Williams decomposition for superprocesses
Yan-Xia Ren∗ Renming Song† and Rui Zhang‡

Abstract
We decompose the genealogy of a general superprocess with spatially dependent branching mechanism with respect to the last individual alive (Williams decomposition). This is a generalization of the main result of Delmas and Hénard [4] where only superprocesses with spatially dependent quadratic branching mechanism were considered. As an application of the Williams decomposition, we prove that, for some superprocesses, the normalized total mass will converge to a point mass at its extinction time. This generalizes a result of Tribe [15] in the sense that our branching mechanism is more general.

AMS 2010 Mathematics Subject Classification: 60J25; 60G55; 60J80.

Keywords and Phrases: superprocesses; Williams decomposition; spatially dependent branching mechanism; genealogy.

1 Introduction
Let $X$ be a superprocess with a spatially dependent branching mechanism. We assume that the extinction time $H$ of $X$ is finite. In this paper we study the genealogical structure of $X$. More precisely, we give a spinal decomposition of $X$ involving the ancestral lineage of the last individual alive, conditioned on $H = h$ with $h > 0$ being a constant. This decomposition is called a Williams decomposition, in analogy with the terminology of Delmas and Hénard [4]. For a superprocess with spatially independent branching mechanism, the spatial motion is independent of the genealogical structure. As a consequence, the law of the ancestral lineage of the last individual alive does not depend on the original motion. Therefore, in this setting, the description of $X$ conditioned on $H = h$ may be deduced from Abraham and Delmas [1] where no spatial motion is taken into account. On the contrary, for a superprocess with nonhomogeneous branching mechanism, the law of the ancestral lineage of the last individual alive should depend on the spatial motion and the extinction time $h$. Delmas and Hénard [4] gave a Williams decomposition for superprocesses with a spatially dependent quadratic branching mechanism given by

$$\Psi(x, z) = \beta(x)z + \alpha(x)z^2,$$

under some conditions on $\beta(x)$ and $\alpha(x)$ (see $(H2)$ and $(H3)$ in [4]). In [4], the Williams decomposition was established for superprocesses with spatially dependent quadratic branching mechanism by using two transformations to change the branching mechanism $\Psi(x, z)$ to a spatially independent one, say $\psi_0$, and then using the genealogy of superprocesses with branching mechanism $\psi_0$ given

∗The research of this author is supported by NSFC (Grant Nos. 11271030 and 11671017)
†Research supported in part by a grant from the Simons Foundation (#429343, Renming Song).
‡The research of this author is supported by NSFC (Grant No. 11601354)
by the Brownian snake. As mentioned in [4], the drawback of the approach in [4] is that one has to restrict to quadratic branching mechanisms with bounded and smooth parameters.

The goal of this paper is to establish a Williams decomposition for more general superprocesses. Our superprocesses are more general in two aspects: first the spatial motion can be a general Markov process and secondly the branching mechanism is general and spatially dependent (see (2.1) below). We will give conditions that guarantee our general superprocesses admit a Williams decomposition. The conditions should be satisfied by a lot of superprocesses. We obtain a Williams decomposition by direct construction. For any fixed constant \( h > 0 \), we first describe the motion of a spine up to time \( h \) and then construct three kinds of immigrations (continuous immigration, jump immigration and immigration at time 0) alone the spine. We prove that, conditioned on \( H = h \), the sum of the contributions of the three types of immigrations has the same distribution as \( X \) before time \( h \), see Theorem 3.5 below. Note that for quadratic branching mechanisms, there is no jump immigration.

As an application of the Williams decomposition, we prove that, for some superprocesses, the normalized total mass will converge to a point mass at its extinction time, see Theorem 3.7 below. This generalizes a result of Tribe [15] in the sense that our branching mechanism is more general.

## 2 Preliminary

### 2.1 Superprocesses and assumptions

In this subsection, we describe the superprocesses we are going to work with and formulate our assumptions.

Suppose that \( E \) is a locally compact separable metric space. Let \( E_\partial := E \cup \{ \partial \} \) be the one-point compactification of \( E \). \( \partial \) will be interpreted as the cemetery point. Any function \( f \) on \( E \) is automatically extended to \( E_\partial \) by setting \( f(\partial) = 0 \).

Let \( D_E \) be the set of all the càdlàg functions from \([0, \infty)\) into \( E_\partial \) having \( \partial \) as a trap. The filtration is defined by \( F_t = F_t^0 \), where \( F_t^0 \) is the natural canonical filtration, and \( F = \bigvee_{t \geq 0} F_t \). Consider the canonical process \( \xi_t \) on \((D_E, \{ F_t \}_{t \geq 0})\). We will assume that \( \xi = \{ \xi_t, \Pi_x \} \) is a Hunt process on \( E \) and \( \zeta := \inf \{ t > 0 : \xi_t = \partial \} \) is the lifetime of \( \xi \). We will use \( \{ P_t : t \geq 0 \} \) to denote the semigroup of \( \xi \). We will use \( \mathcal{B}_0(E) \) (\( \mathcal{B}^+_0(E) \)) to denote the set of (non-negative) bounded Borel functions on \( E \). We will use \( \mathcal{M}_F(E) \) to denote the family of finite measures on \( E \) and \( \mathcal{M}_F(E)^0 \) to denote the family of non-trivial finite measures on \( E \).

Suppose that the branching mechanism is given by

\[
\Psi(x, z) = -\alpha(x)z + b(x)z^2 + \int_{(0, +\infty)} (e^{-zu} - 1 + zu) n(x, dy), \quad x \in E, \quad z > 0, \quad (2.1)
\]

where \( \alpha \in \mathcal{B}_0(E), b \in \mathcal{B}^+_0(E) \) and \( n \) is a kernel from \( E \) to \((0, \infty)\) satisfying

\[
\sup_{x \in E} \int_{(0, +\infty)} (y \wedge y^2) n(x, dy) < \infty. \quad (2.2)
\]

Then there exists a constant \( K > 0 \), such that

\[
|\alpha(x)| + b(x) + \int_{(0, +\infty)} (y \wedge y^2) n(x, dy) \leq K.
\]

Let \( \mathcal{M}_F(E) \) be the space of finite measures on \( E \), equipped with the topology of weak convergence. As usual, \( \langle f, \mu \rangle := \int_E f(x)\mu(dx) \) and \( \| \mu \| := \langle 1, \mu \rangle \). According to [13, Theorem 5.12], there
is a Hunt process \( X = \{\Omega, \mathcal{G}, \mathcal{G}_t, X_t, \mathbb{P}_\mu\} \) taking values in \( \mathcal{M}_F(E) \), such that, for every \( f \in \mathcal{B}_b^+(E) \) and \( \mu \in \mathcal{M}_F(E) \),

\[
- \log \mathbb{P}_\mu \left( e^{-\langle f, X_t \rangle} \right) = \langle u_f(t, \cdot), \mu \rangle,
\]

where \( u_f(t, x) \) is the unique positive solution to the equation

\[
u_f(t, x) + \Pi_x \int_0^t \Psi(\xi_s, u_f(t-s, \xi_s)) ds = \Pi_x f(\xi_t),
\]

where \( \Psi(\xi, z) = 0 \), \( z > 0 \). \( X = \{X_t : t \geq 0\} \) is called a superprocess with spatial motion \( \xi = \{\xi_t, \Pi_x\} \) and branching mechanism \( \Psi \), or sometimes a \((\Psi, \xi)\)-superprocess. In this paper, the superprocess we deal with is always this Hunt realization. For the existence of \( X \), see also [3] and [5].

Define \( v(t, x) := -\log \mathbb{P}_{\delta_x}(\|X_t\| = 0) \), and \( H := \inf\{t \geq 0 : \|X_t\| = 0\} \). It is obvious that \( v(0, x) = \infty \). In this paper, we will consider the critical and subcritical case. More precisely, throughout this paper, we assume that \( X \) satisfy the following uniform global extinction property.

**Assumption (H1)** For any \( t > 0 \),

\[
\sup_{x \in E} v(t, x) < \infty \quad \text{and} \quad \lim_{t \to \infty} v(t, x) = 0.
\]

**Remark 2.1** Note that Assumption (H1) is equivalent to

\[
\inf_{x \in E} \mathbb{P}_{\delta_x}(\|X_t\| = 0) > 0 \quad \text{for all} \quad t > 0 \quad \text{and} \quad \mathbb{P}_{\delta_x}(H < \infty) = \lim_{t \to \infty} \mathbb{P}_{\delta_x}(\|X_t\| = 0) = 1.
\]

**Remark 2.2** If

\[
\Psi(x, z) \geq \bar{\Psi}(z) := bz^2 + \int_0^\infty \left( e^{-yz} - 1 + yz \right) n(dy),
\]

where \( b \geq 0 \), \( \int_0^\infty (y \wedge y^2)n(dy) < \infty \) and \( \bar{\Psi} \) satisfies the Grey condition:

\[
\int_0^\infty \frac{1}{\bar{\Psi}(z)} dz < \infty,
\]

then Assumption (H1) holds.

We also assume that

**Assumption (H2)** For any \( x \in E \) and \( t > 0 \),

\[
w(t, x) := -\frac{\partial v}{\partial t}(t, x)
\]

exists. Moreover, for any \( 0 < r < t \),

\[
\sup_{r \leq s \leq t} \sup_{x \in E} w(s, x) < \infty.
\]

Note that, since \( t \to v(t, x) \) is decreasing, we have \( w(t, x) \geq 0 \). We also use \( v_t \) and \( w_t \) to denote the function \( x \to v(t, x) \) and \( x \to w(t, x) \) respectively.
Example 1 Assume that the spatial motion $\xi$ is conservative, that is $P_t(1) \equiv 1$, and the branching mechanism is spatially independent, that is

$$\Psi(x, z) = \Psi(z) = az + bz^2 + \int_0^\infty (e^{-yz} - 1 + yz)n(dy),$$

where $a \geq 0$, $b \geq 0$ and $\int_0^\infty (y \wedge y^2)n(dy) < \infty$. We also assume that $\Psi$ satisfies the Grey condition:

$$\int_0^\infty \frac{1}{\Psi(z)} dz < \infty.$$

Then $\{\|X_t\|, t \geq 0\}$ is a continuous state branching process with branching mechanism $\Psi(z)$. So $v(t, x) = v(t) < \infty$ does not depend on $x$, and $\lim_{t \to \infty} v(t) = 0$, thus Assumption (H1) holds immediately. Moreover, for $t > 0$, we have that

$$w(t) := -\frac{d}{dt}v(t) = \Psi(v(t)).$$

Thus Assumption (H2) is satisfied. See [10, Theorem 10.1] for more details.

In Section 5, we will give more examples, including some class of superdiffusions, that satisfy Assumptions (H1)-(H2).

2.2 Excursion law of $\{X_t, t \geq 0\}$

We use $\mathbb{D}$ to denote the space of $\mathcal{M}_F(E)$-valued càdlàg functions $t \mapsto \omega_t$ on $(0, \infty)$ having zero as a trap. We use $(\mathcal{A}, \mathcal{A}_t)$ to denote the natural $\sigma$-algebras on $\mathbb{D}$ generated by the coordinate process.

Let $\{Q_t(\mu, \cdot) := P_{\omega_t} (X_t \in \cdot) : t \geq 0, \mu \in \mathcal{M}_F(E)\}$ be the transition semigroup of $X$. Then by [2.3], we have

$$\int_{\mathcal{M}_F(E)} e^{-\langle f, \nu \rangle} Q_t(\mu, d\nu) = \exp\{-\langle V_t f, \mu \rangle\} \quad \text{for } \mu \in \mathcal{M}_F(E) \text{ and } t \geq 0,$$

where $V_t f(x) := u_f(t, x), x \in E$. This implies that $Q_t(\mu_1 + \mu_2, \cdot) = Q_t(\mu_1, \cdot) * Q_t(\mu_2, \cdot)$ for any $\mu_1, \mu_2 \in \mathcal{M}_F(E)$, and hence $Q_t(\mu, \cdot)$ is an infinitely divisible probability measure on $\mathcal{M}_F(E)$. By the semigroup property of $Q_t$, $V_t$ satisfies that

$$V_s V_t = V_{t+s} \quad \text{for all } s, t \geq 0.$$

Moreover, by the infinite divisibility of $Q_t$, each operator $V_t$ has the representation

$$V_t f(x) = \lambda_t(x, f) + \int_{\mathcal{M}_F(E)^0} \left(1 - e^{-\langle f, \nu \rangle}\right) L_t(x, d\nu) \quad \text{for } t > 0, \ f \in B^+_0(E), \tag{2.10}$$

where $\lambda_t(x, dy)$ is a bounded kernel on $E$ and $(1 \wedge \nu(1))L_t(x, d\nu)$ is a bounded kernel from $E$ to $\mathcal{M}_F(E)^0$. Let $Q_t^0$ be the restriction of $Q_t$ to $\mathcal{M}_F(E)^0$. Let $E_0 := \{x \in E : \lambda_t(x, E) = 0 \text{ for all } t > 0\}$.

For $\lambda > 0$, we use $V_t \lambda$ to denote $V_t f$ when the function $f \equiv \lambda$. It then follows from (2.10) that for every $x \in E$ and $t > 0$,

$$V_t \lambda(x) = \lambda_t(x, E) \lambda + \int_{\mathcal{M}_F(E)^0} \left(1 - e^{-\lambda(1, \nu)}\right) L_t(x, d\nu).$$


The left hand side tends to $-\log \mathbb{P}_{\delta_x}(X_t = 0)$ as $\lambda \to +\infty$. Therefore, Assumption (H1) implies that $\lambda_t(x, E) = 0$ for all $t > 0$ and hence $x \in E_0$, which says that $E = E_0$.

