RG limit cycles

K.Bulycheva$^a$, A.Gorsky$^{a,b}$

$^a$ Institute of Theoretical and Experimental Physics, Moscow 117218, Russia
$^b$ Moscow Institute of Physics and Technology, Dolgoprudny 141700, Russia

gorsky@itep.ru
bulycheva@itep.ru

Contribution to the "Pomeranchuk-100" Volume

Abstract

In this review we consider the concept of limit cycles in the renormalization group flows. The examples of this phenomena in the quantum mechanics and field theory will be presented.
1 Generalities

It is usually assumed that the RG flow connects fixed points, starting at a UV repelling point and terminating at a IR attracting point. However it turned out that this open RG trajectory does not exhaust all possibilities and the clear-cut quantum mechanical example of the nontrivial RG limit cycle has been found in [1] confirming the earlier expectations. This example triggered the search for patterns of this phenomena which was quite successful. They have been identified both in the systems with finite number of degrees of freedom [2,3,4,5] and in the field theory framework [6,7,8]. Now the cyclic RG takes its prominent place in the world of RG phenomena however the subject certainly deserves much more study.

The appearance of critical points corresponds to phase transitions of the second kind, hence there exists a natural question concerning the connection of RG cycles with phase transitions. The very phenomenon of the cyclic RG flow has been interpreted in the important paper [7] as a kind of generalization of the BKT phase transitions in two dimensions. One can start from a usual example of an RG flow connecting UV and IR fixed points and then consider a motion in a parameter space which results in a merging of the fixed points. In [7] it was argued that when the parameter goes into the complex region the cyclic behaviour of the RG flow gets manifested and a gap in the spectrum arises. This happens similar to the BKT transition case when a deconfinement of vortices occurs at the critical temperature and the conformal symmetry is restored at lower temperatures. The appearance of the RG cycles can be also interpreted as the peculiar anomaly in the classical conformal group [9]. This anomaly has the origin in some ”falling to the center” UV phenomena which could have quite different reincarnations. We would like to emphasize one more generic feature of the phenomena — the cyclic RG usually occurs in the system with at least two couplings. One of them undergoes the RG cyclic flow while the second determines the period of the cycle.

The collision of the UV and IR fixed points can be illustrated in a quite general manner as follows. Assume that there are two couplings ($\alpha, g$) in the theory and we focus at the renormalization of the coupling which enjoys the following $\beta$-function

$$\beta_g = (\alpha - \alpha_0) - (g - g_0)^2,$$

(1)
which vanishes at the hypersurface in the parameter space

\[ g = g_0 \pm \sqrt{\alpha - \alpha_0}. \]  

(2)

It was argued in [7] that the collision of two roots at \( \alpha = \alpha_0 \) can be interpreted as the collision of UV and IR fixed points. Upon the collision the points move into the complex \( g \) plane and an RG cycle emerges. The period of the cycle can be immediately estimated as

\[ T \propto \int_{g_{UV}}^{g_{IR}} \frac{dg}{\beta(\alpha; g)} \propto \frac{1}{\sqrt{\alpha - \alpha_0}}. \]  

(3)

The phenomena is believed to be generic once the beta–function has the form (1). Note that is was shown that the RG cycles are consistent with the c-theorem [10].

Breaking of the conformal symmetry results in the generation of the mass scale which has non-perturbative nature. Due to the RG cycles the scale is not unique and the whole tower with the Efimov-like scaling gets manifested

\[ E_{n+1} = \lambda E_n, \]  

(4)

where \( \lambda \) is fixed by the period of RG cycle.

In the examples available we could attempt to trace the physical picture behind. It turns out that the origin of two couplings is quite general. One coupling does not break the conformal symmetry which is exact in some subspace of the parameter space. The second coupling plays the role of UV regularization which can be imposed in one or another manner. It breaks the conformal symmetry however some discrete version of the scale symmetry survives which is manifested in the cycle structure. The UV regularization will have different reincarnations in the examples considered: the account of the finite size of the nuclei, contact interaction in the model of superconductivity or the brane splitting in the supersymmetric models.

Historically the first example of this phenomena has been found long time ago by Efimov [11] in the context of nuclear physics. He considered the three-body system when two particles are near threshold and have attractive potential with the third particle. It was shown that two–particle bound states are absent in the spectrum, but there is a tower of the three–particle bound states with the geometrical scaling corresponding to \( \lambda \approx 22, 7 \). The review of the RG interpretation of the Efimov phenomena can be found in [12].
When considering the system with finite number of degrees of freedom the meaning of the RG flows has to be clarified. To this aim some UV cutoff should be introduced. In the first example in [1] the step of the RG corresponds to the integrating out the highest energy level taking into account its correlation with the rest of the spectrum. This approach has a lot in common with the renormalization procedure in the matrix models considered in [13]. The same UV cutoff for formulation of RG procedure has been used for the Russian Doll (RD) model describing the restricted BCS model of superconductivity [14]. In that case the coupling providing the Cooper pairing undergoes the RG cycle while the CP-violating parameter defines the period.

In the second class of examples the UV cutoff is introduced not at high energy scale but at small distances. The RG cycles have been found in the non-relativistic Calogero-like models with \( \frac{1}{r^2} \) potential which enjoys naive conformal symmetry [3, 4, 5]. The RG flow is formulated in terms of the short distance regularization of the model. It is assumed that the wave function with \( E = 0 \) at large \( r \) does not depend on the UV cutoff at small \( r \). This condition yields the equation for the parameter of a cutoff in the regularization potential. This equation has multiple solutions which can be interpreted as the manifestation of the tower of shallow bound states with the Efimov scaling in the regularized Calogero model with attraction. The scaling factor in the tower is determined by the Calogero coupling constant which reflects the remnant of the conformal group upon the regularization.

The list of the field theory examples in different dimensions with the cyclic RG flows is short but quite representative. In two dimensions the explicit example with the RG cycle has been found in some range of parameters in the sin-Gordon model. The cycle manifests itself in the pole structure of the \( S \)-matrix. Efimov-like tower of states corresponds to the specific poles with the Regge-like behavior of the resonance masses [6]

\[
m_n = m_s e^{\frac{n\pi}{h}},
\]

where \( h \) is a certain parameter of the model. Moreover it was argued that the \( S \)-matrix behaves universally under the cyclic RG flows. The tower of Efimov states scales in the same manner as in the quantum mechanical case.

The origination of the cyclic RG behavior in the sine–Gordon model is not surprising. Indeed it was argued in [7] that the famous Berezinsky–Kosterlitz–Thouless (BKT) phase transition in XY system belongs to this
universality class. On the other hand one can map the XY system at the $T$ temperature into the sin-Gordon theory with the parameters:

$$L_{SG} = T(\partial\phi)^2 - 4z \cos \phi,$$

and look at the renormalization of the interaction coupling. The $\beta$-functions read as

$$\beta_u = -2v^2, \quad \beta_v = -2uv,$$

where

$$u = 1 - \frac{1}{8\pi T}, \quad v = \frac{2z}{T\Lambda^2},$$

and $\Lambda$ is the UV cutoff introduced to regularize the vortex core. The form of $\beta$ functions implies the existence of the limit cycle with the following expression for the correlation length:

$$\xi_{BKT} \propto \exp \left( \frac{c}{\sqrt{|T - T_c|}} \right),$$

above the phase transition. This RG behavior gets mapped into the RG cycle in the sine–Gordon model.

The example of the Efimov tower in 2+1 dimensions has been found in \[15\] in the holographic representation. The model is based on the $D3 - D5$ brane configuration and corresponds to the large $N$ 3d gauge theory with fundamentals enjoying $\mathcal{N} = 4$ supersymmetry. In addition the magnetic field and the finite density of conserved charge are present. At strong coupling the gauge theory is described in terms of the probe $N_f$ flavor branes in the nontrivial $AdS_5 \times S^5$ geometry when the $U(1)$ bulk gauge field is added providing the magnetic field in the boundary theory.

The generation of the tower of the Efimov states happens at some value of the “filling fraction” $\nu$ in external magnetic field. The phase transition corresponds to the change of the minimal embedding of the probe $D5$ branes in the bulk geometry with the BKT critical behavior of the order parameter. In that case the order parameter gets identified with the condensate $\sigma$ which behaves as:

$$\sigma \propto \exp \left( -\frac{1}{\nu} \right).$$

Above the phase transition the embedding gets changed and the brane becomes extended in one more coordinate. The scale associated with this extension into new dimension is nothing but the nonperturbative scale amounting
to the mass gap. The phenomena of the cyclic RG flow in this case has the Breitenlohner-Freedman instability as the gravitational counterpart.

