The asymptotic expansion for $n!$ and Lagrange inversion formula.

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Abstract

We obtain an explicit simple formula for the coefficients of the asymptotic expansion for the factorial of a natural number,

$$n! = n^n \sqrt{2\pi n} e^{-n} \left\{ 1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \cdots \right\},$$

in terms of derivatives of powers of an elementary function that we call normalized left truncated exponential function. The unique explicit expression for the $a_k$ that appears to be known is that of Comtet in [6], which is given in terms of sums of associated Stirling numbers of the first kind. By considering the bivariate generating function of the associated Stirling numbers of the second kind, another expression for the coefficients in terms of them follows also from our analysis. Comparison with Comtet’s expression yields an identity which is somehow unexpected if considering the combinatorial meaning of the terms. It suggests by analogy another possible formula for the coefficients, in terms of a normalized left truncated logarithm, that in fact proves to be true. The resulting coefficients, as well as the first ones are identified via the Lagrange inversion formula as the odd coefficients of the inverse of a pair of formal series. This in particular leads to the identification of a couple of simple implicit equations, which permits us to obtain also some recurrences related to the $a_k’s$.

Keywords: Γ function, asymptotic expansions, Lagrange inversion formula, Stirling numbers.
1 Introduction

Consider, for \( t > 0 \)

\[
e^t = \sum_{j \geq 0} \frac{t^j}{j!} \quad \text{or} \quad \sum_{j \geq 0} e^{-t} \frac{t^j}{j!} = 1,
\]

which means that we may think \( e^{-t} \frac{t^j}{j!} \) as the probability \( p_j \) that a random variable \( X_t \) takes the value \( j \). This amounts to say that \( X_t \) has a Poisson distribution with parameter \( t \), whose expectation, variance and characteristic function, \( \varphi_{X_t}(\theta) \) for \( \theta \in \mathbb{R} \) are:

\[
E(X_t) = t \quad \text{Var}(X_t) = t \quad \varphi_{X_t}(\theta) := E(e^{i\theta X_t}) = \sum_{j \geq 0} e^{i\theta j} p_j = \sum_{j \geq 0} e^{-t} \frac{(e^{i\theta}t)^j}{j!} = e^{t(e^{i\theta} - 1)} \quad (1.1)
\]

Take now \( t = n \in \mathbb{N} \), and consider the random variable

\[
Z_n = \frac{X_n - n}{\sqrt{n}}. \quad (1.2)
\]

From the Central Limit Theorem we know that the characteristic function of \( Z_n \) converges pointwise to that of a standard Normal random variable: for \( \theta \in \mathbb{R} \),

\[
E(e^{i\theta Z_n}) \to e^{-\theta^2/2} \quad \text{as} \quad n \to \infty. \quad (1.3)
\]

On the other hand, projecting the series (1.1) with \( t = n \) over \( e^{-i\theta n} \) in \( L^2[-\pi, \pi] \) we obtain

\[
\int_{-\pi}^{\pi} d\theta E(e^{i\theta X_n}) e^{-i\theta n} = \int_{-\pi}^{\pi} d\theta \frac{e^{-n} n^n}{n!} = 2\pi \frac{e^{-n} n^n}{n!}
\]

Recalling the definition (1.2) of \( Z_n \), a change of variables in the first integral above yields

\[
\int_{-\pi}^{\pi \sqrt{n}} d\theta E(e^{i\theta Z_n}) = \int_{-\pi \sqrt{n}}^{\pi \sqrt{n}} d\theta E(e^{i\theta (X_n - n)}) = 2\sqrt{n} \frac{e^{-n} n^n}{n!},
\]

so

\[
n! = \frac{\sqrt{2\pi n} e^{-n} n^n}{\frac{1}{\sqrt{2\pi}} \int_{-\pi \sqrt{n}}^{\pi \sqrt{n}} d\theta E(e^{i\theta Z_n})} \quad (1.4)
\]
If the convergence in (1.3) could be seen to hold uniformly, we would recover Stirling’s formula:

\[ n! \approx \sqrt{2\pi n} e^{-n} n^n \]

It is not difficult to see that this is indeed the case, as has been noticed by several authors, see for instance [12] and the references therein, or Sect 27, exercise 18 of [5]. L. Báez–Duarte considered in [2] the above method to obtain the asymptotics for the general term of a convergent power series of a function \( f \) with non-negative terms, under conditions that guarantee that a uniform Central Limit Theorem for the corresponding random variables holds and, as a particular instance, he obtains the asymptotics for \( p(n) \), the number of partitions of an integer \( n \). The approach has a longer history, and goes back to the articles by W.K. Hayman [9] and E.A. Bender [3].

