Darboux Transformations for SUSY Integrable Systems

Q. P. Liu*  ** and Manuel Mañas***

Departamento de Física Teórica,
Universidad Complutense,
E28040-Madrid, Spain.

Abstract. Several types of Darboux transformations for supersymmetric integrable systems such as the Manin-Radul KdV, Mathieu KdV and SUSY sine-Gordon equations are considered. We also present solutions such as supersolitons and superkinks.

* On leave of absence from Beijing Graduate School, CUMT, Beijing 100083, China
** Supported by Becas para estancias temporales de doctores y tecnólogos extranjeros en España: SB95-A01722297
*** Partially supported by CICYT: proyecto PB95-0401
1 Introduction

Supersymmetric integrable systems constitute a subject of current interest, and as a consequence a number of well known integrable equations have been generalized into the supersymmetric (SUSY) context. We just mention the SUSY versions of sine-Gordon (Di Vecchia and Ferrara(1977), Chaichain and Kulish (1978), Ferrara et al (1978)), Nonlinear Schrödinger (Roelofs and Kersten(1992)), KP (Manin and Radul (1985), Rabin (1991), Mulase (1991)), KdV (Manin and Radul (1985), Mathieu (1988b)) and Boussinesq (Yung (1993), Bellucci et al (1993)). We also point out that there are two different generalizations, namely nonextended ($N = 1$, $N$ being the number of fermionic independent variables) and extended ($N \geq 2$) generalizations. Here we are mainly interested in the former case. So far many of the tools used in the standard theory have been extended to this framework, such as Bäcklund transformations (Chaichain and Kulish (1978)), prolongation theory (Roelofs and Kersten(1992)), Hamiltonian formalism (Oevel and Popowicz (1991), Morosi and Pizzocchero (1993)), Grassmannian description (Ueno et al (1989)), tau function (Ueno et al (1989), Martínez Alonso and Medina (1995)), the relationship with super W algebras (Mathieu (1988a), Bilal and Gervais (1988)), additional symmetries (Das et al (1992), Mañas et al (1994), Stanciu (1994)), etc..

However, there is one that only recently has been considered, we are talking about the so called Darboux transformation, that constitutes a very successful tool in the realm of integrable systems whenever one is interested in the constructions of solutions. The roots of these techniques go back to geometrical studies of the last century. It was in Moutard (1878), see also Athorne and Nimmo (1990), where the two dimensional Schrödinger equation (as we called it today) was considered giving new wave functions and potentials from given ones. This was taken by Darboux (Darboux (1882)) and applied to the one-dimensional case. In fact these results are connected with the theory of conjugate nets (Darboux (1896), Eisenhart (1909)) and it was in Levy (1886) where a transformation of this type was applied for surfaces, and was iterated in Hammond (1920). New transformations for conjugate nets appeared in Jonas (1915) which were called fundamental in Eisenhart (1923) containing the Levy ones in appropriate limits.

In Crum (1955), independently of these geometrical studies, it was presented, for the Schrödinger equation, the iteration of the transformation found by Darboux, giving compact expressions in terms of Wronski determinants. Later on, in Wadati et al. (1975), a new transformation for soliton equations was introduced. This tool was rapidly developed and it was in Matveev (1979) where they were named as Darboux transformations. This name is standard nowadays in the soliton community, however we have seen that is not completely appropriate. In Levi (1988) a further extension of the Darboux transformation was given and some people refer to it as binary or Darboux-Levi transformation, however this is just the fundamental type transformation mentioned above. Finally, we remark that recently (Guil and Mañas (1996), Mañas (1997)) a vectorial formulation of the binary Darboux transformation was given, allowing compact formulae for
Darboux Transformations for SUSY Integrable Systems

The paper Liu (1995) was the first one that considered Darboux transformations for the SUSY KdV system. Later on (Liu and Mañas (1997a), Liu and Mañas (1997b)) extensions of the binary Darboux transformation (fundamental transformation in Geometry) and Darboux transformation appeared. In this paper we want to present the SUSY version of the Darboux transformations. In particular we will consider three important supersymmetric integrable systems, namely the Manin-Radul KdV and its reduction to the Mathieu KdV, and also the SUSY extension of the sine-Gordon equation. For the Manin-Radul KdV and vectorial binary Darboux transformations we improve the presentation of Liu and Mañas (1997a) giving the general transformation for wave functions and a permutability theorem. For the Mathieu KdV equation we present some technical improvements with respect to Liu (1995) and the part regarding the sine-Gordon equation is entirely new. Finally, let us remark that given the character of these proceedings we are not going to give any proof.

