The Cotton tensor in Riemannian spacetimes

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Abstract

Recently, the study of three-dimensional spaces is becoming of great interest. In these dimensions the Cotton tensor is prominent as the substitute for the Weyl tensor. It is conformally invariant and its vanishing is equivalent to conformal flatness. However, the Cotton tensor arises in the context of the Bianchi identities and is present in any dimension \( n \). We present a systematic derivation of the Cotton tensor. We perform its irreducible decomposition and determine its number of independent components as \( n(n^2 - 4)/3 \) for the first time. Subsequently, we exhibit its characteristic properties and perform a classification of the Cotton tensor in three dimensions. We investigate some solutions of Einstein’s field equations in three dimensions and of the topologically massive gravity model of Deser, Jackiw, and Templeton. For each class examples are given. Finally we investigate the relation between the Cotton tensor and the energy-momentum in Einstein’s theory and derive a conformally flat perfect fluid solution of Einstein’s field equations in three dimensions. file cott16.tex, 2004-01-20

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I. INTRODUCTION

The non–linear coupling of gravity to matter in general relativity presents difficult technical problems in attempts to understand the gravitational interaction of elementary particles and strings or to investigate details of the gravitational collapse. Progress in the former area has come mainly from treating quantum fields as propagating on fixed background geometries [1], whereas much of the progress in the latter has come from detailed numerical work [2, 3, 4]. Exact solutions of the relevant matter–gravity equations can play an important role by shedding light on questions of interest in both general relativity and string theory. One is often interested in certain classes of solutions with specified asymptotic properties, the most common of them are the asymptotically flat spacetimes. Recent work in string theory has, via the AdS/CFT conjecture, highlighted the importance of the asymptotically anti–de Sitter spacetimes [5]. The AdS/CFT correspondence relates a quantum field theory in $d$ dimensions to a theory in $d + 1$ dimensions that includes gravity [6, 7]. This is the motivation for looking at the conformally flat spaces and at the spaces of constant curvature. For this reason we decided to review the subject and to collect some old and new results that are nowadays important in the context of anti–de Sitter spacetimes and to present them in a modern language. These results seem presently not to be too well known in the community.

In the theory of conformal spaces the main geometrical objects to be analyzed are the Weyl [8] and the Cotton [9] tensors. It is well known that for conformally flat spaces the Weyl tensor has to vanish. Then the Cotton tensor vanishes, too. However, the Cotton tensor is only conformally invariant in three dimensions.

Recently, the study of three-dimensional spaces is becoming of great interest; for these spaces the Weyl tensor is always zero and the vanishing of the Cotton tensor depends on the type of relation between the Ricci tensor and the energy–momentum tensor of matter. Any three-dimensional space is conformally flat if the Cotton tensor vanishes. If matter is present, the Ricci tensor is related to the energy–momentum tensor of matter by means of the Einstein equations. Then the vanishing of the Cotton tensor imposes severe restrictions on the energy–momentum tensor. The Cotton tensor also plays a role in the context of the Hamiltonian formulation of general relativity, see [10].

The outline of the article is as follows. First we derive the Cotton 2-form in the context of the Bianchi identities. Subsequently we describe its characteristic properties and perform
an irreducible decomposition with respect to the (pseudo-)orthogonal group. This allows us to determine the number of irreducible components in any dimension. Moreover, in four dimensions, we relate the Cotton to the Bach tensor. After that we show how to derive the Cotton 2-form in 3 dimensions by means of a variational procedure. We classify the Cotton 2-form in 3 dimensions by means of its eigenvalues and give examples for all classes. Finally we derive, in 3 dimensions, the conformally flat static and spherically symmetric perfect fluid solution of Einstein’s field equation by means of the relation between the Cotton 2-form and the energy–momentum 2-form of matter.

Our notation and conventions are taken from [11, 12], see also Appendix VIII A for an overview.

II. BIANCHI IDENTITIES AND THE IRREDUCIBLE DECOMPOSITION OF THE CURVATURE

Let $V_n$ be a Riemannian space of $n$ dimensions. We have the coframe $\vartheta^\alpha$ and its dual frame $e_\alpha$ according to

$$\vartheta^\alpha = e_i^\alpha \, dx^i, \quad e_\alpha = e^i_\alpha \partial_i, \quad e_\alpha | \vartheta^\beta = e_i^\alpha e_i^\beta = \delta^\beta_\alpha, \quad (1)$$

where $x^i$ are local coordinates and the $e^i_\alpha$ are the n-bein coefficients. Greek letters $\alpha, \beta, \cdots = 1, \ldots, n$ denote anholonomic, Latin letters $i, j, \cdots = 1, \ldots, n$ holonomic indices. The symbol $\cdot$ stands for the interior product.

We introduce a metric $g$ according to

$$g = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta = g_{ij} \, dx^i \otimes dx^j. \quad (2)$$

With the metric at our disposal we also have the Hodge dual operator. Then we define the $\eta$-basis according to

$$\eta := *1, \quad \eta_\alpha := *\vartheta_\alpha, \quad \eta_{\alpha\beta} := *(\vartheta_\alpha \wedge \vartheta_\beta), \quad \eta_{\alpha\beta\gamma} := *(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma), \cdots \quad (3)$$

Furthermore, we equip our manifold with a symmetric connection ($u$ is an arbitrary vector),

$$\nabla_u e_\alpha = \Gamma^\beta_\alpha (u) \, e_\beta, \quad \Gamma^\beta_\alpha (\partial_i) = \Gamma_i^\alpha_\beta, \quad \Gamma^\beta_\alpha = \Gamma_i^\alpha_\beta \, dx^i. \quad (4)$$
with $d\vartheta^\beta = -\Gamma^\beta_\alpha \wedge \vartheta^\alpha$. In a Riemannian space the connection 1-form can be expressed in terms of the metric and the object of anholonomity $\Omega^\alpha := d\vartheta^\alpha$ and $\Omega_\alpha := g_{\alpha\beta} \Omega^\beta$,

$$\Gamma_{\alpha\beta} = \Gamma_{i\alpha\beta} \, dx^i = \frac{1}{2} d g_{\alpha\beta} + (e_{[\alpha]} d g_{\beta\gamma}) \vartheta^\gamma + e_{[\alpha]} \Omega_{\beta} - \frac{1}{2} (e_{\alpha} e_{\beta} \Omega_{\gamma}) \vartheta^\gamma.$$  

(5)

The curvature 2-form reads

$$R_{\alpha\beta} := d\Gamma_{\alpha\beta} - \Gamma_{\alpha\gamma} \wedge \Gamma_{\gamma\beta} = \frac{1}{2} R_{\mu\nu\alpha\beta} \vartheta^\mu \wedge \vartheta^\nu.$$  

(6)

Since a metric is given, we can lower the second index. Then the curvature 2-form is antisymmetric $R_{\alpha\beta} = - R_{\alpha\beta}$. This can be seen if we choose an orthonormal frame

$$g_{\alpha\beta} = \text{diag}(-1, -1, \ldots, 1, 1, \ldots),$$  

(7)

where $\text{ind}$, the number of negative eigenvalues, denotes the index of the metric. Then $dg = 0$ and, according to (5), the connection is antisymmetric. In turn, from (6) we can infer the antisymmetry of $R_{\alpha\beta}$.

The exterior covariant derivative of an $p$-form $X_{\alpha\beta}^\gamma$ is given by

$$DX_{\alpha\beta} = dX_{\alpha\beta} - \Gamma_{\alpha\gamma} \wedge X_{\beta}^\gamma + \Gamma_{\gamma\beta} \wedge X_{\alpha}^\gamma.$$  

(8)

The Bianchi identities can be formulated concisely with the help of this definition. In a Riemannian space the torsion $T^\alpha = D\vartheta^\alpha$ vanishes. Thus, the first Bianchi identity reads

$$0 = DT^\alpha = DD\vartheta^\alpha = R_{\beta}^\alpha \wedge \vartheta^\beta,$$  

(9)

or, in components,

$$R_{[\alpha\beta\gamma]}^\delta = 0.$$  

(10)

The first Bianchi identity is a (co-)vector valued 3-form with

$$n \left( \begin{array}{c} n \\ 3 \end{array} \right) = \frac{n^2(n-1)(n-2)}{3!}$$  

(11)

independent components that imposes the same number of constraint equations on the components of the curvature. Accordingly, in $n$-dimensions, the curvature 2-form has

$$\left( \begin{array}{c} n \\ 2 \end{array} \right) \left( \begin{array}{c} n \\ 2 \end{array} \right) - n \left( \begin{array}{c} n \\ 3 \end{array} \right) = \frac{n^2(n-1)(n+1)}{12}.$$  

(12)
independent components. For \( n = 3 \), we have 6 independent components and for \( n = 4 \) (the case of GR) 20 independent components.