For $x \in E$, we get from (2.10) that

$$V_t f(x) = \int_{\mathcal{M}_F(E)^0} \left( 1 - e^{-\langle f, \nu \rangle} \right) L_t(x, d\nu) \quad \text{for } t > 0, \quad f \in \mathcal{B}_b^+(E).$$

It then follows from [13] Proposition 2.8 and Theorem A.40 that for every $x \in E$, the family of measures $\{L_t(x, \cdot) : t > 0\}$ on $\mathcal{M}_F(E)^0$ constitutes an entrance law for the restricted semigroup $\{Q_t^0 : t \geq 0\}$. It is known (see [13] Section 8.4) that one can associate with $\{\mathbb{P}_{\delta_x} : x \in E\}$ a family of $\sigma$-finite measures $\{N_x : x \in E\}$ defined on $(\mathcal{D}, \mathcal{A})$ such that $N_x(\{0\}) = 0$,

$$\int_{\mathcal{D}} (1 - e^{-\langle f, \omega \rangle}) N_x(d\omega) = -\log \mathbb{P}_{\delta_x}(e^{-\langle f, X_t \rangle}), \quad f \in \mathcal{B}_b^+(E), \quad t > 0, \quad (2.11)$$

and, for every $0 < t_1 < \cdots < t_n < \infty$, and nonzero $\mu_1, \ldots, \mu_n \in M_F(E), \quad (2.12)$

$$N_x(\omega_{t_1} \in d\mu_1, \cdots, \omega_{t_n} \in d\mu_n) = N_x(\omega_{t_1} \in d\mu_1) \mathbb{P}_{\mu_1}(X_{t_2 - t_1} \in d\mu_2) \cdots \mathbb{P}_{\mu_{n-1}}(X_{t_n - t_{n-1}} \in d\mu_n).$$

This measure $N_x$ is called the Kuznetsov measure corresponding to the entrance law $\{L_t(x, \cdot) : t > 0\}$ or the excursion law for superprocess $X$. For earlier work on excursion law of superprocesses, see [6] [8] [12].

It follows from (2.11) that for any $t > 0$,

$$N_x(\|\omega_t\| \neq 0) = -\log \mathbb{P}_{\delta_x}(\|X_t\| = 0) < \infty. \quad (2.13)$$

3 Main results

In this and the next section we will always assume that Assumptions (H1)-(H2) hold.

Recall that $H := \inf \{t \geq 0 : \|X_t\| = 0\}$. Note that

$$F_H(t) := \mathbb{P}_{\mu}(H \leq t) = \mathbb{P}_{\mu}(\|X_t\| = 0) = e^{-\langle v_t, \mu \rangle}. \quad (3.1)$$

By the continuity of $v(t, x)$ with respect to $t \in (0, \infty)$, we get that for any $t > 0$,

$$\mathbb{P}_{\mu}(H < t) = \lim_{\epsilon \downarrow 0} \mathbb{P}_{\mu}(H \leq t - \epsilon) = \lim_{\epsilon \downarrow 0} e^{-\langle v_{t-\epsilon}, \mu \rangle} = e^{-\langle v_t, \mu \rangle} = \mathbb{P}_{\mu}(H \leq t). \quad (3.2)$$

For $h > 0$, define

$$M^h_t := \frac{\langle w_{t-h}, X_t \rangle e^{-\langle w_{t-h}, X_t \rangle}}{\langle w_h, X_0 \rangle e^{-\langle w_h, X_0 \rangle}}, \quad 0 \leq t < h. \quad (3.3)$$

Then, under $\mathbb{P}_{\mu}$, $\{M^h_t, 0 \leq t < h\}$ is a nonnegative martingale with mean one (see Lemma 4.2 below).

**Theorem 3.1** For any $h > 0$ and $t < h$,

$$\lim_{\epsilon \downarrow 0} \mathbb{P}_{\mu}(A | h \leq H < h + \epsilon) = \mathbb{P}_{\mu}(1_A M^h_t), \quad \forall A \in \mathcal{F}_t.$$
We define, for each \( h > 0 \),
\[
\mathbb{P}_\mu(H = h) := \lim_{\epsilon \downarrow 0} \mathbb{P}_\mu(H < h + \epsilon).
\]

Then, by Theorem 3.1, \( \{X_t, t < h; \mathbb{P}_\mu(H = h)\} \) has the same law as \( \{X_t, t < h; \mathbb{P}_\mu^h\} \), where \( \mathbb{P}_\mu^h \) is a new measure defined via the martingale \( M_t^h \):
\[
\frac{d\mathbb{P}_\mu^h}{d\mathbb{P}_\mu} |_{\mathcal{G}_t} = M_t^h, \quad t < h.
\]

**Corollary 3.2** For any \( A \in \mathcal{G}_t \), we have
\[
\mathbb{P}_\mu(A \cap \{H > t\}) = \int_t^\infty \mathbb{P}_\mu^h(A) F_H(dh).
\]

**Proof:** It follows from Fubini’s theorem that
\[
\int_t^\infty \mathbb{P}_\mu^h(A) F_H(dh) = \int_t^\infty \mathbb{P}_\mu(1_A M_t^h) F_H(dh)
\]
\[
= \int_t^\infty \mathbb{P}_\mu(1_A \langle w_{h-t}, X_t \rangle e^{-(v_{h-t}, X_t)}) dh
\]
\[
= \mathbb{P}_\mu(1_A \int_t^\infty \langle w_{h-t}, X_t \rangle e^{-(v_{h-t}, X_t)} dh)
\]
\[
= \mathbb{P}_\mu(1_A \int_0^\infty \langle w_h, X_t \rangle e^{-(v_h, X_t)} dh)
\]
\[
= \mathbb{P}_\mu(A \cap \{X_t \neq 0\}) = \mathbb{P}_\mu(A \cap \{H > t\}),
\]
where in the fifth equality we use the fact that
\[
\int_0^\infty \langle w_h, X_t \rangle e^{-(v_h, X_t)} dh = \lim_{h \to \infty} e^{-(v_h, X_t)} - \lim_{h \to 0} e^{-(v_h, X_t)} = 1_{\{X_t \neq 0\}}.
\]

For any \( h > 0 \) and \( t \in [0, h) \), we define
\[
Y_t^h := \frac{w(h-t, x)}{w(h, x_0)} e^{-\int_0^t \Psi'_z(x_0, v(h-u, x_u)) du},
\]
where \( \Psi'_z(x, z) = \frac{\partial \Psi(x, z)}{\partial z} \). Then we have the following result whose proof will be given in Section 4.

**Lemma 3.3** Under \( \Pi_x \), \( \{Y_t^h, t < h\} \) is a nonnegative martingale satisfying \( \Pi_x(Y_t^h) = 1 \).

**Remark 3.4** In Example 4, \( w(t, x) \) and \( v(t, x) \) do not depend on \( x \), and for any \( h > 0 \) and \( 0 \leq t < h \), \( Y_t^h \equiv 1 \).

Now we state our main result: the Williams decomposition. We will construct a new process \( \{\Lambda_t^h, t < h\} \) which has the same law as \( \{X_t, t < h; \mathbb{P}_\mu(H = h)\} \).

Let \( \mathcal{F}_{h-} := \bigvee_{t<h} \mathcal{F}_t \). Now we define a new probability measure \( \Pi_x^h \) on \((\mathcal{D}_E, \mathcal{F}_{h-})\) by
\[
\Pi_x^h \bigg|_{\mathcal{F}_t} := Y_t^h, \quad t \in [0, h).
\]
Under $\Pi^h_x$, $(\xi_t)_{0 \leq t < h}$ is a conservative Markov process. If $\nu$ is a probability measure on $E$, we define

$$\Pi^h_\nu := \int_E \Pi^h_x \nu(dx).$$

Then, under $\Pi^h_\nu$, $(\xi_t)_{0 \leq t < h}$ is a Markov process with initial measure $\nu$.

We put

$$H(\omega) := \inf\{t > 0 : \|\omega_t\| = 0\}, \quad \omega \in \mathcal{D}.$$

Let $\xi^h := \{(\xi_t)_{0 \leq t < h}, \Pi^h_\nu\}$, where $\nu(dx) = \frac{u(h,x)}{\mu(h,c,\nu)} \mu(dx)$. Given the trajectory of $\xi^h$, we define three processes as follows:

**Continuous immigration** Suppose that $\mathcal{N}^{1,h}(ds, d\omega)$ is a Poisson random measure on $[0, h) \times \mathcal{D}$ with intensity measure $21_{[0,h)}(s)1_{H(\omega)<h-s}b(\xi_s)\nu(\omega)ds$. Define, for $t \in [0, h)$,

$$X^{1,h,N}_t := \int_{[0,t]} \int_{\mathcal{D}} \omega_{t-s} \mathcal{N}^{1,h}(ds, d\omega). \quad (3.4)$$

**Jump immigration** Suppose that $\mathcal{N}^{2,h}(ds, d\omega)$ is a Poisson random measure on $[0, h) \times \mathcal{D}$ with intensity measure $1_{[0,h)}(s)1_{H(\omega)<h-s} \int_0^\infty g(n(\xi_s, dy))\mathbb{P}_{y\delta_\omega}(X \in d\omega)ds$. Define, for $t \in [0, h)$,

$$X^{2,h,P}_t := \int_0^t \int_{\mathcal{D}} \omega_{t-s} \mathcal{N}^{2,h}(ds, d\omega). \quad (3.5)$$

**Immigration at time 0** Let $\{X^{0,h}_t, 0 \leq t < h\}$ be a process distributed according to the law $\mathbb{P}_\mu(X \in \cdot | H < h)$.

We assume that the three processes $X^{0,h}$, $X^{1,h,N}$ and $X^{2,h,P}$ are independent given the trajectory of $\xi^h$. Define

$$\Lambda^h_t := X^{0,h}_t + X^{1,h,N}_t + X^{2,h,P}_t. \quad (3.6)$$

We write the law of $\Lambda^h$ as $\mathbb{P}_\mu^{(h)}$.

**Theorem 3.5** Under $\mathbb{P}_\mu^{(h)}$, the process $\{\Lambda^h_t, t < h\}$ has the same law as $\{X_t, t < h\}$ conditioned on $H = h$.

If we define $\Lambda^h_t = 0$, for any $t \geq h$, then we get the following result.

**Corollary 3.6** $\{X_t; \mathbb{P}_\mu\}$ has the same finite dimensional distribution as

$$\int_0^\infty \mathbb{P}_\mu^{(h)}(\Lambda^h \in \cdot) F_H(dh).$$

**Proof:** Let $f_k \in \mathcal{B}_b^+(E)$, $k = 1, 2, \ldots, n$ and $0 = t_0 < t_1 < t_2 < \cdots < t_n$. We put $t_{n+1} = \infty$ and define $(t_n, t_{n+1}] := (t_n, \infty)$. We will show that

$$\mathbb{P}_\mu\left( \exp\left\{-\sum_{j=1}^n (f_j, X_{t_j})\right\}\right) = \int_{(0,\infty)} \mathbb{P}_\mu^{(h)}\left( \exp\left\{-\sum_{j=1}^n (f_j, \Lambda^h_{t_j})\right\}\right) F_H(dh).$$
Since $\Lambda_t^h = 0$, for $t \geq h$, we get that

$$\int_{(0, \infty)} P^{(h)}_\mu \left( \exp \left\{ -\sum_{j=1}^n \langle f_j, \Lambda_{t_j}^h \rangle \right\} \right) F_H(dh)$$

$$= \sum_{r=0}^n \int_{(t_r, t_{r+1}]} P^{(h)}_\mu \left( \exp \left\{ -\sum_{j=1}^r \langle f_j, \Lambda_{t_j}^h \rangle \right\} \right) F_H(dh)$$

$$= \sum_{r=0}^n \int_{(t_r, t_{r+1}]} P^{(h)}_\mu \left( \exp \left\{ -\sum_{j=1}^r \langle f_j, X_{t_j} \rangle \right\} ; t_r < H \leq t_{r+1} \right)$$

$$= \mathbb{P}_\mu \left( \exp \left\{ -\sum_{j=1}^n \langle f_j, X_{t_j} \rangle \right\} \right),$$

where the second equality follows from Theorem 3.5, and the third equality follows from Corollary 3.2. The proof is now complete. □

The decomposition (3.6) is called a Williams decomposition or spinal decomposition of the superprocess $\{X_t, t < h\}$ conditioned on $H = h$, and $\xi^h = \{(\xi_t)_{0 \leq t < h}, \Pi^h_\nu\}$ is called the spine of the decomposition. It gives us a tool to study the behavior of the superprocesses $X$ near extinction, see Theorem 3.7 below. To state Theorem 3.7 we need the following assumption:

(H3) For any bounded open set $B \subset E$ and any $t > 0$, the function

$$x \rightarrow -\log \mathbb{P}_{\delta_x} \left( \int_0^t X_s(B^c) \, ds = 0 \right)$$

is finite for $x \in B$ and locally bounded.

**Theorem 3.7** Assume that (H1)-(H3) hold and that for any $\mu \in \mathcal{M}_F(E)$,

$$\lim_{t \uparrow h} \xi_t = \xi_{h-}, \quad \Pi^h_\nu \text{-a.s.,} \quad (3.7)$$

where $\nu(dx) = \frac{w(h,x)}{(w(h),x)}\mu(dx)$. Then there exists an $E$-valued random variable $Z$ such that

$$\lim_{t \uparrow H} \|X_t\| = \delta_Z, \quad \mathbb{P}_\mu \text{-a.s.,}$$

where the limit above is in the sense of weak convergence. Moreover, conditioned on $\{H = h\}$, $Z$ has the same law as $\{\xi_{h-}, \Pi^h_\nu\}$, that is, for any $f \in C_0^b(E)$,

$$\mathbb{P}_\mu f(Z) = \int_0^\infty \Pi^h_\nu(f(\xi_{h-})) F_H(dh). \quad (3.8)$$

Note that, if the martingale $\{Y_t^h, 0 \leq t < h\}$ is uniformly integrable, then condition (3.7) holds. Now we give an example that satisfies Assumption (H3).
Example 2 Assume that $\xi$ is a diffusion on $\mathbb{R}^d$ with infinitesimal generator

$$L = \sum a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum b_j(x) \frac{\partial}{\partial x_j},$$

which satisfies the following two conditions:

(A) (Uniform ellipticity) There exists a constant $\gamma > 0$ such that

$$\sum a_{ij}(x) u_i u_j \geq \gamma \sum u_j^2, \quad x \in \mathbb{R}^d.$$

(B) $a_{ij}$ and $b_j$ are bounded H"older continuous functions.

Suppose that the branching mechanism $\Psi(x, z)$ satisfies that, for some $\alpha \in (1, 2]$ and $c > 0$, $\Psi(x, z) \geq cz^\alpha$ for all $x \in \mathbb{R}^d$.

