A HIGH-ORDER FICTITIOUS-DOMAIN METHOD FOR THE ADVECTION-DIFFUSION EQUATION ON TIME-VARYING DOMAIN

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Abstract. We develop a high-order finite element method to solve the advection-diffusion equation on a time-varying domain. The method is based on a characteristic-Galerkin formulation combined with the $k^{th}$-order backward differentiation formula (BDF-$k$) and the fictitious-domain finite element method. Optimal error estimates of the discrete solutions are proven for $2 \leq k \leq 4$ by taking account of the errors from interface-tracking, temporal discretization, and spatial discretization, provided that the $(k+1)^{th}$-order Runge-Kutta scheme is used for interface-tracking. Numerical experiments demonstrate the optimal convergence of the method for $k = 3$ and 4.

Key words. Free-surface problem, fictitious-domain finite element method, high-order scheme, interface-tracking algorithm, advection-diffusion equation.

AMS subject classifications. 65M60, 65L06, 76R99

1. Introduction. Multiphase flows with time-varying domains have an extremely wide application range in science and engineering. The deformation of fluid phases and consequent complex interactions of multiple time and length scales pose great challenges to the design of accurate, efficient, and simple algorithms for these moving-boundary problems.

In terms of the relative position of the moving domain to the discretization mesh, current numerical methods for moving-boundary problems can be roughly classified into two regimes.

In body-fitted methods, the mesh is arranged to follow the moving phase, which makes it easy to implement the boundary conditions of governing equations. This mesh-domain alignment can also be exploited in numerical analysis to provide thorough error estimates; see [6,7] for some early works on elliptic and Maxwell interface problems. However, these conveniences incur the expenses of mesh regeneration and data migration across the entire computational domain at each time step [1]. Apart from the efficiency issue, another major concern of body-fitted methods is how to maintain certain degree of mesh regularity to prevent ill-conditioning at the presence of abrupt movements and/or large deformations of the phases. A typical example of body-fitted methods is the arbitrary Lagrangian-Eulerian (ALE) approach; see [15,17,24] for some recent advancements of ALE.

In the other regime of unfitted methods, the mesh for the bulk phase is fixed while the moving boundary is allowed to cross the fixed mesh, resulting in irregular grids or cut cells near the boundary where the discretization of governing equations and the enforcement of boundary conditions have to be adjusted appropriately. Popular methods in this regime include the immersed boundary method [28], the immersed interface method (IIM) [20,22], and the extended finite element method (XFEM). As the main ideas of XFEM, the degrees of freedom for the cut cells are increased and penalty terms are added to enforce boundary conditions weakly. Fire and Zilian presented a first-order XFEM method by using backward Euler for time integration [10]. Based on a space-time

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discontinuous Galerkin discretization, Lehrenfeld and Reusken \cite{18} proposed a two-dimensional second-order XFEM scheme, which was extended to three dimensions by Christoph \cite{9}. Recently, Guo \cite{12} analyzed a backward Euler immersed finite element method for solving parabolic moving interface problems. Apart from the aforementioned methods, XFEM ideas can also be found in CutFEM \cite{2,5,13}, the interface-penalty finite element method \cite{25,32}, and the fictitious domain method \cite{3,4,16}.

The fidelity of simulating physical processes on time-varying domains is very much influenced by loci of the domain boundary at each time step. In order to achieve high-order accuracy for the entire solver, an interface-tracking algorithm with at least the same order of accuracy is needed to tracking the moving boundary, even in the case of mild deformations. In the immersed boundary method, the moving boundary is represented by line segments of interface markers and tracked by a fronting tracking method. As such, the overall accuracy is at best second-order \cite{33}. For IIMs, there exist fourth-order schemes for stationary interfaces \cite{21}, but we are not aware of any fourth-order IIMs capable of handling nontrivial deforming boundaries. Within the framework of XFEM, one can design high-order schemes for problems with stationary interfaces (cf. \cite{25}), but still face great challenges in the case of moving boundaries. For other finite element methods of high-order accuracy \cite{12,17,19,23}, it is often assumed that the loci of the boundary curve are known \textit{a priori} at each time step. However, this assumption does not hold for problems with nontrivial boundary movements, even when analytic expressions of these boundary movements are already given.

To the best of our knowledge, there exist no third- and fourth-order methods that can handle moving-boundary problems with large nontrivial deformations of the domain boundaries. A main reason for this absence is that high-order interface tracking has not been coupled to finite elements methods and finite different/volume methods. This is not surprising since third- and fourth-order interface tracking methods have not been available until recently \cite{35}. On the other hand, as the science of multiphase flows evolves toward more and more complex phenomena, there is a growing need of higher-order methods for moving boundary problems.

We answer this need by developing in this work third-order and fourth-order methods for numerically solving the advection-diffusion equation (2.1) on time-varying domains. For the moving boundaries, we track their loci to fourth-order accuracy by the recent cubic MARS method \cite{35}, which features topological modeling of fluids \cite{36} and a rigorous analytic framework for error estimates \cite{33}. For spatial discretization, we adopt a high-order fictitious-domain method on a fixed finite element mesh that covers the full movement range of the deforming domain. For temporal integration, we apply the BDF-$k$ schemes \cite{24} to a Lagrangian form of the advection-diffusion equation where the coordinates are defined by characteristic tracing along the driving velocity field.

The main contribution of this work is the development of a high-order method for solving the advection-diffusion equation on time-varying domains; our method is also established on solid ground of numerical analysis as follows.

(a) We prove the stability of our method under weighted energy norm. As a main difficulty, the composite function $u_k^{n-j} \circ X_r^{n,n-j}$ does not belong to the finite element space at $t_n$ and thus can not serve as a test function, where $u_k^{n-j}$ is the discrete solution at $t_{n-j}$ and $X_r^{n,n-j}$ is the discrete flow map from $t_n$ to $t_{n-j}$. We overcame this difficulty by introducing a modified Ritz projection onto the finite element space.

(b) We present thorough error analysis for our method by taking full consideration of all errors from interface-tracking, spatial discretization, and temporal integration. Our main result is that, if the discrete solution is obtained from the BDF-$k$ scheme, the finite element discretization with piecewise $Q_k$-polynomials, the RK-$k$ ($k$th-order Runge-Kutta) method for
interface-tracking, then the solution errors in approximating regular solutions is $O(\tau^{k-1/2})$
under the energy norm, where $\tau = O(h)$ is the time step size and $k = 2, 3, 4$. In addition, this error estimate can be improved to $O(\tau^k)$ if we use the RK-$(k + 1)$ scheme for interface-tracking.

The rest of this paper is organized as follows. In Section 2, we formalize the model problem and introduce some notations. In Section 3, we discuss the interface-tracking algorithms and derive error estimates between the exact boundary and the approximate boundary. Then we formulate our method as the weak discrete form (4.5) in Section 4 and prove its well-posedness in Section 5. In Section 6, we derive a priori error estimates for the entire solver. In Section 7, we present results of numerical experiments to demonstrate the third- and fourth-order convergence of our method.

2. The model problem. The advection-diffusion equation with initial and boundary conditions is proposed as follows

\[
\frac{\partial u}{\partial t} + w \cdot \nabla u - \Delta u = f \quad \text{in } \Omega_t, \tag{2.1a}
\]
\[
u = 0 \quad \text{on } \Gamma_t, \tag{2.1b}
\]
\[
u(0) = u_0 \quad \text{in } \Omega_0. \tag{2.1c}
\]

where $\Omega_t$ is a time-varying domain in $\mathbb{R}^2$, $\Gamma_t = \partial \Omega_t$ is the boundary of $\Omega_t$, $w(x, t)$ is the velocity of fluid which occupies $\Omega_t$, $u(x, t)$ stands for the tracer transported by the fluid, and $f(x, t)$ stands for the source term distributed in $\mathbb{R}^2$ and has a compact support. The equation has been scaled so that the diffusion coefficient before $\Delta u$ is unit. Since we focus on linear problems, the velocity $w$ is assumed to be given and satisfies $\text{div } w = 0$. For convenience in analysis, we assume that $\Gamma_t$ is $C^r$-smooth for $r \geq 4$ and $w$ can be extended smoothly to the exterior of $\Omega_t$, that is, $w \in C^r(\mathbb{R}^2 \times [0, T])$.

We define the Lagrangian coordinates by the solution to the ordinary differential equations

\[
\frac{dX(t)}{dt} = w(X(t), t) \quad \forall \; t \geq s; \quad X(s) = x_s. \tag{2.2}
\]

Since $w$ is $C^r$-smooth, (2.2) has a unique solution for every $s \geq 0$ and every $x_s \in \mathbb{R}^2$. The graph of $X$ is just the characteristic curve of $u$. To specify the dependence of $X$ on the initial value, the mapping from $x_s$ to $X(t)$ is denoted by

\[
X(t; s, x_s) := X(t), \quad t \geq s.
\]

The physical domain is driven by the fluid and can be defined by means of the flow map $X$

\[
\Omega_t := \{X(t; 0, x) : \; x \in \Omega_0\}. \tag{2.3}
\]

Since any flow map is a diffeomorphism, all topology features of $\Omega_t$ stay the same as those of $\Omega_0$. In the cubic MARS method for interface tracking, a fluid phase with arbitrarily complex topology can be represented by a partially order set of oriented Jordan curves that are pairwise almost disjoint [36]. Thanks to the generality of this representation, it suffices to only consider the case of $\Omega_t$ being simply connected, i.e. $\Gamma_t$ is a positively oriented Jordan curve, since the algorithms and analyses in this work extend straightforwardly to multi-connected domains.

Using (2.2), the material derivative of $u$ is defined by

\[
\frac{du}{dt} = \frac{\partial u}{\partial t} + w \cdot \nabla u. \tag{2.4}
\]
Then (2.1a) can be written into a compact form
\[
\frac{du}{dt} - \Delta u = f \quad \text{in} \; \Omega_t.
\] (2.5)

We introduce some notations for Sobolev spaces and norms. For a domain \( \Omega \subset \mathbb{R}^d \), let \( L^2(\Omega) \) be the space of square-integrable functions on \( \Omega \) and \( L^\infty(\Omega) \) the space of essentially bounded functions. For any integer \( m > 0 \), define
\[
H^m(\Omega) := \left\{ v \in L^2(\Omega) : \frac{\partial^{i+j} v}{\partial x_1^i \partial x_2^j} \in L^2(\Omega), \; 0 \leq i + j \leq m \right\},
\]
\[
W^{m,\infty}(\Omega) := \left\{ v \in L^\infty(\Omega) : \frac{\partial^{i+j} v}{\partial x_1^i \partial x_2^j} \in L^\infty(\Omega), \; 0 \leq i + j \leq m \right\}.
\]
Let \( H^m_0(\Omega) \) be the completion of \( C^\infty_0(\Omega) \) under the norm of \( H^m(\Omega) \). Throughout this paper, vector-valued quantities will be denoted by boldface symbols, such as \( L \). Let \( \partial_x \) valued quantities will be denoted by blackboard bold symbols, such as \( L \).

Let \( X^0(\eta) \) be the completion of \( X \) under the norm of \( L \). Throughout this paper, vector-valued quantities will be denoted by boldface symbols, such as \( L \). Let \( \partial_x \) valued quantities will be denoted by blackboard bold symbols, such as \( L \).

3. Interface-tracking algorithms. We first consider the tracking algorithm for \( \bar{\Omega}_t \) which is the closure of \( \Omega_t \). The purpose is to find a discrete approximation to the continuous mapping \( \{ \Omega_t : 0 \leq t \leq T \} \). Let \( t_n = n\tau \), \( n = 0, 1, \cdots, N \), be a uniform partition of the interval \([0, T] \) where \( T > 0 \) is the final time and \( \tau = T/N \). Write \( X^{n-1,n} := X(t_n; t_{n-1}, \cdot) \) for any \( n > 0 \). The uniqueness of the solution to (2.2) implies that the mapping \( X^{n-1,n} : \bar{\Omega}_{t_n-1} \rightarrow \bar{\Omega}_{t_n} \) is one-to-one. For any \( 1 \leq i \leq n \), the multi-step mapping is defined by
\[
X^{n-i,n} := X^{n-1,n} \circ X^{n-2,n-1} \circ \cdots \circ X^{n-i,n-i+1}, \quad \bar{\Omega}_{t_n} = X^{n-i,n}(\bar{\Omega}_{t_{n-i}}).
\]

We introduce the shorthand notation \( X^{n,i} := (X^{i,n})^{-1} \). The interface-tracking problem is to seek an approximate solution to (2.2) for each initial value \( x \in \Gamma_{t_{n-1}} \). Throughout the following statement, the notation \( f \lesssim g \) means that \( f \leq C g \) with a constant \( C > 0 \) independent of sensitive quantities, such as the segment size \( h \) for interface-tracking, the spatial mesh size \( h \), the time-step size \( \tau \), and the number of time steps \( N \). Moreover, \( f \approx g \) means that \( f \lesssim g \) and \( g \lesssim f \) hold simultaneously.

