An almost complex Chern-Ricci flow

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Abstract. We consider the evolution of an almost Hermitian metric by the (1, 1) part of its Chern-Ricci form on almost complex manifolds. This is an evolution equation first studied by Chu and coincides with the Chern-Ricci flow if the complex structure is integrable and with the Kähler-Ricci flow if moreover the initial metric is Kähler. We find the maximal existence time for the flow in term of the initial data and also give a convergence result. As an example, we study this flow on the (locally) homogeneous manifolds in more detail.

1. Introduction

Let \((M, J)\) be a compact almost complex manifold (without boundary) with \(\dim_{\mathbb{R}} M = 2n\), where \(J\) is the almost complex structure. Then assume that \(g_0\) is an almost Hermitian metric on \((M, J)\), i.e., \(g_0\) is a Riemannian metric satisfying \(g_0(JX, JY) = g_0(X, Y)\) for any vector fields \(X\) and \(Y\). Associated to \(g_0\) is a unique real \((1, 1)\) form \(\omega_0\) defined by \(\omega_0(X, Y) := g_0(JX, Y)\) for any vector fields \(X\) and \(Y\) and vice versa. In what follows, we will not distinguish the two terms.

Since Hamilton [18] introduced the Ricci flow, it has established many deep results in topological, smooth and Riemannian manifolds (see for example [2, 19, 26]). Next, we consider the parabolic flows of metrics on \(M\) starting at \(g_0\) which preserve the Hermitian condition and reveal the information about the structure of \(M\) as a complex manifold. When \(g_0\) is a Kähler metric, i.e., \(d\omega_0 = 0\), the Ricci flow does exactly this and it is called the Kähler-Ricci flow firstly introduced by Cao [3]. The behavior of the Kähler-Ricci flow is deeply intertwined with the complex and algebro-geometric properties of \(M\) (see for example [3, 5, 4, 13, 27, 28, 31, 32, 33, 34, 35, 36, 37, 38, 42, 45, 46, 48, 49, 57, 58, 61]).

If the Hermitian metric \(g_0\) is not Kähler, then there are two types of Ricci curvature which are equal to each other when \(d\omega_0 = 0\). There are also two types of the evolution of the Hermitian metric. One is the Chern-Ricci flow which firstly introduced by Gill [15] when the first Bott-Chern class vanishes and is studied deeply by Tosatti and Weinkove (and Yang) [53, 54, 55]. The Chern-Ricci flow is a natural evolution equation on complex manifolds and its behavior reflects the underlying geometry (see also [12, 16, 17, 22, 23, 25, 62, 64] and references therein). Another type of the evolution of Hermitian metric was introduced by Streets and Tian [39, 40, 41] (see also [24]) and was generalized to almost Hermitian manifolds by Vezzoni [59].

Given \(F \in C^\infty(M, \mathbb{R})\), Chu, Tosatti and Weinkove [10] solved the following Monge-Ampère equation on almost Hermitian manifold

\[
(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{F+b} \omega_0^n \tag{1.1}
\]

\[
\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi > 0, \quad \sup \varphi = 0,
\]

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for a unique \( b \in \mathbb{R} \), where \( \sqrt{-1} \partial \bar{\partial} \varphi = \frac{1}{b} (dJd\varphi)^{(1,1)} \) (see (2.14) in Section 2). When \( J \) is integrable and \( \omega_0 \) is Kähler, (1.1) was solved by Yau [63] to confirm the Calabi conjecture. Tosatti and Weinkove [52] solved (1.1) for any dimension if \( J \) is integrable (see also [7, 51]).

In this paper, we consider the evolution equation for almost Hermitian forms suggested by Chu, Tosatti and Weinkove [10]

\[
\frac{\partial \omega_t}{\partial t} = -\text{Ric}^{(1,1)}(\omega_t), \quad \omega(0) = \omega_0
\]
on almost Hermitian manifold \((M, \omega_0, J)\). Here for convenience, we call \( \text{Ric}^{(1,1)}(\omega) \) is the \((1,1)\) part of the Chern-Ricci form of \( \omega \). This flow coincides exactly with the Chern-Ricci flow if \( J \) is integrable, and the Kähler-Ricci flow if furthermore \( \omega_0 \) is Kähler.

Chu [9] firstly studied the flow (1.2) in the case where there exists an almost Hermitian metric \( \omega_0 \) with \( \text{Ric}^{(1,1)}(\omega_0) = \sqrt{-1} \partial \bar{\partial} F \) for some \( F \in C^\infty(M, \mathbb{R}) \) and proved that the solution to (1.2) exists for all the time and converges to an almost Hermitian metric \( \omega_\infty \) with \( \text{Ric}^{(1,1)}(\omega_\infty) = 0 \) (see more details in Section 7).

In this paper, we characterize the maximal existence time for a solution for the flow (1.2). For this aim, we rewrite the flow as

\[
\frac{\partial \omega_t}{\partial t} = -\text{Ric}^{(1,1)}(\omega_0) + \sqrt{-1} \partial \bar{\partial} \theta(t), \quad \text{with} \quad \theta(t) = \log \frac{\omega_t^n}{\omega_0^n}.
\]

Hence, as long as the flow exists, the solution \( \omega_t \) starting at \( \omega_0 \) must be of the form \( \omega_t = \alpha_t + \sqrt{-1} \partial \bar{\partial} \psi(t) \) with \( \frac{\partial \alpha_t}{\partial t} = \theta(t) \) and \( \alpha_t = \omega_0 - t \text{Ric}^{(1,1)}(\omega_0) \). We define a number \( T = T(\omega_0) \) with \( 0 < T \leq \infty \) by

\[
T := \sup \left\{ t \geq 0 : \exists \phi \in C^\infty(M, \mathbb{R}) \text{ such that } \alpha_t + \sqrt{-1} \partial \bar{\partial} \phi > 0 \right\}.
\]

Note that for any other Hermitian metric \( \omega'_0 = \omega_0 + \sqrt{-1} \partial \bar{\partial} \psi > 0 \) with \( \psi \in C^\infty(M, \mathbb{R}) \), we have \( T(\omega'_0) = T(\omega_0) \). Indeed, (2.27) yields that

\[
\alpha_t + \sqrt{-1} \partial \bar{\partial} \phi = \omega_0 - t \text{Ric}^{(1,1)}(\omega'_0) + \sqrt{-1} \partial \bar{\partial} \left( \phi + t \log \frac{\omega_t^n}{\omega_0^n} \right)
\]

\[
= \omega'_0 - t \text{Ric}^{(1,1)}(\omega'_0) + \sqrt{-1} \partial \bar{\partial} \left( \phi - \psi + t \log \frac{\omega_t^n}{\omega_0^n} \right),
\]
as required.

It is easy to see that a solution to (1.2) cannot exist beyond time \( T \). Indeed, we have

**Theorem 1.1.** There exists a unique maximal solution to the flow (1.2) on \([0, T)\).

In the special case when \( J \) is integrable, this is already known by the result of Tian and Zhang [46] who extended earlier work of Cao [3] and Tsuji [57, 58] when \( \omega_0 \) is Kähler, and by the result of Tosatti and Weinkove [53] (see also [55]) when \( \omega_0 \) is even not Kähler.

We point out that the flow (1.2) is equivalent to the scalar partial differential equation

\[
\frac{\partial}{\partial t} \phi(t) = \log \left( \frac{\alpha_t + \sqrt{-1} \partial \bar{\partial} \phi_t}{\omega_0^n} \right), \quad \alpha_t + \sqrt{-1} \partial \bar{\partial} \phi_t > 0, \quad \phi(0) = 0,
\]
on the same time interval. Indeed, assume that \( \phi_t \) is the solution to (1.4). We set \( \omega_t = \alpha_t + \sqrt{-1} \partial \bar{\partial} \phi_t > 0 \). From (2.27), it follows that

\[
\frac{\partial}{\partial t} \omega_t = \frac{\partial}{\partial t} \left( \alpha_t + \sqrt{-1} \partial \bar{\partial} \phi_t \right) = -\text{Ric}^{(1,1)}(\omega_0) - \text{Ric}^{(1,1)}(\omega_t) + \text{Ric}^{(1,1)}(\omega_0) = -\text{Ric}^{(1,1)}(\omega_t),
\]
i.e., $\omega_t$ is the solution to the flow (1.2).

On the other hand, assume that $\omega_t$ is the solution to (1.2). We set

$$\phi_t = \int_0^t \log \frac{\omega_s}{\omega_0^s} ds$$

with $\phi(0) = 0$. This, together with (2.27), yields

$$\frac{\partial}{\partial t} (\omega_t - \alpha_t - \sqrt{-1} \partial \bar{\partial} \phi_t) = -\text{Ric}^{(1,1)}(\omega_t) + \text{Ric}^{(1,1)}(\omega_0) + \text{Ric}^{(1,1)}(\omega_t) - \text{Ric}^{(1,1)}(\omega_0) = 0,$$

with $(\omega_t - \alpha_t - \sqrt{-1} \partial \bar{\partial} \phi_t)|_{t=0} = 0$, i.e., $\omega_t \equiv \alpha_t + \sqrt{-1} \partial \bar{\partial} \phi_t$. It follows that $\phi_t$ is the solution to the scalar partial differential equation (1.4).

Next, we consider a convergence result about the flow (1.2). Since there is no Bott-Chern cohomology group on almost complex manifold if $J$ is not integrable, the statement of this result may be slightly different.

**Theorem 1.2.** Assume that $(M, \omega_0, J)$ is an almost Hermitian manifold equipped a volume form $\Omega$ with $\text{Ric}^{(1,1)}(\Omega) < 0$. There exists an almost Hermitian metric $\omega_\infty$ with $\text{Ric}^{(1,1)}(\omega_\infty) = -\omega_\infty$, such that for any initial almost Hermitian metric $\omega_0$, the solution to (1.2) exists for all the time and that $\omega(t)/t$ converge smoothly to $\omega_\infty$ as $t \to \infty$.

If the complex structure $J$ is integrable, then the existence of such volume form $\Omega$ is equivalent to the fact that $M$ is Kähler with negative first Chern class, and our theorem coincides precisely with [53, Theorem 1.7], i.e., on Kähler manifold with negative first Chern class, the Chern-Ricci flow, starting at any initial Hermitian metric $\omega_0$, exists for all the time and $\omega(t)/t$ converge smoothly to the unique Kähler-Einstein metric $\omega_{\text{KE}}$ on $M$.

The outline of the paper is as follows. In Section 2, we collect some preliminaries about almost complex geometry. In Section 3, we give the uniform priori estimates of the solution to (3.2) and its time derivative. In Section 4 and Section 5, we give the first and second order estimates of the solution to (3.2) respectively. In Section 6, we get the higher order estimates of the solution to (3.2) and complete the proof of Theorem 1.1. In Section 7, we prove some convergence results including Theorem 1.2. In Section 8, as an example, we study the flow (1.2) on a compact almost Hermitian manifold $M$ whose universal cover is a Lie group $G$ such that if $\pi : G \to M$ is the covering map, then $\pi^* \omega_0$ and $\pi^* J$ are left invariant in more detail.

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2. Preliminaries

In this section, we collect some basic materials about almost Hermitian geometry (see for example [20, 56]).

2.1. The Nijenhuis tensor on almost complex manifolds. Let $(M, J)$ be an almost complex manifold with $\dim \mathbb{R} M = 2n$, where $J$ is the almost complex structure, i.e., $J \in \text{End}(TM)$ with $J^2 = -\text{Id}_TM$. Here $TM$ is the tangent vector bundle of $M$. Denote by $\mathfrak{X}(M)$ the set of all the sections of $TM$, i.e., the set of all the vector fields. Then we have

$$TM \otimes \mathbb{R} \mathbb{C} = T^{1,0} M \oplus T^{0,1} M,$$
where
\[ T^{1,0}M := \left\{ X - \sqrt{-1}JX : \forall X \in TM \right\} \]
and
\[ T^{0,1}M := \left\{ X + \sqrt{-1}JX : \forall X \in TM \right\}. \]
Denote by \( \Lambda^1 M \) the dual of \( TM \). We extend the Nijenhuis tensor \( J \) to any \( p \) form \( \varpi \) by
\[
(J\varpi)(X_1, \cdots, X_p) := (-1)^p \varpi(JX_1, \cdots, JX_p), \quad \forall X_1, \cdots, X_p \in \mathfrak{X}(M).
\]
It is easy to check that for any \((2n - p)\) form \( \xi \) and \( p \) form \( \psi \), there holds
\[
\xi \wedge (J\psi) = (-1)^p (J\xi) \wedge \psi.
\]
We also have
\[
\Lambda^1 M \otimes R C = \Lambda^{1,0} M \oplus \Lambda^{0,1} M,
\]
where
\[
\Lambda^{1,0} M := \left\{ \eta + \sqrt{-1}J\eta : \forall \eta \in \Lambda^1 M \right\}
\]
and
\[
\Lambda^{0,1} M := \left\{ \eta - \sqrt{-1}J\eta : \forall \eta \in \Lambda^1 M \right\}.
\]
It is easy to see that
\[
(T^{1,0}M)^* = \Lambda^{1,0} M, \quad (T^{0,1}M)^* = \Lambda^{0,1} M.
\]
We also denote that
\[
\Lambda^p M := \bigotimes_{r+s=p} \left( \Lambda^{1,0} M \right)^{\otimes r} \otimes \left( \Lambda^{0,1} M \right)^{\otimes s}.
\]
Then we have
\[
\Lambda^p M \otimes R C = \bigoplus_{r+s=p} \Lambda^{r,s} M.
\]
A 2 form \( \zeta \) is a \((1,1)\) form if and only if there holds
\[
\zeta(X, Y) = \zeta(JX, JY), \quad \forall X, Y \in \mathfrak{X}(M).
\]
Indeed, \( \zeta \) is a \((1,1)\) form if and only if
\[
\zeta(X + \sqrt{-1}JX, Y + \sqrt{-1}JY) = \zeta(X - \sqrt{-1}JX, Y - \sqrt{-1}JY) = 0, \quad \forall X, Y \in \mathfrak{X}(M),
\]
as required. Therefore, the \((1,1)\) part of any 2 form \( \xi \), denoted by \( \xi^{(1,1)} \), can be given by
\[
\xi^{(1,1)}(X, Y) = \frac{1}{2} (\xi(X, Y) + \xi(JX, JY)), \quad \forall X, Y \in \mathfrak{X}(M).
\]
We also define
\[
\xi_{ac}(X, Y) := \xi(X, Y) - \xi^{(1,1)}(X, Y) = \frac{1}{2} (\xi(X, Y) - \xi(JX, JY)), \quad \forall X, Y \in \mathfrak{X}(M).
\]
**Definition 2.1.** For any \( X, Y \in \mathfrak{X}(M) \), Nijenhuis tensor \( \mathcal{N} \) is defined by
\[
\mathcal{N}(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].
\]
If \( \mathcal{N} = 0 \), then the Nijenhuis tensor is called integrable.

