Standing waves in a counter-rotating vortex filament pair

Carlos García-Azpeitia

September 24, 2018

Abstract

The distance among two counter-rotating vortex filaments satisfies a beam-type of equation according to the model derived in [15]. This equation has an explicit solution where two straight filaments travel with constant speed at a constant distance. The boundary condition of the filaments is $2\pi$-periodic. Using the distance of the filaments as bifurcating parameter, an infinite number of branches of periodic standing waves bifurcate from this initial configuration with constant rational frequency along each branch.

MSC: 35B10, 35B32

Keywords: Vortex filaments. Periodic solutions. Bifurcation.

Introduction

In [15] is derived a model for the movement of almost-parallel vortex filaments from the three-dimensional Euler equation. This model takes into consideration the interaction between different filaments and an approximation for the self-induction of each filament. The paper [15] presents a first analysis of the finite time collapse of two filaments with negative circulations; close to collapse, the model of vortex filaments as an approximation to the Euler equation loses validity. Later, [11] proves that two filaments with positive...
circulations, and also three filaments with positive circulations near an equilateral triangle, evolve without collapse for all time. On the other hand, the evidence shown in \[2, 3, 13, 16\] suggests that two filaments with opposite circulations develop collapse for many initial configurations. Our aim is to investigate the existence of nontrivial periodic solutions of two vortex filaments with opposite circulations, which evolve without collapse and remains valid within the hypothesis of the model for all time.

The counter-rotating filament pair consists of two filaments with opposite circulations and same strength. In the model deduced in \[15\], the almost parallel filaments are parameterized by

\[(u_j(t, s), s) \in \mathbb{C} \times \mathbb{R}, \quad j = 1, 2,\]

and the distance among the filaments \(w_1 = u_1 - u_2\) satisfies the beam-type of equation

\[\partial_t^2 w_1 - \partial_s^4 w_1 + \partial_s^2 \left( |w_1|^2 w_1 \right). \quad (1)\]

This equation has the explicit solution \(w_1(t, s) = a\) that corresponds to the solution of two straight filaments traveling with speed \(a^{-1}\) at distance \(a\). The aim is to construct \(2\pi/\nu\)-periodic families of standing wave bifurcating from this initial configuration, where the filaments have \(2\pi\)-periodic boundary condition.

The present paper adopts the strategy followed in \[12\] for the wave equation, where bifurcation of periodic solutions is proven to exist using external parameters such as the amplitude, while the frequency is a fixed rational. In \[13\] and \[18\] this result was improved to obtain global bifurcation of periodic solutions in spherical domains. A main difference with our result is that the equation is semilinear and requires special estimates.

**Theorem 1** For each number \(q\), there is an infinite number of non-resonant (Definition \[7\]) amplitudes \(a_0\)'s given by

\[a_0^{-2} := \frac{2}{q} - \frac{1}{k_0^2 q^2}, \quad (2)\]

for some \(k_0 \in \mathbb{N}\). For each of these non-resonant \(a_0\)'s, there is a local continuum of \(2\pi q/p\)-periodic solution bifurcating from the straight filaments with distance \(a_0\), where \(p = qk_0^2 - 1\). The local bifurcation consists of standing waves satisfying the symmetries

\[w_1(t, s) = w_1(-t, s) = w_1(t, -s) = w_1(t, s + 2\pi/k_0) = \bar{w}_1(t + l_0(q\pi/p), s), \quad (3)\]

2
where \( l_0 = 0 \), and the estimate

\[
    w_1(t, s) = a_0 + itb \cos \left( \frac{pt}{q} \right) \cos k_0 s + O_C(b^2),
\]

where \( b \in [0, b_0] \) gives a parameterization of the local bifurcation.

The symmetries imply that the standing waves are even in \( t \) and even and 2\( \pi/k_0 \)-periodic in \( s \). Setting \( w_1 = x + iy \), for \( l_0 = 0 \), the symmetry (3) implies that \( y(t, s) = 0 \), i.e. the orbits of the standing waves are orthogonal to the traveling direction of the filaments. While for \( l_0 = 1 \), this symmetry implies that

\[
    x(t, s) = x(t + (q\pi/p), s), \quad y(t, s) = -y(t + (q\pi/p), s),
\]

i.e. the orbits of the standing waves resemble eight figures.

![Figure 1](image-url)  

Figure 1: Illustration of the two kind of solutions bifurcating from the straight counter-rotating vortex filaments, initially separated by \( a_0 \) and traveling with speed \( a_0^{-1} \). Left: case \( l_0 = 0 \). Right: case \( l_0 = 1 \).

