DYNAMICS OF RELATIVISTIC RECONNECTION

Maxim Lyutikov1,2,3 and Dmitri Uzdensky4
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ABSTRACT

The dynamics of the steady state Sweet-Parker–type reconnection is analyzed in the relativistic regime when energy density in the inflowing region is dominated by the magnetic field. The structure of the reconnection layer (its thickness and inflow and outflow velocities) depends on the ratio of two large dimensionless parameters of the problem: the magnetization parameter \( \sigma \propto 1 \) (the ratio of the magnetic to particle energy densities in the inflowing region) and the Lundquist number \( S \). The inflow velocity may be relativistic (for \( S < \sigma \)) or nonrelativistic (for \( S > \sigma \)), while the outflowing plasma is always moving relativistically. For extremely magnetized plasmas with \( \sigma \gtrsim S^2 \), the inflow four-velocity becomes of the order of the Alfvén four-velocity.

Subject headings: magnetic fields — plasmas — relativity

1. INTRODUCTION

Magnetic reconnection is widely recognized as a very important phenomenon in many laboratory and astrophysical plasmas (Biskamp 2000; Priest & Forbes 2000). It has been studied very extensively over the last 40 years, and very significant progress has been made in understanding this process. However, historically, reconnection was of interest mostly to space physicists studying the solar corona and the Earth’s magnetosphere and to researchers in magnetic confinement fusion. In all these environments, plasma flows are nonrelativistic and the Alfvén velocity is usually much less than the speed of light (equivalently, the magnetic energy density is much smaller than the particle rest-mass energy density). Therefore, it is not surprising that most of the progress on the subject has been made in the nonrelativistic regime.

Over the last decade, however, it has been recognized that magnetic reconnection processes are also of great importance in high-energy astrophysics, in which dynamic behavior is often dominated by superstrong magnetic fields, with energy density \( B^2/(8\pi) \) larger than the rest energy of the matter \( \rho \). The best-studied (but yet not completely understood) case is of magnetized winds from pulsars. Models of the pulsar magnetosphere (Goldreich & Julian 1969; Arons & Scharlemann 1979; Ruderman & Sutherland 1975) predict that near the light cylinder, most of the spin-down luminosity of a pulsar should be in the form of Poynting flux. Other possible examples of relativistic strongly magnetized media include jets emanating from magnetized accretion disks around Galactic black holes and neutron stars as well as active galactic nuclei (AGNs; e.g., Beskin 1997; Lovelace et al. 2002), magnetospheres of magnetars (Thompson & Duncan 1996; Thompson, Lyutikov, & Kulkarni 2002), and gamma-ray bursters (GRBs; M. Lyutikov & R. Blandford 2003, in preparation).

Dissipation of such superstrong magnetic fields may play an important role both for the global dynamics of the system and as a way to produce high-energy emission. Magnetic reconnection has been proposed as the mechanism for the acceleration of pulsar winds (Coroniti 1990; Lyubarsky & Kirk 2001) and GRB outflows (Spruit, Daigène, & Drenkhahn 2001) and as a dissipation mechanism in AGN jets (Romanova & Lovelace 1992), soft gamma-ray repeaters (Thompson & Duncan 1996), and GRBs (M. Lyutikov & R. Blandford 2003, in preparation; Spruit et al. 2001). In the case of pulsar winds, there are strong arguments that effective dissipation of the magnetic field is, in fact, needed to account for the global dynamics of the Crab Nebula (Kennel & Coroniti 1984a, 1984b; see also Michel 1994; Coroniti 1990; Melatos & Melrose 1996; Lyubarsky & Kirk 2001).

This provides the motivation for studying magnetic reconnection in strongly relativistic plasmas (to be defined below). Despite the growing interest in relativistic magnetic reconnection, very little theoretical (let alone experimental!) work has been done on the subject so far. We are aware of only one analytical discussion of relativistic reconnection (Blackman & Field 1994) and two recent numerical works on particle dynamics in relativistic reconnection layers (Zenitani & Hoshino 2001; Larrabee, Lovelace, & Romanova 2003).

In any theoretical analysis of magnetic reconnection, one first makes a number of approximations, e.g., incompressibility, two-dimensionality, and the absence (or presence) of the axial magnetic field. Then one formulates the set of MHD equations in a dimensionless form in which the relative importance of various physical processes is represented by certain dimensionless parameters. After that, one then tries to build a qualitative description of the reconnecting system with a small number of (also dimensionless) characteristic ratios. For example, in the simplest Sweet-Parker model of reconnection (e.g., Priest & Forbes 2000), one assumes incompressibility, uniform and constant resistivity \( \eta \), and so on, and one finds that the principal dimensionless parameter governing the system’s behavior is the Lundquist number \( S \equiv V_A L/\eta \gg 1 \), where \( V_A \) is the upstream Alfvén velocity and \( L \) is the size of the system. The other dimensionless plasma parameter \( \beta \), the ratio of thermal pressure to magnetic field energy density, turns out not to be as...
important, at least in the first approximation. Correspondingly, \( S^{-1} \) plays the role of a small parameter on which all further asymptotic expansions and boundary layer analysis are based. One then seeks to find out how the dimensionless characteristics of the system, such as the reconnection layer’s aspect ratio \( \delta/L \) and the ratio of the incoming velocity to the Alfvén velocity, scale with \( S \) as \( S \to \infty \).

