Research Article

New High-Order Energy Preserving Method of the Fractional Coupled Nonlinear Schrödinger Equations

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The symplectic structure is given for the fractional coupled nonlinear Schrödinger equations. The Fourier spectral method and the fourth-order combination average vector field (AVF) method are applied to discretize the structure, and a new format for the fractional coupled nonlinear Schrödinger equations is obtained. The numerical experiments are showed to illustrate the property of the new format. The new scheme can maintain the energy conservation property better than the classical symplectic scheme.

1. Introduction

The fractional coupled nonlinear Schrödinger (FCNLS) equations are the generalization of the classical Schrödinger equations, which is introduced by Laskin via replacing Brownian motions in Feynman path integrals by the Lévy ones [1]. It involves the fractional Laplacian with the Lévy index $1 < \alpha \leq 2$ instead of the usual one. Constitutive equations for fractional order viscoelastic beam are constructed in the manner of Euler–Bernoulli beam theory by Yıldırım and Alkan [2]. In the special case of $\alpha = 2$, the fractional nonlinear Schrödinger equation can be simplified as the classical Schrödinger equation, which describes the evolution of microscopic particles and has been studied extensively [3–5]. While for $\alpha = 1$, the equation collapses to the relativistic Hartree equation describing the dynamics of boson stars [6–8]. The fractional coupled nonlinear Schrödinger equations have been attracting great interest for scholars. Ortigueira proposed the so-called fractional-centered difference to approximate the fractional Laplacian operator [9] and this method is of the second-order accuracy. Bai et al. proposed the preconditioned modified HSS iteration method [10]. Atangana and Cloot [11] studied the Crank–Nicolson difference scheme for the space fractional variable-order Schrödinger equations [11]. Wei et al. [12] investigated the implicit fully discrete local discontinuous Galerkin methods for the fractional equations and coupled one which involves the Caputo-type time fractional derivative [12, 13]. Ran and Zhang [14] proposed an implicit conservative difference scheme for numerically solving the strongly CNLS equations [14]. Wang et al. [15] studied the difference method for the space fractional coupled nonlinear Schrödinger equations [15]. Wang and Huang [16] constructed the structure-preserving algorithms with the Fourier pseudo-spectral approximation to the spatial fractional operator [16]. Alkan et al. [17] used a sinc-collocation method to approximate the solution of fractional order boundary value problem [17].

The construction of energy conservation schemes plays an important role in the numerical solution of energy conservation partial differential equations. Recently, energy preserving methods for the Hamiltonian system have received much attention [18–20]. Quispel and McLaren [21] constructed the energy preserving scheme of Hamilton system by the average vector field (AVF) method [21], and it can accurately simulate the evolution of the system over a long period of time and preserve the energy conservation of the system. Cai et al. [22] structured a fourth-order AVF method based on the composition technique [22]. At the same time, the second-order AVF method has also been
proposed to solve the multisymplectic structure PDEs, which can also preserve the energy conservation [23]. Based on the composition technique and the AVF method, we proposed a fourth-order energy preserving composition scheme of the multisymplectic structure PDEs [24]. The fractional coupled nonlinear Schrödinger equations have the energy conservation property. Here, we constructed the fourth-order scheme based on the AVF method by using the composition technique, applied the scheme for the fractional coupled nonlinear Schrödinger equations, and compared the energy preserving property and accuracy of new schemes with the other classical schemes to demonstrate the performance of our scheme.

This paper is organized as following. In Section 2, the symplectic structure of the FCNLS equations are proposed. In Section 3, the structures are discretized by the Fourier spectrum method on the spatial direction. Then, the fourth-order combination AVF method, the second-order AVF method, and the symplectic method are used to discrete the time direction, respectively. In Section 4, the numerical experiments are reported. The accuracy and energy conservation property of the fourth-order combination AVF schemes are investigated. At last, we have conclusive remarks in Section 5.