In four dimensions the most famous example of the Efimov tower is the so-called Miransky scaling for the condensate in the magnetic field. In [16] was argued that the chiral condensate is generated in the external magnetic field in the abelian theory with the following behavior:

$$\langle \bar{\Psi}\Psi \rangle \propto \Lambda^3 \exp(-\frac{c}{\sqrt{\alpha - \alpha_{\text{crit}}}}),$$

where $\alpha$ is the fine–structure constant, and $c$ is some parameter of the model.

More recent example [17] of the Efimov tower in four dimensions concerns the Veneziano limit of QCD when $N_f, N_c \to \infty$ while the ratio $x = \frac{N_f}{N_c}$ is fixed. It turns out that this parameter can be considered as the variable in the RG flow which reminds the finite-dimensional examples. At some value of RG scale the tower of condensates gets generated with geometrical Efimov scaling. The period of the RG cycle reads as:

$$T \propto \kappa \sqrt{x_c - x},$$

where $x_c$ is the critical value of the $x$ parameter. Finally the 4d example with the RG cycle has been found in the $\mathcal{N} = 2$ SUSY gauge theory in the $\Omega$-background [8]. In that case the gauge coupling undergoes the RG cycle whose period is determined by the parameter of the $\Omega$-background,

$$T \propto \epsilon^{-1}.\tag{13}$$

The appearance of the RG cycle in this model can be traced from its relation with the quantum integrable systems of the spin chain type.

In this review we provide the reader with the examples of this phenomenon. The list of the systems with finite number of degrees of freedom involves the Calogero model and the relativistic model with the classical conformal symmetry describing the external charge in graphene. Another finite-dimensional example concerns the RD model of the restricted BCS superconductivity. The field theory examples concern the 3d and 4d theories in external fields. We shall focus on their brane representations and use their relations to the finite dimensional integrable systems.
2 RG cycles in non-relativistic quantum mechanics

In this Section we consider the example of the limit cycle in RG in the non-relativistic system with the inverse-square potential, or the Calogero system:

\[ H = \frac{\partial^2}{\partial r^2} - \frac{\mu(\mu - 1)}{r^2}. \]  

(14)

The distinctive feature of the system described by the Hamiltonian (14) is its conformality. Namely, the operators \((H, D, K)\), where \(D\) is the dilation generator and \(K\) is the conformal boost, generate the conformal \(sl_2\) algebra (see Section 4).

The eigenfunctions of (14) having finite energy immediately break this symmetry; more non-trivial is the fact that even the ground state breaks conformality. Namely, the solution to the \(H\psi = 0\) equation is the following:

\[ \psi_0 = c_+ r^\mu + c_- r^{1-\mu}. \]

(15)

This solution is scale-invariant only if one of the coefficients \(c_\pm\) is zero. If both the coefficients are present, they define an intrinsic length scale \(L = (c_+/c_-)^{-1/(2\mu + 1)}\). Requiring that the quantity \(c_+/c_-\) which describes the ground-state solution be invariant under the change of scale,

\[ \frac{c_+}{c_-} = -r_0^{-2\mu + 1} \frac{\gamma - \mu + 1}{\gamma + \mu}, \]

(16)

we arrive at the beta-function for the \(\gamma\) parameter,

\[ \beta_\gamma = \frac{\partial \gamma}{\partial \log r_0} = - (\gamma + \mu)(\gamma - \mu + 1) = \left(\mu - \frac{1}{2}\right)^2 - \left(\gamma - \frac{1}{2}\right)^2, \]

(17)

where \(r_0\) is the RG scale. We can identify \(\gamma = \mu - 1, \gamma = -\mu\) points, i.e. solutions with \(c_+ = 0, c_- = 0\), with UV and IR attractive points of the renormalization group flow [7].

If \(\mu = i\nu\) is imaginary, i.e. the potential is attractive, then the equation (17) allows us to determine the period of the renormalization group:

\[ T = -\int_{-\nu+1}^{\nu} \frac{d\gamma}{\beta_\gamma} = \frac{\pi}{\nu - \frac{1}{2}}. \]

(18)
This means that an infinite number of scales is generated, differing by a factor of \( \exp\left(-\frac{\pi}{\nu - \frac{1}{2}}\right) \). To see this explicitly, we find the solutions to the Schrödinger equation at finite energies. In the attractive potential the solution (15) can be written as:

\[
\psi_0 \propto \sqrt{r} \sin \left( \left( \nu - \frac{1}{2} \right) \log \left( \frac{r}{r_0} \right) + \alpha \right).
\]  

(19)

We observe that this solution oscillates indeterminately in the vicinity of the origin and there is no way to fix the \( \alpha \) constant. To regularize this behaviour, we can break the scale invariance at the level of the Hamiltonian and introduce a regularizing potential. Two most popular regularizations involve the square-well potential [4, 5] or the delta-shell potential [3]. One more choice is to introduce a \( \delta \)-function at the origin [7].

Choosing the square-well regularization,

\[
V(r) = \begin{cases} 
-\frac{\nu(\nu-1)}{r^2}, & r > R, \\
-\lambda, & r \leq R,
\end{cases}
\]  

(20)

we require that the action of the dilatation operator on the wavefunction inside the well and outside it be equal at \( r = R \). This condition amounts to the equation on \( \lambda \),

\[
\sqrt{\lambda} \cot \sqrt{\lambda} = \frac{1}{2} + \nu \cot \left( \nu \log \left( \frac{R}{r_0} \right) \right).
\]  

(21)

The multivalued function \( \lambda(R) \) can be chosen to be continuous [5].

The wavefunction regular at infinity is given as a combination of the Bessel functions [5],

\[
\psi(r, \kappa_m) = \sqrt{r}(-1)^m \left( ie^{-i\nu \frac{\pi}{2}} J_{\nu^2} (\kappa_m r) - ie^{i\nu \frac{\pi}{2}} J_{-\nu^2} (\kappa_m r) \right),
\]  

(22)

where \( \kappa \) is the energy of the state. The spectrum consists of infinitely many shallow bound states with adjacent energies differing by an exponential factor,

\[
\frac{\kappa_{m+1}}{\kappa_m} = e^{-\frac{\pi}{\nu}}.
\]  

(23)

Note that the coordinate enters the wavefunction (22) only in combination with energy, and the spectrum is generated by the dilation operator:
\[ \psi_{m+1} = \exp \left( -\frac{\pi}{\nu} r \partial_r \right) \psi_m. \] (24)

One can think of that relation as that the action of the dilatation operator shifts zeroes of the wave function from the area of \( r < R \) to the area with the inverse square potential, and one step of (24) evolution corresponds to elimination of a single zero in the area with the square-well potential. Since the wave function oscillates infinitely at the origin, the elimination of all the zeroes would require an infinite number of steps, and in this way a whole tower of states gets generated.

3 RG cycle in graphene

In this Section we shall consider the similar problem in 2+1 dimensions which physically corresponds to the external charge in the planar graphene layer. The problem has the classical conformal symmetry and is the relativistic analogue of the conformal non-relativistic Calogero-like system. Due to conformal symmetry we could expect the RG cycles and Efimov-like states in this problem upon imposing the short distance cutoff. The issue of the charge in the graphene plane has been discussed theoretically [19, 20, 21] and experimentally [22, 23]. It was argued that indeed there is the tower of “quasi-Rydberg” states with the exponential scaling [24]. The situation can be interpreted as an atomic collapse phenomena similar to the instability of \( Z > 137 \) superheavy atoms in QED [25].

