RESEARCH ARTICLE

Development of U control chart by variable sample size and sampling interval to improve the statistical properties

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Abstract
It has been proved that the quality control charts with variable sampling schemes are more effective than the classical ones in improving statistical measures. The Average Number of False Alarms (ANF), Average Number of Samples (ANS), Average Number of Inspected Items (ANI), and Adjusted Average Time to Signal (AATS) have been the most important statistical measures always being attending in the evaluation of control charts. Thus, this study was conducted aiming at illustrating a comprehensive analytical review on the U control chart via the statistical measures. To this end, different levels of the possible factors were determined, and the results of calculating the statistical measures and the obtained parameters on the sampling schemes of the U control chart were presented. The results indicated that the variable U control charts were capable of improving the effectiveness of statistical measures, especially for detecting shifts and number of false alarms.

KEYWORDS
AATS, ANF, ANI, ANS, U control chart, variable sampling schemes

1 | INTRODUCTION

Regarding the basis of variable sampling schemes, provided that the drawn sample is close enough to the centerline of the control chart, it would be logical not to consider a shift in the process. Thus, the next sample with smaller sample size or longer sampling interval is drawn on a control chart with more comprehensive control limits. On the contrary, in case the current drawn sample is close to the control limits, a shift has likely occurred in the process. Therefore, the next sample with larger sample size or shorter sampling interval is drawn on a control chart with tighter control limits to identify the potential shift in the process.

The most common variable sampling schemes, known as adaptive schemes, are Variable Sample Size (VSS), Variable Sampling Intervals (VSI), and Variable Sample Size and Sampling Intervals (VSSI), in which sample size, sampling interval, and sample size with sampling interval are determined as the variables, respectively. The statistical performance of sampling schemes for control charts have been examined by a number of researchers (e.g., Khoo et al; Costa & Machado; Saha et al; Shongwea & Malela-Majikaa; Lim et al; Chong et al).

Having proposed the VSI scheme for the X control chart, Reynolds et al evaluated the performance of this type of sampling scheme, that is, VSI, from the statistical aspect. Reynolds et al conducted a comprehensive review between
the VSI scheme and the traditional sampling scheme, that is, Fix Sample Size and Sampling Interval (FSSI), for various shifts in the mean of the process. Comparison among standard deviation, coefficient of variation, the average number of samples to signal, the average time to signal indicated the acceptable performance of the VSI scheme. Similarly, Saccucci et al. investigated the effect of applying the VSI scheme compared with the FSSI scheme on the performance of the Exponentially Weighted Moving Average (EWMA) and the Cumulative Sum (CUSUM). Evaluating the time to signal of VSI-EWMA and VSI-X control charts was also the focus of Yang and Yu’s and Yang and Chen’s studies, respectively, in which implementation of the VSI scheme through measuring the Adjusted Average Time to Signal (AATS) has demonstrated the positive performance of this type of sampling scheme in comparison with the application of a fixed sampling scheme. From the perspective of improving the conditions of different control charts, a number of studies have been carried out, such as Chen and Chiou, Yue and Liu, Nguyen et al., Wan et al., and Ng et al. By presenting a VSI scheme and designing X control chart with two sample sizes, Prabhut al. and Costa conducted studies on the impact of the VSS scheme on the measures and performance of the control chart. Likewise, Castagliola et al. implemented the VSSI scheme and evaluated the X control chart with the VSS scheme. In more practical applications, Nikolaidis et al. concluded that despite the better performance achieved from the statistically and economically aspects, the use of variable control charts does not have much complexity compared to the traditional control charts. Moreover, Faraz and Moghadam investigated the speed of the Hotelling’s T2 control chart in detecting changes in the process by applying the VSS scheme. Designing and evaluating control charts with a VSS scheme have been the focus of some recent research (Amiri et al.; Sabahno et al.; Hassan et al.).

Additionally, Lin and Chou evaluated the VSSI scheme presented by Prabhut al. and analyzed the effect of applying the VSSI method on the X control chart under abnormality and non-normality conditions. Having considered the estimated parameters, Zhou examined VSSI scheme for the X control chart and evaluated Average Time to Signal (ATS) in different states. Although it has been proved to be more efficient in many studies (e.g., Leoni et al; Haq and Khoo; Lee et al.), the VSSI sampling scheme, hence, is more complex compared to the VSS and VSI schemes. Chew and Khaw changed the parameters of MCV control chart to compare the improvement rate in the ATS and Expected Average Time of Signal (EATS) criteria under the VSSI scheme and compared it to other schemes, including VSI and VSS schemes. The results were satisfactory and decreased the ATS and EATS criteria reported under the VSSI sampling scheme. On the other hand, the combination of the control charts with various maintenance and repair policies and reliability has been considered in previous studies such as He et al., Zhong and Ma, Bahria et al., and Farahani et al. in the optimization of the production processes. Therefore, an improvement in the performance of the control chart can significantly enhance the efficiency of a process and should be considered by policymakers.

Much of the Literature in the discipline has mainly examined other types of control charts, especially the X control chart, rather than the U control chart, which is being widely employed today. Moreover, owing to the Literature, the ATS statistical criterion has been implemented to measure the performance of a scheme; whereas, this criterion is applied when the process starts the out-of-control state, which can be considered as a premise to simplify the design of the process. Therefore, according to Faraz et al., the AATS should be employed as an alternative to the ATS. In addition to the AATS and ATS, other statistical criteria, such as ANF, ANI, and ANS, are also significant and will be effective in optimizing the process both statistically and economically. Shojaie-Navokh et al. indicated the superiority of VSSI sampling scheme on the U control chart for economical investigations as compared with other sampling designs such as VSI, VSS, and FSSI. The difference in the present study is that it investigated the performance of VSSI sampling scheme from the statistical point of view. The acceptable performance of the VSSI scheme along with its applicability motivated us to completely assess the statistical features of VSSI-U control chart. An improvement in any of the statistical indices such as ANF, ANS, ANI, and AATS can dramatically affect the performance of the process evaluated by the U control chart. Therefore, as the statistical validity of variable sampling schemes, in particular, VSSI, has not been addressed for U control chart, this study is aimed to evaluate that. In the following sections, the concept and approach of the Markov chain in designing sampling schemes, calculating and introducing statistical measures, the evaluation and comparison of statistical sampling schemes under numerical examples, and the results are presented.

