WELL-POISED HYPERSURFACES

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ABSTRACT. An ideal $I$ is said to be "well-poised" if all of the initial ideals obtained from points in the tropical variety $\text{Trop}(I)$ are prime. This condition was first defined by Nathan Ilten and the third author. We classify all well-poised hypersurfaces over an algebraically closed field. We also study the tropical varieties, singular loci, and associated Newton-Okounkov bodies of these hypersurfaces.

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1. INTRODUCTION

Let $\mathbb{k}[x] = \mathbb{k}[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over an algebraically closed field $\mathbb{k}$ and let $I \subset \mathbb{k}[x]$ be a monomial-free ideal. The tropical variety $\text{Trop}(I)$ associated to $I$ is the set of vectors $\omega \in \mathbb{R}^n$ whose associated ideal of initial forms $\text{in}_\omega(I)$ (see Eq. (1)) also contains no monomials, [8]. The zero locus $V(\text{in}_\omega(I))$ of such an initial ideal is a flat degeneration of the affine variety $V(I) \subseteq \mathbb{A}^n(\mathbb{k})$. When $\text{in}_\omega(I)$ is a prime binomial ideal, the variety $V(\text{in}_\omega(I))$ is an affine (possibly non-normal) toric variety, and we say that $\omega \in \text{Trop}(I)$ defines a toric degeneration of $V(I)$. In this case,
ω is said to be a prime point of Trop(I), and the open face σ of the Gröbner fan of I containing ω in its relative interior (likewise contained in Trop(I)) is a prime cone. In the following, we give Trop(I) the fan structure inherited from the Gröbner fan of I, and we let $\text{in}_\sigma(I)$ denote the initial ideal associated to a relatively open face $\sigma \in \text{Trop}(I)$.

Due to their close connection with polyhedral geometry, prime binomial ideals and their associated toric varieties are often easier to handle than general prime ideals. For example, the Gorenstein property, the Cohen-Macaulay property, the Koszul property, normality of the corresponding variety, and bounds on the Betti numbers can be more easily checked for prime binomial ideals. Moreover, these properties are preserved by flat degeneration, so in this way toric degeneration can be a useful tool for studying both the geometry of the original variety $V(I)$ and its coordinate algebra $\mathbb{k}[x]/I$. In particular, it can be shown that there is a Newton-Okounkov body ([5], [7], [9]) associated to each prime cone of maximal dimension in Trop(I) ([6]) from which many invariants of $V(I)$ can be extracted. Recently, Esco-bar and Harada [2] have shown that maximal prime cones in Trop(I) which share a facet give rise to a wall-crossing phenomenon between their associated Newton-Okounkov bodies. For this reason it is of interest to know when $\text{in}_\omega(I)$ is a prime ideal for every $\omega \in \text{Trop}(I)$. Following work of the third author and Ilten in [4] such an ideal is said to be well-poised. In this paper, we classify all well-poised principal ideals (Theorem 1.1). A description of Newton-Okounkov bodies for well-poised hypersurfaces appears in Section 4.
We write $f = \sum c_i x^{a_i}$ to mean a polynomial in $\mathbb{k}[x]$ with monomial terms $c_i x^{a_i}$, for $c_i \in \mathbb{k}$ and $x^{a_i} = x_1^{a_{i,1}} \cdots x_n^{a_{i,n}}$. The initial form:

\begin{equation}
\text{in}_\omega(f) = \sum_{a_i \in M} c_i x^{a_i}
\end{equation}

for a real vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ is the sum of those monomial terms $c_i x^{a_i}$, where $a_i$ belongs to the set $M$ of those exponents whose inner product with $\omega$ is maximal (see [10]). Likewise, the initial ideal $\text{in}_\omega(I) \subset \mathbb{k}[x]$ is the ideal generated by the initial forms $\{\text{in}_\omega(f) \mid f \in I\}$. If $I$ is principal, say generated by $f \in \mathbb{k}[x]$, then $\text{in}_\omega(I) = \langle \text{in}_\omega(f) \rangle$; with this in mind we get the following definition.

**Definition 1.1.** A polynomial $f$ is said to be well-poised if every initial form which is not a monomial is irreducible.

We introduce the following conventions. We say that that $\gcd(a_i, a_j)$ is the listwise gcd of all entries of the two exponent vectors. We recall the support of a monomial term $x^{a_i}$, denoted $\text{supp}(x^{a_i})$, is the set $\{j : a_{i,j} \neq 0\}$. Several of our results will involve the condition that $\text{supp}(x^{a_i}) \cap \text{supp}(x^{a_j}) = \emptyset$ for two monomials in a polynomial $f$, and so we give the following definition:

**Definition 1.2.** We say a polynomial $f = \sum_{i=1}^n c_i x^{a_i}$ is disjointly supported if $\text{supp}(x^{a_i}) \cap \text{supp}(x^{a_j}) = \emptyset$ for all $c_i, c_j \neq 0$.

The following is our main result.

**Theorem 1.1.** A polynomial $f = \sum_{i \in \mathbb{N}} c_i x^{a_i}$ is well-poised if and only if $f$ is disjointly supported and $\gcd(a_i, a_j) = 1$ for any pair $i, j \in \mathbb{N}$. 
We also give a complete description of two combinatorial invariants of a well-poised hypersurface. We show that the Newton polytope, denoted \( N(f) \), of any well-poised hypersurface contains no interior lattice points (Theorem 1.2), and we give a complete description of the tropical variety \( \text{Trop}(f) \) (Section 4).

**Theorem 1.2.** Let \( f \) be well-poised. Then \( N(f) \) is a simplex. Further, \( N(f) \cap \mathbb{Z}^n \) is precisely the vertex set of \( N(f) \).

In Section 4 we determine the structure of the tropical variety of a polynomial with disjoint supports. We also show that for these disjointly supported hypersurfaces, (and consequently well-poised hypersurfaces), the singular points always lie on a certain coordinate subspace arrangement.