Turn now to the consideration of an electron in graphene which interacts with an external charge. The two-dimensional Hamiltonian reads as,

\[ H_D = v_F \sigma_i p^i + V(r), \quad i = 1, 2. \] (25)

The external charge creates a Coulomb potential,

\[ V(r) = -\frac{\alpha}{r}, \quad r \geq R. \] (26)

As we shall see, the solution in presence of the potential (26) oscillates indefinitely at the origin and needs to be regularized by some cutoff \( R \). Hence close enough to the origin \( r \leq R \) the potential (26) gets replaced by some constant potential \( V_{reg}(r, \lambda(R)) \). The renormalization condition for the \( \lambda \) parameter is that the zero-energy wave function is not dependent on the short-distance
regularization. This condition is chosen similarly to that of the renormalization of the Calogero system (see Section 2). Hence our primary task is to find the zero-energy solution to the Dirac equation,

\[ H_D \psi_0 = 0. \]  

Since the Hamiltonian commutes with the \( J_3 \) operator,

\[ J_3 = i \frac{\partial}{\partial \varphi} + \sigma_3, \quad [H_D, J_3] = 0, \]  

we can look for the solutions of (27) in the form:

\[ \psi_0 = \left( \begin{array}{c} \chi_0(r) \\ \xi_0(r) e^{i \varphi} \end{array} \right), \quad J_3 \psi_0 = \psi_0. \]  

Then in polar coordinates the equation (27) reads as:

\[ \begin{cases} 
- \frac{i}{2} \hbar v_F \left( \partial_r + \frac{1}{r} \right) \xi_0 = -V(r) \chi_0, \\
- \frac{i}{2} \hbar v_F \partial_r \chi_0 = -V(r) \xi_0, 
\end{cases} \]  

which is equivalent to:

\[ \begin{cases} 
\xi_0(r) = i \hbar v_F (V(r))^{-1} \partial_r \chi_0, \\
\partial_r^2 \chi_0 + \left( \frac{1}{r} - \frac{V'(r)}{V(r)} \right) \partial_r \chi_0 + \frac{V^2(r)}{\hbar^2 v_F^2} \chi_0 = 0. 
\end{cases} \]  

For the potential \( V = -\frac{\alpha}{r} \) we get the following equation on \( \chi_0(r) \):

\[ \partial_r^2 \chi_0 + \frac{2}{r} \partial_r \chi_0 + \frac{\beta^2}{r^2} \chi_0 = 0, \quad \beta = \frac{\alpha}{\hbar v_F}. \]  

Supposing that \( \beta^2 = \frac{1}{4} + \nu^2 \) we write the solution as:

\[ \chi_0 = \sqrt{r} \left( c_- \left( \frac{r}{r_0} \right)^{-i \nu} + c_+ \left( \frac{r}{r_0} \right)^{i \nu} \right) \propto \sqrt{r} \sin \left( \nu \log \frac{r}{r_0} + \varphi \right). \]  

We see that this solution shares the properties of the ground-state Calogero wavefunction (15), namely at nonzero \( c_\pm \) it generates its own intrinsic length scale and it oscillates indeterminately at the origin. In order to fix the \( \varphi \) constant we need to introduce a cut-off potential. Hence we consider the solution in the potential:
\[ V(r) = \begin{cases} -\frac{\alpha}{r}, & r > R, \\ -\hbar v_F \frac{\lambda}{R}, & r \leq R. \end{cases} \quad (34) \]

The dilatation operator acts on \( \chi \) as following:

\[ r \partial_r \chi_0 = \left( \frac{1}{2} + \nu \cot \left( \nu \log \frac{r}{r_0} \right) \right) \chi_0. \quad (35) \]

For the constant potential \( V_{\text{reg}} \) we get from (31):

\[ \partial_r^2 \chi_0^{\text{reg}} + \frac{1}{r} \partial_r \chi_0^{\text{reg}} + \frac{\lambda^2}{R^2} \chi_0^{\text{reg}} = 0. \quad (36) \]

Choosing the solution of (36) which is regular at the origin we obtain,

\[ \chi_0^{\text{reg}} \propto J_0 \left( \lambda \frac{r}{R} \right). \quad (37) \]

Computing the action of the dilation operator on the solution in the area of constant potential and equating it to the action of the dilation operator (35) we get the equation on the \( \lambda \) regulator parameter:

\[ \frac{1}{2} + \nu \cot \left( \nu \log \left( \frac{R}{r_0} \right) \right) = -\lambda \frac{J_1(\lambda)}{J_0(\lambda)}. \quad (38) \]

The equation (38) defines \( \lambda \) as a multi-valued function of \( R \). The period of the RG flow corresponds to jump from one branch of the \( \lambda(R) \) function to another.

Now we proceed to find the bound states in the (26) potential. We consider again the Dirac equation,

\[ H_D \psi_\kappa = -\hbar v_F k \psi_\kappa. \quad (39) \]

Then the equation on \( \chi \) analogous to (31) is as following:

\[ \partial_r^2 \chi_\kappa + \frac{2\beta - \kappa r}{\beta - \kappa r} \frac{1}{r} \partial_r \chi_\kappa + \left( \frac{\beta - \kappa}{r - \kappa} \right)^2 \chi_\kappa = 0. \quad (40) \]

Asymptotically when \( r \gg \frac{\beta}{\kappa} \) the solution of (40) regular at infinity is given by the Hankel function,

\[ \chi_\kappa \propto H_0^{(1)}(ikr). \quad (41) \]
At small $r \ll \frac{\beta}{\kappa}$ the solution is not regular at the origin,
\[\chi_\kappa \propto \sqrt{r} \sin \left( \nu \log \frac{r}{r_0} \right), \quad (42)\]
and we are again in need for the regulator potential. Solving again the Dirac equation (39) in presence of the constant potential $V_{\text{reg}}$ and computing the action of the dilatation operator,
\[r \partial_r \chi_{\kappa}^{\text{reg}} = -\left( \lambda - \kappa R \right) \frac{J_1 (\lambda - \kappa R)}{J_0 (\lambda - \kappa R)} \chi_{\kappa}^{\text{reg}}, \quad (43)\]
we can equate (43) to the action of the dilatation operator on (42) and get the equation on the spectrum of the bound states,
\[\frac{1}{2} + \nu \cot (\nu \log (\kappa R)) = -\left( \lambda - \kappa R \right) \frac{J_1 (\lambda - \kappa R)}{J_0 (\lambda - \kappa R)}. \quad (44)\]
This condition gives the spectrum of infinitely many shallow bound states,
\[\kappa_n = \kappa_* \exp \left( -\frac{\pi n}{\nu} \right), \quad \kappa \to \infty. \quad (45)\]

4 Anomaly in the so(2, 1) algebra

Let us make some comments on the algebraic counterpart of the phenomena considered following [9]. As we have mentioned the conformal symmetry is the main player since Hamiltonians under consideration are scale invariant before regularization. Actually this group can be thought of as the example of spectrum generating algebra when the Hamiltonian is one of the generators or is expressed in terms of the generators in a simple manner. This is familiar from the exactly or quasi-exactly solvable problems when the dimension of the representation selects the size of the algebraic part of the spectrum.

Let us introduce the generators of the so(2, 1) conformal algebra $J_1, J_2, J_3$: the Calogero Hamiltonian,
\[J_1 = H = p^2 + V(r), \quad (46)\]
the dilatation generator,
\[J_2 = D = tH - \frac{1}{4}(pr + rp), \quad (47)\]
and the generator of special conformal transformation,

\[ J_3 = K = t^2 H - \frac{t}{2}(pr + rp) + \frac{1}{2}r^2. \]  (48)

They satisfy the relations of the so(2, 1) algebra:

\[ [J_2, J_1] = -iJ_1, \quad [J_3, J_1] = -2iJ_2, \quad [J_2, J_3] = iJ_3. \]  (49)

The singular behavior of the potential at the origin amounts to a kind of anomaly in the so(2, 1) algebra,

\[ A(r) = -i[D, H] + H, \]  (50)

which in d space dimensions can be presented in the following form:

\[ A(r) = -\frac{d-2}{2}V(r) + (r^i \nabla_i)V(r). \]  (51)

The simple arguments imply the following relation

\[ \frac{d}{dt} \langle D \rangle_{\text{ground}} = E_{\text{ground}}, \]  (52)

where the matrix element is taken over the ground state.

It turns out that (52) is fulfilled for the singular potentials in Calogero-like model or in models with contact potential, \( V(r) = g\delta(r) \). The expression for anomaly does not depend on the regularization chosen. Moreover more detailed analysis demonstrates that the anomaly is proportional to the \( \beta \)-function of the coupling providing the UV regularization as can be expected.

A similar calculation of the anomaly for the graphene case can be performed for arbitrary state,

\[ \langle \frac{dD}{dt} \rangle_\psi = \langle \Xi \rangle_\psi = -\int d^2x \psi^*(V(x) + x_i \partial_i V(x))\psi, \]  (53)

which yields using square-well regularization:

\[ \langle \Xi \rangle_\psi = \hbar v_F \frac{\lambda(R)}{R} \int_0^R \frac{r|\psi|^2dr}{\int_0^\infty r|\psi|^2dr}. \]  (54)
It is convenient to use the two-dimensional identity in (51),

$$\nabla \frac{\vec{r}}{r} = 2\pi \delta(\vec{r}),$$

which simplifies the calculation of the anomaly for any normalized bound state,

$$\frac{d}{dt} \langle D \rangle \Psi = -g\pi \int d^2 r \delta(r) |\Psi(r)|^2.$$ 

5 RG cycles in models of superconductivity

In this Section we explain how the cyclic RG flows emerge in truncated models of superconductivity. To this aim we shall first describe the Richardson model and then consider its generalization to the RD model which enjoys the cyclic RG flow. These models are distinguished by the finiteness of the number of fermionic levels. The relation with the integrable many-body systems proves to be quite useful.

