CHARACTERIZATION OF GEODESIC FLOWS ON $T^2$
WITH AND WITHOUT POSITIVE TOPOLOGICAL ENTRPY

EVA GLASMACHERS AND GERHARD KNIEPER

Abstract. In the present work we consider the behavior of the geodesic flow on the unit tangent bundle of the 2-torus $T^2$ for an arbitrary Riemannian metric. A natural non-negative quantity which measures the complexity of the geodesic flow is the topological entropy. In particular, positive topological entropy implies chaotic behavior on an invariant set in the phase space of positive Hausdorff-dimension (horseshoe). We show that in the case of zero topological entropy the flow has properties similar to integrable systems. In particular there exists a non-trivial continuous constant of motion which measures the direction of geodesics lifted onto the universal covering $\mathbb{R}^2$. Furthermore, those geodesics travel in strips bounded by Euclidean lines. Moreover we derive necessary and sufficient conditions for vanishing topological entropy involving intersection properties of single geodesics on $T^2$.

1. Introduction

Let $(T^2, g)$ be a two-dimensional Riemannian torus. By $c_v$ we denote the unique geodesic $c_v : \mathbb{R} \to T^2$ with the initial condition $\dot{c}_v(0) = v \in ST^2$. The geodesic flow on the unit tangent bundle $ST^2$ is given by $\phi^t(v) = \dot{c}_v(t)$. We also consider geodesics on $\mathbb{R}^2$, where $\mathbb{R}^2$ is equipped with the lifted metric. The topological entropy of a continuous dynamical system represents the exponential growth rate of orbits segments distinguishable with arbitrarily fine but finite precision. It therefore describes the total exponential orbit complexity by a single number. Note that due to a theorem of A. Katok [19] for $C^{1+\alpha}$-flows, $\alpha > 0$, on 3-dimensional spaces positive topological entropy and the existence of a horse-shoe are equivalent. We consider in this paper the questions which consequences on the behavior of geodesics on $T^2$ we can expect under the assumption of zero topological entropy and which geometrical restrictions on their behavior forbid high complexity of the geodesic flow. More precisely, we formulate necessary and sufficient conditions for zero topological entropy. In order to state the main theorems we have to define the asymptotic direction of geodesic rays, i.e. geodesics $c : [0, \infty) \to \mathbb{R}^2$. Note that unlike in most text books in our context a geodesic ray does not have to be minimal. Given a geodesic $c : \mathbb{R} \to \mathbb{R}^2$ then we associate to $c$ two geodesic rays given by $c^+ := c|_{[0,\infty)} : [0, \infty) \to \mathbb{R}^2$ and $c^- : [0, \infty) \to \mathbb{R}^2$ with $c^-(t) := c(-t)$.

Date: July 1, 2010.

2000 Mathematics Subject Classification. Primary 37C40, Secondary 53C22, 37C10.

Key words and phrases. topological entropy, geodesic flows on tori.
In the following it will be useful to distinguish the following types of geodesic rays (see also [22]).

**Definition 1.1** (types of geodesic rays). 
Given a complete Riemannian metric on \( \mathbb{R}^2 \). A geodesic ray \( c : [0, \infty) \rightarrow \mathbb{R}^2 \) is called
- bounded, if the set \( c([0, \infty)) \) is bounded.
- escaping, if \( \lim_{t \to \infty} ||c(t)|| = \infty \), where \( || \cdot || \) denotes the Euclidean norm on \( \mathbb{R}^2 \).
- oscillating, if \( c \) is neither bounded nor escaping.

A geodesic \( c : \mathbb{R} \rightarrow \mathbb{R}^2 \) is bounded (escaping or oscillating) if both of its geodesic rays \( c^+ \) and \( c^- \) are bounded (escaping or oscillating).

**Remark.** It will turn out that mixed cases of geodesics will not be relevant. Furthermore, their Liouville measure is zero, as proved by M. Wojtkowski in [22].

**Definition 1.2** (asymptotic direction and rotation number). 
Let \( c : [0, \infty) \rightarrow \mathbb{R}^2 \) be an escaping geodesic ray on \( \mathbb{R}^2 \). Then, if the limit exists,
\[
\delta(c) := \lim_{t \to \infty} \frac{c(t)}{||c(t)||} \in S^1
\]
is called the asymptotic direction of \( c \).

Let \( \pi : S^1 = \{ (x, y) \mid x^2 + y^2 = 1 \} \rightarrow \mathbb{P}_1(\mathbb{R}) \cong \mathbb{R} \cup \{ \infty \} \),
be the canonical projection onto \( \mathbb{P}_1(\mathbb{R}) \) defined by
\[
\pi(s) = \begin{cases} 
\frac{y}{x}, & \text{if } x \neq 0 \\
\infty, & \text{otherwise} 
\end{cases}
\]
Then we call
\[
\rho(c) := \pi \circ \delta(c) \in \mathbb{P}_1(\mathbb{R}) \cong \mathbb{R} \cup \{ \infty \}
\]
the rotation number of \( c \).

Let \( c : [0, \infty) \rightarrow T^2 \) be a geodesic ray such that its lift \( \tilde{c} : [0, \infty) \rightarrow \mathbb{R}^2 \) provides an escaping geodesic ray for which \( \delta(\tilde{c}) \) exists. Then we define
\[
\delta(c) := \delta(\tilde{c}) \text{ and } \rho(c) := \rho(\tilde{c}).
\]
A rotation number \( \rho(c) \) is called rational if \( \rho(c) \in \mathbb{Q} \cup \{ \infty \} \). A direction \( \delta(c) \in S^1 \) is called rational if \( \pi(\delta(c)) \in \mathbb{Q} \cup \{ \infty \} \).

**Remark.**

(a) The definition and the existence of the asymptotic direction are independent of the chosen lift.

(b) A first definition of a rotation number for minimal geodesics induced by the slope of the accompanying Euclidean lines goes back to G. A. Hedlund (see [15]) and H. M. Morse (see [21]). In [7] V. Bangert presents a definition of the rotation number for minimal
geodesics, which coincides with our definition restricted to minimal geodesics.

Now we are able to state our first theorem.

**Theorem I.** Let \( g \) be a Riemannian metric on \( T^2 \) with vanishing topological entropy. Then, on the universal covering \( \mathbb{R}^2 \), every geodesic \( c \) is escaping without self-intersections and the directions \( \delta(c^+) \) and \( \delta(c^-) \) exist. Furthermore, \( \delta(c^+) = -\delta(c^-) \) and \( c \) lies in a bounded Euclidean strip \( S(c) \), where \( S(c) \) is a strip in \( \mathbb{R}^2 \) bounded by two Euclidean lines.

Consider the asymptotic direction and the rotation number for a geodesic \( c_v : \mathbb{R} \to T^2 \) as functions on the unit tangent bundle defined as \( \delta(v) := \delta(c^+_v) \) and \( \rho(v) := \rho(c^+_v) \), if they exist. Obviously \( \delta(c^-_v) = -\delta(-v) \) and \( \rho(c^-_v) = \rho(-v) \).

**Theorem II.** Let \( g \) be a Riemannian metric on \( T^2 \) with vanishing topological entropy. Then the asymptotic direction \( \delta : ST^2 \to S^1 \) and the rotation number \( \rho : ST^2 \to \mathbb{R} \cup \{\infty\} \) are surjective continuous functions invariant under the geodesic flow.

