New light on electromagnetic corrections to the scattering parameters obtained from experiments on pionium

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Abstract

We calculate the electromagnetic corrections needed to obtain isospin invariant hadronic pion-pion s-wave scattering lengths $a_0^I, a_2^I$ from the elements $a_{cc}, a_{0c}$ of the $s$-wave scattering matrix for the $(\pi^+\pi^-, \pi^0\pi^0)$ system at the $\pi^+\pi^-$ threshold. The latter can be extracted from experiments on the $\pi^+\pi^-$ atom (pionium). Our calculation uses energy independent hadronic pion-pion potentials $V_0, V_2$ that satisfactorily reproduce the low-energy phase shifts given by two-loop chiral perturbation theory, with the hadronic mass of the pion taken first as the charged pion mass and then as the neutral pion mass. We also take into account an important relativistic effect whose inclusion influences the corrections considerably.

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1 Introduction

The extraction of the $s$-wave pion-pion scattering lengths $a^I, I = 0, 2$, from the results of the DIRAC experiment currently in progress at CERN [1] requires

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a knowledge of the electromagnetic corrections $a_{ac} - \frac{\sqrt{2}}{3}(a^2 - a^0)$ and $a_{cc} - (\frac{2}{3}a^0 + \frac{1}{3}a^2)$, where $a_{cc}$ and $a_{ac}$ are elements of the well known matrix $K$ of scattering theory in the $s$-wave for the two-channel ($\pi^+\pi^-, \pi^0\pi^0$) system at the $\pi^+\pi^-$ threshold. (The subscript $c$ refers to the $\pi^+\pi^-$ channel, 0 to the $\pi^0\pi^0$ channel.) Previous attempts to calculate these corrections using a potential model have been reported in Refs. [2,3]. In this paper we improve on the calculation of Ref. [3] by using new hadronic potentials obtained by fitting the phase shifts given by two-loop chiral perturbation theory (ChPT) and by including a relativistic modification which turns out to have a significant effect even in the threshold situation we are considering.

This relativistic modification emerged as an important issue in the calculation of electromagnetic corrections for the analysis of low energy $\pi^-p$ scattering data [4]. The treatment of the two-channel ($\pi^-p, \pi^0n$) system closely parallels that of the ($\pi^+\pi^-, \pi^0\pi^0$) system and the coupled relativised Schrödinger equations (RSEs) for the two systems are formally identical. The two-channel RSEs that we now use to model the $s$-wave of the physical ($\pi^+\pi^-, \pi^0\pi^0$) system are therefore given by Eq.(21) of Ref.[4]:

$$\left\{ \frac{1}{2} \frac{d^2}{dr^2} + Q^2 - 2mfV(r) \right\} u(r) = 0,$$

where

$$m = \begin{pmatrix} \frac{1}{2}\mu_c & 0 \\ 0 & \frac{1}{2}\mu_0 \end{pmatrix}, \quad Q = \begin{pmatrix} q_c & 0 \\ 0 & q_0 \end{pmatrix}, \quad f = \begin{pmatrix} f_c & 0 \\ 0 & f_0 \end{pmatrix}.$$  \hspace{1cm} (2)

In Eq.(2), $\mu_c$ and $\mu_0$ are the physical masses of $\pi^\pm$ and $\pi^0$ respectively, while $q_c$ and $q_0$ are the c.m. momenta of the two channels. Further,

$$f_c = \frac{W^2 - 2\mu_c^2}{\mu_c W}, \quad f_0 = \frac{W^2 - 2\mu_0^2}{\mu_0 W},$$  \hspace{1cm} (3)

where $W$ is the total energy in the c.m. frame. Note that $f_c$ and $f_0$ are 1 at the respective thresholds $W = 2\mu_c$ and $W = 2\mu_0$ and increase as $W$ increases.