The second Bianchi identity is

\[
DR_\alpha^\beta = 0, \quad \nabla_{[\lambda}R_{\mu\nu]\alpha^\beta} = 0.
\]  

We now perform the irreducible decomposition of the curvature with respect to the pseudo-orthogonal group [12]:

\[
\begin{align*}
  n = 1 & \quad R_{\alpha\beta} = 0 \\
  n = 2 & \quad R_{\alpha\beta} = \text{Scalar}_{\alpha\beta} \\
  n = 3 & \quad R_{\alpha\beta} = \text{Scalar}_{\alpha\beta} + \text{Ricci}_{\alpha\beta} \\
  n \geq 4 & \quad R_{\alpha\beta} = \text{Scalar}_{\alpha\beta} + \text{Ricci}_{\alpha\beta} + \text{Weyl}_{\alpha\beta}
\end{align*}
\]  

• The \( \text{Scalar}_{\alpha\beta} \)-piece is given by

\[
\text{Scalar}_{\alpha\beta} := -\frac{1}{n(n-1)} R \partial_\alpha \wedge \partial_\beta, \quad R := e_\alpha \lbrack \text{Ric}^\alpha, \quad \text{Ric}_\alpha := e_\beta \rbrack R_\alpha^\beta,
\]  

where \( R \) is the curvature scalar and \( \text{Ric}_\alpha \) the Ricci 1-form. This piece has 1 independent component and is present in any dimension \( n > 1 \). In components we have

\[
\text{Ric}_\alpha = \text{Ric}_{\mu\alpha} \vartheta^\mu, \quad \text{Ric}_{\mu\alpha} = R_{\lambda\mu\alpha}^\lambda, \quad R = R_{\lambda\mu}^\mu^\lambda,
\]  

and

\[
\text{Scalar}_{\mu\nu\alpha\beta} = -\frac{2}{n(n-1)} R g_{[\mu|\alpha} g_{\beta]\nu}.
\]

The Scalar piece enjoys the obvious symmetry

\[
\text{Scalar}_{\alpha\beta} \wedge \partial^\beta = 0, \quad \text{Scalar}_{[\mu\nu\alpha]\beta} = 0.
\]  

• From dimension 3 onwards the tracefree Ricci piece comes into play,

\[
\text{Ricci}_{\alpha\beta} := -\frac{2}{n-2} \partial_\alpha \wedge \partial_\beta \vartheta^\beta, \quad \text{Ricci}_\beta := \text{Ric}_\beta - \frac{1}{n} R \partial_\beta.
\]

It has \( \frac{1}{2}(n+2)(n-1) \) independent components. In index notation this corresponds to

\[
\text{Ricci}_{\mu\nu\alpha\beta} = -\frac{4}{n-2} g_{[\mu|\alpha} \text{Ricci}_{\beta]\nu} , \quad \text{Ricci}_{\alpha\beta} = \text{Ric}_{\alpha\beta} - \frac{1}{n} R g_{\alpha\beta}.
\]

If we contract the first Bianchi identity [31], we find

\[
0 = e_\beta \lbrack (R_\alpha^\beta \wedge \partial^\alpha) \rbrack = \text{Ric}_\alpha \wedge \partial^\alpha,
\]
since $R^\alpha_\alpha = 0$ in a Riemannian space. Thus, $\text{Ric} \vartheta^\mu \wedge \vartheta^\alpha = 0$ or

$$\text{Ric}_{\alpha\beta} = \text{Ric}_{\beta\alpha}, \quad (22)$$

that is, the Ricci tensor is symmetric. This also implies

$$\text{Ricci}_{\alpha\beta} \wedge \vartheta^\alpha = 0. \quad (23)$$

- Finally, in dimension greater than three, the Weyl 2-form emerges according to

$$\text{Weyl}_{\alpha\beta} := R_{\alpha\beta} - \text{Scalar}_{\alpha\beta} - \text{Ricci}_{\alpha\beta}. \quad (24)$$

From the construction it is clear that the Weyl 2-form is totally traceless, i.e.,

$$e_\alpha \lceil \text{Weyl}^{\alpha\beta} = -e_\beta \lceil \text{Weyl}^{\alpha\beta}, \quad e_\alpha \lceil e_\beta \lceil \text{Weyl}^{\alpha\beta} = 0. \quad (25)$$

This property also explains the vanishing of the Weyl 2-form in 3 dimensions. An arbitrary antisymmetric tensor–valued 2-form $A_{\alpha\beta} = 2A_{\beta\alpha} = A_{\mu\nu\alpha\beta} \vartheta^\mu \wedge \vartheta^\nu / 2$ in 3 dimensions has 9 independent components. The condition $e_\alpha \lceil A^{\alpha\beta} = 0$ results in 3 one-forms, i.e., 9 constraint equations that eventually yield the vanishing of all components.

According to [13], we can combine $\text{Scalar}_{\alpha\beta}$ and $\text{Ricci}_{\alpha\beta}$,

$$\text{Scalar}_{\alpha\beta} + \text{Ricci}_{\alpha\beta} = -\frac{2}{n-2} \vartheta[\alpha \wedge L_{\beta}], \quad (26)$$

with

$$L_{\alpha} := e_{\beta} \lceil R^{\alpha}_{\beta} - \frac{1}{2(n-1)} R \vartheta_{\alpha}, \quad (27)$$

i.e., this sum can be expressed in a coherent way in terms of the 1-form $L_{\alpha}$. From $\text{Scalar}_{\alpha\beta}$ and $\text{Ricci}_{\alpha\beta}$ it inherits the property

$$L_{\alpha} \wedge \vartheta^\alpha = 0. \quad (28)$$

We may expand $L_{\alpha}$ in components as

$$L_{\alpha\beta} = L_{\beta\alpha} = \text{Ric}_{\alpha\beta} - \frac{1}{2(n-1)} R g_{\alpha\beta}. \quad (29)$$

This tensor is sometimes called $\textit{Schouten tensor}$. Also the names $\textit{rho tensor}$ or $P_{\alpha\beta}$ can be found in the literature. Then the curvature 2-form can be expressed as

$$R_{\alpha\beta} = \text{Weyl}_{\alpha\beta} - \frac{2}{n-2} \vartheta[\alpha \wedge L_{\beta}], \quad (30)$$

or, in components,

$$\text{Weyl}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{4}{n-2} g_{[\alpha][\gamma} L_{\delta] \beta]. \quad (31)$$
III. COTTON 2-FORM

By applying the exterior covariant derivative to (30), we obtain the following representation of the second Bianchi identity,

$$0 = DR_{\alpha \beta} = DW_{\text{eyl}}_{\alpha \beta} + \frac{2}{n-2} \theta_{[\alpha} \wedge C_{\beta]},$$

(32)

where we encounter the Cotton 2-form

$$C_{\alpha} := DL_{\alpha} = \frac{1}{2} C_{\mu \nu \alpha} \vartheta^{\mu} \wedge \vartheta^{\nu}$$

(33)
or, in components,

$$C_{\alpha \beta \gamma} = 2 \left( \nabla_{[\alpha} \text{Ric}_{\beta \gamma]} - \frac{1}{2(n-1)} \nabla_{[\alpha} \text{Rg}_{\beta \gamma]} \right).$$

(34)

We perform an irreducible decomposition of the Cotton 2-form with respect to the Lorentz group. We can use the decomposition for the torsion, as given in [12], since this is also a vector-valued 2-form. Then we have

$$C^{\alpha} = \begin{array}{c} \text{(1)} C^{\alpha} \\ \text{(2)} C^{\alpha} \\ \text{(3)} C^{\alpha} \end{array} = \begin{array}{c} \text{TENCOT} \quad \text{TRACOT} \quad \text{AXICOT}, \end{array}$$

(35)

$$\frac{1}{2} n^2 (n-1) = \frac{1}{3} n(n^2 - 4) + n + \frac{1}{6} n(n-1)(n-2),$$

where

$$
\begin{align*}
\text{(2)} C^{\alpha} &:= \frac{1}{n-1} \vartheta^{\alpha} \wedge (e_{\beta}] C^{\beta}), \\
\text{(3)} C^{\alpha} &:= \frac{1}{3} e^{[\alpha} (C_{\beta} \wedge \vartheta^{\beta}), \\
\text{(1)} C^{\alpha} &:= C^{\alpha} - \text{(2)} C^{\alpha} - \text{(3)} C^{\alpha},
\end{align*}
$$

