CONVERGENCE IN CAPACITY OF PLURISUBHARMONIC FUNCTIONS WITH GIVEN BOUNDARY VALUES

NGUYEN XUAN HONG, NGUYEN VAN TRAO AND TRAN VAN THUY

Abstract. In this paper, we study the convergence in the capacity of sequence of plurisubharmonic functions. As an application, we prove stability results for solutions of the complex Monge-Ampère equations.

1. Introduction

It is well-known that convergence in the sense of distributions of plurisubharmonic functions does not in general imply convergence of their Monge-Ampère measures. Therefore, it is important to find conditions on sequences of plurisubharmonic functions such that the corresponding Monge-Ampère measures are convergent in the weak* topology.

Bedford and Taylor [3] introduced and studied in 1982 the $C_n$-capacity of Borel sets. Xing [21] proved in 1996 that the complex Monge-Ampère operator is continuous under convergence of bounded plurisubharmonic functions in $C_n$-capacity. He gave a sufficient condition for the weak convergence of complex Monge-Ampère mass of bounded plurisubharmonic functions. Later, Xing [22] studied in 2008 the convergence in the $C_n$-capacity of a sequence of plurisubharmonic functions in the class $E(\Omega)$. Hiep [15] studied in 2010 the convergence in $C_n$-capacity within the class $E(\Omega)$. Recently, Cegrell [8] proved in 2012 that if a sequence of plurisubharmonic functions is bounded from below by a function from the Cegrell class $E(\Omega)$ and convergent in $C_{n-1}$-capacity then the corresponding complex Monge-Ampère measures are convergent in the weak* topology.

The purpose of this paper is to study conditions on a sequence of plurisubharmonic functions which are equivalent to convergence in $C_n$-capacity. Our main result is the following theorem.

Main theorem. Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$, $w \in N^w(\Omega, f)$ such that $\int_{\Omega} (-\rho)(dd^c w)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$. Assume that $\{u_j\} \subset N^w(\Omega, f)$ such that $u_j \to u_0$ a.e. on $\Omega$ as $j \to +\infty$ and $u_j \geq w$ in $\Omega$ for all $j \geq 0$. Then, the following statements are equivalent.

(a) $u_j \to u_0$ in $C_n$-capacity in $\Omega$;

(b) For every $a > 0$, we have

$$\lim_{j \to +\infty} \int_{\Omega} \max \left( \frac{u_j}{a}, \rho \right) (dd^c u_j)^n = \int_{\Omega} \max \left( \frac{u_0}{a}, \rho \right) (dd^c u_0)^n.$$
(c) For every $a > 0$, we have
\[ \lim_{j \to +\infty} \int_{\Omega} \left[ \max \left( \frac{v_j}{a}, \rho \right) - \max \left( \frac{u_j}{a}, \rho \right) \right] (dd^c u_j)^n = 0, \]
where $v_j := (\sup_{k \geq j} u_k)$.

The paper is organized as follows. In Section 2 we recall some notions of pluripotential theory. Section 3 is devoted to the proof of the main theorem. In Section 4 we apply the main theorem to prove a stability result for the solutions of certain complex Monge-Ampère equations.

2. Preliminaries

Some elements of pluripotential theory that will be used throughout the paper can be found in [1]-[22].

**Definition 2.1.** Let $n$ be a positive integer. A bounded domain $\Omega$ in $\mathbb{C}^n$ is called bounded hyperconvex domain if there exists a bounded plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that the closure of the set $\{ z \in \Omega : \varphi(z) < c \}$ is compact in $\Omega$, for every $c \in (-\infty, 0)$.

We denote by $PSH(\Omega)$ the family of plurisubharmonic functions defined on $\Omega$ and $PSH^-(\Omega)$ denotes the set of negative plurisubharmonic functions on $\Omega$. By $MPSH(\Omega)$ denotes the set of all maximal plurisubharmonic functions in $\Omega$.

**Definition 2.2.** Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$. We say that a bounded, negative plurisubharmonic function $\varphi$ in $\Omega$ belongs to $E_0(\Omega)$ if $\{ \varphi < -\varepsilon \} \subset \Omega$ for all $\varepsilon > 0$ and $\int_{\Omega} (dd^c \varphi)^n < +\infty$.

Let $\mathcal{F}(\Omega)$ be the family of plurisubharmonic functions $\varphi$ defined on $\Omega$, such that there exists a decreasing sequence $\{ \varphi_j \} \subset E_0(\Omega)$ that converges pointwise to $\varphi$ on $\Omega$ as $j \to +\infty$ and $\sup_j \int_{\Omega} (dd^c \varphi_j)^n < +\infty$.

We denote by $\mathcal{E}(\Omega)$ the family of plurisubharmonic functions $\varphi$ defined on $\Omega$ such that for every open set $G \supset \Omega$ there exists a plurisubharmonic function $\psi \in \mathcal{F}(\Omega)$ satisfy $\psi = \varphi$ in $G$.

Let $u \in \mathcal{E}(\Omega)$ and let $\{ \Omega_j \}$ be an increasing sequence of bounded hyperconvex domains such that $\Omega_j \subset \Omega_j \subset \Omega$ and $\bigcup_{j=1}^{+\infty} \Omega_j = \Omega$. Put
\[ u_j := \sup \{ \varphi \in PSH^-(\Omega) : \varphi \leq u \text{ in } \Omega \setminus \Omega_j \} \]
and $\mathcal{N}(\Omega) := \{ u \in \mathcal{E}(\Omega) : u_j \not\to 0 \text{ a.e. in } \Omega \}$.

