Continuous-time Quantum Error Correction with Noise-assisted Quantum Feedback

Gerardo Cardona\textsuperscript{1,2}, Alain Sarlette\textsuperscript{2,3} and Pierre Rouchon\textsuperscript{1,2} \textsuperscript{*} \textsuperscript{†} \textsuperscript{‡} \textsuperscript{§}

Abstract

In this article we address the problem of quantum error correction in continuous-time for the three-qubit bit-flip code. This entails rendering a target manifold of quantum states globally attractive. Previous feedback designs could either have unwanted equilibria in closed-loop, or resort to discrete kicks pushing the system away from these bad equilibria to ensure global asymptotic stability. Here we present a new approach, consisting of introducing controls driven by Brownian motions. Unlike the previous methods, the resulting closed-loop dynamics can be shown to stabilize the target manifold exponentially. This exponential property is important to quantify the protection induced by the closed-loop error-correction dynamics against disturbances, i.e. its performance towards enabling robust quantum information processing devices in the future.

1 Introduction

The development of methods for the protection of quantum information in the presence of disturbances is essential to improve existing quantum technologies \cite{Reed2012,Ofek2016}. Quantum error correction (QEC) codes, encode a logical state into multiple physical states. Similarly to classical error correction, this redundancy allows to protect quantum information from disturbances by stabilizing a submanifold of steady states, which represent the nominal logical states \cite{Lidar2013,Nielsen2002}. As long as a disturbance does not drive the system out of the basin of attraction of the original nominal state, the logical information remains unperturbed. To stabilize the nominal submanifold in a quantum system, a syndrome diagnosis stage performs quantum non-destructive (QND) measurement which extract information about the disturbances without perturbing the encoded data. Based on this information, a recovery feedback action restores the corrupted state.

QEC is most often presented as discrete-time op-

\textsuperscript{*}Centre Automatique et Systèmes, Mines-ParisTech, PSL Research University, 60 Bd Saint-Michel, 75006 Paris, France.
\textsuperscript{†}QUANTIC lab, INRIA Paris, rue Simone Iff 2, 75012 Paris, France.
\textsuperscript{‡}Electronics and Information Systems Department, Ghent University, Belgium.
\textsuperscript{§}gerardo.cardona@mines-paristech.fr, pierre.rouchon@mines-paristech.fr, alain.sarlette@inria.fr


Remark 1.1. We will consider concrete instances of \textit{Itô} stochastic differential equations (SDEs) on $\mathbb{R}^n$ of

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where \(W\) is a standard Brownian motion on \(\mathbb{R}^k\), and \(\mu, \sigma\) are regular functions of \(x\) with image in \(\mathbb{R}^n\) and \(\mathbb{R}^{n \times k}\) respectively, satisfying the usual conditions for existence and uniqueness of solutions [Arnold 1974, Kushner 1967] on \(S\), a compact and positively invariant subset of \(\mathbb{R}^n\).

We will use results on stochastic stability [Khasminskii 2011]. Consider (1) with \(\mu(0) = \sigma(0) = 0\). Let \(V(x)\), a nonnegative real-valued twice continuously differentiable function with respect to \(x \in S\) everywhere except possibly at \(x = 0\). Its Markov generator associated with (1) is

\[ AV = \sum_i \mu_i \frac{\partial}{\partial x_i} V + \frac{1}{2} \sum_{i,j} \sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} V, \]

and

\[ \mathbb{E}[V(x_t)] = V(x_0) + \mathbb{E} \left[ \int_0^t AV(x_s)ds \right]. \]

**Theorem (Khasminskii 2011):** If there exists \(r > 0\) such that \(AV(x) \leq -rV(x), \forall x \in S\), then \(V(x)\) is a supermartingale with exponential decay \(\mathbb{E}[V(x_t)] \leq V(x_0) \exp(-rt)\). On the compact set \(S\), it is a stochastic Lyapunov function proving exponential convergence towards the set \(\{x : V(x) = 0\}\).

Analysis in the rest of this paper consists in defining a function \(V\) and constructing controls that ensure exponential decay in the above sense.