(36) (37) (38)
or, in components,

$$
\begin{align*}
\text{(2)} C^{\alpha}_{\mu \nu} &= - \frac{2}{n-1} \delta_{[\mu}^{\alpha} C_{\nu] \beta}^{\beta}, \\
\text{(3)} C^{\alpha}_{\mu \nu} &= \frac{1}{3!} C_{[\mu \nu \beta]} g^{\alpha \beta}, \\
\text{(1)} C^{\alpha}_{\mu \nu} &= C^{\alpha}_{\mu \nu} - \text{(2)} C^{\alpha}_{\mu \nu} - \text{(3)} C^{\alpha}_{\mu \nu}.
\end{align*}
$$

(39) (40) (41)

TENCOT, TRACOT, and AXICOT are the computer algebra names of the pieces of the Cotton 2-form, denoting the tensor, the trace, and the axial pieces, respectively. The number
of independent components of these pieces is given in the third line of (35). They arise as follows: TRACOT corresponds to a scalar-valued 1-form \( C := e_\alpha \| C^\alpha \) with \( n \) independent components. In general, a (co-)vector-valued 2-form has
\[
\binom{n}{2} = \frac{n^2(n-1)}{2}
\]
independent components. AXICOT corresponds to a scalar valued 3-form \((C^\alpha \wedge \vartheta_\alpha)\) and thus has
\[
\binom{n}{3} = \frac{n(n-1)(n-2)}{6}
\]
independent components. Thus, TENCOT is left with
\[
\frac{n^2(n-1)}{2} - \frac{n(n-1)(n-2)}{6} - n = \frac{n}{3}(n-2)(n+2)
\]
independent components.

We now show that in a Riemannian space the trace piece (TRACOT) and the axial piece (AXICOT) vanish. Hence, only the tensor piece (TENCOT) with its \( n(n^2-4)/3 \) independent components survives. This insight seems to be new. For \( n = 3 \), we have 5 and for \( n = 4 \) (the case of GR) 16 independent components.

In order to see the vanishing of AXICOT, we contract the Cotton 2-form with the coframe and use (28):
\[
\vartheta^\alpha \wedge C_\alpha = \vartheta^\alpha \wedge DL_\alpha = -D(\vartheta^\alpha \wedge L_\alpha) = 0,
\]
or
\[
C_{[\mu\nu\alpha]} = \frac{2}{3!} \nabla_{[\mu}L_{\nu\alpha]} = 0.
\]
The second Bianchi identity leads to a vanishing trace of the Cotton 2-form (TRACOT), \( C = e_\alpha \| C^\alpha = 0 \). In order to see this, we contract (32) twice:
\[
0 = e_\beta | DR^{\alpha\beta} = e_\beta | DWeyl^{\alpha\beta} - \frac{n-3}{n-2} C^\alpha - \frac{1}{n-2} \vartheta^\alpha \wedge C,
\]
or
\[
0 = e_\alpha | e_\beta | DR^{\alpha\beta} = e_\alpha | e_\beta | DWeyl^{\alpha\beta} - 2C = -2C,
\]
or
\[
C_{\alpha\beta}^\alpha = \nabla_\alpha \left( Ric_\beta^\alpha - \frac{1}{2} R \delta_\beta^\alpha \right) = 0.
\]
As we see, the second Bianchi identity relates the derivative of the Weyl 2-form to the Cotton 2-form,

$$e_\beta \, DWeyl_{\alpha}^\beta = \frac{n-3}{n-2} \, C_\alpha .$$

(50)

This formula allows us to rewrite the Einstein equation as a Maxwell-like equation for the Weyl tensor, see [14], e.g. The Ricci identity intertwines the derivative of the Cotton 2-form with the Weyl 2-form,

$$DC_\alpha = DDL_\alpha = -R_\alpha^\beta \wedge L_\beta = -Weyl_\alpha^\beta \wedge L_\beta + \frac{2}{n-2} \vartheta_\alpha \wedge L_\beta \wedge L^\beta$$

$$= -Weyl_\alpha^\beta \wedge L_\beta .$$

(51)

Consequently, in 3 dimensions, $C_\alpha$ is a covariantly conserved 2-form, with $DC_\alpha = 0$. Thus it is a candidate for a conserved current that can be derived by means of a variational procedure. The properties of the Cotton tensor are summarized in Table I.

Something similar emerges in 4 dimensions. In Appendix VIII C it is shown that

$$DD^*C_\alpha = -D \left( ^*Weyl_\alpha^\beta \wedge L_\beta \right) .$$

(52)

Thus,

$$B_\alpha := D^*C_\alpha + ^*Weyl_\alpha^\beta \wedge L_\beta =: B_\alpha^\beta \eta_\beta$$

(53)

or, in components,

$$B_{\alpha \beta} = \nabla^\mu C_{\alpha \mu \beta} + L^{\mu \nu} Weyl_{\alpha \mu \beta \nu} ,$$

(54)

is a covariantly conserved 3-form:

$$DB_\alpha = 0 \quad (\nabla_\beta B_\alpha^\beta = 0) .$$

(55)

We recognize the Bach tensor $B_{\alpha \beta}$ [11, 15, 16, 17]. From the symmetry properties of $C_\alpha$, $L_\alpha$, and $Weyl_{\alpha \beta \gamma}$ it follows that

$$B_\alpha \wedge \vartheta^\alpha = 0 \quad (B_\alpha^\alpha = 0) , \quad e_\alpha \, B^\alpha = 0 \quad (B_{[\alpha \beta]} = 0) .$$

(56)

Moreover, it transforms as a conformal density and can be derived from a variational principle. Since in an conformally flat space the Weyl and the Cotton tensors vanish, the vanishing of the Bach tensor is also a necessary (but not sufficient) condition for a four dimensional space to be conformally flat.
**C_α** as a variational derivative

It is well known [18, 19] that \( C_\alpha \) can be obtained by means of varying the 3-dimensional Chern-Simons action

\[
C_{\text{RR}} = -\frac{1}{2} \left( \Gamma_\alpha^\beta \wedge d\Gamma_\beta^\alpha - \frac{2}{3} \Gamma_\alpha^\beta \wedge \Gamma_\beta^\gamma \wedge \Gamma_\gamma^\alpha \right)
\]  
(57)

with respect to the metric keeping the connection fixed. In order to enforce vanishing torsion

\[
T^\alpha := D\vartheta^\alpha = d\vartheta^\alpha - \Gamma_\beta^\alpha \wedge \vartheta^\beta
\]  
(58)

and vanishing nonmetricity

\[
Q_{\alpha\beta} := -Dg_{\alpha\beta} = -dg_{\alpha\beta} + \Gamma_\alpha^\gamma g_{\gamma\beta} + \Gamma_\beta^\gamma g_{\alpha\gamma},
\]  
(59)

we have to apply Lagrange multipliers. Then the total Lagrangian reads

\[
L = C_{\text{RR}} + \lambda_\alpha \wedge T^\alpha + \lambda_{\alpha\beta} \wedge Q_{\alpha\beta},
\]  
(60)

where \( \lambda_\alpha \) is a 1-form and \( \lambda_{\alpha\beta} = \lambda_{\beta\alpha} \) a symmetric 2-form. The corresponding field equations read (for the explicit calculation see Appendix VIII B)

\[
\frac{\delta L}{\delta \lambda_\alpha} = T^\alpha = 0,  
\]  
(61)

\[
\frac{\delta L}{\delta \lambda_{\alpha\beta}} = Q_{\alpha\beta} = 0,  
\]  
(62)

\[
\frac{\delta L}{\delta \Gamma_{\alpha}^{\beta}} = -R_\beta^\alpha - \lambda_\beta \wedge \vartheta^\alpha + 2\lambda_\beta^\alpha = 0,  
\]  
(63)

\[
\frac{\delta L}{\delta \vartheta^\alpha} = D\lambda_\alpha = 0,  
\]  
(64)

\[
\frac{\delta L}{\delta g_{\alpha\beta}} = D\lambda_{\alpha\beta} = 0.  
\]  
(65)

We can solve (63) for its symmetric and its antisymmetric parts,

\[
R_{[\alpha\beta]} + \vartheta_{[\alpha} \wedge \lambda_{\beta]} = 0,  
\]  
(66)

\[
-R_{(\alpha\beta)} + \vartheta_{(\alpha} \wedge \lambda_{\beta)} + 2\lambda_{\alpha\beta} = 0.  
\]  
(67)

