g-function in perturbation theory ∗

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Abstract

We present some explicit computations checking a particular form of gradient formula for a boundary beta function in two-dimensional quantum field theory on a disk. The form of the potential function and metric that we consider were introduced in [16], [18] in the context of background independent open string field theory. We check the gradient formula to the third order in perturbation theory around a fixed point. Special consideration is given to situations when resonant terms are present exhibiting logarithmic divergences and universal nonlinearities in beta functions. The gradient formula is found to work to the given order.

∗This work was supported in part by BSF-American-Israel Bi-National Science Foundation, the Israel Academy of Sciences and Humanities-Centers of Excellence Program, the German-Israel Bi-National Science Foundation.
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1 Introduction

This paper is devoted to the study of gradient property of boundary RG flows in two-dimensional quantum field theory on a surface with boundary. The gradient property means that there exists a metric $G_{ij}$ on the space of boundary conditions and a potential function $g$ such that

$$G_{ij} \beta^j = -\frac{\partial g}{\partial \lambda^i} \tag{1}$$

where $\beta^i$ is a beta function of boundary coupling constant $\lambda^i$.

In the context of bulk RG flows in 2d theories such a gradient property was proved by A. B. Zamolodchikov in [1] (see also [2], [3]). He showed that there exists a potential function $c$ that is constant at a fixed point with the value given by the central charge and that monotonically decreases along the RG flow. Moreover a concrete construction of such $c$-function and the metric such that

$$G_{ij} \beta^j = -\frac{\partial c}{\partial \alpha^i}$$

was given in [1]. Here $\alpha^i$’s are bulk coupling constants.

For the boundary RG flows a similar statement was conjectured in [6] that goes in the literature under the name "g-theorem". A perturbative proof of the conjecture was presented in [7]. The number analogous to the central charge that is supposed to decrease from UV to IR fixed point is called a boundary entropy $g$ and at a fixed point it is defined as follows. Consider a quantum critical system on a cylinder of length $l$ and circumference $r$. A conformally invariant boundary codition at the ends of the cylinder can be represented by a boundary state $|B\rangle$ [8]. In the limit of large $l$ the cylinder partition function has asymptotics

$$Z_{BB}(l,r) = \langle B|e^{-lH}|B\rangle \sim \langle B|0\rangle \langle 0|B\rangle e^{-E_0 l} \tag{2}$$

where $|0\rangle$ is a vacuum state for periodic boundary condition on a cylinder and $E_0$ is the ground state energy. The boundary entropy is then defined as a number

$$g = \langle B|0\rangle = Z_{disk}$$

that equals the value of the disk partition function. More precisely the boundary state is normalized by equating open (strip) and closed string (cylinder) channel representations for the cylinder partition function [6], [9]. (The phase of $|B\rangle$ can be chosen so that $\langle B|0\rangle$ is real and positive and thus $g$ is such as well.)

The perturbative computation presented in [7] in essence goes as follows. The authors consider a theory on a semi-infinite strip perturbed by a single boundary primary operator $\phi$ of dimension $\Delta = 1 - \epsilon$ with $0 < \epsilon << 1$. A Kosterlitz type renormalization scheme (see [10], [11], [12]) is chosen in which the corresponding beta function is

$$\beta(\lambda) = \epsilon \lambda + C \lambda^2 \tag{3}$$

where $C$ is the OPE coefficient of $\phi$ with itself:

$$\phi(\tau)\phi(0) \sim \frac{C\phi(0)}{|\tau|^{1-\epsilon}}.$$
We note that the quadratic term in this beta function is scheme dependent. For example by a suitable coupling constant redefinition one can make the beta function linear. The IR fixed point present in (3) is then pushed to infinity. Employing the scheme that gives (3) allows one to compare the values of the partition function at the nearby fixed points. By carefully dropping terms extensive in the strip width and performing a proper renormalization the authors of [7] arrive at the change in boundary entropy between the fixed points

\[ \frac{\delta g}{g} = -\frac{\pi^2 \epsilon^3}{3C^2} \]  

exact to the order \( \epsilon^3 \).

This formula proved to be useful in the analysis of examples of boundary flows (see e.g. [13], [14] and references therein). However its derivation does not provide us with the potential function and metric of the gradient formula. Also it would be desirable to extend the perturbative analysis to more general flows with several coupling constants running.

Note that the gradient property itself is easy to establish to the third order in perturbation theory. In a certain renormalization scheme [12] the beta functions are

\[ \beta^i = \frac{d\lambda^i}{d \ln L} = \epsilon_i \lambda_i + \sum_{jk} C_{i(jk)} \lambda^j \lambda^k \]  

where \( \epsilon_i = 1 - \Delta_i \), \( C_{i(jk)} = \frac{1}{2}(C_{ijk} + C_{ikj}) \) and \( C_{i(jk)} \) are the boundary 3-point structure constants. Here \( L \) is the position space renormalization scale. The gradient property then follows from the fact that \( C_{i(jk)} \) is totally symmetric in its indices that in its turn follows from the cyclic symmetry of \( C_{ijk} \). Our interest though is in a canonical form of the metric and potential function that would work to all orders similar to the ones that were constructed by A. B. Zamolodchikov for the bulk beta functions.

