QUENCHED LARGE DEVIATIONS FOR
DIFFUSIONS IN A RANDOM GAUSSIAN
SHEAR FLOW DRIFT.

Amine Asselah 1, Fabienne Castell 2
Laboratoire d’Analyse, Topologie et Probabilités. CNRS UMR 6632.
CMI. Université de Provence.
39 rue Joliot Curie.
13453 Marseille Cedex 13. FRANCE.

Abstract. We prove a full large deviations principle in large time, for a
diffusion process with random drift
\[ X_t = W_t + \int_0^t V(X_s) \, ds , \]
where \( V \) is a centered Gaussian shear flow random field independent of the Brownian \( W \). The large
deviations principle is established in a “quenched” setting, i.e. is valid almost
surely in the randomness of \( V \).

Mathematics Subject Classification (1991): 60F10, 60G10.

Key words and Phrases: Diffusions in random shear flow drift. Large
deviations. Parabolic Anderson model.

1 Introduction.

In this paper, we investigate large deviations properties for diffusions
\((X_t, t \geq 0)\) with random drift, solving
\[ X_t = W_t + \int_0^t V(X_s) \, ds , \]  \hspace{1cm} (1)
where \( W \) is a standard Brownian motion in \( \mathbb{R}^2 \), and \( V \) is a centered stationary
solenoidal (i.e. such that \( \text{div}(V) = 0 \)) Gaussian field on \( \mathbb{R}^2 \), independent of
\( W \).

Such a process is a model for diffusion in an incompressible turbulent
flow. As such, it has been discussed thoroughly both in the physics and
mathematics literature (see for instance \cite{1, 2, 4, 5, 8, 9}). These papers deal
with the long time behavior of the process \( X \). More precisely they investigate
the link between the properties of the random drift \( V \), and the convergence
in law of \( X_t \) when \( t \) goes to infinity.

\*E-mail: asselah@cmi.univ-mrs.fr
\*E-mail: castell@cmi.univ-mrs.fr
The model we are working on in this paper, is a very particular case of (1), since \( V \) is assumed to be a shear flow, i.e.

\[
\forall x \in \mathbb{R}^2, \ x = (x_1, x_2), \ V(x) = (0, v(x_1)). \tag{2}
\]

\((v(x_1), x_1 \in \mathbb{R})\) is a centered Gaussian field, with covariance \( K(x_1 - x'_1) \triangleq \langle v(x_1)v(x'_1) \rangle \). This model has the advantage of being easy to handle, since in the shear flow situation, the two coordinates \((X_1,T), (X_2,T)\) of \(X_T\) are just

\[
\begin{cases}
  X_{1,T} = W_{1,T} \\
  X_{2,T} = W_{2,T} + \int_0^T v(W_{1,s}) \, ds
\end{cases} \tag{3}
\]

From the viewpoint of the central limit theorem, this model has been studied in [1, 2], where it is proved that when the covariance function \( K \) decays sufficiently slowly at infinity, the second coordinate \(X_{2,T}\) of \(X\) exhibits a super-diffusive behavior, i.e. for some parameter \( \alpha > 1/2 \) (related to the decay of correlation), \( \frac{1}{T^\alpha} X_{2,T} \) converges in law when \( T \to \infty \).

In [3], the annealed large deviations of the Gaussian shear flow model (1) (2) are established. The result is the following. Let \( P \) denote the annealed law, that is the law of \(X\) integrated over the randomnesses of \(V\) and \(W\). Then, for all Borel set \(A\) of \(\mathbb{R}\), with closure \(\overline{A}\), and interior \(\overset{\circ}{A}\).

\[
- \inf_{x \in \overset{\circ}{A}} L(x) \leq \liminf_{T \to \infty} \frac{1}{T} \log P \left[ \frac{1}{T^{3/2}} X_{2,T} \in A \right] \leq \limsup_{T \to \infty} \frac{1}{T} \log P \left[ \frac{1}{T^{3/2}} X_{2,T} \in A \right] \leq - \inf_{x \in \overline{A}} L(x). \tag{4}
\]

The rate function \( L \) is continuous, with compact level sets and has a unique zero at the origin. Note that the super-diffusive scaling \( T^{3/2} \) does not depend on the decay of correlation, but is intimately linked with the choice of Gaussian statistics for the drift \(V\).

We study here the large deviations of the Gaussian shear flow model in a quenched setting, i.e. almost surely in the environment \(V\). Our main result states that there exists a convex deterministic rate function \( J \) such that a.s. in \(V\) and for all Borel set \(A\) of \(\mathbb{R}\),

\[
- \inf_{x \in \overset{\circ}{A}} J(x) \leq \liminf_{T \to \infty} \frac{1}{T} \log P \left[ \frac{1}{\sqrt{T \log(T)}} X_{2,T} \in A \right] \leq \limsup_{T \to \infty} \frac{1}{T} \log P \left[ \frac{1}{\sqrt{T \log(T)}} X_{2,T} \in A \right] \leq - \inf_{x \in \overline{A}} J(x). \tag{5}
\]

Note that in the scaling \( T^{3/2} \), the Brownian part \(W_2\) does not play any role in the large deviations result, and [3] states a large deviations principle for \(Y_T \triangleq \frac{1}{T^{3/2}} \int_0^T v(B_s) \, ds\), where \(B\) is a Brownian motion independent of \(v\). Here again, the super-diffusive scaling does not depend on the decay of correlation, but on the choice of the Gaussian law for \(v\). This scaling is related to the order of magnitude of a Gaussian field on a box of size \(T\).
Indeed, with probability of order $\exp(-RT)$ ($R$ large), the Brownian motion $(B_s, 0 \leq s \leq T)$ stays in a ball of radius $\sqrt{RT}$, so that in the study of the large deviations of $Y_T$, we can restrict ourselves to trajectories confined to such balls with $R$ large enough. So the effect of the scaling $\sqrt{\log(T)}$ is to deal with a bounded integrand $v/\sqrt{\log(T)}$.

The large deviations upper bound is obtained using the Gärtner-Ellis method, i.e. by considering the quenched behavior of the Laplace transform

$$\Lambda_T(\alpha) = E_0[\exp(\alpha TY_T); \tau_{RT} \geq T] = E_0\left[\exp\left(\alpha \int_0^T \frac{v(B_s)}{\sqrt{\log(T)}} ds\right); \tau_{RT} \geq T\right]. \quad (6)$$

In expression (6), $E_0$ denotes the expectation with respect to $B$, assuming that $B_0 = 0$, and $\tau_{RT}$ is the first exit time of $B$ from the interval $I_{RT} = [-RT; RT]$. As usual, we are led to look at the a.s limit when $T \to \infty$, of the principal eigenvalue of the random operator $\mathcal{L}(f) = -\frac{1}{2}f'' - \alpha \frac{v}{\sqrt{\log(T)}} f$, with Dirichlet conditions on the boundary of $I_{RT}$.

$$\lambda\left(\alpha v/\sqrt{\log(T)}, B(0, RT)\right) = \text{Inf} \left\{ \frac{1}{2} \int (f')^2(x) dx - \int \frac{\alpha v(x)}{\sqrt{\log(T)}} f^2(x) dx : f \in C^\infty_c(B(0, RT)), \int f^2(x) dx = 1 \right\}. \quad (7)$$

Following the image popularized by A.S Sznitman in [11], the main contribution comes from “the regions where the eigenvalue is small”. Thus, a key argument in the study of $\lambda\left(\alpha v/\sqrt{\log(T)}, B(0, RT)\right)$ is a lemma borrowed from [6], which asserts that this principal eigenvalue is comparable with $\text{Min} \lambda(\alpha v/\sqrt{\log(T)}, Q_i)$, where $Q_i$ are balls of fixed size covering $I_{RT}$. This comparison enables one to show that $v$-a.s, $\Lambda(\alpha) \triangleq \lim_{T \to \infty}(\log \Lambda_T(\alpha))/T$ exists, and is deterministic. The upper bound is thus obtained with a rate functional $J$ which is the Legendre transform of $\Lambda$.

On the opposite direction, a first lower bound is obtained using a specific strategy for the path of the Brownian motion: we force it to go “fast” to a region where the field $v$ has a “high” peak, and to remain there until time $T$. The rate function $J_1$ obtained in this way, has a Legendre transform which coincides with $J$. Thus, if $J_1$ were convex, then $J_1 = J$. However, we could not prove convexity of $J_1$. We overcome this problem by adopting the following strategy. We imagine a sequence of scenarios: the $n$-th one corresponds to partitioning $[0, T]$ into $n$ time intervals, in each of which the Brownian motion goes fast to a region where the field $v/\sqrt{\log(T)}$ has a fixed deterministic profile, and stays there during this time interval. To each scenario corresponds a lower bound of the type

$$\lim_{\epsilon \to 0} \liminf_{T \to \infty} \frac{1}{T} \log P_0\left[|Y_T - y| \leq \epsilon\right] \geq -J_n(y).$$
The family of functions \( J_n \) is decreasing, and the limit \( J(y) \triangleq \lim_{n \to \infty} J_n(y) \) is convex. This enables us to identify \( J \) and the upper bound \( J \).

The paper is organized as follows. In section 2, we introduce the notations and state the main result. In section 3, we prove the large deviations upper bound. In section 4, we establish the large deviations lower bound. Finally, section 5 investigates the link between the decay of correlation, and the behavior of the rate function near the origin.

As a concluding remark, we would like to say that the paper is written for a diffusion in \( \mathbb{R}^2 \), but that with a little more work, all could be written in higher dimensions, as soon as the shear flow structure is preserved.

2 Notations and results.

In all the sequel, when \( I \) is a domain of \( \mathbb{R} \), \( M(I) \), and \( M_1(I) \) will denote respectively the set of finite measures on \( I \), and the sets of probability measures on \( I \). \( \mathcal{C}_c^\infty(I), \mathcal{C}(I), H^1_0(I) \) will be respectively the set of infinitely differentiable functions with compact support in \( I \), the set of continuous functions, and the Sobolev space obtained by completion of \( \mathcal{C}_c^\infty(I) \) under the norm 
\[
\|f\|_{H^1_0(I)} = \int_I f^2(x) \, dx + \int_I (f')^2(x) \, dx.
\]
Finally, for all \( p \in [1,\infty] \), \( \|f\|_p \) will denote the norm of the function \( f \) in \( L^p(I) \).

Let \((v(x), x \in \mathbb{R})\) be a centered stationary Gaussian field with values in \( \mathbb{R} \), defined on a probability space \((\mathcal{X}, \mathcal{G}, \nu)\). Brackets will denote the expectation with respect to \( \nu \), so that the covariance function of \( v \) is defined by \( K(x - y) \triangleq <v(x)v(y)> \).

Let \((B_t; t \in [0,1])\) be a standard Brownian motion defined on a probability space \((\Omega, \mathcal{A}, P)\). Expectation with respect to \( P \) is denoted by \( E \).

Our main result is a full large deviations principle for the random variable
\[
Y_T \triangleq \frac{1}{T\sqrt{\log(T)}} \int_0^T v(B_s) \, ds.
\]

Before stating the result, we introduce some assumptions and recall some standard results about the Gaussian field \( v \).

2.1 The Gaussian field.

We assume that \( v \) has a spectral density \( h \) such that for some \( \alpha > 0 \),
\[
\int_{\mathbb{R}} (1 + |\lambda|^\alpha) h(\lambda) \, d\lambda < +\infty.
\]

Then, the covariance \( K(x) = \int_{\mathbb{R}} e^{i\lambda x} h(\lambda) \, d\lambda \) is a continuous function on \( \mathbb{R} \), which attains its maximal value at 0. Moreover, \( K(x) \to 0 \) when \( |x| \to \infty \),
and $K$ is Hölder continuous of order $\alpha$, so that $v$ has a version which is $\beta$-
 Hölder continuous for $0 < \beta < \frac{\alpha}{2}$. Moreover, as it is well known for Gaussian 
fields,

$$
\nu - \text{a.s., } \limsup_{L \to +\infty} \frac{\max\{|v(x)| : x \in [-L, L]\}}{\sqrt{2K(0) \log(L)}} \leq 1.
$$

(9)

We present now a splitting of $v$ into the sum of two Gaussian stationary 
processes, one of which having finite correlation length. This splitting is 
constructed in [7], and goes as follows.

