An Analytic Result for the Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

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Abstract: In the planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, the conformal symmetry constrains multi-loop $n$-edged Wilson loops to be basically given in terms of the one-loop $n$-edged Wilson loop, augmented, for $n \geq 6$, by a function of conformally invariant cross ratios. We identify a class of kinematics for which the Wilson loop exhibits exact Regge factorisation and which leave invariant the analytic form of the multi-loop $n$-edged Wilson loop. In those kinematics, the analytic result for the Wilson loop is the same as in general kinematics, although the computation is remarkably simplified with respect to general kinematics. Using the simplest of those kinematics, we have performed the first analytic computation of the two-loop six-edged Wilson loop in general kinematics.

Keywords: QCD, MSYM, small $x$. 
1. Introduction

In the planar $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, Anastasiou, Bern, Dixon and Kosower (ABDK) [1] proposed an iterative structure for the colour-stripped two-loop scattering amplitude with an arbitrary number $n$ of external legs in a maximally-helicity violating (MHV) configuration. Writing at any loop order $L$, the amplitude $M_n^{(L)}$ as the tree-level amplitude, $M_n^{(0)}$, which depends on the helicity configuration, times a scalar function, $m_n^{(L)}$, 

$$M_n^{(L)} = M_n^{(0)} m_n^{(L)},$$

the proposed iteration formula for the two-loop MHV amplitude $m_n^{(2)}(\epsilon)$ was

$$m_n^{(2)}(\epsilon) = \frac{1}{2} \left[ m_n^{(1)}(\epsilon) \right]^2 + f^{(2)}(\epsilon) m_n^{(1)}(2\epsilon) + C^{(2)} + O(\epsilon).$$

Thus the two-loop amplitude is determined in terms of the one-loop MHV amplitude $m_n^{(1)}(\epsilon)$ evaluated through to $O(\epsilon^2)$ in the dimensional-regularisation parameter $\epsilon = (4 - d)/2$, the constant $C^{(2)} = -\zeta_2/2$, and the function $f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3 \epsilon - \zeta_4 \epsilon^2$, with $\zeta_i = \zeta(i)$ and $\zeta(z)$ the Riemann zeta function.

Subsequently, Bern, Dixon and one of the present authors (BDS) proposed an all-loop resummation formula [2] for the colour-stripped $n$-point MHV amplitude, which implies a tower of iteration formulae, allowing one to determine the $n$-point amplitude at a given number of loops in terms of amplitudes with fewer loops, evaluated to higher orders of $\epsilon$. BDS checked that the ansatz is correct for the three-loop four-point amplitude, by evaluating analytically $m_4^{(3)}(\epsilon)$ through to finite terms, as well as $m_4^{(2)}(\epsilon)$ through to $O(\epsilon^2)$ and $m_4^{(1)}(\epsilon)$ through to $O(\epsilon^4)$. The ansatz has been proven to be correct also for the two-loop five-point amplitude [3, 4], for which $m_5^{(2)}(\epsilon)$ has been computed numerically through to finite terms, as well as $m_5^{(1)}(\epsilon)$ through to $O(\epsilon^2)$.

Using the AdS/CFT correspondence, Alday and Maldacena showed that in the strong-coupling limit the ansatz must break down for amplitudes with a large number of legs [5]. At weak coupling, the computation of the two-loop six-edged Wilson loop [6] led to conclude that either the ansatz or the duality relation between amplitudes and Wilson loops were to break down for two-loop six-point amplitudes. Likewise, there were hints of a failure of the ansatz from the six-point amplitude analysed in the multi-Regge kinematics in a Minkowski region [7, 8, 9]. The accumulating evidence against the ansatz provoked the numerical calculation of $m_6^{(2)}(\epsilon)$ through to finite terms and of $m_6^{(1)}(\epsilon)$ through to $O(\epsilon^2)$, by which the ansatz was demonstrated to fail [10], and where it was shown that the finite pieces of the parity-even part of $m_6^{(2)}(\epsilon)$ are incorrectly determined by the ansatz (although the parity-odd part of $m_6^{(2)}(\epsilon)$ does satisfy the ansatz [11]). In particular, it was shown numerically that the two-loop remainder function, defined as the difference between the two-loop amplitude and the ansatz for it,

$$R_n^{(2)} = m_n^{(2)}(\epsilon) - \frac{1}{2} \left[ m_n^{(1)}(\epsilon) \right]^2 - f^{(2)}(\epsilon) m_n^{(1)}(2\epsilon) - C^{(2)} + O(\epsilon),$$

...
is different from zero for \( n = 6 \), where \( R^{(2)}_{6, WL} \) is a function of the kinematical parameters of the \( n \)-point amplitude, but a constant with respect to \( \epsilon \). However, the analytic form of \( R^{(2)}_{6, WL} \) was not computed.

In the strong-coupling limit, Alday and Maldacena [12] showed that planar scattering amplitudes exponentiate like in the ansatz, and suggested that the vacuum expectation value of the \( n \)-edged Wilson loop could be related to the \( n \)-point amplitude in \( \mathcal{N} = 4 \) SYM. At weak coupling, the agreement between the light-like Wilson loop and the (parity-even part of the) MHV amplitude has been verified for the one-loop four-edged [13] and \( n \)-edged [14] Wilson loops, and for the two-loop four-edged [15], five-edged [16] and six-edged [6, 17] Wilson loops.

