A comparative study of quaternionic rotational Dirac equation and its interpretation

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Abstract

In this study, we develop the generalized Dirac like four-momentum equation for rotating spin-half particles in four-dimensional quaternionic algebra. The generalized quaternionic Dirac equation consists the rotational energy and angular momentum of particle and anti-particle. Accordingly, we also discuss the four vector form of quaternionic relativistic mass, moment of inertia and rotational energy-momentum in Euclidean space-time. The quaternionic four angular momentum (i.e. the rotational analogy of four linear momentum) predicts the dual energy (rest mass energy and pure rotational energy) and dual momentum (linear like momentum and pure rotational momentum). Further, the solutions of quaternionic rotational Dirac energy-momentum are obtained by using one, two and four-component of quaternionic spinor. We also demonstrate the solutions of quaternionic plane wave equation which gives the rotational frequency and wave propagation vector of Dirac particles and anti-particles in terms of quaternionic form.

Keywords: quaternion, four-vector, energy-momentum, rotational motion.
Mathematics Subject Classification: 11R52, 81Q05, 20Gxx
PACS: 03.65.-w, 03.65.Fd, 02.10.Ud

1 Introduction

The results of classical theory have a large impact on the fully appropriate description of matter but this theory does not explain the behavior of subatomic particles. The information about the small-scale behavior of particles created the idea of quantum mechanics. Generally, the quantum physics deals with the description of the state of particles associated with the wave function. The angular momentum is an observable essential element in quantum mechanics, where the quantum particles can involve the orbital angular momentum and intrinsic angular momentum. Further, Erwin Schrödinger developed a fundamental wave equation for non-relativistic microscopic particles but this equation does not applicable to particles moving with
relativistic velocity. To combine the special theory of relativity with quantum mechanics, there has been developed the relativistic quantum mechanics. In the same way, Klein-Gordon and Dirac independently investigated the relativistic wave equations by combining special relativity with quantum mechanics. These relativistic wave equations describe the various phenomena that occur in high energy physics and are invariant under Lorentz transformations. Dirac discussed a relativistic quantum wave equation by using Hamiltonian operator to overcome the difficulties arising in Klein-Gordon equation. As we know that the conservation of energy and angular momentum are one of the mandatory conservation laws to check the validity of Dirac particles. Keeping in mind the conservation laws of energy and angular momentum of a rotating particle, in this paper, we propose a quaternionic Dirac equation that consists not only the energy representation but also shows the angular momentum representation of spin-1/2 particles.

The quaternion number is basically an extension of complex numbers. Although, there are four types of norm-division algebras, i.e. real, complex, quaternion and octonion algebra. The division algebra is defined as an algebra in which all non-zero elements have their inverse under multiplication. Nowadays, the quaternionic algebra is a popular algebra to study the various theories in modern theoretical physics. The quaternionic algebra is associative and commutative under addition but not commutative under multiplication. Thus, this algebra forms a group under multiplication but not an Abelian group, is also called the division ring.

Many researchers have attempted the formulation of usual Dirac equation for free particles in terms of quaternionic algebra. Firstly, Rotelli developed the Dirac equation in quaternionic four fields. The another version of quaternionic Dirac equation has been studied by Rawat et al with the description of quaternionic spinors for positive and negative energy solutions. As such, the quaternionic form of Dirac equation with the connection of supersymmetric quantum mechanics has also been discussed. Besides, the quaternionic algebra has also been used to describe the rotational motion of a rigid body where the quaternionic unit elements are directly connected with matrices of rotational group . Further, the quaternion algebra explained the special theory of relativity, Dirac Lagrangian, superluminal transformations for tachyons, wave equation in curved space-time, electromagnetic-field equations, gravi-electromagnetism, quantum mechanics, particle in a relativistic box and dual magneto-hydrodynamics of dyonic cold plasma. On the other hand, many authors have used hyper-complex algebras to study the several theories in different branches of physics. Chanyal proposed a novel idea on the quaternionic covariant theory of relativistic quantized electromagnetic fields of dyons. Carmeli discussed the various fundamental equations of quantum mechanics viz. Klein-Gordon equation, Schrödinger equation, Weyl equation and the Dirac field equation of rotating particles on . Keeping in mind the properties of rotating spin 1/2 particles, we generalize the Carmeli’s field theory in terms of quaternionic field. The benefit of generalized quaternionic field is that, we can analyze the four-momentum representation of a particle in a single equation i.e. energy (as a scalar component) and momentum (as a vector component). In present study, starting with quaternionic algebra and its representation to SU(2) group (i.e. an isomorphic to orthogonal group SO(3)), we construct the generalized Dirac like energy-momentum equation for rotating...
particles in four-dimensional quaternionic space-time. We define the quaternionic moment of inertia and rotational energy-momentum of a rotating particle by using four-relativistic mass, four-spaces and four-momentum. A novel approach to unified quaternionic Dirac like equation contains the rotational energy (corresponding to the coefficient of quaternionic pure scalar unit element) and the rotational momentum (corresponding to the coefficient of quaternionic pure vector unit elements). Accordingly, the rotational energy and momentum solutions are obtained by using one, two and four-component spinor forms of quaternionic wave function. We have calculated the solutions of rotational energy and rotational momentum for particles with spin up and spin down states. To considering the wave nature of spin half particles (or anti-particles), we have studied the general form of quaternionic plane wave equation and developed the quaternionic form of rotational frequency and wave propagation vector for particles and anti-particles. This theory also point out the conservation law of quaternionic four-momentum of particles.

2 Quaternionic field representation

The quaternionic field ($\mathbb{H}$) is expressed by a linear algebra consist four unit elements known as quaternionic basis ($e_0, e_1, e_2, e_3$). In a quaternionic field, the unit element $e_0$ is used to express scalar field while the other unit elements $e_1, e_2, e_3$ are used to express vector fields. Thus, the quaternionic field algebra can be expressed as

$$\mathbb{H} = e_0(H^S) + e_j(H^V_j), \quad (\forall j = 1, 2, 3)$$

$$\equiv e_0h_0 + e_1h_1 + e_2h_2 + e_3h_3, \quad (2.1)$$

where $(h_0, h_j)$ are the real numbers corresponding to quaternionic scalar-field ($H^S$) and vector-field ($H^V$). If the real part of a quaternion is zero then the quaternion have only vector field components called a pure quaternion as $\mathbb{H} \rightarrow \mathbb{H}_P = (e_1h_1 + e_2h_2 + e_3h_3)$, and if the imaginary part of a quaternion is zero then the quaternion involve only scalar component called a real quaternion as $\mathbb{H} \rightarrow \mathbb{H}_R = e_0h_0$. The addition of two quaternions $(\mathbb{A}, \mathbb{B}) \in \mathbb{H}$ produce a new quaternionic field as

$$\mathbb{A} + \mathbb{B} = (e_0a_0 + e_1a_1 + e_2a_2 + e_3a_3) + (e_0b_0 + e_1b_1 + e_2b_2 + e_3b_3)$$

$$= e_0(a_0 + b_0) + e_1(a_1 + b_1) + e_2(a_2 + b_2) + e_3(a_3 + b_3). \quad (2.2)$$

The quaternions are associative as well as commutative under addition, i.e.

$$\mathbb{A} + \mathbb{B} = \mathbb{B} + \mathbb{A},$$

$$\mathbb{A} + (\mathbb{B} + \mathbb{C}) = (\mathbb{A} + \mathbb{B}) + \mathbb{C}, \quad \forall (\mathbb{A}, \mathbb{B}, \mathbb{C}) \in \mathbb{H}. \quad (2.3)$$
The quaternionic algebra contains additive identity element zero as

$$0 = e_0 0 + e_1 0 + e_2 0 + e_3 0.$$  \hfill (2.4)

