Power-law distribution found in city-scale traffic flow simulation

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Abstract. Origin of a power-law in traffic-volume distribution found in traffic simulations of Kobe city was studied. The traffic distribution which was obtained from a shortest path search with randomized OD (origin-destination) set in Kobe city digital map obeys power-law. The toy model that Cayley tree is embedded in the network is also verified. It is theoretically shown that the traffic distribution with all possible OD set in a Cayley tree obeys power-law like distribution. With randomized OD set, the distribution is diffused from the theoretical point sets. Relationship between these facts and the origin of power-law is discussed.

1. Introduction
Traffic flow model in a simple case, on a single straight lane, has been established\cite{1}. The next topic is to construct an urban scale traffic flow model from the simple traffic model, which is not still well understood in the aspect of the city road network. Attempts to understand the rough behavior of the traffic model with complex city road networks, such as \cite{2, 3}, are still limited in number, therefore it is still difficult to design urban traffic flow based on a systematic and analytical model.

Urban scale traffic, which sometimes spontaneously produce traffic congestions, can be understood as a complex system. Such systems, including our society itself, sometimes produce power-law behavior in such as income distribution \cite{4}, web accesses \cite{5} and so on. Especially, the former is empirically analogous to traffic flow. There are popular roads such as bypasses or highways which are limited in number are used by large number of vehicles, while many local roads such as small roads in front of individual houses are not so common in the transportation. Therefore, there seems to be inverted proportional relationship between number of roads and traffic popularity. We demonstrated such relationships actually appears and the relationship obeys power law robustly in \cite{6}. Our result also suggested the structure of the road network is one of candidates which causes the power-law distribution.

If the network itself causes the power-law behavior, simple mathematic mechanism such as path search is considered to be a candidate that causes the behavior. In the present paper, we demonstrate a simple shortest path search actually causes the power law and further discuss the origin of the power-law appearance, by introducing artificially created network based on our toy model.
2. power-law behavior in Kobe city agent based traffic simulation

In [6], we performed agent-based traffic simulation using SUMO (Simulation of Urban Mobility) and digitized map created by [3] from Zenrin co. ltd.. Each agent is a vehicle. Each road segment (between a intersection and its nearest next intersection) has its own speed limit which is 100, 60, and 30 km/h. Signals are used with SUMO default settings for simplicity. Simulation duration time is 6 hours and 17,500 vehicles are inserted into the simulation. Refer [6] for more detailed simulation settings. The pairs of origin and destination, hereafter we call OD set, is randomly generated by applying uniform distribution within the entire city.

Once executing a simulation with an above setting, SUMO searches shortest time consuming path for each OD based on road length and speed limit without taking interactions between agents into account, before agents start their drive. As a consequence, severe congestions occur in common roads and non-negligible fraction of vehicles stacks, and does not reach their destinations within 6 hours of simulation time, which are only several kilo-meters apart at maximum. This unrealistic situation occurs because each car never avoid congested roads. To avoid this, SUMO also provides re-route algorithm, which searches path based on not only road length and speed limit, but also the time required to go through each road segment. By this iteration, the simulation becomes more realistic; vehicles avoid severely congested roads and traffic flow becomes smoother. In a case that the alternative path may also be congested, the iterations are repeated until the constant number of vehicles appears during the most of the entire simulation time, as shown in Figure 1.

A traffic distribution, which is a histogram of number of road segments vs. number of traffics, without and with seven iterations, of a simulation result with above settings, is shown in Figure 2. Regardless to the presence of iterations, a power-law distribution robustly appears.

3. path search in Kobe-city road network

In Figure 2, the traffic distribution of non-realistic situation without iteration without iterations is more power-law like in entire region, while the other has a cut off in the right-end region. This causes the distribution to distort from pure power law. It suggests that only path search may cause a power-law behavior without taking interactions between vehicles into account. However,
Figure 2. Traffic distribution with SUMO simulator, 1st (left) and 7th (right) iteration, quoted from [6]

this speculation requires validation, because there shall be a distortion in the distribution. As shown in Figure 1, 16,000 vehicles (74% of entire 21,600 vehicles) do not reach their destination until simulation ends due to time limit; the distribution in Figure 2 (left) is not based on either entire path or vehicles.

In purpose of path search, we used Networkx python package. We constructed directed and weighted network by extracting road connection and length from digitized map of Kobe city mentioned above. OD of vehicles are assumed to be uniform distribution, choosing each O and D from all intersections. For simplicity, we assumed uniform speed limit and no traffic signals, therefore the distance is calculated only from road lengths. Dijkstra algorithm implemented in Networkx is used for path search for each OD pair. The number of OD pairs, which correspond to the number of vehicles, is assumed to be 40,000. The entire path search of all OD pairs requires about 30 minutes to complete, using a single core of an usual PC.

We then counted how many times each path contains each road segment, and interpreted it as a number of traffics. The traffic distribution is shown in Figure 3. The distribution obeys a power law. This result evidences that a power-law behavior appears without considering interactions between vehicles at all, and it comes from network structure. Therefore, path search can be used to find out more detailed origin of the power-law behavior.