2. Symplectic Structure of the FCNLS Equations

We consider the fractional coupled nonlinear Schrödinger equations (1 < \alpha \leq 2)

\begin{align}
  iu_t - (-\Delta)^{\alpha/2}u + \rho(|u|^2 + |v|^2)u &= 0, \quad t > 0, \\
  iv_t - (-\Delta)^{\alpha/2}v + \rho(|v|^2 + |u|^2)v &= 0, \quad t > 0,
\end{align}

with the initial conditions \( u(x,0) = u_0(x), v(x,0) = v_0(x) \), where \( i = \sqrt{-1} \), the parameter \( \rho > 0 \) is a real constant. When \( x \in \mathbb{R} \), the fractional Laplacian can be defined by the Fourier transform as

\[ (-\Delta)^{\alpha/2}u(x,t) = -\int_{\mathbb{R}} |\xi|^{\alpha} \hat{u}(\xi,t)e^{ix\xi}d\xi, \]

where the Fourier transform is defined by

\[ \hat{u}(\xi,t) = \frac{1}{2\pi} \int_{\mathbb{R}} u(x,t)e^{-ix\xi}dx. \]

Let \( u(x,t) = p(x,t) + iq(x,t), \quad v(x,t) = w(x,t) + iw(x,t) \), equation (1) can be written as

\begin{align}
  p_t - (-\Delta)^{\alpha/2}q + \rho(p^2 + q^2 + w^2 + \phi^2)q &= 0, \\
  q_t + (-\Delta)^{\alpha/2}p - \rho(p^2 + q^2 + w^2 + \phi^2)p &= 0, \\
  w_t - (-\Delta)^{\alpha/2}\phi + \rho(w^2 + \phi^2 + p^2 + q^2)\phi &= 0, \\
  \phi_t + (-\Delta)^{\alpha/2}w - \rho(w^2 + \phi^2 + p^2 + q^2)w &= 0.
\end{align}

Equation (4) can be transformed as the following symplectic structure

\[ z_t = J \frac{\delta H(z)}{\delta z}, \]

where \( z = (p, q, w, \phi)^T \), and the Hamiltonian function reads

\[ H = \frac{1}{2} \int_{\Omega} \left( (-\Delta)^{\alpha/2}p^2 - ((-\Delta)^{\alpha/2}q)^2 - ((-\Delta)^{\alpha/2}w)^2 - ((-\Delta)^{\alpha/2}\phi)^2 \right) dx + \rho(4) \int_{\Omega} (p^2 + q^2 + w^2 + \phi^2)^2 dx, \]

\[ J = \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  -1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & -1 & 0
\end{pmatrix}. \]

3. Discretization for the FCNLS Equations

3.1. Spatial Discretization for the FCNLS Equations

The fractional Laplacian operator \( \mathcal{L} \) is defined by its symbols in the Fourier space, so we use the Fourier pseudo-spectral method to discretize them. We first construct a discrete approximation of the solution through interpolating trigonometric polynomial of the solution at collocation points and then approximate its fractional derivative in the frequency space based on the symbol of the operator [24–27].

Give a positive even integer \( N \), set the mesh size \( h = (b - a)/N \), and let \( x_j = a + jh, 0 \leq j \leq N - 1 \) be the grid points in \([a, b]\), which are referred to as the Fourier collocation points. Then the interpolation approximation \( I_N p(x) \) denoted by \( p_N(x) \) of the function \( p(x) \) has the form

\[ (I_N p)(x) = p_N(x) = \sum_{k=-N/2}^{N/2} \tilde{p}_k e^{ik\mu(x-a)}, \]

where \( \tilde{p}_k = (1/Nc_k) \sum_{j=0}^{N-1} p(x_j) e^{-ik\mu(x_j-a)} \) and \( \mu = 2\pi/(b-a) \), \( c_k = 1/p \) for \( k < (N/2) \) and \( c_k = 2/p \) for \( k = \pm (N/2) \). Then we approximate the fractional Laplacian of \( p(x) \) by

\[ (-\Delta)^{\alpha/2}p_N(x_j) = -\sum_{k=-N/2}^{N/2} |k\mu|^\alpha \tilde{p}_k e^{ik\mu(x_j-a)}, \]

and similarly, approximate the operator \( \mathcal{L} \) by

\[ \mathcal{L} p_N(x_j) = \sum_{k=-N/2}^{N/2} ik\mu|k\mu|^{(\alpha-2)/2} \tilde{p}_k e^{ik\mu(x_j-a)}. \]