The generalization of reconnection to the relativistic case requires an introduction of one more (in addition to \( S \)) principal dimensionless parameter that should describe how far in the relativistic regime we are. This parameter is \( \sigma \), the ratio of the magnetic field energy density to the plasma’s rest-mass energy density in the inflow region. We are now in a position to define what we mean by relativistic reconnection: we consider a case in which the magnetic energy density in the flowing plasma dominates over the particle energy density: \( \sigma \gg 1 \). During reconnection magnetic energy is dissipated and transformed into plasma thermal energy and later into bulk motion. Since \( \sigma \gg 1 \), the energy per baryon becomes much larger than \( m_pc^2 \), and so we get relativistically hot plasma in the reconnection layer. Expansion of plasma along the reconnection layer will produce relativistic bulk motions downstream. In the relativistic regime, \( \sigma \gg 1 \), the Alfvén velocity becomes relativistic, \( V_\Lambda = [\sigma/(1 + \sigma)]^{1/2} c \approx c \), so that under certain conditions (to be determined later) the upstream flow is expected to be relativistic as well.

Thus, in the case of relativistic reconnection there are two very large dimensionless parameters, \( S \) and \( \sigma \). As we see below, one gets two different regimes depending on the ratio of these parameters: in one regime the incoming (upstream) flow is ultrarelativistic, while in the other it is nonrelativistic.

In this paper we present a relativistic generalization of the simplest model of magnetic reconnection, the Sweet-Parker model, in strongly relativistic plasmas. This is a simple two-dimensional resistive MHD model, presented in Figure 1. We assume that the reconnection layer has a rectangular shape with a width \( L \) and thickness \( \delta \ll L \). The width \( L \) of the reconnection layer is determined by the global system size and thus, for the purposes of studying magnetic reconnection, is a fixed prescribed quantity. Also prescribed are the magnetic field strength and the baryon density and pressure in the ideal-MHD inflow region above and below the reconnection layer. In contrast, the thickness \( \delta \) of the reconnection region, as well as some other parameters such as the plasma inflow and outflow velocity, are not prescribed and need to be calculated as a part of the analysis.

We basically follow the steps of the Sweet-Parker analysis while taking into account relativistic effects, such as relativistic contraction and inertia of the magnetic field. As the plasma enters the reconnection layer, it slows down, coming to a halt at a stagnation point. At the same time, the magnetic energy is dissipated and converted into internal energy of the pair-rich plasma. In the outflowing region the plasma is accelerated by the pressure gradient in the \( x \)-direction, reaching some terminal relativistic velocity \( \gamma \)\( \text{out} \).

We assume that energy losses are not important and that the total energy of a fluid element (or at least a large fraction) stays within this fluid element, providing the corresponding amount of pressure support. Although generally radiative losses may be important, we expect that the reconnecting plasma will be optically thick to Thomson scattering after its temperature becomes weakly relativistic, \( T \geq 20 \) keV. Above this temperature an efficient pair-production process will start (e.g., Goodman 1986) inside the reconnection current layer trapping the radiation. The role of pair production in increasing the optical depth is an interesting question in itself and deserves further study (Thompson 1994). It lies, however, outside the scope of our paper; here we simply assume that the plasma is optically thick inside the reconnection layer and so the released magnetic field energy cannot leave the system and is therefore available for accelerating plasma downstream.

2. RELATIVISTIC RECONNECTION FORMULATION

The basic equations include the relativistic Ohm’s law and the relativistic dynamics, Maxwell’s, and mass-conservation equations (Lichnerowicz 1967):

\[
T^\gamma_j = 0 ,
\]

\[
F^\gamma_j = 0 ,
\]

\[
(\rho u_i^i)_j = 0 ,
\]

where

\[
T^{\gamma j} = (\rho + b^2 + \epsilon) u^j u^l + \left( p + \frac{b^2 + \epsilon^2}{2} \right) g^{ij} - b^i b^j - \epsilon \epsilon^j
\]

is the stress-energy tensor, \( \rho \) is the proper plasma density, \( p \) is the pressure, \( b^2 = h^i h_i \) and \( \epsilon^2 = \epsilon \epsilon_i \) are the plasma proper magnetic and electric energy density times \( 4\pi \), \( u^i = (\gamma, \gamma \mathbf{B}) \) are the plasma four-velocity, Lorentz factor, and three-velocity, \( g^{ij} \) is the metric tensor, \( h_i = \frac{1}{4} \eta_{ijk} u^k \) are the four-vector of the magnetic field, Levi-Civita tensor, and electromagnetic field tensor, and \( \epsilon_i = u^l F_{il} \) is the four-vector of the electric field.

The choice of the stress-energy tensor deserves some discussion. The stress-energy tensor in equation (4) is a full electromagnetic field plus matter tensor; this is not the relativistic magnetohydrodynamic (RMHD) stress-energy...
tensor. The reason is that in the framework of RMHD, it is assumed that one of the electromagnetic invariants is not equal to 0 and the electromagnetic stress-energy tensor can be diagonalized. Equivalently, this implies that there is a reference frame in which the electric field is equal to 0. In the case of resistive RMHD, such a frame may not exist, since generally there are resistive electric fields in the plasma rest frame either along the null magnetic line (this then violates the \( \mathbf{B} \times \mathbf{E} > 0 \) condition of ideal MHD) or aligned with the electric fields (this violates the \( \mathbf{E} \cdot \mathbf{B} = 0 \) condition). In the full relativistic approach, the resistive electric field contributes to the plasma energy density, energy fluxes, and stresses. Nevertheless, we can still define the plasma rest frame by requiring that in that frame the electric fields \( \epsilon \) are only of a resistive nature. We then obtain the stress-energy tensor in equation (4).

