CONTINUOUS RIESZ BASES IN HILBERT $C^*$-MODULES

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ABSTRACT. The paper is devoted to continuous frames and Riesz bases in Hilbert $C^*$-modules. We define a continuous Riesz basis for Hilbert $C^*$-modules and give some results about them.

1. Introduction

Frame theory is nowadays a fundamental research area in mathematics, computer science and engineering with many interesting applications in a variety of different fields. Frames were first introduced by Duffin and Schaeffer [8] in the context of nonharmonic fourier series. Then Daubechieies, Grassman and Mayer [7] reintroduced and developed them. The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by G. Kaiser [13] and independently by Ali, Antoine and Gazeau [1]. These frames are known as continuous frames. For a discussion of continuous frames, we refer to Refs.[17, 16]. Arefijamaal and et al. [3] introduced continuous Riesz bases and give some equivalent conditions for a continuous frame to be a continuous Riesz basis.

One reason to study frames in Hilbert $C^*$-modules is that there are some differences between Hilbert spaces and Hilbert $C^*$-modules. For example, in general, every bounded operator on a Hilbert space has an unique adjoint, while this fact not hold for bounded operators on a Hilbert $C^*$-module. Thus it is more difficult to make a discussion of the theory of Hilbert $C^*$-modules than that of Hilbert spaces in general. We refer the readers to [14], for more details on Hilbert $C^*$-modules. Frank and Larson [9] presented a general approach to the frame theory in Hilbert $C^*$-modules. The theory of frames has been extended from Hilbert spaces to Hilbert $C^*$-modules, see [4, 12, 9, 19, 20].

The paper is organized as follows. First, we recall the basic definitions and some notations about Hilbert $C^*$-modules, and we also give some properties of them. Also, we recall the

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notion of continuous frames in Hilbert $C^*$-modules and their operators. In section 3, we define continuous Riesz bases in Hilbert $C^*$-modules and we give some results about them.

2. Preliminaries

First, we recall some definitions and basic properties of Hilbert $C^*$-modules. We give only a brief introduction to the theory of Hilbert $C^*$-modules to make our explanations self-contained. For comprehensive accounts, we refer to [14, 15]. Throughout this paper, $\mathcal{A}$ shows a unital $C^*$-algebra.

Definition 2.1. A pre-Hilbert module over unital $C^*$-algebra $\mathcal{A}$ is a complex vector space $U$ which is also a left $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle ., . \rangle : U \times U \rightarrow \mathcal{A}$ which is $\mathbb{C}$-linear and $\mathcal{A}$-linear in its first variable and satisfies the following conditions:

(i) $\langle f, f \rangle \geq 0$,
(ii) $\langle f, f \rangle = 0$ iff $f = 0$,
(iii) $\langle f, g \rangle^* = \langle g, f \rangle$,
(iv) $\langle af, g \rangle = a \langle f, g \rangle$,

for all $f, g \in U$ and $a \in \mathcal{A}$.

A pre-Hilbert $\mathcal{A}$-module $U$ is called Hilbert $\mathcal{A}$-module if $U$ is complete with respect to the topology determined by the norm $\|f\| = \|\langle f, f \rangle\|^\frac{1}{2}$.

By [12, Example 2.46], if $\mathcal{A}$ is a $C^*$-algebra, then it is a Hilbert $\mathcal{A}$-module with respect to the inner product $\langle a, b \rangle = ab^*$, \quad (a, b \in \mathcal{A})$.

Example 2.2. [15, Page 237] Let $\ell^2(\mathcal{A})$ be the set of all sequences $\{a_n\}_{n \in \mathbb{N}}$ of elements of a $C^*$-algebra $\mathcal{A}$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ is convergent in $\mathcal{A}$. Then $\ell^2(\mathcal{A})$ is a Hilbert $\mathcal{A}$-module with respect to the pointwise operations and inner product defined by $\langle \{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}} \rangle = \sum_{n=1}^{\infty} a_n b_n^*$.

In the following lemma the Cauchy-Schwartz inequality reconstructed in Hilbert $C^*$-modules.