In order to facilitate the analysis below, we rewrite the above approximation in a matrix form in the physical space. To this end, denoting \( p_j = p(x_j) \) and plugging \( \tilde{p}_k \) into (8) yield

\[ (-\Delta)^{\alpha/2}p_N(x_j) = -\sum_{k=-N/2}^{N/2} |k\mu|^\alpha \left( \sum_{a=0}^{N-1} p_a e^{-ik\mu(x_j-a)} \right) e^{ik\mu(x_j-a)} \]

\[ = \sum_{a=0}^{N-1} p_a \left( -\sum_{k=-N/2}^{N/2} \frac{1}{Nc_k} |k\mu|^\alpha e^{ik\mu(x_j-x_a)} \right) = (D_\alpha^2 p)_j, \]

where \( D_\alpha^2 \) is an \( N \times N \) matrix with elements

\[ j = \int \frac{\delta H(z)}{\delta z}, \]
where $\mathbf{D}_n^i$ is an $N \times N$ matrix with elements
\[
(D_{n}^{i})_{jd} = \sum_{k=-N/2}^{N/2} \frac{1}{Nc_k} ik\mu|k\mu|^{(a-2)/2} e^{ik\mu(x_j-x_k)},
\]
and $\mathbf{p} = (p_0, \ldots, p_{N-1})^T$. Plugging $\mathbf{p}$ into (9) yields
\[
\mathcal{L}\mathbf{P}_N(x_j) = \sum_{k=-N/2}^{N/2} ik\mu|k\mu|^{(a-2)/2} \left( \sum_{d=0}^{N-1} \frac{1}{Nc_k} P_d e^{-i k\mu x_d} \right) e^{i k\mu x_j},
\]
\[
= \sum_{d=0}^{N-1} P_d \left( \sum_{k=-N/2}^{N/2} ik\mu|k\mu|^{(a-2)/2} e^{i k\mu(x_j-x_k)} \right)
= (D_{n}^{i} \mathbf{p}),
\]
where (12) holds.

Applying the Fourier pseudo-spectral method to equation (4) in spatial direction, we can obtain
\[
\frac{dp_j}{dt} = -(D_{n}^{2} q_j) - \rho(p_j^2 + q_j^2 + w_j^2 + \phi_j^2)q_j,
\]
\[
\frac{d\phi_j}{dt} = (D_{n}^{2} \mathbf{w}) + \rho(p_j^2 + q_j^2 + w_j^2 + \phi_j^2)\phi_j,
\]
where $j = 0, \ldots, N-1$.

The combination AVF method is applied to discretize equation (15), then we get
\[
\frac{\mathbf{Z}^{n+1} - \mathbf{Z}^n}{\tau} = c_1 J_N \int_0^1 \nabla H((1-\xi)\mathbf{Z}^n + \xi \mathbf{Z}^{n+1}) d\xi,
\]
where $\mathbf{Z} = (p_0, \ldots, p_{N-1}, q_0, \ldots, q_{N-1}, w_0, \ldots, w_{N-1}, \phi_0, \ldots, \phi_{N-1})^T$.

3.2 Time Discretization for the Symplectic FCNLS Equations.

The corresponding Hamiltonian function is
\[
H(\mathbf{Z}) = -\frac{1}{2} \left( \mathbf{p}^T D_{n}^{2} \mathbf{p} + q^T D_{n}^{2} \mathbf{q} + \mathbf{w}^T D_{n}^{2} \mathbf{w} + \phi^T D_{n}^{2} \phi \right)
\]
\[
+ \frac{\rho}{4} \sum_{j=0}^{N-1} (p_j^2 + q_j^2 + w_j^2 + \phi_j^2)^2.
\]

It also can be written as
\[
\frac{w_j^{n,1} - w_j^n}{\tau} = -c_2 \left( D_2 D_2 \phi_j^{n,1} + \phi_j^{n,1} \right)_j + \frac{\rho}{12} \phi_j^n \left( 3(p_j^n)^2 + 2p_j^n p_j^{n,1} + (p_j^{n,1})^2 \right) \\
+ 3(q_j^n)^2 + 2q_j^n d_j^{n,1} + \left( a_j^{n,1} ight)^2 + 2w_j^n w_j^{n,1} + (u_j^{n,1})^2 + 3\left( \phi_j^n \right)^2 + 2\phi_j^n \phi_j^{n,1} + \\
\left( \phi_j^{n,1} \right)^2 + \phi_j^n \left( (p_j^n)^2 + 2p_j^n p_j^{n,1} + 3(p_j^{n,1})^2 \right) + (q_j^n)^2 + 2q_j^n d_j^{n,1} + 3(d_j^{n,1})^2 \\
\left( u_j^n \right)^2 + 2w_j^n w_j^{n,1} + 3\left( \phi_j^n \right)^2 + 2\phi_j^n \phi_j^{n,1} + 3\left( \phi_j^{n,1} \right)^2 \right),
\]

