Application of a subordination theorem associated with certain new generalized subclasses of analytic and univalent functions

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Abstract
The prime focus of the present work is to investigate some fascinating relations of some analytic and univalent functions using a subordination theorem.

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Introduction
Let $H$ denote the class of normalized analytic functions $f(z)$ having the form:

$$f(z) = z + a_2z^2 + a_3z^3 + ...$$

(1)

in the unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Also, let $S$ denote the subclass of $H$ univalent in $U$. Suppose that $S^*$ denote the subclass of $S$ consisting of the functions $f(z)$ which are starlike in $U$. A function $f(z) \in K$ is said to be convex in $U$ if $f(z) \in S$ satisfies the condition that $zf'(z) \in S^*$. If $f(z) \in H$ satisfies the geometric condition:

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \beta, \quad z \in U$$

for some real $\beta (0 \leq \beta < 1)$, then we say that $f(z)$ belongs to the class $S^*(\beta)$ starlike of order $\beta$, and if $f(z) \in H$ satisfies the geometric condition:

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta, \quad z \in U$$

for some real $\beta (0 \leq \beta < 1)$, then we say that $f(z)$ belongs to the class $K(\beta)$ convex of order $\beta$ (see [1, 2]). Let the function $g(z)$ of the form:

$$g(z) = z + z^3 + z^5 + ... \quad z \in U$$

(2)

be in the class $S^*$ while the function $g(z)$ of the form:

$$g(z) = z + z^2 + z^3 + ... \quad z \in U$$

(3)
be in the class $K$. With reference to (2) and (3), we can write that:

$$g_α(z) = \frac{z}{1 - z^α} = z + \sum_{k=1}^{∞} z^{1+kα} \quad z \in U,$$

(4)

where we consider the principal value of $z^{kα}$ for some real $α (0 < α ≤ 2)$. See Darus and Owa [3] for some properties of functions $f_α(z)$ of the form (4).

Here, we present a more generalized form of (4) such that:

$$g_{α,n}(z) = \frac{A^n z}{(A + B z^α)^n} = z + \sum_{k=1}^{∞} (-1)^k \frac{B^k}{A^k} n_k z^{1+kα} \quad z \in U,$$

(5)

for some real $α (0 < α ≤ 2), -1 < B < A ≤ 1, n ≥ 0$ and $n_k$ is given by $n_k = \prod_{j=1}^{k} \left( \frac{n+j-1}{j} \right)$.

In view of (1) and (5), we introduce a class $H_{α,n}$ of analytic function $f_{α,n}(z)$ which is a convolution (or Hadamard product) of $f(z)$ and $g_{α,n}(f(z) * g_{α,n}(z))$ such that:

$$f_{α,n}(z) = z + \sum_{k=1}^{∞} (-1)^k \frac{B^k}{A^k} n_k a_{k+1} z^{1+kα} \quad z \in U,$$

(6)

In addition, if $f_{α,n}(z) \in H_{α,n}$ satisfies the following condition:

$$\Re \left( \frac{zf'_{α,n}(z)}{f_{α,n}(z)} \right) > γ \quad z \in U,$$

(7)

for some real $α (0 < α ≤ 2), n > 0$, and $γ (0 ≤ γ < 1)$, then $f_{α,n}$ belong to the starlike class $S^*_n(A, B, γ)$ (of order $γ$). Also, if $f_{α,n}(z) \in H_{α,n}$ satisfies the following condition:

$$\Re \left( 1 + \frac{zf''_{α,n}(z)}{f_{α,n}(z)} \right) > γ \quad z \in U,$$

(8)

for some real $α (0 < α ≤ 2), n > 0$, and $γ (0 ≤ γ < 1)$, then $f_{α,n}$ belong to the convex class $K^*_n(A, B, γ)$ (of order $γ$). Here, it is noted that $f_{α,n}(z) \in H_{α,n}(z)$ belong to the convex class $K^*_n(A, B, γ) ⇔ zf'_{α,n}(z)$ belong to the starlike class $S^*_n(A, B, γ)$.

