Abstract. The distances that galactic cosmic ray electrons and positrons can travel are severely limited by energy losses to at most a few kiloparsec, thereby rendering the local spectrum very sensitive to the exact distribution of sources in our galactic neighbourhood. However, due to our ignorance of the exact source distribution, we can only predict the spectrum stochastically. We argue that even in the case of a large number of sources the central limit theorem is not applicable, but that the standard deviation for the flux from a random source is divergent due to a long power law tail of the probability density. Instead, we compute the expectation value and characterise the scatter around it by quantiles of the probability density using a generalised central limit theorem in a fully analytical way. The uncertainty band is asymmetric about the expectation value and can become quite large for TeV energies. In particular, the predicted local spectrum is marginally consistent with the measurements by Fermi-LAT and HESS even without imposing spectral breaks or cut-offs at source. We conclude that this uncertainty has to be properly accounted for when predicting electron fluxes above a few hundred GeV from astrophysical sources.

Keywords: Cosmic ray theory, supernova remnants
The interest in the propagation of galactic cosmic ray (GCR) electrons and positrons from astrophysical sources, like supernova remnants (SNRs), has recently been revived, mostly in light of contemporary, partly “anomalous” measurements [1–6]. In the context of their possible explanation as exotic signals from dark matter annihilation or decay, it is usually believed that astrophysical sources of GCRs only lead to local power law spectra, in contrast to strong features from exotic sources. This however relies on a simplified picture of propagation assuming not only a steady state situation but also a continuous spatial distribution of sources.

On the time and distance scales of GCR propagation, however, astrophysical sources can be considered discrete. Diffusion renders the fluxes originating from point-like and from extended sources like old SNRs indistinguishable. Furthermore, although the details of the injection of particles from SNRs is not fully understood (for a discussion, see ref. [7]), the bulk of the accelerated GCRs is expected to be released with the onset of the radiative phase, that is within a relatively short time.

For nuclear GCRs, the spectrum depends rather weakly on the exact distribution of sources in space and time since protons and nuclei diffuse over distances of kiloparsecs before escaping from the cosmic ray halo, thereby averaging over the distribution of sources on these scales. It is therefore admissible to neglect the small scale distribution and to assume a steady and continuous distribution of sources. Such a distribution function can be obtained by generalising information from radio or x-ray surveys of source candidates [8, 9] and is being used in many computations of GCR fluxes including the most well-known numerical [10–12] and semi-analytical codes [13, 14].

The propagation of leptonic GCRs is, however, dramatically different as electrons and positrons suffer from strong energy losses due to interactions with the galactic magnetic fields and interstellar radiation fields (ISRFs). Therefore, the diffusion-loss length, i.e. the distance electrons and positrons can travel away from the sources without loosing virtually all energy, is much shorter than for protons and nuclei. In addition, this distance is energy dependent such that at high energies, that is above 100 GeV or so, mostly young and nearby sources contribute and the spectrum at these energies strongly depends on the history and spatial distribution of sources within a kiloparsec. Therefore, it is not admissible to neglect the nearby small scale distribution.
It has been proposed [15] to model the electron-positron flux by considering a continuous distribution of sources for distances $\gtrsim 1$ kpc or ages $\gtrsim 10^5$ yr, and to supplement it by the few known nearby SNRs. However, this requires that all nearby sources are known from surveys which seems, at least, challenging. First of all, such studies suffer from various selection effects. Radio surveys, for instance, are insensitive to SNRs of low surface brightness and also young but distant ones [16]. Furthermore, for individual objects it can be difficult to decide whether they can contribute to the local electron-positron flux. For example, it has been suggested that pulsars can accelerate electrons and positrons but it is not clear if these high energy particles can escape efficiently from the surrounding pulsar wind nebulae (PWNe). Recent observations [17], however, suggest that this may be the case. In addition, the prediction of a flux requires accurate distance information. Considering the uncertainty in the distance estimates of some of the known SNRs (for example, for RX J1713.7-3946 distance estimates vary between 1 and 6 kpc [18, 19]) we cannot expect the distances of sources yet to be discovered to be much more reliable. Most importantly however, surveys using electromagnetic radiation will only tell us about sources on the past light cone. The propagation of charged GCRs is however dramatically different from the rectilinear propagation of photons such that much older sources can potentially contribute. Consequently, relying on surveys we might simply miss old but nearby sources which could lead to artificial features in the predicted electron-positron flux (for an illustration of this, see ref. [20]).

It thus seems difficult to determine the complete real distribution of sources. However, as we do not expect temporal variations in the source rate to be too strong, we can generalise the information about sources on the past light cone to earlier times thereby building a model of the statistical distribution in both, position and time. The predictions possible using this statistical information are however only stochastic, that is one can predict the expectation value for the flux from a statistical ensemble of sources. For protons and nuclei, the fluctuations around the expectation value are rather small but for electrons and positrons, the flux from a particular system in the statistical ensemble, that is a particular distribution of sources like the one in the Galaxy, will in general differ from the expectation value. In this sense our ignorance of the true distribution of sources in age and distance introduces an uncertainty into the predicted electron-positron flux, in particular at high energies. This uncertainty can in principle be investigated by Monte Carlo methods [21–23], that is by computing the fluxes from a large number of systems, randomly drawing the source positions and ages from the statistical model. Here we aim at a fully analytical calculation which will allow us to check against possible prejudices, e.g. the gaussianity of the probability density for fluxes. Furthermore, this approach provides the transparency to trace the propagation of model parameters into the final result.

The paper is organised as follows. In section 2 we briefly review the usual setup for propagation of GCR electrons and positrons in a purely diffusive model. The calculation for the expectation value of the flux is outlined in section 3 and the determination of the uncertainties is explained in section 4. We present and discuss our results in section 5 and conclude in section 6.

