Triple Reverse Order Law of Drazin Invertible Operators

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Abstract. In this paper we study the triple reverse-order law \((ABC)^D = C^D B^D A^D\) for the Drazin invertible operators \(A, B\) and \(C\) under the commutative relations \([AB, B] = 0, [BC, B] = 0\) and \([AB, BC] = 0\).

1. Introduction and preliminaries

Let \(X\) and \(Y\) be two infinite dimensional Banach spaces. Denote by \(B(X, Y)\) the Banach space of all bounded linear operators from \(X\) to \(Y\). If \(X = Y\), we will simply write \(B(X)\) instead of \(B(X, X)\). By \(N(T)\) and \(R(T)\), we denote the null space and the range of \(T\), respectively. An operator \(P \in B(X)\) with the property \(P^2 = P\) is called a projection. For any two operators \(T, S \in B(X)\), we define the commutator \([T, S]\) to be \(TS - ST\).

Recall that an operator \(T \in B(X)\) is Drazin invertible if there exists \(S \in B(X)\) that satisfies the following equations

\[
TS = ST, \quad S = STS, \quad T^{k+1}S = T^k. \tag{1}
\]

The third equation in (1) means that \(T - TST\) is nilpotent of index \(k\), in this case we write \(\text{ind}(T) = k\). It is worth pointing out that the Drazin inverse \(S\) of \(T\), when it exists, it is unique. In the sequel, \(S\) will be denoted by \(T\textsuperscript{D}\).

It is also common to cite Koliha’s paper \([6]\) as the pioneering work on generalized Drazin inverses, his definition generalizes (1) by replacing the third equation with the assumption \(T - TST\) is quasi-nilpotent. Drazin invertible as well as generalized Drazin invertible operators have many suitable properties. Mainly, an operator \(T \in B(X)\) is Drazin invertible if and only if 0 is a pole of the resolvent and the spectral projection \(T^\pi\) of \(T\) corresponding to \(\{0\}\) is given by \(T^\pi = I - TT\textsuperscript{D}\). It is extremely useful to mention that \(X = N(T^\pi) \oplus R(T^\pi)\).

Consequently, \(T = T_1 \oplus T_2\) with \(T_1 = T\textsuperscript{D}N\) is invertible and \(T_2 = T\textsuperscript{D}R\) is nilpotent.

Among other things, nilpotent operators of index \(n\) are Drazin invertible with \(T^D = (T^D)^{n+1}T^n = 0\). Projections \(P\) are also Drazin invertible with \(P^D = P\).

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In the literature, it is a common knowledge that if $A, B \in \mathcal{B}(X)$ are invertible then $AB$ is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$, this is often known as the reverse order law for ordinary inverse. However, this rule is not well-adapted to other inverses, such as Drazin inverse. In fact, if $A, B$ and $AB$ are Drazin invertible $(AB)^D = B^D A^D$ is meaningless. This problem was a source of interesting research as operator theorists sought to determine exactly what properties $A$ and $B$ must possess in order to satisfy this equality. Among the many paper which featured the aforesaid problem are [9][11][10]. One can find other related results for various inverses in [2][4] and references therein.

Let $H$ be an infinite dimensional Hilbert space, by $T^*$ we denote the Moore-Penrose inverse of $T \in \mathcal{B}(H)$. With regard to the triple reverse order law for the Moore-Penrose inverses, the authors of [5] obtained necessary and sufficient conditions under which

$$(ABC)^* = C^* B^* A^*,$$

where $A, B, C$ and $ABC$ are Hilbert space operators with closed ranges.

The issue to be discussed in this paper concerns some reverse order law for Drazin invertible operators $A, B$ and $C$ under the commutative relations $[AB, B] = 0$, $[BC, B] = 0$ and $[AB, BC] = 0$. In the light of these relations, we are interested in the relationship between $A, B, C$ and $A^D, B^D, C^D$. Consequently, we provide some necessary and sufficient conditions for which

$$(BCAB)^D = B^D A^D C^D B^D.$$  

Additionally, we obtain several triple reverse order law corresponding to $(ABC)^D$.