A concrete proposal for such a form came about in [16], [18] in the framework of background independent open string field theory that was put forward in [15]. The tentative potential function has the form

\[ g = Z - \sum_i \beta^i \frac{\partial Z}{\partial \lambda^i} \]  

where \[ Z = \int [d\phi] e^{-S} \]  

is the renormalized disk partition function, while the metric is

\[ G_{ij} = \int_0^{2\pi} \frac{r d\theta_1}{2\pi} \int_0^{2\pi} \frac{r d\theta_2}{2\pi} \langle \phi_i(ke^{i\theta_1}) \phi_j(ke^{i\theta_2}) \rangle 2 \sin^2 \left( \frac{\theta_1 - \theta_2}{2} \right). \]  

Here \( \langle ... \rangle \) stands for a nonnormalized correlator that does not include the division by \( Z \), \( r \) is the radius of the disk.

Formula (4) in its covariant form as well as its analysis in conformal perturbation theory first appear in [18]. We find though the alternative computations done in the present paper more explicit. Also we make no reference in our analysis to string theory objects such as conformal ghosts and BRST operator dealing only with field theoretic structures. We hope
that this approach may be more beneficiary in clarifying the field theoretic status of the gradient formula (1), (6), (7).

The approach of [13], [16], [17], [18] was successfully applied not long ago to the analysis of tachyon condensation [23], [24]. A particular exactly calculable model [16] with quadratic perturbation and linear beta functions was considered along the way and it was shown in [24] that (6) monotonically decreases along the RG flow for that model. The model considered in [23], [24] is a sigma model with noncompact target space. A noncompact case in general may require a special care. For instance it is known that the c-theorem can fail in the noncompact situation [19] (see [20] for a recent discussion of such situation). The same concerns application of formulae (6), (7) in the noncompact case. Thus in [24] one of the coordinates was regarded as a space-time volume regulator and was treated in a special way.

In the boundary case one concrete problem in noncompact situation (irrational CFT) arises regarding the value of boundary entropy at a fixed point. For example in the case when we consider a sigma model whose target space is noncompact and translation invariant one may be tempted to define $g$ by dividing $Z_{disk}$ over the infinite space-time volume $V$. It that case however the value of $Z_{disk}/V$ may depend on the value of an exactly marginal boundary coupling as happens for instance in the case of constant $U(1)$ field strength model ([21], [22]). And thus in the field theoretic sense this value cannot be a gradient function† (although it still may define a string space-time effective action).

The paper is organized as follows. In section 2 we perform a third order check of the gradient formula done in the absence of logarithmic divergences. The next three sections deal with cases when low order resonant terms are present each exhibiting a logarithmic divergence and universal nonlinearities in beta functions: second order resonance with nonvanishing linear terms in the beta functions (section 3), quadratic and cubic resonances in the identity coupling beta function (section 4), marginal but not exactly marginal couplings (section 5). In section 6 we conclude by pointing out some open questions. The appendix contains an explicit expression for the first perturbative correction to the local two point function. Although we do not use it in the analysis of the gradient formula this expression complements naturally the perturbative computations done in the paper and may be found of use in the future.

2 Third order computation in the absence of logarithmic divergences

We consider a two-dimensional quantum field theory on a disk $|z|^2 \leq r^2$ whose (Euclidean) action functional has the form

$$S = S_0 - \sum_i \int_0^{2\pi} \frac{r d\theta}{2\pi} \lambda^i \phi_i (re^{i\theta}) - \alpha r$$  \hspace{1cm} (8)

where $S_0$ defines a conformal field theory with conformal boundary conditions and $\phi_i$ are its boundary primary fields with weights $\Delta_i$. For simplicity we assume that $\Delta_i \neq \Delta_j$ if $i \neq j$. The modifications needed for the more general case are straightforward. $\alpha$ is the coupling constant of identity operator.

†I am grateful to Daniel Friedan for the discussion on this point
Throughout this section we will assume that integrals arising in perturbation series in $\lambda^i$’s are all power divergent up to (and including) the third order. We will employ a minimal subtraction scheme. It is easy to show that under these assumptions all beta functions remain linear to the given order. In particular the beta function of the identity operator is $\beta_1 = \alpha$. We will treat the coupling constant $\alpha$ nonperturbatively. Let us show now that its contribution to the gradient formula decouples from the rest of coupling constants. The renormalized disk partition function can be written as

$$Z_{\text{disk}} = Z = \int [d\phi] e^{-S} = e^{\alpha r} \tilde{Z}(r, \lambda). \quad (9)$$

Plugging this into the gradient formula we obtain

$$G_{1i} \beta^i + G_{11} \alpha = \alpha r^2 e^{\alpha r} \tilde{Z} + r e^{\alpha r} \beta^i \frac{\partial \tilde{Z}}{\partial \lambda^i},$$

$$G_{ij} \beta^j + G_{i1} \alpha = -e^{\alpha r} \frac{\partial}{\partial \lambda^i}(\tilde{Z} - \beta^j \frac{\partial \tilde{Z}}{\partial \lambda^j}) + \alpha r e^{\alpha r} \frac{\partial \tilde{Z}}{\partial \lambda^i}. \quad (10)$$

It follows from the definition of the metric (7) that

$$G_{11} = r^2 e^{\alpha r} \tilde{Z},$$

$$G_{1i} = G_{i1} = re^{\alpha r} \frac{\partial \tilde{Z}}{\partial \lambda^i} \quad (12)$$

and thus the first equation in (10) holds identically while the second one (and therefore the whole gradient formula) boils down to

$$\tilde{G}_{ij} \beta^j = -\frac{\partial}{\partial \lambda^i}(\tilde{Z} - \beta^j \frac{\partial \tilde{Z}}{\partial \lambda^j}) \quad (13)$$

where we introduced $\tilde{G}_{ij}$ that is given by formula (11) with the factor $e^{\alpha r}$ omitted. To simplify the notation we will omit below the tilde over $\tilde{Z}$ and $\tilde{G}_{ij}$ but it will be assumed that the factors containing $\alpha$ are everywhere dropped.