Because of (62), the symmetric part of the curvature vanishes,

\[
0 = DQ_{\alpha\beta} = -DDg_{\alpha\beta} = R_\alpha^\gamma g_{\gamma\beta} + R_\beta^\gamma g_{\alpha\gamma} = 2R_{(\alpha\beta)}.
\]  
(68)
Thus, by means of (67)
\[ \lambda_{\alpha\beta} = -\frac{1}{2} \vartheta_{(\alpha} \wedge \lambda_{\beta)} . \] (69)
According to (30), in 3 dimensions, \( R_{\alpha\beta} = -2 \vartheta_{[\alpha} \wedge L_{\beta]} \). We substitute this into (66) and find
\[ \lambda_{\beta} = 2 L_{\beta} , \quad \lambda_{\alpha\beta} = -\vartheta_{(\alpha} \wedge L_{\beta)} . \] (70)
Eventually,
\[ \frac{1}{2} \frac{\delta L}{\delta \vartheta^{\alpha}} = C_{\alpha} , \quad \frac{\delta L}{\delta g_{\alpha\beta}} = -\vartheta^{(\alpha} \wedge C_{\beta)} , \quad - \frac{2}{n-1} \epsilon \delta_{\alpha\beta} \frac{\delta L}{\delta g_{\alpha\beta}} = C_{\alpha} . \] (71)
In the presence of matter, the gravitational field equation is given by \( \delta (L + L_{\text{mat}}) / \delta \vartheta^{\alpha} = 0 \).
Hence, the Cotton 2-form can be coupled to the energy-momentum 2-form of matter
\[ \Sigma_{\alpha} := \frac{\delta L_{\text{mat}}}{\delta \vartheta^{\alpha}} . \] (72)
This is carried out in the topologically massive gravity model of Deser, Jackiw and Templeton (DJT) [18], where the Lagrangian [30] is enriched by a Hilbert-Einstein term and a cosmological term (\( \ell \) is the gravitational constant and \( \theta \) a dimensionless coupling constant):
\[ L_{DJT} = \theta C_{RR} + V_{\text{HE}} + V_{A} + \lambda_{\alpha} \wedge T^{\alpha} + \lambda_{\alpha\beta} Q_{\alpha\beta} + L_{\text{mat}} \]
\[ = \theta C_{RR} - \frac{1}{2\ell} R^\alpha_{\beta\gamma} \wedge \eta_{\alpha\beta} - \frac{\Lambda}{\ell} \eta + \lambda_{\alpha} \wedge T^{\alpha} + \lambda_{\alpha\beta} Q_{\alpha\beta} + L_{\text{mat}} . \] (73)
Then the field equation (64) reads
\[ G_{\alpha} + \Lambda \eta_{\alpha} + \frac{1}{\mu} C_{\alpha} = \ell \Sigma_{\alpha} , \] (74)
where \( G_{\alpha} = \frac{1}{2} \eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} \) is the Einstein 2-form and the DJT coupling constant \( 1/\mu = -2\theta \ell \).
The model of topologically massive gravity in Riemann-Cartan space by Mielke and Baekler [19] considers additionally to the Chern-Simons term for the curvature also a corresponding term for the torsion. This model includes Einstein, Einstein-Cartan, and the DJT field equations as limiting cases.

In three dimensions the Cotton tensor arises from the variation of the topological Chern-Simons term. Recently, a similar procedure was proposed by Jackiw et al. [20, 21, 22] for the case \( n = 4 \) by starting from the corresponding four-dimensional topological Lagrangian
\[ (1/2) \theta R_{\alpha}{}^{\beta} \wedge R^{\beta \alpha} = \theta dC_{RR} , \] where \( \theta \) is an external, prescribed field. The variation with respect to the metric yields a four-dimensional Cotton type tensor that differs from the one in our definition [33] with [27].
TABLE I: Properties of the Cotton 2-form $C_\alpha$ in arbitrary dimensions

| Property | Expression |
|----------|------------|
| $C_\alpha := DL_\alpha$ | Cotton 2-form |
| $L_\alpha := e_\beta R_\alpha^\beta - \frac{1}{2(n-1)} R \vartheta_\alpha$ |
| $\vartheta^\alpha \wedge C_\alpha = 0$ | 1st Bianchi identity |
| $e_\alpha [C_\alpha = 0$ | contracted 2nd Bianchi identity |
| $D\text{Weyl}_{\alpha\beta} = -\frac{2}{n-2} \vartheta_{[\alpha \wedge C_\beta]}$ | 2nd Bianchi identity |
| $DC_\alpha = -\text{Weyl}_{\alpha\beta} \wedge L_\beta$ | Ricci identity |
| $\tilde{C}_\alpha = C_\alpha + (n-2) \sigma_\beta \text{Weyl}_{\alpha\beta}$ | conformal transformation |
| $C_\alpha = \ell D^* [\Sigma_{\alpha} - \frac{1}{n-1} e_\alpha (\Sigma_\beta \wedge \vartheta^\beta)]$ | Einstein equation differentiated |

The Einstein $(n - 1)$-form $G_\alpha$ is equivalent to the 1-form $L_\alpha$ according to

$$G_\alpha = L_\beta \wedge \eta_{\beta\alpha},$$

see [13]. Hence, we may rewrite the DJT-field equation as an differential equation for $L_\alpha$,

$$DL_\alpha + \mu L_\beta \wedge \eta_{\beta\alpha} = \ell \mu \Sigma_\alpha - \mu \Lambda \eta_\alpha.$$  (76)

The Bianchi identities imply full integrability of this system.

In the case of Einstein gravity, the equivalence of $G_\alpha$ and $L_\alpha$ implies a relation between the Cotton 2-form and the energy–momentum $(n - 1)$-form in any dimension. We can solve

$$G_\alpha + \Lambda \eta_\alpha = \ell \Sigma_\alpha$$

for $L_\alpha$ and obtain by covariant exterior differentiation

$$C_\alpha = (-1)^{n-1+\text{ind}} \ell D^* [\Sigma_{\alpha} - \frac{1}{n-1} e_\alpha (\vartheta^\beta \wedge \Sigma_\beta)] .$$

(78)

Note that the cosmological constant $\Lambda$, which induces a constant curvature term, drops out.

We recognize that all vacuum solutions of Einstein’s theory have a vanishing Cotton 2-form. Therefore, via the Bianchi identity, the Weyl 2-form is divergenceless, see [51]. This property considerably simplifies the classification of the Weyl tensor. Petrov type D spacetimes with vanishing Cotton 2-form have been classified in [23].
IV. CONFORMAL CORRESPONDENCE

The conformal correspondence between two \( n \)-dimensional manifolds \( V_n \) and \( \hat{V}_n \) is achieved by means of a conformal transformation of the form \[11, 24\]

\[
\hat{g}_{\alpha\beta} = \exp(2\sigma)g_{\alpha\beta}, \quad \hat{g}^{\alpha\beta} = \exp(-2\sigma)g^{\alpha\beta},
\]

where \( \sigma \) is an arbitrary function. In general, a conformal transformation (79) is not associated with a transformation of coordinates, i.e., with a diffeomorphism of \( V_n \); both metrics in (79) are given in the same coordinate system and frame. Since these transformations preserve angles between corresponding directions, the causal structure of the manifold is preserved. As a rule, indices of quantities with hat are raised and lowered by means of \( \hat{g}_{\alpha\beta} \) or \( \hat{g}^{\alpha\beta} \), respectively, those of untransformed quantities by \( g_{\alpha\beta} \) or \( g^{\alpha\beta} \). The transformed connection reads

\[
\hat{\Gamma}_\alpha^\beta = \Gamma_\alpha^\beta + \left( \delta_\alpha^\beta d\sigma - \vartheta_\alpha \sigma^\beta + \sigma_\alpha \vartheta^\beta \right) =: \Gamma_\alpha^\beta + S_\alpha^\beta,
\]

where a comma denotes partial and a semicolon covariant differentiation. If \( \hat{D} = d + \Gamma_\alpha^\beta + S_\alpha^\beta \) is the exterior covariant derivative with respect to \( \hat{\Gamma}_\alpha^\beta \), the transformed curvature is

\[
\hat{R}_\alpha^\beta = d\hat{\Gamma}_\alpha^\beta - \hat{\Gamma}_\alpha^\gamma \wedge \hat{\Gamma}_\gamma^\beta = R_\alpha^\beta + 2 \vartheta_{[\alpha} \wedge S_{\gamma]} \hat{g}^{\gamma\beta},
\]

with

\[
S_\gamma := D\sigma_\gamma - \sigma_\gamma d\sigma + \frac{1}{2} \sigma^\alpha \sigma_\alpha \vartheta_\gamma.
\]

By contracting (81) with the frame \( e_\beta \), we infer

\[
\hat{L}_\alpha = L_\alpha - (n - 2) S_\alpha, \quad \hat{\text{Weyl}}_\alpha^\beta = \text{Weyl}_\alpha^\beta, \quad \hat{R} = \exp(-2\sigma) \left[ R - 2(n - 1) \sigma^\alpha_\alpha - (n - 1)(n - 2) \sigma_\alpha \sigma^\alpha \right].
\]

The Weyl 2-form is conformally invariant since a conformal transformation does not act on the trace-free part of the curvature. Application of \( \hat{D} \) onto \( \hat{\text{Weyl}}_\alpha^\beta \) yields the transformation behavior of the Cotton 2-form,

\[
\hat{C}_\alpha = C_\alpha + (n - 2) \sigma_\beta \text{Weyl}_\alpha^\beta.
\]

Thus, in \( n = 3 \), where the Weyl 2-form vanishes, the Cotton 2-form becomes conformally invariant.
V. CRITERIA FOR CONFORMAL FLATNESS

In the following paragraphs we investigate the criteria for conformal flatness, i.e., the possibilities to transform the curvature to zero by means of a conformal transformation. We basically follow \[11\]. Since we have seen that the curvature 2-form in 2, 3, and more than 3 dimensions is built up rather differently, we have to investigate these cases separately.

