Quantum speed limit and stability of coherent states in quantum gravity

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Utilizing the program of expectation values in coherent states and its recently developed algorithmic tools, this letter investigates the dynamical properties of cosmological coherent states for Loop Quantum Gravity. To this end, the Quantum Speed Limit is adapted to Quantum Gravity, yielding necessary consistency checks for any proposal of stable families of states. To showcase the strength of the developed tools, they are applied to a prominent model: the Euclidean part of the quantum scalar constraint. We report the variance of this constraint evaluated on a family of coherent states showing that, for short times, this family passes the Quantum Speed Limit test, allowing the transition from one coherent state to another one.

Spectacular results concerning \textit{reduced} quantization of isotropic flat cosmology, so called Loop Quantum Cosmology (LQC) \cite{1}, have been obtained in last decades: For sharply peaked coherent states, the initial singularity of isotropic Universe is resolved \cite{2} and replaced by a Big Bounce \cite{3,4,5,6}. While the peak of such a coherent state follows a trajectory in classical phase space \cite{7}, quantum effects, e.g. relevant for the power spectrum of the cosmic microwave background \cite{8}, are also there.

However, we are left with an important open question, namely, whether a successful quantization of \textit{full} General Relativity would support the aforementioned predictions relevant for LQC. While such a complete theory is still not available, an increasing computational power allows to probe complicated lattice quantum systems \cite{9}. A suitable construction of coherent states on the lattice already exists \cite{10}. Its variant, called gauge (cosmological) coherent states (GCS), seems useful as a litmus paper for LQC. While the Big Bounce is reproduced, deviations from LQC-modified Friedmann equations occur \cite{11}, enabling comparison with discretized General Relativity \cite{12}.

Despite recent progress \cite{13,14}, the dynamics of GCS is poorly understood. In this letter we partially address that important aspect by virtue of \textit{quantum speed limit} \cite{15,16,17,18} — a tool familiar from fundamental considerations concerning time-energy uncertainty relations in quantum mechanics. We reformulate this tool to study “stability” of an arbitrary single-parameter family of “target” states $\Phi_s$, given a generator of evolution $\hat{H}$. We call such a family stable with respect to $\hat{H}$, if $\Phi_s \equiv e^{-i s \hat{H}} \Phi_0$. Afterwards, we touch upon the implementation of the constraints in the quantum domain to argue why stability is important in Loop Quantum Gravity (LQG). Finally, we employ the quantum speed limit, unravelling properties of the effective behaviour of the family of cosmological coherent states.

\textit{Quantum speed limit as consistency check for stability.} — In our setting, the state $\Phi_0$ is both an initial state of evolution rendered by the generator $\hat{H}$, as well as a member of a given family of states $\Phi_s$. However, suspecting that the evolved state $\Phi(s) := e^{-i s \hat{H}} \Phi_0$ does not belong to the family in question, i.e. $\Phi(s) \neq \Phi_s$, we wish to quantify its deviation from $\Phi_s$. To this end we introduce a two-point quantum fidelity

$$W(s, \tau) := |\langle \Phi(s), \Phi_\tau \rangle|^2.$$  \hfill (1)

Note that we label time-evolution by $s$ and $\tau$, not following the standard notation with $t$. We do this for the sake of further consistency with the literature on complexifier coherent states in LQG, where $t$ indicates the spread of a state.

For our purpose, we present a slightly more general formulation of quantum speed limit (QSL):

$$\tau \cdot \Delta_{\Phi_0} \hat{H} / \hbar \geq |A(0, \tau) - A(\tau, \tau)|,$$  \hfill (2)

$$A(s, \tau) := \arccos \left( \sqrt{W(s, \tau)} \right),$$  \hfill (3)

derived in the first section of Supplementary Material. As usual, $\Delta_{\Phi} \hat{O}$ is a standard deviation of an observable $\hat{O}$, given a state $\Psi$. In a typically considered scenario in which stability is given, one just sets $A(\tau, \tau) = 0$. In addition, assuming that the initial and the final states are orthogonal, i.e. $W(0, \tau) = 0$, the bound in (2) equals $\pi/2$. In this way, QSL helps understand how quickly the transition between two (orthogonal) states can happen due to $\hat{H}$.