Every quaternion has its additive inverse which can be expressed in the form of

$$-\mathbb{H} = e_0 (-h_0) + e_1 (-h_1) + e_2 (-h_2) + e_3 (-h_3).$$  \hfill (2.5)

The quaternionic conjugate of given equation (2.1) can be expressed as,

$$\mathbb{H} = e_0 (\mathbb{H}^S) - e_j (\mathbb{H}^V) \equiv e_0 h_0 - e_1 h_1 - e_2 h_2 - e_3 h_3.$$  \hfill (2.6)

As such, the square of a quaternion is written as,

$$\mathbb{H}^2 = \mathbb{H} \circ \mathbb{H} = (h_0^2 + h_1^2 + h_2^2 + h_3^2),$$  \hfill (2.7)

where ‘\circ’ indicated the quaternion multiplication. Here, the quaternionic unit elements ($e_0, e_1, e_2, e_3$) are followed the relations

$$e_0^2 = 1, \ e_1^2 = e_2^2 = e_3^2 = -1, \text{ and } e_0 e_i = e_i e_0 = e_i, \ e_i e_j = -\delta_{ij} e_0 + \epsilon_{ijk} e_k, \ \forall (i, j, k = 1, 2, 3)$$ \hfill (2.8)

where the symbol $\delta_{ij}$ is the Kronecker delta symbol having value one for equal indices and zero for unequal indices, while $\epsilon_{ijk}$ is the Levi Civita three-index symbol taking value $\epsilon_{ijk} = +1$ for cyclic permutation, $\epsilon_{ijk} = -1$ for anti-cyclic permutation and $\epsilon_{ijk} = 0$ for any two repeated indices. The multiplication of two quaternions can be expressed by using Table-1 as,

$$A \circ B = e_0 (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) + e_1 (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2) + e_2 (a_0 b_2 + a_2 b_0 + a_3 b_1 - a_1 b_3) + e_3 (a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1), \hfill (2.9)$$

which can be further simplify as

$$A \circ B = e_0 \left( a_0 b_0 - \vec{a} \cdot \vec{b} \right) + e_j \left( a_0 \vec{b} + b_0 \vec{a} + \left( \vec{a} \times \vec{b} \right)_j \right), \ (\forall j = 1, 2, 3).$$  \hfill (2.10)

The quaternionic multiplication identity element can define as,

$$1 = e_0 1 + e_1 0 + e_2 0 + e_3 0.$$ \hfill (2.11)

Rather, the quaternionic field is associative but non-commutative under multiplication opera-
Table 1: Quaternion multiplication table

| $\circ$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ |
|---------|-------|-------|-------|-------|
| $e_0$   | 1     | $e_1$ | $e_2$ | $e_3$ |
| $e_1$   | $e_1$ | $-1$  | $e_3$ | $-e_2$|
| $e_2$   | $e_2$ | $-e_3$| $-1$  | $e_1$ |
| $e_3$   | $e_3$ | $e_2$ | $-e_1$| $-1$  |

Because the cross product of two vectors are always non-commutative i.e. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. The two quaternions will be commutative only when the cross product of their vectors are zero or the vectors $\vec{a}$ and $\vec{b}$ are parallel to each other. Moreover, the norm of a quaternion $A$ becomes

$$N = \sqrt{\mathbf{A} \circ \mathbf{A}} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}.$$  (2.13)

The equation (2.13) also known the modulus of a quaternion i.e. $|A|$. Every non-zero element of a quaternion has an inverse

$$A^{-1} = \frac{\overline{A}}{|A|}.$$  (2.14)

Besides, a quaternion can also be represented in the form of split quaternionic basis elements $u_0$, $u_0^*$, $u_1$, $u_1^*$ as,

$$A = u_0 a + u_0^* a^* + u_1 b + u_1^* b^*, \quad \forall (a, a^*, b, b^*) \in \mathbb{C}$$

$$\equiv u_0 (h_0 - i h_3) + u_0^* (h_0 + i h_3) + u_1 (h_1 - i h_2) + u_1^* (h_1 + i h_2),$$  (2.15)

where

$$u_0 = \frac{1}{2} (e_0 + i e_3), \quad u_0^* = \frac{1}{2} (e_0 - i e_3), \quad u_1 = \frac{1}{2} (e_1 + i e_2), \quad u_1^* = \frac{1}{2} (e_1 - i e_2),$$

are written in the form of $2 \times 2$ matrix value realization as

$$u_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_0^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \quad u_1^* = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}.$$  (2.16)
Thus, the quaternionic basis represents the following $2 \times 2$ matrix realization as

\[
ed_0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad e_3 \mapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},
\] (2.17)

where the determinant of each quaternionic basis gives unity i.e., $|e_0| = |e_1| = |e_2| = |e_3| = +1$. Interestingly, to considering quaternion basis $(e_0, e_j)$, the algebra of $(e_0, ie_j)$ is isomorphic to the algebra of tau-matrices $(\tau_{e_0}, \tau_{e_j}$ for $j = 1, 2, 3)$ in $SU(2)$ group representation. Here, $\tau_{e_0}$ is $2 \times 2$ identity matrix corresponding to quaternion basis $e_0$ while $\tau_{e_j}$ are corresponding to pure quaternionic basis called Pauli tau-matrices. In given equaation (2.15), the $SU(2)$ representation is the set of all $2 \times 2$ complex matrices of determinant positive one and satisfy $AA^\dagger = A^\dagger A = I_{2\times 2}$, i.e.

\[
A = \begin{pmatrix} h_0 - ih_3 & -(h_2 + ih_1) \\ h_2 - ih_1 & h_0 + ih_3 \end{pmatrix} \mapsto \begin{pmatrix} a & -(ib) \\ (ib)^* & a^* \end{pmatrix},
\] (2.18)

where $|a|^2 + |b|^2 = 1$. For quaternionic rotational group, if $H \in SU(2)$ maps onto $R(H) \in SO(3)$, then we may write

\[
R(H)_{j,k} = \frac{1}{2} \text{Tr}(\tau_{e_j} H \tau_{e_k} H^{-1}), \quad (\forall j, k = 1, 2, 3).
\] (2.19)

As such, the simplest representations of the quaternionic basis can also be expressed by the multiplication of $(-i)$ with $2 \times 2$ Pauli tau-matrices as,

\[
e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \tau_{e_0}, \quad e_1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto -i\tau_{e_1},
\]

\[
e_2 = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \mapsto -i\tau_{e_2}, \quad e_3 = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto -i\tau_{e_3},
\] (2.20)

where $e_j \equiv (-i\tau_{e_j})$ $j = 1, 2, 3$ are also defined a realization of infinitesimal rotations of three dimensions pure quaternionic space. Moreover, the quaternions can be use to study the rotational motion of a spin-1/2 particles because it is identical to rotational $\tau-$matrices [43]. We summarize the properties of quaternionic basis and tau-matrices in Table-2.