Here, we obtain an asymptotic expansion for the denominator in (1.4) in terms of inverse powers of \( n \), and show that it can be explicitly inverted, yielding a simple expression for the coefficients \( a_k \) of the expansion of \( n! \), in terms of an elementary function (see Theorem 2.1 below). To our knowledge, the unique explicit formula for those coefficients is that given by Comtet as an exercise in page 267 of [6], in terms of sums of derangements \( d_3 \) of integers with certain restrictions, related with the Stirling numbers of the first kind. As a corollary of the proof of our first formula, after recalling the bivariate generating function for the associated Stirling number of the second kind \( S_3 \), we obtain a second expression for the coefficients, which is exactly alike to that in [6], and in particular yields an unexpected equality between certain alternating sums of associated Stirling numbers of the first and second kind. This identity suggested another expression for the \( a_k \), in terms of a normalized left truncated logarithm function corresponding exactly with the first one, that we proved to be true (see Theorem 2.7). Finally, we recognize in these two expressions the odd terms of the series obtained by the Lagrange inversion formula applied to appropriate functions. Each of these two inverses satisfy a simple implicit equation. By differentiation we obtain a couple of differential equations that lead to a pair of recurrent relations that could be used also to generate the \( a_k \)'s. One of them is similar to that obtained in [11] using other techniques (see also [7]).

Although we state the results as expressions for the coefficients of the expansion of \( n! \) in inverse powers of \( n \), they could be stated as well as expressions for the coefficients of the asymptotic expansion of the gamma function in inverse powers of \( z \in \mathbb{C} \), for \( |z| \to \infty \) in the region \( \arg(z) \in [0, \pi) \). This
follows at once from the existence of this last expansion. For this and other properties of the gamma, see [1]. Let us also note that the first few coefficients, that in particular appear in most books and tables of special functions (for instance [8] or [10]), can be computed one by one by exponentiating the well known Stirling series for the logarithm of the gamma, or from recurrence relations obtained by different techniques (as in [15] or [11]). We think however that having explicit and simple formulae has an intrinsic interest. In particular, we show that they render some combinatorial identities, and relate with the Lagrange inversion formula for elementary functions, what helps to understand the relation of the expansion with the Lambert W function appearing in [7].

We provide precise statements concerning the asymptotic expansions, and obtain the formulae for the coefficients in the next section. In the last section, we show the relationship with the Lagrange inversion formula, obtain recursive formulae and generalise the identities obtained between sums of $S_3$ and $d_3$.

## 2 The asymptotic expansion and formulae for the coefficients

Let us denote by $\partial^k f$ the $k$th derivative of a function $f$, with respect to its real or complex variable. Our first result is the following formula for the coefficients of the expansion of $n!$ in powers of $\frac{1}{n}$.

**Theorem 2.1.** Let $n \in \mathbb{N}$. Then the coefficients of the expansion

\[
 n! \asymp \sqrt{2\pi n} \ e^{-n} n^n \left( \sum_{k=0}^{\infty} \frac{1}{n^k} a_k \right)
\]

(2.1)

are given by

\[
 a_k = \frac{1}{2^k k!} \ \partial^{2k} \left( G^{-\frac{2k+1}{2}} \right)(0)
\]

(2.2)

where

\[
 G(x) = 2 \ \frac{e^x - 1 - x}{x^2} = 2 \sum_{j \geq 0} \frac{x^j}{(j+2)!}
\]

**Remark 2.2.** Formula (2.1) is to be understood as an asymptotic expansion, that is, for any given $N \geq 0$,

\[
 n! = \sqrt{2\pi n} \ e^{-n} n^n \left( \sum_{k=0}^{N} \frac{1}{n^k} a_k + R_{N+1} \right)
\]
with \( R_{N+1} = O\left(\frac{1}{n^r}\right) \) as \( n \to \infty. \) See for instance [10] or [14] on the subject of asymptotic expansions.

The function \( G \) above is what we call a normalized left truncated exponential. In general we define

**Definition 2.3.** Given a function \( F(x) = \sum f_n x^n \) with \( f_2 \neq 0, \) its normalized left truncated associated function \( F_2 \) is defined as

\[
F_2(x) = \frac{F(x) - f_0 - f_1 x}{f_2 x^2 / 2}
\]

We also obtain another expression for \( a_k \) in terms of the 3-associated Stirling numbers of second kind. Recall that, for \( r \) an integer \( \geq 1, \) \( n, k \) integers \( \geq 0, \) the \( r \)-associated Stirling number of the second kind \( S_r(n, k) \) is the number of partitions of a set of \( n \) elements into \( k \) blocks, all with at least \( r \) elements. The convention is that \( S_r(0, 0) = 1. \) Then we have

**Theorem 2.4.** The coefficients \( a_k \) in (2.1) are also given in terms of the 3-associated Stirling numbers of second kind by

\[
a_k = \sum_{j=0}^{2k} (-1)^j \frac{S_3(2(j + k), j)}{2j + k (j + k)!}
\]  

(2.3)

**Proof of Theorem 2.4** The strategy will consist in expanding the denominator in (1.4) in powers of \( \frac{1}{n} \), and then taking the inverse of the resulting series. From (1.1) and (1.2), for \( \theta \in \mathbb{R}, \)

\[
E(e^{i\theta Z_n}) = e^n \left( e^{\theta^2/n} - 1 - \frac{\theta}{\sqrt{n}} \right) = e^{-\frac{\theta^2}{2} - \theta^2 g\left(\frac{\theta}{\sqrt{n}}\right)}
\]

