Countably-categorical Boolean rings with distinguished ideals

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Abstract We describe and classify countable Boolean rings with finitely many distinguished ideals whose elementary theory is countably categorical. This extends the description by Macintyre and Rosenstein and subsequent authors of countably categorical Boolean algebras with finitely many distinguished ideals. Following Pierce, our classification takes a topological approach using the language of PO systems (partially ordered sets with a distinguished subset) and topological Boolean algebras. We discuss how our findings link with previous results, but the paper is otherwise self-contained.

1 Introduction

A countable structure $M$ for a language $L$ is $\omega$-categorical if $M$ is determined up to isomorphism within the class of countable $L$-structures by its first order properties. By the theorem of Engeler, Ryll-Nardzewski and Svenonius [4,10,11], this is equivalent to saying that for all $r$, the automorphism group of $M$ is almost $r$-transitive on $M$: that is, it has only finitely many orbits on $M^r$.

The problem of describing $\omega$-categorical Boolean algebras with finitely many distinguished ideals arose during the classification by Macintyre and Rosenstein [5] of $\omega$-categorical rings without nilpotent elements. If $C_1,\ldots,C_n$ are closed subsets of a compact Stone space $X$ and $J_i$ is the ideal of $R$ (the Boolean algebra underlying $X$) corresponding to $C_i$ under the Stone correspondence, they showed that the Boolean algebra system $(R,J_1,\ldots,J_n)$ with distinguished ideals is $\omega$-categorical iff $\langle J_1,\ldots,J_n \rangle$ is a finite sub-algebra of the set $H(R)$ of all ideals of $R$, viewed as a Heyting algebra, and $R/J$ has only finitely many atoms for each $J \in \langle J_1,\ldots,J_n \rangle$.

There have been a number of subsequent studies of this condition, mostly from the viewpoint of Heyting algebras. In this paper we adopt a topological approach that looks at partitions of the Stone space, working with partially ordered sets rather than the

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more complex structure of Heyting algebras. This is a natural approach, as Macintyre and Rosenstein’s result originally arose as a question about distinguished closed subsets of a Stone space. We also generalise by considering Boolean rings which may not have an identity rather than just Boolean algebras, as both compact and non-compact Stone spaces arise naturally in the study of \(\omega\)-categorical structures.

First, we recall some definitions.

**Definition 1.1** A PO system is a set \(P\) with an anti-symmetric transitive relation \(<\); equivalently, it is a poset with a distinguished subset \(P_1\), where \(p < p'\) if \(p \in P_1\).

If \(X\) is a topological space and \(S \subseteq X\), we write \(S'\) for the derived set of \(S\), namely the set of all limit points of \(S\), and \(\overline{S}\) for the closure of \(S\). We say that \(S\) is crowded (or dense-in-itself) if it has no isolated points; equivalently, if \(S \subseteq S'\).

A topological Boolean Algebra (TBA) is a Boolean algebra with an additional unary operator \(\prime\) satisfying \((A \cup B)' = A' \cup B'\) and \((A')' \subseteq A';\) and a closure algebra is a Boolean algebra together with an additional closure operator \(\overline{\cdot}\) such that \(A \subseteq \overline{A},\overline{A \cup B} = \overline{A} \cup \overline{B},\) and \(\overline{\overline{A}} = 0\). A TBA \((\mathscr{D}, \prime)\) can also be viewed as a closure algebra \((\mathscr{D}, \prime)\) by setting \(\overline{\mathcal{B}} = B \cup B'\) for \(B \in \mathscr{D}\). If \(X\) is a topological space, then \((2^X, \prime)\) and \((2^X, \overline{\cdot})\) become a closure algebra and a TBA respectively. For \(A \subseteq 2^X\), we write \(\langle A, \prime \rangle\) and \(\langle A, \overline{\cdot} \rangle\) for the closure sub-algebra and sub-TBA of \(2^X\) respectively generated by \(A\), which each include \(X\) itself.

Let \(\mathscr{D}\) be a TBA and \(P\) be a finite PO system; write \(X = 1_{\mathscr{D}}\). We will say that a partition \(\mathcal{X}\) of \(X\) is a complete \(P\)-partition of \(X\) if there is a labelling \(\mathcal{X} = \{X_p \mid p \in P\}\) such that \(X_p = \bigcup_{q < p} X_q\) for all \(p \in P\). A partition \(\mathcal{X}\) of a topological space \(X\) is finite-crowded if \(\mathcal{X}\) is finite and every element of \(\mathcal{X}\) is either finite or crowded.

We will use the term \(\omega\)-Stone space to denote the Stone space of a countable Boolean ring.

If \(\{A_i \mid i \in I\}\) are subsets of \(X\), we say that a partition \(\{X_p \mid p \in P\}\) of \(X\) refines \(\{A_i \mid i \in I\}\) if for each \(i \in I\), we can find \(Q_i \subseteq P\) such that \(A_i = \bigcup_{q \in Q_i} X_q\). Our primary result is the following.

**Theorem 1.2** Let \(J_1, \ldots, J_n\) be distinguished ideals of the countable Boolean ring \(R\), corresponding to closed subsets \(C_1, \ldots, C_n\) of the Stone space \(X\) of \(R\). Then the following are equivalent, writing \(C^{(n)} = \{C_1, \ldots, C_n\}\):

(a) \((R, J_1, \ldots, J_n)\) is \(\omega\)-categorical;
(b) \((C^{(n)}, \prime)\) is finite, and each of its atoms has only finitely many isolated points;
(c) \((C^{(n)}, \overline{\cdot})\) is finite, and its atoms form a finite-crowded partition of \(X\);
(d) the atoms of \((C^{(n)}, \prime)\) form a finite-crowded complete \(P\)-partition of \(X\) for some finite PO system \(P\);
(e) \(C^{(n)}\) can be refined to a finite-crowded complete \(P\)-partition of \(X\) for some finite PO system \(P\).

We will make frequent use of the duality established by Pierce [3] between finite TBAs generated by their closed elements and finite PO systems, and will also use the well-known dualities between finite closure algebras generated by their closed elements, finite posets, and finite Heyting algebras.

In particular, the next result will form an important stepping stone to the proof of the above theorem. If \(C\) is a closed subset of the Stone space of the Boolean ring \(R\), we write \(I(C)\) for the corresponding ideal of \(R\), namely \(I(C) = \{a \in R \mid a \cap C = \emptyset\}\).
Theorem 1.3 Let $P$ be a PO system and $\{X_p \mid p \in P\}$ a complete $P$-partition of the Stone space of the countable Boolean ring $R$.

Then the system $(R, \{I(X_p) \mid p \in P\})$ is $\omega$-categorical iff $\{X_p\}$ is finite-crowded.

These Theorems will lead in turn to the results in Section 5 which classify $\omega$-categorical Boolean rings with distinguished ideals in terms firstly of finite PO systems and secondly of finite posets. The various links with existing results are described in Section 6.

1.1 Notation

If $A$ and $B$ are sets, we write $A \subseteq B$ to denote that $A$ is contained in $B$, $A - B$ for $\{x \in A \mid x \notin B\}$, $A + B$ for the disjoint union of $A$ and $B$, and $|A|$ for the cardinal number of $A$. We write $\mathbb{N}$ and $\mathbb{N}_+$ for the non-negative and positive integers respectively. If $(J_1, \ldots, J_n)$ is an $n$-tuple, we will write $J^{(n)}$ for $(J_1, \ldots, J_n)$ (i.e. when the order matters) and $J''^{(n)}$ for $(J_1', \ldots, J_n')$, and similarly for $C^{(n)}$, $Q^{(n)}$ etc.

If $R$ is a Boolean ring with Stone space $X$, we identify $R$ with the ring of compact open subsets of $X$ under the Stone correspondence, whereby atoms of the Boolean ring correspond to isolated points of the Stone space. $R$ has an identity iff $X$ is compact, and is countable iff $X$ is second countable (i.e. the topology has a countable base of open sets). For $A \in R$, we write $(A)$ for the ideal $\{B \in R \mid B \subseteq A\}$.

2 Prior results for finite $P$-partitions

We will need the following additional definitions and results for complete $P$-partitions when $P$ is finite, which are adapted from [3].

Definition 2.1 Let $P$ be a PO system and let $\mathcal{X} = \{X_p \mid p \in P\}$ be a complete $P$-partition of the Stone space $X$ of the Boolean ring $R$.

For $p, q \in P$, we write $p \preceq q$ to mean $p < q$ or $p = q$; $P^d$ for $\{p \in P \mid p \preceq q\}$; and $P^{d^+} = \{p \in P \mid p > q\}$ for the disjoint union of $A$ and $B$, and $|A|$ for the cardinal number of $A$. We write $\mathbb{N}$ and $\mathbb{N}_+$ for the non-negative and positive integers respectively. If $(J_1, \ldots, J_n)$ is an $n$-tuple, we will write $J^{(n)}$ for $(J_1, \ldots, J_n)$ (i.e. when the order matters) and $J''^{(n)}$ for $(J_1', \ldots, J_n')$, and similarly for $C^{(n)}$, $Q^{(n)}$ etc.