Eq.(1) now replaces Eq.(8) of Ref.[3]. The inclusion of the factor $f$ is of crucial importance for the calculation of the electromagnetic corrections for both the ($\pi^+\pi^-, \pi^0\pi^0$) and ($\pi^-p, \pi^0n$) systems. The first part of Section 3 of Ref.[4] gives the arguments for its inclusion in the RSEs for the latter system. The potential matrix $V$ has the form

$$V = V^{em} + V^h.$$  \hspace{1cm} (4)
The electromagnetic potential matrix has only one nonzero entry \((V_{em})_{cc} = V_{em}\). In the notation of Ref. [4], \(V_{em}\) contains only \(V^{pc}\) (point charge Coulomb) and \(V^{ext}\) (which takes account of the extended charge distributions); the effect of \(V^{rel}\) and \(V^{vp}\) (vacuum polarisation) on the positions and widths of the levels of pionium is treated separately.

The hadronic potential matrix \(V^{h}\) is assumed to be isospin invariant, in accordance with the arguments of Gasser and Leutwyler [5,6], which show that there is practically no isospin breaking of the purely hadronic pion-pion scattering amplitudes. Thus

\[
V^{h} = \begin{pmatrix}
\frac{2}{3}V^0 + \frac{1}{3}V^2 & \frac{\sqrt{2}}{3}(V^2 - V^0) \\
\frac{\sqrt{2}}{3}(V^2 - V^0) & \frac{1}{3}V^0 + \frac{2}{3}V^2
\end{pmatrix}.
\]  

(5)

The determination of the hadronic potentials will be considered in Section 2. Once they are chosen, the coupled equations (1) are integrated numerically to give \(K\) as a function of \(W\). Extrapolation to \(W = 2\mu_c\) gives the matrix

\[
a = K(2\mu_c),
\]

(6)

whose elements \(a_{cc}\), \(a_{0c}\) can be obtained from experiments on pionium. The determination of these quantities from pionium data is reviewed at the beginning of Section 3, which then sets out the results of our calculation of the electromagnetic corrections needed to obtain the hadronic pion-pion scattering lengths \(a^0, a^2\) from \(a_{cc}\) and \(a_{0c}\). In Section 4 we compare our results with those of ChPT.

2 The hadronic pion-pion potentials

We now need to model the hadronic situation and determine the potentials \(V^0\), \(V^2\). Chiral perturbation theory (ChPT) has proved a fruitful effective theory of low energy pion-pion scattering that incorporates the essential constraint of chiral invariance. From the point of view of ChPT, the hadronic mass \(\mu\) of the pion (assumed the same for \(\pi^\pm\) and \(\pi^0\)) is an adjustable parameter and ChPT is able to generate hadronic phase shifts \(\delta^0, \delta^2\) that depend on the value of \(\mu\) that is chosen. This procedure is complicated by the need to fix certain low energy constants using experimental data obtained with physical pions. Nevertheless, Colangelo [7] has provided us with hadronic s-wave phase shifts \(\delta^I(\mu; W)\), \(I = 0, 2\), obtained from two-loop ChPT as functions of \(W\) for a region above the threshold \(W = 2\mu\), for \(\mu = \mu_c\) and for \(\mu = \mu_0\).
The hadronic pion-pion scattering lengths for total isospin $I = 0, 2$ and hadronic mass $\mu = \mu_c, \mu_0$ given by two-loop ChPT (Ref.[7]) and by the potentials $V^0$ and $V^2$ of Fig.1 (all results in fm).

|                  | two-loop ChPT | our potentials |
|------------------|---------------|---------------|
| $a^0(\mu_c)$     | 0.3055        | 0.3004        |
| $a^0(\mu_0)$     | 0.2930        | 0.2892        |
| $a^2(\mu_c)$     | -0.0632       | -0.0632       |
| $a^2(\mu_0)$     | -0.0614       | -0.0613       |

What we have been able to do is to find potentials $V^0, V^2$ that are independent of both the energy $W$ and the hadronic mass $\mu$, which reproduce quite well the two-loop ChPT phase shifts of Ref.[7] for $\mu = \mu_c$ and $\mu = \mu_0$ via the RSEs

$$ \left( \frac{d^2}{dr^2} + q_c^2 - \mu_c f_c V_I(r) \right) u(r) = 0, \quad (7) $$

$$ \left( \frac{d^2}{dr^2} + q_0^2 - \mu_0 f_0 V_I(r) \right) u(r) = 0, \quad (8) $$

with $I = 0, 2$. The important difference from Eq.(6) of Ref.[3] (which applies to $\mu = \mu_c$ only) is the presence of the factors $f_c$ and $f_0$ in Eqs.(7) and (8).

