FRACTIONAL OPTIMAL CONTROL PROBLEMS ON A STAR GRAPH: OPTIMALITY SYSTEM AND NUMERICAL SOLUTION

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Abstract. In this paper, we study optimal control problems for nonlinear fractional order boundary value problems on a star graph, where the fractional derivative is described in the Caputo sense. The adjoint state and the optimality system are derived for fractional optimal control problem (FOCP) by using the Lagrange multiplier method. Then, the existence and uniqueness of solution of the adjoint equation is proved by means of the Banach contraction principle. We also present a numerical method to find the approximate solution of the resulting optimality system. In the proposed method, the $L^2$ scheme and the Grünwald-Letnikov formula is used for the approximation of the Caputo fractional derivative and the right Riemann-Liouville fractional derivative, respectively, which converts the optimality system into a system of linear algebraic equations. Two examples are provided to demonstrate the feasibility of the numerical method.

1. Introduction. This paper presents a formulation and analysis of a fractional optimal control problem (FOCP) on a metric star graph together with the corresponding fractional order optimality system as well as a numerical scheme for such (FOCPs). By a graph, we mean $\mathcal{G} = (V,E)$, consisting of a finite set of nodes $V = \{v_i : i = 0, 1, 2, \ldots, k\}$ and a set of edges $E$ (interpreted as structural elements like strings, heat conducting elements etc.) connecting these nodes. In contrary to discrete graphs, the graph considered in this work (i.e. star graph) is a metric graph [30], in the sense, that the edges are continuous curves. Therefore, each edge $e_i$ is parametrised by an interval $(0, l_i)$, $i = 1, 2, \ldots, k$ and we take $l_i = 1$ for every $i$, i.e., each edge is parametrised by $(0, 1)$. On edge $e_i = \overrightarrow{v_iv'_i}$, $i = 1, 2, \ldots, k$, we consider a local coordinate system with $v_i$ as origin and $x \in (0, l_i)$ as the coordinate. We

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will study the following optimal control problem on a graph $G$:

$$
\text{Min } J(y, u) = \frac{1}{2} \sum_{i=1}^{k} \int_{0}^{1} (y_i(x) - y^d_i(x))^2 dx + \frac{\mu}{2} \sum_{i=1}^{k} u_i^2, \quad (1)
$$

subject to the system of fractional ordinary differential equations along the edges

$$
cD^\alpha_{0,x} y_i(x) = f_i(x, y_i(x)), \quad 0 < x < 1, \quad 1 < \alpha < 2, \quad i = 1, 2, \ldots, k, \quad (2)
$$

and the boundary conditions

$$
y_i(0) = u_i, \quad i = 1, 2, \ldots, k,
$$

$$
y_i(1) = y_j(1), \quad i, j = 1, 2, \ldots, k, \quad i \neq j,
$$

$$
\sum_{i=1}^{k} y'_i(1) = 0. \quad (3)
$$

In (1)-(3), $cD^\alpha_{0,x}$ is the Caputo fractional derivative of order $\alpha$, $y(x) = (y_1(x), y_2(x), \ldots, y_k(x))$ is a state variable, $u = (u_1, u_2, \ldots, u_k) \in \mathbb{R}^k$ is the control on the boundary nodes as shown in Figure 1, $y^d_i(x), \quad i = 1, 2, \ldots, k$ is the desired function on $(0,1)$, $\mu > 0$ is a parameter and $f_i, \quad i = 1, 2, \ldots, k$, are continuously differentiable functions in each variable on $[0,1] \times \mathbb{R}$. Recently in [27], authors proved the existence and uniqueness of the solution (2), with the homogeneous Dirichlet conditions on the boundary nodes in (3). Hence, the present work could be seen as the extension of [27] to fractional optimal control problems on a star graph. Apart from their (nonlinear FBVPs) natural physical significance, the motivation to consider such problems comes from the fact that if we consider optimal control problems driven by fractional wave equations (i.e. hyperbolic equations), then, after a semi-discretization in space, hyperbolic systems may be viewed as sequences of FBVPs. Since fractional wave equations have numerous applications such as charge transmission in amorphous semiconductors [38], propagation of mechanical waves in viscoelastic media [26], etc. it is imperative to initiate the study of optimal control problems for FBVPs.

**Figure 1.** A sketch of the star graph with $k$ edges along with boundary control.
Application of differential equation to the modelling of problems in networked domains, particularly in ecology, engineering, biology etc. has lead to Lumer’s work \[24\] on ramification spaces. Optimal control problem on metric graphs has been studied by various authors. For instance, the general dynamic networks of strings and beams and their control properties were examined by Lagnese et al. in \[18\], while in \[20\], networks of elastic strings were discussed along with optimal control problems and domain decomposition. A nice survey of results for different problems on 1-d flexible multi-structures (i.e., 1-d metric graphs) along with optimal control problems was provided by Dager and Zuazua in \[9\]. Optimal control problems for semi-linear elliptic equations on metric graphs with application to gas network is studied by G. Leugering \[21\]. For more details about differential equations and optimal control problem on metric graphs, as well as their applications, we refer to \[15, 16, 17, 19, 32, 39\] and the references therein.

On the other hand, fractional calculus finds its importance in different fields of science and engineering \[12, 25, 8, 31\]. Fractional order dynamical systems appear in several works, especially in viscoelasticity and in hereditary solid mechanics. Recently, various practical applications of the fractional calculus using Caputo fractional derivative has been addressed by Qureshi et al. in \[6, 33, 34, 35, 36\]. Considerable work has been done in the area of fractional optimal control problem on the real line. For instance, in \[2\], O. P. Aggarwal considered the FOCPs with the Riemann-Liouville derivative and obtained the necessary optimality conditions for FOCP, and also provided the numerical method for solving the resulting equations. This led to an extensive study of optimal control of fractional systems. In \[5\], Almeida et al. obtained the necessary and sufficient optimality condition for FOCPs with the Caputo fractional derivative. T. L Guo \[11\] considered the fractional optimal control problem in the sense of Caputo and obtained the second order optimality condition for FOCPs. Moreover, the FOCPs with JI Huan He’s fractional derivative was considered in \[37\], where the authors derived the necessary and optimality conditions and solved it numerically using Legendre multiwavelet collocation method. For more results on optimal control problems, we refer to \[1, 4, 14, 23, 10, 29\] and reference therein.

