On angles, projections and iterations

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Abstract. We investigate connections between the geometry of linear subspaces and the convergence of the alternating projection method for linear projections. The aim of this article is twofold: in the first part, we show that even in Euclidean spaces the convergence of the alternating method is not determined by the principal angles between the subspaces involved. In the second part, we investigate the properties of the Oppenheim angle between two linear projections. We discuss, in particular, the question of existence and uniqueness of “consistency projections” in this context.

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1 Introduction

The interest in the convergence of sequences of iterates of projections of various types goes back at least to the mid-twentieth century. J. von Neumann’s article \textsuperscript{17} from 1949 can be considered one of the starting points of these investigations. In this article he shows that given a Hilbert space $H$ and two closed subspaces $M, N \subset H$, with corresponding orthogonal projections $P_M$ and $P_N$, respectively, the sequence defined by

$$x_0 \in H \quad x_{2n+1} = P_M x_{2n} \quad \text{and} \quad x_{2n+2} = P_N x_{2n+1}$$

converges in norm to $P_{M \cap N} x_0$ for every initial point $x_0 \in H$. An elementary geometric proof of von Neumann’s theorem can be found in \textsuperscript{14} This result was later generalised to the case of more than two subspaces by I. Halperin in \textsuperscript{8}. In \textsuperscript{9} S. Kayalar and
H. L. Weinert showed that the speed of convergence is determined by the Friedrichs numbers between the subspaces involved. This can be considered a geometric condition controlling the convergence behaviour.

Note that in all these cases the order in which the projections are iterated is of crucial importance. In [1], I. Amemiya and I. Ando asked the question of whether convergence in norm can always be achieved provided that each projection appears infinitely often. This question was finally answered negatively by E. Kopecká and A. Paszkiewicz in [13], where they give an example of three subspaces and an iteration order without convergence in norm. More information concerning this phenomenon can be found in [11] and [12].

In Banach spaces, there are at least two natural generalisations of orthogonal projections: metric projections and linear projections. For the first one, the image is the point inside the subspace minimising the distance to the argument. It turns out that for iterations of metric projections, one cannot expect convergence of the iterates to the metric projection onto the intersection; see, for example, [20].

Recall that a linear mapping $P$ on a Banach space $X$ is called a linear projection if it satisfies the condition that $P^2 = P$. In this case, there are many positive results under the additional assumption that the projections are of norm one. For example, if the Banach space $X$ is uniformly convex, convergence of iterates of norm-one projections was established by R. E. Bruck and S. Reich in [9]. This result was later generalised, for example, to further classes of Banach spaces by C. Badea and Y. I. Lyubich in [2]. A dichtotomy for the speed of convergence of iterations of projections in Banach spaces which are uniformly convex of some power type has been exhibited by C. Badea and D. Seifert in [3]. More results on the convergence of the alternating algorithm for norm-one projections can be found in [7].

In the context of property (T) for certain groups, I. Oppenheim introduced in [19] an angle between linear projections in Banach spaces. This concept was developed further in [18], where a number of sufficient conditions for convergence of iterates of projections are given.

Iterations of non-orthogonal projections in Hilbert spaces, which then necessarily have a norm which is larger than one, are of interest in the context of discrete linear inclusions and of Skorokhod problems; see, for example, [15 21].

Since in most of the above results the convergence behaviour of the iterates of projections in determined or at least influenced by some kind of angle, one might hope that for non-orthogonal projections in Hilbert spaces the situation could be similar. More precisely, in the case of two linear projections, these projections are determined by two subspaces each—the range and the kernel. Moreover, in the case of Euclidean spaces, the concept of principal angles allows to determine the relative position of two subspaces up to an isometry. Therefore one could hope that these data might determine the convergence of the iterates of these projections.

The aim of this article is twofold: in the first part, we show that even in Euclidean spaces the convergence of the iterates is not determined by the principal angles between the subspaces involved. In the second part, we investigate the properties of the Oppenheim angle between two linear projections and provide an example which shows that the modification of the definition of this angle introduced in [18] is indeed necessary.
2 Preliminaries and Notation

2.1 Principal Angles

Principal angles are used to describe the geometric configuration of two subspaces of a real Hilbert space $H$ up to orthogonal mappings. Given two finite-dimensional subspaces $S_1, S_2 \subseteq H$ and denoting by $q$ the minimum of the dimensions of $S_1$ and $S_2$, the principal angles

$$\theta_1, \ldots, \theta_q \in \left[0, \frac{\pi}{2}\right]$$

and the corresponding principal vectors

$$u_1, \ldots, u_q \in S_1 \quad v_1, \ldots, v_q \in S_2$$

are defined inductively by

$$u_k, v_k = \arg \max \{|\langle u_k, v_k \rangle|: \langle u_k, u_i \rangle = \langle v_k, v_i \rangle = \delta_{k,i} \text{ for } i = 1, \ldots, k\}$$

$$\theta_k = \arccos(\sqrt{\lambda_k})$$

for $k = 1, \ldots, q$. The principal angles can also be represented in terms of the orthogonal projections $P_{S_1}$ and $P_{S_2}$ onto $S_1$ and $S_2$, respectively. More precisely,

$$\theta_k = \arccos(\sqrt{\lambda_k}),$$

where $\lambda_1 \geq \ldots \geq \lambda_q$ are the first $q$ eigenvalues of the restriction of $P_{S_2}P_{S_1}$ to $S_2$. This formula allows for a direct computation of the principal angles, thus avoiding the optimisation problems in (1). Moreover, it has the advantage that it also makes sense for infinite-dimensional subspaces; see, for example, [10].

