A Note on the Well-posedness of Prandtl Equations in Dimension Two

Jincheng Gao†  Daiwen Huang‡  Zheng-an Yao†

†School of Mathematics, Sun Yat-sen University, Guangzhou, 510275, P.R.China
‡Institute of Applied Physics and Computational Mathematics, Beijing, 100088, P.R.China

Abstract

In this paper, we investigate the local-in-time well-posedness for the two-dimensional Prandtl equations in weighted Sobolev spaces under the Oleinik’s monotonicity condition. Due to the loss of tangential derivative caused by vertical velocity appearing in convective term, we add with artificial horizontal viscosity term to construct an approximate system that can obtain local-in-time well-posedness results easily. For this approximate system, we construct a new weighted norm for the vorticity to derive a positive life time (independent of artificial viscosity coefficient) and obtain uniform bound for vorticity in this weighted norm. Then, based on compactness argument, we prove the solution of approximate system converging to the solution of original Prandtl equations, and hence, obtain the local-in-time well-posedness result for the Prandtl equations with any large initial data, which improves the recent work [10].

1 Introduction

Throughout this paper, we are concerned with the two-dimensional Prandtl equations, derived by Ludwing Prandtl [1], in a periodic domain \( \mathbb{T} \times \mathbb{R}^+ := \{ (x, y) : x \in \mathbb{R}/\mathbb{Z}, 0 \leq y < +\infty \} \):

\[
\begin{align*}
\partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p &= 0, \\
\partial_x u + \partial_y v &= 0, \\
u|_{y=0} = v|_{y=0} = 0, & \quad \lim_{y \to +\infty} u = U(t, x), \\
u|_{t=0} = u_0.
\end{align*}
\] (1.1)

where the velocity field \((u, v) := (u(t, x, y), v(t, x, y))\) is unknown, and the initial data \(u_0 := u_0(x, y)\) and the outer flow \(U := U(t, x)\) are given and satisfy the compatibility conditions:

\[
u|_{y=0} = 0, \quad \lim_{y \to +\infty} u_0 = U|_{t=0}. \] (1.2)

Furthermore, the given scalar pressure \(p := p(t, x)\) and the outer flow \(U\) satisfy the Bernoulli’s law:

\[
\partial_t U + U \partial_x U = -\partial_x p. \] (1.3)

Note that the Prandtl equations mentioned above arise from the vanishing viscosity limit of Navier-Stokes equations in a domain with Dirichlet boundary condition. This is due to the formation of a
boundary layer, where the solution undergoes a sharp transition from a solution of the Euler system to the zero non-slip boundary condition on boundary of the Navier-Stokes system. This boundary layer satisfies the Prandtl boundary layer equations formally. The first systematic work in rigorous mathematics was obtained by Oleinik [2,3] in which she established the local in time well-posedness of Prandtl equations in dimension two by applying the Crocco transformation under the monotonicity condition on the tangential velocity field in the normal direction to the boundary. For more extensional mathematical results, the interested readers can refer to the classical book finished by Oleinik and Samokhin [4]. In addition to Oleinik’s monotonicity assumption on the velocity field, by imposing a so-called favorable condition on the pressure, Xin and Zhang [5] obtained the existence of global weak solutions to the Prandtl equation. As mentioned in [6–8], the local in time well-posedness of Prandtl equation for the initial data in Sobolev space is an open problem. Then, the researchers in [9] and [10] independently used the nonlinear cancelation method to establish well-posedness theory for the two-dimensional Prandtl equations in the framework of Sobolev spaces. Note that the local-in-time well-posedness result in $H_{s}^{s,\gamma}$ Sobolev framework, obtained by Masmoudi and Wong [10], required there exists a small constant $\delta_0$ such that

$$\sum_{|\alpha|\leq 2} |(1 + y)^{s+\alpha_2}D^\alpha w_0|^2 \leq (2\delta_0)^{-2},$$

and for $s = 4$, they required additionally

$$\|w_0\|_{H_4^{s,\gamma}} \leq C\delta_0^{-1}.$$  

Thus, the main target in this paper is to remove the conditions (1.4)-(1.5). In other words, the local-in-time well-posedness results for the Prandtl equations (1.1) will be established for any large initial data.

Finally, we point out that it is an outstanding open problem to rigorously justify the validity of expansion in the inviscid limit. On one hand, within space of functions that are analytic, Sammartino and Caflisch [11, 12] obtained the well-posedness in the framework of analytic functions without the monotonicity condition on the velocity field and justified the boundary layer expansion for the unsteady incompressible Navier-Stokes equations. On the other hand, for more “realistic” functional settings, Guo and Nguyen [13] justified the boundary layer expansion for the steady incompressible flow with a non-slip boundary condition on a moving plate. This result was generalized to the case of non-moving or with external forcing(cf. [14, 15]). For more results in this direction, the interested readers can refer to [16–18] and references therein.

In this work, we will consider the Prandtl equations (1.1) under Oleinik’s monotonicity assumption:

$$w := \partial_y u > 0.$$  

Under this hypothesis, one must further assume $U > 0$. Let us first introduce some weighted Sobolev spaces for later use. Denoting the vorticity $w := \partial_y u$, we define the weighted Sobolev space $H_{s,\gamma}$ for $w$ by

$$H_{s,\gamma} := \{w : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R} : \|w\|_{H_{s,\gamma}} < \infty\},$$

where the weighted $H_{s,\gamma}$ norm is defined by

$$\|w\|^2_{H_{s,\gamma}} := \sum_{|\alpha| \leq s} \|(1 + y)^{\gamma + \alpha_2}D^\alpha w\|^2_{L^2(\mathbb{T} \times \mathbb{R}^+)},$$

where $D^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$. Here, the main idea is adding an extra weight $(1 + y)$ for each $y$–derivative. This corresponds to the weight $\frac{1}{y}$ in the Hardy-type inequality. Let us define

$$\|w(t)\|^2_{H_{s,\gamma,\sigma}} := \|w(t)\|^2_{H_{s,\gamma}} + \sum_{1 \leq |\alpha| \leq 2} \|(1 + y)^{\gamma + \alpha_2}D^\alpha w(t)\|^2_{L^\infty},$$

(1.6)
and introduce the space

\[ \widetilde{H}^{s,\gamma}_{\sigma,\delta_0} := \{ w : T \times \mathbb{R}^+ \to \mathbb{R} : \| w(t) \|_{H^{s,\gamma}_{\sigma,\delta_0}}^2 < +\infty, \quad (1 + y)^\sigma w \geq \delta_0, \} \]

where \( s \geq 4, \gamma \geq 1, \sigma > \gamma + \frac{1}{2}, \delta_0 \in (0, \frac{1}{2}) \). Now, we can state our main result:

**Theorem 1.1.** Let \( s \geq 4 \) be an even integer, \( \gamma \geq 1, \sigma > \gamma + \frac{1}{2} \) and \( \delta_0 \in (0, \frac{1}{2}) \). Suppose the outer flow \( U \) satisfies

\[ M_U := \sum_{k=0}^{s/2+1} \sup_{0 \leq t \leq T} \| \partial_t^k U \|_{H^{-2k+2}(\mathbb{T})} < +\infty. \tag{1.7} \]

Assume that the initial tangential velocity \( u_0 - U \big|_{t=0} \in H^{s,\gamma-1} \) and the initial vorticity \( w_0 := \partial_y u_0 \in \widetilde{H}^{s,\gamma}_{\sigma,\delta_0} \). Then there exist a times \( T = T(s, \gamma, \sigma, \delta_0, \| w_0 \|_{B^{s,\gamma}_{\sigma,\delta_0}, M_U}) > 0 \) and a unique classical solution \((u, v)\) to the Prandtl equations (1.1)-(1.3) such that

\[ \sup_{0 \leq t \leq T} \| w(t) \|_{H^{s,\gamma}_{\sigma,\delta_0}}^2 \leq C_{s,\gamma,\sigma} \{ 1 + \| w_0 \|_{B^{s,\gamma}_{\sigma,\delta_0}}^8 + M_U^4 \} < +\infty, \]

and

\[ \min_{\mathcal{T} \times \mathbb{R}^+} (1 + y)^\sigma w(t) \geq \delta_0, \]

for all \( t \in [0, T] \).

**Remark 1.1.** Defined the space

\[ H^{s,\gamma}_{\sigma,\delta_0} := \{ w : T \times \mathbb{R}^+ \to \mathbb{R} : \| w \|_{H^{s,\gamma}_{\sigma,\delta_0}}^2 < +\infty, \quad \sum_{\lambda \leq 2} |(1 + y)^{\sigma+\alpha} \partial^\alpha w |^2 \leq \frac{1}{\delta_0^2}, \quad (1 + y)^\sigma w \geq \delta_0 \}, \]

Masmoudi and Wong [10] established the local-in-time well-posedness theory for the Prandtl equations (1.1) if the initial vorticity \( w_0 \) belongs to \( H^{s,\gamma}_{\sigma,\delta_0} \) instead of \( \widetilde{H}^{s,\gamma}_{\sigma,\delta_0} \) (required in Theorem 1.1). In other words, they required the initial vorticity itself, first and second order derivatives with weight in \( L^\infty \)-norm should be controlled by \( \delta_0^{-1} \) rather than being sufficiently large.

**Remark 1.2.** When the well-posedness for the Prandtl equations (1.1) in the \( H^{s,\gamma} \)-framework, Masmoudi and Wong [10] required the condition (1.5) additionally, which we do not need in Theorem 1.1.

We now explain main difficulties of proving Theorem 1.1 as well as our strategies for overcoming them. In order to solve the Prandtl equations (1.1) in certain \( H^s \) Sobolev space, the main difficulty comes from the vertical velocity \( v = -\partial_y^{-1} \partial_x u \) creates a loss of tangential derivative, so the standard energy methods can not apply directly. The main idea of establishing the well-posedness of Prandtl equations (1.1) is to apply the so-called vanishing viscosity and nonlinear cancellation methods. To this end, we consider the following approximate system (or regularized Prandtl equations cf. [10]):

\[ \begin{align*}
\partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon - \varepsilon^2 \partial_x^2 u^\varepsilon - \partial_y^2 u^\varepsilon + \partial_x p^\varepsilon & = 0, \\
\partial_x u^\varepsilon + \partial_y v^\varepsilon & = 0, \\
u^\varepsilon \big|_{t=0} = u_0, \\
u^\varepsilon \big|_{y=0} = v^\varepsilon \big|_{y=0} = 0, \quad \lim_{y \to +\infty} v^\varepsilon(t, x, y) = U(t, x),
\end{align*} \tag{1.8} \]

for any \( \varepsilon > 0 \). Here the quantities \( p^\varepsilon \) and \( U \) satisfy a regularized Bernoulli’s law:

\[ \partial_t U + U \partial_x U = \varepsilon^2 \partial_x^2 U - \partial_x p^\varepsilon. \tag{1.9} \]
We point out that the approximate system (1.8) will turn into the original Prandtl equations (1.1) as the parameter $\varepsilon$ tends to zero. For any $\varepsilon > 0$, the local-in-time well-posedness of approximate system (1.8) is obtained easily in life time $[0,T^\varepsilon]$ ($T^\varepsilon$ may depend on parameter $\varepsilon$), and hence, we hope to prove that the solution of (1.8) in life time $[0,T^\varepsilon]$ will converge to the solution of (1.1) when artificial viscosity $\varepsilon$ tends to zero. For this purpose, we need to prove the time of existence $T^\varepsilon$ stays bounded away from zero. Since the domain considered in this article is periodic, the main part of the boundary layer will vanish or being stable. Thus, the main difficulty to prove $T^\varepsilon$ staying bounded away from zero arises from the vertical velocity $v^\varepsilon$. This can be overcome by the nonlinear cancellation methods developed in [10]. Here we also mention that the readers interested in the vanishing viscosity limit for incompressible Navier-Stokes equations can refer to [13-21] and references therein.

Now, let us explain the main idea to prove $T^\varepsilon$ staying bounded away from zero and the main novelty to relax to additional conditions (1.4) and (1.5) required in [10]. First of all, we derive the weighted $L^2$ estimate for $D^\alpha w^\varepsilon$ for $|\alpha| \leq s$ and $\alpha_1 \leq s - 1$. This works since we are allowed to loss at least one $x-$regularity in these cases. Secondly, we introduce the quantity $g^\varepsilon_s$ (cf. [10]):

$$g^\varepsilon_s := \partial^2_x w^\varepsilon - \frac{\partial_y w^\varepsilon}{w^\varepsilon} \partial^2_y (w^\varepsilon - U)$$

and the weighted norm

$$\|w^\varepsilon\|_{H^s_{\varepsilon,\alpha}((T \times \mathbb{R}^+))}^2 := \|(1 + y)^\gamma g^\varepsilon_s\|_{L^2(T \times \mathbb{R}^+)}^2 + \sum_{|\alpha| \leq s, \alpha_1 \leq s - 1} \|(1 + y)^{\gamma + \alpha_2} D^\alpha w^\varepsilon\|_{L^2(T \times \mathbb{R}^+)}^2,$$

(1.11)

provided $w^\varepsilon := \partial_y u^\varepsilon > 0$. As mentioned in [10], this quantity $g^\varepsilon_s$ can avoid the loss of $x-$derivative by the nonlinear cancellation; in other words, the quantity $g^\varepsilon_s$ in $L^2$-norm will have uniform bound independent of $\varepsilon$. Without the additional condition (1.3) (required in [10]), we need to control the quantity $(1 + y)^{\sigma + \alpha_2} D^\alpha w^\varepsilon (1 \leq |\alpha| \leq 2)$ in $L^\infty$-norm to close the estimate. Due to the weight index $\sigma > \gamma + \frac{3}{2}$, it is not easy to apply the quantity $\|w^\varepsilon\|_{H^s_{\varepsilon,\alpha}}$ to control these quantities by the Sobolev inequality. Define

$$\|w^\varepsilon(t)\|_{B^{s,\sigma}_{\varepsilon}}^2 := \|w^\varepsilon(t)\|_{H^s_{\varepsilon,\alpha}}^2 + \sum_{1 \leq |\alpha| \leq 2} \|(1 + y)^{\sigma + \alpha_2} D^\alpha w^\varepsilon(t)\|_{L^\infty}^2,$$

we may apply the maximum principle of heat equation to control quantities $(1 + y)^{\sigma + \alpha_2} D^\alpha w^\varepsilon (1 \leq |\alpha| \leq 2)$ in $L^\infty$-norm by $\|w^\varepsilon(t)\|_{B^{s,\sigma}_{\varepsilon}}^2$, initial and boundary data, which can be controlled by the quantity $\|w^\varepsilon\|_{H^s_{\varepsilon,\alpha}}$ after using the Sobolev inequality. An important remark is that $\|w^\varepsilon(t)\|_{B^{s,\sigma}_{\varepsilon}}$ is equivalent to $\|w^\varepsilon(t)\|_{B^{s,\sigma}_{\varepsilon}}$ (see (B.3) and (B.4)), which has uniform bound on the life span $[0,T^\varepsilon]$ (see Lemma C.1 in Appendix C). Thus, based on the estimates obtained above, we can choose the life span $T_a$ independent of $\varepsilon$ such the quantity $\|w^\varepsilon(t)\|_{B^{s,\sigma}_{\varepsilon}}$ has uniform bound on $[0, T_a]$. Then we can pass the limit $\varepsilon \to 0^+$ and obtain the existence and uniqueness of solution to the original Prandtl equations (1.1) by the approximate system (1.8). The weighed norm $\|\cdot\|_{B^{s,\sigma}_{\varepsilon}}$ is not only equivalent to norm $\|\cdot\|_{B^{s,\sigma}_{\varepsilon}}$ but also avoids the loss of tangential derivatives without the condition (1.4). This is the main novelty in our paper and help us improve the recent result [10]. But we should point out that the idea, which overcomes the loss of tangential derivative arising by vertical velocity $v = -\partial_y^{-1} \partial_x u$, comes from the nonlinear cancellation method developed in [10].