Recently in \[28\], optimal control of a fractional Sturm-Liouville problem on a star graph has been studied, where the composition of right Caputo and left Riemann-Liouville fractional operators of order $\alpha \in (0,1)$ has been considered. The existence of solutions to boundary optimal control problem and sufficient optimality conditions are obtained. However, the authors have not proposed a numerical scheme for the approximation of the optimal control problem. To the best of our knowledge, no work has been published for the fractional optimal control problem on a metric graph with the Caputo derivative which deals with both the theoretical and numerical aspects of the problem. In this paper, we consider the FOCP on a metric star graph. The adjoint state (defined using right Riemann-Liouville derivative) and optimality system are obtained for the considered optimal control problem. We prove the existence and uniqueness of adjoint system by means of fixed point theory which itself is a new result in the field of fractional calculus with Riemann-Liouville fractional derivatives. Moreover, we propose the numerical method to find the approximate solution of optimality system having left Caputo and right Riemann-Liouville fractional derivatives. The proposed method can be easily applied to find the numerical solution of FBVPs on metric graphs (i.e. state equation) and fractional optimal control problems on intervals (i.e. single edge).
The rest of the paper is organised as follows: In Sec. 2, we recall some basic definitions of fractional calculus and their properties. Then, in Sec. 3.1, we derive the adjoint state and optimality system for FOCP. The existence and uniqueness of the solution of adjoint equation are proved in Sec. 3.2. In Sec. 4, we present a numerical method for the approximation of optimality system. In Sec. 5, we give two examples to demonstrate the feasibility of the proposed numerical method. Sec. 6 concludes the work done and gives a brief idea of the future direction.

2. Preliminaries. First of all, we recall some basic definitions and known results of fractional integrals and fractional derivatives.

Definition 2.1 (Left and right fractional integrals). The left and right fractional integrals of order $\alpha > 0$ for a continuous function $f$ on $[a, b]$ are defined by

$$D_{a,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \left( \int_{a}^{x} (x-t)^{\alpha-1} f(t) dt \right),$$

and

$$D_{x,b}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \left( \int_{x}^{b} (t-x)^{\alpha-1} f(t) dt \right),$$

respectively, where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2 (Left and right Riemann-Liouville fractional derivatives). The left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ for a function $f$ on $[a, b]$ are defined by

$$RLD_{a,x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d^{n}}{dx^{n}} \int_{a}^{x} (x-t)^{n-\alpha-1} f(t) dt \right),$$

and

$$RLD_{x,b}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} (-1)^{n} \left( \frac{d^{n}}{dx^{n}} \int_{x}^{b} (t-x)^{n-\alpha-1} f(t) dt \right),$$

respectively, where the function $f(x)$ has absolutely continuous derivative up to order $(n-1)$, $n-1 \leq \alpha < n$, $n \in \mathbb{N}$.

Definition 2.3 (Caputo fractional derivative). The Caputo fractional derivative of order $\alpha > 0$ for a function $f$ on $[a, b]$ is defined as:

$$CD_{a,x}^{\alpha} f(x) = RLD_{a,x}^{\alpha} \left[ f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^{k} \right],$$

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$.

Remark 1. If function $f$ has $C^{n}$ regularity i.e. $f(x) \in C^{n}[a, b]$, then the Caputo fractional derivative of order $\alpha > 0$ is defined as

$$CD_{a,x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left( \int_{a}^{x} (x-t)^{n-\alpha-1} f^{(n)}(t) dt \right),$$

where $n$ is same as defined in equation (8).

Remark 2. [40] If $\alpha > \beta > 0$ and $f \in C[a, b]$, then
Remark 3. [13] If \( \alpha, \beta > 0 \), then

(i) \( D_{a,x}^{-\alpha}D_{x,b}^{-\beta}f(x) = D_{a,x}^{-\alpha}(D_{x,b}^{-\beta}f(x)) \) and \( D_{a,x}^{-\alpha}D_{x,b}^{-\beta}f(x) = D_{x,b}^{-\alpha}(D_{a,x}^{-\beta}f(x)) \).

(ii) \( RL D_{a,x}^{-\alpha}D_{x,b}^{-\beta}f(x) = D_{a,x}^{-\alpha}(RL D_{x,b}^{-\beta}f(x)) \) and \( RL D_{a,x}^{-\alpha}D_{x,b}^{-\beta}f(x) = D_{x,b}^{-\alpha}(RL D_{a,x}^{-\beta}f(x)) \).

Theorem 2.4. Let \( f_i : [0,1] \times \mathbb{R} \to \mathbb{R}, i = 1,2,\ldots,k \) be continuous functions satisfying the conditions

\[
|f_i(t,x) - f_i(t,x_1)| \leq L_i|x - x_1|, \quad L_i > 0, \quad t \in [0,1],
\]

then BVP (2)-(3) has a unique solution in the product space \( \Omega^k \) if

\[
\left[ \frac{2k}{\Gamma(\alpha + 1)} + \frac{2}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha + 1)} \right] \left( \sum_{i=1}^{k} L_i \right) < 1,
\]

where \( \Omega = C[0,1] \) is the space of all continuous functions on \([0,1]\).

Proof. The proof can be obtained using similar analysis given in Theorem 3.1 of [27], by taking \( l_i = 1 \) for every \( i = 1,2,\ldots,k \). Therefore, the proof is omitted. \( \square \)

Now, we give the following lemma related to integration by parts formula [3] for fractional derivatives which will be used later.

Lemma 2.5 (Fractional integration by parts formula). Let \( \alpha > 0, f,g : [a,b] \to \mathbb{R} \) be \( C^n \) functions, then

\[
\int_a^b g(x) \cdot C D_{x,a}^\alpha f(x) dx = \int_a^b f(x) \cdot RL D_{x,a}^\alpha g(x) dx
\]

\[
+ \sum_{j=0}^{n-1} \left[ RL D_{x,b}^{\alpha+j-n} g(x) \cdot RL D_{x,b}^{\alpha-1-j} f(x) \right]_a^b,
\]

where \( RL D_{x,b}^k g(x) = D_{x,b}^{\alpha-(k)} g(x) \) if \( k < 0 \). Therefore, if \( 1 < \alpha < 2 \), we obtain

\[
\int_a^b g(x) \cdot C D_{x,a}^\alpha f(x) dx = \int_a^b f(x) \cdot RL D_{x,a}^\alpha g(x) dx
\]

\[
+ \left[ D_{x,b}^{\alpha-2} g(x) f'(x) \right]_a^b + \left[ RL D_{x,b}^{\alpha-1} g(x) f(x) \right]_a^b.
\]

3. Main results. In this section, the main results will be presented. The adjoint equation corresponding to (2)-(3) and optimality conditions for optimal control problem (1)-(3) for \( y(x) = (y_1(x), y_2(x), \ldots, y_k(x)) \) and \( u = (u_1, u_2, \ldots, u_k) \) will be derived. We also prove the existence and uniqueness of solutions of the adjoint equation. The solution of adjoint equation is called the adjoint state associated with \( y \).
3.1. Adjoint equation and optimality system for FOCPs on a star graph.

In this subsection, the adjoint equation and necessary and sufficient optimality conditions for the optimal control problem (1)-(3) for \( y(x) = (y_1(x), y_2(x), \ldots, y_k(x)) \) and \( u = (u_1, u_2, \ldots, u_k) \) will be derived. The well known Lagrange multiplier method will be used to obtain the adjoint equation and optimality conditions. To this end, we define the Lagrangian function

\[
\mathcal{L}(y, u, p) = J(y, u) + \sum_{i=1}^{k} \left[ \int_{0}^{1} \left( C^2_{D,0} y_i(x) - f_i(x, y_i(x)) p_i(x) \right) dx \right],
\]

where, \( p_i, i = 1, 2, \ldots, k \) are Lagrange multiplier functions defined on \([0, 1]\), which in \( \mathcal{L} \) are expressed as the vector \( p := (p_1, p_2, \ldots, p_k) \).

