Upper and lower bounds on dynamic risk indifference prices in incomplete markets

Xavier De Scheemaekere

September 8, 2010

Abstract

In the context of an incomplete market with a Brownian filtration and a fixed finite time horizon $T$, this paper proves that for general dynamic convex risk measures, the buyer’s ($p_t^{\text{buyer}}$) and seller’s ($p_t^{\text{seller}}$) risk indifference prices of a bounded contingent claim satisfy

$$p_t^{\text{low}} \leq p_t^{\text{buyer}} \leq p_t^{\text{seller}} \leq p_t^{\text{up}}, \quad \forall t \in [0, T],$$

where $p_t^{\text{low}}$ and $p_t^{\text{up}}$ are the dynamic lower and upper hedging prices, respectively.

Keywords Backward stochastic differential equations · Dynamic convex risk measures · Incomplete markets · Indifference pricing

Mathematics Subject Classification (2010) 60H10 · 91B30 · 91G20

JEL Classification C73 · D52 · G13

1 Introduction

In incomplete markets, arbitrage-free pricing of contingent claims is not unique. The no-arbitrage assumption provides infinitely many equivalent martingale measures and yields an interval of arbitrage-free prices, instead of a unique price (Harrison and Pliska [36], Delbaen and Schachermayer [21]). The reason is that perfect replication is impossible and risk cannot be fully eliminated.

Still, upper and lower hedging prices (El Karoui and Quenez [29], Kramkov [42]) can be charged in order to eliminate all risks. The upper hedging price represents the minimal initial payment needed for the hedging portfolio to attain...
a terminal wealth that is no less than the derivative payoff. This price, however, is excessively high, as it often reduces to the trivial upper bound of the no-arbitrage interval (Eberlein and Jacod [24], Bellamy and Jeanblanc [8]). In order to get more information on the asset value, one possibility is to introduce an optimality criterion that puts more restrictions on the bounds of the price interval.

A few examples include picking martingale measures according to optimal criteria (Bellini and Frittelli [9], Föllmer and Schweizer [31], Gerber and Shiu [34], Goll and Rüschendorf [35]), invoking (exponential) utility indifference arguments (Ankirchner et al. [1], Becherer [7], El Karoui and Rouge [27], Henderson and Hobson [37], Musiela and Zariphopolou [44]), using dynamic risk measures for the optimal design of derivatives (Barrieu and El Karoui [4, 5, 6]), pricing by stress measures (Carr et al. [11]), or good-deal asset price bounds (Cochrane and Saá-Requejo [13]), among many others — more details can be found in Xu [47], or Horst and Müller [39], and the references there in.

In this work, the optimality criterion comes from the risk indifference principle, recently proposed for pricing in incomplete markets (Klöppel and Schweizer [40], Øksendal and Sulem [45], Xu [47]). The (seller’s) dynamic risk indifference price is the initial payment that makes the risk involved for the seller of a contract equal, at any time, to the risk involved if the contract is not sold, with no initial payment. Hence, the resulting price is such that the agent is indifferent between his risk if a transaction occurs and his risk if no transaction occurs.

The abstract risk indifference pricing setting has been studied in Xu [47], where it is shown to generalize utility-based derivative pricing introduced by Hodges and Neuberger [38] and valuation by stress measures of Carr et al. [11].

In a static Markovian framework, and for a particular class of convex risk measures, Øksendal and Sulem [45] study the risk indifference method in a jump diffusion market, using PDE methods. In particular, they prove that the buyer’s ($p_{\text{buyer}}$) and seller’s ($p_{\text{seller}}$) risk indifference prices satisfy

$$p_{\text{low}} \leq p_{\text{buyer}} \leq p_{\text{seller}} \leq p_{\text{up}},$$

(1)

where $p_{\text{low}}$ and $p_{\text{up}}$ are the lower and upper hedging prices, respectively.

In the context of a Brownian filtration, this paper shows that (1) holds for general dynamic risk indifference prices. The result is based on a recent work of Delbaen et al. [22], who provide a representation of the penalty term of general dynamic convex risk measures, and it is obtained by applying backward stochastic differential equation (BSDE) theory.

The paper is organised as follows. Section 2 presents the financial market model and recalls some well-known results on BSDEs and their application in finance. Section 3 introduces the super-replication technique, and the risk indifference principle, for pricing in incomplete markets. Section 4 contains the main result of the paper, that is, the proof that general dynamic risk indifference prices are bounded from below and above by lower and upper hedging prices, respectively. As we shall see, BSDE theory provides a unified mathematical perspective on complete markets, super-replication and indifference prices, and the relationships between them.
2 Notation and preliminaries

2.1 Financial market model

Let a given \( T \in (0, \infty) \) be the fixed finite time horizon of a financial market with two investment possibilities:

(i) A risk free security (e.g. a bond), with unit price \( B_t = 1 \) at all times \( t \in [0, T] \).

(ii) \( n \) risky securities (e.g. stocks), with prices evolving according to the following equation:

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t, \quad S_0 > 0, \quad S_t \in \mathbb{R}^{n \times 1}, \quad t \in [0, T].
\]

In the above, \((W_t)_{t \in [0,T]}\) is a standard \( d \)-dimensional Brownian motion defined on a probability space \((\Omega, F, P)\), \(\{F_t\}_{t \in [0,T]}\) is the augmented filtration generated by \((W_t)_{t \in [0,T]}\), \(\mu_t \in \mathbb{R}^{n \times 1}\) is a \(F_t\)-predictable vector-valued map and \(\sigma_t \in \mathbb{R}^{n \times d}\) is a \(F_t\)-predictable full rank matrix-valued map. \(W\) is described as a column vector of dimension \(d \times 1\), and \(\sigma dW\) has to be understood as a matrix product with dimension \(1 \times 1\) for each \(1 \leq i \leq n\).

Assume that the processes \(\mu\) and \(\sigma\) satisfy

\[
\int_0^T (|\mu_s| + \|\sigma_s\|^2)^2 ds < \infty, \quad P-a.s.
\]

A portfolio in this market is modeled by the \(1 \times n\) row vector \(\pi_t\), representing the amount invested in the risky assets at time \(t\). The dynamics of the corresponding wealth process \(X(t) = X^{(\pi)}_t(t)\) is

\[
\begin{align*}
  dX_t &= \pi_t \frac{dS_t}{S_t} = \pi_t [\mu_t dt + \sigma_t dW_t]; \quad t \in [0, T] \\
  X_0 &= x > 0.
\end{align*}
\]

The portfolio \(\pi_t\) is admissible if it is \(F_t\)-predictable and satisfies

\[
\int_0^T (|\mu_t| \|\pi_t\| S_t + \|\sigma_t\|^2 |\pi_t|^2 |S_t|^2) dt < \infty
\]

and

\[
X_t \geq 0 \quad \text{for } t \in [0, T], \quad P-a.s.
\]

Let \(\Pi\) denote the set of all admissible portfolios.

Suppose that the \(F_T\)-measurable random variable \(\xi = \xi_T\) represents a contingent claim with maturity \(T > 0\), depending on \(S(\cdot)\).