The principal angles between two subspaces define them completely up to a simultaneous rotation. This means that every rotation-invariant function of two subspaces can be written as a function of the principal angles between them; for example, the Dixmier angle and the Friedrichs angle are the smallest and the smallest non-zero principal angle, respectively.

Since the function which maps subspaces to their orthogonal complements commutes with rotations, knowing all the angles between two subspaces $S_1$ and $S_2$ is equivalent to knowing all the angles between $S_1$ and $S_2^\perp$.

We denote by $\Theta(S_1, S_2)$ the ordered tuple of the principal angles between two subspaces $S_1$ and $S_2$.

A more detailed exposition of these angles, where the relations between various approaches to angles between subspaces—including principal angles and directed distances—is examined, can be found in [16, Chapter 5.15].
2.2 The Cross-Ratio of projective points

For four distinct points \(a_1, a_2, a_3, a_4\) of the projective line \(\mathbb{P}^1(\mathbb{R})\), the cross-ratio of these points, denoted by \([a_1, a_2, a_3, a_4] \in \mathbb{R} \cup \{\infty\}\), is defined by

\[
[a_1, a_2, a_3, a_4] = \frac{\det \begin{pmatrix} \lambda_3 & \lambda_1 \\ \mu_3 & \mu_1 \end{pmatrix} \det \begin{pmatrix} \lambda_4 & \lambda_2 \\ \mu_4 & \mu_2 \end{pmatrix}}{\det \begin{pmatrix} \lambda_3 & \lambda_2 \\ \mu_3 & \mu_2 \end{pmatrix} \det \begin{pmatrix} \lambda_4 & \lambda_1 \\ \mu_4 & \mu_1 \end{pmatrix}}
\]

where \( \lambda_i = \infty \) and \(a_i = p(\lambda_i, \mu_i)\), that is, \(\lambda_i, \mu_i\) are the homogeneous coordinates of \(a_i\). In our application, the denominator will never be zero, and so we can always assume that \([a_1, a_2, a_3, a_4] \in \mathbb{R}\). We will need the following formula for the cross-ratio:

**Lemma 1.** Let \(a \in (\mathbb{R}^2)^4\). Then

\[
\frac{\langle a_1, a_3 \rangle \langle a_2, a_4 \rangle}{\langle a_1, a_4 \rangle \langle a_2, a_3 \rangle} = [\text{span}(a_1), \text{span}(a_2), \text{span}(a_3)^\perp, \text{span}(a_4)^\perp]
\]

**Proof.** Using \((x_2, -x_1)\) as homogeneous coordinates for \(\text{span}(x)^\perp\) and the Leibniz formula for the determinant, we can directly obtain this assertion from the definition.

The behaviour of the cross-ratio function with respect to permutations is well known. Since we will use the following lemma later, we state it at this point without proof (which can be found, for example, in [4, pp. 123–126]).

**Lemma 2.** The cross-ratio satisfies

\[
[a, b, c, d] = [d, c, b, a] \quad \text{and} \quad [a, c, b, d] = 1 - [a, b, c, d],
\]

where \(a, b, c, d\) are pairwise distinct one-dimensional subspaces of \(\mathbb{R}^2\).

2.3 The Oppenheim angle between linear projections

Let \(P_1, P_2\) be two bounded linear projections in a Banach space. Assume that there is a bounded linear projection \(P_{12}\) onto the intersection of the images of \(P_1\) and \(P_2\) satisfying

\[
P_{12}P_1 = P_{12} \quad \text{and} \quad P_{12}P_2 = P_{12}.
\]

Using these data, we define

\[
\cos_{P_{12}}(\angle(P_1, P_2)) = \max\{\|P_1(P_2 - P_{12})\|, \|P_2(P_1 - P_{12})\|\}
\]

and

\[
\cos(\angle(P_1, P_2)) = \inf\{\cos_{P_{12}}(P_1, P_2) : P_{12} \text{ projection onto } \text{im} P_1 \cap \text{im} P_2 \text{ satisfying (2)}\}.
\]

In the case of two orthogonal projections \(P_1, P_2\) in a Hilbert space, the above angle coincides with the Friedrichs angle between the images of \(P_1\) and \(P_2\). The subtraction of
the projection $P_{12}$ in the definition above plays the role of the quotient in the definition of the Friedrichs angle. There need not always be such a projection $P_{12}$. Moreover, the intersection of the images of $P_1$ and $P_2$ need not even be complemented. Two projections $P_1$ and $P_2$ are called consistent if such a projection does exist. We also call a projection $P_{12}$ with the above properties a consistency projection. The main interest in this angle lies in the fact that a large Oppenheim angle, that is, a small cosine in the above definition, implies that the iterations $(P_1 P_2)^n$ converge uniformly to a (consistency) projection onto $\text{im} P_1 \cap \text{im} P_2$. For a detailed discussion of these angles, we refer the reader to [13].

