EINSTEIN SOLVMANIFOLDS AS SUBMANIFOLDS OF
SYMMETRIC SPACES

MICHAEL JABLONSKI

Abstract. Lying at the intersection of Ado’s theorem and the Nash embedding theorem, we consider the problem of finding faithful representations of Lie groups which are simultaneously isometric embeddings. Such special maps are found for a certain class of solvable Lie groups which includes all Einstein and Ricci soliton solvmanifolds, as well as all Riemannian 2-step nilpotent Lie groups. As a consequence, we extend work of Tamaru by showing that all Einstein solvmanifolds can be realized as submanifolds (in the submanifold geometry) of a symmetric space.

Recently, Tamaru [Tam11] constructed a collection of explicit examples of Einstein solvmanifolds which arise as submanifolds of symmetric spaces; here by submanifold we mean with the submanifold geometry. There a finite number of examples are produced, for each symmetric space of non-compact type; these examples are also interesting in that they are not totally geodesic. Naturally, it raises the question of whether or not all Einstein solvmanifolds arise this way.

Question. Does every Einstein solvmanifold arise as a submanifold of a symmetric space?

To approach this question, we reframe it into a question on solvable Lie groups. Recall that every symmetric space is a totally geodesic submanifold of the symmetric space $GL(n,\mathbb{R})/O(n,\mathbb{R})$, which can be naturally identified with the group of lower triangular matrices $T(n,\mathbb{R})$ endowed with its left-invariant Einstein metric. This Einstein metric is unique up to scaling. Even stronger than being a submanifold of a symmetric space, those examples by Tamaru can all be realized as subgroups of the group of lower triangular matrices. As every Einstein solvmanifold is isometric to a completely solvable Lie group with left-invariant metric [Lau10], it seems natural to ask the even more delicate question of whether one can achieve an embedding of Einstein solvmanifolds via faithful representations into the lower triangular matrices.

Date: October 25, 2018.
MSC2010: 53C25, 53C30, 53C40, 22E25
This work was supported in part by NSF grant DMS-1612357.
**Definition 0.1.** Consider the group of lower triangular matrices $T(n, \mathbb{R})$ endowed with its left-invariant Einstein metric. Let $S$ be a solvable Lie group with left-invariant metric. By an isomorphic, isometric embedding, we mean a faithful representation

$$\phi : S \to T(n, \mathbb{R})$$

which is an isometric embedding of Riemannian manifolds.

**Theorem 0.2.** Let $S$ be a simply-connected, completely solvable Lie group endowed with a left-invariant Einstein metric. Then there exists $n \in \mathbb{N}$ and an isomorphic, isometric embedding $\phi : S \to T(n, \mathbb{R})$ where $T(n, \mathbb{R})$ is endowed with its Einstein metric.

Essentially what we are describing above is a natural combination of Ado’s Theorem (which says every Lie group is locally linear) and Nash’s Embedding Theorem for Riemannian manifolds. On a solvable Lie group, if an Einstein metric exists it is unique (up to scaling and isometry) and so the symmetric metric on $T(n, \mathbb{R})$ plays the natural rôle of the Euclidean metric on $\mathbb{R}^n$. Furthermore, Einstein metrics are known to have maximal symmetry and this is even more reason to consider them as a preferred choice of metric [GJ18].

One consequence of the theorem above is an affirmative answer to the question at the beginning.

**Corollary 0.3.** Every simply-connected, Einstein solvmanifold can be realized as a submanifold of a symmetric space.

It seems natural to ask if any Lie group with left-invariant metric can be realized as a submanifold (in the submanifold geometry) of some canonical Riemannian Lie group. In some special cases, we can say that this is true.

**Corollary 0.4.** Let $S$ be a simply-connected, completely solvable Lie group endowed with a left-invariant Ricci soliton metric. Then there exists $n \in \mathbb{N}$ and an isomorphic, isometric embedding $\phi : S \to T(n, \mathbb{R})$ where $T(n, \mathbb{R})$ is endowed with its Einstein metric.

