I. INTRODUCTION

A distinctive feature of the Casimir force is its non-additivity. This property enormously complicates the task of computing the Casimir force in arbitrary geometries. Indeed, Casimir himself was able to compute the leading correction to the Casimir energy beyond the commonly employed proximity force approximation. We find that for a sphere of radius \( R \) at distance \( d \) from a plane, \( R \ll d \), the force is given by an asymptotic formula of the form [16], in agreement with recent high-precision numerical computations. We develop a fast-converging numerical scheme for computing the Casimir interaction to high precision, based on spherical partial waves, and we verify that the short-distance formula provides a precise value of the Casimir energy also at large distances.

We study the Casimir interaction between perfectly conducting sphere and plane in the classical regime, i.e., at zero temperature. A major step forward was made a few years later by Lifshitz [17], who derived an exact formula for the Casimir force between two plane-parallel surfaces separated by a sub-nanometer separation. However, the exact formula is not applicable for the Casimir interaction between a sphere and a plate when \( R \ll d \). It is therefore necessary to include in the scattering formula the exact solutions of the Schrödinger equation for the electromagnetic field in the vicinity of the bodies. In principle, the formula allows to estimate the Casimir force between any pair of bodies whose T-operators are either known (as it is the case for planes, spheres or cylinders), or can be computed numerically (for example for periodic rectangular gratings) or explicitly for special shapes (spheres and cylinders) for which the T-operators are known exactly.

Casimir's formula accurately describes the Casimir force between two perfectly conducting parallel plates at distance \( d \) separated by a sub-nanometer vacuum gap, \( R \approx 10^{-15}\text{m} \). However, it is not suited to describe the Casimir force between, for example, a sphere and a plane, even at \( d \approx 10^{-10}\text{m} \). A major advancement, however, is the realization that an exact scattering formula for the Casimir interaction between two perfectly conducting curved surfaces has been found by Lifshitz [17]. We develop a fast-converging numerical scheme for computing the Casimir interaction to high precision, based on spherical partial waves, and we verify that the short-distance formula provides a precise value of the Casimir energy also at large distances.

The scattering formula undoubtedly constitutes a major contribution to the Casimir problem. However, its practical utility has been limited so far because the T-operator is known only for materials with dielectric permittivities \( \varepsilon \ll 1 \) [18]. The exact formula for the Casimir force between a sphere and a plane has been evaluated to yield a high degree of accuracy (see [19] and Refs. therein). Very recently, the exact Casimir interaction between two dielectric objects of arbitrary shapes \( \varepsilon = 1 \) has been computed for the first time [20], and the power of heat transfer between two bodies has been computed [21] has been estimated to be 100, which allows to estimate the moments of the wave function. We address the reader to Refs. [2–5] for a more detailed discussion.
Casimir interaction only for aspects ratios $d/R$ smaller than 0.02 or so. For comparison, it should be considered that in order to increase the magnitude of the force, Casimir experiments use a large sphere at small distances from the plate, with typical aspects ratios of the order of one thousandth. For so small aspect ratios, a precise computation of the Casimir interaction requires multipole orders of several thousands, which are out of reach for now.

From a practical standpoint, the main role of the exact scattering formula has perhaps been to serve as a guide towards systematically deriving approximation schemes in various regimes, going beyond the old PFA. For surfaces carrying corrugations of small amplitude, a systematic perturbative expansion of the Casimir interaction in powers of the small corrugation amplitude has been worked out [22]. Several researchers have instead endeavored to compute curvature corrections to the Casimir interaction, in the experimentally important limit of small surface separations. This is clearly a problem of outstanding practical importance, for the purpose of interpreting current small-distance precision experiments. There are presently two approaches to compute curvature corrections to the PFA. The first one consists in working out the asymptotic small-distance expansion of the exact scattering formula. The method is rigorous, but it has the drawback that the expansion has to be worked out \textit{ab initio} for each model, and for each surface geometry. By following this route, the next-to-leading-order (NTLO) correction to the Casimir energy has been computed for the cylinder/plate and the sphere/plate geometries, initially for a free scalar field obeying Dirichlet (D) boundary conditions (bc) [23], and then for the electromagnetic (em) field with perfect-conductor (P) bc [24]. Later the same approach was applied to a free scalar field obeying D, Neumann (N) and mixed ND bc on two parallel cylinders [25].