For the Nijenhuis tensor, a direct calculation yields
Lemma 2.2. The Nijenhuis tensor $\mathcal{N}$ satisfies

\[ \mathcal{N}(X, Y) = -\mathcal{N}(Y, X), \quad \mathcal{N}(JX, Y) = -J\mathcal{N}(X, Y), \]
\[ \mathcal{N}(X, JY) = -J\mathcal{N}(X, Y), \quad \mathcal{N}(JX, JY) = -\mathcal{N}(X, Y). \]

Moreover, $\mathcal{N}(JX, Y) = \mathcal{N}(X, JY)$.

By Lemma 2.2, for any $(1, 0)$ vector fields $W$ and $V$, it is easy to get

\[ \mathcal{N}(V, W) = -[V, W]^{(0,1)}, \quad \mathcal{N}(V, \overline{W}) = \mathcal{N}(\overline{V}, W) = 0 \]

and

\[ \mathcal{N}(\overline{V}, \overline{W}) = -[\overline{V}, \overline{W}]^{(1,0)} = -[\overline{V}, W]^{(0,1)}. \]

Let $e_1, \ldots, e_n$ be the basis of $T^{1,0}M$ and $\theta^1, \ldots, \theta^n$ be the dual basis of $\Lambda^{1,0}M$, i.e.,

\[ \theta^i(e_j) = \delta^i_j, \quad i, j = 1, \ldots, n. \]

Then we have

\[ \mathcal{N}(e_i, \overline{e}_j) = -[e_i, \overline{e}_j]^{(1,0)} = : N^k_{ij} e_k, \]
\[ \mathcal{N}(e_i, e_j) = -[e_i, e_j]^{(0,1)} = : N^k_{ij} e_k, \]

which implies

\[ \mathcal{N} = \frac{1}{2} N^k_{ij} e_k \otimes (\theta^i \wedge \theta^j) + \frac{1}{2} N^k_{ij} e_k \otimes (\overline{\theta}^i \wedge \overline{\theta}^j). \]

Denote the structure coefficients of Lie bracket by

\[ [e_i, e_j] = : C^k_{ij} e_k - N^k_{ij} e_k, \]
\[ [e_i, e_j] = : C^k_{ij} e_k + C^k_{ji} e_k, \]
\[ [e_i, \overline{e}_j] = : - N^k_{ij} e_k + C^k_{ij} e_k. \]

A direct calculation yields

\[ d\theta^i = -\frac{1}{2} C^i_{kl} \theta^k \wedge \theta^l - C^i_{kl} \overline{\theta}^k \wedge \overline{\theta}^l + \frac{1}{2} N^i_{kl} \overline{\theta}^k \wedge \overline{\theta}^l. \]

From (2.9), we can split the exterior differential operator, $d : \Lambda^* M \otimes \mathbb{C} \rightarrow \Lambda^{*+1} M \otimes \mathbb{C}$, into four components (see for example [1])

\[ d = A + \partial + \overline{\partial} + \overline{A} \]

with

\[ A : \Lambda** M \rightarrow \Lambda^{*+2**-1} M \]
\[ \partial : \Lambda** M \rightarrow \Lambda^{*+1**} M \]
\[ \overline{\partial} : \Lambda** M \rightarrow \Lambda^{**+1} M \]
\[ \overline{A} : \Lambda** M \rightarrow \Lambda^{*-1**+2} M. \]

In terms of these components, the condition $d^2 = 0$ can be written as

\[ A^2 = 0, \]
\[ \partial A + A \partial = 0, \]
\[ A\overline{\partial} + \overline{\partial} A = 0, \]
\[ A\overline{A} + \partial \overline{\partial} + \overline{\partial} A + \overline{A} A = 0. \]
For any $p$ form $\varpi$, there holds

$$\begin{align*}
\partial \varpi + \overline{\varpi}^2 + \overline{A} \partial = & 0, \\
\overline{A} \partial + \partial A = & 0, \\
\overline{A}^2 = & 0.
\end{align*}$$

(2.10)

$$\begin{align*}
(d\varpi)(X_1, \cdots, X_{p+1}) = & \sum_{\lambda=1}^{p+1} (-1)^{\lambda+1} X_\lambda(\varpi(X_1, \cdots, \widehat{X}_\lambda, \cdots, X_{p+1})) \\
& \quad + \sum_{\lambda<\mu} (-1)^{\lambda+\mu} \varpi([X_\lambda, X_\mu], X_1, \cdots, \widehat{X}_\lambda, \cdots, \widehat{X}_\mu, \cdots, X_{p+1})
\end{align*}$$

for any fields $X_1, \cdots, X_{p+1} \in \mathfrak{X}(M)$. For any $\varphi \in C^\infty(M, \mathbb{R})$, from (2.1) and (2.10), a direct computation yields

$$\begin{align*}
(dJd\varphi)(e_i, e_j) = & -2\sqrt{-1}[e_i, e_j]^{(0,1)}(\varphi), \\
(dJd\varphi)(\overline{e}_i, \overline{e}_j) = & 2\sqrt{-1}[\overline{e}_i, \overline{e}_j]^{(1,0)}(\varphi), \\
(dJd\varphi)(e_i, \overline{e}_j) = & 2\sqrt{-1} (e_i \overline{e}_j(\varphi) - [e_i, e_j]^{(0,1)}(\varphi)).
\end{align*}$$

(2.11)

(2.12)

(2.13)

A direct calculation shows that

$$\begin{align*}
\sqrt{-1}\partial \partial \varphi = & \frac{1}{2} (dJd\varphi)^{(1,1)}(\varphi) = \sqrt{-1} \left( e_i \overline{e}_j(\varphi) - [e_i, e_j]^{(0,1)}(\varphi) \right) \theta^i \wedge \overline{\theta}^j.
\end{align*}$$

(2.14)

Thanks to (2.6) and (2.10), we get that

$$
(dJd\varphi)(X, Y) - (dJd\varphi)(JX, JY) = -4 \left( J \circ \mathcal{N}(X, Y) \right)(\varphi), \quad \forall X, Y \in \mathfrak{X}(M),
$$

which, together with (2.3), yields that $dJd\varphi$ is a real $(1, 1)$ form if and only if $J$ is integrable.

For any real $(1, 1)$ form $\eta = \sqrt{-1}\eta^i \overline{\theta}^i$, combining (2.1) and (2.10) gives

$$\begin{align*}
\overline{\partial} \eta = & \frac{\sqrt{-1}}{2} \left( \overline{e}_j \eta_{k\overline{i}} - \overline{e}_i \eta_{k\overline{j}} - C_{k\overline{i}}^{\overline{p}j} \eta_{k\overline{p}} + C_{k\overline{j}}^{\overline{p}i} \eta_{k\overline{p}} + \overline{C}_{i\overline{j}}^{\overline{p}k} \eta_{k\overline{p}} \right) \theta^k \wedge \overline{\theta}^i \wedge \overline{\theta}^j.
\end{align*}$$

(2.15)

**2.2. The almost complex connection on almost complex manifolds.** A connection $D$ is called almost complex if $DJ = 0$, which means that for any $X, Y \in \mathfrak{X}(M)$, we have

$$0 = (DXJ)(Y) = DX(J(Y)) - J(DX Y).$$

(2.16)

Therefore, we can define Christoffel symbols by

$$
D e_a e_j = \Gamma_{\alpha j}^k e_k, \quad D e_a \overline{e}_j = \Gamma_{\alpha j}^k \overline{e}_k, \quad \alpha \in \{1, \cdots, n, \overline{\tau}, \cdots, \overline{\pi}\}, \quad i, j, k \in \{1, \cdots, n\}.
$$

Here we also use the notation $e_{\overline{\tau}} = \overline{e}_i$. The connection form $(\omega^i_j)$ is defined by $\omega^i_j := \Gamma_{k j}^i \theta^k + \Gamma_{k \overline{i}}^j \overline{\theta}^k$.

Then the torsion $\Theta = (\Theta^i)$ is defined by

$$\Theta^i := d\theta^i - \theta^p \wedge \omega^i_p.$$

This implies that

$$\Theta^i(e_j, e_k) = \Gamma_{i j k}^l - \Gamma_{i j k}^j - C_{i j k}^l =: T_{i j k}^l,$$

(2.17)

$$\Theta^i(e_j, \overline{e}_k) = - \Gamma_{i j k}^k \overline{e}_j - C_{i j \overline{k}}^k =: S_{i j \overline{k}}^k,$$

$$\Theta^i(\overline{e}_j, \overline{e}_k) = - d\theta^i(\overline{e}_j, \overline{e}_k) =: N_{i j k}^k.$$

The torsion $\Theta = (\Theta^i)$ can be split into three parts

$$\Theta = \Theta^{2.0} + \Theta^{1.1} + \Theta^{0.2},$$

6
where

\[
\Theta^{2,0} = \left( \frac{1}{2} T_{jk}^i \theta^j \wedge \theta^k \right)_{1 \leq i \leq n},
\]

\[
\Theta^{1,1} = \left( S_{jk}^i \theta^j \wedge \theta^k \right)_{1 \leq i \leq n},
\]

\[
\Theta^{0,2} = \left( \frac{1}{2} N_{jk}^i \theta^j \wedge \theta^k \right)_{1 \leq i \leq n}.
\]

It follows that the \((0, 2)\) part of an almost complex connection is uniquely determined by the Nijenhuis tensor (see for example [20, Theorem 1.1] and [56]).

The curvature form is defined by

\[
\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j.
\]

We define

\[
R_{k\ell j}^i := \Omega^i_j(e_k, e_\ell)
\]

\[
= e_k(\Gamma^i_{\ell j}) - e_\ell(\Gamma^i_{kj}) - C^p_{k\ell} \Gamma^i_{pj} + \overline{N}^p_{k\ell} \Gamma^i_{pj} + \Gamma^i_{kp} \Gamma^p_{\ell j} - \Gamma^i_{kj} \Gamma^p_{p\ell},
\]

(2.19)

\[
R_{k\ell j}^i := \Omega^i_j(e_k, \bar{e}_\ell)
\]

\[
= e_k(\Gamma^i_{\ell j}) - \bar{e}_\ell(\Gamma^i_{kj}) - C^p_{k\ell} \Gamma^i_{pj} - \overline{C}^p_{k\ell} \Gamma^i_{pj} + \Gamma^i_{kp} \Gamma^p_{\ell j} - \Gamma^i_{kj} \Gamma^p_{p\ell},
\]

and

(2.20)

\[
R_{k\ell j}^i := \Omega^i_j(\tau_k, \tau_\ell)
\]

\[
= \tau_k(\Gamma^i_{\ell j}) - \tau_\ell(\Gamma^i_{kj}) + \overline{N}^p_{k\ell} \Gamma^i_{pj} - \overline{C}^p_{k\ell} \Gamma^i_{pj} + \Gamma^i_{kp} \Gamma^p_{\ell j} - \Gamma^i_{kj} \Gamma^p_{p\ell}.
\]

We can write \(\Omega = (\Omega^i_j) = \Omega^{(2,0)} + \Omega^{(1,1)} + \Omega^{(0,2)}\), with

\[
\Omega^{(2,0)} = \left( \frac{1}{2} R_{k\ell j}^i \theta^k \wedge \theta^\ell \right),
\]

\[
\Omega^{(1,1)} = \left( R_{k\ell j}^i \theta^k \wedge \theta^\ell \right),
\]

\[
\Omega^{(0,2)} = \left( \frac{1}{2} R_{k\ell j}^i \overline{\theta}^k \wedge \overline{\theta}^\ell \right).
\]

Then the Chern-Ricci form is \((\sqrt{-1}\Omega^i_j) \in 2\pi c_1(M, J) \in H^2(M, \mathbb{R})\) by the Chern-Weil theory (see for example [21, Chapter 12]), where \(c_1(M, J)\) is the first Chern class of \((M, J)\).