On the contrary, we aim to test whether it is even legitimate to hope for stability, i.e. $W(\tau, \tau) \approx 1$. To this end, using a basic inequality $|x - y| \geq |x| - |y|$, we obtain an equivalent formulation of (2)

$$|A(\tau, \tau)| \geq |A(0, \tau)| - \tau \cdot \Delta_{\Phi_0} \hat{H} / \hbar.$$  \hfill (4)

Therefore, a necessary condition for stability is that the right hand side does not become too positive. Before we apply the above consistency check to Quantum Gravity directly, we shall first ask why stability of coherent states is important in quantization of fully
constrained theories (i.e. where the Hamiltonian is a linear combination of constraints) as happens in the case of General Relativity.

**Classical, strong and weak implementation of constraints.**—Let $C_i$ denote the constraints of the theory\(^1\). In the classical scenario, one can speak about a valid solution only if all $C_i$ are satisfied (i.e. vanish) at a given point of the phase space. How this rule is supposed to be carried over to the quantum level remains an open debate. We recall two known strategies and propose a third option.

First of all, one can avoid the problem by implementing the constraints already at the classical level. Most prominently, in General Relativity one has introduced so called deparametrized models \([19]\), where diffeomorphism and scalar constraints are fixed on the classical level via dust fields. Afterwards, dynamical evolution is described by a true Hamiltonian\(^2\). In this case the question for stable coherent state becomes immediately interesting and asks for applying the consistency check of the QSL.

Alternatively, implementation of some $C_i$ on the quantum level can be sought. Dirac’s quantization proposal suggests to find the kernel of the constraint operator $C_i$ and denote it as the *physical* Hilbert space. Such a strong implementation unfortunately suffers from severe obstacles due to the scalar constraint, since its kernel has not been found so far. Therefore, we consider weak implementation which admits two ways of relaxation, in comparison with the strong implementation. The constraints must hold (at least) on average, and only up to a certain quantum tolerance scale “$t$”. More formally and given some $C_i$, the state $\Psi$ is called physical if it is sharply peaked:

\begin{equation}
\langle \hat{C}_i \rangle _\Psi = 0 + \mathcal{O}(t), \quad \Delta_\Psi \hat{C}_i = 0 + \mathcal{O}(\sqrt{t}),
\end{equation}

Weak implementation of constraints admits that associated symmetries are ultimately broken in the quantum theory, however at the macroscopic level this violation would be extremely difficult to detect. We note in passing that a similar weak implementation has already been suggested for the so-called simplicity constraint of higher dimensional LQG \([20]\).

Physical states subject to weak implementation of the constraints are not automatically invariant, i.e. in general $\Psi \neq \exp(i s \hat{C}_i) \Psi$. This leaves room for two possibilities. In the first one, only gauge-invariant observables $\hat{O}$ are considered while describing physically-relevant quantities. Such observables must commute with the quantum constraints, so that $\hat{O} = e^{i s \hat{C}_i} \hat{O} e^{-i s \hat{C}_i}$ holds. A prime candidate for this is the quantum Gauss constraint of Lattice Gauge Theories, as constructing observables commuting with it is well understood\(^3\). An alternative to gauge-invariant observables (especially when their construction is difficult, e.g. for the scalar constraint) can be the aforementioned weak implementation, in which one would ask for coherent states that are *stable* in the sense of following a sharply peaked trajectory mapping in classical phase space the associated gauge orbit.

It transpires that stable coherent states are of paramount importance for the approach of weak implementation for complicated constraints as well as for the deparametrization approach, where the constraints get replaced by a usual evolution operator. While proposals for coherent states exist in Quantum Gravity, they have so far never been investigated for their stability. This sets the motivation to apply the previous consistency check via the quantum speed limit to LQG as we will do now.

