RG flow equations for the proper-vertices of the background effective average action

Alessandro Codello*

SISSA
via Bonomea 265, I-34136 Trieste, Italy

Abstract

We derive a system of coupled flow equations for the proper-vertices of the background effective average action and we give an explicit representation of these by means of diagrammatic and momentum space techniques. This explicit representation can be used as a new computational technique that enables the projection of the flow of a large new class of truncations of the background effective average action. In particular, these can be single– or bi–field truncations of local or non–local character. As an application we study non–abelian gauge theories. We show how to use this new technique to calculate the beta function of the gauge coupling (without employing the heat kernel expansion) under various approximations. In particular, one of these approximations leads to a derivation of beta functions similar to those proposed as candidates for an “all–orders” beta function. Finally, we discuss some possible phenomenology related to these flows.

*email: codello@sissa.it
# Contents

1 Introduction
2 Flow equations for proper-vertices of the bEAA
   2.1 Construction of bEAA
   2.1.1 Exact flow equation for the bEAA
   2.2 Flow equations for the proper-vertices of the bEAA
   2.2.1 Derivation
   2.2.2 Compact form
   2.3 Diagrammatic and momentum space techniques
      2.3.1 Diagrammatic rules
      2.3.2 Momentum space representation
3 An application: non-abelian gauge theories
   3.1 Truncation ansatz
   3.2 Derivation of the beta functions
      3.2.1 Heat kernel calculation of $\eta_{\bar{A},k}$
      3.2.2 Calculation of $\eta_{\bar{A},k}$ from $\partial_t \bar{\gamma}_k^{(2)}$
      3.2.3 Calculation of $\eta_{a,k}$ and $\eta_{c,k}$
   3.3 Beta functions and non-perturbative predictions
      3.3.1 Beta functions
      3.3.2 Flow and physical observables
4 Discussion and future perspectives
A Momentum space representation of background vertices
   A.1 Perturbative expansion of the heat kernel
   A.2 Derivation of momentum space representation
B Basic relations for non-abelian gauge theories
   B.1 Definitions and conventions
   B.2 Variations and functional derivatives
1 Introduction

The effective average action (EAA) formalism is a promising approach to QFT which recently has seen important developments and applications [1, 2]. The EAA is a $k$ dependent functional that interpolates smoothly between the bare action for $k = \Lambda$ and the standard effective action for $k = 0$; from this point of view it offers a new approach to quantization. When applied to theories with local symmetries, as non-abelian gauge theories [3, 4], non-linear sigma models [5], gravity [6], or membranes [7] it can be implemented using the background field method [3]. This defines the background EAA (bEAA).

In this formalism one usually makes a truncation ansatz for the bEAA and then performs calculations with the aid of heat kernel techniques [8]. Generally, the range of applicability of heat kernel methods is confined to special cases, where the Hessian of the bEAA is of generalized Laplacian type [9]. This technology permits the study of truncations where the bEAA is the sum of local operators; these are interesting from the point of view of renormalizability, but are not enough to capture the full form of the bEAA, which in general contains non-local operators. These non-local terms are of fundamental importance, if one wants to construct RG trajectories that reach the far infrared (IR) $k = 0$. It has been shown in workable examples [10] that only by employing these kind of truncations it is possible to construct an EAA that correctly reproduces the effective action in the limit $k \to 0$. Another important and complementary issue that requires stronger computational tools is the one related to “bi–field” truncations [11]: since (as we will review in section 2) the full bEAA is a functional of both fluctuation and background fields, its RG flow takes place in the enlarged theory space where this functional naturally lives. Bi–field truncations generally give rise to operators that are difficult to treat using heat kernel methods.

In this paper we present a novel formalism based on the hierarchy of coupled flow equations satisfied by the one-particle-irreducible (1PI) proper-vertices of the bEAA. We give an explicit representation of these equations by means of diagrammatic and momentum space techniques. This explicit representation can be used as a new computational technique that enables the projection of the flow of a large new class of truncations of the bEAA. These include both the non-local and bi-field truncations mentioned above.

The hierarchy of flow equations for the proper-vertices contains vertices with both fluctuation and background legs. The non-trivial part of the formalism is the explicit momentum space representation of vertices with background legs, since these contain legs attached to the cutoff action (which is function of the background field). These terms are not present in the non-background formalism of the standard EAA and constitute the necessary contributions
to make the RG flow covariant. A key element in the derivation of this representation is the perturbative expansion of the (un-traced) heat kernel \[^{12}\].

As an application we study non-abelian gauge theories. First we show how standard heat kernel computation of the one-loop beta function of the gauge coupling is reproduced by our formalism, then we extend the study under two approximations, a single- and a bi-field one, to obtain RG improved beta functions. One of these turns out to be very similar to the “all-orders” beta function proposed by Rytov and Sannino \[^{13}\]. Finally, we discuss some possible phenomenology related to these RG flows.

In the second section of the paper we review the basic properties of the bEAA and we derive the hierarchy of coupled flow equations for the proper-vertices and we expose the relative momentum space representation by introducing the diagrammatic rules. The details of the derivation of the non-trivial momentum space representation of vertices with background legs are given in appendix A. In the third section we apply our formalisms to non-abelian gauge theories to show how it works in practice.
2 Flow equations for proper-vertices of the bEAA

In this section we first review the construction of the bEAA and the derivation of the exact RG flow equation it satisfies; successively we derive the hierarchy of flow equations for the proper-vertices of bEAA and we expose their momentum space representation and relative diagrammatic rules.

2.1 Construction of bEAA

The crucial point in the construction of the EAA for theories with local gauge symmetries is, obviously, the preservation of gauge invariance during the RG coarse-graining procedure. A possible way to cut-off field modes covariantly is to define the cutoff using covariant differential operators. But if we try to introduce a cutoff by simply taking it as a function of a covariant differential operator, constructed with the quantum fields, we will spoil the simple one-loop structure of the exact flow that the EAA obeys. To obtain a one-loop like flow equation, the cutoff action has to be quadratic in the quantum fields. Still, the EAA will not be gauge invariant because of the non-covariant coupling of the quantum fields to the source, a problem that affects also the standard effective action. A way out of this is to employ the background field method, as was first done in [3, 14], to define what we will call the background effective average action (bEAA).

The theories that we have in mind are non-abelian gauge theories, gravity and non-linear sigma models. In this paper we will use the language of non-abelian gauge theories to keep the notation as light as possible, but it is clear that everything is readily translated to the other cases.

In the background field formalism [15] the quantum field $A_\mu$ is linearly split between the background field $\bar{A}_\mu$ and the fluctuation field $a_\mu$:

$$A_\mu = \bar{A}_\mu + a_\mu. \quad (1)$$

The cutoff action $\Delta S_k$ is taken to be quadratic in the fluctuation field, while the cutoff kernel $R_k[\bar{A}]$ is constructed employing the background field alone:

$$\Delta S_k[\varphi; \bar{A}] = \frac{1}{2} \int d^dx \varphi R_k[\bar{A}] \varphi, \quad (2)$$

where $\varphi = (a_\mu, \bar{c}, c)$ is the fluctuation multiplet, combining the fluctuation field $a_\mu$ and the ghost fields $\bar{c}$ and $c$. To construct the bEAA we introduce the cutoff action [2] into the
integro-differential definition of the standard background effective action\footnote{Functional derivatives of a functionals depending on many arguments are indicated by the notation $\Gamma^{(n_1,n_2,...)}[...].$}

\[ e^{-\Gamma_k[\varphi; \bar{A}]} = \int D\chi \exp \left( -S[\chi + \varphi; \bar{A}] - \Delta S_k[\chi; \bar{A}] + \int d^d x \Gamma^{(1,0)}_k[\varphi; \bar{A}] \chi \right), \quad (3) \]

where the fluctuation field multiplet $\chi$ has vanishing vacuum expectation value $\langle \chi \rangle = 0$. The role of the cutoff action is to suppress field modes with momenta smaller than the RG scale $k$, the shape of the cutoff kernel must be constructed in a way consistent with this property; more details on this can be found in \cite{1, 2}. With these definitions the bEAA is a functional that interpolates smoothly between the bare action (4) for $k \to \infty$ (fact that can be easily checked) and the full background effective action for $k \to 0$. This fact, together with the exact flow equation bEAA satisfies (that we derive in the next subsection), offers the starting point to develop a formalisms that can be used to define and construct QFT characterized by local gauge symmetries. The computational technique introduced in this paper has to be seen as a tool to pursue this research route.

In (3) $S[\varphi; \bar{A}]$ is the bare action which is the sum of an invariant action (which can be the classical non-abelian gauge theory action), the background gauge-fixing action and the background ghost action:

\[ S[\varphi; \bar{A}] = S[\bar{A} + a] + S_{gf}[a; \bar{A}] + S_{gh}[a, \bar{c}, c; \bar{A}]. \quad (4) \]

The background gauge-fixing action is

\[ S_{gf}[a; \bar{A}] = \frac{1}{2\alpha} \int d^d x \bar{D}_\mu a^\mu \bar{D}_\nu a^\nu, \quad (5) \]

where $\alpha$ is the gauge-fixing parameter, while the background ghost action is

\[ S_{gh}[a, \bar{c}, c; \bar{A}] = \int d^d x \bar{D}_\mu \bar{c} D^\mu c = \int d^d x \bar{D}_\mu \bar{c} \left( \bar{D}^\mu + ga^\mu \right) c. \quad (6) \]

With these definitions, the bEAA (3) is invariant under combined physical\footnote{See appendix B for our conventions.}

\[ \delta_\theta A_\mu = D_\mu \theta \quad \delta_\theta \bar{c} = [\bar{c}, \theta] \quad \delta_\theta c = [c, \theta] \quad \delta_\theta \bar{A}_\mu = 0, \quad (7) \]
and background,
\[ \delta_\theta A_\mu = \delta_\theta \bar{c} = \delta_\theta c = 0 \quad \delta_\theta \bar{A}_\mu = \bar{D}_\mu \theta, \] (8)
gauge transformations:
\[ (\delta_\theta + \bar{\delta}_\theta) \Gamma_k[\varphi; \bar{A}] = 0. \] (9)

For a proof of (9) and for more details about the (background) gauge invariance of the bEAA we refer to the literature [16]. See also [17].

It is possible to define a gauge invariant functional of the background field, that we will call “gauge invariant effective average action” (gEAA), by setting to zero the fluctuation multiplet \( \varphi = 0 \) in the bEAA:
\[ \bar{\Gamma}_k[\bar{A}] \equiv \Gamma_k[0; \bar{A}]. \] (10)

This definition is equivalent to a parametrization of the bEAA as the sum of a functional of the full quantum field \( A_\mu = \bar{A}_\mu + a_\mu \), the gEAA (10), and a “remainder functional” \( \hat{\Gamma}_k[\varphi; \bar{A}] \) (rEAA) which remains a functional of both the fluctuation multiplet and the background field:
\[ \Gamma_k[\varphi; \bar{A}] = \bar{\Gamma}_k[\bar{A} + a] + \hat{\Gamma}_k[\varphi; \bar{A}]. \] (11)

To recover (10) we must have \( \hat{\Gamma}_k[0; \bar{A}] = 0 \). The gEAA defined in this way is a functional invariant under physical gauge transformations (7):
\[ \delta_\theta \bar{\Gamma}_k[\bar{A}] = 0, \] (12)

while the rEAA remains a functional invariant under combined physical and background gauge transformations, as the full bEAA, and is subject to modified Ward-Takahashi identities. We refer to [3] for more details on this point. What is important is that the gEAA flows, in the IR \( k \to 0 \), to the standard gauge invariant effective action of the background field formalism, which can thus be computed using the bEAA formalism.

2.1.1 Exact flow equation for the bEAA

We derive now the exact flow equation satisfied by the bEAA [3]. Differentiating with respect to the “RG time” \( t = \log k/k_0 \) both sides of the integro-differential equation (3) we find:
\[ e^{-\Gamma_k[\varphi; \bar{A}]} \partial_t \Gamma_k[\varphi; \bar{A}] = \int D\chi \left( \partial_t \Delta S_k[\chi; \bar{A}] - \int d^d x \partial_t \Gamma_k^{(1,0)}[\varphi; \bar{A}] \chi \right) \times \]
\[ \times e^{-S[\varphi + \chi; \bar{A}] - \Delta S_k[\chi; \bar{A}] + \int \Gamma_k^{(1,0)}[\varphi; \bar{A}] \chi}. \] (13)
We can re-express the terms on the rhs of (13) as expectation values and use (2) to rewrite (13) as:

\[ \partial_t \Gamma_k[\varphi; \bar{A}] = \langle \partial_t \Delta S_k[\chi; \bar{A}] \rangle - \int d^d x \partial_t \Gamma_k^{(1;0)}[\varphi; \bar{A}] \langle \chi \rangle \]

where we used \( \langle \chi \rangle = 0 \). The two-point function of \( \chi \) in (14) can be written in terms of the inverse Hessian of the bEAA plus the cutoff action (where functional derivatives are taken with respect to \( \varphi \)):

\[ \langle \chi \chi \rangle = \left( \Gamma_k^{(2;0)}[\varphi; \bar{A}] + \Delta S_k^{(2;0)}[\varphi; \bar{A}] \right)^{-1} = \left( \Gamma_k^{(2;0)}[\varphi; \bar{A}] + R_k[\bar{A}] \right)^{-1}, \]

where in the second step we used the fact that the Hessian of cutoff action (2) is the cutoff kernel \( R_k[\bar{A}] \). Inserting (15) back into (14), and writing a functional trace in place of the integral, finally gives:

\[ \partial_t \Gamma_k[\varphi; \bar{A}] = \frac{1}{2} \text{Tr} \left( \Gamma_k^{(2;0)}[\varphi; \bar{A}] + R_k[\bar{A}] \right)^{-1} \partial_t R_k[\bar{A}]. \]

Equation (16) is the exact RG flow equation for the bEAA [3]. The flow equation (16) has the same general properties as the flow equation for the standard effective average action [1]. In the following it will be useful to define the field dependent “regularized propagator”:

\[ G_k[\varphi; \bar{A}] = \left( \Gamma_k^{(2;0)}[\varphi; \bar{A}] + R_k[\bar{A}] \right)^{-1}, \]

to rewrite the flow equation (16) in the following compact form:

\[ \partial_t \Gamma_k[\varphi; \bar{A}] = \frac{1}{2} \text{Tr} G_k[\varphi; \bar{A}] \partial_t R_k[\bar{A}]. \]

The flow equation satisfied by the gEAA is readily obtained combining (11) and (18):

\[ \partial_t \Gamma_k[\bar{A}] = \partial_t \Gamma_k[0; \bar{A}] = \frac{1}{2} \text{Tr} G_k[0; \bar{A}] \partial_t R_k[\bar{A}]. \]

By construction the flow equation for the gEAA respects gauge symmetry, in the sense that the trace on the rhs of (19), the “beta functional”, is a gauge invariant functional of \( \bar{A}_\mu \). Still, it is important to realize that equation (19) is not a closed equation since it involves
the Hessian of the bEAA $\Gamma_k^{(2,0)}[0; \vec{A}]$ taken with respect to the fluctuation field. This fact implies that, even if in the limit $k \to 0$ the gEAA flows to the gauge invariant effective action $\Gamma[\vec{A}] \equiv \Gamma[0; \vec{A}]$, for $k \neq 0$ it is necessary to consider the flow in the extended theory space where the bEAA lives, composed of all functionals of the fields $\varphi$ and $\vec{A}_\mu$ invariant under simultaneous physical and background gauge transformations.

When we set $\varphi = 0$ the Hessian $\Gamma_k^{(2,0)}[0, \vec{A}]$ in (19) becomes “super-diagonal”, since the ghost action is bilinear, and we can immediately perform the multiplet trace in the flow equation. Using the shorthands $\Gamma_{k,aa} = \Gamma_k^{(2,0,0,0)}[0, 0, 0; \vec{A}], \Gamma_{k,cc} = \Gamma_k^{(0,1,1,0)}[0, 0, 0; \vec{A}], \Delta S_{k,aa} = \Delta S_k^{(2,0,0,0)}[0, 0, 0; \vec{A}]$ and $\Delta S_{k,cc} = \Delta S_k^{(0,1,1,0)}[0, 0, 0; \vec{A}]$ we can write the flow equation in the following matrix form:

$$
\partial_t \bar{\Gamma}_k[\vec{A}] = \frac{1}{2} \text{Tr} \left( \begin{array}{ccc}
\Gamma_{k,aa} + \Delta S_{k,aa} & 0 & 0 \\
0 & 0 & \Gamma_{k,cc} + \Delta S_{k,cc} \\
0 & - (\Gamma_{k,cc} + \Delta S_{k,cc}) & 0 \\
\end{array} \right)^{-1} 
\times 
\left( \begin{array}{ccc}
\partial_t \Delta S_{k,aa} & 0 & 0 \\
0 & 0 & \partial_t \Delta S_{k,cc} \\
0 & - \partial_t \Delta S_{k,cc} & 0 \\
\end{array} \right)
$$

$$
= \frac{1}{2} \text{Tr} \left( \begin{array}{ccc}
G_{k,aa} & 0 & 0 \\
0 & 0 & -G_{k,cc} \\
0 & G_{k,cc} & 0 \\
\end{array} \right) 
\left( \begin{array}{ccc}
\partial_t \Delta S_{k,aa} & 0 & 0 \\
0 & 0 & \partial_t \Delta S_{k,cc} \\
0 & - \partial_t \Delta S_{k,cc} & 0 \\
\end{array} \right)
$$

$$
= \frac{1}{2} \text{Tr} \left( \begin{array}{ccc}
G_{k,aa} \partial_t \Delta S_{k,aa} & 0 & 0 \\
0 & G_{k,cc} \partial_t \Delta S_{k,cc} & 0 \\
0 & 0 & G_{k,cc} \partial_t \Delta S_{k,cc} \\
\end{array} \right)
$$

$$
= \frac{1}{2} \text{Tr} \ G_{k,aa} \partial_t \Delta S_{k,aa} - \text{Tr} \ G_{k,cc} \partial_t \Delta S_{k,cc}, \tag{20}
$$

where we also defined $G_{k,aa} = (\Gamma_{k,aa} + \Delta S_{k,aa})^{-1}$ and $G_{k,cc} = (\Gamma_{k,cc} + \Delta S_{k,cc})^{-1}$. In (20) we used the property that the trace over anti-commuting fields carries an additional minus sign.

The last line of equation (20) gives the flow equation for the gEAA in its commonly used form [3):

$$
\partial_t \bar{\Gamma}_k[\vec{A}] = \frac{1}{2} \text{Tr} \left( \Gamma_k^{(2,0,0,0)}[0; \vec{A}] + \Delta S_k^{(2,0,0,0)}[0, 0, 0; \vec{A}] \right)^{-1} \partial_t \Delta S_k^{(2,0,0,0)}[0, 0, 0; \vec{A}] 
- \text{Tr} \left( \Gamma_k^{(0,1,1,0)}[0; \vec{A}] + \Delta S_k^{(0,1,1,0)}[0, 0, 0; \vec{A}] \right)^{-1} \partial_t \Delta S_k^{(0,1,1,0)}[0, 0, 0; \vec{A}] \tag{21}
$$
One can obtain equation (21) as an RG improvement of the one–loop gEAA obtained from (3) by saddle point expansion. One finds:

\[
\Gamma_k[\bar{A}] = S[\bar{A}] + \frac{1}{2} \text{Tr} \log \left( S^{(2)}[\bar{A}] + S_{gf}^{(2,0)}[0,0,0;\bar{A}] + S_{gh}^{(2,0,0,0)}[0,0,0,0;\bar{A}] + \Delta S_{k}^{(2,0,0,0)}[0,0,0,0;\bar{A}] \right) - \text{Tr} \log \left( S_{gh}^{(0,1,1,0)}[0,0,0;\bar{A}] + \Delta S_{k}^{(0,1,1,0)}[0,0,0;\bar{A}] \right) ;
\]

if we now substitute in the traces on the rhs the Hessian of bare action \( S[\varphi; \bar{A}] \) with the Hessians of the bEAA \( \Gamma_k[\varphi; \bar{A}] \) and we differentiate with respect to \( t = \log k/k_0 \) on both sides, we recover the exact RG flow equation (21). We note that one may just substitute \( S[\bar{A}] \) with \( \Gamma_k[\bar{A}] \) to obtain directly a closed equation involving only the gEAA; this seems to suggests that truncations where the rEAA is approximated with the classical gauge-fixing and ghost actions may be a consistent approximation when considering the flow of the gEAA.

The flow equation for rEAA can be deduced differentiating equation (11):

\[
\partial_t \bar{\Gamma}_k[\bar{A}] = \partial_t \bar{\Gamma}_k[\bar{A} + a] - \partial_t \Gamma_k[\varphi; \bar{A}] ,
\]

where we used (16) and (21).

2.2 Flow equations for the proper-vertices of the bEAA

In this section we derive the system of equations governing the RG flow of the proper-vertices of both the bEAA and the gEAA. To obtain these equations we take functional derivatives of the flow equation (16) satisfied by the bEAA with respect to the fields \( \varphi \) and \( \bar{A}_\mu \). When we differentiate with respect to the background field, we have to remember that the cutoff terms present in the flow equation depend explicitly on it: this adds additional terms to the flow equations for the proper-vertices that are not present in the non-background formalism. We will see that these terms are crucial to preserve gauge covariance of the gEAA along the flow.