Let $g$ be the $L^2$-Fourier transform of $\sqrt{h}$. We can assume that $v(x) = \int_{\mathbb{R}} g(x - y) \, dZ(y)$, where $Z$ is a Brownian motion on $\mathbb{R}$. Let $\psi : \mathbb{R} \mapsto [0,1]$ be a smooth even function, such that $\psi = 0$ outside $]-1/2; 1/2[$, and $\psi = 1$ on $[-1/4; 1/4]$. Let $\psi_L(x) \triangleq \psi(\frac{x}{L})$, $g_L(x) \triangleq \psi_L(x)g(x)$, and 
$\tilde{g}_L(x) \triangleq g(x) - g_L(x)$. This splitting of $g$ yields a corresponding splitting of 
v, $v = v_L + \tilde{v}_L$, where

$$
v_L(x) = \int g_L(x - y) \, dZ(y) , \quad \tilde{v}_L(x) = \int \tilde{g}_L(x - y) \, dZ(y).
$$

(10)

$v_L$ and $\tilde{v}_L$ are clearly stationary Gaussian processes. The support of $K_L(x) = \langle v_L(x)v_L(0) \rangle = g_L \ast g_L(x)$ (where $\ast$ denotes the convolution operator), is included in $[-L; L]$.

Note also that $K_L(0) = \langle \tilde{v}_L(0)\tilde{v}_L(0) \rangle = \int (1 - \psi_L(x))^2 g^2(x) \, dx$ tends to 0 
when $L$ goes to infinity.

Moreover, if $f$ denotes the inverse Fourier transform of $f$,

$$
\int |\lambda|^\alpha |\tilde{g}_L(\lambda)|^2 \, d\lambda = \int |\lambda|^\alpha |\tilde{\psi}_L \ast \sqrt{h}|^2 \, d\lambda \\
= \int d\lambda_1 \, d\lambda_2 \, \tilde{\psi}_L(\lambda_1) \tilde{\psi}_L(\lambda_2) \int d\lambda \, |\lambda|^\alpha \sqrt{h}(\lambda - \lambda_1) \sqrt{h}(\lambda - \lambda_2)
$$

But

$$
\int d\lambda |\lambda|^\alpha \sqrt{h}(\lambda - \lambda_1) \sqrt{h}(\lambda - \lambda_2) \leq \prod_{i=1,2} \left( \int d\lambda |\lambda|^\alpha h(\lambda - \lambda_i) \right)^{1/2} \\
\leq C \prod_{i=1,2} \left( |\lambda_i|^\alpha + \int d\lambda |\lambda|^\alpha h(\lambda) \right)^{1/2},
$$

so that

$$
\int |\lambda|^\alpha |\tilde{g}_L(\lambda)|^2 \, d\lambda \leq C \left( \int d\lambda \, (1 + |\lambda|^\alpha)^{1/2} |\tilde{\psi}_L(\lambda)| \right)^2 < \infty,
$$

since $\tilde{\psi}_L$ decreases faster than any polynomial at infinity. Thus, $v_L$ has a 
Hölder continuous version, and so does $\tilde{v}_L$.

2.2 The large deviations principle.

Let us now define the rate function $J$ appearing in the large deviations principle. 
When $f$ is a function of the Sobolev space $H^1(\mathbb{R})$, $K \ast f^2$ is the
continuous function obtained by convolution of the covariance kernel $K$ and $f^2$, so that

$$(K \ast f^2, f^2) = \int_{\mathbb{R}^2} K(x - y) f^2(x) f^2(y) \, dx \, dy = \int_{\mathbb{R}} |\hat{f}^2(\lambda)|^2 h(\lambda) \, d\lambda.$$ 

For any $\alpha \in \mathbb{R}$, let

$$\Lambda(\alpha) \triangleq \text{Sup} \left\{ |\alpha| \sqrt{2(K \ast f^2, f^2)} - \frac{1}{2} \|f'\|_2^2 : f \in H^1(\mathbb{R}), \|f\|_2 = 1 \right\},$$

and for any $y \in \mathbb{R}$

$$J(y) \triangleq \text{Sup} \{ \alpha y - \Lambda(\alpha) : \alpha \in \mathbb{R} \}.$$ (11)

We are now able to state the main result of the paper.

**Theorem 1** Assume (8). Then, $\nu$-a.s, for any measurable subset $E$ of $\mathbb{R}$,

$$\limsup_{T \to \infty} \frac{1}{T} \log P_0[Y_T \in E] \leq - \inf_{y \in E} J(y),$$ (13)

$$\liminf_{T \to \infty} \frac{1}{T} \log P_0[Y_T \in E] \geq - \inf_{y \in \bar{E}} J(y).$$ (14)

$J$ is even, convex, and lower semicontinuous. $J(y) < \infty$ for $|y| < \sqrt{2K(0)}$, and $J(y) = +\infty$ for $|y| > \sqrt{2K(0)}$. Moreover, $J(0) = 0$, and $J$ is increasing on $\mathbb{R}^+$. 

As a corollary of the large deviations for $Y$, we obtain the large deviations for $X_2$ with the same rate function.

**Corollary 2** Assume (8). Then, the estimates (13) and (14) hold when $X_{2,T}/(T \sqrt{\log(T)})$ replaces $Y_T$.

We provide some more informations on $J$, relating the decay of correlation of the field $v$, and the behavior near the origin of $J$.

**Proposition 3** .

1. Assume that for some $\beta \in ]0,1[$, $\limsup_{|x| \to \infty} |x|^{\beta} |K(x)| < \infty$, then

$$\liminf_{y \to 0} \frac{J(y)}{|y|^{4/\beta}} > 0.$$ 

Assume that $K \geq 0$ and $\liminf_{|x| \to \infty} |x|^{\beta} K(x) > 0$ (for some $\beta \in ]0,1[$), or that $\lim_{|x| \to \infty} |x|^{\beta} K(x) > 0$, then

$$\limsup_{y \to 0} \frac{J(y)}{|y|^{4/\beta}} < \infty.$$
2. Assume that for some $\beta > 1$, $\limsup_{|x| \to \infty} |x|^\beta |K(x)| < \infty$, and $\int K(x) \, dx \neq 0$. Then, $\lim_{y \to 0} \frac{J(y)}{y^4}$ exists in $]0, +\infty]$. 

3 Proof of the upper bound.

The aim of this section is to prove (13), and the same estimate for $X_{2,T}$. We begin with the proof of the properties of $J$ stated in theorem 1.

3.1 Proof of the properties of $J$.

$J$ is convex and l.s.c as the supremum of affine functions. $J$ is even because $\Lambda$ is even. We restrict therefore the study of $J$ to $\mathbb{R}^+$. For $y \in \mathbb{R}^+$, 

$$\sup_{\alpha \leq 0} (\alpha y - \Lambda(\alpha)) = \sup_{\alpha \geq 0} (-\alpha y - \Lambda(\alpha)) \leq \sup_{\alpha \geq 0} (\alpha y - \Lambda(\alpha)),$$

so that $\forall y \in \mathbb{R}^+$, $J(y) = \sup \{\alpha y - \Lambda(\alpha), \alpha \geq 0\}$. The monotony of $J$ is thus obvious.

Let us prove now that $J(y) < \infty$ for $|y| < \sqrt{2K(0)}$, and $J(y) = +\infty$ for $|y| > \sqrt{2K(0)}$. For this purpose, note that

$$\sup \{(K \ast f^2, f^2) : f \in H^1(\mathbb{R}), \|f\|_2 = 1\} = K(0). \quad (15)$$

Indeed, on one hand, $\forall f \in H^1(\mathbb{R})$ such that $\|f\|_2 = 1$, $(K \ast f^2, f^2) \leq K(0)$.

On the other hand, let $f_0$ be any function in $H^1(\mathbb{R})$, such that $\|f_0\|_2 = 1$, and let $\lambda > 0$. $f_{0,\lambda}(x) = \sqrt{\lambda}f_0(\lambda x)$ is then a function in $H^1(\mathbb{R})$, such that $\|f_{0,\lambda}\|_2 = 1$. Therefore,

$$\sup \{(K \ast f^2, f^2) : f \in H^1(\mathbb{R}), \|f\|_2 = 1\} \geq (K \ast f_{0,\lambda}^2, f_{0,\lambda}^2) = \int K \left(\frac{x-y}{\lambda}\right) f_0^2(x)f_0^2(y) \, dx \, dy,$$

and (15) follows by letting $\lambda \to \infty$, and dominated convergence. Thus, $\Lambda(\alpha) \leq |\alpha| \sqrt{2K(0)}$, and $\forall y > \sqrt{2K(0)}$,

$$J(y) \geq \sup_{\alpha \geq 0} \left\{\alpha(y - \sqrt{2K(0)})\right\} = +\infty.$$

On the other side, for $0 \leq y < \sqrt{2K(0)}$, (15) allows one to find $f_y$ in $H^1(\mathbb{R})$ such that $\|f_y\|_2 = 1$, and $y < \sqrt{2(K \ast f_y^2, f_y^2)}$. We get then that

$$J(y) \leq \sup_{\alpha \geq 0} \left\{\alpha y - \alpha \sqrt{2(K \ast f_y^2, f_y^2) + \frac{1}{2} \left\|f_y'\right\|_2^2}\right\} = \frac{1}{2} \left\|f_y'\right\|_2^2 < +\infty.$$

Let us now compute $J(0)$. Since $\Lambda$ is even and increasing on $\mathbb{R}^+$,

$$J(0) = -\inf\{\Lambda(\alpha) ; \alpha \in \mathbb{R}\} = -\Lambda(0) = 0.$$
3.2 Large deviations upper bound for $X_{2,T}$.

We are going to prove that (13) implies the same estimate for $X_2$. Let us then assume that (13) holds. Let $\delta > 0$, and let $F_\delta = \{ y : \exists x \in F, |x - y| \leq \delta \}$.

$$P_0 \left[ \frac{X_{2,T}}{T \sqrt{\log(T)}} \in F \right] \leq P_0 \left[ Y_T \in F_\delta \right] + P_0 \left[ \frac{W_{2,T}}{T \sqrt{\log(T)}} \geq \delta \right].$$

But $\lim_{T \to \infty} \frac{1}{T} \log P_0 \left[ \frac{W_{2,T}}{T \sqrt{\log(T)}} \geq \delta \right] = -\infty$. Therefore, (13) yields that $\nu$-a.s., for all closed subset $F$, and all $\delta > 0$,

$$\limsup_{T \to \infty} \frac{1}{T} \log P_0 \left[ \frac{X_{2,T}}{T \sqrt{\log(T)}} \in F \right] \leq - \inf_{y \in F_\delta} J(y).$$

The result follows from the goodness of the rate function $J$, letting $\delta$ go to 0.

3.3 Large deviations upper bound for $Y_T$.

We prove now (13) in theorem 1.

Step 1. Restriction of the problem in a domain of size $T$.

For $R > 0$, let $I_{RT}$ be the interval $]-RT;+RT[$, and let $\tau_{RT}$ be the first time Brownian $B$ exits $I_{RT}$.

**Lemma 4** $\nu$-a.s., for all measurable set $F$ and all $R > 0$,

$$\limsup_{T \to \infty} \frac{1}{T} \log P_0 \left[ Y_T \in F \right] \leq \max \left\{ \limsup_{T \to \infty} \frac{1}{T} \log P_0 \left[ Y_T \in F; \tau_{RT} > T \right], -\frac{R^2}{2} \right\}. \quad (16)$$

**Proof.**

$$P_0(Y_T \in F) \leq P_0 \left[ Y_T \in F; \tau_{RT} > T \right] + P_0 \left[ \sup_{[0,T]} B_s \geq RT \right]. \quad (17)$$

The well known estimate $\limsup_{T \to \infty} \frac{1}{T} \log P_0 \left[ \sup_{[0,T]} |B_s| \geq RT \right] \leq -\frac{R^2}{2}$ yields the result.

Step 2. Spectral estimates of Schrödinger semigroups.

To prove the upper bound, we use the Gärtner-Ellis method, and we have to study the large time asymptotic of

$$\Lambda_T \left( \frac{\alpha v}{\sqrt{\log(T)}}, I_{RT} \right) = E_0 \left[ \exp \left( \int_0^T \frac{\alpha v(B_s)}{\sqrt{\log(T)}} ds \right); \tau_{RT} > T \right]. \quad (18)$$
It is well known that this reduces to study the principal eigenvalue of the random operator \( \mathcal{L}(f) = \frac{1}{2}f'' + \alpha \frac{v}{\sqrt{\log(T)}} f \), with Dirichlet conditions on the boundary of \( I_{RT} \).