For the purpose of the present investigation, we shall call to mind the following definitions and lemmas.

**Definition 1** (Subordination principle) For two functions $f$ and $g$ analytic in $U$, we say that $f$ is subordinate to $g$, and write $f \prec g$ in $U$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1 \quad (z \in U)$, such that $f(z) = g(w(z)).$ It is known that:

$$f(z) \prec g(z) \quad ⇒ f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Furthermore, if the function $g$ is univalent in $U$:

$$f(z) \prec g(z) \quad ⇔ f(0) = g(0) \text{ and } f(U) \subset g(U).$$

(9)

Also, we say that $g(z)$ is superordinate to $f(z)$ in $U$ (see [4–6]).

**Definition 2** (Subordinating factor sequence) A sequence $\{b_k\}_{k=1}^{∞}$ of complex numbers is called subordinating factor sequence if for every univalent function $f(z)$ in $K$, we have the subordination given by:

$$\sum_{k=1}^{∞} a_k b_k z^k \prec f(z) \quad (z \in U, \ a_1 = 1) \text{ (see [4–6])}.$$
Lemma 1 The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if:

$$\Re\left\{1 + 2 \sum_{k=1}^{\infty} b_k z^k\right\} > 0 \quad (z \in U).$$

(11)

The lemma above is due to Wilf [7]. Interested reader can also refer to [4–6].

Lemma 2 Let $s(z)$ ($s(z) \neq 0$) be a univalent function in $U$. Also, let $\mu \neq 0$ be a complex number, then we have that:

$$\Re\left\{1 + z s'(z) s(z) - z s'(z) s(z)\right\} > \max\left\{0, \Re\left(\frac{\mu - 1}{\mu - s(z)}\right)\right\}.\quad (12)

Suppose that $r (r(z) \neq 0)$ satisfies the differential equation:

$$(1 - \mu) (r(z) - 1) + \mu \frac{z r'(z)}{r(z)} < (1 - \mu) (s(z) - 1) + \mu \frac{z s'(z)}{s(z)}, \quad z \in U

(13)

then $r < s$ and $s$ is the best dominant (see [8] among others).

Lemma 3 Let $\omega$ be regular in $H$ with $\omega(0) = 0$. Also, suppose that $|\omega(z)|$ attains its maximum value on the circle $|z| < 1$ at a point $z_0$, then:

$$z_0 \omega'(z_0) = \sigma \omega(z_0),

(14)

where $\sigma$ is any real number and $\sigma \geq 1$ (see [8] among others).

Coefficient inequality

In this section, we consider the coefficient inequalities for function $f_{\alpha,n}(z)$ given by (6) belonging to both classes $S^\alpha_{\alpha,n} (A, B, \gamma)$ and $K_{\alpha,n} (A, B, \gamma)$ in the unit disk $U$.

Theorem 1 Let the function $f_{\alpha,n}(z)$ of the form (6) satisfy the inequality:

$$\sum_{k=1}^{\infty} (ka - \gamma + 1) n_k \frac{|B|^k}{A^k} |a_{k+1}| \leq 1 - \gamma.\quad (15)

Then, $f_{\alpha,n}(z) \in S^\alpha_{\alpha,n} (A, B, \gamma)$ for $0 \leq \gamma < 1$, $0 < \alpha \leq 2$, $-1 \leq B < A \leq 1$, $0 < A \leq 1$ and $n > 0$. The equality holds true for $f_{\alpha,n}(z)$ given by:

$$f_{\alpha,n}(z) = z + \frac{(1 - \gamma) e^{it}}{(ka - \gamma + 1) n_k \frac{|B|^k}{A^k}} z^{k+1} + A (k \geq 1).