2.3. The canonical connection on almost Hermitian manifolds. Assume that \((M, g, J)\) is an almost Hermitian manifold with \(\dim M = 2n\), where a Riemannian metric \(g\) is called a Hermitian metric if it satisfies \(g(X, Y) = g(JX, JY)\), \(\forall X, Y \in \mathfrak{X}(M)\). We also denote the Hermitian metric \(g\) by \(\langle \cdot, \cdot \rangle\). This metric uniquely defines a real \((1, 1)\) form \(\omega(\cdot, \cdot) = g(J\cdot, \cdot)\) and vice versa. An affine connection \(D\) on \(TM\) is called almost Hermitian connection if \(Dg = DJ = 0\).

For any \(X, Y, Z \in \mathfrak{X}(M)\), we have

(2.21)

\[
0 = (D_X g)(Y, Z) = X(Y, Z) - (D_X Y, Z) - (Y, D_X Z).
\]

For the almost Hermitian connection, we have (see for example [20, Theorem 2.1] and [14])

**Lemma 2.3.** Let \((M, g, J)\) be an almost Hermitian manifold with \(\dim M = 2n\). Then for any given vector valued \((1, 1)\) form \(\Phi = (\Phi^i)_{1 \leq i \leq n}\), there exists a unique almost Hermitian connection \(D\) on \((M, g, J)\) such that the \((1, 1)\) part of the torsion is equal to the given \(\Phi\).
If the (1,1) part of the torsion of an almost Hermitian connection vanishes everywhere, then the connection is known as the second canonical connection and was first introduced by Ehrensmann and Libermann [11]. It is also sometimes referred to as the Chern connection, since when $J$ is integrable it coincides with the connection defined in Chern [6], and in Schouten and Datzig [30]. We denote it by $\nabla^C$.

We use the local (1,0) frames as before. We write $\omega$ as

$$\omega = \sqrt{-1} g_{i\bar{j}} \theta^i \wedge \bar{\theta}^j.$$ 

From (2.16), (2.17) and (2.21), we get

$$e_i(e_k, \bar{\tau}_\ell) = \langle \nabla^C_{e_i} e_k, \bar{\tau}_\ell \rangle + \langle e_k, \nabla^C_{e_i} \bar{\tau}_\ell \rangle = \langle \Gamma^p_{ik} e_p, \tau_\ell \rangle + \langle e_k, - \nabla^C_{\bar{\tau}_p} \tau_\ell \rangle,$$

i.e.,

$$\Gamma^p_{ik} = g^{\bar{p}} q e_i g_{k \bar{q}} + g^{\bar{p}} g_k \overline{\nabla^C_{\bar{q}}} \tau_\ell.$$ 

This gives the components of the torsion as

$$T^p_{ik} = \Gamma^p_{ik} - \Gamma^p_{ki} = g^{\bar{p}} q e_i g_{k \bar{q}} + g^{\bar{p}} g_k \overline{\nabla^C_{\bar{q}}} \tau_\ell - g^{\bar{p}} q e_k g_{i \bar{q}} - g^{\bar{p}} g_{i \bar{q}} \overline{\nabla^C_{\bar{q}}} \tau_\ell - C^p_{ik}.$$ 

We also lower the index of torsion and denote it by

$$T_{ik\ell} = T^p_{ik} g_{p \bar{q}} = e_i g_{k \bar{q}} + g_k \overline{\nabla^C_{\bar{q}}} \tau_\ell - e_k g_{i \bar{q}} - g_{i \bar{q}} \overline{\nabla^C_{\bar{q}}} \tau_\ell - C^p_{ik} g_{p \bar{q}}.$$ 

Thanks to (2.15), we obtain

$$\Theta_\omega = \frac{-1}{2} \left( \tau_j g_{k \bar{q}} - \bar{\tau}_i g_{k \bar{q}} - C^p_{k\bar{q}} g_{p \bar{q}} + C^p_{k\bar{q}} g_{p \bar{q}} + C^p_{k\bar{q}} g_{p \bar{q}} \right) \theta^k \wedge \bar{\theta}^i \wedge \bar{\theta}^j = \frac{-1}{2} T_{ijk} \theta^k \wedge \bar{\theta}^i \wedge \bar{\theta}^j.$$ 

By (2.22), it yields that

$$\Gamma^p_{i\bar{p}} = e_i (\log \det g) - C^p_{i\bar{p}}.$$ 

Using (2.18) and (2.23), we can deduce that

$$R_{k\ell} := R_{ik\ell} \ = [e_i, e_\ell]^{(0,1)} (\log \det g) = e_k \left( C^\eta_{k\bar{p}} \right) + e_\ell \left( C^\eta_{i\bar{p}} \right) + C^p_{k\bar{p}} C^\eta_{k\bar{p}} + \nabla^C_{k\bar{p}} C^p_{i\bar{p}}.$$ 

Combining (2.19) and (2.23) implies

$$R_{k\ell} := R_{ik\ell} = - \left( e_i \bar{\tau}_\ell - [e_i, \bar{\tau}_\ell]^{(0,1)} \right) (\log \det g) + \bar{\tau}_\ell \left( C^\eta_{k\bar{p}} \right) + e_k \left( C^\eta_{i\bar{p}} \right) + C^p_{k\bar{p}} C^\eta_{k\bar{p}} - C^p_{k\bar{p}} C^\eta_{i\bar{p}}.$$ 

From (2.20) and (2.23), it follows that

$$R_{k\ell} := R_{ik\ell} = - [e_i, \bar{\tau}_\ell]^{(1,0)} (\log \det g) - N^p_{k\bar{p}} C^\eta_{k\bar{p}} + \bar{\tau}_k (C^\eta_{i\bar{p}}) - \bar{\tau}_\ell (C^\eta_{i\bar{p}}) - C^p_{k\bar{p}} C^\eta_{i\bar{p}}.$$ 

The Chern-Ricci form $\text{Ric}(\omega)$ is defined by

$$\text{Ric}(\omega) := \frac{-1}{2} R_{k\ell} \theta^k \wedge \theta^\ell + \sqrt{-1} R_{k\bar{p}} \theta^k \wedge \bar{\theta}^i + \frac{-1}{2} R_{k\bar{p}} \bar{\theta}^i \wedge \bar{\theta}^j.$$ 

It is a closed real 2-form and furthermore is a closed real (1,1) form if the complex structure is integrable. If $J$ is integrable and $d\omega = 0$, then the Chern-Ricci form coincides exactly with the Ricci form defined by the Levi-Civita connection of $\omega$. Assume that $\tilde{\omega} = \sqrt{-1} g_{k\bar{p}} \theta^k \wedge \bar{\theta}^i$ is another almost Hermitian metric. From (2.11), (2.12), (2.13), (2.24), (2.25) and (2.26), it follows that (see also for example [56, (3.16)])

$$\text{Ric}(\omega) - \text{Ric}(\tilde{\omega}) = - \frac{1}{2} dJ \log \frac{\tilde{\omega}^n}{\omega^n},$$ 

with $\text{Ric}(\omega) \in 2\pi c_1(M, J) \in H^2(M, \mathbb{R})$. Note that in general there exist representatives of $2\pi c_1(M, J)$ which cannot be written in the form $\text{Ric}(\omega) - \frac{1}{2} dJ \text{d}F$ for any $\omega$ and $F$ even when $J$ is integrable (see for example [51, Corollary 2]).
The Chern scalar curvature $R$ is defined by
\[ R := \text{tr}_\omega \text{Ric}(\omega) = \text{tr}_\omega \text{Ric}^{(1,1)}(\omega) = \frac{n \text{Ric}(\omega) \wedge \omega^{n-1}}{\omega^n} = \frac{n \text{Ric}^{(1,1)}(\omega) \wedge \omega^{n-1}}{\omega^n}. \]

For any $\varphi \in C^\infty(M, \mathbb{R})$, we define the canonical Laplacian by
\[ \Delta^C \varphi := \frac{n 
abla^C \varphi \wedge \omega^{n-1}}{\omega^n} = \frac{n (dJ \varphi) \wedge \omega^{n-1}}{2\omega^n} = g^{ji} \left( e_i \overline{e}_j (\varphi) - [e_i, \overline{e}_j]^{(0,1)}(\varphi) \right). \]

Using the second canonical connection $\nabla^C$, it can also be rewritten as
\[ \Delta^C \varphi = g^{ji} \nabla^C_i \nabla^C_j \varphi = g^{ji} \nabla^C_i \nabla^C \varphi \]

since the $(1,1)$ part of the torsion of $\nabla^C$ vanishes. Denote by $\Delta_g$ the Laplace-Beltrami operator of the Riemannian metric $g$. For these two different Laplace operators, we have (see for example [47, Lemma 3.2])
\[ \Delta_g \varphi = 2\Delta^C \varphi + \tau(d\varphi), \]
where
\[ \tau (d\varphi) = 2 \text{Re} \left( T^j_{pq} g^{pq} e^i (\varphi) \right). \]

Given any volume form $\Omega$, there exists an almost Hermitian metric (not unique) $\omega'$ such that $\Omega = \omega'^n$ since we have $f := \log \frac{\Omega}{\omega^n} \in C^\infty(M, \mathbb{R})$ and we can take $\omega' = e^{f/n} \omega$ for example. Hence, from (2.24), (2.25) and (2.26), it follows that we can also define the Ricci form $\text{Ric}(\Omega)$ associated to $\Omega$ by replacing $\det g$ with $\Omega$ in (2.24), (2.25) and (2.26) when it occurs, and also have
\[ \text{Ric}(\omega) - \text{Ric}(\Omega) = -\frac{1}{2} dJ \omega \log \frac{\Omega}{\omega^n}. \]

3. Preliminary estimates

In this section, we give the estimates of $\varphi$ and $\varphi := \frac{\partial}{\partial t} \varphi$. For this aim, we need to prove that the flow (1.2) can be reduced to a parabolic Monge-Ampère equation. Fix $T_0 < T$ and in particular $T_0 < \infty$. By definition of $T$, we can define reference metrics $\hat{\omega}_t$ for $M \times [0, T_0]$ by
\[ \hat{\omega}_t := \alpha_t + \frac{t}{T_0} \sqrt{-1} \partial \overline{\partial} \phi_{T_0} = \frac{T_0 - t}{T_0} \omega_0 + \frac{t}{T_0} \left( \alpha_{T_0} + \sqrt{-1} \partial \overline{\partial} \phi_{T_0} \right) := \sqrt{-1} g_\theta \theta^i \wedge \overline{\theta}^j, \]
where $\phi_{T_0} \in C^\infty(M, \mathbb{R})$ satisfies $\alpha_{T_0} + \sqrt{-1} \partial \overline{\partial} \phi_{T_0} > 0$. Note that the almost Hermitian metrics $\hat{\omega}_t$ vary smoothly on the compact interval $[0, T_0]$ and hence we can deduce uniform estimates on $\hat{\omega}_t$ for $t \in [0, T_0]$. We rewrite $\hat{\omega}_t$ as $\hat{\omega}_t = \omega_0 + t \chi$ with
\[ \chi = \frac{1}{T_0} \sqrt{-1} \partial \overline{\partial} \phi_{T_0} - \text{Ric}^{(1,1)}(\omega_0). \]

We define a volume form $\Omega = \omega_0^n e^{\phi_{T_0}/T_0}$. Note that
\[ \sqrt{-1} \partial \overline{\partial} \log \Omega = \frac{1}{T_0} \sqrt{-1} \partial \overline{\partial} \phi_{T_0} + \sqrt{-1} \partial \overline{\partial} \log \omega_0^n \neq \chi = \frac{\partial \hat{\omega}_t}{\partial t} \]
and
\[ C_0^{-1} \omega_0 \leq \hat{\omega}_t \leq C_0 \omega_0 \]
for some uniform constant $C_0 > 0$.\n
9
Lemma 3.1. A smooth family \( \omega(t) \) of almost Hermitian metrics on \([0, T_0]\) solves the flow (1.2) if and only if there is a family of smooth functions \( \varphi(t) \) for \( t \in [0, T_0) \) such that \( \omega(t) = \tilde{\omega}_t + \sqrt{-1} \partial \overline{\partial} \varphi(t) \), and solve

\[
(3.2) \quad \frac{\partial}{\partial t} \varphi = \log \left( \frac{\tilde{\omega}_t + \sqrt{-1} \partial \overline{\partial} \varphi}{\Omega} \right), \quad \tilde{\omega}_t + \sqrt{-1} \partial \overline{\partial} \varphi > 0, \quad \varphi|_{t=0} = 0.
\]

Proof. We use the ideas from for example [49]. For the “if” direction, we set \( \omega(t) = \tilde{\omega}_t + \sqrt{-1} \partial \overline{\partial} \varphi(t) \). From (2.27), it follows that

\[
\frac{\partial}{\partial t} \Omega = \frac{\partial}{\partial t} \tilde{\omega}_t + \sqrt{-1} \partial \overline{\partial} \left( \frac{\partial}{\partial t} \varphi \right) = \frac{1}{T_0} \sqrt{-1} \partial \overline{\partial} \phi_{T_0} - \text{Ric}^{(1,1)}(\omega_0) - \frac{1}{T_0} \sqrt{-1} \partial \overline{\partial} \log \frac{\omega^n}{\omega_0^n} = \frac{1}{T_0} \sqrt{-1} \partial \overline{\partial} \phi_{T_0} - \text{Ric}^{(1,1)}(\omega_0) - \frac{1}{T_0} \sqrt{-1} \partial \overline{\partial} \phi_{T_0} - \text{Ric}^{(1,1)}(\omega) + \text{Ric}^{(1,1)}(\omega_0),
\]

as required.