**Theorem 1.3.** The singular locus \( S(V(f)) \) of a hypersurface described by a polynomial with disjoint supports is entirely determined by the number and form of the monomials in \( f \). Each monomial describes a set of conditions which give a collection of coordinate subspaces in \( \mathbb{A}^n(\mathbb{k}) \). By choosing one subspace per monomial term, and taking the intersection indexed across all monomials, the resulting coordinate subspace is a component of \( S(V(f)) \). \( S(V(f)) \) is then the union of all such subspaces.

The subspaces corresponding to each monomial are detailed explicitly in the proof.

As an immediate consequence of Theorem 1.3, we are able to easily calculate the codimension of the singular locus in the ambient space. Recall that a domain is normal if it is integrally closed in its field of fractions. When
combined with Serre’s criterion for normality, we get the following corollary:

**Corollary 1.1.** A non-monomial disjointly supported polynomial $f$ in $\mathbb{k}[x]$ is normal if and only if one of the following three conditions hold:

1. $f$ contains a nonzero constant.
2. $f$ contains a monomials of the form $x^1_j$.
3. When $\mathbb{k}$ has characteristic zero, $f$ has 3 monomials that are not constant nor of the form $x^1_j$.
4. When $\mathbb{k}$ has characteristic $p > 0$, $f$ has $3+n$ monomials that are not constant nor of the form $x^1_j$, where $n$ is the number of monomials of the form $x^c_j$ for $c \in \mathbb{N}$.

**Example 1** (Whitney Umbrella). The Whitney Umbrella $V(x_1^2 - x_2^2x_3) \subset \mathbb{A}^3(\mathbb{k})$ is well-poised, and is a well known example of a non-normal variety. By appending several monomials, we also present a pedagogical modification, $V(x_1^2 - x_2^2x_3 + x_4^3x_5^2 + x_6x_7x_8^2) \subset \mathbb{A}^8(\mathbb{k})$. This is the minimal example containing all monomial types used in the proof of Theorem 1.3 that allow for the existence of a singular locus. As a consequence of our classification, not only is this well-poised but also normal.

**Example 2** (E8 Singularity). The Du Val $E_8$ singularity is given by the solution set of $x^2 + y^3 + z^5 = 0$. This is both well-poised and normal by Theorem 1.1 and Corollary 1.1 respectively.

**Example 3** (The Grassmannian $Gr_2(4)$). The Grassmanian is a well-studied and essential object, given by the solution set of $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$. Simple application of our results show that this is well-poised and normal.
Polynomials of the type described in Theorem 1.1 have appeared recently in work of Hausen, Hische, and Wröbel on Mori dream spaces of low Picard number ([3]). The Cox rings of smooth general arrangement varieties of true complexity 2 and Picard number 2 of all of Hausen, Hische, and Wröbel’s types except 14 are examples of well-poised hypersurfaces. Type 14 is well-poised as well, but it is not a hypersurface. As a consequence, any projective coordinate ring of one of these varieties carries a full rank valuation with finite Khovanskii basis, and the varieties themselves have a number of related toric degenerations. The Cox ring of projective variety with a finitely generated and free class group is known to be factorial [1], so we ask the following question.

Question 1. Let $f$ be well-poised. When is the ring $\mathbb{k}[x]/\langle f \rangle$ a UFD?

1.1. Acknowledgements. We thank David Ma and Alston Crowley for many useful conversations. We also thank the UK Math Lab for hosting this project in the spring and fall of 2018. The third author was supported by both the NSF (DMS-1500966) and the Simons Foundation (587209) during this project.

2. The Newton Polytope and Supporting Lemmas

Here we prove the results necessary to establish Theorem 1.1 and Theorem 1.2 in Section 3. These proofs rely on the properties of the Newton polytope of a hypersurface. Terminology and notation are taken from [8].

Definition 2.1. The Newton polytope $N(f)$ of a polynomial $f = \sum c_i x^{a_i}$ is the convex hull of the set $\{a_i : c_i \neq 0\} \subset \mathbb{R}^n$. [8, 61]
Recall that the faces of $N(f)$ are in one-to-one correspondence with the initial forms $in_w(f)$. In particular, each face is of the form $N(in_w(f))$ for some weight vector $w$, and if $N(f)$ is a simplex with no interior lattice points, then every sub-sum of $f = \sum c_ix^{a_i}$ is an initial form of $f$. Also, recall that if $f = pq$ then $N(f) = N(p) + N(q)$ where the right side denotes the Minkowski sum of the Newton polytopes ([10]). The lemmas in this section serve to restrict the combinatorial type of $N(f)$ when $f$ is well-poised.

**Lemma 2.1.** If $f$ is well-poised, then all monomials corresponding to the vertices of $N(f)$ have disjoint supports.

**Proof.** Let $x_j$ be an indeterminant in $f$ and let

$$S := \{a_i : j \in \text{supp}(x^{a_i}), a_i \text{ is a vertex}\}$$

denote the corresponding vertex set in $N(f)$, corresponding to monomials that contain $x_j$. Note that this set exclusively considers vertices and not interior lattice points. We show that this set contains only one element.

First, note that any edge of $N(f)$ corresponds to an initial binomial of $f$. Should any of the vertices $a_i \in S$ share a (potentially degenerate) edge, this would correspond to a factorable initial binomial (or more general form, if the edge is degenerate) of $f$, as the lowest power of $x_j$ could be factored out. Therefore for $f$ to be well-poised, no vertices in $S$ may share an edge. Let $e_j$ be the unit basis vector with respect to the $j^{th}$ coordinate and choose a vertex $a_i$ in $S$ such that the interior product $\langle a_i, e_j \rangle$ is maximal. Now, consider the rays from $a_i$ to its adjacent vertices. Notice that by our above reasoning, $a_i$ cannot connect to any other member of $S$, so all rays connect to vertices
which lie in the subspace of $\mathbb{R}^n$ homeomorphic to $\mathbb{R}^{n-1}$ corresponding to $x_j = 0$.

