Canonical Gauge Equivalences of the sAKNS and sTB Hierarchies

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Abstract

We study the gauge transformations between the supersymmetric AKNS (sAKNS) and supersymmetric two-boson (sTB) hierarchies. The Hamiltonian nature of these gauge transformations is investigated, which turns out to be canonical. We also obtain the Darboux-Bäcklund transformations for the sAKNS hierarchy from these gauge transformations.
I. INTRODUCTION

During the past ten years, the theory of soliton [1–3] has played an important role in theoretical and mathematical physics. Especially, the explorations on the relationship between the integrable models and string theories [4]. On the one hand, several kinds of correlation functions in string theory are governed by the integrable hierarchy equations (e.g. KdV, KP etc.) [4]. On the other hand, the idea of the supersymmetric extensions of the integrable systems [5–7] has motivated people to use them to study the theory of superstrings [8].

Recently, several supersymmetric integrable systems have been proposed and studied (see, for example, [9–17] and references therein). In this paper, we discuss only two of them; the supersymmetric AKNS (sAKNS) hierarchy [13] and the supersymmetric two-boson (sTB) hierarchy [11]. The former was introduced from the study of the reduction scheme in the constrained KP hierarchy [18], and the latter was constructed from the supersymmetric extension of the dispersive long water wave equation [19,20]. Both of them have supersymmetric Lax representations, being bi-Hamiltonian, and have infinite conserved quantities etc. Besides of these properties, these two hierarchies can be related to each other via a gauge transformation [13]. Sometimes, such transformation from one hierarchy to the other is called Miura transformation. However, from our viewpoint, the connection between these two hierarchies has not been totally explored. The purpose of this work is to provide a deeper understanding about the gauge transformations between the sAKNS and the sTB hierarchies.

Our paper is organized as follows: After the introduction of the Lax formulation of the sAKNS hierarchy in Sec. II, we discuss the gauge transformations between the sAKNS and the sTB hierarchies in Sec. III. Sec. IV is devoted to investigate the canonical property of these gauge transformations from the bi-Hamiltonian view point. Our approach follows very closely that of Refs. [21,22] for other systems. We then show, in Sec. V, that the Darboux-Bäcklund transformations (DBTs) for the sAKNS hierarchy itself can be constructed from these gauge transformations. Concluding remarks are presented in Sec. VI.

II. SAKNS HIERARCHY

The sAKNS hierarchy [13] has the Lax operator of the form

\[ L = \partial + \Phi D^{-1} \Psi \]  

(2.1)

which satisfies the hierarchy equations

\[ \frac{\partial L}{\partial t_n} = [L^n, L] \]  

(2.2)

where \( D = \partial_\theta + \theta \partial \) is the supercovariant derivative defined on a (1|1) superspace [23] with coordinates \((x, \theta)\). \( D^{-1} = \theta + \partial_\theta \partial \) is the formal inverse of \( D \), which satisfies \( D^{-1}D = D^{-1}D = 1 \). The coefficients functions \( \Phi \) and \( \Psi \) are superfields with proper parity such that \( L \) is a bosonic operator. It can be proved that (2.2) is consistent with the following equations

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where the conjugate operation \( \ast \) is defined by \((AB)^\ast = (-1)^{|A||B|}B^\ast A^\ast \) for the super-pseudo-differential operator \( A, B \) and \( f^\ast = f \) for arbitrary superfield \( f \). Therefore \( \Phi \) and \( \Psi \) are eigenfunction and adjoint eigenfunction of the hierarchy, respectively.

Since the Lax operator (2.1) is assumed to be homogeneous under \( Z_2 \)-grading, the gradings of the (adjoint) eigenfunction should satisfy \(|\Phi| + |\Psi| = 1\). Here we refer the parity of a superfield \( f \) to be even if \(|f| = 0\) and odd if \(|f| = 1\). There are two cases to be discussed.

