Quantum Mechanical Lorentzian Wormholes in Cosmological Backgrounds

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Abstract

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We present a minisuperspace analysis of a class of Lorentzian
wormholes that evolves quantum mechanically in a background Fried-
man Robertson Walker spacetime. The quantum mechanical wave-
function for these wormholes is obtained by solving the Wheeler-
DeWitt equation for Einstein gravity on this minisuperspace. The
time-dependent expectation value of the wormhole throat radius is
calculated to lowest order in an adiabatic expansion of the Wheeler-
DeWitt hamiltonian. For a radiation dominated expansion, the radius
is shown to relax asymptotically to obtain a value of order the Planck
length while for a deSitter background, the radius is stationary but al-
ways larger than the Planck length. These two cases are of particular
relevance when considering wormholes in the early universe.
1 Introduction

Wormholes are handles in the spacetime topology linking widely separated regions of the universe, or “bridges” joining two or more different spacetimes. Current interest in Lorentzian or Minkowski-signature wormholes has been motivated initially by attempts to understand nontrivial spacetime topologies, topology changing processes, their physical consequences, and their role, if any, in quantum gravity. A survey of recent work on wormhole physics includes topics addressing the fundamental properties of traversable macroscopic wormholes \([1, 2]\), construction of explicit wormhole solutions and corresponding classical \([3]\) and quantum \([4]\) stability analysis, wormholes as time machines and causality violation \([5]\), wormholes in higher-derivative gravity \([6]\), the phenomenon of gravitational squeezing of the vacuum as a natural mechanism for producing the states of weak energy condition violating matter required to prop open the throats of semiclassical microscopic wormholes \([7]\), wormholes as representing one possible mode of fluctuation of the spacetime foam \([8]\), and wormholes as gravitational lenses \([9]\).

Recently, a very simple model of classical dynamic Minkowski signature wormholes connecting Friedman-Robertson-Walker (FRW) spacetimes was constructed with the aim of initiating an investigation of the possible cosmological effects of such objects in the early universe \([10]\). In that work, an alternative solution to the horizon problem was proposed based on the possibility that the early universe did not necessarily inflate, but was populated with a network of microscopic Planck scale wormholes joining otherwise causally disjoint regions of space. The essence of the idea being that these wormholes can allow two-way transmission of signals between spatially separated regions of spacetime and thus permit such regions to come into thermal contact. This wormhole “phenomenology” is based on very elementary considerations. Indeed, to be of cosmological interest, these wormholes need only have stayed open long enough for the radiation, initially located in causally disconnected regions, to traverse the “handle”. Within the class of wormholes treated in \([10]\), a crude estimate for the time required for thermalization can be had considering the following purely illustrative example. First, assume an initial number density \(n(t_{Pl})\) of Planck-sized wormholes of average radius \(L_P\) at the Planck time and let \(n(t_{Pl}) = \gamma n_{Pl}\), where \(\gamma = \text{const.}\) represents a fraction of the Planck number density: \(n_{Pl} = (L_P)^{-3}\). Then, at some later
time \( t \), \( n(t) \sim R^{-3} \), so the wormhole number density is given by

\[
n(t) = \gamma n_{pl} \left( \frac{R_{pl}}{R(t)} \right)^3.
\]  

(1)

Define the volume-filling factor (the volume of space filled by the interior of wormhole mouths) by

\[
v_{\text{fill}}(t) = \frac{\text{number density \times wormhole volume}}{\text{Planck density \times Planck volume}} = \frac{n(t)}{n_{pl}} \left( \frac{l(t)}{L_P} \right)^3.
\]  

(2)

The instantaneous wormhole throat radius \( l(t) = aR \), where \( R \) is the scale factor of the background spacetime and \( a \) is the wormhole radius in the limit of a static background. For a radiation dominated cosmology and with a matter distribution on the throat subject to a particular equation of state (\( P \) is the surface pressure density and \( \sigma \) the surface energy density on the throat) \( P = P(\sigma) \), one can show that the throat decouples from the expansion with \( l \sim R^2 \), so that the filling factor becomes

\[
v_{\text{fill}}(t) = \gamma \left( \frac{R(t)}{R_{pl}} \right)^3.
\]  

(3)

When \( v_{\text{fill}}(T) \approx 1 \), there is one wormhole per unit volume by this equilibrium time \( T \), and the average particle will have traversed at least one wormhole. So, for example, an initial wormhole number density of only \( \gamma \approx 10^{-14} \) allows thermalization of distant spatial regions by the GUT time scale \( T_{\text{GUT}} \sim 10^{-34} \) sec. Of course, different values of \( \gamma \) will lead to different values for the thermalization time scale \( T \).

While the phenomenological application put forward in [10] is novel, it is far from constituting a proof as such, and much more work is clearly required to settle the issue. As a case in point, note that a constant \( \gamma \) implies the total number of wormholes is constant. This need not be the case in general and we expect \( \gamma = \gamma(t) \). This quantity is related to the initial spectrum of wormholes and a first principles calculation of this factor would provide significant insight into the structure of spacetime at the Planck scale. Of equal importance is the fact that the wormholes treated in [10] were strictly classical and if such objects do indeed exist at or near the Planck scale, it is much more likely they will be quantum mechanical in nature. Moreover, one
would like to know how the wormholes evolve in time and how the specifics of their temporal development is affected by the background spacetime. It is this latter issue which shall be taken up in the present paper.

