A unified algorithm for colouring graphs of bounded clique-width

Bruno Courcelle 1, Irène Durand 2, and Michael Raskin * 3

1University of Bordeaux, bruno.courcelle@u-bordeaux.fr, ORCID: 0000-0002-5545-8970

2University of Bordeaux, irene.durand@u-bordeaux.fr, ORCID: 0000-0002-5171-7234

3Technical University of Munich, raskin@{in.tum.de,mccme.ru}, ORCID: 0000-0002-6660-5673

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Abstract

Clique-width is one of the graph complexity measures leading to polynomial special-case algorithms for generally NP-complete problems, e.g. graph colourability. The best two currently known algorithms for verifying $c$-colourability of graphs represented as clique-width terms are optimised towards two different extreme cases, a constant number of colours and a very large number of colours. We present a way to unify these approaches in a single relatively simple algorithm that achieves the state of the art complexity in both cases. The unified algorithm also provides a speed-up for a large number of colours.

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1 Introduction

Clique-width is one of the graph complexity measures leading to polynomial special-case algorithms for generally NP-complete problems. Roughly speaking, clique-width considers binary trees with the vertices of the original graph being the leaves, and measures how many different kinds of vertices can there be in some subtree from the point of view of edges to the vertices outside the subtree. Although computing the clique-width of a graph is NP-hard [4], one might have a good enough clique-width representation of a graph, be it from heuristics (like those implemented in TRAG [3]), algorithms with weaker guarantees [1], special algorithms for restricted graph classes (like graphs with few $P_4$ subgraphs [9]), or even a precise computation (using, for example, SAT solvers [6]) as a per-graph investment to be reused for computing multiple properties of the graph.

One of NP-complete graph problems is graph colourability. While there are multiple notions of graph colouring, in the present paper we consider only assignments of colours to vertices with the ends of each edge having different colours. There are multiple known results on complexity of $c$-colourability for graphs with a known decomposition witnessing a "low" clique-width.

Most of them are based on a natural refinement of the algorithm implied by Courcelle’s theorem for colourability with a fixed number of colours; the same algorithm is sometimes described as a straightforward dynamic programming approach. Specifically, these algorithms traverse the subterms of a clique-width term (a formal representation of a witness of low clique-width) from the leaves towards the root, and for each subterm compute the set of possible colourings of the subgraph corresponding to the subterm. The vertices sharing a label are tracked together. For a fixed clique-width and a fixed number of colours the time per node of the syntax tree of the term is constant.

From the point of view of number of colours there are two limit cases. On
the one hand, the colourability problem is already NP-complete for 3 colours. On the other hand, when computing the chromatic number in the general case, the number of colours could in principle be as large as \( n \), and one could want to limit the impact of the number of colours on complexity. Both these cases have been studied; a fine-grained tight (assuming exponential time hypothesis) bound has been obtained in the former case [8], and an algorithm polynomial in \( n \) for each fixed clique-width (with a degree of the polynomial exponential in the number of colours) has been obtained in the latter [7, 10]. For the latter case there is also a lower bound assuming exponential type hypothesis [5]. However, the algorithms for these two cases are presented from different points of view. This follows naturally from the number of labels being much larger than the number of colours in the one case but much smaller in the other. Thus it remained unclear whether one needs to choose which of the two approaches to use in advance.

In the present paper we describe both sets of optimisations inside a unified framework which allows their simultaneous application. Reformulating the optimisations from the literature and applying them at once yields a single relatively simple algorithm verifying \( c \)-colourability, that naturally achieves the state-of-the-art complexity in both extreme cases. The reformulation is not purely formal as care needs to be taken to make sure each optimisation does not lead to overhead outside of the original area of application. We believe that a uniform representation of the two optimisations side by side makes their corresponding core ideas clearer. The simultaneous application of the optimisations leads to a moderate improvement to the state-of-the-art complexity in case of large number of colours, reducing the constant factor in the degree of polynomial (which is itself exponential in the number of colours).

The rest of the paper is organised as follows. In the next section we remind the definitions related to clique-width decomposition of graphs. In
the following section we present our main result and discuss its relation to the previously known results. Next we describe the details of the algorithm. The following two sections contain the proof of correctness and complexity bounds. We finish with a brief conclusion.