When the market is complete, i.e. \(n = d\), it is well known that the unique dynamic arbitrage-free price of \(\xi\) is given by the Black-Scholes price

\[
p_t^{BS} := E_Q[\xi | F_t],
\]
where $Q = Q^\theta$ is the so-called unique equivalent martingale measure defined as

$$
E_P \left[ \frac{dQ}{dP} | F_T \right] := \exp \left( \int_0^T \theta_s dW_s - \frac{1}{2} \int_0^T |\theta_s|^2 \, ds \right), \quad (2)
$$

and $(\theta_t)_{t \in [0,T]}$ is a suitable $\mathbb{R}^{1 \times d}$-valued process, called the market price of risk, such that $\sigma_t \theta_t^{\top} + \mu_t = 0$.

By Girsanov theorem, $\widetilde{W}_t := W_t - \int_0^T \theta_s \, ds$ is a $Q$-Brownian motion, and the (discounted) price process $S_t$ is a martingale with respect to $Q$ — hence the name equivalent martingale measure (EMM for short).

In the sequel, we identify a probability measure $Q$ equivalent to $P$ with its Radon-Nikodym density $\frac{dQ}{dP}$ and with the predictable process $(\theta_t)_{t \in [0,T]}$ induced by the stochastic exponential, as in (2).

### 2.2 BSDEs and complete markets

Consider the function

$$
g : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^{1 \times d} \to \mathbb{R} : (t, \omega, y, z) \to g(t, \omega, y, z)
$$

satisfying the following assumptions (to simplify the notation, we often write $g(t, y, z)$ instead of $g(t, \omega, y, z)$):

(i) $g$ is Lipschitz in $(y, z)$, i.e. there exists a constant $u > 0$ such that we have $dt \times dP$-a.s., for any $(y_0, z_0), (y_1, z_1) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$,

$$
|g(t, y_0, z_0) - g(t, y_1, z_1)| \leq u (|y_0 - y_1| + |z_0 - z_1|).
$$

(ii) For all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $g(\cdot, y, z)$ is a predictable process such that for any finite $T > 0$, we have $E[\int_0^T (g(t, \omega, y, z))^2 \, dt] < +\infty$ for any $y \in \mathbb{R}$ and $z \in \mathbb{R}^{1 \times d}$.

A BSDE is an equation of the type

$$
- dY_t = g(t, Y_t, Z_t) - Z_t \, dW_t, \\
Y_T = \xi,
$$

where $\xi$ is a random variable in $L^2(\Omega, F_T, P)$. In 1990, Pardoux and Peng [46] showed that for a finite time horizon $T > 0$, there exists a unique solution $(Y_t, Z_t)_{t \in [0,T]}$ consisting of predictable stochastic processes (the former $\mathbb{R}$-valued, the latter $\mathbb{R}^{1 \times d}$-valued) such that $E[\int_0^T Y_t^2 \, dt] < +\infty$ and $E[\int_0^T |Z_t|^2 \, dt] < +\infty$.

El Karoui et al. [25] showed that BSDE theory provides the mathematical background of many problems in finance. In particular, in a complete market, the dynamic arbitrage-free price $p_{BS}^\xi$ of a contingent claim $\xi \in L^2(\Omega, F_T, P)$ is equal to the solution $Y_{BS}^\xi$ of the following BSDE

$$
- dY_{BS}^\xi_t = Z_{BS}^\xi \theta_t^{\top} \, dt - Z_{BS}^\xi \, dW_t, \\
Y_{BS}^\xi_T = \xi.
$$
3 Pricing in incomplete markets

3.1 Upper and lower hedging prices

In incomplete markets, there is no unique EMM and hence no unique method for pricing a given contingent claim in an arbitrage-free way (in our setting, a straightforward example of incomplete market is when \( n < d \); see the works of Cvitanic et al. \[14\], \[15\], \[16\] for more details). As is well known, the maximum arbitrage free price of a contingent claim \( \xi \) is given by

\[
p_{up}^t := \esssup_{\theta \in M} E_{Q^\theta}[\xi | F_t],
\]

where \( M \) is the set of all predictable \( \mathbb{R}^{1 \times d} \)-valued processes \( (\theta_t)_{t \in [0,T]} \) such that \( E[\int_0^T |\theta_t|^2 dt] < +\infty \) and such that \( S_t \) is a martingale with respect to \( Q^\theta \). \( p_{up}^t \) is also referred to as the \textit{dynamic upper hedging price} of \( \xi \) because it satisfies (Kunita \[41\]):

\[
p_{up}^t = \essinf \left\{ X(\pi)(t) \mid \exists \pi \in \Pi \text{ s.t. } X(\pi)(T) \geq \xi, \ P-a.s. \right\}.
\]

This means that \( p_{up}^t \) represents the minimal wealth at time \( t \) needed in order to be able to attain a terminal wealth \( X(\pi)(T) \) which is no less than the guaranteed payoff \( \xi \).

In order to fit into the setting of BSDE theory, we consider the set \( M' = \{ \theta_t | \theta_t \in M \text{ such that } |\theta_t| \leq u_t, \ t \in [0,T] \} \), where \( u_t \) is a positive bounded deterministic process. Abusing notation, we will write \( M \) for \( M' \). The following theorem characterizes the dynamic upper hedging price of \( \xi \) as the solution of a BSDE:

\textbf{Theorem 1 (Chen and Wang \[12\])} Assume that \( \xi \in L^2(\Omega, F_T, P). \) Then \( p_{up}^t \) is the first component of the solution of BSDE

\[
-dY_t = u_t |Z_t| dt - Z_t dW_t,
\]

\[
Y_T = \xi.
\]

Similarly, the dynamic lower hedging price,

\[
p_{low}^t := \essinf_{\theta \in M} E_{Q^\theta}[\xi | F_t],
\]

satisfies

\[
p_{low}^t = \esssup \left\{ X(\pi)(t) \mid \exists \pi \in \Pi \text{ s.t. } X(\pi)(T) \leq \xi, \ P-a.s. \right\}.
\]

It represents the maximal wealth allowed at time \( t \) in order to attain a terminal wealth \( X(\pi)(T) \) which is no more than the promised payoff \( \xi \). Using the same argument as Chen and Wang \[12\], one can show the following theorem:
Theorem 2 Assume that $\xi \in L^2(\Omega, F_T, P)$. Then $p^\text{low}_t$ is the first component of the solution of BSDE

$$
-dY_t = -u_t|Z_t|dt - Z_tdW_t, \quad Y_T = \xi.
$$

If a trader charges the upper hedging price for selling an option, he can trade to eliminate all risks. Likewise, if a trader buys an option for the lower hedging price, he eliminates all risks. However, in general, the gap between $p^\text{up}_t$ and $p^\text{low}_t$ is too wide to make either of them a good candidate for the trading price in an incomplete market.

