A(n, n)-GRADED LIE SUPERALGEBRAS

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Abstract. We determine the Lie superalgebras over fields of characteristic zero that are graded by the root system A(n, n) of the special linear Lie superalgebra psln + 1, n + 1).

1. Introduction

Root space decompositions and gradings by finite root systems have been a fundamental tool in the study of Lie algebras since the classification of the finite-dimensional complex simple Lie algebras by Killing and Cartan over a hundred years ago. Many nonsimple, finite- and infinite-dimensional Lie algebras over arbitrary fields of characteristic zero exhibit a grading by one or more finite root systems. Important examples include the affine Kac-Moody Lie algebras, the toroidal Lie algebras, and the intersection matrix Lie algebras introduced by Slodowy, to name just a few. Any finite-dimensional simple Lie algebra over a field F of characteristic zero which has an ad-nilpotent element (or equivalently by the Jacobson-Morosov theorem, has a copy of sl2(F)) is graded by a finite root system.

To make this concept more precise, Berman and Moody [14] singled out a class of Lie algebras graded by finite root systems. This class1 contains all of the above-mentioned algebras. The definition given by Berman and Moody starts with a finite-dimensional split simple Lie algebra g over a field F of characteristic zero having a root space decomposition $g = h \oplus \bigoplus_{\alpha \in \Delta} g_\alpha$ relative to a split Cartan subalgebra h. Thus, g is the analogue of a complex simple Lie algebra, just as sln(F) is the analogue of sln(C) over the field F. The root system $\Delta$ is one of the finite (reduced) root systems A, B, C, D, E6, E7, E8, F4, or G2. Following Berman and Moody we say...
Definition 1.1. A Lie algebra \( L \) over \( F \) is graded by the (reduced) root system \( \Delta \) if

1. \( L \) contains \( \mathfrak{g} \) as a subalgebra;
2. \( L = \bigoplus_{\alpha \in \Delta \cup \{0\}} L_{\alpha} \), where \( L_{\alpha} = \{ x \in L \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \} \) for \( \alpha \in \Delta \cup \{0\} \); and
3. \( L_0 = \sum_{\alpha \in \Delta} [L_{\alpha},L_{-\alpha}] \).

Assumption (1) may be replaced by the weaker requirement that the derivation algebra of \( L \) contain a copy of \( \mathfrak{g} \) without making a substantial difference. If only conditions (1) and (2) are assumed, then the ideal \( L' := \bigoplus_{\alpha \in \Delta} L_{\alpha} \oplus \left( \sum_{\alpha \in \Delta} [L_{\alpha},L_{-\alpha}] \right) \) is graded by \( \Delta \), and \( L \) acts as derivations on \( L' \). Assumption (2) is the real heart of the matter, for it implies that \( [L_{\alpha},L_{\beta}] \subseteq L_{\alpha+\beta} \) (where \( L_{\alpha+\beta} = 0 \) if \( \alpha + \beta \notin \Delta \cup \{0\} \)), and so \( L \) is graded by the abelian group generated by \( \Delta \).

The Lie algebras in Definition 1.1 have been classified in ([14], [11], [23]) under the assumption that the center of \( L \) is trivial. Their central extensions have been described in [2] and their derivations in [5]. The results in these papers have been used in an essential way to determine the structure of the extended affine Lie algebras (see [12], [13], [1], [4]) and of the intersection matrix Lie algebras (see [14], [11]).

Root decompositions also play a crucial role in the classification of the finite-dimensional complex simple Lie superalgebras (see [19]). Many nonsimple finite- and infinite-dimensional Lie superalgebras exhibit such root decompositions, and to better understand their structure, we introduce a concept analogous to the one above. A finite-dimensional simple complex Lie superalgebra is classical or it is of Cartan type. Here we suppose that \( g \) is a finite-dimensional split simple classical Lie superalgebra over a field \( F \) of characteristic zero (the analogue of one of the classical superalgebras over \( \mathbb{C} \)). Thus, \( g \) has a root space decomposition \( g = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha} \) relative to a split Cartan subsuperalgebra \( \mathfrak{h} \), and the root system \( \Delta \) is one of the following:

(i) \( A(m,n) \),
(ii) \( B(m,n) \),
(iii) \( C(n) \), \( D(m,n) \), \( D(2,1;\mu) \) (\( \mu \in F \setminus \{0,-1\} \)), \( F(4) \), \( G(3) \),
(iv) \( P(n) \), \( Q(n) \).

Such a Lie superalgebra \( g \) can have a nonsplit central extension, as Weyl’s theorem on complete reducibility fails to hold in the superalgebra setting. It is advantageous to take into account this behavior in formulating the notion of a \( \Delta \)-graded Lie superalgebra.

Definition 1.2. Let \( g = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha} \) be a finite-dimensional split simple classical Lie superalgebra over a field \( F \) of characteristic zero relative to a split Cartan subalgebra \( \mathfrak{h} \) (so \( \Delta \) is one of the root systems in (i)). A Lie
superalgebra \( L \) over \( \mathbb{F} \) is said to be \( \Delta \)-graded if it satisfies the following three conditions:

1. \( L \) contains, as a subsuperalgebra, a central cover \( \widetilde{g} \) of \( g \). The subsuperalgebra \( \widetilde{g} \) is called the grading subsuperalgebra of \( L \).

2. Let \( \bar{g}_0 \) be the unique subalgebra of \( \widetilde{g}_0 \) which projects isomorphically onto \( g_0 \) under the cover map (see Remark 2.2 below), and for a Cartan subalgebra \( h \) of \( \bar{g}_0 \), let \( \bar{h} \) be the Cartan subalgebra of \( \widetilde{g}_0 \) which projects isomorphically onto \( h \). Then \( L \) decomposes as

\[
L = \bigoplus_{\alpha \in \Delta \cup \{0\}} L_{\alpha},
\]

where \( \Delta \) is the root system of \( g \) relative to \( h \), and \( L_{\alpha} = \{ x \in L \mid [h,x] = \alpha(h)x \text{ for all } h \in \bar{h} \} \) for \( \alpha \in \Delta \cup \{0\} \); and

3. \( L_0 = \sum_{\alpha \in \Delta} [L_{\alpha}, L_{-\alpha}] \).

Remark 1.3. When \( \Delta \) is as in (ii) or (iii) of \((*)\), or when \( \Delta \) is of type \( A(m,n) \) with \( m \neq n \), the only central cover of \( g \) is \( g \) itself (see [18]). Thus, Definition 1.2 coincides with the notion of a \( \Delta \)-graded Lie superalgebra in [6]–[9] and [16] for those root systems. García and Neher [16] determine the Lie superalgebras graded by the finite root systems \( A_n, B_n, C_n, D_n, E_6, \) and \( E_7 \) over arbitrary commutative superrings by exploiting a 3-grading on the superalgebras and applying Jordan methods. The definition of a \( \Delta \)-graded Lie superalgebra presented in [22, Defn. 1.4] can be seen to be equivalent to the one above. Allowing a central cover of \( g \) to be the grading subsuperalgebra rather than just using \( g \) itself enables the Cheng-Kac superalgebras to be realized in a natural way as \( P(3) \)-graded superalgebras (see [22, Thm. 1.2]) The Cheng-Kac superalgebras provide important examples of conformal superalgebras of finite type ([15] and [17]).

Lie superalgebras graded by the root systems \( B(m,n) \) have been described in [8], while the superalgebras graded by the root systems listed in (iii) of \((*)\) have been classified in [6]. In [9], Benkart and Elduque have determined the Lie superalgebras graded by \( A(m,n) \) for \( m \neq n \). Included in that work are results on the Lie superalgebras containing \( g = \mathfrak{psl}(n+1,n+1) \) and having a root space decomposition of type \( A(n,n) \) relative to a Cartan subalgebra of \( g \). The complexity of the structure of such superalgebras was another motivation for broadening the concept of a \( \Delta \)-graded Lie superalgebra to the one given above. Martínez and Zelmanov [22] have described the Lie superalgebras graded by the root systems of type \( P(n) \) and \( Q(n) \). Thus, all the \( \Delta \)-graded Lie superalgebras, in the sense of Definition 1.2 as well as all of their central extensions, have been determined except for the root systems of type \( A(n,n) \). It is the goal of this present paper to complete the classification by treating the one remaining case of \( A(n,n) \).

Throughout it will be assumed that the ground field \( \mathbb{F} \) has characteristic zero. Usually the field \( \mathbb{F} \) will be omitted from the notation unless it is
needed for clarity. Thus, for example, we will write \( \mathfrak{sl}_n \) rather than \( \mathfrak{sl}_n(\mathbb{F}) \) and \( \mathfrak{sl}(n, n) \) instead of \( \mathfrak{sl}(n, n)(\mathbb{F}) \) or \( \mathfrak{sl}_{n,n}(\mathbb{F}) \) as a notational convenience.

