Category-theoretic recipe for dualities in one-dimensional quantum lattice models

Laurens Lootens, 1,∗ Clement Delcamp, 2 Gerardo Ortiz, 3 and Frank Verstraete 1

1 Department of Physics and Astronomy, Ghent University, Krijgslaan 281, 9000 Gent, Belgium
2 Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Straße 38, 01187 Dresden, Germany
3 Department of Physics, Indiana University, Bloomington, IN 47405, United States of America

We present a systematic approach for generating duality transformations in quantum lattice models. Within our formalism, dualities are completely characterized by equivalent but distinct realizations of a given (possibly non-abelian and non-invertible) symmetry. These different realizations are encoded into fusion categories, and dualities are methodically generated by considering all Morita equivalent categories. The full set of symmetric operators can then be constructed from the categorical data. We construct explicit intertwiners, in the form of matrix product operators, that convert local symmetric operators of one realization into local symmetric operators of its dual. Concurrently, it maps local operators that transform non-trivially into non-local ones. This guarantees that the structure constants of the algebra of all symmetric operators are equal in both dual realizations. Families of dual Hamiltonians, possibly with long range interactions, are then designed by taking linear combinations of the corresponding symmetric operators. We illustrate this approach by establishing matrix product operator intertwiners for well-known dualities such as Kramers-Wannier and Jordan-Wigner, consider theories with two copies of the Ising category symmetry, and present an example with quantum group symmetries. Finally, we comment on generalizations to higher dimensions of this categorical approach to dualities.

SEC. I | Introduction

Since the dawn of mathematics, scholars have been fascinated by the existence of dualities. An early example is the idea of dual polyhedra obtained by interchanging faces and vertices, as described in volume XV of Euclid’s Elements by Isidore of Miletus, whereby the isocahedron is dual to the dodecahedron and the tetrahedron is self-dual [1]. Crucially, dual polyhedra share the same symmetry group. This notion of symmetry, and other abstract generalizations, have remained at the heart of the concept of duality in modern mathematics and physics. Dualities effectively express different ways in which abstract symmetries can establish themselves and their representations can be transmuted into one another.

Dualities play a particularly important role in the field of statistical mechanics and quantum phase transitions, and an essential part of the canon of quantum spin physics consists of constructions such as the Jordan-Wigner transformation [2–6], the Kramers-Wannier duality [7] and generalizations thereof involving a gauging procedure [8–13]. Symmetries, as defined below, are again central stage here, and the corresponding dualities relate theories that implement those symmetries in a different way. In the aforementioned examples, the duality transformation relates local observables that transform trivially under the symmetry operation of one theory to local terms in another theory that transform trivially according to the dual symmetry; an example of such operators would be the local terms of the Hamiltonian. What makes the duality non-trivial is that local operators that do not transform trivially in one theory are mapped to non-local observables in the dual theory, where they also transform non-trivially. Those properties have to hold for any duality transformation.

The symmetries that we consider involve symmetry operators that multiply according to a fusion ring. This includes the usual group-like symmetries, as well as non-invertible symmetries. These symmetry operators are in general non-local in the sense that they are represented as matrix product operators (MPOs) whose multiplication rules are encoded into higher mathematical structures known as fusion categories [14, 15]. Symmetric operators are then defined as operators that commute with these symmetry operators and are said to transform trivially under the symmetry. A well known class of models exhibiting such symmetries are the anyonic chains [16–21]. These can be thought of as generalizations of the Heisenberg model, whereby the spin degrees of freedom are promoted to objects in a fusion category interpreted as topological charges of quasi-particles, and the tensor product of SU(2) representations is replaced by the fusion rules of the category [22]. It is well known that these models satisfy symmetry relations with respect to operators labelled by objects in the input category. These so-called categorical symmetries have received widespread attention in recent years, e.g. [23–32]. In the case that these models are critical, their low-energy physics is described by a conformal field theory, in which these non-local symmetries are identified with the topological defects [33–38]. These quantum spin Hamiltonians can then be understood as the gapless edge theories of a (2+1)d system with topological order [21, 36, 39–42], realizing a lattice version of the well known holographic relation between topological and conformal field theories [43–47].

The main contribution of this paper is to demonstrate that, for the case of one-dimensional quantum lattice systems, there is a systematic approach to constructing du-
alities. In particular, our formulation, which relies on tensor networks [48], explicitly provides MPOs [49, 50] that intertwine local symmetric operators to dual local operators that are symmetric with respect to an equivalent but distinct symmetry. Explicit dual Hamiltonians can then be constructed by taking linear combinations of such symmetric operators. Note that we focus on the case of quantum chains, but a completely equivalent exposition is possible for the case for two-dimensional statistical mechanical models.

In our construction dual theories are given by different choices of module categories over the input fusion category, and these determine the microscopic degrees of freedom. These module categories also appear in the classification of gapped boundaries of (2+1)d topological orders [51, 52], and indeed the interplay between dual quantum models and different choices of gapped boundaries has been investigated in the literature [7, 10–13, 27, 53–55].

More concretely, our recipe for generating duality maps works as follows. Given a symmetry represented as a set of MPOs described by a fusion category, one can construct local symmetric operators that commute with these MPOs. As these local symmetric operators are built from generalized Clebsch-Gordan coefficients, the algebra they generate called the bond algebra [56, 57] is governed by their recoupling theory via generalized 6j-symbols, which give rise to another fusion category we call $\mathcal{D}$. The crux of our approach is that there exist different local operators commuting with different MPO symmetries that are described by the same recoupling theory [15], and therefore satisfy the same algebraic relations. These local symmetric operators define dual models with dual symmetries, while their bond algebras have identical structure constants. Mathematically, these dual local symmetric operators are obtained by different choices of module categories over the fusion category $\mathcal{D}$, and the dual symmetries are said to be Morita equivalent [22]. Crucially, the data of these module categories allows us to explicitly construct MPO intertwiners that map local symmetric operators of one theory to local symmetric operators of its dual.

Note that this construction makes the role of symmetries in dualities very explicit, as the MPO intertwiner only depends on the different choices of module categories; once this MPO is constructed, it will serve as the intertwiner between any two Hamiltonians constructed out of linear combinations of dual bond algebra elements. Note also that the central merit of the tensor network and more specifically of the MPO construction is the fact that the global non-local duality transformations can be implemented in a local way, at the cost of introducing additional virtual vector spaces. This allows us to explicitly construct isometries relating dual Hamiltonians, ensuring their spectra can be related. We also note that the intertwinning MPO completely determines the mapping between dual variables, including in particular the duality of the local order parameter to a string-order parameter which will be associated to intertwining non-symmetric local operators of one theory to its dual one. Finally, note that the procedure of gauging fits into our framework of duality transformations, and our construction recovers the well-known fact that group-like symmetries represented with non-trivial 3-cocycles cannot be gauged as the required module categories do not exist.

An outline of this paper is as follows: first, we give a heuristic exposition of our construction. After reviewing relevant key concepts of category theory, we present a systematic construction of Hamiltonian models with non-trivial dual theories based on the formalism of MPO symmetries, and relate this construction to the bond-algebra approach to dualities. Finally, we present a selection of examples, including the Ising model away $(\mathcal{D} = \mathbb{Z}_2)$ and at $(\mathcal{D} = \text{Ising})$ criticality, the Heisenberg XXZ model with $\mathcal{D} = \text{Ising}^{||} \otimes \text{Ising}$, and quantum IRF-vertex models with $\mathcal{D} = \text{Rep}(\mathfrak{U}_q(\mathfrak{sl}_2))$. In the outlook, we also present first results into extending those ideas to higher dimensions.

### Sec. II Hamiltonian models with categorical symmetries

#### II.A. Heuristics

Before going into technical details, let us motivate our formalism. We consider $(1+1)d$ quantum Hamiltonians—a priori on infinite chains—of the form $H_A = \sum_i h_i$, where for simplicity we restrict $h_i$ to be a two-site interaction. In general, the local terms $h_{A,i}^{\pm}$ are defined in terms of a three-valent tensor $A$ as

$$h_{A,i}^{\pm} = \sum_k A_{i,l}^{\pm} |i',l\rangle \langle i,l|, \quad (1)$$

and from now on we will use the usual graphical notation for describing tensors and their contractions. We are interested in Hamiltonians that fulfill specific symmetry relations. At the level of the tensors $A$, these require the existence of matrix product operators (MPOs) satisfying so-called pulling-through conditions:

$$\begin{array}{c}
A \\
\quad = \\
\quad = \\
\end{array}$$

where red lines depict indices along which MPO tensors get contracted to one another. An elementary example of such tensors $A$ are built from Clebsch-Gordan coefficients of a finite group $G$, which are invariant under the action of any of its generators. The symmetry MPOs are thus labeled by group variables and can be fused together via the group multiplication. More generally, we are interested in MPO symmetries w.r.t. operators that are not necessarily invertible and whose properties are encoded into a so-called fusion category. The tensors $A$ can then...
be thought of as some generalized Clebsch-Gordan coefficients invariant under the action of such operators.

