THE BAKER-RICHTER SPECTRUM AS COBORDISM OF QUASITORIC MANIFOLDS

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ABSTRACT. Baker and Richter construct a remarkable $A_{\infty}$ ring-spectrum $M\Xi$ whose elements possess characteristic numbers associated to quasisymmetric functions; its relations, on one hand to the theory of non-commutative formal groups, and on the other to the theory of omnioriented (quasi)toric manifolds [in the sense of Buchstaber, Panov, and Ray], seem worth investigating.

Introduction: Most of this paper is a draft for a talk JM wishes he had given at the August 2011 conference on toric manifolds [10] at Queen’s University, Belfast (as opposed to the talk he actually gave). Thanks to Thomas Hüttemann for organizing that very interesting meeting, and to Tony Bahri, Martin Bendersky, Fred Cohen, and Sam Gitler for helpful conversations there. Those notes are little more than a collage of conversations, suggestions, and howler-preventing interventions courtesy of Andy Baker, Michiel Hazewinkel, Birgit Richter, and Taras Panov, in the course of the last few years; he is deeply indebted to them all.

He is also extremely grateful to Nitu Kitchloo, for permission to include an appendix by the latter, which outlines some work in progress.

1. Conjectures about the spectrum $M\Xi$

1.1 This is the Thom spectrum defined by A Baker and B Richter [2, with slightly different notation] constructed by pulling back the canonical bundle $\xi \to BU$ along the abelianization map $\Omega\Sigma BT = B\Omega^2\Sigma BT \to BU$.

[An analytic construction of the associated representation $\Xi := \Omega^2\Sigma BT \to U$ might be very interesting.]
They show that $M \Xi$ is an $A_{\infty}$ ring-spectrum, with $M \Xi_*$ torsion-free and concentrated in even degrees, and that the Hurewicz homomorphism

$$M \Xi_* \to H_*(\Omega\Sigma CP^{\infty}) \cong NSymm_*$$

takes values in the (graded) ring of noncommutative symmetric functions [6 §4.2, 8]; it is injective, and becomes an isomorphism after tensoring with $\mathbb{Q}$. Finally, and most striking of all, they show that $M \Xi \otimes \mathbb{Z}_p$ is a wedge of copies of $BP$.

1.2 A (unital) $S$-algebra $A$ defines a cosimplicial algebra

$$A^* : n \to A^{\wedge n}$$

with maps built from its unit and multiplication. In good cases [1] this is a resolution, in a suitable sense, of the sphere spectrum, and its homotopy groups define a cosimplicial algebra

$$A_* \xrightarrow{=} A_* A \otimes_A A_* A \xrightarrow{=} \ldots$$

which leads to the construction of an Adams spectral sequence.

If $A = \mathbb{M}U$, then

$$\pi_* (\mathbb{M}U \wedge \mathbb{M}U) = \mathbb{M}U_* \otimes_{\mathbb{Z}_*} S_*$$

is the product of the Lazard ring with the algebra $S_* \cong H_*(BU, \mathbb{Z})$ of functions on the group of formal power series under composition, and the resulting cosimplicial ring can be interpreted as a presentation

$$\mathbb{M}U_* \to \mathbb{M}U_* \otimes S_* \to (\mathbb{M}U_* \otimes S_*) \otimes_{\mathbb{M}U} (\mathbb{M}U_* \otimes S_*) \to \ldots$$

of (the graded algebra of functions on) the moduli stack of one-dimensional formal group laws. The classical Steenrod augmentation $\mathbb{M}U_* \to \mathbb{Z}$ classifies the additive group law, and its composition

$$\mathbb{M}U_* \to \mathbb{M}U_* \otimes S_* \to \mathbb{Z} \otimes S_* = S_*$$

with the coaction represents inclusion

$$\text{Spec } S_* \to \text{Spec } \mathbb{M}U_*$$

of the orbit of the additive group, under coordinate changes, in the moduli stack of formal groups. It is also the (injective) Hurewicz map

$$\pi_* (\mathbb{M}U) \to H_*(\mathbb{M}U, \mathbb{Z}) \cong \text{Hom}(H^*(BU), \mathbb{Z})$$

which assigns to a manifold, its collection of Chern numbers; and from either point of view it is a rational isomorphism.