Furthermore, it was shown that the \( L \)-loop light-like Wilson loop exhibits a conformal symmetry, and that the solution of the Ward identity for a special conformal boost is the ansatz, augmented, for \( n \geq 6 \), by a function \( R^{(L)}_{n, WL} \) of conformally invariant cross-ratios [16]. Because of the duality between Wilson loops and amplitudes at one and two loops, the function of the conformally invariant cross-ratios, \( R^{(2)}_{n, WL} \), can be identified as the remainder function of Eq. (1.3).

In Refs. [17, 18], the two-loop \( n \)-edged Wilson loop has been given in terms of Feynman-parameter-like integrals. Furthermore, in Ref. [18] a numerical algorithm has been set up, which is valid for the two-loop \( n \)-edged Wilson loop and by which the seven-edged and eight-edged Wilson loops have been computed (although the corresponding MHV amplitudes are not known\(^1\)). Thus, also the remainder functions \( R^{(2)}_{7, WL} \) and \( R^{(2)}_{8, WL} \) of the Wilson loops are known numerically, and the numerical evidence [18] confirms that they are functions of conformally invariant cross-ratios only. However, their analytic form is in general unknown.

In fact, before this letter the remainder function had been computed analytically only in the strong coupling regime: for the two-loop six-edged Wilson loop in the particular kinematic configuration in which the cross ratios coincide [20], and otherwise in a particular kinematic set-up for which only \( 2n \)-edged polygons are allowed [21]. In that set-up, the simplest non-trivial remainder function is the one of the two-loop eight-edged Wilson loop. In the weak coupling regime, and in the same kinematics, \( R^{(2)}_{8, WL} \) has been computed numerically in Ref. [22], where a numerical evidence has been found for a linear relation between the remainder functions at weak and strong couplings.

In this letter, we give a brief account of the first analytic computation at weak coupling of the two-loop six-edged Wilson loop in general kinematics. The computation has been done in the Euclidean region in \( D = 4 - 2\epsilon \) dimensions, where the result is real.

In Sec. 2, we write the two-loop Wilson loop in terms of the one-loop Wilson loop plus a remainder function \( R^{(2)}_{n, WL} \). For \( n = 6 \), \( R^{(2)}_{6, WL} \) is a function of the three conformally invariant cross ratios, \( u_1, u_2, u_3 \). It has been computed numerically in Refs. [10, 17, 18] in arbitrary kinematics, constrained only by momentum conservation. However, it suffices to compute \( R^{(2)}_{6, WL} \) in any kinematical limit which does not modify the analytic dependence of \( R^{(2)}_{6, WL} \) on \( u_1, u_2, u_3 \). To that end, we recall that the \( L \)-loop four-edged Wilson loop is not

\(^1\)The parity-even part of the two-loop \( n \)-point MHV amplitude has been given in terms of integral functions, yet to be evaluated [19].
modified by the Regge limit [13]. In such a limit \( w_4^{(L)} \) undergoes an exact Regge factorisation. So is the case of the five-edged Wilson loop in the multi-Regge kinematics [7, 23]. However, the six-edged Wilson loop is modified by the multi-Regge kinematics: the three conformally invariant cross ratios are not invariant in such a limit [23]. Less constraining Regge limits have been analysed in Ref. [24]. The simplest of those limits to feature an exact Regge factorisation of \( w_6^{(L)} \) is the quasi-multi-Regge kinematics (QMRK) of a pair along the ladder [25, 26].

In Sec. 3, we recall the QMRK of a pair along the ladder for the six-edged Wilson loop, and we show that the QMRK of three-of-a-kind along the ladder [27] for the seven-edged Wilson loop, the QMRK of four-of-a-kind along the ladder [28] for the eight-edged Wilson loop, and in general the QMRK of a cluster of \((n - 4)\)-of-a-kind along the ladder for the \(n\)-edged Wilson loop do not modify the analytic dependence of \( w_n^{(L)} \) on the conformally invariant cross ratios. That is, this class of kinematics exhibits an exact Regge factorisation of \( w_n^{(L)} \). Thus, the result for \( w_n^{(L)} \) in these kinematics is the same as the result in general kinematics, although the computation is remarkably simplified with respect to the same computation in general kinematics. Finally, we note that although in Sec. 4 we apply the analysis of Sec. 3 to the computation of the six-edged two-loop Wilson loop, nothing of what we consider in Sec. 3 is specific to two loops: The analysis of Sec. 3 is valid for any number of loops.

In Sec. 4, we brief on how the Feynman-parameter-like integrals of the two-loop six-edged Wilson loop have been computed in the QMRK of a pair along the ladder, and on the type of functions which appear in the final result. Because of the exact Regge factorisation, the ensuing remainder function is valid in general kinematics. It can be expressed as a linear combination of multiple polylogarithms of uniform transcendental weight four. However, the result is far too long to be reported in this letter. We present it in an electronic form at www.arxiv.org where a text file containing the Mathematica expression for the remainder function is provided.