Recall, for solvable Lie groups that cannot admit an Einstein metric, such as nilpotent Lie groups, a Ricci soliton is the next best thing. For a general metric on a solvable Lie group, we do not know if such embeddings exist. However, we can say more for 2-step nilpotent Lie groups.

**Theorem 0.5.** Every simply-connected, 2-step nilpotent Lie group with left-invariant metric isomorphically, isometrically embeds into some $T(n, \mathbb{R})$ endowed with its Einstein metric.

We suspect the same holds for every nilpotent group with left-invariant metric, but do not have a proof at hand.
Finally, we comment on the compact setting to put our results above in perspective. The natural candidate in which to embed a compact Lie group is $O(n, \mathbb{R})$ with its bi-invariant metric. However, the above results cannot hold in this setting. Any subgroup of $O(n, \mathbb{R})$ would pick up a bi-invariant metric itself with the submanifold geometry, and there are many metrics on a given non-abelian, compact Lie group that are not bi-invariant.

Perhaps there is another, natural candidate besides the bi-invariant metric that one could use for $O(n, \mathbb{R})$; this would be interesting to know.

Acknowledgments. The author would like to thank Megan Kerr and Tracy Payne for conversations in which the question above arose.

1. Preliminaries

Our first observation is that one can reduce to the study of metric Lie algebras, that is Lie algebras endowed with an inner product. Recall, simply-connected Lie groups with left-invariant metrics are in one-to-one correspondence with Lie algebras with inner products.

If one has an isomorphic, isometric embedding $\phi: S \to T(n, \mathbb{R})$, then one immediately has the induced homomorphism between their Lie algebras

$$\phi: s \to t(n, \mathbb{R})$$

which is an isomorphism onto its image and which preserves the inner product on the Lie algebra, i.e.

$$\langle X, Y \rangle_s = \langle \phi_*(X), \phi_*(Y) \rangle_{t(n, \mathbb{R})} \quad \text{for } X, Y \in s.$$  

Conversely, having such a map at the Lie algebra level lifts to a map between the simply-connected Lie groups $S$ and $T(n, \mathbb{R})$.

We will often simply denote either of the inner products on $s$ or $t(n, \mathbb{R})$ by $\langle , , \rangle$ as long as the context is clear. Further, we will denote by $\phi^*\langle , , \rangle$ the induced inner product on $s$ that one gets by restricting the inner product on $t(n, \mathbb{R})$ to the image $\phi_*(s)$; at the group level this corresponds to the submanifold geometry of $\phi(S)$ sitting inside of $T(n, \mathbb{R})$.

The metric on $T(n, \mathbb{R})$. The group $T(n, \mathbb{R})$ has two seemingly natural left-invariant metrics which we describe via the corresponding inner products on $t(n, \mathbb{R})$. First is the Frobenius inner product. This is just the restriction of the usual inner product on $\mathfrak{gl}(n, \mathbb{R})$ given by

$$B(X, Y) = tr \langle XY^t \rangle \quad \text{for } X, Y \in \mathfrak{gl}(n, \mathbb{R}).$$  

Although familiar, there is no geometric way to justify using this metric.

Using the Frobenius inner product, we can describe the Einstein inner product on $t(n, \mathbb{R})$. (We call it the Einstein inner product as the corresponding left-invariant metric on $T(n, \mathbb{R})$ is Einstein.)
Write \( t(n, \mathbb{R}) = a + n \) where \( a \) is the set of diagonal matrices and \( n \) is the set of strictly lower triangular matrices. The Einstein inner product on \( t(n, \mathbb{R}) \) is given by
\[
\langle , \rangle = 2B \quad \text{on } a \times a \\
\langle , \rangle = B \quad \text{on } n \times n
\] (1.2)
with \( a \) and \( n \) orthogonal. The corresponding left-invariant geometry on \( T(n, \mathbb{R}) \) is isometric to \( GL(n, \mathbb{R})/O(n, \mathbb{R}) \) with the symmetric metric and so stands out geometrically.

