JUHL’S FORMULAE FOR GJMS OPERATORS AND Q-CURVATURES

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1. Introduction

GJMS operators and $Q$-curvatures are important objects in conformal geometry which have been studied intensely during the past decade. In [J2], [J3], building on previous work beginning with [J1], Juhl has derived remarkable formulae for GJMS operators and $Q$-curvatures which reveal unexpected algebraic structure. In this paper we give direct proofs of Juhl’s formulae starting from the original construction of [GJMS].

Juhl’s formulae are expressed in terms of quantities arising in the expansion of a Poincaré metric, or equivalently an ambient metric, associated to a given pseudo-Riemannian metric. Let $g$ be a pseudo-Riemannian metric of signature $(p, q)$, $p + q = n \geq 3$, on an $n$-dimensional manifold $M$. A Poincaré metric in normal form relative to $g$ is a metric $g_+$ on $M \times (0, \epsilon)$ of the form

$$g_+ = r^{-2} (dr^2 + h_r),$$

where $h_r$ is a smooth 1-parameter family of metrics on $M$ satisfying $h_0 = g$, for which $\text{Ric}(g_+) + ng_+ = 0$ in the following asymptotic sense. If $n$ is odd, then $\text{Ric}(g_+) + ng_+ = O(r^\infty)$, while if $n$ is even, then $\text{Ric}(g_+) + ng_+ = O(r^{n-2})$ and the tangential trace of $r^{2-n} (\text{Ric}(g_+) + ng_+)$ vanishes at $r = 0$. Set

$$V(r) = \sqrt{\frac{\det h_r}{\det h_0}},$$

and $W(r) = \sqrt{V(r)}$. Let $\delta$ denote the divergence operator on vector fields with respect to $g$, given by $\delta \varphi = \nabla_i \varphi^i$. Define a 1-parameter family $\mathcal{M}(r)$ of second order differential operators on $M$ by

$$\mathcal{M}(r) = \delta(h_r^{-1}d) - U(r),$$

where

$$U(r) = \frac{[\partial^2 - (n-1)r^{-1}\partial_r + \delta(h_r^{-1}d)]W(r)}{W(r)}$$

acts as a zeroth order term. (We write $U(r)$ in the form given in v1 of [J3]. v2 of [J3] expresses it in a different form; see Lemma 8.1 of v2.) Use $\mathcal{M}(r)$ as a generating

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function for second order differential operators \( \mathcal{M}_{2N} \) on \( M \) defined for \( N \geq 1 \) (and \( N \leq n/2 \) if \( n \) is even) by

\[
(1.2) \quad \mathcal{M}(r) = \sum_{N \geq 1} \mathcal{M}_{2N} \frac{1}{(N-1)!^2} \left( \frac{r^2}{4} \right)^{N-1}.
\]

The \( \mathcal{M}_{2N} \) are natural scalar differential operators. Natural scalar invariants \( W_{2N} \) are defined by

\[
(1.3) \quad W(r) = 1 + \sum_{N \geq 1} W_{2N} r^{2N}
\]

for \( N \geq 1 \) (and \( N \leq n/2 \) if \( n \) is even).

Juhl’s formulae involve constants \( n_I, m_I \) which are parametrized by ordered lists \( I = (I_1, \ldots, I_r) \) of positive integers. \( I \) is referred to as a composition of the sum \( |I| = I_1 + I_2 + \cdots + I_r \). Sometimes compositions are written in the form \( (I, a) \) singling out the last entry. In this case the convention is that \( I \) is allowed to be empty but \( a > 0 \). The constants appearing in Juhl’s formulae are:

\[
(1.4)
\begin{align*}
n_I &= (|I| - 1)!^2 \prod_{j=1}^r \frac{1}{(I_j - 1)!^2} \prod_{j=1}^{r-1} \frac{1}{\left( \sum_{k=1}^j I_k \right) \left( \sum_{k=j+1}^r I_k \right)} \\
m_I &= (-1)^r |I|! (|I| - 1)! \prod_{j=1}^r \frac{1}{I_j! (I_j - 1)!} \prod_{j=1}^{r-1} \frac{1}{I_j + I_{j+1}}.
\end{align*}
\]

Empty products are always interpreted as 1. Observe when \( r = 1 \) that \( n_{(N)} = m_{(N)} = 1 \).

Let \( P_{2N} \) denote the GJMS operators, with sign convention determined by \( P_{2N} = \Delta^N + \cdots \) with \( \Delta = \delta(g^{-1}d) \). These are defined for all \( N \geq 1 \) for \( n \) odd and for \( 1 \leq N \leq n/2 \) for \( n \) even. Iterated compositions of the \( P_{2N} \) and the \( \mathcal{M}_{2N} \) are denoted by \( P_{2I} = P_{2I_1} \circ \cdots \circ P_{2I_r} \) and \( \mathcal{M}_{2I} = \mathcal{M}_{2I_1} \circ \cdots \circ \mathcal{M}_{2I_r} \).

Juhl proves four formulae: an explicit formula and a recursive formula each for GJMS operators and for \( Q \)-curvatures. All four formulae are universal in the dimension.

**Theorem 1.1. Explicit formula for GJMS operators.** For \( N \geq 1 \) (and \( N \leq n/2 \) if \( n \) is even),

\[
(1.5) \quad P_{2N} = \sum_{|I|=N} n_I \mathcal{M}_{2I}.
\]

**Theorem 1.2. Recursive formula for GJMS operators.** For \( N \geq 1 \) (and \( N \leq n/2 \) if \( n \) is even),

\[
(1.6) \quad P_{2N} = - \sum_{|I|=N \atop I \neq (N)} m_I P_{2I} + \mathcal{M}_{2N}.
\]
Clearly the explicit formula expresses $P_{2N}$ in terms of the second order building blocks $\mathcal{M}_{2M}$, $M \leq N$. The recursive formula expresses each $P_{2N}$ as a sum of compositions of lower order GJMS operators, modulo the second order term $\mathcal{M}_{2N}$. For $N = 1$ both formulae state that $\mathcal{M}_2 = P_2$, the Yamabe operator. For $N = 2$ the formulae express the Paneitz operator as $P_4 = \mathcal{M}_2^2 + \mathcal{M}_4 = P_2^2 + \mathcal{M}_4$. The principal part of $\mathcal{M}_{2N}$ for $N > 1$ involves curvature, and $\mathcal{M}_{2N} = 0$ for $N > 1$ if $g$ is flat.

Further discussion and specializations of the formulae may be found in [J3].

The $Q$-curvatures are defined in terms of the zeroth order terms of the GJMS operators:

\begin{equation}
Q_{2N}(1) = (-1)^N \left( \frac{n}{2} - N \right) Q_{2N},
\end{equation}

$Q_{2N}$ is defined for all $N \geq 1$ if $n$ is odd and for $1 \leq N \leq n/2$ if $n$ is even. For $n$ even, both sides vanish in the critical case $N = n/2$ and $Q_n$ is defined by an analytic continuation.

**Theorem 1.3. Explicit formula for $Q$-curvatures.** For $N \geq 1$ (and $N \leq n/2$ if $n$ is even),

\begin{equation}
(-1)^N Q_{2N} = \sum_{|I,a| = N} n_{(I,a)} a! (a - 1)! 2^{2a} \mathcal{M}_{2I}(W_{2a}).
\end{equation}

**Theorem 1.4. Recursive formula for $Q$-curvatures.** For $N \geq 1$ (and $N \leq n/2$ if $n$ is even),

\begin{equation}
(-1)^N Q_{2N} = - \sum_{|I,a| = N \atop a < N} m_{(I,a)} (-1)^a P_{2I}(Q_{2a}) + N!(N - 1)! 2^{2N} W_{2N}.
\end{equation}

The explicit formula expresses $Q_{2N}$ in terms of the operators $\mathcal{M}_{2M}$ and the coefficients $W_{2a}$. The recursive formula expresses $Q_{2N}$ in terms of GJMS operators applied to $Q_{2a}$ with $a < N$, modulo the multiple of $W_{2N}$. Observe that the factor $n/2 - N$ in the definition which vanished in the critical case no longer appears. So the $Q$-curvature formulae do not follow immediately from the GJMS operator formulae just by taking constant terms.