The layout of the paper is as follows. We start with the Manin-Radul KdV (MRKdV) by considering the vectorial binary Darboux transformations and the construction of solutions in terms of ordinary determinants of Grammian type, we also give the iteration of the Darboux transformation found in Liu (1995) to get Wronski superdeterminantal expressions for the solutions, here we present a genuine supersoliton. In §3 we study the application of the Darboux transformation for the Mathieu KdV to the SUSY sine-Gordon equation, presenting a superkink.

2 Darboux Transformations for the Manin-Radul KdV Equation

The MRKdV system is defined in terms of three independent variables \( \vartheta, x, t \), where \( \vartheta \in \mathbb{C}_a \) is an odd supernumber, and \( x, t \in \mathbb{C}_c \) are even supernumbers, and two dependent variables \( \alpha(\vartheta, x, t), u(\vartheta, x, t) \), where \( \alpha \) is an odd function taking values in \( \mathbb{C}_a \) and \( u \) is even function with values in \( \mathbb{C}_c \). A basic ingredient is a superderivation defined by \( D := \partial_\vartheta + \partial_\vartheta \partial_x \). The system is

\[
\begin{align*}
\alpha_t &= \frac{1}{4} (\alpha_{xxx} + 3(\alpha D\alpha)_x + 6(\alpha u)_x), \\
u_t &= \frac{1}{4} (u_{xxx} + 6u u_x + 3\alpha_x D u + 3\alpha(D u_x)),
\end{align*}
\]

where we use the notation \( f_x := \partial f / \partial x \) and \( f_t := \partial f / \partial t \).

The following linear system for the wave function \( \psi(\vartheta, x, t) \), that takes values in the Grassmann algebra \( \mathbb{A} = \mathbb{C}_c \oplus \mathbb{C}_n \),

\[
\begin{align*}
L(\psi) &= \psi_{xx} + \alpha D \psi + u \psi = \lambda \psi, \\
\psi_t &= M(\psi) := \frac{1}{2} \alpha (D \psi)_x + \lambda \psi_x + \frac{1}{2} u \psi_x - \frac{1}{4} \alpha x D \psi - \frac{1}{4} u x \psi,
\end{align*}
\]
where the spectral parameter $\lambda \in \mathbb{C}_c$ is an even supernumber, has as its compatibility condition Eqs. (1), and therefore it can be considered as a Lax pair for (1).

### 2.1 Vectorial Binary Darboux Transformations

The linear system (2) is of a scalar nature, $\lambda \in \mathbb{C}_c$, $\psi(\vartheta,x,t) \in A$. Nevertheless, it is possible to give a vector extension of these linear problem. Indeed, we may replace $A$ by an arbitrary linear Grassmann space $\mathcal{E}$ over $A$ and take $b$ as an $\mathcal{E}$-valued eigenfunction, then the spectral parameter can be taken as $\ell \in L(\mathcal{E}_2) \oplus L(\mathcal{E}_\infty)$, an even operator.

Namely, the linear system

$$
\psi_{xx} + \alpha D\psi + u\psi - \ell \psi = 0,
$$

$$
\psi_t - \frac{1}{2} \alpha (D\psi_x) - \ell \psi_x - \frac{1}{2} u\psi_x + \frac{1}{4} \alpha_x D\psi + \frac{1}{4} u_x \psi = 0,
$$

has as its compatibility condition the MRKdV system (1).