To organize the perturbation theory expansion it is convenient to label various terms of order $n$ in $\lambda^i$’s by an upper index ($n$). Thus at the second order we have to prove that

$$G_{ij}^{(0)} \beta^{(1)} = -\partial_i (Z^{(2)} - \beta^{(1)} \partial_j Z^{(2)}). \quad (14)$$

At this order we encounter an integral

$$I_2(\nu) \equiv \int_0^{2\pi} d\theta \frac{\sin^2 \left( \frac{\theta}{2} \right) ^\nu}{2\pi} = \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} \Gamma(1 + \nu)}, \quad \nu > -\frac{1}{2} \quad (15)$$

that for $\nu \neq -1/2$ is defined via analytic continuation that is equivalent to dropping the power divergence. We find then that

$$Z^{(2)} = \frac{1}{8 \sqrt{\pi}} \sum_i (\lambda^i)^2 (2r)^{2\epsilon_i} \frac{\Gamma(\epsilon_i - \frac{1}{2})}{\Gamma(\epsilon_i)} \frac{\Gamma(\epsilon_i + \frac{1}{2})}{\Gamma(1 + \epsilon_i)} \quad (16)$$

$$G_{ij}^{(0)} = \frac{(2r)^{2\epsilon_i} \delta_{ij}}{2 \sqrt{\pi} \Gamma(1 + \epsilon_i)} \quad (17)$$

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Plugging these expressions into the both sides of (14) we find that the equality indeed holds. This computation was also done in [24]. Note that the pole at $\epsilon_i = 1/2$ that is present in $Z^{(2)}$ [16] and that corresponds to a logarithmic running of the identity coupling constant disappears from the $g$ function upon subtracting $\beta^{(1)}_j \partial_j Z^{(2)}$. We will see the same effect at the next order.

At the third order the identity we are supposed to check reads

$$G^{(1)}_{ij} \beta^{(1)}_i = -\partial_i (Z^{(3)} - \beta^{(1)}_i \partial_i Z^{(3)}) \quad (18)$$

A triple correlator of primaries $\phi_i$ is fixed by modular invariance and we encounter the following integral

$$I_3(\nu_1, \nu_2, \nu_3) \equiv \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \left[ \sin^2 \left( \frac{\theta_1}{2} \right) \right]^{\nu_1} \left[ \sin^2 \left( \frac{\theta_2}{2} \right) \right]^{\nu_2} \left[ \sin^2 \left( \frac{\theta_1 + \theta_2}{2} \right) \right]^{\nu_3} = \frac{\Gamma(1 + \nu_1 + \nu_2 + \nu_3)}{\pi^{3/2}} \frac{\Gamma(\nu_1 + \frac{1}{2}) \Gamma(\nu_2 + \frac{1}{2}) \Gamma(\nu_3 + \frac{1}{2})}{\Gamma(\nu_1 + \nu_2 + 1) \Gamma(\nu_2 + \nu_3 + 1) \Gamma(\nu_1 + \nu_3 + 1)} \quad (19)$$

that for values $\nu_i < -\frac{1}{2}$ is defined via analytic continuation. A computation leading to (19) can be found e.g. in the Appendix A of [25]. The integral can be first mapped on the half plane where we obtain a rational integrand, which in its turn can be integrated by standard Feynman parameters technique.

For $Z^{(3)}$ we have an expression

$$Z^{(3)} = \frac{1}{3!} \sum_{ijk} \lambda_i \lambda_j \lambda_k (2r)^{\epsilon_i + \epsilon_j + \epsilon_k} C_{ijk} K_{ijk} \quad (20)$$

where

$$K_{ijk} = \frac{1}{32\pi^2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \left[ \sin^2 \left( \frac{\theta_2}{2} \right) \right]^{\frac{1}{2}(\Delta_i - \Delta_j - \Delta_k)} \left[ \sin^2 \left( \frac{\theta_1 + \theta_2}{2} \right) \right]^{\frac{1}{2}(\Delta_k - \Delta_i - \Delta_j)} \quad (21)$$

and $C_{ijk}$ are the OPE coefficients. This integral has subdivergences whenever

$$\epsilon_i \geq \epsilon_j + \epsilon_k \quad (22)$$

and an overall divergence if

$$\epsilon_i + \epsilon_j + \epsilon_k \leq 1. \quad (23)$$

Assuming that the equalities in (22), (23) do not hold we can evaluate (21) using (19) via analytic continuation

$$K_{ijk} = \frac{\Gamma\left(\frac{1}{2}(1 + \epsilon_i + \epsilon_j + \epsilon_k)\right)}{(4\pi)^{3/2}} \times \frac{\Gamma\left(\frac{1}{2}(\epsilon_j + \epsilon_k - \epsilon_i)\right) \Gamma\left(\frac{1}{2}(\epsilon_i + \epsilon_k - \epsilon_j)\right) \Gamma\left(\frac{1}{2}(\epsilon_i + \epsilon_j - \epsilon_k)\right)}{\Gamma(\epsilon_i) \Gamma(\epsilon_j) \Gamma(\epsilon_k)} \quad (24)$$

The poles in this expression have a natural interpretation in terms of resonances (to be discussed in more detail in the following sections). Thus the poles

$$\epsilon_i = \epsilon_j + \epsilon_k$$

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correspond to a resonance term proportional to $\lambda^i \lambda^k$ in $\beta^i$ while the poles
\[
\epsilon_i + \epsilon_j + \epsilon_k = 1
\]
correspond to resonance terms proportional to $\lambda^i \lambda^j \lambda^k$ in the beta function of identity operator.