(a) \(|\Phi| = 0 \) and \(|\Psi| = 1\)

In this case, the parity of the eigenfunction \( \Phi \) is even, whereas the adjoint eigenfunction \( \Psi \) is odd. Therefore we can parametrize them as

\[
\Phi = \phi_1 + \theta \psi_1, \quad \Psi = \psi_2 + \theta \phi_2.
\]  

Substituting (2.4) into (2.3), we obtain, for example, the \( t_2 \)-flow as

\[
\begin{align*}
\frac{\partial \phi_1}{\partial t_2} &= \phi_{1xx} + 2\phi_1^2\phi_2 + 2\phi_1\psi_1\psi_2 \\
\frac{\partial \psi_1}{\partial t_2} &= \psi_{1xx} + 2\phi_1^2\psi_{2x} + 2\phi_1\phi_2\psi_1 + 2\phi_1\phi_1\psi_2 \\
\frac{\partial \phi_2}{\partial t_2} &= -\phi_{2xx} - 2\phi_1\phi_2 - 2\phi_2\psi_1\psi_2 + 2\phi_1\psi_2\psi_{2x} \\
\frac{\partial \psi_2}{\partial t_2} &= -\psi_{2xx} - 2\phi_1\phi_2\psi_2.
\end{align*}
\]  

(2.5)

In the bosonic limit \((\phi_1, \phi_2 \to 0)\) the AKNS equations \((t_2\text{-flow})\) are recovered. In fact, this can be easily viewed from the bosonic limit of the Lax operator \( L_{\text{bosonic}} = \partial + \phi_1 \partial^{-1} \phi_2 \).

(b) \(|\Phi| = 1 \) and \(|\Psi| = 0\)

In this case, we have odd eigenfunction \( \Phi \) and even adjoint eigenfunction \( \Psi \). Hence, they should be parametrized as

\[
\Phi = \psi_1 + \theta \phi_1, \quad \Psi = \phi_2 + \theta \psi_2
\]  

(2.6)

The hierarchy equations (2.3)(e.g. \( t_2 \)-flow) then become

\[
\begin{align*}
\frac{\partial \phi_1}{\partial t_2} &= \phi_{1xx} + 2\phi_1^2\phi_2 - 2\phi_1\psi_1\psi_2 - 2\phi_2\psi_1\phi_1 \\
\frac{\partial \psi_1}{\partial t_2} &= \psi_{1xx} + 2\phi_1\phi_2\psi_1 \\
\frac{\partial \phi_2}{\partial t_2} &= -\phi_{2xx} - 2\phi_1\phi_2^2 + 2\phi_2\psi_1\psi_2 \\
\frac{\partial \psi_2}{\partial t_2} &= -\psi_{2xx} - 2\phi_2^2\psi_{1x} - 2\phi_2\phi_2\psi_1.
\end{align*}
\]  

(2.7)

which also contain the ordinary AKNS equations in the bosonic limit. However, we want to point out that the Lax operator has no direct bosonic limit in this case.
It has been shown [13] that the hierarchy equations (2.2) are invariant under the supersymmetric transformations:

$$\delta_{\epsilon} \Phi = \epsilon (D^\dagger \Phi), \quad \delta_{\epsilon} \Psi = \epsilon (D^\dagger \Psi)$$

(2.8)

where $\epsilon$ is an odd constant and $D^\dagger \equiv \partial_{\theta} - \theta \partial$. In particular, using the parametrizations (2.4) and (2.6), it is straightforward to show that (2.5) and (2.7) are invariant under the transformations (2.8). In the following sections, the sAKNS Lax operators for the case (a) and case (b) will be denoted by $L_a = \partial + \Phi_a D^{-1} \Psi_a$ and $L_b = \partial + \Phi_b D^{-1} \Psi_b$, respectively and thus $|\Phi_a| = |\Psi_b| = 0$ and $|\Psi_a| = |\Phi_b| = 1$.

III. GAUGE TRANSFORMATIONS AND STB HIERARCHY

Given a sAKNS hierarchy we can construct a nonstandard Lax hierarchy via a gauge transformation. For case (a), let us perform the following transformation

$$G_a : \quad L_a \rightarrow K = \Phi_a^{-1} L_a \Phi_a$$

$$\equiv \partial - (DJ_0) + D^{-1}J_1$$

(3.1)

where both $J_0$ and $J_1$ are odd superfields which can be expressed in terms of $\Phi_a$ and $\Psi_a$ as follows

$$J_0 = -(D \ln \Phi_a), \quad J_1 = \Phi_a \Psi_a.$$  

(3.2)

The hierarchy equations then become

$$\frac{\partial K}{\partial t_n} = [K_{n \geq 1}, K]$$

(3.3)

which is the so-called sTB hierarchy [11]. The first few hierarchy equations are given by

$$\frac{\partial J_0}{\partial t_1} = J_{0x}, \quad \frac{\partial J_1}{\partial t_1} = J_{1x}, \quad \frac{\partial J_0}{\partial t_2} = J_{0xx} - 2J_{1x} - (D(DJ_0)^2), \quad \frac{\partial J_1}{\partial t_2} = -J_{1xx} - 2(J_1(DJ_0))_{xx}$$

(3.4)

e tc. It is easy to show that (3.4) contain the Kaup-Broer hierarchy equations [19, 20] in the bosonic limit. Substituting (3.2) into (3.4) and using the parametrizations (2.4), one can recover the equations (2.5). Moreover, the hierarchy equations (3.3) have been shown [11] to be invariant under the supersymmetric transformations: $\delta_{\epsilon} J_0 = \epsilon (D^\dagger J_0), \delta_{\epsilon} J_1 = \epsilon (D^\dagger J_1)$.

