Effective Actions of IIB Matrix Model on $S^3$

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Abstract

$S^3$ is a simple principle bundle which is locally $S^2 \times S^1$. It has been shown that such a space can be constructed in terms of matrix models. It has been also shown that such a space can be realized by a generalized compactification procedure in the $S^1$ direction. We investigate the effective action of supersymmetric gauge theory on $S^3$ with an angular momentum cutoff and that of a matrix model compactification. The both cases can be realized in a deformed IIB matrix model with a Myers term. We find that the highly divergent contributions at the tree and 1-loop level are sensitive to the UV cutoff. However the 2-loop level contributions are universal since they are only logarithmically divergent. We expect that the higher loop contributions are insensitive to the UV cutoff since 3-dimensional gauge theory is super renormalizable.

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1 Introduction

In our universe, there are many mysteries that remain to be understood such as the selection mechanism of spacetime dimension, gauge groups and matter contents. It is important that we make progress to resolve these problems because we are increasing well informed how our universe is formed. Especially, we focus on the question why the 4-dimensionality of spacetime is selected in our universe.

It is considered that superstring theory provides an effective tool to explain the 4-dimensionality of spacetime. Since superstring theory is a unified theory including the gravity, we may hope to derive all physical predictions from the first principle. Unfortunately, superstring theory suggests 10-dimensional spacetime on the perturbative analysis. We believe that the nonperturbative analysis of superstring theory is needed to explain the 4-dimensionality of our universe. This question may be addressed in the matrix models which are proposed for nonperturbative formulations of superstring theory [1,2].

IIB matrix model is a candidate for the nonperturbative formulation of IIB superstring theory [2,3]. It is defined by the following action:

$$S_{\text{IIB}} = -\text{Tr} \left( \frac{1}{4} [A_\mu, A_\nu]^2 + \frac{1}{2} \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right),$$

(1.1)

where $A_\mu$ is a 10-dimensional vector and $\psi$ is a 10-dimensional Majorana-Weyl spinor field respectively, and both fields are $N \times N$ Hermitian matrices. There are considerable amount of investigations toward understanding the 4-dimensionality of spacetime by using IIB matrix model. For example, we may list the following studies: branched polymer picture [4], complex phase effects [5,6] and mean-field approximations [7–9]. These studies seem to suggest that IIB matrix model predicts 4-dimensionality of spacetime.

But it is difficult to analyze dynamics of IIB matrix model in a generic spacetime. So we would like to understand general mechanisms to single out the 4-dimensionality of spacetime through the studies of concrete examples. We have successfully constructed fuzzy homogeneous spaces using IIB matrix model [10]. We construct the homogeneous spaces as $G/H$ where $G$ is a Lie group and $H$ is a closed subgroup of $G$. When we give a background field to $A_\mu$, we can examine the stability of this matrix configurations by investigating the behavior of the effective action under the change of some parameters of the background. We have investigated the stabilities of fuzzy $S^2$ [11], fuzzy $S^2 \times S^2$ [12], fuzzy $CP^2$ [14] and fuzzy $S^2 \times S^2 \times S^2$ [15] in the past. We have found that IIB matrix model favors the configurations of 4-dimensionality and more symmetric manifolds.

In this paper, we investigate the 3-dimensional sphere $S^3$ configuration. We calculate the effective action of a deformed IIB matrix model by introducing a Myers term up to the 2-loop level on $S^3$. $S^3$ is a simple principle bundle which is locally $S^2 \times S^1$. It has been shown that such a space can be constructed in terms of matrix models [20]. It has been also shown that such a space can be realized by a generalized compactification procedure in the $S^1$ direction [21]. We investigate the effective action of supersymmetric gauge theory on $S^3$ with an angular momentum cutoff and that of a matrix model compactification. The both cases can be realized in a deformed IIB matrix model with a Myers term. We find that the highly divergent contributions at the tree and 1-loop level are sensitive to the UV cutoff. However the 2-loop level contributions are universal since they are only logarithmically divergent. We expect that the higher loop contributions are insensitive to the UV cutoff since 3-dimensional gauge theory is super renormalizable.
The organization of this paper is as follows: In section 2, we review the properties of the $S^3$. In section 3, we investigate the effective action of supersymmetric gauge theory on $S^3$ with an angular momentum cutoff up to the 2-loop level. In section 4, we investigate the corresponding effective action of a matrix model compactification. In section 5, we conclude with discussions.