2. The two-loop Wilson loop

The Wilson loop is defined through the path-ordered exponential,

\[
W[C_n] = \text{Tr} \mathcal{P} \exp \left[ ig \int d\tau \dot{x}^\mu(\tau) A_\mu(x(\tau)) \right],
\]

computed on a closed contour \( C_n \). In what follows, the closed contour is a light-like \( n \)-edged polygonal contour [12]. The contour is such that labelling the \( n \) vertices of the polygon as \( x_1, \ldots, x_n \), the distance between any two contiguous vertices, \( i.e. \), the length of the edge in between, is given by the momentum of a particle in the corresponding colour-ordered scattering amplitude,

\[
p_i = x_i - x_{i+1},
\]

with \( i = 1, \ldots, n \). Because the \( n \) momenta add up to zero, \( \sum_{i=1}^n p_i = 0 \), the \( n \)-edged contour closes, provided we make the identification \( x_1 = x_{n+1} \).
In the weak-coupling limit, the Wilson loop can be computed as an expansion in the coupling. The expansion of Eq. (2.1) is done through the non-abelian exponentiation theorem [29, 30], which gives the vacuum expectation value of the Wilson loop as an exponential,

$$
\langle W[C_n] \rangle = 1 + \sum_{L=1}^{\infty} a^L W_n^{(L)} = \exp \sum_{L=1}^{\infty} a^L W_n^{(L)},
$$

(2.3)

where the coupling is defined as

$$
a = g^2 N \frac{8\pi}{8}. \tag{2.4}
$$

For the first two loop orders, one obtains

$$
w_n^{(1)} = W_n^{(1)}, \quad w_n^{(2)} = W_n^{(2)} - \frac{1}{2} \left( W_n^{(1)} \right)^2. \tag{2.5}
$$

The one-loop coefficient $w_n^{(1)}$ was evaluated in Refs. [13, 14], where it was given in terms of the one-loop $n$-point MHV amplitude,

$$
w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} m_n^{(1)} = m_n^{(1)} - \frac{\zeta_2}{2} + O(\epsilon), \tag{2.6}
$$

where the amplitude is a sum of one-loop two-mass-easy box functions [31],

$$
m_n^{(1)} = \sum_{p,q} F_{2\text{me}}(p, q, P, Q), \tag{2.7}
$$

where $p$ and $q$ are two external momenta corresponding to two opposite massless legs, while the two remaining legs $P$ and $Q$ are massive. The two-loop coefficient $w_n^{(2)}$ has been computed analytically for $n = 4$ [15] and $n = 5$ [16] and numerically for $n = 6$ [17] and $n = 7, 8$ [18].

In Ref. [16] it was established that the Wilson loop fulfils a special conformal Ward identity, whose solution is the ansatz plus, for $n \geq 6$, an arbitrary function of the conformally invariant cross-ratios, defined in Eq. (2.11). Thus, the two-loop coefficient $w_n^{(2)}$ can be written as

$$
w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon) w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_{n, WL}^{(2)} + O(\epsilon), \tag{2.8}
$$

where the constant is the same as in Eq. (1.2), $C_{WL}^{(2)} = C^{(2)} = -\zeta_2^2/2$, and the function $f_{WL}^{(2)}(\epsilon)$ is [15, 18, 32]

$$
f_{WL}^{(2)}(\epsilon) = -\zeta_2 + 7\zeta_3 \epsilon - 5\zeta_4 \epsilon^2. \tag{2.9}
$$

With the two-loop coefficient $w_n^{(2)}$ given by Eqs. (2.8) and (2.9) and the two-loop MHV amplitude given by Eqs. (1.2) and (1.3), the duality between Wilson loops and amplitudes is expressed by the equality of their remainder functions [17, 18],

$$
R_{n, WL}^{(2)} = R_n^{(2)}. \tag{2.10}
$$

Defining the conformally invariant cross ratios as,

$$
u_{ij} = \frac{x_i^2 x_{i+1,j}^2}{x_i^2 x_{i+1,j+1}^2}, \tag{2.11}
$$
for \( n = 6 \), they are \([16]\),
\[
\begin{align*}
 u_{36} &= u_1 = \frac{x_{12}^2 x_{46}^2}{x_{36} x_{41}}, & u_{14} &= u_2 = \frac{x_{12}^2 x_{34}^2}{x_{14} x_{25}}, & u_{25} &= u_3 = \frac{x_{25}^2 x_{36}^2}{x_{26} x_{34}},
\end{align*}
\] (2.12)
where \( x_{ij} = (x_i - x_j)^2 \), and using Eq. (2.2) one sees that \( x_{i,i+2}^2 = s_{i,i+1} \) and \( x_{i,i+3}^2 = s_{i,i+1,i+2} \), where the labels are understood to be modulo 6.