In [7] the existence of standing waves for \( n \) vortex filaments of equal vorticities from a uniformly rotating central configuration is investigated. In the case of two filaments, the distance \( w_1(s, t) \) satisfies the Schrödinger equation

\[
    i \frac{\partial}{\partial s} w_1 = \left( \partial_{ss} w_1 + |w_1|^{-2} w_1 \right),
\]

which has the explicit solution \( ae^{ia^{-2}t} \) that corresponds to the solution where the two filaments rotate with frequency \( a^{-2} \) at distance \( a \). This article proves that the co-rotating filament pair has families of standing waves with amplitudes varying over a Cantor set for irrational diophantine frequencies \( a^{-2} \). In order to solve the small divisor problem that appears due the fact that the standing waves have irrational frequencies,
[7] implements a Nash-Moser procedure. This result is different but complementary to the existence of standing waves with rational frequencies in the counter-rotating filament pair. Indeed, the method in [7] can be used to obtain standing waves with irrational frequencies in the counter-rotating filament pair. The method presented here can be used to obtain standing waves with rational frequencies in the co-rotating filament pair.

Nash-Moser methods for wave, Schrödinger and beam equations have been implemented in [6], [5], [8] and references therein. Different methods which do not involve small divisor problems have been developed to prove existence of periodic solutions. In these methods, the frequency is fixed to a rational or a badly approximated irrational. For rational frequencies, the linear operator has isolated point spectrum, but the kernel associated to the bifurcation problem may have infinite dimension, see [1], [12] and [17]. For strong irrational frequencies, the inverse of the linear operator is bounded, but the inverse lacks compactness, see [4], [5] and [9]. These methods have limited applicability to semilinear beam equations [12, 4], which is the case of our problem, and also in Schrödinger equations, which is the case of the co-rotating vortex filament pair. We recommend [5] for an overview of different applications of these methods to Hamiltonian PDEs.

The proof of our theorem relies on the fact that the inverse operator associated to the bifurcation problem gains two spatial derivatives which compensates the derivatives appearing in the nonlinearity. More precisely, the bifurcation problem is equivalent to solve

\[ L(a)u + \partial_s^2 g(u) = 0, \]

for a perturbation \( u \), where \( L \) is the linearized operator and \( g(u) \) is an analytic nonlinear operator. In the Fourier basis given by \( e^{i(jt+ks)} \), the eigenvalues of \( L \) are

\[ \lambda_{j,k,l}(a) = \left( \frac{pj}{q} \right)^2 - k^4 + (-1)^l a^{-2} k^2, \]

for \((j, k, l) \in \mathbb{Z}^2 \times \mathbb{Z}_2\). The eigenvalues \( \lambda_{\pm 1, \pm k_0, 0} \) are zero at \( a_0 \) and the others satisfy

\[ \lambda_{j,k,l}(a_0) \in (qk_0)^{-2} \mathbb{Z}. \]

Thus, the operator \( L(a) \) can be inverted in the orthogonal complement of the kernel for a neighborhood of \( a_0 \). By choosing \( a_0^{-2} \in (0, 2/q) \), the projected inverse \((PLP)^{-1}\) gains two spatial derivatives due to the sharp estimate

\[ \lambda_{j,k,l}(a) \gtrsim k^2 + |j| . \]
However, the inverse \((PLP)^{-1}\) does not gain extra derivatives and \(\partial_s^2 (PLP)^{-1}\) lacks the necessary compactness to establish the global bifurcation by the classical Rabinowitz approach.

The paper is structured as follows. In Section 1, we present the equation that describes the dynamics of the distance of two straight vortex filaments. In Section 2 the existence of standing waves is obtained by the Lyapunov-Schmidt reduction method. In Section 3 the range equation is solved by the contracting mapping theorem. In Section 4 the bifurcation equation is solved using the symmetries of the problem and the Crandall-Rabinowitz theorem. Existence of traveling waves solutions is discussed in Section 5.

1 Setting the problem

The counter-rotating filament pair consists of two filaments with circulations \(\Gamma_1 = 1\) and \(\Gamma_2 = -1\). According to [15], the equations that describe the dynamics of two almost parallel filaments are

\[
\partial_t u_1 = i \left( \partial_s^2 u_1 - \frac{1}{2} \frac{u_1 - u_2}{|u_1 - u_2|^2} \right),
\]

\[
\partial_t u_2 = i \left( -\partial_s^2 u_2 + \frac{1}{2} \frac{u_2 - u_1}{|u_2 - u_1|^2} \right).
\]

The factor \(1/2\) may be obtained by scaling the dimensions and is useful in the discussion of our analysis.

The coordinates

\[ w_1 = u_1 - u_2, \quad w_2 = u_1 + u_2, \]

represent the distance and the center of mass of two filaments, respectively. In these coordinates, the equations are

\[
\partial_t w_1 = i \partial_s^2 w_2, \quad \partial_t w_2 = i \left( \partial_s^2 w_1 - |w_1|^{-2} w_1 \right).
\]

Therefore, the distance \(w_1\) satisfies the equation

\[
\partial_t^2 w_1 = i \partial_s^2 \partial_t w_2 = -\partial_s^4 w_1 + \partial_s^2 (|w_1|^{-2} w_1),
\]

and the center of mass \(w_2\) can be obtained from \(w_1\) by integration:

\[
w_2(t, s) = i \int_0^t \left( \partial_s^2 w_1 - |w_1|^{-2} w_1 \right) dt + w_2(0, s).
\]
The explicit solution
\[ w_1(t, s) = a, \quad w_2(t, s) = -ia^{-1}t, \]
corresponds to the solution where the filaments travel with constant speed. We look for bifurcation of solution from this initial configuration of the form
\[ w_1(t, s) = a(1 - u(\nu t, s)), \]
where \( u \) is 2\( \pi \)-periodic in \( t \) and \( s \).