**Theorem 3.1.** If \( (y, u) \) is the optimal pair of (1) under the dynamic constraint (2) and the boundary condition (3), then we get the following adjoint equation

\[
RL D_{x,1}^2 p_i(x) - \frac{\partial f_i(x, y_i(x))}{\partial y_i} p_i(x) = -(y_i(x) - y_i^d(x)), \ 0 < x < 1, \ i = 1, 2, \ldots, k,
\]

(11)

together with the boundary conditions

\[
\begin{cases}
\sum_{i=1}^{k} [RL D_{x,1}^{2} p_i(x)]_{x=1} = 0, \\
D_{x,1}^{-(2-\alpha)} p_i(x)|_{x=1} = D_{x,1}^{-(2-\alpha)} p_j(x)|_{x=1}, \ i, j = 1, 2, \ldots, k, \ i \neq j, \\
D_{x,1}^{-(2-\alpha)} p_i(x)|_{x=0} = 0 \ i = 1, 2, \ldots, k
\end{cases}
\]

(12)

and optimality condition

\[
u_i = \frac{1}{\mu_i} \left. RL D_{x,1}^{2} p_i(x) \right|_{x=0}, \ i = 1, 2, \ldots, k.
\]

(13)

Conversely, any control \( \bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k) \in \mathbb{R}^k \), together with \( \bar{y}(x) = (\bar{y}_1(x), \bar{y}_2(x), \ldots, \bar{y}_k(x)) \) and \( \bar{p}(x) = (\bar{p}_1(x), \bar{p}_2(x), \ldots, \bar{p}_k(x)) \) is an optimal solution to problem (1)-(3) if it satisfies (11)-(13).

**Proof.** We have,

\[
\mathcal{L}(y, u, p) = \sum_{i=1}^{k} \left[ \frac{1}{2} \int_{0}^{1} (y_i(x) - y_i^d(x))^2 dx + \frac{\mu_i}{2} u_i^2 \right] \\
+ \sum_{i=1}^{k} \left[ \int_{0}^{1} \left( C^2_{D,0} y_i(x) - f_i(x, y_i(x)) p_i(x) \right) dx \right].
\]

Now, using Lemma 2.5, we obtain

\[
\mathcal{L}(y, u, p)
\]

\[
\begin{aligned}
&= \sum_{i=1}^{k} \left[ \frac{1}{2} \int_{0}^{1} (y_i(x) - y_i^d(x))^2 dx + \frac{\mu_i}{2} u_i^2 \right] + \sum_{i=1}^{k} \left[ \int_{0}^{1} y_i(x) RL D_{x,1}^{2} p_i(x) dx \right] \\
&\quad + \sum_{i=1}^{k} \left[ D_{x,1}^{-(2-\alpha)} p_i(x)|_{x=1} y_i'(1) - D_{x,1}^{-(2-\alpha)} p_i(x)|_{x=0} y_i'(0) + RL D_{x,1}^{2} p_i(x)|_{x=1} y_i(1) \\
&\quad - RL D_{x,1}^{2} p_i(x)|_{x=0} u_i \right] - \sum_{i=1}^{k} \int_{0}^{1} f_i(x, y_i(x)) p_i(x) dx,
\end{aligned}
\]
where we have used the fact that \( y_i(0) = u_i, \ i = 1, 2, \ldots, k. \)

Now, since \( y \) is unconstrained, the derivative of \( \mathcal{L} \) with respect to \( y \) has to vanish at the optimal point, i.e. \( D_y \mathcal{L}(y,u,p)(\hat{y}) = 0 \), where \( \hat{y} = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_k) \) is a solution of

\[
C D_{0,x}^\alpha \hat{y}_i(x) = f_i(x, \hat{y}_i(x)), \quad 0 < x < 1, \ 1 < \alpha < 2, \ i = 1, 2, \ldots, k, \\
\hat{y}_i(0) = 0, \ i = 1, 2, \ldots, k, \\
\hat{y}_i(1) = \hat{y}_j(1), \ i = 1, 2, \ldots, k, \ i \neq j, \\
\sum_{i=1}^k \hat{y}_i'(1) = 0, \ i = 1, 2, \ldots, k. \tag{14}
\]

Hence,

\[
D_y \mathcal{L}(y,u,p)(\hat{y}) = \sum_{i=1}^k \left[ \int_0^1 (y_i(x) - y_i^d(x)) \hat{y}_i(x) dx \right] + \sum_{i=1}^k \left[ \int_0^1 \hat{y}_i(x) RL D_x^\alpha p_i(x) dx \right] \\
+ \sum_{i=1}^k \left[ \left. D_{x,1}^{-(2-\alpha)} p_i(x) \right|_{x=1} - \left. D_{x,1}^{-(2-\alpha)} p_i(x) \right|_{x=0} \right] \hat{y}_i'(0) \\
+ RL D_{x,1}^{\alpha-1} p_i(x) \bigg|_{x=1} - \sum_{i=1}^k \left[ \int_0^1 \frac{\partial f_i(x, y_i(x))}{\partial y_i} \hat{y}_i(x) p_i(x) dx \right] = 0.
\]

Now, since \( \hat{y}_i(1) = \hat{y}_j(1) \), we choose \( p_i(x), i = 1, 2, \ldots, k \) such that

\[
\sum_{i=1}^k \left. [RL D_{x,1}^{\alpha-1} p_i(x) \bigg|_{x=1} \right] = 0, \tag{15}
\]

using \( \sum_{i=1}^k \hat{y}_i'(1) = 0 \), we take

\[
\left. D_{x,1}^{-(2-\alpha)} p_i(x) \right|_{x=1} = \left. D_{x,1}^{-(2-\alpha)} p_j(x) \right|_{x=1}, \ \ i,j = 1, 2, \ldots, k, \ i \neq j \tag{16}
\]

and since \( \hat{y}_i'(0) \) is not specified, we require

\[
\left. D_{x,1}^{-(2-\alpha)} p_i(x) \right|_{x=0} = 0, \ \ i = 1, 2, \ldots, k. \tag{17}
\]

Hence, using (14) and (15)-(17), we obtain

\[
D_y \mathcal{L}(y,u,p)(\hat{y}) = \sum_{i=1}^k \int_0^1 \hat{y}_i(x) \left[ (y_i(x) - y_i^d(x)) + RL D_x^\alpha p_i(x) - \frac{\partial f_i(x, y_i(x))}{\partial y_i} p_i(x) \right] dx = 0,
\]

which gives

\[
RL D_x^\alpha p_i(x) - \frac{\partial f_i(x, y_i(x))}{\partial y_i} p_i(x) = -(y_i(x) - y_i^d(x)), \ i = 1, 2, \ldots, k. \tag{18}
\]

Similarly, we have

\[
D_u \mathcal{L}(y,u,p)(\hat{u}_i) = \mu u_i \hat{u}_i - RL D_{x,1}^{\alpha-1} p_i(x) \big|_{x=0} \hat{u}_i = 0,
\]

which implies

\[
u_i = \frac{1}{\mu} \left. [RL D_{x,1}^{\alpha-1} p_i(x) \bigg|_{x=0}] \right., \ \ i = 1, 2, \ldots, k,
\]
which, in turn, is the required optimality condition (13). The sufficiency part of the theorem directly follows from the convexity of $J$. Therefore, the proof is complete.