Compared with the work of Ref.[3], we were also able to use our experience of constructing potentials for calculating electromagnetic corrections in low energy pion-nucleon scattering to obtain physically more realistic potentials than the double square wells used earlier.

We found that the best fit to the phase shifts was obtained when the range parameter in these potentials was 1.5 fm. Each potential contains three further parameters that were varied to obtain the best fit. Full details of the parameterisation are given in Ref.[8]. We found that it is possible to construct for $I = 0, 2$ potentials $V_I(r)$ that give a quite good fit to the two-loop ChPT phase shifts $\delta_I(\mu; W)$ for $\mu = \mu_c$ and $\mu = \mu_0$, using Eqs.(7) and (8) to generate phase shifts by integrating the regular solution outwards from the origin.

These potentials are plotted in Fig.1. It is impossible to obtain a satisfactory fit to the phase shifts for both values of $\mu$ when the factors $f_c$ and $f_0$ are omitted from Eqs.(7) and (8); this is further significant evidence for the need to include them. What we have been able to do is to capture some of the dynamical behaviour of ChPT by means of a simple potential model involving potentials independent of the energy and the hadronic mass. In Figs.2 and 3 we show how well the potentials reproduce the ChPT phase shifts for $\mu = \mu_c$ and $\mu = \mu_0$ respectively. Table 1 gives a comparison of the hadronic scattering lengths given by two-loop ChPT and by our potentials.
Fig. 1. The potentials $V^0(r)$ and $V^2(r)$ that give the best fit to the two-loop ChPT phase shifts of Ref.[7] for hadronic masses $\mu_c$ and $\mu_0$.

3 From pionium data to hadronic pion-pion scattering lengths

The first step is the extraction of the quantities $a_{cc}$, $a_{0c}$ from the experimental values of the lifetime $\tau$ of the 1s level of pionium and of the difference $\Delta W^{\text{had}}$ between the energies of the 2s and 2p levels. This is considered in Section 2 of Ref.[3], with the results

$$a_{0c}(\text{fm}) = -0.3794 \left[\Gamma(\text{eV})\right]^{1/2},$$  

$$a_{cc}(\text{fm}) = -0.4167 \Delta W^{\text{had}}(\text{eV}).$$
Fig. 2. The phase shifts $\delta^0(\mu_c; W)$ and $\delta^2(\mu_c; W)$ (in degrees) for hadronic mass $\mu_c$ given by two-loop ChPT (crosses) and by the potentials $V^0$ and $V^2$ of Fig.1 (solid curves).

In terms of the lifetime $\tau$, Eq.(9) becomes

$$a_{0c}(\text{fm}) = -0.3078 \ [\tau(\text{fs})]^{-1/2}.$$  \hfill (11)

The slight differences between the numerical constants of Eqs.(9) and (10) and those in Ref.[3] come from a new value of $a_{cc}$ and the direct inclusion of the effect of vacuum polarisation in Eq.(9). These results are based on the work of Ref.[9], which uses general principles of scattering theory, in particular the analytic continuation of $K$ below the $\pi^+\pi^-$ threshold. The basic formalism goes back to Hamilton, Överbö and Tromborg [10] and was worked out systematically for multichannel situations by Rasche and Woolcock [11]. It is
Fig. 3. The phase shifts $\delta^0(\mu_0; W)$ and $\delta^2(\mu_0; W)$ (in degrees) for hadronic mass $\mu_0$ given by two-loop ChPT (crosses) and by the potentials $V^0$ and $V^2$ of Fig.1 (solid curves).

