A FIXED-POINT APPROXIMATION FOR A ROUTING MODEL IN EQUILIBRIUM

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Abstract. We use a method of Luczak [6] to investigate the equilibrium distribution of a dynamic routing model on a network. In this model, there are \( n \) nodes, each pair joined by a link of capacity \( C \). For each pair of nodes, calls arrive for this pair of endpoints as a Poisson process with rate \( \lambda \). A call for endpoints \( \{u, v\} \) is routed directly onto the link between the two nodes if there is spare capacity; otherwise \( d \) two-link paths between \( u \) and \( v \) are considered, and the call is routed along a path with lowest maximum load, if possible. The duration of each call is an exponential random variable with unit mean. In the case \( d = 1 \), it was suggested by Gibbens, Hunt and Kelly in 1990 that the equilibrium of this process is related to the fixed points of a certain equation. We show that this is indeed the case, for every \( d \geq 1 \), provided the arrival rate \( \lambda \) is either sufficiently small or sufficiently large. In either regime, we show that the equation has a unique fixed point, and that, in equilibrium, for each \( j \), the proportion of links at each node with load \( j \) is strongly concentrated around the \( j \)th coordinate of the fixed point.

1. Introduction

We consider a routing model in continuous time, where calls have Poisson arrivals and exponential durations, versions of which were studied earlier in [1, 2, 3, 4, 6, 7, 8, 9]. This is to a large extent a companion paper to Luczak [6], using the same methods and some of the results of that paper; here, our focus is on the properties of the model in equilibrium.

The setting is as follows. For each \( n \in \mathbb{N} \), we have a fully connected communication graph \( K_n \), with node set \( V_n = \{1, \ldots, n\} \) and link set \( E_n = \{\{u, v\} : 1 \leq u < v \leq n\} \). Each link \( \{u, v\} \in E_n \) (which we often denote as \( uv \)) has capacity \( C \in \mathbb{Z}^+ \). Calls arrive as a Poisson process with rate \( \lambda(n) \), where \( \lambda \) is a positive constant, and each arriving call has a pair \( \{u, v\} \) of endpoints, chosen uniformly from among the \( \binom{n}{2} \) possible pairs. The arriving call is routed directly along the link \( uv \), if that link has fewer than \( C \) calls currently using it. If the link \( uv \) is fully loaded, then we pick an ordered
list \((w_1, \ldots, w_d)\) of \(d\) possible intermediate nodes from \(V_n \setminus \{u, v\}\), uniformly at random with replacement, and the call is routed \textit{indirectly} along one of the two-link routes \(uw_1v, \ldots, uw_dv\), chosen to minimise the larger of the current loads on its two links, subject to the capacity constraints. Ties are broken in favour of the first ‘best’ route in the ordered list. If none of the \(d\) two-link paths is available, then the call is lost. Call durations are unit mean exponential random variables, independent of one another and of the arrivals and choices processes. This routing strategy is called the BDAR (Balanced Dynamic Alternative Routing) Algorithm.

This model was introduced by Gibbens, Hunt and Kelly \cite{Gibbens1990} in 1990, in the case \(d = 1\), motivated by a dynamic routing algorithm used at that time by British Telecom. Gibbens, Hunt and Kelly did not analyse the model directly, stating that “It is difficult to analyse the process, even in equilibrium”. Instead, they analysed a simplified version where the graph structure is neglected: there are \(K\) links, and each arriving call chooses one “direct link” and two “alternative links”, uniformly independently at random from the set of all links. If the direct link has spare capacity, then the call is routed along that link; if not, then the call is routed on the two alternative links if both have spare capacity. If a call is accepted onto the two alternative links, then the durations of that call on each of the two links are independent exponential random variables with mean 1. These devices are designed to ensure that the state of this system at time \(t\) can effectively be captured solely by the proportions \(\xi_t(j)\) of links with \(j\) calls, for \(j = 0, \ldots, C\). Gibbens, Hunt and Kelly proved a functional law of large numbers for this simplified system, showing in particular that the vector \(\xi_t\) converges to the solution of a certain differential equation. They suggested that the BDAR network model should behave similarly to their simplified version. They also noted that this behaviour is not always benign: for certain values of the parameters, the differential equation has multiple fixed points, and they gave a heuristic argument indicating that, in some such cases, the time taken to “tunnel” from the neighbourhood of one fixed point to the other is exponential in the number \(K\) of links.

Following the work of Gibbens, Hunt and Kelly, a number of authors studied various aspects of the BDAR model, and/or variants of it. Crametz and Hunt \cite{Crametz2007} and Graham and Méleard \cite{Graham2008} showed that, with suitable assumptions on the initial conditions, the behaviour of the BDAR model follows that of the differential equation over a constant time interval in the case \(d = 1\). These results are considerably extended by Luczak \cite{Luczak2010}, who gives quantitative results, and also treats the case of general \(d \geq 1\).

Several variants on the BDAR model have been considered in the literature. Gibbens, Hunt and Kelly \cite{Gibbens1990} introduce a version, later called the FDAR model, where again \(d\) alternative two-link routes are considered, but now the call is routed along the \textit{first} route in the list where both links have spare capacity, if any. Luczak and Upfal \cite{Luczak2011}, Luczak, McDiarmid and Upfal \cite{Luczak2011}, Anagnostopoluos, Kontoyannis and Upfal \cite{Anagnostopoluos2011}, and Luczak and
McDiarmid [7] consider versions where each link has some of its capacity reserved for indirectly routed calls and (possibly) some for directly routed calls: these versions have the advantage that, for $d \geq 2$, the model exhibits the “power of two choices” phenomenon, leading to far more even allocations of loads across the network. An explicit illustration of this concerns the minimum value of the capacity $C = C(n, \lambda, d)$ required so that few or no calls are lost over a long time interval, which is of order $\log \log n$ if $d \geq 2$, some capacity is reserved for indirectly routed calls, and the BDAR rule is used to select which indirect route to use.

Luczak and McDiarmid [7] prove results about the equilibrium distribution, for a variant of the model with reserved capacity, in the case where the capacity $C$ tends to infinity with $n$, proving in particular that, in that model, which exhibits the power of two choices, the proportion of links of load $j$ falls off doubly exponentially with $j$. To the best of our knowledge, this is the only previous work on the equilibrium distribution of any routing model in this family. In our model, for parameter values where the approximating differential equation has a unique fixed point, it is natural to expect that the proportions $\eta(j)$ are well concentrated, and that their expectations are well-approximated by the corresponding coordinate of this fixed point, but up to now there have been no results along these lines. Analysing the fixed point for our model would show that the proportions of links with each load fall off singly exponentially in the regime with $\lambda$ small. For the regime we study with $\lambda$ large, most links have load $C$ in equilibrium.

In this paper, we prove such a result for two ranges of parameter values. We assume throughout that the number $d$ of choices and the capacity $C$ are fixed positive integers, the arrival rate $\lambda$ is also constant, and $n$ tends to infinity. We consider the cases where $\lambda \leq m_1/d$, and where $\lambda \geq m_2C^2d\log(C^2d)$, for suitable constants $m_1, m_2$. We prove that, in either of these two regimes, the corresponding sequence of Markov chains is rapidly mixing, and that, asymptotically, the vector $\eta$ is well concentrated around the unique fixed point of the differential equation (even more, for each node $v$, the proportion of links incident to $v$ with each load $j$ is well concentrated around the fixed point). This establishes a strong form of the ‘Erlang fixed point approximation’ proposed in [3] in these regimes. See Kelly’s survey [5] for more information. Some cases where $C$, $d$ and $\lambda$ are slowly growing functions of $n$ can also easily be handled by our methods, as in [6]: note that in these circumstances there is no single differential equation approximating the process.

Such ‘law of large numbers in equilibrium’ results have been difficult to prove in settings where there are potentially strong dependencies between system elements (in this context, links). Here we are able to prove that these dependencies are negligible; it turns out that, in a suitable sense, links in certain collections evolve approximately independently of one another.

Our technique was developed by Luczak [6], and uses a general concentration of measure inequality proved in that paper. The heart of our argument
is the analysis of a natural coupling introduced in [6], which in either of
the two regimes we study is contractive. Using this coupling, we prove that
two copies of the Markov chain coalesce quickly. The same coupling is also
used to show that slowly-changing functions of the process (for instance, the
number of links incident with a node \( v \) with load exactly \( j \) at a given time
\( t \), for each node \( v \) and each \( j \in \{0, 1, \ldots, C\} \) are well concentrated at each
time \( t \), with bounds uniform over \( t \). This can then be used to show that
such functions are also well concentrated in equilibrium. (Note that in [6],
no attempt was made to show that the coupling is contractive; instead it
was shown that copies of the process starting close to one another do not
diverge too quickly. This is why the concentration of measure bounds in [6]
are not uniform in time, but only work over a bounded time interval or
an interval of length only slowly increasing with \( n \).) Thanks to the strong
concentration of measure, it is then possible to show that the equilibrium
balance equations for functions of interest factorise approximately, leading
to a fixed point approximation.

We believe that both the concentration of measure inequalities and the
technique as a whole will find further applications. The method is suitable
for proving laws of large numbers in a wide variety of settings, and – as we
demonstrate in this paper – it may sometimes be effective in circumstances
where more standard techniques cannot be used.

Our technique would require only very minor amendments to handle the
FDAR model, where the first available indirect route is chosen, rather than
the ‘best’. The method could also easily be adapted to handle variations
with some capacity reserved for indirectly routed calls, although the details
would be somewhat different. We intend to cover such a variant in future
work, allowing also \( C \) to tend to infinity with \( n \).

For each link \( uv \in E_n \), let \( X_t(uv, 0) \) denote the number of calls in progress
for endpoints \( \{u, v\} \) at time \( t \) which are routed directly along the link \( uv \).
For each pair \( \{u, v\} \) of distinct nodes, and each node \( w \in V_n \setminus \{u, v\} \), let
\( X_t(uvw) \) denote the number of calls in progress at time \( t \) which are routed
along the path \( uvw \) consisting of links \( \{uw, vw\} \), that are in progress at
time \( t \). We call \( X_t = ((X_t(uv, 0))_{u \neq v}, (X_t(uvw))_{u \neq v, w \neq u, v}) \) the load vector
at time \( t \), and let \( S = \{0, 1, \ldots, C\}^{(\frac{n}{2})(n-1)} \) denote the state space, containing
the set of all possible load vectors. Then \( X = (X_t)_{t \geq 0} \) is a continuous-time
discrete-space Markov chain. The chain \( X \), its state space, and functions on
the state space depend on the number \( n \) of nodes, but we will suppress this
dependence in our notation. Given a load vector \( x \in S \) and a link \( uv \in E_n \),
let \( x(uv) \) denote the load of link \( uv \):

\[
x(uv) = x(uv, 0) + \sum_{w \notin \{u,v\}} \left( x(uvw) + x(vuw) \right).
\]

We will also work with a jump chain \( \tilde{X}_t \) corresponding to \( X_t \). The chain
we use is not the standard embedded chain but a slower moving version that
will often not change its state at a given step. Given that the current state, at time $t \in \mathbb{Z}^+$, is $x \in S$, the next event is an arrival with probability

$$p(\lambda, C) = \frac{\lambda}{\lambda + C},$$

and a potential departure with probability $1 - p(\lambda, C) = C/(\lambda + C)$. Given that the event is an arrival, each pair of endpoints $\{u, v\}$ is chosen with probability $1/\binom{n}{2}$, then each $d$-tuple of intermediate nodes is chosen with probability $(n-2)^d$, and the call is routed directly on $uv$ if $x(uv) < C$, and otherwise along the first two-link route among the $d$ selected that minimises the maximum load of a link. Given that the event is a potential departure, the calls currently in the system are enumerated from 1 up to at most $C\binom{n}{2}$, and then a number is chosen uniformly at random from $\{1, \ldots, C\binom{n}{2}\}$. If there is a call assigned to this number, it departs; otherwise nothing happens.

We claim that the discrete jump chain $\hat{X}$ and the continuous-time chain $X$ have the same equilibrium distribution. To see this, note that the $\pi$-vector of the discrete jump chain satisfy $P = \frac{1}{(\lambda + C)^{\binom{n}{2}}} Q + I$. Indeed, given any transition between different states in the continuous chain, with rate $r > 0$, the corresponding entry in $P$ is $r/(\lambda + C)^{\binom{n}{2}}$, by construction; also, the diagonal entries of $P$ are given by $p_{xx} = 1 - \sum_{y \neq x} p_{xy}$, and the diagonal entries of $Q$ are given by $q_{xx} = -\sum_{y \neq x} q_{xy}$. Thus the equilibrium vector $\pi$ of the discrete jump chain, which satisfies $\pi P = \pi$, also satisfies $\pi Q = 0$, and hence is the equilibrium of the continuous-time chain $X$. As our key results concern the equilibrium distribution of the chains, we may and shall work throughout with the discrete jump chain.

The same chain is studied in [6], with no assumptions on $\lambda$, over a bounded time interval. It is shown there that, subject to suitable initial conditions, the chain does stay close to the solution to a certain differential equation over such a time interval.

For a vector $\xi = (\xi(k) : k = 0, \ldots, C)$, let $\xi(\leq j) = \sum_{k=0}^{j} \xi(k)$. Define $F : \mathbb{R}^{C+1} \to \mathbb{R}^{C+1}$ by

$$F_0(\xi) = -\lambda \xi(0) - \lambda g_0(\xi) + \xi(1),$$
$$F_k(\xi) = \lambda \xi(k-1) - \lambda \xi(k) + \lambda g_{k-1}(\xi) - \lambda g_k(\xi) - k \xi(k) + (k+1) \xi(k+1), \quad 0 < k < C,$$
$$F_C(\xi) = \lambda \xi(C-1) + \lambda g_{C-1}(\xi) - C \xi(C),$$

where the functions $g_j$, $j = 0, \ldots, C-1$, are given by

$$g_j(\xi) = 2\xi(C)\xi(j)\xi(\leq j) \sum_{r=1}^{d} (1 - \xi(\leq j))^r (1 - \xi(\leq j-1)^2)^{d-r}$$
$$+ 2\xi(C)\xi(j) \sum_{i=j+1}^{C-1} \xi(i) \sum_{r=1}^{d} (1 - \xi(i))^r (1 - \xi(i-1)^2)^{d-r}. \quad (1.1)$$
As shown, effectively, by Gibbens, Hunt and Kelly [3], these equations describe the evolution of their simplified process. Later, Crametz and Hunt [2], Graham and Méleard [4] and Luczak [6] showed that the same equations describe the evolution of the true process, under suitable initial conditions.