Notice that Eqs. (1) are also the compatibility condition of adjoint linear system:

$$
\phi_{xx} + D(\alpha \phi) + u\phi - \phi m = 0,
$$

$$
\phi_t + \frac{1}{2} \alpha D\phi_x - \phi_x m - \frac{1}{2} (u + D\alpha) \phi_x + \frac{1}{4} D(\alpha_x \phi) + \frac{1}{4} u_x \phi = 0,
$$

where $\phi(\vartheta,x,t) \in \tilde{\mathcal{E}}^*$ is a linear function on the supervector space $\tilde{\mathcal{E}}$, and $m \in L(\mathcal{E}_0) \oplus L(\mathcal{E}_1)$.

In order to construct Darboux transformations for these linear systems we need to introduce an operator, say $V[\psi, \phi] \in L(\mathcal{E}, \tilde{\mathcal{E}})$, bilinear in $\psi$ and $\phi$, defined by the compatible equations

$$
DV[\psi, \phi] = \psi \otimes \phi,
$$

$$
V[\psi, \phi]_t = \ell V[\psi, \phi]_x + V[\psi, \phi]_x m
$$

$$
- D(\psi_x \otimes \phi_x + \frac{1}{2} uDV[\psi, \phi]) - \frac{1}{4} \alpha_x DV[\psi, \phi]
$$

$$
- \frac{1}{2} (D\psi) \otimes ((D\alpha) \phi - \alpha (D\psi)) + \frac{1}{2} \alpha (\psi \otimes \phi_x - \psi_x \otimes \phi)
$$

such that

$$
\ell V[\psi, \phi] - V[\psi, \phi] m = D(\psi_x \otimes \phi - \psi \otimes \phi_x) - \alpha \psi \otimes \phi. \quad (6)
$$

Now we are ready to present the following:

**Theorem 1** Let $\psi_0(\vartheta, x, t) \in \mathcal{V}_0$ be an even vector satisfying Eq. (3) with spectral parameter $\ell_0$, $\phi_0(\vartheta, x, t) \in \mathcal{V}_0^*$ an odd functional solving Eq. (4) with spectral parameter $m_0$ and $V[\psi_0, \phi_0] \in L(\mathcal{V}_0) \oplus L(\mathcal{V}_\infty)$ a non singular even operator,
det \[ V[\psi_0, \phi_0] \text{body} \neq 0 \], defined in terms of the compatible Eqs. (5) and (6). Then, the objects

\[
\hat{\psi} := \psi - V[\psi, \phi_0]V[\psi_0, \phi_0]^{-1}\psi_0, \\
\hat{\phi} := \phi - \phi_0 V[\psi_0, \phi_0]^{-1}V[\psi_0, \phi], \\
\hat{\alpha} = \alpha - 2D^3 \ln \det V[\psi_0, \phi_0], \\
\hat{u} = u + 2\hat{\alpha} D\ln \det V[\psi_0, \phi_0] + 2 \left( \sum_j D(\psi_0)_j \frac{\det V[\psi_0, \phi_0]_j}{\det V[\psi_0, \phi_0]} \right)_x,
\]

where \( V[\psi_0, \phi_0]_j \) is an operator with associated supermatrix obtained from the corresponding one of \( V[\psi_0, \phi_0] \) by replacing the \( j \)-th column by \( \psi_0 \), satisfy the Eqs. (3) and (4) whenever the unhatted variables do. Thus, \( \hat{\alpha} \) and \( \hat{u} \) are new solutions of the MRKdV (1). Moreover,

\[
V[\hat{\psi}, \hat{\phi}] = V[\psi, \phi] - V[\psi, \phi_0]V[\psi_0, \phi_0]^{-1}V[\psi_0, \phi]. \tag{7}
\]

Let us remark that this theorem extends Theorem 1 in our paper (Liu and Mañas (1997a)). In particular we stress the role of the general wave functions \( \psi \) and \( \phi \), and also the formula (7) that gives the path for iteration and it is also deeply connected, in the non SUSY case, with geometrical objects such as points of the transformed manifolds.

