Double-Λ hypernuclei within a Skyrme-Hartree-Fock approach

Neelam Guleria, Shashi K. Dhiman, Radhey Shyam

1 Department of Physics, H. P. University, Shimla 171005, India
2 University Institute of Natural Sciences and Interface Technologies, Himachal Pradesh Technical University, Hamirpur 177001, India
3 Theory Division, Saha Institute of Nuclear Physics, 1/AF Bidhan Nagar, Kolkata - 700064, INDIA and Physics Department, Indian Institute of Technology, Roorkee, India

Received received date
Revised revised date

PACS numbers: 21.10.Dr, 21.30.Fe, 21.60J

Keywords: Study of Double-Lambda hypernuclei, Skyrme-Hartree-Fock model, Lambda-Lambda interaction

1. Introduction

A proper understanding of the properties of double-Λ hypernuclei is important due to several reasons. These systems provide the unique opportunity to obtain information about the hyperon-hyperon (YY) interaction, which is crucial for a complete understanding of the octet of baryons (N, Λ, Σ, Ξ) in a unified way. They also supplement the information about the hyperon-nucleon (YN) interaction that is mostly extracted from the studies of the single-Λ hypernuclei. The knowledge of YN and YY interactions is necessary for making extrapolations to understand the properties of both finite as well as bulk strange hadronic matter and the neutron stars. The study of the ΛΛ hypernuclei is also of interest in connection with the possible existence of the strangeness (S) -2, six-quark H dibaryon resonance with spin parity of 0+ and isospin (1/2)1+. Precisely measured binding energies of double-Λ hypernuclei put a lower limit on the H dibaryon mass. The double-Λ hypernuclei were first observed in the 1960s in the studies.
of stopped Ξ− hyperons in emulsions. Two decades later, modern emulsion-counter
hybrid techniques have been applied in the KEK-E176 experiment where a new
double-Λ hypernucleus event was found. Later on, in another hybrid emul-
sion experiment (KEK-E373) an unambiguous identification of the hypernucleus,
$^6_{\Lambda\Lambda}$He, was made with a precise value of the binding energy of two Λ hyperons. This
is known as the NAGARA event. Recently, in a reanalysis of the double-Λ
hypernuclear data produced in the KEK-E176 and KEK-E373 experiments, results
for the ΛΛ binding energies have been reported for $^6_{\Lambda\Lambda}$He, $^{10}_{\Lambda\Lambda}$Be, $^{12}_{\Lambda\Lambda}$Be, and $^{13}_{\Lambda\Lambda}$B
hypernuclei.

These observations have led to a number of theoretical studies where sev-
eral approaches have been used to investigate the double-Λ hypernuclei. Calcula-
tions have been performed within the three- and four-body cluster models us-
ing the effective interactions or the $G$-matrices. Among other approaches are the
Faddeev, variational Monte-Carlo, and variational six-body calculations. Furthermore, both nonrelativistic and relativistic mean
field (RMF) models have also been used to predict the binding energies of such
nuclei. In Ref. 48 the two-Λ binding energies of several double-Λ nuclei between $^6_{\Lambda\Lambda}$He to $^{13}_{\Lambda\Lambda}$B were calculated within a shell model approach. Furthermore, using the interaction NSC97c of the Nijmegen group, the
calculations for the ΛΛ bond energies were reported in Ref. 49 within a
$G$-matrix approach where the couplings between ΛΛ, ΞN and ΣΣ channels were included.

In these calculations moderate to good success has been achieved in predicting the
two-Λ binding and the ΛΛ bond energies.

The Skyrme-Hartree-Fock (SHF) model provides a self-consistent description
of nuclear ground state properties and it has been shown to be a powerful tool
for investigating the gross properties of the nonstrange nuclei (see, e.g., a recent
review). A clear advantage of this method is that it involves the complete summa-
tion of tadpole diagrams. The extension of this model to describe the single-Λ
hypernuclei was presented in Refs. 54 and 55. For calculating such strange nuclei,
one requires, in addition to the Skyrme $NN$ force, also the Skyrme $\Lambda N$ interaction.
In Ref. 56 the latter was determined from a Bruckner-Hartree-Fock calculation of
the hypernuclear matter using the Nijmegen potentials NSC97a and NSC97f, which
was used in an extended SHF scheme to determine the properties of single-Λ hyper-
nuclei. In this study the binding energies of the hypernuclei were somewhat over-
predicted. In Ref. 57 several sets of the Skyrme $\Lambda N$ interactions were determined
by fitting to the modern data on the binding energies of nearly twenty single-Λ
hypernuclei, which were used in the SHF model to describe the known properties
of such nuclei over a wide mass range.

In this paper, we present an extension of the SHF method of Ref. 57 to the
calculations of the two-Λ binding energies ($B_{\Lambda\Lambda}$) and the ΛΛ bond energies ($\Delta B_{\Lambda\Lambda}$) of the
double-Λ hypernuclei. Unlike a few previous studies within similar approach where
calculations were limited to a few lighter systems, we have applied this method to
investigate the properties of the double-Λ hypernuclei with masses covering essentially the entire range of the periodic table. In fact the SHF method provides an ideal approach for describing the heavier systems.

