Approximations in Bayesian Belief Universes for Knowledge-Based Systems

Frank Jensen & Stig Kjær Andersen
Institute of Electronic Systems
Aalborg University
Fr. Bajersvej 7, DK-9220 Aalborg Ø, Denmark

Abstract
When expert systems based on causal probabilistic networks (CPNs) reach a certain size and complexity, the "combinatorial explosion monster" tends to be present. We propose an approximation scheme that identifies rarely occurring cases and excludes these from being processed as ordinary cases in a CPN-based expert system. Depending on the topology and the probability distributions of the CPN, the numbers (representing probabilities of state combinations) in the underlying numerical representation can become very small. Annihilating these numbers and utilizing the resulting sparseness through data structuring techniques often results in several orders of magnitude of improvement in the consumption of computer resources. Bounds on the errors introduced into a CPN-based expert system through approximations are established. Finally, reports on empirical studies of applying the approximation scheme to a real-world CPN are given.

Keywords: Approximative reasoning, belief network, causal probabilistic network, expert system, knowledge-based system, influence diagram, junction tree, probability propagation, reasoning under uncertainty.

1 Introduction
Expert systems, using causal probabilistic networks (CPNs)\(^1\) for knowledge representation, are reaching the state where it is feasible to handle domains modeled by large-scale networks (e.g., MUNIN [Andersen et al., 1987; Olesen et al., 1989]). When building such large networks, it is (for reasons of practicality) often necessary to introduce approximations besides those inherent in the process of modeling a domain. Two main approaches have been investigated: focusing on the development of an approximative algorithm for propagation of information (e.g., [Henrion, 1989]), and focusing on approximations in the underlying network representation and then using an exact inference algorithm.

The objective of this paper is to present an approximation scheme that takes the latter approach. The scheme is tailored to the Bayesian belief universe approach [Jensen et al., 1989] as used in HUGIN [Andersen et al., 1989]. The method operates by approximations in the quantitative part of the underlying representation, whereas the qualitative structure remains unchanged. Within this framework, we can assess the accuracy of the approximated probabilities, which is not possible with heuristic methods. Application of the method often results in a substantial decrease in the usage of computer resources; the amount of decrease depends on domain characteristics, such as network topology and probability distributions.

It is known that, in general, probabilistic inference in CPNs is \(\mathcal{NP}\)-hard [Cooper, 1987], and exact calculations will eventually become intractable. This fact emphasizes the importance of approximative methods.

A domain model in the causal probabilistic network approach consists of a graph with nodes representing the domain variables and the (directed) arcs representing the causal relations between the domain variables. Conditional probabilities are used to describe the dependency of domain variables given their immediate predecessors (parents). Different inference methods have been developed to propagate information in such a network: If the topology is simple (singly connected) [Pearl, 1986], propagation can be done directly in the CPN; otherwise, a secondary structure for topologies, including nondirected loops [Lauritzen and Spiegelhalter, 1988; Jensen et al., 1989; Shafer and Shenoy, 1988], can be used. Alternatively, for the latter kind of topologies, the inference could also take place in a set of conditioned networks [Suermond and Cooper, 1988] or through manipulation of the network with an arc reversing technique [Shachter, 1988].

The method of Bayesian belief universes splits the inference task into two phases: a compilation phase and a run-time phase. The proposed approxima-
The initial belief tables are calculated as appropriate products of the conditional probability tables [Lauritzen and Spiegelhalter, 1988; Jensen et al., 1989].

- Organize the system as a junction tree: Links between belief universes are introduced, such that a tree with the following property results: For each pair \((U, V)\) of belief universes, each belief universe on the unique path between \(U\) and \(V\) contains the nodes \(U \cap V\). As shown in [Jensen, 1988], a junction tree can be constructed by a maximal spanning-tree algorithm.

All steps except the second are deterministic: There is only one moral graph, and the set of cliques of a triangulated graph is unique. There may be several junction trees, but the differences among them are minor (the major cost of a junction tree is the representation of the belief tables for the belief universes).

The second step is important: A good triangulation can save substantial space and time [Kjærulff, 1990].

Let \(U\) be a belief universe with belief table \(B\), and let \(S \subseteq U\). We can obtain the joint probabilities for \(S\) from \(B\) by summing up all beliefs in \(B\) for \(S\). This operation is called marginalization. In particular, the belief in a single node can be obtained by marginalization of the belief table of any belief universe containing it.

Let \(U\) be a belief universe, and let \(V \subseteq U\). A finding on \(V\) is a subset of the state space of \(V\). The finding is entered into \(U\) through annihilation of the elements in the belief table of \(U\) corresponding to state combinations not in \(V\).