Let $K \in \{ \mathcal{F}, \mathcal{N}, \mathcal{E} \}$. We denote by $K^a(\Omega)$ the subclass of $K(\Omega)$ such that the Monge-Ampère measure $(dd^c \cdot)^n$ vanishes on all pluripolar sets of $\Omega$.

Let $f \in \mathcal{E}(\Omega)$ and $K \in \{ \mathcal{F}^a, \mathcal{N}^a, \mathcal{E}^a, \mathcal{F}, \mathcal{N}, \mathcal{E} \}$. Then we say that a plurisubharmonic function $\varphi$ defined on $\Omega$ belongs to $K(\Omega, f)$ if there exists a function $\psi \in K(\Omega)$ such that $\psi + f \leq \varphi \leq f$ in $\Omega$.

Now we will show that if $u \in \mathcal{N}^a(\Omega, f)$ then the pluripolar part of $(dd^c u)^n$ is carried by $\{ f = -\infty \}$.
Proposition 2.3. Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$. Assume that $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{N}^\alpha(\Omega, f)$ such that $\int_\Omega (-\rho)(dd^c u)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$. Then

$$1_{\{u=-\infty\}}(dd^c u)^n = 1_{\{f=-\infty\}}(dd^c f)^n$$
in $\Omega$.

Proof. Let $v \in \mathcal{F}^\alpha(\Omega)$ such that $v + f \leq u \leq f$ in $\Omega$. By Proposition 4.1 and Lemma 4.12 in [1] we have

$$1_{\{f=-\infty\}}(dd^c f)^n \leq 1_{\{u=-\infty\}}(dd^c u)^n \leq 1_{\{v+f=-\infty\}}(dd^c (v + f))^n = 1_{\{f=-\infty\}}(dd^c f)^n.$$

It follows that

$$1_{\{u=-\infty\}}(dd^c u)^n = 1_{\{f=-\infty\}}(dd^c f)^n$$
in $\Omega$.

The proof is complete. \qed

Proposition 2.4. Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$. Let $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{N}^\alpha(\Omega, f)$ such that $\int_\Omega (-\rho)(dd^c u)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$. Assume that $v \in \mathcal{E}(\Omega)$ such that $v \leq f$ and $(dd^c v)^n \leq (dd^c u)^n$ in $\Omega$. Then $v \leq u$ on $\Omega$.

Proof. Since the measure $1_{\{u=-\infty\}}(dd^c u)^n$ vanishes on all pluripolar subsets of $\Omega$, by Proposition 4.3 in [1] we get

$$(dd^c \max(u, v))^n \geq 1_{\{u>-\infty\}}(dd^c u)^n.$$ 

Hence,

$$1_{\{\max(u, v)>-\infty\}}(dd^c \max(u, v))^n \geq 1_{\{u>-\infty\}}(dd^c u)^n.$$ 

Moreover, by the hypotheses and Proposition 2.3 we have

$$1_{\{\max(u, v)=-\infty\}}(dd^c \max(u, v))^n = 1_{\{u=-\infty\}}(dd^c u)^n.$$ 

Hence, $(dd^c \max(u, v))^n \geq (dd^c u)^n$ in $\Omega$. Therefore, from Theorem 3.6 in [1] it follows that $\max(u, v) = u$ in $\Omega$. Thus, $v \leq u$ in $\Omega$. The proof is complete. \qed

3. PROOF OF THE MAIN THEOREM

In order to prove the main theorem, we need the following auxiliary lemmas.

Lemma 3.1. Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$. Assume that $\rho \in \mathcal{E}_0(\Omega)$ and $u \in \mathcal{N}^\alpha(\Omega, f)$ such that $\int_\Omega (-\rho)(dd^c u)^n < +\infty$. Then for every $v \in \mathcal{E}^\alpha(\Omega, f)$ and for every $\varphi \in \mathcal{E}_0(\Omega)$ with $\varphi \geq \rho$, we have

$$\frac{1}{n!} \int_{\{u<v\}} (v - u)^n (dd^c \varphi)^n + \int_{\{u<v\}} -\varphi (dd^c v)^n \leq \int_{\{u<v\}} -\varphi (dd^c u)^n.$$ 

Proof. For $j \in \mathbb{N}^*$, put $v_j = \max(u, v - \frac{1}{j})$. Because $u \leq v_j \leq f$ in $\Omega$, we have $v_j \in \mathcal{F}^\alpha(\Omega, f)$. By Lemma 3.5 in [1] we have

$$\frac{1}{n!} \int_{\Omega} (v_j - u)^n (dd^c \varphi)^n + \int_{\{u<v_j\}} -\varphi (dd^c v_j)^n \leq \int_{\Omega} -\varphi (dd^c u)^n.$$ 

By Theorem 4.1 in [10] we have $v_j = v - \frac{1}{j}$ in $\{u<v_j\}$. Hence,

$$\frac{1}{n!} \int_{\{u<v_j\}} (v_j - u)^n (dd^c \varphi)^n + \int_{\{u<v_j\}} -\varphi (dd^c v_j)^n = \frac{1}{n!} \int_{\{u<v_j\}} (v_j - u)^n (dd^c \varphi)^n + \int_{\{u<v_j\}} -\varphi (dd^c v_j)^n$$
Therefore, without loss of generality we can assume that \( k = 1 \) and let \( \psi_j = 0 \) for all \( j \).

Assume that \( E \) is the plurisubharmonic function obtained in Lemma 3.2. The proof is complete. \( \square \)

Now, since \( u = v_j \) in \( \{ u > v - \frac{1}{n} \} \) so by Theorem 4.1 in [19] imply that \((dd^c u)^n = (dd^c v_j)^n\) in \( \{ u \geq v \} \cap \{ u > -\infty \} \).