If $g$ is Einstein or locally conformally flat, then there is an invariantly defined Poincaré metric to infinite order also if $n$ is even. It can be written explicitly; see [FG]. In these cases, $P_{2N}$ and $Q_{2N}$ are invariantly defined for all $N \geq 1$ also for $n$ even. For such $g$, Juhl’s formulae and our proofs are valid for all $N$.

The GJMS operators are known to be self-adjoint. This is exhibited by the formulae (1.5) and (1.6), since $\mathcal{M}(r)$ is evidently self-adjoint with respect to $g$ for each $r$ so that the $\mathcal{M}_{2N}$ are all self-adjoint, and since $n_I = n_{I-1}$ and $m_I = m_{I-1}$ where $I^{-1} = (I_r, \ldots, I_1)$. However, the self-adjointness is not obvious from the original GJMS construction. If one desires to understand Juhl’s formulae in terms of the original construction, it is reasonable to start by asking the modest question of how
to see the self-adjointness from that derivation. It turns out that understanding this is the key to unlocking the mysteries of Juhl’s formulae.

The operators $P_{2N}$ were derived in [GJMS] via the Laplacian of the ambient metric associated to $g$. In normal form, this is the metric

$$\tilde{g} = 2\rho dt^2 + 2tdtd\rho + t^2 g_\rho$$

on $\mathbb{R}_+ \times M \times (-\epsilon, \epsilon)$, where $t \in \mathbb{R}_+$, $\rho \in (-\epsilon, \epsilon)$, and $g_\rho = h_r$ with $\rho = -r^2/2$. The asymptotic vanishing of $\text{Ric}(g_\rho) + ng_\rho$ at $r = 0$ translates into asymptotic vanishing of $\text{Ric}(\tilde{g})$ at $\rho = 0$. See [FG] for details. If $f \in C^\infty(M)$ and $\tilde{f} \in C^\infty(M \times (-\epsilon, \epsilon))$ satisfies $\tilde{f}(x,0) = f$, then the GJMS definition is

$$P_{2N} f = \tilde{\Delta}^N(t^{N-n/2}\tilde{f})|_{\rho=0,t=1}$$

where $\tilde{\Delta}$ denotes the Laplacian in the metric $\tilde{g}$. The right-hand side is shown to be independent of the choice of $\tilde{f}$ extending $f$.

It is straightforward to calculate the Laplacian $\tilde{\Delta}$ of a metric of the form (1.10). Evidently there is a term involving the Laplacian $\text{Ric}(g_\rho)$ in the metric $g_\rho$ for fixed $\rho$ acting in the $M$ factor (see (2.1) below). This term is not self-adjoint with respect to $g = g_0$, so $P_{2N}$ obtained by iterating $\tilde{\Delta}$ and restricting to $\rho = 0$ does not appear to be self-adjoint either. This is the reason that self-adjointness of the $P_{2N}$ is not apparent from this construction. However, if we set

$$v(\rho) = \sqrt{\frac{\det g_\rho}{\det g_0}}$$

(so that $v(\rho) = V(r)$ with $\rho = -r^2/2$), then multiplying the volume form for $g$ by $v(\rho)$ gives the volume form for $g_\rho$. It follows that for each $\rho$ the operator $v(\rho)\Delta_{g_\rho}$ is self-adjoint with respect to $g$. Pre- and post-composing a self-adjoint operator with multiplication by a smooth real function gives another self-adjoint operator. Therefore the operator $v^{1/2} \circ \Delta_{g_\rho} \circ v^{-1/2}$ is also self-adjoint with respect to $g$. This motivates consideration of

$$\tilde{\Delta}_v := v^{1/2} \circ \tilde{\Delta} \circ v^{-1/2},$$

as we are guaranteed that the operator acting along $M$ when $\tilde{\Delta}_v$ is written out will be self-adjoint with respect to $g$. Moreover, $\tilde{\Delta}_v^N = v^{1/2} \circ \tilde{\Delta}_v^N \circ v^{-1/2}$ since the middle factors of $v^{\pm 1/2}$ cancel. The pre- and post-multiplications by $v^{\pm 1/2}$ affect neither the extension property nor the restriction back to $\rho = 0$ since $v = 1$ at $\rho = 0$. Hence (1.11) can be rewritten as

$$P_{2N} f = \tilde{\Delta}_v^N(t^{N-n/2}\tilde{f})|_{\rho=0,t=1}.$$

Now a direct calculation which we carry out in §2 shows that

$$\tilde{\Delta}_v(t^2\tilde{f}) = t^{\gamma-2} \left[-2\rho\partial_\rho^2 + (2\gamma + n - 2)\partial_\rho + \tilde{\mathcal{M}}(\rho)\right] \tilde{f},$$

where $\tilde{\Delta}_v$ is the Laplacian in the metric $\tilde{g}$.
where \( \widetilde{\mathcal{M}}(\rho) = \mathcal{M}(r) \), \( \rho = -r^2/2 \). This is the key identity. It explains the previously mysterious appearance of both \( W = \sqrt{V} \) and the generating function \( \mathcal{M}(r) \) in Juhl’s theory. Since 
\[
[-2\rho \partial^2_{\rho} + (2\gamma + n - 2)\partial_{\rho}] \rho^k = c_{k,\gamma,n} \rho^{k-1}
\]
for constants \( c_{k,\gamma,n} \), upon choosing \( \widetilde{f} \) to be independent of \( \rho \) we see that iterating (1.14) and restricting to \( \rho = 0, t = 1 \) gives a formula for \( P_{2N} \) as a linear combination of compositions of the Taylor coefficients of \( \widetilde{\mathcal{M}}(\rho) \), i.e. of the \( \mathcal{M}_{2I} \). Showing that the coefficients in the linear combination are the \( n_I \) reduces to a (rather nontrivial) combinatorial identity which we derive in \( \S 3 \). This proves Theorem 1.1. Theorem 1.3 reduces to an equivalent combinatorial identity upon calculating \( P_{2N} \) using (1.13), (1.14) and taking the extension \( \widetilde{f} \) to be \( v^{1/2} \) rather than 1. This reduction is included in \( \S 2 \) and the proof of the relevant combinatorial identity in \( \S 3 \).

Theorem 1.2 can be derived from Theorem 1.1 by inverting (1.5), viewed as a formal transformation law from the \( \mathcal{M}_{2N} \) to the \( P_{2N} \). A proof in the opposite direction due to Krattenthaler was presented in \( \S 2 \) of [J3] and immediately implies the direction we need here. Likewise, Theorem 1.4 follows from Theorem 1.3 upon inverting (1.8), viewed as a formal transformation from the \( W_{2N} \) to the \( Q_{2N} \). In \( \S 4 \) we review Krattenthaler’s proof of the inversion for the operators following the presentation in [J3] and then present the similar but more complicated proof for the \( Q \)-curvatures.

It is also possible to prove both the explicit and recursive formulae for \( Q \)-curvatures by taking the constant term in the corresponding formula for the GJMS operators and then rewriting by deriving and substituting expressions for the \( \mathcal{M}_{2N}(1) \). This approach is closely related to arguments in [J3], where the scalar invariants \( \mathcal{M}_{2N}(1) \) play a prominent role.