Hence, a control $u$, together with the optimal state $y$ and the adjoint state $p$, is optimal for (1)-(3) if and only if the triplet $(u, y, p)$ satisfies the following optimality system:

\begin{align*}
  cD_{0,x}^\alpha y_i(x) &= f_i(x, y_i(x)) & RLD_{x,1}^\alpha p_i(x) - \partial f_i(x, y_i(x)) \partial y_i (x) &= -(y_i(x) - y_i')
  
  y_i(0) &= u_i & D_{x,1}^{-(2-\alpha)} p_i(x)|_{x=0} &= 0
  
  y_i(1) &= y_j(1) & D_{x,1}^{-(2-\alpha)} p_i(x)|_{x=1} = D_{x,1}^{-(2-\alpha)} p_j(x)|_{x=1} 
  
  & \quad i, j = 1, 2 \ldots, k, i \neq j,
  
  \sum_{i=1}^k y_i(1) &= 0 & \sum_{i=1}^k [RLD_{x,1}^{\alpha-1} p_i(x)]|_{x=1} &= 0.
\end{align*}

(19)

### 3.2. Existence and uniqueness of Lagrange multipliers

In this subsection, we prove the existence and uniqueness of Lagrange multipliers, i.e. the solution of adjoint equation (11) along with boundary conditions (12). In order to prove the existence and uniqueness of the solution of BVP (11)-(12), we prove the existence and uniqueness of Lagrange multipliers, i.e. the solution of the following system:

\begin{align*}
  RLD_{x,1}^\alpha p_i(x) &= g_i(x, p_i(x)), \quad i = 1, 2, \ldots, k, \quad 0 < x < 1, \quad 1 < \alpha < 2,
  
  \sum_{i=1}^k [RLD_{x,1}^{\alpha-1} p_i(x)]|_{x=1} &= 0,
  
  D_{x,1}^{-(2-\alpha)} p_i(x)|_{x=1} = D_{x,1}^{-(2-\alpha)} p_j(x)|_{x=1}, \quad i, j = 1, 2, \ldots, k, \quad i \neq j,
  
  D_{x,1}^{-(2-\alpha)} p_i(x)|_{x=0} = 0 \quad i = 1, 2, \ldots, k,
\end{align*}

(20)

where $g_i, i = 1, 2, \ldots, k$ are continuous functions on $[0, 1] \times \mathbb{R}$. First, we give the following lemma.

**Lemma 3.2.** [13] Assume that $\alpha > 0$, then

\begin{align*}
  D_{x,1}^{-(2-\alpha)} RLD_{x,1}^\alpha p(x) &= p(x) + c_1(1-x)^{\alpha-1} + c_2(1-x)^{\alpha-2} + \ldots + c_n(1-x)^{\alpha-n},
  
  \text{for some } c_i \in \mathbb{R}, \quad i = 1, 2, \ldots, n, \quad n-1 < \alpha < n.
\end{align*}

In view of Lemma 3.2, we prove the following lemma which plays a key role in further analysis.

**Lemma 3.3.** Let $h_i \in C[0,1]$, then the solution of the fractional differential equations

\begin{align*}
  RLD_{x,1}^\alpha p_i(x) = h_i(x), \quad 0 < x < 1, \quad 1 < \alpha < 2, \quad i = 1, 2, \ldots, k,
\end{align*}

(22)
together with the boundary conditions (12) is given by

\[
p_i(x) = \left[ \frac{1}{k\Gamma(\alpha)} \sum_{j=1}^{k} \int_{0}^{1} th_j(t)dt - \frac{1}{\Gamma(\alpha)} \int_{0}^{1} th_i(t)dt \right] (1-x)^{\alpha-1} - \left( \frac{1}{k\Gamma(\alpha-1)} \sum_{j=1}^{k} \int_{0}^{1} th_j(t)dt \right) (1-x)^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_{x}^{1} (t-x)^{\alpha-1} h_i(t)dt.
\]

(23)

Proof. By Lemma 3.2, we have

\[
p_i(x) = c_i^{(1)}(1-x)^{\alpha-1} + c_i^{(2)}(1-x)^{\alpha-2} + D_{x,1}^{-\alpha} h_i(x), \quad i = 1, 2, \ldots, k
\]

\[
= c_i^{(1)}(1-x)^{\alpha-1} + c_i^{(2)}(1-x)^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_{x}^{1} (t-x)^{\alpha-1} h_i(t)dt,
\]

(24)

where \( c_i^{(j)}, i = 1, 2, \ldots, k, j = 1, 2, \) are some constants. Now, using Remark 2 (i) and Remark 3 (ii), we have

\[
D_{x,1}^{-2\alpha} p_i(x) = c_i^{(1)}\Gamma(\alpha)(1-x) + c_i^{(2)}\Gamma(\alpha-1) + D_{x,1}^{-\alpha} h_i(x)
\]

\[
= c_i^{(1)}\Gamma(\alpha)(1-x) + c_i^{(2)}\Gamma(\alpha-1) + \int_{x}^{1} h_i(t)dt.
\]

Now, boundary conditions \( D_{x,1}^{-2\alpha} p_i(x) \big|_{x=0} = 0, i = 1, 2, \ldots, k \) gives

\[
c_i^{(1)}\Gamma(\alpha) + c_i^{(2)}\Gamma(\alpha-1) + \int_{0}^{1} h_i(t)dt = 0, \quad i = 1, 2, \ldots, k.
\]

(25)

Using \( D_{x,1}^{-2\alpha} p_i(x) \big|_{x=1} = D_{x,1}^{-2\alpha} p_j(x) \big|_{x=1}, i, j = 1, 2, \ldots, k, i \neq j, \) we obtain

\[
c_i^{(2)} = c_j^{(2)}, \quad i, j = 1, 2, \ldots, k, \quad i \neq j.
\]

(26)

Also, using Remark 2 (ii), Remark 3 (ii) and Remark 3 (iii), we get

\[
RL D_{x,1}^{-\alpha-1} p_i(x) = c_i^{(1)}\Gamma(\alpha) + D_{x,1}^{-\alpha-1} h_i(x)
\]

\[
= c_i^{(1)}\Gamma(\alpha) + \int_{x}^{1} h_i(t)dt.
\]

Finally, the boundary condition \( \sum_{i=1}^{k} [RL D_{x,1}^{-\alpha-1} p_i(x)] \big|_{x=1} = 0 \) implies

\[
\sum_{i=1}^{k} c_i^{(1)}\Gamma(\alpha) = 0.
\]

(27)

Now, from equations (25) and (26)

\[
\sum_{i=1}^{k} c_i^{(1)}\Gamma(\alpha) + kc_i^{(2)}\Gamma(\alpha-1) + \sum_{i=1}^{k} \int_{0}^{1} h_i(t)dt = 0.
\]

Using (27), we obtain

\[
c_i^{(2)} = -\frac{1}{k\Gamma(\alpha-1)} \sum_{j=1}^{k} \int_{0}^{1} th_j(t)dt, \quad i = 1, 2, \ldots, k.
\]

(28)
After substituting the value of \( c_i^{(2)} \) in equation (25), we get

\[
c_i^{(1)} = \frac{1}{k \Gamma(\alpha)} \sum_{j=1}^{k} \int_{0}^{1} t g_j(t, p_j(t)) dt - \frac{1}{\Gamma(\alpha)} \int_{0}^{1} t g_i(t, p_i(t)) dt, \quad i = 1, 2, \ldots, k. \quad (29)
\]

Substituting the values of \( c_i^{(1)} \) and \( c_i^{(2)} \), \( i = 1, 2, \ldots, k \) in equation (24), we obtain equation (23).