For case (b), we need another gauge transformation to do the job since $|\Phi_b| = 1$ in this case. Let us consider the following transformation
\[ G_b: \quad L_B \rightarrow K = D^{-1}\Psi_b L_b \Psi_b^{-1} D \equiv \partial - (DJ_0) + D^{-1}J_1 \]  

which implies 

\[ J_0 = (D \ln \Psi_b), \quad J_1 = \Phi_b \Psi_b + (D^3 \ln \Psi_b) \]  

and the Lax operator \( K \) still satisfying the hierarchy equations (3.3). Substituting (3.6) into (3.4) and using (2.6), we obtain (2.7) immediately.

In fact, both gauge transformations \( G_a \) and \( G_b \) have their inverse transformations \( H_a \) and \( H_b \), respectively. In other words, for a given sTB Lax operator \( K \), one can perform the following transformation to gauge away the constant term and to obtain the Lax operator \( L_a \):

\[ H_a: K \rightarrow L_a = e^{-\int x(DJ_0)} Ke^{\int x(DJ_0)} \equiv \partial + \Phi_a D^{-1} \Psi_a \]  

where 

\[ \Phi_a = e^{-\int x(DJ_0)}, \quad \Psi_a = J_1 e^{\int x(DJ_0)}. \]

It can be proved that \( L_a \) satisfies (2.2) if \( K \) satisfies (3.3).

Similarly, for case (b), we have

\[ H_b: K \rightarrow L_b = e^{-\int x(DJ_0)} DK D^{-1} e^{\int x(DJ_0)} \equiv \partial + \Phi_b D^{-1} \Psi_b \]  

where 

\[ \Phi_b = (J_1 - J_{0x}) e^{-\int x(DJ_0)}, \quad \Psi_b = e^{\int x(DJ_0)}. \]

Since \( H_a(H_b) \) is the inverse of \( G_a(G_b) \) and vice versa, thus we obtain the correspondences between the sAKNS and sTB hierarchies.

IV. CANONICAL PROPERTY AND HAMILTONIAN STRUCTURES

The discussions presented in the previous section establish the gauge equivalences between the sAKNS and the sTB hierarchies at Lax formulation level. In this section, we would like to discuss the Hamiltonian nature of these gauge transformations. Let us start from the sTB hierarchy.

The Lax equation (3.3) of the sTB hierarchy has a bi-Hamiltonian description as follows

\[ \partial_t \begin{pmatrix} J_0 \\ J_1 \end{pmatrix} = \Theta_1 \begin{pmatrix} \delta H_{n+1} \\ \delta J_0 \\ \delta J_1 \end{pmatrix} = \Theta_2 \begin{pmatrix} \delta H_n \\ \delta J_0 \\ \delta J_1 \end{pmatrix} \]

where the first structure \( \Theta_1 \) and the second structure \( \Theta_2 \) are given by [11]
\[ \Theta_1 = \begin{pmatrix} 0 & -D \\ -D & 0 \end{pmatrix} \]
\[ \Theta_2 = \begin{pmatrix} 2D + 2D^{-1} J_1 D^{-1} - D^{-1} J_0 D^{-1} - D^3 + D(D J_0) - D^{-1} J_1 D \\ D^3 + (D J_0) D + D J_1 D^{-1} J_1 D^2 + D^2 J_1 \end{pmatrix} \]  \hspace{1cm} (4.2)

which have been investigated [11] to be compatible by using the prolongation method [24]. The Hamiltonians \( H_n \) are defined by
\[ H_n = -\frac{1}{n} \text{str} K^n \equiv -\frac{1}{n} \int dx d\theta \text{res} K^n \]  \hspace{1cm} (4.4)

where the res picks up the coefficient of the \( D^{-1} \) term of a super-pseudo-differential operator.