3 Classification of the Convergence Behaviour Through Principal Angles

In all that follows, let $X$ be a Euclidean space. All geometric characterisations of the convergence behaviour we provide in this article are based on the two-dimensional case. Given two subspaces $M, N \subset \mathbb{R}^n$ we use the notation $M \oplus N$ for the direct sum of $M$ and $N$, that is, this notation indicates that $M \cap N = \{0\}$.

3.1 The Two-Dimensional Case

For two two-dimensional projections $P_1, P_2 : \mathbb{R}^2 \to \mathbb{R}^2$, the convergence behaviour of $(P_1 P_2)^n_{n=1}$ is determined by the geometric relation between the nullspaces and ranges of $P_1, P_2$. The question of the convergence behaviour of the sequence of iterates $(P_1 P_2)^n_{n=1}$ is trivial if either of the projections is the identity or the zero mapping. Therefore we restrict ourselves to the case where the range of both projections is a one-dimensional subspace of $\mathbb{R}^2$.

**Proposition 1.** Let $P_1, P_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be two non-trivial projections, that is, neither of them is the identity or zero. We set $R_1 = \text{im} P_1, N_1 = \ker P_1, R_2 = \text{im} P_2$ and $N_2 = \ker P_2$. The composition $P_1 P_2$ has at most one non-zero eigenvalue $\lambda$ and it satisfies

$$\lambda = [R_1, R_2, N_2, N_1].$$

In particular, we have

$$\rho(P_1 P_2) = \frac{s(R_1, N_2)s(R_2, N_1)}{s(R_1, N_1)s(R_2, N_2)},$$

where $s(M, N) := \sqrt{1 - c(M, N)^2}$ for subspaces $M, N \subset \mathbb{R}^2$ and $c(M, N)$ is the Friedrichs number of $M$ and $N$. The iterates $(P_1 P_2)^n$ converge to zero if and only if the geometric condition

$$\frac{s(R_1, N_2)s(R_2, N_1)}{s(R_1, N_1)s(R_2, N_2)} < 1$$

is satisfied.

**Proof.** Let $w_1, v_1, w_2, v_2 \in \mathbb{R}^2$ be such that $P_1 = w_1 v_1^*, P_2 = w_2 v_2^*$. For $k = 1, 2$, set $R_k = \text{im} P_k, N_k = \ker P_k$ and observe that $\text{span}(v_k) = N_k^\perp, \text{span}(w_k) = R_k \neq \{0\}$ and,
since \( P_k \) is a projection,
\[
w_k = P_k(w_k) = w_k \langle v_k, w_k \rangle,
\]
and thus \( 1 = \langle v_k, w_k \rangle \).

Since \( R_1 = \text{span}(w_1) \), any non-zero eigenvector of \( P_1P_2 \) must be a multiple of \( w_1 \). Calculating \( P_1P_2w_1 \), we get
\[
P_1P_2w_1 = w_1v_1^\ast(w_2)v_2^\ast(w_1) = \frac{\langle v_1, w_2 \rangle \langle v_2, w_1 \rangle}{\langle v_1, w_1 \rangle \langle v_2, w_2 \rangle}w_1
\]
and hence, using Lemma 1, we obtain
\[
\lambda = \frac{\langle v_1, w_2 \rangle \langle v_2, w_1 \rangle}{\langle v_1, w_1 \rangle \langle v_2, w_2 \rangle} = \frac{\langle w_1, v_2 \rangle \langle w_2, v_1 \rangle}{\langle w_1, v_1 \rangle \langle w_2, v_2 \rangle}
\]
\[
= \text{span}(w_1), \text{span}(w_2), \text{span}(v_2)\perp, \text{span}(v_1)\perp = [R_1, R_2, N_2, N_1]
\]
and
\[
\rho(P_1P_2) = \frac{|\langle w_1, v_2 \rangle||\langle w_2, v_1 \rangle|}{|\langle w_1, v_1 \rangle||\langle w_2, v_2 \rangle|} = \frac{|(w_1\|v_2\|)(w_2\|v_1\|)|}{|(w_1\|v_1\|)(w_2\|v_2\|)|} = \frac{s(R_1, N_2)s(R_2, N_1)}{s(R_1, N_1)s(R_2, N_2)}
\]
In particular, we see that convergence occurs if and only if the modulus of the above number is smaller than one.

\[\square\]

Remark 1. Note that for two projections \( P_1 \) and \( P_2 \) in \( \mathbb{R}^2 \) with distinct one-dimensional images, the zero mapping is the uniquely determined projection onto the intersection of these ranges. Using the above result, we see that
\[
\frac{s(R_1, N_2)s(R_2, N_1)}{s(R_1, N_1)s(R_2, N_2)} = \rho(P_1P_2) \leq \|P_1P_2\|_2 = \cos \angle_0(P_1, P_2) = \cos \angle(P_1, P_2),
\]
that is, the iterates converge whenever the “cosine” of the Oppenheim angle between the projections is smaller than one. Moreover, the above discussion shows that, in this particular case, there is a relation between the Oppenheim angle and the principal angles.

