BASIC SETS FOR QUASI-ISOLATED BLOCKS OF FINITE GROUPS OF EXCEPTIONAL LIE TYPE

RUWEN HOLLENBACH

Abstract. Let $G$ be a simple, simply connected algebraic group of exceptional type defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : G \to G$. We prove that generalized $e$-Harish-Chandra theory holds for $\mathcal{E}(G^F, s)$ where $s \in G^{*F}$ is semisimple and quasi-isolated. In turn, this yields explicit basic sets for the corresponding quasi-isolated blocks of $G^F$, which we then use to prove a conjecture of Malle and Robinson for these blocks.

Summary

By work of Geck–Hiss [15] and Geck [14], it is known that ordinary basic sets exist for blocks of finite groups of Lie type under some mild conditions on the prime $\ell$. However, their results do not yield explicit basic sets. In this paper, we will determine explicit ordinary basic sets for the quasi-isolated $\ell$-blocks of finite groups of exceptional Lie type under similar conditions on $\ell$ as in [14].

Let $G$ be a simple, simply connected algebraic group of exceptional type defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : G \to G$, or let $G$ be simple, simply connected of type $D_4$ defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : G \to G$ such that $G^F = ^3D_4(q)$. Let $\ell \nmid q$ be a good prime for $G$ and further assume that $\ell \neq 3$ if $G^F = ^3D_4(q)$. We start with an explicit description of the quasi-isolated $\ell$-blocks of $G^F$. This is done by determining all the corresponding so-called $e$-cuspidal pairs. Next we show that a generalized $e$-Harish-Chandra theory (as defined in [12]) holds in every Lusztig series associated to a quasi-isolated element of the dual group $G^{*F}$. Combined with [14] this gives us a basic set for every quasi-isolated block of $G^F$. In the last section we use this to prove a conjecture of Malle and Robinson concerning the number of irreducible Brauer characters in a given block for the $\ell$-blocks of quasi-simple groups of exceptional Lie type, extending their results in [18].

1. Generalized $e$-Harish-Chandra theory

Let $G$ be a connected reductive group defined over $\mathbb{F}_q$ with Frobenius endomorphism $F$. We start by recalling some relevant notions and results. Let $e \geq 1$ be an integer. Suppose that $S$ is an $F$-stable torus of $G$ with complete root datum $\mathcal{S}$ (see [19, Definition 22.10]). Then $S$ is called an $e$-torus if $|S| = \Phi_e(X)^a$ for some non-negative integer $a,$

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where \( \Phi_e \) denotes the \( e \)-th cyclotomic polynomial. A Levi subgroup \( L \) of \( G \) is called \textit{\( e \)-split} if \( L = C_G(S) \) is the centralizer of an \( e \)-torus \( S \) of \( G \).

**Proposition 1.1.** Let \( G \) be a connected reductive group defined over \( \mathbb{F}_q \) with Frobenius endomorphism \( F : G \to G \). If \( L \) is an \( e \)-split Levi subgroup of \( G \), then \( L = C_G(Z^e(L)_{\Phi_e}) \), where \( Z^e(L)_{\Phi_e} \) denotes the \( \Phi_e \)-part of the torus \( Z^e(L) \).

**Proof.** Since \( L \) is \( e \)-split, there exists an \( e \)-torus \( S \) of \( G \) such that \( L = C_G(S) \). Clearly, \( S \subseteq Z^e(L)_{\Phi_e} \). Since \( L = C_G(Z^e(L)) \) (see [11, 1.21 Proposition]), we have

\[
L = C_G(S) \supseteq C_G(Z^e(L)_{\Phi_e}) \supseteq C_G(Z^e(L)) = L.
\]

Hence, \( L = C_G(Z^e(L)_{\Phi_e}) \). \( \square \)

We say an irreducible character \( \chi \) of \( G^F \) is \textit{\( e \)-cuspidal} if \( ^eR^G_L(\chi) = 0 \) for every \( e \)-split Levi subgroup \( L \) contained in a proper parabolic subgroup \( P \subseteq G \), where \( ^eR^G_{L \subseteq P} \) denotes Lusztig restriction (see [11, 11.1 Definition]). Let \( \lambda \in \text{Irr}(L^F) \) for an \( e \)-split Levi subgroup \( L \subseteq G \). Then we call \((L, \lambda)\) an \textit{\( e \)-split pair}. We define a binary relation on \( e \)-split pairs by setting \((M, \zeta) \leq_e (L, \lambda)\) if \( M \subseteq L \) and \( ^eR^G_{M \subseteq G}(\lambda, \zeta) \neq 0 \). Since the Lusztig restriction of a character is in general not a character, but a generalized character, the relation \( \leq_e \) might not be transitive. We denote the transitive closure of \( \leq_e \) by \( \ll_e \). If \((L, \lambda)\) is minimal for the partial order \( \ll_e \), we call \((L, \lambda)\) an \textit{\( e \)-cuspidal pair} of \( G^F \). Moreover, we say \((L, \lambda)\) is a \textit{proper \( e \)-cuspidal pair} if \( L \subsetneq G \) is a proper \( F \)-stable Levi subgroup of \( G \).

Let \( G^* \) be a group in duality with \( G \) with respect to an \( F \)-stable maximal torus \( T \) of \( G \) (see [11, 13.10 Definition]). By results of Lusztig \( \text{Irr}(G^F) \) is a disjoint union of so-called (rational) Lusztig series \( \mathcal{E}(G^F, s) \), where \( s \) runs over the \( G^F \)-conjugacy classes of semisimple elements of the dual group \( G^* \) (see [10, Definition 8.23]).

The following definition can be found in [12, 2.2.1 Definition]. Let \( s \in G^{*F} \) be semisimple. We say that \textit{generalized \( e \)-Harish-Chandra theory} holds in \( \mathcal{E}(G^F, s) \) if, for any \( \chi \in \mathcal{E}(G^F, s) \) there exists an \( e \)-cuspidal pair \((L, \lambda)\) of \( G^F \), uniquely defined up to \( G^F \)-conjugacy, and \( a \neq 0 \) such that

\[
^eR^G_{L \subseteq P} \chi = a \left( \sum_{g \in N_{G^F}(L)/N_{G^F}(L)_{\lambda}} \chi^g \right)
\]

for every parabolic subgroup \( P \subseteq G \) containing \( L \).

Recall the following classical result about the block theory of finite groups of Lie type.

**Theorem 1.2** ([6, 2.2 Théorème], [16, Theorem 3.1]). Let \( s \in G^{*F} \) be a semisimple \( \ell' \)-element. Then we have the following.

(a) The set \( \mathcal{E}_\ell(G^F, s) := \bigcup_{t \in C_{G^F}(s)} \mathcal{E}(G^F, st) \) is a union of \( \ell \)-blocks.

(b) Any \( \ell \)-block contained in \( \mathcal{E}_\ell(G^F, s) \) contains a character of \( \mathcal{E}(G^F, s) \).

Now, \( e \)-cuspidal pairs yield a refinement of Theorem 1.2. Let \( e_\ell(q) \) denote the multiplicative order of \( q \) modulo \( \ell \).

**Theorem 1.3** ([9, Theorem 4.1]). Let \( G \) be a connected reductive group defined over \( \mathbb{F}_q \) with Frobenius endomorphism \( F : G \to G \). Let \( \ell \) be a good prime for \( G \) not dividing \( q \).
Furthermore, assume that \( \ell \) is different from 3 if \( G^F \) has a component of type \( ^3D_4(q) \). Let \( s \in G^{*F} \) be a semisimple \( \ell' \)-element. If \( e = e_s(q) \), then we have the following.

(a) There is a natural bijection

\[
\text{b}_{G^F}(L, \lambda) \leftrightarrow (L, \lambda)
\]

between the \( \ell \)-blocks of \( G^F \) contained in \( E_\ell(G^F, s) \) and the \( e \)-cuspidal pairs \((L, \lambda)\), up to \( G^F \)-conjugation, such that \( s \in L^{*F} \) and \( \lambda \in \mathcal{E}(L^F, s) \), where \( \text{b}_{G^F}(L, \lambda) \) is the unique block containing the irreducible constituents of \( R^G_F(\lambda) \).

(b) If \( B = \text{b}_{G^F}(L, \lambda) \), then \( \text{Irr}(B) \cap \mathcal{E}(G^F, s) = \{ \chi \in \text{Irr}(G^F) \mid (L, \lambda) \ll_e (G, \chi) \} \).

Throughout this work we will work under the following core assumption.

\( (A) \) G is connected reductive defined over \( \mathbb{F}_q \) with Frobenius endomorphism \( F : G \rightarrow G, \ell \nmid q \) is odd and good for \( G \) and \( e = e_s(q) \). If \( G^F \) has a component of type \( ^3D_4 \) then \( \ell \geq 5 \). Furthermore, let \( s \in G^{*F} \) be a semisimple \( \ell' \)-element.

The following theorem shows how the notion of a generalized \( e \)-Harish-Chandra theory holding in a Lusztig series is related to the parametrisation of blocks by the \( e \)-cuspidal pairs. Let \( \mathcal{E}(G^F, (L, \lambda)) := \{ \chi \in \text{Irr}(G^F) \mid (L, \lambda) \leq_e (G, \chi) \} \) be the \( e \)-Harish-Chandra series associated to \((L, \lambda)\).

Note that, even though the statements about \( e \)-cuspidal pairs and \( e \)-Harish-Chandra theory in this section seem like they do not depend on \( \ell \), the proofs of these statements heavily rely on \( \ell \) satisfying the conditions in \( (A) \) as can be seen in the proofs of the results cited in this section.

**Proposition 1.4.** [13, Proposition 2.2.2] Assume \( (A) \) to hold. Then generalized \( e \)-Harish-Chandra theory holds in \( \mathcal{E}(G^F, s) \) if and only if, for any \( e \)-cuspidal pair \((L, \lambda)\) of \( G^F \) with \( \lambda \in \mathcal{E}(L^F, s) \), we have

\[
\mathcal{E}(G^F, (L, \lambda)) = \{ \chi \in \text{Irr}(G^F) \mid (L, \lambda) \ll_e (G, \chi) \}.
\]

**Corollary 1.5.** Assume \( (A) \) to hold. Then generalized \( e \)-Harish-Chandra theory holds in \( \mathcal{E}(G^F, s) \) if and only if

\[
\mathcal{E}(G^F, s) = \bigcup_{(L, \lambda) / G^F} \mathcal{E}(G^F, (L, \lambda)),
\]

where \((L, \lambda)\) runs over the \( G^F \)-conjugacy classes of \( e \)-cuspidal pairs of \( G \) with \( s \in L^{*F} \) and \( \lambda \in \mathcal{E}(L^F, s) \).

**Proof.** By Theorem [13] we have

\[
\mathcal{E}(G^F, s) = \bigcup_{(L, \lambda) / G^F} (\text{Irr}(\text{b}_{G^F}(L, \lambda)) \cap \mathcal{E}(G^F, s))
\]

\[
= \bigcup_{(L, \lambda) / G^F} \{ \chi \in \text{Irr}(G^F) \mid (L, \lambda) \ll_e (G, \chi) \},
\]

where \((L, \lambda)\) runs over the \( G^F \)-conjugacy classes of \( e \)-cuspidal pairs of \( G \) with \( s \in L^{*F} \) and \( \lambda \in \mathcal{E}(L^F, s) \). Since \( \mathcal{E}(G^F, (L, \lambda)) \) is contained in \( \{ \chi \in \text{Irr}(G^F) \mid (L, \lambda) \ll_e (G, \chi) \} \) by definition, the assertion follows from Proposition [13].
We say that the Mackey formula holds for $G^F$, if the Mackey formula
\[ *R_{L \subseteq P}^G \circ R_{M \subseteq Q}^G = \sum_x R_{L \cap x M \subseteq L \cap x Q}^L \circ *R_{L \cap x M \subseteq P \cap x M}^M \circ \text{ad } x, \]
where $x$ runs over a set of representatives of $L^F \setminus S(L, M)^F/M^F$ with $S(L, M) = \{ x \in G \mid L \cap x M \text{ contains a maximal torus of } G \}$, holds for every pair of parabolic subgroups $P$ and $Q$ of $G$ with $F$-stable Levi complements $L$ and $M$ respectively.

**Theorem 1.6** ([13, 2.2.4 Proposition]). Assume (A) to hold. In addition suppose that the centre of $G$ is connected and that the Mackey formula holds for every $L^F$ where $L$ is an $F$-stable Levi subgroup of $G$. Then generalized $e$-Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$.

Since we focus on the case where $G$ is a simple, simply connected group of exceptional type, the only cases for which the Mackey formula is still open are $^{2}E_6(2), E_7(2), E_8(2)$ (see [2, Theorem]). Because of this, we will omit the parabolic subgroups from the subscript of Lusztig induction and restriction throughout this work.