Lemma 2.3. [15, Lemma 15.1.3] (Cauchy-Schwartz inequality) Let $U$ be a Hilbert $C^*$-modules over a unital $C^*$-algebra $\mathcal{A}$. Then

$$\|\langle f, g \rangle\|^2 \leq \|\langle f, f \rangle\| \|\langle g, g \rangle\|,$$

for all $f, g \in U$. 


Definition 2.4. [14, Page 8] Let $U$ and $V$ be two Hilbert $C^*$-modules over a unital $C^*$-algebra $A$. A map $T : U \to V$ is said to be adjointable if there exists a map $T^* : V \to U$ satisfying

$$\langle T f, g \rangle = \langle f, T^* g \rangle$$

for all $f \in U, g \in V$. Such a map $T^*$ is called the adjoint of $T$. By $\text{End}_A^*(U)$ we denote the set of all adjointable maps on $U$.

It is surprising that an adjointable operator is automatically linear and bounded.

Lemma 2.5. [21, Lemma 1.1] Let $U$ and $V$ be two Hilbert $C^*$-modules over a unital $C^*$-algebra $A$ and $T \in \text{End}_A^*(U,V)$ has closed range. Then $T^*$ has closed range and

$$U = \text{Ker}(T) \oplus R(T^*) \quad , \quad V = \text{Ker}(T^*) \oplus R(T)$$

Lemma 2.6. [2, Lemma 0.1] Let $U$ and $V$ be two Hilbert $C^*$-modules over a unital $C^*$-algebra $A$ and $T \in \text{End}_A^*(U,V)$. Then

(i) If $T$ is injective and $T$ has closed range, then the adjointable map $T^*T$ is invertible and

$$\| (T^*T)^{-1} \|^{-1} \leq T^*T \leq \| T \|^2.$$ 

(ii) If $T$ is surjective, then the adjointable map $TT^*$ is invertible and

$$\| (TT^*)^{-1} \|^{-1} \leq TT^* \leq \| T \|^2.$$ 

Now, we introduce continuous frames in Hilbert $C^*$-modules over a unital $C^*$-algebra $A$, and then we give some results for these frames.

Let $\mathcal{Y}$ be a Banach space, $(\mathcal{X}, \mu)$ a measure space, and $f : \mathcal{X} \to \mathcal{Y}$ a measurable function. The integral of the Banach-valued function $f$ has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions (see [6, 22]). Since every $C^*$-algebra and Hilbert $C^*$-module is a Banach space, hence we can use this integral in these spaces. In the following, we assume that $A$ is a unital $C^*$-algebra and $U$ is a Hilbert $C^*$-module over $A$ and $(\Omega, \mu)$ is a measure space.

Definition 2.7. [10, 18] Let $(\Omega, \mu)$ be a measure space and $A$ is a unital $C^*$-algebra. We define,

$$L^2(\Omega, A) = \{ \varphi : \Omega \to A \ ; \ \| \int_{\Omega} (\varphi(\omega))^* \varphi(\omega) d\mu(\omega) \| < \infty \}.$$ 

For any $\varphi, \psi \in L^2(\Omega, A)$, the inner product is defined by $\langle \varphi, \psi \rangle = \int_{\Omega} \langle \varphi(\omega), \psi(\omega) \rangle d\mu(\omega)$ and the norm is defined by $\| \varphi \| = \| \langle \varphi, \varphi \rangle \|^{\frac{1}{2}}$. It was shown in [14] $L^2(\Omega, A)$ is a Hilbert $A$-module.
Continuous frames for Hilbert $\mathcal{A}$-modules are defined as follows.

**Definition 2.8.** [10, 18] A mapping $F : \Omega \to U$ is called a continuous frame for $U$ if

(i) $F$ is weakly-measurable, i.e., for any $f \in U$, the mapping $\omega \mapsto \langle f, F(\omega) \rangle$ is measurable on $\Omega$.