**Quantum gravity on a lattice.**—We recall the basic setup of Loop Quantum Gravity (LQG) on a cubic lattice, the latter composing the discretized, compact spatial slices $\sigma$ of a 4-dim manifold. Passing from classical field theory to quantum theory with finitely many degrees of freedom (with suitable infra-red regulator) is similar to considering Lattice Gauge Theories, albeit, with a few crucial differences: LQG rests on the insights of Sen, Ashtekar, Barbero et al. \([21, 22]\) that gravity can be understood as SU(2)-gauge theory with canonical pair $E^x_\sigma (x), A^y_\sigma (x)$ for $x \in \sigma$, subject to the standard diffeomorphism and scalar constraints and an additional Gauss constraint. Canonical quantization promotes the kinematical phase space to the Hilbert space $\mathcal{H}$, which has tensor-product structure over edges $e$ of a cubic lattice $\gamma$ with $M$ many vertices in each direction (employing periodic boundary conditions):

\begin{equation}
\mathcal{H} = \bigotimes_e \mathcal{H}_e, \quad \mathcal{H}_e = L_2(\text{SU}(2), d\mu_H),
\end{equation}

where, for each edge, $\mathcal{H}_e$ is the space of square-integrable (with Haar measure $\mu_H$) SU(2)-valued functions. Let $f_e \in \mathcal{H}_e$, then the basic operators defined on $\mathcal{H}_e$ are the *holonomy* operators

\begin{equation}
(\hat{h}^{(j)}_{mn}(e)f_e)(g) = D^{(j)}_{mn}(g)f_e(g),
\end{equation}

with $D^{(j)}_{mn}(g)$ being a Wigner-matrix of a group element $g$ in the $2j + 1$-dim, irreducible representation

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\(^1\) For General Relativity, $C_i$ constitutes out of scalar-, diffeomorphism- and Gauss constraint as $C_i$.

\(^2\) Albeit not custom, it is possible to similarly implement the Gauss constraint classically, at the expense of a more complicated phase-space structure.

\(^3\) Hence, when using the GCS (9) later on, no further corrections appear from the quantum Gauss constraint. E.g., Dirac’s quantization would require to change the coherent states which are not in the kernel of the quantum Gauss constraint. Implementing it strongly amounts to an additional term in the next-to-leading order corrections of any expectation value. For further details see the appendix of [13].
of SU(2), and the gauge-covariant flux operators

\[ \left( \hat{P}^K(e) f_e \right)(g) = -\frac{i\hbar \kappa}{2} \left( R^K f_e \right)(g), \]

(8)

with, following the standard notation of [21], \( \kappa = 8\pi G \) being the gravitational coupling constant, \( \beta > 0 \) being the Immirzi parameter, \( \hbar \) being Planck constant and \( c = 1 \). Finally, the right-invariant vector fields are \( R^K \) and their index \( K \) labels a basis of \( \mathfrak{su}(2) \). With the above structure it is possible to promote a discretization of the constraints to quantum operators, albeit current proposals are plagued by discretization artefacts of the order of the finite lattice spacing \( 1/M \).

*Cosmological coherent states.*—To obtain explicit expressions for the variances in order to apply the QSL to LQG, we rely on the extraction of next-to-leading-order quantum corrections with respect to the “semi-classical” parameter \( \tau \geq 0 \), previously referred to as the quantum tolerance scale. As an input we take the GCS [10] sharply peaked over flat isotropic FLRW Universe on a torus (the classical field configuration \( G \in \text{SL}(2, \mathbb{C}) \)), with their building blocks given by

\[
\psi_G^t(g) := \sum_{j \in \mathbb{N}_+/2} (2j + 1)e^{-t(2(j+1)^2 - 1)/\hbar} \mathcal{D}^{(j)}_{mn}(G g^\dagger).
\]

(9)

Here, we chose a lattice oriented along the coordinate axes of the torus and for each edge going in the same direction we pick the same element \( G_1 = n_j e^{-i z_j n_j^I} \), where \( z = \xi - i \eta \) with \( \xi, \eta \geq 0 \), while \( n_j \) is such that \( \tau_1 = n_j \tau_3 n_j^I \) where \( \tau_1 = -i \sigma_I/2 \) with \( \sigma_I \) being the Pauli matrices. Finally, we define the bare isotropic, flat cosmology coherent state as the tensor product:

\[
\Psi_z^t(\{g\}) := (1)^{-3M^2/2} \prod_{I \in \{1,2,3\}} \prod_{k \in \mathbb{Z}_M} \psi_G^t(g_{kI}),
\]

