High frequency limit for gravitational perturbations of cosmological models in modified gravity theories

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Abstract

In general relativity, it has been shown that the effective gravitational stress-energy tensor for short-wavelength metric perturbations acts just like that for a radiation fluid, and thus, in particular, cannot provide any effects that mimic dark energy. However, it is far from obvious if this property of the effective gravitational stress-energy tensor is a specific nature held only in the Einstein gravity, or holds also in other theories of gravity. In particular, when considering modified gravity theories that involve higher order derivative terms, one may expect to have some non-negligible effects arising from higher order derivatives of short-wavelength perturbations. In this paper, we argue this is not the case at least in the cosmological context. We show that when the background, or coarse-grained metric averaged over several wavelengths has FLRW symmetry, the effective gravitational stress-energy tensor for metric perturbations of a cosmological model in a simple class of \(f(R)\) gravity theories, as well as that obtained in the corresponding scalar-tensor theory, takes a similar form to that in general relativity and is in fact traceless, hence acting again like a radiation fluid.

1 Introduction

Our observable universe appears to be homogeneous and isotropic on large scales, but highly inhomogeneous on small scales. It is therefore considerably interesting to consider whether the local inhomogeneities can have any effects on the global dynamics of our universe, in particular, any effect that corresponds to a positive cosmological constant or dark energy. A number of authors have explored this possibility of explaining the present cosmic accelerating expansion by some backreaction effects of the local inhomogeneities \([1]–[18]\). Such a backreaction effect may be described in terms of an effective stress-energy tensor arising from metric as well as matter perturbations.

In general relativity, a consistent expansion scheme for short-wavelength perturbations and the corresponding effective stress-energy tensor were largely developed by Isaacson \([19,20]\), in which the small parameter, say \(\epsilon\), corresponds to the amplitude and at the same time the wavelength of perturbations. Isaacson’s expansion scheme is called the high frequency limit or the short-wavelength approximation. In this expansion, the dominant order of the Einstein equation with respect to this parameter \(\epsilon\) corresponds to the equations of motion for linearized gravitational waves in the ordinary perturbation theory, and is in fact divergent as \(\epsilon^{-1}\). The next order of the expansion of the Einstein equation provides the Einstein equation for the background metric with an effective stress-energy tensor, which is essentially given as minus the second-order Einstein tensor averaged over a spacetime region of several wavelengths of metric perturbations. Since taking a derivative of perturbations corresponds, roughly speaking, to multiplying the inverse of the smallness parameter (or the inverse of the wavelength of perturbations), the effective stress-energy tensor consisting of the square of derivatives of the first-order metric perturbations can have large effects on the background dynamics. Furthermore it can be shown that the effective stress-energy tensor thus constructed is gauge-invariant, hence has a physical meaning. If the effective stress-energy tensor had a term proportional to the background spacetime metric, then it would correspond to adding a cosmological constant to the effective Einstein equations for the background metric, thereby explaining possible origin of dark energy from local inhomogeneities. It has been shown, however, that this effective gravitational stress-energy tensor is traceless and satisfies the weak energy condition, i.e. acts like radiation \([21,22]\), and thus cannot provide any effects that imitate dark energy in general relativity.

However, it is far from obvious if this traceless property of the effective gravitational stress-energy tensor is a nature specific only to the Einstein gravity or is rather a generic property that can hold also in other types of gravity theories. The purpose of this paper is to address this question in a

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simple, concrete model in the cosmological context. Among many, one of the simplest of modified theories so far proposed is the so called \( f(R) \) theory, whose action is a generalization of the Einstein-Hilbert action to an arbitrary function, \( f(R) \), of the scalar curvature \( R \). Since \( f(R) \) gravity contains higher order derivative terms, one can anticipate the effective gravitational stress-energy tensor to be generally modified in the high frequency limit.

Over the past decade, cosmological implications of \( f(R) \) gravity theories have extensively been studied especially in the quest of finding an alternative cosmology to Λ-CDM model. A viable class of the \( f(R) \) theories is summarized in [23][31]. Although it is desirable to examine all these cosmologically favored models, in the present paper, we will restrict our attention to the simplest model \( f(R) = R + cR^2 \) with an eye to applications to analyses of more generic cases, which are left for our future study. This model itself is not considered as a cosmologically favored modified gravity theory for describing the present accelerating universe, but has rather been introduced as a prototype of an inflationary universe model by Starobinsky [32]. However, this simple model can be viewed as the leading term truncation of a more generic class of \( f(R) \) theories that take an analytic form with respect to \( R \) around the vacuum solution \( R = 0 \) and therefore provides, as the first step toward this line of research, a good starting point of our analysis. It would also be interesting to check whether or not a once-excluded model can possibly revive as a cosmologically favored model, due to the inclusion of the backreaction effects of local inhomogeneities.