The equation that satisfies the perturbation \( u \) is
\[ \nu^2 \partial^2_t u = -\partial^4_s u + \frac{1}{a^2} \partial^2_s \left( \frac{1}{1 - \bar{u}} \right). \]

Using a Taylor expansion, this equation is equivalent to
\[ \nu^2 \partial^2_t u = -\partial^4_s u + a^{-2} \partial^2_s \bar{u} + \partial^2_s g(\bar{u}), \]
where
\[ g(\bar{u}) = a^{-2} \frac{\bar{u}^2}{1 - \bar{u}} = a^{-2} \sum_{j=2}^{\infty} \bar{u}^j \]
is analytic for \( |u| < 1 \).

2 The Lyapunov-Schmidt reduction

Hereafter the frequency \( \nu \) is fixed to the rational
\[ \nu = \frac{p}{q}, \]
where \( p \) and \( q \) are relative prime. In order to simplify the analysis of symmetries, the equation is changed to the real coordinates given by \( u = (x, y) \in \mathbb{R}^2 \). In real coordinates, the equation is given by
\[ Lu + \partial^2_s g(u) = 0, \]
where \( L \) is the linear operator
\[ Lu := -\nu^2 \partial^2_t u - \partial^4_s u + a^{-2} R \partial^2_s u, \]
where
\[ R = \text{diag}(1, -1), \]
and \( g(u) = \mathcal{O}(\|u\|^2) \) is analytic for \( |(x, y)| < 1 \).

We present some definitions and useful results about Sobolev spaces before implementing the Lyapunov-Schmidt reduction. We use the inner product in the space \( L^2(T^2; \mathbb{R}^2) \) given by
\[
\langle u_1, u_2 \rangle = \frac{1}{(2\pi)^2} \int_{T^2} u_1 \cdot u_2 \, dt \, ds.
\]
Functions \( u \in L^2(T^2; \mathbb{R}^2) \) have the Fourier representation
\[
u = \sum_{(j,k) \in \mathbb{Z}^2} u_{j,k} e^{i(jt+ks)}, \quad u_{j,k} = \bar{u}_{-j,-k} \in \mathbb{C}.
\]
The Sobolev space \( H^s \) is the subspace of functions in \( L^2 \) with bounded norm
\[
\|u\|_{H^s}^2 = \sum_{(j,k) \in \mathbb{Z}^2} |u_{j,k}|^2 \left( j^2 + k^2 + 1 \right)^s.
\]
This space has the Banach algebra property for \( s > 1 \),
\[
\|uv\|_{H^s} \leq \|u\|_{H^s} \|v\|_{H^s}.
\]
The Banach algebra property implies that the nonlinear operator \( g(u) = \mathcal{O}(\|u\|^2_{H^s}) \) is well defined and continuous for \( \|u\|_{H^s} < 1 \). The Lyapunov-Schmidt reduction is implemented in the Sobolev space of functions with zero average,
\[
H^s_0(T^2; \mathbb{R}^2) = \left\{ u \in H^s(T^2; \mathbb{R}^2) : \int_{T^2} u = 0 \right\}.
\]
The linear operator \( L : D(L) \to H^s_0 \) is continuous when the domain
\[
D(L) = \{ u \in H^s_0 : Lu \in H^s_0 \},
\]
is completed under the graph norm
\[
\|u\|_L^2 = \|Lu\|_{H^s_0}^2 + \|u\|_{H^s_0}^2.
\]
In Fourier basis, the operator \( L : D(L) \rightarrow H^s_0 \) is given by
\[
Lu = \sum_{(j,k) \in \mathbb{Z}^2_0} (\nu^2 j^2 I - k^4 I + a^{-2}k^2 R) \ u_{j,k} e^{i(jt+ks)},
\]
where
\[
\mathbb{Z}^2_0 = \mathbb{Z}^2 \setminus \{(0,0)\}.
\]
Then, the eigenvalues of \( L \) are
\[
\lambda_{j,k,l} = (\nu j)^2 - k^4 + (-1)^l a^{-2}k^2,
\]
for \((j,k,l) \in \mathbb{Z}^2_0 \times \mathbb{Z}_2\). The set of eigenfunctions of \( L \), given by \( e_t e^{i(jt+ks)} \) with \( e_0 = (1,0) \) and \( e_1 = (0,1) \), is orthonormal and complete:
\[
Lu = \sum_{(j,k,l) \in \mathbb{Z}^2_0 \times \mathbb{Z}_2} \lambda_{j,k,l} \langle u, e_t e^{i(jt+ks)} \rangle e_t e^{i(jt+ks)}.
\]
Choosing \( a_0 \) such that
\[
a_0^{-2} = (-1)^{l_0} \left( k_0^2 - \left( \frac{pj_0}{qk_0} \right)^2 \right)
\]
for a fixed \((j_0,k_0,l_0) \in \mathbb{N}^2 \times \mathbb{Z}_2\), we have \( \lambda_{\pm j_0,\pm k_0,l_0} (a_0) = 0 \); the other eigenvalues satisfy
\[
\lambda_{j,k,l}(a_0) = (qj/p)^2 - k^4 + (-1)^{l+l_0} \left( k_0^2 - \left( \frac{pj_0}{qk_0} \right)^2 \right) k^2.
\]
Definition 2 Let \( N \subset \mathbb{Z}^2_0 \times \mathbb{Z}_2 \) be the subset of all lattice points corresponding to zero eigenvalues,
\[
N = \{(j,k,l) \in \mathbb{Z}^2_0 \times \mathbb{Z}_2 : \lambda_{j,k,l} (a_0) = 0\}.
\]
By definition we have that the kernel of \( L(a_0) \) is generated by eigenfunctions \( e_t e^{i(jt+ks)} \) with \((j,k,l) \in N\). Notice that additional sites to \((\pm j_0, \pm k_0, l_0)\) may be present in \( N \) due to resonances.

