METRIZATION OF WEIGHTED GRAPHS

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Abstract. We find a set of necessary and sufficient conditions under which the weight $w : E \to \mathbb{R}^+$ on the graph $G = (V, E)$ can be extended to a pseudometric $d : V \times V \to \mathbb{R}^+$. If these conditions hold and $G$ is a connected graph, then the set $\mathcal{M}_w$ of all such extensions is nonvoid and the shortest-path pseudometric $d_w$ is the greatest element of $\mathcal{M}_w$ with respect to the partial ordering $d_1 \leq d_2$ if and only if $d_1(u, v) \leq d_2(u, v)$ for all $u, v \in V$. It is shown that every nonvoid poset $(\mathcal{M}_w, \leq)$ contains the least element $\rho_{0,w}$ if and only if $G$ is a complete $k$-partite graph with $k \geq 2$ and in this case the explicit formula for computation of $\rho_{0,w}$ is obtained.

Key words: Weighted graph, Metric space, Embedding of graph, Shortest-path metric, Infinite graph, Complete $k$-partite graph.

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1. Introduction

Recall the basic definitions that we adopt here. A graph $G$ is an ordered pair $(V, E)$ consisting of a set $V = V(G)$ of vertices and a set $E = E(G)$ of edges. In this paper we study the simple graphs which are finite, $\text{card}(V) < \infty$, or infinite, $\text{card}(V) = \infty$. Since our graph $G$ is simple we can identify $E(G)$ with a set of two-element subsets of $V(G)$, so that each edge is an unordered pair of distinct vertices. As usual we suppose that $V(G) \cap E(G) = \emptyset$. The edge $e = \{u, v\}$ is said to join $u$ and $v$, and the vertices $u$ and $v$ are called adjacent in $G$. The graph $G$ is empty if no two vertices are adjacent, i.e., if $E(G) = \emptyset$. We use the standard definitions of the path, the cycle, the subgraph and supergraph, see, for example, [1, p. 4, p. 40]. Note only that all paths and cycles are finite and simple graphs.

The following, basic for us, notion is a weighted graph $(G, w)$, i.e., a graph $G = (V, E)$ together with a weight $w : E \to \mathbb{R}^+$ where $\mathbb{R}^+ = [0, \infty)$. If $(G, w)$ is a weighted graph, then for each subgraph $F$ of the graph $G$ define

$$w(F) := \sum_{e \in E(F)} w(e).$$

The last sum may be equal $+\infty$ if $F$ is infinite.
Recall also that a pseudometric $d$ on the set $X$ is a function $d : X \times X \to \mathbb{R}^+$ such that $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. The pseudometric $d$ on $X$ is a metric if, in addition, 

$$d(x, y) = 0 \Rightarrow (x = y)$$

for all $x, y \in X$. Using a pseudometric $d$ on the set $V$ of vertices of the graph $G = (V, E)$ one can simply define a weight $w : E \to \mathbb{R}^+$ by the rule

$$(1.2) \quad w(\{u, v\}) := d(u, v)$$

for all edges $\{u, v\} \in E(G)$. The correctness of this definition follows from the symmetry of $d$.

A legitimate question to raise in this point is whether there exists a pseudometric $d$ such that the given weight $w : E \to \mathbb{R}^+$ is produced as in (1.2). If yes, then we say that $w$ is a metrizable weight.

Figure 1. Here $(Q, w)$ is a weighted quadrilateral with $V(Q) = \{v_1, v_2, v_3, v_4\}$, $E(Q) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}\}$ and $w(\{v_1, v_2\}) = a$, $w(\{v_2, v_3\}) = b$, $w(\{v_3, v_4\}) = c$, $w(\{v_4, v_1\}) = k$.

The above formulated question seems to be converse for the question of embeddings of metrics into weighted graphs. (In the standard terminology one says about the realization of metric spaces by graphs.) This topic is rich and has many applications in various areas, such as psychology, phylogenetic analysis and recent applications from the field of computer science. Some results and references in this direction can be found in [2] and [6].
If \((G, w)\) is a weighted graph with metrizable \(w\), then we shall denote by \(\mathcal{M}_w\) the set of all pseudometrics \(d : V \times V \to \mathbb{R}^+\) such that
\[
d(v_i, v_j) = w(\{v_i, v_j\})
\]
for all \(\{v_i, v_j\} \in E(G)\).

The starting point of our considerations is the following Model Example.

**Theorem 1.3 (Model Example).** Let \((Q, w)\) be a weighted graph depicted by Figure 1. The weight \(w\) on the graph \(Q\) is metrizable if and only if
\[
2 \max\{a, b, c, k\} \leq a + b + c + k.
\]

If \(w\) is metrizable, then for each \(d \in \mathcal{M}_w\) we have the double inequalities
\[
\max\{|b - c|, |a - k|\} \leq d(v_2, v_4) \leq \min\{b + c, a + k\}
\]
and
\[
\max\{|a - b|, |c - k|\} \leq d(v_1, v_3) \leq \min\{a + b, c + k\}.
\]

Conversely, if \(p\) and \(q\) are real numbers such that
\[
\max\{|b - c|, |a - k|\} \leq p \leq \min\{b + c, a + k\}
\]
and
\[
\max\{|a - b|, |c - k|\} \leq q \leq \min\{a + b, c + k\},
\]
then there is \(d \in \mathcal{M}_w\) with
\[
d(v_2, v_4) = p, \quad d(v_1, v_3) = q.
\]

This theorem was proved in [4] and used there as a base to finding of extremally Ptolemeic and extremally non-Ptolemeic metric spaces. The results of the present paper generalize the Model Example to the case of arbitrary (finite or infinite) weighted graphs \((G, w)\).

- Theorem 2.2 gives necessary and sufficient conditions under which a weight \(w\) is metrizable. The key point here is an extension of inequality (1.4) to an arbitrary cycle \(C \subseteq G\).
- Proposition 3.3 claims that for connected \(G\) and metrizable \(w\) the shortest-path pseudometric \(d_w\) belongs to \(\mathcal{M}_w\) and that this pseudometric is the greatest element of \(\mathcal{M}_w\). The reader can observe that the right-side in double inequalities (1.5) and (1.6) are, in fact, \(d_w(v_2, v_4)\) and \(d_w(v_1, v_3)\).
- Theorem 4.3 shows that the least pseudometric in \(\mathcal{M}_w\), (see the left-side in (1.5), (1.6)) exists for each metrizable \(w\) if and only if \(G\) is a complete \(k\)-partite graph with \(k \geq 2\).
- In Theorem 4.36 we show that for complete \(k\)-partite graphs \(G\) with \(k \geq 2\) and with the cardinality of partitions \(\leq 2\) we have the analog of the last part of the Model Example: a symmetric function \(f : V \times V \to \mathbb{R}^+\) belongs to \(\mathcal{M}_w\) if and only if it "lies between" the greatest element of \(\mathcal{M}_w\) and the least one.
Moreover in Theorem 3.11 we describe the structure of connected graphs $G$ which admit strictly positive metrizable weights $w$ such that $\mathcal{M}_w$ does not contain any metrics.

2. Embeddings of weighted graphs into pseudometric spaces

Let $(G, w)$ be a weighted graph and let $u, v$ be vertices belonging to a connected component of $G$. Let us denote $\mathcal{P}_{u,v} = \mathcal{P}_{u,v}(G)$ the set of all paths joining $u$ and $v$ in $G$. Write

\begin{equation}
  d_w(u, v) := \inf \{ w(F) : F \in \mathcal{P}_{u,v} \}
\end{equation}

where $w(F)$ is the weight of the path $F$, see formula (1.1). It is well known for the connected graph $G$ that the function $d_w$ is a pseudometric on the set $V(G)$. This pseudometric will be termed as the weighted shortest-path pseudometric. It coincides with the usual path metric if $w(e) = 1$ for all $e \in E(G)$.

**Theorem 2.2.** Let $(G, w)$ be a weighted graph. The following statements are equivalent.

(i) The weight $w$ is metrizable.

(ii) The equality

\begin{equation}
  w(\{u, v\}) = d_w(u, v)
\end{equation}

holds for all $\{u, v\} \in E(G)$.

(iii) For every cycle $C \subseteq G$ we have the inequality

\begin{equation}
  2 \max_{e \in E(C)} w(e) \leq w(C).
\end{equation}

It seems to be interesting to have conditions under which the set $\mathcal{M}_w$ contains some metrics of a special type. In particular: What are restrictions on the weight $w$ guaranteeing the existence of ultrametrics (or pseudoultrametrics) in the set $\mathcal{M}_w$?

**Remark 2.5.** If $C$ is a 3-cycle, then (2.4) turns to the symmetric form

\[ 2 \max \{ w(e_1), w(e_2), w(e_3) \} \leq w(e_1) + w(e_2) + w(e_3) \]

of the triangle inequality. Thus (2.4) can be considered as a “cyclic generalization” of this inequality.

**Remark 2.6.** Theorem 2.2 evidently holds if $G$ is the null graph, i.e. if $V(G) = \emptyset$. In this case the related metric space $(V, d)$ is empty.