We can conclude now that $\overline{a}$ is the only element in $S$. The intuition is given in Fig. 1. If we suppose there is some other vertex in $S$, $a'$, we see we are forced to choose between convexity or irreducibility, as if these two vertices are not joined by some ray, then the polytope is not convex, but if they are joined, then the ray corresponds to a reducible initial form. Therefore, there can only be one element in $S$, and our monomials will have disjoint supports.

\[\square\]

**Lemma 2.2.** If the monomials corresponding to vertices in $N(f)$ have disjoint supports, then $N(f)$ is a simplex.

**Proof.** We argue by induction on the number of vertices. We first treat the case of a collection of non-zero vertices. The case $k = 2$ is clear, as it will simply be a line segment, so we assume that any polytope with $k$ vertices with disjoint supports is a $k - 1$ simplex. Consider a collection of $k + 1$
points with pairwise disjoint supports. The collection of vertices omitting \( a_i \), denoted \( \{\hat{a}_i\} \), contains \( k \) vertices with disjoint supports and thus form a \( k - 1 \)-simplex. As \( a_i \) has disjoint supports, it is linearly independent of the other vertices and thus not contained within their convex hull. By proceeding for all \( a_i \), we see that no single \( a_i \) is contained within the convex hull of the other vertices, so taking the convex hull of all \( k + 1 \) points results in a \( k \)-polytope with \( k + 1 \)-vertices, which is thus a \( k \)-simplex. If our list of vertices contains zero, we treat zero as our final vertex, and apply the previous induction hypothesis. As all previous \( a_i \) are non-zero and live in \( \mathbb{Z}^n_{\geq 0} \), the vertex at zero will not be contained in the convex hull of the \( (k - 1) \)-simplex given by induction hypothesis. Therefore, the final resulting polytope will be a \( k \)-simplex.  

Lemma 2.4 and Theorem 1.2 require the following lemma, which restricts the number of lattice points in \( N(f) \), when \( f \) is of a specific form. The proof of this lemma is reserved for the end of this section to improve readability.

**Lemma 2.3.** Let \( f \) be a polynomial such that the vertices of \( N(f) \) have disjoint supports. If \( f \) also has \( \gcd(a_i, a_j) = 1 \) for all pairs of vertices \( a_i, a_j \), then \( N(f) \) contains no lattice points besides its vertices.

Now, we prove Lemma 2.4.

**Lemma 2.4.** A binomial \( f = c_i x^{a_i} + c_j x^{a_j} \) is irreducible if and only if \( \gcd(a_i, a_j) = 1 \) and \( \text{supp}(x^{a_i}) \cap \text{supp}(x^{a_j}) = \emptyset \).

**Proof.** Consider the Newton polytope \( L = N(f) \). This is a line segment with end points \( a_i \) and \( a_j \). Now suppose \( f = pq \). We will show that one of
the factors, say \( q \), must be a constant. We must have \( L = N(f) = N(p) + N(q) \). As \( f \) is a binomial in \( \mathbb{k}[x] \),

\[
\max(dim(N(p)), dim(N(q))) \leq dim(N(f)) \leq 1
\]

we conclude that \( N(q) \) is a line segment, a point distinct from the origin, or the origin itself. We assume without loss of generality that \( N(p) \) is a line segment. If \( N(q) \) is a point \( a_0 \), then the monomial \( x^{a_0} \) must divide \( f \), so \( a_i \) and \( a_j \) do not have disjoint supports unless \( a_0 = 0 \). If \( N(p) \) and \( N(q) \) are both lines, they must be colinear, as otherwise, \( N(f) \) would be two-dimensional. The Minkowski sum of two colinear lines with integer endpoints must contain an interior lattice point, corresponding to the sum of one endpoint from each line. However, \( f \) satisfies the form required in Theorem [1.2] and thus contains no interior lattice points. Therefore, \( N(q) \) can only be the point at the origin, meaning that \( q \) must be a constant.

⇒ If \( f \) is irreducible, then \( x^{a_i} \) and \( x^{a_j} \) must have disjoint supports, as if they did not, we could factor out a power of \( x_k \), where \( k \in supp(x^{a_i}) \cap supp(x^{a_j}) \). Now, suppose \( \gcd(a_i, a_j) = d > 1 \), and \( a_i'd = a_i \), and \( a_j'd = a_j \).

Now by rearranging constants, and factoring we get the following:

\[
\begin{align*}
f & = c_i x^{a_i} + c_j x^{a_j} \\
& = c_i x^{a_i'd} - c x^{a_j'd} \\
& = (\sqrt[d]{c} x^{a_i'})^d ((\sqrt[d]{c} x^{a_j'})^d - 1)
\end{align*}
\]
Now, if we relabel \( \frac{\sqrt[d]{cx^i}}{\sqrt[d]{cx^j}} \) as \( z \), the above simplifies to

\[ f = cx^i d (z^d - 1) \]

where \( z^d - 1 \) easily factors as \( \Pi_{1 \leq k < d} (z - \delta_k) \), where \( \delta_k \) is a \( d^{th} \) root of unity.

Now, by resubstituition and distributing, we get:

\[
\begin{align*}
  f &= (\sqrt[d]{cx^i})^d (z^d - 1) \\
  &= (\sqrt[d]{cx^i})^d \Pi_{1 \leq k < d} (z - \delta_k) \\
  &= \Pi_{1 \leq k < d} ((\sqrt[d]{cx^i})z - \delta_k d\sqrt[d]{cx^i}) \\
  &= \Pi_{1 \leq k < d} (\sqrt[d]{ci}x^i - \delta_k d\sqrt[d]{cx^i})
\end{align*}
\]

which gives that \( f \) is factorable.

\[\square\]

Now we return to the proof of Lemma 2.3. Again, we need a helping lemma:

**Lemma 2.5.** For a set of equivalent fractions \( \{ \frac{b_1}{a_1}, \ldots, \frac{b_s}{a_s} \} \), the fractions are equal to \( \frac{\text{gcd}(b_1)}{\text{gcd}(a_1)} \), where \( \text{gcd}(b_1) \) is the listwise gcd of \( b_1, b_2, \ldots, b_s \) and \( \text{gcd}(a_1) \) is the listwise gcd of \( a_1, a_2, \ldots, a_s \).

