INVARIANT RANDOM SUBGROUPS OF LINEAR GROUPS

YAIR GLASNER

With an appendix by Tsachik Gelander and Yair Glasner

ABSTRACT. An invariant random subgroup (IRS) of a countable discrete group \( \Gamma \) is, by definition, a conjugation invariant probability measure on the compact metric space \( \text{Sub}(\Gamma) \) of all subgroups of \( \Gamma \). We denote by \( \text{IRS}(\Gamma) \) the collection of all such invariant measures.

Theorem 0.1. Let \( \Gamma < \text{GL}_n(F) \) be a countable non-amenable linear group with a simple, center free Zariski closure. There exits a non-discrete group topology \( \mathcal{S} \) on \( \Gamma \) such that for every \( \mu \in \text{IRS}(\Gamma) \), \( \mu \)-almost every subgroup \( \langle e \rangle \neq \Delta \in \text{Sub}(\Gamma) \) is open. Moreover there exits a free subgroup \( F < \Gamma \) with the following properties:

- \( F \cap \Delta \) is an infinitely generated free group, for every open subgroup \( \Delta \in \text{Sub}(\Gamma) \).
- \( F : \Delta = \Gamma \) \( \forall \mu \in \text{IRS}(\Gamma) \) and \( \mu \) a.e. \( \langle e \rangle \neq \Delta \in \text{Sub}(\Gamma) \).
- The map
  \[ \Phi : (\text{Sub}(\Gamma), \mu) \to (\text{Sub}(F), \Phi_* \mu) \]
  \[ \Delta \to \Delta \cap F \]
  is an \( F \)-invariant isomorphism of probability spaces, for every \( \mu \in \text{IRS}(\Gamma) \).

A more technical version of this theorem is valid for general countable linear group. We say that an action of \( \Gamma \) on a probability space, by measure preserving transformations, is almost surely non free (ASNF) if almost all point stabilizers are non-trivial.

Corollary 0.2. Let \( \Gamma \) be as in the Theorem above. Then the product of finitely many ANSF \( \Gamma \)-spaces, with the diagonal \( \Gamma \) action, is ASNF.

Corollary 0.3. Let \( \Gamma < \text{GL}_n(F) \) be a countable linear group, \( A \triangleleft \Gamma \) the maximal normal amenable subgroup of \( \Gamma \) - its amenable radical. If \( \mu \in \text{IRS}(\Gamma) \) is supported on amenable subgroups of \( \Gamma \) then in fact it is supported on \( \text{Sub}(A) \). In particular if \( A(\Gamma) = \langle e \rangle \) then \( \Delta = \langle e \rangle \), \( \mu \) a.s.

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1. **Introduction**

Before discussing our main theorem and some corollaries, we need to develop the language of invariant random subgroups.

1.1. **Invariant random subgroups.** Let $\Gamma$ be a discrete countable group and $\text{Sub}(\Gamma)$ the collection of all closed subgroups of $\Gamma$ endowed with the compact Chabauty topology. A basis for this topology is given by sets of the form

$$U_M(C) = \{ D \in \text{Sub}(\Gamma) \mid D \cap M = C \cap M \}.$$

Where $C \in \text{Sub}(\Gamma)$ and $M \subset \Gamma$ is a finite subset. In other words, $\text{Sub}(\Gamma) \subset 2^\Gamma$ inherits its topology from the Tychonoff topology on $2^\Gamma$. The last definition shows that $\text{Sub}(\Gamma)$ is a compact metrizable space. Every subgroup $\Sigma \in \text{Sub}(\Gamma)$ gives rise to two closed subsets, the collection of its subgroups and supergroups

$$\text{Sub}(\Sigma) = \{ \Delta \in \text{Sub}(\Gamma) \mid \Delta \vartriangleleft \Sigma \} \subset \text{Sub}(\Gamma)$$

$$\mathcal{E}_{\text{nv}}(\Sigma) = \{ \Delta \in \text{Sub}(\Gamma) \mid \Sigma \vartriangleleft \Delta \} \subset \text{Sub}(\Gamma).$$
The latter is referred to as the *envelope* of $\Sigma$, it is open whenever $\Sigma$ is finitely generated. The collection of open sets

$$\{\text{Env}(\Sigma) \mid \Sigma \in \text{Sub}(\Gamma) \text{ f.g.}\} \cup \{\text{Sub}(\Gamma) \setminus \text{Sub}(\Sigma) \mid \Sigma \in \text{Sub}(\Gamma) \text{ f.g.}\}.$$ forms a basis for the topology of $\text{Sub}(\Gamma)$.

The group $\Gamma$ acts (continuously from the left) on $\text{Sub}(\Gamma)$ by conjugation $\Gamma \times \text{Sub}(\Gamma) \to \text{Sub}(\Gamma)$ namely $(g, H) \mapsto gHg^{-1}$. The following definition plays a central role in all of our discussion:

**Definition 1.1.** Let $\Gamma$ be a countable group. A *invariant random subgroup* or an *IRS* on $\Gamma$ is a Borel probability-measure $\mu \in \text{Prob}(\text{Sub}(\Gamma))$ which is invariant under the $\Gamma$ action. We denote the space of all invariant random subgroups on $\Gamma$ by $\text{IRS}(\Gamma) = \text{Prob}^\Gamma(\text{Sub}(\Gamma))$. We denote by $\text{IRS}^{NF}(\Gamma)$ these IRS that do not have an atom at $\langle e \rangle$.

As is customary in probability theory, we will sometimes just say that $\Delta$ is an IRS in $\Gamma$ and write $\Delta \ll \Gamma$, when we mean that some $\mu \in \text{IRS}(\Gamma)$ has been implicitly fixed and that $\Delta \in \text{Sub}(\Gamma)$ is a $\mu$-random sample.

**Remark 1.2.** The theory of invariant random subgroups can be developed also in the more general setting where $\Gamma$ is a locally compact second countable group. In that case $\text{Sub}(\Gamma)$ is taken to be the space of all *closed* subgroups in $\Gamma$ and it is again a compact metrizable space. We will not pursue this more general viewpoint here. I refer the interested reader to the paper [ABB+11] and the references therein.

The term “IRS” was introduced in a pair of joint papers with Abért and Virág [AGV13b, AGV13a], however the paper of Stuck-Zimmer [SZ94] is quite commonly considered as the first paper on this subject. That paper provides a complete classification of IRS in a simple Lie group $G$, by showing that every ergodic IRS is supported on a single orbit (i.e. conjugacy class), either of a central normal subgroup or of a lattice. A similar classification is given of the ergodic IRS of any lattice $\Gamma < G$. These are supported either on a finite central subgroup or on the conjugacy class of a finite index subgroup. Recent years have seen a surge of activity in this subject, driven by its intrinsic appeal based on the interplay between group theory and ergodic theory, as well as by many applications that were found. We mention in combinatorics and probability [AGV13b, AL07, LP, Can], representation theory and asymptotic invariants [ABB+11, ABB+, BG04, Rau], dynamics [Bowb, Ver12] group theory [AGV13a, Ver10, Bowa, BGKb, BGKa, GM, KN13, TTD], rigidity [SZ94, Bek07, PT, CPA, CPb, HT, TD, Cre]. Topological analogues of IRS were introduced in [GW14].

The current paper initiates a systematic study of the theory of IRS in countable linear groups. Linear groups initiate a “benchmark” of sorts in group theory. Unlike more restricted families such as abelian or nilpotent groups, they constitute quite general examples of groups, exhibiting a diverse spectrum of behavior and properties. On the other hand, contrary to general groups, they do allow for a rich structure theory. In this capacity, methods and questions that were addressed for linear groups are often pushed further to other groups of “geometric nature” such as convergence groups, hyperbolic and relatively hyperbolic groups, mapping class groups, $\text{Out}(F_n)$ and even just residually finite groups. Undoubtedly many of the methods developed here can be applied in many of these other geometric settings.
One, natural, question which is completely inaccessible for linear groups, or for any other family of groups that is not extremely special, is the study of all subgroups of a given group $\Gamma$. This is out of reach even for lattices in simple Lie groups, where we do have a complete understanding of, seemingly similar, problems such as the classification of all quotient groups or of all finite dimensional representations. An outcome of the Stuck-Zimmer paper [SZ94] is that, in the presence of an invariant measure - an IRS - the situation changes dramatically. For a lattice in a simple Lie group almost every subgroup, with respect to any IRS, is either finite central or of finite index. I would like to draw the attention to this formulation of quantifiers. It will appear frequently, as I attempt to follow a similar path, proving in the setting of a countable linear group $\Gamma$, statements that hold for almost every subgroup with respect to every IRS.

1.2. Examples.

Example 1.3. Normal subgroups: If $N < \Gamma$ is a normal subgroup then the Dirac measure $\delta_N$ supported on the single point $N \in \text{Sub}(\Gamma)$ is an IRS.

Example 1.4. Almost normal subgroups: If $H < \Gamma$ is such that $[\Gamma : N_\Gamma(H)] < \infty$ then we will say that $H$ is an almost normal subgroup. In this case a uniformly chosen random conjugate of $H$ is an IRS. Normal and finite index subgroup are both examples of almost normal subgroups.

Example 1.5. probability-measure preserving actions: If $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is an action on a probability space by measure preserving transformations. The stabilizer $\Gamma_x := \{\gamma \in \Gamma \mid \gamma x = x\} < \Gamma$ of a $\mu$-random point $x \in X$ is an IRS. The probability measure responsible for this IRS is $\Phi_*(\mu)$ where $\Phi : X \to \text{Sub}(\Gamma)$ is the stabilizer map $x \mapsto \Gamma_x$. It was shown in [AGV13a, Proposition 13] that every invariant random subgroup of a finitely generated group is obtained in this fashion:

Lemma 1.6. For every invariant random subgroup $\nu \in \text{IRS}(\Gamma)$ of a finitely generated group there exists a probability-measure preserving action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ such that $\nu = \Phi_*(\mu)$ where $\Phi : X \to \text{Sub}(\Gamma)$ is the stabilizer map $x \mapsto \Gamma_x$.

This was later generalized to the setting of locally compact groups in [ABB$^+$, Theorem 2.4].

1.3. Induction, restriction and intersection. Let $\Sigma < \Gamma$ be a subgroup. The natural, $\Sigma$-invariant, restriction map $R_{\Sigma}^\Gamma : \text{Sub}(\Gamma) \to \text{Sub}(\Sigma)$ given by intersection $\Delta \mapsto \Delta \cap \Sigma$, gives rise to a map on invariant random subgroups $(R_{\Sigma}^\Gamma)_* : \text{IRS}(\Gamma) \to \text{IRS}(\Sigma)$. We will denote this map by $\mu \mapsto \mu|_{\Sigma}$. Thus for any $\mu \in \text{IRS}(\Gamma)$ the original restriction map becomes a $\Sigma$ invariant map of probability spaces:

$$R_{\Sigma}^\Gamma : (\text{Sub}(\Gamma), \mu) \to (\text{Sub}(\Sigma), \mu|_{\Sigma})$$

$$\Delta \mapsto \Delta \cap \Sigma$$

A map in the other direction exists only if $\Sigma$ is of finite index in $\Gamma$. In this case we obtain an induction map $\text{IRS}(\Sigma) \to \text{IRS}(\Gamma)$ given by

$$\mu \mapsto \mu|_\Gamma = \frac{1}{[\Gamma : \Sigma]} \sum_{i=1}^{[\Gamma : \Sigma]} (\gamma_i)_* \mu,$$

where $\gamma_i$ are coset representatives for $\Sigma$ in $\Gamma$. Since the measure $\mu$ is $\Sigma$ invariant this expression does not depend on the choice of these representatives. In the setting of
locally finite groups this construction can be generalized to the case where \( \Sigma \) is a lattice in \( \Gamma \), or more generally a subgroup of co-finite volume. All one has to do is replace the sum by an integral over \( \Gamma/\Sigma \) with respect to the invariant probability measure.

The intersection \( \mu_1 \cap \mu_2 \) of \( \mu_1, \mu_2 \in \text{IRS}(\Gamma) \) is defined as the push forward of the product measure under the map

\[
\text{Sub}(\Gamma) \times \text{Sub}(\Gamma) \rightarrow \text{Sub}(\Gamma)
\]

\[
(\Delta_1, \Delta_2) \mapsto \Delta_1 \cap \Delta_2.
\]

This can also be thought of as restricting the IRS \( \mu_1 \) to a \( \mu_2 \)-random subgroup or vice versa.

1.4. The main theorems. In our main theorem below we adopt the following notation. If \( \text{Sub}(\Gamma) = S_0 \sqcup S_1 \sqcup S_2 \sqcup \ldots \sqcup S_L \) is a partition into \( \Gamma \)-invariant sets and if \( \mu \in \text{IRS}(\Gamma) \) we set \( a_\ell = \mu(S_\ell) \) and \( \mu_\ell(A) = \mu(A \cap S_\ell)/a_\ell \in \text{IRS}(\Gamma) \), so that

\[
\mu = \sum_{\ell=0}^{L} a_\ell \mu_\ell,
\]

is the standard decomposition of \( \mu \) as a convex combination of IRS supported on these parts.

**Theorem 1.7.** (Main theorem, IRS version) Let \( \Gamma < \text{GL}_n(F) \) be a countable linear group with a connected Zariski closure and \( A = A(\Gamma) \) its amenable radical. Then there exists a number \( L = L(\Gamma) \in \mathbb{N} \), proper free subgroups \( \{F_1, F_2, \ldots, F_L\} \subset \text{Sub}(\Gamma) \) and a partition into \( \Gamma \)-invariant subsets \( \text{Sub}(\Gamma) = S_0 \sqcup S_1 \sqcup \ldots \sqcup S_L \). Such that for every \( \mu \in \text{IRS}(\Gamma) \) the following properties hold:

**Amm:** \( \Delta < A \) for \( \mu_0 \)-a.e. \( \Delta \in \text{Sub}(\Gamma) \).

And for every \( 1 \leq \ell \leq L \) and \( \mu_\ell \)-a.e. \( \Delta \in \text{Sub}(\Gamma) \):

**\( \ell \)-Free:** \( F_\ell \cap \Delta \) is a non-abelian infinitely generated free group

**\( \ell \)-Me-Dense:** \( F_\ell \cdot \Delta = \Gamma \).

**\( \ell \)-Isom:** The map

\[
F_\ell^\Gamma : (\text{Sub}(\Gamma), \mu_\ell) \rightarrow (\text{Sub}(F_\ell), \mu_\ell|_{F_\ell})
\]

\[
\Delta \mapsto \Delta \cap F
\]

is an \( (F_\ell \text{-equivariant}) \) isomorphism of probability spaces.

If \( \Gamma/A \) contains no nontrivial commuting almost normal subgroups (see Example 1.4) then one can take \( L = 1 \).

**Remark 1.8.** The free groups in the theorem are usually not finitely generated.

**Remark 1.9.** Property Free of the theorem follows directly from property Isom when the measure has no atoms since such a measure will always give measure zero to the countable subset of finitely generated subgroups of \( F \).

An important corollary of the main theorem is:

**Corollary 1.10.** Let \( \Gamma < \text{GL}_n(F) \) be a countable linear group. If \( \Delta \lhd \Gamma \) is an IRS in \( \Gamma \) that is almost surely amenable, then \( \Delta < A(\Gamma) \) almost surely.

In the Appendix (A), jointly with Tsachik Gelander, we give a short proof for this corollary. This short proof avoids many of the technicalities of the main theorem.

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1Explicitly this means that, after restricting to a conull sets in the domain and range this becomes a measurable, measure preserving bijection with measurable inverse.
as well as all of the projective dynamical argument, I recommend reading it before proceeding to the details of the main proof.

In a previous version of this paper I asked whether the phenomenon that an amenable IRS is always contained in the amenable radical is true in general, not necessarily for linear groups. Recently this statement was proved by a beautiful argument of Bruno Duchesne and Jean Lécureux [DL].

The main Theorem 1.7 assumes its most appealing form when \( A(\Gamma) = \langle e \rangle \) and \( L(\Gamma) = 1 \). As mentioned in the abstract these conditions are satisfied whenever \( \Gamma \) is nonamenable and has a simple, center free, Zariski closure.

Remark 1.11. If \( \Gamma < \text{GL}_n(F) \) is not finitely generated and \( \text{char}(F) > 0 \) then it is possible for an amenable group \( \Gamma \) to have a simple Zariski closure. An example is the group \( \text{PSL}_n(F) \) when \( F \) is a locally finite field, such as the algebraic closure of \( \mathbb{F}_p \). This is the reason why many Tits-alternative type proofs restrict themselves to finitely generated groups in positive characteristic. In Theorem 1.7 we were are able to subsume the amenable case into the statement of the general theorem.

Definition 1.12. Given a representation \( \chi : \Gamma \to \text{GL}_n(\Omega) \) we say that \( \mu \in \text{IRS}(\Gamma) \) is \textit{\( \chi \) non free} if \( \chi(\Delta) \neq \langle e \rangle \) for \( \mu \) almost every \( \Delta \in \text{Sub}(\Gamma) \). We denote the collection of all such IRS by

\[
\text{IRS}^\chi(\Gamma) = \{ \mu \in \text{IRS}(\Gamma) \mid \mu(\text{Sub}(\ker(\chi))) = 0 \}.
\]

Theorem 1.13. (Main theorem, simple version) Let \( \Gamma \) be a countable group and \( \chi : \Gamma \to \text{GL}_n(F) \) a linear representation whose image has a simple, center free, Zariski closure. Then there is a group topology \( \text{St}_\chi \) on \( \Gamma \) such that \( \mu \) almost every subgroup \( \Delta \in \text{Sub}(\Gamma) \) is open for every \( \mu \in \text{IRS}^\chi(\Gamma) \). Moreover there exists a proper free subgroup \( F < \Gamma \) such that the following properties hold for every \( \mu \in \text{IRS}^\chi(\Gamma) \):

- \( \chi \)-Me-Dense: \( F \cdot \Delta = \Gamma \), for \( \mu \) almost every \( \Delta \in \text{Sub}(\Gamma) \)
- \( \chi \)-Non-Disc: \( F \cap \Delta \neq \langle e \rangle \) for every open subgroup \( \Delta \in \text{Sub}(\Gamma) \)
- \( \chi \)-Isom: The restriction map

\[
R^\Gamma_F : \text{(Sub}(\Gamma), \mu) \to \text{(Sub}(F), \mu|_F)
\]

\[
\Delta \mapsto \Delta \cap F
\]

is an \( F \)-invariant isomorphism of probability spaces for every \( \mu \in \text{IRS}^\chi(\Gamma) \).