The rest of this paper is organized as follows. In Section 2, one establishes the a priori estimates for the approximate system (1.8). Some useful inequalities and important equivalent relations will be stated in Appendix A and B. Before we proceed, let us comment on our notation. Through this paper, all constants $C$ may be different from line to line. Subscript(s) of a constant illustrates the dependence of the constant, for example, $C_s$ is a constant depending on $s$ only. Denote by $\partial_y^{-1}$ the inverse of the derivative $\partial_y$, i.e., $(\partial_y^{-1} f)(y) := \int_y^0 f(z) \, dz$. 

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2 A priori estimates

In this section, we will derive a priori estimates (independent of $\varepsilon$), which are crucial to prove the local-in-time well-posedness theory of solutions to original Prandtl equations (1.1). Denote vorticity $w^{\varepsilon} := \partial_y u^{\varepsilon}$, using the the regularized Prandtl equations (1.8), we find that this vorticity satisfies the following evolution equations:

\[
\begin{cases}
\partial_t w^{\varepsilon} + u^{\varepsilon} \partial_x w^{\varepsilon} + v^{\varepsilon} \partial_y w^{\varepsilon} = \varepsilon^2 \partial_x^2 w^{\varepsilon} + \partial_y^2 w^{\varepsilon}, \\
w^{\varepsilon}|_{t=0} = w_0 := \partial_y u_0, \\
\partial_y w^{\varepsilon}|_{y=0} = \partial_x p^{\varepsilon},
\end{cases}
\tag{2.1}
\]

where the velocity field $(u^{\varepsilon}, v^{\varepsilon})$ is given by

\[
u^{\varepsilon}(t, x, y) := U(t, x) - \int_y^{+\infty} w^{\varepsilon}(t, x, \eta)d\eta,
\tag{2.2}
\]

and

\[
u^{\varepsilon}(t, x, y) := -\int_0^y \partial_x u^{\varepsilon}(t, x, \eta)d\eta.
\tag{2.3}
\]

Next, we derive a life existence time $T_0$ (independent of $\varepsilon$) such the quantity $\|w^{\varepsilon}(t)\|_{{B^{\gamma, \sigma}_0}}$ owning a uniform bound. More precisely, we have the following results.

**Theorem 2.1** (a priori estimates). Let $s \geq 4$ be an even integer, $\gamma \geq 1, \sigma > \gamma + \frac{1}{2}, \delta_0 \in (0, \frac{1}{2})$, and $\varepsilon \in (0, 1]$, the smooth solution $(u^{\varepsilon}, v^{\varepsilon}, w^{\varepsilon})$, defined on $[0, T^\varepsilon]$, to the regularized Prandtl equations (2.1)-(2.3). Under the assumptions of Theorem 1.1 there exists a time $T_0 := T_0(s, \gamma, \sigma, \delta_0, \|w_0\|_{{B^{\gamma, \sigma}_0}}, M_U) > 0$ independent of $\varepsilon$ such the following estimates hold on

\[
\Omega(t) := \sup_{0 \leq \tau \leq t} \|w^{\varepsilon}(\tau)\|_{{B^{\gamma, \sigma}_0}}^2 \leq C_{s, \gamma, \sigma}\left\{1 + \|w_0\|_{{B^{\gamma, \sigma}_0}}^8 + M_U^4\right\},
\tag{2.4}
\]

and

\[
\min_{T \times \mathbb{R}^+} (1 + y)^\sigma \|w^{\varepsilon}(t, x, y)\| \geq \delta_0,
\tag{2.5}
\]

for all $t \in [0, \min(T_0, T^\varepsilon)]$.

**Remark 2.1.** After having the results in Theorem 2.1 at hand, we can pass to the limit $\varepsilon \to 0+$ in the regularized Prandtl equations (1.8) and the regularized Bernoulli’s law (1.9). Thus, it is easy to check that the limit functions $(u, v)$ will solve the original Prandtl equations (1.1) with the Bernoulli’s law (1.3) in the classical sense(cf.[10]). On the other hand, the uniqueness of Prandtl equations (1.1) has already been derived in [10] without the condition (1.4). In other words, the local-in-time well-posedness theory of solutions to the Prandtl equations (1.1) in Theorem 1.1 is a direct consequence of Theorem 2.1.

Throughout this section, for any small constant $\delta \in (0, \frac{1}{2})$, we assume a priori assumption

\[
(1 + y)^\sigma \|w^{\varepsilon}(t, x, y)\| \geq \delta, \quad \forall (t, x, y) \in [0, T^\varepsilon] \times T \times \mathbb{R}^+,
\tag{2.6}
\]

holds on. Let us define

\[
\|w^{\varepsilon}(t)\|^2_{{B^{\gamma, \sigma}_0}} := \|w^{\varepsilon}(t)\|^2_{{H^{\gamma}_0}} + Q(t), \quad \text{and} \quad \Omega_0(t) := \sup_{0 \leq \tau \leq t} \|w^{\varepsilon}(\tau)\|^2_{{B^{\gamma, \sigma}_0}},
\tag{2.7}
\]

where $Q(t)$ is defined by

\[
Q(t) := \sum_{1 \leq |\alpha| \leq 2} \|(1 + y)^{\sigma+\alpha_2} D^\alpha w^{\varepsilon}(t)\|_L^2.
\tag{2.8}
\]
2.1. Weighted Energy Estimates

In this subsection, we will derive the uniform weighted estimates for the vorticity \( w^\varepsilon \), which plays an important role for us to find the uniform existence life time. Since one order tangential derivative loss is allowed, we may apply the energy method to establish the weighted estimates for the vorticity \( D^\alpha w^\varepsilon (|\alpha| \leq s, \alpha_1 \leq s - 1) \).

Lemma 2.2. Under the hypotheses of Theorem [2.1], we have the following estimate:

\[
\frac{d}{dt} \sum_{|\alpha| \leq s} \|(1 + y)^{\gamma + \alpha_2} D^\alpha w^\varepsilon \|_{L^2}^2 + \sum_{|\alpha| \leq s - 1} \|(1 + y)^{\gamma + \alpha_2} (\varepsilon \partial_x D^\alpha w^\varepsilon, \partial_y D^\alpha w^\varepsilon) \|_{L^2}^2 \leq \geq C_{s, \gamma, \sigma, \delta} \partial_x^{s+1} U \|_{L^\infty(T)} + C_{s, \gamma, \sigma, \delta} \| (1 + y)^{\sigma + 1} \partial_y w^\varepsilon \|_{L^\infty}^8 + \| w^\varepsilon \|_{H^8, \gamma}^8 \\
+ C_{s, \gamma} (1 + \| w^\varepsilon \|_{H^8, \gamma})^{s-2} \| w^\varepsilon \|_{H^{s-2}, \gamma}^2 + C_s \sum_{k=0}^{s/2} \| \partial_x^k \partial_x \|_{H^{s-2}, \gamma}^2,
\]

where the positive constants \( C_s, C_{s, \gamma} \) and \( C_{s, \gamma, \sigma, \delta} \) are independent of \( \varepsilon \).

Proof. Differentiating the vorticity equation (2.1) with respect to \( x \alpha_1 \) times and \( y \alpha_2 \) times, and multiplying the resulting equality by \( (1 + y)^{\gamma + \alpha_2} D^\alpha w^\varepsilon \), we get after integrating over \( T \times \mathbb{R}^+ \),

\[
\frac{1}{2} \frac{d}{dt} \|(1 + y)^{\gamma + \alpha_2} D^\alpha w^\varepsilon \|_{L^2}^2 + \varepsilon^2 \|(1 + y)^{\gamma + \alpha_2} \partial_x D^\alpha w^\varepsilon \|_{L^2}^2 = J_1 + J_2 + J_3 + J_4,
\]

where \( J_1, J_2, J_3 \) and \( J_4 \) are defined by

\[
J_1 = \int_{T \times \mathbb{R}^+} (1 + y)^{\gamma + \alpha_2} D^\alpha w^\varepsilon \partial_x^2 D^\alpha w^\varepsilon \, dx \, dy,
\]

\[
J_2 = (\gamma + \alpha_2) \int_{T \times \mathbb{R}^+} (1 + y)^{\gamma + \alpha_2} - 1 \varepsilon^2 \| D^\alpha w^\varepsilon \|_{L^2}^2 \, dx \, dy,
\]

\[
J_3 = - \sum_{0 < \beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) \int_{T \times \mathbb{R}^+} (1 + y)^{\gamma + \alpha_2} D^\alpha w^\varepsilon \cdot D^\beta w^\varepsilon \, dx \, dy,
\]

\[
J_4 = - \sum_{0 < \beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) \int_{T \times \mathbb{R}^+} (1 + y)^{\gamma + \alpha_2} D^\alpha w^\varepsilon \cdot D^\beta w^\varepsilon \, dx \, dy.
\]

First of all, integrating by part and applying the Cauchy inequality, we get

\[
J_1 \leq \frac{3}{4} \|(1 + y)^{\gamma + \alpha_2} \partial_y D^\alpha w^\varepsilon \|_{L^2}^2 - \int_T \partial_x^2 D^\alpha w^\varepsilon \partial_y D^\alpha w^\varepsilon \|_{y=0} \, dx + C_{s, \gamma} \| w^\varepsilon \|_{H^{s, \gamma}}^2.
\]

Next, integrating by part and applying the divergence-free condition, it follows

\[
|J_2| \leq C_{s, \gamma} \| \frac{v^\varepsilon}{1 + y} \|_{L^\infty} \|(1 + y)^{\gamma + \alpha_2} D^\alpha w^\varepsilon \|_{L^2}^2.
\]

Using the divergence-free condition, Sobolev and Hardy inequalities, we get for \( \gamma \geq 1 \)

\[
\| \frac{v^\varepsilon}{1 + y} \|_{L^\infty} \leq \| v^\varepsilon + y \partial_x U \|_{L^\infty} + \| y \partial_x U \|_{L^\infty} \leq C\| v^\varepsilon + y \partial_x U \|_{L^2} + \| \partial_x v^\varepsilon + y \partial_x U \|_{L^2} + \| \partial_y (v^\varepsilon + y \partial_x U) \|_{L^2} + \| y \partial_x U \|_{L^\infty} \leq C\| \partial_x w^\varepsilon \|_{L^2} + \| \partial_x (u^\varepsilon - U) \|_{L^2} + \| \partial_x (u^\varepsilon - U) \|_{L^2} + \| \partial_x U \|_{L^\infty(T)} \leq C\| w^\varepsilon \|_{H^{s, \gamma}} + \| \partial_x U \|_{L^\infty(T)}
\]
and hence, $J_2$ can be estimated as

$$|J_2| \leq C_{s, \gamma}(\|\partial_x U\|_{L^\infty(T)} + \|w^\varepsilon\|_{H^{s, \gamma}})\|w^\varepsilon\|_{H^{s, \gamma}}^2.$$

Deal with term $J_3$. Using the Hölder inequality, we get for $0 < \beta \leq \alpha$

$$\left| \int (1 + y)^{\gamma + 2\alpha_2} D^\alpha w^\varepsilon \cdot D^\beta v^\varepsilon \partial_y D^{\alpha - \beta} w^\varepsilon \, dx dy \right| \leq \|(1 + y)^{\gamma + \alpha_2} D^\beta (u^\varepsilon - U) D^{\alpha + \varepsilon_1 - \beta} w^\varepsilon\|_{L^2}(1 + y)^{\gamma + \alpha_2} D^\alpha w^\varepsilon\|_{L^2} \tag{2.11}$$

Using the Moser and Hardy inequalities, it follows

$$\|(1 + y)^{\gamma + \alpha_2} D^\beta (u^\varepsilon - U) D^{\alpha + \varepsilon_1 - \beta} w^\varepsilon\|_{L^2} \leq C_{s, \gamma}\|w^\varepsilon\|_{H^{s, \gamma}}^2. \tag{2.12}$$

Applying the Sobolev and Wirtinger inequalities, we get

$$\|(1 + y)^{\gamma + \alpha_2} \partial_x^\beta U \partial_x^{\alpha_1 + 1 - \beta_1} \partial_y^\alpha w^\varepsilon\|_{L^2} \leq C_{s, \gamma}\|\partial_x^\beta U\|_{L^\infty(T)}\|w^\varepsilon\|_{H^{s, \gamma}}. \tag{2.13}$$

Substituting estimates (2.12) and (2.13) into (2.11), we obtain

$$|J_3| \leq C_{s, \gamma}(\|\partial_x^\beta U\|_{L^\infty(T)} + \|w^\varepsilon\|_{H^{s, \gamma}})\|w^\varepsilon\|_{H^{s, \gamma}}^2.$$

Deal with the term $J_4$. Using the Hölder inequality, we get for $0 < \beta \leq \alpha$

$$\left| \int_{T \times \mathbb{R}^+} (1 + y)^{\gamma + 2\alpha_2} D^\alpha w^\varepsilon \cdot D^\beta v^\varepsilon \partial_y D^{\alpha - \beta} w^\varepsilon \, dx dy \right| \leq \|(1 + y)^{\gamma + \alpha_2} D^\beta (u^\varepsilon + y\partial_x U) D^{\alpha + \varepsilon_2 - \beta} w^\varepsilon\|_{L^2}(1 + y)^{\gamma + \alpha_2} D^\alpha w^\varepsilon\|_{L^2} \tag{2.14}$$

Using the Sobolev and Wirtinger inequalities, it follows

$$\|(1 + y)^{\gamma + \alpha_2} D^\beta (u^\varepsilon + y\partial_x U) D^{\alpha + \varepsilon_2 - \beta} w^\varepsilon\|_{L^2} \leq C\|\partial_x^{\alpha + 1} U\|_{L^\infty(T)}\|w^\varepsilon\|_{H^{s, \gamma}}. \tag{2.15}$$