If we imagine that links behave completely independently, each having load \( j \) at time \( t \) with probability \( \xi(j) \), then \( \lambda g_j(\xi) \) would be proportional to the rate of arrivals which are indirectly routed onto a link with current load \( j \).

The first summand in (1.2) corresponds to the event that the other link on the indirect route considered has load at most \( j \), and the index \( r \) in the summation is the position of this route in the list of \( d \) indirect routes considered: each route ahead of it in the list has to have a link of load greater than \( j \), and each route behind it in the list has to have a link of load at least \( j \). The second summand corresponds to the event that the other link on the route considered has load \( i > j \).

In the idealisation, \( F_k(\xi) \) is proportional to the expected rate of change in the number of links of load \( k \). In the case \( 0 < k < C \), the six terms in the expression for \( F_k(\xi) \) correspond respectively to: arrivals routed directly onto a link of load \( k \), arrivals routed indirectly onto a link of load \( k \), departures from a link of load \( k \), and departures from a link of load \( k + 1 \). In particular, in equilibrium, the expected proportions \( \eta(j) \) of links with each load \( j \) should be close to a solution of the equation \( F(\eta) = 0 \). By symmetry, the same approximation should hold if we instead count the expected proportion of links incident to a given node with load \( j \).

Set \( \Delta^{C+1} = \{ \xi \in [0,1]^{C+1} : \sum_{j=0}^{C} \xi(j) = 1 \} \). It is proved in [6] that, for any \( \xi_0 \in \Delta^{C+1} \), the equation

\[
\frac{d\xi_t}{dt} = F(\xi_t)
\]

has a unique solution starting from \( \xi_0 \), with \( \xi_t \in \Delta^{C+1} \) for all \( t \geq 0 \).

Given a load vector \( x \), node \( v \) and \( k \in \mathbb{Z}^+ \), let \( f_{v,k}(x) = \sum_{w \neq v} 1_{k_{vw}}(x) \) be the number of links in \( x \) with one end \( v \) carrying exactly \( k \) calls. The main theorem of [6] states that, for each node \( v \), and each \( k \in \{0, \ldots, C\} \), over any bounded time interval \([0, t_0]\), the function \( \frac{1}{n} f_{v,k}(X_t) \) closely follows the solution to the differential equation (1.3), with high probability, provided \( n \) is sufficiently large. The result is quantitative, with explicit bounds on the fluctuations, in terms of the parameters of the model and also of a function \( \phi \) measuring how “uniform” the initial state of the system is.

Our results concern the equilibrium distribution of the chain \( X \), or equivalently of the discrete jump chain \( \hat{X} \). We set \( \lambda_0 = \lambda_0(d) = 1/(8d + 4) \), and \( \lambda_1 = \lambda_1(d, C) = 8000C^2d\log(240C^2d) \). Let \( \pi \) denote the equilibrium distribution of the chain, and \( \mu(x_0, t) \) denote the distribution of \( \hat{X}_t \) conditional on \( \hat{X}_0 = x_0 \), for any \( x_0 \in S \) and \( t \geq 0 \). Let \( \hat{Z} \) denote a copy of the discrete jump chain in equilibrium. Our results are as follows.
Theorem 1.1. For $d, C \in \mathbb{N}$, if either $\lambda < \lambda_0(d)$, or $\lambda \geq \lambda_1(d, C)$, then there is $n_0 = n_0(d, C, \lambda)$ such that the following hold for all $n \geq n_0$.

1. The discrete jump chain $\hat{X}$ is rapidly mixing: there are constants $\gamma = \gamma(d, C, \lambda)$ and $K = K(d, C, \lambda)$ such that $d_{TV}(\mu(x_0, t), \pi) \leq K n^2 e^{-t/n^2}$ for all $x_0 \in S$ and all $t \in [0, n^{5/2}]$. In particular, $d_{TV}(\mu(x_0, t), \pi) = o(1)$ for $t \geq 3 \gamma^{-1} n^2 \log n$.

2. There are constants $c_1, c_2 > 0$, depending on $d, C$ and $\lambda$, such that, for each node $v$, each $j \in \{0, \ldots, C\}$, each $t$, and any $a > 0$,

$$
P_\pi \left( \left| f_{v,j}(\bar{Z}_t) - \mathbb{E}_\pi f_{v,j}(\bar{Z}_t) \right| > 2a \right) \leq 3 \exp \left( - \frac{a^2}{c_1 n + c_2 a} \right).
$$

3. There is a unique solution $\eta^* \in \Delta^{C+1}$ to the equation $F(\eta) = 0$.

4. For all nodes $v$, all $j \in \{0, \ldots, C\}$, and all $t$,

$$
\left| \frac{1}{n-1} \mathbb{E}_\pi f_{v,j}(\bar{Z}_t) - \eta^*(j) \right| \leq 160d^2(C + 1)^4 \log n \sqrt{n}.
$$

5. Let $A$ be the event that $|f_{v,j}(\bar{Z}_t) - (n-1)\eta^*(j)| \leq 200d^2(C+1)^4 \sqrt{n} \log n$, for all nodes $v$ and all $j \in \{0, \ldots, C\}$. Then $P_\pi(A) \leq 3Cn^2 e^{-\delta \log^2 n}$ for some constant $\delta = \delta(d, C, \lambda)$.

Note that Theorem 1.1(5) follows immediately from parts (2) and (4), with $a = 30d^2(C + 1)^4 \log n \sqrt{n}$. As remarked earlier, parts (2), (4) and (5) apply equally to the continuous time chain $Z$ in equilibrium, as the equilibrium distributions of the two chains are the same. It is also easy to deduce that the continuous time chain $X$ is rapidly mixing, in time $O(\log n)$, in either of the two regimes considered.

We can deduce immediately from Theorem 1.1(5) that, in either regime, the total number of links in the network of each load $j$ is within $O(n^{3/2} \log n)$ of $\binom{n}{2} \eta^*(j)$. We expect that this error term is not optimal.

Theorem 1.1(3) is not true without some assumptions on the parameters: Gibbens, Hunt and Kelly found that, for $d = 1$ and $C$ sufficiently large, there is a range of $\lambda$ (apparently roughly of order $C$) where there are multiple solutions to the fixed-point equation $F(\eta) = 0$. In such circumstances, we should not expect rapid mixing, or strong concentration of measure in equilibrium. However, it should be possible to improve the functions $\lambda_0$ and $\lambda_1$; the results above are likely to hold for any values of the parameters where the equation $F(\eta) = 0$ does have a unique solution. For instance, in the special case $d = 1$, $C = 1$, the equation has a unique solution for every $\lambda > 0$, and we feel that the results above should hold for the whole range of $\lambda$ in this case. Preliminary calculations indicate that we can improve the bounds on $\lambda$ at the expense of increased complexity in the proofs: we intend to return to this in a subsequent paper, but for now our main purposes are to show that we do indeed have strong concentration of measure in some ranges of the parameters, and to illustrate the power of our methods.
The structure of the paper is as follows. We begin with some preliminary results and definitions in Section 2 including the definition of the key coupling we use. The next two sections are devoted to the analysis of the coupling in the two regimes, $\lambda < \lambda_0(d)$ and $\lambda \geq \lambda_1(d, C)$ respectively. In these sections, we establish rapid mixing of the discrete jump chain, and concentration of measure for the process, both started from a fixed state and in equilibrium. In particular, we prove parts (1) and (2) of Theorem 1.1 in these sections. In Section 5 we use concentration of measure to show that the vector $\zeta$ given by $\zeta(j) = \frac{1}{n-t} \mathbb{E}_x f_{v,j}(\hat{Z}_t)$ satisfies an approximate version of the equation $F(\eta) = 0$. In the final section, we use a contraction argument to prove parts (3) and (4) of Theorem 1.1; we show that the equation $F(\eta) = 0$ has a unique solution $\eta^*$, in either of our two regimes, and that any solution to the approximate version of the equation lies close to $\eta^*$.

2. Preliminaries

Let $\hat{X} = (\hat{X}_t)_{t \in \mathbb{Z}^+}$ be a discrete-time Markov chain with a discrete state space $S$ and transition probabilities $P(x, y)$ for $x, y \in S$. We allow $\hat{X}$ to be lazy, that is we allow $P(x, x) > 0$ for some $x \in S$. We assume that, for each $x \in S$, the set $N(x) = \{ y : P(x, y) > 0 \}$ is finite. Let $\mathbb{P}_{x_0}$ denote the probability measure associated with the chain conditioned on $X_0 = x_0$, and let $\mathbb{E}_{x_0}$ denote the corresponding expectation operator; then $\mathbb{E}_{x_0}[f(\hat{X}_t)]$ is the expectation of the function $f$ with respect to measure $\delta_{x_0} P^t$.

The following concentration of measure result is from [6]. In fact, the version in [6] is more general, catering for the case when the inequalities only hold for states in some “good set” $S_0 \subset S$.

**Theorem 2.1.** Let $P$ be the transition matrix of a discrete-time Markov chain with discrete state space $S$, and let $f : S \to \mathbb{R}$ be a function. Suppose that, for each $x \in S$ and each $i \in \mathbb{Z}^+$, the function $\alpha_{x,i} : S \to \mathbb{R}$ satisfies:

$$\left| \mathbb{E}_x[f(\hat{X}_i)] - \mathbb{E}_y[f(\hat{X}_i)] \right| \leq \alpha_{x,i}(y),$$

for all $y \in S$. Suppose also that the sequence $(\alpha_i : i \in \mathbb{Z}^+)$ of positive constants satisfies:

$$\sup_{x \in S}(P\alpha_{x,i}^2)(x) \leq \alpha_i^2.$$  

(2.2)

Let $t > 0$, and set $\beta = 2\sum_{i=0}^{t-1} \alpha_i^2$. Suppose also that $\bar{\alpha}$ is such that

$$\sup_{0 \leq i \leq t-1} \sup_{x \in S, y \in N(x)} \alpha_{x,i}(y) \leq \bar{\alpha}.$$  

(2.3)

Then, for all $a > 0$,

$$\mathbb{P}_{x_0} \left( |f(\hat{X}_t) - \mathbb{E}_{x_0}[f(\hat{X}_t)]| \geq a \right) \leq 2e^{-a^2/(2\beta + \frac{4}{3} \bar{\alpha} a)}.$$  

(2.4)

Given two load vectors $x, y$, the $\ell_1$-distance between them is

$$\|x - y\|_1 = \sum_{uv \in E_n} |x(uv, 0) - y(uv, 0)| + \sum_{uv \in E_n, u \neq u, v} |x(uvw) - y(uvw)|,$$
the sum of the differences between $x$ and $y$ in loads of all possible routes. Then $\| \cdot \|_1$ is a metric on $S$. For $v \in V_n$, we will also consider the function

$$\|x - y\|_v = \sum_{u \neq v} |x(uv, 0) - y(uv, 0)| + \sum_{u, w} |x(uwv) - y(uwv)| + \sum_{\{u,w\}} |x(uvw) - y(uvw)|,$$

where the sums are over distinct nodes $u, w \neq v$.

Consider the following family of Markovian couplings $(\hat{X}, \hat{Y})$ of pairs of copies $\hat{X}, \hat{Y}$ of the discrete jump chain starting from states $x_0, y_0$ respectively, where $x_0, y_0 \in S$. Let $t \geq 0$, and let $x, y$ be states in $S$. Given that $\hat{X}_{t-1} = x$ and $\hat{Y}_{t-1} = y$, the transition at time $t$ (from state $(\hat{X}_{t-1}, \hat{Y}_{t-1})$ to $(\hat{X}_t, \hat{Y}_t)$) is an arrival in both $\hat{X}$ and $\hat{Y}$, or a potential departure in both $\hat{X}$ and $\hat{Y}$. Given that the transition is an arrival, we choose the same call endpoints and the same $d$-tuple of intermediate nodes in both. Also, given that the transition is a potential departure, we pair calls occupying the same route in both $\hat{X}$ and $\hat{Y}$, and call such pairs matched, as much as possible. We also pair off the remaining calls arbitrarily, as much as possible, in some fashion depending only on $x$ and $y$. (We can pair off all the calls if $\|x\|_1 = \|y\|_1$; otherwise some calls remain unpaired in the process that has more calls.) Then we always choose paired calls for a simultaneous departure; any unpaired calls depart alone. This is achieved by assigning to each pair, and also to each unpaired call, a distinct number in $\{1, \ldots, C(n)\}$. Then, if the transition at time $t$ is a potential departure, we choose a uniformly random number from $\{1, \ldots, C(n)\}$. If the number corresponds to a pair of calls, one in $\hat{X}$ and one in $\hat{Y}$, both depart; if it corresponds to an unpaired call, this call departs and there is no change in the other state; otherwise, nothing happens. The process $\hat{W} = (\hat{W}_t)$ given by $\hat{W}_t = (\hat{X}_t, \hat{Y}_t)$ is a Markov chain adapted to its natural filtration, $\mathcal{G}_t = \sigma(\hat{X}_s, \hat{Y}_s : s \leq t)$.

3. Analysis of coupling – low arrival rate

In this section, we prove that, provided $\lambda$ is sufficiently small, for two coupled copies $\hat{X}$ and $\hat{Y}$ of the discrete jump chain, the distance $\|\hat{X}_t - \hat{Y}_t\|_1$ is contractive. We also prove a similar result for each $\|\hat{X}_t - \hat{Y}_t\|_v$. The aim is firstly to show that the chain is rapidly mixing, and also then to show that quantities of interest are well concentrated, both starting in a fixed state uniformly over long time intervals, and in equilibrium.