The first order correction to the metric has the form
\[
G^{(1)}_{ij} = \sum_k \lambda^k (2r)^{\epsilon_i + \epsilon_j + \epsilon_k} C_{ik}(2r)^{\epsilon_i + \epsilon_j + \epsilon_k} F_{ij}^k
\]
where
\[
F_{ij}^k = \frac{\Gamma(1 + \epsilon_i + \epsilon_j + \epsilon_k) \Gamma(1 + \epsilon_i + \epsilon_k - \epsilon_j) \Gamma(1 + \epsilon_k - \epsilon_i) \Gamma(1 + \epsilon_j)}{4\pi^{3/2} \Gamma(1 + \epsilon_i) \Gamma(1 + \epsilon_j) \Gamma(1 + \epsilon_k)}.
\]

In [26] and everywhere below $C_{ijk}$ denotes the symmetrized OPE coefficients.

From the above formulas one easily obtains the relation
\[
F_{ij}^k = \frac{1}{2} \left[ K_{ijk} \frac{1 - \epsilon_i - \epsilon_j - \epsilon_k}{\epsilon_i \epsilon_j \epsilon_k} \right] \times \epsilon_k (\epsilon_k - \epsilon_i - \epsilon_j).
\]

Using this relation and noting that
\[
\beta^{(1)} = \epsilon_i \lambda^i
\]

it is straightforward to derive
\[
G^{(1)}_{ij} \beta^{(1)} = -\frac{1}{2} \sum_j (2r)^{\epsilon_i + \epsilon_j + \epsilon_k} C_{ijk} K_{ijk} (1 - \epsilon_i - \epsilon_j - \epsilon_k) \lambda^i \lambda^j \lambda^k =
\]
\[
- \frac{\partial}{\partial \lambda^i} (Z^{(3)} - \beta^{(1)} \frac{\partial Z^{(3)}}{\partial \lambda^i}).
\]

We see that in the absence of logarithmic divergences the gradient formula [1], [6], [7] holds to the third order in perturbation theory. Note again that like at the second order the resonance poles [25] corresponding to logarithmic running of the identity coupling constant were subtracted in the final expression for $Z^{(3)}$. This looks suggestive of the fact that this effect may happen to all orders in perturbation theory.

As a final remark in this section let us note that the renormalized partition function satisfies a simple finite size scaling relation
\[
r \frac{dZ}{dr} = \beta^i \frac{\partial Z}{\partial \lambda^i}.
\]

Thus the $g$ function could be written as
\[
g = Z - r \frac{dZ}{dr}.
\]

We will return to this representation of $g$-function in the last section.

6
3 Resonant terms

Let us start with a brief reminder of general facts about perturbation theory resonant terms. Renormalization group equations in general have a form

$$\frac{d\lambda^i}{dt} = D^i_j \lambda^j + h^i(\lambda)$$  \hspace{1cm} (30)

where $D$ is the matrix of anomalous dimensions at the fixed point and $h_i(\lambda)$ contains all the nonlinearities. We assume that the coordinates $\lambda^j, j = 1, \ldots, n$ in the space of the theories are chosen so that $D$ is in its Jordan normal form. The $h^i(\lambda)$ can be written as a formal power series with a typical term of the form

$$h^i_{j_1j_2\ldots j_n}(\lambda^1)^{j_1} \cdot \ldots \cdot (\lambda^n)^{j_n}$$ \hspace{1cm} (31)

where $j_k$’s are nonnegative integers. Different renormalization schemes are related by a formal change of coordinates

$$\lambda'^i(\lambda) = \lambda^i + \xi^i(\lambda).$$

One may try to choose $\xi^i(\lambda)$ such that the transformed equation (30) takes the simplest possible form. In the best case the system (30) can be brought to the linear form

$$\frac{d\lambda'^i}{dt} = D^i_j \lambda'^j.$$ \hspace{1cm} (32)

It is known in the theory of differential equations that obstructions to linearization of the system (30) are the so called resonant monomials. Let $(\epsilon_1, \ldots, \epsilon_n)$ be the set of eigenvalues of the matrix $D$. A monomial of the form (31) is called resonant if

$$\sum_{k=1}^{n} \epsilon_k j_k = \epsilon_i, \quad \sum_{k} j_k \geq 2.$$ \hspace{1cm} (33)

In the context of conformal perturbation theory in 2 dimensions (in the bulk or on the boundary) one can identify coordinates $\lambda^j$ with the coupling constants appearing in (8). The matrix $D$ in this case is diagonal. One can estimate the perturbation expansion divergencies emerging when points of insertion of several operators $\phi_i$ come together via operator product expansion. One sees then that the resonance condition (33) implies a logarithmically divergent counterterm for the $i$-th coupling constant that is proportional to the corresponding monomial (31).

If resonant terms are present in the RHS of (30) those equations cannot be linearized by any choice of coordinates. Assuming that $D = \text{diag}(\epsilon_1, \ldots, \epsilon_n)$ one can prove though (see e.g. [28] chapter 2, theorem 1.5) that there exists a formal change of coordinates such that in the new coordinates the nonlinear parts $h^i(\lambda)$ consist of resonant monomials. In the absence of resonant terms there are stronger results available. The theorems of Poincare and Siegel give sufficient conditions for the existence of an analytic change of coordinates that bring (30) to the form (32). We refer the interested reader to the book [28] for the precise statements of those theorems and more of the mathematical background on normal forms of differential equations. In conformal perturbation theory the resonant terms were discussed in [26], [27]. In the context of background independent string field theory the role of resonances was emphasized in [18].
We see thus that although we did a third order computation in the previous section we have not really tested whether the gradient formula at hand handles universal nonlinearities. To do that we consider in this and the next two sections situations when low order resonant terms are present.