For the ‘only if’ direction, assume that \( \omega \) solves the flow (1.2) on \([0, T_0)\). We define

\[
\varphi(t) = \int_0^t \log \frac{\omega(s)^n}{\Omega} \, ds
\]

for \( t \in [0, T_0) \). We have

\[
(3.3) \quad \frac{\partial}{\partial t} \varphi(t) = \log \frac{\omega(t)^n}{\Omega}, \quad \varphi(0) = 0.
\]

On the other hand, by (2.27), we can deduce

\[
(3.4) \quad \frac{\partial}{\partial t} (\omega - \tilde{\omega}_t) = -\text{Ric}^{(1,1)}(\omega) - \chi = \sqrt{-1} \partial \overline{\partial} \left( \log \frac{\omega^n}{\omega_0^n} - \frac{\phi_{T_0}}{T_0} \right) = \sqrt{-1} \partial \overline{\partial} \log \frac{\omega^n}{\Omega}.
\]

Thanks to (3.3) and (3.4), it follows that

\[
\frac{\partial}{\partial t} (\omega - \tilde{\omega}_t - \sqrt{-1} \partial \overline{\partial} \varphi) = 0, \quad \text{with} \quad (\omega - \tilde{\omega}_t - \sqrt{-1} \partial \overline{\partial} \varphi)|_{t=0} = 0
\]

so that \( \omega = \tilde{\omega}_t + \sqrt{-1} \partial \overline{\partial} \varphi \) and \( \varphi \) is the solution to (3.2).

Standard parabolic theory of partial differential equation yields that there exists a unique maximal solution to (3.2) on some time interval \([0, T_{\text{max}}]\) with \( T_{\text{max}} > 0 \). Assume for a contradiction that \( T_{\text{max}} < T_0 \).

Now we prove uniform estimates for the solution \( \varphi \) to (3.2) up to the maximal time. For later use, we write

\[
\omega(t) = \sqrt{-1} g_\gamma \theta^i \wedge \overline{\theta}^j, \quad \omega_0 = \sqrt{-1} (g_0)_\gamma \theta^i \wedge \overline{\theta}^j.
\]

Lemma 3.2. Assume that \( \varphi(t) \) is the solution to the flow (3.2) on \([0, T_{\text{max}}]\). There exists a positive uniform constant \( C > 0 \), independent of \( t \in [0, T_{\text{max}}) \), such that

1. \( \| \varphi(t) \|_{C^0} \leq C \).
2. \( \| \dot{\varphi}(t) \|_{C^0} \leq C \).
3. \( C^{-1} \omega^n_0 \leq \omega^n \leq C \omega^n_0 \).
Proof. We use the ideas from [46, 53]. For Part (1), set \( \psi = \varphi - At \) for a constant \( A > 0 \) to be determined later. Fix any \( T' \in (0, T_{\text{max}}) \) and assume that \( \psi \) attains a maximum on \( M \times [0, T'] \) at a point \( (x_0, t_0) \) with \( t_0 > 0 \). At this point, the maximum principle yields

\[
0 \leq \frac{\partial}{\partial t} \psi = \log \frac{\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \psi}{\Omega} - A \leq \log \frac{\hat{\omega}_t^n}{\Omega} - A < 0,
\]

provided \( A \) is chosen sufficiently large, a contradiction. Here we use the fact that \( \hat{\omega}_t \) is a smooth family of metrics on \( [0, T_{\text{max}}] \). Since \( T' \in (0, T_{\text{max}}) \) is arbitrary, this yields that the maximum of \( \psi(t) \) is achieved at \( t = 0 \). We get the upper bound for \( \psi \) and hence for \( \varphi \). The lower bound is proved similarly.

As for Part (2), we first consider the upper bound for \( \dot{\varphi} \). We consider the quantity \( Q_1 = t\dot{\varphi} - \varphi - nt \) and have

\[
\left( \frac{\partial}{\partial t} - \Delta^C_\omega \right) Q_1 = t \text{tr}_\omega \chi - n + \text{tr}_\omega (\omega - \hat{\omega}_t) = \text{tr}_\omega (t \chi - \hat{\omega}_t) = -\text{tr}_\omega \omega_0 < 0,
\]

where we use the fact that \( \frac{\partial}{\partial t} \dot{\varphi} = \Delta^C_\omega \dot{\varphi} + \text{tr}_\omega \chi \). This, together with the maximum principle, implies that upper bound for \( Q_1 \) and hence for \( \dot{\varphi} \).

For the lower bound for \( \dot{\varphi} \), we define \( Q_2 := (T_0 - t)\dot{\varphi} + \varphi + nt \) and have

\[
\left( \frac{\partial}{\partial t} - \Delta^C_\omega \right) Q_2 = (T_0 - t) \text{tr}_\omega \chi + n - \Delta^C_\omega \varphi = \text{tr}_\omega (\hat{\omega}_t + (T_0 - t) \chi) = \text{tr}_\omega \hat{\omega}_{T_0} > 0.
\]

Then the lower bound for \( Q_2 \) and hence for \( \dot{\varphi} \) follows from the maximum principle and the fact that \( T_{\text{max}} < T_0 \).

Part (3) follows from Part (2), (3.2) and the definition of \( \Omega \). \qed

4. First order estimate

In this section, we give the first order estimate of the solution \( \varphi \) to (3.2).

**Theorem 4.1.** Assume that \( \varphi(t) \) is the solution to the flow (3.2) on \( [0, T_{\text{max}}] \). There exists a uniform constant \( C > 1 \), independent of \( t \in [0, T_{\text{max}}] \), such that

\[
\sup_{M \times [0, T_{\text{max}}]} |\partial \varphi| \leq C.
\]

**Proof.** We consider the quantity \( Q := e^f(\varphi) |\partial \varphi| \) where \( f \) will be determined later. Fix any \( T' \in (0, T_{\text{max}}) \) and assume that

\[
\sup_{M \times [0, T']} Q = Q(x_0, t_0)
\]

with \( t_0 > 0 \). Since \( g_0 \) is almost Hermitian metric, we choose \( g_0 \)-unitary frame \( e_1, \cdots, e_n \) such that \( g(x_0, t_0) \) is diagonal near \( x_0 \). At the point \( (x_0, t_0) \), the maximum principle yields

\[
0 \geq \left( \Delta^C_\omega - \frac{\partial}{\partial t} \right) Q = e^f \left( \Delta^C_\omega - \frac{\partial}{\partial t} \right) |\partial \varphi| \left( \Delta^C_\omega - \frac{\partial}{\partial t} \right) e^f + 2 \text{Re} \left( g_0^{\gamma \varphi_{\gamma}} (|\partial \varphi|) \overline{\varphi_i}(e^f) \right).
\]

Since \( \omega = \hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi \), at \( (x_0, t_0) \), a direct computation gives

\[
\Delta^C_\omega (e^f) = g_0^{\gamma \varphi_{\gamma}} \left( e_i \overline{e_i} (e^f) - [e_i, \overline{e_i}]^{0(1)} (e^f) \right)
\]

\[
= g_0^{\gamma \varphi_{\gamma}} \left( e^f (f')^2 |e_i(\varphi)|^2 + e^f f'|e_i(\varphi)|^2 + e^f f'' e_i \overline{e_i}(\varphi) - e^f f'[e_i, \overline{e_i}]^{0(1)} (\varphi) \right)
\]

\[
+ 2 \text{Re} \left( g_0^{\gamma \varphi_{\gamma}} (|\partial \varphi|) \overline{\varphi_i}(e^f) \right).
\]
\[= g^{i\bar{j}} e^f \left( (f')^2 + f'' \right) |e_i(\varphi)|^2 + e^f f' \tr_\omega (\omega - \hat{\omega}_t) \]
\[= g^{i\bar{j}} e^f \left( (f')^2 + f'' \right) |e_i(\varphi)|^2 + ne^f f' - e^f f' (\tr_\omega \hat{\omega}_t) \]

and
\[
\Delta^C_\omega \left( \partial \varphi \right)_{g_0}^{2} = g^{i\bar{j}} \left( e_i \overline{e}_i (e_k(\varphi) \overline{e}_k(\varphi)) - |e_i, \overline{e}_i|^{0,1} (e_k(\varphi) \overline{e}_k(\varphi)) \right)
\]
\[= g^{i\bar{j}} \left( |e_i e_k(\varphi)|^2 + |e_i \overline{e}_k(\varphi)|^2 + e_i \overline{e}_k(e_k(\varphi) \overline{e}_k(\varphi)) + e_k(\varphi) e_i \overline{e}_k(\varphi) \right)
\[\quad - g^{i\bar{j}} \left[ |e_i, \overline{e}_i|^{0,1} e_k(\varphi) \overline{e}_k(\varphi) + e_k(\varphi) |e_i, \overline{e}_i|^{0,1} \overline{e}_k(\varphi) \right)
\]
\[= g^{i\bar{j}} \left( |e_i e_k(\varphi)|^2 + |e_i \overline{e}_k(\varphi)|^2 \right) + g^{i\bar{j}} \left( e_k \left( e_k(\varphi) - |e_i, \overline{e}_i|^{0,1} \varphi \right) e_k(\varphi) \right)
\[\quad + g^{i\bar{j}} e_k(\varphi) \left( e_k \left( e_k(\varphi) - |e_i, \overline{e}_i|^{0,1} \varphi \right) \right) - g^{i\bar{j}} |e_i, e_i| \overline{e}_k(\varphi) \overline{e}_k(\varphi)
\]
\[= g^{i\bar{j}} \left( |e_i e_k(\varphi)|^2 + |e_i \overline{e}_k(\varphi)|^2 \right) + g^{i\bar{j}} \left( e_k \left( g_{\bar{i}} - \hat{g}_{\bar{i}} \right) \overline{e}_k(\varphi) \right)
\[\quad + g^{i\bar{j}} e_k(\varphi) \left( g_{k} - \hat{g}_{k} \right) - g^{i\bar{j}} |e_i, e_i| \overline{e}_k(\varphi) \overline{e}_k(\varphi)
\]
\[\quad - g^{i\bar{j}} e_k(\varphi) \left( e_k \left( g_{\bar{i}} - \hat{g}_{\bar{i}} \right) \overline{e}_k(\varphi) - g^{i\bar{j}} e_k(\varphi) e_i \overline{e}_i(\varphi) - g^{i\bar{j}} e_k(\varphi) e_i \overline{e}_i(\varphi) \right)
\]
\[\quad - g^{i\bar{j}} \left[ e_k, |e_i, \overline{e}_i|^{0,1} \left( \varphi \right) e_k(\varphi) - g^{i\bar{j}} e_k(\varphi) \left[ e_k, |e_i, \overline{e}_i|^{0,1} \right] \left( \varphi \right) \right]
\]
\[\geq (1 - \varepsilon) g^{i\bar{j}} \left( |e_i e_k(\varphi)|^2 + |e_i \overline{e}_k(\varphi)|^2 \right) + 2 \Re \left( e_k(\varphi) \overline{e}_k(\varphi) \right)
\]
\[\quad - \frac{C}{\varepsilon} g^{i\bar{j}} \sum_i g^{i\bar{j}} + 2 \Re \left( e_k(\varphi) \overline{e}_k(\varphi) \right),
\]