3.2 Dynamic risk indifference pricing

The starting point of the risk indifference pricing principle is a given dynamic (or conditional) convex risk measure. Coherent risk measures were introduced by Artzner et al. [2] (see also Delbaen [19, 20]). Later, Föllmer and Schied [28, 29], and Frittelli and Rosazza Gianin [32, 33] introduced the class of convex risk measures.

The economic rationale behind the concept of risk measure is the following: Let $\xi(\omega) \in L^\infty(\Omega, F_T, P)$ be a contingent bounded financial position at time $T$, then $\rho_{t,T}(\xi)(\omega)$ may be interpreted as the monetary degree of riskiness of $\xi$ when state $\omega$ occurs (for more details, see the above references, and also Föllmer and Schied [30] or Detlefsen and Scandolo [23]). Here follows the definition:

Definition 3 (Convex risk measure) A convex risk measure $\rho_{t,T}$ on $(\Omega, F_T, P)$ conditional to $(\Omega, F_t, P)$ is a map $\rho_{t,T}: L^\infty(\Omega, F_T, P) \to L^\infty(\Omega, F_t, P)$ satisfying the following properties:

(a) Monotonicity: $\forall \xi, \eta \in L^\infty(\Omega, F_T, P)$, if $\xi \leq \eta$, then $\rho_{t,T}(\xi) \geq \rho_{t,T}(\eta)$.

(b) Translation invariance: $\forall \xi, \eta \in L^\infty(\Omega, F_t, P)$, $\forall \xi \in L^\infty(\Omega, F_T, P)$, $\rho_{t,T}(\xi + \eta) = \rho_{t,T}(\xi) - \rho_{t,T}(\eta)$.

(c) Convexity: $\forall \xi, \eta \in L^\infty(\Omega, F_T, P)$, $\rho_{t,T}(\lambda \xi + (1-\lambda)\eta) \leq \lambda \rho_{t,T}(\xi) + (1-\lambda) \rho_{t,T}(\eta)$, for any $\lambda \in [0, 1]$.

(d) Normalization: $\rho_{t,T}(0) = 0$.

(e) Continuity from above: For any decreasing sequence $(\xi_n)_{n \in \mathbb{N}}$ of elements of $L^\infty(\Omega, F_T, P)$ such that $\xi = \lim_n \xi_n$, the sequence $\rho_{t,T}(\xi_n)$ has the limit $\rho_{t,T}(\xi)$.

(f) Time consistency: for all $s, v$ such that $0 \leq t \leq s \leq v \leq T$, $\rho_{t,v}(\xi) = \rho_{t,s}(\rho_{s,v}(\xi))$, $\forall \xi \in L^\infty(\Omega, F_T, P)$.

(g) $\forall \xi, \eta \in L^\infty(\Omega, F_T, P)$, $\forall A \in F_t$, $\rho_{t,T}(\xi 1_A + \eta 1_A^c) = \rho_{t,T}(\xi) 1_A + \rho_{t,T}(\eta) 1_A^c$.

(h) $c_{t,T}(P) := \text{ess sup}_{\xi \in L^\infty(\Omega, F_T, P)} \{E_P[-\xi|F_t] - \rho_{t,T}(\xi)\} = 0$ for any $t \in [0, T]$. 

6
\( \rho_{t,T} \) is called a normalized time-consistent dynamic convex risk measure. To simplify the notation, we often write \( \rho_t \) instead of \( \rho_{t,T} \).

As in the utility indifference case (initiated by Hodges and Neuberger \( [38] \); see also Davis \( [17] \), Davis et al. \( [18] \) and Barles and Soner \( [3] \)), the risk indifference pricing principle starts from two situations:

(i) If a person sells a contract which guarantees a payoff \( \xi \in L^\infty(\Omega, F_T, P) \) at time \( T \) and receives a payment \( p_t \) for this, then at time \( t \) the minimal risk involved for the seller is

\[
\Phi^\xi_t(X^{(\pi)}_t) + p_t = \inf_{\pi(\cdot) \in \Pi} \rho_t(X^{(\pi)}_t + p_t(T) - \xi).
\]

(ii) If, on the other hand, no contract is sold, and hence no payment is received, then at time \( t \) the minimal risk for the person is

\[
\Phi^0_t(X^{(\pi)}_t) = \inf_{\pi(\cdot) \in \Pi} \rho_t(X^{(\pi)}_T).\]

The dynamic risk indifference price is then defined as follows:

**Definition 4 (Seller’s price)** The seller’s dynamic risk indifference price \( p_t = p_{t}^{\text{seller}} \) of a claim \( \xi \in L^\infty(\Omega, F_T, P) \) is the solution of the equation

\[
\Phi^\xi_t(X^{(\pi)}_t) + p_t = \Phi^0_t(X^{(\pi)}_t), \tag{3}
\]

for \( t \in [0, T] \). Thus \( p_{t}^{\text{seller}} \) is the payment that makes a person, at any time, risk indifferent between selling the contract with liability \( \xi \) and not selling the contract (and not receiving any payment either).

When \( t = 0 \), \( X^{(\pi)}_0(0) = x \) and \( (3) \) reduces to the static risk indifference pricing problem of Øksendal and Sulem \( [45] \).

Now, similarly, let

\[
\Psi^\xi_t(X^{(\pi)}_t) + p_t = \inf_{\pi(\cdot) \in \Pi} \rho_t(\xi - X^{(\pi)}_t + p_t(T)),
\]

and

\[
\Psi^0_t(X^{(\pi)}_t) = \inf_{\pi(\cdot) \in \Pi} \rho_t(X^{(\pi)}_T).\]

The buyer’s dynamic risk indifference price is defined as follows:

**Definition 5 (Buyer’s price)** The buyer’s dynamic risk indifference price \( p_t = p_{t}^{\text{buyer}} \) of a claim \( \xi \in L^\infty(\Omega, F_T, P) \) is the solution of the equation

\[
\Psi^\xi_t(X^{(\pi)}_t) + p_t = \Psi^0_t(X^{(\pi)}_t),
\]

for \( t \in [0, T] \). Thus \( p_{t}^{\text{buyer}} \) is the payment that makes a person, at any time, risk indifferent between buying the contract with payoff \( \xi \) and not buying the contract (and not making any payment either).
We first study in detail the case of the seller’s risk indifference price. By Bion-Nadal [10] and Detlefsen and Scandolo [23], it is known that under the assumptions above and in the setting of a general filtration,

\[ \rho_{t,T}(\xi) = \operatorname{ess} \sup_{Q \in L} \{ E_Q[-\xi|F_t] - c_{t,T}(Q) \}, \]

where

\[ c_{t,T}(Q) := \operatorname{ess} \sup_{\xi \in L^\infty(\Omega,F_T,P)} \{ E_Q[-\xi|F_t] - \rho_{t,T}(\xi) \} \]

is the penalty term associated to \( \rho_{t,T} \), and

\[ L = \{ Q \text{ on } (\Omega,F_T) : Q \sim P, Q = P \text{ on } F_t \}. \]

In particular, we have \( c_t(Q) := c_{t,T}(Q) \geq 0 \). Taking this into consideration, the problem of finding the risk indifference price \( p_{t_{\text{risk}}} \) in (3) amounts to solving the following two zero-sum stochastic differential game problems:

\[
\Phi_t^{\xi}(X_x^{(\pi)}(t) + p_t) = \operatorname{ess} \inf_{\pi(\cdot) \in \Pi} \sup_{Q \in L} \left\{ E_Q[-X_x^{(\pi)}(T) + \xi|F_t] - c_t(Q) \right\},
\]

and

\[
\Phi_t^{0}(X_x^{(\pi)}(t)) = \operatorname{ess} \inf_{\pi(\cdot) \in \Pi} \sup_{Q \in L} \left\{ E_Q[-X_x^{(\pi)}(T)|F_t] - c_t(Q) \right\},
\]

for all \( t \in [0,T] \).