## 2 Continuous-time dynamics of the three-qubit bit-flip code

The general model for a quantum system subject to several measurement channels (see, e.g., Barchielli and Gregoratti 2009) corresponds to Itô stochastic differential equations of the type

\[ d\rho = \sum_k D_{L_k}(\rho)dt + \sqrt{\eta_k} M_{L_k}(\rho)dW_k, \]

\[ dY_k = \sqrt{\eta_k} \text{Trace} \left( (L_k + L_k^\dagger) \rho \right) dt + dW_k. \]

We have used the standard super-operator notation \(D_{L}(\rho) = (L\rho L^\dagger - \frac{1}{2} (L^\dagger L \rho + \rho L L^\dagger)), \quad M_{L}(\rho) = (L \rho + \rho L^\dagger - \text{Trace}(\rho (L + L^\dagger)) \rho)\), where \(L^\dagger\) denotes the complex conjugate transpose of \(L\). The state \(\rho\) belongs to the set of density matrices \(S = \{ \rho \in \mathbb{C}^{n \times n} : \rho = \rho^\dagger, \rho \text{ pos.semi.def.}, \text{Trace}(\rho) = 1\}\) on the Hilbert space of the system \(H \cong \mathbb{C}^{n \times n}\); the \(\{W_k\}\) are independent standard Brownian motions and the \(dW_k\) correspond to the measurement processes of each measurement channel. The \(\eta_k \in [0,1]\) express the corresponding measurement efficiencies, i.e. the ratio between induced disturbance (measurement back-action) and retrieved information, and channels \(k\) with \(\eta_k = 0\) represent pure disturbances.

The simplest way to model the feedback stage consists in applying an infinitesimal unitary operation to the open-loop evolution, \(\rho_{t+dt} = U_t (\rho_t + d\rho_t) U_t^\dagger\), where \(U_t = \exp(-i \sum j H_j u_{t,j} dt)\) with \(H_j\) hermitian operators denoting the control Hamiltonians that can be applied, and each \(u_{t,j}\) a real control input. The fact that \(u_{t,j}\) may contain stochastic processes requires to treat this feedback action with care, we will come back to this in the next section.

### 2.1 Dynamics of the three-qubit bit-flip code

The three-qubit bit-flip code corresponds to a Hilbert space \(H = (\mathbb{C}^2)^{\otimes 3} \cong \mathbb{C}^8\), where \(\otimes\) denotes tensor product (Kronecker product, in matrix representation). We denote \(I_n\) the identity operator on \(\mathbb{C}^n\) and we write \(X_k, Y_k\) and \(Z_k\) the local Pauli operators acting on qubit \(k\), e.g. \(X_2 = I_2 \otimes \sigma_x \otimes I_2\). We denote \(\{0, 1\}\) the usual basis states, i.e. the -1 and +1 eigenstates of the \(\sigma_z\) operator on each individual qubit [Nielsen and Chuang 2002].

The encoding on this 3-qubit system is meant to counter bit-flip errors, which can map a ±1 eigenstate of \(Z_k\) to a ±1 eigenstate for each \(k = 1, 2, 3\). More precisely, the nominal encoding for a logical information 0 (resp. 1) is on the state \(|000\rangle\) (resp. \(|111\rangle\)). A single bit-flip on e.g. the first qubit brings this to \(|100\rangle = |100\rangle\) (resp. \(|011\rangle\)), which by majority vote could be brought back to the nominal encoding.