The quantum mechanics of Minkowski signature wormholes in FRW backgrounds will be addressed within the framework of a minisuperspace analysis applied to a specific class of wormholes resulting from surgically modified FRW cosmologies. The wavefunction of the wormhole will be obtained as a solution of the Wheeler-DeWitt equation. Because no one knows how to solve the Wheeler-DeWitt equation on the full extended superspace of three dimensional metrics (and matter field configurations), it proves extremely useful to examine this equation on minisuperspace configurations, which are restricted configurations described by only a finite number of degrees-of-freedom, with all the other modes of the full superspace frozen out. This is a standard approximation employed in quantum cosmology and is easily adapted for quantizing localized configurations in spacetime. Such analyses have been applied to the study of quantum wormholes in Minkowski, Nordstrom-Reissner and Schwarzschild-deSitter backgrounds [4, 11]. While the Wheeler-DeWitt equation is time-independent, the expanding background provides a natural time parameter which an external observer can employ in order to characterize change in the wormhole’s configuration. For FRW spacetimes, this time parameter is simply the scale factor (or any invertible function of the scale factor).

This paper is organized as follows. In Section II we describe in detail the minisuperspace model we adopt for Lorentzian wormholes. Canonical quantization of the wormhole is carried out in Section III and the Wheeler-DeWitt equation solved in perturbation theory for the wormhole wavefunction. The expectation value of the wormhole throat radius is calculated and is found to be partially dragged along with the expansion when the universe is radiation dominated, although relaxing to attain a value of the order the Planck length for late times. By contrast, a quantum wormhole in a deSitter background maintains a constant throat radius that is always greater than the Planck length. Both these statements hold when there is no matter residing on the wormhole throat. We briefly comment on the effects of including matter in Section IV. Discussion and caveats are presented in Section V. A Green function needed for the solution of a certain nonlocal differential equation is calculated in the Appendix. Units with $G = c = 1$ are used throughout, while $\hbar = L_P^2$, where $L_P$ is the Planck length.
2 Minisuperspace Model for Wormholes

The wormholes we will be considering result from surgically modified Friedman-Robertson-Walker spacetimes. We adopt this technique to make the subsequent analysis tractable, however, we assume the qualitative features of the wormholes are independent of the details of the construction. To construct them, take two copies $M_1, M_2$ of a FRW spacetime with identical scale factors $R(t)$ and spatial curvature constant $\kappa$:

$$ds^2 = -dt^2 + R^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (4)$$

and remove from each an identical four-dimensional region of the form $\Omega_{1,2} = \{r_{1,2} < a\}$. The resulting spacetime contains two disjoint boundaries $\partial \Omega_{1,2} = \{r_{1,2} = a\}$ which are timelike hypersurfaces. An orientation preserving identification $\partial \Omega_1 = \partial \Omega_2 = \partial M$ yields two FRW cosmologies connected by a wormhole whose throat is located on their mutual boundary $\partial \Omega$. This model is an extreme version of a wormhole for which the spacetime curvature in the throat is much greater than that in the regions surrounding the mouths. Here $a = a(\tau)$, is a function describing the dynamics of the wormhole’s throat. It is important to point out that the proper motion of the wormhole throat is in general independent, that is, decoupled, from the background cosmic flow. Nevertheless, at any given instant, the physical radius (see the explicit form of the throat’s three-metric below in (9)) of the wormhole is given by the product $aR$. The wormhole is spherically symmetric and the boundary layer is just the world volume swept out by its throat. This procedure also leads to a wormhole connecting a single FRW spacetime to itself if one identifies the two background geometries, i.e., we have a FRW spacetime with a handle. In this case, the two regions $\Omega_{1,2}$ can be separated by an arbitrarily large distance in an open ($\kappa = 0, -1$) universe. For the classical case, one insures that the modified spacetime is itself a solution of the gravitational field equations by effecting a proper matching of the metric across the boundary layer $\Omega$. These classical wormholes must necessarily involve stress-energy distributions that violate the weak energy condition in the vicinity of the throat. This feature is, however, not an artifact of the construction and holds in general [1,2]. At the quantum mechanical level, the state of the wormhole is described by appropriate solutions of the Wheeler-DeWitt equation, either with, or without the inclusion of matter.
As a consequence of the above construction, the Ricci tensor is everywhere FRW except at the wormhole throat:

\[
R_{\mu}^{\nu}(x) = \left( \begin{array}{cc} Tr \Delta K & 0 \\ 0 & \Delta K_j^j \end{array} \right) \delta(\eta) + R_{\nu}^{(1)\mu}(x) \Theta(\eta) + R_{\nu}^{(2)\mu}(x) \Theta(-\eta),
\]

where \( R_{\nu}^{(1)\mu}, R_{\nu}^{(2)\mu} \) are the Ricci tensors for the two spacetimes \( M_1 \) and \( M_2 \), respectively and \( \Delta K_j^j = (K_j^{(2)i} - K_j^{(1)i}) \) is the jump, or discontinuity, in the extrinsic curvature of the boundary layer (i.e., the throat) in transiting from \( M_1 \) to \( M_2 \). Proper normal distance as measured from the throat is conveniently parametrized in terms of the Gaussian normal coordinate \( \eta \). For the case at hand, reflection symmetry implies \( \Delta K_j^j = 2K_j^{(2)i} \) and spherical symmetry implies that \( K_j^i = \text{diag}(K_r^r, K_\theta^\theta, K_\theta^\theta) \). The gravitational action for the wormhole is given by two contributions, to wit

\[
S_{\text{wormhole}} = S_{\text{throat}} + S_{\text{background}},
\]

where

\[
S_{\text{throat}} = -\frac{1}{8\pi} \int_{\partial M} d^3 x \sqrt{-h} \ Tr(\Delta K),
\]

and

\[
S_{\text{background}} = -\frac{1}{16\pi} \int_{M_2} d^4 x \sqrt{-g} R^{(2)} - \frac{1}{16\pi} \int_{M_1} d^4 x \sqrt{-g} R^{(1)}.
\]