2 Propagation Setup

The propagation of GCR electrons and positrons is governed by the diffusive transport equation [24],

$$\frac{\partial n}{\partial t} - \nabla \cdot (D \cdot \nabla) n - \frac{\partial}{\partial E} (b(E)n) = q.$$  \hspace{1cm} (2.1)
The second and third term on the left hand side describe diffusion, with the diffusion coefficient \( D \), and energy losses, with the energy loss rate \( b(E) \), respectively. We take \( b(E) = b_0 E^2 \), parametrising synchrotron radiation and inverse compton scattering in the Thomson regime.

The right hand side denotes injection of electrons and positrons from the sources. The subdominant secondary production during the propagation of nuclear GCRs is being ignored. As the propagation affects positrons in the same way as electrons, the stochastic effects will be the same for both species, and we consider only the differential density \( n \) of electrons plus positrons. Furthermore, as the stochasticity effects we are interested in only show up at higher energies, we neglect convection and reacceleration which affect the flux below \( \sim 10\) GeV. Finally, solar modulation is expected to modify the spectrum only marginally at the energies in question, so we neglect it, too. We consider the transport in a cylinder of radius \( s_{\max} \) and half-height \( z_{\max} \), however only enforcing the boundary condition in \( z \)-direction, that is \( n(t, s, z_{\max}) \equiv 0 \).

For simplicity, we assume a common spectrum for all sources\(^1\), \( Q(E_0) \), such that the source term factorises, \( q(\mathbf{r}, t, E) = \rho(\mathbf{r}, t)Q(E) \). The Green’s function with respect to the source density \( \rho(\mathbf{r}, t) \) reads,

\[
G(E, \mathbf{r}, t) = (\pi \ell^2)^{-3/2} e^{-r^2/\ell^2} Q \left( \frac{E}{1 - b_0 Et} \right) (1 - b_0 Et)^{-2},
\]

with \( \ell = \ell(E, t) \) the diffusion-loss length,

\[
\ell^2(E, t) = \frac{4D_0}{b_0 (1 - \delta)} \left[ E^{\delta - 1} - \left( \frac{E}{1 - b_0 E t} \right)^{\delta - 1} \right].
\]

To satisfy the above boundary condition, we apply the “mirror-charge” method of ref. [25], yielding,

\[
G(E, \mathbf{r}, t) = \sum_{n=-\infty}^{\infty} G(E, \mathbf{r}_n, t) \quad \text{where} \quad \mathbf{r}_n = \begin{pmatrix} x \\ y \\ (-1)^n z + 2z_{\max} n \end{pmatrix}.
\]

It is useful to factorise the spatial dependence into a dependence on radius \( s \) and \( z \),

\[
G(E, \mathbf{r}, t) = \sum_{n=-\infty}^{\infty} (\pi \ell^2)^{-3/2} e^{-r_n^2/\ell^2} Q \left( \frac{E}{1 - b_0 E t} \right) (1 - b_0 Et)^{-2}
\]

\[
= (\pi \ell^2)^{-1} e^{-s^2/\ell^2} Q \left( \frac{E}{1 - b_0 E t} \right) (1 - b_0 Et)^{-2} \frac{1}{z_{\cr}} \chi \left( \frac{z}{z_{\cr}}, \frac{\ell^2}{z_{\cr}^2} \right),
\]

where the sum has been expressed in terms of the Jacobi theta function, \( \vartheta_3 \),

\[
\chi \left( \hat{z}, \ell^2 \right) = \frac{1}{\pi} \left[ \vartheta_3 \left( \hat{z}, e^{-\ell^2} \right) - \vartheta_3 \left( \hat{z} + \frac{\pi}{2}, e^{-\ell^2} \right) \right],
\]

and \( z_{\cr} = 4z_{\max}/\pi \). As most sources as well as our position are basically in the thin galactic disk, \( z \approx 0 \) and \( \chi(\hat{z}, \ell^2) \to \chi(\ell^2) \equiv \chi(0, \ell^2) \).

\(^1\)In fact, the source parameters, e.g. total power output, spectral indices and cut-off energies, are in general different for different sources. However, this only introduces an additional source of uncertainty which we neglect here. In this sense, the magnitude of the uncertainty due to the stochasticity of source positions and ages is a lower limit on the actual uncertainty.
3 Expectation Value of the Flux

The flux from a source that injected a spectrum \( Q(E) = Q_0 E^{-\gamma} e^{-E/E_{\text{cut}}} \) of electrons and positrons a time \( t \) ago\(^2\) at a distance \( s \) from the observer is given by the Green’s function \( G \) (cf. eq. 2.6) of the diffusion equation,

\[
J_i(E) = \frac{c}{4\pi} G = \frac{c}{4\pi} \left( \pi t^2 \right)^{-1} e^{-s^2/t^2} Q_0 E^{-\gamma} (1 - b_0 Et)^{\gamma - 2} e^{-\frac{E}{E_{\text{cut}}} \frac{1}{1 - b_0 Et}} \frac{1}{z_{\text{cr}}} \chi \left( \frac{t^2}{2z_{\text{cr}}^2} \right), \tag{3.1}
\]

where \( c/(4\pi) \) denotes the “flux factor” for relativistic particles. The flux \( J \) of \( N \) identical sources at distances \( \{s_i\} \) and times \( \{t_i\} \) is the sum of the individual fluxes,

\[
J = \sum_{i=1}^{N} J_i(E) = \frac{c}{4\pi} \sum_{i=1}^{N} \left( \pi t_i^2 \right)^{-1} e^{-s_i^2/t_i^2} Q_0 E^{-\gamma} (1 - b_0 Et_i)^{\gamma - 2} e^{-\frac{E}{E_{\text{cut}}} \frac{1}{1 - b_0 Et_i}} \frac{1}{z_{\text{cr}}} \chi \left( \frac{t_i^2}{2z_{\text{cr}}^2} \right). \tag{3.2}
\]

We note that with the above form of the source spectrum all kinds of astrophysical sources can be modelled, e.g. SNRs or pulsars.