Consider now a fixed point perturbed by 3 fields boundary fields $\phi_i$, $i = 1, 2, 3$ whose anomalous dimensions satisfy a resonant condition

$$\epsilon_3 = \epsilon_1 + \epsilon_2.$$  \hspace{1cm} (34)

We will further assume that the only nonvanishing OPE coefficients are $C_{123}$ and the ones with permuted indices. In the point splitting + minimal subtraction scheme the beta functions have the form

$$\beta^3 = \epsilon_3 \lambda^3 + \frac{1}{\pi} \lambda^1 \lambda^2 C_{123} + \ldots,$$

$$\beta^1 = \epsilon_1 \lambda^1 + \ldots,$$

$$\beta^2 = \epsilon_2 \lambda^2 + \ldots$$  \hspace{1cm} (35)

through the third order in the coupling constants. The quadratic term in the first equation in (35) is universal.

The integral expressions for $K_{123}$ and $F_{13}^2 = F_{31}^2$ \cite{24}, \cite{27} contain a logarithmic divergence. If $\delta$ is a cutoff of the angular variable $\theta$ the divergences are proportional to $-2 \ln \delta$ that in terms of dimensionfull position space cutoff $a$ can be written as $2 \ln(r/a)$. We will employ a minimal type subtraction scheme in which this divergence is subtracted with an additional finite term of the form $2 \ln(r\mu)$ where $\mu$ is a renormalization mass scale.

In this scheme we obtain

$$(Z^{(3)})_{\text{ren}} = \lambda^1 \lambda^2 \lambda^3 (2r)^{2\epsilon_1 + 2\epsilon_2} C_{123} \frac{\Gamma(\epsilon_1 + \epsilon_2 - \frac{1}{2})}{8\pi^{3/2} \Gamma(\epsilon_1 + \epsilon_2)} (2 \ln(r\mu) + g_{12})$$  \hspace{1cm} (36)

where

$$g_{12} \equiv \psi(\epsilon_1 + \epsilon_2 - \frac{1}{2}) - \psi(\epsilon_1) - \psi(\epsilon_2)$$

where $\psi(x) = \frac{d \ln \Gamma(x)}{dx}$ is the logarithmic derivative of the Euler’s Gamma function. To obtain this formula one should expand the integral $I_3$ \cite{19} in $\nu_3$ around $\nu_3 = -\frac{1}{2}$ keeping $\nu_1 = \epsilon_2 - 1/2$ and $\nu_2 = \epsilon_1 - 1/2$ fixed. Then take the limit $\nu_3 \rightarrow -\frac{1}{2}$ subtracting the pole.

Similarly one obtains

$$(F_{13}^2)_{\text{ren}} = \frac{\Gamma(\epsilon_1 + \epsilon_2 + \frac{1}{2})}{4\pi^{3/2} \Gamma(\epsilon_1 + \epsilon_2 + 1)} \left(2 \ln(r\mu) + g_{12} + \frac{1}{\epsilon_1 + \epsilon_2 - 1/2 - \frac{1}{\epsilon_1}} \right)$$  \hspace{1cm} (37)

and the same kind of expression for $(F_{23}^1)_{\text{ren}}$ with $\epsilon_1$ interchanged with $\epsilon_2$.

Note that we can use the renormalization scale $\mu$ to introduce dimensionless couplings $\tilde{\lambda}^i = \mu^{-\epsilon_i} \lambda^i$. It is easy to check then that the renormalized partition function $Z_{\text{ren}} = 1 + Z^{(2)} + (Z^{(3)})_{\text{ren}} + \ldots$ up to the third order satisfies the following renormalization group equation

$$\mu \frac{\partial Z_{\text{ren}}(\tilde{\lambda}, \mu)}{\partial \mu} = \beta^i(\tilde{\lambda}) \frac{\partial Z_{\text{ren}}(\tilde{\lambda}, \mu)}{\partial \tilde{\lambda}^i}$$

with $\beta^i$’s given in (35). Also the simple finite size scaling relation given in \cite{24} holds.
The coefficient $F^3_{12}$ does not require any renormalization (besides the usual subtraction of power divergences) and is given by

$$F^3_{12} = \frac{\Gamma(\epsilon_1 + \epsilon_2 + \frac{1}{2})}{4\epsilon_1 \epsilon_2 \pi^{3/2} \Gamma(\epsilon_1 + \epsilon_2)}.$$  \hspace{1cm} (38)

We are fully equipped now to check the equations

$$G_{1i} \beta^{(1)} = \left[ F^3_{12} \beta^{(1)} \lambda^3 + (F^2_{13})_{\text{ren}} \beta^{(1)} \lambda^2 \right] C_{(123)} (2r)^{2c_3} = -\frac{\partial g^{(3)}}{\partial \lambda^1},$$

$$G_{2j} \beta^{(1)} = \left[ F^3_{21} \beta^{(1)} \lambda^3 + (F^1_{23})_{\text{ren}} \beta^{(1)} \lambda^1 \right] C_{(123)} (2r)^{2c_3} = -\frac{\partial g^{(3)}}{\partial \lambda^2},$$

$$G_{3j} \beta^{(1)} + G_{33} \beta^{(2)} = \left[ (F^1_{32})_{\text{ren}} \beta^{(2)} \lambda^1 + (F^2_{31})_{\text{ren}} \beta^{(1)} \lambda^2 \right] C_{(123)} (2r)^{2c_3} +$$

$$+ G_{3d} \beta^{(2)} = -\frac{\partial g^{(3)}}{\partial \lambda^3}$$  \hspace{1cm} (39)

where

$$g^{(3)} = (Z^{(3)})_{\text{ren}} - \sum_{i=1}^{3} \beta^{(1)} \frac{\partial (Z^{(3)})_{\text{ren}}}{\partial \lambda^i} - \beta^{(2)} \frac{\partial Z^{(2)}}{\partial \lambda^3}. $$

A straightforward computation shows that the equations (39) indeed hold.