We shall first calculate the throat action. The 3-dimensional metric \( h_{ij} \) induced on the throat world-volume is

\[
h_{ij}(x) = \text{diag} \left( -1, a^2(\tau)R^2(t), a^2(\tau)R^2(t)\sin^2\theta \right),
\]

where \( \tau \) denotes the proper time on the wormhole throat and \( t \) refers to the background cosmic time. To evaluate (7), we need the components of the extrinsic curvature tensor

\[
K_{ij} = n_{\mu} \nabla_{(j)} e_{(i)}^{\mu},
\]

where the \( e_{(i)} \) constitute a set of three linearly independent tangent vectors defined along the intrinsic coordinates \( \xi^i \) parametrizing the world-volume hypersurface and \( n^\mu \) is the outward unit normal \( (n^\mu n_\mu = 1) \). The covariant derivative intrinsic to the hypersurface is taken along the \( j^{th} \) coordinate
direction. The throat proper time and the two angles provide a convenient set of intrinsic coordinates: \( \xi^i = (\tau, \theta, \phi) \). As the location of the throat (i.e., its embedding with respect to the background FRW geometry) is \( X^\mu = (t, a, \theta, \phi) \), the tangent vectors are given by \( e^\mu_{(i)} = dX^\mu / d\xi^i \). Since \( n_\mu e^\mu_{(i)} = 0 \) for \( i = 1, 2, 3 \), the unit normal to the throat is given by

\[
n^\mu = \left( \frac{\dot{a}R}{\sqrt{1 - \kappa a^2}}, \frac{1}{R} \sqrt{1 - \kappa a^2 + \dot{a}^2 R^2}, 0, 0 \right). \tag{11}
\]

A straightforward calculation of (10) yields

\[
K^\phi_\phi = K^\theta_\theta = \frac{\dot{a}R'}{\sqrt{1 - \kappa a^2}} + \frac{1}{aR} \sqrt{1 - \kappa a^2 + (\dot{a}R)^2}, \tag{12}
\]

and

\[
K^\tau_\tau = \frac{\ddot{a}R}{\sqrt{1 - \kappa a^2 + (\dot{a}R)^2}} + \frac{2\dot{a}R'}{\sqrt{1 - \kappa a^2}} + \frac{(\kappa a^2)\ddot{a}^2 R}{(1 - \kappa a^2) \sqrt{1 - \kappa a^2 + (\dot{a}R)^2}}, \tag{13}
\]

where \( \dot{a} = da / d\tau \) and \( R' = dR / dt \). Note the appearance of the two time parameters \( \tau \) and \( t \) in (12) and (13). We can always cast the equations in terms of one or the other time parameter by means of the jacobian

\[
e^0_\tau = \frac{dt}{d\tau} = \left( \frac{1 - \kappa a^2 + (\dot{a}R)^2}{1 - \kappa a^2} \right)^{1/2}. \tag{14}
\]

Physically, this amounts to a calibration of the clocks attached to the throat in terms of the comoving clocks, if we choose to express \( \tau \) in terms of \( t \), or vice-versa, if instead we choose to express \( t \) as a function of \( \tau \). We shall have occasion to use (14) below. From (7), (9), (12) and (13), we have \( Tr(\mathcal{K}) = \Delta K^\tau_\tau + 2 \Delta K^\theta_\theta \) and

\[
S_{throat} = \int d\tau [2(a\ddot{a} R^2 + a^2 R\dot{R}) \sinh^{-1} \left( \frac{\dot{a}R}{\sqrt{1 - \kappa a^2}} \right) - \frac{3\dot{a}R'(a^2 R^2)}{\sqrt{1 - \kappa a^2}} - 2aR\sqrt{1 - \kappa a^2 + (\dot{a}R)^2}], \tag{15}
\]

where we have made use of the identity

\[
\frac{\ddot{a}R}{\sqrt{1 - \kappa a^2 + (\dot{a}R)^2}} = \frac{d}{d\tau} \sinh^{-1} \left( \frac{\dot{a}R}{\sqrt{1 - \kappa a^2}} \right) - \frac{\dot{a}\dot{R}}{\sqrt{1 - \kappa a^2 + (\dot{a}R)^2}} - \frac{\kappa a^2 R}{(1 - \kappa a^2) \sqrt{1 - \kappa a^2 + (\dot{a}R)^2}}. \tag{16}
\]
together with an integration by parts. Because \( \frac{\dot{R}}{dt} = \frac{R'}{d\tau} \), we are able to express \( \dot{R} \) in terms of \( R, R', a \) and \( \dot{a} \).