Moreover, by Proposition 2.3 we have
\[
1_{\{u=-\infty\}}(dd^c u)^n = 1_{\{v_j=-\infty\}}(dd^c v_j)^n = 1_{\{f=-\infty\}}(dd^c f)^n \text{ in } \Omega.
\]

Hence, we obtain that \((dd^c u)^n = (dd^c v_j)^n\) in \( \{ u \geq v \} \).

Therefore,
\[
\frac{1}{n!} \int_{\{u<v_j\}} (v_j - u)^n (dd^c \varphi)^n + \int_{\{u<v_j\}} -\varphi (dd^c v)^n \\
\leq \int_{\Omega} -\varphi (dd^c u)^n - \int_{\{u>v\}} -\varphi (dd^c u)^n \\
= \int_{\{u<v\}} -\varphi (dd^c u)^n.
\]

Let \( j \to +\infty \) we obtain that
\[
\frac{1}{n!} \int_{\{u<v\}} (v - u)^n (dd^c \varphi)^n + \int_{\{u<v\}} -\varphi (dd^c v)^n \leq \int_{\{u<v\}} -\varphi (dd^c u)^n.
\]

The proof is complete. \( \square \)

**Lemma 3.2.** Let \( \Omega \) be a bounded hyperconvex domain in \( \mathbb{C}^n \) and let \( \{ u_j \} \subset \mathcal{E}^n(\Omega) \) such that \( u_j \geq u_1 \) for every \( j \geq 1 \) and \( u_j \to u_0 \) in \( C_\alpha \)-capacity in \( \Omega \).

Assume that \( \{ \varphi_j^k \} \), \( k = 1, 2 \), are sequences of uniformly bounded plurisubharmonic functions in \( \Omega \) which converges weakly to a plurisubharmonic function \( \varphi_0^k \) in \( \Omega \). Then \( \varphi_j^1 \varphi_j^2 (dd^c u_j)^n \to \varphi_0^1 \varphi_0^2 (dd^c u_0)^n \) weakly as \( j \to +\infty \).

**Proof.** Without loss of generality we can assume that \( u_j \in \mathcal{F}^n(\Omega) \) and \(-1 \leq \varphi_j^k \leq 0 \) in \( \Omega \) for all \( j \geq 0 \), \( k = 1, 2 \). Put
\[
\psi_j^1 = \frac{(\varphi_j^1 + \varphi_j^2 + 2)^2 + 4}{2}, \quad \psi_j^2 = \frac{(\varphi_j^1 + 2)^2}{2} \quad \text{and} \quad \psi_j^3 = \frac{(\varphi_j^2 + 2)^2}{2}.
\]

It is clear that \( \psi_j^k \in PSH(\Omega) \), \( 0 \leq \psi_j^k \leq 4 \) and \( \psi_j^k \to \psi_0^k \) weakly in \( \Omega \) as \( j \to +\infty \), \( k = 1, 2, 3 \). Since \( \varphi_j^1 \varphi_j^2 = \psi_j^1 - \psi_j^2 - \psi_j^3 \) in \( \Omega \) we obtain by Theorem 3.4 in [22] that
\[
\varphi_j^1 \varphi_j^2 (dd^c u_j)^n \to \psi_0^1 (dd^c u_0)^n - \psi_0^2 (dd^c u_0)^n - \psi_0^3 (dd^c u_0)^n \text{ weakly in } \Omega \text{ as } j \to +\infty. \]

The proof is complete. \( \square \)
Proof of the main theorem. Without loss of generality we can assume that \( f < 0 \) and \(-1 \leq \rho \leq 0 \) in \( \Omega \).

(a) \( \Rightarrow \) (b). Fix \( a > 0 \). Put

\[
\varphi_j := \max \left( \frac{u_j}{a}, \rho \right).
\]

Because

\[
0 \leq \sup_j \int_{\Omega} -\varphi_j (dd^c u_j)^n \leq \int_{\Omega} -\rho (dd^c w)^n < +\infty,
\]

it remains to prove that there exists a subsequence \( \{u_{jk}\} \) of sequence \( \{u_j\} \) such that

\[
\lim_{k \to +\infty} \int_{\Omega} \varphi_{jk} (dd^c u_{jk})^n = \int_{\Omega} \varphi_0 (dd^c u_0)^n.
\]

First we claim that there exists an increasing sequence \( \{j_k\} \subset \mathbb{N}^* \) such that

\[
\lim_{k \to +\infty} \int_{\Omega} \varphi_{j_k} \max \left( 1 + \frac{u_{j_k}}{k}, 0 \right) (dd^c u_{j_k})^n = \int_{\{u_0 > -\infty\}} \varphi_0 (dd^c u_0)^n. \tag{3.1}
\]

Indeed, let \( \chi_k \in C_0^\infty(\Omega) \) such that \( 0 \leq \chi_k \leq \chi_{k+1} \leq 1 \) in \( \Omega \), \( \{\rho \leq -\frac{1}{k}\} \subset \{\chi_k = 1\} \) and

\[
\int_{\{\chi_k < 1\}} (-\rho) (dd^c u_0)^n \leq \frac{1}{k}.
\]

