Partial Quantifier Elimination By Certificate
Clauses

Eugene Goldberg
eu.goldberg@gmail.com

Abstract. In this report\(^1\), we study partial quantifier elimination (PQE) for propositional CNF formulas. PQE is a generalization of quantifier elimination where one can limit the set of clauses taken out of the scope of quantifiers to a small subset of target clauses. The appeal of PQE is twofold. First, PQE can be dramatically simpler than full quantifier elimination. Second, many verification problems (e.g., equivalence checking and model checking) can be solved in terms of PQE. Our approach is based on deriving clauses depending only on unquantified variables that make the target clauses redundant. Proving redundancy of a target clause is done by construction of a “certificate” clause implying the former. We describe a PQE algorithm called START that employs the approach above. To evaluate START, we apply it to invariant generation for a sequential circuit \(N\). The goal of invariant generation is to find an unwanted invariant of \(N\) proving unreachability of a state that is supposed to be reachable. If \(N\) has an unwanted invariant, it is buggy. Our experiments with FIFO buffers and HWMCC-13 benchmarks suggest that START can be used for detecting bugs that are hard to find by existing methods.

1 Introduction

In this paper, we consider the following problem. Let \(F_1(X, Y), F_2(X, Y)\) be propositional formulas in conjunctive normal form (CNF)\(^2\) where \(X, Y\) are sets of variables. Given \(\exists X[F_1 \land F_2]\), find a quantifier-free formula \(F_1^\ast(Y)\) such that \(\exists X[F_1 \land F_2] \equiv F_1^\ast \land \exists X[F_2]\). In contrast to quantifier elimination (QE), only a part of the formula gets “unquantified” here. So, this problem is called partial QE (PQE) [1,2]. We will refer to \(F_1^\ast\) as a solution to PQE. Like SAT, PQE is a way to cope with the complexity of QE. But in contrast to SAT that is a special case of QE (where all variables are quantified), PQE generalizes QE. The latter is just a special case of PQE where \(F_2 = \emptyset\) and the entire formula is unquantified.

\(^1\) Some new results have been produced after publishing this report. We discuss them in Section 12.

\(^2\) Every formula is a propositional CNF formula unless otherwise stated. Given a CNF formula \(F\) represented as the conjunction of clauses \(C_1 \land \cdots \land C_k\), we will also consider \(F\) as the set of clauses \(\{C_1, \ldots, C_k\}\).
The appeal of PQE is twofold. First, it can be much more efficient than QE if \( F_1 \) is a small part of the formula. Second, PQE facilitates the development of new approaches to various verification problems like SAT \([3]\), equivalence checking \([4]\), model checking \([5]\) and so on.

We solve PQE by redundancy based reasoning. Its introduction is motivated by the following observations. First, \( F_1 \land F_2 \Rightarrow F_1^* \) and \( F_1^* \land \exists X[F_1 \land F_2] \equiv F_1^* \land \exists X[F_2] \). Thus, a formula \( F_1^* \) implied by \( F_1 \land F_2 \) becomes a solution as soon as \( F_1^* \) makes the clauses of \( F_1 \) redundant. Second, one can prove clauses of \( F_1 \) redundant\(^3\) one by one. The redundancy of a clause \( C \in F_1 \) can be proved by using \((F_1 \cup F_2) \setminus \{C\}\) to derive a clause \( K \) implying \( C \). We refer to \( K \) as a certificate clause. Importantly, one can produce \( K \) even if \((F_1 \cup F_2) \setminus \{C\}\) does not imply \( C \). This becomes possible if one allows generation of clauses preserving equisatisfiability rather than equivalence.

We implement redundancy based reasoning in a PQE algorithm called START, an abbreviation of Single TARgeT. At any given moment, START proves redundancy of only one clause (hence the name “single target”). START builds the certificate \( K \) above by resolving “local” certificate clauses implying the clause \( C \) in subspaces. Proving redundancy of \( C \) in subspaces where \( F_1 \land F_2 \) is unsatisfiable, in general, requires adding new clauses to \( F_1 \land F_2 \). The added clauses depending only on unquantified variables form a solution \( F_1^* \) to the PQE problem. START is somewhat similar to a SAT-solver with conflict driven learning. A major difference here is that START backtracks as soon as the target clause is proved redundant in the current subspace (even if no conflict occurred).

In this paper, we apply START to the problem of invariant generation that has no general solution yet. We use invariant generation for bug detection as described below. Our objective here is to provide a “proof of concept” for PQE i.e. to give some experimental evidence that PQE is an important direction for research. Let \( N \) be a sequential circuit to verify. As far as reachable states of \( N \) are concerned, one can have bugs of two kinds. A bug of the first kind occurs if a bad state is reachable in \( N \). A bug of the second kind takes place if a required good state (i.e. one that is supposed to be reachable) is unreachable in \( N \). One excludes bugs of the first kind by checking that a set of desired invariants holds. The challenge here is that these invariants may be hard to prove. Bugs of the second kind are currently identified either by testing or by checking if \( N \) has an unwanted invariant. An invariant \( P \) of \( N \) is unwanted if a required good state falsifies \( P \) and so is unreachable in \( N \). If \( P \) holds for \( N \), the latter has a bug of the second kind. The unwanted invariants to check are currently generated manually i.e. are guessed. So, one can easily overlook a bug of the second kind. The main challenge here is to find an unwanted invariant that holds rather than the hardness of proving it true.

The main body of this paper is structured as follows. (Some additional information is given in the appendix.) Sections 2 and 3 provide some basic definitions. A description of START is given in Sections 4-6. In Section 7, we show that PQE

\(^3\) By “proving a clause \( C \) redundant”, we mean showing that \( C \) is redundant after adding (if necessary) some new clauses.
can be used to automatically generate invariants to check for being unwanted. In Section 8, we use START to detect a bug of the second kind in a FIFO buffer that is hard to find by existing methods\(^4\). Section 9 explains how to decide if an invariant is unwanted via test generation. Sections 10 and 11 provide some background and conclusions. Finally, in Section 12, we discuss some new results obtained after the publication of this report.

## 2 Basic Definitions

We assume that every formula is in CNF unless otherwise stated. In this section, when we say “formula” without mentioning quantifiers, we mean “a quantifier-free formula”.

**Definition 1.** Let \( F \) be a formula. Then \( \text{Vars}(F) \) denotes the set of variables of \( F \) and \( \text{Vars}(\exists X[F]) \) denotes \( \text{Vars}(F) \setminus X \).

**Definition 2.** Let \( V \) be a set of variables. An assignment \( \bar{q} \) to \( V \) is a mapping \( V' \to \{0,1\} \) where \( V' \subseteq V \). We will denote the set of variables assigned in \( \bar{q} \) as \( \text{Vars}(\bar{q}) \). We will refer to \( \bar{q} \) as a full assignment to \( V \) if \( \text{Vars}(\bar{q}) = V \). We will denote as \( \bar{q} \subseteq \bar{r} \) the fact that a) \( \text{Vars}(\bar{q}) \subseteq \text{Vars}(\bar{r}) \) and b) every variable of \( \text{Vars}(\bar{q}) \) has the same value in \( \bar{q} \) and \( \bar{r} \).

**Definition 3.** Let \( C \) be a clause (i.e. a disjunction of literals). Let \( H \) be a formula that may have quantifiers, and \( \bar{q} \) be an assignment to \( \text{Vars}(H) \). If \( C \) is satisfied by \( \bar{q} \), then \( C_{\bar{q}} \equiv 1 \). Otherwise, \( C_{\bar{q}} \) is the clause obtained from \( C \) by removing all literals falsified by \( \bar{q} \). Denote by \( H_{\bar{q}} \) the formula obtained from \( H \) by removing the clauses satisfied by \( \bar{q} \) and replacing every clause \( C \) unsatisfied by \( \bar{q} \) with \( C_{\bar{q}} \).

**Definition 4.** Given a formula \( \exists X[F(X,Y)] \), a clause \( C \) of \( F \) is called a quantified clause if \( \text{Vars}(C) \cap X \neq \emptyset \). If \( \text{Vars}(C) \cap X = \emptyset \), the clause \( C \) depends only on free i.e. unquantified variables of \( F \) and is called a free clause.

**Definition 5.** Let \( G,H \) be formulas that may have existential quantifiers. We say that \( G,H \) are equivalent, written \( G \equiv H \), if \( G_{\bar{q}} = H_{\bar{q}} \) for all full assignments \( \bar{q} \) to \( \text{Vars}(G) \cup \text{Vars}(H) \).

**Definition 6.** Let \( F \) be a formula and \( G \subseteq F \) and \( G \neq \emptyset \). Formula \( G \) is redundant in \( \exists X[F] \) if \( \exists X[F] \equiv \exists X[F \setminus G] \).

**Definition 7.** Given a formula \( \exists X[F_1(X,Y) \land F_2(X,Y)] \), the Partial Quantifier Elimination (PQE) problem is to find \( F_1^* \) (\( Y \) such that \( \exists X[F_1 \land F_2] \equiv F_1^* \land \exists X[F_2] \). (So, PQE takes \( F_1 \) out of the scope of quantifiers.) \( F_1^* \) is called a solution to PQE. The case of PQE where \( F_2 = \emptyset \) is called Quantifier Elimination (QE).

**Remark 1.** Let \( C \) be a clause of a solution \( F_1^* \) to the PQE problem above. If \( F_2 \) implies \( C \), then \( F_1^* \setminus \{C\} \) is a solution too.

\(^4\) In Appendix F, we present results showing that START is efficient enough to generate invariants of HWMCC-13 benchmarks. We also give evidence that PQE can be dramatically more efficient than QE.
3 Extended Implication And Blocked Clauses

One can introduce the notion of implication via that of redundancy. Namely, $F \Rightarrow G$, iff $G$ is redundant in $F \land G$ i.e. iff $F \land G \equiv F$. We use this idea to extend the notion of implication via redundancy in a quantified formula.

**Definition 8.** Let $F(X,Y)$ and $G(X,Y)$ be formulas and $G$ be redundant in $\exists X[F \land G]$ i.e. $\exists X[F \land G] \equiv \exists X[F]$. Then $(F \land G)_Y$ and $F_Y$ are equisatisfiable for every full assignment $\vec{y}$ to $Y$. So, we will say that $F$ es-implies $G$ in $\exists X[F \land G]$. (Here “es” stands for “equisatisfiability”.) A clause $C$ is called an es-clause in $\exists X[F \land C]$ if $F$ es-implies $C$ in $\exists X[F \land C]$. One can view es-implication as a weaker version of regular implication.

Note that if $F$ implies $G$, then $F$ also es-implies $G$ in $\exists X[F \land G]$. However, the converse is not true. We will say that $F$ es-implies $G$ without mentioning the formula $\exists X[F \land G]$ if the latter is clear from the context.

**Definition 9.** Let clauses $C',C''$ have opposite literals of exactly one variable $w \in \text{Vars}(C') \cap \text{Vars}(C'')$. Then $C',C''$ are called resolvable on $w$. The clause $C$ having all literals of $C',C''$ but those of $w$ is called the resolvent of $C',C''$. The clause $C$ is said to be obtained by resolution on $w$.

Clauses $C',C''$ having opposite literals of more than one variable are considered unresolvable to avoid producing a tautologous resolvent $C$ (i.e. $C \equiv 1$).

**Definition 10.** Given a formula $\exists X[F(X,Y)]$, let $C$ be a clause of $F$. Let $G$ be the set of clauses of $F$ resolvable with $C$ on a variable $w \in X$. Let $w = b$ satisfy $C$, where $b \in \{0,1\}$. We will call $C$ blocked in $\exists X[F]$ at $w$ if $G$ is redundant in $\exists X[F]$ in subspace $w = b$ (i.e. if $G_{w=b}$ is redundant in $\exists X[F_{w=b}]$).

**Remark 2.** Note that if $G = \emptyset$ or the clauses of $G$ are removed from $\exists X[F]$ as redundant, $C$ meets the original definition of a blocked clause [6]. Definition 10 allows to declare $C$ blocked without removing clauses of $G$ if a proof of their redundancy in $\exists X[F]$ is available. This feature is used by our PQE-solver START (see Remark 4 of Section 6).

