Sobolev regularity of the canonical solutions to $\bar{\partial}$ on product domains

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Abstract

Let $\Omega$ be a product domain in $\mathbb{C}^n$, $n \geq 2$, where each slice has smooth boundary. We observe that the canonical solution operator for the $\bar{\partial}$ equation on $\Omega$ is bounded in $W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+$, $1 < p < \infty$. This Sobolev regularity is sharp in view of Kerzman-type examples.

1 Introduction

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 1$. According to Hörmander’s $L^2$ theory, given a $\bar{\partial}$-closed $(0,1)$ form $f \in L^2(\Omega)$, there exists a unique $L^2$ function that is perpendicular to $\ker(\bar{\partial})$ and solves

$$\bar{\partial}u = f \quad \text{in} \quad \Omega.$$ 

This solution is called the canonical solution (of the $\bar{\partial}$ equation). The $L^2$-Sobolev regularity of the canonical solutions has been investigated through Kohn’s $\bar{\partial}$-Neumann approach for domains with nice regularity and geometry, such as convexity and/or finite type conditions.

The goal of the paper is to give the $L^p$-Sobolev estimate of the canonical solutions on product domains. Here a product domain $\Omega$ in $\mathbb{C}^n$ is a Cartesian product $D_1 \times \cdots \times D_n$ of bounded planar domains $D_j$, $j = 1, \ldots, n$. In particular, $D_j$ needs not be simply-connected. Then $\Omega$ is (weakly) pseudoconvex with at most Lipschitz boundary. The $L^p$ regularity of the canonical solutions on product domains was already thoroughly understood through works of [3–5, 9, 10, 16, 18] and the references therein. In the Sobolev category, combined efforts in [2, 8, 14, 17] have given the existence of a bounded solution operator of $\bar{\partial}$ sending $W^{k+n-2,p}(\Omega)$ into $W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+$, $1 < p < \infty$. Here $W^{k,p}(\Omega)$ is the Sobolev space consisting of functions whose weak derivatives on $\Omega$ up to order $k$ exist and belong to $L^p(\Omega)$. The main theorem is stated as follows.

Theorem 1.1. Let $\Omega := D_1 \times \cdots \times D_n \subset \mathbb{C}^n$, $n \geq 2$, where each $D_j$ is a bounded domain in $\mathbb{C}$ with smooth boundary, $j = 1, \ldots, n$. Given a $\bar{\partial}$-closed $(0,1)$ form $f \in W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+$, $1 < p < \infty$, the canonical solution $Tf$ of $\bar{\partial}u = f$ on $\Omega$ is in $W^{k,p}(\Omega)$. Moreover, there exists a constant $C$ dependent only on $\Omega$, $k$ and $p$ such that

$$\|Tf\|_{W^{k,p}(\Omega)} \leq C\|f\|_{W^{k,p}(\Omega)}.$$ 

The proof of Theorem 1.1 is essentially an observation on a representation formula of the canonical solutions introduced by Li [10], according to which it boils down to the Sobolev estimates of the Bergman projection and canonical solution operators on planar domains. On the other hand,
with an application of a formula of Spencer on planar domains, the Sobolev estimates of these two operators are simply a consequence of a result of Jerison and Kenig in [7]. In Example 1, a datum \( f \) on the bidisc is constructed, such that \( f \in W^{k,q} \) for all \( 1 < q < p \), yet \( \bar{\partial}u = f \) has no \( W^{k,p} \) solutions. This example indicates that the \( \bar{\partial} \) problem does not gain Sobolev regularity on product domains in general, and thus the estimate in Theorem 1.1 is sharp.

Acknowledgement: The author thanks Professor Song-Ying Li for helpful comments and suggestions.

2 Bergman projection and canonical solutions on planar domains

Let \( D \) be a bounded domain in \( \mathbb{C} \) whose boundary \( bD \) is smooth, and \( g \) be the Green’s function on \( D \). In other words, at a fixed pole \( w \in D \),

\[
g(z, w) := -\frac{1}{2\pi} \sup \left\{ u(z) : u \in SH^-(D) \text{ and } \limsup_{\zeta \to w} (u(\zeta) - \log |\zeta - w|) < \infty \right\}, \quad z \in D,
\]

where \( SH^-(D) \) is the collection of negative subharmonic functions on \( D \). It is known ( [6] etc.) that \( g \) is symmetric on the two variables \( z \) and \( w \). Moreover, there exists a harmonic function \( h_w \) on \( D \) with \( h_w = \frac{1}{2\pi} \ln |\cdot - w| \) on \( bD \) such that

\[
g(\cdot, w) = -\frac{1}{2\pi} \ln |\cdot - w| + h_w \quad \text{in} \quad D. \tag{2.1}
\]