Since \( u_j \to u_0 \) in \( C_n \)-capacity in \( \Omega \) as \( j \to +\infty \), so \( \max(u_j, -k) \to \max(u_0, -k) \) in \( C_n \)-capacity as \( j \to +\infty \). By Lemma \( 5.2 \) we have

\[
\varphi_j \max \left( 1 + \frac{u_j}{k}, 0 \right) (dd^c \max(u_j, -k))^n \to \varphi_0 \max \left( 1 + \frac{u_0}{k}, 0 \right) (dd^c \max(u_0, -k))^n
\]

weakly in \( \Omega \) as \( j \to +\infty \). Hence, by Theorem 4.1 in [19] we get

\[
\lim_{j \to +\infty} \int_{\Omega} \chi_k \varphi_j \max \left( 1 + \frac{u_j}{k}, 0 \right) (dd^c u_j)^n
= \lim_{j \to +\infty} \int_{\Omega} \chi_k \varphi_j \max \left( 1 + \frac{u_j}{k}, 0 \right) (dd^c \max(u_j, -k))^n
= \int_{\Omega} \chi_k \varphi_0 \max \left( 1 + \frac{u_0}{k}, 0 \right) (dd^c \max(u_0, -k))^n
= \int_{\Omega} \chi_k \varphi_0 \max \left( 1 + \frac{u_0}{k}, 0 \right) (dd^c u_0)^n.
\]

Because \( \chi_k \max \left( 1 + \frac{u_0}{k}, 0 \right) \nearrow \chi_{\{u_0 > -\infty\}} \) as \( k \to +\infty \) in \( \Omega \), we have

\[
\lim_{k \to +\infty} \int_{\Omega} \chi_k \varphi_0 \max \left( 1 + \frac{u_0}{k}, 0 \right) (dd^c u_0)^n = \int_{\{u_0 > -\infty\}} \varphi_0 (dd^c u_0)^n.
\]

Therefore, there exists an increasing sequence \( \{j_k\} \subset \mathbb{N}^* \) such that

\[
\lim_{k \to +\infty} \int_{\Omega} \chi_k \varphi_{j_k} \max \left( 1 + \frac{u_{j_k}}{k}, 0 \right) (dd^c u_{j_k})^n = \int_{\{u_0 > -\infty\}} \varphi_0 (dd^c u_0)^n. \tag{3.2}
\]
Now, fix $k_0 \in \mathbb{N}^*$. By the proof of the theorem in [8] (see (3.1) in [8]) we have
\[
\liminf_{k \to +\infty} \int_{\Omega} (1 - \chi) \varphi_k \max \left( 1 + \frac{u_j}{k}, 0 \right) (dd^c u_{jk})^n \\
\geq \liminf_{k \to +\infty} \int_{\Omega} (1 - \chi) \varphi_k (dd^c u_{jk})^n \geq \liminf_{k \to +\infty} \int_{\Omega} (1 - \chi_{k_0}) \rho (dd^c u_{jk})^n \\
= \liminf_{k \to +\infty} \left[ \int_{\Omega} \rho (dd^c u_{jk})^n - \int_{\Omega} \chi_{k_0} \rho (dd^c u_{jk})^n \right] \\
= \int_{\Omega} \rho (dd^c u_0)^n - \int_{\Omega} \chi_{k_0} \rho (dd^c u_0)^n \\
\geq \int_{\{ \chi_{k_0} < 1 \}} \rho (dd^c u_0)^n \geq -\frac{1}{k_0}.
\]
Combining this with (3.2) we arrive at
\[
\liminf_{k \to +\infty} \int_{\Omega} \varphi_j \max \left( 1 + \frac{u_j}{k}, 0 \right) (dd^c u_{jk})^n \\
= \lim_{k \to +\infty} \int_{\Omega} \chi \varphi_j \max \left( 1 + \frac{u_j}{k}, 0 \right) (dd^c u_{jk})^n \\
= \int_{\{ u_0 > -\infty \}} \varphi_0 (dd^c u_0)^n.
\]
This proves the claim.

The measure $1_{\{ u_0 > -\infty \}} \varphi_j \max \left( \frac{u_j}{k}, -1 \right) (dd^c u_{jk})^n$ vanishes on all pluripolar subset of $\Omega$, hence by Lemma 5.14 in [6] there exists $h_k \in F^a(\Omega)$ such that
\[
(dd^c h_k)^n = 1_{\{ u_j > -\infty \}} \varphi_j \max \left( \frac{u_j}{k}, -1 \right) (dd^c u_{jk})^n.
\]
Because $(dd^c h_k)^n \leq (dd^c u_{jk})^n$ in $\Omega$ and the measure $(dd^c h_k)^n$ vanishes on all pluripolar subset of $\Omega$, from Corollary 3.2 in [1] we have
\[
h_k \geq u_j \geq w \text{ in } \Omega.
\]
We claim that $h_k \to 0$ in $C^n$-capacity in $\Omega$. Indeed, let $\delta > 0$ and $\psi \in PSH(\Omega)$ with $-1 \leq \psi \leq 0$. By Theorem 3.1 in [1] we have
\[
\int_{\{ h_k < -\delta \}} (dd^c \psi)^n \leq \int_{\{ h_k < \delta \}} (dd^c \psi)^n \leq \frac{1}{\delta^n} \int_{\{ h_k < \delta \}} (dd^c h_k)^n \\
\leq \frac{1}{\delta^n} \int_{\{ u_j > -\infty \}} \varphi_j \max \left( \frac{u_j}{k}, -1 \right) (dd^c u_{jk})^n \\
\leq -\frac{1}{\delta^n} \int_{\{ u_j > -\infty \}} \max \left( \frac{u_j}{k}, \rho \right) (dd^c u_{jk})^n.
\]
Therefore, by Lemma 3.3 in [1] and Proposition 2.3 we obtain that
\[
\int_{\{ h_k < -\delta \}} (dd^c \psi)^n \leq -\frac{1}{\delta^n} \int_{\Omega} \max \left( \frac{u_j}{k}, \rho \right) (dd^c w_{jk})^n + \frac{1}{\delta^n} \int_{\{ u_j = -\infty \}} \rho (dd^c u_{jk})^n \\
\leq -\frac{1}{\delta^n} \int_{\Omega} \max \left( \frac{w}{k}, \rho \right) (dd^c w)^n + \frac{1}{\delta^n} \int_{\{ w = -\infty \}} \rho (dd^c w)^n \\
= -\frac{1}{\delta^n} \int_{\{ w > -\infty \}} \max \left( \frac{w}{k}, \rho \right) (dd^c w)^n.
\]
It follows that
\[ C_n(\{ h_k < -\delta \}) \leq -\frac{1}{\delta^n} \int_{(w > -\infty)} \max \left( \frac{w}{k}, \rho \right) (dd^c w)^n. \]