(2.4)

where, for \( z \in \mathbb{C} \)

\[
g(z) := \frac{e^z - 1 - z - z^2/2}{z^2} = \sum_{k \geq 1} \frac{z^k}{(k + 2)!}
\]

(2.5)

For given \( K \in \mathbb{N}, \) consider a Taylor expansion of order \( K \) in powers of \( z \) of \( e^{-\theta^2 g(z)}: \)

\[
e^{-\theta^2 g(z)} = \sum_{j=0}^{K} \frac{z^j}{j!} \frac{\partial^j}{\partial z^j} \left( e^{-\theta^2 g(z)} \right) \bigg|_{z=0} + R_{K+1}(z)
\]

(2.6)
The remainder $R_{K+1}$ satisfies the estimate

$$|R_{K+1}(z)| \leq \frac{|z|^{K+1}}{(K+1)!} \sup_{\zeta \in [0,z]} |\partial^{K+1}(e^{-\theta^2 g(\zeta)})|$$  \hspace{1cm} (2.7)

Substitution of (2.4) and (2.6) in the denominator $D_n$ in (1.4) yields

$$D_n := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\theta \ E(e^{i\theta} Z_n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\theta \ e^{-\frac{\theta^2}{2}} e^{-\theta^2 g(\frac{i\theta}{\sqrt{n}})}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{K} \int_{-\pi}^{\pi} d\theta \ e^{-\frac{\theta^2}{2}} (\frac{i\theta}{\sqrt{n}})^j \frac{1}{j!} \partial^j \left( e^{-\theta^2 g(z)} \right) \bigg|_{z=0}$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\theta \ e^{-\frac{\theta^2}{2}} R_{K+1}(\frac{i\theta}{\sqrt{n}}).$$  \hspace{1cm} (2.8)

It can be easily seen by induction that, for $j \in \mathbb{N}$ and $z \in \mathbb{C}$

$$\partial^j \left( e^{-\theta^2 g(z)} \right) = e^{-\theta^2 g(z)} P_{j,z}(\theta^2)$$

where $P_{j,y}(\cdot)$ is a polynomial of degree $j$ whose coefficients depend on products of derivatives of $g$ of order up to $j$ evaluated in $z$. From (2.5) it is easy to see that $g$ and all its derivatives are bounded over compacts. Moreover, if we denote by $R(z)$ the real part of $z$, we have

$$|e^{-\theta^2 g(\frac{i\theta}{\sqrt{n}})}| \leq e^{-\theta^2 R(g(\frac{i\theta}{\sqrt{n}}))}$$  \hspace{1cm} (2.9)

and, for $x \in \mathbb{R}$,

$$R(g(ix)) = R \left( \frac{e^{ix} - 1 - ix + x^2/2}{-x^2} \right) = - \left( \frac{\cos x - 1 + x^2/2}{x^2} \right)$$

$$= -\frac{x^2}{4} \partial^4 \cos(ax) \quad \text{for some } a \in [0, 1]$$  \hspace{1cm} (2.10)

From (2.7), (2.9) and (2.10), we can estimate the last integral in (2.8) as follows:

$$\left| \int_{-\pi}^{\pi} d\theta \ e^{-\frac{\theta^2}{2}} R_K(\frac{i\theta}{\sqrt{n}}) \right|$$

$$\leq \frac{1}{(K+1)!} \int_{-\pi}^{\pi} d\theta \ e^{-\frac{\theta^2}{2}} \sup_{|y| \leq \pi} \left| P_{K+1,y}(\theta^2) \right| \leq \frac{C_K}{n^{K+1}},$$  \hspace{1cm} (2.11)
where \( C_K \) is a constant which depends on \( K \) but not on \( n \). Then, observe that the difference of each integral in the sum in (2.8) with respect to the integral in the whole \( \mathbb{R} \) is exponentially small in \( n \): for any \( j \geq 0 \), there is some \( \alpha \) positive,

\[
\left| \int_{-\pi}^{\pi} d\theta e^{-\frac{\theta^2}{2}} \left( \frac{i\theta}{\sqrt{n}} \right)^j \frac{1}{j!} \partial^j \left( e^{-\theta^2 g(x)} \right) \right|_{x=0} = \\
- \int_{-\infty}^{\infty} d\theta e^{-\frac{\theta^2}{2}} \left( \frac{i\theta}{\sqrt{n}} \right)^j \frac{1}{j!} \partial^j \left( e^{-\theta^2 g(x)} \right) \right|_{x=0} \leq \int_{|\theta|>\pi/\sqrt{n}} d\theta e^{-\frac{\theta^2}{2}} \left( \frac{i\theta}{\sqrt{n}} \right)^j \frac{1}{j!} P_{j,0}(\theta^2) \leq e^{-\alpha n} \quad (2.12)
\]

Moreover, if we interchange the derivative with the integral in the second line above, and then compute the integral, we obtain:

\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\theta e^{-\frac{\theta^2}{2}} \left( \frac{i\theta}{\sqrt{n}} \right)^j \frac{1}{j!} \partial^j \left( e^{-\theta^2 g(x)} \right) \right|_{x=0} = \\
\begin{cases} 
0 & \text{if } j \text{ odd} \\
\left( \frac{i}{\sqrt{n}} \right)^j \frac{1}{j!} (j-1)!! \partial^j \left( (1+2g(x))^{-\frac{j+1}{2}} \right) \right|_{x=0} & \text{if } j \text{ even} \quad (2.13)
\end{cases}
\]

Then, for any given positive \( N \in \mathbb{N} \), consider \( K = 2N + 1 \) in (2.8), sum and substract the integrals in the whole line to each term in the sum, estimate the differences using (2.12), evaluate the integrals as in (2.13) and estimate the term with the remainder using (2.11). Rename finally the summation index to obtain:

for any \( N \geq 0 \),

\[
D_n = \sum_{k=0}^{N} \frac{(-1)^k}{n^k} \frac{1}{2^k k!} \partial^{2k} (G^{\frac{-2k+1}{2}})(0) + O\left( \frac{1}{n^{N+1}} \right), \quad (2.14)
\]

where we have used that \( 1 + 2g(x) = G(x) \), as defined in (2.2). Let us shorthand (2.14) above as

\[
D_n \sim \sum_{k\geq0} \frac{(-1)^k}{n^k} \frac{1}{2^k k!} \partial^{2k} (G^{\frac{-2k+1}{2}})(0), \quad (2.15)
\]

and recall we have

\[
n! = \frac{\sqrt{2\pi n} e^{-n} n^n}{D_n}. \quad (2.16)
\]
Consider next the well known Stirling series, which is an asymptotic expansion for large values of $|z|$ of the logarithm of the gamma function in powers of $\frac{1}{z}$ (see for instance Theorem 1.4.2 in [1]). Its coefficients are given in terms of the Bernoulli numbers $B_k$. In particular, it implies
\[
\log(n!) \asymp \log(\sqrt{2\pi n} e^{-n} n^n) + \sum_{k \geq 1} \frac{B_{2k}}{2k(2k-1) n^{2k-1}}
\]
Since the series $S_n$ above contains only odd powers of $\frac{1}{n}$, it follows that
\[
\log\left(\frac{\sqrt{2\pi n} e^{-n} n^n}{n!}\right) \asymp -S_n = S_{-n}.
\]
Therefore,
\[
\frac{1}{D_n} \asymp e^{-S_n} = e^{S_{-n}} = \sum_{k \geq 0} \frac{1}{n^k} \frac{1}{2^{k}k!} \partial^{2k}(G^{\frac{-2k+1}{2}})(0)
\]
and the theorem follows from (2.16).

To prove Theorem 2.4, we will use the generating function of the $r$–associated Stirling numbers already introduced. Namely, (see page 222, exercise 7, Ch. 5 of [6]) : for any $r \geq 1$
\[
H(t, u) := e^{u \left(\frac{t^r}{r} + \frac{t^{r+1}}{(r+1)!} + \cdots \right)} = \sum_{l,k \geq 0} S_r(l,k) u^k t^l \frac{l!}{l!}
\]
Proof of Theorem 2.4  
Observe that, from the first equation in (2.4) we may write the integral in the denominator in (1.4) as
\[
\int_{-\pi \sqrt{n}}^{\pi \sqrt{n}} d\theta \left( e^{i\theta Z_n} \right) = \int_{-\pi \sqrt{n}}^{\pi \sqrt{n}} d\theta e^{-\frac{\theta^2}{2}} e^{nf(\frac{\theta}{\sqrt{n}})},
\]
with $f(z) = e^z - 1 - z - \frac{z^2}{2}$. If we expand
\[
e^{nf(z)} = \sum_{j \geq 0} \frac{z^j}{j!} \partial^j e^{nf}(0),
\]
substitute in (2.18) with \( z = \frac{i\theta}{\sqrt{n}} \) and proceed as in the proof of Theorem 2.1 (that is, consider a finite Taylor expansion with remainder, interchange the sum and derivative with the integral, and observe that the integral may be considered in the whole line with an error smaller than the remainder) we obtain that for any given \( K \),

\[
D_n = \sum_{j=0}^{K} \frac{i^j}{(\sqrt{n})^j j!} \partial^j e^{nf}(0) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\theta \theta^j e^{-\theta^2/2} + R_{K+1}
\]

The remainder can be seen to satisfy \( R_{K+1} = O\left( \frac{1}{n^{(K+1)/2}} \right) \). After computing the integral and renaming the terms, we may shorthand the above as

\[
D_n = \sum_{j=0}^{\infty} \frac{(-1)^j}{n^j 2^j j!} \partial^{(2j)} e^{nf}(0) \tag{2.19}
\]