If the central limit theorem was applicable, at a fixed energy \( E \) the fluxes \( J(E) \) for different realisations of \( N \) sources drawn from the same probability density, \( f_{s,t} \), would follow a normal distribution with mean \( \mu_J \) and standard deviation \( \sigma_J \),

\[
\mu_J = \frac{c}{4\pi} N \mu_G = \frac{c}{4\pi} N \langle G \rangle, \tag{3.3}
\]

\[
\sigma_J = \frac{c}{4\pi} \sqrt{N} \sigma_G = \frac{c}{4\pi} \sqrt{N} \sqrt{\langle G^2 \rangle - \langle G \rangle^2}, \tag{3.4}
\]

were \( \langle G^m \rangle \) denotes the moments of the Green’s function \( G \equiv G(E,s,t) \),

\[
\langle G^m \rangle = \int dg \, f_G(g) g^m. \tag{3.5}
\]

As a function of the random variables \( s \) and \( t \), \( G \) itself is a random variable with the probability density \( f_G \). In the case under consideration, \( s \) and \( t \) are assumed to be independent random variables with probability densities \( f_s \) and \( f_t \), respectively, and thus the joint probability density \( f_{s,t} \) factorises, \( f_{s,t}(s,t) = f_s(s) f_t(t) \).

We assemble a realistic probability density for source distances \( s \) by modelling the Galaxy as a logarithmic spiral \([26]\) and weighting it in such a way that the distribution in galacto-centric radius agrees with what is known from radio-surveys of SNRs \([8]\), see figure 1. The distance distribution is then obtained by transforming to a helio-centric coordinate system and averaging over polar angle. For details, see ref. \([20]\). To implement this analytically complicated expression for \( f_s \), we expand it as a power series in \( s^2 \), see figure 2,

\[
f_s(s) = \sum_{i=0}^{\infty} a_{2i} s^{2i}. \tag{3.6}
\]

In practice, we truncate the series after 14 terms and cut off the source distribution at \( s_{\text{max}} \) to prevent the series from diverging. We emphasise that arbitrary source distributions can be expanded in such a power series.

\(^2\)In the absence of a reliable time-dependent model of GCR escape from a SNR during the Sedov Taylor phase we assume instantaneous injection. We have checked that for reasonable lifetimes \( O(10^4) \) yr this is accurate for all except extremely young and nearby sources which are anyway very unlikely.
Figure 1. The assumed distribution of SNRs in the Galaxy; the cross denotes the position of the Sun in between two spiral arms.

Figure 2. The probability density for the distance of a random SNR from the Sun.

Furthermore, we assume that the probability density for the source ages, e.g. the supernova rate, is constant over the time scales considered, $O(100)$ Myr (electrons and positrons of GeV energies can diffuse over $O(100)$ Myr before losing their energy but are, of course, subject to escape losses). Therefore, we take the sources to be equally distributed up to a maximum time $t_{\text{max}} = 1/(b_0 E_{\text{min}})$, set by the minimum energy $E_{\text{min}}$ to be considered,

$$f_t(t) = \begin{cases} 1/t_{\text{max}} & \text{for } 0 \leq t \leq t_{\text{max}}, \\ 0 & \text{otherwise}. \end{cases} \quad (3.7)$$

Rewriting the $m$-th moment of the Green’s function as an integral over $s$ and $t$, we have,

$$\langle G^m \rangle = \int \mathrm{d}g f_G(g) g^m \quad (3.8)$$

$$= \int_0^{t_{\text{max}}} \mathrm{d}t \int_0^{s_{\text{max}}} \mathrm{d}s f_{s,t}(s,t) G^m(s,t) \quad (3.9)$$

$$= \int_0^{t_{\text{max}}} \mathrm{d}t f_t(t) \int_0^{s_{\text{max}}^2} \mathrm{d}s f_s(s) \frac{1}{2s} G^m(s,t). \quad (3.10)$$

With eq. 3.6, the integral in $s^2$ can be performed and we substitute $t$ for the injection energy $E_0 = E/(1 - b_0 E t)$. For $m = 1$, this gives,

$$\langle G \rangle = \frac{1}{t_{\text{max}}} \frac{1}{2\pi} \sum_{i=0}^{\infty} a_{2i} (-m)^{-1+i} \int_E^{\infty} \frac{\mathrm{d}E_0}{b_0 E^2} \left( \frac{E}{E_0} \right)^i \left[ \Gamma \left( 1 + i, \frac{m s^2}{E} \right) - \Gamma(1 + i) \right]$$

$$\times Q_0 E_0^{-\gamma} e^{-E_0/E_{\text{cut}}} \frac{1}{\ell_{\text{cr}}^2} \chi \left( \frac{t^2}{\ell_{\text{cr}}^2} \right). \quad (3.11)$$

with $\Gamma$ being the (incomplete) gamma function.
Ignoring the cut-off, $E_{\text{cut}} \to \infty$, and expanding the term $\chi(\ell^2)$ for $\ell^2 \ll z_{\text{cr}}^2$ which is only justified for energies above $\sim 50$ GeV for $z_{\text{max}} \simeq 4$ kpc and amounts to ignoring the boundary (condition) in the $z$-direction, $(1/z_{\text{cr}})\chi(\ell^2) \simeq 1/\sqrt{\pi \ell^2}$, we can get analytical estimates for the moments $\langle G^m \rangle$,

$$\langle G^m \rangle = \frac{1}{t_{\text{max}}} \frac{1}{2} \pi^{-3} a_0^m (b_0(1-\delta))^{-1} \sum_{i=0}^{\infty} a_{2i} \Gamma(1+i)m^{-(1+i)} \left( \frac{4D_0}{b_0(1-\delta)} \right)^{1+i-3m/2} \times E^{-(\delta-1)(\frac{2}{3}m-(1+i)) - m\gamma-1} \left\{ C_{ik} - \sum_{k=0}^{i} \frac{(m\sigma_{\text{max}}^2)^k}{k!} D_{ik} \right\}, \quad (3.12)$$