These problems have a direct economic interpretation: Whilst the seller tries to minimize the risk of the transaction over the set \( \Pi \) of admissible financial strategies, the market tries to maximize the corrected expected loss over a set \( L \) of “generalized scenarios” (i.e., probability measures \( Q \)), where correction depends on scenarios.

We now use the following proposition of Delbaen et al. [22] in order to choose a representation of the penalty term \( c_t \) such that the pricing equation (4) holds for any dynamic convex risk measure.

**Proposition 6 (Delbaen, Peng and Rosazza Gianin [22])** Let \( \rho_t \) be a dynamic convex risk measure satisfying the assumption (a)-(h). Then, for any probability measure \( Q \) equivalent to \( P \),

\[ c_t(Q) = E_Q \left[ \int_t^T f(u,\theta_u)du|F_t \right] \]

for some suitable function \( f : [0,T] \times \Omega \times \mathbb{R}^d \rightarrow [0,\infty] \) such that \( f(t,\omega,\cdot) \) is proper, convex, and lower-semicontinuous.

Let \( N \) be the set of all predictable \( \mathbb{R}^{1 \times d} \)-valued processes \( (\theta_t)_{t \in [0,T]} \) such that \( E[\int_0^T |\theta_t|^2 dt] < +\infty \) and such that \( Q^\theta \in L \). Based on proposition 6 we formulate the zero-sum stochastic differential game problem in (4) as follows:
Problem 7 Find $\Phi_\xi^\xi(X^{(\pi)}_x(t))$ and an optimal pair $(\hat{\pi}(\cdot), \hat{\theta}(\cdot)) \in \Pi \times N$ such that

$$\Phi_\xi^\xi(X^{(\pi)}_x(t)) := \operatorname{ess inf}_{\pi(\cdot) \in \Pi} \sup_{\theta(\cdot) \in N} J_t(\pi, \theta) = J_t(\hat{\pi}, \hat{\theta}),$$

and

$$J_t(\pi, \theta) := \mathbb{E}_{Q_\omega} \left[ \xi - X^{(\pi)}_x(T) - \int_t^T f(u, \theta_u) du | F_t \right],$$

for all $t \in [0, T]$, and where $f$ is a predictable function satisfying the assumptions of proposition 6.

4 Upper and lower bounds on dynamic risk indifference prices

In order to study general risk indifference prices, as well as upper and lower hedging prices, in the same (BSDE) setting, we assume that the following assumptions hold in the sequel:

1. $\xi \in L^\infty(\Omega, F_T, P)$;
2. $\forall \theta_t \in N$, there exists a positive bounded deterministic process $u_t$ such that $|\theta_t| \leq u_t, \forall t \in [0, T]$;
3. $\Pi$ is the set of strategies $\pi$ uniformly bounded by a constant $k$.
4. $E[\int_0^T (f(t, \omega, \theta_t))^2 dt] < +\infty$,
5. $dt \times dP - \text{a.s.}, \theta \to f(t, \omega, \theta)$ is continuously differentiable.

The following theorem gives sufficient conditions for solving problem 7:

**Theorem 8** Let $\hat{\pi}(\cdot) \in \Pi$ and $\hat{\theta}(\cdot) \in N$ satisfy the following optimality conditions:

$$X^{(\hat{\pi})}_x(T) \geq X^{(\pi)}_x(T),$$

$\forall \pi(\cdot) \in \Pi$, $P - \text{a.s.},$ and

$$-f(t, \hat{\theta}_t) + z \hat{\theta}_t^T \geq -f(t, \theta_t) + z \theta_t^T,$$

$\forall z \in \mathbb{R}^{1 \times d}, \theta(\cdot) \in N, t \in [0, T), dt \times dP - \text{a.s.}$

Then, for any $t \in [0, T]$, $J_t(\hat{\pi}, \hat{\theta}) = Y^{\hat{\pi}, \hat{\theta}}_t, P - \text{a.s.}$, where $(Y^{\hat{\pi}, \hat{\theta}}_t, Z^{\hat{\pi}, \hat{\theta}}_t)$ is the solution of BSDE

$$-dY^{\hat{\pi}, \hat{\theta}}_t = (-f(t, \hat{\theta}_t) + Z^{\hat{\pi}, \hat{\theta}}_t \hat{\theta}_t^T) dt - Z^{\hat{\pi}, \hat{\theta}}_t dW_t,$$

and $(\hat{\pi}(\cdot), \hat{\theta}(\cdot))$ is an optimal pair for problem 7.
Proof. The existence of \( \hat{\pi}(\cdot) \in \Pi \) and \( \hat{\theta}(\cdot) \in N \) satisfying (5) and (6) follows by applying a predictable selection theorem (see, e.g., proposition 4.1 in Lim and Quenez [43] for a similar argument). Further, it is direct to check that \( J_t(\pi, \theta) \) is equal to the unique solution \( Y_{t, \theta, \pi} \) of the following linear BSDE
\[
-dY_{t, \theta, \pi} = \left( -f(t, \theta) + Z_{t}^{\theta, \pi} \theta_t^\top \right) dt - Z_{t}^{\theta, \pi} dW_t,
\]
(8)
\[
Y_{T, \theta, \pi} = \xi - X_{T}^{(\pi)}(T),
\]
and that \( J_t(\hat{\pi}, \hat{\theta}) \) is equal to the unique solution \( Y_{t, \hat{\theta}, \hat{\pi}} \) of BSDE (7). Combining the optimality condition (6) with the comparison theorem for BSDEs (El Karoui et al. [25]), we have that \( Y_{t, \theta, \pi} \leq Y_{t, \hat{\theta}, \hat{\pi}}, \) \( P \)-a.s., from which we deduce that
\[
\text{ess inf}_{\pi(\cdot) \in \Pi} \sup_{\theta(\cdot) \in N} Y_{t, \theta, \pi} \leq \text{ess inf}_{\pi(\cdot) \in \Pi} Y_{t, \hat{\theta}, \hat{\pi}} = \text{ess sup}_{\theta(\cdot) \in N} Y_{t, \hat{\theta}, \hat{\pi}} \leq \text{ess sup}_{\pi(\cdot) \in \Pi} Y_{t, \theta, \pi}.
\]
Similarly, by (5), we have \( Y_{t, \hat{\theta}, \hat{\pi}} \geq \text{ess inf}_{\pi(\cdot) \in \Pi} Y_{t, \theta, \pi}, \) \( P \)-a.s., and
\[
\text{ess inf}_{\pi(\cdot) \in \Pi} \sup_{\theta(\cdot) \in N} Y_{t, \theta, \pi} \geq \text{ess inf}_{\pi(\cdot) \in \Pi} Y_{t, \hat{\theta}, \hat{\pi}} = \text{ess sup}_{\theta(\cdot) \in N} Y_{t, \hat{\theta}, \hat{\pi}} \geq \text{ess sup}_{\pi(\cdot) \in \Pi} Y_{t, \theta, \pi}.
\]
By uniqueness, it follows that
\[
\text{ess inf}_{\pi(\cdot) \in \Pi} \sup_{\theta(\cdot) \in N} Y_{t, \theta, \pi} = Y_{t, \hat{\theta}, \hat{\pi}}, \ P \text{-a.s.},
\]
that is,
\[
\text{ess inf}_{\pi(\cdot) \in \Pi} \sup_{\theta(\cdot) \in N} J_t(\pi, \theta) = J_t(\hat{\pi}, \hat{\theta}), \ P \text{-a.s.},
\]
which concludes the proof. \( \blacksquare \)

