Discrete Volterra equation via Exp-function Method

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Abstract. In this paper, we utilize the Exp-function method to construct three families of new generalized solitary solutions for the discrete Volterra equation. It is shown that the Exp-function method, with the help of symbolic computation, provides a powerful mathematical tool for solving typical discrete nonlinear evolution equations in physics.

1. Introduction
Seeking exact solutions of nonlinear evolution equations is of important significance in mathematical physics. In the past decades, several powerful methods have been developed to construct many types of exact solutions of nonlinear partial differential equations (PDEs), such as inverse scattering theory[1], Bäcklund transformation[2], the tanh function method[3,4], homogeneous balance method[5,6], multilinear variable separation approach[7,8], Jacobian elliptic function method[9,10], homotopy perturbation method[11,12], variational iteration method[13], a heuristic review on recently developed analytical methods is available on Ref.[14,15]. In recent years, the direct search for exact solutions of PDEs has become more and more attractive partly due to the availability of computer symbolic systems like Maple or Mathematica, which allow us to perform the complicated and tedious algebraic calculations on computer. In particular, one of the most effective direct methods to construct exact solutions of PDEs is the Exp-function method[16-20].

Since the work of Fermi, Pasta and Ulam in the 1950s[21], the investigation of exact solutions of the nonlinear differential-difference equations (DDEs) has played a crucial role in the modelling of many phenomena in different fields, which include condensed matter physics, biophysics and mechanical engineering. One also encounters such systems in numerical simulation of soliton dynamics in high energy physics where they arise as approximations of continuum models. Unlike difference equations which are fully discretized, DDEs are semi-discretized with some (or all) of their spatial variables discretized while time is usually kept continuous.

However, different from the considerable works done on finding exact solutions to PDEs just mentioned above, to our knowledge, only few researchers have investigated exact solutions for DDEs. Recently, the Exp-function method, which was first presented by He[16], was proposed to seek solitary solutions, periodic solutions and compacton-like solutions of nonlinear differential equations. In this paper, we further extend the Exp-function method to the nonlinear differential-difference equations (DDEs). Another section of your paper...
2. **The Exp-function method for DDEs**

The Exp-function method[16-20] can be adapted to solve nonlinear polynomial DDEs. Given a system of DDEs

\[
\frac{\partial u_n}{\partial t} = F(\cdots, u_{n-1}, u_n, u_{n+1}, \cdots) = 0
\]

(1)

where \( u_n = u(n,x,t) \), and \( F \) is assumed to be a polynomial with constant coefficients. The equations are continuous in time, and discretized in the (single) space variable. There are no restrictions on the level of shifts or the degree of nonlinearity. According to the exp-function method[22,23], let us begin with the ansatz for DDEs as follows:

\[
u(n,x,t) = u_n(\xi) = \sum_{n=-f}^{e} a_n \exp(n\xi) + \sum_{m=-q}^{p} b_n \exp(n\xi)
\]

(2)

\[
u_{n-1}(n,x,t) = u_{n-1}(\xi) = \sum_{n=-f}^{e} a_n \exp(n(\xi - id)) + \sum_{m=-q}^{p} b_n \exp(n(\xi - id))
\]

(3)

\[
u_{n+1}(n,x,t) = u_{n+1}(\xi) = \sum_{n=-f}^{e} a_n \exp(n(\xi + id)) + \sum_{m=-q}^{p} b_n \exp(n(\xi + id))
\]

(4)

where \( \xi = dn + c_1x + c_2t + \xi_0 \), \( i \) is a given integral number. The coefficients \( d, c_1, c_2 \) and the phase \( \xi_0 \) are constants to be determined, the integral number \( e, f, p, q \) is given according to the homogeneous balance principle, \( a_n \) and \( b_n \) are unknown constants.

Substituting Eqs. (2)~(4) into Eq. (1), clearing the denominator and setting the coefficients of power terms in \( \exp(j\xi) \), \( j = 1, 2, \cdots \) to zero, a system of nonlinear algebraic equations with \( \exp(d) \) are obtained. From these equations, we can obtain the corresponding undetermined coefficients. Finally, a series of explicit exact solutions of the DDEs (1) are constructed.