The group topology defined above is called the \( \chi \)-stabilizer topology on \( \Gamma \). When \( \chi \) is injective, we just refer to the stabilizer topology \( \text{St} \) on \( \Gamma \). I postpone the formal definition of this topology to Section 3.3 Definition 3.6 below. We call any group \( F < \Gamma \) satisfying the property \textbf{Me-Dense} appearing in the theorem \textit{measurably dense}. Note though that this property is, at least a priori \textit{weaker than actual density in the stabilizer topology}. So far I have not been able to prove the existence of a dense free subgroup \( F \) with respect to the stabilizer topology. Two features stand out that make this theorem considerably stronger than what can be formally deduced from Theorem 1.7 applied to this specific case - the existence of a topology as well as the fact that we no longer require the Zariski closure to be connected. We discuss this further in the next section.

1.5. \textbf{On the requirement for lack of commuting almost normal subgroups.}

In the proof of the Tits alternative and even more so in the papers of Margulis-Soifer [MS79, MS81] a major issue is dealing with representations of groups whose Zariski closure is not connected. This, in turn, allows for statements that do not require
passing to a subgroup of finite index. The issue in the latter paper for example is constructing a profinitely dense free subgroup in every linear group that is not virtually solvable. This line of argument is further developed in [BG07, GG08] as well as in Theorem 1.13 above which does not require the Zariski closure of \( \Gamma \) to be connected. However for general countable linear groups there are simple group theoretic obstructions to the existence of an \textbf{Me-Dense} subgroup. Restrictions which do not apply for for profinitely dense subgroups (or for pro-dense subgroups as defined in the next section).

Assume that a finite index subgroup \( \Gamma^0 < \Gamma \) contains two commuting normal subgroups \( \langle e \rangle \neq N, M \triangleleft \Gamma^0 \). As explained in Example 1.4 both \( M, N \) can appear, with positive probability, as instances of an IRS on \( \Gamma \). Thus is \( F = F_1 \) satisfies condition \textbf{Free} of the theorem both \( F \cap M, F \cap N \) are nontrivial commuting normal subgroups of \( F \cap \Gamma^0 \) which is impossible in a free group. If moreover \( N, N \) are conjugate in \( \Gamma \) we can take \( M, N \) to be two different instances of the same same ergodic IRS so that no free group can satisfy the condition \textbf{Free} with this IRS.

The same example shows that condition \textbf{Me-Dense} too cannot always be satisfied without passing to a subgroup of finite index. Consider a wreath product of the form \( \Gamma = \Sigma \wr \mathbb{Z}/2\mathbb{Z} \), where \( \Sigma \) is any group with infinite abelianization. Let \( \Gamma^0 = \Sigma \times \Sigma < \Gamma \) be the index two normal subgroup and \( M = \Sigma' \times \langle e \rangle, N = \langle e \rangle \times \Sigma' \). Since \( M, N \) are conjugate in \( \Gamma \), the pervious paragraph implies that \( F \cap M = F \cap N = \langle e \rangle \). Thus if condition \textbf{Me-Dense} is satisfied then actually \( \Gamma = F \rtimes M \), and immediately we deduce that \( \Gamma^0 = (F \cap \Gamma^0) \rtimes M \). But this is impossible as \( \Gamma^0/M = \Sigma^{Ab} \times \Sigma \) which is not a free group. This argument can be pushed further to yield the following:

**Proposition 1.14.** The condition on the lack of commuting almost normal subgroups in \( \Gamma/A \) is in fact necessary and sufficient for the theorem to hold with \( L = 1 \).

**Proof.** Let \( \langle e \rangle \neq M, N \triangleleft \Gamma \) be two almost normal subgroups such that \( M, N \not\triangleleft A = A(\Gamma) \) but \( [M, N] < A(\Gamma) \). Still assume by way of contradiction that the theorem holds with \( L = 1 \), setting \( F = F_1 \).

Since almost normal subgroups are special cases of invariant random subgroups, as explained in Example 1.4, we can apply the theorem to them: Both \( M \) and \( N \) appear with positive probability in their corresponding IRS’s so \( F \cdot N = \Gamma \) and \( O := F \cap M \) is a non-abelian free group. But these two equations imply that \( O/A \) is an almost normal subgroup of \( \Gamma/A \). Indeed let \( \Gamma^0 = N_\Gamma(M) \), which is by assumption a finite index subgroup in \( \Gamma \). If we set \( F^0 = F \cap \Gamma^0 \) then \( F^0N \) is of finite index in \( \Gamma \). Clearly \( N < N_\Gamma(OA) \), and \( F^0 < N_\Gamma(O) \) so that together \( \Gamma^0A < N_\Gamma(OA) \).

Now, applying the theorem again to the almost normal subgroup \( OA \) we obtain, \( \Gamma = F \cdot OA = FA \). But \( A \cap F \) is an amenable normal subgroup of \( F \) hence trivial. So that \( \Gamma = F \rtimes A \) and \( \Gamma/A \cong F \) is a free group which of course does not have two commuting almost normal subgroups, a contradiction. \( \square \)

1.6. A hierarchy of group topologies. It is useful to think of the stabilizer topology in analogy with the profinite topology on the group \( \Gamma \). The latter governs all the actions of \( \Gamma \) on finite sets. Hopefully the former will play a similar role with respect to all the invariant random subgroups on \( \Gamma \); or in ergodic theoretic terminology with resect to all the totally non free (TNF) actions of \( \Gamma \) on probability spaces
in the sense of Vershik [Ver10]. One can place the measurable density property Me-Dense within a hierarchy of known properties:

**Definition 1.15.** A subgroup $F < \Gamma$ is called:

- **Me-Dense:** if $F\Delta = \Gamma$ for every $\mu \in IRS(\Gamma)$ and for $\mu$-almost every $\langle e \rangle \neq \Delta \in \text{Sub}(\Gamma)$,
- **pro-dense:** if $F\Delta = \Gamma$ for every $\langle e \rangle \neq \Delta \lhd \Gamma$,
- **profinitely-dense:** if $F\Delta = \Gamma$ for every finite index subgroup $\Delta < \Gamma$.

These notions are organized by decreasing order of generality: every measurably dense subgroup is pro-dense because a normal subgroup is an IRS and it can be arranged to appear with probability one. Every pro-dense subgroup is profinitely dense because a finite index subgroup always contains a further subgroup that is normal of finite index. Let us recall,

**Definition 1.16.** We say that a linear representation $\chi : \Gamma \to \text{GL}_n(\Omega)$, where $\Omega$ is an algebraically closed field, is of almost simple type if the Zariski closure $G = \overline{\chi(\Gamma)}^Z$ is semisimple, center free and for which the conjugation action of $\Gamma$ is transitive on the (necessarily isomorphic) simple factors of the connected component $G^{(0)} = S \times S \times \ldots \times S$. We will say that a group is of almost simple type if it admits a faithful linear representation of almost simple type.

Recall [Bou60, Proposition 1, TGIII.3] that a collection of subgroups $B \subset \text{Sub}(\Gamma)$ forms a basis of identity neighborhoods for a group topology on $\Gamma$ whenever it is conjugation invariant and a filter base. $B$ is a filter base if whenever $\Sigma_1, \Sigma_2 \in B$ there exists $\Theta \in B$ such that $\Theta < \Sigma_1 \cap \Sigma_2$. The profinite topology comes from taking $B$ to be the collection of finite index subgroups. A subgroup is profinitely dense if and only if it is dense in this topology. It follows from the works of Margulis-Soifer [MS79, MS81] that a finitely generated linear group admits a free pro-finitely dense subgroup if and only if it is not virtually solvable. The collection of non-trivial normal subgroups in a group $\Gamma$ forms a base for a non-discrete topology if and only if $\Gamma$ is subdirect irreducible - which is the case for example for linear groups of almost simple type. If this is the case we refer to the topology thus obtained as the normal topology on $\Gamma$. The dense subgroups in the normal topology are exactly the pro-dense subgroups in the sense of the above definition. It follows from the results in [GG08] that for a finitely generated linear group the following conditions are equivalent: (i) the group is of almost simple type, (ii) the normal topology is non-discrete, (iii) the group contains a pro-dense subgroup, (iv) the group contains a free pro-dense subgroup. In that paper we obtain in fact a complete classification of all the countable non-torsion linear groups that admit a pro-dense subgroup, but the non finitely generated case is slightly more complicated as it involves two additional types of groups - the so called groups of affine and diagonal types.

The stabilizer topology, the finest of the three, is also generated by a basis of identity neighborhoods $\text{Base} \subset \text{Sub}(\Gamma)$ consisting of subgroups. A subgroup $F$ would actually be dense in $\text{St}$ if it would satisfy $F\Delta = \Gamma$ for every $\Delta \in \text{Base}$. Thus measurable density property Me-Dense appearing in Theorem 1.13 above is, at least a priori, weaker than density! In fact as far as I can see property Me-Dense does not translate into any topological property in terms of the stabilizer topology. Property Non-Disc however, which holds for all open subgroups, amounts to the
the statement that \( e \in F \setminus \{e\} \) and in particular it shows that \( F \) is not discrete in the stabilizer topology.

The fact that a countable non-amenable linear group of almost simple type admits a free pro-dense subgroup is the most important technical result from [GG08]. Similarly in the current paper, the most important technical result is the existence of the **Me-Dense** free group \( F \) in every group with a simple Zariski closure. Allowing for the shortcoming that this group is not really dense, it does yield all the group theoretic corollaries discussed earlier. Once we have this group at hand all the rest of the statements follow rather directly in Section 3.4. It is interesting to note that the discussion in Section 1.5 shows that it is indeed necessary to further restrict the class of groups from groups of almost simple type to those with a simple center free Zariski closure in order to assure the existence of a **Me-Dense** free subgroup. It seems possible, with some additional work, to obtain a necessary and sufficient condition for the existence of such a subgroup. I will not address this question in this paper, mainly because the notion of an **Me-Dense** subgroup and even the definition of the stabilizer topology still seem somewhat superficial.

The completion of \( \Gamma \) with respect to both profinite and normal topologies is again a topological group. In the profinite case we obtain the famous profinite completion \( \hat{\Gamma} \) which is a compact topological group. There is a natural map \( \iota : \Gamma \to \hat{\Gamma} \) whose kernel is the intersection of all finite index subgroups. For the normal topology the completion \( \hat{\Gamma}_N \) is no longer compact in general still it is a Polish group - complete metrizable and separable. The kernel of the map from \( \Gamma \) to this completion is the intersection of all normal non-trivial subgroups. I have mainly open questions about the completion of \( \Gamma \) with respect to the stabilizer topology. In particular it is not clear if the completion of \( \Gamma \) with respect to the stabilizer topology is a topological group, in general such a completions are only semi-topological semi-groups. I plan to investigate these questions in a sequel paper.

Let me just mention that in some cases the completion is very nice. For example, in this terminology, the content of the Stuck-Zimmer theorem [SZ94] is that for lattices in simple higher rank Lie groups all three topologies mentioned coincide. In particular in this case the completion with respect to the stabilizer topology \( \hat{\Gamma}_{\text{st}} \) coincides with the profinite completion which is a compact group. In fact, modulo the congruence subgroup conjecture, these completions also coincide, up to a finite kernel, with the congruence completion.

1.7. **Probability-measure preserving actions.** In view of Example 1.5 we can restate the main theorems in terms of actions on probability spaces without any reference to IRS. In the main theorem below I choose to pass to a finite index subgroup, instead of restricting to groups with a connected Zariski closure. The proofs are straightforward and are left to the reader.

**Theorem 1.17.** *(Main theorem, p.m.p version)* Let \( \Gamma < \text{GL}_n(F) \) be a countable linear group, \( A = A(\Gamma) \) its amenable radical. Then there exists a finite index subgroup \( \Gamma^0 < \Gamma \), a number \( L \in \mathbb{N} \) and proper free subgroups \( F_1, F_2, \ldots, F_L < \Gamma \) such that for every action, by measure preserving transformations, on a standard Borel probability space, \( \Gamma \actson (X, \mathcal{B}, \mu) \) there exists a measurable partition into \( \Gamma^0 \)-invariant subsets:

\[
X = X_0 \sqcup X_1 \sqcup X_2 \sqcup \ldots \sqcup X_L,
\]

with the following properties:
• $\Gamma_x < A$ for almost every $x \in X_0$.
• $F_\ell \cdot x \supset \Gamma^0 \cdot x$ for every $1 \leq \ell \leq L$ and almost every $x \in X_\ell$.
• $F_\ell \cap \Gamma_x$ is an infinitely generated free group for every $1 \leq \ell \leq L$ and almost every $x \in X_\ell$.
• For every $1 \leq \ell \leq L$ and almost every pair of points $(x, y) \in X^2_\ell$ we have that

$$(F_\ell \cap \Gamma_x = F_\ell \cap \Gamma_y) \Rightarrow (\Gamma_x = \Gamma_y)$$

Moreover if $\Gamma/A$ contains no nontrivial elementwise commuting normal subgroups then we can take $L = 1$ and $\Gamma^0 = \Gamma$ in the above theorem.

Recall from the abstract that an action $\Gamma \curvearrowright (X, B, \mu)$ is almost surely non-free (ASNF) if almost all stabilizers $\Gamma_x$ are non-trivial.

**Theorem 1.18. (simple p.m.p version)** Assume that $\Gamma$ is a countable non-amenable linear group with a simple, center free Zariski closure. Then there exists a free subgroup $F < \Gamma$ with the property that for every ASNF action of $\Gamma$ the two groups $\Gamma$ and $F$ have almost surely the same orbits. In other words the actions of $\Gamma$ and $F$ are orbit equivalent.

**Corollary 1.19. (Actions with amenable stabilizers are free)** Let $\Gamma < \text{GL}_n(F)$ be a countable linear group and $\Gamma \curvearrowright (X, B, \mu)$ a measure preserving action on a standard Borel probability space such that $\Gamma_x$ is almost surely amenable. Then $\Gamma_x < A$ almost surely. In particular if $\Gamma$ has a trivial amenable radical every action with amenable stabilizers is essentially free.

The non trivial intersection of almost every two instances of (possibly different) IRS’s translates to the following:

**Theorem 1.20. (Non freeness of product actions)** Let $\Gamma$ be a countable, non-amenable linear group with a simple center free Zariski closure. If $\alpha_i : \Gamma \to \text{Aut}(X, B, \mu)$, $1 \leq i \leq M$ are ASNF probability measure preserving actions then so is the product action $\Pi_{i=1}^M : \Gamma \to (X^M, B^M, \mu^M)$.

We conclude the introduction by mentioning two places where the results of this paper have been used. In a beautiful paper [TD, Corollary 5.11] Robin Tucker-Drob shows that for a countable linear group $\Gamma$ the condition $A(\Gamma) = \langle e \rangle$ is equivalent to the fact that any probability measure preserving action of $\Gamma$ that is weakly contained in the Bernoulli shift $\Gamma \curvearrowleft [0, 1]^\Gamma$ is actually weakly equivalent to the Bernoulli. This last property is what Tucker-Drob refers to as the *shift minimality* of the group $\Gamma$ which has many other equivalent formulations [TD, Proposition 3.2]. One proof he suggests uses the above Corollary 1.10 but he also gives an alternative proof.

Corollary 1.10 was also used in [AGV13a, Theorem 5] to show that if $\Gamma$ is a finitely generated linear group with $A(\Gamma) = \langle e \rangle$ and $X_n$ is a sequence of Schreier graphs for which the spectral radius of the random walk on $L^2_0(X_n)$ converges to the spectral radius of $\Gamma$ then $X_n$ also converge to $\Gamma$ in the sense of Benjamini-Schramm.

## 2. Background and preliminary proofs

### 2.1. Essential subgroups and the stabilizer topology.

The definition of essential subgroups, and the lemma that follows are, in my opinion, the most important new tools introduced in this paper.
Definition 2.1. Let $\mu \in \text{IRS}(\Gamma)$, a subgroup $\Sigma < \Gamma$ is called $\mu$-essential if $\mu(\text{Env}(\Sigma)) > 0$. In words: if the probability that a $\mu$-random subgroup contains $\Sigma$ (as a subgroup) is positive. A subgroup is called essential if it is $\mu$-essential for some $\mu \in \text{IRS}(\Gamma)$. We denote the collections of $\mu$-essential subgroups and of essential subgroups by

$$E(\mu) := \{ \Sigma \in \text{Sub}(\Gamma) \mid \mu(\text{Env}(\Sigma)) > 0 \} \quad \text{and} \quad E(\Gamma) = \bigcup_{\mu \in \text{IRS}(\Gamma)} E(\mu).$$

An element $\gamma \in \Gamma$ is essential if the cyclic group that it generates is. Note that the essential elements and subgroups of $\Gamma$ associated with an IRS are deterministic objects.

The following simple lemma is the key to much of the theory we develop of IRS in countable groups. In particular it is necessary for the definition of the stabilizer topology. To a large extent it highlights the difference between the theory of IRS in countable and in locally compact groups. Recall that a group is said to satisfy a property locally, if this property holds for every finitely generated subgroup.