If $R$ is a Boolean ring with Stone space $X$, we identify $R$ with the ring of compact open subsets of $X$ under the Stone correspondence, whereby atoms of the Boolean ring correspond to isolated points of the Stone space. $R$ has an identity iff $X$ is compact, and is countable iff $X$ is second countable (i.e. the topology has a countable base of open sets). For $A \in R$, we write $(A)$ for the ideal $\{B \in R \mid B \subseteq A\}$.

An element $A \in R$ is $p$-trim, and we write $t(A) = p$, if $T(A) = \{q \in P \mid q \geq p\}$, with also $|A \cap X_p| = 1$ if $p \in P^d$, and is trim if it is $p$-trim for some $p \in P$.

For $A \in R$, the $\mathcal{X}$-measure of $A$ is the formal sum $\mu(A) = \sum_{p \in F} \eta_{p} p$, where $F = (A)_{\min}$, $\eta_p = 1$ for $p \in F - P^d$ and $\eta_p = |A \cap X_p|$ for $p \in F \cap P^d$. A partition of $A$ into trim sets is a $\mu(A)$-partition if it contains precisely $\eta_{p}$ $p$-trim sets for each $p \in F$, as we shall see, this equates to a minimum decomposition of $A$ into disjoint trim sets. We set $\mu(0) = 0$.

An extended PO system is a triple $[P, L, f]$, where $P$ is a PO system, $L$ is a lower subset of $P$, and $f : L_{\min}^d \to \mathbb{N}_+$, where $L_{\min}^d = L_{\min} \cap P^d$. We will say that $[P, L, f]$ is a finite-crowded extended PO system if $P$ is finite and $P^d \subseteq L_{\min}$.

If $[P, L, f]$ is an extended PO system, then $\mathcal{X}$ is a complete $[P, L, f]$-partition of $X$ if it is a complete $P$-partition such that $X_p$ is compact iff $p \in L$ and $|X_p| = f(p)$ for $p \in L_{\min}^d$.

If $\{X_p \mid p \in P\}$ and $\{Y_p \mid p \in P\}$ are complete $P$-partitions of the Stone spaces $X$ and $Y$ respectively, a homeomorphism $\alpha : X \to Y$ is a $P$-homeomorphism if $X_p\alpha = Y_p$ for all $p \in P$. 

We will now proceed to classify $\omega$-categorical Boolean rings with distinguished ideals in terms of finite PO systems and secondly of finite posets. The various links with existing results are described in Section 6.
Remark 2.2 The broader definitions of “trim” and “semitrim” partitions in [3] apply where $P$ is an infinite PO system or poset and where $\{X_p \mid p \in P\}$ is a partition of a dense subset of $X$ rather than a complete partition of the whole of $X$.

If $\{X_p \mid p \in P\}$ is a complete $P$-partition of $X$ for a PO system $P$, then it follows that $X_p = \bigcup_{q \leq p} X_q$ for all $p \in P$, as $X_p = X_p \cup X'_p$; that $X_p$ has no isolated points if $p \notin P^d$; and that $X_p$ is discrete if $p \in P^d$.

For $A \in R$ and $p \in P^d \cap T(A)_{\min}$, consideration of accumulation points shows that $|A \cap X_p|$ is necessarily finite. Hence $X_p$ is finite iff $X_p$ is compact and $p \in P_{\min}$.

We will use the following results of Apps [3], noting that for finite $P$, a complete $P$-partition is always a “trim” partition (as defined in [3]). In order to make this paper as self-contained as possible, we include proofs of Theorems 2.3 and 2.4. The proof of existence in Theorem 2.4 in the case of finite $P$, which is largely due to Pierce [8], is much simpler than the proof in [3] of the general case of infinite $P$.

Theorem 2.3 ([3, Theorem 3.5]) Let $P$ be a finite PO system and $\mathcal{X}$ and $\mathcal{Y}$ complete $P$-partitions of the Stone spaces $X$ and $Y$ of countable Boolean rings $R$ and $S$ respectively. Suppose $A \in R$ and $B \in S$ are such that $\mu(A) = \mu(B)$. Then there is a $P$-homeomorphism $\alpha : A \rightarrow B$.

Theorem 2.4 ([3, Corollary 6.2]) Let $[P, L, f]$ be a finite extended PO system. Then there is an $\omega$-Stone space $X$ that admits a complete $[P, L, f]$-partition $\mathcal{X}$, and $(X, \mathcal{X})$ is unique up to $P$-homeomorphisms.

3 Proof of Theorem 1.3

We will use the following result twice in what follows.

Proposition 3.1 Let $R$ be a countable Boolean ring with Stone space $X$, and $\mathcal{C}$ a closure subalgebra of $2^X$ generated by its closed elements such that the system $(R, \{U(C) \mid C \in \mathcal{C}\})$ is $\omega$-categorical. Then $\mathcal{C}$ is finite and each of its atoms has only finitely many isolated points.

Proof Let $\Psi$ be the group of homeomorphisms of $X$ that fix each closed element of $\mathcal{C}$. Now $\Psi$ is almost 1-transitive on the compact open subsets of $X$, and so there are only finitely many (say $n$) $\Psi$-invariant closed subsets of $X$, since such are complements of unions of $\Psi$-orbits on the compact opens. But $\mathcal{C}$ is generated as a Boolean algebra by its closed elements, and so $\mathcal{C}$ is finite with at most $2^n$ elements and $\Psi$ fixes every element of $\mathcal{C}$.

If now $A$ is an atom of $\mathcal{C}$ and has at least $k$ isolated points, then we can find compact open sets $\{B_j \mid j \leq k\}$ such that $|B_j \cap A| = j$, and these belong to different orbits of $\Psi$ as $A$ is $\Psi$-invariant. Hence $k$ is bounded and $A$ has only finitely many isolated points, as required. $\square$

To establish the $\omega$-categoricity of many Boolean structures (in this instance, those admitting a finite-crowded partition), it suffices to show that the structure-preserving automorphisms are almost 1-transitive. This idea forms the basis of the proof of Theorem 1.3.
Proof of Theorem 1.3

Let $\mathcal{X} = \{X_p \mid p \in P\}$ be a complete $P$-partition of the Stone space $X$ of the countable Boolean ring $R$, let $L = \{p \in P \mid X_p$ is compact$\}$ and let $\mathcal{C}$ be the closure subalgebra of $2^{\mathcal{X}}$ generated by $\{X_p \mid p \in P\}$.

If the system $(R, \{J_p \mid p \in P\})$ is $\omega$-categorical, where $J_p = I(X_p)$, then so is the system $(R, \{I(C) \mid C \in \mathcal{C}\})$. Hence by Proposition 3.1 $\mathcal{C}$ is finite and each of its atoms has only finitely many isolated points. But the map $P \to \mathcal{C} : p \mapsto X_p$ is injective, since $\prec$ is antisymmetric on $P$. Hence $P$ is finite and $\mathcal{X} = \text{At}(\mathcal{C})$, as $X_p = X_p - \bigcup\{g \in p \mid X_g\}$. But $\mathcal{X}$ is a $P$-partition and so is finite-crowded, if as $p < p$ then $X_p \subseteq X_p'$ and $X_p$ is crowded, if $p \in P$ then $X_p$ is discrete and so finite.

Now suppose that $\mathcal{X}$ is finite-crowded. We must show that the automorphisms of $R$ that preserve each $J_p$ are almost $r$-transitive on $R$ for each $r \geq 1$. Let $X_L = \bigcup_{p \in L} X_p$, which is closed and compact, and choose $E \in R$ such that $X_L \subseteq E$. Fix $r \geq 1$ and let $A^{(r)} \in R$. Set $A_{r+1} = E$ and let $B^{(s)}$ be the atoms of the (finite) Boolean sub-ring of $R$ generated by $A^{(r+1)}$, so that $s \leq 2^{r+1}$. Associate with $A^{(r)}$ the $(r + s + 1)$-tuple $\{\mu(B_1), \ldots, \mu(B_s), K^{(r+1)}\}$, where $K_j = \{i \leq s \mid B_i \subseteq A_j\}$ for $j \leq r + 1$. There are only finitely many possibilities for each $\mu(B_i)$, as if $p \in P^d$ then $|B_i \wedge X_p| \leq |X_p|$, and the latter is finite. Hence there are also only finitely many possibilities for this $(r + s + 1)$-tuple.

Suppose now that $C^{(r)}$ is another $r$-tuple in $R^r$, giving rise to atoms $D^{(s)}$ of the Boolean sub-ring of $R$ generated by $C^{(r+1)}$, where $C_{r+1} = E$, and to the same $(r + s + 1)$-tuple, so that $\mu(D_i) = \mu(B_i)$ and $B_i \subseteq A_j$ iff $D_i \subseteq C_j$ for each $i \leq s$ and $j \leq r + 1$.