Hence, we get
\[ \lim_{k \to +\infty} C_n(\{ h_k < -\delta \}) = 0, \]
for every \( \delta > 0 \). Thus, \( h_k \to 0 \) in \( C_n \)-capacity in \( \Omega \) as \( k \to +\infty \). This proves the claim, and therefore, by (3.1) and the Theorem in [8] we have
\[ 0 \leq \limsup_{k \to +\infty} \int_{\{ u_{jk} > -\infty \}} \varphi_{jk} \max \left( \frac{u_{jk}}{k}, -1 \right) (dd^c u_{jk})^n \]
\[ \leq \limsup_{k \to +\infty} \int_{\Omega} (1 - \chi_{k_0})(dd^c u_{jk})^n + \limsup_{k \to +\infty} \int_{\Omega} \chi_{k_0}(dd^c h_k)^n \]
\[ \leq \frac{1}{k_0}, \]
for all \( k_0 \in \mathbb{N}^* \). Thus,
\[ \lim_{k \to +\infty} \int_{\{ u_{jk} > -\infty \}} \varphi_{jk} \max \left( \frac{u_{jk}}{k}, -1 \right) (dd^c u_{jk})^n = 0. \]

Combining this with (3.1) we arrive at
\[ \lim_{k \to +\infty} \int_{\{ u_{jk} > -\infty \}} \varphi_{jk} (dd^c u_{jk})^n = \int_{\{ u_0 > -\infty \}} \varphi_0 (dd^c u_0)^n. \]

Moreover, by Proposition 2.3 we have
\[ \int_{\{ u_{jk} = -\infty \}} \varphi_{jk} (dd^c u_{jk})^n = \int_{\{ f = -\infty \}} \rho (dd^c f)^n = \int_{\{ u_0 = -\infty \}} \varphi_0 (dd^c u_0)^n. \]

Hence, we get
\[ \lim_{k \to +\infty} \int_{\Omega} \varphi_{jk} (dd^c u_{jk})^n = \int_{\Omega} \varphi_0 (dd^c u_0)^n. \]

(b)\( \Rightarrow \) (c). Fix \( a > 0 \). Since \( u_j \to u_0 \) a.e. in \( \Omega \) as \( j \to +\infty \) so \( v_j \searrow u_0 \) as \( j \nearrow +\infty \). Hence, \( v_j \to u_0 \) in \( C_n \)-capacity in \( \Omega \). Therefore, by the proof of (a)\( \Rightarrow \) (b) and Lemma 3.3 in [11], we have
\[ \int_{\Omega} \max \left( \frac{u_0}{a}, \rho \right) (dd^c u_0)^n = \lim_{j \to +\infty} \int_{\Omega} \max \left( \frac{v_j}{a}, \rho \right) (dd^c v_j)^n \]
\[ \geq \lim_{j \to +\infty} \int_{\Omega} \max \left( \frac{v_j}{a}, \rho \right) (dd^c u_j)^n \]
\[ \geq \lim_{j \to +\infty} \int_{\Omega} \max \left( \frac{u_j}{a}, \rho \right) (dd^c u_j)^n \]
\[ = \int_{\Omega} \max \left( \frac{u_0}{a}, \rho \right) (dd^c u_0)^n. \]

It follows that
\[ \lim_{j \to +\infty} \int_{\Omega} \max \left( \frac{v_j}{a}, \rho \right) (dd^c u_j)^n = \int_{\Omega} \max \left( \frac{u_0}{a}, \rho \right) (dd^c u_0)^n. \]

Therefore, we obtain that
\[ \lim_{j \to +\infty} \int_{\Omega} \left[ \max \left( \frac{v_j}{a}, \rho \right) - \max \left( \frac{u_j}{a}, \rho \right) \right] (dd^c u_j)^n = 0. \]
Note that \( u \nrightarrow v \) in \( \Omega \) as \( j \uparrow +\infty \), we get \( v_j \rightarrow u_0 \) in \( C_0\)-capacity in \( \Omega \). Hence, it is sufficient to prove that \( v_j - u_j \rightarrow 0 \) in \( C_0\)-capacity in \( \Omega \). Let \( K \) be a compact subset of \( \Omega \) and let \( \varepsilon, \delta > 0 \). Without loss of generality we can assume that \( K \subset \{ \rho = 1 \} \). Choose \( \chi \in C_0^\infty(\Omega) \) and \( a > b > 1 \) such that \( 0 \leq \chi \leq 1 \), \( \{ \rho \leq -\varepsilon \} \subset \{ \chi = 1 \}, \{ \chi \neq 0 \} \subset \{ \alpha \rho < -b \} \) and

\[
\frac{a}{b} \int_{\{w > -\infty\}} -\max\left(\frac{w}{a}, \rho\right) (dd^c w)^n < \varepsilon. \tag{3.4}
\]