It is important to observe that the expressions for the hadronic shifts and widths of the levels of hadronic atoms obtained in Ref.[9] are given in terms of the elements of $a$ as defined in Eq.(6). As shown in Ref.[9] there is no significant error involved in evaluating $K$ at the threshold of the charged channel instead of at the pole position corresponding to the bound state. Thus the extraction of $a_{cc}$ and $a_{qc}$ from experimental data is completely separated from the question of the electromagnetic corrections that relate these two numbers to the hadronic $s$-wave scattering lengths $a^I$, $I = 0, 2$. This separation is valuable from a conceptual point of view.
In order to compare our results with field theoretical treatments of the width \( \Gamma \), it is necessary to give our definitions of two-channel scattering amplitudes and to establish the connection with the standard amplitudes used in work on ChPT. We confine ourselves again to the \( s \)-wave. Starting from the unitary matrix \( S \), we define matrices \( T \), \( K \) and \( A \) as follows:

\[
T = -i(S - 1_2)(S + 1_2)^{-1},
\]

\[
K^{-1} = \begin{pmatrix} C_0(\eta_-) & 0 \\ 0 & 1 \end{pmatrix} Q^{1/2}T^{-1}Q^{1/2} \begin{pmatrix} C_0(\eta_-) & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\beta h(\eta_-) & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
A^{-1} = K^{-1} - iQ^{-1}.
\]

The matrix \( Q \) is given in Eq.(2), while \( \beta = \alpha \mu_c, \eta_- = -\beta/2q_c \).

\[
C_0^2(\eta) = \frac{2\pi\eta}{\exp(2\pi\eta) - 1}, h(\eta) = -ln |\eta| + \Re \psi(1 + i\eta).
\]

Eq.(13) is the explicit definition of \( K \), in which the specific low energy behaviour of \( T \) induced by the Coulomb interaction is taken into account. The complex quantity \( A_{cc} \) plays the key role in the formalism of Ref.[9].

If we denote by \( A^x \) the \( s \)-wave part of the matrix of scattering amplitudes used in work on ChPT, the connection with the matrix \( A \) of Eq.(14) is

\[
A = (32\pi W)^{-1} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} A^x \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}.
\]

Note that \( A \) has the dimension of a length, while \( A^x \) is dimensionless. The most satisfactory field theoretical treatment of pionium is given in Gall et al.[12], which uses QCD (including photons) and works with the low energy expansion provided by ChPT. Ref.[12] also gives references to and discusses other field theoretic methods of treating hadronic atoms. In Ref.[13], Gasser, Lyubovitskij and Rusetsky apply the formalism of Ref.[12] to the numerical calculation of \( \Gamma \). A more detailed account of their work is given in Ref.[14]. The expression for \( \Gamma \) given in Refs.[12–14] involves the real part of an amplitude \( A_{thr}^{+-00} \), from which the specific low energy behaviour due to the Coulomb interaction has been removed. Using Eq.(15), we have

\[
A_{0c}(2\mu_c) = \sqrt{2}(64\pi\mu_c)^{-1} A_{thr}^{+-00}.
\]

If we compare the expression for \( \Gamma \) given in Refs.[12–14] with our result in Eq.(9) of Ref.[3], on which Eq.(9) above is based, we find that a term pro-
proportional to \( \ln \alpha \) and the purely hadronic scattering lengths appear in the quantity \( K \) given in Eq.(4) of Ref.[12], while our result has no term involving \( \ln \alpha \) and contains the scattering parameters that include the effect of electromagnetic interaction. As Ivanov et al. [15] have pointed out, these differences come from the same source: the treatment of Coulomb photons. In our method the Coulomb potential is included in the RSEs (1), which are solved exactly. The nonanalytic behaviour in \( \alpha \) is implicitly included in our electromagnetic corrections. The treatments based on scattering theory [9] on the one hand and on field theory [12–14] on the other therefore lead in the end numerically to the same results.