In particular, \( h_w \in C^\infty(D) \) and

\[
g(z, w) = g(w, z) = 0, \quad z \in bD. \tag{2.2}
\]

Given \( f \in L^p(D), 1 < p < \infty \), define

\[
Gf := -4 \int_D g(\cdot, w)f(w)d\nu_w \quad \text{in} \quad D. \tag{2.3}
\]

Here \( d\nu \) is the Lebesgue measure on \( \mathbb{C} \). Then \( Gf \) is the solution to the Dirichlet problem

\[
\begin{cases}
\Delta u = 4f, & \text{in} \quad D; \\
u = 0, & \text{on} \quad bD.
\end{cases}
\]

Moreover, \( G \) is a bounded operator sending \( W^{\alpha-2,p}(D) \) into \( W^{\alpha,p}(D), 1 < p < \infty, \alpha > \frac{1}{p} \). See [7, Theorem 0.3] by Jerison and Kenig. In particular, if \( f \in W^{k-1,p}(D), k \in \mathbb{Z}^+ \cup \{0\}, 1 < p < \infty \), then

\[
\|Gf\|_{W^{k+1,p}(D)} \lesssim \|f\|_{W^{k-1,p}(D)}. \tag{2.4}
\]

Here and throughout the rest of the paper, we say two quantities \( a \) and \( b \) to satisfy \( a \lesssim b \) if there exists a constant \( C \) dependent only possibly on the underlying domain, \( k \) and \( p \) such that \( a \leq Cb \).

The Bergman projection operator \( P \) on a domain \( \Omega \) is the orthogonal projection of \( L^2(\Omega) \) onto the Bergman space \( A^2(\Omega) \), the space of \( L^2 \) holomorphic functions on \( \Omega \). Since \( A^2(\Omega) \) is a
reproducing kernel Hilbert space, there exists a function \( k : \Omega \times \Omega \to \mathbb{C} \), called the Bergman kernel, such that for all \( f \in L^2(\Omega) \),

\[
P f = \int_{\Omega} k(\cdot, w) f(w) d\nu_w \quad \text{in \( \Omega \).}
\]

On a smooth planar domain \( D \), the Bergman kernel \( k \) is related to the Green’s function \( g \) by

\[
k(z, w) = -4\partial_z \partial_{\bar{w}} g(z, w), \quad z \neq w \in D.
\]

See [15, pp 180]. Clearly, \( k(\cdot, w) \in C^\infty(\bar{D}) \) by (2.1).

If \( D \) is simply-connected, the Sobolev boundedness of the Bergman projection \( P \) can be obtained by applying the known Sobolev regularity on the unit disc and the Riemann mapping theorem. On general smooth planar domains, Lanzani and Stein suggested an approach to estimate \( P \) briefly in [11]. For completeness and convenience of the reader, the detail of their approach to the Sobolev regularity of \( P \) is provided below.

**Theorem 2.1.** Let \( D \subset \mathbb{C} \) be a bounded domain with \( C^\infty \) boundary. Then the Bergman projection \( P \) is (or, extends as) a bounded operator on \( W^{k, p}(D) \), \( k \in \mathbb{Z}^+ \cup \{0\} \), \( 1 < p < \infty \). Namely, for any \( f \in W^{k, p}(D) \),

\[
\| P f \|_{W^{k, p}(D)} \lesssim \| f \|_{W^{k, p}(D)}.
\]

**Proof.** We shall need the following Spencer’s formula: for any \( f \in L^2(D) \),

\[
P f + \partial G \bar{\partial} f = f \quad \text{in \( D \),}
\]

where \( G \) is defined in (2.3). Note that \( \partial G \bar{\partial} f \) is well-defined by (2.4). The proof of (2.6) can be found, for instance, in [15, pp. 73-75]. Here we give a short and direct proof. First assume \( f \in C^\infty(\bar{D}) \). Fix \( z \in D \) and let \( D(z; \epsilon) \) be the disc centered at \( z \) with radius \( \epsilon > 0 \). By (2.5) and the Stokes’ theorem,

\[
P f(z) + \partial G \bar{\partial} f(z) = -4 \lim_{\epsilon \to 0} \left( \int_{D \setminus D(z; \epsilon)} \partial_{\bar{w}} \partial_z g(z, w) f(w) d\nu_w - \int_{D \setminus D(z; \epsilon)} \partial_z g(z, w) \partial_{\bar{w}} f(w) d\nu_w \right)
\]

\[
= 2i \int_{bD} \partial_z g(z, w) f(w) dw - 2i \lim_{\epsilon \to 0} \int_{bD(z; \epsilon)} \partial_z g(z, w) f(w) dw =: I_1 + I_2.
\]