Next, from the generating function for \( S_3 \) (2.17) and observing that \( S_3(N, k) = 0 \) if \( k \geq N > 0 \), we obtain

\[
\partial^{(2j)} e^{nf}(0) = \sum_{k \leq 2j} n^k S_3(2j, k)
\]

Substituting in (2.19), and then summing over \( l = k - j \), after recalling that from the combinatorial interpretation of \( S_3 \) it is easy to see that \( S_3(2j, k) = 0 \) if \( k \geq j > 0 \)

\[
D_n = \sum_{j \geq 0, k \leq 2j} (-1)^j \frac{S_3(2j, k)}{n^{j-k} 2^j j!} = \sum_{l \geq 0} \frac{(-1)^l}{n^{l}} \sum_{k=0}^{2l} \frac{(-1)^k S_3(2(k + l), k)}{2^{k+l}(k+l)!}
\]

Concluding as in the proof of 2.1 that \( \frac{1}{D_n} = (2.3) \), we obtain (2.3) \( \square \)

From Theorems 2.1 and 2.3 since the asymptotic expansion of a function is unique, we have that

\[
\frac{1}{2^k k!} \partial^{2k} \left( G^{-\frac{2k+1}{2}} \right)(0) = \sum_{j=0}^{2k} (-1)^j \frac{S_3(2(j + k), j)}{2^{j+k} (j+k)!}
\]

Moreover, we have a second identity that follows from the expression for the coefficients of the expansion for the \( \gamma \) function given in [6], and already
mentioned. It is

\[ \Gamma(x) = x^e \frac{e^{-x}}{\sqrt{2\pi}} \sum_{k \geq 0} \frac{c_k}{k!} x^k, \]

with \( c_k = \sum_{j=0}^{2k} \frac{(-1)^j d_3(2(j+k),j)}{2^{j+k} (j+k)!} \). \hfill (2.20)

Permutations without fixed points (in other words, without cycles of length 1) are known as derangements, and for natural \( r \geq 1 \), \( d_r(n,l) \) is the number of derangements of a set of \( n \) elements, that have \( l \) cycles, all of length \( \geq r \).

Equating the coefficients in (2.3) and the corresponding ones resulting from (2.20) when considering \( n! = n\Gamma(n) \), we get the remarkable combinatorial identity presented in the following proposition.

**Proposition 2.5.** For \( S_3 \) and \( d_3 \) as defined above and for any given positive \( k \in \mathbb{N} \),

\[ \sum_{j=0}^{2k} (-1)^j \frac{S_3(2(j+k),j)}{2^{j+k} (j+k)!} = \sum_{j=0}^{2k} (-1)^j \frac{d_3(2(j+k),j)}{2^{j+k} (j+k)!}. \] \hfill (2.21)

This identity is somehow surprising, since it is needed a precise balance to control the different growth of \( S_3 \) and \( d_3 \). We give a direct proof of a generalization of it in Proposition 3.2. Now, let us compute the generating function for \( d_3(j,k) \).

**Lemma 2.6.** The generating function for \( d_3 \) as defined above is

\[ \Phi(t,u) = (1-t)^{-u} e^{-u \left( t+\frac{t^2}{2} \right)} = \sum_{k,j \geq 0} d_3(j,k) u^k t^j j!, \] \hfill (2.22)

**Proof** We first compute the exponential generating function of the cyclic permutations of length at least 3

\[ \sum_{k \geq 3} (n-1)! \frac{t^n}{n!} = \sum_{k \geq 3} \frac{t^n}{n} = \log (1-t)^{-1} - \left( t + \frac{t^2}{2} \right). \]

By the exponential formula (see [13], Corollary 5.1.6), we obtain that

\[ \phi(t) = (1-t)^{-1} e^{-(t+\frac{t^2}{2})}. \]
is the exponential generating function for the number of derangements of a set of \( n \) elements, whose cycles are all of order at least 3. Then, the generating function
\[
\Phi(t, u) = \phi(t)^u = (1 - t)^{-u} e^{-u(t + \frac{t^2}{2})},
\]
also keeps track of the number of cycles on this kind of derangements, (see [13], Example 5.2.2) obtaining equation (2.22).

By writing the generating function \( H \) of \( S_3 \) (see (2.17)) and \( \Phi \) in the form
\[
H(t, u) = e^u\left(e^{t-1-t-\frac{t^2}{2}}\right) \quad \Phi(t, u) = e^u\left(-\log(1-t)-t-\frac{t^2}{2}\right)
\]
the analogy between them is apparent, and, together with the identity (2.21), suggests that a formula for the \( a_k \) in terms of a normalized left truncated logarithm, in the sense of definition 2.3, may hold. This proves to be true, yielding another formula, exactly alike to (2.1).