The contribution to the action coming from the background (external to the throat) spacetime is

\[
S_{\text{background}} = 3 \int_M dt \frac{dr \, r^2}{\sqrt{1 - \kappa r^2}} (R^2 R'' + RR'^2 + \kappa R),
\]

where \( M = (M_2 - \Omega_2) \cup (M_1 - \Omega_1) \). Note that the integration over \( r \) is formally divergent for the cases \( \kappa = -1, 0 \), corresponding to open universes. To avoid possible technical complications associated with such divergences, we consider for the moment the case when the background FRW spacetime is spatially bounded, though the point we will be making below is independent of the value of \( \kappa \). In this case, the integral over \( r \) is convergent and gives

\[
\int_a^1 \frac{dr \, r^2}{\sqrt{1 - r^2}} = \frac{\pi}{4} + \frac{a}{2} \sqrt{1 - a^2} + \frac{1}{2} \sin^{-1}(a) \equiv f(a),
\]

so that

\[
S_{\kappa=1}^{\text{background}} = 3 \int d\tau f(a) \left( \frac{1 - a^2 + (\dot{a}R)^2}{1 - a^2} \right)^{1/2} (R^2 R'' + RR'^2 + R),
\]

employing the time calibration jacobian in (14). With (15) and (19), we can now easily extract the wormhole lagrangian from

\[
S_{\text{wormhole}} = \int d\tau L_{\text{wormhole}}(a, \dot{a}; R, R'),
\]

and calculate the momentum conjugate to \( a \), \( \Pi = \partial L/\partial \dot{a} \):

\[
\Pi_{\kappa=1} = 2aR^2 \sinh^{-1} \left( \frac{\dot{a}R}{\sqrt{1 - a^2}} \right) - \frac{a^2 R^2 R'}{\sqrt{1 - a^2}} + \frac{3\dot{a}R^2 f(a)}{\sqrt{1 - a^2}} \frac{F_{\kappa=1}(R, R', R'')}{\sqrt{1 - a^2} + (\dot{a}R)^2},
\]

where

\[
F_{\kappa}(R, R', R'') = R^2 R'' + RR'^2 + \kappa R.
\]

As it stands, the transcendental relation (21) in general cannot be inverted in closed form in order to yield \( \dot{a} \) in terms of \( \Pi \) and \( a \). The ‘culprit’ of
this obstruction is the jacobian factor \((dt/d\tau)\), which depends explicitly on the throat velocity \(\dot{a}\) for all values of \(\kappa\). However, provided there exist backgrounds for which \(F^\kappa = 0\), then inversion of this relation in closed form is possible, and we can solve for the throat velocity in terms of the throat momentum (valid for all \(\kappa\)):

\[
\dot{a} = \frac{\sqrt{1 - \kappa a^2}}{R} \sinh \left( \frac{\Pi}{2a R^2} + \frac{a R'}{2\sqrt{1 - \kappa a^2}} \right),
\]

where the canonical momentum \(\Pi\) is that of the wormhole’s throat. Fortunately, there are several physically interesting cases of background FRW cosmologies for which \(F^\kappa\) vanishes identically. Indeed, they are (i) Minkowski space: \(\kappa = 0, R = 1\) [4], (ii) radiation dominated expansion: \(\kappa = 0, R(t) = t^{1/2}\), and (iii) DeSitter space: \(\kappa = 0, R(t) = e^{\sqrt{|\Lambda|/3} t}\), where \(\Lambda\) is a cosmological constant. For this latter case, one of course adds a cosmological constant term to the action in (8). We shall consider wormhole dynamics with \(F^\kappa = 0\) for the remainder of this paper. We should point out that cases (ii) and (iii) are, of course, of particular relevance when treating quantum wormholes in the early universe. Moreover, as the curvature \(\kappa\) is negligible during the early stages of expansion, taking \(\kappa = 0\) is a good approximation during this epoch.

From the lagrangian in (20) the classical Wheeler-DeWitt hamiltonian for the wormhole is obtained by the standard Legendre transform:

\[
H(\Pi, a; R, R') = \Pi \dot{a} - L(a, \dot{a}, R, R') = -2a^2 R R' \sqrt{1 + (\dot{a} R)^2} \sinh^{-1}(\dot{a} R)
\]

\[
+ 2\dot{a} R' (a^2 R^2) + 2a R \sqrt{1 + (\dot{a} R)^2}
\]

\[
= -2a^2 R R' \cosh \left( \frac{\Pi}{2a R^2} + \frac{a R'}{2} \right) \left( \frac{\Pi}{2a R^2} + \frac{a R'}{2} \right)
\]

\[
+ 2a^2 R R' \sinh \left( \frac{\Pi}{2a R^2} + \frac{a R'}{2} \right)
\]

\[
+ 2a R \cosh \left( \frac{\Pi}{2a R^2} + \frac{a R'}{2} \right),
\]

where in the final equality we used (23) to eliminate \(\dot{a}\) in terms of \(\Pi\). As is to be expected, the classical dynamics of the wormhole follows by setting
\( H(\Pi, a; R, R') = 0 \), the vanishing of the hamiltonian being a consequence of the reparametrization invariance of the gravitational action. Note that this hamiltonian depends explicitly on external cosmic time through the scale factor \( R \) and its first derivative \( R' \). Moreover, the canonical momentum of the throat (21) is shifted by an amount directly proportional to time rate of change of the scale factor. The one degree of freedom in this (classical) minisuperspace is the throat radius function \( a \).