Our main use of the lemma below will be in the case $\lambda < \lambda_0(d) = 1/(8d + 4)$, when it indeed states that the distance is contractive under the coupling. We shall also use this lemma in the next section, to show that the distance is only mildly expansive even for much larger $\lambda$: in this context, the result is very similar to a lemma of Luczak [6].
Lemma 3.1. Suppose that \( n \geq \max(d^2, 6) \). Let \( \hat{X} \) and \( \hat{Y} \) be two copies of the discrete jump chain, coupled as in the previous section. Then we have:

\[
\mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_1 \mid G_0) \leq \left( 1 - \frac{1 - (8d + 4)\lambda}{(\lambda + C)(\binom{n}{2})} \right)^t \|\hat{X}_0 - \hat{Y}_0\|_1.
\]

Also, for each node \( v \),

\[
\mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_v \mid G_0) \leq \left( 1 - \frac{1 - (8d + 4)\lambda}{(\lambda + C)(\binom{n}{2})} \right)^t \left( \|\hat{X}_0 - \hat{Y}_0\|_v + \frac{50d^2\lambda}{(\lambda + C)n^3} \|\hat{X}_0 - \hat{Y}_0\|_1 \right).
\]

Proof. We first give bounds on the expected change in distance on one step of the discrete jump chain. So we shall bound \( \mathbb{E}(\|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_1 \mid G_t) \) and \( \mathbb{E}(\|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_v \mid G_t) \), in terms of \( \|\hat{X}_t - \hat{Y}_t\|_1 \) and \( \|\hat{X}_t - \hat{Y}_t\|_v \).

To begin with, suppose that \( \hat{X}_t \) and \( \hat{Y}_t \) differ in one call. Consider first the case where \( \hat{X}_t \) and \( \hat{Y}_t \) are identical except that \( \hat{X}_t \) contains a call on a direct link \( uv \) that is not present in \( \hat{Y}_t \), so \( \hat{X}_t(uv, 0) = \hat{Y}_t(uw, 0) + 1 \).

We consider the expected value of \( \|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_1 \), conditioned on \( G_t \). The distance decreases to 0 on the departure of the single unpaired call, which occurs with probability \( \frac{1}{(\lambda + C)(\binom{n}{2})} \). On any other potential departure, the distance remains equal to 1. On an arrival, the distance remains at most 1 if the pair \( \{u, v\} \) is chosen as endpoints for the arriving call: either \( \hat{X}_t(uv) < C \), in which case the call is routed directly in both \( \hat{X} \) and \( \hat{Y} \), or \( \hat{X}_t(uv) = C = \hat{Y}_t(uw) + 1 \), in which case the call is routed directly in \( \hat{Y} \), and the only difference between \( \hat{X}_{t+1} \) and \( \hat{Y}_{t+1} \) is constituted by a new call routed indirectly in \( \hat{X} \) (or coalescence occurs in the event that the call is not routed at all in \( \hat{Y} \)). If endpoints \( \{w, z\} \) are chosen for the arriving call, where \( \{u, v\} \cap \{w, z\} = \emptyset \), then the new call is routed the same way in both \( \hat{X} \) and \( \hat{Y} \), and the distance remains 1. So the only way that the distance can increase is if an arriving call is for endpoints \( \{w, z\} \), where \( |\{u, v\} \cap \{w, z\}| = 1 \), say \( w = u \). In this case, if \( \hat{X}_t(wz) = \hat{Y}_t(wz) = C \), and \( v \) is among the \( d \) intermediate nodes selected for a possible indirect route, then it is possible for different indirect routes to be chosen – specifically, the route via \( v \) may be chosen in \( \hat{Y} \), and some other route may be preferred in \( \hat{X} \) – and in this case \( \|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_1 = 3 \). The probability that the next transition is an arrival, one of \( u \) and \( v \) is among the selected endpoints \( \{w, z\} \), and the other is among the \( d \) intermediate nodes, is at most

\[
\frac{\lambda}{\lambda + C} \frac{2(n - 2)}{\binom{n}{2}} \frac{d}{n - 2} = \frac{\lambda}{\lambda + C} \frac{2d}{\binom{n}{2}}.
\]

Therefore

\[
\mathbb{E}(\|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_1 \mid G_t) \leq 1 - \frac{1}{(\lambda + C)(\binom{n}{2})} + 2\frac{\lambda}{\lambda + C} \frac{2d}{\binom{n}{2}} = 1 - \frac{1 - 4d\lambda}{(\lambda + C)(\binom{n}{2})}.
\]
We now perform a similar analysis in the case where \( \hat{X}_t \) and \( \hat{Y}_t \) differ by the presence of an extra indirectly routed call in \( \hat{X}_t \), say \( \hat{X}_t(uwv) = \hat{Y}_t(uwv) + 1 \). As before, with probability \( \frac{1}{(\lambda + C)(\frac{d}{2})} \), this extra call departs, and the distance drops to 0, while on any other potential departure the distance between \( \hat{X} \) and \( \hat{Y} \) remains equal to 1. The distance can increase on an arrival, by at most 2, only if one of the links \( uw, vw \) is among the 2d + 1 links whose capacity is (potentially) inspected in deciding how to route the arriving call. The probability of this event is at most \( \frac{2(2d + 1)}{\lambda + C} \). Thus

\[
\mathbb{E}(\|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_1 | G_t) \leq 1 - \frac{1}{(\lambda + C)(\frac{d}{2})} + 2 \frac{\lambda}{\lambda + C} \frac{(8d + 4)\lambda}{(\frac{d}{2})} = 1 - \frac{1}{(\lambda + C)(\frac{d}{2})}.
\]

Overall, we have that, for coupled copies of the chain at distance 1 at time \( t \),

\[
\mathbb{E}(\|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_1 | G_t) \leq 1 - \frac{1}{(\lambda + C)(\frac{d}{2})}.
\]

Hence, for coupled copies \( \hat{X} \) and \( \hat{Y} \) at any distance,

\[
\mathbb{E}(\|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_1 | G_t) \leq \left( 1 - \frac{1}{(\lambda + C)(\frac{d}{2})} \right)^t \|\hat{X}_t - \hat{Y}_t\|_1,
\]

and so

\[
\mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_1 | G_0) \leq \left( 1 - \frac{1}{(\lambda + C)(\frac{d}{2})} \right)^t \|\hat{X}_0 - \hat{Y}_0\|_1. \tag{3.1}
\]

Now we fix a node \( v \), and consider \( \mathbb{E}(\|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_v | G_t) \). We start with the case where the only difference between \( \hat{X}_t \) and \( \hat{Y}_t \) consists of a single call in \( \hat{X} \), directly or indirectly routed, not involving \( v \). In this case, in order for \( \|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_v \) to be non-zero, it must be the case that the next transition is an arrival, in which the loads on some 2d + 1 links are inspected, including one carrying the extra call in \( \hat{X} \), and that \( v \) is among the other at most \( d \) nodes selected. The probability of this event is at most \( \frac{\lambda}{\lambda + C} \frac{2(2d + 1)}{\frac{(d-2}{2})} \). Again, the maximum value of \( \|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_v \) is 2, and so, in this case,

\[
\mathbb{E}(\|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_v | G_t) \leq 2 \frac{\lambda}{\lambda + C} \frac{2(2d + 1)}{\frac{(d-2}{2})} \frac{d}{n-2} \leq \frac{\lambda}{\lambda + C} \frac{12d^2}{\frac{(d-2}{2})(n-2)}.
\]

Now suppose that \( \|\hat{X}_t - \hat{Y}_t\|_v = \|\hat{X}_t - \hat{Y}_t\|_1 = 1 \), and that the only difference between \( \hat{X}_t \) and \( \hat{Y}_t \) is the presence of a single call in \( \hat{X} \), directly or indirectly routed, that does involve \( v \). As usual, the departure of this extra call, which occurs with probability \( \frac{1}{(\lambda + C)(\frac{d}{2})} \), reduces the distance to 0, while no other departure changes \( \|\hat{X} - \hat{Y}\|_v \).
To analyse the arrivals, we distinguish two cases. The first is where \( v \) is an endpoint of the extra call, which utilises a link \( vu \) and possibly a further link \( wu \). In this case, \( \|\hat{X} - \hat{Y}\|_v \) can increase, by at most 2, only if the link \( vw \) is one of the \( 2d + 1 \) links inspected for the arriving call, an event with probability at most \( \frac{\lambda^{2d+1}}{C \binom{\lambda+C}{n-1}} \), or if the arriving call has \( u \) as one endpoint, and \( v \) and \( w \) as two of the potential intermediate nodes, an event with probability at most \( \frac{\lambda}{\lambda+C} \frac{2d+2}{(n-1)2} \). Thus, in this case, for \( n \geq d^2 \),
\[
\mathbb{E}(\|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_v \mid \mathcal{G}_t) \leq 1 - \frac{1}{(\lambda + C) \binom{n}{2}} + 2 \frac{\lambda}{\lambda + C} \frac{2d + 2}{\binom{n}{2}} = 1 - \frac{1 - (8d + 4)\lambda}{(\lambda + C) \binom{n}{2}}.
\]
The other case is where \( v \) is the intermediate node of the extra call, with endpoints \( u \) and \( w \). Here, the distance can increase, by at most 2, only if one of the two links \( vu, vw \) is among the \( 2d + 1 \) links inspected for the arriving call, an event with probability at most \( \frac{\lambda}{\lambda+C} \frac{4d+2}{(n-1)2} \). Thus we have
\[
\mathbb{E}(\|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_v \mid \mathcal{G}_t) \leq 1 - \frac{1}{(\lambda + C) \binom{n}{2}} + 2 \frac{\lambda}{\lambda + C} \frac{4d + 2}{\binom{n}{2}} = 1 - \frac{1 - (8d + 4)\lambda}{(\lambda + C) \binom{n}{2}}.
\]
Therefore we obtain, whenever \( \hat{X} \) and \( \hat{Y} \) are coupled,
\[
\mathbb{E}(\|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_v \mid \mathcal{G}_t) \leq \left( 1 - \frac{1 - (8d + 4)\lambda}{(\lambda + C) \binom{n}{2}} \right) \|\hat{X}_t - \hat{Y}_t\|_v + \frac{\lambda}{\lambda + C} \frac{12d^2}{(n-2) \binom{n}{2}} \|\hat{X}_t - \hat{Y}_t\|_1.
\]
Applying our earlier upper bound (3.1) on \( \mathbb{E} \|\hat{X}_t - \hat{Y}_t\|_1 \), we obtain:
\[
\mathbb{E}(\|\hat{X}_{t+1} - \hat{Y}_{t+1}\|_v \mid \mathcal{G}_t) \leq \left( 1 - \frac{1 - (8d + 4)\lambda}{(\lambda + C) \binom{n}{2}} \right) \mathbb{E} \|\hat{X}_t - \hat{Y}_t\|_v + \frac{\lambda}{\lambda + C} \frac{12d^2}{(n-2) \binom{n}{2}} \left( 1 - \frac{1 - (8d + 4)\lambda}{(\lambda + C) \binom{n}{2}} \right)^t \|\hat{X}_0 - \hat{Y}_0\|_1
\]
and thus, by induction,
\[
\mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_v \mid \mathcal{G}_0) \leq \left( 1 - \frac{1 - (8d + 4)\lambda}{(\lambda + C) \binom{n}{2}} \right)^t \|\hat{X}_0 - \hat{Y}_0\|_v + \frac{\lambda}{\lambda + C} \frac{12d^2 t}{(n-2) \binom{n}{2}} \|\hat{X}_0 - \hat{Y}_0\|_1 \left( 1 - \frac{1 - (8d + 4)\lambda}{(\lambda + C) \binom{n}{2}} \right)^{t-1}
\]
\[
\leq \left( 1 - \frac{1 - (8d + 4)\lambda}{(\lambda + C) \binom{n}{2}} \right)^t \left( \|\hat{X}_0 - \hat{Y}_0\|_v + \frac{50d^2 \lambda t}{(\lambda + C) n^3} \|\hat{X}_0 - \hat{Y}_0\|_1 \right),
\]
for \( n \geq \max(d^2, 6) \), as desired. \(\square\)
The next step is to apply Theorem 2.1 to our Markov chain. For nodes $u$ and $v$, we say that a function $f : S \to \mathbb{R}$ is $(u, v)$-Lipschitz if $|f(x) - f(y)| \leq \|x - y\|_u + \|x - y\|_v$ for all states $x$ and $y$.

**Lemma 3.2.** Set $\rho = 1 - \lambda/\lambda_0(d)$, and assume that $\lambda < \lambda_0(d)$, so that $\rho > 0$. For any nodes $u$ and $v$, let $f$ be a $(u, v)$-Lipschitz function. Then, for all sufficiently large $n$, all $a > 0$, all $t \geq 0$, and all initial states $x_0$,

$$\mathbb{P}_{x_0}(|f(\hat{X}_t) - \mathbb{E}_{x_0}[f(\hat{X}_t)]| \geq a) \leq 2 \exp\left(-\frac{a^2}{40dCn/\rho + 3a}\right).$$

**Proof.** Our aim is to apply Theorem 2.1, so we need to bound the various quantities appearing in that theorem, for our function $f$. The second part of Lemma 3.1 together with the assumption on $f$, allows us to set

$$\alpha_{x,i}(y) = \left(1 - \frac{\rho}{(\lambda + C)}\right)^i \left(\|x - y\|_u + \|x - y\|_v + \frac{100d^2\lambda i}{(\lambda + C)n^3}\right),$$

provided $n$ is sufficiently large.

Let $P$ be the transition matrix of our Markov chain. If $P(x, y) > 0$, then $\|x - y\|_u, \|x - y\|_v \leq \|x - y\|_1 \leq 1$, and moreover, for any state $x$,

$$\sum_{y : \|x - y\|_v = 1} P(x, y) \leq \frac{Cn}{(\lambda + C)}\frac{(n)}{(2)} + \frac{\lambda(d + 2)}{n(\lambda + C)} \leq \frac{d + 2}{n},$$

and the same is true with $v$ replaced by $u$. Therefore

$$(P\alpha^2_{x,i})(x) = \sum_y P(x, y)\alpha_{x,i}(y)^2 \leq 3 \sum_y P(x, y)\left(1 - \frac{\rho}{(\lambda + C)}\right)^{2i} \leq 3 \left(1 - \frac{\rho}{(\lambda + C)}\right)^i \left(2\frac{d + 2}{n} + \frac{10000d^4\lambda^2i^2}{C^2n^6}\right).$$

Set $q = 1 - \frac{\rho}{(\lambda + C)}$. Thus, in Theorem 2.1, we may take

$$\alpha^2_i = 6q^i \left(\frac{d + 2}{n} + \frac{5000d^4\lambda^2i^2}{C^2n^6}\right)$$

and

$$\beta = \sum_{i=0}^{t-1} \alpha^2_i \leq \frac{6d + 12}{n(1 - q)} + \frac{30000d^4\lambda^2}{C^2n^6} \frac{q(q + 1)}{(1 - q)^3}.$$
in a fixed state \( x \), steps, and that \( t \geq 0 \). For sufficiently large \( n \), we have \( \beta \leq 20dCn/\rho \), uniformly for \( t \geq 0 \). Also

\[
\sup_{0 \leq i \leq t-1} \sup_{x \in S, y \in N(x)} \alpha_{x,i}(y) = \max_{i \geq 0} \left( 1 - \frac{\rho}{(\lambda + C)^{n/2}} \right)^i \left( 2 + \frac{100d^2 \lambda i}{Cn^3} \right) = 2,
\]

for sufficiently large \( n \), so we may take \( \alpha = 2 \), uniformly for \( t \geq 0 \).