\[
\frac{\phi_j^{n,1} - \phi_j^n}{\tau} = c_1 \left( D_2 w_j^n + u_j^{n,1} \right)_j + \frac{\rho}{12} w_j^n \left( 3(p_j^n)^2 + 2p_j^n p_j^{n,1} + (p_j^{n,1})^2 \right) \\
+ 3(q_j^n)^2 + 2q_j^n d_j^{n,1} + \left( a_j^{n,1} ight)^2 + 2w_j^n w_j^{n,1} + (u_j^{n,1})^2 + 3\left( \phi_j^n \right)^2 + 2\phi_j^n \phi_j^{n,1} + \\
\left( \phi_j^{n,1} \right)^2 + w_j^n \left( (p_j^n)^2 + 2p_j^n p_j^{n,1} + 3(p_j^{n,1})^2 \right) + (q_j^n)^2 + 2q_j^n d_j^{n,1} + 3(d_j^{n,1})^2 \\
\left( u_j^n \right)^2 + 2w_j^n w_j^{n,1} + 3\left( \phi_j^n \right)^2 + 2\phi_j^n \phi_j^{n,1} + 3\left( \phi_j^{n,1} \right)^2 \right),
\]

\[
\frac{p_j^{n,2} - p_j^{n,1}}{\tau} = -c_2 \left( D_2 \phi_j^{n,1} + q_j^{n,2} \right)_j + \frac{\rho}{12} q_j^n \left( 3(p_j^n)^2 + 2p_j^n p_j^{n,1} + (p_j^{n,1})^2 \right) \\
\left( p_j^{n,2} \right)^2 + 3\left( q_j^{n,1} \right)^2 + 2q_j^{n,1} q_j^{n,2} + \left( a_j^{n,2} \right)^2 + 2w_j^{n,1} w_j^{n,2} + (u_j^{n,2})^2 + 3\left( \phi_j^{n,1} \right)^2 \\
+ 2\phi_j^{n,1} \phi_j^{n,2} + \left( \phi_j^{n,2} \right)^2 + q_j^{n,2} \left( (p_j^n)^2 + 2p_j^n p_j^{n,1} + 3(p_j^{n,1})^2 \right) + (q_j^n)^2 + 2q_j^n d_j^n \\
+ 3\left( q_j^n \right)^2 + (u_j^{n,1})^2 + 2w_j^{n,1} w_j^{n,2} + 3\left( \phi_j^n \right)^2 + 2\phi_j^n \phi_j^{n,2} + 3\left( \phi_j^{n,2} \right)^2 \right),
\]

\[
\frac{q_j^{n,2} - q_j^{n,1}}{\tau} = c_2 \left( D_2 p_j^{n,1} + p_j^{n,2} \right)_j + \frac{\rho}{12} p_j^n \left( 3(p_j^n)^2 + 2p_j^n p_j^{n,1} + (p_j^{n,1})^2 \right) \\
\left( p_j^{n,2} \right)^2 + 3\left( q_j^{n,1} \right)^2 + 2q_j^{n,1} q_j^{n,2} + \left( a_j^{n,2} \right)^2 + 2w_j^{n,1} w_j^{n,2} + (u_j^{n,2})^2 + 3\left( \phi_j^{n,1} \right)^2 \\
+ 2\phi_j^{n,1} \phi_j^{n,2} + \left( \phi_j^{n,2} \right)^2 + p_j^{n,2} \left( (p_j^n)^2 + 2p_j^n p_j^{n,1} + 3(p_j^{n,1})^2 \right) + (q_j^n)^2 + 2q_j^n d_j^n \\
+ 3\left( q_j^n \right)^2 + (u_j^{n,1})^2 + 2w_j^{n,1} w_j^{n,2} + 3\left( \phi_j^n \right)^2 + 2\phi_j^n \phi_j^{n,2} + 3\left( \phi_j^{n,2} \right)^2 \right),
\]

