Definition of local realism weaker than Bell’s compatible with quantum predictions for entangled photon pairs

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Abstract
A definition of local realism is proposed that assumes the reality of the vacuum fields. It is weaker than Bell’s definition of local hidden variables theories. A model is presented for the polarization correlation experiment of two maximally entangled photons created by spontaneous parametric down conversion, that reproduces the quantum predictions. In the model entanglement appears as a correlation between a signal field and a vacuum field.

1 Bell’s local realism
In 2015 the experiments showed for the first time the loophole-free violation of a Bell inequality[1,2]. The result has been interpreted as the “death by experiment for local realism”, this being the hypothesis that “the world is made up of real stuff, existing in space and changing only through local interactions …about the most intuitive scientific postulate imaginable”[3]. This statement, and many others similar, emphasize the relevance of local realism for our understanding of the physical world, in particular the quantum world. In this note it is proposed a new definition of local realism, weaker than Bell’s, that is compatible with the performed experiments involving photon pairs entangled in polarization.

Bell defined “local hidden variables model”, later named “local realistic”, to be any model of an experiment where the results of all measurements may be interpreted according to the formulas
\[ \langle A \rangle = \int \rho(\lambda) d\lambda M(\lambda, A), \langle B \rangle = \int \rho(\lambda) d\lambda M(\lambda, B), \]
\[ \langle AB \rangle = \int \rho(\lambda) d\lambda M(\lambda, A) M(\lambda, B), \]

where \( \lambda \in \Lambda \) is one or several random (“hidden”) variables. \( \langle A \rangle \), \( \langle B \rangle \) and \( \langle AB \rangle \) are the expectation values of the results of measuring the observables \( A, B \) or their product \( AB \). Here we will consider the case that the observables correspond to detection, or not, of some signals (e.g. photons) by two parties usually named Alice and Bob, attaching the values 1 or 0 to the two possibilities. In this case the above expectations correspond to the single and detection rates respectively. The following mathematical conditions are assumed
\[ \rho(\lambda) \geq 0, \int \rho(\lambda) d\lambda = 1, M(\lambda, A) \in \{0, 1\}, M(\lambda, B) \in \{0, 1\}. \]

Eq.(2) corresponds to a “deterministic model” where the statistical aspects derive only from the probabilistic nature of the hidden random variables \( \{\lambda\} \). More general models may be used where the whole integral \([0, 1]\) is substituted for \( \{0, 1\} \) in eq.(2). A constraint of locality is included, namely \( M(\lambda, A) \) is independent of \( M(\lambda, B) \) and \( \rho(\lambda) \) independent of both \( M(\lambda, A) \) and \( M(\lambda, A) \)[4]. From these conditions it is possible to derive empirically testable (Bell) inequalities. The tests are most relevant if the measurements made by Alice and Bob are spatially separated in the sense of relativity theory.

Most relevant for our purposes is the following Clauser-Horne inequality (CH)[5]
\[ \langle A \rangle + \langle B \rangle \geq \langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle, \]

where \( A \) and \( A' \) are two possible observables measured by Alice, say detection, or not, of a photon after crossing a polarizer at two different angles, \( \theta \) and \( \theta' \) respectively, and similarly \( B \) and \( B' \) by Bob with polarizers at angles either \( \phi \) or \( \phi' \). The derivation of the CH inequality is trivial. We consider 4 arbitrary real numbers \( \{x, y, z, w\} \) with domain \([0, 1]\) and it is easy to see that they fulfil the inequality
\[ x + y \geq xy + xw + zw. \]
Then substituting \( \{ M(\lambda, A), M(\lambda, B), M(\lambda, A'), M(\lambda, B') \} \) for \( \{ x, y, z, w \} \), multiplying times \( \rho(\lambda) \) and integrating with respect to \( \lambda \) gives the CH inequality eq.(3). Closely related to that is the Eberhard inequality eq.(5)

\[
\langle AB \rangle \leq \langle AB' \rangle + \langle A'B \rangle + \langle A'B' \rangle,
\]

where \( \langle AB' \rangle = \langle A \rangle - \langle AB' \rangle \) means that Alice detects but Bob does not, and similar for \( \langle A'B \rangle \). This is the inequality violated in the loophole-free experiments commented above\[1], \[2].