Hence, by Theorem 1.1(1) we have, for all sufficiently large \( n \), all \( a > 0 \), all \( t \geq 0 \), and all initial states \( x_0 \),

\[
\mathbb{P}_{x_0} \left( |f(\hat{X}_t) - \mathbb{E}_{x_0}[f(\hat{X}_t)]| \geq a \right) \leq 2e^{-a^2/(2\beta + 4\beta a)} \leq 2\exp \left( -\frac{a^2}{40dCn/\rho + 3a} \right),
\]

as claimed. \( \square \)

Now let \( \hat{X} \) and \( \hat{Z} \) be two copies of the discrete jump chain, with \( \hat{X} \) started in a fixed state \( x_0 \), and \( \hat{Z} \) started in equilibrium, coupled as above. Then we have, for \( \lambda < \lambda_0(d) \) and \( n \) sufficiently large,

\[
\mathbb{E} \|\hat{X}_t - \hat{Z}_t\|_1 \leq \left( 1 - \frac{1 - \lambda/\lambda_0(d)}{(\lambda + C)^{n/2}} \right)^t \mathbb{E} \|\hat{X}_0 - \hat{Z}_0\|_1 \leq \exp \left( -\frac{2(1 - \lambda/\lambda_0(d))}{\lambda + C} \frac{t}{n^2} \right) C \binom{n}{2}.
\]

Hence there are constants \( K \) and \( \gamma \) such that \( \mathbb{E} \|\hat{X}_t - \hat{Z}_t\|_1 \leq Kn^2 e^{-\gamma t/n^2} \). Recall that \( \mu(x_0, t) \) is the distribution of the chain, started in state \( x_0 \), after \( t \) steps, and that \( \pi \) is the equilibrium distribution. We have, for any \( t \),

\[
d_{TV}(\mu(x_0, t), \pi) \leq d_W(\mu(x_0, t), \pi) \leq \mathbb{E} \|\hat{X}_t - \hat{Z}_t\|_1 \leq Kn^2 e^{-\gamma t/n^2}.
\]

Here \( d_W(\cdot, \cdot) \) denotes the Wasserstein distance between the two distributions, which is equal to the infimum, over all couplings between random variables \( U \) and \( V \) having the two distributions, of \( \mathbb{E} \|U - V\|_1 \). The first inequality above holds since \( \|\hat{X}_t - \hat{Z}_t\|_1 \) takes integer values. Thus we have proved Theorem 1.1(1) in the case \( \lambda < \lambda_0(d) \).

We have concentration of measure for the process started from a fixed state, as well as rapid mixing of the process to equilibrium. It is now possible to deduce concentration of measure in equilibrium, as follows. Let \( \hat{Z} \) be a copy of the process in equilibrium, and let \( \hat{X} \) be a copy of the process started in any fixed state \( x \). For nodes \( u \) and \( v \), let \( f \) be a \((u, v)\)-Lipschitz function.
Fix $a > 0$, and choose $t > 0$ so large that $Kn^2e^{-\gamma t/n^2}$ is smaller than both $a/2$ and $\exp\left(-\frac{a^2}{40dCn/\rho + 3a}\right)$. We have

$$|f(\tilde{Z}_t) - \mathbb{E} f(\tilde{Z}_t)| \leq |f(\tilde{Z}_t) - f(\tilde{X}_t)| + |f(\tilde{X}_t) - \mathbb{E} f(\tilde{X}_t)| + |\mathbb{E} f(\tilde{X}_t) - \mathbb{E} f(\tilde{Z}_t)|,$$

and also

$$|\mathbb{E} f(\tilde{X}_t) - \mathbb{E} f(\tilde{Z}_t)| \leq \mathbb{E} |f(\tilde{X}_t) - f(\tilde{Z}_t)| \leq \mathbb{E} \|	ilde{X}_t - \tilde{Z}_t\|_u + \mathbb{E} \|	ilde{X}_t - \tilde{Z}_t\|_v \leq 2Kn^2e^{-\gamma t/n^2} \leq a.$$

Furthermore, $\mathbb{P}(|f(\tilde{Z}_t) - f(\tilde{X}_t)| > 0) \leq \mathbb{P}(\tilde{X}_t \neq \tilde{Z}_t) \leq Kn^2e^{-\gamma t/n^2}$. Thus

$$\mathbb{P}(|f(\tilde{Z}_t) - \mathbb{E} f(\tilde{Z}_t)| > 2a) \leq \mathbb{P}(|f(\tilde{Z}_t) - f(\tilde{X}_t)| > 0) + \mathbb{P}(|f(\tilde{X}_t) - \mathbb{E} f(\tilde{X}_t)| > a) \leq Kn^2e^{-\gamma t/n^2} + 2\exp\left(-\frac{a^2}{40dCn/\rho + 3a}\right) \leq 3\exp\left(-\frac{a^2}{40dCn/\rho + 3a}\right). \tag{3.2}$$

Applying (3.2) to the function $f = f_{v,j}$, for any $j \in \{0, \ldots, C\}$, proves part (2) of Theorem 1.1 in the case $\lambda < \lambda_0$.

Later, we shall use this result in the case $a = \frac{1}{4}\sqrt{n} \log n$, where we have

$$\mathbb{P}(|f(\tilde{Z}_t) - \mathbb{E} f(\tilde{Z}_t)| > \frac{1}{2}\sqrt{n} \log n) \leq 3e^{-\delta \log^2 n}, \tag{3.3}$$

and we may take the constant $\delta > 0$ to be $\rho/800dC$.

### 4. Analysis of coupling – high arrival rate

We now turn attention to our other regime, where the arrival rate is very high. Recall that $\lambda_1(d, C) = 8000C^2d \log(240C^2d)$. Let $R$ be the set of states $x \in S$ such that, for each node $v$, $f_{v,C}(x) \geq (n-1)(1-\frac{1}{60Cd})$. In other words, $R$ is the set of states such that, at each node $v$, there are at most $(n-1)/60Cd$ links that are not fully loaded. Set $s = 3\binom{n}{2} C(\lambda + C^2) \log(240C^2d)$. We shall first show that, for $\lambda \geq \lambda_1(d, C)$, for any starting state, after a “burn-in” period of $s$ steps, with high probability, the chain remains in $R$ for a long period of time.

For this purpose, we need tail estimates on the probability of a sum of Boolean random variables that are not necessarily independent. We use the following special case of a result of Panconesi and Srinivasan [10].

**Theorem 4.1.** Let $W_1, \ldots, W_N$ be indicator random variables such that, for some $\delta \in (0, 1/2)$, and for all subsets $D$ of $\{1, \ldots, N\}$, $\mathbb{P}(W_i = 1$ for all $i \in D) \leq \delta |D|$. Then $\mathbb{P}(\sum_{i=1}^N W_i \geq 2\delta N) \leq e^{-\delta N/3}$.

**Lemma 4.2.** Suppose $\lambda \geq \lambda_1(d, C)$, and let $\kappa > 0$ be any fixed constant. Then $\mathbb{P}(\tilde{X}_t \in R$ for every $t \in [s, n\kappa]) \geq 1 - 2Cn^{\kappa+1}e^{-(n-1)/1500C^2d}$. 

Proof. Fix a node $v$ and a time $t \geq s$. Let $r = \lfloor \frac{n}{2} \frac{\lambda + C}{\lambda} \log (240C^2d) \rfloor = \lfloor s/C \rfloor$. Consider a link $uv$ incident with $v$, and the time intervals $I_C = [t - Cr, t - (C - 1)r], \ldots, I_2 = [t - 2r, t - r], I_1 = [t - r, t]$. Define the events $E_j(u), F_j(u), v \in V_n \setminus \{v\}$, for $j = 1, \ldots, C$, as follows.

- $E_j(u)$: there is at most one arrival during $I_j$ with endpoints $\{u, v\}$;
- $F_j(u)$: there are at least $j$ departures of calls including the link $uv$ during $I_j \cup \cdots \cup I_1 = [t - jr, t)$.

We claim that, if link $uv$ is not fully loaded at time $t$, then one of $E_1(u), \ldots, E_C(u), F_1(u), \ldots, F_C(u)$ occurs. Indeed, suppose none of these events occur, and also that the link was never fully loaded during the interval $I = I_C \cup \cdots \cup I_1 = [t - Cr, t)$; then every arrival during $I$ with endpoints $\{u, v\}$ was routed directly onto the link $uv$, so $E_1(u), \ldots, E_C(u)$ imply that at least $2C$ arriving calls were routed directly onto the link $uv$ during $I$, while $F_C(u)$ states that at most $C - 1$ calls occupying link $uv$ departed during $I$, so there are $C + 1$ more calls on $uv$ at time $t$ than at time $t - Cr$, which is a contradiction, since the link has capacity $C$. Thus, if none of the $E_j(u)$ or $F_j(u)$ occur, then at some point during $I$ the link $uv$ was fully loaded. Consider the last time during $I$ that the link was fully loaded, say at some time $t^*$ during interval $I_j$. Then all calls arriving for endpoints $\{u, v\}$ during $I_{j-1} \cup \cdots \cup I_1$ were routed onto $uv$, so there were at least $2(j - 1)$ arrivals on $uv$ during this interval, and at most $j - 1$ departures over the interval $[t^*, t)$, so there are at least as many calls on $uv$ at time $t$ as there were at time $t^*$, and hence the link is fully loaded at time $t$.

We now seek upper bounds on the probability that many of the events $E_j(u)$ and $F_j(u)$ occur. We start with $E_j(u)$, for any $j \in \{1, \ldots, C\}$. The number $N_j(u)$ of arrivals with endpoints $u$ and $v$ over the interval $I_j$ is a binomial random variable with parameters $(r, n/(\lambda + C))$, of mean $\mu \geq 2 \log (240C^2d)$, and $\mathbb{P}(E_j(u)) = \mathbb{P}(N_j(u) \leq 1) \leq 2e^{-\mu} \leq 1/240C^2d$. Moreover, for each fixed $j$, the family of Boolean random variables $(\mathbb{I}_{E_j(u)}(u))_{u \in V_n}$ is negatively associated, so, for any set $D \subseteq V_n \setminus \{v\}$, the probability that all the variables $(\mathbb{I}_{E_j(u)})_{u \in D}$ are equal to 1 is at most $1/(240C^2d)^{|D|}$. Therefore, by Theorem 12, the probability that at least $(n - 1)/120C^2d$ of them are equal to 1 is at most $e^{-(n - 1)/240C^2d}$. (This application amounts to the fact that negatively associated random variables satisfy the Chernoff-Hoeffding bounds.) Thus the probability that at least $(n - 1)/120Cd$ of the events $E_j(u)$ occur ($u \neq v, j \in \{1, \ldots, C\}$) is at most $Ce^{-(n - 1)/240C^2d}$.

We turn to $F_j(u)$, and first consider the case $j = 1$. The number of departures of calls incident with the node $v$ over the interval $I_1$ of length $r$ is dominated by a binomial random variable with parameters $(r, n/(\lambda + C))$, of mean at most $3C \log (240C^2d)/(\lambda + C)(n - 1) \leq \frac{n - 1}{240Cd}$. The probability that this random variable is at least $(n - 1)/120Cd$ is therefore at most $e^{-(n - 1)/240Cd}$. The number of events $F_1(u)$ that occur is at most twice the number of
departures from links incident with \( v \), so the probability that more than \((n - 1)/600Cd\) of the events \( F_j(u) \) (\( u \neq v \)) occur is at most \( e^{-(n-1)/400Cd} \).

Now fix \( j \in \{2, \ldots, C\} \), and a set \( S \subseteq V_n \setminus \{v\} \). The probability that all the random variables \( (F_j(u))_{u \in S} \) are equal to 1 is at most the probability that there are at least \([j|S|/2]\) departures of calls on some link \( uv \) with \( u \in S \) during the interval \( I_j \) of length \( jr \). (Note that each departure may contribute to at most 2 of the variables.) This probability is at most

\[
\left( \frac{jr}{[j|S|/2]} \right) \left( \frac{C|S|}{(\lambda + C)(\frac{n}{2})} \right)^{[j|S|/2]} \leq \left( \frac{2Cer}{\lambda} \right)^{[j|S|/2]} \leq \left( \frac{1}{480Cd\lambda} \right)^{[j|S|/2]} \leq \left( \frac{1}{480Cd(j-1)^2} \right)^{|S|} .
\]

Thus, for each \( j \geq 2 \), we may take \( \delta = \frac{1}{480Cd(j-1)^2} \) in Theorem 4.1 so the probability that at least \((n - 1)/240Cd(j-1)^2\) of the random variables are equal to 1 is at most \( e^{-(n-1)/1440Cd(j-1)^2} \). Hence, with probability at least \( 1 - Ce^{-(n-1)/1440Cd(j-1)^2} \), at most \( \frac{n-1}{240Cd} \sum_{j=2}^{\infty} 1/(j-1)^2 \leq \frac{n-1}{1500Cd} \) of the events \( F_j(u) \) (\( u \neq v, j = 2, \ldots, C \)) occur.

Thus, with probability at least \( 1 - 2Ce^{-(n-1)/1500Cd} \), at most \((n - 1)/60Cd\) of the events \( E_j \) and \( F_j \) occur, so there are at most \((n - 1)/60Cd\) links at \( v \) that are not full at time \( t \). The result now follows; the probability that there is some node \( v \) and some time \( t \in [s, n^k] \) such that there are fewer than \((n - 1)/60Cd\) links at \( v \) that are not fully loaded at time \( t \) is at most \( 2Cn^{k+1}e^{-(n-1)/1500Cd} \).