**Remark.** If the constant of motion \( \rho \) would be differentiable with non-zero differential almost everywhere, the system would be integrable in the sense of Liouville-Arnold.

\[ \text{From now on we represent } T^2 \text{ as } \mathbb{R}^2 / \mathbb{Z}^2, \text{ where } \mathbb{Z}^2 \text{ acts on } \mathbb{R}^2 \text{ via} \]
\[ \tau_{(m,n)}(x, y) = (x + m, y + n). \]

The non-trivial covering transformations \( \tau_{(m,n)} \) will be called translation elements. Sometimes we will denote \( \tau_{(m,n)} \) only by \( \tau \). A translation element \( \tau \) is called primitive if it is not a nontrivial power of another translation element. Two translation elements \( \tau \) and \( \eta \) are called equivalent if there exist \( k, \ell \in \mathbb{Z} \setminus \{0\} \) such that
\[ \tau^k = \eta^\ell. \]

By \( [\tau] \) we denote the corresponding equivalence classes.

**Definition 1.3.** A geodesic \( c : \mathbb{R} \to \mathbb{R}^2 \) is called an axis if there exists a nontrivial translation element \( \tau \) such that \( \tau c(t) = c(t + l) \) for some \( l \in \mathbb{R} \) and all \( t \in \mathbb{R} \). By \( \delta(\tau) \) we will denote the asymptotic direction of the axes of \( \tau \). An axis \( \alpha \) of a translation element \( \tau^k \) with \( k \geq 2 \) is called a non-primitive axis if \( \alpha(\mathbb{R}) \neq \tau\alpha(\mathbb{R}) \). A geodesic \( c : \mathbb{R} \to T^2 \) is called prime-periodic if its lift is an axis of a primitive translation element \( \tau \).

**Definition 1.4.** For a geodesic ray \( c : [0, \infty) \to \mathbb{R}^2 \) we define
\[ I(c) := \{[\tau] \mid \# \{c([0, \infty)) \cap \eta c([0, \infty])\} = \infty \text{ for some } \eta \in [\tau]\}. \]

We will now present a further characterization of the behavior of geodesics on the universal covering in the case of vanishing topological entropy. By
intersections we always mean transversal intersections.

**Theorem III.** Let \((T^2, g)\) be a Riemannian torus with zero topological entropy. Then \(\#I(c) \leq 1\), for each geodesic ray \(c : [0, \infty) \to \mathbb{R}^2\). Furthermore, geodesics \(c\) with irrational rotation number intersect their translates \(\tau c\) only a finite number of times. Geodesics \(c\) with rational rotation number intersect their translates \(\tau c\) an infinite number of times at most for \(\tau\) with \(\delta(\tau) = \delta(c^+)\).

In a previous version of this paper we have had a weaker formulation of the following theorem. We originally showed applying hyperbolic dynamics and the Curve Shortening flow that the assumption of the theorem implies zero topological entropy. V. Bangert showed to us how to use variational arguments to even conclude flatness. We also note that we obtained flatness under the stronger assumption that no geodesic on the universal covering intersects its translate transversally.

**Theorem IV.** Let \(g\) be a Riemannian metric on \(T^2\). Then flatness of the metric \(g\) is equivalent to the condition that no axis \(c : \mathbb{R} \to \mathbb{R}^2\) on the universal covering intersects any of its translates.

**Remark.**

1) The assumption of Theorem IV is equivalent to the fact that all axes \(c : \mathbb{R} \to \mathbb{R}^2\) are axes of primitive elements.

2) As the theory of minimal geodesics and the Curve Shortening flow extend to symmetric Finsler metrics, all Theorems generalize to symmetric Finsler metrics.

3) M. L. Bialy and L. Polterovich [8] and independently V. Bangert [7] present a formalism for minimal geodesics on \(T^2\) based on a special class of orbits for monotone twist maps. For this setting they formulate the notion of rotation number. However, in general it is not possible to extend their formalism to non-minimal geodesics and arbitrary orbits of twist maps.

The study of monotone twist maps with variational methods and many results about their properties mentioned in [7] and [8] go back to J. N. Mather [20] and independently to S. Aubry and P. Y. Le Daeron [4]. In [1] S. B. Angenent studies orbits of monotone twist maps and their intersection properties in the case that these maps have vanishing topological entropy. He formulates analogous results to parts of the statements of Theorem I and Theorem III.

4) A different approach to understand the relation between the complexity of the geodesic flow and the behavior of geodesics similar to the one in this paper is presented by S. V. Bolotin and P. H. Rabinowitz in [9].

The paper is organized as follows: In the second section we introduce the notion of the topological entropy and discuss the for us relevant properties of
the Curve Shortening flow. In section three we show that vanishing topological entropy implies the non-existence of self-intersections of lifted geodesics and the non-existence of contractible geodesics on $T^2$. Section four deals with geometric conditions which imply via the Curve Shortening flow positive topological entropy. In the case of vanishing topological entropy we exclude the existence of oscillating geodesic rays. Under the same assumption, in section five we prove the existence and regularity of the rotation number. In section six we study the intersection properties of geodesics on the universal covering with their translates. In the last section we provide a characterization for flatness of Riemannian metrics on $T^2$. In particular, we show that no axis on the universal covering intersects its translates iff the torus is flat.

2. Topological entropy and curve shortening

The topological entropy is invariant under topological conjugations and measures as described in the introduction the exponential orbit complexity by a single non-negative number. The precise meaning becomes apparent in the following definition of topological entropy introduced by R. E. Bowen [10].

**Definition 2.1** (Topological entropy). Let $(Y,d)$ be a compact metric space, $\phi^t : Y \to Y$ a continuous flow and $d(\cdot,\cdot)_T$ the dynamical metric defined by $d(v,w)_T := \max_{0 \leq t \leq T} d(\phi^tv,\phi^tw)$ for all $v,w \in Y$. We fix $\varepsilon > 0$. A subset $F \subset Y$ is called $(\phi,\varepsilon)$-separated set of $Y$ with respect to $T$, if for $x_1 \neq x_2 \in F$ it holds $d(x_1,x_2)_T > \varepsilon$. The topological entropy of $\phi^t$ is defined as

$$h_{\text{top}}(\phi) = h_{\text{top}}(\phi^t) = \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \left( \frac{1}{T} \log r_T(\phi,\varepsilon) \right).$$

Here $r_T(\phi,\varepsilon)$ denotes the maximal cardinality of any $(\phi,\varepsilon)$-separated set of $Y$ with respect to $T$.

For more details and properties of the topological entropy see for example [18] or [23].

To prove that certain geometric constellations imply positive topological entropy of the geodesic flow on $ST^2$ we will use the Curve Shortening flow on $T^2$. In the following we will give the precise definition of this flow and state the properties relevant in this context.

Let $(M,g)$ be a Riemannian surface and

$$\Gamma = \{ \gamma : S^1 \to M \text{ smooth immersed closed curve } \}$$

a family of (parameterized) immersed smooth closed curves on $M$. We consider a continuous local semi-flow $\Psi^t : \Gamma \to \Gamma$ with $\Psi^t(\gamma) =: \gamma_t$ and $t \in [0,T_\gamma)$ defined by

$$\frac{\partial \gamma_t}{\partial t} = k_t N_t,$$

where $k_t$ denotes the geodesic curvature of $\gamma_t$ and $N_t$ its unit normal vector. This evolution equation defined in (2.1) is called the Curve Shortening flow.
From the following theorem we will derive the existence of closed geodesics:

**Theorem 2.2** (M. A. Grayson, see [14]). Let $M$ be a smooth Riemannian surface which is convex at infinity, i.e. the convex hull of every compact subset is compact. Let $\gamma_0 : S^1 \to M$ be a smooth curve, embedded in $M$. Then, $\gamma_t : S^1 \to M$ exists for $t \in [0, T)$, for some $T > 0$, satisfying the evolution equation (2.1).

