Null-null components of the generalized Einstein tensor for Lovelock models

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For spherical symmetry, we provide expressions for the radial null-null components of the generalized Einstein tensor $E_{ab}$ for Lovelock models for diagonal $E_{ab}$ in terms of the metric and of the radial null-null components of the Ricci tensor. We show they can be usefully employed for example in obtaining the Birkhoff-like theorem for these models.

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I. INTRODUCTION

Calculations in spherically symmetric spacetimes are far easier if the metric of the spacetime under consideration has time-time component and $r-r$ component ($r$ is areal radius) in the form $g_{tt}g_{rr} = -1$, like the Schwarzschild solution to Einstein’s equations. When one deals with some metric theory of gravity, and starts considering first the case with as much as symmetry and simplicity as possible, one would like then for example to know whether the metric of the spherically symmetric vacuum solutions of the theory is necessarily of this form or not. A general condition for a static spherically symmetric metric to have this form has been given in [1], in terms of the vanishing of the radial null-null components of the Ricci tensor. This condition can equivalently be thought, of course, as the vanishing of the radial null-null components of the Einstein tensor.

For spherically symmetric vacuum solutions to Einstein’s equations, one knows thus that the (static) metric does have the form above; this can also be viewed as a manifestation of Birkhoff’s theorem [2, 3] at work. When one goes, however, to general theories of gravity, what drives the motion is no longer the Einstein tensor but the generalized Einstein tensor. For vacuum solutions, what one knows is thus the vanishing of the radial null-null components of the generalized Einstein tensor (and of the tensor itself, of course), not of these same components for Einstein or Ricci. In this context, the point at issue apparently is to know when the vanishing of those components for generalized Einstein means their vanishing for Ricci.

Here, we investigate this for Lovelock models. From the Birkhoff-like theorem for such models [4–6], one already knows that the metric for spherically symmetric vacuum solutions must have the form above, and thus that for these models the vanishing of the radial null-null components for generalized Einstein must imply their vanishing for Ricci. One interesting thing, however, could be to read this directly from the expression of these components of the generalized Einstein tensor. Our aim is to provide a multi-purpose expression for these components in terms of the metric, or, also, in terms of the same components for Ricci. For vacuum in particular, from the vanishing of this expression one should directly read the vanishing of the radial null-null components of Ricci.

II. STATEMENT OF THE QUESTION

In $D$-dimensional spacetime, we consider gravitational Lagrangians $L$ with $L = L(g^{ab}, R_{abcd})$, being $g^{ab}$ the metric and $R_{abcd}$ the Riemann tensor (latin labels span all the $D$ coordinates, $\{x^a\}$), that is with general dependence on metric and Riemann tensor but with no dependence on derivatives of the latter. The equations of motion for the field $g^{ab}$ we get for an action $I$ with variation $\delta I = \int d^Dx \, \delta(\sqrt{-g}L) - \frac{1}{2} \int d^Dx \, \sqrt{-g} \, T_{ab} \, \delta g^{ab}$, are

\begin{equation}
2 \, E_{ab} = T_{ab},
\end{equation}

where $T_{ab}$ is the energy-momentum tensor and

\begin{equation}
E_{ab} = U_{ab} - 2 \, \nabla^i \nabla^j P_{aijb}
\end{equation}

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is the generalized Einstein tensor, with \( U_{ab} = \frac{1}{2g} \frac{\partial (\sqrt{g} L)}{\partial g^{ab}} \) and \( P_{a}^{ijb} = \frac{\partial L}{\partial R_{ij}^{ab}} \) (see e.g. [7]). \( U_{ab} \) and \( P_{abcd} \) have the same symmetries in their indices as \( g_{ab} \) and \( R_{abcd} \) respectively.

