Quantum Griffiths Singularities in the Transverse-Field Ising Spin Glass

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Abstract

We report a Monte Carlo study of the effects of fluctuations in the bond distribution of Ising spin glasses in a transverse magnetic field, in the paramagnetic phase in the $T \to 0$ limit. Rare, strong fluctuations give rise to Griffiths singularities, which can dominate the zero-temperature behavior of these quantum systems, as originally demonstrated by McCoy for one-dimensional ($d = 1$) systems. Our simulations are done on a square lattice in $d = 2$ and a cubic lattice in $d = 3$, for a gaussian distribution of nearest neighbor (only) bonds. In $d = 2$, where the linear susceptibility was found to diverge at the critical transverse field strength $\Gamma_c$ for the order-disorder phase transition at $T = 0$, the average nonlinear susceptibility $\chi_{nl}$ diverges in the paramagnetic phase for $\Gamma$ well above $\Gamma_c$, as is also demonstrated in the accompanying paper by
Rieger and Young. In $d = 3$, the linear susceptibility remains finite at $\Gamma_c$, and while Griffiths singularity effects are certainly observable in the paramagnetic phase, the nonlinear susceptibility appears to diverge only rather close to $\Gamma_c$. These results show that Griffiths singularities remain persistent in dimensions above one (where they are known to be strong), though their magnitude decreases monotonically with increasing dimensionality (there being no Griffiths singularities in the limit of infinite dimensionality).
I. INTRODUCTION

In his classic paper, Griffiths [1] found that the magnetization of the site-diluted classical Ising ferromagnet exhibits, in the paramagnetic phase, an essential singularity as a function of external magnetic field $H$, at $H = 0$. The singularity occurs for a range of temperatures $T$ above the actual critical temperature $T_c$ of the system, but below the critical temperature of the parent uniform ferromagnet. This effect, known as the Griffiths singularity, arises as a result of fluctuations in the distribution of the disorder in models with quenched randomness. Clusters which are coupled to each other more strongly than the average appear locally ordered, even though the system as a whole is still in the disordered phase. If these (usually rare) “Griffiths clusters” occur with sufficiently large probability, their effects can dominate the properties of the average system.

Griffiths singularities in the static properties of classical site- or bond-diluted random spin systems are quite weak. Consequently, they have not yet been observed in actual experiments or in high precision computer simulations [2]. There have also been studies of the dynamic effects of Griffiths clusters in classical spin systems. Such regions lead to non-exponential relaxation of the temporal spin correlation function in the paramagnetic phase, at temperatures below the $T_c$ of the pure system. Analytical bounds for the diluted ferromagnet [3] and spin glass [4] suggest a relaxation that is slower than not only an exponential in time $t$, but also an exponential in any power of $t$; however, the bound decays faster than any power of $t$. Non-exponential relaxation is observed in Monte Carlo simulations of the symmetric ±J Ising spin glass model [5,6], though the functional form appears to fit a stretched exponential, rather than the asymptotic form obtained from the analytic considerations of dynamic Griffiths singularities [4]. Thus, it is not clear to what extent the observed non-exponential relaxation is related to the rare unfrustrated clusters considered in formulating the analytical bounds, nor is there a well-formed picture of the dominant dynamic Griffiths singularities in these systems.

For random quantum systems the situation is very different, as can be seen from the
work of McCoy \[7\], and as has been recently emphasized by Fisher \[8\]. In quantum systems the dynamics are inextricably linked to statics, so static and dynamic Griffiths singularities are essentially the same phenomenon. Further, Griffiths singularities are greatly enhanced by quantum fluctuations, leading to many cases where system properties are dominated by rare regions of the sample. Thus, for example, the magnetic susceptibility in insulating random antiferromagnets both in one dimensional compounds such as TCNQ salts (e.g. Qn(TCNQ)$_2$) \[9\] and in three dimensional lightly doped semiconductors (e.g. Silicon doped with Phosphorus \[10\] or Boron \[11\]) is dominated at low temperatures by the divergent contribution from rare, weakly coupled spins. Various theoretical studies justifying the observed divergent form exist, based on models of random quantum spin-1/2 antiferromagnets with short range interactions \[12,13\]. Similar explanations have also been suggested for divergent susceptibilities for dimerized spin-1/2 chains with randomness \[14\], and for amorphous metallic systems with magnetic moments \[15\].