Making use of (2.2) we have

\[
I_1 = 2i \partial_z \int_{bD} g(z, w) f(w) dw = 0.
\]

For \( I_2 \), note that by (2.1) there exists some bounded function \( h \) on \( D(z; \epsilon) \) such that

\[
\partial_z g(z, \cdot) = \frac{1}{4\pi (\cdot - z)} + h \quad \text{in \( D(z; \epsilon) \).}
\]

By continuity of \( f \),

\[
I_2 = -2i \lim_{\epsilon \to 0} \int_{bD(z; \epsilon)} \partial_z g(z, w) f(w) dw = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{bD(z; \epsilon)} \frac{f(w)}{w - z} dw = f(z).
\]
Hence (2.6) holds for \( f \in C^\infty(\bar{D}) \). For general \( f \in L^2(D) \), choose \( \{f_m\}_{m=1}^\infty \subset C^\infty(\bar{D}) \) that converges to \( f \) in \( L^2(D) \) norm as \( m \to \infty \) (see, for instance, [6, pp. 268]). Employing a standard density argument, we obtain (2.6) by the trivial boundedness of \( P \) in \( L^2(D) \), and the estimate (2.4) for \( G \) with \( k = 0 \) and \( p = 2 \).

For \( f \in L^p(D), 1 < p < 2 \), one uses (2.6) to extend \( P \) on \( f \) by defining \( Pf := \lim_{m \to \infty} f_m - \partial G\bar{f}_m \), where the family \( \{f_m\}_{m=1}^\infty \subset L^2(D) \) converges to \( f \) in \( L^p(D) \) norm. Note that this limit exists and is independent of the choice of \( \{f_m\}_{m=1}^\infty \) due to (2.4). Moreover, for all \( 1 < p < \infty \), by (2.4) the extended operator \( P \) satisfies

\[
\|Pf\|_{W^{k,p}(D)} \lesssim \|f\|_{W^{k,p}(D)} + \|G\bar{f}\|_{W^{k+1,p}(D)} \lesssim \|f\|_{W^{k,p}(D)} + \|\partial \bar{f}\|_{W^{k-1,p}(D)} \lesssim \|f\|_{W^{k,p}(D)}.
\]

This completes the proof of the theorem.

\( \square \)

Given \( f \in L^p(D), 1 < p < \infty \), define

\[
Tf := \partial Gf \left( = -4\partial \int_D g(\cdot, w)f(w)d\nu_w \right) \text{ in } D. \tag{2.7}
\]

Then \( T \) is the canonical solution of \( \bar{\partial} \) on \( D \). Indeed, by (2) one first has \( \bar{\partial}Tf = \bar{\partial}\partial Gf = f \) on \( D \). On the other hand, for any \( h \in A^2(D) \),

\[
\langle Tf, h \rangle = \langle \bar{\partial}Tf, h \rangle = \langle Tf - PTf, h \rangle = \langle Tf - PTf, Ph \rangle = \langle PTf - PTf, h \rangle = 0,
\]

implying \( Tf \perp A^2(\Omega) \). Here in the first equality we used the fact that \( \bar{\partial}Tf = f \) on \( D \); in the second equality we used (2.6) with \( f \) replaced by \( Tf \); in the third equality we used the fact that \( Ph = h \) when \( h \in A^2(D) \); in the fourth equality we used the projection properties of \( P \), i.e., \( P^* = P = P^2 \). The Sobolev regularity of \( T \) below follows immediately from (2.4) and (2.7).