If $A$ is compact, let $u = r + 1$ and $v = s$. Then $\bigcup_{j \leq u} A_j = \bigcup_{j \leq u} C_j = X$, as $X_L = X$.

If $A$ is not compact, let $u = r + 2$. As $X_L \subseteq E$ and $X_p$ is not compact for $p \notin L$, we can choose $A_u$ and $C_u$ in $R$, disjoint from $A_j$ and $C_j$ respectively for $j \leq r + 1$, such that $T(A_u) = T(C_u) = P - L$. Extending $A_u$ and $C_u$ as necessary, we may further assume that $\bigcup_{j \leq u} A_j = \bigcup_{j \leq u} C_j$. Let $v = s + 1$, $B_v = A_u$ and $D_v = C_u$, so that $K_v = \{v\}$. Then $\mu(B_v) = \mu(D_v) = \sum_{p \in E^d \text{ for } 1\text{-primes } F, F = (P - L)}$.

For each $i \leq v$, apply Theorem 2.3 to find a $P$-homeomorphism $\beta_i : B_i \to D_i$. These can be combined to give a $P$-homeomorphism $\beta$ of $X$, with $B_i\beta = D_i$ for $i \leq v$ and $\beta x = x$ for $x \notin \bigcup_{j \leq u} A_j$. Let $\alpha$ be the corresponding automorphism of $R$; then $A_j\alpha = C_j$ for $j \leq r$, and $\alpha$ will preserve each $J_p$ as $\beta$ preserves each $X_p$. Hence the system $(R, \{J_p \mid p \in P\})$ is $\omega$-categorical, as required. $\square$

4 Proof of Theorem 1.2

We will need the following Lemmas for the proofs both of Theorem 1.2 and of the classification results in Section 4.

Notation: Let $\mathcal{D}$ be a TBA. We write $\mathcal{D}^{(d)} = \{A \in \text{At}(\mathcal{D}) \mid A' \cap A = \emptyset\}$. For $A \in \mathcal{D}$, let $A^c = A \cap A'$ and $A^d = A - A^c$; if $X$ is a topological space and $\mathcal{D}$ is a sub-TBA of $2^X$, then $A^d$ is the set of isolated points in $A$. If $\mathcal{C}$ is a closure algebra or TBA and $N \in \mathcal{C}$, we write $N_{\min}$ for $\{A \in \text{At}(\mathcal{C}) \mid A \subseteq N \wedge \overline{A} = A\}$ and $N_{\max}$ for $1_{\min}$.

Lemma 4.1 Let $\mathcal{D}$ be a TBA and $\mathcal{C}$ a finite closure subalgebra of $(\mathcal{D}, \mathcal{D})$ such that $\mathcal{D} = (\mathcal{D}', \mathcal{D})$. Suppose also that $(A^d)' \subseteq \emptyset$ for each $A \in \text{At}(\mathcal{C})$. Then $A^c \subseteq (A^d)'$ for each $A \in \text{At}(\mathcal{C})$, $\mathcal{D}$ is finite and $\text{At}(\mathcal{D}) = \{A^c, A^d \mid A \in \text{At}(\mathcal{C})\} - \{\emptyset\}$. 


Proof For \( A \in \text{At}(\mathcal{C}) \), we have \( A = A^c + A^d \), so \( A^c \subseteq A' = (A^c)' \), \( A' \cap A^d = A \cap A' \cap A^d = \emptyset \) and so also \( A^c = A - A^d \). Further, \( A^d \) is closed and \( A = A^c + A^d \).

Now let \( \mathcal{D} \) be the (finite) Boolean subalgebra of \( \mathcal{D} \) generated by the non-empty elements of \( \{ A', A^d \mid A \in \text{At}(\mathcal{C}) \} \), which are all disjoint. Then \( \mathcal{D} \) is closed under \( ' \), as \( (A')' = A^c = A^c + A^d \), and \( A \in \mathcal{C} \) for \( A \in \text{At}(\mathcal{C}) \). But \( \mathcal{C} \subseteq \mathcal{D} \), and so \( \mathcal{D} \) is finite, and its atoms are as required. \( \square \)

The next Lemma is a consequence of Pierce [8, Proposition 8.12] and Apps [2, Lemma 3.3]; we include a proof which is based on Naturman [6, Lemma 4.2.7].

Lemma 4.2 Let \( \mathcal{C} \) be a finite closure algebra or TBA which is generated by its closed elements. If \( U \) and \( V \) are distinct atoms of \( \mathcal{C} \), then \( \overline{U} \neq \overline{V} \).

Proof A check shows that elements of \( \mathcal{C} \) have the form \( \bigcup_{1 \leq i \leq r} (C_i \cap (X - D_i)) \), where \( C_i \) and \( D_i \) are closed subsets of \( X \) with \( C_i, D_i \in \mathcal{C} \). Let \( U \) be an atom of \( \mathcal{C} \), hence \( U \subseteq (X - D) \) for some closed sets \( C, D \in \mathcal{C} \). But \( V \cap U = \emptyset \) and so \( V \subseteq (X - C) \cup D \). Hence either \( V \subseteq X \) and \( U \cup C \), or \( V \subseteq D \) and \( U \subseteq X - D \), when \( U \cap V = \emptyset \). In either case we have \( \overline{U} \neq \overline{V} \), as required. \( \square \)

Our third Lemma illustrates the duality between PO systems and finite TBAs generated by their closed elements. If \( P \) is a PO system, we note that \( (2^P,') \) becomes a TBA by setting \( \mathcal{C} = \{ p \in P \mid p < q \text{ some } q \in Q \} \) for \( Q \subseteq P \).

Lemma 4.3 Let \( \mathcal{D} \) be a finite TBA generated by its closed elements and write \( X = 1_\mathcal{D} \).

Then \( \text{At}(\mathcal{D}) \) is a complete \( P \)-partition, \( \{ X_p \mid p \in P \} \) say, of \( X \) for some finite PO system \( P \). The map \( \gamma : Q \rightarrow \bigcup_{q \in Q} X_q \) gives an isomorphism from the TBA \( (2^P,') \) to \( \mathcal{D} \), with lower subsets of \( P \) corresponding to closed elements of \( \mathcal{D} \). Moreover, if \( Q^{(n)} \) are lower subsets of \( P \), then \( \{ Q^{(n)} \} = 2^{2^P} \text{ iff } \langle Q_1, \ldots, Q_n, \gamma_1, \ldots, \gamma_n \rangle = \mathcal{D} \).

Proof Let \( \text{At}(\mathcal{D}) = \{ X_p \mid p \in P \} \), which is a finite partition of \( X \) as \( X \in \mathcal{D} \). Define a relation \( < \) on \( P \) via \( q < p \) iff \( X_q \subseteq X_p \). Now \( X_q \subseteq X_p \) iff \( q < p \). So \( < \) is antisymmetric and \( (P,<) \) is a PO system by Lemma 4.2 Moreover, \( X_p = \bigcup_{q \in P} X_q \) as \( X_p \in \mathcal{D} \), and so \( \{ X_p \mid p \in P \} \) is a complete \( P \)-partition of \( X \). The remaining statements now follow easily, using the fact that \( X_p = \bigcup_{q \subseteq X} X_q \). \( \square \)

We now have all the ingredients required to prove Theorem 1.2.

Proof of Theorem 1.2

(a) \( \Rightarrow \) (b) Immediate from Proposition 5.1
(b) \( \Rightarrow \) (c) Let \( \mathcal{C} = \langle C^{(n)} \rangle \) and let \( \mathcal{D} \) be the sub-TBA of \( 2^X \) generated by \( \mathcal{C} \).
Then \( A^d \) is finite for each \( A \in \text{At}(\mathcal{C}) \). So by Lemma 4.1 \( \mathcal{D} \) is finite, and its atoms form a finite-crowded partition of \( X \), as each non-empty \( A^c \) is crowded for \( A \in \text{At}(\mathcal{C}) \).
(c) \( \Rightarrow \) (d) Follows from Lemma 4.2
(d) \( \Rightarrow \) (e) Immediate.
(e) \( \Rightarrow \) (a) Suppose \( C^{(n)} \) can be refined to a finite-crowded complete \( P \)-partition \( \{ X_p \mid p \in P \} \) of \( X \) for some finite PO system \( P \). Let \( I_p = I(X_p) \). Then each \( C_j \) is a union of some of the \( X_p \), so any automorphism of \( R \) fixing each \( I_p \) will also fix each \( J_j \). By Theorem 1.3 the system \( \langle R, \{ I_p \mid p \in P \} \rangle \) is \( \omega \)-categorical, and hence so also is the system \( \langle R, \mathcal{J}^{(n)} \rangle \). \( \square \)
5 Classification results

Theorem 5.1 There is a natural bijection between isomorphism classes of countable \( \omega \)-categorical Boolean rings with \( n \) distinguished ideals, and isomorphism classes of tuples \( [P, L, f, Q^{(n)}] \) such that \( [P, L, f] \) is a finite-crowded extended PO system, each \( Q_i \) is a lower subset of \( P' \), and \( \langle Q^{(n)}, p' \rangle = 2^P \).