In calculations of the double-Λ hypernuclei, one needs as input the ΛΛ interaction in addition to the $NN$ and $ΛN$ forces. Several phenomenological, meson-exchange motivated, and quark model based forms have been employed for the ΛΛ force in the literature. We have, however, taken phenomenological Skyrme type of parameterizations for this force proposed in Refs. 39 and 66. In Ref. 39 three sets of parameters for the ΛΛ interaction were determined by fitting to the ΛΛ bond energy of $^{13}_{\Lambda\Lambda}B$ ground state ($= 4.8 \pm 0.7$ MeV). In Ref. 66, four additional sets of the parameters were obtained by considering the results of a recent experimental analysis in which a smaller bond energy ($= 0.6 \pm 0.8$ MeV) for the $^{13}_{\Lambda\Lambda}B$ ground state has been reported.

We remark, however, that these forces are too simple and lack several important effects. For example, they neglect the three-body interactions and the possible conversion of the ΛΛ to ΞN and ΣΣ channels. These constitute the important parts of the hyperon-hyperon interaction and neglecting them could bring in significant uncertainty in the calculations. Nevertheless, our aim in this paper is to extend and establish our SHF method for describing the double-Λ hypernuclei. For this purpose we have taken these forces, which were also used in previous SHF type calculations of double-Λ hypernuclei reported in Refs. 39, 66 and 68. Particularly noteworthy are the latter two references where these forces were employed to investigate the fission barriers of double-Λ hypernuclei in the actinide region and the properties of neutron stars, respectively.

2. Formalism

The total energy density functional (EDF) of a double-Λ hypernucleus ($\mathcal{E}^H_{2\Lambda}$) includes contributions from the total energy densities of nucleons (neutron and proton) ($\mathcal{E}_N$) and of hyperons ($\mathcal{E}_Λ$). In addition, $\mathcal{E}^H_{2\Lambda}$ has terms arising from the pairing energy and the center of mass corrections, which are taken to be similar to those described in Ref. 57. $\mathcal{E}_N$ is related to the nucleon Hamiltonian density ($H_N$) as

$$\mathcal{E}_N = \int d^3r H_N(r).$$

The form of $H_N$ is the same as that given in Ref. 57. $\mathcal{E}_Λ$ is given by

$$\mathcal{E}_Λ = \int d^3r H_Λ(r).$$

In Eq. (2) the hyperon Hamiltonian density, $H_Λ$, is the sum of two terms,

$$H_Λ(r) = H_{ΛN}(r) + H_{ΛΛ}(r).$$

In Eq. (3), $H_{ΛN}(r)$ has the same form as that described in Ref. 57 for the case of the single-Λ hypernuclei. The second term, $H_{ΛΛ}$, is attributed to the ΛΛ interaction.
and is given by

\[
H_{\Lambda\Lambda} = \frac{1}{4} \lambda_0 \rho_\Lambda^2 + \frac{1}{8} (\lambda_1 + 3 \lambda_2) \rho_\Lambda \tau_\Lambda + \frac{3}{32} (\lambda_2 - \lambda_1) \rho_\Lambda \nabla^2 \rho_\Lambda
+ \frac{1}{4} \lambda_3 \rho_\Lambda^2 \rho_N, \tag{4}
\]

where \( \rho_\Lambda \) is the hyperon density and \( \tau_\Lambda \) is the corresponding kinetic energy density. \( \rho_N = \rho_p + \rho_n \), is the nucleon density. \( \lambda_0, \lambda_1, \lambda_2, \) and \( \lambda_3 \) are the parameters of the \( \Lambda\Lambda \) force that will be discussed in the next section. The parameter \( \alpha \) in the last term is assumed to be 1/3. In Eq. (4), we have omitted terms corresponding to the \( \Lambda \) spin density.

The wave functions for the proton, neutron and the \( \Lambda \) particle are calculated from the SHF equations;

\[
\left( -\frac{\hbar^2}{2m_q^*} \nabla^2 + V_{NN}(r) + V_q^\Lambda(r) \right) \phi_q(r) = \epsilon_q \phi_q(r), \tag{5}
\]

\[
\left( -\frac{\hbar^2}{2m_\Lambda^*} \nabla^2 + V_\Lambda^\Lambda(r) + V_\Lambda(r) \right) \phi_\Lambda(r) = \epsilon_\Lambda \phi_\Lambda(r), \tag{6}
\]

where \( q \) represents a nucleon (proton or neutron), and \( \epsilon_q \) and \( \epsilon_\Lambda \) are the single-particle energies of the nucleon and the \( \Lambda \) particle, respectively. The purely nuclear mean field potential \([V_{NN}(r)]\), the additional field created by the \( \Lambda \) hyperon that is seen by a nucleon \([V_q^\Lambda(r)]\), and the whole nuclear field experience by a \( \Lambda \) hyperon \([V_\Lambda^\Lambda(r)]\) have the same forms as those given in Ref. 57.