where

$$C_{ik} = \frac{\Gamma \left( m - \frac{1}{2} \gamma - \frac{1}{\delta-1} \right) \Gamma \left( 2 + i - \frac{3}{2} m \right)}{\Gamma \left( 2 + i - \frac{3}{2} m + \frac{m}{2} - \frac{1}{\delta-1} \right)},$$

$$D_{ik} = \Gamma \left( m - \frac{1}{\delta-1} \gamma - \frac{1}{\delta-1} \right) e^{-m\sigma_{\text{max}}^2} U \left( m - \frac{1}{\delta-1} \gamma - \frac{1}{\delta-1} ; -1 - i + k + \frac{3}{2} m, m\sigma_{\text{max}}^2 \right),$$

with $\sigma_{\text{max}}^2 = b_0(1-\delta)/(4D_0)E^{1-\delta} s_{\text{max}}^2$ and Kummer’s confluent hypergeometric function $U(a, b, z)$. In particular, the expectation value of $G$ is,

$$\langle G \rangle = \frac{1}{t_{\text{max}}} \frac{1}{2} \pi^{-3} a_0^m (b_0(1-\delta))^{-1} \Gamma \left( \frac{1}{\delta-1} \gamma \right) \sum_{i=0}^{\infty} a_{2i} \left( \frac{4D_0}{b_0(1-\delta)} \right)^{-\frac{1}{2}+i} E^{-(\delta-1)(\frac{1}{2}-i) - \gamma-1} \times \Gamma(1+i) \left\{ \frac{\Gamma \left( \frac{1}{2} + i \right)}{\Gamma \left( \frac{1}{2} + i + \frac{\gamma-1}{1-\delta} \right)} - e^{-\sigma_{\text{max}}^2} \sum_{k=0}^{i} \frac{(\sigma_{\text{max}}^2)^k}{k!} U \left( 1 - \frac{\gamma}{1-\delta} \frac{1}{2} - i + k, \sigma_{\text{max}}^2 \right) \right\}. \quad (3.15)$$

For a homogeneous distribution of sources in a disk around the observer, $a_0 = 2/s_{\text{max}}^2$ and $a_i \equiv 0$ for $i \geq 1$,

$$\mu_j = \frac{c}{4\pi} N \mu_G \simeq \frac{c}{4\pi} \frac{1}{\sqrt{4D_0 b_0(1-\delta)}} \frac{N}{\pi s_{\text{max}}^2 t_{\text{max}}} \frac{Q_0}{E(1-\delta)/2 - \gamma-1} \Gamma \left( \frac{1}{\delta-1} \gamma \right) \frac{\Gamma \left( \frac{1}{2} + \frac{\gamma-1}{1-\delta} \right)}{\Gamma \left( \frac{1}{2} + \frac{\gamma-1}{1-\delta} \right)}. \quad (3.16)$$

Choosing, for example, $\delta = 0.6$ and $\gamma = 2.2$ this gives a spectrum proportional to $E^{-3}$, close to what has been measured by Fermi-LAT [5].

For $m = 2$, however, the integral in eq. 3.15 and therefore $\langle G^2 \rangle$ diverge because of the finite probability density for having an arbitrarily close source which leads to an arbitrarily large flux. This divergence also occurs for cosmic ray protons and nuclei as was first pointed out [27] in the context of an energy independent diffusion model.

In principle, this could be cured by introducing a minimum distance $s_{\text{min}}$ for the distribution $f_\ell(s)$ or a minimum time $t_{\text{min}}$ for the distribution $f_\ell(t)$. In fact, it has been argued [28] that the absence of very young, nearby sources justifies a lower limit $t_{\text{min}} = 1/\sqrt{4\pi D(E)}\sigma_{\text{SN}}$ in the time integration eq. 3.9, derived from the average distance to the nearest source assuming a homogeneous source density rate $\sigma_{\text{SN}} = N/(\pi s_{\text{max}}^2 t_{\text{max}})$. This statement can either be factual, i.e. no flux from such sources has been observed, or statistical, i.e. the contribution of such sources to the variance is negligible. As to the former statement, the energies
At which such sources contribute significantly in a statistical model. Without regularisation, the standard deviation of the Green’s function therefore does not exist. In addition, the central limit theorem cannot be applied anymore and the standard deviation of the distribution of fluxes does not exist either. As we will see in the next section, the reason that the expectation value is finite but not the variance is that the probability density \( f_G(g) \) has a broad power law tail, \( \propto g^{-\alpha - 1} \) with \( \alpha \leq 1 \). For such cases, a generalised central limit theorem [29] is applicable: If the probability density \( f_G(g) \) behaves like \( |g|^{-\alpha - 1} \) for \( g \to \infty \), the centred and normalised sum \( X_N \) of \( N \) independent and identically distributed (iid) random variables \( G_i \) converges against a stable distribution \( S(\alpha, \beta, 1, 0, 1) \) [30]. In general, the distribution function for \( S \) is not known analytically but can be calculated as the inverse Fourier transform of its characteristic function. To determine the parameters \( \alpha \) and \( \beta \), we need to find the asymptotic behaviour of the probability density \( f_G(g) \) for large \( g \).