**Proof.** Suppose there are only two fractions \( \frac{b_1}{a_1} = \frac{b_2}{a_2} \). Let us define \( n_1, n_2, m_1, m_2 \in \mathbb{Z} \) such that

\[
\begin{align*}
  b_1 &= n_1 \cdot \text{gcd}(b_1, b_2) \\
  b_2 &= n_2 \cdot \text{gcd}(b_1, b_2) \\
  a_1 &= m_1 \cdot \text{gcd}(a_1, a_2)
\end{align*}
\]
This results in \( \gcd(m_1, m_2) = 1 \) and \( \gcd(n_1, n_2) = 1 \). Then we can write

\[
\frac{n_1 \cdot \gcd(b_1, b_2)}{m_1 \cdot \gcd(a_1, a_2)} = \frac{n_2 \cdot \gcd(b_1, b_2)}{m_2 \cdot \gcd(a_1, a_2)}
\]

\[n_1 m_2 = n_2 m_1\]

Since the pairs \( n_1, n_2 \) and \( m_1, m_2 \) are relatively prime, we can use unique prime factorization to conclude \( n_1 = m_1 \) and \( n_2 = m_2 \). Thus,

\[
\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{\gcd(b_1, b_2)}{\gcd(a_1, a_2)}
\]

Let us assume

\[
\frac{b_1}{a_1} = \frac{b_2}{a_2} = \ldots = \frac{b_{s-1}}{a_{s-1}} = \frac{\gcd(b_1, \ldots, b_{s-1})}{\gcd(a_1, \ldots, a_{s-1})} = \frac{b_s}{a_s}
\]

and inductively show these fractions are equivalent to \( \frac{\gcd(b_1, \ldots, b_s)}{\gcd(a_1, \ldots, a_s)} \). These equivalences can be reduced to just

\[
\frac{\gcd(b_1, \ldots, b_{s-1})}{\gcd(a_1, \ldots, a_{s-1})} = \frac{b_s}{a_s}
\]

and thus, the two fractions case tells us they are equivalent to

\[
\frac{\gcd(b_1, \ldots, b_{s-1}, b_s)}{\gcd(a_1, \ldots, a_{s-1}, b_s)} = \frac{\gcd(b_1, \ldots, b_{s-1}, b_s)}{\gcd(a_1, \ldots, a_{s-1}, a_s)} = \frac{\gcd(b_1)}{\gcd(a_j)}
\]

Now, we can finally prove Lemma 2.3. Recall the statement,
Lemma 2.3. Let $f$ be a polynomial such that the vertices of $N(f)$ have disjoint supports. If $f$ also has $\gcd(a_i, a_j) = 1$ for all pairs of vertices $a_i, a_j$, then $N(f)$ contains no lattice points besides its vertices.

Proof of Lemma 2.3. We will prove that the existence of a lattice point in a polytope with vertices $a_1, \ldots, a_k$ having disjoint supports implies that $\gcd(a_i, a_j) > 1$ for some $i$ and $j$. Now suppose an interior lattice point $b$ exists in a polytope with disjoint supports. Then it is of the form

$$b = \sum_{j=1}^{k} p_j a_j.$$

As the point $b$ lies in the convex hull of our polytope, we have the added restriction that

$$\sum_{j=1}^{k} p_j = 1.$$

As $b$ is an integer lattice point, each coordinate must be an integer. By assumption, the collection of all $a_j$ have disjoint supports, so we can break up $b$ into a sum of $b_j$ where each $b_j$ has the same support as $a_j$ and thus $b_{j,i} = p_j a_{j,i}$ for all $i \in \text{supp}(a_j)$. Therefore, $p_j = \frac{b_{j,i}}{a_{j,i}}$. This also gives that for all $s$ supports in a given $a_j$,

$$p_j = \frac{b_{j,1}}{a_{j,1}} = \ldots = \frac{b_{j,s}}{a_{j,s}},$$

which, by our above lemma gives $p_j = \frac{\gcd(b_j)}{\gcd(a_j)}$. We may now rewrite the second summation above as follows:
Now by giving the left above a common denominator of $\Pi_{j=1}^{k-1} \gcd(a_j)$ the above becomes:

$$\frac{\sum_{i=1}^{k-1} \gcd(b_i) \Pi_{j\neq i} \gcd(a_j)}{\Pi_{j=1}^{k-1} \gcd(a_j)} = \frac{\gcd(a_k) - \gcd(b_k)}{\gcd(a_k)}$$

Now once again, this is of the form where we may use the above lemma. For the sake of notation I will relabel the above numerators $B_1$ and $B_2$ so we see that the above fraction reduces to:

$$\frac{\gcd(B_1, B_2)}{\gcd(\Pi_{j=1}^{k-1} \gcd(a_j), \gcd(a_k))}.$$ 

Now if we examine the denominator, this must be greater than 1, as our fraction is less than 1. This would imply that that $\gcd(a_k, a_i) > 1$ for some $i$, thus proving the statement. \qed

3. Proof of Theorem 1.1 and 1.2

We can now proceed with the proofs of Theorem 1.1 and Theorem 1.2. These are largely corollaries of the lemmas given in the previous section, which have been reproduced for readability.
Lemma 2.3. Let $f$ be a polynomial such that the vertices of $N(f)$ have disjoint supports. If $f$ also has $\gcd(a_i, a_j) = 1$ for all pairs of vertices $a_i, a_j$, then $N(f)$ contains no lattice points besides its vertices.

Lemma 2.1. If $f$ is well-poised, then all monomials corresponding to the vertices of $N(f)$ have disjoint supports.

Lemma 2.2. If the monomials corresponding to vertices in $N(f)$ have disjoint supports, then $N(f)$ is a simplex.