3.1. The approximate flow map. Given an approximate domain \( \bar{\Omega}^{n-1} \) of \( \bar{\Omega}_{t_{n-1}} \), the approximate domain \( \bar{\Omega}^n = X^{n-1,n}(\bar{\Omega}^{n-1}) \) is defined by the RK-k scheme for (2.2) [33]. For any \( x^{n-1} \in \Omega^{n-1} \), the point \( x^n = X^{n-1,n}(x^{n-1}) \in \bar{\Omega}^n \) is calculated as follows:
\[
\begin{cases}
  x^{(1)} = x^{n-1}, \\
x^{(i)} = x^{n-1} + \tau \sum_{j=1}^{n_k} a_{ij}^k w(x^{(j)}, t^{(j)}), \quad t^{(j)} = t_{n-1} + c^k_j \tau, \quad 2 \leq i \leq n_k, \\
x^n = x^{n-1} + \tau \sum_{i=1}^{n_k} b_{ki}^k w(x^{(i)}, t^{(i)}).
\end{cases}
\] (3.1)
Then Gronwall’s inequality yields
\[ \phi_{n-1}^{(i)}(x^{n-1}) = x^{(i)}, \quad i = 1, \ldots, n_k. \]

Let \( I \) be the identity map. Then \( X_r^{n-1,n} \) can be represented explicitly as follows
\[ X_r^{n-1,n} = I + \tau \sum_{i=1}^{n_k} b^k_i w(\phi_{n-1}^{(i)}(\cdot), t^{(i)}). \] (3.2)

The multi-step mapping from \( \bar{\Omega}^{n-i} \) to \( \bar{\Omega}^n \) is defined by
\[ X_r^{n-i,n} = X_r^{n-1,n} \circ X_r^{n-2,n-1} \circ \cdots \circ X_r^{n-i,n-i+1}, \quad 1 \leq i \leq n. \] (3.3)

The inverse of \( X_r^{i,n} \) is denoted by \( X_r^{i,n} : (X_r^{i,n})^{-1} \). We only consider the case of \( 1 \leq k \leq r - 1 \).

### 3.2. Useful estimates for \( X_r^{n-i,n} \) and \( X_r^{n-i,n} \)
For any \( t \geq s \geq 0 \) and \( x \in \mathbb{R}^2 \), the Jacobi matrix of \( x \to X(t; s, x) \) is given by
\[ J(t; s, x) := \frac{\partial X(t; s, x)}{\partial x} = I + \int_s^t \nabla w(X(\xi; s, x), \xi) J(\xi; s, x) d\xi. \] (3.4)

Since \( \text{div} \ w = 0 \), it is well-known that \( \text{det}(J) = 1 \) in \( \mathbb{R}^2 \) for all \( t \geq s \) [8]. For convenience, we denote the Jacobi matrices of the discrete flow maps by
\[ J_r^{n-i,n}(x) := J(t_n; t_{n-i}, x), \quad J_r^{n-i,n}(x) := \frac{\partial X_r^{n-i,n}(x)}{\partial x}. \]

The Jacobi matrices of \( X_r^{n,n-i} \), \( X_r^{n-i,n} \) are denoted by \( J_r^{n,n-i} := (J_r^{n-i,n})^{-1} \) and \( J_r^{n-i,n} := (J_r^{n-i,n})^{-1} \), respectively. Clearly \( \text{det}(J_r^{n-i,n}) = \text{det}(J_r^{n,n-i}) = 1 \) in \( \mathbb{R}^2 \).

**Lemma 3.1.** For any \( 0 \leq i \leq k, 0 \leq m \leq n \leq N \), and \( 0 \leq s \leq t \),
\[ \| J_r^{n-i,n} - I \|_{\mathcal{L}_\infty(\mathbb{R}^2)} + \| J_r^{n-i,n} - I \|_{\mathcal{L}_\infty(\mathbb{R}^2)} \lesssim \tau, \]
\[ \| J_r^{n,n} - I \|_{\mathcal{L}_\infty(\mathbb{R}^2)} + \| J_r^{n,n} - I \|_{\mathcal{L}_\infty(\mathbb{R}^2)} \lesssim 1. \] (3.5) (3.6)

**Proof.** For any \( t \geq s \geq 0 \), from [3,4] we know that
\[ \| J(t; s, \cdot) - I \|_{\mathcal{L}_\infty(\mathbb{R}^2)} \lesssim (t-s) \| \nabla w \|_{\mathcal{L}_\infty(\mathbb{R}^2 \times [0,T])} + \| \nabla w \|_{\mathcal{L}_\infty(\mathbb{R}^2 \times [0,T])} \int_s^t \| J(\xi; s, \cdot) - I \|_{\mathcal{L}_\infty(\mathbb{R}^2)} d\xi. \]

Then Gronwall’s inequality yields
\[ \| J(t; s, \cdot) - I \|_{\mathcal{L}_\infty(\mathbb{R}^2)} \lesssim 1, \quad \| J_r^{n-i,n} - I \|_{\mathcal{L}_\infty(\mathbb{R}^2)} \lesssim \tau. \]

The first-order derivatives of \( J(t; s, \cdot) \) are estimated as follows
\[ \left\| \frac{\partial J(t; s, \cdot)}{\partial x_j} \right\|_{\mathcal{L}_\infty(\mathbb{R}^2)} \leq \int_s^t \left\| \nabla w(X(\xi; s, \cdot), \xi) J(\xi; s, \cdot) \right\|_{\mathcal{L}_\infty(\mathbb{R}^2)} d\xi \]
\[ \lesssim 1 + \int_s^t \left\| \frac{\partial J(\xi; s, \cdot)}{\partial x_j} \right\|_{\mathcal{L}_\infty(\mathbb{R}^2)} d\xi, \quad j = 1, 2. \]
Using Gronwall’s inequality again yields \( \left\| \frac{\partial J(t; s, \cdot)}{\partial x_j} \right\|_{L^\infty(\mathbb{R}^2)} \lesssim 1 \). High-order derivatives of \( J(t, s; \cdot) \) can be estimated similarly.

Next we estimate \( \|J_{\tau}^{n-1,n} - \mathbb{I}\|_{L^\infty(\mathbb{R}^2)} \). From (3.1)–(3.2), it is easy to see

\[
J_{\tau}^{n-1,n}(x) = \mathbb{I} + \tau \sum_{i=1}^{n} b_k \nabla w(x^{(i)} - n_{-1}^{(i)} \nabla \phi^{(i)}_{n_{-1}}(x),
\]

(3.7)

where \( \nabla \phi^{(i)}_{n_{-1}} = \mathbb{I} \) and \( \nabla \phi^{(i)}_{n_{-1}} = \mathbb{I} + \tau \sum_{j=1}^{i-1} a_{ij} \nabla w(x^{(j)}, t^{(j)}) \nabla \phi^{(j)}_{n_{-1}} \) for \( i \geq 2 \). The smoothness of \( w \) implies \( \|J_{\tau}^{n-1,n} - \mathbb{I}\|_{L^\infty(\mathbb{R}^2)} \lesssim \tau \). Using (3.3), we easily get \( \|J_{\tau}^{n-1,n} - \mathbb{I}\|_{L^\infty(\mathbb{R}^2)} \lesssim \tau \).

The estimate for \( \|J^{m,n}_{\tau}\|_{W^{r,\infty}(\mathbb{R}^2)} \) is easy by using (3.3). In fact,

\[
\|J^{m,n}_{\tau}\|_{L^\infty(\mathbb{R}^2)} \leq \prod_{j=m+1}^{n} \|J^{i-1,j}_{\tau}\|_{L^\infty(\mathbb{R}^2)} \leq (1 + C\tau)^{n-m} \lesssim 1,
\]

\[
\|\nabla J^{m,n}_{\tau}\|_{L^\infty(\mathbb{R}^2)} \leq \sum_{i=m+1}^{n} \|\nabla J^{i,j-1}_{\tau}\|_{L^\infty(\mathbb{R}^2)} \prod_{j=m+1}^{n} \|J^{i,j}_{\tau}\|_{L^\infty(\mathbb{R}^2)} \lesssim \sum_{i=m+1}^{n} (1 + C\tau)^{\tau} \lesssim 1.
\]

High-order derivatives of \( J^{m,n}_{\tau} \) can be estimated similarly.

**Lemma 3.2.** For \( 0 \leq m \leq n \) and \( \mu = 0, 1 \),

\[
\|X^{m,n}_{\tau} - X^{m,n}\|_{W^{r,\infty}(\mathbb{R}^2)} \lesssim (n - m)\tau^{k+1-\mu}.
\]

Proof. Since \( X^{n-1,n}_{\tau} \) is obtained by the RK-\( k \) scheme for (2.2), standard error estimates for RK schemes yield \( \|X^{n-1,n}_{\tau} - X^{n-1,n}\|_{L^\infty(\mathbb{R}^2)} \lesssim \tau^{k+1} \). Note that

\[
J^{n-1,n}(x) = \nabla X^{n-1,n}(x) = \mathbb{I} + \int_{t_{n-1}}^{t_n} \nabla w(X(s; t_{n-1}, x), s) ds.
\]

Using (3.7) and Taylor’s expansion of \( J^{n-1,n}(x) \) at \( x \), we have \( \|J^{n-1,n} - J^{n-1,n}\|_{L^\infty(\mathbb{R}^2)} \lesssim \tau^k \). It follows from (3.3) that

\[
\|X^{m,n}_{\tau} - X^{m,n}\|_{L^\infty(\mathbb{R}^2)} \leq \sum_{j=m}^{n-1} \left\| (X^{j+1,n}_{\tau} \circ X^{j+1,n}_{\tau} - X^{j+1,n}_{\tau} \circ X^{j+1,n}_{\tau}) \circ X^{m,j}_{\tau} \right\|_{L^\infty(\mathbb{R}^2)}\]

\[
\leq \sum_{j=m}^{n-1} \left\| X^{j+1,n}_{\tau} \right\|_{L^\infty(\mathbb{R}^2)} \left\| X^{j+1,n}_{\tau} - X^{j+1,n}_{\tau} \right\|_{L^\infty(\mathbb{R}^2)} \lesssim (n - m)\tau^{k+1},
\]

where we have applied the intermediate value theorem to each component of \( X^{j+1,n}_{\tau} \) in the second inequality. Furthermore, \( \|J^{m,n}_{\tau} - J^{m,n}\|_{L^\infty(\mathbb{R}^2)} \lesssim (n - m)\tau^k \) can be proven similarly.

**Lemma 3.3.** For any \( 0 \leq m \leq n \leq N \) and \( 0 \leq i \leq k \),

\[
\|J^{n,n-i} - \mathbb{I}\|_{L^\infty(\mathbb{R}^2)} + \|J^{n,n-i} - \mathbb{I}\|_{L^\infty(\mathbb{R}^2)} \lesssim \tau, \quad \|J^{m,n}_{\tau}\|_{L^\infty(\mathbb{R}^2)} \lesssim 1, \quad \|J^{m,n}_{\tau}\|_{L^\infty(\mathbb{R}^2)} \lesssim 1, \quad \|J^{m,n}_{\tau}\|_{L^\infty(\mathbb{R}^2)} \lesssim 1,
\]

(3.8)

\[
\|X^{n,n-i}_{\tau} - X^{n,n-i}\|_{W^{r,\infty}(\mathbb{R}^2)} \lesssim (n - m)\tau^{k+1-\mu}, \quad \mu = 0, 1.
\]

(3.9)
Proof. The identity $\det(J^{n,n-i}) = 1$ and equation (3.6) indicate $\|J^{n,n-i}\|_{L^\infty(\mathbb{R}^2)} \lesssim 1$. Then

$$\|J^{n,n-i} - I\|_{L^\infty(\mathbb{R}^2)} \lesssim \|J^{n,n-i}(I - J^{n,n-i})\|_{L^\infty(\mathbb{R}^2)} \lesssim \|J^{n,n-i} - I\|_{L^\infty(\mathbb{R}^2)} \lesssim \tau.$$  

Since $\|J^{n,n-i} - I\|_{L^\infty(\mathbb{R}^2)} \lesssim \tau$, we also have $\|J^{n,n-i}\|_{L^\infty(\mathbb{R}^2)} \lesssim 1$ and

$$\|J^{n,n-i}\|_{L^\infty(\mathbb{R}^2)} = \|J^{n,n-i}(I - J^{n,n-i})\|_{L^\infty(\mathbb{R}^2)} \lesssim \tau,$$

$$\|J^{n,m}\|_{L^\infty(\mathbb{R}^2)} \leq \prod_{j=m+1}^n \|J^{j,j-1}\|_{L^\infty(\mathbb{R}^2)} \leq (1 + C\tau)^{n-m} \lesssim 1.$$  

Finally, Lemma 3.2 and the boundedness of $J^{n,m}$ indicate that

$$\|X^{n,m} - X^{n,m}\|_{L^\infty(\mathbb{R}^2)} = \|((X^{n,m} \circ X^{m,n} - X^{m,m} \circ X^{m,m}) \circ X^{n,m})\|_{L^\infty(\mathbb{R}^2)}$$

$$= \|X^{n,m} \circ X^{m,n} - X^{n,m} \circ X^{m,n}\|_{L^\infty(\mathbb{R}^2)}$$

$$\leq \|J^{n,m}\|_{L^\infty(\mathbb{R}^2)} \|X^{m,n} - X^{m,m}\|_{L^\infty(\mathbb{R}^2)} \lesssim (n-m)^{\tau^k+1},$$

where we have applied the intermediate value theorem again to each component of $X^{n,m}$ in the last inequality. The gradient of $X^{n,m} - X^{n,m}$ can be estimated similarly

$$\|\nabla(X^{n,m} - X^{n,m})\|_{L^\infty(\mathbb{R}^2)} = \|J^{n,m}(J^{m,n} - J^{m,m})J^{n,m}\|_{L^\infty(\mathbb{R}^2)} \lesssim (n-m)^{\tau^k}.$$  

The proof is finished. \(\Box\)

3.3. The interface-tracking algorithm. In practice, it is unrealistic to track each point in $\Omega^{n-1}$ to obtain $\Omega^n$. We adopt the interface-tracking algorithm in [35] which constructs a $C^2$-smooth boundary with cubic spline interpolations. The purpose is to estimate the errors between the exact boundaries and the approximate boundaries.