In what follows we use both the rest-frame quantities \( (\mathbf{b}, \mathbf{e}, p, \omega, \rho_e^s, \rho) \) and \( \rho_e^s \) for the renormalized magnetic and electric fields, pressure, enthalpy, and charge and mass densities) as well as laboratory quantities \( \mathbf{B}, \mathbf{E}, \rho_e, \) and \( \rho \) (pressure and enthalpy are defined only in the rest frame).

2.1. Relativistic Ohm’s Law

The relativistic Ohm’s law is

\[
j' = \left( j \right) u^l + \frac{1}{4\pi\gamma} F_{jk} u_k , \tag{5}
\]

where \( \eta \) is the plasma resistivity (e.g., Lichnerowicz 1967). In three-dimensional notation (Greek indices 1, 2, 3), this gives

\[
j^0 \equiv \rho_e = \frac{\left( j \cdot \mathbf{B} \right)}{\beta^2} - \frac{1}{4\pi\gamma \beta^2} \mathbf{E} \cdot \mathbf{B} ,
\]

\[
j = \left[ j^0 - \left( j \cdot \mathbf{B} \right) \gamma^2 \beta + \frac{\gamma}{4\pi\gamma} \mathbf{E} + \left( \mathbf{B} \times \mathbf{E} \right) \right] , \tag{6}
\]

which can be written

\[
\left( \beta^{00} - \beta^0 \beta^0 \right) \left( j^0 - \frac{\gamma}{4\pi\gamma} \mathbf{E} \right) = \frac{\gamma}{4\pi\gamma} \left( \mathbf{B} \times \mathbf{E} \right) . \tag{7}
\]

The form in equation (7) of the relativistic Ohm’s law shows that for \( \left( j \cdot \mathbf{B} \right) = 0 \) and \( \mathbf{E} \cdot \mathbf{B} = 0 \), the relativistic effects change the conductivity:

\[
\eta \rightarrow \frac{\eta}{\gamma} . \tag{8}
\]

This can be understood if one notes that in the plasma rest frame, the electric field \( \mathbf{e} = \mathbf{E} / \gamma \), and, since for \( \left( j \cdot \mathbf{B} \right) = 0 \) the current in the rest frame is \( \mathbf{j} \),

\[
j = \frac{\mathbf{e}}{4\pi\gamma} . \tag{9}
\]

2.2. Main Equations

For a stationary flow the energy and momentum flux conservation can be rewritten as

\[
\nabla_a \left[ \gamma^2 (w + b^2) \beta^3 \right] = 0 ,
\]

\[
\nabla \beta \left( \gamma^2 w \beta^3 - p b^3 \right) = F_{\alpha \beta} j_\beta = \rho_e \mathbf{E}^\alpha + \left( \mathbf{j} \times \mathbf{B} \right)^\alpha . \tag{10}
\]

The Maxwell’s equations then become

\[
j = \frac{1}{4\pi} \mathbf{V} \times \mathbf{B} , \quad (\mathbf{V} \times \mathbf{E}) = 0 , \quad \text{div} \mathbf{E} = 4\pi \rho_e . \tag{11}
\]

The equation of continuity is

\[
\text{div} \gamma \beta \rho^s = 0 . \tag{12}
\]

The above equations plus the equation of state form a system of 15 equations for 15 variables, \( \beta, \mathbf{E}, \mathbf{B}, j, \rho_e, \rho, \) and \( p \).

Following the Sweet-Parker model, we neglect possible field-aligned electric fields and currents. Then, the only nonvanishing components of the electric field and current are \( E_z \) and \( j_z \). Since the velocity lies in the \( x-y \) plane from equation (6), it follows that the charge density \( \rho_e \equiv j^0 = 0 \). This is an important simplification, since generally for a relativistic plasma the charge density cannot be neglected.

2.3. Magnetization Parameter

We assume that far in the incoming region, plasma is cold and strongly magnetically dominated. In the incoming region, well outside the reconnection layer, the resistive electric field is vanishing and can be neglected. We introduce a frame-invariant ratio of the rest energy density of the magnetic field and particles:

\[
\sigma = \frac{\beta^0}{\rho^s} > 1 . \tag{13}
\]

This definition of \( \sigma \) is equivalent to the ratio of the Poynting to particle fluxes in the incoming region (e.g., Kennel & Coroniti 1984a, 1984b). For example, for pulsar winds, initially (near the light cylinder), \( \sigma \sim 10^3-10^6 \) (e.g., Arons & Scharlemann 1979).

We have defined the \( \sigma \)-parameter in a frame-invariant way in terms of the rest-frame quantities. An alternative definition involves the laboratory frame fields and densities:

\[
\sigma_l = \frac{B_{in}^2}{\rho} = \gamma_m \sigma . \tag{14}
\]

It is straightforward to express the results in terms of \( \sigma_l \) instead of \( \sigma \).