An alternative route to compute the NTLO correction to PFA assumes that the Casimir energy functional admits a derivative expansion (DE) in powers of derivatives of the surfaces height profiles. The coefficients of the DE are computed by matching the DE with the perturbative expansion of the Casimir-energy functional in the common domain of validity (for details see [26] [27]). An advantage offered by the DE, in comparison with the previous approach, is that once the DE is worked out for a specific model, it can be straightforwardly applied to surfaces of any shape. In [26] the DE was worked out for a D scalar field in the cylinder and sphere/plate geometries, giving results in agreement with the asymptotic small-distance expansion of the scattering formula in [23]. The DE for the em field with P bc, as well as for a scalar field obeying N and mixed DN bc was later worked out in [27], where the DE was also generalized to the case of two curved surfaces. Interestingly, the NTLO correction for the sphere/plate geometry with P bc obtained in [27] by using the DE was in disagreement with the result reported in [24], while the DE predicted an analytic correction $\sim d/R$, a larger logarithmic $\sim d/R \ln(d/R)$ correction had been found in [24]. A successive recalculation by some of the authors of [24] detected a sign mistake in their original computation, and finally led to full agreement with the DE expansion also in em and N cases. The DE for a D and N scalar at zero and finite temperature in any number of space-time dimensions was worked out in [28], while the experimentally important case of dielectric curved surfaces at finite temperature is presented in [29]. It is worth stressing that the NTLO correction predicted by the DE is also in full agreement with the short distance expansion of the exact sphere-plate and sphere-sphere classical Casimir energies both for Dr bc [19] as well as for P bc in four Euclidean dimensions [20]. The DE has been also used to study curvature effects in the Casimir-Polder interaction of a particle with a gently curved surface [30] [31]. The same method has been used very recently to estimate the shifts of the rotational levels of a diatomic molecule due to its van der Waals interaction with a curved dielectric surface [32].

In this paper we study the sphere-plate Casimir interaction for P bc, in the high-temperature (HT) or classical limit. In this limit, the Casimir interaction reduces to the zero-frequency Matsubara-term of the full finite-temperature scattering formula. The zero-frequency (i.e. the classical) term becomes dominant for sphere-plate separations $d$ that are larger than the thermal length $\lambda_T = \hbar c/(k_B T)$ ($\lambda_T = 7.5$ microns at room temperature). A perfect conductor constitutes the idealized limit of a superconductor, i.e. a conductor with perfect Meissner effect [34]. Ohmic metals are better modeled as Drude conductors, since normal metals do not impede static magnetic fields. Quite surprisingly, several short-distance precision experiments (see [34] and Refs. therein) with metallic plates at room temperature are in better agreement with a superconductor-like model (i.e. the plasma model) for the dielectric function of the plates, while a single large distance experiment [35] favors the Drude model. For a thorough discussion of this delicate problem we address the reader to the monograph [1].

The HT limit of the Casimir interaction for P bc has been investigated in [30], where the asymptotic small distance expansion of the scattering formula was shown to reproduce in leading order the PFA. The authors of [30] did not study though corrections to PFA. Determining the form of the the NTLO correction is an interesting problem, for the following reason. In the HT limit, the Casimir interaction for P bc is mathematically equivalent to the sum of the classical Casimir energies for a D or D scalar field (depending on whether the plates are grounded or not) plus a N scalar field. The HT limit of the sphere-plate Casimir interaction for D and Dr bc have been computed exactly not long ago [19]. However, the N and P cases have been intractable so far. Working out the NTLO correction to PFA for these two models is of great interest, because in the HT limit the perturbative kernels for N and P bc display a singular be-
havior for small in-plane momenta, invalidating the DE \[ 19, 28, 42 \]. The DE has been shown to fail also for the plasma model in the HT limit in \[ 37 \]. As a result, the analytic form of the NTLO for N and P bc is so far unknown. A large-scale numerical computation including plasma model in the HT limit in \[ 37 \]. As a result, the DE has been shown to fail also for the \[ 19, 28 \] \[ 42 \]. The DE has been shown to fail also for the NTLO correction for the Dr model is actually a \( \ln^2(d/R) \) form for the NTLO term. However, the data of \[ 38 \] appeared to support a \( \ln^2(d/R) \) also for the Dr model, and from the exact solution in \[ 19 \] we now know that the correct NTLO term is taken with weight 1/2. In Eq. (1), \( \hat{T}^{(j)} \) denotes the T-operator of object \( j \), evaluated for imaginary frequency \( i\xi_n \), and \( \hat{U} \) is the translation operator that translates the scattering solution from the coordinate of one object to the one of the other object. When considered in a plane-wave basis \( |k, Q\rangle \), where \( k \) is the in-plane wave-vector and \( Q = E, M \) is the polarization (\( E \) and \( M \) denote, respectively, transverse magnetic and transverse electric modes), the translation operator \( \hat{U} \) is diagonal, with matrix elements \( e^{-iQ\xi} \) where \( d \) is the minimum distance between the objects, \( \xi_n = \sqrt{k^2 + \xi_n^2/c^2} \), and \( c \) the speed of light. This shows that in the HT limit \( k_B T \gg \hbar c/d \), the free energy is dominated by the first term \( n = 0 \) in the sum Eq. (1):

\[
\mathcal{F}_{HT} = -k_B T \Phi, \quad \Phi = -\frac{1}{2} \text{Tr} \ln[1 - \hat{M}(0)]. \tag{2}
\]

Here, \( \Phi \) is a dimensionless temperature-independent function, depending on the static em response functions of the two bodies. Since the free-energy is proportional to the temperature, the HT (or classical) limit of the Casimir interaction has a purely entropic character.