(10)

where the norm (1) is the same on every edge [13] and \( \mathbb{Z}_M = \{1,2,\ldots,M\} \). These normalized states enjoy many desired properties of GCS (see [11, 13, 23, 24]), in particular, they are sharply peaked for any observable \( \hat{O} \) being a polynomial in the basic operators, in the sense that

\[
\left( \Delta_{\psi_z^t} \hat{O} \right) / \langle \hat{O} \rangle_{\psi_z^t} = \mathcal{O}(t).
\]

(11)

Moreover, if \( z_1 \neq z_2 \), for \( t \to 0 \) we enjoy exponential decay of the overlap, i.e. \( \langle \psi_1^t | \psi_2^t \rangle = \mathcal{O}(e^{-1/t}) \).

Here, we are also able to pose quantitative statements about the variance of the Thiemann regularization of the Euclidean part of the scalar constraint with the lapse function fixed at unity

\[
\hat{C}_E := C_E(\{\hat{h}(e)\}_{e \in \gamma}, \{\hat{P}(e)\}_{e \in \gamma})
\]

(12)

which is a complicated expression in terms of fluxes and holonomy operators, see [25] for a common regularization. Primarily, this is possible due to a very recent and highly non-trivial result concerning the expectation values of monomials in the operators (7) and (8) on a single edge, \( \langle \hat{h}_{ab} R^K_{1...K_N} \rangle_{\psi_G^t} \), where next-to-leading order is included [24]. We postpone exact, compact formulas to the last section of Supplementary Material. Building on this, a feasibly implementable algorithm which allows to compute expectation values in \( \Psi_z^t \) including the next-to-leading order corrections for arbitrary (polynomial) operators has very recently been introduced and tested [25]. A detailed description of the algorithm can be found in Sec. 4 of [25].

With this algorithm it is possible to compute the expectation values of nested operators. In particular, the expectation value of \( \hat{C}_E \) has been reported in [25]. Here, we use this algorithm to compute such a complex object as \( \Delta_{\psi_z^t} \hat{C}_E \). Our implementation [26] applied to \( \hat{C}_E \), defined in (12) with the lapse function fixed at unity, gives the result

\[
\left( \Delta_{\psi_z^t} \hat{C}_E \right)^2 = \frac{3M^3\hbar}{27} \eta \kappa \sin(\xi)^2 \times (17 + 256\eta^2 + (256\eta^2 - 17)\cos(2\xi)) + \mathcal{O}(t).
\]

(13)

We focus on \( \hat{C}_E \) for the sake of simplicity, as our goal is to convey a general message about usefulness of QSL in LQG. We also admit that further optimization of the algorithm’s implementation is necessary to compute the variance of the full scalar constraint.

*QSL for Euclidean LQG.*—Now, all the tools are at hand to apply the QSL to Quantum Gravity. As the cosmological coherent states are the prime proposal for the semi-classical limit of LQG, we will test their validity with respect to the Euclidean part of the scalar constraint. To this end we fix the family of states present in Eq. (1) to be \( \Phi_s = \Psi_z^t(\{s\}) \), and let the generator be \( \hat{H} = \hat{C}_E \). Note that while numerous applications of QSL are known in the literature [27–35], this tool has never before been introduced to Quantum Gravity.

In fact, for the family of coherent states (9), whose overlap is known to decrease exponentially fast with \( t \to 0 \) if \( z \neq z' \), the necessary condition of the QSL can easily be violated. Given a trajectory \( z(\tau) \), such that \( \lim_{\tau \to 0} |z(0) - z(\tau)| \neq 0 \) for all \( \tau > 0 \), we get \( \lim_{\tau \to 0} A(0, \tau) = \pi/2 \). Moreover, since \( \lim_{\tau \to 0} \Delta_{\psi_z^t(\tau)} \hat{C}_E < \infty \), then for \( \tau < \pi \) the negative term on the right hand side of (4) is negligible. Consequently, \( W(\tau, \tau) \approx 0 \), indicating that the state almost immediately leaves the desired trajectory, becoming approximately orthogonal to the state \( \Phi_s \). Hence, the family of such coherent states is not stable.