**Proof of Theorem 2.2.** (i)⇒(ii) Suppose that there is a pseudometric $\rho$ on $V$ such that

\[ w(\{u, v\}) = \rho(u, v) \]
for each \( \{u, v\} \in E(G) \). Then for every sequence of points \( v_1, \ldots, v_n, \ v_1 = u \) and \( v_n = v, \ v_i \in V, \ i = 1, \ldots, n \), the triangle inequality implies

\[
\rho(v_1, v_n) \leq \sum_{i=1}^{n-1} \rho(v_i, v_{i+1}).
\]

Consequently for paths \( F \subseteq G \) joining \( u \) and \( v \), the inequality

\[
w(\{u, v\}) \leq w(F)
\]

holds. Passing in the last inequality to the infimum over the set \( \{w(F) : F \in \mathcal{P}_{u,v}\} \) we obtain

\[
(2.7) \quad \rho(u, v) = w(\{u, v\}) \leq d_w(u, v),
\]

see (2.1). The converse inequality \( w(\{u, v\}) \geq d_w(u, v) \) holds because the path \( (u = v_1, v_2 = v) \) belongs to \( \mathcal{P}_{u,v} \).

(ii) => (iii) Suppose statement (ii) holds. Let \( C \) be an arbitrary cycle in \( G \) and let \( \{u, v\} \in E(C) \) be an edge for which

\[
(2.8) \quad w(\{u, v\}) = \max_{e \in E(C)} w(e).
\]

Deleting the edge \( \{u, v\} \) from the cycle \( C \) we obtain the path \( F := C \setminus \{u, v\} \) joining the vertices \( u \) and \( v \). Using equalities (2.1), (2.3) and (2.8) we conclude that

\[
(2.9) \quad \max_{e \in E(C)} w(e) = d_w(u, v) \leq w(F).
\]

Since \( w(F) = w(C) - w(\{u, v\}) \), (2.9) follows from (2.7).

(iii) => (i) Suppose (iii) is true. If \( G \) is a connected graph, then we can equip \( G \) by the weighted shortest-path pseudometric \( d_w \) so it is sufficient to show that \( d_w \in \mathcal{M}_w \). Let \( \{u, v\} \in E(G) \). In the case where there is no cycle \( C \subseteq G \) such that \( \{u, v\} \in E(C) \) the path \( (u = v_1, v_2 = v) \) is the unique path joining \( u \) and \( v \). Hence, in this case, equality (2.3) follows from (2.1). Let \( P = (u = v_1, \ldots, v_{k+1} = v) \) be an arbitrary \( k \)-path, \( k \geq 2 \), joining \( u \) and \( v \). Then \( C := (u = v_1, \ldots, v_{k+1}, v_{k+2} = u) \) is a \( k + 1 \)-cycle with \( \{u, v\} \in E(C) \). Hence by (2.4) we have

\[
2w(\{u, v\}) \leq 2 \max_{e \in E(C)} w(e) \leq w(C) = w(P) + w(\{u, v\}).
\]

This implies the inequality \( w(\{u, v\}) \leq w(P) \) for all \( P \in \mathcal{P}_{u,v} \). Consequently \( w(\{u, v\}) \leq d_w(u, v) \). The converse inequality is trivial. Thus if \( G \) is connected, then \( d_w \in \mathcal{M}_w \).

Consider now the case of disconnected graph \( G \). Let \( \{G_i : i \in \mathcal{I}\} \) be the set of all components of \( G \) and let \( \{v^*_i : i \in \mathcal{I}\} \) be the subset of \( V(G) \) such that

\[
v^*_i \in V(G_i)
\]

for each \( i \in \mathcal{I} \). We choose an index \( i_0 \in \mathcal{I} \) and fix nonnegative constants \( a_i, \ i \in \mathcal{I} \) such that \( a_{i_0} = 0 \). Let us define the function \( \rho : V(G) \times V(G) \to \mathbb{R}^+ \)
as

\[(2.10) \quad \rho(u, v) = d_{w_i}(u, v)\]

if \(u\) and \(v\) lie in the same component \(G_i\) and as

\[(2.11) \quad \rho(u, v) = a_i + a_j + d_{w_i}(u, v_0^*) + d_{w_j}(v, v_0^*)\]

if \(u \in G_i\) and \(v \in G_j\) with \(i \neq j\). Here \(w_i\) is the restriction of \(w\) on the set \(E(G_i)\) and \(d_{w_i}\) is the weighted shortest-path pseudometric corresponding to the weight \(w_i\). It is easy to see that

\[(2.12) \quad a_i = \rho(v_0^*, v_0)\]

for all \(i \in \mathcal{I}\).

It follows directly from (2.10) and (2.11) that \(\rho\) is a pseudometric on \(V(G)\) and \(\rho \in \mathcal{M}_w\).

\[\square\]

**Remark 2.13.** To obtain the pseudometric \(\rho\) described by formulas (2.10), (2.11) we can consider the supergraph \(G^*\) of \(G\) such that \(V(G^*) = V(G)\) and

\[E(G^*) = E(G) \cup \{v_0^*, v_i^*\} : i \in \mathcal{I} \setminus \{i_0\},\]

see Fig. 2. Then \(G^*\) is a connected graph with the same set of cycles as in \(G\) and all edges \(\{v_i^*, v_{i_0}^*\}\) are bridges of \(G^*\). Now we can extend the weight \(w : E(G) \to \mathbb{R}^+\) to a weight \(w^* : E(G^*) \to \mathbb{R}^+\) by the rule:

\[w^*(\{u, v\}) := \begin{cases} w(\{u, v\}) & \text{if } \{u, v\} \in E(G) \\ a_i & \text{if } \{u, v\} = \{v_i^*, v_{i_0}^*\}, \ i \in \mathcal{I} \setminus \{i_0\}. \end{cases}\]

It can be shown that the pseudometric \(\rho\) is simply the weighted shortest-path pseudometric with respect to the weight \(w^*\).

\[\begin{array}{c}
G \\
\includegraphics[width=0.3\textwidth]{fig1}
\end{array}\quad \begin{array}{c}
G^* \\
\includegraphics[width=0.3\textwidth]{fig2}
\end{array}\]

**Figure 2.** Inclusion of the disconnected \(G\) in the connected \(G^*\). There are no new cycles in \(G^*\).
Let $w_1$ and $w$ be two weights with the same underlying graph $G$. Suppose
the weight $w_1$ is metrizable. What are conditions under which the weight $w$
is also metrizable?

To describe such type conditions we recall the definition of a bridge.

**Definition 2.14.** Let $G$ be a graph and let $e_0 \in E(G)$. For a connected $G$,$e_0$ is a bridge of $G$, if $G - e_0$ is a disconnected graph. If $G$ is disconnected
and $G_0$ is the connected component of $G$ such that $e_0 \in E(G_0)$, then $e_0$ is a bridge of $G$, if $G_0 - e_0$ is disconnected.

Above we denote by $G - e_0$ the edge-deleted subgraph of $G$, see, for example, [1, p. 40].

For weights $w_1$ and $w_2$ on $E(G)$ define a set $w_1 \Delta w_2 \subseteq E(G)$ as

$$w_1 \Delta w_2 = \{ e \in E(G) : w_1(e) \neq w_2(e) \}.$$

**Proposition 2.15.** Let $(G, w_1)$ be a weighted graph with a metrizable
weight $w_1$ and let $E_1$ be a subset of $E(G)$. The following statements are equivalent.

(i) All weights $w : E(G) \to \mathbb{R}^+$ with $w_1 \Delta w \subseteq E_1$ are metrizable.

(ii) Each element $e \in E_1$ is a bridge of the graph $G$.

**Lemma 2.16.** An edge $e \in E(G)$ is a bridge if and only if $e$ is not in $E(C)$
for any cycle $C \subseteq G$.

This lemma is known for the finite graphs $G$, see [5, Theorem 2.3]. The
proof for infinite $G$ is completely analogous, so we omit it here.

**Proof of Proposition 2.15.** The implication (ii)⇒(i) follows directly from con-
dition (iii) of Theorem 2.2 and Lemma 2.16. Conversely, if some $e_0 \in E_1$ is not bridge, then by Lemma 2.16 there is a cycle $C_0 \subseteq G$ such that
$e_0 \in E(C_0)$. Let us define the weight $w_0 : E(G) \to \mathbb{R}$,

$$w_0(e) = \begin{cases} w_1(e) & \text{if } e \neq e_0 \\ 1 + w(C_0) - w(e_0) & \text{if } e = e_0. \end{cases}$$

Then we have $w_1 \Delta w_0 = \{ e_0 \}$ and

$$2w_0(e_0) = 2 + 2w_1(C_0) - 2w_1(e_0) >$$

$$(w_1(C_0) - w_1(e_0) + 1) + (w_1(C_0) - w_1(e_0)) = w_0(C_0).$$

It is clear, that $w_1 \Delta w_0 \subseteq E_1$ but, by Theorem 3.11 the weight $w_0$ is not
metrizable. Thus the implication (i)⇒(ii) follows. □

Recall that acyclic graphs are usually called the forests. Lemma 2.16 implies that a graph $G$ is a forest if and only if all $e \in E(G)$ are bridges of
$G$. Hence as a particular case of Proposition 2.15 we obtain

**Corollary 2.17.** The following conditions are equivalent for every graph $G$.

(i) $G$ is a forest.