\( V_{\Lambda\Lambda}(r) \), which is the field generated by the \( \Lambda\Lambda \) interaction, is given by

\[
V_{\Lambda\Lambda} = \frac{1}{2} \lambda_0 \rho_\Lambda + \frac{1}{8} (\lambda_1 + 3 \lambda_2) \tau_\Lambda + \frac{3}{16} (\lambda_2 - \lambda_1) \nabla^2 \rho_\Lambda
+ \frac{1}{2} \lambda_3 \rho_\Lambda^2 \rho_N. \tag{7}
\]

The last term in Eq. (7) corresponds to the three-body \( \Lambda\Lambda N \) interaction. In actual calculations, this term is dropped.

While the nucleon effective mass remains the same as that described in Ref. 57, the \( \Lambda \) effective mass, \( m_\Lambda^* \), acquires additional terms due to the presence of \( V_{\Lambda\Lambda} \),

\[
\frac{\hbar^2}{2m_\Lambda^*} = \frac{\hbar^2}{2m_\Lambda} + \frac{1}{4} [u_1 + u_2] \rho_N + \frac{1}{8} [\lambda_1 + 3 \lambda_2] \rho_\Lambda, \tag{8}
\]

where \( u_1 \) and \( u_2 \) are the parameters of the \( \Lambda N \) force as defined in Ref. 57. Without the last term this equation is the same as that given in Ref. 57 for the effective mass of the single-\( \Lambda \) hyperon.

The main quantity in \( \Lambda\Lambda \) hypernuclei is the \( \Lambda\Lambda \) bond energy, which is defined as

\[
\Delta B_{\Lambda\Lambda} = B_{\Lambda\Lambda} - 2B_\Lambda, \tag{9}
\]
Fig. 1. (Color online) Comparison of the $A$ dependence of the calculated and experimental separation energies $B_A$ of $1s$, $1p$, $1d$ and $1f$ orbitals of the single-$\Lambda$ hypernuclei. The $NN$ and $\Lambda N$ interactions have been described by parameter sets SLy4 of Ref. 69 and HPΛ2 of Ref. 57, respectively in each case.

where $B_A$ is the separation energy of one $\Lambda$ hyperon from the $A^{-1}Z$ hypernucleus and $B_{AA}$ is that of two $\Lambda$ hyperons from the $A\Lambda Z$ hypernucleus, respectively. The separation energies are evaluated by solving the appropriate Hartree-Fock equations and by using the energy relations given in Ref. 57. In terms of the total binding energies ($E$), the bond energy can be expressed as $\Delta B_{AA} = E(A\Lambda Z) + E(A^{-2}Z) - 2E(A^{-1}Z)$. It is clear that possible uncertainties in the center of mass treatment are mostly canceled in the bond energy.

3. Results and discussions

Before presenting results for $B_{AA}$ and $\Delta B_{AA}$, it would be of interest to discuss our SHF calculations for the single-$\Lambda$ separation energies $B_A$, as they will be used in obtaining $\Delta B_{AA}$ [see, Eq. (9)]. This will be done very briefly here because all the details are given in Ref. 57. In Fig. 1, we show the $A$ dependence of the calculated $B_A$ of $1s$, $1p$, $1d$ and $1f$ shells of various single-$\Lambda$ hypernuclei. The corresponding available experimental data are also shown in this figure. In these calculations forces SLy4 (Ref. 69) and HPΛ2 (Ref. 57) have been used for $NN$ and $\Lambda N$ interactions, respectively. We note that apart from the $A=208$ case, where we see some underbinding of the $1s$ orbital, our calculations are in close agreement with the experimental $B_A$. Thus our SHF model with $\Lambda N$ force HPΛ2 provides a good description of the single-$\Lambda$ separation energies for the lighter as well as heavier systems.

We now proceed to the discussion of the double-$\Lambda$ hypernuclei, which is the main focus of this paper. In calculations of the binding energies of the double-$\Lambda$ hypernuclei, we have used the parameter sets SLy4 and HPΛ2 for the $NN$ and the $\Lambda N$ forces, respectively. For the $\Lambda\Lambda$ force, the three parameter sets (SAA1, SAA2 and SAA3) are reported in Ref. 39 (see Table 1), have been employed.
Table 1. Parameters for the ΛΛ interaction. The last column gives the ranges of "equivalent single Gaussian" potentials.

