Emergent Gravity from Noncommutative Spacetime

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ABSTRACT

We showed before that self-dual electromagnetism in noncommutative (NC) spacetime is equivalent to self-dual Einstein gravity. This result implies a striking picture about gravity: Gravity can emerge from electromagnetism in NC spacetime. Gravity is then a collective phenomenon emerging from gauge fields living in fuzzy spacetime. We elucidate in some detail why electromagnetism in NC spacetime should be a theory of gravity. In particular, we show that NC electromagnetism is realized through the Darboux theorem as a diffeomorphism symmetry $G$ which is spontaneously broken to symplectomorphism $H$ due to a background symplectic two-form $B_{\mu \nu} = (1/\theta)_{\mu \nu}$, giving rise to NC spacetime. This leads to a natural speculation that the emergent gravity from NC electromagnetism corresponds to a nonlinear realization $G/H$ of the diffeomorphism group, more generally its NC deformation. We also find some evidences that the emergent gravity contains the structures of generalized complex geometry and NC gravity. To illuminate the emergent gravity, we illustrate how self-dual NC electromagnetism nicely fits with the twistor space describing curved self-dual spacetime. We also discuss derivative corrections of Seiberg-Witten map which give rise to higher order gravity.

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1 Introduction and Symplectic Geometry

Recently we showed in [1] that self-dual electromagnetism in noncommutative (NC) spacetime is equivalent to self-dual Einstein gravity. For example, $U(1)$ instantons in NC spacetime are actually gravitational instantons [2,3]. This result implies a striking picture about gravity: Gravity can emerge from electromagnetism if the spacetime, at microscopic level, is noncommutative like the quantum mechanical world. Gravity is then a collective phenomenon emerging from gauge fields living in fuzzy spacetime. Similar ideas have been described in [4] and in a recent review [5] that NC gauge theory can naturally induce a gauge theory of gravitation.

In this paper we will show that the “emergent gravity” from NC spacetime is very generic in NC field theories, not only restricted to the self-dual sectors. Since this picture about gravity is rather unfamiliar, though marked evidences from recent developments in string and M theories are ubiquitous, it would be desirable to have an intuitive picture for the emergent gravity. This remarkable physics turns out to be deeply related to symplectic geometry in sharp contrast to Riemannian geometry. Thus we first provide conceptual insights, based on intrinsic properties of the symplectic geometry, on why a field theory formulated on NC spacetime could be a theory of gravity. We will discuss more concrete realizations in the coming sections. We refer [6] for rigorous details about the symplectic geometry.

**Symplectic manifold**: A symplectic structure on a smooth manifold $M$ is a non-degenerate, closed 2-form $\omega \in \Lambda^2(M)$. The pair $(M, \omega)$ is called a symplectic manifold. In classical mechanics, a basic symplectic manifold is the phase space of $N$-particle system with $\omega = \sum dq^i \wedge dp_i$.

A NC spacetime is obtained by introducing a symplectic structure $B = \frac{1}{2} B_{\mu \nu} dy^\mu \wedge dy^\nu$ and then by quantizing the spacetime with its Poisson structure $\theta^{\mu \nu} \equiv (B^{-1})^{\mu \nu}$, treating it as a quantum phase space. That is, for $f, g \in C^\infty(M)$,

$$\{f, g\} = \theta^{\mu \nu} \left( \frac{\partial f}{\partial y^\mu} \frac{\partial g}{\partial y^\nu} - \frac{\partial f}{\partial y^\nu} \frac{\partial g}{\partial y^\mu} \right) \Rightarrow -\frac{i}{\hbar} [\hat{f}, \hat{g}], \quad (1.1)$$

where $\hbar$ is a formal parameter and we sometimes set $\hbar = 1$ by absorbing it in $\theta$.

According to the Weyl-Moyal map [7], the NC algebra of operators is equivalent to the deformed algebra of functions defined by the Moyal $\star$-product, i.e.,

$$\hat{f} \cdot \hat{g} \equiv (f \star g)(y) = \exp \left( \frac{i \hbar}{2} \theta^{\mu \nu} \partial_\mu \partial_\nu \right) f(y)g(z) \bigg|_{y=z} . \quad (1.2)$$

**Symplectomorphism**: Let $(M, \omega)$ be a symplectic manifold. Then a diffeomorphism $\phi : M \to M$ satisfying $\omega = \phi^*(\omega)$ is a symplectomorphism. In classical mechanics, symplectomorphisms are called canonical transformations. A vector field $X$ on $M$ is said to be symplectic if $\mathcal{L}_X \omega = 0$. The Lie derivative along a vector field $X$ satisfies the Cartan’s homotopy formula $\mathcal{L}_X = \iota_X d + d \iota_X$, where $\iota_X$ is the inner product with $X$, e.g., $\iota_X \omega(Y) = \omega(X, Y)$. Since $d\omega = 0$, $X$ is a symplectic vector field if and only if $\iota_X \omega$ is closed. If $\iota_X \omega$ is exact, i.e., $\iota_X \omega = dH$ for any $H \in C^\infty(M)$, $X$ is called the Hamiltonian vector field. So the first cohomology group $H^1(M)$ measures the obstruction for a
symplectic vector field to be Hamiltonian. Since we are interested in simply connected manifolds, e.g., $M = \mathbb{R}^4$, every symplectic vector field would be Hamiltonian.

Through the quantization rule (1.1) and (1.2), one can define NC $\mathbb{R}^4$ by the following commutation relation

$$[y^\mu, y^\nu]_\star = i\theta^{\mu\nu}. \quad (1.3)$$

An important point is that the set of diffeomorphisms generated by Hamiltonian vector fields, denoted as $Ham(M)$, preserves the NC algebra (1.3) since $L_X B = 0$ with $B = \theta^{-1}$ provided $\iota_X B = d\lambda$ where $\lambda$ is an arbitrary function [8, 9]. The symmetry $Ham(M)$ is infinite-dimensional as well as non-Abelian and can be identified with NC $U(1)$ gauge group [8] upon quantization in the sense of Eq.(1.1). The NC gauge symmetry then acts as unitary transformations on an infinite-dimensional, separable Hilbert space $\mathcal{H}$ which is the representation space of the Heisenberg algebra (1.3). This NC gauge symmetry $U_{opt}(\mathcal{H})$ is so large that $U_{opt}(\mathcal{H}) \supset U(N) \ (N \to \infty)$ [10]. In this sense the NC gauge theory is essentially a large $N$ gauge theory. It becomes more explicit on a NC torus through the Morita equivalence where NC $U(1)$ gauge theory with rational $\theta = M/N$ is equivalent to an ordinary $U(N)$ gauge theory [11, 12]. Therefore it is not so surprising that NC electromagnetism shares essential properties appearing in a large $N$ gauge theory such as $SU(N \to \infty)$ Yang-Mills theory or matrix models.

The symplectic manifolds accompany an important property, the so-called Darboux theorem, stating that every symplectic manifold is locally symplectomorphic.

**Darboux theorem**: Locally, $(M, \omega) \cong (\mathbb{C}^n, \sum dq^i \wedge dp_i)$. That is, every $2n$-dimensional symplectic manifold can always be made to look locally like the linear symplectic space $\mathbb{C}^n$ with its canonical symplectic form - Darboux coordinates. More precisely, we will use the Moser lemma [13] describing a cohomological condition for two symplectic structures to be equivalent: Let $M$ be a symplectic manifold of compact support. Given two-forms $\omega$ and $\omega'$ such that $[\omega] = [\omega'] \in H^2(M)$ and $\omega_t = \omega + t(\omega' - \omega)$ is symplectic $\forall t \in [0, 1]$, then exists a diffeomorphism $\phi : M \to M$ such that $\phi^*(\omega') = \omega$. This implies that all $\omega_t$ are related by coordinate transformations generated by a vector field $X$ satisfying $\iota_X \omega_t + A = 0$ where $\omega' - \omega = dA$. In particular we have $\phi^*(\omega') = \omega$ where $\phi$ is the flow of $X$. In terms of local coordinates, there always exists a coordinate transformation $\phi$ whose pullback maps $\omega' = \omega + dA$ to $\omega$, i.e., $\phi : y \mapsto x = x(y)$ so that

$$\frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} \omega_{\alpha\beta}(x) = \omega_{\mu\nu}(y). \quad (1.4)$$

The Darboux theorem leads to an important consequence on the low energy effective dynamics of D-branes in the presence of a background $B$-field. The dynamics of D-branes is described by open string field theory whose low energy effective action is obtained by integrating out all the massive modes, keeping only massless fields which are slowly varying at the string scale. The resulting low energy dynamics is described by the Dirac-Born-Infeld (DBI) action [14]. For a $Dp$-brane in arbitrary
background fields, the DBI action is given by

\[
S = \frac{2\pi}{(2\pi \kappa)^{p+1}} \int d^{p+1}\sigma e^{-\Phi} \sqrt{\det (g + \kappa (B + F))} + O(\sqrt{\kappa} \partial F, \cdots),
\]

(1.5)

where \(\kappa \equiv 2\pi \alpha'\), the size of a string, is a unique expansion parameter to control derivative corrections. But the string coupling constant \(g_s \equiv e^{\langle \Phi \rangle}\) will be assumed to be constant.

The DBI action (1.5) respects several symmetries. The most important symmetry for us is the so-called \(\Lambda\)-symmetry given by

\[
B \to B - d\Lambda, \quad A \to A + \Lambda
\]

(1.6)

for any one-form \(\Lambda\). Thus the DBI action depends on \(B\) and \(F\) only in the gauge invariant combination \(\mathcal{F} \equiv B + F\) as shown in (1.5). Note that ordinary \(U(1)\) gauge symmetry is a special case where the gauge parameters \(\Lambda\) are exact, namely, \(\Lambda = d\lambda\), so that \(B \to B, \quad A \to A + d\lambda\).

Suppose that the two-form \(B\) is closed, i.e. \(dB = 0\), and non-degenerate on the D-brane worldvolume \(M\). The pair \((M, B)\) then defines a symplectic manifold. But the \(\Lambda\)-transformation (1.6) changes (locally) the symplectic structure from \(\omega = B\) to \(\omega' = B - d\Lambda\). According to the Darboux theorem and the Moser lemma stated above, there must be a coordinate transformation such as Eq.(1.4). Thus the local change of symplectic structure due to the \(\Lambda\)-symmetry can always be translated into worldvolume diffeomorphisms as in Eq.(1.4). For some reason to be clarified later, we prefer to interpret the symmetry (1.6) as a worldvolume diffeomorphism, denoted as \(G \equiv Diff(M)\), in the sense of Eq.(1.4). Note that the number of gauge parameters in the \(\Lambda\)-symmetry is exactly the same as \(Diff(M)\). We will see that the Darboux theorem in symplectic geometry plays the same role as the equivalence principle in general relativity.

The coordinate transformation in Eq.(1.4) is not unique since the symplectic structure remains intact if it is generated by a vector field \(X\) satisfying \(\mathcal{L}_X B = 0\). Since we are interested in a simply connected manifold \(M\), i.e. \(\pi_1(M) = 0\), the condition is equivalent to \(\iota_X B + d\lambda = 0\), in other words, \(X \in Ham(M)\). Thus the symplectomorphism \(H \equiv Ham(M)\) corresponds to the \(\Lambda\)-symmetry

\[\text{1}\] The action (1.5) has a worldvolume reparameterization invariance: \(\sigma^\mu \mapsto \sigma'^\mu = f^\mu(\sigma)\) for \(\mu = 0, 1, \cdots, p\). But, this diffeomorphism symmetry is usually gauge-fixed to identify worldvolume coordinates \(\sigma^\mu\) with spacetime ones, i.e., \(x^A(\sigma) = \sigma^A\) for \(A = 0, 1, \cdots, p\). In this static gauge which will be adopted throughout the paper, the induced metric \(g_{\mu\nu}(x(\sigma))\) on the brane reduces to a background spacetime metric, e.g., \(g_{\mu\nu}(\sigma) = \delta_{\mu\nu}\). So we will never refer to this symmetry in our discussion.

\[\text{2}\] Note that the ‘D-manifold’ \(M\) also carries a non-degenerate, symmetric, bilinear form \(g\) which is a Riemannian metric. The pair \((M, g)\) thus defines a Riemannian manifold. If we consider a general pair \((M, g + \kappa B)\), it describes a generalized geometry \([15]\) which continuously interpolates between a symplectic geometry \((|\kappa B g^{-1}| \gg 1)\) and a Riemannian geometry \((|\kappa B g^{-1}| \ll 1)\). The decoupling limit considered in \([11]\) corresponds to the former.

\[\text{3}\] If we consider NC gauge theories on \(M = T^d\) in which \(\pi_1(M) \neq 0\), a symplectic vector field \(X\), i.e. \(\mathcal{L}_X B = 0\), is not necessarily Hamiltonian but takes the form \(X^\mu = \theta^{\mu\nu} \partial_\nu \lambda + \xi^\mu\) where \(\xi^\mu\) is a harmonic one-form, i.e., it cannot be written as a derivative of a scalar globally. This harmonic one-form introduces a twisting of vector bundle or projective module on \(M = T^d\) such that the gauge bundle is periodic only up to gauge transformations \([16]\).
where \( \Lambda = d\lambda \) and so \( \text{Ham}(M) \) can be identified with the ordinary \( U(1) \) gauge symmetry \([8, 17]\). As is well-known, if a vector field \( X_\lambda \) is Hamiltonian satisfying \( i_{X_\lambda} B + d\lambda = 0 \), the action of \( X_\lambda \) on a smooth function \( f \) is given by \( X_\lambda(f) = \{\lambda, f\} \), which is infinite dimensional as well as non-Abelian and, after quantization \([1.1]\), gives rise to NC gauge symmetry.

Using the \( \Lambda \)-symmetry, gauge fields can always be shifted to \( B \) by choosing the parameters as \( \Lambda_\mu = -A_\mu \), and the dynamics of gauge fields in the new symplectic form \( B + dA \) is interpreted as a local fluctuation of symplectic structures. This fluctuating symplectic structure can then be translated into a fluctuating geometry through the coordinate transformation in \( G = \text{Diff}(M) \) modulo \( H = \text{Ham}(M) \), the \( U(1) \) gauge transformation. We thus see that the ‘physical’ change of symplectic structures at a point in \( M \) takes values in \( \text{Diff}_F(M) \equiv \text{G}/\text{Ham}(M) \).

We need an explanation about the meaning of the ‘physical’. The \( \Lambda \)-symmetry \([1.6]\) is spontaneously broken to the symplectomorphism \( H = \text{Ham}(M) \) since the vacuum manifold defined by the NC spacetime \([1.3]\) picks up a particular symplectic structure, i.e.,

\[
\langle B_{\mu\nu}(x) \rangle_{\text{vac}} = (\theta^{-1})_{\mu\nu}.
\]

This should be the case since we expect only the ordinary \( U(1) \) gauge symmetry in large distance (commutative) regimes, corresponding to \( |\kappa Bg^{-1}| \ll 1 \) in the footnote \([2]\) where \( |\theta|^2 \equiv G_{\mu\lambda} G_{\nu\sigma} \theta^{\mu\nu} \theta^{\lambda\sigma} = \kappa^2 |\kappa Bg^{-1}|^2 \ll \kappa^2 \) with the open string metric \( G_{\mu\nu} \) defined by Eq.\((3.21)\) in \([11]\). The fluctuation of gauge fields around the background \([1.7]\) induces a deformation of the vacuum manifold, e.g. \( \mathbb{R}^4 \) in the case of constant \( \theta \)'s. According to the Goldstone’s theorem \([18]\), massless particles, the so-called Goldstone bosons, should appear which can be regarded as dynamical variables taking values in the quotient space \( G/H = \text{Diff}_F(M) \).

Since \( G = \text{Diff}(M) \) is generated by the set of \( \Lambda_\mu = -A_\mu \), so the space of gauge field configurations on NC \( \mathbb{R}^4 \) and \( H = \text{Ham}(M) \) by the set of gauge transformations, \( G/H \) can be identified with the gauge orbit space of NC gauge fields, in other words, the ‘physical’ configuration space of NC gauge theory. Thus the moduli space of all possible symplectic structures is equivalent to the ‘physical’ configuration space of NC electromagnetism.

The Goldstone bosons for the spontaneous symmetry breaking \( G \to H \) turn out to be spin-2 gravitons \([19]\), which might be elaborated by the following argument. Using the coordinate transformation \([1.4]\) where \( \omega' = B + F(x) \) and \( \omega = B \), one can get the following identity \([8]\) for the DBI action \([1.5]\)

\[
\int d^{p+1}x \sqrt{\det(g + \kappa(B + F(x)))} = \int d^{p+1}y \sqrt{\det(\kappa B + h(y))},
\]

where fluctuations of gauge fields now appear as an induced metric on the brane given by

\[
h_{\mu\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta}.
\]

The dynamics of gauge fields is then encoded into the fluctuations of geometry through the embedding functions \( x^\mu(y) \). The fluctuation of gauge fields around the background \([1.7]\) can be manifest by
representing the embedding function as follows

\[ x^\mu(y) \equiv y^\mu + \theta^{\mu\nu} \hat{A}_\nu(y). \] (1.10)

Given a gauge transformation \( A \to A + d\lambda \), the corresponding coordinate transformation generated by a vector field \( X_\lambda \in Ham(M) \) is given by

\[ \delta x^\mu(y) \equiv X_\lambda(x^\mu) = -\{\lambda, x^\mu\} \]

\[ = \theta^{\mu\nu}(\partial_\nu \lambda + \{\hat{A}_\nu, \lambda\}). \] (1.11)

As we discussed already, this coordinate change can be identified with a NC gauge transformation after the quantization (1.1). So \( \hat{A}_\mu(y) \) are NC U(1) gauge fields and the coordinates \( x^\mu(y) \) in (1.10) will play a special role since they are gauge covariant \([20]\) as well as background independent \([21]\).