**Theorem 2.2.** Let \( D \) be a bounded domain in \( \mathbb{C} \) with smooth boundary. For each \( k \in \mathbb{Z}^+ \cup \{0\}, 1 < p < \infty \), the canonical solution operator \( T \) of \( \bar{\partial} \) on \( D \) defined in (2.7) is a bounded operator sending \( W^{k,p}(D) \) into \( W^{k+1,p}(D) \). Namely, for any \( f \in W^{k,p}(D) \),

\[
\|Tf\|_{W^{k+1,p}(D)} \lesssim \|f\|_{W^{k,p}(D)}.
\]

**Remark 2.3.** a). We can further make use of Theorem 2.2 and the Sobolev embedding theorem to conclude that the canonical solution operator \( T \) sends \( W^{k,\infty}(D) \) into \( C^{k,\alpha}(D) \) for all \( 0 < \alpha < 1 \) with

\[
\|Tf\|_{C^{k,\alpha}(D)} \leq C\|f\|_{W^{k,\infty}(D)},
\]

where \( C \) depends only on \( D, k \) and \( \alpha \). In particular, this inequality improves a supnorm estimate in [1].

b). Another well-known solution operator \( \tilde{T} \) of \( \bar{\partial} \) on \( D \) is given in terms of the universal Cauchy kernel as follows.

\[
\tilde{T}f := -\frac{1}{\pi} \int_D \frac{f(w)}{w \cdot} d\nu_w \text{ in } D.
\]

It was proved by Prats in [12] that \( \tilde{T} \) enjoys a similar Sobolev regularity as \( T \) (see also [13] for a much simpler proof using Caldrón-Zygmund’s classical singular integral theory):

\[
\|\tilde{T}f\|_{W^{k+1,p}(D)} \lesssim \|f\|_{W^{k,p}(D)}.
\]
3 Canonical solutions on product domains

Let $\Omega := D_1 \times \cdots \times D_n \subset \mathbb{C}^n$, $n \geq 2$, where each $D_j$ is a bounded planar domain with smooth boundary. Denote by $P_j$ the Bergman projection operator of $D_j, j = 1, \ldots, n$. Then the Bergman projection $P$ of $\Omega$ satisfies

$$P = P_1 \cdots P_n. \quad (3.1)$$

Let $T_j$ be the canonical solution operator on $D_j$ defined in (2.7), with $D$ replaced by $D_j$, $j = 1, \ldots, n$. Given a $\partial$-closed $(0, 1)$ form $f = \sum_{j=1}^n f_j d\zeta_j \in L^p(\Omega)$, it was shown in [10, Theorem 2.5] (or, through a repeated application of (2.6) together with the $\partial$-closedness of $f$) that

$$Tf = T_1f_1 + T_2P_1 f_2 + \cdots + T_nP_1 \cdots P_{n-1} f_n \quad (3.2)$$

is the canonical solution to $\bar{\partial}u = f$ on $\Omega$. The following proposition gives the Sobolev boundedness of $T_j$ and $P_j$ on $\Omega$.

**Proposition 3.1.** Let $\Omega := D_1 \times \cdots \times D_n \subset \mathbb{C}^n$, where each $D_j$ is a bounded domain in $\mathbb{C}$ with smooth boundary, $j = 1, \ldots, n$. Then $T_j$ and $P_j$ are bounded operators in $W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+ \cup \{0\}, 1 < p < \infty$. Namely, for all $f \in W^{k,p}(\Omega)$,

$$\|T_j f\|_{W^{k,p}(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)}; \quad \|P_j f\|_{W^{k,p}(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)}.$$

**Proof.** For simplicity yet without loss of generality, assume $j = 1$ and $n = 2$. Denote by $\nabla_j$ either $\partial_j$ or $\bar{\partial}_j$ in the $z_j$ variable. Since $\partial_1 T_1 = id$ and $\partial_1 P_1 = 0$, we only need to prove for all $k_1, k_2 \in \mathbb{Z}^+ \cup \{0\}$, $k_1 + k_2 = k$,

$$\|\partial_1^{k_1} T_1 \nabla_{z_2}^{k_2} f\|_{L^p(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)}; \quad \|\partial_1^{k_1} P_1 \nabla_{z_2}^{k_2} f\|_{L^p(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)}.$$

In fact, making use of Theorem 2.2 and Fubini Theorem,

$$\|\partial_1^{k_1} T_1 \nabla_{z_2}^{k_2} f\|_{L^p(\Omega)}^p = \int_{D_2} \|\partial_1^{k_1} T_1 \nabla_{z_2}^{k_2} f\|_{L^p(D_1)}^p \, d\nu_{w_2}$$

$$\lesssim \sum_{m=0}^{k_1} \int_{D_2} \|\nabla_{1}^{m_1} \nabla_{z_2}^{k_2} f\|_{L^p(D_1)}^p \, d\nu_{w_2} \lesssim \|f\|_{W^{k,p}(\Omega)}^p.$$

The estimate for $P_1$ is done similarly with an application of Theorem 2.1.