A set of one or more findings is called a case.

A junction tree is said to be consistent if marginalization of two distinct belief universes \(U\) and \(U'\) with respect to some set of nodes \(V\) (contained in both \(U\) and \(U'\)) yield "identical" (i.e., proportional) results. This property is (re)established through the global propagation operation. This operation refers to a local propagation method for transmitting evidence between neighbors in a junction tree.

Absorption is the local propagation method: If we have entered evidence into a belief universe \(V\), then an adjacent belief universe \(U\) absorbs from \(V\) through the following steps:

1. Calculate the belief table for \(U \cap V\) by marginalization of the belief table of \(U\).
2. Calculate the belief table for the same intersection by marginalization of the belief table of \(V\).
3. Multiply the belief table of \(U\) by the ratio of the table achieved by Step 1 and the table achieved by Step 2.

\footnote{Typically, a finding is a statement that a node is known to be in a particular state.}

\footnote{We shall also use the phrase "evidence is entered into \(U\)."}
When absorbing from several neighbors simultaneously, these steps must proceed in "parallel" (implying use of the \textit{same} version of the belief table of \( U \) in Step 1).

Global propagation is described in terms of two operations: \textit{CollectEvidence} and \textit{DistributeEvidence}. \textit{CollectEvidence} is used when evidence from the entire system must be propagated to a single belief universe \( U \): \( U \) asks neighbors to \textit{CollectEvidence}; when they are done, \( U \) absorbs from them. \textit{DistributeEvidence} is used when evidence from a single belief universe \( U \) must propagate to the entire system: \( U \) asks each neighbor to absorb from \( U \) and then \textit{DistributeEvidence} to its other neighbors.

A global propagation operation consists of \textit{CollectEvidence} operation followed by a \textit{DistributeEvidence} operation initiated from an arbitrary belief universe.

\textit{CollectEvidence} has an important property. Assume that we have a consistent and normalized junction tree, and that we enter evidence into some of the belief universes of the junction tree. If we invoke \textit{CollectEvidence} from some belief universe \( U \), then the normalizing constant for the belief table of \( U \), after \textit{CollectEvidence} has terminated, is equal to the \textbf{(prior)} probability of the evidence.

3 The Approximation Scheme

As described in the previous section, the numbers in the belief tables of the belief universes represent probabilities in joint probability distributions. One might expect that excluding the smallest numbers (representing rare state combinations) will lead to substantial improvements in the requirements of computer resources. In this section, we shall investigate some properties of such a scheme.

Assuming we have a consistent junction tree, an \textbf{approximation} is performed in the following way:

1. For each belief universe in the junction tree, we select some elements of its belief table and annihilate those; the rest are left unchanged.
2. The junction tree is made consistent again by a global propagation.
3. [Optional] The belief tables of the belief universes are compressed in order to take advantage of the introduced zeros. (This step will not be described here; see [Jensen and Andersen, 1990] for details.)

How Do We Select the Numbers to Be Annihilated?

As previously mentioned, we are interested in the small numbers. A simple way to do the selection is to use a threshold value to separate the numbers to be annihilated from the numbers to be kept. However, we cannot choose a global threshold value, as the size of tables and their distribution of numbers may vary substantially. So instead we shall use a local threshold value for each table.

We observe that, annihilating an element of a belief table, corresponds to entering a finding that says that the state combinations, corresponding to this element, are "impossible" (or are considered uninteresting). Moreover, the sum of the annihilated elements in a given belief table is the \textbf{probability} of all the state combinations (the finding) corresponding to those elements. This probability is a measure of the (local) error, we commit. We can control this error by choosing a suitable threshold value.

Suppose we want to retain \( 1 - \varepsilon \) of the probability mass of each belief table. Then, a simple method is to compute a threshold value \( \delta \) by repeatedly halving \( \varepsilon \) (using \( \varepsilon \) as the initial value for \( \delta \)) until the sum of the elements less than \( \delta \) is no greater than \( \varepsilon \); these elements will be annihilated (we believe that either all or no elements with the same value in a given table should be eliminated). A more costly method is to sort the elements of the table and to repeat annihilating the smallest number(s) as long as the sum of the annihilated numbers does not exceed \( \varepsilon \).

The global error \( \varepsilon \) (the total amount of probability mass removed) is computed as \( \varepsilon = 1 - \mu \), where \( \mu \) is the normalization constant found during the global propagation step of the approximation algorithm.