**Theorem 2.7.** The coefficients in the asymptotic expansion of \( n! \)
\[
n! = \sqrt{2\pi n} e^{-n} n^n \left( \sum_{k \geq 0} \frac{1}{n^k} a_k \right)
\]
are also given by
\[
a_k = \frac{1}{2^k k!} \delta^{2k}(L^{-\frac{2k+1}{x}})(0) \quad (2.23)
\]
for
\[
L(x) = 2 - \log(1 + x) + x - \frac{2}{x^2} = 2 \sum_{j \geq 0} \frac{x^j}{(j + 2)!}
\]

**Proof** Consider the identity
\[
\Gamma(x) = x^x \int_0^\infty dt \ e^{-xt} t^{x-1} \quad x > 0
\]
After integrating by parts once, and changing variables \( \sqrt{x} (t - 1) \to u \), we obtain
\[
\Gamma(x) = \frac{x^x e^{-x} \sqrt{2\pi}}{\sqrt{x}} \left( \frac{1}{\sqrt{2\pi}} \int_0^\infty dt \ e^{-x\left(\frac{t}{\sqrt{x}} - 1\right)} e^{x\log t} \sqrt{x} \right) \quad (2.24)
\]
\[
= \frac{x^x e^{-x} \sqrt{2\pi}}{\sqrt{x}} \left( \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^\infty du \ e^{-x\left(\frac{u}{\sqrt{x}}\right)} e^{x\log(\frac{u}{\sqrt{x}} + 1)} \right)
\]
\[
= \frac{x^x e^{-x} \sqrt{2\pi}}{\sqrt{x}} I(x)
\]
Define the function
\[ \ell(y) = \frac{\log(1 + y) - y + y^2/2}{y^2} = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k + 2} y^k, \] (2.25)
to obtain, using the expansion
\[ e^{u^2 \ell(y)} = \sum_{j \geq 0} \frac{y^j}{j!} \partial^j e^{u^2 \ell(0)} \]
in (2.24) with \( y = \frac{u}{\sqrt{x}} \)

\[ I(x) = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\infty} du \, e^{-\frac{u^2}{2}} e^{u^2 \ell\left(\frac{u}{\sqrt{x}}\right)} = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\infty} du \, e^{-\frac{u^2}{2}} \sum_{j \geq 0} \left( \frac{u}{\sqrt{x}} \right)^{j} \frac{1}{j!} \partial^j e^{u^2 \ell(0)} \] (2.26)

It is clear that
\[ \left( \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\infty} du \, u^j \, e^{-\frac{u^2}{2}} \left(1 - 2\ell(y)\right) \right) \approx \begin{cases} (1 - 2\ell(y))^{-\frac{j+1}{2}} (j - 1)!! & \text{if } j \text{ even} \\ 0 & \text{otherwise} \end{cases} \]
(in the sense that the difference is exponentially small in \( x \) as \( x \to \infty \), the details are analogous to that in the proof of 2.1). Renaming and reorganizing terms in the sum above, and observing that \( 1 - 2\ell(y) = L(y) \) as defined in (2.23) it follows
\[ I(x) = \sum_{k \geq 0} \frac{1}{2^k k! x^k} \partial^{2k} \left( L^{-\frac{2k+1}{2}} \right)(0) \]

Taking \( x = n \) and using that \( n! = n\Gamma(n) \), we conclude the proof, from (2.24) and the above formula for \( I \).

Since a proof of the expansion (2.20) of Comtet does not seem to be available, we provide next a brief sketch of it, that follows from a slight modification of the arguments in the proof above, exactly as Theorem 2.4 was obtained by modifying the proof of Theorem 2.1.
Proof of (2.20). Write the integral in (2.24) as
\[ I(x) = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\infty} du \, e^{-\frac{u^2}{2} + x h\left(\frac{u}{\sqrt{x}}\right)}, \]
where \( h(y) = \log(1 + y) - y + \frac{y^2}{2} \).
and expand
\[ e^{x h(y)} = \sum_{k \geq 0} \frac{y^k}{k!} \partial^k e^{x h}(0). \]
Substituting in the integral with \( y = \frac{u}{\sqrt{x}} \),
\[
I(x) = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\infty} du \, e^{-\frac{u^2}{2}} \sum_{k \geq 0} \frac{u^k}{k! x^{k/2}} \partial^k e^{x h}(0) \\
= \sum_{k \geq 0} \frac{1}{k! x^{k/2}} \partial^k e^{x h}(0) \left( \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\infty} du \, e^{-\frac{u^2}{2}} u^k \right)
\]
and integrating as before
\[
\frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\infty} du \, e^{-\frac{u^2}{2}} u^k \approx \begin{cases} (k - 1)!! & \text{if } k \text{ even} \\ 0 & \text{otherwise}, \end{cases}
\]
after renaming and collecting terms, we have
\[ I(x) = \sum_{j \geq 0} \frac{1}{x^j 2^j j!} \partial^{2j} e^{x h}(0) \]
Now, from (2.22) and the definition of \( h, \Phi(y, x) = e^{-x h(-y)} \), and
\[
\partial^{2j} e^{x h}(0) = \partial^{2j} e^{x h(y)}|_{y=0} = \partial^{2j} e^{x h(-y)}|_{y=0} = \partial^{2j} \Phi(-x, y)|_{y=0}
\]
Hence, from (2.22)
\[
I(x) = \sum_{j \geq 0} \frac{1}{x^j 2^j j!} \partial^{2j} \Phi(-x, y)|_{y=0} = \sum_{j \geq 0} \frac{1}{x^j 2^j j!} \sum_{k=0}^{2j} d_3(2j, k)(-x)^k,
\]
which changing \( k + l = j \) and proceeding exactly as in the last part of the proof of Theorem 2.4 yields (2.20).
3 Lagrange inversion and recursive formulae.