On the other hand, we have, at \((x_0,t_0)\),
\[
\frac{\partial}{\partial t} |\partial \varphi|_{g_0}^{2} = e_k(\varphi) \overline{e}_k(\varphi) + e_k(\varphi) \overline{e}_k(\varphi)
\]
\[= g^{i\bar{j}} e_k \left( g_{\bar{i}} - \hat{g}_{\bar{i}} \right) \overline{e}_k(\varphi) - e_k(\tr_\omega \Omega) \overline{e}_k(\varphi) + g^{i\bar{j}} e_k(\varphi) \overline{e}_k \left( g_{\bar{i}} - \hat{g}_{\bar{i}} \right) \overline{e}_k(\varphi),
\]
where we use (3.2). At \((x_0,t_0)\), from (4.4) and (4.5), it follows that
\[
\left( \Delta^C_\omega - \frac{\partial}{\partial t} \right) |\partial \varphi|_{g_0}^{2} = g^{i\bar{j}} \left( |e_i e_k(\varphi)|^2 + |e_i \overline{e}_k(\varphi)|^2 \right) - g^{i\bar{j}} e_k \left( g_{\bar{i}} - \hat{g}_{\bar{i}} \right) \overline{e}_k(\varphi) - g^{i\bar{j}} e_k(\varphi) \overline{e}_k \left( g_{\bar{i}} - \hat{g}_{\bar{i}} \right)
\]
\[\quad - g^{i\bar{j}} [e_k, |e_i, \overline{e}_i|^{0,1} \left( \varphi \right) e_k(\varphi) - g^{i\bar{j}} [e_k, |e_i, \overline{e}_i|^{0,1} \left( \varphi \right) e_k(\varphi) - g^{i\bar{j}} e_k(\varphi) \left[ e_k, |e_i, \overline{e}_i|^{0,1} \right] \left( \varphi \right) \right]
\]
\[\quad - 2 \Re \left( e_k(\varphi) \overline{e}_k(\varphi) \right)
\]
\[\geq (1 - \varepsilon) g^{i\bar{j}} \left( |e_i e_k(\varphi)|^2 + |e_i \overline{e}_k(\varphi)|^2 \right)
\]
\[\quad - \frac{C}{\varepsilon} |\partial \varphi|_{g_0}^{2} \sum_i g^{i\bar{j}} + 2 \Re \left( e_k(\varphi) \overline{e}_k(\varphi) \right),
\]
where for the inequality we use the Cauchy-Schwarz inequality and \(\varepsilon \in (0, 1/2)\) and without loss of generality we assume \(|\partial \varphi|_{g_0}^{2} (x_0,t_0) > 1\). At \((x_0,t_0)\), a direct calculation implies
\[
2 \Re \left( g^{i\bar{j}} e_i \left( |\partial \varphi|_{g_0}^{2} \right) \overline{e}_i(e^f) \right)
\]
\[= 2 \Re \left( e^f f' g^{i\bar{j}} e_k(\varphi) e_i \overline{e}_i(\varphi) \overline{e}_k(\varphi) \right) = 2 \Re \left( e^f f' g^{i\bar{j}} e_k(\varphi) g_{\bar{i}} - \hat{g}_{\bar{i}} + |e_i, \overline{e}_i|^{0,1} \left( \varphi \right) \overline{e}_i(\varphi) \right)
\]
\[= 2 \Re \left( e^f f' g^{i\bar{j}} e_k(\varphi) g_{\bar{i}} - \hat{g}_{\bar{i}} + |e_i, \overline{e}_i|^{0,1} \left( \varphi \right) \overline{e}_i(\varphi) \right)
\]
An almost complex Chern-Ricci flow

Tao Zheng

From (4.2), (4.6), (4.7) and (4.8), it yields that, at \((x, t)\),

\[
\begin{align*}
&\geq 2e^f f' |\partial \varphi|^2_{g_0} - 2\text{Re} \left( e^f f' g^i \hat{\nabla}_k (\varphi) \overline{\nabla}_i (\varphi) \right) - \varepsilon e^f \left( f'' + (f')^2 \right) |\partial \varphi|^2_{g_0} - \frac{C_1 e^f}{\varepsilon} |\partial \varphi|^2_{g_0} \sum_i g^i - \left( 1 + 2\varepsilon \right) e^f (f')^2 |\partial \varphi|^2_{g_0} - \left( 1 - \varepsilon \right) e^f \sum_i g^i |e_i e_k (\varphi)|^2,
\end{align*}
\]

where for the inequality we use the Cauchy-Schwarz inequality and \(\varepsilon \in (0, 1/2]\). At \((x_0, t_0)\), from (3.2) and (4.3), it follows that

\[
\begin{align*}
(4.8)\quad (\Delta_{\omega} - \frac{\partial}{\partial t}) e^f f' = &\ e^f (f'' + (f')^2) |\partial \varphi|^2_{g_0} + e^f f' \left( \frac{\Delta \omega}{\partial t} \right) \varphi \\
= &\ e^f (f'' + (f')^2) |\partial \varphi|^2_{g_0} + e^f f' \left( \frac{\partial}{\partial t} \varphi \right).
\end{align*}
\]

From (4.2), (4.6), (4.7) and (4.8), it yields that, at \((x_0, t_0)\),

\[
\begin{align*}
0 \geq &\ e^f (f'' - 3\varepsilon (f')^2) |\partial \varphi|^2_{g_0} - e^f f' |\partial \varphi|^2_{g_0} (\text{tr}_{\omega} \hat{\nabla} \varphi) - \frac{C_1 e^f}{\varepsilon} |\partial \varphi|^2_{g_0} \sum_i g^i - e^f f' \left( \frac{\partial}{\partial t} \varphi \right) \\
&\ - e^f f' \left( \frac{\partial}{\partial t} \right) \varphi - 2e^f \text{Re} (e_k (\log \Omega) \overline{\nabla}_k (\varphi)) + (n + 2)e^f f' |\partial \varphi|^2_{g_0} \\
&\ - 2\text{Re} \left( e^f f' g^i \hat{\nabla}_k e_k (\varphi) \overline{\nabla}_i (\varphi) \right),
\end{align*}
\]

where \(C_1 > 1\) is a uniform constant.

Define

\[
f(\varphi) = \frac{e^{-\Lambda (\varphi - \varphi_0 - 1)}}{A}, \quad \varepsilon = \frac{A e^{\Lambda (\varphi (x_0, t_0) - \varphi_0 - 1)}}{6},
\]

where \(\varphi_0 := \sup_{M \times [0, T_{\text{max}}]} \varphi\). Then, at \((x_0, t_0)\), we have

\[
\begin{align*}
(4.10)\quad f'' - 3\varepsilon (f')^2 &= \frac{A e^{-\Lambda (\varphi (x_0, t_0) - \varphi_0 - 1)}}{2},
\end{align*}
\]

\[
\begin{align*}
(4.11)\quad -e^f f' |\partial \varphi|^2_{g_0} (\text{tr}_{\omega} \hat{\nabla} \varphi) - \frac{C_1 e^f}{\varepsilon} |\partial \varphi|^2_{g_0} \sum_i g^i \\
&\geq e^f |\partial \varphi|^2_{g_0} \left( \sum_i g^i \right) \left( C_0^{-1} - \frac{6C_1}{A} \right) e^{-\Lambda (\varphi (x_0, t_0) - \varphi_0 - 1)},
\end{align*}
\]

and

\[
\begin{align*}
(4.12)\quad 2 \text{Re} \left( e^f f' g^i \hat{\nabla}_k e_k (\varphi) \overline{\nabla}_i (\varphi) \right) \leq -2C_0 e^f f' |\partial \varphi|^2_{g_0} = 2C_0 e^f |\partial \varphi|^2_{g_0} e^{-\Lambda (\varphi (x_0, t_0) - \varphi_0 - 1)},
\end{align*}
\]

where for the inequality we use (3.1) and the fact that \(f' < 0\).
Lemma 4.2. Proof. Denote by $f_1$ from Lemma 3.2 and the same argument in [10] (see also [9]), Theorem 4.1 follows.

Using Theorem 4.1, taking $\varepsilon = 1/2$ in (4.6), we get

\begin{equation}
\Delta^C \square \geq 0
\end{equation}

\begin{equation}
\Delta^C \square \geq 0
\end{equation}

where $C > 0$ is a uniform constant. Combining (4.9), (4.13) and (4.14) implies that, at $(x_0, t_0)$,

\begin{equation}
0 \geq 0
\end{equation}

Note that here the constant $C$ may be larger.

From Lemma 3.2 and the same argument in [10] (see also [9]), Theorem 4.1 follows.

5. Second order estimate

In this section, we deduce the second order estimate of the solution $\varphi$ to (3.2) using the method from [10] in the elliptic setup.

Theorem 5.1. Assume that $\varphi$ is the solution to the flow (3.2). There exists a uniform constant $C > 0$, independent of $t$, such that

\begin{equation}
\sup_{M \times [0, T_{\max})} |\nabla^2 \varphi|_{g_0} \leq C,
\end{equation}

where $\nabla$ is the Levi-Civita connection with respect to the Riemannian metric $g_0$.

Proof. Denote by $\lambda_1(\nabla^2 \varphi) \geq \cdots \geq \lambda_{2n}(\nabla^2 \varphi)$ the eigenvalues of $\nabla^2 \varphi$ with respect to the Riemannian metric $g_0$. Then we have $\Delta_{g_0} \varphi = \sum_{\alpha=1}^{2n} \lambda_\alpha(\nabla^2 \varphi)$, where $\Delta_{g_0}$ is the Laplace-Beltrami operator of $g_0$. Since

$$\omega = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi > 0,$$

it follows that

$$\Delta^C_{\omega_0} \varphi = \tau_{\omega_0} \omega - \tau_{\omega_0} \omega_t > - \tau_{\omega_0} \omega_t > -C,$$

where we use the fact that $\omega_t$ vary smoothly on the compact interval $[0, T_0]$ and hence can be estimated uniformly. This, together with (2.28) and Theorem 4.1, yields

\begin{equation}
\Delta_{g_0} \varphi = 2 \Delta^C_{\omega_0} \varphi + \tau(d \varphi) > -C,
\end{equation}

where $C > 0$ is a uniform constant. As a result, we get

\begin{equation}
\sup_{M \times [0, T_{\max})} |\nabla^2 \varphi|_{g_0} \leq C \max \left\{ \lambda_1(\nabla^2 \varphi), 0 \right\} + C.
\end{equation}

Therefore, it is sufficient to bound $\lambda_1(\nabla^2 \varphi)$ from above in order to prove Theorem 5.1. Denote

$$S_H := \left\{ (x, t) \in M \times [0, T_{\max}) : \lambda_1(\nabla^2 \varphi) > 0 \right\}.$$
Without loss of generality, we assume $S_H \neq \emptyset$. Otherwise, we can get the upper bound of $\lambda_1(\nabla^2 \varphi)$ directly. We consider the quantity

$$H = \log \lambda_1(\nabla^2 \varphi) + \phi(|\partial \varphi|^2) + h(\varphi - \varphi_0),$$

on the set $S_H$, where

$$\phi(s) = -\frac{1}{2} \log \left( 1 - \frac{s}{2K} \right), \quad h(s) = e^{-Ds}$$

with sufficiently large uniform constant $D$ to be determined later. Here recall that $\varphi_0 := \sup_{M \times [0, T_{\text{max}}]} \varphi$, and for convenience, we use notation $K := 1 + \sum_{M \times [0, T_{\text{max}}]} |\partial \varphi|^2_{g_0}$ which is a uniform constant by Theorem 4.1. Note that

$$\phi(|\partial \varphi|^2_{g_0}) \in [0, 2 \log 2]$$

and

$$\frac{1}{4K} \leq \phi' \leq \frac{1}{2K}, \quad \phi'' = 2(\phi')^2.$$

For any $T' \in (0, T_{\text{max}})$, assume that $H$ attains its maximum at $(x_0, t_0)$ on \( \left\{ (x, t) \in M \times [0, T'] : \lambda_1(\nabla^2 \varphi) > 0 \right\} \). Note that $H$ is not smooth in general since the dimension of eigenspace associated to $\lambda_1(\nabla^2 \varphi)$ may be strictly larger than 1. We use a perturbation argument in [10] to deal with this case (see also [43, 44]). Since $g_0$ is almost Hermitian metric, we can choose a coordinate patch $(U; x^1, \ldots, x^{2n})$ centered at $x_0$ such that

1. $U$ is diffeomorphic to a ball $B_2(0) \subset \mathbb{R}^{2n}$ of radius 2 centered at 0.
2. Denote by $\partial_\alpha$ the local vector fields $\partial / \partial x^\alpha$. There holds $g_0(\partial_\alpha, \partial_\beta)|_{x_0} = \delta_{\alpha\beta}, \alpha, \beta = 1, \ldots, 2n$.
3. The almost complex structure $J$ satisfies $J(x_0) = J_0$, where $J_0$ is the standard complex structure on $\mathbb{R}^{2n}$, i.e., $J_0 \partial_2i - \partial_2 = \partial_2$ for $i = 1, \ldots, n$.
4. There holds

$$\partial_\gamma(g_0)_{\alpha\beta}|_{x_0} = 0, \quad \forall \alpha, \beta, \gamma = 1, \ldots, 2n.$$

5. If at $x_0$, we define

$$e_i := \frac{1}{\sqrt{2}} \left( \partial_{2i-1} - \sqrt{-1} \partial_{2i} \right), \quad i = 1, \ldots, n,$$

then these form a frame of $(1, 0)$ vectors at $x_0$, and we have $(g_0)_{\overline{i}j} := g_0(e_i, \overline{e_j}) = \delta_{ij}$, i.e., the frame is $g_0$-unitary. Furthermore, the argument in [10] yields that we can assume $(g_0(x_0, t_0))$ is diagonal with

$$g_0(x_0, t_0) \geq \cdots \geq g_0\pi(x_0, t_0).$$

We extend $e_1, \ldots, e_n$ smoothly to a $g_0$-unitary frame of $(1, 0)$ vectors in a neighborhood of $x_0$, and from now on, the local unitary frame is fixed. Assume that $V_1 \in T_{x_0}M$ is a unit vector, i.e., $g_0(V_1, V_1) = 1$, with

$$\nabla^2 \varphi(V_1, V_1) = \lambda_1(\nabla^2 \varphi)(x_0, t_0).$$

We can construct an orthonormal basis of $T_{x_0}M$, denoted by $V_1, \ldots, V_{2n}$, such that

$$\nabla^2 \varphi(V_\alpha, V_\alpha) = \lambda_\alpha(\nabla^2 \varphi)(x_0, t_0), \quad \alpha = 1, \ldots, 2n.$$
with \( \lambda_1(\nabla^2 \varphi)(x_0, t_0) \geq \cdots \geq \lambda_{2n}(\nabla^2 \varphi)(x_0, t_0) \). We assume \( V_\beta = \sum_{\alpha=1}^{2n} V_\alpha^\beta \partial_\alpha \) for \( \beta = 1, \cdots, 2n \).