We still find these formulae to be quite astonishing. Juhl deserves great credit for their discovery as subtle consequences of the recursive structure of his residue families. Even though we now see that this theory of residue families and their factorization identities is not required for their proofs, this theory, linking ideas from conformal geometry, representation theory and spectral theory, appears deep and fascinating and deserves further exploration.

2. Explicit Formulae

In this section we give the details of the argument outlined in the introduction which reduces Theorem 1.1 to a combinatorial identity and then show how similar reasoning reduces Theorem 1.3 to an equivalent combinatorial identity.

The first task is to establish (1.14) by direct calculation. The inverse of the metric (1.10) is

\[
\tilde{g}^{IJ} = \begin{pmatrix}
0 & 0 & t^{-1} \\
0 & t^{-2}g_{ij} & 0 \\
t^{-1} & 0 & -2\rho t^{-2}
\end{pmatrix}
\]
and $\sqrt{|\det g|} = t^{n+1} \sqrt{|\det g_0|}$. Using this in

$$\bar{\Delta} = \frac{1}{\sqrt{|\det g|}} \partial_t \left( g^{1j} \sqrt{|\det g|} \partial_j \right)$$

gives

$$\bar{\Delta}(t^\gamma \varphi) = t^{\gamma-2} \left[ -2\rho \varphi'' + (2\gamma + n - 2 - 2\rho v'/v) \varphi' + (\Delta_{g_\rho} + \gamma v'/v) \varphi \right]$$

(cf. (3.5) of [GJMS]). Here $v$ is given by (1.12), $'$ denotes $\partial_\rho$, and $\varphi$ is independent of $t$. Set $w = v^{1/2}$ and $\varphi = w^{-1} \psi$. Then $v'/v = 2w'/w$ and

$$\begin{align*}
\varphi' &= w^{-1} \psi' - w^{-2} w' \psi \\
\varphi'' &= w^{-1} \psi'' - 2w^{-2} w' \psi' + (2w^{-3} w'^2 - w^{-2} w'') \psi.
\end{align*}$$

Substituting and simplifying gives

$$w \left[ -2\rho \varphi'' + (2\gamma + n - 2 - 2\rho v'/v) \varphi' + \gamma (v'/v) \varphi \right] = -2\rho w'' + (2\gamma + n - 2) \psi' + w^{-1} \left[ 2\rho w'' - (n - 2) w' \right] \psi.$$

For the remaining term in (2.1) we have

**Lemma 2.1.**

$$w \circ \Delta_{g_\rho} \circ w^{-1} = \delta(g_\rho^{-1} d) - w^{-1} \delta(g_\rho^{-1} dw).$$

(Recall that $\delta$ denotes the divergence operator with respect to $g = g_0$.) The second term on the right-hand side acts as a zeroth order operator.

**Proof.** For fixed $\rho$ it is clear that $w \circ \Delta_{g_\rho} \circ w^{-1}$ and $\delta(g_\rho^{-1} d)$ are second order differential operators whose principal parts agree. We observed in the introduction that the first is self-adjoint with respect to $g$, and clearly this is the case for the second. So their difference is zeroth order. Evaluating on $w$ identifies the zeroth order term. \qed

Multiplying (2.1) by $w$ and substituting (2.2) and (2.3) yields

$$w \bar{\Delta}(t^\gamma w^{-1} \psi) = t^{\gamma-2} \left[ -2\rho w'' + (2\gamma + n - 2 - 2\rho v'/v) \psi' + \left( \delta(g_\rho^{-1} d) - \bar{U}(\rho) \right) \psi \right]$$

where

$$\bar{U}(\rho) = \frac{\left[ -2\rho \partial_\rho^2 + (n - 2) \partial_\rho + \delta(g_\rho^{-1} d) \right] w(\rho)}{w(\rho)}.$$

The chain rule with $\rho = -r^2/2$ shows that $\bar{U}(\rho) = U(r)$ so that $\delta(g_\rho^{-1} d) - \bar{U}(\rho) = \bar{\mathcal{M}}(\rho)$. Hence (2.4) becomes (1.14). This completes the derivation of (1.14).

Set

$$R_k = -2\rho \partial_\rho^2 + 2k \partial_\rho + \bar{\mathcal{M}}(\rho)$$

and note that (1.2) becomes

$$\bar{\mathcal{M}}(\rho) = \sum_{N \geq 1} \mathcal{M}_{2N} \frac{1}{(N-1)!^2} \left( -\frac{\rho}{2} \right)^{N-1}.$$
Iterating (1.14) gives
\[ \tilde{\Delta}^N_v(t^{N-n/2} f) = t^{-N-n/2} \mathcal{R}_{1-N} \mathcal{R}_{3-N} \cdots \mathcal{R}_{N-3} \mathcal{R}_{N-1} f, \]
so we deduce that \( \mathcal{R}_{1-N} \mathcal{R}_{3-N} \cdots \mathcal{R}_{N-3} \mathcal{R}_{N-1} f \vert_{\rho=0} \) depends only on \( f \vert_{\rho=0} \). Taking \( f \) to be independent of \( \rho \), it follows upon expanding the right-hand side that \( \mathcal{R}_{1-N} \mathcal{R}_{3-N} \cdots \mathcal{R}_{N-3} \mathcal{R}_{N-1} \vert_{\rho=0} \) is a linear combination of the compositions \( \mathcal{M}_{2I} \). In the next section we will prove the combinatorial identity
\[ (2.5) \quad \mathcal{R}_{1-N} \mathcal{R}_{3-N} \cdots \mathcal{R}_{N-3} \mathcal{R}_{N-1} \vert_{\rho=0} = \sum_{|I|=N} n_I \mathcal{M}_{2I} \]
which identifies the constants in the linear combination. Theorem 1.1 then follows via (1.13).

We next show that Theorem 1.3, the explicit formula for \( Q \)-curvatures, reduces to a similar combinatorial identity which we will see in the next section is equivalent to (2.5). By definition we have \((-1)^N(n/2 - N)Q_{2N} = P_{2N1} \). Use (1.13) to calculate \( P_{2N1} \), taking \( f = v^{1/2} \) to be the extension of \( f = 1 \). Thus
\[ (-1)^N(n/2 - N)Q_{2N} = \tilde{\Delta}^{N-1}_v(t^{1/2}(t^{N-n/2})) \vert_{\rho=0,t=1}. \]
Equation (2.1) gives
\[ \tilde{\Delta}(t^{N-n/2}) = t^{N-n/2-2}(N-n/2)v'/v = 2t^{N-n/2-2}(N-n/2)w'/w. \]
The factors of \((N-n/2)\) cancel, and it follows that
\[ (2.6) \quad (-1)^N Q_{2N} = -2\tilde{\Delta}^{N-1}_v(t^{N-n/2-2} w') \vert_{\rho=0,t=1}. \]
Iterating (1.14) gives
\[ (2.7) \quad (-1)^N Q_{2N} = -2\mathcal{R}_{1-N} \mathcal{R}_{3-N} \cdots \mathcal{R}_{N-3} (w') \vert_{\rho=0}. \]
Now \( w = 1 + \sum_{a \geq 1} W_{2a}(-2\rho)^a \), so
\[ (2.8) \quad w' = \sum_{a \geq 1} a(-2)^a W_{2a} \rho^{a-1}. \]
As will be shown in the next section, the following is equivalent to (2.5).