The key insight of this paper is the following: Given a tensor $A$, we can often define another tensor $B$ related to $A$ via the existence of an MPO intertwiner:

$$
\begin{array}{c}
\text{\includegraphics[width=1cm]{intertwiner.png}}
\end{array}
$$

(3)

Crucially, tensors $B$ obtained in this way satisfy symmetry conditions of the form \( (2) \) with respect to MPOs whose properties are encoded into a fusion category that is similar—in a sense that is specified further—to that of the symmetry operators of $A$. The Hamiltonian $H_B$ built out of tensors $B$ is then dual to $H_A$. In particular, the intertwining MPO can be used to construct a unitary transformation between $H_A$ and $H_B$, implying a relation between their spectra.

More concretely, given symmetry MPOs encoded into a given fusion category $D$, the tensors $A$ can be constructed from a piece of data known as the $F$-symbols of $D$. The existence of distinct tensors $B$ satisfying the properties outlined above is then guaranteed if there are distinct so-called module categories over $D$. These module categories $\mathcal{M}$ contain data that can be used to construct the aforementioned generalized Clebsch-Gordan coefficients. Analogously to the way ordinary ordinary Clebsch-Gordan coefficients can be recoupled via Wigner 6j-symbols, the generalized Clebsch-Gordan coefficients can be recoupled using the $F$-symbols of $D$ regardless of the choice of $\mathcal{M}$. The fact that the tensors $A$ and $B$ share the same recoupling theory confirms that they are strongly related to one another. A particular manifestation of this relationship can be observed when considering the algebra generated by all local symmetric operators constructed from these generalized Clebsch-Gordan coefficients called the bond algebra. The distinct bond algebra elements corresponding to different choices of $\mathcal{M}$ all possess different MPO symmetries, determined by the Morita dual of $D$ with respect to $\mathcal{M}$ [15].

The bond algebras for these models are isomorphic, since their structure constants are derived from the same recoupling theory. In [57], it was argued that this implies the existence of an isometry relating these two Hamiltonians, motivating the notion of isomorphic bond algebras as a formalization of duality. Our formulation of these concepts allows us to go beyond in that we are able to explicitly construct this unitary transformation from the MPO intertwiners. It turns out that the precise nature of this transformation requires a thorough understanding of the different symmetry sectors of the models, which is provided in a natural way in the categorical formulation.

II.B. Technical preliminaries

As alluded to above, our construction requires two pieces of category theoretical data, namely a spherical fusion category and a module category over it (see [22] for precise definitions). Let $D$ be a (spherical) fusion category. Succinctly, it is a collection of simple objects interpreted as topological charges of various quasi-particles that can fuse with one another. Throughout this manuscript, isomorphism classes of simple objects in $D$ are notated via $\alpha, \beta, \ldots \in \mathcal{D}$ (Greek lowercase letter). The fusion of objects is defined by the so-called monoidal structure $(\otimes, 1, F)$ of $D$, where $\otimes$ is an associative product rule, $1$ is a distinguished object interpreted as the trivial charge and $F$ is an isomorphism $F : \alpha \otimes (\beta \otimes \gamma) \sim (\alpha \otimes \beta) \otimes \gamma$ referred to as the monoidal associator. Denoting by $\mathcal{H}_{\alpha, \beta} := \text{Hom}_D(\alpha \otimes \beta, \gamma) \ni |\alpha \beta \gamma, i\rangle$ the vector space of maps from $\alpha \otimes \beta$ to $\gamma$, we have $\alpha \otimes \beta \simeq \bigoplus \gamma N_{\alpha \beta}^{\gamma \gamma}$, where $N_{\alpha \beta}^{\gamma \gamma} := \dim \mathcal{H}_{\alpha, \beta}$ are the fusion multiplicities. The monoidal associator boils down to a collection of isomorphisms

$$
F_{\delta}^{\gamma \beta} : \bigoplus_{\nu} \mathcal{H}_{\nu, \beta} \otimes \mathcal{H}_{\delta \nu, \gamma} \sim \bigoplus_{\mu} \mathcal{H}_{\alpha, \mu} \otimes \mathcal{H}_{\delta \mu, \gamma}
$$

(4)

that can be depicted as

$$
\begin{array}{c}
\text{\includegraphics[width=7cm]{associator.png}}
\end{array}
$$

(5)

where $i, j, k, l$ label basis vectors in $\mathcal{H}_{\alpha, \beta}$, $\mathcal{H}_{\beta, \gamma}$, $\mathcal{H}_{\alpha, \nu}$ and $\mathcal{H}_{\delta \nu, \gamma}$, respectively.

Let us now consider a (finite semi-simple C-linear) right module category $\mathcal{M}$ over $D$, which is broadly speaking a category with a right action over $D$. Henceforth, isomorphism classes of simple objects in $\mathcal{M}$ are denoted by $A, B, \ldots \in \mathcal{M}$ (Roman capital letters). Simple objects of $\mathcal{M}$ act on the topological charges of $D$ via the module structure $(\a, F^\mathcal{M})$, where $\a := \mathcal{M} \times D \to \mathcal{M}$ is the action and $\a F$ is an isomorphism $\a F : A \a (\alpha \otimes \beta) \sim (A \a \alpha) \a \beta$ referred to as the module associator. For instance, every fusion category $D$ defines a module category over itself, which we refer to as the regular module category. Introducing the notation $\mathcal{V}_{A, \alpha} := \text{Hom}_\mathcal{M}(A \a \alpha, B) \ni |A \alpha B, i\rangle$, the module associator $\a F$ boils down to a collection of isomorphisms

$$
\mathcal{V}_{A, \alpha} \otimes \mathcal{V}_{B, \beta} \sim \bigoplus_{\gamma} \mathcal{V}_{A, \alpha} \otimes \mathcal{V}_{B, \gamma}
$$

(6)

that can be depicted as

$$
\begin{array}{c}
\text{\includegraphics[width=7cm]{module_associator.png}}
\end{array}
$$

(7)
where \( i, j, k, l \) label basis vectors in \( \mathcal{V}_{A,\alpha}^\gamma, \mathcal{H}_{A,\beta}^\gamma, \mathcal{V}_{B,\gamma}^\beta \) and \( \mathcal{V}_{C,\delta}^\beta \), respectively. Notice that we use a different colour (purple) for strands labelled by simple objects in the module category, and we shall refer to those as ‘module strands’. The matrix entries of these isomorphisms, which are referred to as \( \mathcal{F}-\text{symbols} \), can be graphically represented as:

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[scale=0.4]
\node (A) at (0,0) [circle,draw,inner sep=0.2cm] {$A$};
\node (B) at (2,0) [circle,draw,inner sep=0.2cm] {$B$};
\node (C) at (2,1) [circle,draw,inner sep=0.2cm] {$C$};
\node (D) at (4,0) [circle,draw,inner sep=0.2cm] {$D$};
\node (E) at (4,1) [circle,draw,inner sep=0.2cm] {$E$};
\end{tikzpicture}
\end{array}
\end{align*}
\]

(8)

where we also included the matrix entries associated with the inverse \( \mathcal{F} \) of the module associator. Note that we kept the module strands unoriented as the corresponding labels will be summed over in practice. By convention, \( \mathcal{F}-\text{symbols} \) for which the fusion rules are not everywhere satisfied vanish. Crucially, the module associator \( \mathcal{F} \) must satisfy a consistency condition known as the pentagon axiom that involves the monoidal associator \( \mathcal{F} \) and which ensures that the equation

\[
\sum_q \sum_{\mu, i, j, p} (\mathcal{F}_{\delta}^{\alpha\gamma})_{\nu, jk, \mu, il} (\mathcal{F}_{\beta}^{\alpha\gamma})_{\gamma, il} = \sum_q \sum_{\mu, i, j, p} (\mathcal{F}_{\delta}^{\alpha\gamma})_{\nu, jk, \mu, il} (\mathcal{F}_{\beta}^{\alpha\gamma})_{\gamma, il},
\]

(9)

holds for any choice of simple objects and basis vectors. Henceforth, we omit to draw gray patches associated with basis vectors that are being contracted (e.g. \( p \) and \( q \) in the previous equation).

