Generating many Majorana corner modes and multiple phase transitions in Floquet second-order topological superconductors

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Abstract

A $d$-dimensional, $n$th-order topological insulator or superconductor has localized eigenmodes at its $(d - n)$-dimensional boundaries ($n \leq d$). In this work, we apply periodic driving fields to two-dimensional superconductors, and obtain a wide variety of Floquet second-order topological superconducting (SOTSC) phases with many Majorana corner modes at both zero and $\pi$ quasienergies. Two distinct Floquet SOTSC phases are found to be separated by three possible kinds of transformations, i.e., a topological phase transition due to the closing/reopening of a bulk spectral gap, a topological phase transition due to the closing/reopening of an edge spectral gap, or an entirely different phase in which the bulk spectrum is gapless. Thanks to the strong interplay between driving and intrinsic energy scales of the system, all the found phases and transitions are highly controllable via tuning a single hopping parameter of the system. Our discovery not only enriches the possible forms of Floquet SOTSC phases, but also offers an efficient scheme to generate many coexisting Majorana zero and $\pi$ corner modes that may find applications in Floquet quantum computation.

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Floquet topological phases appear in time-periodic driven systems with nontrivial topological properties (see Refs. [1-4] for reviews). Their characteristic features include symmetry classifications unique to nonequilibrium states [5-8], anomalous bulk-boundary correspondence with no static analogy [9-12], and substantial numbers of edge states induced by driving fields [13-17]. The observation of Floquet topological matter in various physical settings including solids [18-21], cold atoms [22-25] and optical systems [26-29] further promotes their applications in ultrafast electronics [30] and topological quantum computing [31].

In recent years, the approach of Floquet engineering has been applied to generate and control higher-order topological phases [32-40]. A gapped topological phase of order $n$ in $d$-spatial dimensions ($d \geq n \geq 1$) usually holds topological edge states along its $(d - n)$-dimensional boundaries (see Refs. [41-44] for reviews). For example, a two-dimensional (2D), second-order topological insulator has localized eigenstates at the corners of the lattice on which it is defined. The conventional topological insulator can thus be viewed as a first-order topological phase. Besides insulating [45-55] and semi-metallic [57-61] setups, periodic driving fields have also been applied to engineer Floquet SOTSC phases [72-80]. The latter could possess two types of symmetry-protected Majorana corner states with zero and $\pi$ quasienergies, which might be adopted in topological quantum computation as substitutions of the more conventional Majorana edge modes. Till now, the studies of Floquet second-order topological superconductors focus on either models with unconventional $d$-wave pairings and up to eight Floquet bands [78-80], or applications in Floquet quantum computation utilizing at most four Majorana corner modes at both zero and $\pi$ quasienergies [73, 74]. It remains unclear whether we could obtain as many as possible the Majorana zero and $\pi$ corner modes together in Floquet SOTSC phases following simple driving protocols, similar to what has been achieved for Floquet second-order topological insulators [45]. Moreover, phase transitions in higher-order topological matter could happen by closing either a gap between the bulk bands (type-I) or a gap between the edge bands (type-II) [81-87]. Following which type of topological phase transition could Floquet Majorana corner modes emerge deserves to be further clarified.

In this work, we couple a $p + ip$ superconductor in two dimensions to time-periodic driving fields, and obtain rich Floquet SOTSC phases with many normal and anomalous Majorana corner modes at zero and $\pi$ quasienergies, respectively. In Sec. II, we introduce our model
FIG. 1. An illustration of the 2D lattice model. The chains A (blue balls) and B (orange balls) are related to the operators $a^\dagger(t)$ and $b^\dagger(t)$ in $\tilde{H}$. A uniform chemical potential $\mu$ is applied to each site. In the Floquet model, the superconducting pairing amplitude $\Delta$ is replaced by $\Delta(t) = \sum_\ell \delta(t-\ell)$. and its driving protocol, discuss the symmetries of the model, and explain how to find its spectrum and eigenmodes under different boundary conditions. In Sec. II we explore the topological phases and phase transitions in our system with gradually increased generality. Floquet SOTSC phases with different numbers of Majorana zero and $\pi$ corner modes are obtained, and they are found to be separated by the closing/reopening of bulk spectral gaps, edge band gaps or even emerging new phases with gapless Floquet spectra. In Sec. III we summarize our results and discuss potential future directions. Throughout this work, we set the Floquet driving period $T = 1$ and the Planck constant $\hbar = 1$. Other system parameters have been properly scaled and set in dimensionless units.

I. MODEL

We start with a lattice model that describes a 2D $p_x + i p_y$ superconductor. It may be viewed as a simplified version of the model introduced in the Eq. (1) of Ref. [73]. A schematic diagram of our model is shown in Fig. 1, whose static Hamiltonian takes the form
\[ \hat{H} = \hat{H}_x + \hat{H}_y, \]  

where

\[ \hat{H}_x = \frac{1}{2} \sum_{m,n} \left[ \mu \left( \hat{a}_{m,n}^\dagger \hat{a}_{m,n} - \frac{1}{2} \right) + J \hat{a}_{m,n}^\dagger \hat{a}_{m+1,n} + \Delta \hat{a}_{m,n} \hat{a}_{m+1,n} \right] + \text{H.c.}, \]  

and

\[ \hat{H}_y = \sum_{m,n} J' \left( \hat{b}_{m,n}^\dagger \hat{b}_{m,n} + \hat{b}_{m,n}^\dagger \hat{b}_{m,n+1} \right) - \sum_{m,n} i \left( \Delta_1 \hat{a}_{m,n} \hat{b}_{m,n} + \Delta_2 \hat{b}_{m,n} \hat{a}_{m,n+1} \right) + \text{H.c..} \]  

Here \( \hat{a}_{m,n}^\dagger (\hat{a}_{m,n}) \) and \( \hat{b}_{m,n}^\dagger (\hat{b}_{m,n}) \) are the creation (annihilation) operators of fermions on the sublattices A and B of a unit cell at the location \( r = (m, n) \) of the 2D lattice. The sublattice structure is originated from the dimerized superconducting pairing amplitudes \( (\Delta_1 \neq \Delta_2) \) along the \( y \)-direction of the lattice. \( J \) and \( J' \) denote the hopping amplitudes along the \( x \)- and \( y \)-directions. \( \mu \) is the chemical potential and \( \Delta \) is the pairing amplitude along the \( x \)-direction. By solving the eigenvalue equation \( \hat{H} |\psi\rangle = E |\psi\rangle \), one can obtain the spectrum and Majorana corner states of the static system, with typical examples shown in Fig. 2. We observe that there are two possible phases, with one of them being topologically trivial and the other one possessing four zero-energy Majorana eigenmodes at the four corners of the lattice. For the parameter domains considered in Fig. 2 these Majorana corner modes emerge once \( |J| > |\mu| \). Similar results are observed in the previous study of a slightly more complicated model [73], where the hopping amplitude \( J' \) and chemical potential \( \mu \) are also dimerized in space. In Ref. [73], it was shown that with a harmonic (sinusoidal) driving field added to the chemical potential \( \mu \), the system could become a Floquet second-order topological superconductor with four Majorana corner modes at both the zero and \( \pi \) quasienergies. Note that the model used in Ref. [73] further assumes the lattice dimerization in the hopping amplitude \( J' \) and chemical potential \( \mu \). In our model above, the hopping
FIG. 2. Typical spectra and Majorana corner modes of the static SOTSC under open boundary conditions along $x$ and $y$ directions. (a) and (b) show the energy spectra $E$ of $\hat{H}$ versus $J$ and $\Delta$, with other system parameters chosen to be $(\mu, J', \Delta_1, \Delta_2) = (0.25\pi, 0.05\pi, 0.2\pi, 0.4\pi)$. (c) and (d) show the probability distributions of the four Majorana corner modes with $E = 0$ in (a) and (b) at $J = 1.5\pi$ and $\Delta = 1.5\pi$, respectively. The lattice sizes are set as $N_x = N_y = 60$ for all panels.

and chemical potential terms are instead uniform along the two spatial dimensions. Besides, the Floquet driving field in Ref. [73] was added to the dimerized chemical potential there. In our case, the driving field will be coupled to the superconducting pairing amplitude as discussed below. The possibility of generating more Floquet Majorana zero/\pi corner modes and inducing multiple topological transitions by periodic driving fields were also not considered in Ref. [73], as the sinusoidal modulation applied there could not achieve such goals.

In this work, we consider a different driving protocol by applying time-periodic kicks to the superconducting pairing amplitude $\Delta$ along the $x$-direction, i.e., by setting $\Delta \rightarrow \Delta(t)$ in Eq. (2) with $\Delta(t) = \Delta \sum_{\ell \in \mathbb{Z}} \delta(t - \ell)$. In one-dimensional Floquet topological superconductors, this type of driving and its physical relevance has been considered in [88]. The Floquet operator of the system, which describes its evolution over a complete driving period (e.g., from $t = 0^- \rightarrow 1^-$) then takes the form

$$ \hat{U} = \hat{U}_2 \hat{U}_1, \quad (4) $$
The Floquet spectrum and eigenstates of \( \hat{U} \) can be obtained from the eigenvalue equation 
\( \hat{U}|\psi\rangle = e^{-iE|\psi\rangle} \), whose solution \( |\psi\rangle \) describes a Floquet state with the quasienergy \( E \in [-\pi, \pi) \) (defined modulus 2\( \pi \)). Under open boundary conditions (OBCs) along both the \( x \) - and \( y \) -directions, we refer to a corner-localized eigenmode of \( \hat{U} \) with the quasienergy \( E = 0 \) \( (E = \pm \pi) \) as a Floquet Majorana zero (\( \pi \)) corner mode. In Sec. [II] we will show that many such Floquet Majorana corner modes could emerge in our system. Moreover, their numbers can be tuned by changing the system parameter across different types of topological phase transitions.