**Proposition 2.2.** Let \( 1 \leq a \leq N \) and let \( f \) be a function on \( M \) (i.e. independent of \( \rho \)). Then
\[ (2.9) \quad \mathcal{R}_{1-N} \mathcal{R}_{3-N} \cdots \mathcal{R}_{N-3} (f \rho^{a-1}) \vert_{\rho=0} = \sum_{|I|=N-a} n_{(I,a)}(a-1)! t^2 (-2)^a - 1 \mathcal{M}_{2I}(f). \]
Substituting (2.8) into (2.7) and applying (2.9) termwise gives
\[ (-1)^N Q_{2N} = \sum_{(I,a)=N} n_{(I,a)} a!(a-1)! t^{2a} \mathcal{M}_{2I}(W_{2a}), \]
which is the explicit formula for \( Q_{2N} \).
For \(n\) even, the above argument applies also for the critical case \(N = n/2\) since \(Q_n\) is defined by removing the factor of \(n/2 - N\). The critical case may also be deduced without this argument of analytic continuation in the dimension by using the realization

\[
(-1)^{n/2}Q_n = -\Delta^{n/2}(\log t)|_{\rho=0, t=1}
\]
derived in [FH]. Namely, write

\[
\Delta^{n/2}(\log t)|_{\rho=0, t=1} = \Delta^{n/2-1}(w\Delta(\log t))|_{\rho=0, t=1}.
\]

Direct calculation gives \(w\Delta(\log t) = 2t^{-2}w'\). So we recover (2.6) and the argument proceeds as above.

### 3. Combinatorial Identities

In this section we derive the combinatorial identities (2.5) and (2.9) to which Theorems 1.1 and 1.3 were reduced above. Begin with (2.5). First change variables:

\[
(3.1) \quad s = -\frac{\rho}{2}, \quad x_N = \frac{M_{2N}}{(N-1)!^2}, \quad X(s) = \widetilde{M}(\rho) = \sum_{N=0}^{\infty} x_{N+1}s^N.
\]

As far as this identity is concerned, \(x_1, x_2, \ldots\) can simply be regarded as noncommuting variables, all of which commute with \(s\). In the new variables, the \(R_k\) become the differential operators

\[
L_k = s \frac{d^2}{ds^2} - k \frac{d}{ds} + X(s),
\]

where \(X(s)\) acts as a zeroth order multiplication operator. We only have to verify the constant term in \(\partial_{\rho}\) of (2.5), which becomes

**Theorem 3.1.** Let \(N \geq 1\). Then

\[
(3.2) \quad L_{1-N}L_{3-N}\cdots L_{N-3}L_{N-1}1|_{s=0} = \sum_{|I|=N} \tilde{n}_I x_{I_1}x_{I_2}\cdots x_{I_r},
\]

where

\[
\tilde{n}_I = \frac{(N-1)!^2}{\prod_{k=1}^{r-1} \left( \sum_{j=1}^k I_j \right) \left( \sum_{j=k+1}^r I_j \right)}.
\]

Set \(\tilde{L}_j = L_{N+1-j}\) so that \(L_{1-N}L_{3-N}\cdots L_{N-3}L_{N-1} = \tilde{L}_N\tilde{L}_{N-1}\cdots \tilde{L}_2\tilde{L}_1\). Since \(\tilde{n}_I = \tilde{n}_{I-1}\), (3.2) can be rewritten as

\[
(3.3) \quad \tilde{L}_N\tilde{L}_{N-1}\cdots \tilde{L}_2\tilde{L}_1|_{s=0} = \sum_{|I|=N} \tilde{n}_I x_{I_r}x_{I_{r-1}}\cdots x_{I_1}.
\]

Fix positive integers \(I_1, \ldots, I_r\), where \(r \geq 1\). We will prove (3.3) by verifying the coefficient of \(x_{I_r}x_{I_{r-1}}\cdots x_{I_1}\) in \(\tilde{L}_N\tilde{L}_{N-1}\cdots \tilde{L}_2\tilde{L}_1|_{s=0}\) for each choice of \(I_1, \ldots, I_r\).
For $1 \leq l \leq r$, set

$$m_l = I_1 + I_2 + \cdots + I_l$$

so that $1 \leq m_1 < m_2 < \cdots < m_{r-1} < m_r$. (The $m_l$ from the previous sections will not appear in this section.) Consider the calculation of $\hat{L}_N \hat{L}_{N-1} \cdots \hat{L}_2 \hat{L}_1$ by successive multiplication from the left. For $1 \leq j \leq N$, $\hat{L}_j \hat{L}_{j-1} \cdots \hat{L}_1$ is a formal power series in $s$ whose coefficients are polynomials in the $x$'s. The only monomials in the $x$'s appearing in $\hat{L}_j \hat{L}_{j-1} \cdots \hat{L}_1$ which can ultimately contribute to the coefficient of $x_{l_1}x_{l_{l_1-1}} \cdots x_{l_l}$ in $\hat{L}_N \hat{L}_{N-1} \cdots \hat{L}_2 \hat{L}_1$ are of the form $x_{l_1}x_{l_{l_1-1}} \cdots x_{l_l}$ for some $l$, $1 \leq l \leq r$. The term $sd^2/ds^2 - (N + 1 - 2k)ds/ds$ in one of the factors $\hat{L}_k$ just reduces the power of $s$ by 1 and multiplies by a constant. The term $X(s)$ is linear in the $x$'s. So in order for a monomial $x_{l_1}x_{l_{l_1-1}} \cdots x_{l_l}$ to appear in the expansion of $\hat{L}_j \hat{L}_{j-1} \cdots \hat{L}_1$, it must be that the zeroth order term $X(s)$ has contributed in exactly $l$ of these $\hat{L}_k$. Thus the differentiation in $s$ terms have contributed in exactly $j - l$ of the $\hat{L}_k$. It follows that the power of $s$ multiplying $x_{l_1}x_{l_{l_1-1}} \cdots x_{l_l}$ is $s^{m_l - l - (j - l)} = s^{m_l - j}$. Hence we have

$$\hat{L}_j \hat{L}_{j-1} \cdots \hat{L}_1 = \sum_{l=1}^{\min(j,r)} c_{j,l} x_{l_1}x_{l_{l_1-1}} \cdots x_{l_l} s^{m_l - j} + \cdots$$

for some constants $c_{j,l}$, where $\cdots$ indicates terms involving monomials in the $x$'s which cannot contribute in the end. The $c_{j,l}$ are defined for $1 \leq j \leq N$, $1 \leq l \leq \min(j,r)$, and we have $c_{1,1} = 1$ and $c_{j,l} = 0$ if $m_l < j \leq N$.

From (3.4) it follows first that the coefficient of $x_{l_1} \cdots x_{l_l}$ in $\hat{L}_N \hat{L}_{N-1} \cdots \hat{L}_2 \hat{L}_1|_{s=0}$ is zero unless $|I| = N$. In fact, taking $j = N$, the term $x_{l_1} \cdots x_{l_l}$ on the right-hand side is multiplied by $s^{m_r - N}$. This vanishes at $s = 0$ unless $m_r = N$, i.e., $|I| = N$. Theorem 3.1 therefore reduces to the statement that $c_{N,r} = \bar{n}_l$ if $|I| = N$. We assume henceforth that $|I| = N$, i.e., $m_r = N$.