We are interested in the classical Casimir interaction \( \mathcal{F}_{HT} \) of a sphere of radius \( R \) placed at a (minimum) distance \( d \) from a plate, but subject to P bc. We take the surface of the plate to coincide with the \((x, y)\) plane of a cartesian coordinate system, whose \( z \) axis passes through the sphere center \( C \) (see Fig.1). We define the aspect ratio \( x \) of the system as \( x = d/R \). According to Eq. (2) the computation of \( \mathcal{F}_{HT} \) involves scattering of static em fields by the two surfaces. Static em fields with \( E \) and \( M \) polarizations represent, respectively, electrostatic and magnetostatic fields which do not mix under scattering by a dielectric surface of any shape. Therefore modes with \( E \) and \( M \) polarizations give separate contributions to the Casimir energy \( \mathcal{F}_{HT} \). Moreover, it is easily seen that in the static limit the em scattering problem is mathematically equivalent to the scattering problem for a free scalar field obeying the Laplace Equation. For perfect conductors, the bc obeyed by the scalar field are as follows. For \( E \) polarization, the scalar field is subjected to either D or Dr bc on the surfaces of the two bodies, depending on whether the plates are grounded or not \[ 19, 39, 40 \], while for \( M \) polarization the scalar field obeys N bc. The dimensionless function \( \Phi^{(P)} \) providing the classical Casimir interaction for P bc can be thus decomposed as the sum of two independent contributions \( \Phi^{(D)/Dr)} \) and \( \Phi^{(N)} \), corresponding respectively to a D/Dr and a N scalar field:

\[
\Phi^{(P)} = \Phi^{(D)/Dr)} + \Phi^{(N)}. \tag{3}
\]

In the limit of vanishing separations \( x \), the Casimir energy approaches the PFA limit:

\[
\Phi_{PFA}^{(D)/Dr)} = \Phi_{PFA}^{(D)/Dr)} = \frac{\Phi_{PFA}^{(P)}}{2} = \frac{\xi(3)}{8x}. \tag{4}
\]

The exact expression of the functions \( \Phi^{(D)/Dr)} \) was determined in \[ 19 \] by taking advantage of the separability of

**II. THE CASIMIR ENERGY IN THE CLASSICAL LIMIT**

We start from the general scattering formula \[ 10, 12 \] for the Casimir free energy of two objects (denoted as 1 and 2) in vacuum:

\[
\mathcal{F} = k_B T \sum_{n \geq 0} \text{Tr} \ln[1 - \hat{M}(i\xi_n)] ,
\]

\[
\hat{M} = \hat{T}^{(1)} \hat{U} \hat{T}^{(2)} \hat{U}. \tag{1}
\]

Here \( k_B \) is Boltzmann’s constant, \( T \) is the temperature, \( \xi_n = 2\pi n k_B T/\hbar \) are the (imaginary) Matsubara frequencies, and the prime in the sum indicates that the \( n = 0 \)
The energy for ungrounded perfect-conductors is accordingly represented as:

$$\Phi^{(P)} = \Phi^{(Dr)} + \Phi^{(D)} + \delta \Phi,$$

while for grounded conductors we write:

$$\Phi^{(P)}|_{gr} = 2 \Phi^{(D)} + \delta \Phi.$$

In the next Section we shall work out an asymptotic formula for $\delta \Phi$, valid in the limit of small separations, while in Sec. IV $\delta \Phi$ shall be computed numerically using the exact scattering formula Eq. (1).

III. A SHORT-DISTANCE FORMULA FOR $\delta \Phi$

Before we start the computation of $\delta \Phi$, it is important to notice that, due to the presence of the trace in the general scattering formula Eq. (1), the Casimir interaction depends only on the equivalence class $[[M]]$ formed by all matrices $M$ that represent the operator $M$, where two elements $M$ and $M'$ of $[[M]]$ differ by a similarity transformation by an invertible matrix $A$: $M' = AMA^{-1}$.

The matrix $M^{(N)}$ for N bc is easily computed in a spherical multipole basis with origin at the sphere center $C$. In this basis the regular and outgoing eigenfunctions of the Laplace Equation have the familiar form $\phi^{(reg)}_{lm} = r^{l} Y_{lm}(\theta, \phi)$, and $\phi^{(out)}_{lm} = r^{-(l+1)} Y_{lm}(\theta, \phi)$, with $l \geq 0$, and $m = -l, \ldots, l$. By rotational symmetry around the azimuthal axis $\hat{\phi}$, the matrix $M^{(N)}$ commutes with $J_z$ and hence is block diagonal. We let $M^{(N|m)}$ the block corresponding to the value $m$ of $J_z$. One finds:

$$\hat{M}^{(N|m)} = \left[ \begin{array}{c}
\frac{l}{l+1} \frac{(l+l')!}{(l+m)!(l'-m)!} \left( \frac{1}{2(1+x)} \right)^{l+l'+1} \\
\end{array} \right],$$

with $|l, l'| \leq |m|$. Apart from the factor $l/(l+1)$, the matrix $M^{(N|m)}$ coincides with the corresponding matrix $M^{(D|m)}$ for D bc:

$$\hat{M}^{(D|m)} = \left[ \begin{array}{c}
\frac{(l+l')!}{(l+m)!(l'-m)!} \left( \frac{1}{2(1+x)} \right)^{l+l'+1} \\
\end{array} \right].$$