2. Preparations

We drawn particular attention in this paper to $2 \times 2$ operator matrices on the Banach space $X \oplus Y$ defined by

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

where $T_1 \in \mathcal{B}(X)$, $T_2 \in \mathcal{B}(Y, X)$, $T_3 \in \mathcal{B}(X, Y)$ and $T_4 \in \mathcal{B}(Y)$. The important point to note here is that every bounded operator on $X \oplus Y$ has the aforementioned form.

We are now going to concern our self with operators $A, B, C \in \mathcal{B}(X)$. If $B$ is Drazin invertible with $\text{ind}(B) = n$ then the Banach space $X$ obeys the following decomposition $X = N(B^n) \oplus R(B^n)$ and $A, B, C$ have these forms

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}. \tag{2}$$

Such that $B_1 \in \mathcal{B}(N(B^n))$ is invertible, $N_1 \in \mathcal{B}(R(B^n))$ is nilpotent, $B^n = B^n \oplus 0$ and $B^D = B_1^{-1} \oplus 0$.

Before going any further we began by the following lemmas which have an adequate amount of properties required.

Lemma 2.1. [6][7] $A, B, C, N \in \mathcal{B}(X)$, requiring $N$ to be nilpotent of index $n$.

(1) If $[N, AN] = 0$ then $AN$ and $NA$ are nilpotent with $\max(\text{ind}(NA), \text{ind}(AN)) \leq n$;

(2) If $[N, NC] = 0$ then $NC$ and $CN$ are nilpotent with $\max(\text{ind}(NC), \text{ind}(CN)) \leq n$;

(3) If $A, B, C$ are Drazin invertible and $[A, B, C]$ are mutual-commutative then $A, B, C, A^D, B^D$ and $C^D$ are all commute with

$$(ABC)^D = A^D B^D C^D = C^D B^D A^D.$$  

Lemma 2.2. [8] For $A \in \mathcal{B}(X), B \in \mathcal{B}(Y, X), C_1 \in \mathcal{B}(Y, X)$ and $C_2 \in \mathcal{B}(X, Y)$. We denote by

$$M_{C_1} = \begin{pmatrix} A & C_1 \\ 0 & B \end{pmatrix}, \quad M_{C_2} = \begin{pmatrix} A & 0 \\ C_2 & B \end{pmatrix}$$

where the two operators $M_{C_1}$ and $M_{C_2}$ are in $\mathcal{B}(X \oplus Y)$.
We thus get
\[
ABC
\]
We next suppose that \(X\) according to the Banach space decomposition \(X\) then
\[
3. \text{Main results}
\]
To sharpen these forms we further assume that
\[
\text{(7)} \text{ Let } A \text{ and } B \text{ are Drazin invertible with ind}(A) = s \text{ and ind}(B) = t. \text{ Then}
\]
\[
M^D_{C_1} = \begin{pmatrix} A^D & X \\ 0 & B^D \end{pmatrix}, \quad M^D_{C_2} = \begin{pmatrix} A^D & 0 \\ Y & B^D \end{pmatrix}
\]
where
\[
X = (A^D)^{t-1} \sum_{n=0}^{t-1} (A^D)^n C_1 B^n |B^n + A^n [\sum_{n=0}^{t-1} A^n C_1 (B^D)^n] (B^D)^2 - A^D C_1 B^D;
\]
and
\[
Y = (B^D)^{t-1} \sum_{n=0}^{t-1} (B^D)^n C_2 A^n |A^n + B^n [\sum_{n=0}^{t-1} B^n C_2 (A^D)^n] (A^D)^2 - B^D C_2 A^D.
\]

**Lemma 2.3.** \(\square\) Let \(A, B \in \mathcal{B}(X)\). If \(AB\) is Drazin invertible then \(BA\) is also Drazin invertible. In this case:
\[
(AB)^D = A((BA)^D) B
\]

3. **Main results**

Let \(A, B, C \in \mathcal{B}(X)\). Suppose that \(B\) is Drazin invertible having index \(n\). First we assume that \([B, AB] = 0, \text{ then } [B^n, AB] = 0\). From \(\square\) it follows that
\[
A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & N_1 \end{pmatrix}, \quad \text{ and } \quad AB = \begin{pmatrix} A_1 B_1 & 0 \\ 0 & A_4 N_1 \end{pmatrix}, \tag{3}
\]
according to the Banach space decomposition \(X = N(B^n) \oplus R(B^n)\). This gives
\[
[A_1, B_1] = 0, \quad [N_1, A_4 N_1] = 0 \quad \text{and} \quad A_2 N_1 = 0. \tag{4}
\]

