The M five brane on a torus

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Abstract. The D-3 brane is examined from the point of view of the wrapped M-theory five brane on a torus. In particular, the S-dual versions of the 3-brane are identified as coming from the different gauge choices of the auxiliary field that is introduced in the PST description of the five brane world volume theory.

1 Introduction

The world volume of the M-theory five brane supports a self dual two form potential. There are well known difficulties in writing down an action formulation for such a field. In certain circumstances however it is possible to write down a covariant action for such a field. The self duality condition is realized through the introduction of an auxiliary field [1, 2, 3]. The role of this field is to introduce a new gauge symmetry that makes half the degrees of freedom associated with the two form to be pure gauge. Gauge fixing then eliminates the unwanted anti self dual degrees of freedom.

There is a conjectured SL(2,Z) duality in IIB string theory [4]. Under the SL(2,Z) transformation, the Neveu Schwarz and Ramond two forms transform as a doublet while the axion-dilaton undergoes a fractional linear transformation. The self dual Ramond Ramond four form however is inert under this transformation. This is the potential that couples to the D3 brane. Thus a D3 brane is mapped to itself. The world volume theory of the D3 brane is a Dirac Born-Infeld theory. Such a theory has been shown to exhibit SL(2,Z) duality properties. As (in the Einstein frame) the complex coupling on the three brane world volume is the string coupling (complexified with the Ramond Ramond scalar) the duality properties of the 3-brane world volume theory may be seen as been induced from the duality properties of the IIB string theory.

IIB string theory reduced on a circle is T-dual to M theory on a torus [5, 6]. This is most easily seen at the level of the supergravity actions. In nine dimensions one may identify the D=11 SUGRA reduced on a torus with the D=10 IIB SUGRA reduced on a circle [7]. The SL(2,Z) symmetry of the IIB theory then becomes associated with the modular group of the torus.

In order to complete the identification one must determine how the branes of the theory become identified. The D3 brane not wrapped on the circle will be identified with the five brane wrapped on the torus. The structure of the paper will be as follows. First we will describe the five brane world volume action and its reduction on a torus. As part of the reduction we will discuss the gauge fixing the PST symmetry. The D3 brane and its reduction will then be described.
in order to identify the two theories it will be necessary to dualize the scalar field on the three brane associated with transverse oscillations in the direction of the compact dimension.

The moduli and fields of the theories will then be identified and shown to be in agreement with supergravity and string/membrane calculations. In particular the role of the self dual field and its realization in the PST formulation will be examined with respect to the SL(2,Z) transformation of the D3 brane.

2 The M-theory five brane

We work with a flat Minkowski background, using a metric, $\eta = \text{diag}(-1, +1, +1, ..)$. The $X^M$ are 11-dimensional space-time coordinates ($M, N = 0..9, 11$). $\sigma^\alpha$ are the coordinates of the brane. The action will also contain a world volume self dual two form gauge field, $B$ whose field strength is as usual given by $H = dB$.

The action for the 5-brane will be written as follows, [12, 3]:

$$S = -\int_{M^6} d^6\sigma \sqrt{-\det(G_{\mu\nu} + i \frac{\hat{H}_{\mu\nu}}{\sqrt{v^\rho v_\rho}})} - \frac{\sqrt{-G} \hat{H}_{\mu\rho} H^\rho_{\mu\rho} v^\rho}{4v^2}$$  (1)

where:

$$\hat{H}_{\mu\nu} = \frac{1}{6} G_{\mu\delta\gamma} G_{\nu\beta\gamma} \epsilon^{\delta\beta\gamma\rho \delta} H_{\delta\rho} v^\delta$$  (2)

and

$$G_{\mu\nu} = dX^M dX^N \eta_{MN}$$  (3)

$G = \det G_{\mu\nu}$; $v$ is a completely auxiliary closed one form field introduced to allow the self-duality condition to be imposed in the action while maintaining Lorentz invariance. Usually the action 1 is written with $v = da$; however this is only locally correct as $v$ is constrained to be closed but not necessarily exact. See the references [2] for a discussion on this Lorentz invariant formulation. Apart from the usual gauge symmetries associated with the gauge potential $B$ and the background field $C$, this action has additional local, so called PST symmetries one of which we will use later to eliminate half the degrees of freedom of the two form gauge field.

$$\delta B = \chi \wedge v$$  (4)