The interface-tracking procedure starts from a partition $\mathcal{P}^0 = \{p^0_j \in \Gamma^0 : j = 0, 1, \cdots, J\}$ of the initial boundary with $p^0_0 = p^0_J$. Let $L$ be the arc length of $\Gamma^0$ and suppose $\Gamma^0$ has a parametrization $\Gamma^0 = \{\chi_0(l) : l \in [0, L]\}$, where $\chi_0 \in C^r([0, L])$ satisfies

$$\chi_0(L_j) = p^0_j, \quad L_j = j\eta, \quad 0 \leq j \leq J.$$  

Here $\eta := L/J$ denotes the segment size for interface tracking. Clearly $\chi_0(0) = \chi_0(L)$. The set of interpolation nodes is defined by $\mathcal{L} = \{L_j : j = 0, 1, \cdots, J\}$.

Algorithm 3.4. Given $n \geq 1$, the interface-tracking algorithm for constructing $\Gamma^n_\eta$ from $\Gamma^{n-1}_\eta$ consists of two steps.

1. Trace forward each marker in $\mathcal{P}^{n-1}$ to obtain the set of markers at $t = t_n$,

$$\mathcal{P}^n = \{p^n_j = X^{n-1,n}(p^{n-1}_j) : j = 0, \cdots, J\}.$$  

2. Compute the cubic spline function $\chi_n \in C^2([0, L])$ based on $\mathcal{L}$ and $\mathcal{P}^n$. Define

$$\Gamma^n_\eta := \{\chi_n(l) : l \in [0, L]\}. \quad (3.10)$$
The exact boundary at \( t_n \) is given by

\[
\Gamma_{t_n} = \{ \chi_n(l) : 0 \leq l \leq L \}, \quad \hat{\chi}_n := X^{0,n} \circ \chi_0. \tag{3.11}
\]

For any \( 1 \leq j \leq J \), by Lemmas 3.1 and 3.3 the arc length of \( \Gamma_{t_n} \) between \( X^{0,n}(p_{j-1}^0) \) and \( X^{0,n}(p_j^0) \) can be estimated formally as follows

\[
\eta = \int_{L_{j-1}}^{L_j} |\dot{\chi}_n| = \int_{L_{j-1}}^{L_j} |\dot{p}_n^{0,n}| \leq (1 + C\tau)^n \int_{L_{j-1}}^{L_j} |\dot{\chi}_0| \lesssim \eta. \tag{3.14}
\]

This means that \( X^{0,n}(P_0) \) provides a quasi-uniform partition of \( \Gamma_{t_n} \). Heuristically, the convergence rate of the interface-tracking algorithm does not deteriorate in the time-evolution process if \( \Gamma_{t_n} \) is smooth and \( \max_{0 \leq j \leq J} |p_j^0 - X^{0,n}(p_j^0)| = o(\eta) \). However, the global high accuracy may deteriorate when \( \Gamma_{t_n} \) suffers a largely deformation or a local \( C^1 \) discontinuity. In order to alleviate the accuracy deterioration, we apply a more elaborate algorithm given by Zhang and Fogelson [35] in numerical experiments. The algorithm adjusts the distance between adjacent markers by creating new markers or removing old markers. Therefore, it can be applied to largely deformed domains even at the presence of dynamic \( C^1 \) discontinuities.

3.4. Error estimates for Algorithm 3.4. Let \( \Gamma_{t_n}^\eta \) be the approximate boundary formed with Algorithm 3.4 and let \( \Omega_{t_n}^\eta \) be the open domain surrounded by \( \Gamma_{t_n}^\eta \), namely, \( \Gamma_{t_n}^\eta = \partial \Omega_{t_n}^\eta \). We are going to estimate the difference between the exact boundary and the tracked boundary. First we clarify three kinds of boundaries and their respective parametrizations

\[
\begin{align*}
\Gamma_{t_n} &= \{ \chi_n(l) : l \in [0, L] \}, \quad \hat{\chi}_n = X^{0,n} \circ \chi_0, \\
\Gamma_n &= \{ \chi_n(l) : l \in [0, L] \}, \quad \hat{\chi}_n = X^{0,n} \circ \chi_0, \quad 1 \leq n \leq N, \tag{3.12}
\end{align*}
\]

Here \( \Gamma_{t_n} \) is the exact material boundary and \( \Gamma_n \) is the approximate boundary obtained by the RK-\( k \) scheme [21]. The starting point for high-order error estimates is the smoothness of \( \Gamma_{t_n} \). However, when \( t \) is large, the smoothness of \( \Gamma_{t_n} \) and \( \Gamma_n \) is not guaranteed even \( w \) is regular [27, 29]. Since the objective of the paper is to study error estimates for numerical solutions, we do not elaborate on the smoothness of \( \Gamma_{t_n} \) and \( \Gamma_n \) and simply assume

\[
(A1) \quad \|\hat{\chi}_n\|_{C^r([0,L])} + \|\hat{\chi}_n\|_{C^r([0,L])} \lesssim 1 \text{ for any } 0 \leq n \leq N.
\]

The assumption is reasonable if the deformation of \( \Gamma_{t_n} \) is not large. Standard error estimates for cubic spline interpolations show that

\[
\|\chi_n - \hat{\chi}_n\|_{C^\mu([0,L])} \lesssim \eta^{1-\mu}, \quad \mu = 0, 1. \tag{3.13}
\]

Theorem 3.5. For any \( 0 \leq n \leq N \) and \( 0 \leq \mu \leq 2 \), there holds

\[
\|\chi_n - \hat{\chi}_n\|_{C^\mu([0,L])} \lesssim \eta^{-\mu}(\eta^4 + \tau^k). \tag{3.14}
\]
Proof. Let \( \hat{x}_{n, \eta} \) be the cubic spline interpolation based on the set of nodes \( \mathcal{L} \) and the set of nodal parameters \( \mathbf{X}^{0,n}(\mathcal{P}^0) \). Step 2 of Algorithm 3.4 indicates that \( \hat{x}_{n, \eta} - x_n \) is the cubic spline interpolation based on \( \mathcal{L} \) and \( \{ \mathbf{X}^{0,n}(p^0_j) - \mathbf{X}^0_{\tau}(p^0_j) : 0 \leq j \leq J \} \). Using Lemma 3.2 and the stability and error estimates for cubic spline interpolations, we have

\[
\| \hat{x}_{n, \eta} - x_n \|_{C^0([0,L])} \lesssim \max_{0 \leq j \leq J} |\mathbf{X}^{0,n}(p^0_j) - \mathbf{X}^0_{\tau}(p^0_j)| \lesssim \tau^k,
\]

\[
\| \hat{x}_{n, \eta} - x_n \|_{C^0([0,L])} \lesssim \eta^{4-\mu} \| \hat{x}_n \|_{C^4([0,L])} \lesssim \eta^{4-\mu}.
\]

The triangular inequality and the inverse estimates indicate that

\[
\| \hat{x}_n - x_n \|_{C^0([0,L])} \lesssim \eta^{-\mu} \| \hat{x}_{n, \eta} - x_n \|_{C^0([0,L])} + \| \hat{x}_{n, \eta} - \hat{x}_n \|_{C^0([0,L])} \lesssim \eta^{-\mu} (\tau^k + \eta^4).
\]

The proof is finished. \( \square \)

**Theorem 3.6.** Suppose \( r \geq \max(k + 1, 4) \). Then for each \( n - k \leq m \leq n \),

\[
\| \mathbf{x}_n - \mathbf{x}^{0,n}_m \|_{C^0([0,L])} \lesssim \eta^{-\mu} \sum_{i=0}^{k-1} \tau_{i+1}^{\min(4,k-i)} + \tau_{k+1}, \quad \mu = 0, 1. \tag{3.15}
\]

**Proof.** The cubic spline function \( \mathbf{x}_n \) is formed with \( \mathcal{L} \) and \( \mathcal{P}^n \) and has the explicit form

\[
\mathbf{x}_n(l) = \alpha_j \frac{(L_j - l)^3}{6\eta} + \alpha_j \frac{(l - L_{j-1})^3}{6\eta} + \left( \frac{p_{j-1}^n - \eta^2}{6} \alpha_{j-1} \right) \frac{L_j - l}{\eta}
\]

\[
+ \left( \frac{p_j^n - \eta^2}{6} \alpha_j \right) \frac{l - L_{j-1}}{\eta} \quad \forall l \in [L_{j-1}, L_j), \tag{3.16}
\]

where \( \alpha^n = [\alpha_1^n, \cdots, \alpha_J^n]^T \) is the solution to the system of algebraic equations

\[
\mathbb{G} \alpha^n = d^n \equiv [d_1^n, \cdots, d_J^n]^T, \quad d_j^n := 3(p_{j+1}^n - p_{j-1}^n - 2p_j^n)/\eta^2, \tag{3.17}
\]

and the matrix is given by

\[
\mathbb{G} = \begin{bmatrix}
2 & 0 & 1/2 & \cdots & 1/2 \\
0 & 2 & 0 & \cdots & 1/2 \\
1/2 & 0 & 2 & \cdots & \vdots \\
& \cdots & \cdots & \cdots & \cdots \\
1/2 & 0 & 2 & 0 & 1/2 \\
1/2 & 1/2 & 0 & 2 & 0 \\
& 1/2 & 1/2 & 0 & 2 \\
\end{bmatrix}_{(2J) \times (2J)}
\]

It is well-known that \( \| \mathbb{G}^{-1} \|_\infty \lesssim 1 \). This yields \( \| \alpha^n \|_\infty \lesssim \| d^n \|_\infty \).

Now we estimate \( \alpha^n - \alpha^m \). For fixed \( s \geq 0 \) and \( p \in \mathbb{R}^2 \), we define a univariate function of \( t \)

\[
\mathbf{w}(t; s, p) := \mathbf{w}(\mathbf{X}(t; s, p), t) \quad \forall t \geq s.
\]

Let \( \mathbf{w}^{(i)}(t; s, p) \) denote the \( i \)-th order derivative of \( \mathbf{w}(t; s, p) \) with respect to \( t \). Since \( \mathbf{w} \in C^r(\mathbb{R}^2 \times I) \), the first-order derivative of \( \mathbf{W} \) can be estimated by the chain rule

\[
\| \mathbf{W}^{(1)}(t; s, p) \| = \left( \| \mathbf{w} \cdot \nabla \mathbf{w} \| + \| \frac{\partial \mathbf{w}}{\partial t} \| \right) \lesssim 1.
\]
High-order derivatives of $W(\cdot; s, p)$ can be estimated similarly. Together with (2.2), this shows
\[
\|X(\cdot; s, p)\|_{C^{r+1}(I, T)} + \|W(\cdot; s, p)\|_{C^r(I, T)} \lesssim 1.
\]
By Lemma 3.1 and Taylor’s expansion of $X(t_n; t_m, p)$ at $t_m$, we have
\[
X_{\tau}^m(p) = X(t_n; t_m, p) + O(\tau^{k+1}) = p + \sum_{i=0}^{k-1} \frac{W(i)(t_m; t_m, p)}{(i+1)!}(t_n - t_m)^{i+1} + O(\tau^{k+1}). \quad (3.18)
\]
Since $p_j = X_{\tau}^m(p_j^m)$, this gives an explicit relation between $d^n_j$ and $d^m_j$
\[
d^n_j - d^m_j = \frac{3}{\eta^2} \left[ X_{\tau}^{m,n}(p_{j+1}^m) - p_{j+1}^m - 2X_{\tau}^{m,n}(p_j^m) + 2p_j^m + X_{\tau}^{m,n}(p_{j-1}^m) - p_{j-1}^m \right]
= \sum_{i=0}^{k-1} \beta_{i,j} (t_n - t_m)^{i+1} + O(\eta^{-2}\tau^{k+1}), \quad (3.19)
\]
where $\beta_{i,j} = 3\eta^{-2} \left[ W(i)(t_m; t_m, p_{j+1}^m) + W(i)(t_m; t_m, p_{j-1}^m) - 2W(i)(t_m; t_m, p_j^m) \right]$. Write $\underline{\beta} = [\beta_{i,1}, \cdots, \beta_{i,J}]^\top$ and $\gamma_i = \Gamma^{-1}\beta_i$. It follows from (3.17) and (3.19) that
\[
\alpha^n - \alpha^m = \Gamma^{-1}(d^n - d^m) = \sum_{i=0}^{k-1} \gamma_i (t_n - t_m)^{i+1} + O(\eta^{-2}\tau^{k+1}). \quad (3.20)
\]
Next we estimate $\chi_n - \chi_m$ and $\chi'_n - \chi'_m$. Clearly (3.17) and the equality $\Gamma \gamma_i = \underline{\beta}_i$ indicate that $\gamma_i$ actually defines a cubic spline function
\[
\zeta_i(l) = \gamma_{i-1,j} \frac{(l_j - l)^3}{6\eta} + \gamma_{i,j} \frac{(l - l_{j-1})^3}{6\eta} + \left( W(i)(t_m; t_m, p_{j+1}^m) - \frac{\eta^2}{6} \gamma_{i,j-1} \right) \frac{l_j - l}{\eta}
+ \left( W(i)(t_m; t_m, p_j^m) - \frac{\eta^2}{6} \gamma_{i,j} \right) \frac{l - l_{j-1}}{\eta} \quad \forall l \in [l_{j-1}, l_j). \quad (3.21)
\]
Substituting (3.18) and (3.20) into (3.16), we immediately get
\[
\chi_n - \chi_m = \sum_{i=0}^{k-1} \frac{(t_n - t_m)^{i+1}}{(i+1)!} \zeta_i + O(\eta^{k+1}), \quad \mu = 0, 1. \quad (3.22)
\]
Taking derivatives of (3.16) and substracting the respective equations for $n$ and $m$, we get
\[
\chi'_n - \chi'_m = \sum_{i=0}^{k-1} \frac{(t_n - t_m)^{i+1}}{(i+1)!} \zeta'_i + O(\eta^{-\mu}\tau^{k+1}), \quad \mu = 0, 1. \quad (3.23)
\]
Now we are ready to prove (3.15). Since $\hat{X}_n(l_j) = p_j^m$ for $0 \leq j \leq J$, $\zeta_i$ is actually the cubic spline interpolation of $W^{(i)}(t_m; t_m, \hat{X}_n(l)), l \in [0, L]$. Since $r \geq k+1 \geq i+2$ and $W^{(i)}(t_m; t_m, \hat{X}_m) \in C^{r-i}([0, L])$. Standard error estimates for cubic spline interpolations show that
\[
\|\zeta_i - W^{(i)}(t_m; t_m, \hat{X}_m)\|_{C^{r-i}([0, L])} \lesssim \eta^{\min(4, r-i) - \mu}. \quad (3.24)
\]
Write $\Theta(\tau, \eta) = \sum_{i=0}^{k-1} \tau^{i+1} \eta^{\min(4, r-i)} + \tau^{k+1}$ for convenience. Combining (3.22)–(3.24) yields

$$\chi^{(\mu)}_n - \chi^{(\mu)}_m = \sum_{i=0}^{k-1} \frac{(t_n - t_m)^{i+1}}{(i+1)!} \frac{d^\mu}{dt^\mu} W^{(i)}(t_m, \tilde{\chi}_m) + O(\eta^{-\mu} \Theta(\tau, \eta)).$$ (3.25)