It is straightforward to get the relation between ordinary and NC field strengths from the identity (1.4):

\[ \left( \frac{1}{B + F(x)} \right)^{\mu\nu} \equiv \left( \theta - \theta \hat{F}(y)\theta \right)^{\mu\nu} \]

\[ \Leftrightarrow \hat{F}_{\mu\nu}(y) = \left( \frac{1}{1 + F\theta} \right)^{\mu\nu}(x), \] (1.12)

where NC electromagnetic fields are defined by

\[ \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \{\hat{A}_\mu, \hat{A}_\nu\}. \] (1.13)

The Jacobian of the coordinate transformation \( y \mapsto x = x(y) \) is obtained by taking the determinant on both sides of Eq.(1.4)

\[ d^{p+1}y = d^{p+1}x \sqrt{\det(1 + F\theta)}(x). \] (1.14)

In addition one can show \([8]\) that the DBI action (1.8) turns into the NC gauge theory with the semi-classical field strength (1.13) by expanding the right-hand side with respect to \( \hbar/\kappa B \) around the background \( B \).

The above argument clarifies why the dynamics of NC gauge fields can be interpreted as the fluctuations of geometry described by the metric (1.9). One may identify \( \partial x^\alpha/\partial y^\mu \equiv e_\mu^\alpha(y) \) with vielbeins on some manifold \( M \) by regarding \( h_{\mu\nu}(y) = e_\mu^\alpha(y)e_\nu^\beta(y)g_{\alpha\beta} \) as a Riemannian metric on \( M \). The embedding functions \( x^\mu(y) \) in (1.10), which are now dynamical fields, subject to the equivalence relation, \( x^\mu \sim x^\mu + \delta x^\mu \), defined by the gauge transformation (1.11), coordinatize the quotient space \( G/H = Diff_F(M) \). As usual, \( y^\mu \) are vacuum expectation values of \( x^\mu \) specifying the background (1.7) and \( \hat{A}_\mu(y) \) are fluctuating (dynamical) coordinates (fields). In this context, the gravitational fields \( e_\mu^\alpha(y) \) or \( h_{\mu\nu}(y) \) correspond to the Goldstone bosons for the spontaneous symmetry breaking (1.7). This is a rough picture showing how gravity can emerge from NC electromagnetism.

So far we are mostly confined to semi-classical limit, say \( \mathcal{O}(\hbar) \) in Eq.(1.2). The semi-classical means here slowly varying fields, \( \sqrt{\kappa |\partial F|} \ll 1 \), in the sense keeping field strengths (without restriction on their size) but not their derivatives. We will consider derivative corrections in the coming sections. This paper is organized as follows.
In section 2, we will revisit the equivalence between ordinary and NC DBI actions shown by Seiberg and Witten \[11\]. We will show that the exact Seiberg-Witten (SW) map in Eqs. (1.12) and (1.14) are a direct consequence of the equivalence after a simple change of variables between open and closed strings as was shown in \[22\, 23\]. This argument illuminates why higher order terms in Eq.(1.2) correspond to derivative corrections \(O(\sqrt{\kappa \partial F})\) in the DBI action (1.5). The leading four-derivative corrections were completely determined by Wyllard \[24\]. We will argue that the SW map with derivative corrections should be obtained from the Wyllard’s result by the same change of variables between open and closed string parameters. Since our main goal in this paper is to elucidate the relation between NC gauge theory and gravity, we will not explicitly check the identities so naturally emerging from well-established relations. Rather they could be regarded as our predictions. According to the correspondence between NC gauge theory and gravity, it is natural to expect that the derivative corrections give rise to higher order gravity, e.g., \(R^2\) gravity.

In section 3, we will newly derive the SW map for the derivative corrections in the context of deformation quantization \[25\, 26\]. The deformation quantization provides a noble approach to reify the Darboux theorem beyond the semiclassical, i.e. \(O(\theta)\), limit. For example, the SW maps, Eq.(1.12) and Eq.(1.14), result from the equivalence in the \(O(\theta)\) approximation between the star products \(*_{\omega'}\) and \(*_{\omega}\) defined by the symplectic forms \(\omega' = B + F(x)\) and \(\omega = B\), respectively \[17\, 27\]. In a seminal paper, M. Kontsevich proved \[26\] that every finite-dimensional Poisson manifold \(M\) admits a canonical deformation quantization. Furthermore he proved that, changing coordinates in a star product, one obtains another gauge equivalent star product. This was explicitly checked in \[28\] by making an arbitrary change of coordinates, \(y^\mu \mapsto x^\mu(y)\), in the Moyal \(\star\)-product (1.2) and obtaining the deformation quantization formula up to the third order. This result is consistent with the SW map in section 2 about derivative corrections. After inspecting the basic principle of deformation quantization, we put forward a conjecture that the emergent gravity from NC electromagnetism corresponds to a nonlinear realization \(G/H\) of the diffeomorphism group or more generally its NC deformation, so meeting a framework of NC gravity \[29\, 30\]. (See also a review \[5\].)

In section 4, we will explore the equivalence between NC \(U(1)\) instantons and gravitational instantons found in \[2\, 3\] to illustrate the correspondence of NC gauge theory with gravity. The emergent gravity reveals a remarkable feature that self-dual NC electromagnetism nicely fits with the twistor space describing curved self-dual spacetime \[31\, 32\]. This construction, which closely follows the results on \(N = 2\) strings \[33\, 34\], will also clarify how the deformation of symplectic (or Kähler) structure on \(\mathbb{R}^4\) due to the fluctuation of gauge fields appears as that of complex structure of the twistor space \[1\]. We observe that our construction is remarkably in parallel with topological D-branes on NC manifolds \[35\], suggesting a possible connection with the generalized complex geometry \[15\].

In section 5, we will generalize the equivalence in section 4 using the background independent formulation of NC gauge theories \[11\, 21\] and show that self-dual electromagnetism in NC spacetime is equivalent to self-dual Einstein gravity \[1\]. This section will also serve to uncover many details in \[1\]. In the course of the construction, it becomes obvious that a framework of NC gravity is in general
needed in order to incorporate the full quantum deformation of diffeomorphism symmetry. We will also discuss in detail the twistor space structure inherent in the self-dual NC electromagnetism.

Finally, in section 6, we will raise several open issues in the emergent gravity from NC spacetime and speculate possible implications for the correspondence between NC gauge theory and gravity.

2 Derivative Corrections and Exact SW Map

We revisit here the equivalence between NC and ordinary gauge theories discussed in [11]. First let us briefly recall how NC gauge theory arises in string theory. The coupling of an open string attached on a $D_p$-brane to massless backgrounds is described by a sigma model of the form

$$S = \frac{1}{2\kappa} \int_{\Sigma} d^2\sigma (g_{\mu\nu}(x)\partial_\sigma x^\mu \partial^\sigma x^\nu - i\kappa \varepsilon^{ab}B_{\mu\nu}(x)\partial_\sigma x^\mu \partial_\tau x^\nu) - i \int_{\partial\Sigma} d\tau A_\mu(x)\partial_\tau x^\mu, \quad (2.1)$$

where string worldsheet $\Sigma$ is the upper half plane parameterized by $-\infty \leq \tau \leq \infty$ and $0 \leq \sigma \leq \pi$ and $\partial\Sigma$ is its boundary. The $\Lambda$-symmetry (1.6), which underlies the emergent gravity, is obvious by rewriting relevant terms into form language, $\int_\Sigma B + \int_{\partial\Sigma} A$, as a simple application of Stokes’ theorem.

We leave the geometry of closed string backgrounds fixed and concentrate, instead, on the dynamics of open string sector. To be specific, we consider flat spacetime with the constant Neveu-Schwarz $B$-field. Here we regard the last term in Eq.(2.1) as an interaction with background gauge fields and define the propagator in terms of free fields $y^\mu(\tau, \sigma)$ subject to the boundary conditions

$$g_{\mu\nu}\partial_\sigma y^\nu + i\kappa B_{\mu\nu}\partial_\tau y^\nu|_{\partial\Sigma} = 0, \quad (2.2)$$

where the worldsheet fields $x^\mu(\tau, \sigma)$ were simply renamed $y^\mu(\tau, \sigma)$ to compare them with another free fields satisfying different boundary conditions, e.g., $g_{\mu\nu}\partial_\sigma x^\nu|_{\partial\Sigma} = 0$. The propagator evaluated at boundary points [11] is

$$\langle y^\mu(\tau)y^\nu(\tau') \rangle = -\frac{\kappa}{2\pi} \left( \frac{1}{G} \right)^{\mu\nu} \log(\tau - \tau')^2 + \frac{i}{2} \theta^{\mu\nu} \epsilon(\tau - \tau') \quad (2.3)$$

where $\epsilon(\tau)$ is the step function. Here

$$G_{\mu\nu} = g_{\mu\nu} - \kappa^2 (Bg^{-1}B)_{\mu\nu}, \quad (2.4)$$

$$\left( \frac{1}{G} \right)^{\mu\nu} = \left( \frac{1}{g + \kappa B} g \frac{1}{g - \kappa B} \right)^{\mu\nu}, \quad (2.5)$$

$$\theta^{\mu\nu} = -\kappa^2 \left( \frac{1}{g + \kappa B} B \frac{1}{g - \kappa B} \right)^{\mu\nu}. \quad (2.6)$$

They are related via the following relation

$$\frac{1}{G} + \frac{\theta}{\kappa} = \frac{1}{g + \kappa B}. \quad (2.7)$$
The metric $G_{\mu\nu}$ has a simple interpretation as an effective metric seen by open strings while $g_{\mu\nu}$ is the closed string metric. Furthermore the parameter $\theta^{\mu\nu}$ can be interpreted as the noncommutativity in a space where embedding coordinates on a $Dp$-brane describe the NC coordinates \( \text{(1.3)} \).

For a moment we will work in the approximation of slowly varying fields relative to the string scale, in the sense of neglecting derivative terms, i.e., $\sqrt{\kappa} \frac{\partial F}{\partial r} \ll 1$, but of no restriction on the size of field strengths. Nevertheless, the field strengths $F$ need not be constant. Indeed the field strength can vary rapidly in the sense of low energy field theory as long as a typical length scale of the varying $F$ is much larger than the string scale. In this limit the open string effective action on a D-brane is given by the DBI action \( \text{(2.8)} \). Seiberg and Witten, however, showed \( \text{(11)} \) that an explicit form of the effective action depends on the regularization scheme of two dimensional field theory defined by the worldsheet action \( \text{(2.1)} \). The difference due to different regularizations is always in a choice of contact terms, leading to the redefinition of coupling constants which are spacetime fields. So theories defined with different regularizations are related each other by the field redefinitions in spacetime.

The usual infinities in quantum field theory also arise in the worldsheet path integral defined by the action \( \text{(2.1)} \) and the theory has to be regularized. Using the propagator \( \text{(2.3)} \) with a point-splitting regularization \( \text{(11)} \) where different operators are never at the same point, the spacetime effective action is expressed in terms of NC gauge fields and has the NC gauge symmetry on the NC spacetime \( \text{(1.3)} \).

In this description, the analog of Eq.(1.5) is

$$\hat{S}(G_s, G, \hat{A}, \theta) = \frac{2\pi}{G_s (2\pi \kappa)^{p+1}} \int d^{p+1}y \sqrt{\det(G + \kappa \hat{F})} + O(\sqrt{\kappa} \hat{D} \hat{F}). \quad (2.8)$$

The action depends only on the open string variables $G_{\mu\nu}, \theta_{\mu\nu}$ and $G_s$, where the $\theta$-dependence is entirely in the $\star$-product in the NC field strength

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i [\hat{A}_\mu, \hat{A}_\nu]_\star. \quad (2.9)$$

The DBI action \( \text{(2.8)} \) is invariant under

$$\hat{\delta}_\chi \hat{A}_\mu = \hat{D}_\mu \hat{\lambda} = \partial_\mu \hat{\lambda} - i [\hat{A}_\mu, \hat{\lambda}]_\star. \quad (2.10)$$

The NC field strength \( \text{(2.9)} \) and the NC gauge transformation \( \text{(2.10)} \) are the quantum version of Eq.(1.13) and Eq.(1.11), respectively, in the sense of Eq.(1.1).

Since the sigma model \( \text{(2.1)} \) respects the $\Lambda$-symmetry \( \text{(1.6)} \), one can absorb the constant $B$-field completely into gauge fields by choosing the gauge parameter $\Lambda_\mu = -\frac{1}{2} B_{\mu\nu} x^\nu$. The worldsheet action is then given by

$$S = \frac{1}{2\kappa} \int_{\Sigma} d^2 \sigma g_{\mu\nu} \partial_\alpha x^\mu \partial^\alpha x^\nu + i \int_{\partial \Sigma} d\tau (A_\mu(x) - \frac{1}{2} B_{\mu\nu} x^\nu) \partial_\tau x^\mu. \quad (2.11)$$

Now we regard the second part as the boundary interaction and define the propagator with the first part with the boundary condition $g_{\mu\nu} \partial_\sigma x^\nu |_{\partial \Sigma} = 0$, resulting in the usual Neumann propagator.
The sigma model path integral using the Neumann propagator with Pauli-Villars regularization, for example, preserves the ordinary gauge symmetry of open string gauge fields \cite{11}. In this case, the spacetime low energy effective action on a single D$p$-brane, which is denoted as $S(g_s, g, A, B)$ to emphasize the background dependence, is given by the DBI action (1.5). Note that the effective action is now expressed in terms of closed string variables $g_{\mu \nu}, B_{\mu \nu}$ and $g_s$.

Since the commutative and NC descriptions arise from the same open string theory depending on different regularizations and the physics should not depend on the regularization scheme, one may expect that

$$\hat{S}(G_s, G, \hat{A}, \theta) = S(g_s, g, A, B) + O(\sqrt{\kappa} \partial F). \quad (2.12)$$

If so, the two descriptions should be related each other by a spacetime field redefinition. Indeed, Seiberg and Witten showed the identity (2.12) and also found the transformation, the so-called SW map, between ordinary and NC gauge fields in such a way that preserves the gauge equivalence relation of ordinary and NC gauge symmetries \cite{11}. The equivalence (2.12) can also be understood as resulting from different path integral prescriptions \cite{36,37} based on the $\Lambda$-symmetry as we discussed above. First of all, the equivalence (2.12) determines the relation between open and closed string coupling constants from the fact that for $F = \hat{F} = 0$ the constant terms in the actions using the two set of variables are the same:

$$G_s = g_s \sqrt{\frac{\det G}{\det (g + \kappa B)}}. \quad (2.13)$$

As was explained in \cite{11}, there is a general description with an arbitrary $\theta$ associated with a suitable regularization that interpolates between Pauli-Villars and point-splitting. This freedom is basically coming from the $\Lambda$-symmetry that imposes the gauge invariant combination of $B$ and $F$ in the open string theory as $F = B + F$. Thus there is a symmetry of shift in $B$ keeping $B + F$ fixed. Given such a symmetry, we may split the $B$-field into two parts and put one in the kinetic part and the rest in the boundary interaction part. By taking the background to be $B$ or $B'$, we should get a NC description with appropriate $\theta$ or $\theta'$ and different $\hat{F}$’s. Hence we can write down a differential equation that describes how $\hat{A}(\theta)$ and $\hat{F}(\theta)$ should change, when $\theta$ is varied, to describe equivalent physics \cite{11}:

$$\delta \hat{A}_\mu(\theta) = -\frac{1}{4} \delta \theta^{\alpha \beta} \left( \hat{A}_\alpha * (\partial_\beta \hat{A}_\mu + \hat{F}_\beta_\mu) + (\partial_\beta \hat{A}_\mu + \hat{F}_\beta_\mu) * \hat{A}_\alpha \right), \quad (2.14)$$

$$\delta \hat{F}_{\mu \nu}(\theta) = \frac{1}{4} \delta \theta^{\alpha \beta} \left( 2 \hat{F}_{\mu \alpha} * \hat{F}_{\nu \beta} + 2 \hat{F}_{\nu \beta} * \hat{F}_{\mu \alpha} - \hat{A}_\alpha * (\hat{D}_\beta \hat{F}_{\mu \nu} + \partial_\beta \hat{F}_{\mu \nu}) \right. \nonumber$$

$$\left. - (\hat{D}_\beta \hat{F}_{\mu \nu} + \partial_\beta \hat{F}_{\mu \nu}) * \hat{A}_\alpha \right). \quad (2.15)$$

An exact solution of the differential equation (2.15) in the Abelian case was found in \cite{27,38}.

The freedom in the description just explained above is parameterized by a two-form $\Phi$ from the viewpoint of NC geometry on D-brane worldvolume. The change of variables for the general case is
given by

\[
\frac{1}{G + \kappa \Phi} + \frac{\theta}{\kappa} = \frac{1}{g + \kappa B}.
\] (2.16)

\[
G_s = g_s \sqrt{\frac{\det(G + \kappa \Phi)}{\det(g + \kappa B)}}.
\] (2.17)

The effective action with these variables are modified to

\[
\hat{S}_\Phi(G_s, G, \hat{A}, \theta) = \frac{2\pi}{G_s(2\pi\kappa)^{\frac{d+1}{2}}} \int d^{p+1}y \sqrt{\det(G + \kappa(\hat{F} + \Phi))}.
\] (2.18)

For every background characterized by \(B, g_{\mu\nu}\) and \(g_s\), we thus have a continuum of descriptions labeled by a choice of \(\Phi\). So we end up with the most general form of the equivalence for slowly varying fields, i.e., \(\sqrt{\kappa} |\frac{\partial F}{\partial F}| \ll 1:\)

\[
\hat{S}_\Phi(G_s, G, \hat{A}, \theta) = S(g_s, g, A, B) + \mathcal{O}(\sqrt{\kappa} \partial F),
\] (2.19)

which was proved in [11] using the differential equation (2.15) and the change of variables, (2.16) and (2.17).