Let $\{X_t, \mathbb{P}_\mu\}$ and $\{\tilde{X}_t, \mathbb{P}_\mu\}$ be a $(\xi, \Psi)$-superprocess and a $(\xi, z^\alpha)$-superprocess respectively. Then, for any open set $B \subset \mathbb{R}^d$,

$$-\log \mathbb{P}_{\delta_x} \left( \exp \left\{ -\lambda \int_0^t X_s(B^c) \, ds \right\} \right) \leq -\log \mathbb{P}_{\delta_x} \left( \exp \left\{ -\lambda \int_0^t \tilde{X}_s(B^c) \, ds \right\} \right) \leq -\log \mathbb{P}_{\delta_x}(R \subset B), \quad \text{(3.9)}$$

where $R$ is the range of $\tilde{X}$, which is the minimal closed subset of $\mathbb{R}^d$ which supports all the measures $\tilde{X}_t$, $t \geq 0$. Thus, we have that

$$-\log \mathbb{P}_{\delta_x} \left( \int_0^t X_s(B^c) \, ds = 0 \right) = \lim_{\lambda \to \infty} -\log \mathbb{P}_{\delta_x} \left( \exp \left\{ -\lambda \int_0^t X_s(B^c) \, ds \right\} \right) \leq -\log \mathbb{P}_{\delta_x}(R \subset B).$$

By [5, Theorem 8.1], $x \to -\log \mathbb{P}_{\delta_x}(R \subset B)$ is continuous in $x \in B$. Therefore the superprocess $X$ satisfies Assumption (H3).

Remark 3.8 Now we consider the superprocess in Example 4. We assume that $\xi$ is a diffusion in $\mathbb{R}^d$ satisfying the conditions in Example 2 and the branching mechanism $\Psi(z)$ satisfies that, for some $\alpha \in (1, 2]$ and $c > 0$, $\Psi(z) \geq cz^\alpha$. Thus, Assumption (H3) holds. Since $Y^h_1 = 1$ and $\Pi^h_\alpha = \Pi_x$, condition (3.7) holds automatically. Therefore, Theorem 3.7 holds and $Z$ has the same law as $\xi_H$, where $\xi_0 \sim \nu(dx) = \mu(dx)/\|\mu\|$. Moreover, $\xi$ and $H$ are independent.

Compared with [15], the example above assumes that the spatial motion $\xi$ is a diffusion, while in [15], the spatial motion is a Feller process. However, in [15], the branching mechanism is binary ($\Psi(z) = z^2$), while in the example above, the branching mechanisms is more general.

4 Proofs of Main Results

We will use $\mathbb{P}_{r, \delta_x}$ to denote the law of $X$ starting from the unit mass $\delta_x$ at time $r > 0$. Similarly, we will use $\Pi_{r, x}$ to denote the law of $\xi$ starting from $x$ at time $r > 0$. First, we give an useful lemma.

Lemma 4.1 Suppose that $f \in \mathcal{B}_b^+(E)$ and $g_i \in \mathcal{B}_b^+(E)$, $i = 1, 2, \cdots, n$. For any $0 < t_1 \leq t_2 \leq \cdots \leq t_n$ and $0 \leq r \leq t_n$, we have

$$\mathbb{P}_{r, \mu} \left( \langle f, X_{t_n} \rangle \exp \left\{ -\sum_{j: t_j \geq r} \langle g_j, X_{t_j} \rangle \right\} \right)$$
\[
\begin{align*}
\int_E \Pi_{r,x} \left( \exp \left\{ - \int_r^{t_n} \Psi_t^f(\xi_u, U_g(u, \xi_u)) \, du \right\} f(\xi_{t_n}) \right) \mu(dx) e^{-(U_g(r), \mu)}, \\
\end{align*}
\]

where

\[ U_g(r, x) := - \log \mathbb{P}_{r, \delta x} \left( \exp \left\{ - \sum_{j:t_j \geq r} \langle g_j, X_{t_j} \rangle \right\} \right). \]

In particular, for any \( f \in \mathcal{B}_b^+(E) \) and \( g \in \mathcal{B}_b^+(E) \), we have

\[ \mathbb{P}_{\delta x} \left( \langle f, X_{t_n} \rangle e^{-(g, X_{t_n})} \right) = \Pi_x \left( \exp \left\{ - \int_0^r \Psi_t^f(\xi_u, u_g(t - u, \xi_u)) \, du \right\} f(\xi_t) \right) e^{-u_g(t, x)}. \]

Proof: By [13, Proposition 5.14], we have that, for \( 0 \leq r \leq t_n \),

\[ - \log \mathbb{P}_{r, \mu} \left( \exp \left\{ - \sum_{j:t_j \geq r} \langle g_j, X_{t_j} \rangle - \theta \langle f, X_{t_n} \rangle \right\} \right) = \langle F_\theta(r, \cdot), \mu \rangle, \]

where \( F_\theta(r, x) \) is the unique bounded positive solution on \([0, t_n] \times E\) of

\[ F_\theta(r, x) + \Pi_{r,x} \int_r^{t_n} \Psi_t^f(\xi_u, F_\theta(u, \xi_u)) \, du = \sum_{j:t_j \geq r} \Pi_{r,x} g_j(\xi_{t_j}) + \theta \Pi_{r,x} f(\xi_{t_n}). \]

Let \( F_\theta'(r, x) := \frac{\partial}{\partial \theta} F_\theta(r, x) \). Then,

\[ \mathbb{P}_{r, \mu} \left( \langle f, X_{t_n} \rangle \exp \left\{ - \sum_{j:t_j \geq r} \langle g_j, X_{t_j} \rangle \right\} \right) = - \frac{\partial}{\partial \theta} e^{-\langle F_\theta(r, \cdot), \mu \rangle} \bigg|_{\theta=0^+} = \langle F_\theta'(r, \cdot), \mu \rangle e^{-(U_g(r), \mu)}. \]

Differentiating both sides of (4.3) with respect to \( \theta \) and then letting \( \theta \to 0 \), we get that

\[ F_\theta'(r, x) + \Pi_{r,x} \int_r^{t_n} \Psi_t^f(\xi_u, U_g(u, \xi_u)) F_\theta'(u, \xi_u) \, du = \Pi_{r,x} f(\xi_{t_n}), \]

which implies that

\[ F_\theta'(r, x) = \Pi_{r,x} \left[ e^{-\int_r^{t_n} \Psi_t^f(\xi_u, U_g(u, \xi_u)) \, du} f(\xi_{t_n}) \right]. \]

Therefore (4.1) holds. \( \square \)

Recall that \( v(t, x) := - \log \mathbb{P}_{\delta x} (\|X_t\| = 0) \) and \( w(t, x) := - \frac{\partial}{\partial \tau} (t, x) \geq 0 \). Recall the definition of \( M_t^h \) in (3.3).

**Lemma 4.2** Under \( \mathbb{P}_\mu \), \( \{M_t^h, t < h\} \) is a nonnegative martingale with \( \mathbb{P}_\mu (M_t^h) = 1 \).

Proof: For any \( h > 0 \) and \( 0 \leq t < h \), by Assumption (H2) and the dominated convergence theorem, we get that

\[ \mathbb{P}_\mu \left[ (w_{h-t}, X_t) e^{-(v_{h-t}, X_t)} \right] = \frac{\partial}{\partial h} \mathbb{P}_\mu e^{-(v_{h-t}, X_t)} = \frac{\partial}{\partial h} e^{-(v_{h}, \mu)} = \langle w_h, \mu \rangle e^{-(v_{h}, \mu)}, \]

where in the second equality, we used the Markov property of \( X \). Thus, it follows that \( \mathbb{P}_\mu (M_t^h) = 1 \).
By the Markov property of $X$, we obtain that, for $s < t < h$,

$$
\mathbb{P}_\mu \left[ (w_{h-t}, X_t)e^{-\langle v_{h-t}, X_t \rangle} \mid \mathcal{G}_s \right] = \mathbb{P}_\mu \left[ (w_{h-t}, X_{t-s})e^{-\langle v_{h-t}, X_{t-s} \rangle} \right] = \langle w_{h-s}, X_s \rangle e^{-\langle v_{h-s}, X_s \rangle},
$$

which implies that, under $\mathbb{P}_\mu$, $\{M^h_t, t < h\}$ is a nonnegative martingale. The proof is complete.  $\square$

**Proof of Theorem 3.1** For any $A \in \mathcal{G}_t$, by the Markov property of $X$,

$$
\mathbb{P}_\mu(A \mid h \leq H < h + \epsilon) = \frac{\mathbb{P}_\mu(A \cap \{h \leq H < h + \epsilon\})}{\mathbb{P}_\mu(h \leq H < h + \epsilon)} = \frac{\mathbb{P}_\mu(1_A \mathbb{P}_{X_t}(h - t \leq H < h - t + \epsilon))}{e^{-\langle v_{h+\epsilon}, \mu \rangle} - e^{-\langle v_h, \mu \rangle}} = \mathbb{P}_\mu(1_A \left( e^{-\langle v_{h+t+\epsilon}, X_t \rangle} - e^{-\langle v_{h-t}, X_t \rangle} \right)) / e^{-\langle v_{h+\epsilon}, \mu \rangle} - e^{-\langle v_h, \mu \rangle}.
$$

By Assumption (H2), we get that

$$
\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left( e^{-\langle v_{h+\epsilon}, \mu \rangle} - e^{-\langle v_h, \mu \rangle} \right) = \langle w_h, \mu \rangle e^{-\langle v_h, \mu \rangle} \quad (4.5)
$$

and

$$
\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left( e^{-\langle v_{h-t+\epsilon}, X_t \rangle} - e^{-\langle v_{h-t}, X_t \rangle} \right) = \langle w_{h-t}, X_t \rangle e^{-\langle v_{h-t}, X_t \rangle} \quad (4.6)
$$

Note that, for $0 < \epsilon < 1$,

$$
\frac{1}{\epsilon} \left( e^{-\langle v_{h-t+\epsilon}, X_t \rangle} - e^{-\langle v_{h-t}, X_t \rangle} \right) \leq \frac{1}{\epsilon} \left( 1 - \exp\{-\langle v_{h-t} - v_{h-t+\epsilon}, X_t \rangle\} \right) \leq \sup_{h - t \leq s \leq h - t + 1} \sup_{x \in E} w(s, x) \langle 1, X_t \rangle.
$$

Thus, it follows from the dominated convergence theorem that

$$
\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{P}_\mu \left( 1_A(e^{-\langle v_{h-t+\epsilon}, X_t \rangle} - e^{-\langle v_{h-t}, X_t \rangle}) \right) = \mathbb{P}_\mu(1_A \langle w_{h-t}, X_t \rangle e^{-\langle v_{h-t}, X_t \rangle}). \quad (4.7)
$$

Thus, by (4.5) and (4.7), we have that

$$
\lim_{\epsilon \downarrow 0} \mathbb{P}_\mu(A \mid h \leq H < h + \epsilon) = \mathbb{P}_\mu(1_A M^h_t).
$$

The proof is now complete.  $\square$

**Proof of Lemma 3.3** By the Markov property of $X$, we get that,

$$
e^{-v(t+s,x)} = \mathbb{P}_{\delta_x}(X_{t+s} = 0) = \mathbb{P}_{\delta_x}(\mathbb{P}_{X_t}(X_s = 0)) = \mathbb{P}_{\delta_x}(e^{-\langle v_s, X_t \rangle}), \quad (4.8)
$$

which implies that $u_{v_t}(t, x) = v(t+s, x)$. By (4.4) with $h = t+s$ and $\mu = \delta_x$, we get that

$$
w(t+s, x)e^{-v(t+s,x)} = \mathbb{P}_{\delta_x}(\langle w_s, X_t \rangle e^{-\langle v_s, X_t \rangle}) = \Pi_x \left( \exp \left\{ - \int_0^t \Psi_x(\xi_u, v(t+s-u, \xi_u)) \, du \right\} w(s, \xi_t) \right) e^{-v(t+s, x)},
$$

where

$$
\frac{1}{h-t} \left( (h-t) - \int_0^{h-t} \Psi_x(\xi_u, v(t+s-u, \xi_u)) \, du \right) \mathbb{P}_\mu(1_A \langle w_{h-t}, X_t \rangle e^{-\langle v_{h-t}, X_t \rangle}).
$$
where in the last equality we used Lemma 4.1 and the fact that \( u_{v_s}(t, x) = v(t + s, x) \). Thus, it follows immediately that

\[
    w(t + s, x) = \Pi_x \left( \exp \left\{ - \int_0^t \Psi_s(\xi_u, v(t + s - u, \xi_u)) \, du \right\} w(s, \xi_t) \right). \tag{4.9}
\]

For \( 0 < s < t \), by the Markov property of \( \xi \), we have that

\[
    \Pi_x \left( w(h - t, \xi_t)e^{-\int_s^t \Psi_s(\xi_u, v(h - u, \xi_u)) \, du} | \mathcal{F}_s \right) = e^{-\int_s^t \Psi_s(\xi_u, v(h - u, \xi_u)) \, du} \Pi_x \left( w(h - t, \xi_t)e^{-\int_s^{t-s} \Psi_s(\xi_u, v(h - u, \xi_u)) \, du} \right).
\]

\[
= e^{-\int_s^t \Psi_s(\xi_u, v(h - u, \xi_u)) \, du} \Pi_x \left( w(h - t, \xi_{t-s})e^{-\int_s^{t-s} \Psi_s(\xi_u, v(h - u, \xi_u)) \, du} \right).
\]

where the last equality above follows from (4.9). The proof is now complete. \( \square \)

4.1 Williams decomposition

**Proof of Theorem 3.5** Let \( f_k \in \mathcal{B}_0^+(E) \), \( k = 1, 2, \ldots, n \) and \( 0 = t_0 < t_1 < t_2 < \cdots < t_n = t < h \). We will show that

\[
    \mathbb{P}_\mu^h \left( \exp \left\{ - \sum_{j=1}^n \langle f_j, X_{t_j} \rangle \right\} \right) = \mathbb{P}_\mu^{(h)} \left( \exp \left\{ - \sum_{j=1}^n \langle f_j, \Lambda_{t_j}^h \rangle \right\} \right).
\]

By the definition of \( \Lambda_{t_j}^h \), we have

\[
\mathbb{P}_\mu^{(h)} \left( \exp \left\{ - \sum_{j=1}^n \langle f_j, \Lambda_{t_j}^h \rangle \right\} \right) = \int_E \frac{w(h, x)}{w(h, \cdot) \mu(x)} \mu(dx) \Pi_x^h \left[ \mathbb{P}_\mu^{(h)} \left( \exp \left\{ - \sum_{j=1}^n \langle f_j, \Lambda_{t_j}^h \rangle \right\} | \xi^h \right) \right]. \tag{4.10}
\]