(ii) There exist constants $A, B > 0$ such that

$$A \langle f, f \rangle \leq \int_{\Omega} \langle f, F(\omega) \rangle \langle F(\omega), f \rangle d\mu(\omega) \leq B \langle f, f \rangle,$$

(2.1)

The constants $A, B$ are called *lower* and *upper* frame bounds, respectively. The mapping $F$ is called *Bessel* if the right inequality in (2.1) holds and is called *tight* if $A = B$.

**Definition 2.9.** [11] A continuous frame $F : \Omega \to U$ is called *exact* if for every measurable subset $\Omega_1 \subseteq \Omega$ with $0 < \mu(\Omega_1) < \infty$, the mapping $F : \Omega \setminus \Omega_1 \to U$ is not a continuous frame for $U$.

**Example 2.10.** Let $U = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \end{pmatrix} : a, b \in \mathbb{C} \right\}$, and $\mathcal{A} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{C} \right\}$ which is a $C^*$-algebra. We define the inner product

$$\langle.,.\rangle : U \times U \to \mathcal{A}$$

$$(M, N) \mapsto M(N)^t.$$  

This inner product makes $U$ a $C^*$-module on $\mathcal{A}$. We consider a measure space $(\Omega = [0, 1], \mu)$ whose $\mu$ is the Lebesgue measure. Also $F : \Omega \to U$ defined by $F(\omega) = \begin{pmatrix} \sqrt{3}\omega & 0 & 0 \\ 0 & 0 & \sqrt{3}\omega \end{pmatrix}$, for any $\omega \in \Omega$.

For each $f = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \end{pmatrix} \in U$, we have

$$\int_{[0,1]} \langle f, F(\omega) \rangle \langle F(\omega), f \rangle d\mu(\omega) = \int_{[0,1]} 3\omega^2 \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} d\mu(\omega)$$

$$= \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} = \langle f, f \rangle.$$  

Therefore $F$ is a continuous tight frame with bounds $A = B = 1$.

The following operators for Bessel mappings and continuous frames in Hilbert $C^*$-modules are defined in [11].
Definition 2.11. Let $F : \Omega \rightarrow U$ be a Bessel mapping. Then
(i) The synthesis operator or pre-frame operator $T_F : L^2(\Omega, A) \rightarrow U$ weakly defined by
\[
\langle T_F \varphi, f \rangle = \int_\Omega \varphi(\omega) \langle F(\omega), f \rangle d\mu(\omega), \quad (f \in U).
\]
(ii) The adjoint of $T$, called the analysis operator $T_F^* : U \rightarrow L^2(\Omega, A)$ is defined by
\[
(T_F^* f)(\omega) = \langle f, F(\omega) \rangle, \quad (\omega \in \Omega).
\]

The pre-frame operator is a well defined, surjective, adjointable $A$-linear map and is bounded with $\|T\| \leq \sqrt{B}$ and the analysis operator $T_F^* : U \rightarrow L^2(\Omega, A)$ is injective and has closed range.

Definition 2.12. Let $F : \Omega \rightarrow U$ be a continuous frame for Hilbert $C^*$-module $U$. Then the frame operator $S_F : U \rightarrow U$ is weakly defined by
\[
\langle S_F f, f \rangle = \int_\Omega \langle f, F(\omega) \rangle \langle F(\omega), f \rangle d\mu(\omega), \quad (f \in U).
\]

In [11] prove that the frame operator $S_F$ is positive, adjointable, selfadjoint and invertible and the lower and the upper bounds of $F$ are respectively $\|S^{-1}\|^{-1}$ and $\|T\|^2$. Now we introduce the concept of the duals of continuous frames in Hilbert $C^*$-modules and give some important properties of continuous frames and their duals.