We say that two such tuples \([P_1, L_1, f_1, Q_{11}, \ldots, Q_{1n}]\) and \([P_2, L_2, f_2, Q_{21}, \ldots, Q_{2n}]\) are isomorphic if there is an order-isomorphism \( \theta : P_1 \to P_2 \) such that \( L_1 \theta = L_2 \), \( Q_{1j} \theta = Q_{2j} \) (all \( j \)) and \( f_2(p \theta) = f_1(p) \) for \( p \in P_1 \).

Proof Suppose first that \( J^{(n)} \) are distinguished ideals of the countable Boolean ring \( R \), corresponding to closed subsets \( C^{(n)} \) of the Stone space \( X \) of \( R \), such that the system \( (R, J^{(n)}) \) is \( \omega \)-categorical. Let \( D \) be the sub-TBA of \( 2^X \) generated by \( C^{(n)} \), with atoms \( \{X_p \mid p \in P\} \), so that \( P \) is a finite PO system and \( \{X_p\} \) is a finite-crowded complete \( P \)-partition of \( X \) (Theorem 1.2). Let \( L = \{p \in P \mid X_p \) is compact\}, which is a lower subset of \( P \) containing \( P' \), and define \( f : P' \to \mathbb{N}_+ \) by \( f(p) = |X_p| \). With these definitions, \( \{X_p\} \) forms a complete \( [P, L, f] \)-partition of \( X \), with \( P' \subseteq \text{L_{min}} \), so that \([P, L, f]\) is a finite-crowded extended PO system. By Lemma 4.3, each closed set \( C_j \) corresponds to a lower subset \( Q_j \), say, of \( P \), and \( Q^{(n)} \) generates \( (2^P, \cdot) \) as a TBA.

Isomorphic \( \omega \)-categorical systems of the form \( (R, J^{(n)}) \) will give rise to homeomorphic tuples of the form \( (X, C^{(n)}) \) where each \( C_j \) is closed, and hence to isomorphic tuples of the form \([P, L, f, Q^{(n)}]\). Moreover, if two \( \omega \)-categorical systems of form \( (R, J^{(n)}) \) give rise to isomorphic tuples of form \([P, L, f, Q^{(n)}]\), then the two systems are isomorphic; as (relabelling as required to obtain the same underlying tuple \([P, L, f, Q^{(n)}]\)) the underlying Stone spaces with their associated \( P \)-partitions are \( P \)-homeomorphic by Theorem 2.2 and this homeomorphism will also map the associated closed sets to each other.

It remains to show that every such \([P, L, f, Q^{(n)}]\) can arise. But if \([P, L, f]\) is a finite-crowded extended PO system, then by Theorem 2.2 we can find an \( \omega \)-Stone space \( X \) and a complete \([P, L, f]\)-partition \( \{X_p \mid p \in P\} \) of \( X \). By Lemma 4.3, \( Q^{(n)} \) gives rise to closed subsets \( C^{(n)} \) of \( X \) such that the atoms of \( \langle C^{(n)}, \cdot \rangle \) are \( \{X_p\} \). In addition, \( \{X_p\} \) is finite-crowded, as \( P' \subseteq \text{L_{min}} \) and therefore discrete sets are compact and finite. Applying Theorem 2.2 the system \( (R, \{I(C_j) \mid j \leq n\}) \) is \( \omega \)-categorical and maps onto \([P, L, f, Q^{(n)}]\), as required. \( \square \)

Remark 5.2 If we restrict to Boolean algebras, we obtain a natural bijection between the isomorphism classes of countable \( \omega \)-categorical Boolean algebras with \( n \) distinguished ideals, and the isomorphism classes of tuples \([P, f, Q^{(n)}]\) where \( P \) is a finite PO system such that \( P' \subseteq \text{P_{min}} \), \( f : P' \to \mathbb{N}_+ \) and \( \langle Q^{(n)}, \cdot \rangle = 2^P \).

Corollary 5.3 Let \( J^{(n)} \) be distinguished ideals of the countable Boolean ring \( R \) with Stone space \( X \) such that \( (R, J^{(n)}) \) is \( \omega \)-categorical. Let \( C \) be the closure subalgebra of \( 2^X \) generated by \( C^{(n)} \) with distinguished closed elements \( C^{(n)} \) corresponding to \( J^{(n)} \).

Then \( (R, J^{(n)}) \) is determined up to isomorphism by the isomorphism class of the tuple \( \langle \mathcal{C}, h, k \rangle \), where \( h : \text{At}(\mathcal{C}) \to \mathbb{N} \) and \( k : \text{At}(\mathcal{C}) \to \{0, 1, 2\} \) are such that for \( D \in \text{At}(\mathcal{C}) \), \( h(D) \) is the number of isolated points in \( D \), and \( k(D) = 0 \) if \( D \) is finite.
k(D) = 1 if D is infinite but bounded (i.e. $\mathcal{T}$ is compact) and $k(D) = 2$ if D is unbounded.

Proof Let $\mathscr{D} = (C^{(n)}, \iota)$ be a sub-TBA of $2^X$. Now for $A \in \text{At}(\mathscr{C})$, $|A|^d = h(A)$, and $A^\circ = \emptyset$ or is bounded or is unbounded depending on whether $k(A) = 0$, 1, or 2 respectively, as $\mathcal{A} = (A^\circ)' = A - A^d$ and $A^d$ is finite (Theorem 1.2). Thus $\mathscr{C}$ satisfies the conditions of Lemma 4.1 and so the isomorphism class of the tuple $\{C, h, k\}$ determines the isomorphism class of $\mathscr{D}$ with the distinguished elements $C^{(n)}$, the size of each discrete finite atom of $\mathscr{D}$, and which of the other atoms of $\mathscr{D}$ are bounded.

By Lemma 4.3 this in turn determines the isomorphism class of a finite PO system $P$ with distinguished lower subsets $Q^{(n)}$ that generate $(2^{P^{\prime}})$, such that the atoms of $\mathscr{D}$ form a complete finite-crowded $P$-partition of $X$, $\{X_p \mid p \in P\}$, say, with $C_j = \bigcup\{X_p \mid p \in Q_j\}$ for each $j$. Let $L = \{p \in P \mid X_p$ is compact $\}$ and let $f(p) = |X_p|$ for $p \in L_{\text{min}}$. We have $p \in L$ iff for some $A \in \text{At}(\mathscr{C})$ either $X_p = A^c$ with $k(A) = 1$, or $X_p = A^d$ in which case $p \in L_{\text{min}}$ and $f(p) = h(A)$. Moreover, $P^d \subseteq L_{\text{min}}$ as $\{X_p\}$ is finite-crowded.

So the isomorphism class of the tuple $\{C, h, k\}$ determines the isomorphism class of a tuple $[P, L, f, Q^{(n)}]$ satisfying the conditions of Theorem 5.1. Hence if two $\omega$-categorical systems of the form $(R, J^{(n)})$ generate isomorphic tuples of the form $\{C, h, k\}$, then they are isomorphic by Theorem 5.1 as required. □

Remark 5.4 We note that for $D \in \text{At}(\mathscr{C})$, $D$ and $\mathcal{T}$ have the same number of isolated points and are both either finite, bounded or unbounded. Using the duality between finite Heyting algebras and finite closure algebras generated by their closed elements, we recover the result of Alaev [1, Theorem 2.5], that a countable $\omega$-categorical Boolean algebra $R$ with distinguished ideals $J^{(n)}$ is uniquely determined by the isomorphism type of $\mathscr{C}$, the Heyting sub-algebra of $H(R)$ generated by $J^{(n)}$ and with distinguished elements $J^{(n)}$, by the number of atoms in $R/J$ for each $j \in \mathscr{C}$, and whether each such $R/J$ is finite or infinite.

With a little more work, it can be shown that every such (abstract) tuple $\{C, h, k\}$ can arise provided (a) $\mathscr{C}$ is finite and generated by its closed elements, (b) if $k(D) = 0$ then $h(D) > 0$ and $\mathcal{T} = D$, and (c) if $D \subseteq \mathcal{T}$ ($D \in \text{At}(\mathscr{C})$) then $k(D) \leq k(E)$. Equivalently, we have the following poset analogue of Theorem 5.1.

**Theorem 5.5** There is a natural bijection between isomorphism classes of countable $\omega$-categorical Boolean rings with $n$ distinguished ideals, and isomorphism classes of tuples $[S, M, F, g, Q^{(n)}]$ where $S$ is a finite poset; $M, Q^1, \ldots, Q_n$ are lower subsets of $S$; $F \subseteq M_{\text{min}}$; $g : S \to \mathbb{N}$ such that $g(s) > 0$ for $s \in F$; and $(Q^{(n)}, \cdot') = 2^S$.