(ii) Every weight $w : E(G) \to \mathbb{R}^+$ is metrizable.
Our final corollary shows that the property of a weight to be metrizable is local.

Corollary 2.18. Let \((G, w_G)\) be a weighted graph. The weight \(w_G\) is metrizable if and only if the restrictions \(w_H = w_G|_{E(H)}\) are metrizable for all finite subgraphs \(H\) of the graph \(G\).

3. Maximality of the weighted shortest-path pseudometric

Let \(G\) be a graph and let \(w\) be a metrizable weight on \(E(G)\). Recall that \(M_w\) is the set of all pseudometrics \(\rho\) on \(V(G)\) satisfying the restriction

\[
\rho(u, v) = w(\{u, v\})
\]

for each \(\{u, v\} \in E(G)\). Let us introduce the ordering relation \(\leq\) on the set \(M_w\) as

\[
(\rho_1 \leq \rho_2) \text{ if and only if } (\rho_1(u, v) \leq \rho_2(u, v))
\]

for all \(u, v \in E(G)\). A reasonable question to ask is whether it is possible to find the greatest and least elements of the partially ordered set \((M_w, \leq)\).

In the present section we show that the shortest-path pseudometric \(d_w\) is the greatest element in \((M_w, \leq)\) for a connected \(G\) and apply this result to the search of metrics in \(M_w\). The existence of the least element of the poset \((M_w, \leq)\) will be discussed in Section 4.

Proposition 3.3. Let \((G, w)\) be a nonempty weighted graph with a metrizable weight \(w\). If \(G\) is connected then the weighted shortest-path pseudometric \(d_w\) belongs to \(M_w\) and this pseudometric is the greatest element of the poset \((M_w, \leq)\), i.e., the inequality

\[
\rho \leq d_w
\]

holds for each \(\rho \in M_w\). Conversely, if the poset \((M_w, \leq)\) contains the greatest element, then \(G\) is connected.

Proof. In fact, for connected \(G\), the membership relation \(d_w \in M_w\) was justified in the third part of the proof of Theorem 2.2. To prove (3.4) see (2.7).

If \(G\) is disconnected and some vertices \(u, v\) lie in distinct components, \(u \in G_i, v \in G_j\), then letting \(a_i, a_j \to +\infty\) in (2.11) we obtain

\[
\sup\{\text{diam}_\rho(A) : \rho \in M_w\} = \infty
\]

for the two-element set \(A = \{u, v\}\). Thus the poset \((M_w, \leq)\) does not contain the greatest element.

Remark 3.5. If \(G\) is a disconnected graph and \(u, v\) belong to distinct connected components of \(G\), then according to (2.1) we can put

\[
d_w(u, v) = +\infty
\]
as for the infimum over the empty set. Under this agreement, the weighted shortest-path pseudometric is also "the greatest element" of \( \mathcal{M}_w \) for the disconnected graphs \( G \).

Recall that connected acyclic graphs are called the trees, so that each tree is a connected forest. The last proposition and Corollary 2.17 imply

**Corollary 3.6.** A graph \( G \) is a tree if and only if each weight \( w : E(G) \to \mathbb{R}^+ \) is metrizable and the inequality

\[
\sup \{ \text{diam}_\rho(A) : \rho \in \mathcal{M}_w \} < \infty
\]

holds for every finite \( A \subseteq V(G) \).

Let \((G, w)\) be a weighted graph with a metrizable \( w \). Then each \( \rho \in \mathcal{M}_w \) is a pseudometric satisfying (3.1). We ask under what conditions does a metric \( \rho \in \mathcal{M}_w \) exist. To this end it is necessary for the weight \( w : E(G) \to \mathbb{R}^+ \) to be strictly positive in the sense that \( w(e) > 0 \) for all \( e \in E(G) \). This trivial condition is also sufficient for the graphs with the vertices of finite degrees. Here, as usual, by the degree of a vertex \( v \) we understand the cardinal number of edges incident with \( v \). More generally we have

**Corollary 3.7.** Let \((G, w)\) be a weighted graph such that each connected component of \( G \) contains at most one vertex of infinite degree. If the weight \( w \) is metrizable and strictly positive, then there is a metric \( \rho \in \mathcal{M}_w \).

**Proof.** Suppose \( G \) is connected and \( w \) is strictly positive and metrizable. Let \( \{u, v\} \notin E(G) \). Without loss of generality we may suppose that the edges of \( G \) which are incident with \( u \) form the finite set \( \{e_1, \ldots, e_n\} \). Then the inequalities

\[
w(F) > \min_{1 \leq i \leq n} w(e_i) > 0
\]

holds for each path \( F \in P_{u, v} \). Thus \( d_w(u, v) > 0 \) for every pair of distinct \( u, v \in V(G) \). It still remains to note that \( d_w \in \mathcal{M}_w \) by Proposition 3.3.

For the case of disconnected \( G \) we can obtain the desirable metric \( \rho \in \mathcal{M}_w \) using (2.10) and (2.11) with strictly positive \( a_i, a_j \).

**Remark 3.8.** The main point of the previous proof is the following: If there is a metric \( \rho \in \mathcal{M}_w \), then the weighted shortest-path pseudometric is also a metric.

The following example shows that the conclusion of Corollary 3.7 is, generally speaking, false for connected graphs \( G \), containing at least two vertices of infinite degree.

**Example 3.9.** Let \((G, w)\) be the infinite weighted graph depicted by Fig. 3 where \( \varepsilon_n = w(\{v_n, u_1\}) = w(\{v_n, u_2\}) \) are real numbers such that

\[
\lim_{n \to \infty} \varepsilon_n = 0
\]

and \( \varepsilon_n > \varepsilon_{n+1} > 0 \) for each \( n \in \mathbb{N} \). Each cycle \( C \) of \( G \) is a quadrilateral of the form \( u_1, v_n, u_2, v_m, u_1 \) with \( n \neq m \). Since \( C \) has two distinct edges
of the maximal weight, inequality (2.4) holds. By Theorem 2.2 it means that $w$ is metrizable. Letting $n, m \to \infty$ and using formula (2.1) we obtain $d_w(u_1, u_2) = 0$. Consequently $d_w$ is a pseudometric so, in accordance with Remark 3.8 there are no metrics in $M_w$.

To describe the characteristic structural properties of graphs $G$ having metrics in $M_w$ for each metrizable $w$:

Given an infinite sequence $\{A_n\}_{n \in \mathbb{N}}$ of sets, we call the upper limit of this sequence, $\limsup_{n \to \infty} A_n$, the set of all elements $a$ such that $a \in A_n$ holds for an infinity of values of the index $n$. We have

\[ \limsup_{n \to \infty} A_n = \bigcap_{k=1}^{\infty} \left( \bigcup_{n=1}^{\infty} A_{n+k} \right). \]

Let $G$ be a graph. A set $F$ of vertices of $G$ is independent if every two vertices in $F$ are nonadjacent.

**Theorem 3.11.** Let $G = (V, E)$ be an infinite connected graph. The following two conditions are equivalent.

(i) There is a strictly positive metrizable weight $w$ such that each $\rho \in M_w$ is not metric but pseudometric only.
(ii) There are two vertices \(u^*, v^* \in V(G)\) and a sequence \(\tilde{F}\) of paths \(F_j, \ j \in \mathbb{N}\), joining \(u^*\) and \(v^*\) such that the upper limit of the sequence \(\{V(F_j)\}_{j \in \mathbb{N}}\) is an independent set.

**Remark 3.12.** It is clear that the relations

\[ u^*, v^* \in \limsup_{n \to \infty} V(F_n) \]

hold for each sequence \(\tilde{F} = \{F_j\}_{j \in \mathbb{N}}\) joining \(u^*\) and \(v^*\). Hence vertices \(u^*\) and \(v^*\) are nonadjacent if condition (ii) of Theorem 3.11 holds.

Using (3.10) we can reformulate the last part of (ii) in the form:

\[(ii_1)\] for every \(e^0 \in E(G)\) there are \(u^0 \in e^0\) and \(i = i(e^0)\) such that

\[ (3.13) \quad u^0 \notin \bigcup_{k=1}^{\infty} V(F_{i+k}). \]

**Lemma 3.14.** Let \(G\) be an infinite connected graph, let \(u^*\) and \(v^*\) be two distinct nonadjacent vertices of \(G\) and let \(\tilde{F} = \{F_j\}_{j \in \mathbb{N}}\) be a sequence of paths joining \(u^*\) and \(v^*\) such that \((ii_1)\) holds. Then there is a subsequence \(\{F_{j_k}\}_{k \in \mathbb{N}}\) of \(\tilde{F}\) such that:

\[(ii_2)\] the equality

\[ (3.15) \quad E(F_{j_l}) \cap E(F_{j_k}) = \emptyset \]

holds whenever \(l \neq k\);

\[(ii_3)\] if a cycle \(C\) is contained in the graph \(\bigcup_{k \in \mathbb{N}} F_{j_k}\),

\[ (3.16) \quad C \subseteq \bigcup_{k \in \mathbb{N}} F_{j_k}, \]

and

\[ (3.17) \quad k_0 = k_0(C) := \min\{k \in \mathbb{N} : E(C) \cap E(F_{j_k}) \neq \emptyset\}, \]

then \(C\) and \(F_{j_{k_0}}\) have at least two common edges.