We shall call \((V, \psi_0, \phi_0)\) as transformation data. The composition of two vectorial Darboux transformations yield a new Darboux transformation, and as it is shown in next proposition they commute as they can be expressed as a vectorial Darboux transformation:

**Proposition 1** The vectorial Darboux transformation with transformation data \((V_1 \oplus V_2, (\psi_{0,(1)}, \psi_{0,(2)}), (\phi_{0,(1)}, \phi_{0,(2)})\)) coincides with the following composition of Darboux transformations:

1. First transform with data \((V_2, \psi_{0,(2)}, \phi_{0,(2)})\), and denote the transformation by ‘\('.
2. On the result of this transformation apply a second one with data \((V_\infty, \psi'_{r,(\infty)}, \phi'_{r,(\infty)})\).

A similar theorem in a completely different framework, namely discrete integrable systems: multidimensional quadrilateral lattices, appears in Doliwa et al (1997).

### 2.2 Wronski Superdeterminants Representation of Iterated Darboux Transformations

A Darboux transformation for the MRKdV equation is (Liu (1995))
Proposition 2 Let $\psi$ be a solution of (2) and $\theta_0$ be a particular solution with $\lambda = \lambda_0$. Then, the quantities defined by

$$
\hat{\psi} := (D + \delta_0)\psi, \quad \delta_0 := -\frac{D\theta_0}{\theta_0}, \quad (\theta_0 : \text{even})
$$

$$
\hat{\alpha} := -\alpha - 2\partial\delta_0,
\hat{u} := u + (D\alpha) + 2\delta_0(\alpha + \partial\delta_0)
$$

satisfy

$$
\hat{L}\hat{\psi} = \lambda\hat{\psi},
\partial_t\hat{\psi} = \hat{M}\hat{\psi},
$$

where $\hat{L}$ and $\hat{M}$ are obtained from $L$ and $M$ by replacing $\alpha$ and $u$ with $\hat{\alpha}$ and $\hat{u}$, respectively.

As a consequence of this Proposition we conclude that $\hat{u}$ and $\hat{\alpha}$ are new solutions of the MRKdV system (1). We remark that, as usual, the Darboux transformation can be viewed as a gauge transformation:

$$
\psi \rightarrow T_0\psi,
L \rightarrow \hat{L} = T_0LT_0^{-1},
M \rightarrow \hat{M} = \partial_tT_0 \cdot T_0^{-1} + T_0MT_0^{-1},
T_0 := D + \delta_0.
$$

To construct Crum type transformation, let us start with $n$ solutions $\theta_i$, $i = 0, ..., n - 1$, of equation (2) with eigenvalues as $\lambda = k_i$, $i = 0, ..., n - 1$. To make sense, we choose the $\theta_i$ in such way that its index indicates its parity: those with even indices are even and with odd indices are odd variables. We use $\theta_0$ to do our first step transformation and then $\theta_i, i = 1, \ldots, n - 1$, are transformed to new solutions $\theta_i[1]$ of the transformed linear equation and $\theta_0$ goes to zero. Next step can be effected by using $\theta_1[1]$ to form a Darboux operator and at this time $\theta_1[1]$ is lost. We can continue this iteration process until all the seeds are mapped to zero. In this way, we have

Proposition 3 Let $\theta_i, i = 0, \ldots, n - 1$, be solutions of the linear system (2) with $\lambda = k_i, i = 0, \ldots, n - 1$, and parities $p(\theta_i) = (-1)^i$, then after $n$ iterations of the Darboux transformation of Proposition 1, one obtains a new Lax operator

$$
\hat{L} = T_nLT_n^{-1}, \quad T_n = D^n + \sum_{i=0}^{n-1} a_i D^i,
$$

where the coefficients $a_i$ of the gauge operator $T_n$ are defined by

$$
(D^n + \sum_{i=0}^{n-1} a_i D^i)\theta_j = 0, \quad j = 0, \ldots, n - 1. \quad (8)
$$
The explicit form of the transformed field variables is given by $\hat{L} = T_nLT_n^{-1}$ from where it follows that the new fields $\hat{\alpha}$ and $\hat{u}$ can be written as

$$\hat{\alpha} = (-1)^n \alpha - 2\partial a_{n-1},$$

$$\hat{u} = u - 2\partial a_{n-2} - a_{n-1}((-1)^n \alpha + \hat{\alpha}) + \frac{1 - (-1)^n}{2} D\alpha.$$