2. Universal central extensions

A central extension of a Lie superalgebra \( L \) is a pair \((\hat{L}, \pi)\) consisting of a Lie superalgebra \( \hat{L} \) and a surjective Lie superalgebra homomorphism \( \pi: \hat{L} \to L \) (preserving the grading), whose kernel lies in the center of \( \hat{L} \). If \( \hat{L} \) is perfect \((\hat{L} = [\hat{L}, \hat{L}] )\), then \( \hat{L} \) is said to be a cover or covering of \( L \), and \( \pi \) is referred to as the cover map or covering homomorphism. Any perfect Lie superalgebra \( L \) has a unique (up to isomorphism) covering \((\hat{L}, \hat{\pi})\) which is universal, called the universal covering superalgebra or the universal central extension of \( L \). Thus, any covering of \( L \) is isomorphic to a quotient of \( \hat{L} \) by a central ideal. A superalgebra is centrally closed if \( \hat{L} = L \). When the universal central extensions of two perfect Lie superalgebras \( L_1 \) and \( L_2 \) are isomorphic (or equivalently, when \( L_1/Z(L_1) \cong L_2/Z(L_2) \)), the superalgebras \( L_1 \) and \( L_2 \) are said to be centrally isogenous.

The universal central extensions of the basic classical simple Lie superalgebras (that is, all the classical Lie superalgebras except for \( \mathfrak{p}(n) \) and \( \mathfrak{q}(n) \)) have been obtained in \([18]\) under the additional hypothesis that the kernel of the covering map is an even subspace. But this condition is superfluous:

**Lemma 2.1.** Let \( \mathfrak{g} \) be a basic classical simple Lie superalgebra.

(i) If \( \mathfrak{g} \) is not of type \( A(n,n) \), then \( \mathfrak{g} \) is centrally closed.

(ii) If \( \mathfrak{g} = \mathfrak{psl}(n+1,n+1) \), then its universal central extension \( \hat{\mathfrak{g}} = \text{uce}(\mathfrak{g}) \) is given by

\[
\hat{\mathfrak{g}} = \begin{cases} 
\mathfrak{sl}(n + 1, n + 1) & \text{for } n \geq 2, \\
"\text{D}(2,1;-1)" & \text{for } n = 1 
\end{cases} \quad \text{ (notation of [18]).}
\]

**Proof.** The universal central extension \( \hat{\mathfrak{g}} = \text{uce}(\mathfrak{g}) \) is a vector space direct sum of \( \mathfrak{g} \) and a quotient of \( \mathfrak{g} \wedge \mathfrak{g} \), so \( \dim \hat{\mathfrak{g}} < \infty \). Let \( \pi : \hat{\mathfrak{g}} \to \mathfrak{g} \) denote the projection.

Now \( \hat{\mathfrak{g}} \) is a \( \mathfrak{g} \)-module by means of \( x.z = [\hat{x},\hat{z}] \) for any \( x \in \mathfrak{g} \) and \( \hat{z} \in \hat{\mathfrak{g}} \), where \( \hat{x} \) is any element in \( \hat{\mathfrak{g}} \) which projects onto \( x \). As a \( \mathfrak{g} \)-module, \( \hat{\mathfrak{g}} \) has a composition series having an adjoint factor and trivial factors. If \( \mathfrak{g} \) is not of type \( A(n,n) \) for some \( n \geq 1 \), then by the complete reducibility results of \([6,7]\), \( \hat{\mathfrak{g}} = \ker \pi \oplus \mathfrak{s} \) for a suitable \( \mathfrak{g} \)-module \( \mathfrak{s} \). But \( [\hat{\mathfrak{g}},\mathfrak{s}] = \mathfrak{g} \cdot \mathfrak{s} \subseteq \mathfrak{s} \), so \( \mathfrak{s} \) is an ideal of \( \hat{\mathfrak{g}} \) and \( \hat{\mathfrak{g}} = [\hat{\mathfrak{g}},\hat{\mathfrak{g}}] \subseteq \mathfrak{s} \), since \( \ker \pi \) is central. Therefore \( \ker \pi = 0 \).

When \( \mathfrak{g} = \mathfrak{psl}(n+1,n+1) \) for \( n \geq 1 \), then \( \mathfrak{g}_0 = \mathfrak{sl}_{n+1} \oplus \mathfrak{sl}_{n+1} \) is semisimple so, as a \( \mathfrak{g}_0 \)-module, \( \hat{\mathfrak{g}}_0 = \mathfrak{s}_1 \oplus (\ker \pi \cap \hat{\mathfrak{g}}_0) \) for some \( \mathfrak{g}_0 \)-submodule \( \mathfrak{s}_1 \). But \( [\mathfrak{g}_0,\mathfrak{s}_1] = \mathfrak{g}_0 \cdot \mathfrak{s}_1 \subseteq \mathfrak{s}_1 \) and \( \hat{\mathfrak{g}}_0 = [\hat{\mathfrak{g}}_0,\hat{\mathfrak{g}}_0] = [\mathfrak{g}_0,\mathfrak{s}_1] \subseteq \mathfrak{s}_1 \). Therefore, \( \ker \pi \subseteq \mathfrak{g}_0 \), and the results of \([18]\) can be used to complete the proof. \( \square \)
Remark 2.2. For every basic classical Lie superalgebra \( g \), there is a unique
subalgebra \( \overline{g}_0 \) of \( \overline{g}_0 = uc(g)_0 \) such that the covering homomorphism \( \pi : \overline{g} \rightarrow g \) restricts to an isomorphism \( \overline{g}_0 \rightarrow g_0 \), since for \( \mathfrak{psl}(n + 1, n + 1) \), \( \overline{g}_0 \) is the
direct sum of \( \overline{g}_0 := [g_0, g_0] \), which is a direct sum of two copies of \( \mathfrak{s}_2 \),
and the center \( Z(g) \). In fact, the same result applies to any cover of \( g \). The even
part of the ‘strange’ Lie superalgebra \( D(2, 1; -1) \) is the direct sum of two
copies of \( \mathfrak{s}_2 \) and a three-dimensional center.

The situation for the classical simple Lie superalgebras \( g \) of type \( P(n) \)
and \( Q(n) \) is completely analogous. For both of them, the even part is simple
(isomorphic to \( \mathfrak{s}_2 \)), so the even part of any cover \( \overline{g} \) is the direct sum of
its center and of \( [g_0, g_0] \), which is the required unique subalgebra.

Lemma 2.3. Let \( \mathfrak{s} \) and \( \mathfrak{g} \) be two perfect Lie superalgebras such that \( \mathfrak{s} \) is a
subsuperalgebra of \( \mathfrak{g} \) and \( \mathfrak{s} \) is centrally closed. Let \( \overline{g} \) be a cover of \( g \) with
associated projection \( \pi : \overline{g} \rightarrow g \). Then there is a unique subsuperalgebra \( \overline{s} \) of
\( \overline{g} \) such that the restriction of \( \pi \) to \( \overline{s} \) is an isomorphism from \( \overline{s} \) onto \( s \).

Proof. Because \( \pi : \pi^{-1}(s) \rightarrow s \) is a cover of \( s \), and \( s \) is centrally closed,
\( \pi^{-1}(s) = \mathfrak{s} \oplus \overline{s} \) for subsuperalgebras \( \mathfrak{s} \) and \( \overline{s} \) with \( \mathfrak{s} \subseteq \ker \pi \subseteq Z(\overline{g}) \). Thus
\( \pi|_{\overline{s}} : \overline{s} \rightarrow s \) is an isomorphism. Moreover, \( \overline{s} = \left[ \pi^{-1}(s), \pi^{-1}(s) \right] \), which
demonstrates the uniqueness of \( \overline{s} \). \( \square \)

Lemma 2.4. Let \( L \) be a Lie superalgebra with an abelian subsuperalgebra \( \mathfrak{h} \)
such that \( L = \bigoplus_{\alpha \in \Delta \cup \{0\}} L_\alpha \), where \( \Delta \) is a finite subset of \( \mathfrak{h}^* \setminus \{0\} \); \( L_\alpha = \{ x \in L \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h} \} \) for any \( \alpha \in \Delta \); and \( L_0 = \sum_{\alpha \in \Delta} [L_\alpha, L_{-\alpha}] \). Let
\( \pi : \widehat{L} \rightarrow L \) be the universal central extension of \( L \). Then \( \widehat{L} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \widehat{L}_\alpha \),
where \( \widehat{L}_\alpha = \{ x \in \widehat{L} \mid [h', x] = \alpha(\pi(h'))x \ \forall h' \in \pi^{-1}(h) \} \) for any \( \alpha \in \Delta \), and
\( \widehat{L}_0 = \sum_{\alpha \in \Delta} [\widehat{L}_\alpha, \widehat{L}_{-\alpha}] \). In particular \( \ker \pi \subseteq \widehat{L}_0 \), and the restriction of \( \pi \) to
\( \widehat{L}_\alpha \) is a linear isomorphism \( \pi_\alpha : \widehat{L}_\alpha \rightarrow L_\alpha \) for any \( \alpha \in \Delta \).