Lemma 2.4. A binomial $f = c_i x^{a_i} + c_j x^{a_j}$ is irreducible if and only if $\gcd(a_i, a_j) = 1$ and $\text{supp}(x^{a_i}) \cap \text{supp}(x^{a_j}) = \emptyset$.

With these lemmas presented, we prove Theorem 1.2, which we will use in the proof of Theorem 1.1. Recall the statement:

Theorem 1.2. Let $f$ be well-poised. Then $N(f)$ is a simplex. Further, $N(f) \cap \mathbb{Z}^n$ is precisely the vertex set of $N(f)$.

Proof. If $f$ is well-poised, then by Lemma 2.1 the vertices of $N(f)$ have disjoint supports satisfying the first condition. Additionally by Lemma 2.2, $N(f)$ is a simplex. Since $N(f)$ is a simplex, for any two vertices $x^{a_i}, x^{a_j}$ there is an edge connecting them corresponding to the irreducible initial form $c_i x^{a_i} + c_j x^{a_j}$. Then by Lemma 2.4 any pair of vertices have $\gcd(a_i, a_j) = 1$. We may now apply Lemma 2.3 to conclude there are no interior lattice points, and consequently all monomial terms in $f$ must correspond to vertices. \qed

Theorem 1.1. A polynomial $f = \sum_{i \in \mathbb{N}} c_i x^{a_i}$ is well-poised if and only if $f$ is disjointly supported and $\gcd(a_i, a_j) = 1$ for any pair $i, j \in \mathbb{N}$. 
Proof. \( \Rightarrow \) This is a consequence of the proof of Theorem 1.2, which gives all monomial terms in \( f \) must correspond to vertices of \( N(f) \), where all vertices are disjointly supported and pairwise of the form that \( \gcd(a_i, a_j) = 1 \).

\( \Leftarrow \) By Lemma 2.2, the Newton polytope \( N(f) \) is a simplex. Each edge of a simplex corresponds to a binomial initial form, which is irreducible by Lemma 2.4. Consider an arbitrary initial form \( \text{in}_\omega(f) \) of \( f \), and suppose it can be written, \( \text{in}_\omega(f) = g_1 g_2 \). We show without loss of generality that \( g_1 \) is a constant. We have noted above that any binomial initial form of \( f \) is irreducible, so if \( \text{in}_\omega(f) \) is a binomial we are finished. Therefore consider the case where \( \text{in}_\omega(f) \) must have at least 3 terms (i.e. is a trinomial or bigger). Therefore, there are at least two nonconstant monomials with disjoint support, say \( c_i x^{a_i} \) and \( c_j x^{a_j} \). Let \( w \) be defined by

\[
  w = \begin{cases} 
    1 & x_n \text{ is supported by } x^{a_i} \text{ or } x^{a_j} \\
    0 & \text{otherwise.}
  \end{cases}
\]

This gives the initial form

\[
  \text{in}_w(g_1g_2) = c_i x^{a_i} + c_j x^{a_j}.
\]

By [8, Lemma 2.4.6] for some \( \epsilon \in \mathbb{R}_{\geq 0} \), we have

\[
  \text{in}_w(g_1g_2) = \text{in}_w(g_1) \text{in}_w(g_2) = \text{in}_{\omega+\epsilon w}(f).
\]

So then we have

\[
  \text{in}_w(g_1) \text{in}_w(g_2) = c_i x^{a_i} + c_j x^{a_j}.
\]
Since $c_i x^{a_i} + c_j x^{a_j}$ is a binomial initial form of $f$, we have that $c_i x^{a_i} + c_j x^{a_j}$ is irreducible. Thus we must have that one of $\text{in}_w(g_1)$ or $\text{in}_w(g_2)$ is constant - assume without loss of generality that $\text{in}_w(g_1)$ is constant. Since $w$ is taken from $\mathbb{R}_{\geq 0}^n$, if $\text{in}_w(g_1)$ is constant, then $g_1$ itself is constant. Therefore $\text{in}_w(f)$ is irreducible, so $f$ is well-poised. $\square$

4. THE TROPICAL VARIETY

Let $f = \sum_{i=1}^{K} c_i x^{a_i}$ be a disjointly supported polynomial with no constant term. In this section we explicitly construct the faces of the Gröbner fan of the principal ideal $\langle f \rangle$ whose support is $\text{Trop}(f)$.

Let $\mathbf{1}$ be the vector of 1’s, then $\ell_i := \langle \mathbf{1}, a_i \rangle = \sum_{j=1}^{n} a_{ij} > 0$ for all $c_i \neq 0$. By letting $\ell$ be $\text{LCM}\{\ell_i \mid c_i \neq 0\}$ and $v_f$ be the vector with entry $\frac{\ell_i}{\ell}$ at any index in $\text{supp}(a_i)$ and 0 otherwise, we see that $\langle v_f, a_i \rangle = \langle v_f, a_j \rangle > 0 \ \forall i, j$.

It follows that $f$ is homogeneous with respect to $v_f$, and that the support of the Gröbner fan of $f$ is all of $\mathbb{R}^n$ (see [10, Proposition 1.12]). We let $L_f$ denote the homogeneity space of $f$, in particular $L_f = \{ u \mid \langle u, a_i \rangle = \langle u, a_j \rangle, \ 1 \leq i < j \leq K \}$.

Moreover, for each $a_i$ with $c_i \neq 0$ we let $w_i$ be the vector with entry 0 for $j \notin \text{supp}(a_i)$ and $-1$ for $j \in \text{supp}(a_i)$.

**Proposition 1.** Let $S \subseteq [K]$ and let $f_S = \sum_{j \in S} c_j x^{a_j}$, and let $C_S := \{ \omega \mid \text{in}_w(f) = f_S \} \in G(f)$, then:

$$C_S = L_f + \sum_{i \notin S} \mathbb{R}_{\geq 0} w_i.$$
Proof. Let $\omega \in L_f + \sum_{i \notin S} \mathbb{R}_{>0} w_i$ and consider $\text{in}_\omega(f)$. Without loss of generality we may assume that $\omega = \sum_{i \in S} n_i w_i$ with $n_i > 0$. The polynomial $f$ has disjoint supports, so $\langle \omega, a_i \rangle$ is 0 if $i \in S$ and $< 0$ if $i \notin S$. It follows that $\text{in}_\omega(f) = f_S$. This proves that $L_f + \sum_{i \notin S} \mathbb{R}_{>0} w_i \subseteq C_S$.