Interpreting the diagrams in eq. (8) as the non-vanishing components of four-valent tensors, it follows from eq. (9) that these tensors can be used to define a tensor network representation [14, 58–61] of the ground state subspace of the string-net with input data \( \mathcal{D} \) [62–65]. Crucially, these tensors exhibit symmetry conditions with respect to non-trivial MPOs defined by tensors whose non-vanishing components are of the form

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[scale=0.4]
\node (A) at (0,0) [circle,draw,inner sep=0.2cm] {$A$};
\node (B) at (2,0) [circle,draw,inner sep=0.2cm] {$B$};
\node (C) at (0,1) [circle,draw,inner sep=0.2cm] {$C$};
\node (D) at (2,1) [circle,draw,inner sep=0.2cm] {$D$};
\end{tikzpicture}
\end{array}
\end{align*}
\]

(10)

The symmetry conditions then ensure that these operators can be freely deformed throughout the tensor network away from their endpoints according to the pulling-through condition

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[scale=0.4]
\node (A) at (0,0) [circle,draw,inner sep=0.2cm] {$A$};
\node (B) at (2,0) [circle,draw,inner sep=0.2cm] {$B$};
\node (C) at (2,1) [circle,draw,inner sep=0.2cm] {$C$};
\node (D) at (4,0) [circle,draw,inner sep=0.2cm] {$D$};
\node (E) at (4,1) [circle,draw,inner sep=0.2cm] {$E$};
\end{tikzpicture}
\end{array}
\end{align*}
\]

(11)

These symmetry operators, whose properties are encoded into another fusion category \( \mathcal{C} \equiv \mathcal{D}_M^* \), known as the Morita dual of \( \mathcal{D} \) with respect to \( \mathcal{M} \) [22], can then be used to characterize degenerate ground states and create anyonic excitations of the topological string-net model. In this context, different choices of module categories \( \mathcal{M} \) yields different tensor network representations of the same topological model [15]. Interestingly, given a string-net model, it is possible to define distinct tensor network representations across different regions of the underlying lattice via the introduction of intertwining MPOs. Crucially, these intertwining operators can be fused to the hereinafter symmetry operators associated with either representation in an associative way. Analogously to the symmetry operators, these can be freely moved through the lattice ensuring that the tensor network representations are locally indistinguishable.

II.C. Categorically symmetric local operators

Before addressing our construction per se, let us consider the following situation: Let \( \mathcal{D} \) be a fusion category and \( \mathcal{M} \) a module category over it. Invoking the graphical calculus sketched above, every tree-like diagram of the form

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[scale=0.4]
\node (A) at (0,0) [circle,draw,inner sep=0.2cm] {$A$};
\node (B) at (2,0) [circle,draw,inner sep=0.2cm] {$\tilde{C}$};
\node (C) at (4,0) [circle,draw,inner sep=0.2cm] {$\tilde{B}$};
\end{tikzpicture}
\end{array}
\end{align*}
\]

(12)

labelled by simple objects in \( \mathcal{I}_D \) and \( \mathcal{I}_M \) such that the vector spaces \( \mathcal{V}_{A,\tilde{\alpha}}^\tilde{C} \) and \( \mathcal{V}_{\tilde{B},\tilde{\beta}}^\tilde{C} \) are non-trivial can be interpreted as a state in a Hilbert space according to

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[scale=0.4]
\node (A) at (0,0) [circle,draw,inner sep=0.2cm] {$A$};
\node (B) at (2,0) [circle,draw,inner sep=0.2cm] {$\tilde{C}$};
\end{tikzpicture}
\end{array}
\end{align*}
\]

(13)

Diagrams of the form

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}[scale=0.4]
\node (A) at (0,0) [circle,draw,inner sep=0.2cm] {$A$};
\node (B) at (2,0) [circle,draw,inner sep=0.2cm] {$\tilde{C}$};
\end{tikzpicture}
\end{array}
\end{align*}
\]

(14)
This is a generalization of the usual anyonic chain construction [16-21]. Note that the definition of these operators only depends on \( \mathcal{D} \), whereas the Hilbert space is specified by a choice of \( \mathcal{M} \), suggesting that distinct choices of \( \mathcal{D} \)-module categories should yield dual models.

Let us now formalize this construction. Given a spherical fusion category \( \mathcal{D} \) and a \( \mathcal{D} \)-module category \( \mathcal{M} \), we are interested in local operators acting on (total) Hilbert spaces of the form

\[
\mathcal{H} = \bigoplus \bigoplus \bigotimes V_{i+\frac{1}{2}} \quad \equiv \bigoplus \bigoplus \bigotimes A_{i-2} A_{i-1} A_i A_{i+1}
\]

with \( V_{i+\frac{1}{2}} := \text{Hom}_\mathcal{M}(A_i \otimes A_{i+\frac{1}{2}}, A_{i+1}) \). Throughout the manuscript, we shall implicitly work with infinite chains, unless otherwise stated. Importantly, the Hilbert space (16) is typically not a tensor product of local Hilbert spaces. We choose the convention that unlabelled module strands denote the morphism

\[
\equiv \sum_{A \in \mathcal{I}_\mathcal{M}} \sum_i A_i A \in \text{End}_\mathcal{M}(A \bigoplus A)
\]

where the second sum is over basis vectors in the endomorphism spaces \( \text{End}_\mathcal{M}(A) \). As suggested by the definition, we shall also consider (fermionic) module categories whose simple objects may have non-trivial endomorphism algebras, but from now on we shall take all endomorphism spaces to be isomorphic to \( \mathbb{C} \) unless otherwise stated. In the same spirit, unlabelled gray patches as depicted below are provided by

\[
A_\alpha B_\beta \equiv \sum_i A_\alpha |A_i B_\beta\rangle,
\]

where \( |A_i B_\beta\rangle \in V_{A \alpha} \) and \( \langle A_\alpha B_\beta | \in (V_{A \alpha}^B)^* \) are basis vectors, for any \( i = 1, \ldots, \dim \mathcal{C} \). Given these nomenclatures, we consider local operators \( b_{a;i}^M \) of the form

\[
b_{a;i}^M \equiv \sum_{\alpha, \beta, \gamma, j} b_{a}(|\alpha, \beta, \gamma, j\rangle \langle j, \gamma, \beta, \alpha|)
\]

where \( b_{a}(|\alpha, \beta, \gamma, j\rangle \langle j, \gamma, \beta, \alpha|) \in \mathbb{C} \). In virtue of eqs. (8,17,18), the operator on the r.h.s. is such that

\[
\left( (A \alpha C, i | \otimes (C \beta B, i) \right) b_{a;i}^M (| C \beta B, i \rangle \otimes | A \alpha C, i \rangle)
\]

(20)

where the contribution of the \( \mathcal{F} \)-symbols

\[
\sum_{k} (\mathcal{F}_{B}^{A \alpha B \delta} \gamma_{j k} (\mathcal{F}_{B}^{A \alpha B \delta} \gamma_{j k})_{C,il} = (21)
\]

matches that in eq. (15).

We commented earlier that the tensors whose non-vanishing components evaluate to the \( \mathcal{F} \)-symbols satisfy symmetry conditions translating into pulling-through conditions of the form depicted in eq. (11). It follows from the definition of \( b_{a;i}^M \) in terms of these tensors that they commute with the corresponding symmetry MPOs [16, 17, 21]. These non-local operators and their properties being encoded into (fusion) categories—in contrast to mere groups for instance—this justifies why we refer to the \( b_{a;i}^M \) defined in this section as categorically symmetric operators. We point out that we restrict ourselves to two-site operators purely for simplicity, and that operators of any size can be constructed in this way.

II.D. Bond algebras and duality

Given an input category \( \mathcal{D} \) and the symmetric local operators \( b_{a;i}^M \), we would like to argue that the different representations of the \( b_{a;i}^M \) associated with any choice of \( \mathcal{D} \)-module category \( \mathcal{M} \) provide a way of constructing dual
models. The concept of quantum duality was formalized in [57] introducing the notion of bond algebras. This formulation turns out to be particularly suited to our construction.