### 3 Quaternionic rotational energy-momentum

Let us start with the quaternionic relativistic mass of Dirac particles (e.g. electrons) that can be expressed in terms of four-masses [44],

\[
M = e_0 E c^2 + e_1 \frac{p_1}{v_1} + e_2 \frac{p_2}{v_2} + e_3 \frac{p_3}{v_3}
\]

\[
\simeq e_0 m_0 + \sum_{j=1}^{3} e_j m_j,
\] (3.1)
| Properties          | Quaternion basis | Tau-matrices                  |
|---------------------|------------------|-------------------------------|
| Square              | $e_0^2 = 1$, $e_j^2 = -1$ | $(\tau_{e_0})^2 = 1$, $(-i\tau_{e_j})^2 = -1$ |
| Determinant         | 1                | 1                             |
| Eigen values        | $\pm i$          | $\pm i$                       |
| Trace               | 0                | 0                             |
| Multiplication      | $e_i e_j = -\delta_{ij}e_0 + e_{ijk}e_k$ | $\tau_{e_i} \tau_{e_j} = \delta_{ij} \tau_{e_0} + i\epsilon_{ijk} \tau_{e_k}$ |
| Commutation         | $\{ e_i, e_j \} = 2\epsilon_{ijk} e_k$ | $\{ \tau_{e_i}, \tau_{e_j} \} = 2i\epsilon_{ijk} \tau_{e_k}$ |
| Anti-commutation    | $\{ e_i, e_j \} = -2\epsilon_{0ij}$ | $\{ \tau_{e_i}, \tau_{e_j} \} = 2\delta_{ij} \tau_{e_0}$ |

Table 2: Properties of quaternionic basis and $\tau$–matrices

where $m_0 \sim E_0/c^2$ indicates the rest mass and $m_1$, $m_2$, $m_3$ are the moving masses of particle having velocities $v_1$, $v_2$, $v_3$ corresponding to quaternionic basis $e_1$, $e_2$, $e_3$, respectively. It should be notice that the quaternionic scalar field associated with the coefficient of $e_0$ while the quaternionic vector field associated with the coefficient of $e_j$. If vector part is zero then the real quaternion expresses only the rest mass ($m_0$), and if rest mass-energy of a particle is zero then the pure quaternion experiences the motion of particle. As such, the quaternionic space-time position ($\mathbb{R}$) can also be expressed as

$$\mathbb{R} = e_0 r_0 + e_1 r_1 + e_2 r_2 + e_3 r_3 ,$$

(3.2)

where $r_0$ is the scalar component considering as time, while $r_1$, $r_2$, $r_3$ are the three spacial components in Euclidean space-time. Now, the moment of inertia in quaternionic form becomes,

$$I = \mathbb{M} (\mathbb{R} \circ \mathbb{R}) = \mathbb{M} \left( r_0^2 + r_1^2 + r_2^2 + r_3^2 \right)$$

$$= \mathbb{M} R^2 .$$

(3.3)

Thus, we have

$$I = e_0 I_0 + e_1 I_1 + e_2 I_2 + e_3 I_3 ,$$

(3.4)

where $I_0 = m_0 R^2$ is considered the moment of inertia along scalar axis $e_0$ while $I_1 = m_1 R^2$, $I_1 = m_2 R^2$ and $I_3 = m_3 R^2$ are the moment of inertia along pure-quaternionic axes $e_1$, $e_2$, $e_3$, respectively. Keeping in mind the rotational analog of translation motion, we can define the quaternionic four-angular momentum as,

$$L = \mathbb{R} \circ \mathbb{P} ,$$

(3.5)
where \( P \rightarrow \{e_0 p_0, e_1 p_1, e_2 p_2, e_3 p_3\} \) is the quaternionic linear four-momentum. Thus, from equation (3.3), we get
\[
L = e_0 (r_0 m_0 c - r_1 m_1 v_1 - r_2 m_2 v_2 - r_3 m_3 v_3)
+ e_1 (r_0 m_1 v_1 + r_1 m_0 c + r_2 m_3 v_3 - r_3 m_2 v_2)
+ e_2 (r_0 m_2 v_2 + r_2 m_0 c + r_3 m_1 v_1 - r_1 m_3 v_3)
+ e_3 (r_0 m_3 v_3 + r_3 m_0 c + r_1 m_2 v_2 - r_2 m_1 v_1) ,
\]
(3.6)

It can be reduces as,
\[
L = e_0 L_0 + e_1 L_1 + e_2 L_2 + e_3 L_3
= e_0 (L_{00} - L_{11} - L_{22} - L_{33}) + e_1 (L_{01} + L_{10} + L_{23} - L_{32})
+ e_2 (L_{02} + L_{20} + L_{31} - L_{13}) + e_3 (L_{03} + L_{30} + L_{12} - L_{21}) ,
\]
(3.7)

where \( L_0 \) is purely a scalar component can be indicated by quaternionic rotational energy \( (E_0) \), while the vector components \( (L_j, j = 1, 2, 3) \) indicated the quaternionic rotational angular-momentum. Now, the quaternionic rotational energy \( E_0 \sim L_0 \) can be expressed by
\[
E_0 = r_0 p_0 - (\vec{r} \cdot \vec{p}) , \quad \text{(coefficient of } e_0).
\]
(3.8)

Here, in quaternionic formalism the quaternionic rotational energy consists a couple of energy. We may consider the first scalar term \( (r_0 p_0) \) represents the rest mass-energy while the second scalar term \( (\vec{r} \cdot \vec{p}) \) represents the moving (projectional) energy. In the same way, the quaternionic angular momentum can then be written as,
\[
L_j = (r_0 \vec{p} + p_0 \vec{r}) + (\vec{r} \times \vec{p})_j , \quad \text{(coefficient of } e_j) .
\]
(3.9)

In equation (3.9), the first term indicates the longitudinal component (irrotational momentum) while the second term indicates the transverse component (rotational momentum) of quaternionic four-momentum. Therefore, we may describe the quaternionic form of rotational energy-momentum as following specific cases:

**Case-1:** Conditionally, for the pure quaternion field the scalar coefficient is taken zero i.e., \( r_0 = p_0 = 0 \). Then, we get the usual three dimensional form of rotational energy and angular momentum in vector field, i.e.
\[
| E_0 | \simeq (\vec{r} \cdot \vec{p}) , \quad \text{(pure rotational energy)} \quad \text{(3.10)}
\]
\[
\vec{L} \simeq (\vec{r} \times \vec{p}) , \quad \text{(pure angular momentum)} .\quad \text{(3.11)}
\]

**Case-2:** If pure quaternion part is zero i.e. \( \vec{r} = \vec{p} = 0 \), then we get the pure scalar field as
\[
E_0 \simeq r_0 p_0 , \quad \vec{L} \simeq 0 ,
\]
(3.12)
which shows that there will be only rest mass-energy having no rotating motion.

**Case-3:** If the quaternionic variables $\mathbb{R}$ and $\mathbb{P}$ are mutually interchange as $\mathbb{R} \to \mathbb{P}$ and $\mathbb{P} \to \mathbb{R}$, then the rotational energy ($E_0$) remains unaffected but the angular momentum ($\vec{L}$) become changed, i.e.,

$$E_0 (\mathbb{R} \leftrightarrow \mathbb{P}) = r_0 p_0 - (\vec{r} \cdot \vec{p}) \;,$$

$$\vec{L} (\mathbb{R} \leftrightarrow \mathbb{P}) = (p_0 \vec{r} + r_0 \vec{p}) - (\vec{r} \times \vec{p}) \;.$$  \hspace{1cm} (3.13)

### 4 Quaternionic Rotational Dirac (QRD) equation

Let us start with the Dirac equation for free particle as

$$\left( \vec{\alpha} \cdot \vec{p} - \beta mc^2 \right) \psi = 0 \;,$$

where $\vec{\alpha}$ and $\beta$ are the Dirac matrices. Now, to check the energy and momentum relations of an electron rotating in quaternionic space-time, we can extend the Dirac equation (4.1) in term of QRD equation form, i.e.

$$(\mathcal{A} \circ \mathbb{L} - \mathcal{B} \lambda^2 \mathbb{I}) \circ \Psi = 0 \;,$$

where $\mathcal{A}$, $\mathbb{L}$, $\mathcal{B}$, $\mathbb{I}$ and $\Psi$ are quaternionic variables considering for the rotational analogy of $\vec{\alpha}$, $\vec{p}$, $\beta$, $m$ and $\psi$, respectively. For rotating particles the speed of light $c$ can be replaced by the maximum speed $\lambda$ \[42\] as $\lambda = \frac{c \left( \frac{M}{R} \right)^{\frac{1}{2}}}{\chi} = \frac{c}{\pi}$ for rotating particle, where $R = \sqrt{r_0^2 + r_1^2 + r_2^2 + r_3^2}$.