Lemma 2.2. (The locally essential lemma) Let $\Gamma$ be a countable group and $\mu \in \text{IRS}(\Gamma)$. Then $\mu$-almost every $\Delta \in \text{Sub}(\Gamma)$ is locally essential.

Proof. Let $H_1, H_2, \ldots \in \text{Sub}(\Gamma)$ be an enumeration of the finitely generated subgroups of $\Gamma$ that are not $\mu$-essential. Then by definition $\mu(\text{Env}(H_i)) = 0$ for every $i$. But

$$\{ \Delta \in \text{Sub}(\Gamma) \mid \Delta \text{ is not locally essential} \} = \bigcup_i \text{Env}(H_i),$$

which has measure zero as a countable union of null sets. $\square$

Definition 2.3. We say that a subgroup $\Delta \in \text{Sub}(\Gamma)$ is locally essential if it is locally essential for some $\mu \in \text{IRS}(\Gamma)$. We further say that $\Delta$ is locally essential recurrent if in addition to being locally essential, for every finitely generated subgroup $\Sigma < \Delta$ and for every group element $\gamma \in \Gamma$ the corresponding set of return times

$$N(\Delta, \text{Env}(\Sigma), \gamma) = \{ n \in \mathbb{N} \mid \gamma^n \Delta \gamma^{-n} \in \text{Env}(\Sigma) \} = \{ n \in \mathbb{N} \mid \gamma^{-n} \Sigma \gamma^n < \Delta \}$$

is infinite. If $I \subset \text{IRS}(\Gamma)$ is a collection of invariant random subgroup we set

$$\text{Ler}^I(\Gamma) = \{ \langle e \rangle \neq \Delta \in \text{Sub}(\Gamma) \mid \Delta \text{ is locally essential recurrent for some } \mu \in I \} \quad \text{and} \quad \text{Ler}(\Gamma) = \text{Ler}^{\text{IRS}(\Gamma)}(\Gamma).$$

When the group is understood we often omit it from the notation. Note that we have explicitly excluded the trivial group: $\langle e \rangle \notin \text{Ler}$.

The locally essential lemma will play an important role in the definition of the stabilizer topology in Section 3.3. While the final definition is more technical we can think of $\text{Ler}$ as a first approximation for a (sub-)basis for this topology. The locally essential lemma, combined with Poincaré recurrence, then imply that $\Delta \in \text{Ler}$ for every $\mu \in \text{IRS}(\Gamma)$ and for $\mu$-almost every we have $\langle e \rangle \neq \Delta \in \text{Sub}(\Gamma)$. 


2.2. Covering of IRS. Throughout this section $\Gamma$ is any countable group, $\mu \in IR\S(\Gamma)$, $I \subset IR\S(\Gamma)$ is a (measurable) set of invariant random subgroups. We denote by $\mathcal{E}$ the essential subgroups relative to the situation, that is either $\mathcal{E} = \mathcal{E}(\mu)$ or $\mathcal{E} = \mathcal{E}(I)$ as the case might be. $\mathcal{F}, \mathcal{H} \subset \mathcal{E}$ will always denote measurable subsets of $\mathcal{E}$. We will refer to $\mathcal{F}, \mathcal{H}$ as families of essential subgroups.

**Definition 2.4.** A family of essential subgroups $\mathcal{F}$ covers $\mu$, if $\mu$-almost every $\Delta \in \text{Sub}(\Gamma)$ contains some $\Sigma \in \mathcal{F}$. $\mathcal{F}$ covers $I$ if it covers every $\mu \in I$.

**Definition 2.5.** Assume that $\mathcal{F}$ covers $\mu$ as in the previous definition. We say that a family of essential subgroups $\mathcal{H}$ refines this cover and write $\mathcal{F} <_{\mu} \mathcal{H}$, if for every $\Sigma \in \mathcal{F}$ and for $\mu$-almost every $\Delta \in \mathcal{E}(\Sigma)$ there exists $\Theta \in \mathcal{H}$ such that $\Sigma < \Theta < \Delta$. Similarly if $I \subset IR\S(\Gamma)$ is covered by $\mathcal{F}$ we will say that $\mathcal{H}$ refines this cover and write $\mathcal{F} <_{I} \mathcal{H}$ if $\mathcal{F} <_{\mu} \mathcal{H}, \ \forall \mu \in I$.

**Definition 2.6.** We say that a family of essential subgroups $\mathcal{F}$ is monotone if it is covered under passing to larger essential subgroups. Namely $\mathcal{F}$ is monotone if, together with every $\Sigma \in \mathcal{F}$ the collection $\mathcal{F}$ contains all the groups in $\mathcal{E}(\Sigma) \cap \mathcal{E}$. For example, the collections of infinite essential subgroups is monotone while the collection of finite essential subgroups is usually not. The following lemma is useful in the construction of refined covers.

**Lemma 2.7.** Let $\mu \in IR\S(\Gamma)$, and $\mathcal{F}, \mathcal{H} \subset \mathcal{E}(\mu)$ two (measurable) families of $\mu$-essential subgroups. Assume that $\mathcal{F}$ is monotone and covers $\mu$. If every $\Sigma \in \mathcal{F}$ is contained in some $\Theta \in \mathcal{H}$. Then $\mu$ is covered by $\mathcal{H}$ and $\mathcal{F} <_{\mu} \mathcal{H}$.

**Proof.** Assume, by way of contradiction that for some $\Xi \in \mathcal{F}$ the set

$$Y = Y(\Xi) = \mathcal{E}(\Xi) \setminus \left( \bigcup_{\Theta \in \mathcal{H} \cap \mathcal{E}(\Xi)} \mathcal{E}(\Theta) \right)$$

is of positive measure. Since $\mathcal{F}$ does cover we can find some $\Sigma \in \mathcal{F}$ such that $\mu(\mathcal{E}(\Sigma) \cap Y) > 0$. Consider the essential subgroup $\Sigma' := \bigcap_{\Delta \in \mathcal{E}(\Sigma) \cap Y} \Delta > \Sigma$. By definition $\Sigma' > \Sigma$ so by monotonicity $\Sigma' \in \mathcal{F}$. So that there exists some $\Theta \in \mathcal{H}$ with $\Theta > \Sigma'$. But this implies that $\mathcal{E}(\Theta) \subset \mathcal{E}(\Sigma') = Y \cap \mathcal{E}(\Sigma) \subset Y$ in contrast to the definition of $Y$. This shows that $\mathcal{F} <_{\mu} \mathcal{H}$. The fact that $\mathcal{H}$ covers follows upon taking $\Xi = \langle e \rangle$ in the argument above. \[\square\]

Assume that $\Delta = \Gamma_x \triangleleft \Gamma$ is realized as the stabilizer of a $\mu$-random point for some probability-measure preserving action $\Gamma \curvearrowright (X,\mathcal{B},\mu)$ as in Example (1.5) of Section 1.2. For any subgroup $\Sigma < \Gamma$ we denote $\text{Fix}(\Sigma) = \{ x \in X \mid \sigma x = x \ \forall \sigma \in \sigma \}$ and for any subset $O \subset X$ we denote $\Gamma_O = \{ g \in \Gamma \mid gx = x \text{ for almost all } x \in O \}$. Then, up to nullsets, $\mathcal{E}(\Sigma) = \{ \Gamma_x \mid x \in \text{Fix}(\Sigma) \}$ and the essential subgroups are exactly these subgroups $\Sigma < \Gamma$ such that $\mu(\text{Fix}(\Sigma)) > 0$. An essential covering by the essential subgroups $\{ \Sigma_1, \Sigma_2, \ldots \}$ is just a covering (up to nullsets) of the form $X = \cup_i \text{Fix}(\Sigma_i)$.

2.3. Projective dynamics. Our main theorem is reminiscent of the Tits alternative. Indeed the main part of the proof lies in the construction of the dense free subgroup $F$ from Theorem 1.13. Following the geometric strategy put forth by Tits in [Tit72] and further developed in [MS79], [MS81], [BG07] and many other papers this free group is constructed by first finding a projective representation
exhibiting rich enough dynamics and then playing, ping-pong on the corresponding projective space. In particular the paper [BG07] describes how to go about this when the group $\Gamma$ fails to be finitely generated. In this case the topological field can no longer be taken to be a local field, but rather a countable extension of a local field - which is no longer locally compact. The corresponding projective space in this case is no longer compact, but it is still a bounded complete metric space and the dynamical techniques used in the ping pong game still work. Following [GG08, Section 7.2] we give a short survey of the necessary projective dynamics that will be needed. We refer the reader to [BG03, Section 3], [BG07, Sections 3 and 6] for the proofs. Section 6 in the last mentioned paper deals explicitly with the non finitely generated setting.

Let $k$ be a local field $\|\cdot\|$ the standard norm on $k^n$, i.e. the standard Euclidean norm if $k$ is Archimedean and $\|x\| = \max_{1 \leq i \leq n} |x_i|$ where $x = \sum x_i e_i$ when $k$ is non-Archimedean and $(e_1, \ldots, e_n)$ is the canonical basis of $k^n$. Let $K$ be an extension field of finite or countable degree over $k$, by [Lan02, XII, 4, Th. 4.1, p. 482] the absolute value on $k$ extends to $K$. The norm on both fields extends in the usual way to $\Lambda^2 k^n$ (resp. $\Lambda^2 K^n$). The standard metric on $\mathbb{P}^n(K^n)$ is defined by $d([v], [w]) = \|v - w\|/\|v\|/\|w\|$, where $[v]$ denotes the projective point corresponding to $v \in K^n$. All our notation will refer to this metric, and in particular if $R \subset \mathbb{P}^n(K^n)$ and $\epsilon > 0$ we denote the $\epsilon$ neighborhood of $R$ by $(R)_\epsilon = \{ x \in \mathbb{P}(K^n) \mid d(x, R) < \epsilon \}$.

The projective space $\mathbb{P}^n(k^n)$ is embedded as a compact subset of the complete metric space $\mathbb{P}(K^n)$.

Let $U < \text{PGL}_n(K)$ be the subgroup preserving the standard norm and hence also the metric on $\mathbb{P}^n(K^n)$ and $A^+ = \{ \text{diag}(a_1, a_2, \ldots, a_n) \in \text{PGL}_n(K) \mid |a_1| \geq |a_2| \geq \ldots \geq |a_n| \}$. The Cartan decomposition (see [BT72, Section 7]) is a product decomposition $\text{PGL}_n(K) = U A^+ U$, namely each $g \in \text{PGL}_n(K)$ as a product $g = k_g a_g k'_g$ where $k_g, k'_g \in U$ and $a_g \in A^+$. This is an abuse of notation because only the $a_g$ component is uniquely determined by $g$, but we will be careful not to imply uniqueness of $k_g, k'_g$. A calculation shows that a diagonal element $a \in A^+$ acts as an $|a_1/a_n|^{1/2}$-Lipschitz transformation on $\mathbb{P}(K^n)$ which immediately implies:

**Lemma 2.8.** ([BG03, Lemma 3.1]) Every projective transformation $g \in \text{PGL}_n(K)$ is bi-Lipschitz on $\mathbb{P}(K^n)$.

For $\epsilon \in (0, 1)$, we call a projective transformation $g \in \text{PGL}_n(K)$ $\epsilon$-contracting if there exist a point $v_g \in \mathbb{P}(K^n)$, called an attracting point of $g$, and a projective hyperplane $H_g < \mathbb{P}(K^n)$, called a repelling hyperplane of $g$, such that

$$g(\mathbb{P}(K^n) \setminus (H_g)_\epsilon) \subset (v_g)_\epsilon.$$  

Namely $g$ maps the complement of the $\epsilon$-neighborhood of the repelling hyperplane into the $\epsilon$-ball around the attracting point. We say that $g$ is $\epsilon$-very contracting if both $g$ and $g^{-1}$ are $\epsilon$-contracting. A projective transformation $g \in \text{PGL}_n(K)$ is called $(r, \epsilon)$-proximal for some $r > 2\epsilon > 0$, if it is $\epsilon$-contracting with respect to some attracting point $v_g \in \mathbb{P}(K^n)$ and some repelling hyperplane $H_g$, such that $d(v_g, H_g) \geq r$. The transformation $g$ is called $(r, \epsilon)$-very proximal if both $g$ and $g^{-1}$ are $(r, \epsilon)$-proximal. Finally, $g$ is simply called proximal (resp. very proximal) if it is $(r, \epsilon)$-proximal (resp. $(r, \epsilon)$-very proximal) for some $r > 2\epsilon > 0$. The attracting point $v_g$ and repelling hyperplane $H_g$ of an $\epsilon$-contracting transformation are in general not uniquely defined. Still under rather mild conditions it is possible to find a canonical choice of $v_g$ and $H_g$ which are fixed by $g$ and all of its powers:
Lemma 2.9. ([BG07, Lemma 3.1]) Let \( \epsilon \in (0, \frac{1}{3}) \). There exist two constants \( c_1, c_2 \geq 1 \) (depending only on the field \( K \)) such that if \( g \in \text{PGL}_n(K) \) is an \((r, \epsilon)\)-proximal transformation with \( r \geq c_1 \epsilon \) then it must fix a unique point \( \tau_g \) inside its attracting neighborhood and a unique projective hyperplane \( \overline{H}_g \) lying inside its repelling neighborhood\(^2\). Moreover, if \( r \geq c_1 \epsilon^{2/3} \), then the positive powers \( g^n \), \( n \geq 1 \), are \((r - 2c, c_2 \epsilon^{7/3})\)-proximal transformations with respect to these same \( \tau_g \) and \( \overline{H}_g \).

Proof. The proof of this lemma in [BG07, Section 3.4] is done for a local field \( k \). Most of the proof carries over directly to the larger field \( K \), except for the existence of a \( g \)-fixed hyperplane \( \overline{H}_g \subset R(g) \). That part of the proof uses a compactness argument, which clearly does not work of the larger field \( K \).

We consider the action of the transpose \( g' \) on the projective space over the dual \( \mathbb{P}((K^n)^*) \). Points are can be identified with hyperplanes in \( \mathbb{P}((K^n)^*) \) and vice-versa so if we prove that \( g' \) is still \((r, \epsilon)\) contracting with the same constants the first part of the proof in Breuillard-Gelander’s proof would apply and give rise to a fixed point in the dual projective space, corresponding to the desired fixed hyperplane. Given vectors \( v, w \in K^n \) and linear functionals \( f, h \in (K^n)^* \), all of them of norm one, we will use the following formulas

\[
\begin{align*}
    d([v], [w]) &= \| v \wedge w \| \\
    d([v], [\ker f]) &= \| f(v) \| \\
    \text{Hd}([\ker f], [\ker h]) &= \max_{x \in \ker h} \frac{|f(x)|}{\|x\|} = \| f \wedge h \| = d_{\mathbb{P}((K^n)^*)}([f], [h]) \\
\end{align*}
\]

(2.1)

Where \( \text{Hd} \) denotes the Hausdorff distance. All distances are measured in \( \mathbb{P}(K^n) \) except in the very last expression, where the contrary is explicitly noted. The first two equations appear in [BG03, Section 3]. The first and last equalities in the last line follow directly from the second line. To check the middle equality in the third line let us choose a basis so that \( h = (h_1, 0, 0, \ldots, 0) \), \( f = (f_1, f_2, f_3, \ldots, f_n) \). In this basis the inequality we wish to prove becomes

\[
\max_{(x_2, \ldots, x_n) \in K^{n-1}} \frac{|f_2x_2 + f_3x_3 + \cdots + f_nx_n|}{\|x\|} = \| f_2 \hat{e}_1 \wedge \hat{e}_2 + \cdots + f_n \hat{e}_1 \wedge \hat{e}_n \|.
\]

Which is indeed true.

Let \((H, v)\) be a hyperplane and unit vector in \( K^n \) realizing the fact that \( g \) is \((r, \epsilon)\)-proximal. We claim that the transpose \( g' \) is \((r, \epsilon)\)-proximal with respect to the dual hyperplane point pair \((L, f)\). Explicitly \( f \) is a norm one linear functional with \( \ker(f) = H \); and \( L = \{ l \in (K^n)^* \mid l(v) = 0 \} \). The fact that \( g \) is \( \epsilon \)-contracting with respect to \((H, v)\) translates, using Equations (2.1), into

\[
|f(u)| \geq \epsilon \|u\| \implies \|gu \wedge v\| \leq \epsilon \|gu\|.
\]

Assume that \( \phi \) is any norm one functional with \( d_{\mathbb{P}((K^n)^*)}([\phi], [L]) > \epsilon \). Considering \( v \) as a functional on \((K^n)^*\) with kernel \( L \), the second line in (2.1) gives \( |\phi(v)| > \epsilon \). The same equation, now with \( \phi \) considered as a functional on \( K^n \) implies that \( d_{\mathbb{P}((K^n)^*)}([\ker(\phi)], [v]) > \epsilon \). Since \( g \) is by assumption \( \epsilon \) contracting this means that

\[
\epsilon > \text{Hd}(g^{-1} \ker(\phi), H) = \text{Hd}(\ker(g' \phi), H) = \max_{x \in H} \left\{ \left| \frac{|g' \phi(x)|}{\|x\|} \right| \right\} = d([g' \phi], [f]).
\]

\(^2\)by this we mean that if \( v, H \) are any couple of a pointed a hyperplane with \( d(v, H) \geq r \) s.t. the complement of the \( \epsilon \)-neighborhood of \( H \) is mapped under \( g \) into the \( \epsilon \)-ball around \( v \), then \( \tau_g \) lies inside the \( \epsilon \)-ball around \( v \) and \( \overline{H}_g \) lies inside the \( \epsilon \)-neighborhood around \( H \)
Which is exactly the desired \( \epsilon \)-contraction.