While above we have shown that there is a serious risk of violation of (4), we now prove that, at least for very short times, this risk is mitigated by natural choice for \( z(\tau) \) which parametrizes the semi-classical family. To this end, we write

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4 An implementation of the algorithm as a free-to-use Mathematica code is available at [26].
\[ \eta(\tau) = 2tp(\tau)/(M^2\hbar\kappa) \] and \[ \xi(\tau) = c(\tau)/M, \]
where \( (p(\tau), c(\tau)) \) is a trajectory on the 1-dimensional phase space \( \{p,c\} = \kappa\beta/6 \), rendered by the generator \( C_E(p,c) = (6M^2/\kappa)\sqrt{\eta}\sin(c/M)^2 \) being a leading order of the average of \( \hat{C}_E \) evaluated on flat isotropic cosmology.

Looking at very short times \( \tau << t \) we get (see the second section of Supplementary Material)

\[
(\partial_\tau A(0,\tau)|_{\tau=0})^2 = \frac{3}{2^7 \eta \kappa h} \sin(\xi)^2 \times
\]
\[
(16 + 256\eta^2 + (256\eta^2 - 16)\cos(2\xi)) + O(t)
\]
\[
= \left( \Delta_{\psi_{z(0)}}\hat{C}_E \right)^2 / h^2 - \frac{3}{2^6 \eta \kappa h} \sin(\xi)^4 + o(t).
\]

Since by definition \( A(0,0) = 0 \) while Eq. (14) implies \( \Delta_{\psi_{z(0)}}\hat{C}_E / h \geq \partial_\tau A(0,\tau)|_{\tau=0} \), we fascinatingly observe that the right hand side of (4) is negative for very short times. Hence, evolution due to \( \hat{C}_E \) obeys the necessary condition for sharp peakedness, i.e. \( W(\tau,\tau) \approx 1 \) is not automatically ruled out. In other words, the time to transfer from one coherent state to the next one is sufficient, at least for \( \tau << t \).

However, albeit the QSL necessity check is passed, we point out that stability of the GCS family is not automatically proven.

Further applications— While the main message of this letter is the consistency check for semi-classical families, the potential for applying QSL does not end here. E.g., the QSL in Eq. (2) can be used to make further conclusions for sharply peaked states with \( t << \tau \), which are peaked on huge fluxes. By the latter we mean that \( p(0) \) and \( c(0) \) scale with \( t \) in such a way that \( \xi(0), \eta(0) \neq 0 \) are both independent of \( t \). The regime \( t << 1 \) then corresponds to the semi-classical limit with huge fluxes and small relative variances. Now, there exists \( \tau \) making the left hand side of (2) arbitrary small. Hence, as already explained, the stability directly depends on \( W(0,\tau) \), and due to orthogonality of GCS there is a risk that \( W(0,\tau) \approx 0 \).

However, for the toy model driven by \( \hat{C}_E \) we show at the end of the second section of Supplementary Material that \( z(\tau) = z(\sqrt{\tau}t) \), with \( z \) otherwise independent of \( t \). This implies that effectively the flow parameter rescales with \( \sqrt{t} \) and the evolution freezes in the limit \( t \to 0 \). Thus, for fixed \( \tau \) we get both \( W(0,\tau) \to 1 \) and \( W(\tau,\tau) \to 1 \). For short times we do have a family with infinitely huge fluxes, which in the limit \( t \to 0 \) is stable. Further details on when such a rescaling occurs for other operators, as well as its implications for finite-time stability of generalized coherent states can be found in [12, 36].