Using irreducible representation of $SU(2)$ group, we may write the rotational Dirac matrices $\mathcal{A}$ with quaternionic structure as

$$\mathcal{A} = (D^0 (A), D^j (A)) \; , \; (\forall j = 1, 2, 3) \; ,$$

where the matrices $D^0 (A)$ and $D^j (A)$ are quaternionic $D-$matrices, can be define as

$$D^0 (A) = \begin{pmatrix} \tau_{e_0} & 0 \\ 0 & \tau_{e_0} \end{pmatrix} \; , \; D^j (A) = \begin{pmatrix} 0 & \tau_{e_j} \\ \tau_{e_j} & 0 \end{pmatrix} \; , \; (\forall j = 1, 2, 3) \; ,$$

and the $\mathcal{B}$–matrix is

$$\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; .$$
Now, using quaternion multiplication the first term $\mathbb{A} \circ \mathbb{L}$ of QRD equation (4.2) can be expressed by

$$
\mathbb{A} \circ \mathbb{L} = e_0 \left[ D^0 (A) E_0 - \lambda D^1 (A) L_1 - \lambda D^2 (A) L_2 - \lambda D^3 (A) L_3 \right] \\
+ e_1 \left[ \lambda D^0 (A) L_1 + D^1 (A) E_0 + \lambda D^2 (A) L_3 \right] \\
+ e_2 \left[ \lambda D^0 (A) L_2 + D^2 (A) E_0 + \lambda D^3 (A) L_1 \right] \\
+ e_3 \left[ \lambda D^0 (A) L_3 + D^3 (A) E_0 + \lambda D^1 (A) L_2 \right],
$$

(4.6)

which can further reduces as

$$
\mathbb{A} \circ \mathbb{L} = e_0 \left[ D^0 (A) E_0 - \lambda \left( \overrightarrow{D} (A) \cdot \overrightarrow{L} \right) \right] \\
+ e_j \left[ \lambda D^0 (A) L_j + D^j (A) E_0 + \lambda \left( \overrightarrow{D} (A) \times \overrightarrow{L} \right)_j \right], \quad \forall \ (j = 1, 2, 3).
$$

(4.7)

Correspondingly, the second term of QRD equation (4.2) i.e. $\mathcal{B} \lambda^2 \mathbb{I}$ can be written as,

$$
\mathcal{B} \lambda^2 \mathbb{I} = e_0 \mathcal{B} \lambda^2 I_0 + e_1 \mathcal{B} \lambda^2 I_1 + e_2 \mathcal{B} \lambda^2 I_2 + e_3 \mathcal{B} \lambda^2 I_3.
$$

(4.8)

Therefore, from equations (4.7) and (4.8) the generalized QRD equation can be expressed as

$$
[e_0 \{ D^0 (A) E_0 - \lambda \left( \overrightarrow{D} (A) \cdot \overrightarrow{L} \right) \} - \mathcal{B} \lambda^2 I_0] \\
+ [e_j \{ \lambda D^0 (A) L_j + D^j (A) E_0 + \lambda \left( \overrightarrow{D} (A) \times \overrightarrow{L} \right)_j \} - \mathcal{B} \lambda^2 I_j] \circ \Psi = 0.
$$

(4.9)

Interestingly, the QRD equation (4.9) consist both scalar and vector components that gives not only the rotational energy but also gives the angular momentum of electrons.

Real quaternionic field (corresponding to $e_0$) $\mapsto$ Dirac rotational energy,

Pure quaternionic field (corresponding to $e_j$) $\mapsto$ Dirac rotational momentum.

To explain the dual nature of quaternion unified rotational energy-momentum solutions, we start with quaternionic spinor $\Psi$ with scalar and vector fields as $\mathbb{I}$,

$$
\Psi = e_0 \Psi_0 + e_1 \Psi_1 + e_2 \Psi_2 + e_3 \Psi_3 \\
= (\Psi_0 + e_1 \Psi_1) + e_2 (\Psi_2 - e_1 \Psi_3) \\
= \Psi_a + e_2 \Psi_b,
$$

(4.10)

where $\Psi_a = (\Psi_0 + e_1 \Psi_1)$ and $\Psi_b = (\Psi_2 - e_1 \Psi_3)$. Further, for two and four components form, the quaternionic spinors can be written as

$$
\Psi = \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix}, \quad \text{(two component form),}
$$

(4.11)
and
\[
\Psi = \begin{pmatrix}
\Psi_0 \\
\Psi_1 \\
\Psi_2 \\
\Psi_3
\end{pmatrix}, \quad \text{(four component form).} \tag{4.12}
\]

Now, in next section we shall use one, two and four-component spinors to determine the solutions of quaternionic rotational energy and angular momentum.

### 5 The energy solutions of QRD equation

In order to attempt the energy solutions of QRD equation, we equate the scalar components (coefficient of $e_0$) in given equation (4.9) as
\[
\left[D^0(A) E_0 - \lambda \left( \mathbf{D} \cdot \mathbf{\hat{L}} \right) - \mathcal{B} \lambda^2 I_0 \right] \Psi = 0 , \tag{5.1}
\]

substituting the value of $D^0(A)$, $\mathbf{D}(A)$ and $\mathcal{B}$ from equations (4.4) and (4.5), and obtain the following matrix form
\[
\begin{pmatrix}
E_0 - \lambda^2 I_0 & -i\lambda \left( \mathbf{\hat{e}} \cdot \mathbf{\hat{L}} \right) \\
-i\lambda \left( \mathbf{\hat{e}} \cdot \mathbf{\hat{L}} \right) & E_0 + \lambda^2 I_0
\end{pmatrix}
\begin{pmatrix}
\Psi_a \\
\Psi_b
\end{pmatrix} = 0 , \tag{5.2}
\]

which gives
\[
(E_0 - \lambda^2 I_0) \Psi_a - i\lambda \left( \mathbf{\hat{e}} \cdot \mathbf{\hat{L}} \right) \Psi_b = 0 , \tag{5.3}
\]
\[
(E_0 + \lambda^2 I_0) \Psi_b - i\lambda \left( \mathbf{\hat{e}} \cdot \mathbf{\hat{L}} \right) \Psi_a = 0 , \tag{5.4}
\]

where $\mathbf{\hat{e}} \rightarrow (e_1, e_2, e_3)$. Equations (5.3) and (5.4) are represented the positive and negative energies of a rotating Dirac particle, respectively. The values of $\Psi_a$ and $\Psi_b$ are coupled with the function of rotational energy and angular momentum $(E_0, \mathbf{\hat{L}})$, i.e.
\[
\Psi_0(E_0, \mathbf{\hat{L}}) = \frac{\lambda}{E_0 - \lambda^2 I_0} i \left( \mathbf{\hat{e}} \cdot \mathbf{\hat{L}} \right) \Psi_2(E_0, \mathbf{\hat{L}}) , \tag{5.5}
\]
\[
\Psi_1(E_0, \mathbf{\hat{L}}) = \frac{\lambda}{E_0 - \lambda^2 I_0} i \left( \mathbf{\hat{e}} \cdot \mathbf{\hat{L}} \right) \Psi_3(E_0, \mathbf{\hat{L}}) , \tag{5.6}
\]
\[
\Psi_2(E_0, \mathbf{\hat{L}}) = \frac{\lambda}{E_0 + \lambda^2 I_0} i \left( \mathbf{\hat{e}} \cdot \mathbf{\hat{L}} \right) \Psi_0(E_0, \mathbf{\hat{L}}) , \tag{5.7}
\]
\[
\Psi_3(E_0, \mathbf{\hat{L}}) = \frac{\lambda}{E_0 + \lambda^2 I_0} i \left( \mathbf{\hat{e}} \cdot \mathbf{\hat{L}} \right) \Psi_1(E_0, \mathbf{\hat{L}}) , \tag{5.8}
\]
which yields,

\[
\Psi_A(E_0, \vec{L}) - i\Omega_+(E_0) \left[ \vec{e} \cdot \vec{L} \right] \Psi_{A+2}(E_0, \vec{L}) = 0, \ (A = 0, 1) \text{ (for positive rotational energy)}
\]

\[
\Psi_A(E_0, \vec{L}) - i\Omega_-(E_0) \left[ \vec{e} \cdot \vec{L} \right] \Psi_{A-2}(E_0, \vec{L}) = 0, \ (A = 2, 3) \text{ (for negative rotational energy).
}\]