The random quantum Ising model in a transverse field, which we study here, undergoes a $T = 0$ quantum phase transition from a spin-glass ordered phase at low field, to a disordered phase at high field. While this is true in all dimensions at $T = 0$, most is known about the model in $d = 1$ \[16–18\]. In this case, Griffiths singularities dominate both the disordered phase as well as the critical region. Among the more spectacular results is the demonstration \[7,16\] (more than 25 years ago) that the average linear magnetic susceptibility diverges in the disordered (paramagnetic) phase as the transverse field is reduced from a large value, well before the transition to the ordered phase. Further, the dynamical exponent, $z$, characterizing the scaling of the temporal and spatial correlation lengths at the phase transition, is found to be infinite (the correlation time scales exponentially with the spatial correlation length). As shown by Fisher \[18\], there is a significant difference between average and typical spin-spin correlation functions, and the former are dominated by Griffiths singularities coming from rare events. Note that for the linear chain ($d = 1$) there is no frustration even when the exchanges are of random sign. The $d = 1$ transverse-field Ising spin glass is simply mapped onto an equivalent random-exchange ferromagnet by a variable change that
interchanges the $\sigma^z = 1/2$ and $\sigma^z = -1/2$ states at the appropriate sites. In higher dimensions, effects of Griffiths singularities are expected on general grounds to be weaker for both the spin glass and the random ferromagnet. However, as argued by Thill and Huse [19], divergences due to Griffiths singularities do survive for any finite $d$, although they may be confined to very high-order susceptibilities. It is the quenched disorder that produces the Griffiths singularities, so the frustration present in the spin glass is not essential to the phenomenon. However, the spin glass is in a certain sense the simplest system to study, since there is a standard no-parameter model, namely the Edwards-Anderson model with Gaussian-distributed exchange couplings with variance one and mean zero. A model random ferromagnet has an additional parameter to set, namely the strength of the random variations in the exchanges relative to the mean exchange. Although it would be interesting to also study the random ferromagnet, here we confine our attention to the spin glass.

Numerical simulations of the quantum Ising spin glass in $d = 2$ [20] as well as in $d = 3$ [21] suggest that in both cases, the scaling at the $T = 0$ quantum phase transition is characterized by a finite dynamical exponent (so that the relation between temporal and spatial correlation lengths is of the conventional power-law type). However, the average linear susceptibility is found to be divergent [20] at the phase transition in $d = 2$ (where the spin-glass ordered phase, and hence the transition occurs only at $T = 0$), whereas in $d = 3$, which has a transition both at $T = 0$ and at nonzero $T$, the average linear susceptibility remains finite at the transition [21]. The non-linear susceptibility is found to be divergent (as expected) in both cases; however, the exponent characterizing the divergence is found to be considerably larger than the one obtained in experiments on a related system, the diluted, dipolar coupled, Ising system, LiHo$_x$Y$_{1-x}$F$_4$ [22].

The present work has thus been motivated on a number of grounds. Firstly, it would be of interest to see whether the strong effects of Griffiths singularities seen in $d = 1$ are a purely one-dimensional phenomenon (as many unusual properties turn out to be), or are more general. Secondly, a systematic study of Griffiths singularities in higher dimensions would help ascertain their importance in quantum systems without any theoretical assump-
tions or biases. (Previous studies in higher dimensions are based on the applicability of certain schemes - e.g. phenomenological arguments in the case of Thill and Huse [19] and convergence of perturbative renormalization schemes in the case of quantum antiferromagnets [13].) Finally, it would be of interest to investigate the possibility that strong Griffiths singularities may be complicating the comparison between numerical simulations on the transverse Ising spin glass and the experiments on the diluted dipolar system. Here we study both the two- and three-dimensional quantum Ising spin glass; the accompanying paper by Rieger and Young [23] is a substantially more thorough study of the two-dimensional case. The previous simulation studies of these systems [20,21] focussed on the quantum phase transition; here, to study the quantum Griffiths singularities, we instead focus on the paramagnetic phase.