Given two states \( x \) and \( y \), we call a link \( uv \) unbalanced if \( x(uv) \neq y(uv) \), and let \( A_{x,y} \) be the set of unbalanced links. In order to analyse the coupling in the case of high arrival rate, we use a specially tailored distance function.

We define \( d(x, y) \) to be the sum of two contributions. One is a multiple of the sum of the differences between numbers of indirect links on all possible routes, and the other gives a contribution from each unbalanced link:

\[
d(x, y) = (4C + 1) \sum_{uv, w} |x(uwv) - y(uwv)| + \sum_{uv \in A_{x,y}} (C - \min(x(uv), y(uv))) .
\]

Note that \( d(x, y) = 0 \) if and only if \( x = y \), and that \( d(\cdot, \cdot) \) is symmetric. Also, it is easy to see that \( d(\cdot, \cdot) \) satisfies the triangle inequality. (Indeed, it suffices to check that, for each link \( uv \), \( (C - \min(x(uv), y(uv))) \mathbb{1}_{x(uv) \neq y(uv)} \) satisfies the triangle inequality \( d(x, z) \leq d(x, y) + d(y, z) \); this is immediate if any two of \( x(uv) \), \( y(uv) \) and \( z(uv) \) are equal, and otherwise two of the three terms are equal and the other is smaller.) Thus \( d(\cdot, \cdot) \) is a metric.

We also define a “localised” version of the distance. Fix a node \( v \), and define

\[
d_v(x, y) = (4C + 1) \left[ \sum_{u, w} |x(uuw) - y(uuw)| + \sum_{u} |x(uvw) - y(uvw)| \right]
\]
\[ + \sum_u (C - \min(x(uv), y(uv))) \mathbb{I}_{x(uv) \neq y(uv)}. \]

We couple \( \hat{X} \) and \( \hat{Y} \) as in Section 2. Recall in particular that we pair calls
for departure occupying the same route, as far as possible – and that we call
such pairs of calls *matched* – and then do any further possible pairing for
departures in an arbitrary way depending only on the current states.

Before analysing how the distance behaves under a step of the coupling,
we give a brief sketch, aimed at motivating the choice of distance function
and the strategy of the proof. We imagine that \( \lambda \) is very large, and so we
expect that most links will be fully loaded (in the formal proof below, this
is reflected in the assumption that the starting states are in \( R \)). Suppose
first that two states \( x \) and \( y \) differ by one directly routed call, on link \( uv \),
present in \( x \) but not in \( y \). Coalescence between the states may occur on the
departure of this unpaired call, but if \( \lambda \) is large, then in the interim it is
quite likely that the differing loads on \( uv \) between the two states will result
in some arriving call either being routed on a different indirect route in the
two states, or being accepted in one state and rejected in the other (below,
we describe in detail the two types of *bad arrival* that may cause this). The
other possibility for coalescence is for the load \( x(uv) \) on link \( uv \) to be \( C \),
and an arriving call with endpoints \( u \) and \( v \) to be accepted onto link \( uv \) in
\( \hat{Y} \), and rejected completely in \( \hat{X} \): we call such an arrival a *good arrival*; for
the probability of such an arrival to be suitably large, we need the network
to be very full, so that once a call arrives for endpoints \( u \) and \( v \) that can
be routed directly in \( y \) but not \( x \), it is quite unlikely for it to succeed in
being routed indirectly. If \( x(uv) \) is less than \( C \), then immediate coalescence
at the next step is impossible: however in this case, any call arriving with
endpoints \( u \) and \( v \) will be routed directly in both states, and this moves
us closer to our goal. We represent this in our distance function via the
term \( (C - \min(x(uv), y(uv))) \) for the unbalanced link \( uv \), which decreases
by 1 on such an arrival, which we consider under the same umbrella as
the good arrival that results in immediate coalescence. A side-effect of this
choice is that departures of matched calls on link \( uv \) (*bad departures of
type 1*) increase the distance, but these are rare and their contribution to
the expected change in distance is relatively small.

Now suppose that the two states differ by the presence of one indirectly
routed call, say on route \(uvw \) in \( x \). Again, in our analysis we cannot wait
for this call to depart, as in the interim it will give rise to other unmatched
indirect calls on “bad arrivals”. In this case, we move toward coalescence in
stages: we deem it to be progress for calls to arrive with endpoints \( \{u, w\} \)
or \( \{w, v\} \), increasing the loads of the two links \( uw \) and \( wv \), and ultimately
providing extra direct calls (in the proof below, these are *covered* direct
calls) in \( \hat{Y} \) on each of the two links via good arrivals as before. Once the
two covered calls in \( \hat{Y} \) are in place, all links are balanced, so in the coupling
every arrival is routed the same way in the two states; in particular, no bad
Lemma 4.3. (a) For all \( t \geq 1 \), under the coupling,

\[
\mathbb{E}(d(\tilde{X}_t, \tilde{Y}_t) \mid \mathcal{G}_{t-1}) \leq \left(1 - \frac{1}{10C\lambda(n^2)}\right)d(\tilde{X}_{t-1}, \tilde{Y}_{t-1}),
\]

on the event that \( \tilde{X}_{t-1} \) and \( \tilde{Y}_{t-1} \) are both in \( R \).

(b) For all \( t \geq 1 \), under the coupling,

\[
\mathbb{E}(d_v(\tilde{X}_t, \tilde{Y}_t) \mid \mathcal{G}_{t-1}) \leq \left(1 - \frac{1}{10C\lambda(n^2)}\right)d_v(\tilde{X}_{t-1}, \tilde{Y}_{t-1}) + \frac{300d^2C}{n^3}d(\tilde{X}_{t-1}, \tilde{Y}_{t-1}),
\]

on the event that \( \tilde{X}_{t-1} \) and \( \tilde{Y}_{t-1} \) are both in \( R \).

Proof. (a) We fix states \( x \) and \( y \) in \( R \), and consider

\[
\mathbb{E}(d(\tilde{X}_t, \tilde{Y}_t) - d(\tilde{X}_{t-1}, \tilde{Y}_{t-1}) \mid \tilde{X}_{t-1} = x, \tilde{Y}_{t-1} = y).
\]

Let \( a(x, y) = |A_{x,y}| \) be the number of unbalanced links. Let \( b(x, y) = \sum_{uv, w} |x(uwv) - y(uwv)| \) be the number of unmatched indirect calls.

In the coupling, we call an unmatched direct call in one of the two states – say on link \( uv \) in state \( x \) – covered if \( x(uv) \leq y(uv) \). Note that, on each link \( uv \), unmatched direct calls are present in at most one of the two states, and if these calls are covered then we can associate each of the covered calls to an unmatched indirectly routed call occupying the link in the other state; each unmatched indirectly routed call is associated with at most two covered calls, so the number \( c(x, y) \) of covered calls is at most twice the number \( b(x, y) \) of unmatched indirect calls.

We consider in turn each of the various ways in which the distance can change on the event at time \( t \). First we consider arrivals. An arrival can only change the distance if there is some unbalanced link \( uv \) that is inspected during the arrival (either because the endpoints of the call are \( \{u, v\} \), or because an indirect route including the link \( uv \) is considered). In all other cases, the arriving call will be routed the same way in both \( \tilde{X} \) and \( \tilde{Y} \), onto some balanced link(s), and there will be no change to the distance. Moreover, if the call is for endpoints \( \{u, v\} \) and \( x(uv) = y(uv) < C \), then the distance will not change. If \( x(uv) = y(uv) = C \), then the distance can only change if either the call is routed differently in the two states, or if the call is assigned to an indirect route using an unbalanced link; for either of these to occur,
there has to be an indirect route considered where one of the two links in the route is unbalanced – say it has lower load in $y$ – and the other link on the route is not fully loaded in $y$. We now analyse all the possibilities for an arrival that may change the distance, labelling them as “good” or “bad” according to whether they decrease or increase the distance, respectively.

A good arrival is an arrival for a pair of nodes that are the endpoints of some unbalanced link $uv$ – say with $x(uv) > y(uv)$ – and such that each of the links $uv_1, \ldots, uv_d$ is fully loaded in $x$, where $w_1, \ldots, w_d$ are the intermediate nodes considered. We claim that the distance decreases by 1 on a good arrival. If $x(uv) < C$, then the arriving call is routed directly on $uv$ in both $\tilde{X}$ and $\tilde{Y}$. The only change to the distance is in the term $(C - \min(x(uv), y(uv)))$, which is decreased by 1. If $x(uv) = C$, then the call is routed directly onto $uv$ in $\tilde{Y}$, and not routed at all in $\tilde{X}$, as all the routes considered are unavailable. Hence $x(uv)$ is unchanged and $y(uv)$ is increased by 1, resulting in a decrease by 1 of $(C - \min(x(uv), y(uv))) \mathbb{1}_{x(uv) \neq y(uv)}$, while no other term contributing to the distance is affected. The probability of a good arrival, conditional on $\tilde{X}_{t-1} = x$ and $\tilde{Y}_{t-1} = y$, is at least

$$\frac{\lambda a(x, y)}{(\lambda + C)(n/2)} \left(1 - \frac{1}{60Cd}\right)^d \geq \frac{\lambda a(x, y)}{(\lambda + C)(n/2)} \left(1 - \frac{1}{60C}\right) \geq \frac{\lambda a(x, y)}{60(\lambda + C)(n/2)}.$$ 

Here we used the fact that $x \in R$, so that the proportion of links at $u$ that are not fully loaded in $x$ is at most $1/60Cd$.

A bad arrival of type 1 is an arrival for endpoints $\{u, v\}$, where $uv$ is an unbalanced link – again say with $x(uv) > y(uv)$ – but such that at least one of the links $uv_j$ ($j \in \{1, \ldots, d\}$) inspected for indirect routing is not fully loaded. In the case where $x(uv) = C$, such an arrival is routed directly onto the link $uv$ in $\tilde{Y}$, and may be routed onto the 2-link route $uwv$ in $\tilde{X}$. In the worst case, this introduces an extra indirect route in $\tilde{X}$, adding $4C + 1$ to the distance, and also causes both links $uv_j$ and $w_jv$ to become unbalanced, giving a new contribution of up to $C$ to the distance for each of those two links. On the other hand, the contribution from link $uv$ decreases by 1, so the total increase in distance from a bad arrival of type 1 is at most $6C$. The probability of a bad arrival of type 1, conditional on $\tilde{X}_{t-1} = x$ and $\tilde{Y}_{t-1} = y$, is at most $\frac{\lambda a(x, y)}{(\lambda + C)(n/2)} \cdot \frac{1}{60Cd}$, as in the analysis for a good arrival.

A bad arrival of type 2 is an arrival for endpoints $\{u, v\}$ where the link $uv$ is fully loaded in both $x$ and $y$, and where at least one indirect route $uwv$ is considered that consists of an unbalanced link $uv$ – say with $x(uv) > y(uw)$ – and a link $uv$ that is not fully loaded in $y$. On such an arrival, it may be the case that different indirect routes are chosen in $\tilde{X}$ and in $\tilde{Y}$ (or an indirect route is chosen in $\tilde{Y}$, and the call is rejected in $\tilde{X}$). As in the analysis of a bad arrival of type 1, the introduction of a new indirect call in one state may increase the distance by up to $6C + 1$, so the maximum increase in distance on a bad arrival of type 2 is $12C + 2 \leq 14C$. The conditional probability of
A bad arrival of type 2 is at most
\[ \frac{2a(x, y) n - 1}{60Cd} d \left( \frac{\lambda}{(\lambda + C)} \right) \frac{1}{n - 2} \leq \frac{\lambda a(x, y)}{(\lambda + C)} \frac{1}{(\lambda + C)(\frac{n}{2})} 20C, \]
since there are at most \(2a(x, y)\) choices of an unbalanced link \(uw\), together with one of its ends \(u\) to be an endpoint for the arriving call, then at most \((n - 1)/60Cd\) choices for a node \(v\) such that \(uv\) is not fully loaded in \(y\) – since \(y \in R\) – and \(d\) choices for the place where the indirect route \(uwv\) is in the list of potential indirect routes.

The distance between \(\hat{X}\) and \(\hat{Y}\) can change on a departure for two reasons: either some matched pair of calls departs, changing the load on some unbalanced link \(uv\), or a pair of unmatched calls or a single unpaired call departs. If an unmatched direct call, say in \(x\), departs, and the link it occupies has higher load in \(x\) than in \(y\), then the distance does not increase. So the only case in which the distance increases on the departure of an unmatched direct call is if the call is covered. We shall see that the departure of an unmatched indirectly routed call always decreases the distance.

A bad departure of type 1 is a departure of a pair of matched calls using some unbalanced link(s) \(uv\). A bad departure can increase the distance by at most 2 (if an indirect call departs and both links on the call are unbalanced). The conditional probability of a bad departure of type 1 is at most \(\frac{Ca(x, y)}{(\lambda + C)(\frac{n}{2})}\), as there are at most \(Ca(x, y)\) pairs of matched calls using unbalanced links.

A bad departure of type 2 is a departure of a covered direct call on some link \(uv\). Such a departure may lead to an increase in distance of up to \(C\), either because the departure unbalances the link, creating a new contribution of \((C - \min(x(uw), y(uv)) + 1)\), or because the link is already unbalanced and its contribution increases by 1. The conditional probability of a bad departure of type 2 is equal to \(\frac{c(x, y)}{(\lambda + C)(\frac{n}{2})}\). It is possible for two bad departures of type 2 to happen together, if two covered direct calls in the different states are paired. Such a pair of departures leads to an increase of up to \(2C\) in the distance, but accounts for 2 of the \(c(x, y)\) covered direct calls; thus the pairing of these calls to depart together gives the same contribution to the expected increase in distance as would be obtained by having the two departures occupy separate “departure slots”.