For $|\alpha| \leq s - 1$, we apply the Moser and Hardy inequalities to get for $i = 1, 2$,

$$\|(1 + y)^{\gamma + \alpha_2} D^\beta (u^\varepsilon + y\partial_x U) D^{\alpha + \varepsilon_2 - \beta} w^\varepsilon\|_{L^2} \leq C\|D^{\varepsilon_i}(u^\varepsilon + y\partial_x U)\|_{H^{s - 2, i - 2}}\|w^\varepsilon\|_{H^{s - 1, \gamma}} \leq C\|w^\varepsilon\|_{H^{s, \gamma}}^2. \tag{2.16}$$

Similar, for $|\alpha| = s$ and $\alpha_1 \leq s - 1$, we get for $\beta_2 \geq 1$

$$\|(1 + y)^{\gamma + \alpha_2} D^\beta (u^\varepsilon + y\partial_x U) D^{\alpha + \varepsilon_2 - \beta} w^\varepsilon\|_{L^2} \leq C\|w^\varepsilon\|_{H^{s, \gamma}}^2. \tag{2.17}$$

and for $\beta_2 = 0$

$$\|(1 + y)^{\gamma + \alpha_2} D^\beta (u^\varepsilon + y\partial_x U) D^{\alpha + \varepsilon_2 - \beta} w^\varepsilon\|_{L^2} \leq \|(1 + y)^{\gamma + \alpha_2} \partial_x^\beta (u^\varepsilon + y\partial_x U) \partial_x^{\alpha_1 - \beta_1} \partial_y^\alpha w^\varepsilon\|_{L^2} \tag{2.18}$$

Substituting the estimates (2.15)-(2.18) into (2.14), it follows

$$|J_4| \leq C_{s, \gamma}(\|\partial_x^{\alpha + 1} U\|_{L^\infty(T)} + \|w^\varepsilon\|_{H^{s, \gamma}})\|w^\varepsilon\|_{H^{s, \gamma}}^2.$$

A Note on the Well-posedness of Prandtl Equations
Plugging the estimates of $J_1$ through $J_4$ into the equality (2.19), we get

\[
\frac{1}{2} \frac{d}{dt} \|(1 + y)^{\gamma + \alpha_2} D^\alpha w^\varepsilon \|_{L_2}^2 + \varepsilon^2 \|(1 + y)^{\gamma + \alpha_2} \partial_x D^\alpha w^\varepsilon \|_{L_2}^2 + \frac{3}{4} \|(1 + y)^{\gamma + \alpha_2} \partial_y D^\alpha w^\varepsilon \|_{L_2}^2 \leq - \int_T D^\alpha w^\varepsilon \partial_y D^\alpha w^\varepsilon \big|_{y=0} dx + C_{s,\gamma}(1 + \|\partial_x^{\varepsilon+1} U\|_{L^\infty(T)} + \|w^\varepsilon\|_{H^\gamma}) \|w^\varepsilon\|_{H^\gamma}^2 \tag{2.19}
\]

which, together with the relation [13,7], yields directly

\[
\frac{1}{2} \frac{d}{dt} \|(1 + y)^{\gamma + \alpha_2} D^\alpha w^\varepsilon \|_{L_2}^2 + \varepsilon^2 \|(1 + y)^{\gamma + \alpha_2} \partial_x D^\alpha w^\varepsilon \|_{L_2}^2 + \frac{3}{4} \|(1 + y)^{\gamma + \alpha_2} \partial_y D^\alpha w^\varepsilon \|_{L_2}^2 \leq - \int_T D^\alpha w^\varepsilon \partial_y D^\alpha w^\varepsilon \big|_{y=0} dx + C_{s,\gamma,\sigma,\delta}(1 + \|\partial_x^{\varepsilon+1} U\|_{L^\infty(T)} + \|w^\varepsilon\|_{H^\gamma}) \|w^\varepsilon\|_{H^\gamma}^2 \tag{2.20}
\]

Note the boundary term above can be estimated as follows (cf. [10]) for $|\alpha| \leq s - 1$,

\[
\left| \int_T D^\alpha w^\varepsilon \partial_y D^\alpha w^\varepsilon \big|_{y=0} dx \right| \leq \frac{1}{12} \|(1 + y)^{\gamma + \alpha_2 + 1} \partial_y D^\alpha w^\varepsilon \|_{L_2}^2 + C \|w^\varepsilon\|_{H^\gamma}^2 \tag{2.21}
\]

and for $|\alpha| = s$,

\[
\left| \int_T D^\alpha w^\varepsilon \partial_y D^\alpha w^\varepsilon \big|_{y=0} dx \right| \leq \left\{ \begin{array}{ll}
\frac{1}{12} \|(1 + y)^{\gamma + \alpha_2} \partial_y D^\alpha w^\varepsilon \|_{L_2}^2 + H(t), \ & \alpha_2 = 2k, k \in \mathbb{N}; \\
\frac{1}{12} \|(1 + y)^{\gamma + \alpha_2 + 1} \partial_y^{\alpha_2 + 1} w^\varepsilon \|_{L_2}^2 + H(t), \ & \alpha_2 = 2k + 1, k \in \mathbb{N};
\end{array} \right. \tag{2.22}
\]

where $H(t) = C_s \sum_{j=0}^{s/2} \|\partial_x^j u^\varepsilon\|_{H^{s-2N}(T)}^2 + C_{s,\gamma}(1 + \|u^\varepsilon\|_{H^{s-1}(T)})^{s-2} \|w^\varepsilon\|_{H^\gamma}^2$. Thus, plugging the estimates (2.21) and (2.22) into (2.20), and summing over $\alpha$, we complete the proof of this lemma.

Next, define $a^\varepsilon := \frac{\partial_x w^\varepsilon}{w^\varepsilon}$ and $g^\varepsilon_s := \partial_x^s w^\varepsilon - a^\varepsilon \partial_x^s (w^\varepsilon - U)$, we are going to derive the $L^2$ estimate for $(1 + y)^{g^\varepsilon_s}$ by using the standard energy methods. As mentioned in [10], this quantity $g^\varepsilon_s$ will avoid the loss of $x$-derivative by a nonlinear cancellation. By routine checking, it is easy to justify that the quantity $g^\varepsilon_s$ satisfies the evolution equation (cf. [10])

\[
\begin{align*}
(\partial_t + u^\varepsilon \partial_x + v^\varepsilon \partial_y - \varepsilon^2 \partial_x^2 - \partial_y^2) g^\varepsilon_s \\
= & 2\varepsilon^2 \left\{ \partial_x^{\varepsilon+1} (u^\varepsilon - U) - \partial_x w^\varepsilon \partial_x^2 (u^\varepsilon - U) \right\} \partial_x a^\varepsilon + 2g^\varepsilon_s \partial_y a^\varepsilon - g^\varepsilon_s \partial_x^2 U \\
& - \sum_{j=1}^{s-1} \left( \begin{array}{c}
s \\
j
\end{array} \right) g^\varepsilon_{s-j+1} \partial_x^{s-j} u^\varepsilon - \sum_{j=1}^{s-1} \left( \begin{array}{c}
s \\
j
\end{array} \right) \partial_x^{s-j} u^\varepsilon \partial_x^j \partial_y w^\varepsilon - a^\varepsilon \partial_x^{s-j+1} U \\
& + a^\varepsilon \sum_{j=0}^{s-1} \left( \begin{array}{c}
s \\
j
\end{array} \right) \partial_x^j (u^\varepsilon - U) \partial_x^{s-j+1} U,
\end{align*}
\tag{2.23}
\]

where $g^\varepsilon_k := \partial_x^k w^\varepsilon - a^\varepsilon \partial_x^k (w^\varepsilon - U)$.

Now, we are going to derive the following weighted energy estimate for $g^\varepsilon_s$:

**Lemma 2.3.** Under the hypothesis of Theorem 2.1, we have the following estimate:

\[
\frac{d}{dt} \|(1 + y)^{\gamma + \alpha_2} g^\varepsilon_s \|_{L_2}^2 + \varepsilon^2 \|(1 + y)^{\gamma + \alpha_2} \partial_x g^\varepsilon_s \|_{L_2}^2 + \|(1 + y)^{\gamma + \alpha_2} \partial_y g^\varepsilon_s \|_{L_2}^2 \leq C \|\partial_x^{\varepsilon+1} p^\varepsilon \|_{L^2(T)} + C_{s,\gamma,\sigma,\delta} \|\partial_x^{\varepsilon+1} U \|_{L^\infty(T)} + C_{s,\gamma,\sigma,\delta}(1 + Q^4(t) + \|w^\varepsilon\|_{H^\gamma}^6),
\]

where the quantity $Q(t)$ is defined in (2.8), the positive constants $C$ and $C_{s,\gamma,\sigma,\delta}$ are independent of $\varepsilon$. 


Proof. Multiplying the equation (2.23) by \((1 + y)^{2\gamma} g_x^\varepsilon\) and integrating over \(\mathbb{T} \times \mathbb{R}^+\), we have
\[
\frac{1}{2} \frac{d}{dt} \|(1 + y)^\gamma g_x^\varepsilon\|_{L^2}^2 + \varepsilon^2 \|(1 + y)^\gamma \partial_x g_x^\varepsilon\|_{L^2}^2 = \sum_{i=1}^{8} K_i, \tag{2.24}
\]
where the terms \(K_i (i = 1, \ldots, 8)\) are defined by
\[
K_1 = \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} g_x^\varepsilon \partial_y g_x^\varepsilon \, dxdy, \quad K_2 = - \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} g_x^\varepsilon (u^\varepsilon \partial_x g_x^\varepsilon + v^\varepsilon \partial_y g_x^\varepsilon) \, dxdy,
\]
\[
K_3 = 2\varepsilon^2 \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} g_x^\varepsilon \{ \partial_x^{\gamma+1} (u^\varepsilon - U) - \frac{\partial_v w^\varepsilon}{w^\varepsilon} \partial_x^\varepsilon (u^\varepsilon - U) \} \partial_x g_x^\varepsilon \, dxdy,
\]
\[
K_4 = 2 \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} |g_x^\varepsilon|^2 \partial_y g_x^\varepsilon \, dxdy, \quad K_5 = \sum_{j=1}^{s-1} \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} g_x^{j+1} g_x^\varepsilon \partial_x^{-j} u^\varepsilon \, dxdy,
\]
\[
K_6 = \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} g_x^\varepsilon \partial_x^\varepsilon U \, dxdy, \quad K_7 = \sum_{j=0}^{s-1} \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} g_x^\varepsilon \partial_x^{-j} (u^\varepsilon - U) \partial_x^{-j+1} U \, dxdy,
\]
\[
K_8 = \sum_{j=1}^{s-1} \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} \partial_x^{-j} v^\varepsilon \{ \partial_x^{\varepsilon} \partial_y g_x^\varepsilon - a^\varepsilon \partial_x^\varepsilon g_x^\varepsilon \} g_x^\varepsilon \, dxdy.
\]
Deal with the term \(K_1\). Integrating by part in the \(y\)--variable, we get
\[
K_1 = \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} g_x^\varepsilon \partial_y^2 g_x^\varepsilon \, dxdy
\]
\[
= \int_{\mathbb{T}} g_x^\varepsilon \partial_y g_x^\varepsilon |_{y=0} \, dx - \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} |\partial_y g_x^\varepsilon|^2 \, dxdy - 2\gamma \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma-1} g_x^\varepsilon \partial_y g_x^\varepsilon \, dxdy
\]
\[
= -\frac{3}{4} \|(1 + y)^\gamma \partial_y g_x^\varepsilon\|_{L^2}^2 - \int_{\mathbb{T}} g_x^\varepsilon \partial_y g_x^\varepsilon |_{y=0} \, dx + C \|(1 + y)^{\gamma-1} g_x^\varepsilon\|_{L^2}^2.
\]
Due to the definition of \(g_x^\varepsilon\) and boundary condition \(\partial_y w^\varepsilon |_{y=0} = \partial_x p^\varepsilon\), it follows
\[
\partial_y g_x^\varepsilon |_{y=0} = \partial_x^{\gamma+1} p^\varepsilon + \frac{\partial_x^2 w^\varepsilon}{w^\varepsilon} \partial_x U |_{y=0} - \frac{\partial_y w^\varepsilon}{w^\varepsilon} g_x^\varepsilon |_{y=0}.
\]
Using the Hölder and trace inequalities, we get
\[
| \int_{\mathbb{T}} g_x^\varepsilon \partial_x^{\gamma+1} \partial_y w^\varepsilon |_{y=0} \, dx | \leq \| g_x^\varepsilon \|_{y=0} \|L^2(\mathbb{T})\| \| \partial_x^{\gamma+1} \partial_y w^\varepsilon \|_{L^2(\mathbb{T})}
\]
\[
\leq \sqrt{2} \| g_x^\varepsilon \|_{L^2} \| \partial_y g_x^\varepsilon \|_{L^2} \| \partial_x^{\gamma+1} \partial_y w^\varepsilon \|_{L^2(\mathbb{T})}
\]
\[
\leq \frac{1}{8} \| \partial_y g_x^\varepsilon \|_{L^2}^2 + C \| \partial_x^{\gamma+1} \partial_y w^\varepsilon \|_{L^2(\mathbb{T})} \| g_x^\varepsilon \|_{L^2}^2.
\]
Due to \((1 + y)^\sigma w^\varepsilon \geq \delta\), it follows \(w^\varepsilon \geq \delta\), and hence, we get, after using \(\partial_y w^\varepsilon |_{y=0} = \partial_x p^\varepsilon\),
\[
| \int_{\mathbb{T}} g_x^\varepsilon \partial_y w^\varepsilon g_x^\varepsilon |_{y=0} \, dx | \leq \frac{1}{8} \| \partial_y g_x^\varepsilon \|_{L^2}^2 + C \delta \| \partial_x p^\varepsilon \|_{L^\infty(\mathbb{T})} \| g_x^\varepsilon \|_{L^2}^2.
\]
and
\[
| \int_{\mathbb{T}} g_x^\varepsilon \frac{\partial_y w^\varepsilon}{w^\varepsilon} \partial_x U |_{y=0} \, dx | \leq \delta^{-1} \| \partial_x U \|_{L^\infty(\mathbb{T})} \| g_x^\varepsilon \|_{y=0} \|L^2(\mathbb{T})\| \| \partial_y w^\varepsilon \|_{y=0} \|L^2(\mathbb{T})\|
\]
\[
\leq 2 \delta^{-1} \| \partial_x U \|_{L^\infty(\mathbb{T})} \| g_x^\varepsilon \|_{L^2} \| \partial_y g_x^\varepsilon \|_{L^2} \| \partial^2 w^\varepsilon \|_{L^2} \| \partial_y w^\varepsilon \|_{L^2} \| \partial^2 w^\varepsilon \|_{L^2}
\]
\[
\leq \frac{1}{8} \| \partial_y g_x^\varepsilon \|_{L^2}^2 + C \delta \| \partial_x U \|_{L^\infty(\mathbb{T})} \| g_x^\varepsilon \|_{L^2} \| \partial_y w^\varepsilon \|_{L^2} \| \partial_y w^\varepsilon \|_{L^2}^2.
\]
Thus, the term $K_1$ can be estimated as follows

$$K_1 \leq -\frac{1}{2} \|(1 + y)^\gamma \partial_y g_\alpha\|^2_{L^2} + C_{\gamma,\delta}(1 + \|\partial_x^{\sigma+1} p\|^2_{L^2(\mathbb{R}^+)} + \|\partial_y U\|^2_{L^\infty(\mathbb{R}^+)})(1 + \|w^\varepsilon\|^2_{H_{s,\gamma}^{\gamma}}).$$