2 | MARKOV CHAIN APPROACH

Designing variable sampling schemes is based on Markov chain concepts and the transition states in a process. Given this approach, to predict the action of a system in the future, considering the current state of the system would be sufficient,
that is, the previous states does not have any impact on the future states. Hence, if $X_i$ is a random variable defined in the probability region, then:

$$P \left[ X_{n+1} = j | X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n \right] = P \left[ X_{n+1} = j | X_n = i_n \right]$$

(1)

So, the probability of transition ($p_{ij}$) in the Markov chain is:

$$P \left[ X_{n+1} = j | X_n = i \right] = P \left[ X_1 = j | X_0 = i \right] = p_{ij}$$

(2)

Where:

$$1 \leq i, j \leq N \quad 0 \leq p_{ij} \leq 1, i, j \in S$$

$$1 \leq i, j \leq N \quad \sum p_{ij} = 1, i, j \in S$$

Therefore, the Markov chain for sampling schemes can be defined by a transfer probability matrix ($P$), where the element is located in row $i$ and column $j$. The transition probability matrix with $n$ state is expressed as follows:

$$P = \begin{bmatrix}
  p_{11} & p_{12} & \cdots & p_{1j} & \cdots & p_{1n} \\
  p_{21} & p_{22} & \cdots & p_{2j} & \cdots & p_{2n} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots & p_{in} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  p_{n1} & p_{n2} & \cdots & p_{nj} & \cdots & p_{nn}
\end{bmatrix}$$

In designing sampling schemes based on Markov chain concepts, at each sampling stage, according to the process state (in-control or out-of-control) and other states, the process consists of several transition states and one absorbing state. To design sampling schemes, the method employed in the present study was based on the studies conducted by Costa,34 Faraz and Moghadam,20 Faraz et al,32 and Shojaie-Navokh et al.33

Therefore, considering the warning limits of $WL_i$ and the control limits of $UCL_i$ in each sampling step, the U control chart with one of the following states ($i = 1, 2$) will occur by application of the VSSI sampling scheme.

$$\begin{cases}
0 \leq U < WL_i \\
WL_i \leq U < UCL_i \\
U \geq UCL_i \text{ (False alarm)}
\end{cases}$$

In – control states

$$\begin{cases}
0 \leq U < WL_i \\
WL_i \leq U < UCL_i \\
U \geq UCL_i \text{ (True alarm)}
\end{cases}$$

Out – of – control states

For instance, by considering the study of Shojaie-Navokh et al33 and Abdella et al,35 the diagram of different process states, in the VSSI scheme can be seen in Figure 1.

Generally, it is assumed that the duration in which the process remains under control (until the incidence of the reasonable deviation) follows an exponential distribution with the parameter of $\lambda$. The mean of the process is shown by $u_0$ and $u_+$ for in-control and out-of-control states, respectively. Moreover, regarding the Poisson’s accumulative distribution function of $F(d,u)$ and based on Figure 1, each $p_{ij}$ can be defined by:

$$p_{11} = F \left( WL_1, u_0 \right) \cdot e^{-\lambda h_1}$$

$$p_{12} = F \left( UCL_1, u_0 \right) \cdot e^{-\lambda h_1} - p_{11}$$

$$p_{13} = e^{-\lambda h_1} - p_{11} - p_{12}$$

$$p_{14} = F \left( WL_1, u_+ \right) \cdot \left( 1 - e^{-\lambda h_1} \right)$$
\( p_{15} = F(\text{UCL}_1, u_+) \cdot (1 - e^{-\lambda h_1}) - p_{14} \)
\( p_{16} = 1 - e^{-\lambda h_1} - p_{14} - p_{15} \)
\( p_{21} = p_{31} = F(\text{WL}_2, u_0) \cdot e^{-\lambda h_1} \cdot e^{-\lambda h_2} \)
\( p_{22} = p_{32} = F(\text{UCL}_2, u_0) \cdot e^{-\lambda h_2} - p_{21} \)
\( p_{23} = p_{33} = e^{-\lambda h_1} \cdot p_{21} - p_{22} \)
\( p_{24} = p_{34} = F(\text{WL}_2, u_+) \cdot (1 - e^{-\lambda h_2}) \)
\( p_{25} = p_{35} = F(\text{UCL}_2, u_+) \cdot (1 - e^{-\lambda h_2}) - p_{24} \)
\( p_{26} = p_{36} = 1 - e^{-\lambda h_2} - p_{24} - p_{25} \)
\( p_{44} = F(\text{WL}_1, u_+) \)
\( p_{45} = F(\text{UCL}_1, u_+) - p_{44} \)
\( p_{46} = 1 - p_{44} - p_{45} \)
\( p_{54} = F(\text{WL}_2, u_+) \)
\( p_{55} = F(\text{UCL}_2, u_+) - p_{54} \)
\( p_{56} = 1 - p_{54} - p_{55} \)

### 3 Statistical Measures for Performance Evaluation of Sampling Schemes

As illustrated in Figure 2, according to the quality cycle provided by Lorenzen and Vance \(^3\) and Shojaie-Navokh et al. \(^3\) the process starts from the in-control state, where the occurrence of the assignable cause leads to a shift in the mean score of the process to an out-of-control state.

Contrary to the industrial trends, process regulations may not be possible in programs such as public health to return them into the in-control state. In many of the public health programs, the warning alarm probably continues after the first warning alarm by the control chart. Kenett and Pollak \(^4\), Chen et al. \(^5\) and Sasikumar and Devi \(^6\) presented methods to calculate this phenomenon.

The average time from the start of the production until the dispatched signal by the control chart after the process shift is equal to the Average Time of Cycle ATC. This measure can be calculated based on the properties of the
FIGURE 2 The quality cycle

exponential distribution and Markov chain approach. The expected number of experiments at each stage was obtained from the following equation:

$$b(I - Q)^{-1}$$  \hspace{1cm} (3)

where $Q$ is a square matrix by deleting the elements corresponding to the absorbing state of the transition probability matrix $P$, $I$ is the identity matrix, and $b$ is the initial probabilities vector. Thus, ATC can be computed as follows:

$$ATC = b(I - Q)^{-1}h$$  \hspace{1cm} (4)

where $h$ is the vector of interval sampling vector of different process states. Further, assuming that the average time that the process remains in the in-control state is $\lambda^{-1}$ ($\lambda$ is the exponential distribution parameter). The mean time from the occurrence of an assignable cause to the time the control chart detects an out-of-control signal, evaluated by the AATS measure:

$$AATS = ATC - \lambda^{-1}$$  \hspace{1cm} (5)

The AATS is the newest measure implemented to compare the effectiveness of different sampling schemes and shows the control chart sensitivity in detecting shifts in the process. Therefore, with the smaller AATS, the performance of the control chart is better.

The use of variable control charts reducing the AATS, moreover, is able to decrease the ANS and ANI. The ANS and ANI of various variable control charts are less or equal in comparison with the FSSI sampling scheme. The reduction in measures is also indicated the efficiency of the control chart and would have an impact on costs. The values of ANI and ANS for sampling schemes were obtained as follows:

$$ANI = b(I - Q)^{-1}n$$  \hspace{1cm} (6)

$$ANS = b(I - Q)^{-1}s$$  \hspace{1cm} (7)

where $n$ and $s$ are vectors of the number of inspected items and the number of samples, respectively. In designing control charts with variable sampling methods, since the AATS, ANS, and ANI are focused more, increasing the rate of false alarms is a possibility, and researchers have always been suspicious to employ these types of control charts. To this end, the current study also attended to compare the sampling schemes based on the ANF. Reduction in the ANF, as highlighted by Fallahnejhad et al., Amiri et al., Faraz and Saniga, Shojaee et al. and Katebi et al., has improved the performance of the control chart. Being similar to other measures based on the Markov chain concepts, The formula for calculating this measure with the vector of false alarms $f$ is as follows:

$$ANF = b(I - Q)^{-1}f$$  \hspace{1cm} (8)