**Proposition 1.** Given a formula $\exists X[F(X,Y)]$, let $C$ be a clause blocked in $\exists X[F]$ at $w \in X$. Then $C$ is redundant in $\exists X[F]$ i.e. $\exists X[F] \equiv \exists X[F \setminus \{C\}]$. So, $C$ is es-implied by $F \setminus \{C\}$ in $\exists X[F]$.

Proofs of the propositions are given in Appendix A.

4 A Simple Example Of How START Operates

In this paper, we introduce a PQE algorithm called START (an abbreviation of Single TARgeT). In this section, we give a taste of START by a simple example. Figure 1 describes how START operates on the problem shown in lines 1-6. (Figure 1 and Figures 6,7,8 of the appendix are built using a version of START generating execution traces. A Linux binary of this version can be downloaded from [7].)
Find $F_i^*(Y)$ such that
\[
\exists X[F_i \land F_2] \equiv F_i^* \land \exists X[F_2]
\]
\[
Y = \{y_1\}, X = \{x_2, x_3\}
\]
\[
F_1 = \{C_1\}, C_1 = \overline{y}_1 \lor x_3
\]
\[
F_2 = \{C_2, C_3\}, C_2 = y_1 \lor x_2,
\]
\[
C_4 = y_1 \lor \overline{x}_1
\]

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First, \textit{START} picks $C_1$, the only quantified clause of $F_1$. We will refer to $C_1$ as the \textbf{target clause}. Then \textit{START} invokes a procedure called \textit{PrvRed} to prove $C_1$ redundant (lines 8–26). The algorithm of \textit{PrvRed} is somewhat similar to that of a SAT-solver [8]. \textit{PrvRed} makes decision assignments and runs \textit{BCP} (Boolean Constraint Propagation). Besides, \textit{PrvRed} uses the notion of a \textbf{decision level} that consists of a decision assignment and implied assignments derived by \textit{BCP}. (The decision level number 0 is an exception. It has only implied assignments.) On the other hand, there are a few important differences. In particular, \textit{PrvRed} has a richer set of backtracking conditions, a conflict being just one of them.

\textit{PrvRed} starts the decision level number 1 by making assignment $y_1 = 0$. Then it runs \textit{BCP} to derive assignments $x_2 = 1$ and $x_3 = 0$ from clauses $C_2$ and $C_3$ that became \textbf{unit} (i.e. have only one unassigned variable). At this point, a conflict occurs since $C_1$ is falsified (lines 11–16). Then \textit{PrvRed} generates conflict clause $C_4 = y_1$. It is built like a regular conflict clause [8]. Namely, $C_4$ is obtained by resolving $C_1$ with $C_2$ and $C_3$ to eliminate the variables whose values were derived by \textit{BCP} at decision level 1. The clause $C_4$ \textbf{certifies} that $C_1$ is redundant in $\exists X[F_1 \land F_2]$ in subspace $y_1 = 0$. We call a clause like $C_4$ \textbf{a certificate}. Note that $C_1$ becomes redundant only after adding $C_4$ to the formula, because $C_4$ itself is involved in the derivation of $C_4$. We will refer to the certificates one has to add to the formula as \textbf{participant certificates}.

After generating $C_4$, like a SAT-solver, \textit{PrvRed} backtracks to the smallest decision level where $C_4$ is unit (i.e. level 0) and derives the assignment $y_1 = 1$. Then the target $C_1$ is blocked at variable $x_2$ (lines 19–25). The reason is that $C_2$, the only clause resolvable with $C_1$ on $x_2$, is satisfied by $y_1 = 1$. At this point, \textit{PrvRed} generates the clause $K_1 = \overline{y}_1 \lor \overline{x}_2$. It implies $C_1$ in subspace $y_1 = 1$, thus certifying its redundancy there. (The construction of $K_1$ is explained in Example 1 of Subsection 5.3. Importantly, the target $C_1$ \textit{is not used} in generation of $K_1$.) By resolving $K_1$ and $C_4 = y_1$, \textit{PrvRed} builds the final certificate $K_2 = \overline{x}_2$ for the decision level 0. \textit{PrvRed} derives $K_2$ from $K_1$ like a SAT-solver derives a

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Fig. 1: \textit{START}, an example of operation

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\textbf{participant certificates}. The participant certificates depending only on free variables form a solution to the PQE problem.
conflict clause from a clause falsified at a conflict level. That is $K_2$ is built by resolving out variables of $K_1$ assigned by values derived at the current decision level. In our case, it is the variable $y_1$. Since $K_1$ and $K_2$ are derived without using the target clause $C_1$, one does not have to add them to the formula. They just “witness” the redundancy of $C_1$. We will refer to them as witness certificates.

$K_2$ implies $C_1$ in the entire space and thus is a global certificate. So, $START$ removes $C_1$ from $F_1$ (line 28). Since now $F_1$ does not have quantified clauses, $START$ terminates. It returns the current $F_1 = \{C_4\}$ as a solution $F_1^*(Y)$ to the PQE problem. That is $\exists X[C_1 \land F_2] \equiv C_4 \land \exists X[F_2]$.

5 Description Of $START$

In this section, we describe $START$ in more detail. A proof of correctness of $START$ is given in Appendix E. For the sake of simplicity, in the current version of $START$, the witness certificates are not added to the formula and so are not reused\(^5\).

5.1 The main loop of $START$

The main loop of $START$ is shown in Fig. 2. $START$ accepts formulas $F_1(X,Y)$, $F_2(X,Y)$ and set $Y$ and outputs formula $F_1^*(Y)$ such that $\exists X[F_1 \land F_2] \equiv F_1^* \land \exists X[F_2]$. The loop begins with picking a quantified clause $C_{trg} \in F_1$ that is the target clause to be proved redundant (line 2). If $F_1$ has no quantified clauses, it is the solution $F_1^*(Y)$ returned by $START$ (lines 3-5). Otherwise, $START$ initializes the assignment $\vec{q}$ to $X \cup Y$ and invokes a procedure called $PrvRed$ to prove $C_{trg}$ redundant (lines 6-7). $PrvRed$ returns a clause $K$ implying $C_{trg}$ and thus certifying its redundancy. If $K$ is an empty clause (i.e. has no literals), $F$ is unsatisfiable. Then $PrvRed$ returns $K$ as a solution to the PQE problem (line 8). Otherwise, $K$ consists of (some) literals of $C_{trg}$. Besides, $K$ is redundant in $\exists X[K \land (F_1 \cup F_2 \setminus \{C_{trg}\})]$. So, $C_{trg}$ is redundant in $\exists X[C_{trg} \land (F_1 \cup F_2 \setminus \{C_{trg}\})]$ and $START$ removes it from $F_1$ (line 9). In the process of deriving the certificate $K$ above, $PrvRed$ may add participant certificates to $F_1$. If an added certificate clause $K'$ is quantified, $PrvRed$ will be called at a later iteration of the main loop to prove $K'$ redundant.

5.2 Description of $PrvRed$

The pseudo-code of $PrvRed$ is shown in Fig 3. Let $F$ denote $F_1 \land F_2$. The objective of $PrvRed$ is to prove the current target clause $C_{trg}$ redundant in

\(^5\) In practice, witness certificates are derived in subspaces where the formula is satisfiable. So, reusing them should boost the pruning power of $START$ in those subspaces.
\[ \exists X[F] \] in the subspace specified by an assignment \( \vec{q} \) to \( X \cup Y \). The reason why one needs \( \vec{q} \) is that \( \text{PrvRed} \) can be called recursively in subspaces to prove redundancy of some “local” target clauses (Section 6).

First, in line 1, \( \text{PrvRed} \) stores the initial value of \( \vec{q} \). (It is used in line 10 to limit the backtracking of \( \text{PrvRed} \).) Besides, \( \text{PrvRed} \) initializes the assignment queue \( Q \). The main work is done in a loop similar to that of a SAT-solver [8]. The operation of \( \text{PrvRed} \) in this loop is partitioned into two parts separated by the dotted line.

The first part (lines 3-7) starts with checking if the assignment queue \( Q \) is empty. If so, a decision assignment \( v = b \) is picked and added to \( Q \) (lines 4-5). Here \( v \in (X \cup Y) \) and \( b \in \{0, 1\} \). The variables of \( Y \) are the first to be assigned by \( \text{PrvRed} \) so \( X \) is empty, only if all variables of \( Y \) are assigned. If \( v \in \text{Vars}(C_{\text{try}}) \), then \( v = b \) is picked so as to falsify the corresponding literal of \( C_{\text{try}} \). (\( C_{\text{try}} \) is obviously redundant in subspaces where it is satisfied.)

Then \( \text{PrvRed} \) calls the BCP procedure. If BCP identifies a backtracking condition, it returns a certificate clause \( K_{\text{bct}} \) implying \( C_{\text{try}} \) in the current subspace. (Here, “bct” stands for “backtracking” because \( K_{\text{bct}} \) is the reason for backtracking.) After BCP, \( \text{PrvRed} \) goes to the second part of the loop where the actual backtracking is done. If no backtracking condition is met, a new iteration begins.

The certificate \( K_{\text{bct}} \) returned by BCP depends on the backtracking condition. BCP identifies three of them: a) a conflict, b) \( C_{\text{try}} \) is implied in subspace \( \vec{q} \) by an existing clause, and c) \( C_{\text{try}} \) is blocked in subspace \( \vec{q} \). In the first case, \( K_{\text{bct}} \) is a clause falsified in the current subspace \( \vec{q} \). In the second case, \( K_{\text{bct}} \) is a clause that BCP made unit and that shares its only literal with \( C_{\text{try}} \). (Such a clause implies \( C_{\text{try}} \) in the current subspace \( \vec{q} \).) In the third case, \( K_{\text{bct}} \) is generated by \( \text{PrvRed} \) as described in the next subsection.

\( \text{PrvRed} \) starts the second part (lines 8-12) with a procedure called \( \text{Lrn} \) that uses \( K_{\text{bct}} \) to build another certificate \( K \) implying \( C_{\text{try}} \) in subspace \( \vec{q} \). Generation of \( K \) from \( K_{\text{bct}} \) is similar to how a SAT-solver generates a conflict clause from a falsified clause [8]. Namely, when building \( K \), \( \text{Lrn} \) resolves out the variables whose value was derived at the decision level where the backtracking condition occurred. If \( C_{\text{try}} \) was used to generate \( K \) i.e. the latter is a participant certificate,

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6 The goal of \( \text{START} \) is to derive free clauses making the quantified clauses of \( F_1 \) redundant in \( \exists X[F_1 \land F_2] \). Assigning variables of \( X \) after those of \( Y \) guarantees that, when generating a new clause, the variables of \( X \) are resolved out before those of \( Y \).
$K$ is added to $F_1$ (line 9). This guarantees that $\text{PrvRed}$ adds only clauses implied by the current formula. (The only es-clauses generated by $\text{PrvRed}$ and described in the next subsection are used solely to generate witness certificates. So, a witness certificate $K$ is, in general, es-implied rather than implied by the formula $F$ in $\exists X[K \land F]$. For that reason, in the current version of $\text{START}$, witness certificates are not added to the formula. In one special case, to avoid adding a witness certificate, $\text{PrvRed}$ has to derive and add to the formula a special clause. This case is described in Appendix D.)

After generating $K$, $\text{PrvRed}$ backtracks (line 10). The assignment $\hat{q}_{init}$ sets the limit of backtracking. If $\text{PrvRed}$ reaches this limit, $C_{trg}$ is proved redundant in the required subspace and $\text{PrvRed}$ terminates (line 11). Otherwise, an assignment is derived from $K$ and added to the queue $Q$ (line 12). This is similar to the backtracking of a SAT-solver to the smallest decision level where the last conflict clause is unit. So, an assignment can be derived from this clause by $BCP$. More information can be found in Appendix B.