\text{Proof} Suppose that the function $f_{\alpha,n}(z)$ given by (6) satisfies (15), then:

$$\left|\frac{zf_{\alpha,n}'(z)}{f_{\alpha,n}(z)} - 1\right| = \frac{\sum_{k=1}^{\infty} (-1)^k kan_k \frac{|B|^k}{A^k} |a_{k+1}| z^k}{1 + \sum_{k=1}^{\infty} (-1)^k n_k \frac{|B|^k}{A^k} |a_{k+1}| z^k}

\leq \frac{\sum_{k=1}^{\infty} kan_k |B|^k |a_{k+1}| |z|^k}{1 - \sum_{k=1}^{\infty} n_k |B|^k |a_{k+1}| |z|^k}

\leq \frac{\sum_{k=1}^{\infty} kan_k |B|^k |a_{k+1}| |z|^k}{1 - \sum_{k=1}^{\infty} n_k |B|^k |a_{k+1}| |z|^k} \leq 1 - \gamma.

This shows that $f_{\alpha,n}(z) \in S^\alpha_{\alpha,n} (A, B, \gamma)$, and this ends the proof.\qed
Corollary 1 Let the function \( f_{a,n}(z) \) of the form (6) satisfy the inequality:
\[
\sum_{k=1}^{\infty} (k\alpha + 1)n_k \frac{|B|^k}{A^k} |a_{k+1}| \leq 1.
\]
Then, \( f_{a,n}(z) \in S_{a,n}^+(A, B, 0) \).

Theorem 2 Let the function \( f_{a,n}(z) \) of the form (6) satisfy the inequality:
\[
\sum_{k=1}^{\infty} (k\alpha - \gamma + 1)n_k \frac{|B|^k}{A^k} |a_{k+1}| \leq 1 - \gamma.
\]
Then, \( f_{a,n}(z) \in K_{a,n}(A, B, \gamma) \) for \( 0 \leq \gamma < 1 \), \( 0 < \alpha \leq 2 \), \(-1 \leq B < A \leq 1 \), \( 0 < A \leq 1 \) and \( n > 0 \). The equality holds true for \( f_{a,n}(z) \) given by:
\[
f_{a,n}(z) = z + \frac{(1 - \gamma) e^{i\pi}}{(k\alpha + 1) (k\alpha - \gamma + 1) n_k \frac{|B|^k}{A^k}} z^{1+k\alpha} \quad (k \geq 1).
\]

Proof The proof is similar to that of Theorem 1.

Corollary 2 Let the function \( f_{a,n}(z) \) of the form (6) satisfy the inequality:
\[
\sum_{k=1}^{\infty} (k\alpha + 1)^2 n_k \frac{|B|^k}{A^k} |a_{k+1}| \leq 1.
\]
Then, \( f_{a,n}(z) \in K_{a,n}(A, B, 0) \).

Remark 1 Putting \( A = n = 1 \) and \( B = -1 \) in Theorems 1 and 2, we obtain the results obtained by Darus and Owa [[3], Theorems 3 and 4].

Next, we present some subordination results.

Some subordination results
Our prime objective here is to establish sufficient conditions for functions belonging to the analytic class \( S_{a,n}^+(A, B, \gamma) \).

Theorem 3 Suppose that the function \( f_{a,n}(z) \) is as defined in (6). Let \( 0 < \alpha \leq 2 \), \( n > 0 \), \( \sigma \neq -1 \) and \( \mu \) be a non-zero complex number in \( U \) such that:
\[
\Re \left\{ 1 + \frac{z[1 - \sigma(1 - 2\alpha)]}{(1 - z)(1 + \sigma z)} \right\} \geq \max \left\{ 0, \Re \left( \frac{\mu - 1}{\mu} \left( \frac{1 + \sigma z}{1 - z} \right) \right) \right\}.
\]
If
\[
(1 - \mu) \left( f_{a,n}'(z) - 1 \right) + \mu \left( \frac{zf_{a,n}''(z)}{f_{a,n}'(z)} \right) < (1 - \mu) \left( \frac{1 + \sigma z}{1 - z} \right) - 1 + \mu \left( \frac{(1 + \sigma z)}{(1 + \sigma z)(1 - z)} \right)
\]
holds true, then \( f_{a,n}(z) \in S_{a,n}^+(A, B, \gamma) \).