Proof. The superalgebra \( \widehat{L} \) is an \( \mathfrak{h} \)-module in a natural way. If for \( h \in \mathfrak{h} \), \( m(t) \) denotes the minimal polynomial of \( \text{ad} h \) on \( L \), then \( m(t) \) divides \( \prod_{\alpha \in \Delta \cup \{0\}} (t - \alpha(h)) \), and \( m(\text{ad} h') \widehat{L} \subseteq Z(\widehat{L}) \) for any \( h' \in \pi^{-1}(h) \). Thus the
minimal polynomial of \( \text{ad} h' \) divides \( tm(t) \). This implies \( \widehat{L} = \widehat{L}_0 \oplus (\bigoplus_{\alpha \in \Delta} \widehat{L}_\alpha) \)
where the \( \widehat{L}_\alpha \)'s are as required, and where \( \widehat{L}_0 = \{ x \in \widehat{L} \mid (\text{ad} h')^2 x = 0 \) for all \( h' \in \pi^{-1}(h) \} \). Now \( L' := (\bigoplus_{\alpha \in \Delta} \widehat{L}_\alpha) \oplus (\sum_{\alpha \in \Delta} [\widehat{L}_\alpha, \widehat{L}_{-\alpha}]) \) is a perfect
subsuperalgebra of \( \widehat{L} \) with \( \pi(L') = L \). Since \( L' + \ker \pi = \widehat{L} \) and \( \ker \pi \subseteq Z(\widehat{L}) \),
we have \( \widehat{L} = [\widehat{L}, \widehat{L}] = [L', L'] = L' \), and the lemma follows. \( \square \)
3. Jordan supersystems

Let $\Delta$ be a finite subset of an abelian group $\Gamma$ with $0 \notin \Delta$. Suppose $\mathcal{J} = (J_\alpha, \alpha \in \Delta)$ is a family of $\mathbb{Z}_2$-graded vector spaces with even linear maps $\psi_{\alpha, \beta} : J_\alpha \otimes J_\beta \to J_{\alpha + \beta}$ whenever $\alpha, \beta, \alpha + \beta \in \Delta$, and linear maps $\psi_{\alpha, -\alpha} : J_\alpha \otimes J_{-\alpha} \to \bigoplus_{\beta \in \Delta} \text{End}(J_\beta)$ for all $\alpha \in \Delta$ such that $-\alpha \in \Delta$. For any $\alpha, \beta \in \Delta$ and $x_\alpha \in J_\alpha$, $x_{-\alpha} \in J_{-\alpha}$, $x_\beta \in J_\beta$, $\psi_{\alpha, -\alpha}(x_\alpha \otimes x_{-\alpha})(x_\beta)$ will denote the action on $x_\beta$ of the $\text{End}(J_\beta)$-component of $\psi_{\alpha, -\alpha}(x_\alpha \otimes x_{-\alpha})$.

Then $\mathcal{J}$ is said to be a Jordan supersystem if it contains all $\psi_{\alpha, -\alpha}(x_\alpha \otimes x_{-\alpha})$. $K(\mathcal{J}) = \bigoplus_{\gamma \in \Delta \cup \{0\}} J_\gamma$, where $J_0 = \sum_{\alpha \in \Delta} \psi_{\alpha, -\alpha}(J_\alpha \otimes J_{-\alpha})$, becomes a Lie superalgebra under the natural product

$$[x_\alpha, y_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \notin \Delta \cup \{0\} \\ \psi_{\alpha, \beta}(x_\alpha \otimes y_\beta) & \text{if } \alpha, \beta, \alpha + \beta \in \Delta \cup \{0\} \\ x_\alpha(y_\beta) & \text{if } \alpha = 0, \beta \in \Delta \\ (-1)^{|x_\alpha||y_\beta|} y_\beta(x_\alpha) & \text{if } \alpha \in \Delta, \beta = 0 \\ x_\alpha y_\beta - (-1)^{|x_\alpha||y_\beta|} y_\beta x_\alpha & \text{if } \alpha = \beta = 0. \end{cases}$$

The even and odd parts of $K(\mathcal{J})$ are given by

$$K(\mathcal{J})_0 = \bigoplus_{\gamma \in \Delta \cup \{0\}} (J_\gamma)_0 \quad \text{and} \quad K(\mathcal{J})_1 = \bigoplus_{\gamma \in \Delta \cup \{0\}} (J_\gamma)_1.$$

Equivalently, $\mathcal{J}$ is a Jordan supersystem if it consists of the nonzero homogeneous components of a $\Gamma$-graded Lie superalgebra $L = \bigoplus_{\gamma \in \Delta \cup \{0\}} L_\gamma$ with $[x_\alpha, y_\beta] = \psi_{\alpha, \beta}(x_\alpha y_\beta)$ for $\alpha, \beta, \alpha + \beta \in \Delta$, $x_\alpha \in L_\alpha$, $y_\beta \in L_\beta$; and $[[x_\alpha, x_{-\alpha}], y_\beta] = \psi_{\alpha, -\alpha}(x_\alpha \otimes x_{-\alpha})(y_\beta)$ for $\alpha, -\alpha, \beta \in \Delta$.

A homomorphism of Jordan supersystems $\mathcal{J} = (J_\alpha, \alpha \in \Delta)$ and $\mathcal{J}' = (J'_\alpha, \alpha \in \Delta)$ is a family of $\mathbb{Z}_2$-graded linear mappings $\eta = (\eta_\alpha, \alpha \in \Delta)$, with $\eta_\alpha : J_\alpha \to J'_\alpha$, such that the induced mapping $K(\mathcal{J}) \to K(\mathcal{J}')$ is a Lie superalgebra homomorphism. Applying the same arguments as in [11, Sec. 1.4], we deduce:

1. There exists a universal such Lie superalgebra; that is, there exists a $\Gamma$-graded Lie superalgebra $U_{\mathcal{J}} = \bigoplus_{\gamma \in \Delta \cup \{0\}} (U_{\mathcal{J}})_\gamma$ and a homomorphism $\nu : (J_\alpha, \alpha \in \Delta) \to ((U_{\mathcal{J}})_\alpha, \alpha \in \Delta)$ of Jordan supersystems such that for any $\Gamma$-graded Lie superalgebra $L = \bigoplus_{\gamma \in \Delta \cup \{0\}} L_\gamma$ and any homomorphism of Jordan supersystems $\eta : (J_\alpha, \alpha \in \Delta) \to (L_\alpha, \alpha \in \Delta)$, there is a unique homomorphism of $\Gamma$-graded Lie superalgebras $\theta : U_{\mathcal{J}} \to L$ such that $\theta \nu_\alpha = \eta_\alpha$ for any $\alpha \in \Delta$. Moreover, $\nu_\alpha$ is a linear isomorphism for every $\alpha \in \Delta$, and $(U_{\mathcal{J}})_0 = \sum_{\alpha \in \Delta} [(U_{\mathcal{J}})_\alpha, (U_{\mathcal{J}})_{-\alpha}]$. 


2. Let $L = \bigoplus_{\gamma \in \Delta \cup \{0\}} L_{\gamma}$ and $L' = \bigoplus_{\gamma \in \Delta \cup \{0\}} L'_{\gamma}$ be two perfect $\Gamma$-graded Lie superalgebras. If the Jordan supersystems $(L_{\alpha}, \alpha \in \Delta)$ and $(L'_{\alpha}, \alpha \in \Delta)$ are isomorphic, and if $L_{0} = \sum_{\alpha \in \Delta} [L_{\alpha}, L_{-\alpha}]$ and $L'_{0} = \sum_{\alpha \in \Delta} [L'_{\alpha}, L'_{-\alpha}]$, then $L$ and $L'$ are centrally isogenous. Moreover, if $L = (L_{\alpha}, \alpha \in \Delta)$ is the corresponding Jordan supersystem, then $U_{L}$ is a central cover of $L$ [11 Prop. 1.5].

3. Let $L = \bigoplus_{\gamma \in \Delta \cup \{0\}} L_{\gamma}$ be a $\Gamma$-graded Lie superalgebra, and let $L = (L_{\alpha}, \alpha \in \Delta)$ be the corresponding Jordan supersystem. If there exists an abelian subalgebra $h \subseteq (L_{0})_{0}$ such that $\Delta \subseteq h^{*}$ and $L_{\alpha} = \{ x \in L \mid [h, x] = \alpha(h)x \ \forall h \in h \}$ for any $\alpha \in \Delta \cup \{0\}$, and if $L_{0} = \sum_{\alpha \in \Delta} [L_{\alpha}, L_{-\alpha}]$, then $U_{L}$ is centrally closed [11 Prop. 1.6]. In particular, $U_{L}$ is the universal central cover of $L$.