We extend \( V_1, \cdots, V_{2n} \) to be vector fields in a neighborhood of \( x_0 \) by taking the components to be constant. To use the perturbation argument, we define a smooth section \( B = B_{\alpha\beta} dx^\alpha \otimes dx^\beta \) of \( T^*M \otimes T^*M \) near \( x_0 \), where

\[
B_{\alpha\beta} = \delta_{\alpha\beta} - V_1^\alpha V_1^\beta.
\]

It is easy to deduce that the eigenvalues of \( (B_{\alpha\beta}) \) are \( 0, 1, \cdots, 1 \) and that \( V_1 \) is the eigenvector associated to the eigenvalue 0, i.e., \( B(V_1, V_1) = 0 \). Consider the endomorphism \( \Phi = (\Phi^\alpha_\beta) \) of \( TM \) defined by

\[
\Phi^\alpha_\beta = g_0^{\alpha\gamma} \nabla^2_{\gamma\beta} \varphi - g_0^{\alpha\gamma} B_{\gamma\beta},
\]

and denote its eigenvalues by

\[
\lambda_1(\Phi) \geq \cdots \geq \lambda_{2n}(\Phi).
\]

Since \( B = (B_{\alpha\beta}) \) is nonnegative, we can deduce that \( \lambda_1(\Phi) \leq \lambda_1(\nabla^2 \varphi) \) in the neighborhood of \((x_0, t_0)\). In particular, we have

\[
\lambda_1(\Phi)(x_0, t_0) = \lambda_1(\nabla^2 \varphi)(x_0, t_0), \quad \Phi(V_\alpha)(x_0, t_0) = \lambda_\alpha(\Phi)(x_0, t_0)V_\alpha, \quad \alpha = 1, \cdots, 2n
\]

and hence

\[
\lambda_1(\Phi)(x_0, t_0) > \lambda_2(\Phi)(x_0, t_0) \geq \cdots \geq \lambda_{2n}(\Phi)(x_0, t_0).
\]

This yields that \( \lambda_1(\Phi) \) is smooth in a neighborhood of \((x_0, t_0)\). In the following, we write \( \lambda_\alpha \) for \( \lambda_\alpha(\Phi) \) for short. We can apply the maximum principle to the quantity

\[
\tilde{H} = \log \lambda_1(\Phi) + \phi(|\partial \varphi|^2_{g_0}) + h(\varphi - \varphi_0),
\]

which still obtains a local maximum at \((x_0, t_0)\).

We need the first and second derivatives of \( \lambda_1 \) at \((x_0, t_0)\) as follows.

**Lemma 5.2** (Chu, Tosatti and Weinkove [10]; 2016). At \((x_0, t_0)\), we have

\[
(5.7) \quad \lambda_1^{\alpha\beta} := \frac{\partial \lambda_1}{\partial \Phi^\alpha_\beta} = V_1^\alpha V_1^\beta.
\]

\[
(5.8) \quad \lambda_1^{\alpha\beta,\gamma\delta} := \frac{\partial^2 \lambda_1}{\partial \Phi^\alpha_\beta \partial \Phi^{\gamma_\delta}} = \sum_{\mu > 1} \frac{V_1^\alpha V_1^\beta V_1^\gamma V_1^\delta}{\lambda_1 - \lambda_\mu},
\]

where the Greek indices \( \alpha, \beta, \gamma, \mu, \cdots \) go from 1 to 2n, unless otherwise indicated.

By Lemma 3.2 and (3.2), the arithmetic-geometry mean inequality yields

\[
(5.9) \quad \text{tr}_\omega \omega_0 \geq c,
\]

for a uniform constant \( c > 0 \). We also assume that, at \((x_0, t_0)\), there holds \( \lambda_1 \gg K \geq 1 \).

**Lemma 5.3.** At \((x_0, t_0)\), we have

\[
(5.10) \quad \left( \Delta^C_\omega - \frac{\partial}{\partial t} \right) \lambda_1 \geq 2 \sum_{\alpha > 1}^2 \hat{g}^{i\iota} \frac{|e_i(\varphi_{V_\alpha} V_1)|^2}{\lambda_1 - \lambda_\alpha} + g^{pp} g^{qq} |V_1(g_{p\varphi})|^2
\]

\[
- 2g^{i\iota} [V_1, e_i] V_1 \tau_i(\varphi) - 2g^{i\iota} [V_1, \tau_i] V_1 \tau_i(\varphi) - C \lambda_1 \sum_i g^{i\iota},
\]

where we write

\[
\varphi_{\alpha\beta} := \nabla^2_{\alpha\beta} \varphi, \quad \varphi_{V_\alpha V_\beta} := \varphi_{\gamma\delta} V_\gamma^\alpha V_\delta^\beta = \nabla^2 \varphi(V_\alpha, V_\beta).
\]
Proof. At \((x_0, t_0)\), noting that \(g(x_0, t_0)\) is diagonal, by (5.7) and (5.8), a direct computation yields
\[
(\Delta^C - \partial_t) \lambda_1 = g^{i\bar{i}} \alpha\beta \gamma\delta e_i(\Phi^\gamma_\delta) \bar{e}_i(\Phi^\alpha_\beta) + g^{i\bar{i}} \alpha\beta e_i \bar{e}_i (\Phi^\alpha_\beta)
\]
\[
- g^{i\bar{i}} \alpha\beta [e_i, \bar{e}_i]^{(0,1)} (\Phi^\alpha_\beta) - \lambda_1^{\alpha\beta} \frac{\partial}{\partial t} (\Phi^\alpha_\beta)
\]
\[
= g^{i\bar{i}} \alpha\beta e_i (\varphi_\gamma_\beta) \bar{e}_i (\varphi_\alpha_\beta) + g^{i\bar{i}} \alpha\beta e_i \bar{e}_i (\varphi_\alpha_\beta) + g^{i\bar{i}} \lambda_1^{\alpha\beta} \varphi_\gamma_\beta e_i \bar{e}_i (g^\alpha_\gamma_0) - g^{i\bar{i}} \lambda_1^{\alpha\beta} B_\gamma_\beta e_i \bar{e}_i (g^0_\alpha) - g^{i\bar{i}} \lambda_1^{\alpha\beta} \varphi_\alpha_\beta
\]
\[
\geq 2 \sum_{\alpha > 1} g^{i\bar{i}} \left| e_i (\varphi_{i\bar{i}V_i}) \right|^2 + g^{i\bar{i}} e_i \bar{e}_i (\varphi_{i\bar{i}V_i}) - g^{i\bar{i}} [e_i, \bar{e}_i]^{(0,1)} (\varphi_{i\bar{i}V_i})
\]
\[- \varphi_{i\bar{i}V_i} - C \lambda_1 \sum_i g^{i\bar{i}}.\]

Since \(\varphi_{i\bar{i}V_i} = V_i V_i (\varphi) - (\nabla_i V_i) (\varphi)\) for any \(\alpha\) and \(\beta\), we have
\[
g^{i\bar{i}} \left( e_i \bar{e}_i - [e_i, \bar{e}_i]^{(0,1)} \right) (\varphi_{i\bar{i}V_i}) = g^{i\bar{i}} \left( e_i \bar{e}_i - [e_i, \bar{e}_i]^{(0,1)} \right) (V_i V_i (\varphi) - (\nabla_i V_i) (\varphi)).
\]

From now on, we write \(G\) for term bounded by \(C \lambda_1 \sum_i g^{i\bar{i}}\) which may change from line to line. At \((x_0, t_0)\), noting that \(|\partial \varphi|_{g_0} \leq C\), a direct computation yields
\[
g^{i\bar{i}} \left( e_i \bar{e}_i - [e_i, \bar{e}_i]^{(0,1)} \right) (\nabla_i V_i) (\varphi)
\]
\[
= g^{i\bar{i}} e_i (\nabla_i V_i) \bar{e}_i (\varphi) - g^{i\bar{i}} (\nabla_i V_i) [e_i, \bar{e}_i]^{(0,1)} (\varphi) + G
\]
\[
= g^{i\bar{i}} (\nabla_i V_i) e_i \bar{e}_i (\varphi) - g^{i\bar{i}} (\nabla_i V_i) [e_i, \bar{e}_i]^{(0,1)} (\varphi) + G
\]
\[
= g^{i\bar{i}} (\nabla_i V_i) \left( e_i \bar{e}_i (\varphi) - [e_i, \bar{e}_i]^{(0,1)} (\varphi) \right) + G
\]
\[
= g^{i\bar{i}} (\nabla_i V_i) (g_{\alpha\beta} - g_{\alpha\bar{i}}) + G
\]
\[
= g^{i\bar{i}} (\nabla_i V_i) (g_{\alpha\beta}) + G
\]
and
\[
g^{i\bar{i}} \left( e_i \bar{e}_i V_i (\varphi) - [e_i, \bar{e}_i]^{(0,1)} V_i (\varphi) \right)
\]
\[
= g^{i\bar{i}} V_i (e_i \bar{e}_i (\varphi) - [e_i, \bar{e}_i]^{(0,1)} (\varphi)) - 2g^{i\bar{i}} [V_i, e_i] V_i \bar{e}_i (\varphi) - 2g^{i\bar{i}} [V_i, \bar{e}_i] V_i e_i (\varphi) + G
\]
\[
= g^{i\bar{i}} V_i (g_{\alpha\beta} - g_{\alpha\bar{i}}) - 2g^{i\bar{i}} [V_i, e_i] V_i \bar{e}_i (\varphi) - 2g^{i\bar{i}} [V_i, \bar{e}_i] V_i e_i (\varphi) + G
\]
\[
= g^{i\bar{i}} V_i (g_{\alpha\beta}) - 2g^{i\bar{i}} [V_i, e_i] V_i \bar{e}_i (\varphi) - 2g^{i\bar{i}} [V_i, \bar{e}_i] V_i e_i (\varphi) + G,
\]
where we use the fact that \(\dot{\omega}_t\) vary smoothly on \(M \times [0, T_0]\) and hence can be estimated uniformly.

From (3.2), we get
\[
g^{i\bar{i}} (\nabla_i V_i) (g_{\alpha\beta}) = (\nabla_i V_i) (\dot{\varphi}) + (\nabla_i V_i) (\log \Omega).
\]
Applying \(V_i\) twice to (3.2) implies
\[
g^{i\bar{i}} V_i (g_{\alpha\bar{i}}) = g^{i\bar{i}p} g^{\bar{q}p} |V_i (g_{\alpha\bar{q}})|^2 + V_i V_i (\dot{\varphi}) + V_i V_i (\log \Omega).
\]
Combining (5.9), (5.12), (5.13), (5.14), (5.15) and (5.16) yields
\[
g^{i\bar{i}} e_i \bar{e}_i (\varphi_{i\bar{i}V_i}) - g^{i\bar{i}} [e_i, \bar{e}_i]^{(0,1)} (\varphi_{i\bar{i}V_i})
Lemma 5.4. Thanks to (5.11) and (5.17), we get (5.10). Therefore, it follows that, at $(x, t)$, we use the Cauchy-Schwarz inequality for $\lambda > 0$.

Proof. At $(x_0, t_0)$, we define

$$[V_1, e_i] = \sum_{\alpha=1}^{2n} \tau_{\alpha} V_\alpha$$

for some $\tau_{\alpha} \in \mathbb{C}$ uniformly bounded. This yields

$$||[V_1, e_i] V_1 \xi_i(\varphi) + [V_1, \xi_i] V_1 (\varphi)\| \leq C \sum_{\alpha=1}^{2n} \|V_\alpha V_1 e_i(\varphi)\|.$$

Note that

$$V_\alpha V_1 e_i(\varphi) = V_\alpha e_i V_1(\varphi) + V_\alpha [V_1, e_i] (\varphi) = e_i V_\alpha V_1(\varphi) + [V_\alpha, e_i] V_1(\varphi) + V_\alpha [V_1, e_i] (\varphi) = e_i (\varphi V_\alpha V_1) + e_i (\nabla V_\alpha V_1(\varphi) + [V_\alpha, e_i] V_1(\varphi) + V_\alpha [V_1, e_i] (\varphi).$$

Therefore, it follows that, at $(x_0, t_0)$,

$$(5.19) \quad 2 \sum_{\alpha=1}^{2n} \frac{g^{\bar{i}i} ( [V_1, e_i] V_1 \xi_i(\varphi) + [V_1, \xi_i] V_1 e_i(\varphi))}{\lambda_1}$$

where we use the Cauchy-Schwarz inequality for $\varepsilon \in (0, 1/2]$ and (5.2) to get, at $(x_0, t_0)$,

$$\sum_{\alpha=1}^{2n} \frac{\lambda_1 - \lambda_\alpha}{\lambda_1} = (2n - 1) - \sum_{\alpha=1}^{2n} \frac{\lambda_\alpha (\nabla^2 \varphi) - 1}{\lambda_1} = 2n + \frac{2n - 1}{\lambda_1} - \frac{\Delta u}{\lambda_1} \leq 4n - 1 + C/\lambda_1 \leq C,$$

by the assumption that $\lambda_1 (\nabla^2 \varphi) \gg K \geq 1$. 