Under periodic boundary conditions (PBCs) along both the \( x \) - and \( y \) -directions, we can apply Fourier transformations to the creation and annihilation operators, i.e.,

\[
\hat{a}_r = \frac{1}{\sqrt{S}} \sum_k e^{i\mathbf{k} \cdot \mathbf{r}} \hat{a}_k, \quad \hat{a}^\dagger_r = \frac{1}{\sqrt{S}} \sum_k e^{-i\mathbf{k} \cdot \mathbf{r}} \hat{a}^\dagger_k, \\
\hat{b}_r = \frac{1}{\sqrt{S}} \sum_k e^{i\mathbf{k} \cdot \mathbf{r}} \hat{b}_k, \quad \hat{b}^\dagger_r = \frac{1}{\sqrt{S}} \sum_k e^{-i\mathbf{k} \cdot \mathbf{r}} \hat{b}^\dagger_k.
\]  

(7)

Here \( S = N_x N_y \) denotes the total number of unit cells, with \( N_x \) (\( N_y \)) being the number of cells along the \( x \) (\( y \)) direction of the lattice. The unit cell index is \( \mathbf{r} = (m, n) \), where \( m = 1, 2, ..., N_x - 1, N_x \) and \( n = 1, 2, ..., N_y - 1, N_y \). We thus identify \( \hat{a}_r = \hat{a}_{m,n} \) and \( \hat{b}_r = \hat{b}_{m,n} \) for the second-quantized operators. The 2D quasimomentum is \( \mathbf{k} = (k_x, k_y) \), with \( k_x = -\pi, -\pi + \frac{2\pi}{N_x}, ..., -\pi + \frac{2\pi (N_x - 2)}{N_x}, -\pi + \frac{2\pi (N_x - 1)}{N_x} \) \( (k_y = -\pi, -\pi + \frac{2\pi (N_y - 1)}{N_y}, ..., -\pi + \frac{2\pi (N_y - 2)}{N_y}, -\pi + \frac{2\pi (N_y - 1)}{N_y}) \) being the quasimomentum along the \( x \) (\( y \)) direction. Therefore we also identify \( \hat{a}_k = \hat{a}_{k_x, k_y} \) and \( \hat{b}_k = \hat{b}_{k_x, k_y} \). After the Fourier transformation, we find the components \( \hat{H}_x(t) \) and \( \hat{H}_y \) of the driven Hamiltonian \( \hat{H}(t) = \hat{H}_x(t) + \hat{H}_y \) as

\[
\hat{H}_x(t) = \frac{1}{2} \sum_k \hat{\Psi}^\dagger_k H_x(k, t) \hat{\Psi}_k, \quad \hat{H}_y = \frac{1}{2} \sum_k \hat{\Psi}^\dagger_k H_y(k) \hat{\Psi}_k.
\]  

(9)
Here $\hat{\Psi}_k^\dagger = (\hat{a}_k^\dagger, \hat{a}_{-k}^\dagger, \hat{b}_k^\dagger, \hat{b}_{-k}^\dagger)$ is the creation operator in the Nambu basis. The Hamiltonian matrices $H_x(k, t)$ and $H_y(k)$ in the Nambu spinor representation are given by

\begin{align}
H_x(k, t) &= \sigma_0 \otimes [\Delta(t) \sin k_x \sigma_y + (\mu + J \cos k_x) \sigma_z], \\
H_y(k) &= J' [(1 + \cos k_y) \sigma_x + \sin k_y \sigma_y] \otimes \sigma_z + [\Delta_2 \sin k_y \sigma_x + (\Delta_1 - \Delta_2 \cos k_y) \sigma_y] \otimes \sigma_x,
\end{align}

where $\sigma_{x,y,z}$ are Pauli matrices and $\sigma_0$ is the two by two identity matrix. In the momentum space, the Floquet operator now takes the form

\begin{equation}
U(k) = U_2(k)U_1(k) = e^{-iH_2(k)}e^{-iH_1(k)},
\end{equation}

where

\begin{align}
H_1(k) &= \Delta(t) \sin k_x \sigma_0 \otimes \sigma_y, \\
H_2(k) &= (\mu + J \cos k_x) \sigma_0 \otimes \sigma_z + J' [(1 + \cos k_y) \sigma_x + \sin k_y \sigma_y] \otimes \sigma_z \\
&\quad + [\Delta_2 \sin k_y \sigma_x + (\Delta_1 - \Delta_2 \cos k_y) \sigma_y] \otimes \sigma_x.
\end{align}

The Floquet spectrum and eigenstates of the system in $k$-space are then obtained by solving the $4 \times 4$ eigenvalue equation $U(k)|\psi_j(k)\rangle = e^{-iE_j(k)}|\psi_j(k)\rangle$, yielding four quasienergy bands $E_j(k)$ for $j = 1, 2, 3, 4$. When there exists $j$ and $l$ such that $E_j(k) = E_l(k) = 0$ or $E_j(k) = E_l(k) = \pm \pi$, the bulk Floquet spectral gap may close at the center or boundary of the first quasienergy Brillouin zone $E \in [-\pi, \pi)$. In Sec. II we will explore the Floquet spectra and gap closing conditions of our system for cases with gradually increased complexity. We will further relate the gap closing/reopening transitions of the bulk Floquet spectra with topological phase transitions accompanied by the adjustment of Floquet corner modes under OBCs.

The topological properties of the system is closely related to the symmetries of $U(k)$. Upon unitary transformations, we can express the Floquet operator in a pair of symmetric time frames \cite{89,92} as

\begin{equation}
U_1(k) = U_1^{1/2}(k)U_2(k)U_1^{1/2}(k), \quad U_2(k) = U_2^{1/2}(k)U_1(k)U_2^{1/2}(k).
\end{equation}
When the hopping amplitude $J' = 0$, it is not hard to verify that both $U_1(k)$ and $U_2(k)$ possess the chiral symmetry $\Gamma = \sigma_z \otimes \sigma_x$ in the sense that $\Gamma U_\alpha(k) \Gamma = U_\alpha^{-1}(k)$ for $\alpha = 1, 2$. Since unitary transformations do not change the quasienergy spectrum, the chiral symmetry $\Gamma$ enforces the degeneracy of the eigenmodes of $U(k)$ when their quasienergies are equal to zero or $\pm \pi$ under the condition $J' = 0$. This will be the situation in the first three cases explored in Sec. I. When $J' \neq 0$, the chiral symmetry $\Gamma$ is broken. But $U_1(k)$ and $U_2(k)$ still possess the particle-hole symmetry $C = \sigma_0 \otimes \sigma_x K$, enforcing the Floquet spectrum of $U(k)$ to be symmetric with respect to the quasienergy $E = 0$. With the particle-hole symmetry only, we may obtain at most four Floquet Majorana corner modes at the quasienergies zero and $\pm \pi$ when a rectangle geometry and the OBCs along both $x$ and $y$ directions are taken for the system. This will be the case encountered in Sec. IID.

In higher-order topological phases, the change of topological corner modes can be induced not only by closing/reopening a bulk spectrum gap, but also by closing/reopening a gap between different edge bands. The former (latter) is usually called a type-I (type-II) topological phase transition [81–87]. To unveil the possibility of generating these two types of phase transitions in Floquet systems, we will also consider the solution of the system under the PBC (OBC) along $x$ ($y$) direction of the lattice, or vice versa. In the former case, we can apply the Fourier transformation only along the $x$-direction, i.e.,

\[
\hat{a}_{m,n} = \frac{1}{\sqrt{N_x}} \sum_{k_x} e^{i k_x m} \hat{a}_{k_x,n}, \quad \hat{a}^\dagger_{m,n} = \frac{1}{\sqrt{N_x}} \sum_{k_x} e^{-i k_x m} \hat{a}^\dagger_{k_x,n}, \quad \tag{16}
\]

\[
\hat{b}_{m,n} = \frac{1}{\sqrt{N_x}} \sum_{k_x} e^{i k_x m} \hat{b}_{k_x,n}, \quad \hat{b}^\dagger_{m,n} = \frac{1}{\sqrt{N_x}} \sum_{k_x} e^{-i k_x m} \hat{b}^\dagger_{k_x,n}. \quad \tag{17}
\]

The Floquet operator under the PBC (OBC) along $x$ ($y$) direction then takes the form

\[
\hat{U}(k_x) = \hat{U}_2(k_x) \hat{U}_1(k_x), \quad \tag{18}
\]

where

\[
\hat{U}_1 = e^{-\frac{i}{2} \sum_{m,n} \Delta (\hat{a}_{k_x,n} \hat{a}_{k_x,n}^\dagger e^{-ik_x} + \hat{b}_{k_x,n} \hat{b}_{k_x,n}^\dagger e^{-ik_x} + \text{H.c.})}, \quad \tag{19}
\]

\[
\hat{U}_2 = e^{-i \hat{H}_y(k_x) - \frac{i}{2} \sum_{m,n} \left[ \mu (\hat{a}_{k_x,n}^\dagger \hat{a}_{k_x,n} + \hat{b}_{k_x,n}^\dagger \hat{b}_{k_x,n} - 1) + Je^{ik_x} (\hat{a}_{k_x,n}^\dagger \hat{a}_{k_x,n} + \hat{b}_{k_x,n}^\dagger \hat{b}_{k_x,n}) + \text{H.c.} \right]}. \quad \tag{20}
\]

and
\[ \hat{H}_y(k_x) = \sum_{k_x, n} \left[ J^\prime \left( \hat{a}^{\dagger}_{k_x, n} \hat{b}_{k_x, n} + \hat{b}^{\dagger}_{k_x, n} \hat{a}_{k_x, n+1} \right) - i \left( \Delta_1 \hat{a}_{k_x, n} \hat{b}_{-k_x, n} + \Delta_2 \hat{b}_{k_x, n} \hat{a}_{-k_x, n+1} \right) \right] + \text{H.c.} \]  

(21)

The Floquet spectrum and eigenstates are then obtained by diagonalizing \( \hat{U}(k_x) \) at all different quasimomenta \( k_x \in [-\pi, \pi] \). Similarly, under the PBC (OBC) along \( y \) (\( x \)) direction of the lattice, we can apply the Fourier transformation along the \( y \)-direction, i.e.,

\[
\hat{a}_{m, n} = \frac{1}{\sqrt{N_y}} \sum_{k_y} e^{ik_y n} \hat{a}_{m, k_y}, \quad \hat{a}^{\dagger}_{m, n} = \frac{1}{\sqrt{N_y}} \sum_{k_y} e^{-ik_y n} \hat{a}^{\dagger}_{m, k_y}, \quad (22)
\]

\[
\hat{b}_{m, n} = \frac{1}{\sqrt{N_y}} \sum_{k_y} e^{ik_y n} \hat{b}_{m, k_y}, \quad \hat{b}^{\dagger}_{m, n} = \frac{1}{\sqrt{N_y}} \sum_{k_y} e^{-ik_y n} \hat{b}^{\dagger}_{m, k_y}, \quad (23)
\]