Extend the definition of the $c_{j,l}$ to $0 \leq j \leq N$, $0 \leq l \leq r$ by defining $c_{0,0} = 1$ and $c_{j,l} = 0$ if $0 \leq j < l \leq r$ or if $l = 0$ and $1 \leq j \leq N$. We claim that these constants satisfy the recursion relation:

$$c_{j+1,l} = -(m_l - j)(N - m_l - j)c_{j,l} + c_{j,l-1}$$

for $0 \leq j \leq N - 1$, $1 \leq l \leq r$. For $1 \leq j \leq N - 1$ and $1 \leq l \leq \min(j + 1, r)$ this follows by applying $\hat{L}_{j+1}$ to (3.4). For $j = 0$, $l = 1$ both sides are 1, and for all the other values both sides vanish. Now extend the definition of the $c_{j,l}$ to $j > N$, $0 \leq l \leq r$ by setting $c_{j,0} = 0$ for $j > N$ and by requiring that (3.5) hold for $j \geq N$, $1 \leq l \leq r$. The resulting $c_{j,l}$ are defined for $j \geq 0$, $0 \leq l \leq r$, and (3.5) holds for $j \geq 0$, $1 \leq l \leq r$.

Define generating functions

$$F_l(y) = \sum_{j=0}^{\infty} \frac{c_{j,l}}{(j!)^2} y^j, \quad 0 \leq l \leq r.$$
The definitions of the \( c_{j,0} \) and \( c_{0,l} \) show that

\[
F_0 = 1 \quad \text{and} \quad F_l(0) = 0, \quad 1 \leq l \leq r.
\]

The recursion (3.5) turns into a differential equation relating \( F_l \) and \( F_{l-1} \). For fixed positive integral \( N \) as above, define ordinary differential operators

\[
\mathcal{D}_m = y(1+y) \frac{d^2}{dy^2} + [1 - (N-1)y] \frac{d}{dy} + m(N-m).
\]

**Lemma 3.2.** Let

\[
u_j = \sum_{j=0}^{\infty} \frac{u_j}{(j!)^2} y^j, \quad f = \sum_{j=0}^{\infty} \frac{f_j}{(j!)^2} y^j
\]

be formal power series. Then \( \mathcal{D}_m u = f \) if and only if

\[
u_{j+1} = -(m-j)(N-m-j)u_j + f_j, \quad j \geq 0.
\]

The proof is to substitute the expansions into the equation and to compare coefficients of like powers of \( y \). Comparing (3.5) and (3.8) then gives immediately

\[
\mathcal{D}_m F_l = F_{l-1}, \quad 1 \leq l \leq r.
\]

Now \( \mathcal{D}_m \) has a regular singularity at \( y = 0 \) with indicial root 0 of multiplicity 2. By general Frobenius theory or just by staring at (3.8), there exists a unique formal power series solution of \( \mathcal{D}_m u = 0 \) with \( u(0) = 1 \). Also, for any formal power series \( f \) there exists a unique formal power series solution \( u \) to \( \mathcal{D}_m u = f \) with \( u(0) = 0 \). In particular, (3.6) and (3.9) together characterize the functions \( F_l \). Combining the solutions of the homogeneous and inhomogeneous problems shows that for any \( f \) there is a unique solution \( u \) to \( \mathcal{D}_m u = f \) with \( u(0) \) any prescribed value.

Since the \( y^N \) coefficient of \( F_r(y) \) is \( c_{N,r}/(N!)^2 \), the above considerations show that the statement \( c_{N,r} = \bar{n}_I \) to which Theorem 3.1 reduced is a consequence of the following.

**Proposition 3.3.** Let \( r \geq 1 \) and \( 1 \leq m_1 < m_2 < \cdots < m_r = N \). Define formal power series \( F_l(y) \) for \( 0 \leq l \leq r \) by (3.6) and (3.9). Then \( F_r \) is a polynomial of degree \( N \) and its \( y^N \) coefficient is

\[
\left[ N^2 \prod_{i=1}^{r-1} m_i(N-m_i) \right]^{-1}.
\]

**Remarks.** It follows easily from the discussion below (or from the definition of the \( c_{j,l} \)) that \( F_l \) is a polynomial of degree \( \leq m_l \). For \( l < r \) it often happens that \( \deg F_l < m_l \). It is easily seen from the definition of the \( c_{j,l} \) (or from (3.5)) that the lowest power of \( y \) occurring in \( F_l \) with nonzero coefficient is \( y^{l} \), and its coefficient is 1.

We prove Proposition 3.3 by expressing the \( F_l(y) \) in terms of special solutions of the differential equations. Let \( P_m \) denote the formal power series defined by

\[
\mathcal{D}_m P_m = 0, \quad P_m(0) = 1.
\]
Then \( P_m = P_{N-m} \) since \( D_m = D_{N-m} \). Clearly \( P_0(y) = 1 \). We claim that if \( m \) is an integer satisfying \( 0 \leq m \leq N \), then \( P_m \) is a polynomial of degree \( \min(m, N-m) \). This is clear from (3.8) with \( f = 0 \) since the multiplicative factor first vanishes when \( j = \min(m, N-m) \). Up to a simple linear change of independent variable and overall multiplicative factor, the \( P_m \) are particular instances of Jacobi polynomials.

Next observe that the same reasoning applies if \( f \) is a polynomial of degree \(< \min(m, N-m) \): the unique solution \( u \) with \( u(0) \) any prescribed value is a polynomial of degree \( \leq \min(m, N-m) \). The multiplicative factor \((m-j)(N-m-j)\) also vanishes for \( j = \max(m, N-m) \). Again the same reasoning shows that if \( f \) is a polynomial of degree \(< \max(m, N-m) \), then \( u \) is a polynomial of degree \( \leq \max(m, N-m) \). In particular, if \( m \neq N/2 \) the conditions

\[
D_m Q_m = P_m, \quad Q_m(0) = 0
\]

uniquely determine a polynomial \( Q_m \) of degree \( \leq \max(m, N-m) \). Again \( Q_m = Q_{N-m} \).

In the special case \( m = 0 \), we have

**Lemma 3.4.** The \( y^N \) coefficient of \( Q_0 \) is \( N^{-2} \).

**Proof.** We have \( P_0 = 1 \). So (3.8) with \( j = 0 \) and \( u_0 = 0 \) gives \( u_1 = 1 \). Setting \( m = 0 \) and iterating (3.8) for higher \( j \) gives

\[
u_j = (j-1)! (N-j+1)(N-j+2) \cdots (N-1).
\]

Hence \( u_N = (N-1)!^2 \). The result now follows from (3.7). \( \square \)

**Proof of Proposition 3.3.** Begin by observing that the definition of the \( F_l \) and the conclusion both remain unchanged if any \( m_l \) is replaced by \( N - m_l \). We use this observation to redefine some of the \( m_l \). Namely, if \( 1 \leq l \leq r - 1 \) and \( m_l \) satisfies the two conditions that \( m_l > N/2 \) and for no \( k \) is it the case that \( m_k = N - m_l \), then we replace \( m_l \) by \( N - m_l \). The new sequence of \( m_l \) need no longer be increasing but that will be irrelevant; it suffices to prove the statement of the theorem with the \( F_l \) defined using these \( m_l \). It is still the case that all \( m_l \) are distinct, and we now have the property that if for some \( l \) one has \( m_l > N/2 \), then necessarily there is \( k < l \) for which \( N - m_l = m_k \).

For convenience, let us set \( m_0 = 0 \) and enlarge the set of \( m \)'s to include \( m_0 \). Then \( m_0 = 0 \) and \( m_r = N \) are both in our enlarged set of \( m \)'s, and now the property stated above that if \( m_l > N/2 \), then there is \( k < l \) for which \( N - m_l = m_k \) holds also for \( l = r \).