Our main results hold for both geometries on \( T(n, \mathbb{R}) \), but we are primarily interested in the symmetric space/Einstein geometry.

**Picking a scale.** Up to scaling, there is only one Einstein metric on \( T(n, \mathbb{R}) \). However, for the above theorems, one actually has to pick a scale. Instead, one can equivalently look for representations \( \phi : s \to t(n, \mathbb{R}) \) which satisfy
\[
\phi^* \langle , \rangle_{t(n, \mathbb{R})} = c\langle , \rangle_s \quad \text{for some } c \in \mathbb{R}.
\] (1.3)
Then one simply chooses the “right” Einstein metric on the left so as to make \( c = 1 \) on the right. In the work that follows, we will find \( \phi \) which satisfies Eqn. [1.3]

**Structure of Einstein solvmanifolds.** Let \( S \) be a solvable Lie group which admits an Einstein metric with Lie algebra \( s \). Then \( s = a + n \) where \( n \) is the nilradical and \( a \) is an abelian subalgebra acting reductively on \( n \). Further, under the Einstein metric, \( a \perp n \).

From here going forward, we assume that \( S \) is completely solvable - i.e \( \text{ad } X \) has real eigenvalues for \( X \in s \). For such an \( s \) with Einstein metric, it is known that there exists some \( A \in a \) such that \( D = \text{ad } A \) has positive eigenvalues on \( n \); consequently, \( s \) has no center. Furthermore, \( \text{ad } a \) acts by symmetric transformations (relative to the Einstein inner product).

For a detailed discussion of the structure of Einstein solvmanifolds, we refer the interested reader to [Lau10].

2. **Isomorphically embedding Einstein solvmanifolds**

To prove our main results, we actually work in a slightly more general setting. We consider metric Lie algebras \( (s, \langle , \rangle) \) satisfying the following conditions:

(i) \( s = a + n \) is completely solvable
(ii) \( n \) is the nilradical of \( s \)
(iii) \( a \) is an abelian subalgebra acting fully-reducibly on \( n \)
(iv) \( a \) acts by symmetric transformations on \( s \)
(v) there is some \( A \in a \) such that \( D = \text{ad } A \) has positive eigenvalues on \( n \)
Remark 2.1. As \( \text{ad} \, a \) acts symmetrically on \( s \), its weight spaces are orthogonal. Furthermore, the last item implies \( a \perp n \).

Theorem 2.2. Let \( S \) be a simply-connected, completely solvable Lie group with metric Lie algebra satisfying the above conditions (i)-(v). There exists an isomorphic, isometric embedding \( \phi : S \to T(n, \mathbb{R}) \), up to scaling, where \( T(n, \mathbb{R}) \) is the group of lower triangular matrices endowed with the Einstein metric given in Eqn. 1.1.

Remark 2.3. Given the structure of Einstein solvmanifolds outlined in Section 1, clearly the above theorem implies Theorem 0.2.

The eigenspace decomposition of \( D \) on \( n \) gives a grading to the nilradical
\[
\mathfrak{n} = \mathfrak{n}_{\lambda_1} \oplus \cdots \oplus \mathfrak{n}_{\lambda_k},
\]
where \( 0 < \lambda_1 < \cdots < \lambda_k \) are the eigenvalues of \( D \).

Remark 2.4. We recall that the grading above satisfies \( [\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j} \). A useful consequence that we note for later is that \( \mathfrak{n}_{\lambda_k} \subset \mathfrak{i} \), the center of \( \mathfrak{n} \).

2.1. Strategy. We begin with an outline of the proof of Theorem 2.2. The first step is to reduce the problem to studying what happens on the nilradical. Step 2 will be to solve the problem in the special case that there exists a faithful representation which simply rescales the metric on each eigenspace. On the nilradical, our strategy is to use induction on the number of eigenvalues of \( D \). Finally, the last step will be to reduce to this special case.