5.1 Richardson model versus Gaudin model

Let us recall the truncated BCS-like Richardson model of superconductivity [26] with some number of doubly degenerated fermionic levels with the energies $\epsilon_{j\sigma}, j = 1, \ldots, N$. It describes the system of a fixed number of the Cooper pairs. It is assumed that several energy levels are populated by Cooper pairs while levels with the single fermions are blocked. The Hamiltonian reads as

$$H_{BCS} = \sum_{j,\sigma=\pm}^N \epsilon_{j\sigma} c_{j\sigma}^\dagger c_{j\sigma} - G \sum_{jk} c_{j+}^\dagger c_{j-} c_{k-} c_{k+},$$

where $c_{j\sigma}$ are the fermion operators and $G$ is the coupling constant providing the attraction leading to the formation of the Cooper pairs. In terms of the hard-core boson operators it reads as

$$H_{BCS} = \sum_j \epsilon_j b_j^\dagger b_j - G \sum_{jk} b_j^\dagger b_k,$$

where

$$[b_j^\dagger, b_k] = \delta_{jk} (2N_j - 1), \quad b_j = c_{j-} c_{j+}, \quad N_j = b_j^\dagger b_j.$$
The eigenfunctions of the Hamiltonian can be written as,

$$|M\rangle = \prod_i^M B_i(E_i)|\text{vac}\rangle, \quad B_i = \sum_j^N \frac{1}{\epsilon_j - E_i} b^\dagger_j,$$

(60)

provided the Bethe ansatz equations are fulfilled,

$$G^{-1} = -\sum_j^N \frac{1}{\epsilon_j - E_i} + \sum_j^M \frac{2}{E_j - E_i}.$$  

(61)

The energy of the corresponding states reads as:

$$E(M) = \sum_i E_i.$$  

(62)

It was shown in [27] that the Richardson model is exactly solvable and closely related to the particular generalization of the Gaudin model. To describe this relation it is convenient to introduce the so-called pseudospin \(sI(2)\) algebra in terms of the creation-annihilation operators for the Cooper pairs,

$$t_j^- = b_j, \quad t_j^+ = b^\dagger_j, \quad t_j^0 = N_j - 1/2.$$  

(63)

The Richardson Hamiltonian commutes with the set of operators \(R_i\),

$$R_i = -t_i^0 - 2G \sum_{j \neq i}^N \frac{t_i t_j}{\epsilon_i - \epsilon_j},$$  

(64)

which are identified as the Gaudin Hamiltonians,

$$[H_{BCS}, R_j] = [R_i, R_j] = 0.$$  

(65)

Moreover the Richardson Hamiltonian itself can be expressed in terms of the operators \(R_i\) as:

$$H_{BCS} = \sum_i \epsilon_i R_i + G \left(\sum_i R_i\right)^2 + \text{const.}$$  

(66)

The number \(N\) of the fermionic levels coincides with the number of sites in the Gaudin model and the coupling constant in the Richardson Hamiltonian corresponds to the "twisted boundary condition" in the Gaudin model.
Bethe ansatz equations for the Richardson model \[61\] exactly coincide with the ones for the generalized Gaudin model. It was argued in \[2\] that the Bethe roots correspond to the excited Cooper pairs that is natural to think about the solution to the Baxter equation as the wave function of the condensate. In terms of the conformal field theory Cooper pairs correspond to the screening operators \[28\].

For the nontrivial degeneracies of the energy levels \(d_j\) the BA equations read as:

\[
G^{-1} = -\sum_{j}^{N} \frac{d_j}{\varepsilon_j - E_i} + \sum_{j \neq i}^{M} \frac{2}{E_j - E_i}.
\]

(67)

5.2 Russian Doll model of superconductivity and twisted XXX spin chains

The important generalization of the Richardson model describing superconductivity is the so-called RD model \[2\]. It involves the additional dimensionless parameter \(\alpha\) and the RD Hamiltonian reads as:

\[
H_{RD} = \frac{2}{N} \sum_{i}^{N} (\epsilon_i - G) N_i - \bar{G} \sum_{j<k}^{M} (e^{i\alpha} b_k^+ b_j + e^{-i\alpha} b_j^+ b_k),
\]

(68)

with two dimensionful parameters \(G, \eta\) and \(\bar{G} = \sqrt{G^2 + \eta^2}\). In terms of these variables the dimensionless parameter \(\alpha\) has the following form:

\[
\alpha = \arctan \left( \frac{\eta}{G} \right).
\]

(69)

It is also useful to consider two dimensionless parameters \(g, \theta\) defined as \(G = gd\) and \(\eta = \theta d\) where \(d\) is the level spacing. The RD model reduces to the Richardson model in the limit \(\eta \rightarrow 0\).

The RD model turns out to be integrable as well. Now instead of the Gaudin model the proper counterpart is the generic quantum twisted XXX spin chain \[29\]. The transfer matrix of such spin chain model \(t(u)\) commutes with the \(H_{RD}\) which itself can be expressed in terms of the spin chain Hamiltonians.

The equation defining the spectrum of the RD model reads as:

\[
e^{2i\alpha} \prod_{l=1}^{N} \frac{E_i - \varepsilon_l + i\eta}{E_i - \varepsilon_l - i\eta} = \prod_{j \neq i}^{M} \frac{E_i - E_j + 2i\eta}{E_i - E_j - 2i\eta},
\]

(70)
and coincides with the BA equations for the spin chain.

Taking the logarithm of the both sides of the equation (70) we obtain:

\[ \alpha + \pi Q_i + \sum_{l=1}^{N} \arctan \left( \frac{\eta}{E_i - \varepsilon_l} \right) - \sum_{j=1}^{M} \arctan \left( \frac{2\eta}{E_i - E_j} \right) = 0. \quad (71) \]

Note that here we have added an arbitrary integer term to account for generically multivalued arctangent function.

The RG step amounts to integrating out the \( N \)-th degree of freedom in the RD model, or equivalently to integrating out the \( N \)-th inhomogeneity in the XXX chain. This results into renormalization of the twist. From (71) it is easy to see that:

\[ \arctan \left( \frac{\eta}{G_N} \right) - \arctan \left( \frac{\eta}{G_{N-1}} \right) = \sum_{i=1}^{M} \arctan \left( \frac{2\eta}{E_i - \varepsilon_N} \right). \quad (72) \]

When \( M = 1 \) it implies that:

\[ G_{N-1} - G_N = \frac{G_N^2 + \eta^2}{\varepsilon_N - G_N - E}, \quad (73) \]

which is a discrete version of the (1) equation. Of course the same relation can be derived from the RD Hamiltonian (68). If we consider the wavefunction \( \psi = \sum_{i}^{N} \psi_i b_i^\dagger |0\rangle \), the Schrödinger equation for a state with one Cooper pair amounts to:

\[ (\varepsilon_i - G - E) \psi_i = (G + i\eta) \sum_{j=1}^{i-1} \psi_j + (G - i\eta) \sum_{j=i+1}^{N} \psi_j. \quad (74) \]

Integration out the \( N \)-th degree of freedom amounts to expressing \( \psi_M \) in terms of the other modes,

\[ \psi_N = \frac{G + i\eta}{\varepsilon_N - G - E} \sum_{j=1}^{N-1} \psi_j, \quad (75) \]

and substituting it back into the Schrödinger equation (74). The \( G_{N-1} \) constant in the resulting equation will differ from the initial \( G_N \) value as in (73).
The key feature of the RD model is the multiple solutions to the gap equation. The gaps are parameterized as follows:

$$\Delta_n = \frac{\omega}{\sinh t_n}, \quad t_n = t_0 + \frac{\pi n}{\theta}, \quad n = 0, 1, \ldots,$$  \hspace{1cm} (76)

where $t_0$ is the solution to the following equation:

$$\tan(\theta t_0) = \frac{\theta}{g}, \quad 0 < t_0 < \frac{\pi}{\theta},$$  \hspace{1cm} (77)

and $\omega = dN$ for equal level spacing. Here $E^2 = \varepsilon^2 + |\Delta|^2$. This behavior can be derived via the mean field approximation [14]. The gap with minimal energy defines the ground state, and the other values of the gap describe excitations. In the limit $\theta \to 0$ the gaps $\Delta_n > 0 \to 0$ and

$$t_0 = \frac{1}{\theta}, \quad \Delta_0 = 2\omega \exp\left(-\frac{1}{\theta}\right),$$  \hspace{1cm} (78)

therefore the standard BCS expression for the gap is recovered. At the weak coupling limit the gaps behave as:

$$\Delta_n \propto \Delta_0 e^{-\frac{\alpha}{\theta}}.$$  \hspace{1cm} (79)

In terms of the solutions to the BA equations the multiple gaps correspond to the choices of the different branches of the logarithms, i.e. to different choices of the integer $Q$ parameter in (70).