A good departure is the departure of an unmatched indirect call on some route \(uwv\). This yields a decrease in distance of \(4C + 1\) due to the reduction by 1 of \(|x(uwv) - y(uwv)|\). There may be an increase of up to \(C\) in the contributions from each of the two links \(uw\) and \(vw\), but overall there is a decrease in distance of at least \(2C + 1\) on a good departure of type 2. The conditional probability of a good departure is equal to \(\frac{b(x, y)}{(\lambda + C)(\frac{n}{2})}\). As above, two good departures may happen simultaneously, or indeed a good departure may be paired with a bad departure of type 2 in the other state, but making such pairings does not affect the expected change in distance.
Summing the contributions to the expected change in distance from all the possible types of good or bad arrivals and departures, we have

\[
\mathbb{E} \left( d(\hat{X}_t, \hat{Y}_t) - d(\hat{X}_{t-1}, \hat{Y}_{t-1}) \mid \hat{X}_{t-1} = x, \hat{Y}_{t-1} = y \right) \\
\leq \frac{1}{(\lambda + C)^n/2} \left[ (-1) \frac{59}{60} \lambda a(x, y) + (6C) \frac{\lambda a(x, y)}{60C} + (14C) \frac{\lambda a(x, y)}{20C} \\
+ (2)Ca(x, y) + (C)c(x, y) + (-2C - 1)b(x, y) \right] \\
\leq \frac{1}{(\lambda + C)^n/2} \left[ \lambda a(x, y) \left( - \frac{59}{60} + \frac{1}{10} + \frac{7}{10} + \frac{2C}{\lambda} \right) + b(x, y)(2C - 2C - 1) \right] \\
\leq - \frac{1}{4C + 1} \frac{1}{(\lambda + C)^n/2} d(x, y).
\]

The final inequality follows since \( d(x, y) \leq Ca(x, y) + (4C + 1)b(x, y), \) and

\[
\frac{59}{60} - \frac{8}{10} - \frac{2C}{\lambda} \geq \frac{1}{10} \geq \frac{C}{(4C + 1)\lambda}.
\]

Hence we have

\[
\mathbb{E} \left( d(\hat{X}_t, \hat{Y}_t) \mid \mathcal{G}_{t-1} \right) \leq \left( 1 - \frac{1}{4C + 1} \frac{1}{(\lambda + C)^n/2} \right) d(\hat{X}_{t-1}, \hat{Y}_{t-1}) \\
\leq \left( 1 - \frac{1}{10C\lambda(n/2)} \right) d(\hat{X}_{t-1}, \hat{Y}_{t-1}),
\]

on the event that \( \hat{X}_{t-1} \) and \( \hat{Y}_{t-1} \) are in \( R \), as desired.

(b) We now bound the expected change \( \Delta d_v = \mathbb{E} \left( d_v(\hat{X}_t, \hat{Y}_t) - d_v(\hat{X}_{t-1}, \hat{Y}_{t-1}) \right) \) in \( d_v \), conditional on \( \hat{X}_{t-1} = x \) and \( \hat{Y}_{t-1} = y \), where \( x \) and \( y \) are states in \( R \). Let \( a_v(x, y) \) be the number of unbalanced links \( uv \) incident with \( v \). Let \( b_v(x, y) \) be the number of unmatched indirect calls incident with \( v \), and let \( c_v(x, y) \) be the number of covered direct calls in \( x \) or \( y \) on links incident with \( v \). Note that, for the same reason as in (a), \( c_v(x, y) \leq 2b_v(x, y) \).

First, we bound the contribution to \( \Delta d_v \) due to an arriving call inspecting an unbalanced link \( uv \) not incident with \( v \). Such an arrival can lead to an increase in \( d_v \) of at most \( 10C + 2 \) (if the call is for endpoints \( \{u, v\} \), and is routed via \( w \) in one of \( \hat{X} \) and \( \hat{Y} \) but not the other, giving two extra calls on indirect routes including \( v \), and potentially creating two new unbalanced links at \( v \)). The probability of such an arrival is at most

\[
\frac{(2d + 1)a(x, y)\lambda}{(\lambda + C)^n/2} \cdot \frac{d}{n - 2} \leq \frac{9d^2\lambda}{n} \frac{d(x, y)}{(\lambda + C)^n/2}.
\]

Note that \( d_v \) only changes on a departure if the call uses a link incident with \( v \) on which there is some contribution to \( d_v(x, y) \), either because the link is unbalanced or because it carries an unmatched indirectly routed call.
We now go through the various types of good and bad arrivals and departures identified in the proof of part (a), focussing on the contribution to $\Delta d_v$ due to differences between the loads in $x$ and $y$ on routes incident to $v$.

A good arrival is an arrival with endpoints \{u, v\}, where $uv$ is an unbalanced link incident to $v$, such that all the links $uw_1, \ldots, uw_d$ are fully loaded. Such an arrival leads to a decrease by 1 in $d_v$, and has probability at least $\frac{59}{60} \frac{\lambda a_v(x, y)}{(\lambda + C)(\frac{n}{2})}$, provided $x$ and $y$ are in $R$.

A bad arrival of type 1 is an arrival with endpoints \{u, v\}, where $uv$ is an unbalanced link incident to $v$, say with $y(uv) < x(uv)$, such that at least one link $uv_j$ is not fully loaded in $y$. Such an arrival leads to an increase in $d_v$ by at most $2C$, and has probability at most $\frac{\lambda a_v(x, y)}{(\lambda + C)(\frac{n}{2})} \frac{1}{20C}$.

A bad arrival of type 2 can take two different forms. In each, there are three nodes $u, v, w$, where the three links joining these nodes consist of: one link that is fully loaded in both $x$ and $y$, one unbalanced link with, say, greater load in $x$, and a third link that is not fully loaded in $y$. The node $v$ is incident to the unbalanced link and one of the other two links. The arrival is for the two endpoints of the fully loaded link, and the third node is one of the intermediate nodes chosen. A bad arrival of type 2 can increase $d_v$ by up to $10C + 2$, and has probability at most

$$a_v(x, y) \frac{2(n - 1)}{60Cd} \frac{\lambda}{(\lambda + C)(\frac{n}{2})} \frac{d}{n - 2} \leq \frac{\lambda a_v(x, y)}{(\lambda + C)(\frac{n}{2})} \frac{1}{20C}.$$

A bad departure of type 1 is a departure of matched calls on the same route, affecting one or two unbalanced links incident with $v$. Such a departure increases $d_v$ by at most 2, and has probability at most $\frac{\lambda a_v(x, y)}{(\lambda + C)(\frac{n}{2})}$.

A bad departure of type 2 is a departure of an unmatched indirect call incident with $v$, say on link $uv$ in $x$. Such a departure can lead to an increase in $d_v$ of at most $C$, and has probability at most $\frac{\lambda c_v(x, y)}{(\lambda + C)(\frac{n}{2})} \leq 2 \frac{\lambda b_v(x, y)}{(\lambda + C)(\frac{n}{2})}$.

A good departure is a departure of an unmatched indirect call incident with $v$ in one state. This departure reduces $d_v$ by at least $(4C + 1) - 2C = 2C + 1$, and the probability of such a departure is at least $\frac{\lambda b_v(x, y)}{(\lambda + C)(\frac{n}{2})}$.

Combining these contributions, we find that, for $x, y \in R$,

$$\mathbb{E}(d_v(\tilde{X}_t, \tilde{Y}_t) - d_v(\tilde{X}_{t-1}, \tilde{Y}_{t-1}) \mid \tilde{X}_{t-1} = x, \tilde{Y}_{t-1} = y)$$

\[\leq \frac{1}{(\lambda + C)(\frac{n}{2})} \left[ a_v(x, y) \left( (-1) \frac{59\lambda}{60} + (5C) \frac{\lambda}{60C} + (10C + 2) \frac{\lambda}{20C} + (2)C \right) 
\quad + (C)c_v(x, y) - (2C + 1)b_v(x, y) + (10C + 2) \frac{9d^2\lambda}{n} d(x, y) \right] \]

\[\leq \frac{1}{(\lambda + C)(\frac{n}{2})} \left[ - \frac{\lambda}{4} a_v(x, y) - b_v(x, y) + \frac{108d^2C\lambda}{n} d(x, y) \right] \]

\[\leq \frac{1}{(\lambda + C)(\frac{n}{2})} \left[ - \frac{1}{4C + 1} a_v(x, y) + \frac{108d^2C\lambda}{n} d(x, y) \right] \]
Then we have from Lemma 4.3 that, for \( t_n \leq 3 \), then they do not get too far apart during the “burn-in” period. Recall that we apply Lemma 3.1 to show that, if the processes start close together, then they do not get too far apart during the “burn-in” period. □

We now use an inductive argument to establish an analogue to Lemma 3.1 for this regime. One issue we face in applying Lemma 4.3 is that the processes may not start in the set \( R \) (or they may exit the set straight away with reasonable probability); all we know is that they remain in \( R \) with high probability after the “burn-in” period of \( s \) steps. To deal with this, we apply Lemma 3.1 to show that, if the processes start close together, then they do not get too far apart during the “burn-in” period. Recall that \( s = 3 \left( \binom{n}{2} \frac{C(\lambda + C)}{\lambda} \right) \log(240C^2d) \).

**Lemma 4.4.** Suppose \( \lambda \geq \lambda_1(d, C) \), and \( \kappa > 0 \). Then for all sufficiently large \( n \), and all \( t \in [s, n^\kappa] \), we have

\[
\mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_1 | G_0) \leq 5Ce^{40dC^2 \log(240C^2d)} \left( 1 - \frac{1}{10C\lambda \binom{n}{2}} \right)^{t-s} \|\hat{X}_0 - \hat{Y}_0\|_1 + 2C^2n^{\kappa+3}e^{-(n-1)/1500C^3d},
\]

and, for each node \( v \),

\[
\mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_v | G_0) \leq 5Ce^{40dC^2 \log(240C^2d)} \left( 1 - \frac{1}{10C\lambda \binom{n}{2}} \right)^{t-s} \times \left( \|\hat{X}_0 - \hat{Y}_0\|_v + \left( \frac{75d^2C^2 \log(240C^2d)}{n} + \frac{400d^2C(t-s)}{n^3} \right) \|\hat{X}_0 - \hat{Y}_0\|_v \right) + 4C^2n^{\kappa+3}e^{-(n-1)/1500C^3d}.
\]

**Proof.** From Lemma 3.1 with \( t = s \), we obtain that

\[
\mathbb{E}(\|\hat{X}_s - \hat{Y}_s\|_1 | G_0) \leq \left( 1 + \frac{8d+4}{\binom{n}{2}} \right)^s \|\hat{X}_0 - \hat{Y}_0\|_1 \leq e^{40dC^2 \log(240C^2d)}\|\hat{X}_0 - \hat{Y}_0\|_1,
\]

and, for each node \( v \),

\[
\mathbb{E}(\|\hat{X}_s - \hat{Y}_s\|_v | G_0) \leq e^{40dC^2 \log(240C^2d)} \left( \|\hat{X}_0 - \hat{Y}_0\|_v + \frac{50d^2\lambda s}{(\lambda + C)n^3} \|\hat{X}_0 - \hat{Y}_0\|_1 \right).
\]

For \( t \geq s \), let \( B_t \) be the event that \( \hat{X}_r \) and \( \hat{Y}_r \) are both in \( R \) for all \( r \in [s, t] \). Then we have from Lemma 4.3 that, for \( t \geq s \),

\[
\mathbb{E}(d(\hat{X}_{t+1}, \hat{Y}_{t+1}) I_{B_{t+1}} | G_t) \leq \left( 1 - \frac{1}{10C\lambda \binom{n}{2}} \right) d(\hat{X}_t, \hat{Y}_t) I_{B_t};
\]

\[
\mathbb{E}(d(\hat{X}_t, \hat{Y}_t) I_{B_t} | G_t) \leq \left( 1 - \frac{1}{10C\lambda \binom{n}{2}} \right) d(\hat{X}_t, \hat{Y}_t) I_{B_t}.
\]
\[ \mathbb{E}(d_v(\hat{X}_{t+1}, \hat{Y}_{t+1}) \mathbb{1}_{B_{t+1}} \mid \mathcal{G}_t) \leq \left(1 - \frac{1}{10C\lambda(n)^2}\right)d_v(\hat{X}_t, \hat{Y}_t)\mathbb{1}_{B_t} + \frac{300d^2C}{n^3}d(\hat{X}_t, \hat{Y}_t)\mathbb{1}_{B_t}. \]

From these it follows, as in the proof of Lemma 3.1, that
\[ \mathbb{E}(d(\hat{X}_t, \hat{Y}_t)\mathbb{1}_{B_t} \mid \mathcal{G}_s) \leq \left(1 - \frac{1}{10C\lambda(n)^2}\right)^{t-s}d(\hat{X}_s, \hat{Y}_s); \]
\[ \mathbb{E}(d_v(\hat{X}_t, \hat{Y}_t)\mathbb{1}_{B_t} \mid \mathcal{G}_s) \leq \left(1 - \frac{1}{10C\lambda(n)^2}\right)^{t-s}d_v(\hat{X}_s, \hat{Y}_s) + \frac{300d^2C(t-s)}{n^3}d(\hat{X}_s, \hat{Y}_s)\left(1 - \frac{1}{10C\lambda(n)^2}\right)^{t-s-1}, \]
provided \( n \) is sufficiently large. Using also that \( \|x - y\|_1 \leq d(x, y) \leq 5C\|x - y\|_1 \) and \( \|x - y\|_v \leq d_v(x, y) \leq 5C\|x - y\|_v \) for each \( v \), we now have
\[ \mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_1\mathbb{1}_{B_t} \mid \mathcal{G}_s) \leq 5C\left(1 - \frac{1}{10C\lambda(n)^2}\right)^{t-s}\|\hat{X}_s - \hat{Y}_s\|_1; \]
\[ \mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_v\mathbb{1}_{B_t} \mid \mathcal{G}_s) \leq 5C\left(1 - \frac{1}{10C\lambda(n)^2}\right)^{t-s}\left(\|\hat{X}_s - \hat{Y}_s\|_v + \frac{400d^2C(t-s)}{n^3}\|\hat{X}_s - \hat{Y}_s\|_1\right), \]
provided \( n \) is sufficiently large. Combining these inequalities with (4.2) and (4.3) gives us that, for \( s \leq t \leq n^\kappa \), provided \( n \) is sufficiently large,
\[ \mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_1\mathbb{1}_{B_t} \mid \mathcal{G}_0) \leq 5Ce^{40dC\log(240C^2d)}\left(1 - \frac{1}{10C\lambda(n)^2}\right)^{t-s}\|\hat{X}_0 - \hat{Y}_0\|_1; \]
\[ \mathbb{E}(d_v(\hat{X}_t - \hat{Y}_t)\mathbb{1}_{B_t} \mid \mathcal{G}_0) \leq 5Ce^{40dC\log(240C^2d)}\left(1 - \frac{1}{10C\lambda(n)^2}\right)^{t-s}\times\left(\|\hat{X}_0 - \hat{Y}_0\|_v + \left(\frac{75d^2C\log(240C^2d)}{n} + \frac{400d^2C(t-s)}{n^3}\right)\|\hat{X}_0 - \hat{Y}_0\|_1\right). \]

Noting that \( \|x - y\|_1 \leq \lambda(n)^2 \) and \( \|x - y\|_v \leq Cn \) for any states \( x \) and \( y \), we also have from Lemma 4.2 that, for \( s \leq t \leq n^\kappa \),
\[ \mathbb{E}(\|\hat{X}_t - \hat{Y}_t\|_1\mathbb{1}_{B_t} \mid \mathcal{G}_0) \leq 2C^2n^{\kappa + 3}e^{-(n-1)/1500C^3d}, \]
Lemma 4.5. Suppose \( t \leq \text{indicator function of the event that link } uv \) for the discrete jump chain starting in a fixed state, for \( \lambda \) \( \pi \) equilibrium, and

\[
E(||X_t - \hat{Y}_t||_1 | G_0) \leq 4C^2n^{\kappa + 2}e^{-(n-1)/1500C^3d},
\]

and the result follows. \( \square \)

It follows from (4.2) and (4.1) that, provided \( \lambda \geq \lambda_1(d, C) \), there are constants \( K \) and \( \varepsilon > 0 \) such that, for \( n \) sufficiently large and \( 0 \leq t \leq n^{5/2} \),

\[
E(||X_t - \hat{Y}_t||_1 | G_0) \leq Ke^{-t/n^2}||\tilde{X}_0 - \tilde{Y}_0||_1. \tag{4.4}
\]

Theorem 1.1(1), for \( \lambda \geq \lambda_1(d, C) \), then follows.