5.3 Generation of clause $K_{bct}$ when $C_{trg}$ is blocked

Let $C_{trg}$ get blocked in $\exists X[F]$ in the current subspace $\hat{q}$ during $BCP$. So, $C_{trg}$ is redundant in $\exists X[F]$ in this subspace. Then a clause $K_{bct}$ is generated as described in Proposition 2 where $(K_{bct})_{\hat{q}} \Rightarrow (C_{trg})_{\hat{q}}$ and $K_{bct}$ is redundant in $\exists X[K_{bct} \land (F \setminus \{C_{trg}\})]$. Thus, $K_{bct}$ certifies redundancy of $C_{trg}$ in subspace $\hat{q}$ and is returned by $BCP$ as the reason for backtracking (line 6 of Fig 3). This is the only case of backtracking where the clause $K_{bct}$ returned by $BCP$ is es-implied rather than implied by $F$ in $\exists X[K_{bct} \land F]$.

**Proposition 2.** Given a formula $\exists X[F(X,Y)]$, let $C_{trg} \in F$. Let $\hat{q}$ be an assignment to $X \cup Y$ that does not satisfy $C_{trg}$. Let $C_{trg}$ be blocked in $\exists X[F]$ at $w \in X$ in subspace $\hat{q}$ where $w \notin \text{Vars}(\hat{q})$. Let $l(w)$ be the literal of $w$ present in $C_{trg}$. Let $K'$ denote the longest clause falsified by $\hat{q}$. Let $K''$ be a clause formed from $l(w)$ and a subset of literals of $C_{trg}$ such that every clause of $F_{\hat{q}}$ unresolvable with $(C_{trg})_{\hat{q}}$ on $w$ is unresolvable with $(K'')_{\hat{q}}$ too. Let $K_{bct} = K' \lor K''$. Then $(K_{bct})_{\hat{q}} \Rightarrow (C_{trg})_{\hat{q}}$ and $K_{bct}$ is redundant in $\exists X[K_{bct} \land (F \setminus \{C_{trg}\})]$.

**Example 1.** Let us recall the example of Section 4. Here we have a formula $\exists X[F]$ where $X = \{x_1, x_2, x_3\}$, $Y = \{y_1\}$, $F = C_1 \land C_2 \land C_3 \land C_4$, $C_1 = \overline{\overline{y}_1} \lor x_2$, $C_2 = \overline{y}_1 \lor x_2$, $C_3 = \overline{x}_3$, $C_4 = y_1$. In subspace $y_1 = 1$, the target clause $C_1$ is blocked at $x_2$ and hence is redundant. ($C_1$ can be resolved on $x_2$ only with $C_2$ that is satisfied by $y_1 = 1$.) This redundancy can be certified by the clause $K_1 = \overline{\overline{y}_1} \lor C_2$ implying $C_1$ in subspace $y_1 = 1$. The clause $K_1$ is constructed as $K' \lor K''$ of Proposition 2. Here $K' = \overline{y}_1$ is the clause falsified by the assignment $y_1 = 1$. The clause $K'' = C_2$ has the same literal of the blocked variable $x_2$ as the target clause $C_1$. (Formula $F$ has no clauses unresolvable with $C_1$ on $x_2$. So, $K''$ needs no more literals.)

**Remark 3.** Let $C_{trg}$ of Proposition 2 be unit in subspace $\hat{q}$ (and $w$ be the only unassigned variable of $C_{trg}$). Then $K''$ reduces to $l(w)$ and $K_{bct} = K' \lor l(w)$.
6 The Case When The Target Clause Becomes Unit

In this section, we describe what `PrvRed` does when the current target clause \(C_{trg}\) becomes unit. (Since `PrvRed` first assigns variables of \(Y\), the unassigned variable of \(C_{trg}\) is in \(X\) i.e. quantified.) In this case, `PrvRed` recursively calls itself to prove redundancy of every clause resolvable with \(C_{trg}\). A concrete example is given in Appendix C.

Figure 4 shows the fragment of BCP invoked when the current target \(C_{trg}\) becomes unit. Let \(x \in X\) denote the only unassigned variable of \(C_{trg}\). Assume for the sake of clarity that \(C_{trg}\) contains the positive literal of \(x\). At this point a SAT-solver would derive the assignment \(x = 1\) because \(C_{trg}\) is falsified under assignment \(x = 0\). However, the goal of `PrvRed` is to prove \(C_{trg}\) redundant rather than find a satisfying assignment. The fact that \(C_{trg}\) is falsified in a subspace says nothing about whether it is redundant there.

So, BCP invokes procedure `Rcrs` that recursively calls `PrvRed` for every clause resolvable with \(C_{trg}\) on \(x\). The name `Rcrs` abbreviates “recurse”. This call can have two outcomes. First, `Rcrs` may return a clause \(K_{bct}\) that is falsified by \(\vec{q}\). (This is possible only if \(F\) is unsatisfiable in subspace \(\vec{q}\).) Then BCP returns \(K_{bct}\) as the reason for backtracking (line 12). Second, `Rcrs` proves the clauses resolvable with \(C_{trg}\) on \(x\) redundant and returns a set \(G\) of certificates. For each clause \(C\) resolvable with \(C_{trg}\) on \(x\), the set \(G\) contains a certificate of redundancy of \(C\) in subspace \(\vec{q} \cup \{x = 1\}\). At this point, \(C_{trg}\) is blocked at \(x\) in subspace \(\vec{q}\). So, a certificate \(K_{bct}\) is built using Proposition 2 (line 13). It is returned by BCP as the reason for backtracking.

Remark 4. Every clause \(C\) resolvable with \(C_{trg}\) on \(x\) and proved redundant in subspace \(\vec{q} \cup \{x = 1\}\) is temporarily removed from the formula \(F\) until backtracking. Since \(C\) is proved redundant only locally, one has to return it to \(F\) after backtracking. Nevertheless, \(C_{trg}\) remains blocked in subspace \(\vec{q}\) and hence redundant there (see Remark 2 of Section 3).

7 Invariant Generation For Bug Detection

In this section, we discuss using PQE for bug detection by invariant generation. An invariant \(P\) of a sequential circuit \(N\) is a formula satisfied by every reachable state of \(N\). So, the states falsifying \(P\) are unreachable in \(N\). We will call an invariant local if it holds in some time frames. To distinguish between local invariants and those holding in every time frame we will call the latter global. When we say “invariant” without a qualifier we mean a global invariant.
7.1 Two kinds of bugs

Let $N$ be a sequential circuit. Let $P_1(S), \ldots, P_n(S)$ be invariants that must hold for $N$ where $S$ is the set of state variables. That is, these are desired invariants of $N$. One can view the aggregate invariant $P_1 \land \cdots \land P_n$ as a specification $Sp$ for $N$. We will say that $\vec{s}$ is a bad state (respectively a good state) if $Sp(\vec{s}) = 0$ (respectively $Sp(\vec{s}) = 1$). As far as reachable states are concerned, $N$ can have two kinds of bugs. A bug of the first kind occurs when a bad state is reachable in $N$. A bug of the second kind takes place when a good state that is supposed to be reachable is unreachable in $N$. Informally, a bug of the first kind (respectively the second kind) indicates that the set of reachable states is “larger” (respectively “smaller”) than it should be.

To prove that $N$ has no bugs of the first kind, it suffices to show that the aggregate invariant $Sp$ holds for $N$. Note that this does nothing to identify bugs of the second kind. Indeed, let $N_{triv}$ be a circuit looping in an initial state $s_{init}$ satisfying $Sp$. Then $Sp$ holds for $N_{triv}$. However, $N_{triv}$ has bugs of the second kind (assuming that a correct implementation has to reach states other than $s_{init}$). A straightforward way to identify bugs of the second kind is to compute the set of all unreachable states of $N$. If this set contains a state that is supposed to be reachable, $N$ has a bug of the second kind. Unfortunately, computing such a set can be prohibitively hard.

Note that one cannot prove the existence of a bug of the second kind by testing: the unreachability of a state cannot be established by a counterexample. However, testing can point to the possibility of such a bug (see Section 9). An important method for finding bugs of the second kind in a circuit $N$ is to identify its unwanted invariants. We will call $Q$ an unwanted invariant if it is falsified by a state $\bar{s}$ that is supposed to be reachable. If $Q$ holds for $N$, then $\bar{s}$ is unreachable and $N$ has a bug of the second kind. Currently, unwanted invariants are detected via checking a list of expected events [9]. (If an event of this list never occurs, $N$ has an unwanted invariant.) This list is formed manually. So, in a sense, unwanted invariants are simply guessed. For instance, one can check if $N$ reaches a state where a state variable $s_i \in S$ changes its initial value. If not, then $N$ has an unwanted invariant, assuming that states with both values of $s_i$ are supposed to be reachable in $N$. (For the circuit $N_{triv}$ above, this unwanted invariant holds for every state variable.) The problem with guessing unwanted invariants is that, in general, they are as unpredictable as bugs.

In this paper, we consider an approach to finding bugs of the second kind where invariants are generated automatically in a systematic way. The necessary condition for an invariant $Q$ to be unwanted is $Sp \not\Rightarrow Q$. (If $Sp \Rightarrow Q$, then $Q$ is a desired invariant of $N$.) So, the overall idea is to generate invariants of $N$ not implied by $Sp$ and check if any of them is unwanted. In some cases, the designer can tell if $Q$ is an unwanted invariant. Otherwise, one needs to find a bug-exposing test as explained in Section 9. In general, an invariant specifies only a subset of unreachable states of $N$. So, it can be generated much more efficiently than the entire set of unreachable states.
7.2 Invariant generation by PQE

Let us show how one can generate invariants by PQE. First, we consider the generation of a local invariant that holds in \( k \)-th time frame. So, a state falsifying such an invariant is unreachable in \( k \) transitions. Then we show that a local invariant can be used to generate global invariants. Let formulas \( I \) and \( T \) specify the initial states and the transition relation of \( N \) respectively. Let \( F_k \) denote the formula obtained by unfolding \( N \) for \( k \) time frames. That is \( F_k = I(S_0) \land T(S_0, S_1) \land \cdots \land T(S_{k-1}, S_k) \) where \( S_j \) denotes the state variables of \( j \)-th time frame, \( 0 \leq j \leq k \). (For the sake of simplicity, in \( T \), we omit the combinational i.e. unlatched variables of \( N \).)

Let \( H_k(S_k) \) be a solution to the PQE problem of taking a clause \( C \) out of \( \exists S_{k-1}[F_k] \) where \( S_{k-1} = S_0 \cup \cdots \cup S_{k-1} \). That is \( \exists S_{k-1}[F_k] \equiv H_k \land \exists S_{k-1}[F_k \setminus \{C\}] \). Since \( F_k \) implies \( H_k \), the latter is a local invariant of \( N \) holding in \( k \)-th time frame. Note that performing full QE on \( \exists S_{k-1}[F_k] \) produces the strongest local invariant specifying all states unreachable in \( k \) transitions. Computing this invariant can be prohibitively hard. PQE allows to build a collection of weaker local invariants \( H_k \) each specifying only a subset of states unreachable in \( k \) transitions. Computation of such invariants can be dramatically more efficient since PQE can be much easier than QE.

One can use \( H_k \) to find global invariants as follows. The fact that \( H_k \) is not a global invariant does not mean that every clause of \( H_k \) is not a global invariant either. On the contrary, the experiments presented in Appendix F showed that even for small \( k \), a large share of clauses of \( H_k \) were a global invariant. (To find out if a clause \( Q \in H_k \) is a global invariant, one can simply run a model checker to see if \( Q \) holds.)

7.3 Using Invariant Generation

One of possible ways to use invariant generation is to take out clauses according to some coverage metric. The intuition here is based on the two observations below. Let \( Q \) be an invariant obtained by taking a clause \( C \) out of \( \exists S_{k-1}[F_k] \). The first observation is that the states falsifying \( Q \) are unreachable due to the presence of \( C \). So, if a part of the circuit \( N \) is responsible for a bug of the second kind and \( C \) is related to this part, taking out \( C \) may produce an unwanted invariant. This observation is substantiated in the next section. The second observation is that by taking out different clauses one generates different invariants "covering" different parts of the circuit \( N \). An example of a coverage metric is presented in the next section. There we take out the clauses containing an unquantified variable of \( \exists S_{k-1}[F_k] \) (i.e. a state variable of the \( k \)-th time frame). One can view such a choice of clauses as a way to cover the design in terms of latches.