2 Graphs and clique-width

Graphs are finite, undirected without loops and multiple edges. A vertex may have a label in \([k] := \{1, \ldots, k\}\) (the graph is \(k\)-labelled) and/or a colour in \([c]\) (the graph is \(c\)-coloured). Of course the choice of specific finite sets of labels and colours only matters when talking about specific graphs and specific colourings, i.e. in the examples.

Certain \(c\)-coloured labeled graphs can be constructed from basic graphs \(a\) where \(a \in [k]\), and the following operations:

- \(\oplus\) constructing the union of two disjoint graphs,
- the unary operation \(\text{add}_{a,b}\) for \(a, b \in [k], a < b\) that adds an edge between each \(a\)-labelled vertex and each \(b\)-labelled vertex (unless they are adjacent),
- the unary operation \(\text{relab}_{a\rightarrow b}\) changes every vertex label \(a\) into \(b\).

The type of \(G\), denoted by \(\pi(G)\), is the set of labels of its vertices.

From these operation symbols, we can build terms that denote \(k\)-labelled graphs. When discussing a term and the graph it represents, we often want to refer to specific vertices regardless of their labels, which can coincide with each other or vary depending on the subterm in question. In this case we use the notation \(1(x)\) where \(x\) is an arbitrary vertex name. We expect vertex names to be unique. An example is:

\[
t = \text{add}_{1,2}(\text{relab}_{3\rightarrow 1}(\text{add}_{1,3}(1(w) \oplus 3(w')) \oplus 2(x) \oplus \text{add}_{1,3}(1(y) \oplus 3(z))))
\]
that denotes the graph $G$ obtained from the path $w - w' - x - y - z$ by addition of the edge $w - x$, where the labels of $w, w', x, y, z$ are respectively $1, 1, 2, 1, 3$ in the nullary symbols that create them. We call such terms \textit{clique-width terms}, but as we do not consider any other kinds of terms, we omit the “clique-width” in the present paper. Each term denotes a vertex labelled graph $\text{val}(t)$ whose vertices are those specified by the nullary symbols of $t$. (No two occurrences of nullary symbols denote the same vertex.) The term’s \textit{width} is the number of labels that occur in $t$.

Using a standard convention, we will denote in the same way a function symbol and the graph operation it defines. Hence, $\text{relab}_{a\to b}(t)$ is a term if $t$ is a term and $\text{relab}_{a\to b}(G)$ denotes a vertex labelled graph if $G$ denotes a vertex labelled graph.

We denote by $t/u$ the \textit{subterm of $t$ issued from position $u$}. In the above example of term $t$, we have $t/u = 1(w) \oplus 3(w')$ if $u$ is the first position that is an occurrence of $\oplus$. Since no nullary symbol has two occurrences in a term, for any two positions $u$ and $u'$ in any term $t$, we have $t/u \neq t/u'$.

The \textit{clique-width} of a graph $G$ without labels, denoted by $\text{cwd}(G)$, is the least width of a term $t$ that denotes some vertex labelling of $G$. Such a term is said to be \textit{optimal}.

We will consider algorithms about graphs that are given by defining terms. The time computations of these algorithms will depend on the widths of the input terms. It is thus better to specify the input graphs by optimal terms (see however the conclusion). But, deciding whether $\text{cwd}(G) \leq k$ is true is an NP-complete problem (if $k$ is part of the input \cite{4}).

### 3 Main result

\textbf{Theorem 1.} Given a clique-width term of width $k$ and $m$ operation symbols (including nullary ones), $c$-colourability of $G$ can be checked in time $O(m \times \min((c + 1)^{2k} \times 2^{2k} \times \log c, (2^c - 2)^k \times c^3) \times k^3)$. A more precise formula is
Remark 1. For \( c = 3 \) we obtain \( O(m \times 6^k \times k^3) \), matching the \( O^*(2^c-2)^k \) bound [8]. Here \( O^*() \) denotes \( O() \) up to factors polynomial in \( k, c, \) and \( n \). For \( c = \Theta(n) \) we obtain \( O(m \times (n+1)^{2k} \times 2^{2k} \times k^3) \), matching the \( n^{O(2^k)} \) bound [10]. Moreover, the algorithm provided in [10] contains an operation with worst-case complexity \( \Omega(n^{3-2k}) \). We obtain better worst case complexity by generalising an optimisation from [8].