2 Derivatives on $S^3$

In this section, we construct the derivatives on a 3-dimensional sphere: $S^3$. First of all, we review basic properties of an $S^3$ [16–19]. $S^3$ is defined by the following condition involving four Cartesian coordinates $x_n$:

$$\sum_{n=1}^{4} x_n^2 = 1.$$  \hfill (2.1)

It is more convenient to introduce the following parametrization:

$$u = x_1 + ix_2 = \cos \theta e^{i\phi}, \quad v = x_3 + ix_4 = \sin \theta e^{i\tilde{\phi}},$$  \hfill (2.2)

where $0 \leq \theta \leq \pi/2$, $0 \leq \phi < 2\pi$ and $0 \leq \tilde{\phi} < 2\pi$. The $S^3$ has an $SO(4)$ symmetry while the $S^2$ has an $SO(3)$ symmetry. It is a well-known fact that the $SO(4)$ is isomorphic to $SU(2) \times SU(2)$. Let $J_{1i}$ and $J_{2i}$ denote the generators of each $SU(2)$ subgroup respectively, where $i = 1, 2, 3$. We define the generators of the $SO(4)$ rotation group by the following linear combinations of the $SU(2)$ generators:

$$M_i = J_{1i} + J_{2i} = -i\epsilon_{ijk} x^j \frac{\partial}{\partial x^k},$$

$$N_i = J_{1i} - J_{2i} = i \left( x_4 \frac{\partial}{\partial x^i} - x_i \frac{\partial}{\partial x_4} \right).$$  \hfill (2.3)

In fact, the operators $M_i$ represent the rotations around the $x^i$ directions and the operators $N_i$ represent the rotations in the $x_i$-$x_4$ planes. They obey the following commutation relations:

$$[M_i, M_j] = i\epsilon_{ijk} M^k, \quad [N_i, N_j] = i\epsilon_{ijk} N^k, \quad [M_i, N^j] = i\epsilon_{ijk} N^k.$$  \hfill (2.4)

$J_{1i}$ and $J_{2i}$ obey the following commutation relations:

$$[J^1_{1i}, J_{1j}] = 0, \quad [J_{13}, J_{1\pm}] = \pm J_{1\pm}, \quad [J_{1+}, J_{1-}] = 2J_{13},$$

$$[J^2_{2i}, J_{2j}] = 0, \quad [J_{23}, J_{2\pm}] = \pm J_{2\pm}, \quad [J_{2+}, J_{2-}] = 2J_{23},$$  \hfill (2.5)

where $J_{1\pm} = J_{11} \pm iJ_{12}$ and $J_{2\pm} = J_{21} \pm iJ_{22}$. We can express these operators in terms of the coordinates system (2.2). For example,

$$J_{13} = -\frac{i}{2} \left( \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \tilde{\phi}} \right), \quad J_{23} = -\frac{i}{2} \left( \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \tilde{\phi}} \right).$$  \hfill (2.6)

The raising and lowering operators are

$$J_{1\pm} = \frac{1}{2} e^{\pm i\phi} e^{\pm i\tilde{\phi}} \left( \mp \frac{\partial}{\partial \theta} + i \tan \theta \frac{\partial}{\partial \phi} - i \cot \theta \frac{\partial}{\partial \tilde{\phi}} \right),$$

$$J_{2\pm} = \frac{1}{2} e^{\pm i\phi} e^{\pm i\tilde{\phi}} \left( \mp \frac{\partial}{\partial \theta} + i \tan \theta \frac{\partial}{\partial \phi} + i \cot \theta \frac{\partial}{\partial \tilde{\phi}} \right).$$  \hfill (2.7)
The $S^3$ is isomorphic to the $SU(2)$ group manifold as an element of $SU(2)$ is represented by the Pauli matrices $\sigma_i$: $U = x_4 1 + i \sum_{i=1}^3 x_i \sigma_i$. When we impose the condition of the unitarity: $U^* U = U U^* = 1$ and speciality: $\det U = 1$, we obtain the equation (2.1).