| SET  | λ₀   | λ₁   | μ   |
|------|------|------|-----|
|      | (MeV fm³) | (MeV fm⁵) | (fm) |
| SΛΛ1 | -312.6 | 57.5  | 0.61 |
| SΛΛ2 | -437.7 | 240.7 | 1.05 |
| SΛΛ3 | -831.8 | 922.9 | 1.49 |
| SΛΛ1' | -37.9 | 14.1  | 0.61 |
| SΛΛ3' | -156.4 | 347.2 | 1.49 |

these sets the density dependent terms of Eq. (4) (λ₂ and λ₃) were ignored because the p wave contributions do not take part in the lowest single-particle level. The parameters λ₀ and λ₁ were determined by fitting to the bond energy ΔBΛΛ = 4.8 ± 0.7 MeV of the 13ΛΛB ground state. The difference between the three parameter sets is the interaction range μ = √−λ₁/λ₀, which is evaluated from the equivalent single-Gaussian potential assumption. In addition, two more sets of the ΛΛ force parameters were obtained in Ref. 66 (sets SΛΛ1' and SΛΛ3' of Table 1) by fitting to a weaker bond energy ΔBΛΛ = 0.6 ± 0.8 for the 13ΛΛB ground state reported in Ref. 67. This results from taking into account the excited state of the daughter single-Λ hypernucleus 13ΛC* in the decay channel. We have used these 5 sets of the ΛΛ forces in our study. The parameter sets SΛΛ1 and SΛΛ3 of Ref. 66 that correspond to a repulsive ΛΛ interaction, have also been used in few cases. However, these sets produce unrealistic bond energies in our calculations so we do not discuss them here.

In Table 2 we show our results for the double-Λ binding energies for a number of hypernuclei that are obtained by using sets SΛΛ1, SΛΛ2 and SΛΛ3 for the ΛΛ force. In each case, the parameter sets SLy4, and HPA2 were employed for NN and ΛN interactions, respectively. In this table we have also listed the BΛ (A−1ΛZ), obtained with the same NN and ΛN forces. It is seen that BΛΛ increases with mass number of the hypernucleus. This is in agreement with the observations made in the RMF calculations of the double-Λ hypernuclei in Refs. 45 and 46. We further note that for the first two lightest mass hypernuclei, BΛΛ depends rather strongly on the ΛΛ force. However, with increasing mass this dependence becomes less stronger.

For the purpose of comparison we have also shown in Table 2 the single-Λ separation energies BΛ for the same systems. We see that for double-Λ systems heavier than mass 10, the BΛΛ is nearly twice of the BΛ. However, for the lighter systems, this is not so. It is shown in Ref. 46 that BΛΛ is related to the Λ single-particle energies (εΛ) as BΛΛ = 2εΛ − ER, where ER is the rearrangement energy that quantifies the core polarization. Therefore, results shown in Table 2 indicate that the core polarization effects may contribute significantly to the binding energy for lighter systems. More discussion on the core polarization effect is presented
Table 2. $B_{AA}$ of various hypernuclei ($A_{AA}Z$) of various hypernuclei ($A_{AA}Z$) calculated using sets SLy4, HPΛ2 of Ref.57, for NN, and ΛN interactions, respectively, and the three parameter sets SΛΛ1, SΛΛ2, and SΛΛ3 for the ΛΛ force. For comparison the separation energies $[B_{Λ(A−1)Z}]$ of the single-Λ hyperon calculated with forces SLy4, and HPΛ2 are also shown.

| Hypernuclei ($A_{AA}Z$) | $B_A$ (MeV) | $B_{SΛΛ1}^{AA}$ (MeV) | $B_{SΛΛ2}^{AA}$ (MeV) | $B_{SΛΛ3}^{AA}$ (MeV) | $B_{Exp}^{(exp.)}$ (MeV) | Ref., Event |
|--------------------------|-------------|-----------------------|-----------------------|-----------------------|--------------------------|-------------|
| $^6_{AA}$He              | 7.12        | 11.88                 | 9.25                  | 7.60                  | 6.91 ± 0.16              | Ref[24], NAGARA            |
| $^{10}_{AA}$Be           | 10.76       | 19.78                 | 18.34                 | 15.19                 | 14.94 ± 0.13*            | Ref[20]                |
| $^{11}_{AA}$Be           | 10.80       | 20.55                 | 19.26                 | 16.27                 | 20.49 ± 1.15             | Ref[24], HIDA             |
| $^{12}_{AA}$Be           | 10.91       | 21.10                 | 19.97                 | 17.18                 | 22.23 ± 1.15             | Ref[24]                |
| $^{13}_{AA}$B            | 11.02       | 21.21                 | 20.26                 | 17.76                 | 23.30 ± 0.70             | Ref[24], E176             |

*This value has been obtained from the experimentally deduced value 11.90 ± 0.13 MeV by adding 3.04 MeV for the $2^+$ excitation energy, assuming equal $2^+$ core excitation energies in $^9\Lambda Be$ and $^{10}_{AA}Be$.

Towards the end of this section.

For $^6_{AA}$He and $^{10}_{AA}$Be, the $B_{AA}$ calculated with force SΛΛ3 that has the largest range, reproduce the corresponding experimental values the best. The other two sets lead to larger binding energies for these hypernuclei. This could indicate that a longer range ΛΛ force leads to a lesser binding of two Λs to a lighter core. Nevertheless, it should be emphasized that description of the lightest nuclei $^6_{AA}$He and $^{10}_{AA}$Be may be less reliable in the SHF approach. The cluster or the shell model methods should be more appropriate for these cases.