In all the sequel, when \( D \) is a bounded domain of \( \mathbb{R} \), and \( V : D \mapsto \mathbb{R} \) is a bounded measurable function, we will write \( \lambda(V, D) \) for the principal eigenvalue of the operator \( \frac{1}{2}\Delta + V \), with Dirichlet boundary condition on \( D \).

\[
\lambda(V, D) \triangleq \inf \left\{ \frac{1}{2} \int_D (f')^2(x) \, dx - \int_D V(x)f^2(x) \, dx : f \in C_0^\infty(D), \|f\|_2 = 1 \right\}
\]

Since any sequence \((f_n)\) which is bounded in \( H^1_0(D) \) has a subsequence which converges strongly in \( L^2(D) \) and weakly in \( H^1_0(D) \), one also has

\[
\lambda(V, D) = \min \left\{ \frac{1}{2} \int_D (f')^2(x) \, dx - \int_D V(x)f^2(x) \, dx : f \in H^1_0(D), \|f\|_2 = 1 \right\}
\]

In these notations, the task at hand is to study the behavior for large \( T \) of \( \lambda(\alpha v/\sqrt{\log(T)}, I_{RT}) \). To this end, we recall proposition 1 of [3], which compares this eigenvalue, with the minimum of the principal eigenvalues in balls of fixed size.

**Lemma 5** (Proposition 1 of [3]).

\( \forall r \geq 2 \), there exists a continuous \( 2\pi \)-periodic function \( \Phi_r : \mathbb{R} \mapsto \mathbb{R}^+ \), with support included in \( \cup_{k \in \mathbb{Z}} ((2k + 1)\pi r + 1, 1] \), such that for all \( R > r \), for all \( \theta \in I_{2r} \),

\[
\lambda(V - \Phi_r^\theta, I_R) \geq \min \left\{ \lambda(V, z + I_{2r+1}) : z \in (2r\mathbb{Z}) \cap I_{R+r} \right\},
\]

where \( \Phi_r^\theta(x) = \Phi_r(x - \theta) \).

Moreover, \( \frac{1}{|I_r|} \int_{I_r} \Phi_r(x) \, dx \leq \frac{K}{r} \), where the constant \( K \) is independent of \( r \).

We deduce from this the following lemma.

**Lemma 6** There exists a constant \( K \) such that \( \nu \)-a.s., for all \( r \geq 2 \), \( \forall \alpha \in \mathbb{R}, \forall R > 0 \)

\[
\limsup_{T \to \infty} \frac{1}{T} \log \Lambda_T \left( \frac{\alpha v}{\sqrt{\log(T)}}, I_{RT} \right) 
\]

\[
\leq \frac{K}{r} - \liminf_{T \to \infty} \min \left\{ \lambda \left( \frac{\alpha v}{\sqrt{\log(T)}}, z + I_{2r+1} \right) : z \in (2r\mathbb{Z}) \cap I_{RT+r} \right\}.
\]
Proof. We use the same trick as in [8] and [9]. Let \( \Phi_r \) be the function introduced in lemma \( 5 \). By periodicity of \( \Phi_r \), \( \frac{1}{|I_r|} \int_{I_r} \Phi_r(\theta + B_s) d\theta = \frac{1}{|I_r|} \int_{I_r} \Phi_r(\theta) d\theta \leq \frac{K_r}{r} \). By Jensen inequality, we obtain then that

\[
\Lambda_T \left( \frac{\alpha v}{\sqrt{\log(T)}} , I_{RT} \right) \leq \exp \left( \frac{KT}{r} \right) \frac{1}{|I_r|} \int_{I_r} d\theta \Lambda_T \left( \frac{\alpha v}{\sqrt{\log(T)}} - \Phi_r^{\theta}, I_{RT} \right).
\]

We use then the usual bounds on Schrödinger semigroups in terms of their principal eigenvalue (see for instance theorem 1.2 in chapter 3 of [11]).

\[
\Lambda_T \left( \frac{\alpha v}{\sqrt{\log(T)}} , I_{RT} \right) \leq C e^{KT/r} \left( 1 + \sqrt{T \sup_{\theta \in I_r} \lambda(\frac{\alpha v}{\sqrt{\log(T)}}, \Phi_r^{\theta} , I_{RT})} \right) \exp \left( -T \inf_{\theta \in I_r} \lambda(\frac{\alpha v}{\sqrt{\log(T)}}, \Phi_r^{\theta} , I_{RT}) \right)
\]

\[
\leq C e^{KT/r} \left( 1 + \sqrt{T(\|\Phi_r\|_{\infty}^2 + \max_{I_{RT}} |\alpha v|^2)} \right) \exp \left( -T \min \left\{ \lambda(\frac{\alpha v}{\sqrt{\log(T)}}, z + I_{2r+1}) : z \in 2r \mathbb{Z} \cap I_{RT+r} \right\} \right)
\]

The conclusion follows from (9) and (19). \( \blacksquare \)

Step 3. \( \nu \)-a.s. behavior of \( \min \left\{ \lambda(\frac{\alpha v}{\sqrt{\log(T)}}, z + I_{2r+1}) : z \in 2r \mathbb{Z} \cap I_{RT+r} \right\} \).

This is done via a Borel-Cantelli argument. Using the stationarity of \( v \), note that the random variables \( \lambda(\frac{\alpha v}{\sqrt{\log(T)}}, z + I_{2r+1}) : z \in 2r \mathbb{Z} \cap I_{RT+r} \) have the same law. The next lemma gives some estimates for this law.

Lemma 7 Let \( c \triangleq \min \{ \frac{1}{2} \int (f')^2 dx : f \in H_0^1(I_1), \int f^2 = 1 \} \). Let us define for all \( x \in \mathbb{R} \), and \( r > 0 \)

\[
J_r(x) \triangleq \begin{cases} \inf \left\{ \frac{1}{2} (\frac{1}{2} \int (f')^2 - x)^2 : f \in H_0^1(I_r) , \int f^2 = 1 \right\} & \text{if } x < \frac{c}{r^2} , \\ 0 & \text{otherwise.} \end{cases}
\]

Then, \( \forall r > 0, \forall x \in \mathbb{R} \),

\[
\lim_{T \to \infty} \frac{1}{\log(T)} \log \nu \left[ \lambda \left( \frac{\alpha v}{\sqrt{\log(T)}}, I_r \right) \leq x \right] = - \frac{J_r(x)}{\alpha^2}.
\]

Proof. Let \( f \) be any function in \( H_0^1(I_r) \) such that \( \int f^2 = 1 \). Then

\[
\nu \left[ \lambda \left( \frac{\alpha v}{\sqrt{\log(T)}}, I_r \right) \leq x \right] \geq \nu \left[ (\alpha v, f^2) \geq \sqrt{\log(T)} \left( \frac{1}{2} \|f'\|_2^2 - x \right) \right].
\]
But \((\alpha v, f^2) \sim N(0, \alpha^2(K \ast f^2, f^2))\), so that
\[
\liminf_{T \to \infty} \frac{1}{\log(T)} \log \nu \left[ \lambda \left( \frac{\alpha v}{\sqrt{\log(T)}}, I_r \right) \leq x \right] \\
\geq \begin{cases} 
0 & \text{for } x \geq \frac{1}{2} \|f'\|_2^2 \\
- \frac{(\frac{1}{2} \|f'\|_2^2 - x)^2}{2 \alpha^2 (K \ast f^2, f^2)} & \text{for } x < \frac{1}{2} \|f'\|_2^2.
\end{cases}
\]

Taking the supremum over all functions \(f \in H^1_0(I_r)\) such that \(\|f\|_2 = 1\), yields
\[
\liminf_{T \to \infty} \frac{1}{\log(T)} \log \nu \left[ \lambda \left( \frac{\alpha v}{\sqrt{\log(T)}}, I_r \right) \leq x \right] \geq - \frac{J_r(x)}{\alpha^2}. \tag{22}
\]

We are now going to prove the upper bound. To this end, note that \(\lambda(\cdot, I_r) : \mathcal{C}(I_r) \to \mathbb{R}\) is continuous (the topology in \(\mathcal{C}(I_r)\) being given by the supremum norm). Indeed, first \(\lambda(\cdot, I_r)\) is u.s.c as infimum of continuous functions. Secondly, we prove the lower semicontinuity: let then \((v_n, n \in \mathbb{N})\) be a sequence in \(\mathcal{C}(I_r)\) converging to \(v\). For all \(n \in \mathbb{N}\), let \(f_n\) realize the infimum in \(\lambda(v_n, I_r)\). Since \(\lambda(v_n, I_r) \leq - \operatorname{Min}_{I_r} v_n\), and \(\|v_n - v\|_{\infty} \to 0\), the sequence \((f_n)\) is bounded in \(H^1_0(I_r)\), and admits therefore a subsequence converging strongly in \(L^2(I_r)\) and weakly in \(H^1_0(I_r)\) to a function \(f \in H^1_0(I_r)\). One obtains then that \(\|f\|_2 = \lim \|f_n\|_2 = 1\), \(\liminf \|f_n'\|_2 \geq \|f'\|_2\), and \(\lim(v_n, f_n^2) = (v, f^2)\), so that \(\liminf_{n \to \infty} \lambda(v_n, I_r) \geq \frac{1}{2} \|f'\|_2^2 - (v, f^2) \geq \lambda(v, I_r)\).

Therefore, for all \(r > 0\), \(x, \alpha \in \mathbb{R}\), \(F^{\alpha, x}_r \triangleq \{ u \in \mathcal{C}(I_r), \lambda(\alpha u, I_r) \leq x \}\) is a closed subset of \(\mathcal{C}(I_r)\), and
\[
\nu \left[ \lambda \left( \frac{\alpha v}{\sqrt{\log(T)}}, I_r \right) \leq x \right] \geq \nu \left[ \frac{v}{\sqrt{\log(T)}} \in F^{\alpha, x}_r \right].
\]

We now use the large deviations in \(\mathcal{C}(I_r)\) of the Gaussian field \(v/\sqrt{\log(T)}\) to deduce that
\[
\limsup_{T \to \infty} \frac{1}{\log(T)} \nu \left[ \lambda \left( \frac{\alpha v}{\sqrt{\log(T)}}, I_r \right) \leq x \right] \\
\leq - \inf \left\{ K^*_r(u) : u \in \mathcal{C}(I_r), \lambda(\alpha u, I_r) \leq x \right\},
\]
where
\[
K^*_r(u) \triangleq \sup \left\{ (u, \mu) - \frac{1}{2} (K \ast \mu, \mu) : \mu \in \mathcal{M}(I_r) \right\}. \tag{23}
\]

Note that
\[
K^*_r(u) = \sup_{\mu \in \mathcal{M}(I_r)} \sup_{m \in \mathbb{R}} \left\{ m(u, \mu) - \frac{m^2}{2} (K \ast \mu, \mu) \right\} \\
= \sup_{\mu \in \mathcal{M}(I_r)} \left\{ \frac{(u, \mu)^2}{2(K \ast \mu, \mu)} \right\} \text{ (with the convention } 0_0 = 0\). \tag{24}
\]
Hence $\forall \alpha \in \mathbb{R}, K^*_r(\alpha u) = \alpha^2 K^*_r(u)$, and
\[
\limsup_{T \to \infty} \frac{1}{\log(T)} \nu \left[ \lambda \left( \frac{\alpha v}{\sqrt{\log(T)}}, I_r \right) \leq x \right]
\leq -\frac{1}{\alpha^2} \inf \left\{ K^*_r(u) : u \in \mathcal{C}(I_r), \lambda(u, I_r) \leq x \right\},
\]

It remains now to show that
\[
\inf \left\{ K^*_r(u) : u \in \mathcal{C}(I_r), \lambda(u, I_r) \leq x \right\} \geq J_r(x). \quad (25)
\]

We can restrict ourselves to the case where $x < \frac{\epsilon}{R}$. Let $u \in \mathcal{C}(I_r)$ be such that $\lambda(u, I_r) \leq x$. Let $f_u \in H^1_0(I_r)$ be such that $\|f_u\|_2 = 1$ and $\lambda(u, I_r) = \frac{1}{2} \|f_u\|_2^2 - (u, f_u^2)$. It follows from (24) that $K^*_r(u) \geq \frac{(u, f_u^2)^2}{2(K^* f_u^2, f_u^2)}$. But
\[
(u, f_u^2) = -\lambda(u, I_r) + \frac{1}{2} \|f_u'\|_2^2 \geq \frac{1}{2} \|f_u'\|_2^2 - x.
\]

Moreover, $x < \frac{\epsilon}{R} \leq \frac{1}{2} \|f_u'\|_2^2$ by definition of the constant $c$. Thus,
\[
K^*_r(u) \geq \frac{\left( \frac{1}{2} \|f_u'\|_2^2 - x \right)^2}{2(K^* f_u^2, f_u^2)} \geq J_r(x).
\]

Taking the infimum over functions $u$ such that $\lambda(u, I_r) \leq x$ yields then (25).