If the degeneracy of the levels is $d_n$ then the RD model gets modified a little bit and is related to the higher spin XXX spin chain. The local spins $s_i$ are determined by the corresponding higher pair degeneracy $d_i$ of the $i$-th level,

$$s_i = d_i / 2,$$  \hspace{1cm} (80)

and the corresponding BA equations read as:

$$e^{2i\alpha} \prod_{l=1}^{N} \frac{E_i - \varepsilon_i + id_l + i\eta}{E_i - \varepsilon_i - id_l - i\eta} = \prod_{j \neq i}^{M} \frac{E_i - E_j + 2i\eta}{E_i - E_j - 2i\eta},$$  \hspace{1cm} (81)
5.3 Cyclic RG flows in the RD model

The RD model of truncated superconductivity enjoys the cyclic RG behavior [2]. The RG flows can be treated as the integrating out the highest fermionic level with appropriate scaling of the parameters using the procedure developed in [1]. The RG equations read as (73):

\[ g_{N-1} = g_N + \frac{1}{N}(g_N^2 + \theta^2), \quad \theta_{N-1} = \theta_N. \]  

(82)

At large \( N \) limit the natural RG variable is identified with \( s = \log(N/N_0) \) and the solution to the RG equation is:

\[ g(s) = \theta \tan \left( \theta s + \tan^{-1} \left( \frac{g_0}{\theta} \right) \right). \]  

(83)

Hence the running coupling is cyclic,

\[ g(s + \lambda) = g(s), \quad g(e^{-\lambda}N) = g(N), \]  

(84)

with the RG period,

\[ \lambda = \frac{\pi}{\theta}, \]  

(85)

and the total number of the independent gaps in the model is:

\[ N_{\text{cond}} \propto \frac{\theta}{\pi} \log N. \]  

(86)

The multiple gaps are the manifestations of the Efimov-like states. The sizes of the Cooper pairs in the \( N \)-th condensates also have the RD scaling. The cyclic RG can be derived even for the single Cooper pair.

What is going on with the spectrum of the model during the period? It was shown in [14] that it gets reorganized. The RG flow experiences discontinuities from \( g = +\infty \) to \( g = -\infty \) when a new cycle gets started. At each jump the lowest condensate disappears from the spectrum,

\[ \Delta_{N+1}(g = +\infty) = \Delta_N(g = -\infty), \]  

(87)

indicating that the \((N + 1)\)-th state wave function plays the role of \( N \)-th state wave function at the next cycle (see (75)).

The same behavior can be derived from the BA equation [14]. To identify the multiple gaps it is necessary to remind that the solutions to the BA
Figure 1: Besides the triality shown on the picture, a bispectral duality acts on RS/Calogero and XXX/Gaudin sides of the correspondence. Being originated from three-dimensional mirror-symmetry [30], this duality interchanges coordinates with Lax eigenvalues in the classical systems, and inhomogeneities with twists in quantum ones.

equations are classified by the integers $Q_i, i = 1, \ldots, M$ parameterizing the branches of the logarithms. If one assumes that $Q_i = Q$ for all Bethe roots then this quantum number gets shifted by one at each RG cycle and was identified with the integer parameterizing the solution to the gap equations,

$$\Delta_Q \propto \Delta_0 \exp^{-\lambda Q}. \quad (88)$$

At the large $N$ limit the BA equations of the RD model reduce to the BA equation of the Richardson-Gaudin model with the rescaled coupling,

$$G_Q^{-1} = \eta^{-1}(\alpha + \pi Q), \quad (89)$$

which can be treated as the shifted boundary condition in the generalized Gaudin model parameterized by an integer. Let us emphasize that the unusual cyclic RG behavior is due to the presence of two couplings in the RD model.

6 Triality in the integrable models and RG cycles

In this Section we summarize several dualities between the integrable models and consider the realizations of the cyclic RG flows in these systems. The question is motivated by the close relationship between the restricted BCS
models and spin chains. Actually there are three different families of models related with each other by the particular identifications of phase spaces and parameters. The first family concerns the system of fermions (Richardson-Russian Doll) which develop superconducting gap. The second family involves the spin systems of twisted inhomogeneous Gaudin-XXX-XXZ type and their generalizations. The third family involves the Calogero-Ruijsenaars (CR) chain of the integrable many body systems.

We look for the answers on the following questions

- What is the condition yielding the RG equation for some coupling in each family?
- What is the RG variable?
- What determines the period of the cycle?

In the superconducting system at RG step one decouples the highest energy level and looks at the renormalization of the interaction coupling constant. The RG time is identified with the number of energy levels \( t = \log N \). The period of RG is defined by the T-asymmetric parameter of RD model.

In the spin chain model the RG step corresponds to the ”integrating out” one ”highest” inhomogeneity with the corresponding renormalization of the twist. The period of the RG flow is fixed by the Planck constant in the quantum spin chain. In the bispectral dual spin chain \([33]\) one now ”integrates out” one of the twists and ”renormalizes” the inhomogeneity. Since the Planck constant gets inverted upon bispectrality \( \hbar_{spin} \to \hbar_{spin}^{-1} \) the period of the RG cycle gets inverted as well. Note that the RG equation in the superconducting model can be mapped into BAE in the spin chain \([14]\).

The condition yealding the RG equation corresponds to the independence of the Bethe root on the RG step.

For two-body system with attractive rational potential one can define the RG condition as the continuity of the zero-energy wave function under the changing the cutoff scale at small \( r \). This condition imposes the RG equation at the cutoff UV coupling constant. This RG equation has the cycle with the period

\[
T_{Cal} = \frac{\pi}{\nu - \frac{1}{2}},
\]

as was shown in Section \([2]\).
The Quantum-Classical (QC) duality [30, 31] relates the quantum spin chain systems and the classical Calogero-type systems. Through the QC correspondence, the rational Gaudin model can be linked with the rational Calogero system spin chain inhomogeneities being the Calogero coordinates, and the twist in the spin chain (which is a single variable in our case) being the Lax matrix eigenvalue. It is also possible to make a bispectrality transformation of rational Calogero model, which interchanges Lax eigenvalues with coordinates. This means that now the Calogero coordinates correspond to the twists at the spin chain side. In this case the Calogero coupling gets inverted which means that the period of the RG cycle gets inverted as well.

To consider the mapping of RG cycles in the Calogero system and the spin chain we need the generalization of QC duality to the quantum-quantum case. The spectral problem in Calogero model has been identified with the KZ equation involving the Gaudin Hamiltonian,

$$\frac{d}{dz_i} \Psi = H_{gaud} \Psi + \lambda \Psi.$$  \hbox{(91)}

Since we formulate RG condition on the Calogero side for the $E = 0$ state, the inhomogeneous term in the KZ equation is absent. The simplest test of the mapping of the RG cycles under QQ duality concerns the identifications of the periods. On the spin chain side it is identified with the Planck constant while at the Calogero side the period is defined by the coupling constant. The following identification holds for QC duality [31]:

$$\hbar_{\text{spin}} = \nu,$$  \hbox{(92)}

which implies that the periods of the cycles at the Calogero and spin chain sides match.

The Efimov-like tower in these families have the following interpretations. In the superconducting system it corresponds to the family of the gaps $\Delta_n$ with the Efimov scaling responsible of the scale symmetry broken down to the discrete subgroup. In the spin chain it corresponds to the different branches of the solutions to the BAE which can be also interpreted in terms of the allowed set of twists. Finally in the CR family it corresponds to the family of the shallow bound states near the continuum threshold.
7 RG cycles in Ω-deformed SUSY gauge theories

In this Section we shall explain how the RG flows in Ω-deformed SUSY gauge theories can be reformulated in terms of the brane moves. Why the very RG cycles could be expected in the deformed gauge theories? The answer is based on the identification of the quantum spin chains in one or another context in the SUSY gauge theory. Once such quantum spin chain has been found we can apply the results of the previous sections where the place of the RG cycles in the spin chain framework has been clarified.