Exactly as in the previous section, we also obtain concentration of measure for the discrete jump chain starting in a fixed state, for \( \lambda \geq \lambda_1(d, C) \).

Lemma 4.5. Suppose \( \lambda \geq \lambda_1(d, C) \). For any nodes \( u \) and \( v \), let \( f \) be a \((u, v)\)-Lipschitz function. Then, for all sufficiently large \( n \), all \( a > 0 \), all \( t \leq n^\kappa \), and all initial states \( x_0 \),

\[
\mathbb{P}_{x_0}\left(|f(\tilde{X}_t) - \mathbb{E}_{x_0}[f(\tilde{X}_t)]| \geq a\right) \leq 2 \exp\left(\frac{-a^2}{(2\lambda n + a) \exp(1000dC^2 \log(240C^2d))}\right).
\]

The proof of the lemma is in essence the same as for Lemma 3.2. We use the bounds on \( \sigma_t^2 \) derived above, where we have a different form of the bound according to whether \( t \leq s \) or \( s < t \leq n^\kappa \).

Having established this lemma, we proceed exactly as before. We have concentration of measure for the discrete jump chain started from a fixed state, and we have rapid mixing of the process to equilibrium. From these, it follows that we have concentration of measure in equilibrium for any \((u, v)\)-Lipschitz function \( f \):

\[
\mathbb{P}(|f(\tilde{Y}_t) - \mathbb{E}f(\tilde{Y}_t)| > 2a) \leq 3 \exp\left(\frac{-a^2}{(2\lambda n + a) \exp(1000dC^2 \log(240C^2d))}\right),
\]

where \( \tilde{Y}_t \) denotes a copy of the discrete jump chain in equilibrium. Applying the result to \( f = f_{u, j} \) yields Theorem 1.1(2) in the case \( \lambda \geq \lambda_1(d, C) \).

For \( a = \frac{1}{4} \sqrt{n} \log n \), we obtain, for sufficiently large \( n \),

\[
\mathbb{P}(|f(\tilde{Y}_t) - \mathbb{E}f(\tilde{Y}_t)| > \frac{1}{2} \sqrt{n} \log n \leq 3e^{-\delta \log^2 n}, \tag{4.5}
\]

where we may take \( \delta \) equal to \( 32\lambda \exp(1000dC^2 \log(240C^2d)) \).

5. The process in equilibrium

In this section, as earlier, we use \( \tilde{Z} \) to denote a copy of the process in equilibrium, and \( \pi \) to denote the equilibrium distribution of the process. Given a link \( uv \) and an integer \( j \in \{0, \ldots, C\} \), let \( \Pi_{uv}^j : S \to \{0, 1\} \) be defined by \( \Pi_{uv}^j(x) = 1 \) if \( x(uv) = j \) and \( \Pi_{uv}^j(x) = 0 \) otherwise. Note that \( \Pi_{uv} = \Pi_{uv}^j \) and that \( \Pi_{uv}^j \) is identically 0 for each \( u \) and \( j \). Also, \( \Pi_{uv}^j \) is the indicator function of the event that link \( uv \) has load at most \( j \).
Let \( P \) be the transition matrix of the Markov chain \( \hat{X} \). For a function \( f : S \rightarrow \mathbb{R} \), define \( (Pf) : S \rightarrow \mathbb{R} \) by \( (Pf)(x) = \sum_y P(x,y)f(y) \). By standard theory of Markov chains, for each \( t \geq 0 \), each \( v \in V_n \) and each \( j \in \{0, \ldots, C\} \),

\[
\mathbb{E}_\pi[(Pf_{v,j})(\hat{Z}_t) - f_{v,j}(\hat{Z}_t)] = 0.
\]

For \( x \in S \), we have

\[
(Pf_{v,0})(x) - f_{v,0}(x) = \frac{1}{(\lambda+C)(n/2)}( - \lambda f_{v,0}(x) - \lambda g_{v,0}(x) + f_{v,1}(x)),
\]

\[
(Pf_{v,j})(x) - f_{v,j}(x) = \frac{1}{(\lambda+C)(n/2)}(\lambda f_{v,j-1}(x) - \lambda f_{v,j}(x) + \lambda g_{v,j-1}(x) - \lambda g_{v,j}(x) - jj f_{v,j}(x) + (j+1)f_{v,j+1}(x)) \quad (0 < j < C),
\]

\[
(Pf_{v,C})(x) - f_{v,C}(x) = \frac{1}{(\lambda+C)(n/2)}(\lambda f_{v,C-1}(x) + \lambda g_{v,C-1}(x) - Cf_{v,C}(x)),
\]

where the \( g_{v,j} \), representing contributions due to alternatively routed arrivals with one end \( v \), are given, for \( j = 0, \ldots, C-1 \), by:

\[
\frac{1}{(n-2)^d} \left[ \sum_{r=1}^{d} \sum_{u,w} C_{u,v}^{C-1} \sum_{v',w'}^{r-1} \prod_{s=1}^{r-1} (1 - \mathbb{I}_{uv,wv} \mathbb{I}_{uw,vw}) \prod_{s=r+1}^{d} (1 - \mathbb{I}_{uw,vw} \mathbb{I}_{uv,wv}) 
+ \sum_{r=1}^{d} \sum_{u,w} C_{u,v}^{C-1} \sum_{v',w'}^{r-1} \prod_{s=1}^{r-1} (1 - \mathbb{I}_{uv',wv'} \mathbb{I}_{uw',v'w'}) \prod_{s=r+1}^{d} (1 - \mathbb{I}_{uv',v'w'} \mathbb{I}_{uw',wv'}) 
+ \sum_{r=1}^{d} \sum_{u,v',w_r} C_{u,v'}^{C-1} \sum_{v''}^{r-1} \prod_{s=1}^{r-1} (1 - \mathbb{I}_{uv',v''} \mathbb{I}_{v',w_r}) \prod_{s=r+1}^{d} (1 - \mathbb{I}_{v',w_r} \mathbb{I}_{uv',v''}) 
+ \sum_{r=1}^{d} \sum_{u,v',v''} C_{u,v'}^{C-1} \sum_{v'}^{r-1} \prod_{s=1}^{r-1} (1 - \mathbb{I}_{uv',v''} \mathbb{I}_{v',v''}) \prod_{s=r+1}^{d} (1 - \mathbb{I}_{v',v''} \mathbb{I}_{uv',v''}) \right]. \quad (5.2)
\]

Here, \( \sum_{u,w} \) denotes the sum over all \( u \neq v \), and over all \( w_1, \ldots, w_d \) such that each \( w_r \neq u, v \), and \( \sum_{u,v',w_r} \) denotes the sum over all \( u \neq v, v' \neq u, v \) and over all \( w_1, \ldots, w_r-1, w_r+1, \ldots, w_d \) such that each \( w_j \neq u, v' \).

Recall that \( \Delta^{C+1} = \{ \xi \in [0,1]^{C+1} : \sum_{j=0}^{C} \xi(j) = 1 \} \). Define the vector \( \zeta \in \Delta^{C+1} \) to have coordinates \( \zeta(j) = \frac{1}{n-1} \mathbb{E}_\pi f_{v,j}(\hat{Z}_t) \), for \( j = 0, \ldots, C \). Note that, by symmetry and stationarity, \( \zeta \) is independent of \( v \) and \( t \).

Taking expectations in (5.2), we now have, for all \( v \) and \( t \),

\[
0 = -\lambda \zeta(0) - \frac{\lambda}{n-1} \mathbb{E}_\pi g_{v,0}(\hat{Z}_t) + \zeta(1),
\]

\[
0 = \lambda \zeta(j-1) - \lambda \zeta(j) + \frac{\lambda}{n-1} \mathbb{E}_\pi g_{v,j-1}(\hat{Z}_t) - \frac{\lambda}{n-1} \mathbb{E}_\pi g_{v,j}(\hat{Z}_t) - j \zeta(j) + (j+1) \zeta(j+1) \quad (0 < j < C),
\]
0 = \lambda \zeta (C - 1) + \frac{\lambda}{n - 1} \mathbb{E}_\pi g_{v,C-1}(\hat{Z}_t) - C \zeta (C).

Summing these equations over \( j = 0, \ldots, k \) yields, for \( k = 0, \ldots, C - 1 \),

\[
0 = -\lambda \zeta (k) - \frac{\lambda}{n - 1} \mathbb{E}_\pi g_{v,k}(\hat{Z}_t) + (k + 1) \zeta (k + 1). \tag{5.3}
\]

We claim that each \( \frac{1}{n - 1} \mathbb{E}_\pi g_{v,j}(\hat{Z}_t) \) is close to \( g_j(\zeta) \), where the \( g_j \) are as defined in (1.2). We state two properties of the process in equilibrium, whose proofs are practically identical to two proofs in [6], except that they use the concentration of measure results from the previous sections. For distinct nodes \( u \) and \( v \), and \( j, k \in \{0, \ldots, C\} \), we define

\[
\phi_{u,v,j,k}^1 = \frac{1}{n - 2} \sum_w \mathbb{I}_{uw}^j t^k_{uw} - \frac{1}{(n - 2)^2} \sum_w \mathbb{I}_{uw}^j \sum_{w' \neq u,v} \mathbb{I}_{wv}^k;
\]

\[
\phi_{u,v,j}^2 = \frac{1}{n - 2} (f_{u,j} - f_{v,j}).
\]

Then we set \( \phi^1 = \max_{u,v,j,k} \phi_{u,v,j,k}^1 \) and \( \phi^2 = \max_{u,v,j} \phi_{u,v,j}^2 \), where the maximisations are over distinct nodes \( u \) and \( v \) and, where appropriate, \( j, k \in \{0, \ldots, C\} \). Finally, let \( \tilde{\phi} = \max(\phi^1, \phi^2) \). The function \( \tilde{\phi} \) was used in [6] as a measure of how “uniform” a state is. We shall show that the expected value of \( \tilde{\phi}(\hat{Z}_t) \) is small. For a state to have a low value of \( \phi_1 \) means that, for each fixed \( u \) and \( v \), if we choose a uniformly random “intermediate node” \( w \), then the loads on the two links \( uw \) and \( wv \) are nearly independent. To have a small value of \( \phi_2 \) means that all nodes have similar distributions of loads on the links incident with them. In a state where \( \tilde{\phi} \) is small, the complex functions \( g_{v,j} \) above can be “approximately factorised”, so that they can be approximated by products of functions similar to the \( f_{v,j} \). Then, applying our concentration inequalities, Lemmas 3.2 and 4.5 to various functions \( h \), we can show that the expectation of the product is close to the product of the expectations. This phase of the proof is almost identical to the proof of Lemma 6.2 in [4] — indeed, that proof goes through whenever the distribution of states satisfies a concentration inequality of the same type as ours. As the argument is somewhat lengthy, we omit the proof.

**Lemma 5.1.** Suppose that either \( \lambda < \lambda_0(d) \) or \( \lambda \geq \lambda_1(d,C) \). For all sufficiently large \( n \), for all \( v \in V_u \) and \( j \in \{0, \ldots, C\} \),

\[
\left| \mathbb{E}_\pi [g_{v,j}(\hat{Z}_t)] - (n - 1) g_j(\zeta) \right| \leq 8d^2(C + 1)^3 n \mathbb{E}_\pi [\tilde{\phi}(\hat{Z}_t)] + 16d^2 (C + 1) \sqrt{n} \log n.
\]

The only real difference between the proof of Lemma 5.1 above and that of Lemma 6.2 in [4] is that we use Lemma 3.2 or Lemma 4.5 in place of the concentration of measure result used in [4], which applied to a process starting in a fixed state, and running for a bounded period.

It remains to bound \( \mathbb{E}_\pi [\hat{\phi}] \). The proof of the following lemma is identical to the argument in Section 8 of [4]. Indeed, that proof goes through for any
distribution of states that respects the symmetry of the complete graph, and
is such that \((u, v)\)-Lipschitz functions are well-concentrated.