Remark 2. If a graph has \( n \) vertices, a representing term has \( n \) nullary operations for vertex addition and \( n - 1 \) union operations. Furthermore, between two union operations or after the last union operation, it is easy to optimise the unary operations to first add edges in \( O(k^2) \) operations, then relabel in \( O(k) \) operations, as after any relabelling there is an unused label that can be used as a buffer. Thus \( m = O(n \cdot k^2) \), and also \( m - n = \Omega(n) \).

Our proof is based on an algorithm using the optimisations from the cited constructions. On the base level, the algorithm enumerates the possible configurations of colours corresponding to labels, computing this for all the subterms from the nullary symbols up to the entire term. The graph is colourable if we find any valid configuration for the entire term.

There are two main optimisations for the low colour count case [8]. The first optimisation is faster enumeration of colourings of a union of two graphs by considering all overapproximations of the sets of used colours for each label. This allows to look for identical label-colour relations for the two subgraphs instead of choosing compatible pairs of relations. The complexity of the operation becomes linear in the number of possible label-colour relations, instead of quadratic or cubic with other approaches. Unlike [8], we implement this approach in a way compatible with large numbers of colours. The second optimisation is based on lookahead, namely computing which edge addition operations outside a subterm affect the vertices corresponding to the nullary symbols inside the term. We call such additional small
pieces of data computed for each subterm before starting the main part of
the algorithm annotations, following [2]. The algorithm uses the fact that
if the vertices with label $a$ will get connected to some other vertices, there
must be at least one colour that is not used for any vertex with the label $a$.

The main optimisation for the case of a large number of colours is iden-
tification of colourings differing only in permutation of colours. It is quite
similar to the optimisation used in [7, 10]. This is achieved by storing for
each set of labels the number of colours corresponding to these labels. We
hope that our presentation is closer to the basic approach and thus more
natural than the previous presentation of the case of large number of colours.

We now proceed to define the algorithm precisely.

4 Algorithm

The algorithm takes as the input a clique-width term $t$ of width $k$ with $n$
nullary symbols, and the number of colours $c$. In this section we assume
that the term $t$ and the number of colours $c$ are fixed. We will only consider
subterms $t/u$ of $t$.

We start with an outline of the algorithm. The basic approach is to
compute the set of all possible colourings for each subterm, starting with
the nullary operations in the leaves and going towards the root; for each
operation we compute the set of colourings based on the colourings of the
arguments. As all the vertices sharing the same label also share all the edges
left to account for, the colouring is stored as sets of labels for each colour.
To avoid unnecessary repetition of the work, we only store the colourings
up to permutation of the colours, i.e. as a multiset of set of colours. To
reduce the effort spent on the dead-ends, we apply the constraints related
to the edges as soon as both vertices appear within the subterm after a $\oplus$
operation without waiting for the $\text{add}$ operation. Moreover, once all the
edges related to some label have been processed, we stop keeping track of
this label. To optimise the union operation we observe that the colouring for the entire graph adds some colours to some labels in comparison to the colourings of each subgraph. We call this overapproximation and observe that finding a common overapproximation can be done by computing all overapproximations of the schemes on both sides and then computing the intersection.

We now proceed with a more detailed definition. The main notion in the algorithm is that of a label-colouring scheme, or simply a scheme. Such a scheme is a multiset of sets of labels. Roughly speaking, a scheme denotes the number of colours corresponding to some sets of labels. For example, if we care about labels $a$ and $b$, let us assume that the vertices with label $a$ are of colours 1, 2, 3 and vertices with label $b$ are of colours 1 and 2. This means that colours 1 and 2 are used for labels $a$ and $b$, while the colour 3 is only used for $a$. The scheme for that case would be $\{2 : \{a, b\}, \{a\}\}$.

The algorithm aims to compute the set of all the schemes describing the colourings of $\text{val}(t)$ and check if there are any.

Before performing the main part of the work, the algorithm computes some data we call annotations for each subterm. These annotations are described using the following notions.

**Definition 1.** A label $a$ is used in a subterm $t/u$ if some vertex of the labeled graph $\text{val}(t/u)$ has label $a$.

A label $a$ is a boundary label in a subterm $t/u$ of a term $t$ if there is a vertex $x$ labeled $a$ in $\text{val}(t/u)$ and there is a vertex $y$ in $\text{val}(t)$ connected with $x$ in $\text{val}(t)$, such that $y$ is not in $\text{val}(t/u)$. In other words, a label is a boundary label if there are some edges from a vertex with this label to vertices added later.