3 Quantum Dynamics of the Wormhole

As we are interested in the quantum mechanical features of Minkowski-signature wormholes, let us proceed directly to quantize canonically the class of wormholes constructed above. Invoking the prescription

\[
\Pi \to -i\hbar \frac{\partial}{\partial a},
\]

promotes the classical constraint \( H = 0 \) to a differential equation

\[
H \left( -i\hbar \frac{\partial}{\partial a}, a; R, R' \right) \Psi(a, t) = 0,
\]

where \( \Psi(a, t) \) is the time-dependent wavefunction of the wormhole. As pointed out above, the throat momentum \( \Pi \) is shifted relative to the \( \sinh^{-1} \) factor by an amount proportional to the time rate of change of the background spacetime, \( R' \). This term represents the coupling between the wormhole throat and the background. This shift results in a complex Wheeler-DeWitt hamiltonian operator

\[
H = H_r + i H_i
\]

where

\[
H_r = \left[ -(2a^2RR')x \cosh(x) + (2a^2RR') \sinh(x) + (2aR) \cosh(x) \right] \cos(y) + (2a^2RR') \sinh(x) y \sin(y),
\]

is the real part and

\[
H_i = \left[ (2a^2RR')x \sinh(x) - (2a^2RR') \cosh(x) - (2aR) \sinh(x) \right] \sin(y) + (2a^2RR') \cosh(x) y \cos(y)
\]
is the imaginary part. Here, \( x = aR'/2 \) and the operator \( \mathbf{y} = \frac{L_p^2}{2iaR^2} \partial_a \), with \( L_p = \hbar^{1/2} \) the Planck length, and we have adopted the factor ordering prescription such that \( \mathbf{y} \) always stands to the right of \( x \). In the absolute static limit (e.g., case (i)), \( R' = 0 \) so \( x = 0 \), and we obtain (after setting \( R = 1 \), without loss of generality)

\[
H_r = 2a \cos \left( \frac{L_p^2}{2a} \frac{\partial}{\partial a} \right) \quad (30)
H_i = 0, \quad (31)
\]

that is, the hamiltonian is purely real. Solutions of the Wheeler-DeWitt equation with the hamiltonian in (30,31) provide the starting point for a time-independent quantum analysis of wormholes in a Minkowski background [4].

Although we have not been able to find exact solutions of (26) for \( R' \neq 0 \), we can obtain approximate solutions for an adiabatic expansion of the hamiltonian. To this end, we expand \( H_r + iH_i \) to order \( R' \), and put

\[
H = H_r^{(0)} + i R' H_i^{(1)}
\]

where

\[
H_r^{(0)} = 2aR \cos(y),
H_i^{(1)} = -(3a^2R) \sin(y) + (2a^2R) y \cos(y). \quad (33)
\]

We solve for the wormhole wavefunction to \( O(R') \) in perturbation theory:

\[
\Psi(a, t) = \Psi^{(0)}(a, t) + R' \Psi^{(1)}(a, t), \quad (34)
\]

where \( \Psi^{(0)} \) is the exact zeroth-order solution which satisfies \( H_r^{(0)} \Psi^{(0)}(a, t) = 0 \). To this order, the Wheeler-DeWitt equation implies a relation between the zeroth and first order parts of the wavefunction:

\[
H_r^{(0)} \Psi^{(1)}(a, t) = -i H_i^{(1)} \Psi^{(0)}(a, t). \quad (35)
\]

We make use of the fact that there is an infinite set of exact zeroth-order solutions

\[
\Psi_m^{(0)}(a, t) = C_m e^{-(m+\frac{1}{2})\pi a^2 R^2/L_p^2}, \quad (36)
\]

which is an obvious generalization of the wavefunction employed in [4]. The integer index \( m = 0, 1, 2, \cdots \), negative values being discarded as the wavefunction is not normalizable for \( m < 0 \). Since

\[
y \Psi_m^{(0)}(a, t) = -(m + \frac{1}{2})\pi \Psi_m^{(0)}(a, t), \quad (37)
\]
inserting (36) into (35) leads to
\[
\cos(y) \Psi^{(1)}_m(a, t) = (-)^{m+1} \frac{3ia}{2} \Psi^{(0)}_m(a, t). \tag{38}
\]
To solve this equation, we assume a product ansatz
\[
\Psi^{(1)}_m(a, t) = g(a, t) \Psi^{(0)}_m(a, t), \tag{39}
\]
which when substituted into (38) and employing the identity \[15\]
\[
\cos(y)g(a)f(a) = [\cos(y)g(a)][\cos(y)f(a)] - [\sin(y)g(a)][\sin(y)f(a)], \tag{40}
\]
leads to the equation
\[
\sin(y)g(a, t) = -\frac{3i}{2} a. \tag{41}
\]
This nonlocal inhomogeneous differential equation can be solved using Green function techniques. The details of the calculation of the Green function for the operator \(\sin(y)\) are relegated to the Appendix. The result is that
\[
g(u) = \frac{3L_P}{2iR} \int_0^\infty G(u - u') \sqrt{u'} \, du' + \sum A_m e^{-m\pi u}, \tag{42}
\]
where \(u = a^2 R^2 / L_P^2 \geq 0\) and the Green function \(G(u - u')\) is calculated explicitly in (A10,A11).

To proceed, we take the \(A_m = 0\) (i.e., vanishing homogeneous solution) and note that the Green function (A10,11) has the form of a “kink” or topological soliton with center at \(u > 0\):
\[
\lim_{u' \to \pm \infty} G(u - u') = \begin{cases} -\sqrt{\frac{\pi}{2}}, & \text{for } u' > u \\ +\sqrt{\frac{\pi}{2}}, & \text{for } u' < u \end{cases}. \tag{43}
\]
While (42) is exact, the integral cannot be reduced to known elementary functions. Nevertheless, a reasonable approximation can be had by taking
\[
G(u - u') \sim \sqrt{\frac{\pi}{2}} \left( \Theta(u - u') - \Theta(u' - u) \right), \tag{44}
\]
which does incorporate the essential kink structure and asymptotic behavior of \(G\). Doing so, we obtain
\[
g(a, t) = -i\sqrt{2\pi} a^3 (R(t)/L_P)^2, \tag{45}
\]
after discarding an infinite constant \[^{[14]}\].