Conclusions.— In recent years, the conceptual side of LQG has developed a lot. However, most if not virtually all practical questions in the field require to perform computations of drastically huge complexity. As a part of efforts directed towards this objective of making complex objects in LQG computable, we have recently established an algorithm [25] offering a severe simplification of the computations with the coherent states in LQG. In this letter we employ these complex techniques to explore new physics within LQG. In particular, our methodology, for the first time, allows to probe the evolution of the coherent states. To this end, we brought quantum speed limit, usually discussed within standard quantum theory, and after slight adjustments applied it to the evolution of the coherent states. However, the inequality offered by the quantum speed limit is informative only if one is able to compute or measure the variance of the generator of the evolution. Since we show that with the techniques developed, it is now possible to access such a complex quantity, a thorough discussion of the physics underlying the evolution of the coherent states is possible. The physical problem at hand, which is the main theme of this letter is the following. We know that a coherent state will evolve to a state which is not exactly coherent. On the other hand, classical evolution on the phase space, by construction, would mean that in the quantum description we stay in the family of coherent states, just their parameters evolve appropriately. Then, what is the deviation between such quantum and classical evolution? With this letter we provide a first attempt towards answering such a question, which from a computational perspective is extremely difficult. Interestingly our finding is that the deviation between both states, at least for short times, is infinitesimal. This conclusion, \( a \) \( p \) \( r \) \( i \) \( o \) \( r \) \( e \) \( r \) \( a \) \( p \) \( r \) \( i \) \( o \) \( r \) \( e \) is neither obvious nor even expected, due to the fact that the overlap between the coherent states decays exponentially. With more computational effort it will be interesting to extend these considerations to the various quantum regularizations of the Lorentzian part.

We have seen that the QSL can serve as a powerful tool to rule out candidates for semi-classical families and indicate those for which stability is still possible. The findings stemming from our methodology also raise the hope for full LQG to align with previous investigations in LQC, where the only genuinely quantum investigations happened so far [1]. In particular, by minimally coupling isotropic gravity to a free scalar field, which could serve as a clock, the authors of [3–5] deparametrized the system to obtain an evolution operator on a 1-particle Hilbert space. In this way, they were maximally close to the setting described above and, after rigorous analytical and numerical work, it transpired that standard Gaussian states remain stable under such evolution [3–6]. In the language of the current manuscript, these states would obviously pass the QSL test. These results nurtured the conjecture that the same might be true for more elaborate models. However, due to a tremendous complexity, this conjecture has not only never been tested, but also feasible tools to probe it were basically missing. Our work opens up the latter possibility. It also allows one to follow up on earlier investigations such as [37], enabling tests of minisuperspace models. A first con-
sistency check of the assumed stability in these models can be an attractive starting point for future work.

In fact, pinpointing a suitable candidate for semiclассical models will create a crucial fundamental for the years to come: it is predicted that LQG will soon start to enter the realm of quantum simulations [38, 39]. Several platforms, for example those utilizing linear optics [40], adiabatic (quantum) computation [41], Nuclear Magnetic Resonance techniques [42] or superconducting qubits [43] are considered. Hence, before the era of quantum simulations in LQG, preparatory work - like in this letter - aiming at making complex objects in LQG computable (by both classical and quantum machines) shall be in focus.

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A GENERALIZED DERIVATION OF QUANTUM SPEED LIMIT

Even though derivation of quantum speed limit can easily be found in the literature (consult, for example, a recent pedagogical review [44]), we shall repeat it here because in the main text we use a non-standard variant of the QSL. Perhaps, the inequality to be presented, even though conceptually as simple as the typical QSL, was never in the focus due to a more general assumption we shall make concerning one of the quantum states in question.

One starts with the Heisenberg uncertainty relation for two operators \( \hat{A} \) and \( \hat{H} \)
\[
\Delta_s \hat{A} \Delta_s \hat{H} \geq \frac{1}{2} |\langle \Phi(s), [\hat{A}, \hat{H}] \Phi(s) \rangle|, \tag{15}
\]
one of which being the Hamiltonian. We simplify the notation, so that \( \Delta_s \) denotes \( \Delta_{\Phi(s)} \), where \( \Phi(s) = e^{-is\hat{H}/\hbar} \Phi_0 \) is the evolved initial state \( \Phi(0) = \Phi_0 \).

Clearly, \( \Delta_s \hat{H} = \Delta_0 \hat{H} \), so that it does not depend on \( s \).