A pair of labels $(a, b)$ is pending in a subterm $t/u$ if there are non-connected vertices $x$ and $y$ in $\text{val}(t/u)$ with labels $a$ and $b$ and there is an edge between $x$ and $y$ in $\text{val}(t)$. In other words, a pair of labels is pending if
there are edges to be added later (above in the syntactic tree) between some vertices with these labels.

**Example 1.** Consider the term \( \text{relab}_{1 \rightarrow 3}(\text{add}_{1,2}(1(x) \oplus 2(y)) \oplus 3(z)) \).

Each nullary symbol as a subterm has its label as the only used (and boundary) label, and no pending pairs. The subterm \( 1(x) \oplus 2(y) \) has a pending pair \((1, 2)\), and only 1 is a boundary label (as \( y \) is only connected to \( x \) also defined inside the same subterm). Of course both 1 and 2 are used labels.

The subterm \( \text{add}_{1,2}(1(x) \oplus 2(y)) \oplus 3(z) \) has two pending pairs, \((1, 2)\) and \((1, 3)\) and no boundary labels as it contains all the vertices of the containing term. The entire term naturally has no boundary labels, and after the relabeling only 2 and 3 are used labels.

For each position \( u \) in the term \( t \), the algorithm precomputes the sets of used labels, of boundary labels, and of pending pairs of labels.

We now define formally the notion of scheme.

**Definition 2.** Assume that we have a proper colouring of \( G = \text{val}(t) \) and consider \( \text{val}(t/u) \).

Let \( B(u) \) be the set of boundary labels of \( t/u \).

For each colour \( i \), let \( C(i) \) the set of boundary labels \( a \in B(u) \) such that some vertex has label \( a \) and colour \( i \). (Cf \( R_u \) in introduction).

A scheme is a multiset of sets of labels. The description of the chosen colouring at \( u \) is the multiset of all \( C(i) \).

For a scheme \( s \) and a set of labels \( L \) we let \( s(L) \) denote the multiplicity of \( L \) in \( s \), in particular, when \( L \) is not in \( s \) we have \( s(L) = 0 \).

**Remark 3.** No boundary label \( a \) can occur in all sets \( C(i) \) for some \( u \) (because no colour would be left for the vertices that become adjacent to the \( a \)-vertices later on).

The sum of all the multiplicities in the description of a colouring is always \( c \), as each colour adds one set (or increases its multiplicity).
scheme describing a colouring also describes many others, at least those obtained by permuting colours but possibly more.

**Example 2.** We let \( \{2 : \{a, b\}, \{b, c\}, 3 : \emptyset\} \) denote the following scheme: twice \( \{a, b\} \), once \( \{b, c\} \) and three colours do not colour any vertex with a boundary label. As expected, we have \( c = 2 + 1 + 3 = 6 \).

The main part of the algorithm computes a set of schemes for every position \( u \) in \( t \) based on the subterm \( t/u \). The schemes contain as elements some sets of boundary labels. The aim is that every valid colouring of \( \text{val}(t) \) restricted to \( \text{val}(t/u) \) is described by some scheme for \( t/u \). Of course we do not guarantee that each scheme describes a restriction of some valid colouring of \( \text{val}(t) \). For the sake of simplicity, we only require that each scheme overapproximates a scheme describing some valid colouring of \( \text{val}(t/u) \) in the following sense.

**Definition 3.** A single-step overapproximation of a scheme \( s \) is obtained by reducing the multiplicity of some set of labels \( L \) in \( s \) by one, by picking some label \( a \), and by increasing the multiplicity of the set \( L \cup \{a\} \) by one. Additionally, the resulting scheme must still have a set of labels with non-zero multiplicity not containing \( a \). An overapproximation of a scheme \( s \) is any scheme reachable from \( s \) by some number of single-step overapproximations.

**Remark 4.** For a description of some label-colour relation, a single-step overapproximation describes the result of adding one more colour corresponding to the label \( a \).

Overapproximation increases the weight of a scheme \( \{..., m_i : L_i, ...\} \) defined as the sum of the \( m_i(1 + |L_i|) \).

The algorithm ensures that each scheme in the set for \( t/u \) overapproximates some scheme describing a valid colouring of \( \text{val}(t/u) \). The computation is defined recursively based on the symbol at position \( u \).

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For a nullary symbol with a label \(a\), the algorithm returns \(\{\{a\}, (c-1) : \emptyset\}\). In other words, there is just one scheme, and it contains one colour corresponding to the set of labels \(\{a\}\) (and \(c-1\) unused colours).