The resulting Floquet operator takes the form

\[
\hat{U}(k_y) = \hat{U}_2(k_y) \hat{U}_1(k_y), \quad (24)
\]

where

\[
\hat{U}_1 = e^{-\frac{i}{2} \sum_{m, k_y} \Delta \left( \hat{a}_{m, k_y} \hat{a}_{m+1, -k_y} + \hat{b}_{m, k_y} \hat{b}_{m+1, -k_y} + \text{H.c.} \right)}, \quad (25)
\]

\[
\hat{U}_2 = e^{-i \hat{H}_y(k_y) - \frac{i}{2} \sum_{m, k_y} \left( \mu \left( \hat{a}^{\dagger}_{m, k_y} \hat{a}_{m, k_y} + \hat{b}^{\dagger}_{m, k_y} \hat{b}_{m, k_y} - 1 \right) - J \left( \hat{a}^{\dagger}_{m, k_y} \hat{a}_{m+1, k_y} + \hat{b}^{\dagger}_{m, k_y} \hat{b}_{m+1, k_y} \right) \right) + \text{H.c.}} \quad (26)
\]

and

\[
\hat{H}_y(k_y) = \sum_{m, k_y} J^\prime \left( \hat{a}^{\dagger}_{m, k_y} \hat{b}_{m, k_y} + \hat{b}^{\dagger}_{m, k_y} \hat{a}_{m, k_y} e^{ik_y} \right) \quad (27)
\]

\[
-\sum_{m, k_y} i \left( \Delta_1 \hat{a}_{m, k_y} \hat{b}_{m, -k_y} + \Delta_2 \hat{b}_{m, k_y} \hat{a}_{m, -k_y} e^{-ik_y} \right) + \text{H.c.}
\]

The Floquet spectrum and eigenstates can now be found by diagonalizing \( \hat{U}(k_y) \) at all different quasimomenta \( k_y \in [-\pi, \pi] \). The spectrum of \( \hat{U}(k_x) \) and \( \hat{U}(k_y) \) could provide useful information for us to understand different types of phase transitions in Floquet second-order topological superconductors, as will be shown in the following section.
II. RESULTS

In this section, we study the emerging Floquet SOTSC phases in our system with gradually increased complexity. We start with a minimal model which yet possesses rich Floquet SOTSC phases, many Majorana zero/π corner modes and different classes of topological phase transitions in Sec. IIA. How will these intriguing states and transitions be modified by the presence of finite chemical potential \( \mu \) and intracell superconducting pairing \( \Delta_1 \) are further explored in Secs. IIB and IIC. The most general case with nonvanishing hopping amplitude \( J' \) is finally discussed in Sec. IID.

A. Case 1: \( \mu = J' = \Delta_1 = 0 \)

We start with a simplest construction that allows us to have Floquet SOTSC phases with Majorana corner modes in our system. Assuming the chemical potential \( \mu \), hopping amplitude \( J' \) and intracell pairing amplitude \( \Delta_1 \) along the \( y \)-direction to be zero, we arrive at following Floquet operator in momentum space from Eqs. (12)–(14), i.e.,

\[
U(k) = e^{-i[J \cos k_x \sigma_0 \otimes \sigma_z + \Delta_2 (\sin k_y \sigma_x - \cos k_y \sigma_y) \otimes \sigma_z]} e^{-i \Delta \sin k_x \sigma_0 \otimes \sigma_y}.
\] (28)

It is clear that the tensor product matrices \( \sigma_0 \otimes \sigma_z, \sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_x \) and \( \sigma_0 \otimes \sigma_y \) are anti-commute with one another. We can thus apply the Taylor expansion to each exponential term in Eq. (28). After recombining relevant terms in the expansion, we find the quasienergy band dispersions

\[
E_\pm(k) = \pm \arccos \left[ \cos(\Delta \sin k_x) \cos \left( \sqrt{J^2 \cos^2 k_x + \Delta_2^2} \right) \right].
\] (29)

This gives us two quasienergy bands, with each of them being twofold degenerate. Moreover, the dispersions \( E_\pm(k) \) are independent of \( k_y \), which means that all the Floquet bands of our system in Case 1 are flat along \( k_y \). Since the two pairs of bands are symmetric with respect to the zero quasienergy, they could touch with each other either at \( E_\pm(k) = 0 \) (center of the quasienergy Brillouin zone) or at \( E_\pm(k) = \pm \pi \) (edge of the quasienergy Brillouin zone). The Floquet spectrum could then become gapless if
\[
\cos(\Delta \sin k_x) \cos \left( \sqrt{J^2 \cos^2 k_x + \Delta^2} \right) = \pm 1, 
\]
whose solution yields the phase boundary equation in parameter space

\[
\frac{p^2 \pi^2}{\Delta^2} + \frac{q^2 \pi^2 - \Delta^2}{J^2} = 1, \quad p, q \in \mathbb{Z}. 
\]

That is, when the system parameters satisfy this equation, the bulk quasienergy spectrum becomes gapless at either \( E = 0 \) or \( E = \pi \). To investigate gap-closing transitions induced by the change of system parameters, we introduce the following quasienergy gap functions

\[
F_0 \equiv \frac{1}{\pi} \min_{k \in \text{BZ}} |E_{\pm}(k)|, \quad F_\pi \equiv \frac{1}{\pi} \min_{k \in \text{BZ}} ||E_{\pm}(k)| - \pi|. 
\]

Here the \( 1/\pi \) in front is a scaling factor that restricts the ranges of both \( F_0 \) and \( F_\pi \) to \([0, 1]\).

It is clear that \( F_0 = 0 \) (\( F_\pi = 0 \)) once the spectrum gap closes at the quasienergy zero (\( \pi \)). In Fig. 3(a), we show the gap functions \( F_0 \) and \( F_\pi \) versus the hopping amplitude \( J \) for a typical set of system parameters in Case 1. For our choice of system parameters, we have \( p = 0 \), and according to Eq. (31) the phase transition points appear at \( J = \sqrt{q^2 \pi^2 - \Delta^2} \) for \( q \in \mathbb{Z} \) assuming \( q^2 \pi^2 \geq \Delta^2 \). They are coincide with the locations where \( F_0 \) or \( F_\pi \) vanish in Fig. 3(a). Moreover, we observe a series of gap closing transitions with the increase of \( J \), and more such transitions are expected to happen at larger values of \( J \), which are yet absent in the static limit of the system. Therefore, the periodic driving allows us create multiple gap-closing/reopening transitions in the quasienergy spectrum of the Floquet system.

In Figs. 3(b) and 3(c), we report the Floquet spectrum of the system at different hopping amplitudes \( J \) under the PBCs along both \( x, y \) directions (denoted by PBCXY) and under the open (periodic) boundary condition along \( x \) (\( y \)) direction (denoted by OBCX, PBCY). The gap-closing points at \( E = 0 \) and \( \pm \pi \) in the bulk spectrum are found to be the same as those observed in Fig. 3(a). Meanwhile, some curves representing the dispersion of edge bands are observed in Fig. 3(c) due to the open boundary condition taken along the \( x \)-direction. Note that the quasienergies of edge states in Fig. 3(c) are not equal to zero and \( \pm \pi \), even though they look very close to them in some parameter regions. In Fig. 3(d), we present the quasienergies under the periodic (open) boundary condition along \( x \) (\( y \)) direction of the lattice (denoted by PBCX, OBCY). Interestingly, despite the bulk gap-closing points
FIG. 3. Gap functions and Floquet spectrum versus $J$ under different boundary conditions in the Case 1. (a) Gap functions under PBCXY. (b) Floquet spectrum under PBCXY. (c) Floquet spectrum under OBCX, PBCY. (d) Floquet spectrum under PBCX, OBCY. The crossing points between the vertical dotted lines and the horizontal axis show the bulk gap-closing points predicted by Eq. (31). Other system parameters are set as $(\Delta_1, \Delta_2) = (0.5\pi, 0.2\pi)$. The number of cells along $x$ and $y$ directions are $N_x = N_y = 60$.

already seen in Fig. 3(a), we observe another gap-closing point at $E = 0$ around $J = 0$. This point goes beyond the prediction of the bulk phase boundary according to Eq. (31). As we will soon notice, this point corresponds to a closing of the quasienergy gap between different edge bands under the given boundary condition. It will thus be related to a type-II topological phase transition \[87\] in our Floquet SOTSC system.

To deepen our understanding of the gap-closing transitions observed in Case 1, we show the Floquet spectrum of the system under the OBCs along both $x$ and $y$ directions (denoted by OBCXY) in Fig. 4. We find that more and more eigenmodes with $E = 0$ or $\pm \pi$ emerge following each gap-closing transition of the bulk as predicted by Eq. (31). In addition, the numbers of these eigenmodes are integer multiples of four, which strongly suggest that they are Floquet Majorana zero/$\pi$ modes localized around the four corners of the lattice. In Fig. 3(a)–(e), we report the probability distributions of these zero/$\pi$ modes in the lattice for a typical case [(\(J, \Delta_1, \Delta_2\) = (2.5\pi, 0.5\pi, 0.2\pi)], with their quasienergies displayed in Figs. 4(b) and 4(c). The results confirm that they are indeed Majorana corner modes in the Floquet system. Compared with the Floquet SOTSC phases found in Ref. [73], we could now obtain
FIG. 4. Floquet spectrum versus $J$ under OBCs in the Case 1. (a) Quasienergies at different $J$ under OBCs along both $x$ and $y$ directions (OBCXY). The crossing points between the vertical dotted lines and the horizontal axis show the bulk gap closing points predicted by Eq. (31). The numbers in red color denote the numbers of Floquet corner modes at zero and $\pi$ quasienergies. (b) and (c) show the absolute values of quasienergies of the first fifteen and last eleven Floquet eigenstates indexed by $j$ at $J = 2.5\pi$. Other system parameters are $(\Delta_1, \Delta_2) = (0.5\pi, 0.2\pi)$. The number of cells along $x$ and $y$ directions are $N_x = N_y = 60$. The total number of Floquet eigenstates is $N = 14400$.

many quartets of Majorana zero and $\pi$ corner modes, which may give more room for the realization of topological qubits and the operation of Floquet quantum computing protocols as suggested previously [73]. Besides, with the increase of $J$, we could in principle obtain unbounded numbers of Floquet zero/$\pi$ Majorana corner modes in the thermodynamic limit. This demonstrates again one key advantage of Floquet engineering, i.e., to generate many topological nontrivial states and topological phase transitions in a controlled manner.