It is well-known that \( f(R) \) gravity is equivalent to a scalar-tensor theory, which contains the coupling of the scalar curvature \( R \) to a scalar field \( \phi \) in a certain way [33][34][35]. The Brans-Dicke theory [36] is one of the simplest examples. Therefore our analysis can be performed, in principle, either (i) by first translating a given \( f(R) \) theory into the corresponding scalar-tensor theory and then inspecting the stress-energy tensor for the scalar field \( \phi \), or (ii) by directly dealing with metric perturbations of the \( f(R) \) theory. One may expect that the former approach is much easier than the latter metric approach, as one has to deal with metric perturbations of complicated combinations of the curvature tensors in the latter case. Nevertheless we will take the both approaches. In fact, in the metric approach, by directly taking up perturbations of the scalar curvature \( R \), the Ricci tensor \( R_{\alpha\beta} \) and the Riemann tensor \( R^{\gamma}_{\alpha\beta\delta} \) involved in a given \( f(R) \) theory, we can learn how to generalize our present analysis of a specific class of \( f(R) \) gravity to analyses of other, different types of modified gravity theories that cannot even be translated into a scalar-tensor theory, such as the Gauss-Bonnet gravity.

The theory we consider in the present paper contains higher order derivative terms. As in the case of general relativity, we can consider short-wavelength metric perturbations with a small parameter \( \epsilon \) and expand the field equations with respect to \( \epsilon \). In contrast to the Einstein gravity, the dominant part of the field equations for this theory is of order \( O(\epsilon^{-3}) \). In \( O(\epsilon^{-1}) \), we have equations of motion for linearized gravitational waves. In \( O(1) \) we obtain equations for the background metric with a source term arising from short-wavelength perturbations. This source term contains a number of higher order derivatives of metric perturbations. However, as one cannot have any meaningful notion of stress-energy for gravitational waves in a local sense (at least within a wavelength), we have to take a suitable spacetime average over several wavelengths. Also, since we are interested in backreaction effects on the cosmological dynamics, we assume that our background metric takes the form of the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. We also impose that in the limit to the Einstein gravity, i.e., \( f(R) \rightarrow R \) (as \( \epsilon \rightarrow 0 \)), the field equations for our \( f(R) \) theory reduce to those for the Einstein gravity in the corresponding order of the expansion parameter. At this point, a number of terms that involve higher order derivatives of metric perturbations vanish by the spacetime averaging procedure and the assumption of the background FLRW symmetry. Eventually, besides the terms corresponding to the Isaacson’s formula in the Einstein gravity, only a few terms that contain higher order derivatives of metric perturbations can remain in the effective stress-energy tensor for short-wavelength perturbations in our modified gravity. Furthermore, the resultant effective stress-energy tensor is shown to be traceless as in the Einstein gravity case.

We briefly comment on the previous work along the similar line. Since the effective stress-energy tensor can be used to measure energy flux carried out by gravitational radiation from astrophysical sources, such as inspiral binary systems, it can be used to test various modified gravity theories by the near future gravitational wave detectors [37][38]. For this purpose, the effective stress-energy tensor for gravitational radiation has been derived by Sopuerta and Yunes [40] by applying Isaacson’s scheme to the field equations of dynamical Chern-Simons theory. A more general formalism to compute the effective stress-energy tensor from the effective action, which can apply to a wide class of modified gravity theories, has been proposed by Stein and Yunes [47]. As a concrete example, the formula has been applied to dynamical Chern-Simons gravity as well as theories with dynamical scalar fields coupled to higher-order curvature invariants. It has been shown that in these modified
theories the stress-energy tensor for gravitational radiations reduces, at future null infinity, to that in the Einstein gravity. Berry and Gair [18] also have derived the effective stress-energy tensor for gravitational waves in the $f(R)$ gravity which is analytic around the vacuum $R = 0$. However, since the main concern in these studies is mainly to test alternative theories in the astrophysical context by using gravitational wave detectors, the formulas mentioned just above have been formulated for asymptotically flat spacetimes (as energy flux of gravitational waves needs to be evaluated at future null infinity) and therefore do not appear to apply to cosmological models. In contrast, our analysis will proceed by exploiting the cosmological setup that our background metric possesses the FLRW symmetry.

In the next section, before going into the effective stress-energy tensor in modified gravity theories, we will first briefly summarize the high frequency limit in general relativity. In Sec. [3] we consider the high frequency limit in $f(R)$ gravity theory. Based on the Isaacson’s scheme we expand the field equations for $f(R) = R + c R^2$ theory and first derive the general expression of the effective stress-energy tensor for gravitational perturbations in our $f(R)$ gravity. Then, assuming that our background metric has the FLRW symmetry and also that the resulting equations reduce to the corresponding equations for the Einstein gravity in the limit $c \to 0$, we see that the effective stress-energy tensor whose expression is significantly simplified, is in fact traceless as in the Einstein gravity case. As briefly mentioned above, when a given $f(R)$ gravity is translated into the corresponding scalar-tensor theory, the scalar field $\phi$, which expresses an extra-degree of freedom in the $f(R)$ theory, possess a non-trivial potential term. [Compare with the earlier work by Lee [19] on a computation of the effective stress-energy tensor in a scalar-tensor theory with vanishing potential term.] In Sec. [4] we will make sure that the effective stress-energy tensor in Brans-Dicke theory is consistent with that in our $f(R)$ gravity. Sec. [5] is devoted to summary and points to future research.

Our signature convention for $g_{ab}$ is $(-, +, +, +)$. We define the Riemann tensor by $R_{abcd} \omega_d = 2\nabla_a \nabla_b \omega_c$ and the Ricci tensor by $R_{ab} = R_{abcd} \omega^d$ as in Wald’s book [20].

## 2 High frequency limit in general relativity

In this section we introduce our notation by recapitulating Isaacson’s expansion scheme for short-wavelength gravitational perturbations in general relativity.