Since \( \partial_s \Phi(s) = -\frac{i}{\hbar} \hat{H} \Phi(s) \) it follows that
\[
\tau \cdot \Delta_0 \hat{H} / \hbar \geq \left| \arccos \left( \sqrt{\langle \Phi_0, \hat{A} \Phi_0 \rangle} \right) - \arccos \left( \sqrt{\langle \Phi(\tau), \hat{A} \Phi(\tau) \rangle} \right) \right|.
\]

This is a more general variant of QSL. As already mentioned, the standard QSL follows if \( \Upsilon = \Phi_0 \), so that \( \langle \Phi_0, \hat{A} \Phi_0 \rangle = 1 \), and consequently the first term inside the absolute value vanishes. Furthermore, in the special case in which the final state \( \Phi(\tau) \) is orthogonal to the initial state \( \Phi_0 \), the right hand side of (17) equals \( \pi/2 \). In that case, the transition time between the two orthogonal states is at least [15]
\[
\tau \geq \frac{\hbar \pi}{2 \Delta_0 \hat{H}}. \tag{18}
\]

Finally, let us observe that our QSL in the main text follows from (17) if we set \( \Upsilon = \Phi_\tau \). While \( \Phi_0 \) remains to play a role of the initial state, \( \Upsilon \) is neither an initial nor final state of the evolution, i.e. neither \( \Phi_0 \) nor \( \Phi(\tau) \). Instead, \( \Upsilon \) represents some target state \( \Phi_\tau \), i.e. the state we wish to obtain during the evolution.

\[
\Delta_s \hat{A} \Delta_0 \hat{H} \geq \frac{\hbar}{2} |\partial_s \langle \Phi(s), \hat{A} \Phi(s) \rangle|. \tag{16}
\]

From now on, everything depends on the choice of \( \hat{A} \). In the standard treatment \( \hat{A} \) is taken to be the projector on the initial state \( \Phi_0 \). In such a way it is possible to track the evolution from the initial to the final state. In our treatment we generalize this approach and let \( \hat{A} \) be the projector on an arbitrary state \( \Upsilon \). Still, we have that \( \hat{A}^2 = \hat{A} \). Therefore, we can easily rewrite the previous inequality into
\[
\Delta_0 \hat{H} \geq \frac{\hbar}{2} \frac{|\partial_s \langle \Phi(s), \hat{A} \Phi(s) \rangle|}{\sqrt{\langle \Phi(s), \hat{A} \Phi(s) \rangle (1 - \langle \Phi(s), \hat{A} \Phi(s) \rangle)}}.
\]

Since
\[
\frac{1}{2} \int_0^\tau ds \frac{\partial_s f(s)}{\sqrt{f(s)(1 - f(s))}} = \arccos(\sqrt{f(0)}) - \arccos(\sqrt{f(\tau)}),
\]
we can integrate both sides of the above inequality with respect to \( s \) from 0 to \( \tau \), enter with the integral inside the absolute value on the right hand side, and obtain
\[
\tau \Delta_0 \hat{H} / \hbar \geq \left| \arccos \left( \sqrt{\langle \Phi_0, \hat{A} \Phi_0 \rangle} \right) - \arccos \left( \sqrt{\langle \Phi(\tau), \hat{A} \Phi(\tau) \rangle} \right) \right|. \tag{17}
\]

DERIVATION OF EQ. (14) IN THE MAIN TEXT AND FURTHER DETAILS

We first recall that the classical Euclidean part of the scalar constraint is
\[
C_E(p, c) = \frac{6 M^2}{\kappa} \sqrt{p} \sin(c/M)^2, \tag{19}
\]
while the Poisson bracket on the reduced phase space reads
\[
\{p, c\} = \kappa \beta / 6. \tag{20}
\]