The Lyapunov-Schmidt reduction separates the kernel and the range equations using the projections
\[
Qu = \sum_{(j,k,l) \in N} u_{j,k,l} e_t e^{i(jt+ks)}, \quad Pu = (I - Q)u.
\]
Setting 
\[ u = v + w, \quad v = Qu, \quad w = Pu, \]
equation (9) is equivalent to the kernel equation
\[ QLQv + Q\partial_s^2 g(v + w) = 0, \] (14)
and the range equation
\[ PLPw + P\partial_s^2 g(v + w) = 0. \] (15)

**Proof of Theorem 1.** The proof is split in three propositions. In Proposition 5 we use the contraction mapping theorem to prove that the range equation has a unique solution 
\[ w(v, a) \in H^s_0 \]
defined in a neighborhood of \((0, a_0)\), where \( w = O(||v||^2_{H^s_0}) \). Using this solution in the kernel equation we obtain the bifurcation equation
\[ QLQv + Q\partial_s^2 g(v + w(v, a)) = 0, \] (16)
which is defined in a neighborhood of \((0, a_0) \in \ker L(a_0) \times \mathbb{R} \).

Proposition 8 proves that for each fixed positive \( q \) there is an infinite number of non-resonant amplitudes \( a_0 \in (0, 2/q) \) with \( j_0 = 1 \) and \( l_0 = 0 \). In Proposition 10, using the symmetries and a non-resonant amplitude \( a_0 \), the bifurcation equation is reduced to a subspace of dimension one within the kernel. Then, the existence of the local bifurcation is obtained by the Crandall-Rabinowitz theorem, which gives the estimates 
\[ v(t, a) = be^{i\theta} \cos t \cos k_0 s + O(b^2) \] and \( a = a_0 + O(b^2) \) for \( b \in [0, b_0] \).

Estimates in Propositions 5 and 10 imply that
\[ w_1(t, s) = a + (v + w)(pt/q, s) = a_0 + b^{j_0} \cos(pt/q) \cos k_0 s + O_{H^6_0}(b^2). \] (17)
The regularity of the solutions is obtained by the embedding \( H^6_0 \subset C^4 \) for \( s \geq 6 \). The symmetries of \( u = v + w \) follow from the symmetries of \( v \) in Propositions 10, i.e.
\[ u(t, s) = u(-t, s) = u(t, -s) = u(t, s + 2\pi/k_0) = Ru(t + l_0 \pi, s). \]
Finally, the symmetries of \( w_1 \) in the theorem follow from the symmetries of \( u \) after rescaling the period. ■
3 The range equation

In this section, the range equation is solved as a fixed point \( w(a, v) \in H^s_0 \) of the operator

\[
Kw = -(PLP)^{-1} \partial^2_s g(w + v, a).
\]

The key element in the proof consists in showing that

\[
(PLP)^{-1} \partial^2_s : H^s \to H^s_0
\]

is well defined and bounded. Once this result is established, the solution is obtained by an application of the contraction mapping theorem to the nonlinear operator

\[
Kw = \mathcal{O}(\epsilon^{-1} \|w\|^2_{H^s_0}) : B_\rho \subset H^s_0 \to H^s_0.
\]

**Lemma 3** Assume that \( 2\epsilon < a^{-2}/2 < 1/q - 2\epsilon \) and \( |a^{-2} - a^{-2}_0| \lesssim \epsilon \). Then, we have the estimate

\[
|\lambda_{j,k,l}(a_0)| \gtrsim \epsilon (k^2 + |j|) \text{ for } (j, k, l) \in \mathbb{N}^c.
\]