By the construction of \( \Lambda_{t_j}^h \), we have

\[
\mathbb{P}_\mu^{(h)} \left( \exp \left\{ - \sum_{j=1}^n \langle f_j, \Lambda_{t_j}^h \rangle \right\} | \xi^h \right) = \mathbb{P}_\mu \left( \exp \left\{ - \sum_{j=1}^n \langle f_j, X_{t_j} \rangle \right\} \right) \times \mathbb{P}_\mu^{(h)} \left( \exp \left\{ - \sum_{j=1}^n \langle f_j, X_{t_j}^{1,h,N} \rangle \right\} | \xi^h \right) \times \mathbb{P}_\mu^{(h)} \left( \exp \left\{ - \sum_{j=1}^n \langle f_j, X_{t_j}^{2,h,P} \rangle \right\} | \xi^h \right)
\]

\[
= (I) \times (II) \times (III). \tag{4.11}
\]

Define, for \( s < h \),

\[
J_s(h, x) := - \log \mathbb{P}_\delta \left[ e^{-\sum_{j=1}^n \langle f_j, X_{t_j-s} \rangle 1_{s \leq t_j}; \|X_{h-s}\| = 0} \right]. \tag{4.12}
\]
We first deal with part (I). By (4.12), we have

\[ J_0(h, x) = -\log \mathbb{P}_{\delta_x} \left( \exp \left\{ - \sum_{j=1}^{n} (f_j, X_{t_j}) \right\}; \|X_h\| = 0 \right), \tag{4.13} \]

By (3.2), \( \mathbb{P}_\mu (H < h) = \mathbb{P}_\mu (H \leq h) = e^{-(\nu(H), \mu)} \). Thus we have

\[ (I) = e^{(\nu(H), \mu)} e^{-J_0(h, \cdot, \mu)}. \tag{4.14} \]

Next we deal with part (II). By the definition of \( X^{1,h,N} \) and Fubini’s theorem, we have

\[
\sum_{j=1}^{n} (f_j, X_{t_j}^{1,h,N}) = \sum_{j=1}^{n} \int_0^t \int_D (f_j, \omega_{t_{j-s}}) 1_{s < t_{j}} N^{1,h}(ds, d\omega) \\
= \int_0^t \int_D \sum_{j=1}^{n} (f_j, \omega_{t_{j-s}}) 1_{s < t_{j}} N^{1,h}(ds, d\omega). \tag{4.15}
\]

Therefore,

\[
(II) = \mathbb{P}_\mu^{(h)} \left( \exp \left\{ - \int_0^t \int_D \sum_{j=1}^{n} (f_j, \omega_{t_{j-s}}) 1_{s < t_{j}} N^{1,h}(ds, d\omega) \right\} | \mathcal{F}_h \right) \\
= \exp \left\{ - \int_0^t 2b(\xi_s) ds \int_\mathbb{D} \left( 1 - e^{-\sum_{j=1}^{n} (f_j, \omega_{t_{j-s}}) 1_{s < t_{j}}} \right) 1_{H(H) < h-s} N_{\xi_s}(d\omega) \right\}. 
\]

By the dominated convergence theorem, we obtain that, for \( s \neq t_j, j = 1, 2, \ldots, n, \)

\[
\int_\mathcal{D} \left( 1 - e^{-\sum_{j=1}^{n} (f_j, \omega_{t_{j-s}}) 1_{s < t_{j}}} \right) 1_{H(H) < h-s} N_{\xi_s}(d\omega) \\
= \int_\mathcal{D} \left( 1 - e^{-\sum_{j=1}^{n} (f_j, \omega_{t_{j-s}}) 1_{s < t_{j}}} \right) 1_{\|X_{h-s}\| = 0} N_{\xi_s}(d\omega) \\
= \lim_{\theta \to \infty} \int_\mathcal{D} \left( 1 - e^{-\sum_{j=1}^{n} (f_j, \omega_{t_{j-s}}) 1_{s < t_{j}}} \right) e^{-\theta \|X_{h-s}\|} N_{\xi_s}(d\omega) \\
= \lim_{\theta \to \infty} \int_\mathcal{D} \left( 1 - e^{-\sum_{j=1}^{n} (f_j, \omega_{t_{j-s}}) 1_{s < t_{j}}} \right) N_{\xi_s}(d\omega) - \int_\mathcal{D} \left( 1 - e^{-\theta \|X_{h-s}\|} \right) N_{\xi_s}(d\omega) \\
= \lim_{\theta \to \infty} - \log \mathbb{P}_{\delta_{\xi_s}} e^{-\sum_{j=1}^{n} (f_j, X_{t_{j-s}}) 1_{s < t_{j}} - \theta \|X_{h-s}\|} + \log \mathbb{P}_{\delta_{\xi_s}} e^{-\theta \|X_{h-s}\|} \\
= - \log \mathbb{P}_{\delta_{\xi_s}} \left[ e^{-\sum_{j=1}^{n} (f_j, X_{t_{j-s}}) 1_{s < t_{j}}}; \|X_{h-s}\| = 0 \right] + \log \mathbb{P}_{\delta_{\xi_s}} (\|X_{h-s}\| = 0) \\
= J_s(h, \xi_s) - v(h - s, \xi_s).
\]

Hence,

\[ (II) = \exp \left\{ - \int_0^t 2b(\xi_s) \left( J_s(h, \xi_s) - v(h - s, \xi_s) \right) ds \right\}. \tag{4.16} \]

Now we deal with (III). Using arguments similar to those leading to (4.15), we get that

\[
\sum_{j=1}^{n} (f_j, X_{t_j}^{2,h,P}) = \int_0^t \int_D \sum_{j=1}^{n} (f_j, \omega_{t_{j-s}}) 1_{s < t_{j}} N^{2,h}(ds, d\omega).
\]
Thus,

$$\text{(III)} = \mathbb{P}_{\mu}^{(h)} \left\{ \exp \left\{ - \int_0^t \sum_{j=1}^n (f_j, \omega_{t_j-s}) \mathbf{1}_{s \leq t_j} \mathcal{N}^{2,h}(ds, d\omega) \right\} | \xi^h \right\}$$

$$= \exp \left\{ - \int_0^t ds \int_0^\infty y_n(\xi_s, dy) \mathbb{P}_{y_{\xi_s}} \left[ \left(1 - e^{-\sum_{j=1}^n (f_j, X_{t_j-s}) \mathbf{1}_{s \leq t_j}} \mathbf{1}_{H \leq h-s}\right) \right] \right\}$$

$$= \exp \left\{ - \int_0^t ds \int_0^\infty y_n(\xi_s, dy) \left(e^{-y_n(h-s, \xi_s)} - e^{-y_n J_s(h, \xi_s)}\right) \right\}. \quad (4.17)$$

Recall that

$$\Psi'_z(x, z) = -\alpha(x) + 2b(x)z + \int_0^\infty y(1 - e^{-yz})n(x, dy).$$

Combining (4.16) and (4.17), we get that

$$\text{(II) \times (III)}$$

$$= \exp \left\{ - \int_0^t \left(2b(\xi_s) J_s(h, \xi_s) + \int_0^\infty y \left(1 - e^{-y J_s(h, \xi_s)}\right) n(\xi_s, dy)\right) ds \right\}$$

$$\times \exp \left\{ \int_0^t \left(2b(\xi_s) v(h - s, \xi_s) - \int_0^\infty y \left(1 - e^{-y h - s, \xi_s}\right) n(\xi_s, dy)\right) ds \right\}$$

$$= \exp \left\{ - \int_0^t \Psi'_z(\xi_s, J_s(h, \xi_s)) ds \right\} \times \exp \left\{ \int_0^t \Psi'_z(\xi_s, v(h - s, \xi_s)) ds \right\}. \quad (4.18)$$

By (4.18), (4.14) and (4.18), we get that, for \( h > t, \)

$$\Pi_x^h \left[ \mathbb{P}_{\mu}^{(h)} \left( \exp \left\{ - \sum_{j=1}^n (f_j, \Lambda_{t_j}^h) \right\} | \xi^h \right) \right]$$

$$= e^{\langle v(h,:), \mu \rangle} e^{-\langle J_0(h,:), \mu \rangle} \Pi_x^h \left[ \exp \left\{ - \int_0^t \Psi'_z(\xi_s, J_s(h, \xi_s)) ds \right\} \times \exp \left\{ \int_0^t \Psi'_z(\xi_s, v(h - s, \xi_s)) ds \right\} \right]$$

$$= e^{\langle v(h,:), \mu \rangle} e^{-\langle J_0(h,:), \mu \rangle} \Pi_x \left[ w(h - t, \xi_t) \exp \left\{ - \int_0^t \Psi'_z(\xi_s, J_s(h, \xi_s)) ds \right\} \right].$$

So, by (4.10), we obtain that

$$\mathbb{P}_{\mu}^{(h)} \left( \exp \left\{ - \sum_{j=1}^n (f_j, \Lambda_{t_j}^h) \right\} \right)$$

$$= e^{\langle v(h,:), \mu \rangle} e^{-\langle J_0(h,:), \mu \rangle} \int_E \Pi_x \left[ w(h - t, \xi_t) \exp \left\{ - \int_0^t \left( \Psi'_z(\xi_s, J_s(h, \xi_s)) \right) ds \right\} \right] \mu(dx). \quad (4.19)$$

Now we calculate \( J_s(h, x) \) defined in (4.12). For \( 0 \leq s < t < h, \) by the Markov property of \( X, \) we have that

$$J_s(h, x) = - \log \mathbb{P}_{\delta_x} \left[ e^{-\sum_{j=1}^n (f_j, X_{t_j-s}) \mathbf{1}_{s \leq t_j}} \mathbb{P}_{X_{t-s}}(\|X_{h-t}\| = 0) \right]$$

$$= - \log \mathbb{P}_{\delta_x} \left[ e^{-\sum_{j=1}^n (f_j, X_{t_j-s}) \mathbf{1}_{s \leq t_j} - \langle v(h-t,:), X_{t-s} \rangle} \right]$$

$$= - \log \mathbb{P}_{\delta_x} \left[ e^{-\sum_{j=1}^n (f_j, X_{t_j}) \mathbf{1}_{s \leq t_j} - \langle v(h-t,:), X_t \rangle} \right]. \quad (4.20)$$
Using Lemma 4.1 with $r = 0$, we have that
\[
e^{-\langle J_0(h,\cdot),\mu \rangle} \int_E \Pi_x \left[ w(h-t,\xi_t) \exp \left\{ - \int_0^t \left( \Psi_z(\xi_s, J_h(h,\xi_s)) \right) ds \right\} \right] \mu(dx)
\]
\[
= \mathbb{P}_\mu \left[ \langle w(h-t,\cdot), X_t \rangle \exp \left\{ - \sum_{j=1}^n \langle f_j, X_{t_j} \rangle - \langle v(h-t,\cdot), X_t \rangle \right\} \right].
\]
Thus, by (4.19), we get that
\[
\mathbb{P}_\mu^{(h)} \left( \exp \left\{ - \sum_{j=1}^n \langle f_j, \Lambda_t \rangle \right\} \right) = \mathbb{P}_\mu \left[ \exp \left\{ - \sum_{j=1}^n \langle f_j, X_{t_j} \rangle \right\} M_t \right].
\]
Now, the proof is complete. 

\[\square\]

4.2 The behavior of $X_t$ near extinction

Recall that, for any $\mu \in \mathcal{M}_F(E)$, $\xi^h = \{ (\xi_t)_{0 \leq t < h}, \Pi_t^h \}$, where $\nu(dx) = \frac{w(h,x)}{(w(h,\cdot))} \mu(dx)$.

Lemma 4.3 Suppose that Assumptions (H1)-(H3)) hold and that for any $\mu \in \mathcal{M}_F(E)$,

\[
\lim_{t \uparrow h} \xi_t = \xi_{h-}, \quad \Pi_t^h \cdot a.s.,
\]

where $\nu(dx) = \frac{w(h,x)}{(w(h,\cdot))} \mu(dx)$. Then, for any $h > 0$,

\[
\lim_{t \uparrow h} \frac{\Lambda_t^h}{\| \Lambda_t^h \|} = \delta_{\xi_{h-}}, \quad \mathbb{P}_\mu^{(h)} \cdot a.s.
\]

Proof: By the decomposition (3.6), we have

\[
\Lambda_t^h := X_t^{0,h} + X_t^{1,h,N} + X_t^{2,h,P}.
\]

Define

\[
H_0 := \inf\{ t \geq 0 : X_t^{0,h} = 0 \} \quad \text{and} \quad H(\Lambda^h) := \inf\{ t \geq 0 : \Lambda_t^h = 0 \}.
\]

Then by the definition of $X^{0,h}$, we have $H_0 < h$. By Theorem 3.5, $H(\Lambda^h) = h$. It follows that

\[
\lim_{t \uparrow h} \frac{X_t^{0,h}}{\| \Lambda_t^h \|} = 0, \quad \mathbb{P}_\mu^{(h)} \cdot a.s. \quad (4.21)
\]

Note that $E_0$ is a compact separable metric space. According to Exercise 9.1.16 (iii), $C_b(E_0; \mathbb{R})$, the space of bounded continuous $\mathbb{R}$-valued functions $f$ on $E_0$, is separable. Therefore, $C_b^+(E)$, the space of nonnegative bounded continuous $\mathbb{R}$-valued functions $f$ on $E$, is also a separable space. It suffices to prove that, for any $f \in C_b^+(E)$,

\[
\mathbb{P}_\mu^{(h)} \left( \lim_{t \uparrow h} \frac{\langle f_h, X_t^{1,h,N} \rangle + \langle f_h, X_t^{2,h,P} \rangle}{\| \Lambda_t^h \|} = 0 \right) = 1, \quad (4.22)
\]

15
where \( f_h(x) = f(x) - f(\xi_{h-}) \). Note that
\[
P_{\mu}^{(h)} \left( \lim_{t \uparrow h} \frac{\langle f_h, X_t^{1,h,N} \rangle + \langle f_h, X_t^{2,h,P} \rangle}{\| \Lambda_t^h \|} = 0 \right) = P_{\mu}^{(h)} \left[ P_{\mu}^{(h)} \left( \lim_{t \uparrow h} \frac{\langle f_h, X_t^{1,h,N} \rangle + \langle f_h, X_t^{2,h,P} \rangle}{\| \Lambda_t^h \|} = 0 | \xi_h \right) \right].
\]

Therefore, it suffices to prove that, for any \( f \in C_b^+ (E) \),
\[
P_{\mu}^{(h)} \left( \lim_{t \uparrow h} \frac{\langle f_h, X_t^{1,h,N} \rangle + \langle f_h, X_t^{2,h,P} \rangle}{\| \Lambda_t^h \|} = 0 | \xi_h \right) = 1, \quad P_{\mu}^{(h)} \text{-a.s.} \tag{4.23}
\]

**Step 1** We first prove that given \( \xi_h \),
\[
\lim_{t \uparrow h} \frac{\langle f_h, X_t^{1,h,N} \rangle}{\| \Lambda_t^h \|} = 0, \quad P_{\mu}^{(h)} \text{-a.s.} \tag{4.24}
\]

Note that given \( \xi_h \),
\[
\langle f_h, X_t^{1,h,N} \rangle := \int_0^t \int_D \langle f_h, \omega_{t-s} \rangle \mathcal{N}^{1,h}(ds,d\omega),
\]
where \( \mathcal{N}^{1,h}(ds,d\omega) \) is a Poisson random measure on \([0,h) \times D\) with intensity measure
\[
21_{(0,h)}(s)1_{H(\omega)<h-s} b(\xi_s) \mathcal{N}_{\xi_s}(d\omega)ds.
\]

Let \( I_1 \) be the support of the measure \( \mathcal{N}^{1,h} \). Note that \( I_1 \) is a random subset of \([0,h) \times D\).