**Proof.** For every \(e \in E(G)\) define a set

\[ N(e) := \{j \in \mathbb{N} : E(F_j) \ni e\}. \]

Condition \((iii_1)\) implies that \(N(e)\) is finite for each \(e \in E(G)\). Now we construct a subsequence \(\{F_{j_k}\}_{k \in \mathbb{N}}\) by induction. Write \(j_1 := 1\) and

\[ M_1 := \bigcup_{e \in F_{j_1}} N(e). \]

Since all \(N(e)\) are finite, the set \(M_1 \subseteq \mathbb{N}\) is also finite. Let \(j_2\) be the least natural number in the set \(\mathbb{N} \setminus M_1\). Write

\[ M_2 := \bigcup_{e \in F_{j_2}} N(e), \quad j_3 = \min\{m : m \in \mathbb{N} \setminus (M_1 \cup M_2)\}; \]
\[ M_3 := \bigcup_{e \in F_{j_3}} N(e), \quad j_4 = \min\{m : m \in \mathbb{N} \setminus (M_1 \cup M_2 \cup M_3)\} \]

and so on. It is plain to show that (3.15) holds for distinct \( j_k \) and \( j_e \). Thus

the subsequence \( \{F_{j_k}\}_{k \in \mathbb{N}} \) satisfies (ii2). To construct a subsequence of \( \tilde{F} \)

which satisfies simultaneously (ii2) and (ii3), note that condition (ii1) re-
mains valid when one passes from the sequence \( \tilde{F} \) to any of its subsequences.

Hence, without loss of generality, we may assume that \( \{F_{j_k}\}_{k \in \mathbb{N}} = \tilde{F} \).

To define a new subsequence \( \{F_{j_k}\}_{k \in \mathbb{N}} \) we again use the induction. Put

\( j_1 := 1 \). Suppose \( j_k \) are defined for \( k = 1, \ldots, n - 1 \). By (ii1) for every \( e \in F_{j_{n-1}} \) there are \( i(e) \in \mathbb{N} \) and \( u \in e \) such that

\[ u \notin \bigcup_{k=1}^{\infty} V(F_{i(e)+k}). \]

Define

\[ j_n := 1 + \max_{e \in F_{j_{n-1}}} i(e). \]

Note that \( j_n > j_{n-1} \).

Suppose that a cycle \( C \) is contained in \( \bigcup_{k \in \mathbb{N}} F_{j_k} \) where \( \{F_{j_k}\}_{k \in \mathbb{N}} \) is above
constructed subsequence of \( \tilde{F} \) and \( k_0 = k_0(C) \) is defined by (3.17) but \( F_{j_{k_0}} \)
contains the unique edge \( e = \{u, v\} \) from \( E(C) \). Let \( e_1, e_2 \in E(C) \) be the
distinct edges which are adjacent to \( e \). The uniqueness of \( e \), (3.16) and
(3.17) imply the relations

\[ e_1 \in \bigcup_{k > k_0} E(F_{j_k}) \quad \text{and} \quad e_2 \in \bigcup_{k > k_0} E(F_{j_k}). \]

If \( e_1 = \{u_1, v_1\} \) and \( e_2 = \{u_2, v_2\} \), then

\[ u \in \{u_1, v_1, u_2, v_2\} \quad \text{and} \quad v \in \{u_1, v_1, u_2, v_2\} \]

so that from (3.20) we obtain

\[ u \in \bigcup_{k > k_0} V(F_{j_k}) \quad \text{and} \quad v \in \bigcup_{k > k_0} V(F_{j_k}) \]

contrary to (3.18) and (3.19). \( \square \)

**Proof of Theorem 3.11** (i)\( \Rightarrow \) (ii) Let \( w \) be a strictly positive metrizable
weight such that each \( \rho \in \mathfrak{M}_w \) is a pseudometric only. Hence \( d_w \) is not
metric, so for some distinct \( u^*, v^* \in V(G) \) we have

\[ d_w(u^*, v^*) = 0. \]

From the definition of \( d_w \) it follows at once that there is a sequence \( \tilde{F} = \{F_i\}_{i \in \mathbb{N}}, F_i \in \mathcal{P}_{u^*, v^*}, i \in \mathbb{N}, \) such that

\[ \lim_{i \to \infty} w(F_i) = 0. \]
Since all $F_i$ are finite, we may suppose, taking a subsequence of $\tilde{F}$ if it is necessary, that

\[(3.23) \min_{e \in E(F_j)} w(e) > \sum_{i=1}^{\infty} w(F_{i+j})\]

for all $j \in \mathbb{N}$. We claim that condition (ii) is fulfilled by $\tilde{F}$.

Assume there is $e_0 = \{u_0, v_0\} \in E(G)$ such that $u_0, v_0 \in \bigcup_{k=1}^{\infty} V(F_{i+k})$ for each $i \in \mathbb{N}$. Since all $F_i$ are paths joining $u^*$ and $v^*$, there is an $u_0v_0$-walk in the graph

\[(3.24) \tilde{G}_i := \bigcup_{k=1}^{\infty} F_{i+k}.\]

It is well known that if there is an $xy$-walk in a graph, then there is also a path joining $x$ and $y$ in the same graph, see, for example [1, p. 82]. Let $P_i$ be a path joining $u^0$ and $v^0$ in $\tilde{G}_i$. The weight $w$ is metrizable. Consequently we may use Theorem 2.2. Equalities (2.1)–(2.3) imply

$$w(\{u^0, v^0\}) \leq w(P_i) \leq \sum_{k=1}^{\infty} w(F_{i+k}).$$

Letting $i \to \infty$ and using (3.22), (3.23) we obtain

$$w(\{u^0, v^0\}) \leq \lim_{i \to \infty} \sum_{k=1}^{\infty} w(F_{i+k}) = 0,$$

contrary to the condition: $w(e) > 0$ for all $e \in E(G)$.

(ii)$\Rightarrow$(i) Let $G$ be a graph satisfying condition (ii). In view of Lemma 3.14 we may suppose that (ii$_2$) and (ii$_3$) are also fulfilled with $\{F_{j_k}\}_{k \in \mathbb{N}} = \tilde{F}$ where $\tilde{F}$ is the sequence of paths appearing in (ii). To verify condition (i) it suffices, by Proposition 3.3, to find a metrizable weight $w : V(G) \to \mathbb{R}^+$ such that $w(e) > 0$ for all $e \in E(G)$ and

$$d_w(u^*, v^*) = 0$$

for some distinct vertices $u^*$ and $v^*$.

In the rest of the proof we will construct the desired weight $w$.

Let us consider the graph

$$\tilde{G} = \bigcup_{i \in \mathbb{N}} F_i,$$

cf. (3.24). Denote by $m(F_i), i \in \mathbb{N}$, the number of edges of $F_i$. Let $\{\varepsilon_i\}_{i \in \mathbb{N}}$ be a decreasing sequence of positive real numbers such that $\lim_{i \to \infty} \varepsilon_i = 0$.
and that the sequence \( \{ \frac{\varepsilon_i}{m(F_i)} \}_{i \in \mathbb{N}} \) is also decreasing. Define a weight \( w_1 : E(\tilde{G}) \to \mathbb{R}^+ \) as
\[
(3.25) \quad w_1(e) := \frac{\varepsilon_i}{m(F_i)}, \quad \text{if } e \in E(F_i).
\]
This definition is correct because, by (ii_2), the edge sets \( E(F_i) \) and \( E(F_j) \) are disjoint for distinct \( i \) and \( j \).

Let us note that the weight \( w_1 \) is metrizable. It follows from Theorem 2.2 because (ii_3), (3.25) and the decrease of the sequence \( \{ \frac{\varepsilon_i}{m(F_i)} \}_{i \in \mathbb{N}} \) imply inequality (2.4) for every cycle \( C \) in \( \tilde{G} \). (As has been stated above, (2.4) holds if there are two distinct edges \( e_1, e_2 \) in \( C \) such that \( w(e_1) = w(e_2) = \max_{e \in E(C)} w(e) \). To find these \( e_1 \) and \( e_2 \) we can use (ii_3).)