**Remark 4.** Using the definition of Riemann-Liouville derivative and direct computations, it can be easily shown that if \( p = (p_i)_{i=1}^{k} \) is given by (23), then \( p = (p_i)_{i=1}^{k} \) also satisfy (22) along with boundary conditions (12).

Now, we use the fixed point theory to prove the existence and uniqueness of solutions of BVP (20)-(21). For \( 1 < \alpha < 2 \), we define the weighted space

\[
X = C_{2-\alpha}([0,1]) = \{ p \in C([0,1]) : (1-x)^{2-\alpha} p \in C([0,1]) \}
\]

equipped with the norm

\[
\| p \|_X = \sup_{x \in [0,1]} (1-x)^{2-\alpha} |p(x)|.
\]

Then, \((X, \| \cdot \|_X)\) is a Banach space and, accordingly, the product space \( X^k = X \times X \times \cdots X \times X \) \( k \)-times, \( \| \cdot \|_{X^k} \) is a Banach space with norm

\[
\| (p_1, p_2, \ldots, p_k) \|_{X^k} = \sum_{i=1}^{k} \| p_i \|_X \text{ for } (p_1, p_2, \ldots, p_k) \in X^k.
\]

In view of Lemma 3.3, we define the operator \( T : X^k \rightarrow X^k \), associated with the BVP (20)–(21), by

\[
T(u_1, u_2, \ldots, u_k)(t) := (T_1(u_1, u_2, \ldots, u_k)(t), \ldots, T_k(u_1, u_2, \ldots, u_k)(t)),
\]

where \( i \)th component of \( T \) is defined as

\[
T_i(p_1, p_2, \ldots, p_k)(x) = \left[ \frac{1}{k \Gamma(\alpha)} \sum_{j=1}^{k} \int_{0}^{1} t g_j(t, p_j(t)) dt - \frac{1}{\Gamma(\alpha)} \int_{0}^{1} t g_i(t, p_i(t)) dt \right] (1-x)^{\alpha-1}
\]

\[
- \left( \frac{1}{k \Gamma(\alpha-1)} \sum_{j=1}^{k} \int_{0}^{1} t g_j(t, p_j(t)) dt \right) (1-x)^{\alpha-2}
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_{x}^{1} (t-x)^{\alpha-1} g_i(t, p_i(t)) dt.
\]

**Remark 5.** It is clear that BVP (20)-(21) has a solution if and only if \( T \) has a fixed point in \( X^k \).

In the forthcoming analysis, we introduce the following hypothesis:

**Hypothesis 1 (H1).** There exists a nonnegative function \( a_i(x) \in C_{2-\alpha}([0,1]) \), \( i = 1, 2, \ldots, k \), such that for \( 0 \leq x < 1 \), \( p, p_1 \in \mathbb{R} \), we have

\[
|g_i(x, p) - g_i(x, p_1)| \leq (1-x)^{2-\alpha} a_i(x) |p - p_1|, \quad i = 1, 2, \ldots, k.
\]

The following theorem proves the existence and uniqueness of solution of BVP (20)-(21).
**Theorem 3.4.** Assume that the hypothesis $(H1)$ holds. Then BVP (20)-(21) has a unique solution in $X^k$ if

$$\sum_{j=1}^k |a_j|x \left( \frac{2\Gamma(\alpha - 1)}{\Gamma(\alpha + 1)} + \frac{2}{\Gamma(\alpha + 1)} + \frac{k\Gamma(\alpha - 1)}{\Gamma(\alpha)\Gamma(\alpha + 1)} + \frac{k\Gamma(\alpha - 1)}{\Gamma(2\alpha - 1)} \right) < 1.$$ 

**Proof.** We shall prove that $T$ is a contraction map. To this end, let $p = (p_1, p_2, \ldots, p_k)$, $q = (q_1, q_2, \ldots, q_k) \in X^k$, $x \in [0, 1)$. Then, by equation (30), we have

$$|(1 - x)^{2-\alpha}T_i p(x) - (1 - x)^{2-\alpha}T_i q(x)|$$

$$\leq \left[ \frac{1}{k\Gamma(\alpha)} \sum_{j=1}^k \int_0^1 t|g_j(t, p_j(t) - g_j(t, q_j(t))| dt \right] (1 - x)$$

$$+ \left[ \frac{1}{\Gamma(\alpha)} \int_0^1 t|g_i(t, p_i(t) - g_i(t, q_i(t))| dt \right] (1 - x)$$

$$+ \left[ \frac{1}{k\Gamma(\alpha - 1)} \sum_{j=1}^k \int_0^1 t|g_j(t, p_j(t) - g_j(t, q_j(t))| dt \right]$$

$$+ \left[ \frac{1}{\Gamma(\alpha)} \int_x^1 (t-x)^{\alpha-1}|g_i(t, p_i(t)) - g_i(t, q_i(t))| dt \right] (1 - x)^{2-\alpha}.$$ 

Now, using $(H1)$ and $x < 1$, we can write

$$|(1 - x)^{2-\alpha}T_i p(t) - (1 - x)^{2-\alpha}T_i q(t)|$$

$$\leq \left[ \frac{1}{k\Gamma(\alpha)} \sum_{j=1}^k \|p_j - q_j\| x \|a_j\| x + \frac{1}{\Gamma(\alpha)} \|p_i - q_i\| x \|a_i\| x \right] \int_0^1 t(1-t)^{\alpha-2} dt$$

$$+ \frac{1}{\Gamma(\alpha)} \|p_i - q_i\| x \|a_i\| x \int_0^1 (t-x)^{\alpha-1}(1-t)^{\alpha-2} dt$$

$$+ \frac{1}{k\Gamma(\alpha - 1)} \sum_{j=1}^k \|p_j - q_j\| x \|a_j\| x \int_0^1 t(1-t)^{\alpha-2} dt$$

$$= \frac{\Gamma(\alpha - 1)}{k\Gamma(\alpha)\Gamma(\alpha + 1)} \sum_{j=1}^k \|p_j - q_j\| x \|a_j\| x + \frac{\Gamma(\alpha - 1)}{\Gamma(\alpha)\Gamma(\alpha + 1)} \|p_i - q_i\| x \|a_i\| x$$