Consider the set of all symmetric local operators $\{b_{a,i}^{\mathcal{M}}\}_{a,i}$ as defined previously and let us refer to them as bonds. These bonds define an algebra $\mathcal{A}\{b_{a,i}^{\mathcal{M}}\}$ known as the bond algebra, generated by taking all possible finite products of all possible bonds, as well as the identity operator:

$$\{id, b_{a,i}^{\mathcal{M}}, b_{a,i}^{\mathcal{M}}b_{b,j}^{\mathcal{M}}, b_{a,i}^{\mathcal{M}}b_{b,j}^{\mathcal{M}}b_{c,k}^{\mathcal{M}}, \ldots\}. \quad (22)$$

In general, elements of the bond algebra as defined above are not all linearly independent but we can find a basis $\{O^\mathcal{M}_x\}$ so that bonds and products of bonds can be decomposed into it. By definition, these basis elements satisfy operator product expansions

$$O^\mathcal{M}_x O^\mathcal{M}_y = \sum_z f^\mathcal{M}_{xy} O^\mathcal{M}_z, \quad (23)$$

where $f^\mathcal{M}_{xy}$ are the structure constants of the bond algebra so that bond operators with the same structure constants are isomorphic.

Given our definition of the bonds in eq. (19), products of bonds are computed by invoking the recoupling theory encoded into eq. (9) of tensors (8) thought as some generalized Clebsch-Gordan coefficients. For a choice of basis of the bond algebra, repeated use of eq. (9) can thus be used to explicitly compute the structure constants. Crucially, this recoupling theory is invariant under a change of $\mathcal{D}$-module category $\mathcal{M}$. Indeed, it is manifest from eq. (9) that recoupling does not involve the objects in $\mathcal{M}$ and only depends on the monoidal structure of $\mathcal{D}$ via its $F$-symbols. As such the structure constants of the bond algebras associated with our models only depend on a choice of $\mathcal{D}$:

$$f^\mathcal{M}_{xy} = f^\mathcal{M}_{xy}(F). \quad (24)$$

Consequently, the bonds $\{b_{a,i}^{\mathcal{M}}\}$ generate isomorphic bond algebras for any choice of $\mathcal{D}$-module category $\mathcal{M}$. It follows from the results in [57] that categorically symmetric local operators that only differ by the choice of $\mathcal{D}$-module category are dual to one another, formalizing the intuition that dualities are maps between local operators preserving their algebraic relations.

In our formalism, the existence of such an isomorphism between the two bond algebras is ensured by the fact that we have access to MPO intertwiners mapping bonds associated with distinct module categories onto one another. In the scenario where one of the modules is taken to be the regular module category, these intertwining operators admit particularly simple expressions in terms of tensors whose non-vanishing components evaluate to the $F$-symbols:

$$\langle A, C \mid f_{ij}^{\mathcal{M}} \rangle = (\mathcal{F}_{BC}^{\alpha \gamma} \delta_{ij})_{A,ik}. \quad (25)$$

Pulling such an intertwiner operator through bonds associated with the regular module category yields bonds associated with $\mathcal{M}$ according to

$$\mathcal{B}_i = \sum_a J_a b_{a,i}^{\mathcal{M}}. \quad (27)$$

These operators implement the bond algebra isomorphism and therefore the duality at the level of the local tensors, where the non-locality is captured by the fact that the virtual bond dimension of this operator is nontrivial. Intertwiners between bond algebras obtained from two generic module categories can be obtained from the ones above.

To appreciate how the bond-algebraic formulation of duality works, let us consider two Hamiltonians $\mathcal{H}_A = \sum_i \mathcal{B}_i$ whose local terms are constructed by taking some linear combination of the bonds

$$\mathcal{H}_A = \bigoplus_i \mathcal{H}_{A,i} \quad \text{and} \quad \mathcal{H}_B = \bigoplus_i \mathcal{H}_{B,i}. \quad (28)$$

This defines two dual Hamiltonians $\mathcal{H}_A$ and $\mathcal{H}_B$, acting on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, where the Hamiltonian $\mathcal{H}_A$ is constructed by taking the regular $\mathcal{D}$-module category and $\mathcal{H}_B$ is built from an arbitrary $\mathcal{D}$-module category $\mathcal{M}$. The MPO symmetries of these models are then given by the Morita duals $\mathcal{D}_B \cong \mathcal{D}$ and $\mathcal{D}_M \cong \mathcal{C}$, respectively, and the Hamiltonians are transformed into one another by action of the MPO intertwiner. To understand the action of the duality map at the level of the Hilbert spaces, we consider the models on finite size rings. The presence of MPO symmetries indicates that the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ can be decomposed into direct sums of $n$ sectors:

$$\mathcal{H}_A = \bigoplus_{i} \mathcal{H}_{A,i} \quad \text{and} \quad \mathcal{H}_B = \bigoplus_{i} \mathcal{H}_{B,i}, \quad (29)$$

where $i$ roughly labels all possible charges under the MPO symmetry, as well as all symmetry twisted boundary conditions. Consequently, the Hamiltonians are block diagonal and decompose as

$$\mathcal{H}_A = \bigoplus_{i} \mathcal{H}_{A,i} \quad \text{and} \quad \mathcal{H}_B = \bigoplus_{i} \mathcal{H}_{B,i}. \quad (29)$$

The Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ need not be the same dimension, as the fusion categories describing the MPO symmetries $\mathcal{D}$ and $\mathcal{C}$ are typically not (monoidally) equivalent, but the number of sectors is the same for both. Mathematically, this is guaranteed by the fact that the
fusion categories $\mathcal{D}$ and $\mathcal{C}$ are Morita equivalent; the sectors are given by the monoidal centers of these fusion categories, which are equivalent for Morita equivalent fusion categories [22]. At the level of the MPO symmetries, the monoidal center can be constructed from the tube algebra [14, 66], the central idempotents of which correspond to the different sectors in the model. The dimension of these central idempotents is in general different for the two models, which is reflected in the difference in Hilbert space dimension.

The fact that these models have isomorphic bond algebras implies the existence of a set of unitary transformations:

$$
\mathcal{U}_i: \mathcal{H}_{A,i} \times \mathcal{H}_{B,i}^\text{aux} \to \mathcal{H}_{B,i} \times \mathcal{H}_{B,i}^\text{aux},
$$

s.t.

$$
\mathcal{U}_i(\mathcal{H}_{A,i} \otimes \mathcal{1}_{A,i})\mathcal{U}_i^\dagger = \mathcal{H}_{B,i} \otimes \mathcal{1}_{B,i},
$$

where the auxiliary Hilbert spaces $\mathcal{H}_{B,i}^\text{aux}$ are chosen to account for the potential mismatch in Hilbert space dimension between $\mathcal{H}_{A,i}$ and $\mathcal{H}_{B,i}$; a prototypical instance where this occurs is in dualities obtained by reducing or enlarging a gauge symmetry. The existence of such unitary transformations has been discussed in [57] and implies that up to degeneracies, the Hamiltonians $\mathcal{H}_A$ and $\mathcal{H}_B$ have the same spectrum, but its explicit construction for generic models has not been obtained. In our formalism however, these unitary transformations can be explicitly constructed from the MPO intertwiners together with the knowledge of their interaction with the MPO symmetries of the two models. A detailed description requires an analysis of the sectors of these two models in terms of the idempotents of tube algebras and generalizations thereof involving MPO intertwiners.

Finally, the local operators in these theories admit a similar characterization in terms of tube algebras, and are able to change the sector by acting on a state in a process that is equivalent to the fusion of anyons. A detailed exposition of these aspects will be presented elsewhere.

**Sec. III | Examples**

In this section we illustrate the previous construction for various choices of input category theoretical data and bonds building up a Hamiltonian.

### III.A. Transverse field Ising model

Let $\mathcal{D} = \text{Vec}_{\mathbb{Z}_2}$ be the category of $\mathbb{Z}_2$-graded vector spaces, where we write the simple objects of $\text{Vec}_{\mathbb{Z}_2}$ as $\{1,m\}$. The non-trivial fusion rules read $\mathbb{1} \otimes \mathbb{1} \simeq \mathbb{1} \simeq m \otimes m$ and $\mathbb{1} \otimes m \simeq m \simeq m \otimes \mathbb{1}$. We are interested in the model

$$
\mathcal{H} = -J \sum_i \mathbb{E}_{1,i}^M - g \sum_i \mathbb{E}_{2,i}^M.
$$

[31]

where $\gamma \in \mathcal{I}_\mathcal{D}$ is uniquely specified by the fusion rules. For this example, the module associator evaluates to the identity morphism for any choice of module category.

