but not Kähler, whereas the metric $g^- = |F^-|g$ is Kähler but not Einstein.

If $n_1 \neq 0$ then the $x$–dependence in $u$ in (4.25) is given by a logarithm of a quartic with one repeated real root, and two other roots. Without loss of generality we may translate $x$ to set this repeated root to $x = 0$. In the ASD case we have

$$\partial_2 \lambda = 0, \quad \partial_x \lambda = -\frac{1}{2\sqrt{2\ell}} \lambda^{-1}$$

and (4.4) is implied by (4.3) and (4.5). Now the transformation

$$x = \frac{1}{Z}, \quad z = \frac{1}{2}(X + iY), \quad u = \frac{U}{2} - 2\log Z$$

reduces (4.5) to the $SU(∞)$ Toda equation (1.2) for $U(X, Y, Z)$. The corresponding solution is separable [21, 27]

$$U(X, Y, Z) = U_1(Z) + U_2(X, Y), \quad \text{where} \quad U_1(Z) = \log (\alpha Z^2 + \beta Z + \gamma)$$

and $U_2(X, Y)$ satisfies the Liouville equation

$$(U_2)_{XX} + (U_2)_{YY} + 2\alpha e^{U_2} = 0.$$  

(4.34)

Here $\alpha, \beta, \gamma$ are constants which depend on $(n_1, n_2, n_3)$. The resulting metric is

$$g = \frac{1}{Z^2} \left( V(\alpha Z^2 + \beta Z + \gamma)^2 h_3 + V^{-1}(d\psi + \phi)^2 \right)$$

(4.35)

where the three-metric $h_3$ is Einstein so its curvature is constant and

$$V = -\frac{1}{4\Lambda} \left( \frac{\beta Z + 2\gamma}{\alpha Z^2 + \beta Z + \gamma} \right).$$

(4.36)

If $\alpha = 0$ then $U_2$ is harmonic and can be set to zero by a coordinate transformation. The metric $g_3$ is then hyperbolic if $\beta \neq 0$ or flat if $\beta = 0$. The case $(\alpha \neq 0, \gamma = 0, \beta/\alpha = 4\Lambda)$ corresponds to the $\bar{\mathbb{C}P}^2$ solution (1.3). In this case $n_1 = 0$ and $|F^-|$ is constant.

**General case and ambi-Kähler surfaces of Calabi type.**

More generally, the geometry (4.27) corresponds to a Kähler surface of Calabi type, as described in [2]. To see this, define

$$\hat{g} = (k_1x + k_2)^{-2}g$$

(4.37)

and define the co-ordinate $y$, and the function $Q(y)$, by

$$(n_1k_2 - n_2k_1)y = \frac{n_1x + n_2}{k_1x + k_2},$$

$$(k_1x + k_2)^{-4}W(x)^{-1} = \frac{1}{4}\ell^2(n_1k_2 - k_1n_2)Q(y)$$

(4.38)
where we here assume that \( n_1 k_2 - k_1 n_2 \neq 0 \). Then
\[
\hat{g} = yd\Sigma^2 + yQ^{-1}dy^2 + y^{-1}Q(d\psi + \phi)^2
\]  
(4.39)

where \( Q(y) \) is a product of two quadratic polynomials in \( y \) and
\[
d\phi = dv_{\Sigma}, \quad ds_{\Sigma}^2 = \frac{1}{2} \ell^2 (n_1 k_2 - n_2 k_1) ds^2(M_2). \quad (4.40)
\]

This form of the metric is a special case of the metric appearing in equation (10) of [2].

It remains to consider the case for which the solution preserves 3/4 of the supersymmetry. The analysis in this case proceeds using spinorial geometry techniques analogous to those used to prove that there are no solutions preserving 31 supersymmetries in IIB supergravity [14]. In particular, by introducing a \( \text{Spin}(4) \)-invariant inner product on the space of spinors, a 3/4 supersymmetric solution must have spinors orthogonal, with respect to this inner product, to a normal spinor. By applying appropriately chosen \( \text{Spin}(4) \) gauge transformations, this normal spinor can be reduced to the simple canonical form as that adapted for the Killing spinor in [8]. In this gauge, a simplified basis for the space of spinors can be chosen. On evaluating the integrability conditions on this basis, one finds that \( F = 0 \), and the metric must be locally isometric to \( H^4 \).