4 | SENSITIVITY ANALYSIS

The performance of the VSSI, VSS, VSI, and FSSI schemes were calculated and compared to the ANF, ANS, ANI, and AATS measures for the U control chart by using various numerical examples. The values of $b$, $h$, $n$, $s$, and $f$ employed to calculate the statistical criteria. The method implemented to compute the control limits ($UCL_i$) and the warning limits
employed, the range for choosing values were $h_2$ and sampling interval of 2 to control a process with an estimated $\lambda$. Suppose that the quality control operator uses FSSI-U control chart with a sample size of 3 for practical reasons and better comparison of VSSI-U control chart, the values of obtained parameters for the VSSI, VSS, VSI, and FSSI schemes are respectively as follows: $(n_1, n_2, h_1, h_2, WL_1, WL_2, UCL_1, UCL_2), (n_1, n_2, h_0, WL_1, WL_2, UCL_1, UCL_2), (n_0, h_1, h_2, WL, UCL)$, and $(n_0, h_0, UCL)$. The performance of the VSSI, VSS, VSI, and FSSI schemes through estimating different values for $u_0, u_1, \lambda, n_0,$ and $h_0$ and diverse combinations of these values entailed the positive performance of the VSSI scheme; for instance, a production process was monitored with a VSSI-U control chart. For practical reasons and better comparison of VSSI-U control chart with FSSI-U control chart, suppose that the quality control operator uses FSSI-U control chart with a sample size of 3 and sampling interval of 2 h to control a process with an estimated $\lambda$ of 0.02. The calculations showed that this control chart can detect a change with the size of 1.7 due to a shift in the process average from $n_0 = 1.5$ to $u_1 = 3.2$ in 17.98 h or 1079 min (Table A11). This means that it will take two working shifts for the operator to detect an out-of-control state. In the meantime, the produced products will not comply with the customer’s desire giving rise to a huge latency for the organization. For better control of the process and decline the risks, Table A2 proposes the following scheme to alter the sampling process:

$$n_1 = 3, n_2 = 25, h_1 = 2, h_2 = 0.1.$$  

**Table 1** Results of applying the VSSI scheme when $u_1 = 3.2$ and $\lambda = 0.04$

| $u_0$ | $h_0$ | $n_0$ | Value | Obtained parameters | $\lambda$ | $n_0$ | Value | Obtained parameters | $\lambda$ | $n_0$ | Value | Obtained parameters | $\lambda$ | $n_0$ | Value | Obtained parameters |
|-------|-------|-------|-------|----------------|----------|-------|-------|----------------|----------|-------|-------|----------------|----------|-------|-------|----------------|
| 0.5   | 1     | 3     | 0.74  | (3,9,1,0,1,32, 0.97,2,54,1,68) | 0.02     | (1,3,8,1,1,91, 1.32,4,04,2,54) | 11.14  | (1,3,8,1,1,91, 1.32,4,04,2,54) | 5.31     | (3,6,8,1,1,32, 1.08,2,54,1,94) |
| 6     | 0.68  |       |       | (6,9,1,0,5,1,08, 0.97,1,94,1,68) | 0.12     | (1,6,8,1,1,91, 1.08,4,04,1,94) | 16.72  | (1,6,8,1,1,91, 1.08,4,04,1,94) | 5.01     | (6,6,8,1,1,08, 1.08,1,94,1,94) |
| 9     | 0.53  |       |       | (9,9,1,0,2,0,97, 0.97,1,68,1,68) | 0.17     | (1,9,8,1,1,91, 0.97,4,04,1,68) | 28.68  | (1,9,8,1,1,91, 0.97,4,04,1,68) | 5.54     | (6,9,8,1,1,08, 0.97,1,94,1,68) |
| 1     | 2     | 3     | 2.47  | (3,26,2,0,1,1,1,1,15, 1.39,3,89,1,98) | 0.02     | (1,3,8,2,3,00, 2.15,6,00,2,89) | 15.91  | (1,3,8,2,3,00, 2.15,6,00,2,89) | 5.79     | (3,26,8,2,2,1,5, 1.39,3,89,1,98) |
| 6     | 1.69  |       |       | (5,26,2,0,1,1,8, 1.39,3,24,1,98) | 0.02     | (1,6,8,2,3,00, 1.82,6,00,3,04) | 24.01  | (1,7,8,2,3,00, 1.76,6,00,2,89) | 5.79     | (3,26,8,2,2,1,5, 1.39,3,89,1,98) |
| 9     | 1.54  |       |       | (7,26,2,0,1,1,76, 1.39,2,89,1,98) | 0.10     | (1,9,8,2,3,00, 1.67,6,00,2,67) | 29.37  | (1,9,8,2,3,00, 1.67,6,00,2,67) | 5.64     | (7,26,8,2,1,7,6, 1.39,2,89,1,98) |
| 1.8   | 3     | 3     | 9.05  | (3,32,3,0,1,3,35, 2.27,5,67,2,99) | 0.01     | (1,3,8,3,4,48, 3.33,5,81,5,67) | 38.73  | (3,5,8,3,3,35, 3.00,5,67,4,80) | 7.78     | (3,3,8,3,3,35, 2.27,5,67,2,99) |
| 6     | 4.99  |       |       | (6,32,3,0,1,2,90, 2.27,4,54,2,99) | 0.05     | (1,6,8,3,4,48, 2.90,8,51,4,54) | 43.00  | (3,6,8,3,3,35, 2.90,5,67,4,54) | 6.64     | (6,3,8,3,2,9, 2.27,4,54,2,99) |
| 9     | 4.99  |       |       | (6,32,3,0,1,2,90, 2.27,4,54,2,99) | 0.05     | (1,9,8,3,4,48, 2.69,8,51,4,04) | 46.82  | (1,1,0,3,3,4,48, 2.65,8,51,3,92) | 6.64     | (6,3,8,3,2,9, 2.27,4,54,2,99) |
| 2.8   | 4     | 3     | 39.20 | (3,49,4,0,1,4,73, 3.28,7,63,4,00) | 0.01     | (1,3,8,4,6,15, 4.73,11,17,7,63) | 121.04 | (3,1,5,8,4,47, 3.66,7,63,4,96) | 14.80    | (3,49,8,4,4,7,3, 3.28,7,63,4,00) |
| 6     | 33.14 | (4,49,4,0,1,4,47, 3.28,6,98,4,00) | 0.03     | (1,6,8,4,6,15, 4.17,11,17,6,22) | 99.08    | (4,7,8,4,4,47, 4.06,6,98,5,96) | 6.30    | (6,3,8,4,2,9, 2.27,4,54,2,99) |
| 9     | 16.16 | (8,49,4,0,1,3,98, 3.28,5,76,4,00) | 0.08     | (1,9,8,4,6,15, 3.92,11,17,5,59) | 96.91    | (7,9,8,4,4,06, 3.92,5,96,5,59) | 9.05    | (8,49,8,4,3,98, 3.28,5,76,4,00) |
By this scheme, the mentioned change can be detected in 3.02 h or about 181 min. VSSI sampling scheme can identify the out-of-control state 898 min earlier; giving rise to 83% improvement compared to the conventional condition. This clearly shows how the VSSI scheme can considerably decrease the time required to detect out-of-control products. In this scheme, the flexibility in the sample size and sampling intervals resulted in a significant increment in the efficiency of the U control chart. The higher the cost of the products, the higher the advantages of this control scheme. The rest of the statistical criteria are shown below whose improvement can be observed as the consequence of the VSSI scheme application.

\[ \text{FSSI} : \text{ANF} = 0.11, \text{ANI} = 101.97, \text{ANS} = 33.99. \]
\[ \text{VSSI} : \text{ANF} = 0.01, \text{ANI} = 34.33, \text{ANS} = 11.01. \]

Also, when the VSS scheme is implemented, the values of the statistical criteria will be \( \text{AATS} = 4.74, \text{ANF} = 0.02, \text{ANI} = 55.53, \text{ANS} = 27.37 \); and if using VSI scheme, the values are \( \text{AATS} = 8.69, \text{ANF} = 0.03, \text{ANI} = 51.14, \text{ANS} = 17.05 \) (see Tables A5 and A8).