Lemma 8 allows one to prove

**Lemma 8** $\forall \alpha \in \mathbb{R}$, and $\forall r \geq 2$, let
\[
\Lambda(\alpha, r) \triangleq \sup \left\{ |\alpha| \sqrt{2(K^* f^2, f^2)} - \frac{1}{2} \|f'\|_2^2 : f \in H^1_0(I_{2r+1}), \|f\|_2 = 1 \right\}. \quad (26)
\]

Then, $\forall \alpha \in \mathbb{R}$, $\forall R > 0$ and $\forall r \geq 2$, $\nu$-a.s.,
\[
\liminf_{T \to \infty} \nu \left[ \lambda \left( \frac{\alpha v}{\sqrt{\log(T)}}, z + I_{2r+1} \right) : z \in (2rZ) \cap I_{RT+r} \right] \geq -\Lambda(\alpha, r). \quad (27)
\]

**Proof.** We use Borel-Cantelli lemma. We assume that $\Lambda(\alpha, r) < \infty$, otherwise there is nothing to prove. Let $\epsilon > 0$ be fixed.
\[
\nu \left[ \min \left\{ \lambda \left( \frac{\alpha v}{\sqrt{\log(T)}}, z + I_{2r+1} \right) : z \in (2rZ) \cap I_{RT+r} \right\} \leq -\Lambda(\alpha, r) - \epsilon \right]
\leq \sum_{z \in (2rZ) \cap I_{RT+r}} \nu \left[ \lambda \left( \frac{\alpha v}{\sqrt{\log(T)}}, z + I_{2r+1} \right) \leq -\Lambda(\alpha, r) - \epsilon \right]
\leq C(1 + \frac{RT}{r}) \nu \left[ \lambda \left( \frac{\alpha v}{\sqrt{\log(T)}}, I_{2r+1} \right) \leq -\Lambda(\alpha, r) - \epsilon \right] \text{ by stationarity.}
Thus, by lemma 7,
\[
\limsup_{T \to \infty} \frac{1}{\log(T)} \log \nu \left[ \min_{z \in (2rZ) \cap I_{RT+r}} \lambda \left( \frac{\alpha v}{\sqrt{\log(T)}}, z + I_{2r+1} \right) \right] \leq -\Lambda(\alpha, r) - \epsilon
\]
\[
\leq 1 - \frac{J_{2r+1}(\Lambda(\alpha, r) - \epsilon)}{\alpha^2}.
\]
(28)

We claim that
\[
x < -\Lambda(\alpha, r) \iff J_{2r+1}(x) > \alpha^2.
\]
(29)

The only point to note in order to prove (29) is that the infimum in (20), and the supremum in (26) are actually reached, since again any majorizing sequence will be bounded in $H_{10}(I_{2r+1})$, and $f \in L^2(I_{2r+1}) \mapsto (K \ast f^2, f^2)$ is continuous. Hence,
\[
x < -\Lambda(\alpha, r)
\]
\[
\iff \forall f \in H_{0}(I_{2r+1}), \|f\|_2 = 1, \quad |\alpha| \sqrt{2(K \ast f^2, f^2)} < \frac{1}{2} \|f'||_2^2 - x
\]
\[
\iff \forall f \in H_{1}(I_{2r+1}), \|f\|_2 = 1,
\]
\[
\left\{ \begin{array}{l}
\frac{1}{2} \|f'||_2^2 - x > 0 \\
(\frac{1}{2} \|f'||_2^2 - x)^2 > \alpha^2
\end{array} \right.
\]
\[
\iff J_r(x) > \alpha^2.
\]

It follows then from (28), (29), and Borel-Cantelli lemma applied along the sequence $T_n = 2^n$, that $\forall \alpha \in \mathbb{R}, \forall r \geq 2, \forall R > 0, \nu$-a.s.,
\[
\liminf_{n \to \infty} \min \left\{ \lambda \left( \frac{\alpha v}{\sqrt{\log(T_n)}}, z + I_{2r+1} \right) : z \in (2rZ) \cap I_{RT_n+r} \right\} \geq -\Lambda(\alpha, r),
\]

To end the proof of lemma 8, note that for $T$ sufficiently large, and $n$ such that $T_n \leq T < T_{n+1},$
\[
\min \left\{ \lambda \left( \frac{\alpha v}{\sqrt{\log(T_n)}}, z + I_{2r+1} \right) : z \in (2rZ) \cap I_{RT_n+r} \right\}
\]
\[
\geq \min \left\{ \lambda \left( \frac{\alpha v}{\sqrt{\log(T_{n+1})}}, z + I_{2r+1} \right) : z \in (2rZ) \cap I_{RT_{n+1}+r} \right\} - \frac{\max |\alpha v|}{\log(2n^2)}.
\]

The last term is $\nu$-a.s. of order $1/n$ by (2).

Concerning lemma 8, we would like to underline that using the decorrelation properties of the field $v$, and Borel Cantelli inverse lemma, it is possible to prove that $-\Lambda(\alpha, r)$ is in fact the a.s. limit when $T \to \infty$ of
\[
\min \left\{ \lambda \left( \frac{(\alpha v)^2}{\sqrt{\log(T)}}, z + I_{2r+1} \right) : z \in (2rZ) \cap I_{RT+r} \right\},
\]
At this point, putting lemma 6 and lemma 8 together, we have proved that there exists $K > 0$ such that: $\forall r \geq 2, \forall R > 0, \forall \alpha \in \mathbb{R}, \nu$-a.s.,

$$\limsup_{T \to \infty} \frac{1}{T} \log \Lambda_T \left( \frac{\alpha v}{\sqrt{\log(T)}}, I_{RT} \right) \leq \frac{K}{r} + \Lambda(\alpha, r).$$

Taking the limit $r \to \infty$ along subsequences, we obtain that $\nu$-a.s., $\forall \alpha \in \mathbb{Q}$, $\forall R \in \mathbb{Q}^+$,

$$\limsup_{T \to \infty} \frac{1}{T} \log \Lambda_T \left( \frac{\alpha v}{\sqrt{\log(T)}}, I_{RT} \right) \leq \Lambda(\alpha).$$

(30)

Step 4. Conclusion.

It is now routine to obtain from (30) the weak large deviations upper bound (i.e. the upper bound for compact sets). (13) follows then from the exponential tightness of $Y$ (lemma 10).

Lemma 9 (weak large deviations upper bound).

$\nu$-a.s., $\forall y \in \mathbb{R}$,

$$\lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log P_0 [Y_T \in [y - \epsilon, y + \epsilon]] \leq -\mathcal{J}(y).$$

Proof. We treat only the case $y > 0$. By lemma 4, $\forall \epsilon < y$, $\forall \alpha > 0$, and $\forall R > 0$

$$\limsup_{T \to \infty} \frac{1}{T} \log P_0 [Y_T \in [y - \epsilon, y + \epsilon]]
\leq \max \left[ \frac{-R^2}{2}, \limsup_{T \to \infty} \frac{1}{T} \log P_0 [Y_T \in [y - \epsilon, y + \epsilon]; \tau_{RT} \geq T] \right]
\leq \max \left[ \frac{-R^2}{2}, -\alpha(y - \epsilon) + \limsup_{T \to \infty} \frac{1}{T} \log E_0 \left[ e^{\alpha Y_T} ; \tau_{RT} \geq T \right] \right]$$

Therefore, $\nu$-a.s., $\forall y > 0$, $\forall \epsilon < y$, $\forall R \in \mathbb{Q}^+$,

$$\limsup_{T \to \infty} \frac{1}{T} \log P_0 [|Y_T - y| \leq \epsilon]
\leq \max \left[ \frac{-R^2}{2}, -\sup \left\{ \alpha(y - \epsilon) - \Lambda(\alpha) : \alpha \in \mathbb{Q}^+ \right\} \right].$$

Note that by continuity of $\Lambda$, the supremum on $\mathbb{Q}^+$, is a supremum on $\mathbb{R}^+$. Thus, (13) is obtained by taking the limit $R \to \infty$, then $\epsilon \to 0$, and by using the lower semi-continuity of $\mathcal{J}$.

Lemma 10 (exponential tightness).

$\nu$-a.s., $\forall L > \sqrt{2K(0)}$,

$$\limsup_{T \to \infty} \frac{1}{T} \log P_0 [|Y_T| > L] \leq -\frac{L^2}{2}.$$
Proof. Let $L > \sqrt{2K(0)}$ be fixed.

$$P_0 [|Y_T| > L] \leq P_0 [\tau LT \leq T] + \mathbb{I}_{\frac{\max_{t \in T}|\cdot|}{\sqrt{\log(T)}} > L} \leq C \exp(-\frac{L^2 T}{2}) + \mathbb{I}_{\frac{\max_{t \in T}|\cdot|}{\sqrt{\log(T)}} > L}.$$  

By (9), $\nu$-a.s., the indicator is null for $T$ sufficiently large. Therefore, $\forall L > \sqrt{2K(0)}$, $\nu$-a.s.,

$$\limsup_{T \to \infty} \frac{1}{T} \log P_0 [|Y_T| > L] \leq -\frac{L^2}{2}.$$  

Inverting the “$\forall L$” and the “$\nu$-a.s”, is easily done using the monotony of $L \mapsto P_0(|Y_T| > L)$.

4 Proof of the lower bound.

Here, we prove (14), from which the same assertion for $X_2$ is easily deduced.

4.1 a.s. behavior of the field with finite correlation length.

As explained in the introduction, the lower bound is obtained by forcing the Brownian motion to spend a certain amount of time in boxes where the field $v/\sqrt{\log(T)}$ has a fixed profile. We need therefore to describe the a.s. behavior of this random field. This is done in the following lemma, assuming that $K$ has compact support.

Lemma 11 Assume that $K$ has compact support in $I_L$ for some $L > 0$. Let $\epsilon > 0$, $r > L$ and let $u$ be any function in $C(I_r)$ such that $K^*(u) < 1$. Then $\nu$-a.s., for $T$ sufficiently large, $\exists z \in 2r \mathbb{Z} \cap I_T/\log(T)$ such that

$$\max_{y \in z + I_r} \left| \frac{v(y)}{\sqrt{\log T}} - u(y - z) \right| \leq \epsilon.$$  

Proof.

$$\nu \left[ \forall z \in 2r \mathbb{Z} \cap I_T/\log(T), \left\| \frac{v(\cdot)}{\sqrt{\log T}} - u(\cdot - z) \right\|_{\infty, z + I_r} \geq \epsilon \right]$$

$$\leq \nu \left[ \forall z \in 4r \mathbb{Z} \cap I_T/\log(T), \left\| \frac{v(\cdot)}{\sqrt{\log T}} - u(\cdot - z) \right\|_{\infty, z + I_r} \geq \epsilon \right]$$
Since $K$ has compact support in $I_L$, and $r > L$, the random variables $(v(z + I_r)/\sqrt{\log T}, z \in 4r\mathbb{Z} \cap I_{T/\log(T)})$ are independent. Thus,

$$\nu \left[ \forall z \in 2r\mathbb{Z} \cap I_{T/\log(T)}, \left\| \frac{v(\cdot)}{\sqrt{\log T}} - u(\cdot - z) \right\|_{\infty, z + I_r} \geq \epsilon \right] \leq \prod_{z \in 4r\mathbb{Z} \cap I_{T/\log(T)}} \nu \left[ \left\| \frac{v(\cdot)}{\sqrt{\log T}} - u(\cdot - z) \right\|_{\infty, z + I_r} \geq \epsilon \right] \leq \left( \nu \left[ \left\| \frac{v(\cdot)}{\sqrt{\log T}} - u \right\|_{\infty, I_r} \geq \epsilon \right] \right)^2 \frac{T}{4r\log(T)} + 1$$

Let $\eta > 0$ be such that $K^*(u) + \eta < 1$. Using the large deviations estimates of $v/\sqrt{\log(T)}$, we obtain that for $T$ sufficiently large,

$$\nu \left[ \left\| \frac{v(\cdot)}{\sqrt{\log T}} - u \right\|_{\infty, I_r} \geq \epsilon \right] \leq 1 - T^{-K^*(u) - \eta}.$$  

Thus, for $T$ sufficiently large,

$$\nu \left[ \forall z \in 2r\mathbb{Z} \cap I_{T/\log(T)}, \left\| \frac{v(\cdot)}{\sqrt{\log T}} - u(\cdot - z) \right\|_{\infty, z + I_r} \geq \epsilon \right] \leq \exp \left( - \left( 2 \frac{T}{4r\log(T)} + 1 \right) T^{-K^*(u) - \eta} \right) \sim \exp \left( - \frac{T^{-1} - K^*(u) - \eta}{2r\log(T)} \right).$$

The result follows by Borel-Cantelli lemma applied along the sequence $T_n = n$.