Finally, we are left to understand the transition at $J = 0$ in Fig. 4(a), following which four Floquet Majorana corner modes appear at $E = 0$. However, the bulk Floquet spectral gap remains open throughout this transition, as observed clearly in Fig. 3(b). By diagonalizing the $\hat{U}(k_x)$ in Eq. (18), we obtain the Floquet spectrum of the system versus $k_x$ under the PBCX, OBCY, as reported in Figs. 6(a)-(c) for $J = 0$, $J = \sqrt{\pi^2 - \Delta_2^2}$ and $J = \sqrt{(2\pi)^2 - \Delta_2^2}$, respectively. We observe that while the bulk gaps close at $E = \pm \pi$ and $E = 0$ in the latter two cases, only edge states develop crossings at $E = 0$ when $J = 0$. For comparison, we obtain the Floquet spectrum versus the quasimomentum $k_y$ under the
FIG. 5. Probability distributions of Floquet corner modes in the Case 1 with zero and $\pi$ quasienergies in panels (a)–(c) and panels (d)–(e), respectively. Other system parameters are chosen to be $(J, \Delta, \Delta_2) = (2.5\pi, 0.5\pi, 0.2\pi)$. The number of cells along $x$ and $y$ directions are $N_x = N_y = 60$.

OBCX, PBCY by diagonalizing the $\hat{U}(k_y)$ in Eq. (24), as reported in Figs. 6(d)–(f). We again find the closings of bulk spectrum gaps at $E = \pm \pi$ and $E = 0$ for $J = \sqrt{\pi^2 - \Delta_2^2}$ and $J = \sqrt{(2\pi)^2 - \Delta_2^2}$, respectively. These are the bulk transition points predicted by Eq. (31). Yet, the spectrum is found to be well gapped at $E = 0$ for $J = 0$ in Figs. 6(d).

Putting together, we conclude that under the condition in Eq. (31), Floquet zero/$\pi$ corner modes emerge following type-I topological phase transitions (with bulk-band touchings) in the Case 1. Meanwhile, the Floquet zero corner modes could also appear following a type-II higher-order topological phase transition (with edge-band touchings) at $J = 0$. There are thus two types of topological phase transitions in our Floquet SOTSC system, with each of them being able to generate more Floquet Majorana corner modes. This is another key difference between our system and that explored before [73].

The model presented in the Case 1 thus forms a “minimal” model of Floquet second-order topological superconductors with rich and different types of topological phase transitions, along with unbounded numbers of Floquet Majorana zero/$\pi$ corner modes. All of them are induced by a simple time-periodic driving protocol applied to the superconducting pairing amplitude. In the following subsections, we explore more general situations and check how the introducing of extra onsite, pairing and hopping terms could affect the corner modes and the phase transitions found in our system.
FIG. 6. Floquet spectrum versus $k_x$ ($k_y$) in the Case 1 under PBCX, OBCY (OBCX, PBCY) in panels (a)–(c) [(d)–(f)]. The gray dots, red circles and blue stars highlight the bulk states, states localized around the left edge and the right edge of the lattice. The value of hopping amplitude is set to $J = 0$ for panels (a), (d), to the first bulk gap-closing point at $E = \pm \pi$ for panels (b), (e), and to the first bulk gap-closing point at $E = 0$ for panels (c), (f). Other system parameters are set as $(\Delta, \Delta_2) = (0.5\pi, 0.2\pi)$.

B. Case 2: $J' = \Delta_1 = 0$

In a slightly more general situation, we allow the system to possess a finite chemical potential $\mu$. From Eqs. (12)–(14), the resulting Floquet operator in momentum space now takes the form

$$U(k) = e^{-i[\mu J \cos k_x] \sigma_0 \otimes \sigma_z + \Delta_2 (\sin k_y \sigma_x - \cos k_y \sigma_y) \otimes \sigma_x]} e^{-i \Delta \sin k_x \sigma_0 \otimes \sigma_y}. \quad (33)$$

Similar to the Case 1, the tensor product matrices $\sigma_0 \otimes \sigma_z$, $\sigma_x \otimes \sigma_x$, $\sigma_y \otimes \sigma_x$ and $\sigma_0 \otimes \sigma_y$ are anti-commute with one another. Therefore, we can obtain the bulk quasienergy dispersion relation as

$$E_{\pm}(k) = \pm \arccos \left[ \cos(\Delta \sin k_x) \cos \left( \sqrt{(\mu + J \cos k_x)^2 + \Delta_2^2} \right) \right]. \quad (34)$$

There are again two pairs of quasienergy bands with each of them been twofold degenerate and flat along $k_y$. Setting $E_{\pm}(k) = 0$ or $\pm \pi$, we find the gapless condition of Floquet
spectrum to be
\[
\cos(\Delta \sin k_x) \cos \left( \sqrt{(\mu + J \cos k_x)^2 + \Delta_2^2} \right) = \pm 1,
\] (35)
yielding the phase boundary equation in the parameter space \((\Delta, \mu, J, \Delta_2)\) as
\[
\frac{p^2 \pi^2}{\Delta^2} + \left( \frac{\sqrt{q^2 \pi^2 - \Delta_2^2} \pm \mu}{J^2} \right)^2 = 1, \quad p, q \in \mathbb{Z}.
\] (36)

In contrast to the Eq. (31), we see that the general effect of a nonvanishing \(\mu\) is to split each transition point of the Case 1 into two distinct points separated by a distance \(\sim 2\mu\) on the phase diagram. We would thus expect richer gap closing/reopening transitions in the parameter space of Case 2 compared with the Case 1. Meanwhile, the gap functions in Case 2 share the same forms with Eq. (32) of the Case 1, as the systems possess two pairs of twofold-degenerate Floquet bands in both cases.

In Fig. 7(a), we present the gap functions \(F_0\) and \(F_\pi\) versus \(J\) for a typical set of system parameters in Case 2. The first interesting observation is that each gap-closing transition point located originally at \(J = \sqrt{q^2 \pi^2 - \Delta_2^2}\) (for \(p = 0, q \in \mathbb{Z}\)) in Fig. 3(a) now splits into two points residing at \(J = \sqrt{q^2 \pi^2 - \Delta_2^2} \pm \mu\) in Fig. 7(a), as also predicted by the phase boundary Eq. (36). Therefore, the presence of a finite chemical potential \(\mu\) endues the system with richer transition patterns in its spectrum. This is further confirmed by the quasienergies with respect to \(J\) under three different boundary conditions, as reported in Figs. 7(b)–(d). The bulk gap-closing points in each figure are found to be consistent with the prediction of Eq. (36). Notably, we observe in Fig. 7(d) a gapless point in the Floquet spectrum at \(J = \mu\), which is not captured by Eq. (36). We will see that this point is again related to a touching of the edge bands instead of the bulk spectrum, similar to what we have encountered at \(J = 0\) in the Case 1.

The spectrum of our system in the Case 2 under OBCs is shown in Fig. 8. We observe that following each gap-closing transition, there are indeed more eigenmodes emerge at the quasienergy zero or \(\pi\). Their numbers in each phase are denoted by the numbers in red in Fig. 8(a). Similar to the Case 1, here the numbers of zero and \(\pi\) Floquet eigenmodes are both integer multiples of four, implying that they are fourfold degenerate Majorana corner states in Floquet SOTSC phases. However, across each bulk gap-closing transition,
FIG. 7. Gap functions and Floquet spectrum versus $J$ under different boundary conditions in the Case 2. (a) Gap functions under PBCXY. (b) Floquet spectrum under PBCXY. (c) Floquet spectrum under OBCX, PBCY. (d) Floquet spectrum under PBCX, OBCY. The crossing points between the vertical dotted lines and the horizontal axis show the bulk gap closing points predicted by Eq. (36). Other system parameters are set as $(\mu, \Delta, \Delta_2) = (0.25\pi, 0.5\pi, 0.2\pi)$. The number of cells along $x$ and $y$ directions are $N_x = N_y = 60$.

the number of Floquet zero or $\pi$ eigenmodes now only changes by 4 instead of 8. This is originated from the splitting of each gapless point into two in the phase diagram with a nonvanishing $\mu$. Following each of the new generated transitions, the change of Floquet zero/$\pi$ eigenmode number is just half of the original transition in the Case 1. We will further digest this point by investigating the momentum space spectrum of the system later in this subsection. Besides, we observe monotonic increases of the numbers of zero and $\pi$ Floquet eigenmodes with the increase of $J$ following the consecutive spectrum transitions. It implies that we could also obtain a great deal of Floquet zero and $\pi$ corner modes at large $J$ in the thermodynamic limit when $\mu \neq 0$. Meanwhile, four Floquet zero corner modes emerge from a transition at $J = \mu$, which is not captured by the bulk phase boundary Eq. (36). As mentioned before, we will trace it back to the touching of edge state bands, i.e., a type-II topological phase transition.

To confirm that the zero and $\pi$ Floquet eigenmodes observed in the spectrum are indeed Majorana corner modes, we plot their probability distributions in the lattice for a typical
FIG. 8. Floquet spectrum versus $J$ under OBCs in the Case 2. (a) Quasienergies at different $J$ under OBCXY. The crossing points between the vertical dotted lines and the horizontal axis show the bulk gap-closing points predicted by the Eq. (36). The numbers in red color denote the numbers of Floquet corner modes at zero and $\pi$ quasienergies. (b) and (c) show the absolute values of quasienergies of the first twelve and last eleven Floquet eigenstates indexed by $j$ at $J = 2\pi$. Other system parameters are $(\mu, \Delta, \Delta_2) = (0.25\pi, 0.5\pi, 0.2\pi)$. The number of cells along $x$ and $y$ directions are $N_x = N_y = 60$. The total number of Floquet eigenstates is $N = 14400$.

To understand the difference between the anomalous transition observed at $J = \mu$ and the other bulk gap-closing transitions, we present the spectra of the system under the PBCX, OBCY versus $k_x$ and under the OBCX, PBCY versus $k_y$ at three critical points $J = \mu$, $J = \sqrt{\pi^2 - \Delta_2^2} - \mu$ and $J = \sqrt{(2\pi)^2 - \Delta_2^2} - \mu$ in Fig. 10. These spectra are obtained by diagonalizing $\hat{U}(k_x)$ and $\hat{U}(k_y)$ in Eqs. (18) and (24), respectively. We find that for the
FIG. 9. Probability distributions of Floquet corner modes in the Case 2 with quasienergies zero and $\pi$ in the panels (a)–(b) and panels (c)–(d), respectively. Other system parameters are set as $(J, \mu, \Delta, \Delta_2) = (2\pi, 0.25\pi, 0.5\pi, 0.2\pi)$. The number of cells along $x$ and $y$ directions of the lattice are $N_x = N_y = 60$.

transitions at $J = \sqrt{\pi^2 - \Delta^2} - \mu$ and $J = \sqrt{(2\pi)^2 - \Delta^2} - \mu$, the bulk Floquet bands of the system indeed touch at $E = \pm \pi$ and $E = 0$ in Figs. 10(b), 10(e) and 10(c), 10(f). This is consistent with the prediction of the bulk phase boundary in Eq. (36). Besides, a comparison between the Figs. 10(b)–(c) and Figs. 6(b)–(c) also helps us to understand why the change of zero/$\pi$ corner mode numbers is four instead of eight across each topological transition when $\mu \neq 0$. That is, in the Case 1 the Floquet bands touch at two distinct points along $k_x$ [$k_x = 0, \pi$ in Figs. 6(b)–(c)] at the bulk phase transition point. Meanwhile, in the Case 2 the Floquet bands only touch at a single point along $k_x$ [$k_x = 0$ in Figs. 10(b)–(c)] at each bulk phase transition point, yielding a smaller change in the number of Majorana corner modes across the transition.