Let $g_{ab}$ be the metric with linear perturbation $\delta g_{ab}$; it is described by $g_{ab} = g_{ab}^{(0)} + \delta g_{ab}$ with $g_{ab}^{(0)}$ being the background metric including the backreaction from perturbations. The amplitude of $\delta g_{ab}$ is of order $\delta g_{ab} \sim O(\epsilon)$ with $\epsilon$ being the small parameter, which also corresponds to the wavelength $\lambda$ of perturbations compared with the background characteristic curvature radius, $L$. The order of derivatives of $\delta g_{ab}$ are

$$\nabla_a \nabla_b \cdots \nabla_m \delta g_{bc} \sim O\left(\frac{\epsilon}{(\lambda/L)^m}\right) \sim O(\epsilon^{1-m}),$$

where $\nabla_a$ denotes the covariant derivative with respect to $g_{ab}^{(0)}$, so that $\nabla_a g_{bc}^{(0)} = 0$. We may bear in mind perturbations of the form $h \sim \epsilon \sin(x/\lambda)$ and $\lambda/L \sim O(\epsilon)$. In what follows we normalize $L \sim 1$.

The inverse metric takes the form

$$\lambda \sim \epsilon \sin(x/\lambda) = \epsilon^{-2} \sin(x/\lambda),$$

where $\lambda$ contains terms such as those that are expressed as $g^{-1} \nabla g, \nabla g^{-1} \nabla g$ and $g^{-1} \nabla g \nabla g$.

The general relationship between the Ricci tensor of $g_{ab}$ and that of $g_{ab}^{(0)}$, namely

$$R_{ab} = R_{ab}[g^{(0)}] + 2\nabla_a C^a_{bc} + 2C^a_{bc} C^d_{bca},$$

where $C^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{bc}[g^{(0)}] = g^{ad}(\nabla_bg_{dc} + \nabla_c g_{bd} - \nabla_d g_{bc})/2$. Since $R_{ab}$ contains terms such as those that are expressed as $g^{-1} \nabla g, \nabla g^{-1} \nabla g$ and $g^{-1} \nabla g \nabla g$, we can find

$$R^{(n)}_{ab} [h] \sim O(\epsilon^{n-2}),$$

where $n$ is the number of $\delta g_{ab}$ included in $R_{ab}$. We also find

$$R^{(n)}[h] \sim G^{(n)}_{ab} [h] \sim O(\epsilon^{n-2}),$$

where $R \equiv g^{ab} R_{ab}, G_{ab} \equiv R_{ab} - g_{ab} R/2$ is the Einstein tensor, and $R^{(n)}[h]$ and $G^{(n)}_{ab} [h]$ do not contain $R_{ab}[g^{(0)}]$.

The Einstein equation is

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = \kappa^2 T_{ab}^{(0)},$$

(2.5)
where $\kappa^2 = 8\pi G$ and $T_{ab}^{(0)}$ is the stress-energy tensor for the background matter fields. In the following, for simplicity, we focus on metric perturbations and ignore perturbations of the matter fields in $T_{ab}^{(0)}$. We can find the dominant terms, $O(\epsilon^{-1})$, of the Einstein equation as

$$G_{ab}^{(1)} [h] = R_{ab}^{(1)} [h] - \frac{1}{2} g_{ab}^{(0)} R^{(1)} [h] = 0,$$

or simply

$$R_{ab}^{(1)} [h] = 0. \tag{2.6}$$

This is equivalent to the equation for linearized gravitational waves in the ordinary perturbation theory. Next, in the order of $O(1)$, we find

$$G_{ab} [g]^{(0)} = \kappa^2 T_{ab}^{(0)} + \kappa^2 T_{ab}^{\text{eff}}, \tag{2.7}$$

where the effective gravitational stress-energy tensor, $T_{ab}^{\text{eff}}$, is given by

$$\kappa^2 T_{ab}^{\text{eff}} = - \left\langle c_{ab}^{(2)} [h] \right\rangle = - \left\langle R_{ab}^{(2)} [h] - \frac{1}{2} g_{ab}^{(0)} g^{cd} R_{cd}^{(2)} [h] \right\rangle = \left\langle \frac{1}{4} \nabla_a h^{TT} \nabla_b h^{TT} \right\rangle. \tag{2.8}$$

Here and in the following $\langle \cdots \rangle$ denotes taking a spacetime average over several wavelengths of perturbations. Here the indices are raised and lowered with $g^{(0)ab}$ and $g_{(0)ab}$. For the expression of the third-line, the transverse-traceless gauge, $\nabla_c h^c = 0$, denoted by $h^{TT}$, and $(\ref{3.6})$ have been used (see also Appendix. A). We can check that $T_{ab}^{\text{eff}}$ is traceless, i.e., it acts like radiation:

$$\kappa^2 T_{ab}^{\text{eff}} h^a_a = 0, \tag{2.9}$$

from (2.6) and (A.2). Thus, in particular, it cannot provide any effects that mimic dark energy in general relativity. For more mathematically rigorous treatments of short-wavelength perturbations and the effective stress-energy tensor, see [21][22].