On the other hand, for hypernuclei $^{11}_{AA}$Be, $^{12}_{AA}$Be, and $^{13}_{AA}$B, $B_{AA}$s calculated with sets SΛΛ1 and SΛΛ2 do not differ much from each other and reproduce the experimental data better in comparison to those obtained with set SΛΛ3. The SHF method is expected to be relatively better suited to describe these systems.

It would be interesting to compare our $B_{AA}$ with the available corresponding results obtained in other theoretical approaches. For the nucleus $^{11}_{AA}$Be, results for the binding energy are available in both the shell model as well as the cluster model methods. We note that while both $B_{SΛΛ1}^{AA}$ and $B_{SΛΛ2}^{AA}$ of this hypernucleus are larger than that the value (18.40 MeV) predicted by the shell model calculation of Ref[48] by about 5-10%, they are comparable to that calculated (19.81 MeV) in a αωαΛΛ five-body cluster model. Furthermore, $B_{AA}$s are similar to that obtained (19.46 MeV) recently within a quark mean-field model. For the $^{12}_{AA}$Be and $^{13}_{AA}$B hypernuclei, our binding energies are about 10-15% larger than the values predicted by both the shell model and the quark mean-field model. It is worth noticing that our results are in agreement with the corresponding experimental data within the statistical error except for the $^{13}_{AA}$B case where our calculations underpredict the data.
In Table 3, we display our results for the bond energy $\Delta B_{\Lambda\Lambda}$ calculated with $\Lambda\Lambda$ forces $S_{\Lambda\Lambda}1$, $S_{\Lambda\Lambda}2$ and $S_{\Lambda\Lambda}3$. These are obtained by using Eq. (9), where the $B_\Lambda$ values have been taken from the Ref. 57 (also shown in Fig. 1). The experimental points are from Refs. 24, 67. It is seen that generally $\Delta B_{\Lambda\Lambda}$ decreases with increasing $A$. It is further noted that for more complex (heavier) double-$\Lambda$ hypernuclei the bond energies are smaller. This points to the fact that for heavier systems the binding energies $B_{\Lambda\Lambda}$ are closer to the twice of $B_\Lambda$, which is seen already in Table 2. These results are similar to those obtained in the SHF and RMF calculations of Refs. 39 and 45, respectively.

Table 3. Bond energies $\Delta B_{\Lambda\Lambda}$ calculated with $\Lambda\Lambda$ interactions $S_{\Lambda\Lambda}1$, $S_{\Lambda\Lambda}2$ and $S_{\Lambda\Lambda}3$. In each case the parameter sets SLy4 and HPΛ2 were employed for the $NN$ and $\Lambda N$ interactions, respectively. The total baryon number of the double-$\Lambda$ hypernucleus is represented by $A$ in the first column.

| \(A\) | \(\Delta B_{\Lambda\Lambda}\) (MeV) | \(\Delta B_{\Lambda\Lambda}\) (exp.) (MeV) |
|-------|---------------------------------|---------------------------------|
| 6     | 2.36 4.99 6.64                  | 3.82 ± 1.72                    |
| 9     | 2.17 4.71 5.5                   |                                 |
| 10    | 1.72 3.64 4.27                  | 1.3 ± 0.4                      |
| 11    | 0.90 2.74 3.91                  | 2.27 ± 1.23                    |
| 12    | 0.71 1.97 2.44                  | 0.6 ± 0.8                      |
| 13    | 0.69 1.71 2.32                  |                                 |
| 30    | 0.55 1.33 1.56                  |                                 |
| 50    | 0.50 1.01 1.11                  |                                 |
| 58    | 0.45 0.99 1.08                  |                                 |
| 92    | 0.33 0.73 0.91                  |                                 |
| 140   | 0.31 0.58 0.77                  |                                 |
| 210   | 0.25 0.42 0.48                  |                                 |

In Fig. 2(a) the $A$ dependence of $\Delta B_{\Lambda\Lambda}$ is shown in some more details. We see that whereas the decrease of $\Delta B_{\Lambda\Lambda}$ with increasing $A$ is quite steep for lighter hypernuclei, it is gradual for the medium mass and heavier systems. We further note that with the $S_{\Lambda\Lambda}1$ force, the agreement between the calculated and the experimental bond energy for the last two data points is somewhat better as compared to that obtained with sets $S_{\Lambda\Lambda}2$ and $S_{\Lambda\Lambda}3$ - the bond energy determined with set $S_{\Lambda\Lambda}3$ is farthest from the data for these points. However, given the large statistical errors in the data points it is premature to draw any definite conclusion about the preference of one parameter set over the other. More experimental data, particularly for heavier double-$\Lambda$ hypernuclear systems are needed to extract an unambiguous information about the $\Lambda\Lambda$ force from such calculations.