In the remainder of this proof, we always assume that \( \xi_h \) is given. Since \( f \in C_b^+ (E) \), for any \( \epsilon > 0 \), there exists \( \delta_1 > 0 \), depending on \( \xi_{h-} \), such that \( |f(x) - f(\xi_{h-})| \leq \epsilon \) for all \( |x - \xi_{h-}| \leq \delta_1 \).

It follows from the fact that \( \xi_{h-} = \lim_{t \uparrow h} \xi_s \) there exists \( \delta_2 \in (0,h) \), depending on \( \xi_{h-} \), such that \( |\xi_s - \xi_{h-}| < \delta_1/2 \) for all \( s \in (h - \delta_2, h) \). Let \( B := B(\xi_{h-}, \delta_1) = \{ x \in E : |x - \xi_{h-}| < \delta_1 \} \). Then, for any \( t \in (h - \delta_2/2, h) \), we have
\[
|\langle f_h, X_t^{1,h,N} \rangle| = |\langle f_h 1_{B}, X_t^{1,h,N} \rangle + \langle f_h 1_{B^c}, X_t^{1,h,N} \rangle|
\leq \epsilon (1, X_t^{1,h,N}) + 2\| f \|_\infty (1_{B^c}, X_t^{1,h,N})
\leq \epsilon (1, \Lambda_t^h) + 2\| f \|_\infty \int_0^{h-\delta_2} \int_D \langle 1, \omega_{t-s} \rangle \mathcal{N}^{1,h}(ds,d\omega)
+ 2\| f \|_\infty \int_{h-\delta_2}^t \int_D \langle 1_{B^c}, \omega_{t-s} \rangle \mathcal{N}^{1,h}(ds,d\omega)
= : \epsilon (1, \Lambda_t^h) + 2\| f \|_\infty J_1(t) + 2\| f \|_\infty J_2(t). \tag{4.25}
\]

It follows that
\[
\frac{|\langle f_h, X_t^{1,h,N} \rangle|}{\| \Lambda_t^h \|} \leq \epsilon + 2\| f \|_\infty J_1(t) + 2\| f \|_\infty J_2(t). \tag{4.26}
\]

First we deal with \( J_1 \). For \( s \in (0,h-\delta_2) \) and \( t \in (h-\delta_2/2, h) \), we have \( t-s > \delta_2/2 \). Thus, for \( t \in (h-\delta_2/2, h) \), we have
\[
J_1(t) = \int_0^{h-\delta_2} \int_{H(\omega)<h-s} \langle 1, \omega_{t-s} \rangle \mathcal{N}^{1,h}(ds,d\omega) = \sum_{(s,\omega) \in (I_1 \cap S_1)} \langle 1, \omega_{t-s} \rangle,
\]
where
\[ S_1 := \{(s, \omega) : s \in [0, h - \delta_2), w(\delta_2/2) \neq 0 \text{ and } H(\omega) < h - s\}. \] (4.27)

Notice that
\[
\int_{S_1} 21_{[0,h)}(s)1_{H(\omega) < h-s}b(\xi_s)N_{\xi_s}(d\omega)ds \\
\leq 2K \int_{h-\delta_2}^{h} \left| N_{\xi_s}(w(\delta_2/2) \neq 0)ds \\
= 2K \int_{h-\delta_2}^{h} v(\delta_2/2, \xi_s)ds \leq 2Kh\|v_{\delta_2/2}\|_{\infty} < \infty,
\] (4.28)

which implies that given \( \xi^h \),
\[ \mathcal{N}_{1,h}(S_1) < \infty, \quad \mathcal{P}_{\mu}^{(h)} \text{-a.s.} \]

That is, given \( \xi^h \), \#\{\( I_1 \cap S_1 \) \} < \infty, \mathcal{P}_{\mu}^{(h)} \text{-a.s.} For any \((s, \omega) \in (I_1 \cap S_1)\), we have \( s + H(\omega) < h \), which implies that \( H_1 := \max(s, \omega) \in (I_1 \cap S_1)(s + H(\omega)) < h \). Thus, for any \( t \in (H_1, h) \), \( J_1(t) = 0 \), which implies that given \( \xi^h \),
\[ \lim_{t \uparrow h} \frac{J_1(t)}{\|A^h_t\|} = 0, \quad \mathcal{P}_{\mu}^{(h)} \text{-a.s.} \] (4.29)

To deal with \( J_2 \), we define
\[ D_1 := \{\omega : \exists u \in (0, \delta_2), \text{such that } \langle 1_{B^c}, \omega_u \rangle > 0\}, \quad \text{and} \quad S_2 = [h - \delta_2, h) \times D_1. \] (4.30)
Then,
\[ J_2(t) = \sum_{(s, \omega) \in (I_1 \cap S_2)} \langle 1_{B^c}, \omega_{t-s} \rangle 1_{s<t}. \]
We claim that \#\{\( I_1 \cap S_2 \) \} < \infty. Then using arguments similar to those leading to (4.29), we can get that given \( \xi^h \),
\[ \lim_{t \uparrow h} \frac{J_2(t)}{\|A^h_t\|} = 0, \quad \mathcal{P}_{\mu}^{(h)} \text{-a.s.} \] (4.31)

Now we prove the claim. It suffices to prove that given \( \xi^h \)
\[
\int_{S_2} 21_{[0,h)}(s)1_{H(\omega) < h-s}b(\xi_s)N_{\xi_s}(d\omega)ds < \infty.
\] (4.32)

Note that
\[
\int_{S_2} 21_{[0,h)}(s)1_{H(\omega) < h-s}b(\xi_s)N_{\xi_s}(d\omega)ds \leq 2K \int_{h-\delta_2}^{h} N_{\xi_s}(D_1)ds.
\]
For \( \omega \in D_1 \), we have
\[
D_1 = \{\omega \in D : \exists u \in (0, \delta_2), \text{such that } \langle 1_{B^c}, \omega_u \rangle > 0\} = \left\{ \omega \in D : \int_0^{\delta_2} \langle 1_{B^c}, \omega_u \rangle du > 0 \right\} \subset \left\{ \omega \in D : \int_0^{\delta_2} \langle 1_{B^c}, \omega_u \rangle du > 0 \right\}.
\]
Thus,
\[ N_x(D_1) \leq N_x \left( \int_0^{\delta_2} (1_{B^c}, \omega_u) \, du > 0 \right) \]
\[ = \lim_{\lambda \to \infty} N_x \left( 1 - \exp \left\{ -\lambda \int_0^{\delta_2} (1_{B^c}, \omega_u) \, du \right\} \right) \]
\[ = \lim_{\lambda \to \infty} -\log P_{d_x} \left( \exp \left\{ -\lambda \int_0^{\delta_2} (1_{B^c}, X_u) \, du \right\} \right) \]
\[ = -\log P_{d_x} \left( \int_0^{\delta_2} (1_{B^c}, \omega_u) \, du = 0 \right). \]  
(4.33)

Combining (4.33) and Assumption (H3), we get
\[ \int_{S_2} 21_{[0,h)}(s)1_{H(\omega)<h-s}b(\xi_s)N_{\xi_s}(d\omega)ds \leq 2K\delta_2 \sup_{x \in B(\xi_s, \delta_3/2)} \left[ -\log P_{d_x} \left( \int_0^{\delta_2} (1_{B^c}, \omega_u) \, du = 0 \right) \right] < \infty. \]

Combining (4.26), (4.29) and (4.31), we get (4.24).

**Step 2** Next we prove that given \( \xi^h, \)
\[ \lim_{t \uparrow h} \frac{\langle f_h, \lambda_{2,h,P} \rangle}{\| \lambda_{t}^h \|} = 0, \quad P^{(h)}_\mu\text{-a.s.} \]  
(4.34)

Note that given \( \xi^h, \)
\[ \langle f_h, \lambda_{2,h,P} \rangle := \int_0^t \int_D \langle f_h, \omega_{t-s} \rangle \lambda^{2,h}(ds, d\omega), \]
where \( \lambda^{2,h}(ds, d\omega) \) is a Poisson random measure on \([0, h) \times \mathbb{D}\) with intensity measure
\[ 1_{[0,h)}(s)1_{H(h)<h-s} \int_0^\infty y\mu(\xi_s, dy)P_{\delta^2}(X \in d\omega)ds. \]

Let \( I_2 \) be the support of the measure \( \lambda^{2,h} \). Note that \( I_2 \) is a random countable subset of \([0, h) \times \mathbb{D}\). Using arguments similar to those leading to (4.25), we get that
\[ \langle f_h, X_{t}^{2,h,P} \rangle \leq \epsilon(1, \lambda_{t}^h) + 2\| f \|_\infty \int_0^{h-\delta_2} \int_D \langle 1_{B^c}, \omega_{t-s} \rangle \lambda^{2,h}(ds, d\omega) \]
\[ + 2\| f \|_\infty \int_0^{t} \int_D \langle 1_{B^c}, \omega_{t-s} \rangle \lambda^{2,h}(ds, d\omega) \]
\[ = \epsilon(1, \lambda_{t}^h) + 2\| f \|_\infty \sum_{(s, \omega) \in (I_2 \cap S_1)} \langle 1_{B^c}, \omega_{t-s} \rangle + 2\| f \|_\infty \sum_{(s, \omega) \in (I_2 \cap S_2)} \langle 1_{B^c}, \omega_{t-s} \rangle \]
\[ = \epsilon(1, \lambda_{t}^h) + 2\| f \|_\infty J_3(t) + 2\| f \|_\infty J_4(t), \]
where \( S_1 \) and \( S_2 \) are the set defined in (4.27) and (4.30). It follows that
\[ \frac{|\langle f_h, X_{t}^{2,h,P} \rangle|}{\| \lambda_{t}^h \|} \leq \epsilon + 2\| f \|_\infty J_3(t) + 2\| f \|_\infty J_4(t), \]  
(4.35)
So, to prove (4.34), we only need to prove that
\[
\lim_{t \uparrow h} J_3(t) = 0, \quad \mathbb{P}^{(h)}_{\mu} \text{-a.s.,}
\] (4.36)
and
\[
\lim_{t \uparrow h} J_4(t) = 0, \quad \mathbb{P}^{(h)}_{\mu} \text{-a.s.}
\] (4.37)

Note that
\[
\int_{S_1} 1_{(0,h)}(s) 1_{H(\omega) < h-s} \int_0^\infty y n(\xi_s, dy) \mathbb{P}_{y \delta_\xi_s}(X \in d\omega) ds
\]
\[
\leq \int_0^{h-\delta_2} \int_0^\infty y n(\xi_s, dy) \mathbb{P}_{y \delta_\xi_s}(X_{\delta_2/2} \neq 0) ds
\]
\[
\leq \int_0^{h-\delta_2} v(\delta_2/2, \xi_s) \int_0^1 y^2 n(\xi_s, dy) ds + \int_0^{h-\delta_2} \int_1^\infty y n(\xi_s, dy) ds
\]
\[
\leq K h(\|v_{\delta_2/2}\|_{\infty} + 1),
\] (4.38)

where in the second inequality we used the fact that
\[
\mathbb{P}_{y \delta_\xi_s}(X_{\delta_2/2} \neq 0) = 1 - \mathbb{P}_{y \delta_\xi_s}(X_{\delta_2/2} = 0) = 1 - e^{-y v(\delta_2/2, \xi_s)} \leq y v(\delta_2/2, \xi_s).
\]
Thus, \(N^{2/h}(S_1) < \infty\), a.s., which implies that (4.36).