Given an arbitrary case, we can determine if it is one of the cases that have been \textit{completely} excluded from consideration by detecting a zero normalization constant. The probability of such a case occurring (assuming the assessed conditional probabilities are correct) is \( \varepsilon \).

For each remaining case, some of the state combinations supporting the case may have been eliminated. The accumulated probability for those state combinations determines the error on the posterior probabilities as shown in the following.

How Good Is the Approximation?

Assume that we have approximated the belief universes and have propagated the approximations throughout the junction tree. We now have a consistent junction tree.

Let \( A \) denote the approximation performed, and let \( F \) denote a set of findings to be entered into the (consistent) approximated junction tree. Entering such a set of findings is a common operation when using the junction tree (or rather the underlying CPN) as an expert system. After \( F \) has been entered, and the junction tree has been made consistent by propagation, we want to query the system for probabilities of the form \( P(H|F) \), where \( H \) is some hypothesis.\(^7\) However, the probabil-

\(^7\)This method is used in Hugin [Andersen et al., 1989].

\(^8\)In a real application, the CPN might model the relationships between some diseases and the associated symptoms; \( F \) then would be the set of symptoms found, \( H \) typically would be of the form "the patient has disease \( X \)," and \( P(H|F) \) would denote the probability that
probability \( P(H|F) \) is not available; instead, we get the probability \( P(H|F, A) \) (that is, the probability for \( H \) given the findings \( F \) and the approximation \( A \)).

We therefore want to find an upper bound on \( |P(H|F) - P(H|F, A)| \):

\[
|P(H|F) - P(H|F, A)| = |P(H|F, A)P(A|F) + P(H|F, \overline{A})P(\overline{A}|F) - P(H|F, A)|
\]

\[
= |P(H|F, A)[P(A|F) - 1] + P(H|F, \overline{A})P(\overline{A}|F)|
\]

\[
= P(\overline{A}|F)[P(H|F, \overline{A}) - P(H|F, A)]
\]

\[
\leq P(\overline{A}|F)
\]

The quantity \( P(\overline{A}|F) \) can be rewritten as

\[
P(F \cap \overline{A}) = \frac{P(F \cap \overline{A})}{P(F \cap \overline{A}) + P(F \cap A)} \leq \epsilon
\]

where \( \epsilon = P(\overline{A}) \) and \( \mu = P(F|A) \). These quantities are known; \( \epsilon \) is the approximation error found at approximation time, and \( \mu \) is the normalization constant found during propagation of \( F \). Unfortunately, \( \mu \) is almost always small (\( \ll \epsilon \)), so this upper bound is not a good indicator of the approximation error.

In practice, however, \( F \) is almost always of the form \( f_1 \cap \ldots \cap f_n \), where \( f_i \) \((1 \leq i \leq n)\) states that “node \( X_i \) is in state \( y_i \).” Thus

\[
P(F \cap \overline{A}) \leq \min\{P(f_1 \cap \overline{A}), \ldots, P(f_n \cap \overline{A})\}
\]

We can compute these quantities for all combinations of nodes and states at approximation time (the space required to store these quantities is small).

Although this gives us a better upper bound for the approximation error, it is, however, strictly a worst-case bound, and we may have to rely on empirical studies to determine the actual errors. In the next section, we shall investigate this issue for a real application.

### 4 An Application

We shall use a network from the MUNIN knowledge base to study the effect of the proposed approximation scheme on a real-world CPN.

The domain of MUNIN is electromyography, a technique for diagnosing peripheral muscle and nerve disorders. We have chosen a network describing disorders in the median nerve.\(^9\) On the basis of four electromyographic findings, this model is capable of diagnosing three local nerve lesions and one diffuse disorder in the median nerve in the arm. The CPN contains 57 nodes; the disease nodes each have between three and five states, and the finding nodes have from 15 to 21 states.

The specification of the conditional probability tables requires 8126 numbers, of which 67.1 percent are assessed as zeros; however, most of these numbers have been generated by local models from a much smaller set of parameters, which has been assessed by domain experts [Andreassen et al., 1987].

An explanation of the domain concepts, as well as a description of the medical performance, can be found in [Andreassen et al., 1989; Olesen et al., 1989].