The above change of variables between open and closed string parameters is independent of dynamical gauge fields and so one can freely use them independently of local dynamics to express two different descriptions with the same string variables [22]. For example, we get from Eq. (2.16)

\[
G \equiv g + \kappa(B + F) = (1 + F\theta) \left(G + \kappa(\Phi + F)\right) \frac{1}{1 + \frac{2}{\kappa}(G + \kappa \Phi)}
\] (2.20)

where

\[
F(x) = \left(\frac{1}{1 + F\theta} F\right) (x).
\] (2.21)

The equivalence (2.19), using the identity (2.20), immediately leads to the dual description of the NC DBI action via the exact SW map [22, 23]

\[
\int d^{p+1}y \sqrt{\det(G + \kappa(\Phi + \hat{F}))} = \int d^{p+1}x \sqrt{\det(1 + F\theta)} \sqrt{\det(G + \kappa(\Phi + F))} + \mathcal{O}(\sqrt{\kappa} \partial F).
\] (2.22)

Note that the commutative action in Eq. (2.22) is exactly the same as the DBI action obtained from the worldsheet sigma model using \(\zeta\)-function regularization scheme [37]. The equivalence (2.22) was also proved in [39] in the framework of deformation quantization.

One can expand both sides of Eq. (2.22) in powers of \(\kappa\). \(\mathcal{O}(1)\) implies that there is a measure change between NC and commutative descriptions, which is exactly the same as Eq. (1.14). In other words, the coordinate transformation, \(y^\mu \rightarrow x^\mu(y)\), between commutative and NC descriptions depends on the dynamical gauge fields. Since the identity (2.22) must be true for arbitrary small \(\kappa\).
substituting Eq. (1.14) into Eq. (2.22) leads to the relation (1.12), but now for the NC field strength (2.9). We thus see that the embedding coordinates \( x^\mu(y) \) are always defined by (1.10) independently of the choice \( \Phi \). This is consistent with the fact [21] that the covariant coordinates \( x^\mu(y) \) are background independent.

So far we have ignored derivative terms containing \( \partial^\mu F \). However the left hand side of Eq. (2.22) contains infinitely many derivatives from the star commutator in \( \hat{F} \) which have to generate such derivative terms, though we had taken the ordinary product neglecting a potential NC ordering. So we need to carefully look into the identity (2.22) to what extent the equivalence holds. In fact it is easily inferred from the SW map in [40] that the left hand side of Eq. (2.22) contains infinitely many higher order derivative terms. The derivative corrections are coming from \( F_{\mu\nu}^{(n,m)} \) for \( n < m \) with the notation in [40] (see the figure 1 and section 3.2). This can also be inferred from the previous argument related to the SW map (1.12) which does not incorporate any derivative corrections and precisely corresponds to \( F_{\mu\nu}^{(n,n)} \) in [40]. As we discussed there, Eq. (1.12) is the SW map for the semi-classical field strength (1.13) and the DBI action (1.8) is equivalent to the semi-classical DBI action [8] where field strengths are given by (1.13) rather than (2.9). It is thus obvious that the equivalence (2.22) is still true with the field strength (1.13) in the approximation of slowly varying fields.

So there must be more terms with derivative corrections on the right hand side of Eq. (2.22) if one insists to keep the NC field strength (2.9). To find the derivative corrections systematically, however, one has to notice that there are another sources giving rise to them. The NC description has two dimensionful parameters, \( \theta \) and \( \kappa \), which control derivative corrections. The parameter \( \kappa \) takes into account stringy effects coming from massive modes in worldsheet conformal field theory, as indicated in Eq. (2.8), while the noncommutativity parameter \( \theta \) does the effects of NC spacetime in worldvolume field theory. Which one becomes more important depends on a scale we are probing.

We are interested in the Seiberg-Witten limit [11], \( \kappa \to 0 \), keeping all open string variables fixed. In this limit, \(|\theta|^2 \equiv G_{\mu\lambda}G_{\nu\sigma}\theta^{\mu\nu}\theta^{\lambda\sigma} = \kappa^2|\kappa Bg^{-1}|^2 \gg \kappa^2\), using the metric \( G_{\mu\nu} \) in the background independent scheme, i.e., \( \Phi = -B \) in Eq. (2.16). This implies that the noncommutativity effects in the SW limit are predominant compared to stringy effects. So we will neglect the stringy effects such as \( \mathcal{O}(\sqrt{\kappa\hat{D}\hat{F}}) \) in Eq. (2.8) [41]. But we have to keep \( \kappa\hat{F} \) since \( \hat{F} \) could be arbitrarily large. The stringy corrections in NC gauge theory have been discussed in several papers [42] based on the SW equivalence between ordinary and NC gauge theories.

An ordering problem in NC spacetime has to be taken into account. A unique feature is that translations in NC directions are basically gauge transformations, i.e., \( e^{ik\cdot y}f(y)e^{-ik\cdot y} = f(y+k\cdot \theta) \). This immediately implies that there are no local gauge-invariant observables in NC gauge theory [43]. It turns out that NC gauge theories allow a new type of gauge invariant objects, the so-called open Wilson lines, which are localized in momentum space [44]. Attaching local operators which transform in the adjoint representation of gauge transformations to an open Wilson line also yields gauge invariant operators [43]. For example, the NC DBI action carrying a definite momentum \( k \) is
given by
\[ \widehat{S}_k^\Phi(G_s, G, \widehat{A}, \theta) = \frac{2\pi}{G_s(2\pi\kappa)^{n+1}} \int d^{n+1}y \sqrt{\det(G + \kappa(\widehat{F} + \Phi))} W(y, C_k) \ast e^{ik \cdot y}, \] (2.23)
where \( W(y, C_k) \) is a straight Wilson line with momentum \( k \) with path \( C_k \) and \( L_\ast \) is defined as smearing the operators along the Wilson line and taking the path ordering with respect to \( \ast \)-product. We refer [27] for more informations useful for Eq.(2.23). The DBI action (2.18) corresponds to \( \widehat{S}_k^\Phi = 0 \) without regard to the NC ordering.

Let us now turn to the commutative description. Unlike the NC case, there is only one dimensionful parameter, \( \kappa \), to control derivative corrections. So the derivative corrections due to \( \theta \) and \( \kappa \) in NC gauge theory all appear as stringy corrections from the viewpoint of commutative description and they are intricately entangled. The derivative correction to the DBI action (1.5) has been calculated by Wyllard [24] using the boundary state formalism and it is given by
\[ S_W(g_s, g, A, B) = \frac{2\pi}{g_s(2\pi\kappa)^{n+1}} \int d^{n+1}x \sqrt{\det G} \left( 1 + \frac{\kappa^4}{96} (-G^\mu_1^\nu_1 G^\mu_2^\nu_3 G^\rho_4^\rho_1 G^\rho_2^\rho_3 S^\rho_1^\rho_2^\rho_3^\rho_4) + \cdots \right), \] (2.24)
where \( G_{\mu\nu} \) is a non-symmetric metric defined by (2.20) and \( \tilde{G}^{\mu\nu} \) is its inverse, i.e., \( G^{\mu\lambda} G_{\lambda\nu} = \delta^\mu_\nu \). The tensor
\[ S_{\rho_1 \rho_2 \mu_1 \mu_2} = \partial_{\rho_1} \partial_{\rho_2} F_{\mu_1 \mu_2} + \kappa G^{\nu_1 \nu_2} \left( \partial_{\rho_1} F_{\nu_1 \nu_1} \partial_{\rho_2} F_{\mu_2 \nu_2} - \partial_{\rho_1} F_{\nu_2 \nu_1} \partial_{\rho_2} F_{\mu_1 \nu_2} \right) \] (2.25)
may be interpreted as the Riemann tensor for the nonsymmetric metric \( G_{\mu\nu} \) and \( S_{\rho_1 \rho_2} = G^{\mu_1 \mu_2} S^\rho_1^\rho_2^\rho_1^\rho_2 \) as the Ricci tensor [24].

As we reasoned before, the SW equivalence between ordinary and NC gauge theories has to be general regardless of a specific limit under our consideration. So this should be the case even after including derivative corrections in ordinary and NC theories. Let us denote these corrections by \( \Delta S_{DBI} \) and \( \Delta \widehat{S}_{DBI} \), respectively. The SW equivalence in general means that
\[ S_{DBI} + \Delta S_{DBI} = \widehat{S}_{DBI} + \Delta \widehat{S}_{DBI}. \] (2.26)
We already argued that we will neglect the NC correction \( \Delta \widehat{S}_{DBI} \) in the SW limit. We will discuss later to what extent we can do it. The equivalence (2.26) in this limit then reduces to [41]
\[ S_{DBI}|_{SW} + \Delta S_{DBI}|_{SW} = \widehat{S}_{DBI}|_{SW}. \] (2.27)
Recall that the exact SW map (2.22) was obtained by the equivalence (2.19) with the simple change of variables between open and closed string parameters defined by Eqs.(2.16) and (2.17). This change of variables should be true even with derivative corrections since they are independent of local dynamics. As illustrated in Eq.(2.22), the description of both DBI actions in terms of the same string variables has provided a great simplification to identify SW maps. Thus we will equally use the
open string variables for the derivative corrections in Eq. \((2.24)\), where the metric \(G_{\mu\nu}\) will be replaced by Eq. \((2.20)\). Since we are commonly using the open string variables for both descriptions, the SW limit in \((2.27)\) simply means the zero slope limit, i.e. \(\kappa \to 0\). So it is straightforward to extract the SW maps with derivative corrections by expanding both sides of Eq. \((2.27)\) in powers of \(\kappa\).

Though the general case with \(\Phi\) does not introduce any complication, we will work in the background independent scheme where \(\Phi = -B = -1/\theta\), for definiteness. In this case, the metric \(G_{\mu\nu}\) has a simple expression

\[
G_{\mu\nu} = \kappa \left( g^{\theta^{-1}} - \frac{\kappa}{\theta G \theta} \right)_{\mu\nu}, \quad G^{\mu\nu} = \frac{1}{\kappa} \left( \theta g^{-1} \left( 1 - \frac{\kappa}{g G \theta} \right)^{-1} \right)_{\mu\nu},
\]

where we have introduced an effective metric \(g_{\mu\nu}\) induced by dynamical gauge fields

\[
g_{\mu\nu} = \delta_{\mu\nu} + (F\theta)_{\mu\nu}, \quad (g^{-1})^{\mu\nu} \equiv g^{\mu\nu} = \left( \frac{1}{1 + F\theta} \right)^{\mu\nu},
\]

which will play a role in our later discussions. (Unfortunately we have abused many metrics, \((g, G, h, G, g)\). We hope it does not cause many confusions in discriminating them.) Noting that \(g^{\theta^{-1}} = B + F\), Eq.\((1.4)\) implies that the effective metric \(g_{\mu\nu}\) is not independent of the induced metric \(h_{\mu\nu}\) in Eq.\((1.9)\) but related as follows:

\[
h_{\mu\nu}(y) = e^{\alpha}(y) e^{\beta}(y) g_{\alpha\beta}, \quad (\theta g^{-1})^{\alpha\beta}(y) = \theta^{\mu\nu} e^{\alpha}(y) e^{\beta}(y).
\]

Identifying \(e^{\alpha}(y) \equiv \partial x^{\alpha}/\partial y^{\mu}\) with vielbeins on some emergent manifold \(M\), it is suggestive that the Darboux theorem can be interpreted as the equivalence principle in symplectic geometry.

Let us start with \(O(1)\) terms from both sides in Eq.\((2.27)\). Note that the factor \(\kappa^4\) in front of the derivative corrections in Eq.\((2.24)\) is precisely cancelled by the factors from the metric \(G^{-1}\) in Eq.\((2.28)\), and thus they already give rise to the \(O(1)\) contribution. Taking this into account, we get the following SW map\(^4\)

\[
\int d^{p+1}y L \left[ W(y, C_k) \right] * e^{ik}\gamma = \int d^{p+1}x \sqrt{\det g} \left( 1 - \frac{1}{96} g^{\mu_1 \rho_1} g^{\mu_2 \rho_2} (g^{-1})^{\rho_1 \rho_2} (g^{-1})^{\rho_2 \rho_3} S_{\rho_1 \rho_2 \rho_3 \rho_4} + \frac{1}{192} (g^{-1})^{\rho_1 \rho_2} (g^{-1})^{\rho_2 \rho_3} S_{\rho_1 \rho_2} S_{\rho_3 \rho_4} + \cdots \right),
\]

where

\[
S_{\rho_1 \rho_2 \rho_3 \rho_4} = \partial_{\rho_1} \partial_{\rho_2} g_{\rho_3 \rho_4} - g^{\mu \nu} \left( \partial_{\rho_1} g_{\mu \nu_1} \partial_{\rho_2} g_{\nu_2 \mu_2} + \partial_{\rho_2} g_{\mu \nu_1} \partial_{\rho_1} g_{\nu_2 \mu_2} \right)
\]

and \(S_{\rho_1 \rho_2} \equiv (g^{-1})^{\mu_1 \mu_2} S_{\rho_1 \rho_2 \mu_1 \mu_2}\). Note that \(S_{\rho_1 \rho_2 \mu_1 \mu_2}\) and \(S_{\rho_1 \rho_2}\) are symmetric with respect to \(\rho_1 \leftrightarrow \rho_2\). This map constitutes a generalization of the previous measure transformation \((1.14)\). Our result is an

\(^4\)Although we use the momentum space representation for the manifest gauge invariance, the actual comparison with the commutative description is understood to be made in coordinate space using the formula \((2.16)\) in \([27]\).
the approximation (2.27) up to corrections, namely, the form invariance \[49\]. It is then obvious that the leading correction, \(\Delta \hat{S}_{DBI}\), in the NC gauge theory starts with \(O(\kappa^4)\) as the commutative one. So we can safely believe the approximation (2.27) up to \(O(\kappa^3)\). Beyond that, we have to take into account \(\Delta \hat{S}_{DBI}\) \[42\]. In order to find higher order SW map, it is thus enough to expand the metric \(G^{-1}\) in powers of \(\kappa\):

\[
G^{\mu\nu} = \left( \frac{1}{\kappa} \theta g^{-1} + \frac{1}{\kappa} \frac{1}{g G^t \theta} + \ldots \right)^{\mu\nu}
\]

with \(g^t = (1 + \theta F)\). We keep \(\theta\) in the second term without cancelation with the denominator since it will be combined with \(F_{\mu\nu}\) in the Riemann tensor to make \(g_{\mu\nu}\). After straightforward calculation, we get

\[
\int d^{d+1}y L_* \left[ \text{Tr} \, G^{-1} \hat{F}(y) W(y, C_k) \right] \ast e^{ik \cdot y} = \int d^{d+1}x \sqrt{\text{det}g} \left[ \text{Tr} \, G^{-1}(g^{-1} F) \left( 1 - \frac{1}{96} g^{\mu_1 \mu_2 \rho_1 \rho_2} (\theta g^{-1})^{\rho_2 \rho_1} (\theta g^{-1})^{\rho_2 \rho_3} S_{\rho_1 \rho_2 \mu_1 \mu_2} S_{\rho_3 \rho_4 \mu_3 \mu_4} \right) + \frac{1}{192} (\theta g^{-1})^{\rho_4 \rho_3} (\theta g^{-1})^{\rho_2 \rho_3} S_{\rho_1 \rho_2 S_{\rho_3 \rho_4}} \right.
\]

\[
\left. - \frac{1}{24} g^{\mu_2 \mu_3} (\theta g^{-1})^{\rho_2 \rho_3} \left\{ \left( \frac{1}{g G^t \theta} \right)^{\mu_2 \mu_3} (\theta g^{-1})^{\rho_2 \rho_4} + g^{\mu_4 \mu_1} \left( \frac{1}{g G^t \theta} \right)^{\rho_4 \rho_1} \right\} S_{\rho_1 \rho_2 \mu_1 \mu_2} S_{\rho_3 \rho_4} + \frac{1}{48} (\theta g^{-1})^{\rho_2 \rho_3} \left\{ \left( \frac{1}{g G^t \theta} \right)^{\mu_2 \mu_1} (\theta g^{-1})^{\rho_2 \rho_4} + g^{\mu_4 \mu_1} \left( \frac{1}{g G^t \theta} \right)^{\rho_4 \rho_1} \right\} S_{\rho_1 \rho_2 \mu_1 \mu_2} S_{\rho_3 \rho_4} + \frac{1}{24} g^{\mu_2 \mu_3} (\theta g^{-1})^{\rho_2 \rho_3} \left( \frac{1}{g G^t \theta} \right)^{\nu_1 \nu_2} \left( \partial_{\rho_1} g_{\mu_1 \nu_1} \partial_{\rho_2} g_{\nu_2 \mu_2} - \partial_{\rho_2} g_{\mu_1 \nu_1} \partial_{\rho_1} g_{\nu_2 \mu_2} \right) S_{\rho_3 \rho_4 \mu_3 \mu_4} \right.
\]

\[
\left. - \frac{1}{48} g^{\mu_2 \mu_1} (\theta g^{-1})^{\rho_2 \rho_3} \left( \frac{1}{g G^t \theta} \right)^{\nu_1 \nu_2} \left( \partial_{\rho_1} g_{\mu_1 \nu_1} \partial_{\rho_2} g_{\nu_2 \mu_2} - \partial_{\rho_2} g_{\mu_1 \nu_1} \partial_{\rho_1} g_{\nu_2 \mu_2} \right) S_{\rho_3 \rho_4} \right].
\]

---

5 Here we would like to put forward an interesting observation. It was well-known \[47\] that the leading derivative corrections in bosonic string theory start with two-derivatives, whose exact result including all orders in \(F\) was recently obtained in \[48\]. Thus, if we were adopted the bosonic result with an assumption that the NC part (2.23) were common for bosonic string and superstring theories (that would be wrong), we would definitely be on a wrong way. So the perfect agreement in the identity (2.31) is quite surprising since Eq. (2.23) already singles out the superstring result prior to the bosonic one, though that was \textit{a priori} not clear. It was also shown in \[46\] that the result in \[48\] for the bosonic string is not invariant under the SW map. All these seem to imply that the bosonic string needs to incorporate an effect of tachyons from the outset both in commutative and in NC descriptions, as suggested in \[46\]. See Mukhi and Suryanarayana in \[38\] for a relevant discussion.
We see that the left hand side (and also the first term on the right hand side) of Eq.(2.34) identically vanishes since $\text{Tr} G^{-1} \hat{F} = 0$. Thus the identity (2.34) implies that the right hand side must be a total derivative. We will not check it but leave it as our prediction. We note that the commutative description in terms of open string variables can be solely expressed in terms of $g_{\mu\nu}$ (after rewriting $g^{-1} F = (1 - g^{-1}) \theta^{-1}$).