To understand the electromagnetic corrections we must distinguish between the physical situation, in which the electromagnetic interaction is present and the pions have their observed masses, and the hadronic situation, in which the electromagnetic interaction is switched off. The pioneering work of Gasser and Leutwyler [5,6], already used in Section 1, shows that, at first order in the low energy expansion of ChPT, in the hadronic situation the only isospin breaking effect due to the difference between the quark masses \( m_u \) and \( m_d \) is a very small shift in the \( \pi^0 \) mass. It is therefore a very good approximation to take the hadronic situation as one in which all the pions have the same mass \( \mu \) and the \( s \)-wave amplitudes are isospin invariant.

As noted in Section 2, for ChPT at the hadronic level, the value of \( \mu \) is a parameter that is not determined by the theory. (This does not mean that its value can be chosen completely arbitrarily.) Thus ChPT contains dynamical information about \( s \)-wave hadronic \( \pi \pi \) scattering for a whole range of hadronic masses of the pions. The potentials constructed in Section 2 capture a lot of this dynamical information for a range of masses around \( \mu_0 \). They were constructed in order to be able to incorporate the effects of the electromagnetic interaction (the Coulomb potential for extended charges and the difference between \( \mu_0 \) and \( \mu_c \)) into a potential model of the physical situation that enables the electromagnetic corrections to be calculated. That model is just the two-channel RSEs(1); it is simple and hopefully will give a reasonably reliable estimate of the corrections.

We now define the hadronic quantities \( a_{cc}^h(\mu) , a_{0c}^h(\mu) \) by

\[
  a_{cc}^h(\mu) = \frac{2}{3} a^0(\mu) + \frac{1}{3} a^2(\mu) , \quad a_{0c}^h(\mu) = \frac{\sqrt{2}}{3} (a^2(\mu) - a^0(\mu)).
\]  

(17)

As stated in Section 3 of Ref.[3], numerical experience shows that the quantities least sensitive to small variations of the hadronic potentials \( V^I \) are

\[
  \Delta a_{cc}(\mu) = a_{cc}^h(\mu) - a_{cc} , \quad \Delta_0(\mu) = a_{0c}^h(\mu)/a_{0c} - 1.
\]  

(18)
The term ‘electromagnetic corrections’ will in future mean $\Delta a_{cc}(\mu)$ and $\Delta a_{0c}(\mu)$. Note that the hadronic quantities $a^I(\mu), I = 1, 2, a^h_{cc}(\mu)$ and $a^h_{0c}(\mu)$ depend explicitly on $\mu$. They are given by an equation like Eq.(7) or Eq.(8), with $\mu$ replacing $\mu_c$ or $\mu_0$ and a momentum variable $q(\mu; W)$ corresponding to the mass $\mu$. However, the physical scattering parameters $a_{cc}$ and $a_{0c}$ are given by the RSEs(1) and do not depend on $\mu$. The $a^I(\mu)$ given by the potentials of Section 2 are listed in the second column of Table 1 for $\mu = \mu_c$ and $\mu = \mu_0$. They lead to the results

$$a^h_{cc}(\mu_c) = 0.1791 \text{fm}, \quad a^h_{0c}(\mu_c) = -0.1714 \text{fm}, \quad (19)$$

$$a^h_{cc}(\mu_0) = 0.1721 \text{fm}, \quad a^h_{0c}(\mu_0) = -0.1651 \text{fm}. \quad (20)$$

The procedure for determining the elements of $K$ at the threshold of the charged channel was described briefly at the end of Section 1. The coupled RSEs(1) were integrated numerically to obtain $K$ as a function of $W$, as described in Section 3 of Ref.[16]. The matrix $K$ was then extrapolated to $W = 2\mu_c$ to obtain $a$. The results are:

$$a_{cc} = 0.1773 \text{fm}, \quad a_{0c} = -0.1726 \text{fm}, \quad a_{00} = 0.0576 \text{fm}. \quad (21)$$

From Eqs.(19)-(21) the electromagnetic corrections of Eq.(18) are then

$$\Delta a_{cc}(\mu_c) = 0.0018 \text{fm}, \quad \Delta a_{0c}(\mu_c) = -0.0068, \quad (22)$$

$$\Delta a_{cc}(\mu_0) = -0.0052 \text{fm}, \quad \Delta a_{0c}(\mu_0) = -0.0436. \quad (23)$$

Eqs.(22) and (23) are the final results of our calculations of the electromagnetic corrections in the potential model. They supersede the results in Refs.[2,3], where much more primitive hadronic potentials were used and the relativistic factors $f_c, f_0$ were not taken into account.