4. Let $L = \bigoplus_{\gamma \in \Delta \cup \{0\}} L_{\gamma}$ and $L' = \bigoplus_{\gamma \in \Delta \cup \{0\}} L'_{\gamma}$ be two $\Gamma$-graded Lie superalgebras, and assume $(\eta_{\alpha}, \alpha \in \Delta)$ is a family of $\mathbb{Z}_{2}$-graded linear isomorphisms $\eta_{\alpha} : L_{\alpha} \to L'_{\alpha}$ such that for any $\alpha, \beta, \alpha + \beta \in \Delta$, $\eta_{\alpha + \beta}([x_{\alpha}, y_{\beta}]) = \eta_{\alpha}(x_{\alpha}), \eta_{\beta}(y_{\beta})$ for $x_{\alpha} \in L_{\alpha}, y_{\beta} \in L_{\beta}$. If for any $\alpha \in \Delta$,

$$L_{\alpha} = \sum_{\delta, \gamma \in \Delta, \delta \gamma \neq \alpha} [L_{\delta}, L_{\gamma}],$$

then $\eta$ is an isomorphism of Jordan supersystems [11 Prop. 1.8].

As a noteworthy example, let $\Delta = \{ \pm 1 \}(\subseteq \mathbb{Z})$ and let $J$ be a Jordan superalgebra. Then $J = (J_{1}, J_{-1})$, where $J_{1} = J_{-1} = J$, is a Jordan super-system with

$$\psi_{1,-1}(x \otimes y)(z) = (xy)z + x(yz) - (-1)^{|x||y|}y(xz)$$

$$\psi_{1,-1}(x \otimes y)(w) = -(xy)w + x(yw) - (-1)^{|x||y|}y(wx)$$

for any $x, z \in J_{1}$ and $y, w \in J_{-1}$. The associated Lie algebra $K(J)$ is the well-known (centerless) Tits-Kantor-Koecher superalgebra of $J$ (see [11 1.10–1.14] and the references therein), which we denote by TKK($J$).

Also, if $J = (J_{1}, J_{-1})$ is a Jordan pair, then $U_{L}$ coincides with the Lie algebra TKK($J$) (functorial Tits-Kantor-Koecher algebra) considered in [24], since both algebras satisfy the same universal property [24 Thm. 2.7].

4. $A(n, n)$-grarded Lie superalgebras, $n \geq 2$

Let $V = V_{0} \oplus V_{1}$ be a $\mathbb{Z}_{2}$-graded $\mathbb{F}$-vector space with dim $V_{0} = n + 1 = V_{1}$. Then End $V = (\text{End } V)_{0} \oplus (\text{End } V)_{1}$, where $Xv \in V_{+}^{2}$ (subscript mod 2) for $X \in (\text{End } V)_{0}$ and $v \in V_{1}$, and End $V$ is a Lie superalgebra under the supercommutator product $[X, Y] = XY - (-1)^{|X||Y|}YX$. The transformations $X$ having supertrace $\text{str}(X) = \text{tr}_{V_{0}}(X) - \text{tr}_{V_{1}}(X) = 0$ form a subsuperalgebra. Choosing a basis of $V$ that respects the decomposition, we
may identify that subsuperalgebra with the Lie superalgebra $\mathfrak{sl}(n+1, n+1)$. The identity matrix spans the center of $\mathfrak{sl}(n+1, n+1)$, and factoring out the center gives the classical simple Lie superalgebra $\mathfrak{psl}(n+1, n+1)$.

We assume a basis for $V_\emptyset$ consists of vectors numbered $1, \ldots, n+1$, and one for $V_1$ by $\overline{1}, \ldots, \overline{n+1}$, and number the rows and columns of matrices in $\mathfrak{sl}(n+1, n+1)$ accordingly by the indices in $\mathcal{I} = \{1, \ldots, n+1, \overline{1}, \ldots, \overline{n+1}\}$. Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g} = \mathfrak{psl}(n+1, n+1)$ consisting of the diagonal matrices in $\mathfrak{sl}(n+1, n+1)$ modulo the center. Thus, any element in $\mathfrak{h}$ is uniquely the class, modulo the center, of a diagonal matrix 

$$
\hat{a}_n = \begin{bmatrix}
\varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n & 0 & 0 \\
0 & 0 & \cdots & 0 & \varepsilon_{n+1} & 0 \\
0 & 0 & \cdots & 0 & 0 & \varepsilon_{n+2} \\
\varepsilon_{n+1} & \varepsilon_{n+2} & \cdots & \varepsilon_{2n-1} & 0 & 0 \\
0 & 0 & \cdots & 0 & \varepsilon_{2n} & 0 \\
\varepsilon_{n+2} & \varepsilon_{n+3} & \cdots & \varepsilon_{2n+1} & 0 & 0
\end{bmatrix}
$$

modulo the center, of a diagonal matrix $$diag(a_1, \ldots, a_{n+1}, a_{\overline{1}}, \ldots, a_{\overline{n+1}})$$ with $a_1 + \cdots + a_{n+1} = 0 = a_{\overline{1}} + \cdots + a_{\overline{n+1}}$.

Then the root system of $\mathfrak{psl}(n+1, n+1)$ relative to $\mathfrak{h}$ is

$$
\Delta = \{ \varepsilon_i - \varepsilon_j \mid i \neq j, \; i, j \in \mathcal{I} \}, \quad (4.1)
$$

where $\varepsilon_i$ takes the class of a diagonal matrix $diag(a_1, \ldots, a_{n+1}, a_{\overline{1}}, \ldots, a_{\overline{n+1}})$ (with $a_1 + \cdots + a_{n+1} = 0 = a_{\overline{1}} + \cdots + a_{\overline{n+1}}$) to $a_i$. Thus, $\varepsilon_1 + \cdots + \varepsilon_{n+1} = 0 = \varepsilon_{\overline{1}} + \cdots + \varepsilon_{\overline{n+1}}.$

Throughout the remainder of the section it will always be assumed that $n \geq 2$.

Let $\hat{\mathfrak{g}} = \mathfrak{sl}(n+1, n+1)$, which is the universal central extension of $\hat{\mathfrak{g}} = \mathfrak{psl}(n+1, n+1)$, and let $\pi: \hat{\mathfrak{g}} \to \mathfrak{g}$ be the natural projection. The unique subalgebra of $\hat{\mathfrak{g}}_0$ which projects isomorphically onto $\mathfrak{g}_0$ is $[\hat{\mathfrak{g}}_0, \hat{\mathfrak{g}}_0] \cong \mathfrak{sl}_{n+1} \oplus \mathfrak{sl}_{n+1}$. Identify $\mathfrak{h}$ with the Cartan subalgebra of $[\hat{\mathfrak{g}}_0, \hat{\mathfrak{g}}_0]$ formed by its diagonal matrices (which are the diagonal matrices satisfying $a_1 + \cdots + a_{n+1} = 0 = a_{\overline{1}} + \cdots + a_{\overline{n+1}}$ as above). Thus $\mathfrak{h}' = \mathcal{F} \mathfrak{z} \oplus \mathfrak{h}$ is the Cartan subalgebra of $\hat{\mathfrak{g}}$ consisting of the diagonal matrices, where $\mathcal{F}$ is the $2(n+1) \times 2(n+1)$ identity matrix. Extend $\varepsilon_i$ to $\mathfrak{h}'$ by imposing the condition $\varepsilon_i(z) = 1$ for any $i$. Hence on $\mathfrak{h}'$, we have $\varepsilon_1 + \cdots + \varepsilon_{n+1} = \varepsilon_{\overline{1}} + \cdots + \varepsilon_{\overline{n+1}}$, but this expression does not equal 0.

Let $L$ be an $A(n,n)$-graded Lie superalgebra. Then, by definition, either $\hat{\mathfrak{g}}$ or $\hat{\mathfrak{g}}_0$ is the grading subsuperalgebra in $L$. In either event, $L$ is a $\hat{\mathfrak{g}}$-module (this is obvious if $\hat{\mathfrak{g}} \subseteq L$, and otherwise, $L$ is a $\mathfrak{g}$-module, and hence a $\hat{\mathfrak{g}}$-module through $\pi$). Write the action of $\hat{\mathfrak{g}}$ on $L$ by $a.x$ for $a \in \hat{\mathfrak{g}}$ and $x \in L$. Thus, $a.x = [a, x]$ if $\hat{\mathfrak{g}} \subseteq L$ and $a.x = [\pi(a), x]$ otherwise.

The next lemma is fundamental in our work. To avoid complicated expressions, $e_{ij}$ ($i,j \in \mathcal{I}$) will denote the matrix with a 1 in place $(i,j)$ and 0’s elsewhere if we are working in $\hat{\mathfrak{g}}$ or the class of that matrix modulo the center in $\mathfrak{g}$, depending on the context. Thus, for $i \neq j$, $\hat{\mathfrak{g}} e_{i,-j} = \mathcal{F} e_{ij}$ and $\mathfrak{g} e_{i,-j} = \mathcal{F} e_{ij}$ also.