The following lemma will be very useful:

**Lemma 9** Assume that \( \hat{\theta}(\cdot) = \hat{\theta}(\cdot, \pi(\cdot)) \) and \( \hat{\pi}(\cdot) \) satisfy the optimality conditions (7) and (8) \( \forall \pi(\cdot) \in \Pi \), and assume that the function \( \pi \to \theta(\pi) \) from \( \mathbb{R}^{1 \times n} \) into \( \mathbb{R}^{1 \times d} \) is a continuously differentiable function. Then \( Q_\theta \) is an EMM, where \( \tilde{\theta}(\cdot) = \hat{\theta}(\cdot, \hat{\pi}(\cdot)) \).

**Proof.** Consider the map
\[
J_{t, \theta}^{\pi, \theta} : \mathbb{R}^{1 \times d} \to L^2(\Omega, F_t, dP \times dt)
\]
\[
: \theta \to E_{Q_\theta} \left[ \xi - X_{T}^{(\pi)}(T) - \int_{t}^{T} f(u, \theta) du | F_t \right],
\]
and note that \( J_{t, \theta}^{\pi, \theta} = Y_{t, \theta}^{\theta} \) is the first component of the solution of BSDE
\[
-dY_{t}^{\theta} = \left( -f(t, \theta) + Z_{t}^{\theta} \theta_t^\top \right) dt - Z_{t}^{\theta} dW_t,
\]
and that \( J_{t, \theta}^{\pi, \theta} = Y_{t, \theta}^{\theta} \) is the first component of the solution of BSDE
\[
-dY_{t}^{\theta} = \left( -f(t, \theta) + Z_{t}^{\theta} \theta_t^\top \right) dt - Z_{t}^{\theta} dW_t,
\]

\[
Y_{T}^{\theta} = \xi - X_{T}^{(\pi)}(T).
\]
For a differentiable function \( g : \mathbb{R}^n \rightarrow \mathbb{R}^m : x \rightarrow g(x) \), let \( \nabla_x g \) denote the gradient matrix of \( g \) with respect to \( x \), i.e. \( (\nabla_x g)_{i,j} = \frac{\partial g^i(x)}{\partial x^j} \) for each \( 0 \leq i \leq n, \quad 0 \leq j \leq m \). By Proposition 2.4 of El Karoui et al. [25] on the continuity and differentiability of BSDEs with respect to a parameter (which may be extended to multi-dimensional parameters), the map \( \theta \rightarrow Y^\theta_t \) is differentiable in \( \theta \), with derivatives given by \( \nabla_\theta Y^\theta_t \), the first component of the solution of the following \( d \)-dimensional BSDE

\[
-d\nabla_\theta Y^\theta_t = (\nabla_\theta f(t, \theta_t) + (Z^\theta_t)^\top + \nabla_\theta Z^\theta_t \theta_t^\top) dt - \nabla_\theta Z^\theta_t dW_t,
\]

\[
\nabla_\theta Y^\theta_T = \nabla_\theta (\xi - X^\theta(T)).
\]

The first order condition for a maximum point of the map \( \nabla_\theta Y^\theta_t \) yields that \( \nabla_\theta Y^\theta_t \) evaluated at \( \hat{\theta} = \tilde{\theta}(\cdot, \pi(\cdot)) \) must be equal to zero, i.e. \( \nabla_\theta Y^\theta_t = 0 \), almost surely, for all \( t \in [0, T] \), where \( \nabla_\theta Y^\theta_t \) is the solution of

\[
-d\nabla_\theta Y^\theta_t = (\nabla_\theta f(t, \hat{\theta}_t) + (\hat{Z}^\theta_t)^\top + \nabla_\theta \hat{Z}^\theta_t \hat{\theta}_t^\top) dt - \nabla_\theta \hat{Z}^\theta_t dW_t,
\]

\[\nabla_\theta Y^\theta_T = 0.\]

By Girsanov theorem, we can define the \( Q_\hat{\theta} \)-Brownian motion as \( \hat{W}_t := W_t - \int^t_0 \hat{\theta}_s ds \), and rewrite the above equation as

\[
-d\nabla_\theta Y^\theta_t = (\nabla_\theta f(t, \hat{\theta}_t) + (\hat{Z}^\theta_t)^\top) dt - \nabla_\theta \hat{Z}^\theta_t d\hat{W}_t,
\]

\[\nabla_\theta Y^\theta_T = 0.\]

Taking the conditional expectation, we conclude that

\[\nabla_\theta f(t, \hat{\theta}_t) + (\hat{Z}^\theta_t)^\top = 0, \quad dt \times dP - a.s. \tag{9}\]

Now, consider the map

\[\tilde{J}_t^{\pi, \hat{\theta}} : \mathbb{R}^{1 \times n} \rightarrow L^2(\Omega, F_t, dP \times dt)\]

\[\quad : \pi \rightarrow E_{Q^{\hat{\theta}(\pi)}} \left[ \xi - X^{\hat{\theta}(\pi)}(T) - \int^T_t f(u, \hat{\theta}(\pi)) du | F_t \right],\]

and note that \( \tilde{J}_t^{\pi, \hat{\theta}} = Y^\hat{\theta}_t \). By the same argument, we must have that \( \nabla_x Y^\hat{\theta}_t \) evaluated at \( \hat{\theta} = \tilde{\theta}(\cdot, \tilde{\pi}(\cdot)) \) must be equal to 0, i.e. \( \nabla_x Y^\hat{\theta}_t = 0 \), almost surely, for all \( t \in [0, T] \), where \( \nabla_x Y^\hat{\theta}_t \) is the solution of the following \( n \)-dimensional BSDE

\[
-d\nabla_x Y^\hat{\theta}_t = (\nabla_x \hat{\theta}_t (\nabla_\theta f(t, \hat{\theta}_t) + (\hat{Z}^\theta_t)^\top) + \nabla_x \hat{Z}^\theta_t \hat{\theta}_t^\top) dt - \nabla_x \hat{Z}^\theta_t dW_t,
\]

\[\nabla_x Y^\hat{\theta}_T = \nabla_x (\xi - X^{\hat{\theta}(\pi)}(T)).\]