We next suppose that \([B, BC] = 0\), thus \([B^n, BC] = 0\) with respect to \(\square\)
\[
B = \begin{pmatrix} B_1 & 0 \\ 0 & N_1 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & 0 \\ C_3 & C_4 \end{pmatrix} \quad \text{ and } \quad BC = \begin{pmatrix} B_1 C_1 & 0 \\ 0 & N_1 C_4 \end{pmatrix}. \tag{5}
\]
Continuously on \(X = N(B^n) \oplus R(B^n)\). Hence:
\[
[B_1, C_1] = 0, \quad [N_1, N_1 C_4] = 0 \quad \text{ and } \quad N_1 C_3 = 0. \tag{6}
\]
We thus get \(ABC = \begin{pmatrix} A_1 B_1 C_1 & 0 \\ 0 & A_4 N_1 C_4 \end{pmatrix}\).

To sharpen these forms we further assume that \([AB, BC] = 0\), then:
\[
[A_1, C_1] = 0 \quad \text{ and } \quad [A_4 N_1, N_1 C_4] = 0. \tag{7}
\]
This yields that \(A_1, B_1\) and \(C_1\) are pairwise commutative. Nevertheless \(A, B\) and \(C\) are not necessary commutative (e.g. \(AC \neq CA\)).

The following lemma is essential to prove certain results.

**Lemma 3.1.** Let \(A, C, N \in \mathcal{B}(X)\), where \(N\) is nilpotent.
\[
(1) \text{ If } [N, AN] = 0 \text{ and } [AN, ANC] = 0 \text{ then } CAN \text{ is also nilpotent;}
\]
\[
(2) \text{ If } [N, NC] = 0 \text{ and } [AN, NC] = 0 \text{ then } NCA \text{ is also nilpotent;}
\]
Further, by 

\[ (1) \quad \text{Proof.} \]

Theorem 3.2. Let \( A \)

(3) If only one element of \( A \)

(2) If only one element of \( A \)

(1) If only one element of \( A \)

\[ \text{is Drazin invertible, then each of the following statements hold:} \]

\[ (\text{ABC})^D = (\text{ABC})^D B B^D = B B^D (\text{ABC})^D = (\text{ABC} B^D)^D B^D = (B D^2)^D (\text{ABC} B^D)^D = (B D^2)^D (\text{ABC} B^D)^D; \]

\[ (i) \quad \text{ABC}(\text{ABC} B^D)^D \text{ and } ABC - (\text{ABC})^D (\text{ABC} B^D)^D B^D \text{ are nilpotent;} \]

\[ (ii) \quad (\text{ABC} B^D)^D = (\text{ABC})^D B = B (\text{ABC})^D; \]

\[ (iii) \quad (\text{ABC} B^D)^D = (\text{ABC})^D B = B (\text{ABC})^D; \]

\[ (iv) \quad [\text{ABC} B^D, \text{ABC} (\text{ABC} B^D)] = 0; \]

\[ (v) \quad B B^D (\text{ABC})^D = (\text{ABC})^D B B^D = 0. \]

We can now formulate our first main result.

**Theorem 3.2.** Let \( A, B, C \in \mathcal{B}(X), B \text{ is Drazin invertible with } B, AB \text{ and } BC \text{ are all commute. Write} \]

\[ \mathcal{A} = \{ ABC, BCA, CAB, ABCB, BCAB, AB C B^D, B^D ABC, A B B^D C, B D^2 C A B C, B C A B^D, C A B B^D, A C B B^D, B B^D A C B C \}; \]

\[ \mathcal{B} = \{ B, B D, B B^D, A B, B C, A B C, A C B D, B B D (A B C D), (A B C D) B B^D \}. \]

(1) If only one element of \( \mathcal{A} \) is Drazin invertible, then all elements of \( \mathcal{A} \) are Drazin invertible.

(2) If only one element of \( \mathcal{A} \) is Drazin invertible, then all elements of \( \mathcal{B} \) commute.