This will be the action that we will double dimensionally reduce on $T^2$. And so we send, $M^6 \to M^4 \times T^2$ and $M^{11} \to M^8 \times T^2$. We will identify

$$(X^{11}, X^9) = (\sigma^4, \sigma^5) = (y^1, y^2)$$  (5)

Where $(y^1, y^2)$ are the coordinates on the space-time torus. In these coordinates we will identify $y^1 = y^1 + 1$ and $y^2 = y^2 + 1$. We will drop all functional dependence of the fields on the compact coordinates, that is taking only the zero modes. $m, n = 0..8$ will be the non compact space-time indices, $i, j = 1, 2$
will be torus coordinate indices and $\mu, \nu = 0..3$ will be the coordinates of the non-wrapped 5-brane world volume. The space-time metric will be written as

$$\eta_{MN} \rightarrow \eta_{mn} \oplus \eta_{ij}$$

(6)

This truncates the space-time Kaluza Klein fields associated with the torus. This is because we are only interested in the M-5 brane/D-3 relationship. Such Kaluza-Klein fields in the M-theory picture are associated with the wrapped D and fundamental string in IIB. We will take $\eta_{mn}$ to be flat Minkowski metric and take the metric on the torus to be given by

$$\eta_{ij} dy^i \otimes dy^j = V \tau_2^2 (dy^1 \otimes dy^1 + \tau_1 dy^2 \otimes dy^1 + \tau_1 dy^1 \otimes dy^2 + |\tau|^2 dy^2 \otimes dy^2)$$

(7)

$\tau = \tau_1 + i\tau_2$ is the complex structure of the torus and $V$ is the area of the torus. The reduction of the brane metric $G$ from 3 follows.

Similarly, we reduce the world volume gauge field as follows:

$$B = B_{(0)} dy^1 \wedge dy^2 + B_{(1)}i \wedge dy^1 + B_{(2)}$$

(8)

so that

$$H = J + F_i \wedge dy^i + L dy^1 \wedge dy^2$$

(9)

There are two distinct possibilities for the auxiliary one form under this decomposition. One may take $v$ to be in $T^2$ only or in $M^4$ only. The two choices must be physically equivalent. The restriction simply corresponds to a partial gauge fixing. In what follows we will take $v$ to be a member of the first cohomology on $T^2$. We will consider the specific choices $v = dy^1$ and $v = dy^2$. These two independent gauge choices are what will eventually generate the S-duality on the 3-brane. Should we put $v$ in $M^4$, for example $v = dt$ then the SL(2,Z) symmetry of the 3-brane will become manifest in the action but we will lose manifest Lorentz invariance. This will give an action of type given in [9], [10].

The relationship between the formulation of the reduced action and the different gauge choices for the PST one form was discussed in [10, 11].

Our goal will be to compare with the D-3 brane, hence it is natural to express the six dimensional determinant appearing in the action as a four dimensional determinant. For this purpose one may use the well known identities:

$$\det \begin{pmatrix} L & P \\ Q & J \end{pmatrix} = \det \begin{pmatrix} L - QJ^{-1}P & 0 \\ 0 & J \end{pmatrix}$$

(10)

and

$$\det(A \oplus B) = \det(A)\det(B)$$

(11)

We have

$$\det M = \det(M_{ij})\det(M_{\mu\nu} - M_{\mu i}(M^{-1})^{ij}M_{j\nu})$$

(12)

This gives the following action:
\[ S_{5-2} = - \int_{T^2} \int_{M^4} \sqrt{\eta} \sqrt{-\det(G_{\mu\nu} + i\alpha^i(v)^* F_{(i)\mu\nu} - \beta(v)^* J^i_{\mu} J^i_{\nu}) + \frac{1}{2} F_i \wedge F_j \gamma^{ij}(v)} \]  

where \( \alpha^i(v) \) and \( \beta(v) \) and \( \gamma(v)^{ij} \) are constants that remain to be evaluated and will be dependent on our choice of \( v \).

However, before evaluating them we will put the \( \sqrt{\eta} \) inside the determinant. This becomes \( \eta^{1/4} \) inside the determinant. We will then carry out a Weyl scaling so that we absorb this factor into the rescaled metric. That is

\[ X' = X \eta^{1/8} \quad G'_{\mu\nu} = G_{\mu\nu} \eta^{1/4} \]  

We then rewrite the action in this rescaled metric taking care with factors of \( \eta \). The \( T^2 \) integral is trivial.