Let \( d_{w_1} \) be the weighted shortest-path pseudometric generated by the weight \( w_1 \). The condition \( \lim_{i \to \infty} \varepsilon_i = 0 \) implies \( d_{w_1}(u^*, v^*) = 0 \). Indeed, we have
\[
d_{w_1}(u^*, v^*) \leq \inf_{i \in \mathbb{N}} w_1(F_i) \leq \lim_{i \to \infty} \sum_{e \in E(F_i)} w_1(e) = \lim_{i \to \infty} \frac{\varepsilon_i}{m(F_i)} = \lim_{i \to \infty} \varepsilon_i = 0.
\]

Let \( e^0 = \{u^0, v^0\} \in E(G) \) with \( u^0, v^0 \in V(\tilde{G}) \). We will use (ii_2) to prove the inequality
\[
(3.26) \quad d_{w_1}(u^0, v^0) > 0.
\]
Condition (ii_1) implies at least one from the relations
\[
u^0 \notin \bigcup_{k=1}^{\infty} V(F_{i+k}), \quad v^0 \notin \bigcup_{k=1}^{\infty} V(F_{i+k})
\]
for sufficiently large \( i \). Suppose, for instance, that there is \( i_0 = i_0(e^0) \in \mathbb{N} \) such that
\[
u^0 \notin V(F_i) \quad \text{if } i > i_0.
\]
Let \( F \) be an arbitrary path in \( \tilde{G} \) joining \( u^0 \) and \( v^0 \) and let \( e \in E(F) \) be the edge incident with the end \( u^0 \). From (3.27) follows
\[
e \in \bigcup_{i=1}^{i_0} E(F_i).
\]
Using this relation, (3.25) and the decrease of the sequence \( \{ \frac{\varepsilon_i}{m(F_i)} \}_{i \in \mathbb{N}} \) we obtain
\[
d_{w_1}(u^0, v^0) \geq \frac{\varepsilon_{i_0}}{m(F_{i_0})} > 0,
\]
so that (3.26) holds.

Write
\[
V' := V(G) \setminus V(\tilde{G}).
\]
If $V' = \emptyset$, then the desirable strictly positive weight $w : E(G) \to \mathbb{R}^+$ can be obtained as

$$w(\{u, v\}) := d_{w_1}(u, v), \quad \{u, v\} \in E(G)$$

because as was shown above $d_{w_1}(u, v) > 0$ for each $\{u, v\} \in E(G)$. The weight $w$ is metrizable because it is a “restriction” of the pseudometric $d_{w_1}$. It is easy to show also that

$$d_{w_1}(u, v) = d_w(u, v)$$

for all $u, v \in V(G)$. Indeed, since $d_{w_1} \in \mathcal{M}_w$ the inequality

$$d_{w_1}(u, v) \leq d_w(u, v)$$

follows from Proposition 3.3. To prove the converse inequality note that

$$\mathcal{P}_{u, v}(\tilde{G}) \subseteq \mathcal{P}_{u, v}(G)$$

because $\tilde{G}$ is a subgraph of $G$. Consequently,

$$d_w(u, v) = \inf\{w(F) : F \in \mathcal{P}_{u, v}(G)\} \leq \inf\{w(F) : F \in \mathcal{P}_{u, v}(\tilde{G})\} = \inf\{w_1(F) : F \in \mathcal{P}_{u, v}(\tilde{G})\} = d_{w_1}(u, v).$$

Equality (3.28) implies, in particular, that $d_w(\{u^*, v^*\}) = 0$. Let us consider now the case where

$$V' = V(G) \setminus V(\tilde{G}) \neq \emptyset.$$

Let $v'$ be a fixed point of the set $V'$. Define a distance function $d'$ on the set $V(G)$ as $d'(v, v) = 0$ for all $v \in V(G)$ and as

$$d'(u, v) := \begin{cases} 1 & \text{if } u = v^*, \ v = v' \\ 1 & \text{if } u, v \in V', \ u \neq v \\ d_{w_1}(u, v) & \text{if } u, v \in V(\tilde{G}) \\ d_{w_1}(u, v^*) + 1 & \text{if } u \in V(\tilde{G}), \ v = v' \\ 2 & \text{if } u = v^*, \ v \in V', \ v \neq v' \\ d_{w_1}(u, v^*) + 2 & \text{if } u \in V(\tilde{G}), \ v \in V', \ u \neq u^*, \ v \neq v'. \end{cases}$$

Note that the past three lines in the right side of (3.30) can be rewritten in the form:

$$d'(u, v) = d'(u, v^*) + d'(v^*, v') + d'(v', v)$$

if $u \in V(\tilde{G})$ and $v \in V'$. The last equality and (3.30) imply that $d'$ is a pseudometric. Writing

$$w(\{u, v\}) = d'(u, v)$$

for all $\{u, v\} \in E(G)$ we obtain the weighted graph $(G, w)$ with $d' \in \mathcal{M}_w$. The weight $w$ is strictly positive since, by (3.30), $d'(u, v) \geq 1$ if $u \neq v$ and $\{u, v\} \cap V' \neq \emptyset$ and, by (3.26), $d'(u, v) > 0$ if $u \neq v$ and $u, v \in V(\tilde{G})$, and $\{u, v\} \in E(G)$.

Proposition 3.3 implies that

$$d_w(u, v) = w(\{u, v\})$$
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for each \( \{u, v\} \in E(G) \). To complete the proof, it suffices to observe that 
\[ d_w(u^*, v^*) = 0. \]
To see this we can use (3.29) with \( u = u^* \) and \( v = v^* \). \( \square \)

For the case of disconnected graphs \( G \) we have the following

Proposition 3.32. Let \((G, w)\) be a disconnected weighted graph with the strictly positive metrizable \( w \). Then there is a pseudometric \( \rho \in \mathcal{M}_w \) which is not metric. Moreover let \( G_i \) be connected components of \( G \) and let \( w_i \) be the restrictions of the weight \( w \) on the sets \( E(G_i) \). Then the following statements are equivalent.

(i) There exists a metric in \( \mathcal{M}_w \).

(ii) The shortest-path pseudometrics \( d_{w_i} \) are metrics for all \( G_i \).

Proof. Set in (2.11) \( a_i = 0 \) for some \( i \neq i_0 \). Then, by (2.12), \( \rho \) is not a metric. If all \( d_{w_i} \) are metrics, then to obtain a metric in \( \mathcal{M}_w \) it is sufficient to put \( a_i > 0 \) for all \( i \neq i_0 \). \( \square \)

4. The least element in \( \mathcal{M}_w \)

We wish characterize the structure of the graphs \( G \) for which the set \( \mathcal{M}_w \) contains the least pseudometric \( \rho_{0,w} \) as soon as \( w \) is metrizable. To this end, we recall the definition of \( k \)-partite graph.

Definition 4.1. Let \( G \) be a simple graph and let \( k \) be a cardinal number. The graph \( G \) is \( k \)-partite if the vertex set \( V(G) \) can be partitioned into \( k \) nonvoid disjoint subsets, or parts, in such a way that no edge has both ends in the same part. A \( k \)-partite graph is complete if any two vertices in different parts are adjacent.

We shall say that \( G \) is a complete multipartite graph if there is a cardinal number \( k \geq 1 \) such that \( G \) is complete \( k \)-partite, cf. [3, p. 14].

Remark 4.2. It is easy to prove that each nonempty complete \( k \)-partite graph \( G \) is connected if \( k \geq 2 \). Each 1-partite graph \( G \) is an empty graph.

Theorem 4.3. The following conditions are equivalent for each nonempty graph \( G \).

(i) For every metrizable weight \( w : E(G) \to \mathbb{R}^+ \) the poset \( (\mathcal{M}_w, \leq) \) contains the least pseudometric \( \rho_{0,w} \), i.e., the inequality

\[
\rho_{0,w}(u, v) \leq \rho(u, v)
\]

holds for all \( \rho \in \mathcal{M}_w \) and all \( u, v \in V(G) \);

(ii) \( G \) is a complete \( k \)-partite graph with \( k \geq 2 \).

If condition (ii) holds and \( w \) is a metrizable weight, then for each pair of distinct nonadjacent vertices \( u, v \) we have

\[
d_w(u, v) = \inf_{\alpha \neq \alpha_0} \inf_{p \in X_\alpha} \left( w(\{u, p\}) + w(\{p, v\}) \right),
\]
and

\[ \rho_{0,w}(u,v) = \sup_{\alpha \neq \alpha_0} \sup_{p \in X_\alpha} \max \{ w(\{u,v\}) - w(\{p,v\}) \} \]

where \( \{X_\alpha : \alpha \in \mathcal{I} \} \) is a partition of \( G \) and \( \alpha_0 \in \mathcal{I} \) is the index such that \( u, v \in X_{\alpha_0} \).

**Remark 4.7.** Formulas \((1.5)\) and \((4.6)\) give the generalization of double inequality \((1.5)\) for an arbitrary complete \( k \)-partite graph, with \( k \geq 2 \). The quadrilateral \( Q \) depicted by Figure 1 is evidently a complete bipartite graph.

**Lemma 4.8.** Let \( G \) be a connected nonempty graph. The following conditions are equivalent

(i) For each metrizable \( w : E(G) \to \mathbb{R}^+ \) the poset \((\mathcal{M}_w, \leq)\) contains the least pseudometric \( \rho_{0,w} \).