If $\omega \in C_S$, then $\omega$ weights each term of $f_S$ equally. We let $k = \langle \omega, a_j \rangle$, where $j$ is any element of $S$ and $k_i = \langle \omega, a_i \rangle$ for $i \notin S$. Observe that $k_i < k$, and that $\omega - \sum_{i=1}^K (k - k_i) \frac{1}{t_i} w_i$ weights all monomials of $f$ equally. It follows that $\omega - \sum (k - k_i) \frac{1}{t_i} w_i \in L_f$. As $k - k_i > 0$, we conclude that $\omega \in L_f + \sum_{i \notin S} \mathbb{R}_{>0} w_i$, and that $C_S \subseteq L_f + \sum_{i \notin S} \mathbb{R}_{>0} w_i$.

\[
\square
\]

Each $f_S$ is a polynomial which corresponds to a face of the Newton polytope of $f$ and likewise, to a face of the Gröbner fan. Observe that by definition the tropical variety $\text{Trop}(f)$ is the union of the faces $C_S$ where $|S| \geq 2$.

To complete the description of each cone $C_S$ we compute a basis for $L_f$. First, we observe that $v_f \in L_f$, and for any $\lambda \in L_f$ there is some $q$ such that $\langle \lambda - qv_f, a_i \rangle = 0$ for all $i \in [K]$. The space $N_f = \{\lambda' \mid \langle \lambda', a_i \rangle = 0\}$ is certainly contained in $L_f$, so it follows that $v_f$ and a basis of $N_f$ suffice to give a basis of $L_f$. For a basis of $N_f$ we take the integral vectors $v_{i,j} = a^i_1 e^j_i - a^j_1 e^i_j$; for $2 \leq j \leq k_i$, where $e^i_j$ is the $j$-th elementary basis vector from the support of $a_i$.

Theorem 1 of [6] gives a recipe for producing a full rank valuation with associated Newton-Okounkov given a prime cone from a tropical variety. It is required to choose a a linearly independent set of vectors from the cone which span a full dimensional subcone. For the cone $C_S$ with $|S| = 2$ we
select the set $W_S$ composed of the basis \{v_f, \ldots, v_{i,j}, \ldots\} \subset L_f$, and the extremal vectors $w_i$ for $i \in S^c$. Observe that if the sets $S$ and $S'$ differ by a single index, then $W_S$ and $W_{S'}$ differ by a single vector. We let $M_S$ be the matrix with rows equal to the elements of $W_S$.

\[
\begin{bmatrix}
v_f \\
v_2,i \\
v_3,i \\
v_{k,i} \\
w_i \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
\ldots & \ell_{k,i} & \ell_{k,i} & \ell_{k,i} & \cdots & \ell_{k,i} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\ldots & a_{2} & -a_{i} & 0 & \cdots & 0 & \cdots \\
\ldots & a_{3} & 0 & -a_{i} & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\ldots & a_{k_i} & 0 & 0 & -a_{i} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\ldots & -1 & -1 & -1 & \cdots & -1 & \cdots \\
\vdots
\end{bmatrix}
\]

By [6, Proposition 4.2], the matrix $M_S$ defines a full rank valuation $v_S : \mathbb{k}[x]/\langle f \rangle \to \mathbb{R}^{n-1}$. The image $S(\mathbb{k}[x]/\langle f \rangle, v_S) \subseteq \mathbb{R}^{n-1}$ of $v_S$ is the semigroup generated by the columns of $M_S$ under addition. In particular, $v_S(x_{ij})$ is the $ij$-th column of $M_S$. If $i \in S^c$, $v_S$ sends $x_{ij}$ to the $j$-th column of the block displayed above. If $i \in S$, $v_S$ sends $x_{ij}$ to the $j$-th column of the block displayed above, except the $-1$ entries are $0$.

The convex hull $P(\mathbb{k}[x]/\langle f \rangle, v_S) \subseteq \mathbb{R}^{n-1}$ of $S(\mathbb{k}[x]/\langle f \rangle, v_S) \subseteq \mathbb{R}^{n-1}$ is called the Newton-Okounkov cone of $v_S$. For each choice of $S$ with $|S| = 2$, there is a flat family $\pi_S : E_S \to \mathbb{A}^1(\mathbb{k})$ such that the coordinate ring of the fiber $\pi_S^{-1}(c)$ for $c \neq 0$ is $\mathbb{k}[x]/\langle f \rangle$ and the coordinate ring of the fiber
\pi_S^{-1}(0) is the affine semigroup algebra \( \mathbb{k}[S(\mathbb{k}[x]/\langle f \rangle, v_S)] \). In particular, \( \mathbb{k}[S(\mathbb{k}[x]/\langle f \rangle, v_S)] \cong \mathbb{k}[x]/\langle in_\omega(f) \rangle \) for any \( \omega \in C_S \).

If there is a vector \( \mathbf{d} \in \mathbb{Z}^n_> \) such that \( \langle \mathbf{d}, a_i \rangle \) is a fixed integer for all monomial exponents \( a_i \) appearing in \( f \), we say that \( f \) is homogeneous with respect to \( \mathbf{d} \). For example, if \( f \) is homogeneous in the classical sense we may take \( \mathbf{d} \) to be the all 1’s vector. Assuming a fixed \( \mathbf{d} \) has been chosen, the algebra \( \mathbb{k}[x]/\langle f \rangle \) is positively graded, that is it can be expressed as direct sum of finite dimensional vector spaces \( A_N \):

\[
\mathbb{k}[x]/\langle f \rangle \cong \bigoplus_{N \geq 0} A_N.
\]

In this setting, the projective variety \( X = \text{Proj}(\mathbb{k}[x]/\langle f \rangle) \) carries a flat de-generation to the projective toric variety \( X_S = \text{Proj}(\mathbb{k}[S(\mathbb{k}[x]/\langle f \rangle, v_S)]) \). It is possible that \( X_S \) is non-normal, however the normalization is the projective toric variety associated to the Newton-Okounkov body \( \Delta(\mathbb{k}[x]/\langle f \rangle, v_S) \).