For the edge addition between labels \(a\) and \(b\), the same set is used as computed for the subterm that is the only argument of the operation.

For label renaming from \(a\) to \(b\) we modify the set of schemes computed for the only argument of the operation. First for each scheme the algorithm modifies each element by replacing \(a\) with \(b\) in the set (if present). If \(b\) is already present, \(a\) is just removed; if \(a\) is not present, no change is applied.

In the process some of the label sets can become equal; in this case their multiplicities are added together. For example, \(\{a\}, \{b\}\) becomes \(2 : \{b\}\). Afterwards, the algorithm removes all the schemes with every element containing \(b\). In other words, if \(b\) is a boundary label it cannot use all the colours.

The most elaborate operation is union.

In this case the subterm \(t/u\) has two subterm arguments, \(t/u_1\) and \(t/u_2\). Let the sets of schemes corresponding to these subterms be \(S_1\) and \(S_2\). These sets of schemes describe some of the valid colourings of \(\text{val}(t/u_1)\) and \(\text{val}(t/u_2)\), including all restrictions of valid colourings of \(\text{val}(t)\). The aim is to enumerate the schemes corresponding to the valid colourings of \(\text{val}(t/u)\) (with all the restrictions of valid colourings of \(\text{val}(t)\) included).

This is performed via the following steps. First, the overapproximations are computed. The algorithm computes all overapproximations of schemes in \(S_1\) and \(S_2\) that use all the boundary labels of \(u/t_1\) and \(u/t_2\) and no other labels. This is done by depth-first search using single-step overapproximation as edge relation. More precisely, if some labels are missing we first add the missing labels one by one in a fixed order. There are multiple ways to add each given label, and we consider all of them. Once all labels are used in the scheme, we consider all the possible single-step overapproximations.
This yields sets $S_1$ and $S_2$. Remember that by definition of overapproximation, as long as the initial schemes do not contain any labels using all the colours, the same is true for the overapproximations. Then $S_1 \cap S_2$ is computed. In other words, all the schemes simultaneously overapproximating some schemes from $S_1$ and $S_2$ are enumerated. Next schemes with colouring violation, i.e. some element set containing both labels of some pending pair for $t/u$, are removed. The pending pairs are obtained from the annotations. Then for each remaining schemes all labels that are not boundary labels for $t/u$ are removed from all the elements of the scheme. Boundary labels have also been precomputed as a part of the annotations. If two elements of the scheme become the same set, their multiplicities are combined. The resulting set of schemes is returned.

Consider an example where we have the sets of schemes $\{\{2 : \{a\}, \emptyset\}\}$ and $\{\{\{b\}, 2 : \emptyset\}\}$, a pending pair $(a, b)$, and only $b$ is a boundary label after the union (i.e. after the union vertices with label $b$ have some external connections but vertices with the label $a$ do not). Among the overapproximations of the scheme from the first set there are $\{2 : \{a\}, \emptyset\}$, $\{\{a\}, \{a, b\}, \emptyset\}$, $\{2 : \{a\}, \{b\}\}$, as well as some others. But, for example, $\{2 : \{a\}, \{a, b\}\}$ is not an overapproximation as $a$ is in every set. Of the above mentioned overapproximations, $\{\{a\}, \{a, b\}, \emptyset\}$ and $\{2 : \{a\}, \{b\}\}$ are common overapproximations for both sets of schemes. The former, however, is removed for colouring violation. Removing $a$ in the elements of the latter ($a$ is not a boundary label) yields the scheme $\{2 : \emptyset, \{b\}\}$. Actually, this is the only scheme we would obtain if we checked all the overapproximations. The resulting set of schemes is $\{\{2 : \emptyset, \{b\}\}\}$.

Once the sets of schemes corresponding to all subterms of $t$ are calculated, the algorithm verifies if the set of schemes for the term $t$ is empty. A valid colouring of $\text{val}(t)$ exists if and only if the set of schemes for the term $t$ is not empty.
5 Correctness

Consider a subterm \( t/u \). Consider some colouring of the graph \( \text{val}(t/u) \). A *boundary set* of a colour \( Q \) is the set of all boundary labels of \( t/u \) having at least one vertex of the colour \( Q \). A *description* of the colouring is a scheme such that each set of boundary labels \( L \) has multiplicity equal to the number of colours with boundary set equal to \( L \).