The effective stress-energy tensor (2.8) can be shown to be gauge-invariant [21]. In fact, the expression of the right-hand side of (2.8) is given by manifestly gauge-invariant part of $h_{ab}^{TT}$. For this purpose, one can introduce the polarization tensors $\epsilon^{(+)}_{ab}$, as usual, and decompose the metric perturbation accordingly $h_{ab}^{TT} = \epsilon_{ab}^{(+)} h^{(+)\prime} + \epsilon_{ab}^{(\times)} h^{(\times)\prime}$. In the cosmological context, one is concerned with the Friedmann-Lemaître-Robertson-Walker (FLRW) metric,

$$ds^2 = -dt^2 + a(t)^2 \gamma_{ij} dx^i dx^j. \tag{2.10}$$

with $d\sigma^2 = \gamma_{ij} dx^i dx^j$ being the metric of 3-dimensional constant curvature space. So, it may be more convenient to impose the transverse-traceless condition with respect to this FLRW time-slicing, i.e., $h_{0i}^{TT} = 0, h_{0i}^{TT} = 0, \nabla_a h^{TT} a_i = 0, h_{TT}^{TT} = 0$. This condition completely fix the gauge freedom and (2.8) is written by the gauge invariant variable $h_{ij}^{TT}$ as $(\frac{1}{4} \nabla_\alpha h^{TT}) (\nabla_\beta h^{TT} \nabla_\beta h^{TT})$.

### 3 High frequency limit in $f(R)$ gravity

The general action for the $f(R)$ gravity is given by

$$S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} f(R) + \int d^4 x L_M, \tag{3.1}$$

where $L_M$ is the Lagrangian for matter fields, such as perfect fluid in the cosmological context. Varying this action with respect to the metric, we have the field equations

$$G_{ab}^{(R)} \equiv G_{ab} + \kappa^2 T_{ab}^{(0)} - \frac{1}{2} g_{ab}^{(0)} F + g_{ab}^{(0)} \nabla_i F + g_{cd}^{(0)} \nabla_c \nabla_d F = \kappa^2 T_{ab}^{(0)}, \tag{3.2}$$

where $\dot{F} \equiv f - R, \ddot{F} \equiv d\dot{F}/dR$, and $T_{ab}^{(0)}$ denotes the matter stress-energy tensor.

The field equations in the $f(R)$ gravity have terms consisting of higher order derivatives of $R$, and the order of those derivatives are higher than that of $R$:

$$\nabla_{a_1} \nabla_{a_2} \cdots \nabla_{a_m} R^{(n)} [h] \sim O\left(\epsilon^{n-2-m}\right). \tag{3.3}$$
Therefore it is expected that the effect of the short-wavelength approximation would be enhanced. In order to see whether this is the case, from now on we restrict our attention to the following concrete model

\[ f(R) = R + cR^2, \]  

where \( c \) is a constant. This model has been considered for the first time in the context of inflationary universe \[ 32. \] The field equations are

\[ G_{ab}^{(R)} ≡ G_{ab} + 2c \left( R R_{ab} - \frac{1}{4} g_{ab} R^2 - \nabla_a \nabla_b R + g_{ab} g^{cd} \nabla_c \nabla_d R \right) = \kappa^2 T_{ab}^{(0)}. \]  

As in Isaacson’s formula reviewed in the previous section, we expand the above equations with respect to the small parameter \( \epsilon \). Then, the dominant part of the order \( O(\epsilon^{-3}) \), in which we have the following equations

\[ \nabla_a \nabla_b R^{(1)}[t] - g_{ab}^{(0)} \Box R^{(1)}[t] = 0. \]  

By contracting with \( g^{(0)ab} \), we immediately have

\[ \nabla_a \nabla_b R^{(1)}[t] = 0. \]  

Next, for the order \( O(\epsilon^{-2}) \), we have

\[ R^{(1)}[t] R_{ab}^{(1)}[t] - \frac{1}{4} g_{ab}^{(0)} \left( R^{(1)}[t] \right)^2 - \nabla_a \nabla_b R^{(2)}[t] + g_{ab}^{(0)} \Box R^{(2)}[t] = 0. \]  

By dotting with \( g^{(0)ab} \), we have

\[ \Box R^{(2)}[t] = 0, \]  

\[ \nabla_a \nabla_b R^{(2)}[t] = R^{(1)}[t] R_{ab}^{(1)}[t] - \frac{1}{4} g_{ab}^{(0)} \left( R^{(1)}[t] \right)^2. \]

Note that since we are working in the short-wavelength approximation, we find \( g^{(0)ab} R^{(1)}_a = R^{(1)}[t] \) in \( O(\epsilon^{-1}) \), which is different from calculation in ordinary perturbation theory, where in general \( g^{(0)ab} R^{(1)}_a \neq R^{(1)}[t] \). For \( O(\epsilon^{-1}) \), we have

\[ R^{(1)}[t] + \frac{c}{2} \left( R^{(1)}[t] \right)^2 - 6c \left( \Box R^{(3)}[t] - R_{cd} [g^{(0)}] \Box h^{cd} \right) + 8c h^{ab} \nabla_a \nabla_b R^{(2)}[t] = 0. \]

Again, by dotting with \( g^{(0)ab} \), we have

\[ R^{(1)}[t] + \frac{c}{2} \left( R^{(1)}[t] \right)^2 - 6c \left( \Box R^{(3)}[t] - R_{cd} [g^{(0)}] \Box h^{cd} \right) + 8c h^{ab} \nabla_a \nabla_b R^{(2)}[t] = 0. \]