We will identify the \( a_k \)'s in (2.2) and (2.23) as factors of the odd coefficients of a pair of formal power series, with the aid of the Lagrange inversion formula, that we recall next in a suitable form. Let \( R(x) \) be a formal power series of the form \( R(x) = xT(x) \), where \( T(x) \) is a formal power series with non-zero constant term. Denote by \( R^{(-1)} \) the substitutional inverse of \( R \), i.e., \( S = R^{(-1)} \) means \( S(x)T(S(x)) = S(xT(x)) = x \). Let \( S(x) = \sum_{k=1}^{\infty} s_k \frac{x^k}{k!} \). The Lagrange inversion formula gives a simple recipe to compute the coefficients of the series \( S \) (see for example [4]).

\[
\begin{align*}
    s_k &= \partial^{k-1} \left( T^{(-1)} \right) (0) = \partial^{k-1} \left( T^{-k} \right) (0). \tag{3.1}
\end{align*}
\]

Define the following exponential formal power series

\[
\begin{align*}
    B(x) &= \sum_{k=1}^{\infty} b_k \frac{x^k}{k!} := \left( x \left( \frac{e^x - 1 - x}{x^2/2} \right)^{1/2} \right)^{(-1)} \tag{3.2} \\
    &= (\left( 2e^x - 2 - 2x \right)^{1/2})^{(-1)} \\
    C(x) &= \sum_{k=1}^{\infty} c_k \frac{x^k}{k!} := \left( x \left( \frac{-\log(1 + x) + x}{x^2/2} \right)^{1/2} \right)^{(-1)} \tag{3.3} \\
    &= (\left( -2\log(1 + x) + 2x \right)^{1/2})^{(-1)}.
\end{align*}
\]

Observe that from equations (2.2) and (2.23), and the Lagrange inversion formula (3.1) we identify

\[
a_k = \frac{b_{2k+1}}{2^k k!} = \frac{c_{2k+1}}{2^k k!} = (2k + 1)!! \tilde{b}_{2k+1} = (2k + 1)!! \tilde{c}_{2k+1},
\]

where \( \tilde{b}_k := b_k/k! \) and \( \tilde{c}_k := c_k/k! \). Then, the coefficients of the formal power series \( B(x) \) and \( C(x) \) coincide at odd powers. A much stronger result holds indeed.

**Proposition 3.1.** The formal power series \( B(x) \) and \( C(x) \) differ only at the coefficient of the quadratic term. More precisely, we have

\[
C(x) - B(x) = \frac{x^2}{2}.
\]
Proof. From equations (3.2) and (3.3) we obtain the implicit equations

\[ e^{B(x)} - 1 - B(x) = \frac{x^2}{2} \]  (3.4)

and

\[ C(x) - \frac{x^2}{2} = \log(1 + C(x)). \]  (3.5)

Taking log at both sides of (3.4),

\[ B(x) = \log \left(1 + B(x) + \frac{x^2}{2}\right), \]  (3.6)

which is equivalent to (3.5) by the change \( C(x) = B(x) + \frac{x^2}{2} \). All the operations to obtain respectively (3.6) and (3.5) from (3.2) and (3.3) are reversible in the context of formal power series. Then, the result follows.

Explicit expressions for the coefficients of the series \( B(x) \) and \( C(x) \) can be obtained by using Lagrange inversion formula, leading to the following generalization of the identity (2.21).

**Proposition 3.2.** The following identity holds for every \( k \neq 2 \)

\[
\sum_{j=0}^{k-1} (-1)^j \frac{S_3(k+2j-1, j)}{(k+1)(k+3)\ldots(k+2j-1)} = \sum_{j=0}^{k-1} (-1)^{k+j-1} \frac{d_3(k+2j-1, j)}{(k+1)(k+3)\ldots(k+2j-1)}.
\]

Proof. By the Lagrange inversion formula we have

\[
b_k = \partial^{k-1} \left( \frac{e^x - 1 - x}{x^2/2} \right)^{-k/2} \bigg|_{x=0} = \partial^{k-1} \left( 1 + 2 \frac{e^x - 1 - x - x^2/2}{x^2} \right)^{-k/2} \bigg|_{x=0}
\]
By the binomial identity

\[
\left(1 + 2 \frac{e^x - 1 - x - x^2/2}{x^2}\right)^{-k/2} = \sum_{j=0}^{\infty} \left(\frac{-k/2}{j}\right) 2^j \frac{(e^x - 1 - x - x^2/2)^j}{x^{2j}}
\]

\[
= \sum_{j=0}^{\infty} (-1)^j k(k+2)\ldots(k+2(j-1)) \frac{1}{x^{2j}} j! \sum_{l \geq 3j} S_3(l, j) \frac{x^l}{l!}.
\]  