Recall that an element $s$ of a connected reductive group $G$ is called quasi-isolated if $C_G(s)$ is not contained in any proper Levi subgroup $L \subsetneq G$. If even $C_G^0(s)$ is not contained in any proper Levi subgroup $L \subsetneq G$, then $s$ is called isolated.

**Theorem A.** Assume (A) to hold with $G$ a simple, simply connected group of exceptional type or $G^F = ^3D_4(q)$ and $s \in G^*F$ a semisimple, quasi-isolated element. Then generalized $e$-Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$.

Apart from the cases handled by Theorem 1.6, this has also been proved for $e = 1$ and $e = 2$ (unless $G = E_6$ or $E_7$ and $s$ semisimple, quasi-isolated of order 6) in [17]. Note that we do not need to assume that the Mackey formula holds in Theorem A. Even though the general proof uses the Mackey formula, we can use a different method for the cases $^{2}E_6(2), E_7(2), E_8(2)$ where it is still not proved.

The reason we restrict our attention to the quasi-isolated elements are the results of Bonnafé–Rouquier [3] and Bonnafé–Dat–Rouquier [11]. They show that most questions about blocks of groups of Lie type can be reduced to questions about quasi-isolated blocks of smaller subgroups. In view of minimal counterexamples to block-theoretic questions, it therefore suffices to study the quasi-isolated setting.

Theorem A will be proved case-by-case for the groups in question. For the reader’s convenience we recall the classification of the quasi-isolated elements here (see [11, Proposition 4.3 and Table 3]).

**Proposition 1.7** (Bonnafé). Let $G$ be a simple, exceptional algebraic group of adjoint type. Then the conjugacy classes of semisimple, quasi-isolated elements $1 \neq s \in G$, their orders, the root system of their centraliser $C_G(s)$, and the group of components $A(s) := C_G(s)/C_G^0(s)$ are as given in Table 7.

The order of $s$ is denoted by $o(s)$. 
The first step is to determine the $e$-cuspidal pairs. To this end we need the following result.

**Theorem 1.8** ([9, Theorem 4.2.]). Assume (A) to hold. Then an element $\chi \in \mathcal{E}(G^F, s)$ is $e$-cuspidal if and only if it satisfies the following conditions.

(a) $Z^e(C_{G^F}^e(s))_{\Phi_e} = Z^e(G^*)_{\Phi_e}$ and

(b) $\chi$ corresponds to a $C_{G^*}(s)^F$-orbit of an $e$-cuspidal unipotent character of $C_{G^*}^e(s)^F$ by Jordan decomposition (see Theorem [10, Corollary 15.14]).

Using this result we can show that the assertion of Theorem A is immediate for certain numbers $e \in \mathbb{N}$.

Let $\delta(G^F) = \{ e \in \mathbb{N} \mid \exists \text{ a proper } e\text{-cuspidal pair } (L, \lambda) \text{ of } G^F \}$. For a semisimple element $s \in G^{*F}$ we define $\delta(G^F, s) := \{ e \in \mathbb{N} \mid \exists \text{ a proper } e\text{-cuspidal pair } (L, \lambda) \text{ of } G^F \text{ with } \lambda \in \mathcal{E}(L^F, s) \}$. We say an integer $e$ is **relevant** for a semisimple element $s \in G^{*F}$ if it occurs in $\delta(G^F, s)$.

The following easy conclusion justifies this terminology.
Proposition 1.9. Let $G$ be a connected reductive group defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : G \to G$. Let $s \in G^F$ be semisimple. If $e$ is not relevant for $s$, then an $e$-Harish-Chandra theory holds in $E(G^F, s)$.

Proof. If $e$ is not relevant for $E(G^F, s)$, then, by definition, every character in $E(G^F, s)$ is $e$-cuspidal. In other words, if $(L, \lambda)$ is an $e$-cuspidal pair of $G$ with $\lambda \in E(L^F, s)$, then $L = G$. Since, clearly

$$E(G^F, s) = \bigcup_{\chi \in E(G^F, s)} E(G^F, (G, \chi))$$

the assertion follows by Corollary 1.5. □

Next we will show that we can determine the relevant integers for the Lusztig series associated to quasi-isolated elements by only looking at unipotent data.

Proposition 1.10. Let $G$ be a connected reductive group, $s \in G$ a semisimple element and $L \subseteq G$ a Levi subgroup of $G$ containing $s$. Then $L \cap C^o_G(s)$ is a Levi subgroup of $C^o_G(s)$ and every Levi subgroup of $C^o_G(s)$ is of that form. In particular, $L \cap C^o_G(s) \subseteq C^o_G(s)$ is $e$-split if and only if $L \subseteq G$ is $e$-split.

Proof. As a semisimple element, $s$ lies in at least one maximal torus $S$ of $L$, which then is a maximal torus of $G$. Since $s \in L$, one of these tori lies in $L$. Let $S$ be such a maximal torus of $L$. Now, $Z(L)$ lies in every maximal torus of $L$; in particular, $Z(L)$ lies in $S$. In other words, we have $Z(L) \subseteq S \subseteq C^o_G(s)$. As $L = C_G(Z^o(L))$ (see 11.1 1.21 Proposition), we have $L \cap C^o_G(s) = C_{C^o_G(s)}(Z^o(L))$. Since $Z^o(L)$ is a torus of $C^o_G(s)$, $C_{C^o_G(s)}(Z^o(L))$ is a Levi subgroup of $C^o_G(s)$, proving the first part.

Let $M$ be a Levi subgroup of $C^o_G(s)$. Then $M = C_{C^o_G(s)}(Z^o(M))$. Now, $L = C_G(Z^o(M))$ is a Levi subgroup such that $M = L \cap C^o_G(s)$. The second part follows from Proposition 1.1. □

Proposition 1.11. Assume $(A)$ to hold with $s \in G^F$ a semisimple, quasi-isolated $\ell'$-element. Then $\delta(G^F, s) = \delta(C^o_{G^*(s)}(F^s), 1)$.

Proof. Let $(L, \lambda)$ be the proper $e$-cuspidal pair of $G$ with $\lambda \in E(L^F, s)$. Let $L^*$ denote the dual of $L$ in $G^*$. To prove the assertion we show that Jordan decomposition yields a $C_{L^*}(s)^F$-orbit of proper unipotent $e$-cuspidal pairs.

By [8, Proposition 1.4], $L^*$ is $e$-split. Hence $L^* = C_{G^*}(Z^o(L^*)_F)$ by Proposition 1.11. Since $s \in G^*$ is quasi-isolated, we know that $C^o_{G^*}(s) \not\subseteq L^*$. It follows that $C^o_{L^*}(s) = L^* \cap C^o_{G^*}(s) \not\subseteq C^o_{G^*}(s)$ is a proper subgroup. Furthermore, by Proposition 1.11, $C^o_{L^*}(s)$ is an $e$-split Levi subgroup of $C^o_{G^*}(s)$ and by condition. Moreover, $\lambda$ corresponds to a $C_{L^*}(s)^F$-orbit of $e$-cuspidal unipotent characters of $C_{L^*}(s)^F$ by condition (ii) of Theorem 1.8. Hence, $\delta(G^F, s) \subseteq \delta(C^o_{G^*(s)}(F^s), 1)$.

Conversely, let $(M, \chi)$ be a proper $e$-cuspidal pair of $C^o_{G^*}(s)$ with $\chi \in E(C^o_{G^*}(s), 1)$. By Proposition 1.10, there is a proper $e$-split Levi $L^* \subseteq G^*$ such that $L \cap C^o_{G^*}(s) = M$. If $\lambda$ is the character in $E(L^F, s)$ mapped to $\chi$ by Jordan decomposition, then $(L, \lambda)$ is an $e$-cuspidal pair by Theorem 1.8. □
2. RELEVANT $e$-CUSPIDAL PAIRS

Let $G$ be a simple, simply connected group of exceptional type defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : G \to G$ or let $G$ be simple, simply connected of type $D_4$ defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : G \to G$ such that $G^F = 3D_4(q)$.

The tables in this section are the key ingredient in the proof of Theorem A. The layout of the Tables 2, 4, 6, 8 and 10 is based on the layout of the tables in [17].

Note that we do not include tables for every relevant integer $e$. However, the missing tables are Ennola duals of the ones in this section and they can be obtained fairly easily. This follows from the fact that Ennola duality of finite groups of Lie type interacts nicely with Lusztig induction and restriction (see [5] and especially [5, 3.3 Theorem]). The Ennola dual cases are $e = 1 \leftrightarrow e = 2$, $e = 3 \leftrightarrow e = 6$, $e = 5 \leftrightarrow e = 10$, $e = 7 \leftrightarrow e = 14$, $e = 9 \leftrightarrow e = 18$, $e = 15 \leftrightarrow e = 30$.

More importantly, note that it follows from Theorem 1.6 that a generalized $e$-Harish-Chandra theory holds for $F_4(q)$, and for $E_6(q)$ unless when $q = 2$. However, since we need the $e$-cuspidal pairs of these groups in Section 3 on the Malle–Robinson conjecture, we reprove [13, 2.2.4 Proposition] for them along the way.

In addition to the Tables 2, 4, 6, 8 and 10, Tables 3, 5, 7 and 9 contain the decomposition of $R_{L}^F(\lambda)$ into its irreducible constituents for every $e$-cuspidal pair $(L, \lambda)$ for which $R_{L}^F(\lambda)$ is not uniform. These constituents are parametrized via Jordan decomposition (see [10, Corollary 15.14]). Since the semisimple element will always be clear from the context, we omit it from the parametrization and denote every irreducible constituent by the corresponding unipotent character. Except for the unipotent characters of classical groups (where we use the common notation using partitions and symbols), we use the notation of Chevie [20].

Remark 2.1. The $e$-cuspidal pairs of $G^F$ for $e = 1 \leftrightarrow e = 2$ were already determined by Kessar and Malle in [17] except for the pairs associated to quasi-isolated elements of order 6 when $G^F = E_6(q)$ or $E_7(q)$.

2.1. $e$-CUSPIDAL PAIRS OF $F_4$. Let $G$ be simple, simply connected of type $F_4$ defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : G \to G$. In this case, $e$ is relevant for some quasi-isolated semisimple $1 \neq s \in G^F$ if and only if $e \in \{1, 2, 3, 4, 6\}$. By Remark 2.1 and the remark at the beginning of this section about Ennola duality, it remains to determine the $e$-cuspidal pairs for $e = 3$ and $e = 4$.

Theorem 2.2. Let $e = e_L(q) \in \{3, 4\}$. For any quasi-isolated semisimple element $1 \neq s \in G^F$, the $e$-cuspidal pairs $(L, \lambda)$ of $G$ with $\lambda \in E(L^F, s)$ (up to $G^F$-conjugacy), and the order of their relative Weyl groups $W = W_{G^F}(L, \lambda)$ are as indicated in Table 2. In particular, generalized $e$-Harish-Chandra theory holds in every case.

| No. | $C_G(s)^F$ | $e$ | $L^F$ | $C_{L^F}(s)^F$ | $\lambda$ | $|W|$ |
|-----|------------|-----|-------|----------------|----------|------|

Table 2. Quasi-isolated blocks in $F_4(q)$
Let $\pi_{uni}$ denote the projection from the space of class functions onto the subspace of uniform functions (see [11, 12.11 Definition]). The image of a class function under $\pi_{uni}$ can be explicitly computed using [11, 12.12 Proposition].

**Proof.** The $e$-cuspidal pairs can be determined with Chevie using Theorem [L8] and Proposition [1.10]. The fact that a generalized $e$-Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$ is proved using Corollary [1.3].

Let $e = 3$. By Table [2], we see that every 3 cuspidal pair $(L, \lambda)$ is of the form $(G, \chi)$, or $L$ is a proper Levi subgroup of $G$ of type $A$ and $\lambda$ is a uniform character. Since Lusztig induction is transitive (see [11, 11.5 Transitivity]) and $\lambda$ is uniform, $R^G_L(\lambda)$ is uniform as well. Hence, we can determine the decomposition of $R^G_L(\lambda)$ using the formula for the uniform projection. For any semisimple, quasi-isolated element $s \in G^F$, we see that the constituents of $R^G_L(\lambda)$ for the 3-cuspidal pairs $(L, \lambda)$ with $\lambda \in \mathcal{E}(L^F, s)$ given in Table [2] exhaust $\mathcal{E}(G^F, s)$. Thus, a generalized 3-Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$.