The representations. To prove the theorem, we will start with a familiar representation and modify it a piece at a time. Let \( \mathfrak{s} \) be a solvable Lie algebra as above and consider the adjoint representation
\[
\phi = \text{ad} : \mathfrak{s} \to \mathfrak{gl}(\mathfrak{s}).
\]
As \( D = \text{ad} \, A \) is non-singular on \( \mathfrak{n} \) for some \( A \in \mathfrak{a} \), by hypothesis, \( \mathfrak{s} \) has no center and we have that \( \phi \) is a faithful representation.

Consider \( D = \text{ad} \, A \) where the eigenvalues are positive on \( \mathfrak{n} \). Now choose an ordered basis of \( \mathfrak{s} \) which is a union of bases of the weight spaces of \( \text{ad} \, \mathfrak{a} \) and ordered so that the lower eigenspaces of \( D \) come first. In this way, we see that \( \phi(\mathfrak{s}) \subset \mathfrak{t}(n, \mathbb{R}). \)

Starting with such a representation, we will modify it by composing with automorphisms and also adding such representations together.

For example, given two representations \( \phi_1 : \mathfrak{s} \to \mathfrak{gl}(V_1) \) and \( \phi_2 : \mathfrak{s} \to \mathfrak{gl}(V_2) \), we may build a new representation
\[
\phi_1 + \phi_2 : \mathfrak{s} \to \mathfrak{gl}(V_1) \times \mathfrak{gl}(V_2) \subset \mathfrak{gl}(V_1 + V_2).
\]
If either \( \phi_1 \) is non-singular, so is \( \phi_1 + \phi_2 \).
Remark 2.5. As the coordinates of \( \phi_1 \) and \( \phi_2 \) lie in different blocks, they are orthogonal relative to both the Frobenius and Einstein inner products. Likewise, for representations into \( T(n, \mathbb{R}) \times T(m, \mathbb{R}) \subset T(n+m, \mathbb{R}) \), the coordinates of \( \phi_1 \) and \( \phi_2 \) are orthogonal. Moreover, \(|\phi_1(X) + \phi_2(X)| = |\phi_1(X)| + |\phi_2(X)|\) - this property will be essential in the work that follows.

3. Step 1: Reduce to the nilradical

Consider \( s = a \oplus n \). In the following sections, we will build a family of representations \( \phi_t \) of \( s \), for \( t \in \mathbb{R} \) large, such that

(i) \( \langle \phi_t(a), \phi_t(n) \rangle = 0 \);
(ii) On \( a \), we have that \( \phi_t^* \langle \cdot, \cdot \rangle \) is constant on \( a \times a \) as \( t \in \mathbb{R} \) varies;
(iii) On \( n \), we have that \( \phi_t \) solves the following

\[ \phi_t^* \langle \cdot, \cdot \rangle = t \langle \cdot, \cdot \rangle \] on \( n \times n \).

Using such a family of representations, we demonstrate how to prove Theorem 2.2.

What remains is to alter the inner product on \( \phi_t(a) \); we can do this by adding on blocks to our representation for only the elements of \( \phi(a) \). More precisely, given \( \phi_t : s \rightarrow gl(V) \), then we may consider

\[ gl(V) \subset gl(V) \oplus gl(W) \subset gl(V \oplus W) \].

Replace \( \phi_t \) with some \( \Phi_t \) such that on \( n \) we have \( \Phi_t = \phi_t \), but on \( a \) we have that

\[ \Phi_t(A) = \phi_t(A) + diag(A) \],

where \( diag(A) \) is an element of the diagonal in \( gl(W) \). Observe, \( \Phi_t \) is still an isomorphism onto its image; all that changes will be the inner product on the image of \( a \). We will do this using \( W = a \).

Building \( \Phi_t \) on \( a \). On \( a \times a \), consider the symmetric, bilinear form \( \langle \phi_t', \phi_t' \rangle \). For large \( c \in \mathbb{R} \), we have

\[ c\langle \cdot, \cdot \rangle - \langle \phi_t', \phi_t' \rangle \]

is a positive definite, symmetric, bilinear form - i.e. an inner product - on \( a \). As such, there is an element \( g \in GL(a) \) such that

\[ \langle g', g' \rangle = c\langle \cdot, \cdot \rangle - \langle \phi_t', \phi_t' \rangle \].