Again, by Girsanov theorem, we define the \( Q_\hat{\theta} \)-Brownian motion as \( \hat{W}_t := W_t - \int^T_0 \hat{\theta}_s ds \) and we obtain that

\[
-d\nabla_{\pi} Y^\hat{\theta}_t = \nabla_\theta f(t, \hat{\theta}_t) + Z^\theta_t \theta_t^\top \nabla_\pi \hat{\theta}_t dt - \nabla_\pi Z^\theta_t d\hat{W}_t,
\]

\[\nabla_{\pi} Y^\hat{\theta}_T = \nabla_\pi (\xi - \tilde{X}^{\hat{\theta}(\pi)}(T)),\]
where \( \tilde{X} \) is the portfolio process with respect to \( \tilde{W} \). Combining the first order condition for a minimum point in \( \tilde{\pi} \) with \( 9 \), we obtain that

\[
\nabla_\pi (\tilde{X}_x(\pi)(T)) = \nabla_\pi (x + \int_0^T (\pi_t \mu_t S_t + \tilde{\theta}_t \sigma_t \pi_t S_t) dt + \int_0^T \sigma_t \pi_t S_t d\tilde{W}_t)
\]

\[
= 0.
\]

Since the wealth process \( X \) is almost surely nonnegative for any time \( t \in [0, T] \), we conclude that \( dt \times dP - a.s., \forall t \in [0, T] \),

\[
\mu_t + \tilde{\theta}_t \sigma_t = 0,
\]

which concludes the proof. \( \blacksquare \)

We can now show that the stochastic differential game problem \( 7 \) which involves the supremum with respect to the set of all probability measures \( Q \) equivalent to \( P \), can be reduced to a single stochastic control problem with respect to the (narrower) set of EMM.

**Theorem 10** Assume that \( \tilde{\theta}(\cdot) \) and \( \tilde{\pi}(\cdot) \) satisfy the assumptions of lemma \( 9 \). Then

\[
\text{ess inf}_{\pi(\cdot) \in \Pi \theta(\cdot) \in N} \sup_{Q_\theta} E_{Q_\theta} \left[ \xi - X_x^{(\pi)}(T) - \int_t^T f(u, \theta_u) du | F_t \right]
\]

\[
= \text{ess sup}_{\theta(\cdot) \in M} E_{Q_\theta} \left[ \xi - \int_t^T f(u, \theta_u) du | F_t \right],
\]

for all \( t \in [0, T] \).

**Proof.** By lemma \( 9 \) the probability measure \( Q_{\tilde{\theta}} \) induced by \( \tilde{\theta}(\cdot) = \tilde{\theta}(\cdot, \tilde{\pi}(\cdot)) \) is an EMM, so we have

\[
\text{ess inf}_{\pi(\cdot) \in \Pi \theta(\cdot) \in N} \sup_{Q_\theta} E_{Q_\theta} \left[ \xi - X_x^{(\pi)}(T) - \int_t^T f(u, \theta_u) du | F_t \right]
\]

\[
= \text{ess inf}_{\pi(\cdot) \in \Pi} E_{Q_{\tilde{\theta}}} \left[ \xi - X_x^{(\tilde{\pi})}(T) - \int_t^T f(u, \tilde{\theta}_u) du | F_t \right]
\]

\[
= E_{Q_{\tilde{\theta}}} \left[ \xi - X_x^{(\tilde{\pi})}(T) - \int_t^T f(u, \tilde{\theta}_u) du | F_t \right]
\]

\[
= E_{Q_{\tilde{\theta}}} \left[ \xi - \int_t^T f(u, \tilde{\theta}_u) du | F_t \right]
\]

\[
\leq \text{ess sup}_{\theta(\cdot) \in M} E_{Q_\theta} \left[ \xi - \int_t^T f(u, \theta_u) du | F_t \right].
\]
Conversely, since $M \subset N$, we always have that
\[
\begin{align*}
\operatorname{ess inf} & \sup_{\pi(\cdot) \in \Pi_{\Theta(\cdot) \in N}} E_{Q^{\omega}} \left[ \xi - X^{(\pi)}_x(T) - \int_t^T f(u, \theta_u) du | F_t \right] \\
\geq & \operatorname{ess inf} \sup_{\pi(\cdot) \in \Pi_{\Theta(\cdot) \in M}} E_{Q^{\omega}} \left[ \xi - X^{(\pi)}_x(T) - \int_t^T f(u, \theta_u) du | F_t \right] \\
= & \operatorname{ess sup}_{\theta(\cdot) \in M} E_{Q^{\omega}} \left[ \xi - \int_t^T f(u, \theta_u) du | F_t \right],
\end{align*}
\]
from which the claim follows. \(\blacksquare\)

Define, \(\forall \xi \in L^\infty(\Omega, F_t, P)\),
\[
\rho_t^M(\xi) := \operatorname{ess sup}_{\theta(\cdot) \in M} E_{Q^{\omega}} \left[ \xi - \int_t^T f(u, \theta_u) du | F_t \right].
\]

The following corollary states that the buyer’s and seller’s risk indifference prices can be formulated in terms of \(\rho_t^M\).

**Corollary 11** Let \(\rho_t^M\) be defined as above. Then
\[
p_t^{\text{seller}} = \rho_t^M(-\xi),
\]
and
\[
p_t^{\text{buyer}} = -\rho_t^M(\xi).
\]

**Proof.** By definition, \(p_t = p_t^{\text{seller}}\) is such that
\[
\operatorname{ess inf}_{\pi(\cdot) \in \Pi} \rho_t(X^{(\pi)}_{x+p_t}(T) - \xi) = \operatorname{ess inf}_{\pi(\cdot) \in \Pi} \rho_t(X^{(\pi)}_x(T)).
\]

By the translation invariance property of \(\rho_t\), it follows that
\[
p_t^{\text{seller}} = \operatorname{ess inf}_{\pi(\cdot) \in \Pi} \rho_t(X^{(\pi)}_x(T) - \xi) - \operatorname{ess inf}_{\pi(\cdot) \in \Pi} \rho_t(X^{(\pi)}_x(T)).
\]

The dual representation of dynamic convex risk measures and theorem [10] imply that
\[
p_t^{\text{seller}} = \rho_t^M(-\xi) - \rho_t^M(0).
\]

The first part of the claim follows because \(\rho_t^M\) is a normalized risk measure. Since the buyer’s risk indifference price is defined as
\[
\operatorname{ess inf}_{\pi(\cdot) \in \Pi} \rho_t(X^{(\pi)}_{x+p_t}(T)) = \operatorname{ess inf}_{\pi(\cdot) \in \Pi} \rho_t(-X^{(\pi)}_x(T)),
\]

13
a similar argument yields
\[ p^\text{seller}_t = -\left( \text{ess inf}_{\pi(\cdot) \in \Pi} \rho_t(\xi - X_{x+p_t}^\pi(T)) - \text{ess inf}_{\pi(\cdot) \in \Pi} \rho_t(-X_x^\pi(T)) \right) \]
\[ = -(\rho^M_t(\xi) - \rho^M_t(0)) \]
\[ = -\rho^M_t(\xi). \]

Hence, in incomplete markets, risk indifference prices can be formulated in terms of a risk measure \( \rho^M_t \) which depends on the incompleteness of the market — since it involves the supremum with respect to the class of EMM.