Let \( \psi_j \in \mathcal{E}_0(\Omega) \) with \( \psi_j \geq \rho \) such that

\[
C_n(K \cap \{v_j - u_j > 2\delta\}) < \int_{K \cap \{v_j - u_j > 2\delta\}} (dd^c \psi_j)^n + \varepsilon. \tag{3.5}
\]

Note that \( u_j \leq v_j \) in \( \Omega \) for all \( j \geq 1 \). From the hypotheses we have

\[
0 \leq \limsup_{j \to +\infty} \int_{\{u_j < v_j - \delta\} \cap \{u_j > -b\}} \chi(dd^c u_j)^n
\]

\[
\leq \frac{1}{\delta} \limsup_{j \to +\infty} \int_{\{u_j < v_j - \delta\} \cap \{u_j > -b\}} \chi(v_j - u_j)(dd^c u_j)^n
\]

\[
\leq \frac{a}{\delta} \limsup_{j \to +\infty} \int_\Omega \left[ \max\left(\frac{v_j}{a}, \rho\right) - \max\left(\frac{u_j}{a}, \rho\right) \right] (dd^c u_j)^n
\]

\[
= 0.
\]

It follow that

\[
\lim_{j \to +\infty} \int_{\{u_j < v_j - \delta\} \cap \{u_j > -b\}} \chi(dd^c u_j)^n = 0.
\]

By Lemma 3.3 in [1] and Proposition 2.3 we have

\[
\limsup_{j \to +\infty} \int_{\{u_j < v_j - \delta\} \cap \{u_j > -\infty\}} \chi(dd^c u_j)^n
\]

\[
= \limsup_{j \to +\infty} \int_{\{u_j < v_j - \delta\} \cap \{-\infty < u_j < -b\}} \chi(dd^c u_j)^n
\]

\[
\leq \limsup_{j \to +\infty} \int_{\{u_j > -\infty\}} -\max\left(\frac{u_j}{b}, \frac{a \rho}{b}\right) (dd^c u_j)^n
\]

\[
\leq \frac{a}{b} \limsup_{j \to +\infty} \int_\Omega -\max\left(\frac{u_j}{a}, \rho\right) (dd^c u_j)^n + \frac{a}{b} \int_{\{u_j = -\infty\}} \max\left(\frac{u_j}{a}, \rho\right) (dd^c u_j)^n
\]

\[
\leq \frac{a}{b} \int_\Omega -\max\left(\frac{w}{a}, \rho\right) (dd^c w)^n + \frac{a}{b} \int_{\{w = -\infty\}} \max\left(\frac{w}{a}, \rho\right) (dd^c w)^n
\]

\[
= \frac{a}{b} \int_{\{w > -\infty\}} -\max\left(\frac{w}{a}, \rho\right) (dd^c w)^n.
\]

Therefore, by (3.4) we get

\[
\limsup_{j \to +\infty} \int_{\{u_j < v_j - \delta\} \cap \{u_j > -\infty\}} \chi(dd^c u_j)^n < \varepsilon. \tag{3.6}
\]

Now, by Proposition 2.3 and Lemma 3.1 we have

\[
\int_{K \cap \{v_j - u_j > 2\delta\}} (dd^c \psi_j)^n \leq \int_{\{u_j < v_j - 2\delta\}} (dd^c \psi_j)^n
\]

\[
\leq \frac{1}{\delta^n} \int_{\{u_j < v_j - 2\delta\}} (v_j - \delta - u_j)^n (dd^c \psi_j)^n
\]
subsets of \( \Omega \), applying Proposition 5.1 in [14] we see that there are 
\[ \bigcup \{ \{ v_j \in \Omega : v_j > \delta \} \} \]
Proposition 2.3 we have 
The uniqueness imply from Proposition 2.4. From the hypotheses and 
Proof.
there exists a unique 
\( u \)
Every nonnegative Borel measures 
\( w \in N \)
\( \Rightarrow \)
Lemma 4.1. In this section, we prove a generalization of Cegrell and Ko/suppresslodziej’s stability 

\[ \limsup_{j \to +\infty} \int_{K \cap \{ v_j - u_j > 2\delta \}} (dd^c \psi_j)^n \]
Combining this with (3.5) we get 
\[ \limsup_{j \to +\infty} \int_{K \cap \{ v_j - u_j > 2\delta \}} (dd^c \psi_j)^n \leq \frac{n!}{\delta^n} \int_{\Omega} -\max(\rho, -\varepsilon)(dd^c w)^n + \frac{n!\varepsilon}{\delta^n} \]
Combining this with (3.5) we get 
\[ \limsup_{j \to +\infty} C_n(\{ K \cap \{ v_j - u_j > 2\delta \} \}) \leq \frac{n!}{\delta^n} \int_{\Omega} -\max(\rho, -\varepsilon)(dd^c w)^n + \left( \frac{n!}{\delta^n} + 1 \right) \varepsilon. \]
Let \( \varepsilon \searrow 0 \) we obtain that 
\[ \lim_{j \to +\infty} C_n(\{ K \cap \{ v_j - u_j > 2\delta \} \}) = 0. \]
Thus, \( v_j - u_j \to 0 \) in \( C_n \)-capacity in \( \Omega \). The proof is complete. \( \square \)

4. Application

In this section, we prove a generalization of Cegrell and Ko/suppresslodziej’s stability 