Now identify the diagonal in \( GL(W) \) with \( a \) (which we can do as they have the same dimension) and consider the map

\[ \Phi_t(A) = \phi_t(A) + g(A) \in gl(V) + gl(W) \].
By construction, we have that for large $c \in \mathbb{R}$ we can always achieve
\begin{equation}
\Phi^*_t \langle , \rangle = c \langle , \rangle \text{ on } \mathfrak{a} \times \mathfrak{a} \text{ for all large } t \in \mathbb{R}.
\end{equation}
Note, here $c$ and $t$ are independent of each other.
Choosing $c$ and $t$ large enough so that $\Phi_t$ exists and solves Eqn. 3.1 and taking the max of these two we may assume $c = t$ and now we have
\begin{equation}
\Phi^*_t \langle , \rangle = t \langle , \rangle
\end{equation}
on all of $\mathfrak{s} \times \mathfrak{s}$, i.e. we have embedded the metric on $\mathfrak{s}$ up the scaling.

4. Step 2: The nilradical in the special case

In this section we will prove the theorem starting with the following special hypothesis:

there exists a representation $\phi$ such that $\phi^* \langle , \rangle$ leaves the eigenspaces of $D$ orthogonal and simply rescales $\langle , \rangle$ on each eigenspace.

The reduction to this special case will be carried out in the following section.

Remark 4.1. We will be using induction on the number of eigenvalues of $D$ and assuming the hypothesis holds for all such $\mathfrak{s}$ with $k$ or fewer eigenvalues of $D$.

Consider the orthogonal eigenspace decomposition of $\mathfrak{n}$ relative to the positive derivation $D$

$$\mathfrak{n} = \mathfrak{n}_{\lambda_1} \oplus \cdots \oplus \mathfrak{n}_{\lambda_k}.$$ 

As $\mathfrak{n}_{\lambda_k}$ is an ideal, we may consider the quotient algebra $\mathfrak{n}^{(2)} = \mathfrak{n}/\mathfrak{n}_{\lambda_k} \simeq \mathfrak{n}_{\lambda_1} \oplus \cdots \oplus \mathfrak{n}_{\lambda_{k-1}}$. We endow $\mathfrak{n}^{(2)}$ with the inner product from $\mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_{k-1}$.

Likewise, we may define and endow with inner products the following

\begin{equation}
\begin{align*}
\mathfrak{n}^{(3)} &= \mathfrak{n}/(\mathfrak{n}_{\lambda_{k-1}} \oplus \mathfrak{n}_{\lambda_k}) \simeq \mathfrak{n}_{\lambda_1} \oplus \cdots \oplus \mathfrak{n}_{\lambda_{k-2}} \\
\mathfrak{n}^{(4)} &= \mathfrak{n}/(\mathfrak{n}_{\lambda_{k-2}} \oplus \cdots \oplus \mathfrak{n}_{\lambda_k}) \simeq \mathfrak{n}_{\lambda_1} \oplus \cdots \oplus \mathfrak{n}_{\lambda_{k-3}} \\
& \vdots \\
\mathfrak{n}^{(k)} &= \mathfrak{n}/(\mathfrak{n}_{\lambda_2} \oplus \cdots \oplus \mathfrak{n}_{\lambda_k}) \simeq \mathfrak{n}_{\lambda_1}
\end{align*}
\end{equation}

We define $\mathfrak{n}^{(1)} := \mathfrak{n}$ for convenience.

Observe that $D$ induces a positive derivation on each $\mathfrak{n}^{(i)}$ whose eigenvalues are the first $k+1-i$ eigenvalues of $D$ on $\mathfrak{n}$. For simplicity, we will denote each of these derivations by $D$. Further, $a(t) = exp(tD) \text{ is an automorphism of } \mathfrak{n}^{(i)} \text{ which acts as scalar multiplication by } e^{t\lambda}$ on the eigenspace $\mathfrak{n}_{\lambda}$. 