4. results in artificial road networks

In [6], we also showed the power-law behavior on randomly generated road network. Furthermore, in the previous study, we pointed out contrast of common and local roads may cause reverse-proportional relationship, which is a power-law distribution with exponent $\Gamma = -1$.

More quantitatively, we propose a toy model that a finite Cayley tree (or Bethe lattice) is embedded in the road network. An example of Cayley tree map that is used in this experiment is shown in Figure 4 (top). Assume that the network has one common road, and it branches into $n_r$ local roads at the top of the tree. Consider $n_v$ vehicles with uniform OD distributed in the top of the tree for simplicity. In general, local roads share $n_v$ vehicles with their number $n_r$, therefore their traffic $T$ shall be $T = n_v/n_r$ in average. The only common road is used by all these vehicles, then its traffic shall be $T = n_v$. During branching of the tree, the number of local roads $n'_r$ shall be smaller than $n_r$, here, the traffic still obeys $T = n_v/n'_r$. Thus The traffic shall obey a reverse-proportional relationship $T \propto 1/n'_r$, which is a power-law distribution with exponent $\Gamma = -1$. However, this speculation does not taking it into account, that there are vehicles with O or D in the way-half of the tree. Verification of this toy model with the uniform
distribution is required.

Assuming uniform distribution, it is possible to theoretically derive how many vehicles go on each edge. By removing one edge from Cayley graph can always separate the graph into completely isolated two graphs. When the smaller graph has \( n \) nodes, the other has \( N - n \) nodes, where \( N \) is the number of nodes of the entire graph. The vehicles that go through this edge, the traffic \( T \), can be derived as

\[
T = n(N - n).
\]

The number of roads \( N_k \) that is connected to the nodes that have distance \( k \) from the center is given by

\[
N_k = 2z(z - 1)^{k-1}.
\]

(1)

Note that it is multiplied by two because the graph is bi-directional. The traffic \( T_k \) on each of these roads is given by

\[
T_k = n_k(N - n_k) = \frac{(z - 1)^{h-k+1} - 1}{z - 2}
\]

(2)

because

\[
n_k = \frac{(z - 1)^{h-k+1} - 1}{z - 2}
\]

(3)

and

\[
N = \frac{z(z - 1)^h - 2}{z - 2}
\]

(4)

where \( n_k \) is the number of nodes contained in the higher part of the tree separated by the present edge, and \( h \) is the height of the tree.

We performed an exhaustive path search, by applying a complete set of OD from all nodes to all the other nodes. This is the same assumption with uniform OD set with infinite number of samples. The exhaustive traffic distribution is shown as dots in Figure 4 (bottom left). Note that there are six points, each corresponds to each layers of the tree. The theoretical curve

**Figure 3.** Traffic distribution in Kobe, with simple traffic path search, using Dijkstra method and randomized OD set, not a result of the traffic simulation.
Figure 4. Top: An example of the Cayley tree with coordination number $z = 4$ and height $h = 3$. Bottom left: Points represent the theoretical discrete traffic distribution in a Cayley tree with coordination number $z = 6$, height $h = 3$, and exhaustive OD set. The solid line represents the distribution as continuum function obtained by interpreting the discrete distribution. The dashed line represents the power-law distribution with exponent -1. The theoretical distribution is power law like. Bottom right: traffic distribution in the same map as bottom left, but with a randomized OD set of 40,000 pairs.

The result that path search caused the power-law behavior strongly suggests that the behavior is originated by the network derived in equation 1 and 2 is parametrically drawn and displayed as a solid curve for validation in Figure 4 (bottom left). A dashed line with exponent of -1 is also drawn to guide your eyes.

Also the traffic distribution with randomized OD set with 40,000 pair is shown in Figure 4 (bottom right). Note the points are diffused due to randomness. The actual road network is considered to be a combination of trees with various heights and coordination numbers. Therefore the traffic distribution shall be a superposition of these diffused distributions. It is considered to converge into a power-law like distribution.

5. summary and discussion
In the present study, we showed that simple shortest path search can cause power-law behavior in traffic distribution, using the Kobe city map and a Cayley tree. The result that path search caused the power-law behavior strongly suggests that the behavior is originated by the network
structure itself, not interactions between vehicles. Also, the theoretical consideration of the power-law behavior in the Cayley tree suggests that the non-uniformity or diversity of trees that are embedded in the road network caused the power-law.

The power-law behavior robustly appears and it seems to be an aspect of an actual traffic flow. To reconstruct a realistic traffic simulation, it requires power-law appearance. The result that simple path search can cause the behavior, suggests that it is possible to construct a well-characterized urban scale traffic simulation with only such a simple algorithm, without generating detailed agents. Eventually, so much detailed information of all the agents are not required to obtain general features and responses of urban scale traffic flow. Therefore, in the future study, we are going to implement and validate a path iteration algorithm as a simplified version of iteration algorithms that are used in traffic simulators. The densities of congestions can be calculated from traffic using fundamental diagram, and the mean time required to go through each road segments, shall allow to reroute the path.

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