(3) If only one element of \( \mathcal{A} \) is Drazin invertible, then each of the following statements hold:

\[ (i) \quad (A B C)^D = (A B C)^D B B^D = B B^D (A B C)^D = (A B C B^D)^D B^D = (B D^2)^D (A B C B^D)^D = (B D^2)^D (A B C B^D)^D; \]

\[ (ii) \quad [\text{ABC}, \pi] \quad \text{and } ABC - (\text{ABC})^D (\text{ABC} B^D)^D B^D \text{ are nilpotent;} \]

\[ (iii) \quad (\text{ABC} B^D)^D = (\text{ABC})^D B = B (\text{ABC})^D; \]

\[ (iv) \quad [\text{ABC} B^D, \text{ABC} (\text{ABC} B^D)] = 0; \]

\[ (v) \quad B B^D (\text{ABC})^D = (\text{ABC})^D B B^D = 0. \]
Proof. (1) Formulas $[3]$ and $[5]$ provided the forms of $A, B, C$ and $ABC$. Note that $\{A_i, B_i, C_i\}$ are mutually commutative, $[N_1, A_4 N_4] = 0$, $[N_1, N_1 C_4] = 0$ and $[A_4 N_4, N_1 C_4] = 0$. Hence, from Lemma $3.1$, $A_4 N_4 C_4$ is nilpotent. Further, $ABC$ is Drazin invertible $\iff A_1 B_1 C_1$ is Drazin invertible $\iff A_1 C_1 = (A_1 B_1 C_1) B_1^{-1}$ is Drazin invertible ($\text{since } [A_1 B_1 C_1, B_1^{-1}] = 0$).

Also, we have $CAB = \begin{pmatrix} C_4 A_1 C_4 & 0 & 0 \\ C_3 A_1 C_3 & C_4 A_4 N_4 \\ C_2 A_1 C_2 \\ C_1 A_1 C_1 \end{pmatrix}$ and $BCA = \begin{pmatrix} B_1 C_1 A_1 & B_1 C_1 A_2 \\ B_1 C_1 A_2 \\ 0 & N_1 C_4 A_4 \\ 0 & N_1 C_4 A_4 \end{pmatrix}$.

By Lemma $3.1$, $C_4 A_4 N_4$ and $N_1 C_4 A_4$ are nilpotent. Again, $CAB$ and $BCA$ are Drazin invertible if and only if $C_1 A_1$ is Drazin invertible. In this case

$$(ABC)^D = \begin{pmatrix} (A_1 C_1)^D B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix};$$

$$(CAB)^D = \begin{pmatrix} (A_1 C_1)^D B_1^{-1} & 0 \\ C_3 A_1 ((A_1 C_1)^D B_1^{-1}) & 0 \\ 0 & 0 \end{pmatrix};$$

$$(BCA)^D = \begin{pmatrix} (A_1 C_1)^D B_1^{-1} & 0 \\ C_1 ((A_1 C_1)^D B_1^{-1}) A_2 \\ 0 & 0 \end{pmatrix}.$$ We deduce that Drazin invertibility of each element of $A$ lies in Drazin invertibility of $A_1 C_1$.

(2) The set $\{B, AB, BC\}$ is commutative, then from $[4]$, $[6]$ and $[7]$, the set $\{A_1, B_1, C_1\}$ is also commutative and $[N_1, A_4 N_4] = [N_1, N_1 C_4] = [A_4 N_4, N_1 C_4] = 0$. So clearly

$$N_1 A_4 N_4 C_4 = A_4 N_1 N_1 C_4 = A_4 N_1 C_4 N_1,$$

which means that $[A_4 N_1 C_4, A_4 N_1] = 0$, hence $[ABC, AB] = 0$. Besides this, $[ABC, BC] = 0$ as well. On the other hand all the element of $B$ can be written as diagonal matrix forms, and this imply that all the elements of $B$ commute.

(3) Observe that, $ABCB^D = AB^D BC = ABB^D C$

$$ABCB^D = \begin{pmatrix} A_1 C_1 & 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (ABCB^D)^D = \begin{pmatrix} (A_1 C_1)^D & 0 \\ 0 & 0 \end{pmatrix}.$$ In addition, $ABC(ABB^D C)^n = \begin{pmatrix} A_1 B_1 C_1 (A_1 C_1)^n & 0 \\ 0 & A_4 N_1 C_4 \end{pmatrix}$ is nilpotent. Finally, we can verify by a simple computation the other equalities. \hfill \Box

The following theorem gives a partial solution of the reverse order law for the triple product $ABC$.