(3.7)

The last equality follows from equation (2.17) for \(r = 3\), after writing the exponential in power series of the exponent, and equating powers of \(u\), to obtain

\[
\frac{(e^x - 1 - x - x^2/2)^j}{j!} = \sum_{l \geq 3j} S_3(l, j) \frac{x^l}{l!}.
\]  

(3.8)

Interchanging sums in (3.7) we get

\[
\sum_{l=0}^{\infty} \sum_{3j \leq l} (-1)^j \frac{k(k+2)\ldots(k+2(j-1))}{l(l-1)\ldots(l-2j+1)} \frac{1}{S_3(l, j)} x^{l-2j} \frac{x^l}{(l-2j)!}.
\]  

(3.9)

The coefficient of \(x^{k-1}/(k-1)!\) of this series, obtained by making \(l-2j = k-1\), is equal to

\[
b_k = \sum_{j=0}^{k-1} (-1)^j \frac{k(k+2)\ldots(k+2(j-1))}{k(k+1)\ldots(k+2j-1)} S_3(k+2j-1, j)
\]

\[
= \sum_{j=0}^{k-1} (-1)^j \frac{S_3(k+2j-1, j)}{(k+1)(k+3)\ldots(k+2j-1)}.
\]

Notice that from equation 2.22, proceeding as to obtain (3.8)

\[
\frac{(- \log(1-x) - x - x^2/2)^j}{j!} = \sum_{l \geq 3j} d_3(l, j) \frac{x^l}{l!}.
\]

From that we easily obtain

\[
\frac{(\log(1+x) - x + x^2/2)^j}{j!} = \sum_{l \geq 3j} (-1)^{l-j} d_3(l, j) \frac{x^l}{l!}.
\]  

(3.10)
By expanding the binomial
\[
\left(1 - 2\frac{\log(1 + x) - x + x^2/2}{x^2}\right)^{k/2} = \left(\frac{-\log(1 + x) + x}{x^2/2}\right)^{k/2},
\]
using equation (3.10), and proceeding exactly as for the computation of \(b_k\), we get
\[
c_k = \sum_{j=0}^{k-1} (-1)^{k+j-1} \frac{d_3(k + 2j - 1, j)}{(k+1)(k+3)\ldots(k+2j-1)}.
\]
The result then follows from Proposition 3.1.

Now we state and prove recursive formulae to obtain \(b_k\), \(\tilde{b}_k\), \(c_k\), and \(\tilde{c}_k\).

**Proposition 3.3.** The sequences \(b_k\), \(c_k\), \(\tilde{b}_k\) and \(\tilde{c}_k\) satisfy the following recursive formulae

\[
b_k = -\frac{1}{k+1} \left( \binom{k}{2} b_{k-1} + \sum_{j=1}^{k-2} \binom{k}{j} b_{j+1} b_{k-j} \right) \tag{3.11}
\]
\[
\tilde{b}_k = -\frac{1}{k+1} \left( \frac{k-1}{2} \tilde{b}_{k-1} + \sum_{j=1}^{k-2} (j+1) \tilde{b}_{j+1} \tilde{b}_{k-j} \right) \tag{3.12}
\]
\[
c_k = \frac{1}{k+1} \left( k c_{k-1} - \sum_{j=1}^{k-2} \binom{k}{j} c_{j+1} c_{k-j} \right) \tag{3.13}
\]
\[
\tilde{c}_k = \frac{1}{k+1} \left( \tilde{c}_{k-1} - \sum_{j=1}^{k-2} (j+1) \tilde{c}_{j+1} \tilde{c}_{k-j} \right) \tag{3.14}
\]
with \(b_1 = c_1 = \tilde{b}_1 = \tilde{c}_1 = 1\).

**Proof.** Formulae (3.12) and (3.14) follow from formulae (3.11) and (3.13) respectively. Computing the derivative in both sides of equations (3.6) and (3.5), after clearing up we obtain
\[
B'(x)B(x) = x - (x^2/2)B'(x) \tag{3.15}
\]
\[
C'(x)C(x) = xC(x) + x. \tag{3.16}
\]
Recurrence (3.11) (respectively (3.13)) is obtained by equating coefficients of \(x^k/k!\) in both sides of the resulting series in (3.15) (respectively (3.16)).
Observe that (3.14) is similar to the recurrence obtained in [11] (see also [7]). We list a few terms of both formal power series $B(x)$ and $C(x)$,

\[
B(x) = x - x^2/6 + x^3/36 - x^4/270 + x^5/4320 + x^6/17010 + \ldots \\
C(x) = x + x^2/3 + x^3/36 - x^4/270 + x^5/4320 + x^6/17010 + \ldots .
\]

The coefficients $a_k's$ can be readily computed as well from formulae (2.2) or (2.23), using for instance MAPLE. In particular, we checked that the first twenty terms coincide with that of [15].

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