Since the bi-Hamiltonian structure is one of the most important properties of an integrable system, it is quite natural to ask whether the gauge transformations discussed above are canonical or not. To see this, from the gauge transformation \( H_a \), we can obtain the linearized map \( H'_a \) and its transposed map \( H'^{\dagger}_a \) as follows
\[ H'_a = \begin{pmatrix} -\Phi_a D^{-1} & 0 \\ \Psi_a D^{-1} & \Phi_a^{-1} \end{pmatrix}, \quad H'^{\dagger}_a = \begin{pmatrix} D^{-1} \Phi_a & -D^{-1} \Psi_a \\ 0 & \Phi_a^{-1} \end{pmatrix} \]  \hspace{1cm} (4.5)

where \( \Phi_a \) and \( \Psi_a \) are related to \( J_0 \) and \( J_1 \) via Eq. (3.6) (or Eq. (3.8)). A straightforward calculation shows that

\[ H'_a \Theta_1 H'^{\dagger}_a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv P_a \]  \hspace{1cm} (4.6)

\[ H'_a \Theta_2 H'^{\dagger}_a = \begin{pmatrix} -\Phi_a D^{-2} \Phi_a D - D \Phi_a D^{-2} \Phi_a & D^2 + D \Phi_a D^{-2} \Psi_a + \Phi_a D^{-2} (D \Psi_a) \\ -2 \Phi_a D^{-2} \Phi_a \Psi_a D^{-2} \Phi_a & +2 \Phi_a D^{-2} \Phi_a \Psi_a D^{-2} \Psi_a \\ D^2 + \Psi_a D^{-2} \Phi_a D + (D \Psi_a) D^{-2} \Phi_a & -\Psi_a D^{-2} (D \Psi_a) - (D \Psi_a) D^{-2} \Psi_a \\ +2 \Psi_a D^{-2} \Phi_a \Psi_a D^{-2} \Phi_a & -2 \Psi_a D^{-2} \Phi_a \Psi_a D^{-2} \Psi_a \end{pmatrix} \equiv Q_a \]  \hspace{1cm} (4.7)

where \( P_a \) and \( Q_a \) are just the first and the second Hamiltonian structures obtained in [14]. Moreover, it has been shown [14] that \( P_a \) and \( Q_a \) are compatible through the method of prolongation and describe the hierarchy equations (2.2) as follows

\[ \partial_{t_n} \left( \begin{array}{c} \Phi_a \\ \Psi_a \end{array} \right) = P_a \left( \begin{array}{c} \frac{\delta H_{n+1}}{\delta \Phi_n} \\ \frac{\delta H_{n+1}}{\delta \Psi_n} \end{array} \right) = Q_a \left( \begin{array}{c} \frac{\delta H_n}{\delta \Phi_n} \\ \frac{\delta H_n}{\delta \Psi_n} \end{array} \right) \]  \hspace{1cm} (4.8)

where the Hamiltonian \( H_n \) are defined by \( H_n = -\frac{1}{n} \text{str} L_a^n \). Hence, the gauge transformation \( H_a \) (or \( G_a \)) is a canonical map.

Next, let us turn to the gauge transformation \( H_b \). From (3.9), the linearized map \( H'_b \) and its transposed map \( H'^{\dagger}_b \) can be constructed as follows
\[ H'_b = \begin{pmatrix} -\Phi_b D^{-1} - \Psi_b^{-1} \partial \Psi_b^{-1} \\ \Psi_b D^{-1} \end{pmatrix}, \quad H'^{\dagger}_b = \begin{pmatrix} \partial \Psi_b^{-1} + D^{-1} \Phi_b - D^{-1} \Psi_b \\ \Psi_b^{-1} \end{pmatrix} \]  \hspace{1cm} (4.9)

where \( \Phi_b \) and \( \Psi_b \) are related to \( J_0 \) and \( J_1 \) via (3.6) (or (3.10)). Using (4.3), we can obtain two Poisson structures of the sAKNS hierarchy for the case (b). After some algebras, we have
\[
H_b' \Theta_1 H_b^\dagger = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv -P_b \tag{4.10}
\]

\[
H_b' \Theta_2 H_b^\dagger = \begin{pmatrix} -\Phi_b D^{-2} (D \Phi_b) - (D \Phi_b) D^{-2} \Phi_b & D^2 + \Phi_b D^{-2} \Psi_b D + (D \Phi_b) D^{-2} \Psi_b \\ -2 \Phi_b D^{-2} \Phi_b \Psi_b D^{-2} \Phi_b & 2 \Phi_b D^{-2} \Phi_b \Psi_b D^{-2} \Psi_b \end{pmatrix} \equiv -Q_b \tag{4.11}
\]

which imply that the hierarchy equations (2.2) for case (b) can be written as

\[
\partial_t \begin{pmatrix} \Phi_b \\ \Psi_b \end{pmatrix} = P_b \begin{pmatrix} \frac{\delta H_{n+1}}{\delta \Phi_b} \\ \frac{\delta H_{n+1}}{\delta \Psi_b} \end{pmatrix} = Q_b \begin{pmatrix} \frac{\delta H_n}{\delta \Phi_b} \\ \frac{\delta H_n}{\delta \Psi_b} \end{pmatrix}. \tag{4.12}
\]