**Proof.** The inequality \(|pj/q - k^2| \geq 1/q\) is true unless \( pj/q = k^2 \). In the case that \( pj/q = k^2 \), then \( \lambda_{j,k,l}(a_0) = \pm a^{-2}_0 k^2 \) and

\[
|\lambda_{j,k,l}(a_0)| \gtrsim 2\epsilon k^2 \gtrsim 2\epsilon (k^2 + |j|).
\]

We may assume that \( j \geq 0 \), since the case \( j \leq 0 \) follows by analogy. For the case \( |pj/q - k^2| \geq 1/q \) and \( j \geq 0 \), we have

\[
|\lambda_{j,k,l}(a_0)| = |pj/q + \mu_k| |pj/q - \mu_k|,
\]

where \( \mu_k = \sqrt{k^4 + a^{-2}_0 k^2} \). Since \( \lim_{k \to \infty} |k^2 - \mu_k| = a^{-2}_0 /2 \), then

\[
|pj/q - \mu_k| \geq |pj/q - k^2| - |\mu_k - k^2| \geq \frac{1}{q} - \left( \frac{1}{2} a^{-2}_0 + \epsilon \right) > \epsilon,
\]

for \(|j| + |k| \geq M\) with \( M \) big. Therefore, we have the estimate

\[
|\lambda_{j,k,l}(a_0)| \geq \epsilon |pj/q + \mu_k| \geq c \epsilon (k^2 + |j|)
\]

We can adjust the constant \( \epsilon \) such that the estimate

\[
|\lambda_{j,k,l}(a)| \geq c \epsilon (k^2 + |j|)
\]
is true for all \((j, k, l) \in N^c\). Since \(|a^{-2} - a_0^{-2}| < c\varepsilon\) and
\[ L(a) = L(a_0) \pm (a^{-2} - a_0^{-2}) \partial^2_s , \]
we conclude that
\[ |\lambda_{j,k,l}(a)| \geq |\lambda_{j,k,l}(a_0)| - c\varepsilon k^2 \geq c\varepsilon (k^2 + |j|) . \]

\[ \square \]

**Lemma 4** Assume that \(2\varepsilon < a_0^{-2}/2 < 1/q - 2\varepsilon\) and \(|a^{-2} - a_0^{-2}| \lesssim \varepsilon\). The linear operator \((PLP)^{-1} \partial^2_s\) is continuous with
\[ \|(PLP)^{-1} \partial^2_s w\|_{H^s_0} \lesssim \varepsilon^{-1} \|w\|_{H^s_0} . \] (19)

**Proof.** By the previous lemma \(|\lambda_{j,k,l}(a)| \gtrsim \varepsilon k^2\) for \((j, k, l) \in N^c\). Then, the estimate
\[ \|(PLP) w\|_{H^s_0} \gtrsim \varepsilon \|\partial^2_s P w\|_{H^s_0} \]
holds true with \(w \in H^s_0\). Applying this estimate to \((PLP)^{-1} w \in D(L) \subset H^s_0\), we obtain
\[ \|\partial^2_s (PLP)^{-1} w\|_{H^s_0} \lesssim \varepsilon^{-1} \|Pw\|_{H^s_0} . \]
Since \(H^s_0 \subset C^4\) for \(s \geq 6\), then \(\partial^2_s\) and \((PLP)^{-1}\) commute. Therefore, the operator \((PLP)^{-1} \partial^2_s : PH^s \to PH^s_0\) is well define and bounded by \(O(\varepsilon^{-1})\).

\[ \square \]

**Proposition 5** Assume \(a_0^{-2} \in (0, 2/q)\). There is a unique continuous solution \(w(v,a) \in H^s_0\) of the range equation defined for \((v,a)\) in a small neighborhood of \((0,a_0) \in \ker L(a_0) \times \mathbb{R}\) such that
\[ \|w(v,a)\|_{H^s_0} \lesssim \varepsilon^{-1} \|v\|^2 , \] (20)
for small \(\varepsilon\).

**Proof.** By the Banach algebra property of \(H^s\), the operator
\[ g(w) = O(\|w\|_{H^s_0}^2) : B_{\rho} \to H^s \]
is well defined in the domain $B_\rho = \{ w \in H^s_0 : \| w \|_{H^s_0} < \rho \}$ for $\rho < 1$. Since $a_0^{-2} \in (0, 2/q)$, we can chose a small enough $\varepsilon$ such that the hypothesis of the previous lemma hold true. Therefore,

$$Kw = -\partial_s^2 (PLP)^{-1} g(w + v, a) = \mathcal{O}(\varepsilon^{-1} \| w \|_{H^s_0}^2)$$

is well defined and continuous. Moreover, it is a contraction for $\rho$ of order $\rho = \mathcal{O}(\varepsilon)$. By the contraction mapping theorem, there is a unique continuous fixed point $w(v, a) \in B_\rho$. The estimate $\| w(v, a) \|_{H^s_0} \leq \varepsilon^{-1} \| v \|^2$ is obtained from

$$\| Kw \|_{H^s_0} \lesssim \varepsilon^{-1} \left( \| w \|_{H^s_0}^2 + \| v \|^2 \right).$$

**Remark 6** Since $\lambda_{j,k} \geq \varepsilon (k^2 + j)$, the domain $D(L)$ is compactly contained in $H^s_0$. However, we cannot prove the global bifurcation by the classical Rabinowitz theorem because $(PLP)^{-1} \partial_s^2 g(u)$ is not compact, but only continuous. This lack of compactness is the reason why we cannot obtain the regularity by bootstrapping arguments. Instead, the regularity of the solutions is obtained using the Sobolev embedding $H^s_0 \subset C^4$ for $s \geq 6$.