3. FLOW ALONG THE VELOCITY SEPARATRIX

In nonrelativistic reconnection an important model problem is the flow of plasma along the velocity separatrix \( x = 0 \) (e.g., Priest & Forbes 2000). In this case a simple equation relating the magnetic field strength and inflow velocity can be obtained using only Ohm’s law (eq. [7]). Solving this equation for a given velocity profile (or vice versa for a given magnetic field) would allow us to find an example of velocity and magnetic distributions. The relativistic Ohm’s law (eq. [7]) along the velocity separatrix can be written as

\[
j(y) \equiv j_z (x = 0, y) = \frac{\gamma(y)}{4\pi\eta} (E_z - \beta_y B_x) . \tag{15}
\]

For a more compact notation, we introduce \( B(\gamma) \equiv B_z (x = 0, y) > 0 \) and also \( \beta(\gamma) \equiv -\beta_y (x = 0, y) > 0 \) (here we changed the sign so that \( \beta \) is positive for convenience).
Then, the above equation for Ohm’s law can be rewritten as
\[
j(y) \equiv j_c(x = 0, y) = \frac{\gamma(y)}{4\pi\eta} [E_z + \beta(y)B(y)] .
\]  
(16)

In a steady case, Maxwell’s equation gives \( \mathbf{V} \times \mathbf{E} = 0 \), and hence, using \( \partial / \partial z = 0 \),
\[
E_z(x, y) = \text{const} \equiv E .
\]  
(17)

In the ideal region above the layer, Ohm’s law gives
\[
E = -\beta_{in}B_{in} ,
\]  
(18)
where \( B_{in} \equiv B(y \gg \delta) \) and \( \beta_{in} \equiv \beta(y \gg \delta) \).

Using the first of the Maxwell equations (11) in the laboratory frame, we find
\[
\eta \frac{\partial B}{\partial y} = -\gamma(E + \beta B) ,
\]  
(19)
or, using our expression for \( E \),
\[
\eta \frac{\partial B}{\gamma} = \beta_{in} - \beta B ,
\]  
(20)
where \( \beta \equiv B / B_{in} \). This equation should be supplemented by the boundary conditions \( B(y = 0) = 0, \beta(y = 0) = 0, \beta(y \to \pm \infty) = \pm 1, \) and \( \beta(y \to \pm \infty) = \pm \beta_{in} \). Equation (20) relates the two functions, inflow velocity and magnetic field, along the separatrix. For example, for a given \( \beta(y) \), the two boundary conditions for the first-order ordinary differential equation (20) determine the solution \( B(y) \) and put a constraint on the parameters \( \eta, \beta_{in}, \) and \( \delta \).

For example, equation (20) can be resolved for \( B(y) \) for a given \( \beta(y) \):
\[
B(y) \equiv \frac{B(y)}{B_{in}} = \frac{\beta_{in}}{\eta} e^{-\gamma \left[ \int^{y} \beta(y') \gamma(y') dy' \right] / \eta} \times \int^{y} e^{\gamma \left[ \int^{y} \beta(y') \gamma(y') dy' \right] / \eta} \gamma(y') dy' .
\]  
(21)

The boundary condition \( B(y = 0) = 0 \) is automatically satisfied, while the condition \( B(y = \delta) = 1 \) serves as an eigenvalue problem for \( \delta(\eta, \beta_{in}) \).

As an example, consider a case in which the inflow four-velocity, \( u = \gamma / \beta \), is a linear function of distance along the \( y \)-axis:

\[
u = \begin{cases} u_{in} \frac{y}{\delta}, & |y| < \delta, \\ u_{in}, & |y| > \delta . \end{cases}
\]  
(22)

By definition, \( \delta \) is the scale on which the inflow velocity changes from initial \( u = u_{in} \) to 0.

Introducing the dimensionless parameter \( Y \equiv \delta / \eta \) (\( Y = c\delta / \eta \) in dimensional units) and rescaling the coordinate \( y \) by \( \delta \), \( \tilde{y} = y / \delta \), the magnetic field is then determined by
\[
\frac{B(\tilde{y} \leq 1)}{B_{in}} = \frac{u_{in} Y}{\sqrt{1 + u_{in}^2}} \int_{0}^{\tilde{y}} e^{-u_{in}^2 \tilde{y}^2 / 2} \int_{0}^{\tilde{y}} e^{u_{in}^2 \tilde{y}^2 / 2} \sqrt{1 + \tilde{y}^2 u_{in}^2} \tilde{y} .
\]  
(23)

Parameter \( Y \) here is an implicit function of \( u_{in} \) given by the condition \( B(1) = B_{in} \). The numerical solution \( Y(u_{in}) \) is plotted in Figure 2. The corresponding magnetic field profiles for different values of \( u_{in} \) are plotted in Figure 3.

Several important conclusions can be drawn from this exercise: (1) Both the four-velocity and magnetic field have the same typical scale \( \sim \delta \). (2) In both the nonrelativistic \( (u_{in} \ll 1) \) and strongly relativistic \( (u_{in} \gg 1) \) regimes, the ratio \( Y = c\delta / \eta \) asymptotically becomes inversely proportional to the four-velocity of the incoming flow, \( Y \sim 1 / u_{in} \), in agreement with the Sweet-Parker theory (in the nonrela-
tivistic case). Asymptotic scalings of $Y(u_{in})$ for $u_{in} \ll 1$ and $u_{in} \gg 1$ are analyzed in the Appendix. (3) In the case of strongly relativistic inflow, there is a thin sublayer near the neutral point, $y = 0$, with thickness $\delta_{SR} \sim \delta/u_{in}$, where the flow becomes nonrelativistic. Outside of this sublayer, the function $B(y)$ approaches a universal shape in the limit $u_{in} \to \infty$. Inside the sublayer, the magnetic field becomes linear:

$$B \sim \frac{\gamma}{\eta}B_{in} \text{ for } y \ll \frac{\delta}{u_{in}} .$$

A typical magnetic field in the nonrelativistic sublayer is

$$B_{SR} \sim \frac{\delta}{\eta u_{in}}B_{in} \sim \frac{B_{in}}{u_{in}} \ll B_{in} ,$$

while the typical current density is

$$j_{SR} \sim \frac{B_{in}}{\eta} \sim \frac{\gamma_{SR}B_{in}}{\gamma_{in}} .$$

Here lies a qualitative difference between the nonrelativistic and relativistic reconnection layer. In the nonrelativistic Sweet-Parker theory, the thickness of the layer is defined by the magnitude of the current density at the center $y = 0$. In the case of relativistic inflow ($\gamma_{SR} \gg 1$), the current flowing in the bulk of the flow $j \sim B_{SR}/\delta$ is much stronger than the typical current on the midplane.