Finally, we realize that at $J = \mu$, the edge state bands develop a touching at $k_x = 0$ under the PBCX, OBCY in Fig. 10(a), while no edge states and band touchings are observed under the OBCX, PBCY in Fig. 10(d). This observation confirms that the four Floquet Majorana zero corner modes indeed emerge out of a type-II topological phase transition mediated by an edge-band touching at $J = \mu$ in Fig. 9(a). Therefore, we also encounter two types of topological phase transitions in the Case 2, after which more Floquet Majorana zero/$\pi$ corner modes appear with the increase of the hopping amplitude $J$ in our system. This tendency
FIG. 10. Floquet spectrum versus $k_x (k_y)$ in the Case 2 under PBCX, OBCY (OBCX, PBCY) in panels (a)–(c) [(d)–(f)]. The gray dots, red circles and blue stars highlight the bulk states, states localized around the left edge and the right edge of the lattice. The value of hopping amplitude is set to $J = \mu$ for the panels (a), (d), to the first bulk gap-closing point at $E = \pm \pi$ for the panels (b), (e), and to the first bulk gap-closing point at $E = 0$ for the panels (c), (f). Other system parameters are chosen to be $(\mu, \Delta, \Delta_2) = (0.25\pi, 0.5\pi, 0.2\pi)$.

will continue in more general situations as will be discussed in the following subsection.

C. Case 3: $J' = 0$

We now consider the case with only $J' = 0$. In this case, all the pairing terms $(\Delta, \Delta_1, \Delta_2)$ are switched on. Following Eqs. (12)–(14), the Floquet operator in momentum space takes the form

$$U(k) = e^{-i(\mu + J \cos k_x)\sigma_0 \otimes \sigma_z + [\Delta_2 \sin k_y \sigma_x + (\Delta_1 - \Delta_2 \cos k_y)\sigma_y] \otimes \sigma_y} e^{-i\Delta \sin k_x \sigma_0 \otimes \sigma_y}. \quad (37)$$

Using again the anticommuting nature of tensor product matrices $\sigma_0 \otimes \sigma_z$, $\sigma_x \otimes \sigma_x$, $\sigma_y \otimes \sigma_x$, $\sigma_0 \otimes \sigma_y$ and the Taylor expansion, we find the quasienergy dispersions of $U(k)$ to be...
\( E_{\pm}(k) = \pm \arccos \left[ \cos(\Delta \sin k_x) \cos \left( \sqrt{(\mu + J \cos k_x)^2 + \Delta_1^2 + \Delta_2^2 - 2\Delta_1 \Delta_2 \cos k_y} \right) \right] \). (38)

As before, we obtain two pairs of twofold-degenerate Floquet bands. However, the quasienergies \( E_{\pm}(k) \) now depend on both the quasimomenta \( k_x \) and \( k_y \) due to the nonvanishing \( \Delta_1 \). Setting \( E_{\pm}(k) \) to zero or \( \pm \pi \), we obtain the gapless condition of the spectrum as

\[
\cos(\Delta \sin k_x) \cos \left( \sqrt{(\mu + J \cos k_x)^2 + \Delta_1^2 + \Delta_2^2 - 2\Delta_1 \Delta_2 \cos k_y} \right) = \pm 1. \tag{39}
\]

Without loss of generality, we now assume all system parameters \( (\mu, J, \Delta, \Delta_1, \Delta_2) \) in the Case 3 to be positive. One can then deduce from Eq. (39) that the gap between the two sets of Floquet bands vanishes at \( E = 0 \) or \( \pi \) when the inequality

\[
q^2 \pi^2 - (\Delta_1 + \Delta_2)^2 \leq \left( \mu \pm J \sqrt{1 - \frac{p^2 \pi^2}{\Delta^2}} \right)^2 \leq q^2 \pi^2 - (\Delta_1 - \Delta_2)^2 \tag{40}
\]

is satisfied for \( p, q \in \mathbb{Z} \). This relation suggests a rather different situation compared with the Cases 1 and 2, i.e., the Floquet spectrum could now become gapless at the quasienergy zero or \( \pi \) in finite domains of the parameter space. Therefore, our model in the Case 3 could possess gapless phases with semimetal-like Floquet band structures. Two different Floquet SOTSC phases may then be separated by a finite gapless region instead of a gap-closing point in the spectrum. In the meantime, we can still define the gap functions as the Eq. (32) in Case 1.

In Fig. 11(a), we plot the gap functions \( F_0 \) and \( F_\pi \) versus \( J \) for a typical set of system parameters in the Case 3. Interestingly, we observe that with a finite intracell pairing \( \Delta_1 \), the gap-closing points in the Case 2 now broaden into regions of finite widths along the \( J \)-axis. The boundaries of these gapless bulk regions are highlighted by the vertical dotted lines in Fig. 11, which are further determined theoretically by Eq. (40). Therefore, in the presence of a finite \( \Delta_1 \), the transitions between different Floquet SOTSC phases can be mediated by other phases with gapless Floquet spectra instead of isolated critical points in the parameter space. The existence of these gapless phases is further confirmed by the Floquet spectrum presented in Figs. 11(b)–(d) under different boundary conditions. We could thus obtain richer patterns of Floquet phases and spectral transitions with a nonvanishing \( \Delta_1 \). This is
FIG. 11. Gap functions and Floquet spectrum versus $J$ under different boundary conditions in the Case 3. (a) Gap functions under PBCXY. (b) Floquet spectrum under PBCXY. (c) Floquet spectrum under OBCX, PBCY. (d) Floquet spectrum under PBCX, OBCY. The crossing points between the vertical dotted lines and the horizontal axis show the bulk gap-closing points predicted by the Eq. (40). Other system parameters are set as $(\mu, \Delta, \Delta_1, \Delta_2) = (0.25\pi, 0.5\pi, 0.2\pi, 0.4\pi)$. The number of cells along $x$ and $y$ directions are $N_x = N_y = 60$.

distinct from the situation considered previously, where the gapless phases are not observed in a harmonically driven setting [73]. Despite the many transitions occurred with the increase of $J$, we observe another isolated transition at $J = \mu$ that is not described by Eq. (40). As discussed in previous subsections, this transition is also expected to be originated from the touching of edge state bands instead of bulk bands.

In Fig. 12, we report the Floquet spectrum of the system under OBCs along both directions. Following either the transition at $J = \mu$ or the transition mediated by a gapless phase at $E = 0$ or $E = \pm \pi$, we obtain more eigenmodes at the quasienergy zero or $\pi$ with the increase of $J$, whose numbers are highlighted in red in Fig. 12(a). The $4\mathbb{Z}$-quantization of these eigenmode numbers again suggest that they are localized states around the four corners of the lattice. Similar to the previous two cases, we could foresee unboundedly many quartets of eigenmodes at $E = 0$ and $\pm \pi$ with $J \to \infty$ in the thermodynamic limit. This demonstrates a controllable generation of these topological modes in our Floquet SOTSC model under a simple driving protocol. The main impact of a finite pairing amplitude
FIG. 12. Floquet spectrum versus $J$ under OBCs in the Case 3. (a) Quasienergies at different $J$ under OBCXY. The crossing points between the vertical dotted lines and the horizontal axis show the bulk gap-closing points predicted by the Eq. (40). The numbers in red color denote the numbers of Floquet corner modes at zero and $\pi$ quasienergies. (b) and (c) show the absolute values of quasienergies of the first sixteen and last sixteen Floquet eigenstates indexed by $j$ at $J = 3\pi$. Other system parameters are $(\mu, \Delta, \Delta_1, \Delta_2) = (0.25\pi, 0.5\pi, 0.2\pi, 0.4\pi)$. The number of cells along $x$ and $y$ directions are $N_x = N_y = 60$. The total number of Floquet eigenstates is $N = 14400$.

$\Delta_1 \neq 0$ is thus to generate gapless regions between different Floquet SOTSC phases, while preserving the zero and $\pi$ Floquet eigenmodes in the gapped phases of the bulk. Besides, four Floquet zero modes emerge through the transition at $J = \mu$, where the bulk spectrum is expected to be gapped due to Eq. (40). This transition will also be understood as a type-II transition following the closing of a gap between the edge bands at $E = 0$.

In Fig. 13 we plot the probability distributions of the first twelve and the last twelve of Floquet eigenmodes in Figs. 12(b) and 12(c), respectively. It is clear that they are indeed Floquet Majorana corner modes at the quasienergies zero and $\pm \pi$. We could thus obtain many such quartets of Majorana modes in our Floquet SOTSC system with $\Delta_1 \neq 0$. Furthermore, their appearances follow transitions over gapless phases instead of critical transition points of the system (except at $J = \mu$), which are different from those happened in the Cases 1 and 2. The large numbers of Majorana corner modes found here may also allow us to implement Floquet quantum computing protocols in more general situations [73].

We are left to understand the transition at $J = \mu$. Similar to what we have done in the last
two subsections, we obtain the Floquet spectrum of $\hat{U}(k_x)$ and $\hat{U}(k_y)$ in Eqs. (18) and (24) under the PBCX, OBCY and OBCX, PBCX, respectively, and present them in Figs. 14(a)–(c) and 14(d)–(f) for $J = \mu$, $J = \sqrt{\pi^2 - (\Delta_1 + \Delta_2)^2} - \mu$, and $J = \sqrt{(2\pi)^2 - (\Delta_1 + \Delta_2)^2} - \mu$. In the latter two cases, the transitions happen through the touching of bulk bands at $E = \pm \pi$ and $E = 0$ with the quasimomentum $(k_x, k_y) = (0, \pi)$. This coincides with the prediction of the bulk phase boundary in Eq. (40). They are thus conventional type-I topological phase transition in the Floquet SOTSC system. Instead, we find that the Floquet spectrum versus $k_y$ is gapped for $J = \mu$, while it is gapless at $E = 0$ through a touching of edge state bands at $k_x = \pi$. Combining this observation with the emerging zero corner modes after the first transition in Fig. 12(a), we conclude that the transition at $J = \mu$ is indeed a type-II topological phase transition induced by an edge band touching, and it is not modified by the presence of a finite intracell pairing strength $\Delta_1$ along the $y$ direction of the lattice. It deserves to be mentioned that the edge bands in Fig. 14(a) have vanishing net chiralities even though they traverse the band gap, yielding Floquet bands with zero Chern numbers.