- Let $\mathcal{M} = \text{Vec}_{\mathbb{Z}_2}$ be the regular $\text{Vec}_{\mathbb{Z}_2}$-module category. As per eq. (17), we have $\mathbb{1} \otimes \mathbb{1} \simeq \mathbb{1} \otimes m \simeq m \otimes \mathbb{1}$. Imposing hom-spaces to be non-trivial in the definition eq. (16) of the total Hilbert space $\mathcal{H}$ constrains objects $\{\alpha\}$ in $\mathcal{I}_\mathcal{D}$ to be determined by a choice of objects $\{A\}$ in $\mathcal{I}_\mathcal{M}$ via the fusion rules. Since hom-spaces in $\mathcal{M}$ are all one-dimensional, it follows that $\mathcal{H}$ is isomorphic to $\bigotimes_i (\mathbb{C} \oplus \mathbb{C})$, where $\mathbb{C} \oplus \mathbb{C} \simeq \mathbb{C}[-1] \simeq \mathbb{C}^2$, such that the physical spins are located in the ‘middles’ of the module strands. The operator $\mathbb{E}_{1,1,i}^M$ acts on the strand $i$ as $|1/m\rangle \to |m/1\rangle$, whereas the operator $\mathbb{E}_{2,1,i}^M$ projects out states whose strands $i-1$ and $i+1$ have distinct labelling objects, and acts as the identity operator otherwise. It follows that in the Pauli $Z$ basis, the Hamiltonian (31) reads

$$
\mathcal{H} = -J \sum_i \left( Z_{i-\frac{1}{2}} Z_{i+\frac{1}{2}} + Z_{i-\frac{1}{2}} Z_{i+\frac{1}{2}} \right),
$$

which we recognize as the transverse field Ising model [67].

- Let us now consider the category $\mathcal{M} = \text{Vec}$, which is a $\text{Vec}_{\mathbb{Z}_2}$-module category via the forgetful functor $\text{Vec}_{\mathbb{Z}_2} \to \text{Vec}$. We notate the unique simple object in $\text{Vec}$ via $\mathbb{1} \simeq \mathbb{C}$ such that $1 \alpha = \alpha$ for any $\alpha \in \mathcal{I}_\mathcal{D}$. According to eq. (16) the total Hilbert space boils down to

$$
\mathcal{H} = \bigotimes_i \text{Hom}_{\mathcal{M}}(\mathbb{1} \otimes \mathbb{1}, \mathbb{1}) \simeq \bigotimes_i \mathbb{C}^2.
$$

It follows immediately from the definitions of $\mathbb{E}_{1,1,i}$ and $\mathbb{E}_{2,1,i}$ that in the Pauli $Z$ basis the Hamiltonian (31) now reads

$$
\mathcal{H} = -J \sum_i \left( X_{i-\frac{1}{2}} X_{i+\frac{1}{2}} + g Z_{i-\frac{1}{2}} \right),
$$

which we recognize as the Kramers-Wannier dual of the Hamiltonian given in eq. (33).
Denoting the model associated with the regular Vec$_{Z_2}$-module category as $\mathcal{H}_A$ and the other one by $\mathcal{H}_B$, their duality implies the existence of a unitary transformation mapping one to the other, as discussed above. On closed boundary conditions, these Hamiltonians are block diagonal in the four symmetry sectors, corresponding to even/odd charge under the global $Z_2$ symmetry and periodic/antiperiodic boundary conditions. These sectors are all of the same dimension and as such no auxiliary Hilbert spaces are required to construct a unitary transformation relating the individual blocks of the Hamiltonians. Interestingly, this unitary transformation, as provided by the MPO intertwiners, interchanges the even (odd), periodic (antiperiodic) sector of $\mathcal{H}_A$ with the periodic (antiperiodic), even (odd) sector of $\mathcal{H}_B$, illustrating the subtle interplay of duality transformations with the symmetry properties of the Hamiltonians.

- In order to investigate another duality of the Ising model, it is convenient to think of the input category $\mathcal{D}$ as the category $s\text{Vec}$ of super vector spaces, which is equivalent, as a fusion category, to Vec$_{Z_2}$. We denote the simple objects of $s\text{Vec}$ as $\{1, \psi\}$. Let $\mathcal{M} = s\text{Vec}/(\psi \simeq 1)$ be the (fermionic) $s\text{Vec}$-module category whose unique simple object $1$ satisfies $1 \triangleleft 1 \simeq 1$ and $1 \triangleleft \psi \simeq C^{01} \cdot 1$, where $C^{01}$ is the purely odd one-dimensional super vector space. It follows that the local Hilbert spaces $V_{i+\frac{1}{2}}$ are now given by

$$V_{i+\frac{1}{2}} = \text{Hom}_\mathcal{M}(1 \triangleleft 1, \mathbb{I}) \oplus \text{Hom}_\mathcal{M}(1 \triangleleft \psi, \mathbb{I})$$

$$\simeq C^{110} \oplus C^{01} = C^{111},$$

(36)

where $C^{111}$ is super vector space with one even and odd basis vector. We identify the basis vector in $C^{110}$ with the empty state $|\varnothing\rangle$ and the basis vector in the odd vector space $C^{01}$ with $c_i |\varnothing\rangle$ where $c_i$ is a fermionic operator satisfying $\{c_i, c^\dagger_j\} = \delta_{ij}$ and $\{c_i, c_j\} = 0$. Replacing $m$ by $\psi$ in eq. (32), we notice that the first two terms in the definition of $\mathcal{D}_{1,1}$ act on states

$$(c_{i-\frac{1}{2}}^\dagger)^{n_{1-\frac{1}{2}}} (c_{i+\frac{1}{2}}^\dagger)^{n_{1+\frac{1}{2}}} |\varnothing\rangle \in C^{111} \otimes C^{111}$$

(37)

via $c_{i-\frac{1}{2}}^\dagger c_{i+\frac{1}{2}}^\dagger$ and $c_{i-\frac{1}{2}}^\dagger c_{i+\frac{1}{2}}^\dagger$, respectively, and similarly for the last two terms, whereas the operator $b_{2,1}$ acts as

$$\frac{1}{2} \left( (-1)^{n_{1-\frac{1}{2}}} + (-1)^{n_{1+\frac{1}{2}}} \right).$$

It follows that the Hamiltonian (31) now reads

$$H = -J \sum_i \left( c_{i-\frac{1}{2}}^\dagger c_{i+\frac{1}{2}} + c_{i-\frac{1}{2}}^\dagger c_{i+\frac{1}{2}} + \text{h.c.} - g(2c_{i+\frac{1}{2}}^\dagger c_{i+\frac{1}{2}} - 1) \right),$$

(38)

which we recognize as the Jordan-Wigner dual of the Hamiltonian given in eq. (35).

There exists a fairly straightforward generalization of this class of examples obtained by replacing the spherical fusion category $\mathcal{D}$ by the category Vec$_G$ of $G$-graded vector spaces, with $G$ an arbitrary finite group. This is the input data of topological gauge theories known as (untwisted) Dijkgraaf-Witten theories [68], whose lattice Hamiltonian incarnations were studied by Kitaev in [69]. Indecomposable module categories over Vec$_G$ are well understood and are labelled by pairs $(L, [\psi])$ with $L \subset G$ a subgroup of $G$ and $[\psi]$ a cohomology class in $H^2(L, C^\times)$. Applying our construction for various choices of module categories, we expect to recover dual versions of the Ising-like Hamiltonians investigated in [71]. As for Vec$_{Z_2}$, two particularly interesting choices of Vec$_G$-module categories are provided by $\mathcal{M} = \text{Vec}$ and $\mathcal{M} = \text{Vec}_G$. The MPOs intertwining the corresponding tensor network representations of the Vec$_G$ string-net model are particularly simple:

$$= 1, \quad \forall g_1, g_2 \in G,$$

where the black dots represent group delta functions. As mentioned earlier, these intertwining operators map the bonds of the Hamiltonian associated with $\mathcal{M} = \text{Vec}_G$ to those of the Hamiltonian associated with $\mathcal{M} = \text{Vec}$. Invoking the property

$$|= g \rangle \mapsto |gh\rangle$$

and $L_h : |g\rangle \mapsto |hg\rangle$, we can readily confirm that this operator implements the Kramers-Wannier duality of the Ising-like Hamiltonians. Equivalently, this tensor network can be thought of as implementing a gauging map. In particular, applying such an operator to product states yield GHZ-like states thus providing a non-local map relating short- and long-range order. A further generalization is obtained by considering the fusion category Vec$_G^\sigma$ that only differs from Vec$_G$ by its monoidal associator, which now evaluates to some normalized representative of a cohomology class $[\sigma]$ in $H^3(G, U(1))$. Module categories over this category are also well understood [70]. Interestingly, as long as $[\sigma]$ is not the trivial class, it is not possible to choose Vec as a module category. This corresponds precisely to the “anomaly” which presents an obstruction to gauging the corresponding quantum models.