4 **Resonances in the identity coupling beta function**

So far we considered the cases when the beta function of the identity operator is exactly linear: $\beta_1 = \alpha$. In this section we will consider two cases when $\beta_1$ is of a more general form

$$\beta_1 = \alpha + h(\lambda^i).$$ \hspace{1cm} (40)

In this case one of the gradient equations

$$G_{1i} \beta^i + G_{11} \beta_1 = -\frac{\partial g}{\partial \alpha}$$  \hspace{1cm} (41)

holds identically while the remaining set of equations takes the form

$$G_{ij} \beta^j = -\frac{\partial}{\partial \lambda^i} (Z - \beta^j \frac{\partial Z}{\partial \lambda^j}) + rZ \frac{\partial h}{\partial \lambda^i}$$  \hspace{1cm} (42)

Again the factors $e^{\alpha r}$ can be dropped on both sides of the equation.

In section 2 we noted two resonances of the identity coupling. One happens at the second order in perturbation expansion when a coupling of dimension $\Delta = \frac{1}{2}$ is present while another one takes place at the third order when we have 3 couplings whose anomalous dimensions satisfy

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = 1.$$ \hspace{1cm} (43)

In the first case let us restrict our attention only to the identity coupling and a coupling constant $\lambda$ of the dimension $1/2$ operator. The beta functions then are readily shown to be

$$\beta_1 = \alpha + \frac{\lambda^2}{2\pi} + \ldots,$$

$$\beta_\lambda = \frac{\lambda}{2} + \ldots$$

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through the second order in couplings. The expression \((Z^{(2)})_{\text{ren}} - \beta^{(1)} \partial_\lambda (Z^{(2)})_{\text{ren}}\) vanishes and the gradient formula (42) at the leading order boils down to the equation
\[
G^{(0)}_{\lambda\mu} \beta^{(1)} = r \partial_\lambda \beta^{(2)}
\]
that upon using (17) is readily found to be correct. At the next order in perturbation expansion our analysis done in section 2 is still valid because the additional term \(Z^{(2)}\) next contributes at the 4th order.

Let us now look at the second case when there are 3 couplings \(\lambda^i, i = 1, 2, 3\) whose anomalous dimensions satisfy (43). By analyzing the behavior of integral (21) in the regions \(\theta_1, 2 \to 0, 2\pi\) we find that it contains a logarithmic divergence
\[
K_{123}^* = \frac{\Gamma(\nu_1 + \frac{1}{2})\Gamma(\nu_2 + \frac{1}{2})\Gamma(\nu_3 + \frac{1}{2})}{8\pi^{3/2}\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma(-\nu_3)},
\]
(44)
and \(\nu_2, \nu_3\) are defined by cyclic permutation. The beta functions thus have the form
\[
\beta^i = \epsilon_i \lambda^i + \ldots, \quad i = 1, 2, 3,
\]
\[
\beta_1 = \alpha + 2K_{123}^* \lambda^1 \lambda^2 \lambda^3 + \ldots.
\]
At the third order the equation we need to check is
\[
\sum_{j=1}^{3} C^{(1)}_{ij} \beta^{(1)}_j = r \frac{\partial \beta^{(3)}_1}{\partial \lambda_1}.
\]
(46)
(The term \((Z^{(3)})_{\text{ren}} - \beta^{(1)} \partial_\lambda (Z^{(3)})_{\text{ren}}\) vanishes similarly to the \(\Delta = 1/2\) resonance case above.) Using (26), (27) we find that (46) holds.

5 Marginal but not exactly marginal couplings

As a final case of universal beta function nonlinearities consider the case when marginal but not exactly marginal couplings are present. Let us restrict our attention to a model situation when we have perturbation by only three operators whose anomalous dimensions vanish: \(\epsilon_1 = \epsilon_2 = \epsilon_3 = 0\). Assume also for simplicity that only \(C_{123}\) and its cyclic permutations are nonvanishing. In that case we have
\[
\beta^i = \sum_{k,j=1}^{3} \frac{1}{2\pi} C_{ijk} \lambda^j \lambda^k + \mathcal{O}(\lambda^3)
\]
(47)
with the first term being scheme independent. It follows from (10) that the \(Z^{(2)}\) correction to the partition function vanishes. The renormalized value of \(Z^{(3)}\) can be computed the following way. It is represented by the integral
\[
Z^{(3)} = \lambda^1 \lambda^2 \lambda^3 C_{(123)} K_{123},
\]
\[
K_{123} = \frac{1}{64\pi^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{d\theta_1 d\theta_2 d\theta_3}{\sin \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\theta_2 - \theta_3}{2} \right) \sin \left( \frac{\theta_3 - \theta_1}{2} \right)},
\]
(48)
The integral at hand is up to a constant factor the integrated three-tachyon amplitude on a disk. Another way of looking at such an integrated amplitude is that it gives the volume of Möbius group $\text{PSL}(2, \mathbb{R})$. More precisely $K_{123} = \frac{1}{\pi^3} \text{Vol}(\text{PSL}(2, \mathbb{R}))$. Regulating the integral in (48) by requiring that the distance between any two points is greater than $\delta$ one finds that the Möbius volume is linearly divergent

$$\text{Vol}(\text{PSL}(2, \mathbb{R})) = \frac{3\pi \ln 2}{\delta} - \frac{\pi^2}{2}$$

and the renormalized value of the volume is $-\frac{\pi^2}{2}$ \cite{29, 30}. Using this value we obtain

$$(Z^{(3)})_{\text{ren}} = -\frac{1}{2\pi} \lambda^1 \lambda^2 \lambda^3 C_{(123)}.$$ 

Using this expression, $G^{(0)}_{ij} = \delta_{ij}/2$ and formula (47) we find that the gradient formula considered at the first nonvanishing order

$$G^{(0)}_{ij} \beta^{j(2)} = -\frac{\partial}{\partial \lambda^i} Z^{(3)}$$

is correct.