This space is a homogeneous space as we recall the following relation:

$$SO(4)/SU(2) = SU(2) \sim S^3.$$  \hspace{1cm} (2.8)

The homogeneous space is represented as $G/H$, where $G$ is a Lie group and $H$ is a closed subgroup of $G$. We can construct the $S^3$ by using the subgroup $SU(2)$ of the $SO(4)$. Let us choose the basis vectors as follows:

$$\mathbf{e}^{(1)}_1 = (x_4, x_3, -x_2, -x_1),$$
$$\mathbf{e}^{(1)}_2 = (-x_3, x_4, x_1, -x_2),$$
$$\mathbf{e}^{(1)}_3 = (x_2, -x_1, x_4, -x_3),$$

where $(a, b, c, d)$ denotes the Cartesian components of a vector. We find that the components of a vector $V$ is denoted by $V_i^{(1)} = \mathbf{V} \cdot \mathbf{e}_i^{(1)}$, and then the derivatives $\partial_i^{(1)}$ along these axes are denoted by

$$\partial_i^{(1)} = -2i J_{1i}, \quad \left[ \partial_i^{(1)}, \partial_j^{(1)} \right] = 2\epsilon_{ijk} \partial_k^{(1)}.$$  \hspace{1cm} (2.10)

The derivatives on the $S^3$ is constructed by a Lie algebra of the $SU(2)$. On the other hand, we can choose another different basis vectors as follows:

$$\mathbf{e}^{(2)}_1 = (x_4, -x_3, x_2, -x_1),$$
$$\mathbf{e}^{(2)}_2 = (x_3, x_4, -x_1, -x_2),$$
$$\mathbf{e}^{(2)}_3 = (-x_2, x_1, x_4, -x_3).$$

In the same way, we can construct another set of derivatives on $S^3$

$$\partial_i^{(2)} = 2i J_{2i}, \quad \left[ \partial_i^{(2)}, \partial_j^{(2)} \right] = -2\epsilon_{ijk} \partial_k^{(2)}.$$  \hspace{1cm} (2.12)

In the parameterization (2.2) of the $S^3$, we find the Laplacian on the $S^3$ is as follows:

$$\triangle_3 = \frac{1}{\sin \theta \cos \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \cos \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\cos^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.$$  \hspace{1cm} (2.13)

It is easy to recognize that the Casimir operators act as the Laplacian on the $S^3$:

$$J_1^2 = J_2^2 = -\frac{1}{4} \triangle_3.$$  \hspace{1cm} (2.14)

We may consider an eigenvalue equation of the Laplacian on the $S^3$ as follows:

$$\triangle_3 Y(\theta, \phi, \tilde{\phi}) = -\lambda Y(\theta, \phi, \tilde{\phi}),$$  \hspace{1cm} (2.15)

where $Y(\theta, \phi, \tilde{\phi})$ represents the solutions of this equation. We can solve this equation by separating the variables completely: $Y(\theta, \phi, \tilde{\phi}) = \Theta(\theta) \exp(\i m \phi + \i \tilde{m} \tilde{\phi})$. We find the following differential equation:

$$\left( 1 - z^2 \right) \frac{d^2 P(z)}{dz^2} - 2z \frac{dP(z)}{dz} + \left( \frac{1}{4} \lambda - \frac{m^2 + \tilde{m}^2}{2} \frac{1}{1 - z^2} + \frac{m^2 - \tilde{m}^2}{2} \frac{z}{1 - z^2} \right) P(z) = 0,$$  \hspace{1cm} (2.16)
where \( z = \cos 2\theta \) and \( P(z) = \Theta(\theta) \). The \( m \) and \( \tilde{m} \) take integers because \( \exp(\text{i}m\phi + \text{i}\tilde{m}\tilde{\phi}) \) has a periodicity of \( 2\pi \). If we consider the case \( m = \tilde{m} \equiv k \), we obtain the Legendre’s differential equation:

\[
(1 - z^2) \frac{d^2P(z)}{dz^2} - 2z \frac{dP(z)}{dz} + \left( \frac{1}{4} \lambda - \frac{k^2}{1 - z^2} \right) P(z) = 0. \tag{2.17}
\]