**Theorem 3.3.** Let $A, B, C \in B(X)$. If $B, AB, BC, C$ are Drazin invertible and $B, AB, BC$ are all commute, then $ABC$ is Drazin invertible and the following reverse order laws conditions are equivalent.

(i) $(ABC)^D = C^D (AB)^D$;

(ii) $((AB)^D ABC)^D = C^D (AB)^D AB$;

(iii) $(ABC)^D AB = C^D (AB)^D AB$.

**Proof.** If $B$ is Drazin invertible and $\{B, AB, BC\}$ are mutually commutative, then by $[4]$ and $[5]$: $AB = \begin{pmatrix} A_1 B_1 & 0 & 0 \\ 0 & A_4 N_1 \end{pmatrix}$, $C = \begin{pmatrix} C_1 & 0 \\ C_3 & C_4 \end{pmatrix}$ and $ABC = \begin{pmatrix} A_1 B_1 C_1 & 0 \\ 0 & A_4 N_1 C_4 \end{pmatrix}$.
From the proof of [11] Theorem 3.1 \( AB \) is Drazin invertible if and only if \( A_1 \) is Drazin invertible. In this case

\[
(AB)^D = \begin{pmatrix} A_1^DB_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}.
\]

Also the Drazin invertibility of \( BC \) implies that \( C_1 \) is Drazin invertible. Now since \( C \) and \( C_1 \) are Drazin invertible then by Lemma 2.2 \( C_4 \) is also Drazin invertible.

By assuming that \( \text{ind}(C_1) = s \) and \( \text{ind}(C_4) = t \), we can assert that \( C^D = \begin{pmatrix} C_1^D & 0 \\ Y & C_4^D \end{pmatrix} \), where

\[
Y = (C_4^D)^2 \left( s \sum_{i=0}^{s-1} (C_4^D)^i C_3 C_1^T + t \sum_{i=0}^{t-1} C_4^i C_3 (C_1^D)^i (C_1^D)^2 - C_4^D C_3 C_1^D \right).
\]

Also, from Lemma [3.1] \( AB_1C_1 \) and \( C_4 \) are nilpotent \( \{ A_1, B_1, C_1 \} \) are mutually commutative and \( A_1, B_1, C_1 \) are all Drazin invertible. Hence, \( ABC \) is also Drazin invertible and

\[
(ABC)^D = \begin{pmatrix} A_1^DB_1^{-1} & C_1 \\ 0 & 0 \end{pmatrix}.
\]

Now let’s mention that

\[
(AB)^DABC = \begin{pmatrix} A_1^DB_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1B_1C_1 & 0 \\ 0 & A_4N_1C_4 \end{pmatrix} = \begin{pmatrix} A_1^DB_1C_1 & 0 \\ A_1A_4N_1C_4 & 0 \end{pmatrix},
\]

\[
((AB)^DABC)^D = \begin{pmatrix} A_1^DB_1C_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ (since } [C_1, A_1A_1^D] = 0 \text{ and } A_1A_1^D \text{ is a projection)}
\]

\[
C^D(AB)^D = \begin{pmatrix} C_1^D & 0 \\ Y & C_3^D \end{pmatrix} \begin{pmatrix} A_1^DB_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1^D A_1^DB_1^{-1} & 0 \\ YA_1^DB_1^{-1} & 0 \end{pmatrix},
\]

\[
C^D(AB)^DAB = \begin{pmatrix} C_1^D A_1^DB_1 & 0 \\ YA_1^DB_1 & 0 \end{pmatrix}.
\]

We can deduce that \( (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow YA_1^D = 0 \). \( \square \)

A similar observation gives the following theorem and its proof will be omitted.

**Theorem 3.4.** Let \( A, B, C \in \mathcal{B}(X) \). If \( A, B, AB, BC \) are Drazin invertible and \( B, AB, BC \) are all commute, then \( ABC \) is Drazin invertible and the following reverse order laws conditions are equivalent.

(i) \( (ABC)^D = (BC)^D A^P \);
(ii) \( (ABC(BC)^D)^D = (BC)^D (BC)A^P \);
(iii) \( (BC(ABC)^D) = (BC)^D (BC)A^P \).