2.2.1 Derivation

To start, we take a functional derivative of the flow equation (16) with respect to the fluctuation field multiplet, or with respect to the background field, to obtain the following flow equations for the one-vertices of the bEAA:

\[
\partial_t \Gamma_k^{(1,0)}[\varphi; \bar{A}] = -\frac{1}{2} \text{Tr} G_k[\varphi; \bar{A}] \Gamma_k^{(3,0)}[\varphi; \bar{A}] G_k[\varphi; \bar{A}] \partial_t R_k[\bar{A}]
\]
perform a Taylor expansion of the functional

\[ \partial_t \Gamma_k^{(0;1)}[\varphi; \bar{A}] = -\frac{1}{2} \text{Tr} G_k[\varphi; \bar{A}] \left( \Gamma_k^{(2;1)}[\varphi; \bar{A}] + R_k^{(1)}[\bar{A}] \right) G_k[\varphi; \bar{A}] \partial_t R_k[\bar{A}] + \frac{1}{2} \text{Tr} G_k[\varphi; \bar{A}] \partial_t R_k^{(1)}[\bar{A}] . \]  

(24)

Note that in the second equation of (24), where we differentiated with respect to the background field, there are additional terms containing functional derivatives of the cutoff kernel \( R_k[\bar{A}] \). Taking further derivatives of equation (24), with respect to both the fluctuation field multiplet and background field, gives the following flow equations for the two-vertices\(^3\):

\[ \partial_t \Gamma_k^{(2;0)} = \text{Tr} G_k \Gamma_k^{(3;0)} G_k \Gamma_k^{(3;0)} G_k \partial_t R_k - \frac{1}{2} \text{Tr} G_k \Gamma_k^{(4;0)} G_k \partial_t R_k \]

\[ \partial_t \Gamma_k^{(1;1)} = \text{Tr} G_k \left( \Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \Gamma_k^{(3;0)} G_k \partial_t R_k + \text{Tr} G_k \Gamma_k^{(3;0)} G_k \left( \Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \partial_t R_k \]

\[ -\frac{1}{2} \text{Tr} G_k \Gamma_k^{(3;1)} G_k \partial_t R_k - \frac{1}{2} \text{Tr} G_k \Gamma_k^{(3;0)} G_k \partial_t R_k^{(1)} \]

\[ \partial_t \Gamma_k^{(0;2)} = \text{Tr} G_k \left( \Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \left( \Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \partial_t R_k + \frac{1}{2} \text{Tr} G_k \partial_t R_k^{(2)} . \]  

(25)

Proceeding in this way we generate a hierarchy of flow equations for the proper-vertices \( \Gamma_k^{(n;m)}[\varphi; \bar{A}] \) of the bEAA. In general the flow equation for \( \Gamma_k^{(n;m)}[\varphi; \bar{A}] \) involves proper-vertices up to \( \Gamma_k^{(n+2;m)}[\varphi; \bar{A}] \) and functional derivatives of the cutoff kernel up to \( R_k^{(m)}[\bar{A}] \).

Note that, as they stand in (24) and (25), every equation of the hierarchy contains the same information as the original flow equation (16). To profit of the vertex expansion we perform a Taylor expansion of the functional \( \Gamma_k[\varphi; \bar{A}] \) around \( \varphi = 0 \) and \( \bar{A}_\mu = 0 \):

\[ \Gamma_k[\varphi; \bar{A}] = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \int_{x_1 \ldots x_n y_1 \ldots y_m} \gamma_k^{(n;m)} \varphi x_1 \ldots \varphi x_n \bar{A} y_1 \ldots \bar{A} y_m , \]  

(26)

where we defined the zero-field proper-vertices as follows:

\[ \gamma_k^{(n;m)} \equiv \Gamma_k^{(n;m)}[0; 0] x_1 \ldots x_n y_1 \ldots y_m . \]  

(27)

\(^3\)Here and in other equations of this section we omit, for clarity, to write explicitly the arguments of the functionals when these are understood.
If we evaluate the hierarchy of flow equations at $\varphi = 0$ and $\bar{A}_\mu = 0$ it becomes an infinite system of coupled integro-differential equations for the zero-field proper-vertices $\gamma^{(n;m)}_k$. From the Taylor expansion (26) that defines these vertices, we see that this system can be used to project the RG flow of all terms of the bEAA which are analytic in the fields $\varphi$ and $\bar{A}_\mu$. In particular these terms can be of non-local character.

Note that in the above considerations is not necessary to expand around the zero-background configuration $\bar{A}_\mu = 0$; one can choose to expand around any background, preferably where one is able to perform computations. An example is a constant magnetic field configuration or, in the gravitational case, a sphere or an Einstein space.

As for the bEAA, we can derive a hierarchy of flow equations for the proper-vertices of the gEAA. In this case the functional depends only on the background field. Taking a functional derivative of (19) with respect to this field gives the following flow equation for the one-vertex of the gEAA:

$$\partial_t \bar{\Gamma}^{(1)}_k[A] = -\frac{1}{2} \text{Tr} G_k[0, A] \left( \Gamma^{(2;1)}_k[0, A] + R^{(1)}_k[A] \right) G_k[0, A] \partial_t R_k[A]$$

$$+ \frac{1}{2} \text{Tr} G_k[0, A] \partial_t R^{(1)}_k[A].$$

(28)

Note that since $\partial_t \bar{\Gamma}^{(0;1)}_k[0; A] = \partial_t \bar{\Gamma}^{(1)}_k[A]$, equation (28) is the same as the second equation in (24), but with $\varphi = 0$. A further derivative of (28) with respect to $\bar{A}_\mu$ gives the flow equation for the two-vertex of the gEAA:

$$\partial_t \bar{\Gamma}^{(2)}_k = \text{Tr} G_k \left( \Gamma^{(2;1)}_k + R^{(1)}_k \right) G_k \left( \Gamma^{(2;1)}_k + R^{(1)}_k \right) G_k \partial_t R_k$$

$$- \frac{1}{2} \text{Tr} G_k \left( \Gamma^{(2;2)}_k + R^{(2)}_k \right) G_k \partial_t R_k$$

$$- \text{Tr} G_k \left( \Gamma^{(2;1)}_k + R^{(1)}_k \right) G_k \partial_t R^{(1)}_k + \frac{1}{2} \text{Tr} G_k \partial_t R^{(2)}_k.$$

(29)

As for (28), this equation is equal to the last equation in (25) if we set $\varphi = 0$ and use $\partial_t \Gamma^{(0;2)}_k[0; A] = \partial_t \bar{\Gamma}^{(2)}_k[A]$. As previously stated, the terms coming from functional derivatives of the cutoff kernel (that are present in the background formalism but not in the non-background one) are crucial in preserving the covariant character of the flow of the gEAA and its vertices.

As for the bEAA, we can perform a Taylor expansion of the gEAA analogous to (26) and
define the zero-field proper-vertices
\[ \bar{\gamma}_{k,x_1...x_n}^{(n)} \equiv \bar{\Gamma}_{k}^{(n)}[0]_{x_1...x_n}, \]
(30)
to turn the hierarchy of flow equations for the proper-vertices of the gEAA into an infinite system of coupled integro-differential equations for the vertices \( \bar{\gamma}_{k}^{(m)} \).

### 2.2.2 Compact form

We introduce now a compact notation to rewrite the flow equations for the proper-vertices just derived. If we introduce the formal operator
\[ \tilde{\partial}_t = (\partial_t R_k - \eta_k R_k) \frac{\partial}{\partial R_k}, \]
(31)
where \( \eta_k \) is the multiplet matrix of anomalous dimensions, we can rewrite the flow equation for the bEAA (16) as:
\[ \partial_t \Gamma_k[\varphi; \bar{A}] = \frac{1}{2} \text{Tr} \tilde{\partial}_t \{ \Gamma_k^{(3;0)}[\varphi; \bar{A}] G_k[\varphi; \bar{A}] \}, \]
(32)
In (32) we used the following simple relations:
\[ \tilde{\partial}_t G_k = -G_k \partial_t R_k G_k \quad \tilde{\partial}_t \log G_k = G_k^{-1} \tilde{\partial}_t G_k = G_k \partial_t R_k. \]

In this way, we can rewrite the flow equation for the one-vertices of the bEAA (24) in the following compact form:
\[ \partial_t \Gamma_k^{(1;0)}[\varphi; \bar{A}] = \frac{1}{2} \text{Tr} \tilde{\partial}_t \left\{ \Gamma_k^{(3;0)}[\varphi; \bar{A}] G_k[\varphi; \bar{A}] \right\}, \]
\[ \partial_t \Gamma_k^{(0;1)}[\varphi; \bar{A}] = \frac{1}{2} \text{Tr} \tilde{\partial}_t \left\{ \left( \Gamma_k^{(2;1)}[\varphi; \bar{A}] + R_k^{(1)}[\bar{A}] \right) G_k[\varphi; \bar{A}] \right\}, \]
(33)
while the flow equations for the two-vertices of the bEAA (25) read now:
\[ \partial_t \Gamma_k^{(2;0)} = -\frac{1}{2} \text{Tr} \tilde{\partial}_t \left\{ \Gamma_k^{(3;0)} G_k \Gamma_k^{(3;0)} G_k \right\} + \frac{1}{2} \text{Tr} \tilde{\partial}_t \left\{ \Gamma_k^{(4;0)} G_k \right\}, \]
\[ \partial_t \Gamma_k^{(1;1)} = -\frac{1}{2} \text{Tr} \tilde{\partial}_t \left\{ \left( \Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \Gamma_k^{(3;0)} G_k \right\} + \frac{1}{2} \text{Tr} \tilde{\partial}_t \left\{ \Gamma_k^{(3;1)} G_k \right\}, \]
\[ \partial_t \Gamma_k^{(0;2)} = -\frac{1}{2} \text{Tr} \tilde{\partial}_t \left\{ \left( \Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \left( \Gamma_k^{(2;1)} + R_k^{(1)} \right) G_k \right\}. \]
\[ \partial_t \gamma_k^{(1;0)} = -\frac{1}{2} \]
\[ \partial_t \gamma_k^{(0;1)} = -\frac{1}{2} \]

Figure 1: Diagrammatic representation of the flow equations for \( \partial_t \gamma_k^{(1;0)} \) and \( \partial_t \gamma_k^{(0;1)} \) as given in (24).

\[ + \frac{1}{2} \text{Tr} \partial_t \left\{ \left( \Gamma_k^{(2;2)} + R_k^{(2)} \right) G_k \right\}. \] (34)

This compact form turns out to be very useful since the flow equations (33) and (34) contain fewer terms than their counter-parts (24) and (25), and are thus much more manageable when employed in actual computations. The same reasoning applies to all subsequent equations of the hierarchy and extend to the flow equations for the zero-field proper-vertices \( \gamma_k^{(n;m)} \). The flow equations for the proper-vertices of the gEAA are just those for the bEAA evaluated at \( \varphi = 0 \) and we won’t repeat them here. We will see in the next section how the flow equation for the zero-field proper-vertices just defined can be turned into a powerful computational device to perform computations in the bEAA framework.

### 2.3 Diagrammatic and momentum space techniques

In this section we introduce a useful diagrammatic representation to organize the various contributions to the flow equations for the zero-field proper-vertices \( \gamma_k^{(n;m)} \) and we expose the momentum space rules that enables us to calculate these contributions explicitly. As we said before, all these results are valid for any theory with local gauge invariance. For this reason we try to maintain our notation as general as possible.

As we will see, there are some non-trivial technical steps that have to be made in order to write explicit momentum space flow equations for zero-field proper-vertices with some background legs, i.e. containing the vertices \( \gamma^{(n;m)} \) with \( m > 0 \). This issue is related to the dependence of the cutoff kernel \( R_k[\bar{A}] \) on the cutoff operator which is constructed using the background field. The functional derivatives of the cutoff kernel \( R_k^{(m)}[\bar{A}] \) can be calculated as terms of a Taylor series expansion of the cutoff kernel with respect to the background field. We will perform this expansion using the perturbative expansion for the (un-traced) heat kernel developed in [12] and reviewed in appendix A.
2.3.1 Diagrammatic rules

We start to introduce the diagrammatic rules used to represent the hierarchy of flow equations for the zero-field proper-vertices. In particular, we show how to represent equations (24) and (25) graphically. The virtue of diagrammatic techniques is that they allow to switch from coordinate space to momentum space straightforwardly.

When considering vertices with background lines, it is useful to introduce the “tilde” bEAA 4 defined by:

\[
\tilde{\Gamma}_k[\varphi; \bar{A}] = \Gamma_k[\varphi; \bar{A}] + \Delta S_k[\varphi; \bar{A}],
\]

and to define the related “tilde” zero-field proper-vertices

\[
\tilde{\gamma}^{(n;m)}_k = \gamma^{(n;m)}_k + \Delta S^{(n;m)}_k[0;0].
\]

Obviously, for every \( n \) the relation \( \tilde{\gamma}^{(n;0)}_k = \gamma^{(n;0)}_k \) holds. Hence, if we want, we can re-phrase the flow equations for the zero-field proper-vertices solely in terms of the tilde vertices \( \tilde{\gamma}^{(n;m)}_k \).

We represent the zero-field regularized propagator \( G_k[0;0] \) with an internal continuous line, the cutoff insertions \( \partial_t R_k[0] \) are indicated with a crossed circle and the zero-field proper-vertices \( \tilde{\gamma}^{(n;m)}_k \) are represented as vertices with \( n \) external continuous lines (fluctuation legs) and \( m \) external thick wavy lines (background legs). Note that \( \Delta S^{(n;m)}_k[0;0] = 0 \) if \( n > 2 \) since the cutoff action is quadratic in the fluctuation fields. This diagrammatic rules are summarized graphically as follows:

Finally, to every closed loop we associate a coordinate or a momentum \( \int \Omega \int_q \) integral (\( \Omega \) is the space-time volume), together with the factor \( \partial_t R_k - \eta R_k \). Here the anomalous dimension \( \eta_k \) pertains to the fields present in the cutoff action. The application of these diagrammatic rules to the flow equations (24) for the zero-field one-vertices \( \partial_t \gamma^{(1;0)}_k \) and \( \partial_t \gamma^{(0;1)}_k \) gives the representation of Figure 1, while the flow equations (25) for the zero-field two-vertices \( \partial_t \gamma^{(2;0)}_k \), \( \partial_t \gamma^{(1;1)}_k \) and \( \partial_t \gamma^{(0;2)}_k \) can be represented as in Figure 2.

---

4Note that \( \tilde{\Gamma}_k[\varphi; \bar{A}] \) is the actual Legendre transform of the functional generator of connected correlation functions \( W_k[J; \bar{A}] \).

5We define \( \int_x \equiv \int d^d x \) and \( \int_q \equiv \int d^d q / (2\pi)^d \).
\[ \partial_t \gamma_k^{(2;0)} = \quad -\frac{1}{2} \quad -\frac{1}{2} \]

\[ \partial_t \gamma_k^{(1;1)} = \quad -\frac{1}{2} \quad -\frac{1}{2} \]

\[ \partial_t \gamma_k^{(0;2)} = \quad -\frac{1}{2} \quad -\frac{1}{2} \quad +\frac{1}{2} \]

Figure 2: Diagrammatic representation of the flow equations for the vertices \( \partial_t \gamma_k^{(2;0)} \), \( \partial_t \gamma_k^{(1;1)} \) and \( \partial_t \gamma_k^{(0;2)} \) as in equation (25).

As explained in the previous section, it is sometimes useful to work with the set of flow equations for the bEAA (33) and (34) written employing the formal operator \( \tilde{\partial}_t \) defined in (31). In this case there is no explicit insertion of the cutoff term \( \partial_t R_k[0] \) in the loops, but to every loop is now associated an integration together with the action of this formal operator, i.e. \( \int_x \tilde{\partial}_t \) in coordinate space or \( \Omega \int_q \tilde{\partial}_t \) in momentum space. In this way, the flow equations (33) for the zero-field one-vertices \( \partial_t \gamma_k^{(1;0)} \) and \( \partial_t \gamma_k^{(0;1)} \) can be represented as in Figure 3, while equations (34) for the zero-field two-vertices \( \partial_t \gamma_k^{(2;0)} \), \( \partial_t \gamma_k^{(1;1)} \) and \( \partial_t \gamma_k^{(0;2)} \) can be represented as in Figure 4.

Note that in this last representation, the flow equations for the different one-vertices or two-vertices assume a more symmetric form with respect to each other. This reflects in a computational advantage, especially when considering the flow equation for the two-vertex \( \gamma_k^{(0;2)} \), where the additional terms involving cutoff kernel vertices are accounted for by the use of the tilde vertices and by the action of the operator \( \tilde{\partial}_t \). We will see in section 3.2.2 that this fact is very useful.

2.3.2 Momentum space representation

We are now in the position to write down the flow equations for the zero-field proper-vertices \( \gamma_k^{(nm)} \) of the bEAA in momentum space. The only non-standard step is to write the momentum space representation of the terms involving functional derivatives of the cutoff kernel
\[ \partial_t \gamma_k^{(1;0)} = \frac{1}{2} \]
\[ \partial_t \gamma_k^{(0;1)} = \frac{1}{2} \]

Figure 3: Diagrammatic representation of the flow equation for the one-vertices \( \partial_t \gamma_k^{(1;0)} \) and \( \partial_t \gamma_k^{(0;1)} \) as given in (33).

\( R_k[\bar{A}] \) with respect to the background field, i.e. the momentum representation of the vertices \( \gamma_k^{(2;1)} \) and \( \gamma_k^{(2;2)} \). The zero-field proper-vertex \( \gamma_k^{(3;0)} \) is represented graphically by the following diagram:

Here we use capital letters to indicate general composite indices, in the case of non-abelian gauge theories these have to be interpreted as \( A = a_\alpha, B = b_\beta, C = c_\gamma \), while, for example, in the gravitational context they have to interpreted as \( A = \alpha_\beta, B = \gamma_\delta, C = \epsilon_\kappa \). Note that each index is associated with a momentum variable, so that \( A, B, C \) are the indices of the related momenta \( q, p, -q - p \) respectively. Note also that we always define ingoing momenta as being positive. The two-fluctuations one-background zero-field proper-vertex \( \tilde{\gamma}_k^{(2;1)} = \gamma_k^{(2;1)} + \Delta S_k^{(2;1)}[0; 0] \) is represented graphically by the diagram:

The non-trivial technical step is to write the explicit momentum space representation for the zero-field proper-vertex \( \tilde{\gamma}_k^{(2;1)} \). We start by stating the final result and we postpone the details of the derivations to appendix A:

\[ [\tilde{\gamma}_k^{(2;1)}]^{ABC} = [\gamma_k^{(2;1)}]^{ABC} + [l_{q,-q-p,p}]^{ABC} R_{q+p,q}^{(1)} \]
\[ \partial_t \gamma_k^{(2;0)} = -\frac{1}{2} + \frac{1}{2} \]
\[ \partial_t \gamma_k^{(1;1)} = -\frac{1}{2} + \frac{1}{2} \]
\[ \partial_t \gamma_k^{(0;2)} = -\frac{1}{2} + \frac{1}{2} \]

Figure 4: Diagrammatic representation of the flow equations for the two-vertices \( \partial_t \gamma_k^{(2;0)}, \partial_t \gamma_k^{(1;1)} \) and \( \partial_t \gamma_k^{(0;2)} \) as in equation (34).

In (37) we introduced the “cutoff operator action” \( L[\varphi; \vec{A}] \), defined as that action whose Hessian with respect to \( \varphi \) is the cutoff operator and we defined its vertices as

\[
\Pi_{x_1 \ldots x_n y_1 \ldots y_m}^{(n;m)} \equiv L^{(n;m)}[0; 0]_{x_1 \ldots x_n y_1 \ldots y_m}.
\]

Furthermore,

\[
R_{q+p,q}^{(1)} \equiv \frac{R_{q+p} - R_q}{(q + p)^2 - q^2}
\]

(38)

represents the first finite-difference derivative of the cutoff shape function \( R_q \equiv R_k(q^2) \). The momentum space representation (37) is of crucial importance since it gives, together with the generalization to higher vertices given in the following, access to the computational use of the flow equations for the zero-field proper-vertices \( \gamma_k^{(n;m)} \). The four-fluctuation vertex \( \gamma_k^{(4;0)} \) is represented graphically as:

Note that here we are giving the four-vertex only for a particular combination of momenta, which is not the most general one, since this will be the case that we will use in section 3. The two-fluctuations two-backgrounds vertex \( \tilde{\gamma}_k^{(2;2)} = \gamma_k^{(2;2)} + \Delta S_k^{(2;2)}[0; 0] \) is represented instead by the following diagram:
The momentum space representation of \( \tilde{\gamma}^{(2;2)}_k \), derived in appendix B, turns out to be:

\[
[\tilde{\gamma}^{(2;2)}_k]^{ABCD} = [\tilde{\gamma}^{(2;2)}_{q,-q,p,-p}]^{ABCD} + \left[ \tilde{\gamma}^{(2;2)}_{q,-q,p,-p} \right]^{ABM} [\tilde{\gamma}^{(2;1)}_{q+p,-q,-p}]^{MCD} R^{(2)}_{q+p,q}.
\]

Note that now the cutoff operator action \( L[\phi; \bar{A}] \) enters both as a four-vertex and as a product of two three-vertices, and that this time we need to consider the second finite-difference derivative of the cutoff shape function \( R^{(2)}_q \):

\[
R^{(2)}_{q+p,q} = \frac{2}{(q + p)^2 - q^2} \left[ \frac{R_{q+p} - R_q}{(q + p)^2 - q^2} - R'_q \right].
\]

The version of (39) with general momenta is given in appendix B. Equations (37) and (39) are the example of the kind of relations needed to obtain the explicit momentum space representation of the flow equations for the zero-field proper-vertices of the bEAA. With the technique of appendix B one can find the explicit representation for all the other zero-field proper-vertices \( \tilde{\gamma}^{(n;m)}_k \).

The finite-difference derivatives (38) and (40) can be expanded for small external momenta as follows:

\[
R^{(1)}_{q+p,q} = R'_q + p \cdot q R''_q + \frac{1}{2} p^2 R'''_q + \frac{2}{3} (p \cdot q)^2 R^{(3)}_q + O(p^3)
\]

\[
R^{(2)}_{q+p,q} = R''_q + \frac{2}{3} p \cdot q R^{(3)}_q + \frac{1}{3} p^2 R^{(3)}_q + \frac{1}{3} (p \cdot q)^2 R^{(4)}_q + O(p^3).
\]

As we will see in section 3, the “correction terms” in (41), proportional to \( p, p^2 \), or higher, are those needed to make the flow of the zero-field proper-vertices \( \tilde{\gamma}^{(m)}_k = \gamma^{(0;m)}_k \) fully transverse, as they should be by construction.