Proof Suppose that we let:
\[
r(z) = f_{a,n}'(z) \quad \text{and} \quad s(z) = \frac{1 + \sigma z}{1 - z}.
\]
(17)
Then,
\[
\Re \left\{ 1 + \frac{zs''(z)}{s'(z)} - \frac{zs'(z)}{s(z)} \right\} \geq \max \left\{ 0, \Re \left( \frac{\mu - 1}{\mu} \left( \frac{1 + \sigma z}{1 - z} \right) \right) \right\} = \max \left\{ 0, \Re \left( \frac{\mu - 1}{\mu} s(z) \right) \right\}
\]
and
\[
(1 - \mu)(r(z) - 1) + \mu \frac{zr'(z)}{r(z)} = (1 - \mu) \left( f'_{\alpha,n}(z) - 1 \right) + \mu \left( \frac{zf''_{\alpha,n}(z)}{f'_{\alpha,n}(z)} \right)
\]
\[
< (1 - \mu) \left( \frac{1 + \sigma z}{1 - z} - 1 \right) + \mu \left( \frac{(1 + \sigma)z}{(1 + \sigma)(1 - z)} \right) = (1 - \mu) \left( s(z) - 1 \right) + \mu \frac{zs'(z)}{s(z)}.
\]  
\tag{18}

Using Lemma 2 in (18), then we obtain the desired result.

\[\square\]

**Theorem 4** Let the analytic function \( f_{\alpha,n}(z) \) be defined as in (6). Suppose that \( f_{\alpha,n}(z) \) satisfies the condition that:
\[
\Re \left\{ \frac{zf''_{\alpha,n}(z)}{f'_{\alpha,n}(z)} \right\} < -\frac{1 + \sigma}{2(1 - \sigma)}, \quad \sigma \neq -1.
\]  
\tag{19}

Then, for \( 0 < \alpha \leq 2, \ n > 0 \) and \( \sigma > 1, \ f_{\alpha,n}(z) \in S^*_\alpha,n(A,B,\gamma). \)

**Proof** Setting:
\[
f'_{\alpha,n}(z) = \frac{1 + \sigma \omega(z)}{1 - \omega(z)}, \quad \omega(z) \neq 1.
\]

Then, \( \omega \) is regular in \( U \), and since \( \sigma \neq -1 \), then \( \omega(0) = 0 \). Also, it follows that:
\[
\Re \left\{ \frac{zf''_{\alpha,n}(z)}{f'_{\alpha,n}(z)} \right\} = \Re \left\{ \frac{(1 + \sigma)\omega'(z)}{(1 - \omega(z))(1 + \sigma \omega(z))} \right\} < \frac{\sigma + 1}{2(\sigma - 1)}, \quad \sigma \neq -1.
\]

Next, we show that \( |\omega(z)| < 1 \). So, let there exists a point \( z_0 \in U \) such that for \( |z| \leq |z_0|: \)
\[
\max |\omega(z)| = |\omega(z)| = 1.
\]

Then, appealing to Lemma 3 and setting \( \omega(z_0) = e^{i\theta}, \ z_0 \omega'(z_0) = \delta e^{i\theta} \) and for \( \delta \geq 1, \ \sigma > 1 \), we have that:
\[
\Re \left\{ \frac{zf''_{\alpha,n}(z)}{f'_{\alpha,n}(z)} \right\} \geq -\frac{1 + \sigma}{2(1 - \sigma)} \quad z \in U.
\]

which negates the hypothesis (19).