Lemma 4.1. Let \( \Omega \) be a bounded hyperconvex domain in \( \mathbb{C}^n \) and let \( f \in \mathcal{E}(\Omega) \), 
\( w \in N^a(\Omega, f) \) such that \( \int_{\Omega} (-\rho)(dd^c w)^n < +\infty \) for some \( \rho \in \mathcal{E}_0(\Omega) \). Then for 
every nonnegative Borel measures \( \mu \) in \( \Omega \) such that 
\[ (dd^c f)^n \leq \mu \leq (dd^c w)^n, \]
there exists a unique \( u \in N^a(\Omega, f) \) such that \( u \geq w \) and \( (dd^c u)^n = \mu \) in \( \Omega \).
Proof. The uniqueness imply from Proposition 2.4. From the hypotheses and 
Proposition 2.3 we have 
\[ 1_{\{ w = -\infty \} } \mu = 1_{\{ f = -\infty \} } (dd^c f)^n \text{ in } \Omega. \]
Let \( \{ \Omega_j \} \) be a sequence of bounded hyperconvex domains such that \( \Omega_j \Subset \Omega_{j+1} \Subset \Omega \) and \( \Omega = \bigcup_{j=1}^{+\infty} \Omega_j \). Because the measure \( 1_{\{ w > -\infty \} } \mu \) vanishes on all pluripolar 
subsets of \( \Omega \), applying Proposition 5.1 in [14] we see that there are \( u_j \in N^a(\Omega_j, f) \) such that 
\[ (dd^c u_j)^n = 1_{\Omega_j \cap \{ w > -\infty \} } \mu + 1_{\Omega_j \cap \{ f = -\infty \} } (dd^c f)^n \text{ in } \Omega_j. \]
By Proposition 2.4 we have \( w \leq u_{j+1} \leq u_j \leq f \) on \( \Omega_j \). Put \( u := \lim_{j \to +\infty} u_j \).

Then \( w \leq u \leq f \) and \((dd^c u)^n = \mu\) in \( \Omega \). Moreover, since \( w \in N^a(\Omega, f)\), we get \( u \in N^a(\Omega, f)\). The proof is complete.

**Proposition 4.2.** Let \( \Omega \) be a bounded hyperconvex domain in \( \mathbb{C}^n \) and let \( f \in \mathcal{E}(\Omega) \). Assume that \( w \in N^a(\Omega, f) \) such that \( \int_\Omega (-\rho)(dd^c w)^n < +\infty \) for some \( \rho \in \mathcal{E}_0(\Omega) \). Then for every sequence of nonnegative Borel measures \( \{\mu_j\} \) that converges weakly to a non-negative Borel measure \( \mu_0 \) in \( \Omega \) and satisfies

\[
(dd^c f)^n \leq \mu_j \leq (dd^c w)^n \quad \text{for all } j \geq 0,
\]

there exist unique \( u_j \in N^a(\Omega, f) \) such that \( u_j \geq w \), \((dd^c u_j)^n = \mu_j \) and \( u_j \to u_0 \) in \( C_n \)-capacity in \( \Omega \).

**Proof.** By Lemma 4.1 there exist unique \( u_j \in N^a(\Omega, f) \) such that \( u_j \geq w \) and \((dd^c u_j)^n = \mu_j \) in \( \Omega \). Since \( u_j \geq w \), the sequence \( \{u_j\} \) is compact in \( L^1_{loc}(\Omega) \). Let \( u \) be a cluster point and let \( \{u_{j_k}\} \) be a subsequence of the sequence \( \{u_j\} \) such that \( u_{j_k} \to u \) a.e. in \( \Omega \). Put \( v_k := (sup_{j \geq k} u_j)^+ \). We claim that

\[
\lim_{k \to +\infty} \int_\Omega \left[ \max \left( \frac{v_k}{a}, \rho \right) - \max \left( \frac{u_{j_k}}{a}, \rho \right) \right] (dd^c u_{j_k})^n = 0,
\]

for every \( a > 0 \). Indeed, let \( \varepsilon > 0 \). Choose \( \chi \in C_0^\infty(\Omega) \) such that \( 0 \leq \chi \leq 1 \) and \( \{\chi = 1\} \subset \{\rho < -\varepsilon\} \). By Proposition 2.3 we have that the measure \( \int_{\{f > -\varepsilon\}} \chi(dd^c w)^n \) vanishes on all pluripolar subsets of \( \Omega \). By Lemma 3.1 in \( \mathbb{S} \) we get

\[
0 \leq \limsup_{k \to +\infty} \int_\Omega \left[ \max \left( \frac{v_k}{a}, \rho \right) - \max \left( \frac{u_{j_k}}{a}, \rho \right) \right] \chi(dd^c u_{j_k})^n
\]

\[
= \limsup_{k \to +\infty} \int_\Omega \left[ \max \left( \frac{v_k}{a}, \rho \right) - \max \left( \frac{u_{j_k}}{a}, \rho \right) \right] 1_{\{f > -\varepsilon\}} \chi(dd^c u_{j_k})^n
\]

\[
\leq \limsup_{k \to +\infty} \int_\Omega \left[ \max \left( \frac{v_k}{a}, \rho \right) - \max \left( \frac{u_{j_k}}{a}, \rho \right) \right] 1_{\{f > -\varepsilon\}} \chi(dd^c w)^n
\]

\[
= \int_\Omega \left[ \max \left( \frac{u}{a}, \rho \right) - \max \left( \frac{u_{j_k}}{a}, \rho \right) \right] 1_{\{f > -\varepsilon\}} \chi(dd^c w)^n = 0.
\]