Let $e = 4$. Let $(L, \lambda)$ be a 4-cuspidal pair in Table [2]. Then $\lambda$ is a uniform character, except for the two 4-cuspidal pairs in the line numbered 2. So the decomposition of $R^G_L(\lambda)$ can be determined using the formula for the uniform projection again, except for the two exceptions, for which we need to use a different method. For further reference we will explain this method in detail in the case $(L, \lambda) = (B_2, (12, 0))$. In this case $\pi_{uni}(R^G_L(\lambda)) = \frac{1}{4}[(1234, 012) - (123, 02) + (023, 12) - (0124, 123) + (0123, 124) - (23, 0) + (14, 0) - (02, 3) + (01, 4) + (023, 0) - (014, 0) + (0123, 2)] - \frac{1}{4}[(03, 2) - (012, 23) - (04, 1) - (01234, 12)] \in \frac{1}{4} \mathbb{Z} \mathcal{E}(G^F, s)$. Since $R^G_L(\lambda)$ is a generalized character, there exists an element $\gamma \in \mathbb{Q} \mathcal{E}(G^F, s)$ which is orthogonal to the space of uniform class functions of $G^F$, such that $R^G_L(\lambda) = \pi_{uni}(R^G_L(\lambda)) + \gamma \in \mathbb{Z} \mathcal{E}(G^F, s)$. A basis for the subspace of $\mathbb{Q} \mathcal{E}(G^F, s)$ orthogonal

|   | $A_2(q)A_2(q)$ | 3 | $\Phi_4^e$ | $\Phi_3^e$ | 1 | 9 |
|---|----------------|---|------------|------------|---|---|
| 2 | $B_4(q)$       | 3 | $\Phi_3A_2(q)$ | $\Phi_1\Phi_3A_1(q)$ | $C_{G^*}(s)^F$ | $\phi_{11}, \phi_2$ | 6 |
| 3 |               |   | $G^F$     |            |          | 13 chars. | 1 |
| 4 | $C_3(q)A_1(q)$ | 3 | $\Phi_3A_2(q)$ | $\Phi_1\Phi_3A_1(q)$ | $C_{G^*}(s)^F$ | $\phi_{11}, \phi_2$ | 6 |
| 5 |               |   | $G^F$     |            |          | 12 chars. | 1 |
| 6 | $A_3(q)A_1(q)$ | 3 | $\Phi_3A_2(q)$ | $\Phi_1\Phi_3A_1(q)$ | $C_{G^*}(s)^F$ | $\phi_{11}, \phi_2$ | 3 |
| 7 |               |   | $G^F$     |            |          | 4 chars. | 1 |
| 8 | $A_3(q)A_1(q)$ | 4 | $\Phi_4B_2(q)$ | $\Phi_1\Phi_4A_1(q)$ | $C_{G^*}(s)^F$ | $\phi_{11}, \phi_2$ | 4 |
| 9 |               |   | $G^F$     |            |          | 8 chars. | 1 |

|   | $A_2(q)A_2(q)$ | 3 | $\Phi_4^e$ | $\Phi_3^e$ | 1 | 9 |
|---|----------------|---|------------|------------|---|---|
| 2 | $B_4(q)$       | 4 | $\Phi_4B_2(q)$ | $\Phi_4A_1(q)A_1(q)$ | $C_{G^*}(s)^F$ | $\phi_{11}, \phi_2$ | 14 |
| 3 |               |   | $G^F$     |            |          | (12, 0), (01, 2) | 4 |
| 4 | $C_3(q)A_1(q)$ | 4 | $\Phi_4B_2(q)$ | $\Phi_4A_1(q)A_1(q)$ | $C_{G^*}(s)^F$ | $\phi_{11}, \phi_2$ | 4 |
| 5 |               |   | $G^F$     |            |          | (13, 1), (013, 13), (014, 12) | 1 |
| 6 | $A_3(q)A_1(q)$ | 4 | $\Phi_4B_2(q)$ | $\Phi_4A_1(q)A_1(q)$ | $C_{G^*}(s)^F$ | $\phi_{22}, \phi_{11}, \phi_2$ | 4 |
| 7 |               |   | $G^F$     |            |          | (04, 1), (0124, 12) | 1 |
| 8 | $A_3(q)A_1(q)$ | 4 | $\Phi_4B_2(q)$ | $\Phi_4A_1(q)A_1(q)$ | $C_{G^*}(s)^F$ | $\phi_{11}, \phi_2$ | 4 |
| 9 |               |   | $G^F$     |            |          | (04, 1), (0124, 12) | 1 |
to the space of uniform class functions is given by

\[ \varphi_1 = \frac{1}{4} \left( ((1234, 012) - (0124, 123) + (0123, 124) - (01234, 12)) \right), \]

\[ \varphi_2 = \frac{1}{4} \left( ((123, 02) - (023, 12) + (012, 23) - (0123, 2)) \right), \]

\[ \varphi_3 = \frac{1}{4} \left( ((124, 01) - (014, 12) + (012, 14) - (0124, 1)) \right), \]

\[ \varphi_4 = \frac{1}{4} \left( ((23, 0) - (03, 2) + (02, 3) - (023, )) \right), \]

\[ \varphi_5 = \frac{1}{4} \left( ((14, 0) - (04, 1) + (01, 4) - (014, )) \right). \]

By the Mackey formula we know that \( \|R^G_L(\lambda)\|^2 = |W_G^e(L, \lambda)| = 4 \) and since \( \|R^G_L(\lambda)\|^2 = \|\pi_{uni}(R^G_L(\lambda))\|^2 + \|\gamma\|^2 \), it follows that \( \gamma = -\varphi_1 + \varphi_2 + \varphi_4 - \varphi_5 \). Hence, \( R^G_L(\lambda) = -(03, 2) + (012, 23) + (04, 1) + (01234, 12) \). The same method yields the decomposition of \( R^G_L(\lambda) \) for \( (L, \lambda) = (B_2, (01, 2)) \). With this we have established the decomposition for every 4-cuspidal pair in Table 3.

We see that the constituents of \( R^G_L(\lambda) \) for the 4-cuspidal pairs associated to a given semisimple, quasi-isolated element \( s \in G^{r_F} \) exhaust \( \mathcal{E}(G^{r_F}, s) \).

**Table 3. Decomposition of non-uniform \( R^G_L(\lambda) \)**

| No. | \( e \) | \( \lambda \) | \( \pm R^G_L(\lambda) \) |
|-----|---|------|-----------------|
| 2   | 4 | \((12, 0)\) | \(-(03, 2) + (012, 23) + (04, 1) + (01234, 12)\) |
|     |   | \((01, 2)\) | \(-(023, 12) + (0124, 123) + (23, 0) + (014, 0)\) |

2.2. \( e \)-cuspidal pairs of \( E_6 \). Let \( G \) be a simple, simply connected group of type \( E_6 \) defined over \( \mathbb{F}_q \) with Frobenius endomorphism \( F : G \to G \). Then \( G^{r_F} = E_{6,sc}(q) \) or \( 2E_{6,sc}(q) \). In this case, \( e \) is relevant for some quasi-isolated \( 1 \neq s \in G^{r_F} \) if and only if \( e \in \{1, 2, 3, 4, 5, 6, 10\} \). Since the center of \( G \) is disconnected, the situation is slightly more complicated.

In the tables, we write \( C_{G^s}(s)^F = C_{G^s}(s)^F/(C_{G^s}(s)^F/C_{G^s}(s)^F) \) to indicate whether or not a given centraliser is connected. A star in the first column next to the number of the line indicates that the quotient \( C_{G^s}(s)^F/C_{G^s}(s)^F \) acts non-trivially on the unipotent characters of \( C_{G^s}(s)^F \). To demonstrate the adjustments, we take line 4 of Table 4 for \( e = 3 \) as an example.

First, the star indicates that the \( F \)-stable points of the component group act non-trivially on the 14 unipotent character of \( C_{G^s}(s)^F = \Phi_4^2 D_4(q) \). It can be shown that there are two orbits of order 3 and 8 trivial orbits. Thus, by Jordan decomposition, \( |\mathcal{E}(G^{r_F}, s)| = 26 \). Now, \( C_{L^s}(s)^F/C_{L^s}(s)^F \) acts obviously trivially on the one unipotent character (which is the trivial character) of the torus \( C_{L^s}(s)^F = \Phi_4^4 \). Hence the induction of that character to \( C_{L^s}(s)^F \) yields 3 irreducible constituents. We denote them by \( 1^{(1)}, 1^{(2)} \) and \( 1^{(3)} \).

In general, if \( C_{G^s}(s)^F/C_{G^s}(s)^F \) acts trivially on a given unipotent character of \( C_{G^s}(s)^F \), the induction of that character always yields 3 irreducible characters of \( C_{G^s}(s)^F \). In Table 4 we indicate this by adding a superscript from 1 to 3 to that unipotent character.
Theorem 2.3. Let $e = e_i(q) \in \{3, 4, 5, 6\}$. For any quasi-isolated semisimple element $1 \neq s \in G^s$, the $e$-cuspidal pairs $(L, \lambda)$ of $G$ with $\lambda \in \mathcal{E}(L^F, s)$ (up to $G^F$-conjugacy), and the order of their relative Weyl groups $W = W_{G^F}(L, \lambda)$ are as indicated in Table 4. In particular, generalized $e$-Harish-Chandra theory holds in every case.

| No. | $C_G(s)^F$ | $e$ | $L^F$ | $C_{L^F}(s)^F$ | $\lambda$ | $|W|$ |
|-----|-------------|-----|-------|----------------|-----------|-----|
| 1$^*$ | $\Phi_1^i. A_1(q)^4, 3$ | 1 | $\Phi_1^r$ | $\Phi_1^r$ | 1 | 48 |
| 2 | $\Phi_3. A_1(q) A_1(q^4). 3$ | 1 | $\Phi_1^r. A_2(q)^2$ | $\Phi_1^r. \Phi_3^r$ | 1 | 12 |
| 3 | $\Phi_1 \Phi_2. A_1(q)^2 A_1(q^2)^2$ | 1 | $\Phi_1^r. A_1(q)^2$ | $\Phi_1^r. \Phi_3^r$ | 1 | 8 |
| 1$^*$ | $A_2(q)^4, 3$ | 3 | $\Phi_3^r$ | $\Phi_3^r$ | 1 | 81 |
| 2 | $A_2(q^4)^3$ | 3 | $\Phi_3^r. A_2(q)$ | $\Phi_3^r. \Phi_3^r$ | 1 | 18 |
| 3 | $A_2(q) A_2(q^2)$ | 3 | $\Phi_3. D_1(q)$ | $\Phi_3. A_2^3(q)$ | $\Phi_1, \Phi_2, \Phi_3$ | 3 |
| 4$^*$ | $\Phi_1^r. D_1(q). 3$ | 3 | $\Phi_3. A_2(q)^2$ | $\Phi_3^r. \Phi_3. 3$ | $1^{(3)}$, $1^{(3)}$, $1^{(3)}$ | 6 |
| 5$^*$ | $\Phi_1^r. D_1(q). 3$ | 3 | $\Phi_3. A_2(q)^2$ | $\Phi_3^r. \Phi_3. 3$ | $1^{(3)}$, $1^{(3)}$, $1^{(3)}$ | 6 |
| 6 | $\Phi_3. D_4(q). 3$ | 3 | $\Phi_3^r. D_4(q)$ | $\Phi_3^r. D_4(q)$ | $3^{D_4[-1]}$ | 6 |
| 7 | $\Phi_1 \Phi_2. D_4(q)$ | 3 | $\Phi_3. A_2(q)^2$ | $\Phi_3^r. \Phi_3. 3$ | $1^{(3)}$, $1^{(3)}$, $1^{(3)}$ | 6 |
| 8 | $\Phi_1 \Phi_2. D_4(q)$ | 3 | $\Phi_3. A_2(q)^2$ | $\Phi_3^r. \Phi_3. 3$ | $1^{(3)}$, $1^{(3)}$, $1^{(3)}$ | 6 |
| 9 | $\Phi_1 \Phi_2. D_4(q)$ | 3 | $\Phi_3. A_2(q)^2$ | $\Phi_3^r. \Phi_3. 3$ | $1^{(3)}$, $1^{(3)}$, $1^{(3)}$ | 6 |
| 10 | $\Phi_1 \Phi_2. D_4(q)$ | 3 | $\Phi_3. A_2(q)^2$ | $\Phi_3^r. \Phi_3. 3$ | $1^{(3)}$, $1^{(3)}$, $1^{(3)}$ | 6 |
| 11 | $\Phi_1 \Phi_2. D_4(q)$ | 3 | $\Phi_3. A_2(q)^2$ | $\Phi_3^r. \Phi_3. 3$ | $1^{(3)}$, $1^{(3)}$, $1^{(3)}$ | 6 |
| 1 | $A_5(q) A_1(q)$ | 4 | $\Phi_2. A_3(q)$ | $\Phi_1. A_2(q)^2$ | $\Phi_1. A_2(q)^2$ | 4 chars. | 4 |
| 2 | $A_5(q) A_1(q)$ | 4 | $\Phi_2. A_3(q)$ | $\Phi_1. A_2(q)^2$ | $\Phi_1. A_2(q)^2$ | 4 chars. | 4 |
| 3$^*$ | $\Phi_1^r. D_4(q). 3$ | 4 | $\Phi_3. D_4(q)$ | $\Phi_3. D_4(q)$ | $3^{D_4[-1]}$ | 72 |
| 4$^*$ | $\Phi_1^r. D_4(q). 3$ | 4 | $\Phi_3. D_4(q)$ | $\Phi_3. D_4(q)$ | $3^{D_4[-1]}$ | 72 |
| 5$^*$ | $\Phi_1 \Phi_2. D_4(q)$ | 4 | $\Phi_3. A_2(q)^2$ | $\Phi_3. A_2(q)^2$ | $1^{(3)}$, $1^{(3)}$, $1^{(3)}$ | 6 |
| 6 | $A_5(q) A_1(q)$ | 5 | $\Phi_1. A_1(q)^3$ | $\Phi_1. A_1(q)^3$ | $\Phi_1. A_1(q)^3$ | 4 chars. | 4 |
| 7 | $A_5(q) A_1(q)$ | 5 | $\Phi_1. A_1(q)^3$ | $\Phi_1. A_1(q)^3$ | $\Phi_1. A_1(q)^3$ | 4 chars. | 4 |
| 8$^*$ | $\Phi_3. A_1(q)^2$ | 6 | $\Phi_2. A_2(q)$ | $\Phi_1. A_1(q)^3$ | $\Phi_1. A_1(q)^3$ | 4 chars. | 4 |
| 9 | $\Phi_3. A_1(q)^2$ | 6 | $\Phi_2. A_2(q)$ | $\Phi_1. A_1(q)^3$ | $\Phi_1. A_1(q)^3$ | 4 chars. | 4 |
| 10 | $\Phi_3. A_1(q)^2$ | 6 | $\Phi_2. A_2(q)$ | $\Phi_1. A_1(q)^3$ | $\Phi_1. A_1(q)^3$ | 4 chars. | 4 |
| 11 | $\Phi_3. A_1(q)^2$ | 6 | $\Phi_2. A_2(q)$ | $\Phi_1. A_1(q)^3$ | $\Phi_1. A_1(q)^3$ | 4 chars. | 4 |