Within this set of Lagrangians, we consider the subset of Lovelock Lagrangians (LL) \([8–10]\). These Lagrangians can be defined as

\[
L_{m} = \frac{1}{2m} \delta_{a_{1}a_{2}a_{3}a_{4}}^{d_{1}d_{2}d_{3}d_{4}} R_{c_{1}c_{2}}^{a_{1}b_{1}} R_{c_{2}d_{2}}^{a_{2}b_{2}} \cdots R_{c_{m}d_{m}}^{a_{m}b_{m}} = \frac{1}{m} P_{ab}^{cd} R_{cd}^{ab},
\]

(3)

in terms of the order parameter \( m = 1, 2, \ldots \). Here, \( \delta_{a_{1}a_{2}a_{3}a_{4}}^{d_{1}d_{2}d_{3}d_{4}} \) are D-dimensional “permutation tensors” \([11]\) of rank 2m, and the equality comes either differentiating directly the expression for \( L(m) \) or making use of Euler’s theorem, being \( L(m) \) a homogeneous function of degree \( m \) in \( R_{ab} \). \( L(1) = \frac{1}{2} \delta_{ab} R_{ab} = R \) is the Einstein-Hilbert Lagrangian of general relativity, given by the scalar curvature \( R \).

\( L(2) = \frac{1}{4} \delta_{a_{1}a_{2}a_{3}a_{4}}^{b_{1}b_{2}b_{3}b_{4}} R_{c_{1}c_{3}}^{a_{1}b_{1}} R_{c_{2}c_{4}}^{a_{2}b_{2}} \) or making use of Euler’s theorem, being \( L(m) \) a homogeneous function of degree \( m \) in \( R_{ab} \).

Denoting \( U_{ab(m)} \), \( P_{a}^{ijb} \), and \( E_{ab(m)} \) the tensors entering eq. \( (2) \) for general LL of order \( m \), the equations of motion \( (1) \) read \( 2 E_{ab(m)} = T_{ab} \) with

\[
E_{ab(m)} = U_{ab(m)} - 2 \nabla^{i} \nabla^{j} P_{aijb(m)} = U_{ab(m)} = -\frac{1}{2} g_{ab} L_{m} + \frac{\partial L_{m}}{\partial g^{ab}}
\]

\[
= -\frac{1}{2} g_{ab} L_{m} + \frac{\partial L_{m}}{\partial R_{ij}^{kl}} \frac{\partial R_{ij}^{kl}}{\partial g^{ab}} = -\frac{1}{2} g_{ab} L_{m} + P_{ij}^{kl(m)} \frac{1}{2} \left( \delta_{ab} R_{bij} + \delta_{ij} R_{ab} \right)
\]

\[
= -\frac{1}{2} g_{ab} L_{m} + \frac{1}{2} \left( P_{kj}^{ij(m)} R_{kij} + P_{ij}^{kj(m)} R_{kij} \right) = -\frac{1}{2} g_{ab} L_{m} + P_{ij}^{kl(m)} R_{kij},
\]

(4)

where the last equality follows from turning out the two terms in round brackets of l.h.s. equal \([12]\). These expressions are only second order in the derivatives of the metric. Essential to this, is the second equality. It is obtained thanks to the crucial property of these Lagrangians of having \( P_{ab(m)}^{cd} \) with zero divergence on each of the indices:

\[
\nabla_{c} P_{abcd(m)} = 0,
\]

(5)

and the same for any other index. For \( m = 1 \), we get

\[
E_{ab(1)} = -\frac{1}{2} g_{ab} L_{(1)} + P_{ij}^{kl(1)} R_{kij}^{ab} = -\frac{1}{2} g_{ab} R + \frac{1}{2} \delta_{ij} R_{kij}^{ab} = -\frac{1}{2} g_{ab} R + R_{ab},
\]

(6)

so that \( E_{ab(1)} \) is the Einstein tensor, and the equations of motion \( 2 E_{ab(1)} = T_{ab} \) are the Einstein equations (in Planck units, and without a \( 1/16\pi \) factor in the l.h.s.). For the Gauss-Bonnet case, we get

\[
E_{ab(2)} = -\frac{1}{2} g_{ab} L_{(2)} + P_{ij}^{kl(2)} R_{kij}^{ab} = -\frac{1}{2} g_{ab} R + \frac{1}{2} \delta_{ij} R_{kij}^{ab} = -\frac{1}{2} g_{ab} R + 2 R_{ab},
\]

(7)

The Lagrangians are chosen with cosmological constant \( \Lambda = 0 \). The effects of any additional term \( \Lambda g_{ab} \) with \( \Lambda \neq 0 \) in the Lagrangian, can conveniently be described leaving the Lagrangian as it is, i.e. without cosmological term, and introducing, among the sources, a cosmological ideal fluid with stress-energy tensor \( (T_{\Lambda})_{ab} = -2 \Lambda g_{ab} \).