**Lemma 4.2.** If $\hat{\mathfrak{g}}$ is the grading subsuperalgebra of an $A(n,n)$-graded Lie superalgebra $L$, then the central element $z$ acts trivially on $L$. Equivalently, for any $\alpha \in \Delta$,

$$
L_\alpha = \{ x \in L \mid h'.x = \alpha(h')x \; \forall \; h' \in \mathfrak{h}' \}.
$$

(4.3)
We conclude that \( z \in L_{\epsilon_1 - \epsilon_j} \).

However, for the calculation in the first paragraph of the proof, \( \epsilon \leq 1, j \leq n + 1 \) different from \( i \) and \( j \) (which is possible since \( n \geq 2 \)). Then \( \epsilon_i - \epsilon_j = 1 \pm (\epsilon_k - \epsilon_T) \not\in \Delta \) and hence \( [e_{k,k}, e_{k,k}]_n L_{\epsilon_1 - \epsilon_j} = 0 \). But \( [e_{k,k}, e_{k,k}] = e_{k,k} + e_{k,k} \in \mathfrak{h}' \setminus \mathfrak{h} \), with

\[
(e_i - \epsilon_j)((e_{k,k}, e_{k,k})) = 0,
\]

so that \( (4.3) \) holds for \( \alpha = \epsilon_i - \epsilon_j \). The same argument works for \( \alpha = \epsilon_T - \epsilon_T \).

Now without loss of generality, it is enough to check \( (4.3) \) for \( \alpha = \epsilon_1 - \epsilon_T \). If \( n \geq 3 \), one can use that \( \epsilon_1 - \epsilon_T \pm (\epsilon_2 - \epsilon_T) \not\in \Delta \), and proceed as above.

Therefore, for the rest of the proof, \( n \) will be assumed to be 2. Here \( \epsilon_1 - \epsilon_T + \epsilon_2 - \epsilon_T \not\in \Delta \), so \( e_{\epsilon_1 - \epsilon_T} L_{\epsilon_1 - \epsilon_T} = 0 \), while \( \epsilon_1 - \epsilon_T - (\epsilon_2 - \epsilon_T) = \epsilon_1 - \epsilon_2 \in \Delta \). Thus,

\[
(e_{22} + e_{\epsilon_T \epsilon_T})(z, L_{\epsilon_1 - \epsilon_T}) = (e_{\epsilon_T \epsilon_T})(z, L_{\epsilon_1 - \epsilon_T})
\]

by the calculation in the first paragraph of the proof.

Similarly, \( (e_{11} + e_{\epsilon_T \epsilon_T})(z, L_{\epsilon_1 - \epsilon_T}) = 0 \). Hence, since \( z \in L_{\epsilon_1 - \epsilon_T} \), too for any \( z \in L_{\epsilon_1 - \epsilon_T} \),

\[
0 = ((e_{11} - e_{22}) - (e_{\epsilon_T \epsilon_T} - e_{\epsilon_T \epsilon_T}))(z, z) = (\epsilon_1 - \epsilon_T)((e_{11} - e_{22}) - (e_{\epsilon_T \epsilon_T} - e_{\epsilon_T \epsilon_T}))(z, z) = 2 z.x.
\]

We conclude that \( z, L_{\epsilon_1 - \epsilon_T} = 0 \) also, thus proving the lemma.

\[ \square \]

**Lemma 4.4.** With

\[
L(-1) = \bigoplus_{1 \leq i, j \leq n + 1} L_{\epsilon_i - \epsilon_j},
\]

\[
L(0) = L_0 = L_0 \bigoplus \left( \bigoplus_{1 \leq i \neq j \leq n + 1} L_{\epsilon_i - \epsilon_j} \right) \bigoplus \left( \bigoplus_{1 \leq i \neq j \leq n + 1} L_{\epsilon_i - \epsilon_T} \right), \quad \text{and}
\]

\[
L(1) = \bigoplus_{1 \leq i, j \leq n + 1} L_{\epsilon_i - \epsilon_T},
\]

the associated decomposition \( L = L(-1) \oplus L(0) \oplus L(1) \) is a 3-grading of \( L \).

**Proof.** The only problem is to show that \( [L(1), L(1)] = 0 = [L(1), L(1)] \), and this is clear for \( n \geq 3 \). However, for \( n = 2 \), \( \epsilon_i - \epsilon_T + \epsilon_j - \epsilon_T \) is not in \( \Delta \) for any \( i, j \). However, for \( n = 2 \), \( \epsilon_i - \epsilon_T + \epsilon_j - \epsilon_T \) is not in \( \Delta \) and \( \epsilon_i + \epsilon_j \not\in \Delta \) for any \( i, j \). However, for \( n = 2 \), \( \epsilon_i - \epsilon_T + \epsilon_j - \epsilon_T \) is not in \( \Delta \) and \( \epsilon_i + \epsilon_j \not\in \Delta \) for any \( i, j \).

Without loss of generality, it suffices to prove that \( [L_{\epsilon_1 - \epsilon_T}, L_{\epsilon_2 - \epsilon_T}] = 0 \).
Set $\hat{\Delta} := \{ \varepsilon_i - \varepsilon_j \mid i \neq j \in \{1, 2, 3, \overline{3}\} \}$. Then
\[
\hat{L} = \sum_{\alpha \in \Delta} L_\alpha + \sum_{\alpha \in \Delta} [L_\alpha, L_{-\alpha}]
\]
is a subsuperalgebra of $L$ which is $\mathfrak{sl}(3,1)$-graded. To see this, observe that $\mathfrak{sl}(3,1)$ embeds naturally in $\mathfrak{sl}(3,3)$ (using the indices $\{1, 2, 3, \overline{3}\}$), and $\mathfrak{sl}(3,3)$ projects through $\pi$ onto $\mathfrak{psl}(3,3)$. The composition gives an embedding of $\mathfrak{sl}(3,1)$ into $\mathfrak{psl}(3,3)$, because $\mathfrak{sl}(3,1)$ is simple. Hence $\mathfrak{sl}(3,1)$ is embedded in $L$, and Lemma 4.2 guarantees that the root spaces work properly for the natural Cartan subalgebra of $\mathfrak{sl}(3,1)$, which is contained in $\mathfrak{h}'$.

By the results in [Gi Sec. 3] on $A(m,n)$-graded superalgebras for $m \neq n$, in particular by the specific form of the multiplication of these algebras [Gi (3.1)], it follows that $L_{\varepsilon_1 - \varepsilon_3} = e_{3\overline{3}} L_{\varepsilon_1 - \varepsilon_3}$. Then because $\varepsilon_1 - \varepsilon_3 + \varepsilon_2 - \varepsilon_\overline{3} = -2\varepsilon_3 - \varepsilon_\overline{3} \notin \Delta$, we have $[L_{\varepsilon_1 - \varepsilon_3}, L_{\varepsilon_2 - \varepsilon_\overline{3}}] = 0$, and thus
\[
[L_{\varepsilon_1 - \varepsilon_3}, L_{\varepsilon_2 - \varepsilon_\overline{3}}] = [e_{3\overline{3}} L_{\varepsilon_1 - \varepsilon_3}, L_{\varepsilon_2 - \varepsilon_\overline{3}}] = [L_{\varepsilon_1 - \varepsilon_3}, e_{3\overline{3}} L_{\varepsilon_2 - \varepsilon_\overline{3}}].
\]
Therefore, it is enough to prove that $e_{3\overline{3}} L_{\varepsilon_2 - \varepsilon_\overline{3}} = 0$. But now using the indices $\{2, \overline{2}, \overline{3}, 3\}$, we get an embedding of $\mathfrak{sl}(1, 3)$ in $\mathfrak{psl}(3,3)$ as above, which shows that $L_{\varepsilon_2 - \varepsilon_\overline{3}} = e_{3\overline{3}} L_{\varepsilon_2 - \varepsilon_\overline{3}}$. Hence
\[
eq \subseteq [e_{3\overline{3}}, e_{3\overline{3}}], L_{\varepsilon_2 - \varepsilon_\overline{3}} + e_{3\overline{3}}, (e_{3\overline{3}} L_{\varepsilon_2 - \varepsilon_\overline{3}})
\]
\[
= 0,
\]
since $[e_{3\overline{3}}, e_{3\overline{3}}] = 0$ and $e_{3\overline{3}} L_{\varepsilon_2 - \varepsilon_\overline{3}} = 0$, because $\varepsilon_3 + \varepsilon_\overline{3} - \varepsilon_\overline{3} = \varepsilon_3 + 2\varepsilon_\overline{3} \notin \Delta$. 