II. THEORETICAL CONSIDERATIONS

In this section, we will briefly review the model, as well as the basic concepts concerning the effect of rare strongly-coupled clusters. In the latter discussion, we follow the description of Thill and Huse [19].

The model we study is the quantum Ising spin glass in a transverse field, which is governed by the Hamiltonian:

\[ H_{qm} = - \sum_{\langle i,j \rangle} J_{ij} \sigma_i^z \sigma_j^z - \Gamma \sum_i \sigma_i^x, \]  

where \( \sigma_i^\alpha \)'s are the Pauli spin matrices representing the quantum mechanical spin-1/2 at site \( i \), and the exchange interactions \( J_{ij} \) between the nearest neighbors on a d-dimensional hypercubic lattice are taken to be independent quenched random variables with a gaussian distribution with mean zero and variance unity. The second term represents the coupling to a transverse field \( \Gamma \) applied along the \( x \)-axis. The ground state of the above model, in any dimension \( d \geq 1 \), undergoes a phase transition at \( \Gamma = \Gamma_c \) from a spin-glass ordered state at low field, to a disordered paramagnetic state at high field. For \( d > d_l \), where \( d_l \) is the lower
critical dimension for the classical Ising spin glass (currently believed to be between 2 and 3) [24], such a transition exists also at finite temperature (see Figure 1), close enough to which the critical behavior is that of the classical model; the quantum behavior is recovered in the $T \to 0$ limit.

As explained in previous work [20,21,25], the quantum mechanical Hamiltonian (1) above may be approximately mapped onto an equivalent classical Hamiltonian in $(d+1)$ dimensions given by:

$$H_{cl} = - \sum_{<ij>,\tau} J_{ij} S_i(\tau) S_j(\tau) - \sum_{i,\tau} S_i(\tau) S_i(\tau + 1),$$  \hspace{1cm} (2)

where $S_i(\tau)$ are classical Ising variables taking on the values $\pm 1$, with roman indices referring to sites on a $d$-dimensional hypercubic lattice composed of the space dimensions of the original quantum mechanical Hamiltonian, and the index $\tau$ referring to the $(d+1)$-th imaginary time dimension obtained by transforming the quantum Hamiltonian to a path integral. Note that Eq. (2) has nearest-neighbor ferromagnetic couplings in the imaginary time direction, and further the random couplings in the $d$ space dimensions are independent of $\tau$. Our simulations are done on the classical model of Eq. (2) on hypercubic lattices in (3+1) and (2+1) dimensions for a range of spatial sizes $L$ and “time” sizes $L_\tau$. Periodic boundary conditions are applied in both spatial and imaginary time directions. In the transformation to the classical Hamiltonian, the length $L_\tau$ corresponds to the inverse temperature of the quantum Hamiltonian (so $T \to 0$ corresponds to $L_\tau \to \infty$), and the temperature $T_{cl}$ of the classical statistical mechanical problem corresponds to the strength of the transverse field, $\Gamma$, with higher $T_{cl}$ corresponding to larger $\Gamma$. (For details see Guo et al. [21,25] or Rieger and Young [20]).
FIG. 1. Schematic phase diagrams of the transverse-field quantum spin glasses in \(d < d_l\) (e.g. \(d = 2\)) and \(d > d_l\) (e.g. \(d = 3\)). The thick portion along the horizontal axis (\(T = 0\)) indicates the quantum Griffiths phase. \(\Gamma_c\) is the critical transverse field, while the average nonlinear susceptibility becomes finite only when \(\Gamma\) exceeds \(\Gamma_{nl}\). (Also shown are the variables of the equivalent classical model). Our results indicate that \(\Gamma_{nl}\) remains slightly larger than \(\Gamma_c\) for \(d = 3\).