The example in this section is an illustration only, invoked in order to demonstrate a possible relation between $\delta$ and $\beta_{SR}$. We use here only Ohm’s law to establish such a connection; in reality, the velocity profile needs to be determined self-consistently by solving the entire two-dimensional problem, including the equation of motion.

4. RELATIVISTIC SWEET-PARKER MODEL

In this section we use Ohm’s law and the conservation laws for energy and particle flux and do not solve the momentum equation. All the estimates below are order of magnitude or, more precisely, for how they scale with the two parameters $S$ and $\sigma$.

Estimating the current in the reconnection layer,

$$j_{c} \sim \frac{B}{\delta} \sim \frac{\gamma_{SR}\beta_{SR}B_{in}}{\eta} ,$$

we find

$$\beta_{SR} \sim \frac{\gamma_{SR}}{\gamma_{in}} \sim \frac{\eta}{\delta c}$$

(we have restored the velocity of light here to make the right-hand side explicitly dimensionless). Introducing the relativistic Lundquist number,

$$S = \frac{L_{c}}{\eta} \gg 1 ,$$

equation (28) can be rewritten

$$\beta_{SR} \sim \frac{\gamma_{SR}}{\gamma_{in}} \sim \frac{L}{\delta}S .$$

This is the first basic equation of the model.

The energy flux and particle conservation give

$$\gamma_{SR}^{2}\beta_{SR}(1 + \sigma)\rho_{SR}^{*}L = \gamma_{out}^{2}\beta_{out}^{*}\rho_{out}^{*} \delta ,$$

$$\gamma_{SR}\rho_{SR}^{*}\beta_{SR}L = \gamma_{out}\rho_{out}^{*}\beta_{out}^{*} \delta .$$

In writing down these equations we have assumed that the magnetic energy is fully spent on the acceleration of baryons in the downstream flow. This assumes that all the electron-positron pairs created in the reconnection layer have annihilated.

Then, it follows that

$$\gamma_{SR}(1 + \sigma) = \gamma_{out} ,$$

$$\delta \sim \frac{1}{S\beta_{SR}^{*}} , \quad \frac{\rho_{SR}^{*}\beta_{SR}}{\rho_{out}^{*}\beta_{out}^{*}} = (1 + \sigma)\delta .$$

Equation (32) relates the velocity in the outflowing region to the inflow velocity and magnetization parameter. For small $\sigma \ll 1$, equation (32) reproduces the familiar nonrelativistic result that the outflow velocity is of the order of the inflowing Alfvén speed: $\beta_{out} \sim (2\sigma)^{1/2} = B/c(\pi \rho)^{1/2} = V_{A}/c$. In the relativistic regime, $\sigma \gg 1$, the outflow velocity is always relativistic, so that $\beta_{out} \sim 1$. It is remarkable that in relativistic reconnection, the Lorentz factor of the outflowing plasma is much larger than the Lorentz factor of the Alfvén waves in the inflow region $\gamma_{SR} \sim 1$, so that $\gamma_{out} \sim \sigma \gamma_{in} = \gamma_{SR} \gamma_{in}$, where $\gamma_{SR} \sim \sqrt{\sigma}$.

Equations (32) and (33) are general relations for four unknown quantities: $\delta$, $\gamma_{out}$, $\sigma_{in}$, and $\rho_{SR}^{*}/\rho_{out}^{*}$. In order to resolve this system, we need to use the momentum equations and find the structure of the flow inside the reconnection region, but this is a difficult task. An alternative way to proceed is to assume incompressibility of the plasma in its rest frame; $\rho_{SR}^{*} = \rho_{out}^{*}$ (e.g., Blackman & Field 1994). The assumption of incompressibility is justified as long as the inflowing plasma velocity is smaller than the fast magneto-sonic velocity, which in our case is similar to the Alfvén velocity. In addition, a significant longitudinal field component $u_{L}$ also contributes to plasma incompressibility. This is most clearly seen for cold plasma, when fast velocity coincides with Alfvén velocity. Consider, for example, a one-dimensional steady motion of cold plasma along the $y$-axis carrying a substantial longitudinal field (with magnitude $b_{z}$ in the plasma rest frame) that slightly dominates over the reconnecting field. The equation of motion $\partial_{x}(u + b_{z}^{2})u + \frac{b_{z}^{2}}{2} = 0$ can be simplified using mass conservation, $\partial_{x}(\rho^{*}u) = 0$, and magnetic flux conservation, $\partial_{y}(b_{z}u) = 0$, to give

$$\rho^{*}u\partial_{x}u + \frac{1}{2}\partial_{y}\rho^{*}u = 0 ,$$

where we used $b_{z}^{2} = \sigma\rho^{*}$. Equation (34) shows that the first term, which is related to plasma compressibility through $\partial_{x}u/u = -\partial_{x}\rho^{*}/\rho^{*}$, becomes comparable to the second only when $u \sim \sqrt{\sigma} \sim u_{A}$.