Note also that
\[
d_{\mathcal{P}(K^n)}([f],[L]) = d_{\mathcal{P}(K^n)}([f],[\ker f]) = |f(v)| = d_{\mathcal{P}(K^n)}([\ker f],v) \geq r.
\]
So that \( g^t \) is indeed \((r,\epsilon)\) contracting with respect to \(([L],[f])\). Applying the Breuillard-Gelander proof we obtain a unique \( g^t \) fixed point \([\mathbf{v}] \in ([f])_\epsilon\) which, under the additional assumptions on \((r,\epsilon)\) is also a common attracting point for all powers of \( g^t \). We set \( \overline{H}_g = \ker(\eta) \). Clearly this is a \( g \) fixed subspace and it is easy to verify that it serves as a common repelling hyperplane for all the powers of \( g \).

In what follows, whenever we add the article the (or the canonical) to an attracting point and repelling hyperplane of a proximal transformation \( g \), we shall mean these fixed point \( \tau_g \) and fixed hyperplane \( \overline{H}_g \) obtained in Lemma 2.9. Moreover, when \( r \) and \( \epsilon \) are given, we shall denote by \( A(g), R(g) \) the \( \epsilon \)-neighborhoods of \( \tau_g, \overline{H}_g \) respectively. In some cases, we shall specify different attracting and repelling sets for a proximal element \( g \), denoting them by \( \mathcal{A}(g), \mathcal{R}(g) \subset \mathcal{P}(K^n) \) respectively. This means that
\[
g(\mathcal{P}(K^n) \setminus \mathcal{R}(g)) \subset \mathcal{A}(g).
\]
If \( g \) is very proximal and we say that \( \mathcal{A}(g), \mathcal{R}(g), \mathcal{A}(g^{-1}), \mathcal{R}(g^{-1}) \) are specified attracting and repelling sets for \( g, g^{-1} \) then we shall always require additionally that
\[
\mathcal{A}(g) \cap (\mathcal{R}(g) \cup \mathcal{A}(g^{-1})) = \mathcal{A}(g^{-1}) \cap (\mathcal{R}(g^{-1}) \cup \mathcal{A}(g)) = \emptyset.
\]

Very proximal elements are important as the main ingredients for the construction of free groups using the famous ping-pong lemma:

**Lemma 2.10. (The ping-pong Lemma)** Suppose that \( \{g_i\}_{i \in I} \subset \text{PGL}_n(K) \) is a set of very proximal elements, each associated with some given attracting and repelling sets for itself and for its inverse. Suppose that for any \( i \neq j \), \( i, j \in I \) the attracting set of \( g_i \) (resp. \( g_i^{-1} \)) is disjoint from both the attracting and repelling sets of both \( g_j \) and \( g_j^{-1} \), then the \( g_i \)'s form an independent set, i.e. they freely generate a free group.

A set of elements which satisfy the condition of Lemma 2.10 with respect to some given attracting and repelling sets will be said to form a ping-pong set (or a ping-pong tuple).

A novel geometric observation of the papers [BG03, BG07] is that an element \( g \in \text{PGL}_n(K) \) is \( \epsilon \)-contracting, if and only if it is \( \epsilon' \)-Lipschitz on an arbitrarily small open set. This equivalence comes with an explicit dependence between the two constants, as summarized in the lemma below:

**Lemma 2.11. ([BG03, Proposition 3.3 and Lemmas 3.4 and 3.5]).** There exists some constant \( c \), depending only on the local field \( k < K \), such that for any \( \epsilon \in (0, \frac{1}{4}) \) and \( d \in (0,1) \),

- if \( g \in \text{PGL}_n(K) \) is \((r,\epsilon)\)-proximal for \( r > c_1 \epsilon \), then it is \( \epsilon d^2 \)-Lipschitz outside the \( d \)-neighborhood of the repelling hyperplane,
- Conversely, if \( g \in \text{PGL}_n(K) \) is \( \epsilon^2 \)-Lipschitz on some open neighborhood then it is \( \epsilon \)-contracting.

Here \( c_1 \) is the constant given by Lemma 2.9 above.
Remark 2.12. In [BG03] equivalence between the two properties is established via a third property, namely that $|(a_2)g/(a_1)g| < c\epsilon^2$. In that paper the Proposition and two Lemmas are stated for a local field $k$, but it is easy to see that the proofs for $K$ are identical. One thing to notice is that in the proof that $\epsilon$-contraction implies $|a_2(g)/a_2(g)| < \epsilon^2/|\pi|$ (the converse implication of [BG03, Proposition 3.3]), the constant does depend on the field in a non-trivial way. Indeed $|\pi|$ here is the absolute value of the uniformizer of the (non-Archimedean in this case) local field. However when one passes to a larger field the the situation only gets better as the norm of the uniformizer can only grow. If ultimately the value group of $K$ is non-discrete one can do away with $|\pi|$ altogether (taking $|\pi| = 1$ as it were).

The above Lemma will enable us to obtain contracting elements - we just have to construct elements with good Lipschitz constants on an arbitrarily small open set. But what we really need, in order to play ping-pong are very-proximal elements. To guarantee this we need to assume that our group is large enough.

Definition 2.13. A group $G \leq \text{PGL}_n(K)$ is called irreducible if it does not stabilize any non-trivial projective subspace. It is called strongly irreducible if the following equivalent conditions hold

- It does not stabilize any finite union of projective hyperplanes,
- Every finite index subgroup is irreducible,
- The connected component of the identity in its Zariski closure is irreducible.

We will say that a projective representation $\rho : G \rightarrow \text{PGL}_n(K)$ is reducible or strongly irreducible if the image of the representation has the same property.

The equivalence of the first two conditions is clear. The equivalence with the third property follows from the fact that fixing a projective subspace is a Zariski closed condition. It turns out that if a group is strongly irreducible and it contains contracting elements then the existence of very proximal elements is guaranteed.

Lemma 2.14. (See [BG03, Proposition 3.8 (ii) and (iii)]) Suppose that $G \leq \text{PGL}_n(K)$ is a group which acts strongly irreducibly on the projective space $\mathbb{P}(K^n)$. Then there are constants $\epsilon(G), r(G), c(G) > 0$ such that for every $\lambda \in \text{PGL}_n(K)$ which is locally $\epsilon$-Lipschitz for some $\epsilon < \epsilon(G)$ there exist $f_1, f_2 \in G$ such that $\lambda f_1 \lambda^{-1} f_2$ is $(r(G), c(G)\epsilon)$-very proximal.

Combining the above two lemmas we obtain.

Lemma 2.15. Suppose that $\Sigma \leq \text{PGL}_n(K)$ is a group which acts strongly irreducibly. Then there are constants $\epsilon(\Sigma), r(\Sigma), c(\Sigma) > 0$ such that for every $\lambda \in \text{PGL}_n(K)$ which is locally $\epsilon$-Lipschitz for some $\epsilon < \epsilon(\Sigma)$ there exist $f_1, f_2 \in \Sigma$ such that $\lambda f_1 \lambda^{-1} f_2$ is $(r(\Sigma), c(\Sigma)\sqrt{\epsilon})$-very proximal.

Finally, we will need the following two elementary lemmas.

Lemma 2.16. A projective linear transformation $[B] \in \text{PGL}_n(K)$ that fixes $n + 1$ points in general position is trivial.

Proof. By definition, $n + 1$ vectors in $K^n$ represent projective points in general position if any $i$ of them span an $i$-dimensional subspace as long as $i \leq n$. If
Lemma 2.17. Assume that $\rho : \Gamma \to \text{PGL}_n(K)$ is a totally irreducible projective representation. Then every orbit $\rho(\Gamma)v$ contains a set of $n + 1$ points in general position.

Proof. Without loss of generality we assume the field is algebraically closed. Let $H = \overline{\rho(\Gamma)^o}$ be the Zariski closure, $H^{(0)}$ the connected component of the identity and $\Gamma^{(0)} = \rho^{-1}(\rho(\Gamma) \cap H^{(0)}(K))$. Strong irreducibility is equivalent to the irreducibility of $\rho(\Gamma^{(0)})$, thus the $\rho(\Gamma^{(0)})$-orbit of every non trivial vector $v \in k^n$ contains a basis $B := \{v = v_1, \ldots, v_n\}$. Let $V_i := \text{span}\{v_1, v_2, \ldots, \hat{v}_i, \ldots, v_n\}$ - the $i^{th}$ element missing. Assume by way of contradiction that $\rho(\Gamma^{(0)})v \subset \bigcup_{i=1}^n V_i$. Since taking a vector into a subspace is a Zariski closed condition we obtain, upon passing to the Zariski closure that

$$H^{(0)}(K) = \bigcup_{i=1}^n H_i,$$

where $H_i = \{h \in H^{(0)}(K) \mid h(v) \in V_i(K)\}$

This is a union of Zariski closed sets, and since $H^{(0)}$ is Zariski connected we have $H^{(0)}(K) = H_i$ for some $i$ contradicting irreducibility of $H^{(0)}$. \hfill $\square$

3. Proof of the Main theorem

3.1. Reduction to the simple case. In this chapter we prove the Main theorem 1.7 assuming the validity of its the simple version 1.13. We start with lemmata. As mentioned in Remark 1.11, a group with a simple Zariski closure might be amenable. If it is not amenable though, it has no nontrivial normal amenable subgroups:

Lemma 3.1. If $\Gamma < \text{GL}_n(F)$ is a nonamenable linear group with a Zariski closure that is simple and has no finite normal subgroups, then $A = A(\Gamma) = \langle e \rangle$.

Proof. Let $H = \Gamma^Z$ be the Zariski closure. Since $\overline{\Gamma^Z}$ is normal in $H$ our assumptions imply $\overline{\Gamma^Z} > H^{(0)}$ whenever $A(\Gamma) \neq \langle e \rangle$. Assume by contradiction that the former case holds. By Schur’s theorem [Weh73, Theorem 4.9] every torsion linear group is locally finite and hence in particular amenable; since by assumption this is not the case for $\Gamma$ we can fix an element of infinite order $\gamma \in \Gamma$. The group $\Theta = \langle \gamma, A \rangle$, being amenable by cyclic is still amenable and its Zariski closure still contains $H^{(0)}$. The group $\Theta$ cannot contain a non-trivial normal solvable subgroup $S \triangleleft \Theta$. Indeed if $S$ were such a subgroup then $\left(\overline{S^Z}\right)^{(0)} \prec H^{(0)}$ would be a Zariski closed normal solvable subgroups, since the Zariski closure of a solvable subgroup is still solvable. Clearly there is no such group. Now, the Tits alternative [Tit72, Theorem 2] implies that $\Theta$ is a locally finite group contrary to the existence of the infinite order element $\gamma$. \hfill $\square$

Lemma 3.2. If $\Delta \triangleleft \Gamma$ is an IRS in a countable group $\Gamma$ that is a.s. finite then the centralizer $Z_{\Gamma}(\Delta)$ is of finite index in $\Gamma$ a.s.

Proof. The collection of finite subgroups of $\Gamma$ is countable. Any ergodic measure that is supported on a countable set has finite orbits which in our case means that $\Delta$ has a finite conjugacy class a.s. Thus $N_{\Gamma}(\Delta)$ is of finite index a.s. The claim follows since $Z_{\Gamma}(\Delta)$ is the kernel of the action of $N_{\Gamma}(\Delta)$ on the finite group $\Delta$. \hfill $\square$
Clearly F-index normalizer in \( \mu_2 \) is contained in the amenable radical. By definition \( \mu_2 = \mu_2^\text{Amm} \) is automatic. Similarly \( \mu_2^\text{Amm} \) is supported on \( S \) and \( S \cap \ker(\chi) = 0 \) so that \( \mu_2 \in \text{IRS}^\text{Amm}(\Gamma) \) for every \( 1 \leq \ell \leq L \). Thus properties \( \ell \text{-Me-Dense} \) and \( \ell \text{-Isom} \) in theorem 1.7 follow directly from the corresponding properties in Theorem 1.13. For property \( \ell \text{-Free} \), we can decompose \( \mu_\ell = \mu_\ell^\text{At} + \mu_\ell^\text{NA} \) to an atomic part and a non-atomic part. From Remark 1.9 property \( \chi \text{-Free} \) follows directly from \( \chi \text{-Isom} \) for the non-atomic part. Therefore we may assume that \( \mu_\ell \) is completely atomic. But this means that every ergodic component is supported on a finite orbit, or in other words that the IRS is supported on subgroups with finite index normalizers. So \( \Delta \cap F \) also has a finite index normalizer in \( F \), \( \mu_\ell \) almost surely. Since \( F \) itself is not finitely generated (by the proof of Theorem 1.13) it follows easily from properties of free groups that so is \( \Delta \cap F \).

Finally if \( L \geq 2 \) then \( \Gamma / A \) contains two commuting normal subgroups \( \ker(\chi) \) and \( \cap_{\ell=2}^L \ker(\chi) \). Completing the proof. \( \square \)
3.2. A good projective representation. Following Tits’ geometric strategy for construction of free groups in [Tit72], further developed in [MS79], [MS81], [BG07] the first step in proving Theorem 1.13 is to fix a projective representation $\rho : \Gamma \to \PGL_n(K)$ over a topological field $K$ with enough elements that exhibit proximal dynamics on $P(K^n)$. In particular Section 6 of the last mentioned paper describes how to go about this when the group $\Gamma$ fails to be finitely generated - in which case $K$ can no longer be taken to be a local field. When this happens it is crucial at least to have a Zariski dense finitely generated subgroup $\Pi < \Gamma$ whose image under the representation will fall into the $\PGL_n(k)$ for some local subfield $k < K$.

Lemma 3.3. Let $\Gamma < \GL_n(\Omega)$ be a non-amenable group with simple center free Zariski closure. Then there exists a non-amenable finitely generated subgroup $\Pi < \Gamma$ that has the same Zariski closure $\Pi^Z = \Gamma^Z$. Hence $\Pi$ is also non-amenable with a simple center free Zariski closure.

Proof. Let $H = \Gamma^Z$ and $H^{(0)}$ the Zariski connected component of the identity in $H$. Since $[H : H^{(0)}] < \infty$ it is enough to find a finitely generated subgroup of $\Pi < \Gamma$ such that $\Pi^Z > H^{(0)}$.

Let $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ be elements of $\Gamma$ generating a subgroup $\Pi'$ with the property that $I = \overline{\langle \gamma_1, \gamma_2, \ldots, \gamma_m \rangle}$ is of maximal possible dimension. By Schur’s theorem [Weh73, Theorem 4.9] every torsion linear group is locally finite and in particular amenable. Since $\Gamma$ is by assumption nonamenable it contains an element $\gamma_1 \in \Gamma$ of infinite order. The group $\overline{\langle \gamma_1 \rangle}^Z$ is at least one dimensional and hence, also $\dim(I) \geq 1$.

Now by our maximality assumption, for every $\gamma \in \Gamma$ we have $\left( I, \gamma \right)^{(0)} = I^{(0)}$ and in particular $\gamma$ normalizes $I^{(0)}$. Since $\Gamma$ has a simple center free Zariski cosure that implies $I^{(0)} = H^{(0)}$.

A group is amenable if and only if all of its finitely generated subgroups are amenable. If every finitely generated group containing $\Pi$ were amenable this would imply that $\Gamma$ is amenable too, as the union of all these groups. Thus, after possibly replacing $\Pi$ with a larger finitely generated subgroup, we may always assume that $\Pi$ is non-amenable. This concludes the proof.$\square$

Proposition 3.4. Let $\Pi$ be a finitely generated group and $\chi : \Pi \to \GL_n(f)$ be a representation with a simple center free Zariski closure. Then there is a number $r > 0$, a local field $k$ an embedding $f \hookrightarrow k$, an integer $n$, and a faithful strongly irreducible projective representation $\phi : H(k) \to \PGL_n(k)$ defined over $k$, such that for any $\epsilon \in (0, \frac{1}{2})$ there is $g \in \Pi$ for which $\chi(g) \in H^{(0)}(k)$ and $\phi \circ \chi(g)$ acts as an $(r, \epsilon)$-very proximal transformation on $P(k^n)$. We set $\rho := \phi \circ \chi : \Gamma \to \PGL_n(k)$.

Proof. See [BG07, Theorem 4.3] (also [GG08, Theorem 7.6]) for a much more general statement. The faithfulness follows from the simplicity of the Zariski closure.$\square$

Let $\chi : \Gamma \to \GL_n(F)$ be the simple representation given in Theorem 1.13. If $\Gamma$ is finitely generated we apply the above theorem with $f = F, \Pi = \Gamma$ and fix once and for all the local field $k = K$, the representation $\rho : \Gamma = \Pi \to \PGL_n(k) = \PGL_n(K)$ and an element $g \in \Gamma$ such that $\rho(g) \in \left( \Gamma^Z \right)^{(0)}$ and $\rho(g)$ is $(r, \epsilon)$-very proximal where $\epsilon$ is chosen so as to satisfy the conditions of Lemma 2.9. Namely
Let $r \geq c_1 \varepsilon^{2/3}$, where $c_1$ is the constant, given in that lemma. We denote by $\tau_{g^{\pm 1}}, \Pi_{g^{\pm 1}}$ the attracting and repelling points and hyperplanes associated to $g$ in that lemma.

If $\Gamma$ is not finitely generated then we apply Lemma 3.3 and fix, once and for all, a non-amenable finitely generated subgroup $\Pi < \Gamma$ such that $\chi(\Pi)^g = \chi(\Pi)^g$. Let $f < F$ be the finitely generated subfield generated by the matrix coefficients of $\chi(\Pi)$. Applying Proposition 3.4 to $\chi|_\Pi : \Pi \rightarrow PGL_n(f)$ we obtain a number $r > 0$, a local field $k$ an embedding $f \hookrightarrow k$, an integer $n$, and a faithful strongly irreducible projective representation $\phi : H(k) \rightarrow PGL_n(k)$ defined over $k$, such that for any $\epsilon \in (0, \frac{1}{2})$ there is $g \in \Pi$ for which $\chi(g) \in H^0(k)$ and $\phi \circ \chi(g)$ acts as an $(r, \epsilon)$-very proximal transformation on $\mathbb{P}(k^n)$. Let $\rho := \phi \circ \chi : \Pi \rightarrow PGL_n(k)$.