Each block $\hat{M}^{(N|m)}$ contributes separately to the Casimir energy, and we denote by $\Phi^{(N)}_m$ the corresponding contribution to $\Phi^{(N)}$. Of course, opposite values of $m$ give identical contributions to the Casimir energy, i.e. $\Phi^{(N)}_m = \Phi^{(N)}_{-m}$. We can thus write $\Phi^{(N)}$ as:

$$\Phi^{(N)} = 2 \sum_{m \geq 0} \phi^{(N)}_m,$$

where the prime again denotes that the $m = 0$ term is taken with a weight 1/2 and

$$\phi^{(N)}_m = -\frac{1}{2} \text{Tr} \ln[1 - \hat{M}^{(N|m)}].$$
A. Contribution of the modes with \( m = 0 \).

Luckily enough the contribution \( \Phi_0^{(N)} \) of the \( m = 0 \) modes can be computed exactly. By a similarity transformation with the diagonal matrix \( A_{ll'} = (l+1)\delta_{ll'} \), the matrix \( M_0^{(N)(0)} \) in Eq. (11) is transformed to the matrix \( \tilde{M}_0^{(N)(0)} \):

\[
\tilde{M}_0^{(N)(0)} = \frac{l}{l'+1} \left( \frac{l+l'}{l!l'!} \right)^{l+l'+1} , \tag{15}
\]

with \( l, l' \geq 0 \). The first row of the matrix \( \tilde{M}_0^{(N)(0)} \) is zero, while its \( l \)-th row with \( l = 1, 2, \cdots \) is identical to the \((l-1)\)-th row of the matrix \( M_l^{(D)(0)} \) in Eq. (12) with its first column deleted: \( \tilde{M}_0^{(N)(0)} = M_{l-1,l'+1}^{(D)(0)} \), \( l = 1, 2, \cdots \), \( l' = 0, 1, 2, \cdots \). By a second similarity transformation with the upper diagonal matrix \( \tilde{A}(Z) \):

\[
\tilde{A}_l(Z) = Z^{l'-l} \left( \frac{l'}{l'-l!} \right) \tag{16}
\]

with \( \tilde{A}^{-1}(Z) = \tilde{A}(-Z) \), the matrix \( \tilde{M}_0^{(N)(0)} \) is transformed into a lower triangular matrix \( M_0^{(N)(0)} \), with diagonal elements equal to \( \tilde{M}_0^{(N)(0)} = Z^{2l+3}, l = 0, 1, 2, \cdots \). This implies at once:

\[
\Phi_0^{(N)} = -\frac{1}{2} \sum_{l \geq 0} \ln[1 - Z^{2l+3}] . \tag{17}
\]

This result can be contrasted with the analogous formula for \( D \) be [19]:

\[
\Phi_0^{(D)} = -\frac{1}{2} \sum_{l \geq 0} \ln[1 - Z^{2l+1}] . \tag{18}
\]

B. Contribution of modes with \( m \neq 0 \).

Unfortunately, the contributions \( \Phi_m^{(N)} \) of the modes with \( m \neq 0 \) cannot be computed exactly. By using the technique of Refs. [23][25] it is however possible to prove a short-distance formula for \( \Phi_m^{(N)} \), or more precisely for the difference \( \delta \Phi_m = \Phi_m^{(N)} - \Phi_m^{(D)} \). We start by expanding the logarithm in Eq. (14):

\[
\Phi_m^{(N/D)} = \frac{1}{2} \sum_{s=0}^{\infty} \frac{1}{s+1} \left( \prod_{i=0}^{s} \sum_{l_i, l_i' = [m]} \prod_{i=0}^{s} M_{l_i, l_i'+1}^{(N/D)(m)} \right) , \tag{19}
\]

where \( l_{s+1} = l_0 \). Next, for \( 0 < i \leq s \) we perform on the indices \( l_i \) the shift: \( l_i = l + l_i' \), where we set \( l := l_0 \). For small separations \( x \ll 1 \), the Casimir energy is dominated [23][25] by multipoles such that:

\[
l \sim 1/x , \quad |l_i'| \sim 1/\sqrt{x} , \quad |m| \sim 1/\sqrt{x} . \tag{20}
\]