We can solve this equation by a series expansion. We obtain Legendre polynomials as the solutions with the eigenvalues: \( \lambda = n(n+2) \) where \( n \) is a positive integer. We can subsequently operate the raising and lowering operators \( J^1_\pm \) and \( J^2_\pm \) to the Legendre polynomials and obtain the complete solutions of (2.16) for \( m \neq \tilde{m} \). In this way we find that the solutions of the differential equation (2.16) are the spherical harmonics on the \( S^3 \):

\[
Y^{n}_{m\tilde{m}}(\theta, \phi, \tilde{\phi}) = \sqrt{n + 1} e^{\text{i}m\phi + \text{i}\tilde{m}\tilde{\phi}} d_n \frac{\pi}{2} \frac{\pi}{2} (m + \tilde{m}) \frac{\pi}{2} (m - \tilde{m}) (2\theta), \tag{2.18}
\]

where the \( d \) functions are given in terms of the Jacobi polynomials [22,23], \( -n/2 \leq (m+\tilde{m})/2 \leq n/2 \) and \( -n/2 \leq (m - \tilde{m})/2 \leq n/2 \). When the \( J^1_{13} \) and \( J^2_{23} \) operate on the spherical harmonics \( Y^{n}_{m\tilde{m}}(\Omega) \) on the \( S^3 \), we obtain eigenvalues \( (m + \tilde{m})/2 \) and \( (m - \tilde{m})/2 \) respectively. They satisfy the following orthogonality condition:

\[
\int d\Omega Y^{n}_{m_{1}m_1}^* (\Omega) Y^{n_2}_{m_{2}m_2} (\Omega) = \delta_{n_1 n_2} \delta_{m_1 m_2} \delta_{\tilde{m}_1 \tilde{m}_2}. \tag{2.19}
\]

We define the volume element \( d\Omega \) as follows:

\[
d\Omega = \frac{1}{2\pi^2} \sin \theta \cos \theta \ d\theta \ d\phi \ d\tilde{\phi}. \tag{2.20}
\]

### 3 Effective action on \( S^3 \)

In this section, we investigate the effective action of a deformed IIB matrix model on the \( S^3 \) background. In order to obtain \( S^3 \) as classical solutions of IIB matrix model, we deform the action of IIB matrix model by adding a Myers term as follows:

\[
S = -\text{Tr} \left( \frac{1}{4} [A_\mu, A_\nu]^2 + \frac{1}{2} \bar{\psi} \Gamma^\mu [A_\mu, \psi] - \frac{i}{3} f_{\mu\nu\rho} [A^\mu, A^\nu] A^\rho \right), \tag{3.1}
\]

where \( f_{\mu\nu\rho} \) is the structure constant of \( SU(2) \). When we assume \( \psi = 0 \), we obtain the equation of motion for \( A_\mu \):

\[
[A_\mu, [A^\mu, A_\nu]] - if_{\mu\nu\rho} [A^\mu, A^\rho] = 0. \tag{3.2}
\]

The nontrivial classical solutions of this equation of motion are

\[
A_a = t_a, \quad \text{other} \ A_\mu = 0, \tag{3.3}
\]

where \( t_a \)'s satisfy the Lie algebra. To calculate the effective action of the deformed IIB matrix model, we decompose the matrices \( A_\mu \) and \( \psi \) into the backgrounds and the quantum fluctuations as follows:

\[
A_\mu = p_\mu + a_\mu, \quad \psi = \chi + \varphi, \tag{3.4}
\]
where $p_\mu$ and $\chi$ are the backgrounds, and $a_\mu$ and $\varphi$ are the quantum fluctuations. When we expand the action (3.1) around the quantum fluctuations, we also add the gauge fixing term and the Faddeev-Popov ghost term as follows:

$$
S_{g.f.} = -\frac{1}{2} \text{Tr} \left( [p_\mu, a_\mu]^2 \right),
$$

$$
S_{F.P.} = \text{Tr} \left( b \left[ p_\mu, [p_\mu + a_\mu, c] \right] \right),
$$

(3.5)

where $c$ and $b$ are ghosts and anti-ghosts, respectively. In this way we find that the following action:

$$
\tilde{S} = S + S_{g.f.} + S_{F.P.}
$$

$$
= -\text{Tr} \left( \frac{1}{4} [p_\mu, p_\nu]^2 + \frac{1}{2} \bar{\chi} \Gamma^\mu [p_\mu, \chi] - \frac{i}{3} f_{\mu\nu\rho} [p_\mu, p_\nu] p_\rho 
- a_\nu \left[ p_\mu, [p_\mu, p_\nu] \right] + \frac{1}{2} \left( \bar{\chi} \Gamma^\nu [p_\mu, \varphi] - if_{\mu\nu\rho} [p_\mu, p_\nu] a_\rho \right) 
+ \frac{1}{2} [p_\mu, a_\mu]^2 + [p_\mu, p_\nu] [a_\mu, a_\nu] - b \left[ p_\mu, [p_\mu, c] \right] + \frac{1}{2} \bar{\varphi} \Gamma_\mu [p_\mu, \varphi] + \bar{\chi} \Gamma^\mu [a_\mu, \varphi] + if_{\mu\nu\rho} a_\mu [p_\rho, a_\nu] 
+ [p_\mu, a_\mu] [a_\mu, a_\nu] - b \left[ p_\mu, [a_\mu, c] \right] + \frac{1}{2} \bar{\varphi} \Gamma_\mu [a_\mu, \varphi] - \frac{i}{3} f_{\mu\nu\rho} [a_\mu, a_\nu] a_\rho + \frac{1}{4} [a_\mu, a_\nu]^2 \right) \right).
$$

(3.6)

Since we investigate the effective action on the $S^3$ background, we may substitute the derivatives $i \partial^{(1)}_i$ on the $S^3$ for the backgrounds as follows:

$$
p_i = i \partial^{(1)}_i = \alpha J_i, \quad \text{other} \quad p_\mu = 0,
\chi = 0,
$$

(3.7)

where $\alpha$ is a scale factor and $i = 1, 2, 3$. In [20], the authors have found that the bosonic part $A_\mu$ of IIB matrix model can be interpreted as differential operators on the principle bundles. One of the goals of our investigations is to obtain deeper understanding of such an interpretation through a concrete example: $S^3$.

When the backgrounds is a flat space, we expand the quantum fluctuations by a plane wave. Since we consider the $S^3$ background, it is natural to expand the quantum fluctuations by a spherical harmonics on the $S^3$:

$$
a_\mu = \sum_{nmn} a_{\mu mn} Y_{mn} (\Omega),
\varphi = \sum_{nmn} \varphi_{mn} Y_{mn} (\Omega),
$$

(3.8)

where $a_{\mu mn}^n$ and $\varphi_{mn}^n$ are expansion coefficients. In the same way, ghosts and anti-ghosts fields are expanded by the spherical harmonics on the $S^3$:

$$
c = \sum_{nmn} c_{mn} Y_{mn} (\Omega),
\quad
b = \sum_{nmn} b_{mn} Y_{mn} (\Omega),
$$

(3.9)
The structure constants \( f_{\mu \nu \rho} \) are given as follows:

\[
f_{ijk} = \alpha \epsilon_{ijk}, \quad \text{other} \quad f_{\mu \nu \rho} = 0.
\] (3.10)

The gauge fixed action around the backgrounds \( (3.7) \) take the following form:

\[
\tilde{S} = \begin{aligned}
- \operatorname{Tr} & \left( \frac{1}{4} [p_i, p_j]^2 - \frac{i}{3} f_{ijk} [p^i, p^j] p^k \\
&+ \frac{1}{2} [p_i, a_\mu]^2 - b [p_i, [p^i, c]] + \frac{1}{2} \varphi \Gamma^i [p_i, \varphi] \\
&+ [p_i, a_\mu] [a^i, a^\mu] - b [p_i, [a^i, c]] + \frac{1}{2} \varphi \Gamma^\mu [a_\mu, \varphi] - \frac{i}{3} f_{ijk} [a^i, a^j] a^k + \frac{1}{4} [a_\mu, a_\nu]^2 \right).
\end{aligned}
\] (3.11)

Using the above action \( \tilde{S} \), we can evaluate the effective action \( W \) of the deformed IIB matrix model on the \( S^3 \) background as follows in a background gauge method:

\[
W = - \log \int da \, d\varphi \, dc \, db \, e^{-\tilde{S}}.
\] (3.12)

Firstly, we evaluate the effective action at the tree level

\[
W_{\text{tree}} = - \operatorname{Tr} \left( \frac{1}{4} [p_i, p_j]^2 - \frac{i}{3} f_{ijk} [p^i, p^j] p^k \right)
= - \frac{\alpha^4}{6} \operatorname{Tr} (J_{1i})^2.
\] (3.13)

In the last step, we substitute the derivatives: \( i \partial^{(1)}_i = \alpha J_{1i} \) on the \( S^3 \) for the backgrounds \( p_i \) of the deformed IIB matrix model. We evaluate the effective action of the deformed IIB matrix model by taking the continuous limit as follows:

\[
\operatorname{Tr} X \longrightarrow \int d\Omega \, \langle \Omega | X | \Omega \rangle.
\] (3.14)