Table 4. Quasi-isolated blocks in $E_6(q)$
Proof. Suppose \( q > 2 \). For \( q = 2 \) the assertion will follow from Proposition 2.8. As for \( F_4(q) \), the key step is to determine \( R_L^G(\lambda) \) for the \( e \)-cuspidal pairs \((L, \lambda)\) in Table 4. Except for the pairs given in Table 5, \( \lambda \) is uniform, so \( R_L^G \) can be determined using the formula for the uniform projection. For the 3-cuspidal pairs \((\Phi_3, A_2(q^2), 1^{(i)}) (i = 1, \ldots, 3)\) and the 6-cuspidal pairs \((\Phi_6, A_2(q^2), 1^{(i)}) (i = 1, \ldots, 3)\), we are not able to determine \( R_L^G(\lambda) \) (see Remark 2.4). However the methods used in the proof of Theorem 2.2 give enough information to prove that an \( e \)-Harish-Chandra theory holds in the Lusztig series related to the \( e \)-cuspidal pairs above. For the 3-cuspidal pair \((\Phi_3, 3D_4(q), 3D_4[-1])\) we use a slightly different argument. Let \( s \in G^{*F} \) be semisimple, quasi-isolated with \( C_{G^*}(s)^F = \Phi_3, 3D_4(q), 3 \). By Table 4 and Theorem 1.8, \( \mathcal{E}(G^F, s) \) decomposes into two blocks, namely \( b_{G^F}(\Phi_3, 1) \), which contains \( \mathcal{E}(G^F, (\Phi_3, 1)) \) and \( b_{G^F}(\Phi_3, 3D_4(q), 3D_4[-1]) \) which contains \( \mathcal{E}(G^F, (\Phi_3, 3D_4(q), 3D_4[-1])) \). Since any two different blocks, seen as subsets of \( \text{Irr}(G^F) \cup \text{IBr}(G^F) \), are disjoint, we have
\[
\mathcal{E}(G^F, (\Phi_3, 3D_4(q), 3D_4[-1])) \subseteq \mathcal{E}(G^F, s) \setminus \mathcal{E}(G^F, (\Phi_3, 1))
\]
and the latter is equal to \( \{3D_4[-1]^{(1)}, 3D_4[-1]^{(1)}, 3D_4[-1]^{(2)}\} \). Since \( R_{\Phi_3, 3D_4(q)}^G(3D_4[-1]) \) has norm 3, it follows that \( R_{\Phi_3, 3D_4(q)}^G(3D_4[-1]) = 3D_4[-1]^{(0)} + 3D_4[-1]^{(1)} + 3D_4[-1]^{(2)} \). Hence, an \( e \)-Harish-Chandra theory holds in \( \mathcal{E}(G^F, s) \), which proves the assertion. \( \square \)

Remark 2.4. The reason we are not able to determine \( R_L^G(\lambda) \) in the cases numbered 4* and 3* in Table 5 is the following. Every constituent of \( R_L^G(1^{(i)}) \) in those lines is an element of an orbit of order 3. However, we are not able to determine which element of this orbit is the right constituent. We only know that it has to be one of the three. This is indicated by adding an superscript \( (i) \) to the constituents.

**Table 5. Decomposition of the non-uniform \( R_L^G(\lambda) \)**

| No. | \( e \) | \( \lambda \) | \( \pm R_L^G(\lambda) \) |
|-----|-------|----------|------------------|
| 4*  | 3     | \( 1^{(i)} \) | \((01, 123)^{(i)} + (0123, 1234)^{(i)} + (02, 13)^{(i)} + (01, 23)^{(i)} + (1, 3)^{(i)} + (0, 4)^{(i)} \) |
| 7   | 3     | \( 3D_4[-1] \) | \( 3D_4[-1]^{(0)} + 3D_4[-1]^{(1)} + 3D_4[-1]^{(2)} \) |
| 3*  | 6     | \( 1^{(i)} \) | \((01, 123)^{(i)} + (0123, 1234)^{(i)} + (12, 03)^{(i)} + (1, 3)^{(i)} + (0, 4)^{(i)} + (0123, )^{(i)} \) |

The analogue of Table 4 for \( ^2E_6(q) \) can be obtained as follows. The \( e = 3 \) part of the table for \( ^2E_6(q) \) is the Ennola dual of the \( e = 6 \) part of Table 4 and vice-versa. The \( e = 10 \) part is the Ennola dual of the \( e = 5 \) part and the \( e = 4 \) part is the Ennola dual of the \( e = 4 \) part of Table 4. Similarly, the analogue of Table 5 for \( ^2E_6(q) \) can be obtained via Ennola duality.

2.3. \( e \)-cuspidal pairs of \( E_7 \). Let \( G \) be a simple, simply connected group of type \( E_7 \) defined over \( \mathbb{F}_q \) with Frobenius endomorphism \( F : G \to G \). In this case, \( e \) is relevant for some quasi-isolated semisimple \( 1 \neq s \in G^{*F} \) if and only if \( e \in \{1, 2, 3, 4, 5, 6, 7, 9, 12, 14, 18\} \). By Remark 2.1 and Ennola duality, it remains to determine the \( e \)-cuspidal pairs for
$e = 3, 4, 5, 7, 9, 12$. Since the center of $G$ is disconnected, we encounter the same issues as in Section 2.2.

**Theorem 2.5.** Let $e = e_{e}(q) \in \{3, 4, 5, 7, 9, 12\}$. For any quasi-isolated semisimple element $1 \neq s \in G^{F}$, the $e$-cuspidal pairs $(L, \lambda)$ of $G$ with $\lambda \in \mathcal{E}(L^{F}, s)$ (up to $G^{F}$-conjugacy), and the order of their relative Weyl groups $W = W_{G^{F}}(L, \lambda)$ are as indicated in Table 6. In particular, generalized $e$-Harish-Chandra theory holds in every case.

