Solution of five dimensional Dirac equation with Asymptotic Iteration in the case of spin symmetry

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Abstract The relativistic energies and wave function of particle s wave spin $\frac{1}{2}$ which is governed by separable non central potential in five dimensions are obtained determined using AIM in the case of spin symmetry. The separable five dimensional shape invariant potentials consist of radial potentials of Hulthen and Manning Rosen angular potentials of $\theta_1$, $\theta_2$, $\theta_3$, and $\theta_4$. The relativistic energies are numerically calculated from relativistic energy equation using MatLab software.

1. Introduction
Exact solution for relativistic energy can be obtained from Dirac equation. Dirac equation works in half spin particle such as electron and positron. Like Klein-Gordon equation [1] Dirac equation [2] can always be reduced to a Schrödinger type equation [3]. Many types of potential also been done for specific system, Ring-Shaped like potential [4], Hulthen potential [5], Eckart potential [6] and so on. Here an electron works on Manning-Rosen [7] and Hulthen potential. Manning-Rosen used to explain system of diatomic molecules and Hulthen potential is one of the most important short-range potentials which behaves like a Coloumb potential for small values of $r$ and decreases exponentially for large values of $r$. The higher dimensional field theory have been studied of many authors. Variation of dimensional case have been obtained in D-dimensional[8] for Dirac equation. Nowadays AIM used for many research to find exact solution[9-10]. AIM can be used in many potential Here Hulthen potential works on radial, and Manning-Rosen potential works on angular called $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$. Hulthen potential and Manning rosen potentials used in this research are:

\begin{align}
V(r) &= -\frac{V_0 e^{-ar}}{1 - e^{-ar}} \\
V(\theta_1) &= \frac{V_1}{\sin^2 \theta_1} - V_2 \cot \theta_1 \\
V(\theta_2) &= \frac{V_1'}{\sin^2 \theta_2} \\
V(\theta_3) &= \frac{V_1''}{\sin^2 \theta_2} - V_2' \cot \theta_3 \\
V(\theta_4) &= -V_2'' \cot \theta_4
\end{align}
2. Asymptotic Iteration Method

AIM solve homogeneous linear second order differential equation to get exact solution of the form

\[ y_n''(x) = k_0(x)y_n'(x) + s_0(x)y_n(x) \]  

(5)

Where \( k_0 \neq 0, s_0(x) \) is coefficient differential equation and \( n \) clarifying quantum number. We differentiate Eq (5) with respect to \( x \):

\[ y_n'''(x) = k_1(x)y_n'(x) + s_1(x)y_n(x) \]  

(6)

Where

\[ k_1(x) = k_0'(x) + k_0^2(x) + s_0(x) \]
\[ s_1(x) = s_0'(x) + s_0k_0 \]  

(7)

And the second derivative of Eq (5) gives:

\[ y_n'''(x) = k_2(x)y_n'(x) + s_2(x)y_n(x) \]  

(8)

Where

\[ k_2(x) = k_1'(x) + k_1k_0(x) + s_1(x) \]
\[ s_1(x) = s_0'(x) + s_0k_1(x) \]  

(9)

It can iterate up to \((i+1)h\) and \((i+2)h\) derivatives, with \( i = 1, 2, 3... \)

So we have

\[ y_n^{(i+1)}(x) = k_{i-1}(x)y_n'(x) + s_{i-1}(x)y_n(x) \]
\[ y_n^{(i+2)}(x) = k_i(x)y_n'(x) + s_i(x)y_n(x) \]  

(10)

Where

\[ k_i(x) = k_{i-1}'(x) + k_{i-1}k_0(x) + s_{i-1}(x) \]
\[ s_i(x) = s_{i-1}'(x) + s_{i-1}k_{i-1}(x) \]  

(11)

From Eq (13) we got relation:

\[ \frac{y_n^{(i+2)}(x)}{y_n^{(i+1)}(x)} = \frac{k_i[y_n'(x) + \frac{s_i}{k_i}y_n(x)]}{k_{i-1}[y_n'(x) + \frac{s_{i-1}}{k_{i-1}}y_n(x)]} \]  