We may consider that this limit corresponds to a semi-classical limit. We evaluate the trace of the Casimir operator as:

\[
\operatorname{Tr} (J_{1i})^2 \longrightarrow - \frac{1}{4} \int d\Omega \, \langle \Omega | \triangle_3 | \Omega \rangle
= \frac{1}{4} \int d\Omega \sum_{nm\tilde{m}} n(n + 2) \langle \Omega | n, m, \tilde{m} \rangle \langle n, m, \tilde{m} | \Omega \rangle
= \frac{1}{4} \sum_n n(n + 2)(n + 1)^2.
\] (3.15)

In the second line, we have used the complete set of the eigenstates for the Laplacian on the \( S^3 \): \( \sum_{nm\tilde{m}} | n, m, \tilde{m} \rangle \langle n, m, \tilde{m} | = 1 \). In the last line, we have used the fact that \( Y_{nm\tilde{m}}^n(\Omega) = \langle \Omega | n, m, \tilde{m} \rangle \) and the degeneracy factors coming from \( m \) and \( \tilde{m} \) are \( (n + 1)^2 \). Therefore,

\[
W_{\text{tree}} \longrightarrow - \frac{\alpha^4}{24} \sum_n n(n + 2)(n + 1)^2.
\] (3.16)
Since \( n \) takes a positive integer, the summation over \( n \) is formally divergent. It is necessary to impose a cutoff at a some large but finite \( n \). When impose a cutoff at \( n = l \), the tree level effective action is evaluated as follows:

\[
W_{\text{tree}} \rightarrow -\frac{\alpha^4}{24} \sum_{n=1}^{l} n(n+2)(n+1)^2 \sim -\frac{\alpha^4}{24} l^5.
\]  
(3.17)

Secondly, we evaluate the effective action at the 1-loop level as follows:

\[
W_{1-\text{loop}} = \frac{1}{2} \text{Tr} \log \left( P_i^2 \delta^{\mu\nu} \right) - \text{Tr} \log \left( P_i^2 \right) \\
- \frac{1}{4} \text{Tr} \log \left[ \left( P_i^2 + \frac{i}{2} F_{ij} \Gamma^{ij} \right) \left( \frac{1 + \Gamma_{11}}{2} \right) \right],
\]  
(3.18)

where

\[
[p_i, X] = P_i X, \\
[f_{ij}, X] = F_{ij} X, \quad i f_{ij} = [p_i, p_j].
\]  
(3.19)

The first and the second terms are the bosonic contributions while the third term is the fermionic contribution. We have to include a projection operator \((1 + \Gamma_{11})/2\) in the fermionic part because we consider a 10-dimensional Majorana-Weyl spinor field. We expand the third term of the equation (3.18) into the power series of \( P_i \) and \( F_{ij} \). In this way, we obtain the leading term of the 1-loop level effective action as follows:

\[
W_{1-\text{loop}} \sim -\text{Tr} \left( \frac{1}{P_i^2} \right)^2 F_{ij} F^{ji} = \text{Tr} \frac{2\alpha^2}{P_i^2}.
\]  
(3.20)

When we impose the same cutoff procedure: \( n \leq l \), we obtain the effective action at the 1-loop level as follows:

\[
W_{1-\text{loop}} \rightarrow 8 \sum_{n=1}^{l} \frac{(n+1)^2}{n(n+2)} \sim 8 l.
\]  
(3.21)

Finally, we evaluate the effective action at the 2-loop level due to planar diagrams. We describe the detailed calculations of the 2-loop effective action in appendix B. The effective action at the 2-loop level is:

\[
W_{2-\text{loop}} = \frac{2304}{\alpha^4} \sum_{n_1n_2n_3} \frac{(n_1 + 1)(n_2 + 1)(n_3 + 1)}{n_1(n_1 + 2)n_2(n_2 + 2)n_3(n_3 + 2)}.
\]  
(3.22)