Hence, we conclude that \( |\omega(z)| < 1 \) for all \( z \in U \) and:
\[
f'_{\alpha,n}(z) < \frac{1 + \sigma z}{1 - z}, \quad \sigma \neq 1, \ z \in U.
\]

and this obviously ends the proof.

\[\square\]

**Application of a subordination theorem**

Let \( \mathcal{S}^*_\alpha,n(A,B,\gamma) \) and \( \mathcal{K}^*_\alpha,n(A,B,\gamma) \) denote the classes of functions \( f_{\alpha,n} \in H_{\alpha,n} \) whose coefficients satisfy conditions (15) and (16), respectively. We note that \( \mathcal{S}^*_\alpha,n(A,B,\gamma) \subseteq \mathcal{S}^*_\alpha,n(A,B,\gamma) \) and \( \mathcal{K}^*_\alpha,n(A,B,\gamma) \subseteq \mathcal{K}^*_\alpha,n(A,B,\gamma) \). Here, we consider an application of the subordination result given in Lemma 1 to both classes \( \mathcal{S}^*_\alpha,n(A,B,\gamma) \) and \( \mathcal{K}^*_\alpha,n(A,B,\gamma) \).
Theorem 5 Let $f_{α,n}(z) \in S^{\gamma}_{α,n}(A,B,\gamma)$. If $0 \leq \gamma < 1$, $0 < α \leq 2$, $-1 \leq B < A \leq 1$, $0 < A \leq 1$ and $n > 0$, then:

$$ n(α - γ + 1)|B| \quad (f_{α,n} * g_α) (z) < g_α(z) $$

(20)

for every function $g_α$ in $K_α$ and:

$$ \forall (f_{α,n}(z)) > -\frac{[nα|B| + (1 - γ)(A + n|B|)]}{n(α - γ + 1)|B|} $$

(21)

The constant factor:

$$ n(α - γ + 1)|B| \quad \frac{2}{[nα|B| + (1 - γ)(A + n|B|)]} $$

in the subordination result (20) is sharp.

Proof Let $f_{α,n} \in S^{\gamma}_{α,n}(A,B,\gamma)$ and let $g_α$ be any function in $K_α$. Then:

$$ n(α - γ + 1)|B| \quad (f_{α,n} * g_α) (z) < g_α(z) $$

(20)

Thus, by Definition 2, the subordination result (20) will hold true if:

$$ \forall \left\{ \frac{n(α - γ + 1)|B|}{2[nα|B| + (1 - γ)(A + n|B|)]} a_k \right\}^∞_{k=1} $$

is a subordinating factor sequence, with $a_1 = 1$, appealing to Lemma 1, this is equivalent to:

$$ \forall \left\{ 1 + \sum_{k=1}^∞ \frac{n(α - γ + 1)|B|}{[nα|B| + (1 - γ)(A + n|B|)]} a_k z^{α(k-1)α+1} \right\} > 0 \ (z \in U). $$

(22)

Since $n_k (kα - γ + 1) \frac{|B|^k}{A^k}$ is an increasing function of $k \ (k \geq 1)$, we have that:

$$ \forall \left\{ 1 + \sum_{k=1}^∞ \frac{n(α - γ + 1)|B|}{[nα|B| + (1 - γ)(A + n|B|)]} a_k z^{α(k-1)α+1} \right\} $$

$$ = \forall \left\{ 1 + \frac{n(α - γ + 1)|B|}{M} z + \frac{A}{M} \sum_{k=2}^∞ n(α - γ + 1) \frac{|B|}{A} a_k z^{α(k-1)α+1} \right\} $$

$$ \geq 1 - \frac{n(α - γ + 1)|B|}{M} r - \frac{A}{M} \sum_{k=2}^∞ n_k - 1 (k - 1)α - γ + 1 \frac{|B|}{A^k} a_k r^{(k-1)α+1} $$

$$ > 1 - \frac{n(α - γ + 1)|B|}{[nα|B| + (1 - γ)(A + n|B|)]} r - \frac{A}{[nα|B| + (1 - γ)(A + n|B|)]} r = 1 - r > 0 $$

(23)

where $M = [nα|B| + (1 - γ)(A + n|B|)]$.