Deal with the term $K_2$. Integrating by part and applying the divergence-free condition, it follows

$$|K_2| = \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} g_\alpha^\varepsilon (u^\varepsilon \partial_x g_\alpha^\varepsilon + v^\varepsilon \partial_y g_\alpha^\varepsilon) dx dy \leq C_\gamma \|\varepsilon^{\gamma} \partial_y g_\alpha^\varepsilon\|^2_{L^2},$$

which, along with inequality (3.13), yields directly

$$|K_2| \leq C_{\gamma,\sigma,\delta}(1 + \|\partial_x^\sigma U\|_{L^\infty(\mathbb{R}^+)} + (1 + y)^{\sigma+1} \|\partial_y w^\varepsilon\|_{L^\infty}) \|w^\varepsilon\|^3_{H_{s,\gamma}^{\gamma}}.$$

Deal with the term $K_3$. First of all, using the Hölder inequality, we get

$$| \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} g_\alpha^\varepsilon \frac{\partial_x w^\varepsilon}{w^\varepsilon} \partial_x (u^\varepsilon - U) \partial_x a^\varepsilon dx dy | \leq \| \partial_x w^\varepsilon \|_{L^\infty} \| (1 + y) \partial_x a^\varepsilon \|_{L^\infty} \| (1 + y)^{\gamma-1} \partial_x (u^\varepsilon - U) \|_{L^2} \| (1 + y)^{\gamma} g_\alpha^\varepsilon \|_{L^2}. \quad (2.25)$$

Due to the fact $\partial_x a^\varepsilon = \frac{\partial_x w^\varepsilon}{w^\varepsilon} - \frac{\partial_y w^\varepsilon \partial_x w^\varepsilon}{(w^\varepsilon)^2}$, we get, after using the inequality $(1 + y)^{\sigma} w^\varepsilon \geq \delta$, we get

$$\| \partial_x w^\varepsilon \|_{L^\infty} \leq \delta^{-1} \| (1 + y)^{\sigma} \partial_x w^\varepsilon \|_{L^\infty} \quad \quad (2.26)$$

and

$$\| (1 + y) \partial_x a^\varepsilon \|_{L^\infty} \leq \delta^{-1} \| (1 + y)^{\sigma+1} \partial_x w^\varepsilon \|_{L^\infty} + \delta^{-2} \| (1 + y)^{\sigma+1} \partial_y w^\varepsilon \|_{L^\infty} \| (1 + y)^{\gamma} \partial_x w^\varepsilon \|_{L^\infty}. \quad (2.27)$$

Substituting the inequalities (2.26) and (2.27) into (2.25), and applying inequality (3.9), we obtain

$$| \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} g_\alpha^\varepsilon \frac{\partial_x w^\varepsilon}{w^\varepsilon} \partial_x (u^\varepsilon - U) \partial_x a^\varepsilon dx dy | \leq C_{\gamma,\sigma,\delta}(1 + \|\partial_x^\sigma U\|^2_{L^\infty(\mathbb{R}^+)} + Q^2(t)) \|w^\varepsilon\|^2_{H_{s,\gamma}^{\gamma}}. \quad (2.28)$$

On the other hand, by virtue of the Hölder inequality, we get

$$\left| \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} g_\alpha^\varepsilon \partial_x^{\sigma+1} (u^\varepsilon - U) \partial_x a^\varepsilon dx dy \right| \leq \| (1 + y)^{\sigma+1} \partial_x w^\varepsilon \|_{L^\infty} \| (1 + y)^{\gamma-\sigma-1} \frac{\partial_x^{\sigma+1} (u^\varepsilon - U)}{w^\varepsilon} \|_{L^2} \| (1 + y)^{\gamma} g_\alpha^\varepsilon \|_{L^2}. \quad (2.29)$$

Due to the fact $(1 + y)^{\sigma} w^\varepsilon \geq \delta$, it follows

$$\| (1 + y)^{\sigma+1} \partial_x w^\varepsilon \partial_y a^\varepsilon \|_{L^\infty} \leq \| (1 + y)^{\sigma+1} \partial_x w^\varepsilon \|_{L^\infty} + \delta^{-1} \| (1 + y)^{\sigma} \partial_x w^\varepsilon \|_{L^\infty} \| (1 + y)^{\sigma+1} \partial_y w^\varepsilon \|_{L^\infty} \quad (2.30)$$

By routine checking, it follows

$$\partial_x g_\alpha^\varepsilon = w^\varepsilon \partial_y \left\{ \frac{\partial_x^{\sigma+1} (u^\varepsilon - U)}{w^\varepsilon} \right\} - \partial_x a^\varepsilon \partial_x^\sigma (u^\varepsilon - U),$$

and hence, we get after applying the Hardy inequality

$$\| (1 + y)^{\gamma-\sigma-1} \frac{\partial_x^{\sigma+1} (u^\varepsilon - U)}{w^\varepsilon} \|_{L^2} \leq C_{\gamma,\sigma} \left\{ \| \frac{\partial_x^{\sigma+1} U}{w^\varepsilon} \|_{L^2(\mathbb{T})} \right\} + \| (1 + y)^{\gamma-\sigma} \partial_y \left\{ \frac{\partial_x^{\sigma+1} (u^\varepsilon - U)}{w^\varepsilon} \right\} \|_{L^2} \right\} \leq C_{\gamma,\sigma,\delta} \left\{ \| \partial_x^{\sigma+1} U \|_{L^2(\mathbb{T})} \right\} + \| (1 + y)^{\gamma} \partial_x g_\alpha^\varepsilon \|_{L^2} + \| (1 + y) \partial_x a^\varepsilon \|_{L^\infty} \| (1 + y)^{\gamma-1} \partial_x^\sigma (u^\varepsilon - U) \|_{L^2} \right\}. \quad (2.31)
Substituting inequalities (2.30) and (2.31) into (2.29), and applying the inequality (B.9), it follows
\[
\left| \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} g_s^\varepsilon \partial^{s+1}_x (u^\varepsilon - U) \partial_x a^\varepsilon dxdy \right|
\leq \frac{1}{4} \|(1 + y)^\gamma \partial_x g_s^\varepsilon \|^2_{L^2} + C_{\gamma, \sigma, \delta} (1 + \|\partial^{s+1}_x U\|^2_{L^2(\mathbb{T})} + Q^2(t))(1 + \|w^\varepsilon\|^2_{H^{s, \gamma}}).
\]
This and the inequality (2.28) imply directly
\[
|K_3| \leq \frac{1}{2} \varepsilon^2 \|(1 + y)^\gamma \partial_x g_s^\varepsilon \|^2_{L^2} + C_{\gamma, \sigma, \delta} (1 + \|\partial^{s+1}_x U\|^2_{L^2(\mathbb{T})} + Q^2(t))(1 + \|w^\varepsilon\|^2_{H^{s, \gamma}}).
\]
Deal with the term $K_4$. Indeed, it is easy to get
\[
|K_4| = 2 \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} |g_s^\varepsilon| |\partial_y a^\varepsilon| dxdy \leq C \|\partial_y a^\varepsilon\|_{L^\infty} \|(1 + y)^\gamma g_s^\varepsilon\|^2_{L^2}.
\] (2.32)
Due to the fact $\partial_y a^\varepsilon = \frac{\partial^2_w}{w^\varepsilon} - (\frac{\partial_w}{w^\varepsilon})^2$, we obtain
\[
\|\partial_y a^\varepsilon\|_{L^\infty} \leq \delta^{-1} \|(1 + y)^\sigma \partial^2_w w^\varepsilon\|_{L^\infty} + \delta^{-2} \|(1 + y)^\sigma \partial_y w^\varepsilon\|_{L^\infty}.
\]
This and the inequality (2.32) give immediately
\[
|K_4| \leq C\delta (1 + Q(t))\|w^\varepsilon\|^2_{H^{s, \gamma}}.
\]
Deal with the term $K_5$. Using the H"{o}lder inequality, it follows
\[
|K_5| \leq C_s \|\partial^{s-j}_x U\|_{L^\infty(\mathbb{T})} \|(1 + y)^\gamma g_{s+j}^\varepsilon\|_{L^2} \|(1 + y)^\gamma g_s^\varepsilon\|_{L^2}.
\]
This and the inequalities (B.11) and (B.12) imply directly
\[
|K_5| \leq C_{s, \gamma, \sigma} (1 + \|\partial^{s+1}_x U\|^2_{L^2(\mathbb{T})} + Q(t))(1 + \|w^\varepsilon\|^3_{H^{s, \gamma}}).
\]
Similarly, it is easy to deduce
\[
|K_6| \leq C_{\gamma, \delta} (1 + \|\partial^{s}_x U\|^2_{L^2(\mathbb{T})} + Q(t))\|w^\varepsilon\|^2_{H^{s, \gamma}}.
\]
Deal with the term $K_7$. Using the H"{o}lder and Hardy inequalities, we get for $j = 0, \ldots, s - 1$
\[
|K_7| \leq C_s \|\partial^{s-j+1}_x U\|_{L^\infty(\mathbb{T})} \|(1 + y)^\gamma a^\varepsilon\|_{L^\infty} \|(1 + y)^\gamma \partial^j_x (u^\varepsilon - U)\|_{L^2} \|(1 + y)^\gamma g_s^\varepsilon\|_{L^2}
\leq C_{s, \gamma} \|\partial^{s+1}_x U\|_{L^\infty(\mathbb{T})} \|(1 + y)^\gamma a^\varepsilon\|_{L^\infty} \|w^\varepsilon\|^2_{H^{s, \gamma}},
\]
and hence, using the fact $\|(1 + y)^\gamma a^\varepsilon\|_{L^\infty} \leq \delta^{-1} \|(1 + y)^\sigma \partial_y w^\varepsilon\|_{L^\infty}$, we get
\[
|K_7| \leq C_{s, \gamma, \delta} (\|\partial^{s+1}_x U\|^2_{L^\infty(\mathbb{T})} + Q(t))\|w^\varepsilon\|^2_{H^{s, \gamma}}.
\]
Deal with the term $K_8$. For the case $j = 1$, it is easy to check that
\[
\int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} \partial^{s-1}_x (\partial_{xy} w^\varepsilon - a^\varepsilon \partial_x w^\varepsilon) g_s^\varepsilon dxdy
= \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma+1} \frac{\partial^{s-1}_x (\partial_{xy} w^\varepsilon + y \partial^2_x U)}{1 + y} (\partial_{xy} w^\varepsilon - a^\varepsilon \partial_x w^\varepsilon) g_s^\varepsilon dxdy
- \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma+1} \frac{y \partial^2_x U}{1 + y} (\partial_{xy} w^\varepsilon - a^\varepsilon \partial_x w^\varepsilon) g_s^\varepsilon dxdy.
\]
By virtue of the Hölder inequality, it follows
\[
\left| \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma+1} \frac{\partial_x^{s-1} v^c}{1 + y} \left( \partial_{xy} w^c - a^c \partial_x w^c \right) g_s^c \, dx \, dy \right| \\
\leq (1 + y)^{\gamma+1} \| \partial_{xy} w^c \|_{L^\infty} + (1 + y) a^c \| \partial x w^c \|_{L^\infty} \| (1 + y)^{\gamma} \partial_x w^c \|_{L^\infty} \\
\times \| \partial_x^{s-1} v^c + y \partial_x^s U \|_{L^2} \| (1 + y)^\gamma g_s^c \|_{L^2}
\] (2.33)

and
\[
\left| \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma+1} \frac{\partial_x^s v^c}{1 + y} \left( \partial_{xy} w^c - a^c \partial_x w^c \right) g_s^c \, dx \, dy \right| \\
\leq (1 + y)^{\gamma+1} \| \partial_{xy} w^c \|_{L^2} + (1 + y) a^c \| \partial x w^c \|_{L^2} \| (1 + y)^\gamma \partial_x w^c \|_{L^2} \\
\times \| \partial_x^s U \|_{L^\infty(T)} \| \partial^\gamma g_s^c \|_{L^2}.
\] (2.34)

Due to the fact \( \| (1 + y) a^c \|_{L^\infty} \leq \delta^{-1} \| (1 + y)^{\sigma+1} \partial_y w^c \|_{L^\infty} \), applying the inequality (B.10) and Sobolev inequality to inequalities (2.33) and (2.34), we get
\[
\left| \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} \partial_x^{s-1} v^c (\partial_{xy} w^c - a^c \partial_x w^c) g_s^c \, dx \, dy \right| \leq C_{\gamma, \delta} (1 + \| \partial_x^s U \|_{L^\infty(T)}^2 + Q(t))(1 + \| w^c \|_{H^{s, \gamma}}^3).
\] (2.35)

On the other hand, by virtue of the Hölder inequality and estimate (B.13), we get for \( j = 2, \ldots, s - 1 \),
\[
\left| \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} \partial_x^{s-j} v^c (\partial_x \partial_y w^c - a^c \partial_x^j w^c) g_s^c \, dx \, dy \right| \\
\leq (1 + y)^{\gamma+1} \| \partial_x^{s-j} \partial_y w^c \|_{L^2} + (1 + y) a^c \| \partial x w^c \|_{L^2} \| (1 + y)^\gamma \partial_x w^c \|_{L^2} \\
\times \| \partial_x^s U \|_{L^\infty(T)} \| \partial^\gamma g_s^c \|_{L^2} \\
\leq C_{\gamma, \sigma, \delta} (1 + \| \partial_x^s U \|_{L^\infty(T)}^2 + Q(t)) \| w^c \|_{H^{s, \gamma}}^3.
\]

This and the inequality (2.35) imply
\[
|K_s| \leq C_{s, \gamma, \sigma, \delta} (1 + \| \partial_x^s U \|_{L^\infty(T)}^2 + Q(t))(1 + \| w^c \|_{H^{s, \gamma}}^3).
\]

Substituting the estimates of \( K_1 \) through \( K_8 \) into equality (2.24), we complete the proof of lemma.