3. The quasi-multi-Regge kinematics of a cluster along the ladder

As we remarked in the Introduction, it suffices to compute \( R_{6,WL}^{(2)} \) in any kinematical limit which does not modify the analytic dependence of \( R_{6,WL}^{(2)} \) on \( u_1, u_2, u_3 \). The simplest of these limits to feature an exact Regge factorisation of \( u_6^{(2)} \), and in fact in general of \( u_6^{(L)} \), is the QMRK of a pair along the ladder \([25, 26]\). In those kinematics, the outgoing gluons are strongly ordered in rapidity, except for a central pair of gluons along the ladder, while their transverse momenta are all of the same size. In the physical region, defining 1 and 2 as the incoming gluons and 3, 4, 5, 6 as the outgoing gluons, the ordering can be chosen as
\[
y_3 \gg y_4 \simeq y_5 \gg y_6; 
\end{align*}
\] (3.1)
where the particle momentum \( p \) is parametrised in terms of the rapidity \( y \) and the azimuthal angle \( \phi \), \( p = (|p_\perp| \cosh y, |p_\perp| \cos \phi, |p_\perp| \sin \phi, |p_\perp| \sinh y) \). We shall work in the Euclidean region, where the Wilson loop is real. There the Mandelstam invariants are taken as all negative, and in the QMRK of a pair along the ladder they are ordered as follows,
\[
-s_{12} \gg -s_{34}, -s_{56}, -s_{345}, -s_{123} \gg -s_{23}, -s_{45}, -s_{61}, -s_{234}.
\] (3.2)
Introducing a parameter \( \lambda \ll 1 \), the hierarchy above is equivalent to the rescaling
\[
\{s_{34}, s_{56}, s_{123}, s_{345}\} = \mathcal{O}(\lambda), \quad \{s_{23}, s_{45}, s_{61}, s_{234}\} = \mathcal{O}(\lambda^2).
\] (3.3)
It is easy to see that in this limit the three conformally invariant cross-ratios (2.12) do not take trivial limiting values \([24]\),
\[
\begin{align*}
 u_1 \to u_1^{QMRK} &= \frac{s_{45}}{(p_4^+ + p_5^-)(p_4^- + p_5^+)} = \mathcal{O}(1), \\
 u_2 \to u_2^{QMRK} &= \frac{|p_3\perp|^2 p_5^- p_6^-}{(|p_3\perp + p_4\perp|^2 + p_5^+ p_4^-)(p_4^+ + p_5^+ p_6^-)} = \mathcal{O}(1), \\
 u_3 \to u_3^{QMRK} &= \frac{|p_6\perp|^2 p_5^+ p_4^-}{p_3^+ (p_4^- + p_5^+)(|p_3\perp + p_4\perp|^2 + p_5^+ p_4^-)} = \mathcal{O}(1).
\end{align*}
\] (3.4)
A similar analysis can be carried through for the seven-edged Wilson loop, \( w_7^{(L)} \). We have verified that the simplest limit to feature an exact Regge factorisation is the QMRK of three-of-a-kind along the ladder \([27]\). In the physical region, the outgoing gluons are strongly ordered in rapidity, except for a cluster of three along the ladder,
\[
y_3 \gg y_4 \simeq y_5 \simeq y_6 \gg y_7; 
\end{align*}
\] (3.5)
In the Euclidean region, the Mandelstam invariants are ordered as follows,
\[-s_{12} \gg -s_{123}, -s_{345}, -s_{567}, -s_{712}, -s_{34}, -s_{67} \gg \]
\[\gg -s_{23}, -s_{45}, -s_{56}, -s_{71}, -s_{234}, -s_{456}, -s_{671}. \]  
(3.6)

Through a parameter \( \lambda \ll 1 \), the hierarchy above is equivalent to the rescaling
\[\begin{align*}
\{s_{123}, s_{345}, s_{567}, s_{712}, s_{34}, s_{67}\} &= \mathcal{O}(\lambda), \\
\{s_{23}, s_{45}, s_{56}, s_{71}, s_{234}, s_{456}, s_{671}\} &= \mathcal{O}(\lambda^2).
\end{align*}\]  
(3.7)

Using Eq. (2.11), and the fact that \( x_{ij}^2 = s_{i\ldots j} \), it is easy to see that the seven cross ratios of the eight-edged Wilson loop do not take trivial limiting values under the rescaling (3.7),
\[\{u_{14}, u_{25}, u_{36}, u_{47}, u_{51}, u_{62}, u_{73}\} = \mathcal{O}(1).\]  
(3.8)

Thus, the dependence of \( w_7^{(L)} \) on the seven cross ratios is not modified by the QMRK of three-of-a-kind along the ladder (3.6), and hence \( w_7^{(L)} \) undergoes an exact Regge factorisation in this limit.

The same pattern unfolds for the eight-edged Wilson loop, \( w_8^{(L)} \). The simplest limit to feature an exact Regge factorisation is the QMRK of four-of-a-kind along the ladder [28]. In the physical region, the outgoing gluons are strongly ordered in rapidity, except for a cluster of four along the ladder,
\[y_3 \gg y_4 \simeq y_5 \simeq y_6 \gg y_8; \quad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{6\perp}| \simeq |p_{7\perp}| \simeq |p_{8\perp}|.\]  
(3.9)

In the Euclidean region, the Mandelstam invariants are ordered as follows,
\[-s_{12} \gg -s_{1234}, -s_{3456}, -s_{123}, -s_{345}, -s_{678}, -s_{812}, -s_{34}, -s_{87} \gg \]
\[\gg -s_{2345}, -s_{4567}, -s_{234}, -s_{456}, -s_{67}, -s_{23}, -s_{45}, -s_{56}, -s_{67}, -s_{81}.\]  
(3.10)

Through the parameter \( \lambda \ll 1 \), the hierarchy above corresponds to the rescaling
\[\begin{align*}
\{s_{1234}, s_{3456}, s_{123}, s_{345}, s_{678}, s_{812}, s_{34}, s_{87}\} &= \mathcal{O}(\lambda), \\
\{s_{2345}, s_{4567}, s_{234}, s_{456}, s_{67}, s_{23}, s_{45}, s_{56}, s_{67}, s_{81}\} &= \mathcal{O}(\lambda^2).
\end{align*}\]  
(3.11)

It is easy to check that all the twelve cross ratios of the eight-edged Wilson loop do not take trivial limiting values under the rescaling (3.11),
\[\{u_{14}, u_{25}, u_{36}, u_{47}, u_{58}, u_{61}, u_{72}, u_{83}, u_{15}, u_{26}, u_{37}, u_{48}\} = \mathcal{O}(1).\]  
(3.12)

Thus, the dependence of \( w_8^{(L)} \) on the twelve cross ratios is not modified by the QMRK of four-of-a-kind along the ladder (3.10), and hence \( w_8^{(L)} \) undergoes an exact Regge factorisation in this limit.