In the continuous-time model (3), bit-flip errors occurring with a probability \(\gamma_k dt \ll 1\) during a time interval \([t, t + dt]\) are modeled by disturbance channels, with \(L_{k+3} = \sqrt{\eta_k} X_k\) and \(\eta_{k+3} = 0, k = 1, 2, 3\). Possible bit-flips are tracked by so-called syndromes, continuously comparing the \(\sigma_z\) value of pairs of qubits. The associated measurements correspond in (3) to \(L_k = \sqrt{T_k} S_k\) for \(k = 1, 2, 3\), with \(S_1 = Z_2 Z_3\), \(S_2 = Z_1 Z_3\), \(S_3 = Z_1 Z_2\) and \(\Gamma_k\) representing the measurement strength. This yields the following open-loop model:

\[ d\rho = \sum_{k=1}^3 \Gamma_k D_{S_k}(\rho) dt + \sqrt{\eta_k \Gamma_k} M_{S_k}(\rho) dW_k \]

\[ + \sum_{s=1}^3 \gamma_s DX_s(\rho) dt. \]

We further define the operators:

\[ \Pi C = \frac{1}{4} (I_8 + \sum_{k=1}^3 S_k) \]
\[ \Pi_j = X_j \Pi C X_j, j \in \{1, 2, 3\}, \]

corresponding to orthogonal projectors onto the various joint eigenspaces of the measurement syndromes. The first one \(\Pi C\) projects onto the nominal code \(C = \text{span}(|000\rangle, |111\rangle)\) (+1 eigenspace of all the \(S_k\)), whereas \(\Pi_j\) projects onto the subspace where qubit \(j\) is flipped with respect to the two others. For each
Lemma 2.1. Consider equation (3) with \( \gamma_s = 0 \) for \( s \in \{1, 2, 3\} \).

(i) For each \( k \in \{C, 1, 2, 3\} \), the subspace population \( p_{t,k} \) is a martingale i.e. \( \mathbb{E}(p_{t,k}|p_0,k) = p_{0,k} \) for all \( t \geq 0 \).

(ii) For a given \( p_0 \), if there exists \( \bar{k} \in \{C, 1, 2, 3\} \) such that \( p_{0,k} = 1 \), \( p_{0,\bar{k}} = 0 \) for all \( k \neq \bar{k} \), then \( p_0 \) is a steady state of (3).

(iii) The Lyapunov function

\[
V(\rho) = \sum_{k \in \{C, 1, 2, 3\}} \sum_{k' \neq k} \sqrt{p_{k}p_{k'}},
\]

decreases exponentially as

\[
\mathbb{E}[V(p_{t})] \leq e^{-rt}V(p_0)
\]

for all \( t \geq 0 \), with rate

\[
r = 4 \min\{1, 2, 3\} \eta_k \Gamma_k.
\]

In this sense the system exponentially approaches the set of invariant states described in point (ii).

Proof. The first two statements are easily verified, we prove the last one. The variables \( \xi_j = \sqrt{p_j}, j \in \{1, 2, 3, C\} \) satisfy the following SDE’s:

\[
d\xi_C = -2\xi_C \left( \sum_{k \in \{1, 2, 3\}} \eta_k \Gamma_k (1 - \xi_k^2 - \xi_l^2)^2 \right) dt
\]
\[+ 2\xi_C \left( \sum_{k \in \{1, 2, 3\}} \sqrt{\eta_k \Gamma_k (1 - \xi_k^2 - \xi_l^2)} dW_k \right),
\]

\[
d\xi_{j \neq C} = -2\xi_j \left( \eta_j \Gamma_j (1 - \xi_j^2 - \xi_l^2)^2 \right)
\]
\[+ \sum_{k \in \{1, 2, 3\} \setminus j} \eta_k \Gamma_k (\xi_k^2 + \xi_l^2)^2 dt
\]
\[+ 2\xi_j \left( \sqrt{\eta_j \Gamma_j (1 - \xi_j^2 - \xi_l^2)} dW_j
\]
\[- \sum_{k \in \{1, 2, 3\} \setminus j} \sqrt{\eta_k \Gamma_k (\xi_k^2 + \xi_l^2)} dW_k \right).
\]

Then

\[
V = \sum_{k \in \{C, 1, 2, 3\}} \sum_{k' \neq k} \xi_k \xi_{k'}.
\]