If $T$ is finite, then $\gamma_t$ converges to a point. If $T$ is infinite, then the curvature of $\gamma_t$ converges to zero in the $C^\infty$ norm, i.e. there exists a subsequence $t_n$ such that $\gamma_{t_n}$ converges to a closed geodesic.

The assumption that $M$ is convex at infinity ensures that the set of limit curves exists since an evolving curve $\gamma_t$ cannot leave a compact set. In our special case of the torus we will directly show that the embedded curve $\gamma_t$ stays in a compact set on the universal covering $\mathbb{R}^2$ of $T^2$ or in a compact set on some unbounded cylinder $C$ for all $t \in [0, T)$. Then, by excluding that $\gamma_t$ converges to a point we conclude the existence of a closed geodesic. As the length $l_t$ of the curve $\gamma_t$ fulfills

$$\frac{dl_t}{dt} = -\int k_t^2(s)ds,$$

the length is a decreasing function of $t$. This is why this flow is called Curve Shortening.

An important fact is that embedded curves never become singular, unless they shrink to a point, as proved by M. A. Grayson [14].

Furthermore we will apply the following consequence of the maximum-principle for parabolic differential equations several times:

**Theorem 2.3** (S. B. Angenent, see [2]). Let $\gamma_0, \eta_0 : [0, 1] \to M$ be two curve-segments and $\{\gamma_t \mid 0 < t < T\}, \{\eta_t \mid 0 < t < T\}$ solutions of Curve Shortening which satisfy

$$\partial\gamma_t \cap \eta_t = \partial\eta_t \cap \gamma_t = \emptyset$$

for all $t \in [0, T)$. Then the number of intersections of the solutions $\gamma_t$ and $\eta_t$ is a finite and nonincreasing function of $t \in (0, T)$. The solutions intersect only transversally except at a discrete set of times $\{t_j\} \subset (0, T)$, and at each $t_j$ the number of intersections of $\gamma_t$ and $\eta_t$ decreases.

3. **Contractible Closed Geodesics and Self-intersections**

In 1998 J. Denvir and R. S. MacKay [12] showed the following result as a conclusion of their study on geodesically convex surfaces of negative Euler characteristic.

**Theorem 3.1.** Let $g$ be a Riemannian metric on $T^2$ with a simple closed contractible geodesic $c$. Then $g$ has positive topological entropy.

This theorem allows us to conclude the following lemma:

**Lemma 3.2.** Let $g$ be a Riemannian metric on $T^2$ and $c : \mathbb{R} \to \mathbb{R}^2$ a lift of a geodesic with two self-intersections such that $c(t_1) = c(t_2)$ and $c(t_3) = c(t_4)$ with $t_1 < t_2 < t_3 < t_4$. Then $g$ has positive topological entropy.
Proof. We use similar ideas as S. Angenent in [3]. Let $\gamma_0 : S^1 \to \mathbb{R}^2$ be a simple closed contractible and smooth curve such that $\gamma_0$ and $c([t_1, t_4])$ do not intersect on $\mathbb{R}^2$ and such that $c[t_1, t_4]$ lies in the bounded connected component of $\mathbb{R}^2 \setminus \gamma_0(S^1)$. We apply the curve-shortening flow to $\gamma_0$ and $c$. Note that $c([t_1, t_4])$ is a constant solution. Assume, that $\gamma_t$ shrinks to a point. Then the solution $\gamma_t$ has to pass $c([t_1, t_4])$ under the flow. Consider $\tilde{t}$ with $\gamma_{\tilde{t}}(S^1) \cap c([t_1, t_4]) \neq \emptyset$ such that $\tilde{t}$ is the first time at which the segments meet. By Theorem 2.3 new intersections under the flow can only appear in the endpoints $c(t_1)$ or $c(t_4)$. Consider a small $\varepsilon > 0$ such that $t_1 + \varepsilon < t_2$ and $t_4 - \varepsilon > t_3$. As $c(t_1) = c(t_2)$ and $c(t_3) = c(t_4)$ the curve $\gamma_{\tilde{t}}$ meets the geodesic segment $c([t_1 + \varepsilon, t_4 - \varepsilon])$ also at $s \in (t_1 + \varepsilon, t_4 - \varepsilon)$, in contradiction to Theorem 2.3. Hence, as $\gamma_t$ cannot pass $c([t_1, t_4])$ it will not shrink to a point.

Furthermore, $\gamma_t$ will stay in a bounded set $K \subset \mathbb{R}^2$ for all $t$: We consider two non-equivalent primitive translation elements $\tau, \eta$ and two corresponding axes $\alpha_\tau$ and $\alpha_\eta$. For $k \in \mathbb{N}$ large enough the curve $\gamma_0$ lies in the open strip between $\tau^k\alpha_\eta$ and $\tau^{-k}\alpha_\eta$, and in the open strip between $\eta^k\alpha_\tau$ and $\eta^{-k}\alpha_\tau$. The intersection $K$ of the closures of these strips defines a region from which $\gamma_t$ cannot escape, because since $\gamma_0$ does not intersect any of the bounding minimal geodesics, by Theorem 2.3 $\gamma_t$ will not intersect them for all $t > 0$.

Now we apply M. A. Grayson’s Theorem 2.2. Since $\gamma_t$ will not shrink to a point and stays in a compact set for all times, there exists a simple closed geodesic on $\mathbb{R}^2$. By Theorem 3.1 $g$ has positive topological entropy.

\[ \begin{array}{c}
\tau^{-k}\alpha_\eta \\
\gamma_0 \\
\tau^k\alpha_\eta \\
\eta^k\alpha_\tau \\
\eta^{-k}\alpha_\tau \\
K
\end{array} \]

Figure 1. Illustration of the two loops of $c$ and $\gamma_0$ in the proof of Lemma 3.2.

In the sequel we will make use of the following standard recurrence theorem (see for instance [23], page 157).

Theorem 3.3. Let $X$ be a manifold and $\mu$ a probability measure invariant under a continuous flow $\phi^t : X \to X$. Then almost all $p \in X$ are recurrent, i.e. there exists a sequence $t_n$ with $t_n \to \infty$ as $n \to \infty$ such that

\[ \lim_{n \to \infty} \phi^{t_n}p = p. \]

Furthermore, if $p$ is recurrent all points on the orbit $\phi^t(p)$ are recurrent as well. If $\mu$ is positive on open sets, the set of recurrent points is dense.
Theorem 3.4. Let \( g \) be a Riemannian metric on \( T^2 \) with zero topological entropy. Then no lift of a geodesic has a self-intersection.

Proof. Let \( c : \mathbb{R} \to \mathbb{R}^2 \) be a lift of a geodesic and assume that \( c \) has a self-intersection. Then there exist \( t_0 < t_1 \in \mathbb{R} \) with \( c(t_0) = c(t_1) \). Let \( \dot{c}(t_0) = w \) denote the initial condition of \( c \) in \( t_0 \). Assume first that \( c \) is recurrent. Then there exists an increasing sequence of times \( (t_n)_{n \in \mathbb{N}} \) and a sequence \( \tau_n \) of translation elements such that \( D\tau_n \dot{c}(t_n) = w_n \to w \) for \( n \to \infty \), see Figure 2. For \( n \) large enough and \( t_n \) much larger than \( t_1 \) there exist \( t \) near \( t_n \) and \( s > 0 \) such that \( \tau_n c(t) = \tau_n c(t+s) \). Hence, \( c(t) = c(t+s) \). As \( t_0 < t_1 < t < t+s \) by Lemma 3.2 the metric \( g \) has positive topological entropy, in contradiction to the assumption. Let \( c \) be an arbitrary geodesic. As recurrent geodesics are dense, there exists a sequence of recurrent geodesics \( c_n \) with \( \dot{c}_n(t_0) = w_n \) such that \( w_n \) converges to \( w \). Since \( c \) has a self-intersection, i.e. \( c(t_0) = c(t_1) \), the continuous dependency implies that for \( n \) large enough the recurrent geodesic \( c_n \) has a self-intersection. Then by Lemma 3.2 the metric \( g \) has positive topological entropy in contradiction to the assumption. \( \square \)

The following theorem is due to V. Bangert in [6]

Theorem 3.5. Let \( g \) be a complete Riemannian metric on \( \mathbb{R}^2 \). Then the existence of a bounded geodesic ray \( c : [0, \infty) \to \mathbb{R}^2 \) implies the existence of a simple closed geodesic.