**Table 6. Quasi-isolated blocks of $E_{7}(q)$**

| No. | $C_{G_{e}}(s)^{F}$ | $e$ | $L^{F}$ | $C_{L_{e}}(s)^{F}$ | $\lambda$ | $|W|$ |
|-----|-------------------|-----|---------|-------------------|----------|-----|
| 1*  | $\Phi_{1}.A_{2}(q)^{3}.2$ | 1   | $\Phi_{7}$ | $\Phi_{7}$ | 1         | 432 |
| 2   | $\Phi_{2}.A_{2}(q).A_{2}(q^{2})^{2}$ | 1   | $\Phi_{1}^{2}.A_{1}(q)^{2}$ | $\Phi_{1}^{2}.\Phi_{2}^{2}$ | 1         | 36  |
| 3   | $\Phi_{2}.A_{2}(q).A_{2}(q^{4})^{2}$ | 1   | $\Phi_{1}^{2}.A_{1}(q)^{2}$ | $\Phi_{1}^{2}.\Phi_{2}^{2}$.2$A_{2}(q)$ | 1         | 12  |
| 4   |                      |     | $\Phi_{1}.D_{4}(q)$ | $\Phi_{1}^{2}.\Phi_{2}^{2}$.2$A_{2}(q)$ | $\phi_{21}$ | 6   |
| 1   | $A_{7}(q).2$ | 3   | $\Phi_{1}^{2}.\Phi_{2}^{2}$.2$A_{2}(q)$ | $\Phi_{7}^{2}.A_{1}(q)$ | $\phi_{11}, \phi_{2}$ | 36  |
| 2   | | 3 | $\Phi_{2}.A_{3}(q)$ | $\Phi_{7}^{2}.A_{1}(q)$ | $\phi_{211}$ | 6  |
| 3   |                      |     | $G^{F}$ | $\Phi_{7}.A_{1}(q)$ | $(\phi_{1}^{(1)}, \phi_{2})$ | 1  |
| 4   | $2A_{7}(q).2$ | 3   | $\Phi_{3}.A_{2}(q).A_{1}(q^{2})$ | $\Phi_{1}^{2}.\Phi_{2}^{2}.A_{1}(q)^{2}$ | $3D_{4}[−1]$ | 6 |
| 5   |                      |     | $G^{F}$ | $\Phi_{7}.A_{1}(q)$ | $3D_{4}[−1]$ | 6 |
| 6   | $\Phi_{1}.E_{6}(q).2$ | 3   | $\Phi_{3}^{2}$ | $\Phi_{3}^{2}$.2$D_{4}(q)$ | $\phi_{11}, \phi_{2}$ | 6  |
| 7   |                      |     | $G^{F}$ | $\Phi_{3}^{2}$.2$D_{4}(q)$ | $\phi_{11}, \phi_{2}$ | 6  |
| 8   |                      |     | $G^{F}$ | $\Phi_{3}^{2}$.2$D_{4}(q)$ | $\phi_{11}, \phi_{2}$ | 6  |
| 9   | $\Phi_{2}.E_{6}(q).2$ | 3   | $\Phi_{3}^{2}.A_{1}(q^{2})$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $1^{(1,2)}$ | 72 |
| 10  |                      |     | $G^{F}$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $1^{(1,2)}$ | 72 |
| 11* | $A_{3}(q)^{2}A_{1}(q).2$ | 3   | $\Phi_{1}^{2}.\Phi_{2}^{2}$.2$A_{2}(q)$ | $\Phi_{7}^{2}.A_{1}(q)$ | $\phi_{11}, \phi_{2}$ | 18 |
| 12* |                      |     | $G^{F}$ | $\Phi_{7}^{2}.A_{1}(q)$ | $\phi_{211}$ | 6  |
| 13* |                      |     | $G^{F}$ | $\Phi_{7}^{2}.A_{1}(q)$ | $\phi_{211}$ | 6  |
| 14  | $A_{3}(q^{2})A_{1}(q).2$ | 3   | $\Phi_{1}^{2}.\Phi_{3}^{2}.2$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $\phi_{11}, \phi_{2}$ | 6  |
| 15  |                      |     | $G^{F}$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $\phi_{11}, \phi_{2}$ | 6  |
| 16* | $\Phi_{1}.A_{2}(q)^{3}.2$ | 3   | $\Phi_{3}^{2}$ | $\Phi_{3}^{2}$.2$D_{4}(q)$ | $\phi_{11}, \phi_{2}$ | 6  |
| 17  | $\Phi_{2}.A_{2}(q).A_{2}(q^{2})^{2}$ | 3   | $\Phi_{3}^{2}.A_{1}(q^{2})^{2}$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $1^{(1,2)}$ | 72 |
| 18  |                      |     | $G^{F}$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $1^{(1,2)}$ | 72 |
| 19* | $\Phi_{1}.D_{4}(q).A_{1}(q)^{2}.2$ | 3   | $\Phi_{3}^{2}.A_{3}(q)$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $1^{(1,2)}$ | 72 |
| 20* |                      |     | $G^{F}$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $1^{(1,2)}$ | 72 |
| 21  | $\Phi_{2}.D_{4}(q).A_{1}(q)^{2}.2$ | 3   | $\Phi_{3}^{2}.A_{3}(q)$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $1^{(1,2)}$ | 72 |
| 22  |                      |     | $G^{F}$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $1^{(1,2)}$ | 72 |
| 23* | $\Phi_{2}.D_{4}(q).A_{1}(q)^{2}.2$ | 3   | $\Phi_{3}^{2}.A_{3}(q)$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $1^{(1,2)}$ | 72 |
| 24* |                      |     | $G^{F}$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $1^{(1,2)}$ | 72 |
| 25  | $\Phi_{1}.D_{4}(q).A_{1}(q)^{2}.2$ | 3   | $\Phi_{3}^{2}.A_{3}(q)$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $1^{(1,2)}$ | 72 |
| 26  |                      |     | $G^{F}$ | $\Phi_{7}^{2}.\Phi_{2}^{2}.A_{6}.A_{1}(q)^{2}$ | $1^{(1,2)}$ | 72 |
| 1   | $A_{7}(q).2$ | 4   | $\Phi_{1}^{3}.A_{1}(q)$ | $\Phi_{1}^{3}.\Phi_{2}^{3}.2$ | $1^{(1,2)}$ | 32 |
| 2   |                      |     | $G^{F}$ | $\Phi_{1}^{3}.A_{1}(q)$ | $\phi_{11}, \phi_{2}$ | 4  |
| 3   |                      |     | $G^{F}$ | $\Phi_{1}^{3}.A_{1}(q)$ | $\phi_{11}, \phi_{2}$ | 4  |
| 4   | $2A_{7}(q).2$ | 4   | $\Phi_{1}^{3}.A_{1}(q)$ | $\Phi_{1}^{3}.\Phi_{2}^{3}.2$ | $1^{(1,2)}$ | 32 |
| 5   |                      |     | $G^{F}$ | $\Phi_{1}^{3}.A_{1}(q)$ | $\phi_{11}, \phi_{2}$ | 4  |
| Row | Condition | G | C_{G^*}(s)^F | Character Count | Value |
|-----|-----------|---|--------------|----------------|-------|
| 6   | \Phi_1.E_6(q).2 | 4 | G^F | C_{G^*}(s)^F | 4 chars. | 1 |
| 7   | \Phi_1.E_6(q).2 | 4 | \Phi_1^*A_1(q)^3 | \Phi_1^*\Phi_1^*A_1(q)^2 | 11(1) | 96 |
| 8   | \Phi_1.E_6(q).2 | 4 | G^F | C_{G^*}(s)^F | 20 chars. | 1 |
| 9   | \Phi_1.E_6(q).2 | 4 | \Phi_1^*A_1(q)^3 | \Phi_1^*\Phi_1^*A_1(q)^2 | 11(1) | 96 |
| 10  | \Phi_2^2E_6(q).2 | 4 | \Phi_2^2A_1(q)^3 | \Phi_2^2\Phi_2^2A_1(q)^2 | 20 chars. | 1 |
| 11  | \Phi_2^2E_6(q).2 | 4 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 12  | \Phi_2^2E_6(q).2 | 4 | \Phi_2^2A_1(q)^3 | \Phi_2^2\Phi_2^2A_1(q)^2 | 20 chars. | 1 |
| 13* | A_3(q)A_1(q).2 | 4 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 14* | A_3(q)A_1(q).2 | 4 | \Phi_1^*A_1(q)^3 | \Phi_1\Phi_1^*\Phi_1^*A_1(q)^2 | \phi_{11}, \phi_2 | 16 |
| 15* | A_3(q)A_1(q).2 | 4 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 16  | A_3(q)A_1(q).2 | 4 | \Phi_1^*A_1(q)^3 | \Phi_1\Phi_1^*\Phi_1^*A_1(q)^2 | \phi_{11}, \phi_2 | 16 |
| 17* | A_3(q)A_1(q).2 | 4 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 18* | A_3(q)A_1(q).2 | 4 | \Phi_1^*A_1(q)^3 | \Phi_1\Phi_1^*\Phi_1^*A_1(q)^2 | \phi_{11}, \phi_2 | 16 |
| 19* | A_3(q)A_1(q).2 | 4 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 20  | A_3(q)A_1(q).2 | 4 | \Phi_1^*A_1(q)^3 | \Phi_1\Phi_1^*\Phi_1^*A_1(q)^2 | \phi_{11}, \phi_2 | 16 |
| 21  | A_3(q)A_1(q).2 | 4 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 22  | A_3(q)A_1(q).2 | 4 | \Phi_1^*A_1(q)^3 | \Phi_1\Phi_1^*\Phi_1^*A_1(q)^2 | \phi_{11}, \phi_2 | 16 |
| 23  | A_3(q)A_1(q).2 | 4 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 24  | A_3(q)A_1(q).2 | 4 | \Phi_1^*A_1(q)^3 | \Phi_1\Phi_1^*\Phi_1^*A_1(q)^2 | \phi_{11}, \phi_2 | 16 |
| 25  | A_3(q)A_1(q).2 | 4 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 26  | A_3(q)A_1(q).2 | 4 | \Phi_1^*A_1(q)^3 | \Phi_1\Phi_1^*\Phi_1^*A_1(q)^2 | \phi_{11}, \phi_2 | 16 |
| 27  | A_3(q)A_1(q).2 | 4 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 28* | \Phi_1.D_4(q)A_1(q)^2:2 | 4 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 29* | \Phi_1.D_4(q)A_1(q)^2:2 | 4 | \Phi_1^*A_1(q)^3 | \Phi_1\Phi_1^*\Phi_1^*A_1(q)^2 | \phi_{11}, \phi_2 | 16 |
| 30  | \Phi_1.D_4(q)A_1(q)^2:2 | 4 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 31* | \Phi_1.D_4(q)A_1(q)^2:2 | 4 | \Phi_1^*A_1(q)^3 | \Phi_1\Phi_1^*\Phi_1^*A_1(q)^2 | \phi_{11}, \phi_2 | 16 |
| 32* | \Phi_1.D_4(q)A_1(q)^2:2 | 4 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 33  | \Phi_1.D_4(q)A_1(q)^2:2 | 4 | \Phi_1^*A_1(q)^3 | \Phi_1\Phi_1^*\Phi_1^*A_1(q)^2 | \phi_{11}, \phi_2 | 16 |
| 1   | A_7(q).2 | 5 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 2   | A_7(q).2 | 5 | \Phi_1\Phi_5.A_2(q) | \Phi_1\Phi_5.A_2(q) | \phi_{11}, \phi_2 | 16 |
| 3   | A_7(q).2 | 5 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 4   | A_7(q).2 | 5 | \Phi_1\Phi_5.A_2(q) | \Phi_1\Phi_5.A_2(q) | \phi_{11}, \phi_2 | 16 |
| 1   | A_7(q).2 | 7 | G^F | C_{G^*}(s)^F | 4 chars. | 4 |
| 2   | A_7(q).2 | 7 | \Phi_1\Phi_7 | \Phi_1\Phi_7 | \phi_{11}, \phi_2 | 16 |
Table 7. Decomposition of the non-uniform $R^2_T(\lambda)$

| No. | $e$ | $\lambda$ | $\pm R^2_T(\lambda)$ |
|-----|-----|------------|------------------------|
| 4   | 3   | $\phi_{11}$ | $\phi_{11}^{(i)} + \phi_{2411}^{(i)} + \phi_{3221}^{(i)} + \phi_{142}^{(i)} + \phi_{62}^{(i)} + \phi_{71}^{(i)}$ |
|     |     | $\phi_2$    | $\phi_{16}^{(i)} + \phi_{2411}^{(i)} + \phi_{3431}^{(i)} + \phi_{431}^{(i)} + \phi_5^{(i)} + \phi_8^{(i)}$ |
| 7   | 3   | $3D_4[-1]$  | $D_4:3^{(0)} + D_4:3^{(4)} + D_4:111^{(0)} + D_4:111^{(4)} - D_4:21^{(0)} - D_4:21^{(4)}$ |
| 19* | 3   | $\phi_2 \otimes \phi_2$ | $((013, 123) \otimes \phi_2 \otimes \phi_2)^{(i)} + ((0123, 1234) \otimes \phi_2 \otimes \phi_2)^{(i)} + ((02, 13) \otimes \phi_2 \otimes \phi_2)^{(i)} + ((01, 23) \otimes \phi_2 \otimes \phi_2)^{(i)} + ((0, 4) \otimes \phi_2 \otimes \phi_2)^{(i)}$ |
|     |     | $\phi_{11} \otimes \phi_2$ | $((013, 123) \otimes \phi_1 \otimes \phi_2)^{(i)} + (0123, 1234) \otimes \phi_1 \otimes \phi_2)^{(i)} + ((02, 13) \otimes \phi_1 \otimes \phi_2)^{(i)} + ((01, 23) \otimes \phi_1 \otimes \phi_2)^{(i)} + ((01, 3) \otimes \phi_1 \otimes \phi_2)^{(i)} + ((0, 4) \otimes \phi_1 \otimes \phi_2)^{(i)}$ |
|     |     | $\phi_{11} \otimes \phi_{11}$ | $((013, 123) \otimes \phi_1 \otimes \phi_1)^{(i)} + ((0123, 1234) \otimes \phi_1 \otimes \phi_1)^{(i)} + ((02, 13) \otimes \phi_1 \otimes \phi_1)^{(i)} + ((01, 23) \otimes \phi_1 \otimes \phi_1)^{(i)} + ((01, 3) \otimes \phi_1 \otimes \phi_1)^{(i)} + ((0, 4) \otimes \phi_1 \otimes \phi_1)^{(i)}$ |
| 21  | 3   | $\phi_{11}$ | $((123, 0) \otimes \phi_1)^{(i)} + ((0123, 123) \otimes \phi_2)^{(i)} + ((13, \otimes \phi_1)^{(i)} + (0123, 13) \otimes \phi_1)^{(i)} + ((04, \otimes \phi_1)^{(i)} + (0123, 3) \otimes \phi_1)^{(i)}$ |
|     |     | $\phi_2$    | $((123, 0) \otimes \phi_2)^{(i)} + ((0123, 123) \otimes \phi_2)^{(i)} + ((13, \otimes \phi_2)^{(i)} + ((01, 3) \otimes \phi_2)^{(i)} + ((04, \otimes \phi_2)^{(i)} + (0123, 3) \otimes \phi_2)^{(i)}$ |

Proof. Similar to the proof of Theorem 2.3. □
2.4. e-cuspidal pairs of $E_8$. Let $G$ be simple, simply connected of type $E_8$ defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : G \to G$. In this case, $e$ is relevant for some quasi-isolated $1 \neq s \in G^{*F}$ if and only if $e \in \{1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 14, 18, 20\}$. By Remark 2.1 and Ennola duality, it remains to determine the e-cuspidal pairs for $e = 3, 4, 5, 7, 9, 12, 20$.

**Theorem 2.6.** Let $e \in \{3, 4, 5, 7, 9, 12, 20\}$. For any quasi-isolated semisimple element $1 \neq s \in G^{*F}$, the e-cuspidal pairs $(L, \lambda)$ of $G$ with $\lambda \in E(L^F, s)$ (up to $G^F$-conjugacy), and the order of their relative Weyl groups $W = W_{G^F}(L, \lambda)$ are as indicated in Table 8. In particular, generalized e-Harish-Chandra theory holds in every case.