Now, we must recall the reader that the Berezinian or superdeterminant of an even matrix, say $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, is

$$\text{sdet} M = \frac{\det (A - BD^{-\infty}C)}{\det D} = \frac{\det A}{\det (D - CA^{-\infty}B)}.$$

To obtain the explicit expressions, we have to find out the $a_{2k-2}$ and $a_{2k-1}$ by solving the linear system (8). In the even case $n = 2k$ which is most interesting, we have

$$a_{2k-2} = -\frac{\text{sdet} \hat{W}}{\text{sdet} W}, \quad a_{2k-1} = D \ln \text{sdet} W,$$

where

$$a^{(0)} = (a_0, a_2, \ldots, a_{2k-2}), \quad a^{(1)} = (a_1, a_3, \ldots, a_{2k-1}),$$

$$\theta^{(0)} = (\theta_0, \theta_2, \ldots, \theta_{2k-2}), \quad \theta^{(1)} = (\theta_1, \theta_3, \ldots, \theta_{2k-1}),$$

$$b^{(i)} = \partial^k \theta^{(i)}, \quad W^{(i)} = \begin{pmatrix} \theta^{(i)} \\ \partial \theta^{(i)} \\ \vdots \\ \partial^{k-1} \theta^{(i)} \end{pmatrix}, \quad i = 0, 1,$$

$$W := \begin{pmatrix} W^{(0)} & W^{(1)} \\ DW^{(0)} & DW^{(1)} \end{pmatrix}, \quad \hat{W} := \begin{pmatrix} \hat{W}^{(0)} & \hat{W}^{(1)} \\ D\hat{W}^{(0)} & D\hat{W}^{(1)} \end{pmatrix}$$

and $\hat{W}^{(0)}$ and $\hat{W}^{(1)}$ are obtained from the matrices $W^{(0)}$ and $W^{(1)}$ by replacing the last rows with $b^{(0)}$ and $b^{(1)}$, respectively. It should be noticed that the supermatrix $W$ is even and has a Wronski type structure.

Summarizing the above results, we now have the following

**Theorem 2** Let $\alpha, u$ be a solution of (1) and $\{\theta_j\}_{j=0}^{n-1}$ be a set of $n(= 2k)$ solutions of the associated linear system (2), such that the parity is $p(\theta_j) = (-1)^j$. Then, we have new solutions $\hat{\alpha}, \hat{u}$ of (1) given by

$$\hat{\alpha} = \alpha - 2D^3 \ln \text{sdet} W,$$

$$\hat{u} = u + 2\partial \left(\frac{\text{sdet} \hat{W}}{\text{sdet} W}\right) + (\alpha + \hat{\alpha}) D \ln \text{sdet} W.$$
Let us remark that our Darboux transformations are useful even outside the MRKdV system. Indeed, the most obvious application is to the SUSY KP equation (Ueno et al (1989)) which is a closed system obtained from supersymmetric KP hierarchy. Since the Lax pair is essentially the one we had for MRKdV, our Darboux transformations can be used directly in this context.

Solutions of the MRKdV are found rarely, see for example Radul (1988), Ibort et al (1996). In Liu and Mañas (1997a) and Liu and Mañas (1997b) one can find examples of solutions, in particular in Liu and Mañas (1997b) we gave an interesting solution which behaves like a genuine supersoliton.