III.B. Critical Ising model

Let $\mathcal{D} = \text{Ising}$ be the fusion category with simple objects $\{1, \psi\} \oplus \{\sigma\}$. The non-trivial fusion rules read $\psi \otimes \psi \simeq 1$, $\sigma \otimes \psi \simeq \sigma$ and $\sigma \otimes \sigma \simeq 1 \oplus \psi$ such that $d_1 = 1 = d_{\psi}$ and $d_{\sigma} = \sqrt{2}$. The non-trivial $F$-symbols are then provided
by
\[ F^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad F^{\psi\psi\sigma} = F^{\psi\sigma\psi} = (-1). \] (40)

We are interested in the model
\[ \mathcal{H} = -\sum_i \mathcal{E}_i^{\mathcal{M}} \] (41)
defined by the bond
\[ \mathcal{E}_i^{\mathcal{M}} = \sigma\sigma\psi \sigma\sigma\psi \equiv \sigma\sigma\psi \sigma\sigma\psi, \]
where \( \gamma = \uparrow \psi \) is a shorthand defined via the diagrams above.

- Let \( \mathcal{M} = \text{Ising} \) be the regular Ising-category. As per eq. (17), we have
\[ \equiv \sigma\sigma\psi \sigma\sigma\psi = 1 \psi \sigma \sigma \psi \].

Given the definition of \( \mathcal{E}_i^{\mathcal{M}} \) and the fusion rules in Ising, the Hamiltonian acts on the effective total Hilbert space
\[ \mathcal{H}^\text{eff} = C \left[ \begin{array}{c|c|c} \uparrow & \sigma & \sigma \\ \hline \downarrow & \psi & \sigma \\ \hline \sigma & \sigma & \sigma \end{array} \right] \oplus C \left[ \begin{array}{c|c|c} \uparrow & \sigma & \sigma \\ \hline \sigma & \sigma & \sigma \\ \hline \sigma & \psi & \sigma \end{array} \right]. \] (44)

Let us focus on either one of the two Hilbert spaces appearing in this decomposition. First, notice that all the hom-spaces are one-dimensional. We then identify a module strand labelled by \( \sigma \) as the presence of a defect and \( C[-] \) as the local Hilbert space of a physical spin, which we locate as before in the middle of the corresponding strand. The operator \( \mathcal{E}_i^{\mathcal{M}} \) acts differently depending on whether the module strand at the site \( i \) is labelled by \( \uparrow \psi \) or \( \sigma \). In the former case, it follows from the definition of the F-symbols that \( \mathcal{E}_i^{\mathcal{M}} \) acts on the strand \( i \) as \( |\psi\rangle \mapsto \frac{1}{2}(|\psi\rangle + |\psi\rangle) \) and identically on the strands \( i-1 \) and \( i+1 \) labelled by \( \sigma \). In the latter case, we notice that the first term in the definition of \( \mathcal{E}_i^{\mathcal{M}} \) acts as the identity or the zero operator depending on whether the strands \( i-1 \) and \( i+1 \) have matching labelling objects, and vice versa for the second term. Putting everything together, we obtain in the Pauli \( X \) basis the Hamiltonian
\[ \mathcal{H} = -\sum_i \left( |\sigma\rangle \langle \sigma| \otimes Z_i \otimes |\sigma\rangle \langle \sigma| + X_i \otimes |\sigma\rangle \langle \sigma| \otimes X_{i+2} \right) \]
\[ = -\sum_i (Z_i + X_i X_{i+1}), \] (45)

which we recognize as the critical transverse field Ising model. Taking into account the second Hilbert space in the decomposition (44) of \( \mathcal{H} \), we obtain a direct sum of two copies of the model.

- Let \( \mathcal{M} = \text{Ising}/(\psi \simeq 1) \) be the (fermionic) Ising-module category obtained via fermion condensation of \( \psi \) (see [72, 73] for detailed construction). This module category has two simple objects denoted by \( \uparrow \psi \) such that \( \text{End}_{\mathcal{M}}(\uparrow \psi) \simeq C \) and \( \text{End}_{\mathcal{M}}(\uparrow \beta) \simeq C^{11} \). The non-trivial fusion rules are indicated in the table below
\[
\begin{array}{c|ccc}
\langle & \uparrow & \downarrow & \psi \\
\hline
\psi & \uparrow & \circ & \psi \\
\downarrow & \uparrow & \circ & \beta \\
\psi & \uparrow & \circ & \beta \\
\end{array}
\]
(46)

Components of the module associator \( \mathcal{F} \) are even isomorphisms and a non-exhaustive list of non-vanishing \( \mathcal{F} \)-symbols is given by
\[
(q^{\beta \sigma \sigma})_{\psi,\psi} = -(q^{\beta \sigma \sigma})_{\psi,\psi} = -i(q^{\beta \sigma \sigma})_{\psi,\psi} = \frac{1}{d_{\sigma}},
\]
\[
(q^{\beta \sigma \beta})_{\psi,\psi} = -(q^{\beta \sigma \beta})_{\psi,\psi} = i(q^{\beta \sigma \beta})_{\psi,\psi} = \frac{1}{d_{\sigma}},
\]
\[
(q^{\beta \sigma \psi})_{\psi,\psi} = (q^{\beta \sigma \psi})_{\psi,\psi} = (q^{\beta \sigma \psi})_{\psi,\psi} = 1,
\]
where the labels ‘\( \psi \)’ and ‘\( \sigma \)’ refer to the oddness and evenness of the basis vectors. As per eq. (17), we have
\[ \equiv \uparrow \psi \sigma \sigma \psi \equiv \uparrow \psi \sigma \sigma \psi = \uparrow \psi \sigma \sigma \psi \equiv \uparrow \psi \sigma \sigma \psi. \]
(47)

Henceforth, we assume that \( \sigma \equiv \downarrow \psi \equiv \uparrow \beta \equiv \psi \equiv \sigma \equiv \beta \equiv \psi \equiv \sigma \). Notice that the odd basis vector in \( \mathcal{V}_{\beta,\sigma}^\mathcal{F} \) can be obtained from the even basis vector by acting with the odd element in \( \text{End}_{\mathcal{M}}(\beta) \), we can fix basis vectors in split spaces to be even. Using a slightly abusing notation, it follows that the Hamiltonian acts on the effective total Hilbert space
\[ \mathcal{H}^\text{eff} = C \left[ \begin{array}{c|c|c} \uparrow & \sigma & \sigma \\ \hline \sigma & \sigma & \sigma \\ \hline \sigma & \psi & \sigma \end{array} \right] \oplus C \left[ \begin{array}{c|c|c} \uparrow & \sigma & \sigma \\ \hline \sigma & \sigma & \sigma \\ \hline \sigma & \psi & \sigma \end{array} \right]. \] (48)

at which point our derivation closely follows that of [72]. Focusing on either one of the Hilbert spaces appearing in the decomposition above, we identify the module strand labelled by \( \uparrow \psi \) as a vacancy and \( C[-] \) as the local Hilbert space of a physical fermion. Analogously to the previous choice of \( \mathcal{M} \), we distinguish two actions for the operator \( \mathcal{E}_i^{\mathcal{M}} \) depending on the labelling of the module strand \( i \). If the labelling of the strand \( i \) takes value in \( \text{End}_{\mathcal{M}}(\beta) \), it follows from \( \psi \psi \simeq C^{01} \cdot 1 \) and the evenness of \( \mathcal{F} \) that \( \mathcal{E}_i^{\mathcal{M}} \) acts as the fermion parity operator. If the strand \( i \) is labelled by \( \uparrow \psi \), the definition of
the $^F$-symbols guarantees that the operator $b^M_i$ acts on the strands $i-1$ and $i+1$ as the hopping or pairing operator, e.g. $|e,o\rangle \mapsto \frac{1}{2}(|e,o\rangle + |o,e\rangle - |e,o\rangle + |o,e\rangle) = |o,e\rangle$. Putting everything together, we obtain
\[
H_{\text{eff}} = -\sum_i (c_i^1 c_{i+1}^1 + c_i^1 c_{i+1}^1 + \text{h.c.} - 2c_i^1 c_i^1 - 1) \quad (49)
\]
which is the Jordan-Wigner dual of the Hamiltonian given in eq. (45).