First, we shall briefly review the Ω-deformation of the SUSY gauge theories. Then we make some general comments concerning the realization of the gauge theories as the worldvolume theories on D-branes to explain how the parameters of the gauge theory are identified with the brane coordinates.

7.1 Four-dimensional Ω-deformed gauge theory

The Bethe ansatz equations can be encountered not only in the models of superconductivity, but also in gauge theories. The quantum XXX spin chain governs the moduli space of vacua of an Ω-deformed four-dimensional theory in the Nekrasov–Shatashvili limit, i.e. when one of the deformations is chosen to be zero: $\epsilon_2 = 0, \epsilon_1 = \epsilon$. Since the quantum XXX spin chain displays a cyclic RG behaviour, as we have seen in the Section 5, it is interesting to identify this phenomenon in the four-dimensional gauge theory.

Consider a four-dimensional $\mathcal{N} = 2$ theory with matter hypermultiplet, which has a vanishing $\beta$-function, i.e. when $N_f = 2N_c$. This theory is dual to a classical inhomogeneous twisted XXX chain, in a sense that the Seiberg-Witten curve for the gauge theory coincides with the spectral curve for the spin chain. The twist of the spin chain is identified with the modular parameter of the curve and with the complexified coupling of the gauge theory, the inhomogeneities of the spin chain get mapped into masses of the hypermultiplets. For more information on the correspondence between classical integrable systems and gauge theories the reader can consult [36].

The Ω-deformation is introduced to regularize the instanton divergence in the partition function of the gauge theory [37]. We can consider the four-dimensional theory as a reduction of the six-dimensional $\mathcal{N} = 1$ theory with metric:
\[ ds^2 = 2dzd\bar{z} + (dx^m + \Omega^{mn}x_n dz + \tilde{\Omega}^{mn}x_n d\bar{z})^2, \quad m = 1, \ldots, 4, \]  

(93)
i.e. we can consider the theory on a four-dimensional space, fibered over a two-dimensional torus. One can imagine the \( \epsilon_{1,2} \) deformation parameters as chemical potentials for the rotations in two orthogonal planes in four-dimensional Euclidean space. One can also think that the Euclidean \( \mathbb{R}^4 \) space gets substituted by a sphere \( S^4 \) with finite volume.

The non-trivial \( \Omega \)-deformation modifies the correspondence between gauge theories and integrable systems. Namely, in the Nekrasov-Shatashvili limit the \( \Omega \)-deformed gauge theory corresponds to a quantum XXX spin chain with \( \epsilon \) playing the role of the Planck constant [35]. This deformed gauge theory also appears to be dual to the two-dimensional effective theory on a worldsheet of a non-abelian string [39].

Consider \( \Omega \)-deformed \( \mathcal{N} = 2 \) SQCD with \( SU(L) \) gauge group, \( L \) fundamental hypermultiplets with masses \( m_i^f \) and \( L \) antifundamental hypermultiplets with masses \( m_i^{af} \). Let us denote the set of the eigenvalues of the adjoint scalar in the vector multiplet by \( \vec{a} \). We can expand the deformed partition function around \( \epsilon = 0 \) to identify the prepotential and effective twisted superpotential,

\[ \log Z(\vec{a}, \epsilon_1, \epsilon_2) \sim \frac{1}{\epsilon_1 \epsilon_2} F(\vec{a}, \epsilon) + \frac{1}{\epsilon_2} W(\vec{a}, \epsilon). \]  

(94)

The effective twisted superpotential is a multivalued function, with the branch fixed by the set of integers \( \vec{k} \):

\[ W(\vec{a}, \epsilon) = \frac{1}{\epsilon} F(\vec{a}, \epsilon) - 2\pi i \vec{k} \cdot \vec{a}, \quad \vec{k} \in \mathbb{Z}^L. \]  

(95)
The equation on vacua,

\[ \frac{\partial W(\vec{a}, \epsilon)}{\partial \vec{a}} = \vec{n}, \quad \vec{n} \in \mathbb{Z}^L, \]  

(96)
provides the condition on \( \vec{a} \),

\[ \vec{a} = \vec{m}^f - \epsilon \vec{n}. \]  

(97)

This theory admits the existence of non-abelian strings probing the four-dimensional space-time. The two-dimensional worldsheet theory of the non-abelian string involves \( L \) fundamental chiral multiplets with twisted masses
$M_i^F$ and $L$ antifundamental multiplets with twisted masses $M_i^{AF}$, which are identified as:

$$M_i^F = m_i^f - \frac{3}{2}\epsilon, \quad M_i^{AF} = m_i^{af} + \frac{1}{2}\epsilon.$$  (98)

The two-dimensional theory also contains an adjoint chiral multiplet with mass $\epsilon$. The rank of the gauge group $N$ (or equivalently the number of non-abelian strings) is given in terms of $\vec{n}$ vector by the relation:

$$N + L = \sum_{i=1}^{L} n_i.$$  (99)

The modular parameters of the four-dimensional and the two-dimensional theories are related as:

$$\tau_{2d} = \tau_{4d} + \frac{1}{2}(N + 1).$$  (100)

The effective twisted worldsheet superpotential is given in terms of the four-dimensional superpotential:

$$\mathcal{W}_{4d}\left(a_i = m_i^f - \epsilon n_i, \epsilon\right) - \mathcal{W}_{4d}\left(a_i = m_i^f - \epsilon, \epsilon\right) = \mathcal{W}_{2d}\left(\{n_i\}\right).$$  (101)

The two-dimensional superpotential depends on the set of eigenvalues of the adjoint scalar in vector representation $\lambda_i$, $i = 1, \ldots, N$. The set of equations $\partial\mathcal{W}_{2d}/\partial\lambda = 0$ appears to be equivalent to the Bethe ansatz equations for the XXX spin chain:

$$\prod_{i=1}^{L} \left( \frac{\lambda_j - M_i^F}{\lambda_j - M_i^{AF}} \right) = \exp\left(2\pi i \tau_{4d} \prod_{k\neq j}^{N} \left( \frac{\lambda_j - \lambda_k - \epsilon}{\lambda_j - \lambda_k + \epsilon} \right) \right).$$  (102)

The Planck constant in the spin chain is identified with the $\epsilon$ deformation parameter. The complexified coupling parameter plays the role of twist in the spin chain. The renormalization of the spin chain amounts to decoupling of one fundamental and one anti-fundamental chiral multiplet. In the four-dimensional theory it corresponds to the decrease of the number of flavors $N_f \to N_f - 2$ simultaneously with reducing the rank of the gauge group $N_c \to N_c - 1$. Therefore the theory remains conformal. The renormalization
of the coupling constant analogous to (73) derived from the relation (102) for \( N = 1 \) is:

\[
\exp (2\pi i (\tau_L - \tau_{L-1})) = \frac{\lambda - M_L^F}{\lambda - M_L^{AF}}.
\]  

If we choose the masses to be equidistant with spacing \( \delta m \), the change in the coupling constant during one step of RG flow is:

\[
\exp (2\pi i (\tau_L - \tau_{L-1})) \propto \frac{\epsilon}{\delta m}.
\]

Hence a number of nonperturbative scales emerges in a theory, analogously to the generation of the Efimov scaling in the Calogero model. These scales correspond to multiple gaps in the superconducting model:

\[
\Delta_n \propto \Delta_0 \exp \left( -\frac{n \delta m}{\epsilon} \right).
\]

Note that the emergence of cyclic RG evolution is a feature caused by the \( \Omega \)-deformation, since in a non-deformed theory a decoupling of the heavy flavor does not lead to any cyclic dynamics.

### 7.2 3d gauge theories and theories on the brane world-volumes

Let us briefly explain the main points concerning the geometrical engineering of the gauge theories on the \( D \)-branes suggesting the reader to consult the details in the review paper [40]. The \( Dp \) brane is the \((p+1)\)-dimensional hypersurface in the ten-dimensional space-time which supports the \( U(1) \) gauge field. This feature provides the possibility to built up the gauge theories with the desired properties. Let us summarize the key elements of the ”building procedure”.

- A stack of coinciding \( N \) \( D \)-branes supports \( U(N) \) gauge theory with the maximal supersymmetry.

- Displacing some branes from the stack in the transverse direction corresponds to the Higgs mechanism in the \( U(N) \) gauge theory and the distance between branes corresponds to the Higgs vev.
Figure 2: Hanany-Witten move. Here vertical lines are $NS_5$ branes, horizontal lines are $D3$ branes, and circles are $D5$ branes. When a $D5$ brane is moved through a sequence of $NS_5$ branes the linking number between them is conserved hence additional $D3$ branes appear.