We are now ready to write the flow equations for the zero-field proper-vertices of the bEAA in their full momentum space form. We will present solely the flow equations for the two-vertices, equations (25) and (34), since it is from these equations that in the next section we will extract the beta functions of the gauge coupling. All other equations can be derived
using the rules stated in the previous subsection and their generalizations.

We find the following momentum space representation for the first equation in (25), describing the flow of the zero-field fluctuation-fluctuation two-vertex:

\[
[\partial_t \gamma^{(2;0)}_{p,-p}]^{AB} = \Omega \int_q (\partial_t R_q - \eta R_q)[G_q]^{12}[\gamma^{(2;1)}_{q,-q-p,p}]^{2A3}[G_{q+p}]^{34}[\gamma^{(3;0)}_{q+p,-q,-p}]^{4B5}[G_q]^{51}
- \frac{1}{2} \Omega \int_q (\partial_t R_q - \eta R_q)[G_q]^{12}[\gamma^{(3;0)}_{q,-q,p,-p}]^{2AB3}[G_q]^{31}.
\] (42)

In (42) and in the following relations \(\eta\) is the multiplet matrix of anomalous dimensions of the fluctuation fields in \(\varphi\). Note also that we are using the generalized notation for the indices introduced before and we use just integers to denote dummy indices. With respect to the first equation in Figure 2, the first line in (42) is the contribution from the first diagram, while the second line is the contribution from the second one. The second equation in (25), describing the flow of the fluctuation-background zero-field two-vertex, takes the following form:

\[
[\partial_t \gamma^{(1;1)}_{p,-p}]^{AB} = \Omega \int_q (\partial_t R_q - \eta R_q)[G_q]^{12}[\gamma^{(2;1)}_{q,-q-p,p}]^{2A3}[G_{q+p}]^{34}[\gamma^{(3;0)}_{q+p,-q,-p}]^{4B5}[G_q]^{51}
- \frac{1}{2} \Omega \int_q (\partial_t R_q - \eta R_q)[G_q]^{12}[\gamma^{(3;1)}_{q,-q,p,-p}]^{2AB3}[G_q]^{31}
- \Omega \int_q [\gamma^{(2,1)}_{q,-q-p,p} (\partial_t R^{(1)}_{q+p,q} - \eta R^{(1)}_{q+p,q})]^{4A1}
\times [G_{q+p}]^{12}[\gamma^{(3;0)}_{q+p,-q,-p}]^{2B3}[G_q]^{34}.
\] (43)

In (43) there are now two tilde vertices since, referring to the second equation in Figure 2, there is a background line attached to, respectively, a three-vertex, a four-vertex and to the factor \(\partial_t R_k[A]\). The contribution from these three diagrams are respectively the first, second and third lines of (43). The last equation of (25) takes the following form:

\[
[\partial_t \gamma^{(0;2)}_{p,-p}]^{AB} = \Omega \int_q (\partial_t R_q - \eta R_q)[G_q]^{12}[\gamma^{(2;1)}_{q,-q-p,p}]^{2A3}[G_{q+p}]^{34}[\gamma^{(2;1)}_{q+p,-q,-p}]^{4B5}[G_q]^{51}
- \frac{1}{2} \Omega \int_q (\partial_t R_q - \eta R_q)[G_q]^{12}[\gamma^{(2;2)}_{q,-q,p,-p}]^{2AB3}[G_q]^{31}
- \Omega \int_q [\gamma^{(2,1)}_{q,-q,-p} (\partial_t R^{(1)}_{q+p,q} - \eta R^{(1)}_{q+p,q})]^{1A2}[G_{q+p}]^{23}[\gamma^{(2;1)}_{q+p,-q,-p}]^{3B4}[G_q]^{41}
+ \frac{1}{2} \Omega \int_q [\gamma^{(2,2)}_{q,-q,p,-q} (\partial_t R^{(1)}_{q+p,q} - \eta R^{(1)}_{q+p,q})]^{1AB2}[G_q]^{21}
\]
\[ + [\zeta_{q,-q-p,p}^{(2;1)}]^{1A3} [\zeta_{q+p,-q,-p}^{(2;1)}]^{3B2} (\partial_t R_{q+p,q}^{(2)} - \eta R_{q+p,q}^{(2)}) [G_q]^{21} \] . \tag{44}

Equation (44) represents the flow of the zero-field background-background two-vertex of the bEAA and thus every term is written in terms of the tilde vertices and of the cutoff action \( L[\varphi; \bar{A}] \). As we explained earlier, these equations are very general and can be adapted to every theory with local gauge symmetry.

In terms of the compact representation introduced in subsection 2.2.2 using the formal operator \( \tilde{\partial}_t \) defined in (31), the flow equations for the zero-field two-vertices of the bEAA are given in equation (34). These are represented graphically in Figure 4 and all three have the same overall structure. The flow of the zero-field fluctuation-fluctuation two-vertex has the following momentum space representation:

\[
[\partial_t \zeta_{p,-p}^{(2,0)}]^{AB} = -\frac{1}{2} \Omega \int_q \tilde{\partial}_t \left\{ [\zeta_{q,-q-p,p}^{(3,0)}]^{4A1} [G_{q+p}]^{12} [\zeta_{q+p,-q,-p}^{(3,0)}]^{2B3} [G_q]^{34} \right\} \\
+ \frac{1}{2} \Omega \int_q \tilde{\partial}_t \left\{ [\zeta_{q,-q,p,-p}^{(4,0)}]^{1AB2} [G_q]^{21} \right\} . \tag{45}
\]

The second equation in (34), shown in Figure 4, expresses the flow of the zero-field fluctuation-background two-vertex and differs from (45) in two tilde vertices:

\[
[\partial_t \zeta_{p,-p}^{(1;1)}]^{AB} = -\frac{1}{2} \Omega \int_q \tilde{\partial}_t \left\{ [\zeta_{q,-q-p,p}^{(2;1)}]^{4A1} [G_{q+p}]^{12} [\zeta_{q+p,-q,-p}^{(2;1)}]^{2B3} [G_q]^{34} \right\} \\
+ \frac{1}{2} \Omega \int_q \tilde{\partial}_t \left\{ [\zeta_{q,-q,p,-p}^{(3;1)}]^{1AB2} [G_q]^{21} \right\} . \tag{46}
\]

Finally, the compact form for the flow of the zero-field background-background two-vertex is:

\[
[\partial_t \zeta_{p,-p}^{(0;2)}]^{AB} = -\frac{1}{2} \Omega \int_q \tilde{\partial}_t \left\{ [\zeta_{q,-q-p,p}^{(2;1)}]^{4A1} [G_{q+p}]^{12} [\zeta_{q+p,-q,-p}^{(2;1)}]^{2B3} [G_q]^{34} \right\} \\
+ \frac{1}{2} \Omega \int_q \tilde{\partial}_t \left\{ [\zeta_{q,-q,p,-p}^{(2;2)}]^{1AB2} [G_q]^{21} \right\} . \tag{47}
\]

Note that in equation (47) all zero-field proper-vertices are tilde vertices. As already said, equation (47), as equation (44), represents also the flow of the zero-field proper-vertex \( \bar{\gamma}_k^{(2)} \) of the gEAA since we have that \( \partial_t \bar{\gamma}_k^{(2)} = \partial_t \zeta_{k}^{(0;2)} \).

Thus the flow equation for the zero-field proper-vertices of the bEAA are formally as those of the standard EAA when written in terms of the formal operator \( \tilde{\partial}_t \) but with tilde
vertices in place of the standard vertices. All the non-trivial dependence on the cutoff kernel is in this way hidden in the dependence of the tilde vertices on it. This turns out to be a very useful property in actual computations. One can thus naively draw the diagrams as in the non-background formalism with the only caveat that the vertices are tilde vertices and that these can be represented using the rules derived in subsection 2.3.1. Remember also that \( \tilde{\gamma}^{(n;0)} = \gamma^{(n;0)} \) and \( \tilde{\gamma}^{(m)} = \gamma^{(0;m)} \). Clearly, these are only the first equations of the respective hierarchies and the results exposed in this section are valid for all the subsequent equations of the hierarchy, for both the zero-field proper-vertices of the bEAA and of the gEAA.

Equations (42-47), together with the momentum space rules of subsection 2.3.1, are the main result of this paper. A lot of information about the flow of both the bEAA and of the gEAA can be extracted already from the flow of the zero-field two-vertices described by these equations.

The results of this section, when combined with the flow equations of the previous one, constitute the basis for a concrete framework in which all truncations of the bEAA which are analytic in the fields can be treated. In particular, one can consider a “curvature expansion” where the gEAA is expanded in powers of the (generalized) curvatures, and where the running is encoded in \( k \)-dependent structure functions.\(^6\) As we said, the methods presented in this section can be extended to gravity, non-linear sigma models and membranes.

Finally, we remark that truncations like \( \tilde{\Gamma}_k[A] = \int W(F^2) \), where \( W \) is an arbitrary function of the invariant \( F^2 \equiv F_{\mu\nu}F^{\mu\nu} \), in non-abelian gauge theories \([18]\), or like \( \tilde{\Gamma}_k[g] = \int \sqrt{g}f(R) \), in gravity \([19]\), cannot be treated with the present method and one needs to resort to other techniques to treated them. To be more precise, one can treat only polynomial approximations to these functions since we ultimately set the background gauge field, or the background metric, to zero.

\(^6\)See \([10]\) for a first one-loop application in the context of gravity.
3 An application: non-abelian gauge theories

In this section we apply the formalism developed in section 2 to non-abelian gauge theories with gauge group $SU(N)$. We derive the beta function for the gauge coupling and we show how standard results obtained by heat kernel methods can be recovered in our new framework. Then we calculate the anomalous dimensions of the fluctuation and ghost fields, $\eta_{a,k}$ and $\eta_{c,k}$, and we use them to “close” the beta function of $g_k$, obtaining new RG improved forms for this. Finally, we study the phenomenology associated to these RG flows.

3.1 Truncation ansatz

As a simple application of the formalism, we consider a truncation ansatz for the gEAA which is the RG improvement of the classical action:

$$\bar{\Gamma}_k[A] = I[A] \equiv \frac{1}{4} \int d^4x F^a_{\mu\nu} F^{a\mu\nu}.$$  \hspace{1cm} (48)

The quantum gauge field can be expressed in terms of the background and fluctuation fields as follows:

$$A_\mu = Z^{1/2}_{\bar{A},k} \bar{A}_\mu + Z^{1/2}_{a,k} a_\mu ,$$  \hspace{1cm} (49)

where we rescaled the gauge fields to account for their renormalization. $Z_{a,k}$ is the running wave-function renormalization of the fluctuation field while $Z_{\bar{A},k}$ is the running wave-function renormalization of the background field. In the background field formalism the gauge coupling is related to the wave-function renormalization of the background gauge field $[15]$:

$$g_k = Z^{-1/2}_{\bar{A},k} .$$  \hspace{1cm} (50)

It is thus sufficient to determine the scale dependence of $Z_{\bar{A},k}$ to find the running of $g_k$. One defines the anomalous dimension of the background field:

$$\eta_{\bar{A},k} = - \partial_t \log Z_{\bar{A},k} = - g_k^2 \partial_t Z_{\bar{A},k} ;$$  \hspace{1cm} (51)

then the beta function of the gauge coupling is:

$$\partial_t g_k = \frac{1}{2} \eta_{\bar{A},k} g_k .$$  \hspace{1cm} (52)
Inserting (49) into equation (203) from appendix B gives the following expansion for the gEAA (48):

\[
\begin{align*}
\Gamma_k[Z_{A,k}^{1/2}A + Z_{a,k}^{1/2}a] &= Z_{A,k}\Gamma_k[A] + Z_{A,k}^{1/2}Z_{a,k}^{1/2} \int d^4x F^{a}_{\mu\nu}(D_{\mu}a_{\nu})^a + \\
&\quad + \frac{1}{2}Z_{a,k} \int d^4x a_{\mu}^a \left[ (-D^2)^{ab}g^{\mu\nu} + 2 f^{abc}F_{c\mu\nu} + D^{a\mu}D^{b\nu}\right] a_{\nu}^b + \\
&\quad + g_k Z_{a,k}^{3/2} f^{abc} \int d^4x (D_{\mu}a_{\nu})^a a_{\mu}^b a_{\nu}^c + \\
&\quad + \frac{1}{4} g_k^2 Z_{a,k}^2 f^{abc} f^{ade} \int d^4x a_{\mu}^b a_{\nu}^c a_{\mu}^d a_{\nu}^e. 
\end{align*}
\]

(53)

We used (50) to set \( g_k Z_{A,k}^{1/2} = 1 \) in the covariant derivatives and in the second term of the second line. Quantities with a bar are constructed with the background gauge field of (53). We approximate the rEAA to be the sum of the bare background gauge-fixing action (5) and of the bare background ghost action (6) with rescaled fluctuation and ghost fields:

\[
\hat{\Gamma}_k[Z_{A,k}^{1/2}a, Z_{c,k}^{1/2}c, Z_{A,k}^{1/2}A] = Z_{a,k}S_{gf}[a; A] + Z_{c,k}S_{gh}[a, c; A].
\]

(54)

With these definitions our truncation comprises three running couplings \( \{g_k, Z_{a,k}, Z_{c,k}\} \), or three anomalous dimensions \( \{\eta_{\bar{A},k}, \eta_{a,k}, \eta_{c,k}\} \), where the anomalous dimensions of the fluctuation and ghost fields are defined as:

\[
\eta_{a,k} = -\partial_t \log Z_{a,k} \quad \eta_{c,k} = -\partial_t \log Z_{c,k}.
\]

(55)

We can say that \( g_k \) parametrizes the RG evolution of the gEAA, while \( Z_{a,k} \) and \( Z_{c,k} \) account for the influence of the RG evolution of the full bEAA. This truncation is an example of bi–field truncation in the nomenclature of [11] since the ansatz has a non-trivial \( k \)–dependence in both \( \hat{\Gamma}_k[A] \) and \( \hat{\Gamma}_k[a, \bar{c}, c; A] \). More general bi–field truncations will include, for example, terms like the fluctuation gluon mass. Even if these truncations can be treated with the present methods we will not treat these cases for the moment.

### 3.2 Derivation of the beta functions

In this subsection we calculate the anomalous dimensions of the background, fluctuation and ghost fields. In subsection 3.2.1 we review the standard heat kernel calculation of \( \eta_{\bar{A},k} \), while

\[7\text{Again we used } g_k Z_{A,k}^{1/2} = 1.\]
in subsection 3.2.2 we re-derive it by the methods presented in section 2. In subsection 3.2.3 we use the methods of section 2 to calculate $\eta_{a,k}$ and $\eta_{c,k}$.

3.2.1 Heat kernel calculation of $\eta_{\bar{A},k}$

We review how to calculate the beta function $\partial_t Z_{\bar{A},k}$ of the wave-function renormalization of the background field, using the standard method based on the local heat kernel expansion [2, 3].

If we take a scale derivative of the expansion (53) and we evaluate at $a_\mu = 0$, we find:

$$\partial_t \Gamma_k[Z^{1/2}_{\bar{A},k}] = \partial_t Z_{\bar{A},k} \frac{1}{4} \int d^d x \, F_{\mu\nu}^a F^{a\mu\nu}. \quad (56)$$

From (56) we see that to extract $\partial_t Z_{\bar{A},k}$ we need to consider the flow equation (21) for the gEAA, which we rewrite here for convenience

$$\partial_t \Gamma_k[\bar{A}] = \frac{1}{2} \text{Tr} \left( \Gamma^{(2,0,0,0)}_k[0,0,0;\bar{A}] + \Delta S^{(2,0,0,0)}_k[0,0,0;\bar{A}] \right)^{-1} \partial_t \Delta S^{(2,0,0,0)}_k[0,0,0;\bar{A}]$$

$$- \text{Tr} \left( \Gamma^{(0,1,1,0)}_k[0,0,0;\bar{A}] + \Delta S^{(0,1,1,0)}_k[0,0,0;\bar{A}] \right)^{-1} \partial_t \Delta S^{(0,1,1,0)}_k[0,0,0;\bar{A}], \quad (57)$$

and extract, from the functional traces on rhs, the coefficient of the invariant $\frac{1}{4} \int \bar{F}^2$.

As a first step to evaluate the rhs of (57), we make the following choices for the cutoff kernels of the fluctuation and ghost fields:

$$\Delta S^{(2,0,0,0)}_k[0,0,0;\bar{A}] = Z_{a,k} R_k[\bar{A}]$$

$$\Delta S^{(0,1,1,0)}_k[0,0,0;\bar{A}] = Z_{c,k} R^{gh}_k[\bar{A}], \quad (58)$$

Note that, consistently with the previous discussion, we introduced in the respective cutoff kernels (58) the wave-function renormalization of the fluctuation fields $Z_{a,k}$ and $Z_{c,k}$. We specify the differential operators that will be the arguments of the cutoff kernels $R_k[\bar{A}]$ and $R^{gh}_k[\bar{A}]$ in a moment.

Second, we calculate the Hessians present in the denominators of the traces on the rhs of the flow equation (57). Using the general decomposition of the bEAA given in (11), these can be written as follows:

$$\Gamma^{(2,0,0,0)}_k[a,c;\bar{A}] = \hat{\Gamma}^{(2)}_k[\bar{A} + a] + \hat{\Gamma}^{(2,0,0,0)}_k[a,c;\bar{A}]$$

$$\Gamma^{(0,1,1,0)}_k[a,c;\bar{A}] = \hat{\Gamma}^{(0,1,1,0)}_k[a,c;\bar{A}]. \quad (59)$$
From (53) one can read-off the Hessian of the gEAA:

\[ \tilde{\Gamma}^{(2)}_k [\bar{A} + a]^{ab\mu\nu} = Z_{a,k} \left[ (-D^2)^{ab} g^{\mu\nu} + 2 f^{abc} F_{c\mu\nu} + D^{ac\mu} D^{cb\nu} \right] , \]  

(60)

where \((-D^2)^{ab} \equiv -D^a_{\mu} D^b_{\nu}\). Note that we can write the second term as \(2 f^{abc} F_{c\mu\nu} = -2(F^{\mu\nu})^{ab}\). The Hessians of the rEAA (54) are just the Hessians of the bare background gauge-fixing and background ghost actions:

\[ \tilde{\Gamma}^{(2,0,0;0)}_k[a, \bar{c}, c; \bar{A}] = Z_{a,k} S_{g_{gf}}^{(2,0)}[a; \bar{A}] \]
\[ \tilde{\Gamma}^{(0,1,1;0)}_k[a, \bar{c}, c; \bar{A}] = Z_{c,k} S_{g_{gh}}^{(0,1,1;0)}[a, \bar{c}, c; \bar{A}] , \]  

(61)

from (5) and (6) we can easily read off the following forms:

\[ \tilde{\Gamma}^{(2,0,0;0)}_k[a, \bar{c}, c; \bar{A}]^{ab\mu\nu} = -\frac{1}{\alpha} Z_{a,k} \bar{D}^{ac\mu} \bar{D}^{cb\nu} \]
\[ \tilde{\Gamma}^{(0,1,1;0)}_k[a, \bar{c}, c; \bar{A}]^{ab} = -Z_{c,k} \bar{D}^{ac}_{\mu} D^{cb\mu} . \]  

(62)

In the flow equation (57) we need the Hessian (59) evaluated at \(a_{\mu} = \bar{c} = c = 0\); combining (60) and (62) we find:

\[ \Gamma^{(2,0,0;0)}_k[0, 0, 0; \bar{A}]^{ab\mu\nu} = Z_{a,k} \left[ (-\bar{D}^2)^{ab} g^{\mu\nu} + 2 f^{abc} F_{c\mu\nu} + \left( 1 - \frac{1}{\alpha} \right) \bar{D}^{ac\mu} \bar{D}^{cb\nu} \right] \]
\[ \Gamma^{(0,1,1;0)}_k[0, 0, 0; \bar{A}]^{ab} = Z_{c,k} \delta^{ab} (-\bar{D}^2) . \]  

(63)

From now on we set the gauge-fixing parameter to \(\alpha = 1\) in order to make the first Hessian in (63) proportional to the Laplace-type differential operator

\[ (D_T^{\mu\nu})^{ab} = (-D^2)^{ab} g^{\mu\nu} - 2(F^{\mu\nu})^{ab} . \]  

(64)