Definition 2.13. [11] Let $F : \Omega \rightarrow U$ be a continuous Bessel mapping. A continuous Bessel mapping $G : \Omega \rightarrow U$ is called a dual for $F$ if
\[
f = \int_\Omega \langle f, G(\omega) \rangle F(\omega) d\mu(\omega), \quad (f \in U),
\]
or
\[
\langle f, g \rangle = \int_\Omega \langle f, G(\omega) \rangle \langle F(\omega), g \rangle d\mu(\omega), \quad (f, g \in U).
\]
In this case $(F, G)$ is called a dual pair. If $T_F$ and $T_G$ denote the synthesis operators of $F$ and $G$, respectively, then (2.5) is equivalent to $T_FT_G^* = I_U$. The condition
\[
\langle f, g \rangle = \int_\Omega \langle f, G(\omega) \rangle \langle F(\omega), g \rangle d\mu(\omega), \quad (f, g \in U),
\]
is equivalent
\[
\langle f, g \rangle = \int_\Omega \langle f, F(\omega) \rangle \langle G(\omega), g \rangle d\mu(\omega), \quad (f, g \in U),
\]
because $T_F T_G^* = I_U$ if and only if $T_G T_F^* = I_U$.

Also, by reconstruction formula we have

$$f = S^{-1} S f = S^{-1} \int_\Omega \langle f, F(\omega) \rangle F(\omega) d\mu(\omega) = \int_\Omega \langle f, F(\omega) \rangle S^{-1} F(\omega) d\mu(\omega),$$

and

$$f = S S^{-1} f = \int_\Omega \langle S^{-1} f, F(\omega) \rangle F(\omega) d\mu(\omega) = \int_\Omega \langle f, S^{-1} F(\omega) \rangle F(\omega) d\mu(\omega).$$

Then $S^{-1} F$ is a dual for $F$, which is called canonical dual.

3. Continuous Riesz bases in Hilbert $C^*$-modules

In this section, we introduce the concept of continuous Riesz bases in Hilbert $C^*$-modules and give some important properties of them. First, we give the notion of a Riesz-type frame that is introduced in [11].

**Definition 3.1.** Let $F : \Omega \to U$ be a continuous frame for Hilbert $C^*$-module $U$. If $F$ has only one dual, we call $F$ a Riesz-type frame.

**Theorem 3.2.** [11, Theorem 3.4] Let $F : \Omega \to U$ be a continuous frame for Hilbert $C^*$-module $U$ over a unital $C^*$-algebra $A$. Then $F$ is a Riesz-type frame if and only if the analysis operator $T_F^* : U \to L^2(\Omega, A)$ is onto.

**Definition 3.3.** Let $U$ be a Hilbert $C^*$-module $U$ over a unital $C^*$-algebra $A$. A Bessel mapping $F : \Omega \to U$ is called a $\mu$-complete if

$$\{ f \in U : \langle f, F(\omega) \rangle = 0 \text{ a.e. } ]\} = \{0\}.$$

Now, we define a continuous Riesz basis for Hilbert $C^*$-modules.

**Definition 3.4.** Let $U$ be a Hilbert $C^*$-module $U$ over a unital $C^*$-algebra $A$. A mapping $F : \Omega \to U$ is called a continuous Riesz basis for Hilbert $C^*$-module $U$, if the following conditions are satisfied:

1. $F$ is weakly-measurable, i.e., for any $f \in U$, the mapping $\omega \mapsto \langle f, F(\omega) \rangle$ is measurable on $\Omega$.
2. $F$ is $\mu$-complete.
3. There are two constants $A, B > 0$ such that

$$A \left\| \int_{\Omega_1} |\varphi(\omega)|^2 d\mu(\omega) \right\|^2 \leq \left\| \int_{\Omega_1} \varphi(\omega) F(\omega) d\mu(\omega) \right\| \leq B \left\| \int_{\Omega_1} |\varphi(\omega)|^2 d\mu(\omega) \right\|^2,$$

where $\varphi$ is a function from $\Omega$ to $\mathbb{C}$.
for every \( \varphi \in L^2(\Omega, A) \) and measurable subset \( \Omega_1 \subseteq \Omega \) with \( \mu(\Omega_1) < +\infty \).