- To reduce the SUSY one imposes some boundary conditions at some coordinates using other types of branes or rotates some branes.

- All geometrical characteristics of the brane configurations have the meaning of parameters of the gauge theory like couplings or vevs of some operators in the gauge theory on their worldvolumes.

- If we move some brane through another one the brane of smaller dimension could be created. The Hanany-Witten move is the simplest example (see fig. 2).

- Since generically we have branes of different dimensions in the configuration, for example, $N$ $D2$ branes and $M$ $D4$ branes we have simultaneously $U(N)$ 2+1 dimensional gauge theory and $U(M)$ dimensional 4+1 dimensional theory on the brane worldvolumes. These theories coexist simultaneously hence there is highly nontrivial interplay between two gauge theories.

Let us explain now how these brane rules can be used to engineer the gauge theories which are related with the quantum spin chains. Our main example is a 3d $\mathcal{N} = 2$ quiver gauge theory.

The brane configuration relevant for this theory is built as follows. We have $M$ parallel $NS_5$ branes extended in (012456), $N_i$ $D3$ branes extended in (0123) between $i$-th and $(i+1)$-th $NS_5$ branes, and $M_i$ $D5$ branes extended in (012789) between $i$-th and $(i+1)$-th $NS_5$ branes (see table 3). From this brane configuration we obtain the $\prod_i^M U(N_i)$ gauge group on the $D3$ branes worldvolume with $M_i$ fundamentals for the $i$-th gauge group. The distances between the $i$-th and $(i+1)$-th $NS_5$ branes yield the complexified gauge
coupling for $U(N_i)$ gauge group while the coordinates of the $D5$ branes in the (45) plane correspond to the masses of fundamentals. The positions of the $D3$ branes on (45) plane correspond to the coordinates on the Coulomb branch in the quiver theory. The additional $\Omega$ deformation reduces the theory with $\mathcal{N} = 4$ SUSY to the $\mathcal{N} = 2^*$ theory, i.e. an $\mathcal{N} = 2$ theory with massive adjoint. It is identified as 3$d$ gauge theory when the distance between NS5 is assumed to be small enough. We assume that one coordinate is compact that is the theory lives on $\mathbb{R}^2 \times S^1$.

The mapping of the gauge theory data into the integrability framework goes as follows. In the NS limit of the $\Omega$-deformation the twisted superpotential in 3$d$ gauge theory on the $D3$ branes gets mapped into the Yang-Yang function for the XXZ chain [35]. The minimization of the superpotential yields the equations describing the supersymmetric vacua and in the same time they are the Bethe ansatz equations for the XXZ spin chain, generally speaking the nested Bethe ansatz equations. That is $D3$ branes are identified with the Bethe roots which are distributed according to the ranks of the gauge groups at each of $M$ steps of nesting $\prod_i^M U(N_i)$. The positions of the $D5$ branes in the (45) plane correspond to the inhomogeneities in the XXZ spin chain. The anisotropy of the XXZ chain is defined by the radius of the compact dimensions while the parameter of the $\Omega$ deformation plays the role of the Planck constant in the XXZ spin chain. At small radius the XXZ spin chain turns to the XXX spin chain. The twists in the spin chain correspond to the coordinates of the NS5 branes in the (78) plane, and the Fayet–Iliopoulos parameters in the three-dimensional theory [30].

One step of the RG flow corresponds to elimination of one inhomogeneity in the spin chain resulting in renormalization of the twists. In the three-dimensional theory this means that the integration of one massive flavor leads to the renormalization of the FI parameters. In terms of the transformations of the brane configurations this process receives transparent geometrical interpretation:
• The RG step is the removing of one $D5$ brane which amounts to the renormalization of the position of $NS5$ branes or twists.

• The period of the RG cycle is fixed by the number of $NS5$ branes \cite{34}, since it was identified with the Planck constant in the spin chain.

• At some scale the twists flow from $+\infty$ to $-\infty$.

8 Conclusion

Are there any general lessons which we could learn for the quantum field theory from the very existence of the cyclic RG flows? The most important point is that there is some fine structure at the UV scale which is reflected in the Efimov tower with the BKT scaling behavior. Moreover the cyclic flows imply the interplay between the UV and IR cutoffs in the theory which usually was attributed to the noncommutative theory. This mixing presumably could shed the additional light on the dimensional transmutation phenomena in the field theory and provide the examples for the simultaneous generation of the multiple scales.

The presence of two parameters in RG is quite common however probably some additional properties of these parameters are required. In particular in many (although not all) examples the period of the cycle is fixed by the “filling fraction” in some external field which could be magnetic field or parameter of $\Omega$ background. The latter has the meaning of the Planck constant in the auxiliary finite dimensional integrable model. This could suggest that the very issue can be formulated purely in terms of the quantum phase space since the Planck constant can be interpreted as the external field applied to the classical phase space.

Actually we could expect the relation of RG cycles with some refinement of the path integral in quantum mechanics. As an aside remark note that the attempt to get the rigorous mathematical formulation of the renormalization of the QFT leaded to the motivic generalization of the path integral. It corresponds to some fine structure at the regulator scale which has some similarities with the discussion above. The RG cycle in the quantum rational Calogero model implies the intimate relation with the knot theory since the knot invariants at the rational Calogero coupling are the characteristics of the Calogero spectrum (cf. \cite{34}).
As we already mentioned, cyclic renormalization dynamics is connected with BKT–pairing of partons in two-dimensional model. One could wonder whether this connection is universal. One four-dimensional example of such pairing has to be mentioned. It is bion condensation in 3+1 dimensions. The RG analysis of the model involving the gas of bions and electrically charged W-bosons has been considered in [12] where the RG flows involves the fugacities for electric and magnetic components and the coupling constant. The coupled set of the RG equations has been solved explicitly in the self-dual case and the solution to the RG equations for the fugacities obtained in [12] is identical to the solution for the coupling in the RD model upon the analytic continuation. The period of the RG in the solution above is fixed by the RG invariant which has been identified with the product of the UV values of the electric and magnetic fugacities $y_e \times y_m$. The similarity between the RG behavior is not accidental since the mapping of the gauge theory and the perturbed XY model has been found in [12].

We would like to emphasize that the investigation of various aspects of limit cycles in RG dynamics still remains on its early stage and there is a considerable number of open questions. The RG cycles can have numerous applications to different aspects of mathematical physics. In this case the RG dynamics is considered as an example of non-trivial dynamical system.

The work of A.G. and K.B. was supported in part by grants RFBR-12-02-00284 and PICS-12-02-91052. The work of K.B. was also supported by the “Dynasty” fellowship. A.G. thanks FTPI at University of Minnesota where the part of this work has been done for the hospitality and support. We would like to thank N. Nekrasov and F. Popov for useful discussions and comments.

References

[1] S. Glazek and K. Wilson, “Limit cycles in quantum theories,” Phys.Rev.Lett. 89 (2002) 23401, arXiv:hep-th/0203088

[2] A. LeClair, J. M. Roman and G. Sierra, “Russian doll renormalization group and superconductivity,” Phys. Rev. B 69, 20505 (2004) arXiv:cond-mat/0211338
[3] E. Braaten, H. -W. Hammer, “Universality in few-body systems with large scattering length,” Phys. Rept. 428, 259-390 (2006), arXiv:cond-mat/0410417.

[4] M. Bawin and S. A. Coon, “The Singular inverse square potential, limit cycles and selfadjoint extensions,” Phys. Rev. A 67, 042712 (2003), arXiv:quant-ph/0302199.

E. Braaten and D. Phillips, “The Renormalization group limit cycle for the 1/r**2 potential,” Phys. Rev. A 70, 052111 (2004), arXiv:hep-th/0403168.

[5] S. R. Beane, P. F. Bedaque, L. Childress, A. Kryjevski, J. McGuire and U. van Kolck, “Singular potentials and limit cycles,” Phys. Rev. A 64, 042103 (2001), arXiv:quant-ph/0010073.

[6] A. Leclair, J. M. Roman and G. Sierra, “Russian doll renormalization group, Kosterlitz-Thouless flows, and the cyclic sine-Gordon model,” Nucl. Phys. B 675, 584 (2003), arXiv:hep-th/0301042.

[7] D. B. Kaplan, J. -W. Lee, D. T. Son and M. A. Stephanov, “Conformal-Loss,” Phys. Rev. D 80, 125005 (2009), arXiv:0905.4752 [hep-th].

[8] A. Gorsky, “SQCD, Superconducting Gaps and Cyclic RG Flows,” arXiv:1202.4306 [hep-th].