We need now to choose the cutoff operator, the eigenvalues of which, we compare to the RG scale \(k\) to separate the fast field modes from the slow field modes. Without introducing any running coupling in the cutoff action, apart for the wave-function renormalization of the fluctuation fields, there are two possible choices in the gauge sector. Looking back at equation (63), we see that we can take as cutoff operator the covariant Laplacian \(-\bar{D}^2\) or the full Laplace-type differential operator \(\bar{D}_T\). We call cutoff actions constructed in this way, respectively, type I and type II. In the ghost sector there is no such freedom and we choose as cutoff operator, in both cases, the covariant Laplacian \(-\bar{D}^2\).
We start by considering the type II case. In view of (63) we can rewrite the flow equation (57) as follows:
\[
\partial_t \bar{\Gamma}_k[\bar{A}] = \frac{1}{2} \text{Tr}_{1c} \frac{\partial_t R_k(\bar{D}_T) - \eta_{a,k} R_k(\bar{D}_T)}{\bar{D}_T + R_k(\bar{D}_T)} - \text{Tr}_{0c} \frac{\partial_t R_k(-\bar{D}^2) - \eta_{c,k} R_k(-\bar{D}^2)}{-\bar{D}^2 + R_k(-\bar{D}^2)},
\]
where we emphasized that the traces are also over space-time as well as color indices. We proceed by employing the local heat kernel expansion (for the operators \(\bar{D}_T\) and \(-\bar{D}^2\)) to expand the traces on the rhs side of equation (65) in terms of gauge invariant operators. The functional trace of a Laplace-type second order differential operator \(\mathcal{O} = -\bar{D}^2 + U\) can be expanded as:
\[
\text{Tr} \ h(\mathcal{O}) = \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} B_{2n}(\Delta) Q_{\frac{d}{2}-n}[h],
\]
where the first heat kernel coefficients \(B_{2n}(\mathcal{O})\) are:
\[
B_0(\mathcal{O}) = \int d^d x \ \text{tr} \ 1
\]
\[
B_2(\mathcal{O}) = \int d^d x \ \text{tr} \ U
\]
\[
B_4(\mathcal{O}) = \int d^d x \ \text{tr} \left( \frac{1}{2} U^2 + \frac{1}{6} D^2 U + \frac{1}{12} [D_\mu, D_\nu][D^\mu, D^\nu] \right),
\]
and the \(Q\)-functionals are defined as:
\[
Q_n[h] = \begin{cases} 
\frac{1}{\Gamma(n)} \int_0^\infty dz \ z^{n-1} h(z) & n > 0 \\
(-1)^n h^{(n)}(0) & n \leq 0.
\end{cases}
\]
For more details on the applications of the heat kernel expansion to the calculation of functional traces see [3]. The invariant \(\frac{1}{4} \int F^2\) is contained in the heat kernel coefficients \(B_4(\bar{D}_T)\) and \(B_4(-\bar{D}^2);\) a use of equation (66) gives the following:
\[
\partial_t \bar{\Gamma}_k[\bar{A}] \big|_{\int F^2} = \frac{1}{(4\pi)^{d/2}} \left\{ \frac{1}{2} B_4(\bar{D}_T) Q_{\frac{d}{2}-2} \left[ (\partial_t R_k - \eta_{a,k} R_k) G_k \right] \\
- B_4(-\bar{D}^2) Q_{\frac{d}{2}-2} \left[ (\partial_t R_k - \eta_{c,k} R_k) G_k \right] \right\},
\]
where we introduced the regularized propagator
\[
G_k(z) = \frac{1}{z + R_k(z)}.
\]
For the differential operator $\tilde{D}_T$ we find the following heat kernel coefficient:\footnote{Note that $\text{tr} U^2 \equiv U^{a\mu\nu} U_{a\mu\nu}$.}

$$B_4(\tilde{D}_T) = \int d^d x \left\{ \frac{1}{2} \text{tr} U^2 + \frac{1}{12} \text{tr} \left[ D_\mu, D_\nu \right] \left[ D^\mu, D^\nu \right] \right\}$$

$$= \int d^d x \left\{ \frac{1}{2} \left( 2 f^{abc} \bar{F}_{c\mu\nu} \right) \left( 2 f^{bad} \bar{F}_{d\nu\mu} \right) + \frac{1}{12} \left( - f^{abc} \bar{F}_{c\mu\nu} \right) \left( - f^{bad} \bar{F}_{d\nu\mu} \right) \right\}$$

$$= \frac{24 - d}{12} N \int d^d x \bar{F}_{\mu\nu} \tilde{F}^{\mu\nu} , \quad (71)$$

where we used $U^{a\mu\nu} = 2 f^{abc} \bar{F}_{c\mu\nu}$, $[\tilde{D}_\mu, \tilde{D}_\nu]^{ab} = - f^{abc} \tilde{F}_{c\mu\nu}$ and $f^{abc} f^{ab\ell} = N \delta^{cd}$. The ghost operator $\bar{D}^2$, i.e. the Laplacian, has instead the following heat kernel coefficient:

$$B_4(-\bar{D}^2) = \int d^d x \left\{ \frac{1}{12} \text{tr} \left[ D_\mu, D_\nu \right] \left[ D^\mu, D^\nu \right] \right\} = - \frac{N}{12} \int d^d x \bar{F}_{\mu\nu} \bar{F}^{\mu\nu} . \quad (72)$$

Inserting (71) and (72) in (69), comparing with (56) and using (51) finally gives:

$$\eta_{A,k} = - \frac{g_s^2 N}{(4\pi)^{d/2}} \left\{ \frac{24 - d}{6} Q_{d/2} \left[ (\partial_t R_k - \eta_{a,k} R_k) G_k \right] + \frac{1}{3} Q_{d/2} \left[ (\partial_t R_k - \eta_{c,k} R_k) G_k \right] \right\} . \quad (73)$$

Equation (73) gives the type II anomalous dimension of the background field, within the truncation specified by equations (53) and (54), in the gauge $\alpha = 1$, in arbitrary dimension and for general cutoff shape function $R_k(z)$.

When instead we employ the type I cutoff, the gauge trace in the flow equation (65) becomes:

$$\frac{1}{2} \text{Tr}_{1c} \left\{ \partial_t R_k(-\bar{D}^2) - \eta_{a,k} R_k(-\bar{D}^2) \right\} \bar{D}^2 - 2 \bar{F} + R_k(-\bar{D}^2) . \quad (74)$$

To collect all terms proportional to $\frac{1}{4} \int \bar{F}^2$ we expand the denominator in powers of the curvature:

$$\left[ -\bar{D}^2 - 2 \bar{F} + R_k(-\bar{D}^2) \right]^{-1} = g_{\mu\nu} G_k(-\bar{D}^2) + 2 G_k(-\bar{D}^2) \bar{F}_{\mu\nu} G_k(-\bar{D}^2)$$

$$+ 4 G_k(-\bar{D}^2) \bar{F}_{\mu\alpha} G_k(-\bar{D}^2) \bar{F}^\alpha\nu G_k(-\bar{D}^2) + O(\bar{F}^3) . \quad (75)$$

When we insert back the expansion (75) in the trace (74), the third term we obtain is already proportional to $\frac{1}{4} \int \bar{F}^2$:

$$\text{Tr}_{1c} \left\{ \left[ \partial_t R_k(-\bar{D}^2) - \eta_{a,k} R_k(-\bar{D}^2) \right] G_k(-\bar{D}^2) \bar{F}_{\mu\alpha} \bar{F}^\alpha\nu \right\} \int \bar{F}^2 =$$
\[
= \frac{1}{(4\pi)^{d/2}} Q^{\frac{d}{2}} \left[ (\partial_t R_k - \eta_{a,k} R_k) G^3_k \right] \left( N \int d^d x \bar{F}^a_{\mu\nu} F^{a\mu\nu} \right),
\]

the second term we obtain is zero when traced over space-time indices, while the first term we obtain generates a contribution proportional to \(\frac{1}{4} \int \bar{F}^2\) when expanded using the heat kernel:

\[
\text{Tr}_{1c} \left[ (\partial_t R_k (-D^2) - \eta_{a,k} R_k (-D^2)) G_k (-D^2) \right] \bigg|_{\bar{F}^2} = \frac{1}{(4\pi)^{d/2}} Q^{\frac{d}{2}} - 2 \left[ (\partial_t R_k - \eta_{a,k} R_k) G_k \right] \left( -\frac{d}{12} N \int d^d x \bar{F}^a_{\mu\nu} F^{a\mu\nu} \right).
\]

Note that with respect to (72) the heat kernel coefficient in (77) contains an additional factor \(d\), since the trace is over spin one fields. Combining the contributions (75) and (77) gives the type I anomalous dimension of the background field:

\[
\eta_{\bar{A},k} = -\frac{g_k^2 N}{(4\pi)^{d/2}} \left\{ -\frac{d}{6} Q^{\frac{d}{2}} - 2 \left[ (\partial_t R_k - \eta_{a,k} R_k) G_k \right] + 8 Q^{\frac{d}{2}} \left[ (\partial_t R_k - \eta_{a,k} R_k) G^3_k \right] \right. \\
\left. + \frac{1}{3} Q^{\frac{d}{2}} - 2 \left[ (\partial_t R_k - \eta_{c,k} R_k) G_k \right] \right\}.
\]

As (73), this relation is valid within the truncation (53) and (54), in the gauge \(\alpha = 1\), in arbitrary dimension and for general cutoff shape function \(R_k(z)\).

Equation (73) and (78) are the main results of this subsection and represent the RG running of the wave-function renormalization of the background field in the two schemes, type I and II. In subsection 3.3 we will specify the cutoff shape function \(R_k(z)\) in order to obtain explicit forms for the anomalous dimensions.

As a final comment, we remark that to calculate \(\partial_t Z_{\bar{A},k}\) for general values of the gauge-fixing parameter \(\alpha\) (possibly scale dependent) using the heat kernel expansion, we need to be able to evaluate functional traces containing the following non-minimal operator:

\[
(D_T^{\mu\nu})^{ab} = (-D^2)^{ab} g^{\mu\nu} + \left( 1 - \frac{1}{\alpha} \right) D^{ac\mu} D^{cb\nu} - 2(F^{\mu\nu})^{ab}.
\]

A way to handle this is to use the re-summation technique proposed in [9]; otherwise, one can perform the calculation by decomposing the gauge field into its spin components as was done in [18].
$$\partial_t \tilde{\gamma}^{(2)}_k = -\frac{1}{2} \text{(a)} + \frac{1}{2} \text{(b)} + \text{(d)} - \text{(d)}$$

Figure 5: Diagrammatic representation of the compact form of the flow equation for the zero-field two-point function $\tilde{\gamma}^{(2)}_k$ of the gEAA after the field multiplet decomposition within the truncation specified by equations (53) and (54). Thick wavy lines represent the background field $\tilde{A}_\mu$, light wavy lines represent the fluctuation field $a_\mu$, while dotted lines represent the ghost fields $\bar{c}$ and $c$.

### 3.2.2 Calculation of $\eta_{A,k}$ from $\partial_t \tilde{\gamma}^{(2)}_k$

We show how to calculate the anomalous dimension of the background field from the flow equation for the zero-field two-point function $\tilde{\gamma}^{(2)}_k$ of the gEAA employing the techniques introduced in section 2.

In particular, we are going to work with the flow equation for $\tilde{\gamma}^{(2)}_k$ in its compact form given in equation (47). After the field multiplet decomposition, this equation can be represented diagrammatically as shown in Figure 5. In formulas, we can write the flow equation as:

$$\begin{align*}
[\partial_t \tilde{\gamma}^{(2)}_{p,-p}]^{mn\mu\nu} &= -\frac{1}{2} \Omega \int_q \partial_t [a_{p,q}]^{mn\mu\nu} + \frac{1}{2} \Omega \int_q \partial_t [b_{p,q}]^{mn\mu\nu} + \Omega \int_q \partial_t [c_{p,q}]^{mn\mu\nu} \\
&- \Omega \int_q \partial_t [d_{p,q}]^{mn\mu\nu} .
\end{align*}$$

(79)

Here $\Omega = (2\pi)^d \delta(0)$ is a volume factor coming from the momentum integrals. Tensor products in the integrands are:

$$\begin{align*}
[a_{p,q}]^{mn\mu\nu} &= [\tilde{\gamma}^{(2,0,0,1)}_{q,-q,-p,p}]^{\alpha\beta\mu\nu} [G_{q+p}]^{bc\gamma\delta} [\tilde{\gamma}^{(2,0,0,1)}_{q+p,-q,-p,-q}]^{cdn\gamma\delta\nu} [G_{q}]^{r\delta} \delta \alpha \\
[b_{p,q}]^{mn\mu\nu} &= [\tilde{\gamma}^{(2,0,0,2)}_{q,-q,p,-p}]^{\alpha\beta\mu\nu} [G_{q}]^{ba\beta} \delta \beta \\
[c_{p,q}]^{mn\mu\nu} &= [\tilde{\gamma}^{(0,1,1,1)}_{q,-q,-q,p}]^{\alpha\mu\nu} [G_{q+p}]^{bc\gamma\delta} [\tilde{\gamma}^{(0,1,1,1)}_{q+p,-q,p,-q}]^{cdn\gamma\delta\nu} [G_{q}]^{r\delta} \delta c \\
[d_{p,q}]^{mn\mu\nu} &= [\tilde{\gamma}^{(0,1,1,2)}_{q,-q,p,-p}]^{\alpha\beta\mu\nu} [G_{q}]^{bc\beta} \delta a .
\end{align*}$$

(80)

---

*In this subsection, as in the next we omit to write the scale index $k$ on functionals and on couplings to simplify the notation.*
The “tilde” gauge vertices in (80) are the following:

\[
\tilde{\gamma}^{(2,0,0;1)}_{q,-q,-p,p} ab_{m\alpha\beta\mu} = Z_a \left[ I^{(3)}_{q,-q,-p,p} \right]_{ab_{m\alpha\beta\mu}} + Z_a \left[ S_{gf}^{(2;1)}_{q,-q,-p,p} \right]_{ab_{m\alpha\beta\mu}} R^{(1)}_{q+p,p} ,
\]

(note that \( \tilde{\gamma}^{(2,0,0;1)}_{q,-q,-p,p} = \tilde{\gamma}^{(2,0,0;1)}_{q+p,-q,-p} \)) and:

\[
\tilde{\gamma}^{(2,0,0;2)}_{q,-q,-p,p} abmn_{\alpha\beta\mu\nu} = Z_c \left[ S_{gh}^{(0,1,1;1)}_{q,-q,-p,p} \right]_{abmn_{\alpha\beta\mu\nu}} + Z_c \left[ l^{(1;1;1)}_{q,-q,-p,p} \right]_{abmn_{\alpha\beta\mu\nu}} R^{(2)}_{q+p,q} .
\]

The “tilde” ghost vertices are instead:

\[
\tilde{\gamma}^{(0,1,1;1)}_{q,-q,-p,p} ab_{m\mu} = Z_c \left[ S_{gh}^{(0,1,1;1)}_{q,-q,-p,p} \right]_{ab_{m\mu}} + Z_c \left[ l^{(1;1;1)}_{q,-q,-p,p} \right]_{ab_{m\mu}} R^{(1)}_{q+p,q} \]

and

\[
\tilde{\gamma}^{(0,1,1;2)}_{q,-q,-p,p} abmn_{\mu\nu} = Z_c \left[ S_{gh}^{(0,1,1;2)}_{q,-q,-p,p} \right]_{abmn_{\mu\nu}} + Z_c \left[ l^{(1;1;1)}_{q,-q,-p,p} \right]_{abmn_{\mu\nu}} R^{(2)}_{q+p,q} .
\]

The momentum space representations of the vertices of the functionals \( I, S_{gf} \) and \( S_{gh} \) appearing in these equations can be found in appendix B. The vertices \( l \) of the cutoff action will be given after we make the cutoff action specifications. We will soon see that these vertices, when reinserted in equation (79), will correctly conspire to make the flow of \( \partial_t \bar{\gamma}^{(2)} \) correctly transverse.

To proceed we first need to construct the regularized propagators that enter the integrands in (80). The fluctuation field regularized propagator is:

\[
\left[ G^{-1}_p \right]^{\alpha\beta \, ab} = \left[ \gamma^{(2,0,0;0)}_{p,-p} + R_p \right]^{\alpha\beta \, ab} .
\]

Within our truncation, given by equations (53) and (54), we have:

\[
\left[ \gamma^{(2,0,0;0)}_{p,-p} \right]^{ab_{\alpha\beta}} = \left[ \gamma^{(2,0,0;0)}_{p,-p} \right]^{ab_{\alpha\beta}} + Z_a \left[ l^{(2)}_{p,-p} \right]^{ab_{\alpha\beta}} + Z_a \left[ S_{gf}^{(2,0;0)}_{p,-p} \right]^{ab_{\alpha\beta}} = \Omega Z_a \left[ \delta^{ab} \right]^{\alpha\beta} = \Omega Z_a \delta^{ab} \left[ \left( 1 - P \right)^{\alpha\beta} + \frac{1}{\alpha} P^{\alpha\beta} \right] ,
\]

(86)
where we used the two-vertices of $I$ and $S_{gf}$ from appendix B and we introduced the longitudinal $P^{\alpha\beta} = p^{\alpha} p^{\beta} / p^2$ and transverse $(1 - P)^{\alpha\beta}$ projectors. Inserting (86) in (85) gives the following form:

\[
[G_p^{-1}]^{ab\alpha\beta} = \Omega Z_a \delta^{ab} p^2 \left( (1 - P)^{\alpha\beta} + \frac{1}{\alpha} P^{\alpha\beta} \right) + [R_p]^{ab\alpha\beta}.
\]  

(87)

We need now to specify the tensor structure of the cutoff kernel; there are two basic choices:

\[
[R_p]^{ab\alpha\beta} = \Omega Z_a \delta^{ab} \left( (1 - P)^{\alpha\beta} + \frac{1}{\alpha} P^{\alpha\beta} \right) R_p,
\]  

(88)

\[
[R_p]^{ab\alpha\beta} = \Omega Z_a \delta^{ab} g^{\alpha\beta} R_q.
\]  

(89)

Using the following relation that can be easily verified,

\[
[1a + (1 - P)b + Pc]^{-1} = \frac{1}{a + b} (1 - P) + \frac{1}{a + c} P,
\]

we can invert equation (87) to obtain the explicit expression for the regularized propagator. In the case that we are using the cutoff kernel as defined in (88), we find the form:

\[
[G_p]^{ab\alpha\beta} = (\Omega Z_a)^{-1} \left[ \delta^{ab} \frac{1}{p^2 + R_p} (1 - P)^{\alpha\beta} + \delta^{ab} \frac{\alpha}{p^2 + R_p} P^{\alpha\beta} \right],
\]  

(90)

while in the case we are using the cutoff kernel as defined in (89), we get instead:

\[
[G_p]^{ab\alpha\beta} = (\Omega Z_a)^{-1} \left[ \delta^{ab} \frac{1}{p^2 + R_p} (1 - P)^{\alpha\beta} + \delta^{ab} \frac{\alpha}{p^2 + \alpha R_p} P^{\alpha\beta} \right].
\]  

(91)

In this paper we will consider the choice corresponding to (89), since this is the minimal cutoff kernel choice we can make.\(^{10}\) It’s useful to define the transverse and longitudinal regularized propagators:

\[
G_T(z) = \frac{1}{z + R_k(z)} \quad G_L(z) = \frac{\alpha}{z + \alpha R_k(z)}.
\]  

(92)

Note that the transverse regularized propagator is equal to the regularized propagator defined \(^{10}\)In (88), we are introducing in the cutoff kernel the gauge-fixing parameter, if this is running it will give rise to additional terms on the rhs of the flow equation, generated by \(\partial_t R_k\), proportional to \(\partial_t \alpha_k\). A similar choice has been made in [20].
in (70), i.e. \( G_{T,k}(z) = G_k(z) \). We can now write (91) as follows:

\[
[G_p]^{ab\alpha\beta} = (\Omega Z_a)^{-1} \left[ \delta^{ab} (1 - P)^{\alpha\beta} G_T^p + \delta^{ab} P^{\alpha\beta} G_L^p \right]. \tag{93}
\]

Note that \( G_{L,k}(z) = 0 \) if \( \alpha = 0 \) and that \( G_{L,k}(z) = G_{T,k}(z) = G_k(z) \) if \( \alpha = 1 \); from now on we set \( \alpha = 1 \) so that the tensor structure of the regularized propagator is proportional to the identity. The ghost regularized propagator, defined by

\[
[G_p^{-1}]^{ab} = [\gamma^{(0,1,1,0)}_{p,-p} + R_p]^{ab},
\]

is easily obtained from

\[
[\gamma^{(0,1,1,0)}_{p,-p}]^{ab} = Z_c [S^{(0,1,1,0)}_{gh p,-p}]^{ab} = \Omega Z_c \delta^{ab} p^2, \tag{94}
\]

together with the minimal cutoff kernel choice

\[
[R_p^c]^{ab} = \Omega Z_c \delta^{ab} R_p, \tag{95}
\]

in the form:

\[
[G_p^c]^{ab} = (\Omega Z_c)^{-1} \delta^{ab} \frac{1}{p^2 + R_p} = (\Omega Z_c)^{-1} \delta^{ab} G_p. \tag{96}
\]

This completes the construction of the regularized propagators needed in the flow equation (79).

We need now to specify the cutoff operator we employ to cutoff the field modes. Remember from section 2 that the the cutoff operator action, here \( L[a; \bar{A}] \) in the gauge sector and \( L[\bar{c}, c; \bar{A}] \) in the ghost sector, is just the action whose Hessian is the cutoff operator. As in the previous subsection, there are two basic options in the gauge sector, which we called type I and type II, where the cutoff operator is taken to be, respectively, the gauge Laplacian \(-\bar{D}^2\) or the operator \(\bar{D}_T\).

We start to consider the type I case, where the gauge cutoff operator action is the following:

\[
L[a; \bar{A}] = \frac{1}{2} \int d^d x \bar{D}_\mu a_\nu \bar{D}^\mu a_\nu;
\tag{97}
\]

one can easily check that the Hessian of this action is the gauge Laplacian:

\[
L^{(2,0)}[0; \bar{A}]_{xy}^{ab\alpha\beta} = g^{\alpha\beta} \int d^d z \bar{D}_x^{ac} \delta_{xz} \bar{D}_y^{cb} \delta_{zy} = -g^{\alpha\beta} \bar{D}_x^{ac} \bar{D}_y^{cb} \delta_{xy} = (-\bar{D}^2_x)^{ab} g^{\alpha\beta} \delta_{xy}. \tag{98}
\]
The ghost cutoff operator is the gauge Laplacian in both type I and II cases; the ghost cutoff operator action is then simply\textsuperscript{11}

\[ L[\bar{c}, c; \bar{A}] = \int d^d x \bar{D}_\mu \bar{c} \bar{D}^\mu c. \] (99)

Again we have \( L^{(1,1,0)}[0, 0; \bar{A}]_{ab} = (-\bar{D}_{x}^2)^{ab} \delta_{xy} \). The vertices of the actions \( (97) \) and \( (99) \) can be found in appendix B.