**n = 2**

In \( n = 2 \) the only non-vanishing curvature piece is the curvature scalar \( R \). Its behavior under conformal transformation is given by

\[
\hat{R} = \exp(-2\sigma) \left( R - \frac{2}{n} \sigma^i_{;i} \right) = 0. \tag{87}
\]

Thus,

\[
\hat{R} = 0 \iff \sigma^i_{;i} = \frac{R}{2}. \tag{88}
\]

This is a scalar wave equation for the conformal factor \( \sigma \) with \( R \) as source. Since the wave equation always has a solution, we conclude that all 2-dimensional spaces are conformally flat.

**n \geq 3**

For more than 2 dimensions we start from \[30\], namely

\[
R_{\alpha\beta} = \text{Weyl}_{\alpha\beta} - \frac{2}{n-2} \theta_{[\alpha} \wedge L_{\beta]}. \tag{89}
\]

Since the Weyl 2-form is conformally invariant it cannot be transformed to zero by means of a conformal transformation. Consequently, the vanishing of the Weyl 2-form is a necessary condition for conformal flatness.

The \( L_\alpha \) 1-form transforms according to

\[
\hat{L}_\alpha = L_\alpha - (n-2) S_\alpha. \tag{90}
\]

We can transform \( L_\alpha \) to zero if there is a function \( \sigma \) such that

\[
L_\alpha = (n-2) S_\alpha. \tag{91}
\]

This will impose a differential restriction on \( L_{ij} \). By means of \[32\], we rewrite the latter equation as a differential equation for \( \sigma_{,i} \),

\[
D\sigma_{,i} = \sigma_{,i} \sigma_{,j} \partial^j - \frac{1}{2} \sigma_{,j} \sigma_{,j} \partial_i + \frac{1}{n-2} L_i. \tag{92}
\]
If we apply the covariant derivative to both sides of (92), we obtain a necessary condition for the integrability,

$$-R^{i}_{\ j} \sigma_j = DD\sigma_i = \sigma_{,j} D\sigma_i \wedge \vartheta^j - \sigma^j D\sigma_{,j} \wedge \vartheta_i + \frac{1}{n-2} C_i.$$  \hspace{1cm} (93)

This becomes a necessary and sufficient condition of integrability if the dependence on \( \sigma_{,i} \) can be eliminated, see [11, 25]. Thus we substitute \( D\sigma_i \) from (92) into (93):

$$-R^{i}_{\ j} \sigma_j = -\frac{2}{n-2} L[i \wedge \vartheta_j] \sigma^j + \frac{1}{n-2} C_i.$$  \hspace{1cm} (94)

Using the decomposition (30) of the curvature, we finally arrive at

$$-(n-2) \text{Weyl}^{i}_{\ j} \sigma_j = C_i.$$  \hspace{1cm} (95)

For \( n = 3 \), the Weyl 2-form is zero and \( C_\alpha = 0 \) is the integrability condition for the conformal factor. Thus, if the Cotton 2-form is zero, the space is conformally flat. Conversely, if the space is conformally flat, there is a conformal transformation such that \( \hat{R}_{\alpha\beta} = 0 \Leftrightarrow \hat{L}_\alpha = 0 \Rightarrow \hat{C}_\alpha = 0 \). Since the Cotton 2-form is conformally invariant in 3 dimensions, we find \( C_\alpha = 0 \). Hence, the vanishing of the Cotton 2-form is the necessary and sufficient condition for a \( V_3 \) to be conformally flat.

In more than 3 dimensions the vanishing of the Weyl 2-form is a necessary condition for conformal flatness. Thus, also in dimensions greater than 3, \( C_\alpha = 0 \) is the integrability condition for the conformal factor. However, for \( n > 3 \), the contracted second Bianchi identity (47) implies the vanishing of the Cotton 2-form when the Weyl 2-form is zero. Hence, the vanishing of the Weyl 2-form is also the sufficient condition for conformal flatness.

VI. CLASSIFICATION OF THE COTTON 2-FORM IN 3D

A vector-valued 2-form in 3 dimensions has 9 independent components, the same as the number of components of a \( 3 \times 3 \) matrix. A mapping between these two can be achieved by means of the Hodge dual. The Hodge dual of a vector-valued 2-form in 3 dimensions is a vector-valued 1-form with the same number of independent components. Its components form a 2nd rank tensor (“matrix”),

$$C_{\alpha\beta} := e_\alpha|^{*}C_\beta = *(C_\beta \wedge \vartheta_\alpha)$$  \hspace{1cm} (96)
or, in components,

$$C_{\alpha}^{\beta} = \nabla_{\mu} \left( \text{Ric}_{\nu \alpha} - \frac{1}{4} R g_{\nu \alpha} \right) \eta^{\mu \nu \beta}. \tag{97}$$

This alternative representation of the Cotton 2-form, often called Cotton-York tensor \[26\] (even though it was already discussed explicitly by ADM \[10\]), can only be defined in three dimensions. Sometimes it appears under the name Bach tensor in the literature, see \[27\], e.g. This seems to be a misnomer.

The Cotton tensor is tracefree

$$C_{\alpha}^{\alpha} = e_{\alpha} \star C^{\alpha} = \star (C^{\alpha} \wedge \vartheta_{\alpha}) = 0. \tag{98}$$

In the 3 dimensions, the 2nd Bianchi identity \[32\] amounts to \(\vartheta_{[\alpha} \wedge C_{\beta]} = 0\). In view of the definition \(96\), we infer that the Cotton tensor is symmetric \(C_{\alpha \beta} = C_{\beta \alpha}\). Introducing this symmetry explicitly into \(97\), we obtain the alternative representation

$$C^{\alpha \beta} = C^{\beta \alpha} = \eta^{\mu \nu (\alpha} \nabla_{\mu} \text{Ric}_{\nu \beta)}. \tag{99}$$

We now perform a classification of the Cotton tensor with respect to its eigenvalues. The corresponding generalized eigenvalue problem reads:

$$\left( C^{\alpha \beta} - \lambda g^{\alpha \beta} \right) V_{\beta} = 0, \quad C^{[\alpha \beta]} = 0, \quad C^{\alpha \beta} g_{\alpha \beta} = 0. \tag{100}$$

By lowering one index, we can reformulate this as ordinary eigenvalue problem for the matrix \(C_{\alpha}^{\beta}\). However, in that case, the symmetry \(C^{\alpha \beta} = C^{\beta \alpha}\) is no longer manifest:

$$\left( C_{\alpha}^{\beta} - \lambda \delta_{\alpha}^{\beta} \right) V_{\beta} = 0, \quad C_{\alpha}^{\alpha} = 0. \tag{101}$$

A. Euclidean signature

The case of Euclidean signature is simple: the generalized eigenvalue problem reduces to an ordinary one. As a real symmetric matrix \(C^{\alpha \beta}\) possesses 3 real eigenvalues and the eigenvectors form a basis. With respect to this basis, \(C^{\alpha \beta}\) takes a diagonal form. Since \(C^{\alpha \beta}\) is tracefree, the sum of the eigenvalues is zero. Consequently, we can distinguish 3 classes:

- Class A

  Three distinct eigenvalues: \(\lambda_{1} \neq \lambda_{2}\) and \(\lambda_{3} = -(\lambda_{1} + \lambda_{2})\).
• Class B
  Two distinct eigenvalues: \( \lambda_1 = \lambda_2 \neq 0, \lambda_3 = -2\lambda_1. \)

• Class C
  One distinct eigenvalue: \( \lambda_1 = \lambda_2 = \lambda_3 = 0. \)
  In the present context of Euclidean signature, this implies \( C_{\alpha\beta} = 0. \)

B. Lorentzian signature

In the case of an indefinite metric, the roots of the characteristic polynomial
\[
\det \left( C^{\alpha\beta} - \lambda g^{\alpha\beta} \right) = 0
\]
(102)
may be complex. Accordingly, the matrix \( C_{\alpha\beta} \) is no longer symmetric and in the equivalent ordinary eigenvalue problem
\[
\det \left( C_{\alpha\beta} - \lambda \delta_{\alpha\beta} \right) = 0
\]
(103)
complex eigenvalues occur, too. This point seems to have been overlooked by the authors of [28]. Consequently, the classification will not be as simple as it was the case for the Euclidean metric.