More important consequence of Eq.(2.34) is the following. Let us take the metric $G^{\mu\nu}$ out from the integration on both sides. Since Eq.(2.34) is an identity valid for any arbitrary $G^{\mu\nu}$, the coefficients of $G^{\mu\nu}$ must be equal too. Then the left hand side has the form

$$
\int d^{p+1}y L_{s} \left[ \hat{F}_{\mu\nu}(y) W(y, C_k) \right] * e^{ik \cdot y}.
$$

(2.35)

We can thus derive the exact SW map of Eq.(2.35) from the coefficient of $G^{\mu\nu}$ on the right hand side of Eq.(2.34), up to fourth-order derivative. We will not give the explicit form since it is rather lengthy but directly readable from Eq.(2.34). This map has to correspond to the inverse of the exact (inverse) SW map

$$
F_{\mu\nu}(k) = \int d^{p+1}y L_{s} \left[ \sqrt{\det (1 - \theta \hat{F})} \left( \frac{1}{1 - \hat{F} \theta} \hat{F} \right) (y) W(y, C_k) \right] * e^{ik \cdot y}
$$

(2.36)

which was conjectured in [27] and immediately proved in [38].

As we argued above, we can continue this procedure using the expansion (2.33) up to $O(\kappa^3)$ without including $\Delta S_{DBI}$. At each step, we get exact SW maps including all powers of gauge fields and $\theta_{\mu\nu}$. Up to our best knowledge, this was never achieved even for the $O(1)$ result (2.31). (But see [39] for a formal solution based on the Kontsevich’s formality map.) Such a great simplification is due to the use of the same string variables using the formula (2.20), originally suggested in [22]. So let us ponder upon possible sources to ruin the conversion relations (2.16) and (2.17). If quantum corrections are included, the effect of renormalization group flow of coupling constants might be incorporated into Eq.(2.17). But this is only true for asymmetric running of a dilaton field in commutative and NC theories, which seems not to be the case. Another source may be a possibility that gauge field dynamics modifies either $g_{\mu\nu}, G^{\mu\nu}$ or $\theta_{\mu\nu}$, themselves. As was explained in the previous section and will be shown later, the dynamics of gauge fields induces the deformation of background geometry, but this kind of modification is entirely encoded in $g_{\mu\nu}$ or $h_{\mu\nu}$, as indicated in Eq.(2.30). Then the variables in Eqs.(2.16) and (2.17) in general appear as non-dynamical parameters. Thus the change of variable (2.20) seems to be quite general independently of gauge field dynamics. If this is so, we may go much further using the conjectured higher-order derivative corrections in [46].
3 Deformation Quantization and Emergent Geometry

In classical mechanics, the set of possible states of a system forms a Poisson manifold. The observables that we want to measure are the smooth functions in $C^\infty(M)$, forming a commutative (Poisson) algebra. In quantum mechanics, the set of possible states is a projective Hilbert space $\mathcal{H}$. The observables are self-adjoint operators, forming a NC C*-algebra. The change from a Poisson manifold to a Hilbert space is a pretty big one.

A natural question is whether the quantization such as Eq. (1.1) for general Poisson manifolds is always possible with a radical change in the nature of the observables. The problem is how to construct the Hilbert space for a general Poisson manifold, which is in general highly nontrivial. Deformation quantization was proposed in [25] as an alternative, where the quantization is understood as a deformation of the algebra of classical observables. Instead of building a Hilbert space from a Poisson manifold and associating an algebra of operators to it, we are only concerned with the algebra $A$ to deform the commutative product in $C^\infty(M)$ to a NC, associative product. In flat phase space such as the case we have considered up to now, it is easy to show that the two approaches have one to one correspondence (1.2) through the Weyl-Moyal map [7].

Recently M. Kontsevich answered the above question in the context of deformation quantization [26]. He proved that every finite-dimensional Poisson manifold $M$ admits a canonical deformation quantization and that changing coordinates leads to gauge equivalent star products. We briefly recapitulate his results which will be crucially used in our discussion.

Let $A$ be the algebra over $\mathbb{R}$ of smooth functions on a finite-dimensional $C^\infty$-manifold $M$. A star product on $A$ is an associative $\mathbb{R}[[h]]$–bilinear product on the algebra $A[[h]]$, a formal power series in $\hbar$ with coefficients in $C^\infty(M) = A$, given by the following formula for $f, g \in A \subset A[[h]]$:

$$ (f, g) \mapsto f \star g = fg + \hbar B_1(f, g) + \hbar^2 B_2(f, g) + \cdots \in A[[h]] $$

where $B_i(f, g)$ are bidifferential operators. There is a natural gauge group which acts on star products. This group consists of automorphisms of $A[[h]]$ considered as an $\mathbb{R}[[h]]$–module (i.e. linear transformations $A \to A$ parameterized by $h$). If $D(h) = 1 + \sum_{n\geq 1} h^n D_n$ is such an automorphism where $D_n : A \to A$ are differential operators, it acts on the set of star products as

$$ \star \to \star', \quad f(h) \star' g(h) = D(h) \left( D(h)^{-1}(f(h)) \star D(h)^{-1}(g(h)) \right) $$

---

6 A Poisson manifold is a differentiable manifold $M$ with skew-symmetric, contravariant 2-tensor (not necessarily nondegenerate) $\theta = \theta^{\mu\nu} \partial_\mu \wedge \partial_\nu \in \Lambda^2 TM$ such that $\{ f, g \} = \langle \theta, df \otimes dg \rangle = \theta^{\mu\nu} \partial_\mu f \partial_\nu g$ is a Poisson bracket, i.e., the bracket $\{ \cdot, \cdot \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a skew-symmetric bilinear map satisfying 1) Jacobi identity: $\{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = 0$ and 2) Leibniz rule: $\{ f, gh \} = g \{ f, h \} + \{ f, g \} h$, $\forall f, g, h \in C^\infty(M)$. Poisson manifolds appear as a natural generalization of symplectic manifolds where the Poisson structure reduces to a symplectic structure if $\theta$ is nongenerate.
for \( f(h), g(h) \in \mathcal{A}[[h]] \). This is evident from the commutativity of the diagram

\[
\begin{array}{ccc}
\mathcal{A}[[h]] \times \mathcal{A}[[h]] & \overset{*}{\longrightarrow} & \mathcal{A}[[h]] \\
D(h) \times D(h) \downarrow \quad & & \quad \downarrow D(h) \\
\mathcal{A}[[h]] \times \mathcal{A}[[h]] & \overset{*'}{\longrightarrow} & \mathcal{A}[[h]]
\end{array}
\]

We are interested in star products up to gauge equivalence. This equivalence relation is closely related to the cohomological Hochschild complex of algebra \( \mathcal{A} \) \([26]\), i.e. the algebra of smooth polyvector fields on \( M \). For example, it follows from the associativity of the product (3.1) that the symmetric part of \( B_1 \) can be killed by a gauge transformation which is a coboundary in the Hochschild complex, and that the antisymmetric part of \( B_1 \), denoted as \( B^{-}_1 \), comes from a bivector field \( \alpha \in \Gamma(M, \Lambda^2 TM) \) on \( M \):

\[
B^{-}_1(f, g) = \langle \alpha, df \otimes dg \rangle.
\] (3.3)

In fact, any Hochschild coboundary can be removed by a gauge transformation \( D(h) \), so leading to the gauge equivalent star product (3.2). The associativity at \( O(h^2) \) further constrains that \( \alpha \) must be a Poisson structure on \( M \), in other words, \([\alpha, \alpha]_{SN} = 0\), where the bracket is the Schouten-Nijenhuis bracket on polyvector fields (see \([26]\) for the definition of this bracket and the Hochschild cohomology). Thus, gauge equivalence classes of star products modulo \( O(h^2) \) are classified by Poisson structures on \( M \). It was shown \([26]\) that there are no other obstructions to deforming the algebra \( \mathcal{A} \) up to arbitrary higher orders in \( h \).

For an equivalence class of star products for any Poisson manifold, Kontsevich arrived at the following general results.

Theorem 1.1 in \([26]\): The set of gauge equivalence classes of star products on a smooth manifold \( M \) can be naturally identified with the set of equivalence classes of Poisson structures depending formally on \( h \)

\[
\alpha = \alpha(h) = \alpha_1 h + \alpha_2 h^2 + \cdots \in \Gamma(M, \Lambda^2 TM)[[h]], \quad [\alpha, \alpha]_{SN} = 0 \in \Gamma(M, \Lambda^3 TM)[[h]] \quad (3.4)
\]

modulo the action of the group of formal paths in the diffeomorphism group of \( M \), starting at the identity diffeomorphism.

Theorem 2.3 in \([26]\): Let \( \alpha \) be a Poisson bi-vector field in a domain of \( \mathbb{R}^d \). The formula

\[
f \ast g = \sum_{n=0}^{\infty} \hbar^n \sum_{\Gamma \in G_n} w_{\Gamma} B_{\Gamma, \alpha}(f, g) \quad (3.5)
\]

defines an associative product. If we change coordinates, we obtain a gauge equivalent star product.

The formula (3.5) has a natural interpretation in terms of Feynman diagrams for the path integral of a topological sigma model \([50]\).
The simplest example of a deformation quantization is the Moyal product \( \star \) for the Poisson structure on \( \mathbb{R}^d \) with constant coefficients \( \alpha^\mu\nu = i\theta^\mu\nu/2 \). If \( \alpha^\mu\nu \) are not constant, a global formula is not yet available but can be perturbatively computed by the prescription given in [26]. Up to the second order, this formula can be written as follows

\[
f \star g = fg + \hbar \alpha^{ab} \partial_a f \partial_b g + \frac{\hbar^2}{2} \alpha^{a_1b_1} \alpha^{a_2b_2} \partial_{a_1} \partial_{a_2} f \partial_{b_1} \partial_{b_2} g \]
\[
+ \frac{\hbar^2}{3} \alpha^{a_1b_1} \partial_{b_1} \alpha^{a_2b_2} (\partial_{a_1} \partial_{a_2} f \partial_{b_2} g + \partial_{a_1} \partial_{a_2} g \partial_{b_2} f) + O(\hbar^3). \tag{3.6}
\]

Now we are ready to promote the properties such as the Darboux theorem discussed in section 1 to the framework of deformation quantization. Since \( \omega \) and \( \omega' \) in Eq. (1.4) are related by diffeomorphisms, according to the Theorem 2.3, the two star products \( \star_\omega \) and \( \star_{\omega'} \) defined by the Poisson structures \( \omega^{-1} \) and \( \omega'^{-1} \), respectively, should be gauge equivalent. Conversely, if we make an arbitrary change of coordinates, \( y^\mu \mapsto x^\alpha(y) \), in the Moyal \( \star \)-product \((1.2)\), which is nothing but Kontsevich’s star product \((3.5)\), with the constant Poisson bi-vector, we get a new star product defined by a Poisson bi-vector \( \alpha(h) \). But the resulting star product has to be gauge equivalent to the Moyal product \((1.2)\) and \( \alpha(h) \) should be determined by the original Poisson bi-vector \( \theta^\mu\nu \). This is the general statement of the Theorem 2.3, which was explicitly checked by Zotov in [28] where he obtained the deformation quantization formula up to the third order.

We copy the result in [28] for completeness and for our later use.

\[
f_M \star g = fg + \hbar \alpha^{ab} \partial_a f \partial_b g
\]
\[
+ \hbar^2 \left[ \frac{1}{2} \alpha^{a_1b_1} \alpha^{a_2b_2} \partial_{a_1} \partial_{a_2} f \partial_{b_1} \partial_{b_2} g + \frac{1}{3} \alpha^{a_1b_1} \partial_{b_1} \alpha^{a_2b_2} (\partial_{a_1} \partial_{a_2} f \partial_{b_2} g + \partial_{a_1} \partial_{a_2} g \partial_{b_2} f) \right]
\]
\[
+ \hbar^3 \left[ \frac{1}{6} \alpha^{a_1b_1} \alpha^{a_2b_2} \alpha^{a_3b_3} \partial_{a_1} \partial_{a_2} \partial_{a_3} f \partial_{b_1} \partial_{b_2} \partial_{b_3} g \right]
\]
\[
+ \frac{1}{3} \alpha^{a_1b_1} \partial_{b_1} \alpha^{a_2b_2} \partial_{b_2} \partial_{a_1} \alpha^{a_3b_3} (\partial_{a_2} \partial_{b_3} f \partial_{a_3} g - \partial_{a_2} \partial_{b_3} g \partial_{a_3} f)
\]
\[
+ \left( \frac{2}{3} \alpha^{a_1b_1} \partial_{b_1} \alpha^{a_2b_2} \partial_{a_2} \partial_{a_3} f \partial_{b_2} \partial_{b} \partial_{a_3} \right) \partial_{a_2} \partial_{b_3} f \partial_{a_3} \partial_{a_1} g
\]
\[
+ \frac{1}{6} \alpha^{a_1b_1} \alpha^{a_2b_2} \partial_{b_1} \partial_{b_2} \alpha^{a_3b_3} (\partial_{a_1} \partial_{a_2} \partial_{a_3} f \partial_{b_2} \partial_{b_3} g - \partial_{a_1} \partial_{a_2} \partial_{a_3} g \partial_{b_3} f)
\]
\[
+ \frac{1}{3} \alpha^{a_1b_1} \partial_{b_1} \alpha^{a_2b_2} \alpha^{a_3b_3} (\partial_{a_1} \partial_{a_2} \partial_{a_3} f \partial_{b_2} \partial_{b_3} g - \partial_{a_1} \partial_{a_2} \partial_{a_3} g \partial_{b_2} \partial_{b_1} f) \right] + O(\hbar^4) \tag{3.7}
\]

where\(^7\)

\[
\alpha^{ab} = \left( i \frac{\theta^\mu\nu}{2} \frac{\partial x^a}{\partial y^\mu} \frac{\partial x^b}{\partial y^\nu} + \hbar^2 \frac{i}{3} \frac{\theta^{\mu_1\nu_1} \theta^{\mu_2\nu_2} \theta^{\mu_3\nu_3}}{\partial y^{\mu_1} \partial y^{\mu_2} \partial y^{\mu_3}} \frac{\partial^3 x^a}{\partial y^{\nu_1} \partial y^{\nu_2} \partial y^{\nu_3}} \frac{\partial^3 x^b}{\partial y^{\nu_1} \partial y^{\nu_2} \partial y^{\nu_3}} \right)
\]
\[
+ \frac{2}{9} S^{a_1a_2a_3} \partial_{a_1} \partial_{a_2} \partial_{a_3} \alpha^{ab} + \theta^{\mu_1\nu_1} \theta^{\mu_2\nu_2} \frac{\partial^2 x^{a_1}}{\partial y^{\mu_1} \partial y^{\nu_2}} \frac{\partial^2 x^{b_1}}{\partial y^{\nu_1} \partial y^{\mu_2}} \partial_{a_1} \partial_{b_1} \alpha^{ab} \right) + O(\hbar^3) \tag{3.8}
\]

\(^7\)We scale \( \theta^{ij} \to \frac{i}{2} \theta^\mu\nu \) in [28] to be compatible with the definition \((1.2)\) and we denote \( \partial_a \equiv \frac{\partial}{\partial x^a} \).
and $S^{abc}$ is given by

$$S^{abc} = \theta_{\mu_1\nu_1} \theta_{\mu_2\nu_2} \left( \frac{\partial^2 x^a}{\partial y^{\mu_1} \partial y^{\nu_1}} \frac{\partial x^b}{\partial y^{\mu_2}} \frac{\partial x^c}{\partial y^{\nu_2}} + \frac{\partial^2 x^c}{\partial y^{\mu_1} \partial y^{\nu_2}} \frac{\partial x^a}{\partial y^{\mu_2}} \frac{\partial x^b}{\partial y^{\nu_1}} + \frac{\partial^2 x^b}{\partial y^{\mu_1} \partial y^{\nu_2}} \frac{\partial x^c}{\partial y^{\mu_2}} \frac{\partial x^a}{\partial y^{\nu_1}} \right). \quad (3.9)$$

The differential operator in the automorphism (3.2) necessary for obtaining Eqs. (3.7) and (3.8) is the following

$$D(h) = 1 + \frac{\hbar^2}{16} \left[ \theta_{\mu_1\nu_1} \theta_{\mu_2\nu_2} \left( \frac{\partial^2 x^a}{\partial y^{\mu_1} \partial y^{\nu_1}} \frac{\partial x^b}{\partial y^{\mu_2}} \frac{\partial x^c}{\partial y^{\nu_2}} \right) + 2 \frac{1}{9} S^{abc} \frac{\partial a}{\partial y} \frac{\partial b}{\partial y} \frac{\partial c}{\partial y} \right] + O(h^3). \quad (3.10)$$

Note that $f \star_M g \equiv D(h)(D(h)^{-1}(f) \star D(h)^{-1}(g))$ in Eq.(3.7) is the Moyal star product (1.2) but after a change of coordinates it becomes equivalent to the general Kontsevich star product (3.6) up to the gauge equivalence map (3.10), thus checking the Theorem 2.3. Also notice that

$$[f, g]_\star = f \star g - g \star f = 2\hbar \alpha^{ab} \frac{\partial a}{\partial y} f \frac{\partial b}{\partial y} g + O(h^3) \quad (3.11)$$

since $O(h^3)$ is symmetric with respect to $f \leftrightarrow g$.