$$+ \frac{1}{k\Gamma(\alpha + 1)} \sum_{j=1}^k \|p_j - q_j\| x \|a_j\| x + \frac{\Gamma(\alpha - 1)}{\Gamma(2\alpha - 1)} \|p_i - q_i\| x \|a_i\| x (1 - x)^{\alpha}$$

$$\leq \left[ \frac{\Gamma(\alpha - 1)}{k\Gamma(\alpha)\Gamma(\alpha + 1)} + \frac{1}{k\Gamma(\alpha + 1)} \right] \sum_{j=1}^k \|p_j - q_j\| x \|a_j\| x$$

$$+ \left[ \frac{\Gamma(\alpha - 1)}{\Gamma(\alpha)\Gamma(\alpha + 1)} + \frac{\Gamma(\alpha - 1)}{\Gamma(2\alpha - 1)} \right] \|p_i - q_i\| x \|a_i\| x.$$
\[
\begin{align*}
\left[ \frac{\Gamma(\alpha - 1)}{k \Gamma(\alpha) \Gamma(\alpha + 1)} + \frac{1}{k \Gamma(\alpha + 1)} \right] & \| p_i - q_i \| x \| a_i \| x \\
+ \left[ \frac{\Gamma(\alpha - 1)}{k \Gamma(\alpha) \Gamma(\alpha + 1)} + \frac{1}{k \Gamma(\alpha + 1)} \right] \sum_{j=1, j \neq i}^{k} \| p_j - q_j \| x \| a_j \| x,
\end{align*}
\]

where we have used that
\[
\int_0^1 t(1-t)^{\alpha-2} \, dt = \frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)} \quad \text{and} \quad \int_x^1 (t-x)^{\alpha-1}(1-t)^{\alpha-2} \, dt = \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-1)} (1-x)^{2\alpha-2}.
\]

Hence,
\[
\| T_i p - T_i q \| x \leq \left[ \frac{2\Gamma(\alpha - 1)}{k \Gamma(\alpha) \Gamma(\alpha + 1)} + \frac{2}{k \Gamma(\alpha + 1)} + \frac{\Gamma(\alpha - 1)}{\Gamma(\alpha) \Gamma(\alpha + 1)} + \frac{\Gamma(\alpha - 1)}{\Gamma(2\alpha - 1)} \right] \\
\times \left( \sum_{j=1}^{k} |a_j| x \right) \left( \sum_{j=1}^{k} \| p_j - q_j \| x \right)
\]
\[
= \left[ \frac{2\Gamma(\alpha - 1)}{k \Gamma(\alpha) \Gamma(\alpha + 1)} + \frac{2}{k \Gamma(\alpha + 1)} + \frac{\Gamma(\alpha - 1)}{\Gamma(\alpha) \Gamma(\alpha + 1)} + \frac{\Gamma(\alpha - 1)}{\Gamma(2\alpha - 1)} \right] \\
\times \left( \sum_{j=1}^{k} |a_j| x \right) \| p - q \| x^k.
\]

Now, from equation (32), it follows that
\[
\| T p - T q \| x^k = \sum_{i=1}^{k} \| T_i p - T_i q \| x
\]
\[
\leq \left[ \frac{2\Gamma(\alpha - 1)}{\Gamma(\alpha) \Gamma(\alpha + 1)} + \frac{2}{\Gamma(\alpha + 1)} + \frac{k\Gamma(\alpha - 1)}{\Gamma(\alpha) \Gamma(\alpha + 1)} + \frac{k\Gamma(\alpha - 1)}{\Gamma(2\alpha - 1)} \right] \\
\times \left( \sum_{j=1}^{k} |a_j| x \right) \| p - q \| x^k.
\]
Since \( \sum_{j=1}^{k} |a_j| x \left( \frac{2\Gamma(\alpha - 1)}{\Gamma(\alpha) \Gamma(\alpha + 1)} + \frac{2}{\Gamma(\alpha + 1)} + \frac{k\Gamma(\alpha - 1)}{\Gamma(\alpha) \Gamma(\alpha + 1)} + \frac{k\Gamma(\alpha - 1)}{\Gamma(2\alpha - 1)} \right) < 1, \) we deduce that \( T \) is a contraction map. As a result of Banach’s contraction principle, BVP (20)-(21) has a unique solution. \( \square \)

Now, in particular, if we take
\[
g_i(x, p_i(x)) = \frac{\partial f_i(x, y_i(x))}{\partial y_i} p_i(x) - (y_i(x) - y_i^d(x)),
\]
where \( y_i(x), \ i = 1, 2, \ldots, k \) is the solution of (2)-(3) and
\[
a_i(x) = \left| \frac{\partial f_i(x, y_i(x))}{\partial y_i} \right| \frac{1}{(1-x)^{2-\alpha}}, \quad i = 1, 2, \ldots, k,
\]
then, clearly \( a_i(x) \in C_{2-\alpha}[0,1] \) and (31) is also satisfied. Hence, using Theorem 3.4, we deduce that BVP (11)-(12) has a unique solution.
4. Numerical method for solving FOCPs. In this section, we provide the numerical approximation of the optimal control \( u = (u_1, u_2, \ldots, u_k) \), the state variable \( y(x) = (y_1(x), y_2(x), \ldots, y_k(x)) \) and the adjoint state \( p(x) = (p_1(x), p_2(x), \ldots, p_k(x)) \), satisfying the optimality system (19). The \( L^2 \) scheme and Gr"unwald-Letnikov formula will be used for the approximation of Caputo fractional derivative and right Riemann-Liouville derivative respectively. Let \( \Delta x = 1/N \) be the step size with \( x_n = n\Delta x, \ n = 0, 1, \ldots, N, \ x_0 = 0 \) and \( x_N = 1 \).

The Gr"unwald-Letnikov formula for the approximation of right Riemann-Liouville derivative is defined as [22]

\[
[RLD_{x,1}^\alpha p_i(x)]_{x=x_n} \approx \frac{1}{\Delta x^\alpha} \sum_{r=0}^{N-n} w_r^{(\alpha)} p_i(x_{n+r})
\]

and the right shifted (one shift) Gr"unwald-Letnikov formula is defined as

\[
[RLD_{x,1}^\alpha p_i(x)]_{x=x_n} \approx \frac{1}{\Delta x^\alpha} \sum_{r=0}^{N-n+1} w_r^{(\alpha)} p_i(x_{n+r-1}),
\]

where \( w_r^{(\alpha)} = (-1)^r \left( \begin{array}{c} \alpha \\ r \end{array} \right) \).