In the following, we will present a classification of \( C_{\alpha\beta} \). The tracefree condition \( \mu_1, \mu_2, \mu_3 \), in orthonormal coordinates, reads explicitly
\[
C_1^1 + C_2^2 + C_3^3 = 0.
\]
(104)
Accordingly, we can eliminate \( C_3^3 \), e.g., from \( \mu_1, \mu_2 \). Then the secular determinant reads
\[
\det \begin{vmatrix}
  C_1^1 - \lambda & C_1^2 & C_1^3 \\
  -C_1^2 & C_2^2 - \lambda & C_2^3 \\
  -C_1^3 & C_2^3 & -C_1^2 - C_2^2 - \lambda
\end{vmatrix} = 0,
\]
(105)
with the 5 matrix elements \( C_1^2, C_1^3, C_2^2, C_2^3 \). We compute the determinant and order according to powers of \( \lambda \),
\[
\lambda^3 + b \lambda + c = 0,
\]
(106)
where
\[
b := -(C_1^1)^2 - C_1^1 C_2^2 - (C_2^2)^2 + (C_1^2)^2 + (C_1^3)^2 - (C_2^3)^2,
\]
(107)
\[
c := [(C_1^1)^2 C_2^2 + C_1^1 (C_2^2)^2 + C_1^1 (C_2^3)^2 + C_1^1 (C_2^3)^2]
+ (C_1^2)^2 C_2^2 + 2 C_1^2 C_3^2 C_3^3 - (C_1^3)^2 C_2^2 \]
(108)
The roots of (106) are given by
\[
\lambda_1 = A, \quad \lambda_2 = -\frac{A}{2} + i\frac{\sqrt{3}}{2} B, \quad \lambda_3 = -\frac{A}{2} - i\frac{\sqrt{3}}{2} B,
\]
with
\[
A := \frac{D^2 - 12b}{6D}, \quad B := \frac{D^2 + 12b}{6D}, \quad D := \left(-108c + 12\sqrt{12b^3 + 81c^2}\right)^{1/3}. \tag{110}
\]

A cubic polynomial with real coefficients has at least one real root and the complex roots have to be complex conjugates. The Jordan normal forms of the Cotton tensor read:

| “Petrov”-type | Jordan form | Segre notation | eigenvalues |
|---------------|-------------|----------------|-------------|
| I             | \[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & -\lambda_1 - \lambda_2
\end{pmatrix}
\] | [111] | \(\lambda_1 \neq \lambda_2, \lambda_3 = -\lambda_1 - \lambda_2\) |
| D             | \[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & -2\lambda_1
\end{pmatrix}
\] | [(11)1] | \(\lambda_1 = \lambda_2 \neq 0, \lambda_3 = -2\lambda_1\) |
| II            | \[
\begin{pmatrix}
\lambda_1 & 1 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & -2\lambda_1
\end{pmatrix}
\] | [21] | \(\lambda_1 = \lambda_2 \neq 0, \lambda_3 = -2\lambda_1\) |
| N             | \[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] | [(21)] | \(\lambda_1 = \lambda_2 = \lambda_3 = 0\) |
| III           | \[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\] | [3] | \(\lambda_1 = \lambda_2 = \lambda_3 = 0\) |
| O             | \[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] |             |             |
This parallels exactly the Petrov classification of the Weyl tensor in 4 dimensions \[29\]. This comes about since the Weyl tensor in 4D is equivalent to a (complex) $3 \times 3$ tracefree matrix, as $C_{\alpha \beta}$ in 3D; for a similar classification of $C_{\alpha \beta}$, see \[30\].

Since one eigenvalue is real, types D and II with only one independent eigenvalue $\lambda_1 = \lambda_2 = -2\lambda_3$ are always real. For class I, besides the real eigenvalue, two complex eigenvalues may occur. In that case, they are complex conjugated. Therefore, class I can be subdivided into class I with 3 real eigenvalues, [111], and class I’ with one real and two complex conjugated eigenvalues, [1z$\bar{z}$]. By performing a kind of null rotation, we can also give a real form for class I’:

$$
\begin{bmatrix}
\text{Re } z & \text{Im } z & 0 \\
-\text{Im } z & \text{Re } z & 0 \\
0 & 0 & -2 \text{ Re } z
\end{bmatrix}
\begin{bmatrix}
[1z\bar{z}] \\
\lambda_1 = -2 \text{ Re } z, \lambda_2 = z, \lambda_3 = \bar{z}.
\end{bmatrix}
$$

We can now specify simple criteria for deciding to which of these classes the Cotton tensor $C_{\alpha \beta}$ belongs. First determine the eigenvalues.

1. Three different eigenvalues (2 independent)
   
   (a) all real $\Rightarrow$ Class I
   
   (b) one real, two complex $\Rightarrow$ Class I’

2. Two different eigenvalues (1 independent $\lambda_1 = \lambda_2 = -2\lambda_3$)
   
   (a) $(C_{\alpha \beta} - \lambda_1 \delta_\alpha^\beta)(C_{\beta \gamma} + \frac{1}{2} \lambda_1 \delta_\beta^\gamma) = 0 \Rightarrow$ Class D
   
   (b) else $\Rightarrow$ Class II

3. All eigenvalues zero
   
   (a) $C_{\alpha \beta} = 0 \Rightarrow 0$
   
   (b) $C_{\alpha \beta} C_{\beta \gamma} = 0 \Rightarrow$ Class N
   
   (c) else $\Rightarrow$ Class III
VII. EXAMPLES

We now give examples in order to demonstrate explicitly that all classes presented are non-empty indeed. All results have been checked by means of computer algebra, see Appendix VIII D for an explicit sample program.

• Class I’

The generic example is the (1 + 2)D static and spherically symmetric spacetime, given in an orthonormal coframe with signature (+ − −) by

$$\vartheta^0 = \sqrt{\psi} dt, \ \vartheta^1 = \frac{dr}{\sqrt{\psi}}, \ \vartheta^2 = r d\varphi, \ \psi = \psi(r). \quad (111)$$

The Cotton tensor and its eigenvalues read, here ()′ = d/dr:

$$C_{\alpha \beta} = \frac{\sqrt{\psi} \psi'''}{4} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \lambda_1 = 0, \lambda_2 = -\lambda_3 = i \frac{\sqrt{\psi} \psi'''}{4}. \quad (112)$$

A well-known example is the 3D analog to the Reissner-Nordström solution, a solution of the 3D Einstein-Maxwell equation [31]:

$$\psi = \Lambda r^2 - q^2 \ln r - M. \quad (113)$$

• Class I

In [32], eq.(4.1), the following solution for the vacuum DJT field equation is given:

The orthonormal coframe with signature (− ++) reads

$$\vartheta^0 = a_0 (d\psi + \sin \theta \ d\phi), \quad (114)$$

$$\vartheta^1 = a_1 (- \sin \psi \ d\theta + \cos \psi \cosh \theta \ d\phi), \quad (115)$$

$$\vartheta^2 = a_2 (\cos \psi \ d\theta + \sin \psi \cosh \theta \ d\phi), \quad (116)$$

where the DJT field equations are fulfilled provided

$$a_0 + a_1 + a_2 = 0, \quad \mu = -\frac{a_0^2 + a_1^2 + a_2^2}{a_0 a_1 a_2}. \quad (117)$$

Then the Cotton tensor reads

$$C_{\alpha \beta} = -4 \frac{a_1^2 + a_1 a_2 + a_2^2}{(a_1 + a_2)a_1^2 a_2^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{-a_1}{a_1 + a_2} & 0 \\ 0 & 0 & \frac{-a_2}{a_1 + a_2} \end{pmatrix}. \quad (118)$$
The eigenvalues can be read off from the diagonal. For $a_1 = a_2$, the Cotton tensor degenerates to class D. The solution eq.(4.6) in [32] is analogous to the present case.