Considering any null field \( n^{a} \) in vacuum (vac) or in vacuum with cosmological constant (cosmovac), we have \( T_{ab} n^{a} n^{b} = 0 \), and thus \( E_{ab(m)} n^{a} n^{b} = 0 \). Considering, in particular, some piece of spacetime with spherical symmetry with \( l^{a} \) denoting a generic radial null vector field, this means \( E_{ab(m)} l^{a} l^{b} = 0 \).

Taking \( m = 1 \), from \( (6) \) we see that \( E_{ab(1)} l^{a} l^{b} = 0 \) is manifestly equivalent to \( R_{ab} l^{a} l^{b} = 0 \). In the \( m = 2 \) case, and even more so for \( m > 2 \), the dependence on \( R_{abcd} \) in \( E_{ab(m)} \) is more involved, so that the equivalence above is not manifest, if present at all (but we know this must actually be the case from Birkhoff-like theorem for Lovelock models). Our aim is to try to work out a general expression for \( E_{ab(m)} l^{a} l^{b} \) in terms of the components of the metric for Lovelock models for diagonal \( E_{ab(m)} \), somehow generalizing what is done in certain derivations (in [2], for instance) of Birkhoff’s theorem in general relativity. The idea/hope is that this expression can be put also in a form such that \( E_{ab(m)} l^{a} l^{b} = 0 \) turns out to be manifestly equivalent to \( R_{ab} l^{a} l^{b} = 0 \), and that, among other possible uses, it can be exploited to derive the Birkhoff-like theorem for Lovelock models.
III. CALCULATION

We are going to provide an expression for $E_{ab(m)}$ $l^a l^b$ in terms of the metric for spherically symmetric configurations. This expression turns out to be the same in the static and non-static cases, provided $E_{ab(m)}$ is diagonal in spherical coordinates (for a static configuration, this is the case already; for a non-static configuration, the meaning of this is to require $E_{rr(m)} = E_{rt(m)} = 0$). Let us consider first the static case. The general metric for a spherically symmetric, static spacetime can be written as

$$ds^2 = -A\ dt^2 + B\ dr^2 + r^2\ d\Omega^2 = -A\ dt^2 + B\ dr^2 + r^2\ \sum_A h_{AA}(x)\ (dx^A)^2 \quad (8)$$

where $A = A(r)$ and $B = B(r)$, being $r$ the areal radius, and where the $x^A$, $A = 3, 4, ..., D$, are chosen to be the usual angular coordinates parametrising the $(D - 2)$-dimensional manifold orthogonal to $(t, r)$ in diagonal form $(d\Omega^2 = d\alpha^2_1 + \sin^2\alpha_1(d\alpha^2_2 + \sin^2\alpha_2(d\alpha^2_3 + \sin^2\alpha_3(...)))$, with $(\alpha_1, ..., \alpha_{D-2}) = x^A$ and the sum over $A$ is explicitly indicated. The radial null vector field we consider is rescaled to have components

$$l^t = \sqrt{B}; \ l^r = \pm \sqrt{A}. \quad (9)$$

The general radial null vector field $n^a$ is expressed in terms of $l^a$ as $n^a = \mu(x^a)\ l^a$, with $\mu$ a function. What we have to do, is to find an expression for $E_{ab(m)}\ l^a l^b$ in terms of $r$, $A$ and $B$.

From (11) we get

$$E_{ab(m)}\ l^a l^b = P_{kl(m)}^{ij} R_{aij}^{\ kl} l^a l^b, \quad (10)$$

and, from the definition of $P_{ab(m)}^{cd}$,

$$E_{ab(m)}\ l^a l^b = \frac{m}{2m} \delta_{kl}^{ij} d_{m, d_m} c_{m, d_m} r_{c_{2, d_2}} ... r_{c_{m, d_m}} R_{aij}^{\ kl} l^a l^b. \quad (11)$$