(ii) For every two distinct nonadjacent vertices \( u \) and \( v \) and each \( p \in V(G) \) with \( u \neq p \neq v \) we have either

\[ \{u,p\} \in E(G) \& \{v,p\} \in E(G) \]

or

\[ \{u,p\} \notin E(G) \& \{v,p\} \notin E(G). \]

**Proof.** (i)\(\Rightarrow\)(ii) Suppose condition (ii) does not hold. Let \( v_1, v_2, v_3 \) be distinct vertices of \( G \) such that \( v_1 \) and \( v_2 \) are nonadjacent and \( v_2 \) and \( v_3 \) are also nonadjacent but \( \{v_3,v_1\} \in E(G) \). Define the weight \( w \) as \( w(e) = 1 \) for all \( e \in E(G) \). Let \( \rho_1 \) and \( \rho_2 \) be the distance functions on \( V(G) \) such that:

\[ \rho_1(v_2,v_1) = \rho_1(v_1,v_2) = \rho_1(u,u) = 0 \text{ for all } u \in V(G) \text{ and } \rho_1(u,v) = 1 \text{ otherwise; } \]

\[ \rho_2(v_3,v_1) = \rho_2(v_2,v_3) = \rho_2(u,u) = 0 \text{ for all } u \in V(G) \text{ and } \rho_2(u,v) = 1 \text{ otherwise.} \]

It is easy to see that every triangle in \((V(G), \rho_1)\) or in \((V(G), \rho_2)\) is an isosceles triangle having the third side shorter or equal to the common length of the other two sides. Hence \( \rho_1 \) and \( \rho_2 \) are pseudometrics and even pseudoultrametrics. Furthermore \( \rho_1 \) and \( \rho_2 \) belong to \( \mathcal{M}_w \). Suppose that there is the least pseudometric \( \rho_{0,w} \) in \( \mathcal{M}_w \). Then we obtain the contradiction

\[ 1 = \rho_{0,w}(v_2,v_3) \leq \rho_{0,w}(v_2,v_1) + \rho_{0,w}(v_1,v_3) \]

\[ \leq (\rho_1 \land \rho_2)(v_2,v_1) + (\rho_1 \land \rho_2)(v_1,v_3) = 0 + 0 = 0. \]

(ii)\(\Rightarrow\)(i) Suppose condition (ii) holds. Let \( w \) be a metrizable weight. For each pair \( u, v \) of vertices of \( G \) write:

\[ \rho_0(u,v) = 0 \text{ if } u = v; \quad \rho_0(u,v) = w\{u,v\} \text{ if } \{u,v\} \in E(G); \]

and

\[ \rho_0(u,v) := \sup_{P \in \mathcal{P}_{u,v}} \max_{v \in P} (2w(e) - w(P))_+ \]
if \( \{u, v\} \not\in E(G) \) and \( u \neq v \). Here we use the notation

\[
t_+ := \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}
\]

We claim that \( \rho_0 \) is the least element of \((\mathcal{M}_w, \leq)\). To show this it suffices to prove the triangle inequality for \( \rho_0 \). Indeed, in this case \( \rho_0 \) is a pseudometric and \( \rho_0 \in \mathcal{M}_w \) by the definition of \( \rho_0 \). Moreover if \( \rho \) is an arbitrary pseudometric from \( \mathcal{M}_w \), then \eqref{2.4} implies:

\[
2w(e) \leq w(P) + \rho(u, v)
\]

for all distinct \( u, v \in V(G) \), all \( P \in \mathcal{P}_{u,v} \) and all \( e \in P \). Consequently we have

\[
2w(e) - w(P) \leq \rho(u, v),
\]

\[
(2w(e) - w(P))_+ \leq (\rho(u, v))_+ = \rho(u, v),
\]

\[
(4.10) \quad \sup_{P \in \mathcal{P}_{u,v}} \max_{e \in P} (2w(e) - w(P))_+ \leq \rho(u, v).
\]

The last inequality and \eqref{4.9} imply \eqref{4.4} with \( \rho_{0,w} = \rho_0 \). Thus \( \rho_0 \) is the least pseudometric in \( \mathcal{M}_w \).

Let us turn to the triangle inequality for \( \rho_0 \). Let \( x, y, z \) be some distinct vertices of \( G \). Since \( w \) is metrizable, the definition of \( \rho_0 \) implies this inequality if \( \{x, y\}, \{y, z\} \) and \( \{z, x\} \) belong to \( E(G) \). Let \( \{x, y\} \not\in E(G) \). In accordance with condition (ii), either both \( \{y, z\} \) and \( \{z, x\} \) are edges of \( G \) or both \( \{y, z\} \) and \( \{z, x\} \) are not edges of \( G \).

Suppose \( \{y, z\}, \{z, x\} \in E(G) \). The three-point sequence \( P_1 := (x, z, y) \) is a path joining \( x \) and \( y \). Consequently by \eqref{4.9} we obtain

\[
\rho_0(x, y) \geq \max_{e \in P_1} (2w(e) - w(P_1))_+ = |w(\{x, z\}) - w(\{z, y\})|.
\]

Thus

\[
(4.11) \quad \rho_0(x, y) + \min(\rho_0(x, z), \rho_0(z, y)) \geq \max(\rho_0(x, z), \rho_0(z, y))
\]

To prove the inequality

\[
\rho_0(x, y) \leq \rho_0(x, z) + \rho_0(z, y)
\]

it is sufficient to show

\[
(4.12) \quad \max_{e \in P} (2w(e) - w(P))_+ \leq \rho_0(x, z) + \rho_0(z, y)
\]

for each path \( P \) joining \( x \) and \( y \). This inequality is trivial if its left part equals zero. In the opposite case, \eqref{4.12} can be rewritten in the form

\[
2\max_{e \in P} w(e) \leq w(P) + w(\{x, z\}) + w(\{z, y\}).
\]

Applying \eqref{2.4} we see that the last inequality holds, so \eqref{4.12} follows. (Note that inequality \eqref{2.4} holds for each closed walk in \( G \) if it holds for each cycle in \( G \).)
It is slightly more difficult to prove the triangle inequality for $\rho_0$ when
\begin{equation}
\{y, z\} \notin E(G), \quad \{z, x\} \notin E(G) \quad \text{and} \quad \{z, y\} \notin E(G).
\end{equation}

To this end, we establish first the following lemma.

**Lemma 4.14.** Let $(G, w)$ be a connected, weighted graph with a metrizable $w$, let condition (ii) of Lemma 4.8 hold and let $x, y$ be distinct non-adjacent vertices of $G$. Then, for every $P \in P_{x,y}$ there is $v \in V(G)$ with \{v, x\}, \{v, y\} $\in E(G)$ and such that
\begin{equation}
\max_{e \in P}(2w(e) - w(P)) \leq |w(\{x, v\}) - w(\{v, y\})|.
\end{equation}

**Proof.** Let $P = (x, v_1, \ldots, v_n, y)$ be a path joining $x$ and $y$ in $G$. We claim that there is a path $(x, v, y)$ in $G$ such that (4.15) holds. It is trivial if $n = 1$ or if the left part in (4.15) is zero. So we may suppose that $n \geq 2$ ($v_1 \neq v_2$) and
\begin{equation}
2 \max_{e \in P} w(e) > w(P).
\end{equation}

Condition (ii) of Lemma 4.8 implies
\begin{equation}
\{v_1, y\} \in E(G) \quad \text{and} \quad \{x, v_n\} \in E(G),
\end{equation}
see Figure 4. For convenience we write
\[ M := \max_{e \in P} w(e). \]

![Figure 4](image-url)

**Figure 4.** The path $P$ joining $x$ and $y$ with $n = 4$ (heavily drawn lines), and two additional edges $\{x, v_1\}, \{y, v_1\}$ (dotted lines).

Let us prove (4.15). If $M = w(\{x, v_1\})$, then
\[
2M - w(P) = w(\{x, v_1\}) - \left(\sum_{i=1}^{n-1} w(\{v_i, v_{i+1}\}) + w(\{v_n, y\})\right) \\
\leq w(\{x, v_1\}) - w(\{v_1, y\})
\]
because \( w \) is metrizable and so we have the “triangle inequality”

\[
w(\{v_1, y\}) \leq \sum_{i=1}^{n-1} w(\{v_i, v_{i+1}\}) + w(\{v_n, y\}).
\]

Hence the path \((x, v_1, y)\) satisfies (4.15) with \( v = v_1 \). Similarly if \( M = (\{v_n, y\}) \), then the desired path is \((x, v_n, y)\).

Suppose now that

\[
(4.18) \quad w(\{v_1, y\}) \geq w(\{x, v_1\}) \quad \text{and} \quad M = \max_{1 \leq i \leq n-1} w(\{v_i, v_{i+1}\}).
\]

Since \( w \) is metrizable, applying (2.4) to the cycle \((v_1, v_2, \ldots, y, v_1)\) we obtain

\[
w(\{v_1, y\}) + w(\{v_n, y\}) + \sum_{i=1}^{n-1} w(\{v_i, v_{i+1}\}) \geq 2M.
\]

Consequently

\[
w(\{v_1, y\}) - w(\{x, v_1\}) \geq 2M - w(P).
\]

Thus \((x, v_1, y)\) satisfies (4.15) with \( v = v_1 \) if (4.18) holds. Similarly (4.15) holds with \( v = v_n \) if

\[
(4.19) \quad w(\{v_n, x\}) \geq w(\{y, v_n\}) \quad \text{and} \quad M = \max_{1 \leq i \leq n-1} w(\{v_i, v_{i+1}\}).
\]

It still remains to find \((x, v, y)\) satisfying (4.15) if

\[
(4.20) \quad w(\{v_1, y\}) \leq w(\{x, v_1\}), \quad w(\{v_n, x\}) \leq w(\{y, v_n\})
\]

and \( M = \max_{1 \leq i \leq n-1} w(\{v_i, v_{i+1}\}) \).