Now for any fixed $l \in \mathcal{I} = \{1, \ldots, n + 1, \overline{1}, \ldots, \overline{n+1}\}$, there is an embedding of $\mathfrak{sl}(n + 1, n)$ into both $\mathfrak{sl}(n + 1, n + 1)$ and $\mathfrak{psl}(n + 1, n + 1)$ using the rows and columns indexed by the elements in $\mathcal{I} \setminus \{l\}$. Consider
\[
\Delta_{(l)} = \{ \varepsilon_i - \varepsilon_j \mid i \neq j, \ i, j \in \mathcal{I} \setminus \{l\} \}, \quad \text{and}
\]
\[
L_{(l)} = \left( \bigoplus_{\alpha \in \Delta_{(l)}} L_\alpha \right) \oplus \left( \sum_{\alpha \in \Delta_{(l)}} [L_\alpha, L_{-\alpha}] \right).
\]
Then $L_{(l)}$ contains the image of $\mathfrak{sl}(n + 1, n)$ (under the embedding above) and, thanks to Lemma 4.2, $L_{(l)}$ is a $A(n, n - 1)$-graded Lie superalgebra. By [Gi Thm. 3.10], there is a unital associative superalgebra $A_{(l)}$ such that $L_{(l)}$ is centrally isogenous to $[\mathfrak{gl}(n + 1, n) \otimes A_{(l)}, \mathfrak{gl}(n + 1, n) \otimes A_{(l)}]$. In particular, we may assume that
\[
L_{\varepsilon_i - \varepsilon_j} = e_{ij} \otimes A_{(l)}
\]
(a copy of $A_{(l)}$) for $i \neq j \neq l \neq i$, and
\[
[e_{ij} \otimes a, e_{jk} \otimes b] = (-1)^{|a||j|+|k|} e_{ik} \otimes ab \quad (4.5)
\]
Lemma 4.6. The unital associative superalgebra \( A_{(l)} \) is independent of \( l \).

Proof. Assume \( l \neq l' \) and take \( i \neq j \in I \setminus \{l, l'\} \) (which can be done since \( I \) contains at least 6 elements). Then \( L_{\epsilon_i - \epsilon_j} = e_{ij} \otimes A_{(l)} = e_{ij} \otimes A_{(l')} \), so there is a linear \((\mathbb{Z}_2\text{-graded})\) bijection \( \varphi_{l,l'}^{ij} : A_{(l)} \to A_{(l')} \) such that \( \varphi_{l,l'}^{ij}(1) = 1 \).

However, for distinct values \( i, j, j' \in I \setminus \{l, l'\} \), we have

\[
[e_{ij} \otimes x, e_{jj'} \otimes 1] = (-1)^{|x|(|j| + |j'|)} e_{ij'} \otimes x
\]

for any homogeneous \( x \in A_{(l)} \). The same can be said for elements \( x' \in A_{(l')} \). Consequently,

\[
\varphi_{l,l'}^{ij} = \varphi_{l,l'}^{ij'}.
\]

That is, \( \varphi_{l,l'}^{ij} \) does not depend on \( j \); and in the same way, it does not depend on \( i \) either. Therefore, there is a linear \((\mathbb{Z}_2\text{-graded})\) isomorphism \( \varphi_{l,l'} : A_{(l)} \to A_{(l')} \).

Moreover, when \( i, j, k \in I \setminus \{l, l'\} \) are all different, and \( x, y \) are homogeneous elements of \( A_{(l)} \), we have

\[
[e_{ij} \otimes x, e_{jk} \otimes y] = (-1)^{|x|(|j| + |k|)} e_{ik} \otimes xy,
\]

and the same equation holds for \( x', y' \in A_{(l')} \). As a result,

\[
\varphi_{l,l'}(xy) = \varphi_{l,l'}(x) \varphi_{l,l'}(y),
\]

which implies \( \varphi_{l,l'} \) is an isomorphism that allows us to identify \( A_{(l)} \) and \( A_{(l')} \) for any \( l \neq l' \) in \( I \).

Setting \( A = A_{(1)} \) and making these identifications, we have

\[
L_{\epsilon_i - \epsilon_j} = e_{ij} \otimes A
\]

for any \( i \neq j \) in \( I \). Moreover, the product is given by

\[
[e_{ij} \otimes a, e_{jk} \otimes b] = (-1)^{|a|(|j| + |k|)} e_{ik} \otimes ab
\]

for all distinct indices \( i, j, k \in I \) and all homogeneous elements \( a, b \in A \).

It follows from these considerations, that \( \left( L_{\alpha}, \alpha \in \Delta \right) \) (where \( \Delta \) is of type \( A(n, n), (n \geq 2) \) as in \((\mathbf{11})\)) is a Jordan supersystem isomorphic to the Jordan supersystem consisting of the root spaces of the Lie superalgebra \( \mathfrak{sl}_{n+1,n+1}(A) := [\mathfrak{gl}(n + 1, n + 1) \otimes A, \mathfrak{gl}(n + 1, n + 1) \otimes A] \). The results on Jordan supersystems in Section 3 immediately imply our main result in this section:
Theorem 4.7. Let \( n \geq 2 \), and let \( L \) be a Lie superalgebra over a field \( \mathbb{F} \) of characteristic zero graded by the root system \( A(n,n) \). Then there is a unital associative superalgebra \( A \) such that \( L \) is centrally isogenous to

\[
\mathfrak{sl}_{n+1,n+1}(A) = [\mathfrak{gl}(n+1,n+1) \otimes A, \mathfrak{gl}(n+1,n+1) \otimes A].
\]

Conversely, for any unital associative superalgebra \( A \), any Lie superalgebra centrally isogenous to \( \mathfrak{sl}_{n+1,n+1}(A) \) is \( A(n,n) \)-graded.

For the converse, note that by Lemma 2.4, any central extension of an \( A(n,n) \)-graded Lie superalgebra is also \( A(n,n) \)-graded, as are its central quotients.

Remark 4.8. The universal central extension of the Lie superalgebra \( \mathfrak{sl}_{m+1,n+1}(A) \), for \( m + n \geq 3 \), has been determined in [20 Thm. 2]. It is the so-called Steinberg Lie superalgebra \( \mathfrak{sl}_{m+1,n+1}(A) \).

5. \( A(1,1) \)-Graded Lie Superalgebras

In this final section, let \( \mathfrak{g} = \mathfrak{psl}(2,2) \) and let \( \tilde{\mathfrak{g}} \) be any central cover of \( \mathfrak{g} \). The unique subalgebra of \( \tilde{\mathfrak{g}} \) which projects isomorphically onto \( \mathfrak{g} \) is \( \tilde{\mathfrak{g}}_0 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \). Adopting the same notation as in the previous section, we have that the root system here is

\[
\Delta = \{ \pm 2\varepsilon_1, \pm 2\varepsilon_T, \pm \varepsilon_1 \pm \varepsilon_T \}
\]

since \( \varepsilon_1 + \varepsilon_2 = 0 = \varepsilon_T + \varepsilon_T \). Moreover, the root spaces are given by

\[
\begin{align*}
\mathfrak{g}_{2\varepsilon_1} &= \mathfrak{F}e_{12}, & \mathfrak{g}_{-2\varepsilon_1} &= \mathfrak{F}e_{21}, & \mathfrak{g}_{2\varepsilon_T} &= \mathfrak{F}e_{1T}, & \mathfrak{g}_{-2\varepsilon_T} &= \mathfrak{F}e_{2T}, \\
\mathfrak{g}_{\varepsilon_1 + \varepsilon_T} &= \mathfrak{F}e_{12} \oplus \mathfrak{F}e_{T2}, & \mathfrak{g}_{-\varepsilon_1 - \varepsilon_T} &= \mathfrak{F}e_{21} \oplus \mathfrak{F}e_{1T}, \\
\mathfrak{g}_{\varepsilon_1 - \varepsilon_T} &= \mathfrak{F}e_{1T} \oplus \mathfrak{F}e_{2T}, & \mathfrak{g}_{-\varepsilon_1 + \varepsilon_T} &= \mathfrak{F}e_{T1} \oplus \mathfrak{F}e_{2T}.
\end{align*}
\]

Since \( \tilde{\mathfrak{g}}_\alpha \) projects isomorphically onto \( \mathfrak{g}_\alpha \) for any \( \alpha \in \Delta \) (Lemma 2.1), we will denote the unique element in \( \tilde{\mathfrak{g}} \) which projects onto \( e_{ij} \in \mathfrak{g} \) (for \( i \neq j \in \{1,2, \overline{1}, \overline{2}\} \) by \( e_{ij} \) also. Thus

\[
[\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{g}}_0] = \text{span}_\mathbb{F} \{e_{12}, e_{21}, [e_{12}, e_{21}], e_{1T}, e_{2T}, [e_{1T}, e_{2T}]\}
\]

and

\[
\varepsilon_1([e_{12}, e_{21}]) = 1 = \varepsilon_T([e_{1T}, e_{2T}]), \quad \varepsilon_T([e_{12}, e_{21}]) = 0 = \varepsilon_1([e_{1T}, e_{2T}]).
\]