We impose the same the cutoff procedure of the summations over \( n_1, n_2 \) and \( n_3 \):

\[
\sum_{n_1=1}^{l} \sum_{n_2=1}^{l} \sum_{n_3=1}^{l} \frac{(n_1 + 1)(n_2 + 1)(n_3 + 1)}{n_1(n_1 + 2)n_2(n_2 + 2)n_3(n_3 + 2)} \equiv f \left( l \right).
\]  
(3.23)

where we recall the following selection rules that \( |n_1 - n_2| \leq n_3 \leq n_1 + n_2 \) and \( n_1 + n_2 + n_3 \) must be even numbers. To the leading logarithmic order, we can analytically evaluate it:

\[
f \left( l \right) \sim \pi^2 \frac{2}{4} \log l,
\]  
(3.24)
We illustrate the comparison between the numerical evaluation and the analytic expression (3.24) in Fig. 1. We find that the analytic expression is valid to the leading logarithmic order as the slope of the two lines are identical. We thus conclude that the 2-loop level effective action is

$$ W_{2\text{-loop}} \sim \frac{576\pi^2}{\alpha^4} \log l. \quad (3.25) $$

In this way we can summarize the effective action with an angular momentum cutoff $l$ up to the 2-loop level:

$$ W \sim -\frac{\alpha^4}{24} l^5 + 8l + \frac{576\pi^2}{\alpha^4} \log l. \quad (3.26) $$

While the tree and 1-loop contributions are highly divergent, the 2-loop contribution is only logarithmically divergent. This result is consistent with the fact that 3-dimensional gauge theory is super renormalizable and we expect that the higher loop contributions are finite. Since $1/\alpha^4$ acts as the loop expansion parameter, we need to assume $\alpha \sim \mathcal{O}(1)$. We then conclude that the effective action of the deformed IIB matrix model on the $S^3$ is stable against the quantum corrections as it is dominated by the tree level contribution.

## 4 IIB matrix model compactification on $S^3$

There is an interesting construction of an $S^3$ background in matrix models recently. In [21], the authors have concluded that the $S^3$ is realized by three matrices. They have proved the following two relations between the vacua of different gauge theories:

(i) The theory around each vacuum of super Yang-Mills theory on $S^2$ (SYM$_{S^2}$) is equivalent to the theory around a certain vacuum of a matrix model.

(ii) The theory around each vacuum of the super Yang-Mills theory on $S^3/Z_k$ (SYM$_{S^3/Z_k}$) is equivalent to the theory around a certain vacuum of SYM$_{S^2}$ with periodic identifications.
They selected the following nontrivial vacua of $\text{SYM}_{S^2}$:

\[
\Phi = \frac{\mu}{2} \begin{pmatrix}
\alpha_1 & N_1 \\
\alpha_2 & N_2 \\
& \ddots \\
& & \alpha_T & N_T
\end{pmatrix},
\]

(4.1)

\[
A_\theta = 0,
\]

\[
A_\phi = \begin{cases}
\frac{1}{\mu} (1 - \cos \theta) \Phi & \text{for } 0 \leq \theta < \frac{\pi}{2} + \epsilon \\
-\frac{1}{\mu} (1 + \cos \theta) \Phi & \text{for } \frac{\pi}{2} - \epsilon < \theta \leq \pi
\end{cases},
\]

(4.2)

where $\alpha_s$'s ($s = 1, \cdots, T$) parameterize monopole charges:

\[
q_{st} = \frac{1}{2} (\alpha_s - \alpha_t),
\]

(4.3)

and all $\alpha_s$'s are different. Additionally we have the following relation

\[
N_1 + \cdots + N_T = \tilde{N}.
\]

(4.4)

The radius of the $S^2$ is fixed to be $1/\mu$. The fields in $\text{SYM}_{S^2}$ are split into the blocks of $N_s \times N_t$ rectangular matrices around the above nontrivial vacua.

On the other hand, a vacuum of a matrix model is represented as follows:

\[
Y_i = -\mu L_i,
\]

(4.5)

where

\[
L_i = \begin{pmatrix}
L_i^{[j_1]} & N_1 \\
& \ddots \\
& & L_i^{[j_T]}
\end{pmatrix}.
\]

(4.6)
The $L_i$ is a reducible $\hat{N}$-dimensional representation of $SU(2)$, and obeys the following commutation relation:

$$[L_i, L_j] = i\epsilon_{ijk}L^k,$$  \hspace{1cm} (4.7)

where

$$(2j_1 + 1)N_1 + (2j_2 + 1)N_2 + \cdots + (2j_T + 1)N_T = \hat{N}. \hspace{1cm} (4.8)$$

$L_{[j s]} (s = 1, \cdots, T)$ is the $(2j_s + 1) \times (2j_s + 1)$ spin $j_s$ representation of $SU(2)$, and obeys the commutation relation:

$$\left[ L_{[j s]}^i, L_{[j s]}^j \right] = i\epsilon_{ijk}L_{[j s]}^k. \hspace{1cm} (4.9)$$

The Casimir operator of $L_{[j s]}^i$ is that

$$L_{[j s]}^i L_{[j s]}^i = j_s (j_s + 1) \mathbf{1}_{2j_s + 1}. \hspace{1cm} (4.10)$$

This vacuum (4.6) can be interpreted as a set of coincident $N_s$ fuzzy spheres with the radii $\mu\sqrt{j_s(j_s + 1)}$, where all the fuzzy spheres are concentric.