For small \( x \) the discrete sums over \( l \) and \( l_i' \) in Eq. (19) can be replaced by integrations (this corresponds to taking the leading term in the Abel-Plana summation formula):

\[
\Phi_m^{(N/D)} = \frac{1}{2} \int_0^\infty dl \left[ M_{l,l}^{(N/D)(m)} + \sum_{s=1}^{\infty} \frac{1}{s+1} \left( \prod_{i=1}^{s} \int_{-\infty}^{\infty} dl_i' \prod_{i=0}^{s} M_{l+l_i', l_i'+1}^{(N/D)(m)} \right) \right] , \tag{21}
\]

and we set \( l_0 = l_{s+1} = 0 \). In writing the above Equation, we considered that the integration over \( l \) extending from \( |m| \) to \( \infty \) can be replaced by an integration from zero to \( \infty \) because, according to Eq. (20), in the limit of small separations \( m \) is negligibly small compared to \( l \). We similarly replaced the integration over \( l_i' \) extending from \( |m| - l \) to \( \infty \) by an integration from \( -\infty \) to \( \infty \). Compared to \( l_i' \), \( (|m| - l) \) can be identified with \( -\infty \). Next, we observe that by virtue of Eq. (20) the numbers \( l + l_i', l \pm m \) are all large integers for small \( x \) and therefore the factorials in Eqs. (11) and (12) can be computed using Stirling’s formula:

\[
\ln n! = \left( n + \frac{1}{2} \right) \ln n - n + \frac{1}{2} \ln 2\pi + \frac{1}{12n} + \cdots \tag{22}
\]

At this point, we Taylor expand the difference \( \delta M_{l+l_i', l_i'+1}^{(m)} = M_{l+l_i', l_i'+1}^{(N)(m)} - M_{l+l_i', l_i'+1}^{(D)(m)} \) among the matrices \( M_0^{(N)} \) and \( M_0^{(D)} \) in powers of \( \sqrt{x} \) (powers of \( \sqrt{x} \) are reckoned according to the estimates in Eqs. (20)). Up to terms of order \( x^2 \) we find:

\[
\delta M_{l+l_i', l_i'+1}^{(m)} = \frac{1}{\sqrt{4\pi l}} \left( \frac{l}{l+1} - 1 \right) \exp \left[ -2xl - \frac{(l_i' - l_i'+1)^2}{4l} - \frac{m^2}{l} \right] + o(x^2) . \tag{23}
\]

On the other hand, by taking the Taylor expansion of Eq. (12) we find:

\[
M_{l+l_i', l_i'+1}^{(D)(m)} = \frac{1}{\sqrt{4\pi l}} \exp \left[ -2xl - \frac{(l_i' - l_i'+1)^2}{4l} - \frac{m^2}{l} \right] + o(x) . \tag{24}
\]


The two formulae above confirm correctness of the estimates in Eq. (20). There is a tricky but important point to stress here: following the logic of the Taylor expansion, one might find appropriate to replace the factor \([l/(l + 1) - 1]\) in the r.h.s. of Eq. (23) by its first order Taylor approximation \([l/(l + 1) - 1] = -1/l + o(x^2)\).

The problem with this substitution is that it leads to an infra-red divergence in the integral over \(l\). To avoid this problem, we keep the complete factor \([l/(l + 1) - 1]\) in Eq. (23).

Starting from Eq. (21), and making use of Eqs. (23) and (24), we obtain the following expression for \(\delta \Phi_m\):

\[
\delta \Phi_m = \frac{1}{2} \int_0^\infty \frac{dl}{\sqrt{4\pi l}} \left\{ \left( \frac{l}{l+1} - 1 \right) \exp \left( -2x l - \frac{m^2}{l} \right) + \frac{1}{2} \sum_{s=1}^\infty \frac{1}{s+1} \left( \frac{l}{l+1} \right)^{s+1} - 1 \right\}
\]

\[
\times \left( \prod_{i=1}^s \int_{-\infty}^{\infty} \frac{dl_i}{\sqrt{4\pi l_i}} \right) \prod_{i=0}^s \exp \left[ -2x l - \frac{(l_i - l_i+1)^2}{4l} - \frac{m^2}{l} \right] \right\} + o(x),
\]

(25)

Performing the gaussian integrals over \(l_i\), we then obtain the following estimate for \(\delta \Phi_m\) accurate to order \(x^{1/2}\):

\[
\delta \Phi_m^{(1/2)} = \frac{1}{2} \sum_{s=0}^\infty \frac{1}{(s+1)^{3/2}} \int_0^\infty \frac{dl}{\sqrt{4\pi l}} \left[ \left( \frac{l}{l+1} \right)^{s+1} - 1 \right] \exp \left[ (s+1) \left( -2x l - \frac{m^2}{l} \right) \right].
\]

(26)

The sum over \(s\) can be expressed in terms the polylogarithm function \(Li_n(z) = \sum_{k=1}^\infty z^k/k^n\):

\[
\delta \Phi_m^{(1/2)} = \frac{1}{2} \int_0^\infty \frac{dl}{\sqrt{4\pi l}} \left\{ Li_{3/2} \left[ \frac{l}{l+1} \exp \left( -2x l - \frac{m^2}{l} \right) \right] - Li_{3/2} \left[ \exp \left( -2x l - \frac{m^2}{l} \right) \right] \right\}.
\]