#### 4.1 Junction Trees

Based on different triangulations of the median-nerve CPN, we have created four junction trees, yielding different starting points for approximation. We have used a maximum-cardinality search [Taranjan and Yannakakis, 1984] and two heuristic search strategies that minimize the clique cardinality (the min-size heuristic) and the size of the state space of the nodes in the cliques (the min-weight heuristic), respectively; see [Kjærulff, 1990] for details.

| Clique Size | Max-Card 1 | Max-Card 2 | Min-Size | Min-Weight |
|-------------|------------|------------|----------|-----------|
| Number of Cliques | | | | |
| 14 | 1 | - | - | - |
| 13 | 2 | - | - | - |
| 10 | 1 | - | - | - |
| 9 | 1 | - | - | - |
| 8 | 4 | 6 | 3 | 3 |
| 7 | 4 | 7 | 2 | 2 |
| 6 | 2 | 4 | 5 | 4 |
| 5 | 9 | 2 | 7 | 9 |

| Total State-space (10^6) | 4849 | 10.7 | 1.6 | 1.6 |
|-------------------------|------|------|-----|-----|
| Zeros (Percent) | - | 93 | 71 | 77 |
| Max State-space (10^6) | - | 4.0 | 0.45 | 0.54 |

Table 1: Statistics of junction trees for the median-nerve knowledge base generated from different triangulations.

Table 1 summarizes key parameters of junction trees, based on different triangulations. We have obtained two maximal-cardinality searches using differ-
ent starting nodes. However (for obvious reasons), we consider only the second one, referred to as "max-card," in the following subsections. The data in Table 1 apply to the initial consistent (i.e., after initialization) junction trees before any approximation or compression has been done.

4.2 Effect on Resources
We shall focus on two aspects of resources: (1) the propagation time needed to make the junction tree consistent after a set of findings has been entered, and (2) the storage space needed to represent the knowledge base in a suitable compact form (see Jensen and Andersen, 1990) for details.

The global error $e$, defined in Section 3, is used to characterize the approximation; we shall use the term total removed probability mass to refer to this value.

![Figure 1: The effect of compression on required storage space and propagation time for the median-nerve knowledge base.](image1)

![Figure 2: The relation between required storage space and propagation time for the median-nerve knowledge base for various approximations. The line corresponds to a linear relationship between propagation time and storage space.](image2)

![Figure 3: The space requirement as a function of the probability mass removed for different junction trees. The arrows indicate the storage requirements for unapproximated but compressed junction trees.](image3)

The time and space measurements reported are for an implementation of HUGIN [Andersen et al., 1989] in C for a Sun 3 workstation; however, we are only interested in relative improvements, so the space and (in particular) time units should be regarded as arbitrary.

Figure 1 illustrates the effect of the initial compression on required storage space and propagation time for three different junction trees. As expected, the gain varies according to the different ratio of zeros in the junction trees (see Table 1).

Figure 2 shows the relation between propagation time and storage space needed for the three different triangulation methods at different approximations. The total removed probability mass ($e$) varied between 0.001 and 1 percent. At each data point, the corresponding approximated and compressed runtime system was created, and the time and space
characteristics were measured. We observe a linear relationship between propagation time and storage space needed; thus, we characterize resource requirements in term of storage space only.

The resource requirements for approximated junction trees as a function of the total removed probability mass is the subject of Figure 3. Each data point in this figure corresponds to a data point in Figure 2, except for points corresponding to \( e > 2 \) percent. The values corresponding to no approximation for the compressed junction trees are also indicated.

We observe that, for \( e \) less than \( \sim 0.1 \) percent, the approximation is equally efficient for the three junction trees. For each junction tree, \( e = 0.25 \) percent yields about one order of magnitude in reduction of the required space. However, for a sufficiently large value of \( e \), the differences between the junction trees disappear.

Table 2 shows the effect of the method applied to the different junction trees at \( e = 0.1 \) percent.

| Triangulation Method | Max Card | Min Size | Min Weight |
|----------------------|----------|----------|------------|
|                      |          |          |            |
| Space                |          |          |            |
| Initial (Mbytes)     | 46       | 8.5      | 7.2        |
| Approx. (Mbytes)     | 0.95     | 0.71     | 0.60       |
| Reduction            | 0.989    | 0.916    | 0.916      |

Table 2: The effect of approximation and compression on junction trees generated from the median-nerve CPN.

4.3 Effect on the Quality

Whenever we commit ourselves to making an approximation, we want to know the risk that we will make serious errors. Unfortunately, the basis on which we calculate the theoretical worst-case error bounds might be too coarse, and it is highly unlikely that the worst-case situation will appear in a real application. If we had some method that could warn us when the situation was questionable, we might take the risk and make approximations beyond the magnitude imposed by a given worst-case error bound.

We shall use our median-nerve knowledge base, and shall make a diagnosis on the basis of a set of findings, thus showing how our theoretical estimate on upper bounds on errors compares to practical values.