Many of the components of the Riemann tensor are vanishing. Writing explicitly only the non-vanishing terms in (11), we are left with

$$E_{ab(m)}\ l^a l^b = \frac{m}{2m} \left\{ \sum_{a_2, b_2, ..., a_m, b_m} R_{a_2 b_2}^{\ a_2 b_2} ... R_{a_m b_m}^{\ a_m b_m} R_{trtr}^{\ trtr} l^t l^t \right. + 2^{m+1} \sum_{a_2, a_3, b_2, ..., a_m, b_m} R_{a_2 b_2}^{\ a_2 b_2} ... R_{a_m b_m}^{\ a_m b_m} R_{trt}^{\ trt} l^t l^t \right. + 2^{m+1} \sum_{a_2, a_3, b_2, ..., a_m, b_m} R_{a_2 b_2}^{\ a_2 b_2} ... R_{a_m b_m}^{\ a_m b_m} R_{ttt}^{\ ttt} l^t l^t \right. + 2 \sum_{a_2, a_3, b_2, ..., a_m, b_m} R_{a_2 b_2}^{\ a_2 b_2} ... R_{a_m b_m}^{\ a_m b_m} R_{ttt}^{\ ttt} l^t l^t \right. + \sum_{a_2, a_3, b_2, ..., a_m, b_m} R_{a_2 b_2}^{\ a_2 b_2} ... R_{a_m b_m}^{\ a_m b_m} R_{ttr}^{\ ttr} l^t l^t \right. + 2 \sum_{a_2, a_3, b_2, ..., a_m, b_m} R_{a_2 b_2}^{\ a_2 b_2} ... R_{a_m b_m}^{\ a_m b_m} R_{rtr}^{\ rtr} l^t l^t \right. + \sum_{a_2, a_3, b_2, ..., a_m, b_m} R_{a_2 b_2}^{\ a_2 b_2} ... R_{a_m b_m}^{\ a_m b_m} R_{rtr}^{\ rtr} l^t l^t \right. + \sum_{a_2, a_3, b_2, ..., a_m, b_m} R_{a_2 b_2}^{\ a_2 b_2} ... R_{a_m b_m}^{\ a_m b_m} R_{rtr}^{\ rtr} l^t l^t \right. \right\} \quad (12)$$

for $m \geq 2$, and
\[ E_{ab(1)} \ l^a l^b = \frac{1}{2} \left\{ 2 \delta_{r} R_{rtt} \ l^t l^t + 2 \sum_{\alpha} \delta_{\alpha} R_{\alpha \ tot} \ l^t l^t \right\} + \frac{1}{2} \left\{ t \leftrightarrow r \right\} \]
\[ = R_{rtt} \ l^t l^t + \sum_{\alpha} R_{\alpha \ tot} \ l^t l^t + R_{r tr} \ l^t l^t + \sum_{\alpha} R_{\alpha \ r ar} \ l^t l^t. \]  
(13)

In expressions (12,13) and in what follows, greek indices denote specific angular components and no convention of sum on repeated indices is assumed for them. The factors in front of sum symbols account for the different combinations of indices producing always a same term. The ' \( \leftrightarrow \) ' symbol in the sums means that the sums are taken with all indices different.

Now, the terms with \( R_{rtt} \) and \( R_{r tr} \) cancel each other as well as the terms with \( R_{\alpha \ r gb} \) and \( R_{\alpha \ gb} \), since \( R_{rtt} \ l^t l^t = g^{tt} R_{rtt} \ l^t l^t = -g^{tt} R_{r tr} \ l^t l^t = -R_{r tr} \ l^t l^t \) and \( R_{\alpha \ r gb} \ l^t l^t = g^{tt} R_{\alpha \ r gb} \ l^t l^t = -g^{tt} R_{\alpha \ r gb} \ l^t l^t = -R_{\alpha \ r gb} \ l^t l^t \) respectively, due to (11), which gives

\[ g^{tt} \ l^t l^t + g^{tt} \ l^t l^t = 0. \]  
(14)

The expressions (12,13) then become

\[ E_{ab(m)} \ l^a l^b = m \sum_{\alpha_1,\alpha_2,\ldots,\alpha_m, \beta_1, \beta_2, \ldots, \beta_m} R_{\alpha_1 \beta_1 \ldots \alpha_m \beta_m} \left( R_{\alpha \ tot} \ l^t l^t + R_{\alpha \ r ar} \ l^t l^t \right), \]  
(15)

with the understanding that the \( m = 1 \) case is covered setting to 1 the factors ahead of round brackets. We get