Now we apply (3.18) to $X^m \circ \chi_m$. Together with (3.25) and (3.13), the result implies

$$|X_n - X^m \circ \chi_m| \leq \sum_{i=0}^{k-1} \tau^{i+1} |W^{(i)}(t_m, \tilde{\chi}_m) - W^{(i)}(t_m, \chi_m)| + \Theta(\tau, \eta) \lesssim \Theta(\tau, \eta).$$

Since $w \in C^{k+1}(\mathbb{R}^2 \times [0, T])$, similar to (3.18), we also have

$$\nabla_p X^m(p) = \mathbb{I} + \sum_{i=0}^{k-1} \frac{(t_n - t_m)^{i+1}}{(i+1)!} \nabla_p W^{(i)}(t_m; \tau_m, p) + O(\tau^{k+1}).$$

Using (3.25) and (3.13), the derivative of $X_n - X^m \circ \chi_m$ can be estimated similarly

$$|X'_n - (X^m \circ \chi_m)'| \lesssim \sum_{i=0}^{k-1} \tau^{i+1} \left|\frac{d}{dt} W^{(i)}(t_m, \tilde{\chi}_m) - \frac{d}{dt} W^{(i)}(t_m, \chi_m)\right| + \eta^{-1} \Theta(\tau, \eta)$$

$$\lesssim \sum_{i=0}^{k-1} \tau^{i+1} |\tilde{\chi}'_m - \chi'_m| + \eta^{-1} \Theta(\tau, \eta) \lesssim \eta^{-1} \Theta(\tau, \eta).$$

The proof is finished. \[\square\]

4. The fictitious-domain finite element method. The purpose of this section is to propose high-order fictitious-domain finite element methods for solving (2.1) on a fixed mesh. The BDF-$k$ schemes for $2 \leq k \leq 4$ and the characteristics-based discretization will be used.

4.1. Finite element spaces. First we take an open square $D \subset \mathbb{R}^2$ such that $\Omega_n \cup \Omega_n^\circ \cup \Omega_n^\circ \subseteq D$ for all $0 \leq t \leq T$ and $0 \leq n \leq N$. Let $T_h$ be the uniform partition of $D$ into closed squares of side-length $h$. It generates a cover of $\Omega_n$ and a cover of $\Gamma_n$, respectively, given by

$$T_h := \{ K \in T_h : \text{area}(K \cap \Omega_n) > 0 \}, \quad T_h^\circ := \{ K \in T_h : \text{length}(K \cap \Gamma_n) > 0 \}.$$

The cover $T_h^\circ$ generates a fictitious domain which is denoted by

$$\tilde{\Omega}^\circ := \text{interior}(\cup_{K \in T_h^\circ} K), \quad \tilde{\Gamma}^\circ := \partial \tilde{\Omega}^\circ.$$

Let $E_h$ be the set of all edges in $T_h$. The set of interior edges of boundary elements is denoted by

$$E_{h,B} := \{ E \in E_h : \exists K \in T_h^\circ, \text{s.t.} \ E \subset \partial K \cap \tilde{\Gamma}^\circ \}.$$

The finite element spaces on $D$ and on the fictitious domain are, respectively, defined by

$$V(k, T_h) := \{ v \in H^1(D) : v|_K \in Q_k(K), \forall K \in T_h \}, \quad V(k, T_h^\circ) := \{ v|_{\tilde{\Omega}^\circ} : v \in V(k, T_h) \},$$

where $Q_k$ is the space of polynomials whose degrees are no more than $k$ for each variable. The space of piecewise regular functions over $T_h^\circ$ is defined by

$$H^m(T_h^\circ) := \{ v \in L^2(\tilde{\Omega}^\circ) : v|_K \in H^m(K), \forall K \in T_h^\circ \}, \quad m \geq 1.$$
4.2. The discrete problem. For an interior edge $E \in \mathcal{E}_h$, let $K_1, K_2 \in \mathcal{T}_h$ be the two elements sharing $E$. The jump of a function $v$ across $E$ is defined by

$$[v](x) = \lim_{\varepsilon \to 0^+} [v(x - \varepsilon n_{K_1})n_{K_1} + v(x - \varepsilon n_{K_2})n_{K_2}] \quad \forall x \in E,$$

where $n_{K_1}, n_{K_2}$ are the unit outward normals on $\partial K_1$ and $\partial K_2$, respectively. We define four bilinear forms on $H^{k+1}(\mathcal{T}_h^\eta) \cap H^1(\tilde{\Omega}^\eta)$ as follows

$$A_h^\eta(u, v) := \int_{\Omega^\eta} \nabla u \cdot \nabla v + \int_{\Gamma^\eta} (v \partial_n u + w \partial_n v),$$

$$S_h^\eta(u, v) := -\int_{\Gamma^\eta} (v \partial_n u + w \partial_n v),$$

$$J_0^\eta(u, v) := \frac{\gamma_0}{h} \int_{\Gamma^\eta} u v,$$

$$J_1^\eta(u, v) := \gamma_1 \sum_{E \in \mathcal{E}_h^{\eta, B}} \sum_{l=1}^{k} \int_E \left[ \partial_l^n u \right] \left[ \partial_l^n v \right],$$

where $\gamma_0$ and $\gamma_1$ are positive constants, $J_0^\eta$ is called boundary penalty, $J_1^\eta$ is called boundary-zone penalty, and $\partial_l^n v$ denotes the $l$-th order derivative of $v$ in the normal direction $n$ of $E$. Here $J_0^\eta$ is used to impose the Dirichlet boundary condition of $u_h^\eta$ weakly, $J_1^\eta$ is used to enhance the stability of $u_h^\eta$ (see section 5). In (4.2), $\partial_n v$ denotes the directional derivative of $v$ and $n$ is the unit outward normal to $\Gamma^\eta$.

The discrete approximation to problem (2.1) is to seek $u_h^\eta \in V(k, \mathcal{T}_h^\eta)$ such that

$$\tau^{-1} \left( \Lambda^k U_h^\eta, v_h \right)_{\Omega^\eta} + \omega_h^\eta(u_h^\eta, v_h) = (f^n, v_h)_{\Omega^\eta} \quad \forall v_h \in V(k, \mathcal{T}_h^\eta),$$

where $f^n = f(t_n)$, $U_h^\eta = [U_h^{n-k, n}, \ldots, U_h^{n, n}]^T$, $U_h^{m,n} := u_h^m \circ X_r^m$ for any $n \geq m \geq k$, and $(\cdot, \cdot)_{\Omega^\eta}$ stands for the inner product on $L^2(\Omega^\eta)$. Moreover, $\tau^{-1} \Lambda^k$ stands for the BDF-$k$ finite-difference
operator which is defined by (cf. [24])

$$\Lambda^k u_h^n = \sum_{i=0}^{k} \lambda^k_i u_h^{n-i,n}.$$  \hfill (4.6)

The coefficients $\lambda^k_i$ for $k = 1, \cdots, 4$ are listed in Table 4.1.

Table 4.1: Coefficients of the BDF-$k$ scheme (cf. e.g. [24]).

| $k$ | $\lambda^1_i$ | $\lambda^2_i$ | $\lambda^3_i$ | $\lambda^4_i$ |
|-----|---------------|----------------|---------------|---------------|
| 1   | 1             | -1             | 0             | 0             |
| 2   | 3/2           | -2             | 1/2           | 0             |
| 3   | 11/6          | -3             | 3/2           | -1/3          |
| 4   | 25/12         | -4             | 3             | -4/3          | 1/4          |

Throughout the paper, we extend each $v_h \in V(k,T^n)$ to the exterior of $\tilde{\Omega}^n$ such that the extension, denoted still by $v_h$, belongs to $V(k,T_h)$ and vanishes at all degrees of freedom outside of $\tilde{\Gamma} \cup \tilde{\Omega}^n$. It is easy to verify that

$$\|v_h\|_{L^2(D)} \lesssim \|v_h\|_{L^2(\tilde{\Omega}^n)}, \quad \|v_h\|_{H^1(D)} \lesssim \|v_h\|_{H^1(\tilde{\Omega}^n)}.$$  \hfill (4.7)

5. The well-posedness of the discrete problem. First- and second-order methods have been studied extensively in the literature. To study higher-order methods ($k \geq 3$), first we make a mild assumption on the finite element mesh.

(A2) There exist an integer $I > 0$ and a constant $\gamma > 0$ independent of $h$ and $\tau$ such that, for any $K \in T^n_{h,B}$, one can find at most $I$ elements $\{K_j\}_{j=1}^I \subset T^n_h$ satisfying that $K_1 = K$, $K_j \cap K_{j+1}$ is an interior edge of $T^n_h$, and that $K_1 \cap \Omega^n_\eta$ contains a disk of radius $\gamma h$ (see Fig. 5.1).

We remark that the assumption does not require how boundary intersects $T_h$ and is less restrictive than generally used in the literature [13,14,25,26]. Since the domain is time-varying, too strong assumptions are inappropriate. In fact, (A2) can be satisfied if $T_h$ is fine and the deformation of the domain is moderate.

5.1. Trace inequalities and norm equivalence. Now we give some useful results on trace inequalities and norm equivalence. Similar results can be found in [13,14,26] under various stronger
assumptions.

**Lemma 5.1.** Suppose that \( \Omega^n_h \) is a Lipschitz domain. For any \( K \in T_h \), we have
\[
\|v\|_{L^2(\partial K)} + \|v\|_{L^2(K \cap \Gamma^n_h)} \lesssim h^{-1/2} \|v\|_{L^2(K)} + h^{1/2} \|v\|_{H^1(K)} \quad \forall v \in H^1(K).
\] (5.1)

The upper bound for \( \|v\|_{L^2(\partial K)} \) is a standard result. The proof for the upper bound for \( \|v\|_{L^2(K \cap \Gamma^n_h)} \) can be found in [11, Lemma 1 and Appendix A]. To study the well-posedness of the discrete problems, we define the mesh-dependent norms
\[
\|v\|_{\Omega^n_h} = \left( |v|^2_{H^1(\Omega^n_h)} + h^{-1} \|v\|_{L^2(\Gamma^n_h)}^2 + h \|\nabla v\|^2_{L^2(\Gamma^n_h)} \right)^{1/2},
\]
\[
\|v\|_{T^n_h} = \left( |v|^2_{H^1(\Omega^n_h)} + \mathcal{F}_0^n(v, v) + \mathcal{F}_1^n(v, v) \right)^{1/2},
\]
\[
\|v\|_{*,\tilde{\Omega}^n} = \left( |v|^2_{H^1(\tilde{\Omega}^n)} + h^{-1} \|v\|_{L^2(\tilde{\Omega}^n)}^2 \right)^{1/2}.
\]
Clearly \( \|\cdot\|_{T^n_h} \) is a norm on \( H^1(\Omega^n_h) \cap H^{k+1}(\mathcal{T}_h) \).