3.2 The three-dimensional case: Some additional information might be needed

Also in the three-dimensional case, we restrict ourselves to non-trivial projections, that is, we exclude both the identity mapping and the zero mapping. In this section let \( P_1, P_2 \) be two non-trivial projections in \( \mathbb{R}^3 \). In order to simplify the notation, we use the abbreviations
\[
R_1 = \text{im} P_1, \quad N_1 = \ker P_1, \quad R_2 = \text{im} P_2 \quad \text{and} \quad N_2 = \ker P_2.
\]
We start our investigation of the convergence behaviour with the observation that the problem is at its core still a two-dimensional problem. We call an eigenvector of the composition \( P_1P_2 \) non-trivial if the corresponding eigenvalue is neither zero nor one.
**Lemma 3.** For every non-trivial eigenvector \( v \) of \( P_1 P_2 \), there exists a two-dimensional subspace \( E_v \) such that the intersections of \( E_v \) with \( R_1, R_2, N_1, N_2 \) are all one-dimensional with trivial pairwise intersection and \( v \in E_v \).

**Proof.** Let \( \lambda \) be the eigenvalue corresponding to \( v \) (that is \( P_1 P_2 v = \lambda v \)). Set \( w = P_2(v) \), \( E_v = \text{span}(w, v) \) (\( \lambda \neq 0 \) implies \( w \neq 0 \)). Then
\[
\lambda v \in R_1 \cap E_v, \quad \lambda v - w = P_1(w) - w \in N_1 \cap E_v, \\
w \in R_2 \cap E_v \quad \text{and} \quad w - v = P_2(v) - v \in N_2 \cap E_v.
\]
Note that \( w \in R_1 \) or \( v \in R_2 \) would mean that \( w = v = \lambda v \), while \( w \in N_1 \) or \( v \in N_2 \) would mean that \( P_1 P_2 v = 0 \), both contrary to the assumption that \( v \) is non-trivial. Therefore \( v \) and \( w \) are linearly independent, all the intersections are one-dimensional and \( E_v \) is two-dimensional. Also using this, we can easily reproduce the triviality of the pairwise intersections.

The above lemma implies, in particular, that we still can interpret the ranges and kernels of the projections under consideration as projective points. Moreover, the cross-ratio of these points still carries the vital information regarding the convergence.

**Lemma 4.** Let \( v \) be an eigenvector corresponding to the non-trivial eigenvalue \( \lambda \) of the operator \( P_1 P_2 \) and \( E_v \) the associated subspace established in Lemma 3. Set
\[
R'_1 = R_1 \cap E_v, \quad R'_2 = R_2 \cap E_v, \quad N'_1 = N_1 \cap E_v \quad \text{and} \quad N'_2 = N_2 \cap E_v
\]
Then
\[
\lambda = [R'_1, R'_2, N'_2, N'_1],
\]
where the cross-ratio is meant to be taken on \( E_v \).

**Proof.** First, we show that \( P_k(E_v) \subseteq E_v \) for \( k \in \{1, 2\} \). To this aim, first observe that Lemma 3 implies that \( E_v = R'_k \oplus N'_k \). Given \( x \in E_v \), we choose \( r \in R'_k, n \in N'_k \) such that \( x = r + n \). Then
\[
P_k(x) = P_k(r) + P_k(n) = r \in E_v.
\]
Hence the mappings \( P_k' : E_v \to E_v, x \mapsto P_k(x) \) are well-defined projections for \( k \in \{1, 2\} \).

Since, by construction, \( \lambda \) is the non-trivial eigenvalue of \( P_1 P_2' \), we can use Proposition 1 to complete the proof:
\[
\lambda = [\text{im } P'_1, \text{im } P'_2, \ker P'_2, \ker P'_1] = [R'_1, R'_2, N'_2, N'_1].
\]

A plane with the properties of \( E_v \) in Lemma 3 is conversely always associated to a non-trivial eigenvector:

**Lemma 5.** Let \( P_1, P_2 \) be projections and let \( E \) be a two-dimensional subspace of \( \mathbb{R}^3 \) such that the intersections of \( E \) with \( R_1, R_2, N_1, N_2 \) are all one-dimensional with trivial pairwise intersection. Then \( E \cap R_1 \) is an eigenspace of \( P_1 P_2 \) corresponding to a non-trivial eigenvalue.
Proof. As in the proof of Lemma 4 above, the well-definedness of the two projections \( P'_k : E \to E, x \mapsto P_k(x) \) follows from \( E = (R_k \cap E) \oplus (N_k \cap E) \) for \( k \in \{1, 2\} \). Since \( R'_1 = R(P'_1) \) is one-dimensional, it is an eigenspace of \( P_1 P_2' \) corresponding to the eigenvalue

\[
\lambda = [\text{im} \ P'_1, \text{im} \ P'_2, \ker \ P'_2, \ker \ P'_1] = [R_1 \cap E, R_2 \cap E, N_2 \cap E, N_1 \cap E].
\]