| No. | $C_{G^*}(s)^F$ | $e$ | $L^F$ | $C_{L^*}(s)^F$ | $\lambda$ | $|W|$ |
|-----|----------------|-----|-------|----------------|----------|-----|
| 1   | $E_7(q)A_1(q)$ | 3   | $\Phi_3.A_2(q)$ | $\Phi_1.\Phi_3^2.A_1(q)$ | $\phi_1, \phi_{22}$ | 1296 |
| 2   | $\Phi_3.3D_4(q)A_2(q)$ | 3   | $\Phi_3.3D_4(q)A_2(q)$ | $\Phi_1.\Phi_3^2.D_4(q)A_1(q)$ | $\phi_1.3D_4[-1] \otimes \phi_{11}$ | 6 |
| 3   | $\Phi_3.E_6(q)$ | 3   | $\Phi_3.E_6(q)$ | $\Phi_3.\Phi_3^2.E_6(q)$ | $\phi_1.3D_4[-1] \otimes \phi_2$ | 6 |
| 4   | $\Phi_3^2$ | 3   | $\Phi_3^2$ | $\Phi_3^2.3D_4(q)$ | 4 chars. | 6 |
| 5   | $E_6(q)A_2(q)$ | 3   | $\Phi_3^2$ | $\Phi_3^2.3D_4(q)$ | 20 chars. | 1 |
| 6   | $D_5(q)A_3(q)$ | 3   | $\Phi_3.A_2(q)^2$ | $\Phi_3.A_2(q)^2$ | 1 | 1944 |
| 7   | $\Phi_3^2$ | 3   | $\Phi_3^2$ | $\Phi_3.3D_4(q)$ | $\phi_{81.6}, \phi_{81.10}, \phi_{90.8}$ | 3 |
| 8   | $G^F$ | 3   | $G^F$ | $\Phi_3.A_2(q)^2$ | 3 |
| 9   | $\Phi_3.E_6(q)$ | 3   | $\Phi_3.E_6(q)$ | $\Phi_1.\Phi_3^2.A_3(q)A_1(q)^2$ | $\phi_{11} \otimes \phi_{11}, \phi_{2} \otimes \phi_{2}$ | 18 |
| 10  | $\Phi_3^2$ | 3   | $\Phi_3^2$ | $\Phi_3^2.3D_4(q)$ | 4 chars. | 9 |
| 11  | $\Phi_3.D_5(q)$ | 3   | $\Phi_3.D_5(q)$ | $\Phi_1.\Phi_3^2.D_4(q)$ | 4 chars. | 3 |
| 12  | $C_{G^*}(s)^F$ | 3   | $C_{G^*}(s)^F$ | $\Phi_3.A_2(q)^2$ | 4 chars. | 1 |
| 13  | $\Phi_3^2$ | 3   | $\Phi_3^2$ | $\Phi_3.3D_4(q)$ | 6 |
| 14  | $G^F$ | 3   | $G^F$ | $\Phi_3^2$ | 40 chars. | 1 |
| 15  | $\Phi_3.E_6(q)$ | 3   | $\Phi_3.E_6(q)$ | $\Phi_1.\Phi_3.A_3(q)A_1(q)^2$ | 3 |
| 16  | $A_4(q)^2$ | 3   | $\Phi_3.A_2(q)^2$ | $\Phi_1.\Phi_3^2.A_4(q)A_1(q)$ | 10 chars. | 18 |
| 17  | $\Phi_3.E_6(q)$ | 3   | $\Phi_3.E_6(q)$ | $\Phi_1.\Phi_3^2.A_4(q)A_1(q)$ | 4 chars. | 3 |
| 18  | $G^F$ | 3   | $G^F$ | $\Phi_3.A_2(q)^2$ | 4 chars. | 3 |
| 19  | $A_5(q)A_2(q).A_1(q)$ | 3   | $\Phi_3.A_2(q)$ | $\Phi_1.\Phi_3.A_3(q)A_1(q)$ | $\phi_{11} \otimes \phi_{11}, \phi_{2} \otimes \phi_{2}$ | 54 |
| 20  | $\Phi_3.E_6(q)$ | 3   | $\Phi_3.E_6(q)$ | $\Phi_1.\Phi_3.A_4(q)A_1(q)$ | 4 chars. | 3 |
| 21  | $A_7(q)A_1(q)$ | 3   | $\Phi_3.A_2(q)^2$ | $\Phi_1.\Phi_3^2.A_1(q)$ | 4 chars. | 18 |
| 22  | $\Phi_3.E_6(q)$ | 3   | $\Phi_3.E_6(q)$ | $\Phi_1.\Phi_3^2.A_1(q)$ | $\phi_{11} \otimes \phi_{311}, \phi_{2} \otimes \phi_{311}$ | 3 |
| 23  | $G^F$ | 3   | $G^F$ | $\Phi_3.A_2(q)^2$ | $\phi_{4211} \otimes \phi_{11}, \phi_{4211} \otimes \phi_{2}$ | 1 |
|   | $A_5(q)$ | 3 | $\Phi_1^3, A_2(q)$ | $\Phi_1^3, \Phi_3^4 \Phi_3^3 A_5(q)$ | 162 |
|---|----------|---|---------------------|---------------------------------|-----|
| 1 | $D_8(q)$ | 3 | $\Phi_3, E_6(q)$ | $\Phi_1^3, E_6(q)$ | 3 |
| 2 | $E_6(q)$ | 3 | $G^F$ | $C_{G^F}(s)^F$ | 1 |
| 3 | $E_7(q)$ | 3 | $D_4(q)$ | $\Phi_1^3, A_4(q)$ | 4 chars | 4 |
| 4 | $E_8(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_1(q)^* \Phi_3^4 \Phi_3^3 A_2(q)$ | 4 chars |
| 5 | $E_6(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 6 | $E_7(q) A_2(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 7 | $E_6(q) A_2(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 8 | $E_7(q) A_2(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 9 | $E_6(q) A_2(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 10 | $D_5(q) A_3(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 11 | $E_7(q) A_2(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 12 | $E_6(q) A_2(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 13 | $E_7(q) A_2(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 14 | $E_6(q) A_2(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 15 | $E_7(q) A_2(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 16 | $E_6(q) A_2(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 17 | $E_7(q) A_2(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 18 | $E_6(q) A_2(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 19 | $E_7(q) A_2(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 20 | $E_6(q) A_2(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 21 | $E_7(q) A_2(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 22 | $E_6(q) A_2(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 23 | $E_7(q) A_2(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 24 | $E_6(q) A_2(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 25 | $E_7(q) A_2(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 26 | $E_6(q) A_2(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 27 | $E_7(q) A_2(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 28 | $E_6(q) A_2(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 29 | $E_7(q) A_2(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 30 | $E_6(q) A_2(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 31 | $E_7(q) A_2(q)$ | 4 | $D_4(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 32 | $E_6(q) A_2(q)$ | 4 | $D_6(q)$ | $\Phi_1^3, A_2(q)$ | 4 chars | 2 |
| 34 | 4 | $\Phi_4^2 D_6(q)$ | $\Phi_1 \Phi_2 \Phi_3 \Phi_4^- A_3(q) A_1(q)$ | $\phi_{11} \otimes \phi_{22}$, $\phi_2 \otimes \phi_{22}$ | 4 | 8 chars. | 1 |
| 35 | 4 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 36 | 4 | $^2 A_7(q) A_1(q)$ | $\Phi_1^2 D_4(q)$ | $\Phi_1 \Phi_2 \Phi_4^- A_3(q) A_1(q)$ | $\phi_{11}, \phi_2$ | 32 | 8 chars. | 1 |
| 37 | 4 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 38 | 4 | $\Phi_4^2 D_6(q)$ | $\Phi_1 \Phi_2 \Phi_4^- A_3(q) A_1(q)$ | $\phi_{11}, \phi_2$ | 4 | 8 chars. | 1 |
| 39 | 4 | $A_8(q)$ | $\Phi_4 A_1(q^2)^2$ | $\Phi_1 \Phi_2 \Phi_4^- A_3(q) A_1(q)$ | 1 | 32 | 8 chars. | 1 |
| 40 | 4 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 41 | 4 | $\Phi_4 D_6(q)$ | $\Phi_1 \Phi_2 \Phi_4^- A_3(q) A_1(q)$ | 1 | 32 | 8 chars. | 1 |
| 42 | 4 | $A_8(q)$ | $\Phi_4^2 A_1(q^2)^2$ | $\Phi_1 \Phi_2 \Phi_4^- A_3(q) A_1(q)$ | 4 | 8 chars. | 1 |
| 43 | 4 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 44 | 4 | $\Phi_4^2 D_6(q)$ | $\Phi_1 \Phi_2 \Phi_4^- A_3(q) A_1(q)$ | | | | |
| 1 | 5 | $E_7(q) A_1(q)$ | $\Phi_5 A_4(q)$ | $\Phi_1 \Phi_5 A_2(q) A_1(q)$ | 6 chars. | 10 | 92 chars. | 1 |
| 2 | 5 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 3 | 5 | $E_6(q) A_2(q)$ | $\Phi_5 A_4(q)$ | $\Phi_1 \Phi_5 A_2(q) A_1(q)$ | 6 chars. | 5 | 60 chars. | 1 |
| 4 | 5 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 5 | 5 | $D_5(q) A_3(q)$ | $\Phi_5 A_4(q)$ | $\Phi_1 \Phi_5 A_3(q)$ | 5 chars. | 5 | 75 chars. | 1 |
| 6 | 5 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 7 | 5 | $A_4(q)$ | $\Phi_5^E$ | $\Phi_5^E$ | 1 | 25 | 4 chars. | 5 |
| 8 | 5 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 9 | 5 | $A_5(q) A_2(q) A_1(q)$ | $\Phi_5 A_4(q)$ | $\Phi_1 \Phi_5 A_2(q) A_1(q)$ | 6 chars. | 5 | 36 chars. | 1 |
| 10 | 5 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 11 | 5 | $A_7(q) A_1(q)$ | $\Phi_5 A_4(q)$ | $\Phi_1 \Phi_5 A_1(q)$ | 6 chars. | 5 | 14 chars. | 1 |
| 12 | 5 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 13 | 5 | $A_8(q)$ | $\Phi_5 A_4(q)$ | $\Phi_1 \Phi_5 A_3(q)$ | 5 chars. | 5 | 5 chars. | 1 |
| 14 | 5 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 15 | 5 | $D_5(q) A_3(q)$ | $\Phi_5 A_4(q)$ | $\Phi_1 \Phi_5 A_3(q)$ | 5 chars. | 10 | 60 chars. | 1 |
| 16 | 5 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 17 | 5 | $\Phi_7 A_1(q)$ | $\Phi_5^E$ | $\Phi_7^E$ | 1 | 14 | 104 chars. | 1 |
| 1 | 7 | $E_7(q) A_1(q)$ | $\Phi_1 \Phi_7 A_1(q)$ | $\Phi_1 \Phi_7 A_1(q)$ | $\phi_{11} \otimes \phi_2$ | 14 | 124 chars. | 1 |
| 2 | 7 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 3 | 7 | $A_7(q) A_1(q)$ | $\Phi_1 \Phi_7 A_1(q)$ | $\Phi_1 \Phi_7 A_1(q)$ | $\phi_{11}, \phi_2$ | 7 | 30 chars. | 1 |
| 4 | 7 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 5 | 7 | $\Phi_7 A_1(q)$ | $\Phi_1 \Phi_7 A_1(q)$ | $\Phi_1 \Phi_7 A_1(q)$ | $\phi_{11}, \phi_2$ | 7 | 16 chars. | 1 |
| 6 | 7 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 7 | 7 | $D_8(q)$ | $\Phi_7^E$ | $\Phi_7^E$ | 1 | 14 | 104 chars. | 1 |
| 8 | 7 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 1 | 8 | $E_7(q) A_1(q)$ | $\Phi_8^2 D_4(q)$ | $\Phi_8 A_1(q)^2 A_1(q^2)$ | 8 chars. | 8 | 88 chars. | 1 |
| 2 | 8 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 3 | 8 | $E_6(q) A_2(q)$ | $\Phi_8^2 D_4(q)$ | $\Phi_1 \Phi_2 \Phi_8 A_2(q)$ | $\phi_{11}, \phi_2, \phi_3$ | 8 | 66 chars. | 1 |
| 4 | 8 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 5 | 8 | $^2 E_6(q) a_2 A_2(q)$ | $\Phi_8^2 D_4(q)$ | $\Phi_1 \Phi_2 \Phi_8 A_2(q)$ | $\phi_{11}, \phi_2, \phi_3$ | 8 | 66 chars. | 1 |
| 6 | 8 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 7 | 8 | $D_5(q) A_3(q)$ | $\Phi_8^2 D_4(q)$ | $\Phi_8 A_3(q)$ | 5 chars. | 8 | 60 chars. | 1 |
| 8 | 8 | $G^F$ | $C_{G^*}(s)^F$ | | | | |
| 9 | 8 | $^2 D_5(q) a_3 A_3(q)$ | $\Phi_8^2 D_4(q)$ | $\Phi_1 \Phi_8 A_3(q)$ | 5 chars. | 8 | 60 chars. | 1 |
\begin{tabular}{|c|c|c|c|c|}
\hline
No. & \( e \lambda \) & \( \pm R_E^x(\lambda) \) & \\
\hline
1 & 10 & \((2A_4(q^2)) \) & \(8 \) & \( G^F \) & \(\Phi_1 \Phi_2 \Phi_4 \Phi_8 \) & \(\Phi_1 \Phi_2 \Phi_4 \Phi_8 \) & \(C_{G^*}(s)^F\) & \(60 \text{ chars.}\) & \(1\) & \\
11 & 2 & \(E_7(q)A_1(q) \) & \(9 \) & \( G^F \) & \(\Phi_1 \Phi_9 \cdot A_1(q) \) & \(\Phi_{11} \cdot \Phi_{21} \cdot \Phi_3 \) & \(\phi_{11} \cdot \phi_{21} \cdot \phi_{3} \) & \(116 \text{ chars.}\) & \(1\) & \\
12 & 3 & \(E_6(q)A_2(q) \) & \(9 \) & \( G^F \) & \(\Phi_9 \cdot A_2(q) \) & \(\Phi_{11} \cdot \Phi_{21} \cdot \Phi_3 \) & \(\phi_{11} \cdot \phi_{21} \cdot \phi_{3} \) & \(63 \text{ chars.}\) & \(1\) & \\
1 & 4 & \(A_8(q) \) & \(9 \) & \( G^F \) & \(\Phi_9 \cdot A_2(q) \) & \(\Phi_{11} \cdot \Phi_{21} \cdot \Phi_3 \) & \(\phi_{11} \cdot \phi_{21} \cdot \phi_{3} \) & \(9 \) & \(1\) & \\
2 & 5 & \(E_7(q)A_1(q) \) & \(12 \) & \( G^F \) & \(\Phi_1 \cdot A_1(q) \cdot A_1(q^3) \) & \(\Phi_{11} \cdot \Phi_{21} \cdot \Phi_3 \) & \(\phi_{11} \cdot \phi_{21} \cdot \phi_{3} \) & \(124 \text{ chars.}\) & \(1\) & \\
2 & 6 & \(E_6(q)A_2(q) \) & \(12 \) & \( G^F \) & \(\Phi_9 \cdot A_2(q) \cdot A_2(q) \) & \(\Phi_{11} \cdot \Phi_{21} \cdot \Phi_3 \) & \(\phi_{11} \cdot \phi_{21} \cdot \phi_{3} \) & \(54 \text{ chars.}\) & \(1\) & \\
1 & 7 & \(E_6(q)A_2(q) \) & \(12 \) & \( G^F \) & \(\Phi_9 \cdot A_2(q) \cdot A_2(q^2) \) & \(\Phi_{11} \cdot \Phi_{21} \cdot \Phi_3 \) & \(\phi_{11} \cdot \phi_{21} \cdot \phi_{3} \) & \(12 \) & \(3\) & \\
1 & 8 & \(E_6(q)A_2(q) \) & \(12 \) & \( G^F \) & \(\Phi_9 \cdot A_2(q) \cdot A_2(q^2) \) & \(\Phi_{11} \cdot \Phi_{21} \cdot \Phi_3 \) & \(\phi_{11} \cdot \phi_{21} \cdot \phi_{3} \) & \(96 \text{ chars.}\) & \(1\) & \\
2 & 9 & \(E_6(q)A_2(q) \) & \(12 \) & \( G^F \) & \(\Phi_9 \cdot A_2(q) \cdot A_2(q^2) \) & \(\Phi_{11} \cdot \Phi_{21} \cdot \Phi_3 \) & \(\phi_{11} \cdot \phi_{21} \cdot \phi_{3} \) & \(1\) & \(5\) & \\
2 & 10 & \(E_6(q)A_2(q) \) & \(12 \) & \( G^F \) & \(\Phi_9 \cdot A_2(q) \cdot A_2(q^2) \) & \(\Phi_{11} \cdot \Phi_{21} \cdot \Phi_3 \) & \(\phi_{11} \cdot \phi_{21} \cdot \phi_{3} \) & \(1\) & \(1\) & \\
\hline
\end{tabular}