Since the map (3.10) is explicitly known, we can now solve the gauge equivalence (3.2). First let us represent the coordinates $x^\mu(y)$ as in Eq.(1.10) to study its consequence from gauge theory point of view. The equivalence (3.7) immediately leads to

$$[x^\mu, x^\nu]_\star = i(\theta - \theta \hat{F}(y)\theta)^{\mu\nu} = 2D(h)^{-1}(\alpha^{\mu\nu}) \quad (3.12)$$

where the left hand side is the Moyal product (1.2). As a check, one can easily see that Eq.(3.12) is trivially satisfied if Eqs.(3.8) and (3.10) are substituted for the right hand side with $\hbar = 1$. Note that Eq.(3.12) is an exact result since the higher order terms in Eq.(3.11) identically vanish.

By our construction, the new Poisson structure

$$\alpha^{\mu\nu}(x) = \frac{i}{2} \left( \frac{1}{B + F} \right)^{\mu\nu}(x) = \frac{i}{2} (\theta g^{-1})^{\mu\nu}(x) \quad (3.13)$$

belongs to the same equivalence class as constant $\theta^{\mu\nu} = (1/B)^{\mu\nu}$, but now depends on dynamical gauge fields. Thus, if it is determined how the map $D(h)$ depends on the coordinate transformations as in Eq.(3.10), one can in principle calculate exact SW maps from Eq.(3.12) up to a desired order. As it should be, Eq.(3.12) reduces to Eq.(1.12) at the leading order where $D(h) \approx 1$. In general, it definitely contains derivative corrections coming from the higher-order terms in $D(h)$.

---

8For a comparison with these literatures, $D(h)^{-1} : A[x[[\hbar]] \rightarrow A_y[[\hbar]]$ is understood as $D \equiv D(h) \circ \rho^*$ and $x^\mu(y) = D y^\mu$ in their notation since Eq.(3.10) is already including the coordinate transformation $\rho^*$.

9The leading derivative corrections calculated from Eq.(3.12) are four derivative terms consistently with Eqs.(2.34) and (2.35) which are based on superstring theory. As was mentioned in the footnote[5] the bosonic string case starts with two derivative terms. It is not so clear how to reproduce the bosonic string result [48] within the deformation quantization scheme by incorporating tachyons. It would be an interesting future work.
identity (3.12) defines the exact SW map with derivative corrections and corresponds to a quantum deformation of Eq.(1.4) or equivalently Eq.(1.12). Incidentally, we can also get the inverse SW map from Eq.(3.13) by solving Eq.(3.8) (at least perturbatively) which is of the form $\alpha^{\mu\nu} = \frac{1}{2}[x^\mu, x^\nu]_\star + \text{terms with derivatives of } \alpha^{\mu\nu}$. Thus, getting a full quantum deformation reduces to the calculation of $\alpha(h)$ or $D(h)$, as done up to $O(h^2)$ in (3.8) and (3.10).

The above construction definitely shows that the deformation quantization is a NC deformation of the diffeomorphism symmetry (1.4). Since NC gravity is based on a NC deformation of the diffeomorphism group [29, 5], we expect the emergent gravity may be a NC gravity in general. We will find further evidences for this connection.

As was shown in [27, 39], using the exact SW map (3.12) together with (2.16) and (2.17), it is possible to prove the SW equivalence (2.19), or more generally, Eq.(2.26). Conversely, we showed in section 2 that the SW map (3.12) at the leading order directly results from the SW equivalence (2.22). As we checked above, Eq.(3.12) is a direct consequence of the gauge equivalence (3.2) between the star products $\star_{\omega'}$ and $\star_\omega$ defined by the symplectic forms $\omega' = B + F(x)$ and $\omega = B$, respectively. One might thus claim that the SW equivalence (2.26) is just the statement of the gauge equivalence (3.2) between star products.

We would like to point out some beautiful picture working in these arguments. First note that symplectic (or more generally Poisson) structures in a gauge equivalence class are related to each other by the diffeomorphism symmetry, which is realized as the gauge equivalence (3.2) after deformation as illustrated in Eq.(3.7). This is precisely the statement of Theorem 1.1. We realize from the argument in section 1 that the gauge equivalence (3.2) is also related to the $\Lambda$-symmetry (1.6) where the local deformation of symplectic structure is due to the dynamics of gauge fields who live in NC spacetime (1.3), as shown in (3.13). Thus the dynamics of gauge fields appears as the local deformation of symplectic structures which always belong to the same gauge equivalence class, so it can entirely be translated into the diffeomorphism symmetry according to the Theorem 2.3.

But notice that not all diffeomorphism does deform the symplectic structure. For example, if the diffeomorphism is generated by a vector field $X_\lambda$ satisfying $\mathcal{L}_{X_\lambda}B = 0$, i.e. $X_\lambda \in Ham(M)$, it does not change the symplectic structure $\omega = B$. Let us recall the argument about the Moser lemma in section 1. For $\omega' = \omega + dA$, there is a flow $\phi$ generated by a vector field $X$ such that $\phi^\star(\omega') = \omega$. But the gauge transformation $A \rightarrow A + d\lambda$ only affects the vector field as $X \rightarrow X + X_\lambda$ where $\iota_{X_\lambda}\omega + d\lambda = 0$. The action of $X_\lambda$ on a smooth function $f$ is given by $X_\lambda(f) = \{\lambda, f\}$ and, upon quantization (1.1), $\hat{X}_\lambda(\hat{f}) = -i[\lambda, \hat{f}]_\star$, which is exactly the NC $U(1)$ gauge transformation.

Also note that the gauge equivalence (3.2) is defined up to the following inner automorphism [17]

$$f(h) \rightarrow \lambda(h) \star f(h) \star \lambda(h)^{-1} \quad (3.14)$$

or its infinitesimal version is

$$\delta f(h) = i[\lambda, f]_\star \quad (3.15)$$

The above similarity transformation definitely does not change star products. For $f(h) = x^\mu(y)$,
Eq. (3.15) is equal to the NC gauge transformation (2.10) with the definition $\lambda(h) \equiv \hat{\lambda}$ since $[y^\mu, \hat{\lambda}] = i\theta^{\mu\nu} \partial_\nu \hat{\lambda}$. This is a quantum deformation of Eq. (1.11).

In consequence, the $U(1)$ gauge symmetry is realized as the symplectomorphism $Ham(M)$ on a symplectic manifold $M$ and, upon quantization (3.1), it appears as the inner automorphism (3.14), which is the NC $U(1)$ gauge symmetry [8, 17, 39, 51].

If the $\Lambda$-symmetry (1.6) happens to be an exact gauge symmetry, a puzzle arises. If this is the case, two symplectic structures $\omega' = B + F(x)$ and $\omega = B$ are related by the local gauge symmetry (1.6) and thus the gauge fields should be physically unobservable. But we know well that the physical configuration space of (NC) gauge theory is nontrivial. The puzzle can be resolved by noticing that the NC spacetime (1.3) is a background induced by a (homogeneous) condensation of gauge fields. Consequently, the $\Lambda$-symmetry is spontaneously broken to the ordinary $U(1)$ gauge symmetry since the background (1.7) preserves only the latter. The spontaneous symmetry breaking (1.7) therefore explains why gravity is physically observable in spite of the gauge symmetry (1.6).

Now we are fairly ready to speculate a whole picture about the emergent gravity from NC spacetime. The $U(1)$ gauge theory defined by (1.5) respects the $\Lambda$-symmetry (1.6) since the underlying sigma model (2.1) clearly respects this symmetry. The $\Lambda$-symmetry is mapped to $G = Diff(M)$ via the Darboux theorem and is realized as the gauge equivalence (3.2) after NC deformation. The ordinary $U(1)$ gauge symmetry appears as the symplectomorphism $H = Ham(M) \subset Diff(M)$, which is realized as the diffeomorphism symmetry $Diff(M)$ via the Darboux theorem while the $U(1)$ gauge symmetry appears as the symplectomorphism $Ham(M)$. The symmetry breaking (1.7) therefore explains why gravity is physically observable in spite of the gauge symmetry (1.6).
emerging from a nonlinear realization $G/H$ should be in general spin-2 gravitons [19]. According to the conjecture, the gravitational fields $e_\mu^\alpha(y)$ in Eq. (2.30) might be identified with the Goldstone bosons for the spontaneous symmetry breaking (1.7). We already gave a supporting argument in section 1 that the dynamics of NC gauge fields appears as the fluctuation of geometry through general coordinate transformations in $G = \text{Diff}(M)$. We will see that a NC gauge theory describes an emergent geometry in the way that the fluctuation of gauge fields in NC spacetime (1.3) induces a deformation of the vacuum manifold, e.g. $\mathbb{R}^d$ for constant $\theta^{\mu\nu}$.

It should be very important to completely determine the structure of emergent gravity based on the framework of the nonlinear realization $G/H$ [19] (including a full quantum deformation). Unfortunately this goes beyond the present scope. Instead we will confirm the conjecture by considering the self-dual sectors for ordinary and NC gauge theories. We will see that so beautiful structures about gravity, e.g. the twistor space [31], naturally emerge from this construction. Since the emergent gravity seems to be very generic if the conjecture is true anyway, we believe that the correspondence between self-dual NC electromagnetism and self-dual Einstein gravity is enough to strongly guarantee the conjecture.

\section{NC Instantons and Gravitational Instantons}

To illustrate the correspondence of NC gauge theory with gravity, we will explore in this section the equivalence found in [2, 3] between NC $U(1)$ instantons and gravitational instantons. To make the essence of emergent gravity clear as much as possible, we will neglect the derivative corrections and consider the usual NC description with $\Phi = 0$. The semi-classical approximation, or slowly varying fields, means that the Moyal star product (1.2) is approximated only to the first order, $O(\theta)$, in which the NC field strength (2.9) is replaced by Eq. (1.13). In next section we will consider the effect of derivative corrections using the background independent formalism of NC gauge theory [11, 21], namely, with $\Phi = -B$. This section will be mostly a mild extension of the previous works [2, 3] with more focus on the emergent gravity and the relation to the twistor space.

Let us consider electromagnetism in the NC spacetime (1.3). The action for the NC $U(1)$ gauge theory in flat Euclidean $\mathbb{R}^4$ is given by

\[ \tilde{S}_{\text{NC}} = \frac{1}{4} \int dy \tilde{F}_{\mu\nu} \star \tilde{F}^{\mu\nu}. \] (4.1)

Contrary to ordinary electromagnetism, the NC $U(1)$ gauge theory admits non-singular instanton solutions satisfying the NC self-duality equation [52],

\[ \tilde{F}_{\mu\nu}(y) = \pm \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} \tilde{F}^{\lambda\sigma}(y). \] (4.2)

When we consider NC instantons, the ADHM construction depends only on the combination $\mu^\alpha = \theta^{\mu\nu} \eta^{(\pm)\alpha}_{\mu\nu}$ [11, 53] for anti-self-dual (ASD) (with + sign) and self-dual (SD) (with - sign) instantons.
where $\eta^{(+)}_{\mu\nu} = \eta^a_{\mu\nu}$ and $\eta^{(-)}_{\mu\nu} = \bar{\eta}^a_{\mu\nu}$ are three $4 \times 4$ SD and ASD 't Hooft matrices [2]. If the instanton is ASD in the NC spacetime satisfying $\theta_{\mu\nu}(\eta^{(-)})^a = 0$, the ADHM equation then gets a nonvanishing deformation, which puts a non-zero minimum size of NC instantons. In this case, the small instanton singularities are eliminated and the instanton moduli space is thus non-singular [52]. However, if the instanton is SD, the deformation is vanishing. Thus the small instanton singularity is not eliminated and the instanton moduli space is still singular. The so-called localized instantons in this case are generated by shift operators [54].

As was explained in section 2, the NC gauge theory (4.1) has an equivalent dual description through the SW map in terms of ordinary gauge theory on commutative spacetime [11]. Applying the maps (1.12) and (1.14) to the action (4.1), one can get the commutative nonlinear electrodynamics [22, 23] equivalent to Eq.(4.1) in the semi-classical approximation:

$$S_C = \frac{1}{4} \int d^4x \sqrt{\det g} g^{\mu\lambda} g^{\sigma\nu} F_{\mu\nu} F_{\lambda\sigma},$$

(4.3)

where the effective metric $g_{\mu\nu}$ [55] was defined in Eq.(2.29). It was shown in [2] that the self-duality equation for the action $S_C$ is given by

$$F_{\mu\nu}(x) = \pm \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} F_{\lambda\sigma}(x),$$

(4.4)

with the definition (2.21). Note that Eq.(4.4) is nothing but the exact SW map (1.12) of the NC self-duality equation (4.2).

A general strategy was suggested in [2] to solve the self-duality equation (4.4). To be specific, consider self-dual NC $\mathbb{R}^4$, i.e., $\theta_{\mu\nu} \bar{\eta}^a_{\mu\nu} = 0$, with the canonical form $\theta_{\mu\nu} = \frac{\varepsilon}{2} \eta^{3}_{\mu\nu}$. Take a general ansatz for the SD $F^+_{\mu\nu}$ and the ASD $F^-_{\mu\nu}$ as follows

$$F^\pm_{\mu\nu}(x) = f^a(x) \eta^\pm_{\mu\nu},$$

(4.5)

$^{10}$Here we would like to correct an incorrect statement, Proposition 3.1 in [56], to remove a disagreement with existing literatures, especially, with [40]. See the comments in page 11. The Proposition 3.1 states that the terms of order $n$ in $\theta$ in the NC Maxwell action (4.1) via SW map form a homogeneous polynomial of degree $n + 2$ in $F$ without derivatives of $F$. The proposition is also inconsistent with our general result about derivative corrections in section 2 and 3. This disagreement was recently pointed out in [57].

The proposition was based on a wrong observation that the derivation acting on the $\theta$'s appearing in star products always gives rise to total derivatives. That is not true in general. For example, let us consider the following derivation with respect to $\theta_{\mu\nu}$:

$$\delta \frac{\delta}{\delta \theta_{\mu\nu}} (f * g * h)(y) = i \left( \partial_{\mu} f * \partial_{\nu} g * h + \partial_{\mu} f * g * \partial_{\nu} h + f * \partial_{\nu} g * \partial_{\mu} h \right)(y),$$

where $f, g, h \in C^\infty(M)$ are rapidly decaying functions at infinity and are assumed to be $\theta$-independent. The above derivation cannot be rewritten as a total derivative. If it were the case, it would definitely imply a wrong result: $\int d^4y (f * g * h)(y) = \int d^4y (f g h)(y)$. This is not true for the triple or higher multiple star product.
where \( f^a \)'s are arbitrary functions. Then the equation (4.4) is automatically satisfied. Next, solve the field strength \( F_{\mu\nu} \) in terms of \( F_{\mu\nu}^\pm \):

\[
F_{\mu\nu}(x) = \left( \frac{1}{1 - F_{\pm}^\theta} \right)^\mu_{\nu}(x). \tag{4.6}
\]

Substituting the ansatz (4.5) into Eq. (4.6), we get

\[
F_{\mu\nu} = \frac{1}{1 - \phi} f^a \eta^a_{\mu\nu} + \frac{2\phi}{\zeta(1 - \phi)} \eta^3_{\mu\nu}, \quad \text{for SD case}, \tag{4.7}
\]

\[
F_{\mu\nu} = \frac{1}{1 - \phi} f^a \bar{\eta}^a_{\mu\nu} - \frac{2\phi}{\zeta(1 - \phi)} \eta^3_{\mu\nu}, \quad \text{for ASD case}, \tag{4.8}
\]

where \( \phi \equiv \frac{\zeta^2}{4} \sum_{a=1}^{3} f^a(x) f^a(x) \).\[1]\)

For the ASD case (4.8), we get the instanton equation in [11] (see also [60]):

\[
F_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} = \frac{1}{4} (F \tilde{F}) \theta_{\mu\nu}^+ \tag{4.9}
\]

since

\[
F \tilde{F} \equiv \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = -\frac{16\phi}{\xi^2(1 - \phi)}, \tag{4.10}
\]

while, for the SD case (4.7),

\[
F_{\mu\nu}(x) = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}(x). \tag{4.11}
\]

Interestingly, using the inverse metric

\[
(g^{-1})_{\mu\nu} = \frac{1}{\sqrt{\text{det} g}} \left( \frac{1}{2} g_{\lambda\lambda} \delta_{\mu\nu} - g_{\mu\nu} \right),
\]

Eq. (4.9) can be rewritten as the self-duality in a curved space described by the metric \( g_{\mu\nu} \):

\[
F_{\mu\nu}(x) = -\frac{1}{2} \varepsilon^{\lambda\sigma\rho\tau} g_{\mu\lambda} g_{\nu\sigma} F_{\rho\tau}(x). \tag{4.12}
\]

It is interesting to compare this with the SD case (4.11). It should be remarked, however, that the self-duality in (4.12) cannot be interpreted as a usual self-duality equation in a fixed background since the four-dimensional metric used to define Eq. (4.12) depends in turn on the \( U(1) \) gauge fields.