8 An Experiment With FIFO Buffers

In this section, we describe an experiment with FIFO buffers. Our objective here is twofold. First, we explain how bug detection by invariant generation works
on a practical example. Second, we want to show that even the current version of START whose performance can be dramatically improved can address an important practical problem. In Appendix F, we apply START to invariant generation for HWMCC-13 benchmarks. We also use these benchmarks to compare PQE with QE and START with DS-PQE, our previous PQE-solver [1]. In this section, such a comparison is done on FIFO buffers. A Linux binary of START and a sample of formulas used in the experiments can be downloaded from [7]. In all experiments, we used a computer with Intel Core i5-8265U CPU of 1.6 GHz.

8.1 Buffer description

In this section, we give an example of bug detection by invariant generation for a FIFO buffer called Fifo. Let \( n \) be the number of elements of Fifo and Data denote the data buffer of Fifo. Let each \( Data[i], i = 1, \ldots, n \) have \( p \) bits and be an integer where \( 0 \leq Data[i] < 2^p \). A fragment of the Verilog code describing Fifo is shown in Fig 5. This fragment has a buggy line marked with an asterisk. In the correct version without the marked line, a new element \( dataIn \) is added to \( Data \) if the write flag is on and Fifo holds less than \( n \) elements. Since \( Data \) can have any combination of numbers, all \( Data \) states are supposed to be reachable.

However, due to the bug, the number \( Val \) cannot appear in \( Data \). (Here \( Val \) is some constant \( 0 < Val < 2^p \). We assume that the buffer elements are initialized to 0.) So, Fifo has a bug of the second kind since it cannot reach states where an element of \( Data \) equals \( Val \). This bug is hard to detect by random testing because it is exposed only if one tries to add \( Val \) to Fifo. Similarly, it is virtually impossible to guess an unwanted invariant of Fifo exposing this bug unless one knows exactly what this bug is.

8.2 Bug detection by invariant generation

Let \( N \) be a circuit implementing Fifo. Let \( S \) be the set of state variables of \( N \) and \( S_{data} \subset S \) be the subset corresponding to the data buffer Data. We used START to generate invariants of \( N \) as described in the previous section. Note that an invariant \( Q \) depending only on \( S_{data} \) is an unwanted one. If \( Q \) holds for \( N \), some states of \( Data \) are unreachable. Then Fifo has a bug of the second kind since every state of \( Data \) is supposed to be reachable. To generate invariants, we used the formula \( F_k = I(S_0) \land T(S_0, S_1) \land \cdots \land T(S_{k-1}, S_k) \) introduced in Subsection 7.2. Here \( I \) and \( T \) describe the initial states and the transition relation of \( N \) respectively and \( S_j \) is the set of state variables in \( j \)-th time frame. First, we used START to generate local invariants \( H_k \). Namely, \( H_k \) was obtained by taking a clause \( C \) out of \( \exists S_{k-1}[F_k] \) where \( S_{k-1} = S_0 \cup \cdots \cup S_{k-1} \). That is,
∃S_{k-1}[F_k] \equiv H_k \land \exists S_{k-1}[F_k \setminus \{C\}]. We picked clauses to take out as described in Subsection 7.3. Namely, we took out only clauses containing an unquantified variable (i.e. a state variable of the k-th time frame). The time limit for solving the PQE problem of taking out a clause was set to 10 sec.

For each clause Q of every local invariant H_k generated by PQE, we checked if Q was a global invariant. Namely, we used a publicly available version of IC3 [10,11] to verify if the invariant Q held. If so, and Q depended only on variables of S_{data}, N had an unwanted invariant. Then we stopped invariant generation. The results of the experiment are given in Table 1. (In the experiment, we considered buffers with 32-bit elements.) Let us use the first line of Table 1 to explain its structure. The first two columns show the size of Fifo implemented by N and the number of latches in N (8 and 300). The third column gives the number k of time frames (i.e. 5). The value 13 shown in the fourth column is the number of clauses taken out of ∃S_{k-1}[F_k] before an unwanted invariant was generated. That is, 13 was the number of PQE problems for START to solve.

Table 1: FIFO buffer with n elements of 32 bits. Time limit is 10 sec. per PQE problem

| buff. size (n) | latches | time frames | clauses taken out | local single clause invariants | tot. run time (s) |
|---------------|---------|-------------|-------------------|-------------------------------|------------------|
| 8             | 300     | 5           | 10                | 8                            | 25               |
| 8             | 300     | 10          | 11                | 4                            | 1               |
| 10            | 500     | 3           | 26                | 18                           | 25               |
| 10            | 500     | 10          | 17                | 2                            | 25               |

Let C be a clause taken out of the scope of quantifiers by START. Every free clause Q generated when taking out C was stored as a local single-clause invariant. The fifth column shows that when solving the 13 PQE problems above, START generated 10 free clauses forming 10 local single-clause invariants. These invariants held in k-th time frame (where k=5). The next two columns show how many invariants out of 10 IC3 proved false or true globally (8 and 2). The last column gives the total run time (25 sec).

For all four instances of Fifo listed in Table 1, the invariants generated by START had one asserting that Fifo cannot reach a state where an element of Data equals Val. This invariant was produced when taking out a clause of F_k related to the buggy line of Fig. 5 (confirming the intuition of Subsection 7.3.) When picking a clause to take out, i.e. a clause containing a state variable of k-th time frame, one could make a good choice by pure luck. To address this issue, we picked clauses to take out randomly and performed 10 different runs of invariant generation. For each line of Table 1, the columns four to eight actually describe the average value of 10 runs.

8.3 Comparing PQE and QE

To contrast PQE and QE, we used a high-quality tool CADET [12,13] to perform QE on formulas ∃S_{k-1}[F_k]. That is, instead of taking a clause out of ∃S_{k-1}[F_k] by PQE, we applied CADET to perform full QE on this formula. As mentioned in Subsection 7.2, performing QE on ∃S_{k-1}[F_k] produces the strongest local invariant specifying all states unreachable in k transitions. CADET failed to
finish QE on $\exists S_{k-1}[F_k]$ with the time limit of 600 sec. On the other hand, START finished 63% of the PQE problems of taking a clause out of $\exists S_{k-1}[F_k]$ in the time limit (i.e. under 10 sec). This shows that PQE can be dramatically more efficient than QE if only a small part of the formula gets unquantified.

8.4 START versus DS-PQE

We repeated the experiment above using DS-PQE instead of START. DS-PQE is our previous PQE-solver [1] based on the machinery of D-sequent [14,15]. DS-PQE solved only 2% of the PQE problems in the time limit of 10 sec. (as opposed to 63% by START) and failed to generate an unwanted invariant.

9 Identifying Unwanted Invariants

Sometimes it is easy to see that an invariant $Q$ is unwanted (e.g. an invariant of Fifo depending only on variables of $S_{data}$ is obviously unwanted). However, in general, to show that $Q$ is unwanted, one needs to find a bug-exposing test (or be-test for short.) Let $\bar{t}$ denote a test $(\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_{k-1})$ for a circuit $N$. Here $\bar{s}_0$ is an initial state of $N$ and $\bar{s}_i$, $0 \leq i < k$ is a full assignment to the combinational input variables of $N$ in $i$-th time frame. (Recall that so far, for the sake of simplicity, we omitted combinational variables in the description of $N$.)

Let $(\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_k)$ be the trace produced by the test $\bar{t}$ above (i.e. $N$ moves from state $\bar{s}_i$ to $\bar{s}_{i+1}$ under input $\bar{x}_i$). We will say that $\bar{t}$ is a be-test for an invariant $Q$ if it is a counterexample for $Q$ in a correct version $N'$ of $N$. That is $\bar{t}$ produces a trace $(\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_k)$ in $N'$ where $\bar{s}_k$ falsifies $Q$. Consider, for instance, the invariant $Q$ stating that Fifo cannot have the number $Val$ in $j$-th element of its data buffer. Let $\bar{t} = (\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_{k-1})$ be a test such that when applied to a correct design, $\bar{t}$ would make $Val$ appear in the $j$-th element of the data buffer. Then $\bar{t}$ is a be-test for $Q$.

Finding a be-test is based on the following idea. Let an invariant clause $Q$ be extracted from a formula $H_k$ obtained by taking a clause $C$ out of $\exists S_{k-1}[F_k]$ as described above. As we mentioned in Remark 1, $\exists S_{k-1}[F_k] = H_k \land \exists S_{k-1}[F_k \setminus \{C\}]$ holds even if the clauses implied by $F_k \setminus \{C\}$ are removed from $H_k$. So, we will assume that $F_k \setminus \{C\} \nRightarrow Q$. Then there is an assignment $\bar{p}$ satisfying $(F_k \setminus \{C\}) \land \overline{Q}$. One can view $\bar{p}$ as an execution trace of $N$ when $C$ is removed from $F_k$.

Let $\bar{t}^x = (s^x_0, s^x_1, \ldots, s^x_{k-1})$ be the test where the variables are assigned as in $\bar{p}$. One can make two claims about $\bar{t}^x$. First, if $Q$ is an unwanted invariant, $\bar{t}^x$ can be very close to a be-test. Second, if $Q$ is a desired invariant, $\bar{t}^x$ is a high-quality test for $N$ that can be used e.g. in regression testing. The first claim is due to $\bar{t}^x$ being extracted from $\bar{p}$ falsifying $Q$ and satisfying all clauses of $F_k$ but $C$. The second claim is due to $\bar{t}^x$ being able to detect modifications of $N$ breaking $Q$. One can try to produce a be-test from $\bar{t}^x$ either “manually” or automatically generating small variations of $\bar{t}^x$. 
Consider, for example, the unwanted invariant $Q$ stating that the number $Val$ cannot appear in $j$-th element of the data buffer of $Fifo$. For every example of Table 1, we built the test $t^* = (s^*_{0,0}, x^*_0, \ldots, x^*_{k-1})$ extracted from $\overline{p}$ satisfying $(F_k \setminus \{C\}) \land \overline{Q}$. In every case, $t^*$ turned out to be different from a be-test only in one bit.

10 Some Background

In this section, we discuss some research relevant to PQE and invariant generation. Information on BDD and SAT based QE can be found in [16,17] and [18,19,20,21,22,23,24,25,12] respectively. Making clauses of a formula redundant by adding resolvents is routinely used in pre-processing [26,27] and in-processing [28] phases of QBF/SAT-solving. Identification and removal of blocked clauses is also an important part of formula simplification [29]. The difference of our approach from these techniques is twofold. First, our approach employs redundancy based reasoning rather than formula optimization. So, for instance, to make a target clause redundant, $START$ can add a lot of new clauses making the formula larger. Second, these techniques try to identify non-trivial conditions under which a clause $C$ is redundant in the entire space. In our approach, one branches to reach a subspace where proving $C$ redundant is trivial. Proving redundancy of $C$ in the entire space is achieved by merging the results of different branches.

The predecessor of the approach based on certificate clauses is the machinery of dependency sequents (D-sequents) [14,15]. A D-sequent is a record stating redundancy of a clause in a quantified formula. A potential flaw of this machinery is that to reuse a learned D-sequent, one has to keep some contextual information [30], which can make D-sequent reusing expensive. On the other hand, the reuse of certificate clauses does not require to store any contextual information.

To the best of our knowledge, the existing procedures generate only particular classes of invariants. For instance, they generate invariants relating internal points of circuits to check for equivalence [31] or loop invariants [32]. Another example of special invariants are clauses generated by $IC3$ to make an invariant $P$ inductive [10]. The problem here is that the closer $P$ to an inductive invariant, the fewer invariant clauses $IC3$ generates to make $P$ inductive. For instance, for the circuit $N_{\text{triv}}$ mentioned in Subsection 7.1 that loops in an initial state, every true desired invariant $P_i$ is already inductive. Hence, $IC3$ will not generate any new invariant clauses and will not produce an unwanted invariant even though $N_{\text{triv}}$ is obviously buggy. In Appendix F.3, we experimentally compare invariants generated by $IC3$ and $START$.