Let us consider the new path \( F = (x, u_1, \ldots, u_n, y) \) such that \( u_1 = v_n, u_2 = v_{n-1}, \ldots, u_n = v_1 \), see Fig. 5. Condition (4.20) implies that

\[
M = \max_{1 \leq i \leq n-1} w(\{u_i, u_{i+1}\}) = \max_{e \in F} w(e)
\]

and, moreover, that \( w(F) \leq w(P) \). Hence it suffices to prove the inequality

\[
\max_{e \in F} (2w(e) - w(F)) \leq |w(\{x, v\}) - w(\{v, y\})|
\]

Figure 5. The new path \( F \) is a modification of the old path \( P \).
for a 2-path \((x, v, y)\) in \(G\). We can make it in a way analogous to that was used under consideration of restriction (4.18) if
\[
w(\{u_1, y\}) \geq w(\{x, u_1\}).
\]

To complete the proof, it suffices to observe that the last inequality can be rewritten as
\[
w(\{v_n, y\}) \geq w(\{x, v_n\})
\]
which follows from (4.20). \qed

Continuation of the proof of Lemma 4.8. It still remains to prove the inequality
\[
\rho_0(x, y) \leq \rho_0(x, z) + \rho_0(z, y)
\]
if \(x, y, z\) are distinct vertices such that
\[
\{x, y\} \notin E(G), \quad \{x, z\} \notin E(G) \quad \text{and} \quad \{z, y\} \notin E(G).
\]
It follows from Lemma 4.14 that
\[
\rho_0(x, y) = \sup_v |w(\{x, v\}) - w(\{v, y\})|
\]
where the supremum is taken over the set of all vertices \(v\) such that
\[
\{x, v\}, \{v, y\} \in E(G).
\]
Condition (ii), relations (4.22) and relations (4.24) give the membership relation
\[
\{z, v\} \in E(G).
\]
Thus the weight function \(w\) is defined at the “point” \(\{z, v\}\). Hence
\[
|w(\{x, v\}) - w(\{v, y\})| \leq |w(\{x, v\}) - w(\{v, z\})| + |w(\{v, z\}) - w(\{v, y\})| \leq \rho_0(x, z) + \rho_0(z, y).
\]
These inequalities and (4.23) imply (4.21). \qed

Recall that a subgraph \(H\) of a graph \(G\) is induced if \(E(H)\) consists of all edges of \(G\) which have both ends in \(V(H)\).

**Proposition 4.25.** A nonnull graph is complete multipartite if and only if it has no induced subgraphs depicted by Figure 6.

**Proof.** Suppose \(G\) is a complete \(k\)-partite graph. If \(k = 1\), then all subgraphs of \(G\) are empty. Let \(k \geq 2\). If \(u\) and \(v\) are vertices of \(G\) such that \(\{u, v\} \notin E(G)\), then there exists a part \(V_1\) in the partition of \(V(G)\) such that
\[
u \in V_1 \quad \text{and} \quad v \in V_1.
\]
If \(p\) is a vertex of \(G\) and \(u \neq p \neq v\), then either \(p \in V_1\) or there is a part \(V_2 \neq V_1\) such that \(p \in V_2\). Using Definition 4.1 we obtain that \(\{p, v\} \notin E(G)\) and \(\{p, u\} \notin E(G)\) if \(p \in V_2\) or, in the opposite case \(p \in V_1\), that \(\{p, v\} \in E(G)\) and \(\{p, u\} \in E(G)\).
Assume now that $G$ has no induced subgraphs depicted by Figure 6. Let us define a relation $\sim$ on the set $V(G)$ as

\[(u \sim v) \iff (\{u, v\} \notin E(G)).\]

Relation $\sim$ is evidently symmetric. Since simple graphs contain no loops, we have $\{u, u\} \notin E(G)$ for each $u \in V(G)$. Consequently $\sim$ is reflexive. Moreover if $\{u, v\} \notin E(G)$ and $\{v, p\} \notin E(G)$, then we obtain $\{u, p\} \notin E(G)$. Thus $\sim$ is transitive, so this is an equivalence relation. The set $V(G)$ is partitioned by the relation $\sim$ on the disjoint parts $V_i$, $i \in I$, where $I$ is an index set. It follows directly from (4.26) that no edge of $G$ has both ends in the same part. Hence $G$ is a $k$-partite graph with $k = \text{card} I$. Finally note that $\{u, v\} \in E(G)$ if and only if the relation $u \sim v$ does not hold. Consequently $G$ is a complete $k$-partite graph.

This proposition implies

**Lemma 4.27.** Let $G$ be a nonempty graph. Condition (ii) of Lemma 4.8 holds if and only if $G$ is a complete $k$-partite graph with $k \geq 2$.

It still remains to prove the next lemma.

**Lemma 4.28.** Let $G$ be a nonempty graph. If condition (i) of Theorem 4.3 holds, then $G$ is connected.

**Proof.** Let $w : E(G) \to \mathbb{R}^+$ be a weight such that the equality

\[(4.29) \quad w(e) = 1\]

holds for all $e \in E(G)$. It is clear that $w$ is metrizable. Let $\{u_1, v_1\}$ be an edge of $G$. If $G$ is disconnected, then there are two connected components...
$G_1$ and $G_2$ of $G$ such that

$$u_1 \in V(G_1), \quad v_1 \in V(G_1) \quad \text{and} \quad V(G_1) \cap V(G_2) = \emptyset.$$  

Let $p$ be a vertex of $G_2$. Using formulas (2.10), (2.11) with zero constants $a_i$ we can find some pseudometrics $\rho_1, \rho_2 \in M_w$ for which

\begin{equation}
(4.30) \quad \rho_1(u_1, p) = 0 \quad \text{and} \quad \rho_2(v_1, p) = 0.
\end{equation}

If condition (i) of Theorem 4.3 holds, then for the least pseudometric $\rho_{0,w}$ in $M_w$ we have the inequalities

$$\rho_{0,w}(u_1, p) \leq \rho_1(u_1, p) \quad \text{and} \quad \rho_{0,w}(v_1, p) \leq \rho_2(v_1, p).$$

These inequalities, the triangle inequality and (4.30) imply

$$\rho_{0,w}(u_1, v_1) \leq \rho_1(u_1, p) + \rho_2(v_1, p) = 0.$$  

Since $\rho_{0,w} \in M_w$, it implies $w(e) = 0$ for $e = \{u_1, v_1\}$, contrary (4.29).  

\begin{proof}[Proof of Theorem 4.3] Suppose that condition (i) of the theorem holds. By Lemma 4.28, $G$ is a connected graph and so we can use Lemma 4.8. Applying this lemma we obtain the equivalence of its condition (ii) with condition (i) of Theorem 4.3. By Lemma 4.27 condition (ii) of Lemma 4.8 implies condition (ii) of the Theorem 4.3.

Conversely, suppose that condition (ii) of Theorem 4.3 holds. Using Lemma 4.27 we see that condition (ii) of Lemma 4.8 holds. Moreover condition (ii) of Theorem 4.3 implies that $G$ is connected, see Remark 4.2. Hence by Lemma 4.8 we obtain condition (i) of the theorem. Thus we have the implication (ii) $\Rightarrow$ (i) in Theorem 4.3.

Assume now that $G$ is complete $k$-partite graph with $k \geq 2$ and $w$ is metrizable. Let $u$ and $v$ be some distinct nonadjacent vertices of $G$. Then we have $u, v \in X_{\alpha_0}$ for some $\alpha_0 \in I$. We must to prove equalities (4.5) and (4.6). For every vertex $p \notin X_{\alpha_0}$ the sequence $(u, p, v)$ is a path joining $u$ and $v$. Consequently the inequality

\begin{equation}
(4.31) \quad d_w(u, v) \leq \inf_{\alpha \neq \alpha_0, p \in X_\alpha} \inf_{\alpha \in I} |w(\{u, p\}) + w(\{p, v\})|
\end{equation}

follows from (2.1). To prove the converse inequality it is sufficient to show that for every $F \in P_{u,v}$ there is $p \in X_\alpha$, $\alpha \neq \alpha_0$, such that

\begin{equation}
(4.32) \quad w(F) \geq w(\{u, v\}) + w(\{p, v\}).
\end{equation}

Since $u$ and $v$ are nonadjacent, the length (the number of edges) of $F$ is more or equal 2 for every $F \in P_{u,v}$. If the length of $F$ is 2, then the ”inner” vertex of $F$ does not belong to $X_{\alpha_0}$ so we have (4.31). Let $(u = v_0, v_1, \ldots, v_n = v)$ belong to $P_{u,v}$ and $n \geq 3$. If $v_1 \in X_{\alpha_0}$, then $\alpha \neq \alpha_0$ and $\{u, v\} \in E(G)$ because $G$ is a complete $k$-partite graph. Since $w$ is metrizable, statement (ii) of Theorem 2.2 implies

\begin{equation}
(4.33) \quad w(\{v_1, v\}) \leq w(F')
\end{equation}
where \( F' \) is the part \((v_1, \dotsc, v_n)\). It is clear that
\[
w(F) = w(\{u, v_1\}) + w(F').
\]
Consequently (4.33) implies (4.32) with \( p = v_2 \). Equality (4.5) follows.