Define polynomials \( p_l, \ 0 \leq l \leq r, \) as follows:

\[
\begin{align*}
p_l &= P_{m_l} & \text{if} \ & m_l \leq N/2 \\
p_l &= Q_{m_l} & \text{if} \ & m_l > N/2.
\end{align*}
\]

Clearly \( \deg p_l \leq m_l \).
Claim: There are constants \( a_{j,l} \) for \( 0 \leq l \leq r \), \( 0 \leq j \leq l \), satisfying

\[
F_l = \sum_{j=0}^{l} a_{j,l} p_j, \quad 0 \leq l \leq r
\]  

(3.10)

\[ a_{0,l} = \left[ \prod_{j=1}^{l} m_j(N - m_j) \right]^{-1}, \quad 0 \leq l \leq r - 1 \]  

(3.11)

\[ a_{r,r} = \left[ \prod_{l=1}^{r-1} m_l(N - m_l) \right]^{-1}. \]  

(3.12)

In (3.11) and (3.12) an empty product is interpreted as 1.

Proposition 3.3 follows immediately from the Claim. In fact, all \( p_j \) for \( 0 \leq j \leq r - 1 \) have degree \(< N\) and \( p_r = Q_0 \) has degree \( = N \) by Lemma 3.4. Thus (3.10) for \( l = r \) together with (3.12) show that \( F_r \) has degree \( = N \). Only \( p_r = Q_0 \) contributes to its \( y^N \) coefficient, which by Lemma 3.4 is \( N^2 a_{r,r} \).

The Claim is proved by induction on \( l \). It is clear for \( l = 0 \) since \( F_0 = p_0 = 1 \). Suppose that the Claim is established for \( l - 1 \) and assume first that \( l < r \). The argument is slightly different for the last induction step passing from \( l = r - 1 \) to \( l = r \).

Now \( F_l \) is defined by

\[
D_{m_l} F_l = F_{l-1} = \sum_{j=0}^{l-1} a_{j,l-1} p_j, \quad F_l(0) = 0.
\]

For each \( j \), \( 0 \leq j \leq l - 1 \), we will solve \( D_{m_j} u_j = p_j \), \( u_j(0) = 0 \), with \( u_j \) a linear combination of the \( p_k \), \( 0 \leq k \leq l \). Then \( F_l = \sum_{j=0}^{l-1} a_{j,l-1} u_j \) is of the desired form.

The construction of the \( u_j \)'s is based on the observation

\[
D_{m_l} = D_{m_j} + [m_l(N - m_l) - m_j(N - m_j)].
\]

(3.13)

Consider different cases for \( j \). If \( m_j \leq N/2 \) and \( m_j(N - m_j) \neq m_l(N - m_l) \), then \( p_j = P_{m_j} \) solves \( D_{m_j} p_j = 0 \). Hence (3.13) gives

\[
D_{m_l} \left( [m_l(N - m_l) - m_j(N - m_j)]^{-1} p_j \right) = p_j.
\]

Correct the value at \( y = 0 \) by subtracting a multiple of the solution of the homogeneous equation: set

\[ u_j = [m_l(N - m_l) - m_j(N - m_j)]^{-1} (p_j - P_{m_l}). \]

Clearly \( u_j \) solves the equation and the initial condition. Now \( P_{m_l} \) is of the form \( p_k \) for some \( k \) with \( 1 \leq k \leq l \); if \( m_l \leq N/2 \) then \( P_{m_l} = p_l \), while if \( m_l > N/2 \), then \( P_{m_l} = p_k \), where \( k < l \) is the index such that \( N - m_l = m_k \). Thus we have constructed \( u_j \) of
the desired form in this case. Note that if \( j = 0 \), then \( m_j = 0 \) and our solution is 
\[ u_0 = [m_l(N - m_l)]^{-1}(p_0 - P_{m_l}). \]
The coefficient of \( p_0 \) is \([m_l(N - m_l)]^{-1}\), and \( p_0 \) has coefficient zero when any of the \( u_j \) with \( j > 0 \) is expressed as a linear combination of the \( p_i \).

Next consider the construction of \( u_j \) in case \( m_j \leq N/2 \) but \( m_j(N - m_j) = m_l(N - m_l) \). This case might not occur at all, and if it does it can occur for only one \( j \). Since \( j < l \) we have \( m_j \neq m_l \), so it must be that \( m_l > N/2 \) and \( m_j = N - m_l \). Therefore \( p_j = P_{m_j} \) and \( p_l = Q_{m_l} \). Since \( D_{m_l}Q_{m_l} = P_{m_l} = P_{m_j} \) and \( Q_{m_l}(0) = 0 \), we just take \( u_j = Q_{m_l} = p_l \). \( p_0 \) does not occur in the expression of this \( u_j \) as a linear combination of the \( p_i \).

The remaining possibility is \( m_j > N/2 \). Now we need to solve \( D_{m_l}u_j = p_j = Q_{m_l} \). Once again we apply (3.13) to conclude that

\[ D_{m_l}Q_{m_l} = D_{m_j}Q_{m_j} + [m_l(N - m_l) - m_j(N - m_j)]Q_{m_j} = P_{m_j} + [m_l(N - m_l) - m_j(N - m_j)]Q_{m_j}. \]

Since \( j < l \) it is impossible that \( m_l = N - m_j \). Therefore \( m_l(N - m_l) - m_j(N - m_j) \neq 0 \). Arguing exactly as in the first case above we conclude that we can solve \( D_{m_j}v_j = P_{m_j} \), \( v_j(0) = 0 \), with \( v_j \) a linear combination of the \( p_k \) for \( 1 \leq k \leq l \). Then we take

\[ u_j = [m_l(N - m_l) - m_j(N - m_j)]^{-1}(Q_{m_j} - v_j) = [m_l(N - m_l) - m_j(N - m_j)]^{-1}(p_j - v_j). \]

Once again, \( p_0 \) has coefficient zero when \( u_j \) is expressed as a linear combination of the \( p_i \).

This concludes the induction step for \( l < r \): \( F_l = \sum_{j=0}^{l-1} a_{j,l-1}u_j \) is of the desired form. Since \( p_0 \) only entered in the construction of \( u_0 \), and its coefficient in \( u_0 \) was \([m_l(N - m_l)]^{-1}\), we have

\[ a_{0,l} = [m_l(N - m_l)]^{-1}a_{0,l-1}. \]

Thus (3.11) follows by induction as well.

Finally consider the last inductive step, passing from \( r - 1 \) to \( r \). Now \( m_l = N \) so \( N - m_l = m_0 = 0 \). We again divide \( \{ j : 0 \leq j \leq r - 1 \} \) into the same three cases as above and solve for the \( u_j \) using the same methods. The difference now is that \( j = 0 \) occurs in the second case instead of the first, since \( m_0(N - m_0) = m_r(N - m_r) \). So \( u_0 = Q_0 = p_r \). In no other \( u_j \) does \( p_r \) occur with nonzero coefficient. From \( F_r = \sum_{j=0}^{r-1} a_{j,r-1}u_j \) we therefore deduce \( a_{r,r} = a_{0,r-1} \), which gives (3.12). \( \square \)

This completes the proof of Theorem 3.1 and thus of (2.5). It remains to prove Proposition 2.2. It is evident upon expanding the \( R_k \)’s that the left-hand side of (2.9) is a linear combination of \( M_{2l}(f) \). Again make the change of variables (3.1). Then
(2.9) becomes
\[ L_{1-N}L_{3-N} \cdots L_{N-3}(s^{a-1})|_{s=0} = \sum_{|I|=N-a} n_{(I,a)}(a-1)^2(I_1-1)^2 \cdots (I_r-1)^2 x_I \]
\[ = \sum_{|I|=N-a} \tilde{n}_{(I,a)} x_I. \]
But this is equivalent to (3.2), which stated
\[ L_{1-N}L_{3-N} \cdots L_{N-3}L_{N-1}|_{s=0} = \sum_{|J|=N} \tilde{n}_{J} x_J = \sum_{|I|=N} \tilde{n}_{(I,a)} x_I x_a, \]
as one sees upon evaluating \( L_{N-1}1 = X(s) = \sum_{a \geq 1} x_a s^{a-1} \).