Setting $K = k \otimes_f F$, the absolute value on the local field $k$ extends to an absolute value on the extension field $K$ [Lan92, XII, 4, Th. 4.1, p. 482]. The new field $K$ is not locally compact any more and if the original field $k$ was non-Archimedean then the corresponding extension of the discrete valuation to $K$ is a real valuation that need no longer be discrete. However $K$ is still a complete valued field. The representation $\phi$, defined as it is over $k$, extends to a representation which we will still call by the same name $\phi : H(K) \rightarrow PGL_n(K)$ which gives rise to an extension $\rho := \phi \circ \chi : \Gamma \rightarrow PGL_n(K)$.

### 3.3. Definition of the stabilizer topology.

From now on let us fix a group $\Gamma$, as specified in Theorem 1.13 and the projective representation $\rho : \Gamma \rightarrow PGL_n(K)$ constructed in the Section 3.2. The relevant class of invariant random subgroups appearing in Theorem 1.13 is:

$$\text{IRS}_X(\Gamma) = \{ \mu \in \text{IRS}(\Gamma) \mid \chi(\Delta) \neq \langle e \rangle \text{ for } \mu\text{-almost every } \Delta \in \text{Sub}(\Gamma) \}$$

Let $\mathcal{E}^{f,g} :$ be the collection of finitely generated IRS$^X$-essential subgroups. Namely these subgroups that are finitely generated and $\mu$-essential for some $\mu \in \text{IRS}^X$. By the locally essential lemma 2.2 the, countable collection of subgroups $\mathcal{E}^{f,g}$ covers IRS$^X$. All of our main theorems will follow from the following technical statement, asserting the existence of a sufficiently good refinement of this cover.

**Theorem 3.5.** There exists a collection of subgroups $\mathcal{H} = \{ \Theta_1, \Theta_2, \ldots \} \subset \mathcal{E}^{f,g}$ and a list of elements $\{ f(p, q, \gamma) \mid p, q \in N, \gamma \in \Gamma \}$ with the following properties.

1. $\mathcal{E}^{f,g} <_{\text{IRS}} \mathcal{H}$, (see Definition 2.5).
2. $\{ f(p, q, \gamma) \mid p, q \in N, \gamma \in \Gamma \}$ freely generate a free group.
3. $f(p, q, \gamma) \in (\Theta_\gamma)^g \Theta_p$, $\forall p, q \in N, \gamma \in \Gamma$.

Theorem 3.5 now enables us to define a basis of identity neighborhoods that will give rise to the stabilizer topology. A collection of subgroups Base $\subset \text{Sub}(\Gamma)$ forms a basis of identity neighborhoods of a topology if it satisfies the following properties (i) It is conjugation invariant (ii) It is a filter base, namely for every $\Delta_1, \Delta_2 \in \text{Base}$ there exits some $\Delta_3 \in \text{Base}$ with $\Delta_3 < \Delta_1 \cap \Delta_2$ (see [Bou00, Proposition 1, TGIII.3]). A collection of subgroups $\text{SB}$ forms a sub-basis of identity neighborhoods if is merely closed to conjugation. Such a sub-basis yields a basis consisting of all possible intersections of finitely many subgroups taken from $\text{SB}$. The resulting topology will be discrete if and only if there are subgroups $\{ \Delta_1, \Delta_2, \ldots, \Delta_M \} \subset \text{SB}$ with $\cap_{i=1}^M \Delta_i = \langle e \rangle$.

**Definition 3.6.** Let $\Gamma$ be as in Theorem 1.13. We define the **stabilizer topology** on $\Gamma$ as the topology arising from the sub-basis of identity neighborhoods $\text{SB} \subset \text{Sub}(\Gamma)$.
consisting of all subgroups $\Delta \in \operatorname{Sub}(\Gamma)$ for which there exists some $\mu \in \operatorname{IRS}^\chi(\Gamma)$ satisfying the following properties:

**St-Ler:** $\Delta \in \mathcal{Ler}(\mu)$. By definition this means that every finitely generated subgroup $\Sigma < \Delta$ is $\mu$-essential and that for every $\gamma \in \Gamma$ the set of return times

$$N(\gamma, \Delta, \mathcal{E}nv(\Sigma)) = \{ m \in \mathbb{Z} \mid \gamma^m \Delta \gamma^{-m} \in \mathcal{E}nv(\Sigma) \} = \{ m \in \mathbb{Z} \mid \gamma^{-m} \Sigma \gamma^m < \Delta \}$$

is infinite.

**St-Cov:** Every conjugate of $\Delta$ contains some $\Theta \in \mathcal{H}$ as a subgroup.

We define

$$\text{Base} = \left\{ \bigcap_{i=1}^M \Delta_i \ \bigg| \ M \in \mathbb{N}, \Delta_1, \Delta_2, \ldots, \Delta_M \in \mathcal{SB} \right\} \subset \operatorname{Sub}(\Gamma)$$

to be the basis of identity neighborhoods coming from this sub-basis. We call the associated topology the $\chi$-stabilizer topology and denote it $\mathcal{S}t^\chi(\Gamma)$. If $\chi$ is injective, we will just refer to the stabilizer topology and denote it $\mathcal{S}t(\Gamma)$.

**Proposition 3.7.** (Almost every subgroup is open) For $\Gamma$ as above, $\mu$-almost every subgroup $\langle e \rangle \neq \Delta \in \operatorname{Sub}(\Gamma)$ is open, with respect to every $\mu \in \operatorname{IRS}^\chi(\Gamma)$.

**Proof.** Fix $\mu \in \operatorname{IRS}^\chi(\Gamma)$, we show a little more, that $\mu$-almost every nontrivial subgroup is in $\mathcal{SB}$. Given any finitely generated subgroup $\Sigma < \Gamma$ and $g \in \Gamma$ we set

$$\operatorname{Rec}(\Sigma, g) = \{ \Delta \in \mathcal{E}nv(\Sigma) \mid |N(\Delta, \mathcal{E}nv(\Sigma), g)| = \infty \} \cup (\operatorname{Sub}(\Gamma) \setminus \mathcal{E}nv(\Sigma))$$

$$\text{LE} = \{ \Delta \in \operatorname{Sub}(\Gamma) \mid \Delta \text{ is } \mu \text{ locally essential} \}$$

$$\text{InH} = \bigcap_{\gamma \in \Gamma} \gamma \left( \bigcup_{\Theta \in \mathcal{H}} \mathcal{E}nv(\Theta) \right) \gamma^{-1}$$

By the locally essential lemma 2.2 we know that $\mu(\text{LE}) = 1$. Given this $\mu(\mathcal{E}nv(\Sigma)) > 0$ for every finitely generated subgroup so that Poincaré recurrence implies that $\mu(\operatorname{Rec}(\Sigma, g)) = 1$ for every $g \in \Gamma$. Finally $\mu(\text{InH}) = 1$ by (1) of Theorem 3.5. The intersection of all of these these:

$$\text{LE} \cap \text{InH} \cap \left( \bigcap_{\Sigma, g} \operatorname{Rec}(\Sigma, g) \right)$$

consists of open sets and is of full measure as a countable intersection of conull sets. \qed

3.4. **Proof of the main theorem.** In this section we prove theorem 1.13 assuming Theorem 3.5. The rest of the paper will be dedicated to the proof of the latter.

**Proof.** We first verify the equation appearing in property $\chi$-Me-Dense of Theorem 1.13:

$$F \Delta = \Gamma, \quad \forall \mu \in \operatorname{IRS}^\chi(\Gamma) \text{ and for } \mu\text{-almost every } \Delta \in \operatorname{Sub}(\Gamma).$$

Fix $\mu \in \operatorname{IRS}^\chi$ and some $\gamma \in \Gamma$. By Property (1) of Theorem 3.5, for $\mu$-almost every $\Delta \in \operatorname{Sub}(\Gamma)$ there is a $p \in \mathbb{N}$ such that $\Theta_p < \Delta$. Since $\mu$ is conjugation
invariant, there is also a.s. some \( q \in \mathbb{N} \) such that \( \gamma^{-1} \Theta_q \gamma < \Delta \). Set \( f = f(p, q, \gamma) \).

Now by Property (3) \( f = \theta_q \gamma \theta_p \) for some \( \theta_p \in \Theta_p, \theta_q \in \Theta_q \) so that we can write
\[
\gamma = f \theta_p^{-1} (\gamma^{-1} \theta_q \gamma) \in f \Theta_p (\gamma^{-1} \Theta_q \gamma) \subset f \Delta \subset F \Delta.
\]

Note that this does not show that \( F \) is dense in the stabilizer topology.

To show that this topology is not discrete we have to show that the intersection of finitely many subgroups \( \{ \Delta_1, \ldots, \Delta_m \} \subset SB \) is never trivial. We will prove this by induction on \( m \), starting with \( m = 2 \).

Let \( \Delta_1, \Delta_2 \in SB \), by \( St\text{-}Cov \), for \( i \in \{1, 2\} \) we have \( \mu_i \in IRS^X(\Gamma) \), \( \Theta_{p(i)} \in \mathcal{H} \) with \( \Theta_i < \Delta_i \). Let us find two elements \( f_i \in \Delta_i \cap F \), where \( F \) is the free group whose existence is guaranteed in Theorem 3.5. Explicitly we can take \( f_i = f(p(i), p(i), e) \), this choice further guarantees that \( f_1, f_2 \) freely generate a free group. As \( \Delta_1 \) is locally essential each \( f_i \) is a \( \mu_i \)-essential element. By \( St\text{-}Ler \), we can find \( 0 \neq m_1 \in N(f_1, \Delta_2, EnV((f_2))) \) and \( 0 \neq m_2 \in N(f_2, \Delta_1, EnV((f_1))) \), which by definition means that \( f_2 \in f_1^{m_1} \Delta_2 f_1^{-m_1} \) and \( f_1 \in f_2^{m_2} \Delta_1 f_2^{-m_2} \). Combining the two we have:
\[
[f_1^{m_1}, f_2^{m_2}] = f_1^{-m_1} f_2^{-m_2} f_1^{m_1} f_2^{m_2} \in \Delta_1 \cap \Delta_2.
\]

Since \( f_1, f_2 \) are independent this is a non-trivial element and \( \Delta_1 \cap \Delta_2 \neq \langle e \rangle \).

Consider now the general case \( m > 2 \). We let \( \mu_i \) be IRS according to which \( \Delta_i \) is locally essential recurrent and let \( f_i \in \Delta_i \cap F \) be \( m \) independent elements as above.

Assume by induction that:

- There is an element \( e \neq g_1 \in \langle f_1, f_2, \ldots, f_{m-1} \rangle \) which is not contained in the group \( \langle f_2, f_3, \ldots, f_{m-1} \rangle \). Such that \( g_1 \in \bigcap_{i=1}^{m-1} \Delta_i \).
- There is an element \( e \neq g_2 \in \langle f_2, f_3, \ldots, f_m \rangle \) which is not contained in the subgroup \( \langle f_2, f_3, \ldots, f_m \rangle \). Such that \( g_2 \in \bigcap_{i=2}^{m} \Delta_i \).

Repeating the argument we had in the two generator case there are integers \( m_1, m_2 \) such that
\[
[g_1^{-m_1}, g_2^{-m_2}] = g_1^{-m_1} g_2^{-m_2} g_1^{m_1} g_2^{m_2} \in \bigcap_{i=1}^{m} \Delta_i.
\]

Where the inclusion in \( \Delta_i \) is automatic for \( 2 \leq i \leq m-1 \) and follows for \( i \in \{1, m\} \) just in the case \( m = 2 \). For the sake of the induction it is easy to verify that this new element is not contained in \( \langle f_1, f_2, \ldots, f_{m-1} \rangle \cup \langle f_2, f_3, \ldots, f_m \rangle \). This completes the proof that the topology is non-discrete and also proves Property \( \chi\text{-}Non\text{-}Disc \) of the theorem.

Let \( \mu_1, \mu_2 \in IRS^X(\Gamma) \). By Proposition 3.7 \( \mu_i \)-almost every subgroup of \( \Gamma \) is open. In fact it is even true that almost every subgroup is in \( SB \). Thus for \( \mu_1 \times \mu_2 \) almost every pair \( (\Delta_1, \Delta_2) \in \text{Sub}(\Gamma)^2 \) both subgroups are open and hence, by the previous paragraph, so is their intersection \( \Delta_1 \cap \Delta_2 \). This immediately implies that \( \mu_1 \cap \mu_2 \in IRS^X(\Gamma) \), where \( \mu_1 \cap \mu_2 \) is the intersection IRS, as defined in Section 1.3.

To establish property \( \chi\text{-}Isom \) we have to show that the restriction map is an isomorphism.

\[
\Phi : (\text{Sub}(\Gamma), \mu) \rightarrow (\text{Sub}(F), \mu|_F)
\]

\[
\Delta \mapsto \Delta \cap F
\]

This is clearly a measure preserving \( F \)-invariant surjective map. By Souslin’s theorem [Kec95, Corollary 15.2] it is enough to show that \( \Phi \) is essentially injective on \( \text{Sub}(\Gamma) \) in order to establish an isomorphism. In other words we wish to show that \( \Phi(\Delta) \neq \Phi(\Delta') \) for \( \mu \times \mu \) almost every \( (\Delta, \Delta') \in \text{Sub}(\Gamma)^2 \) such that \( \Delta \neq \Delta' \).
Applying Theorem 3.5 (1) to the IRS $\nu$ we conclude that $\mu \times \mu$-almost surely, there is some $\Theta \in \mathcal{H}$ with $\Theta < \Delta \cap \Delta'$. Since by assumption $\Delta \neq \Delta'$, after possibly switching the roles of $\Delta, \Delta'$ we can find an element $\delta \in \Delta \setminus \Delta'$. Repeating the argument from the proof for property $\chi$-Me-Dense, we can find a non-trivial element of $F$ in the coset

$$f \in F \cap \delta \Theta \subset F \cap \delta(\Delta \cap \Delta') = (F \cap \Delta) \cap (F \cap \delta \Delta').$$

Since two cosets of the same group are disjoint $f \in (F \cap \Delta) \setminus (F \cap \delta \Delta') = \Phi(\Delta) \setminus \Phi(\Delta')$. Establishing the injectivity.

Note 3.8. The dynamical interpretation of the argument used in the proof of property $\chi$-Me-Dense, that was used twice in the above proof is as follows. By Lemma 1.6 every $\mu \in \text{IRS}^Y$ arises as the point stabilizer of a probability-measure preserving action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$. Since $\mathcal{H}$ covers we can find, for every $\gamma \in \Gamma$ and for almost every $x \in X$, some $p, q$ such that $\Gamma_x > \Theta_p$ and $\Gamma_{\gamma x} > \Theta_q$. Hence $\gamma x = \theta_p \gamma \theta_q x$, $\forall \theta_p \in \Theta_p, \theta_q \in \Theta_q$. Theorem 3.5 above provides a free subgroup with generators in each and every one of these double cosets. Thus for every probability-measure preserving action of $\Gamma$ with the property that $\chi(\Gamma) \neq \langle e \rangle$ a.s. $\Gamma$ and $F$ have the same orbits a.s.

The rest of the paper is dedicated to the proof of Theroem 3.5. The proof proceeds in a few steps in which we construct successive refinements for our cover of the collection $\text{IRS}^Y$.

3.5. Infinite essential subgroups. Assume that $\mu \in \text{IRS}^Y(\Gamma)$. Our first goal is to show that $\mu$ admits a cover by essential subgroups whose $\rho$-image is infinite. In this section we assume either that $\text{char}(F) = 0$ or that $\Gamma$ is finitely generated. General countable groups in positive characteristic will be treated separately in the next section. Assume by way of contradiction that there is no such essential cover. In this case we shall further assume that $|\rho(H)| < \infty$ for every $\mu$-essential subgroup - just by conditioning on the event: $\{\Delta \in \text{Sub}(\Gamma) \mid |\rho(H)| < \infty, \forall H \in \mathcal{E}(\mu) \cap \text{Sub}(\Delta)\}$ which we assumed has positive probability. The locally essential lemma 2.2 now implies that $\rho(\Delta)$ is locally finite, $\mu$-almost surely. It follows from the theory of linear groups that there exists a number $M$ such that $\rho(\Delta)$ admits a finite index subgroup $\Lambda(\Delta)$ that fixes a point $s(\Delta) \in \mathcal{P}(k^n)$, almost surely. Indeed if $\Gamma$ is linear in characteristic zero then every locally finite subgroup is virtually abelian, with a bound on the index of the abelian subgroup by Schur’s theorem [Weh73, Corollary 9.4] and the claim follows from Mal’cev’s Theorem [Weh73, Theorem 3.6]. If $\Gamma$ is finitely generated it follows from [Weh73, Corollary 4.8] that there is a bounded index subgroup consisting of unipotent elements, which is therefore unipotent.

Let $g \in \Gamma$ be any element whose $\rho$ image is very proximal and also satisfies the condition $r > c_i\epsilon^{2/3}$ appearing in Lemma 2.9. The existence of one such element is guaranteed at the end of Section 3.2. Adhering to the notation set in Section 2.3 we denote by $\tau_g, \overline{\tau}_g, \tau_g^{-1}, \overline{\tau}_g^{-1}$ the canonical attracting points and repelling hyperplanes of $g, g^{-1}$ respectively.