**Remark 3.5.** Let \( F : \Omega \to U \) be a continuous Riesz basis for Hilbert \( C^* \)-module \( U \). Define

\[
T : L^2(\Omega, A) \longrightarrow U
\]

\[
\varphi \mapsto \int_{\Omega} \varphi(\omega) F(\omega) d\mu(\omega)
\]

Then \( T \) is well-defined, adjointable map with \( T^*f = \{ \langle f, F(\omega) \rangle \}_{\omega \in \Omega} \) and bounded such that

\[
A\|\varphi\|^2 \leq \|T\varphi\|^2 \leq B\|\varphi\|^2.
\]

Hence \( F \) is a continuous Bessel mapping. Also by \( \mu \)-completeness of \( F \) we have

\[
\text{Ker}(T^*) = \{ f \in U \mid \langle f, F(\omega) \rangle = 0 \quad \forall \omega \in \Omega \} = \{0\},
\]

so by lemma 2.5, \( R(T) = \text{Ker}(T^*)^\perp = U \). Then \( T \) is onto and by [11, theorem 2.15], \( F \) is a continuous frame for \( U \).

**Definition 3.6.** Let \( U \) be a Hilbert \( C^* \)-module \( U \) over a unital \( C^* \)-algebra \( A \). A Bessel mapping \( F : \Omega \to U \) is said to be \( L^2 \)-independent if for \( \varphi \in L^2(\Omega, A) \),

\[
\int_{\Omega} \varphi(\omega) F(\omega) d\mu(\omega) = 0
\]

implies that \( \varphi(\omega) = 0 \), for each \( \omega \in \Omega \).

We give the following result.

**Theorem 3.7.** Let \( F : \Omega \to U \) be a continuous frame for Hilbert \( C^* \)-module \( U \) over a unital \( C^* \)-algebra \( A \) with bounds \( A, B > 0 \). Then the following are equivalent:

(i) \( F \) is a continuous Riesz basis.

(ii) \( F \) is \( \mu \)-complete and \( L^2 \)-independent.

**Proof.** (i) \( \implies \) (ii) Let \( F \) be a continuous Riesz basis and \( \int_{\Omega} \varphi(\omega) F(\omega) d\mu(\omega) = 0 \) for some \( \varphi \in L^2(\Omega, A) \). Since

\[
A\|\int_{\Omega} |\varphi(\omega)|^2 d\mu(\omega)\| \leq \|\int_{\Omega} \varphi(\omega) F(\omega) d\mu(\omega)\|^2 = 0,
\]

so

\[
\langle \{\varphi(\omega)\}_{\omega \in \Omega}, \{\varphi(\omega)\}_{\omega \in \Omega} \rangle = \int_{\Omega} |\varphi(\omega)|^2 d\mu(\omega) = 0.
\]

Hence \( \{\varphi(\omega)\}_{\omega \in \Omega} = 0 \) and \( \varphi = 0 \) i.e. \( F \) is \( L^2 \)-independent.
(ii) \implies (i) Let \( F \) be a \( L^2 \)-independent continuous frame for Hilbert \( C^* \)-module \( U \) with bounds \( A, B > 0 \). For \( \varphi \in L^2(\Omega, A) \) and measurable subset \( \Omega_1 \subseteq \Omega \) with \( \mu(\Omega_1) < +\infty \), put \( f = \int_{\Omega_1} \varphi(\omega) F(\omega) d\mu(\omega) \). Then we have,

\[
f = \int_{\Omega_1} \varphi(\omega) F(\omega) d\mu(\omega) = \int_{\Omega} \varphi(\omega) \chi_{\Omega_1}(\omega) F(\omega) d\mu(\omega).
\]

Also \( f = \int_{\Omega} \langle f, S^{-1}F(\omega) \rangle F(\omega) d\mu(\omega) \) where \( S \) is the continuous frame operator of \( F \). Since \( F \) is \( L^2 \)-independent, so

\[
\varphi(\omega) \chi_{\Omega_1}(\omega) = \langle f, S^{-1}F(\omega) \rangle, \quad (\omega \in \Omega).
\]

and by [11, corollary 2.11],

\[
B^{-1} \langle f, f \rangle \leq \langle S^{-1}f, f \rangle \leq A^{-1} \langle f, f \rangle,
\]

and so

\[
A \| \langle S^{-1}f, f \rangle \| \leq \| \langle f, f \rangle \| \leq B \| \langle S^{-1}f, f \rangle \|.
\]