4. Recursive Formulæ

In this section we present the proofs of Theorems 1.2 and 1.4. First consider Theorem 1.2. Since \( n_{(N)} = 1 \), (1.5) can be written as
\[ P_{2N} = M_{2N} + \sum_{|I|=N, I \neq (N)} n_{I} M_{2I}. \]
The second term on the right-hand side only involves \( M_{2M} \) with \( M < N \). Thus this is a polynomial lower-triangular system, and it follows that there are constants \( a_I \) determined inductively by inverting this relation so that
\[ M_{2N} = P_{2N} + \sum_{|I|=N, I \neq (N)} a_{I} P_{2I}. \]
Observe that (1.6) is another relation of this same form. §2 of [J3] presents a proof due to Krattenthaler that (1.5) and (1.6) are inverse relations in the other direction. Specifically, Krattenthaler showed that if \( M_{2N} \) are defined by
\[ \bar{M}_{2N} = \sum_{|I|=N} m_{I} P_{2I}, \]
then
\[ P_{2N} = \sum_{|I|=N} n_{I} \bar{M}_{2I}. \]
Our desired identity (1.6) follows from the uniqueness of the inverse. Concretely, from (4.2) one deduces \( \bar{M}_{2N} = P_{2N} + \sum_{|I|=N, I \neq (N)} a_{I} P_{2I} \) by precisely the same inductive inversion as for the \( M_{2N} \). Hence \( \bar{M}_{2N} = M_{2N} \), and (1.6) follows.

We review Krattenthaler’s proof of (4.2) as presented in §2 of [J3] as a warm-up for the proof of Theorem 1.4. Substitution of (4.1) into (4.2) shows that (4.2) is equivalent to
\[ P_{2N} = \sum_{|I|=N} \sum_{|J_1|=I_1, \ldots, |J_r|=I_r} n_{I} m_{J_1} \cdots m_{J_r} P_{2J_1} \cdots P_{2J_r}. \]
The coefficient of \( P_{2N} \) on the right-hand side is 1, so one is reduced to showing that for \( K = (K_1, \ldots, K_s) \) with \( s > 1 \), the coefficient of \( P_{2K} \) in (4.3) vanishes. Given \( K \), the choice of \( J \)'s corresponds to a choice of subset \( A = \{a_1, \ldots, a_{r-1}\} \) of
[s − 1] = \{1, \ldots, s − 1\} (including the empty set) of cardinality r − 1, which we order by 1 ≤ a_1 < a_2 < \ldots < a_{r−1} ≤ s − 1. The parameterization is

\begin{align*}
J_1 &= (K_1, \ldots, K_{a_1}), & J_2 &= (K_{a_1+1}, \ldots, K_{a_2}), & \ldots, \\
J_{r−1} &= (K_{a_{r−2}+1}, \ldots, K_{a_{r−1}}), & J_r &= (K_{a_{r−1}+1}, \ldots, K_s).
\end{align*}

(4.4)

The J’s determine I by \(I = (|J_1|, \ldots, |J_r|)\). The coefficient of \(P_{2K_1} \cdots P_{2K_s}\) is then

\begin{equation}
\sum_{A \subset [s−1]} n_I m_{J_1} \cdots m_{J_r},
\end{equation}

(4.5)

so (4.2) reduces to showing that this vanishes for all \((K_1, \ldots, K_s)\) with \(s > 1\).

Sums such as (4.5) can be evaluated using the following ingenious lemma of Kratenthaler.

**Lemma 4.1.** Let \(s > 1\) and let \(K_1, \ldots, K_s \in \mathbb{N}\). Set \(|K| = \sum_{j=1}^s K_j\). For \(A = \{a_1, \ldots, a_{r−1}\} \subset [s−1]\), define \(J_1, \ldots, J_r\) and \(I\) as above. Then

\begin{equation}
\sum_{A \subset [s−1]} (-1)^r I_1 \cdots I_{r−1}(I_r + X) \cdot \frac{\prod_{a \in A}(K_a + K_{a+1} + Y\chi(a = s − 1))}{\prod_{i=1}^{r−1}(\sum_{k=1}^i I_k)(\sum_{k=i+1}^r I_k)}
\end{equation}

(4.6)

\[= \frac{X(|K| - K_s) + Y(K_s + X)}{|K| - K_1}.\]

Here \(\chi(S) = 1\) if \(S\) is true and \(\chi(S) = 0\) otherwise. \(X\) and \(Y\) are formal variables; the identity holds as polynomials in \(X\) and \(Y\).

This is Lemma 2.1 in [J3]. The proof is by induction on \(s\), decomposing the set of subsets \(A \subset [s]\) according to their last element. The proof is not at all obvious, but the real ingenuity was to introduce the variables \(X\) and \(Y\) and to find the identity (4.6) amenable to a proof by induction. For the purposes of this paper it suffices to know (4.6) in the case \(X = Y\). An examination shows that the proof by induction actually applies to this case directly; it is not necessary for our purposes to introduce both independent variables \(X\) and \(Y\). We rewrite the identity for the case \(X = Y\) in the form we will need it in the proof of Theorem 1.4 Setting \(X = Y = −b\) and replacing \(K_s\) by \(K_s + b\) in (4.6) gives

\begin{equation}
\sum_{A \subset [s−1]} (-1)^r I_1 \cdots I_r \frac{\prod_{a \in A}(K_a + K_{a+1})}{\prod_{i=1}^{r−1}(\sum_{k=1}^i I_k)(\sum_{k=i+1}^r I_k + b)} = \frac{b|K|}{|K| - K_1 + b}.
\end{equation}

(4.7)

This holds also for \(s = 1\), since in that case both sides are \(-K_1\). As usual, empty products are interpreted as 1. The form (4.7) seems natural: the induction hypothesis arises naturally in its proof by induction and the function \(\chi(a = s−1)\) does not appear.
We use Lemma 4.1 to finish the proof of (4.2). Substitution of the definitions of \( n_I \) and the \( m_{J_i} \) into (4.3) shows that
\[
\sum_{A \subseteq \{s-1\}} n_{I}m_{J_1} \cdots m_{J_r} = (-1)^s(|K| - 1)! \prod_{j=1}^{s} \frac{1}{K_j!} \frac{1}{K_j + K_{j+1}} \cdot \Sigma,
\]
where \( \Sigma \) is the expression occurring on the left-hand side of (4.6) with \( X = Y = 0 \). Lemma 4.1 (or (4.7) with \( b = 0 \)) shows that this vanishes. Thus (4.2) follows, and hence also Theorem 1.2.