By construction, \( R'_1 \) is also an eigenspace of \( P_1 P_2 \) corresponding to \( \lambda \), and since the arguments in the cross-ratio function are pairwise distinct, \( \lambda \notin \{0, 1\} \) (see, for example, [4, Proposition 6.1.3]). \( \square \)

In order to formulate a characterisation of convergence in three dimensions, we need a geometric lemma regarding the connection of angles and directed distances for subspaces. Recall that for subspaces \( M, N \subseteq X \) the directed distance \( \delta(M, N) \) is defined by

\[
\delta(M, N) = \sup \{d(x, N) : x \in M, \|x\| = 1\}.
\]

A simple computation shows that \( \delta(M, N) = \sup \{\|P_{N^\perp} x\| : x \in M, \|x\| = 1\} \).

Lemma 6. Let \( H \) be a real Hilbert space over, let \( S_1, S_2, V \) be three one-dimensional, pairwise distinct subspaces of \( H \) such that \( V \subseteq S_1 \oplus S_2 \), and let \( W \) be another subspace of \( H \) such that \((S_1 \oplus S_2) \cap W = \{0\}\). Then

\[
\frac{s(S_1, V)}{s(S_2, V)} = \frac{\delta(S_1, V \oplus W)}{\delta(S_2, V \oplus W)}.
\]

Proof. Let \( k \in \{1, 2\} \) and \( s_k \in S_k \) such that \( \|s_k\| = 1 \). Note that

\[
s(s_k, V) = \|P_{V^\perp}(s_k)\| \quad \text{and} \quad \delta(s_k, V \oplus W) = \|P_{(V \oplus W)^\perp}(s_k)\|
\]

because both \( S_k \) and \( V \) are one-dimensional. From

\[
N(P_{V^\perp}) = V \subseteq V \oplus W = N(P_{(V \oplus W)^\perp})
\]

we may conclude that \( P_{(V \oplus W)^\perp} = P_{(V \oplus W)^\perp} P_{V^\perp} \). Since the subspaces \( S_1, S_2, V \) are pairwise distinct and \( S_1 \subseteq S_2 \oplus V \), we may pick \( c \in \mathbb{R} \) and \( v \in V \) such that \( s_1 = c s_2 + v \). Then,

\[
P_{V^\perp}(s_1) = P_{V^\perp}(c s_2) + P_{V^\perp}(v) = c P_{V^\perp}(s_2).
\]

Comparing the norms of these expressions, we see that \( P_{V^\perp}(s_1) = c P_{V^\perp}(s_2) \) with a number \( c \) satisfying \( |c| = \frac{s(S_1, V)}{s(S_2, V)} \). Therefore, we may conclude that

\[
\delta(S_1, V \oplus W) = \|P_{(V \oplus W)^\perp}(s_1)\| = \|P_{(V \oplus W)^\perp} P_{V^\perp}(s_1)\| = \|c P_{(V \oplus W)^\perp}(s_2)\| = |c| \| P_{(V \oplus W)^\perp}(s_2)\| = \frac{s(S_1, V)}{s(S_2, V)} \delta(S_2, V \oplus W).
\]

Finally, as \((S_1 \oplus S_2) \cap W = \{0\}\) and hence \( S_2 \not\subseteq V \oplus W \), we have \( \delta(S_2, V \oplus W) \neq 0 \) which finishes the proof. \( \square \)
Proposition 2. Let $P_1, P_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be two projections with two-dimensional images. There is at most one non-trivial eigenvalue and if it exists it satisfies the equation

$$|\lambda| = \frac{\delta(N_2, R_1)\delta(N_1, R_2)}{\delta(N_1, R_1)\delta(N_2, R_2)}.$$

In particular, the iterates $(P_1 P_2)^n$ converge if and only if

$$\frac{\delta(N_2, R_1)\delta(N_1, R_2)}{\delta(N_1, R_1)\delta(N_2, R_2)} < 1,$$

that is, the convergence is determined by the angles between the ranges and kernels.

Proof. Assuming the existence of at least one non-trivial eigenvector $v$ with corresponding eigenvalue $\lambda$, we see that $N_1 \cap N_2 = \{0\}$, the only possible choice of $E_v$ is $N_1 \oplus N_2$, and $R_1 \cap R_2 \cap E_v = \{0\}$. We set

$$R_1' = R_1 \cap E_v, \quad R_2' = R_2 \cap E_v \quad \text{and} \quad Z = R_1 \cap R_2.$$

Then, by comparing dimensions, we observe that $R_1 = R_1' \oplus Z$ and $R_2 = R_2' \oplus Z$. Using Lemmata 4 and 6, we conclude that

$$|\lambda| = \frac{s(N_2, R_1')}{s(N_1, R_1')} \frac{s(N_1, R_2')}{s(N_2, R_2')} = \frac{\delta(N_2, R_1' \oplus Z)}{\delta(N_1, R_1' \oplus Z)} \frac{\delta(N_1, R_2' \oplus Z)}{\delta(N_2, R_2' \oplus Z)},$$

as claimed.