We say that two such tuples $[S_1, M_1, F_1, g_1, Q_1, \ldots, Q_n]$ $(i = 1, 2)$ are isomorphic if there is an order-isomorphism $\theta : S_1 \to S_2$ such that $M_1 \theta = M_2$, $F_1 \theta = F_2$, $Q_{i,j} \theta = Q_{2,j}$ (all $j$) and $g_2(s\theta) = g_1(s)$ for $s \in S_1$.

Proof Immediate from Proposition 5.7 below, Theorem 5.1 and the dualities firstly between finite closure algebras generated by their closed elements and finite posets, and secondly between finite TBAs generated by their closed elements and finite PO systems. □

We will first need the following Proposition, following on from Lemma 4.1 which extends a closure algebra $\mathscr{C}$ to a TBA by splitting relevant atoms $A \in \text{At}(\mathscr{C})$ into "new" atoms $A^\circ$ and $A^d$. 


Proposition 5.6 Let $C$ be a finite closure algebra, and $F, H$ subsets of $\text{At}(C)$ such that $F \subseteq H \cap C_{\text{min}}$.

Then there is a TBA $\mathcal{D}$ and a closure algebra injection $\alpha: C \to (\mathcal{D}, -)$ such that $\mathcal{D} = \langle C_{\alpha \cdot} \rangle$, and for each $A \in \text{At}(C)$ $(\langle A \alpha \rangle)^d \neq \emptyset$, with also:

$$(A \alpha)^c \neq \emptyset \text{ iff } A \notin F \quad (A \alpha)^d \neq \emptyset \text{ iff } A \in H$$

The pair $(\mathcal{D}, \alpha)$ is unique up to isomorphism: if $(\mathcal{E}, \beta)$ is another such pair, then there is a TBA-isomorphism $\theta: \mathcal{D} \to \mathcal{E}$ such that $A \alpha \theta = A \beta$ for all $A \in C$.

Proof Let $G = \text{At}(C) - F$, so that $\text{At}(C) = G \cup H$. To show existence, let $\mathcal{D}$ be the TBA with atoms $\{ A_c \mid A \in G \} \cup \{ A_d \mid A \in H \}$, and for $A \in \text{At}(C)$ let $A_c = \emptyset$ if $A \in F$ and $A_d = \emptyset$ if $A \notin H$. For $A \in \text{At}(C)$ define:

$$\begin{align*}
(A_c)^d & = \bigcup \{ B_c, B_d \mid B \in \text{At}(C), B \subseteq A \} - A_d \text{ for } A \in G; \\
(A_d)^c & = \emptyset \text{ for } A \in H; \\
A \alpha & = A_c \cup A_d.
\end{align*}$$

Then $(A \alpha)^c = A_c, (A \alpha)^d = A_d$, and $\overline{A \alpha} = (A_c)^d \cup A_d = (A \alpha)^d \cup (A \alpha) = \overline{A \alpha}$ for all $A \in \text{At}(C)$, so that $\alpha$ is a closure algebra injection; we need the condition $F \subseteq C_{\text{min}}$ to ensure that the definition of $(A \alpha)^d$ remains valid for $A \in F$ (when $A_c = \emptyset$). Moreover, $A_c = (A \alpha) \cap (A \alpha)^d \in \langle C_{\alpha \cdot} \rangle$ and so $\mathcal{D} = \langle C_{\alpha \cdot} \rangle$.

For uniqueness, if $\mathcal{D}$ satisfies the required conditions then the atoms of $\mathcal{D}$ are precisely $\{(A \alpha)^c \mid A \in G\} \cup \{(A \alpha)^d \mid A \in H\}$, by Lemma 4.3. Moreover, $(A \alpha)^c = \overline{A \alpha} - (A \alpha)^d$, as $\overline{A \alpha} = \overline{A \alpha}$, and $([A \alpha]^d)^d = \emptyset$. So if $(\mathcal{E}, \beta)$ is another pair satisfying the conditions, then we can define a TBA-isomorphism $\theta: \mathcal{D} \to \mathcal{E}$ such that $([A \alpha]^c)^d \theta = ([A \beta]^c)^d$ for $A \in G$ and $([A \alpha]^d)^d \theta = ([A \beta]^d)^d$ for $A \in H$, with $A \alpha \theta = A \beta \theta$ as required. □

Proposition 5.7 For each $n \geq 1$, there is a natural bijection between the isomorphism classes of:

(a) “TBA-systems” $[\mathcal{D}, L, f, Q(n)]$ such that $\mathcal{D}$ is a finite TBA; $L, Q_1, \ldots, Q_n$ are closed elements of $\mathcal{D}$; $\mathcal{D}(n) \subseteq L_{\text{min}}$; $f: \mathcal{D}(n) \to \mathbb{N}_+$; and $[Q(n)]^d = \mathcal{D}$;

(b) “CA-systems” $[C, M, f, g, Q(n)]$ such that $C$ is a finite closure algebra; $Q_1, \ldots, Q_n$, $M$ are closed elements of $\mathcal{C}$; $F \subseteq M_{\text{min}}$; $g: \text{At}(C) \to \mathbb{N}$ such that $g(s) > 0$ for $s \in F$; and $[Q(n)]^d = \mathcal{C}$.

Remark 5.8 It is easier for this Proposition to work with closure algebras and TBAs rather than with the associated posets and PO systems.

We note that if $[\mathcal{D}, L, f, Q(n)]$ is a TBA-system and $B \in \text{At}(\mathcal{D})$, then either $B \subseteq B'$ and $B'^d = \emptyset$, or $B \cap B' = \emptyset$ (so that $B \in \mathcal{D}(n)$) with $B \subseteq B' \subseteq L$ and $B' = \emptyset$.

Proof We construct each way maps $\alpha$ and $\beta$ between the two sets of systems. Firstly, let the image under $\alpha$ of the TBA-system $[\mathcal{D}, L, f, Q(n)]$ be $[\mathcal{D}, L, \mathcal{D}(n), f, Q(n)]$, where:

$$\begin{align*}
\hat{\mathcal{D}} & = \langle Q(n), - \rangle, \text{ the closure subalgebra of } \mathcal{D} \text{ generated by } Q(n); \\
\hat{L} & = \bigcup \{ A \in \text{At}(\mathcal{D}) \mid A \subseteq L \}; \\
\hat{\mathcal{D}(n)} & = \{ A \in \text{At}(\mathcal{D}) \mid A = A^d \}; \\
\hat{f}(A) & = f(A^d) \text{ for } A \in \text{At}(\mathcal{D}), \text{ setting } f(\emptyset) = 0.
\end{align*}$$
Now for \(B \in \mathcal{D}\), \(B^d\) is a finite union of elements of \(\mathcal{D}^{(d)}\), so \(B^d\) is closed and \((B^d)' = \emptyset\). Also, \(\mathcal{D} = \langle \mathcal{D}' \rangle\), so by Lemma \(1.1\), each \(A^c\) and \(A^d\) for \(A \in \text{At}(\mathcal{D})\) is either empty or an atom of \(\mathcal{D}\). Hence \(\hat{f}\) is well-defined as \(A^d \in \mathcal{D}^{(d)}\) if \(A^d \neq \emptyset\), and so also \(\mathcal{D}^{(d)} \subseteq \hat{\mathcal{L}}_{\text{min}}\).

It follows easily that \([\mathcal{D}, \hat{L}, \mathcal{D}^{(d)}, \hat{f}, Q^{(n)}]\) is a CA-system.

Suppose next that \([\mathcal{C}, M, F, g, Q^{(n)}]\) is a CA-system. As we are working with isomorphism classes, we may assume (using Proposition \(\text{[43]}\) and setting \(H = \{A \in \text{At}(\mathcal{C}) \mid g(A) > 0\}\)) that there is a TBA \(\hat{\mathcal{C}}\) containing \(\mathcal{C}\) as a closure subalgebra such that \(\hat{\mathcal{C}} = \langle \mathcal{C}' \rangle\), with

\[A^c \neq \emptyset \text{ iff } A \notin F \quad A^d \neq \emptyset \text{ iff } A \in H \quad (A^d)' = \emptyset \text{ for all } A \in \mathcal{C}\]

Write \(H^d = \{A^d \mid A \in H\}\) and let \([\mathcal{C}, M, F, g, Q^{(n)}]\) be such that \(\mathcal{C} = [\mathcal{C}, M, F, g, Q^{(n)}]\), where:

\[\hat{M} = M \cup H^d;\]

\[\hat{g}(A^d) = g(A) \text{ for } A \in H.\]

Since \(Q^{(n)} = \mathcal{C}\), it follows that \(Q^{(n)} = \mathcal{C}\). Moreover, \(\hat{M}\) is closed in \(\mathcal{C}\) as \(M\) is closed in \(\mathcal{C}\), and \(\mathcal{C}^d = H^d \subseteq \hat{\mathcal{L}}_{\text{min}}\), with \(H^d\) being the domain of \(\hat{g}\). Therefore \([\mathcal{C}, M, F, g, Q^{(n)}]\) is a TBA-system.