We turn now to the proof of Theorem 1.4. Recall that the scalar invariants \( W_{2N} \) are defined by (1.3). It will be convenient to introduce
\[
\mathcal{W}_{2N} = 2^{2N} N!(N - 1)! W_{2N}, \quad N \geq 1
\]
so that (1.8) takes the form
\[
(-1)^N Q_{2N} = \sum_{(|I,b)|=N} n_{(I,b)} \mathcal{M}_{2I}(\mathcal{W}_{20})
\]
and (1.9) becomes
\[
\mathcal{W}_{2N} = \sum_{(|L,d)|=N} \sum_{(|I,b)|=d} m_{(L,d)}(-1)^d P_{2L}(Q_{2d}).
\]
Substitution of (4.8) for each \((-1)^d Q_{2d}\) shows that (1.9) is equivalent to
\[
\mathcal{W}_{2N} = \sum_{(|L,d)|=N} \sum_{(|I,b)|=d} m_{(L,d)} n_{(I,b)} P_{2L} \mathcal{M}_{2I}(\mathcal{W}_{20}).
\]
The term on the right-hand side with \( L = I = \emptyset \) is \( \mathcal{W}_{2N} \), so it suffices to prove
\[
\sum_{(|L,d)|=N} \sum_{(|I,b)|=d} m_{(L,d)} n_{(I,b)} P_{2L} \mathcal{M}_{2I} = 0
\]
for each fixed \( b \) such that \( 1 \leq b < N \). Substitution of (1.6) for each \( \mathcal{M}_{2I_j} \) rewrites this as
\[
\sum_{(|L,d)|=N} \sum_{(|I,b)|=d} \sum_{|J_1|=I_1, \ldots, |J_r|=I_r} m_{(L,d)} n_{(I,b)} m_{J_1} \cdots m_{J_r} P_{2L} P_{2J_1} \cdots P_{2J_r} = 0.
\]
Fix \( K_1, \ldots, K_s \) with \( s \geq 1 \) and each \( K_j \geq 1 \) and consider the coefficient of \( P_{2K_1} \cdots P_{2K_s} \) in (4.9). We must have \( L = (K_1, \ldots, K_p) \) for some \( p, 0 \leq p \leq s \). Each \( |J_i| \geq 1, \) although \( r = 0 \) is allowed corresponding to \( p = s \). For \( p < s \), the choice of \( J_i \)'s corresponds to a choice of subset \( A = \{a_1, \ldots, a_{r-1}\} \) of \( [s - p - 1] = \{1, \ldots, s - p - 1\} \) (including the empty set) of cardinality \( r-1 \), which we order by \( 1 \leq a_1 < a_2 < \ldots < a_{r-1} \leq s - p - 1 \). Here
\[
J_1 = (K_{p+1}, \ldots, K_{p+a_1}), \quad J_2 = (K_{p+a_1+1}, \ldots, K_{p+a_2}), \ldots,
J_{r-1} = (K_{p+a_{r-2}+1}, \ldots, K_{p+a_{r-1}}), \quad J_r = (K_{p+a_{r-1}+1}, \ldots, K_s).
\]
For \( p = s - 1 \), the only possibility for \( A \) is the empty set, in which case \( J_1 = (K_s) \). The \( J \)'s determine \( I \) by \( I = (|J_1|, \ldots, |J_r|) \) as above. The coefficient of \( P_{2K_1} \cdots P_{2K_s} \) is then

\[
(4.11) \quad m_{(K,b)} + \sum_{p=0}^{s-1} m_{(L,|K|-|L|+b)} \sum_{A \subset [s-p-1]} n_{(I,b)}m_{J_1} \cdots m_{J_r}.
\]

So Theorem 1.4 reduces to showing that this vanishes for all \( b \geq 1 \) and all \((K_1, \ldots, K_s)\) with \( s \geq 1 \).

We use Lemma 4.1 in the form (4.7) to evaluate the inner sum. Set \( K'_j = K_{p+j} \) for \( 1 \leq j \leq s - p \). Substitution of (1.4) for \( n_{(I,b)} \) and the \( m_{J_i} \) shows that

\[
(4.12) \quad \sum_{A \subset [s-p-1]} n_{(I,b)}m_{J_1} \cdots m_{J_r}
= (-1)^{s-p}(\frac{|K'| + b - 1}{|K'|})! \prod_{j=1}^{s-p} \frac{1}{K'_j!(K'_j - 1)!} \prod_{j=1}^{s-p-1} \frac{1}{K'_j + K'_{j+1}} \cdot \Sigma
\]

where

\[
\Sigma = \sum_{A \subset [s-p-1]} (-1)^r I_1 \cdots I_r \frac{\prod_{a \in A} (K'_a + K'_{a+1})}{\prod_{l=1}^{r-1} (\sum_{k=1}^{l} I_k)(\sum_{k=l+1}^{r} I_k + b)}.
\]

Replacement of \( s \) by \( s - p \) and \( K_j \) by \( K'_j \) in (4.7) shows that

\[
(4.13) \quad \Sigma = -\frac{b|K'|}{|K'|-K_{p+1} + b}.
\]

Substitute (4.13) into (4.12) and multiply by \( m_{(L,|K|-|L|+b)} \). One obtains

\[
(4.14) \quad m_{(L,|K|-|L|+b)} \sum_{A \subset [s-p-1]} n_{(I,b)}m_{J_1} \cdots m_{J_r}
= (-1)^{s+1}(\frac{|K| + b}{|K|})!(\frac{|K| + b - 1}{|K|})! \prod_{j=1}^{s} \frac{1}{K_j!(K_j - 1)!} \prod_{j=1}^{s-1} \frac{1}{K_j + K_{j+1}} \cdot R_p
\]

where

\[
R_p = \frac{K_p + K_{p+1}}{(\sum_{i=p}^{s} K_i + b)(\sum_{i=p+1}^{s} K_i + b)(\sum_{i=p+2}^{s} K_i + b)}, \quad 1 \leq p \leq s - 1
\]

and

\[
R_0 = \frac{1}{(\sum_{i=1}^{s} K_i + b)(\sum_{i=2}^{s} K_i + b)}.
\]

Empty sums are interpreted as 0.
Set \( b = K_{s+1} \) and substitute (4.14) into (4.11). After cancellation of factors in common with \( m(K, b) \), one finds that the vanishing of (4.11) is equivalent to
\[
\sum_{p=1}^{s-1} \frac{K_p + K_{p+1}}{(\sum_{i=p}^{s+1} K_i)(\sum_{i=p+1}^{s+1} K_i)(\sum_{i=p+2}^{s+1} K_i)} = \frac{1}{K_{s+1}} - \frac{1}{(\sum_{i=1}^{s+1} K_i)(\sum_{i=2}^{s+1} K_i)}.
\]
This is proved by induction on \( s \). For \( s = 1 \) the sum on the left-hand side is empty and the right-hand side vanishes. Suppose the identity holds for \( s \). Write
\[
\sum_{p=2}^{s} \frac{K_p + K_{p+1}}{(\sum_{i=p}^{s+2} K_i)(\sum_{i=p+1}^{s+2} K_i)(\sum_{i=p+2}^{s+2} K_i)} = \frac{K_1 + K_2}{(\sum_{i=1}^{s+2} K_i)(\sum_{i=2}^{s+2} K_i)(\sum_{i=3}^{s+2} K_i)} + \sum_{p=2}^{s} \frac{K_p + K_{p+1}}{(\sum_{i=p}^{s+2} K_i)(\sum_{i=p+1}^{s+2} K_i)(\sum_{i=p+2}^{s+2} K_i)}
\]
and use the induction hypothesis on the second term on the right-hand side to obtain that the above equals
\[
\frac{K_1 + K_2}{(\sum_{i=1}^{s+2} K_i)(\sum_{i=2}^{s+2} K_i)(\sum_{i=3}^{s+2} K_i)} + \frac{1}{K_{s+2}(K_{s+1} + K_{s+2})} \left( \frac{1}{(\sum_{i=2}^{s+2} K_i)(\sum_{i=3}^{s+2} K_i)} - \frac{1}{(\sum_{i=1}^{s+2} K_i)(\sum_{i=2}^{s+2} K_i)} \right) = \frac{1}{K_{s+2}(K_{s+1} + K_{s+2})} - \frac{1}{(\sum_{i=2}^{s+2} K_i)(\sum_{i=3}^{s+2} K_i)}.
\]
This completes the proof of the vanishing of (4.11) and thus also of Theorem 1.4.

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