Figure 4 displays the results of entering a typical case into various approximated junction trees. The probability of the case is \( 4.1 \times 10^{-4} \).

The observed error in the beliefs caused by the approximation is shown as a function of the total removed probability mass \( (e) \). The figure shows observed errors in the beliefs of states representing exact beliefs between 0.9189 and 0.0005. The worst-case error bound (Section 3) for each approximation and case also has been computed. We observe that the difference between the worst-case bound and the worst measured absolute error is about three orders of magnitude for \( e \leq 0.1 \) percent.

Figure 5 shows triples of the worst-case bound (filled square), maximal observed error (diamond), and average observed error (open square) for 18 different randomly generated cases as a function of the case-specific normalizing constant, \( \mu_{case} \). The approximation used corresponds to a decrease in resource requirements by a factor of four relative to an unapproximated but compressed junction tree.

Figure 5 shows that the observed errors on computed beliefs for the displayed cases are much smaller than that predicted by the worst-case error bound derived in Section 3. This difference shows that it is very unlikely, by picking a randomly generated case with a given \( \mu_{case} \), to get the
The ratio between the worst-case bound and the maximal observed error is three orders of magnitude larger than the maximal observed error, and average observed error in the beliefs of the states of the disorder nodes used for the case in Figure 4 are shown for 18 different cases.

Worst-case configuration. In the present CPN, the ratio between the worst-case bound and the maximal observed error is three orders of magnitude larger than the maximal observed error. Decreasing the normalizing constant (\( \mu_{\text{case}} \)) implies increasing the error in beliefs for the specific case, as well as for the worst-case error. When \( \mu_{\text{case}} \) approaches zero, the error in beliefs approaches one, corresponding to excluding the case from the domain model.

These empirical studies show, that if we have a specific hypothesis in mind (for example the diagnosis of a local nerve lesion at the wrist) and a set of test cases which provides us with a span of \( \mu_{\text{case}} \), we can get empirical values for the actual expected error in a specific case, given \( \mu_{\text{case}} \).

Given a specific approximation \( \varepsilon \), we would have the following situations: If we insert a set of findings, and the theoretical worst-case error bound are below an accepted level, we can use the approximated junction tree. If we insert a set of findings which already has been taken out of the domain model by "zeroing out," the violation on the model will be recognized by a zero normalizing constant, and we have to use a less approximated junction tree. If we insert a set of findings yielding an unacceptably high worst-case error, we have to rely on empirical studies, such as those above, to estimate the error based on \( \mu_{\text{case}} \), and on basis of this, decide whether to fall back on a less approximated junction tree or accept the risk of committing an error. This approach allows us to obtain a graceful degradation of the quality of diagnoses as the limit of the approximation is reached.

For the median-nerve knowledge base and the focus on the hypothesis of a lesion at the wrist, a demand of 0.01 as the upper limit of error in a state, would allow us set the alert threshold as low as \( \mu_{\text{case}} = 10^{-7} \) for \( \varepsilon = 2 \times 10^{-4} \).

5 Conclusion

We have presented a scheme for approximation in the numerical part of a CPN-based expert system. Our approach eliminates the (small) numbers representing probabilities of rare combinations of findings, thereby preventing these findings from being treated as ordinary findings in the expert system. The approximation has two effects: (1) we may gain several orders of magnitude in improvement of resource usage, and (2) we may lose some accuracy in the computed beliefs. However, we can estimate case-specific upper bounds for the errors made on the computed beliefs, although these bounds may be too pessimistic, as the studies reported in Section 4 show.

If the case has been completely excluded by the approximation process, we will detect it by finding a zero normalizing constant during propagation; if the case is one of the common cases, we know that the computed beliefs can be trusted to a large degree. The problematic cases are the ones that have a nonzero probability outside the "trusted range" of probabilities (remember that the probability of a case is equal to the normalization constant found during propagation). We suggest that, when a problematic case occur, we should reenter the case into a less approximated (maybe even a nonapproximated) junction tree; however, this solution should rarely be necessary.

It would be nice to find an upper bound on the error of beliefs that is better (and still easily computable) than is the one presented in Section 3. Calculation of this bound involves the errors made on individual findings. We might be able to do better if we considered two or more findings simultaneously; however, a straightforward approach would require \( O(s^n) \) space, where \( s \) is the total number of states in the nodes, and \( n \) is the number of findings considered.

There might be a clever technique to avoid considering all these combinations of findings and at the same time to provide a better error bound. We shall leave this topic for future research.

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