Noting that

\[2(1 - \xi_k^2 - \xi_l^2)\] and \[2(\xi_k^2 + \xi_l^2)\]
just correspond to \( 1 \pm \text{Trace}(\rho S_k) \), we only have to keep track of \( \pm \) signs in the various terms to compute

\[
AV = -2 \sum_{k \in \{C, 1, 2, 3\}} \sum_{j \in \{C, 1, 2, 3\} \setminus k} \xi_j \xi_k \sum_{l \in \{1, 2, 3\}} \epsilon_{j,k,l} \eta_l \Gamma_l
\]

where, for each pair \( (j, k) \), the selector \( \epsilon_{j,k,l} \in \{0, 1\} \) equals 1 for two \( l \) values, namely \( \epsilon_{C,k,l} = \epsilon_{k,C,l} = 1 \) if \( l \neq k \) \( \in \{1, 2, 3\} \) and \( \epsilon_{j,k,3} = \epsilon_{j,k,1} = 1 \) for \( j, k \in \{1, 2, 3\} \). This readily leads to \( AV \leq -4 \min_{k \in \{1, 2, 3\}} (\eta_l \Gamma_l) V \).

The above Lyapunov function describes the convergence of the state towards \( \text{Trace}(\Pi_C \rho) = 1 \), for a random subspace \( k \in \{C, 1, 2, 3\} \) chosen with probability \( p_{0,k} \). This is the equivalent, for invariant subspaces, of our previous result in Cardona et al. [2018] for a measurement featuring invariant isolated states. In a similar way, we now address how to render one particular subspace globally attractive, here the one associated to nominal codewords and with projector \( \Pi_C \).

3 Error correction via noise-assisted feedback stabilization

3.1 Controller design

Error correction requires to design a control law satisfying two properties:

- Drive any initial state \( \rho_0 \) towards a state with support only on the nominal codespace \( C = \text{span}\{000, 111\} \). This comes down to making \( \text{Trace}(\Pi_C \rho_0) \) converge to 1.

- For \( \text{Trace}(\Pi_C \rho_0) = 1 \) and in the presence of disturbances \( \gamma_s \neq 0 \), minimize the distance between \( \rho_t \) and \( \rho_0 \) for all \( t \geq 0 \).

We now directly address the first point, the second one will be discussed in the sequel.

As mentioned in the introduction, this problem has already been considered before, yet without proof of exponential convergence. Towards establishing such proof, we introduce a key novelty into the feedback signal: we drive it by a stochastic process. In Cardona et al. [2018], we have shown that using \( u(\rho) dt = f(\rho) dt + \kappa(\rho) dB \), with nonzero gain \( \kappa \) assigned to the Brownian motion present in \( dB \), enables global exponential stabilization of a qubit. The reasoning behind this strategy is that in the associated SDE, Itô calculus imposes to keep second-order terms of the feedback action \( U_t = \exp(-i \sum_j H_j u_j dt) \) itself, and the latter can exponentially destabilize a spurious equilibrium.

In the present paper we introduce an even simpler strategy, which we call noise-assisted quantum feedback, and where the control input consists of pure noise with state-dependent gain. Explicitly, we take

\[
u_j dt = \kappa_j(\rho) dB_j,
\]
with \( B_j(t) \) a Brownian motion independent of any \( W_k(t) \). As control Hamiltonians we take \( H_j = X_j \), thus rotating back the bit-flip actions. The closed-loop dynamics in Itô sense then writes:

\[
d\rho = \sum_{k=1}^{3} \Gamma_k D_{S_k}(\rho)dt + \sqrt{\eta_k} \Gamma_k M_{S_k}(\rho)dW_k
\]

\[
+ \sum_{s=1}^{3} \gamma_s D_{X_s}(\rho)dt
\]

\[
+ \sum_{j=1}^{3} -i\kappa_j(\rho) [X_j, \rho]dB_j + \kappa_j(\rho)^2 D_{X_j}(\rho)dt.
\]

(6)

The remaining task is to design the gains \( \kappa_j \), which in general can follow some dynamic control logic.