**Lemma 5.2.** Let assumption (A2) be satisfied. Then for any \( v_h \in V(k, T^n_h) \),
\[
\|v_h\|_{L^2(\tilde{\Omega}^n)} \lesssim \|v_h\|_{L^2(\Omega^n_h)} + h^2 \mathcal{F}_1^n(v_h, v_h),
\] (5.2)
\[
\|v_h\|_{\Omega^n_h} \lesssim \|v_h\|_{T^n_h} \simeq \|v_h\|_{*,\tilde{\Omega}^n}.
\] (5.3)

**Proof.** For each \( K \in T^n_hB \), by assumption (A2), there exist (at most) \( l \) elements \( K_1 = K, K_2, \cdots, K_l \) such that \( E_j = K_j \cap K_{j+1} \) is an interior edge of \( T^n_h \) and \( K_f \cap \Omega^n_h \) contains a disk of radius \( \gamma h \). From [26, Lemma 5.1], we have
\[
\|\nabla^\mu v_h\|_{L^2(K_f)} \lesssim \|\nabla^\mu v_h\|_{L^2(K_{f+1})} + \sum_{l=1}^k h^{2(l-\mu)+1} \int_{E_j} \left[ \partial_n^l v_h \right]^2, \quad j = 1, \cdots, I - 1, \quad \mu = 0, 1.
\]
Since \( K_f \cap \Omega^n_h \) contains a disk of radius \( \gamma h \), the norm equivalence shows
\[
\|\nabla^\mu v_h\|_{L^2(K_f)} \lesssim \|\nabla^\mu v_h\|_{L^2(K_{f+1})} + \sum_{j=1}^{I-1} \sum_{l=1}^k h^{2(l-\mu)+1} \int_{E_j} \left[ \partial_n^l v_h \right]^2
\]
\[
\lesssim \|\nabla^\mu v_h\|_{L^2(K_f \cap \Omega^n_h)} + \sum_{j=1}^{I-1} \sum_{l=1}^k h^{2(l-\mu)+1} \int_{E_j} \left[ \partial_n^l v_h \right]^2, \quad \mu = 0, 1.
\] (5.4)

Now we sum up (5.4) for all \( K \in T^n_hB \). Letting \( \mu = 0 \) yields (5.2) and letting \( \mu = 1 \) yields
\[
\sum_{K \in T^n_hB} |v_h|^2_{H^1(K)} \lesssim |v_h|^2_{H^1(\Omega^n_h)} + \mathcal{F}_1^n(v_h, v_h).
\] (5.5)

This shows \( \|v_h\|_{*,\tilde{\Omega}^n} \lesssim \|v_h\|_{T^n_h} \).
For any $E \in \mathcal{E}_{h,B}^n$ and $1 \leq l \leq k$, the norm equivalence and the inverse estimate show
\[
h^{2l-1} \int_E \left[ \partial_{n,l} v_h \right]^2 \lesssim h^{2l} \left\| \nabla^l v_h \right\|_{L^\infty(K_E)} \lesssim h^{2l-2} \left\| \nabla^l v_h \right\|_{L^2(K_E)} \lesssim v_h \left\| \nabla v_h \right\|_{L^2(K_E)},
\]
where $K_E \in \mathcal{T}_{h,B}^n$ satisfies $E \subset \partial K_E$. This shows $\mathcal{J}_n(v_h, v_h) \lesssim \|v_h\|_{H^1(\Omega^n)}^2$. Then we have $\|v_h\|_{\mathcal{T}_{h}^n} \approx \|v_h\|_{\mathcal{D}_{h}^{\gamma}}$. The proof for $\|v_h\|_{\mathcal{T}_{h}^n} \approx \|v_h\|_{\mathcal{D}_{h}^{\gamma}}$ is similar and omitted here. \(\square\)

5.2. The well-posedness of the discrete problem. First we prove the coercivity and continuity of the discrete bilinear form $\mathcal{A}_h^n$.

**Lemma 5.3.** Suppose $\gamma_0$ is large enough and $\gamma_1 > 0$. Then for any $u_h, v_h \in V(k, T_{h}^n)$,
\[
\mathcal{A}_h^n(v_h, v_h) \gtrsim \|v_h\|_{\mathcal{T}_{h}^n}^2, \quad \mathcal{A}_h^n(u_h, v_h) \lesssim \|u_h\|_{\mathcal{T}_{h}^n} \|v_h\|_{\mathcal{T}_{h}^n}.
\]

**Proof.** The definition of $\mathcal{A}_h^n$ shows
\[
\mathcal{A}_h^n(v_h, v_h) = \|v_h\|_{H^1(\Omega^n)}^2 + \mathcal{J}_n(v_h, v_h) + \mathcal{J}_0^n(v_h, v_h) + \mathcal{J}_1^n(v_h, v_h). \tag{5.6}
\]
For any $\varepsilon > 0$, the Cauchy-Schwarz inequality shows
\[
|\mathcal{J}_n(v_h, v_h)| \lesssim \varepsilon h \sum_{K \in \mathcal{T}_{h}^n} \|\partial_{n,l} v_h\|_{L^2(\Gamma_K)}^2 + \frac{1}{\varepsilon h} \sum_{K \in \mathcal{T}_{h,B}^n} \|v_h\|_{L^2(\Gamma_K)}^2. \tag{5.7}
\]
From Lemma 5.1, the inverse inequality and (5.5), we know that
\[
h \sum_{K \in \mathcal{T}_{h,B}^n} \|\partial_{n,l} v_h\|_{L^2(\Gamma_K)}^2 \leq C \sum_{K \in \mathcal{T}_{h,B}^n} |v_h|_{H^1(K)}^2 \leq C_0 \left[|v_h|_{H^1(\Omega^n_H)}^2 + \mathcal{J}_1^n(v_h, v_h)\right]. \tag{5.8}
\]
Taking $\varepsilon = 1/(2C_0)$ in (5.7) and inserting the result into (5.6), we get
\[
\mathcal{A}_h^n(v_h, v_h) \geq \frac{1}{2} |v_h|_{H^1(\Omega^n_H)}^2 + \left(\gamma_0 - \varepsilon^{-1}\right) \mathcal{J}_0^n(v_h, v_h) + \frac{1}{2} \mathcal{J}_1^n(v_h, v_h).
\]
The coercivity of $\mathcal{A}_h^n$ is obtained by setting $\gamma_0 \geq 2\varepsilon^{-1} = 4C_0$.

Similarly, the Cauchy-Schwarz inequality shows
\[
|\mathcal{A}_h^n(u_h, v_h)| \leq \|u_h\|_{\mathcal{T}_{h}^n} \|v_h\|_{\mathcal{T}_{h}^n} + \gamma_0^{-1/2} \left(h \sum_{K \in \mathcal{T}_{h,B}^n} \|\partial_{n,l} u_h\|_{L^2(\Gamma_K)}^2 \right)^{1/2} \frac{1}{\gamma_0} \mathcal{J}_0^n(u_h, v_h)^{1/2} + \gamma_0^{-1/2} \left(h \sum_{K \in \mathcal{T}_{h,B}^n} \|\partial_{n,l} v_h\|_{L^2(\Gamma_K)}^2 \right)^{1/2} \frac{1}{\gamma_0} \mathcal{J}_0^n(u_h, u_h)^{1/2}.
\]
Using (5.8), we obtain the continuity of $\mathcal{A}_h^n$. \(\square\)

Since (4.5) is a linear problem for given $\{U_{h}^{n-i,n} : i = 1, \ldots, k\}$, by Lemma 5.3 and the Lax-Milgram lemma, problem (4.5) has a unique solution $u_h^n$ in each time step.
5.3. The modified Ritz projection. Since the computational domain is varying, in general, we have to deal with the problem that \( u_h^{n-j} \notin V(k, T_n^h) \) for \( 1 \leq j \leq k \) when proving the stability and convergence of discrete solutions. To overcome this difficulty, we define a modified Ritz projection operator \( \mathcal{P}_h^n : Y(\Omega_h^n) \to V(k, T_n^h) \) as follows

\[
\mathcal{A}_h^n(\mathcal{P}_h^n w, v_h) = a_h^n(w, v_h) \quad \forall v_h \in V(k, T_n^h),
\]

where \( a_h^n(w, v) := \mathcal{A}_h^n(w, v) - \mathcal{J}_1^n(w, v) \) and \( Y(\Omega_h^n) := \{ v \in H^1(\Omega_h^n) : \| v \|_{\Omega_h^n} < \infty \} \).

**Lemma 5.4.** For any \( v \in Y(\Omega_h^n) \), there holds

\[
\| \mathcal{P}_h^n w \|_{\Omega_h^n} + \| \mathcal{P}_h^n w \|_{T_n^h} \lesssim \| w \|_{\Omega_h^n} \quad \forall w \in Y(\Omega_h^n).
\]

**Proof.** Taking \( v_h = \mathcal{P}_h^n w \) in \((5.9)\)

and using Lemma 5.3 we find that

\[
\| \mathcal{P}_h^n w \|_{T_n^h} \lesssim \mathcal{A}_h^n(\mathcal{P}_h^n w, \mathcal{P}_h^n w) = a_h^n(w, \mathcal{P}_h^n w) \lesssim \| w \|_{\Omega_h^n} \| \mathcal{P}_h^n w \|_{\Omega_h^n}.
\]

Together with Lemma 5.2 this shows \( \| \mathcal{P}_h^n w \|_{T_n^h} \lesssim \| w \|_{\Omega_h^n} \). The estimate for \( \| \mathcal{P}_h^n w \|_{\Omega_h^n} \) is easily obtained by using Lemma 5.2 again.

To estimate the error \( w - \mathcal{P}_h^n w \), we shall use the the Scott-Zhang interpolation operator \( \mathcal{I}_h \): \( H^1(D) \to V(k, T_h) \) (cf. [30]). For any element \( K \in T_h \) and any edge \( E \in E_h \),

\[
\| \mathcal{I}_h v - v \|_{H^1(K)} \lesssim h^{k-l}|v|_{H^{k+1}(D_K)}, \quad 0 \leq l \leq k,
\]

\[
\| \mathcal{I}_h v - v \|_{H^1(E)} \lesssim h^{k-l+1/2}|v|_{H^{k+1}(D_E)},
\]

where \( D_A \) is the union of all elements having non-empty intersection with \( A = K \) or \( A = E \).

**Lemma 5.5.** For any \( w \in H^{k+1}(D) \), there holds

\[
\| w - \mathcal{I}_h w \|_{\Omega_h^n} + \| w - \mathcal{I}_h w \|_{T_h^n} \lesssim h^k|w|_{H^{k+1}(D)}.
\]

**Proof.** Write \( w_h = \mathcal{I}_h w \) for convenience. From \((5.10)\)–\((5.11)\), we have

\[
\mathcal{J}_1(w - w_h, w - w_h) \lesssim h^k \sum_{E \in E_h^n} |w|_{H^{k+1}(D_E)}^2 \lesssim h^k |w|_{H^{k+1}(\Omega_h^n)}^2.
\]

For any \( K \in T_h^n \) and \( \Gamma_K = \Gamma_h \cap K \), from Lemma 5.1 and inequality \((5.10)\), we deduce that

\[
\| w - w_h \|_{L^2(\Gamma_K)} \lesssim h^{-1/2} \| w - w_h \|_{L^2(K)} + h^{1/2} |w - w_h|_{H^1(K)} \lesssim h^{k-1/2} |w|_{H^{k+1}(D_K)};
\]

\[
\| \nabla_h (w - w_h) \|_{L^2(\Gamma_K)} \lesssim h^{-1/2} |w - w_h|_{H^1(K)} + h^{1/2} |w - w_h|_{H^2(K)} \lesssim h^{k-1/2} |w|_{H^{k+1}(D_K)}.
\]

This shows

\[
\sum_{K \in T_h^n} \left( h^{-1} \| w - w_h \|_{L^2(\Gamma_K)}^2 + h \| \nabla_h (w - w_h) \|_{L^2(\Gamma_K)}^2 \right) \lesssim h^k |w|_{H^{k+1}(D)}^2.
\]

The proof is finished by inserting \((5.10)\), \((5.13)\), \((5.14)\) into the definitions of \( \| \cdot \|_{T_h^n} \) and \( \| \cdot \|_{\Omega_h^n} \).

**Lemma 5.6.** For any \( w \in H^{k+1}(D) \), there holds

\[
\| w - \mathcal{P}_h^n w \|_{\Omega_h^n} + \| w - \mathcal{P}_h^n w \|_{T_h^n} \lesssim h^k |w|_{H^{k+1}(D)}.
\]
Proof. By the definition of $P_h^n$ and the fact $J_1^n(w, P_h^n w - I_h w) = 0$, we have
\[ \|P_h^n w - I_h w\|_{T_h}^2 \lesssim s_h^n (P_h^n w - I_h w, P_h^n w - I_h w) = s_h^n (w - I_h w, P_h^n w - I_h w). \]
Then Lemma 5.3 and (5.3) imply $\|P_h^n w - I_h w\|_{\Omega_h^n} \lesssim \|P_h^n w - I_h w\|_{T_h} \lesssim \|w - I_h w\|_{T_h}$. The proof is finished by using the triangle inequality and Lemma 5.5.

Lemma 5.7. Let assumptions (A2) be satisfied. Then
\[ \|w - P_h^n w\|_{L^2(\Omega_h^n)} \lesssim h \|w\|_{\Omega_h^n} \quad \forall w \in Y(\Omega_h^n), \]  \[ \|w - P_h^n w\|_{L^2(\Omega_h^n)} \lesssim h^{k+1} |w|_{H^{k+1}(\Omega_h^n)} \quad \forall w \in H^{k+1}(\Omega_h^n). \]