### 4 Uncertainty of the Flux

The cumulative distribution function \( F_G(g) \), i.e. the probability to find \( G < g \), is given by

\[
F_G(g) = \int_{t_0}^{t_{\text{max}}} dt \, f_t(t) \int_{s_{\text{min}}(E,g,t)}^{s_{\text{max}}(E,g,t)} ds \, f_s(s),
\]

(4.1)

where

\[
D_{G < g} = \{(t, s) \mid 0 \leq s \leq s_{\text{max}}, 0 \leq t \leq t_{\text{max}}, G \leq g \}.
\]

(4.2)

With \( G \equiv G(E, s, t) \), \( G < g \) can be transformed into a condition on \( s \),

\[
s^2 > s_{\text{min}}^2 \equiv -t^2 \log g + \ell^2 \log \left[ (\pi \ell^2)^{-3/2} Q_0 E^{-\gamma} (1 - b_0 E t)^{\gamma - 2} e^{-E/(1 - b_0 E t) E_{\text{cut}}} \right],
\]

(4.3)

see figure 3, and hence

\[
F_G = \int_{t_0}^{t_{\text{max}}} dt \, f_t(t) \int_{s_{\text{min}}(E,g,t)}^{s_{\text{max}}(E,g,t)} ds \, f_s(s)
\]

(4.4)

\[
= \int_{t_0}^{t_\ast} dt \, f_t(t) \int_{s_{\text{min}}(E,g,t)}^{s_{\text{max}}(E,g,t)} ds \, f_s(s) + \int_{t_{\ast}}^{t_{\text{max}}} dt \, f_t(t) \int_{s_{\text{min}}(E,g,t)}^{s_{\text{max}}(E,g,t)} ds \, f_s(s),
\]

(4.5)

where \( s_{\text{min}} \geq 0 \) for \( 0 \leq t \leq t_\ast \) and \( s_{\text{min}} < 0 \) for \( t > t_\ast \). The probability density can then be obtained by differentiating,

\[
f_G = \frac{dF_G(g)}{dg} = \int_{t_0}^{t_\ast} dt \, f_t(t) \frac{\ell^2}{g} f_s(s_{\text{min}}) = \frac{1}{2} \frac{1}{t_{\text{max}}} \frac{1}{g} \sum_{i=0}^{\infty} a_{2i} \int_{t_0}^{t_\ast} dt \, \ell^2 (s_{\text{min}}^2)^i.
\]

(4.6)

We now substitute,

\[
\chi^2 = \frac{b_0 (1 - \delta)}{4D_0} E^{1-\delta} \ell^2 = 1 - (1 - b_0 E t)^{1-\delta},
\]

(4.7)
and find

\[
f_{G} = \frac{1}{2} \left( \sum_{i=0}^{\infty} a_{2i} \left( \frac{4D_{0}}{b_{0}(1-\delta)} \right)^{1+i} (b_{0}(1-\delta))^{-1} E^{(\delta-1)(1+i)-1} \right.\]
\[
\times \left[ \int_{0}^{\lambda_{2}^2} d\lambda^2 (1 - \lambda^2)^{-\delta/(\delta-1)} (\lambda^2)^{1+i} \right.\]
\[
\times \left. \left( -\log g + \log \left( \frac{4\pi D_{0}}{b_{0}(1-\delta)} \right)^{-3/2} Q_{0} E^{-\gamma} (1 - \lambda^2)^{2-\gamma} e^{-\frac{E}{E_{cut}} (1-\lambda^2)^{-1/2}} \right) - \frac{3}{2} \log \lambda^2 \right) i, \tag{4.8}\]

where \( \lambda_{2}^2 = 1 - (1 - b_{0} E_{cut})^{1-\delta} \). For \( g \to \infty \), we anticipate \( \lambda^2 \leq \lambda_{2}^2 \ll 1 \) and therefore neglect the \((1 - \lambda^2)\) terms,

\[
f_{G} \simeq \frac{1}{2} \left( \sum_{i=0}^{\infty} a_{2i} \left( \frac{4D_{0}}{b_{0}(1-\delta)} \right)^{1+i} (b_{0}(1-\delta))^{-1} E^{(\delta-1)(1+i)-1} \right.\]
\[
\times \left[ \int_{0}^{\lambda_{2}^2} d\lambda^2 (\lambda^2)^{1+i} \left( -\log g + \log A - \frac{3}{2} \log \lambda^2 \right) i, \tag{4.9}\]

where \( A = ((4\pi D_{0})/(b_{0}(1-\delta)))^{3/2} Q_{0} E^{-\gamma} e^{-E/E_{cut}} \). With the same approximation, we solve \( s_{\min}^2 = 0 \) for \( \lambda_{2}^2 \),

\[
\lambda_{2}^2 \simeq \frac{b_{0}(1-\delta)}{4\pi D_{0}} E^{1-\delta-\frac{2}{3} \gamma} Q_{0}^{2/3} g^{-2/3} e^{-\frac{2}{3} E/E_{cut}}, \tag{4.10}\]

and we finally compute

\[
f_{G} \simeq \frac{1}{2} \left( \sum_{i=0}^{\infty} a_{2i} \left( \frac{3}{2} \right)^{i} (2 + i)^{-(1+i-\pi^{-1}(2+i)) \Gamma(1+i, (2+i)(\delta-1)) \log E} \right.\]
\[
\times \left. E^{-\delta+(\delta-1-\frac{2}{3} \gamma)(2+i)} Q_{0}^{\frac{2}{3}(2+i)} g^{-\frac{2}{3}(2+i)} e^{-\frac{2}{3}(2+i)E/E_{cut}} \right) - \frac{3}{2} \log \lambda^2 \right) i, \tag{4.11}\]

For \( g \to \infty \), the non-zero term with the smallest \( i \) determines the asymptotic behaviour, such that for \( a_{0} \neq 0 \), we have

\[
f_{G} \simeq \frac{1}{2} \left( \sum_{i=0}^{\infty} a_{2i} \left( \frac{3}{2} \right)^{i} (2 + i)^{-(1+i-\pi^{-1}(2+i)) \Gamma(1+i, (2+i)(\delta-1)) \log E} \right.\]
\[
\times \left. E^{-\delta+(\delta-1-\frac{2}{3} \gamma)(2+i)} Q_{0}^{\frac{2}{3}(2+i)} g^{-\frac{2}{3}(2+i)} e^{-\frac{2}{3}(2+i)E/E_{cut}} \right) - \frac{3}{2} \log \lambda^2 \right) i, \tag{4.12}\]

The asymptotic behaviour of the distribution function \( F_{G} \) for \( g \to \infty \) is therefore

\[
1 - F_{G}(g) \sim w g^{-4/3} \quad \text{with} \quad w = \frac{3}{4} \frac{1}{t_{\max}^{2}} \frac{1}{8\pi^{2} D_{0}^{2}} a_{0} E^{-\delta-\frac{4}{3} \gamma} Q_{0}^{4/3} e^{-\frac{4}{3} E/E_{cut}}, \tag{4.13}\]

and \( F_{G} = 0 \) for \( g < 0 \), see figure 4.