(5.9)

Equation (5.9) shows the complex behavior of quaternionic quantum wave function associated with the interaction between quaternion spin and orbital angular momentum \((\vec{e} \cdot \vec{L})\). It should be noticed that the imaginary unit \(i\) placed with quaternionic basis to represent the rotational matrices, so that, \(ie_j \equiv \tau_j, \ j = 1, 2, 3\). Here, \(\Omega_{\pm}(E_0) = \frac{\lambda}{E_0 + \lambda^2 I_0}\) is a constant used for a quaternionic rotational energy (i.e. \(\Omega_+(E_0)\) corresponding to positive energy and \(\Omega_-(E_0)\) corresponding to negative energy).

### 5.1 One component solutions

In this case, we consider \(\Psi_0 = 1, \ \Psi_1 = 0\) for spin up and \(\Psi_0 = 0, \ \Psi_1 = 1\) for spin down positive energy solutions. Then, we obtain

\[
\Psi \rightarrow \Psi_{\uparrow}^{+}(E_0, \vec{L}) = N^E_+ \left( 1 + e_2 \frac{i\lambda \left( \vec{e} \cdot \vec{L} \right)}{E_0 + \lambda^2 I_0} \right), \quad (5.10)
\]

\[
\Psi \rightarrow \Psi_{\uparrow}^{-}(E_0, \vec{L}) = N^E_+ e_1 \left( 1 + e_2 \frac{i\lambda \left( \vec{e} \cdot \vec{L} \right)}{E_0 + \lambda^2 I_0} \right), \quad (5.11)
\]

where the normalization constant \(N^E_+ = \frac{E_0 + \lambda^2 I_0}{\sqrt{(E_0 + \lambda^2 I_0)^2 + \lambda^2 L_j^2}}\). Similarly, for negative energy solutions with spin up and spin down states, we get

\[
\Psi \rightarrow \Psi_{\downarrow}^{+}(E_0, \vec{L}) = N^E_+ \left( e_2 \frac{i\lambda \left( \vec{e} \cdot \vec{L} \right)}{E_0 - \lambda^2 I_0} + e_2 \right), \quad (5.12)
\]

\[
\Psi \rightarrow \Psi_{\downarrow}^{-}(E_0, \vec{L}) = N^E_- e_1 \left( \frac{i\lambda \left( \vec{e} \cdot \vec{L} \right)}{E_0 - \lambda^2 I_0} + e_2 \right). \quad (5.13)
\]

where \(N^E_- = \frac{E_0 - \lambda^2 I_0}{\sqrt{(E_0 - \lambda^2 I_0)^2 + \lambda^2 L_j^2}}\). It should be noticed that in one component formalism, all positive and negative energy (spin up and spin down) spinors associated with quaternionic basis with the fields corresponding to particle and antiparticle.
5.2 Two component solutions

The two component solutions corresponding to positive and negative energy with spin up state are expressed in quaternionic form as,

\[
\Psi \rightarrow \Psi^{\uparrow+} (E_0, \vec{L}) = N_+^E \begin{pmatrix}
1 \\
\frac{i\lambda(\vec{e} \cdot \vec{L})}{E_0 + \lambda^2 I_0}
\end{pmatrix}
\]

\[
\Psi \rightarrow \Psi^{\uparrow-} (E_0, \vec{L}) = N_-^E \begin{pmatrix}
\frac{i\lambda(\vec{e} \cdot \vec{L})}{E_0 - \lambda^2 I_0} \\
1
\end{pmatrix}
\] (5.14)

Similarly, for positive and negative energy with spin down states,

\[
\Psi \rightarrow \Psi^{\downarrow+} (E_0, \vec{L}) = N_+^E e_1 \begin{pmatrix}
1 \\
\frac{-i\lambda(\vec{e} \cdot \vec{L})}{E_0 + \lambda^2 I_0}
\end{pmatrix} \simeq -iN_+^E \begin{pmatrix}
\frac{-i\lambda(\vec{e} \cdot \vec{L})}{E_0 + \lambda^2 I_0} \\
1
\end{pmatrix} .
\]

\[
\Psi \rightarrow \Psi^{\downarrow-} (E_0, \vec{L}) = -N_-^E e_1 \begin{pmatrix}
\frac{-i\lambda(\vec{e} \cdot \vec{L})}{E_0 - \lambda^2 I_0} \\
1
\end{pmatrix} \simeq iN_-^E \begin{pmatrix}
\frac{-i\lambda(\vec{e} \cdot \vec{L})}{E_0 - \lambda^2 I_0} \\
1
\end{pmatrix} .
\] (5.15)

Notice that, the quaternionic basis element \( e_3 \) shows the diagonal matrix which does not able to change the state of orientation of spin-1/2 particles, while the basis \( e_1 \) and \( e_2 \) show off-diagonal matrices whose can transform the state of orientation of spin-1/2 particles. The quaternionic unified of two component solution can now be written as,

\[
\Psi_{\text{Unified}}(E_0, \vec{L}) \simeq N_+^E \left\{ \begin{pmatrix}
1 \\
\frac{i\lambda(\vec{e} \cdot \vec{L})}{E_0 + \lambda^2 I_0}
\end{pmatrix} - i \begin{pmatrix}
\frac{-i\lambda(\vec{e} \cdot \vec{L})}{E_0 + \lambda^2 I_0} \\
1
\end{pmatrix} \right\} \\
+ N_-^E \left\{ \begin{pmatrix}
\frac{i\lambda(\vec{e} \cdot \vec{L})}{E_0 - \lambda^2 I_0} \\
1
\end{pmatrix} + i \begin{pmatrix}
\frac{-i\lambda(\vec{e} \cdot \vec{L})}{E_0 - \lambda^2 I_0} \\
1
\end{pmatrix} \right\} .
\] (5.18)

The unified spinor-function of two component solution shows a complex behavior of quaternionic field where the each component predicts the energy solution for particles (or anti particles) with their spinor.