To prove (4.6) we now return to lemmas 4.8, 4.14, 4.27. By the assumption \( G \) is a complete \( k \)-partite graph. Hence, by Lemma 4.27, condition (ii) of Lemma 4.8 holds. This condition implies that
\[
(4.34) \quad \rho_{0,w}(u, v) = \sup_{F \in \mathcal{P}_{u,v}} \max_{e \in F} (2w(e) - w(F))_+ + \text{see (4.9) and (4.10).}
\]
Using Lemma 4.14 we obtain that for every \( F \in \mathcal{P}_{u,v} \) there is \( p \in V(G) \) with \( \{u, p\}, \{p, v\} \in E(G) \) and such that
\[
\max_{e \in F} (2w(e) - w(F))_+ \leq |w(\{u, p\}) - w(\{p, v\})|.
\]
Consequently we have
\[
\rho_{0,w}(u, v) \leq \sup_{\alpha \neq \alpha_0} \sup_{p \in X_\alpha} \sup_{a \in I} |w(\{u, p\}) - w(\{p, v\})|
\]
for every two distinct nonadjacent vertices \( u, v \). The converse inequality follows from (4.34). Indeed, for every path \( F \) of the form \((u, p, v)\) we have
\[
|w(\{u, p\}) - w(\{p, v\})| = \max_{e \in F} (2w(e) - w(F))_+.
\]

Recall that the star is a complete bipartite graph \( G \) with a bipartition \((X, Y)\),
\[
V(G) = X \cup Y, \quad X \cap Y = \emptyset
\]
such that \( \text{card } X = 1 \) or \( \text{card } Y = 1 \).

**Corollary 4.35.** The following conditions are equivalent for each nonempty graph \( G \).

(i) Every weight \( w : E(G) \to \mathbb{R}^+ \) is metrizable and the poset \((\mathcal{M}_w, \leq)\) contains the least pseudometric \( \rho_{0,w} \).

(ii) \( G \) is a star.

**Proof.** Let condition (i) hold. Then, by Theorem 4.3, \( G \) is complete \( k \)-partite with \( k \geq 2 \) and by Corollary 2.17 \( G \) is acyclic. Each \( k \)-partite graph with \( k \geq 3 \) contains a 3-cycle (triangle). Hence we have \( k = 2 \), i.e. \( G \) is bipartite. If \( (X, Y) \) is a bipartition of \( G \) with
\[
\text{card } X \geq 2 \quad \text{and} \quad \text{card } Y \geq 2,
\]
then we can find some vertices
\[
x_1, x_3 \in X \quad \text{and} \quad x_2, x_4 \in Y.
\]
Since \( G \) is a complete bipartite graph, \( G \) contains the quadrilateral \( Q \), see Fig. 1. Consequently we have \( \text{card } X = 1 \) or \( \text{card } Y = 1 \). Thus \( G \) is a star.

Conversely suppose \( G \) is a star. Then \( G \) is acyclic, so using Corollary 2.17 we obtain that every weight \( w \) is metrizable. Since stars are complete
bipartite graphs, Theorem 4.3 implies the existence of the least pseudometric $\rho_{0,w} \in \mathcal{M}_w$.

**Theorem 4.36.** The following conditions (i) and (ii) are equivalent for each nonempty graph $G$.

(i) For each metrizable weight $w : E(G) \to \mathbb{R}^+$ the set $\mathcal{M}_w$ contains the least pseudometric $\rho_{0,w}$ and this set contains also all symmetric functions $f : V(G) \times V(G) \to \mathbb{R}^+$ which lie between $\rho_{0,w}$ and the shortest-path pseudometric $d_w$, i.e., which satisfy the double inequality

\[
(4.37) \quad \rho_{0,w}(u, v) \leq f(u, v) \leq d_w(u, v)
\]

for all $u, v \in V(G)$.

(ii) $G$ is a complete $k$-partite graph with a partition $\{X_\alpha : \alpha \in \mathcal{I}\}$ such that $\text{card } \mathcal{I} = k \geq 2$ and $\text{card } X_\alpha \leq 2$ for each part $X_\alpha$.

**Proof.** (i)$\Rightarrow$(ii) Suppose that (i) holds. By Theorem 4.3 $G$ is a complete $k$-partite graph with $k \geq 2$. Assume that there is a part $X_{a_0}$ such that $\text{card}(X_{a_0}) \geq 3$. Let $v_1, v_2, v_3$ be some pairwise distinct elements of $X_{a_0}$ and let $w$ be the weight such that $w(e) = 1$ for each $e \in E(G)$. Define functions $\rho_1$ and $\rho_2$ on the set $V(G) \times V(G)$ as

\[
\rho_1(u, v) = \begin{cases} 
0 & \text{if } u = v \\
1 & \text{if } u \neq v
\end{cases} \quad \text{and} \quad \rho_2(u, v) = \begin{cases} 
1 & \text{if } \{u, v\} \in E(G) \\
0 & \text{if } \{u, v\} \notin E(G)
\end{cases}.
\]

It is clear that $\rho_1 \in \mathcal{M}_w$. To prove that $\rho_2 \in \mathcal{M}_w$ it is sufficient to verify the triangle inequality

\[
(4.38) \quad \rho_2(u, v) \leq \rho_2(u, s) + \rho_2(v, s)
\]

for all $u, v, s \in V(G)$. If (4.38) does not hold, then $\rho_2(u, v) = 1$ and $\rho_2(u, s) = \rho_2(v, s) = 0$. Consequently we have that

\[
(4.39) \quad \{u, v\} \in E(G) \text{ and } \{u, s\} \notin E(G) \text{ and } \{v, s\} \notin E(G).
\]

The relation $\{u, s\} \notin E(G)$ and $\{v, s\} \notin E(G)$ imply that $u, s$ belong to a part $X_u$, similarly $v, s$ belong to a part $X_v$. Since $s \in X_u \cap X_v$ we obtain that $X_u = X_v$, hence $\{u, v\} \notin E(G)$ contrary to the first membership relation in (4.39). Thus (4.38) holds for all $u, v, s \in V(G)$, so $\rho_2 \in \mathcal{M}_w$. The function $f : V(G) \times V(G) \to \mathbb{R}^+$ defined as

\[
f(v_1, v_2) = f(v_2, v_1) = 1, \quad f(v_1, v_3) = f(v_3, v_1) = f(v_3, v_2) = f(v_2, v_3) = 0
\]

and

\[
f(u, v) = \rho_1(u, v)
\]

for $(u, v) \in (V(G) \times V(G)) \setminus \{(v_1, v_2), (v_2, v_1), (v_2, v_3), (v_3, v_2), (v_1, v_3), (v_3, v_2)\}$ satisfies the double inequality

\[
\rho_2(u, v) \leq f(u, v) \leq \rho_1(u, v)
\]
that implies (4.37). Hence by (i) we must have
\[ f(v_1, v_2) \leq f(v_1, v_3) + f(v_3, v_2) \]
that contradicts the definition of the function \( f \). Thus the inequality \( \text{card } X_\alpha \leq 2 \) holds for each part \( X_\alpha \).

\[ G \]

Figure 7. A complete 3-partite graph \( G \) satisfies condition (ii) of Theorem 4.36.

\[(ii) \Rightarrow (i)\]

Suppose that condition (ii) holds. Since \( k \geq 2 \) and \( G \) is a complete \( k \)-partite graph, Theorem 4.3 provides the existence of the least pseudometric \( \rho_{0,w} \) for each metrizable weight \( w \). Let \( f : V(G) \times V(G) \to \mathbb{R}^+ \) be a symmetric function such that (4.37) holds for all \( u, v \in V(G) \). The double inequality (4.37) implies that \( f \) is nonnegative and \( f(u, v) = w(\{u, v\}) \) for all \( \{u, v\} \in E(V) \) and \( f(u, u) = 0 \) for all \( u \in V(G) \). Consequently to prove that \( f \in M_w \) it is sufficient to obtain the triangle inequality
\[
(4.40) \quad f(u, v) \leq f(u, p) + f(p, v)
\]
for all \( u, v, p \in V(G) \). We may assume \( u, v \) and \( p \) are pairwise disjoint, otherwise (4.40) is trivial. Since \( \text{card } X_\alpha \leq 2 \) for each part \( X_\alpha \), at most one pair from the vertices \( u, v \) and \( p \) are nonadjacent. If \( \{u, v\} \notin E(G) \), then using (4.37) we obtain
\[
f(u, v) \leq d_w(u, v) \leq d_w(u, p) + d_w(p, v) = w(\{u, p\}) + w(\{p, v\}) \leq f(u, p) + f(p, v).\]
Similarly if \( \{u, p\} \notin E(G) \) or \( \{p, v\} \notin E(G) \), then we have
\[
f(u, v) \leq \rho_{0,w}(u, v) \leq \rho_{0,w}(u, p) + \rho_{0,w}(p, v) = f(u, p) + f(p, v).\]
Inequality (4.40) follows and we obtain condition (i). \( \Box \)

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