Our solution is

\[ \hat{\alpha} = -2\partial a_1, \quad \hat{u} = -2\partial a_0. \]

with

\[
a_0 = f - k(\gamma_+^{(1)} + \gamma_-^{(0)} - \gamma_-^{(0)} \gamma_+^{(1)})g - \partial k(c_+^{(0)} \gamma_-^{(0)} - c_-^{(0)} \gamma_+^{(1)})fg,
\]

\[
a_1 = (k(c_+^{(0)} - c_-^{(0)} \gamma_+^{(1)}) + \partial (c_+^{(0)} \gamma_-^{(0)} - c_-^{(0)} \gamma_+^{(0)}))g,
\]

where

\[
f := -k \left( \frac{c_+^{(0)} \exp(\eta) - c_-^{(0)} \exp(-\eta)}{c_+^{(0)} \exp(\eta) + c_-^{(0)} \exp(-\eta)} \right),
\]

\[
g := \frac{2}{(c_+^{(0)} \exp(\eta) + c_-^{(0)} \exp(-\eta)) \left( \gamma_+^{(0)} \exp(\eta) + \gamma_-^{(0)} \exp(-\eta) \right)}.\]

here \( \eta = kx + k^3t, k \in \mathbb{C} \) and \( c_\pm^{(i)}, \gamma_\pm^{(i)} \) are supernumbers with parities indicated by the superfix.

Notice that our solution can be understood as a supersoliton which has the KdV soliton, \(-2\partial f\), as its body, and that the choice \( c_+^{(0)} = \gamma_+^{(0)} \) and \( c_-^{(0)} = \gamma_-^{(0)} \) gives the solution found in Ibort et al (1996).
In the figures we plot, in the real $x$-$t$ plane, the functions $f_x, g_x$ and $(fg)_x$ that appear in the construction of the solution corresponding to the data: $k = 1.1$, $c_+^{(0)} = 1$, $c_-^{(0)} = 1.5$, $\gamma_+^{(0)} = 1.2$ and $\gamma_+^{(0)} = 2$.

3 Darboux Transformations for the Mathieu KdV and SUSY Sine-Gordon Equations

The Mathieu KdV equation reads (Manin and Radul (1985), Mathieu (1988b))

$$\alpha_t = \frac{1}{4}(\alpha_{xx} + 3\alpha D\alpha)_x,$$

which is obtained from (1) by setting $u = 0$, being the Lax operator

$$L = \partial^2 + \alpha D.$$

In Liu (1995) one of the authors presented a preliminary version of

**Proposition 4** If

$$\psi_{xx} + \alpha D\psi = \lambda \psi,$$

and $\psi_0$ is an even solution of above equation with $\lambda = \lambda_0$ such that the constants

$$J(\psi_0) = \psi_0^2 + 2(D\psi_0)x D\psi_0 - \lambda_0 \psi_0^2,$$

$$I(\psi_0, \psi) = (\lambda - \lambda_0) (D^{-1}((D\psi_0)\psi)) + \lambda_0 \psi_0 \psi - \psi_{0,x} \psi_x$$

vanish, then

$$\hat{\psi} := \psi_0^{-1} D^{-1}(\psi_0 D\psi - (D\psi_0)\psi), \quad \hat{\alpha} := \alpha - 4D^3 \ln \psi_0.$$
satisfy
\[ \hat{\psi}_{xx} + \hat{\alpha} D \hat{\psi} = \lambda \hat{\psi}, \]

Compared with the result of Liu (1995), the above Proposition is an improved version in the sense that the constant \( I(\psi_0, \psi) \) has a much simpler structure. An application of this Darboux transformation is to the supersymmetric sine-Gordon system, which reads (Inami and Kanno (1991))

\[ DD_t \Phi = 2 \cosh(2\Phi) \tag{10} \]

where \( D_t = \frac{\partial}{\partial t} + \theta_t \frac{\partial}{\partial t} \) and \( \theta_t \) is another Grassmann odd variable.