Tao Zheng
An almost complex Chern-Ricci flow
At \((x_0, t_0)\), since we have
\[
0 \geq \left( \Delta_\omega \frac{\partial}{\partial t} \right) \bar{H} = \frac{1}{\lambda_1} \left( \Delta_\omega \frac{\partial}{\partial t} \right) \lambda_1 - \frac{g^{|e_i(\lambda_1)|^2}}{\lambda_1^2} + \phi' \left( \Delta_\omega \frac{\partial}{\partial t} \right) |\partial \varphi|_{g_0}^2 + \phi'' g^{|e_i(\varphi)|_{g_0}}^2 + D e^{-D(\varphi)} \phi + D^2 e^{-D(\varphi)} |\partial \varphi|_{g},
\]
and a direct computation yields \(e_i(\lambda_1) = e_i(\varphi V_i)\), the inequality (5.18) follows from (4.16), (5.10) and (5.19).

In what follows, we use \(C_D\) to denote the constant depending on the initial data and \(D\), which is a uniform constant when \(D\) is determined. We split up into different cases.

**Case 1:** Assume that
\[
\|g_{\pi}(x_0, t_0)\| \leq D^2 e^{-D(\varphi(x_0, t_0) - \varphi_0)} g_{\pi}(x_0, t_0).
\]
Since \(\bar{H}\) attains maximum at \((x_0, t_0)\), we have \(e_i(\bar{H}) = 0\), i.e.,
\[
e_i(\varphi V_i) \lambda_1 = D e^{-D(\varphi - \varphi_0)} e_i(\varphi) - \phi' e_i |\partial \varphi|_{g_0}^2.
\]
Taking \(\varepsilon = 1/2\), combining (5.21) and the Cauchy-Schwarz inequality yields
\[
0 \geq -6 \left( \sup_{M \times [0, T_{\text{max}}]} |\partial \varphi|_{g_0}^2 \right) D^2 e^{-2D(\varphi - \varphi_0)} \sum_i g^\varphi i - 2(\phi')^2 g^\varphi i e_i |\partial \varphi|_{g_0}^2.
\]
From (5.18) and (5.22), it follows that
\[
0 \geq -6 \left( \sup_{M \times [0, T_{\text{max}}]} |\partial \varphi|_{g_0}^2 \right) D^2 e^{-2D(\varphi - \varphi_0)} \sum_i g^\varphi i + \frac{\phi'}{2} \sum_p g^\varphi i \left( |e_i e_p(\varphi)|^2 + |e_i \bar{e}_p(\varphi)|^2 \right) - D e^{-D(\varphi - \varphi_0)} \left( \Delta_\omega \frac{\partial}{\partial t} \right) \varphi - C \sum_i g^\varphi i,
\]
where we discard some positive terms. Thanks to (3.2) and (5.20), we can deduce
\[
C_D \geq g_{\pi}(x_0, t_0) \geq \cdots \geq g_{\pi}(x_0, t_0) \geq C_D^{-1}.
\]
This, together with Lemma 3.2 and (5.23), yields
\[
\sum_{i, p} \left( |e_i e_p(\varphi)|^2 + |e_i \bar{e}_p(\varphi)|^2 \right) (x_0, t_0) \leq C_D,
\]
where we also use
\[
-D e^{-D(\varphi - \varphi_0)} \Delta_\omega \varphi = -D e^{-D(\varphi - \varphi_0)} g^\varphi \left( g_{\pi} - \hat{g}_{\pi} \right)
\]
\[ \geq -nD e^{-D(\varphi-\varphi_0)} + C_0^{-1} D e^{-D(\varphi-\varphi_0)} \sum_i g^{\tilde{\alpha}}_i. \]

From Theorem 4.1 and (5.24), it follows that \( \lambda_1(x_0, t_0) \) and hence \( \tilde{H} \) is bounded from above. This completes the proof of Case 1.

**Case 2**: At \((x_0, t_0)\), assume that

\[ \sum_{i} g^{\tilde{\alpha}}_i |e_i e_p(\varphi)|^2 + |e_i e_p(\varphi)|^2 \geq 6 \left( \sup_{M \times [0, T_{\max}]} |\partial \varphi|^2_{g_0} \right) D^2 e^{-2D(\varphi-\varphi_0)} \sum_i g^{\tilde{\alpha}}_i. \]

By the same argument as in Case 1, we also have (5.23). From Lemma 3.2, (5.23), (5.25) and (5.26), it follows that, at \((x_0, t_0)\),

\[ 0 \geq C^{-1} \sum_p g^{\tilde{\alpha}}_i (|e_i e_p(\varphi)|^2 + |e_i e_p(\varphi)|^2) + (C^{-1}D - C) \sum_i g^{\tilde{\alpha}}_i - C_D. \]

We choose \( D \) sufficiently large such that \( C_0^{-1} D - C > 1 \). Then from (3.2), we can deduce the lower and upper bounds of \( g^{\tilde{\alpha}}_i \) and applying the same argument as in Case 1 to (5.27) completes the proof of Case 2.

**Case 3**: At \((x_0, t_0)\), neither Case 1 nor Case 2 holds.

In this case, we need to estimate

\[ (2 - \varepsilon) \sum_{\alpha > 1} g^{\tilde{\alpha}}_i \frac{|e_i (\varphi V_\alpha V_1)|^2}{\lambda_1 (\lambda_1 - \lambda_\alpha)} + g^{\tilde{\alpha}}_i g^{\tilde{\gamma}}_q |V_1 (g_p q)|^2 \lambda_1 - (1 + \varepsilon) g^{\tilde{\alpha}}_i |e_i (\varphi V_\alpha V_1)|^2 \]

in Lemma 5.4. For this aim, we define

\[ I := \left\{ i : g^{\tilde{\alpha}}_i (x_0, t_0) > D^3 e^{-2D(\varphi(x_0, t_0)-\varphi_0)} g_0\mathfrak{m}(x_0, t_0) \right\}. \]

Since Case 1 does not hold, we have \( 1 \in I \). Without loss of generality, we denote \( I = \left\{ 1, \ldots, j \right\} \).

By the similar argument of [10, Lemma 5.5], we get

**Lemma 5.5.** At \((x_0, t_0)\), for any \((0, 1/2]\), we have

\[ -(1 + \varepsilon) \sum_{i \in I} g^{\tilde{\alpha}}_i |e_i (\varphi V_\alpha V_1)|^2 \geq - \sum_{i \in I} g^{\tilde{\alpha}}_i - g^{\tilde{\alpha}}_i |e_i (|\partial \varphi|^2_{g_0})|^2 \]

Assume that \( \lambda_1(x_0, t_0) \geq C_D / \varepsilon^3 \), where \( D \) and \( \varepsilon \) will be chosen uniformly later. The similar arguments in [10, Lemma 5.6, Lemma 5.7, Lemma 5.8] yield

**Lemma 5.6.** At \((x_0, t_0)\), for any \((0, 1/6]\), we have

\[ (2 - \varepsilon) \sum_{\alpha > 1} g^{\tilde{\alpha}}_i \frac{|e_i (\varphi V_\alpha V_1)|^2}{\lambda_1 (\lambda_1 - \lambda_\alpha)} + g^{\tilde{\alpha}}_i g^{\tilde{\gamma}}_q |V_1 (g_p q)|^2 \lambda_1 - (1 + \varepsilon) \sum_{i \in I} g^{\tilde{\alpha}}_i |e_i (\varphi V_\alpha V_1)|^2 \]

\[ \geq -3 \varepsilon \sum_{i \in I} g^{\tilde{\alpha}}_i |e_i (\varphi V_\alpha V_1)|^2 \lambda_1^2 - C \varepsilon \sum_i g^{\tilde{\alpha}}_i. \]

Since \( \partial \tilde{H}(x_0, t_0) = 0 \), for any \( \varepsilon \in (0, 1/6] \), the Cauchy-Schwarz inequality yields

\[ -3 \varepsilon \sum_{i \in I} g^{\tilde{\alpha}}_i |e_i (\varphi V_\alpha V_1)|^2 \lambda_1^2 \]
An almost complex Chern-Ricci flow

Proof of Theorem 1.1.

In this section, using the priori estimates established earlier, we get the higher order priori estimates on both sides of (3.2), we get a solution to (1.2) on \([0, T]\). Taking \(\sqrt{-T\delta}\) on both sides of (3.2), we get a solution to (1.2) on \([0, T]\). Since \(T_0 < T\) is arbitrary, we get a solution to (1.2) on \([0, T]\). Uniqueness follows from the uniqueness of the solution to (3.2). As mentioned before, the flow (1.2) cannot extend beyond \(T\). \(\square\)

6. Proof of the maximum time existence theorem

In this section, using the priori estimates established earlier, we get the higher order priori estimates and complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By Theorem 5.1, there exists a uniform constant \(C > 0\) such that

\[
C^{-1} \omega_0 \leq \omega \leq C \omega_0.
\]

From the higher order estimates of [9] using the \(C^{2, \alpha}\) estimate of [8] (see also [50]), it follows that for each \(k = 0, 1, 2, \cdots\), there exists uniform constants \(C_k > 0\) such that

\[
\|\varphi(t)\|_{C^k(\omega_0)} \leq C_k.
\]

These uniform estimates on \([0, T_{\text{max}}]\) imply that we can take limits of \(\varphi(t)\) and get a solution on \([0, T_{\text{max}}]\). Applying the standard parabolic short time existence theory we get a solution a little beyond \(T_{\text{max}}\), a contradiction. Hence there exists a solution \(\varphi\) to (3.2) on \([0, T_0]\). Taking \(\sqrt{-T\delta}\) on both sides of (3.2), we get a solution to (1.2) on \([0, T_0]\). Since \(T_0 < T\) is arbitrary, we get a solution to (1.2) on \([0, T]\). Uniqueness follows from the uniqueness of the solution to (3.2). As mentioned before, the flow (1.2) cannot extend beyond \(T\). \(\square\)

7. Some convergence results

In this section, we consider some convergence results of the flow (1.2). Firstly, let us recall the case when there exists a volume form \(\Omega\) with \(\text{Ric}^{(1,1)}(\Omega) = 0\) considered by Chu [9]. The similar argument as in Lemma 3.1, yields that
Lemma 7.1 (Chu [9]; 2016). Assume that there exists a volume form $\Omega$ with $\text{Ric}^{(1,1)}(\Omega) = 0$. A smooth family $\omega(t)$ of almost Hermitian metrics on $[0, \infty)$ solves the flow (1.2) if and only if there is a family of smooth functions $\varphi(t)$ for $t \in [0, \infty)$ solves

$$\frac{\partial}{\partial t} \varphi = \log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi}{\Omega} \right)^n, \quad \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi > 0, \quad \varphi(0) = 0,$$

with $\omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$.

Chu [9] shows that $\varphi(t)$ converge to $\varphi_{\infty}$ smoothly as $t \to \infty$ and $\omega_{\infty} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{\infty} > 0$ with $\text{Ric}^{(1,1)}(\omega_{\infty}) = 0$.

If the complex structure $J$ is integrable, then this assumption is equivalent to the fact that $c^{BC}(M) = 0$, and the convergence result belongs to Gill [15] who used the crucial zero order estimate from [52].

Secondly, we assume that there exists a volume form $\Omega$ with $\text{Ric}^{(1,1)}(\Omega) < 0$, which is equivalent to the fact that $M$ is Kähler manifold with negative first Chern class if $J$ is integrable. By Theorem 1.1, the flow (1.2) exists for all the time and we denote by $\tilde{\omega}(s)$ its solution.

Suppose that $t = \log(s + 1)$ and $\omega = \frac{\tilde{\omega}}{s + 1}$. We get a new metric which solves

$$\frac{\partial}{\partial t} \omega = -\text{Ric}^{(1,1)}(\omega) - \omega, \quad \omega(0) = \omega_0,$$

for $t \in [0, \infty)$. We claim that the flow (7.1) is equivalent to the parabolic Monge-Ampère equation

$$\frac{\partial}{\partial t} \varphi = \log \left( \frac{\hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi}{\Omega} \right)^n - \varphi, \quad \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi > 0, \quad \varphi|_{t=0} = 0,$$

where $\hat{\omega} = -\text{Ric}^{(1,1)}(\Omega) + e^{-t} \left( \text{Ric}^{(1,1)}(\Omega) + \omega_0 \right)$ with

$$\frac{\partial}{\partial t} \hat{\omega} = -\text{Ric}^{(1,1)}(\Omega) - \hat{\omega}, \quad \hat{\omega}|_{t=0} = \omega_0.$$

Indeed, assume that $\omega$ is the solution to (7.1). By (7.1) and (7.3), we know that

$$\frac{\partial}{\partial t} (\omega - \hat{\omega}) = -(\omega - \hat{\omega}) + \sqrt{-1} \partial \bar{\partial} \log \frac{\omega^n}{\tilde{\omega}},$$

where $	ilde{\omega}$ is the solution to (7.1). We define $\varphi(t)$ by

$$\varphi(t) := e^{-t} \int_0^t e^s \log \frac{\omega(s)^n}{\Omega} ds,$$

for any $t \in [0, \infty)$. This function $\varphi(t)$ satisfies

$$\frac{\partial}{\partial t} \varphi = \log \frac{\omega^n}{\Omega} - \varphi, \quad \varphi(0) = 0.$$

Thanks to (7.4) and (7.5), we can deduce that

$$\frac{\partial}{\partial t} (e^t (\omega - \hat{\omega} - \sqrt{-1} \partial \bar{\partial} \varphi)) = 0, \quad (\omega - \hat{\omega} - \sqrt{-1} \partial \bar{\partial} \varphi)|_{t=0} = 0,$$

which implies $\omega = \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi > 0$ for all $t \geq 0$.