Here is our main theorem:

**Theorem 12** Let \( p^\text{low}_t \) and \( p^\text{up}_t \) be the lower and upper hedging prices of \( \xi \) defined in Section 3.1 and let \( p^\text{buyer}_t \) and \( p^\text{seller}_t \) be the buyer’s and seller’s risk indifference prices of \( \xi \) defined is Section 3.2. Then, \( \mathbb{P} - \text{a.s.}, \forall t \in [0, T] \),
\[ p^\text{low}_t \leq p^\text{buyer}_t \leq p^\text{seller}_t \leq p^\text{up}_t. \]

**Proof.** By theorem 1, we know that the upper hedging price \( p^\text{up}_t \) is equal to the solution \( Y^\text{up}_t \) of BSDE
\[ -dY^\text{up}_t = u_t |Z^\text{up}_t| dt - Z^\text{up}_t dW_t, \]
\[ Y^\text{up}_T = \xi. \]

On the other hand, by corollary 11,
\[ p^\text{seller}_t = \rho^M_t(-\xi) \]
\[ = \text{ess sup}_{\theta(\cdot) \in \mathcal{M}} E_{Q_\theta} \left[ \xi - \int_t^T f(u, \theta_u) du |F_t \right] \]
\[ = \text{ess sup}_{\theta(\cdot) \in \mathcal{M}} Y^\text{seller}_t, \]
where \((Y^\text{seller}_t, Z^\text{seller}_t)\) is the solution of
\[ -dY^\text{seller}_t = (-f(t, \theta_t) + Z^\text{seller}_t \theta_t^\top) dt - Z^\text{seller}_t dW_t, \]
\[ Y^\text{seller}_T = \xi. \]

By the comparison theorem for BSDEs (see El Karoui et al. 25), it follows that \( p^\text{seller}_t = \hat{Y}^\text{seller}_t \), where
\[ -d\hat{Y}^\text{seller}_t = (\text{ess sup}_{\theta(\cdot) \in \mathcal{M}} (-f(t, \theta_t) + \hat{Z}^\text{seller}_t \theta_t^\top)) dt - \hat{Z}_t^\text{seller} dW_t, \]
\[ \hat{Y}^\text{seller}_T = \xi. \]
Since \( f \in [0, +\infty] \) and \( |\theta_t| \leq u_t, -f(t, \theta_t) + z\theta_t^T \leq |z| u_t \ \forall z \in \mathbb{R}^{1 \times d} \), and
\[
\operatorname{ess sup}_{\theta(\cdot) \in M} (-f(t, \theta_t) + Z_t^{seller} \theta_t^T) \leq |Z_t^{seller}| u_t, dt \times dP - a.s.
\]

The comparison theorem for BSDEs implies that \( Y^{upper}_t \geq \hat{Y}^{seller}_t \ P - a.s. \), which proves the last inequality. By theorem (2), \( p_t^{low} \) is equal to the solution \( Y^{low}_t \) of BSDE
\[
-dY^{low}_t = -u_t |Z^{low}_t| dt - Z^{low}_t dW_t, \\
Y^{low}_T = \xi.
\]

On the other hand, by a similar argument,
\[
p_t^{buyer} = -p_t^{M}(\xi) = -\operatorname{ess sup}_{\theta(\cdot) \in M} E_Q \left[ -\xi - \int_t^T f(u, \theta_u) du \right] = \operatorname{ess inf}_{\theta(\cdot) \in M} E_Q \left[ \xi + \int_t^T f(u, \theta_u) du \right] = \operatorname{ess inf}_{\theta(\cdot) \in M} Y^{buyer}_t,
\]

where \((Y^{buyer}_t, Z^{buyer}_t)\) is the solution of
\[
-dY^{buyer}_t = (f(t, \theta_t) + Z^{buyer}_t \theta_t^T) dt - Z^{buyer}_t dW_t, \\
Y^{buyer}_t = \xi.
\]

By the comparison theorem for BSDEs, we have that \( p_t^{buyer} = \hat{Y}^{buyer}_t \), where
\[
-d\hat{Y}^{buyer}_t = (\operatorname{ess inf}_{\theta(\cdot) \in M} (f(t, \theta_t) + \hat{Z}^{buyer}_t \theta_t^T)) dt - \hat{Z}^{buyer}_t dW_t, \\
\hat{Y}^{buyer}_t = \xi.
\]

Since \( f(t, \theta_t) + z\theta_t^T \geq -|z| u_t \ \forall z \in \mathbb{R}^{1 \times d} \), it follows from the comparison theorem for BSDEs that \( Y^{low}_t \leq \hat{Y}^{buyer}_t \ P - a.s. \). It remains to prove the second inequality, i.e.
\[
\operatorname{ess sup}_{\theta(\cdot) \in M} Y^{seller}_t \geq \operatorname{ess inf}_{\theta(\cdot) \in M} Y^{buyer}_t.
\]

This follows from the fact that
\[
\operatorname{ess sup}_{\theta(\cdot) \in M} Y^{seller}_t - \operatorname{ess inf}_{\theta(\cdot) \in M} Y^{buyer}_t \\
\geq \operatorname{ess sup}_{\theta(\cdot) \in M} (Y^{seller}_t - Y^{buyer}_t) \\
= \operatorname{ess sup}_{\theta(\cdot) \in M} -2Y^0_t \\
= -2\operatorname{ess inf}_{\theta(\cdot) \in M} Y^0_t = 0,
\]
where \((Y^0_t, Z^0_t)\) is the solution of

\[
-dY^0_t = f(t, \theta_t)dt - Z^0_t \tilde{W}_t,
\]

\[
Y^0_T = 0,
\]

and \(\tilde{W}\) is a \(Q_\theta\)-Brownian motion. \(\blacksquare\)

Finally, we observe that in complete markets, i.e. when there is a unique EMM, super-replication prices and risk indifference prices all reduce to the single Black-Scholes price, that is, \(p^\text{low}_t = p^\text{buyer}_t = p^\text{seller}_t = p^\text{up}_t = p^\text{BS}_t\), \(\forall t \in [0, T]\).