Bell definition of local realistic models involves the implicit assumption that detections are instantaneous events, or at least that we may ignore the activation time of the detectors. If we included the detection time we should substitute the following for eqs.(1)

\[
P_a = \int \rho(\lambda) d\lambda \int_1^T F_a(\lambda, t, A) dt, P_b = \int \rho(\lambda) d\lambda \int_1^T F_b(\lambda, t, B) dt, P_{ab} = \int \rho(\lambda) d\lambda \int_1^T F_a(\lambda, t, A) dt \int_1^T F_b(\lambda, t', B) dt'.
\]

I point out that the functions \( F_a(\lambda, t, A) \) and \( F_b(\lambda, t, B) \) need not be positive definite, but their time averages cannot be negative, they being detection probabilities for a given \( \lambda \). Therefore they have constraints similar to eqs.(2) and lead to identical consequences, as may be seen by substituting \( T^{-1} \int F_a(\lambda, t, A) dt \) for \( M(\lambda, A) \). The form eq.(6) would be more convenient for the formulation of our “local realism” in section 3.

Before proceeding with our argument I shall shortly review the treatment within the Weyl-Wigner formalism of the polarization correlation measurement of two maximally entangled photons produced via spontaneous parametric down conversion (SPDC).

\section{Polarization correlation of entangled photon pairs in the Weyl-Wigner formalism}

The WW formalism was developed for non-relativistic quantum mechanics, where the basic observables involved are positions, \( \hat{x}_j \), and momenta, \( \hat{p}_j \), of the particles\[5], \[7]. It may be trivially extended to quantum optics provided that we interpret \( \hat{x}_j \) and \( \hat{p}_j \) to be the sum and the difference of the creation,
\( \hat{a}_j \), and annihilation, \( \hat{a}_j \), operators of the \( j \) normal mode of the radiation. That is

\[
\hat{x}_j = \frac{c}{\sqrt{2 \omega_j}} (\hat{a}_j + \hat{a}_j^\dagger), \quad \hat{p}_j = \frac{\hbar \omega_j}{\sqrt{2c}} (\hat{a}_j - \hat{a}_j^\dagger)
\]

\[
\Rightarrow \hat{a}_j = \frac{1}{\sqrt{2}} \left( \frac{\omega_j}{c} \hat{x}_j + \frac{i c}{\hbar \omega_j} \hat{p}_j \right), \quad \hat{a}_j^\dagger = \frac{1}{\sqrt{2}} \left( \frac{\omega_j}{c} \hat{x}_j - \frac{i c}{\hbar \omega_j} \hat{p}_j \right) . \tag{7}
\]

Here \( \hbar \) is Planck constant, \( c \) the velocity of light and \( \omega_j \) the frequency of the normal mode. In the following I will use units \( \hbar = c = 1 \). For the sake of clarity I shall represent the operators in a Hilbert space with a ‘hat’, e. g. \( \hat{a}_j, \hat{a}_j^\dagger \) and the amplitudes in the WW formalism without ‘hat’, e. g. \( a_j, a_j^* \).

For an introduction to the subject see [9].

The connection with the Hilbert-space formalism is made via the Weyl transform as follows. For any trace class operator \( \hat{M} \) of the former we define its Weyl transform to be a function of the field operators \( \{ \hat{a}_j, \hat{a}_j^\dagger \} \), that is

\[
W_{\hat{M}} = \frac{1}{(2 \pi)^n} \prod_{j=1}^{n} \int_{-\infty}^{\infty} d\lambda_j \int_{-\infty}^{\infty} d\mu_j \exp \left[ -2i \lambda_j \text{Re}a_j - 2i \mu_j \text{Im}a_j \right] 
\times \text{Tr} \left\{ \hat{M} \exp \left[ i \lambda_j \left( \hat{a}_j + \hat{a}_j^\dagger \right) + i \mu_j \left( \hat{a}_j - \hat{a}_j^\dagger \right) \right] \right\} . \tag{8}
\]