Conversely, assume that $\varphi$ is the solution to (7.2). Taking $\sqrt{-1} \partial \bar{\partial} \varphi$ on the both sides of (7.2), we know that $\omega = \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi$ is the solution to (7.1).
Note that $\hat{\omega}$ converge smoothly to $-\text{Ric}^{(1,1)}(\Omega) > 0$ as $t \to \infty$. Hence there exists a uniform constant $C_0 > 0$ such that

\begin{equation}
C_0^{-1} \omega_0 \leq \hat{\omega} \leq C_0 \omega_0.
\end{equation}

It is easy to see that Theorem 1.2 follows from

**Theorem 7.2.** Let $\varphi(t)$ be the solution to (7.2) on $M \times [0, \infty)$. Then $\varphi$ converge smoothly to $\varphi_\infty$ as $t \to \infty$, and we have

\begin{equation}
\text{Ric}^{(1,1)}(\varphi_\infty) = -\omega_\infty,
\end{equation}

where $\omega_\infty = -\text{Ric}^{(1,1)}(\Omega) + \sqrt{-1} \partial \bar{\partial} \varphi_\infty > 0$.

**Proof.** The proof consists of several steps as follows.

*Step 1:* We deduce the uniform estimates for $\varphi$ and $\dot{\varphi} := \frac{\partial}{\partial t} \varphi$ using the method from [3, 46, 57] (see also [53]). A simple maximum principle argument yields that $|\varphi| \leq C$ for a uniform constant $C > 0$ independent of $t$. A direct computation gives

\[
\left( \frac{\partial}{\partial t} - \Delta^C \right) \dot{\varphi} = -\dot{\varphi} - \text{tr}_\omega \left( \text{Ric}^{(1,1)}(\Omega) + \hat{\omega} \right)
= -\dot{\varphi} - n + \Delta^C \varphi - \text{tr}_\omega \left( \text{Ric}^{(1,1)}(\Omega) \right),
\]

i.e.,

\[
\left( \frac{\partial}{\partial t} - \Delta^C \right) (\dot{\varphi} + \varphi) = -n - \text{tr}_\omega \left( \text{Ric}^{(1,1)}(\Omega) \right).
\]

At the minimum $(x_0, t_0)$ of $\varphi + \dot{\varphi}$, without loss of generality, we can assume $t_0 > 0$, otherwise we can get the lower bound of $\dot{\varphi}$ directly. At $(x_0, t_0)$, we have $-\text{tr}_\omega \left( \text{Ric}^{(1,1)}(\Omega) \right) \leq n$. This, together with $\text{Ric}^{(1,1)}(\Omega) < 0$ and the arithmetic-geometric means inequality, yields

\[
\dot{\varphi} + \varphi = \log \frac{\omega^n}{\Omega} = \log \frac{\omega^n}{-\text{Ric}^{(1,1)}(\Omega)^n} + \log \frac{-\text{Ric}^{(1,1)}(\Omega)^n}{\Omega} \geq \log \left( \frac{-\text{Ric}^{(1,1)}(\Omega)^n}{\Omega} \right) \geq -C
\]

at this point and hence everywhere. Since $|\varphi| \leq C$, we get $\dot{\varphi} \geq -C$.

Since $e^t \left( \text{Ric}^{(1,1)}(\Omega) + \hat{\omega} \right) = \text{Ric}^{(1,1)}(\Omega) + \omega_0$, a direct computation yields

\[
\left( \frac{\partial}{\partial t} - \Delta^C \right) (\dot{\varphi} + \varphi + nt - e^t \dot{\varphi}) = \text{tr}_\omega \omega_0 > 0.
\]

This, together with the maximum principle, implies that there holds $\dot{\varphi} \leq Cte^{-t} \leq Ce^{-t/2}$ for $t \geq 1$. The uniform upper bound of $\varphi + \dot{\varphi} = \log \frac{\omega^n}{\Omega}$ and the arithmetic-geometric means inequality yield that

\begin{equation}
\text{tr}_\omega \omega_0 \geq n \left( \frac{\omega_0}{\omega^n} \right)^{1/n} = n \left( \frac{\Omega}{\omega_0} \frac{\omega_0}{\omega^n} \right)^{1/n} \geq c,
\end{equation}

for a uniform constant $C > 0$.

Given (7.6) and (7.8), we can deduce the first and second order estimates.
Step 2: We need the estimate \( \sup_{M \times [0, \infty)} |\partial \varphi|_{g_0} \leq C \) for a uniform constant \( C > 0 \). To see this, we just need to follow the proof of Theorem 4.1 word for word except replacing (4.5) by
\[
\frac{\partial}{\partial t} |\partial \varphi|^2_{g_0} = e_k(\varphi) \overline{e}_k(\varphi) + e_k(\varphi) \overline{e}_k(\varphi) \\
= g^i \overline{e}_k(g_{ji}) \varphi_k(\varphi) - e_k(\varphi) (\log \Omega) \varphi_k(\varphi) + g^i \overline{e}_k(g_{ji}) - e_k(\varphi) \overline{e}_k(\log \Omega),
\]
In the process of calculation of \((\Delta^C - \frac{\partial}{\partial t}) |\partial \varphi|^2_{g_0}\), the term \(-2e_k(\varphi) \overline{e}_k(\varphi)\) is harmless. Then all other arguments are the same and hence we get the first order estimate.

Step 3: We need the third order estimate \( \sup_{M \times [0, \infty)} |\nabla^2 \varphi|_{g_0} \leq C \), where \( \nabla^2 \varphi \) is the Hessian with respect to the Levi-Civita connection of \( g_0 \). To see this, we still need to follow the proof of Theorem 5.1 word for word except replacing (5.15) and (5.16) by
\[
g^i \overline{V}_i V_i g_{ji} = (\nabla V_i V_i) (\varphi) + (\nabla V_i V_i) (\log \Omega) + (\nabla V_i V_i) (\varphi).
\]
and
\[
g^i V_i g_{ji} = g^{ip} g^{jq} |V_i (g_{ip})|^2 + V_i V_i (\varphi) + V_i V_i (\log \Omega) + V_i V_i (\varphi).
\]
Note that the new term \( V_i V_i (\varphi) - (\nabla V_i V_i) (\varphi) = \varphi V_i V_i \) is harmless and absorbed by \( G \), as required.

Step 4: The higher order estimates follows from [9].

Step 5: Convergence result. The second order estimate implies that
\[
C^{-1} \omega_0 \leq \omega \leq C \omega_0
\]
for a uniform constant \( C > 0 \). This yields that
\[
\left( \frac{\partial}{\partial t} - \Delta^C \right) (e^t \varphi) = -\text{tr}_\omega \left( \text{Ric}^{(1,1)}(\Omega) + \omega_0 \right) \geq -C.
\]
This, together with the maximum principle, implies that \( \varphi \geq -(1 + t)e^{-t} \geq -Ce^{-t/2} \).

We can deduce that \( \varphi \) converge uniformly to zero exponentially fast, and hence \( \varphi \) converge uniformly exponentially fast to a continuous limit function \( \varphi_\infty \). From the higher order uniform estimates of \( \varphi \), it follows that \( \varphi_\infty \) is smooth and
\[
\varphi \rightarrow \varphi_\infty
\]
in smooth topology. Therefore, by passing to the limits in (7.5), we can deduce that the limit metric \( \omega_\infty = -\text{Ric}^{(1,1)}(\Omega) + \sqrt{-1} \partial \partial \varphi_\infty \) satisfies
\[
\log \frac{\omega_\infty^n}{\Omega} = \varphi_\infty.
\]
Taking \( \sqrt{-1} \partial \partial \varphi_\infty \) on the both sides of this equation, we get
\[
\text{Ric}^{(1,1)}(\omega_\infty) = -\omega_\infty,
\]
as required.

Step 6: We prove that \( \omega_\infty \) is independent of the initial metric \( \omega_0 \). Assume that the normalized flow (7.1) starts at another almost Hermitian metric \( \omega'_0 \). The same argument as above implies that there exists a smooth function \( \varphi'_\infty \) such that
\[
(-\text{Ric}^{(1,1)}(\Omega) + \sqrt{-1} \partial \partial \varphi'_\infty)^n = e^{\varphi'_\infty} \Omega, \quad \omega'_\infty := -\text{Ric}^{(1,1)}(\Omega) + \sqrt{-1} \partial \partial \varphi'_\infty > 0.
\]
Thanks to (7.9) and (7.10), we get
\[
\omega_\infty' = (\omega_\infty + \sqrt{-1} \partial \bar{\partial} \phi)^n = e^{\phi} \omega_\infty^n,
\]
where we denote \(\phi := \varphi'_\infty - \varphi_\infty\). Applying the maximum principle to (7.11) shows that \(\phi \equiv 0\), i.e.,
\[
\omega'_\infty = \omega_\infty,
\]
as desired. \(\square\)

8. An example

In this section, as an example, we study the flow (1.2) on the (locally) homogeneous manifolds in more detail. We note that the flow (1.2) can be seen as a \((p,q)\) flow given in [22] and hence we can use the method in [22].

Let \((M,J,\omega_0)\) be a compact almost Hermitian manifold whose universal cover is a Lie group \(G\) such that if \(\pi : G \rightarrow M\) is the covering map, then \(\pi^* \omega_0\) and \(\pi^* J\) are left invariant. For example, we can take \(M = G/\Gamma\), where \(\Gamma\) is a compact discrete subgroup of \(G\), including solvmanifolds and nilmanifolds. Then the solution to (1.2) on \(M\) is obtained by pulling down the corresponding flow solution on the Lie group \(G\), which stays left invariant. The flow (1.2) on \(G\) becomes an ordinary differential equation for a family of almost Hermitian metrics \(\omega(t)\) with respect to the fixed complex structure \(J\) on the Lie algebra \(\mathfrak{g}\) of the Lie group \(G\).

It is sufficient to consider the flow (1.2) on Lie group \(G\). Since the Chern-Ricci form \(p\) of a left invariant almost Hermitian metric \(\omega\) defined by
\[
p(X,Y) = -\frac{1}{2} \text{tr} J \text{ad}[X,Y] + \frac{1}{2} \text{tr} \text{ad} J[X,Y], \quad \forall X,Y \in \mathfrak{g}
\]
depends only on \(J\) (see for example [29] or [60, Proposition 4.2]), the flow (1.2) becomes
\[
\frac{\partial}{\partial t} \omega(t) = -2p^{(1,1)}, \quad \omega(0) = \omega_0
\]
with solution \(\omega(t)\) given by
\[
\omega_t := \omega(t) = \omega_0 - 2tp^{(1,1)}.
\]
We define
\[
p^{(1,1)} = \omega_0(P_0 \cdot, \cdot) = \omega_t(P(t) \cdot, \cdot),
\]
which implies that
\[
P_t := P(t) = (\text{Id} - 2tP_0)^{-1} P_0.
\]
We call \(P_0\) (resp. \(P_t\)) the Chern-Ricci operator of \(\omega_0\) (resp. \(\omega_t\)). It follows that the maximal existence time \(T\) is given by
\[
T = \begin{cases} 
\infty, & \text{if } P_0 \leq 0, \\
1/(2p_+), & \text{otherwise},
\end{cases}
\]
where \(p_+\) is the maximal positive eigenvalue of the Chern-Ricci operator \(P_0\) of \(\omega_0\).

Let \(p_1, \cdots, p_{2n}\) of eigenvalues of \(P_0\) with the orthonormal basis \(e_1, \cdots, e_{2n}\), i.e.,
\[
\omega_0(e_i, e_j) = \delta_{ij}, \quad P_0(e_\alpha) = p_\alpha e_\alpha, \quad \alpha = 1, \cdots, 2n.
\]
A direct calculation yields that the scalar curvature \(R(\omega_t)\) of \(\omega_t\) is given by
\[
R(\omega_t) = \text{tr}_{\omega_t} p^{(1,1)} = \text{tr} P_t = \sum_{\alpha=1}^{2n} \frac{p_\alpha}{1 - 2tp_\alpha}.
\]
From (8.3), it follows that
\[
\frac{d}{dt} R(\omega_t) = 2 \sum_{\alpha=1}^{2n} \frac{2p_\alpha^2}{(1 - 2tp_\alpha)^2} \geq 0.
\]
This implies that \( R(\omega_t) \) is strictly increasing unless \( P_t \equiv 0 \), i.e., \( \omega_t \equiv \omega_0 \), and that the Chern scalar curvature must blow up in finite singularities, i.e., if \( T < \infty \), then there holds
\[
\int_0^T R(\omega_t) dt = \infty.
\]
Also we note that
\[
R(\omega_t) \leq \frac{C}{T - t},
\]
for a uniform constant \( C > 0 \). We remark that (8.4) is the claim of [37, Conjecture 7.7] for the Kähler-Ricci flow on general compact Kähler manifolds (see [17] for the Chern-Ricci flow on general compact Hermitian manifolds).

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An almost complex Chern-Ricci flow

Tao Zheng

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