Equation \( (79) \) is a tensor equation from which we can obtain two independent scalar equations after we contract it with the projectors \( (1 - P)^{\mu\nu} \) and \( P^{\mu\nu} \). We start by considering the projection in the transverse direction; in particular, we find the following results for the integrands \( (80) \):

\[
(1 - P)^{\mu\nu}[a_{p,q}]^{mn\mu\nu} = (\Omega Z_a)^{-2}(1 - P)^{\mu\nu}[\gamma^{(2,0,0;1)}_{q,-q,-p,p}]^{abm\alpha\beta\mu}[\gamma^{(2,0,0;1)}_{q+p,-q,-p,0}]^{ban\beta\alpha\nu}G_qG_{q+p}
= N\delta^{mn}\left\{8(d-1)p^2 - 4dq^2(1 - x^2) \left(1 + R^{(1)}_{q+p,q}\right)^2\right\}G_qG_{q+p}
\]

\[
(1 - P)^{\mu\nu}[b_{p,q}]^{mn\mu\nu} = (\Omega Z_a)^{-1}(1 - P)^{\mu\nu}[\gamma^{(2,0,0;2)}_{q,-q,-p,p}]^{aamn\alpha\alpha\mu\nu}G_q
= N\delta^{mn}\left\{2d(d-1)(1 + R^r_q) + 4dq^2(1 - x^2)R^{(2)}_{q+p,q}\right\}G_q
\]

\[
(1 - P)^{\mu\nu}[c_{p,q}]^{mn\mu\nu} = (\Omega Z_c)^{-2}(1 - P)^{\mu\nu}[\gamma^{(0,1,1;1)}_{q,-q,-p,p}]^{abm\mu}[\gamma^{(0,1,1;1)}_{q+p,-q,-p}]^{ban\nu}G_qG_{q+p}
= N\delta^{mn}\left\{4q^2(1 - x^2) \left(1 + R^{(1)}_{q+p,q}\right)^2\right\}G_qG_{q+p}
\]

\[
(1 - P)^{\mu\nu}[d_{p,q}]^{mn\mu\nu} = (\Omega Z_c)^{-1}(1 - P)^{\mu\nu}[\gamma^{(0,1,1;2)}_{q,-q,-p,p}]^{aamn\mu\nu}G_q
= N\delta^{mn}\left\{2d(d-1)(1 + R^r_q) + 4dq^2(1 - x^2)R^{(2)}_{q+p,q}\right\}G_q. \] (100)

In the first line of every contraction we used the fact that in the gauge \( \alpha = 1 \) the regularized propagators are proportional to the identity, while in the second line of every contraction we performed the tensor products between the “tilde” vertices. Here \( x \) is the cosine of the angle between the vectors \( p \) and \( q \). We evaluated group factors as usual using \( f^{abc}f^{abd} = N\delta^{cd} \).

Note also that all factors \( \Omega Z_a \) and \( \Omega Z_c \) simplified in the final result.

The transverse component of the lhs of the flow equation \( (79) \) is:

\[
(1 - P)_{\alpha\beta}[\partial q^{(2)}_{p,-p}]^{ab\alpha\beta} = \Omega \delta^{ab}(d - 1)\partial_q Z_A p^2; \] (101)

\textsuperscript{11}Note that \( L[\bar{c}, c; \bar{A}] = S_{gh}[0, \bar{c}, c; \bar{A}] \).
inserting the contractions $\{100\}$ into the transverse component of the flow equation $\{79\}$, gives the following explicit relation:

\[
(d - 1) \partial_t Z_A p^2 = -4(d - 1)N \int_q \partial_t \{G_q G_{q+p}\} \\
-2Nd \int_q q^2(1 - x^2) \partial_t \left\{ G_q \left[ G_{q+p} \left( 1 + R_{q+p,q}^{(1)} \right)^2 - R_{q+p,q}^{(2)} \right] \right\} \\
+Nd(d - 1) \int_q \partial_t \left\{ G_q(1 + R_q') \right\} \\
+4N \int_q q^2(1 - x^2) \partial_t \left\{ G_q \left[ G_{q+p} \left( 1 + R_{q+p,q}^{(1)} \right)^2 - R_{q+p,q}^{(2)} \right] \right\} \\
-2N(d - 1) \int_q \partial_t \left\{ G_q(1 + R_q') \right\}.
\] (102)

In (102) the first three lines can be traced back to the gauge diagrams $(a)$ and $(b)$ of Figure 5, while the last two lines come from the ghost diagrams $(c)$ and $(d)$. This equation is the projected form of the flow equation for $\bar{\gamma}^{(2)}$ within the truncation of the bEAA we are considering and with the functional trace explicitly evaluated. The rhs of (102) contains contributions of arbitrary order in $p^2$: from the heat kernel perspective this is equivalent to the re-summation of all contributions proportional to the invariants of the form $\int \tilde{F}_{\mu \nu}(-\bar{D}^2)^n \bar{F}^{\mu \nu}$ present in the coefficients $B_{2n}$ for any $n \in \mathbb{N}$.

To extract the beta function $\partial_t Z_A$ we need those terms on the rhs of (102) proportional to $p^2$. The first integral in equation (102) is needed only in the limit $p \to 0$, since its coefficient is already of order $p^2$; it’s easy to see that it can be rewritten as a $Q$-functional in the following way:

\[
\int_q \partial_t \left\{ G_q G_{q+p} \right\} = -2 \int_q (\partial_t R_k - \eta R_k) G^2_q G_{q+p} \\
= -\frac{2}{(4\pi)^{d/2}} Q_{\frac{d}{2}} \left\{ (\partial_t R_k - \eta R_k) G^2_k \right\} + O(p^2).
\] (103)

The integrals in the second and fourth lines of (102) have the same form; this is an important fact since it can be shown that the following relation holds in arbitrary dimension:

\[
\int_q q^2(1 - x^2) \partial_t \left\{ G_q \left[ G_{q+p} \left( 1 + R_{q+p,q}^{(1)} \right)^2 - R_{q+p,q}^{(2)} \right] \right\} = \\
= \frac{d - 1}{(4\pi)^{d/2}} \left\{ -\frac{1}{2} Q_{\frac{d}{2} - 1} [ (\partial_t R_k - \eta R_k) G_k ] + \frac{1}{12} Q_{\frac{d}{2} - 2} [ (\partial_t R_k - \eta R_k) G_k ] p^2 \right\}.
\]
\[-\frac{1}{120} Q_{\frac{d}{2}-3} \left[ (\partial_t R_k - \eta R_k) G_k \right] p^4 \right\} + O(p^6). \tag{104}\]

Relation (104) can be checked by inserting a sufficiently smooth cutoff shape function and calculating both sides of it in a Taylor expansion in \(p^2\). Relations like (103) and (104) serve as a dictionary to transform the explicit integrals on the rhs of the flow equation (102) into \(Q\)-functionals. With the aid of these relations we can extract from the rhs of equation (102) all the terms of order \(p^2\) and a comparison with the lhs finally leads to the type I anomalous dimension of the background field:

\[
\eta A = -\frac{g^2 N}{(4\pi)^{d/2}} \left\{ 8 Q_{\frac{d}{2}} \left[ (\partial_t R_k - \eta_a R_k) G_k^2 \right] - \frac{d}{6} Q_{\frac{d}{2}-2} \left[ (\partial_t R_k - \eta_a R_k) G_k \right] + \frac{1}{3} Q_{\frac{d}{2}-2} \left[ (\partial_t R_k - \eta_c R_k) G_k \right] \right\},
\]

which is precisely equation (78). With this result we succeeded in showing that the \(p^2\) terms of (102) indeed correspond exactly to the coefficient of the \(\frac{1}{4} \tilde{F}^2\) term of the functional trace that we calculated previously using the heat kernel expansion.

We consider now the type II cutoff. This means that in the above derivation the gauge cutoff operator action has to be replaced by the following action:

\[
L[a; \bar{A}] = I[\bar{A} + a] + S_{gf}[a; \bar{A}], \tag{105}
\]

where the action \(I[A]\) has been defined in (48) and the gauge fixing action in (5). One can check that \(L_0^{(2;0)}[0; \bar{A}] = \bar{D}T\) as it should. Diagrams \((a)\) and \((b)\) now contract to:

\[
(1 - P)_{\mu\nu}[a_{p,q}]^{mn,\mu\nu} = N \delta^{mn} \left[ 8(d - 1)p^2 - 4dq^2(1 - x^2) \right] \left[ 1 + R_{q+p,q}^{(1)} \right]^2 G_q G_{q+p}
\]

\[
(1 - P)_{\mu\nu}[b_{p,q}]^{mn,\mu\nu} = N \delta^{mn} \left\{ 2d(d - 1)(1 + R_q') \right.
+ \left[ 8(d - 1)p^2 - 4dq^2(1 - x^2) \right] R_{q+p,q}^{(2)} \right\} G_q. \tag{106}
\]

In the ghost sector we don’t make any changes. The first two terms of equation (102) are now replaced by:

\[
- N \int_q \left[ 4(d - 1)p^2 + 2dq^2(1 - x^2) \right] \partial_t \left\{ G_q \left[ G_{q+p} \left( 1 + R_{q+p,q}^{(1)} \right)^2 - R_{q+p,q}^{(2)} \right] \right\}. \tag{107}
\]
By using the following relation, that can be checked as before,

$$\int_q \partial_t \left\{ G_q \left[ G_{q+p} \left( 1 + R_{q+p,q}^{(1)} \right)^2 - R_{q+p,q}^{(2)} \right] \right\} =$$

$$= \frac{1}{(4\pi)^{d/2}} \left\{ -Q_{\frac{d}{2}-2} [ (\partial_t R_k - \eta R_k)G_k ] + \frac{1}{6} Q_{\frac{d}{2}-3} [ (\partial_t R_k - \eta R_k)G_k ] \frac{p^2}{k^2} \right\} + O \left( \frac{p^4}{k^4} \right),$$

(108)

to expand the rhs of (107) one arrives at the expression for the type II anomalous dimension of the background field:

$$\eta_A = -g^2 N (4\pi)^{d/2} \left\{ \frac{24 - d}{6} Q_{\frac{d}{2}-2} [ (\partial_t R_k - \eta_a R_k)G_k ] + \frac{1}{3} Q_{\frac{d}{2}-2} [ (\partial_t R_k - \eta_c R_k)G_k ] \right\}.$$

Again, as a further confirmation of our methods, we have re-derived the result (73) obtained before employing the heat kernel expansion.

Along the same lines one can study the longitudinal equation obtained from (79) by contraction with $P_{\mu\nu}$ and make an important check. In fact one finds, in both type I and II cases, that the equation (in particular the rhs) turns out to be identically zero; this is expected, since, by construction, the flow of the zero-field proper-vertices $\gamma^{(2)}$ of the gEAA is transverse. Another check one can make is to control that the terms of order $p^0$ in equation (102) cancel each other, since otherwise they will generate a non-gauge invariant mass term for the background field. Using (104) and the following relation,

$$\int_q \partial_t \left\{ G_q (1 + R'_{q}) \right\} = -\frac{1}{(4\pi)^{d/2}} \left( \frac{4}{3} \right) Q_{\frac{d}{2}-1} [ (\partial_t R_k - \eta R_k)G_k ] ,$$

(109)

to evaluate equation (102) at $p = 0$, gives:

$$-2Nd \left( -\frac{1}{2} \frac{d-1}{(4\pi)^{d/2}} Q_{\frac{d}{2}-1} [ (\partial_t R_k - \eta_a R_k)G_k ] \right) + Nd(d-1) \left( -\frac{1}{(4\pi)^{d/2}} Q_{\frac{d}{2}-1} [ (\partial_t R_k - \eta_a R_k)G_k ] \right)$$

$$+ 4N \left( -\frac{1}{2} \frac{d-1}{(4\pi)^{d/2}} Q_{\frac{d}{2}-1} [ (\partial_t R_k - \eta_c R_k)G_k ] \right) - 2Nd(d-1) \left( -\frac{1}{(4\pi)^{d/2}} Q_{\frac{d}{2}-1} [ (\partial_t R_k - \eta_c R_k)G_k ] \right)$$

$$= \left[ -2 \left( -\frac{1}{2} \right) + (-1) \right] \frac{Nd(d-1)}{(4\pi)^{d/2}} Q_{\frac{d}{2}-1} [ (\partial_t R_k - \eta_a R_k)G_k ]$$

$$+ \left[ 4 \left( -\frac{1}{2} \right) - (-2) \right] \frac{Nd(d-1)}{(4\pi)^{d/2}} Q_{\frac{d}{2}-1} [ (\partial_t R_k - \eta_c R_k)G_k ] = 0 .$$

As we see the $p^0$ contributions in both the gauge and ghost sectors correctly cancel each other. The same can be checked in case of the type II cutoff. We remark that these are nontrivial checks of the formalism.
It this framework it is not difficult to relax the condition $\alpha = 1$. One does not need to change anything apart considering the regularized propagator in its general form (93) and using the $\alpha$ dependent gauge-fixing vertices from appendix B. Work in this direction will be material for a future studies [21].

The aim of this subsection was to show how the method introduced in section 2 can be used to reproduce the results obtained with the aid of the local heat kernel expansion, and to show how the different cutoff choices are reflected in this formalism. In this way we made an important test of the method. We remark that this way of evaluating the flow equation for the gEAA is very general and can be applied to any general truncation ansatz, in particular to those that cannot be treated with the aid of the local heat kernel expansion. We move now to perform a new application of the technique by calculating $\eta_{a,k}$ and $\eta_{c,k}$ in the next subsection.

3.2.3 Calculation of $\eta_{a,k}$ and $\eta_{c,k}$

In this subsection we calculate the anomalous dimensions of the fluctuation and ghost fields; these are related to the scale derivatives of wave-function renormalization constants, $\partial_t Z_a$ and $\partial_t Z_c$, by the relations:

$$
\begin{align*}
\eta_a &= -\partial_t \log Z_a, \\
\eta_c &= -\partial_t \log Z_c.
\end{align*}
$$

(110)

We will extract $\partial_t Z_a$ and $\partial_t Z_c$ form, respectively, the flow equations for the zero-field two-point functions $\gamma^{(2,0,0;0)}$ and $\gamma^{(0,1,1;0)}$ obtained from the flow equation (42) of section 2 after performing the field multiplet decomposition.

The flow equation for $\gamma^{(2,0,0;0)}$, within the truncation represented by equations (53) and
is given in Figure 6. In formulas this equation can be written as follows:

\[ [\partial_t \gamma_{p,-p}^{(2,0,0)}]^{mn \mu \nu} = \Omega \int \left( \partial_t R_q - \eta_a R_q \right) [a_{p,q}]^{mn \mu \nu} - \frac{1}{2} \Omega \int \left( \partial_t R_q - \eta_a R_q \right) [b_{p,q}]^{mn \mu \nu} \]

\[ -2\Omega \int \left( \partial_t R_q - \eta_c R_q \right) [c_{p,q}]^{mn \mu \nu}, \tag{111} \]

where the tensor products entering it are:

\[ [a_{p,q}]^{mn \mu \nu} = [G_p]^{ab \alpha \beta} [\gamma_{q,-q-p}^{(3,0,0)}]^{bcn \beta \gamma \mu} [G_{q+p}]^{cd \gamma \delta} [\gamma_{q+p,-q,-p}^{(3,0,0)}]^{den \delta \nu} [G_q]^{ea \epsilon \alpha} \]

\[ [b_{p,q}]^{mn \mu \nu} = [G_p]^{ab \alpha \beta} [\gamma_{q,-q-p}^{(4,0,0)}]^{bcn \beta \gamma \mu} [G_{q+p}]^{cd \gamma \delta} [\gamma_{q+p,-q,-p}^{(4,0,0)}]^{den \delta \nu} [G_q]^{ea \epsilon \alpha} \]

\[ [c_{p,q}]^{mn \mu \nu} = [C_p]^{ab \gamma \delta} [\gamma_{q,-q-p}^{(1,1,1,0)}]^{mbc \mu} [C_q]^{cd \gamma \delta} [\gamma_{q+p,-q,-p}^{(1,1,1,0)}]^{nde \nu} [G_{q+p}]^{ea \epsilon \alpha}, \tag{112} \]

and the vertices the following:

\[ [\gamma_{p_1,p_2,p_3}^{(3,0,0)}]^{abm \alpha \beta \mu} = g Z_a^3/2 [I_p^{(3)}]^{abm \alpha \beta \mu} \]

\[ [\gamma_{p_1,p_2,p_3,p_4}^{(4,0,0)}]^{abm \alpha \beta \mu} = g^2 Z_a [I_p^{(4)}]^{abm \alpha \beta \mu} \]

\[ [\gamma_{p_1,p_2,p_3,p_4}^{(1,1,1,0)}]^{mab \mu} = Z_c^{1/2} [C_p^{(1,1,1,0)}]^{mab \mu}. \tag{113} \]

Diagram (d) in Figure 6 is identically zero, since in our truncation there is no term bilinear both in the ghost and in the gauge fields, and we omitted to write it in (111). The vertices of the actions in (113) are given in appendix B.

To deal with scalar equations we project equation (111) using the orthogonal projectors \((1 - P)_{\mu \nu}\) and \(P_{\mu \nu}\). In this way we obtain two independent equations describing, respectively, the flow of the transverse and longitudinal components of \(\gamma^{(2,0,0)}\). We will use the transverse equation to extract \(\partial_t Z_a\). We mention here that the longitudinal equation can be used to extract the running of \(\alpha\), if one considers a truncation with running gauge parameter [21]. Applying the transverse projector to the lhs of the flow equation (111) and using (86) gives:

\[ (1 - P)_{\mu \nu} [\partial_t \gamma_{p,-p}^{(2)} + \partial_t \gamma_{p,-p}^{(2,0,0)}]^{mn \mu \nu} = \Omega \delta^{mn} (d - 1) \partial_t Z_a p^2, \tag{114} \]

where we used the trace \((1 - P)\alpha_a = d - 1\). Equation (114) shows that we can extract \(\partial_t Z_a\) as the coefficient of the \(p^2\) term of the transverse projection of equation (111). Acting on the first integrand in (112) with the transverse projector gives:

\[ (1 - P)_{\mu \nu} [a_{p,q}]^{mn \mu \nu} = g^2 \Omega Z_a (\Delta/2) \left\{ -5(d - 1)p^2 - 2(d - 1)p \cdot q \right\} \]
The group factor here is $f_{amb} f_{bna} = -N \delta^{mn}$, while, as before, the variable $x$ is the cosine of the angle between $p$ and $q$. The transverse contribution from the second integrand in (112) is:

$$ (1 - P)_{\mu\nu}[b_{p,q}]^{mn}_{\mu\nu} = g^2 \Omega Z_a(-2N \delta^{mn}) \left\{ -(d-1)^2 \right\} (G_q)^2, $$

(116)

where the group factor is $f_{abm} f_{ban} + f_{abn} f_{amb} = -2N \delta^{mn}$. From (112) we find the transverse contribution from the ghost diagram (c):

$$ (1 - P)_{\mu\nu}[c_{p,q}]^{mn}_{\mu\nu} = g^2 \Omega Z_a(-N \delta^{mn}) \left\{ -(1-x^2)q^2 \right\} (G_q)^2 G_{q+p}, $$

(117)

where the group factor is $f_{abm} f_{ban} = -N \delta^{mn}$.

Note that each integrand (115-117) (or diagram) is proportional to $g^2 Z_a$ since the fluctuation three-vertex comes with a factor $gZ_a^{3/2}$, the four-vertex with a factor $g^2 Z_a^2$, while the regularized gauge propagators come with a power of $Z_a^{-1}$ and a gauge cutoff kernel insertion with a power of $Z_a$. In the ghost diagrams the three-vertex gives a factor $gZ_a^{1/2} Z_c$, there is no four-vertex, the regularized ghost propagator has a factor of $Z_c^{-1}$ and the ghost cutoff kernel insertion has a power of $Z_c$. Also, all the space-time volume factors $\Omega$ on both sides of equation (111) delete each other.

Once we insert equations (115), (116) and (117) back in (111) we obtain the explicit flow of the vertex $\gamma^{(2,0,0;0)}_{p,-p}$, to all orders in the external momenta $p$. The momentum integrals in (111) can be performed using spherical coordinates with the $z$-axis along the vector $p$:

$$ \int_q \to \frac{S_{d-1}}{(2\pi)^d} \int_0^\infty dq q^{d-1} \int_{-1}^1 dx \left( 1 - x^2 \right)^{\frac{d-3}{2}}, $$

(118)

where $S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ is the volume of the $d$-dimensional sphere. We also change variable $z = q^2$ in the radial integral so that:

$$ \int_0^\infty dq q^{d-1} \to \frac{1}{2} \int_0^\infty dz z^{\frac{d}{2}-1}. $$

(119)

After expanding the transverse projection of equation (111) in powers of $p$ the $x$-integrals can be easily performed; from the terms proportional to $p^2$ we can extract $\partial_t Z_a$ and from
Figure 7: Graphical representation of the flow equation for the ghost-ghost zero-field proper vertex $\gamma^{(0,1,1:0)}_k$ from (122).

de these we obtain the anomalous dimension of the fluctuation field:

$$
\eta_a = \frac{g^2 N}{(4\pi)^{d/2}} \left\{-5Q_{1 \frac{d}{2}} \left[ (\partial_t R_k - \eta_a R_k) G^3_k \right] - (3d - 1)Q_{\frac{d}{2}+1} \left[ (\partial_t R_k - \eta_a R_k) G^2_k G'_{k} \right] \\
- (3d - 1)Q_{\frac{d}{2}+2} \left[ (\partial_t R_k - \eta_a R_k) G^2_k G''_{k} \right] + Q_{\frac{d}{2}+1} \left[ (\partial_t R_k - \eta_c R_k) G^2_k G'_{k} \right] \\
+ Q_{\frac{d}{2}+2} \left[ (\partial_t R_k - \eta_c R_k) G^2_k G''_{k} \right] \right\},
$$

(120)

where we wrote everything in terms of $Q$-functionals. Equation (120) is valid in arbitrary dimension and for every cutoff shape function, and gives implicitly the anomalous dimension of the fluctuation field, in the gauge $\alpha = 1$, as part of a linear system for the variables $\eta_a$ and $\eta_c$. In the following we derive an analogous equation for the anomalous dimension $\eta_c$ of the ghost fields, that together with (120), can be used to solve for $\eta_a$ and $\eta_c$ in function of $g$.

We now calculate the scale derivative of the ghost wave-function renormalization $\partial_t Z_c$.