To prove (4.37) we only need to show that, given \(\xi^h\),
\[
\int \int_{S_2} \int_0^\infty y n(\xi_s, dy) \mathbb{P}_{y \delta_\xi_s}(X \in d\omega) ds < \infty.
\] (4.39)

In fact,
\[
\int \int_{S_2} \int_0^\infty y n(\xi_s, dy) \mathbb{P}_{y \delta_\xi_s}(X \in d\omega) ds
\]
\[
\leq \int_0^{h} \int_0^\infty y n(\xi_s, dy) \mathbb{P}_{y \delta_\xi_s} \left( \int_0^{\delta_2} \langle 1_{B^c}, X_u \rangle du > 0 \right) ds
\]
\[
\leq \int_0^{h} \int_1^\infty y n(\xi_s, dy) ds + \int_0^{h-\delta_2} \left( - \log \mathbb{P}_{\delta_\xi_s} \left( \int_0^{\delta_2} \langle 1_{B^c}, X_u \rangle du = 0 \right) \right) \int_0^1 y^2 n(\xi_s, dy) ds
\]
\[
\leq K h + K h \sup_{x \in B(\xi^h_{\delta_2/2})} \left[ - \log \mathbb{P}_{\delta_\xi_s} \left( \int_0^{\delta_2} \langle 1_{B^c}, X_u \rangle du = 0 \right) \right] < \infty,
\]
where in the second inequality, we used the fact that
\[
\mathbb{P}_{y \delta_\xi_s} \left( \int_0^{\delta_2} \langle 1_{B^c}, X_u \rangle du > 0 \right) = 1 - \exp \left\{ y \log \mathbb{P}_{\delta_\xi_s} \left( \int_0^{\delta_2} \langle 1_{B^c}, \omega_u \rangle du = 0 \right) \right\}
\]
\[
\leq - y \log \mathbb{P}_{\delta_\xi_s} \left( \int_0^{\delta_2} \langle 1_{B^c}, X_u \rangle du = 0 \right).
\]

The proof is now complete.
**Proof of Theorem 3.7.** Since \( \{X_t, t \geq 0\} \) is a Hunt process, \( t \rightarrow X_t \) is right continuous, which implies that

\[
\left\{ \lim_{t \uparrow H} \frac{X_t}{\|X_t\|} \text{ exists} \right\} = \left\{ \lim_{t \in \mathcal{Q} \uparrow H} \frac{X_t}{\|X_t\|} \text{ exists} \right\},
\]

where \( \mathcal{Q} \) is the set of all rational numbers in \([0, \infty)\). And, note that

\[
H = \inf \{ t \in \mathcal{Q} : \|X_t\| = 0 \}.
\]

Thus, by Corollary 3.6 and Lemma 4.3, we get that

\[
\mathbb{P}_\mu \left[ \lim_{t \in \mathcal{Q} \uparrow H} \frac{X_t}{\|X_t\|} \right] = \int_{\mathcal{Q}} \mathbb{P}_\mu \left[ \lim_{t \in \mathcal{Q} \uparrow H} \frac{\Lambda_t^h}{\|\Lambda_t^h\|} \right] F_H(dh) = 1.
\]

Let \( V := \lim_{t \uparrow H} \frac{X_t}{\|X_t\|} \). Then, for any \( f \in \mathcal{B}_b^+(E) \), by Lemma 4.3,

\[
\mathbb{P}_\mu [\exp \{-\langle f, V \rangle\}] = \mathbb{P}_\mu \left[ \lim_{t \in \mathcal{Q} \uparrow H} \exp \left\{ -\frac{\langle f, X_t \rangle}{\|X_t\|} \right\} \right] = \int_0^\infty \lim_{t \in \mathcal{Q} \uparrow h} \mathbb{P}_\mu^{(h)} \left[ \exp \left( -\frac{\langle f, \Lambda_t^h \rangle}{\|\Lambda_t^h\|} \right) \right] F_H(dh) = \int_0^\infty \Pi_x^h [\exp(-f(\xi_{h-}))] F_H(dh).
\]

Thus, \( V \) is a Dirac measure of the form \( V = \delta_Z \) and the law of \( Z \) satisfies (3.8). The proof is now complete.

\[\square\]

**5 Examples**

In this section, we will list some examples that satisfy Assumptions (H1) and (H2). The purpose of these examples is to show that Assumptions (H1) and (H2) are satisfied in a lot of cases. We will not try to give the most general examples possible.

**Example 3** Suppose that \( P_t \) is conservative and preserves \( C_b(E) \). Let \( A \) be the infinitesimal generator of \( P_t \) in \( C_b(E) \) and \( \mathcal{D}(A) \) be the domain of \( A \). Also assume that

\[
\Psi(x, z) = -\alpha(x)z + b(x)z^2,
\]

where \( \sup_{x \in E} \alpha(x) \leq 0 \) and \( \inf_{x \in E} b(x) > 0 \) and \( 1/b \in \mathcal{D}(A) \). Then by Remark 2.2, we know that Assumption (H1) is satisfied. One can check that

\[
\left( \frac{b^{-1}(\xi_t)}{b^{-1}(x)} e^{-\int_0^t (b(\xi_s)A(1/b)(\xi_s)) ds}, t \geq 0 \right)
\]

is a positive martingale under \( \Pi_x \). Thus we define another probability measure \( \Pi_x^{1/b} \) by

\[
\Pi_x^{1/b} = \Pi_x \frac{b^{-1}(\xi_t)}{b^{-1}(x)} e^{-\int_0^t (b(\xi_s)A(1/b)(\xi_s)) ds}, t \geq 0.
\]
Let $A^{1/b}$ be the infinitesimal generator of $\xi$ under $\Pi^{1/b}$. If $-\alpha(x) - b(x)A(1/b)(x) \in D(A^{1/b})$, then it follows from [4] (3.10) and Lemma 4.9 that $w(t,x)$ exists and satisfies

$$
w(t,x) \leq \frac{1}{\inf_{x \in E} b(x)} e^{ct} \frac{\beta_0^2 \beta_0 t}{(e^{\beta_0 t} - 1)^2},
$$

where $c, \beta_0$ are positive constants. Using this, one can check that Assumption (H2) is satisfied. This example shows that our result covers Delmas and Hénard [4, Corollary 4.14].

Now we give some examples of superprocesses, with general branching mechanisms, satisfying Assumptions (H1) and (H2).

Recall that the general form of branching mechanism is given by

$$
\Psi(x,z) = -\alpha(x)z + b(x)z^2 + \int_0^\infty (e^{-yz} - 1 + yz)n(x,dy).
$$

By (2.2), there exists $K > 0$, such that

$$
|\alpha(x)| + b(x) + \int_0^\infty (y \wedge y^2)n(x,dy) \leq K.
$$

Thus we have

$$
|\Psi(x,z)| \leq 3K(z + z^2), \quad x \in \mathbb{R}^d. \tag{5.1}
$$

In the next two examples, we always assume that $E = \mathbb{R}^d$ and that $\Psi$ satisfies (2.7) and the following condition: for any $M > 0$, there exist $c > 0$ and $\gamma_0 \in (0,1]$ such that

$$
|\Psi(x,z) - \Psi(y,z)| \leq c|x - y|^{\gamma_0}, \quad x, y \in \mathbb{R}^d, z \in [0,M]. \tag{5.2}
$$

By Remark 2.2, condition (2.7) implies that Assumption (H1) is satisfied. Therefore, in the following examples, we only need to check that Assumption (H2) is satisfied.

**Example 4** Assume that the spatial motion $\xi$ is a diffusion on $\mathbb{R}^d$ satisfying the conditions in Example 4. The branching mechanism $\Psi$ is of the form in (2.1) and satisfies (2.7) and (5.2). Then the $(\xi, \Psi)$-superprocess $X$ satisfies Assumptions (H1) and (H2).

We now proceed to prove the second assertion of the example above.

**Lemma 5.1** For $f \in B_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the function $t \to P_t f(x)$ is differentiable on $(0, \infty)$. Furthermore, there exists a constant $c$ such that for any $t \in (0,1]$, $x \in \mathbb{R}^d$ and $f \in B_b(\mathbb{R}^d)$,

$$
\left| \frac{\partial}{\partial t} P_t f(x) \right| \leq c\|f\|_\infty t^{-1}. \tag{5.3}
$$

**Proof:** For $t \in (n, n + 1]$, $P_t f(x) = P_{t-n} (P_n f)(x)$. Thus, we only need to prove the differentiability for $t \in (0,1]$. It follows from [11] IV. (13.1)] that

$$
\left| \frac{\partial}{\partial t} p(t,x,y) \right| \leq c_1 t^{-\frac{d}{2}} e^{-c_2 |x-y|^2 t}. \tag{5.4}
$$

Thus by the dominated convergence theorem we have that for all $t \in (0,1]$ and $x \in \mathbb{R}^d$, 

$$
\frac{\partial}{\partial t} P_t f(x) = \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p(t,x,y) f(y) \, dy.
$$
and that for all \( t \in (0, 1], \ x \in \mathbb{R}^d \) and bounded Borel function \( f \) on \( \mathbb{R}^d \),
\[
\frac{\partial}{\partial t} P_t f(x) \leq c_3 \| f \|_{\infty} t^{-1}.
\]
The proof is now complete.

\[\square\]

**Lemma 5.2** Assume that \( f_s(x) \) is uniformly bounded in \( (s, x) \in [0, 1] \times \mathbb{R}^d \), that is, there is a constant \( L > 0 \) so that, for all \( s \in [0, 1] \) and \( x \in \mathbb{R}^d \), \( |f_s(x)| \leq L \). Then there is a constant \( c \) such that for any \( t \in (0, 1] \) and \( x, x' \in \mathbb{R}^d \),
\[
\left| \int_0^t P_{t-s} f_s(x) \, ds - \int_0^t P_{t-s} f_s(x') \, ds \right| \leq cL(|x - x'| \wedge 1).
\]

**Proof:** It follows from [11, IV.(13.1)] that there exist constants \( c_1, c_2 > 0 \) such that for all \( t \in (0, 1] \) and \( x, x' \in \mathbb{R}^d \),
\[
|\nabla_x p(t, x, y)| \leq c_1 t^{-d+1} e^{-\frac{c_2|x-y|^2}{t}}.
\]

Thus
\[
|p(t, x, y) - p(t, x', y)| \leq c_3 ((t^{-1/2}|x - x'| \wedge 1)t^{-d/2} (e^{-\frac{c_4|x-y|^2}{t}} + e^{-\frac{c_4|x'-y|^2}{t}}).
\]

Hence for any \( t \in (0, 1] \) and \( x, x' \in \mathbb{R}^d \),
\[
\left| \int_0^t P_{t-s} f_s(x) \, ds - \int_0^t P_{t-s} f_s(x') \, ds \right| \leq c_5 L \int_0^1 s^{-1/2} |x - x'| \, ds \leq c_6 L |x - x'|.
\]

\[\square\]

**Lemma 5.3** Assume that \( f_s(x) \) satisfies the following conditions:

(i) There is a constant \( L \) so that, for all \( (s, x) \in [0, 1] \times \mathbb{R}^d \), \( |f_s(x)| \leq L \).

(ii) For any \( t_0 \in [0, 1] \), \( \lim_{s \to t_0} \sup_{x \in \mathbb{R}^d} |f_s(x) - f_{t_0}(x)| = 0 \).

(iii) There exist constants \( s_0 \in (0, 1), C > 0 \) and \( \gamma \in (0, 1] \) such that for all \( s \in [0, s_0] \) and \( x, x' \in \mathbb{R}^d \) with \( |x - x'| \leq 1 \),
\[
|f_s(x) - f_s(x')| \leq C |x - x'|^\gamma.
\]

Then, \( t \to \int_0^t P_{t-s} f_s(x) \, ds \) is differentiable on \( (0, s_0) \), and for \( t \in [0, s_0] \),
\[
\frac{\partial}{\partial t} \int_0^t P_{t-s} f_s(x) \, ds = \int_0^t \frac{\partial}{\partial t} P_{t-s} f_s(x) \, ds + f_t(x).
\]

**Proof:** Let \( G(t, x) := \int_0^t P_{t-s} f_s(x) \, ds \). First, we will show that for any \( x \in \mathbb{R}^d \),
\[
\lim_{t \downarrow 0} t^{-1} \int_0^t P_{t-s} f_s(x) \, ds = f_0(x).
\]

Since \( f_0 \in C_0(\mathbb{R}^d) \), we have \( \lim_{s \to 0} P_s f_0(x) = f_0(x) \), which implies that
\[
\lim_{t \to 0} t^{-1} \int_0^t P_{t-s} f_0(x) \, ds = \lim_{t \to 0} t^{-1} \int_0^t P_s f_0(x) \, ds = f_0(x).
\]
Thus, it suffices to prove that
\[
\lim_{t \to 0} t^{-1} \int_0^t P_{t-s}(f_s - f_0)(x) \, ds = 0. \tag{5.11}
\]

Notice that
\[
t^{-1} \int_0^t |P_{t-s}(f_s - f_0)(x)| \, ds \leq \sup_{s \leq t} \|f_s - f_0\|_\infty \to 0,
\]
as \( t \to 0 \). Thus, (5.10) is valid.

For any \( 0 < t < t + r < s_0 \), by the definition of \( G(t, x) \),
\[
\frac{1}{r} (G(t + r, x) - G(t, x)) = \frac{1}{r} \int_0^t \left( P_{t+r-s}f_s(x) - P_{t-s}f_s(x) \right) \, ds + \frac{1}{r} \int_t^{t+r} P_{t+r-s}f_s(x) \, ds
\]
\[
= \int_0^t \frac{P_{t+r-s}f_s(x) - P_{t-s}f_s(x)}{r} \, ds + \frac{1}{r} \int_0^r P_{r-s}f_{t+s}(x) \, ds
\]
\[
= (I) + (II).
\]

By (5.10), we have
\[
\lim_{r \downarrow 0} (II) = f_t(x). \tag{5.12}
\]

Now we deal with part (I). For \( 0 < t < t + r < s_0 \), using (5.21), we obtain that
\[
\left| \frac{P_{t+r-s}f_s(x) - P_{t-s}f_s(x)}{r} \right| = \left| \int_{\mathbb{R}^d} \frac{p(t + r - s, x, y) - p(t - s, x, y)}{r} (f_s(y) - f_s(x)) \, dy \right|
\]
\[
\leq c_3 \int_{\mathbb{R}^d} \left| f_s(y) - f_s(x) \right| |(t - s)^{-\frac{d}{2}} e^{-\frac{c_4 |x - y|^2}{t-s}} dy
\]
\[
\leq c_5 \int_{\mathbb{R}^d} |x - y|^\gamma (t - s)^{-\frac{d}{2}} e^{-\frac{c_4 |x - y|^2}{t-s}} dy
\]
\[
\leq c_6 (t - s)^{\gamma/2 - 1}. \tag{5.13}
\]

Thus, using the dominated convergence theorem, we get that, for any \( 0 \leq t < t + r < s_0 \),
\[
\lim_{r \downarrow 0} (I) = \int_0^t \lim_{r \downarrow 0} \frac{P_{t+r-s}f_s(x) - P_{t-s}f_s(x)}{r} \, ds = \int_0^t \frac{\partial}{\partial t} P_{t-s}f_s(x) \, ds. \tag{5.14}
\]

Combining (5.12) and (5.14), we get that
\[
\lim_{r \downarrow 0} \frac{G(t + r, x) - G(t, x)}{r} = \int_0^t \frac{\partial}{\partial t} P_{t-s}f_s(x) \, ds + f_t(x).
\]

Using similar arguments, we can also show that
\[
\lim_{r \downarrow 0} \frac{G(t, x) - G(t - r, x)}{r} = \int_0^t \frac{\partial}{\partial t} P_{t-s}f_s(x) \, ds + f_t(x).
\]

Thus, (5.9) follows immediately. The proof is now complete.
Recall that \( v(s, \cdot) \) is a bounded function and

\[
v(t + s, x) + \int_0^t P_{t-u}(\Psi_{s+u})(x) \, du = P_t(v_s)(x),
\]

where

\[
\Psi_u(x) = \Psi(x, v(u, x)). \tag{5.15}
\]

**Lemma 5.4** For any \( s > 0 \), there is a constant \( c(s) \) such that for \( t \in [0, 1/2) \) and \( x, y \in \mathbb{R}^d \),

\[
|v_{t+s}(x) - v_{t+s}(y)| \leq c(s)|x - y|.
\]

Moreover, \( c(s) \) is decreasing in \( s > 0 \).