In order \( O(1) \), as in Isaacson’s formula in general relativity, we have the field equations for the background metric with the backreaction source term: \( G_{ab}^{(f(R))} [g^{(0)}] = \kappa^2 T_{ab}^{(0)} + \kappa^2 T_{ab}^{\text{eff}} \), where

\[ \kappa^2 T_{ab}^{\text{eff}} = - \left( R_{ab}^{(2)}[t] - \frac{1}{2} \left( g_{ab}^{(0)} R^{(2)}[t] + h_{ab} R^{(1)}[t] \right) \right) \]

\[ + 2c \left( R^{(1)}[t] R_{ab}^{(3)}[t] + R[g^{(0)}] R^{(2)}[t] \right) \]

\[ + \left( R^{(2)}[t] \right) \left( R_{ab}^{(0)} + R^{(2)}[t] \right) + \left( R^{(3)}[t] - h^{cd} R_{cd} [g^{(0)}] \right) R_{ab}^{(1)}[t] \right) \]

\[ - \frac{c}{2} \left[ g_{ab}^{(0)} \left( 2 R^{(1)}[t] R^{(3)}[t] - 2 R^{(1)}[t] h^{cd} R_{cd} [g^{(0)}] + 2 R^{(0)} R^{(2)}[t] \right) \right] \]

\[ + 2 h_{ab} \left( R[g^{(0)}] + R^{(2)}[t] \right) R^{(1)}[t] \]

\[ + 2 c \left( \left( g_{ab}^{(0)} R^{(3)}[t] - h_{ab} g^{(0)cd} \right) \left( \nabla_c \nabla_d R^{(3)}[t] - R_{e_f} [g^{(0)}] \nabla_e \nabla_d h^{e_f} \right) \right) \]

\[ + \left( g_{ab}^{(0)} h^{cd} - h_{ab} R^{(0)cd} \right) \nabla_c \nabla_d R^{(2)}[t] \right) \].
This is the expression of the stress-energy tensor for short-wavelength metric perturbations on the generic background metric $g^{(0)}_{\mu\nu}$ in our $f(R)$ gravity.

From now on, we consider in the cosmological context. We assume that our background is spatially homogeneous and isotropic, that is, our background metric possesses the FLRW symmetry and therefore takes the form of $\ell ^{2.10}$. Then, thanks to this background symmetry we can explicitly solve equations of the form $\nabla_a\nabla_bS(t,\vec{x}) = 0$, such as (3.7) (see Appendix. 13). Equation (3.7) (the equations of motion of $O(\epsilon^{-3})$) is solved to yield

$$R^{(1)}[h] = \text{const}. \quad (3.14)$$

Taking the average, we find

$$R^{(1)}[h] = \langle \text{const.} \rangle = \langle R^{(1)}[h] \rangle = 0. \quad (3.15)$$

Then, the equations (3.10) (those of $O(\epsilon^{-2})$) become

$$\nabla_a\nabla_bR^{(2)}[h] = 0. \quad (3.16)$$

Again using the result in Appendix. 13 we find

$$R^{(2)}[h] \equiv S_1 = \text{const}. \quad (3.17)$$

By using (3.15) and (3.16), the equation (3.12) (of $O(\epsilon^{-1})$) immediately yields

$$\Box R^{(3)}[h] - R_{ab}[g^{(0)}] \Box h^{ab} = 0, \quad (3.18)$$

and the equations (3.11) (of $O(\epsilon^{-1})$) reduce to

$$\left(1 + 2cR[g^{(0)}] + 2cS_1\right)R^{(1)}_{ab}[h] = 2c \left(\nabla_a\nabla_bR^{(3)}[h] - R_{cd}[g^{(0)}]\nabla_a\nabla_bh^{cd}\right). \quad (3.19)$$

The effective stress-energy tensor (3.13) is then expressed as

$$\kappa^2 T^{\text{eff}}_{ab} = -\left\langle \left(1 + 2cR[g^{(0)}]\right)R^{(2)}_{ab}[h] - \frac{1}{2}g^{(0)}_{ab}S_1 \right. \right.
+ 2c\left. \left. \left\langle R^{(3)}[h] - h^{cd}R_{cd}[g^{(0)}]\right\rangle R^{(1)}_{ab}[h] + S_1 \left(R_{ab}[g^{(0)}] + R^{(2)}_{ab}\right)\right\rangle \right.
- \frac{c}{2}g^{(0)}_{ab} \left(2R[g^{(0)}] + S_1\right)S_1

+ 2c \left. \left(-g^{(0)}_{ab}h^{cd}\right) \left(\nabla_a\nabla_dR^{(3)}[h] - R_{ef}[g^{(0)}]\nabla_e\nabla_fh^{ef}\right)\right\rangle. \quad (3.20)$$

Since the effective stress-energy tensor in the $R^2$ model should reduce to that in general relativity when $c = 0$, we choose $S_1 \equiv R^{(3)}[h] = 0$. Then, (3.20) becomes

$$\kappa^2 T^{\text{eff}}_{ab} = -\left\langle \left(1 + 2cR[g^{(0)}]\right)R^{(2)}_{ab}[h] + 2c \left(R^{(3)}[h] - h^{cd}R_{cd}[g^{(0)}]\right)R^{(1)}_{ab}[h] \right. \right.
- \left. g^{(0)}_{ab} \left(1 + 2cR[g^{(0)}]\right)h^{cd}R^{(1)}_{cd}[h]\right\rangle