In particular, the proposition states that $T_j$ does not lose Sobolev regularity. This estimate of $T_j$ is also the best that one can expect when $n \geq 2$. This is because $T_j$ only improves the regularity in the $z_j$ direction and has no smoothing effect on the rest of the variables.

**Theorem 3.2.** Let $\Omega := D_1 \times \cdots \times D_n \subset \mathbb{C}^n$, $n \geq 1$, where each $D_j$ is a bounded domain in $\mathbb{C}$ with smooth boundary, $j = 1, \ldots, n$. The Bergman projection $P$ is (or, extends as) a bounded operator in $W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+ \cup \{0\}, 1 < p < \infty$. Namely, for any $f \in W^{k,p}(\Omega)$,

$$\|P f\|_{W^{k,p}(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)}.$$

**Proof of Theorem 1.1 and Theorem 3.2:** The proof to Theorem 1.1 is a direct consequence of Proposition 3.1 and (3.2); the proof to Theorem 3.2 is a direct consequence of Proposition 3.1 and (3.1).

\[ \square \]
Denote by $\Delta^2$ the bidisc in $\mathbb{C}^2$. The following Kerzman-type example demonstrates that the \(\bar{\partial}\) problem in general does not improve the Sobolev regularity. In this sense the Sobolev estimate of the canonical solution operator in Theorem 1.1 is sharp.

**Example 1.** For each \(k \in \mathbb{Z}^+ \cup \{0\}\) and \(1 < p < \infty\), consider \(f = (z_2 - 1)^{\frac{2-k}{p}}dz_1\) on \(\Delta^2\) if \(p \neq 2\), or \(f = (z_2 - 1)^{k-1}\log(z_2 - 1)dz_1\) on \(\Delta^2\) if \(p = 2\), \(\frac{1}{2}\pi < \arg(z_2 - 1) < \frac{3}{2}\pi\). Then \(f \in W^{k,q}(\Delta^2)\) for all \(1 < q < p\), and is \(\bar{\partial}\)-closed on \(\Delta^2\). However, there does not exist a solution \(u \in W^{k,p}(\Delta^2)\) to \(\bar{\partial}u = f\) on \(\Delta^2\).

**Proof.** One can directly verify that \(f \in W^{k,q}(\Delta^2)\) for all \(1 < q < p\) and is \(\bar{\partial}\)-closed on \(\Delta^2\). Suppose there exists some \(u \in W^{k,p}(\Delta^2)\) satisfying \(\bar{\partial}u = f\) on \(\Delta^2\). Then \(u = (z_2 - 1)^{\frac{2-k}{p}}z_1 + h \in W^{k,p}(\Delta^2)\) for some holomorphic function \(h\) on \(\Delta^2\). For each \((r, z_2) \in U := (0, 1) \times \Delta \subset \mathbb{R}^3\), consider

\[
v(r, z_2) := \int_{|z_1| = r} u(z_1, z_2)dz_1.
\]

By Fubini theorem and Hölder inequality,

\[
\|\partial^k_2 v\|_{L^p(U)}^p = \int_U \left( \int_{|z_1| = r} \partial^k_2 u(z_1, z_2)dz_1 \right)^p dv_z_2 dr = \int_{|z_2| < 1} \int_0^1 r \int_0^{2\pi} \left| \partial^k_2 u(re^{i\theta}, z_2) \right|^p d\theta dv_z_2 \leq \|u\|_{W^{k,p}(\Delta^2)}^p < \infty.
\]

Thus \(\partial^k_2 v \in L^p(U)\).

On the other hand, by Cauchy’s theorem, for each \((r, z_2) \in U\),

\[
\partial^k_2 v(r, z_2) = C_{k,p} \int_{|z_1| = r} (z_2 - 1)^{-\frac{2}{p}}\bar{z}_1dz_1 = C_{k,p}(z_2 - 1)^{-\frac{2}{p}} \int_{|z_1| = r} \frac{r^2}{\bar{z}_1}dz_1 = 2\pi C_{k,p}r^2 i(z_2 - 1)^{-\frac{2}{p}}
\]

for some non-zero constant \(C_{k,p}\) depending only on \(k\) and \(p\). However, \(r^2(z_2 - 1)^{-\frac{2}{p}} \notin L^p(U)\). This is a contradiction!

\(\square\)

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