Proof. We will prove the lemma by the duality technique. Consider the auxiliary problem
\[ -\Delta z = w - P_h^n w \quad \text{in } \Omega_h^n, \quad z = 0 \quad \text{on } \Gamma_h^n. \]  \[ \text{(5.17)} \]
From Lemma 3.3 and Theorem 3.5, we know that $\Gamma_h^n$ is $C^2$-smooth and its parametrization satisfies $\|\chi_n\|_{C^2([0,1], I)} \approx 1$. The regularity result for elliptic equations indicates that
\[ \|z\|_{H^2(\Omega_h^n)} \leq C \|w - P_h^n w\|_{L^2(\Omega_h^n)}. \]  \[ \text{(5.18)} \]
Multiplying both sides of the elliptic equation with $w - P_h^n w$ and integrating by parts, we have
\[ \|w - P_h^n w\|_{L^2(\Omega_h^n)}^2 = \int_{\Omega_h^n} \nabla z \cdot \nabla (w - P_h^n w) - \int_{\Gamma_h^n} \frac{\partial z}{\partial n} (w - P_h^n w) = a_h(w - P_h^n w, w). \]  \[ \text{(5.19)} \]
Let $\tilde{z} \in H^2(D)$ be the Sobolev extension of $z$ to the exterior of $\Omega_h^n$. There exists a constant $C$ depending only on $\Omega_h^n$ such that
\[ \|\tilde{z}\|_{H^2(D)} \leq C \|z\|_{H^2(\Omega_h^n)} \leq C \|w - P_h^n w\|_{L^2(\Omega_h^n)}. \]
Let $\tilde{z}_h \in V(1, T_h)$ be the Scott-Zhang interpolation of $\tilde{z}$. The arguments similar to Lemma 5.5 show
\[ \|\tilde{z} - \tilde{z}_h\|_{\Omega_h^n}^2 + \sum_{E \in E_h} \frac{h}{E} \|\nabla (\tilde{z} - \tilde{z}_h)\|_{E_h}^2 \lesssim h^2 \|\tilde{z}\|_{H^2(D)}^2 \lesssim h^2 \|w - P_h^n w\|_{L^2(\Omega_h^n)}^2. \]  \[ \text{(5.20)} \]
Inserting $a_h(w - P_h^n w, \tilde{z}_h)$ into (5.19) leads to
\[ \|w - P_h^n w\|_{L^2(\Omega_h^n)}^2 = a_h(w - P_h^n w, \tilde{z} - \tilde{z}_h) + J_1^n(P_h^n w, \tilde{z}_h). \]  \[ \text{(5.21)} \]
By (5.20), the first term on the right-hand side is estimated as follows
\[ a_h(w - P_h^n w, \tilde{z} - \tilde{z}_h) \lesssim \|w - P_h^n w\|_{\Omega_h^n} \|\tilde{z} - \tilde{z}_h\|_{\Omega_h^n} \lesssim h \|w - P_h^n w\|_{\Omega_h^n} \|w - P_h^n w\|_{L^2(\Omega_h^n)}. \]  \[ \text{(5.22)} \]
Note that each edge $E \in \mathcal{E}_h$ is parallel either to the $x$-axis or the $y$-axis so that $\hat{z}_h|_E$ is a linear function of either $x$ or $y$. Using (5.20) and Lemma 5.4, we deduce that

$$\left| \mathcal{J}_1^h(P_h^n w, \hat{z}_h) \right| = \sum_{E \in \mathcal{E}_h} h \int_E \left[ \partial_n(P_h^n w) \partial_n \hat{z}_h \right] = \sum_{E \in \mathcal{E}_h} h \int_E \left[ \partial_n(P_h^n w) \partial_n (\hat{z}_h - \bar{z}) \right] \leq \left( \sum_{E \in \mathcal{E}_h} h \left\| \partial_n(P_h^n w) \right\|_{L^2(E)}^2 \right)^{1/2} \left( \sum_{E \in \mathcal{E}_h} h \left\| \partial_n(\hat{z}_h - \bar{z}) \right\|_{L^2(E)}^2 \right)^{1/2} \lesssim h \left\| P_h^n w \right\|_{T_{\infty}} \left\| w - P_h^n w \right\|_{L^2(\Omega^n)} \lesssim h \left\| w \right\|_{\Omega^n} \left\| w - P_h^n w \right\|_{L^2(\Omega^n)}. \tag{5.23}$$

Inserting (5.22) and (5.23) into (5.21) and using Lemma 5.4, we get (5.15).

Next let $\tilde{w} \in H^{k+1}(D)$ be the Sobolev extension of $w \in H^{k+1}(\Omega^n)$. Since $\left\| \partial_n \tilde{w} \right\| = 0$ on any $E \in \mathcal{E}_h$, using (5.21), (5.20), and Lemma 5.6, we find that

$$\left\| w - P_h^n w \right\|_{L^2(\Omega^n)} \leq \left\| w - P_h^n w \right\|_{\Omega^n} \left\| \hat{z} - \bar{z} \right\|_{\Omega^n} + \sum_{E \in \mathcal{E}_h} h \int_E \left[ \partial_n(\tilde{w} - P_h^n \tilde{w}) \partial_n (\hat{z} - \bar{z}) \right] \lesssim h^{k+1} \left\| \tilde{w} \right\|_{H^{k+1}(D)} \left\| w - P_h^n w \right\|_{L^2(\Omega^n)}.$$ This finishes the proof. \(\square\)

**5.4. The stability of the discrete solutions.** First we cite the telescope formulas of BDF schemes from [24 Section 2 and Appendix A].

**Lemma 5.8.** Let $1 \leq k \leq 4$ and $\alpha = \delta_{k,3} + \delta_{k,4}$ where $\delta_{ij}$ is the Kronecker delta. Then

$$\left( \Lambda^k U_h^n \right) \left( u_h^n + \alpha \Lambda^k U_h^n \right) = \sum_{i=1}^{k+1} \left( \Phi_i^k(U_h^n) \right)^2 - \sum_{i=1}^{k} \left( \Phi_i^k(U_h^n) \right)^2,$$

where $\Phi_i^k(U_h^n) = \sum_{j=1}^i c_{i,j} U_{h}^{n+1-j,n}$, $\Phi_i^k(U_h^n) = \sum_{j=1}^i c_{i,j} U_{h}^{n-j,n}$, and the parameters $c_{i,j}$, $1 \leq j \leq i \leq k+1$, are given in [24 Table 2.2].

**Theorem 5.9.** Suppose $2 \leq k \leq 4$, $O(\eta^{4/k}) = O(h) = \tau \leq h$, and that the penalty parameter $\gamma_0$ in $\mathcal{A}_h^n$ is large enough. Let $u_h^n$, $n \geq k$, be the solution to the discrete problem (4.5) based on the pre-calculated initial values $\{u_h^0, \ldots, u_h^{k-1}\}$. There is an $h_0 > 0$ small enough such that, for any $h \in (0, h_0)$ and $m \geq k$,

$$\left\| u_h^n \right\|_{L^2(\Omega^n)}^2 + \sum_{n=k}^m \tau \left\| u_h^n \right\|_{T_n}^2 \lesssim \sum_{n=k}^m \tau \left\| f^n \right\|_{L^2(\Omega^n)}^2 + \sum_{i=0}^{k-1} \left( \left\| u_h^i \right\|_{L^2(\Omega^n)}^2 + \tau \left\| u_h^i \right\|_{H^1(\bar{\Omega})}^2 \right). \tag{5.24}$$

**Proof.** Without loss of generality, we fix the $k$ in (5.24) to be 4 in the following discussion. The proof for other cases are similar. For convenience, we write $\tilde{U}^{n-i,n}_h = P_h^n(U_h^{n-i,n}) \in V(k, T_h^n)$ for $i > 1$. The discrete problem for $k = 4$ has the form

$$(\Lambda^4 U_h^n, v_h)_{\Omega^n} + \tau \mathcal{A}_h^n(u_h^n, v_h) = \tau (f^n, v_h)_{\Omega^n} \quad \forall v_h \in V(k, T_h^n). \tag{5.25}$$

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Since $U_{n-1,n}^n \notin V(k, T_k)$, we choose $v_h = 2u_h^n - \tilde{U}_{n-1,n}^n$ as a test function in (5.25). From equation (4.6), we know that $A^1 U_{h,n}^n = u_h^n - U_{n-1,n}^n$. An application of Lemma 5.8 shows

$$\sum_{i=1}^{5} \left\| \Psi_i(U_{h,n}^n) \right\|^2_{L^2(\Omega_{\gamma}^n)} - \sum_{i=1}^{4} \left\| \Phi_i(U_{h,n}^n) \right\|^2_{L^2(\Omega_{\gamma}^n)} + \tau \varphi_h^n(u_h^n, 2u_h^n - \tilde{U}_{n-1,n}^n) = \tau A_1^n + A_2^n,$$  \hspace{1cm} (5.26)

where $A_1^n = (f^n, 2u_h^n - \tilde{U}_{n-1,n}^n)_{\Omega_{\gamma}^n}$ and $A_2^n = (A^1 U_{h,n}^n, \tilde{U}_{n-1,n}^n - U_{n-1,n}^n)_{\Omega_{\gamma}^n}$. Here we take it for granted that $u_h^{n-j} \in V(k, T_k)$ as in (4.7).

Note that $\Phi_i^1(U_{h,n}^n) = \Psi_i^1(U_{h,n}^n) \circ X^n_{n-1}$. By Lemmas 3.1 and A.2 we find that

$$\left\| \Phi_i^1(U_{h,n}^n) \right\|^2_{L^2(\Omega_{\gamma}^n)} \leq (1 + C \tau) \left\| \Phi_i^1(U_{h,n}^n) \right\|^2_{L^2(\Omega_{\gamma}^n)} + C \tau \sum_{j=1}^i \left\| u_h^{n-j} \right\|^2_{L^2(\Omega_{n-1}^n)}.$$  \hspace{1cm} (5.27)

From (A.4) and (4.7) we have $\left\| U_{h,n}^{n-1} \right\|_{H^1(\Gamma_{\gamma}^n)} \leq h^{-1/2} \left\| u_h^{n-1} \right\|_{H^1(\Gamma_{n-1})}$, and from (5.1) we have $\left\| \partial_n u_h^n \right\|_{L^2(\Gamma_{\gamma}^n)} \leq h^{-1/2} \left\| u_h^{n-1} \right\|_{H^1(\Gamma_{n-1})}$. For any $\varepsilon \in (0, 1)$, by the definitions of $\mathcal{P}_h^n$ and the Cauchy-Schwarz inequality, we deduce that

$$\varphi_h^n(u_h^n, 2u_h^n - \tilde{U}_{n-1,n}^n) = 2 \varphi_h^n(u_h^n, u_h^n) + \varphi_h^n(u_h^n, U_{n-1,n}^n)$$

$$\geq \frac{3}{2} \left\| u_h^n \right\|^2_{H^1(\Omega_{\gamma}^n)} - \frac{5}{2\varepsilon h} \left\| u_h^n \right\|^2_{L^2(\Gamma_{\gamma}^n)} - C \varepsilon \left\| u_h^n \right\|^2_{H^1(\Omega_{\gamma}^n)} - \frac{1}{2} \left\| U_{n-1,n}^n \right\|^2_{H^1(\Omega_{\gamma}^n)}$$

$$\geq \frac{3}{2} \left\| u_h^n \right\|^2_{H^1(\Omega_{\gamma}^n)} - \frac{5}{2\varepsilon h} \left\| u_h^n \right\|^2_{L^2(\Gamma_{\gamma}^n)} - \frac{1}{2} \left\| U_{n-1,n}^n \right\|^2_{H^1(\Omega_{\gamma}^n)},$$  \hspace{1cm} (5.28)

Since $U_{n-1,n}^n \in Y(\Omega_{\gamma}^n)$, we infer from (5.15) and the Cauchy-Schwarz inequality that

$$A_1^n \leq 2 \left\| f^n \right\|^2_{L^2(\Omega_{\gamma}^n)} + \left\| u_h^n \right\|^2_{L^2(\Omega_{\gamma}^n)} + 2 \left\| U_{h,n}^{n-1} \right\|^2_{L^2(\Omega_{\gamma}^n)} + C \tau^2 \left\| U_{h,n}^{n-1} \right\|^2_{\Omega_{\gamma}^n},$$  \hspace{1cm} (5.29)

$$A_2^n \leq \frac{C}{\varepsilon} \sum_{j=0}^{4} \left\| U_{h,n}^{n-j,n} \right\|^2_{L^2(\Omega_{\gamma}^n)} + \varepsilon h \left\| U_{h,n}^{n-1,n} \right\|^2_{\Omega_{\gamma}^n}.$$  \hspace{1cm} (5.30)

Substituting (5.27)–(5.30) into (5.26) and using $O(h) = \tau \leq h \ll \varepsilon \ll 1$ lead to

$$\sum_{i=1}^{4} \left\| \Psi_i^1(U_{h,n}^n) \right\|^2_{L^2(\Omega_{\gamma}^n)} - \sum_{i=1}^{4} \left\| \Phi_i^1(U_{h,n}^n) \right\|^2_{L^2(\Omega_{\gamma}^n)} + \frac{3}{2} \tau \left\| u_h^n \right\|^2_{\hat{T}_h}$$

$$\leq 2 \left\| f^n \right\|^2_{L^2(\Omega_{\gamma}^n)} + C \tau \sum_{j=0}^{4} \left\| u_{h,n}^{n-j} \right\|^2_{L^2(\Omega_{n-1}^n)} + \frac{5\tau}{2\varepsilon h} \left\| u_h^n \right\|^2_{L^2(\Gamma_{\gamma}^n)} + C \varepsilon \tau \left( \left\| u_{h,n}^2 \right\|_{H^1(\Omega_{\gamma}^n)} + \left\| u_{h,n}^{n-1} \right\|^2_{H^1(\Omega_{n-1}^n)} \right)$$

$$+ \frac{1 + C \varepsilon}{2} \tau \left\| U_{h,n}^{n-1,n} \right\|^2_{H^1(\Omega_{\gamma}^n)} + \frac{\gamma_0 + 3\varepsilon^{-1}\tau}{2h} \left\| U_{h,n}^{n-1,n} \right\|^2_{L^2(\Omega_{\gamma}^n)} + C \tau \sum_{j=0}^{4} \left\| U_{h,n}^{n-j,n} \right\|^2_{L^2(\Omega_{\gamma}^n)}.$$

(5.31)
By Lemma 3.3 and arguments similar to [19], we have the following lemma. The proof is omitted.