Using the Hamilton’s equations of motion we can thus expand the solution for short times:
\[
p(\tau) \approx p(0) + \tau \{C_E, p\} + \frac{\tau^2}{2} \{C_E, \{C_E, p\}\} = \quad (21)
\[
= p - M \beta \sqrt{p} \sin(2c/M) \tau + \beta^2 1 - \cos(2c/M) \frac{\tau^2}{4M},
\]
with \( p \equiv p(0) \), and similarly for \( c \). Translating these solutions to the variables of the GCS

\[
\eta = \frac{2t}{M^2 \hbar \kappa \beta} p, \quad \xi = c/M, \tag{22}
\]

leads to its respective expansion for short times

\[
\eta(\tau) \approx \eta - \sqrt{\frac{3\beta \tau^2}{\kappa \hbar}} \sin(2\xi) + \beta \tau^2 \frac{1 - \cos(2\xi)}{2\kappa \hbar}, \tag{23}
\]

In the next step we take \( \arccos(\cdot) \) in order to arrive at an approximate form of \( A(0, \tau) \). Lastly, we take the limit \( t \to 0 \) to get the leading order expansion:

\[
A(0, \tau) = A(0, 0) + \tau (\partial_\tau A(0, \tau))|_{\tau=0} + O(\tau^2), \tag{27}
\]

\[
A(0, 0) = 0, \tag{28}
\]

\[
(\partial_\tau A(0, \tau))|_{\tau=0}^2 = \frac{3}{2} M^3 \beta \sin(\xi)^2 \tau^2 (16 + 256\eta^2 + (16 + 256\eta^2) \cos(2\xi)) + O(t). \tag{29}
\]

Note that \( W(0, \tau) \) simply follows from the definition of the overlap.

We can also use Eqs. (23) and (24) to re-obtain the equations of motion written in terms of \( \eta \) and \( \xi \):

\[
\frac{\partial \eta}{\partial \tau} = -\frac{\sqrt{2\beta}}{\sqrt{\kappa \hbar}} \sin(2\xi) \tag{30}
\]

\[
\frac{\partial \xi}{\partial \tau} = \frac{\sqrt{\beta}}{2\eta \kappa \hbar / t} \sin(\xi)^2 \tag{32}
\]

While making a substitution \( \tilde{t} = \tau \sqrt{t} \), we can see that the equations become independent of \( t \). Given the initial conditions \( \eta(0), \xi(0) \) we therefore get a unique solution \( \eta(\sqrt{t}), \xi(\sqrt{t}) \). Consequently, we find that in the limit \( t \to 0 \) the evolution freezes.

## EXPECTATION VALUES

Here we collect the expectation values mentioned in the main text, which are a vital ingredient of the algorithm presented in Sec. 4 in [25]. Working in the spherical basis \( \tau^K \) with \( K_1, ..., K_N \in \{ \pm 1 \} \), we have [24]:

\[
\langle \hat{r}^{(j)}_{a} R^{K_1}_{1} ... R^{K_N}_{N} \rangle_{\psi^{(t)}} = \langle 1 \rangle \left( \frac{i \eta}{t} \right)^N D^{(1)}_{-K_1-S_1}(n_1)...D^{(1)}_{-K_N-S_N}(n_N) \sum_{c=-j}^{j} \left( \eta^{S_1} ... \delta_{aa'}^{S_N} e^{-i \gamma^{a}_j t} \right)
\]

\[
+ \frac{t}{2\eta} \left[ \delta_{aa'} \delta_0^{S_1} ... \delta_0^{S_N} N \left( \frac{N+1}{2\eta} - \coth(\eta) \right) + i \sum_{A=1}^{N} \delta_0^{S_A} ... \delta_0^{S_B} ... \delta_0^{S_N} \left( 1 - s_A \tanh \left( \frac{N}{2} \right) \right) D^{(j)}_{-s_A-S_L}(n_{A}) \right] \left( e^{-i \gamma^{b}_j t} e^{-i \chi^{b}_j t} D^{(j)}_{cb}(n_{B}) \right) + O(t^2),
\]

As in the main text, by \( D^{(j)}_{mn} \) we denote Wigner matrices in the \( j \)th representation. Moreover, the overlap reads

\[
\langle 1 \rangle := \langle \hat{r}^{(0)}_{00} \rangle_{\psi^{(t)}} = \sqrt{\frac{\pi 2\eta e^{\eta^2/4}}{t^3 \sinh(\eta)}} e^{t^2/4}. \tag{35}
\]