The pattern above generalises to the \( n \)-edged Wilson loop, \( w_n^{(L)} \). We illustrate briefly how the QMRK of a cluster of \((n - 4)\)-of-a-kind along the ladder features the exact Regge factorisation of \( w_n^{(L)} \). In the physical region, the outgoing gluons are strongly ordered in rapidity, except for a cluster of \((n - 4)\) along the ladder,
\[y_3 \gg y_4 \simeq \ldots \simeq y_{n-1} \gg y_n; \quad |p_{3\perp}| \simeq \ldots \simeq |p_{n\perp}|.\]  
(3.13)
In order to display the exact Regge factorisation of $w_n^{(L)}$, we deal with the cases of an even, $n = 2r$, and an odd, $n = 2r + 1$, number of edges separately. Through a parameter $\lambda \ll 1$, the hierarchy of the Mandelstam invariants implied by Eq. (3.13) can be rendered by requiring that $s_{12} = O(1)$ and by the rescaling

$$s_{12...j} = O(\lambda), \quad 3 \leq j \leq r,$$
$$s_{34...j} = O(\lambda), \quad 4 \leq j \leq r + 2,$$

(3.14)

for any $n$ and $r \geq 3$. In addition,

$$s_{j...2r} = O(\lambda), \quad r + 2 \leq j \leq 2r - 1, \quad r \geq 3,$$
$$s_{j...12} = O(\lambda), \quad r + 4 \leq j \leq 2r, \quad r \geq 4,$$

(3.15)

for $n = 2r$, with the labels understood to be modulo $2r$, and

$$s_{j...2r+1} = O(\lambda), \quad r + 2 \leq j \leq 2r,$$
$$s_{j...12} = O(\lambda), \quad r + 4 \leq j \leq 2r + 1,$$

(3.16)

for $n = 2r + 1$ and $r \geq 3$, with the labels understood to be modulo $2r + 1$. All other invariants rescale to be $O(\lambda^2)$. It is easy to check that for $n = 6, 7, 8$, Eqs. (3.14)-(3.16) reproduce the rescaling of Eqs. (3.3), (3.7) and (3.11).

In order to compare with the rescaling of Eqs. (3.3), (3.7) and (3.11), it is convenient to classify the cross ratios as follows,

$$u_{1j} = \frac{x_{1j+1}^2 x_{2j}^2}{x_{1j}^2 x_{2j+1}^2} = \frac{s_{1...j}s_{2...j-1}}{s_{1...j-1}s_{2...j}}, \quad j = 4, \ldots, r + 1,$$
$$u_{2j} = \frac{x_{2j+1}^2 x_{3j}^2}{x_{2j}^2 x_{3j+1}^2} = \frac{s_{2...j}s_{3...j-1}}{s_{2...j-1}s_{3...j}}, \quad j = 5, \ldots, r + 2,$$

(3.17)

$$\vdots$$

$$u_{2r+1j} = \frac{x_{2r+1j+1}^2 x_{1j}^2}{x_{2r+1j}^2 x_{1j+1}^2} = \frac{s_{2r+1...j}s_{1...j-1}}{s_{2r+1...j-1}s_{1...j}}, \quad j = 3, \ldots, r,$$

for $n = 2r + 1$, and

$$u_{1j} = \frac{x_{1j+1}^2 x_{2j}^2}{x_{1j}^2 x_{2j+1}^2} = \frac{s_{1...j}s_{2...j-1}}{s_{1...j-1}s_{2...j}}, \quad j = 4, \ldots, r + 1,$$
$$\vdots$$

$$u_{rj} = \frac{x_{rj+1}^2 x_{rj+1}^2}{x_{rj}^2 x_{rj+1}^2} = \frac{s_{r...j}s_{r+1...j-1}}{s_{r...j-1}s_{r+1...j}}, \quad j = r + 3, \ldots, 2r,$$
$$u_{r+1j} = \frac{x_{r+1j+1}^2 x_{r+2j}^2}{x_{r+1j}^2 x_{r+2j+1}^2} = \frac{s_{r+1...j}s_{r+2...j-1}}{s_{r+1...j-1}s_{r+2...j}}, \quad j = r + 4, \ldots, 2r,$$