Figure 1 shows the schematic phase diagram of the quantum mechanical model for \(d < d_l\) and \(d > d_l\), where \(d_l\) is the lower critical dimension of the classical short range Ising spin glass, currently believed to be between \(d = 2\) and \(d = 3\) [24]. (Also shown are the variables of the equivalent classical model). Based on earlier work for the nearest-neighbor spin glass with gaussian-distributed bonds with zero mean and unit variance [20,21], we have:

\[
T_c^{cl}(d = 2) = 3.275 \pm 0.025 \quad \text{and} \quad T_c^{cl}(d = 3) = 4.32 \pm 0.03
\]

(3)

for the equivalent classical model (2).

Next, we examine the effect of Griffiths singularities in quantum Ising spin-glass systems,
following Thill and Huse [19]. Consider a rare strongly-coupled or less-frustrated cluster (at $T = 0$) with local critical transverse field $\Gamma_c^{\text{loc}}$ higher than the critical $\Gamma_c$ of the whole system. The probability of finding such a cluster with linear length scale $L$ is exponentially small in the cluster volume:

$$P(L) \sim \exp(-c_1 L^d),$$

(4)

where $c_1$ is a nonnegative constant depending continuously on the difference between $\Gamma_c^{\text{loc}}$ and $\Gamma_c$; $c_1$ vanishes only for $\Gamma_c^{\text{loc}} = \Gamma_c$.

We are interested in the disordered phase where the applied transverse field $\Gamma > \Gamma_c$, but $\Gamma_c^{\text{loc}} > \Gamma$ such that the rare cluster is locally ordered. This situation may be examined in terms of the equivalent classical spin system (2). It is easily seen that the anomalous bond configuration of the rare cluster in the spatial hyperplane is replicated along the imaginary time direction. Thus we have a one dimensional chain of special spins in this classical spin system. As argued by Thill and Huse [19], this cluster can be modeled by a 1D Ising model at $T^{cl} = 1$, with the Ising spins $S(\tau) = \pm 1$ describing the two degenerate classical ordered states that the cluster flips between as $\tau$ varies. The ferromagnetic exchange coupling, $K$, between adjacent Ising spins along the imaginary time direction is the action to flip the cluster and is proportional to the droplet volume in the spatial hyperplane: $K \sim c_2 L^d/2$, where $c_2$ depends continuously on the difference between $\Gamma$ and $\Gamma_c^{\text{loc}}$ and is positive for $\Gamma < \Gamma_c^{\text{loc}}$ when the cluster is ordered. As usual, the correlation length of this 1D Ising model along the imaginary time direction, $\xi_\tau$, is proportional to $\exp(2K)$, thus grows exponentially with the cluster volume $L$:

$$\xi_\tau \sim \exp(c_2 L^d).$$

(5)

As a consequence, the probability of a spin being in a cluster with $\Gamma_c^{\text{loc}}$ and imaginary time correlation length greater than $\xi_\tau$ varies as $\sim \xi_\tau^{-\lambda}$, with $\lambda = c_1/c_2$. The exponent $\lambda$ depends on both $\Gamma$ and $\Gamma_c^{\text{loc}}$. For a given transverse field $\Gamma$ the clusters that are most likely to produce large correlations will be those with the $\Gamma_c^{\text{loc}}$ that minimizes this exponent; let us
call the minimum value $\lambda_{\text{min}}(\Gamma)$. It is this exponent $\lambda_{\text{min}}(\Gamma)$ that characterizes the tail of the distribution of strongly-coupled clusters that produce Griffiths singularity effects. In particular, we will be studying the distribution of the local nonlinear susceptibility, which is proportional to $\xi^3$. Thus the probability of a spin having a local nonlinear susceptibility larger than $x$ falls off as $Q(x) \sim x^{-\lambda_{\text{min}}(\Gamma)/3}$, for large $x$. We do observe such a power-law tail of the distribution in our simulations, with the exponent varying continuously with $\Gamma$, as expected.