In Fig. 2(b), we present a comparison of the $A$ dependence of $\Delta B_{\Lambda\Lambda}$ obtained
Fig. 2. (Color online) (a) Comparison of the $A$ dependence of the bond energy $\Delta B_{\Lambda \Lambda}$ obtained by using parameter set SAA1, SAA2, and SAA3 for the $\Lambda \Lambda$ force. The $NN$ and $\Lambda N$ interactions have been described by parameter sets SLy4 and HPΛ2, respectively in each case. (b) Same as in (a) for $\Lambda \Lambda$ force parameter sets SAA1, SAA1’, SAA3’. The experimental points are taken from Refs. 24, 67.

with parameters sets SAA1’ and SAA3’ of Ref. 66 and SAA1. We see that set SAA3’ leads to the $\Delta B_{\Lambda \Lambda}$ that are larger in magnitude and fall less steeply with increasing $A$ as compared to those obtained with set SAA1. On the other hand, bond energies produced by set SAA1’ are comparable with those of set SAA1 for $A > 50$. However, for $A < 50$ the difference between the two is quite large. It should be remarked that the $\Delta B_{\Lambda \Lambda}$ calculated with parameter sets SAA3’ and SAA1’ for $A$ around 10 are larger than the value $0.6 \pm 0.8$ MeV, to which they are fitted to in Ref. 66. This can be understood from the fact that in the fitting procedure of Ref. 66, the adopted $NN$ and $\Lambda N$ interactions were different from those used in our calculations. As will be shown later on, the calculated $\Delta B_{\Lambda \Lambda}$ shows strong dependence over $\Lambda N$ interaction.

In Fig. 3, we show the sensitivity of $\Delta B_{\Lambda \Lambda}$ to the $\Lambda N$ force. In these calculations parameters sets SLy4 and SAA1 have been used for the $NN$ and $\Lambda \Lambda$ interactions, respectively. For the $\Lambda N$ force we have used sets HPΛ2, NA1 and OΛ1 of Ref. 57. It may be recalled here that while the set HPΛ2 provides a good agreement with the experimental binding energies of the $\Lambda$ single-particle states of all the orbitals in the entire mass range of hypernuclei (see Fig. 1), the parameter sets OΛ1 and NA1 slightly overestimate the data for the lighter nuclei (we refer to Ref. 57 for a detailed discussion of this points). In Fig. 3, we note that even the currently available sparse data clearly favor the parameter set HPΛ2.

It should, however, be added that the bond energy as defined by Eq. (9) is more applicable to those cases where the core nuclei $A-1 Z$ have zero spin. In case of the nonzero spin core nucleus, the $B_\Lambda$ appearing in Eq. (9) is actually an average
of the binding energies of the spin-doublet states. The bond energy is also influenced by the structural changes that are caused to the core nucleus due to the Λ-core interaction (e.g., the core polarization). The core polarization effects are significant for ΛN potentials that are strongly polarizing. However, the ΛΛ forces used in this study have been extracted by fitting the data with SHF calculation where the used ΛN interactions lead to small core polarization energy in $^{12}\Lambda B$. Furthermore, the spin-doublet splitting in $^{12}\Lambda B$ is probably small. Nevertheless, an alternative definition of the ΛΛ bond energy is suggested in Ref. 34, where it is essentially determined by the strength of the ΛΛ interaction. However, the prevailing uncertainties in this interaction may also creep into the bond energies calculated within this alternative method.

4. Summary and conclusions

In conclusion, we have calculated the binding energies and the bond energies of the light to heavy double-Λ hypernuclei within a Skyrme-Hartree-Fock model. This is an extension of the model used earlier to describe successfully the binding energies of the Λ single particle states of both lighter as well as heavier single-Λ hypernuclei. For the NN interaction the parameter set SLy4 of Ref. [69] has been used while for the ΛN force the parameter set HPA2 of Ref. [67] has been employed. These sets provide a reasonable overall description of the single-Λ hypernuclear data. For the ΛΛ force, parameter sets SΛΛ1, SΛΛ2 and SΛΛ3 of Ref. [39] as well as SΛΛ1’ and SΛΛ3’ of Ref. [66] were used. Since in this work our aim has been more to establish our SHF model for the description of the double-Λ hypernuclei, we selected the ΛΛ force parameters that are already available in the literature. In future, efforts will be made to determine corresponding force parameters by refitting the data using
our $NN$ and $\Lambda N$ interactions. Nevertheless, even such a force will not be free from uncertainties.

We have calculated the binding energies of a number of double-$\Lambda$ hypernuclei where some experimental information is available. We showed that SHF calculations done with empirical Skyrme type $\Lambda\Lambda$ forces without the density dependent terms, provide a reasonable description of the $\Lambda\Lambda$ hypernuclear systems. Our results for the two-$\Lambda$ binding energies of the hypernuclear systems $^{11}_\Lambda\Lambda$Be, $^{12}_\Lambda\Lambda$Be, and $^{13}_\Lambda\Lambda$B are comparable to those obtained within other approaches such as the shell model, the cluster model and the quark mean-field model. However, in the present calculation it has not been possible to reproduce simultaneously the experimental binding energies of all the known double-$\Lambda$ hypernuclei with any one set of the $\Lambda\Lambda$ potential. Further studies are required to obtain more precise information about this force as compared to what is available now.