[9] G. N. J. Ananos, H. E. Camblong, C. Gorrichategui, E. Hernandez and C. R. Ordonez, “Anomalous commutator algebra for conformal quantum mechanics,” Phys. Rev. D 67, 045018 (2003), arXiv:hep-th/0205191.

H. E. Camblong and C. R. Ordonez, “Renormalization in conformal quantum mechanics,” Phys. Lett. A 345, 22 (2005), arXiv:hep-th/0305035.

S. Moroz and R. Schmidt, “Nonrelativistic inverse square potential, scale anomaly, and complex extension,” Annals Phys. 325, 491 (2010), arXiv:0909.3477 [hep-th].

G. N. J. Ananos, H. E. Camblong and C. R. Ordonez, “SO(2,1) conformal anomaly: Beyond contact interactions,” Phys. Rev. D 68, 025006 (2003), arXiv:hep-th/0302197.

[10] T. L. Curtright, X. Jin and C. K. Zachos, “RG flows, cycles, and c-theorem folklore,” Phys. Rev. Lett. 108, 131601 (2012) arXiv:1111.2649 [hep-th].
[11] V. Efimov, “Energy levels arising from resonant two-body forces in a three-body system,” Phys. Lett. B33, 563 (1970).
V. Efimov, “Energy levels of three resonantly interacting particles,” Nucl. Phys. A210, 157 (1973).

[12] H.-W. Hammer and L. Platter, “Efimov physics from a renormalization group perspective,” Phil. Trans. Roy. Soc. Lond. A 369, 2679 (2011), arXiv:1102.3789 [nucl-th].

[13] E. Brezin, J. Zinn-Justin, “Renormalization group approach to matrix models,” Phys. Lett. B288, 54-58 (1992), arXiv:hep-th/9206035

[14] A. Anfossi, A. Leclair and G. Sierra, “The elementary excitations of the exactly solvable Russian doll BCS model of superconductivity,” Journal of Statistical Mechanics: 05011 (2005), arXiv:cond-mat/0503014 [cond-mat.supr-con].

[15] K. Jensen, A. Karch, D. T. Son and E. G. Thompson, “Holographic Berezinskii-Kosterlitz-Thouless Transitions,” Phys. Rev. Lett. 105, 041601 (2010), arXiv:1002.3159 [hep-th].

[16] V. A. Miransky, “Dynamics of Spontaneous Chiral Symmetry Breaking and Continuum Limit in Quantum Electrodynamics,” Nuovo Cim. A 90, 149 (1985).
V. P. Gusynin, V. A. Miransky and I. A. Shovkovy, “Catalysis of dynamical flavor symmetry breaking by a magnetic field in (2+1)-dimensions,” Phys. Rev. Lett. 73, 3499 (1994) [Erratum-ibid. 76, 1005 (1996)] [hep-ph/9405262].

[17] D. Arean, I. Iatrakis, M. Jarvinen and E. Kiritsis, “The discontinuities of conformal transitions and mass spectra of V-QCD,” JHEP 1311, 068 (2013), arXiv:1309.2286 [hep-ph].

[18] A. Gorsky and F. Popov, ”Atomic collapse in graphene and cyclic RG flow”, arXiv:1312.7399

[19] A. Shytov, M. Katsnelson and L. Levitov, “Vacuum Polarization and Screening of Supercritical Impurities in Graphene,” Phys. Rev. Lett. 99, 236801 (2007), arXiv:0705.4663 [cond-mat.mes-hall].
[20] V. Pereira, V. Kotov and A. Castro Neto, “Supercritical Coulomb Impurities in Gapped Graphene,”, Phys.Rev. B78, 085101 (2008), arXiv:0803.4195 [cond-mat.mes-hall].

[21] M. Fogler, D. Novikov and B. Shklovskii, “Screening of a hypercritical charge in graphene,” Phys. Rev. B 76, 233402 (2007), arXiv:0707.1023 [cond-mat.mes-hall].

[22] Y. Wang et al., Nat. Phys 8, 653 (2012).

[23] Y. Wang et al., Science 340, 734 (2013).

[24] A. Shytov, M. Katsnelson and L. Levitov, “Atomic Collapse and Quasi-Rydberg States in Graphene,” Phys. Rev. Lett. 99, 246802 (2007), arXiv:0708.0837 [cond-mat.mes-hall].

[25] Y. Pomeranchuk and Y. Smorodinsky, J.Phys. USSR, 9,97 (1945).
Y. Zeldovich and V. Popov, Sov.Phys.Usp. 14, 673 (1972).

[26] R. Richardson, ”A restricted class of exact eigenstates of the pairing-force Hamiltonian”, Phys. Lett 3, (1963) 277.
M. C. Cambiaggio, A. M. F. Rivas and M. Saraceno, “Integrability of the pairing hamiltonian,” Nucl.Phys. A 624, 157 (1997), arXiv:nucl-th/9708031.

[27] M. C. Cambiaggio, A. M. F. Rivas and M. Saraceno, “Integrability of the pairing hamiltonian,” Nucl.Phys. A 624, 157 (1997), arXiv:nucl-th/9708031.

[28] G. Sierra, “Conformal field theory and the exact solution of the BCS Hamiltonian,” Nucl. Phys. B 572, 517 (2000), arXiv:hep-th/9911078.
M. Asorey, F. Falceto and G. Sierra, “Chern-Simons theory and BCS superconductivity,” Nucl. Phys. B 622, 593 (2002), arXiv:hep-th/0110266.

[29] C. Dunning and J. Links, ”Integrability of the Russian doll BCS model”, Nucl.Phys. B702 (2004) 481, arXiv:cond-mat/0406234 [cond-mat.stat-mech].

[30] D. Gaiotto and P. Koroteev, On Three Dimensional Quiver Gauge Theories and Integrability, JHEP 1305, 126 (2013) arXiv:1304.0779 [hep-th].
[31] A. Gorsky, A. Zabrodin and A. Zotov, “Spectrum of Quantum Transfer Matrices via Classical Many-Body Systems,” arXiv:1310.6958 [hep-th].

[32] A. Veselov, *Calogero quantum problem, Knizhnik-Zamolodchikov equation, and Huygens principle*, Theor.Math.Phys. 98, i.3 (1994) 368-376.

[33] M. R. Adams, J. Harnad, J. Hurtubise, Lett. Math. Phys., Vol. 20, Num. 4, 299-308 (1990).
E. Mukhin, V. Tarasov, A Varchenko, “Bispectral and (glN, glM) dualities, discrete versus differential”, Advances in Mathematics, Volume 218, 2008 216-265;

[34] K. Bulycheva and A. Gorsky, “BPS states in the Omega-background and torus knots,” arXiv:1310.7361 [hep-th].

[35] N. Nekrasov and S. Shatashvili, “Supersymmetric vacua and Bethe ansatz,” Nucl.Phys.B, Proc.Suppl.192–193 2009:91–112 arXiv:0901.4744 [hep-th].
N. A. Nekrasov and S. L. Shatashvili, “Quantization of Integrable Systems and Four Dimensional Gauge Theories,” arXiv:0908.4052 [hep-th].

[36] A. Gorsky, A. Mironov, “Integrable Many-Body Systems and Gauge Theories,” arXiv:hep-th/0011197.

[37] N. Nekrasov, “Seiberg-Witten Prepotential From Instanton Counting,” Adv.Theor.Math.Phys.7:831-864 (2004), arXiv:hep-th/0206161.

[38] M. Shifman, A. Yung, “Supersymmetric Solitons and How They Help Us Understand Non-Abelian Gauge Theories,” Rev. Mod. Phys. 79, 1139 (2007). hep-th/0703267.

[39] N. Dorey, T. Hollowood, S. Lee, “Quantization of Integrable Systems and a 2d/4d Duality,” arXiv:1103.5726 [hep-th].
N. Dorey, “The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms,” JHEP 9811, 005 (1998) hep-th/9806056.

[40] A. Giveon and D. Kutasov, “Brane dynamics and gauge theory,” Rev. Mod. Phys. 71, 983 (1999) hep-th/9802067.
[41] P. Bedaque, H. Hammer and U. van Kolck, “Renormalization of the three-body system with short-range interaction”, Phys.Rev.Lett 82 (1999) 463, arXiv:nucl-th/9809025.

[42] E. Poppitz, M. Unsal, ”‘Seiberg-Witten and ‘Polyakov-like’ magnetic bion confinements are continuously connected,” JHEP 1107, 082 (2011). arXiv:1105.3969 [hep-th].