Below we show that for a given value of $S$, the inflowing velocity increases with the magnetization parameter $\sigma$, reaching the Alfvén velocity at $\sigma \sim S^{2}$. For such strong magnetization, the compressibility of plasma becomes important. For any $\sigma \ll S^{2}$, the incompressibility is expected to be a good approximation.
The assumption of incompressibility gives in the relativistic limit \( \sigma \gg 1 \)

\[ \beta_\text{in} \gamma_\text{in} L = \gamma_\text{out} \delta, \tag{35} \]

\[ \beta_\text{in} = \frac{\delta}{L}, \tag{36} \]

(here we dropped \( \beta_\text{out} \) because we expect an ultrarelativistic outflow with \( \beta_\text{out} \approx 1 \)).

Equation (36) and the approximate Ohm’s law (eq. [30]) form a system of two equations for the thickness of the reconnection layer \( \delta \) and the inflow velocity \( \beta_\text{in} \). There are two generic regimes (limiting cases) of relativistic reconnection: (1) relativistic inflow, \( \gamma_\text{in} \gg 1 \), and (2) nonrelativistic inflow, \( \beta_\text{in} \ll 1 \).

4.1. Nonrelativistic Inflow: \( \beta_\text{in} \ll 1 \)

Using equations (36) and (30) we find

\[ \beta_\text{in} \sim \frac{L}{\delta} \frac{1}{S} \sim \frac{\sigma \delta}{L}. \tag{37} \]

Thus,

\[ \delta \sim \frac{1}{\sqrt{\sigma S}} \sim \frac{\beta_\text{in}}{\sigma} \sim \frac{1}{\sqrt{\gamma_\text{in} \beta_\text{in}}}, \tag{38} \]

and finally,

\[ \beta_\text{in} \sim \sqrt{\frac{\sigma}{S}} \sim \sqrt{\frac{2}{3}} \gamma_\text{A}, \tag{39} \]

where \( \gamma_\text{A} = (2\sigma)^{1/2} \) is the Lorentz factor of the Alfvén wave velocity in the incoming region. Since by assumption \( \beta_\text{in} \ll 1 \), the nonrelativistic inflow velocity is realized for \( \sigma \ll S \).

Thus, for a given \( \sigma \) the inflow velocity is inversely proportional to the square root of the Lundquist number, similar to the classical nonrelativistic Sweet-Parker model.

4.2. Relativistic Sub-Alfvénic Inflow: \( 1 \ll \gamma_\text{in} \ll (2\sigma)^{1/2} \)

When \( \beta_\text{in} \approx 1 \) from equation (36), it follows that

\[ \delta \sim \frac{1}{\sigma} \quad \gamma_\text{out} \sim \frac{\sigma^2}{S}, \quad \gamma_\text{in} \sim \frac{\sigma}{S}. \tag{40} \]

For consistency we need \( \sigma \gg S \) (since we have always assumed relativistic motion, \( \gamma \gg 1 \)).

The ratio of the inflowing plasma Lorentz factor to the Alfvén-wave Lorentz factor is

\[ \frac{\gamma_\text{in}}{\gamma_\text{A}} \sim \frac{\sqrt{\sigma/2}}{S}. \tag{41} \]

Since the inflowing plasma should be sub-Alfvénic for the incompressibility assumption to hold, it is required that

\[ \sigma \ll 2S^2. \tag{42} \]

Thus, the incompressible relativistic inflow case is applicable for

\[ S \ll \sigma \ll 2S^2. \tag{43} \]

4.3. Relativistic Alfvénic Inflow: \( \gamma_\text{in} \sim (2\sigma)^{1/2} \)

If \( \sigma \geq 2S^2 \) the required inflow velocity becomes of the order of the Alfvén velocity, and the assumption of incompressibility should break down. Since the inflow velocity cannot exceed the Alfvén velocity for causal reasons, we assume that \( \gamma_\text{in} \sim \gamma_\text{A} = (2\sigma)^{1/2} \). The parameters of the reconnection layer then become

\[ \gamma_\text{out} \sim \sqrt{2\sigma}^{3/2}, \quad \delta \sim \frac{1}{\sqrt{2\sigma S}}, \quad \frac{\beta_\text{in}^*}{\rho_\text{in}^*} \sim \frac{\sqrt{\sigma}}{\sqrt{2S}} \geq 1. \tag{45} \]

5. BOHM DIFFUSION

In § 4 we have derived how the flow variables (first of all the inflow velocity) scale with two parameters of the model, \( S \) and \( \sigma \). For astrophysical applications it is often useful to have a qualitative estimate of the maximum possible reconnection rate in a given system. The maximum reconnection rate corresponds to the maximum value for resistivity and thus the minimal Lundquist number. This resistivity can be estimated using Bohm’s arguments that the maximum diffusion coefficient in magnetized plasmas cannot be much larger than \( r_L v \), where \( r_L \) is the Larmor radius and \( v \) is the typical velocity of electrons (of the order of the speed of light in our case). Thus,

\[ \eta \sim \frac{e^2}{\omega_B}, \tag{46} \]

and we find

\[ S \sim \frac{L}{r_L}, \tag{47} \]

\[ \delta \sim \frac{r_L}{\beta_\text{in} \gamma_\text{in}}, \tag{48} \]

\[ \frac{\beta_\text{in}^2 \gamma_\text{in}}{\sigma} \sim \frac{r_L}{L}. \tag{49} \]

Note that, in this case, as \( \beta_\text{in} \) approaches the velocity of light, the reconnection layer becomes microscopically thin. One can expect that the fluid picture will become inapplicable at this point.