4.2 Lower bounds for $Y_T$, with fixed profiles of the field.

From lemma [4], we know that the field can be close to $u$ with $K^*_r(u) < 1$, in a region a size $r$. Thus, for $n$ integer, let

$$\mathcal{U}(n, r) \equiv \left\{ \bar{u} \in \mathcal{C}(I_r)^n ; \text{Max}_i \bar{u}(i) < 1 \right\},$$

be the $n$-tuples of admissible profiles. A lower bound for $P_0 [ |Y_T - y| < \epsilon ]$ is obtained by dividing $[0, T]$ into $n$ time intervals of length $\alpha_i T$ ($0 \leq \alpha_i \leq 1, \sum_{i=1}^n \alpha_i = 1$). In each time interval, we force the Brownian motion to go “fast” (say in a time of order $T/ \log(T)$) from $I_1$ to a region in $I_{T/\log(T)}$, where the field $v/\sqrt{\log(T)}$ is close to $u$, to remain there during $\alpha_i T$, and then to return fast (in time of order $T/ \log(T)$) to $I_1$.

Before stating the lower bound obtained in this way, we introduce some notations. For any integer $n$, and any $r \in [0, \infty]$, define

$$\mathcal{D}(n, r) \equiv \left\{ (\bar{\alpha}, \bar{f}) \in [0, 1]^n \times H^1(I_r)^n ; \sum_{i=1}^n \alpha_i = 1, \forall 1 \leq i \leq n, \|f_i\| = 1 \right\},$$

$$\mathcal{D}(n) \equiv \mathcal{D}(n, \infty), \text{ and for } (\bar{\alpha}, \bar{f}) \in \mathcal{D}(n), I_n(\bar{\alpha}, \bar{f}) \equiv \frac{1}{2} \sum_{i=1}^n \alpha_i \|f_i\|^2_2.$$
Lemma 12 Assume that $K$ has compact support in $I_L$. Then, $\forall r > L$, $\forall \epsilon > 0$, $\forall n \in \mathbb{N}$, $\forall \vec{u} \in \mathcal{U}(n, r)$, $\nu$-a.s., $\forall y \in \mathbb{R}$,

$$\lim \inf_{T \to \infty} \frac{1}{T} \log P_0 \left[ |Y_T - y| < \epsilon \right] \geq \inf_{\vec{z} \in \mathcal{B}(y, \epsilon)} \inf_{(\vec{\alpha}, \vec{f}) \in \mathcal{D}(n, r)} \left\{ I_n(\vec{\alpha}, \vec{f}) : \sum_{i=1}^{n} \alpha_i(u_i, f_i^2) = z \right\}.$$  \hspace{1cm} (31)

**Proof.** We begin with some more notations.

For $0 < S < T$, we will write $L^T_S$ for the occupation measure of $B$ between $S$ and $T$, $L^T_S = \frac{1}{T-S} \int_{S}^{T} \delta_{B_s} \, ds$.

Let us fix $\epsilon > 0$, $n \in \mathbb{N}$, $\vec{u} \in \mathcal{U}(n, r)$. Lemma 11 associates to $(\epsilon, \vec{u})$ a full $\nu$-measure set $A$ and a vector $\vec{z} = (z_1, \cdots, z_n)$ of points in $2r\mathbb{Z} \cap I_T/\log(T)$, such that when $v \in A$, and $T$ is sufficiently large,

$$\forall i \in \{1, \cdots, n\}, \ \left\| \frac{v}{\sqrt{\log(T)}} - \bar{u}_i \right\|_{\infty, z_i + Ir} \leq \frac{\epsilon}{6},$$  \hspace{1cm} (32)

where $\bar{u}_i(\cdot) \triangleq u_i(\cdot - z_i)$.

Now, let us fix $\vec{u} \in \mathcal{D}(n, r)$ such that $\sum_{i=1}^{n} \alpha_i(u_i, f_i^2) = y$. We set $T_0 = 0$, $T_i = \sum_{j=1}^{i} \alpha_j T$, and $\Delta = T/\log(T)$.

$$|Y_T - y| \leq \sum_{i=1}^{n} \left| \frac{1}{T} \int_{T_{i-1}}^{T_i} \frac{v(B_s)}{\sqrt{\log(T)}} \, ds \right|$$

$$+ \sum_{i=1}^{n} \alpha_i(1 - \frac{2}{\log(T)}) \left| \left( L_{T_i-\Delta}^{T_i+\Delta}; \frac{v}{\sqrt{\log(T)}} \right) - (f_i^2, u_i) \right|$$

$$+ \sum_{i=1}^{n} \left| \frac{1}{T} \int_{T_i-\Delta}^{T_i} \frac{v(B_s)}{\sqrt{\log(T)}} \, ds \right| + 2 \frac{|y|}{\log(T)}$$

Therefore, for $T$ sufficiently large, $2 \frac{|y|}{\log(T)} \leq \frac{\epsilon}{5}$, and

$$P_0 \left[ |Y_T - y| < \epsilon \right] \geq P_0 \left[ \forall i \in \{1, \cdots, n\}, |B_{T_i-1}| \leq 1; \left| \frac{1}{T} \int_{T_{i-1}}^{T_i+\Delta} \frac{v(B_s)}{\sqrt{\log(T)}} \, ds \right| < \frac{\epsilon}{60}; \right.$$  \hspace{1cm} (33)

$$|B_{T_i-1+\Delta} - z_i| \leq 1; \tau_{z_i + Ir} \circ \theta_{T_i-1+\Delta} > T_i - \Delta;$$

$$\left| \left( L_{T_i-\Delta}^{T_i+\Delta}; \frac{v}{\sqrt{\log(T)}} \right) - (f_i^2, u_i) \right| < \frac{\epsilon}{5};$$

$$\left| \frac{1}{T} \int_{T_i-\Delta}^{T_i} \frac{v(B_s)}{\sqrt{\log(T)}} \, ds \right| < \frac{\epsilon}{60}$$

$$\right]$$

But, on $\{ \tau_{z_i + Ir} \circ \theta_{T_i-1+\Delta} > T_i - \Delta \}$, and for $v \in A$,

$$\left| \left( L_{T_i-\Delta}^{T_i+\Delta}; \frac{v}{\sqrt{\log(T)}} \right) - (f_i^2, u_i) \right|$$

$$\leq \left\| \frac{v}{\sqrt{\log(T)}} - \bar{u}_i \right\|_{\infty, z_i + Ir} + \left| \left( L_{T_i-\Delta}^{T_i+\Delta}; \bar{u}_i \right) - (f_i^2, \bar{u}_i) \right|$$

$$\leq \frac{\epsilon}{6} + \left| \left( L_{T_i-\Delta}^{T_i+\Delta}; \bar{u}_i \right) - (f_i^2, \bar{u}_i) \right|$$

$$\leq \frac{\epsilon}{6} + \left| \left( L_{T_i-\Delta}^{T_i+\Delta}; \bar{u}_i \right) - \left( f_i^2, \bar{u}_i \right) \right|.$$
The Markov property applied recursively at times $T_{i-1}$ yields then
\[ P_0 [ |Y_T - y| < \epsilon ] \geq \prod_{i=1}^{n} U_i, \quad (33) \]

where
\[ U_i = \inf_{|z| \leq 1} P_z \left[ \frac{1}{T} \int_0^\Delta \frac{v(B_s)}{\sqrt{\log(T)}} ds < \frac{\epsilon}{6n}; |B_{\Delta} - z_i| \leq 1; \right. \]
\[ \tau_{z_i + I_r} \circ \theta_\Delta > \alpha_i T - \Delta; \]
\[ \left. \left( L_{\alpha_i T - \Delta} \circ \tilde{u}_i - \left( \tilde{f}_i^2, \tilde{u}_i \right) \right) < \frac{\epsilon}{6}; \right. \]
\[ \frac{1}{T} \int_0^{\alpha_i T} \frac{v(B_s)}{\sqrt{\log(T)}} ds < \frac{\epsilon}{6n}; |B_{\alpha_i T} \circ \theta_\Delta| \leq 1 \]

Now, it follows from Markov property applied successively at times $\alpha_i T - \Delta$ and $\Delta$, that for all $i \in \{1, \cdots, n\}$,
\[ U_i \geq V_i W_i X_i, \quad (34) \]

with
\[ V_i = \inf_{|z| \leq 1} P_z \left[ \frac{1}{T} \int_0^\Delta \frac{v(B_s)}{\sqrt{\log(T)}} ds < \frac{\epsilon}{6n}; |B_{\Delta} - z_i| \leq 1 \right], \]
\[ W_i = \inf_{z \in z_i + I_r} P_z \left[ \tau_{z_i + I_r} > \alpha_i T - 2\Delta; \left( L_{\alpha_i T - 2\Delta} \circ \tilde{u}_i - \left( \tilde{f}_i^2, \tilde{u}_i \right) \right) < \frac{\epsilon}{6} \right], \]
\[ X_i = \inf_{z \in z_i + I_r} P_z \left[ \frac{1}{T} \int_0^\Delta \frac{v(B_s)}{\sqrt{\log(T)}} ds < \frac{\epsilon}{6n}; |B_{\Delta}| \leq 1 \right]. \]

**Estimates for $W_i$.** By translation invariance,
\[ W_i = \inf_{z \in I_1} P_z \left[ \tau_{I_r} > \alpha_i T - \frac{2T}{\log(T)}; \left( L_{\alpha_i T - \frac{2T}{\log(T)}} \circ \tilde{u}_i - \left( f_i^2, u_i \right) \right) < \frac{\epsilon}{6} \right]. \]

It follows then from the large deviations for the occupation measure that for all $i \in \{1, \cdots, n\}$,
\[ \liminf_{T \to \infty} \frac{1}{T} \log W_i \geq -\frac{\alpha_i}{2} \| f_i' \|_2^2. \quad (35) \]

**Estimates for $V_i$ and $X_i$.** We are now going to show that
\[ \liminf_{T \to \infty} \frac{1}{T} \log V_i \geq 0, \quad \text{and} \quad \liminf_{T \to \infty} \frac{1}{T} \log X_i \geq 0. \quad (36) \]

Since $V_i$ and $X_i$ are treated in the same way, we give only the proof for $V_i$. Let $z \in I_1$. Since $|z_i| \leq T/\log(T)$, we have
\[ P_z \left[ |B_{T/\log(T)} - z_i| \leq 1 \right] = \int_{|y + z - z_i| \leq 1} \exp \left( -\frac{y^2}{2T/\log(T)} \right) \frac{dy}{\sqrt{2\pi T/\log(T)}} \geq \frac{1}{\sqrt{2\pi T/\log(T)}} \exp \left( -\frac{(T/\log(T) + 1)^2}{2T/\log(T)} \right). \]

18
Moreover,
\[
P_z \left[ \frac{1}{T} \int_0^T \frac{v(B_s)}{\sqrt{\log(T)}} \, ds \geq \frac{\epsilon}{6n} \right]
\leq P_z \left[ \tau_T \leq \frac{T}{\log(T)} + \frac{\max_{\mathcal{I}_T} |v|}{\sqrt{\log(T)}} > \frac{\epsilon \log(T)}{6n} \right]
\leq P_0 \left[ \tau_T - 1 \leq \frac{T}{\log(T)} + \frac{\max_{\mathcal{I}_T} |v|}{\sqrt{\log(T)}} > \frac{\epsilon \log(T)}{6n} \right]
\leq C \exp \left( -\frac{(T-1)^2 \log(T)}{2T} \right) + \frac{\max_{\mathcal{I}_T} |v|}{\sqrt{\log(T)}} \geq \frac{\epsilon \log(T)}{6n}.
\]

Since
\[
P_z \left[ \frac{1}{T} \int_0^T \frac{v(B_s)}{\sqrt{\log(T)}} \, ds < \frac{\epsilon}{6n}, \right]
\geq \left| B_{\frac{T}{\log(T)}} - z_i \right| < 1 - P_0 \left[ \tau_T - 1 \leq \frac{T}{\log(T)} + \frac{\max_{\mathcal{I}_T} |v|}{\sqrt{\log(T)}} > \frac{\epsilon \log(T)}{6n} \right],
\]
we obtain
\[
V_i \geq \frac{1}{\sqrt{2\pi T/\log(T)}} \exp \left( -\frac{(T/\log(T)+1)^2}{2T/\log(T)} \right) - C \exp \left( -\frac{(T-1)^2 \log(T)}{2T} \right)
- \frac{\max_{\mathcal{I}_T} |v|}{\sqrt{\log(T)}} \geq \frac{\epsilon \log(T)}{6n}.
\]

By (3), the indicator is null for $T$ sufficiently large, and we get (36) for $V_i$.