\[
\frac{u_j^{n,2} - u_j^{n,1}}{\tau} = -c_2 \left( D_2 \phi_j^{n,1} + \phi_j^{n,2} \right)_j + \frac{\rho}{12} \phi_j^n \left( 3(p_j^n)^2 + 2p_j^n p_j^{n,1} + (p_j^{n,1})^2 \right) \\
\left( p_j^{n,2} \right)^2 + 3\left( q_j^{n,1} \right)^2 + 2q_j^{n,1} q_j^{n,2} + \left( a_j^{n,2} \right)^2 + 2w_j^{n,1} w_j^{n,2} + (u_j^{n,2})^2 + 3\left( \phi_j^{n,1} \right)^2 \\
+ 2\phi_j^{n,1} \phi_j^{n,2} + \left( \phi_j^{n,2} \right)^2 + q_j^{n,2} \left( (p_j^n)^2 + 2p_j^n p_j^{n,1} + 3(p_j^{n,1})^2 \right) + (q_j^n)^2 + 2q_j^n d_j^n \\
+ 3\left( q_j^n \right)^2 + (u_j^{n,1})^2 + 2w_j^{n,1} w_j^{n,2} + 3\left( \phi_j^n \right)^2 + 2\phi_j^n \phi_j^{n,2} + 3\left( \phi_j^{n,2} \right)^2 \right) + \frac{\rho}{12} \phi_j^n \left( 3(p_j^n)^2 + 2p_j^n p_j^{n,1} + (p_j^{n,1})^2 \right)
\]
\[
\begin{align*}
\frac{\phi_j^{n_2} - \phi_j^{n_1}}{\tau} &= c_2 \left( \frac{D_2^z w_{j}^{n_1} + w_{j}^{n_2}}{2} \right) + \frac{\rho}{12} \left( u_{j}^{n_1} \left( 3(p_j^{n_1})^2 + 2p_{j_1}^{n_1} p_{j_2}^{n_2} + (p_j^{n_1})^2 + 3(q_j^{n_1})^2 + 2q_{j_1}^{n_1} q_{j_2}^{n_2} + (q_j^{n_1})^2 \right) + u_{j}^{n_2} \left( 3(p_j^{n_2})^2 + 2p_{j_1}^{n_2} p_{j_2}^{n_1} + (p_j^{n_2})^2 + 3(q_j^{n_2})^2 + 2q_{j_1}^{n_2} q_{j_2}^{n_1} + (q_j^{n_2})^2 \right) \right) \\
\frac{p_{j_1}^{n_1} - p_{j_1}^{n_2}}{\tau} &= -c_3 \left( \frac{D_2^z q_{j_2}^{n_2} + q_{j_2}^{n_1}}{2} \right) + \frac{\rho}{12} \left( p_{j_2}^{n_1} \left( 3(p_{j_2}^{n_1})^2 + 2p_{j_1}^{n_1} p_{j_2}^{n_2} + (p_{j_2}^{n_1})^2 + 3(q_{j_2}^{n_1})^2 + 2q_{j_1}^{n_1} q_{j_2}^{n_2} + (q_{j_2}^{n_1})^2 \right) \right) \\
\frac{q_{j_2}^{n_1} - q_{j_2}^{n_2}}{\tau} &= c_3 \left( \frac{D_2^z p_{j_2}^{n_2} + p_{j_2}^{n_1}}{2} \right) + \frac{\rho}{12} \left( p_{j_2}^{n_2} \left( 3(p_{j_2}^{n_2})^2 + 2p_{j_1}^{n_2} p_{j_2}^{n_1} + (p_{j_2}^{n_2})^2 + 3(q_{j_2}^{n_2})^2 + 2q_{j_1}^{n_2} q_{j_2}^{n_1} + (q_{j_2}^{n_2})^2 \right) \right) \\
\frac{w_j^{n_1} - w_j^{n_2}}{\tau} &= -c_3 \left( \frac{D_2^z \phi_{j_2}^{n_2} + \phi_{j_2}^{n_1}}{2} \right) + \frac{\rho}{12} \left( \phi_{j_2}^{n_1} \left( 3(p_{j_2}^{n_1})^2 + 2p_{j_1}^{n_1} p_{j_2}^{n_2} + (p_{j_2}^{n_1})^2 + 3(q_{j_2}^{n_1})^2 + 2q_{j_1}^{n_1} q_{j_2}^{n_2} + (q_{j_2}^{n_1})^2 \right) \right) \\
\frac{\phi_j^{n_2} - \phi_j^{n_1}}{\tau} &= c_3 \left( \frac{D_2^z w_{j}^{n_2} + w_{j}^{n_1}}{2} \right) + \frac{\rho}{12} \left( u_{j}^{n_2} \left( 3(p_j^{n_2})^2 + 2p_{j_1}^{n_2} p_{j_2}^{n_1} + (p_j^{n_2})^2 + 3(q_j^{n_2})^2 + 2q_{j_1}^{n_2} q_{j_2}^{n_1} + (q_j^{n_2})^2 \right) \right) \\
\end{align*}
\]
Table 1: The \( L^\infty \) errors and the order of convergence of three different schemes with \( h = 0.02 \) and different time steps at \( t = 1.6 \).