References

[1] Ankirchner, S., Imkeller, P., Popier, A., 2008. Optimal cross hedging of insurance derivatives. Stochastic Anal. Appl. 26, 679–709

[2] Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., 1999. Coherent measures of risk. Math. Finance 9, 203–228

[3] Barles, G., Soner, H.M., 1998. Option pricing with transaction costs and a nonlinear Black-Scholes equation. Finance Stoch. 2, 369–397

[4] Barrieu, P., El Karoui, N., 2004. Optimal derivatives design under dynamic risk measures. In: Mathematics of Finance, Contemporary Mathematics, A.M.S. Proceedings 351, 13–26

[5] Barrieu, P., El Karoui, N., 2005. Inf-convolution of risk measures and optimal risk transfer. Finance Stoch. 9, 269–298

[6] Barrieu, P., El Karoui, N., 2009. Pricing, hedging and optimally designing derivatives via minimization of risk measures. In: Carmona, R. (ed.). Indifference pricing: Theory and applications, Princeton University Press

[7] Becherer, D., 2006. Bounded solutions to backward sde’s with jumps for utility optimization and indifference hedging. Ann. Appl. Probab. 16, 2027–2054

[8] Bellamy, N., Jeanblanc, M., 2000. Incompleteness of markets driven by a mixed diffusion. Finance Stoch. 4, 209–222

[9] Bellini, F., Frittelli, M., 2002. On the existence of minimax martingale measures. Math. Finance 12, 1–21

[10] Bion-Nadal, J., 2009. Time-consistent dynamic risk processes. Stoch. Process. Their Appl. 119, 633–654

[11] Carr, P., Geman, H., Madan, D., 2001. Pricing and hedging in incomplete markets. J. Financial Econ. 62, 131–167
[12] Chen, Z., Wang, B., 2000. Infinite time interval BSDEs and the convergence of g-martingales. J. Austral. Math. Soc. (Series A) 69, 187–211

[13] Cochrane, J.H., Saá-Requejo, J.: Beyond arbitrage: Good-deal asset price bounds in incomplete markets. J. P. E. 108, 79–119 (2000)

[14] Cvitanic, J., Karatzas, I., 1992. Convex duality in constrained portfolio optimization. Ann. Appl. Probab. 2, 767–818

[15] Cvitanic, J., Karatzas, I., 1993. Hedging contingent claims with constrained portfolios. Ann. Appl. Probab. 3, 652–681

[16] Cvitanic, J., Karatzas, I., Soner, M., 1998. Backward stochastic differential equations with constraints on the gains-process. Ann. Probab. 26, 1522–1551

[17] Davis, M., 1997. Option pricing in incomplete markets. In: Demtser, M. and Pliska, S. (eds). Mathematics of derivative securities, Cambridge University Press

[18] Davis, M., Panas, V.C., Zariphopoulou, T., 1993. European option pricing with transactions costs. SIAM J. Control Optim. 31, 470–493

[19] Delbaen, F., 2000. Coherent risk measures. Lecture Notes, Scuola Normale Superiore, Pisa, Italy

[20] Delbaen, F. 2002. Coherent risk measures on general probability spaces. In: Sandmann, K., Schönbucher, P. J. (eds.) Advances in Finance and Stochastics, 1–37, Springer

[21] Delbaen, F., Schachermayer, W., 1994. A general version of the fundamental theorem of asset pricing. Mathematische. Ann. 300, 463–520

[22] Delbaen, F., Peng, S., Rosazza Gianin, E., 2009. Representation of the penalty term of dynamic concave utilities. Finance Stoch. 14, 449–472

[23] Detlefsen, K., Scandolo, G., 2005. Conditional and dynamic convex risk measures. Finance Stoch. 9, 539–561

[24] Eberlein, E., Jacod, J., 1997. On the range of options prices. Finance Stoch. 1, 131–140

[25] El Karoui, N., Peng, S., Quenez, M.-C., 1997. Backward stochastic differential equations in finance. Math. Finance 7, 1–71

[26] El Karoui, N., Quenez, M.-C., 1995. Dynamic programming and pricing of contingent claims in incomplete markets. SIAM J. Control Optim. 33, 29–66

[27] El Karoui, N., Rouge, R., 2000. Pricing via utility maximization and entropy. Math. Finance 10, 259–276
[28] Föllmer, H., Schied, A., 2002. Convex measures of risk and trading constraints. Finance Stoch. 6, 429–447

[29] Föllmer, H., Schied, A., 2002. Robust preferences and convex measures of risk. In: Sandmann, K., Schönbucher, P.J. (eds.) Advances in Finance and Stochastics, 39–56, Springer

[30] Föllmer, H., Schied, A., 2002. Stochastic finance, An introduction in discrete time. De Gruyter Studies in Mathematics, vol. 27

[31] Föllmer, H., Schweizer, M., 1991. Hedging of contingent claims under incomplete information. In: Davis, M.H.A., Elliott, R.J. (eds.), Applied Stochastic Analysis, 389–414, Gordon and Breach

[32] Frittelli, M., Rosazza Gianin, E. 2002. Putting order in risk measures. J. Bank. Finance 26, 1473–1486

[33] Frittelli, M., Rosazza Gianin, E., 2004. Dynamic convex risk measures. In: Szegö, G. (ed) Risk measures for the 21st century, 227–248, Wiley

[34] Gerber, H.U., Shiu, E.S.W., 1994. Option pricing by Esscher transforms (with discussion). Trans. Soc. Actuaries 46, 99–191

[35] Goll, T., Rüschendorf, L., 2001. Minimax and minimal distance martingale measures and their relationship to portfolio optimization. Finance Stoch. 5, 557–581

[36] Harrison, J.M., Pliska, S.R., 1981. Martingales and stochastic integrals in the theory of continous trading. Stoch. Process. Their Appl. 11, 215–260

[37] Henderson, V., Hobson, D., 2008. Utility indifference pricing: An overview. In: Carmona, R. (ed.) Indifference pricing: Theory and Applications, Princeton University Press

[38] Hodges, S. D., Neuberger, A., 1989. Optimal replication of contingent claim under transaction costs. Review of Future Markets 8, 222–239

[39] Horst, U., Müller, M., 2007. On the spanning property of risk bonds priced by equilibrium. Mathematics of Operations Research 32, 784–807

[40] Klöppel, S., Schweizer, M., 2007. Dynamic indifference valuation via convex risk measures. Math. Finance 17, 599–627

[41] Kunita, H., 2004. Representation of martingales with jumps and application to mathematical finance. In: Stochastic Analysis and Related Topics. Advanced Studies in Pure Mathematics, vol. 41, pp. 209–232

[42] Kramkov, D., 1996. Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. Probab. Theory. Related Fields 105, 459–479
[43] Lim, T., Quenez, M.-C., 2010. Portfolio optimization in a defaults model under full/partial information. Preprint arXiv: 1003.6002v1. Available on http://arxiv.org

[44] Musiela, M., Zariphopolou, T., 2004. An example of indifference prices under exponential preferences. Finance Stoch. 8, 229–239

[45] Øksendal, B., Sulem, A., 2009. Risk indifference pricing in jump diffusion markets. Math. Finance 19, 619–637

[46] Pardoux, E., Peng, S., 1990. Adapted solution of a backward stochastic differential equation. Systems Control Lett. 14, 55–61

[47] Xu, M., 2005. Risk measure pricing and hedging in incomplete markets. Ann. Finance 2, 51–71