We claim that $\rho(\gamma)$ fixes $\tau_g$ for every essential element $\gamma \in \Gamma$. Indeed, by the definition of an essential element, $\mathcal{E}n\nu(\gamma) := \{\Delta \in \text{Sub}(\Gamma) \mid \gamma \in \Delta\}$ is a set of positive measure. Thus by Poincaré recurrence, for almost all $\Delta \in \mathcal{E}n\nu(\gamma)$ the
sequence of return times

\[ N(\Delta, \mathcal{E}\text{nv}(\gamma)) = \{ n_k \mid g_n^k \Delta g_n^{-k} \in \mathcal{E}\text{nv}(\gamma) \} = \{ n_k \mid g_n^{-k} \gamma^k g_n \in \Delta \}, \]

is infinite. Fix such a recurrent point \( \Delta \in \mathcal{E}\text{nv}(\gamma) \) which is, at the same time, locally finite. Many such exist because both properties hold with probability one \( \) the first by Poincaré recurrence, the second by the locally essential lemma 2.2.

Now we follow the dynamics of the same elements on the projective space. Let \( \Lambda = \Lambda(\Delta) < \Delta \) be the finite index subgroup of \( \Delta \) fixing the point \( s = s(\Delta) \in \mathbb{P}(K^n) \) as constructed in the beginning of this section. Let \( \Omega := \Delta \cdot s \) be (the finite) orbit of this point. Assume first that \( \Omega \cap \mathcal{P}_g = \emptyset \). In this case \( \rho(g^n) \omega \xrightarrow{n \to \infty} v_g \) for every \( \omega \in \Omega \) and consequently

\[
\rho(\gamma)(v_g) = \rho(\gamma) \left( \lim_{k \to \infty} \rho(g_n^k) s \right) = \lim_{k \to \infty} \rho(g_n^k) (g_n^{-k} \gamma^k g_n) \in \lim_{k \to \infty} \rho(g_n^k) \Omega = \{ v_g \}. 
\]

Consider now the general case when \( \Omega \cap \mathcal{P}_g \neq \emptyset \). In the next two paragraphs we will exhibit a sequence of very proximal elements \( g_n \) who satisfy \( \Omega \cap \mathcal{P}_g = \emptyset \), and \( \lim_{n \to \infty} v_{g_n} = v_g \). By the previous paragraph the essential element \( \gamma \) fixes \( v_{g_n} \) and since the fixed point set of \( \rho(\gamma) \) is closed we conclude that \( \rho(\gamma)v_g = v_g \). As desired.

We can find an element \( \sigma \in \Gamma \) such that the following conditions hold:

- \( \rho(\sigma)(\Omega) \cap \mathcal{P}_g = \emptyset. \)
- \( \rho(\sigma)v_g \notin \mathcal{P}_g. \)
- \( \rho(\sigma^{-1})v_{g^{-1}} \notin \mathcal{P}_{g^{-1}}. \)

Indeed we will even find such an element in \( \Gamma^{(0)} \) - the intersection of \( \Gamma \) with the connected component of its Zariski closure. The strong irreducibility of \( \rho(\Gamma) \) is equivalent to the irreducibility of \( \rho(\Gamma^{(0)}) \) (see Definition 2.13). Thus for every \( \omega \in \mathbb{P}(K^n) \) the collection \( D_\omega := \{ \sigma \in \Gamma^{(0)} \mid \rho(\sigma) \cdot \omega \notin \mathcal{P}_g \} \) is non empty and Zariski open. Since \( \Gamma^{(0)} \) is Zariski connected the sets \( D_\omega \) are open and dense. A similar dense open set can be constructed for the last condition \( E := \{ \sigma \in \Gamma^{(0)} \mid \rho(\sigma^{-1})v_{g^{-1}} \notin \mathcal{P}_{g^{-1}} \} \). Our desired element can be chosen as any element in the intersection \( \sigma \in (\cap_{E \in \Omega} D_\omega) \cap D_{g^{-1}} \cap E \). Having chosen such a \( \sigma \) let \( d \) be the minimal distance attained in all the above relations, namely \( d(\rho(\sigma)\omega, \mathcal{P}_g) > d \forall \omega \in \Omega \), \( d(\rho(\sigma)v_g, \mathcal{P}_g) > d \), \( d(\rho(\sigma^{-1})v_{g^{-1}}, \mathcal{P}_{g^{-1}}) > d \).

Now consider the sequence of elements \( g_n := g^n\sigma \). If \( \mathcal{A}(g^n), \mathcal{R}(g^n), \mathcal{A}(g^{-n}), \mathcal{R}(g^{-n}) \) are attracting and repelling neighborhoods for \( \rho(g^n) \) then \( \mathcal{A}(g_n) = \mathcal{A}(g^n), \mathcal{R}(g_n) = \rho(\sigma^{-1})(\mathcal{R}(g^n)), \mathcal{A}(g_n^{-1}) = \rho(\sigma^{-1})(\mathcal{A}(g^n)), \mathcal{R}(g_n^{-1}) = \mathcal{R}(g^{-n}) \) will be attracting and repelling neighborhoods for \( g_n \). For large enough values of \( n \) we can assume that the original neighborhoods are arbitrarily small \( \mathcal{A}(g^n) \subset (v_g)_\epsilon = (v_g)_\epsilon, \mathcal{R}(g^n) \subset (g_n)_\epsilon, \) etc'. Where \( (\cdot)_\epsilon \) stands for the \( \epsilon \)-neighborhood. In particular if \( d \) is the bound constructed in the previous paragraph and we choose \( n \) large enough so that \( d > \epsilon/2 \). The above inequalities will imply immediately that

\[
\mathcal{A}(g_n) \cap \mathcal{R}(g_n) = \mathcal{A}(g_n^{-1}) \cap \mathcal{R}(g_n^{-1}) = \mathcal{R}(g_n) \cap \Omega = \emptyset.
\]

The first two ensure that \( \rho(g_n) \) is again very proximal. The last one shows that \( \Omega \cap \mathcal{A}(g_n) = \emptyset \) for every \( n \geq N \). Thus as mentioned above we have \( \rho(\lambda)v_{g_n} = v_{g_n} \)}. 

3.6. Non finitely generated groups in positive characteristic. Here we treat the case where the group $\Gamma$ is linear in positive characteristic and non finitely generated. Just as in the previous section it is enough to prove that the only IRS supported on locally finite subgroups of $\Gamma$ is the trivial IRS.

The geometric strategy employed in the previous section to show that such IRS cannot exist is quite general. Loosely speaking two geometric properties of the action on the projective plane were used:

1. Every locally finite subgroup fixes a point (or a finite number thereof)
2. There are enough proximal elements

The problem is that when $\Gamma$ is linear over a field of positive characteristic and, at the same time, fails to be finitely generated the action on the projective space $\mathbb{P}(K^n)$ no longer satisfies condition (1) above. For example if $F < K$ is a locally finite field then $\text{PGL}_n(F) < \text{PGL}_n(K)$ is locally finite, but does not fix any projective point in $\mathbb{P}(K^n)$. We assumed explicitly that $\rho(\Gamma)$ is not an amenable group and in particular it cannot be of this form. Still $\rho(\Gamma)$ may contain many such locally finite subgroups and we have to show that the IRS is not supported on these.

I will use the action of $\text{PGL}_n(K)$ on its (affine) Bruhat-Tits building $X = X(K^n)$ and show that $X$ does satisfy the desired conditions. Note that in this generality the building $X$ is neither locally finite nor simplicial. This is due, respectively, to the facts that the residue field might no longer be finite and the value group might fail to be discrete. Bruhat-Tits buildings in this generality were treated by Bruhat-Tits [BT72, BT84] but a geometric axiomatization of such affine buildings was given only later in the thesis of Anne Parreau [Par00].

I will follow the notation in the excellent survey paper [RTW14, Section 1]. In particular we set $V = K^n$ and denote by $\mathcal{N}(V, K)^{\text{diag}}$ the set of all diagonalizable non-Archimedean norms on $V$ (these are discussed in some detail in [Wei95]) and we identify the building $X = \mathcal{X}(V, K) = \mathcal{N}(V, K)^{\text{diag}}/\sim$ with the collection of all such norms modulo homothety $||\cdot|| \sim c ||\cdot||$, $c > 0$. A norm is diagonalized by a basis $\xi = \{e_1, \ldots, e_n\}$ of $V$, if it is of the form $\sum a_i e_i \| c = \max_i \{c^e | a_i|\}$ for some vector $\xi = (c_1, \ldots, c_n)$. Thus the collection of all norms diagonalized by a given basis is identified with $\{\|\cdot\|_c \mid \xi \in \mathbb{R}^n\} \cong \mathbb{R}^n$. The image of these Euclidean spaces in $\mathcal{X}(V, K)$ are the apartments of the building, they are $n - 1$ dimensional spaces of the form $A_\xi = \mathbb{R}^n / \langle (1, 1, \ldots, 1) \rangle$. The important features for us are that the building $X$ is still a $\text{CAT}(0)$ space (see [RTW14, Section 1.1.3]) and it still has a finite dimensional Tits boundary. In fact, since the base field is complete, the boundary $\partial X$ is naturally identified with the spherical building of $\text{PGL}_n(K)$. Recall that the zero skeleton $\partial X^0$ of the spherical building consists of non-trivial proper subspaces of $V$. The higher dimensional simplexes correspond to flags of
such subspaces. We will say that a vertex of $\partial X$ is of type $i$ if it corresponds to an $i$-dimensional subspace.

These conditions are important because they are exactly these needed for the following theorem of Caprace-Monod.

**Proposition 3.9.** ([CM09, Corollary 3.4]) Let $X$ be a CAT(0) space with finite dimensional Tits boundary. Then every locally finite group acting on $X$ fixes a point in $X \sqcup \partial X$.

Thus we may assume that almost every $\Delta \in \text{Sub}(\Gamma)$ fixes some point $s(\Delta) \in X \sqcup \partial X$. Since the action of $\text{PGL}_n(K)$ on $\partial X$ preserves types we may, and shall, assume that whenever $s(\Delta) \in \partial X$ then it is actually a vertex in the spherical building, of type $i(\Delta)$.

Let us write $\mu$ as a convex combination of IRS with disjoint supports as follows

$$\mu = c_X \mu_X + c_1 \mu_1 + c_H \mu_H$$

where:

- $\mu_X$-almost every $\Delta \in \text{Sub}(\Gamma)$ has a fixed point inside $X$.
- $\mu_1$-almost every $\Delta \in \text{Sub}(\Gamma)$ does not have a fixed point in $X$ but it does fix a vertex of type 1 in $\partial X$.
- $\mu_H$-almost every $\Delta \in \text{Sub}(\Gamma)$ fixes neither a point in $X$ nor a vertex of type 1 in $\partial X$. So that $s(\Delta) \in \partial X$ is a vertex of higher type.

For $\mu_X$ almost very $\Delta \in \text{Sub}(\Gamma)$ we insist that $s(\Delta) \in X$. Similarly for $\mu_1$ almost every $\Delta$ we take $s(\Delta)$ to be a vertex of type 1. We will treat each of these measures seperately. Note that only $\mu_X$ can be trivial as the other two give measure zero to the trivial group. Therefore what we have to prove is that $\mu_X(\{(e)\}) = 1$ and that the other two measures cannot exist and hence $c_X = 1, c_1 = c_H = 0$. The measure $\mu_1$ is easiest, $\mu_1$-almost every $\Delta \in \text{Sub}(\Gamma)$ fixes a point in $P(V)$ and we argue exactly as in the previous section.

For $\mu_X$ the argument is very similar. Let $g \in \Gamma$ be an element whose $\rho$ image is $(r, e)$-very proximal with attracting point $\tau_g$ and repelling hyperplane $\Pi_g$ in the projective plane $P(V)$. Let $\tilde{v}_g = \langle v_g \rangle \in \partial X$ be the associated $g$ fixed point in the spherical building $\partial X$. $\rho(g)$ will act as a hyperbolic isometry on the building $X$. Just as in the previous section it would be enough to show that $\tilde{v}_g$, and hence also $\tau_g$ is fixed by every $\mu_X$-essential element $\gamma \in \Gamma$.

Pick $\Delta \in \mathcal{E}_{\text{inv}}(\gamma)$ that is simultaneously locally finite and recurrent in the sense that $g^{-n} \gamma g^n \in \Delta$ infinitely often. Let $s = s(\Delta) \in X$ be the corresponding fixed point. The element $\rho(g)$ acts as a hyperbolic element on $X$ with attracting point $\tilde{v}_g \in \partial X$. Thus for every sequence $n_i \in N(\Delta, \mathcal{E}_{\text{inv}}(\gamma)), g)$ the sequence $\rho(g^{n_i})s$ consists of $\rho(\Delta)$ fixed points converging to the attracting point $\rho(g^{n_i})s \xrightarrow{i \to \infty} \tilde{v}_g$. We conclude using the fact that set of $\rho(\gamma)$ fixed points is closed.

We turn to $\mu_H$, which is the hardest of the three. The following Lemma shows that $\mu_H$ cannot be supported on totally reducible subgroups. We first recall the definition.

**Definition 3.10.** A subspace of $V$ is called **totally reducible** if it decomposes as a direct sum of irreducible $\rho(\Delta)$ modules. We say that $\rho(\Delta)$ is totally reducible if $V$ itself is totally reducible as a $\rho(\Delta)$ module.

**Lemma 3.11.** Let $K$ be a complete valued field, $\Delta$ a locally finite group and $\rho : \Delta \to \text{PGL}_n(K)$ a totally reducible representation, all of whose irreducible compnenets are of dimension $\geq 2$. Then $\rho(\Delta)$ fixes a point in the affine Bruhat-Tits building $X$ of $\text{PGL}_n(K)$.
Proof. Assume that $V = W_1 \oplus W_2 \oplus \ldots \oplus W_l$ is the decomposition into $\rho(\Delta)$-irreducible representations. Since each $W_i$ is of dimension at least two we consider the affine building $Y = Y_1 \times Y_2 \times \ldots \times Y_l$ corresponding to the group $\text{PGL}(W_1) \times \text{PGL}(W_2) \times \ldots \times \text{PGL}(W_l)$. The boundary of this product is the spherical join of the boundaries of the individual factors $\partial Y = \partial Y_1 \ast \partial Y_2 \ast \ldots \ast \partial Y_l$, see [BH99, Definition 5.13, Corollary 9.11]. $\partial Y$ embeds as a sub-building of $\partial X$. Since the group $\rho(\Delta)$ preserves the direct sum decomposition, it acts on this building. Being locally finite as it is, Proposition 3.9 implies that $\rho(\Delta)$ has a fixed point in $Y \cup \partial Y$. If the fixed point is on the boundary we may assume again that it is a vertex. But $\partial Y^0 = \partial Y_1^0 \cup \ldots \cup \partial Y_l^0$ so that in fact we have a fixed vertex in $\partial Y_i$ for some $i$. This corresponds to a proper invariant subspace of $W_i$ which is impossible since the action of $\rho(\Delta)$ on $W_i$ is irreducible. So there is a $\rho(\Delta)$ fixed point $y = (y_1, y_2, \ldots, y_l) \in Y$. Each $y_i$ corresponds to a homothety class of norms, let $\|\|_i$ be a norm on $W_i$ representing this homothety class. Consider the subspace of $X$ consisting of all homothety classes of norms that coincide with these fixed norms on each $W_i$.

$$R = \{\|\| \in N(V, K)^{\text{diag}} \mid \exists (d_1, d_2, \ldots, d_l) \in \mathbb{R}^l \text{ s.t. } \|w\| = e^{d_i} \|w\|_i, \forall w \in W_i\} / \sim$$

This is clearly a convex $\rho(\Delta)$ invariant subset of the building isometric to a Euclidean space of dimension $l - 1$. Since $\rho(\Delta)$ fixes $l$ points on the boundary of this Euclidean space it must fix the whole boundary. Hence it acts on $R$ by translations. This give a homomorphism $\rho(\Delta) \to \mathbb{R}^{l-1}$ which must be trivial as $\rho(\Delta)$ is locally finite. Thus $\rho(\Delta)$ fixes $R$ pointwise and in particular it fixes many points in $X$ in contradiction to our definition of the measure $\mu_H$.

This last argument is illustrated in figure 1, inside one apartment, of the simplicial building associated with $\text{PGL}_4(k)$ over a local field. The basis corresponding to this apartment $\{e_1, e_2, e_3, e_4\}$ was chosen to respect the splitting $W_1 = \langle e_1, e_2 \rangle, W_2 = \langle e_3, e_4 \rangle$ as well as the individual fixed norms, which are here just the standard norms on these two subspaces. As is customary in the simplicial setting the homothety classes of norms are represented by homothety classes of unit balls with respect to these norms. The vertices thus correspond to lattices over the integer rings.

I will now aim at a contradiction, to the existence of $\mu_H$, by showing that $\mu_H$ almost every subgroup is totally reducible. As in Proposition 3.7, the locally essential lemma 2.2 combined with Poincaré recurrence show that $\mu_H$ almost every subgroup is locally essential recurrent.

It is a well known fact from ring theory that the a module is totally reducible if and only if it is generated by its irreducible components (see for example [Wel73, Theorem 1.4]). Thus the subspace $W = W(\Delta) < V$ generated by all the irreducible submodules of $\rho(\Delta)$ is a maximal $\rho(\Delta)$-totally reducible submodule. Assume toward contradiction that $W \neq V$ is not the whole space. We start by finding a finitely generated subgroup with the same maximal totally reducible subspace.

Lemma 3.12. There is a finitely generated subgroup $\Sigma < \Delta$ such that $W$ is also a maximal totally reducible $\rho(\Sigma)$-module.

Proof. Let $W = \oplus_i W_i$ be the decomposition of $W_i$ into irreducible representations. Let $\Sigma_1 < \Sigma_2 < \ldots$ be finitely generated subgroups ascending to $\Delta$. Within each $W_i$
we can find some $W_{i,j} < W_i$ which is irreducible as a $\rho(\Sigma_j)$-module. Furthermore we can clearly arrange that $W_{i,j} \leq W_{i,j+1}$, $\forall i,j$. By dimension considerations this process must stabilize. And it must stabilize with $W_{i,j} = W_i$, $\forall j \geq J$ since a module that is invariant under every $\Sigma_j$ must be $\Delta$-invariant too. Thus the maximal totally reducible subspace of $\Sigma$ contains $W$.