Remark 2. We conclude this section with a few observations concerning the validity of the above characterisation of convergence of the iterates.

1. Using the behaviour of the cross-ratio with respect to permutations of its arguments stated in Lemma 2, we obtain that for the case of two linear projections $P_1, P_2$ on $\mathbb{R}^3$ with one-dimensional ranges, we also see that there is at most one non-trivial eigenvalue and if it exists, then it satisfies the equation

$$|\lambda| = \frac{\delta(R_1, N_2)\delta(R_2, N_1)}{\delta(R_1, N_1)\delta(R_2, N_2)}.$$

In particular, the iterates $(P_1 P_2)^n$ converge if and only if

$$\frac{\delta(R_1, N_2)\delta(R_2, N_1)}{\delta(R_1, N_1)\delta(R_2, N_2)} < 1.$$

This shows that also in this case, convergence of the iterates is determined by the angles between the ranges and the kernels.
2. Copying the above arguments, it is possible to show the same characterisation for projections on Hilbert spaces, where both have either one-dimensional images or one-dimensional kernels.

3. The “mixed case”, that is, the case where one projection has a one-dimensional range and the other projection has a one-dimensional kernel, is more complicated. Let \( P_1 \) and \( P_2 \) be two projections on \( \mathbb{R}^3 \) and assume that \( P_1 \) has a one-dimensional range and \( P_2 \) has a two-dimensional one. Using the results of this section, we may conclude that there is a unique non-trivial eigenvalue \( \mu \) of

\[
P(\text{im} \ P_1, \ker \ P_1)P(\ker \ P_2, \text{im} \ P_2)
\]

and that the unique non-trivial eigenvalue of \( P_1P_2 \) is \( \lambda = 1 - \mu \). So, if

\[
\delta(\text{im} \ P_1, \ker \ P_2)\delta(\text{im} \ P_2, \ker \ P_1) < 1.
\]

and the cross-ratio \( [\text{im} \ P_1, \ker \ P_2, \text{im} \ P_2, \ker \ P_1] > 0 \), we have convergence of the iterates \( (P_1P_2)^n \). In order to show that condition (3) is not enough, we consider the following example. Consider the vectors

\[
w_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad w_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad w'_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}
\]

and set \( S_i = \text{span}(w_i) \) for \( i = 1, 2; S_i = \text{span}(w_i)\perp \) for \( i = 3, 4; S'_4 = \text{span}(s'_4)\perp \). A direct computation shows that

\[
\Theta(S_4, S_i) = \Theta(S'_4, S_i) \quad \text{for} \quad i = 1, 2, 3.
\]

On the other hand, setting \( V_i = S_i \cap (S_1 \oplus S_2), V'_4 = S'_4 \cap (S_1 \oplus S_2), \) we get

\[
\frac{1}{2} = [V_1, V_2, V_3, V_4] = -[V_1, V_2, V_3, V'_4].
\]

Using the above arguments, we see that the sequence \( (P(S_1, S_4)(P(S_3, S_2))^n)_{n=1}^{\infty} \) converges whereas \( ((P(S_1, S'_4)P(S_3, S_2))^n)_{n=1}^{\infty} \) does not. Since all the principal angles are the same in both cases, they do not determine the convergence behaviour on their own.

### 3.3 Higher dimensions: Angles are not enough

For dimensions higher than three, a characterisation using only the principal angles cannot work. We show this by giving a counterexample. It is somewhat similar to the one given in Remark 2, but in contrast to the situation there, in higher dimensions it seems to be unclear how to overcome the problem. The counterexample is built by combining two two-dimensional examples in a specific way. In order to do this, we first
need a simple observation on operator matrices. For two operators $T_1 : H_1 \to H_1$, $T_2 : H_2 \to H_2$ on Hilbert spaces $H_1$ and $H_2$, we denote the operator matrix

$$
\begin{bmatrix}
T_1 & 0 \\
0 & T_2
\end{bmatrix} : H_1 \oplus H_2 \to H_1 \oplus H_2
$$

\((h_1, h_2) \mapsto (T_1(h_1), T_2(h_2))\)

by $T_1 \oplus T_2$. We consider the case of four projections $P_1$ and $P_2$ on $H_1$, and $P_3$ and $P_4$ on $H_2$. Then $P_1 \oplus P_3$ and $P_2 \oplus P_4$ are projections on $H_1 \oplus H_2$ satisfying

$$
im P_1 \oplus P_3 = \im P_1 \oplus \im P_3, \quad \ker P_1 \oplus P_3 = \ker P_1 \oplus \ker P_3,$$

$$
im P_2 \oplus P_4 = \im P_2 \oplus \im P_4 \quad \text{and} \quad \ker P_2 \oplus P_4 = \ker P_2 \oplus \ker P_4$$

as can be seen by a direct computation. Moreover, the spectrum of $(P_1 \oplus P_3)(P_2 \oplus P_4)$ is just the union of the spectra of $P_1 P_2$ and $P_2 P_4$.