3. **Exact traveling wave solutions of the discrete Volterra equation**

Itoh[24] has studied this extended version of the Lotka-volterra equation

\[
\frac{\partial u_n}{\partial t} = \sum_{r=1}^{k-1} (u_{n+r} - u_{n-r})u_n
\]

(5)

And Hereman et al[25] have researched its conserved densities, symmetries and the associate fluxes. For \( k=2 \), it reduces to the Volterra equation

\[
\frac{\partial u_n}{\partial t} = u_n(u_{n+1} - u_{n-1}).
\]

(6)

Let \( u_n = u_n(\xi_n), \xi_n = dn + ct + \xi_0 \), then Eq.(6) becomes
According to the homogeneous balance principle, which leads to the result in the forms of (2)~(4), \( e = p; f = q \).

3.1 For simplicity, we set \( e = p = 1 \) and \( f = q = 1 \), so the forms of (2)~(4) become

\[
\begin{align*}
  u_n &= a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi) \\
  \exp(\xi) + b_0 + b_{-1} \exp(-\xi) \\
\end{align*}
\]

(8)

\[
\begin{align*}
  u_{n-1} &= a_1 \exp(\xi - d) + a_0 + a_{-1} \exp(-\xi + d) \\
  \exp(\xi - d) + b_0 + b_{-1} \exp(-\xi + d) \\
\end{align*}
\]

(9)

\[
\begin{align*}
  u_{n+1} &= a_1 \exp(\xi + d) + a_0 + a_{-1} \exp(-\xi - d) \\
  \exp(\xi + d) + b_0 + b_{-1} \exp(-\xi - d) \\
\end{align*}
\]

(10)

Substituting Eq.(8)~(10) into Eq.(7), and by the help of Maple, clearing the denominator and setting the coefficients of power terms like \( \exp(j\xi), j = 1, 2, \cdots \) to zero yield a system of algebraic equations, we obtain the following exact solutions:

Case 1:

\[
\begin{align*}
  c &= a_1(e^d - 1) \\
  a_0 &= \frac{a_1b_0(e^d - e^d + 1)}{e^d}, a_{-1} = \frac{a_1b_0^2e^d}{(e^d + 1)^2}, b_{-1} = \frac{b_0^2e^d}{(e^d + 1)^2}. \\
\end{align*}
\]

(11)

where \( a_1 \) and \( b_0 \) are arbitrary constants. According to (8) and (11), we can obtain

\[
\begin{align*}
  u_{n1} &= \frac{a_1 \exp(\xi) + \frac{a_1b_0(e^d - e^d + 1)}{e^d} + \frac{a_1b_0^2e^d}{(e^d + 1)^2} \exp(-\xi)}{\exp(\xi) + b_0 + \frac{b_0^2e^d}{(e^d + 1)^2} \exp(-\xi)}. \\
\end{align*}
\]

(12)

Where \( \xi = dn + \frac{a_1(e^d - 1)}{e^d} t + \xi_0 \), \( a_1 \) and \( b_0 \) are arbitrary constants, which are new generalized solitary solutions.

3.2 We set \( e = p = 2 \) and \( f = q = 1 \), so the forms of (2)~(4) become

\[
\begin{align*}
  u_n &= \frac{a_2 \exp(2\xi) + a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{\exp(2\xi) + b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \\
\end{align*}
\]

(13)

\[
\begin{align*}
  u_{n-1} &= \frac{a_2 \exp(2\xi - 2d) + a_1 \exp(\xi - d) + a_0 + a_{-1} \exp(-\xi + d)}{\exp(2\xi - 2d) + b_1 \exp(\xi - d) + b_0 + b_{-1} \exp(-\xi + d)}. \\
\end{align*}
\]