\[ E_{ab(m)} \ l^a l^b = m \sum_{\alpha_1,\alpha_2,\ldots,\alpha_m, \beta_1, \beta_2, \ldots, \beta_m} R_{\alpha_1 \beta_1 \ldots \alpha_m \beta_m} \left( \frac{1}{2rB} \right)^m (AB)^{m-1} \left( \frac{1}{2rB} \right)^m (AB)^{m-1} \]
\[ = m \left( D - 2 \right) \left( D - 3 \right) \ldots \left( D - 2m + 2 \right) \left( D - 2m + 1 \right) \left( D - 2m \right) \left( \frac{1}{r^2} \left( 1 - \frac{1}{B} \right) \right)^{m-1} \left( \frac{1}{2rB} \right)^m (AB)^{m-1} \]
\[ = m \left( D - 2 \right)! \left( \frac{1}{r^2} \left( 1 - \frac{1}{B} \right) \right)^{m-1} \left( \frac{1}{2rB} \right)^m (AB)^{m-1} \]
\[ = m \left( 2m - 1 \right)! \left( \frac{D - 2}{2m - 1} \right) \left( \frac{1}{r^2} \left( 1 - \frac{1}{B} \right) \right)^{m-1} \left( \frac{1}{2rB} \right)^m (AB)^{m-1}. \]  
(16)

where use has been made of the fact that the components \( R_{\alpha \ tot} \), \( R_{\alpha \ r ar} \) and \( R_{\alpha \ beta} \) do not depend on \( \alpha \) and \( \beta \) \((\alpha \neq \beta)\), with their expressions given by \( R_{\alpha \ tot} = \frac{1}{r} A'_{2B} \), \( R_{\alpha \ r ar} = \frac{1}{r} B' \) and \( R_{\alpha \ beta} = \frac{1}{r} \left( 1 - \frac{1}{B} \right) \), with the prime denoting differentiation with respect to \( r \).

This is the expression for \( E_{ab(m)} \ l^a l^b \) we obtain assuming the configuration is static. For vac or cosmovac, from \( E_{ab(m)} \ l^a l^b = 0 \) we get \((a) (AB)' = 0 \) or \((b) B = 1 \). Case \((b)\) corresponds to Minkowski spacetime. This can be seen from the fact that \( B(r) \) is related to the (generalised) Misner-Sharp mass \( M \) in inside \( r \) through \( M = C(m) r^{D-1-2m}(1 - \frac{r}{B})^m \) [14,16], with \( C(m) \) a constant, and thus \( B = 1 \) means \( M = 0 \) inside \( r \). Case \((a)\) gives \( AB \) = \( const \), which means, through rescaling of the time coordinate, \( AB = 1 \), encompassing both cases.

In the non-static case, i.e. assuming \( A = A(t, r) \) and \( B = B(t, r) \) in (8), the calculation of \( E_{ab(m)} l^a l^b \) gets modified since some of the expressions for the components of Riemann tensor change with respect to the static case (due to \( A \neq 0 \) and \( B \neq 0 \), where the dot denotes differentiation with respect to \( t \)), producing also some components which are no longer vanishing. These latter are the \( R_{\alpha \ r ar} = \frac{1}{r} B' \)) and those related to these by symmetries. In calculations (12,13), the effect of those no-longer-vanishing components is to give place to the additional terms \( E_{tr(m)} \ l^t l^t \) and \( E_{rt(m)} \ l^t l^t \), and, when \( m \geq 2 \), to additional terms in the expressions for \( E_{tr(m)} \ l^t l^t \) and \( E_{rt(m)} \ l^t l^t \).

From (4), the expression for \( E_{tr(m)} \) is

\[ E_{tr(m)} = -\frac{1}{2} g_{tr} L_{(m)} + P_{ijt} R_{k (m)}^{ij} = \frac{1}{m} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_m, \beta_1, \beta_2, \ldots, \beta_m} R_{\alpha_1 \beta_1 \ldots \alpha_m \beta_m} \left( \frac{1}{2m} \right)^m \]
\[ = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_m, \beta_1, \beta_2, \ldots, \beta_m} R_{\alpha_1 \beta_1 \ldots \alpha_m \beta_m} \left( \frac{1}{2m} \right)^m \]
\[ \left( \frac{1}{r^2} \left( 1 - \frac{1}{B} \right) \right)^{m-1} \left( \frac{1}{2rB} \right)^m (AB)^{m-1}. \]  
(17)
where the second equality comes from $g_{rt} = 0$, due to the form of the metric. Here, writing again explicitly only the non-vanishing terms and proceeding as before, we get