Therefore, (22) holds true in $U$ and this obviously proves the inequality (20) while (21) follows by taking:

$$ g_α (z) = \frac{z}{1 - z^α} \in K_α $$
in (20). Now, suppose that we consider the function $q_{\alpha,n}(z)$ of the form:

$$q_{\alpha,n}(z) = z - \frac{1 - \gamma}{n(\alpha - \gamma + 1)\frac{|B|}{A}}z^{\alpha + 1}$$

which belongs to the class $S^*_\alpha(A,B,\gamma)$. Then, using (20), we have that:

$$\frac{n(\alpha - \gamma + 1)|B|}{2[n\alpha|B| + (1 - \gamma)(A + n|B|)]} q_{\alpha,n}(z) \preceq \frac{z}{1 - z^\alpha} \quad (z \in U)$$

which can easily be verified that for $0 \leq \gamma < 1$, $0 < \alpha \leq 2$, $-1 \leq B < A \leq 1$, $0 < A \leq 1$, $n \geq 0$ and $|z| \leq r$:

$$\min \left\{ \Re \left( \frac{n(\alpha - \gamma + 1)|B|}{2[n\alpha|B| + (1 - \gamma)(A + n|B|)]} q_{\alpha,n}(z) \right) \right\} = -\frac{1}{2} \quad (z \in U)$$

and this evidently completes the proof of Theorem 5. For various choices of the parameters involved, several interesting results are obtained. Given below are few instances. □

**Corollary 3** Let $f_{\alpha,n}(z) \in S^*_\alpha(1,1,\gamma)$. Then:

$$\frac{n(\alpha - \gamma + 1)}{2(\alpha - 2\gamma + 2)} \left( f_{\alpha,n} * g_{\alpha} \right)(z) < g_{\alpha}(z)$$

for every function $g_{\alpha}$ in $K_{\alpha}$ and:

$$\Re \left( f_{\alpha,n}(z) \right) > -\frac{(\alpha - 2\gamma + 2)}{(\alpha - \gamma + 1)}.$$

The constant factor:

$$\frac{n(\alpha - \gamma + 1)}{2(\alpha - 2\gamma + 2)}$$

is sharp.

**Corollary 4** Let $f_{\alpha,1}(z) \in S^*_\alpha(1,1,\gamma)$. Then:

$$\frac{\alpha - \gamma + 1}{2(\alpha - 2\gamma)} \left( f_{\alpha,1} * g_{\alpha} \right)(z) < g_{\alpha}(z)$$

for every function $g_{\alpha}$ in $K_{\alpha}$ and:

$$\Re \left( f_{\alpha,1}(z) \right) > -\frac{(\alpha - 2\gamma)}{(\alpha - \gamma + 1)}.$$

The constant factor:

$$\frac{\alpha - \gamma + 1}{2(\alpha - 2\gamma)}$$

is sharp.

**Corollary 5** [9, 10] Let $f_{1,1}(z) \in S^*_1(1,1,\gamma)$. Then:

$$\frac{2 - \gamma}{2(3 - 2\gamma)} \left( f_{1,1} * g_{1} \right)(z) < g_{1}(z)$$

for every function $g_{1}$ in $K_{1}$ and:

$$\Re \left( f_{1,1}(z) \right) > -\frac{(3 - 2\gamma)}{(2 - \gamma)}.$$

The constant factor:

$$\frac{2 - \gamma}{2(3 - 2\gamma)}$$
is sharp.

Corollary 6 [9–11] Let \( f_{1,1}(z) \in S_{1,1}^{\alpha}(1,-1,0) \). Then:
\[
\frac{1}{3} (f_{1,1} * g_1)(z) < g_1(z)
\]
for every function \( g_1 \) in \( K_1 \) and:
\[
\Re (f_{1,1}(z)) > -\frac{3}{2}.
\]

Theorem 6 Let \( f_{\alpha,n}(z) \in K_{\alpha,n}(A,B,\gamma) \). If \( 0 \leq \gamma < 1, 0 < \alpha \leq 2, -1 \leq B < A \leq 1 \) and \( n > 0 \), then:
\[
\frac{n(\alpha + 1)(\alpha - \gamma + 1)|B|}{2 \left[ n\alpha(\alpha + 1)|B| + (1 - \gamma)(A + n(\alpha + 1)|B|) \right]} (f_{\alpha,n} * g_\alpha)(z) < g_\alpha(z)
\]
for every function \( g_\alpha \) in \( K_\alpha \) and:
\[
\Re (f_{\alpha,n}(z)) > -\frac{n\alpha(\alpha + 1)|B| + (1 - \gamma)(A + n(\alpha + 1)|B|)}{n(\alpha + 1)(\alpha - \gamma + 1)|B|}.
\]