11 Conclusions

We consider partial quantifier elimination (PQE) on propositional CNF formulas with existential quantifiers. PQE allows to unquantify a part of the formula. We
present a PQE algorithm called START employing redundancy based reasoning via the machinery of certificate clauses. To prove a target clause \( C \) redundant, START derives a clause implying \( C \), thus “certifying” its redundancy. The version of START we describe here can still be drastically improved. We show that PQE can be used to generate invariants of a sequential circuit. The goal of invariant generation is to find an unwanted invariant of this circuit indicating that the latter is buggy. Bugs causing unwanted invariants can be easily overlooked by the existing methods. We applied START to identify a bug in a FIFO buffer by generating an unwanted invariant of this buffer. We also showed that even the current version of START is good enough to generate invariants for HWMCC-13 benchmarks. Our experiments suggest that START can be used for detecting hard-to-find bugs in real-life designs.

12 Discussing Some New Results

Some new results have been obtained after publishing this technical report. In this section we discuss new developments and their relation to the machinery of certificate clauses. In Subsection 12.1, we describe some limitations on reusing witness certificates. The problem of generation of duplicate participant certificates is discussed in Subsection 12.2. Subsection 12.3 presents some clarification on comparing START with DS-PQE. In Subsection 12.4, we give a few concluding thoughts.

12.1 Limitations on reusing certificates

In this subsection, we briefly discuss some limitations on reusing witness certificates. Let \( K \) be a witness certificate stating redundancy of a clause \( C \) in \( \exists X[F] \) in subspace \( \vec{q} \). As we mentioned earlier, the advantage of deriving \( K \) instead of a D-sequent is that the former can be added to \( \exists X[F] \) (because \( \exists X[F] \equiv \exists X[F \land K] \)). Adding \( K \) to \( \exists X[F] \) makes redundancy of \( C \) in subspace \( \vec{q} \) obvious and so enables safe reusing of \( K \) as a proof of redundancy of \( C \) in subspace \( \vec{q} \).

However, adding witness certificates poses the following problem. The certificate \( K \) above can be represented as \( K' \lor K'' \) where \( K' \) is a clause falsified by \( \vec{q} \) and clause \( K'' \) consists of literals of \( C \). If \( K'' \) contains all the literals of \( C \), then adding \( K \) to \( \exists X[F] \) does not make sense (because \( K \) is implied by \( C \)). Of course, one can always use \( K \) as a proof of redundancy in subspace \( \vec{q} \) without adding it to \( \exists X[F] \). However, in this case, safe reusing of \( K \) requires storing some contextual information like it is done with D-sequents. The theory of reusing D-sequents (that also applies to certificate clauses) is described in [33].

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As we mentioned in Sections 4 and 5, START derives participant and witness certificates. Participant certificates must be added to the formula to make the current target clause redundant. Since participant certificates are essentially conflict clauses they can be safely reused. On the other hand, adding witness certificates is optional and their reusing is not trivial. Importantly, the overwhelming majority of certificates derived by START are witness certificates.
12.2 Generation of duplicate certificates

When running START on multiple practical and crafted problems, we found out that START often generates duplicate participant certificates. Suppose, for instance, that START proved a target clause $C$ redundant in $\exists X[F]$ in the current subspace and temporarily removed $C$ from the formula. When proving another target clause $C'$, START may add to the formula a conflict clause (certifying the redundancy of $C'$ in the current subspace) that is identical to $C$. Since, in general, the new copy of $C$ needs to be proved redundant too, generation of duplicate target clauses may lead to looping$^8$.

The reason for looping is that START processes one target clause at a time and so uses a collection of search trees rather than a single one. (On the other hand, DS-PQE processes all target clauses together, which allows it to maintain a single search tree thus avoiding looping.) There are at least two ways to prevent looping. First, one can just relax the requirement to process only one target clause at a time. Suppose, for instance, that a duplicate of a target clause $C$ earlier proved redundant is generated when proving redundancy of $C'$. Then it makes sense to prove $C$ and $C'$ redundant together. In other words, instead of choosing between two extreme paradigms of processing all clauses together (DS-PQE) or one clause at a time (START), one can consider joint processing of subsets of target clauses. The second way to avoid looping is to completely avoid adding new clauses with quantified variables. (So, no duplicates are generated and looping becomes impossible.) However, this requires a radical overhaul of the current approach to PQE solving [34].

12.3 On comparison of START with DS-PQE

Let $C$ be a clause to be taken out of the scope of quantifiers in $\exists X[F(X,Y)]$. Let $\vec{y}$ be a full assignment to $Y$. Let $C$ be redundant in $\exists X[F]$ in subspace $\vec{y}$. A trivial case of redundancy occurs when $F \setminus \{C\}$ implies $C$ in subspace $\vec{y}$. (In general, this is not the case i.e., redundancy of $C$ in $\exists X[F]$ in subspace $\vec{y}$ does not entail $F \setminus \{C\} \Rightarrow C$ in this subspace.) Even though we did not describe this fact, START actually employs a SAT-based procedure for identifying trivial redundancies. How this is done is described in [35]. On the other hand, DS-PQE does not use any procedure for identifying trivial redundancies. This partly explains the poor performance of DS-PQE in comparison with START reported in Subsection 8.4 and Appendix F.1. Our later results presented in [35] show that DS-PQE supplied with a procedure for identifying trivial redundancies performs much better on the examples we used in this report.

12.4 A few concluding thoughts

Despite the remarks above, we believe that processing one target clause at a time is an appealing idea substantiated experimentally. The remaining issue to

$^8$ In Appendix E, to prove the completeness of START, we introduce its modification that does not loop. However, this modification is not efficient and hence impractical.
be resolved is to find a way for preventing START from generation of duplicate target clauses (thus eliminating the possibility of looping).

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Appendix

A Proofs

Lemma 1 is used in the proof of Proposition 1.

Lemma 1. Given a formula $\exists X[F(X)]$, let $C$ be a clause blocked in $\exists X[F]$ at $w$. Then $\exists X[F] \equiv \exists X[F \setminus \{C\}]$ i.e. $C$ is redundant in $\exists X[F]$.

Proof. Let us prove that $F$ and $F \setminus \{C\}$ are equisatisfiable (and so $\exists X[F] \equiv \exists X[F \setminus \{C\}]$). The satisfiability of $F$ obviously implies that of $F \setminus \{C\}$. Let us show that the converse is true as well. Let $\bar{x}$ be a full assignment to $X$ satisfying $F \setminus \{C\}$. If $\bar{x}$ falsifies $C$, it satisfies $F$ and our proof is over. Now assume that $\bar{x}$ falsifies $C$ and hence falsifies $F$. Let $\bar{x}_{fl}$ be the assignment obtained from $\bar{x}$ by flipping the value of $\bar{x}$ on $C$, because $\bar{x}$ falsifies $C$. Let $\bar{x}_{fl}$ satisfies $C$. Let $G$ be the set of clauses of $F$ resolvable with $C$ on $w$. Let $w' = b$ satisfy $C$ where $b \in \{0, 1\}$. (So, $w$ is assigned $b$ in $\bar{x}_{fl}$, because $\bar{x}$ falsifies $C$.)

First, let us show that $\bar{x}_{fl}$ satisfies $F \setminus G$. Assume the contrary i.e. $\bar{x}_{fl}$ falsifies a clause $D$ of $F \setminus G$. (Note that $D$ is different from $C$ because the latter is satisfied by $\bar{x}_{fl}$.) Assume that $D$ does not contain the variable $w$. Then $D$ is falsified by the assignment $\bar{x}$ and hence the latter does not satisfy $F \setminus \{C\}$. So we have a contradiction. Now, assume that $D$ contains $w$. Then $D$ is resolvable with $C$ on $w$ and $D \in G$. So $D$ cannot be in $F \setminus G$ and we have a contradiction again.

Since $\bar{x}_{fl}$ satisfies $F \setminus G$, then $(F \setminus G)_{w = b}$ is satisfiable. By definition of a blocked clause (see Definition 10), $G$ is redundant in $\exists X[F]$ in subspace $w = b$. So formula $F_{w = b}$ is satisfiable. Hence $F$ is satisfiable too.

Proposition 1. Given a formula $\exists X[F(X,Y)]$, let $C$ be a clause blocked in $\exists X[F]$ at $w \in X$. Then $C$ is redundant in $\exists X[F]$ i.e. $\exists X[F] \equiv \exists X[F \setminus \{C\}]$. So, $C$ is es-implied by $F \setminus \{C\}$ in $\exists X[F]$.

Proof. One needs to show that for every full assignment $\bar{y}$ to $Y$, formulas $F_{\bar{y}}$ and $(F \setminus \{C\})_{\bar{y}}$ are equisatisfiable. If $\bar{y}$ satisfies $C$, it is trivially true. Assume that $\bar{y}$ does not satisfy $C$. From Definition 10 it follows that if $C$ is blocked in $\exists X[F]$ at a variable $w$, then $C_{\bar{y}}$ is blocked in $(\exists F[X])_{\bar{y}}$ at $w$. Then from Lemma 1 if follows that $C_{\bar{y}}$ is redundant in $(\exists F[X])_{\bar{y}}$.

Proposition 2. Given a formula $\exists X[F(X,Y)]$, let $C_{try} \in F$. Let $\bar{q}$ be an assignment to $X \cup Y$ that does not satisfy $C_{try}$. Let $C_{try}$ be blocked in $\exists X[F]$ at $w \in X$ in subspace $\bar{q}$ where $w \notin Vars(\bar{q})$. Let $l(w)$ be the literal of $w$ present in $C_{try}$. Let $K'$ denote the longest clause falsified by $\bar{q}$. Let $K''$ be a clause formed from $l(w)$ and a subset of literals of $C_{try}$ such that every clause of $F_{\bar{q}}$ unresolvable with $(C_{try})_{\bar{q}}$ on $w$ is unresolvable with $(K'')_{\bar{q}}$ too. Let $K_{bct} = K' \lor K''$. Then $(K_{bct})_{\bar{q}} \Rightarrow (C_{try})_{\bar{q}}$ and $K_{bct}$ is redundant in $\exists X[K_{bct} \land (F \setminus \{C_{try}\})]$.

Proof. The fact that $(K_{bct})_{\bar{q}} \Rightarrow (C_{try})_{\bar{q}}$ trivially follows from the definition of $K_{bct}$. The latter equals $K' \lor K''$ where $K'$ is falsified by $\bar{q}$ and $K''$ consists
only of (some) literals of $C_{\text{try}}$. Now we prove that the clause $K_{\text{bct}}$ is redundant in $\exists X [K_{\text{bct}} \land (F \setminus \{C_{\text{try}}\})]$. Let $H$ denote $F \setminus \{C_{\text{try}}\}$. One needs to show that for every full assignment $\bar{y}$ to $Y$, $(K_{\text{bct}} \land H)\bar{y}$ and $H\bar{y}$ are equisatisfiable. If $\bar{y}$ satisfies $K_{\text{bct}}$, it is trivially true. Let $\bar{y}$ not satisfy $K_{\text{bct}}$. (This means that the variables of $\text{Vars}(\bar{y}) \cap \text{Vars}(\bar{q})$, if any, are assigned the same value in $\bar{y}$ and $\bar{q}$.) The satisfiability of $(K_{\text{bct}} \land H)\bar{y}$ obviously implies that of $H\bar{y}$. Below, we show in three steps that the converse is true as well. First, we introduce an assignment $\bar{p}_{fl}$ such that $\bar{y} \subseteq \bar{p}_{fl}$ and $\bar{q} \subseteq \bar{p}_{fl}$. (Here 'fl' stands for 'flipped'.) Second, we prove that $\bar{p}_{fl}$ satisfies $F_{\bar{q}} \setminus G_{\bar{q}}$ where $G$ is the set of clauses resolvable with $C_{\text{try}}$. Third, we show that the satisfiability of $F_{\bar{q}} \setminus G_{\bar{q}}$ and the fact that $C_{\text{try}}$ is blocked imply that $(K_{\text{bct}} \land H)\bar{y}$ is satisfiable.