Putting together, we find two types of topological phase transitions in the Cases 1–3. With the increase of the hopping amplitude $J$, more and more Floquet Majorana zero/$\pi$ corner
FIG. 14. Floquet spectrum versus $k_x$ ($k_y$) in the Case 3 under PBCX, OBCY (OBCX, PBCX) in panels (a)–(c) [(d)–(f)]. The gray dots, red circles and blue stars highlight the bulk states, states localized around the left edge and the right edge of the lattice. The value of hopping amplitude is set to $J = \mu$ for panels (a), (d), to the first bulk gap-closing point at $E = \pm \pi$ for panels (b), (e), and to the first bulk gap-closing point at $E = 0$ for panels (c), (f). Other system parameters are set as $(\mu, \Delta, \Delta_1, \Delta_2) = (0.25\pi, 0.5\pi, 0.2\pi, 0.4\pi)$.

modes emerge within our system, and richer patterns of topological phases and transitions appear in the cases with nonvanishing $\mu$ and $\Delta_1$. We now arrive at the stage to discuss what will happen if a finite hopping amplitude $J'$ along the $y$ direction is turned on. This will be our task in the following subsection, in which we treat the effect of $J'$ essentially as a perturbation.

D. Case 4: General situation

We finally discuss the general case with $J' \neq 0$. The Floquet operator of our model in momentum space now takes the form of Eq. (12), i.e.,

\begin{equation}
U(k) = e^{-i\left((\mu+J\cos k_x)\sigma_0\otimes\sigma_z+\Delta_2\sin k_y\sigma_x+(\Delta_1-\Delta_2\cos k_y)\sigma_y\right)\otimes\sigma_z+J'(1+\cos k_y)\sigma_z+\sin k_y\sigma_0\otimes\sigma_y}\times e^{-i\Delta\sin k_x}\sigma_0\otimes\sigma_y.
\end{equation} 

(41)
FIG. 15. Gap functions and Floquet spectrum versus $J$ under different boundary conditions in the Case 4. (a) Gap functions under PBCXY. (b) Floquet spectrum under PBCXY. (c) Floquet spectrum under OBCX, PBCY. (d) Floquet spectrum under PBCX, OBCY. Other system parameters are set as $(\mu, \Delta, \Delta_1, \Delta_2, J') = (0.25\pi, 0.5\pi, 0.2\pi, 0.4\pi, 0.05\pi)$. The number of cells along $x$ and $y$ directions are $N_x = N_y = 60$.

In this case, a simple expression of the quasienergy bands cannot be obtained from the Taylor expansion of $U(k)$, as the tensor products of Pauli matrices $\sigma_x \otimes \sigma_z$ and $\sigma_y \otimes \sigma_z$ are not anticommuting with all other matrices. We thus devolve to numerical calculations of the spectrum and states. Note that the gap functions $F_0$ and $F_\pi$ for the Case 4 should be defined more generally as

$$
F_0 \equiv \frac{1}{\pi} \min_{k \in BZ} \min_j |E_j(k)|, \quad F_\pi \equiv \frac{1}{\pi} \min_{k \in BZ} \min_j |E_j(k)| - \pi,
$$

where $j = 1, 2, 3, 4$ labels all possible quasienergies of $U(k)$ at each $k$.

In Fig. 15(a), we show the gap functions versus $J$ in the Case 4 for a typical set of system parameters. We see that with a nonvanishing but small $J'$, the general pattern of spectral transitions in the Case 3 is preserved. That is, with the increase of $J$, the system could enter a series of gapped Floquet phases in the bulk (with $F_0 \neq 0$ and $F_\pi \neq 0$), which are separated by multiple gapless phases (with $F_0 = 0$ or $F_\pi = 0$). The main difference, caused by a finite $J'$, is that the regions of gapless phases are broadened in the parameter space. This is also
confirmed by the Floquet spectrum versus $J$ plotted under different boundary conditions in Figs. 15(b)–(d). Therefore, we expect the gapped Floquet SOTSC phases to be robust to certain amounts of hoppings along the $y$ directions. Notably, the anomalous transition at $J = \mu$, which is not related to the closing of a bulk spectral gap is also observed in Fig. 15(d), which implies that this type-II topological transition remains intact when $J' \neq 0$.

To further understand the fate of Floquet Majorana zero/π corner modes in the presence of a finite $J'$, we plot the spectrum under OBCs along both the two spatial dimensions in Fig. 16(a). The results suggest that with the increase of $J$, there are still eigenmodes with zero and π quasienergies coming out of the gap-closing transition at $J = \mu$ and the transitions through other gapless regions. Meanwhile, in the gapped phases, we notice that the spectra become broadened around $E = 0$ and $E = \pm \pi$ compared with the Cases 1–3. In Figs. 16(b) and 16(c), we show the absolute values of quasienergy for the first sixteen and the last sixteen eigenstates in the spectrum at $J = 3\pi$. In this case, we only find four eigenmodes with the quasienergies $E = 0$ and $\pm \pi$, which means that not all the zero and π Floquet Majorana corner modes are robust to the perturbation introduced by $J'$. A possible explanation of this observation is that with $J' \neq 0$, the chiral symmetry $\Gamma = \sigma_z \otimes \sigma_x$ of the
Floquet system is broken. Yet, the particle-hole symmetry $C = \sigma_0 \otimes \sigma_x K$ is preserved as mentioned in Sec. I, which enables the system to possess at most four Floquet Majorana corner modes at the quasienergies zero and $\pi$. However, since we do not expect a spectral gap-closing between bulk or edge bands when $J'$ goes from zero up to a small value ($0.05\pi$ here), the emerging eigenmodes away from but close to $E = 0$ and $\pm\pi$ in Figs. 16(b) and 16(c) should still have profiles localized around the four corners of the lattice, since they have no chances to mix up with other bulk or edge states under the perturbation.

In Figs. 17(a)–(c) and 17(d)–(f), we present the probability distributions of the first twelve and the last twelve eigenstates in Figs. 16(b) and 16(c), respectively. Indeed, we find that all these modes are localized around the four corners of the lattice, even though only eight of them are Majorana zero and $\pi$ corner modes of the Floquet SOTSC phase [in Figs. 17(a) and 17(d)]. The fourfold-degenerate corner modes with $E \neq 0, \pm\pi$ are not topologically protected, as their quasienergies can vary with the change of $J'$. It remains an interesting issue to explore whether these non-topological corner modes could also find applications in Floquet quantum computing schemes. In our calculations, we have also checked the effects of dimerized hopping amplitude and chemical potential on our results. These dimerization
terms are introduced in the same way as the $\delta J$ and $\delta \mu_0$ used in the Eq. (1) of Ref. [73]. Our numerical results suggest that the main observations about Floquet spectrum and corner modes presented in this subsection hold when these dimerization terms are introduced to our model as small perturbations.

III. DISCUSSION AND CONCLUSION

In this work, we explored the generation of many Majorana zero/$\pi$ corner modes and multiple phase transitions in Floquet second-order topological superconductors. By applying time-periodic kicking to the pairing amplitude of a generic 2D $p_x + ip_y$ superconductor with dimerized superconducting pairing, we obtain rich Floquet SOTSC phases with arbitrarily many Majorana corner modes at zero and $\pi$ quasienergies in principle in the thermodynamic limit. Moreover, two different Floquet SOTSC phases are found to be separated by either a type-I topological phase transition with the closing/reopening of a bulk spectral gap, a type-II topological phase transition with the closing/reopening of an edge band gap, or a third phase with gapless quasienergy spectrum. The multiple quartets of Floquet Majorana zero/$\pi$ corner modes are found to be protected by the chiral and particle-hole symmetries of the system. When the chiral symmetry is lifted by a perturbation that couples different sites along the $y$-direction, four out of the many Majorana zero and $\pi$ corner modes could still survive, which represent the most stable higher-order topological modes in the system. Our work thus extends the present understanding of topological phases and transitions in Floquet SOTSC setups. It further provides an efficient means to generate many Majorana corner modes, which may give us more choices and freedoms to design topological quantum computing schemes.

From the perspective of bulk-corner correspondence, there should exist a pair of integer-quantized topological invariants that can characterize the two types of Floquet Majorana corner modes at zero and $\pi$ quasienergies in our system, similar to what has been identified before for Floquet second-order topological insulators [45]. In practice, due to the complicated patterns of phase transitions and Floquet corner modes in our model, the topological winding numbers introduced in Ref. [45] could not fully characterize all the Majorana corner modes and bulk-corner correspondence in our system. In future work, it remains an interesting issue to explore suitable topological invariants that could predict the numbers of
Majorana corner modes in different Floquet SOTSC phases found here.

In our Floquet model, the periodic driving is introduced via a delta-kicked superconducting pairing amplitude. We expect similar results as those reported in Sec. II if the delta-kick is replaced by a piecewise quench applied successively to $\Delta$ and the other terms of $\hat{H}$ within each driving period. With a sinusoidal driving field, much fewer Floquet zero and $\pi$ Majorana corner modes and more restricted patterns of topological phase transitions are expected in our system, similar to what has been identified before in Ref. [73]. Meanwhile, in the presence of two sinusoidal drivings with different frequencies, there are still opportunities to find rich Floquet phases with many topological boundary states and multiple topological transitions [93]. It is thus interesting to consider the Floquet engineering of SOTSC phases by multi-frequency harmonic driving fields in future studies.

In our system, we have numerically checked that when the time-periodic delta kicking is applied to the hopping amplitude $J$ instead of the pairing amplitude $\Delta$, similar results could be obtained as those reported in Sec. II. That is, we can now find many Floquet Majorana zero and $\pi$ corner modes together with multiple topological phase transitions with the increase of $\Delta$ when $J$ is kicked. Interestingly, substantial differences were observed with the periodic delta kicking applied to the pairing versus hopping amplitudes in the chaotic dynamics of interacting Floquet models [94]. Therefore, the signature of quantum chaos in many-body Floquet SOTSC systems under different kicking protocols deserves to be revealed in future studies.