The cosimplicial algebra $\pi_* \mathbb{M}U^* \otimes \mathbb{Q}$ is thus a resolution of $\pi_*^S \otimes \mathbb{Q} = \mathbb{Q}$; it is a cosimplicial presentation of the stack over $\mathbb{Q}$ defined by the action of the group of formal diffeomorphisms on itself[1].

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[1] If $G$ is a group object, then the category $[G/G]$ defined by its translation action is equivalent to the category with one object and its identity morphism.
1.3 It would be nice to have a similar description for \( A = M \Xi \), when
\[
\pi_*(M \Xi \wedge M \Xi) = M \Xi \star M \Xi \cong M \Xi \otimes_{\mathbb{Z}} \text{NSymm}_*,
\]
but we don’t yet have a good description of the coaction maps in the cosimplicial ring it defines. I am indebted to Michiel Hazewinkel for suggesting the following possibility:

**Conjecture:** The Hurewicz homomorphism
\[
M \Xi \star M \Xi \to H_*(M \Xi \wedge M \Xi) \cong \text{NSymm}_* \otimes_{\mathbb{Z}} \text{NSymm}_*
\]
is a homomorphism of Hopf algebras, with target the Novikov double \([9]\) of the Hopf algebra defined by the diagonal
\[
\Delta_{BFK} Z(t) = \text{res}_{u=0} Z(u) \otimes (u - Z(t))^{-1}
\]
\([4 \S 2.4]\) on the ring
\[
\text{NSymm}_* = Z(Z_i \mid i \geq 1), \quad Z(t) = \sum_{i \geq 0} Z_i t^{i+1}
\]
of noncommutative symmetric functions.

1.4 In further work Baker and Richter construct \([3]\) an injective homomorphism \( \lambda_{BR} \)
\[
z \mapsto c + \sum_{i > 0} z_i c^i + 1 : M \Xi^* B \mathbb{T} \cong M \Xi(*) \to H_* M \Xi[[c]] \cong \text{NSymm}_*[c]
\]
of Hopf algebras, where \( c \) is a central element corresponding to the Chern class for line bundles in ordinary cohomology. The diagram
\[
\begin{array}{ccc}
M \Xi & \longrightarrow & \text{NSymm}_* \\
\downarrow & & \downarrow \\
M \Xi \otimes_{\mathbb{Q}} \lambda_{BR} \otimes_{\mathbb{Q}} & \longrightarrow & \text{NSymm}_* \otimes_{\mathbb{Q}}
\end{array}
\]
lets us regard the coefficients \( z_i \) of their logarithm as elements of \( M \Xi_{2i} \otimes \mathbb{Q} \).

1.5 **Conjecture:** With the left vertical homomorphism defined by the natural coaction, the diagram
\[
\begin{array}{ccc}
M \Xi^* B \mathbb{T} & \longrightarrow & \lambda_{BR} \text{NSymm}_*[c] \\
\downarrow & & \downarrow \\
M \Xi^* B \mathbb{T} \otimes_{M \Xi_*} M \Xi & \cong M \Xi^* B \mathbb{T} \otimes_{\mathbb{Z}} \text{NSymm}_* \longrightarrow (\text{NSymm}_* \otimes_{\mathbb{Z}} \text{NSymm}_*)[[c]]
\end{array}
\]
commutes; where
\[
\Delta_{BFK}(1 \otimes c) := 1 \otimes Z(c) : \text{NSymm}_*[c] \to (\text{NSymm}_* \otimes_{\mathbb{Z}} \text{NSymm}_*)[[c]].
\]
Moreover, \( \lambda_{BR} \otimes_{\mathbb{Q}} \) maps \( \pi_* M \Xi^* \otimes_{\mathbb{Q}} \) isomorphically to the resolution
\[
\begin{array}{ccc}
\mathbb{Q} & \longrightarrow & \text{NSymm}_* \otimes_{\mathbb{Q}} (\text{NSymm}_* \otimes_{\mathbb{Q}} \text{NSymm}_*) \longrightarrow \ldots
\end{array}
\]
2. Characteristic numbers for quasitoric manifolds

2.1 A complex-oriented $2m$-dimensional quasitoric manifold $M$ has a simple quotient polytope $P$, with an omniorientation [5 §5.31] defined by a characteristic map $\Lambda$ [5 §5.10] from the ordered set of vertices of the simplicial $(m-1)$-sphere $K_P$ bounding the dual simplicial complex $P^*$ [5 §1.10], to a free abelian group $\Theta$ with generators $\theta_1, \ldots, \theta_m$.