Now let $U = W \oplus U_1 \oplus \ldots \oplus U_l$ be the maximal totally reducible subspace for $\rho(\Sigma_j)$. For each $U_i$ we can find some $\delta_i \in \Delta$ such that $\rho(\delta_i)U_i \cap W \neq \langle 0 \rangle$. Thus none of the spaces $U_i$ can be contained in the maximal totally reducible subspace for $\Sigma := \langle \Sigma_j, \delta_1, \delta_2, \ldots, \delta_s \rangle$ and the maximal totally reducible subspace for this last group must be $W$ itself. □

$\rho(\Gamma)$ contains a very proximal element $g$. Since $\rho(\Gamma)$ is strongly irreducible we can assume, after possibly replacing $g$ by a conjugate, that $W \not< H_g$ and $v_g \not\in W$. But this immediately implies that

$$\lim_{n \to \infty} d(\rho(g^n)W, v_g) = 0.$$  

In particular there exists some $N \in \mathbb{N}$ such that for every $n > N$ we have $W \not\subseteq \langle W, \rho(g^n)W \rangle$. Now consider the measure preserving dynamics on $\text{Sub}(\Gamma)$. Let $\Sigma < \Delta$ is the finitely generated subgroup provided by Lemma 3.12. Since $\Delta$ was chosen to be locally essential we know that $\mu(\mathcal{E}_\text{nv}(\Sigma)) > 0$. Now by the recurrence assumption the set of return times $N(\Delta, \mathcal{E}_\text{nv}(\Sigma), g) = \{ n \in \mathbb{N} \mid g^n \Delta g^{-n} \in \mathcal{E}_\text{nv}(\Sigma) \}$ is infinite. And for every such such return time $n$ we have

$$\sigma \rho(g^n)W = \rho(g^n)\rho(g^{-n}\sigma g^n)W = \rho(g^n)W \quad \forall \sigma \in \Sigma.$$  

Moreover since conjugation by $g$ induces an isomorphism between $W$ and $\rho(g^n)W$ as $\Sigma$-modules the latter is again totally reducible. Thus if we choose $n > N$ which is also in $N(\Delta, \mathcal{E}_\text{nv}(\Sigma), g)$ we have a subspace $W' = \langle W, \rho(g)W \rangle$ that is $\Sigma$-invariant, totally reducible as a $\rho(\Sigma)$ module and strictly larger than $W$. In contradiction to our construction of $\Sigma$. This contradiction finishes the proof.

3.7. A nearly Zariski dense essential cover. Let $H = \overline{\rho(\Gamma)}^Z(K)$ be the $K$ points of the Zariski closure of $\rho(\Gamma)$ and $H^{(0)}$ the connected component of the
identity in $H$. Our goal in this section is to find an essential cover by finitely generated subgroups $\Sigma$ such that $\rho(\Sigma)^Z$ contains $H^{(0)}$. For this fix $\mu \in \text{IRS}^\chi$. As a first step we claim that $\rho(\Delta)^Z > H^{(0)}$ almost surely (compare [ABB^+, Theorem 2.6]).

Let $h = \text{Lie}(H)(K)$ be the Lie algebra and $\text{Gr}(h)$ the Grassmannian over $h$. By taking the Lie algebra of the random subgroup we obtain a random point in this Grassmannian

$$\Psi : \text{Sub}(\Gamma) \to \text{Gr}(h) \quad \Delta \mapsto \left[\text{Lie} \left(\rho(\Delta)^Z\right)\right]$$

The push forward of the IRS gives rise to a $\rho(\Gamma)$-invariant measure on $\text{Gr}(h)$. Since $\rho(\Gamma)$ is Zariski dense and $H^0$ is simple, it follows from Furstenberg’s lemma [Fur76, Lemma 2] that such a measure can be supported only on the $\Gamma^{(0)}$-invariant points $\{h, \{0\}\}$. By the previous subsection we already know that $\rho(\Delta)$ is almost surely infinite so that in fact $\psi(\Delta) = h$ almost surely. But this means $\dim(\rho(\Delta)^Z) = \dim(H)$. Hence $\rho(\Delta)^Z$ itself has to be a subgroup of maximal dimension and it contains $H^{(0)}$ - even in positive characteristic.

By the results of Sections 3.5, 3.6 $\rho(\Delta)$ is not locally finite a.s. Since $\rho(\Delta)$ has a simple Zariski closure this immediately implies that in fact it is not even amenable. Thus by Lemma 2.7 it is enough to find, for every such essential subgroup $\Sigma$, a larger essential subgroup $\Sigma < \Theta$ containing a very proximal element. By Lemma 2.15, in order to establish the existence of a very proximal element, it is enough to find an element with a small enough Lipschitz constant on an arbitrarily small open set. Let $(\epsilon', r', c') = (\epsilon(\Sigma), r(\Sigma), c(\Sigma))$ be the constants, governing the connection between the proximality and Lipschitz constants, provided by that Lemma. Recall also that we have fixed an $(r, \epsilon)$-very proximal element $g \in \Pi$ together with our linear representation. Since $\Sigma$ is strongly irreducible, there exists an element $x \in \Sigma$ such that $x$ moves the attracting point $\pi_{g^{-1}}$ of $g^{-1}$ outside the repelling hyperplane $\overline{H}_g$ of $g$. Set $d = d(x \pi_{g^{-1}}, \overline{H}_g)$ to be the projective distance
between the two. Now consider the element
\[ y = g^m x g^{-m}, \]
I claim that if \( m > m_0 \) for some \( m_0 \) then \( y \) is \( \epsilon' \)-Lipschitz on a small neighborhood of \( \overline{\varpi}_{g^{-1}} \). It follows from Lemma 2.9 that for a large enough \( m \) the element \( \rho(g^{-m}) \) is \( \eta \)-very proximal with the same attracting point and repelling hyperplane as \( \rho(g^{-1}) \); where \( \eta > 0 \) can be chosen arbitrarily small. By Lemma 2.11 \( \rho(g^{-m}) \) is \( C_1 \eta^2 \)-Lipschitz on some small open neighborhood \( O \), away from \( \overline{\Pi}_{g^{-1}} \), for some constant \( C_1 \) depending only on \( \rho(g) \). If we take \( O \) to be small enough and \( m \) large enough then \( d \left( \rho(x g^{-m})(O), \overline{\Pi}_g \right) > d/2 \). By Lemma 2.11 again, the element \( \rho(g^m) \) is \( C_2 \epsilon^2 \)-Lipschitz on \( \rho(x g^{-m})(O) \). Now since \( \rho(x) \) is bi-Lipschitz with some constant \( C_3 \) depending on \( \rho(x) \), it follows that \( \rho(y) = \rho(g^m x g^{-m}) \) is \( C_4 \epsilon^4 \)-Lipschitz on \( O \), where \( C_4 \) depends only on \( \rho(y) \) and \( \rho(x) \). If we require also \( \epsilon' \leq C_4^{-1/4} \) then \( \rho(y) \) is \( \epsilon' \)-Lipschitz on \( O \) as we claimed. The dynamics of the action of the element \( \rho(g^m x g^{-m}) \) on \( P(K) \) is depicted on the left side of Figure 2.

![Figure 2. The dynamics of \( \rho(g^m x g^{-m}) \) on the \( P(K) \) and \( \text{Sub}(\Gamma) \).](image)

Now we analyze the action of the same element \( g^m x g^{-m} \) on \( \text{Sub}(\Gamma) \). Poincaré recurrence yields infinitely many values of \( m \) for which
\[
\mu \left( \mathcal{E}_\text{nv}(\Sigma) \cap \mathcal{E}_\text{nv}(g^m \Sigma g^{-m}) \right) = \mu \left( \mathcal{E}_\text{nv}(\Sigma, g^m \Sigma g^{-m}) \right) > 0.
\]
We choose \( m \in \mathbb{N} \) for which this is true and at the same time the reasoning of the previous paragraph applies so that \( y = g^m x g^{-m} \) is \( \epsilon' \)-Lipschitz contained in the essential subgroup \( \Theta := (\Sigma, g^m \Sigma g^{-m}) \). Since by assumption \( \rho(\Sigma) \) is strongly irreducible Lemma 2.15 provides elements \( f_1, f_2 \in \Sigma \) such that \( a_{\Theta} := y f_1 y^{-1} f_2 \in \Theta \) is \( (\epsilon', \epsilon' \sqrt{\epsilon'}) \)-very proximal. Which completes the proof. Let \( \mathcal{A}(a_{\Theta}), \mathcal{R}(a_{\Theta}) \) be attracting and repelling neighborhoods for the elements we have just constructed. By our construction all the groups \( \Theta \) are again finitely generated, as they are generated by two finitely generated subgroups. Hence if
\[
\mathcal{F} := \left\{ \Theta \in \mathcal{E} \mid \begin{array}{l} \rho(\Theta) \text{ is strongly irreducible} \\
\exists a \in \Theta \text{ with } \rho(a) \text{ very proximal and } \rho(a) \in \left( \rho(\Theta)^2 \right)^{(0)} \\
\Theta \text{ is finitely generated} \end{array} \right\}
\]
We have just shown that \( \mathcal{F} \) covers IRS\(^{\times}(\Gamma) \).
3.9. Putting the very proximal elements \( \{a_\Sigma\} \) in general position. In the construction of the very proximal elements \( \{a_\Sigma \mid \Sigma \in \mathcal{F}\} \) in Section 3.8 no connection was established between these different elements. As a first step towards arranging them we would like, each one of them separately, to form a ping-pong pair with our fixed very proximal element \( g \in \Gamma \). Moreover we would like this to be realized with repelling and attracting neighborhoods:

\[
\mathcal{R}(g^{\pm 1}), \mathcal{A}(g^{\pm 1}), \mathcal{R}(a_\Sigma^{\pm 1}), \mathcal{A}(a_\Sigma^{\pm 1}),
\]

such that \( \mathcal{R}(g^{\pm 1}), \mathcal{A}(g^{\pm 1}) \) are independent of \( \Sigma \). This will be done, at the expense of refining the cover again \( \mathcal{G} <_{\text{IRS}} \Gamma \mathcal{F} \).

Let \( h \in \Pi \) be any very proximal element satisfying the ping-pong lemma conditions with \( g \). Assume that \( \mathcal{R}(g^{\pm 1}), \mathcal{A}(g^{\pm 1}), \mathcal{R}(h^{\pm 1}), \mathcal{A}(h^{\pm 1}) \) are the corresponding attracting and repelling neighborhoods. For the sake of this proof only we will call an element \( h' \) good if it is very proximal with attracting and repelling neighborhoods contained in these of \( h \) as follows: \( \mathcal{A}(h'^{\pm 1}) \subset \mathcal{A}(h) \), \( \mathcal{R}(h'^{\pm 1}) \subset \mathcal{R}(h^{-1}) \). Such a good element is indeed good for us: the pair \( \{h',g\} \) satisfies the conditions of the ping-pong lemma, with the given attracting and repelling neighborhood.

Let \( \mathcal{G} \subset \mathcal{F} \) be the collection of essential subgroups in \( \mathcal{F} \) that contain a good element. Since \( \mathcal{F} \) is monotone Lemma 2.7 implies that it is enough to show that any given \( \Sigma \in \mathcal{F} \) is contained in a larger subgroup \( \Theta \in \mathcal{G} \). So fix \( \Sigma \in \mathcal{F} \) and let \( a = a_\Sigma \) be the associated very proximal element with attracting and repelling neighborhoods \( \mathcal{A}(a_\Sigma^{\pm 1}), \mathcal{R}(a_\Sigma^{\pm 1}) \). We can always assume that \( a, h \) are in general position with respect to each other, that is to say that some high enough powers of them play ping-pong. Indeed conjugation just moves the attracting points and repelling hyperplanes around \( \tau_{\sigma a^\pm \sigma^{-1}} = \rho(\sigma)\tau_{a^\pm}, \Pi_{\sigma a^\pm \sigma^{-1}} = \rho(\sigma)\Pi_{a^\pm} \), and since by assumption \( \Sigma \) is strongly irreducible we can choose \( \sigma \in \Sigma \) that will bring \( a \) to a general position with respect to \( h \).

Now as \( a, h \) are in general position we claim that \( h'ah^{-l} \) is good for any sufficiently large \( l \). This is statement is clear upon observation that

\[
\begin{align*}
\mathcal{A}(h'ah^{-l}) &= \rho(h')\mathcal{A}(a^\pm) \\
\mathcal{R}(h'ah^{-l}) &= \rho(h')\mathcal{R}(a^\pm)
\end{align*}
\]

Consider for example the second equation. Since \( a \) and \( h \) are in general position we know that \( d(\tau_{a^{-1}}(h')^{-1}) > 0 \). Since \( h \) satisfies the conditions of Lemma 2.9 we can choose \( l \) large enough so that \( \mathcal{A}(h^{-l}) \subset (\tau_{h^{-1}})_{d/2} \) and hence \( \mathcal{A}(h^{-l}) \cap (\mathcal{R}(a) \cup \mathcal{R}(a^{-1})) = \emptyset \). Now the equation above implies that

\[
\mathcal{R}(h'ah^{-l}) = \rho(h')\mathcal{R}(a^\pm) \subset \rho(h')((\mathcal{P}(k^n) \setminus \mathcal{A}(h^{-l})) \subset \mathcal{R}(h') \subset \mathcal{A}(h^{-l}).
\]

Where the inclusion before last follows from the fact that \( \rho(h^{-l})((\mathcal{P}(k^n) \setminus \mathcal{R}(h')) \subset \mathcal{A}(h^{-l}) \). The second case follow in a similar fashion.

On the space \( \mathbf{Sub}(\Gamma) \) we repeat the argument from the previous section. By Poincaré recurrence the group \( \Theta := \langle \Sigma, h'\Sigma h^{-l} \rangle \) is again essential for infinitely many values of \( l \). We choose one such value to make \( \Theta \) essential and at the same time large enough so that \( a_{\Theta} := h'ah^{-l} \) is good. This completes the proof that \( \mathcal{G} \)
Proof. Consider the closed sets satisfy the conditions of the ping-pong lemma and are therefore independent. The conditions of the ping-pong lemma 2.10 with respect to the given neighborhoods. G covers IRS 32 YAIR GLASNER

3.10. Poincaré recurrence estimates for multiple measures. Before proceeding to the proof of Theorem 3.5. We will need the following quantitative version of Poincaré recurrence.

Lemma 3.14. (Poincaré recurrence) Let (X, B, µ, T) be a probability measure preserving action of Z = ‹T›, A ∈ B a set of positive measure and ϵ > 0. Then there exists a number n = n(A, µ, ϵ) ∈ N, such that for every N ∈ N we have

$$\mu \left( A \setminus \left( \bigcup_{i=N}^{N+n-1} T^i A \right) \right) < \epsilon.$$
Proof. Let $V_m = \{ x \in A \mid T^m x \in A \text{ but } T^l x \notin A \ \forall 1 \leq l < m \}$. Clearly these sets are measurable and disjoint. Poincaré recurrence implies that their union covers all of $A$ up to $\mu$-nullsets. This enables us to cover the forward $(T)$-orbit of $\mathcal{E}n(\Sigma)$ by a so called Kakutani-Rokhlin tower (see figure 3.10):

$$
\bigcup_{n \in \mathbb{N}} T^n A = [V_1] \sqcup [V_2 \cup T^1 V_2] \sqcup [V_3 \cup T^1 V_3 \cup T^2 V_3] \sqcup \ldots
$$

Where all the sets are disjoint and equality is up to null sets. Denoting the tail of this tower by

$$
\text{Tail}(m) = [V_{m+1} \cup \ldots \cup T^m V_{m+1}] \sqcup [V_{m+2} \cup \ldots \cup T^{m+1} V_{m+2}] \sqcup \ldots,
$$

we choose $n = n(T, \mu, \epsilon)$ to be the minimal number such that $\mu(\text{Tail}(n)) < \epsilon$. The lemma now follows from the obvious inclusion

$$
A \setminus \left( \bigcup_{i=N}^{N+n-1} T^i A \right) \subset T^{-N} (\text{Tail}(n)).
$$

$\square$

3.11. An independent set. We proceed to our last refinement: a collection $\mathcal{H} \subset \mathcal{G}$ of finitely generated essential subgroups together together with a set of associated very proximal elements $B := \{ b_\Theta \in \Theta \mid \Theta \in \mathcal{H} \}$ which satisfy the following two conditions:

- **Ind:** $B = \{ b_\Theta \mid \Theta \in \mathcal{H} \}$ forms an independent set.
- **Cov:** $\mathcal{H}$ covers IRS$^V$. 

To achieve this, we can no longer appeal to Lemma 2.7 as our new requirement \textbf{Ind} can no longer be verified one group at a time. To obtain an independent set we apply Proposition 3.13 with a function $\psi: \mathbb{Z} \to \mathcal{G}$ satisfying the following property:

\textbf{Rpt:} For every $(n, \Sigma) \in \mathbb{N} \times \mathcal{G}$ there is some index $N = N(n, \Sigma)$ such that $\psi(i) = \psi(i + 1) = \psi(i + 2) = \ldots = \psi(i + n - 1) = \Sigma$.

to obtain a sequence subgroups $\Theta_j = \langle \Sigma, g^j \Sigma g^{-j} \rangle$ and an associated sequence of elements

$$\{b_{\Theta_j} = g^j a_{\psi(j)} g^{-j} \in \Theta_j \mid j \in \mathbb{N} \}.$$  

Proposition 3.13 guarantees that the elements $\{b_{\Theta_j} \mid j \in \mathbb{N}\}$ form a ping-pong tuple and are hence independent. It is not necessarily true that every $\Theta_j$ is essential but to construct $\mathcal{H}$ and $B$ we take only the indexes $j$ for which this is the case:

$$\mathcal{H} = \{ \Theta_j \mid j \in \mathbb{Z}, \Theta_j \text{ is } \mu \text{-essential for some } \mu \in \text{IRS}^X \}$$

$$B = \{ b_{\Theta} \mid \Theta \in \mathcal{H} \}.$$  

Clearly $\mathcal{H} \subset \mathcal{G}$. Indeed each $\Theta \in \mathcal{H}$ is finitely generated as it is generated by two finitely generated subgroups; it is strongly irreducible as it contains a strongly irreducible subgroup and it contains the very proximal element $b_{\Theta}$. The independence condition (\textbf{Ind}) for the elements of $B$ follows directly from Proposition 3.13.