5.3 Four component solutions

Like one and two component solutions, we can extent it into four component solutions. In this case, we obtain the quaternionic four component solutions for positive energy with spin up and down states as

\[
\Psi \rightarrow \Psi^{\uparrow+} (E_0, \vec{L}) \simeq N_+^E \begin{pmatrix}
1 \\
0 \\
\frac{i\lambda(\vec{e} \cdot \vec{L})}{E_0 + \lambda^2 I_0} \\
0
\end{pmatrix} , \quad \Psi \rightarrow \Psi^{\downarrow+} (E_0, \vec{L}) \simeq N_+^E \begin{pmatrix}
0 \\
1 \\
\frac{-i\lambda(\vec{e} \cdot \vec{L})}{E_0 + \lambda^2 I_0} \\
0
\end{pmatrix} .
\] (5.19)
Similarly, the negative energy solutions for spin up and down states are

\[
\Psi \rightarrow \Psi_{E}^{-}(E_{0}, \vec{L}) \simeq N_{E}^{-} \begin{pmatrix}
\frac{i\lambda(\vec{\gamma} \cdot \vec{L})}{E_{0} - \lambda^{2}I_{0}} \\
0 \\
1 \\
0
\end{pmatrix}, \quad \Psi \rightarrow \Psi_{E}^{-}(E_{0}, \vec{L}) \simeq N_{E}^{-} \begin{pmatrix}
0 \\
-\frac{i\lambda(\vec{\gamma} \cdot \vec{L})}{E_{0} - \lambda^{2}I_{0}} \\
0 \\
1
\end{pmatrix} \quad (5.20)
\]

Interestingly, the one, two and four component solutions are isomorphic to each other. In quaternionic formalism, the scalar term \((\vec{e} \cdot \vec{L})\) can be used as the rotational helicity of particle, which represents the quaternionic spin-orbit interaction energy. In right hand rotational helicity the direction of spin is along to the direction of quaternionic angular momentum, while for left hand rotational helicity the direction of spin is opposite to the direction of quaternionic angular momentum.

6 The momentum solutions of QRD equation

To discuss the momentum like solutions of QRD equation \(4.9\), we compare the quaternionic vector coefficient \(e_{j} (\forall j = 1, 2, 3)\) as,

\[
\left[ \lambda D^{0} (A) L_{j} + D^{j} (A) E_{0} + \lambda \left( \vec{D} \times \vec{L} \right)_{j} - \mathcal{B} \lambda^{2} I_{j} \right] \Psi = 0 \quad (6.1)
\]

Now, putting the values of \(D^{0} (A), D^{j} (A)\) and \(\mathcal{B}\) from equations \(4.4\) and \(4.5\) in given equation \(6.1\), we get

\[
\begin{pmatrix}
\lambda (L_{j} - \lambda I_{j}) \\
i e_{j} E_{0} + \lambda \left( \vec{e} \times \vec{L} \right)_{j}
\end{pmatrix}
\begin{pmatrix}
\Psi_{a} \\
\Psi_{b}
\end{pmatrix} = 0 \quad (6.2)
\]

which gives

\[
\lambda (L_{j} - \lambda I_{j}) \Psi_{a} + i \left[ e_{j} E_{0} + \lambda \left( \vec{e} \times \vec{L} \right)_{j} \right] \Psi_{b} = 0 \quad (6.3)
\]

\[
\lambda (L_{j} + \lambda I_{j}) \Psi_{b} + i \left[ e_{j} E_{0} + \lambda \left( \vec{e} \times \vec{L} \right)_{j} \right] \Psi_{a} = 0 \quad (6.4)
\]

Equations \(6.3\) and \(6.4\) are identical to vector analogy of Dirac’s energy equations called generalized quaternionic angular momentum equations which can describe respectively the angular momentum of a electron and positron. Substituting the values of \(\Psi_{a}\) and \(\Psi_{b}\), we obtain the
following equations:

\[
\Psi_0(E_0, \vec{L}) = \frac{1}{i \lambda (L_j - \lambda I_j)} \left[ e_j E_0 + \lambda \left( \vec{e} \times \vec{L} \right)_j \right] \Psi_2(E_0, \vec{L}) , \tag{6.5}
\]

\[
\Psi_1(E_0, \vec{L}) = \frac{1}{i \lambda (L_j - \lambda I_j)} \left[ e_j E_0 + \lambda \left( \vec{e} \times \vec{L} \right)_j \right] \Psi_3(E_0, \vec{L}) , \tag{6.6}
\]

\[
\Psi_2(E_0, \vec{L}) = \frac{1}{i \lambda (L_j + \lambda I_j)} \left[ e_j E_0 + \lambda \left( \vec{e} \times \vec{L} \right)_j \right] \Psi_0(E_0, \vec{L}) , \tag{6.7}
\]

\[
\Psi_3(E_0, \vec{L}) = \frac{1}{i \lambda (L_j + \lambda I_j)} \left[ e_j E_0 + \lambda \left( \vec{e} \times \vec{L} \right)_j \right] \Psi_1(E_0, \vec{L}) , \tag{6.8}
\]

In the simplified form the above equations reduce to,

\[
\Psi_\lambda(E_0, \vec{L}) + i \Omega_+(\vec{L}) \left[ e_j E_0 + \lambda \left( \vec{e} \times \vec{L} \right)_j \right] \Psi_{\lambda+2}(E_0, \vec{L}) = 0 , (\lambda = 0, 1)
\]

\[
\Psi_\lambda(E_0, \vec{L}) + i \Omega_-(\vec{L}) \left[ e_j E_0 + \lambda \left( \vec{e} \times \vec{L} \right)_j \right] \Psi_{\lambda-2}(E_0, \vec{L}) = 0 , (\lambda = 2, 3) . \tag{6.9}
\]

Here, \( \Omega_\pm(\vec{L}) = \frac{1}{\lambda (L_j \pm \lambda I_j)} \) is a rotational variable, can be used as \( \Omega_+(\vec{L}) \) for particle and \( \Omega_-(\vec{L}) \) for anti-particle angular momentum. The term \( \left( \vec{e} \times \vec{L} \right) \) shows a directional interaction between quaternion spin and orbital angular momentum. Now, using equations (6.5)-(6.8) we can analysis one, two and four component solutions for QRD equation.

### 6.1 One component solutions

We can start the angular momentum solutions with spin up state as \( \Psi_0 = 1 \) and \( \Psi_1 = 0 \) and spin down states as \( \Psi_0 = 0 \) and \( \Psi_1 = 1 \) of Dirac-particle, and obtain

\[
\Psi \rightarrow \Psi^{+\dagger}(E_0, \vec{L}) = N^+_\pm \left( \frac{i \left[ e_j E_0 + \lambda \left( \vec{e} \times \vec{L} \right)_j \right]}{\lambda (L_j + \lambda I_j)} \right), \tag{6.10}
\]

\[
\Psi \rightarrow \Psi^{-\dagger}(E_0, \vec{L}) = N^-_\pm \left( \frac{i \left[ e_j E_0 + \lambda \left( \vec{e} \times \vec{L} \right)_j \right]}{\lambda (L_j + \lambda I_j)} \right), \tag{6.11}
\]

where \( N^+_\pm = \sqrt{\frac{\lambda (L_j + \lambda I_j)}{[\lambda (L_j + \lambda I_j)]^2 - \left[ e_j E_0 + \lambda \left( \vec{e} \times \vec{L} \right)_j \right]^2}} \). Correspondingly, the angular momentum solutions for rotating Dirac anti-particle, we take \( \Psi_2 = 1 \) and \( \Psi_3 = 0 \) for spin up state and \( \Psi_2 = 0 \)
and $\Psi_3 = 1$ for spin down states, then

$$\Psi \rightarrow \Psi^{\uparrow-}(E_0, \vec{L}) = N_L^+ \left( -i \frac{e_j E_0 + \lambda (\vec{\sigma} \times \vec{L})}{\lambda (L_j - \lambda I_j)} + e_2 \right),$$  \hspace{1cm} (6.12)$$

$$\Psi \rightarrow \Psi^{\downarrow-}(E_0, \vec{L}) = N_L^- \left( -e_1 \frac{e_j E_0 + \lambda (\vec{\sigma} \times \vec{L})}{\lambda (L_j - \lambda I_j)} - e_2 e_1 \right),$$  \hspace{1cm} (6.13)$$

where $N_L^+ = \frac{\lambda (L_j - \lambda I_j)}{\sqrt{[\lambda (L_j - \lambda I_j)]^2 - e_j E_0 + \lambda (\vec{\sigma} \times \vec{L})}}$. Like energy solutions, the quaternionic angular momentum solutions of one component describe the rotational momentum field for Dirac particle and anti-particle.