(3.18)

$$\vdots$$

$$u_{2rj} = \frac{x_{2rj+1}^2 x_{1j}^2}{x_{2rj}^2 x_{1j+1}^2} = \frac{s_{2r...j}s_{1...j-1}}{s_{2r...j-1}s_{1...j}}, \quad j = 3, \ldots, r - 1,$$
for $n = 2r$, where we use the fact that $u_{i,i+1} = u_{i,i+2} = 0$. Without further imposing the Gram-determinant constraints, the counting yields $n(n - 5)/2$ conformally invariant cross ratios, in agreement with Ref. [18]. It is easy to check that for $n = 6, 7, 8$, Eqs. (3.17) and (3.18) generate the cross ratios of Eqs. (2.12), (3.8) and (3.12). Then by inspection one can see that the cross ratios (3.17) and (3.18) do not take limiting values under the rescaling of Eqs. (3.3), (3.7) and (3.11). Thus, the dependence of $w_n^{(L)}$ on the cross ratios (3.17) and (3.18) is not modified by the QMRK of a cluster of $(n - 4)$-of-a-kind along the ladder (3.13), and hence $w_n^{(L)}$ undergoes an exact Regge factorisation in this limit.

Finally, we note that the exact Regge factorisation of the $n$-edged Wilson loops, with $n = 4, 5$, may be dealt with as a degenerate case of the QMRK of a cluster of $(n - 4)$-of-a-kind along the ladder. Namely, for $n = 4$ one obtains the QMRK of a cluster of zero particles along the ladder, which is the standard Regge limit [13], and for $n = 5$ the QMRK of a cluster of one particle along the ladder, which is the multi-Regge kinematics of five particles [7, 23].

4. The two-loop six-edged Wilson loop

In Ref. [18] an expression for a generic two-loop $n$-edged Wilson loop as a sum of Euler-type integrals was presented, similar to Feynman parameter integrals appearing in the computation of Feynman integrals. In Sec. 3 we argued that the Wilson loops are Regge exact in the QMRK where $(n - 4)$ gluons are emitted along the ladder. Hence, it follows that it is sufficient to compute the individual integrals in the QMRK to obtain the Wilson loop in general kinematics.

In the present work we concentrate exclusively on the two-loop six-edged Wilson loop, $w_6^{(2)}$, which is the first case where the remainder function is non zero. Hence, an analytic computation of $w_6^{(2)}$ is equivalent to an analytic computation of the two-loop six-point remainder function $R_{W6}^{(2)}$. We start from the parametric representations for the diagrammatic contributions to the two-loop $n$-edged Wilson loop derived in Ref. [18], and we derive appropriate Mellin-Barnes representations for them using the standard formula,

$$
\frac{1}{(A + B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \, \Gamma(-z) \Gamma(\lambda + z) \frac{A^z}{B^{\lambda+z}},
$$

where the contour is chosen such as to separate the poles in $\Gamma(\ldots - z)$ from the poles in $\Gamma(\ldots + z)$. Note that in our case $\lambda$ is in general an integer plus an off-set corresponding to the dimensional regulator $\epsilon$. In order to resolve the singularity structures in $\epsilon$, we apply the strategy based on the Mellin-Barnes representation and given in Refs. [33, 34, 35, 36]. To this effect, we apply the codes MB [37] and MBresolve [38] and obtain a set of Mellin-Barnes integrals that can be safely expanded in $\epsilon$ under the integration sign. Then we proceed to take the QMRK limit defined by Eq. (3.2) by applying MBasymptotics [39] to extract the leading QMRK behavior of each Mellin-Barnes integral, and barnesroutines [40] to perform integrations that can be done by corollaries of Barnes lemmas. To be more explicit, in Ref. [18] the six-edged Wilson loop was expressed as,

$$
w_6^{(2)} = C \sum_i f_i(p_k),
$$
where \( f_i(p_k) \) denote the parametric integrals of Ref. [18] depending on the external momenta \( p_k \). The prefactor \( C \) is defined by

\[
C = 2a^2 \mu^4 \Gamma(1 + \epsilon) e^{\gamma_E \epsilon} \left[ \Gamma(1 + \epsilon) e^{\gamma_E \epsilon} \right]^2. \tag{4.3}
\]

and the scale \( \mu^2 \) is given in terms of the Wilson loop scale, \( \mu^2_{WL} = \pi e^{\gamma_E} \mu^2 \). The Regge exactness of \( w_6^{(2)} \) enables us to write

\[
w_6^{(2)} = C \sum_i f_i^{(1)}(p_k), \tag{4.4}
\]

where \( f_i^{(1)}(p_k) \) denotes the leading asymptotic behavior of \( f_i(p_k) \) in the QMRK limit defined by the scaling (3.3),

\[
f_i(p_k) = f_i^{(1)}(p_k) + \mathcal{O}(\lambda). \tag{4.5}
\]

In Section 3 we considered the QMRK limit where gluons 1 and 2 are incoming, \( i.e., \) where \( s_{12} \) is the largest invariant. Of course there are five additional ways in which we could have defined the limit, corresponding to the cyclic permutations of the external gluons, \( e.g., \) we could have considered the QMRK limit defined by the scaling,

\[
\{ s_{45}, s_{61}, s_{234}, s_{123} \} = \mathcal{O}(\lambda), \quad \{ s_{34}, s_{56}, s_{12}, s_{345} \} = \mathcal{O}(\lambda^2), \tag{4.6}
\]

where \( \lambda \ll 1 \) and \( s_{23} = \mathcal{O}(1) \). Note that this limit is incompatible with the limit (3.3), \( i.e., \) terms that are \( \mathcal{O}(\lambda) \) in one limit can be large in another limit, and vice-versa. However, the Regge exactness of the Wilson loop allows us to iterate this procedure and to repeat the previous argument starting from Eq. (4.4) and to take the limit (4.6). Then we arrive at

\[
w_6^{(2)} = C \sum_i f_i^{(2)}(p_k), \tag{4.7}
\]

where \( f_i^{(2)}(p_k) \) is the leading asymptotic behavior of \( f_i^{(1)}(p_k) \) in the limit (4.6),

\[
f_i^{(1)}(p_k) = f_i^{(2)}(p_k) + \mathcal{O}(\lambda). \tag{4.8}
\]