$$E_{rt(m)} = m \left( -2 \sum_{\alpha, \beta, \alpha_2, \ldots, \alpha_m, \beta_m \neq} R^\beta_{\beta \beta_1} R^\alpha_{\alpha \beta_2} \cdots R^\alpha_{\alpha \beta_m} R^\alpha_{\beta r} \right) + 2 \sum_{\alpha, \beta, \alpha_2, \ldots, \alpha_m, \beta_m \neq} R^\beta_{\beta \beta_1} R^\alpha_{\alpha \beta_2} \cdots R^\alpha_{\alpha \beta_m} R^\alpha_{\beta r} + \sum_{\alpha, \alpha_2, \beta_2, \ldots, \alpha_m, \beta_m \neq} R^\alpha_{\alpha \beta_2} \cdots R^\alpha_{\alpha \beta_m} R^\alpha_{\beta r},$$

(18)

for $m \geq 2$, and

$$E_{rt(1)} = \sum_{\alpha} R^\alpha_{\beta r}.$$

(19)

The first two terms in round brackets in (18) cancel, and expressions (18,19) are thus given by

$$E_{rt(m)} = m \sum_{\alpha, \alpha_2, \beta_2, \ldots, \alpha_m, \beta_m \neq} R^\alpha_{\alpha \beta_2} \cdots R^\alpha_{\alpha \beta_m} R^\alpha_{\beta r},$$

(20)

with the $m = 1$ case corresponding to all the factors before $R^\alpha_{\beta r}$ set to 1. Using the explicit expressions of $R^\alpha_{\alpha \beta}$ and $R^\alpha_{\beta r}$, this gives

$$E_{rt(m)} = m (D - 2)(D - 3)\ldots(D - 2m) \left[ \frac{1}{r^2} \left( 1 - \frac{1}{B} \right) \right]^{m-1} \frac{1}{2r} \frac{\dot{B}}{B}$$

(21)

(compare with equation (13) in [6]).

From this expression, we have that when, in the spherical coordinates we consider, $E_{ab}$ is diagonal, as in vac or cosmovac solutions, from $E_{rt(m)} = 0$ it follows $B = 1$ or $\dot{B} = 0$, that means $\dot{B} = 0$, and thus $B = B(r)$ even if the configuration we are considering is actually not static. Thus, for diagonal $E_{ab}$, the expressions (12,13) remain unchanged going to the non-static case. Using (14), which holds true also in the non-static case, we come again to equation (15). Now, since the explicit expressions for $R^\alpha_{\alpha \beta}$, $R^\alpha_{\beta r}$ and $R^\alpha_{\alpha \beta}$ are left unchanged by any $A \neq 0$ (and $\dot{B} \neq 0$), the final expression of $E_{ab(m)} l^a l^b$ in the non-static case is still equation (16).

We can give to equation (16) a slightly different form. Since $E_{ab}$ diagonal implies $R^\alpha_{\alpha r} = 0$, and thus $R_{tt} = 0$, we have

$$R_{ab} l^a l^b = R_{tt} l^t l^t + R_{rr} l^r l^r = R_{tt} l^t l^t + \sum_{\alpha} R^\alpha_{\alpha \alpha \alpha} l^t l^t + \sum_{\alpha} R^\alpha_{\alpha \alpha r} l^r l^r = \sum_{\alpha} \alpha_{\alpha \alpha \alpha} l^t l^t + \sum_{\alpha} R^\alpha_{\alpha \alpha r} l^r l^r = \frac{D - 2}{2} \frac{1}{rB} (AB),$$

(22)

where we used (14) again, and the expression for $R_{ab} l^a l^b$ is the same in both the static and non-static cases. Equation (16) can then be expressed also as

$$E_{ab(m)} l^a l^b = m \frac{(D - 3)!}{(D - 2m - 1)!} \left[ \frac{1}{r^2} \left( 1 - \frac{1}{B} \right) \right]^{m-1} R_{ab} l^a l^b = m \frac{(2m - 1)!}{(D - 3)(2m - 1)} \left[ \frac{1}{r^2} \left( 1 - \frac{1}{B} \right) \right]^{m-1} R_{ab} l^a l^b,$$

(23)

with $B = B(r)$ both in the static and non-static case.