(14)

\[
\begin{align*}
  u_{n+1} &= \frac{a_2 \exp(2\xi + 2d) + a_1 \exp(\xi + d) + a_0 + a_{-1} \exp(-\xi - d)}{\exp(2\xi + 2d) + b_1 \exp(\xi + d) + b_0 + b_{-1} \exp(-\xi - d)}. \\
\end{align*}
\]

(15)

Substituting Eq.(13)~(15) into Eq.(7), and by the help of Maple, clearing the denominator and setting the coefficients of power terms like \( \exp(j\xi), j = 1, 2, \cdots \) to zero yield a system of algebraic equations, we obtain the following exact solutions:

Case 2:
\[
c = \frac{a_2 (e^{2d} - 1)}{e^d},
\]
\[
a_0 = -\frac{\left[ a_2^2 b_1^2 (e^{3d} + e^{4d} + e^{5d} + e^{6d} + 1) - a_1 a_2 b_1 (e^{4d} + e^{5d} + 1) + a_1^2 (e^{2d} + 4d - e^{3d} + e^{4d} + e^d) \right]}{a_2 (e^d + 1)^3 (e^d - 1)^4},
\]
\[
b_0 = -\frac{\left[ a_2^2 b_1^2 (e^{4d} + e^{5d} + e^{6d} + e^d + 1) - a_1 a_2 b_1 (e^{4d} + 2e^{5d} + 2e^d + 1) + a_1^2 (e^{2d} + e^d) \right]}{a_2 (e^d + 1)^3 (e^d - 1)^4},
\]
\[
b_1 = -\frac{\left[ a_2^2 a_1 b_1 (e^{2d} + e^d + 1) + a_1^3 (2e^{2d} - e^d + 2) - a_1^3 e^d \right]}{a_2^2 (e^d + 1)^3 (e^d - 1)^4},
\]
\[
a_1 = -\frac{\left[ a_2^2 a_1 b_1 (e^{2d} + e^d + 1) + a_1^3 (2e^{2d} - e^d + 2) - a_1^3 e^d \right]}{a_2^2 (e^d + 1)^3 (e^d - 1)^4},
\]

where \( a_1, a_2 \) and \( b_1 \) are arbitrary constants. According to (13) and (16), we can obtain
\[
u_{n2} = \frac{a_2 \exp(2\xi) + a_1 \exp(\xi) + A + B \exp(-\xi)}{\exp(2\xi) + b_1 \exp(\xi) + C + D \exp(-\xi)},
\]

where
\[
A = -\frac{\left[ a_2^2 b_1^2 (e^{6d} + e^{4d} + 1) - a_1 a_2 b_1 (e^{4d} + e^{5d} + 1) + a_1^2 (e^{5d} + e^{4d} - e^{3d} + e^{2d} + e^d) \right]}{a_2 (e^d + 1)^3 (e^d - 1)^4},
\]
\[
B = -\frac{\left[ a_2^2 b_1^2 (e^{4d} + e^d + 1) + a_1^3 (2e^{2d} - e^d + 2) - a_1^3 e^d \right]}{a_2^2 (e^d + 1)^3 (e^d - 1)^4},
\]
\[
C = -\frac{\left[ a_2^2 b_1^2 (e^{2d} + e^d + 1) - a_1 a_2 b_1 (e^{2d} + 2e^{3d} + 2e^d + 1) + a_1^2 (e^{2d} + e^d) \right]}{a_2^2 (e^d + 1)^3 (e^d - 1)^4},
\]
\[
D = -\frac{\left[ a_2^2 a_1 b_1 (e^{2d} + e^d + 1) + a_1^3 (2e^{2d} - e^d + 2) - a_1^3 e^d \right]}{a_2^2 (e^d + 1)^3 (e^d - 1)^4},
\]

\[
\xi = \frac{dn + a_1 (e^{2d} - 1)}{e^d} t + \xi_0, \quad a_1, a_2 \text{ and } b_1 \text{ are arbitrary constants, which are new generalized solitonary solutions.}
\]