Now we show that \( \langle S^{-1}f, f \rangle = \int_{\Omega_1} |\varphi(\omega)|^2 d\mu(\omega) \).

\[
\langle S^{-1}f, f \rangle = \langle f, S^{-1}f \rangle = \int_{\Omega_1} \langle f, \varphi(\omega)S^{-1}F(\omega) \rangle d\mu(\omega)
\]

\[
= \int_{\Omega_1} \langle f, S^{-1}F(\omega) \rangle \varphi(\omega)^* d\mu(\omega)
\]

\[
= \int_{\Omega_1} \varphi(\omega) \varphi(\omega)^* d\mu(\omega) = \int_{\Omega_1} |\varphi(\omega)|^2 d\mu(\omega).
\]

Therefore,

\[
A \| \int_{\Omega_1} |\varphi(\omega)|^2 d\mu(\omega) \| \leq \| \int_{\Omega_1} \varphi(\omega) F(\omega) d\mu(\omega) \|^2 \leq B \| \int_{\Omega_1} |\varphi(\omega)|^2 d\mu(\omega) \|,
\]

i.e. \( F \) is a continuous Riesz basis for \( U \) with bounds \( A, B \). \( \square \)

**Theorem 3.8.** Let \( F : \Omega \to U \) be a continuous frame for Hilbert \( C^* \)-module \( U \) over a unital \( C^* \)-algebra \( A \). If \( F \) is a continuous Riesz basis for \( U \), then it is a continuous exact frame.

**Proof.** Let \( \Omega_1 \subseteq \Omega \) be a measurable subset of \( \Omega \) with \( 0 < \mu(\Omega_1) < \infty \). For \( \varphi = \chi_{\Omega_1} \in L^2(\Omega, A) \) we have,

\[
\| \int_{\Omega_1} F(\omega) d\mu(\omega) \|^2 = \| \int_{\Omega_1} \chi_{\Omega_1}(\omega) F(\omega) d\mu(\omega) \|^2
\]
\[
\geq A \| \int_{\Omega_1} |\chi_{\Omega_1}(\omega)|^2 d\mu(\omega) \|
\]
\[
= A \| \mu(\Omega_1) \| > 0.
\]

Hence \( \int_{\Omega_1} F(\omega) d\mu(\omega) \neq 0 \).

Now suppose that \( F : \Omega \setminus \Omega_1 \to U \) is a continuous frame for \( U \). Then by completeness of \( F \mid_{\Omega \setminus \Omega_1} \) there exists \( \varphi_0 \in L^2(\Omega \setminus \Omega_1, A) \) such that
\[
\int_{\Omega_1} F(\omega) d\mu(\omega) = \int_{\Omega \setminus \Omega_1} \varphi_0(\omega) F(\omega) d\mu(\omega).
\]

Define \( \varphi : \Omega \to A \) where
\[
\varphi(\omega) = \begin{cases} 
\varphi_0(\omega) & \omega \in \Omega \setminus \Omega_1 \\
1 & \omega \in \Omega_1.
\end{cases}
\]

Then \( \varphi \in L^2(\Omega, A) \) and
\[
\int_{\Omega} \chi_{\Omega_1}(\omega) F(\omega) d\mu(\omega) = \int_{\Omega} \varphi(\omega) F(\omega) d\mu(\omega),
\]
so \( \int_{\Omega} (\varphi - \chi_{\Omega_1})(\omega) F(\omega) d\mu(\omega) = 0 \). Hence \( L^2 \)-independent shows that \( \varphi = \chi_{\Omega_1} \) and so \( \varphi_0 = 0 \).

Therefore
\[
\int_{\Omega_1} F(\omega) d\mu(\omega) = \int_{\Omega \setminus \Omega_1} \varphi_0(\omega) F(\omega) d\mu(\omega) = 0,
\]
which is a contradiction. \( \square \)

**Proposition 3.9.** Let \( F : \Omega \to U \) be a continuous Bessel mapping for Hilbert \( C^* \)-module \( U \) over a unital \( C^* \)-algebra \( A \) with pre-frame operator \( T \). Suppose that \( F \) is \( \mu \)-complete and the mapping
\[
V : L^2(\Omega, A) \to L^2(\Omega, A)
\]
\[
\varphi \mapsto \int_{\Omega} \varphi(\omega) \langle F(\omega), F(\cdot) \rangle d\mu(\omega)
\]
defines a bounded, adjointable and invertible operator. Then \( F \) is a continuous Riesz basis for \( U \).