The cohomology of $M$ can be naturally identified [5 §5.2.2, §6.5, 7] with a quotient

$$k^*(K_P) \otimes_{P(\Theta^*)} \mathbb{Z},$$

of the Stanley-Reisner face ring [5 §3.1, 3.4]; it is an algebra over the symmetric algebra $P(\Theta^*)$ on the $\mathbb{Z}$-dual of $\Theta$ via

$$\Lambda^*: P(\Theta^*) \to P(V_P^*) \to k^*(K_P)$$

(where $V_P$ is the free abelian group generated by the vertices of $K_P$, and $\mathbb{Z}$ is a $P(\Theta^*)$-algebra via augmentation).

2.2 The order on the vertex set of $K_P$ embeds it as the initial segment of the natural numbers $\mathbb{N}$, identifying $V_P$ with a subgroup of a free abelian group $\mathcal{V}$ on a countable set of generators. A similar identification embeds $\Theta$ in another free abelian group $\mathcal{O}$ on a countable set of generators, defining an extension

$$\overline{\Lambda}: \mathcal{V} \to \mathcal{O}$$

of $\Lambda$ by $\overline{\Lambda}(i) = \theta_i$ when $i$ is not a vertex of $K_P$. The resulting homomorphism

$$P(\mathcal{O}^*) \to P(\mathcal{V}^*) = P(V_P^*) \otimes P(\mathcal{V}_{-P}^*)$$

makes

$$\overline{k}^*(K_P) := k^*(K_P) \otimes_{\mathbb{Z}} P(\mathcal{V}_{-P})$$

into an algebra over the polynomial ring $P(\mathcal{O})$ generated by a countable sequence of variables, such that

$$H^{2*}(M, \mathbb{Z}) \cong \overline{k}^*(K_P) \otimes_{P(\mathcal{O})} \mathbb{Z}.$$

2.3 For an ordered partition

$$\textbf{m} = m_1 + \cdots + m_r$$

of $m$, let $[\textbf{m}]_P$ be the image in $H^{2m}(M, \mathbb{Z})$ of the formal sum

$$[\textbf{m}] := \sum_{i_1 < \cdots < i_r} x_{i_1}^{m_1} \cdots x_{i_r}^{m_r}.$$
where \( x_i \) is the polynomial generator corresponding to \( i \in \mathcal{V} \) [6 §4]. If \( i \) is sufficiently small, \( x_i \) corresponds to a vertex of \( K_P \); otherwise, it is a kind of dummy element, and is killed by \(- \otimes_{P(\mathcal{G})} \mathbb{Z}\).

### 2.4

More generally, if \( M \) and \( N \) are almost-complex quasitoric manifolds of dimension \( m, n \) respectively, with quotient polytopes \( P, Q \), then the product
\[
k^*(K_P) \otimes_{\mathbb{Z}} k^*(K_Q) \cong k^*(K_P * K_Q)
\]
of face rings is naturally isomorphic to the face ring of the join
\[
K_P * K_Q \cong K_{P \times Q}
\]
of the simplicial spheres \( K_P \) and \( K_Q \) [5 §2.13].

**Claim:** the corresponding isomorphism
\[
H^*(M) \otimes H^*(N) \to H^*(M \times N)
\]
sends \([m]_P \otimes [n]_Q\) to
\[
[m + n]_{P \times Q} := [m_1 + \cdots + m_r + n_1 + \cdots + n_s]_{P \times Q}.
\]

**Proof:** \([m + n]_{P \times Q}\) is the image in \( H^{2(m+n)}(M \times N)\) of
\[
\sum_{i_1 < \cdots < i_{r+s}} x_{i_1}^{m_1} \cdots x_{i_{r+s}}^{n_s},
\]
summed over strings \( i_1 < \cdots < i_{r+s} \) of elements of the disjoint union of the vertex sets of \( P^* \) and \( Q^* \). For a monomial of this sort to have a nontrivial image in the top-dimensional cohomology of \( M \times N \), the elements of the set \( \{x_{i_1}, \ldots, x_{i_r}\} \) must be vertices of \( K_P \), and those of \( \{x_{i_{r+1}}, \ldots, x_{i_{r+s}}\} \) must be vertices of \( K_Q \). The image of the sum is thus the product of the images of the sums \([m]_P\) and \([n]_Q\) (modulo the identification of the top-dimensional cohomology group of \( P \) (resp. \( Q \)) with the integers). \(\square\)

This construction associates to a \(2m\)-dimensional complex-oriented quasitoric manifold \( M \), a homomorphism
\[
[m]_P : \mathbb{Q}\text{Symm}^m \to \mathbb{Z}
\]
of abelian groups (ie a noncommutative symmetric function \( \mathbb{M} \)), which sends \( M \times N \) to a noncommutative symmetric function
\[
\mathbb{M} \times \mathbb{N} = \mathbb{M} \bullet \mathbb{N}
\]
equal to the product of the noncommutative symmetric functions \( \mathbb{M} \) and \( \mathbb{N} \).