Substitute the estimates into (5.31), apply Lemma 5.2 and take the sum of the inequalities over 4 \leq n \leq m. After proper arrangements and combinations, we end up with

\[
\sum_{i=1}^{4} \left\| \Psi_i(U_h^{m}) \right\|_{L^2(\Omega_1^m)}^2 + \frac{3}{2} \sum_{n=4}^{m} \left\| u_h^n \right\|_{H^1(\Omega_1^m)}^2 \leq \sum_{n=4}^{m} \tau \left( C \left\| u_h^n \right\|_{L^2(\Omega_1^m)}^2 + \frac{1 + C \varepsilon}{2} \frac{1 + 10/\varepsilon_0}{\tau} \right) \left( f^n(u_h^n, u_h^n) \right) + \sum_{n=4}^{m} \left( \left\| \psi_4(U_h^{m}) \right\|_{L^2(\Omega_1^m)}^2 \right) ^2. \tag{5.32}
\]

From [24 Table 2.2], we know \( \psi_4(U_h^{m}) = c_4 m u_h^n = 0.06 u_h^n \). Finally, we choose \( \varepsilon \) small enough and \( \gamma_0 \) large enough such that \( C \varepsilon + 10/\varepsilon_0 \leq 1 \). Then the proof is finished by using Lemma 5.2 and Gronwall’s inequality. \( \square \)

**6. A priori error estimates.** The purpose of this section is to prove the error estimates between the exact solution and the finite element solution. For convenience, we define \( Q_T = \{(x, t): x \in \Omega_t, t \in [0, T]\} \), and

\[
L^\infty(0, T; H^m(\Omega_t)) = \{ v \in L^2(Q_T) : \text{esssup}_{t \in [0, T]} \left\| v(t, \cdot) \right\|_{H^m(\Omega_t)} < +\infty \}, \quad m \geq 0.
\]

Since \( \overline{\Omega_0} \setminus \Omega_t \neq \emptyset \) in general, we follow Lehrenfeld and Olshanskii [19] to extend the solution \( u \) to the exterior of \( \Omega_t \). By [31 Chapter 6], there is an extension operator \( E_0: H^{k+1}(\Omega_0) \rightarrow H^{k+1}(\mathbb{R}^2) \) such that

\[
(E_0w)|_{\Omega_0} = w, \quad \|E_0w\|_{H^{k+1}(\mathbb{R}^2)} \lesssim \|w\|_{H^{k+1}(\Omega_0)} \quad \forall w \in H^{k+1}(\Omega_0).
\]

Since \( X(t; s, \cdot) \) is one-to-one, its inverse is denoted by \( X(s; t, \cdot) \). Then \( \Omega_0 = X(0; t, \Omega_t) \). We can define an extension operator from \( H^{k+1}(\Omega_t) \) to \( H^{k+1}(\mathbb{R}^2) \) by \( E_v := [E_0 \circ X(t; 0, \cdot)] \circ X(0; t, \cdot) \).

The global extension operator \( E: L^\infty(0, T; H^{k+1}(\Omega_t)) \rightarrow L^\infty(0, T; H^{k+1}(\mathbb{R}^2)) \) is defined by

\[
(Ev)(\cdot, t) = E_v(\cdot, t) \quad \forall t \in [0, T].
\]

By Lemma 5.3 and arguments similar to [19], we have the following lemma. The proof is omitted.

**Lemma 6.1.** There is a constant \( C > 0 \) depending only on \( \Omega_0 \) and \( \|w\|_{W^{\infty}([0, T])} \) such that, for any \( v \in L^\infty(0, T; H^{k+1}(\Omega_t)) \cap H^{k+1}(Q_T) \),

\[
\|Ev\|_{H^{k+1}(\mathbb{R}^2)} \leq C \|v\|_{H^{k+1}(\Omega_t)}, \quad 1 \leq m \leq k + 1,
\]

\[
\|Ev\|_{H^{k+1}(\mathbb{R}^2)} \leq C \|v\|_{H^{k+1}(\Omega_t)}, \quad 1 \leq m \leq k + 1.
\]

Furthermore, for \( v \in L^\infty(0, T; H^{m}(\Omega_t)) \) satisfying \( \partial_t v \in L^\infty(0, T; H^{m-1}(\Omega_t)) \), it holds

\[
\|\partial_t (Ev)\|_{H^{m-1}(\mathbb{R}^2)} \leq C \left( \|v\|_{H^{m}(\Omega_t)} + \|\partial_t v\|_{H^{m-1}(\Omega_t)} \right), \quad 1 \leq m \leq k + 1.
\]
Let \( u \) be the exact solution to (2.1). For convenience, we abuse the notation and denote the extension \( E u \) still by \( u \) throughout this section. Write \( u^n := u(t_n) \), \( U^{m,n} := u^m \circ X^{n,m} \), and \( U^n := [U^{n-k,n}, \ldots, U^{n,n}]^\top \). By (2.5), \( u^n \) satisfies the semi-discrete equation

\[
\frac{1}{\tau} \Lambda^k U^n - \Delta u^n = \bar{f}^n + R^n \quad \text{in } \Omega_n^k,
\]

where \( \bar{f}^n = \frac{du}{dt}(t_n) - \Delta u^n \) and \( R^n = \frac{1}{\tau} \Lambda^k U^n - \frac{du}{dt}(t_n) \). Multiplying both sides of the equation by \( v_h \in V(k, T_h) \) and using integration by parts, we obtain

\[
\frac{1}{\tau} \Lambda^k U^n, v_h)_{\Omega_n^k} + a_h^n(u^n, v_h) = (\bar{f}^n + R^n, v_h)_{\Omega_n^k} + \int_{\Gamma_n^k} u^n \left( \frac{\gamma_0}{h} v_h - \partial_n v_h \right). \tag{6.1}
\]

Now we present the main theorem of this section.

**Theorem 6.2.** Let the assumptions in Theorem 5.9 be satisfied. Suppose that the exact solution satisfies \( u \in L^\infty(0, T; H^{k+1}(\Omega_t)) \cap H^k(Q_T) \) and \( \partial_t u \in L^\infty(0, T; H^2(\Omega_t)) \). Moreover, suppose the pre-calculated initial solutions satisfy

\[
\|u^i - u_h^i\|_{L^2(\Omega_n^i)} + \tau \|u^i - u_h^i\|_{\Omega_n^i}^2 \lesssim \tau^{2k}, \quad i = 0, 1, \ldots, k - 1. \tag{6.2}
\]

Then for any \( k \leq m \leq N \),

\[
\|u^m - u_h^m\|_{L^2(\Omega_n^m)} + \left( \sum_{n=k}^m \tau \|u^n - u_h^n\|_{\Omega_n^m}^2 \right)^{1/2} \lesssim \tau^{k-1/2}.
\]

**Proof.** We only prove the theorem for \( k = 4 \). The proofs for other cases are similar. Write \( \rho^n := u^n - P_h^n u^n \) and \( \theta_h^n := P_h^n u^n - u_h^n \). By Lemmas 5.6, 5.7, it suffices to estimate \( \theta_h^n \).

Define \( \Theta_h^n := [\Theta_h^{n-k,n}, \ldots, \Theta_h^{n,n}]^\top \) and \( \zeta^n := [\zeta^{n-k,n}, \ldots, \zeta^{n,n}]^\top \) where \( \Theta_h^{m,n} = \theta_h^m \circ X_{\tau}^{n,m} \) and \( \zeta^{m,n} = (u^m \circ X_{\tau}^{n,m} - u^m \circ X^{n,m}) - \rho^n \circ X^{n,m} \). Subtracting (4.5) from (6.1) and using (5.9), we find that

\[
\frac{1}{\tau} (\Lambda^4 \Theta_h^n, v_h)_{\Omega_n^k} + a_h^n(\theta_h^n, v_h) = A_1 + A_2, \tag{6.3}
\]

where

\[
A_1 = \frac{1}{\tau} (\Lambda^4 \zeta^n, v_h)_{\Omega_n^k} + (R^n - \bar{f}^n, v_h)_{\Omega_n^k}, \quad A_2 = \int_{\Gamma_n^k} \left( \gamma_0 h^{-1} v_h - \partial_n v_h \right) u^n.
\]

Taking \( \nu_h = 2\theta_h^n - P_h^n \Theta_h^{-1,n} \) and using arguments similar to (5.31), we find that, for any \( \varepsilon > 0 \),

\[
\frac{1}{\tau} \sum_{i=1}^4 \left\| \Psi_i^4(\Theta_h^n) \right\|_{L^2(\Omega_n^i)}^2 - \frac{1}{\tau} \sum_{i=1}^4 \left\| \Psi_i^4(\Theta_h^{-1,n}) \right\|_{L^2(\Omega_n^{i-1})}^2 + \frac{3}{2} \|\theta_h^n\|_{2,k}^2
\]

\[
\leq A_1 + A_2 + \frac{\varepsilon h}{\tau} \left\| \Theta_h^{-1,n} \right\|_{2,k}^2 + \frac{C h}{\varepsilon \tau} \sum_{j=0}^4 \left\| \Theta_h^{j,n} \right\|_{L^2(\Omega_n^j)}^2 + \frac{5}{2\varepsilon h} \|\theta_h^n\|^2_{L^2(\Gamma_n^k)} + \frac{5\varepsilon h}{2} \|\partial_n \theta_h^n\|^2_{L^2(\Gamma_n^k)}
\]

\[
+ C \sum_{j=1}^4 \left\| \Theta_h^{j-1,n} \right\|_{H^1(\Omega_n^j)}^2 + \frac{1}{2} \left\| \Theta_h^{-1,n} \right\|_{H^1(\Omega_n^k)}^2 + \frac{\varepsilon h}{2} \left\| \Theta_h^{-1,n} \right\|_{H^1(\Gamma_n^k)}^2 + \frac{\gamma_0 + \varepsilon^{-1}}{2h} \left\| \Theta_h^{-1,n} \right\|_{L^2(\Gamma_n^k)}^2. \tag{6.4}
\]
By Lemma A.1, Lemma 5.7, and Taylor’s formula, we have
\[ \|A^4 u^n\|_{L^2(\Omega^n)} \lesssim \tau^5 \|u^n\|_{H^1(D)} + \sum_{i=0}^4 \|\beta^{n-i}\|_{L^1(\Omega^{n-i})} \lesssim (\tau^5 + h^5) \|u^n\|_{H^5(D)}, \]
\[ \|R^n\|_{L^2(\Omega^n)}^2 = \int_{\Omega^n} \left| \frac{4}{4\tau} \int_{t_n}^{t_{n-1}} (t_n - \xi)^4 \frac{d^5}{dt^5} u(X(\xi; t, x), \xi) \right|^2 d\xi \lesssim \tau^7 \|u\|_{H^7(D \times (t_{n-1}, t_n))}^2. \]
Moreover, using Lemma A.3 and the identity \( \hat{f}^n = f^n \) in \( \Omega_t \), we find that
\[ \left| (\hat{f}^n - f^n, v_h)_{\Omega^n} \right| = \left| (\hat{f}^n - f^n, v_h)_{\Omega^n \setminus \Omega_t} \right| \lesssim \tau^4 h^{-1/2} \|\hat{f}^n - f^n\|_{H^2(D)} \|v_h\|_{L^2(\Omega^n)} \tag{6.5} \]
Since \( u^n = 0 \) on \( \Gamma_t \), we have \( u^n \circ \tilde{\chi}_n \equiv 0 \) and deduce from Theorem 3.5 that
\[ \|u^n\|_{L^2(\Gamma_t^n)} = \left( \int_0^L (u^n \circ \chi_n - u^n \circ \tilde{\chi}_n)^2 \right)^{1/2} \lesssim \int_0^L |\chi_n - \tilde{\chi}_n|^2 \int_0^1 |\nabla u^n(\xi \chi_n + (1 - \xi) \tilde{\chi}_n)|^2 d\xi \lesssim \tau^8 \|u^n\|_{H^1(D)}. \tag{6.6} \]
The inverse estimate and the assumption \( \tau = O(h) \) show
\[ |A_1| \lesssim (\tau^4 \|u^n\|_{H^5(D)} + \tau^{7/2} \|u^n\|_{H^5(D \times (t_{n-1}, t_n))} + \tau^{7/2} \|\hat{f}^n - f^n\|_{H^2(D)}) \|v_h\|_{L^2(\Omega^n)}, \tag{6.7} \]
\[ |A_2| \lesssim h^{-1} \|u^n\|_{L^2(\Gamma^n)} \|v_h\|_{L^2(\Gamma^n)} \lesssim h^{-1} \|u^n\|_{H^1(D)} \|v_h\|_{L^2(\Gamma^n)}. \tag{6.8} \]
Moreover, \( v_h \) can be estimated by using Lemmas 5.2, 5.3, and 5.4 and satisfies
\[ \|v_h\|_{L^2(\Omega^n)}^2 \lesssim \|v_h\|_{L^2(\Omega^n)}^2 + h^2 \|H_{\lambda}^n(v_h, v_h) \lesssim \|\Theta_h^n\|_{L^2(\Omega^n)}^2 + \|\Theta_h^{n-1,n}\|_{L^2(\Omega^n)}^2 + h^2 \|\Theta_h^{n-1,n}\|_{L^2(\Omega^n)}^2, \tag{6.9} \]
\[ \|v_h\|_{L^2(\Gamma_t^n)}^2 \lesssim h \|H_{\lambda}^n(v_h, v_h) + h \|\Theta_h^{n-1,n}\|_{L^2(\Omega^n)}^2. \tag{6.10} \]
Inserting est-A1–(6.10) into (6.4) yields
\[
\sum_{i=1}^4 \left\| \Psi_i^4(\Theta_h^n) \right\|_{L^2(\Omega^n)}^2 - \sum_{i=1}^4 \left\| \Psi_i^4(\Theta_h^{n-1}) \right\|_{L^2(\Omega^{n-1})}^2 + \frac{3}{2} \tau \|\theta^n_h\|_{T_h^n}^2 \leq C \tau^8 \left( \|u^n\|_{H^5(D \times (t_{n-1}, t_n))} + \tau \|u^n\|_{H^5(D)} + \|u^n\|_{H^1(D)} + \|\hat{f}^n - f^n\|_{H^2(D)} \right) + C[\varepsilon + (\gamma_0 - 1) \tau] \|\theta_h^{n-1,n}\|_{T_h^n}^2 + \varepsilon \tau \|\Theta_h^{n-1,n}\|_{L^2(\Omega^n)}^2 + C \tau \sum_{j=0}^4 \left( \|\Theta_h^{n,j,n}\|_{L^2(\Omega^n)}^2 + \|\theta_h^{n-1,n}\|_{L^2(\Omega^{n-1})}^2 \right) + \frac{\tau}{2} \|\Theta_h^{n-1,n}\|_{H^1(\Omega^n)}^2 + \frac{\tau}{2} \|\Theta_h^{n-1,n}\|_{H^1(\Gamma_t^n)}^2 + \frac{\tau}{2} \|\Theta_h^{n-1,n}\|_{H^1(\Omega^{n-1})}^2.
\]
Letting \( 1 - \varepsilon \ll \gamma_0 \) and using arguments similar to (5.32), we obtain
\[
\|\theta_h^n\|_{L^2(\Omega^n)}^2 + \tau \sum_{n=4}^m \|\theta_h^n\|_{T_h^n}^2 \lesssim \tau^7 (\tau \|u^n\|_{H^5(D \times [0,T])} + (1 + \tau) \|u^n\|_{L^\infty(0,T;H^5(D))} + \|u^n\|_{L^\infty(0,T;H^2(D))}) + \sum_{i=0}^3 \left( \|\theta_h^i\|_{L^2(\Omega^n)}^2 + \|\theta_h^i\|_{T_h^n}^2 \right).
\]
The proof is finished by using Lemmas 5.6, 5.7, 6.1, and the assumption (6.2).