Putting together (3), (34), (35), (36), and taking the supremum over admissible $(\vec{\alpha}, \vec{f})$, we have proved that $\forall r > L, \forall \epsilon > 0, \forall n \in \mathbb{N}, \forall \vec{u} \in \mathcal{U}(n, r)$, $\nu$-a.s., $\forall y \in \mathbb{R}$,
\[
\liminf_{T \to \infty} \frac{1}{T} \log P_0 \left[ |Y_T - y| < \epsilon \right] \geq - \inf_{(\vec{\alpha}, \vec{f}) \in \mathcal{D}(n, r)} \left\{ I_n(\vec{\alpha}, \vec{f}) : \sum_{i=1}^n \alpha_i (u_i, f_i^2) = y \right\}.
\]

This in turn implies easily (31).

### 4.3 Realizing the supremum over countably many profiles.

We would like now to take the supremum over functions $u_1, \ldots, u_n$. Here, we have to be a little careful, since the “$\nu$-a.s” appearing in (31) depends on the functions $u_1, \ldots, u_n$. This problem would be overcome using the separability of $\mathcal{C}(\mathcal{I}_r)$, if the function $K^*_r$ were continuous. This is not the case everywhere. However, assume for a moment that we could take the
supremum over admissible functions \( u_i \). We would obtain that \( \nu \)-a.s.,

\[
\lim_{T \to \infty} \frac{1}{T} \log P_0(|Y_T - y| < \epsilon) \\
\geq - \inf_{\vec{u} \in \mathcal{U}(n,r)} \inf_{(\vec{\alpha}, \vec{f}) \in \mathcal{D}(n,r)} \left\{ I_n(\vec{\alpha}, \vec{f}) : |\sum \alpha_i(u_i, f_i^2) - y| < \epsilon \right\}
\]

\[
= - \inf_{(\vec{\alpha}, \vec{f}) \in \mathcal{D}(n,r)} \left\{ I_n(\vec{\alpha}, \vec{f}) : \exists \vec{u} \in \mathcal{U}(n,r), |\sum \alpha_i(u_i, f_i^2) - y| < \epsilon \right\}
\]

\[
= - \inf_{(\vec{\alpha}, \vec{f}) \in \mathcal{D}(n,r)} \left\{ I_n(\vec{\alpha}, \vec{f}) : \inf \{ \max_i K^*_i(u_i) : |\sum \alpha_i(u_i, f_i^2) - y| < \epsilon \} < 1 \right\}
\]

We are thus led to show that the infimum of \( \max_i K^*_i(u_i) \) on the set \( \{ \vec{u} \in \mathcal{C}(\vec{I}_r)^n : |\sum \alpha_i(u_i, f_i^2) - y| < \epsilon \} \) can actually be reached on a countable subset of \( \mathcal{C}(\vec{I}_r)^n \).

**Lemma 13.**

- \( \forall f \in L^2(I_r), \ K \ast f^2 \in \mathcal{C}(\vec{I}_r), \ and\)

\[
K^*_i(K \ast f^2) = \frac{1}{2}(K \ast f^2, f^2).
\]

(37)

- \( \forall n \in \mathbb{N}, \forall (\vec{\alpha}, \vec{f}) \in \mathcal{D}(n,r), \ and \forall y \in \mathbb{R}, \)

\[
\inf_{\vec{u} \in \mathcal{C}(\vec{I}_r)^n} \left\{ \max_i K^*_i(u_i) : \sum \alpha_i(u_i, f_i^2) = y \right\} = \frac{|y|^2}{2 \left( \sum_{i=1}^{n} \alpha_i \sqrt{(K \ast f_i^2, f_i^2)} \right)^2},
\]

(38)

with the convention \( 0/0 = 0 \). Moreover, the infimum in (38) is reached for functions \( (\vec{u}_1, \cdots, \vec{u}_n) \) defined in the following way. Let \( I_0 = \{ i; \alpha_i = 0 \} \).

- If \( \sum_{i=1}^{n} \alpha_i \sqrt{(K \ast f_i^2, f_i^2)} = 0, \) take

\[
\begin{cases}
\vec{u}_i \equiv 0, & \text{for } i \in I_0; \\
\vec{u}_i \equiv \frac{y}{\alpha_i |I_0|}, & \text{for } i \notin I_0.
\end{cases}
\]

- If \( \sum_{i=1}^{n} \alpha_i \sqrt{(K \ast f_i^2, f_i^2)} > 0, \) take

\[
\begin{cases}
\vec{u}_i \equiv 0, & \text{for } i \in I_0; \\
\vec{u}_i = \frac{y}{\sum_{j=1}^{n} \alpha_j \sqrt{(K \ast f_j^2, f_j^2)}} \sqrt{(K \ast f_i^2, f_i^2)} K \ast f_i^2, & \text{for } i \notin I_0.
\end{cases}
\]
Let $\mathbb{D}_1$ be a dense countable subset of $L^2(I_r)$, and let
\[
\mathbb{D} \triangleq \{ s(K \ast g^2) : g \in \mathbb{D}_1, s \in \{-1; 1\} \} \cup \{ u \equiv q, q \in \mathbb{Q} \}.
\]
$\mathbb{D}$ is a countable subset of $C(\bar{I}_r)$, and $\forall n \in \mathbb{N}$, $\forall (\vec{\alpha}, \vec{f}) \in \mathbb{D}(n, r)$, $\forall y \in \mathbb{R}$, $\forall \epsilon > 0$,
\[
\inf_{\vec{u} \in \mathbb{C}(I_r)^n} \{ \max_i K^*_r(u_i) : |\sum \alpha_i(u_i, f_i^2) - y| < \epsilon \}
= \inf_{\vec{u} \in \mathbb{D}^n} \{ \max_i K^*_r(u_i) : |\sum \alpha_i(u_i, f_i^2) - y| < \epsilon \}
\tag{39}
\]
\[\text{Proof.}
\]
\[\text{Proof of (37).}
\]
\[
K^*_r(K \ast f^2) = \sup \{ (\mu, K \ast f^2 \ast \mu) : \mu \in M(I_r) \}
= \sup \left\{ \frac{1}{2}(K \ast f^2, f^2) - \frac{1}{2}(K \ast \mu, \mu) : \mu \in M(I_r) \right\},
\]
by the change of variable $\mu \rightarrow \mu + f^2 \, dx$. Thus $K^*_r(K \ast f^2) = \frac{1}{2}(K \ast f^2, f^2)$.
\[\text{Proof of (38).}
\]
First of all, note that $\sum \alpha_i(\vec{u}_i, f_i^2) = y$, so that
\[
\inf_{\vec{u} \in \mathbb{C}(I_r)^n} \{ \max_i K^*_r(u_i) : \sum \alpha_i(u_i, f_i^2) = y \}
\leq \max_{i \notin I_0} K^*_r(\vec{u}_i) = \max_{i \notin I_0} K^*_r(\vec{u}_i).
\]
If $\sum \alpha_i(\sqrt{(K \ast f_i^2, f_i^2)} = 0$, then $(K \ast f_i^2, f_i^2) = 0$ for any $i \notin I_0$. In this situation, $K^*_r(1) \geq \frac{(1, f_i^2)}{2(K \ast f_i^2, f_i^2)} = +\infty$, for any $i \notin I_0$. Thus,
\[
\max_{i \notin I_0} K^*_r(\vec{u}_i) = \begin{cases} 0 & \text{if } y = 0 \\ +\infty & \text{if } y \neq 0 \end{cases} = \frac{y^2}{2(\sum \alpha_i \sqrt{(K \ast f_i^2, f_i^2)})^2}.
\]
If $\sum \alpha_i(\sqrt{(K \ast f_i^2, f_i^2)} > 0$, then
\[
\max_{i \notin I_0} K^*_r(\vec{u}_i) = \max_{i \notin I_0} \frac{y^2}{\sqrt{(\sum \alpha_i(\sqrt{(K \ast f_i^2, f_i^2)})^2(K \ast f_i^2, f_i^2))}} \leq \frac{y^2}{2(\sum \alpha_i \sqrt{(K \ast f_i^2, f_i^2)})^2}, \text{ by (37)}.
\]
It remains now to show that
\[
\inf_{\vec{u} \in \mathbb{C}(I_r)^n} \{ \max_i K^*_r(u_i) : \sum \alpha_i(u_i, f_i^2) = y \} \geq \frac{y^2}{2(\sum \alpha_i \sqrt{(K \ast f_i^2, f_i^2)})^2}.
\]
First, note that
\[
\inf_{\vec{u} \in \mathbb{C}(I_r)^n} \{ \max_i K^*_r(u_i) : \sum \alpha_i(u_i, f_i^2) = y \}
= \inf_{\vec{y} \in \mathbb{R}^n, \sum \alpha_y = y} \inf_{\vec{u} \in \mathbb{C}(I_r)^n} \inf_{\vec{u} \in \mathbb{C}(I_r)^n} \{ \max_i K^*_r(u_i) : \forall i, (u_i, f_i^2) = y_i \}
\geq \inf_{\vec{y} \in \mathbb{R}^n, \sum \alpha_y = y} \max_i \inf_{\vec{u} \in \mathbb{C}(I_r)} \{ K^*_r(u_i) : u_i \in \mathbb{C}(I_r), (u_i, f_i^2) = y_i \}
\]
\[21\]
For \((u_i, f_i^2) = y_i\), \(K^*_r(u_i) \geq \frac{(u_i, f_i^2)^2}{2(K \ast f_i^2, f_i^2)} = \frac{y_i^2}{2(K \ast f_i^2, f_i^2)}\), so that
\[
\inf_{\vec{u} \in C(\bar{I}_r)} \{\max_i K^*_r(u_i) : \sum \alpha_i(u_i, f_i^2) = y\} \\
\geq \inf_{\vec{y} \in \mathbb{R}^n} \left\{\max_i \frac{y_i^2}{2(K \ast f_i^2, f_i^2)} : \sum \alpha_i y_i = y\right\}.
\]

Now, for \(\sum \alpha_i y_i = y\),
\[
|y| \leq \max_i \left[\frac{|y_i|}{\sqrt{(K \ast f_i^2, f_i^2)}}\right] \sum \alpha_i \sqrt{(K \ast f_i^2, f_i^2)}.
\]
Thus,
\[
\inf_{\vec{u} \in C(\bar{I}_r)} \{\max_i K^*_r(u_i) : \sum \alpha_i(u_i, f_i^2) = y\} \geq \frac{y^2}{2(\sum \alpha_i \sqrt{(K \ast f_i^2, f_i^2)})^2}.
\]

This ends the proof of (38).

(39) is a straightforward consequence of (38), of the expression of the minimizing functions \(\bar{u}_i\), and of the continuity of \(f \in L^2(I_r) \mapsto (K \ast f^2, f^2)\), and \(f \in L^2(I_r) \mapsto K \ast f^2 \in C(\bar{I}_r)\).

Performing now in lemma 12 the supremum over functions \(u_i \in D\), then over \(r \in \mathbb{Q}\), we have thus shown that when \(K\) has compact support, \(\nu\)-a.s., \(\forall n \in \mathbb{N}, \forall \epsilon > 0, \forall y \in \mathbb{R}\),
\[
\liminf_{T \to \infty} \frac{1}{T} \log P_0(|Y_T - y| < \epsilon) \geq -J_n(y).
\]
(40)

where
\[
J_n(y) \triangleq \inf \left\{I_n(\vec{\alpha}, \vec{f}) : (\vec{\alpha}, \vec{f}) \in D_n(y)\right\} \\
D_n(y) \triangleq \left\{(\vec{\alpha}, \vec{f}) \in D(n) : \frac{|y|}{\sqrt{2}} < \sum \alpha_i \sqrt{(K \ast f_i^2, f_i^2)}\right\}.
\]

4.4 Identifying the rate function.

Now, our aim is to characterize the limit \(n \to \infty\) in (40).