Comparing the experiments shown in Tables 1 and A1–A11, it can be highlighted that any changes in the process regarding the VSSI scheme, while other statistical criteria are decreased, would be possible to be detected more quickly. Although, the performance of the VSS and VSI schemes is not as sufficient as the VSSI scheme, however, compared to the FSSI scheme, a significant reduction in the statistical criteria is noticeable. While Shojaie-Navokh et al.\textsuperscript{33} showed that the VSSI sampling scheme can decrease the production costs in the use of the U control chart, based on Woodall,\textsuperscript{46} an improvement in the cost performance of the control chart may weaken its statistical performance. However, the comprehensive investigation of the statistical criteria in this research indicated that the use of the VSSI sampling scheme can significantly improve the statistical features of the U control chart.

5 | CONCLUSION

The present study was carried out to develop variable sampling schemes attributed to the U control chart. The proposed schemes were examined using a statistical approach. The main goal was to achieve an effective and efficient tool to monitor the processes, in which the quality of the manufactured products was classified by product characteristics conforming or nonconforming determined in both groups. The criteria employed to evaluate the statistical performance of the proposed schemes were the AATS, ANF, ANS, and ANI, respectively. In designing variable sampling schemes, the range of sample size or sampling interval is unpredictable. Variation in sampling may seem unpleasant from an operational viewpoint; however, significant statistical improvement has been generated in the process. In this study, the design of the sampling schemes and evaluation of statistical measures were done based on the Markov chain approach and the transition probability of different process states.

Various numerical experimental compositions were developed to evaluate the performance of the U control chart and compare the efficiency of the sampling schemes. By comparing the numerical results to those of other schemes, the VSSI scheme, in which two sample sizes and two sampling intervals were implemented, has led to more reduction in the statistical measures and has increased the efficiency in the U control chart. Therefore, by changing the sampling method, the efficiency of the U control chart can be significantly increased; while the period of time to reach the signal, the false alarms, and the sample taken and inspected can be decreased. Furthermore, according to the advantage of the VSSI scheme over other schemes, especially the FSSI scheme, the practical applications are recommended to reach more accuracy and better results.

The performance of the U control chart under the variable sampling scheme can be also examined under the influence of other statistical criteria such as the Probability of False Alerts (PFA) and the Conditional Expected Delay (CED) (see e.g. Reference 38). Various investigations will definitely help in understanding other features of the variable sampling scheme. It would also be possible to improve the performance of the U control chart under other sampling methods as well as applying other optimization methods or upgrading the methods proposed. Moreover, it should be considered in future studies that the quality of a process or product often includes several inter-related qualitative features. Therefore, evaluation of the multivariable control charts with the use of VSSI under different statistical indices (as mentioned in this research) can be considered.
CONFLICT OF INTEREST
The authors declare no potential conflict of interest.

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### TABLE A1  The results of applying the VSI sampling scheme when $u_0 = 1.5$ and $\lambda = 0.04$

| $u_0$ | $h_0$ | $n_0$ | Value | Obtained parameters | Value | Obtained parameters | Value | Obtained parameters | Value | Obtained parameters |
|-------|-------|-------|-------|---------------------|-------|---------------------|-------|---------------------|-------|---------------------|
| 2.5   | 1     | 3     | 3.01  | (3,25,1,0.1,2.91, 1.99,5.04,2.72) | 0.01  | (1,3,8,1,3.95,2.91, 7.62,5.04) | 50.9  | (2,7,8,1,3.23, 2.43,5.83,3.81) | 9.89  | (3,25,1,0.1,2.91, 1.99,5.04,2.72) |
| 6     | 2     | 0.70  | (4,25,1,0.1,2.72, 1.99,4.56,2.72) | 0.03  | (1,6,8,1,3.95,2.50, 7.62,4.00) | 50.9  | (2,7,8,1,3.23, 2.43,5.83,3.81) | 9.14  | (4,17,8,1,2.72, 2.09,4.56,2.99) |
| 9     | 2.21  | (7,25,1,0,1.24, 1.99,3.81,2.72) | 0.09  | (1,9,8,1,3.95,2.32, 7.62,3.54) | 58.6  | (2,9,8,1,3.23, 2.32,5.83,3.54) | 7.72  | (7,17,8,1,2.43, 2.09,3.81,2.99) |
| 3.5   | 2     | 3     | 2.47  | (3,25,2,0.1,2.91, 1.99,5.04,2.72) | 0.01  | (1,3,8,2,3.95,2.91, 7.62,5.04) | 24.2  | (2,4,8,2,3.23, 2.72,5.83,4.56) | 6.80  | (3,17,8,2,2.91, 2.09,5.04,2.99) |
| 6     | 2.32  | (4,25,2,0.1,2.72, 1.99,4.56,2.72) | 0.02  | (1,6,8,2,3.95,2.50, 7.62,4.00) | 28.5  | (2,7,8,2,3.23, 2.43,5.83,3.81) | 6.37  | (4,17,8,2,2.72, 2.09,4.56,2.99) |
| 9     | 2.13  | (7,25,2,0.1,2.43, 1.99,3.81,2.72) | 0.09  | (1,9,8,2,3.95,2.32, 7.62,3.54) | 33.7  | (2,9,8,2,3.23, 2.32,5.83,3.54) | 5.85  | (7,17,8,2,2.43, 2.09,3.81,2.99) |
| 4.7   | 3     | 3     | 2.25  | (3,25,3,0.1,2.91, 1.99,5.04,2.72) | 0.01  | (1,3,8,3,3.95,2.91, 7.62,5.04) | 15.1  | (1,3,8,3,3.95, 2.91,7.62,5.04) | 5.47  | (2,17,8,3,2.33, 2.09,5.83,2.99) |
| 6     | 2.17  | (4,25,3,0.1,2.72, 1.99,4.56,2.72) | 0.02  | (1,6,8,3,3.95,2.50, 7.62,4.00) | 20.9  | (1,7,8,3,3.95, 2.43,7.62,3.81) | 5.25  | (4,17,8,3,2.72, 2.09,4.56,2.99) |
| 9     | 2.07  | (7,25,3,0,6,2.43, 1.99,3.81,2.72) | 0.08  | (1,9,8,3,3.95,2.32, 7.62,3.54) | 25.3  | (2,9,8,3,3.23, 2.32,5.83,3.54) | 5.02  | (7,17,8,3,2.43, 2.09,3.81,2.99) |
| 6     | 4     | 3     | 2.36  | (3,25,4,0.1,2.91, 1.99,5.04,2.72) | 0.01  | (1,3,8,4,3.95,2.91, 7.62,5.04) | 10.6  | (1,3,8,4,3.95, 2.91,7.62,5.04) | 4.74  | (2,17,8,4,3.23, 2.09,5.83,2.99) |
| 6     | 2.31  | (4,25,4,0.6,2.72, 1.99,4.56,2.72) | 0.02  | (1,6,8,4,3.95,2.50, 7.62,4.00) | 16.3  | (1,6,8,4,3.95, 2.50,7.62,4.00) | 4.68  | (4,17,8,4,2.72, 2.09,4.56,2.99) |
| 9     | 2.18  | (7,25,4,1,3.24, 1.99,3.81,2.72) | 0.08  | (1,9,8,4,3.95,2.32, 7.62,3.54) | 21.4  | (1,9,8,4,3.95, 2.32,7.62,3.54) | 4.56  | (7,17,8,4,2.43, 2.09,3.81,2.99) |