3.3 We set \( e = p = 2 \) and \( f = q = 2 \), so the forms of (2) ~ (4) become
\[
u_n = \frac{a_2 \exp(2\xi) + a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi) + a_{-2} \exp(-2\xi)}{\exp(2\xi) + b_0 + b_{-2} \exp(-2\xi)},
\]
\[
u_{n-1} = \frac{a_2 \exp(2\xi - 2d) + a_1 \exp(\xi - d) + a_0 + a_{-1} \exp(-\xi + d) + a_{-2} \exp(-2\xi + 2d)}{\exp(2\xi - 2d) + b_0 + b_{-2} \exp(-2\xi + 2d)},
\]
\[
u_{n-1} = \frac{a_2 \exp(2\xi + 2d) + a_1 \exp(\xi + d) + a_0 + a_{-1} \exp(-\xi - d) + a_{-2} \exp(-2\xi - 2d)}{\exp(2\xi + 2d) + b_0 + b_{-2} \exp(-2\xi - 2d)}.
\]

Substituting Eqs. (18) ~ (20) into Eq. (7), and by the help of Maple, clearing the denominator and setting the coefficients of power terms like \( \exp(j\xi) \), \( j = 1, 2, \cdots \) to zero yield a system of algebraic equations, we obtain the following exact solutions:
Case 3:

\[ c = a_2 \left( e^{2d} - 1 \right) \left( e^d \right)^3, a_0 = \frac{a_2^3 e^d \left[ (e^{2d} - 1)^2 + e^d (e^{2d} + 1) \right]}{a_2^3 (e^d + 1)^2 (e^d - 1)^4}, \]

\[ a_{-1} = \frac{a_2^3 e^{3d}}{a_2^3 (e^d + 1)^2 (e^d - 1)^4}, a_{12} = \frac{a_2^3 e^{3d}}{a_2^3 (e^d + 1)^2 (e^d - 1)^4}, \]

\[ b_0 = \frac{a_2^3 e^{2d} (e^d - 1)^2 (e^d - 1)^4}{a_2^3 (e^d + 1)^2 (e^d - 1)^4}, b_{-2} = \frac{a_2^3 e^{6d}}{a_2^3 (e^d + 1)^2 (e^d - 1)^4}. \]

where \( a_1 \) and \( a_2 \) are arbitrary constants. According to (18) and (21), we can obtain

\[ u_{a3} = \frac{a_2 \exp(2\xi) + a_1 \exp(\xi) + E + F \exp(-\xi) + G \exp(-2\xi)}{\exp(2\xi) + H + I \exp(-2\xi)}. \]  

Where

\[ E = \frac{a_2^2 e^d \left[ (e^{2d} - 1)^2 + e^d (e^{2d} + 1) \right]}{a_2^2 (e^d + 1)^2 (e^d - 1)^4}, F = \frac{a_2^3 e^d}{a_2^2 (e^d + 1)^2 (e^d - 1)^4}, G = \frac{a_2^3 e^{3d}}{a_2^2 (e^d + 1)^2 (e^d - 1)^4}, \]

\[ H = \frac{a_2^3 e^{2d} (e^d - 1)^2 (e^d - 1)^4}{a_2^2 (e^d + 1)^2 (e^d - 1)^4}, I = \frac{a_2^3 e^{6d}}{a_2^3 (e^d + 1)^2 (e^d - 1)^4}, \]

\[ \xi = dn + \frac{a_2^3 (e^{2d} - 1)}{e^d} t + \xi_0, \]

\( a_1 \) and \( a_2 \) are arbitrary constants, which are new generalized solitonary solutions.

So the suggested Exp-function method can obtain easily the generalized solitonary solutions for the discrete Volterra equation.

4. Conclusion

In this paper, we have utilized the Exp-function method to study the discrete Volterra equation. As a result, some new explicit exact generalized solitonary solutions of the discrete Volterra equation have been obtained. So we think the Exp-function method will become a promising and powerful new method for a lot of DDEs.

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