**Theorem 3.5.** Let \( A, B, C \in \mathcal{B}(X) \). If \( A, B, C, AB, BC \) are Drazin invertible and \( B, AB, BC \) are all commute, then the following reverse order laws conditions are equivalent:

(i) \( (BCAB)^D = B^P A^P C^P B^D \);
(ii) \( (ABB^D)^C = B^B A^P C^P B^D \);
(iii) \( B(BCAB)^DB = BB^P A^P C^P B^D B \).

**Proof.** The Drazin invertibility of \( A, B, C, AB, BC \) combined with the commutativity conditions of \( B, AB, BC \) provided the following matrix forms

\[
A^D = \begin{pmatrix} A_1^D & X \\ 0 & A_4^D \end{pmatrix}, \quad C^D = \begin{pmatrix} C_1^D & 0 \\ Y & C_4^D \end{pmatrix},
\]

(8)
with
\[ \begin{align*}
X &= (A_1^n)^{k-1} \left( A_1^n A_2 A_1^n \right) + A_2^n \left( A_1^n A_2 A_1^n \right)^2 - A_1^n A_2 A_1^n, \\
Y &= (C_1^n)^{k-1} \left( C_1^n C_2 C_1^n \right) + C_2^n \left( C_1^n C_2 C_1^n \right)^2 - C_1^n C_2 C_1^n.
\end{align*} \]

Here \( \text{ind}(A_1) = s_1, \text{ind}(A_2) = t_1, \text{ind}(C_1) = s_2 \) as well as \( \text{ind}(C_2) = t_2 \). Also \( BCAB = \begin{pmatrix} C_1(B_1)^2 A_1 & 0 \\ 0 & N_1 C_4 A_4 N_1 \end{pmatrix} \).

Certainly, \( N_1 C_4 A_4 N_1 \) is nilpotent and \( (BCAB)^D = \begin{pmatrix} (C_1(B_1)^2 A_1)^D & 0 \\ 0 & 0 \end{pmatrix} \). Moreover, \( ABB^D C = \begin{pmatrix} A_1 C_4 & 0 \\ 0 & 0 \end{pmatrix} \) and \( (ABB^D) C = \begin{pmatrix} A_1 C_4 & 0 \\ 0 & 0 \end{pmatrix} \).

By a simple calculation, we can obtain the following:

\[ \begin{align*}
B^D A^D C^D B^D &= \begin{pmatrix} A_1^D (B_1)^2 C_1^D + (B_1)^D X Y B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \\
B^D A^D C^D B^D &= \begin{pmatrix} A_1^D C_1^D + X Y & 0 \\ 0 & 0 \end{pmatrix}, \\
B(BCAB)^D B &= \begin{pmatrix} (C_1^D A_1^D) & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*} \]

This gives the following equivalences \( (i) \iff (ii) \iff (iii) \iff XY = 0. \)

In the following theorem, we get a first glimpse of \( (ABC)^D = C^D B^D A^D \).

**Theorem 3.6.** Let \( A, B, C \in \mathcal{B}(X) \). If \( A, B, C, AB, BC \) are Drazin invertible and \( B, AB, BC \) are all commute, then \( ABB^D, B^D BC, ABC \) are all Drazin invertible. Furthermore, the following reverse order law conditions are equivalent:

1. \( (ABC)^D = C^D B^D A^D \);  
2. \( C^D (AB)^D = C^D B^D A^D = (BC)^D A^D \);  
3. \( BB^D C^D B^D A^D = C^D B^D A^D = C^D B^D A^D BB^D \);  
4. \( (ABB^D)^D (B^D BC) = C^D B^D A^D \);  
5. \( A^D B^D BC^D A^D ABB = C^D B^D A^D \);  
6. \( B^n C^D B^D A^D = BB^n C^D B^D A^D \) and \( C^D B^D A^D B^n = C^D B^D A^D B^n \).