\textbf{Proof.} See the proof of Theorem \[23\].

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
No. & \( e \lambda \) & \( \pm R_E^x(\lambda) \) & \\
\hline
2 & 3 & \(^3D_4[-1] \otimes \phi_{11} \) & \(D_4: 111 \otimes \phi_{11} - D_4: 21 \otimes \phi_{11} + D_4: 3 \otimes \phi_{11} + D_4: 111 \otimes \phi_{11} - D_4: 21 \otimes \phi_{11} + D_4: 3 \otimes \phi_{11} \) & \\
3 & 3 & \(^3D_4[-1] \otimes \phi_{2} \) & \(D_4: 111 \otimes \phi_{2} - D_4: 21 \otimes \phi_{2} + D_4: 3 \otimes \phi_{2} + D_4: 111 \otimes \phi_{2} - D_4: 21 \otimes \phi_{2} + D_4: 3 \otimes \phi_{2} \) & \\
6 & 3 & \(^3D_4[-1] \) & \(D_4: 3 \otimes \phi_{111} - D_4: 3 \otimes \phi_{21} + D_4: 3 \otimes \phi_{3} + D_4: 111 \otimes \phi_{111} - D_4: 111 \otimes \phi_{21} + D_4: 111 \otimes \phi_{3} - D_4: 21 \otimes \phi_{111} + D_4: 21 \otimes \phi_{21} - D_4: 21 \otimes \phi_{3} \) & \\
12 & 3 & \((013, 124) \) & \((013, 124) \otimes \phi_{111} - (013, 124) \otimes \phi_{22} + (013, 124) \otimes \phi_{4} \) & \\
3 & 3 & \((02, 14) \) & \((02, 14) \otimes \phi_{1111} - (02, 14) \otimes \phi_{22} + (02, 14) \otimes \phi_{4} \) & \\
3 & 3 & \((0124, \) & \((0124, \) \otimes \phi_{1111} - (0124, \) \otimes \phi_{22} + (0124, \) \otimes \phi_{4} \) & \\
3 & 3 & \((01234, 1) \) & \((01234, 1) \otimes \phi_{1111} - (01234, 1) \otimes \phi_{22} + (01234, 1) \otimes \phi_{4} \) & \\
29 & 3 & \((013, 124) \) & \((0234, 1235) + (012346, 123457) - (124, 034) - (013, 145) + (124, 016) + (013, 127) \) & \\
3 & 3 & \((02, 14) \) & \((1234, 0136) + (01235, 12347) - (013, 235) + (14, 05) + (02, 17) \) & \\
3 & 3 & \((0124, \) & \((1234, \) + (0123457, 123) - (012346, 13) - (0145, \) + (01235, 3) + (0127, \) \) & \\
3 & 3 & \((01234, 1) \) & \((01345, 1) + (01234567, 1234) - (01246, 1) - (012345, 23) + (01237, 1) + (01234, 4) \) & \\
\hline
\end{tabular}
\end{table}
2.5. \( e \)-cuspidal pairs of \( G_2(q) \) and \( 3D_4(q) \). Let \( G \) be a simple, simply connected group of type \( G_2 \) or \( D_4 \) defined over \( \mathbb{F}_q \) with Frobenius endomorphism \( F : G \to G \) such that \( G^F = G_2(q) \) or \( G^F = 3D_4(q) \) respectively. In these cases, \( e \) is relevant for some quasi-isolated \( 1 \neq s \in G^F \) if and only if \( e \in \{1, 2, 3, 6\} \). It remains to determine the \( e \)-cuspidal pairs for \( e = 3 \).

**Theorem 2.7.** Let \( e = 3 \). For any quasi-isolated semisimple element \( 1 \neq s \in G^F \), the \( e \)-cuspidal pairs \((L, \lambda)\) of \( G \) with \( \lambda \in \mathcal{E}(L^F, s) \) (up to \( G^F \)-conjugacy), and the order of their relative Weyl groups \( W = W_{GF}(L, \lambda) \) are as indicated in Table 10. In particular, generalized \( e \)-Harish-Chandra theory holds in every case.

**Table 10.** Quasi-isolated blocks of \( G_2(q) \) and \( 3D_4(q) \)

| No. | \( G^F \) | \( C_{G^*}(s)^F \) | \( e \) | \( L^F \) | \( C_{L^*}(s)^F \) | \( \lambda \) | \( W \) |
|-----|---------|-----------------|-----|--------|----------------|-----|-----|
| 1   | \( G_2(q) \) | \( A_2(q) \)    | 3   | \( \Phi_3 \) | \( \Phi_3 \)    | 1   | 3   |
| 2   | \( 3D_4(q) \) | \( A_1(q)A_1(q^2) \) | 3   | \( \Phi_1\Phi_3.A_1(q) \) | \( \Phi_1\Phi_3.A_1(q) \) | \( \phi_{11}, \phi_2 \) | 2   |

**Proof.** See the proof of Theorem 2.2 \( \square \)

**The cases \( 2E_6(2) \), \( E_7(2) \) and \( E_8(2) \)**

Note that these groups do not have semisimple elements of even order. Furthermore, note that the Mackey Formula holds for \( e = 1 \) regardless of \( q \) since 1-split Levi subgroups are contained in \( F \)-stable parabolic subgroups. In this case, Lusztig induction is just ordinary Harish-Chandra induction. Consequently, the proofs of the previous section still hold for \( e = 1 \) for these groups.

**Proposition 2.8.** The assertion of the Theorems 2.3, 2.7 and 2.6 are still valid when \( q = 2 \).

**Proof.** The only thing missing was to check that generalized \( e \)-Harish-Chandra theory holds in every case of the corresponding tables.

\( 2E_6(2) \): In this case it remains to check the conjecture for the blocks corresponding to centralizers of type \( A_2^3 \) and \( D_4 \). Let \((L, \lambda)\) be an \( e \)-cuspidal pair for a semisimple, quasi-isolated element with centralizer of type \( A_2^3 \). From the tables it follows that either \( L = G \) or that \( \lambda \) is uniform. Hence the decomposition of \( R_{L^*}^G(\lambda) \) can be determined without using the Mackey formula, so the proof of Theorem 2.3 still works.

Now, let \((L, \lambda)\) be an \( e \)-cuspidal pair corresponding to a quasi-isolated element \( s \in G^F \) with \( C_{G^*}(s)^F = \Phi_2^3.D_4(2)^3.3 \). If \( e = 2 \) there are two 2-cuspidal pairs \((L_1, \lambda_1) = \( \Phi_2^6, 1 \) \) and \((L_2, \lambda_2) = \( \Phi_2^3.D_4(2), (02, 13) \) \). Since \( \lambda_1 \) is uniform, we can decompose \( R_{L_1^*}^G(\lambda_1) \) without
using the Mackey formula. For the second pair we use the following argument. We observe that \(\pi_{\text{uni}}(R_{L_2}^G(\lambda_2)) \in \frac{1}{2}\mathcal{Z}(G^F, s)\). Since \(R_{L_2}^G(\lambda_2) \in \mathcal{Z}(G^F, s)\) is a generalized character, there exists an element \(\gamma \in \mathcal{Q}(G^F, s)\) which is orthogonal to the space of uniform class functions of \(G^F\), such that \(\pi_{\text{uni}}(R_{L_2}^G(\lambda_2)) + \gamma \in \mathcal{Z}(G^F, s)\). Furthermore, we know that \(R_{L_1}^G(\lambda_1)\) and \(R_{L_2}^G(\lambda_2)\) do not have any irreducible constituents in common because their constituents lie in different blocks by Theorem 1.3(a). In this particular case this already determines the constituents of \(\gamma\). Without knowing the norm of \(R_{L_2}^G(\lambda_2)\), we are unfortunately not able to determine the multiplicities of the individual constituents. However, it is enough for our purposes to know the constituents.

A similar argument is needed for \(e = 3\) (and \(e = 6\)). There are four 3-cuspidal pairs \((L_i, \lambda_i), i = 1, \ldots, 4\) with \(L := L_1 = L_2 = L_3 = \Phi_{3, 2}A_2(2)\) and \(L_4 = G^F\). Again, we are able to determine the constituents of \(R_{L_i}^G(\lambda_i)\) for \(i = 1, 2, 3\) (the case \(i = 4\) being trivial). In addition to the arguments used for \(e = 2\) above, we know that \(\lambda_1 + \lambda_2 + \lambda_3\) is uniform. Therefore, \(R_{L_i}^G(\lambda_1 + \lambda_2 + \lambda_3)\) is also uniform by transitivity of Lusztig induction (see 11.5 Transitivity). The same arguments as for \(e = 2\) yield that a generalized \(e\)-Harish-Chandra theory holds.

For the quasi-isolated elements \(s \in G^*\) with \(C_{G^*}(s)^F = \Phi_6 \cdot 3D_4(2)\). We argue the same way: either \(\lambda\) is uniform; \(\lambda\) is an \(e\)-cuspidal character of \(G^F\) already; or we can determine the constituents of \(R_{L_i}^G(\lambda)\) without using the Mackey formula, as for the other 3-cuspidal pairs.

\(E_7(2)\): In this case it remains to consider the blocks corresponding to centralizers of type \(A_5 + A_2\). Let \((L, \lambda)\) be an \(e\)-cuspidal pair corresponding to one of these blocks. Checking the tables we see that either \(\lambda\) is uniform or \(L = G\) and \(\lambda\) is an \(e\)-cuspidal character of \(G^F\). Thus we can determine \(R_{L_i}^G(\lambda)\) without the Mackey formula and the proof of Theorem 2.5 works.

\(E_8(2)\): The only cases to consider are the ones corresponding to centralizers of type \(A_5, A_4 \times A_4\) and \(E_6 \times A_2\). For every \(e\)-cuspidal pair \((L, \lambda)\) corresponding to the first or second centralizer type, \(\lambda\) is uniform. Hence, we can determine the decomposition of \(\mathcal{E}(G^F, s)\) without the Mackey-formula. For the last centralizer type we use the same arguments as for the troublesome cases of \(2E_6(2)\).