Clearly \(\hat{\mathcal{C}} = \mathcal{C}\). Fix \(A \in \text{At}(\mathcal{C})\). Then \(A = A^d \text{ iff } A^c = \emptyset \text{ iff } A \in F\), so \(\mathcal{C}^{(d)} = F\).

Also, \(A \subseteq \hat{M}\) iff \(A^c \subseteq M\) iff \(A \in F\) or \(A \subseteq M\), so that \(\hat{M} = M\). Finally, \(\hat{g}(A) = g(A^d) = g(A)\) if \(A \in H\) and is zero if \(A \notin H\). Hence \([\mathcal{C}, M, F, g, Q^{(n)}]\) is a TBA-system.

Conversely, if \([\mathcal{D}, L, f, Q^{(n)}]\) is a TBA-system, then we can take \(\hat{\mathcal{D}}\) to be \(\mathcal{D}\) as it satisfies the requirements of Proposition \(1.3\). If \(B \in \mathcal{D}^{(d)}\), we can find \(A \in \text{At}(\mathcal{D})\) such that \(A^d = B\), so that \(\hat{f}(B) = \hat{f}(A^d) = f(A^d) = f(B)\). Likewise, if \(B \in \text{At}(\mathcal{D})\), we can find \(A \in G\) such that \(A^c = B\), and then \(B \subseteq \hat{L}\) iff \(A \subseteq L\) iff \(B \subseteq L\) if \(B \subseteq L\), and so \(\hat{L} = L\).

Hence \([\mathcal{D}, L, f, Q^{(n)}]\) is isomorphic to \([\mathcal{D}, L, f, Q^{(n)}]\). But if two TBA-systems are isomorphic then so are their images under \(\alpha\), and similarly for CA-systems. The result follows. \(\square\)

6 Links with previous results

We have used some results from Apps \(\text{[3]}\) concerning complete finite \(P\)-partitions of Stone spaces of Boolean rings. These results are stated in Section \(\text{[3]}\) with simplified proofs appropriate for the case of finite \(P\) included in the final section.

The other relevant work to date in this area has concerned Boolean algebras; the case where \(X\) is the compact Stone space of the Boolean algebra \(R\), and there are closed subsets \(C^{(n)}\) of \(X\) corresponding to distinguished ideals \(J^{(n)}\) of \(R\). Let \(\mathcal{C}\) be the closure subalgebra of \(2^X\) generated by \(C^{(n)}\), and let \(\mathcal{C}\) be the corresponding Heyting subalgebra \(I(\mathcal{C})\) of \(H(R)\) with distinguished elements \(J^{(n)}\). Here, if \(I\) and \(J\) are ideals of \(R\), then we set \(I \cdot J = I \cap J, I + J = \{A + B \mid A \in I, B \in J\}\) and \(I \to J = \{A \in R \mid (B \leq A) \wedge (B \in I \Rightarrow B \in J)\}\).

The result of Macintyre and Rosenstein \(\text{[5]}\) Theorem 7), proved in the context of Heyting algebras, translates to the equivalence of statements \([a]\) and \([b]\) in Theorem \(\text{[3.2]}\) as elements of \(\mathcal{C}\) correspond exactly to the closed elements of \(\mathcal{C}\); \(\mathcal{C}\) is finite if \(\mathcal{C}\) is finite; and an atom and its closure have the same isolated points.
Countably-categorical Boolean rings with distinguished ideals

Apps [2] Theorem C], as part of an investigation into \( \omega \)-categorical finite extensions of certain Boolean powers of groups, showed that that for any finite PO system \( P \) there is a compact \( \omega \)-Stone space \( X \) that admits a complete \( P \)-partition \( \mathcal{X} = \{X_p \mid p \in P\} \) such that \( |X_p| = 1 \) if \( p \in P^d \); that \( \{X, \mathcal{X}\} \) is unique up to \( P \)-homeomorphisms; and that the corresponding Boolean algebra system is \( \omega \)-categorical. (We note that [2 Theorem C] concerns the more general situation of Stone spaces acted on by a finite group; the above stated result follows by taking the group action to be trivial.) By splitting each finite partition element into singletons, this result, together with [2 Theorem B], leads easily to the “if” statement in Theorem 1.3 and to the equivalence of statements (a) to (d) of Theorem 1.2 for the case of Boolean algebras.

A Boolean algebra \( R \) is pseudo-indecomposable (PI) if for all \( A \in R \), either \( (A) \cong R \) or \( (1 - A) \cong R \); \( R \) and \( R/J \) is PI if \( (B) \) is PI. Pal’chunov [2] extended this definition to cover Boolean algebras with distinguished ideals, termed \( I \)-algebras. He showed that any \( \omega \)-categorical \( I \)-algebra can be decomposed as a finite direct product of PI \( I \)-algebras, and he identified propositions which serve as axioms for the PI \( I \)-algebras.

Touraille [3] defined a Heyting-sa algebra to be a Heyting algebra with an additional unary operation \( sa() \) satisfying certain conditions. For such an algebra \( \mathcal{F} \), he defined \( M(\mathcal{F}) \) to be the set of maximal elements \( J \in \mathcal{F} - \{\}\) such that \( sa(J) = J \). The Heyting algebra \( H(R) \) of ideals of \( R \) becomes a Heyting-sa-algebra with the definition \( sa(J) = \{B \in R \mid B/J \) is atomless in \( R/J\} \). Let \( \mathcal{F}_R \) be its sub-algebra generated by \( J^{(n)} \) with distinguished elements \( J^{(n)} \), and for \( J \in M(\mathcal{F}_R) \) let \( \eta_R(J) = \{n \in \mathbb{N}_+ \cup \{\infty\} \mid \text{the number of atoms in } R/J \} \). A consequence of Touraille’s work is that the isomorphism type of an \( \omega \)-categorical \( I \)-algebra is uniquely determined by the pair \((\mathcal{F}_R, \eta_R)\), and that every possible pair \((\mathcal{F}, \eta)\) can arise, where \( \mathcal{F} \) is a finite Heyting-sa-algebra generated by some distinguished elements and satisfying an extra condition, and \( \eta \) is a function from \( M(\mathcal{F}) \) to \( \mathbb{N}_+ \) (see [1 Theorem 2.2, Proposition 2.3]). Touraille [3] also showed that finite Heyting-sa-algebras correspond exactly firstly to finite TBA's generated by their closed elements and secondly to finite PO systems. Under this correspondence \( \mathcal{F}_R \) becomes \( \mathcal{D} \), the sub-TBA of \( 2^X \) generated by \( C^{(n)} \) and with distinguished elements \( C^{(n)} \), corresponding to a finite PO system \( P; M(\mathcal{F}_R) \) becomes the finite minimal closed sets in \( \mathcal{D} \), corresponding to \( P^d \); and \( \eta \) becomes \( f: P^d \to \mathbb{N}_+ \). Hence Touraille’s result for Heyting-sa-algebras yields Theorem 5.4 for Boolean algebras.

Alaev [1] further developed the results of Pal’chunov and Touraille and described \( \omega \)-categorical \( I \)-algebras via their decomposition into PI \( I \)-algebras. He showed that \( \omega \)-categorical PI \( I \)-algebras correspond precisely to pairs of the form \((H, \varepsilon)\), where \( H \) is a finite Heyting algebra with \( n \) distinguished non-zero generating elements (for some \( n \)), \( H - \{\} \) has a greatest element \( H_0 \) (say), and \( \varepsilon \notin \{0, 1\} \). Under this correspondence, the Heyting algebra \( \mathcal{G} \) is isomorphic to \( H; R/H_0 \) is atomless or has 2 elements depending on whether \( \varepsilon \) takes the value 0 or 1; and \( R/J \) is atomless for all ideals \( J \subseteq H_0 \). He showed further [1 Corollary 2.2] that any \( \omega \)-categorical \( I \)-algebra \( R \) is isomorphic to a product \( R_1 \times R_2 \times \cdots \times R_k \), where \( R_k \) has type \((H_k, \varepsilon_k)\) and \((H_k, \varepsilon_k)\) is as above, and that a minimal decomposition of this form is unique. He also proved that an \( \omega \)-categorical \( I \)-algebra is uniquely determined by \( \mathcal{G} \) together with information about the size and number of atoms of \( R/J \) for each \( J \in \mathcal{G} \) (see Remark 5.4).