The final observation needed for the example is that principal angles between two direct sums of subspaces are nothing but the combined principal angles between the individual subspaces in both summands.

**Example 1.** We construct a counterexample in $\mathbb{R}^4$. More precisely, we construct four projections $P_1, P_2, P'_1, P'_2$ such that all the principal angles between the ranges and null-spaces of $P_1, P_2$ on one hand, and $P'_1, P'_2$ on the other, are identical, but $\rho(P_1 P_2) = 0$, while $\rho(P'_1 P'_2) = 2$.

First, we fix $\varphi \in (0, \frac{\pi}{2})$. For $\theta \in (0, \frac{\pi}{2})$, we set

$$
e(\theta) = \begin{pmatrix}
\cos(\theta) \\
\sin(\theta)
\end{pmatrix}, \quad S(\theta) = \text{span}(e(\theta)) \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$

Now let $s = \pm 1$ and set

$$
P^{1}_{1} = P(S(0), S(\frac{\pi}{2} + \varphi)) \quad \text{and} \quad P^{1}_{2,s} = P(S(\frac{\pi}{2} + s\varphi), S(\frac{\pi}{2})).$$

Since $\im(P^{1}_{2,1}) = \ker(P^{1}_{1})$ we have $P^{1}_{1} P^{1}_{2,1} = 0$. Calculating the unique non-trivial eigenvalue of $P^{1}_{1} P^{1}_{2,-1}$, we get

$$|\lambda| = \rho(P^{1}_{1} P^{1}_{2,-1}) = \frac{|\langle e(0), e(\frac{\pi}{2} - \varphi) \rangle \langle e(\frac{\pi}{2} + \varphi), e(\frac{\pi}{2} - \varphi) \rangle|}{\langle e(0), e(\frac{\pi}{2} + \varphi) \rangle \langle e(\frac{\pi}{2} - \varphi), e(\frac{\pi}{2} - \varphi) \rangle} = \frac{\cos(\frac{\pi}{2} - 2\varphi)}{\cos(\varphi) \cos(\frac{\pi}{2} - \varphi)} = \frac{\cos(2\varphi)}{\cos(\varphi) \sin(\varphi)} = 2 \frac{\cos(\varphi) \sin(\varphi)}{\cos(\varphi) \sin(\varphi)} = 2.$$

In order to construct the other pair of projections, we set

$$
P^{2}_{1} = P(S(0), S(\frac{\pi}{2} + \varphi)) \quad \text{and} \quad P^{2}_{2,s} = P(S(\frac{\pi}{2} - s\varphi), S(0)).$$

Since $R(P^{2}_{1}) \subseteq N(P^{2}_{2,s})$ we have $\rho(P^{2}_{1} P^{2}_{2,s}) = 0.$
We combine these projections by setting
\[ P_1 := P_1^1 \oplus P_1^2, \quad \text{and} \quad P_{2,s} := P_{2,s}^1 \oplus P_{2,s}^2, \]
for \( s = \pm 1 \). Since the principal angles between direct sums of subspaces are just the combination of the principal angles between the individual spaces, the principal angles between the ranges and kernels of \( P_1 \) and \( P_{2,-1} \) are the same as the ones between the ranges and kernels of \( P_1 \) and \( P_{2,1} \). On the other hand, we have \( \rho(P_1 P_{2,1}) = 0 \) and \( \rho(P_1 P_{2,-1}) = 2 \), that is, although the principal angles agree, nevertheless the convergence behaviour is vastly different.

### 4 Some remarks on angles between linear projections

Let \( P_1, P_2 \) be two bounded linear projections in a Banach space. Recall that in order to define the Oppenheim angle between \( P_1 \) and \( P_2 \), we need a projection \( P_{12} \) which satisfies \( P_{12} P_1 = P_{12} \) and \( P_{12} P_2 = P_{12} \). As noted in Remark 2.6 in [18, p. 346] such a projection need not be unique. Since in [18] no example is given, we now give a simple example illustrating this phenomenon.