(Continues)
### Table A2

The results of applying the VSS sampling scheme when $u_0 = 0.1$ and $\lambda = 0.02$.

| $\lambda$ | $h_0$ | $n_0$ | Value   | Obtained parameters               | $\mu$ | $\lambda$ | Value   | Obtained parameters               | $\mu$ | $\lambda$ | Value   | Obtained parameters               |
|----------|-------|-------|---------|-----------------------------------|-------|----------|---------|-----------------------------------|-------|----------|---------|-----------------------------------|
| 0.05     | 3     | 4.50  | (3.25,3.01,2.91, 1.99,5.04,2.72) | 0.01  | (1.38,3.395, 2.91,7.62,5.04)     | 26.46  | (2.48,3.3,2.3, 2.72,5.83,4.56)   | 6.10  | (3.25,3.2,91, 1.99,5.04,2.72)    |
| 0.1      | 4     | 6.02  | (3.25,4.01,2.91, 1.99,5.04,2.72) | 0.00  | (1.38,4.395, 2.91,7.62,5.04)     | 22.72  | (2.48,4.3,2.3, 2.72,5.83,4.56)   | 4.03  | (3.25,4.2,91, 1.99,5.04,2.72)    |
| 6        | 4.20  | 3     | (4.25,3.01,2.72, 1.99,4.56,2.72) | 0.02  | (1.68,4.395, 2.50,7.62,4.00)     | 29.43  | (2.78,3.3,2.3, 2.43,5.83,3.81)   | 5.74  | (4.17,8,3,2,72, 2.09,4.56,2.99)  |
| 9        | 3.78  | 2.27  | (7.25,2.0,1.243, 1.99,3.81,2.72) | 0.08  | (1.98,8,3.95, 2.32,7.62,3.54)    | 34.86  | (2.98,8,3,2.3, 2.32,5.83,3.54)   | 5.13  | (7.17,8,2,2,43, 2.09,3.81,2.99)  |

### Table A3

The results of applying the VSS sampling scheme when $u_0 = 3.2$ and $\lambda = 0.04$.

| $u_0$ | $h_0$ | $n_0$ | Value   | Obtained parameters               | $\mu$ | $\lambda$ | Value   | Obtained parameters               | $\mu$ | $\lambda$ | Value   | Obtained parameters               |
|-------|-------|-------|---------|-----------------------------------|-------|----------|---------|-----------------------------------|-------|----------|---------|-----------------------------------|
| 0.5   | 1     | 3     | 0.97    | (3.9,1.32, 0.97,5.41,1.68)       | 0.05  | (1.3,1.91, 1.32,4.04,2.54)       | 35.43  | (1.3,1.91, 1.32,4.04,2.54)       | 25.97  | (3.9,1.32, 0.97,5.41,1.68)       |
| 6     | 0.71  | 1     | (6,1,8, 1,08,1.94,1.94)          | 0.28  | (1.6,4,1, 1.08,4.04,1.94)        | 46.44  | (1.6,1,91, 1.08,4.04,1.94)       | 25.71  | (6,1,8, 1,08,1.94,1.94)          |
| 9     | 0.71  | 2     | (6,9,1.8, 0.97,1.94,1.68)       | 0.40  | (1.9,1,91, 0.97,4.04,1.68)       | 68.75  | (1.9,1,91, 0.97,4.04,1.68)       | 25.27  | (6,9,1.8, 0.97,1.94,1.68)       |
| 1     | 2     | 3.95  | (3,26,2,15, 1,39,3.89,1.98)      | 0.02  | (1.3,3,00, 2.15,6,00,3.89)       | 24.75  | (1.3,3,00, 2.15,6,00,3.89)       | 13.97  | (3,26,2,15, 1,39,3.89,1.98)      |
| 6     | 2.44  | 1     | (5,2,6,1,89, 1,39,3,24,1.98)     | 0.03  | (1.6,3,00, 1.82,6,00,3.04)       | 33.48  | (1,7,3,00, 1.76,6,00,2.89)       | 13.72  | (5,2,6,1,89, 1,39,3,24,1.98)     |
| 9     | 1.89  | (7,2,6,1,76, 1,39,2,89,1.98)     | 0.12  | (1.9,3,00, 1.67,6,00,2.67)       | 39.24  | (1,9,3,00, 1.67,6,00,2.67)       | 13.45  | (7,2,6,1,76, 1,39,2,89,1.98)     |
| 1.8    | 3     | 10.74 | (3,32,3,35, 2.27,5,67,2.99)      | 0.01  | (1.3,4,48, 3.38,5,81,5,67)       | 48.79  | (1,5,4,48, 3.00,8,51,4,80)       | 11.91  | (3,32,3,35, 2.27,5,67,2.99)      |
| 6     | 6.61  | (6,3,2,90, 2.27,4,54,2,99)       | 0.05  | (1.6,4,48, 2.90,8,51,4,54)       | 51.76  | (1,6,4,48, 2.90,8,51,4,54)       | 10.54  | (6,3,2,90, 2.27,4,54,2,99)       |
| 9     | 6.61  | (6,3,2,90, 2.27,4,54,2,99)       | 0.05  | (1.9,4,48, 2.70,8,51,4,04)       | 52.69  | (1,10,4,48, 2.65,8,51,3,92)      | 10.54  | (6,3,2,90, 2.27,4,54,2,99)       |
| 2.8    | 4     | 42.46 | (3,4,9,4,73, 3.28,7,63,4,00)     | 0.01  | (1.3,6,15, 4.73,11,17,7,63)      | 131.66 | (3,15,4,73, 3.66,7,63,4,96)      | 16.87  | (3,4,9,4,73, 3.28,7,63,4,00)     |
| 6     | 35.28 | (4,4,9,4,73, 3.28,6,98,4,00)     | 0.03  | (1.6,6,15, 4.17,11,17,6,22)      | 109.59 | (4,7,4,47, 4.06,6,98,5,96)       | 15.07  | (4,4,9,4,73, 3.28,6,98,4,00)     |
| 9     | 19.22 | (8,4,9,3,98, 3.28,5,76,4,00)     | 0.08  | (1.9,6,15, 3.92,11,17,5,59)      | 113.85 | (7.9,4,06, 3.92,5,96,5,59)       | 11.06  | (8,4,9,3,98, 3.28,5,76,4,00)     |
### Table A4
The results of applying the VSS sampling scheme when \( u_0 = 1.5 \) and \( \lambda = 0.04 \)