Using (16) (or (23)) for vac or cosmovac in the non-static configuration, from $E_{ab(m)} l^a l^b = 0$ we get $(AB)' = 0$, that is $AB = f(t)$, with $A = A(t, r)$, $B = B(r)$ and $f$ a function of $t$ alone. The metric we have is $ds^2 = $
\[ -\frac{\Omega}{2}\frac{\partial}{\partial t} dt^2 + B(r) dr^2 + r^2 d\Omega^2. \] Changing the t-coordinate to \( \tilde{t} \) with \( d\tilde{t} = \sqrt{f} dt \), we get \( ds^2 = -\frac{\Omega}{2}\frac{\partial}{\partial \tilde{t}} d\tilde{t}^2 + B(r) dr^2 + r^2 d\Omega^2 \), thus reducing to \( AB = 1 \) with \( A = A(r) \) and \( B = B(r) \), exactly as in the case of the static configuration.

While deriving expressions (11) and (23), we have thus shown the following propositions:

**Proposition 1.** Let consider a region with spherically symmetric geometry and a Lovelock model of order \( m \). If the generalized Einstein tensor \( E_{ab(m)} \) turns out to be diagonal in the spherical coordinates, then its two radial null-null components \( E_{ab(m)} \) (which are equal) can be expressed according to formulae (10) and (23).

**Proposition 2.** Let consider a region with spherically symmetric geometry. Any vac or cosmovac solution to Lovelock of order \( m \) can be expressed in the form \( ds^2 = -A(r) dt^2 + \frac{\Omega}{2\sqrt{f(r)}} dr^2 + r^2 d\Omega^2 \). That is, Birkhoff-like theorem for Lovelock of order \( m \).

### IV. COMMENTS AND CONCLUSIONS

We have provided, in equations (16) and (23), expressions for the radial null-null components \( E_{ab(m)} \) of generalized Einstein tensor for Lovelock of order \( m \). And shown they can be used to derive the Birkhoff-like theorem for these models.

The validity of Birkhoff-like theorem means that any spherically symmetric vac or cosmovac solution to Lovelock of order \( m \) is static, and its metric can be put in the form \( g_{tt} = -1 \). A general condition, given in [1], for a spherically symmetric static metric to have this form, is as mentioned the vanishing of the radial null-null components of Ricci tensor \( R_{ab} \) \( \ell \Omega^b \). We expect that, for spherically symmetric vac or cosmovac solutions to Lovelock of order \( m \), \( R_{ab} \ell \Omega^b = 0 \). But, this is precisely what happens, since, from expression (23), \( E_{ab(m)} \ell \Omega^b \) vanishes when \( R_{ab} \ell \Omega^b \) vanishes.

In the approach we have described, the validity of Birkhoff-like theorem is read in the expression for \( E_{ab(m)} \ell \Omega^b \); in that \( B = B(r) \) and in that the vanishing of \( E_{ab(m)} \ell \Omega^b \) means \( \langle AB \rangle = 0 \). Even a ‘minimal’ departure from LL gives troubles. Considering, for example, \( L = f(R) = R^2 \), thus restricting to this particular term of the Gauss-Bonnet Lagrangian, we have \( L = \frac{1}{2} \delta_{ab}^d \ R_{cd} = \frac{1}{2} \frac{\partial L}{\partial R_{cd}} \ R_{cd} = \frac{1}{2} R_{ab} R_{cd} \) being \( \partial R_{cd} = \delta_{cd} \ R. \) \( L \) is in the form \( L = Q^d_{ab} R_{cd} \) with \( Q_{ab} \) still being polynomial (linear indeed) in the components of Riemann and having the same symmetries of Riemann as in LL eq. (3); the only change is the relaxing of the condition \( \delta_{cd} \) on the divergence of \( Q_{abcd} \). The new expression for \( E_{ab(m)} \ell \Omega^b \) one obtains following the lines here, replacing eq. (11), contains indeed 4-th order derivatives of the metric, analogously to what happens for the equations of motion (an account of Birkhoff-like theorems in \( f(R) \) theories can be found in [11]; an investigation of the conditions which theories with equations of motion of order larger than 2 in the derivatives of the metric should obey for Birkhoff-like theorem to hold, is in [18, 19].

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