Lemma 14.

1. \(\forall n \in \mathbb{N}, \forall y \in \mathbb{R}\),
\[
J(y) \leq J_{n+1}(y) \leq J_n(y).
\]
(42)

2. \(\forall n \in \mathbb{N}, \forall \alpha \in [0, 1], \forall y_1, y_2 \in \mathbb{R}\),
\[
J_{2n}(\alpha y_1 + (1 - \alpha)y_2) \leq \alpha J_n(y_1) + (1 - \alpha)J_n(y_2).
\]
(43)
3. If $\mathcal{J}_1^*$ denotes the Fenchel-Legendre transform of $\mathcal{J}_1$, $\mathcal{J}_1^{**} = \mathcal{J}$.

4. Let $\mathcal{J}(y) = \lim_{n \to \infty} \mathcal{J}_n(y)$, and $\bar{\mathcal{J}}(y) = \sup_{\epsilon > 0} \inf_{z: \|z - y\| \leq \epsilon} \mathcal{J}(z)$ the greater l.s.c. minorant of $\mathcal{J}$. Then $\bar{\mathcal{J}} = \mathcal{J}$.

**Proof of 1.** From the large deviations upper bound, we have $\mathcal{J}(y) \leq \mathcal{J}_n(y)$ for all $n$.

For any $(\alpha, f) \in \mathcal{D}_n(y)$, $\bar{\beta} \triangleq (\bar{\alpha}, 0)$ and $\bar{g} \triangleq (\bar{f}, f_1)$ are such that $(\bar{\beta}, \bar{g}) \in \mathcal{D}_{n+1}(y)$, so that $\mathcal{J}_{n+1}(y) \leq I_{n+1}(\bar{\beta}, \bar{g}) = I_n(\bar{\alpha}, \bar{f})$. Taking the infimum over $\mathcal{D}_n(y)$ yields $\mathcal{J}_{n+1}(y) \leq \mathcal{J}_n(y)$.

**Proof of 2.** In the same way, let $\alpha \in [0, 1]$ and $y_1, y_2 \in \mathbb{R}$ be fixed. For any $(\tilde{\beta}, \tilde{f}) \in \mathcal{D}_n(y_1)$, and any $(\tilde{\gamma}, \tilde{g}) \in \mathcal{D}_n(y_2)$, $\tilde{\lambda} \triangleq (\alpha \tilde{\beta}, (1 - \alpha) \tilde{\gamma})$ and $\tilde{h} \triangleq (\tilde{f}, \tilde{g})$ are such that $(\tilde{\lambda}, \tilde{h}) \in \mathcal{D}_{2n}(\alpha y_1 + (1 - \alpha) y_2)$. Thus,

$$J_{2n}(\alpha y_1 + (1 - \alpha) y_2) \leq I_{2n}(\tilde{\lambda}, \tilde{h}) = \alpha I_n(\tilde{\beta}, \tilde{f}) + (1 - \alpha) I_n(\tilde{\gamma}, \tilde{g}).$$

Taking the infimum over elements of $\mathcal{D}_n(y_1)$ and $\mathcal{D}_n(y_2)$, leads to $\mathcal{J}_1$.

**Proof of 3.** Let us now compute the Legendre transform of $\mathcal{J}_1$. First of all, note that

$$J_1(y) = \inf_{f \in H^1, \|f\|_2 = 1} \left\{ \frac{1}{2} \|f'\|_2^2 : |y| < \sqrt{2(K \ast f^2, f^2)} \right\}.$$

Therefore,

$$J_1^*(\alpha) = \sup_{y \in \mathbb{R}} \{ \alpha y - J_1(y) \}
= \sup_{f \in H^1, \|f\|_2 = 1} \sup_{y \in \mathbb{R}} \left\{ \alpha y - \frac{1}{2} \|f'\|_2^2 : |y| < \sqrt{2(K \ast f^2, f^2)} \right\}
= \sup_{f \in H^1, \|f\|_2 = 1} \left\{ |\alpha| \sqrt{2(K \ast f^2, f^2)} - \frac{1}{2} \|f'\|_2^2 \right\}
= \Lambda(\alpha).$$

Hence, $J_1^{**}(y) = \Lambda^*(y) = J(y)$.

**Proof of 4.** Taking the limit in $\mathcal{J}_1$, we obtain that for all $y \in \mathbb{R}$, $\mathcal{J}(y) \leq \mathcal{J}(y) \leq J_1(y)$. Since $\mathcal{J}$ is l.s.c, we also have $\mathcal{J}(y) \leq \mathcal{J}(y) \leq J_1(y) \leq \mathcal{J}_1(y)$. Since $\mathcal{J}_1^{**} = J$, the preceding inequality implies that $\mathcal{J}(y) = \mathcal{J}^{**}$. Now, taking the limit in $\mathcal{J}_1$, we see that $\mathcal{J}$ is convex, and so is $\mathcal{J}$. $\mathcal{J}$ being convex and l.s.c., $\mathcal{J} = \mathcal{J}^{**} = J$.

**Lemma 15.** Assume that $K$ has compact support. Then, $\nu$-a.s., $\forall y \in \mathbb{R}$,

$$\lim_{\epsilon \to 0} \liminf_{T \to \infty} \frac{1}{T} \log P_0 \left[ |Y_T - y| \leq \epsilon \right] \geq -\mathcal{J}(y).$$
Proof. Taking the limit \( n \to \infty \) in (40) yields that \( \nu \)-a.s., \( \forall \epsilon > 0, \forall y \in \mathbb{R} \),

\[
\liminf_{T \to \infty} \frac{1}{T} \log P_0 \left[ |Y_T - y| \leq \epsilon \right] \geq -J(y).
\]

Let \( z \) be any point in \( B(y, \epsilon) \), and let \( \eta > 0 \) be such that \( B(z, \eta) \subset B(y, \epsilon) \).

\[
\liminf_{T \to \infty} \frac{1}{T} \log P_0 \left[ |Y_T - y| \leq \epsilon \right] \geq \liminf_{T \to \infty} \frac{1}{T} \log P_0 \left[ |Y_T - z| \leq \eta \right] \geq -J(z).
\]

Taking the supremum in \( z \in B(y, \epsilon) \), and letting \( \epsilon \) go to 0, leads to

\[
\lim \liminf_{T \to \infty} \frac{1}{T} \log P_0 \left[ |Y_T - y| \leq \epsilon \right] \geq -\tilde{J}(y) = -J(y).
\]

\[
\begin{align*}
\text{4.5 The general case.} \\
\text{We are now going to prove the lower bound in the general case, i.e. under assumption (8) for the covariance } K. \text{ To this end, we use the decomposition of } v = v_L + \tilde{v}_L \text{ (cf section 2.2 and equation (10)). Let } Y = Y_L + \tilde{Y}_L \text{ the corresponding decomposition of } Y. \text{ Let } \epsilon > 0 \text{ and } L \text{ sufficiently large so that } \\
\sqrt{2K_L(0)} < \frac{\epsilon}{2}. \text{ Then,}
\end{align*}
\]

\[
P_0 \left[ |Y_T - y| < \epsilon \right] \geq P_0 \left[ |Y_{L,T} - y| < \epsilon/2 \right] - P_0 \left[ |\tilde{Y}_{L,T}| \geq \epsilon/2 \right].
\]

But,

\[
P_0 \left[ |\tilde{Y}_{L,T}| \geq \epsilon/2 \right] \leq P_0 \left[ \tau_{RT} > T; |\tilde{Y}_{L,T}| \geq \epsilon/2 \right] + P_0 \left[ \tau_{RT} \leq T \right] \leq \frac{1}{\max_{0 \leq s \leq \epsilon/2}} P_0 \left[ \tau_{RT} \leq T \right].
\]

Thus, \( \forall \epsilon > 0, \text{ and } L \text{ sufficiently large, } \nu \)-a.s.,

\[
\liminf_{T \to \infty} \frac{1}{T} \log P_0 \left[ |\tilde{Y}_{L,T}| \geq \epsilon/2 \right] \geq -\infty.
\]

Therefore, by (40), \( \forall \epsilon > 0, \forall L \text{ sufficiently large, } \nu \)-a.s., \( \forall y \in \mathbb{R}, \forall n \in \mathbb{N} \)

\[
\liminf_{T \to \infty} \frac{1}{T} \log P_0 \left[ |Y_T - y| < \epsilon \right] \geq -J^L_n(y),
\]

where \( J^L_n(y) \triangleq \inf_{(\tilde{\alpha}, \tilde{f}) \in D^L_n(y)} I_n(\tilde{\alpha}, \tilde{f}) \),

\[
D^L_n(y) \triangleq \left\{ (\alpha, \tilde{f}) \in D(n); \frac{|y|}{\sqrt{2}} < \sum \alpha_i \sqrt{(K_L \ast f^2_i, f^2_i)} \right\}.
\]

We are now going to prove that \( \forall n \text{ and } \forall y, \limsup_{L \to \infty, L \in \mathbb{Q}} J^L_n(y) \leq J_n(y) \),

and we can assume that \( J_n(y) < \infty \). Let \( \eta > 0 \), and \( (\tilde{\alpha}, \tilde{f}) \in D_n(y) \) be such that \( I_n(\tilde{\alpha}, \tilde{f}) \leq J_n(y) + \eta \). Since \( K_L \) converges almost everywhere to \( K \)
when $L \to \infty$, \(\forall i, \left( K_L \ast f_i^2, f_i^2 \right) \to \left( K \ast f_i^2, f_i^2 \right) \) by Lebesgue dominated convergence theorem. Thus, for $L$ sufficiently large, \((\bar{a}, \bar{f}) \in D_n^L(y), \) and $J_n^L(y) \leq J_n(y) + \eta$. Therefore, letting first $L \to \infty$, then $n \to \infty$ in (44), we obtain that $\nu$-a.s., $\forall \epsilon > 0, \forall y \in \mathbb{R},$

$$\liminf_{T \to \infty} \frac{1}{T} \log P_0 \left[ |Y_T - y| < \epsilon \right] \geq -J(y).$$

As usual, this in turn implies the same bound with $\bar{J}$ in place of $J$. To conclude the proof of (44), note that the results of lemma [4] are independent of the support of $K$, so that we have $\bar{J} = \bar{J}$.

### 4.6 Lower bound for $X_{2,T}$

As for the upper bound, the lower bound for $Y_T$ yields straightforwardly the same lower bound for $X_{2,T}$, since

$$P_0 \left[ \frac{X_{2,T}}{T \sqrt{\log(T)}} - y < \epsilon \right] \geq P_0 \left[ |Y_T - y| < \epsilon/2 \right] - P_0 \left[ \left| \frac{W_{2,T}}{T \sqrt{\log(T)}} \right| \geq \epsilon/2 \right],$$

and $\lim_{T \to \infty} \frac{1}{T} \log P_0 \left[ \left| \frac{W_{2,T}}{T \sqrt{\log(T)}} \right| \geq \epsilon/2 \right] = -\infty$.

### 5 Properties of the rate function.

The aim of this section is to prove proposition [3] linking the behavior of $K$ at infinity, with the behavior of $\mathcal{J}$ near the origin. Note that since the functions normalizing $\mathcal{J}$ are convex and continuous, and since $\mathcal{J}_{11^*} = \mathcal{J}$, to prove proposition [3] it is enough to prove the corresponding assertions for $\mathcal{J}_1$.