= -\left\langle \left(1 + 2cR[g^{(0)}]\right)R^{(2)}_{ab}[h] + 2c \left(R^{(3)}[h] - h^{cd}R_{cd}[g^{(0)}]\right)R^{(1)}_{ab}[h]\right\rangle, \quad (3.21)$$

where we have used (3.19) in the first equality above, and

$$\left\langle h^{cd}R^{(1)}_{cd}[h]\right\rangle = 0. \quad (3.22)$$

in the second equality so as to make the above expression compatible with that of general relativity in the $c = 0$ case. The expression, (3.21), is our main result of this section. From $R^{(3)}[h] = 0$ and (3.22), we see

$$\left\langle g^{(0)ab}R^{(2)}_{ab}[h]\right\rangle = \left\langle R^{(2)}[h] + h^{ab}R^{(1)}_{ab}[h]\right\rangle = 0. \quad (3.23)$$

Then using this and $R^{(1)}[h] = 0$, we can find that $\kappa^2 T^{\text{eff}}_{ab}$ is in fact traceless:

$$\kappa^2 T^{\text{eff}}_{ab} = 0. \quad (3.24)$$
4 The high frequency limit in scalar-tensor theory

In the previous section, the scalar curvature \( R \) and the Ricci tensor \( R_{ab} \) are taken up directly in the metric formalism of the \( f(R) \) gravity. It is well-known that any \( f(R) \) gravity theory is included in Brans-Dicke theory, which is one of the simplest examples of scalar-tensor theory \([30,35]\). In this section, we will see that the results obtained in the previous section are indeed consistent with those obtained within the corresponding scalar-tensor theory.

The action of Brans-Dicke theory \([30]\) is

\[
S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \phi R - \frac{\omega_{BD}}{2\phi} \nabla^a \phi \nabla_a \phi - V(\phi) \right\} + \int d^4x L_M ,
\]

where \( \omega_{BD} \) is a constant called the Brans-Dicke parameter and \( \phi \) is a dimensionless scalar field, and \( L_M \) denotes the Lagrangian for matter fields, which can in general couple to the metric \( g_{ab} \) as well as the scalar field \( \phi \). Then the equations of motion for \( \phi \) and \( g_{ab} \) are, respectively, obtained as

\[
\Box \phi + \frac{\phi}{2\omega_{BD}} \left( - \frac{\omega_{BD}}{\phi^2} \nabla^a \phi \nabla_a \phi + R - 2\partial_\phi V(\phi) \right) = \kappa^2 T^{(0)}_{\phi \phi} ,
\]

\[
\phi G_{ab} - \frac{\omega_{BD}}{\phi} \left( \nabla_a \phi \nabla_b \phi - \frac{g_{ab}}{2} \nabla^c \phi \nabla_c \phi \right) - \nabla_a \nabla_b \phi + g_{ab} \left( g^{cd} \nabla_c \nabla_d \phi + V(\phi) \right) = \kappa^2 T^{(0)}_{ab} ,
\]

where \( T^{(0)}_{\phi \phi} \) and \( T^{(0)}_{ab} \) are the stress-energy tensor for matter fields obtained by taking variations of \( \phi \) and \( g_{ab} \), respectively.

The \( f(R) \) gravity of the metric formalism, \([31]\), can be cast into the form of the above Brans-Dicke theory by setting

\[
\phi = F(R) \equiv \frac{df(R)}{dR} , \quad \omega_{BD} = 0 , \quad V = \frac{F(R)R - f(R)}{2} .
\]

In this case, as one can find \( R - 2\partial_\phi V = 0 \), the equations of motion for \( \phi \) and \( g_{ab} \) just given above become respectively

\[
\Box \phi - \frac{1}{2\phi} \nabla^a \phi \nabla_a \phi = \kappa^2 T^{(0)}_{\phi \phi} ,
\]

\[
G^{ST}_{ab} \equiv \phi \left( R_{ab} - \frac{1}{2} g_{ab} R \right) - \nabla_a \nabla_b \phi + g_{ab} \left( g^{cd} \nabla_c \nabla_d \phi + V(\phi) \right) = \kappa^2 T^{(0)}_{ab} .
\]

From now we consider short-wavelength perturbations for \( \phi : \phi = \phi_0 + \delta \phi \). We also assume that there is no coupling of matter fields with the second-order derivatives of \( \phi \), so that there are no non-vanishing terms of order \( O(\epsilon^{-1}) \) in the stress-energy tensor for matter fields. Then, the equation of motion for \( \phi \) of \( O(\epsilon^{-1}) \) is

\[
\Box \delta \phi = 0 ,
\]

and the equations of motion for \( g_{ab} \) of \( O(\epsilon^{-1}) \) are

\[
\phi_0 \left( R_{ab}^{(1)}[h] - \frac{1}{2} g_{ab}^{(0)} R^{(1)}[h] \right) = \nabla_a \nabla_b \phi - g_{ab}^{(0)} \Box \delta \phi .
\]

Contracting with \( g^{(0)ab} \), we have

\[
R^{(1)}[h] = \frac{3}{\phi_0} \Box \delta \phi = 0 ,
\]

where we have used \([4.7]\). From this equation, we can immediately find

\[
R_{ab}^{(1)}[h] = \frac{1}{\phi_0} \nabla_a \nabla_b \delta \phi .
\]

The equations of motion in \( O(1) \) are given by \( G^{ST}_{ab} [g^{(0)}, \phi_0] = \kappa^2 T_{ab}^{(0)} + \kappa^2 T_{ab}^{\text{eff}} \), where

\[
\kappa^2 T_{ab}^{\text{eff}} \equiv - \left( \phi_0 R_{ab}^{(2)}[h] + \delta \phi R_{ab}^{(1)}[h] - g_{ab}^{(0)} h^{cd} \nabla_c \nabla_d \delta \phi \right) .
\]

Here we would like to emphasise that so far we have made no assumptions concerning the form of \( f(R) \) or the symmetry of our background metric \( g_{ab}^{(0)} \); the above expression, \([4.11]\), applies to the generic \( f(R) \) theory with an arbitrary background metric.