The generalised central limit theorem [31] then states that the centred and normalised sum \( X_{N} \),

\[
X_{N} = \frac{1}{\sqrt{N}} \left( \sum_{i=1}^{N} X_{i} - u_{N} \right), \tag{4.14}\]
weakly converges to the stable distribution $S(\alpha, \beta, 1, 0, 1)$ with $\alpha = 4/3$ and $\beta = 1$. The normalisation constants are

$$u_N = N\mu_G \quad \text{and} \quad v_N = \left( \frac{\pi w}{2\Gamma\left(\frac{4}{3}\right)\sin\left(\frac{\pi}{2}\frac{4}{3}\right)} \right)^{3/4} N^{3/4}. \quad (4.15)$$

The expectation value $\mu_J$ for the flux on Earth is therefore still $c/(4\pi)N\mu_G$ but instead of the standard deviation, we use quantiles $x_q$ of the stable distribution to define uncertainty intervals,

$$I_{68\%} = \left[ \left( \mu_J + \frac{c}{4\pi} v_N x_{16\%} \right), \left( \mu_J + \frac{c}{4\pi} v_N x_{84\%} \right) \right], \quad (4.16)$$

$$I_{95\%} = \left[ \left( \mu_J + \frac{c}{4\pi} v_N x_{2.5\%} \right), \left( \mu_J + \frac{c}{4\pi} v_N x_{97.5\%} \right) \right]. \quad (4.17)$$

For $S(4/3, 1, 1, 0, 1)$ the $2.5\%$, $16\%$, $84\%$ and $97.5\%$ quantiles are approximately $-3.6$, $-2.6$, $1.0$ and $8.7$, respectively. The stable distribution is asymmetric with respect to its expectation value $x = 0$ and so are the uncertainty bands with respect to $\mu_J$. In particular, the expectation value is larger than most of the particular realisations of the ensemble and in this sense not representative which testifies to the long power law tail of $f_G$. We therefore define a most likely flux $J_m$ through the maximum $x_{\text{max}} \simeq -2.0$ of the stable distribution,

$$J_m = \frac{c}{4\pi} (u_N + v_N x_{\text{max}}) = \frac{c}{4\pi} \left( N\mu_G + \left( \frac{\pi w}{2\Gamma\left(\frac{4}{3}\right)\sin\left(\frac{\pi}{2}\frac{4}{3}\right)} \right)^{3/4} N^{3/4} x_{\text{max}} \right). \quad (4.18)$$

The energy dependence of both, the quantiles of $J$ and the most likely flux $J_m$, is contained in $v_N \propto w^{3/4} \propto E^{-\gamma-3\delta/4}e^{-E/E_{\text{cut}}}$, which is harder than the expectation value $\mu_J$, so the uncertainty is growing with energy, as expected. We note the peculiar dependence of the uncertainty relative to the expectation value, i.e. $v_N/u_N$, on the source and propagation
Table 1. Summary of parameters.

| Diffusion Model | Source Distribution | Source Model |
|-----------------|--------------------|--------------|
| $D_0$ $10^{25}$ cm$^2$s$^{-1}$ | $t_{\text{max}}$ $3 \times 10^8$ yr from $E_{\text{min}} \simeq 1$ GeV | $Q_0$ $7.8 \times 10^{49}$ GeV$^{-1}$ fit to absolute $e^+ + e^-$ flux |
| $\delta$ 0.6 | $N$ $4.4 \times 10^6$ from supernova rate, $s_{\text{max}}$ and $t_{\text{max}}$ | $\gamma$ 2.2 |
| $z_{\text{max}}$ 3 kpc ISRF and $B$ energy densities | $b_0$ $10^{-16}$ GeV$^{-1}$ s$^{-1}$ | $s_{\text{max}}$ 10 kpc |

parameters. In particular, the uncertainty increases as $Q_0N^{3/4}$ whereas the expectation goes like $Q_0N$. The relative fluctuations therefore decrease with the number of sources as $N^{-1/4}$ which is slower than would have been the case for the normal central limit theorem where $\sigma_J/\mu_J \propto N^{-1/2}$.

We note that although the normalisations of the quantiles of $J$ and of the fluctuations estimated from the square root of the regularised variance $\langle (\delta N)^2 \rangle^{1/2}$ in Ref. [28] are different, their energy dependence, $v_N \propto E^{-\gamma-38/4}$, is the same. This is due to the particular choice for the regulator $t_{\text{min}}$ which corresponds to a cut-off of the probability density at $J_c$. It turns out that $J_c$ must itself be a quantile of the probability density and for sufficiently large $J_c$ it can be shown that the energy-dependence of $\langle (\delta N)^2 \rangle^{1/2}$ reduces to $v_N$. Unfortunately, the definition of $t_{\text{min}}$ does not make manifest which quantile $\langle (\delta N)^2 \rangle^{1/2}$ corresponds to.

5 Discussion

As an example of the analytical results, we consider the interstellar electron-positron flux from SNRs distributed according to eqs. 3.6 and 3.7. The parameters used for the source and diffusion model are summarised in table 1. Figures 5 and 6 show the calculated expectation value and the uncertainty bands, defined by the quantiles of the stable distribution, for $E_{\text{cut}} \to \infty$. As explained above, the spectra are unreliable below $\sim 10$ GeV as convection and acceleration have been ignored.