6 Some open questions

Although the gradient formula \cite{11, 12, 17} survived all the checks that we performed above, that is certainly encouraging, many issues remain open. Below we point out to some of them.

First note that in all our checks we employed a version of minimal subtraction scheme that in general is known to behave very generously towards various Ward identities. The precise scheme dependence of the gradient formula at hand still needs to be clarified.

Another issue is the relation of this formula to the Affleck and Ludwig’s computation \cite{7}. Consider a cylinder partition function in the presence of non-scale-invariant boundary conditions. In this case we can still define the boundary conditions via a boundary state $|\overline{B}\rangle$ and the asymptotic \cite{22} yields a quantity $\ln \langle B|0 \rangle$ that in general contains a term corresponding to a free-energy per unit length of the boundary

$$\ln \langle B|0 \rangle = -rf_B + \log g(r). \quad (49)$$

The extensive free energy piece is non-universal and for large $r$ dominates over the second piece. The last one is believed to contain universal information and to interpolate between the UV $\ln g(0)$ and the IR $\ln g(\infty)$ values of boundary entropy at the corresponding fixed points.

While universality of function $g(r)$ still remains a subtle issue (see \cite{31} for some discussion) the free energy piece certainly needs to be subtracted. The authors of \cite{7} carefully drop similar extensive terms in their computation.

In the case of the gradient formula at hand as was already noted at the end of section 2 the potential function can be represented in the form

$$g = Z - r \frac{dZ}{dr}. \quad (50)$$

One may be tempted then to think that the role of the second term in (50) is to subtract the boundary free energy extensive piece. For that to be the case however one should have
used ln Z in place of Z in (30). At the level of 3rd order computations considered in this paper the only difference between Z and ln Z in (30) comes in the treatment of the identity coupling. It is not hard to track that to the given order in perturbation nothing would change in the above checks of the main gradient formula (42) had we replace d

\(Z\) with \(\ln Z\). However equation (41) for the derivative with respect to the identity coupling that previously was true identically would stop holding. The whole issue needs further clarification. It is not excluded that like in the case of bulk gradient formula when various potential functions are possible [4] (see also [5] section 6.2) the boundary potential function \(g(r)\) in (49) are essentially different off-criticality extensions of the boundary entropy.

Finally and obviously a nonperturbative proof of ”g-theorem” remains to be much desired.

Acknowledgments

I am particularly grateful to Daniel Friedan for numerous discussions on the boundary gradient formula. It is also a pleasure to thank Alexander Zamolodchikov for useful discussions and for reading the draft version of the paper.

A Two-point function

In this appendix we give an explicit expression for the first order correction to the local two point function in the minimal subtraction scheme. It is given by the integral

\[
G_{12}(\theta_{12}) = \sum_k \lambda_k r^{\epsilon_1 + \epsilon_2 + \epsilon_k} \langle \phi_1(e^{i\theta_1}) \phi_2(e^{i\theta_2}) \rangle = \sum_k \left[ 4 \sin^2 \left( \frac{\theta_{12}}{2} \right) \right]^{\frac{1}{2}(\Delta_k - \Delta_1 - \Delta_2)} \lambda_k r^{\epsilon_1 + \epsilon_2 + \epsilon_k} C_{12k} \times
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} d\phi \left[ 4 \sin^2 \left( \frac{\phi}{2} \right) \right]^{\frac{1}{2}(\Delta_1 - \Delta_2 - \Delta_k)} \left[ 4 \sin^2 \left( \frac{\phi + \theta_{12}}{2} \right) \right]^{\frac{1}{2}(\Delta_2 - \Delta_1 - \Delta_k)}.
\]

(A.1)

The integral can be expressed via associated Legendre functions of the first kind which in their turn are expressed via hypergeometric functions and after simplifications we obtain

\[
G_{12}(\theta_{12}) = \sum_k \lambda_k (2r)^{\epsilon_1 + \epsilon_2 + \epsilon_k} \frac{C_{12k}}{8\pi} B \left( \frac{\epsilon_2 + \epsilon_k - \epsilon_1}{2}, \frac{\epsilon_1 + \epsilon_2 - \epsilon_k}{2} \right)
\]

\[
\left[ \sin^2 \left( \frac{\theta_{12}}{2} \right) \right]^{(\epsilon_1 + \epsilon_2 + \epsilon_k - 2)/2} F_1 \left( \frac{\epsilon_2 + \epsilon_k - \epsilon_1}{2}, \frac{\epsilon_1 + \epsilon_2 - \epsilon_k}{2}; \frac{1}{2}; 1 - \sin^2 \left( \frac{\theta_{12}}{2} \right) \right)
\]

(A.2)

where \(B\) stands for the Euler’s beta function. In the resonance limit \(\epsilon_2 \to \epsilon_1 + \epsilon_k\) we see the properly normalized logarithmic divergence. (This provides a simple check of the answer.)