If we restrict the form of \( f(R) \) to be \([34]\), then by inspecting the expansions \( \phi = \phi_0 + \delta \phi + \cdots \) and \( F(R) = 1 + 2\epsilon R = \left( 1 + 2\epsilon R[g^{(0)}] \right) + 2\epsilon \left( R^{(3)}[h] - h^{cd} R_{cd}[g^{(0)}] \right) + \cdots \), we find

\[
\phi_0 = 1 + 2\epsilon R[g^{(0)}] , \quad \delta \phi = 2\epsilon \left( R^{(3)}[h] - h^{cd} R_{cd}[g^{(0)}] \right) .
\]
Using these and (4.10), we have
\[ \kappa^2 T_{ab}^{\text{eff}} = - \left( 1 + 2cR[g^{(0)}] \right) R_{ab}^{(2)}[h] + 2c \left( R^{(3)}[h] - h^{cd}R_{cd}[g^{(0)}] \right) R_{ab}^{(1)}[h] - g_{ab} \phi_0 h^{cd} R_{cd}^{(1)}[h] \right). \] (4.13)

Now we work on the cosmological situation so that the background metric has the FLRW symmetry. Provided that the limit \( c \to 0 \) should reproduce results in the case of the Einstein gravity, we finally obtain
\[ \kappa^2 T_{ab}^{\text{eff}} = - \left( 1 + 2cR[g^{(0)}] \right) R_{ab}^{(2)}[h] + 2c \left( R^{(3)}[h] - h^{cd}R_{cd}[g^{(0)}] \right) R_{ab}^{(1)}[h] \right), \] (4.14)
where we have used (4.22), derived under the FLRW symmetry in Sec. 3. We see that the expression (4.14) above is precisely the same as (5.21) derived within the metric formalism of the \( f(R) \) gravity. This verifies our methods of Sec. 5 for dealing with short-wavelength perturbations of the \( f(R) \) gravity within the metric formalism.

5 Summary

We have addressed the effective gravitational stress-energy tensor for short-wavelength perturbations in the simple class of \( f(R) \) gravity of \( R^2 \) type in the cosmological context. As in the Isaacson’s formula for the Einstein gravity reviewed in Sec. 2, we have obtained the field equations for the background metric with a backreaction source term \( T_{ab}^{\text{eff}} \) in order \( O(1) \) of the small parameter \( \epsilon \). Reflecting the fact that our \( f(R) \) theory contains higher order derivative terms, the source term or the effective stress-energy tensor \( T_{ab}^{\text{eff}} \) takes, as given in (4.13), a complex form that contains, in principle, terms of fourth order derivatives, schematically expressed as \( \langle \nabla h \nabla h \nabla h \nabla h \rangle \). The resultant expression, (4.13), of the effective stress-energy tensor, in fact, applies to any background metric \( g_{ab}^{(0)} \); Until this point, no symmetry assumption on the background metric has been used. Then, by imposing that our background has the FLRW symmetry, we have derived our effective stress-energy tensor for short-wavelength metric perturbations in cosmological models. At this point, thanks to the background FLRW symmetry and the spacetime averaging over several wavelengths, the expression of our effective stress-energy tensor has been significantly reduced to have the simple form, (5.24). We have also shown that the obtained effective stress-energy tensor is traceless, so that it acts like a radiation fluid as in the Einstein gravity case and thus, in particular, cannot mimic dark energy.

Since any \( f(R) \) gravity theory is known to be equivalent to a scalar-tensor theory, we have cast our \( f(R) \) theory into the corresponding scalar-tensor theory. Then, within the scalar-tensor theory, we have derived the effective stress-energy tensor for short-wavelength perturbations of the scalar field and checked consistency with the stress-energy tensor obtained within the metric perturbations of the original \( f(R) \) theory.

Although we have focused on the \( R^2 \) model especially about the FLRW background, in Sec. 4 we have pushed forward our calculations with a general \( f(R) \) gravity about an arbitrary background as far as possible, and have not used the property of the \( R^2 \) model about the FLRW background, up to (4.11). We can immediately note that (4.11) does not involve any terms of fourth order derivatives but has only terms of the square of first order derivatives of perturbations \( h_{ab} \) and \( \delta \phi \). This result obtained within the framework of the scalar-tensor theory indicates that the higher order derivatives could vanish also in the metric framework of general \( f(R) \) gravity theory for a generic background. However, to see whether this is indeed the case needs further involved calculation, and is beyond the scope of this paper. This is left open for future study.