**Proof.** \( AB, BC \) and \( ABC \) have the matrix forms:

\[
AB = \begin{pmatrix} A_1 B_1 & 0 \\ 0 & A_4 N_1 \end{pmatrix}, \quad BC = \begin{pmatrix} B_1 C_1 & 0 \\ 0 & N_1 C_4 \end{pmatrix} \quad \text{and} \quad ABC = \begin{pmatrix} A_1 B_1 C_1 & 0 \\ 0 & A_4 N_1 C_4 \end{pmatrix}.
\]

Of course, \( A_4 N_1, N_1 C_4 \) and \( A_4 N_1 C_4 \) are nilpotent. Moreover, \( A \) and \( AB \) are Drazin invertible (resp, \( C \) and \( BC \)) then \( A_1 \) and \( A_4 \) (resp, \( C_1 \) and \( C_4 \)) are Drazin invertible. Hence, it is easy to verify that \( AB \) is Drazin invertible. In this case, we obtain

\[
(AB)^D = \begin{pmatrix} A_1^D B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad (BC)^D = \begin{pmatrix} B_1^{-1} C_1^D & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad (ABC)^D = \begin{pmatrix} A_1^D B_1^{-1} C_1^D & 0 \\ 0 & 0 \end{pmatrix}.
\]

On the other hand \( A^D, C^D \) can be written as in [5]. So we get

\[
C^D B^D A^D = \begin{pmatrix} A_1^D B_1^{-1} C_1^D \\ Y B_1^{-1} A_1^D \end{pmatrix}, \quad (ABC)^D = \begin{pmatrix} A_1^D B_1^{-1} C_1^D \\ Y B_1^{-1} X \end{pmatrix}.
\]
Equivalent conditions of \((ABC)^D = C^DB^D A^D\) are:
\[
\begin{align*}
C_1^D X &= 0 \\
Y A_1^D &= 0. \\
Y B_1^{-1} X &= 0
\end{align*}
\]
Note that
\[
C^D (AB)^D = C^DB^D A^D BB^D = \begin{pmatrix} C_1^D A_1^{-1} B_1^{-1} & 0 \\ Y A_1^D B_1^{-1} & 0 \end{pmatrix},
\]
and
\[
(BC)^D A^D = BB^D C^DB^D A^D = \begin{pmatrix} B_1^{-1} C_1^D A_1^D & C_1^D B_1^{-1} X \\ 0 & 0 \end{pmatrix}.
\]

Therefore, \((2) \iff (3) \iff (4) \iff (5)\). Finally, \((1) \iff (4) \iff (5)\).

Inserting the reverse order law of \(AB\) in Theorem 3.3 yields the following corollary.

**Corollary 3.7.** Let \(A, B, C \in \mathcal{B}(X)\) be such that \(A, B, C, AB, BC\) are Drazin invertible and \(B, AB, BC\) are all commute. If \((AB)^D = B^D A^D\) then the following reverse order law conditions are equivalent:

(i) \((ABC)^D = C^DB^D A^D\);
(ii) \(((AB)^D ABC)^D = C^DB^D A^D AB\);
(iii) \((ABC)^D AB = C^DB^D A^D AB\).

**Proof.** The reverse order law condition \((AB)^D = B^D A^D\) is equivalent to \(X = 0\). Thus \(C^DB^D A^D = \begin{pmatrix} A_1^D B_1^{-1} C_1^D & 0 \\ Y B_1^{-1} A_1^D & 0 \end{pmatrix}\), and the equality \((ABC)^D = C^DB^D A^D\) is equivalent to \(Y A_1^D = 0\).

\[
C^DB^D A^D AB = \begin{pmatrix} C_1^D A_1^D & 0 \\ Y A_1^D & 0 \end{pmatrix}
\quad \text{and} \quad
(AB)^D ABC = \begin{pmatrix} A_1^D C_1^D & 0 \\ 0 & 0 \end{pmatrix}.
\]

Hence, \(((AB)^D ABC)^D = C^DB^D A^D AB \iff Y A_1^D = 0\).
Also, \((ABC)^D AB = C^DB^D A^D AB \iff Y A_1^D = 0\). Which complete the proof.
In a similar pattern using the reverse order law of $BC$ in Theorem 3.4, we obtain:

**Corollary 3.8.** Let $A, B, C \in \mathcal{B}(X)$ be such that $A, B, C, AB, BC$ are Drazin invertible and $B, AB, BC$ all commute. If $(BC)^D = C^DB^D$ then the following reverse order law conditions are equivalent:

(i) $(ABC)^D = C^DB^PA^D$;
(ii) $(ABC(BC)^D)^D = BCC^DB^PA^D$;
(iii) $BC(ABC)^D = BCC^DB^PA^D$.

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