**Step 1.** Let $\bar{p}$ denote a full assignment to $X \cup Y$ such that $\bar{y} \subseteq \bar{p}$ and $\bar{p}$ satisfies $H_{\bar{q}}$. If $\bar{p}$ satisfies $(K_{\text{bct}} \land H)_{\bar{q}}$, our proof is over. Assume that $\bar{p}$ falsifies $(K_{\text{bct}} \land H)_{\bar{q}}$. This means that $\bar{q} \subseteq \bar{p}$. Let $w = b$ denote the assignment to $w$ in $\bar{p}$. Denote by $\bar{p}_{fl}$ the assignment obtained from $\bar{p}$ by flipping the value of $w$ from $b$ to $\bar{b}$. Denote by $\bar{q}_{\text{ext}}$ the extension of $\bar{q}$ such that $\bar{q}_{\text{ext}} \subseteq \bar{p}_{fl}$. Besides, due to the assignment $w = \bar{b}$, both $\bar{q}_{\text{ext}}$ and $\bar{p}_{fl}$ satisfy $K_{\text{bct}}$ and $C_{\text{try}}$.

**Step 2.** Let $G$ denote the set of clauses of $F$ resolvable with $C_{\text{try}}$ on $w$. Then $G_{\bar{q}}$ is the set of clauses of $F_{\bar{q}}$ resolvable with $(C_{\text{try}})_{\bar{q}}$ on $w$. Let us show that $\bar{p}_{fl}$ satisfies $F_{\bar{q}} \setminus G_{\bar{q}}$. Assume the contrary i.e. there is a clause $D \in F_{\bar{q}} \setminus G_{\bar{q}}$ falsified by $\bar{p}_{fl}$. First, assume that $D$ does not contain $w$. Then $D$ is falsified by $\bar{p}$ as well. So, $\bar{p}$ falsifies $F_{\bar{q}}$ and hence $H_{\bar{q}}$ (because $(C_{\text{try}})_{\bar{q}}$ is satisfied by $\bar{p}_{fl}$ and so is different from $D$). Thus, we have a contradiction. Now, assume that $D$ contains the literal $\overline{l(w)}$. Then it is resolvable with clause $(K_{\text{bct}})_{\bar{q}}$. This means that $D$ is resolvable with $(C_{\text{try}})_{\bar{q}}$ too. (By our assumption, every clause of $F_{\bar{q}}$ resolvable with $(C_{\text{try}})_{\bar{q}}$ is resolvable with $(K_{\text{bct}})_{\bar{q}}$ too.) Then $D$ cannot be in $F_{\bar{q}} \setminus G_{\bar{q}}$ and we have a contradiction.

**Step 3.** Since $\bar{p}_{fl}$ satisfies $F_{\bar{q}} \setminus G_{\bar{q}}$, the formula $F_{\bar{q}_{\text{ext}}} \setminus G_{\bar{q}_{\text{ext}}}$ is satisfiable. The same applies to $(F \setminus G)_{\bar{q}_{\text{ext}}}$, since $C_{\text{try}}$ is blocked in $\exists X[F]$ at $w$ in subspace $\bar{q}$, it is also blocked in $\exists X[F]$ in subspace $\bar{y} \cup \bar{q}$. Then $(F \setminus G)_{\bar{q}_{\text{ext}}}$ es-implies $G_{\bar{q}_{\text{ext}}}$ (see Definition 10) and $F_{\bar{q}_{\text{ext}}}$ is satisfiable too. Since $K_{\text{bct}}$ is satisfied by $\bar{q}_{\text{ext}}$, then $(K_{\text{bct}} \land F)_{\bar{q}_{\text{ext}}}$ is satisfiable. Hence $(K_{\text{bct}} \land F)_{\bar{q}}$ is satisfiable and so is $(K_{\text{bct}} \land H)_{\bar{y}}$.

**B Backtracking By START**

When a SAT-solver encounters a conflict, it generates a conflict clause and backtracks to the smallest decision level where this clause is unit. So, an assignment can be derived from this clause. In contrast to a SAT-solver, the goal of a PQE-solver is to prove a target clause $C_{\text{try}}$ redundant rather than find a satisfying assignment. So, START backtracks slightly differently from a SAT-solver. After START derives a certificate $K$ proving $C_{\text{try}}$ in the current subspace, it backtracks to the smallest decision level at which the conditional of the derived certificate $K$ (rather than $K$ itself) is unit.
Definition 11. Let $K$ be a certificate stating the redundancy of clause $C_{trg}$ in a subspace. The clause consisting of the literals of $K$ not shared with $C_{trg}$ is called the conditional of $K$.

If the conditional of $K$ is empty, $K$ implies $C_{trg}$ in the entire space. Otherwise, $K$ implies $C_{trg}$ only in subspaces $\vec{q}$ where the conditional of $K$ is falsified by $\vec{q}$. One can derive an implied assignment from $K$ when its conditional is unit like this is done by a SAT-solver when a clause becomes unit.

Example 2. Consider the example of Fig. 6 showing the operation of $START$. This figure gives only the relevant part of formula $F_2$ and the relevant fragment of the execution trace. $PrvRed$ begins proving the target clause $C_1 = x_2 \lor x_4$ redundant by the decision assignment $y_1 = 0$. Then it calls $BCP$ that derives $x_3 = 1$ from the clause $C_2$. At this point, $C_3$ becomes the unit clause $x_4$ implying $C_1$. So, $BCP$ returns $C_3$ as the reason for backtracking (i.e. as the clause $K_{bc}$ in line 6 of Fig. 3). Then the $Lrn$ procedure generates the final certificate $K_1 = y_1 \lor x_4$ by resolving $C_3$ and $C_2$ to drop the non-decision variable $x_3$ assigned at level 1 (line 14).

The conditional of $K_1$ is the unit clause $y_1$ because the literal $x_4$ is shared by $K_1$ and the target clause $C_1$. $PrvRed$ backtracks to level 0, the smallest level where the conditional of $K_1$ is unit. (Note that $K_1$ itself is not unit at level 0). Then $PrvRed$ runs $BCP$ and derives the assignment $y_1 = 1$ from $K_1$ even though $K_1$ is not unit at level 0. This derivation is possible because $K_1$ certifies that the redundancy of $C_1$ in subspace $y_1 = 0$ is already proved.

As we mentioned above, in the general case, after deriving a certificate $K$, $PrvRed$ backtracks to the smallest decision level where the conditional of $K$ is unit. The assignment derived from $K$ is added to the assignment queue $Q$ (lines 10 and 12 of Fig. 3). If $K$ shares no literals with $C_{trg}$, the $PrvRed$ procedure backtracks as a regular SAT-solver, i.e. to the smallest decision level where $K$ is unit.
C Operation Of \textit{START} When $C_{trg}$ Becomes Unit

In Section 6, we described how \textit{START} operates when the current target clause $C_{trg}$ becomes unit. In this appendix, we give a concrete example. Consider solving the PQE problem shown in Fig. 7 by lines 1-6. First, $C_1$ is picked as a clause to prove. We will refer to it as the \textbf{primary} target assuming that it makes up target level $A$. After decision assignment $y_1 = 0$, the clause $C_1$ turns into unit clause $x_2$ (lines 9-10). Denote the current assignment (i.e. $y_1 = 0$) as $\vec{q}$. At this point, a SAT-solver would simply derive the assignment $x_2 = 1$.

However, the goal of \textit{PrvRed} is not to check if $F_1 \land F_2$ is satisfiable but to prove $C_1$ redundant. The fact that $C_1$ is falsified in subspace $\vec{q} \cup \{x_2 = 0\}$ does not say anything about whether $C_1$ is redundant there.

So, \textit{PrvRed} creates a new target level (referred to as level $B$). It consists of the clauses resolvable with $C_1$ on $x_2$. Suppose all clauses of this level are redundant in subspace $\vec{q} \cup \{x_2 = 1\}$. Then according to Definition 10, $C_1$ is blocked (and hence redundant) in $\exists X[F_1 \land F_2]$ in subspace $\vec{q}$. In our case, level $B$ consists only of $C_2$. So, \textit{PrvRed} recursively calls itself to prove redundancy of $C_2$ in subspace $\vec{q} \cup \{x_2 = 1\}$ (lines 15-20). Note that $C_2$ is blocked at $x_3$ in this subspace since $C_3$ (the clause resolvable with $C_2$ on $x_3$) is satisfied by $y_1 = 0$. Then using Proposition 2, \textit{PrvRed} derives the certificate $K' = y_1 \lor x_3$ asserting the redundancy of $C_2$. The latter is temporarily removed from the formula as redundant (see Remark 4). At this point, the second activation of \textit{PrvRed} terminates.

Then the first activation of \textit{PrvRed} undoes target level $B$ and assignment $x_2 = 1$. The clause $C_2$ is restored in the formula (lines 21-23). Now, the primary target $C_1$ is blocked at $x_2$, since $C_2$ is proved redundant in subspace $\vec{q} \cup \{x_2 = 1\}$. Using Proposition 2, \textit{PrvRed} derives the certificate $K'' = y_1 \lor x_2$ proving redundancy of $C_1$ in the entire space.

D Certificate Generation When A Conflict Occurs

In this appendix, we discuss in more detail the generation of a certificate by the \textit{Lrn} procedure when a conflict occurs. As before, we denote $F_1 \land F_2$ by $F$. Let
\( C_{\text{try}} \) be the current target clause. Let \( K_{\text{bct}} \) be the clause of \( F \) falsified in this conflict. (Here, we use the notation of Figure 3 describing the \textit{PrvRed} procedure).

First, consider the case when \( K_{\text{bct}} \neq C_{\text{try}} \). Then \textit{Lrn} generates a certificate \( K \) as described in Subsection 5.2. Namely, it starts with \( K_{\text{bct}} \) gradually resolving out literals assigned at the conflict level by non-decision assignments. Since \( C_{\text{try}} \) is not involved in derivation of \( K \), the latter is a witness certificate.

Now, consider the case when \( K_{\text{bct}} = C_{\text{try}} \). If, no relevant assignment is derived from a witness certificate, \textit{Lrn} generates the resulting certificate \( K \) as described above. Since \( C_{\text{try}} \) is involved in derivation of \( K \), the latter is a participant certificate that is added to the formula. If an assignment relevant to the conflict is derived from a witness certificate, \textit{Lrn} acts differently. Namely, it derives a witness certificate \( K \) and a special clause \( \hat{K} \) that is added to the formula. (For the sake of simplicity, we did not mention this fact in the pseudo-code of the \textit{PrvRed} procedure.)

Figure 8 illustrates adding a special clause. Here \( C_1 = \overline{x}_2 \vee x_3 \) is the target clause. In the branch \( y_1 = 0 \), \textit{PrvRed} proves \( C_1 \) redundant by deriving a witness certificate \( K_1 = y_1 \vee \overline{x}_2 \) (lines 9-13). Then \textit{PrvRed} backtracks to level 0 and runs \textit{BCP} to derive \( y_1 = 1 \) from \( K_1, x_2 = 1 \) from \( C_2 \) and \( x_3 = 0 \) from \( C_3 \). At this point, \( C_1 \) is falsified i.e. a conflict occurs. Assume we construct a certificate \( K_2 = \overline{x}_2 \) by resolving \( C_1 \) with \( C_2, C_3, \) and \( K_1 \) (i.e. with the clauses from which the relevant assignments were derived). Then we have a problem. On one hand, \( K_2 \) is a participant certificate that has to be added to \( F \) since the target clause \( C_1 \) was involved in building \( K_2 \). On the other hand, \( K_2 \) may not be implied by \( F \) since a witness certificate \( K_1 \) was involved in producing \( K_2 \). (A witness certificate \( K \) is, in general, only es-implied by \( F \) in \( \exists X[K \land F] \).) This breaks the invariant maintained by \textit{START} that only clauses implied by \( F \) are added to it.

The \textit{Lrn} procedure addresses the problem above as follows. First, it generates a clause \( \hat{K} = \overline{x}_1 \) that is falsified in the current subspace and so “replaces” \( C_1 \) as the reason for the conflict. \( \hat{K} \) is built without using witness certificates and so \textit{can} be added to \( F \). It is obtained by resolving \( C_1 \) with \( C_2 \) and \( C_3 \) and is added to \( F \) (lines 19-23). Then \textit{Lrn} derives the certificate \( K_2 = \overline{x}_2 \) by resolving \( \hat{K} \) and \( K_1 \). The clause \( K_2 \) certifies the redundancy of the target clause \( C_1 \) in the entire
space. Note that $K_2$ was derived using $\hat{K}$ instead of the target clause $C_1$. So, it is a witness certificate that does not have to be added to the formula.