It is straightforward to see how this procedure iterates for the remaining four cyclic permutations of Eq. (3.3). Finally, we arrive at a set of multiple Mellin-Barnes integrals \( f_i^{(6)}(p_k) \) of a much simpler type than the original ones. After applying our procedure, all integrals are at most threefold and all of them are explicitly dependent on the conformal cross ratios only, because the cross ratios are the only combination of invariants that are invariant under all the cyclic permutations of Eq. (3.3). However, note that the coefficients of the integrals do not depend only on the conformal cross ratios, but also on logarithms of Mandelstam invariants, which arise when expanding the MB integrals in a QMRK limit of the type defined in Eq. (3.3), as can be easily seen from the following example,

\[
\frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} dz \Gamma(-z) \Gamma(z) \lambda^z \left( \frac{s_{12}}{s_{23}} \right)^z = \ln \frac{s_{12}}{s_{23}} + \mathcal{O}(\lambda). \tag{4.9}
\]
Finally, we checked numerically that the sum of the Mellin-Barnes integrals in the QMRK, $f_i^{(6)}(p_k)$, is equal to the sum of all the original parametric integrals, the latter being evaluated numerically using FIESTA [41, 42].

The resulting Mellin-Barnes integrals are then evaluated by directly closing contours and summing up residues or by exchanging a Mellin-Barnes integration with an integral of Euler type. The infinite sums which appear in the intermediate steps of the computation are typically generalised harmonic sums [43, 44] as well as multiple binomial sums [45, 46]. The convergence of the series requires the conformal cross ratios to be less than 1, and in the following we concentrate on this kinematic region, within the Euclidean region. Details on the explicit computation of the integrals will be presented in a forthcoming publication [47]. Here it suffices to say that, except for the contribution coming from the hard diagram with six light-like edges, all the integrals can be expressed in terms of harmonic polylogarithms [48] in one conformal cross ratio. In turn, the six-edged hard diagram constitutes the bulk of the final result, and can be written as a linear combination of Goncharov’s multiple polylogarithms [49], whose arguments are functions of conformal cross ratios. These polylogarithms are defined by the iterated integration,

$$G(\vec{w}; z) = \int_0^z \frac{dt}{t-a} G(\vec{w}'; t) \quad \text{and} \quad G(\vec{0}_n; z) = \frac{1}{n!} \ln^n z,$$

where we define $\vec{w} = (a, \vec{w}')$, and for $z = 1$ they are manifestly real, if all the elements in the weight vector $\vec{w}$ are either greater than 1 or negative. The number of elements of $\vec{w}$ is called the (transcendental) weight of $G(\vec{w}; z)$. The polylogarithms we obtain can be divided into several classes, corresponding to the elements $w_i$ of the weight vector,

1. $w_i = 1/u_j, 1/(1-u_j), (1-u_j)/(1-u_j-u_k)$.

   It is easy to see that in this case $w_i > 1$ or $w_i < 0$, for $0 < u_i, u_j < 1$.

2. $w_i = 1/(u_i + u_j)$.

   In this case $w_i$ could be smaller than 1, i.e., the polylogarithms can develop an imaginary part. However, we checked numerically that the imaginary parts cancel in the final answer.

3. $w_i = 1/u_{jkl}, 1/v_{jkl}$, where we define

$$u_{jkl}^{(\pm)} = \frac{1 - u_j - u_k + u_l \pm \sqrt{(u_j + u_k - u_l - 1)^2 - 4 (1-u_j)(1-u_k) u_l}}{2 \ (1-u_j) \ u_l},$$

$$v_{jkl}^{(\pm)} = \frac{u_k - u_l \pm \sqrt{-4 u_j u_k u_l + 2 u_k u_l + u_k^2 + u_l^2}}{2 \ (1-u_j) \ u_k}.$$

A comment is in order about the square roots in Eq. (4.11). It turns out that the square roots become complex for certain values of the conformal cross ratios inside the unit cube, but they always come in pairs such that the sum of the two contributions is real. To
emphasize this property, we introduce the following notation,
\[ G(\ldots, u_{ijk}, \ldots; z) = G(\ldots, u_{ijk}^{(+)}, \ldots; z) + G(\ldots, u_{ijk}^{(-)}, \ldots; z), \]
\[ H(\vec{w}; 1/u_{ijk}) = H(\vec{w}; 1/u_{ijk}^{(+)}) + H(\vec{w}; 1/u_{ijk}^{(-)}), \quad (4.12) \]
and similarly for \( v_{ijk}^{(\pm)} \).