By hypothesis, on each $\mathfrak{n}^{(i)}$ we have faithful representation $\phi_i$ which simply rescales each eigenspace. If we precompose with $a(t)$, then for large $t$ we see
that the scaling on the last eigenspace \( n_{\lambda} \) is the dominant scaling - relatively speaking, the lower eigenspaces have their inner product essentially shrunk as small as we like.

Now we consider

\[ \phi = \phi_1 \circ a(t_1) + \cdots + \phi_{k-1} \circ a(t_{k-1}). \]

as in Eqn. 2.2. Choosing large \( t_i \), we can make all the eigenspaces scaled by the same amount. And further that we can get any scaling past a certain point; i.e. we can solve

\[ \langle \phi_v, \phi_w \rangle = t \langle v, w \rangle, \]

for any large \( t \in \mathbb{R} \). In the notation of the previous section, this representation is \( \phi_t \).

Next we show how to reduce to this special case.

5. Step 3: Reducing to the special case

In this section, we will complete the proof of the main theorem by constructing a representation which keeps eigenspaces of \( D \) orthogonal and simply rescales the inner product on each eigenspace.

Consider the solvable algebra \( s = a \ltimes n \). As \( s \) has no center, its adjoint representation

\[ \phi = \text{ad} : s \to \mathfrak{gl}(s). \]

is faithful.

Choose an (ordered) orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( s \) which consists of a union of bases of the eigenspaces \( n_{\lambda_i} \) of \( D \) which are ordered to have the lesser eigenvalues first, starting with \( a = n_0 \). By making this choice, we have that \( \phi(s) \) is in the set of lower triangular matrices \( T(n, \mathbb{R}) \).

Using the inner product on \( s \), we have the Frobenius inner product on \( \mathfrak{gl}(s) \) given by

\[ \langle A, B \rangle = \text{tr}(AB^t) = \sum_i \langle Ae_i, Be_i \rangle, \]

for \( A, B \in \mathfrak{gl}(s) \) and an orthonormal basis \( \{e_i\} \) of \( s \). We consider the corresponding Einstein metric on \( T(n, \mathbb{R}) \) as above, see Section II.

Notice that \( \phi(a) \) consists of diagonal matrices while \( \phi(n) \) consists of strictly lower triangular matrices. Furthermore, let \( n_\alpha \) and \( n_\beta \) be two distinct weight spaces of \( \text{ad} \ a \). Since \( [n_\alpha, n_\beta] \subset n_{\alpha+j} \), we see that for \( \alpha \neq \beta \), \( \phi(n_\alpha) \) and \( \phi(n_\beta) \) share no common entries and so

\[ \phi(n_\alpha) \perp \phi(n_\beta) \]

with respect to the Frobenius and Einstein inner products.
Furthermore, consider \( e_i \in n_{\lambda_k} \), the highest eigenspace of \( D \). Since \( e_i \) is central in \( n \), we are able to easily compute \( \phi(e_i) \), as follows. Let \( a = \dim a \) and \( \{ A_1, \ldots, A_a \} \) be our orthonormal basis of \( a \). Then

\[
\phi(e_i) = \sum_{1 \leq j \leq a} \lambda(A_j) E_{i,j}
\]

where \( \lambda(A_j) \) is the weight of \( \text{ad} \ A_j \) acting on \( n_{\lambda_k} \). Observe, we have produced a faithful representation with the property that it rescales the metric on the weight spaces of \( \text{ad} \ a \) acting on \( n_{\lambda_k} \). More precisely, on each weight space we have

\[
\phi^* \langle \ , \ \rangle = c \langle \ , \ \rangle,
\]

where \( c = \sqrt{\sum \lambda(A_j)^2} \).

Now we just need to do two things:

(i) adjust the representation so that on \( n_{\lambda_k} \) the metric is simply rescaling, and

(ii) adjust the representation so that on \( n_{\lambda_1} \oplus \cdots \oplus n_{\lambda_{k-1}} \) the metric simply rescales on each factor.

We achieve both using a common idea.