There are many options for designing \( \kappa_j \) — its only essential role is to “shake” the state when it is close to Trace \((\Pi \rho) = 0\), since the open-loop behavior already ensures stochastic convergence to either Trace \((\Pi \rho) = 0\) or Trace \((\Pi \rho) = 1\). The following hysteresis-based control law depends only on the variables \( p_{t,k} \) and should not be too hard to implement in practice. Select some \( 0 < \alpha_j < 1 \) and \( 0 < \beta_j < \alpha_j \) for \( j \in \{1,2,3\} \), and a constant \( c > 0 \).

- If \( \min (1 - p_{t,C} , 3p_{t,j} - 3p_{t,C}) \geq \alpha_j \), then take \( \kappa_j(t) = \frac{\alpha_j}{\gamma_j} \).
- If \( \min (1 - p_{t,C} , 3p_{t,j} - 3p_{t,C}) \leq \beta_j \), then take \( \kappa_j(t) = 0 \).
- When entering or moving in the hysteresis region, i.e. the values of \( \rho \) not covered by the above two cases: keep the previous value of \( \kappa_j(t) \).

The principle of this feedback control is sketched on Figure 1. Its rigorous analysis follows right after.

### 3.2 Convergence proof

Since we leave the system in open-loop when it is close to the target Trace \((\rho \Pi) = 1\), we express the distance to the target manifold \( \mathcal{C} \) with the Lyapunov function \( V(\rho) = \sqrt{1 - \text{Trace}(\rho \Pi)} \) which is similar to the open-loop Lyapunov function for Trace \((\rho \Pi) \) close to 1. We then have the following main result.

**Theorem 3.1.** Consider system (9) with \( \gamma_s = 0, s \in \{1,2,3\} \) and feedback gains \( (\kappa_j) \) designed as explained above. Then

\[
E[V(\rho_t)] \leq V(\rho_0)e^{-rt}, \forall t \geq 0,
\]

with the exponential convergence rate estimated as:

\[
r = \min \left( \frac{c}{2} \cdot \frac{8}{3} \left( 1 - \max (\alpha_1, \alpha_2, \alpha_3, \frac{3+\sum_j \alpha_j/3}{4}) \right)^2 \right) 
\]

\[
\cdot \min_j (\eta_j, \Gamma_j).
\]

Figure 1: Abstract sketch of our error-correction controller, representing only the first qubit. The target is the north pole of the sphere, whereas the south pole represents the spurious open-loop equilibria with Trace \((\Pi \rho) = 1\). The three control regions are: above the horizontal purple dashed line \( (\kappa_1 = 0) \); below the horizontal red dashed line \( (\kappa_1 = \sqrt{\frac{\max_j \Gamma_j}{\eta_j}}) \) for \( j = 1 \); and the hysteresis region in between. If a bit-flip \( X_1 \) brings the state from the north pole towards the south pole (green arrow), then we start driving the system with noise, “shaking it away” from the spurious open-loop equilibria Trace \((\Pi \rho) = 1\), until it stochastically reaches the purple dashed line (black trajectory). When the trajectory crosses the purple dashed line, we switch the noise off and the system evolves in open loop, according to Lemma 2.3, thus stochastically converging towards one of the poles (gray trajectory). If the stochastic open-loop motion stays above the red dashed line for all \( t \), then it must converge towards the north pole i.e. the target; if instead the motion crosses the red dashed line, we again switch on the drive with noise.
Proof. The proof consists in showing that $V(p_t)$ on $S$ is an exponential supermartingale under any control gain $\kappa = (\kappa_1, \kappa_2, \kappa_3)$ allowed by the design of section 3.1.

We consider two overlapping regions of the state-space: $Q_{>\beta} := \bigcup_{j=1}^{3} \left\{ \left( \rho : 1 - p_c > \beta_j \right) \cap \{ \rho : p_j - p_c > \beta_j/3 \} \right\}$ and $Q_{<\alpha} = S \setminus Q_{>\alpha} := \bigcap_{j=1}^{3} \left\{ \left( \rho : 1 - p_c < \alpha_j \right) \cup \{ \rho : p_j - p_c < \alpha_j/3 \} \right\}$. As illustrated by figure 1, $S = Q_{>\beta} \cup Q_{<\alpha}$ with the hysteresis zone being $Q_{>\beta} \cap Q_{<\alpha}$. Then we analyze how the diffusion behaves on each feedback region, by computing its infinitesimal generator. By design of the hysteresis, well-posedness of the solution then follows from standard arguments on the construction of solutions of SDE’s. There remains to check that $\mathcal{A}(V) \leq -rV$.