| \( u_0 \) | \( n_0 \) | \( n_0 \) | \( \lambda \) | \( \mu_0 \) | \( h_0 \) | AATS | Obtained parameters | Value | ANF | Obtained parameters | Value | ANI | Obtained parameters | Value | ANS | Obtained parameters | Value |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2.5 | 1 | 3 | 4.21 | (3,25,2.91,1.99, 5.04,2.72) | 0.02 | (1,3.95,2.91, 7.62,5.04) | 81.37 | (1,7.95,2.43, 7.62,3.81) | 29.21 | (3,25,2.91, 1.99,5.04,2.72) |
| 6 | 3.63 | (4,25,2.72,1.99, 4.56,2.72) | 0.06 | (1,6.95,2.50, 7.62,4.00) | 81.37 | (1,7.95,2.43, 7.62,3.81) | 28.63 | (4,25,2.72, 1.99,4.56,2.72) |
| 9 | 2.72 | (7,25,2.43,1.99, 3.81,2.72) | 0.19 | (1,9.95,2.32, 7.62,3.54) | 94.88 | (1,9.95,2.32, 7.62,3.54) | 27.72 | (7,25,2.43, 1.99,3.81,2.72) |
| 3.5 | 2 | 3 | 3.90 | (3,25,2.91,1.99, 5.04,2.72) | 0.01 | (1,3.95,2.91, 7.62,5.04) | 34.66 | (1,4.35,2.72, 7.62,4.56) | 14.45 | (3,25,2.92, 1.99,5.03,2.72) |
| 6 | 3.37 | (4,25,2.72,1.99, 4.56,2.72) | 0.04 | (1,6.95,2.50, 7.62,4.00) | 40.22 | (1,7.95,2.43, 7.62,3.81) | 14.19 | (4,25,2.72, 1.99,4.56,2.72) |
| 9 | 2.67 | (7,25,2.43,1.99, 3.81,2.72) | 0.12 | (1,9.95,2.32, 7.62,3.54) | 47.90 | (1,9.95,2.32, 7.62,3.54) | 13.84 | (7,25,2.43, 1.99,3.81,2.72) |
| 4.7 | 3 | 3 | 3.54 | (3,25,2.91,1.99, 5.04,2.72) | 0.01 | (1,3.95,2.91, 7.62,5.04) | 20.02 | (1,3.95,2.91, 7.62,5.04) | 9.51 | (3,25,2.91, 1.99,5.04,2.72) |
| 6 | 3.03 | (4,25,2.72,1.99, 4.56,2.72) | 0.03 | (1,6.95,2.50, 7.62,4.00) | 26.91 | (1,6.95,2.50, 7.62,4.00) | 9.34 | (4,25,2.72, 1.99,4.56,2.72) |
| 9 | 2.51 | (7,25,2.43,1.99, 3.81,2.72) | 0.10 | (1,9.95,2.32, 7.62,3.54) | 32.46 | (1,9.95,2.32, 7.62,3.54) | 9.17 | (7,25,2.43, 1.99,3.81,2.72) |
| 6 | 4 | 3 | 3.39 | (3,25,2.91,1.99, 5.04,2.72) | 0.01 | (1,3.95,2.91, 7.62,5.04) | 13.42 | (1,3.95,2.91, 7.62,5.04) | 7.10 | (3,25,2.91, 1.99,5.04,2.72) |
| 6 | 2.94 | (4,25,2.72,1.99, 4.56,2.72) | 0.03 | (1,6.95,2.50, 7.62,4.00) | 19.60 | (1,6.95,2.50, 7.62,4.00) | 6.99 | (4,25,2.72, 1.99,5.56,2.72) |
| 9 | 2.57 | (7,25,2.43,1.99, 3.81,2.72) | 0.09 | (1,9.95,2.32, 7.62,3.54) | 25.22 | (1,9.95,2.32, 7.62,3.54) | 6.89 | (7,25,2.43, 1.99,3.81,2.72) |

### Table A5
The results of applying the VSS sampling scheme when \( u_0 = 1.5 \) and \( \mu_0 = 3.2 \)

| \( \lambda \) | \( h_0 \) | \( n_0 \) | \( \lambda \) | \( \mu_0 \) | \( h_0 \) | AATS | Obtained parameters | Value | ANF | Obtained parameters | Value | ANI | Obtained parameters | Value | ANS | Obtained parameters | Value |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 1 | 3 | 2.41 | (3,25,2.91,1.99, 5.04,2.72) | 0.05 | (1,3.95,2.91, 7.62,5.04) | 147.58 | (1,4.35,2.72, 7.62,5.04) | 102.41 | (3,25,2.91, 1.99,5.04,2.72) |
| 6 | 2.07 | (4,25,2.72,1.99, 4.56,2.72) | 0.17 | (1,6.95,2.50, 7.62,4.00) | 171.38 | (1,7.95,2.43, 7.62,3.81) | 102.07 | (4,25,2.72, 1.99,4.56,2.72) |
| 9 | 1.61 | (7,25,2.43,1.99, 3.81,2.72) | 0.57 | (1,9.95,2.32, 7.62,3.54) | 193.06 | (1,9.95,2.32, 7.62,3.54) | 101.61 | (7,25,2.43, 1.99,3.81,2.72) |
| 0.02 | 2 | 3 | 4.74 | (3,25,2.91,1.99, 5.04,2.72) | 0.02 | (1,3.95,2.91, 7.62,5.04) | 55.53 | (1,4.35,2.72, 7.62,4.56) | 27.37 | (3,25,2.91, 1.99,5.04,2.72) |
| 6 | 4.10 | (4,25,2.72,1.99, 4.56,2.72) | 0.06 | (1,6.95,2.50, 7.62,4.00) | 62.34 | (1,7.95,2.43, 7.62,3.81) | 27.05 | (4,25,2.72, 1.99,4.56,2.72) |
| 9 | 3.20 | (7,25,2.43,1.99, 3.81,2.72) | 0.19 | (1,9.95,2.32, 7.62,3.54) | 72.70 | (1,9.95,2.32, 7.62,3.54) | 26.60 | (7,25,2.43, 1.99,3.81,2.72) |