Our formulas derived in Sec. 5 deal directly with the scalar curvature \( R \) and the Ricci tensor \( R_{ab} \), and therefore should be able to apply to similar analyses of other modified gravity theories which contain higher order curvature terms composed of \( R, R_{ab}, \) and \( R^c_{\;bcd} \) and which cannot even be cast in the form of a scalar-tensor theory. It would be interesting to consider an extension of our present work to a wide class of modified gravity theories with high-rank curvatures.

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A Perturbation formulas in Transverse-Traceless gauge

The conditions of transverse-traceless waves are $\nabla_a h^{TT}{}^a = h^{TT}{}^a = 0$. The Ricci tensors for $h^{TT}{}_{ab}$ are

$$\begin{align*}
C^{(1)}_{ab}[h^{TT}] &= 0, \\
R^{(1)}_{ab}[h^{TT}] &= \nabla_c C^{(1)}_{ca}[h^{TT}], \\
&= -\frac{1}{2} \nabla^a h^{TT}, \\
R^{(2)}_{ab}[h^{TT}] &= 2\nabla [c C^{(2)}_{ba}] - C^{(1)}_{db} C^{(1)}_{ca}[h^{TT}] \\
&= -\frac{1}{2} h^{TT} \nabla_c \left( \nabla_b h^{TT} + \nabla_a h^{TT} - \nabla_a h^{TT} \right) \\
&\phantom{= \frac{1}{2}} + \frac{1}{2} \nabla_b \left( h^{TT} \nabla_c h^{TT} + \nabla_a h^{TT} - \nabla_a h^{TT} \right) \\
&\phantom{= \frac{1}{2}} - \frac{1}{4} \left( \nabla^{TT}_{ab} + \nabla_a h^{TT} \right) \left( \nabla_c h^{TT} + \nabla_a h^{TT} - \nabla^{TT}_{ca} \right),
\end{align*}$$

and the scalar curvatures for $h^{TT}{}_{ab}$ are

$$\begin{align*}
R^{(1)}[h^{TT}] &= g^{(0)ab} R^{(1)}_{ab}[h^{TT}] \\
&= 0, \\
R^{(2)}[h^{TT}] &= g^{(0)ab} R^{(2)}_{ab}[h^{TT}] - h^{TT} a^{(1)} R^{(1)}_{ab}[h^{TT}] \\
&= \frac{3}{4} \nabla^{TT} \nabla_a h^{TT} - \frac{1}{2} \nabla^a h^{TT} \nabla_c h^{TT} + h^{TT} a^{(3)} \nabla h^{TT},
\end{align*}$$

where we have used $[\nabla_a, \nabla_b] h^{TT} = R^{(e)ab} [g^{(0)}] h^{TT} + R^{(e)ab} [g^{(0)}] h^{TT} = O(\epsilon)$. From \(A.4\) and \(A.5\), we find $\nabla h^{TT} = 0$. Using these we find

$$\begin{align*}
\langle R^{(2)}_{ab}[h^{TT}] \rangle &= -\frac{1}{4} \nabla_a h^{TT} \nabla_b h^{TT}, \\
\langle R^{(2)}[h^{TT}] \rangle &= 0.
\end{align*}$$

B Solution of $\nabla_a \nabla_b S(t, \vec{x}) = 0$

We solve the equation

$$\nabla_a \nabla_b S(t, \vec{x}) = 0,$$

such as \((B.1)\), in the FLRW background spacetime:

$$ds^2 = -dt^2 + a^2(t) \gamma_{ij} dx^i dx^j.$$

Since we are interested in the expanding universe, in what follows we assume that the scale factor $a(t)$ is dynamical, i.e., $a(t) \neq const$. The $(0,0)$, $(0,i)$ and $(i,j)$ components of \((B.1)\) are

$$\begin{align*}
\dot{S}(t, \vec{x}) &= 0, \\
\dot{S}(t, \vec{x}) - \frac{\dot{a}(t)}{a(t)} \partial_0 S(t, \vec{x}) &= 0, \\
\partial_i \partial_j S(t, \vec{x}) - a(t) \dot{a}(t) \gamma_{ij} S(t, \vec{x}) - \Gamma^{k[3]}_{ij} \partial_k S &= 0,
\end{align*}$$

where the dot denotes the derivative with respect to the cosmic time. The solution of \((B.3)\) is

$$S(t, \vec{x}) = c_1(\vec{x}) t + c_2(\vec{x}),$$

where $c_1(\vec{x})$ and $c_2(\vec{x})$ are arbitrary functions of $\vec{x}$. Since $\dot{a} \neq 0$, Equation \((B.4)\) becomes

$$\begin{align*}
\left( \frac{a(t)}{\dot{a}(t)} - t \right) \partial_i c_1(\vec{x}) = \partial_i c_2(\vec{x}).
\end{align*}$$

Therefore, either

$$c_1, c_2 = const.$$

or $\partial_i c_2 / \partial_i c_1 = const$. In the latter case, by shifting $t \rightarrow t - const.$, we have $\dot{a}/a = 1/t$, and therefore have $a(t) \propto t$; the behavior of the background FLRW universe is determined. This is, in our present
context, too restrictive, and for this reason, we should take the former case; \(c_1, c_2 = \text{const.}\). Then, equation (B.5) for \(i = j\) becomes
\[
a(t) \dot{a}(t) \gamma_{ij} c_1 = 0, \tag{B.9}
\]
which immediately implies
\[
c_1 = 0. \tag{B.10}
\]
Therefore we find the solution of (B.4) to be
\[
S = \text{const.} \tag{B.11}
\]

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