5 Supersymmetric Solutions with \( SU(2) \times U(1) \) symmetry

A class of supersymmetric solutions with \( SU(2) \times U(1) \) symmetry was constructed in [20]. In this section, we investigate the relationship between these solutions, and those found in this work, and in [8]. The solutions in [20] take \( \ell = 1 \), with metric and gauge field strength:
\[
ds^2 = \frac{r^2 - s^2}{\Omega(r)} dt^2 + (r^2 - s^2)(\sigma_1^2 + \sigma_2^2) + \frac{4s^2 \Omega(r)}{r^2 - s^2} \sigma_3^2
\]
\[
F = d\left( \left( \frac{P^2 + s^2}{r^2 - s^2} - \frac{2rs}{r^2 - s^2} \right) \sigma_3 \right) \quad (5.1)
\]
where
\[
\Omega(r) = (r^2 - s^2)^2 + (1 - 4s^2)(r^2 + s^2) - 2Mr + P^2 - Q^2 \quad (5.2)
\]
and
\[
\sigma_1 + i\sigma_2 = e^{-i\psi}(d\theta + i \sin \theta d\phi), \quad \sigma_3 = d\psi + \cos \theta d\phi. \quad (5.3)
\]

The constants \( M, s \) are real, and \( P, Q \) are either both real or both imaginary. There are two possibilities for supersymmetric solutions. For half-supersymmetric solutions we take
\[
M = Q\sqrt{4s^2 - 1}, \quad P = -s\sqrt{4s^2 - 1} \quad (5.4)
\]
whereas for quarter-supersymmetric solutions
\[
M = 2sQ, \quad P = -\frac{1}{2}(4s^2 - 1). \quad (5.5)
\]

\(^2\)In the special case for which \( n_1 k_2 - k_1 n_2 = 0 \), \( \hat{g} \) corresponds to the product of two Riemann surfaces.
5.1 Half-Supersymmetric $SU(2) \times U(1)$-symmetric Solutions

For the half-supersymmetric solutions satisfying (5.4), there are two cases, according as $s^2 < \frac{1}{4}$ and $s^2 > \frac{1}{4}$. We begin with the case $s^2 < \frac{1}{4}$; in this case $P$ and $Q$ are both imaginary. It is useful to set

$$s = \frac{1}{2} \cos \mu, \quad M = \frac{k}{\sqrt{2}} \sin \mu, \quad P = -\frac{i}{2} \sin \mu \cos \mu, \quad Q = -\frac{i}{\sqrt{2}} k$$

(5.6)

for real constants $\mu, k$. To proceed we introduce new co-ordinates $\sigma$ and $\psi'$ by

$$r = s \left( \frac{1 + \sigma^2}{1 - \sigma^2} \right)$$

(5.7)

and

$$d\psi' = d\psi + f(\sigma) d\sigma$$

(5.8)

where

$$f(\sigma) = \left( \frac{2 \sigma^2 \cos^4 \mu}{(1 - \sigma^2)^2} + \frac{1}{2} \sin^2 \mu \cos^2 \mu (\sigma + \sigma^{-1})^2 - \sqrt{2} k \sin \mu \cos \mu (\sigma^{-2} - \sigma^2) + k^2 (\sigma^{-1} - \sigma)^2 \right) \times \left( \frac{2 \sin \mu \cos \mu (\sigma^{-1} + \sigma)}{(\sigma^2 - 1)} + 2 \sqrt{2} k \sigma^{-1} \right).$$

(5.9)

After some manipulation, the resulting metric can be written as

$$ds^2 = \frac{1}{2} \left( \frac{2 \sqrt{2}}{\sigma^2 - 1} d\sigma - \frac{1}{\sqrt{2}} (\sigma^{-1} + \sigma) B + (\sigma^{-1} - \sigma) \xi \right)^2 + \frac{\sigma^2}{(1 - \sigma^2)^2} ds^2_{GT}$$

(5.10)

where $ds^2_{GT}$ is the metric of the Berger sphere given by

$$ds^2_{GT} = \cos^2 \mu \left( \cos^2 \mu (\sigma_3')^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

(5.11)

and

$$B = \sin \mu \cos \mu \sigma_3', \quad \xi = k \sigma_3'$$

(5.12)

where

$$\sigma_3 = d\psi' + \cos \theta d\phi.$$

(5.13)

The gauge field strength is also given (now using the conventions of [8]) by

$$F = d \left( -\frac{1}{4} (\sigma^2 + \sigma^{-2}) B + \frac{1}{2 \sqrt{2}} (\sigma^{-2} - \sigma^2) \xi \right).$$

(5.14)

\footnote{In order to match the $s^2 < \frac{1}{4}$ solution of [20] to those found in the classification of section 4.2 in [8] we take the gauge field strength here to be real, and the coefficient multiplying $F$ in the Killing spinor equation is also real. This is to be compared with the $s^2 < \frac{1}{4}$ solution of [20], where the coefficient multiplying the gauge field strength in the Killing spinor equation is imaginary, but the gauge field strength is also imaginary.}
As expected, it follows that the half-supersymmetric solutions of [20] with \( s^2 < \frac{1}{4} \) correspond to a special case of the half-supersymmetric solutions found previously in section 4.2 of [8]. To obtain the \( s^2 > \frac{1}{4} \) solutions in (5.1) one makes an analytic continuation of the parameters (5.6) of the form

\[
\mu \to i\mu, \quad k \to ik .
\]

(5.15)

It is clear that when this is done, the gauge field strength becomes imaginary. However, when this analytic continuation is applied to the solution (5.10), because the function \( f \) given in (5.9) becomes imaginary, it follows that \( \sigma_3' \) becomes complex, because the coordinate \( \psi' \) is complex. Furthermore, the term

\[
\frac{2\sqrt{2}}{\sigma^2 - 1} d\sigma - \frac{1}{\sqrt{2}} (\sigma^{-1} + \sigma)B + (\sigma^{-1} - \sigma)\xi
\]

(5.16)

which appears in the metric (5.10) also becomes complex under the analytic continuation. So the analytically continued solution need not lie within the class of solutions constructed in section 3 of [8], or in the analysis of the previous section, because in these cases, it is assumed that the metric is real.