Following [8, Corollary 4.7], the Newton-Okounkov body \( \Delta(\mathbb{k}[x]/\langle f \rangle, v_S) \) is obtained from \( M_S \) by dividing the \( ij \)-th column by the degree of \( x_{ij} \) assigned by \( \mathbf{d} \), and taking the convex hull of the resulting column vectors.

**Example 4.** We compute the matrices \( M_S \) for \( f = x + y^2 + zw \in \mathbb{k}[x, y, z, w] \).

Let \( x, y^2 \), and \( zw \) be the \( i = 1, 2 \) and 3 monomials, respectively. The space \( L_f \subset \mathbb{Q}^4 \) can be generated by the vectors \( \mathbf{v}_f = (2, 1, 1, 1) \) and \( \mathbf{v}_{2,3} = (0, 0, 1, -1) \).

From this we deduce that \( A = \mathbb{k}[x, y, z, w]/\langle f \rangle \) is graded by the semigroup in \( \mathbb{Z}^2 \) generated by \( (1, 0), (1, 1), \) and \( (1, -1) \). The third row of \( M_S \) is \( \mathbf{w}_i \),
where \( \{i\} = S^c \):

\[
M_{12} = \begin{bmatrix}
2 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & -1 & -1
\end{bmatrix}
\]

\[
M_{13} = \begin{bmatrix}
2 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 \\
0 & -1 & 0 & 0
\end{bmatrix}
\]

\[
M_{23} = \begin{bmatrix}
2 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0
\end{bmatrix}
\]

For each \( S \), the columns of \( M_S \) generate a semigroup in \( \mathbb{Z}^3 \) whose semigroup algebra is the coordinate ring of a toric degeneration of \( \mathbb{C}[x, y, z, w]/\langle f \rangle \).

5. The Singular Locus

We give a complete description of the singular locus of the hypersurface \( V(f) \) for disjointly supported \( f \). Specifically, as a consequence of Theorem [1], we can classify the singular loci of all well-poised hypersurfaces. We show that under certain conditions the codimension of the singular locus is greater than or equal to the number of monomial terms in \( f \), and thus, by Serre’s Criterion, give an easy process for checking the normality of the hypersurface. First, allow us to work through an illuminating example, which motivates the description and illustrates the proof.

Example [1]. Let \( f = x_1^2 - x_2^2x_3 + x_4^3x_5^5 + x_6x_7x_8^7 \). Computing the singular locus is by definition finding all points on the surface such that \( \nabla f = 0 \),
and so we get the following:

\[(2x_1, 2x_2x_3, x_2^2, 3x_4^2x_5, 5x_4^3x_6^5, x_7x_8^7, x_6x_8^7, 7x_6x_7x_8^6) = 0\]

Notice that as a consequence of being disjointly supported these conditions can be broken up by corresponding monomial, and solved independently. Each monomial gives a collection of subspaces with satisfy the condition, described below:

1. From \(x_1^2\), we have the subspace
   \[V_1 = \{ z | x_1 = 0 \} \]

2. From \(x_2^2x_3\) we have the subspace
   \[V_2 = \{ z | x_2 = 0 \} \]

3. From \(x_4^3x_5^5\), we have the subspaces
   \[V_4 = \{ z | x_4 = 0 \} \text{ and } V_5 = \{ z | x_5 = 0 \} \]

4. From \(x_6x_7x_8^7\), we have the subspaces
   \[V_8 = \{ z | x_8 = 0 \} \text{ and } V_0 = \{ z | x_6, x_7 = 0 \} \]

Now, if we chose one subspace per monomial, and take the intersection of these subspaces across all monomials, we are left with a subspace space that satisfies the Jacobian condition for each monomial. There are four subspaces obtained in this way. Two of these spaces are four dimensional corresponding to the intersection of \(V_1, V_2, V_4, V_8\) and \(V_1, V_2, V_5, V_8\). The other two are
three dimensional, corresponding to $V_1$, $V_2$, $V_4$, $V_0$ and $V_1$, $V_2$, $V_5$, $V_0$. Each component has codimension of at least 3, so we satisfy the $R_1$ condition of Serre. As all hypersurfaces are $S_2$, this hypersurface is normal.

As in this example, the structure of singular locus for a general well-poised $f$ is determined by the monomials in $f$. As such we can provide a complete description of the singular locus of a well-poised hypersurface.

**Theorem 1.3.** The singular locus $S(V(f))$ of a hypersurface described by a polynomial with disjoint supports is entirely determined by the number and form of the monomials in $f$. Each monomial describes a set of conditions which give a collection of coordinate subspaces in $\mathbb{A}^n(\mathbb{k})$. By choosing one subspace per monomial term, and taking the intersection indexed across all monomials, the resulting coordinate subspace is a component of $S(V(f))$. $S(V(f))$ is then the union of all such subspaces.

The subspaces described by the monomial terms give an obvious way to calculate the codimension of the singular locus. This lets us determine normality of well-poised (or more generally disjointly supported) hypersurfaces easily, giving the following corollary:

**Corollary 1.1.** A non-monomial disjointly supported polynomial $f$ in $\mathbb{k}[x]$ is normal if and only if one of the following three conditions hold:

1. $f$ contains a nonzero constant.
2. $f$ contains a monomials of the form $x_1^1$.
3. When $\mathbb{k}$ has characteristic zero, $f$ has 3 monomials that are not constant nor of the form $x_1^1$. 
(4) When \( k \) has characteristic \( p > 0 \), \( f \) has \( 3+n \) monomials that are not constant nor of the form \( x_j^1 \), where \( n \) is the number of monomials of the form \( x_j^{cp} \) for \( c \in \mathbb{N} \).