**Remark 6.3.** The error estimate in Theorem 6.2 is sub-optimal. In view of (6.5)–(6.7), we find that the loss of one half order is due to the \( k \)-th-order interface-tracking algorithm. In fact, the optimal error estimate can be recovered if we use the RK-\((k + 1)\) scheme for interface-tracking.

**7. Numerical experiments.** Now we use two numerical experiments with \( k = 3, 4 \), respectively, to validate the theoretical analysis. The exact solution and the flow velocity are set by

\[
\begin{align*}
    u(x, t) &= e^{-t} \sin(\pi x_1) \sin(\pi x_2), \\
    w &= \cos \frac{\pi t}{4} \left( \sin^2(\pi x_1) \sin(2\pi x_2), -\sin^2(\pi x_2) \sin(2\pi x_1) \right)^	op.
\end{align*}
\]

The initial domain \( \Omega \) is a disk of unit radius, and the final domain \( \Omega_T \) is stretched into a snake-like domain (see Fig. 7.1). In real computations, we apply the cubic MARS algorithm in [35] to track the boundary. The algorithm is a slight modification of Algorithm 3.4 by creating new markers or removing old markers when necessary. Throughout the section, we choose \( \gamma_0 = 800 \) and \( \gamma_1 = 1/\gamma_0 \).

![Fig. 7.1: The approximate domains \( \Omega_n^\eta \) at \( t_n = 0, 1/2, 1, \) and 2, respectively (\( h = 1/16 \)).](image)

The solution errors are measured by the norm

\[
e^N = \left( \|u(\cdot, T) - u^N_h\|_{L^2(\Omega^N_n)}^2 + \tau \sum_{n=k}^{N} |u - u^N_h|_{H^1(\Omega^N_n)}^2 \right)^{1/2}.
\]

To simplify the computation, we set the pre-calculated initial values by the exact solution, namely,

\[
u^j_h = u(t_j), \text{ for } 0 \leq j \leq k - 1.
\]

Numerical results for \( k = 3, 4 \) are shown in Table 7.1. They show that optimal convergence rates, \( e^N \sim \tau^k \), are obtained for both the third- and fourth-order methods.

| \( h = \tau \) | \( e^N \) | rate | \( e^N \) | rate |
|---|---|---|---|---|
| 1/16 | 3.47e-05 | - | 2.43e-06 | - |
| 1/32 | 4.31e-06 | 3.01 | 9.90e-08 | 4.62 |
| 1/64 | 4.25e-07 | 3.34 | 4.56e-09 | 4.44 |
| 1/128 | 4.93e-08 | 3.11 | 2.34e-10 | 4.29 |

Table 7.1: Convergence rates for \( k = 3 \) (the middle column) and \( k = 4 \) (the right column).
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[36] Q. Zhang and Z. Li, *Boolean algebra of two-dimensional continua with arbitrarily complex topology*, Math. Comput., 89 (2020): 2333-2364.
Appendix A. Useful estimates for the pull-back maps.

In this appendix, we prove some useful estimates concerning the pull-back maps \( v \to v \circ X^{n,n-i}_\tau \) and \( v \to v \circ X^{n,n-i}_\tau \) for a function \( v \). Assume \( 1 \leq n \leq N \) and \( 0 \leq i \leq k \) throughout the appendix.

**Lemma A.1.** Let \( \Omega \subset D \) satisfy \( X^{p,n-i}_\tau(\Omega) \subset D \) and \( X^{n,n-i}_\tau(\Omega) \subset D \). There exists a constant \( C > 0 \) independent of \( n \) and \( \tau \) such that, for any \( v \in H^1(D) \),

\[
\| \nabla^\mu (v \circ X^{n,n-i}_\tau) \|_{L^2(\Omega)}^2 \leq (1 + C\tau) \| \nabla^\mu v \|_{L^2(X^{p,n-i}_\tau(\Omega))}^2, \quad \mu = 0, 1, \tag{A.1}
\]

\[
\| v \circ X^{n,n-i}_\tau - v \circ X^{n,n-i}_\tau \|_{L^2(\Omega)} \lesssim \tau^{k+1} \| v \|_{H^1(D)}, \tag{A.2}
\]

Moreover, if \( \eta = O(\tau^{k/4}) \), then for any \( v_h \in V(k, T_h) \),

\[
\| v_h \circ X^{n,n-i}_\tau \|_{L^2(T^n_\tau)} \leq (1 + C\tau) \| v_h \|_{L^2(T^n_\tau)} + C\tau^{k+1} h^{-2} \| v_h \|_{H^1(D)}, \tag{A.3}
\]

\[
\| v_h \circ X^{n,n-i}_\tau \|_{H^1(T^n_\tau)} \lesssim h^{-1} \| v_h \|_{H^1(D)}. \tag{A.4}
\]

**Proof.** Changing variables of integration and using Lemma 3.3, we have, for \( \mu = 0, 1 \),

\[
\| \nabla^\mu (v \circ X^{n,n-i}_\tau) \|_{L^2(\Omega)}^2 = \int_{X^{n,n-i}_\tau(\Omega)} \left| (J^{n,n-i}_\tau \nabla)^\mu v \right|^2 \det(J^{n,n-i}_\tau) \leq (1 + C\tau) \| \nabla^\mu v \|_{L^2(X^{n,n-i}_\tau(\Omega))}^2.
\]

To prove (A.2), we use Lemma 3.1 and get

\[
\| v \circ X^{n,n-i}_\tau - v \circ X^{n,n-i}_\tau \|_{L^2(\Omega)} \leq \| \nabla v \|_{L^2(D)} \| X^{n,n-i}_\tau - X^{n,n-i}_\tau \|_{L^\infty(\mathbb{R}^2)} \lesssim \tau^{k+1} \| v \|_{H^1(D)}.
\]

Suppose \( \eta = O(\tau^{k/4}) \) and note from Theorem 3.6 that \( |X^{n}_\tau - (X^{n-i,n}_\tau \circ X^{n-i}_\tau)| \lesssim \tau^{1+3k/4} \).

Together with Lemma 3.3, this shows

\[
|X^{n}_\tau|^{-1} \left| X^{n}_\tau - (X^{n-i,n}_\tau \circ X^{n-i}_\tau) \right| |X^{n-i}_\tau|^{-1} \lesssim 1 + C\tau.
\]

We deduce that then we have

\[
\| v_h \circ X^{n,n-i}_\tau \|_{L^2(T^n_\tau)} \leq (1 + C\tau) \int_0^L \| v_h \circ X^{n,n-i}_\tau \|_{L^2(T^n_\tau)} \| X^{n,n-i}_\tau \|_{L^\infty(\mathbb{R}^2)} \lesssim 1 + \eta^3 \lesssim 1. \tag{A.5}
\]

Note from (3.13) that \( |X^{n}_\tau| \leq \left| X^{n}_\tau \right| + \left| X^{n}_\tau - X^{n}_\tau \right| \lesssim 1 + \eta^3 \lesssim 1. \) By Theorem 3.6 and the inverse estimate, we deduce that

\[
\int_0^L \left( \| v_h \circ X^{n,n-i}_\tau \circ X^{n}_\tau \|_{L^2(D)} - \| v_h \circ X^{n,n-i}_\tau \|_{L^2(D)}^2 \right) \lesssim \int_0^L \left| X^{n,n-i}_\tau - (X^{n,n-i}_\tau \circ X^{n}_\tau) \right| \| \nabla v_h \|_{L^2(D)} \| \theta \|_{L^\infty(D)} \| (1 - \theta) \|_{L^\infty(D)} \| \nabla v_h \|_{L^2(D)} \lesssim \tau^{k+1} h^{-2} \| v_h \|_{H^1(D)}. \tag{A.6}
\]

Then (A.3) follows from (A.5), (A.6), and the following identity

\[
\| v_h \circ X^{n,n-i}_\tau \|_{L^2(T^n_\tau)}^2 = \int_0^L \| v_h \circ X^{n,n-i}_\tau \|_{L^2(D)}^2 \| X^{n}_\tau \|_{L^\infty(D)} + \int_0^L \left( \| v_h \circ X^{n,n-i}_\tau \|_{L^2(D)}^2 - \| v_h \circ X^{n,n-i}_\tau \|_{L^2(D)}^2 \right) \| X^{n}_\tau \|_{L^\infty(D)}.
\]
The proof of (A.4) is easy. Using Lemma 3.1 and Lemma 5.1 we immediately get

\[ |v_h \circ X^{n, n-i}_\tau|^2_{H^1(\Omega_h^n)} = \int_{X^{n, n-i}_\tau(\Gamma_h^n)} \left| (\nabla X^{n, n-i}_\tau \circ X^{n, n-i}_\tau) \nabla v_h \right|^2 \leq (1 + C\tau) \int_{X^{n, n-i}_\tau(\Gamma_h^n)} |\nabla v_h|^2. \]

Using the scaling technique and the norm equivalence shows (A.4).

Lemma A.2. Assume \( \eta = O(\tau^{k/4}) \). There exists a constant \( C > 0 \) independent of \( n \) and \( \tau \) such that, for any \( 1 \leq i \leq l \leq k \) and \( \mu = 0, 1 \),

\[ \left\| \nabla^\mu (v_h \circ X^{n, n-i}_\tau) \right\|^2_{L^2(\Omega_h^n)} \leq (1 + C\tau) \left\| \nabla^\mu v_h \right\|^2_{L^2(\Omega_h^n)} + C\tau^{k+1} h^{-1} \left\| \nabla^\mu v_h \right\|^2_{L^2(D)}. \]  

(A.7)

Proof. Using Lemma 3.1 and (A.1), we have

\[ \left\| v_h \circ X^{n, n-i}_\tau \right\|^2_{L^2(\Omega_h^n)} = \left\| v_h \circ X^{n, n-i}_\tau \right\|^2_{L^2(X^{n, n-i}(\Omega_h^n))} + \left\| v_h \circ X^{n, n-i}_\tau \right\|^2_{L^2(\Omega_h^n \setminus X^{n, n-i}(\Omega_h^n))} \]

\[ \leq (1 + C\tau) \left( \left\| v_h \circ X^{n, n-i}_\tau \right\|^2_{L^2(\Omega_h^n)} + \left\| v_h \right\|^2_{L^2(X^{n, n-i}(\Omega_h^n))} + \left\| v_h \right\|^2_{L^2(\Omega_h^n \setminus X^{n, n-i}(\Omega_h^n))} \right). \]

By Theorem 3.6 and the inverse inequality, we know that

\[ \left\| v_h \right\|^2_{L^2(X^{n, n-i}(\Omega_h^n) \setminus X^{n, n-i}(\Omega_h^n))} = \sum_{K \in T_h} \int_{X^{n, n-i}(\Omega_h^n) \setminus X^{n, n-i}(\Omega_h^n)} |v_h|^2 \]

\[ \lesssim h^{k+1} \sum_{K \cap X^{n, n-i}(\Omega_h^n) \neq \emptyset} \left\| v_h \right\|^2_{L^\infty(K)} + h^{k+1} \sum_{K \cap X^{n, n-i}(\Gamma_h^n) \neq \emptyset} \left\| v_h \right\|^2_{L^\infty(K)} \]

\[ \lesssim h^{k+1} h^{-1} \left\| v_h \right\|^2_{L^2(D)}. \]  

(A.8)

This yields (A.7) for \( \mu = 0 \). The proof for the case of \( \mu = 1 \) is similar.

Lemma A.3. Assume \( \eta = O(\tau^{k/4}) \) and \( r \geq 1 \). For any \( v_h \in V(k, T_h) \) and \( v \in H^r(D) \),

\[ \left\| v_h \right\|^2_{L^2(\Omega_h^n)} \lesssim h^{k+1} \left\| v_h \right\|^2_{L^2(D)}, \quad \left\| v \right\|^2_{L^2(\Omega_h^n)} \lesssim \sigma(r) \left\| v \right\|^2_{H^r(D)}, \]

where \( \sigma(r) = \max(h, \tau^r) \) if \( r = 1 \) and \( \sigma(r) = \tau^r \) for \( r \geq 2 \).

Proof. It is easy to obtain the first inequality from the proof of (A.8). Now we prove the second one for different values of \( r \). If \( r \geq 2 \), Theorem 3.6 and the injection \( H^r(D) \hookrightarrow L^\infty(D) \) shows

\[ \left\| v \right\|^2_{L^2(\Omega_h^n)} \lesssim \text{area}(\Omega_h^n(\Omega_h^n)) \left\| v \right\|^2_{L^\infty(D)} \lesssim \tau^k \left\| v \right\|^2_{H^r(D)}. \]

If \( r = 1 \), the Scott-Zhang interpolation in (5.10) and the first inequality lead to

\[ \left\| v \right\|^2_{L^2(\Omega_h^n)} \lesssim \left\| v - I_h v \right\|^2_{L^2(D)} + \left\| I_h v \right\|^2_{L^2(\Omega_h^n(\Omega_h^n))} \lesssim (1) \left\| v \right\|^2_{H^1(D)}. \]

The proof is finished.