The pair \((H, \varepsilon)\) corresponds to a PO system \( P \) with a unique minimum element \( p \), such that \( p \in P^d \) iff \( \varepsilon = 1 \), where \( H \) is isomorphic to the Heyting algebra of upper subsets of \( P \). The minimum decomposition \( R_1 \times R_2 \times \cdots \times R_k \) corresponds to a decomposition of \( X \) into disjoint trim sets (see Proposition [6, (b)]), with \( H_k = \{q \in P \mid q \geq p_k \} \).
for some \( p_k \in P_{\min} \) and \( \varepsilon_k = 1 \) iff \( p_k \in P^d \). We note the usual order reversal between ideals (open sets) in \( H(R) \) and closed sets / partition elements in the Stone space.

Underpinning the above discussion are the two parallel sets of dualities mentioned in the introduction:

(a) The dualities between finite closure algebras generated by their closed elements, finite posets, and finite Heyting algebras;

(b) The dualities between finite TBAs generated by their closed elements, finite PO systems, and finite Heyting sa-algebras.

7 Proofs of Theorems 2.3 and 2.4

7.1 Proof of Theorem 2.3

We first recall some basic properties of complete \( P \)-partitions, where \( P \) is a finite PO system.

**Proposition 7.1** Let \( P \) be a finite PO system and let \( \{ X_p \mid p \in P \} \) be a complete \( P \)-partition of the Stone space \( X \) of the Boolean ring \( R \). Then:

(a) \([3, \text{Proposition 2.13(iii)}] \) \( x \in X_p \) iff \( x \) has a neighbourhood basis of \( p \)-trim sets;

(b) \([3, \text{Proposition 3.2}] \) each \( A \in R \) has a \( \mu(A) \)-partition;

(c) if \( A \in R \) is \( p \)-trim and \( q \geq p \), then we can write \( A = B + C \), where \( B \) is \( p \)-trim and \( C \) is \( q \)-trim.

**Proof** (a) Suppose \( A \in R \) contains \( x \in X_p \); we need to find a \( p \)-trim neighbourhood of \( x \) contained in \( A \). If \( p \in P^d \), reduce \( A \) if necessary to assume that \( A \cap X_p = \{ x \} \). For each \( q \neq p \) we have \( x \notin X_q \), so we can find \( B_q \in R \) with \( x \in B_q \subseteq A \) and \( B_q \cap X_q = 0 \). Then \( \bigcap_{q \neq p} B_q \) is the required \( p \)-trim neighbourhood of \( x \), as if \( q \geq p \) then all neighbourhoods of \( x \) meet \( X_q \). The converse follows easily, as if \( x \in X_q \) \( (q \neq p) \), then \( x \) cannot also have a neighbourhood base of \( p \)-trim sets.

(b) Let \( F = T(A)_{\min} \), and choose a finite subset \( G \) of \( A \) such that \( G \cap X_p = A \cap X_p \) for \( p \in P \cap P^d \) and \( |G \cap X_p| = 1 \) for \( p \in P - P^d \). For each \( x \in G \), use (a) to choose a \( p_x \)-trim neighbourhood \( A_x \) of \( x \), where \( x \in X_{p_x} \), ensuring that the \( A_x \) are disjoint.

Let \( \mathcal{B} = \{ A_x \mid x \in G \} \) and \( E = \bigcup \mathcal{B} \); we see immediately that \( \mathcal{B} \) is a \( \mu(A) \)-partition of \( E \), as \( T(E) = T(A) \) and \( \mu(E) = \mu(A) \).

Let \( B = A - E \). For each \( y \in B \), use (a) to find a trim set \( B_y \subseteq B \) containing \( y \). By compactness, we can find a finite subset \( \{ B_{y_1}, \ldots, B_{y_m} \} \) of trim sets that cover \( B \). Let \( \mathcal{C} = \{ C_i \} \) be the atoms of the finite Boolean ring generated by \( \{ B_{y_j} \} \). For each \( C \in \mathcal{C} \), we have \( C \subseteq B_{y_j} \) for some \( j \), and we can find \( r \in F \) with \( r \leq t(B_{y_j}) \). Find \( D \in \mathcal{B} \) such that \( t(D) = r \); then \( D \cup C \) is still \( r \)-trim, as \( T(C) \subseteq T(D) \), and if \( r \in F \cap P^d \) then \( C \cap X_r = \emptyset \) by construction of \( \mathcal{B} \). So we can replace \( D \) with \( D \cup C \) to obtain a \( \mu(A) \)-partition of \( E \cup C \). Proceeding in this way, we obtain a \( \mu(A) \)-partition of \( A \).

(c) Choose \( x \in A \cap X_q \) and a \( q \)-trim \( C \subseteq A \) such that \( x \in C \), ensuring that \( (A - C) \cap X_p \neq \emptyset \) if \( q = p \) (so \( X_p \) is crowded). Let \( B = A - C \), which is \( p \)-trim. \( \square \)

**Remark 7.2** If \( \mathcal{B} \) is a partition of \( A \) into trim sets, \( B \in \mathcal{B} \) and \( B \cap G \neq \emptyset \) (where \( G \) is as above), then \( |B \cap G| = 1 \) and \( t(B) = T(B \cap C) \) as \( T(G) = T(A)_{\min} \). So \( \mathcal{B} \) has a subset \( \mathcal{C} \) such that \( \mathcal{C} \) is a \( \mu(A) \)-partition of \( \bigcup \mathcal{C} \). Hence any minimum decomposition of \( A \in R \) into disjoint trim subsets has the form specified in (b) above.
To prove Theorem 2.4 we will use the following version of Vaught’s Theorem, as cited in [9, 1.1.3].

**Theorem 7.3 (Vaught)** Suppose \( \sim \) is a relation between elements of countable Boolean algebras \( R \) and \( S \) such that:

(a) \( 1_R \sim 1_S \);
(b) if \( A \sim 0_S \), then \( A = 0_R \); and vice versa;
(c) if \( A \sim (B_1 + B_2) \) (\( B_1, B_2 \in S \)), then we can write \( A = A_1 + A_2 \), with \( A_i \in R \) and \( A_1 \sim B_1 \) (\( i = 1, 2 \)); and vice versa.

Then there is an isomorphism \( \alpha : R \to S \) such that each \( A \in R \) can be expressed as \( A = A_1 + \ldots + A_n \) where \( A_i \sim A_i \alpha \) for all \( i \leq n \).

**Proof of Theorem 7.3**

Let \( P \) be a finite PO system and \( \mathcal{X} = \{X_p \mid p \in P\} \) and \( \mathcal{Y} = \{Y_p \mid p \in P\} \) be complete \( P \)-partitions of the Stone spaces \( X \) and \( Y \) of countable Boolean rings \( R \) and \( S \) respectively. Suppose \( A \in R \) and \( B \in S \) are such that \( \mu(A) = \mu(B) \). We must find a \( P \)-homeomorphism between \( A \) and \( B \).

Define a relation between the Boolean algebras \((A)\) and \((B)\) by setting \( C \sim D \iff \mu(C) = \mu(D) \), for \( C \subseteq A \) and \( D \subseteq B \). Clearly \( C \sim 0 \iff C = 0 \), and \( A \sim B \).

We must show that \( \sim \) satisfies criterion (C) of Theorem 7.3. By symmetry, it is enough to consider the case where \( C \subseteq A, C \sim D \) and \( D = D_1 + D_2 \), where \( D_1, D_2 \subseteq B \). If \( \mathcal{C} \) and \( \mathcal{D} \) are partitions of subsets of \( A \) and \( B \) respectively with each partition element trim, we write \( \mathcal{C} \sim \mathcal{D} \) to mean that there is a type-preserving bijection from \( \mathcal{C} \) to \( \mathcal{D} \).

Let \( \mathcal{R}_i \) be a \( \mu(D_i) \)-partition of \( D_i \) (\( i = 1, 2 \)) and let \( \mathcal{A} \) be a \( \mu(C) \)-partition of \( C \).

By Remark 2.2 \( \mathcal{R}_1 \cup \mathcal{R}_2 \) must contain a subset \( \mathcal{R}_3 \), say, such that \( \mathcal{A} \sim \mathcal{R}_3 \); let \( \mathcal{R}_4 = \mathcal{R}_3 \cup \mathcal{R}_2 - \mathcal{R}_3 \). For each \( E \in \mathcal{R}_4 \), we can find \( F \in \mathcal{A} \) such that \( t(F) \leq t(E) \); but (counting again) we cannot have \( t(F) = t(E) \) in \( P^d \), and so \( t(F) < t(E) \).

We can therefore apply Proposition 2.3 repeatedly to obtain a partition \( \mathcal{A} \) of \( C \) such that \( \mathcal{A} \sim \mathcal{R}_3 \cup \mathcal{R}_4 = \mathcal{R}_1 \cup \mathcal{R}_2 \). Rearranging, we obtain a partition \( \mathcal{A}_1 \cup \mathcal{A}_2 \) of \( C \) such that elements of \( \mathcal{A}_1 \cup \mathcal{A}_2 \) are disjoint from each other and \( \mathcal{A}_i \sim \mathcal{R}_i \) for \( i = 1, 2 \).