(12)

Asymptotic aspect of the method makes large \( i(>0) \), Eq (14) becomes

\[ \frac{s_i}{k_i} = \frac{s_{i-1}}{k_{i-1}} \equiv \Delta \]  

(13)

Eq (14) can reduced to simpler equation

\[ \frac{y_n^{(i+2)}(x)}{y_n^{(i+1)}(x)} = \frac{k_i}{k_{i-1}} \]  

(14)
The integration of Eq (16) is:

$$y_n^{(i+2)}(x) = C e^{\int_{k_{i-1}}^{k_i}}$$

(15)

Where \( C \) is constant outcome from integration. Inserting Eq (11) and Eq (13) to Eq (15) become

$$y_n^{(i+1)}(x) = C2_{i-1}e^{\int_{[\alpha(x)+\lambda_i(x)]}^{u_i}}$$

(16)

Inserting Eq (16) to Eq (10), the first order differential equation is obtained as:

$$y'_n + \alpha y_n(x) - C e^{\int_{k_i(x)+\lambda_i(x)}^{u_i}} = 0$$

(17)

This first order differential equation general solution can be write as:

$$y_n(x) = e^{-\int_{[\alpha(x)+\lambda_i(x)]}^{u_i}} \left[ C' + C e^{\int_{k_i(x)}^{u_i} dx} \right]$$

(18)

For given potential, we convert five dimensional Dirac equation to the form of Eq (5). Than determine \( k_0 \) and \( s_0 \). Using Eq (11) \( s_i(x) \) and \( k_i(x) \) are obtained. Energy eigenvalue we get from

$$k_i(x)s_{i-1}(x) - k_{i-1}(x)s_i = \Delta_i = 0, \ i = 1,2,3...$$

(19)

Where \( k \) is the iteration number, and radial quantum number \( n \) is equal with iteration number \( k \) for this case

3. **Bound State Solution**

In natural unit \((\hbar = c = 1)\) the Dirac equation in case of symmetry spin for half spin particle of rest mass \( M \) is written as

$$-\nabla^2 + 2V(E+M)\psi = (E^2 - M^2)\psi$$

(20)

Where \( E \) represent relativistic energy of half spin particle. In case of spin symmetry vector potential \( V(x) \) assuming equal with scalar potential \( S(x) \). We propose a solvable potential model which is expressed as

$$V(r, \theta_1, \theta_2, ..., \theta_{D-1}) = \frac{\hbar^2}{2m} \left[ V(r) \frac{1}{r^2} \left( \frac{V(\theta_1)}{\sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3 ... \sin^2 \theta_{D-1}} + \frac{V(\theta_2)}{\sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_4 ... \sin^2 \theta_{D-1}} + ... + \frac{V(\theta_{D-1})}{\sin^2 \theta_1 \sin^2 \theta_2 ... \sin^2 \theta_{D-1}} \right) \right]$$

(21)

And we use \( D \) dimensional Laplacian in hyper-spherical coordinates given as

$$\nabla^2 = \frac{1}{r^{D-3}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left[ \frac{1}{\sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3 ... \sin^2 \theta_{D-1}} \left( \frac{\partial^2}{\partial \theta_1^2} \right) + \frac{1}{\sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_4 ... \sin^2 \theta_{D-1}} \left( \frac{\partial}{\partial \theta_2} \right) + ... + \frac{1}{\sin^2 \theta_1 \sin^2 \theta_2 ... \sin^2 \theta_{D-1}} \left( \frac{\partial^2}{\partial \theta_{D-1}^2} \right) \right]$$

(22)
With \( \psi(r, \theta_m) = R(r)Y(\theta_m) \), and \( Y(\theta_m) = P_1(\theta_1)P_2(\theta_2)P_3(\theta_3)P_4(\theta_4) \) by inserting Eq (21) and (22) to Eq (20) become
\[
\left[ -\frac{\nabla^2_r}{r^2} - \frac{\nabla^2_{\theta_m}}{r^2} + \frac{V(\theta)}{r^2}(E + M) \right] RY = \left( E^2 - M^2 \right) RY
\]  
(23)

We separated Eq (23) to one dimensional equation of radial, \( \theta_1, \theta_2, \theta_3, \) and \( \theta_4 \).