• Class D

An example is the 3D Gödel solution (signature $(+−−)$), see [33] eq.(4.1):

$$\vartheta^0 = \left(\frac{3}{\mu}\right) \left[(dt - 2(\sqrt{r^2 + 1} - 1) d\phi\right],$$ (119)

$$\vartheta^1 = \left(\frac{3}{\mu}\right) \frac{dr}{\sqrt{r^2 + 1}},$$ (120)

$$\vartheta^2 = \left(\frac{3}{\mu}\right) r d\phi,$$ (121)

with

$$C_{\alpha \beta} = \left(\frac{\mu}{3}\right)^3 \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_1 = \lambda_2 = -\frac{1}{2} \lambda_3 = \left(\frac{\mu}{3}\right)^3. \quad (122)$$

This is a vacuum solution of the DJT model as well as a solution of the 3D Einstein equation with matter.

Many of the other solutions known for the DJT field equation are also of Class D:

- The squashed 3-sphere solutions by Nutku and Baekler [32], eq.(4.10) and eq.(4.1), eq.(4.6) for a special choice of parameters (see above).

- The topologically massive planar universe with constant twist of Percacci et al. [33], eq.(3.20).

- The perfect fluid solution of Gürses [34], eq.(6).

- The DJT-black hole solution of Nutku [35], eq.(24).

- The recent black hole solution by Moussa et al. [36], eq.(4).

• Class N

Inspired by the corresponding $(1+3)$D metrics, we start with the ansatz

$$\vartheta^0 = dt + dx, \quad \vartheta^1 = dt - dx, \quad \vartheta^2 = dy,$$ (123)

with the non-orthonormal metric

$$g = \vartheta^0 \otimes \vartheta^1 + \psi \vartheta^0 \otimes \vartheta^1 - \vartheta^2 \otimes \vartheta^2, \quad \psi = \psi(y). \quad (124)$$
The Cotton tensor, in this frame, reads \(( ()' = d/dy)\):

\[
C^\alpha_{\beta} = \psi''' \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_1 = \lambda_2 = \lambda_3 = 0.
\] (125)

The vacuum DJT field equation reduces to

\[
\frac{1}{\mu} \psi''' - \psi'' = 0,
\] (126)

with the general solution

\[
\psi = Ay + B e^{\mu y} + C.
\] (127)

In an orthonormal coframe with signature \((-+++)\) and \(A = C = 0\) and \(B = 1\), coframe and Cotton tensor can be brought into the more familiar form

\[
\vartheta^0 = e^{\mu y/2} \left[ (1 + \frac{1}{2} e^{-\mu y}) dt + (1 - \frac{1}{2} e^{-\mu y} dx) \right],
\] (128)

\[
\vartheta^1 = \frac{1}{2} e^{-\mu y/2} (dt - dx),
\] (129)

\[
\vartheta^2 = dy.
\] (130)

The Cotton tensor, with all eigenvalues being zero, reads

\[
C^\alpha_{\beta} = \frac{\mu^3}{2} \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_1 = \lambda_2 = \lambda_3 = 0.
\] (131)

Another class N solution is given in [33], eq.(4.9).

We have found no (sensible) solutions to the Einstein or DJT field equations which are of Class II or III. However, it is easy to find metrics for which the Cotton tensor is in general of class I but may degenerate to classes II or III. Just in order to show that these classes are nonempty, we will sketch corresponding examples:

- **Class II**

  The following coframe (signature \((-+++)\)),

  \[
  \vartheta^0 = e^{-2y} dt + dx, \quad \vartheta^1 = e^y dx, \quad \vartheta^2 = dy,
  \] (132)

  yields a Cotton tensor which, in general, is of class I. However, on the surface \(y = 0\), it degenerates to class II.
Class III

The Cotton tensor for the following coframe (signature \((-+++)\)) is also of class I in general:

\[
\vartheta^0 = (x - t) \, dt, \quad \vartheta^1 = (x + t) \, dx, \quad \vartheta^2 = dy. \tag{133}
\]

On the hypersurface given by \(x = t(\sqrt{13} + 3)/2\), the Cotton tensor degenerates to class III.

Class 0

All conformally flat solutions.

Conformally flat perfect fluid solution

As an application of the relation between energy-momentum 2-form and Cotton 2-form and as an example for a class 0 solution, we will derive the spherically symmetric, conformally flat, perfect fluid solution to Einstein’s field equation. We use the ansatz

\[
\vartheta^0 = N(r) \, dt, \quad \vartheta^i = dr/F(r), \quad \vartheta^2 = r \, d\phi, \tag{134}
\]

with signature \((-+++)\). The energy–momentum of the perfect fluid is given by

\[
\Sigma_\alpha = [\rho(r) + p(r)] \, u^\alpha \, \eta_{\beta} + p \, \eta_\alpha, \tag{135}
\]

where \(u^\alpha\) is the 4-velocity of the fluid elements which, in an orthonormal coframe, is given by \(u^\alpha = (1, 0, 0)\). By using \((138)\), we find

\[
C_0 = - \left\{ \frac{F}{2N} \, (2 \partial_r [N (p + \rho)] - N \partial_r \rho) \right\} \vartheta^0 \wedge \vartheta^i, \tag{136}
\]

\[
C_1 = 0, \tag{137}
\]

\[
C_2 = -\frac{F}{2} \partial_r \rho \, \vartheta^1 \wedge \vartheta^2. \tag{138}
\]

Consequently, we have to demand constant energy density \(\rho = \text{const}\) for a conformally flat solution with \(C_\alpha = 0\). By using \(\rho = \text{const}\), we infer from \((136)\)

\[
N(r) = \frac{c_1}{\rho + p(r)}, \tag{139}
\]

where \(c_1\) is an integration constant. The 0-component of the Einstein field equation \((141)\) yields

\[
F^2(r) = c_2 - (\ell \rho + \Lambda) \, r^2. \tag{140}
\]
The remaining components of the field equation are fulfilled provided

$$\frac{dp}{dr} = \frac{(\ell \rho - \Lambda)(p + \rho) r}{F^2}.$$  \hspace{1cm} (141)

This ordinary differential equation can be integrated yielding ($c_3$ is another integration constant)

$$p = \frac{c_3 F (\ell \rho + \Lambda) + (c_3)^2 \ell \Lambda + \rho F^2}{(c_3)^2 \ell^2 - F^2}.$$  \hspace{1cm} (142)

Finally, the solution is given by (134,139,140,142), compare the solutions in [37].

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**VIII. APPENDIX**

**A. Conventions**

Our index notation is based on the conventions of Schouten [11] and for exterior calculus we refer to [12]. For quick and easy reference, we display our conventions for index positions and signs of the Christoffel symbol, the Riemann tensor, and the Ricci tensor (holonomic indices $i,j,\cdots = 0,1,2,3$). The sign of the Ricci tensor is the same as those of the $L_{ij}$ tensor and the Cotton tensor. In particular, the Ric$_{ij}$ sign introduces a relative sign between the $L_{ij}$ tensor and the Weyl tensor in the decomposition of the curvature:

$$\nabla_i T^k_j = \partial_i T^k_j - \Gamma^k_{ij} T^l_j + \Gamma^k_{i\ell} T^\ell_j,$$  \hspace{1cm} (143)

$$+ R_{ijk}^\ell = \partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} + \Gamma^m_{im} \Gamma^\ell_{jk} m - \Gamma^m_{jm} \Gamma^\ell_{ik} m,$$  \hspace{1cm} (144)

$$+ \text{Ric}_{jk} = R_{ijk}^i,$$  \hspace{1cm} (145)

$$+ R = R_{ij}^j,$$  \hspace{1cm} (146)

$$\text{Weyl}_{ijk\ell} = R_{ijk\ell} + \frac{4}{n-2} g_{i[i} L_{\ell]j]},$$  \hspace{1cm} (147)

An extensive comparison between the various conventions can be found in [39].
B. Variation of the Chern-Simons Lagrangian

We consider the Lagrangian

$$C_{RR} = - \frac{1}{2} \left( \Gamma_{\alpha}^{\beta} \wedge d\Gamma_{\beta}^{\alpha} - \frac{2}{3} \Gamma_{\alpha}^{\beta} \wedge \Gamma_{\beta}^{\gamma} \wedge \Gamma_{\gamma}^{\alpha} \right). \quad (148)$$

The variation of this Chern-Simons Lagrangian, which only depends on the connection, turns out to be

$$\delta C_{RR} = - \delta \Gamma_{\alpha}^{\beta} \wedge R_{\beta}^{\alpha} + \frac{1}{2} d \left( \Gamma_{\alpha}^{\beta} \wedge \delta \Gamma_{\beta}^{\alpha} \right). \quad (149)$$