Let us compare the dynamic Griffiths singularities for the $T = 0$ quantum random paramagnetic phase and the classical paramagnetic phase at $T > 0$. In both cases the probability of a locally-ordered cluster is exponential in the volume of the cluster as in (4). However, at $T > 0$ a cluster can flip by thermal activation as well as by quantum tunnelling. The free energy barrier for thermally-activated flipping is at most proportional to the cross-section of the cluster, so the correlation time grows no faster than $\exp(c_3 L^{d-1}/T)$, ($c_3$ is another constant) in contrast to the result (5) for $T = 0$. The large-time tail of the resulting distribution of local relaxation times decays faster than any power of time. Thus the power-law tails discussed above are unique to the $T = 0$ quantum case where the Griffiths clusters can flip only by quantum tunnelling.

III. MEASUREMENTS OF GRIFFITHS SINGULARITIES

In the disordered (paramagnetic) phase, Griffiths singularities arise as a result of clusters which have finite correlation in space and rare sets of stronger-than-average bonds; these Griffiths clusters are strongly correlated in imaginary time. In fact, one obtains a power law decay in the average imaginary time spin autocorrelation function above $\Gamma_c$, the power law provides a measure of the strength of the Griffiths singularities. The $T = 0$ linear susceptibility to a uniform magnetic field oriented along the Ising axis is proportional to the time-integral of the average spin-autocorrelation function. For the $(1+1)$-dimensional quantum Ising spin glass case, the uniform linear susceptibility is divergent for a range
of $\Gamma > \Gamma_c$ in the paramagnetic phase, due to strong Griffiths singularities. For $d = (2 + 1)$ the linear susceptibility is divergent at $\Gamma_c$ but may not be divergent for $\Gamma$ just above $\Gamma_c$ \cite{20, 23}. For the $(3 + 1)$-dimensional system, on the other hand, we find that this uniform linear susceptibility is finite even at the critical point \cite{21}. Thus we see that the linear susceptibility gets progressively less divergent with increasing $d$. Nevertheless, it remains of interest to see what is obtained in the case of the average non-linear susceptibility. It is the non-linear susceptibility that is directly related to spin-glass order and it is always divergent at $\Gamma_c$.

Since Griffiths singularities originate from local, strongly coupled clusters, we can study them by directly measuring various average local susceptibilities, obtained from the distribution functions of the single-site local magnetizations. In our simulations, we measure the single-site linear and nonlinear susceptibility, defined by

$$\chi_{i,l} = \frac{1}{L_{\tau}} \left[ \langle m_i^2 \rangle \right]_{av},$$

$$\chi_{i,nl} = \frac{1}{L_{\tau}} \left[ \langle m_i^4 \rangle - 3 \langle m_i^2 \rangle^2 \right]_{av},$$

respectively, with $m_i = \sum_{\tau=1}^{L_{\tau}} S_i(\tau)$.

The histograms (distributions) of both $\chi_{i,l}$ and $\chi_{i,nl}$, at various $\Gamma > \Gamma_c$ ($T^{cl} > T_c^{cl}$ in the equivalent classical system) are obtained for a sequence of of points in the paramagnetic phase for both the $(2+1)$- and the $(3+1)$-dimensional case, for a sequence of sizes, in which the distances are generally scaled in the same way as done for the study of the critical behavior \cite{21}, ($L_{\tau} \sim L^z$), though this is not essential and is not followed by Rieger and Young \cite{23}. The simulations done are performed on massively-parallel MasPar computers available at Princeton (with 4096 processors) and at AT&T Bell Laboratories (with 16384 processors), taking advantage of the fast nearest-neighbor message-passing connections among processors of the machines. In $d = 3$ case the number of independent samples simulated ranges from 1024 for $L = 4$ and $L_{\tau} = 6$, to 64 for $L = 16$ and $L_{\tau} = 38$. Since we are measuring local quantities on each site (there are $L^d$ sites per sample), the resulting histogram represents
quite a substantial amount of data even for the largest sizes where we have the fewest number of samples. Up to $2.5 \times 10^5$ Monte Carlo steps were taken for equilibration and measurement.