Relative to the system \( \Pi = \{ \alpha = 2\varepsilon_1, \beta = -\varepsilon_1 + \varepsilon_T \} \) of simple roots, \( \varepsilon_1 + \varepsilon_T = \alpha + \beta \) and \( 2\varepsilon_T = \alpha + 2\beta \). Consequently, \( \tilde{\mathfrak{g}} \) can be graded by the ‘height in \( \alpha \)’, so that

\[
\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(-1) \oplus \tilde{\mathfrak{g}}(0) \oplus \tilde{\mathfrak{g}}(1).
\]
with multiplication given by

\[
\bar{g}(-1) = \bar{g}(-2) \oplus \bar{g}(-\varepsilon_1 + \varepsilon_\bar{T}) \oplus \bar{g}(-\varepsilon_\bar{T})
\]

\[
\bar{g}(0) = \bar{g}_{\varepsilon_1 - \varepsilon_\bar{T}} \oplus \bar{g}_0 \oplus \bar{g}_{-\varepsilon_1 + \varepsilon_\bar{T}}
\]

\[
\bar{g}(1) = \bar{g}_{2\varepsilon_1} \oplus \bar{g}_{\varepsilon_1 + \varepsilon_\bar{T}} \oplus \bar{g}_{2\varepsilon_\bar{T}}
\]

Set \( h = [e_{12}, e_{21}] + [e_{1\bar{T}}, e_{2\bar{T}}] \), \( e = e_{12} + e_{1\bar{T}} \), and \( f = e_{21} + e_{2\bar{T}} \). Then \( h, e, f \) span a copy of \( \mathfrak{s}\mathfrak{l}_2 \) inside \( \overline{[\mathfrak{g}_0, \mathfrak{g}_0]} \) (the “diagonal” subalgebra of \( \overline{[\mathfrak{g}_0, \mathfrak{g}_0]} = \mathfrak{s}\mathfrak{l}_2 \oplus \mathfrak{s}\mathfrak{l}_2 \)). The gradation spaces above can be described alternatively by

\[
\mathfrak{g}(i) = \{ x \in \mathfrak{g} \mid [h, x] = 2ix \}, \quad i = 0, \pm 1.
\]

Assume now that \( L \) is an \( A(1,1) \)-graded Lie superalgebra with grading subsuperalgebra \( \mathfrak{g} \), a central cover of \( \mathfrak{psl}(2,2) \). Then \( L \) also has a decomposition:

\[
L = L(-1) \oplus L(0) \oplus L(1),
\]

where

\[
L(-1) = L_{-2\varepsilon_1} \oplus L_{-(\varepsilon_1 + \varepsilon_\bar{T})} \oplus L_{-2\varepsilon_\bar{T}} = \{ x \in L \mid [h, x] = -2x \},
\]

\[
L(0) = L_{\varepsilon_1 - \varepsilon_\bar{T}} \oplus L_0 \oplus L_{-\varepsilon_1 + \varepsilon_\bar{T}} = \{ x \in L \mid [h, x] = 0 \},
\]

\[
L(1) = L_{2\varepsilon_1} \oplus L_{\varepsilon_1 + \varepsilon_\bar{T}} \oplus L_{2\varepsilon_\bar{T}} = \{ x \in L \mid [h, x] = 2x \}.
\]

Observe that

\[
L_{\varepsilon_1 - \varepsilon_\bar{T}} = [e_{12}, e_{21}], L_{\varepsilon_1 - \varepsilon_\bar{T}} \]

\[
= [e_{12}, [e_{21}, L_{\varepsilon_1 - \varepsilon_\bar{T}}]] \quad \text{as} \quad [e_{12}, L_{\varepsilon_1 - \varepsilon_\bar{T}}] \subseteq L_{3\varepsilon_1 - \varepsilon_\bar{T}} = 0
\]

\[
\subseteq [L_{2\varepsilon_1}, L_{-(\varepsilon_1 + \varepsilon_\bar{T})}] \subseteq [L(1), L(-1)],
\]

and also

\[
L_{-\varepsilon_1 + \varepsilon_\bar{T}} \subseteq [L(1), L(-1)],
\]

\[
[L_{\varepsilon_1 - \varepsilon_\bar{T}}, L_{-\varepsilon_1 + \varepsilon_\bar{T}}] \subseteq [L(0), [L(1), L(-1)]] \subseteq [L(1), L(-1)].
\]

Therefore, since \( L_0 = \sum_{\gamma \in \Delta} [L_\gamma, L_{-\gamma}] \), it follows that

\[
L(0) = [L(1), L(-1)],
\]

Consequently, by the Tits-Kantor-Koecher construction for Jordan superalgebras (see, for instance the last paragraph of Section 3, [11] Thm. 1.10 and 1.11) and the references therein, or [16]) \( J = L(1) \) is a Jordan superalgebra with multiplication given by

\[
x \cdot y = \frac{1}{2}[[x, f], y]
\]

and unit element \( e := e_{12} + e_{1\bar{T}} \). Moreover, \( j := \mathfrak{g}(1) \) is a Jordan subsuperalgebra of \( J \) under this product. Let

\[
e_1 := e_{12}, \quad e_2 := e_{1\bar{T}}, \quad x := e_{2\bar{T}}, \quad \text{and} \quad y := e_{2\bar{T}}.
\]

Then \( \{e_1, e_2, x, y\} \) is an homogeneous basis of \( j \) and, relative to the multiplication in \( [51], j \) is (isomorphic to) the Jordan superalgebra \( \mathfrak{M} := \mathfrak{M}_{1,1}(\mathbb{F})^+ = \mathfrak{M}_0 \oplus \mathfrak{M}_1 \), where \( \mathfrak{M}_0 = \{ \left( \begin{array}{cc} a & \theta \\ \theta & d \end{array} \right) \mid a, d \in \mathbb{F} \} \), and \( \mathfrak{M}_1 = \{ \left( \begin{array}{cc} 0 & h \\ c & 0 \end{array} \right) \mid h \in \mathbb{F} \} \).
The product in \( \mathfrak{M} \) is simply the (super)symmetrized matrix multiplication

\[
X \cdot Y = \frac{1}{2} \left( XY + (-1)^{|X||Y|} YX \right).
\]

(5.3)

The isomorphism can be realized explicitly by matching the basis elements of \( j \) with matrix units in \( \mathfrak{M} \) according to

\[
e_1 = (1 0 \ 0 \ 0), \quad e_2 = (0 1 \ 0 \ 0), \quad x = (0 1 \ 0 \ 0), \quad y = (0 0 \ 1 0).
\]

(5.4)

Thus,

\[
e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 \cdot e_2 = 0 \quad \text{ and } \quad x \cdot y = e_1 - e_2 = -y \cdot x, \quad e_1 \cdot x = \frac{1}{2} x = e_2 \cdot x, \quad e_1 \cdot y = \frac{1}{2} y = e_2 \cdot y.
\]

(5.5)

Therefore, by the same arguments as in [11, 1.14], \( L \) is centrally isogenous to the centerless Tits-Kantor-Koecher Lie superalgebra \( \text{TKK}(J) \), and the first part of the next (and last) theorem is established:

**Theorem 5.6.** The \( \Lambda(1,1) \)-graded Lie superalgebras are precisely those Lie superalgebras which are centrally isogenous to a Tits-Kantor-Koecher Lie superalgebra \( \text{TKK}(J) \) of a Jordan superalgebra \( J \) which contains \( \mathfrak{M}_{1,1}(\mathbb{F})^+ \) as a unital subsuperalgebra.

**Proof.** Only the converse remains to be proven. Let \( J \) be a Jordan superalgebra. Set \( j = \mathfrak{M}_{1,1}(\mathbb{F})^+ \), and assume that \( j \) is a unital subsuperalgebra of \( J \). It has to be shown that \( T = \text{TKK}(J) \) is \( \Lambda(1,1) \)-graded. Recall that \( T = T(-1) \oplus T(0) \oplus T(1) \), where \( T(1) = J \) and \( T(-1) = \mathfrak{J} \) is another copy of \( J \). Consider the Jordan supersystem \( \mathfrak{J} = (T(i), i = \pm 1) \). Also, let \( t = \text{TKK}(j) = t(-1) \oplus t(0) \oplus t(1) \), which is isomorphic to \( \text{psl}(2,2) \), and let \( \mathfrak{j} = (t(i), i = \pm 1) \) be the corresponding Jordan supersystem. The inclusion \( j \subset J \) gives a homomorphism of Jordan supersystems so, by Section 3, there is a unique homomorphism of 3-graded Lie superalgebras

\[\nu : U_{\mathfrak{j}} \rightarrow U_{\mathfrak{j}}(-1) \oplus U_{\mathfrak{j}}(0) \oplus U_{\mathfrak{j}}(1) \rightarrow T.\]

Moreover, the universal Lie superalgebra \( U_{\mathfrak{j}} \) is centrally closed and centrally isogenous to \( t \), and \( \ker \nu \subseteq U_{\mathfrak{j}}(0) \). Hence \( [\ker \nu, t(\pm 1)] = 0 \), so \( \ker \nu \subseteq Z(U_{\mathfrak{j}}) \).