Remark. We note that this result can be also obtained combining Curve Shortening with arguments from topological dynamics. For more details about this approach see [13] where a slightly weaker result has been obtained.

Corollary 3.6. Let \( g \) be a Riemannian metric on \( T^2 \). Assume there exists a bounded geodesic ray \( c \) on the universal covering \( \mathbb{R}^2 \). Then the metric \( g \) has positive topological entropy.

4. Central Geometric Argument

In this section we present a fundamental geometric constellation on the universal covering of \( T^2 \) and prove that it implies positive topological entropy for the Riemannian metric \( g \) on \( T^2 \). In the further sections we will use this argument several times.
Definition 4.1. A continuous curve $c : I \to \mathbb{R}^2$, for $I \subset \mathbb{R}$, is called a broken geodesic if there exists a finite set $W \subset I$ such that $c$ is geodesic on $I \setminus W$. We call $W$ the set of vertices.

Lemma 4.2 (Fundamental Lemma). Let $g$ be a Riemannian metric on $T^2$ and $\alpha : \mathbb{R} \to \mathbb{R}^2$ a minimal axis of the translation element $\tau$. Let $c_1 : [0, a] \to \mathbb{R}^2$ and $c_2 : [0, b] \to \mathbb{R}^2$ be two geodesic segments with endpoints on $\alpha$ and $c_1((0, a)) \cap \alpha(\mathbb{R}) = \emptyset$, $c_2((0, b)) \cap \alpha(\mathbb{R}) = \emptyset$. Assume that there exists a translation element $\eta$, with $\eta(\alpha)(\mathbb{R}) \cap \alpha(\mathbb{R}) = \emptyset$ such that $\eta \alpha(\mathbb{R}) \cap c_1([0, a]) \neq \emptyset$ and $\eta^{-1} \alpha(\mathbb{R}) \cap c_2([0, b]) \neq \emptyset$. Then the metric $g$ has positive topological entropy.

Proof. We choose $k \geq 1$ such that
$$ \eta^k \alpha(\mathbb{R}) \cap c_1([0, a]) \neq \emptyset \quad \text{but} \quad \eta^{k+1} \alpha(\mathbb{R}) \cap c_1([0, a]) = \emptyset. $$

Analogously we choose $l \geq 1$ such that
$$ \eta^{-l} \alpha(\mathbb{R}) \cap c_2([0, b]) \neq \emptyset \quad \text{but} \quad \eta^{-l-1} \alpha(\mathbb{R}) \cap c_2([0, b]) = \emptyset. $$

W.l.o.g. let $l \geq k$. Then
$$ \alpha(\mathbb{R}) \cap \eta^{l+1} c_2([0, b]) = \emptyset \quad \text{but} \quad \eta^{l} \alpha(\mathbb{R}) \cap \eta^{l+1} c_2([0, b]) \neq \emptyset. $$

By $S$ we denote the geodesic strip bounded by $\alpha$ and $\eta^{l+1} \alpha$. In the next step we use the segments $c_1$ and $\eta^{l+1} c_2$ to construct suitable broken geodesics $\sigma_1$ and $\sigma_2$ which intersect. We distinguish two cases:

Case 1) There exists $\tilde{n} \in \mathbb{Z}$ such that $\tau^{\tilde{n}} c_1$ and $\eta^{l+1} c_2$ intersect. In this case we will denote $\sigma_1 := \tau^{\tilde{n}} c_1$ and $\sigma_2 := \eta^{l+1} c_2$.

Case 2) For all $\tilde{n} \in \mathbb{Z}$ the segments $\tau^{\tilde{n}} c_1$ and $\eta^{l+1} c_2$ do not intersect. Using $c_1$ and $\eta \alpha$ we will construct a broken geodesic segment $\sigma_1$ with endpoints on $\alpha$ which intersects $\eta^{l+1} c_2$, see Figure 4. We choose $n \in \mathbb{Z}$.

![Figure 3. Illustration of the translated segments $c_1$ and $\eta^{l+1} c_2$ in the proof of the Fundamental Lemma 4.2](image)
\[\eta \alpha \eta \alpha \eta^k \alpha \eta^{l+1} \alpha\]

\textbf{Figure 4.} The construction of intersecting broken geodesic segments \(\sigma_1\) and \(\sigma_2\) in the proof of the Fundamental Lemma 4.2.

such that on \(\eta \alpha (\mathbb{R})\) the set of intersections \(\eta^{l+1} c_2 \cap \eta \alpha (\mathbb{R})\) lies between the finite sets \(M_+ = \tau^n c_1 \cap \eta \alpha (\mathbb{R})\) and \(M_- = c_1 \cap \eta \alpha (\mathbb{R})\).

Let \(\eta \alpha ([t_1, t_2])\) be the smallest segment of \(\eta \alpha\) such that \(\eta \alpha ([t_1, t_2]) \cap \eta \alpha \eta^j \alpha \eta^{l+1} \alpha \neq \emptyset \neq \eta \alpha ([t_1, t_2]) \cap M_-\).

Let \(A\) be the union of the bounded connected components of \(S \setminus (c_1([0, a]) \cup \tau^n c_1([0, a]) \cup \eta \alpha ([t_1, t_2]))\).

We consider the broken geodesic \(\sigma_1([0, \tilde{a}])\) endowed with a new parametrization consisting of \(\eta \alpha ([t_1, t_2])\) and segments of \(\tau^n c_1\) and \(c_1\), such that it separates \(\tilde{A}\) from \(S \setminus A\). By construction \(\sigma_1([0, \tilde{a}])\) and \(\sigma_2([0, \tilde{b}]) = \eta^{l+1} c_2([0, \tilde{b}])\) intersect.

Let \(m \in \mathbb{N}\) be large enough such that

\[(4.1) \quad (\sigma_1 \cup \sigma_2) \cap \tau^{mi} (\sigma_1 \cup \sigma_2) = \emptyset \quad \text{for all} \quad i \in \mathbb{Z} \setminus \{0\}.
\]

For fixed \(j \in \mathbb{N}\) and a finite sequence of \(j\) elements \((a_0, ..., a_{j-1}) = a(j)\) with \(a_k \in \{1, 2\}\) we consider the set

\[A(a(j)) := \bigcup_{k=0}^{j-1} \tau^{mk} \sigma a_k .\]

For each \(a(j)\) let \(\gamma_0^{a(j)} : \mathbb{R} \to S \setminus A(a(j))\) be a smooth curve such that

\[\tau^{mj} \gamma_0^{a(j)} = \gamma_0^{a(j)} ,\]

see Figure 5. Furthermore, we can choose these curves such that the length of the segment of \(\gamma_0^{a(j)}\) joining \(\gamma_0^{a(j)}(0)\) and \(\tau^{jm} \gamma_0^{a(j)}(0)\) is less than \(j b_2\) and larger than \(j b_1\) for some universal constants \(0 < b_1 < b_2\).