We have given the corrections for two possible values of the hadronic mass of the pion. However it was also shown by Gasser and Leutwyler [5,6] that the mass of the pion in the hadronic situation is very close to $\mu_0$. It follows that the hadronic starting point, with respect to which the electromagnetic corrections need to be calculated, has a pion mass $\mu_0$. The results in Eq.(23) are therefore the true electromagnetic corrections. These need always to be calculated with $\mu_0$ as the hadronic mass of the pion.
4 Discussion

We now compare our results with those obtained from the work of Knecht and Urech [17], who use the low energy expansion of ChPT and include the effect of the electromagnetic interaction. At lowest order in ChPT the hadronic scattering lengths are

\[ a^0(\mu_0) = \frac{7\mu_0}{32\pi F^2}, \quad a^2(\mu_0) = -\frac{\mu_0}{16\pi F^2}, \]  

(24)

giving

\[ a_{cc}^h(\mu_0) = \frac{\mu_0}{8\pi F^2}, \quad a_{0c}^h(\mu_0) = -\frac{3\sqrt{2}\mu_0}{32\pi F^2}, \]  

(25)

where \( F \) is the pion decay constant in the chiral limit (\( \approx 88 \text{ MeV} \)). Using Eq.(15) above and Eqs.(2.14)-(2.16) of Ref.[17], we find that

\[ a_{cc} = (2\mu_0^2 - \mu_0^2)_{\mu_0^{-1}}, \quad a_{0c} = -\frac{\sqrt{2}(4\mu_0^2 - \mu_0^2)}{32\pi F^2}_{\mu_0^{-1}}, \]  

(26)

for the physical situation, with the pions having their observed masses. Combining these results, the lowest order calculation of Ref.[17] gives the corrections

\[ \Delta a_{cc}(\mu_0) = -0.0139 \text{ fm}, \quad \Delta a_{0c}(\mu_0) = -0.0533. \]  

(27)

At the one-loop level, only the amplitude for \( \pi^+\pi^- \to \pi^0\pi^0 \) is calculated in Ref.[17]. Using Eq.(15) again, the results in Eqs.(5.2) and (5.6) of Ref.[17] yield the quantities

\[ a_{0c}^h(\mu_0) = -0.1570 \text{ fm }, \quad a_{0c} = -0.1675 \text{ fm }, \]  

(28)

from which we deduce that

\[ \Delta a_{0c}(\mu_0) = -0.0625, \]  

(29)

to be compared with the leading order result in Eq.(27). These numbers have been calculated using the values of the low energy constants \( \tilde{\ell}_i, i = 1 - 4, \) and \( \mathcal{K}_1^{\pm 0}, \mathcal{K}_2^{\pm 0} \) given in Ref.[17].
To compare the results in Eqs.(23) (present work) and (29) (calculated from Ref.[17]) we separate the effect of the pion mass difference from that of the Coulomb interaction. The correction due to the mass difference only is

$$\Delta_{0c}^{md}(\mu_0) = -0.0443 \text{ (present work)}, \quad \Delta_{0c}^{md}(\mu_0) = -0.0561 \text{ (Ref.[17])}. \quad (30)$$

The uncertainty in the second number, based on the quoted uncertainties in $\bar{l}_i, i = 1 - 4$, is 0.0015. Moreover, the change from the lowest order result to the one-loop result is small. The number from our present work is subject to an uncertainty that comes from the possible difference between the purely hadronic potential that we have used and the effective hadronic potential in the presence of the electromagnetic interaction. This difference can be thought of in a field theory picture as the result of electromagnetic mass differences in the internal lines of Feynman diagrams. Assuming that these effects are correctly accounted for in Ref.[17], it is likely that the ChPT result gives a more reliable estimate of the mass difference correction than does our potential model.