**Proof of Theorem.** This proof details an explicit computation of the singular locus. The process emulates the example given above the theorem. We start by noting that if \( f \) has monomials with disjoint supports, when we examine \( \nabla f = 0 \), we see that the term corresponding to \( \frac{\partial f}{\partial x_j} = 0 \) is dependant entirely on the monomial in \( f \) containing \( x_j \). Therefore the conditions forcing \( \nabla f = 0 \) can be determined by independently determining the conditions that force \( \nabla x^{a_i} = 0 \) for a given monomial \( x^{a_i} \) in \( f \). Below are the possible forms of monomial, and the subspace of \( \mathbb{R}^n \) forcing \( \nabla x^{a_i} = 0 \) for such monomials. The example presents the minimal case for the form of each monomial. For the sake of notation, we assume our indeterminates are indexed in the lexicographic order they appear in \( f \).

1. \( x^{a_i} = c \): The subspace is empty and \( V(f) \) has no singular locus. This is seen by noting for a point to be singular and on the surface, it must satisfy both \( \nabla f = 0 \) and \( f = 0 \). Inspection shows this is impossible when \( f \) contains both a constant and monomials with disjoint supports.
2. \( x^{a_i} = x_i \): the subspace is empty and \( V(f) \) has no singular locus. This results in \( \nabla x_1 = 1 \) which means \( \nabla f \neq 0 \).
3. \( x^{a_i} = x_i^{p_i} \) with \( p_i > 1 \): This gives \( \frac{\partial f}{\partial x_i} = p_i x_i^{p_i-1} \) and therefore we have the obvious subspace:
\[
V = \{ z \in \mathbb{A}^n(\mathbb{k}) | x_i = 0 \}
\]
Note that when working over characteristic $p$, we will not have this subspace.

(4) $x^{a_i} = x_i x_{i+1}^{p_1} \ldots x_{i+m}^{p_m}$ with $p_j > 1$: Here

$$\nabla x^{a_i} = (x_{i+1}^{p_1} \ldots x_{i+m}^{p_m}, \ldots p_j x_i x_{i+1}^{p_1} \ldots x_{i+j}^{p_j-1} \ldots x_{i+m}^{p_m}, \ldots)$$

and so all subspaces are given by

$$V_j = \{ z \in A^n(k) | x_{i+j} = 0, 0 < j \leq m \}$$

Note that if we are working over non-zero characteristic, the mixed nature of the terms means that regardless of exponents, we still have all subspaces listed. This note applies for the following two cases also.

(5) $x^{a_i} = x_i \ldots x_{i+k} x_{i+k+1}^{p_1} \ldots x_{i+k+m}^{p_m}$ with $p_j > 1$: The Jacobian here is entirely analogous to the previous case for $x_{i+k+1}$ to $x_{i+k+m}$ and so we have the obvious subspaces by analog:

$$V_j = \{ z \in z \in A^n(k) | x_{i+k+j} = 0, 0 < j \leq m \}.$$

We also have the added subspace

$$V_0 = \{ z \in A^n(k) | x_i, \ldots, x_{i+k} = 0 \}$$

resulting from the conditions imposed by the partials $\frac{\partial}{\partial x_i}$ to $\frac{\partial}{\partial x_{i+k}}$.

(6) $x^{a_i} = x_i^{p_0} \ldots x_{i+m}^{p_m}$ with $p_j > 1$: This is again analogous to the previous examples and therefore we have subspaces given by

$$V_j = \{ z \in z \in A^n(k) | x_{i+j} = 0, 0 < j \leq m \}$$
By choosing one subspace per monomial term, and taking the intersection indexed across all monomials, the resulting coordinate subspace is a component of $S(V(f))$. Any point in the singular locus must be contained in one such component by construction. $S(V(f))$ is then the union of all such hyperplanes.

\[\square\]

**Proof of Corollary.** We appeal to Serre’s criterion. All hypersurfaces are $S_2$, and so it suffices to check the $R_1$ condition for $f$. First, if $f$ contains any monomial of the form (1) or (2) above, our space has no singular locus and obviously satisfies $R_1$, and is therefore normal.

It is worth noting that each monomial term above (disregarding (1) and (2)) imposes a condition that fixes a particular variable or subset of variables at zero. By the note on item (3), when we work over a field of characteristic $p$, we are missing one subspace for each term of the form $x_j^{cp}$, and thus do not fix any variables. Each fixed variable increases the codimension of $S(V(f))$. Satisfying $R_1$ is equivalent to checking that the singular locus has codimension $\geq 2$ in the hypersurfaces and by extension, codimension $\geq 3$ in the ambient space. By the above characterizations, if $f$ has more than three monomial terms not of the forms previously discussed, then $V(f)$ is normal. \[\square\]

**Example [1]**. We have already examined the enhanced version of the Whitney Umbrella above, but now we examine the classical Whitney Umbrella. This polynomial has two monomial terms, one of type (3) above and the other of type (4). Here, we intersect the two subspaces $V_1 = \{ z \in \mathbb{A}^3(k) | x_1 =$
\[ V_2 = \{ z \in \mathbb{A}^3(\mathbb{k}) \mid x_2 = 0 \} \], which gives us a single copy of \( \mathbb{A}(\mathbb{k}) \) as the singular locus.

**Example 2** The E8 singularity exclusively contains monomials of the form of type (3) described in the theorem above. The intersection of the three corresponding subspaces is clearly the origin, which indeed is the entire singular locus of this surface. Additionally, the E8 singularity has three monomials of the necessary form, and therefore is normal.

**Example 3** This surface is described entirely of monomials of type (5) described above, with no higher order terms, meaning that each monomial corresponds to exactly one subspace. These subspaces only intersect at the origin, which is the only singular point of this surface. Additionally, this surface is normal.

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