After having computed the contributions from the individual integrals, we can easily extract the remainder function \( R_{WL,6}^{(2)} \) by subtracting the contribution from our computation, thus obtaining the first fully analytic representation of \( R_{WL,6}^{(2)} \) in the Euclidean region. Note that although we performed the computation in the QMRK of a pair along the ladder, as introduced in Sec. 3, the Regge exactness of the Wilson loop in this limit ensures that our expression is valid in general kinematics. The final result for the remainder function can be expressed as a linear combination of multiple polylogarithms of uniform transcendental weight four\(^2\). Because the result is rather lengthy, we present it in an electronic form at www.arxiv.org where a text file containing the Mathematica expression for the remainder function is provided.

We have checked numerically that our result is completely symmetric in its arguments. Furthermore, we have checked analytically that the expression satisfies the constraints imposed by the multi-Regge and the collinear limits. Note that the vanishing in these limits is non trivial, since the expression of \( R_{WL,6}^{(2)} \) in general kinematics involves polylogarithms whose arguments are ratios of cross ratios, which can be \( \mathcal{O}(1) \) in the limit. However, all those contributions exactly cancel when approaching the limit. Finally, we have checked numerically at several points that our results agree with the numerical results of Ref. [18].

In the particular case where all three conformal cross ratios are equal, we find that,
\[ R_{WL,6}^{(2)}(1, 1, 1) = -\frac{\pi^4}{36} \simeq -2.70581\ldots, \quad (4.13) \]
\[ \lim_{u \to \infty} R_{WL,6}^{(2)}(u, u, u) = -\frac{\pi^4}{144} \simeq -0.67645\ldots, \]
in agreement, within numerical errors, with the values quoted in Ref. [18]. Similarly, the asymptotic behavior for \( u \to 0 \) is given by,
\[ \lim_{u \to 0} R_{WL,6}^{(2)}(u, u, u) = \frac{\pi^2}{8} \ln^2 u + \frac{17\pi^4}{1440} + \mathcal{O}(u). \quad (4.14) \]

Further results for the special case where all three conformal ratios are equal are summarized in Fig. 1. Note that even though our numerical evaluation is for the moment limited to \( 0 < u_i \leq 1 \), we can still compute the asymptotic value when all conformal cross ratios are equal and large by expanding the Mellin-Barnes integrals around \( u = \infty \) before taking residues. We find perfect agreement with the numerical value quoted in Ref. [18], which deviates slightly from the asymptotic value obtained from the analytic expression of the remainder function proposed in Ref. [20].

\(^2\)In the present version of the remainder function, we had to extend the definition of the transcendental weight to include the imaginary roots, which are present in Eq. (4.11). However, given that the remainder function is real, we cannot exclude that in a suitable basis those imaginary roots could disappear.
Figure 1: The remainder function $R_{(2)}^6(u,u,u)$ for $0 < u \leq 1$. The points represent the numerical values given in Ref. [18].

We conclude this section by making some comments on the numerical evaluation of Goncharov’s multiple polylogarithms. Up to weight two, Goncharov’s polylogarithms can be expressed in terms of ordinary logarithms and dilogarithms, e.g.:

$$G(a;z) = \ln \left(1 - \frac{z}{a}\right),$$
$$G(a,b;z) = \text{Li}_2 \left(\frac{b-z}{b-a}\right) - \text{Li}_2 \left(\frac{b}{b-a}\right) + \log \left(1 - \frac{z}{b}\right) \log \left(\frac{z-a}{b-a}\right).$$

However, our result involves polylogarithms up to weight four. We observe that in these cases the polylogarithms can be evaluated in a numerically stable and fast way, even for complex arguments, by writing them as an iterated integration of polylogarithms of weight two,

$$G(a,b,c;z) = \int_0^z \frac{dt_1}{t_1-a} G(b,c;t_1),$$
$$G(a,b,c,d;z) = \int_0^z \frac{dt_1}{t_1-a} \int_0^{t_1} \frac{dt_2}{t_2-b} G(c,d;t_2),$$

and using Mathematica’s native \texttt{NIntegrate} command to perform the integration, provided that the integrals converge.

5. Conclusions

In this letter we have identified a class of kinematics for which the multi-loop $n$-edged Wilson loop exhibits exact Regge factorisation and which leave invariant the analytic form of the Wilson loop. In those kinematics, the analytic result for the Wilson loop is the same as in general kinematics, although the computation is remarkably simplified with respect to general kinematics. Using the simplest of those kinematics, the QMRK of a pair

\footnote{Note that these expressions are valid for generic values of the parameters $a$ and $b$. For the limiting cases where the parameters approach 0 or 1 some care is needed, e.g., $G(0;z) = \ln z$, whereas $\lim_{a \to 0} \ln \left(1 - \frac{z}{a}\right)$ is divergent.}
along the ladder, we have performed the first analytic computation of the two-loop six-
edged Wilson loop in general kinematics. The computation has been done in the Euclidean
region, where the result is real. Except for the contribution coming from the hard diagram
with six light-like edges, the result can be expressed in terms of harmonic polylogarithms
in one conformal cross ratio. In turn, the six-edged hard diagram, which constitutes the
bulk of the final result, can be written as a linear combination of Goncharov’s multiple
polylogarithms, whose arguments are functions of conformal cross ratios. Finally, the
remainder function can be expressed as a linear combination of multiple polylogarithms of
uniform transcendental weight four. Details on the explicit computation of the integrals
will be presented in a forthcoming publication [47].

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