The wormhole wavefunction, to the order we are working, is therefore given by

\[
\Psi_{mn}(a, t) = \Psi_{mn}^{(0)}(a, t) \left(1 - i\sqrt{2\pi}a^3 R'(R/L_p)^2\right),
\]

(46)

where

\[
\Psi_{mn}^{(0)}(a, t) = C_{mn} \left[ e^{-(m+\frac{1}{2})\pi(\frac{aR}{L_p})^2} - e^{-(n+\frac{1}{2})\pi(\frac{aR}{L_p})^2}\right],
\]

(47)

and is manifestly complex, the imaginary part being induced by the time dependence of the background spacetime. The leading order factor is simply a linear combination of the exact zeroth-order solutions so chosen that \(\Psi_{mn}^{(0)}(0) = 0, (m \neq n)\) the condition required for hermiticity of \(H^{(0)}\) \[^{[4]}\].

With the FRW wormhole wavefunction in hand, we proceed to calculate the normalization constant \(C_{mn}\) and the expectation value of the wormhole throat radius. From

\[
\int_0^\infty da |\Psi_{mn}(a, t)|^2 = 1,
\]

(48)

we find after an elementary integration that

\[
C_{mn} = \frac{L_p}{2R} \left(\frac{1}{\sqrt{2m+1}} + \frac{1}{\sqrt{2n+1}} - \frac{2}{\sqrt{n+m+1}}\right)
\]

\[
+ \frac{15L_p^3 R^2}{8\pi^2 R^3} \left( (2m+1)^{-7/2} + (2n+1)^{-7/2} - 2(n+m+1)^{-7/2}\right).
\]

(49)

In a similar fashion, we find for the matrix element of the throat radius

\[
\langle aR | \Psi_{mn} \rangle = \langle aR \rangle_{mn} = R \langle a \rangle_{mn} = R \int_0^\infty da a |\Psi_{mn}(a, t)^0| = \frac{L_p}{\pi} \left[ \frac{1}{\sqrt{2m+1}} + \frac{1}{\sqrt{2n+1}} - \frac{2}{\sqrt{n+m+1}}\right]
\]

\[
\times \left\{ 1 + \frac{L_p^2}{\pi^2} \left( \frac{R'}{R} \right)^2 f_{(m,n)} \right\}
\]

where the numerical factor \(f_{(m,n)}\) is defined by

\[
f_{(m,n)} = 12 \left[ \frac{1}{(2m+1)^2} + \frac{1}{(2n+1)^2} - \frac{2}{(n+m+1)^2}\right] - \frac{15}{4} \frac{1}{(2m+1)^{7/2}} + \frac{1}{(2n+1)^{7/2}} - \frac{2}{(n+m+1)^{7/2}}.
\]

(51)
Thus, to lowest order, the average value of the throat radius is of the order the Planck length with corrections coming from the explicit time dependence of the background spacetime. From (50), we can see that for static backgrounds, we recover the result that the wormhole is quantum mechanically stabilized against collapse, with an average radius \( L_P \) \(^4\). However, when the background depends on time, the wavefunction becomes manifestly time dependent and the wormhole throat radius need no longer be stationary. Indeed, already at lowest order, we find that the radius is modified from its static value. Thus, for radiation dominated expansion (RDE), \( (\frac{\dot{R}}{R})^2 = \frac{1}{4t^2} \), while for a deSitter cosmology, \( (\frac{\dot{R}}{R})^2 = |\Lambda|/3 \). As \( f(m,n) > 0 \), we see from (50) that the expansion augments the magnitude of the matrix element. It is noteworthy that for the radiation dominated case, the perturbation to the throat radius expectation value vanishes as \( t \rightarrow \infty \) so that the wormhole throat relaxes from initially larger values to assume its static value. For the deSitter example, the first order perturbation is actually time independent; in summary:

\[ <aR>_{t} \rightarrow <aR>_{static} \approx L_P, \quad \text{RDE}, \quad (52) \]
\[ <aR>_{t} > <aR>_{static} \quad \text{deSitter}. \quad (53) \]

These calculations indicate that the expectation value for the wormhole radius is larger than its static value when the background is expanding, indicating that the throat gets partially dragged open with the expansion. This is perhaps not too surprising as the wormhole is gravitationally coupled to the background. However, the extent of the dragging cannot be known \( a \text{ priori} \), and we clearly see that it depends crucially on the form of the background scale factor.

4 Comoving Wormholes

When matter is added to the throat, even approximate solutions of the Wheeler-DeWitt equation are difficult to obtain. The variant of the WKB approximation scheme described in \([12]\), designed to handle non-quadratic hamiltonians, fails to be of use for our hamiltonian because the classical energy relation

\[ H (\Pi, a; R, R') = E \quad (54) \]
with \( H \) as given in (24), cannot be inverted in closed form so as to express \( \Pi = \Pi(E, a; R, R') \), the latter relation needed for calculating the WKB wavefunction. Nevertheless, we can make an exact statement, at least at the classical level, concerning the state of motion and type of matter needed such that the wormhole throat is comoving, that is, coupled to the Hubble flow.