Using the isometry of $L^2$: $f \mapsto f_\lambda = \sqrt{\lambda} f(\lambda \cdot)$, note that

$$\mathcal{J}_1(y) = \inf_{f \in H^1, \|f\|_2 = 1} \left\{ \frac{\lambda^2}{2} \|f'\|_2^2 : \iint K(\frac{x-z}{\lambda^2})f^2(x)f^2(z) \, dx \, dz > \frac{y^2}{2} \right\}, \forall \lambda > 0$$

$$= \inf_{f \in H^1, \|f\|_2 = 1} \left\{ \frac{\lambda^2}{2} \|f'\|_2^2 : \iint K \left( \frac{x-z}{\lambda^2} \right) f^2(x)f^2(z) \, dx \, dz > \frac{y^2}{2} \right\}.$$

**Case** $\limsup_{|x| \to \infty} |K(x)| |x|^\beta < \infty$ for some $\beta \in \{0, 1\}$.

Since $K$ is bounded, there exists a constant $C$ such that $K(x) \leq C|x|^{-\beta}$. It follows then from (45) that

$$\mathcal{J}_1(y) \geq \inf_{f \in H^1(\mathbb{R}), \|f\|_2 = 1} \inf_{\lambda > 0} \left\{ \frac{\lambda^2}{2} \|f'\|_2^2 : C\lambda^\beta (I_\beta(f^2), f^2) > \frac{y^2}{2} \right\},$$

where $I_\beta$ is the Riesz operator defined by $I_\beta(f)(x) \triangleq \int_{\mathbb{R}} \frac{f(y)}{|x-y|^\beta} \, dy.$
Taking the infimum in $\lambda$ leads to

$$J_{1}(y) \frac{1}{|y|^\frac{2}{\beta}} \geq C \Inf \left\{ \frac{\|f^r\|_2^2}{(I_\beta(f^2), f^2)^{\frac{2}{\beta}}} : f \in H^1(\mathbb{R}), \|f\|_2 = 1 \right\}.$$  \hspace{1cm} (46)

Now, for $p \in ]1, \frac{1}{1-\beta}[,$ $I_\beta$ is continuous from $L^p(\mathbb{R})$ to $L^r(\mathbb{R})$, for $1/r = 1/p - (1 - \beta)$ (see for instance theorem 1 pp 119 in [10]). Therefore, for any $f \in H^1(\mathbb{R})$ such that $\|f\|_2 = 1$, and for any $p \in ]1, \frac{1}{1-\beta}[,$

$$(I_\beta(f^2), f^2) \leq \|f^2\|_r \|I_\beta(f^2)\|_r \text{ where } \frac{1}{r} + \frac{1}{r'} = 1,$$

$$\leq C \|f^2\|_r \|f^2\|_p \text{ by continuity of } I_\beta,$$

$$\leq C \|f\|_{\infty}^{2(1 - \frac{1}{r'} - \frac{1}{p})} \text{ since } \int f^2 = 1.$$ 

Note that by Sobolev embedding theorem, any function $f \in H^1(\mathbb{R})$ belongs to $L^\infty(\mathbb{R})$, and $\|f\|_{\infty} \leq C \|f\|_2^{1/2} \|f^r\|_2^{1/2}$ for some constant $C \in ]0, \infty[.$ Thus, for any $f \in H^1(\mathbb{R})$ such that $\|f\|_2 = 1,$

$$(I_\beta(f^2), f^2) \leq C \|f\|_2^\beta.$$ 

Therefore, the infimum in (46) is strictly positive, and it is clearly finite.

Let us now turn to the converse inequality, and let us assume that $K \geq 0$ and $l = \liminf_{|x| \to \infty} K(x)|x|^\beta > 0.$ The change of variable $\lambda = \gamma|y|^\frac{2}{\beta}$ in (43) leads to

$$J_{1}(y) \frac{1}{|y|^\frac{2}{\beta}} = \Inf_{f \in H^1(\mathbb{R}), \|f\|_2 = 1} \Inf_{\gamma > 0} \left\{ \frac{\gamma^2}{2} \|f^r\|_2 : \frac{1}{|y|^2} \int \int K\left(\frac{x-z}{\gamma|y|^\frac{2}{\beta}}\right)f^2(x)f^2(z)dx dz > \frac{1}{2} \right\}.$$ 

Let $f \in H^1(\mathbb{R}), \|f\|_2 = 1$, and $\gamma > 0$ be such that $l\gamma^\beta(I_\beta(f^2), f^2) > \frac{1}{2}$. By Fatou lemma,

$$l\gamma^\beta(I_\beta(f^2), f^2) \leq \liminf_{|y| \to 0} \frac{1}{|y|^2} \int \int K\left(\frac{x-z}{\gamma|y|^\frac{2}{\beta}}\right)f^2(x)f^2(z)dx dz,$$

and thus, for any $(f, \gamma)$ with $\|f\|_2 = 1$, and $l\gamma^\beta(I_\beta(f^2), f^2) > \frac{1}{2},$

$$\limsup_{|y| \to 0} \frac{J_{1}(y)}{|y|^\frac{2}{\beta}} \leq \frac{\gamma^2}{2} \|f^r\|_2$$

Therefore,

$$\limsup_{|y| \to 0} \frac{J_{1}(y)}{|y|^\frac{2}{\beta}} \leq C \Inf \left\{ \frac{\|f^r\|_2^2}{(I_\beta(f^2), f^2)^{\frac{2}{\beta}}} : f \in H^1(\mathbb{R}), \|f\|_2 = 1 \right\} < \infty.$$ 

Note that using Lebesgue dominated convergence theorem in place of Fatou lemma, the same result holds, as soon as $\lim_{|x| \to \infty} K(x)|x|^\beta > 0.$ This concludes the proof of point 1. of proposition [3].
Case $\limsup_{|x| \to \infty} |K(x)||x|^\beta < \infty$ for some $\beta > 1$, $\int K(x) \, dx \neq 0$.
In this situation,

$$K \in L^1(\mathbb{R}), \quad \forall \delta \in ]0, \beta - 1[, \quad \int K^2(x)|x|^{1+2\delta} \, dx < \infty. \quad (47)$$

Therefore, by dominated convergence,

$$\int \int \frac{1}{\lambda} K\left(\frac{x-y}{\lambda}\right)f^2(x)f^2(y) \to \left(\int K(x) \, dx\right) \|f\|_4^4, \quad (48)$$

Note that this implies that $\bar{K} \triangleq \int K(x) \, dx > 0$, and thus $\bar{K} > 0$.

Moreover, we know from standard results in functional analysis (see for instance [10]) that

**Lemma 16** If $f \in H^1(\mathbb{R})$, then $\forall p \in [2, +\infty]$, $f \in L^p(\mathbb{R})$, and

$$\|f\|_p \leq C \|f\|_2^{\frac{1}{2} + \frac{1}{p}} \|f'\|_2^{\frac{1}{2} - \frac{1}{p}},$$

where $C$ is a constant depending only on $p$. Moreover, $\forall \delta \in ]0, 1[$, there exists a constant $C$ such that

$$\left(\int \mathbb{R} \frac{\|f(\cdot + t) - f(\cdot)\|_2^2}{|t|^{1+2\delta}} \, dt\right)^{\frac{1}{2}} \leq C (\|f\|_2 + \|f'\|_2).$$

Thus, for all $f \in H^1(\mathbb{R})$, $\|f\|_2 = 1$, $\forall \delta \in ]0, 1\cap]0, \beta - 1[$, $\forall \lambda > 0$,

$$\begin{align*}
\left|\int \int \frac{1}{\lambda} K\left(\frac{x-y}{\lambda}\right) f^2(x)f^2(y) \, dx \, dy - \left(\int K(x) \, dx\right) \int f^4(x) \, dx\right| \\
\leq \int dx f^2(x) \int dz |K(z)| \left|\int f^2(x) - f^2(x + \lambda z)\, dx\right| \\
\leq 2 \|f\|_\infty \int dz |K(z)| \left|\int dx f(x + \lambda z) - f(x)\right| \\
\leq 2 \|f\|_\infty \|f\|_4^2 \int |K(z)| \left|\int f(\cdot + \lambda z) - f(\cdot)\right|_2 \, dz \\
\leq 2\lambda^\delta \|f\|_\infty \|f\|_4^2 \left(\int K^2(z)|z|^{1+2\delta} \, dz\right)^{1/2} \left(\int \frac{\|f(\cdot - z) - f(\cdot)\|_2^2}{|z|^{1+2\delta}} \, dz\right)^{1/2} \\
\leq C \lambda^\delta \|f'\|_2 (1 + \|f'\|_2) \text{ by lemma 16.} \quad (49)
\end{align*}$$

Now, the change of variable $\lambda = \gamma |y|^2$ in (48) gives

$$\frac{J_1(y)}{y^4} = \inf_{f \in H^1(\mathbb{R}), \|f\|_2 = 1} \frac{\gamma^2}{2} \|f'\|_2^2 : \frac{1}{y^2} \int \int K\left(\frac{x-z}{\gamma y^2}\right)f^2(x)f^2(z) \, dx \, dz > \frac{1}{2}\right).$$

Let us fix $(f, \gamma) \in H^1(\mathbb{R}) \times ]0, \infty[$ such that $\|f\|_2 = 1$, and $\gamma \bar{K} \|f\|_4^4 > 1/2$. (48) implies that

$$\lim_{y \to 0} \frac{J_1(y)}{y^4} \leq \frac{\gamma^2}{2} \|f'\|_2^2.$$

Taking the infimum in $\gamma$ first, then in $f$, we obtain

$$\lim_{y \to 0} \frac{J_1(y)}{y^4} \leq \frac{I}{8 \bar{K}^2};$$
where $I \triangleq \text{Inf} \left\{ \frac{\|f\|^2_2}{\|f\|^4_4} : f \in H^1(\mathbb{R}), \|f\|_2 = 1 \right\} \in \mathbb{R}, +\infty$, by lemma 16.

For the opposite direction, we begin by rewriting the first equality in (45) with $\lambda = y^2$:

$$J_1(y) = \text{Inf} \left\{ \frac{\|f\|^2_2}{\|f\|^4_4} : \frac{1}{y^2} \int \int K\left(\frac{x-z}{y^2}\right) f^2(x) f^2(z) \, dx \, dz \geq \frac{1}{2} \right\}.$$ 

Let $\eta > 0$ and for each $y$, let $f_y$ satisfying the above constraints and $\frac{1}{2} \|f_y\|^2_2 \leq \frac{\|f_y\|^4_4}{y^4} + \eta$. Since $\limsup_{y \to 0} \frac{J_1(y)}{y^4} < \infty$, we also have $\limsup_{y \to 0} \|f'_y\|_2 < \infty$. Moreover, by (49),

$$K \|f_y\|^4_4 - \frac{1}{y^2} \int \int K\left(\frac{x-z}{y^2}\right) f^2_y(x) f^2_y(z) \, dx \, dz \leq C|y|^{2\delta} \|f_y\|_2 (1 + \|f_y\|_2),$$

so that $\lim_{y \to 0} \left| K \|f_y\|^4_4 - \frac{1}{y^2} \int \int K\left(\frac{x-z}{y^2}\right) f^2_y(x) f^2_y(z) \, dx \, dz \right| = 0$. Thus,

$$\liminf_{y \to 0} K \|f_y\|^4_4 \geq \frac{1}{2}.$$ 

Now, by definition of $I$, $\|f'_y\|^2_2 \geq I \|f_y\|^8_8$. Thus, $\liminf_{y \to 0} \frac{1}{2} \|f'_y\|^2_2 \geq \frac{I}{8K^2}$. This ends the proof of point 4. of proposition 1.

Acknowledgements. We would like to thank Francis Comets for having given us the right scaling.

References

[1] M. Avellaneda, A. Majda. Mathematical models with exact renormalization for turbulent transport. Commun. Math. Phys. 131 (1990), pp 381-429.

[2] M. Avellaneda, A. Majda. Mathematical models with exact renormalization for turbulent transport II. Commun. Math. Phys. 146 (1992), pp 139-204.

[3] F. Castell, F. Pradeilles. Annealed large deviations for diffusions in a random Gaussian shear flow drift. Stoc. Proc. and Appl. 94 (2001), pp 171-197.

[4] R. Carmona, Transport properties of Gaussian velocity fields. in Real and Stochastic Analysis. Probab. Stochastics Series. CRC, Boca Raton, FL (1997), pp 9-63.
[5] R. Carmona, L. Xu. *Homogenization for time dependent 2-D incompressible Gaussian flows*. Ann. Appl. Probab. **7** (1997), no 1, pp 265-279.

[6] J. Gärtner, W. König. *Moment asymptotics for the continuous parabolic Anderson model*. Ann. Appl. Probab. **10** (2000), no 1, 192–217.

[7] J. Gärtner, W. König, S. A. Molchanov. *Almost sure asymptotics for the continuous parabolic Anderson model*. Probab. Theory Related Fields **118** (2000), no 4, 547–573.

[8] C. Landim, S. Olla, H.T. Yau. *Convection-diffusion equation with space-time ergodic random flow*. Probab. Theory Related Fields **112** (1998), no 2, 203–220.

[9] S. Olla. *Homogenization of diffusion processes in random fields*. Cours de l’Ecole Polytechnique (1994).

[10] E. M. Stein. *Singular integral and differentiability properties of functions*. Princeton Mathematical series. No 30. Princeton University Press, Princeton, N.J. 1970.

[11] A. S. Sznitman. *Brownian motion, obstacles and random media*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.