Consider for each \(j \in \mathbb{N}\) the cylinder

\[C_j := \mathbb{R}/(\tau^{jm})\]

where \((\tau^{jm})\) is the subgroup of \(\mathbb{Z}^2\) generated by \(\tau^{jm}\). In particular, the projection of \(S\) onto \(C_j\) defines an annulus. By assumption the curves \(\gamma_0^{a(j)}\)
Figure 5. Simplified illustration of the translates of $\sigma_1$, $\sigma_2$, and $\gamma_0^{(a)}$ in the proof of the Fundamental Lemma. The figure shows the projection of closed curves $\gamma_0^{(a(j))}: \mathbb{R} \to C_j \setminus B(a(j))$, where $B(a(j))$ is the projection of $A(a(j))$ onto $C_j$. By construction the interior angles at the vertices $w_i$ of the connected component of $C_j \setminus B(a(j))$ containing $\gamma_0^{(a(j))}$ are all less than $\pi$. Applying the curve shortening flow to the smooth curve $\gamma_0^{(a(j))}$ we obtain smooth curves $\gamma_t^{(a(j))}: \mathbb{R} \to C_j \setminus B(a(j))$ which stay in $C_j$ and do not intersect $B(a(j))$. This follows from Theorem 2.3 and the fact that due to their homotopy class $\gamma_t^{(a(j))}$ never become singular. Since the curves $\gamma_0^{(a(j))}$ are not contractible, M. A. Grayson’s Theorem 2.2 implies that the curvature of $\gamma_t^{(a(j))}$ converges to zero. This yields the existence of a closed geodesic on $C_j$.

Let $D_i$ be the bounded connected component of $S \setminus \sigma_i$ for $i \in \{1, 2\}$. Let $B(x, \varepsilon) \subset D_1 \cap D_2$ be a geodesic ball with radius $\varepsilon > 0$. Obviously, by construction the curves $\gamma_t^{(a(j))}$ do not intersect $\gamma_{mk}^{(a(j))}B(x, \varepsilon)$ for $0 \leq k \leq j - 1$ for all sequences $a(j)$ and $\gamma_t^{(a(j))}$ will not intersect $\gamma_{mk}^{(a(j))}B(x, \varepsilon)$ for all $0 \leq k \leq j - 1$ and $t \in (0, T)$ by the properties of the Curve Shortening flow. By this construction for fixed $j$ and different sequences $a(j)$ we get $2^j$ different closed geodesics on $C_j$ of length between $b_1j$ and $b_2j$. We project the constructed closed geodesics onto $C_1$. Identifying $\alpha$ and $\eta^{l+1}\alpha$ we get a torus $\tilde{T}^2$ on which we still have at least $2^j$ different closed geodesics. By construction the initial conditions of these closed geodesics form a $(\phi, \varepsilon)$-separated set which grows exponentially for $j \to \infty$. By Definition 2.1 this implies positive topological entropy for the Riemannian metric on $\tilde{T}^2$. As $\tilde{T}^2$ is a finite cover of $T^2$ also the Riemannian metric $g$ on $T^2$ has positive topological entropy.

Remark. For the proof of the Fundamental Lemma it is not necessary that $\alpha$ is minimal. It suffices the condition $\alpha \cap \eta^{k+1}\alpha = \emptyset$ for the special $k \in \mathbb{Z}$ used in the proof.
Corollary 4.3. Let \((T^2, g)\) be a Riemannian torus. The existence of a contractible closed geodesic on \(T^2\) implies the existence of non-primitive axes for all translation elements \(\tau\).

Proof. Considering an arbitrary translation element and replacing in the proof of the Fundamental Lemma \(4.2\) the segments \(\sigma_1\) and \(\sigma_2\) by a closed geodesic we can use a similar construction, as presented starting in \((4.1)\), to produce a non-primitive axis. □

Using the recurrence of the geodesic flow on \(T^2\) we obtain the following refinement of the Fundamental Lemma.

Theorem 4.4. Let \((T^2, g)\) be a Riemannian torus. Let \(c : \mathbb{R} \to \mathbb{R}^2\) be a geodesic and \(\alpha\) a minimal axis such that \(c(a), c(b) \in \alpha(\mathbb{R})\) for \(a < b\). If there exists a translation element \(\tau\) with \(\tau(\alpha(\mathbb{R})) \cap \alpha(\mathbb{R}) = \emptyset\) and \(\tau(\alpha(\mathbb{R})) \cap c([a, b]) \neq \emptyset\), then \(g\) has positive topological entropy.

Proof. Let \(c : \mathbb{R} \to \mathbb{R}^2\) be a geodesic and \(\alpha\) a minimal axis such that \(c(a), c(b) \in \alpha(\mathbb{R})\) for \(a < b\) and \(\tau\) the translation element with \(\tau(\alpha(\mathbb{R})) \cap \alpha(\mathbb{R}) = \emptyset\) and \(\tau(\alpha(\mathbb{R})) \cap c([a, b]) \neq \emptyset\). We will now prove that this implies positive topological entropy. Consider the two halfplanes given by the connected components of \(\mathbb{R}^2 \setminus \tau(\alpha(\mathbb{R}))\) and denote by \(A\) the one containing \(\alpha(\mathbb{R})\) and by \(A'\) the other one. By assumption there exists \(t_0 \in [a, b]\) such that \(c(t_0) \in A'\).

We can assume that \(c([b, \infty))\) does not intersect \(\tau(\alpha(\mathbb{R}))\) since this would immediately imply positive topological entropy by the Fundamental Lemma \(4.2\).

We can also assume that \(c\) is recurrent, since the recurrent geodesics are dense. In particular, there exists an increasing sequence \((t_n)_{n \in \mathbb{N}}\) and a sequence \(\tau_n\) of translates such that \(D\tau_n c(t_n) \to c(t_0)\) for \(n \to \infty\). But then for \(n\) large enough \(c([b, t_n])\) must intersect the translates \(\tau_n \alpha(\mathbb{R})\) and \(\tau_n \alpha(\mathbb{R})\). Then, \(c([b, t_n])\) fulfills the assumption of the Fundamental Lemma \(4.2\) and hence, \(g\) has positive topological entropy and there exist non-primitive axes of \(\tau\). □

Corollary 4.5. Let \((T^2, g)\) be a Riemannian torus. Let \(c : [0, \infty) \to \mathbb{R}^2\) be a geodesic ray on the universal covering \(\mathbb{R}^2\) which intersects a minimal axis \(\alpha\) and disjoint translate infinitely often. Then \(g\) has positive topological entropy.

Lemma 4.6. Let \(g\) be a Riemannian metric on \(T^2\). Then the existence of an oscillating geodesic ray \(c\) on \(\mathbb{R}^2\) implies positive topological entropy for the metric \(g\).

Proof. Due to the definition of an oscillating geodesic ray there exist sequences \(t_n, s_n\) tending to infinity and a compact set \(K \subset \mathbb{R}^2\) such that \(c(s_n) \in K\) and \(\|c(t_n)\| \to \infty\) for \(n \to \infty\). W.l.o.g. we can assume that

\[
\lim_{n \to \infty} \frac{c(t_n)}{\|c(t_n)\|} = z
\]

for some \(z \in S^1\). Let \(\alpha : \mathbb{R} \to \mathbb{R}^2\) be a minimal axis such that \(z \neq \pm \delta(\alpha)\). Furthermore, we can choose \(\alpha\) and a disjoint translate \(\alpha'\) in such a way that \(K\) and \(\{c(t_n)\mid n \geq n_0\}\) are contained in the two different halfplanes of
\[ \mathbb{R}^2 \setminus (\alpha(\mathbb{R}) \cup \alpha'(\mathbb{R})) \] for sufficiently large \( n_0 \). This implies that the geodesic ray \( c : [0, \infty) \to \mathbb{R}^2 \) will intersect \( \alpha \) and \( \alpha' \) infinitely often. By Corollary 4.5 the metric \( g \) has positive topological entropy.