Now, we describe the \( L^2 \) scheme for the approximation of Caputo fractional derivative. By using (9), the Caputo fractional derivative of order \( 1 < \alpha < 2 \) at \( x = x_n \) is defined by

\[
[CD_{0,x}^\alpha y_i(x)]_{x=x_n} = \frac{1}{\Gamma(2-\alpha)} \sum_{r=0}^{n-1} \left( \int_{x_r}^{x_{r+1}} (x_n - t)^{1-\alpha} y_i''(t) dt \right)
\]

\[
= \frac{1}{\Gamma(2-\alpha)} \sum_{r=0}^{n-1} \left( \int_{x_r}^{x_{r+1}} t^{1-\alpha} y_i''(x_n - t) dt \right).
\]

On each subinterval \([x_r, x_{r+1}]\), \( y_i''(x_n - t) \) is approximated by

\[
y_i''(x_n - t) \approx \frac{y_i(x_{n-r+1}) - 2y_i(x_{n-r}) + y_i(x_{n-r-1})}{\Delta x^2},
\]

which leads to the following \( L^2 \) scheme

\[
[CD_{0,x}^\alpha y_i(x)]_{x=x_n} = \frac{\Delta x^{-\alpha}}{\Gamma(3-\alpha)} \sum_{r=0}^{n-1} b_r[y_i(x_{n-r+1}) - 2y_i(x_{n-r}) + y_i(x_{n-r-1})],
\]

where \( b_r = (r+1)^{2-\alpha} - r^{2-\alpha} \).

Finally, for the right fractional integral, we use the following approximation [7]:

\[
D_{x,1}^{-\alpha} p_i(x)|_{x=x_n} \approx \sum_{r=1}^{N} v_r^{(\alpha)} p_i(x_r),
\]

where \( v_r^{(\alpha)} \) is defined as

\[
v_r^{(\alpha)} = \frac{1}{\Gamma(2-\alpha)} \sum_{r=0}^{n-1} \left( \int_{x_r}^{x_{r+1}} t^{1-\alpha} y_i''(x_n - t) dt \right).
\]
where the coefficients \( v_{n,r}^{(\alpha)} \) have the form

\[
v_{n,r}^{(\alpha)} = \frac{\Delta x^{\alpha}}{\Gamma(\alpha + 2)} \times \\
\begin{cases} 
0 & \text{for } n = N \text{ and } r = N, \\
(N - n - 1)^{\alpha + 1} - (N - n)^{\alpha + 1} + (N - n)^{\alpha}(\alpha + 1) & \text{for } n < N \text{ and } r = N, \\
(r - n + 1)^{\alpha + 1} - 2(r - n)^{\alpha + 1} + (r - n - 1)^{\alpha + 1} & \text{for } n < N \text{ and } n < r < N, \\
1 & \text{for } n < N \text{ and } r = n.
\end{cases}
\]

Now, substituting equations (33)-(37) into optimality system (19), we obtain

\[
\sum_{r=0}^{n-1} \frac{1}{\Gamma(3 - \alpha)} b_r [y_i(x_{n-r+1}) - 2y_i(x_{n-r}) + y_i(x_{n-r-1})] = f_i(x_n, y_i(x_n)) \Delta x^{\alpha}, 
\]

\( n = 1, 2, \ldots, N - 1, \quad i = 1, 2, \ldots, k, \) (38)

\[
y_i(x_0) - \frac{1}{\Delta x^{\alpha - 1}} \sum_{r=0}^{N} (-1)^r \binom{\alpha - 1}{r} p_i(x_r) = 0, \quad i = 1, 2, \ldots, k, \]

\( y_i(x_N) - \frac{1}{k} \sum_{j=1}^{k} y_j(x_{N-1}) = 0, \quad i = 1, 2, \ldots, k, \) (40)

\[
\sum_{r=1}^{N-n+1} (-1)^r \binom{\alpha}{r} p_i(x_{n+r-1}) - \Delta x^{\alpha} \frac{\partial f_i(x_{n}, y_i(x_n))}{\partial y_i} p_i(x_n) = -\Delta x^{\alpha} (y_i(x_n) - y_i^d(x_n)), \quad n = 1, 2, \ldots, N, \quad i = 1, 2, \ldots, k, \]

\( \sum_{i=1}^{N-1} \left[ (r + 1)^{3-\alpha} - 2r^{3-\alpha} + (r - 1)^{3-\alpha} \right] p_i(x_r) \]

\( + \left[ (N - 1)^{3-\alpha} - N^{3-\alpha} + N^{2-\alpha}(3 - \alpha) \right] p_i(x_N) = 0, \quad i = 1, 2, \ldots, k, \) (42)

\[
\sum_{i=1}^{k} [p_i(x_{N-1}) - (\alpha - 1)p_i(x_N)] = 0, \quad (43)
\]

where we have taken the approximation

\[
y_i'(1) \approx \frac{y_i(x_N) - y_i(x_{N-1})}{\Delta x}
\]

and used the fact that \( y_i(x_N) = y_i(x_N), \ i, j = 1, 2, \ldots, k, \ i \neq j. \) Now, substituting the value of \( p_1(x_N) \) from equation (43) into equation (41), we get

\[
\sum_{r=0}^{N-1} (-1)^r \binom{\alpha}{r} p_1(x_r) + (-1)^N \binom{\alpha}{N} \left[ \frac{1}{\alpha - 1} \sum_{i=1}^{k} p_i(x_{N-1}) \right] \\
- (-1)^N \binom{\alpha}{N} \left[ \sum_{i=2}^{k} p_i(x_N) \right] - \Delta x^{\alpha} \frac{\partial f_1(x_{1}, y_1(x_1))}{\partial y_1} p_1(x_1) \\
= -\Delta x^{\alpha} (y_1(x_1) - y_1^d(x_1)), \quad (44)
\]
the state variable \( y \).

The problem is solved for different values of \( N \). 

\[ \sum_{r=0}^{N} (-1)^r \binom{\alpha}{r} p_i(x_r) - \Delta x^\alpha \frac{\partial f_i(x_1, y_i(x_1))}{\partial y_i} p_i(x_1) = -\Delta x^\alpha (y_i(x_1) - y^d_i(x_1)), \quad (45) \]

\( i = 2, 3, \ldots, k \)

and

\[ \sum_{r=0}^{N-n+1} (-1)^r \binom{\alpha}{r} p_i(x_{n+r-1}) - \Delta x^\alpha \frac{\partial f_i(x_n, y_i(x_n))}{\partial y_i} p_i(x_n) = -\Delta x^\alpha (y_i(x_n) - y^d_i(x_n)), \]

\( n = 2, \ldots, N, \quad i = 1, 2, \ldots, k. \) (46)

Hence, equations (38)-(40), (42) and (44)-(46) provide a set of \((2kN + 2k)\) equations in \((2kN + 2k)\) unknowns \((y_i(x_n)\) and \(p_i(x_n)\), \(i = 1, 2, \ldots, k, \quad n = 0, 1, \ldots, N)\) which can be solved using any standard technique. Once the solution is known, the optimal control \( u = (u_1, u_2, \ldots, u_k) \) can be obtained using equation (13).

5. **Examples.** In this section, two examples will be presented in order to validate the numerical method given in Section 4. We also prove that the solutions converge as the step size \( \Delta x \) is decreased or \( N \) is increased. The convergence is taken in the sense that the change (measured with respect to some norm) in the state variable \( y_i \) decreases with increasing values of \( N \).