On the other hand, due to the large uncertainties in $K_1^{\pm 0}, K_2^{\pm 0}$, the Coulomb correction in the ChPT calculation has a large uncertainty; it is $-0.0064(82)$. The potential model gives a small correction, $+0.0007$, with no significant uncertainty. This result lies within the ChPT error band and removes the uncertainty.

Combining these two results we think that a realistic value of the total electromagnetic correction is

$$\Delta_{0c}(\mu_0) = -0.055(5). \quad (31)$$

The error given in Eq.(31) is a generous one; it may be smaller. There is some uncertainty due to the neglect of higher orders in the chiral expansion and we have allowed the possibility of a larger uncertainty from the constants $\bar{l}_i, i = 1 - 4$. Since the DIRAC experiment [1] aims to deliver a result for $\tau$ that will give a value of $a_{0c}$ with an error of 5%, the uncertainty in $\Delta_{0c}(\mu_0)$ is much smaller than the expected experimental error.

This means that, if the DIRAC experiment achieves its proposed accuracy, it will be able to give a result for $a_{0c}^{h}(\mu_0)$ with an accuracy comparable with that of the present estimates from ChPT. Recent work of Colangelo, Gasser and Leutwyler [18], in which two-loop ChPT is combined with the Roy equations and phenomenological data from pion-pion scattering, gives $a^0 - a^2 = 0.265(4)$. This value is in units $\mu_c^{-1}$ and is based on a hadronic pion mass of $\mu_c$. From the two loop values in Table 1 we obtain

$$a^0(\mu_0) - a^2(\mu_0) = a^0(\mu_c) - a^2(\mu_c) - 0.0143 \text{ fm} \quad (32)$$
If we convert the Colangelo et al. value to fm and assume that Eq.(32) is also true for these results, we then obtain

\[ a_{0c}^h(\mu_0) = -\frac{\sqrt{2}}{3}(a^0(\mu_0) - a^2(\mu_0)) = -0.170(3) \text{ fm} \]  

(33)

Ref.[18] gives an error for \( a_{0c}^h(\mu_c) \) of only 1.8%. However, it needs to be remembered that the calculations were made within the framework of an isospin invariant effective hadronic theory in which the pions have the mass \( \mu_c \) and that the phenomenological data used have not been corrected in any way for the presence of the electromagnetic interaction. The same remarks also apply to the determination of the low energy constants needed to make numerical predictions from ChPT. The error of 1.8% could therefore be an underestimate.

The value \( a_{0c}^h(\mu_0) = -0.170(3) \text{ fm} \), combined with the result in Eq.(31), leads via Eq.(11) to the value \( \tau = 2.9 \text{ fs} \), which agrees with the result of Ref.[14]. We regard this value as an indication of the lifetime that the DIRAC experiment is measuring. The important result of Ref.[17] and the present paper is that the electromagnetic correction in Eq.(31) is now determined with sufficient precision for a measurement of \( \tau \) with an accuracy of 10% to give a value of the hadronic quantity \( a_{0c}^h(\mu_0) \) with an accuracy of not much more than 5%. We have also emphasised that true hadronic information needs to be obtained with a pion mass \( \mu_0 \). In particular, this mass needs to appear in the formal results of hadronic ChPT. Also, the phenomenological hadronic data used to fix the low energy constants and, at a later stage, to refine the hadronic scattering lengths, need to be corrected for the effect of the electromagnetic interaction. This is an important long-term programme and until it is implemented our knowledge of the hadronic scattering lengths will remain limited. On the other hand, since the electromagnetic correction given in Eq.(31) is now well determined, the DIRAC experiment will give a clean value of \( a_{0c}^h(\mu_0) \), whose error comes almost entirely from the error in the experimental result.

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