**Proof:** Let \( e(s) := \frac{1 + s}{2} \). Note that \( t + e(s) \in (e(s), 1) \). Thus

\[
v(t + s, x) + \int_0^{t+e(s)} P_{t+e(s)-u}(\Psi(\cdot, v_{s-e(s)}(\cdot))(x)) \, du = P_{t+e(s)}(v_{s-e(s)})(x).
\]

It follows from (5.16) that there exists a constant \( c_1 \) such that for all \( x, y \in \mathbb{R}^d \),

\[
|P_{t+e(s)}(v_{s-e(s)})(x) - P_{t+e(s)}(v_{s-e(s)})(y)|
\leq c \|v_{s-e(s)}\|_{\infty}((t+e(s))^{-1/2}|x-y| \wedge 1)
\leq c \|v_{s-e(s)}\|_{\infty}(t+e(s))^{-1/2}|x-y|
\leq c \|v_{s/2}\|_{\infty}(e(s))^{-1/2}|x-y|. \tag{5.16}
\]

Since \( v(s-e(s) + u, x) \leq v(s-e(s), x) \leq v(s/2, x) \), we have for \( u > 0 \),

\[
\|\Psi(\cdot, v_{s-e(s)}(\cdot))(\cdot)\|_{\infty} \leq 3K(\|v_{s/2}\|_{\infty} + \|v_{s/2}\|_{\infty}^2).
\]

Applying Lemma 5.2, we get that there is a constant \( c_2 > 0 \) such that for \( t \in [0, 1/2) \) and \( x, y \in \mathbb{R}^d \),

\[
\left|\int_0^{t+e(s)} P_{t+e(s)-u}(\Psi(\cdot, v_{s-e(s)}+u(\cdot))(x)) \, du - \int_0^{t+e(s)} P_{t+e(s)-u}(\Psi(\cdot, v_{s-e(s)}+u(\cdot))(y)) \, du\right|
\leq c_2 3K(\|v_{s/2}\|_{\infty} + \|v_{s/2}\|_{\infty}^2)(|x-y| \wedge 1). \tag{5.17}
\]

The conclusions of the lemma now follow immediately from (5.16) and (5.17). \( \square \)

**Lemma 5.5** The function \( \Psi_u(x) \) given by (5.15) satisfies the following two properties:

1. For any \( u_0 > 0 \),

\[
\lim_{u \to u_0} \sup_{x \in \mathbb{R}^d} |\Psi_u(x) - \Psi_{u_0}(x)| = 0;
\]

2. For \( t_0 \in (0, 1), \) there exists a constant \( c > 0 \) such that for any \( |x - x'| < 1, \) \( s > t_0 \) and \( t \in [0, 1/2] \),

\[
|\Psi_{s+t}(x) - \Psi_{s+t}(x')| \leq c|x - x'|^{\gamma_0}.
\]
Proof: (1) For \( z_1 < z_2 \in [0, a] \), we can easily check that
\[
|\Psi(x, z_1) - \Psi(x, z_2)|
\leq |\alpha(x)||z_1 - z_2| + b(x)|z_1^2 - z_2^2| + \int_0^\infty |e^{-yz_1} + yz_1 - e^{-yz_2} - yz_2| n(x, dy)
\leq K(1 + 2a)|z_1 - z_2| + \int_0^\infty (2 \wedge (ya))|z_1 - z_2|n(x, dy) \leq K(3 + 3a)|z_1 - z_2|,
\] (5.18)
where in the second inequality above we use the fact that
\[
\frac{d}{dx}(e^{-x} + x) = 1 - e^{-x} \leq 2 \wedge x.
\]
Thus, for \( |u - u_0| \leq u_0/2 \), we have that
\[
|\Psi_u(x) - \Psi_{u_0}(x)| \leq 3K(1 + \|v_{u_0/2}\|_\infty)|v_u(x) - v_{u_0}(x)|.
\] (5.19)
Thus, it suffices to show that \( t \mapsto v_t(x) \) is continuous on \((0, \infty)\) uniformly in \( x \).

It follows from Lemma 5.4 that, for any \( t > 0 \), \( x \mapsto v_t(x) \) is uniformly continuous, thus
\[
\lim_{r \downarrow 0} \|P_rv_t - v_t\|_\infty = 0.
\]
For \( r > 0 \) and \( t > 0 \), we have that
\[
|v_t(x) - v_{t+r}(x)| \leq |P_rv_t(x) - v_t(x)| + \int_0^r P_{r-u}(\Psi_{t+u})(x) du
\leq \|P_rv_t - v_t\|_\infty + 3K(\|v_t\|_\infty + \|v_t\|_\infty^2)r \to 0, \quad r \downarrow 0,
\]
where in the last inequality we used (5.1) and the fact that \( v_{t+u}(x) \leq v_t(x) \).

The proof of \( \lim_{r \downarrow 0} \|v_t - v_{t-r}\|_\infty = 0 \) is similar and omitted. The proof of part (1) is now complete.

(2) For any \( s > t_0 \) and \( t \in [0, 1/2] \), \( v(t + s, x) \leq \|v_{t_0}\|_\infty \). By our assumption on \( \Psi \), there exist \( c_1 > 0 \) and \( \gamma_0 \in (0, 1] \) such that for \( |x-y| \leq 1, s > t_0 \) and \( t \in [0, 1/2] \),
\[
|\Psi(x, v_{s+t}(x)) - \Psi(y, v_{s+t}(x))| \leq c_1|x-y|^\gamma_0.
\]
By Lemma 5.4 there exists \( c_2 = c_2(t_0) \) such that for \( s > t_0 \) and \( t \in [0, 1/2] \),
\[
|v_{s+t}(x) - v_{s+t}(y)| \leq c_2|x-y|.
\]
Thus, for \( |x-y| \leq 1, s > t_0 \), and \( t \in [0, 1/2] \),
\[
|\Psi_{s+t}(x) - \Psi_{s+t}(y)| \leq |\Psi(x, v_{s+t}(x)) - \Psi(y, v_{s+t}(x))| + |\Psi(y, v_{s+t}(x)) - \Psi(y, v_{s+t}(y))|
\leq |\Psi(x, v_{s+t}(x)) - \Psi(y, v_{s+t}(x))| + 3K(1 + \|v_{t_0}\|_\infty)|v_{s+t}(x) - v_{s+t}(y)|
\leq c_1|x-y|^\gamma_0 + 3K(1 + \|v_{t_0}\|_\infty)c_2|x-y|
\leq c_3|x-y|^\gamma_0.
\] (5.20)
The proof of (2) is now complete. \( \square \)
Lemma 5.6 The function $t \to v_t(x)$ is differentiable in $(0, \infty)$, and for any $s > 0$ and $t \in [0, 1/2)$, $w(t + s, x) = -\frac{\partial}{\partial t} v_{t+s}(x)$ satisfies that

$$w(t + s, x) = -\frac{\partial}{\partial t} P_t(v_s)(x) + \int_0^t \frac{\partial}{\partial t} P_{t-u}(\Psi_{s+u})(x) \, du + \Psi_{t+s}(x).$$

(5.21)

Moreover, $t \to w(t, x)$ is continuous and for any $s_0 > 0$, $\sup_{s>s_0} \sup_{x \in \mathbb{R}^d} w(t, x) < \infty$.

Proof: For any $t, s > 0$,

$$v(t + s, x) + \int_0^t P_{t-u}(\Psi_{s+u})(x) \, du = P_t(v_s)(x).$$

Thus, combining Lemmas 5.1, 5.3 and 5.5, (5.21) follows immediately.

For fixed $t \in (0, 1/2)$, we deal with the three parts on right hand side of (5.21) separately.

Since $t \to v(t, x)$ is continuous, the function $s \to \Psi_{t+s}(x) = \Psi(x, v(t + s, x))$ is continuous and, by 5.11,

$$\sup_{s>t_0} |\Psi_{t+s}(x)| \leq 3K(\|v_0\|_\infty + \|v_0\|_\infty^2) < \infty.$$  

(5.22)

By (5.26),

$$\sup_{s>t_0} |\frac{\partial}{\partial t} P_t(v_s)(x)| \leq c_4 \|v_0\|_\infty t^{-1} < \infty.$$  

(5.23)

By 5.13 and Lemma 5.5 (2), we get that, for any $s > t_0$,

$$\sup_{s>t_0} \sup_{x \in \mathbb{R}^d} |\int_0^t \frac{\partial}{\partial t} P_{t-u}(\Psi_{s+u})(x) \, du| < \infty.$$  

(5.24)

Combining (5.22) - (5.24), we get that, for $t_0 > 0$,

$$\sup_{s=t_0} \sup_{x \in \mathbb{R}^d} w(t + s, x) < \infty,$$

which implies that, for any $s_0 > 0$, $\sup_{s>s_0} \sup_{x \in \mathbb{R}^d} w(t, x) < \infty$. 

Now we give an example of a superprocess with discontinuous spatial motion and general branching mechanism such that Assumptions (H1) and (H2) are satisfied.

Example 5 Suppose that $B = \{B_t\}$ is a Brownian motion in $\mathbb{R}^d$ and $S = \{S_t\}$ is an independent subordinator with Laplace exponent $\varphi$, that is

$$\mathbb{E} e^{-\lambda S_t} = e^{-t \varphi(\lambda)}, \quad t > 0, \lambda > 0.$$  

The process $\xi_t = B_{S_t}$ is called a subordinate Brownian motion in $\mathbb{R}^d$. Subordinate Brownian motions form a large class of Lévy processes. When $S$ is an $(\alpha/2)$-stable subordinator, that is, $\varphi(\lambda) = \lambda^{\alpha/2}$, $\xi$ is a symmetric $\alpha$-stable process in $\mathbb{R}^d$. Suppose that $\Psi$ is of the form in (2.1) satisfying (2.7) and (5.2). Suppose further that $\varphi$ satisfies the following conditions:

1. $\int_{0}^{1} \frac{\varphi(r^2)}{r} \, dr < \infty$.

2. There exist constants $\delta \in (0, 2)$ and $a_1 \in (0, 1)$ such that

$$a_1 \lambda^{\delta/2} \varphi(r) \leq \varphi(\lambda r), \quad \lambda \geq 1, r \geq 1.$$  

(5.25)
then $X$ satisfies Assumptions (H1) and (H2).

Now we proceed to prove the second assertion of the example above. The arguments are similar to that for the second assertion of Example 4. Without loss of generality, we will assume that $\varphi(1) = 1$. First we introduce some notation. Put $\Phi(r) = \varphi(r^2)$ and let $\Phi^{-1}$ be the inverse function of $\Phi$. For $t > 0$ and $x \in \mathbb{R}^d$, we define

$$\rho(t, x) := \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-1}\right) \left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-d}.$$  

For $t > 0, x \in \mathbb{R}^d$ and $\beta, \gamma \in \mathbb{R}$, we define

$$\rho^\beta(t, x) := \Phi^{-1}(t^{-1})^{-\gamma}(|x|^\beta \wedge 1)\rho(t, x), \quad t > 0, x \in \mathbb{R}^d.$$  

Let $p(t, x, y) = p(t, x - y)$ be the transition density of $\xi$ and let $\{P_t : t \geq 0\}$ be the transition semigroup of $\Phi$. It is well known that $\{P_t : t \geq 0\}$ satisfies the strong Feller property, that is, for any $t > 0$, $P_t$ maps bounded Borel functions on $\mathbb{R}^d$ to bounded continuous functions on $\mathbb{R}^d$.

Now we list some other properties of the semigroup $\{P_t : t \geq 0\}$ which will be used later.