| \( t = 1.6 \) | AVF scheme | Order | Symplectic scheme | Order | Combination AVF scheme | Order |
|------------|------------|-------|-------------------|-------|------------------------|-------|
| \( \tau = 0.08 \) | 1.2743e-01 | —     | 1.2876e-01 | —     | 3.2165e-02 | —     |
| \( \tau = 0.04 \) | 2.9558e-02 | 2.1082 | 2.9890e-02 | 2.1071 | 2.600e-03 | 3.6292 |
| \( \tau = 0.02 \) | 7.2589e-03 | 2.0259 | 7.3525e-03 | 2.0235 | 1.9171e-04 | 3.7618 |
| \( \tau = 0.01 \) | 1.8076e-03 | 2.0058 | 1.8304e-03 | 2.0064 | 1.2805e-05 | 3.9044 |

Table 2: The mass values of three different schemes with \( \tau = 0.02 \) at \( t = 20 \).

| \( \alpha = 1.2 \) | Initial value | AVF scheme | Symplectic scheme | Combination AVF scheme |
|----------------|--------------|------------|-------------------|------------------------|
| 10             | 9.99561      | 10.00003   | 9.9971            |
| 10             | 9.99598      | 10.00000   | 9.9958            |
| 10             | 9.99825      | 10.00017   | 9.9996            |
| 10             | 9.99917      | 10.00000   | 9.99997           |

**Theorem 1.** System (18) can preserve the energy conservation property.

**Proof.** To the first equation in scheme (18), taking the scaling product of both sides of the equation with respect to \( \int_0^1 V H((1 - \xi)Z_0 + \xi Z_1) d\xi \), we can get

\[
\frac{c_1}{h} \int_0^1 (Z_1 - Z_0) \nabla H((1 - \xi)Z_0 + \xi Z_1) d\xi = 0,
\]

\[
\frac{c_1}{h} \int_0^1 \frac{d}{d\xi} H((1 - \xi)Z_0 + \xi Z_1) d\xi = 0.
\]

(20)

From the fundamental theorem of calculus, we can obtain

\[
\frac{c_1}{h} (H(Z_1) - H(Z_0)) = 0.
\]

(21)

The second and third equations in scheme (18) are the same as the first equation. Therefore, the new fourth composition scheme (18) can preserve the energy conservation.

We also apply the following second-order AVF method to discretize equation (15):

\[
\frac{Z^{n+1} - Z^n}{\tau} = J_N \int_0^1 \nabla H((1 - \xi)Z^n + \xi Z^{n+1}) d\xi,
\]

(22)

where \( \tau \) is the time step. The following symplectic scheme of the FCNLS equations is also given to compare with the AVF method of symplectic FCNLS equations [28–30]:

\[
\frac{Z^{n+1} - Z^n}{\tau} = J_N \nabla H\left(\frac{Z^n + Z^{n+1}}{2}\right).
\]

(23)

4. Numerical Experiment

In this section, numerical experiments for the fractional coupled nonlinear Schrödinger equations with periodic boundary conditions are presented to investigate the relative energy error and accuracy of convergence. The energy error is defined as

\[
E(t) = \frac{\mathcal{T}(Z^n) - \mathcal{T}(Z^0)}{\mathcal{T}(Z^0)},
\]