We prove by induction on subterm structure that all schemes for each subterm \( t/u \) are overapproximations of descriptions of some valid colourings of the subgraph \( \text{val}(t/u) \), and descriptions of restrictions of all valid colourings of the entire graph \( \text{val}(t) \) belong to the set of schemes computed for \( t/u \). More precisely, each scheme for subterm \( t/u \) is an overapproximation of some scheme describing a valid colouring of the induced subgraph in the full graph \( \text{val}(t) \) for the vertices of \( \text{val}(t/u) \).

In case of a single vertex there is just a single label and a single colour used. It is clear that the condition holds.

In case of an edge addition, the set of vertices does not change and the edge has been already taken into account before by induction assumption.

In case of relabeling from \( a \) to \( b \) there are two cases. Consider that the same future edge additions apply to the vertices labeled \( a \) and \( b \), as they share a label after relabeling. Therefore either \( b \) is a boundary label after relabeling, and then both \( a \) and \( b \) are boundary labels before the relabeling; or \( b \) is not a boundary label and then neither is \( a \). If neither \( a \) nor \( b \) are boundary labels, the relabeling does not change the descriptions of any colourings, and the algorithm does not change the set of schemes. If \( a \) and \( b \) are boundary labels, it is straightforward to verify that the for each colouring of \( \text{val}(t/u) \) taking a describing scheme before relabeling and applying the label replacement to the scheme provides the same result as relabeling the graph first then taking the description. It remains to show that the schemes we drop are not descriptions of colourings of the entire graph \( \text{val}(t) \). But
indeed a boundary label cannot have vertices of all colours. Note that we might remove overapproximations of descriptions of some valid colourings, but we do not promise to keep all the overapproximations.

The last case is the union operation. Consider a colouring of $\text{val}(t/u)$. Each colour is used for some labels. The same colour is used for two subsets of labels in $\text{val}(t/u_1)$ and $\text{val}(t/u_2)$. Thus we can find a common overapproximation for the descriptions of the two restrictions. The condition that an overapproximation cannot make a label correspond to all the colours is not violated, as each label we care about is either a boundary label for $t/u$ and does not use all the colours, or a boundary label in $t/u_1$ or $t/u_2$ but not in $t/u$ and is in some pending pair of labels, thus not using any colours used by the second label in the pair. Moreover, the description of this colouring only differs from this common overapproximation by dropping non-boundary labels in each element of the scheme.

Now we show that each scheme $s$ that is not removed is an overapproximation of the description of some colouring of $\text{val}(t/u)$. Consider the scheme $\tilde{s}$ such that $s$ was obtained by ignoring non-boundary labels in $\tilde{s}$. The scheme $\tilde{s}$ was obtained from two such schemes $s_1$ and $s_2$ for the subgraphs $\text{val}(t/u_1)$ and $\text{val}(t/u_2)$. Without loss of generality assume that $s_1$ and $s_2$ are exact descriptions are not overapproximations, as overapproximation is transitive. Consider some valid colourings $C_1$ and $C_2$ of $\text{val}(t/u_1)$ and $\text{val}(t/u_2)$ with descriptions $s_1$ and $s_2$. Such colourings exist by induction assumption. Picking an arbitrary colour to be assigned to the additional label for each single-step overapproximation, we can give each colour a set of labels larger than the set of corresponding labels according to $C_1$ so that each set of labels $L$ is used $s(L)$ times. The same applies to $C_2$. As the colours can be permuted, we assume without loss of generality that we obtain the same function from colours to sets of labels. In this case we can just unite the colourings $C_1$ and $C_2$ and obtain a valid colouring of induced
subgraph corresponding to the vertices of $\text{val}(t/u)$. Indeed, each edge in
this subgraph either connects the vertices on the same side of the union
(then the colours must be different as $C_1$ and $C_2$ are valid), or on different
sides. In the latter case the corresponding labels are boundary labels in $t/u_1$
and $t/u_2$, and this pair of labels is pending for $t/u$, thus $\tilde{s}$ would have been
removed. Note that this case analysis is per edge, not per pair of labels; the
same pair of labels might correspond to edges of both types.

It is straightforward to verify that the description of the restriction of
a colouring of the full graph $\text{val}(t)$ will not be removed, as it cannot have
boundary labels using all the colours nor pending pairs sharing a colour.