**Example 1.**

\[ \text{Min } J(y, u) = \frac{1}{2} \sum_{i=1}^{3} \int_0^1 (y_i(x) - x)^2 dx + \frac{1}{2} \sum_{i=1}^{3} u_i^2, \quad (47) \]

subject to the edgewise system constraints

\[ CD_0^\alpha y_i(x) = x^i, \quad 0 < x < 1, \quad i = 1, 2, 3, \quad (48) \]

and the boundary conditions

\[ y_i(0) = u_i, \quad i = 1, 2, 3, \]

\[ y_1(1) = y_2(1) = y_3(1), \]

\[ \sum_{i=1}^{3} y_i'(1) = 0. \quad (49) \]

Note that in this example, \( y_i^d(x) = x, \mu = 1, f_i(x, y_i(x)) = x^i, \quad i = 1, 2, 3 \) and \( k = 3 \), i.e. star graph with three edges. Hence, the optimality system is given as

\[
\begin{align*}
C D_0^\alpha y_i(x) &= x^i \\
y_i(0) &= u_i = RL D_{x,1}^\alpha p_i(x) \big|_{x=0} \\
y_1(1) &= y_2(1) = y_3(1) = D_{x,1}^{(2-\alpha)} p_i(x) \big|_{x=0} = 0. \\
D_{x,1}^{(2-\alpha)} p_i(x) \big|_{x=1} &= D_{x,1}^{(2-\alpha)} p_j(x) \big|_{x=1} \\
&\text{if } i, j = 1, 2, 3, \quad i \neq j, \\
\sum_{i=1}^{3} y_i'(1) &= 0 \\
\sum_{i=1}^{3} [RL D_{x,1}^{\alpha-1} p_i(x)] \big|_{x=1} &= 0. \\
\end{align*}
\]

(50)

The problem is solved for different values of \( N \) and \( \alpha \). Figure 2 and Table 1 show the state variable \( y(x) = (y_1(x), y_2(x), y_3(x)) \) and the control \( u = (u_1, u_2, u_3) \),
respectively for $\alpha = 3/2$ for different values of $N$. From Figure 2 it is clear that state variables converge as the value of $N$ increases, while Table 1 shows the convergence of control variable to $u = (1.802, 1.732, 1.688)$. Figure 3 and Table 2 show the state variable $y(x) = (y_1(x), y_2(x), y_3(x))$ and the control $u = (u_1, u_2, u_3)$ respectively for $N = 64$ for different values of fractional order $\alpha$, where it can be observed that the amplitudes of $y_i(x), i = 1, 2, 3$, decrease as $\alpha$ is decreased.

| $N$  | $u_1$  | $u_2$  | $u_3$  |
|------|--------|--------|--------|
| 32   | 1.867  | 1.792  | 1.749  |
| 64   | 1.834  | 1.762  | 1.718  |
| 128  | 1.817  | 1.746  | 1.702  |
| 256  | 1.808  | 1.738  | 1.694  |
| 512  | 1.804  | 1.734  | 1.690  |
| 1024 | 1.802  | 1.732  | 1.688  |

Figure 2. Convergence of $y_i(x), i = 1, 2, 3$ for the optimality system (50) for $\alpha = 3/2$. 

(a) Convergence of $y_1(x)$ for $\alpha = 3/2$

(b) Convergence of $y_2(x)$ for $\alpha = 3/2$

(c) Convergence of $y_3(x)$ for $\alpha = 3/2$
Table 2. Control variable $u = (u_1, u_2, u_3)$ for different fractional order $\alpha$ with $N = 64.$

| $\alpha$ | $u_1$  | $u_2$  | $u_3$  |
|----------|--------|--------|--------|
| 1.2      | 0.2017 | 0.1959 | 0.1910 |
| 1.4      | 0.1894 | 0.1824 | 0.1778 |
| 1.6      | 0.1775 | 0.1703 | 0.1662 |
| 1.8      | 0.1666 | 0.1598 | 0.1563 |
| 2        | 0.1572 | 0.1511 | 0.1482 |

![Graphs](a) $y_1(x)$ for $N = 64$  
(b) $y_2(x)$ for $N = 64$  
(c) $y_3(x)$ for $N = 64$

Figure 3. State variables $y_i(x)$, $i = 1, 2, 3$, for different fractional order $\alpha$ for the optimality system (50) with $N = 64.$

Example 2.

$$Min\ J(y, u) = \frac{1}{2} \sum_{i=1}^{3} \int_0^1 (y_i(x) - x)^2 dx + \frac{1}{2} \sum_{i=1}^{3} u_i^2,$$  
subject to the edgewise system constraints

$$c D_{0,x}^\alpha y_i(x) = y_i(x), \quad 0 < x < 1, \ i = 1, 2, 3,$$
and the boundary conditions
\[ y_i(0) = u_i, \quad i = 1, 2, 3, \]
\[ y_1(1) = y_2(1) = y_3(1), \]
\[ \sum_{i=1}^{3} y'_i(1) = 0. \]  
(53)

Note that in this example, \( y^d_i(x) = x, \mu = 1, f_i(x, y_i(x)) = y_i(x), \) \( i = 1, 2, 3, \) and \( k = 3. \) Hence, the optimality system is given as

\[
\begin{align*}
C D_{0,x}^\alpha y_i(x) &= y_i(x) \\
y_i(0) &= u_i = RL D_{x,1}^{\alpha-1} p_i(x) \bigg|_{x=0} \\
y_i(1) &= y_j(1) \\
D_{x,1}^{(2-\alpha)} p_i(x) \bigg|_{x=0} &= 0, \\
D_{x,1}^{(2-\alpha)} p_i(x) \bigg|_{x=1} = D_{x,1}^{(2-\alpha)} p_j(x) \bigg|_{x=1} \\
&\quad i, j = 1, 2, 3, \quad i \neq j, \\
\sum_{i=1}^{3} y'_i(1) &= 0 \\
\sum_{i=1}^{3} [RL D_{x,1}^{\alpha-1} p_i(x)]_{x=1} &= 0. 
\end{align*}
\]  
(54)

The problem is solved for different values of \( N. \) Figure 4 shows the state variable

![Figure 4](image)

(a) Convergence of \( y_1(x) \) for \( \alpha = 3/2 \)

(b) Convergence of \( y_2(x) \) for \( \alpha = 3/2 \)

(c) Convergence of \( y_3(x) \) for \( \alpha = 3/2 \)

**Figure 4.** Convergence of \( y_i(x), \ i = 1, 2, 3 \) for the optimality system (54) for \( \alpha = 3/2. \)
\[ y(x) = (y_1(x), y_2(x), y_3(x)) \] for \( \alpha = 3/2 \) for different values of \( N \), where it is clear that state variables converge as the value of \( N \) increases.

6. Conclusion. In this paper, we have extended our recent work \[27\] to fractional optimal control problems on a star graph. Therefore, this paper could be seen as the bridge between the fractional optimal control problems from euclidean domain to topologically complicated domains. The adjoint state and the optimality system are obtained for FOCP. Existence and uniqueness results are proved for the solution of the adjoint equation. For the numerical solution of the optimality system, we consider the \( L^2 \) scheme and Grünwald Letnikov formula for the approximation of fractional derivatives which reduces the optimality system into the system of algebraic equations. Two examples have been shown to demonstrate the applicability of the presented method. In future, we will consider fractional optimal control problems for time dependent semi-linear problems such as wave and heat equations on a metric graph.

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