It is well-known that there is no nontrivial solution to (A)SD equation in ordinary \( U(1) \) gauge theory. Since the SD instanton satisfies Eq. (4.11), the exact SW map of localized instantons is thus

\[\text{One can rigorously show that the smooth function } \phi \text{ for the ASD case (4.8) satisfies the inequality, } 0 \leq \phi < 1. \text{ The proof is done by noticing that}\]

\[
\frac{1}{2} \varepsilon^{\mu\lambda\alpha\beta} \sqrt{\text{det} g(g^{-1} F)_{\mu\nu}(g^{-1} F)_{\alpha\beta}} = -\frac{16\phi}{\xi^2(1 - \phi)}
\]

since the left-hand side is negative definite unless zero and \( \phi \) is definitely non-negative.
either trivial or very singular. This result is consistent with [61]. From now on, we thus focus on the ASD instantons.

Since the field strength \( (4.8) \) is given by a (locally) exact two-form, i.e., \( F = dA \), we impose the Bianchi identity for \( F_{\mu\nu} \),

\[
\varepsilon_{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0. \tag{4.13}
\]

In the end Eq.\( (4.13) \) leads to general differential equations governing \( U(1) \) instantons [2]. The equation \( (4.13) \) was explicitly solved in [2, 11] for the single instanton case. It was found in [2] that the effective metric \( (2.29) \) for the single \( U(1) \) instanton is related to the Eguchi-Hanson (EH) metric [58], the simplest asymptotically locally Euclidean (ALE) space, given by

\[
ds^2 = \left( 1 - \frac{t^4}{\varrho^4} \right)^{-1} \varrho^2 (\sigma_x^2 + \sigma_y^2) + \varrho^2 \left( 1 - \frac{t^4}{\varrho^4} \right) \sigma_z^2 \tag{4.14}
\]

where \( \sigma_i \) are the \( SU(2) \) left-invariant 1-forms satisfying \( d\sigma_i + \varepsilon_{ijk} \sigma_j \wedge \sigma_k = 0 \). The metric \( (4.14) \) can be transformed to the Kähler metric form \( (4.2) \) in [2] by the following coordinate transformation [59]:

\[
\begin{align*}
r^2 (\sigma_x^2 + \sigma_y^2) &= |dz_1|^2 + |dz_2|^2 - r^{-2} |\bar{z}_1 dz_1 + \bar{z}_2 dz_2|^2; \\
r^2 \sigma_z^2 &= -\frac{1}{4r^2} (\bar{z}_1 dz_1 + \bar{z}_2 dz_2 - z_1 d\bar{z}_1 - z_2 d\bar{z}_2)^2, \tag{4.15}
\end{align*}
\]

where

\[
\varrho^4 = r^4 + t^4 \tag{4.16}
\]

and \( r^2 = |z_1|^2 + |z_2|^2 \) is the embedding coordinate in field theory.

The EH metric \( (4.14) \) has a curvature that reaches a maximum at the ‘origin’ \( \varrho = t \), falling away to zero in all four directions as the radius \( \varrho \) increases. The apparent singularity in Eq.\( (4.14) \) at \( \varrho = t \) (which is the same singularity appearing at \( r = 0 \) in the instanton solution constructed in [2, 11]) is only a coordinate singularity, provided that \( \psi \) is assigned the period \( 2\pi \) rather than \( 4\pi \) (where \( \sigma_z = \frac{1}{2} (d\psi + \cos \theta d\phi) \)). Since the radial coordinate runs down only as far as \( \varrho = t \), there is a minimal 2-sphere \( S^2 \) of radius \( t \) described by the metric \( t^2 (\sigma_x^2 + \sigma_y^2) \). This degeneration of the generic three dimensional orbits to the two dimensional sphere is known as a ‘bolt’ [62]. As we mentioned above, the NC parameter \( \zeta \) in the gauge theory settles the size of NC \( U(1) \) instantons and removes the singularity of instanton moduli space coming from small instantons. The parameter \( \zeta \) is related to the parameter \( t^2 \) in the EH metric \( (4.14) \) as \( t^2 = \zeta \bar{t}^2 \) with a dimensionless constant \( \bar{t} \) and so to the size of the ‘bolt’ in the gravitational instantons [2]. Unfortunately, since \( \varrho = t \) corresponds to the origin \( r = 0 \) of the embedding coordinates, this nontrivial topology is not visible in the gauge theory description, as was pointed out in [63]. However, we see that the dynamical approach where a manifold is emerging from dynamical gauge fields, as in Eq.\( (1.9) \), reveals the nontrivial topology of the D-brane submanifold.

It would be useful to briefly summarize the work [63] since it seems to be very related to ours although explicit solutions are different from each other (see section 4.2 in [63]). Braden and Nekrasov
constructed $U(1)$ instantons using the deformed ADHM equation defined on a *commutative* space $X$. They showed that the resulting gauge fields are singular unless one changes the topology of the spacetime and that the $U(1)$ gauge field can have a non-trivial instanton charge if the spacetime contains non-contractible two-spheres. They thus argued that $U(1)$ instantons on NC $\mathbb{R}^4$ correspond to non-singular $U(1)$ gauge fields on a commutative Kähler manifold $X$ which is a blowup of $\mathbb{C}^2$ at a finite number of points. Also they speculated that the manifold $X$ for instanton charge $k$ can be viewed as a spacetime foam with $b_2 \sim k$.

Now let us show the equivalence between $U(1)$ instantons in NC spacetime and gravitational instantons [3]. In other words, Eq.(4.2) or Eq.(4.4) describes gravitational instantons obeying the SD equations [64]

$$R_{abcd} = \pm \frac{1}{2} \varepsilon_{abef} R^{ef}_{\ \ cd}, \quad (4.17)$$

where $R_{abcd}$ is a curvature tensor. The instanton equation (4.9) can be rewritten using the metric (2.29) as follows

$$g_{13} = g_{24}, \quad g_{14} = -g_{23},$$

$$g_{\mu\nu} = 4 \sqrt{\det g_{\mu\nu}} \quad (4.18)$$

with $\sqrt{\det g_{\mu\nu}} = g_{11}g_{33} - (g_{13}^2 + g_{14}^2)$ and $g_{12} = g_{34} = 0$ identically. We will show that Eq.(4.18) reduces to the so-called complex Monge-Ampère equation [65] or the Plebański equation [66], which is the Einstein field equation for a Kähler metric [3].

To proceed with the Kähler geometry, let us introduce the complex coordinates and the complex
gauge fields

$$z_1 = x^2 + ix^1, \quad z_2 = x^4 + ix^3, \quad (4.19)$$

$$A_{z_1} = A^2 - iA^1, \quad A_{z_2} = A^4 - iA^3. \quad (4.20)$$

In terms of these variables, Eq.(4.9) are written as

$$F_{z_1 z_2} = 0 = F_{\bar{z}_1 \bar{z}_2}, \quad (4.21)$$

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = -\frac{i\zeta}{4} F, \quad (4.22)$$

where $F = -4(F_{z_1 \bar{z}_1} F_{z_2 \bar{z}_2} + F_{z_1 \bar{z}_2} F_{z_2 \bar{z}_1})$. Note that Eq.(4.21) is the condition for a holomorphic vector bundle, but the so-called stability condition (4.22) is deformed by noncommutativity. (See Chap.15 in [67].)

One can easily see that the metric $g_{\mu\nu}$ is a Hermitian metric [3]. That is,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{ij} dz_i d\bar{z}_j, \quad i, j = 1, 2. \quad (4.23)$$

If we let

$$\omega = \frac{i}{2} g_{ij} dz_i \wedge d\bar{z}_j \quad (4.24)$$
be the Kähler form, then the Kähler condition is \( d\omega = 0 \), or, for all \( i, j, k \),
\[
\frac{\partial g_{ij}}{\partial z^k} = \frac{\partial g_{kj}}{\partial z^i}. \tag{4.25}
\]
The Kähler condition (4.25) is then equivalent to the Bianchi identity (4.13) since
\[
\omega = -(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) + \frac{\zeta}{2} F. \tag{4.26}
\]
Thus the metric \( g_{ij} \) is a Kähler metric and thus we can introduce a Kähler potential \( K \) defined by
\[
g_{ij} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^j}. \tag{4.27}
\]
The Kähler potential \( K \) is related to the integrability condition of Eq.(4.21) (defining a holomorphic vector bundle):
\[
A_{z_i} = 0, \quad A_{\bar{z}_i} = 2i \partial \bar{z}_i (K - \bar{z}_k z_k). \tag{4.28}
\]
Let us rewrite \( g_{\mu\nu} \) as
\[
g_{\mu\nu} = \frac{1}{2} (\delta_{\mu\nu} + \tilde{g}_{\mu\nu}). \tag{4.29}
\]
Then, from Eq.(4.18), one can easily see that
\[
\sqrt{\det \tilde{g}_{\mu\nu}} = 1. \tag{4.30}
\]
Note that the metric \( \tilde{g}_{\mu\nu} \) is also a Kähler metric:
\[
\tilde{g}_{ij} = \frac{\partial^2 \tilde{K}}{\partial z^i \partial \bar{z}^j}. \tag{4.31}
\]
The relation \( \det \tilde{g}_{\mu\nu} = (\det \tilde{g}_{ij})^2 \) definitely leads to the Ricci-flat condition
\[
\det \tilde{g}_{ij} = 1. \tag{4.32}
\]
Therefore the metric \( \tilde{g}_{\mu\nu} \) is both Ricci-flat and Kähler, which is the case of gravitational instantons [3]. For example, if one assumes that \( \tilde{K} \) in Eq.(4.31) is a function solely of \( r^2 = |z_1|^2 + |z_2|^2 \), Eq.(4.32) can be integrated to give [59]
\[
\tilde{K} = \sqrt{r^4 + t^4 + t^2 \log \frac{r^2}{\sqrt{r^4 + t^4 + t^2}}}. \tag{4.33}
\]
This leads precisely to the EH metric (4.14) after the coordinate transformation (4.15). We thus confirmed that the instanton equation (4.9) is equivalent to the Einstein field equation for Kähler metrics.

The above arguments can be elegantly summarized as the hyper-Kähler condition in the following way [3]. Let us consider the line element defined by the metric \( \tilde{g}_{\mu\nu} \)
\[
ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu \equiv \tilde{\sigma}_\mu \otimes \tilde{\sigma}_\mu. \tag{4.34}
\]
It is easy to check that $\tilde{\sigma}_1 \land \tilde{\sigma}_2 \land \tilde{\sigma}_3 \land \tilde{\sigma}_4 = d^4x$, in other words, $\sqrt{\det g_{\mu\nu}} = 1$. We then introduce the triple of Kähler forms as follows,

$$\tilde{\omega}^a = \frac{1}{2} \eta^a_{\mu\nu} \tilde{\sigma}^\mu \land \tilde{\sigma}^\nu, \quad a = 1, 2, 3.$$  \hspace{1cm} (4.35)

One can easily see that

$$\omega \equiv \tilde{\omega}^2 + i \tilde{\omega}^1 = dz_1 \land dz_2, \quad \tilde{\omega} \equiv \tilde{\omega}^2 - i \tilde{\omega}^1 = d\bar{z}_1 \land d\bar{z}_2,$$

$$\Omega \equiv -\tilde{\omega}^3 = i/2 (dz_1 \land d\bar{z}_1 + dz_2 \land d\bar{z}_2) + \zeta F.$$ \hspace{1cm} (4.36)

It is obvious that $d\tilde{\omega}^a = 0, \forall a$. This means that the metric $\tilde{g}_{\mu\nu}$ is hyper-Kähler [3], which is an equivalent statement as Ricci-flat Kähler in four dimensions.

Eq. (4.36) shows how dynamical gauge fields living in NC spacetime induce a deformation of background geometry through gravitational instantons, thus realizing the emergent geometry we claimed before. We see that, if we turn off either gauge fields or noncommutativity (to be precise, a commutative limit $\zeta \to 0$), we simply arrive at flat $\mathbb{R}^4$. But, if we turn on both gauge fields and noncommutativity, the background geometry, say flat $\mathbb{R}^4$, is nontrivially deformed and we arrive at a curved manifold. For instance, hyper-Kähler manifolds emerge from NC instantons. Actually this picture also implies that the flat $\mathbb{R}^4$ has to be interpreted as emergent from the homogeneous gauge field condensation (1.7) [1].

$\mathbb{R}^4$ is the simplest hyper-Kähler manifold, viewed as the quaternions $\mathbb{H} \simeq \mathbb{C}^2$. Hyper-Kähler manifold is a manifold equipped with infinitely many ($S^2$-family) of Kähler structures. This $S^2$-family corresponds to the number of inequivalent choices of local complex structures on $\mathbb{R}^4$. Since there is no preferred complex structure, it is democratic to consider $\mathbb{R}^4 \simeq \mathbb{C}^2$ with the set of all possible local complex structures simultaneously at each point, in other words, a $\mathbb{P}^1 = S^2$ bundle over $\mathbb{R}^4$. The total space of this $\mathbb{P}^1$ bundle is the twistor space $Z$ [31]. Since gravitational instantons are also hyper-Kähler manifolds, they also carry a $\mathbb{P}^1$-family of Kähler structures. So one can similarly construct the corresponding twistor space $Z$ describing curved self-dual spacetime as a $\mathbb{P}^1$ bundle over a hyper-Kähler manifold $M$ [31, 32]. The twistor space $Z$ may also be viewed as a fiber bundle over $\mathbb{P}^1$ with a fiber being $M$.

Now we will show that the equivalence of NC instantons with gravitational instantons perfectly fits with the geometry of the twistor space describing curved self-dual spacetime. This construction, which closely follows the results on $N = 2$ strings [33, 34], will clarify how the deformation of symplectic (or Kähler) structure on $\mathbb{R}^4$ due to the fluctuation of gauge fields appears as that of complex structure of the twistor space $Z$ [1].

Consider a deformation of the holomorphic (2,0)-form $\omega = dz_1 \land dz_2$ as follows

$$\Psi(t) = \omega + it\Omega + \frac{t^2}{4} \tilde{\omega}$$ \hspace{1cm} (4.37)
where the parameter $t$ takes values in $\mathbb{P}^1$. Note that $\Omega$ is a $(1,1)$ form because of Eq. (4.21). One can easily see that $d\Psi(t) = 0$ due to the Bianchi identity $dF = 0$ and

$$\Psi(t) \wedge \Psi(t) = 0$$

(4.38)

since Eq. (4.38) is equivalent to Eq. (4.9). Since the two-form $\Psi(t)$ is closed and degenerate, the Darboux theorem asserts that one can find a $t$-dependent map $(z_1, z_2) \rightarrow (Z_1(t; z_i, \bar{z}_i), Z_2(t; z_i, \bar{z}_i))$ such that

$$\Psi(t) = dZ_1(t; z_i, \bar{z}_i) \wedge dZ_2(t; z_i, \bar{z}_i).$$

(4.39)

When $t$ is small, one can solve (4.39) by expanding $Z_i(t; z, \bar{z})$ in powers of $t$ as

$$Z_i(t; z, \bar{z}) = z_i + \sum_{n=1}^{\infty} \frac{t^n}{n} p^i_n(z, \bar{z}).$$

(4.40)

By substituting this into Eq. (4.37), one gets at $O(t)$

$$\partial_z p^i_1 = 0,$$

(4.41)

$$\epsilon_{ik} \partial_{\bar{z}_j} p^k_i dz^i \wedge d\bar{z}^j = i\Omega.$$  

(4.42)

Eq. (4.41) can be solved by setting $p^i_1 = 1/2 e^{ij} \partial_{\bar{z}_j} \tilde{K}$ and then $\Omega = i/2 \partial_z \partial_{\bar{z}} \tilde{K} dz^i \wedge d\bar{z}^j$. The real-valued smooth function $\tilde{K}$ is the Kähler potential of $U(1)$ instantons in Eq. (4.31). In terms of this Kähler two-form $\Omega$, Eq. (4.38) leads to the complex Monge-Ampère or the Plebański equation, Eq. (4.32),

$$\Omega \wedge \bar{\Omega} = \frac{1}{2} \omega \wedge \bar{\omega},$$

(4.43)

that is, $\det(\partial_z \partial_{\bar{z}} \tilde{K}) = 1$.

When $t$ is large, one can introduce another Darboux coordinates $\tilde{Z}_i(t; z_i, \bar{z}_i)$ such that

$$\Psi(t) = t^2 d\tilde{Z}_1(t; z_i, \bar{z}_i) \wedge d\tilde{Z}_2(t; z_i, \bar{z}_i)$$

(4.44)

with expansion

$$\tilde{Z}_i(t; z, \bar{z}) = \tilde{z}_i + \sum_{n=1}^{\infty} \frac{t^{-n}}{n} \tilde{p}^i_n(z, \bar{z}).$$

(4.45)

One can get the solution (4.37) with $\tilde{p}^i_1 = -1/2 e^{ij} \partial_{\bar{z}_j} \tilde{K}$ and $\Omega = i/2 \partial_z \partial_{\bar{z}} \tilde{K} dz^i \wedge d\bar{z}^j$.