We have chosen to plot the integrated histogram, $Q(\chi_{i, nl})$, of the single-site nonlinear susceptibility, $\chi_{i, nl}$, on a double-logarithmic plot. Fig. 2 shows such a plot for the (3+1)-dimensional anisotropic classical spin model, at $T^{cl} = 4.8$, which is in the paramagnetic phase, well above its critical point ($T^{cl}_c \approx 4.3$). The plots are size dependent for small L, but converge for large sizes to an asymptotic limiting distribution with a power-law tail, as expected. Similar power-law behavior is obtained for the single-site linear susceptibility as well [25], with different, but related exponents. If the probability distribution, $P(\chi)$, of the nonlinear susceptibility has a power law tail with $P(\chi) \sim \chi^{-s-1}$, then its integral $Q(x) = \int_x^\infty dy P(y)$ behaves as $Q(x) \sim x^{-s}$ for large $x$, with a slope $-s$ in our figures, where we plot $Q(\chi_{i, nl})$ vs. $\chi_{i, nl}$ on logarithmic scales.

![FIG. 2](image)

FIG. 2. Log-log plot of the integrated histogram of the single-site nonlinear susceptibility $\chi_{i, nl}$ for the (3+1)-dimensional system at a simulation temperature $T^{cl} = 4.8$. The straight line is parallel to a least-square fit to the linear region in the plot for the largest samples, and has a slope -2.6.

The slope $s$ of the large $\chi_{i, nl}$ tail of the distribution determines whether the average local
nonlinear susceptibility diverges or not. The average local nonlinear susceptibility is

\[ \chi_{\text{loc},nl} = \int \chi_{i,nl} P(\chi_{i,nl}) d\chi_{i,nl} \]

\[ = - \int \chi_{i,nl} \frac{dQ(\chi_{i,nl})}{d\chi_{i,nl}} d\chi_{i,nl} \]  

(8)

which diverges when \( s \leq 1 \). Clearly, the average local nonlinear susceptibility is not divergent at \( T_{cl} = 4.8 \), where \( s \approx 2.6 \).

FIG. 3. As in Figure 2, but for a temperature \( T_{cl} = 4.5 \). Note the decrease in slope \( s \) with increasing sample size. The slope approaches close to the critical value \( s = 1 \) where the average local nonlinear susceptibility diverges.
FIG. 4. As in Figures 2 and 3, but now $T_{cl} = 4.4$. Note that the slope has not converged for the largest sizes, suggesting that $s < 1$ as $L$ and $L_\tau \to \infty$.

We find that the large size asymptotic slope $s$ decreases continuously as $T_{cl}$ is decreased, and Fig. 3 and 4 plot the integrated histogram at $T_{cl} = 4.5$ and $T_{cl} = 4.4$ respectively. As can be seen, it takes larger and larger size samples for the slope to converge as $T_{cl}$ is approached, and the data at $T_{cl} = 4.4$ have not yet converged for our largest size. Nevertheless, because the apparent slope decreases monotonically with increasing sample size, we are able to put an upper bound on its magnitude, and can conclude from Fig. 4 that the average local nonlinear susceptibility appears to diverge somewhat above $T_{cl} = 4.4$, which is close to, but above our estimated $T_{cl} = 4.32 \pm 0.03$. The same conclusion may be inferred from a plot of the slope $s$ as a function of temperature $T_{cl}$, as shown in Fig. 5.
FIG. 5. Plot of the slope of the integrated histogram of the average local nonlinear susceptibility as a function of temperature $T_{cl}$ for the (3+1)-dimensional quantum Ising spin glass. The data suggest that the slope drops below 1 in the paramagnetic phase just above the critical point $T_{cl}$ shown.