Based on the estimates obtained in Lemmas 2.2 and 2.3, we have the estimate:
\[
\frac{d}{dt} \| w^c \|_{H^{s, \gamma}}^2 + \varepsilon^2 \sum_{\{\alpha \leq s\}} \| (1 + y)^{\gamma+\alpha_2} \partial_x D^\alpha w^c \|_{L^2}^2 + \varepsilon^2 \| (1 + y)^\gamma \partial_x g^c_s \|_{L^2}^2 \\
+ \sum_{\{\alpha \leq s\}} \| (1 + y)^{\gamma+\alpha_2} \partial_x \partial_y D^\alpha w^c \|_{L^2}^2 + \| (1 + y)^\gamma \partial_y g^c_s \|_{L^2}^2
\] (2.36)

\[
\leq C_s \| \partial_x^{s+1} p^c \|_{L^2(T)}^4 + \sum_{k=0}^{s/2} \| \partial_x^k \partial_y p^c \|_{L^{s-2k}(T)}^4 + C_{s, \gamma, \sigma, \delta} \| \partial_x^{s+1} U \|_{L^\infty(T)}^4 \\
+ C_{s, \gamma, \sigma, \delta} \{ 1 + Q^4(t) + \| w^c \|_{H^{s, \gamma}}^{2s+4} \}.\]

Using the regularized Bernoulli’s law (1.9), we get
\[
\sum_{k=0}^{s/2} \| \partial_x^k \partial_y p^c \|_{H^{s-2k}(T)} \leq C_s \{ 1 + \sum_{k=0}^{s/2+1} \| \partial_x^k U \|_{H^{s-2k+2}(T)}^2 \}^2.
\] (2.37)
Using the Sobolev inequality for dimension one, it follows
\[
\|\partial_{x}^{s+1}u\|_{L^{\infty}(T)} \leq C\|\partial_{x}^{s+1}\|_{H^{1}(T)}.
\]
This and the inequality (2.37) imply
\[
\|\partial_{x}^{s+1}k^4\|_{L^{2}(T)} + \|\partial_{x}^{s+1}U\|_{L^{\infty}(T)} + \sum_{k=0}^{s/2} \|\partial_{t}^{k}\partial_{x}p^\epsilon\|_{H^{s-2k}(T)}^{2} \leq C_{s}\{1 + \sum_{k=0}^{s/2+1} \|\partial_{t}^{k}U\|_{H^{s-2k+2}(T)}^{2}\}^{4} \leq C_{s}(1 + Mu)^{4}.
\]
(2.38)

Substituting estimate (2.38) into (2.36), and integrating the resulting inequality over \([0, t]\), it follows
\[
\sup_{0 \leq \tau \leq t} \|w^{\epsilon}(\tau)\|_{H_{\gamma}^{s}}^{2} \leq \|w_{0}\|_{H_{\gamma}^{s}}^{2} + C_{s,\gamma,\sigma,\delta}(1 + Mu)^{4}t + C_{s,\gamma,\sigma,\delta}\int_{0}^{t} \{1 + Q^{4}(\tau) + \|w^{\epsilon}(\tau)\|_{H_{\gamma}^{s}}^{2}\}d\tau,
\]

or equivalently,
\[
\sup_{0 \leq \tau \leq t} \|w^{\epsilon}(\tau)\|_{H_{\gamma}^{s}}^{2} \leq \|w_{0}\|_{H_{\gamma}^{s}}^{2} + C_{s,\gamma,\sigma,\delta}(1 + Mu)^{4}t + C_{s,\gamma,\sigma,\delta}\Omega_{g}(t)^{4}t,
\]
(2.39)

for all \(s \geq 4\) and \(\gamma \geq 1\).

2.2. Weighted \(L^{\infty}\) Estimates for Lower Order Terms

In this subsection, we will establish the estimate for the quantity \(Q(t)\) to close the estimate. Since the weight index \(\sigma > \gamma + \frac{1}{2}\), we cannot close this \(L^{\infty}\) estimate by the Sobolev inequality and weighted energy estimates directly. Similar to [10], we apply the maximum principle of heat equation to control the quantity \(Q(t)\) by its initial data, boundary condition and quantity \(\Omega_{g}(t)\). Note that the boundary condition without weight can be controlled by \(\Omega_{g}(t)\) owning to \(\gamma \geq 1\).

**Lemma 2.4.** Under the hypotheses of Theorem [2.4] we have the following estimates:

\[
\sup_{0 \leq \tau \leq t} Q(\tau) \leq e^{C_{s,\gamma,\sigma,\delta}(1 + Mu + \Omega_{g}(t))t}\{\|w_{0}\|_{H_{\gamma}^{s}}^{2} + C_{s,\gamma,\sigma,\delta}(1 + Mu)^{4}t + C_{s,\gamma,\sigma,\delta}\Omega_{g}(t)^{4}t\},
\]
(2.40)

and
\[
(1 + y)^{\sigma}w^{\epsilon}(t, x, y) \geq (1 + y)^{\sigma}w_{0}(x, y) - Ct(1 + Mu + \Omega(t)).
\]
(2.41)

**Proof.** Let us define
\[
I(t) := \sum_{1 \leq |\alpha| \leq 2} \|\partial_{t}^{\alpha}D_{x}^{\alpha}w^{\epsilon}(t)\|_{L^{\infty}(\Omega)}^{2},
\]
similar to [10], we may check that the quantity \(I(t)\) satisfies:
\[
\{\partial_{t} + u^{\epsilon}\partial_{x} + v^{\epsilon}\partial_{y} - \epsilon^{2}\partial_{x}^{2} - \partial_{y}^{2}\}I(t) \leq C_{s,\gamma,\sigma,\delta}\{1 + \|\partial_{x}^{2}U\|_{L^{\infty}(\Omega)}^{2} + \|(1 + y)^{\sigma+1}\partial_{y}w^{\epsilon}\|_{L^{\infty}(\Omega)}^{2} + \|w^{\epsilon}\|_{H_{\gamma}^{s}}^{2}\}I(t).
\]
This and the maximum principle in Lemma [A.4] will give directly for all \(t \in [0, T^{\epsilon}]\)
\[
\|I(t)\|_{L^{\infty}(\Omega)} \leq \max\{e^{C_{s,\gamma,\sigma,\delta}(1 + Mu + \Omega_{g}(t))t}\|I(0)\|_{L^{\infty}(\Omega)}, \max_{\tau \in [0, t]} \{e^{C_{s,\gamma,\sigma,\delta}(1 + Mu + \Omega_{g}(t))(t-\tau)}\|I(\tau)\|_{y=0}\|L^{\infty}(T)\}\},
\]
or equivalently
\[
\sup_{0 \leq \tau \leq t} Q(\tau) \leq e^{C_{s,\gamma,\sigma,\delta}(1 + Mu + \Omega_{g}(t))t}(\|I(0)\|_{L^{\infty}(\Omega)} + \|I(t)\|_{y=0}\|L^{\infty}(T)\}).
\]
(2.42)
By virtue of \( s \geq 4 \), we apply the Sobolev inequality \((A.3)\) to get
\[
\|I(t)\|_{L^\infty(\mathbb{T})} \leq C\|w^\varepsilon\|^2_{H^s_y\gamma}. 
\]
This and the inequality \((2.42)\) yield directly
\[
\sup_{0 \leq \tau \leq t} Q(\tau) \leq e^{C_{s,\gamma,\sigma,\delta}(1+M_U+\Omega_g(t))t}(Q(0) + \|w^\varepsilon\|^2_{H^s_y\gamma}). 
\]
Substituting the estimate \((2.39)\) into inequality \((2.43)\), we get the inequality \((2.40)\).

Finally, using the first equation of \((2.1)\), it follows
\[
\|(1 + y)^\sigma \partial_t w^\varepsilon\|_{L^\infty} \leq e^2\|(1 + y)^\sigma \partial_x^2 w^\varepsilon\|_{L^\infty} + \|(1 + y)^\sigma \partial_y^2 w^\varepsilon\|_{L^\infty} + \|(1 + y)^\sigma \partial_x w^\varepsilon\|_{L^\infty}\|\varepsilon\|_{L^\infty}
\]
\[
\leq C(1 + \|\partial_x U\|^2_{L^\infty(\mathbb{T})} + Q(t) + \|w^\varepsilon\|^2_{H^s_y\gamma}). 
\]
Due to the basic fact
\[
w^\varepsilon(t, x, y) - w_0(x, y) = \int_0^t \partial_\tau w^\varepsilon(\tau, x, y)d\tau,
\]
we get after using \((2.44)\)
\[
(1 + y)^\sigma w^\varepsilon(t, x, y) \geq (1 + y)^\sigma w_0(x, y) - \int_0^t \|(1 + y)^\sigma \partial_\tau w^\varepsilon(\tau)\|_{L^\infty} d\tau
\]
\[
\geq (1 + y)^\sigma w_0(x, y) - Ct(1 + \sup_{0 \leq \tau \leq t} \|\partial_\tau U(\tau)\|^2_{L^\infty(\mathbb{T})} + \sup_{0 \leq \tau \leq t} \{\|w^\varepsilon(\tau)\|^2_{H^s_y\gamma} + Q(\tau)\})
\]
\[
\geq (1 + y)^\sigma w_0(x, y) - Ct(1 + M_U + \Omega(t)). 
\]
Therefore, we complete the proof of this lemma. \(\square\)

From the estimates \((2.39), (2.40)\) and \((2.41)\), we have the estimates
\[
\Omega_g(t) \leq 2e^{C_{s,\gamma,\sigma,\delta}(1+M_U+\Omega_g(t))t}\{\|w_0\|^2_{B^{s,\gamma,\sigma}_0} + C_{s,\gamma,\sigma,\delta}(1+M_U)^4t + C_{s,\gamma,\sigma,\delta}\Omega_g(t)^s\}, 
\]
and
\[
(1 + y)^\sigma w^\varepsilon(t, x, y) \geq (1 + y)^\sigma w_0(x, y) - Ct(1 + M_U + \Omega(t)). 
\]
The advantage of estimates \((2.45)\) and \((2.46)\) is that the constants \(C\) and \(C_{s,\gamma,\sigma,\delta}\) are independent of the artificial viscosity \(\varepsilon\).

2.3. Proof of Theorem \((2.1)\)

Based on the estimates obtained so far, we can complete the proof of Theorem \((2.1)\) in this subsection. First of all, for any \(\varepsilon > 0\), we can apply standard energy method to gain the regularity propagates from the initial data (see estimates \((C.1)\) and \((C.2)\) in Lemma \((C.1)\), that is to say on \([0, T^\varepsilon]\), we have
\[
\Omega(T) = \sup_{0 \leq t \leq T} \|w^\varepsilon(t)\|^2_{B^{s,\gamma,\sigma}_0} < +\infty. 
\]
Moreover, we can also get from the initial data that \((2.6)\) is valid on \([0, T^\varepsilon]\) (possibly by taking \(T^\varepsilon\) smaller). An important remark is that if \(\Omega(T_{ax}) < +\infty\), the solution can be continued on \([0, T_{bx}], T_{bx} > T_{ax}\) with \(\Omega(T_{bx}) < +\infty\). This and the estimates \((2.4) - (2.5)\) can guarantee that the solution can be continued on an interval of time independent of \(\varepsilon\). Thus, it suffices to verify the estimates \((2.1)\) and \((2.5)\).
For two constants $R$ and $\delta$, which will be defined later, we define

$$T^*_\varepsilon := \sup \{ T \in [0,1] \mid \Omega(t) \leq R, \quad (1+y)^\sigma w^\varepsilon(t,x,y) \geq \delta, \quad \forall (t,x,y) \in [0,T] \times \mathbb{T} \times \mathbb{R}^+ \}.$$  \hspace{1cm} (2.47)

Recall the relations (see (5.5) and (5.6))

$$\Omega(t) \leq C_{\gamma,\sigma,\delta} \| \partial_x^s U \|_{L^\infty(T)}^4 + \Omega(t)^2,$$

and

$$\Omega(t) \leq C_{\gamma,\sigma,\delta}(1 + \Omega(t)^2),$$

and hence, we get for all $t \leq T^*$, after using the inequality (2.45),

$$\Omega(t) \leq C_{s,\gamma,\sigma}(1 + \Omega(t)^4 + M_U^4 + C_{s,\gamma,\sigma,\delta}(1 + M_U)^8) t^2 + C_{s,\gamma,\sigma,\delta}(1 + \Omega(t)^2)^2 t^2 \times e^{C_{s,\gamma,\sigma,\delta}(1 + M_U + \Omega(t)^2) t}.$$  \hspace{1cm} (A.2)

Then, we may conclude for $T \leq T^* \varepsilon$

$$\Omega(T) \leq C_{s,\gamma,\sigma}(1 + \Omega(t)^4 + M_U^4) + C_{s,\gamma,\sigma,\delta}(1 + M_U)^8 T + C_{s,\gamma,\sigma,\delta}(1 + R^2)^2 T \times e^{C_{s,\gamma,\sigma,\delta}(1 + M_U + R^2) T}.$$  \hspace{1cm} (A.1)

Choose constants $R = 8C_{s,\gamma,\sigma}(1 + \Omega(t)^4 + M_U^4)$ and $\delta = \frac{\delta^*}{2}$, we get

$$\Omega(T_1) \leq 4C_{s,\gamma,\sigma}(1 + \Omega(t)^4 + M_U^4) = \frac{R}{2},$$

where $T_1 := \min \left\{ \frac{\ln 2}{C_{s,\gamma,\sigma,\delta}(1 + M_U + R^2)}; \frac{1 + \Omega(t)^4 + M_U^4}{C_{s,\gamma,\sigma,\delta}(1 + M_U)^8}; \frac{1 + \Omega(t)^4 + M_U^4}{C_{s,\gamma,\sigma,\delta}(1 + R^2)^2} \right\}$. It follows from (2.46)

$$\min_{T \times \mathbb{R}^+} \Omega(t) \geq \delta_0 = 2\delta, \quad t \in [0,T_2],$$

where $T_2 := \min \{ T_1, \frac{\delta_0}{C_U} \}$. Obviously, we conclude that there exists a time $T_2 > 0$ depending only on $s, \gamma, \sigma, \delta_0, M_U$ and the initial data $\| w_0 \|_{\mathbb{H}^{s,\gamma,\sigma}}$ (hence independent of $\varepsilon$) such that for all $T \leq \min \{ T_2, T^* \varepsilon \}$, the estimates (2.4) and (2.5) hold on. Of course, it holds that $T_2 \leq T^* \varepsilon$. Otherwise, our criterion about the continuation of the solution would contradict the definition of $T^* \varepsilon$ in (2.47). Then, taking $T_\alpha = T_2$, we obtain the estimate (2.5) and close the a priori assumption (2.6). Therefore, we complete the proof of Theorem 2.1.

### A. Calculus Inequalities

In this appendix, we will introduce some basic inequality that be used frequently in this paper. For the proof in detail, the interested readers can refer to [1].