3.1 Radial Energy Equation

From the separation of Eq (23) one dimensional radial equation written as
\[
\frac{1}{R} \left[ -r^2 \nabla^2_r R + r^2 V(r)(E + M) - r^2 (E^2 - M^2) \right] = \lambda_4
\]  
(24)

Where \( \lambda_4 \) is constant and \( r \) is distance a half particle with source of potential field. \( \nabla_r \) represent of radial laplacian D dimensional written as
\[
\nabla^2_r = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right)
\]  
(25)

By using \( R(r) = \frac{\chi(r)}{r^{(D-1)/2}} \), \( \coth\left( \frac{ar}{2} \right) = 1 - 2z \) and \( \chi(z) = (z)^\delta (1-z)^\gamma f_n(z) \) and inserting Eq (1) and Eq (25) to Eq (24) we have
\[
z(1-z)f_n''(z) + \left[ 1 + 2\delta - (2\delta + 2\gamma + 2)z \right] f_n'(z) + \left[ (\delta + \gamma)^2 - \delta - \gamma + A_n \right] f_n(z) = 0
\]  
(26)

Where
\[
\delta = \sqrt{E_s - B_s} \quad \quad \gamma = \sqrt{B_s + E_s}
\]  
(27)

\[
A_n = 2 - \lambda_4 \quad \quad B_s = \frac{V_0(E + M)}{2\alpha^2} \quad \quad E_s = \frac{V_0}{2\alpha^2} (E + M) + \frac{(E^2 - M^2)}{\alpha^2}
\]  
(28)

Eq (26) can reduce to AIM type homogeneous linear second order differential equation. \( s_0 \) and \( k_0 \) are obtained from Eq (26), so we can easy to find \( \Delta_n \) form Eq (11) and (19) represent as
\[
(\delta + \gamma + n)(\delta + \gamma + (n+1)) - A_{sn} = \Delta_0 = 0
\]  
(29)

Eq (29) expand mathematically. By inserting Eq (27) and (28) to equation (29) we have
\[
\left[ \sqrt{2 - \lambda_4} + \frac{1}{4} \left( n + \frac{1}{2} \right) \right] + \frac{4 \sqrt{V_0(E + M)}}{2\alpha^2} = 4 \sqrt{V_0(E + M)} + \frac{(E^2 - M^2)}{\alpha^2}
\]  
(30)

3.2 \( \theta_1 \) Equation

The result of separation Eq (22) in variable \( \theta_1 \) written as
\[
\frac{1}{P_1} \left( \frac{d^2 P_1}{d\theta^2} \right) - V(\theta_1)(E + M) = \lambda_1
\]

(31)

Where \( \lambda_1 \) is constant and \( P_1 \) is an equation respect to \( \theta_1 \). Eq (31) can reduce to homogeneous linear second order differential equation by inserting Eq (2) and consider \( \cot \theta_1 = (1 - 2z_1)^\gamma \), \( P_1 = z_1^\delta (1 - z_1)^\gamma f_n(z_1) \), than solve second differential of \( P_1 \) and we have

\[
z_1 (1 - z_1) f''_n + \left[ 1 + 2\delta - z_1 (2\delta + 2\gamma + 2) \right] f'_n - \left[ (\delta + \gamma)^2 + \delta + \gamma + V_1(E + M) \right] f_n = 0
\]

(32)

We can determine \( s_0 \) and \( k_0 \) from Eq (32). With Eq (19) we have

\[
(\delta + \gamma + n)(\delta + \gamma + n + 1) + V_1(E + M) = \Delta_3 = 0
\]

(33)

Can be rewritten as

\[
(\delta + \gamma + n + 1)^2 - (\delta + \gamma + n + 1) + V_1(E + M) = \Delta_n = 0
\]

(34)