In figure 5 we also show the result of 50 runs of a Monte Carlo calculation of the electron-positron flux with the same source and propagation model. Indeed, the fluxes from individual realisations of the source distribution fluctuate between the uncertainty bands. We emphasise that the probability density for the fluxes is not a Gaussian but a more general stable function. We have checked the statistical interpretation of the uncertainty bands, i.e. the fluxes of 68% and 95% of the source realisations lie within the respective bands. As mentioned above, due to the asymmetry of the probability density $f_G$ around its expectation value $\mu_G$, most fluxes are smaller than $\mu_G$ but the ones that are larger deviate more strongly.

What most of the fluxes have in common is a roll-over at a few TeV compared to the expectation value. This well-known propagation cut-off is different from a source cut-off and reflects the fact that the defacto youngest source has a finite age $t_{\text{min}}$. Demanding $\ell^2 > 0$ then implies a maximum energy $1/(b_0t_{\text{min}})$, provided that the source $i$ is close enough.
Figure 5. Interstellar flux of GCR electrons and positrons from an ensemble of source distributions, see eqs. 3.6 and 3.7. The solid line denotes the expectation value for the sum of fluxes from $N$ discrete, transient sources without cut-off, and the dotted line shows the most likely flux, corresponding to the maximum of the probability density for fluxes. The coloured bands quantify the uncertainty due to our ignorance of the real distribution of individual sources, in the sense that the fluxes of 68% and 95% of the source realisations should lie within the respective bands. The fluxes of 50 realisations of $N$ individual sources from a Monte Carlo calculation are shown by the thin grey lines.

Figure 6. Same as figure 5, but instead of the fluxes from the Monte Carlo calculation, we show the electron-positron flux as measured by Fermi-LAT [5], and from the low energy (LE) [4] and high energy (HE) [6] analyses by HESS. The error bars on the data are statistical only, the shaded bands denote the systematic uncertainty and the diagonal arrows show the error due to the scale uncertainty.
Figure 7. Same as figure 5 but assuming a source spectrum with cut-off energy $E_{\text{cut}} = 20\,\text{TeV}$.

Figure 8. Same as figure 6, but assuming a source spectrum with cut-off energy $E_{\text{cut}} = 20\,\text{TeV}$. 
to considerably contribute to the total electron-positron flux. In most cases, a few young sources determine the propagation cut-off.

In figure 6 we compare our results to the electron-positron flux as measured by Fermi-LAT [5], and HESS [4, 6]. The calculated expectation value fits the Fermi-LAT data but overshoots the HESS data, in particular those from the high energy analysis. Choosing a softer injection, i.e. $\gamma > 2.2$, or imposing a stronger energy dependence for the diffusion coefficient, i.e. $\delta > 0.6$, would make the spectrum softer improving the fit to the HESS data but making the fit to the Fermi-LAT data worse. Within the uncertainty the model might still reproduce the data, although only the lower edge of the 95% uncertainty band is marginally consistent with the HESS data.

The situation can be easily improved on by reinstating the intrinsic cut-off in the source spectrum, $E_{\text{cut}}$. For the bulk of (mature) SNRs, the cut-off energy is naturally expected to lie in the TeV range [32]. Figures 5 and 6 show the calculated expectation value and the uncertainty bands for $E_{\text{cut}} = 20$ TeV. The cut-off improves the fit to the HESS data such that all data lie in the 68% uncertainty band.

6 Conclusion

We have investigated the expectation value in the electron-positron flux from a statistical ensemble of astrophysical sources as well as the uncertainty introduced by our ignorance of the distances and ages of individual sources. We have considered the transport of GCR electrons and positrons in the usual diffusive setup with the same power law injection spectrum for all sources. Their average spatial distribution is modelled combining the distribution of SNRs in galacto-centric radius with a model of the spiral structure. For this as for any other source distribution, the expectation value can be calculated as a sum of power laws in energy where the coefficients are given by the Taylor coefficients of the source distance distribution. We found that the probability density $f_G$ of the Green’s function $G$ has a long power law tail. Consequently, the standard deviation is not defined and the probability density for the flux is not a Gaussian but a more general stable distribution. We have quantified the uncertainty around the expectation value by the quantiles of the stable distribution and have also calculated the most-likely flux. We find that the level of fluctuations between different members of the ensemble grows with energy as expected and that most fluxes exhibit a propagation cut-off at a few TeV. In the case without source cut-off the uncertainty interval is in agreement with data from Fermi-LAT and marginally consistent with the measurements by HESS. This can be improved by invoking a source cut-off, e.g. at $E_{\text{cut}} = 20$ TeV. The analytical formulae we have provided allow to consider different types of astrophysical sources with arbitrary spatial distribution functions. We emphasise that the uncertainty in the flux is inherent to the propagation of GCR electrons and positrons and our ignorance of the positions and ages of individual sources. As such it needs to be considered when predicting their fluxes from astrophysical sources.

References

[1] PPB-BETS Collaboration, S. Torii et. al., High-energy electron observations by PPB-BETS flight in Antarctica, arXiv:0809.0760.

[2] PAMELA Collaboration, O. Adriani et. al., An anomalous positron abundance in cosmic rays with energies 1.5-100 GeV, Nature 458 (2009) 607–609, [arXiv:0810.4995].
[3] J. Chang et al., An excess of cosmic ray electrons at energies of 300-800 GeV, Nature 456 (2008) 362–365.

[4] H.E.S.S. Collaboration, F. Aharonian et al., The energy spectrum of cosmic-ray electrons at TeV energies, Phys. Rev. Lett. 101 (2008) 261104, [arXiv:0811.3894].

[5] The Fermi LAT Collaboration, A. A. Abdo et al., Measurement of the Cosmic Ray e+ plus e- spectrum from 20 GeV to 1 TeV with the Fermi Large Area Telescope, Phys. Rev. Lett. 102 (2009) 181101, [arXiv:0905.0025].