**Lemma A.1 (Hardy Type Inequalities).** Let function $f : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}$.

(i) if $\lambda > -\frac{1}{2}$ and $\lim_{y \to +\infty} f(x,y) = 0$, then

$$\|(1+y)^\lambda f\|_{L^2(T \times \mathbb{T} \times \mathbb{R}^+)} \leq \frac{2}{2\lambda + 1} \|(1+y)^{\lambda+1} \partial_y f\|_{L^2(T \times \mathbb{T} \times \mathbb{R}^+)}.$$  \hspace{1cm} (A.1)

(ii) if $\lambda < -\frac{1}{2}$, then

$$\|(1+y)^\lambda f\|_{L^2(T \times \mathbb{T} \times \mathbb{R}^+)} \leq \sqrt{-\frac{1}{2\lambda + 1}} \| f \|_{L^2(T \times \mathbb{T} \times \mathbb{R}^+)} - \frac{2}{2\lambda + 1} \|(1+y)^{\lambda+1} \partial_y f\|_{L^2(T \times \mathbb{T} \times \mathbb{R}^+)}.$$  \hspace{1cm} (A.2)
Lemma A.2 (Sobolev-Type Inequality). Let the proper function \( f : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R} \). Then there exists a universal constant \( C > 0 \) such that

\[
\| f \|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} \leq C \left\{ \| f \|_{L^2(\mathbb{T} \times \mathbb{R}^+)} + \| \partial_x f \|_{L^2(\mathbb{T} \times \mathbb{R}^+)} + \| \partial_y f \|_{L^2(\mathbb{T} \times \mathbb{R}^+)} \right\}. \tag{A.3}
\]

Next, we state the Morse type inequality that will be used frequently when we deal with the convective term. For the sake of brevity, we omit the proof for inequality (A.4) since it can be guaranteed by the Sobolev inequality (A.3).

Lemma A.3 (Morse-Type Inequality). Let \( f \) and \( g \) be proper functions, \( \gamma \in \mathbb{R} \) and an integer \( s \geq 3 \), we have for all \( |\alpha + \tilde{\alpha}| \leq s \)

\[
\|(1 + y)^{\gamma + \alpha + \tilde{\alpha}} (D^\alpha f \cdot D^{\tilde{\alpha}} g) (t, \cdot)\|_{L^2} \leq C \|f(t)\|_{H^{\gamma, \gamma_1}} \|g(t)\|_{H^{\gamma, \gamma_2}}, \tag{A.4}
\]

where \( \gamma_1, \gamma_2 \in \mathbb{R} \) with \( \gamma_1 + \gamma_2 = \gamma \), and \( C > 0 \) is a universal constant.

Finally, let us recall the maximum principle for bounded solutions to parabolic equations (cf. [10]).

Lemma A.4 (Maximum Principle for Parabolic Equations). Let \( \varepsilon \geq 0 \). If \( H \in C([0, T]; C^2(\mathbb{T} \times \mathbb{R}^+)) \cap C^1([0, T]; C^0(\mathbb{T} \times \mathbb{R}^+)) \) is a bounded function which satisfies the differential inequality:

\[
\{ \partial_t + b_1 \partial_x + b_2 \partial_y - \varepsilon^2 \partial_{xx} - \partial_{yy} \} H \leq fH \quad \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+,
\]

where the coefficients \( b_1, b_2 \) and \( f \) are continuous and satisfy

\[
\left\| \frac{b_2}{1 + y} \right\|_{L^\infty([0, T] \times \mathbb{T} \times \mathbb{R}^+)} < +\infty \quad \text{and} \quad \|f\|_{L^\infty([0, T] \times \mathbb{T} \times \mathbb{R}^+)} \leq \lambda,
\]

then for any \( t \in [0, T] \),

\[
\sup_{T \times \mathbb{R}^+} H(t) \leq \max\{e^{\lambda t}\|H(0)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)}, \max_{\tau \in [0, t]} \{e^{\lambda(t-\tau)}\|H(\tau)\|_{|y=0}\}_{L^\infty(\mathbb{T})}\}.
\]

B Almost Equivalence of Weighted Norms

In this section, we will state some estimates that will be used in section 2. First of all, we derive the relation between \( g^\varepsilon_s \) and \( \partial_x^\varepsilon w^\varepsilon \) as follows.

Lemma B.1. Let \( s \geq 4 \) be an even integer, \( \gamma \geq 1, \sigma \geq \gamma + \frac{1}{2} \), and \( \varepsilon \in (0, 1) \), the smooth solution \( (u^\varepsilon, v^\varepsilon, w^\varepsilon) \), defined on \([0, T^\varepsilon]\), to the regularized Prandtl equations (2.1) - (2.3). There exists a small constant \( \delta \in (0, 1) \) such that \( (1 + y)^{\sigma} w^\varepsilon \geq \delta, \forall (t, x, y) \in [0, T^\varepsilon] \times \mathbb{T} \times \mathbb{R}^+ \), then it holds on

\[
\|(1 + y)^\gamma g^\varepsilon_s \|_{L^2} \leq C_{\gamma, \delta} (1 + \|(1 + y)^{\sigma + 1} \partial_y w^\varepsilon \|_{L^\infty}) \|(1 + y)^\gamma \partial_x^\varepsilon w^\varepsilon \|_{L^2}, \tag{B.1}
\]

and

\[
\|(1 + y)^\gamma \partial_x^\varepsilon w^\varepsilon \|_{L^2} \leq C_{\gamma, \sigma, \delta} (1 + \|\partial_x^\varepsilon U\|_{L^\infty(T)}) + \|(1 + y)^{\sigma + 1} \partial_y w^\varepsilon \|_{L^\infty}) \|w^\varepsilon\|_{H^{\gamma, \gamma}_x}, \tag{B.2}
\]

where \( g^\varepsilon_s := \partial_x^\varepsilon w^\varepsilon - \frac{\partial_{yy} w^\varepsilon}{w^\varepsilon} \partial_x^\varepsilon (w^\varepsilon - U) \).
Proof. Using the definition $g_\varepsilon^y = \partial_2^s w^\varepsilon - \frac{\partial_y w^\varepsilon}{w^\varepsilon} \partial_2^s (u^\varepsilon - U)$ and Hardy inequality \eqref{A.2}, it follows

$$
\|(1 + y)\gamma g_\varepsilon^y\|_{L^2} \leq \|(1 + y)\gamma^\varepsilon \partial_2^s w^\varepsilon\|_{L^2} + \|(1 + y)\gamma^\varepsilon \partial_y w^\varepsilon\|_{L^\infty} \|(1 + y)^{\varepsilon - 1} \partial_2^s (u^\varepsilon - U)\|_{L^2}
\leq \|(1 + y)\gamma^\varepsilon \partial_2^s w^\varepsilon\|_{L^2} + \left( \begin{array}{c}
(1 + y)^{\varepsilon + 1} \partial_y w^\varepsilon \\
(1 + y)^{2\varepsilon}
\end{array} \right) \|(1 + y)\gamma^\varepsilon \partial_2^s (u^\varepsilon - U)\|_{L^2}
\leq \|(1 + y)\gamma^\varepsilon \partial_2^s w^\varepsilon\|_{L^2} + C_{\gamma,\varepsilon} \|(1 + y)^{\varepsilon + 1} \partial_y w^\varepsilon\|_{L^\infty} \|(1 + y)\gamma^\varepsilon \partial_2^s w^\varepsilon\|_{L^2},
$$

where we have used $(1 + y)^{\varepsilon} w^\varepsilon \geq \delta$, and hence, we get

$$
\|(1 + y)\gamma^\varepsilon g_\varepsilon^y\|_{L^2} \leq C_{\gamma,\varepsilon} \|(1 + y)^{\varepsilon + 1} \partial_y w^\varepsilon\|_{L^\infty} \|(1 + y)\gamma^\varepsilon \partial_2^s w^\varepsilon\|_{L^2}.
$$

This implies inequality \eqref{B.1}. On the other hand, we get from the definition of $g_\varepsilon^y$ that

$$
\|(1 + y)\gamma^\varepsilon \partial_2^s w^\varepsilon\|_{L^2} \leq \|(1 + y)\gamma^\varepsilon g_\varepsilon^y\|_{L^2} + \|(1 + y)\gamma^\varepsilon \partial_y w^\varepsilon\frac{\partial_2^s (u^\varepsilon - U)}{w^\varepsilon}\|_{L^2}
$$

By routine checking, it follows the relation $g_\varepsilon^y = w^\varepsilon \partial_y \left( \frac{\partial_2^s (u^\varepsilon - U)}{w^\varepsilon} \right)$, and hence using $u^\varepsilon|_{y=0} = 0$, we get

$$
\partial_2^s (u^\varepsilon - U) \frac{\partial_2^s (u^\varepsilon - U)}{w^\varepsilon} = - \frac{\partial_2^s (u^\varepsilon - U)}{w^\varepsilon} + \int_0^y g_\varepsilon^y \frac{d\xi}{w^\varepsilon}.
$$

This and the condition $(1 + y)^{\varepsilon} w^\varepsilon \geq \delta$ yield directly

$$
\|(1 + y)\gamma^\varepsilon \partial_y w^\varepsilon \frac{\partial_2^s (u^\varepsilon - U)}{w^\varepsilon}\|_{L^2}
\leq \delta^{-1} \|\partial_2^s U\|_{L^\infty(T)} \|(1 + y)\gamma^\varepsilon \partial_y w^\varepsilon\|_{L^2} + \|(1 + y)^{\varepsilon + 1} \partial_y w^\varepsilon\|_{L^\infty} \|(1 + y)^{\varepsilon - 1} \partial_2^s (u^\varepsilon - U)\|_{L^2}
\leq C\|\partial_2^s U\|_{L^\infty(T)} \|(1 + y)\gamma^\varepsilon \partial_y w^\varepsilon\|_{L^2} + C_{\gamma,\varepsilon} \|(1 + y)^{\varepsilon + 1} \partial_y w^\varepsilon\|_{L^\infty} \|(1 + y)\gamma^\varepsilon g_\varepsilon^y\|_{L^2}.
$$

Plugging inequality \eqref{B.4} into \eqref{B.3}, we obtain the inequality \eqref{B.2}. □

Based on the inequalities \eqref{B.1}, \eqref{B.2}, and the definitions of $H^{s,\gamma}$ and $H^{\gamma}\varepsilon$, we can establish the following estimates, which are important relation for us to obtain the well-posedness for the Prandtl equations in Sobolev space.

**Lemma B.2.** Let $s \geq 4$ be an even integer, $\gamma \geq 1$, $\sigma \geq \gamma + \frac{1}{2}$, and $\varepsilon \in (0,1)$, the smooth solution $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$, defined on $[0,T^\varepsilon]$, to the regularized Prandtl equations \eqref{2.1}-\eqref{2.3}. There exists a small constant $\delta \in (0,1)$ such that $(1 + y)^{\varepsilon} w^\varepsilon \geq \delta$, $\forall (t, x, y) \in [0,T^\varepsilon] \times T \times \mathbb{R}^+$, then it holds on \eqref{Omega},

$$
\Omega(t) \leq C_{\gamma,\sigma,\delta} \|\partial_2^s U\|^2_{L^\infty(T)} + C_{\gamma,\sigma,\delta} (1 + \Omega_g(t)^2),
$$

and

$$
\Omega_g(t) \leq C_{\gamma,\sigma,\delta} (1 + \Omega(t)^2).
$$

**Proof.** Using the definition of $\|\cdot\|_{H^{s,\gamma}\varepsilon}$ and estimate \eqref{B.2}, it follows

$$
\|w^\varepsilon\|_{H^{s,\gamma}\varepsilon} \leq C_{\gamma,\sigma,\delta} \left( 1 + \|\partial_2^s U\|_{L^\infty(T)} + \|(1 + y)^{\varepsilon + 1} \partial_y w^\varepsilon\|_{L^\infty} \right) \|w^\varepsilon\|_{H^{s,\gamma}\varepsilon},
$$

which along with Cauchy inequality implies directly

$$
\|w^\varepsilon(t)\|^2_{H^{s,\gamma}\varepsilon} \leq C_{\gamma,\sigma,\delta} \left( 1 + \|\partial_2^s U\|_{L^\infty(T)}^2 + \|(1 + y)^{\varepsilon + 1} \partial_y w^\varepsilon\|_{L^\infty}^2 \right) \|w^\varepsilon\|^2_{H^{s,\gamma}\varepsilon}
\leq C_{\gamma,\sigma,\delta} \|\partial_2^s U\|^2_{L^\infty(T)} + C_{\gamma,\sigma,\delta} (1 + \|w^\varepsilon(t)\|^4_{H^{s,\gamma}\varepsilon}).
$$


This yields inequality (B.5). Similarly, it follows from inequality (B.1) that
\[ \|w^\varepsilon\|_{H^\gamma_0} \leq C_{\gamma,\sigma,\delta}(1 + \|_1 + (1 + y)^{\sigma + 1}\|_{\gamma,\sigma,\delta}) \|w^\varepsilon\|_{H^{\gamma,\sigma,\delta}}, \] (B.8)
and hence, we obtain
\[ \|w^\varepsilon(t)\|_{H^{\gamma,\sigma,\delta}}^2 \leq C_{\gamma,\sigma,\delta}(1 + \|w^\varepsilon(t)\|_{H^{\gamma,\sigma,\delta}}). \] Then we prove the inequality (B.6).

Next, we establish some estimates for the quantity \((u^\varepsilon, v^\varepsilon, g^\varepsilon_k)\) in weighted \(L^2\)-norm.

**Lemma B.3.** Let \(s \geq 4\) be an even integer, \(\gamma \geq 1, \sigma \geq \gamma + \frac{1}{2},\) and \(\varepsilon \in (0, 1),\) the smooth solution \((u^\varepsilon, v^\varepsilon, w^\varepsilon),\) defined on \([0, T^\varepsilon]\), to the regularized Prandtl equations (2.1) - (2.3). There exists a small constant \(\delta \in (0, 1)\) such that \((1 + y)^\sigma w^\varepsilon \geq \delta,\) \(\forall (t, x, y) \in [0, T^\varepsilon] \times T \times \mathbb{R}^+\), then it holds on:

(i) For all \(k = 0, 1, \ldots, s,\)
\[ \|(1 + y)^{\gamma - 1} \partial^k_t(u^\varepsilon - U)\|_{L^2} \leq C_{\gamma,\sigma,\delta}(1 + \|\partial^k_t U\|_{L^\infty(T)} + \|1 + y\|^{\sigma + 1}\|\partial_y w^\varepsilon\|_{L^\infty})\|w^\varepsilon\|_{H^{\gamma,\sigma,\delta}}, \] (B.9)

(ii) For all \(k = 0, 1, \ldots, s - 1,\)
\[ \|\partial^k_t v^\varepsilon + y\partial^k_{xx} \|_{L^2} \leq C_{\gamma,\sigma,\delta}(1 + \|\partial^k_t U\|_{L^\infty(T)} + \|1 + y\|^{\sigma + 1}\|\partial_y w^\varepsilon\|_{L^\infty})\|w^\varepsilon\|_{H^{\gamma,\sigma,\delta}}, \] (B.10)

(iii) For all \(k = 1, 2, \ldots, s,\)
\[ \|(1 + y)^{\gamma} g^\varepsilon_k\|_{L^2} \leq C_{\gamma,\delta}(1 + \|\partial^k_t U\|_{L^\infty(T)} + \|1 + y\|^{\sigma + 1}\|\partial_y w^\varepsilon\|_{L^\infty})\|w^\varepsilon\|_{H^{\gamma,\sigma,\delta}}. \] (B.11)

**Proof.** (i) It follows from Hardy inequality (A.1) that \(\|(1 + y)^{\gamma - 1}\partial^k_x(u^\varepsilon - U)\|_{L^2} \leq C_{\gamma}(1 + y)^{\gamma}\|\partial^k_x w^\varepsilon\|_{L^2},\) and hence inequality (B.9) is a direct consequence of the inequality (B.7).