Let \( C_i = \bigcup \mathcal{A}_i \), so that \( \mu(C_i) = \mu(D_i) \) (\( i = 1, 2 \)), to complete this step.

So by Theorem 2.3 there is an isomorphism \( \beta : (A) \to (B) \) such that \{\( C \in (A) \mid C \sim C \beta \} \) disjointsy generates \((A)\). Let \( \alpha : A \to B \) be the Stone space homeomorphism induced by \( \beta \). If \( x \in A \cap X_p \), then \( x \) has a neighbourhood base \( \mathcal{V}_x \) of \( p \)-trim sets by Proposition 2.3(a) and we may assume that \( C \sim C \beta \) for each \( C \in \mathcal{V}_x \). But \( C \) is \( p \)-trim iff \( \mu(C) = 1 \) if \( C = \beta \) is \( p \)-trim. Hence \( \alpha x \) has a neighbourhood base of \( p \)-trim sets, so \( \alpha x \in B \cap Y_p \) (Proposition 2.3(a) again), and \( \{X_p \cap A \} \subseteq Y_p \cap B \). Equality follows by considering \( \alpha^{-1} \), so \( \alpha \) is a \( P \)-homeomorphism, as required. \( \square \)

### 7.2 Theorem 2.4: proof of uniqueness

**Proof of uniqueness in Theorem 2.4**

Suppose that \([P, L, f] \) is a finite extended PO system and that \( \{X_p \mid p \in \mathcal{P}\} \) is a complete \([P, L, f] \)-partition of the Stone space \( X \) of the countable Boolean ring \( R \).

Let \( F = L_{\min} \), \( G = (P - L)_{\min} \) and \( Q = F \cup G \). It suffices to show that we can write \( R = \bigoplus_{p \in \mathcal{P}} \{\bigoplus_{j \leq n_p} (A_{pj})\} \), where \( A_{pj} \) is \( p \)-trim and \( n_p = 1 \) if \( p \in L_{\min} \). \( P ; n_p = \infty \) if \( p \in G \); then \( F, G, Q \) and each \( n_p \) are determined by \([P, L, f] \), and any two \( p \)-trim sets are \( P \)-homeomorphic (Theorem 2.3).
To see this, find $A \in R$ such that $\bigcup_{p \in L} X_p \subseteq A$. Using Proposition [2, 3, 4] write $A = A_1 + \ldots + A_n$, where each $A_j$ is trim. Discarding any redundant $A_j$'s as necessary and reducing $A$ accordingly, we may assume that $T(A_j) \cap L \neq \emptyset$ for each $j$, so that $L \subseteq T(A)$ and $T(A)_{\min} = L_{\min} = F$. Then $A$ admits a $\mu(A)$-partition, where $\mu(A) = \sum_{p \in P} \nu_p \cap p$ and $\nu_p$ are as above, as $[A \cap X_p] = f(p)$ for $p \in F \cap P^d = L_{\min}^d$.

If $L = P$, then $A = X$ and we are done. Otherwise, let $S = \{B \in R \mid B \cap A = \emptyset\}$, so that $R = (A) \oplus S$. By a routine argument we can find disjoint $\{B_n \in S \mid n \geq 1\}$ that generate $S$, with $T(B_n) \subseteq P - L$ for all $n$. As $X_p$ is not compact for $p \in P - L$, we can find $m_1$ such that $T(B_1 \cup \ldots \cup B_{m_1}) = P - L$. By “clumping together” the $B_n$ in this way, we can find disjoint $\{C_n \in S \mid n \geq 1\}$ that generate $S$ such that $T(C_n) = P - L$ for each $n$. Now split each $C_n$ into $q$-trim sets, one for each $q \in G$, to give the desired decomposition of $S$ as $\bigoplus_{p \in G} \{\bigcup_{j \geq 1}(A_{pj})\}$, where each $A_{pj}$ is $p$-trim. □

7.3 Theorem 22.4 proof of existence

Let $D_1$ and $D_0$ denote the Cantor set and the Cantor set minus a point respectively, whose associated Boolean rings are countable atomless with and without a 1 respectively. The following Lemma and proof of existence of complete $[P, L, f]$-partitions for finite $P$ are based on proofs by Pierce [8, Theorem 4.6] which handle the compact cases; we have extended these to cover the non-compact cases as well.

Lemma 7.4 Let $W$ be an $\omega$-Stone space and $C$ a non-empty closed subset of $W$. Then we can find an $\omega$-Stone space $X$ and a homeomorphism $\alpha: W \to W_\alpha \subseteq X$ such that $W_\alpha$ is closed in $X$ and $\alpha \cap W_\alpha = C_\alpha$ where $Y = X - W_\alpha$. Moreover, we can arrange for $Y$ to be infinite discrete or crowded, and for $\alpha$ to be either compact (provided $C$ is compact) or non-compact.

Proof Case 1: $Y$ is to be crowded.

Let $D = D_0$ if $C$ is compact and $Y$ is to be compact (case 1A), and let $D = D_1$ if $Y$ is to be non-compact (case 1B). Select $x_0 \in D$, and define the subset $X$ of $W \times D$ by $X = (W \times \{x_0\}) \cup (C \times D)$; this is a closed subset of a Stone space, and so is itself a Stone space. Define $\alpha: W \to X : y \mapsto (y, x_0)$ which is a homeomorphism onto $W \times \{x_0\}$. We have $Y = X - W_\alpha = C \times (D - \{x_0\})$, so $\alpha = C \times D$, which is compact (case 1A) or non-compact (case 1B), $\alpha \cap W_\alpha = C \times \{x_0\} = C_\alpha$, and neither $D - \{x_0\}$ nor $Y$ has any isolated points, as required.

Case 2: $Y$ is to be infinite discrete.

Let $V = \emptyset$ if $C$ is compact and $Y$ is to be compact (case 2A), and let $V = \{2, 3, \ldots\}$ if $Y$ is to be non-compact (case 2B). Let $\{x_n \mid n \geq 1\}$ be a countable dense discrete subset of $C$, let $Z = \{0\} \cup \{1/n \mid n \geq 1\} \cup V$, which is compact in case 2A, and define the closed subset $X$ of $W \times Z$ by $X = (W \times \{0\}) \cup Y$, where

$Y = \{(x_m, 1/n) \mid m < n\} \cup \{\{x_1\} \times V\}$

Define $\alpha: W \to X : y \mapsto (y, 0)$ which is a homeomorphism onto $W \times \{0\}$, with $Y = X - W_\alpha$ being countably discrete. Then $\alpha = (C \times \{0\}) \cup Y$, $\alpha \cap W_\alpha = C_\alpha$, and $Y$ is compact in case 2A (being a closed subset of the compact $W \times Z$) and is non compact in case 2B.

Finally, each of the spaces $X$ so constructed is a closed subset of a product of spaces, each with a countable base, and so itself has a countable base; hence each $X$ is an $\omega$-Stone space. □
This Lemma enables the main induction step required to construct an $\omega$-Stone space that admits a complete $[P, L, f]$-partition.

Proof of existence in Theorem 2.4 by induction on $|P|$. If $|P| \geq 1$, there are 4 cases. Let $P = \{p\}$.

If $P^d = \emptyset$, we can take $X$ to be $D_1$ if $L = P$ or $D_0$ if $L = \emptyset$.

If $P^d = \{p\}$, we can take $X$ to be a set of $f(p)$ discrete points if $L = P$ or a countably infinite set of discrete points if $L = \emptyset$.

Suppose now that the result holds for $|P| \leq k$, and that $|P| = k + 1$. Pick a maximal $p \in P$; let $Q = P - \{p\}$, $M = L - \{p\}$, and let $g$ be the restriction of $f$ to $M_{\text{min}} \cap Q^d \subseteq L_{\text{min}}$. By the induction hypothesis, we can find an $\omega$-Stone space $W$ that admits a complete $[Q, M, g]$-partition $\{X_q \mid q \in Q\}$.

Let $Q_1 = \{q \in Q \mid q < p\}$ and $C = \bigcup_{q \in Q_1} X_q$, which is closed in $W$. If $Q_1 = \emptyset$ (i.e. $p \in P_{\text{min}}$), we can take $X = W \cup X_p$, where $X_p$ is determined as for when $P = \{p\}$.

Otherwise, use Lemma 7.4 to find an $\omega$-Stone space $X$ and a homeomorphism $\alpha : W \to W \alpha \subseteq X$ such that $W \alpha$ is closed in $X$; $X_p \cap W \alpha = C$, where $X_p = X - W \alpha$; $X_p$ is compact if $p \in L$ (as if $p \in L$, then $Q_1 \subseteq L$ and so $C$ is compact); and $X_p$ is crowded if $p \notin P^d$ and $X_p$ is infinite and discrete if $p \in P^d$. An easy check now shows that $\{X_q \mid q \in P\}$ is the required complete $[P, L, f]$-partition of $X$. \qed

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