The constant factor:
\[
\frac{n(\alpha + 1)(\alpha - \gamma + 1)|B|}{2 \left[ n\alpha(\alpha + 1)|B| + (1 - \gamma)(A + n(\alpha + 1)|B|) \right]}
\]
in the subordination result (24) cannot be replaced by a larger one, and the proof of which is similar to that of Theorem 3.

Corollary 7 Let \( f_{\alpha,n}(z) \in K_{\alpha,n}(1,-1,\gamma) \). Then:
\[
\frac{n(\alpha + 1)(\alpha - \gamma + 1)}{2 \left[ n\alpha(\alpha + 1) + (1 - \gamma)(1 + n(\alpha + 1)) \right]} (f_{\alpha,n} * g_\alpha)(z) < g_\alpha(z)
\]
for every function \( g_\alpha \) in \( K_\alpha \) and:
\[
\Re (f_{\alpha,n}(z)) > -\frac{n\alpha(\alpha + 1) + (1 - \gamma)(1 + n(\alpha + 1))}{n(\alpha + 1)(\alpha - \gamma + 1)}.
\]

The constant factor:
\[
\frac{n(\alpha + 1)(\alpha - \gamma + 1)}{2 \left[ n\alpha(\alpha + 1) + (1 - \gamma)(1 + n(\alpha + 1)) \right]}
\]
cannot be replaced by a larger one.

Corollary 8 Let \( f_{\alpha,1}(z) \in K_{\alpha,1}(1,-1,\gamma) \). Then:
\[
\frac{(\alpha + 1)(\alpha - \gamma + 1)}{2 \left[ \alpha(\alpha + 1) + (1 - \gamma)(\alpha + 2) \right]} (f_{\alpha,1} * g_\alpha)(z) < g_\alpha(z)
\]
for every function \( g_\alpha \) in \( K_\alpha \) and:
\[
\Re (f_{\alpha,1}(z)) > -\frac{\alpha(\alpha + 1) + (1 - \gamma)(\alpha + 2)}{\alpha + 1)(\alpha - \gamma + 1)}.
\]

The constant factor:
\[
\frac{(\alpha + 1)(\alpha - \gamma + 1)}{2 \left[ \alpha(\alpha + 1) + (1 - \gamma)(\alpha + 2) \right]}
\]
cannot be replaced by a larger one.
**Corollary 9** [9, 10] Let \( f_{1,1}(z) \in \mathcal{K}_{1,1}(1, -1, \gamma) \). Then:

\[
\frac{2 - \gamma}{5 - 3\gamma} (f_{1,1} * g_1)(z) < g_1(z)
\]

(30)

for every function \( g_1 \) in \( K_1 \) and:

\[
\Re (f_{1,1}(z)) > -\frac{5 - 3\gamma}{2(2 - \gamma)}.
\]

(31)

The constant factor:

\[
\frac{2 - \gamma}{5 - 3\gamma}
\]

cannot be replaced by a larger one.

**Corollary 10** [9, 10] Let \( f_{1,1}(z) \in \mathcal{K}_{1,1}(1, -1, 0) \). Then:

\[
\frac{2}{5} (f_{1,1} * g_1)(z) < g_1(z)
\]

(32)

for every function \( g_1 \) in \( K_1 \) and:

\[
\Re (f_{1,1}(z)) > -\frac{5}{4}.
\]

(33)

The constant factor:

\[
\frac{2}{5}
\]

cannot be replaced by a larger one.

For further illustrations on the applications of the subordination result stated in Lemma 1, interested reader can see [4, 6, 8–11].

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