(ii) Due to the Hardy inequality (A.2) and divergence-free condition, it follows
\[ \|\partial^k_t v^\varepsilon + y\partial^k_{xx} \|_{L^2} \leq C\|\partial^k_t (\partial_y v^\varepsilon + \partial_x U)\|_{L^2} \leq C\|\partial^k_{xx} (u^\varepsilon - U)\|_{L^2}, \]
and hence, we get inequality (B.10) after using inequality (B.9).

(iii) For the case \(s = 1, 2, \ldots, s - 1,\) we get after using \((1 + y)^{\sigma} w^\varepsilon \geq \delta,\)
\[ \|(1 + y)^{\gamma} g^\varepsilon_k\|_{L^2} \leq \|(1 + y)^{\gamma}\|_{L^2} + \|\partial^k_t w^\varepsilon\|_{L^2} + \|\partial^k_y w^\varepsilon\|_{L^\infty}\|(1 + y)^{\gamma - 1}\|_{L^2} = \leq \|(1 + y)^{\gamma}\|_{L^2} + \|\partial^k_t w^\varepsilon\|_{L^2} \leq C_{\gamma,\delta}(1 + \|\partial^k_t U\|_{L^\infty(T)} + \|1 + y\|^{\sigma + 1}\|\partial_y w^\varepsilon\|_{L^\infty})\|w^\varepsilon\|_{H^{\gamma,\sigma,\delta}}. \]

This and the Hardy inequality yield directly
\[ \|(1 + y)^{\gamma} g^\varepsilon_k\|_{L^2} \leq C_{\gamma,\delta}(1 + \|\partial^k_t U\|_{L^\infty(T)} + \|1 + y\|^{\sigma + 1}\|\partial_y w^\varepsilon\|_{L^\infty})\|w^\varepsilon\|_{H^{\gamma,\sigma,\delta}}. \]

Therefore, we complete the proof of this lemma.

Finally, we establish some estimates for the quantity \((u^\varepsilon, v^\varepsilon)\) in \(L^\infty\)-norm.

**Lemma B.4.** Let \(s \geq 4\) be an even integer, \(\gamma \geq 1, \sigma \geq \gamma + \frac{1}{2},\) and \(\varepsilon \in (0, 1),\) the smooth solution \((u^\varepsilon, v^\varepsilon, w^\varepsilon),\) defined on \([0, T^\varepsilon]\), to the regularized Prandtl equations (2.1) - (2.3). There exists a small constant \(\delta \in (0, 1)\) such that \((1 + y)^{\sigma} w^\varepsilon \geq \delta,\) \(\forall (t, x, y) \in [0, T^\varepsilon] \times T \times \mathbb{R}^+\), then it holds on:

(i) For all \(k = 0, 1, \ldots, s - 1,\)
\[ \|\partial^k_t w^\varepsilon\|_{L^\infty} \leq C_{\gamma,\sigma,\delta}(1 + \|\partial^k_t U\|_{L^\infty(T)} + \|1 + y\|^{\sigma + 1}\|\partial_y w^\varepsilon\|_{L^\infty}(1 + \|w^\varepsilon\|_{H^{\gamma,\sigma,\delta}}) \] (B.12)

(ii) For all \(k = 0, 1, \ldots, s - 2,\)
\[ \|\partial^k_t v^\varepsilon + y\partial^k_{xx} \|_{L^2} \leq C_{\gamma,\sigma,\delta}(1 + \|\partial^k_t U\|_{L^\infty(T)} + \|1 + y\|^{\sigma + 1}\|\partial_y w^\varepsilon\|_{L^\infty})\|w^\varepsilon\|_{H^{\gamma,\sigma,\delta}}. \] (B.13)
Due to the fact this lemma. is consequence of Sobolev inequality (A.3), estimates (B.9) and (B.10). Thus, we complete the proof of

\[ \text{Let} \]

In this section, we state the local in time well-posedness theory for the regularized Prandtl equations

C Existence for the Regularized Prandtl Equations

By virtue of the Sobolev inequality (A.3) and estimate (B.9), it follows

\[ \| \partial_x^k u^\varepsilon \|_{L^2} \leq C(\| \partial_x^k (u^\varepsilon - U) \|_{L^2} + \| \partial_x^{k+1} (u^\varepsilon - U) \|_{L^2} + \| \partial_x^k \partial_y u^\varepsilon \|_{L^2} + \| \partial_x^k U \|_{L^\infty(T)}) \]

Due to the fact \( U = \int_0^{+\infty} w^\varepsilon dy \), we get

\[ \| U \|_{L^2(T)} \leq C_\gamma \| (1 + y)^\gamma w^\varepsilon \|_{L^2}, \]

and hence, we get after using the Sobolev and Wirtinger inequalities for \( k = 0, 1, \ldots, s - 1 \),

\[ \| \partial_x^k U \|_{L^\infty(T)} \leq C(\| \partial_x^k U \|_{L^2(T)} + \| \partial_x^{k+1} U \|_{L^2(T)}) \leq C_{\gamma}(1 + y)^\gamma w^\varepsilon \|_{L^2} + \| \partial_x^k U \|_{L^\infty(T)}. \]

Submitting inequality (B.15) into (B.14), we obtain the inequality (B.12). Finally, the inequality (B.13) is consequence of Sobolev inequality (A.3), estimates (B.9) and (B.10). Thus, we complete the proof of this lemma.

C Existence for the Regularized Prandtl Equations

In this section, we state the local in time well-posedness theory for the regularized Prandtl equations (2.1)-(2.3). More precisely, we have the following results:

Lemma C.1. Let \( s \geq 4 \) be an even integer, \( \gamma \geq 1, \sigma > \gamma + \frac{1}{2}, \delta_0 \in (0, \frac{1}{2}) \) and \( \varepsilon \in (0, 1) \). If the vorticity \( w_0 \in \dot{H}^{s,\gamma}_{\sigma,2\delta_0} \), \( U \) and \( p^\varepsilon \) are given and satisfy the regularized Bernoulli’s law (1.9) and the regularity assumption (1.7), then there exist a time

\[ T^\varepsilon := T(s, \gamma, \sigma, \delta_0, \varepsilon, \| w_0 \|_{B^{s,\gamma}}, M_U) > 0, \]

and a solution \( w^\varepsilon \), to the regularized vorticity system (2.1)-(2.3), satisfying the estimates:

\[ \Omega(t) := \sup_{0 \leq \tau \leq t} \| w^\varepsilon(\tau) \|^2_{B^{s,\gamma}, \varepsilon} \leq C_\varepsilon(1 + \| w_0 \|^2_{B^{s,\gamma}, \varepsilon}) < +\infty, \]

and

\[ (1 + y)^\sigma w^\varepsilon(t, x, y) \geq c_\varepsilon \delta_0, \]

for all \( (t, x, y) \in [0, T^\varepsilon] \times \mathbb{T} \times \mathbb{R}^+ \), here \( C_\varepsilon, c_\varepsilon \) are positive constants and \( c_\varepsilon \in (0, 1) \).

Proof. We only establish the a priori estimates (C.1) and (C.2), and the well-posedness results of the regularized vorticity system (2.1)-(2.3) can be obtained immediately(cf.[10]).

Step 1: \( H^{s,\gamma} \)-estimates. Differentiating the regularized vorticity equation (2.1) with differential operator \( D^\alpha(\| \alpha \| \leq s) \), we get

\[ \{ \partial_t + u^\varepsilon \partial_x + v^\varepsilon \partial_y - \varepsilon^2 \partial_x^2 - \partial_y^2 \} D^\alpha w^\varepsilon = -[D^\alpha, u^\varepsilon \partial_x] w^\varepsilon - [D^\alpha, v^\varepsilon \partial_y] w^\varepsilon. \]

Multiplying this equation by \( (1 + y)^{2\gamma + 2\alpha} D^\alpha w^\varepsilon \) and integrating over \( \mathbb{T} \times \mathbb{R}^+ \), it follows

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma + 2\alpha} |D^\alpha w^\varepsilon|^2 dx dy + \varepsilon^2 \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma + 2\alpha} |\partial_x D^\alpha w^\varepsilon|^2 dx dy \]

\[ = (\gamma + \alpha_2) \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma + 2\alpha - 1} |D^\alpha w^\varepsilon|^2 dx dy + \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma + 2\alpha} 2 \partial_y |D^\alpha w^\varepsilon| \cdot D^\alpha w^\varepsilon dx dy \]

\[ - \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma + 2\alpha} [D^\alpha, u^\varepsilon \partial_x] w^\varepsilon \cdot D^\alpha w^\varepsilon dx dy - \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma + 2\alpha} [D^\alpha, v^\varepsilon \partial_y] w^\varepsilon \cdot D^\alpha w^\varepsilon dx dy, \]

\[ C(1 + y)^\gamma w^\varepsilon \|_{L^2(T)} \leq C_\gamma \| (1 + y)^\gamma w^\varepsilon \|_{L^2}. \]
where we have used the divergence-free condition.

First of all, we deal with the case $|\alpha| \leq s$ and $\alpha_1 \leq s - 1$ in (C.3). Similar to (2.19), we get

$$
\frac{1}{2} \frac{d}{dt} \left\| (1 + y)^{\gamma+\alpha_2} D^\alpha w^\varepsilon \right\|_{L^2}^2 + \varepsilon^2 \left\| (1 + y)^{\gamma+\alpha_2} \partial_x D^\alpha w^\varepsilon \right\|_{L^2}^2 + \frac{3}{4} \left\| (1 + y)^{\gamma+\alpha_2} \partial_y D^\alpha w^\varepsilon \right\|_{L^2}^2
\leq - \int_T D^\alpha w^\varepsilon \partial_y D^\alpha w^\varepsilon |_{y=0}^\varepsilon dx + C_{s,\gamma} (1 + \| \partial_x^{s+1} U \|_{L^\infty(\mathbb{T})} + \| w^\varepsilon \|_{H^{s,\gamma}}) \| w^\varepsilon \|_{H^{s,\gamma}}^2
$$

(C.4)

Note the boundary term above can be estimated as follows (cf. [10]) for $|\alpha| \leq s - 1$,

$$
\left| \int_T D^\alpha w^\varepsilon \partial_y D^\alpha w^\varepsilon |_{y=0}^\varepsilon dx \right| \leq \frac{1}{12} \left\| (1 + y)^{\gamma+\alpha_2+1} \partial_y^2 D^\alpha w^\varepsilon \right\|_{L^2}^2 + C \| w^\varepsilon \|_{H^{s,\gamma}}^2
$$

(C.5)

and for $|\alpha| = s$,

$$
\left| \int_T D^\alpha w^\varepsilon \partial_y D^\alpha w^\varepsilon |_{y=0}^\varepsilon dx \right| \leq \begin{cases} \frac{1}{12} \left\| (1 + y)^{\gamma+\alpha_2} \partial_y D^\alpha w^\varepsilon \right\|_{L^2}^2 + G(t), & \alpha_2 = 2k, k \in \mathbb{N}; \\ \frac{1}{12} \left\| (1 + y)^{\gamma+\alpha_2+1} \partial_x^{k+1} \partial_y^{\alpha_2+2} w^\varepsilon \right\|_{L^2}^2 + G(t), & \alpha_2 = 2k + 1, k \in \mathbb{N}; \end{cases}
$$

(C.6)

where

$$
G(t) = C_s \sum_{\ell=0}^{s/2} \| \partial_t^{\ell} \partial_x^{s/2} p^\varepsilon \|_{H^{s-2\ell}(\mathbb{T})}^2 + C_{s,\gamma} (1 + \| w^\varepsilon \|_{H^{s,\gamma}})^{s-2} \| w^\varepsilon \|_{H^{s,\gamma}}^2.
$$

Thus, plugging the estimates (C.5) and (C.6) into (C.4), and summing over $\alpha$, we get

$$
\frac{d}{dt} \sum_{|\alpha| \leq s} \left\| (1 + y)^{\gamma+\alpha_2} D^\alpha w^\varepsilon \right\|_{L^2}^2 + \sum_{|\alpha| \leq s} \left\| (1 + y)^{\gamma+\alpha_2} (\varepsilon \partial_x D^\alpha w^\varepsilon, \partial_y D^\alpha w^\varepsilon) \right\|_{L^2}^2
\leq C_{s,\gamma} \| \partial_x^{s+1} U \|_{L^\infty(\mathbb{T})}^2 + C_s \sum_{k=0}^{s/2} \| \partial_t^{\ell} \partial_x^{s/2} p^\varepsilon \|_{H^{s-2\ell}(\mathbb{T})}^2 + C_{s,\gamma} (1 + \| w^\varepsilon \|_{H^{s,\gamma}}).
$$

(C.7)

Next, we deal with the case $|\alpha| = s$ and $\alpha_1 = s$ in (C.3), and hence, it follows

$$
\frac{1}{2} \frac{d}{dt} \int_{T \times \mathbb{R}^+} (1 + y)^{2\gamma} |\partial_x^{s} w^\varepsilon|^2 dx dy + \varepsilon^2 \int_{T \times \mathbb{R}^+} (1 + y)^{2\gamma} |\partial_x^{s+1} w^\varepsilon|^2 dx dy
= (\gamma + \alpha_2) \int_{T \times \mathbb{R}^+} (1 + y)^{2\gamma-1} \partial_x^{s} \partial_y w^\varepsilon \partial_x^{s} w^\varepsilon dx dy + \int_{T \times \mathbb{R}^+} (1 + y)^{2\gamma} \partial_x^{s} \partial_x^{s} w^\varepsilon \partial_x^{s} w^\varepsilon dx dy
- \int_{T \times \mathbb{R}^+} (1 + y)^{2\gamma} [\partial_x^{s}, \partial_x^{s}] w^\varepsilon \partial_x^{s} w^\varepsilon dx dy - \int_{T \times \mathbb{R}^+} (1 + y)^{2\gamma} [\partial_x^{s}, \partial_x^{s}] w^\varepsilon \cdot \partial_x^{s} w^\varepsilon dx dy,
$$