5. Existence and Regularity of the Rotation Number

In the proof of Theorem I we will need the following property of minimal geodesics:

**Remark.** As already mentioned, the study of minimal geodesics on \( T^2 \) goes back to H. M. Morse [21], G. A. Hedlund [15], and V. Bangert [7]. Central results are that for each \( r \in \mathbb{R} \cup \{ \infty \} \) there exists a minimal geodesic \( c : \mathbb{R} \to \mathbb{R}^2 \) with asymptotic direction \( r \) and that furthermore there exists a constant \( D > 0 \), such that for each minimal geodesic \( c : \mathbb{R} \to \mathbb{R}^2 \) there exists a Euclidean line \( l_c \), and for each Euclidean line \( l_c \) there exists a minimal geodesic \( c \) such that

\[
d(l_c, c(t)) \leq \frac{D}{2}, \quad \text{for all } t \in \mathbb{R}.
\]

Furthermore, we can assume that \( D \) is larger than the diameter of the fundamental domain. As shown by V. Bangert [7], for irrational rotation numbers the set of minimal geodesics with this rotation number is totally ordered, i.e., all these minimal geodesics have pairwise no intersections with each other. For the set of minimal geodesics with a fixed rational rotation number the subset of axes is ordered. Two minimal axes with the same rotation number bounding a strip containing no further minimal axes are called neighboring mininals.

**Proof of Theorem I.** Combining Corollary 3.6, Lemma 4.6, and Theorem 3.4 we conclude that in the case of vanishing topological entropy all geodesic rays are escaping and have no self-intersections on the universal covering \( \mathbb{R}^2 \).

If for \( c^+ \) the asymptotic direction does not exist, there exist two accumulation points \( z_1 \neq z_2 \in S^1 \) for the quotient \( \frac{c^+(t)}{\|c^+(t)\|} \) as \( t \) tends to \( \infty \). Choose a rational direction \( z \in S^1 \) which lies in the connected component of \( S^1 \setminus \{z_1, z_2\} \), which is not larger than a half-circle. Let \( \alpha \) be a minimal axis of the primitive translation element \( \tau \) with \( \delta(\tau) = z \). Then the geodesic \( c : [0, \infty) \to \mathbb{R}^2 \) must intersect \( \alpha(\mathbb{R}) \) and a given disjoint translate \( \eta(\alpha(\mathbb{R})) \) with \( [\eta] \neq [\tau] \) (which is a minimal axis of \( \tau \) as well) infinitely often. But then, by Corollary 4.5 the metric \( g \) has positive topological entropy in contradiction to the assumption.

Assume that \( \delta(c^+) \neq -\delta(c^-) \). Choose \( z \in S^1 \) rational such that \( \pm z \) both lie in one connected component of \( S^1 \setminus \{\delta(c^+), \delta(c^-)\} \). Let \( \alpha \) be an axis with \( \delta(\alpha) = z \) which intersects \( c \). Then \( \alpha \) divides \( \mathbb{R}^2 \) in two halfplanes. There is precisely one of those halfplanes denoted by \( A \) for which there exists \( t_0 > 0 \) such that \( c(t) \in A \) for \( |t| \geq t_0 \). Choose a disjoint translate \( \tau \alpha \) which is contained in \( A \) as well. Then there exist \( a < b \) such \( c(a), c(b) \in \tau(\alpha(\mathbb{R})) \) and \( c[a, b] \cap \alpha(\mathbb{R}) \neq \emptyset \) which contradicts Theorem 4.4.

We will now show that there exists a Euclidean strip \( S(c) \) with direction \( \delta(c^+) \) such that \( c(\mathbb{R}) \subset S(c) \). We already know that the asymptotic directions exist and that \( \delta(c^+) = -\delta(c^-) \). Assume that \( c \) is not contained in a
Euclidean strip. Consider an arbitrary minimal geodesic with asymptotic
direction $\delta(c^+)$. As minimal geodesics are accompanied by Euclidean lines
with the same direction and as $c$ is not bounded by any strip with direc-
tion $\delta(c^+)$, $c$ also intersects an infinite number of translates of this geodesic.
Consider $D > 0$ as introduced in the previous remark. Choose a translation
element $\tau$, a time $t_0 > 0$ and a minimal geodesic $b$ with asymptotic direc-
tion $\delta(c^+)$ such that $c[0, t_0]$ crosses $b$ and $\tau b$, $c(0)$ and $c(t_0)$ lie in different
halfplanes of $\mathbb{R}^2 \setminus (b(\mathbb{R}) \cup \tau b(\mathbb{R}))$ and such that there fits a Euclidean strip
$R$ with direction $\delta(c^+)$ and width $4D$ between $b$ and $\tau b$, see Figure 6. Then

![Figure 6](image)

Figure 6. Illustration of the argumentation in the second
part of the proof of Theorem I.

there exists a Euclidean strip $S$ with width $3D$ and a rational asymptotic
direction $z$ near $\delta(c^+)$ on $S^1$ but $z \neq \delta(c^+)$ such that $c(0)$ and $c(t_0)$ lie in different
halfplanes of $\mathbb{R}^2 \setminus (b(\mathbb{R}) \cup \tau b(\mathbb{R}))$ and such that there fits a Euclidean strip
$R$ with direction $\delta(c^+)$ and width $4D$ between $b$ and $\tau b$, see Figure 6. Then

Now we like to prove regularity properties of the rotation number. We
begin with the following lemma which holds for all Riemannian metrics.

**Lemma 5.1.** For all $r \in \mathbb{R} \cup \{\infty\}$ and $x \in T^2$ there exists $v = v(r) \in ST^2$
such that the corresponding geodesic $c_v : \mathbb{R} \to T^2$ fulfills $c_v(0) = \pi(v) = x$, 
$c_v(0) = v$ and for the lift of $c_v$ on the universal covering it holds $\rho(c_v^+) = r$. 
Let $E_x = \{ v \in S_xT^2 \mid \rho(v) \text{ exists} \}$. Then, for fixed $x \in T^2$ the map

$$f_x : E_x \to \mathbb{R} \cup \{\infty\} \text{ with } v \mapsto \rho(v)$$

is surjective.

**Proof.** We fix $r \in \mathbb{R} \cup \{\infty\}$ and $x \in T^2$. By the remark at the beginning of
this section there exists a minimal geodesic $\gamma$ with rotation number $r$. We
assume that $x \notin \gamma(\mathbb{R})$, otherwise we set $\gamma = c$ after a reparameterization
such that $c(0) = x$ and so we choose $v = \dot{c}(0)$.

We consider minimal geodesic segments $c_t$ connecting $x$ and $\gamma(t)$ on the universal covering such that $c_t(0) = x$ with $c_t(0) = w_t \in S_x \mathbb{R}^2$. As $S_x \mathbb{R}^2$ is compact there exists a sequence $t_n$ with $t_n \to \infty$ such that $w_{t_n}$ converges to $w$. By its minimality the segment $c_t$ intersects $\gamma(R)$ only once in $\gamma(t)$.

Let $\tau \gamma$ be a translate of $\gamma$ such that $x$ lies in the bounded strip $S$ between $\gamma(R)$ and $\tau \gamma(R)$. Repeating the minimality arguments we conclude that $c_t$ and $\tau \gamma(R)$ have no intersections for all $t$. By the continuous dependence on the initial conditions also the limit geodesic ray $c^+$ with $c^+(0) = x$ and $\dot{c}^+(0) = w$ will not intersect $\tau \gamma(R)$ and $\gamma(R)$.

Hence, the geodesic ray $c^+$ lies in the geodesic strip $S$ with rotation number $r$. This implies the existence of the rotation number and even that $\rho(c^+) = r$.