The proof consists mainly of two steps

1. Compute with (3) the value of $\mathcal{A}(V)$ for any value of the control gain-vector $\kappa$. We have formally

$$\mathcal{A}V = \frac{c}{2(1-p_c)^2} \left( \sum_{j=1}^{3} \kappa_j^2 (1-p_c)(p_j-p_c) + \sum_{k=1}^{3} \eta_k \Gamma_k \left( (1 - \text{Trace}(S_k \rho))^2 p_c^2 \right) + \sum_{j=1}^{3} \frac{1}{c} \left( \text{Trace}(\iota[X_j, \rho] \Pi_C) \right)^2 \right). \quad (7)$$

The last line can only improve convergence because $\iota \Pi_C[X_j, \rho] \Pi_C$ is Hermitian. We will drop it from now on.

2. Show that $\mathcal{A}(V) \leq -rV(\rho)$ for any $\rho$ with the feedback gains $\kappa(\rho)$ following the hysteresis logic.

- The second member of (7) satisfies the following inequality for $\rho \in Q_{<\alpha}$:

$$\mathcal{A}V \leq -\frac{c}{2(1-p_c)^2} \left( \sum_{j=1}^{3} \eta_j \Gamma_j \left( 1 - \text{Trace}(S_j \rho) \right)^2 \right) \leq -\frac{c}{2(1-p_c)^2} \text{min}_{j} (\eta_j \Gamma_j) \left( \frac{16}{3} (1-p_c) \right)^2 \leq -\frac{8}{3} \left( 1 - \text{max} (\alpha_1, \alpha_2, \alpha_3, \frac{3 \sum_j \alpha_j/3}{\beta_j}) \right)^2 \cdot \text{min}_{j} (\eta_j \Gamma_j) V. \quad (8)$$

In the first inequality we have used that by construction of the feedback $\kappa_j = 0$ when $p_j - p_c \leq 0$, and used again that $1 - \text{Trace}(S_j \rho) = 2(1 - p_c - p_j)$ with $p_1 + p_2 + p_3 = 1 - p_c$; in the last line we have computed a lower bound on $p_c$ when we know that, for each $j$, either $p_c > 1 - \alpha_j$ or $p_c > p_j - \alpha_j/3$.

- For $\rho \in S \setminus Q_{<\alpha}$, necessarily, for at least one $j$, $\kappa_j = \sqrt{\frac{2 \alpha_j}{\beta_j}}$ and $p_c < 1$. Thus we have, from the first term in (7),

$$\mathcal{A}V \leq -\frac{c}{2} \text{min}_{j \in \{1,2,3\}} (\eta_j \Gamma_j) V.$$

We can repeat the same analysis with $Q_{>\alpha}$ replaced by $S \setminus Q_{>\beta}$, and $S \setminus Q_{<\alpha}$ replaced by $Q_{>\beta}$, obtaining the same or better bounds; this covers the hysteresis behavior with standard arguments. Thus for any $\rho \in S$ and any compatible control, we have $\mathcal{A}(V) \leq -rV$.

Typically one would take $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ very close to 1, in a situation where the $\eta_j \Gamma_j$ are all equal. Then the convergence rate from the theorem simplifies to

$$r = \frac{1}{3}(1-\alpha)^2 \eta \Gamma.$$

4 On the protection of quantum information

It is well-known in control theory that exponential stability gives an indication of robustness against unmodeled dynamics. In the present case, this concerns the first control goal, namely stabilization of $\rho_t$ close to the nominal subspace $\mathcal{C}$ in the presence of bit-flip errors $\gamma_s \neq 0$. About the second control goal, namely keeping the dynamics on $\mathcal{C}$ close to zero, the analysis of the previous section is less telling.