### 6.2 Two component solutions

For the study of quaternionic two component angular momentum solutions, we extant one component angular momentum solutions into two component solution by using equation (4.11). Thus, the quaternionic two component solutions corresponding to spin up and spin down states of particle and anti-particle are expressed by,

$$\Psi \rightarrow \Psi^{\uparrow+}(E_0, \vec{L}) = N_L^+ \left( -i \frac{e_j E_0 + \lambda (\vec{\sigma} \times \vec{L})}{\lambda (L_j + \lambda I_j)} \right),$$  \hspace{1cm} (6.14)$$

$$\Psi^{\uparrow-}(E_0, \vec{L}) = N_L^- \left( -e_1 \frac{e_j E_0 + \lambda (\vec{\sigma} \times \vec{L})}{\lambda (L_j + \lambda I_j)} \right),$$  \hspace{1cm} (6.15)$$

$$\Psi^{\downarrow+}(E_0, \vec{L}) = -i N_L^+ \left( i \frac{e_j E_0 + \lambda (\vec{\sigma} \times \vec{L})}{\lambda (L_j + \lambda I_j)} \right),$$  \hspace{1cm} (6.16)$$

$$\Psi^{\downarrow-}(E_0, \vec{L}) = i N_L^- \left( i \frac{e_j E_0 + \lambda (\vec{\sigma} \times \vec{L})}{\lambda (L_j - \lambda I_j)} \right).$$  \hspace{1cm} (6.17)$$

The two component solutions of quaternionic angular momentum show that how spinors are associated with the rotational motion of particle and anti-particle.
6.3 Four component solutions

Further, we also may write the four component solutions of quaternionic angular momentum for spin up and spin down states of Dirac-particle as,

\[
\Psi \rightarrow \Psi^{\uparrow}(E_0, \vec{L}) \simeq N_+^L \begin{pmatrix}
1 \\
0 \\
-i \left[ e_j E_0 + \lambda (\vec{e} \times \vec{L})_j \right] \\
\lambda (L_j + \lambda I)
\end{pmatrix},
\]

(6.18)

\[
\Psi^{\downarrow}(E_0, \vec{L}) \simeq N_+^L 
\begin{pmatrix}
0 \\
1 \\
0 \\
i \left[ e_j E_0 + \lambda (\vec{e} \times \vec{L})_j \right] \\
\lambda (L_j - \lambda I)
\end{pmatrix},
\]

(6.19)

and for anti-particle, we obtain

\[
\Psi \rightarrow \Psi^{\downarrow}(E_0, \vec{L}) \simeq N_-^L \begin{pmatrix}
-i \left[ e_j E_0 + \lambda (\vec{e} \times \vec{L})_j \right] \\
\lambda (L_j - \lambda I) \\
0 \\
1 \\
0
\end{pmatrix},
\]

(6.20)

\[
\Psi^{\uparrow}(E_0, \vec{L}) \simeq N_-^L \begin{pmatrix}
0 \\
1 \\
0 \\
i \left[ e_j E_0 + \lambda (\vec{e} \times \vec{L})_j \right] \\
\lambda (L_j - \lambda I)
\end{pmatrix}.
\]

(6.21)

Thus, in quaternionic formalism, one, two and four component angular momentum solutions are isomorphic to each other. The interesting part in quaternionic description for Dirac equation is, it shows not only rotational energy solution but also shows the rotational momentum solutions for considering the four-dimensional Euclidean space-time.

7 Quaternionic angular frequency and wave propagation vector

We know that an electron rotate in a permissible orbit exhibits the wave-nature. In order to calculate the angular frequency and wave propagation vector of electron and positron in quaternionic space-time, let us start with the quaternionic wave function \( \Psi \) consisting to \( SU(2) \) group elements as \([42]\),

\[
\Psi = \sum_{J=\frac{1}{2}}^{\infty} (2J + 1) \sum_{M=-J}^{J} T^{\mu \nu}_{M} \varphi_{S,M}^{J} \forall (\mu = 0, 1, 2, 3),
\]

(7.1)
where $J$, $M$ and $S$ are denoted the total angular momentum, magnetic quantum number due to total angular momentum and spin quantum number, respectively. Here $(2J + 1)$ defines the discrete value of possible total angular momentum called the statistical weight, $T^j_M$ is a quaternionic variable associated with $SU(2)$ group and $\mathcal{D}^j_{S=\pm \frac{1}{2},M}$ is the Dirac spinor. Thus, in quaternionic form, we have

$$\Psi_0 = \sum_{J=\frac{1}{2}}^{\infty} (2J + 1) \sum_{M=-J}^{J} T^j_M \mathcal{D}^j_{\frac{1}{2},M}, \quad \text{(corresponding to $e_0$)} \quad (7.2)$$

$$\Psi_1 = \sum_{J=\frac{1}{2}}^{\infty} (2J + 1) \sum_{M=-J}^{J} T^j_M \mathcal{D}^j_{\frac{1}{2},M}, \quad \text{(corresponding to $e_1$)} \quad (7.3)$$

$$\Psi_2 = \sum_{J=\frac{1}{2}}^{\infty} (2J + 1) \sum_{M=-J}^{J} T^j_M \mathcal{D}^j_{\frac{1}{2},M}, \quad \text{(corresponding to $e_2$)} \quad (7.4)$$

$$\Psi_3 = \sum_{J=\frac{1}{2}}^{\infty} (2J + 1) \sum_{M=-J}^{J} T^j_M \mathcal{D}^j_{\frac{1}{2},M}, \quad \text{(corresponding to $e_3$)} \quad (7.5)$$

Using above quaternionic rotational wave functions on equations (5.5) - (5.8), we obtain

$$i\hbar T^0_M - \lambda^2 I_0 T^0_M - i\lambda \left( \vec{e} \cdot \vec{L} \right) T^2_M = 0, \quad (7.6)$$

$$i\hbar T^1_M - \lambda^2 I_0 T^1_M - i\lambda \left( \vec{e} \cdot \vec{L} \right) T^3_M = 0, \quad (7.7)$$

$$i\hbar T^2_M - \lambda^2 I_0 T^2_M - i\lambda \left( \vec{e} \cdot \vec{L} \right) T^0_M = 0, \quad (7.8)$$

$$i\hbar T^3_M - \lambda^2 I_0 T^3_M - i\lambda \left( \vec{e} \cdot \vec{L} \right) T^1_M = 0, \quad (7.9)$$

where we used energy operator $E_0 \sim i\hbar \frac{\partial}{\partial t}$. On the other hand, we may consider the general plane wave solution of equation (7.1) in terms of quaternionic form as,

$$T^\mu_M = G^\mu_M \exp \left[ -\frac{i}{\hbar} (\mathbb{P} \cdot \mathbb{R}) \right], \quad (7.10)$$