We conclude that the algorithm indeed computes some set of overapprox-
imations of descriptions of colourings. It remains to observe that for the
full term $t$ we have some overapproximations of descriptions of colourings,
including all exact descriptions of colourings. This set is empty iff there is
no colouring. This concludes the proof of correctness.

**Remark 5.** The correctness proof is essentially constructive. In other
words, if we have a scheme for a term and the corresponding schemes for
all the subterms, the correctness proof explains how to combine the trivial
colourings of single-vertex graphs represented by the leaf subterms into a
colouring for the entire graph.

To be able to obtain intermediate schemes, we can store the first jus-
tification of inclusion for each intermediate scheme computed during the
algorithm.

### 6 Complexity

We use sequences of boolean values of length $k$ to represent sets of labels.
We use PATRICIA trie data structure [11] on strings of length $k$ with inte-
ger labels in leaves to represent schemes, and also PATRICIA tries for some
string encoding of schemes to represent sets of schemes (any set data structure on strings with operations taking time proportional to string traversal time is suitable here). We store the pointers to the arguments and to the parent operation for each position in the term (this is trivial to compute in linear time).

The annotations can be precomputed in the time $m \times k^2$.

Each scheme has at most $c$ entries, as we do not need to keep elements with zero multiplicity, and at most $2^k$ entries as this is the number of sets of labels. The total number of schemes using $k'$ labels is bounded both by $\binom{c+2^k-1}{2^k} \leq (c+1)^{2^k}$ counting them as ordered partitions of $c$ into $2^k$ summands, and by $(2^c-2)^k$ as each label has a non-empty set of colours and also cannot use all the colours. Let $D_s$ be $\min(c, 2^k)$ and $N_s$ be $\min((c+1)^{2^k}, (2^c-2)^k)$. When encoding a scheme, each set of labels needs $l_s = k + \log c$ bits.

For the main computation, it is easy to see that the union operation is the most expensive one. The overapproximations are computed by traversing a graph. The graph can be stratified by the set of labels used; for each size of the label set there is at most one possible set of used labels (as we add labels in a fixed order). It is clear that the number of schemes for the largest set of labels is larger than the total number for the smaller sets, so we bound the number of vertices by $2N_s$. The degree of the nodes in the graph is at most $D_s \times k$, as we pick a set of labels with at least one colour, then pick a label to add. Overall the traversal takes $2N_s D_s k$ operations on schemes. Intersection takes just $3N_s$ operations on schemes considering lookup/insertion on sets of schemes an operation on schemes. Removal of non-boundary labels takes at most $N_s k$ operations on schemes. Removing invalid schemes takes $N_s k^2$ operations on schemes, as we need to consider up to $\frac{k(k-1)}{2} + k$ reasons to remove a scheme; the reason can be either an overflow in a label or a pending pair of labels. Each operation on schemes takes time at most proportional
to $D_s l_s$ steps. The total time is $O(N_s(D_s + k)k \times D_s l_s)$.

Given that we have $m$ operations, precomputations are much cheaper than the main part, and the total runtime of the algorithm is $O(m \times \min((c + 1)^k, (2^c - 2)^k) \times (\min(c, 2^k) + k) \times k \times \min(c, 2^k) \times (k + \log c))$.

### 7 Conclusion and further work

We have shown that a single algorithm can verify $c$-colourability of a graph provided as a clique-width decomposition with performance matching the state of the art both in the few-colours and many-colours cases.

A natural question arises whether a modification of this algorithm can be used to count the number of possible colourings. We conjecture that this can be done without significantly exceeding the complexity of the cited algorithms from the literature.

As the most complicated and the most expensive operations is the union operation, it might be of interest to see what simplifications and optimisations can be achieved for linear clique-width terms, i.e. when the second subgraph in each union has exactly one vertex.

The algorithm presented optimises the worst case complexity of computing the sets of schemes. Empirical evidence shows that for some graphs there are few schemes corresponding to each subterm, making the approach based on precomputing all overapproximations less efficient than a naive approach based on enumerating all pairs of schemes. In this approach, for a pair of schemes we solve a matching-like problem to figure out which colour used for colouring the first subgraph correspond to which colour in the colouring of the second subgraph. One can use either a brute force approach or some algorithms based on maximum flow and similar considerations (e.g. [12]) to enumerate the possible matchings. We believe it is an interesting question to find some natural class of graphs where such an approach guarantees a better upper bound on the runtime.
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