5.2 Quarter-Supersymmetric \( SU(2) \times U(1) \)-symmetric Solutions

Next, we consider the quarter-supersymmetric solutions in [20], satisfying (5.5). In this case, we begin by considering the solution of the previous section (4.27), with \( M_2 = S^2 \),

\[
ds_{3M_2}^2 = \frac{1}{k_1 n_3 + n_1 k_3 - k_2 n_2} (d\theta^2 + \sin^2 \theta d\phi^2)
\]

(5.17)

and assume that

\[
k_1 n_1 > 0, \quad k_1 n_3 + n_1 k_3 - k_2 n_2 > 0, \quad k_1 n_2 - k_2 n_1 \neq 0 .
\]

(5.18)

Then, set \( \ell = 1 \), and change co-ordinates from \( x \) to \( r \) where

\[
r = \frac{s}{(k_1 n_2 - k_2 n_1)} (2k_1 n_1 x + k_1 n_2 + k_2 n_1)
\]

(5.19)

with

\[
s^2 = \frac{(k_1 n_2 - k_2 n_1)^2}{8k_1 n_1 (k_1 n_3 + n_1 k_3 - k_2 n_2)}
\]

\[
M^2 = \frac{1}{32(n_1 k_1)^3} (k_1 n_3 + n_1 k_3 - n_2 k_2)^{-3} (k_1 n_2 - k_2 n_1)^2
\]

\[
\times (-2k_1 n_1^2 k_3 - k_1 n_2^2 + n_1 n_2^2 + 2k_1^2 n_1 n_3)^2
\]

\[
P = \frac{1}{4n_1 k_1 (k_1 n_3 + n_1 k_3 - n_2 k_2)} (n_1^2 k_3^2 + k_1^2 n_2^2 - 2n_1 n_3 k_3^2 - 2k_1 k_3 n_1^2)
\]

\[
Q = \frac{1}{4n_1 k_1 (k_1 n_3 + n_1 k_3 - n_2 k_2)} (-n_1^2 k_2^2 + k_1^2 n_2^2 - 2n_1 n_3 k_1^2 + 2k_1 k_3 n_1^2)
\]

(5.20)

After some manipulation, one recovers the quarter supersymmetric solutions in [20], satisfying (5.5); it is straightforward to see that the gauge field strengths also match.
Hence, although the conditions obtained in the previous section by considering the
integrability of the Killing spinor equations are necessary for the enhancement of super-
symmetry, they are not sufficient. In particular, an explicit construction of the Killing
spinors will produce additional conditions on the constants $n_i, k_i$ which are sufficient to
ensure supersymmetry enhancement.

We shall conclude by outlining a more geometric approach where the difference be-
tween the necessary and sufficient conditions becomes apparent. This approach follows [9],
where it has been used to characterise four dimensional Riemannian manifolds conformally
equivalent to hyper-Kahler.

Equation (4.1) defines a connection $D$ on a rank four complex vector bundle over $M_4$,
and Killing spinors are in a 1-1 correspondence with the parallel sections of this connection.
The necessary conditions for the enhanced supersymmetry analysed in Section 4 arise
from commuting the covariant derivatives, and thus computing the curvature $R = [D, D]$ of $D$. This curvature, which is essentially given by (4.11), can be thought of as a four
by four matrix, and the necessary conditions which we have computed are equivalent to
the statement that this matrix has two–dimensional kernel. Thus there are two linearly
independent solutions to

$$R\epsilon = 0. \quad \text{(5.21)}$$

To satisfy the sufficient conditions for the SUSY enhancement we need to make sure that
all differential consequences of the formula (5.21) hold which, by the Frobenius theorem,
then guarantees that the kernel of $R$ is parallel w.r.t the connection $D$. We differentiate
(5.21) and use the parallel property of sections $D\epsilon = 0$ to produce a sequence of matrix
algebraic conditions

$$R\epsilon = 0, \quad (D R)\epsilon = 0, \quad (D^2 R)\epsilon = 0, \quad \ldots .$$

We stop the process once the differentiation does not produce new equations. For enhanced
supersymmetry the rank of the extended matrix $\{R, DR, \ldots \}$ should be two. Therefore,
as the rank of $R$ is two when the necessary conditions hold, it is sufficient to demand that
the condition $(D R)\epsilon = 0$ holds identically. Thus the vanishing of all 3 by 3 minors of the
matrix $\{R, DR\}$ will give the sufficiency conditions for the existence of two Killing spinors.

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