**Proof of Theorem 1.4.** If \(s = 1\), the assertion follows from 16 and 17. If \(1 \neq s \in G^*\) is semisimple and quasi-isolated, then the assertion follows from Theorem 2.2, Theorems 2.3, 2.5, 2.6, 2.7 and Proposition 2.8. \(\Box\)

**Block distributions**

Let, as before, \(G\) be a simple, simply connected group of exceptional type defined over \(\mathbb{F}_q\) with Frobenius endomorphism \(F: G \to G\) or let \(G\) be simple, simply connected of type \(D_4\) defined over \(\mathbb{F}_q\) with Frobenius endomorphism \(F: G \to G\) such that \(G^F = 3D_4(q)\). Let \(\ell \nmid q\) be a good prime for \(G\) and let \(\ell \neq 3\) if \(G^F = 3D_4(q)\). If \(1 \neq s \in G^*\) is a quasi-isolated \(\ell\)-element, then the \(\ell\)-block distribution of \(\mathcal{E}(G^F, s)\) is known.

For \(e_4(q) = 1, 2\) this can be read off from the tables in 17. However, the \(\ell\)-block distribution differs from the one given there. By Theorem 1.3 every line of their tables (instead of just every numbered line) yields a different block-type. For the other relevant \(e_4(q)\)'s the block distribution can be read off from the Tables in this section.
3. ON THE MALLE–ROBINSON CONJECTURE

Let $H$ be a finite group. If $N \triangleleft K \subseteq H$ are two subgroups of $H$, we call the quotient $K/N$ a section of $H$. The sectional $\ell$-rank $s(H)$ of a finite group $H$ is then defined to be the maximum of the ranks of elementary abelian $\ell$-sections of $H$. Note that $s(K/N) \leq s(H)$ for every section $K/N$ of $H$.

**Conjecture** (Malle–Robinson, [18, Conjecture 1]). Let $B$ be an $\ell$-block of a finite group $H$ with defect group $D$. Then

$$l(B) \leq \ell^s(D).$$

If strict inequality holds, we say that the conjecture holds in strong form. Since the defect groups of a given block $B$ are conjugate and therefore isomorphic to each other, we often write $s(B)$ instead of $s(D)$.

**Definition 3.1.** If $U$ is a union of blocks (regarded as subsets of $\text{Irr}(H) \cup \text{IBr}(H)$) we set $\text{Irr}(U) = \bigcup_{B \in U} \text{Irr}(B)$ and $\text{IBr}(U) = \bigcup_{B \in U} \text{IBr}(B)$.

A subset $A \subseteq \mathbb{Z}\text{IBr}(U)$ is called a generating set for $U$ if it generates $\mathbb{Z}\text{IBr}(U)$ as a $\mathbb{Z}$-module and it is called a basic set for $U$ if it is a basis of $\mathbb{Z}\text{IBr}(U)$ as a $\mathbb{Z}$-module. Let $H^\circ = \{h \in H \mid \ell \nmid o(h)\}$ denote the set of $\ell$-regular elements of $H$. A subset $C \subseteq \text{Irr}(G)$ is called an ordinary generating (respectively basic) set for $U$ if the set $C^\circ = \{\psi^\circ \mid \psi \in C\}$ consisting of the restrictions of the irreducible characters in $C$ to $H^\circ$ is a generating (respectively basic) set for $U$.

Let $k(B) := |\text{Irr}(B)|$ and $l(B) := |\text{IBr}(B)|$. Note that every basic set of a block $B$ has cardinality $l(B)$.

We return to our initial setting. Let $G$ be a connected reductive group defined over $\mathbb{F}_q$ with Frobenius endomorphism $F : G \to G$. Let $\ell \nmid q$ be a good prime for $G$.

By Theorem [12] we know that $\mathcal{E}_\ell(G^F, s)$ is a union of $\ell$-blocks of $G^F$. However, we can say even more about $\mathcal{E}_\ell(G^F, s)$. We define $\mathcal{E}(G^F, s) := \{\chi^\circ \mid \chi \in \mathcal{E}(G^F, s)\}$.

**Theorem 3.2** ([14, Theorem A]). Assume that $\ell$ is a good prime for $G$ not dividing the order of $(Z(G)/Z^\circ(G))_F$ (the largest quotient of $Z(G)$ on which $F$ acts trivially). Let $s \in G^{s,F}$ be a semisimple $\ell'$-element. Then $\mathcal{E}(G^F, s)$ is an ordinary basic set for the union of blocks $\mathcal{E}_\ell(G^F, s)$.

**Remark 3.3.** Let $B$ be an $\ell$-block contained in $\mathcal{E}_\ell(G^F, s)$ for some semisimple $\ell'$-element $s \in G^{s,F}$. It follows that a basic set for $B$ is then given by $\text{Irr}(B) \cap \mathcal{E}(G^F, s)$.

**Definition 3.4.** (a) The $\ell$-blocks contained in $\mathcal{E}_\ell(G^F, s)$ for a semisimple, quasi-isolated $\ell'$-element $s \in G^{s,F}$ are called quasi-isolated.

(b) Let $H = G^F/Z$, for some subgroup $Z \subseteq Z(G^F)$. A block of $H$ is said to be quasi-isolated if it is dominated by a quasi-isolated block of $G^F$. Furthermore, it is said to be unipotent if the block dominating it is unipotent.

**Theorem 3.5.** Assume (A) to hold with $G$ a simple, simply connected group of exceptional type or $G^F = 3D_4(q)$. Then the Malle–Robinson conjecture holds for all quasi-isolated $\ell$-blocks of $G^F$ and of $G^F/Z(G^F)$. 

Proof. The conjecture holds for the unipotent blocks of $G^F$ and $G^F/Z(G^F)$ by [18] Proposition 6.10]. Suppose $B = b_{G^F}(L, \lambda)$ is a non-unipotent, quasi-isolated block of $G^F$. By the results in Section 2 and Theorem 3.2, we conclude that $l(B) = |E(G^F, (L, \lambda))|$. These cardinalities are known and can be determined with the tables in Section 2 (see the Tables 2, 4, 6, 8, and 10). Let $D$ be a defect group of $B$ associated to a maximal $B$-subpair as in [19] Lemma 4.16]. In particular, $Z(L)_G^F$ is the unique maximal abelian normal subgroup of $D$ (note that in our case $G = G_b$ in their notation). The structure of $Z(L)_G^F$ can be read off from the tables in Section 2 as well. We see that $l(B) \leq |s(Z(L)_G^F)|$ in every case. With this, the assertion is proved for the quasi-isolated $\ell$-blocks of $G^F$.

Suppose that $B$ is a non-unipotent, quasi-isolated block of $H = G^F/Z(G^F)$ with defect group $D$. Let $B$ be the quasi-isolated block of $G^F$ that dominates $B$. The order of $Z(G^F)$ is either 1 or a bad prime for $G$. Thus, $Z(G^F)$ is an $\ell'$-subgroup by our assumption on $\ell$. By [21] (9.9) Theorem 1, $l(\bar{B}) = l(B)$ and $\bar{D}$ is of the form $DZ(G^F)/Z(G^F)$, for a defect group $D$ of $B$. Note that $DZ(G^F)$ is a direct product since the two groups commute and $D \cap Z(G^F) = \{1\}$. It follows that $DZ(G^F)/Z(G^F) \cong D$, i.e. $s(\bar{D}) = s(D)$. Thus, the conjecture holds for $\bar{B}$ since it holds for $B$. □

The reason we focused on the quasi-isolated blocks are the results of Bonnafé–Rouquier [3] and more recently Bonnafé–Dat–Rouquier [1]. Using their results, we will prove that the $\ell$-blocks of finite quasi-simple groups of exceptional Lie type do not yield minimal counterexamples to the Malle–Robinson conjecture when $\ell$ is a good prime for the underlying algebraic group.

Definition 3.6. Let $H$ be a finite group and let $B$ be an $\ell$-block of $H$. Then $(H, B)$ is called a minimal counterexample to the Malle–Robinson conjecture if

(a) the conjecture does not hold for $B$, and

(b) the conjecture holds for all $\ell$-blocks $B'$ of groups $K$ with $|K/Z(K)|$ strictly smaller than $|H/Z(H)|$ having defect groups isomorphic to those of $B$.

We also say that $B$ is a minimal counterexample if the group $H$ is understood.

Theorem B. Let $H$ be a finite quasi-simple group of exceptional Lie type. Let $\ell$ be a prime and let $B$ be an $\ell$-block of $H$. Then $B$ is not a minimal counterexample to the Malle–Robinson conjecture for $\ell > 5$ if $H = E_8(q)$ and for $\ell \geq 5$ otherwise.

Proof. Suppose that $(H, B)$ is a minimal counterexample to the Malle–Robinson conjecture. Let $D$ be a defect group of $B$. By [18] Proposition 6.4, $H$ is not an exceptional covering group of a finite group of exceptional Lie type. Hence, $H = G^F/Z$, where $G$ is a simple, simply connected group of exceptional type and $Z \subseteq Z(G^F)$ is a central subgroup. By [18], Proposition 6.1, $\ell$ does not divide $q$. Let $B'$ be the unique block of $G^F$ dominating $B$ and let $D'$ be a defect group of $B'$. Recall from the proof of Theorem 3.5 that $l(B) = l(B')$ and $s(D) = s(D')$. By [1] Theorem 7.7, $B'$ is Morita equivalent to an $\ell$-block $b$ of a subgroup $N$ of $G^F$ and their defect groups are isomorphic. In particular, $l(B') = l(b)$ and $s(B') = s(b)$. If $s$ is not quasi-isolated, then $N$ is a proper subgroup. By the minimality of $(H, B)$, $B$ is therefore a quasi-isolated block of $H$. So, the assertion follows from Theorem 3.5. □
References

[1] C. Bonnafé, J.-F. Dat, and R. Rouquier. Derived categories and Deligne-Lusztig varieties II. *Ann. of Math.* (2), 185(2):609–670, 2017.
[2] C. Bonnafé and J. Michel. Computational proof of the Mackey formula for $q > 2$. *J. Algebra*, 327:506–526, 2011.
[3] C. Bonnafé and R. Rouquier. Catégories dérivées et variétés de Deligne-Lusztig. *Publ. Math. Inst. Hautes Études Sci.*, 97:1–59, 2003.
[4] C. Bonnafé. Quasi-isolated elements in reductive groups. *Comm. Algebra*, 33(7):2315–2337, 2005.
[5] M. Broué, G. Malle, and J. Michel. Generic blocks of finite reductive groups. *Astérisque*, 212:7–92, 1993. Représentations unipotentes génériques et blocs des groupes réductifs finis.
[6] M. Broué and J. Michel. Blocs et séries de Lusztig dans un groupe réductif fini. *J. Reine Angew. Math.*, 395:56–67, 1989.
[7] M. Cabanes and M. Enguehard. Unipotent blocks of finite reductive groups of a given type. *Math. Z.*, 213(3):479–490, 1993.
[8] M. Cabanes and M. Enguehard. On unipotent blocks and their ordinary characters. *Invent. Math.*, 117(1):149–164, 1994.
[9] M. Cabanes and M. Enguehard. On blocks of finite reductive groups and twisted induction. *Adv. Math.*, 145(2):189–229, 1999.
[10] M. Cabanes and M. Enguehard. *Representation theory of finite reductive groups*, volume 1 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2004.
[11] F. Digne and J. Michel. *Representations of Finite Groups of Lie Type*. London Mathematical Society Student Texts. Cambridge University Press, 1991.
[12] M. Enguehard. Sur les $l$-blocs unipotents des groupes réductifs finis quand $l$ est mauvais. *J. Algebra*, 230(2):334–377, 2000.
[13] M. E. Enguehard. Towards a Jordan decomposition of blocks of finite reductive groups. *ArXiv e-prints*, November 2013.
[14] M. Geck. Basic sets of Brauer characters of finite groups of Lie type. II. *J. London Math. Soc.* (2), 47(2):255–268, 1993.
[15] M. Geck and G. Hiss. Basic sets of Brauer characters of finite groups of Lie type. *J. Reine Angew. Math.*, 418:173–188, 1991.
[16] G. Hiss. Regular and semisimple blocks of finite reductive groups. *J. London Math. Soc.* (2), 41(1):63–68, 1990.
[17] R. Kessar and G. Malle. Quasi-isolated blocks and Brauer’s height zero conjecture. *Ann. of Math.* (2), 178(1):321–384, 2013.
[18] G. Malle and G. R. Robinson. On the number of simple modules in a block of a finite group. *J. Algebra*, 475:423–438, 2017.
[19] G. Malle and D. Testerman. *Linear Algebraic Groups and Finite Groups of Lie Type*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2011.
[20] J. Michel. The development version of the *chevie* package of *gap3*. *J. Algebra*, 435:308–336, 2015.
[21] G. Navarro. *Characters and blocks of finite groups*, volume 250 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

FB Mathematik, TU Kaiserslautern, 67663 Kaiserslautern, Germany.

E-mail address: hollenbach@mathematik.uni-kl.de