Thus \( \nu(U_{\mathfrak{j}}) \cong U_{\mathfrak{j}}/\ker \nu \) is a central cover of \( t \) (\( t \) is centerless).

Therefore \( T \) contains \( \tilde{\mathfrak{g}} = \nu(U_{\mathfrak{j}}) \), which is a central cover of \( \mathfrak{g} \), and \( \tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(-1) \oplus \tilde{\mathfrak{g}}(0) \oplus \tilde{\mathfrak{g}}(1) \) as above, where \( \tilde{\mathfrak{g}}(1) = j \).

For \( a \in J = T(1), \) let \( \overline{a} \) denote the copy of \( a \) in \( \mathfrak{J} = T(-1) \). Then for \( a, b, c \in J \) we have

\[
[[a, \overline{b}], c] = 2((a \cdot b) \cdot c + a \cdot (b \cdot c) - b \cdot (a \cdot c),
\]

(5.7)

while

\[
[[a, \overline{b}], \overline{c}] = 2(- (a \cdot b) \cdot c + a \cdot (b \cdot c) - b \cdot (a \cdot c)).
\]

(5.8)
Take the basis of $j = \mathfrak{M}_{1,1}(\mathbb{F})^+$ formed by the elements in $\mathfrak{M}_{1,1}(\mathbb{F})^+$. Then $e_1$ and $e_2$ are orthogonal idempotents in $J$ with $e_1 + e_2 = 1$, so $J$ has the Peirce decomposition:

$$J = J_0 \oplus J_1 \oplus J_2,$$

where $J_i = \{ v \in J \mid e_1 \cdot v = \frac{1}{2} v \}$.

Now, the Lie superalgebra $t = \text{TKK}(j)$ is isomorphic to $\mathfrak{psl}(2,2)$ via the homomorphism which makes the following assignments:

$$(t_0) = \mathfrak{psl}(2,2)$$

Since $\mathfrak{g}(1) = t(1) = j$ and $\mathfrak{g}(-1) = \mathfrak{t}(-1) = j$, $\mathfrak{g}$ is generated, as a Lie superalgebra, by the elements $e_1, e_2, x, y, \bar{e}_1, \bar{e}_2, \bar{x}, \bar{y}$, and $[\bar{g}_0, \bar{g}_0]$ is the span of $\{e_1, \bar{e}_1, [e_1, e_1], e_2, \bar{e}_2, [e_2, e_2]\}$. That is,

$$[\bar{g}_0, \bar{g}_0] = \text{span}_F \{e_1, \bar{e}_1, [e_1, e_1]\} \oplus \text{span}_F \{e_2, \bar{e}_2, [e_2, e_2]\},$$

the direct sum of two copies of $\mathfrak{sl}_2$, with Cartan subalgebra $\mathfrak{h} = F[e_1, \bar{e}_1] \oplus F[e_2, \bar{e}_2]$. Therefore, it suffices to show that relative to the action of $\mathfrak{h}$, the Lie superalgebra $T$ decomposes as

$$T = T_0 \oplus \left( \bigoplus_{\gamma \in \Delta} T_\gamma \right),$$

where

$$T_0 = \sum_{\gamma \in \Delta} [T_\gamma, T_{-\gamma}].$$

But from (5.7) and (5.8), it follows that:

$$J_2 \subseteq T_{2e_1}, \quad J_0 \subseteq T_{2\bar{e}_1}, \quad J_1 \subseteq T_{e_1 + e_2}, \quad \bar{J}_2 \subseteq T_{-2e_1}, \quad \bar{J}_0 \subseteq T_{-2\bar{e}_1}, \quad \bar{J}_1 \subseteq T_{-e_1 + e_2},$$

Since $T(0) = [T(1), T(-1)] = [J, \bar{J}]$, the containments in (5.12) are in fact equalities and

$$T(0) = [J, \bar{J}] = T_0 \oplus T_{2(e_1 - \bar{e}_1)} \oplus T_{-(e_1 - \bar{e}_1)} \oplus T_{e_1 - \bar{e}_1} \oplus T_{-e_1 + \bar{e}_1}.$$ 

But $T_{2(e_1 - \bar{e}_1)} = [J_2, \bar{J}_0] = 0$ because of the usual properties of the Peirce decomposition in (5.9); namely, $J_2 \cdot J_0 = 0 = (J_2, J, J_0)$ (where $(, , )$ denotes the associator: $(a, b, c) = (a \cdot b) \cdot c - a \cdot (b \cdot c)$). Hence (5.10) is obtained, while (5.11) follows immediately from the condition $T(0) = [T(1), T(-1)]$. □

**Remark 5.13.** The results in Section 3 show that, given a unital Jordan superalgebra $J$ and the associated Jordan supersystem $\mathcal{J} = (J_1, J_{-1})$ (where $J_{\pm 1} = J$), then $U_{\mathcal{J}}$ is the universal central extension of TKK($J$). Therefore, any A(1,1)-graded Lie superalgebra is a central quotient of a Lie superalgebra $U_{\mathcal{J}}$, for $\mathcal{J}$ as above, where $J$ is a Jordan superalgebra which contains $\mathfrak{M}_{1,1}(\mathbb{F})^+$ as a unital subsuperalgebra.
Remark 5.14. It is easy to see that the Jordan superalgebras \( \mathfrak{M}_{n,n}(\mathbb{F})^+ \), \( n \geq 2 \), contain \( \mathfrak{M}_{1,1}(\mathbb{F})^+ \), so their Tits-Kantor-Koecher superalgebras are \( A(1,1) \)-graded.

The simple Jordan superalgebra

\[
J_4(\mathbb{F}) = \{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mid a, b, c \in \mathfrak{M}_4(\mathbb{F}), \ b = -b^t, c = c^t \} \subset \mathfrak{M}_{4,4}^+(\mathbb{F})
\]

(where \( t \) is the transpose) contains \( \mathfrak{M}_{1,1}(\mathbb{F})^+ \) as a unital subsuperalgebra. This containment may be realized explicitly in terms of matrix units by the following assignments (compare the related embedding in [7, Sec. 5]):

\[
e_1 = E_{1,1} + E_{2,2} + E_{5,5} + E_{6,6},
\]

(5.15)

\[
e_2 = E_{3,3} + E_{4,4} + E_{7,7} + E_{8,8},
\]

\[
x = E_{1,7} + E_{2,8} - E_{3,5} - E_{4,6},
\]

\[
y = 2(E_{7,1} + E_{8,2} + E_{5,3} + E_{6,4}).
\]

Thus, these elements satisfy the multiplication relations in (5.5). The Tits-Kantor-Koecher superalgebra of \( J_4(\mathbb{F}) \) is the simple Lie superalgebra \( P(3) \), so \( P(3) \) is \( A(1,1) \)-graded. In fact, much more is true. For an arbitrary unital associative commutative superalgebra \( \Phi \) and an even derivation \( d : \Phi \to \Phi \), the Cheng-Kac Jordan superalgebra \( JCK(\Phi, d) \) contains \( J_4(\mathbb{F}) \), hence also \( \mathfrak{M}_{1,1}(\mathbb{F})^+ \). Therefore, the Cheng-Kac Lie superalgebra \( CK(\Phi, d) \), which is the Tits-Kantor-Koecher superalgebra of \( JCK(\Phi, d) \), (see [15, 17, 21]) is \( A(1,1) \)-graded. As every \( P(3) \)-graded Lie superalgebra is centrally isogenous to some \( CK(\Phi, d) \) by [22], every \( P(3) \)-graded superalgebra is \( A(1,1) \)-graded too.

The simple Jordan superalgebra

\[
J_{0,4}(\mathbb{F}) = \{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mid a, b \in \mathfrak{M}_4(\mathbb{F}) \} \subset \mathfrak{M}_{4,4}^+(\mathbb{F})
\]

also contains \( \mathfrak{M}_{1,1}(\mathbb{F})^+ \) as a unital subsuperalgebra, for example by taking \( e_1 \) and \( e_2 \) as in (5.15) and

\[
x = E_{1,7} + E_{2,8} + E_{5,3} + E_{6,4},
\]

\[
y = 2(E_{7,1} + E_{8,2} + E_{3,5} + E_{4,6}),
\]

so its Tits-Kantor-Koecher Lie superalgebra, which is \( Q(3) \), is \( A(1,1) \)-graded.

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