For a nonrelativistic inflow velocity, the assumption of Bohm diffusion gives

\[ \delta \sim \sqrt{\frac{r_L L}{\sigma}}, \quad \frac{\beta_\text{in}}{\rho_\text{in}} \sim \sqrt{\frac{\sigma r_L}{L}}, \tag{50} \]

while for a relativistic inflow velocity,

\[ \delta \sim \frac{r_L}{\gamma_\text{in}}, \quad \gamma_\text{in} \sim \frac{r_L}{L}, \tag{51} \]

which requires \( \sigma r_L / L > 1 \). In the case of relativistic inflow velocity and under the assumption of Bohm diffusion, the thickness of the reconnection layer becomes smaller than the external gyroradius.

Equations (49) and the two limiting cases (50) and (51) give useful “order-of-magnitude” estimates of the potential efficiency of reconnection in relativistic plasma.

6. DISCUSSION

We have considered the dynamics of the relativistic Sweet-Parker reconnection under the assumption that the
inflow region’s energy density is dominated by the magnetic field. We have found three generic regimes depending on the ratio of the magnetization parameter \( \sigma \) to the Lundquist number \( S \): (1) nonrelativistic inflow velocity, \( \sigma < S \); (2) relativistic sub-Alfvénic inflow velocity, \( S \ll \sigma \ll 2S^2 \); (3) relativistic Alfvénic inflow velocity, \( \sigma \geq 2S^2 \). For the first two regimes, the plasma flow can be assumed incompressible, while for the Alfvénic inflow velocity, compressibility is important.

An apparent drawback of our approach is that we did not solve in a self-consistent way both the momentum and energy equations. This is a common flaw of many models of reconnection based on the Sweet-Parker approach. One can say that the role of the energy equation is to determine how compressible the plasma is. Conventionally, the simplest Sweet-Parker model does not include the energy balance equation with all its subtleties arising from possible radiation and conducting cooling effects; instead, it just replaces the energy balance equation with the incompressibility condition. In the absence of strong cooling or when a strong axial magnetic field component is present, the compressibility effects are indeed not important, at least in a rough, order-of-magnitude analysis. One then combines the incompressibility condition with the momentum equation and Ohm’s law to arrive at the Sweet-Parker reconnection scaling. In our analysis we also assume incompressibility, but then we use the energy conservation instead of the momentum equation; in the absence of energy losses, these two approaches are, of course, equivalent and lead to the same results.

In spite of the incompressibility assumption, our approach represents a step forward in understanding relativistic reconnection as compared to the view of Blackman & Field (1994). Using the same incompressibility assumption, they were able to determine only the ratio of inflow and outflow velocities, while we find both these quantities separately, expressed in terms of external magnetization and the Lundquist number. We have also found a possible structure of the relativistic reconnection layer.

For astrophysical applications, we were able to provide order-of-magnitude estimates of reconnection rates in relativistic plasma (eq. [49]), which should be used instead of the often ad hoc assumptions of reconnection rates in strongly magnetized plasmas of pulsar winds and AGN jets. An important result is that under certain conditions, the inflow velocity may become relativistic, ensuring very efficient dissipation of magnetic energy.

Finally, we give a short comparison of the present work with that of Larrabee et al. (2003), which in many ways can be considered as complementary. Unlike the present paper, which treats the hydrodynamic behavior of plasma and the structure of the reconnection layer in the “\( \gamma \)-direction,” that of Larrabee et al. (2003) deals mostly with the “\( x \)-direction” structure of the infinitely thin (in the \( \gamma \)-direction) current sheet and with particle trajectories, corresponding energy gains, and spectra using a kinetic treatment of particle orbits in the X-type reconnection point.

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**APPENDIX**

**ASYMPTOTIC SCALING OF \( Y(u_{in}) \)**

In this appendix we investigate the asymptotic behavior of the function \( Y(u_{in}) \) in the case of the four-velocity field \( u(y) \) described by equation (22) for two limiting regimes, \( u_{in} \to 0 \) and \( u_{in} \to \infty \).

We start by rewriting equation (22) in terms of the rescaled coordinate \( \tilde{y} = y/\delta \):

\[
\frac{\partial B(\tilde{y})}{\partial \tilde{y}} = Y\gamma (\beta_{in} - \beta \tilde{B}) \tag{A1}
\]

Now let us integrate this equation from 0 to 1 and use the boundary conditions \( \dot{B}(0) = 0 \) and \( \dot{B}(1) = 1 \). We get

\[
\dot{B}(1) - \dot{B}(0) = 1 = Y\beta_{in} \int_0^1 \gamma(\tilde{y}) \, d\tilde{y} - Y \int_0^1 u(\tilde{y}) \dot{B}(\tilde{y}) \, d\tilde{y} \tag{A2}
\]

For the velocity field given by equation (22), i.e., \( u(\tilde{y}) = u_{in} \tilde{y} \), we then have

\[
Y^{-1} = \beta_{in} \int_0^1 \gamma(\tilde{y}) \, d\tilde{y} - u_{in} \int_0^1 \tilde{y} \dot{B}(\tilde{y}) \, d\tilde{y} \tag{A3}
\]

Using the relationship \( \gamma = (1 + u^2)^{1/2} \), the integral in the first term on the right-hand side can be computed exactly, with the result

\[
\int_0^1 \gamma(\tilde{y}) \, d\tilde{y} = \frac{1}{u_{in}} \int_0^{u_{in}} \frac{1}{\sqrt{1 + u^2}} \, du = \frac{\sqrt{1 + u_{in}^2}}{2} + \frac{1}{2u_{in}} \ln \left( u_{in} + \sqrt{1 + u_{in}^2} \right) \tag{A4}
\]