The most general stress-energy tensor which gives rise to two identical FRW spaces connected by a \( \delta \)-layer wormhole is

\[
T^{\mu \nu}(x) = S^{\mu \nu}(x) \delta(\eta) + T^{(1)\mu \nu}(\eta) + T^{(2)\mu \nu}(\eta),
\]

(55)

where \( T^{(1)\mu \nu} = T^{(2)\mu \nu} \) is one of the standard perfect fluid source terms leading to (4) and

\[
S^{\mu \nu} = \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} d\eta \ T^{\mu \nu}(x),
\]

(56)

is the surface stress energy on the wormhole throat. In components

\[
S^i_j = \text{diag}(-\sigma, P, P)
\]

(57)

where \( \sigma, P \) denote the surface energy and pressure densities. The throat matter action and lagrangian are

\[
S_{\text{matter}} = -\int d^3x \sqrt{-h} \sigma = -4\pi \int d\tau a^2 R^2 \sigma = \int d\tau L_{\text{matter}}.
\]

(58)

The Wheeler-DeWitt hamiltonian for the combined system of wormhole plus matter \( H_{\text{total}} = H_{\text{throat}} + H_{\text{matter}} \), where

\[
H_{\text{matter}} = -L_{\text{matter}} = 4\pi a^2 R^2 \sigma.
\]

(59)

In order that the wormhole be comoving with the background expansion, \( \dot{a} = 0 \), since the radius \( l(t) = aR(t) \) is then simply proportional to the scale factor. This is also reflected in the fact that \( dt/d\tau = 1 \) if and only if \( a \) is constant, i.e., throat proper time is identical to cosmic time when there is no relative motion between the throat and background. For \( \dot{a} = 0 \) the classical hamiltonian constraint, obtained after summing (24) and (59), reduces to

\[
2aR + 4\pi a^2 R^2 \sigma = 0,
\]

(60)

that is, the surface energy density on the throat must be

\[
\sigma = -\frac{1}{2\pi aR} < 0
\]

(61)
which is negative definite and scales as $R^{-1}$ in time. According to this, the wormhole mouth will expand just as rapidly as the Hubble flow and for all scale factors. However, in order for this motion to be realized, the matter on the throat must be “adjusted” to have the surface energy density specified in (61) as well as obey the equation of state $P = -\frac{\sigma}{2}$. We suspect this will also hold at the quantum level, that is, a quantum comoving wormhole, one for which $\langle aR \rangle = \text{constant} \times R$, will require nontrivial stress-energy located at the throat. We can turn this argument around and conclude that in general, the wormhole will not be comoving, but will exhibit motion relative to the background expansion.

This result, that a classical comoving wormhole entails weak energy condition (WEC) violating matter should be compared to the results of Roman [17], who analyzed a class of Morris-Thorne type metrics representing Lorentzian wormholes embedded in a deSitter inflationary background. Those classical wormholes were shown to inflate in step with the background and require a nontrivial stress energy tensor violating the WEC.

5 Discussion

In this paper we have investigated a simple model of Minkowski-signature wormholes attached to FRW spacetimes. The quantum gravitational dynamics of these wormholes is reduced to the quantum mechanics of a single variable, namely, the throat radius, by solving the Wheeler-DeWitt equation over minisuperspace. When quantizing wormholes in static backgrounds via the Wheeler-DeWitt prescription [4, 11], one comes up against the problem of time. That is, the resulting wavefunction is time-independent, despite the fact that the wormhole may well evolve classically. This makes it awkward to discuss the quantum stability since the calculation of transitions between allowed quantum states of the wormhole lies beyond the scope of the formalism. However, for time-dependent backgrounds one can treat the evolution of quantum wormholes in an obvious and natural way. For the case of FRW backgrounds, the temporal parameter is provided by the scale factor. Moreover, FRW cosmologies are the relevant spacetimes in which to analyze early universe wormholes.

We have calculated the average wormhole radius for both radiation dom-
inated and deSitter inflationary scale factors. It is noteworthy that even in the absence of matter at the throat, the wormholes are stabilized quantum mechanically against both collapse and blowing-up. For radiation domination, the wormhole radius decreases monotonically from initially large values to attain a minimum of order the Planck length. For the inflationary case, the lowest-order calculation reveals the radius to be held fixed at a value always larger than the Planck length, by an amount proportional to the absolute value of the cosmological constant (vacuum energy density). These two cases indicate that the wormhole throat gets dragged open by the background expansion, although there is relative motion, or "slippage", between the wormhole and the ambient Hubble flow. This result is encouraging as it weighs in favor of the ideas set forth in [10].

A limitation of our calculation is the fact that the wormholes treated here have zero-length handles. Of course, we by no means claim that all (Planck-scale) wormholes would be of this form, but we believe that the qualitative aspects of their evolution should not be overly sensitive to this trait. Nevertheless, it would be interesting to check this by quantizing finite handle wormholes as described, for example, by Morris-Thorne type metrics.

Perhaps a more severe limitation of our calculation is due to the fact that we have not employed the full superspace description for the wormhole dynamics. The truncation from an infinite number of degrees of freedom down to a single degree of freedom may appear drastic, but it is the only viable option if we wish to be able to make a definite calculation. Working in the full superspace is obviously preferable, but is currently an intractable problem.

Finally, when formulating a time-dependent quantum mechanics of wormholes via the Wheeler-DeWitt approach, as we have done here, we find the Hamiltonian operator picks up many new terms not present in the static background limit, e.g., compare (28,29) to (30,31). These additional terms are functions of the expansion velocity $R'$ and complicate the task of solving for the wavefunction. Nevertheless, we have been able to calculate the wavefunction perturbatively employing $R'$ as a small parameter. Naturally, one would like to get at information lying beyond the restraints of perturbation theory. To this end, we would like to remark that the wave equation (26) may lend itself to approximate, but non-perturbative, solutions based on methods borrowed from the calculus of finite differences [18]. This follows from the observation that the nonlocal operators appearing there can be expressed in
terms of finite translations acting on minisuperspace. These points will be explored elsewhere.