Note that the minus sign appearing in the front of \(P_b\) and \(Q_b\) in (4.10) and (4.11) is due to the fact that the parity of the gauge operator of the gauge transformation \(H_b\) is odd. Therefore, by the identity \(\text{str} AB = (-1)^{|A||B|} \text{str} BA\), the Hamiltonians \(H_n\) in (4.12) are equal to the minus ones for the sTB hierarchy. We follow the same line in [13] to investigate the Jacobi-identity for \(P_b\) and \(Q_b\) by using the prolongation method. It turns out that \(P_b\) and \(Q_b\) are compatible and indeed define a bi-Hamiltonian structure of the associated hierarchy. Hence, just like \(H_a\), the gauge transformation \(H_b\) is canonical as well.

To sum up, the canonical property of the gauge transformations between the sAKNS and sTB hierarchies can be summarized as follows

\[
H_i' \Theta_1 H_i^\dagger = (-1)^{|H_i|} P_i, \quad H_i' \Theta_2 H_i^\dagger = (-1)^{|H_i|} Q_i, \quad i = a, b. \tag{4.13}
\]

V. DARBOUX-BÄCKLUND TRANSFORMATIONS

Having constructed the canonical gauge transformations between the sAKNS and sTB hierarchies, now we would like to use these gauge transformations to derive the DBTs for the sAKNS hierarchy itself.

Given a sAKNS Lax operator, say \(L_a\), we can perform the gauge transformation \(G_a\) followed by \(H_b\) to obtain the Lax operator \(L_b\) as follows

\[
L_a \xrightarrow{G_a} K \xrightarrow{H_b} L_b \tag{5.1}
\]

That is, using (3.1) and (3.3), we can define the gauge operator \(T(\Phi_a) = \Phi_a D \Phi_a^{-1}\) such that

\[
L_a \rightarrow L_b = TL_a T^{-1} \equiv \partial + \Phi_b D^{-1} \Psi_b \tag{5.2}
\]

where the (adjoint) eigenfunctions are related by

\[
\Phi_b = \Phi_a (\Phi_a \Psi_a + (D^3 \ln \Phi_a)) \tag{5.3}
\]

\[
\Psi_b = \Phi_a^{-1} \tag{5.4}
\]
Notice that although the gauge transformation (5.3) preserves the form of the Lax operator and the Lax formulations, however, the parity of the transformed (adjoint) eigenfunction has been changed due to the fact that the parity of the gauge operator $S_{ab}$ is odd. Thus, strictly speaking, the gauge transformation (5.3) is not a DBT but “quasi-DBT”.

On the other hand, we can construct another quasi-DBT from $L_b$ to $L_a$ as follows

$$L_b \xrightarrow{G_b} K \xrightarrow{H_a} L_a$$

which is triggered by the gauge operator $S(\Psi_b) = \Psi_b^{-1}D^{-1}\Psi_b$ such that

$$L_b \rightarrow L_a = SL_bS^{-1} \equiv \partial + \Phi_aD^{-1}\Psi_b$$

where

$$\Phi_a = \Psi_b^{-1} \quad (5.7)$$

$$\Psi_a = \Phi_b(\Phi_b\Psi_b + (D^3\ln\Psi_b)). \quad (5.8)$$

Note that both quasi-DBTs (5.3) and (5.6) are canonical since they are constructed out from the canonical transformations $G_i$ and $H_i$. We also remark that the form of the gauge operator $T$ was first considered in [25] for studying the DBT for the Manin-Radul super KdV equation [5].