In the strongly relativistic limit \( u_{in} \to \infty \), this expression can be expanded as

\[
\int_0^1 \gamma(\tilde{y}) \, d\tilde{y} = \frac{u_{in}}{2} + \frac{\ln u_{in}}{2u_{in}} + O(1) \tag{A5}
\]
As for the second term, we note that in the strongly relativistic limit there is a thin nonrelativistic boundary sublayer of thickness \( \delta_{nr} \sim \delta/u_{in} \); inside this sublayer, \( B(y) \) behaves linearly, with a slope that, as we see below, is inversely proportional to \( u_{in} \). However, what is important for us here, outside of this infinitesimally thin sublayer, is that the function \( \tilde{B}(\tilde{y}) \) approaches a certain universal shape, \( \tilde{B}_{rel}(\tilde{y}) \), in the limit \( u_{in} \to \infty \). This conclusion follows from our numerical solutions presented in Figure 3. Ignoring the nonrelativistic sublayer’s contribution to the integral in the second term on the right-hand side of equation (A3), we can then estimate this integral as

\[
\int_0^1 \tilde{y} \tilde{B}(\tilde{y}) \, d\tilde{y} \to \int_0^1 \tilde{y} \tilde{B}_{rel}(\tilde{y}) \, d\tilde{y} \quad \text{as } u_{in} \to \infty .
\]  

(A6)

Combining this result with the result in equation (A5) derived for the first term on the right-hand side of equation (A3), we can finally write the following expression:

\[
Y = \frac{1}{A_{rel}u_{in} + (1/2) \ln u_{in} + C} ,
\]

where we used the identity \( \int_0^1 \tilde{y} \, d\tilde{y} = \frac{1}{2} \)

(A7)

and defined

\[
A_{rel} \equiv \int_0^1 \tilde{y} [1 - \tilde{B}_{rel}(\tilde{y})] \, d\tilde{y} = \text{const} .
\]

(A8)

From our numerical solutions with very high (of the order of 10^3) values of \( u_{in} \), we find \( A_{rel} \approx 0.027 \) and \( C = -1 \). Thus, we get

\[
Y(u_{in} \to \infty) \approx \frac{37}{u_{in}} \left( 1 - \frac{18.5 \ln u_{in}}{u_{in}} + \ldots \right) .
\]

(A9)

Note that, because of the rather large (18.5) numerical coefficient multiplying the logarithmic factor, this logarithmic correction does not become negligible until one considers very large values of \( u_{in} \), of the order of 10^3 or more. This is the reason why the plot in Figure 2 (with \( u_{in} \) up to 200) does not show a very good agreement with the simple power law \( Y \sim 1/u_{in} \). Note, however, that the numerical results for values of \( u_{in} \) in the range between 200 and 2000 are in excellent agreement with the predicted asymptotic behavior in equation (A7), as demonstrated separately in Figure 4.

Now let us consider the nonrelativistic regime, \( u_{in} \ll 1 \). In this limit, the expression (A4) for the integral entering the first term on the right-hand side of equation (A3) tends to 1 [this is, of course, a trivial result since in this limit \( \gamma(\tilde{y}) \approx 1 \)]. Then, since \( \tilde{\gamma}_{nr} \approx u_{in} \) in this regime, this first term can be evaluated simply as \( u_{in} \).

At the same time, just as in the strongly relativistic case, the function \( \tilde{B}(\tilde{y}) \) also approaches a certain universal profile, as can be seen in Figure 3 [we call this profile \( \tilde{B}_{nr}(\tilde{y}) \)]. Then, the integral entering the second term on the right-hand side of equation (A3) asymptotically approaches a constant value,

\[
\int_0^1 \tilde{y} \tilde{B}(\tilde{y}) \, d\tilde{y} \to \frac{1}{2} - A_{nr} = \text{const} ,
\]

(A10)

Note that, because of the rather large (18.5) numerical coefficient multiplying the logarithmic factor, this logarithmic correction does not become negligible until one considers very large values of \( u_{in} \), of the order of 10^3 or more. This is the reason why the plot in Figure 2 (with \( u_{in} \) up to 200) does not show a very good agreement with the simple power law \( Y \sim 1/u_{in} \). Note, however, that the numerical results for values of \( u_{in} \) in the range between 200 and 2000 are in excellent agreement with the predicted asymptotic behavior in equation (A7), as demonstrated separately in Figure 4.

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At the same time, just as in the strongly relativistic case, the function \( \tilde{B}(\tilde{y}) \) also approaches a certain universal profile, as can be seen in Figure 3 [we call this profile \( \tilde{B}_{nr}(\tilde{y}) \)]. Then, the integral entering the second term on the right-hand side of equation (A3) asymptotically approaches a constant value,
where we defined $A_{nr}$ in a manner similar to $A_{rel}$:

$$A_{nr} \equiv \int_0^1 \tilde{y}[1 - B_{nr}(\tilde{y})] d\tilde{y}.$$  

(A11)

Thus, we see that in the nonrelativistic limit, the function $Y(u_{in})$ asymptotically approaches an inverse power law:

$$Y(u_{in}) \simeq \frac{1}{u_{in} (1/2) + A_{nr}} \simeq \frac{1.71}{u_{in}},$$  

(A12)

where we used the value of $A_{nr} = 0.0855$ computed using our numerical solution for the case $u_{in} = 0.01$. The resulting asymptotic behavior in the nonrelativistic regime agrees very well with the numerical results, as can be seen in Figure 2.

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