(27)

Combining the above formula with the exact expressions of \(\Phi_0^{(N)}\) (Eq. (17)) and \(\Phi_0^{(D)}\) (Eq. (18)) we obtain for \(\delta \Phi\) the approximate small distance formula:

\[
\delta \Phi^{(0)} = -\frac{1}{2} \sum_{l \geq 0} \ln \left( 1 - \frac{Z^{2l+3}}{1 - Z^{2l+3}} \right) + 2 \sum_{m > 0} \delta \Phi_m^{(1/2)}.
\]

(28)

We expect that this formula for \(\delta \Phi\) is accurate to order \(x^0\). We shall later see that, despite the assumption \(x \ll 1\) made in its derivation, the above formula provides a very precise value of \(\delta \Phi\) also for relatively large separations (see Fig. 2).

C. Expressions at small distances

With exact expressions for the Casimir energies in the D and Dr models, one can compute explicitly the interaction in the limit of short distances \(x \ll 1\). This limit corresponds to \(Z\) close to unity, and one can compute the series in Eqs. (5) and (6) using the Abel-Plana formula. We set \(Z = \exp(-\mu)\), and then expand for small \(\mu\), where \(\mu = \ln[1 + x + \sqrt{2(x^2 + x)}]\). The resulting analytical expressions for the Casimir interaction were worked out in [19], and we reproduce them here for the convenience of the reader:

\[
\Phi^{(D)} = \frac{\zeta(3)}{4\mu^2} - \frac{1}{24} \ln \mu \left( 1 - \frac{1}{6} + \gamma_0 \right) + \frac{7}{5760} \mu^2 + o(\mu^4),
\]

(29)

\[
\Phi^{(D)} = \Phi^{(D)} - \frac{1}{12} \ln \mu + o(\mu^4),
\]

(30)

with \(\gamma_0 = 0.0874485, \gamma_1 = 1.270362, \gamma_2 = 1.35369\). We used \(\mu\) as a variable for the expansion, for it provides a very accurate result also at larger distances. Both the D and Dr energies depend only on \(\ln \mu\) and even powers of \(\mu\). This implies that the energies depend only on \(\ln x\) and integer powers of \(x\). In particular, the force for the D case, once expanded in \(x\), is a Laurent series starting from \(1/x^2\). However, for the Dr case there are logarithmic terms in the force as well. The leading correction to PFA is the same \(\ln \mu\) term for both models, and its coefficient is in agreement with the DE. Interestingly, for practically relevant separations the subleading double logarithmic term in Eq. (30) dominates over the leading logarithmic term, and therefore the D and Dr energies display rather different behaviors.

To work out the leading correction to PFA of the N energy, we start from Eq. (28). It is convenient to use for \(\delta \Phi_m^{(1/2)}\) the expression in Eq. (26). In the limit of vanishing separation, the sum over the angular index \(m\) can be replaced by an integration over \(m\) extending from \(-\infty\) to \(\infty\). Performing the straightforward gaussian integral we find:

\[
\delta \Phi_{as} = \frac{1}{4} \sum_{s=0}^{\infty} \frac{1}{(s+1)^2} \int_0^\infty dl \left[ \left( \frac{l}{l+1} \right)^{s+1} - 1 \right] e^{-2(s+1)x}.
\]
\[ \delta \Phi_{as} = -\frac{1}{16} \ln^2 x + o(\ln x) . \] (32)

Since in the D model the leading correction to the PFA is a \( \ln x \) term (see Eq. (29)), the leading correction to the PFA for the N model coincides with the \( \ln^2 x \) term of \( \delta \Phi \):

\[ \Phi^{(N)} = \frac{\xi(3)}{8x} - \frac{1}{16} \ln^2 x + o(\ln x) . \] (33)

Earlier we pointed out that the leading correction to the PFA for the Dr and the D model is the same \( \ln x \) term. It then follows from Eq. (3) that the \( \ln^2 x \) term of \( \Phi^{(N)} \) represents also the leading correction to the PFA for P bc:

\[ \Phi^{(P)} = \frac{\xi(3)}{4x} - \frac{1}{16} \ln^2 x + o(\ln x) . \] (34)

Thus our analytical results provide a rigorous proof of the \( \ln^2(x) \) form of the leading curvature correction to PFA, in accordance with indications obtained from the high-precision numerical data of \( [38] \).

**IV. NUMERICAL COMPUTATION OF \( \delta \Phi \)**

The HT limit of the (ungrounded) sphere-plate Casimir energy with P bc was computed in \([38]\) by a large-scale numerical computation of the exact scattering formula Eq. (1) using the standard spherical basis with origin at the sphere center \( C \). The computation in \([38]\) included up to 5000 partial wave orders, which allowed the authors of Ref. \([38]\) to accurately estimate the functions \( \Phi^{(P)} \) and \( \Phi^{(Dr)} \) for aspect ratios \( x \geq 2 \times 10^{-3} \).