The only term in our truncation (54) that contains the ghost fields and the related wave-function renormalization is the following:

$$
Z_c \int d^d x \bar{D}_\mu \bar{c} \left( \bar{D}_\mu + gZ^{1/2}_a a^\mu \right) c.
$$

(121)

We can extract $\partial_t Z_c$ from the flow equation for the ghost-ghost zero-field proper-vertex $\gamma^{(0,1,1:0)}$. This equation is obtained from the flow equation (42) for the zero-field proper-vertex $\gamma^{(2:0)}$ after the field multiplet decomposition; it is represented graphically in Figure 7 and reads as follows:

$$
[\partial_t \gamma^{(0,1,1:0)}_{p,-p}]^{mn} = \Omega \int_q (\partial_t R_q - \eta_a R_q) [e_{p,q}]^{mn} + \Omega \int_q (\partial_t R_q - \eta_c R_q) [f_{p,q}]^{mn},
$$

(122)

where the integrands are:

$$
[e_{p,q}]^{mn} = [G_q]^{ab \alpha \beta} \left[ \gamma^{(1,1,1:0)}_{q,p,-q,-p} \right] bmc \beta \left[ G_{q+p} \right] cdf \left[ \gamma^{(1,1,1:0)}_{-q,q+p,-p} \right] d \delta \gamma \left[ G_{q} \right] ea \gamma \alpha
$$

$$
[f_{p,q}]^{mn} = [G_q]^{ab \alpha \beta} \left[ \gamma^{(1,1,1:0)}_{-q,-p,q,p} \right] bmc \gamma \left[ G_{q+p} \right] cdf \left[ \gamma^{(1,1,1:0)}_{q,-q,-p,-p} \right] d \delta \gamma \left[ G_{q} \right] ea .
$$

(123)
The only vertex we need is thus the following:

\[ [\gamma^{(1,1,1)}_{p_1,p_2,p_3}]^{mab\mu} = gZ_c^{1/2}[\zeta^{(1,1,1)}_{ghp_1,p_2,p_3}]^{mab\mu}, \]

which can be found in appendix B. In equation (122) only the ghost-ghost-gluon vertex appears since the action (121) is at most trilinear in the fluctuation and ghost fields, thus the second term in the flow equation (42) vanishes. Inserting (121) in the lhs of (122) gives:

\[ [\partial_t \gamma^{(0,1,1,0)}_{p,-p}]^{mn} = \Omega \delta^{mn} \partial_t Z_c p^2. \]

As before we perform the tensor products in (123) and we evaluate the integrals in (122); we then extract \( \partial_t Z_c \) from the term of order \( p^2 \); after writing everything in terms of \( Q \)-functionals, we obtain the anomalous dimension of the ghost fields in the following form:

\[ \eta_c = \frac{g^2 N}{(4\pi)^{d/2}} \left\{ -Q_{d/2} \left[ (\partial_t R - \eta_a R) G_k^3 \right] - Q_{d+1} \left[ (\partial_t R - \eta_a R) G_k^2 G'_{k} \right] \\
+ Q_{d+1} \left[ (\partial_t R - \eta_c R) G_k^2 G'_{k} \right] \right\}. \]

Equation (125) is the ghost anomalous dimensions in the gauge \( \alpha = 1 \) (within the truncation we are considering) and is valid for general cutoff shape function and dimension. Equations (120) and (125) are the main results of this subsection.

### 3.3 Beta functions and non-perturbative predictions

In this subsection we study the different ways one has to “close” the beta function for the gauge coupling derived in the previous subsection and we make a comparison with other approaches. Then we study some possible phenomenology related to these RG flows.

#### 3.3.1 Beta functions

To obtain an explicit form for the anomalous dimension of the background field,

\[ \eta_{\bar{A},k} = -\partial_t \log Z_{\bar{A},k}, \]

given in equation (73) for type II cutoff or in equation (78) for type I cutoff, we need to specify the cutoff shape function \( R_k(z) \). We will consider the theta– or optimized–cutoff
shape function:

\[ R_k(z) = Z_k(k^2 - z)\theta(k^2 - z), \tag{127} \]

(where \( Z_k \) represents \( Z_{a,k} \) or \( Z_{c,k} \)) since it allows us to perform the integrals in the \( Q \)-functionals analytically for every value of the dimension \( d \). A simple integration gives the following form:

\[ Q_n[(\partial_t R_k - \eta R_k)G^m_k G'_k] = \frac{k^{2(n+1-m)}}{\Gamma(n+2)} \left[ 2(n+1) - \eta \right] \tag{128} \]

(where \( \eta \) stands for \( \eta_{a,k} \) or \( \eta_{c,k} \)). If we insert (128) in the type I anomalous dimension (78), we find:

\[ \bar{\eta}_{A,k} = \frac{g_k^2 N k^{d-4}}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} \right)} \left[ -\frac{192 - d(d-2)}{6d} + \frac{192 - d^2(d+2)}{6d(d+2)} \eta_{a,k} + \frac{1}{3} \eta_{c,k} \right]; \tag{129} \]

while if we insert (128) in the type II anomalous dimension (73) we get:

\[ \bar{\eta}_{A,k} = \frac{g_k^2 N k^{d-4}}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} \right)} \left[ -(26 - d)(d-2) + \frac{24 - d}{6} \eta_{a,k} + \frac{1}{3} \eta_{c,k} \right]. \tag{130} \]

Note that only the type II anomalous dimension has the one–loop term proportional to \( 26 - d \), while type I becomes positive already at \( d \approx 7.174 \); for \( d \to \infty \) both coefficients go as \( d^2/12 \); at \( d = 2 \) the type II coefficient is zero while the type I is negative. See Figure 8 for a comparison.

Equation (129) and (130) show that the anomalous dimension of the background field \( \eta_{A,k} \) is determined by the anomalous dimensions of the fluctuation and ghost fields

\[ \eta_{a,k} = -\partial_t \log Z_{a,k}, \quad \eta_{c,k} = -\partial_t \log Z_{c,k}. \tag{131} \]

This reflects the fact that the flow of the gEAA (here represented by \( \eta_{A,k} \)) is not closed but depends on the flow of the full bEAA (here represented by \( \eta_{a,k} \) and \( \eta_{c,k} \)). The anomalous dimensions of the fluctuation and of the ghost fields are given, respectively, in equations (120) and (125). The \( Q \)-integrals that we need now are:

\[ Q_n[(\partial_t R_k - \eta R_k)G^m_k G'_k] = 0 \]
\[ Q_n[(\partial_t R_k - \eta R_k)G^m_k G''_k] = -\frac{k^{2(n+1-m)}}{\Gamma(n)}; \tag{132} \]
with these we find the following system:

\[ \eta_{a,k} = \frac{g_k^2 N k^{d-4}}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} \right)} \left[ -\frac{8(d+6)}{d(d+2)} + \frac{20}{d(d+2)} \eta_{a,k} \right] \]  
(133)

\[ \eta_{c,k} = \frac{g_k^2 N k^{d-4}}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} \right)} \left[ -\frac{4}{d} + \frac{4}{d(d+2)} \eta_{a,k} \right] . \]  
(134)

We want to remember to the reader that we are working in the gauge $\alpha = 1$ and that (133) and (134) generally depend on the gauge-fixing parameter $\alpha$. Contrary to $\eta_{A,k}$, the anomalous dimensions $\eta_{a,k}$ and $\eta_{c,k}$ are the same for both cutoff types. Note also that the one–loop terms in (133) and (134) are always negative.

In the background field formalism the gauge coupling is related to the wave-function renormalization of the background field by (50), i.e. $g_k = Z_{A,k}^{-1/2}$. Thus the beta function for the gauge coupling can be obtained from the anomalous dimensions of the background field by the following simple relation:

\[ \partial_t g_k = \frac{1}{2} \eta_{A,k} g_k . \]  
(135)

One can now shift to dimensionless coupling $\tilde{g}_k = k^{\frac{d-4}{2}} g_k$ to find:

\[ \partial_t \tilde{g}_k = \frac{1}{2} (d - 4 + \eta_{A,k}) \tilde{g}_k . \]  
(136)

Even if the previous equations are valid in any dimension, in the following we will consider
the physical interesting case $d = 4$ where $g_k = \tilde{g}_k$.

Owing to the structure of both (129) and (130) we can write the following general form for the background field anomalous dimension in a given scheme:

$$\eta_{\bar{A}, k} = (2a + b \eta_{a, k} + b' \eta_{c, k}) g_k^2,$$

(137)

together with the following values for the coefficients in the two cutoff schemes:

$$a_I = -\frac{11}{3} \frac{N}{(4\pi)^2}, \quad b_I = \frac{2}{3} \frac{N}{(4\pi)^2}, \quad b'_I = \frac{1}{3} \frac{N}{(4\pi)^2},$$

(138)

$$a_{II} = -\frac{11}{3} \frac{N}{(4\pi)^2}, \quad b_{II} = \frac{10}{3} \frac{N}{(4\pi)^2}, \quad b'_{II} = \frac{1}{3} \frac{N}{(4\pi)^2}.$$

(139)

As expected we recover the scheme independent one–loop result $a_I = a_{II} = -\frac{11}{3} \frac{N}{(4\pi)^2}$. The coefficients $b$ depend instead on the cutoff type; however, these coefficients are gauge dependent and may have closer values in another gauge such as the Landau $\alpha = 0$. Note also that $b'_I = b'_{II}$ since the ghost contributions are the same in both schemes.

Since the anomalous dimensions of the fluctuation and ghost fields (133) and (134), to lowest order, are proportional to $g_k^2$, the only term on the lhs of (137) of order $g_k^2$ is the first. To this order we recover the standard one–loop beta function of the gauge coupling:

$$\partial_t g_k = a g_k^3 + O(g_k^5) = -\frac{11}{3} \frac{N}{(4\pi)^2} g_k^3 + O(g_k^5).$$

(140)

Since the beta function is negative at small coupling, non-abelian gauge theories are asymptotically free in $d = 4$ [22]. From this we see that the one–loop approximation is equivalent to set $\eta_{a,k} = \eta_{c,k} = 0$ in (137). For a discussion of the physical mechanism behind (140) asymptotic freedom we refer to [23].

In the general case, to determine $\eta_{\bar{A},k}$ from equation (137) we need to calculate or specify the anomalous dimensions $\eta_{a,k}$ and $\eta_{c,k}$ as functions of $g_k$. As we said, this is the actual manifestation of the fact that the flow of the gEAA ($g_k$) is not closed but is given in terms of the flow of the bEAA ($\eta_{a,k}$ and $\eta_{c,k}$). We discuss now two different approximations that allow us to obtain a closed form for $\eta_{\bar{A},k}$ and thus an improved form for the beta function of the gauge coupling.

The first approximation, or improvement, consists in identifying the anomalous dimension of the background field with the anomalous dimension of the fluctuation field, and by setting
the anomalous dimension of the ghost fields to zero:

$$\eta_{A,k} = \eta_{a,k} \quad \eta_{c,k} = 0 \, .$$  \hfill (141)

Previous applications of the bEAA to non-abelian gauge theories \[3, 18\] used this identification. As we will see in a moment, this identification becomes exact in the supersymmetric limit. If we insert now (141) in (137) we find the following linear system for the variable $\eta_{A,k}$:

$$\eta_{A,k} = (2a + b \eta_{A,k}) \, g_k^2 \, , \quad \hfill (142)$$

which is easily solved to yield:

$$\eta_{A,k} = \frac{2a}{1 - bg_k^2} g_k^2 \, , \quad \hfill (143)$$

or, using (135), the following beta function for the gauge coupling:

$$\partial_t g_k = a - bg_k^2 g_k^3 \, . \quad \hfill (144)$$

The beta function (144) is a rational function of the gauge coupling; this shows how the identification (141) implements the re-summation of an infinite number of perturbative contributions. One clearly sees the presence of a singularity in the beta function (144) at $g_k^2 = 1/b$. We will see in the next subsection a possible physical consequence of this fact. The beta function (144) in the two schemes is shown in Figure 9. In the spirit of [11] this is a single-field truncation.

Ryttov and Sannino (RS) \[13\] proposed an “all orders” beta function for the gauge coupling that has exactly the structure found in (144), but with the following coefficients:

$$a_{RS} = -\frac{11}{3} \frac{N}{(4\pi)^2} \quad \text{and} \quad b_{RS} = \frac{34}{11} \frac{N}{(4\pi)^2} \, , \quad \hfill (145)$$

chosen in order to match the one- and two-loop results for the beta function upon Taylor expansion in $g_k$. The inspiration behind the ansatz of RS comes from the knowledge of the exact beta function found by Novikov-Shifman-Vainshtein-Zakharov (NSVZ) \[24\] in the supersymmetric $\mathcal{N} = 1$ version of the model. The NSVZ beta function is also of the form (144) but with coefficients $a_{NSVZ} = -3 \frac{N}{(4\pi)^2}$ and $b_{NSVZ} = 2 \frac{N}{(4\pi)^2}$. RS postulated the existence of a perturbative massless scheme where all the perturbative orders are reproduced by the choice (145); here we note the interesting fact that the RS beta function is very similar to our type II beta function, as clearly shown in Figure 9. We see that the bEAA formalism offers
a framework where beta functions of the form (144) can be obtained from first principles. The linear relation for the anomalous dimension of the background field postulated in [25] to derive the RS beta function, is here a consequence of the structure of the exact RG flow equation for the gEAA. As a future work, it will be interesting to reproduce the exact results for supersymmetric beta functions of [24] using functional RG methods.

When the coupling is small $g_k \ll 1$ we can expand the beta function (144) to find:

$$\partial_t g_k = a g_k^3 + ab g_k^5 + O\left(g_k^7\right). \quad (146)$$

As we already noticed, the one-loop contributions are scheme independent $a_I = a_{II} = a_{RS}$ and equal to the perturbative result, while the two-loop contributions are instead different. The RS coefficient is, by construction, equal to the perturbative result $a_{RS} b_{RS} = - \frac{g_k^4}{(4\pi)^4} \frac{102}{9} N^2$ [15], while we find $a_I b_I = - \frac{g_k^4}{(4\pi)^4} \frac{22}{9} N^2$ and $a_{II} b_{II} = - \frac{g_k^4}{(4\pi)^4} \frac{110}{9} N^2$. Note that the type I coefficient is 79% smaller than the two-loop coefficient, while type II coefficient is just 8% bigger. These two-loop coefficients are expected not to be equal to the perturbative result, since this is scheme independent only in massless schemes, while the bEAA formalisms implements a kind of mass dependent regularization [2].

The second way we can obtain a closed beta function for the gauge coupling, is to first calculate the anomalous dimensions $\eta_{a,k}$ and $\eta_{c,k}$ and then reinsert them back in (137). This
means that we are considering the flow of the full bEAA which takes place in the enlarged theory space of all functionals of the fluctuation fields \( a_\mu, \bar{c}, c, \) and of the background field \( \bar{A}_\mu. \) In the truncation we are considering, given by equations (53) and (54), this is the three-dimensional space parametrized by the coupling constants \( \{ g_k, Z_{a,k}, Z_{c,k} \}. \) In the spirit of [11] this is a bi–field truncation; see also [26] for a discussion of this enlarged class of truncations.

We have seen that the anomalous dimensions of the fluctuation field and of the ghost fields are determined by a linear system; we will solve this to obtain \( \eta_{a,k} \) and \( \eta_{c,k} \) as functions of the gauge coupling \( g_k \) and then we will reinsert them back in equation (137) to obtain \( \eta_{A,k}; \) from this we then obtain a new closed form for \( \partial_t g_k. \)

In the most general case (compatible with our truncation), the anomalous dimensions \( \eta_{a,k} \) and \( \eta_{c,k} \) are determined by the following linear system;

\[
\begin{align*}
\eta_{a,k} &= (A + B \eta_{a,k} + C \eta_{c,k}) g_k^2 \\
\eta_{c,k} &= (A' + B' \eta_{a,k} + C' \eta_{c,k}) g_k^2.
\end{align*}
\]

(147)

The coefficients in (147) do not depend on the scheme I or II, but they depend on the gauge and on \( R_k(z). \) In particular, setting \( d = 4 \) in (133) and (134) gives the following values for the coefficients:

\[
\begin{align*}
A &= -\frac{10}{3} \frac{N}{(4\pi)^2} \\
B &= \frac{5}{6} \frac{N}{(4\pi)^2} \\
C &= 0,
\end{align*}
\]

\[
\begin{align*}
A' &= -\frac{N}{(4\pi)^2} \\
B' &= \frac{1}{6} \frac{N}{(4\pi)^2} \\
C' &= 0.
\end{align*}
\]

(148)

As a note, we mention that the scheme independent coefficients \( A \) and \( A' \) become, respectively, \( -\frac{13-3\alpha}{3} \) and \( -\frac{3-\alpha}{2} \) when \( \alpha \neq 1 \) and the coefficients \( C \) and \( C' \) become non-zero. A system analogous to (147) was studied within the non-background EAA approach to non-abelian gauge theories in [20]. Considering that the anomalous dimensions in the rhs of (147) are at least of order \( g_k^2 \) we find, to lowest order, the following forms:

\[
\begin{align*}
\eta_{a,k} &= -\frac{10}{3} \frac{g_k^2 N}{(4\pi)^2} + O(g_k^4) \\
\eta_{c,k} &= -\frac{g_k^2 N}{(4\pi)^2} + O(g_k^4).
\end{align*}
\]

(149)

The terms on the rhs of (149) are scheme independent and agree with the ones of [20]. To
solve the system (147) we rewrite it as a matrix equation:

\[ \bar{\eta}_k = (\bar{A} + M \bar{\eta}_k) g_k^2, \]  

(150)

where we defined:

\[ \bar{\eta}_k = \begin{pmatrix} \eta_{a,k} \\ \eta_{c,k} \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} A \\ A' \end{pmatrix}, \quad M = \begin{pmatrix} B & C \\ B' & C' \end{pmatrix}. \]  

(151)

The linear system (150) is easily solved:

\[ \bar{\eta}_k = (1 - g_k^2 M)^{-1} \bar{A}, \]  

(152)

or more explicitly:

\[ \eta_{a,k} = \frac{A(1 - C'g_k^2) + ACg_k^2}{(1 - Bg_k^2)(1 - C'g_k^2) + B'Cg_k^2 g_k^2}, \]

\[ \eta_{c,k} = \frac{A'(1 - Bg_k^2) + B'g_k^2}{(1 - Bg_k^2)(1 - C'g_k^2) + B'Cg_k^2 g_k^2}. \]  

(153)

We can now turn back to the beta function for the gauge coupling and “close” it by inserting the functions (153) in (137). Under the condition \( C = C' = 0 \), we find the following RG improved result for the beta function of the gauge coupling:

\[ \partial_t g_k = \frac{a + c g_k^2 + d g_k^4}{1 - bg_k^2} g_k^2, \]  

(154)

with coefficients:

\[ c = \frac{1}{2} (bA + b'A' - 2aB), \quad d = \frac{b'}{2} (AB' - A'B). \]  

(155)

Using the values (138) and (148) we find:

\[ c_I = \frac{16}{9} \frac{N^2}{(4\pi)^4}, \quad d_I = \frac{5}{108} \frac{N^3}{(4\pi)^6}, \]

\[ c_{II} = -\frac{8}{3} \frac{N^2}{(4\pi)^4}, \quad d_{II} = \frac{5}{108} \frac{N^3}{(4\pi)^6}. \]  

(156)

Note that only the coefficients \( c \) are different in the two schemes. The beta function (154)
clearly shows the influence of the fluctuation couplings, here represented by $Z_{a,k}$ and $Z_{c,k}$, on the flow of the gauge coupling $g_k$. We remember that the anomalous dimensions we used to obtain the closed form for the beta function of $g_k$ was calculated in the gauge $\alpha = 1$, in a future work \cite{21} we will study the dependence on the gauge-fixing parameter, considering in particular the Landau gauge $\alpha = 0$.

Beta functions of the form \cite{154} are obtained as Padé approximations of the perturbative beta function \cite{27}. By employing data at three–loops one finds a beta function of the form \cite{154} with coefficients given by:

$$a_{\text{Padé}} = -b_1, \quad b_{\text{Padé}} = \frac{b_2}{b_1}, \quad c_{\text{Padé}} = -b_1 + \frac{b_0 b_2}{b_1}, \quad d_{\text{Padé}} = 0,$$

where $b_1, b_2, b_3$ are the one–, two– and three–loop coefficients. The bi–field beta functions are compared with the Padé re-summed one in Figure 10. When making this comparisons one has to remember that starting from the three–loop coefficients these become scheme dependent even when employing massless renormalization schemes. Finally, the coefficient $d$ becomes non-zero when one considers data at four–loops, thus our beta function \cite{154} contains re-summed information up to four–loops.
3.3.2 Flow and physical observables

We now integrate the beta functions derived in the previous subsection. We will be able to do these steps analytically. We start to consider the single–field beta function (144); a simple integration of this gives:

\[ t = -\frac{1}{2a} \left( \frac{1}{g_k^2} - \frac{1}{g_0^2} \right) - \frac{b}{a} \log \frac{g_k}{g_0}, \] (157)

where \( t = \log \frac{k}{k_0} \), with \( k_0 \) a reference scale where \( g_k = g_{k_0} \). Equation (157) is transcendental and cannot be inverted analytically except in the one–loop case \( b = 0 \). In this last case one obtains:

\[ g_k^2 = \frac{g_0^2}{1 - 2a \log \frac{k}{k_0}}. \] (158)

Since \( a = -\frac{11}{3} \frac{N}{(4\pi)^2} < 0 \), in the UV limit \( k \to \infty \) the theory is asymptotically free, i.e. \( g_k \to 0 \).