**Lemma 5.7** For $f \in B_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the function $t \to P_t f(x)$ is differentiable on $(0, \infty)$. Furthermore, there exists a constant $c$ such that for any $t \in (0, 1)$, $x \in \mathbb{R}^d$ and $f \in B_b(\mathbb{R}^d)$,

$$\left|\frac{\partial}{\partial t} P_t f(x)\right| \leq c\|f\|_\infty t^{-1}. \quad (5.26)$$

**Proof:** For $t \in (n, n + 1]$, $P_t f(x) = P_{t-n}(P_n f)(x)$. Thus, we only need to prove the differentiability for $t \in (0, 1]$. It follows from [9, Lemma 3.1(a) and Theorem 3.4] that

$$\left|\frac{\partial}{\partial t} p(t, x)\right| \leq c_1 \rho(t, x). \quad (5.27)$$

By [9, Lemma 2.6(a)], we have

$$\int_{\mathbb{R}^d} \rho(t, x) dx < c_2 t^{-1}, \quad t \in (0, 1]. \quad (5.28)$$

Thus by the dominated convergence theorem we have that for all $t \in (0, 1]$ and $x \in \mathbb{R}^d$,

$$\frac{\partial}{\partial t} P_t f(x) = \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p(t, x, y) f(y) dy,$$

and that for all $t \in (0, 1]$, $x \in \mathbb{R}^d$ and bounded Borel function $f$ on $\mathbb{R}^d$,

$$\left|\frac{\partial}{\partial t} P_t f(x)\right| \leq c_3 \|f\|_\infty t^{-1}.$$  

The proof is now complete. \qed

**Lemma 5.8** Assume that $f_s(x)$ is uniformly bounded in $(s, x) \in [0, 1] \times \mathbb{R}^d$, that is, there is a constant $L > 0$ so that, for all $s \in [0, 1]$ and $x \in \mathbb{R}^d$, $|f_s(x)| \leq L$. Then there is a constant $c$ such that for any $t \in (0, 1]$ and $x, x' \in \mathbb{R}^d$,

$$\left|\int_0^t P_{t-s} f_s(x) \, ds - \int_0^t P_{t-s} f_s(x') \, ds\right| \leq cL(|x - x'|^{\delta/2} \wedge 1).$$

27
Assume that Lemma 5.9

Combining (5.30) and (5.31), we immediately get the desired conclusion. Thus for all $t \in (0,1]$ and $x, x' \in \mathbb{R}^d$,

\[ |p(t,x) - p(t,x')| \leq c_1 \left( (\Phi^{-1}(t^{-1})|x-x'|) \wedge 1 \right) t \left( \rho(t,x) + \rho(t,x') \right). \tag{5.29} \]

Thus using (5.28) we get that for any $\varphi$,

\[ |\varphi(\lambda r)| \leq c_2 L \int_0^t \left( (\Phi^{-1}(s^{-1})|x-x'|) \wedge 1 \right) ds. \tag{5.30} \]

When $|x-x'| < 1$, $\Phi(|x-x'|^{-1}) \geq \Phi(1) = 1$. Thus,

\[ \int_0^t \left( (\Phi^{-1}(s^{-1})|x-x'|) \wedge 1 \right) ds \leq |x-x'| \int_0^1 \Phi^{-1}(s^{-1})ds + (\Phi(|x-x'|^{-1}))^{-1}. \]

It is well known that $\varphi$, the Laplace exponent of a subordinator, satisfies

\[ \varphi(\lambda r) \leq \lambda \varphi(r), \quad \lambda \geq 1, r > 0. \]

Using this, we immediately get that

\[ \Phi^{-1}(\lambda r) \geq \lambda^{1/2} \Phi^{-1}(r), \quad \lambda \geq 1, r > 0. \]

For $s \in ([\Phi(|x-x'|^{-1})^{-1}, 1]$, by taking $r = s^{-1}$ and $\lambda = s \Phi(|x-x'|^{-1})$ in the display above, we get

\[ \Phi^{-1}(s^{-1}) \leq |x-x'|^{-1/2}(\Phi(|x-x'|^{-1}))^{-1/2}. \]

Therefore

\[ |x-x'| \int_0^1 \Phi^{-1}(s^{-1})ds \leq (\Phi(|x-x'|^{-1}))^{-1/2} \int_0^1 s^{-1/2}ds \leq c_3 (\Phi(|x-x'|^{-1}))^{-1/2}. \]

Consequently for all $t \in (0,1]$ and $x, x' \in \mathbb{R}^d$ with $|x-x'| < 1$, we have

\[ \int_0^t \left( (\Phi^{-1}(t^{-1})|x-x'|) \wedge 1 \right) ds \leq c_4 (\Phi(|x-x'|^{-1}))^{-1/2}. \]

By taking $r = 1$ and $\lambda = |x-x'|^{-1}$ in (5.28), we get

\[ a_1 |x-x'|^{-\delta} \leq \Phi(|x-x'|^{-1}). \]

Thus for all $t \in (0,1]$ and $x, x' \in \mathbb{R}^d$ with $|x-x'| < 1$, we have

\[ \int_0^t \left( (\Phi^{-1}(s^{-1})|x-x'|) \wedge 1 \right) ds \leq c_4 a_1^{-1/2} |x-x'|^{\delta/2}. \tag{5.31} \]

Combining (5.30) and (5.31), we immediately get the desired conclusion.

**Lemma 5.9** Assume that $f_s(x)$ satisfies the following conditions:

(i) There is a constant $L$ so that, for all $(s,x) \in [0,1] \times \mathbb{R}^d$, $|f_s(x)| \leq L$. 

28
(ii) For any $t_0 \in [0, 1]$, \( \lim_{s \to t_0} \sup_{x \in \mathbb{R}^d} |f_s(x) - f_{t_0}(x)| = 0. \)

(iii) There exist constants $s_0 \in (0, 1)$, $\gamma \in (0, \delta/2]$ and $C > 0$ such that for all $s \in [0, s_0]$ and $x, x' \in \mathbb{R}^d$ with $|x - x'| \leq 1$,

\[
|f_s(x) - f_s(x')| \leq C|x - x'|^\gamma. \tag{5.32}
\]

Then, $t \to \int_0^t P_{t-s}f_s(x) \, ds$ is differentiable on $(0, s_0)$, and for $0 \leq t < s_0$,

\[
\frac{\partial}{\partial t} \int_0^t P_{t-s}f_s(x) \, ds = \int_0^t \frac{\partial}{\partial t} P_{t-s}f_s(x) \, ds + f_t(x). \tag{5.33}
\]

**Proof:** Let $G(t, x) := \int_0^t P_{t-s}f_s(x) \, ds$. For any $0 < t < t + r < s_0$, by the definition of $G(t, x)$,

\[
\frac{1}{r} (G(t + r, x) - G(t, x))
\]

\[
= \frac{1}{r} \int_0^t \left( P_{t+r-s}f_s(x) - P_{t-s}f_s(x) \right) \, ds + \frac{1}{r} \int_r^{t+r} P_{t+r-s}f_s(x) \, ds
\]

\[
= \int_0^t \frac{P_{t+r-s}f_s(x) - P_{t-s}f_s(x)}{r} \, ds + \frac{1}{r} \int_0^r P_{t-r}f_{t+s}(x) \, ds
\]

\[
:= (I) + (II).
\]

Using the same arguments as those leading to (5.10), we get

\[
\lim_{t \downarrow 0} t^{-1} \int_0^t P_{t-s}f_s(x) \, ds = f_0(x),
\]

which implies that

\[
\lim_{r \downarrow 0} (II) = f_t(x). \tag{5.34}
\]

Now we deal with part $(I)$. For $0 < t < t + r < s_0$, using (5.27), we obtain that

\[
\left| \frac{P_{t+r-s}f_s(x) - P_{t-s}f_s(x)}{r} \right|
\]

\[
= \left| \int_{\mathbb{R}^d} \frac{p(t + r - s, x, y) - p(t - s, x, y)}{r} (f_s(y) - f_s(x)) \, dy \right|
\]

\[
\leq c_3 \int_{\mathbb{R}^d} |f_s(y) - f_s(x)| \rho(t - s, x - y) \, dy
\]

\[
\leq c_4 \int_{\mathbb{R}^d} \rho_0^\gamma (t - s, x - y) \, dy
\]

\[
\leq c_5 (t - s)^{-1} \Phi^{-1}((t - s)^{-1})^{-\gamma}, \tag{5.35}
\]

where in the last inequality we used [9 Lemma 2.6(a)]. It follows from [9 Lemma 2.3] that

\[
\int_0^t (t - s)^{-1} \Phi^{-1}((t - s)^{-1})^{-\gamma} \, ds \leq c_6 \Phi^{-1}(t^{-1})^{-\gamma}. \tag{5.36}
\]

Thus, using the dominated convergence theorem, we get that, for any $0 \leq t < t + r < s_0$,

\[
\lim_{r \downarrow 0} (I) = \int_0^t \lim_{r \downarrow 0} \frac{P_{t+r-s}f_s(x) - P_{t-s}f_s(x)}{r} \, ds = \int_0^t \frac{\partial}{\partial t} P_{t-s}f_s(x) \, ds. \tag{5.37}
\]
Combining (5.34) and (5.37), we get that
\[
\lim_{r \downarrow 0} \frac{G(t+r,x) - G(t,x)}{r} = \int_0^t \frac{\partial}{\partial t} P_{t-s} f_s(x) \, ds + f_t(x).
\]

Using similar arguments, we can also show that
\[
\lim_{r \downarrow 0} \frac{G(t,x) - G(t-r,x)}{r} = \int_0^t \frac{\partial}{\partial t} P_{t-s} f_s(x) \, ds + f_t(x).
\]

Thus, (5.33) follows immediately. The proof is now complete.

Lemma 5.10 For any \( s > 0 \), there is a constant \( c(s) \) such that for \( t \in [0,1/2) \) and \( x,y \in \mathbb{R}^d \),
\[
|v_{t+s}(x) - v_{t+s}(y)| \leq c(s)|x - y|^{\delta/2}.
\]
Moreover, \( c(s) \) is decreasing in \( s > 0 \).

Proof: The proof of this lemma is similar as that of Lemma 5.4. We use Lemma 5.8 instead of Lemma 5.2. Here we omit the details.

Lemma 5.11 The function \( \Psi_u(x) \) satisfies the following two properties:

1. For any \( u_0 > 0 \),
\[
\lim_{u \to u_0} \sup_{x \in \mathbb{R}^d} |\Psi_u(x) - \Psi_{u_0}(x)| = 0;
\]

2. For \( t_0 \in (0,1) \), there exists a constant \( c > 0 \) and \( \gamma_1 \in (0,\delta/2] \) such that for any \( |x - x'| \leq 1 \), \( s > t_0 \) and \( t \in [0,1/2) \),
\[
|\Psi_{s+t}(x) - \Psi_{s+t}(x')| \leq c|x - x'|^{\gamma_1}.
\]

Proof: The proof of part (1) is exactly the same as that of part (1) of Lemma 5.5.

Using arguments similar to that in the proof of part (2) of Lemma 5.3 and using Lemma 5.10 instead of Lemma 5.4 we can get the result in part (2). Here we omit the details.

Lemma 5.12 The function \( t \to v_t(x) \) is differentiable in \( (0,\infty) \), and for any \( s > 0 \) and \( t \in [0,1/2) \), \( w(t+s,x) = -\frac{\partial}{\partial t} v_{t+s}(x) \) satisfies that
\[
w(t+s,x) = -\frac{\partial}{\partial t} P_t(v_s)(x) + \int_0^t \frac{\partial}{\partial t} P_{t-u}(\Psi_{s+u})(x) \, du + \Psi_{t+s}(x).
\]

(5.38)

Moreover, \( t \to w(t,x) \) is continuous and for any \( s_0 > 0 \), \( \sup_{s > s_0} \sup_{x \in \mathbb{R}^d} w(s,x) < \infty \).

Proof: Combining Lemmas 5.7, 5.9 and 5.11 and using arguments similar to that in the proof of Lemma 5.6, Lemma 5.12 follows immediately.

Remark 5.13 Actually, by the same arguments and the results from [9], one check that in the example above, we could have replaced the subordinate Brownian motion by the non-symmetric jump process considered there, which contains the non-symmetric stable-like process discussed in [2].
Let $L$ be as in Example 2. Let $E$ be a bounded smooth domain in $\mathbb{R}^d$ and let $p(t,x,y)$ be the Dirichlet heat kernel of $L$ in $E$. It follows from [7, Theorem 2.1, p. 247] that there exist $c_i > 0, i = 1, 2, 3, 4$, such that for all $t \in (0,1]$,

$$\left| \frac{\partial}{\partial t} p(t,x,y) \right| \leq c_1 t^{\frac{d-1}{2}} e^{-\frac{c_3 |x-y|^2}{t}},$$

and

$$|\nabla_x p(t,x,y)| \leq c_3 t^{\frac{d+1}{2}} e^{-\frac{c_3 |x-y|^2}{t}}.$$

Using these instead of (5.4) and (5.5), and repeating the arguments for Example 4, we can get the following example.

**Example 6** Assume that $E$ be is bounded smooth domain in $\mathbb{R}^d$ and that the spatial motion is $\xi^E$, which is the diffusion $\xi$ of Example 2 killed upon exiting $E$. The branching mechanism $\Psi$ is of the form in (2.1) and satisfies (2.7) and (5.2) on $E$. Then the $(\xi^E, \Psi)$-superprocess $X$ satisfies Assumptions (H1) and (H2).

**References**

[1] Abraham, R. and Delmas, J.-F.: Williams decomposition of the Lévy continuum random tree and simultaneous extinction probability for populations with neutral mutations. *Stochastic Process. Appl.* 119 (2009), 1124–1143.

[2] Chen, Z.-Q. and Zhang, X.: Heat kernels and analyticity of non-symmetric jump diffusion semigroups. *Probab. Theory Relat. Fields*, DOI 10.1007/s00440-015-0631-y.

[3] Dawson, D. A.: *Measure-Valued Markov Processes*. Springer-Verlag, 1993.

[4] Delmas, J. F. and Hénard, O.: A Williams decomposition for spatially dependent super-processes. *Electron. J. Probab.* 18 (2013), 1–43.

[5] Dynkin, E. B.: Superprocesses and partial differential equations. *Ann. Probab.* 21 (1993), 1185–1262.

[6] Dynkin, E. B. and Kuznetsov, S. E.: $\mathbb{N}$-measure for branching exit Markov system and their applications to differential equations. *Probab. Theory Rel. Fields* 130 (2004), 135–150.

[7] Garroni, M. G. and Menaldi, J.-L.: *Green functions for second order parabolic integro-differential problems*. Longman, Harlow, 1992.

[8] El Karoui, N. and Roelly, S.: Propriétés de martingales, explosion et représentation de Lévy-Khintchine d’une classe de processus de branchement à valeurs mesures. *Stoch. Proc. Appl.* 38 (1991), 239–266.

[9] Kim, K., Song, R. and Vondracek, Z.: Heat kernels of non-symmetric jump processes: beyond the stable case. *arXiv:1606.02005*

[10] Kyprianou, A. E.: *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer.

[11] Ladyzenskaja, O. A., Solomnikov, V. A. and Ural’ceva, N. N.: *Linear and Quasi-linear Equations of Parabolic Type*. American Math. Soc., Providence, Rhode Island, 1968.

[12] Li, Z.: Skew convolution semigroups and related immigration processes. *Theory Probab. Appl.* 46 (2003), 274–296.

[13] Li, Z.: *Measure-Valued Branching Markov Processes*. Springer, Heidelberg, 2011.

[14] Stroock, D. W.: *Probability Theory. An Analytic View*. 2nd ed. Cambridge University Press, Cambridge, 2011.

[15] Tribe, R.: The behavior of superprocesses near extinction. *Ann. Probab.* 20 (1992), 286-311.
Yan-Xia Ren: LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing, 100871, P.R. China. Email: yxren@math.pku.edu.cn

Renming Song: Department of Mathematics, University of Illinois, Urbana, IL 61801, U.S.A. Email: rsong@math.uiuc.edu

Rui Zhang: School of Mathematical Sciences, Capital Normal University, Beijing, 100048, P.R. China. Email: zhangrui27@cnu.edu.cn