(24)

where \( \mathcal{T}(Z^0) \) is the initial energy, and \( \mathcal{T}(Z^n) \) is the energy value at \( t_n = nr \). The maximal module error of numerical solution and exact solution is defined as

\[
L^\infty (\tau) = \max_{1 \leq j \leq N} |u_j^n - u(x_j, t_n)|.
\]

(25)

The order of convergence is defined as

\[
\text{Ratio} = \frac{\log(L^\infty (\tau)/L^\infty (\tau/2))}{\log(2)}.
\]

(26)

We consider the FCNLS equations. Equation (1) has the initial conditions such as

\[
u(x, 0) = \sech(x + d_0) e^{iV_0 x},
\]

(27)

\[
u(x, 0) = \sech(x - d_0) e^{-iV_0 x},
\]

where \( d_0 = 1, V_0 = 2, \rho = 2, x \in [-20, 20], N = 200, \) and \( \tau = 0.01 \).

Since it is difficult to find the exact solution of fractional coupled Schrödinger equation and calculate the convergence order of the system, we use the following fractional Schrödinger equation to calculate the convergence order of the system

\[
iu_t - (-\Delta)^{\alpha/2} u + 2|u|^2 u = 0,
\]

(28)

where \( \alpha = 2 \). The exact solution can be given by

\[
u(x, t) = \sech(x - 4t) e^{i(2\alpha - 3t)}.
\]

(29)

First, we consider the symplectic structure of FCNLS equations. The discrete energy corresponding to (15) can be expressed as

\[
\mathcal{T}(Z) = -\frac{1}{2} \left( p^T D_s^\alpha p + q^T D_s^\alpha q + w^T D_s^\alpha w + \phi^T D_s^\alpha \phi \right)
\]

\[
+ \sum_{j=0}^{N-1} \left( p_j^2 + q_j^2 + w_j^2 + \phi_j^2 \right)^2,
\]

(30)
Figure 1: Numerical solutions of $|u| + |v|$ in $t \in [0, 12]$ at $\alpha = 1.8$.

Figure 2: Numerical solutions of $|u| + |v|$ in $t \in [0, 12]$ at $\alpha = 1.4$.

Figure 3: Energy errors for the fourth-order combination AVF method in $t \in [0, 20]$ with different values of $\alpha$.

Figure 4: Energy errors for the second-order AVF method in $t \in [0, 20]$ with different values of $\alpha$. 
and the discrete mass corresponding to (15) can be written as

$$\mathcal{M}(Z) = \frac{\rho}{4} \sum_{j=0}^{N-1} \left( \rho_j^2 + q_j^2 + w_j^2 + \phi_j^2 \right)^2 \tag{31}.$$ 

Table 1 shows the error of numerical solution and exact solution and the order of convergence of the different schemes with different time steps at $t = 1.6$. It is easy to see that the orders of convergence of the fourth-order energy preserving combination scheme are almost equal to 4, and the orders of convergence of AVF scheme are almost equal to 2. The new fourth energy preserving combination scheme is more accurate than other two schemes. Table 2 shows the mass values of three numerical schemes with different $\alpha$. One can observe that three numerical schemes all can preserve the conservation of mass approximately, and the symplectic scheme has the best mass preserving property, followed by the combination AVF scheme, and finally the AVF scheme. These data can be obtained by the Matlab programs based on the symplectic scheme and the combination AVF scheme of the paper.

From Figures 1 and 2, we can get that the numerical solution obtained from the fourth-order combination AVF scheme can well simulate the waves. The energy errors of the fourth-order combination AVF method with different $\alpha$ values are plotted in Figure 3. The error is up to $10^{-13}$. Figure 4 shows the energy error obtained from the second-order AVF method with two different $\alpha$ values. The error is up to $10^{-14}$. Figure 5 shows the energy error obtained from the second-order symplectic method with different $\alpha$ values. The error is only up to $10^{-4}$. We can get that the fourth-order combination AVF method and second-order AVF method preserve energy conservation of the system more accurately.

5. Concluding Remarks

We construct symplectic fourth-order combination AVF formats of the fractional coupled nonlinear Schrödinger equations and compare the new high order scheme with the second-order AVF formats and symplectic formats. Numerical results show that the new fourth-order combination AVF formats have fourth-order accuracy. The new high order scheme can accurately simulate evolution of the equations and maintain the energy conservation well.

Data Availability

These data can be obtained by the Matlab programs based on symplectic scheme and the combination AVF scheme of the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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