Here is how one handles the general case when $K_{bct} = C_{trg}$ and a witness certificate is involved in the conflict. First, one produces a special clause $\hat{K}$. It is obtained by resolving $C_{trg}$ with clauses of $F$ from which relevant assignments were derived. This process stops when the assignment derived from a witness certificate is reached. Then $\hat{K}$ is added to the formula $F$. (This can be done since witness certificates are not used in derivation of $\hat{K}$.) After that, $Lrn$ generates a certificate $K$ starting with $\hat{K}$ as a clause falsified in the current subspace. (That is, $K$ replaces $C_{trg}$ as the cause of the conflict.) Since $C_{trg}$ is not involved in generation of $K$, the latter is a witness certificate.

E Correctness of \textit{START}

In this appendix, we give a proof that \textit{START} is correct. Let \textit{START} be used to take $F_1$ out of the scope of quantifiers in $\exists X [F_1(X,Y) \land F_2(X,Y)]$. We will denote $F_1 \land F_2$ by $F$. In Subsection E.1, we show that \textit{START} is sound. Subsection E.2 discusses the problem of generating duplicate clauses by \textit{START} and describes a solution to this problem. In Subsection E.3, we show that the versions of \textit{START} that do not produce duplicate clauses are complete.

E.1 \textit{START} is sound

In its operation, \textit{START} adds participant certificates and removes target clauses from $F$. Denote the initial formula $F$ as $F_{ini}$. Let $\vec{y}$ be a full assignment to the variables of $Y$ (i.e. unquantified ones). Below, we demonstrate that for every subspace $\vec{y}$, \textit{START} preserves the equisatisfiability between the $F_{ini}$ and the current formula $F$. That is, $\exists X [F_{ini}] \equiv \exists X [F]$. Then we use this fact to show that \textit{START} produces a correct solution.

First, consider adding participant certificates by \textit{START}. As we mention in Section 5, every clause added to $F$ is implied by $F$. (If a clause $K$ is es-implied by $F$ in $\exists X [K \land F]$, it is used only as a witness certificate and is not added to $F$.) So, adding clauses cannot break equisatisfiability of $F$ and $F_{ini}$ in a subspace $\vec{y}$.

Now, we consider removing target clauses from $F$ by \textit{START}. A target clause $C_{trg}$ is permanently removed from the formula only if a certificate $K$ implying $C_{trg}$ in the entire space is derived. $K$ is obtained by resolving clauses of the current formula $F$ and witness certificates (if any). Derivation of $K$ is correct due to correctness of Proposition 2 (describing generation of clauses that are es-implied rather than implied by the formula) and soundness of resolution. So, removing $C_{trg}$ from $F$ cannot break equisatisfiability of $F$ and $F_{ini}$ in some subspace $\vec{y}$.

The fact that $\exists X [F_{ini}] \equiv \exists X [F]$ entails that \textit{START} produces a correct solution. Indeed, \textit{START} terminates when the current formula $F_1$ does not contain a quantified clause. So, the final formula $F$ can be represented as $F_1(Y) \land F_2(X,Y)$. Then $\exists X [F_1^{mi} \land F_2^{mi}] \equiv F_1 \land \exists X [F_2]$. \textit{START} does not add any clauses to $F_2$. 
Hence, the final and initial formulas $F_2$ are identical. So, $\exists X[F_1^{ini} \land F_2^{ini}] \equiv F_1^* \land \exists X[F_2^{ini}]$ where $F_1^*$ is the final formula $F_1$.

E.2 Avoiding generation of duplicate clauses

The version of $PrvRed$ described in Sections 4-6 may generate a duplicate of a quantified clause that is currently proved redundant. To avoid generating duplicates one can modify $START$ as follows. (We did not implement this modification due to its inefficiency. We present it just to show that the problem of duplicates can be fixed in principle.) We will refer to this modification as $START^*$.

Suppose $PrvRed$ generated a quantified clause $C$ proved redundant earlier. This can happen only when all variables of $Y$ are assigned because they are assigned before those of $X$. Then $START^*$ discards the clause $C$, undoes the assignment to $X$, and eliminates all recursive calls of $PrvRed$. That is $START^*$ returns to the original call of $PrvRed$ made in the main loop (Fig. 2, line 7). Let $C_{trg}$ be the target clause of this call of $PrvRed$ and $\bar{y}$ be the current (full) assignment to $Y$. At this point $START^*$ calls an internal SAT-solver to prove redundancy of $C_{trg}$ in subspace $\bar{y}$. This goal is achieved by this SAT-solver via generating a witness or participant certificate implying $C_{trg}$ in subspace $\bar{y}$ (see below). After that, $PrvRed$ goes on as if it just finished line 10 of Figure 3.

Let $B(Y)$ denote the longest clause falsified by $\bar{y}$. Suppose the internal SAT-solver of $START^*$ proves $F_{\bar{y}}$ unsatisfiable. (Recall that $F$ denotes $F_1 \land F_2$.) Then the clause $B$ is a certificate of redundancy of $C_{trg}$ in $F_{\bar{y}}$. If $C_{trg}$ is involved in proving $F_{\bar{y}}$ unsatisfiable, $B$ is a participant certificate. The $PrvRed$ procedure adds $B$ to $F$ to make $C_{trg}$ redundant in subspace $\bar{y}$. If $F_{\bar{y}}$ is proved unsatisfiable without using $C_{trg}$, then $B$ is a witness certificate that is not added to $F$.

Suppose that $F_{\bar{y}}$ is satisfiable. Then the internal SAT-solver above derives an assignment $\bar{p}$ satisfying $F_{\bar{y}}$ where $\bar{y} \subseteq \bar{p}$. Note that $\bar{y}$ does not satisfy $C_{trg}$ since, otherwise, $PrvRed$ would have already proved redundancy of $C_{trg}$ in subspace $\bar{y}$. Hence, $\bar{p}$ satisfies $C_{trg}$ by an assignment to a variable $w \in X$. Then $PrvRed$ derives a witness certificate $K$ equal to $B \lor \ell(w)$ where $\ell(w)$ is the literal of $w$ present in $C_{trg}$. It is not hard to show that $K$ is indeed a certificate. First, it implies $C_{trg}$ in subspace $\bar{y}$ certifying its redundancy there. Second, $K$ is implied by $F \setminus \{C_{trg}\}$ in $\exists X[K \land (F \setminus \{C_{trg}\})]$.

E.3 $START$ is complete

In this subsection, we show the completeness of the versions of $START$ that do not generate duplicate clauses. (An example of such a version is given in the previous subsection). The completeness of $START$ follows from the fact that

- the number of times $START$ calls the $PrvRed$ procedure (to prove redundancy of the current target clause) is finite;
- the number of steps performed by one call of $PrvRed$ is finite.

So, $START$ always terminates. First, let us show that $PrvRed$ is called a finite number of times. By our assumption, $START$ does not generate quantified
clauses seen before. So, the number of times PrvRed is called in the main loop of START (see Figure 2) is finite. PrvRed recursively calls itself when the current target clause $C_{trg}$ becomes unit. The number of such calls is finite (since the number of clauses that can be resolved with $C_{trg}$ on its unassigned variable is finite). The depth of recursion is finite here. Indeed, before a new recursive call is made, the unassigned variable $w \in X$ of $C_{trg}$ is assigned and $X$ is a finite set. Summarizing, the number of recursive calls made by PrvRed invoked in the main loop of START is finite.

Now we prove that the number of steps performed by a single call of PrvRed is finite. (Here we ignore the steps taken by recursive calls of PrvRed.) Namely, we show that PrvRed examines a finite search tree. The number of branching nodes of the search tree built by PrvRed is finite because $X \cup Y$ is a finite set. Let us show that PrvRed indeed builds a tree. That is PrvRed does not have “holes” and always reaches a leaf i.e. a node where a backtracking condition is met. Below, we list the four kinds of leaves reached by PrvRed. (The backtracking conditions are identified by the BCP procedure called by PrvRed.) Let $\vec{q}$ specify the current assignment to $Y \cup X$. A leaf of the first kind is reached when the target clause $C_{trg}$ becomes unit in subspace $\vec{q}$. Then BCP calls the Rcrs procedure (line 11 of Fig. 4) and PrvRed backtracks. PrvRed reaches a leaf of the second kind when BCP finds a clause of $F$ implying $C_{trg}$ in subspace $\vec{q}$. A leaf of the third kind is reached when BCP identifies a clause falsified by $\vec{q}$ (i.e. a conflict occurs). PrvRed reaches a leaf of the fourth kind when the target clause $C_{trg}$ is blocked in $\exists X[F]$ in subspace $\vec{q}$.

If $F$ is unsatisfiable in subspace $\vec{q}$, PrvRed always reaches a leaf before all variables of $Y \cup X$ are assigned. (Assigning all variables without a conflict, i.e. without reaching a leaf of the third kind, would mean that $F$ is satisfiable in subspace $\vec{q}$.) Let us show that if $F$ is satisfiable in subspace $\vec{q}$, PrvRed also always reaches a leaf before every variable of $Y \cup X$ is assigned. (That is before a satisfying assignment is generated.) Let $\vec{p}$ be an assignment satisfying $F$ where $\vec{q} \subseteq \vec{p}$. Consider the worst case scenario. That is all variables of $Y \cup X$ but some variable $w$ are already assigned in $\vec{q}$ and no leaf condition is encountered yet. Assume that no literal of $w$ is present in the target clause $C_{trg}$. Since $\vec{q}$ contains all assignments of $\vec{p}$ but that of $w$, $C_{trg}$ is satisfied by $\vec{q}$. Recall that PrvRed does not make decision assignments satisfying $C_{trg}$ (see Subsection 5.2). So, $C_{trg}$ is satisfied by an assignment derived from a clause $C$. Then $C$ implies $C_{trg}$ in subspace $\vec{q}$ and a leaf of the second kind must have been reached. So, we have a contradiction.

Now assume that $C_{trg}$ has a literal $l(w)$ of $w$. Note that since PrvRed assigns variables of $Y$ before those of $X$, then $w \in X$. Since $C_{trg}$ is not implied by a clause of $F$ in subspace $\vec{q}$, all the literals of $C_{trg}$ but $l(w)$ are falsified by $\vec{q}$. Let us show that $C_{trg}$ is blocked in $\exists X[F]$ at $w$ in subspace $\vec{q}$. Assume the contrary i.e. there is a clause $C$ resolvable with $C_{trg}$ on $w$ that contains the literal $l(w)$ and is not satisfied yet. That is all the literals of $C$ other than $l(w)$ are falsified by $\vec{q}$. Then $\vec{p}$ cannot be a satisfying assignment because it falsifies either $C_{trg}$ or $C$
(depending on how the variable $w$ is assigned). So, we have a contradiction. Thus, $C_{trg}$ is blocked at $w$ in subspace $\vec{q}$ and hence a leaf of the fourth kind is reached.

F Experiments With HWMCC-13 Benchmarks

In this appendix, we describe experiments with multi-property benchmarks of the HWMCC-13 set [36]. (We use this set because the multi-property track has been discontinued in HWMCC since 2013.) Each benchmark consists of a sequential circuit $N$ and invariants that are supposed to hold for $N$. One can view the conjunction of those invariants as a specification $Sp$ for $N$. In the experiments, we used $START$ to generate invariants of $N$ not implied by $Sp$. Similarly to the experiment of Section 8, the formula $F_k = I(S_0) \land T(S_0, S_1) \land \cdots \land T(S_{k-1}, S_k)$ was used to generate invariants. The number $k$ of time frames was in the range of $2 \leq k \leq 10$. Specifically, we set $k$ to the largest value in this range where $|F_k|$ did not exceed 500,000 clauses. We discarded the benchmarks with $|F_2| > 500,000$. So, in the experiments, we used 112 out of the 178 benchmarks of the set.