The effects of disorder and boundary confinements form major challenges in the experimental study of Majorana bound states in topological superconductors [95, 96]. In the present work, we focus on the theoretical possibility of generating many Majorana zero/$\pi$ corner modes and multiple topological phase transitions with the help of Floquet driving fields. For completeness, we have also numerically checked the effects of disorder on Majorana corner modes for the different cases considered in Sec. II. We found that for the Cases 1–3, the Floquet Majorana corner modes are robust to disorder in the forms of $\sum_{m,n} \delta \mu_{m,n}(\hat{a}_{m,n} \hat{a}_{m,n} + \hat{b}_{m,n} \hat{b}_{m,n})$, $\sum_{m,n} \delta J_m (\hat{a}_{m,n} \hat{a}_{m+1,n} + \hat{b}_{m,n} \hat{b}_{m+1,n} + \text{H.c.})$, $\sum_{m,n} \delta \Delta_n (\hat{a}_{m,n} \hat{a}_{m+1,n} + \hat{b}_{m,n} \hat{b}_{m+1,n} + \text{H.c.})$, and $\sum_{m,n} (i \delta \Delta_1,n \hat{a}_{m,n} \hat{b}_{m,n} + i \delta \Delta_2,n \hat{b}_{m,n} \hat{a}_{m,n+1} + \text{H.c.})$. Here $\delta \mu_{m,n}$, $\delta J_m$, $\delta \Delta_n$, $\delta \Delta_1,n$ and $\delta \Delta_2,n$ vary randomly in space over different sites and bonds. The values for each of them are taken separately from a uniform distribution with the range $[-W,W]$ (we take $W = 1/20$ in our numerical calculations). Note that
these forms of disorder do not break the chiral symmetry of the model. The many Floquet Majorana corner modes in our system are found to be well localized around the corners so long as they are well separated in quasienergies from the other bulk or edge states in the presence of disorder. In the Case 4, the four Majorana zero/π corner modes are further found to be robust to weak disorder in the form of \( \sum_{m,n} \delta J'_n (\hat{a}_{m,n} \hat{b}_{m,n} + \hat{b}_{m,n}^\dagger \hat{a}_{m,n+1} + \text{H.c.}) \), where \( \delta J'_n \) varies over bonds along the \( y \)-direction and taking values randomly in a uniform distribution with the range \([-W, W]\). Other forms of disorder and boundary confinements may also change the fate of Majorana corner modes in our system, which deserve to be investigated more thoroughly in a potential future work.

Finally, it would be interesting to explore the engineering of Floquet SOTSC phases in other symmetry classes, in higher spatial dimensions and with many-body interactions. The application of the multiple Floquet zero/π Majorana corner modes found here to different quantum computing protocols also deserves to be investigated in detail.

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[1] J. Cayssol, B. Dóra, F. Simon, and R. Moessner, Floquet topological insulators, Phys. Status Solidi RRL 7, 101 (2013).
[2] A. Eckardt, Colloquium: Atomic quantum gases in periodically driven optical lattices, Rev. Mod. Phys. 89, 011004 (2017).
[3] F. Harper, R. Roy, M. S. Rudner, and S. Sondhi, Topology and Broken Symmetry in Floquet Systems, Annu. Rev. Condens. Matter Phys. 11, 345 (2020).
[4] M. Rudner and N. Lindner, Band structure engineering and non-equilibrium dynamics in Floquet topological insulators, Nat. Rev. Phys. 2, 229 (2020).
[5] F. Nathan and M. S. Rudner, Topological singularities and the general classification of Floquet-Bloch systems, New J. Phys. 17, 125014 (2015).

[6] A. C. Potter, T. Morimoto, and A. Vishwanath, Classification of Interacting Topological Floquet Phases in One Dimension, Phys. Rev. X 6, 041001 (2016).

[7] R. Roy and F. Harper, Periodic table for Floquet topological insulators, Phys. Rev. B 96, 155118 (2017).

[8] J. Yu, R.-X. Zhang, and Z.-D. Song, Dynamical symmetry indicators for Floquet crystals, Nat. Commun. 12, 5985 (2021).

[9] M. S. Rudner, N. H. Lindner, E. Berg, and M. Levin, Anomalous Edge States and the Bulk-Edge Correspondence for Periodically Driven Two-Dimensional Systems, Phys. Rev. X 3, 031005 (2013).

[10] P. Titum, E. Berg, M. S. Rudner, G. Refael, and N. H. Lindner, Anomalous Floquet-Anderson Insulator as a Nonadiabatic Quantized Charge Pump, Phys. Rev. X 6, 021013 (2016).

[11] L. Zhou and J. Gong, Recipe for creating an arbitrary number of Floquet chiral edge states, Phys. Rev. B 97, 245430 (2018).

[12] Z. Zhang, P. Delplace, and R. Fleury, Superior robustness of anomalous non-reciprocal topological edge states, Nature 598, 293-297 (2021).

[13] D. Y. H. Ho and J. Gong, Quantized Adiabatic Transport In Momentum Space, Phys. Rev. Lett. 109, 010601 (2012).

[14] Q.-J. Tong, J.-H. An, J. Gong, H.-G. Luo, and C. H. Oh, Generating many Majorana modes via periodic driving: A superconductor model, Phys. Rev. B 87, 201109(R) (2013).

[15] L. Zhou, H. Wang, D. Y. H. Ho, and J. Gong, Aspects of Floquet bands and topological phase transitions in a continuously driven superlattice. Eur. Phys. J. B 87, 204 (2014).

[16] Z.-Z. Li, C.-H. Lam, and J. Q. You, Floquet engineering of long-range p-wave superconductivity: Beyond the high-frequency limit, Phys. Rev. B 96, 155438 (2017).

[17] L. Zhou and J. Gong, Floquet topological phases in a spin-1/2 double kicked rotor, Phys. Rev. A 97, 063603 (2018).

[18] Y. H. Wang, H. Steinberg, P. Jarillo-Herrero, and N. Gedik, Observation of Floquet-Bloch States on the Surface of a Topological Insulator, Science 342, 453-457 (2013).

[19] K. Yang, L. Zhou, W. Ma, X. Kong, P. Wang, X. Qin, X. Rong, Y. Wang, F. Shi, J. Gong, and J. Du, Floquet dynamical quantum phase transitions, Phys. Rev. B 100, 085308 (2019).
[20] J. W. McIver, B. Schulte, F.-U. Stein, T. Matsuyama, G. Jotzu, G. Meier, and A. Cavalleri, Light-induced anomalous Hall effect in graphene, Nat. Phys. **16**, 38-41 (2020).

[21] B. Chen, S. Li, X. Hou, F. Ge, F. Zhou, P. Qian, F. Mei, S. Jia, N. Xu, and H. Shen, Digital quantum simulation of Floquet topological phases with a solid-state quantum simulator, Photon. Res. **9**, 81-87 (2021).

[22] G. Jotzu, M. Messer, R. Desbuquois, M. Lebrat, T. Uehlinger, D. Greif, and T. Esslinger, Experimental realization of the topological Haldane model with ultracold fermions, Nature **515**, 237-240 (2014).

[23] N. Fläschner, B. S. Rem, M. Tarnowski, D. Vogel, D.-S. Lühmann, K. Sengstock and C. Weitenberg, Experimental reconstruction of the Berry curvature in a Floquet Bloch band, Science **352**, 1091-1094 (2016).

[24] L. Asteria, D. T. Tran, T. Ozawa, M. Tarnowski, B. S. Rem, N. Fläschner, K. Sengstock, N. Goldman, and C. Weitenberg, Measuring quantized circular dichroism in ultracold topological matter, Nat. Phys. **15**, 449-454 (2019).

[25] K. Wintersperger, C. Braun, F. N. Ünal, A. Eckardt, M. D. Liberto, N. Goldman, I. Bloch, and M. Aidelsburger, Realization of an anomalous Floquet topological system with ultracold atoms, Nat. Phys. **16**, 1058-1063 (2020).

[26] T. Kitagawa, M. A. Broome, A. Fedrizzi, M. S. Rudner, E. Berg, I. Kassal, A. Aspuru-Guzik, E. Demler, and A. G. White, Observation of topologically protected bound states in photonic quantum walks, Nat. Commun. **3**, 882 (2012).

[27] M. C. Rechtsman, J. M. Zeuner, Y. Plotnik, Y. Lumer, D. Podolsky, F. Dreisow, S. Nolte, M. Segev, and A. Szameit, Photonic Floquet topological insulators, Nature **496**, 196-200 (2013).

[28] W. Hu, J. C. Pillay, K. Wu, M. Pasek, P. P. Shum, and Y. D. Chong, Measurement of a Topological Edge Invariant in a Microwave Network, Phys. Rev. X **5**, 011012 (2015).

[29] S. Mukherjee, A. Spracklen, M. Valiente, E. Andersson, P. Öhberg, N. Goldman, and R. R. Thomson, Experimental observation of anomalous topological edge modes in a slowly driven photonic lattice, Nat. Commun. **8**, 13918 (2017).

[30] T. Oka and S. Kitamura, Floquet Engineering of Quantum Materials, Annu. Rev. Condens. Matter Phys. **10**, 387-408 (2019).