(C.8)

Using the inequality (2.10), it follows

$$
\left\| (\gamma + \alpha_2) \int_{T \times \mathbb{R}^+} (1 + y)^{2\gamma-1} \partial_x^{s} \partial_y w^\varepsilon | \partial_x^{s} w^\varepsilon |^2 dx dy \right\|
\leq \frac{\varepsilon}{1 + y} \left\| \int_{T \times \mathbb{R}^+} (1 + y)^{\gamma} \partial_x^{s} w^\varepsilon \right\|_{L^2}^2
\leq C_{s,\gamma} \left( \| \partial_x U \|_{L^\infty(\mathbb{T})} + \| w^\varepsilon \|_{H^{s,\gamma}} \right) \left( 1 + y \right)^\gamma \| \partial_x^{s} w^\varepsilon \|_{L^2}^2.
$$

(C.9)
Integrating by parts and applying Cauchy inequality, we get
\[
\int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} \partial_y^2 \partial_x^s w^\varepsilon \cdot \partial_x^s w^\varepsilon \, dx \, dy
\]
\[
= \int_{\mathbb{T}} \partial_y \partial_x^s w^\varepsilon \cdot \partial_x^s w^\varepsilon \big|_{y=0} \, dx - \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} |\partial_y \partial_x^s w^\varepsilon|^2 \, dx \, dy
\]
\[
- \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma - 1} \partial_y \partial_x^s w^\varepsilon \cdot \partial_x^s w^\varepsilon \, dx \, dy + C \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma - 2} |\partial_x^s w^\varepsilon|^2 \, dx \, dy
\]
\[
\leq - \frac{3}{4} \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} |\partial_y \partial_x^s w^\varepsilon|^2 \, dx \, dy + C \int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma - 2} |\partial_x^s w^\varepsilon|^2 \, dx \, dy
\]
\[
+ \int_{\mathbb{T}} \partial_y \partial_x^s w^\varepsilon \cdot \partial_x^s w^\varepsilon \big|_{y=0} \, dx.
\]
(C.10)

Using the boundary condition \(\partial_y w^\varepsilon|_{y=0} = p^\varepsilon\) and Sobolev inequality, it follows
\[
\int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} \partial_y^2 \partial_x^s w^\varepsilon \cdot \partial_x^s w^\varepsilon \, dx \, dy
\]
\[
\leq \frac{\sqrt{2}}{4} \|\partial_x^{s+1} p^\varepsilon\|_{L^2(\mathbb{T})} \|\partial_y \partial_x^s w^\varepsilon\|^2_{L^2} + (1 + y)^{\gamma - 1} \|\partial_x^s w^\varepsilon\|^2_{L^2}.
\]
(C.11)

By virtue of Hölder inequality, we get
\[
|\int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} [\partial_x^s, u^\varepsilon \partial_x] w^\varepsilon \cdot \partial_x^s w^\varepsilon \, dx \, dy|
\]
\[
\leq \sum_{1 \leq k \leq s} \left( y^{\gamma} \right) \|\partial_x^k (u^\varepsilon - U) \partial_x^{s+1-k} w^\varepsilon\|_{L^2} (1 + y)^{\gamma} \|\partial_x w^\varepsilon\|_{L^2}
\]
\[
+ \sum_{1 \leq k \leq s} (1 + y)^{\gamma} \|\partial_x^k U \partial_x^{s+1-k} w^\varepsilon\|_{L^2} (1 + y)^{\gamma} \|\partial_x w^\varepsilon\|_{L^2}.
\]

Using the Hardy and Morse type inequalities, it follows for \(1 \leq k \leq s\)
\[
(1 + y)^{\gamma} \|\partial_x^k (u^\varepsilon - U) \partial_x^{s+1-k} w^\varepsilon\|_{L^2} \leq \|\partial_x (u^\varepsilon - U)\|_{H^{s-1,0}} \|\partial_x w^\varepsilon\|_{H^{s-1,\gamma}} \leq C \|w^\varepsilon\|^2_{H^{s,\gamma}},
\]
and applying Wirtinger inequality, it follows
\[
(1 + y)^{\gamma} \|\partial_x^k U \partial_x^{s+1-k} w^\varepsilon\|_{L^2} \leq C \|\partial_x^k U\|_{L^\infty(\mathbb{T})} \|\partial_x w^\varepsilon\|_{H^{s,\gamma}}.
\]

Thus, we can conclude the estimate
\[
|\int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} [\partial_x^s, u^\varepsilon \partial_x] w^\varepsilon \cdot \partial_x^s w^\varepsilon \, dx \, dy| \leq C_s \|\partial_x^k U\|_{L^\infty(\mathbb{T})} \|\partial_x w^\varepsilon\|_{H^{s,\gamma}}. \quad \text{(C.12)}
\]

Applying the Hölder inequality, it follows
\[
\int_{\mathbb{T} \times \mathbb{R}^+} (1 + y)^{2\gamma} [\partial_x^s, u^\varepsilon \partial_y] w^\varepsilon \cdot \partial_x^s w^\varepsilon \, dx \, dy
\]
\[
\leq \sum_{1 \leq k \leq s} \left( y^{\gamma} \right) \|\partial_x^k (u^\varepsilon + y \partial_x U) \partial_x^{s-k} \partial_y w^\varepsilon\|_{L^2} (1 + y)^{\gamma} \|\partial_x^s w^\varepsilon\|_{L^2}
\]
\[
+ \sum_{1 \leq k \leq s} (1 + y)^{\gamma} \|\partial_x^{k+1} U\|_{L^\infty(\mathbb{T})} \|\partial_x^{s-k} \partial_y w^\varepsilon\|_{L^2} (1 + y)^{\gamma} \|\partial_x^s w^\varepsilon\|_{L^2}.
\]
(C.13)
For $1 \leq k \leq s - 1$, using the Hardy and Morse type inequalities, it follows

$$
\|{(1 + y)^\gamma \partial_x^k (v^\varepsilon + y \partial_x U) \partial_x^{s-k} \partial_y \varepsilon}\|_{L^2} \leq C \|\partial_x (v^\varepsilon + y \partial_x U)\|_{H^{s-2,1}} \|{(1 + y) \partial_y \varepsilon w^\varepsilon}\|_{H^{s-2,\gamma}}
$$

(C.14)

and

$$
\|{(1 + y)^\gamma \partial_x^s (v^\varepsilon + y \partial_x U) \partial_y \varepsilon}\|_{L^2} \leq \|{(1 + y)^{\gamma - 1} \partial_x^s (v^\varepsilon + y \partial_x U)\|_{L^2}} \|{(1 + y) \partial_y \varepsilon w^\varepsilon}\|_{L^\infty} \leq C_\gamma \|{(1 + y)^\gamma \partial_x^{s+1} \varepsilon w^\varepsilon}\|_{L^2} \|\varepsilon w^\varepsilon\|_{H^{2,0}}.
$$

(C.15)

Substituting the estimates (C.14) and (C.15) into (C.13), and using the Cauchy inequality, we get

$$
\left| \int_{\mathbb{T}^2} (1 + y)^{2\gamma} |\partial_x^s \varepsilon w^\varepsilon |^2 dxdy \right| \leq \frac{1}{4} \varepsilon^2 \|{(1 + y)^\gamma \partial_x^{s+1} \varepsilon w^\varepsilon}\|_{L^2}^2 + C_s \|\partial_x^{s+1} U\|_{L^\infty}^2 + C_{s,\gamma,\varepsilon} \|{(1 + \|\varepsilon w^\varepsilon\|_{H^{s,\gamma}})}^4.
$$

(C.16)

Substituting the estimates (C.9), (C.11), (C.12), (C.16) into (C.8), we obtain

$$
\frac{d}{dt} \|\varepsilon w^\varepsilon\|_{H^{s,\gamma}}^2 + \varepsilon^2 \|\partial_x \varepsilon w^\varepsilon\|_{H^{s,\gamma}}^2 + \|\partial_y \varepsilon w^\varepsilon\|_{H^{s,\gamma}}^2 \leq C_{s,\gamma} \|{(1 + M U)}^2 + C_{s,\gamma,\varepsilon} \|\varepsilon w^\varepsilon\|_{H^{s,\gamma}}^2,
$$

and hence, we conclude that

$$
1 + \|\varepsilon w^\varepsilon(t)\|_{H^{s,\gamma}}^2 \leq \frac{1 + \|\varepsilon w_0\|_{H^{s,\gamma}}^2}{\{1 - \frac{2}{\varepsilon^2} \max \{C_{s,\gamma} \|{(1 + M U)}^2, C_{s,\gamma,\varepsilon}\} \{1 + \|\varepsilon w_0\|_{H^{s,\gamma}}^2\} \varepsilon^{-2} t\}^{s/2}},
$$

(C.17)

as long as $t < \frac{2}{\varepsilon^2} \max \{C_{s,\gamma} \|{(1 + M U)}^2, C_{s,\gamma,\varepsilon}\} \{1 + \|\varepsilon w_0\|_{H^{s,\gamma}}^2\}$.

Step 2: $L^\infty$ estimates. Denote $B_\alpha := (1 + y)^{\sigma + \alpha_2} D^\alpha w^\varepsilon$ and $I := \sum_{1 \leq |\alpha| \leq 2} |B_\alpha|^2$, it is easy to check that $I$ satisfies the evolution equation

$$
\left( \partial_t + u^\varepsilon \partial_x + v^\varepsilon \partial_y - \varepsilon^2 \partial_x^2 - \partial_y^2 \right) I = -2 \sum_{1 \leq |\alpha| \leq 2} \{\varepsilon^2 |\partial_x B_\alpha|^2 + |\partial_y B_\alpha|^2\} + 2 \sum_{1 \leq |\alpha| \leq 2} \{Q_\alpha B_\alpha \partial_y B_\alpha + R_\alpha |B_\alpha|^2 + S_\alpha B_\alpha\},
$$

where the quantities $Q_\alpha, R_\alpha$ and $S_\alpha$ are given explicitly by

$$
Q_\alpha := -\frac{2(\sigma + \alpha_2)}{1 + y}, \quad R_\alpha := \frac{\sigma + \alpha_2}{1 + y} \varepsilon + \frac{(\sigma + \alpha_2)(\sigma + \alpha_2 + 1)}{(1 + y)^2},
$$

and

$$
S_\alpha := -\sum_{0 < \beta \leq \alpha} C_{\alpha,\beta}(1 + y)^{\beta_2} \{D^\beta u^\varepsilon B_{\alpha - \beta + \epsilon_1} + \frac{D^\beta u^\varepsilon B_{\alpha - \beta + \epsilon_2}}{1 + y}\}.
$$

By routine checking, we get that

$$
|Q_\alpha| \leq C_\sigma, \quad |R_\alpha| \leq C_\sigma \|\partial_x U\|_{L^\infty(\mathbb{T})} + \|\varepsilon w^\varepsilon\|_{H^{s,\gamma}},
$$
and
\[ |S_\alpha| \leq C_\sigma \left( \| \partial_x^2 u \|_{L^\infty(\Omega)} + \| w^\varepsilon \|_{H^{s,\gamma}} \right) \sum_{0 < \beta \leq \alpha} \{|B_{\alpha-\beta+e_1}| + |B_{\alpha-\beta+e_2}| \}.
\]

Then, we can verify that the quantity \( I \) satisfies
\[ (\partial_t + u^\varepsilon \partial_x + v^\varepsilon \partial_y - \varepsilon^2 \partial_x^2 - \partial_y^2)I \leq C_\sigma \left( 1 + \| \partial_x^2 u \|_{L^\infty(\Omega)} + \| w^\varepsilon \|_{H^{s,\gamma}} \right) I,
\]
and hence, we apply the maximum principle in Lemma A.4 to get
\[ \sup_{0 \leq \tau \leq t} Q(\tau) \leq e^{C_\sigma \left( 1 + \| \partial_x^2 u \|_{L^\infty(\Omega)} + \| w^\varepsilon \|_{H^{s,\gamma}} \right) t} (\| I(0) \|_{L^\infty(\Omega)} + \| I(t) \|_{H^{s,\gamma}}) \].
By virtue of \( s \geq 4 \), we apply the Sobolev inequality (A.3) to get
\[ \| I(t) \|_{H^{s,\gamma}} \leq C \| w^\varepsilon \|_{H^{s,\gamma}}^2.
\]
This and the inequality (C.18) yield directly
\[ \sup_{0 \leq \tau \leq t} Q(\tau) \leq e^{C_\sigma \left( 1 + \sup_{0 \leq \tau \leq t} \| \partial_x^2 u(\tau) \|_{L^\infty(\Omega)} + \sup_{0 \leq \tau \leq t} \| w^\varepsilon(\tau) \|_{H^{s,\gamma}} \right) t} (Q(0) + \| w^\varepsilon \|_{H^{s,\gamma}}^2).
\]

Step 3: Life span time. Taking \( T_1 = \frac{(1 - \frac{1}{2})^{s-2}}{2 \max \{ C_{s,\gamma}(1 + M_u^2)^2 C_{s,\gamma,s} \} \{ 1 + \| w_0 \|_{H^{s,\gamma}}^2 \} \frac{\gamma - 2}{s+2} } \), we get by using (C.17)
\[ \sup_{0 \leq \tau \leq t_1} \| w^\varepsilon(\tau) \|_{H^{s,\gamma}}^2 \leq 2(1 + \| w_0 \|_{H^{s,\gamma}}^2).
\]
Taking \( T_2 = \min \{ T_1, \frac{\ln 2}{C_s(3 + M_u^2 + \sqrt{2} \| w_0 \|_{H^{s,\gamma}})} \} \), it follows from (C.19)
\[ \sup_{0 \leq t \leq t_2} Q(t) \leq 4(1 + Q(0) + \| w_0 \|_{H^{s,\gamma}}^2).
\]
Taking \( T_3 = \min \{ T_1, T_2, \frac{\delta_1}{C_s(\gamma + M_u^2 + 4Q(0) + 6 \| w_0 \|_{H^{s,\gamma}})} \} \), we get by using (C.21)
\[ (1 + y)^\sigma w^\varepsilon(t, x, y) \geq (1 + y)^\sigma w_0(x, y) - \delta_0 \geq 2\delta_0 - \delta_0 = \delta_0.
\]
Then, we have chosen the life span time \( T_\alpha := T_3 \) such that the estimates (C.20)-(C.22) hold on. Finally, we point out that we can use the local existence results established above to extend the solution(defined on \([0, T_\alpha])\) step by step to the time interval \([0, T^c]\) such that the estimates (C.1) and (C.2) hold on. Therefore, we complete the proof of this lemma.

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