It is interesting to observe a jump in the short-distance behavior of (A.2) that happens at the value \(\epsilon_3 = 1/2\): for \(\epsilon_3 < 1/2\), \(G_{12} \sim (\theta_{12})^{\epsilon_1 + \epsilon_2 + \epsilon_3 - 1}\) while for \(\epsilon_3 > 1/2\), \(G_{12} \sim (\theta_{12})^{\epsilon_1 + \epsilon_2 - \epsilon_3}\).

References

[1] A. B. Zamolodchikov, "Irreversibility" of the flux of the renormalization group in a 2d field theory, JETP Lett. 43 (1986) 730-732.

\footnote{I would like to thank Alexander Zamolodchikov for stressing this point to me}
[2] A. B. Zamolodchikov, *Renormalization group and perturbation theory about fixed points in two-dimensional field theory*, Sov. J. Nucl. Phys. 46 (1987) 1090.

[3] A. W. W. Ludwig and J. L. Cardy, *Perturbative evaluation of the conformal anomaly at new critical points with applications to random systems*, Nucl. Phys. B285 (1987) 687-718.

[4] A. Cappelli, D. Friedan, J. I. Latorre, *c-theorem and the spectral representation*, Nucl. Phys. B352 (1991) 616-670.

[5] D. Friedan, *A tentative theory of large distance physics*, hep-th/0204131.

[6] I. Affleck and A. W. Ludwig, *Universal noninteger "ground state degeneracy" in critical quantum systems*, Phys. Rev. Lett. 67 (1991) 161.

[7] I. Affleck and A. W. Ludwig, *Exact conformal field theory results on the multichannel Kondo effect: single fermion Green’s function, self-energy, and resistivity*, Phys. Rev. B48 (1993) 7297.

[8] J. L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, Nucl. Phys. B324, (1989) 581-596.

[9] S. Elitzur, E. Rabinovici, G. Sarkissian, *On Least Action D-Branes*, Nucl. Phys. B541 (1999) 246-264; hep-th/9807161.

[10] J. M. Kosterlitz, *The critical properties of the two-dimensional xy model*, J. Phys. C, Vol. 7, 1974, p. 1046.

[11] A. Ludwig, *Critical behavior of the two-dimensional random q-state Potts model by expansion in (q − 2)*, Nucl. Phys. B285 (1987) 97-142.

[12] J. L. Cardy, *Conformal invariance and statistical mechanics*, in Les Houches session XLIX, 1988, *Fields, strings and critical phenomena*, Eds. E. Brezin and J. Zinn-Justin, Elsevier, New York, 1989.

[13] A. Recknagel, D. Roggenkamp and V. Schomerus, *On relevant boundary perturbations of unitary minimal models*, Nucl. Phys. B588 (2000) 552-564; hep-th/0003110.

[14] K. Graham, I. Runkel and G. M. T. Watts, *Renormalisation group flows of boundary theories*, hep-th/0010082.

[15] E. Witten, *On background independent open-string field theory*, Phys. Rev. D46 (1992) 5467-5473; hep-th/9208027.

[16] E. Witten, *Some computations in background independent off-shell string theory*, Phys.Rev. D47 (1993) 3405-3410; hep-th/9210065.

[17] S. Shatashvili, *Comment on background independent open string theory*, Phys.Lett. B311 (1993) 83-86; hep-th/9303143.

[18] S. Shatashvili, *On the problems with background independence in string theory*, hep-th/9311177.
[19] J. Polchinski, *Scale and conformal invariance in quantum field theory*, Nucl. Phys. **B303** (1988) 226-236.

[20] A. Adams, J. Polchinski and E. Silverstein, *Don't Panic! Closed String Tachyons in ALE Spacetimes*, JHEP **0110** (2001) 029; hep-th/0108075.

[21] E. Fradkin and A. Tseytlin, *Nonlinear electrodynamics from quantized strings* Phys. Lett. **B163** (1985) 123.

[22] C. G. Callan, C. Lovelace, C. R. Nappi and S. A. Yost, *String loop corrections to beta functions*, Nucl. Phys. **B288** (1987) 525-550.

[23] A. A. Gerasimov and S. L. Shatashvili, *On exact tachyon potential in open string field theory*, JHEP **0010** (2000) 034; hep-th/0009103.

[24] D. Kutasov, M. Marino and G. Moore, *Some exact results on tachyon condensation in string field theory*, JHEP **0010** (2000) 045; hep-th/0009148.

[25] S. A. Frolov, *On off-shell structure of open string sigma model*, JHEP **0108** (2001) 020; hep-th/0104042.

[26] A. B. Zamolodchikov, *Integrable field theory from conformal field theory*, Advanced Studies in Pure Mathematics 19, 1989, pp. 641-674.

[27] A. B. Zamolodchikov, *Two-point correlation function in scaling Lee-Yang model*, Nucl. Phys. **B348**, 619 (1991).

[28] S.-N. Chow, C. Z. Li, D. Wang, *Normal forms and bifurcation of planar vector fields*, Cambridge Univ. press 1994.

[29] A. A. Tseytlin, *Renormalization of Möbius infinities and partition function representation for the string theory effective action*, Phys. Lett. **B202** (1988) 81.

[30] J. Liu and J. Polchinski, *Renormalization of the Möbius volume*, Phys. Lett. **203B** (1988) 39.

[31] P. Dorey, I. Runkel, R. Tateo and G. Watts, *g-function flow in perturbed boundary conformal field theories*, Nucl. Phys. **B578** (2000) 85-122; hep-th/9909216.