### III.C. Heisenberg XXZ model

Let $\mathcal{D} = \text{Ising}^{op} \otimes \text{Ising}$ be the Deligne tensor product of two copies of $\text{Ising}$, such that the simple objects are of the form $(\alpha_1, \alpha_2) \equiv \alpha_1 \otimes \alpha_2$ with $\alpha_1, \alpha_2 \in \mathcal{I}_{\text{Ising}}$. The fusion rules are obtained from those of $\text{Ising}$ according to $(\alpha_1 \otimes \alpha_2) \otimes (\beta_1 \otimes \beta_2) = (\beta_1 \otimes \alpha_1) \otimes (\alpha_2 \otimes \beta_2)$ and the $F$-symbols are given by
\[
F^{(\alpha_1 \otimes \alpha_2)(\beta_1 \otimes \beta_2)(\gamma_1 \otimes \gamma_2)}_{(\delta_1 \otimes \delta_2)} = F^{\gamma_1 \beta_1 \alpha_1}_{\delta_1} \otimes F^{\alpha_2 \beta_2}_{\delta_2}, \quad (50)
\]
where the $\text{Ising}$ $F$-symbols on the r.h.s. were defined in eq. (40). We are interested in the model
\[
H = -J\sum_i b^M_{1,i} - J\sum_i b^M_{2,i} + Jg\sum_i b^M_{3,i} \quad (51)
\]
whose bond algebra is generated by the operators
\[
b^M_{a,i} = \begin{pmatrix} 1 & \psi & \sigma \\ \psi & \sigma \end{pmatrix}, \quad \begin{cases}
\gamma_1 = (1 \otimes \psi) \otimes (1 \otimes \psi)
\gamma_2 = (1 \otimes \psi) \otimes (1 \otimes \psi)
\gamma_3 = (1 \otimes \psi) \otimes (1 \otimes \psi)
\end{cases}, \quad (52)
\]
where $1 \otimes \psi$ is defined via eq. (42) and we introduced the shorthand $\sigma \sigma = \sigma \otimes \sigma$ that we shall use in diagrams.

- Let $\mathcal{M} = \text{Ising}^{op} \otimes \text{Ising}$ be the regular module category. We have
  \[
  (\alpha_1 \otimes \alpha_2) \otimes (\beta_1 \otimes \beta_2) = \alpha_1 \otimes (\beta_1 \otimes \alpha_2 \otimes \beta_2) \quad (53)
  \]
  so that, given the definition of $b^M_{a,i}$, the Hamiltonian acts on an effective total Hilbert space that decomposes into four terms as
  \[
  \mathcal{H} = C \begin{pmatrix}
  \ldots & \ldots & \ldots & \ldots \\
  \ldots & \ldots & \ldots & \ldots \\
  \ldots & \ldots & \ldots & \ldots \\
  \ldots & \ldots & \ldots & \ldots \\
  \ldots & \ldots & \ldots & \ldots \\
  \ldots & \ldots & \ldots & \ldots \\
  \ldots & \ldots & \ldots & \ldots \\
  \ldots & \ldots & \ldots & \ldots \\
  \end{pmatrix}
  \quad (57)
  \]
  where $1 \otimes 2$ refer to the two basis vectors in $V_{\sigma,\sigma}$. The operator $b^M_{1,i} + b^M_{2,i} - g^M_{3,i}$ acts on the first vector space appearing in this decomposition in the Pauli $Z$ basis as $Z_{i-\frac{1}{2}}X_{i+\frac{1}{2}} + X_{i-\frac{1}{2}}Z_{i+\frac{1}{2}} - gY_{i-\frac{1}{2}}Y_{i+\frac{1}{2}}$, whereas it acts on the second term as $Z_{i-\frac{1}{2}}X_{i+\frac{1}{2}} + X_{i-\frac{1}{2}}Z_{i+\frac{1}{2}} - gZ_{i-\frac{1}{2}}Z_{i+\frac{1}{2}}$. Conjugating every other site of these spin chains via a Hadamard matrix yields an effective Hamiltonian $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with
\[
\mathcal{H}_1 \equiv -J\sum_i (X_{i-\frac{1}{2}}X_{i+\frac{1}{2}} + Z_{i-\frac{1}{2}}Z_{i+\frac{1}{2}} + gY_{i-\frac{1}{2}}Y_{i+\frac{1}{2}}), \quad (58)
\]
which we recognize as the XXZ Heisenberg model \[74\], and
\[
H_2 \equiv -J \sum_i \left( X_i X_{i+1} + Z_i Z_{i+1} - g(X_i X_{i+2} + Z_i Z_{i+2}) \right). \tag{59}
\]

We can readily check via repeated usage of Kramers-Wannier duality that these Hamiltonians are indeed dual to the coupled chains provided in eq. \(55\) and eq. \(56\), respectively \[57, 75\].

- Let \( \mathcal{M} = \text{Ising}/(\psi \simeq 1) \) be the (fermionic) Ising-module category obtained from the module category Ising via fermion condensation. The underlying category was described in the previous example and the module structure is such that \( 1 \triangleleft (\sigma, \sigma) \simeq \mathbb{C}^{1 \uparrow1 \downarrow} \cdot 1, 1 \triangleleft (\psi, 1) \simeq \mathbb{C}^{0 \uparrow1 \downarrow} \cdot 1 \) and \( \beta \triangleleft (\sigma, \sigma) \simeq \mathbb{C}^{2 \uparrow2 \downarrow} \cdot \beta \). Amongst others, we have the following \( F \)-symbols:
\[
\begin{align*}
F^1_{1}(\sigma, \sigma)(\sigma, \sigma)_{1,ee} &= \sigma_{1,ee} = \frac{1}{d_\sigma}, \\
F^1_{1}(\sigma, \sigma)(\psi, 1)_{1,oe} &= i \sigma_{1,oe} = \frac{1}{d_\sigma}, \\
F^1_{1}(\sigma, \sigma)(\psi, \psi)_{1,ee} &= \sigma_{1,ee} = \frac{1}{d_\sigma}.
\end{align*}
\]

It follows from the fusion rules that the Hamiltonian acts on the total Hilbert space
\[
\mathcal{H}^{\text{eff}} = \mathbb{C} \left[ \cdots \frac{1}{\sigma} \frac{1}{\sigma} \frac{1}{\sigma} \frac{1}{\sigma} \frac{1}{\sigma} \cdots \right] \oplus \mathbb{C} \left[ \cdots \frac{1}{\beta} \frac{1}{\sigma} \frac{1}{\beta} \frac{1}{\sigma} \frac{1}{\beta} \cdots \right],
\]
where as before \( e \oplus o \) refer to the purely even and odd basis vectors in \( \mathbb{C}^{1 \uparrow1 \downarrow} \). Applying the same techniques as previously, up to local unitaries, we find an effective Hamiltonian \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \) with
\[
\begin{align*}
H_1^{\text{eff}} &= -J \sum_i \left( 2c_i^{\dagger} \frac{1}{2} c_{i+\frac{1}{2}} + 2c_i^{\dagger} \frac{1}{2} c_{i-\frac{1}{2}} \\
&\quad + g(2c_i^{\dagger} \frac{1}{2} c_{i-\frac{1}{2}} - 1) (2c_i^{\dagger} \frac{1}{2} c_{i+\frac{1}{2}} - 1) \right) \
H_2^{\text{eff}} &= -J \sum_i \left( 2c_i^{\dagger} c_{i+1} + 2c_i^{\dagger} c_i \right) \\
&\quad + 2g(2c_i^{\dagger} c_i - 1) c_{i+2} + 2c_i^{\dagger} (2c_i^{\dagger} c_i - 1) c_i \right)
\end{align*}
\tag{61}
\]
i.e. the Jordan-Wigner duals of the Hamiltonians \(58\) and \(59\), respectively. We note that \( \mathcal{H}_1 \) corresponds to the standard \textit{Kogut-Susskind} prescription for discretizing a Dirac fermion \[9\].