Here $G^\mu_M$ is a constant, $\mathbb{P}$ and $\mathbb{R}$ are usual quaternionic four-momentum and four-space, respectively. The scalar and vector components of quaternionic equation (7.10) are

$$T^0_M = G^0_M \exp \left[ -\frac{i}{\hbar} (E_0 t - \vec{p} \cdot \vec{r}) \right], \quad \text{(Coefficient of $e_0$)} \quad (7.11)$$

$$T^a_M = G^a_M \exp \left[ -\frac{i}{\hbar} \left( ct \vec{p} + \frac{E_0}{c} \vec{r} - (\vec{p} \times \vec{r}) \right) \right], \quad \text{(Coefficient of $e_a$)} \quad (7.12)$$
where \( a = 1, 2, 3 \). Now, substituting equations (7.11) and (7.12) and their time derivative in equations (7.0) - (7.9), we found

\[
\begin{align*}
&(-\hbar \omega - \lambda^2 I_0) G^{0J}_M - i\lambda \left( \vec{\omega} \cdot \vec{L} \right) G^{2J}_M = 0 , \quad (7.13) \\
&(-\hbar \omega - \lambda^2 I_0) G^{1J}_M - i\lambda \left( \vec{\omega} \cdot \vec{L} \right) G^{3J}_M = 0 , \quad (7.14) \\
&(-\hbar \omega + \lambda^2 I_0) G^{2J}_M - i\lambda \left( \vec{\omega} \cdot \vec{L} \right) G^{0J}_M = 0 , \quad (7.15) \\
&(-\hbar \omega + \lambda^2 I_0) G^{3J}_M - i\lambda \left( \vec{\omega} \cdot \vec{L} \right) G^{1J}_M = 0 . \quad (7.16)
\end{align*}
\]

Equations (7.13) and (7.14) are manifested to negative energy solution while equations (7.15) and (7.16) are manifested to positive energy of Dirac particle. These dual-energy equations can be reduced in 4 \times 4 matrix form as,

\[
\begin{pmatrix}
-\hbar \omega - \lambda^2 I_0 & 0 & -i\lambda \left( \vec{\omega} \cdot \vec{L} \right) & 0 \\
0 & -\hbar \omega - \lambda^2 I_0 & 0 & -i\lambda \left( \vec{\omega} \cdot \vec{L} \right) \\
-i\lambda \left( \vec{\omega} \cdot \vec{L} \right) & 0 & -\hbar \omega + \lambda^2 I_0 & 0 \\
0 & -i\lambda \left( \vec{\omega} \cdot \vec{L} \right) & 0 & -\hbar \omega + \lambda^2 I_0
\end{pmatrix}
\begin{pmatrix}
G^{0J}_M \\
G^{1J}_M \\
G^{2J}_M \\
G^{3J}_M
\end{pmatrix} = 0 , \quad (7.17)
\]

which gives

\[
\omega : \rightarrow \omega_{\pm} = \pm \sqrt{\lambda^2 I_0^2 - \hbar^2 \left( \vec{\omega} \cdot \vec{\nabla}_\Theta \right)^2} \frac{1}{\hbar} . \quad (7.18)
\]

Here, we used angular momentum \( \vec{L} \rightarrow -i\hbar \vec{\nabla}_\Theta \), where \( \vec{\nabla}_\Theta \) shows the rotational analog of nabla operator corresponding to Euler angles \((\theta, \phi, \psi)\)\(^{15}\). The term \( \omega_{\pm} \) represented the angular frequency of the Dirac like particle while the \( \omega_- \) represented the angular frequency of its anti-particle. Similarly, for the general solution of angular momentum equations, we substitute equations (7.11) and (7.12) in equations (6.5) - (6.8), and obtain

\[
\begin{align*}
\lambda (L_j - \lambda I_j) G^{0J}_M + i \left[ e_j \left( \vec{c} \hat{k} \right) + \lambda \left( \vec{\omega} \times \vec{L} \right) \right] G^{2J}_M = 0 , \quad (7.19) \\
\lambda (L_j - \lambda I_j) G^{1J}_M + i \left[ e_j \left( \vec{c} \hat{k} \right) + \lambda \left( \vec{\omega} \times \vec{L} \right) \right] G^{3J}_M = 0 , \quad (7.20) \\
\lambda (L_j + \lambda I_j) G^{2J}_M + i \left[ e_j \left( \vec{c} \hat{k} \right) + \lambda \left( \vec{\omega} \times \vec{L} \right) \right] G^{0J}_M = 0 , \quad (7.21) \\
\lambda (L_j + \lambda I_j) G^{3J}_M + i \left[ e_j \left( \vec{c} \hat{k} \right) + \lambda \left( \vec{\omega} \times \vec{L} \right) \right] G^{1J}_M = 0 . \quad (7.22)
\end{align*}
\]
where the wave propagation vector $\vec{k} \sim \frac{\hbar}{E}$. These equations also can be written in $4 \times 4$ matrix form as,

$$
\begin{pmatrix}
A_- & 0 & B & 0 \\
0 & A_- & 0 & B \\
B & 0 & A_+ & 0 \\
0 & B & 0 & A_+
\end{pmatrix}
\begin{pmatrix}
G_{MI}^0 \\
G_{MI}^1 \\
G_{MI}^2 \\
G_{MI}^3
\end{pmatrix} = 0 ,
$$

(7.23)

along with

$$A_{\pm} = \lambda (L_j \pm \lambda I_j) ,\; B = i \left[ e_j \left( c \vec{k} \right) + \lambda \left( \vec{\theta} \times \vec{L} \right) \right] .
$$

(7.24)

Therefore, from equation (7.23) we obtain

$$\vec{k} :\mapsto \vec{k}_\pm = \pm \frac{ie_j \lambda \sqrt{(-\hbar^2 \nabla^2 - \lambda^2 I_j)} + i\hbar \lambda \left( \vec{\theta} \times \vec{\nabla} \Theta \right) j}{c} ,\; (\forall j = 1, 2, 3) .
$$

(7.25)

Equation (7.25) represented an expression for wave vector $\left( \vec{k} \right)$ corresponding to quaternionic angular momentum that propagates along quaternionic basis $e_j$. Accordingly, $\vec{k}_+$ can represent the wave propagation for Dirac like particle while $\vec{k}_-$ can represent the wave propagation corresponding to the anti-particle. Here, we should be notice that $(\omega, \vec{k})$ shows the quaternionic four-wave vector for Euclidean space.

8 Conclusion

In the present work, the generalized Dirac equation for rotating particle has been demonstrated in term of quaternionic division algebra. Split quaternion is the another variety of quaternion. The interesting part of split quaternion is that, it can use not only the wave-mechanism (Schrödinger theory) but also use for matrix-mechanism (Heisenberg theory) of quantum formalism. The matrix realization of split quaternion shows the Pauli’s spin state of fermions or anti-fermions. Therefore, to visualizing the rotational properties of Dirac-like particles split quaternions or quaternions algebra can be used for Euclidean space-time. In quaternionic field, we have discussed four angular momentum consisted rotational energy and rotational momentum of an electron in four dimensional Euclidean space-time. The connection between rotational matrices (tau-matrices) along with quaternionic basis $(e_0, e_1, e_2, e_3)$ has been described. Further, the quaternionic four-masses have been associated with the rest mass corresponding to quaternionic scalar basis and the moving mass corresponding to quaternionic vector basis given by equation (3.1). We have also written the quaternionic relativistic four-space, moment of inertia and angular momentum. The components of quaternionic resultant rotational energy and rotational momentum are established in compact and simple manner given by equations (3.8) and (3.9). We have investigated a new form of QRD equation (4.9) that unifies the rotational form of Dirac-energy and Dirac- angular momentum of a particle in a single framework.
The solutions of QRD equation has been represented in one, two and four components form which described the rotational motion of Dirac particle and anti-particle with spin up and down states. Further, we also have discussed a general form of quaternionic wave function and its plane wave solutions in terms of quaternionic field. The components of rotational energy solution of quaternionic wave function leads to rotational frequency (7.18) while the components of rotational momentum solution of quaternionic wave leads to wave propagation vector (7.25) for Dirac spin-1/2 particles (or anti-particles). Interestingly, the QRD equation for the rotating particles also fulfills the conservation of quaternionic four rotational-momentum. Therefore, we can conclude that the benefit of quaternionic formalism is very important to study the rotating quantum particles as fermions in four-dimensional Euclidean space-time. This theory can be extending to discuss the behavior of rotating other subatomic particles in terms of quaternionic field.

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