**Lemma 5.2.** Suppose that either \(\lambda < \lambda_0(d)\) or \(\lambda \geq \lambda_1(d, C)\). Then, for
sufficiently large \(n\),
\[
\mathbb{E}_\pi \tilde{\phi} (\hat{Z}_t) \leq 4 \log n \sqrt{n}.
\]

Combining Lemmas 5.1 and 5.2, we obtain that, if either \(\lambda < \lambda_0(d)\) or \(\lambda \geq \lambda_1(d, C)\), then for
all sufficiently large \(n\), for all \(v \in V_n\) and \(j \in \{0, \ldots, C\}\),
\[
|\mathbb{E}_\pi [g_{v,j}(\hat{Z}_t)] - (n-1)g_j(\zeta)| \leq 36d^2(C + 1)^3 \sqrt{n} \log n.
\]
Combining this with (5.3) gives that, for \(j = 0, \ldots, C - 1\),
\[
|\lambda \zeta(j) - \lambda g_j(\zeta) + (j+1)\zeta(j+1)| \leq 40\lambda d^2(C + 1)^3 \frac{\log n}{\sqrt{n}}.
\]

6. Approximate solutions to \(F(\eta) = 0\)

The aim of this section is to prove parts (3) and (4) of Theorem 1.1,
showing that, in either of the two regimes we are considering, the equation
\(F(\eta) = 0\) has a unique solution, and any approximate solution, i.e., a vector
\(\zeta\) satisfying (5.4), lies close to this solution.

Luczak \[6\] shows, for each \(k \in \{0, \ldots, C\}\) and each pair \(\xi, \eta\) in \(\Delta^{C+1}\), that
\[
|g_k(\xi) - g_k(\eta)| \leq 3d^2(C + 1)^2 \|\xi - \eta\|_{\infty};
\]
we begin this section by obtaining
a sharper version of this Lipschitz-like inequality, which can be sharpened
further in the two regimes we study.

For \(0 \leq b \leq a \leq 1\), we set
\[
H[a, b] = (a-b) \sum_{r=1}^{d} (1-a^2)^{r-1}(1-b^2)^{d-r},
\]
so that, for \(\xi \in \Delta^{C+1}\), and \(k \in \{0, \ldots, C - 1\}\),
\[
g_k(\xi) = 2\xi(C)\xi(\leq k)H[\xi(\leq k), \xi(\leq k-1)] + 2\xi(C)\xi(k) \sum_{i=k+1}^{C-1} H[\xi(\leq i), \xi(\leq i-1)];
\]
here we used the fact that \(\xi(j) = \xi(\leq j) - \xi(\leq j-1)\) for each \(j\). Observe that
\(H[a, b] \leq d(a-b)\). We also see that
\[
\left| \frac{\partial H[a, b]}{\partial a} \right| = \left| \sum_{r=1}^{d} (1-a^2)^{r-1}(1-b^2)^{d-r} - 2a(a-b) \sum_{r=2}^{d} (r-1)(1-a^2)^{r-2}(1-b^2)^{d-r} \right|
\]
\[
\leq d + 2 \sum_{r=2}^{d} (r-1)a^2(1-a^2)^{r-2} \leq 3d,
\]
and similarly $|\frac{\partial H[a,b]}{\partial b}| \leq 3d$. Therefore, whenever $0 \leq b \leq a \leq 1$ and $0 \leq b' \leq a' \leq 1$, we have $|H[a,b] - H[a',b']| \leq 3d(|a - a'| + |b - b'|)$. 

We write, for any two vectors $\xi, \eta \in \Delta^{C+1}$, and any $k \in \{0, \ldots, C - 1\}$,

$$g_k(\xi) - g_k(\eta) = 2[\xi(C) - \eta(C)]\left\{\xi(\leq k)H[\xi(\leq k),\xi(\leq k-1)] + \xi(k)\sum_{i=k+1}^{C-1} H[\xi(\leq i),\xi(\leq i-1)]\right\}$$

$$+ 2\eta(C)[\xi(\leq k) - \eta(\leq k)]H[\xi(\leq k),\xi(\leq k-1)]$$

$$+ 2\eta(C)\eta(\leq k) [H[\xi(\leq k),\xi(\leq k-1)] - H[\eta(\leq k),\eta(\leq k-1)]$$

$$+ 2\eta(C)\eta(k) \sum_{i=k+1}^{C-1} H[\xi(\leq i),\xi(\leq i-1)]$$

$$+ 2\eta(C)\eta(k) \sum_{i=k+1}^{C-1} [H[\xi(\leq i),\xi(\leq i-1)] - H[\eta(\leq i),\eta(\leq i-1)]].$$

So, using our bounds on $H[a,b]$ and $|H[a,b] - H[a',b']|$, we have

$$|g_k(\xi) - g_k(\eta)| \leq 2|\xi(C) - \eta(C)|d\left\{\xi(\leq k)\xi(k) + \xi(k)\sum_{i=k+1}^{C-1} \xi(i)\right\}$$

$$+ 2\eta(C)|\xi(\leq k) - \eta(\leq k)|d\xi(k)$$

$$+ 2\eta(C)\eta(\leq k)3d(|\xi(\leq k) - \eta(\leq k)| + |\xi(\leq k-1) - \eta(\leq k-1)|)$$

$$+ 2\eta(C)|\xi(k) - \eta(k)|d\sum_{i=k+1}^{C-1} \xi(i)$$

$$+ 2\eta(C)\eta(k)3d \sum_{i=k+1}^{C-1} (|\xi(\leq i) - \eta(\leq i)| + |\xi(\leq i-1) - \eta(\leq i-1)|)$$

$$\leq d\xi(k)||\xi - \eta||_1\xi(\leq C - 1) + \eta(C)d\left[\xi(k)||\xi - \eta||_1 + 6||\xi - \eta||_1\eta(\leq k)$$

$$+ 2||\xi(k) - \eta(k)||\xi(\leq C - 1) + 6C\eta(k)||\xi - \eta||_1\right].$$

Summing over $k = 0, \ldots, C - 1$, we have that, for any vectors $\xi, \eta \in \Delta^{C+1}$,

$$\|g(\xi) - g(\eta)\|_1 \leq d||\xi - \eta||_1\xi(\leq C - 1)^2 + \eta(C)d\left(||\xi - \eta||_1\xi(\leq C - 1) + 6C||\xi - \eta||_1\eta(\leq C - 1)$$

$$+ 2||\xi - \eta||_1\xi(\leq C - 1) + 6C||\xi - \eta||_1\eta(\leq C - 1)\right)$$

$$\leq (1 + \eta(C)(12C + 3))d||\xi - \eta||_1\max(\xi(\leq C - 1), \eta(\leq C - 1)). \quad (6.1)$$
At this point, we separate the calculations for small and large $\lambda$. Suppose first that $\lambda < \lambda_0(d) = 1/(8d + 4)$. In this case, we simplify the bound (6.1) above to obtain, for any $\xi$ and $\eta$ in $\Delta^{C+1}$, that

$$\|g(\xi) - g(\eta)\|_1 \leq (1 + \eta(C)(12C + 3))d\|\xi - \eta\|_1.$$ 

Suppose also that $\eta$ satisfies the fixed-point equation

$$-\lambda(j) - \lambda g_j(\eta) + (j + 1)\eta(j + 1) = 0 \quad (j = 0, \ldots, C - 1). \quad (6.2)$$

Noting that $g_j(\eta) \leq 2d\eta(j)\eta(C)$ for all $j$, we obtain

$$\eta(j + 1) \leq \frac{\eta(j)\lambda(1 + 2d\eta(C))}{j + 1}, \quad \text{and hence} \quad \eta(j) \leq \left(\frac{\lambda(1 + 2d\eta(C))}{j!}\right)^j,$$

for each $j$. For $\lambda \leq \lambda_0(d) = 1/(8d + 4)$, we apply this inequality to obtain $\eta(C) \leq 1/4^C C! \leq 1/4$, and then apply it again to obtain

$$(12C + 3)\eta(C) \leq \frac{12C + 3}{8^C C!} < 2.$$ 

Thus, for any $\xi \in \Delta^{C+1}$, and any $\eta \in \Delta^{C+1}$ satisfying (6.2), we have

$$\|g(\xi) - g(\eta)\|_1 \leq 3d\|\xi - \eta\|_1,$$

provided $\lambda \leq \lambda_0(d)$.

Now let $\zeta$ be any vector in $\Delta^{C+1}$ satisfying (6.1), and $\eta$ any vector in $\Delta^{C+1}$ satisfying (6.2). We have, for $j = 0, \ldots, C - 1$,

$$(j + 1)|\zeta(j + 1) - \eta(j + 1)| \leq \lambda(|\zeta(j) - \eta(j)| + |g_j(\zeta(j) - g_j(\eta(j))|) + 40\lambda d^2(C + 1)^3\log n.$$ 

Summing over $j = 0, \ldots, C - 1$ now gives, provided $\lambda < \lambda_0(d) = 1/(8d + 4)$,

$$\|\zeta - \eta\|_1 \leq 2 \sum_{j=0}^{C-1} |\zeta(j + 1) - \eta(j + 1)| \leq 2\lambda(\|\zeta - \eta\|_1 + \|g(\zeta) - g(\eta)\|_1) + 80\lambda d^2(C + 1)^4\log n \sqrt{n},$$

$$\leq (6d + 2)\lambda\|\zeta - \eta\|_1 + 80\lambda d^2(C + 1)^4\log n \sqrt{n},$$

$$\leq \frac{3}{4}\|\zeta - \eta\|_1 + 80\lambda d^2(C + 1)^4\log n \sqrt{n},$$

and so

$$\|\zeta - \eta\|_1 \leq 320\lambda d^2(C + 1)^4\log n \sqrt{n}.$$ 

If $\eta$ and $\zeta$ are two elements of $\Delta^{C+1}$ satisfying (6.2), then we can apply exactly the same argument with the term $40\lambda d^2(C + 1)^3\log n \sqrt{n}$, coming from the right-hand-side of (6.1), replaced by zero, and we deduce that $\|\zeta - \eta\| = 0$ and so $\zeta = \eta$: this amounts to an application of the contraction mapping theorem. This establishes part (3) of Theorem 1.1 stating that there is a unique solution $\eta^*$ of (6.2), in this range of $\lambda$. 

We now conclude that any \( \zeta \in \Delta^{C+1} \) satisfying (5.4) lies within \( \ell_1 \) distance \( 320\lambda d^2(C+1)^4 \log n \sqrt{n} \) of \( \eta^* \), and so in particular, if \( \tilde{Z}_t \) is a copy of the process in equilibrium, then
\[
\left| \frac{1}{n-1} E_n f_{n,j}(\tilde{Z}_t) - \eta^*(j) \right| \leq 320\lambda d^2(C+1)^4 \log n \sqrt{n}
\]
for each \( j \), and this implies part (4) of Theorem 1.1 in this regime.

For \( \lambda \geq \lambda_1(d,C) \), we take the same general approach. In this case, we deduce from (6.1) that, for any \( \eta \) and \( \zeta \) in \( \Delta^{C+1} \),
\[
\|g(\zeta) - g(\eta)\|_1 \leq 16Cd\|\zeta - \eta\|_1 \max(\zeta(\leq C-1), \eta(\leq C-1)).
\]
Now we assume that \( \eta \) satisfies the fixed-point equation (6.2), and \( \zeta \) satisfies the approximate version (5.4). We have that, for \( j = 0, \ldots, C - 1 \),
\[
\eta(j) \leq \frac{j + 1}{\lambda} \eta(j + 1) \leq \frac{C}{\lambda} \eta(j + 1),
\]
and so \( \eta(C - i) \leq (C/\lambda)^i \eta(C) \), for \( i = 1, \ldots, C \). Thus
\[
\eta(C) \geq \frac{1}{\sum_{i=0}^C (C/\lambda)^i} \geq 1 - \frac{C}{\lambda},
\]
and so \( \eta(\leq C-1) \leq C/\lambda \). Essentially the same is true for \( \zeta \):
\[
\zeta(j) \leq \frac{C}{\lambda} \zeta(j + 1) + 40\lambda d^2(C+1)^4 \log n \sqrt{n},
\]
which leads to \( \zeta(\leq C - i) \leq (C/\lambda)^i \zeta(C) + 80\lambda d^2(C+1)^3 \log n \sqrt{n} \) for each \( i = 0, \ldots, C \), and hence
\[
\zeta(\leq C-1) \leq \frac{C}{\lambda} + 80\lambda d^2(C+1)^4 \log n \sqrt{n} \leq \frac{2C}{\lambda},
\]
for sufficiently large \( n \). Therefore we obtain that
\[
\|g(\zeta) - g(\eta)\|_1 \leq \frac{32C^2d}{\lambda} \|\zeta - \eta\|_1 \leq \frac{1}{4} \|\zeta - \eta\|_1,
\]
whenever \( \lambda \geq \lambda_1(d,C) = 8000C^2d \log(240C^2d), \eta \in \Delta^{C+1} \) satisfies (5.2) and \( \zeta \in \Delta^{C+1} \) satisfies (5.4). Now we note that, for \( j = 0, \ldots, C - 1 \),
\[
|\zeta(j) - \eta(j)| \leq \frac{j + 1}{\lambda} |\zeta(j + 1) - \eta(j + 1)| + |g_j(\zeta) - g_j(\eta)| + 40d^2(C+1)^3 \log n \sqrt{n}.
\]
Summing over \( j = 0, \ldots, C - 1 \) yields
\[
\|\zeta - \eta\|_1 \leq 2 \sum_{j=0}^{C-1} |\zeta(j) - \eta(j)|
\leq \frac{2C}{\lambda} \|\zeta - \eta\|_1 + 2\|g(\zeta) - g(\eta)\| + 80d^2(C+1)^4 \log n \sqrt{n}
\leq \frac{1}{4} \|\zeta - \eta\|_1 + \frac{1}{4} \|\zeta - \eta\|_1 + 80d^2(C+1)^4 \log n \sqrt{n}.
\]
Hence, provided $\lambda \geq \lambda_1(d, C)$, $\eta$ satisfies (6.2) and $\zeta$ satisfies (5.4),

$$\|\zeta - \eta\|_1 \leq 160d^2(C + 1)^4 \frac{\log n}{\sqrt{n}}.$$ 

The same technique shows that there is only one solution $\eta^\ast \in \Delta^{C+1}$ to the fixed-point equation (6.2). Since the vector $\zeta$ given by $\zeta(j) = \frac{1}{n-1} \mathbb{E} f_{v,j}(\hat{Z}_t)$ ($j = 0, \ldots, C$), where $\hat{Z}$ is in equilibrium, lies in $\Delta^{C+1}$ and satisfies (5.4), we then obtain that

$$\left| \frac{1}{n-1} \mathbb{E} f_{v,j}(\hat{Z}_t) - \eta^\ast(j) \right| \leq 160d^2(C + 1)^4 \frac{\log n}{\sqrt{n}}.$$ 

This gives Theorem 1.1(3) and (4) in this regime, completing the proof.

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