Motivated by the above discussions, we may have true DBTs by considering the hierarchy equations (2.2) associated with the Lax operator

$$L = \partial + \Phi_1D^{-1}\Psi_1 + \Phi_2D^{-1}\Psi_2$$

with parity $|\Phi_1| = |\Psi_2| = 0$ and $|\Psi_1| = |\Phi_2| = 1$. Let us consider the DBT triggered by the eigenfunction $\Phi_1$ as follows

$$L \rightarrow \hat{L} = TLT^{-1}, \quad T(\Phi_1) \equiv \Phi_1D\Phi_1^{-1} \quad (5.10)$$

$$\equiv \partial + \hat{\Phi}_1D^{-1}\hat{\Psi}_1 + \hat{\Phi}_2D^{-1}\hat{\Psi}_2$$

where the transformed (adjoint) eigenfunctions are given by

$$\hat{\Phi}_1 = \Phi_1(D\Phi_1^{-1}\Phi_2) = (T(\Phi_1)\Phi_2)$$

$$\hat{\Psi}_1 = \Phi_1^{-1}(D^{-1}\Phi_1\Psi_2) = (S(\Phi_1)\Psi_2)$$

$$\hat{\Phi}_2 = \Phi_1(\Phi_1\Psi_1 - \Phi_2\Psi_2 + (D^3\ln\Phi_1) + (D\Phi_1^{-1}\Phi_2)(D^{-1}\Phi_1\Psi_2)) = (T(\Phi_1)L\Phi_1)$$

$$\hat{\Psi}_2 = \Phi_1^{-1} \quad (5.11)$$

with parity $|\hat{\Phi}_1| = |\hat{\Psi}_2| = 0$ and $|\hat{\Psi}_1| = |\hat{\Phi}_2| = 1$.

Moreover, we can consider the DBT triggered by the adjoint eigenfunction $\Psi_2$ as follows

$$L \rightarrow \hat{L} = SLS^{-1}, \quad S(\Psi_2) \equiv \Psi_2^{-1}D^{-1}\Psi_2 \quad (5.12)$$

$$\equiv \partial + \hat{\Phi}_1D^{-1}\hat{\Psi}_1 + \hat{\Phi}_2D^{-1}\hat{\Psi}_2$$

where
\begin{align*}
\hat{\Phi}_1 &= \Psi_2^{-1} \\
\hat{\Psi}_1 &= (\Phi_2 \Psi_2 - \Phi_1 \Psi_1 + (D^3 \ln \Psi_2) + (D^{-1} \Psi_2 \Phi_1)(D \Psi_1 \Psi_2^{-1}))\Psi_2 = - (T(\Psi_2)L^*\Psi_2) \\
\hat{\Phi}_2 &= \Psi_1^{-1}(D^{-1} \Psi_2 \Phi_1) = (S(\Psi_2) \Phi_1) \\
\hat{\Psi}_2 &= \Psi_2(D \Psi_1^{-1} \Psi_1) = (T(\Psi_2) \Psi_1) 
\end{align*}

with parity \(|\hat{\Phi}_1| = |\hat{\Psi}_2| = 0\) and \(|\hat{\Psi}_1| = |\hat{\Phi}_2| = 1\).

Finally, we would like to mention that the above scheme can be generalized to a class of
supersymmetric hierarchies which have Lax operators of the form

\begin{equation}
L = \partial + \sum_{i=1}^{n} (\Phi_{2i-1} D^{-1} \Psi_{2i-1} + \Phi_{2i} D^{-1} \Psi_{2i}), \quad (n \geq 1)
\end{equation}

with parity \(|\Phi_{2i-1}| = |\Psi_{2i}| = 0\) and \(|\Phi_{2i}| = |\Psi_{2i-1}| = 1\). The gauge operators of the DBTs
then can be constructed from the even (adjoint) eigenfunctions as
\[ T_i = \Phi_{2i-1} D \Phi_{2i-1}^{-1} \] or
\[ S_i = \Psi_{2i}^{-1} D^{-1} \Psi_{2i} \] which not only preserve the Lax formulations but also the parity content
of the (adjoint) eigenfunctions in the Lax operator.

**VI. CONCLUDING REMARKS**

We have established the gauge equivalences between the sAKNS and sTB hierarchies. We have also shown that the gauge transformations connecting these two hierarchies are canonical, in the sense that the bi-Hamiltonian structure of the sAKNS hierarchy is mapped to the bi-Hamiltonian structure of the sTB hierarchy according to Eq. (4.13). Using these
gauge transformations, the (quasi) DBTs for the sAKNS hierarchy and its generalizations
can be constructed, which turns out to be canonical as well. Some other topics such as
iterated DBTs, soliton solutions and nonlocal conserved charges of these hierarchies are
worth further investigation [26]. We leave this work to a future publication.

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