We extend $c^+$ on $T^2$ to the geodesic $c : \mathbb{R} \to T^2$ with $c(0) = x$ and $\dot{c}(0) = w$. The choice $v = w$ fulfills the required properties.

**Proof of Theorem II.** By Lemma 5.1 the rotation number is surjective. We have to show its continuity. As the topological entropy vanishes, by Theorem I the asymptotic directions $\delta(v)$ and $\delta(-v)$ exist and it holds $\delta(v) = -\delta(-v)$ for all $v \in S \mathbb{R}^2$. Here, we think of the asymptotic directions as a function on the unit tangent bundle of $\mathbb{R}^2$. Assume that $\delta$ is not continuous at $v_0 \in S \mathbb{R}^2$. Let $c$ be the geodesic with the initial condition $\dot{c}(0) = v_0$ and let $\delta(c^+) = z$. As $\delta$ is not continuous at $v_0$, there exists a sequence $v_n \to v_0$ such that $\delta(v_n)$ does not converge to $z$. Since $S^1$ is compact there exists a subsequence $v_n_k$ with $\delta(v_{n_k}) \to \tilde{z} \neq z$. We will again denote this subsequence by $v_n$ and the geodesics corresponding to $v_n$ by $c_n$, i.e. $\dot{c}_n(0) = v_n$.

Consider the two Euclidean rays $r : [0, \infty) \to \mathbb{R}^2$ and $\tilde{r} : [0, \infty) \to \mathbb{R}^2$ given by $r(t) = tz$ and $\tilde{r}(t) = t\tilde{z}$. Choose a minimal axis $\gamma : \mathbb{R} \to \mathbb{R}^2$ and a disjoint translate $\gamma' : \mathbb{R} \to \mathbb{R}^2$ such that the two halfplanes $H_1$ and $H_2$ which are two of the three connected components of $\mathbb{R}^2 \setminus (\gamma(R) \cup \gamma'(R))$ have the following properties: First, there exists an $\varepsilon > 0$ such that $B(c(0), \varepsilon) \subset H_1$. Second, the sets $H_2 \cap r[0, \infty)$ and $H_1 \cap \tilde{r}[0, \infty)$ are unbounded, see Figure 7.
$c_n(0) \in H_1$ and $c_n(t_1) \in H_2$ for a suitable $t_1 > 0$. Moreover, $\delta(v_n) \to \tilde{z}$ as $n \to \infty$ yields the existence of $n_1 \geq n_0$ such that the set $H_1 \cap c_n[0, \infty)$ is unbounded for each $n \geq n_1$. In particular, for a given $n \geq n_1$ we will find $t_2 \geq t_1$ such that $c_n(t_2) \in H_1$. Such a geodesic $c_n$ crosses the pair of minimal axes $\gamma$ and $\gamma'$ at least twice which implies by Theorem 4.3 positive topological entropy in contradiction to the assumption in Theorem II. Hence, the asymptotic direction is continuous and its continuity implies the continuity of the rotation number. 

6. Further Conditions for Vanishing Topological Entropy

In this section we show that vanishing topological entropy implies strong restrictions of the intersections of geodesics on the universal cover with their translates.

Proof of Theorem III. As $h_{\text{top}}(g) = 0$, by Theorem I the asymptotic direction $\delta(c)$ exists for all $c$ and for each $c$ there exists a Euclidean strip $S(c)$ bounding $c$. Assume there exists a geodesic ray $c : [0, \infty) \to \mathbb{R}$ with $\#I(c) \geq 2$. Consider a translation element $\tau$ with $\#\{c \cap \tau c\} = \infty$ and $\delta(\tau) \neq \pm \delta(c)$, e.g. it does not leave $S(c)$ invariant. Let $k \in \mathbb{Z}$ be large enough such that the distance between $S(c)$ and $\tau^k(S(c))$ is larger than $4D$ with $D > 0$ introduced in the remark at the beginning of the previous section. We denote the Euclidean strip between $S(c)$ and $\tau^k(S(c))$ by $E$. For each $l \in \{0, \ldots, k\}$ we denote the connected component of $\mathbb{R}^2 \setminus \tau^l c(\mathbb{R})$ which contains $\tau^{l+1}c(\mathbb{R})$ by $A_{\tau^l c}$ and by $B_{\tau^l c}$ the other one. As $c(\mathbb{R}) \subset S(c)$ it follows that $\tau^k c(\mathbb{R}) \subset \tau^k(S(c))$. Consider $t_0 \in [0, \infty)$. As $\#\{c \cap \tau c\} = \infty$ there exist $t_1, t_1 > t_0$ with $c(t_1) = \tau c(t_1)$ such that $\tau c$ passes in $c(t_1)$ from $B_c$ to $A_c$. Analogously there exist $t_2, t_2 > \max\{t_1, \tilde{t}_1\}$ with $\tau c(t_2) = \tau^2 c(t_2)$ such that $\tau^2 c$ passes in $\tau c(t_2)$ from $B_{\tau c}$ to $A_{\tau c}$. By this construction we get a broken geodesic segment

$$\alpha[0, t_k] := c[t_0, t_1] \cup \tau c[t_1, t_2] \cup \tau^2 c[t_2, t_3] \cup \cdots \cup \tau^{k-1}c[t_{k-1}, t_k]$$

connecting $c$ and $\tau^k c$.

Consider $s_0 > t_k$ and analogously to the previous construction there exist $s_1, \tilde{s}_1 > s_0$ with $\tau^k c(s_1) = \tau^{k-1}c(\tilde{s}_1)$ such that $\tau^{k-1}c$ passes in $\tau^k c(s_1)$ from $A_{\tau^k c}$ to $B_{\tau^k c}$. Analogously we get the broken geodesic segment

$$\beta[0, s_k] := \tau^k c[s_0, s_1] \cup \tau^{k-1}c[\tilde{s}_1, s_2] \cup \tau^{k-2}c[\tilde{s}_2, s_3] \cup \cdots \cup \tau c[\tilde{s}_{k-1}, s_k].$$

Gluing these two geodesic segments we obtain the broken geodesic segment

$$V_1 := \alpha([t_0, t_k]) \cup \tau^k c([\tilde{t}_k, s_0]) \cup \beta([s_0, s_k])$$

which connects $S(c)$, $\tau^k(S(c))$, and $S(c)$. We call the unbounded connected component of $\mathbb{R}^2 \setminus \{S(c) \cup V_1\}$ which is not a Euclidean halfplane the exterior of $V_1 \cup S(c)$. By construction the exterior angles of $V_1 \cup S(c)$ are smaller than $\pi$. Analogously we construct a broken geodesic $V_2$ connecting $\tau^k(S(c))$, $S(c)$, and $\tau^k(S(c))$ such that the exterior angles of $V_2 \cup \tau^k S(c)$ are smaller than $\pi$. Consider two minimal axes $\xi_1$ and $\xi_2$ of $\tau$ bounding a geodesic strip $G$ such that $V_1, V_2 \subset G$. As the distance of $S(c)$ and $\tau^k(S(c))$ is larger than $4D$ consider three ordered translates of minimal axes denoted by $\alpha_1$, $\alpha_2$, and $\alpha_3$ with a rational asymptotic direction near $\delta(c^+)$ such that $\alpha_i \cap G \subset E$ for
all \(i \in \{1,2,3\}\). Then, there exist subsegments \(c_1\) and \(c_2\) of \(V_1\) and \(V_2\) with endpoints on \(\alpha_2\) intersecting \(\alpha_1\) or \(\alpha_3\), respectively. These broken geodesic segments \(c_1\) and \(c_2\) fulfill by construction the assumptions of the segments in the Fundamental Lemma 4.2. Hence, \(h_{\text{top}}(g) > 0\) in contradiction to the assumption. Furthermore, by construction we conclude, that geodesics \(c\) with irrational rotation number intersect their translates \(\tau c\) only a finite number of times. Geodesics \(c\) with rational rotation number intersect their translates \(\tau c\) an infinite number of times at most for \(\tau\) with \(\delta(\tau) = \delta(c^+)\). \(\square\)

7. Characterization of Flatness

In order to prove Theorem IV we first prove the following lemma due to V. Bangert which is of independent interest.