**Problem:** What is the noncommutative symmetric function \( \mathbb{CP}^n \) defined by complex projective \( n \)-space, with its usual toric structure (and the \( n \)-simplex as associated polytope)?
2.5 This might be paraphrased as saying that the Davis-Januszkiewicz construction defines a ring homomorphism from the algebra generated by the monoid, under join, of certain omnioriented simplicial spheres, to the free graded associative algebra $\text{NSymm}_*$; in other words, something like a coordinate patch for a noncommutative space of quasitoric manifolds.

3. Appendix, by Nitu Kitchloo:

A quasitoric manifold $M$ of dimension $2m$ admitting an action of a torus $T$ of rank $m$ is associated with a polytope $P$. Assume $F = \{f_i\}$ is the set consisting of the co-dimension one faces $f_i$ of $P$. The data required to construct $M$ involves a collection of primitive characteristic weights $\lambda_i \in \pi_1(T)$, indexed on the set $F$.

Let $\hat{T}$ denote the torus $(S^1)^F$, of rank given by the cardinality of $F$, with a canonical set of generating circles indexed by the faces $f_i$. Let $H \subset \hat{T}$ denote the kernel of the map $\lambda: \hat{T} \rightarrow T$, defined by $\lambda(\exp(tf_i)) = \exp(t\lambda_i)$.

The procedure for constructing $M$ is described as follows: Notice that $\hat{T}$ acts on $\mathbb{C}^F$ in a canonical way via Hamiltonian symplectomorphisms. This induces an action of $H$. Let $\mathcal{H}$ denote the Lie algebra of $H$. Let $\varphi: \mathbb{C}^F \rightarrow \mathcal{H}^*$ denote the moment map of the $H$ action. Let the ”Moment angle complex” $Z(P)$ denote the preimage of a regular value. The manifold $M$ is defined as the orbit space: $M = Z(P)/H$. From this it follows easily that:

3.1 Claim: $M = Z(P)/H$ has stable tangent bundle classified by the composite map:

$$\tau(M): Z(P)/H \rightarrow BH \rightarrow B\hat{T} \rightarrow BU(F),$$

where $BU(F)$ denotes the group of unitary transformations of $\mathbb{C}^F$, with maximal torus $\hat{T}$.

Let us now try to find a natural Thom spectrum that is the receptacle for the cobordism class of $M$. Firstly notice that there is a commutative diagram:

$$\begin{array}{ccc}
B\hat{T} & \xrightarrow{\omega} & \Omega\Sigma BU(1) \\
\downarrow & & \downarrow \varphi \\
BU(F) & \xrightarrow{\varphi} & BU
\end{array}$$

where $\varphi: \Omega\Sigma BU(1) \rightarrow BU$ is the $A_\infty$ extension of the inclusion map $BU(1) \rightarrow BU$, and the map $B\hat{T} \rightarrow \Omega\Sigma BU(1)$ is the inclusion of the $|F|$-th James filtration. We conclude:
3.2 Corollary: Let $\mathcal{M}^{\Xi}$ denote the Thom spectrum of $-\varphi$, then the cobordism class of $M$ belongs to $\pi_{2m}\mathcal{M}^{\Xi}$.

3.3 Remark: Notice that since $\varphi$ is an $A_{\infty}$ map, the Thom spectrum of $-\varphi$ is equivalent to the Thom spectrum $M^{\Xi}$ of $\varphi$, as seen easily from the following commutative diagram:

$$
\begin{array}{ccc}
\Omega \Sigma BU(1) & \xrightarrow{-\text{Id}} & \Omega \Sigma BU(1) \\
\downarrow^{\varphi} & & \downarrow^{-\varphi} \\
BU & \xrightarrow{\text{Id}} & BU
\end{array}
$$

In particular, the algebraic procedure described in Section 2 is indeed a method of computing the characteristic numbers of the tangent bundle of $M$.

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