### 4 The bifurcation equation

In this section, the bifurcation equation is solved by an application of the Crandall-Rabinowitz theorem to the case of non-resonant $a_0$'s with $a_0^{-2} \in (0, 2/q)$.

**Definition 7** An $a_0$ is non-resonant for the lattice point $(j_0, k_0, l_0) \in \mathbb{N}^2 \times \mathbb{Z}_2$ if

$$N \cap (j_0 \mathbb{Z} \times k_0 \mathbb{Z} \times \mathbb{Z}_2) = \{ (\pm j_0, \pm k_0, l_0) \}. \quad (21)$$

**Proposition 8** For each $q$, there is an infinite number of non-resonant $a_0$'s for $k_0 \in \mathbb{N}$, $j_0 = 1$ and $l_0 = 0$, given by

$$a_0^{-2} = \frac{2}{q} - \frac{1}{k_0^2 q^2}, \quad p = qk_0^2 - 1. \quad (22)$$
Proof. First we fix positive numbers $p$ and $q$. The condition

$$a_0^{-2} = (-1)^l \left( k^2 - \left( \frac{pj}{qk} \right)^2 \right) \in (0, 2/q)$$

holds for the infinite number of lattice points $(j, k, l)$ with $jp = qk^2 - 1$ and $l = 0$. Then

$$a_0^{-2} = k^2 - \left( k - \frac{1}{qk} \right)^2 = \frac{2}{q} - \frac{1}{k^2q^2}.$$  

By Proposition [3], there is a finite number of elements $(j_m, k_m, l_m)$ corresponding to a non-resonant amplitude $a_0$. That is,

$$a_0^{-2} = (-1)^{l_m} \left( k_m^2 - \left( \frac{pj_m}{qk_m} \right)^2 \right)$$

for $m \in \{0, \ldots, M\}$. Therefore, there is an infinite number of $a_0^{-2} \in (0, 2/q)$ with a finite number of resonances. We say that $(j_0, k_0)$ is a maximal lattice point if $j_m < j_0$ or $k_m < k_0$ when $j_m = j_0$ for $m \neq 0$. Let $(j_0, k_0)$ be a maximal lattice point such that

$$a_0^{-2} = (-1)^{l_0} \left( k_0^2 - \left( \frac{pj_0}{qk_0} \right)^2 \right),$$

then one has that

$$N \cap \left( j_0 \mathbb{Z} \times k_0 \mathbb{Z} \times \mathbb{Z}_2 \right) = \{ \pm j_0, \pm k_0, l_0 \}.$$  

Therefore, there is an infinite number of non-resonant $a_0$’s with $a_0^{-2} \in (0, 2/q)$.

The choice of a maximal $j_0$ is equivalent to choose a maximal $p_0 =pj_0$. That is, we have $a_0^{-2} = (-1)^{l_0} \left( k_0^2 - \left( \frac{p_0}{qk_0} \right)^2 \right)$ for the numbers $p_0$ and $q$ and

$$N \cap \left( \mathbb{Z} \times k_0 \mathbb{Z} \times \mathbb{Z}_2 \right) = \{ \pm 1, \pm k_0, l_0 \}.$$  

Therefore, for each fixed $q$, and possibly different numbers $p_0$, there is an infinite number of non-resonant amplitudes $a_0^{-2} \in (0, 2/q)$ with $j_0 = 1$ and $l_0 = 0$.  

Remark 9 The choice of maximal $p_0$ leads to the choice of a minimal period $2\pi q/p_0$ for the bifurcation. This argument is similar to the argument used in [14] for the wave equation.
To apply the Crandall-Rabinowitz theorem we need to reduce the bifurcation equation to a subspace of dimension one. This is attained by exploiting the equivariance of the problem. The equation is equivariant under the action of the group $G = \mathbb{Z}_2 \times O(2) \times O(2)$ given by

$$\rho(\tau, \sigma)u(t, s) = u(t + \tau, s + \sigma),$$

for the abelian part, and

$$\rho(\kappa_1)u(t, s) = u(-t, s), \quad \rho(\kappa_2)u(t, s) = u(t, -s), \quad \rho(\kappa_3)u(t, s) = Ru(t, s),$$

for the reflections. By the uniqueness of $w(v, a)$, the bifurcation equation has the same equivariant properties that the differential equation. This property is used in the following proposition to reduce the bifurcation equation to a subspace of dimension one.