It follows that

\[
0 \leq \limsup_{k \to +\infty} \int_\Omega \left[ \max \left( \frac{v_k}{a}, \rho \right) - \max \left( \frac{u_{j_k}}{a}, \rho \right) \right] (dd^c u_{j_k})^n
\]

\[
\leq \limsup_{k \to +\infty} \int_\Omega \left[ \max \left( \frac{v_k}{a}, \rho \right) - \max \left( \frac{u_{j_k}}{a}, \rho \right) \right] (1 - \chi)(dd^c u_{j_k})^n
\]

\[
\leq \limsup_{k \to +\infty} \int_{\{\rho \geq -\varepsilon\}} \left[ - \max \left( \frac{v_k}{a}, \rho \right) - \max \left( \frac{u_{j_k}}{a}, \rho \right) \right] (dd^c u_{j_k})^n
\]

\[
\leq 2 \int_\Omega - \max(\rho, -\varepsilon)(dd^c w)^n.
\]

Let \( \varepsilon \searrow 0 \) we obtain \( 1.1 \). This proves the claim, and therefore, by the main theorem we get \( u_{j_k} \to u \) in \( C_n \)-capacity in \( \Omega \) as \( k \to +\infty \). Hence, by \( \mathbb{S} \) we have \((dd^c u)^n = \mu_0\) in \( \Omega \). It is clear that \( u \in N^a(\Omega, f) \). From the uniqueness of \( u_0 \) we get \( u = u_0 \). Thus, \( u_{j_k} \to u_0 \) a.e. in \( \Omega \). It follows that \( u_j \to u_0 \) a.e. in \( \Omega \). Similarly, we get

\[
\lim_{j \to +\infty} \int_\Omega \left[ \max \left( \frac{v_j}{a}, \rho \right) - \max \left( \frac{u_j}{a}, \rho \right) \right] (dd^c u_j)^n = 0,
\]
for every $a > 0$, where $v_j := (\sup_{k \geq j} u_k)^*$. Now, again by the main theorem we get $u_j \to u_0$ in $C_n$-capacity in $\Omega$. The proof is complete. \hfill $\square$

References

[1] P. Ahag, U. Cegrell, R. Czyż and P. H. Hiep, Monge-Ampère measures on pluripolar sets, J. Math. Pures Appl., 92 (2009), 613–627.
[2] E. Bedford and B.A. Taylor, The Dirichlet problem for a complex Monge-Ampère operator, Invent. Math., 37 (1976), 1–44.
[3] E. Bedford and B. A. Taylor, A new capacity for plurisubharmonic functions, Acta Math., 149 (1982), 1–40.
[4] Z. Błocki, On the $L^p$-stability for the complex Monge-Ampère operator, Michigan Math. J., 42 (1995), 269–275.
[5] U. Cegrell, Pluricomplex energy, Acta Math., 180 (1998), 187–217.
[6] U. Cegrell, The general definition of the complex Monge-Ampère operator, Ann. Inst. Fourier (Grenoble), 54, 1 (2004), 159–179.
[7] U. Cegrell, A general Dirichlet problem for the complex Monge-Ampère operator, Ann. Polon. Math., 94 (2008), 131–147.
[8] U. Cegrell, Convergence in Capacity, Canad. Math. Bull., 55, 2 (2012), 242–248.
[9] U. Cegrell and S. Kołodziej, The equation of complex Monge-Ampère type and stability of solutions, Math. Ann., 334 (2006), 713–729.
[10] V. Guedj and A. Zeriahi, Stability of solutions to complex Monge-Ampère equations in big cohomology classes, Mathematical Research Letters, 19 (2012), Number 5, 1025–1042.
[11] L. M. Hai and N. X. Hong, Subextension of plurisubharmonic functions without changing the Monge-Ampère measures and applications, Ann. Polon. Math., 112 (2014), 55–66.
[12] L. M. Hai, N. X. Hong and T. V. Dung, Subextension of plurisubharmonic functions with boundary values in weighted pluricomplex energy classes, Complex Var. Elliptic Equ., 60, Issue 11 (2015), 1580–1593.
[13] L. M. Hai, P. H. Hiep, N. X. Hong and N. V. Phu, The Monge-Ampère type equation in the weighted pluricomplex energy class, International Journal of Mathematics, 25, No. 5 (2014), 1450042 (17 pages).
[14] L. M. Hai, N. V. Trão and N. X. Hong, The complex Monge-Ampère equation in unbounded hyperconvex domains in $\mathbb{C}^n$, Complex Var. Elliptic Equ., 59 (2014), no. 12, 1758–1774.
[15] P. H. Hiep, Convergence in capacity and applications, Math. Scand., 107 (2010), 90–102.
[16] N. X. Hong, Monge-Ampère measures of maximal subextensions of plurisubharmonic functions with given boundary values, Complex Var. Elliptic Equ., 60 (2015), no. 3, 429–435.
[17] N. X. Hong, The locally $F$-approximation property of bounded hyperconvex domains, J. Math. Anal. Appl., 428 (2015), 1202–1208.
[18] M. Klimek, Pluripotential Theory, The Clarendon Press Oxford University Press, New York, 1991, Oxford Science Publications.
[19] N. V. Khue and P. H. Hiep, A comparison principle for the complex Monge-Ampère operator in Cegrell’s classes and applications, Trans. Am. Math. Soc., 361, 10 (2009), 5539–5554.
[20] S. Kołodziej, The Complex Monge-Ampère Equation and Pluripotential Theory, Memoirs of AMS., 840 (2005).
[21] Y. Xing, Continuity of the complex Monge-Ampère operator, Proc. Amer. Math. Soc., 124, 2 (1996), no. 2, 457–467.
[22] Y. Xing, Convergence in capacity, Ann. Inst. Fourier (Grenoble), 58 (2008), no. 5, 1839–1861.

Department of Mathematics, Hanoi National University of Education, 136 Xuan Thuy Street, Cau Giay District, Hanoi, Vietnam
E-mail address: xuanhongdhsp@yahoo.com, ngvtrao@yahoo.com and thuyhum@gmail.com