(Continues)
### TABLE A5 (Continued)

| $\lambda$ | $h_0$ | $n_0$ | \(\text{Value} \quad \text{Obtained parameters}\) | $\lambda$ | $h_0$ | $n_0$ | \(\text{Value} \quad \text{Obtained parameters}\) |
|----------|-------|-------|---------------------------------------------|----------|-------|-------|---------------------------------------------|
| 0.05     | 3     | 3     | 6.78 (3.25, 2.91, 1.99, 5.04, 2.72)       | 0.05     | 3     | 3     | 6.78 (3.25, 2.91, 1.99, 5.04, 2.72)       |
|          | 6     | 5.91  | (4.25, 2.72, 1.99, 4.56, 2.72)            |          | 6     | 5.91  | (4.25, 2.72, 1.99, 4.56, 2.72)            |
|          | 9     | 4.69  | (7.25, 2.43, 1.99, 3.81, 2.72)            |          | 9     | 4.69  | (7.25, 2.43, 1.99, 3.81, 2.72)            |
| 0.1      | 4     | 3     | 8.29 (3.25, 2.91, 1.99, 5.04, 2.72)       | 0.1      | 4     | 3     | 8.29 (3.25, 2.91, 1.99, 5.04, 2.72)       |
|          | 6     | 7.34  | (4.25, 2.72, 1.99, 4.56, 2.72)            |          | 6     | 7.34  | (4.25, 2.72, 1.99, 4.56, 2.72)            |
|          | 9     | 6.01  | (7.25, 2.43, 1.99, 3.81, 2.72)            |          | 9     | 6.01  | (7.25, 2.43, 1.99, 3.81, 2.72)            |

### TABLE A6 The results of applying the VSI sampling scheme when $u_0 = 3.2$ and $\lambda = 0.04$

| $u_0$ | $h_0$ | $n_0$ | \(\text{Value} \quad \text{Obtained parameters}\) | $u_0$ | $h_0$ | $n_0$ | \(\text{Value} \quad \text{Obtained parameters}\) |
|-------|-------|-------|---------------------------------------------|-------|-------|-------|---------------------------------------------|
| 0.5   | 1     | 3     | 0.81 (1.0, 1.13, 2.52)                      | 0.5   | 1     | 3     | 0.81 (1.0, 1.13, 2.52)                      |
|       | 6     | 0.69  | (1.05, 1.08, 1.94)                         |       | 6     | 0.69  | (1.05, 1.08, 1.94)                         |
|       | 9     | 0.53  | (1.02, 0.97, 1.68)                         |       | 9     | 0.53  | (1.02, 0.97, 1.68)                         |
| 1     | 2     | 3     | 2.97 (2.0, 1.25, 3.89)                     | 1     | 2     | 3     | 2.97 (2.0, 1.25, 3.89)                     |
|       | 6     | 1.96  | (2.0, 1.18, 3.04)                         |       | 6     | 1.96  | (2.0, 1.18, 3.04)                         |
|       | 9     | 1.58  | (2.0, 1.67, 2.67)                         |       | 9     | 1.58  | (2.0, 1.67, 2.67)                         |
| 1.8   | 3     | 18.94 | (3.0, 1.35, 5.67)                         | 1.8   | 3     | 18.94 | (3.0, 1.35, 5.67)                         |
|       | 6     | 6.88  | (3.0, 1.90, 4.54)                         |       | 6     | 6.88  | (3.0, 1.90, 4.54)                         |
|       | 9     | 6.88  | (3.0, 1.69, 4.04)                         |       | 9     | 6.88  | (3.0, 1.69, 4.04)                         |
| 2.8   | 4     | 188.77| (4.0, 1.47, 3.76)                         | 2.8   | 4     | 188.77| (4.0, 1.47, 3.76)                         |
|       | 6     | 72.41 | (4.0, 1.47, 6.22)                         |       | 6     | 72.41 | (4.0, 1.47, 6.22)                         |
|       | 9     | 25.16 | (4.0, 1.92, 5.59)                         |       | 9     | 25.16 | (4.0, 1.92, 5.59)                         |

### TABLE A7 The results of applying the VSI sampling scheme when $u_0 = 1.5$ and $\lambda = 0.04$

| $u_0$ | $h_0$ | $n_0$ | \(\text{Value} \quad \text{Obtained parameters}\) | $u_0$ | $h_0$ | $n_0$ | \(\text{Value} \quad \text{Obtained parameters}\) |
|-------|-------|-------|---------------------------------------------|-------|-------|-------|---------------------------------------------|
| 2.5   | 1     | 3     | 14.42 (1.0, 1.29, 5.04)                    | 2.5   | 1     | 3     | 14.42 (1.0, 1.29, 5.04)                    |
|       | 6     | 5.81  | (1.0, 1.25, 4.00)                         |       | 6     | 5.81  | (1.0, 1.25, 4.00)                         |
|       | 9     | 2.82  | (1.0, 1.23, 3.54)                         |       | 9     | 2.82  | (1.0, 1.23, 3.54)                         |
| 3.5   | 2     | 3     | 5.88 (2.0, 1.92, 5.04)                    | 3.5   | 2     | 3     | 5.88 (2.0, 1.92, 5.04)                    |
|       | 6     | 3.48  | (2.0, 1.50, 4.00)                         |       | 6     | 3.48  | (2.0, 1.50, 4.00)                         |
|       | 9     | 2.43  | (2.0, 1.32, 3.54)                         |       | 9     | 2.43  | (2.0, 1.32, 3.54)                         |
| 4.7   | 3     | 3.05  | (3.0, 1.29, 5.04)                         | 4.7   | 3     | 3.05  | (3.0, 1.29, 5.04)                         |
|       | 6     | 2.48  | (3.0, 1.50, 4.00)                         |       | 6     | 2.48  | (3.0, 1.50, 4.00)                         |
|       | 9     | 2.20  | (3.0, 1.32, 3.54)                         |       | 9     | 2.20  | (3.0, 1.32, 3.54)                         |
| 6     | 4     | 3     | 2.55 (4.0, 1.92, 5.04)                    | 6     | 4     | 3     | 2.55 (4.0, 1.92, 5.04)                    |
|       | 6     | 2.41  | (4.0, 1.50, 4.00)                         |       | 6     | 2.41  | (4.0, 1.50, 4.00)                         |
|       | 9     | 2.30  | (4.0, 1.32, 3.54)                         |       | 9     | 2.30  | (4.0, 1.32, 3.54)                         |
TABLE A8  The results of applying the VSI sampling scheme when \( u_0 = 1.5 \) and \( u_+ = 3.2 \)