[31] R. W. Bomantara and J. Gong, Simulation of Non-Abelian Braiding in Majorana Time Crystals, Phys. Rev. Lett. **120**, 230405 (2018).
[32] M. Sitte, A. Rosch, E. Altman, and L. Fritz, Topological Insulators in Magnetic Fields: Quantum Hall Effect and Edge Channels with a Nonquantized $\theta$ Term, Phys. Rev. Lett. 108, 126807 (2012).
[33] F. Zhang, C. L. Kane, and E. J. Mele, Surface State Magnetization and Chiral Edge States on Topological Insulators, Phys. Rev. Lett. 110, 046404 (2013).
[34] R.-J. Slager, L. Rademaker, J. Zaanen, and L. Balents, Impurity-bound states and Green’s function zeros as local signatures of topology, Phys. Rev. B 92, 085126 (2015).
[35] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Quantized electric multipole insulators, Science 357, 61-66 (2017).
[36] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Electric multipole moments, topological multipole moment pumping, and chiral hinge states in crystalline insulators, Phys. Rev. B 96, 245115 (2017).
[37] J. Langbehn, Y. Peng, L. Trifunovic, F. von Oppen, and P. W. Brouwer, Reflection-symmetric second-order topological insulators and superconductors, Phys. Rev. Lett. 119, 246401 (2017).
[38] Z. Song, Z. Fang, and C. Fang, $(d-2)$-Dimensional edge states of rotation symmetry protected topological states, Phys. Rev. Lett. 119, 246402 (2017).
[39] F. Schindler, A. M. Cook, M. G. Vergniory, Z. Wang, S. S. P. Parkin, B. A. Bernevig, and T. Neupert, Higher-order topological insulators, Sci. Adv. 4, eaat0346 (2018).
[40] M. Geier, L. Trifunovic, M. Hoskam, and P. W. Brouwer, Second-order topological insulators and superconductors with an order-two crystalline symmetry, Phys. Rev. B 97, 205135 (2018).
[41] M. Kim, Z. Jacob, and J. Rho, Recent advances in 2D, 3D and higher-order topological photonics. Light Sci. Appl. 9, 130 (2020).
[42] F. Schindler, Dirac equation perspective on higher-order topological insulators, J. Appl. Phys. 128, 221102 (2020).
[43] L. Trifunovic and P. W. Brouwer, Higher-Order Topological Band Structures, Phys. Status Solidi B 258, 2000090 (2021).
[44] B. Xie, H.-X. Wang, X. Zhang, P. Zhan, J.-H. Jiang, M. Lu and Y. Chen, Higher-order band topology, Nat. Rev. Phys. 3, 520-532 (2021).
[45] R. W. Bomantara, L. Zhou, J. Pan, and J. Gong, Coupled-wire construction of static and Floquet second-order topological insulators, Phys. Rev. B 99, 045441 (2019).
[46] R. Seshadri, A. Dutta, and D. Sen, Generating a second-order topological insulator with multiple corner states by periodic driving, Phys. Rev. B 100, 115403 (2019).

[47] Y. Peng and G. Refael, Floquet Second-Order Topological Insulators from Nonsymmorphic Space-Time Symmetries, Phys. Rev. Lett. 123, 016806 (2019).

[48] M. Rodriguez-Vega, A. Kumar, and B. Seradjeh, Higher-order Floquet topological phases with corner and bulk bound states, Phys. Rev. B 100, 085138 (2019).

[49] T. Nag, V. Juričič, and B. Roy, Out of equilibrium higher-order topological insulator: Floquet engineering and quench dynamics, Phys. Rev. Research 1, 032045(R) (2019).

[50] B. Huang, and W. V. Liu, Floquet Higher-Order Topological Insulators with Anomalous Dynamical Polarization, Phys. Rev. Lett. 124, 216601 (2020).

[51] H. Hu, B. Huang, E. Zhao, and W. V. Liu, Dynamical Singularities of Floquet Higher-Order Topological Insulators, Phys. Rev. Lett. 124, 057001 (2020).

[52] J. Pan and L. Zhou, Non-Hermitian Floquet second order topological insulators in periodically quenched lattices, Phys. Rev. B 102, 094305 (2020).

[53] A. K. Ghosh, G. C. Paul, and A. Saha, Higher order topological insulator via periodic driving, Phys. Rev. B 101, 235403 (2020).

[54] Y. Meng, G. Chen, and S. Jia, Second-order topological insulator in a coinless discrete-time quantum walk, Phys. Rev. A 102, 012203 (2020).

[55] W. Zhu, Y. D. Chong, and J. Gong, Floquet higher-order topological insulator in a periodically driven bipartite lattice, Phys. Rev. B 103, L041402 (2021).

[56] R. V. Bhat and S. Bera, Out of equilibrium chiral higher order topological insulator on a π-flux square lattice, J. Phys.: Condens. Matter 33, 164005 (2021).

[57] S. Franca, F. Hassler, and I. C. Fulga, Simulating Floquet topological phases in static systems, SciPost Phys. Core 4, 007 (2021).

[58] R.-X. Zhang and Z.-C. Yang, Tunable fragile topology in Floquet systems, Phys. Rev. B 103, L121115 (2021).

[59] W. Zhu, Y. D. Chong, and J. Gong, Symmetry analysis of anomalous Floquet topological phases, Phys. Rev. B 104, L020302 (2021).

[60] L. Zhou, Floquet Second-Order Topological Phases in Momentum Space, Nanomaterials 11, 1170 (2021).
[61] W. Zhu, H. Xue, J. Gong, Y. Chong, and B. Zhang, Time-periodic corner states from Floquet higher-order topology, Nat. Commun. 13, 11 (2022).

[62] J. Jin, L. He, J. Lu, E. J. Mele, and B. Zhen, Floquet Quadrupole Photonic Crystals Protected by Space-Time Symmetry, Phys. Rev. Lett. 129, 063902 (2022).

[63] S. Franca, F. Hassler, and I. C. Fulga, Topological reflection matrix, Phys. Rev. B 105, 155121 (2022).

[64] A. K. Ghosh, T. Nag, and A. Saha, Systematic generation of the cascade of anomalous dynamical first- and higher-order modes in Floquet topological insulators, Phys. Rev. B 105, 115418 (2022).

[65] Z. Ning, B. Fu, D.-H. Xu, and R. Wang, Tailoring quadrupole topological insulators with periodic driving and disorder, Phys. Rev. B 105, L201114 (2022).

[66] Y. Lei, X.-W. Luo, and S. Zhang, Second-order topological insulator in periodically driven optical lattices, Opt. Express 30, 24048-24061 (2022).

[67] B. Huang, V. Novičenko, A. Eckardt, and G. Juzeliūnas, Floquet chiral hinge modes and their interplay with Weyl physics in a three-dimensional lattice, Phys. Rev. B 104, 104312 (2021).

[68] B.-Q. Wang, H. Wu, and J.-H. An, Engineering exotic second-order topological semimetals by periodic driving, Phys. Rev. B 104, 205117 (2021).

[69] W. Zhu, M. Umer, and J. Gong, Floquet higher-order Weyl and nexus semimetals, Phys. Rev. Research 3, L032026 (2021).

[70] S. Ghosh, K. Saha, and K. Sengupta, Hinge-mode dynamics of periodically driven higher-order Weyl semimetals, Phys. Rev. B 105, 224312 (2022).

[71] X.-L. Du, R. Chen, R. Wang, and D.-H. Xu, Weyl nodes with higher-order topology in an optically driven nodal-line semimetal, Phys. Rev. B 105, L081102 (2022).

[72] K. Plekhanov, M. Thakurathi, D. Loss, and J. Klinovaja, Floquet second-order topological superconductor driven via ferromagnetic resonance, Phys. Rev. Research 1, 032013(R) (2019).

[73] R. W. Bomantara and J. Gong, Measurement-only quantum computation with Floquet Majorana corner modes, Phys. Rev. B 101, 085401 (2020).

[74] R. W. Bomantara, Time-induced second-order topological superconductors, Phys. Rev. Research 2, 033495 (2020).

[75] Y. Peng, Floquet higher-order topological insulators and superconductors with space-time symmetries, Phys. Rev. Research 2, 013124 (2020).
[76] S. Chaudhary, A. Haim, Y. Peng, and G. Refael, Phonon-induced Floquet topological phases protected by space-time symmetries, Phys. Rev. Research 2, 043431 (2020).
[77] D. Vu, R.-X. Zhang, Z.-C. Yang, and S. D. Sarma, Superconductors with anomalous Floquet higher-order topology, Phys. Rev. B 104, L140502 (2021).
[78] A. K. Ghosh, T. Nag, and A. Saha, Floquet generation of a second-order topological superconductor, Phys. Rev. B 103, 045424 (2021).
[79] A. K. Ghosh, T. Nag, and A. Saha, Floquet second order topological superconductor based on unconventional pairing, Phys. Rev. B 103, 085413 (2021).
[80] A. K. Ghosh, T. Nag, and A. Saha, Dynamical construction of quadrupolar and octupolar topological superconductors, Phys. Rev. B 105, 155406 (2022).
[81] F. Liu and K. Wakabayashi, Novel Topological Phase with a Zero Berry Curvature, Phys. Rev. Lett. 118, 076803 (2017).
[82] Y.-B. Yang, K. Li, L.-M. Duan, and Y. Xu, Type-II quadrupole topological insulators, Phys. Rev. Research 2, 033029 (2020).
[83] M. Ezawa, Edge-corner correspondence: Boundary-obstructed topological phases with chiral symmetry, Phys. Rev. B 102, 121405(R) (2020).
[84] K. Asaga and T. Fukui, Boundary-obstructed topological phases of a massive Dirac fermion in a magnetic field, Phys. Rev. B 102, 155102 (2020).
[85] C.-A. Li, B. Fu, Z.-A. Hu, J. Li, and S.-Q. Shen, Topological Phase Transitions in Disordered Electric Quadrupole Insulators, Phys. Rev. Lett. 125, 166801 (2020).
[86] E. Khalaf, W. A. Benalcazar, T. L. Hughes, and R. Queiroz, Boundary-obstructed topological phases, Phys. Rev. Research 3, 013239 (2021).
[87] W. Jia, X.-C. Zhou, L. Zhang, L. Zhang, X.-J. Liu, Unified characterization for higher-order topological phase transitions, arXiv:2209.10394 (2022).
[88] L. Zhou, Non-Hermitian Floquet topological superconductors with multiple Majorana edge modes, Phys. Rev. B 101, 014306 (2020).
[89] J. K. Asbóth and H. Obuse, Bulk-boundary correspondence for chiral symmetric quantum walks, Phys. Rev. B 88, 121406(R) (2013).
[90] J. K. Asbóth, B. Tarasinski, and P. Delplace, Chiral symmetry and bulk-boundary correspondence in periodically driven one-dimensional systems, Phys. Rev. B 90, 125143 (2014).
[91] D. Y. H. Ho and J. Gong, Topological effects in chiral symmetric driven systems, Phys. Rev. B 90, 195419 (2014).

[92] L. Zhou and Q. Du, Floquet topological phases with fourfold-degenerate edge modes in a driven spin-1/2 Creutz ladder, Phys. Rev. A 101, 033607 (2020).

[93] X. Liu, S. Tan, Q.-h. Wang, L. Zhou, and J. Gong, Floquet band engineering with Bloch oscillations, arXiv:2208.05260 (2022).

[94] D. Roy and T. Prosen, Random matrix spectral form factor in kicked interacting fermionic chains, Phys. Rev. E 102, 060202(R) (2020).

[95] G. Kells, D. Meidan, and P. W. Brouwer, Near-zero-energy end states in topologically trivial spin-orbit coupled superconducting nanowires with a smooth confinement, Phys. Rev. B 86, 100503(R) (2012).

[96] D. Roy, N. Bondyopadhaya, and S. Tewari, Topologically trivial zero-bias conductance peak in semiconductor Majorana wires from boundary effects, Phys. Rev. B 88, 020502(R) (2013).