We can illustrate both control goals by simulation. We set an initial condition $\rho_0 = \langle \psi \rangle \langle \psi \rangle$, $\langle \psi \rangle \in \mathcal{C}$ and simulate 100 closed-loop trajectories under the feedback law of section 3.1. We compare the average evolution of this encoded qubit with a single physical qubit subject to a $\sigma_x$ decoherence of the same strength, since this is the situation that the bit-flip code is meant to improve. Parameter values and simulation results are shown on Figure 2.

Regarding the first control goal, we observe that the controller indeed confines the mean evolution to a small neighborhood of $\mathcal{C}$, for all times, as expected from our analysis. Regarding the second criterion, the distance between $\rho_t$ and $\rho_0$ cannot be confined to a small value for all times. Indeed, majority vote can decrease the rate of information corruption but not totally suppress it; as corrupted information is irreversibly lost, $\rho_t$ progressively converges towards an equal distribution of logical 0 and logical 1. However, for the protected 3-qubit code, this information loss is much slower than for the single qubit; this indicates that the 3-qubit code with our feedback law indeed improves on its components.

In our feedback design, making $\alpha_j$ closer to 0 would improve the convergence rate estimate in Theorem 3.1; however, this also has a negative effect on the codeword fidelity, since it means that we turn on the noisy drives more often. Analytically computing the optimal trade-off is the subject of ongoing work. Simulations clearly show that intermediate values of the constants deliver better overall results. One would typically expect as
Figure 2: In black: mean fidelity of the logical qubit w.r.t. $\Pi_C$ under active feedback. In gray: mean fidelity of the logical qubit towards $\rho_0$ under active quantum feedback. In blue, for comparison: mean fidelity towards $\rho_0$ of a single physical qubit without measurement/control and subject to bit-flip disturbances with $\gamma = 1/100$. Fidelity is Trace($\rho \rho_0$) and roughly represents $1$-distance. The initial state is chosen as $\rho_0 = |\psi\rangle \langle \psi|$, $|\psi\rangle = \cos(\theta)|000\rangle + e^{i\varphi}\sin(\theta)|111\rangle$ for the 3-qubit code and $|\psi\rangle = \cos(\theta)|0\rangle + e^{i\varphi}\sin(\theta)|1\rangle$ for the single qubit, with $\theta = \pi/8$, $\varphi = \pi/2$. Simulation parameters are $\Gamma_j = 1$, $\gamma_j = 1/100$, $\eta_j = 0.5$, and for the feedback law $\beta_j = 0.1$, $\alpha_j = 0.95$, $c = 1$.

a rule of thumb, that logical error rate induced by the control noise should be chosen of comparable order as the natural errors $\gamma_s$.

5 Conclusions

We have approached continuous-time quantum error correction in the same spirit as Ahn et al. [2002], and showed how introducing Brownian motion to drive control fields yields exponential stabilization of the nominal codeword manifold. The main idea relies on the fact that the SDE in open loop stochastically converges to one of a few steady-state situations, but on the average does not move closer to any particular one. It is then sufficient to activate noise only when the state is close to a bad equilibrium, in order to induce global convergence to the target ones. This general idea can be extended to other systems with this property, and in particular to more advanced error-correcting schemes. In the same line, while we have proposed particular controls with hysteresis, proving a similar property with smoother control gains should not be too different. The convergence rate obtained is dependent on our choice of Lyapunov function and on the values of $\alpha_j$; from parallel investigation it seems possible to get a closed-loop convergence rate arbitrarily close to the measurement rate.

However, unlike in classical control problems, the key performance indicator is not how fast we approach the target manifold. Instead, what matters is how well, in presence of disturbances, we preserve the encoded information. Towards this goal, we should refrain from disturbing the system with feedback actions; accordingly, we have noticed that taking $\alpha_j$ closer to 1 can improve the codeword fidelity, despite leading to a slower convergence rate estimate. A theoretical analysis of information-protection capabilities is the subject of ongoing work.

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