[6] H.E.S.S. Collaboration, F. Aharonian et al., Probing the ATIC peak in the cosmic-ray electron spectrum with H.E.S.S, Astron. Astrophys. 508 (2009) 561, [arXiv:0905.0105].

[7] D. Caprioli, E. Amato, and P. Blasi, The contribution of supernova remnants to the galactic cosmic ray spectrum, Astropart. Phys. 33 (2010) 160–168, [arXiv:0912.2964].

[8] G. L. Case and D. Bhattacharya, A New Σ – D Relation and Its Application to the Galactic Supernova Remnant Distribution, Astrophys. J. 504 (1998) 761, [astro-ph/9807162].

[9] D. R. Lorimer, The Galactic population and birth rate of radio pulsars, astro-ph/0308501.

[10] I. V. Moskalenko and A. W. Strong, Production and propagation of cosmic-ray positrons and electrons, Astrophys. J. 493 (1998) 694–707, [astro-ph/9710124].

[11] “GALPROP.” GALPROP website. http://galprop.stanford.edu.

[12] C. Evoli, D. Gaggero, D. Grasso, and L. Maccione, Cosmic-Ray Nuclei, Antiprotons and Gamma-rays in the Galaxy: a New Diffusion Model, JCAP 0810 (2008) 018, [arXiv:0807.4730].

[13] D. Maurin, F. Donato, R. Taillet, and P. Salati, Cosmic Rays below Z=30 in a diffusion model: new constraints on propagation parameters, Astrophys. J. 555 (2001) 585–596, [astro-ph/0101231].

[14] D. Maurin, R. Taillet, and F. Donato, New results on source and diffusion spectral features of Galactic cosmic rays: I- B/C ratio, Astron. Astrophys. 394 (2002) 1039–1056, [astro-ph/0206286].

[15] T. Kobayashi, Y. Komori, K. Yoshida, and J. Nishimura, The Most Likely Sources of High Energy Cosmic-Ray Electrons in Supernova Remnants, Astrophys. J. 601 (2004) 340–351, [astro-ph/0308470].

[16] D. A. Green, Some statistics of Galactic SNRs, Mem. Soc. Ast. It. 76 (2005) 534–541, [astro-ph/0505428].

[17] A. Bamba, T. Anada, T. Dotani, K. Mori, R. Yamazaki, K. Ebisawa, and J. Vink, X-ray Evolution of Pulsar Wind Nebulae, Astrophys. J. Lett. 719 (Aug., 2010) L116–L120, [arXiv:1007.3203].

[18] K. Koyama, K. Kinugasa, K. Matsuzaki, M. Nishiuchi, M. Sugizaki, K. Torii, S. Yamauchi, and B. Aschenbach, Discovery of Non-Thermal X-Rays from the Northwest Shell of the New SNR RX J1713.7–3946: The Second SN 1006?, PASJ 49 (1997) L7–L11, [astro-ph/9704140].

[19] P. Slane, B. M. Gaensler, T. M. Dame, J. P. Hughes, P. P. Plucinsky, and A. Green, Nonthermal X-Ray Emission from the Shell-Type Supernova Remnant G347.3-0.5, Astrophys. J. 525 (1999) 357–367, [astro-ph/9906364].

[20] M. Ahlers, P. Mertsch, and S. Sarkar, On cosmic ray acceleration in supernova remnants and the FERMI/PAMELA data, Phys. Rev. D80 (2009) 123017, [arXiv:0909.4060].

[21] M. Pohl and J. A. Esposito, Electron acceleration in SNR and diffuse gamma-rays above 1 GeV, Astrophys. J. 507 (1998) 327, [astro-ph/9806160].

[22] A. W. Strong and I. V. Moskalenko, A 3D time-dependent model for galactic cosmic rays and
gamma-rays, astro-ph/0106505.

[23] S. P. Swordy, *Stochastic Effects on the Electron Spectrum above TeV Energies*, in *Proceedings of the 28th International Cosmic Ray Conference, Tsukuba, Japan*, vol. 4 of *International Cosmic Ray Conferences*, p. 1989, July, 2003.

[24] V. L. Ginzburg (ed.), V. A. Dogiel, V. S. Berezinsky, S. V. Bulanov, and V. S. Ptuskin, *Astrophysics of cosmic rays*. North-Holland, Amsterdam, Netherlands, 1990.

[25] E. A. Baltz and J. Edsjo, *Positron Propagation and Fluxes from Neutralino Annihilation in the Halo*, *Phys. Rev.* D59 (1998) 023511, [astro-ph/9808243].

[26] J. P. Vallée, *The Spiral Arms and Interarm Separation of the Milky Way: An Updated Statistical Study*, *Astron. J.* 130 (2005) 569–575.

[27] M. A. Lee, *A statistical theory of cosmic ray propagation from discrete galactic sources*, *Astrophys. J.* 229 (1979) 424.

[28] V. S. Ptuskin, F. C. Jones, E. S. Seo, and R. Sina, *Effect of random nature of cosmic ray sources Supernova remnants on cosmic ray intensity fluctuations, anisotropy, and electron energy spectrum*, *Adv. Space Res.* 37 (2006) 1909.

[29] B. V. Gendenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, Cambridge, MA, USA, 1954. Trans. and annotated by K. L. Chung.

[30] J. P. Nolan, *Stable Distributions - Models for Heavy Tailed Data*. Birkhäuser, Boston, 2010. In progress, Chapter 1 online at academic2.american.edu/~jpnolan.

[31] V. V. Uchaikin and V. M. Zolotarev, *Chance and Stability. Stable Distributions and Their Applications*. VSP, Utrecht, 1999.

[32] S. P. Reynolds, *Supernova Remnants at High Energy*, *Annu. Rev. Astron. Astrophys.* 46 (2008) 89.