When \( b \neq 0 \) we can make an implicit plot of \( t(g_k) \) using (157); the result for the three cases (type I, type II and RS) together with the one–loop case are shown in Figure 11. One sees that the type II and RS flows are very similar. The improved flows appear to be convex functions (contrary to the perturbative result) and have a minimum at the value where the beta function (144) has the pole, i.e. at \( g_k^2 = \frac{1}{b} \). Note the important fact that even if the beta functions have a pole, the flow they generate is instead smooth. But one cannot invert \( t(g_k) \) to obtain \( g_k(t) \) for every value of \( t \in \mathbb{R} \). If we choose to be continuously connected to the asymptotic free regime, when we lower the RG scale toward the IR we reach the point

Figure 11: Implicit plot for the single–field flow of \( g_k \) from (157). From top: RS, type II, type I and one-loop flows.
$t(b^{-1/2})$ after which we cannot proceed further, i.e. the curve $g_k(t)$ cannot be extended. This signals that our single coupling truncation breaks down (and one needs to consider more sophisticated truncations to proceed further toward the IR \cite{31}). Following \cite{28} we can interpret $t(b^{-1/2})$ as the point where bound states are formed and identify the relative mass scale $M$ as (twice) the mass gap. With this identification, using (157), we find the following relation for the mass gap as a function of $k_0$ and $g_0$:

$$M = k_0 \left( g_0 b \right)^{\frac{1}{2a}} e^{\frac{1}{a^2} \left( \frac{1}{g_0 b} - b \right)} .$$  \hspace{1cm} (159)$$

This estimate for the mass gap is plotted in Figure 12. For $k_0 = 91.19$ GeV and $N = 3$ one finds $M_I = 0.61$ GeV and $M_{II} = 1.79$ GeV, to be compared with the estimate $M_{RS} = 1.69$ GeV of \cite{28}. As for the beta functions, the type II estimate and the one based on the RS beta function are similar, while a strong scheme dependence is observed in the type I case.

In the same way one can integrate the bi–field beta function (154) to obtain the following analytical relation:

$$t = -\frac{1}{2a} \left( \frac{1}{g_k^2} - \frac{1}{g_0^2} \right) - \frac{ab + c}{a^2} \log \frac{g_k}{g_0} + \frac{ab + c}{4a^2} \log \frac{a + cg_k^2 + dg_k^2}{a + cg_0^2 + dg_0^2}$$

$$+ \frac{abc - 2ad + c^2}{4a^2 \sqrt{c^2 - 4ad}} \log \frac{(c + 2dg_k^2 - \sqrt{c^2 - 4ad}) (c + 2dg_0^2 + \sqrt{c^2 - 4ad})}{(c + 2dg_k^2 + \sqrt{c^2 - 4ad}) (c + 2dg_0^2 - \sqrt{c^2 - 4ad})} .$$  \hspace{1cm} (160)$$

Figure 12: The mass gap as a function of $g_0$ (with $k_0 = 91.19$ GeV) for the single–field flows compared to the flow proposed by RS from \cite{159}. From top-left the RS, type II and type I curves.
The implicit plot of (160) is shown in Figure 13. Obviously if we set \(c = d = 0\) we recover (157). As before, we can estimate the mass gap \(M\), the results is now:

\[
M = k_0 \left( g_0^2 b \right)^{\frac{abk}{2a^2}} e^{\frac{1}{2a^2} \left( \frac{1}{g_0^2} - b \right) \left( a + c/b + d/b^2 \right) \frac{abk}{a^2}} \times \frac{\left( c + 2d/b - \sqrt{c^2 - 4ad} \right) \left( c + 2dg_0^2 + \sqrt{c^2 - 4ad} \right)}{\left( c + 2d/b + \sqrt{c^2 - 4ad} \right) \left( c + 2dg_0^2 - \sqrt{c^2 - 4ad} \right)}.
\] (161)

Evaluating (161) for the different cutoff choices gives Figure 14. For \(k_0 = 91.19\) GeV and \(N = 3\) one now finds the estimates \(M_I = 0.54\) GeV and \(M_{II} = 0.85\) GeV, which are nearer to each other but smaller than the RS or the lattice predictions.

To conclude, we discuss the RG scale dependence of a physical observable \(\mathcal{O}_k(g_k)\). In particular, if this is RG invariant \(\frac{d}{dk} \mathcal{O}_k(g_k) = 0\), then it’s flow is governed by the beta function \(\beta(g_k) = \partial_t g_k\) of the coupling constant. If the observable has dimension \(d_\mathcal{O}\) the we can set \(\mathcal{O}_k(g_k) = k^{d_\mathcal{O}} \tilde{\mathcal{O}}(g_k)\) and obtain:

\[
d_\mathcal{O} \tilde{\mathcal{O}}(g_k) + \beta(g_k) \partial_y \tilde{\mathcal{O}}(g_k) = 0.
\] (162)

It’s easy to integrate (162) to find:

\[
\mathcal{O}_k = \mathcal{O}_{k_0} e^{-d_\mathcal{O} \int_{g_0}^{g_k} \frac{dg}{\beta(g)}},
\] (163)

Figure 13: Implicit plot for the bi–field flow of \(g_k\) from (160). From the top: RS, type II, type I and one-loop flows. Note that the type I and II are closer than in Figure 11.
Figure 14: The mass gap $M$ as a function of $g_0$ (with $k_0 = 91.19$ GeV) for the (from top-left) RS and bi–field flows.

where $t = \int_{g_0}^{g_k} \frac{dg}{\beta(g)}$. If we insert (157) into (163) we find:

$$O_k = O_{k_0} \left( \frac{g_k}{g_0} \right)^{d \phi_{a a} \frac{b + c}{2} \left( \frac{1}{g_k} - \frac{1}{g_0} \right)} \left( \frac{a + c g_k^2 + d g_k^4}{a + c g_0^2 + d g_0^4} \right)^{d \phi \frac{a b + c}{4 a^2}} \times$$

$$\times \left[ \frac{c + 2 dg_k^2 - \sqrt{c^2 - 4ad}}{c + 2 dg_0^2 + \sqrt{c^2 - 4ad}} \right] \left[ \frac{c + 2 dg_k^2 + \sqrt{c^2 - 4ad}}{c + 2 dg_0^2 - \sqrt{c^2 - 4ad}} \right].$$

Even in this case observables are smooth functions of the coupling constant. The simplest observable is the invariant scale $M$; we can recover (161) by setting $d \phi = 1$, $g_k^2 = 1/b$, $O_k = M$ and $O_{k_0} = k_0$ in (165). Other possible observables have different canonical dimensions, but in general observables turn out to be smooth functions of the coupling constant in spite of the singular behavior of the RG improved (or re-summmed) beta functions. Thus this kind of beta functions are consistent approximations to the RG flow that can be applied to the study of the IR physics of non-abelian gauge theories.
4 Discussion and future perspectives

In this paper we introduced a method that enables the projection of the RG flow equations of a large new class of truncations of the background effective average action (bEAA). Our method is based on the explicit momentum space representation of the hierarchy of flow equations satisfied by the proper-vertices of the bEAA. The key step in our construction is the determination of the explicit momentum space form of vertices with background legs, since these are related to functional derivatives of the cutoff action and must be treated with care. We showed how these vertices indeed have a simple form related to the vertices of the cutoff action, the action defined by having the cutoff operator as Hessian, and contains finite difference derivatives of the cutoff shape function. We also gave simple diagrammatic rules that provide a representation of the flow equations for the proper-vertices of the bEAA that can be used in actual computations. Our method is very general and can be applied to a variety of new interesting truncations of the bEAA, with applications to non-abelian gauge theories, nonlinear sigma models, gravity and membranes; or any other theory characterized by local gauge symmetries.

As a first application, we studied a bi–field truncation of the bEAA for $SU(N)$ non-abelian gauge theories. Employing bi–field truncations we can explore the dependency of the bEAA on all its arguments: fluctuation and background fields. We considered the three-dimensional subspace of theory space parametrized by the three anomalous dimensions $\{\eta_{\bar{A},k}, \eta_{a,k}, \eta_{c,k}\}$. We showed how all the results obtained with the aid of the local heat kernel expansion, usually employed in this framework, are reproduced by our technique. The structure of the exact flow equation for the bEAA implies that $\eta_{\bar{A},k}$ is linearly related to both $\eta_{a,k}$ and $\eta_{c,k}$; in the background field method, the beta function of the gauge coupling is related to $\eta_{\bar{A},k}$, thus to obtain $\partial_t g_k$ one needs $\eta_{a,k}$ and $\eta_{c,k}$. Usually one imposes $\eta_{a,k} = \eta_{\bar{A},k}$ and $\eta_{c,k} = 0$, in order to treat the problem by using only a single–field truncation. Instead, we calculated $\eta_{a,k}$ and $\eta_{c,k}$ directly and showed how they can be determined as a function of $g_k$ by solving a linear system; in this way we obtained a new RG improved form for the beta function of the gauge coupling. We then discussed some phenomenological implications of the flow so obtained, which has similarities with re-summed perturbative beta functions. As a future extension of this study, one can consider more general bi–field truncations by allowing the gauge fluctuation mass and gauge-fixing parameter to run [21].

In another application one can study bi–field truncations in the context of quantum gravity and check how Asymptotic Safety, up to now observed in single–field truncations, is robust to the bi–field extension [29].
Still, as work done in the non-background formalism [30] has shown, the non-trivial part of the bEAA has a non-local structure. With the method presented here we can project a large class of non-local truncations. In a gauge invariant formalism, the finite part of the effective action is encoded in “structure functions”, which are generally functions of the covariant Laplacian, and act on local invariants. As an example, in non-abelian gauge theories, one can consider the following ansatz for the gauge invariant part of the bEAA:

$$\bar{\Gamma}_k[A] = \int d^4x F^{\alpha}_{\mu\nu} f_k(-D^2) F^{\alpha\mu\nu} + O(F^3),$$

(166)

where $f_k$ is a running structure function which encodes non-trivial physical information. By applying our algorithm we can obtain a RG flow equation for the running structure function, which will be of the form:

$$\partial_t f_k = \mathcal{F}^{d}_{R_k} (g_k, f_k, f'_k, f''_k) .$$

(167)

By following the flow of the running structure function $f_k$ from the UV, where the theory is asymptotically free, down to the IR, we can obtain the full finite non-local effective action, from which we can extract relevant physical information. Similar applications can be made in the context of gravity, extending the results obtained using the non-local heat kernel expansion [10], in the context of non-linear sigma models or in the context membrane theory.

For these reasons, among others, the computational strategy based on the hierarchy of flow equations for the zero-field proper-vertices of the bEAA, when combined with the momentum space rules that we derived, is a promising tool for further studies of the bEAA in its full generality.

Acknowledgments

I would like to thank M. Reuter for stimulating discussions and O. Zanusso, M. Demmel, R. Percacci for careful reading the manuscript.
A Momentum space representation of background vertices

In this appendix we give the details of the derivation of the momentum space representation of vertices of the bEAA with background legs.

A.1 Perturbative expansion of the heat kernel

We shortly review the perturbative expansion for the heat kernel as developed in [12], to which we refer for more details. This expansion will be used in the next subsection to derive the momentum space representation of background vertices.

The heat kernel $K_{s}(x,y)$ satisfies the following partial differential equation with boundary condition:

$$(\partial_{s} + \Delta) K_{s}(x,y) = 0 \quad K^{0}(x,y) = \delta(x-y),$$

where $\Delta = -D_{\mu}D^{\mu} + U$ is the Laplacian operator constructed using the covariant derivatives $D_{\mu}$ and $U$ is a potential term. We decompose the the Laplacian as the sum of a “non-interacting” Laplacian $-\partial^{2}$ and of an interaction $V$ in the following way:

$$\Delta = -\partial^{2} + V.$$  

The potential $V$ contains $U$ and all terms proportional to the gauge connection $A_{\mu}$. Two examples are the flat space Laplacian $\Delta = -\partial^{2} + U$, where simply $V = U$, or the abelian gauge Laplacian where $V$ contains all terms that vanish for $A_{\mu} = 0$:

$$\Delta = -D_{\mu}D^{\mu} = - (\partial_{\mu} + iA_{\mu})(\partial^{\mu} + iA^{\mu}) = -\partial^{2} - 2A_{\mu}\partial^{\mu} - \partial_{\mu}A^{\mu} + A_{\mu}A^{\mu} + U.$$

To derive the perturbative series for the heat kernel we need first to calculate the heat kernel\footnote{Here and in the following we use the compact notation $K_{s}^{x,y} = K^{s}(x,y)$ and $\delta_{xy} = \delta^{(d)}(x-y)$. We also define $\int_{x} = \int d^{d}x$ and $\int_{q} = \int \frac{d^{d}q}{(2\pi)^{d}}$.} $K_{0,xy}^{s}$ of the operator $-\partial^{2}$, around which we will perform the expansion. From equation (168) we see that it satisfies the following equation with boundary condition:

$$(\partial_{s} - \partial_{x}^{2}) K_{0,xy}^{s} = 0 \quad K^{0}_{0,xy} = \delta_{xy};$$

(170)
the solution of (170) is the standard Gaussian:

\[ K_{0,xy}^{s} = \frac{1}{(4\pi s)^{d/2}} e^{-\frac{(x-y)^2}{4s}}. \]  

Equation (171) is the solution around which we will construct the perturbative expansion. Note that the heat kernel (171) satisfies:

\[ K_{0,xy}^{s_{1}+s_{2}} = \int_{z} K_{0,xz}^{s_{1}} K_{0,zy}^{s_{2}}. \]  

(172)

To derive the perturbative expansion of \( K_{xy}^{s} \) around \( K_{0,xy}^{s} \) we define the operator \( U_{xy}^{s} = \int_{z} K_{0,xx}^{-s} K_{zy}^{s} \). Using (168) and (170) we find it satisfies the following equation:

\[
\begin{align*}
\partial_{s} U_{xy}^{s} &= \int_{z} \left[ \partial_{s} K_{0,xx}^{-s} K_{zy}^{s} + K_{0,xx}^{-s} \partial_{s} K_{zy}^{s} \right] \\
&= \int_{z} \left[ K_{0,xx}^{-s} (-\partial_{z}^2) K_{zy}^{s} - K_{0,xx}^{-s} \Delta_{z} K_{zy}^{s} \right] \\
&= -\int_{z} K_{0,xx}^{-s} V_{z} K_{zy}^{s} \\
&= -\int_{zw} K_{0,xx}^{-s} V_{z} K_{0,ww}^{s} U_{wwy}^{s}.
\end{align*}
\]  

(173)

We know that equation (173) is solved by Dyson’s series:

\[ U_{xy}^{s} = T \exp \left\{ -\int_{0}^{s} dt \int_{z} K_{0,xx}^{-t} V_{z} K_{0,zy}^{t} \right\}, \]

where the exponential is time-ordered with respect to \( s \). Using (172) we immediately find:

\[ K_{xy}^{s} = \int_{z} K_{0,xx}^{s} T \exp \left\{ -\int_{0}^{s} dt \int_{w} K_{0,ww}^{-t} V_{w} K_{0,wy}^{t} \right\}. \]  

(174)

Rescaling the integration variable in (174) as \( t \to t/s \) and using (172) gives the final formula for the perturbative expansion of the heat kernel:

\[
\begin{align*}
K_{xy}^{s} &= \int_{z} K_{0,xx}^{s} T \exp \left\{ -s \int_{0}^{1} dt \int_{w} K_{0,ww}^{-st} V_{w} K_{0,wy}^{st} \right\} \\
&= K_{0,xy}^{s} - s \int_{0}^{1} dt \int_{z} K_{0,xx}^{s(1-t)} V_{z} K_{0,zy}^{st} + \ldots
\end{align*}
\]

\[ ^{13}\text{Not to be confused with the potential } U. \]

58
\[ +s^2 \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_{zw} K_{0,xx}^{s(1-t_1)} V_z K_{0,zw}^{s(t_1-t_2)} V_w K_{0,wy}^{st_2} + O(V^3). \tag{175} \]

In the following section we will use this relation to find an explicit momentum space representation for the functional derivatives of the cutoff action.

### A.2 Derivation of momentum space representation

In this section we derive the momentum space representation of background vertices, in particular we show how to obtain relations (37) and (39) used in section 2.3 to obtain the momentum space representation of the flow equations for the zero-field proper-vertices of the bEAA.

We need to calculate the momentum space representation of the cutoff vertices \( \Delta S_k^{(2;0)} [0; 0] \) and \( \Delta S_k^{(2;2)} [0; 0] \). From the definition of the cutoff action \( \Delta S_k \) we see that

\[ \Delta S_k^{(2;0)} [0; \bar{A}]_{xy} = \int_{zw} \delta_{xz} R_k[\bar{A}]_{zw} \delta_{wy} = R_k[\bar{A}]_{xy} \tag{176} \]

The cutoff kernel \( R_k[\bar{A}] \) is a function of the cutoff operator \( L^{(2;0)} [0; \bar{A}] \), constructed as the Hessian of the cutoff operator action \( L[a; \bar{A}] \). For example, if the cutoff operator is the gauge Laplacian, then:

\[ L^{(2;0)} [0; \bar{A}]_{xy} = \int_z \bar{D}_z \delta_{xz} \bar{D}_z^\mu \delta_{zy} = - \int_z \delta_{xz} \bar{D}_z^\mu \delta_{zy} = - \bar{D}_x^2 \delta_{xy}. \]

Thus, after making a Laplace transform, we can write the cutoff kernel in terms of the heat kernel of the cutoff operator \( R_k[\bar{A}]_{xy} = \int_0^\infty ds \tilde{R}_k(s) K^s[\bar{A}]_{xy} \),

\[ R_k[\bar{A}]_{xy} = \int_0^\infty ds \tilde{R}_k(s) K^s[\bar{A}]_{xy} \tag{177} \]

where the heat kernel can be written in terms of the Hessian of the cutoff operator action:

\[ K^s[\bar{A}] = \exp \{-sL^{(2;0)} [0; \bar{A}] \} \tag{178} \]

Inserting (177) in equation (176) and setting the background field to zero after differentiating with respect to it one time, gives the following representation for the cutoff vertex with one

\[ \text{In the gravitational case one must additionally care about factors of } \sqrt{g}. \]

\[ \text{For simplicity we consider } R_k^{AB} \text{ having tensor structure proportional to } \delta^{AB}. \]
background leg in terms of the heat kernel:

\[
\Delta S^{(2;1)}_k[0; 0]^{ABC}_{xy} = \frac{\delta R_k[A]_{xy}^{AB}}{\delta A^C_z} = \left. \int_0^\infty ds \frac{\delta K^s[A]_{xy}^{AB}}{\delta A^C_z} \right|_{A=0}.
\]  

(179)

We can now use the perturbative expansion for the heat kernel obtained in the previous section, equation (175), to write the last term of (179) as follows:

\[
\hat{\int}_0^\infty ds \tilde{R}_k(s) \frac{\delta K^s[A]_{xy}^{AB}}{\delta A^C_z} \bigg|_{A=0} = \int_0^\infty ds \tilde{R}_k(s)(-s) \int_0^1 dt K^{s(1-t)}_{0,xw} L^{(2;1)[0; 0]}_{wuz} K^{st}_{0,xy}.
\]

(180)

In (180) we omitted to explicitly write the coordinate integrals and we wrote the flat space heat kernels as \(K^s_{0,xy} = K^s_{0,xy}\delta^{AB}\), where \(K^s_{0,xy}\) is given in equation (171). Going to momentum space and inserting (180) in (179) gives the following representation for the cutoff vertex:

\[
\Delta S^{(2;1)}_k[0; 0]^{ABC}_{p_1,p_2,p_3} = \int_0^\infty ds \tilde{R}_k(s)(-s) \int_0^1 dt K^{s(1-t)}_{0,p_1} \int_0^1 dt e^{-s(1-t)p^2_2} L^{(2;1)[0; 0]}_{p_1,p_2,p_3}^{ABC} e^{-stp^2_2},
\]

(181)

where the cutoff operator vertices are defined as:

\[
l^{(n;m)}_{x_1 \ldots x_n y_1 \ldots y_m} 
\]

Here we used the following simple momentum space representation for the flat space heat kernel \(K^s_{0,p} = e^{-sp^2}\). It is left to evaluate the double integral in (181). This can be done with the aid of the Q-functionals (see the appendix A of [8]):

\[
Q_n[h](z + a) = \int_0^\infty ds \tilde{h}(s) s^{-n} e^{-sa},
\]

(182)

\[
Q_n[h](z + a) = \begin{cases} 
\frac{1}{f(n)} \int_0^\infty dz z^{n-1} h(z + a) & n > 0 \\
(-1)^n h^{(n)}(a) & n \leq 0
\end{cases}. 
\]

(183)
Using (183) we find:

\[ \int_0^\infty ds \tilde{R}_k(s)(-s)e^{-s(1-t)p_1^2-stp_2^2} = -Q_1[R_k(z + s(1-t)p_1^2 + stp_2^2)] = R_k(s(1-t)p_1^2 + stp_2^2). \]  

(184)

Now the parameter integral is easily evaluated:

\[ \int_0^1 dt R_k'(s(1-t)p_1^2 + stp_2^2) = \frac{R_k(p_2^2) - R_k(p_1^2)}{p_2^2 - p_1^2}. \]  

(185)

If we introduce the first finite-difference derivative, defined as

\[ f^{(1)}_{p_1,p_2} = \frac{f(p_2^2) - f(p_1^2)}{p_2^2 - p_1^2}, \]

we can finally write, for the cutoff vertex with one external background leg (181), the following momentum space representation:

\[ \Delta S^{(2;1)}_{AB}[0; 0]_{x,y} = \left[ l^{(2;1)}_{p_1,p_2,p_3} \right]^{ABC} \tilde{R}_k^{(1)}_{p_1,p_2}. \]  

(186)

We just need now to consider (186) with the momentum values \( p_1 = q, p_2 = -q - p \) and \( p_3 = p \) to prove the relation given in equation (37):

\[ \Delta S^{(2;1)}_{i_{q,-q-p}p} = \left[ l^{(2;1)}_{q,-q-p,p} \right]^{ABC} R_k^{(1)}_{q+p,q}. \]  

(187)

Along the same lines we can derive the momentum space representation for the cutoff vertex with two external background legs. In place of (179) we have now

\[ \Delta S^{(2;2)}_{ABCD}[0; 0]_{x,y} = \frac{\delta^2 R_k[A]^{AB}_{xy}}{A} = \int_0^\infty ds \tilde{R}_k(s) \frac{\delta^2 K^s[A]^{AB}_{xy}}{A}. \]  

(188)

Using the perturbative expansion (175) gives the following expansion for the second functional derivative of the cutoff kernel:

\[ \frac{\delta^2 R_k[A]^{AB}_{xy}}{A} = \int_0^\infty ds \tilde{R}_k(s) \frac{\delta^2 K^s[A]^{AB}_{xy}}{A}. \]  