According to Inami and Kanno (1991), the linear problems are

\[ D\psi_1 + \lambda\psi_3 + \psi_4 = 0, \]
\[ D\psi_2 - \lambda\psi_3 + \psi_4 = 0, \]
\[ D\psi_3 + 2(D\Phi)\psi_3 + \psi_1 + \psi_2 = 0, \]
\[ D\psi_4 - 2(D\Phi)\psi_4 + \lambda(\psi_1 - \psi_2) = 0, \]

and

\[ D_t\psi_1 + \exp(2\Phi)\psi_3 - \lambda^{-1} \exp(-2\Phi)\psi_4 = 0, \]
\[ D_t\psi_2 + \exp(2\Phi)\psi_3 + \lambda^{-1} \exp(-2\Phi)\psi_4 = 0, \]
\[ D_t\psi_3 + \lambda^{-1} \exp(-2\Phi)(\psi_1 - \psi_2) = 0, \]
\[ D_t\psi_4 - \lambda^{-1} \exp(2\Phi)(\psi_1 + \psi_2) = 0. \]

Introducing \( \tilde{\psi}_1 := \psi_1 - \psi_2 \) and \( \tilde{\psi}_2 := \psi_1 + \psi_2 \), we easily see that the above linear system can be written as follows:

\[ \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}_x = 2 \begin{pmatrix} -(D\Phi)D & \lambda \\ \lambda & (D\Phi)D \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} \tag{11} \]

and

\[ D_t \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\lambda^{-1} \exp(-2\Phi)D \\ \lambda^{-1} \exp(2\Phi)D & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}. \tag{12} \]

Differentiating (11) we get

\[ \tilde{\psi}_{1,xx} = -\alpha_1 D\tilde{\psi}_1 + 4\lambda^2 \tilde{\psi}_1, \]
\[ \tilde{\psi}_{2,xx} = \alpha_2 D\tilde{\psi}_2 + 4\lambda^2 \tilde{\psi}_2 \]

where \( \alpha_1 := \gamma_x - \gamma D\gamma \) and \( \alpha_2 := -\gamma_x - \gamma D\gamma \) with \( \gamma = 2D\Phi \).

Now a slight modification of the previous Darboux transformation yields
Theorem 3 If $\Phi$ is a solution of the SUSY sine-Gordon equation (10) and $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are particular solutions of (11) and (12) with spectral parameter $\lambda = \lambda_0$, such that $J(\tilde{\psi}_1) = J(\tilde{\psi}_2) = 0$, then

$$\hat{\Phi} = \Phi + \ln \frac{\tilde{\psi}_1}{\tilde{\psi}_2},$$

is a new solution of (10).

Remarks

1. The method we used is a generalization of the one by Wadati et al (1975) for the classical sine-Gordon equation. We also notice that this idea was used in Nimmo (1993) for the two-dimensional sine-Gordon equation of Konopelchenko and Rogers.
2. We may also obtain Darboux type transformations for the SUSY modified KdV equation (Mathieu (1988b)).
3. We conjecture that Crum type iteration of the above Darboux transformation will be represented in terms of Pfaffians instead of Wronskians.

The application of this theorem to the most simple case, namely $\Phi = i \pi / 4$ yields the following interesting SUSY extension of the kink solution. Namely, we have the following solution

$$\Phi = \frac{i \pi}{4} + \ln \frac{1 + A \exp(-4\eta)}{1 - A \exp(-4\eta)}$$

where

$$A(\theta, \theta_t) := A_0(1 - 2i \theta \theta_t) + A_1(\theta + i \lambda^{-1} \theta_t) + c^{-1}A_0(1 - 2i \theta \theta_t) + cA_1(\theta - i \lambda^{-1} \theta_t),$$

$$\eta(x, t) := \lambda x + \lambda^{-1} t,$$

with $A_0$ and $c$ even supernumbers with nonvanishing body, $\lambda \in \mathbb{C}$ and $A_1$ an odd supernumber.

This solution can be thought as a superkink, in fact its body is

$$\Phi_{\text{body}} = \frac{i \pi}{4} + \ln \frac{1 + c_{\text{body}} \exp(-4\eta)}{1 - c_{\text{body}} \exp(-4\eta)},$$

that after taking the appropriate Wick rotation goes into the standard kink solution of the sine-Gordon equation. Obviously, the soul of this solution is far from trivial.

Acknowledgement. The present paper is the extended version of the talk presented at the meeting by one of us (MM). A number of the results were obtained during or after the meeting. MM would like to thank the organizers for the hospitality and coverage of local expenses; and also Bolsa de Viaje Complutense 1997.
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