As the effects of the Griffiths singularities lead to a convergent average local nonlinear susceptibility for most of the paramagnetic phase in the (3+1)-dimensional case, whereas strong effects were obtained in (1+1)-dimension [18], we have also simulated the quantum Ising spin glass in a transverse field in $d = 2$, the (2+1)-dimensional case. In Fig. 6, we show our results for two temperatures $T_{cl} = 3.6$ and $T_{cl} = 3.5$ in the paramagnetic phase (here $T_{c}^{cl} = 3.275 \pm 0.025$) [20]. In this case, the data clearly show a slope less than the critical value $s = 1$ at the lower temperature, signalling a divergent average local nonlinear susceptibility at a temperature almost 8% above $T_{c}^{cl}$. Our data are consistent with the more detailed data of Rieger and Young [23] in the adjoining paper (though we go to somewhat larger sizes), within statistical errors. If we make a linear interpolation of the slope between the two temperatures shown, we obtain the temperature for the onset of a divergent nonlinear susceptibility as $T_{nl}^{cl} = 3.55$ which compares well with their estimate, $T_{nl}^{cl} = 3.56$. 


FIG. 6. Log-log plot of the integrated histogram of the single-site nonlinear susceptibility $\chi_{i,nl}$ for the (2+1)-dimensional quantum Ising spin glass, for two simulation temperatures $T_{cl} = 3.5$ (a) and $T_{cl} = 3.6$ (b) in the paramagnetic phase. The average nonlinear susceptibility diverges for slope $s \leq 1$, as happens at the lower temperature in (a).
IV. DISCUSSION AND CONCLUDING REMARKS

We have performed extensive Monte Carlo simulations of the quantum Ising spin glass in a transverse field in dimensions $d = 2$ and $d = 3$ in the paramagnetic phase, to measure the effects of Griffiths singularities. The simulations are performed on a classical anisotropic $(d + 1)$-dimensional model, which approximates the path integral of the quantum system, with spin-glass interactions in the $d$ space dimensions and ferromagnetic interactions along the imaginary time dimension. We find clear evidence for Griffiths singularities in both dimensions for a range of transverse fields (represented by the simulation temperatures in the classical system) in the paramagnetic phase. In particular, we find that the distribution of local nonlinear (and linear) susceptibilities has a long power-law tail, $P(\chi) \sim \chi^{-s-1}$ for large enough sizes, and this power-law exponent $s$ decreases continuously as the spin-glass transition is approached. This leads to a divergent average local nonlinear susceptibility when $s \leq 1$, over a range of the paramagnetic phase just above the critical point, as in the one-dimensional case. However, the range where this divergence occurs decreases monotonically with increasing dimensionality, being about 8% of $T_{cl}^d$ for $d = 2$, and about 2% of $T_{cl}^d$ for $d = 3$. Nevertheless, the effects are clearer to see than in classical systems, because the Griffiths clusters occur as line defects in the path-integral representation of quantum systems, as opposed to point defects in purely classical models.

Here we have focussed on the effects of rare Griffiths clusters in the paramagnetic phase. One can also ask about rare clusters at the critical point: which, if any, of the divergences at the critical point are due only to rare clusters? How do the Griffiths singularities in the paramagnetic phase match onto the scaling behavior at the critical point? For $d > (1 + 1)$ we know very little about these issues. Rieger and Young [23] show some data for the distribution of the local linear susceptibility at the critical point for $d = (2 + 1)$. There is a long tail and it appears possible that the typical (i.e., median) susceptibility is not divergent. If this is correct, then the divergence of the linear susceptibility is solely due to rare regions. Another related question is the universality of the Griffiths singularities. We have found for
our model for $d = (3 + 1)$ that the nonlinear susceptibility does diverge in the paramagnetic phase. Will this be true for all three-dimensional quantum Ising spin-glass systems or could another model with the same critical behavior not show this divergence? Some discussion of this question is in Thill and Huse [19], but further work on this issue is needed to provide an answer.

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