5.1. **Tweaking the metric on** \( n_{\lambda_k} \). Next we demonstrate how to alter the inner product on \( n_{\lambda_k} \). Consider a lower triangular matrix \( L \) which is the identity on \( a \oplus n_{\lambda_1} \oplus \cdots \oplus n_{\lambda_{k-1}} \), while on \( n_{\lambda_k} \) need only preserve the weight spaces of \( \text{ad} \ a \) on \( n_{\lambda_k} \) - so it is block lower triangular on \( n_{\lambda_k} \).

Let \( C_L \) denote conjugation by \( L \) in \( \mathfrak{gl}(s) \). The composition

\[
C_L \circ \phi
\]

is still a faithful representation. The images of the weight spaces \( n_\alpha \) of \( \text{ad} \ a \) are still orthogonal relative to the Frobenius and Einstein inner products. Further, for \( Z_1, Z_2 \in n_{\lambda_k} \) in the same weight space, we see that

\[
\langle C_L \circ \phi(Z_1), C_L \circ \phi(Z_2) \rangle_{(n,R)} = c \langle L(Z_1), L(Z_2) \rangle_n,
\]

where \( c = \sqrt{\sum \lambda(A_j)^2} \) (cf. Eqn. 5.1) and the inner product on the left is either the Frobenius or the Einstein inner product (as they coincide on the lower triangular matrices).

**Remark 5.1.** The constant \( c \) varies depending on which weight space one is looking at, but clearly we can choose the lower triangular matrix \( L \) so that all weight spaces have the same constant. Below we go even further.

For a vector space \( V \) with inner product \( \langle \ , \ \rangle \), recall that every symmetric, positive definite, bilinear form on \( V \) can be represented as

\[
\langle L(\cdot), L(\cdot) \rangle \quad \text{for} \ L \in \text{GL}(V) \text{ a lower triangular matrix.}
\]
Thus we can alter our faithful representation $\phi$ to have any prescribed inner product on $n_{\lambda_k}$ so long as the weight spaces of the ad $a$ action remain orthogonal.

We apply the above to $n^{(2)} \simeq n_{\lambda_1} \oplus \cdots \oplus n_{\lambda_{k-1}}$. For $c \in \mathbb{R}$, consider the symmetric, bilinear form

$$c\langle \cdot, \cdot \rangle_n - \langle \phi \cdot, \phi \cdot \rangle_{t(n,R)}$$

on $n_{\lambda_1} \oplus \cdots \oplus n_{\lambda_{k-1}}$. For $c \in \mathbb{R}$ large, this symmetric, bilinear form is positive definite and with the property that the weight spaces $n_\alpha$ of ad $a$ are orthogonal. So there is a lower triangular matrix $L$ of $GL(n_{\lambda_1} \oplus \cdots \oplus n_{\lambda_{k-1}})$ acting only on the last eigenspace (see Eqn. 5.2) such that

$$\langle L \cdot, L \cdot \rangle_n = c\langle \cdot, \cdot \rangle_n - \langle \phi \cdot, \phi \cdot \rangle_{t(n,R)}$$

when restricted to the last eigenspace $n_{\lambda_{k-1}}$.

Let $\phi_1 = \phi$ be the faithful representation given above which simply rescales the metric on the highest eigenspace $n_{\lambda_k}$, and $\phi_2$ the corresponding representation on $n^{(2)} \simeq n_{\lambda_1} \oplus \cdots \oplus n_{\lambda_{k-1}}$. Then the sum of representations

$$\phi_1 + C_L \circ \phi_2$$

has the property that it simply rescales on the last two eigenspaces $n_{\lambda_{k-1}}$ and $n_{\lambda_k}$.

Continuing this process for the remaining $n^{(i)}$, we are able to build a representation such that the pullback metric simply rescales on each eigenspace, as desired.

6. Beyond Einstein Metrics

In the above, we prove the existence of an isomorphic, isometric embedding for a special class of metrics, see Theorem 2.2. Although Ricci solitons do not fall into this class, Corollary 0.4 follows immediately from the Einstein case as Ricci solitons can be realized as a subgroup with the submanifold geometry of an Einstein space [Lau11].