The transform is invertible that is

\[
\hat{M} = \frac{1}{(2 \pi)^n} \prod_{j=1}^{n} \int_{-\infty}^{\infty} d\lambda_j \int_{-\infty}^{\infty} d\mu_j \exp \left[ i \lambda_j \left( \hat{a}_j + \hat{a}_j^\dagger \right) + i \mu_j \left( \hat{a}_j - \hat{a}_j^\dagger \right) \right] 
\times \prod_{j=1}^{n} \int_{-\infty}^{\infty} d\text{Re}a_j \int_{-\infty}^{\infty} d\text{Im}a_j W_{\hat{M}} \left\{ a_j, a_j^* \right\} \exp \left[ -2i \lambda_j \text{Re}a_j - 2i \mu_j \text{Im}a_j \right] .
\]

The transform is linear, that is if \( f \) is the transform of \( \hat{f} \) and \( g \) the transform of \( \hat{g} \), then the transform of \( \hat{f} + \hat{g} \) is \( f + g \).

The WW formalism in quantum optics may be useful because it is just quantum optics, therefore the predictions for experiments are the same as in the more usual Hilbert-space formalism. Also the calculations are generally no more involved and it suggests a physical picture in terms of random variables and stochastic processes. In particular the counterparts of creation
and annihilation operators look like random amplitudes. This is the basis of the argument presented in the next section.

We shall use the WW formalism in the Heisenberg picture, where the evolution appears in the observables (usually functions of the fields), that evolution resembling classical stochastic processes. It is straightforward to get the (Wigner) function corresponding to any quantum state. In particular the vacuum state, represented by the density matrix $|0⟩⟨0|$ in the Hilbert space formalism, is associated to the following Wigner function

$$W_0 = \prod_j \frac{2}{\pi} \exp(-2|a_j|^2).$$

The crucial assumption for the new definition of local realism proposed in this paper is the assumption that the vacuum fields are real when we interpret eq. (9) as a (positive) probability distribution. Hence the picture that emerges is that the quantum vacuum of the electromagnetic field (also named zeropoint field, ZPF) consists of stochastic fields with a probability distribution independent for every mode, having a Gaussian distribution with mean energy $\frac{1}{2}\hbar\omega$ per mode. However the rest of this section is devoted to calculations and no assumption will be made as to whether the vacuum fields are real.

In the following I will apply the WW formalism to the “signal” and “idler” fields produced via spontaneous parametric down-conversion (SPDC) in a nonlinear crystal. The fields may be written for a single mode,

$$E^+_s = a_s \left(1 + \frac{1}{2}|C|^2\right) + Ca^*_s, \quad E^+_i = a_i \left(1 + \frac{1}{2}|C|^2\right) + Ca^*_i, \quad |C| \ll 1,$$

where we ignore the space-time dependence and the vector character of the fields and use a single mode simplification. The terms of order $|C|$ are produced by the interaction of the laser field and the vacuum fields, $a_s$ and $a_i$ entering the crystal (see e.g. [9]). We work in the complex representation of the fields, $E^+(E^-)$ being the positive (negative) frequency component that should be multiplied times the factor $\exp(-i\omega t)$ (resp. $\exp(i\omega t)$). The two components are complex conjugates of each other. The parameter $C$ is proportional to the coupling constant between the laser and the corresponding vacuum field in the crystal and eq. (10) is an approximation to order $|C|^2$. We may perform a change from $C$ to the new parameter $D = \left(1 + \frac{1}{2}|C|^2\right)^{-1}C$. 


whence eqs. (10) become

\[ E_s^+ = \left( 1 + \frac{1}{2} |C|^2 \right) [a_s + D a_s^*], \]

\[ E_i^+ = \left( 1 + \frac{1}{2} |C|^2 \right) [a_i + D a_i^*], |D| << 1, \]  

(11)

and ignore the constant global factor \((1 + \frac{1}{2} |C|^2) \sim 1\) because we will be interested in calculating relative detection rates. Up to here we have not taken into account the vector character of the field, that will be essential in the following. Thus we will include two unit vectors, say horizontal \(h\) and vertical \(v\).