Where

\[
\delta = \sqrt{\frac{V_3 i(E + M)}{4} + \lambda_1} \quad \gamma = \sqrt{-\frac{V_3 i(E + M)}{4} + \lambda_1}
\]

(35)

By inserting Eq (35) to Eq (34) we have

\[
\lambda_1 = \frac{\left[ V_3 i(E + M) \right]^2}{4 \left[ -V_3 i(E + M) + \frac{1}{4} \left( n + \frac{1}{2} \right) \right]^2} + \left[ -\frac{V_3 i(E + M)}{4} + \frac{1}{4} \left( n + \frac{1}{2} \right) \right]^2
\]

(36)

3.3 \( \theta_2 \) Equation

Variable \( \theta_2 \) from separation of Eq (23) written as

\[
\frac{\lambda_1}{\sin^2 \theta_2} + \frac{1}{P_2} \left( \frac{1}{\sin \theta_2} \frac{d}{d\theta_2} \frac{\sin \theta_2}{d\theta_2} \right) - V(\theta_2)(E + M) = \lambda_2
\]

(37)

By using \( P_2 = \frac{Q_2(\theta_2)}{\sin^{3/2} \theta_2} \), \( \cot \theta_2 = i(1 - 2z_2) \), and \( Q_2(z_2) = z_2^\alpha (1 - z_2)^\beta f_n(z_2) \) Eq (45) become

\[
z_2 (1 - z_2) f''(z_2) = \left[ 1 + 2\alpha - z_2 (2\alpha + 2\beta) \right] f'(z_2) + \left[ (\alpha + \beta)^2 + \alpha + \beta + \lambda_1 - V'(E + M) + \frac{1}{4} \right] f_n(z_2)
\]

(38)

Where

\[
\alpha = \sqrt{\left[ \frac{1}{4} - \lambda_1 \right]^2 / 4} \quad \beta = \sqrt{\left[ \frac{1}{4} - \lambda_1 \right]^2 / 4}
\]

(39)

We can find \( \Delta_n \) from Eq (11), and Eq (19):

\[
\Delta_n = (\alpha + \beta + n + 1)^2 - (\alpha + \beta + n + 1) + \lambda_1 - V'(E + M) + \frac{1}{4}
\]

(40)
So we have eigenvalue from $\theta_2$ equation

$$\lambda_2 = \frac{1}{4} \left[ \sqrt{-\lambda_1 + V'(E + M)} - \left( n + \frac{1}{2} \right) \right]^2 \quad (41)$$

3.4 $\theta_3$ Equation

Separation variable of Eq (25) for $\theta_3$ is Eq (49)

$$\frac{\lambda_2}{\sin^2 \theta_3} + \frac{1}{P_3} \left( \frac{1}{\sin^2 \theta_3} \frac{d}{d\theta_3} \left( \sin^2 \theta_3 \frac{dP_3}{d\theta_3} \right) \right) - V(\theta_3)(E + M) = \lambda_3$$  

(42)

With the same way, consider $P_3 = \frac{Q_3 \left( \theta_3 \right)}{\sin^{2/2} \theta_3}$, $\cot \theta_3 = i(1 - 2z_3)$, $Q_3 = z_3^{\alpha} (1 - z_3)^{\beta} f_n(z_3)$ we reduce Eq (42) to be homogeneous linear second order differential equation

$$z_3(1 - z_3)f_n'' + [2\alpha + 1 - z_3(2 + 2\alpha + 2\beta)]f_n' - (\alpha + \beta - 4V_1(E + M) + 4\lambda_2)f_n = 0 \quad (43)$$

Using Eq (11) and Eq (19) we have

$$\Delta_n = (\alpha + \beta + n + 1) - (\alpha + \beta + n + 1) - 4V_1(E + M) + 4\lambda_2 \quad (44)$$

Where

$$\alpha = \sqrt{V_2'(E + M) - (\lambda_3 + 1)} \quad \beta = -\sqrt{V_2'(E + M) - (\lambda_3 + 1)} \quad (45)$$