We describe three experiments. In every experiment, we generated local invariants $H_k$ by taking out a clause of $\exists S_{k-1}[F_k]$. The objective of the first experiment was to demonstrate that $START$ could compute $H_k$ for realistic designs. We also showed in this experiment that PQE could be much easier than QE and that $START$ outperforms our previous PQE-solver called DS-PQE. The second experiment demonstrated that a clause $Q$ of a local invariant $H_k$ generated by $START$ was often a global invariant not implied by the specification $Sp$. (As we mentioned in Section 7, the necessary condition for an invariant $Q$ to be unwanted is $Sp \not\Rightarrow Q$.) Note that the circuits of the HWMCC-13 set are “anonymous”. So, we could not decide if $Q$ was an unwanted invariant. Our goal was to show that $START$ was good enough to generate invariants not implied by $Sp$. (Then one could check those invariants for being unwanted as described in Section 9.) As in the experiment of Section 8, we took out only clauses containing a state variable of the $k$-th time frame. The choice of the next clause to take out was made according to the order in which clauses were listed in $F_k$. In the third experiment, we showed that $START$ generates invariants that are different from those produced by $IC3$. 


F.1 Experiment 1

Table 2: START and DS-PQE.

| pqe solver | total solved | unsolved |
|------------|--------------|----------|
| start      | 5,418        | 3,102    |
| ds-pqe     | 5,418        | 1,285    |

In this experiment, for each benchmark out of 112 mentioned above we generated PQE problems of taking a clause out of ∃S_{k-1}[F_k]. Some of them were trivially solved by pre-processing. The latter eliminated the blocked clauses of F_k that could be easily identified and ran BCP launched due to the unit clauses specifying the initial state. We generated up to 50 non-trivial problems per benchmark ignoring those solved by pre-processing. (For some benchmarks the total number of non-trivial problems was under 50.)

We compared START with DS-PQE introduced in [1] that is based on the machinery of D-sequents. The relation of D-sequents and certificates is briefly discussed in Section 10. In contrast to START, DS-PQE proves redundancy of many targets at once, which can lead to generating very deep search trees. To make the experiment less time consuming, we limited the run time of START to 5 sec. per PQE problem. The results are shown in Table 2. The first column gives the name of a PQE solver. The second column shows the total number of PQE problems we generated for the 112 benchmarks. The last two columns give the number of problems solved and unsolved in the time limit. Table 2 shows that START solved 57% of the problems within 5 sec. For 92 benchmarks out of 112, at least one PQE problem generated off ∃S_{k-1}[F_k] was solved by START in the time limit. This is quite encouraging since many solved PQE problems had more than a hundred thousand variables and clauses. Table 2 also shows that START drastically outperforms DS-PQE.

To contrast PQE and QE, we used CADET [12,13] to perform QE on 112 formulas ∃S_{k-1}[F_k]. That is, instead of taking a clause out of ∃S_{k-1}[F_k] by PQE, we applied CADET to perform full QE on this formula. (As mentioned in Subsection 7.2, performing QE on ∃S_{k-1}[F_k] produces the strongest local invariant specifying all states unreachable in k transitions.) Our choice of CADET was motivated by its high performance. CADET is a SAT-based tool that solves QE implicitly via building Skolem functions. In the context of QE, CADET often scales better than BDDs [16,37]. CADET solved only 32 out of 112 QE problems with the time limit of 600 sec. For many formulas ∃S_{k-1}[F_k] that CADET failed to solve in 600 sec., START solved all 50 PQE problems generated off ∃S_{k-1}[F_k] in 5 sec. So, PQE can be much easier than QE if only a small part of the formula gets unquantified.

F.2 Experiment 2

The second experiment was an extension of the first experiment. Namely, for each clause Q of a local invariant H_k generated by PQE we used IC3 to verify if Q was a global invariant. If so, we checked if Sp ≠ Q held.
Similarly to the first experiment, to make the experiment less time consuming, we set the time limit of 5 sec. per PQE problem. Besides, we imposed the following constraints. (Even with those constraints, the run time of the experiment was about 4 days.) First, we stopped START even before the time limit if it generated more than 5 free clauses. Second, the time limit for IC3 was set to 30 sec. Third, instead of constraining the number of PQE problems per benchmark, we limited the total number of free clauses generated for a benchmark. Namely, processing a benchmark terminated when this number exceeded 100.

A sample of 9 benchmarks out of the 112 we used in the experiment is shown in Table 3. Let us explain the structure of this table by the benchmark 6s380 (the first line of the table). The name of this benchmark is shown in the first column. The second column gives the number of latches (5,606). The number of invariants that should hold for 6s380 is provided in the third column (897). So, the specification $Sp$ of 6s380 is the conjunction of those 897 invariants. The fourth column shows that the number $k$ of time frames for 6s380 was set to 2 (since $|F_{23}| > 500,000$). The value 46 shown in the fifth column is the total number of clauses taken out of $\exists S_{k-1}[F_k]$ i.e. the number of PQE problems. (We keep using the index $k$ here assuming that $k = 2$ for 6s380.)

Let $C$ be a clause taken out of the scope of quantifiers by START. Every free clause $Q$ generated by START was stored as a local single-clause invariant. The sixth column shows that taking clauses out of the scope of quantifiers was terminated when 101 local single-clause invariants were generated. (Because the total number of invariants exceeded 100.) Each of these 101 local invariants held in $k$-th time frame. The following three columns show how many of those 101 local invariants were true globally. IC3 finished every problem out of 101 in the time limit. So, the number of undecided invariants was 0. The number of invariants IC3 proved false or true globally was 49 and 52 respectively. The last column gives the number of global invariants not implied by $Sp$. For 6s380, this number is 0.

For 109 benchmarks out of the 112 we used in the experiments, START was able to generate local single-clause invariants that held in $k$-th time frame. For 100 benchmarks out of the 109 above, the invariants $H_k$ generated by START contained global single-clause invariants. For 89 out of these 100 benchmarks, there were global invariants not implied by the specification $Sp$. Those invariants were meant to be checked if any of them was unwanted.

### Table 3: A sample of HWMCC-13 benchmarks

| name   | latches | invar. of $Sp$ | time frames | clauses taken out | local single-clause invariants | gen. invariants | global invariants | not implied by $Sp$ |
|-------|---------|---------------|-------------|-------------------|-------------------------------|-----------------|-------------------|---------------------|
| 6s380 | 5,606   | 897           | 2           | 46                | 101                           | 0               | 49                | 0                   |
| 6s175 | 1,566   | 952           | 4           | 30                | 101                           | 0               | 9                 | 32                  |
| 6s292 | 1,190   | 247           | 5           | 21                | 104                           | 44              | 0                 | 60                  |
| 6s136 | 32      | 6             | 6           | 218               | 101                           | 0               | 90                | 11                  |
| 6s275 | 3,196   | 340           | 4           | 29                | 105                           | 15              | 75                | 75                  |
| 6s325 | 2,686   | 387           | 9           | 30                | 104                           | 0               | 14                | 90                  |
| 6s372 | 1,124   | 34            | 10          | 159               | 101                           | 60              | 41                | 0                   |

- Similarly to the first experiment, to make the experiment less time consuming, we set the time limit of 5 sec. per PQE problem.
- We imposed the following constraints: (Even with those constraints, the run time of the experiment was about 4 days.)
- First, we stopped START even before the time limit if it generated more than 5 free clauses.
- Second, the time limit for IC3 was set to 30 sec.
- Third, instead of constraining the number of PQE problems per benchmark, we limited the total number of free clauses generated for a benchmark. Namely, processing a benchmark terminated when this number exceeded 100.

Let $C$ be a clause taken out of the scope of quantifiers by START. Every free clause $Q$ generated by START was stored as a local single-clause invariant. The sixth column shows that taking clauses out of the scope of quantifiers was terminated when 101 local single-clause invariants were generated. (Because the total number of invariants exceeded 100.) Each of these 101 local invariants held in $k$-th time frame. The following three columns show how many of those 101 local invariants were true globally. IC3 finished every problem out of 101 in the time limit. So, the number of undecided invariants was 0. The number of invariants IC3 proved false or true globally was 49 and 52 respectively. The last column gives the number of global invariants not implied by $Sp$. For 6s380, this number is 0.

For 109 benchmarks out of the 112 we used in the experiments, START was able to generate local single-clause invariants that held in $k$-th time frame. For 100 benchmarks out of the 109 above, the invariants $H_k$ generated by START contained global single-clause invariants. For 89 out of these 100 benchmarks, there were global invariants not implied by the specification $Sp$. Those invariants were meant to be checked if any of them was unwanted.
F.3 Experiment 3

When proving an invariant \( P \), \( IC3 \) conjoins it with clauses \( Q_1, \ldots, Q_m \) to make \( P \land Q_1 \land \cdots \land Q_m \) inductive. If \( IC3 \) succeeds, every \( Q_i \) is an invariant. Moreover, \( Q_i \) may be an unwanted invariant. Arguably, the cause of efficiency of \( IC3 \) is that \( P \) is often close to an inductive invariant. So, \( IC3 \) needs to generate a relatively small number of clauses \( Q_i \) to make the constrained version of \( P \) inductive. However, as we mentioned in Section 10, this nice feature of \( IC3 \) drastically limits the set of unwanted invariants it can produce. In this subsection, we substantiate this claim by an experiment. In this experiment, we picked the HWMCC-13 benchmarks for which one could prove all pre-defined invariants \( P_1, \ldots, P_n \) within a time limit. Namely, for every benchmark we formed the specification \( Sp = P_1 \land \cdots \land P_n \) and ran \( IC3 \) to prove \( Sp \) true.

We selected the benchmarks that \( IC3 \) solved in less than 1000 sec. (In addition to dropping the benchmarks not solved in 1000 sec., we discarded those where \( Sp \) failed because some invariants \( P_i \) were false). Let \( Sp^* \) denote the inductive version of \( Sp \) produced by \( IC3 \) when proving \( Sp \) true. That is, \( Sp^* \) is \( Sp \) conjoined with the invariant clauses generated by \( IC3 \). For each of the selected benchmarks we generated invariants by \( START \) exactly as in Experiment 2. That is, we stopped generation of local single clause invariants when their number exceeded 100. Then we ran \( IC3 \) to identify local invariants that were global as well. After that we checked which of the global invariants generated by \( START \) were not implied by \( Sp \) and \( Sp^* \).

Table 4: Invariants of \( START \) and \( IC3 \)

| name | latches | invars in Sp | glob, invrants | glob, invars not impl. by Sp | glob, invars not impl. by Sp* |
|------|---------|-------------|----------------|-----------------------------|-----------------------------|
| 6s135 | 2,460   | 340         | 90             | 85                          | 91                          |
| 6s325 | 1,756   | 301         | 101            | 101                         | 96                          |
| ex1  | 130     | 33          | 29             | 21                          | 19                          |
| ex2  | 212     | 32          | 93             | 61                          | 32                          |
| 6s106 | 135     | 17          | 100            | 56                          | 83                          |
| 6s266 | 3,141   | 1           | 0              | 0                           | 0                           |
| ex3  | 61      | 3           | 2              | 2                           | 2                           |
| ex4  | 64      | 3           | 3              | 3                           | 3                           |
| 6s209 | 5,569   | 2           | 73             | 72                          | 66                          |
| 6s143 | 104     | 1           | 15             | 17                          | 17                          |
| 6s143 | 260     | 1           | 103            | 83                          | 27                          |
| 6s113 | 3,141   | 1           | 1              | 1                           | 1                           |
| 6s202 | 170     | 1           | 94             | 71                          | 65                          |
| Total |         |             |                |                             |                             |
|       | 586     | 532         |                |                             |                             |

The results of the experiment are shown in Table 4. The first three columns of this table are the same as in Table 3. They give the name of a benchmark, the number of latches and the number of invariants \( P_1, \ldots, P_n \) to prove. (The actual names of examples \( ex1, \ldots, ex4 \) in the HWMCC-13 set are \( pdtsarmultip, bobtuintmulti, nusmvdmec1d3multi, nusmvdmec2d3multi \).) The next two columns of Table 4 are the same as the last two columns of Table 3. They show the number of local invariant clauses that turn out to be global invariants and the number of global invariants that were not implied by \( Sp \). The last column gives the number of global invariants that were not implied by \( Sp^* \). The last row of the table shows that in 532 cases out of 586 the invariants not implied by \( Sp \) were not implied by \( Sp^* \) either. So, in 90% of cases \( START \) generated invariant clauses different from those of \( IC3 \).