**Proposition 10** Let $a_0^{-2} \in (0, 2/q)$ be a non-resonant amplitude for the lattice point $(1, k_0, l_0) \in \mathbb{N}^2 \times \mathbb{Z}_2$. The bifurcation equation has a local continuum of $2\pi q/p$-periodic solution bifurcating from the initial configuration with amplitude $a_0$. These solutions satisfy the estimates

$$v(t, s) = be_{l_0} \cos t \cos k_0 s + O(b^2), \quad a = a_0 + O(b^2),$$

and symmetries

$$v(t, s) = v(-t, s) = v(t, -s) = v(t, s + 2\pi/k_0) = Rv(t + l_0 \pi, s).$$

**Proof.** In the Fourier basis, the action of $G$ is given by

$$\rho(\varphi)u_{j,k} = e^{ij\varphi}u_{j,k}, \quad \rho(\theta)u_{j,k} = e^{ik\theta}u_{j,k},$$

for the abelian part and

$$\rho(\kappa_1)u_{j,k} = u_{-j,k}, \quad \rho(\kappa_2)u_{j,k} = u_{j,-k}, \quad \rho(\kappa_3)u_{j,k} = Ru_{j,k},$$

for the reflections. Setting $u_{j,k} = (u_{j,k,0}, u_{j,k,1})$, the irreducible representations correspond to the subspaces generated by $(u_{j,k,1}, u_{j,-k,1}) \in \mathbb{C}^2$. Indeed, the linear operator $L$ has blocks $\lambda_{j,k,l}I$ in these irreducible representations, which is predicted by Schur’s lemma.

Set the irreducible representation

$$(u_1, u_2) = (u_{1,k_0,l_0}, u_{1,-k_0,l_0}).$$
The action of the group in this representation is
\[ \rho(\varphi)(u_1, u_2) = e^{i\varphi}(u_1, u_2), \quad \rho(\theta)(u_1, u_2) = (e^{ik_0\theta}u_1, e^{-ik_0\theta}u_2), \]
and
\[ \rho(\kappa_1)(u_1, u_2) = (\bar{u}_2, \bar{u}_1), \quad \rho(\kappa_2)(u_1, u_2) = (u_2, u_1), \quad \rho(\kappa_3)(u_1, u_2) = (-1)^{l_0}(u_1, u_2). \]
Therefore, the group
\[ S = \langle \kappa_1, \kappa_2, (l_0\pi, \kappa_3), (\pi, \pi/k_0) \rangle \]
has fixed point space \((u_1, u_2) = (b, b)\) for \(b \in \mathbb{R}\) in this representation.

Set
\[ \ker L^S(a_0) := \ker L(a_0) \cap \text{Fix}(S). \]
The bifurcation equation
\[ QLQw + Q\partial_s^2g(v + w(v, a)) : \ker L^S(a_0) \times \mathbb{R} \to \ker L^S(a_0) \quad (25) \]
is well defined by the equivariant properties. Since for a non-resonant amplitude \(a_0\), the kernel consist of the subspace \((u_1, u_2) = (b, b)\) for \(b \in \mathbb{R}\), then the kernel in the fixed point space of \(S\) is generated by the simple eigenfunction
\[ \sum_{(j,k,l) \in \mathbb{N}} e_l e^{i(jt + ks)} = 4e_{l_0} \cos j_0 t \cos k_0 s. \]
Therefore,
\[ \ker L^S(a_0) = \{ be_{l_0} \cos j_0 t \cos k_0 s : b \in \mathbb{R} \}. \]
Since \(\ker L^S(a_0)\) has dimension one, the local bifurcation for \(a\) close to \(a_0\) follows from the Crandall-Rabinowitz theorem applied to the bifurcation equation \((25)\). It is only necessary to verify that \(\partial_a L(a) (e_{l_0} \cos j_0 t \cos k_0 s)\) is not in the range of \(L\). This follows from
\[ \partial_a L(a) (e_{l_0} \cos j_0 t \cos k_0 s) = -2a^{-3}k_0^2 Re_{l_0} (\cos j_0 t \cos k_0 s) \in \ker L(a_0). \]
The estimates \(a = a_0 + \mathcal{O}(b)\) and
\[ v(t, s) = be_{l_0} \cos j_0 t \cos k_0 s + \mathcal{O}(b^2) \]
are consequence of the Crandall-Rabinowitz theorem. Moreover, the \(S^1\)-action of the element \(\varphi = \pi/j_0\) in the kernel generated by \(e_{l_0} \cos j_0 t \cos k_0 s\) is given by \(\rho(\varphi) = -1\). This symmetry implies that the bifurcation equation is odd and \(a = a_0 + \mathcal{O}(b^2)\). \(\blacksquare\)
5 Traveling waves

The irreducible representation \((u_{1,k_0,l_0}, u_{1,-k_0,l_0})\) has another isotropy group given by

\[ T = \langle \kappa_1 \kappa_2, (l_0 \pi, \kappa_3), (\varphi, -\varphi/k_0) \rangle. \]

This isotropy group has a one dimensional fixed point space corresponding to \((u_1, u_2) = (b, 0)\) for \(b \in \mathbb{R}\). Solutions with isotropy group \(T\) are traveling waves of the form \(u(\nu t + s)\) for \(k_0 = 1\).