To prove Theorem 0.5, one simply needs to observe the well-known fact that every metric, 2-step nilpotent Lie algebra has a “good” rank 1 extension. More precisely, let $n$ be 2-step nilpotent with center $z$. Let $v$ be the orthogonal complement to $z$. Then we may define a derivation $D$ of $n$ which is the identity on $v$ and twice the identity on $z$. Observe, we can extend the inner product on $n$ to one on $s = \mathbb{R}D \ltimes n$ so that $D \perp n$. In this way, $D$ is symmetric and the theorem follows from Theorem 2.2.
7. Final thoughts

Question 7.1. In the proof of Theorem 2.2, we showed that one can find $\phi$ which solves
\[ \phi^* \langle , \rangle = c \langle , \rangle, \]
for any large $c \in \mathbb{R}$. But can this be achieved for any $c \in \mathbb{R}$?

It is unclear what happens even in the Heisenberg case. Using naïve attempts, one can easily get $c = 1$, but it is unclear if a small value of $c$ is possible or if there is even a minimum.

Question 7.2. Is the main theorem true for homogeneous Einstein spaces in general?

Even though there is still no classification of homogeneous Einstein spaces, in the non-compact setting a substantial amount is known about their structure and so one might reasonably expect the results to hold true in general, see [LL14] [AL17] [JP17].

Our construction above is likely far from efficient.

Question 7.3. What is the optimal dimension for achieving the above isomorphic, isometric embeddings?

Regarding the dimension of our target space, we note that if we fix $\dim S$ then there is a common $n = n(\dim S) \in \mathbb{N}$ such that an isomorphic, isometric embedding of $S$ into $T(n, \mathbb{R})$ exists for all $S$ with the given fixed dimension and satisfying the conditions of Theorem 2.2. This follows from noting that there are a finite number of partitions $\{\lambda_1, \ldots, \lambda_k\}$ of $\dim S$ and that $n$ was determined by nothing more than the dimensions of $\mathfrak{a}$ and of the weights spaces $\mathfrak{n}_{\lambda_i}$ of the $\mathfrak{a}$-action.

As a final remark, we note that in [Ker14] Kerr constructs examples of Einstein solvmanifolds. There it is claimed that those examples cannot arise as submanifolds of symmetric spaces, but as Kerr pointed out to us, what is proven is that those examples cannot arise as totally geodesic submanifolds, and so the results there are not at odds with the present work.

References

[AL17] Romina M. Arroyo and Ramiro A. Lafuente, The Alekseevskii conjecture in low dimensions, Math. Ann. 367 (2017), no. 1-2, 283–309. MR 3606442

[GJ18] Carolyn Gordon and Michael Jablonski, Einstein solvmanifolds have maximal symmetry, Journal of Differential Geometry - in press (2018).

[JP17] Michael Jablonski and Peter Petersen, A step towards the Alekseevskii conjecture, Math. Ann. 368 (2017), no. 1-2, 197–212. MR 3651571

[Ker14] Megan M. Kerr, New examples of non-symmetric Einstein solvmanifolds of negative Ricci curvature, Ann. Global Anal. Geom. 46 (2014), no. 3, 281–291. MR 3263202
[Lau10] Jorge Lauret, *Einstein solvmanifolds are standard*, Ann. of Math. (2) 172 (2010), no. 3, 1859–1877.

[Lau11] ———, *Ricci soliton solvmanifolds*, J. Reine Angew. Math. 650 (2011), 1–21.

[LL14] Ramiro Lafuente and Jorge Lauret, *Structure of homogeneous Ricci solitons and the Alekseevskii conjecture*, J. Differential Geom. 98 (2014), no. 2, 315–347. MR 3263520

[Tam11] Hiroshi Tamaru, *Parabolic subgroups of semisimple Lie groups and Einstein solvmanifolds*, Math. Ann. 351 (2011), no. 1, 51–66. MR 2824845