We will use the symmetry given by equation 4 to eliminate half the degrees of freedom contained in the gauge fields. For the choice \( v = dy^L \) we gauge away \( F_{(L)} \) and \( L_{12} \). This leaves only one vector gauge field in the action, with field strength \( F_\ast \), and one two form gauge field, with field strength \( J \). The PST part of the action will then contribute a total derivative that we shall be able to identify it with an axion coupling. We will now write the action in its final form as follows:

\[ S_{5-2} = - \int_{M^4} \sqrt{-\det(G'_{\mu\nu} + i\alpha(v)^* F_{\mu\nu} - \beta^* J_{\mu} J_{\nu}) + \frac{1}{2} F_i \wedge F_j \gamma^{ij}(v)} \]  

We now consider the two natural independent gauge choices for \( v \) and evaluate the coefficients, \( \alpha, \beta \) and \( \gamma \).

For \( v = dy_1 \):

\[ \alpha = \sqrt{\tau_2} \quad \beta = \eta^{3/4} \quad \gamma = -\frac{\tau_1}{|\tau|^2} \]  

for \( v = dy_2 \):

\[ \alpha = \sqrt{\tau_2} \quad \beta = \eta^{3/4} \quad \gamma = \tau_1 \]  

Note that the vector fields couple only to the complex structure of the torus. That is the couplings are completely determined by the shape of the torus and are independent of its size. Different choices of \( v \) give different couplings. The opposite is true for the two form fields. The coupling for the two form field is independent of the choice of \( v \) and is dependent only on the area of the torus. Note that this is frame dependent statement that is reliant on the Weyl rescaling. Combining \( \tau = \tau_1 + i\tau_2 \) we see the different choices of \( v \) generate the transformation \( \tau \to -\frac{\tau}{\tau} \) in the vector field couplings. This corresponds to one of
the generators of SL(2,\mathbb{Z}) the modular group of the torus. The other generator will arise from an integral shift in \(\tau_1\) which will cause a trivial shift in the total derivative term. Later when we compare with the 3 brane on \(S^1\), we will identify the complex structure of the torus with the axion-dilaton and the area of the torus will be related to the radius of the compact dimension as given in [5, 6].

3 The D-3 brane

Starting with the 10 dimensional IIB three brane action in 10 dimensions [12], we will directly reduce the action on a circle. The action in the Einstein frame, including an axion coupling, is given by:

\[
S_3 = - \int d^4\sigma \sqrt{-\text{det}(G_{\mu\nu} + e^{-2\phi}F_{\mu\nu}) + \frac{1}{2}C_0 F \wedge F} \tag{18}
\]

We will reduce this action directly implying we will not identify any of the brane coordinates with the compact dimension. Hence, we will write \(X^9 = X^9 + 1 = \phi\) and so decompose the background metric \(g_{mn} \to g_{mn} \oplus R^2\) where \(R\) is the circumference of the compact dimension. That is as before we truncate out the space time Kaluza Klein field. (On the M-theory side this corresponds to truncating the wrapped membrane).

This gives for the induced world volume metric, pulling out the dependence on \(\phi\):

\[
G_{\mu\nu} \to G_{\mu\nu} + R^2\partial_\mu \phi \partial_\nu \phi \tag{19}
\]

The world volume gauge field is left invariant.

So the final reduced action for the three brane becomes:

\[
S_{3,(S^1)} = - \int d^4\sigma \sqrt{-\text{det}(G_{\mu\nu} + e^{-2\phi}F_{\mu\nu} + R^2\partial_\mu \phi \partial_\nu \phi) + \frac{1}{2}C_0 F \wedge F} \tag{20}
\]

We wish to compare the wrapped 5-brane with different choices of \(v\) with the 3-brane and its S-dual. The S-dual 3-brane is determined by dualizing the vector field on the brane using the same method as described below for dualizing the scalar field.

The action with the vector field dualized takes the same form (this is not true when the full supersymmetric action is used) but the axion dilaton, \(\lambda = C_0 + ie^{-\phi}\) is inverted in the dual action. That is the usual, \(\lambda \to \frac{1}{\lambda}\), [12, 13].

In order to exactly identify the reduced 3-brane action with the 5-brane wrapped action we will first need to do a world volume duality transformation on the field \(\phi\). This is in the spirit of [14] whereby world volume dual actions are associated with the M-theory picture of the brane.