We have also studied the $A$ dependence of the $\Lambda\Lambda$ bond energy in the ground state of the double-$\Lambda$ hypernuclei. We observe that the currently available limited experimental data for such hypernuclei do not allow to distinguish between the $\Lambda\Lambda$ forces used in this study. However, they show a significant selectivity for the $\Lambda N$ force where the set HP$\Lambda^2_2$ is favored. We acknowledge that the $\Lambda\Lambda$ forces used by us are too simple. Moreover, we have not considered the core polarization effects, which depend on the $\Lambda N$ interaction. However, calculations made with more realistic $\Lambda\Lambda$ potentials in Ref. arrive at similar conclusions. At the present stage of our knowledge on hyperon-nucleon and hyperon-hyperon interactions any fine tuning of these forces would clearly be premature.

A systematic study of the data over a large mass range - as is done in the present paper, is necessary for deriving constraints on various interactions and density functionals. More experimental information on the double-$\Lambda$ hypernuclei over a wide mass range is, therefore, clearly required. The mean field method, on the other hand, may come to its limit for very light nuclei, but experience with SHF calculations on nonstrange nuclei do not show dramatic failures of the method for such systems. Rather they are surprisingly successful even in mass-4 region. Thus this method is quite robust. Our work demonstrates that the Skyrme-Hartree-Fock model can be used as a workable theoretical framework for investigating the properties of both single- and double-$\Lambda$ hypernuclei over a wide mass region.

5. Acknowledgments

One of the authors (NG) would like to thank the theory division of the Saha Institute of Nuclear Physics for the kind hospitality during her several visits there. This work has been supported by the Council of Scientific and Industrial Research (CSIR), India.
References