**Lemma 7.1.** Let \((T^2, g)\) be a Riemannian torus. Suppose there exists a primitive translation element \(\tau\) such that its minimal axes do not foliate \(\mathbb{R}^2\). Then there exists \(k \geq 2\) and a non-primitive axis of \(\tau^k\).

**Proof.** If the minimal axes of \(\tau\) do not foliate \(\mathbb{R}^2\), then there exist two neighboring minimal axes \(c_1, c_2 : \mathbb{R} \to \mathbb{R}^2\) of \(\tau\). Consider the cylinder \(\mathbb{R}^2/\langle \tau \rangle = C\) and the projections \(c_1, c_2 : \mathbb{R} \to C\) of \(c_1\) and \(c_2\) which are closed geodesics of equal length \(\ell\). By assumption all closed curves between \(c_1\) and \(c_2\) in the same homotopy class have length strictly larger than \(\ell\). Choose locally convex neighborhoods \(C_1\) and \(C_2\) about \(c_1\) and \(c_2\). (Take for instance minimal geodesic loops homotopic to \(c_1\) and \(c_2\), respectively, close to \(c_1\) and \(c_2\).) There exists \(\varepsilon > 0\) such that for all closed curves \(\gamma\) homotopic to \(c_1\) with \(L(\gamma) \leq \ell + \varepsilon\) we obtain that

\[
\gamma \text{ is contained in } C_1 \cup C_2 . \quad (\ast)
\]

Using the methods in [5], pp. 87/88 we obtain a constant \(A > 0\) such that for all \(n > 0\) there exists a closed geodesic \(a_n\) between \(c_1\) and \(c_2\) with the following properties:

(i) \(a_n\) is homotopic to \(c_1 : [0, n\ell] \to C\)

(ii) \(L(a_n) \leq n\ell + A\)
Choose \( n \geq 2 \) large enough such that \( \frac{A}{n} < \varepsilon \). Then there does not exist a curve \( b_n \) homotopic to \( c_1 : [0, \ell] \to C \) whose \( n \)-th iterate is equal to \( a_n \). Otherwise using property (ii),

\[
L(b_n) = \frac{1}{n} L(a_n) \leq \ell + \frac{A}{n} \leq \ell + \varepsilon.
\]

But by (*) this would imply that \( b_n \) and hence \( a_n \) is contained in \( C_0 \cup C_1 \) which contradicts property (iii). Hence, \( a_n \) is a non-primitive axis of \( \tau^n \).

\( \square \)

**Proof of Theorem IV.** Obviously the flatness of the metric on \( T^2 \) implies that no periodic geodesic (and in fact no geodesic) intersects its translates. Conversely, assume that no axis intersects its translates. Then Lemma A.1 implies that for each primitive translation element \( \tau \) the associated minimal axes foliate \( \mathbb{R}^2 \). Since all axes are minimal the result of N. Innami in [17] implies that \( (T^2, g) \) does not have conjugate points. Then by E. Hopf’s theorem [10] the metric \( g \) is flat.

\( \square \)

**Acknowledgement.** The authors like to thank Sigurd Angenent for explaining to us the essential features of the Curve Shortening flow which is a crucial technique in this paper. We are also grateful to Victor Bangert for explaining to us his variational methods which led to Theorem IV.

**References**

[1] S. B. Angenent, *Monotone recurrence relations, their Birkhoff orbits and topological entropy*, Ergod. Th. and Dynam. Sys., 10 (1990), 15-41.

[2] S. B. Angenent, *Parabolic equations for curves on surfaces, Part II. Intersections, blow-up, and generalized solutions*, Annals of Mathematics, 133 (1991), 171-215.

[3] S. B. Angenent, *Self-intersecting Geodesics and Entropy of the Geodesic Flow*, preprint 2005, [http://www.math.wisc.edu/~angenent/preprints.html](http://www.math.wisc.edu/~angenent/preprints.html)

[4] S. Aubry, P. Y. Le Daeron, *The discrete Frenkel-Kontorova Model and its Generalizations*, Physica, 8D (1983), 381-422.

[5] V. Bangert, *Closed Geodesics on Complete Surfaces*, Math. Ann. 251 (1980), 83-96.

[6] V. Bangert, *Geodesics and Totally Convex Sets on Surfaces*, Invent. math. 63 (1981), 507-517.

[7] V. Bangert, *Mather Sets for Twist Maps and Geodesics on Tori*, Dynamics Reported 1 (1988), 1-56.

[8] M. L. Bialy and L. V. Polterovich, *Geodesic Flows on the two-dimensional Torus and Phase Transitions ‘Commensurability-Noncommensurability’*, English translation: Functional Anal. Appl. 20 (1986) no. 4, 260-266.

[9] S. V. Bolotin and P. H. Rabinowitz, *Some geometrical conditions for the existence of chaotic geodesics on a torus*, Ergod. Th. and Dynam. Sys. 22 (2002), 1407-1428.

[10] R. Bowen, *Entropy for Group Endomorphisms and Homogeneous spaces*, Trans. of Am. Math. Soc. 153 (1971), 401-414.

[11] R. Bowen, *Entropy-expansive maps*, Trans. of Am. Math. Soc. 164 (1972), 323-331.

[12] J. Denvir and R. S. MacKay, *Consequences of contractible geodesics on surfaces*, Trans. Amer. Math. Soc. 350 (1998), no. 11, 4553-4568.

[13] E. Glasmachers, *Characterization of Riemannian metrics on \( T^2 \) with and without positive topological entropy*, Thesis 2007, [http://www-brs.ub.rub.de/ metahtml/HSS/Diss/GlasmachersEva/](http://www-brs.ub.rub.de/metahtml/HSS/Diss/GlasmachersEva/)

[14] M. A. Grayson, *Shortening embedded curves*, Ann. of Math. (2) 129 (1989), no. 1, 71-111.

[15] G. A. Hedlund, *Geodesics on a two-dimensional Riemannian manifold with periodic coefficients*, Ann. of Math. 33 (1932), 719-739.
[16] E. Hopf, *Closed surfaces without conjugate points*, Proc. Natl. Acad. Sci. 34 (1948), 47-51.

[17] N. Innami, *Families of geodesics which distinguish flat tori*, Math. J. Okayama Univ. 28 (1986), 207-217.

[18] A. Katok, B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press (1995).

[19] A. Katok, *Lyapunov Exponents, Entropy and Periodic Orbits for Diffeomorphisms*, Inst. Hautes Études Sci. Publ. Math. 51 (1980), 137-173.

[20] J. N. Mather, *Existence of quasi-periodic orbits for twist homeomorphisms of the annulus*, Topology, 21 (1982), 457-67.

[21] H. M. Morse, *A fundamental class of geodesics on any closed surface of genus greater than one*, Trans. Amer. Math. Soc. 26 (1924), 25-60.

[22] M. Wojtkowski, *Oscillating geodesics on two-dimensional manifolds*, Asterisque 51 (1978), 443-356.

[23] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, Springer-Verlag, New York, Berlin, Heidelberg (1982).

Faculty of Mathematics, Ruhr University Bochum, 44780 Bochum, Germany
E-mail address: eva.glasmachers@rub.de, gerhard.knieper@rub.de