We will dualize the scalar field \(\phi\) by replacing its field strength \(d\phi\) with \(l\) and then adding an additional constraint term to the action \(S_c = H \wedge (d\phi - l)\). \(H\) is a lagrange multiplier ensuring that \(l = d\phi\). To find the dual we first find the equations of motion for \(\phi\) and solve. This implies \(dH = 0\) which means we may
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locally write $H = dB$. Then we must find the equations of motion for $l$ and solve in terms of $H$. We simplify the problem by working in the frame in which $F$ is in Jordan form with eigenvalues $f_1$ and $f_2$. $l_i$ are the components of $l$ and $h_i$ are the components of the dual of $H$. The equations of motion for $l$ are:

$$h_1 = \frac{-(1 + f_2)}{\sqrt{-\det M}} l_1 R^2 \quad h_2 = \frac{(1 + f_2^2)}{\sqrt{-\det M}} l_2 R^2$$

$$h_3 = \frac{(1 - f_1^2)}{\sqrt{-\det M}} l_3 R^2 \quad h_4 = \frac{(1 - f_2^2)}{\sqrt{-\det M}} l_4 R^2$$

where

$$M_{\mu\nu} = G_{\mu\nu} + F_{\mu\nu} + R^2 l_\mu l_\nu$$

We then invert these equations to solve for $l_i$. The solutions are:

$$l_1 = \frac{(f_1^2 - 1)}{\sqrt{-\det M} R^2} h_1 \quad l_2 = \frac{-(f_1^2 - 1)}{\sqrt{-\det M} R^2} h_2$$

$$l_3 = \frac{(1 + f_1^2)}{\sqrt{-\det M} R^2} h_3 \quad l_4 = \frac{(1 + f_2^2)}{\sqrt{-\det M} R^2} h_4$$

Where

$$\tilde{M}_{\mu\nu} = G_{\mu\nu} + i F_{\mu\nu} - \frac{1}{R^2} (*H)_{\mu} (*H)_{\nu}$$

When we substitute these equations into the action we find, reinstating dilaton dependence and the axion term:

$$S_D = -\int d^4 \sqrt{-\det \left( G_{\mu\nu} + ie^{-\phi} F_{\mu\nu} - \frac{1}{R^2} (*H)_{\mu} (*H)_{\nu} \right) + \frac{1}{2} C_0 F \wedge F}$$

We wish to compare with the usual M-theory predictions given in [5] concerning the relationship between the moduli of the IIB theory in 9 dimensions with the geometrical properties of the torus used in the M-theory compactification. By comparing the actions for the reduced five brane and the reduced/dualized three brane on identifies the moduli in the different theories. The scaling of the metric given in equation 14 implies

$$G^B_{\mu\nu} = Area(T^2)^{1/2} G^M_{\mu\nu}$$

From both the coefficient in front of $F$ in the determinant and the coefficient in front of the $F \wedge F$ term, we identify the axion-dilaton of the IIB theory (in the 10 dimensional Einstein frame) with the complex structure of the torus.

$$\lambda = C_0 + ie^{-\phi} = \tau$$

From comparing the coefficient in front of $*H$, the radius of the the 10th dimension in IIB becomes:

$$R_B = Area(T^2)^{-\frac{3}{4}}$$
We have identified the gauge field on the reduced 5-brane with the gauge field on the reduced D-3. The dualized scalar on the D-3 brane becomes identified with the three form on the reduced M-5 brane.

4 Conclusions/discussion

The identification of the moduli given above are in agreement with [5, 6] as one would expect. The elegant aspect is how the S-dual versions of the three brane appear from different gauge choices of the auxiliary field in the five brane. In the discussion above no attempt has been made to discuss the full supersymmetric versions of the actions used. This has been explored in detail in [11]. There it was shown that the conclusions of the above are unaffected by the presence of Fermions though the dualization procedure becomes a great deal more involved due to the presence of extra interaction terms that make the dual action take on a different form. Nevertheless the different gauge choices of the auxiliary field still allow one to generate the different dual versions once the correct field redefinitions are made. See [11] for the details.

Another aspect that has not been explored here is the validity of the identification at the quantum level. As is known, global considerations make the construction of partition functions for chiral gauge fields problematical [15, 16]. Given that classically one identifies the coupling on the 3 brane with the modular parameter of the torus, this suggests that the duality properties of the 3-brane partition function reported in [17] are manifest as modular properties of the 5-brane partition function. Related ideas have been discussed in [18].

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