The $t$-dependent Darboux coordinates $Z_i(t; z, \bar{z})$ and $\tilde{Z}_i(t; z, \bar{z})$ correspond to holomorphic coordinates on two local charts, where the 2-form $\Psi(t)$ becomes the holomorphic $(2,0)$-form, of the dual projective twistor space $Z$ as a fiber bundle over $S^2$ with a fiber $M$, a hyper-Kähler manifold. Here we regard $t$ as a parameter of deformation of complex structure on $M$. The coordinate charts can be consistently glued together along the equator on $\mathbb{P}^1$ as $(Z', t') = (t^{-1} f(t; Z), -t^{-1})$ and so the complex structure is extended over $Z$. Therefore the Darboux coordinates are related by a $t$-dependent symplectic transformation on an overlapping coordinate chart as $f_i(t; Z(t)) = t \tilde{Z}_i(t)$.
In this way, the complex geometry of the twistor space $Z$ encodes all the information about the Kähler geometry of self-dual 4-manifolds $M$ emerging from NC gauge fields. This twistor construction clarifies the nature of emergent gravity; the gauge fields act as a deformation of the complex structure of twistor space or the Kähler structure of self-dual 4-manifold. In this way gauge fields in NC spacetime manifest themselves as a deformation of background geometry, which is consistent with the picture observed below Eq. (4.36). Thus we should think of the twistor space as already incorporating the backreaction of NC instantons. This picture is remarkably similar to that in [68] where placing D1-branes (as instantons in gauge theory) in twistor space is interpreted as blowing up points in four dimensions dubbed as spacetime foams and the Kähler blowups in four dimensions are encoded in the twistor space as the backreaction of the D1-branes. Under the twistor correspondence, each $\mathbb{P}^1$ in $Z$ corresponds to a point on $M$. In particular, D1-branes which wrap $\mathbb{P}^1$s correspond to Kähler blowups in four dimensions via the Penrose transform [68]. This can be interpreted as the backreaction of the D1-branes in the twistor space, which is precisely our picture if the D1-branes are identified with NC $U(1)$ instantons, which are in turn gravitational instantons.

The above construction is also very similar to topological D-branes on NC manifolds in [35] which can be understood in terms of generalized complex geometry [15]. Especially, see section 6 of the first paper in [35] where Eq. (6) corresponds to our (4.9) or (4.43). This coincidence might be expected at the outset since the generalized complex geometry incorporates symplectic structures as well as usual complex structures (see the footnote 2) and the emergent gravity is essentially based on a NC deformation of symplectic structures. We will more exploit this relation in [69].

5 NC Self-duality and Twistor Space

In section 4, we ignored derivative corrections whose explicit forms are given in section 2 and 3. Furthermore we used the usual NC description with $\Phi = 0$, which is not background independent, i.e., $\theta$-dependent [21]. As a result, we separately considered two kinds of NC instantons; Nekrasov-Schwarz instantons [52] and localized instantons [54]. In particular, the SW map of localized instantons generated by shift operators was shown to be trivial, i.e., $F_{\mu\nu}^\pm = 0$ [61], not to probe the geometry by localized instantons. (This may be an artifact of the semi-classical approximation.) Therefore the background independent formulation of NC gauge theory [11, 21] might be more effective to have a unified description for all possible backgrounds and to implement a possible effect of derivative corrections.

In this section, we will generalize the equivalence in section 4 using the background independent formulation of NC gauge theories and show that self-dual electromagnetism in NC spacetime is equivalent to self-dual Einstein gravity, uncovering many details in [1]. In particular, we will discuss in detail the twistor space structure inherent in the self-dual NC electromagnetism. As a great bonus, the background independent formulation clearly reveals a picture that the NC gauge theory and gravity correspondence may be understood as a large $N$ duality.
To see this picture, consider the SW map (2.22) at $O(\kappa^2)$ for the background independent case with $\Phi = -B$ where $B_{\mu\nu} = (1/\theta)_{\mu\nu}$:

$$
\frac{1}{4G_s} \int d^4y (\hat{F} - B)^2 = \frac{1}{4G_s} \int d^4x \sqrt{\det g} g^{\mu\lambda} g^{\sigma\nu} B_{\mu\nu} B_{\lambda\sigma}.
$$

(5.1)

Although the right hand side is neglecting derivative corrections, we will now use the full NC field strength (2.9) to examine the derivative corrections. Later on, we will use only the left hand side which can be rewritten in terms of closed string variables only as follows

$$
\frac{1}{4G_s} \int d^4y (\hat{F} - B)_{\mu\nu} \star (\hat{F} - B)^{\mu\nu} = -\frac{\pi^2}{gs\kappa^2} g_{\mu\lambda} g_{\nu\sigma} \text{Tr}_H [x^\mu, x^\nu] [x^\lambda, x^\sigma]
$$

(5.2)

where we made a replacement $\frac{1}{(2\pi)^4} \int \frac{d^4y}{\theta} \leftrightarrow \text{Tr}_H$ using the Weyl-Moyal map [7]. The covariant, background-independent coordinates $x^\mu$ are defined by (1.10) and they are now operators on an infinite-dimensional, separable Hilbert space $H$, which is the representation space of the Heisenberg algebra (1.3). The NC gauge symmetry in Eq.(5.2) then acts as unitary transformations on $H$, i.e.,

$$
x^\mu \rightarrow x'^\mu = U x^\mu U^\dagger.
$$

(5.3)

This NC gauge symmetry $U_{\text{cpt}} (H)$ is so large that $U_{\text{cpt}} (H) \supset U(N) (N \rightarrow \infty)$ [10]. In this sense the NC gauge theory in Eq.(5.2) is essentially a large $N$ gauge theory. Note that the second expression in Eq.(5.2) is a large $N$ version of the IKKT matrix model which describes the nonperturbative dynamics of type IIB string theory [70].

Now let us apply the gauge equivalence in Eq.(3.2) or Eq.(3.7) for the adjoint action of $x^\mu$ with respect to star product:

$$
[x^\mu, \hat{f}]_* = 2\hbar D(h)^{-1}(\alpha^{\mu\nu}(x) \frac{\partial f}{\partial x^\nu}) + O(\hbar^3)
$$

$$
\approx 2\hbar \alpha^{\mu\nu}(x) \frac{\partial f}{\partial x^\nu} + O(\hbar^3)
$$

(5.4)

where $\hat{f} \equiv D(h)^{-1}(f)$. If $\hat{f} = x^\nu$, we recover Eq.(3.12).

Beyond the semi-classical approximation, Eq.(5.4) is not reduced to usual vector fields since there are infinitely many derivatives as shown in Eqs.(3.7) and (3.10). But it is important to notice the following properties (see the footnote 6), where we use the operator notation using the Weyl-Moyal map (1.2) for definiteness

$$
[\hat{x}^\mu, \hat{f} \hat{g}] = [\hat{x}^\mu, \hat{f}] \hat{g} + \hat{f} [\hat{x}^\mu, \hat{g}],
$$

$$
[\hat{x}^\mu, [\hat{x}^\nu, \hat{f}]] - [\hat{x}^\nu, [\hat{x}^\mu, \hat{f}]] = [[\hat{x}^\mu, \hat{x}^\nu], \hat{f}].
$$

(5.5)

(5.6)
These properties show that \(\text{ad}_{x^\mu} \equiv [x^\mu, \cdot]_*\) generally satisfy the property of vector fields or Lie derivatives even after quantum deformation. (Also note that \(D_\mu \equiv -iB_{\mu\nu}x^\nu\) is a covariant derivative in NC gauge theory.) Indeed this kind of vector fields was already defined in terms of twisted diffeomorphisms \([29, 30, 5]\), where a vector field on \(\mathbb{R}^d\) becomes a higher-order differential operator acting on fields in \(\mathcal{A}\). We thus see that Eq. (5.4) defines generalized vector fields according to Eqs. (5.5) and (5.6).

The appearance of NC gravity framework in our context might be anticipated. We observed in section 3 that the emergent gravity is related to a NC deformation of diffeomorphism symmetry (1.4). In other words, the Darboux theorem in symplectic geometry can be regarded as the equivalence principle in general relativity, but in general we need a NC version of the equivalence principle since we now live in the NC phase space (1.3). Actually, this is an underlying principle of NC gravity [29]. We will more clarify in [69] the emergent gravity from the viewpoint of NC gravity.

Let us now return to the semi-classical limit \(\mathcal{O}(\hbar)\). In this limit,

\[
[x^\mu, f]_* \approx i\hbar\theta^{\alpha\beta} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial f}{\partial y^\beta} = i\hbar\{x^\mu, f\} \equiv V^\mu[f].
\] (5.7)

As expected, the adjoint action of \(x^\mu\) with respect to star product reduces to a vector field \(V_\mu \in T\mathcal{M}\) on some emergent four manifold \(\mathcal{M}\). This is precisely the limit in section 4 that NC electromagnetism reduces to Einstein gravity for the (A)SD sectors. Note that NC gauge fields \(\widehat{A}_\mu(y)\) are in general arbitrary, so they generate arbitrary vector fields \(V^\mu \in T\mathcal{M}\) according to the map (5.7) and \([x^\mu, f]_* = i\theta^{\mu\nu} \partial_\nu f\) when \(\widehat{A}_\mu = 0\). One can easily check that

\[
(\text{ad}_{x^\mu} \text{ad}_{x^\nu} - \text{ad}_{x^\nu} \text{ad}_{x^\mu})[f] = \text{ad}_{[x^\mu, x^\nu]}[f] = [V^\mu, V^\nu][f]
\] (5.8)

where the right-hand side is defined by the Lie bracket between vector fields in \(T\mathcal{M}\). Note that the gauge transformation (5.3) naturally induces coordinate transformations of frame fields

\[
V^{\mu\alpha} \rightarrow V'^{\mu\alpha} = \frac{\partial y'^\alpha}{\partial y^\beta} V^{\mu\beta}.
\] (5.9)

This leads to a consistent result [1] that the gauge equivalence due to (5.3) corresponds to the diffeomorphic equivalence between the frame fields \(V^\mu\).

Let us look for an instanton solution of Eq. (5.2). Since the instanton is a Euclidean solution with a finite action, the instanton configuration should approach a pure gauge at infinity. Our boundary condition is \(\widehat{F}_{\mu\nu} \rightarrow 0\) at \(|y| \rightarrow \infty\) for the instanton configuration. Thus one has to remove parts due to backgrounds from the action in Eq. (5.2). One can easily achieve this by defining the self-duality equation as follows [1]

\[
\text{ad}_{[x^\mu, x^\nu]}_* = \pm \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} \text{ad}_{[x^\lambda, x^\sigma]}_*
\]

\[
\Leftrightarrow [V_\mu, V_\nu] = \pm \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma}[V_\lambda, V_\sigma],
\] (5.10)
where we used Eq. (5.8). From the above definition, it is obvious that the constant part in $[x^\mu, x^\nu] = -i(\theta (\hat{F} - B) \theta)^{\mu\nu}$, i.e. $i\theta^{\mu\nu}$, can be dropped. Note that, for nondegenerate $\theta^{\mu\nu}$'s, the first self-duality equation in Eq. (5.10) reduces to Eq. (4.2). An advantage of background independent formulation is that Eq. (5.10) holds for an arbitrary non-degenerate $\theta^{\mu\nu}$ and there is no need to specify a background.

It is obvious that the vector fields preserve the volume form $\varepsilon_4$, i.e., $\mathcal{L}_{V_\mu}\varepsilon_4 = 0$, where $\mathcal{L}_{V_\mu}$ is the Lie derivative along $V_\mu$ since all the vector fields $V_\mu$ are divergence free, i.e. $\partial_\alpha V_\mu^\alpha = 0$. Incidentally, this is simply the Liouville theorem in symplectic geometry [6]. In consequence, instanton configurations are mapped to the volume preserving diffeomorphism, $SDiff(M)$, satisfying Eq. (5.10).

So we arrive at the result of Ashtekar et al. [71]. Their result is summarized as follows [72]. Let $\mathcal{M}$ be an oriented 4-manifold and let $V_\mu$ be vector fields on $\mathcal{M}$ forming an oriented basis for $T\mathcal{M}$. Then $V_\mu$ define a conformal structure $[G]$ on $\mathcal{M}$. Suppose that $V_\mu$ preserve a volume form on $\mathcal{M}$ and satisfy the self-duality equation

$$[V_\mu, V_\nu] = \pm \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} [V_\lambda, V_\sigma].$$

(5.11)

Then $[G]$ defines an (anti-)self-dual and Ricci-flat metric.

The (inverse) metric determined by the vector fields in Eq. (5.7) is then given by [71]

$$G^{\alpha\beta} = \det V^{-1} V_\mu^\alpha V_\nu^\beta \delta^{\mu\nu},$$

(5.12)

where the background spacetime metric was taken as $g_{\mu\nu} = \delta_{\mu\nu}$ for simplicity.

Motivated by the similarity of Eq. (5.11) to the self-duality equation of Yang-Mills theory, Mason and Newman showed [73] that, if we have a reduced Yang-Mills theory where the gauge fields take values in the Lie algebra of $SDiff(M)$, which is exactly the case for the action (5.2) through the map (5.7), Yang-Mills instantons are actually equivalent to gravitational instantons [12]. We showed that this is the case for NC electromagnetism. See also related works [74, 75].

Since the second expression in Eq. (5.2) is the bosonic part of the IKKT matrix model [70], our current result is consistent with the claim that the IKKT matrix model is a theory of gravity (or type IIB string theory). See also recent works [76] addressing this issue directly from IKKT matrix model. In addition the result in [77] obviously indicates the existence of 4-dimensional massless gravitons in NC gauge theory, which supports our claim about the emergent gravity from NC electromagnetism [13].

It is a priori not obvious that the self-dual electromagnetism in NC spacetime is equivalent to the self-dual Einstein gravity. Therefore it should be helpful to have explicit nontrivial examples to appreciate how it works. It is not difficult to find them from Eq. (5.11), which was already done for the Gibbons-Hawking metric [78] in [72] and for the real heaven solution [79] in [80].

---

[12] In their approach, there exists a tetrad freedom which was referred to as a (metric-preserving) gauge transformation. We also showed that the NC $U(1)$ gauge symmetry [5.3] appears as the diffeomorphic equivalence [5.9] between the metrics on $\mathcal{M}$.

[13] We are grateful to S. Nagaoka for drawing our attention to their paper.
The Gibbons-Hawking metric \cite{78} is a general class of self-dual, Ricci-flat metrics with the triholomorphic \(U(1)\) symmetry which describes a particular class of ALE and asymptotically locally flat (ALF) instantons. Let \((a_i, U), \ i = 1, 2, 3,\) are smooth real functions on \(\mathbb{R}^3\) and define \(V_i = -a_i \frac{\partial}{\partial x^i} + \frac{\partial}{\partial \tau}\) and \(V_4 = U \frac{\partial}{\partial \tau},\) where \(\tau\) parameterizes circles and the Killing vector \(\partial/\partial \tau\) generates the triholomorphic \(U(1)\) symmetry. Eq.\((5.11)\) then becomes the equation \(\nabla U + \nabla \times \vec{a} = 0\) and the metric whose inverse is \((5.12)\) is given by
\[
ds^2 = U^{-1} (d\tau + \vec{a} \cdot d\vec{x})^2 + U d\vec{x} \cdot d\vec{x},
\]
where \(\vec{x} \in \mathbb{R}^3.\)

The real heaven metric \cite{79} describes four dimensional hyper-Kähler manifolds with a rotational Killing symmetry which is also completely determined by one real scalar field \cite{81}. The vector fields \(V_\mu\) in this case are given by \cite{80}
\[
V_1 = \frac{\partial}{\partial x^1} - \partial_2 \psi \frac{\partial}{\partial \tau},
V_2 = \frac{\partial}{\partial x^2} + \partial_1 \psi \frac{\partial}{\partial \tau},
V_3 = e^{\psi/2} \left( \sin\left(\frac{\tau}{2}\right) \frac{\partial}{\partial x^3} + \partial_3 \psi \cos\left(\frac{\tau}{2}\right) \frac{\partial}{\partial \tau} \right),
V_4 = e^{\psi/2} \left( \cos\left(\frac{\tau}{2}\right) \frac{\partial}{\partial x^3} - \partial_3 \psi \sin\left(\frac{\tau}{2}\right) \frac{\partial}{\partial \tau} \right),
\]
where the rotational Killing vector is given by \(c_i \partial_i \psi \partial/\partial \tau\) with constants \(c_i\) \((i = 1, 2)\) and the function \(\psi\) is independent of \(\tau.\) Eq.\((5.11)\) is then equivalent to the three-dimensional continual Toda equation \((\partial_1^2 + \partial_2^2) \psi + \partial_3^2 e^\psi = 0\) and the metric is determined by Eq.\((5.12)\) as
\[
ds^2 = \left(\partial_3 \psi\right)^{-1} (d\tau + a^i dx^i)^2 + \left(\partial_3 \psi\right) (e^{\psi/2} dx^i dx^i + dx^3 dx^3)
\]
where \(a^i = \varepsilon^{ij} \partial_j \psi.\)

The canonical structures, in particular, complex and Kähler structures, of the self-dual system \((5.11),\) have been fully studied in a beautiful paper \cite{82}. The arguments in \cite{82} are essentially the same as ours leading to Eq.\((4.43)\). It was also shown there how the Plebański’s heavenly equations \cite{66} can be derived from Eq.\((5.11)\). It should be interesting to recall \cite{83} that Eq.\((5.11)\) can be reduced to the \textit{sdiff} \((\Sigma_g)\) chiral field equations in two dimensions, where \textit{sdiff} \((\Sigma_g)\) is the area preserving diffeomorphisms of a Riemann surface of genus \(g.\)

Now we will study in detail the structure of twistor space inherent in Eq.\((5.11).\) Define holomorphic vector fields \(\mathcal{V}\) and \(\mathcal{W}\) locally by
\[
\mathcal{V} = V_2 + i V_1 = f_i \frac{\partial}{\partial z_i}, \quad \mathcal{W} = V_4 + i V_3 = g_i \frac{\partial}{\partial z_i},
\]

34
where $f_i, g_i (i = 1, 2)$ are complex functions on $\mathcal{M}$. In terms of these vector fields, Eq. (5.11) reduces to the following triple

$$[\mathcal{V}, \mathcal{W}] = 0, \quad [\mathcal{V}, \bar{\mathcal{W}}] = [\mathcal{W}, \mathcal{W}].$$

(5.17)

Substituting (5.16) into (5.17), we find that the equations are satisfied identically if

$$\frac{\partial f_i}{\partial \bar{z}_j} = \frac{\partial g_i}{\partial \bar{z}_j} = 0.$$  

(5.18)

So we can construct a hypercomplex (or hyper-Kähler) structure on $\mathcal{M}$ locally out of four holomorphic functions $f_i$ and $g_i$, or globally out of two holomorphic vector fields [72].