If we send these two beams to the incoming channels of a polarizer beam splitter, we shall have in the outgoing channels

\[ E_A^+ = (a_s + D a_s^*) v + i (a_i + D a_i^*) h, \]

\[ E_B^+ = (a_s + D a_s^*) h + i (a_i + D a_i^*) v \]  

(12)

where I work to order \(O(|D|^2)\). These equations represent ‘two photons entangled in polarization’ as seen in the Weyl-Wigner formalism. These beams will arrive at the Alice and Bob polarization analyzers put at angles \(\theta\) and \(\phi\) with the vertical respectively. Hence the beams emerging from them will have moduli

\[ E_A^+ = (a_s + D a_s^*) \cos \theta + i (a_i + D a_i^*) \sin \theta, \]

\[ E_B^+ = (a_s + D a_s^*) \sin \phi + i (a_i + D a_i^*) \cos \phi, \]  

(13)

and polarizations at angles \(\theta\) and \(\phi\) with the vertical, respectively. The fields produced by the interaction of the laser with the nonlinear crystal are of order \(|D|\) and these are the terms able to produce detection events. Thus it is convenient to define the partial fields

\[ E_{A1}^+ = D [a_i^* \cos \theta + ia_s^* \sin \theta], E_{B1}^+ = D [ia_i^* \sin \phi + a_s^* \cos \phi]. \]  

(14)

The single, \(P_A\) and \(P_B\), and coincidence, \(P_{AB}\), rates in the WW formalism may be got translating the well known detection rules of the Hilbert space via the Weyl transform eq. (8)[9]. They are given from the fields, modulo a common proportionality constant, as follows
\( P_A = \langle I_{A1} \rangle, P_B = \langle I_{B1} \rangle, P_{AB} = \langle (I_A - \langle I_A \rangle)I_{B1} \rangle + \langle I_{A1}(I_B - \langle I_B \rangle) \rangle \), \hspace{1cm} (15)

where 
\( I_A = \left| E_A^+ \right|^2, I_B = \left| E_B^+ \right|^2, I_{A1} = \left| E_{A1}^+ \right|^2, I_{B1} = \left| E_{B1}^+ \right|^2. \) \hspace{1cm} (16)

We are assuming ideal detectors, but for real detectors \( P_A \) and \( P_B \) should be multiplied times the detection efficiencies \( \eta_A \) and \( \eta_B \), and \( P_{AB} \) multiplied times \( \eta_A \eta_B \).

For the single rate of Alice we get from eq.(15)

\[
P_A = \langle |E_{A1}^+|^2 \rangle = |D|^2 |a_s \cos \theta + a_i \sin \theta|^2 \\
= |D|^2 \langle |a_s|^2 \cos^2 \theta + |a_i|^2 \sin^2 \theta \rangle = \frac{1}{2} |D|^2,
\]

\hspace{1cm} (17)

where \( \langle |a_s|^2 \rangle = \langle |a_i|^2 \rangle = 1/2 \) and \( \langle a_s^2 \rangle = \langle a_i^2 \rangle = 0 \), taking eq.(9) into account. The single rate of Bob is the same, that is \( P_B = P_A \).

In order to get the joint detection rate it may be realized that \( I_{A1} \) and \( I_{B1} \) are of order \( |D|^2 \), whence for a calculation to that order it is enough to take \( I_A \) and \( I_B \) to order zero in eq.(5). It is easy to see that this is equivalent to substitute for the actual intensities the following intensities \( I_{A0} \) and \( I_{B0} \)

\[
I_{A0} = |a_s \cos \theta + ia_i \sin \theta|^2 \Rightarrow \langle I_{A0} \rangle = \frac{1}{2}, \]
\[
I_{B0} = |ia_s \sin \phi + a_i \cos \phi|^2 \Rightarrow \langle I_{B0} \rangle = \frac{1}{2}, \hspace{1cm} (18)
\]

in eq.(15). Thus we get

\[
P_{AB} = \langle I_{A0}I_{B1} \rangle + \langle I_{A1}I_{B0} \rangle - \langle I_{A0} \rangle \langle I_{B1} \rangle - \langle I_{A1} \rangle \langle I_{B0} \rangle,
\]