| \( \lambda \) | \( h_0 \) | \( n_0 \) | Value | Obtained parameters | Value | Obtained parameters | Value | Obtained parameters | Value | Obtained parameters |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 1 | 3 | 4.58 (1.0,1.2,91,5.04) | 0.07 (8,1,2.91,5.04) | 74.80 (8,1,2.91,5.04) | 24.94 (8,1,2.91,5.04) |
| | 6 | 2.42 (1.0,1.2,50,4.00) | 0.29 (8,1,2.50,4.00) | 120.04 (8,1,2.50,4.00) | 20.01 (8,1,2.50,4.00) |
| | 9 | 1.50 (1.0,1.2,33,3.54) | 1.01 (8,1,2.33,3.54) | 161.67 (8,1,2.33,3.54) | 17.96 (8,1,2.33,3.54) |
| 0.02 | 2 | 3 | 8.69 (2.0,1.2,91,5.04) | 0.03 (8,2,2.91,5.04) | 51.14 (8,2,2.91,5.04) | 17.05 (8,2,2.91,5.04) |
| | 6 | 4.64 (2.0,1.2,50,4.00) | 0.14 (8,2,2.50,4.00) | 72.71 (8,2,2.50,4.00) | 12.12 (8,2,2.50,4.00) |
| | 9 | 2.96 (2.0,1.2,33,3.54) | 0.50 (8,2,2.33,3.54) | 90.68 (8,2,2.33,3.54) | 10.08 (8,2,2.33,3.54) |
| 0.05 | 3 | 3 | 12.82 (3.0,1.2,91,5.04) | 0.01 (8,3,2.91,5.04) | 37.26 (8,3,2.91,5.04) | 12.42 (8,3,2.91,5.04) |
| | 6 | 6.90 (3.0,1.2,50,4.00) | 0.05 (8,3,2.50,4.00) | 44.96 (8,3,2.50,4.00) | 7.49 (8,3,2.50,4.00) |
| | 9 | 4.44 (3.0,1.2,33,3.54) | 0.19 (8,3,2.33,3.54) | 49.05 (8,3,2.33,3.54) | 5.45 (8,3,2.33,3.54) |
| 0.1 | 4 | 3 | 17.00 (4.0,1.2,91,5.04) | 0.01 (8,4,2.91,5.04) | 32.42 (8,4,2.91,5.04) | 10.81 (8,4,2.91,5.04) |

TABLE A9  The results of applying the FSSI sampling scheme when \( u_+ = 3.2 \) and \( \lambda = 0.04 \)

| \( u_0 \) | \( h_0 \) | \( n_0 \) | Value | Obtained parameters | Value | Obtained parameters | Value | Obtained parameters | Value | Obtained parameters |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 1 | 3 | 1.12 (2.54) | 0.35 (2.54) | 78.35 (2.54) | 26.12 (2.54) |
| | 6 | 4.05 (3.04) | 0.23 (3.04) | 87.14 (3.04) | 14.52 (3.04) |
| | 9 | 0.71 (1.68) | 2.21 (1.68) | 121.39 (1.68) | 121.39 (1.68) |
| 1 | 2 | 3 | 4.05 (3.89) | 0.23 (3.89) | 43.57 (3.89) | 14.52 (3.89) |
| | 6 | 4.05 (3.04) | 0.23 (3.04) | 87.14 (3.04) | 14.52 (3.04) |
| | 9 | 2.24 (2.67) | 0.96 (2.67) | 121.57 (2.67) | 13.62 (2.67) |
| 1.8 | 3 | 3 | 26.99 (5.67) | 0.08 (5.67) | 51.99 (5.67) | 17.33 (5.67) |
| | 6 | 12.20 (4.54) | 0.29 (4.54) | 74.41 (4.54) | 12.40 (4.54) |
| | 9 | 12.20 (4.04) | 0.29 (4.04) | 111.61 (4.04) | 12.40 (4.04) |
| 2.8 | 4 | 3 | 235.73 (7.63) | 0.05 (7.63) | 195.54 (7.63) | 65.18 (7.63) |
| | 6 | 87.70 (6.22) | 0.14 (6.22) | 169.05 (6.22) | 28.18 (6.22) |
| | 9 | 36.00 (5.59) | 0.38 (5.59) | 137.25 (5.59) | 15.25 (5.59) |

TABLE A10  The results of applying the FSSI sampling scheme when \( u_0 = 1.5 \) and \( \lambda = 0.04 \)

| \( u_+ \) | \( h_0 \) | \( n_0 \) | Value | Obtained parameters | Value | Obtained parameters | Value | Obtained parameters | Value | Obtained parameters |
|---|---|---|---|---|---|---|---|---|---|---|
| 2.5 | 1 | 3 | 23.30 (5.04) | 0.11 (5.04) | 144.90 (5.04) | 48.30 (5.04) |
| | 6 | 8.69 (4.00) | 0.46 (4.00) | 202.16 (4.00) | 33.69 (4.00) |
| | 9 | 3.63 (3.54) | 1.61 (3.54) | 257.66 (3.54) | 28.63 (3.54) |
| 3.5 | 2 | 3 | 13.06 (5.04) | 0.05 (5.04) | 57.09 (5.04) | 19.03 (5.04) |
| | 6 | 6.30 (4.00) | 0.22 (4.00) | 93.89 (4.00) | 15.65 (4.00) |
| | 9 | 3.33 (3.54) | 0.79 (3.54) | 127.48 (3.54) | 14.16 (3.54) |
| 4.7 | 3 | 3 | 7.58 (5.04) | 0.03 (5.04) | 32.58 (5.04) | 10.86 (5.04) |
| | 6 | 4.47 (4.00) | 0.15 (4.00) | 58.93 (4.00) | 9.82 (4.00) |
| | 9 | 2.88 (3.54) | 0.51 (3.54) | 83.63 (3.54) | 9.29 (3.54) |
| 6 | 4 | 3 | 5.27 (5.04) | 0.03 (5.04) | 22.70 (5.04) | 7.57 (5.04) |
| | 6 | 3.65 (4.00) | 0.11 (4.00) | 42.97 (4.00) | 7.16 (4.00) |
| | 9 | 2.77 (3.54) | 0.38 (3.54) | 62.47 (3.54) | 6.94 (3.54) |
The results of applying the FSSI sampling scheme when $u_0 = 1.5$ and $u_+ = 3.2$

| $\lambda$ | $h_0$ | $n_0$ | Value | Obtained parameters | Value | Obtained parameters | Value | Obtained parameters |
|----------|-------|-------|-------|--------------------|-------|--------------------|-------|--------------------|
| 0.01     | 1     | 3     | 8.99  | (5.04)             | 0.44  | (5.04)             | 326.96| (5.04)             |
|          | 6     |       | 4.06  | (4.00)             | 1.85  | (4.00)             | 624.35| (4.00)             |
|          | 9     |       | 2.02  | (3.54)             | 6.53  | (3.54)             | 918.15| (3.54)             |
| 0.02     | 2     | 3     | 17.98 | (5.04)             | 0.11  | (5.04)             | 101.97| (5.04)             |
|          | 6     |       | 8.12  | (4.00)             | 0.46  | (4.00)             | 174.37| (4.00)             |
|          | 9     |       | 4.04  | (3.54)             | 1.61  | (3.54)             | 243.17| (3.54)             |
| 0.05     | 3     | 3     | 27.00 | (5.04)             | 0.03  | (5.04)             | 47.00 | (5.04)             |
|          | 6     |       | 12.21 | (4.00)             | 0.11  | (4.00)             | 64.42 | (4.00)             |
|          | 9     |       | 6.09  | (3.54)             | 0.41  | (3.54)             | 78.26 | (3.54)             |
| 0.1      | 4     | 3     | 36.08 | (5.04)             | 0.01  | (5.04)             | 34.56 | (5.04)             |
|          | 6     |       | 16.37 | (4.00)             | 0.04  | (4.00)             | 39.55 | (4.00)             |
|          | 9     |       | 8.20  | (3.54)             | 0.13  | (3.54)             | 40.94 | (3.54)             |