(189)
Finally, inserting (193) in (191) and combining with (190), gives the following momentum space representation for (188):

\[ +2 \int_0^\infty ds \tilde{R}_k(s) s^2 \int_0^1 dt_1 \int_0^{t_1} dt_2 K^{s(1-t_1)}_0,0 \int_0^{s(t_1-t_2)} K^{s(t_1-t_2)}_0,0 L^{(2;1)}[0;0]_{\text{cmd}} R^{s(t_1-t_2)}_0,0,ty. \]  

When we insert (189) into (188) and shift to momentum space, the first contribution in (189), like in the previous case, takes the form:

\[ [j^{(2;2)}_{p_1,p_2,p_3,p_4}]^{ABCD} R^{(1)}_{p_4,p_1}. \]  

The second contribution takes instead the following form:

\[ [j^{(2;1)}_{p_1,-p_1-p_2,p_2}]^{ABM} [j^{(2;1)}_{p_3,p_1+p_2,p_4}]^{CMD} \times \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^\infty ds \tilde{R}_k(s) s^2 e^{-s(1-t_1)p_1^2-s(t_1-t_2)(p_1+p_2)^2-s_2t_2p_2^4}. \]  

We can calculate the double integral in (191) using the properties of the Q-functionals as before:

\[ \int_0^\infty ds \tilde{R}_k(s) s^2 e^{-s(1-t_1)p_1^2-s(t_1-t_2)(p_1+p_2)^2-s_2t_2p_2^4} = 
\quad = Q_{-2}[R_k(z+s(1-t_1)p_1^2+s(t_1-t_2)(p_1+p_2)^2+s_2t_2p_2^4)] 
\quad = R'_k(s(1-t_1)p_1^2+s(t_1-t_2)(p_1+p_2)^2+s_2t_2p_2^4), \]  

and

\[ 2 \int_0^1 dt_1 \int_0^{t_1} dt_2 R_k'(s(1-t_1)p_1^2+s(t_1-t_2)(p_1+p_2)^2+s_2t_2p_2^4) = 
\quad = \frac{2}{(p_1+p_2)^2-p_4^2} \left[ R_k((p_1+p_2)^2) - R_k(p_4^2) - R_k(p_1^2) - R_k(p_2^2) \right]. \]  

Finally, inserting (193) in (191) and combining with (190), gives the following momentum space representation for (188):

\[ \Delta S_k^{(2;2)}[0;0]^{ABCD}_{p_1,p_2,p_3,p_4} = [j^{(2;2)}_{p_1,p_2,p_3,p_4}]^{ABCD} R^{(1)}_{p_4,p_1} + 
\quad + [j^{(2;1)}_{p_1,-p_1-p_2,p_2}]^{ABM} [j^{(2;1)}_{p_3,p_1+p_2,p_4}]^{CMD} \frac{2}{(p_1+p_2)^2-p_4^2} \left[ R^{(1)}_{p_1,p_2,p_1} - R^{(1)}_{p_4,p_1} \right]. \]  

To recover relation (39) we set \(p_1 = -p_2 = q\) and \(p_3 = -p_4 = p\) in (194) so that:

\[ \Delta S_k^{(2;2)}[0;0]^{ABCD}_{q,-q,p,-p} = [j^{(2;2)}_{q,-q,p,-p}]^{ABCD} R^q_0 + [j^{(2;1)}_{q,p,-q,p}]^{ABM} [j^{(2;1)}_{q+p, q,-p}]^{CMD} R^{(2)}_{q+p,q}. \]  

62
where \( R'_p \equiv R'(p^2) \). In (195) we defined the second finite-difference derivative of a function as:

\[
    f^{(2)}_{p+q,q} = \frac{2}{(p+q)^2 - p^2} \left[ f^{(1)}_{p+q,p} - f'_p \right].
\]  

(196)

This concludes the derivation of relations (37) and (39) needed in section 2.3 to write the explicit momentum space representation for the flow equations of the zero-field proper-vertices of the bEAA.
B  Basic relations for non-abelian gauge theories

In this appendix we review our conventions and we collect the relevant formulae for functional variations and derivatives that are used in the paper.

B.1 Definitions and conventions

We consider the Lie groups $SU(N)$; we pick up a representation such that the group elements are represented by matrices $R = e^{-\theta t^a}$, where $\theta = -i \theta^a t^a$ are the (infinitesimal) group parameters, the indices $a, b, \ldots$ run from one to $\dim SU(N) = N^2 - 1$ and the $\dim R \times \dim R$ matrices $t^a$ are the generators of the Lie algebra of $SU(N)$ in the given representation. The generators satisfy the following commutation relations $[t^a, t^b] = if^{abc} t^c$; in a general representation the structure constants $f^{abc}$ are antisymmetric in the first two indices $f^{abc} = -f^{bac}$.

The covariant derivative is defined as $D_\mu = \partial_\mu + gA_\mu$, where $g$ is the gauge coupling constant and $A_\mu$ is the Lie algebra valued connection. The components of the connection are defined by $A_\mu = -i A^a_\mu t^a$ and one may write the covariant derivative as:

$$D_\mu = \partial_\mu - i A^a_\mu t^a. \quad (197)$$

The gauge field strength $F_{\mu\nu}$ is the curvature of the gauge connection; it can be defined as the commutator of covariant derivatives acting on matter fields $\phi$, $[D_\mu, D_\nu] \phi = F_{\mu\nu} \phi$, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ the field strength. In component form $F_{\mu\nu} = -i A^a_{\mu} t^a$ and we have:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \quad (198)$$

In the adjoint representation the structure constants are related to the generators $f^{abc} = i (t^a_{\text{ad}})^{ab}$ and are completely antisymmetric and we have $\dim \text{ad} = \dim SU(N)$. In this representation the covariant derivative (197) becomes:

$$D^{ab}_\mu = \partial_\mu \delta^{ab} - i A^c_\mu (t^c_{\text{ad}})^{ab} = \partial_\mu \delta^{ab} + g f^{acb} A^c_\mu. \quad (199)$$

We also have the following relation relating the commutator of the covariant derivatives and the field strength $[D_\mu, D_\nu]^{ab} = -f^{abc} F^c_{\mu\nu}$.

Under an infinitesimal gauge transformation $R \approx 1 - \theta$ matter fields and the connection transform as:

$$\delta_\theta \phi = [\phi, \theta] \quad \delta_\theta A_\mu = D_\mu \theta \quad \delta_\theta F_{\mu\nu} = [F_{\mu\nu}, \theta]. \quad (200)$$
In components we have instead\footnote{Note that \(-iA^c_{(t)}{^a}{_c}{^b}{_b} = i\theta^c(t{^a}{_c}){^a}{_b}A^b_{-}\).}

\[
\delta_\theta \phi^i = i\theta^a(t^a_R)^{ij} \phi^j \quad \delta_\theta A^a_\mu = D^a_\mu \theta^b + \partial_\mu \theta^a + f^{abc} A^b_\mu \theta^c \quad \delta_\theta F^a_{\mu\nu} = i\theta^c(t^c_{ad}){^{ab}}F^b_{\mu\nu} = f^{abc} F^b_{\mu\nu} \theta^c .
\]

(201)

where \(i, j, ...\) to the dimension of the representation of the matter fields.

### B.2 Variations and functional derivatives

We consider the functional:

\[
I[A] = \frac{1}{4} \int d^d x F^a_{\mu\nu} F^{a\mu\nu} ,
\]

(202)

which can be easily shown to be gauge invariant \(\delta_\theta I[A] = 0\). Expanding (202) around a background configuration \(A_\mu = \bar{A}_\mu + a_\mu\) gives the following:

\[
I[\bar{A} + a] = I[\bar{A}] + \int d^d x \bar{F}^{a\mu\nu} (\bar{D}_a \alpha_\nu)^a \\
\quad + \frac{1}{2} \int d^d x a_\mu^a \left[-D^2 g^{\mu\nu} + 2 f^{abc} F^{c\mu\nu} + D^{ac\mu} D^{cb\nu} \right] a^b_\nu \\
\quad + g f^{abc} \int d^d x (\bar{D}_a \alpha_\mu)^a a^b_\mu a^c_\nu + \frac{1}{4} g^2 f^{abc} f^{ade} \int d^d x a^b_\mu a^c_\nu a^d_\alpha a^e_\nu .
\]

(203)

One may use \(2 f^{abc} F^{c\mu\nu} = 2i F^{c\mu\nu} (t^c_{ad}){^{ab}} = -2 F^{\mu\nu}\) to write the differential operator in the quadratic term of (203) as \(-\bar{D}^2 g^{\mu\nu} - 2 F^{\mu\nu} + \bar{D}^{\mu} \bar{D}^{\nu}\). Note that the background gauge-fixing action (5) and the background ghost action (6) are already in their varied form since they are by construction quadratic in the fields.

We now calculate the functional derivatives of the functional (202), of the gauge-fixing action (5) and of the ghost action (6) that are needed in the flow equations for both the bEAA and the gEAA. The Hessian of \(I[A]\) at \(A_\mu = 0\) becomes:

\[
[I^{(2)}_{p_1, p_2}]^{\alpha\beta ab} = (2\pi)^d \delta_{p_1 + p_2} \delta^{ab} \left[ -g^{\alpha\beta} (p_1 \cdot p_2) + \frac{1}{2} \left( p_1^\alpha p_2^\beta + p_1^\beta p_2^\alpha \right) \right] .
\]

(204)

Note that in both (204), as in all the following formulas, the indices and the momentum variables are related in the precise order they appear. The third functional derivative of
which gives the gauge three-vertex, is:
\[
\left[I_{p_1,p_2,p_3}^{(3)}\right]^{\alpha\beta\gamma\, abc} = (2\pi)^d \delta^{\alpha\beta\gamma}_{p_1+p_2+p_3} \frac{1}{d} \int d^d x \left( \tilde{g} f^{abc} \left[ g^{\alpha\beta}(p_2-p_1)^\gamma + g^{\beta\gamma}(p_3-p_2)^\alpha + g^{\gamma\alpha}(p_1-p_3)^\beta \right] \right).
\] 

The fourth functional derivative of (202), which gives the gauge four-vertex, is:
\[
\left[f_{p_1,p_2,p_3,p_4}^{(4)}\right]^{\alpha\beta\gamma\delta\, abcd} = (2\pi)^d \delta^{\alpha\beta\gamma\delta}_{p_1+p_2+p_3+p_4} \frac{1}{d} \int d^d x \left( \tilde{g} f^{abcd} \left[ f^{e\alpha\beta\gamma\delta}_{abcdef} \right] \right). 
\]

We will use the vertices (205) and (206) to construct the zero-field proper vertices of the bEAA with both fluctuation and background legs. This is possible, since for a gauge invariant action as is (202), the following property holds:
\[
\frac{\delta I[A]}{\delta A^a_\mu} = \frac{\delta I[\bar{A} + a]}{\delta a^a_\mu} = \frac{\delta I[\bar{A} + a]}{\delta A^a_\mu}. \tag{207}
\]

Next, we need the vertices coming from the gauge-fixing action (5), which in component form reads:
\[
S_{gf}[a; \bar{A}] = \frac{1}{2\alpha} \int d^d x \left( \partial_\mu a^a_\mu + g f^{abc} \bar{A}^b_\mu a^c_\nu \right) \left( \partial_\nu a^a_\nu + g f^{ade} \bar{A}^d_\nu a^e_\mu \right), \tag{208}
\]

The second functional derivative of (208) with respect to the fluctuation field gives rise to:
\[
\left[S_{gf}^{(2;0)}\right]^{\alpha\beta\gamma\, ab} = -(2\pi)^d \delta^{\alpha\beta\gamma}_{p_1+p_2+p_3} \frac{1}{d} \int d^d x \left( \tilde{g} f^{abc} \left[ g^{\gamma\alpha}(p_1-p_2)^\beta + g^{\alpha\gamma}(p_2-p_3)^\beta \right] \right). \tag{209}
\]

The mixed functional derivatives give the fluctuation-fluctuation-background vertex:
\[
\left[S_{gf}^{(2;1)}\right]^{\alpha\beta\gamma\delta\, abcd} = (2\pi)^d \delta^{\alpha\beta\gamma\delta}_{p_1+p_2+p_3} \frac{1}{d} \int d^d x \left( \tilde{g} f^{abcd} \left[ f^{e\alpha\beta\gamma\delta}_{abcdef} \right] \right). \tag{210}
\]

Four mixed functional derivatives give the fluctuation-fluctuation-background-background vertex:
\[
\left[S_{gf}^{(2;2)}\right]^{\alpha\beta\gamma\delta\, abcd} = (2\pi)^d \delta^{\alpha\beta\gamma\delta}_{p_1+p_2+p_3+p_4} \frac{1}{d} \int d^d x \left( \tilde{g} f^{abcd} \left[ f^{e\alpha\beta\gamma\delta}_{abcdef} \right] \right). \tag{211}
\]

The vertices (210) and (211) will be used in section 3.2.2. The ghost action (6), when written
out explicitly, reads:

\[ S_{gh}[a, \bar{c}, c; \bar{A}] = \int d^d x \left( \partial_\mu \bar{c}^a + f^{abc} \bar{A}^b \partial^c \right) \left( \partial^\mu c^a + f^{ade} A^{d\mu} c^e + g f^{ade} a^{d\mu} c^e \right). \]  \hspace{1cm} (212)

Note that (212) generates the three-vertices ghost-ghost-fluctuation and ghost-ghost-background but only the four-vertex ghost-ghost-background-background. The two three-vertices differ by a factor of two since the background field enters both covariant derivatives while the fluctuation field does not. We have:

\[ [S_{gh}^{(1,1,1;0)}]^{\alpha abc} = -(2\pi)^d \delta_{p_1+p_2+p_3} i g f^{abc} \frac{\alpha}{p_2} \]  \hspace{1cm} (213)

and

\[ [S_{gh}^{(0,1,1;1)}]^{\gamma abc} = (2\pi)^d \delta_{p_1+p_2+p_3} i f^{abc} (p_2 - p_1)^\gamma. \]  \hspace{1cm} (214)

Finally the four-vertex is:

\[ [S_{gh}^{(0,1,1;2)}]^{\gamma \delta abcd} = (2\pi)^d \delta_{p_1+p_2+p_3+p_4} g^{\gamma \delta} \left( f^{eac} f^{ebd} + f^{ead} f^{ebc} \right). \]  \hspace{1cm} (215)

The gauge cutoff operator action in the type I case is

\[ L[a; \bar{A}] = \frac{1}{2} \int d^d x \bar{D}_\mu a^\mu; \]  \hspace{1cm} (216)

its vertices are:

\[ [L_{p_1,p_2,p_3}^{(2;1)}]^{\alpha \beta \gamma abc} = (2\pi)^d \delta_{p_1+p_2+p_3} i f^{abc} g^{\alpha \beta} (p_2 - p_1)^\gamma \]

\[ [L_{p_1,p_2,p_3,p_4}^{(2;2)}]^{\alpha \beta \gamma \delta abcd} = (2\pi)^d \delta_{p_1+p_2+p_3+p_4} g^{\alpha \beta} g^{\gamma \delta} \left( f^{eac} f^{ebd} + f^{ead} f^{ebc} \right). \]  \hspace{1cm} (217)

The ghost cutoff operator action is:

\[ L[\bar{c}, c; \bar{A}] = \int d^d x \bar{D}_\mu c \bar{D}^\mu c; \]  \hspace{1cm} (218)

since \( L[\bar{c}, c; \bar{A}] = S_{gh}[0, \bar{c}, c; \bar{A}] \) we have:

\[ [L_{p_1,p_2,p_3}^{(1,1;1)}]^{\alpha abc} = [S_{gh}^{(0,1,1;1)}]^{\alpha abc} \]

\[ [L_{p_1,p_2,p_3,p_4}^{(1,1;2)}]^{\gamma \delta abcd} = [S_{gh}^{(0,1,1;2)}]^{\gamma \delta abcd}. \]  \hspace{1cm} (219)

With these we derived all variations and vertices that we use in sections 3.2.2 and 3.2.3.
References

[1] J. Berges, N. Tetradis and C. Wetterich, Phys. Rep. 363 (2002) 223.

[2] H. Gies, Lect. Notes Phys. 852 (2012) 287, hep-ph/0611146.

[3] M. Reuter and C. Wetterich, Nucl. Phys. B 417 (1994) 181.

[4] M. Reuter, Phys. Rev. D 53 (1996) 4430, hep-th/9511128; M. Reuter, hep-th/9602012; M. Reuter, Mod. Phys. Lett. A 12 (1997) 2777, hep-th/9604124.

[5] A. Codello and R. Percacci, Phys. Lett. B 672 (2009) 280, arXiv:0810.0715; R. Percacci and O. Zanusso, Phys. Rev. D 81 (2010) 065012, arXiv:0910.0851; M. Fabbrichesi, R. Percacci, A. Tonero and O. Zanusso, Phys. Rev. D 83 (2011) 025016, arXiv:1010.0912; R. Flore, A. Wipf and O. Zanusso, arXiv:1207.4499.

[6] M. Reuter, Phys. Rev. D 57 (1998) 971, hep-th/9605030.

[7] A. Codello and O. Zanusso, Phys. Rev. D 83 (2011) 125021, arXiv:1103.1089; A. Codello, N. Tetradis and O. Zanusso, arXiv:1212.4073.

[8] A. Codello, R. Percacci and C. Rahmede, Annals Phys. 324 (2009) 414, arXiv:0805.2909.

[9] D. Benedetti, K. Groh, P.F. Machado and F. Saueressig, JHEP 1106 (2011) 079, arXiv:1012.3081.

[10] A. Codello, Annals Phys. 325 (2010) 1727, arXiv:1004.2171; A. Satz, A. Codello and F.D. Mazzitelli, Phys. Rev. D 82 (2010) 084011, arXiv:1006.3808; A. Codello, New J. Phys. 14 (2012) 015009, arXiv:1108.1908.

[11] E. Manrique and M. Reuter, Annals Phys. 325 (2010) 785, arXiv:0907.2617; E. Manrique, M. Reuter and F. Saueressig, Annals Phys. 326 (2011) 463, arXiv:1006.0099; E. Manrique, M. Reuter and F. Saueressig, Annals Phys. 326 (2011) 440, arXiv:1003.5129.

[12] A. Codello and O. Zanusso, J. Math. Phys. 54 (2013) 013513, arXiv:1203.2034.

[13] T.A. Ryttov and F. Sannino, Phys. Rev. D 78 (2008) 065001, arXiv:0711.3745.

[14] M. Reuter and C. Wetterich, Nucl. Phys. B 391 (1993) 147; M. Reuter and C. Wetterich, Nucl. Phys. B 408 (1993) 91; M. Reuter and C. Wetterich, Nucl. Phys. B 427 (1994) 291.
[15] L.F. Abbott, Nucl. Phys. B 185 (1981) 189; L.F. Abbott, Acta Phys. Polon. B 13 (1982) 33.

[16] U. Ellwanger, Phys. Lett. B 335 (1994) 364, hep-th/9402077; D.F. Litim and J.M. Pawlowski, Nucl. Phys. Proc. Suppl. 74 (1999) 325, hep-th/9809020; D.F. Litim and J.M. Pawlowski, hep-th/9901063; D.F. Litim and J.M. Pawlowski, Phys. Lett. B 546 (2002) 279, hep-th/0208216; F. Freire, D.F. Litim and J.M. Pawlowski, Phys. Lett. B 495 (2000) 256, hep-th/0009110.

[17] P.M. Lavrov and I.L. Shapiro, arXiv:1212.2577.

[18] M. Reuter and C. Wetterich, hep-th/9411227; M. Reuter and C. Wetterich, Phys. Rev. D 56 (1997) 7893, hep-th/9708051; H. Gies, Phys. Rev. D 66 (2002) 025006, hep-th/0202207; H. Gies, Phys. Rev. D 68 (2003) 085015, hep-th/0305208.

[19] A. Codello, R. Percacci and C. Rahmede, Int. J. Mod. Phys. A 23 (2008) 143-150, arXiv:0705.1769; P. Machado and F. Saueressig, Phys. Rev. D 77 (2008) 124045, arXiv:0712.0445.

[20] U. Ellwanger, M. Hirsch and A. Weber, Z. Phys. C 69 (1996) 687, hep-th/9506019.

[21] A. Codello, in preparation.

[22] D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343; H. D. Politzer, Phys. Rev. 30 (1973) 1346.

[23] A. Nink and M. Reuter, JHEP 1301 (2013) 062, arXiv:1208.0031.

[24] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B 229 (1983) 381.

[25] C. Pica and F. Sannino, Phys. Rev. D 83 (2011) 116001, arXiv:1011.3832.

[26] O.J. Rosten, arXiv:1106.2544.

[27] M. Gockeler, R. Horsley, A.C. Irving, D. Pleiter, P.E.L. Rakow, G.Schierholz and H.Stuben, Phys. Rev. D 73 (2006) 014513, hep-ph/0502212.

[28] F. Sannino and J. Schechter, Phys. Rev. D 82 (2010) 096008, arXiv:1009.0265.

[29] A. Codello, G. D’Odorico and C. Pagani, Phys. Rev. D 89 (2014) 8, 081701, arXiv:1304.4777.
[30] U. Ellwanger, M. Hirsch and A. Weber, Eur. Phys. J. C 1 (1998) 563, hep-ph/9606468; L. von Smekal, R. Alkofer and A. Hauck, Phys. Rev. Lett. 79 (1997) 3591, hep-ph/9705242; R. Alkofer and L. von Smekal, Phys. Rept. 353 (2001) 281, hep-ph/0007355; J.M. Pawlowski, D.F. Litim, S. Nedelko and L. von Smekal, Phys. Rev. Lett. 93 (2004) 152002, hep-th/0312324; C.S. Fischer and H. Gies, JHEP 0410 (2004) 048, hep-ph/0408089; C.S. Fischer and J.M. Pawlowski, Phys. Rev. D 75 (2007) 025012, hep-th/0609009; C.S. Fischer, A. Maas and J.M. Pawlowski, Annals Phys. 324 (2009) 2408, arXiv:0810.1987; C.S. Fischer and J.M. Pawlowski, Phys. Rev. D 80 (2009) 025023, arXiv:0903.2193; L. Fister and J.M. Pawlowski, arXiv:1301.4163.