**Example 2.** We consider the projections
\[ P_1 : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, y, z) \mapsto (x + y, 0, z) \]
and
\[ P_2 : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, y, z) \mapsto (0, x + y, z). \]
The intersection of the images of \( P_1 \) and of \( P_2 \) is the \( z \)-axis, the projections
\[ P_{12} : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, y, z) \mapsto (0, 0, z) \]
and
\[ P'_{12} : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, y, z) \mapsto (0, 0, x + y + z) \]
are both projections onto \( \text{im} P_1 \cap \text{im} P_2 \). Observe that
\[ P_{12} P_1 = P_{12}, \quad P_{12} P_2 = P_{12}, \quad P'_{12} P_1 = P'_{12} \quad \text{and} \quad P'_{12} P_2 = P'_{12}. \]
So both projections are admissible in the definition of the Oppenheim angle. A direct computation shows that
\[ \|P_1 (P_2 - P_{12})\|_1 = \|P_2 (P_1 - P_{12})\|_1 = 1 \]
but
\[ \|P_1 (P_2 - P'_{12})\|_1 = \|P_2 (P_1 - P'_{12})\|_1 = 2 \]
This shows that these projections result in different values for the Oppenheim angle. For the Euclidean norm these two projections result in the same Oppenheim angle. Taking on the other hand
\[ P''_{12} : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, y, z) \mapsto (0, 0, x + 2x + 2y) \]
and \( P'_{12} \), we obtain different angles for the Euclidean norm as well. Note that in the first case, we even have \( \|P_{12}\|_1 = \|P'_{12}\|_1 = 1 \).
In infinite dimensional Banach spaces, even the question of whether two projections $P_1$ and $P_2$ are consistent, that is, if there is a projection $P_{12}$ onto the intersection of the ranges of $P_1$ and $P_2$ such that $P_{12}P_1 = P_{12}P_2 = P_{12}$, is of interest. Note that there are complemented subspaces with the property that their intersection is no longer complemented. In other words, it might happen that not only there is no projection satisfying the above condition, but that there is no bounded projection at all.

On the positive side, we can mention the following result of R. E. Bruck and S. Reich:

**Proposition 3** (Theorem 2.1 in [5, p. 464]). Let $X$ be a uniformly convex space and let $P_1, \ldots, P_k$ be linear norm-one projections onto subspaces $Y_1, \ldots, Y_k$. Then the strong limit $\lim_{n \to \infty} (P_k P_{k-1} \cdots P_1)^n x$ exists for each $x \in X$ and defines a norm-one-projection onto the intersection $Y_1 \cap \ldots \cap Y_k$.

Using this proposition together with the uniqueness of norm-one projections in smooth spaces, we obtain the following simple result:

**Proposition 4.** Let $X$ be a uniformly convex and smooth Banach space and let $P_1$ and $P_2$ be two norm-one projections in $X$. Then these projections are consistent, that is, there is a projection $P_{12}$ onto the intersection of the ranges of $P_1$ and $P_2$ with the property that $P_{12}P_1 = P_{12}P_2 = P_{12}$.

**Proof.** By Proposition 3 the limits

$$P_{12} x = \lim_{n \to \infty} (P_1 P_2)^n x \quad \text{and} \quad P'_{12} x = \lim_{n \to \infty} (P_2 P_1)^n x$$

both define a norm-one projection onto $\text{im} P_1 \cap \text{im} P_2$. These projections satisfy

$$P'_{12} P_1 x = \lim_{n \to \infty} (P_2 P_1)^n P_1 x = \lim_{n \to \infty} (P_2 P_1)^{n-1} P_2 P_1^2 x = \lim_{n \to \infty} (P_2 P_1)^n x = P'_{12} x$$

and

$$P_{12} P_2 x = \lim_{n \to \infty} (P_1 P_2)^n P_2 x = \lim_{n \to \infty} (P_1 P_2)^{n-1} P_1 P_2^2 x = \lim_{n \to \infty} (P_1 P_2)^n x = P_{12} x$$

for all $x \in X$. Since in smooth Banach spaces norm-one projections are unique (see, for example, Theorem 6 in [6, p. 356]), we necessarily have $P_{12} = P'_{12}$ and this projection has the required properties.

Note that we cannot drop the assumption that $X$ is smooth. This can be seen in the following four-dimensional example.

**Example 3.** We consider the space $\mathbb{R}^4$ equipped with the norm

$$\|(x, y, z, w)\| = \sqrt{x^2 + y^2 + z^2 + w^2 + |x| + |y| + |z| + |w|}$$

which turns it into a uniformly convex but non-smooth space. Consider the projections

$$P_1(x, y, z, w) = \left( x, y - \frac{z}{4}, 0, 0 \right) \quad \text{and} \quad P_2(x, y, z, w) = (0, y, 0, w)$$

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which, by
\[
\|P_1(x, y, z, w)\| = \sqrt{x^2 + \left(y - \frac{z}{4}\right)^2 + |x| + \left|y - \frac{z}{4}\right|}
\leq \sqrt{x^2 + y^2 + \frac{z^2}{4} + |x| + |y| + \frac{|z|}{4}}
\leq \sqrt{x^2 + y^2 + z^2 + w^2 + |x| + |y| + \frac{|z|}{2} + |w|}
\]
are both norm-one projections. Moreover,
\[
(P_1 P_2)(x, y, z, w) = (0, y, 0, 0) \quad \text{and} \quad (P_2 P_1)(x, y, z, w) = \left(0, y - \frac{z}{4}, 0, 0\right)
\]
and since both elements are already in the intersection of the ranges, the sequence of iterates is constant in both cases. Hence we get \(\lim_{n \to \infty} (P_1 P_2)^n \neq \lim_{n \to \infty} (P_1 P_2)^n\). Moreover, note that neither one of these projections has the properties required in the definition of the Oppenheim angle.

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