**Proof.** Since \( F \) is Bessel, so the synthesis operator \( T \) is well-defined and bounded and adjointable with \( T^* f = \{ \langle f, F(\omega) \rangle \}_{\omega \in \Omega} \) for \( f \in U \).

Also \( T^* T = V \), because
\[
(T^* T)(\varphi) = T^*(\int_{\Omega} \varphi(\omega) F(\omega) d\mu(\omega))
\]
\[
\begin{align*}
\langle \int_{\Omega} \varphi(\omega)F(\omega)d\mu(\omega), F(\gamma) \rangle & \vphantom{\int} = \{ \int_{\Omega} \varphi(\omega)F(\omega)d\mu(\omega) \} \gamma \in \Omega \\
\langle \int_{\Omega} \varphi(\omega)F(\omega)d\mu(\omega), F(\gamma) \rangle & \vphantom{\int} = \{ \int_{\Omega} \varphi(\omega)F(\omega)d\mu(\omega) \} \gamma \in \Omega.
\end{align*}
\]

Since \( T \) is bounded, so there exist \( B > 0 \) such that \( \| T\varphi \|^2 \leq B\| \varphi \|^2 \) i.e.

\[
\| \int_{\Omega} \varphi(\omega)F(\omega)d\mu(\omega) \|^2 \leq B\| \varphi \|^2.
\]

Since \( T^*T \) is positive, so

\[
\| \int_{\Omega} \varphi(\omega)F(\omega)d\mu(\omega) \|^2 = \| T\varphi \|^2
\]

\[
= \| \langle T^*T\varphi, \varphi \rangle \|
\]

\[
= \| \langle (T^*T)^2\varphi, (T^*T)^2\varphi \rangle \|
\]

\[
= \| (T^*T)^2\varphi \|^2 \geq \| (T^*T)^{-1} \|^2 \| \varphi \|^2.
\]

Therefore \( F \) is continuous Riesz basis with lower and upper bounds \( \| (T^*T)^{-1} \|^2 \), \( B \) respectively.

\[\square\]

**Theorem 3.10.** Let \( F : \Omega \to U \) be a continuous frame for Hilbert \( C^* \)-module \( U \) over a unital \( C^* \)-algebra \( A \) with pre-frame operator \( T \). Then \( F \) is a continuous Riesz basis for \( U \) if and only if \( F \) is a Riesz-type frame.

**Proof.** \((\implies)\) Let \( F_1 \neq F_2 \) are two duals of \( F \). Then for each \( f \in U \),

\[
\int_{\Omega} \langle f, F_1(\omega) - F_2(\omega) \rangle F(\omega)d\mu(\omega) = \int_{\Omega} \langle f, F_1(\omega) \rangle F(\omega)d\mu(\omega) - \int_{\Omega} \langle f, F_2(\omega) \rangle F(\omega)d\mu(\omega)
\]

\[
= f - f = 0.
\]

Since \( F \) is continuous Riesz basis, so is \( L^2 \)-independent and

\[
\langle f, F_1(\omega) - F_2(\omega) \rangle = 0 \implies \langle f, F_1(\omega) \rangle = \langle f, F_2(\omega) \rangle \quad (\omega \in \Omega).
\]

Therefore \( F_1 = F_2 \).

\((\impliedby)\) Let \( F \) be Riesz-type frame and \( \varphi \in L^2(\Omega, A) \) such that \( \int_{\Omega} \varphi(\omega)F(\omega)d\mu(\omega) = 0 \).

Since \( F \) is Riesz-type, so \( R(T^*) = L^2(\Omega, A) \). Also \( L^2(\Omega, A) = Ker(T) \oplus R(T^*) \).