For these traveling waves, the PDE becomes the ODE

\[ -u'' - \nu^2 u + a^{-2} Ru + g(u) = 0. \quad (26) \]

The spectrum of the linear operator associated to the bifurcation problem is

\[ \lambda_j = j^2 - \nu^2 + (-1)^j a^{-2}. \]

Actually, the global bifurcation of traveling waves for filaments has been proven in [10] applying equivariant degree theory to the reduced ODE. In a similar manner, one can prove the following theorem.

**Theorem 11** The equation (1) has a global bifurcation of traveling waves starting from the initial configuration \(u = a\) with frequency

\[ \nu_0 = \sqrt{1 + (-1)^j a^{-2}} \in \mathbb{R}^+. \]

The local bifurcation can be parameterized by \(b\) with the estimate \(\nu(b) = \nu_0 + \mathcal{O}(b^2)\) and

\[ u(\nu t + s) = a + b^j \cos (\nu t + s) + \mathcal{O}_C(b^2). \]

Observe that the set of traveling waves forms a two-dimensional family parameterized by amplitude \(a\) and frequency \(\nu\), while standing waves exist for an infinite number of local and continuous curves that are parameterized by amplitude \(a\) and have fixed rational frequency \(\nu\).

**Acknowledgement.** The author is grateful to W. Craig and H. Kielhöfer for useful discussions related to this project. This project is supported by PAPIIT-UNAM grant IA105217.
References

[1] H. Amann, E. Zehnder. *Nontrivial Solutions for a Class of Nonresonance Problems and Applications to Nonlinear Differential Equations*. Ann. Scuol. Norm. Sup. Pisa Cl. Sci (4), 8 (1980), pp. 539–603.

[2] V. Banica, E. Faou, E. Miot. *Collision of almost parallel vortex filaments*. Comm. Pure Appl. Math. 70 (2016) 378-405.

[3] V. Banica, E. Miot. *Global existence and collisions for symmetric configurations of nearly parallel vortex filaments*. Ann. Inst. H. Poincaré Anal. Non Linéaire 29(5) (2012) 813–832.

[4] D. Bambusi. *Lyapunov Center Theorem For Some Nonlinear PDEs: A Simple Proof*. Ann. Scuola Norm. Sup. Pisa Cl. Sci 4 (1999) 823–837.

[5] M. Berti. *Nonlinear Oscillations of Hamiltonian PDEs*. Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, 2007.

[6] J. Bourgain. *Construction of periodic solutions of nonlinear wave equations in higher dimension*. Geometric and Functional Analysis. 5(4) (1995) 629-639.

[7] W. Craig, C. García-Azpeitia, C-R. Yang. *Standing waves in near-parallel vortex filaments*. Communications in Mathematical Physics, 350 (2017) 175-203.

[8] W. Craig and C.E. Wayne. *Newton’s method and periodic solutions of nonlinear wave equations*. Commun. Pure Appl. Math., 46(11) (1993) 1409–1498.

[9] R. de La LLave. *Variational methods for quasi-periodic of partial differential equations*. Hamiltonian Systems and Celestial Mechanics. World Scientific Pub Co Pte Lt, 2000, pp. 214-228.

[10] C. García-Azpeitia, J. Ize. *Bifurcation of periodic solutions from a ring configuration in the vortex and filament problems*. J. Differential Equations 252 (2012) 5662-5678.

[11] C. Kenig, G. Ponce, L. Vega. *On the interaction of nearly parallel vortex filaments*. Commun. Math. Phys., 243(3) (2003) 471–483.
[12] H. Kielhöfer. Bifurcation of periodic solutions for a semilinear wave equation. Journal of Mathematical Analysis and Applications 68 (1979) 408–420.

[13] H. Kielhöfer. Nonlinear Standing and Rotating Waves on the Sphere. Journal of Differential Equations 166 (2000) 402-442.

[14] H. Kielhöfer. Bifurcation Theory, An Introduction with Applications to Partial Differential Equations. Applied Mathematical Sciences, Springer, 2012.

[15] R. Klein, A. Majda, K. Damodaran. Simplified equations for the interaction of nearly parallel vortex filaments. J. Fluid Mech. 288 (1995) 201-248.

[16] P. Newton. The N-vortex problem. Analytical techniques. Applied Mathematical Sciences, 145. Springer-Verlag, New York, 2001.

[17] P. Rabinowitz. Free Vibration for a Semilinear Wave Equation. Comm. Pure. Appl. Math. 31 (1978) 31-68.

[18] S. Rybicki. Periodic solutions of vibrating strings. Degree theory approach. Annali di Matematica pura ed applicata 179 (2001) 197–214.