1. R. H. Dalitz, P. H. Fowler, A. Montwill, J. Pniewski and J. A. Zakrzewski, Proc. R. Soc. London Ser. A 426, 1 (1989).
2. O. Hashimoto and H. Tamura, Prog. Part. Nucl. Phys. 57, 564 (2006).
3. J. Schaffner, C. B. Dover, A. Gal, C. Greiner, and H. Stöcker, Phys. Rev. Lett. 71, 1328 (1993).
4. J. Schaffner-Bielich, Nucl. Phys. A 804, 309 (2008).
5. R. L. Jaffe, Phys. Rev. Lett. 38, 195 (1977).
6. P. J. Mulders and A. W. Thomas, J. Phys. G 9, 1159 (1983).
7. S. R. Beane et al., Phys. Rev. Lett. 106, 162001 (2011).
8. T. Inoue et al., Phys. Rev. Lett. 106, 162002 (2011).
9. P. E. Shanahan, A. W. Thomas, and R. D. Young, Phys. Rev. Lett. 107, 092004 (2011).
10. J. Haidenbauer and U.-G. Meissner, Phys. Lett. B 706, 100 (2011).
11. J. Haidenbauer and U.-G. Meissner, Nucl. Phys. A 881, 44 (2012).
12. S. R. Beane et al., Mod. Phys. Lett. A 26, 2587 (2011).
13. T. Inoue, PoS LATTICE2012, 144 (2012), arXiv:1308.6404.
14. J. Haidenbauer, Few-Body Systems 54, 85 (2013).
15. R. Shyam, O. Scholten, and A. W. Thomas, Phys. Rev. C 88, 025209 (2013).
16. T. F. Carames and A. Valcarce, Int. J. Mod. Phys. E 22, 1330004 (2013).
17. J. K. Ahn et al., Phys. Rev. C 62, 055201 (2000).
18. H. Takahashi et al., Phys. Rev. Lett. 87, 212502 (2001).
19. C. J. Yoon et al., Phys. Rev. C 75, 022201 (2007).
20. M. Danysz et al., Nucl. Phys. 49, 121 (1963).
21. J. Prowse, Phys. Rev. Lett. 17, 782 (1966).
22. S. Aoki et al., Phys. Rev. Lett. 65, 1729 (1990).
23. S. Aoki et al., Prog. Theor. Phys. 85, 1287 (1991).
24. K. Nakazawa et al., Nucl. Phys. A 835, 207 (2010).
25. R. H. Dalitz and G. Rajasekaran, Nucl. Phys. 50, 450 (1964).
26. S. Ali and A. R. Bodmer, Phys. Lett. B24, 343 (1967).
27. H. Bando, Prog. Theor. Phys. 67, 699 (1982).
28. K. Ikeda, H. Bando and T. Motoba, Prog. Theor. Phys. Suppl. 81, 147 (1985).
29. A. R. Bodmer and Q. N. Usmani, Nucl. Phys. A 463, 221c (1987).
30. A. R. Bodmer and Q. N. Usmani, Nucl. Phys. A 468, 653 (1987).
31. Y. Yamamoto, H. Takaki and K. Ikeda, Progr. Theo. Phys. 86, 867 (1991).
32. Y. Yamamoto, Nucl. Phys. A 547, 233c (1992).
33. H. Himeno, T. Sakuda, S. Nagata, and Y. Yamamoto, Prog. Theor. Phys 89, 109 (1993).
34. E. Hiyama, M. Kamimura, T. Motoba, T. Yamada, and Y. Yamamoto, Phys. Rev. C 66, 024007 (2002).
35. N. Filikhin, A. Gal, Phys. Rev. C 65, 041001(R) (2002).
36. Q. N. Usmani, A. R. Bodmer and B. Sharma, Phys. Rev. C 70, 061001(R) (2004).
37. H. Nemura, S. Shinmura, Y. Akaishi and K. S. Myint, Phys. Rev. Lett. 94, 022502 (2005).
38. D. E. Lansky, Y. A. Lurie and Y. Shirikov, Z. Phys. A 357, 95 (1997).
39. D. E. Lansky, Phys. Rev. C 58, 3351 (1998).
40. M. Rufa, H. Stöcker, J. Mahrun, P. G. Reinhard, and W. Greiner, J. Phys. G 13, 143 (1987).
41. M. Rufa, J. Schaffner, J. Mahrun, H. Stöcker, W. Greiner and P.-G. Reinhard, Phys. Rev. C 42, 2469 (1990).
42. J. Mares and Z. Zoška, Z. Phys. A 333, 209 (1989).
43. J. Mares and Z. Zofka, Z. Phys. A 345, 47 (1993).
44. J. Schaffner, C. B. Dover, A. Gal, C. Greiner, D. J. Milner, Ann. Phys. (N.Y.) 235, 35 (1994).
45. S. Marcos, R. J. Lombard, and J. Mares, Phys. Rev. C 57, 1178 (1998).
46. H. Shen, F. Yang and H. Toki, Prog. Theor. Phys. 115, 325 (2006).
47. J. N. Hu, A. Li, H. Shen and H. Toki, arXiv:1310.3602 [nucl-th].
48. A. Gal and D. J. Millener, Phys. Lett. B 701, 342 (2011).
49. I. Vidana, A. Ramos and A. Polls, Phys. Rev. C 70, 024306 (2004).
50. D. Vautherin and D. M. Brink, Phys. Rev. C 5, 626 (1972).
51. J. Erler, P. Klipfel and P-G. Reinhardt, J. Phys. G: Nucl. & Part. Phys., 38, 033101 (2011).
52. K. Saito, and A. W. Thomas, Phys. Lett. B327, 9 (1994).
53. T. Miyatsu, T. Katayama, and K. Saito, Phys. Lett. B709, 242 (2012).
54. M. Rayet, Ann. of Phys. 102, 226 (1976);
55. M. Rayet, Nucl. Phys. A 367, 381 (1981).
56. I. Vidana, A. Polls, A. Ramos, and H.-J. Schulze, Phys. Rev. C 64, 044301 (2001).
57. N. Guleria, S. K. Dhiman and R. Shayam, Nucl. Phys. A 886, 71 (2012).
58. J. Caro, C. Garcia-Recio and J. Nieves, Nucl. Phys. A 646, 299 (1999).
59. J. Haidenbauer, T. Hippchen, K. Holinde, B. Holzenkamp, V. Mull, and J. Speth, Phys. Rev. C 45, 931 (1992).
60. J. Haidenbauer, K. Holinde, V. Mull and J. Speth, Phys. Rev. C 46, 2158 (1992).
61. C. Albertus, J. E. Amaro and J. Nieves, Phys. Rev. Lett. 89, 032501 (2002).
62. I. R. Afman, B. F. Gibson, Phys. Rev. C 67, 017001 (2003).
63. T. Fernandez-Carames, A. Valcarce, and P. Gonzalez, Phys. Rev. D 72, 054008 (2005).
64. Y. Fujimura, Y. Suzuki, C. Nakamoto, Prog. Part. Nucl. Phys. 58, 439 (2007).
65. H.-J. Schulze and T. Rijken, Phys. Rev. C 88, 024322 (2013).
66. F. Minato and S. Chiba, Nucl. Phys. A 856, 55 (2011).
67. S. Aoki et al., Nucl. Phys. A 828, 191 (2009).
68. L. Mornas, Eur. Phys. J. A 24, 293 (2005).
69. E. Chabanat, P. Bonche, P. Haensel, J. Meyer, and R. Schaeffer, Nucl. Phys. A 635, 231 (1998).
70. E. Hiyama, M. Kamimura, Y. Yamamoto, and T. Matoba, Phys. Rev. Lett. 104, 212502 (2010).
71. E. Hiyama, M. Kamimura, Y. Yamamoto, T. Motoba, and T. A. Rijken, Prog. Theo. Phys. Suppl. 185, 152 (2010).
72. E. Hiyama, Few-Body Syst. bf 53, 189 (2012).
73. Y. Yamamoto, T. Motoba, H. Himeno, K. Ikeda, and S. Nagata, Prog. Theor. Phys. Suppl. 117, 361 (1994).