\hspace{1cm} (19)

where \( I_{A1} \) and \( I_{B1} \) were given in eq.(16) and for the latter two terms see eqs.(17) and (18). The first term of eq.(19) may be got starting from

\[
\langle I_{A0}I_{B1} \rangle = \langle E_A^+ E_{A0}^- E_B^+ E_{B1}^- \rangle \hspace{1cm} (20)
\]

(I remember that in the WW formalism the field amplitudes are c-numbers and therefore they commute). In order to obtain the expectation eq.(20) we take into account that the fields have the mathematical properties of Gaussian random variables (although at the moment I do not support any
physical interpretation). Now I apply a well known property of the average of the product of four Gaussian random variables, that is

$$\langle I_{A0} I_{B1} \rangle = \langle E_{A0}^+ E_{A0}^- \rangle \langle E_{B1}^+ E_{B1}^- \rangle + \langle E_{A0}^+ E_{B1}^- \rangle \langle E_{A0}^- E_{B1}^+ \rangle + \langle E_{A0}^- E_{B1}^+ \rangle \langle E_{A0}^+ E_{B1}^- \rangle$$

Let us define

$$\langle E_{A0}^+ E_{A0}^- \rangle \langle E_{B1}^+ E_{B1}^- \rangle = \langle E_{A0}^+ E_{B1}^- \rangle \langle E_{A0}^- E_{B1}^+ \rangle + \langle E_{A0}^- E_{B1}^+ \rangle \langle E_{A0}^+ E_{B1}^- \rangle + \langle E_{A0}^+ E_{B1}^- \rangle \langle E_{A0}^- E_{B1}^+ \rangle + \langle E_{A0}^- E_{B1}^+ \rangle \langle E_{A0}^+ E_{B1}^- \rangle = \langle E_{A0}^+ E_{B1}^- \rangle \langle E_{A0}^- E_{B1}^+ \rangle + \langle E_{A0}^- E_{B1}^+ \rangle \langle E_{A0}^+ E_{B1}^- \rangle\right) = 1/2 |D|^2.$$ 

The calculation of $$\langle I_{A0} I_{B1} \rangle$$ is similar and eq.(19) leads to

$$P_{AB} = \frac{1}{2} |D|^2 \cos^2(\theta + \phi).$$ (22)
of the incoming radiation is different from zero, including both signal and vacuum fields. This is the basic assumption in this paper and it is sufficient in order to construct a local model for the experiment discussed in section 2. In free space the vacuum radiation arriving at a point will be isotropic on the average, whence its associated mean Poynting vector would be nil and only the signal radiation from the source would produce photocounts. A problem remains because the vacuum fields are fluctuating so that the net vector Poynting also fluctuates. This problem is important and it will be discussed elsewhere, but if we ignore it for the moment our hypotheses allow constructing a realistic local model as shown in the following.

In order to get a model I assume that the WW amplitudes $a_s$ and $a_i$ are real amplitudes of two modes of the radiation entering the crystal, having the (positive) probability distribution eq.(9). Thus the model is defined by all equations of the previous section. The rest of this section is devoted to the physical interpretation of these equations.

Firstly we comment on the fact that the model seems not plausible. For instance the single detection rate $P_A$ is associated to the partial field $E_{A1}$, eq.(14), rather than to the full field $E_A$, eq.(14), arriving at Alice detector. So, how may the detector discover what is the part of the field able to activate it?. The answer of this and many other questions would require a more sophisticated model. Here I will simply state that the purpose of this paper is to show that there are local models for the experiment and the plausibility of the model is irrelevant for the proof. However more realistic models are worthwhile that should be studied elsewhere.