In the next step, we enforce vanishing torsion and nonmetricity by means of respective Lagrange multiplier terms:

$$L = C_{RR} + \lambda_{\alpha} \wedge T^{\alpha} + \lambda^{\alpha\beta} \wedge Q_{\alpha\beta}. \quad (150)$$

The variation then yields

$$\delta L = \delta C_{RR} + \delta \lambda_{\alpha} \wedge T^{\alpha} + \lambda_{\alpha} \wedge \delta T^{\alpha} + \delta \lambda^{\alpha\beta} \wedge Q_{\alpha\beta} + \lambda^{\alpha\beta} \wedge \delta Q_{\alpha\beta}$$

$$= - \delta \Gamma_{\alpha}^{\beta} \wedge R_{\beta}^{\alpha} + \delta \lambda_{\alpha} \wedge T^{\alpha} + \lambda_{\alpha} \wedge \left( d\delta \vartheta^{\alpha} + \delta \Gamma_{\beta}^{\alpha} \wedge \vartheta^{\beta} + \Gamma_{\beta}^{\alpha} \wedge \delta \vartheta^{\beta} \right)$$

$$+ \delta \lambda^{\alpha\beta} \wedge Q_{\alpha\beta} + \lambda^{\alpha\beta} \wedge \left( -d\delta g_{\alpha\beta} + \delta \Gamma_{\alpha}^{\gamma} g_{\gamma\beta} + \Gamma_{\alpha}^{\gamma} \delta g_{\gamma\beta} + \delta \Gamma_{\beta}^{\gamma} g_{\alpha\gamma} + \Gamma_{\beta}^{\gamma} \delta g_{\alpha\gamma} \right)$$

$$+ \frac{1}{2} d \left( \Gamma_{\alpha}^{\beta} \wedge \delta \Gamma_{\beta}^{\alpha} \right)$$

$$= - \delta \Gamma_{\alpha}^{\beta} \wedge R_{\beta}^{\alpha} + \delta \lambda_{\alpha} \wedge T^{\alpha} + \delta \lambda^{\alpha\beta} \wedge Q_{\alpha\beta} + \frac{1}{2} d \left( \Gamma_{\alpha}^{\beta} \wedge \delta \Gamma_{\beta}^{\alpha} \right) + \lambda_{\alpha} \wedge D \delta \vartheta^{\alpha}$$

$$- \delta \Gamma_{\alpha}^{\beta} \wedge \lambda_{\alpha} \wedge \vartheta^{\beta} - \lambda^{\alpha\beta} \wedge D \delta g_{\alpha\beta} + \delta \Gamma_{\alpha}^{\beta} \wedge \left( \lambda_{\beta}^{\alpha} + \lambda^{\alpha\beta} \right)$$

$$= \delta \lambda_{\alpha} \wedge T^{\alpha} + \delta \lambda^{\alpha\beta} \wedge Q_{\alpha\beta} + \delta \vartheta^{\alpha} \wedge D \lambda_{\alpha} + \delta g_{\alpha\beta} D \lambda^{\alpha\beta}$$

$$- \delta \Gamma_{\alpha}^{\beta} \wedge \left( R_{\beta}^{\alpha} + \lambda_{\beta} \wedge \vartheta^{\alpha} - \lambda^{\alpha\beta} - \lambda_{\beta}^{\alpha} \right)$$

$$- d \left( -\lambda_{\alpha} \wedge \delta \vartheta^{\alpha} + \frac{1}{2} \Gamma_{\alpha}^{\beta} \wedge \delta \Gamma_{\beta}^{\alpha} - \lambda^{\alpha\beta} \delta g_{\alpha\beta} \right). \quad (151)$$

C. Proof of eq.(52)

By means of the Ricci identity and the decomposition of the curvature we have

$$DD^{*}C_{\alpha} = - R_{\alpha}^{\beta} \wedge *C_{\beta}$$

$$= - \text{Weyl}_{\alpha}^{\beta} \wedge *C_{\beta} - \text{Ricci}_{\alpha}^{\beta} \wedge *C_{\beta} - \text{Scalar}_{\alpha}^{\beta} \wedge *C_{\beta}. \quad (152)$$

For $p$-forms $\phi, \psi$ of the same degree, there holds $*\phi \wedge \psi = *\psi \wedge \phi$. By means of $\vartheta^{\alpha} \wedge *\phi = (-1)^{p-1} (e^{\alpha} \wedge \varphi)$ we can prove that $\text{Scalar}_{\alpha\beta} \wedge *C^{\alpha} = 0$. Performing a “partial integration”
we arrive at

\[ DD^*C_\alpha = -D \left( ^*\text{Weyl}_\alpha^\beta \wedge L^\beta \right) + \left( D^*\text{Weyl}_\alpha^\beta \right) \wedge L^\beta - ^*\text{Ricci}_\alpha^\beta \wedge C^\beta. \]  

(153)

Next, we use the “double duality relations” for the irreducible pieces of the curvature,

\[ ^*\text{Weyl}_{\alpha\beta} = \text{Weyl}_{\mu\nu} \frac{1}{2} \eta^{\mu\nu}_{\alpha\beta}, \]  

(154)

\[ ^*\text{Ricci}_\alpha^\beta = -\text{Ricci}_{\mu\nu} \frac{1}{2} \eta^{\mu\nu}_{\alpha\beta}, \]  

(155)

\[ ^*\text{Scalar}_{\alpha\beta} = \text{Scalar}_{\mu\nu} \frac{1}{2} \eta^{\mu\nu}_{\alpha\beta}. \]  

(156)

Together with eqs. (32) and (26), we obtain

\[ \left( D^*\text{Weyl}_\alpha^\beta \right) \wedge L^\beta = \frac{1}{2} \eta^{\mu\nu}_{\alpha\beta} D\text{Weyl}_{\mu\nu} \wedge L^\beta = -\frac{1}{2} \eta^{\mu\nu}_{\alpha\beta} \partial_\mu \wedge C_\nu \wedge L^\beta \]

\[ = -\frac{1}{2} \eta^{\mu\nu}_{\alpha\beta} \partial_\mu \wedge L^\beta \wedge C_\nu = -^*\text{Ricci}_\nu^\alpha \wedge C_\nu + ^*\text{Scalar}_\nu^\alpha \wedge C_\nu \]

\[ = ^*\text{Ricci}_\alpha^\nu \wedge C_\nu. \]  

(157)

Substituting this into (153) completes the proof.

D. Computer algebra

We present a Reduce-Excalc program for the computation of the Cotton 2-form in 3 dimensions and for testing of the DJT-field equations. The components \( C_\alpha^\beta \) are also assigned to a 3\( \times \)3 matrix. This matrix is then treated by the appropriate linear algebra packages, as, for instance, the Reduce package “Normform” which allows the determination of the Jordan form. For an introduction into the use of Reduce-Excalc in gravity theories see [38].

% file cotton.exi  chh 2003-08-08
% Calculation of Cotton tensor form given coframe/metric, n=3

load excalc ;

% Definition of coframe/metric
% eq.(4.7) in Percacci et al., Ann Phys (NY) 176 (1987) 344
coframe $o(0) = \exp(\mu x/3)(d t + 2\exp(-\mu x/3) * d y)$,
    $o(1) = d x$,
    $o(2) = d y$
with metric $g = o(0)*o(0)-o(1)*o(1)-o(2)*o(2)$ ;

frame e ;

% calculation of curvature

pform riem2(a,b) = 2;
riemannconx chris1 ;
chris1(a,b) := chris1(b,a) ;
riem2(-a,b) := d chris1(-a,b) - chris1(-a,c) ^ chris1(-c,b) ;

% calculation of $L_a$ and Cotton

pform ll1(a)=1,cotton2(a)=2;
ll1(a) := $e(-b) _| riem2(a,b)$
- 1/4 * ($e(-c) _| (e(-d) _| riem2(c,d))$) * $o(a)$ ;
cotton2(a) := d ll1(a) + chris1(-b,a) ^ ll1(b) ;

% Definition of Cotton tensor
pform cotmat(a,b) = 0 ;
cotmat(a,b) := #(cotton2(a)^$o(b)) ;

% Definition of Cotton matrix
matrix cotm(3,3) $
for a:= 1:3$ do {for b := 1:3 do{
cotm(a,b) := cotmat(-(a-1),b-1) }}$

% Definition Einstein 2-form
pform einstein2(a) =2;
einstein2(a) := (1/2) * #(o(a)^o(b)^o(c)) ^ riem2(-b,-c);

% Test of DJT field equations
pform null(a)=2;
null(a) := einstein2(a)+(1/mu)*cotton2(a);

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