If we introduce the following $\mathbb{P}^1$-family of vector fields parameterized by $t \in \mathbb{P}^1$,

$$\mathcal{L} = \mathcal{V} + t\mathcal{W}, \quad \mathcal{N} = \mathcal{W} - t\mathcal{V},$$

(5.19)

the self-dual Einstein equations (5.11) are more compactly written as

$$[\mathcal{L}, \mathcal{N}] = 0$$

(5.20)

with the volume preserving constraint

$$\mathcal{L}\varepsilon_4 = 0 = \mathcal{N}\varepsilon_4.$$  

(5.21)

Eq. (5.20) can be interpreted as a Lax pair form of curved self-dual spacetime.

It is easy to see that Eq. (5.17) defines a hyper-Kähler structure on $\mathcal{M}$. We construct a $t$-dependent two-form $\Psi(t)$ on $\mathcal{M}$ by contracting the volume form $\varepsilon_4$ by $\mathcal{L}$ and $\mathcal{N}$

$$\Psi(t) = \varepsilon_4(\cdot, \cdot, \mathcal{L}, \mathcal{N}) = \omega + it\Omega + t^2\bar{\omega}.$$  

(5.22)

where

$$\omega = \varepsilon_4(\cdot, \cdot, \mathcal{V}, \mathcal{W}),$$

$$i\Omega = \varepsilon_4(\cdot, \cdot, \mathcal{W}, \bar{\mathcal{W}}) - \varepsilon_4(\cdot, \cdot, \mathcal{V}, \bar{\mathcal{V}}),$$

$$\bar{\omega} = \varepsilon_4(\cdot, \cdot, \bar{\mathcal{V}}, \mathcal{W}).$$

(5.23)

Note that we need the property (5.20) to make sense of Eq. (5.22). The two-form $\Psi(t)$ in (5.22) is an exact analogue of Eq (4.37) and the resulting consequences are exactly parallel to section 4. Nevertheless it will be useful to understand parallel arguments with section 4 for this purely geometrical setting since it seems to be very powerful for later applications.

It is straightforward to prove (see Eq.(8) in [82]) using the Cartan’s homotopy formula $\mathcal{L}_X = i_X d + d i_X$ that $\Psi(t)$ is closed, i.e., $d\Psi(t) = 0$. We see from the proof that Eq. (5.21) is analogous to the Bianchi identity. We can thus define on $\mathcal{M}$ the three non-degenerate symplectic forms

$$\omega^a = \left( \omega^1 = -\frac{i}{2}(\omega - \bar{\omega}), \omega^2 = \frac{1}{2}(\omega + \bar{\omega}), \omega^3 = -\Omega \right).$$

(5.24)
\( \omega^a \) are also three Kähler forms compatible with three complex structures on \( \mathcal{M} \) (see section IV. A and B in [82]) and thus define the hyper-Kähler structure on \( \mathcal{M} \). Therefore any metric defined by Eq.(5.11) with the constraint (5.21) should be hyper-Kähler, as also shown in section 4.

Since the two-form \( \Psi(t) \) is closed and degenerate for any \( t \in \mathbb{P}^1 \), one can introduce holomorphic coordinates in a natural fashion via the Darboux theorem on each coordinate chart on \( \mathbb{P}^1 \) such that \( \Psi(t) \) is a holomorphic \((2,0)\)-form on the local chart. For example, Eq.(4.39) in a neighborhood of \( t = 0 \) (the south pole of \( \mathbb{P}^1 \)) and Eq.(4.44) in a neighborhood of \( t = \infty \) (the north pole of \( \mathbb{P}^1 \)). But they can be consistently patched along the equator of \( \mathbb{P}^1 \) in such a way that the total space \( Z \), the twistor space, including \( \mathbb{P}^1 \) becomes a three-dimensional complex manifold as we explained in section 4.

We know from Eq.(5.23) that \( \Omega \) is rank 4 while \( \omega \) and \( \bar{\omega} \) are both rank 2. As a direct consequence, we immediately get Eq.(4.43) [82]. In terms of local coordinates, it reduces to the complex Monge-Ampère equation [65] or the Plebański equation [66]. Since \( \Omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \zeta(B + F) \) in Eq.(5.23) can always serve as a symplectic form on both coordinate charts (note that it is rank 4), two sets of coordinates at \( t = 0 \) and \( t = \infty \) should be related to each other by a canonical transformation (where we refer the canonical transformation in a more general sense). A beautiful fact was shown in [82] that the canonical transformation between them is generated by the Kähler potential appearing in the complex Monge-Ampère equation or the Plebański equation. In other words, the Kähler potential is a generating function of canonical transformations or a transition function of three-dimensional complex manifold \( Z \) as a holomorphic vector bundle [32].

Finally we would like to discuss an interesting fact that Eq.(5.11) can be reduced to \( sdiff(\Sigma_g) \) chiral field equations in two dimensions [83]. We will directly show using canonical transformations that the Husain’s equation [83] (where we denote \( \Lambda_x = \partial_x \Lambda, \ \Lambda_{xq} = \partial_x \partial_q \Lambda, \) etc.)

\[
\Lambda_{xx} + \Lambda_{yy} + \Lambda_{xq} \Lambda_{yp} - \Lambda_{xp} \Lambda_{yq} = 0
\] (5.25)

is equivalent to the first heavenly equation [66], which is a governing equation of self-dual Einstein gravity. This implies that the self-dual system (5.11) is deeply related to two dimensional \( SU(\infty) \) chiral models [74, 75]. An interesting implication of this connection will be briefly discussed in next section.

Although the first heavenly equation was also obtained in [83] by a different reduction of Eq.(5.11), an explicit canonical transformation between them was not available there. In the course of derivation, we will find an interesting symplectic structure of Eq.(5.11) which was also noticed in [84] from a different approach. Ours is more straightforward.

By complex coordinates, \( u = x + iy, \ v = q + ip \), Eq.(5.25) reads as

\[
\Lambda_{u\bar{u}} - (\Lambda_{uv} \Lambda_{\bar{u}\bar{v}} - \Lambda_{u\bar{v}} \Lambda_{\bar{u}v}) = 0.
\] (5.26)

We will now apply a similar strategy as the Appendix in [85]. Define two functions \( A = \Lambda_u \) and
\( B = \Lambda \bar{u} \). Eq. (5.26) is then equivalent to

\[
A\bar{u} - (A u B_\bar{v} - A_\bar{v} B_v) = 0 \quad (5.27)
\]

\[
A\bar{u} = B u. \quad (5.28)
\]

Instead of looking on \( A \) as a function of \((u, v, \bar{u}, \bar{v})\), we take \( A \) as a coordinate and look on \( f \equiv \bar{u} \) and \( g \equiv B \) as functions of \((\xi^1 \equiv A, \xi^2 \equiv u, \tilde{\xi}^1 \equiv v, \tilde{\xi}^2 \equiv \bar{v})\). This is a canonical transformation which is well-defined as long as \( A\bar{u} \neq 0 \).

It is convenient to denote coordinates by \( \xi^A, \tilde{\xi}^A, A = 1, 2 \) for compact notation and to use the antisymmetric tensors \( \epsilon^{AB} \) and \( \epsilon^{\tilde{A}\tilde{B}} \) to raise indices in a standard way, e.g. \( \xi^A = \epsilon^{AB} \xi_B, \tilde{\xi}^A = \epsilon^{\tilde{A}\tilde{B}} \tilde{\xi}_B \). It is easy to show that, after the above coordinate transformation, Eq. (5.27) and Eq. (5.28) are transformed to (after applying a series of chain rules)

\[
\epsilon^{\tilde{A}\tilde{B}} \partial_{\tilde{A}} f \partial_{\tilde{B}} g = 1 \quad (5.29)
\]

and

\[
\epsilon^{AB} \partial_A f \partial_B g = 1, \quad (5.30)
\]

respectively. Thus the Husain’s equation is reduced to the two Poisson bracket relations which relate \( f \) and \( g \) to \( \xi^A \) and \( \tilde{\xi}^A \) by canonical transformations.

One can show that Eqs. (5.29) and (5.30) lead to the result [84] that there is a function \( K \) such that

\[
\partial_A f \partial_B g - \partial_B f \partial_A g = \partial_A \partial_B K. \quad (5.31)
\]

Using some relation between Poisson brackets (Eq. (11) in [84]), we arrive at the result that \( K \) satisfies the Plebański equation

\[
\epsilon^{\tilde{A}\tilde{B}} \partial_{\tilde{A}} \partial_A K \partial_B \partial_B K = \epsilon_{AB}. \quad (5.32)
\]

This completes the proof that Eq. (5.25) is equivalent to the first heavenly equation (5.32).

### 6 Discussion

Let us briefly recapitulate our main results. A basic reason for the emergent gravity from NC spacetime is that the \( \Lambda \)-symmetry (1.6) can be regarded as a par with \( Diff(M) \), which results from the Darboux theorem in symplectic geometry. The spontaneous symmetry breaking (1.7) also comes into play for the emergent gravity. In general the emergent gravity needs to incorporate NC deformations of the diffeomorphism symmetry since it should be defined on NC spacetime (1.3). In this context, the gauge equivalence (3.2) in deformation quantization might be interpreted as a quantum equivalence principle.

We have derived the exact SW maps with derivative corrections in two ways; from the SW equivalence (2.27) and from the deformation quantization (3.2). But they should be the same since the
SW equivalence (2.26) is the equivalent statement as the gauge equivalence (3.2) as we showed in the semi-classical limit. It should be interesting, in its own right, to explicitly check the consistency between two different approaches for the derivative corrections.

We showed in section 4 and 5 that the self-dual Einstein gravity is emerging from self-dual NC electromagnetism neglecting derivative corrections, i.e., defined with the Poisson bracket (1.1). We thus expect that the derivative corrections give rise to higher order gravity, e.g., $R^2$ gravity. It should be important to precisely determine the form of the higher order gravity. Since the emergent gravity is in general a full quantum deformation of $Diff(M)$, it might modestly be identified with a NC gravity [29], as we argued in section 5. If this is the case, the SW maps in section 2 and 3 including derivative corrections may be related to those of NC gravity. Our construction in section 4 and 5 also implies that we need a NC deformation of twistor space [86] to describe general nonlinear gravitons in NC gravity.

Recently, it was found [87] that NC field theory is invariant under the twisted Poincaré symmetry where the action of generators is now defined by the twisted coproduct in the deformed Hopf algebras. We think that the twisted Poincaré symmetry, especially the deformed Hopf algebra and quantum group structures, will be important to understand the NC field theory and gravity correspondence since underlying symmetries are always an essential guide for physics. Actually this symmetry plays a prominent role to construct NC gravity [29, 5].

Unlike the homogeneous background (1.7), there could be an inhomogeneous condensation of $B$-fields in a vacuum. In this case, we expect a nontrivial curved spacetime background, e.g., a Ricci-flat Einstein manifold instead of flat $\mathbb{R}^4$ and we need a quantization on general symplectic (or Poisson) manifolds [5]. Our approach suggests an intriguing picture for an inhomogeneous background, for example, specified by

$$\langle B'_{\mu\nu}(x) \rangle_{\text{vac}} = (\theta^{-1})_{\mu\nu}(x).$$

(6.1)

One may regard $B'_{\mu\nu}(x)$ as coming from an inhomogeneous gauge field condensation on a constant $B$-background, say, $B'_{\mu\nu}(x) = (B + F_{\text{back}}(x))_{\mu\nu}$. For instance, if $F_{\text{back}}(x)$ is an instanton, our result implies that the vacuum manifold (6.1) is a Ricci-flat Kähler manifold. From NC gauge theory point of view, this corresponds to the description of NC gauge theory in instanton backgrounds [88]. Therefore the NC gauge theory with nonconstant NC parameters $\theta^{\mu\nu}(x)$ may be interpreted as that defined by the usual Moyal star product (1.2) but around a nonperturbative solution described by $F_{\text{back}}(x)$. The gravity picture in this case corresponds to a (perturbative) NC gravity on a curved manifold. It will be interesting to see whether this reasoning can shed some light on NC gravity.

Recently we suggested in [89] a very simple toy model for emergent gravity. We claimed that (2+1)-dimensional NC field theory for a real scalar field in large NC limit $\theta \to \infty$ is equivalent to two dimensional string theory via $c = 1$ Matrix model. See [90] for a field theory discussion from this aspect. This claim is based on the well-known relation [91]

Real field on NC $\mathbb{R}^2$ (or $\Sigma_g$) $\iff$ $N \times N$ Hermitian matrix at $N \to \infty$,  

(6.2)
where $\Sigma_g$ is a Riemann surface of genus $g$ which can be quantized via deformation quantization. In two dimensions, a symplectic 2-form $\omega$ is a volume form and Hodge-dual to a real function. So symplectomorphism is equal to area-preserving diffeomorphism (APD). (In higher dimensions, symplectomorphism is much smaller than volume-preserving diffeomorphism.) We observed in Eq. (1.11) that symplectomorphism can be identified with NC gauge symmetry. In two dimensions, we thus have the relation: Symplectomorphism = APD = NC gauge symmetry. So, if there is a NC field theory which is gauge invariant, the NC field theory is then APD invariant and thus we expect an emergent gravity in two dimensions from this NC field theory.

We can infer the nature of two-dimensional emergent gravity from four-dimensional case. Noting that the electromagnetic 2-form $F$ acts as a deformation of the symplectic (or Kähler) structure, it is natural to guess that a real scalar field plays the same role in two dimensions. Since the Kähler potential behaves as a generating function of canonical transformations as we observed in section 5, it is also plausible that the real scalar field is a generating function of APDs and acts as a Kähler potential. We hope to discuss this interesting correspondence in a separate publication.

In section 5, we showed that the Husain’s equation (5.25) is equivalent to the first heavenly equation (5.32). Here we note that Eq. (5.25) is the $SU(N)$ self-dual Yang-Mills equation in the limit $N \to \infty$ [92], which implies that $SU(N \to \infty)$ Yang-Mills instantons are gravitational instantons too. This interesting fact is also coming from the relation (6.2) since the gauge fields in $SU(N)$ Yang-Mills theory on $\mathbb{R}^4$ are all $N \times N$ Hermitian matrices and thus they can be mapped to real scalar fields on a six-dimensional space $\mathbb{R}^4 \times \Sigma_g$. This seems to imply that the AdS/CFT duality [93] might be deeply related to the NC field theory and gravity correspondence.

In order to more extensively understand the nature of emergent gravity, it is useful to consider couplings with matter fields. To do this, we need to know the SW maps for currents and energy-momentum tensors for matter fields. These were obtained in [94] at leading order. It turned out [55, 23] that the gravitational coupling with matter fields is not universal unlike as general relativity. It deserves to ask more study, especially for experimental verifications.

The emergent gravity from NC gauge theory discussed in this paper may have interesting implications to string theory and black-hole physics. We briefly discuss possible implications citing relevant literatures.

It was argued [95] that tachyon condensation at the fixed points of noncompact nonsupersymmetric orbifolds, e.g. $\mathbb{C}^2/\Gamma$, drives these orbifolds to flat space or supersymmetric ALE spaces. But ALE spaces are $U(1)$ instantons in flat NC $\mathbb{R}^4$ [2, 3]. Does it imply that the closed string tachyon condensation can be understood as an open string tachyon condensation? The picture in [96] may be useful for this problem.

Microscopic black hole entropy in string theory [97] was derived by counting the degeneracy of BPS soliton bound states, mostly involved with instanton moduli space. If we simply assume that the instanton moduli space is coming from NC $U(1)$ instantons, then the counting of the degeneracy is just the counting of all possible hyper-Kähler geometries inside the black hole horizon, according to
our picture. This is very reminiscent of the Mathur’s program for black hole entropy [98].

We showed that the equivalence between NC $U(1)$ instantons and gravitational instantons could be beautifully understood in terms of the twistor space. We think that the equivalence and its twistor space structure should have far-reaching applications to Nekrasov’s instanton counting [99] and topological strings for crystal melting [100].

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