Then

\[
\varphi \in Ker(T) = R(T^*)^\perp = \{0\},
\]

Therefore \( F_1 = F_2 \).
then $\varphi = 0$ and so $F$ is $L^2$-independent. Therefore $F$ is a continuous Riesz basis.

**Corollary 3.11.** Let $F : \Omega \to U$ be a continuous frame for Hilbert $C^*$-module $U$ over a unital $C^*$-algebra $A$. If $F$ is a Riesz-type frame, then it is a continuous exact frame.

Due to Theorem 3.10, the converse of the Proposition 3.9 holds as follows.

**Corollary 3.12.** Let $F : \Omega \to U$ be a continuous Riesz basis for Hilbert $C^*$-module $U$ over a unital $C^*$-algebra $A$ with bounds $A, B > 0$ and pre-frame operator $T$. Then $F$ is $\mu$-complete and the mapping

$$V : L^2(\Omega, A) \to L^2(\Omega, A)$$

$$\varphi \mapsto \int_\Omega \varphi(\omega) \langle F(\omega), F(\cdot) \rangle d\mu(\omega)$$

defines a bounded, adjointable and invertible operator.

**Proof.** Let $F$ be a continuous Riesz basis for $U$ with bounds $A, B > 0$. Then the synthesis operator $T$ satisfies $\|T\| \leq \sqrt{B}$. Also,

$$(T^* T)(\varphi) = T^* \left( \int_\Omega \varphi(\omega) F(\omega) d\mu(\omega) \right)$$

$$= \{ \int_\Omega \varphi(\omega) F(\omega) d\mu(\omega), F(\gamma) \} \gamma \in \Omega$$

$$= \{ \int_\Omega \varphi(\omega) \langle F(\omega), F(\gamma) \rangle d\mu(\omega) \} \gamma \in \Omega.$$  

Then $V = T^* T$. Moreover, $F$ is Riesz-type and $T^*$ is onto. Then by lemma 2.6, $V$ is adjointable and invertible operator and

$$\|(T^* T)^{-1}\|^{-1} \leq V \leq \|T^*\|^2 \leq B.$$  

According to the Theorem 3.10, in the next corollary we show the relation between two continuous Riesz bases for a Hilbert $C^*$-module $U$.

**Corollary 3.13.** Let $F, G : \Omega \to U$ be two continuous Riesz bases for Hilbert $C^*$-module $U$ over a unital $C^*$-algebra $A$ and $T_F, T_G, S_G$ are the pre-frame operator of $F$, the pre-frame operator of $G$ and the frame operator of $G$ respectively. Then there exists an invertible operator $K \in \text{End}^*(U)$ such that $G = S_G K^* F$. 

Proof. Let $f \in U$ such that $(T_G T_F^*)f = 0$. Then $T_G((T_F^*f)(\omega)) = 0$ for all $\omega \in \Omega$ and $
abla_\Omega \langle f, F(\omega) \rangle G(\omega) d\mu(\omega) = 0$.

Since $G$ is $L^2$-independence, so $\langle f, F(\omega) \rangle = 0$ for all $\omega \in \Omega$ and by completeness of $F$, we have $f = 0$. This shows that $T_G T_F^*$ is injective. Also $F$ a is Riesz-type frame, so $T_F^*$ is onto. Since $G$ is a continuous frame so $T_G$ is onto. Hence $T_G T_F^*$ is onto and so is invertible.

Put $K := (T_G T_F^*)^{-1}$. Then for any $f, g \in U$,

$$
\langle f, g \rangle = \langle K^{-1} K f, g \rangle
= \langle T_F^* K f, T_G^* g \rangle
= \int_\Omega \langle K f, F(\omega) \rangle \langle G(\omega), g \rangle d\mu(\omega)
= \int_\Omega \langle f, K^* F(\omega) \rangle \langle G(\omega), g \rangle d\mu(\omega).
$$

Thus $K^* F$ is a dual of $G$. But $G$ is a Riesz-type frame, then $S_G^{-1} G = K^* F$ and hence $G = S_G K^* F$. \hfill \Box

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