Earlier we saw that the classical Casimir energy for ungrounded perfect conductors is the sum of the energies for a Dr scalar plus a N scalar (see Eq. (9)). The (normalized) energy \( \Phi^{(Dr)} \) can be computed exactly in the sphere-plate geometry (see Eq. (10)), while for N bc an exact formula exists for \( m = 0 \) modes. In the previous Section we derived an asymptotic formula, Eq. (28), valid for small-distances, for the difference \( \delta \Phi \) among the HT Casimir energies for N and D bc. In this Section the energy-difference \( \delta \Phi \) is computed numerically, using the exact scattering formula Eq. (1). As we shall see, \( \delta \Phi \) can be computed very efficiently by using a basis of *bipshperical multipoles* \([19]\).

Bispherical coordinates \( (\mu, \eta, \phi) \) \([41]\) are defined by \((x, y, z) = a(\sin \eta \cos \phi, \sin \eta \sin \phi, \sinh \mu)/(\cosh \mu - \cos \eta)\), where \( a \) identifies the focus \( F \) of the sphere defined by \( \mu = \mu_1 > 0 \) (see Fig.1). The sphere has radius \( R = a/\sinh \mu_1 \), and \( L = a \coth \mu_1 \) is the distance of its center \( C \) from the \( \mu = 0 \) plane. The Laplace Equation is separable in bispherical coordinates, and its regular and outgoing eigenfunctions are:

\[ \phi_{lm}^{reg/out} = \sqrt{\cosh \mu - \cos \eta} Y_{lm}(\eta, \phi) \exp[\pm(l + 1/2)\mu] , \] (35)

for \( l \geq 0, m = -l, \ldots, l \). Relative to the sphere (plane) outgoing and regular eigenfunctions correspond, respectively to the upper (lower) and lower (upper) sign in the exponential. Scattering solutions can be expanded in these eigenfunctions. It is a simple matter to verify that in the bispherical basis of Eq. (35) the translation matrix \( U \) is diagonal with elements \( U_{lm'\mu'} = Z^{l+1/2} \delta_{l \mu} \delta_{m m'} \), where \( Z = \exp(-\mu_1) \). For D bc the \( T \)-matrix for both the plane and sphere are minus the identity operator. Therefore, in the bispherical basis the \( M^{(D)} \) matrix for D bc is diagonal with elements \( M_{lm'\mu'}^{(D)} = Z^{2l+1} \delta_{l \mu} \delta_{m m'} \), and thus evaluation of the scattering formula Eq. (1) is straightforward yielding the result quoted in Eq. (3).

The case of Dr bc is more elaborate, as one has to remove the contribution of monopoles from the \( m = 0 \) block. Details can be found in \([19]\). For N bc the \( T \)-matrix of the \( \mu = 0 \) plane is equal to the identity operator. However, the \( T \)-matrix of the sphere is unfortunately non-diagonal. Of course, the \( T \)-matrix is still block diagonal with respect to the angular index \( m \), and it is convenient to decompose its blocks as \( T^{(2m)} = 1 + \delta T^{(N|m)} \). By an explicit computation in the bispherical basis, it is found that the matrix \( \delta T^{(N|m)} \) satisfies the linear system

\[ B^{(m)} \delta T^{(N|m)} = -2 \sinh \mu_1 1 , \] (36)

where \( B^{(m)} \) is the matrix of elements

\[ B^{(m)}_{ll'} = [(2l + 1) \cosh \mu_1 + \sinh \mu_1] \delta_{ll'} - (l - m) \delta_{l,l+1} - (l' + m) \delta_{l+1,l'} . \] (37)

with \( l, l' \geq |m| \). The linear system Eq. (36) cannot be solved analytically, but it can be easily solved numerically after truncation in the multipole order \( l, l' < l_{max} \).

At this point it would seem that nothing is really gained by using bispherical multipoles, because we still face the problem of computing determinants of infinite-dimensional matrices, as we had to do anyhow in the standard base of spherical multipoles. Indeed the situation seems even worse now, because earlier at least the matrix \( M^{(N|m)} \) had a simple expression (see Eq. (11)), while now the matrix \( \delta T^{(N|m)} \) has to be itself computed numerically by solving an infinite-dimensional linear system. This shortcoming of bispherical coordinates is however rewarded by the crucial advantage of a much faster rate of convergence with respect to the maximum value \( l_{max} \) of the multipole index \( l \). To see this, consider the expression of the \( M \) matrix for N bc in bispherical coordinates:

\[ M_{ll'}^{(N|m)} = Z^{2l+1} (\delta_{ll'} + \delta T_{ll'}^{(m)}) , \] (38)

with \( l, l' \geq |m| \). When this expression is substituted into the scattering formula Eq. (1), it is easy to factor out
the D contribution, and one ends up with the following exact representation for the energy-difference \( \delta \Phi \) defined in Eq. (39):

\[
\delta \Phi = - \sum_{m \geq 0} \frac{\ln[1 + V^{(m)} \delta T_{N}^{(m)}]}{1 - e^{\mu_{1}(2l+1)}} \delta_{ll'}.
\]