By inserting Eq (45) to Eq (44) we obtain

$$\lambda_3 = -\frac{\left[ V_2'(E + M) \right]^2}{\left[ V_2(E + M) - (\lambda_3 + 1) \right]^2} - \frac{4}{4} \quad (46)$$

3.5 $\theta_4$ Equation

The last equation is for variable $\theta_4$

$$\frac{1}{\sin^3 \theta_4} \frac{d}{d\theta_4} \left( \sin^3 \theta_4 \frac{dP_4}{d\theta_4} \right) - \left( V(\theta_4)(E + M) - \lambda_4 + \frac{\lambda_3}{\sin^2 \theta_4} \right) P_4 = 0 \quad (47)$$

With $P_4 = \frac{Q_4}{\sin^{2/2} \theta_4}$, $\cot \theta_4 = i - 2iz_4$, and $Q_4 = z_4^{\delta} (1 - z_4)^{\gamma} f_n(z_4)$ we have

$$z_4(1 - z_4)f_n''(z_4) + [1 + 2\delta - z_4(2 + 2\delta + 2\gamma)]f_n'(z_4) + \left[ - (\delta + \gamma)^2 - \gamma - \delta + \lambda_3 + \frac{3}{4} \right] f_n(z_4) = 0 \quad (48)$$

Than we find $\Delta_n$

$$(\delta + \gamma + n + 1)^2 - (\delta + \gamma + n + 1) - \lambda_3 - \frac{3}{4} = \Delta_n = 0 \quad (49)$$

Where
\[ \delta = \frac{\sqrt{V_0(E + M) + \left(\frac{9}{4} + \lambda_4\right)}}{4} \quad \gamma = \frac{-V_0(E + M) + \left(\frac{9}{4} + \lambda_4\right)}{4} \] (50)

By inserting Eq (50) to Eq (49) the eigenvalue represent as

\[ \left[\sqrt{\lambda_4 + 1 - \left(\frac{n + \frac{1}{2}}{2}\right)}\right]^2 + \frac{1}{4}V_0(E + M)^2 = \frac{9}{4} = \lambda_4 \] (51)

### 4. Result and Conclusion

By inserting Eq (36) to Eq (41), than Eq (41) to Eq (46), Eq (46) to (51), and Eq (51) to Eq (30), we have as shown in Table 1

| V1 | V2 | Vak | Vak1 | Vvak | V3 | N0 | N1 | N2 | N3 | N4 | alpha | M | Energy |
|----|----|-----|-----|------|----|----|----|----|----|----|-------|---|--------|
| 1  | 0  | 0   | 0   | 0    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.70554 |
| 2  | 0  | 0   | 0   | 0    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.71991 |
| 3  | 0  | 0   | 0   | 0    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.73098 |
| 4  | 0  | 1   | 0   | 0    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.68491 |
| 5  | 0  | 2   | 0   | 0    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.68838 |
| 6  | 0  | 3   | 0   | 0    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.69407 |
| 7  | 0  | 0   | 1   | 0    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.84598 |
| 8  | 0  | 0   | 2   | 0    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.85014 |
| 9  | 0  | 0   | 0   | 3    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.85383 |
| 10 | 0  | 0   | 0   | 4    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.90994 |
| 11 | 0  | 0   | 0   | 5    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.91249 |
| 12 | 0  | 0   | 0   | 3    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.91474 |
| 13 | 0  | 0   | 0   | 4    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.97549 |
| 14 | 0  | 0   | 0   | 5    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.97599 |
| 15 | 0  | 0   | 0   | 0    | 0  | 1  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.9764 |
| 16 | 0  | 0   | 0   | 0    | 0  | 2  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.99404 |
| 17 | 0  | 0   | 0   | 0    | 0  | 0  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.9943 |
| 18 | 0  | 0   | 0   | 0    | 0  | 0  | 1  | 1  | 1  | 1  | 0.5   | 5 | -4.99454 |

The amount of energy must be not too far from \( M \). Five-dimensional with AIM can use to solve eigenvalue of Dirac equation. Nevertheless five-dimensional equation is interesting for the next study.

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