III.D. Quantum IRF-Vertex models

This last example is slightly beyond the framework employed so far as the input data does not quite meet all the requirements to be a fusion category. However, our construction still largely applies allowing us to define dual models with categorical symmetries. Let \( \mathcal{D} = \text{Rep}(U_q(\mathfrak{sl}_2)) \) be the representation category of the quantum group defined as the \( q \)-deformed universal enveloping algebra \( U_q \) of the Lie algebra \( \mathfrak{sl}_2 \). We will restrict to the case where \( q \) is not a root of unity. Isomorphism classes of simple objects in \( \mathcal{D} \) are labeled by half-integer spins \( j \in \frac{1}{2} \mathbb{N} \), with fusion rules given by
\[
J_1 \otimes J_2 \simeq \bigoplus_{j=|J_1-J_2|} J_3.
\tag{63}
\]

The \( F \)-symbols of this category are well known and those required for our derivations can be found for instance in \[76\]. In the limit \( q \to 1 \), these boil down to the so-called \( 6j \)-symbols of SU(2). We are interested in the model
\[
\mathcal{H} = \sum_i E_i^{M},
\tag{64}
\]
whose bond algebra is generated by the operator
\[
E_i^{M} = \begin{array}{c}
1 \\
\beta
\end{array}
\quad \begin{array}{c}
\beta \\
1
\end{array}
\quad \begin{array}{c}
1 \\
\beta
\end{array}
\quad \begin{array}{c}
\beta \\
1
\end{array}
\quad \begin{array}{c}
1 \\
\beta
\end{array}
\quad \begin{array}{c}
\beta \\
1
\end{array}
\quad \cdots,
\tag{65}
\]

- Let \( \mathcal{M} = \text{Rep}(U_q(\mathfrak{sl}_2)) \) be the regular module category. We have
\[
\mathcal{M} = \bigoplus_{j \in \frac{1}{2} \mathbb{N}} j 
\quad \text{and the Hilbert space is given by summing over all possible spin configurations such that neighbouring sites are labelled by spins that differ by } \frac{1}{2}. \quad \text{Using a slightly abusing notation, we have}
\]
\[
\mathcal{H}^{\text{eff}} = \mathbb{C} \left[ \cdots \frac{1}{\beta} \frac{1}{\beta} \frac{1}{\beta} \frac{1}{\beta} \frac{1}{\beta} \cdots \right] \oplus \mathbb{C} \left[ \cdots \frac{1}{\beta} \frac{1}{\beta} \frac{1}{\beta} \frac{1}{\beta} \frac{1}{\beta} \cdots \right] \tag{66}
\]
\[
\mathcal{H} = \sum_i E_i^{M},
\]
\[
\langle \psi_{j_1-j_1, j_1-j_1} | \mathcal{M} | j_1-j_1, j_1-j_1 \rangle = \delta_{j_1-j_1, j_1-j_1} \sqrt{|j_1-j_1| q |j_1-j_1| q}
\tag{67}
\]
where \( |j \rangle_q \) is defined as \( |j \rangle_q = (q^j - q^{-j})/(q - q^{-1}) \). This Hamiltonian is the quantum analogue of the well-known interacting round a face (IRF) statistical model.
Let $\mathcal{M} = \text{Vec}$, where we again denote the unique simple object of $\text{Vec}$ as $\mathbb{1}$. The action is given by $\mathbb{1} \otimes \mathbb{j} \simeq (2j + 1) \cdot \mathbb{1}$ and the $\mathcal{F}$-symbols can be found in [76]. In the limit $q \to 1$, these boil down to the Clebsch-Gordan coefficients of $SU(2)$. The total Hilbert space is now given by $\mathcal{H} \simeq \bigotimes_{i=1}^{q} C^2$ with $C^2 \simeq \text{Hom}_M(\mathbb{1} \otimes \mathbb{j}, \mathbb{1})$ and the Hamiltonian becomes

\begin{equation}
\hat{H} = -\frac{1}{2} \sum_i \left( X_{i-\frac{1}{2}} X_{i+\frac{1}{2}} + Y_{i-\frac{1}{2}} Y_{i+\frac{1}{2}} + q + q^{-1} \right) (Z_{i-\frac{1}{2}} Z_{i+\frac{1}{2}} - 1),
\end{equation}

which is the quantum analogue of the 6-vertex statistical model. Note that it coincides with the XXZ model up to an additive constant.

IV. OUTLOOK

We proposed a systematic framework to define dual (1+1)d quantum models displaying categorical symmetries. We illustrated our construction with examples that are simple and yet non-trivial. The dualities as constructed above are all realised as non-local transformations on the Hilbert space, implemented by MPO intertwining with non-trivial virtual bond dimension. Although we focused on recovering well-known duality relations, this approach is systematic and the problem of constructing new duality relations and Hamiltonians realizing them is equivalent to the classification of module categories and the determination of their associators.

IV.A. Extensions

Within the same mathematical framework, our study can be extended in several directions: Firstly, we can account for choices of boundary conditions and establish how such choices interact with the duality relations. Secondly, we can systematically define the order and disorder parameters of the (1+1)d models. As evoked in the main text, these are related to the anyons of the topological phase that shares the same input category theoretical data. Moreover, we can readily compute how these operators transform under the duality relations via the intertwining matrix product operators. Thirdly, our construction naturally applies for Hamiltonians with longer range interactions as well. Indeed, relying on the recoupling theory of the underlying tensors, we can readily define bonds simultaneously acting on a larger number of sites. Finally, duality relations of classical statistical mechanics models in terms of their transfer matrices can be investigated using the same tools. These studies will be reported in a follow-up manuscript.

The last example we considered in the main text, namely the IRF-Vertex correspondence, suggests possible extensions beyond the current framework. In particular, it seems possible to relax some of the defining properties of a spherical fusion category whilst still being able to consistently define dual models. For instance, it would be interesting to derive Hamiltonian models with symmetries associated with categories of representations over Lie groups (see for instance [77] for a discussion about module categories over $\text{Rep}(\text{SL}(2))$).

IV.B. Higher dimensions

An even more tantalizing direction consists in generalizing our construction to (2+1)d quantum models. Formally, this amounts to considering a categorification of the present construction, whereby the input category theoretical data are provided by a spherical fusion 2-category and a choice of a finite semi-simple module 2-category over it [78, 79]. The first step of such a generalization then amounts to defining the higher-dimensional analogues of the tensors given in eq. (8). In the simple case where the input spherical fusion category is that associated with the (3+1)d toric code, we distinguish two tensor network representations that satisfy symmetry conditions w.r.t. string- and membrane-like operators, respectively [80, 81]. Furthermore, it was shown in [80] that the corresponding intertwining tensor network realizes the Kramers-Wannier duality. Mimicking the example in sec. III A, we should thus be able to use these tensor network representations to reconstruct the (2+1)d Ising model and Wegner’s $\mathbb{Z}_2$-gauge theory [7, 10–13].

A general framework for defining such higher-dimensional tensors and deriving the corresponding symmetry conditions is outlined in [82] with an emphasis on generalizations of the (3+1)d toric code for arbitrary finite groups. Akin to the lower-dimensional analogues, we distinguish two canonical tensor network representations associated with module 2-categories labelling the so-called rough and smooth gapped boundaries of the topological models. Interestingly, we recover the tensor networks implementing gauging maps considered in [83] as the intertwining operators between these tensor network representations. Following the notations of eq. (39), these gauging maps are of the form

Just as the duality map corresponding to gauging (1+1)d systems relates paramagnetic eigenstates to long-range ordered GHZ states (see end of sec. III A), the (2+1)d gauging procedure maps symmetry protected topologically ordered states to states exhibiting intrinsic topological order [84]. It will be very interesting to consider those questions from the point of view of fusion 2-categories, consider duality relations beyond this canonical gauging procedure, and explicitly construct (2+1)d Hamiltonian models related by such generalizations.
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