As for the covering condition (\textbf{Cov}), this is where our uniform estimates on Poincaré recurrence come into play. Fix $\Sigma \in \mathcal{G}$, $\epsilon > 0$ and let $\mu \in \text{IRS}^X(\Gamma)$ be a measure according to which $\Sigma$ is $\mu$-essential. Usually there will be uncountably many such measures since $\mathcal{H}$ is merely countable while IRS$^X(\Gamma)$ is not! Still we can apply Lemma 3.14 to the dynamical system $(X = \text{Sub}(\Gamma), B, \mu, g)$ and to the positive measure set $A = \mathcal{E}\text{nv}(\Sigma)$. Let $n = n(A, \mu, \epsilon) = n(\Sigma, \mu, \epsilon)$ be the constant given by that lemma so that for every $N \in \mathbb{Z}$ we have

$$\mu \left\{ \mathcal{E}\text{nv}(\Sigma) \setminus \bigcup_{j=N}^{N+n-1} \mathcal{E}\text{nv}(\langle \Sigma, g^j \Sigma g^{-j} \rangle) \right\} < \epsilon.$$  

By our construction of the function $\psi$ there exists an index $N = N(n, \Sigma)$ such that $\Theta_j = \langle \Sigma, g^j \Sigma g^{-j} \rangle \in \mathcal{H}$ for every $N(n, \Sigma) \leq j \leq N(n, \Sigma) + n - 1$. Or at least for every such $j$ such that $\Theta_j$ is essential. Since anyhow these that are non essential have envelopes of measure zero we have

$$\mu \left\{ \mathcal{E}\text{nv}(\Sigma) \setminus \bigcup_{\Theta \in \mathcal{H}} \mathcal{E}\text{nv}(\Theta) \right\} \leq \mu \left\{ \mathcal{E}\text{nv}(\Sigma) \setminus \bigcup_{j=N}^{N+n-1} \mathcal{E}\text{nv}(\Theta_j) \right\} < \epsilon.$$  

Since $\mu, \epsilon, \Sigma$ are arbitrary we deduce that $\mathcal{H}$ covers IRS$^X$ as promised.

3.12. \textbf{Conclusion.} Here we complete the proof of Theorem 3.5 thereby completing the proof of the main theorem. Let $\mathcal{H} = \{ \Theta_1, \Theta_2, \ldots \}$, $B = \{ b_1, b_2, \ldots \}$ be the groups and very proximal elements constructed in the previous section - with a new enumeration making the indices consecutive. No further refinement of $\mathcal{H}$ the collection of essential subgroups will be needed; what we do want is a much richer free subgroup containing representatives of all double cosets of the form $\Theta_q \gamma \Theta_p$.

For this fix a bijection

\begin{align*}
\mathbb{N} & \to \mathbb{N}^2 \times \Gamma \\
q & \mapsto (p(q), q(k), \gamma(k))
\end{align*}
We proceed to prove Theorem 3.5 by induction on $k$. Assume that after $k$ steps of induction we have constructed group elements $\{f_i \in \Gamma \mid 1 \leq i \leq k\}$ and a sequence of numbers $\{n_p(k) \in \mathbb{N} \mid p \in \mathbb{N}\}$ with the following properties:

- $\rho(f_i)$ is very proximal for each $1 \leq i \leq k$.
- $n_p(k) \geq n_p(k-1)$, $\forall p \in \mathbb{N}$.
- $f_i \in \Theta_q(\gamma(i))\Theta_p(i)$, $\forall 1 \leq i \leq k$.

The elements

$$B(k) := \{f_1, f_2, \ldots, f_k, b_1^{n_1(k)}, b_2^{n_2(k)}, \ldots\},$$

form an (infinite) ping-pong tuple - satisfying the conditions of Lemma 2.10.

For the basis of the induction we take $k = 0, n_p(0) = 1$ and $B(0) = B$. Now assume that we already have all of the above for $k-1$ and let us show how to carry out the induction. Set $p = \rho(k), q = \rho(k), \gamma = \gamma(k)$. Since $\Theta_p, \Theta_q \in \mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ both these groups act strongly irreducibly on the projective space. So there exist elements $x = x(k) \in \Theta_p, y = y(k) \in \Theta_q$ such that:

$$\rho(y^{-1}x) b_p \not \in \overline{b_q}$$  \hspace{1cm} (3.2)

Indeed the set of elements in $(\Theta_p \cap G^{(0)}) \times (\Theta_q \cap G^{(0)})$ that satisfy each one of the above conditions is Zariski open and non-trivial by the strong irreducibility. Hence their intersection is non-empty.

We define

$$f_k := b_q^l y^{-1}x b_p^1.$$  

Denote by $(\cdot)_\epsilon$ an $\epsilon$-neighborhood in $\mathbb{P}(k^n)$. We claim that for a small given $\epsilon > 0$ if $l_1, l_2$ are large enough then there exists some $0 < \epsilon' < \epsilon$ such that

$$\rho(f_k) \left( \mathbb{P}^{n-1}(k) \setminus \left[ (\overline{b_q})_\epsilon \setminus (\overline{b_p})_\epsilon \right] \right) \subset \left( (\overline{b_q})_{\epsilon'} \setminus (\overline{b_p})_{\epsilon'} \right)$$

$$\rho(f_k^{-1}) \left( \mathbb{P}^{n-1}(k) \setminus \left[ (\overline{b_p})_{\epsilon'} \setminus (\overline{b_q})_{\epsilon'} \right] \right) \subset \left( (\overline{b_q})_{\epsilon} \setminus (\overline{b_p})_{\epsilon} \right).$$  \hspace{1cm} (3.3)

Indeed we can write

$$\rho(f_k) \left( \mathbb{P}^{n-1}(k) \setminus (\overline{b_p})_{\epsilon} \right) \subset \rho(b_q^l y^{-1}x) \left( (\overline{b_p})_{\epsilon_1} \right)$$

$$\subset \rho(b_q^l) \left( (\rho(y^{-1}x) b_p)_{\epsilon_2} \right)$$

$$\subset \rho(b_q^l) \left( (\mathbb{P}^{n-1} \setminus (\overline{b_p})_{\epsilon_2}) \right)$$

$$\subset (\overline{b_q})_{\epsilon} \setminus (\overline{b_p})_{\epsilon}.$$  

Where, since $b_p$ is very proximal, the first inclusion holds for arbitrarily small values of $\epsilon_1$, whenever $l_1$ is large enough. Consequently $\epsilon_2$ can be set arbitrarily small by continuity of $\rho(y^{-1}x)$ on the projective plane. In view of the first line in Equation (3.2) we can choose $\epsilon_2 < \frac{1}{2}d(\rho(y^{-1}x) b_p, \overline{b_p})$ which accounts the third line. Now by choosing $l_2$ to be large enough, $b_q^l$ becomes very proximal with attracting and repelling neighborhoods arbitrarily close to $\overline{b_q}$ and $\overline{b_p}$ respectively. In particular we can arrange for the image to be contained in $(\overline{b_p})_{\epsilon}$ as indicated. The fact that
we can exclude from the image a ball of some positive radius \( \epsilon' > 0 \) comes from the fact that \( \pi_{b_q} \) is a fixed point for the homeomorphism \( \rho(b_q^2) \) and that, by the above observations, we may assume that \( \epsilon_2 < \frac{1}{2}d(\pi_{b_q}, \pi_{b_q}) \).

![Figure 5. Final refinement](image)

And on the other hand
\[
\rho(f_k)((\mathcal{P}_{b_p})_{\epsilon'}) \subset \rho(b_q^2 y \gamma x)((\mathcal{P}_{b_p})_{\epsilon_1})
\]
\[
\subset \rho(b_q^2)((\rho(y \gamma x)\mathcal{P}_{b_p})_{\epsilon_2})
\]
\[
\subset \rho(b_q^2)(\mathcal{P}^{n-1} \setminus (\pi_{b_q})_{\epsilon_2})
\]
\[
\subset (\pi_{b_q})_{\epsilon} \setminus (\pi_{b_q})_{\epsilon'}
\]

Where the first equation can be satisfied, for arbitrarily small values of \( \epsilon_1 \), if we assume that \( \epsilon' \) is small enough; by the fact that \( H_{b_p} \) invariant under the continuous map \( b_p^1 \) and that the latter map admits some global Lipschitz constant on the whole of \( \mathbb{P}(k^n) \). The second inclusion follows, for arbitrarily small values of \( \epsilon_2 \), using the Lipschitz nature of \( \rho(y \gamma x) \) combined with the ability to bound \( \epsilon_1 \). Now using the third line in Equation 3.2 we can choose \( \epsilon_2 < d(\pi_{b_q}, \rho(y \gamma x)\mathcal{P}_{b_p}) \). Finally the last line follows from the previous one and from the fact that \( \pi_{b_q} \) is \( \rho(b_q) \)-invariant.

These last two calculations combine to yield the first Equation in 3.3. The second equation there follows in a completely symmetric fashion.

Equations 3.3, applied to a small enough value of \( \epsilon \), imply the existence of \( \epsilon' > 0 \) and \( l_1, l_2 \in \mathbb{N} \) such that \( \rho(f_k) \) becomes a very proximal element with attracting and repelling neighborhoods as follows:
\[
\mathcal{R}(f_k) \subset \mathcal{R}(b_p) \setminus (\mathcal{P}_{b_p})_{\epsilon'}
\]
\[
\mathcal{A}(f_k) \subset \mathcal{A}(b_q) \setminus (\pi_{b_q})_{\epsilon'}
\]
\[
\mathcal{R}(f_k^{-1}) \subset \mathcal{R}(b_q^{-1}) \setminus (\mathcal{P}_{b_q^{-1}})_{\epsilon'}
\]
\[
\mathcal{A}(f_k^{-1}) \subset \mathcal{A}(b_p^{-1}) \setminus (\pi_{b_p^{-1}})_{\epsilon'}
\]

Since the elements \( b_p, b_q \) are very proximal, and in addition satisfy the conditions of Lemma 2.9, powers of them will have the same attracting points and repelling hyperplanes, and we can arrange for the attracting and repelling neighborhoods to be in an arbitrarily small neighborhoods of these points and planes respectively.
Thus choose \( n_p(k) > n_p(k-1), n_q(k) > n_q(k-1) \) that give rise to \( R^{\pm n_p(k)}(H^{\pm n_r(k)}) \subset (\overline{H^{\pm n_p(k)}})^c, A^{\pm n_r(k)}(H^{\pm n_p(k)}) \subset (\overline{H^{\pm n_p(k)}})^c \), for every \( r \in \{p, q\} \). For every other \( r \in \mathbb{N} \setminus \{p, q\} \) we just set \( n_r(k) = n_r(k-1) \).

We leave it to the reader to verify that the induction holds. The only thing that really requires verification is the ping-pong conditions for:

\[
\{f_1, f_2, \ldots, f_k, b^{n_{1}}(k), b^{n_{2}}(k), \ldots, b^{n_{p}}(k), \ldots, b^{n_{q}}(k), \ldots, \},
\]

which follows quite directly based only on the induction hypothesis and the specific repelling and attracting neighborhoods we have found for \( f^{\pm 1}, b^{\pm n_p(k)}, b^{\pm n_q(k)} \).

**Appendix A. A relative version of Borel density theorem - By Tsachik Gelander and Yair Glasner.**

In this appendix we prove a relative version of the Borel Density Theorem for Invariant Random Subgroups in countable linear groups.

**Theorem A.1.** (Relative version of the Borel density theorem) Let \( \Gamma < \text{GL}_n(F) \) be a countable linear group with a simple, center free, Zariski closure. Then every non-trivial invariant random subgroup \( \Delta \subset \Gamma \) is Zariski dense a.s.

**Proof.** We first claim that \( \Delta \) is either trivial or infinite a.s. Assume that \( \Delta \) is finite with positive probability, in this case we may further assume that \( \Delta \) is finite a.s. by conditioning on the event \( \text{Fin} := \{ \Delta \mid |\Delta| < \infty \} \subset \text{Sub}(\Gamma) \). Since the invariant measure is supported on the countable set \( \text{Fin} \), every ergodic component is a finite conjugacy class. Thus the IRS is supported on finite subgroups with finite index normalizers. Passing to the Zariski closure we conclude that \( |G : N_G(\Delta)| < \infty \) and since the normalizer of a finite group is algebraic \( N_G(\Delta) > G(0) \) a.s. Since \( G(0) \) is center free \( \Delta = \langle e \rangle \).

We apply Theorem 3.4 to the group \( \Gamma \); obtaining a local field \( F < k \), a faithful, unbounded, strongly irreducible projective representation \( \rho : G(k) \to \text{PGL}_n(k) \) defined over \( k \). Let \( g = \text{Lie}(G) \) be the Lie algebra and \( \text{Gr}(g) \) the Grassmannian over \( g \). By taking the Lie subalgebra corresponding to the Zariski closure of the random subgroup we obtain a random point in this Grassmannian

\[
\Psi : \text{Sub}(\Gamma) \to \text{Gr}(g)(k)
\]

\[
\Delta \mapsto \text{Lie}(\overline{\Delta})
\]

The push forward of the IRS gives rise to a \( \rho(\Gamma) \)-invariant measure on \( \text{Gr}(g) \). Since \( \rho(\Gamma) \) is unbounded, it follows from Furstenberg’s lemma [Fur76, Lemma 2], applied in each dimension separately, that such a measure is supported on a disjoint union of linear projective subspaces. But \( \Gamma \) itself is Zariski dense and \( G \) is simple, so the only (ergodic) possibilities are the Dirac \( \delta \) measures supported on the trivial and the whole Lie algebra: \( \{g, \{0\}\} \). In the first case \( \overline{\Delta} \) is open and unbounded and hence equal to \( G \) (c.f. [Sha99]). In the second case \( \overline{\Delta} \) is finite and hence trivial by the previous paragraph. \( \square \)

As a consequence we obtain an alternative direct proof of Corollary 1.10, which bypasses the main theorem of the paper, which we restate for the convenience of the readers.
Corollary A.2. Let $\Gamma < \text{GL}_n(F)$ be a countable linear group. If $\Delta \rhd \Gamma$ is an IRS in $\Gamma$ that is almost surely amenable, then $\Delta < \text{A}(\Gamma)$ almost surely.

Proof. (of Corollary A.2) Let $\Gamma$ be a linear group and $\Delta \rhd \Gamma$ an IRS that is a.s. amenable. We wish to show that $\Delta < \text{A}(\Gamma)$ is contained in the amenable radical a.s. At first we assume that $\Gamma$ is finitely generated.

Let $\Gamma$ be a linear group and $\Delta \rhd \Gamma$ an IRS that is almost surely amenable, then $\Delta < \text{A}(\Gamma)$ almost surely. Let $\gamma \in \Gamma$ be an element in $\Delta$ so that $\Delta < \text{A}(\Gamma)$ is contained in the normal amenable subgroup $\Gamma^o \subset \Gamma$ which is again, almost surely amenable. Since $\Gamma$ is finitely generated we may deduce from the above paragraphs that $\Delta_i := \Delta \cap \gamma_i \rhd \Gamma_i$ is almost surely contained in $A(\Gamma_i)$. Since an essential element element in $\Gamma_i$ is contained in $\Delta_i$ with positive probability this means that every essential element in $\Gamma_i$ is contained in $A(\Gamma_i)$. Now essential elements in $\Gamma$ are eventually essential in $\Gamma_i$ so, denoting by $E$ the collection of essential elements in $\Gamma$ we have
\[ E \subset \liminf_n A(\Gamma_i) := \bigcup_n \bigcap_{i \geq n} A(\Gamma_i) \]
\[ \liminf A(\Gamma_i) \text{ is amenable as an ascending union of amenable groups. It is also normal since if } \gamma \in \Gamma_m \text{ then for every } \rho \geq \max\{m, n\} \text{ we have } \cap_{i \geq n} A(\Gamma_i) \subset A(\Gamma_{\rho}) \text{ so that } \gamma \big( \bigcap_{i \geq n} A(\Gamma_i) \big) \gamma^{-1} \subset \gamma A(\Gamma_{\rho}) \gamma^{-1} = A(\Gamma_{\rho}) \text{ and consequently} \]

\[ \gamma \left( \bigcap_{i \geq n} A(\Gamma_i) \right) \gamma^{-1} \subset \bigcap_{i \geq \max\{m, n\}} A(\Gamma_i). \]
Being an amenable normal subgroup \( \liminf_n A(\Gamma_n) \) is contained in the amenable radical \( A(\Gamma) \) which implies that every essential element is contained in \( A(\Gamma) \). Lemma 2.2, applied only to cyclic groups, shows that \( \Delta \) almost surely contains only essential elements, thus with probability one \( \Delta < A(\Gamma) \) and the Corollary is proved. \( \square \)

Remark: Very recently Bruno Duchesne and Jean Lécureux established a far reaching and very elegant generalization of Corollary 1.10, showing that the same statement holds for (not necessarily linear) locally compact groups (privet communication [DL]).

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YAIR GLASNER. Department of Mathematics. Ben-Gurion University of the Negev. P.O.B. 653, Be’er Sheva 84105, Israel. yairgl@math.bgu.ac.il

TSACHIK GELANDER Department of Mathematics. Weizmann Institute of Science. Rehovot 76100, Israel. tsachik.gelander@weizmann.ac.il