(40)

where \( V^{(m)} \) is the diagonal matrix of elements:

\[
V_{ll'}^{(m)} = \frac{1}{1 - e^{\mu_{1}(2l+1)}} \delta_{ll'}.
\]

The exponential in the denominator of \( V_{ll'}^{(m)} \) shows that the multipoles contributing to \( \delta \Phi \) are those with \( l, l' \leq 1/\mu_{1} \). For small \( x \), \( \mu_{1} = - \ln Z = \ln[1 + x + \sqrt{x(2 + x)}]^{-1} \simeq \sqrt{2x} \) and then we see that the order \( l \) of the relevant partial waves scales like \( \sqrt{R/d} \), which is only the square root of the multipole order \( l_{\text{max}} \sim R/d \) (see Eq. (20)) needed in the spherical basis.

To demonstrate the fast rate of convergence of the scattering formula in bispherical coordinates, we take as an example \( x = 2 \times 10^{-3} \), which is the smallest aspect ratio considered in [38]. By including in the scattering formula 5000 partial (spherical) waves, the authors of [38] computed \( \Phi_{(P)} = 146.812 \) and \( \Phi_{(Dr)} = 74.5962 \). On the other hand, using the exact formula in Eq. (9) we find \( \Phi_{(D)} = 75.2936 \). From Eq. (9) we then get \( \delta \Phi = -3.07737 \). In Table I we quote the values of \( \delta \Phi \) obtained from Eq. (39) with inclusion of bispherical multipoles of order \( l \leq l_{\text{max}} \) for \( l_{\text{max}} = 20, 40, 80, 120 \). As it can be seen, \( \delta \Phi \) converges quickly, and already with \( l_{\text{max}} = 80 \) the error made by using \( \delta \Phi \) is as small as \( 1.2 \times 10^{-5} \). With \( l_{\text{max}} = 120 \) we reproduce the value computed in [38] using 5000 partial waves. It is interesting to compare the numerical value of \( \delta \Phi \) with the estimate provided by the asymptotic formula Eq. (28). Evaluation of Eq. (28) gives \( \delta \Phi_{(0)} = -3.068 \), which differs from the numerical value of \( \delta \Phi \) by less than 0.3 %.

| \( l_{\text{max}} \) | 20  | 40  | 80  | 120 |
|---------------------|-----|-----|-----|-----|
| \( \delta \Phi \)   | -2.92435 | -3.06243 | -3.07725 | -3.07737 |

TABLE I: Numerical values of \( \delta \Phi \) for aspect ratio \( x = 2 \times 10^{-3} \) obtained from the scattering formula in bispherical coordinates Eq. (39) with inclusion of multipoles of order \( l \leq l_{\text{max}} \). The value of \( \delta \Phi \) for \( l_{\text{max}} = 120 \) is in perfect agreement with the value obtained in [38] using spherical multipoles up to \( l_{\text{max}} = 5000 \).

In Fig. 2 we plot \( \delta \Phi \) as a function \( \log_{10}(x) \). The dots were computed using the scattering formula for \( \delta \Phi \) in a bispherical basis Eq. (39). The fast convergence of Eq. (39) allowed us to accurately compute \( \delta \Phi \) for aspect ratios as small as \( x = 10^{-5} \) by using less than 1000 partial waves. The solid line Fig. 2 was computed using the asymptotic formula for \( \delta \Phi \), Eq. (28). It can be seen that the asymptotic formula Eq. (28) provides a precise estimate of \( \delta \Phi \) over the entire range of aspect ratios displayed in the Figure, up to the fairly large value \( x = 0.1 \).

In Fig. 2 we display a plot of \( \beta^{(P)}(x) \), where dots represent our numerical data, while the solid line is computed using the small-distance formula Eq. (28) for \( \delta \Phi \).
V. CONCLUSIONS

We studied the Casimir interaction between a sphere and a plate, both perfectly conducting, in the classical limit of high temperatures. We worked out an analytical formula for the energy, valid for sufficiently small separations. Taking the asymptotic expansion of the small-distance formula we found a $\ln^2(d/R)$ correction in the energy, beyond the commonly used proximity force approximation. The $\ln^2(d/R)$ form of the correction is in agreement with a fit of large-scale numerical data. We developed a fast-converging numerical scheme for computing the Casimir energy, based on a system of bispherical partial waves. In bispherical coordinates, convergence of the exact scattering formula is achieved at multipole order $l_{\text{max}} \simeq \sqrt{R/d}$, while in the standard approach based on spherical multipoles convergence is achieved only at order $l_{\text{max}} \simeq R/d$. Using the bispherical basis, we could accurately compute the Casimir energy for very small aspects ratio $d/R = 10^{-5}$. Comparison with the high-precision numerical data shows that the analytical small-distance formula precisely estimates the energy also for fairly large values of the aspect ratio.

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