Now we discuss the crucial aspect of locality. Firstly it is obvious that the model violates the Bell inequality, otherwise it would be impossible to agree with the quantum predictions in the discussed experiment. The violation is explained taking into account that some of the conditions of Bell local realism, see section 1, are no fulfilled. I propose a weaker form of locality that rest upon the existence of real vacuum fields and the condition of positivity does not agree with Bell’s proposal eq.(2) in general. For the sake of clarity let us rewrite eq.(19) in a form more similar to eqs.(1), exhibiting explicitly the meaning of the ensemble average, that is

$$\langle X \rangle \equiv \int X(\lambda)\rho(\lambda)\,d\lambda.$$  \hfill (23)

Hence eq.(19) may be rewritten
\[ P_{AB} = \int \rho(\lambda) f(\lambda) d\lambda, \quad f(\lambda) \equiv (I_{A0}(\lambda) - \langle I_{A0} \rangle) I_{B1}(\lambda) + I_{A1}(\lambda) (I_{B0}(\lambda) - \langle I_{B0} \rangle). \]

(24)

It is obvious that \( I_{B1}(\lambda) \) is positive for any \( \lambda \), it being a density. However the difference \( I_{A0}(\lambda) - \langle I_{A0} \rangle \) will be positive for some values of \( \lambda \) but negative for other values, it being the difference of a function minus its average. Therefore the quantity \( (I_{A0}(\lambda) - \langle I_{A0} \rangle) I_{B1}(\lambda) \) is not positive definite and the same is true for the whole term within square bracket in eq.(24). However at a closer look shows that the quantity that really should multiply \( \rho(\lambda) \) is indeed positive definite. It consists of the modulus of the sum of Poynting vectors of all radiation arriving at the detector, including the ZPF. If there was no pumping laser the modulus of the Poynting vector associated to detector would be some intensity \( I'_{A0}(\lambda) \). Therefore we should subtract all contributions of the vacuum fields, whose net Poynting vector is zero, including \( I'_{A0}(\lambda) \). Actually all vacuum fields except \( I'_{A0}(\lambda) \) are implicitly subtracted because they do not appear in eq.(24), but \( I'_{A0} \) had not been subtracted and it should. However the detector “cannot know” the value of \( I'_{A0} \), that would have arrived if there were no pump, a counterfactual value. Therefore the best choice is to subtract the mean as we made in eq.(24). I stress that the correlation between \( I_{A0}(\lambda) \) and \( I_{B1}(\lambda) \) is essential for the prediction of the model, but no correlation can exist between \( I_{B1}(\lambda) \) and \( \langle I_{A0} \rangle \) the latter being a (fixed) number. And this is correct because in the case of no pump \( I_{B1} \) would be nil whence \( P_{AB} \) would be just the product of single rates, \( P_A P_B \). This term is of order \(|D|^4\) and it has not been included in eq.(24) that is correct to order \(|D|^2\).

There is still a problem. As said above the quantity \( f(\lambda) \) eq.(24) is sometimes positive but other times negative because in our interpretation \( f(\lambda) \) represents the component of the Poynting vector arriving at the detector in the direction of the signal field. But sometimes there is net radiation coming in that direction and other times radiation coming from the back side (and from other directions that I will ignore here for simplicity of the argument). If we assume that the detector is activated by any net Poynting vector then we should average \(|f(\lambda)|\) rather than \( f(\lambda) \) and the model prediction would not agree with the quantum one. The problem is alleviated taking into account that detection is not instantaneous but takes some time interval \( T \) for the activation. As a consequence we should include the time in our model and assume that the detection rate is not really eq.(24) but a modification including time averages as in eq.(6). It is plausible that the time integral
will be now positive with few exceptions. However this problem and the one associated to the fluctuations of the Poynting vector in free space would produce some modifications, presumably small, in the predictions that should be studied more carefully.

It is interesting to study more closely the “quantum” correlation eq. (24) qualified as strange from a classical point of view because it is a consequence of the phenomenon of entanglement. The origin of eq. (24) is the correlation between the signal produced in the crystal, say $I_{B1}(\lambda)$ and the part $I_{A0}(\lambda)$ of the ZPF. With reference to eqs. (14) and (18) we see that if $I_{A0}(\lambda) - \langle I_{A0} \rangle$ is positive for some value of $\lambda$ then $I_{B1}(\lambda)$ is large but if the former is negative then the latter is small. Therefore the net result is that the product is positive on the average. The interesting feature is that it is a correlation between the fluctuation of the vacuum field arriving at Alice after crossing the nonlinear crystal and the signal field arriving at Bob and vice versa. This is entanglement as seen in the light of our model, that is a correlation between fluctuations involving the vacuum fields.

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