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A Appendix

We wish to obtain an explicit representation for $G$, defined as the inverse, over $C^\infty$ function space, of the nonlocal differential operator $\sin\left(\frac{d}{du}\right)$, that is,

$$\sin\left(\frac{d}{du}\right) G(u, u') = \delta(u - u'), \quad (A1)$$

where

$$\sin\left(\frac{d}{du}\right) = \frac{d}{du} - \frac{1}{3!} \frac{d^3}{du^3} + \frac{1}{5!} \frac{d^5}{du^5} - \cdots \quad (A2)$$

As the operator (A2) is linear, it proves useful to introduce the Fourier transform $\tilde{G}$ of $G$ via

$$G(u, u') = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \tilde{G}(p) e^{-ip(u-u')}, \quad (A3)$$

whereby using the identity

$$\sin\left(\frac{d}{du}\right) e^{-ipu} = \sin(-ip)e^{-ipu}, \quad (A4)$$

we have that the transform of the Green function is

$$\tilde{G}(p) = \frac{i}{\sinh(p)}. \quad (A5)$$

The problem of computing $G$ thus reduces to carrying out the integration in (A3) with $\tilde{G}$ as given in (A5). There are two cases to consider:
A.1 case (1): $\Delta = (u - u') > 0$

We go to the complex $p$-plane and consider the following contour built up from closed intervals and semicircular arcs centered about the origin: $\Gamma_{R,\delta} = [-R, -\delta] + \gamma_{\delta} + [\delta, R] + \gamma_{R}$, where $\gamma_{\delta}$ is a semicircle of radius $\pi > \delta > 0$, and $\gamma_{R}$ a semicircle of radius $R$ with $(M + 1)\pi > R > M\pi$ ($M >> 1$), both located in the lower half-plane, as indicated in Fig. A. The inequalities are imposed so that the contour avoids hitting any of the poles. Since $\sinh(x + iy) = \sinh(x) \cos(y) + i \cosh(x) \sin(y)$, the integrand in (A3) has simple poles located on the imaginary axis: $z = \pm im\pi$ for $m = 0, 1, 2, \cdots$. Thus, by the residue calculus [16], we can calculate the integral taken around the complete contour in the clockwise sense

$$\oint_{\Gamma_{R,\delta}} \frac{dz}{\sqrt{2\pi}} \frac{ie^{-iz\Delta}}{\sinh(z)} = \sqrt{2\pi} \sum_{m=1}^{M} (-e^{-\pi\Delta})^m. \quad (A6)$$

The Fourier integral representation for $G$ (A3) results after taking the two limits $\delta \to 0$ and $R \to \infty$. Denoting by $I_{R}$ the contribution to (A6) coming from the large semicircular path $\gamma_{R}$, it is straightforward to show that

$$0 \leq |I_{R}| \leq R \int_{0}^{\pi} \frac{d\theta}{\sqrt{2\pi}} \frac{e^{-\Delta R \sin \theta}}{\cosh^2(\theta) - \cos^2(\theta) \sin^2(\theta)^{1/2}} \to 0, \quad (A7)$$

as $R \to \infty$ for every $\theta \in [0, \pi]$. On the other hand, the contribution $I_{\delta}$ to (A6) coming from the small semicircle $\gamma_{\delta}$ centered about the origin gives

$$\lim_{\delta \to 0} I_{\delta} = -\sqrt{\frac{\pi}{2}}. \quad (A8)$$

Putting together (A3) and (A6-8), we have that

$$\sqrt{2\pi} \sum_{m=1}^{\infty} (-e^{-\pi\Delta})^m = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \frac{ie^{-ip\Delta}}{\sinh(p)} + \lim_{\delta \to 0} I_{\delta} + \lim_{R \to \infty} I_{R}, \quad (A9)$$

so that

$$G(\Delta) = \sqrt{\frac{\pi}{2}} - \frac{\sqrt{2\pi} e^{-\pi\Delta}}{1 + e^{-\pi\Delta}}, \quad (A10)$$

where in the limit $M \to \infty$, $R \to \infty$ and $\delta \to 0$ the contour $\Gamma_{R,\delta}$ maps to the real line joined to an infinite semicircle in the lower half-plane.
A.2 case (2): $\Delta = (u - u') < 0$

This time, we close the contour in the upper half-plane and in the counterclockwise sense, i.e., $\Gamma_{R,\delta} = [-R, -\delta] + \gamma_{\delta} + [\delta, R] + \gamma_{R}$, where now $\gamma_{\delta}$ and $\gamma_{R}$ are semicircular arcs in the upper half-plane. The steps and limits involved in computing $G$ are entirely analogous to those of the previous case, with the final result that

$$G(\Delta) = -\sqrt{\frac{\pi}{2}} + \frac{\sqrt{2\pi} e^{\pi \Delta}}{1 + e^{\pi \Delta}}. \tag{A11}$$

The Green function (A10,11) is bounded for $|\Delta| \to \infty$ and $G(0) = 0$. From (A1), the general solution of $\sin \left( \frac{d}{du} \right) g(u) = f(u)$ is therefore

$$g(u) = g_{\text{homogeneous}}(u) + \int du' G(u - u') f(u'), \tag{A12}$$

where the homogeneous part of the solution is given by

$$g_{\text{homogeneous}} = \sum A_m e^{-m\pi u}. \tag{A13}$$
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Figure Caption Fig. A. The contour $\Gamma_{R,\delta}$ for case (1).