In order to prove the above two relations, they expand the theories around various vacua using appropriate spherical harmonics respectively. The spherical harmonics $Y_{JM\tilde{M}}$ on an $S^3$ is relevant for $\text{SYM}_{S^3/Z_k}$ ($J = n/2$, $M = (m + \tilde{m})/2$ and $\tilde{M} = (m - \tilde{m})/2$ in our convention) where $J = 0, 1/2, 1, \cdots$, $M = -J, -J + 1, \cdots, J - 1, J$ and $\tilde{M} = -J, -J + 1, \cdots, J - 1, J$. The monopole harmonic function $\tilde{Y}_{JM\tilde{M}}$, the harmonic function on a set of fuzzy spheres with different radii, is used to expand around the background of $\text{SYM}_{S^2}$, where $J = |q_{st}|, |q_{st}| + 1, |q_{st}| + 2, \cdots$, and $M = -J, -J + 1, \cdots, J - 1, J$. For a matrix model, fuzzy sphere harmonics $\hat{Y}_{JM\tilde{M}}$, the harmonic function on a set of fuzzy spheres with different radii, is used where $J = |j_s - j_t|, |j_s - j_t| + 1, \cdots, j_s + j_t$ and $M = -J, -J + 1, \cdots, J - 1, J$. The equivalence (i) is proved when the following conditions are imposed on the parameters of the vacua of $\text{SYM}_{S^2}$ and the vacua of a matrix model:

$$j_s - j_t = \frac{1}{2}(\alpha_s - \alpha_t) = q_{st},$$
$$j_s, j_t \to \infty. \hspace{1cm} (4.11)$$

The equivalence (ii) is proved under the following conditions

$$\alpha_s = sk, \hspace{0.5cm} N_s = N, \hspace{1cm} s = 1, \cdots, \infty. \hspace{1cm} (4.12)$$

Additionally, they identify the fuzzy spheres by imposing the periodicity on the $(s, t)$ blocks and by factoring out the overall factor. Combining the equivalences (i) and (ii), they have concluded that the theory around the trivial vacua of $\text{SYM}_{S^3/Z_k}$ is equivalent to the theory around the vacua of a matrix model. In order to draw the above conclusion, the following conditions are necessary:

$$j_s - j_t = \frac{k}{2}(s - t) = q_{st}, \hspace{0.5cm} N_s = N,$$
$$j_s, j_t \to \infty, \hspace{0.5cm} s, t = 1, \cdots, \infty. \hspace{1cm} (4.13)$$
The condition that $j_s - j_t = k(s - t)/2$ can be also written as $2j_s + 1 = N_0 + ks$, where $N_0$ is a positive integer. The condition that $j_s \to \infty$ corresponds to $N_0 \to \infty$.

We make use of the work [21] to make a connection between super Yang-Mills theory on an $S^3$ background and IIB matrix model on an $S^3$ background. We expand the deformed IIB matrix model action (3.1) around the backgrounds in an analogous way in section 3. We introduce the matrix $Y_i$ as the backgrounds of the deformed IIB matrix model:

$$p_i = Y_i = -\mu L_i = \beta L_i, \quad \text{other } p_\mu = 0,$$

where $\beta$ is a scale factor and $i = 1, 2, 3$. We make a mode expansion of the quantum fluctuations using the fuzzy sphere harmonics:

$$a^{(s,t)}_\mu = \sum_{J=|j_s-j_t|}^{j_s+j_t} \sum_{M=-J}^{J} a^{(s,t)}_{\mu JM} \hat{Y}^{(j_s,j_t)}_{JM},$$

$$\varphi^{(s,t)} = \sum_{J=|j_s-j_t|}^{j_s+j_t} \sum_{M=-J}^{J} \varphi^{(s,t)}_{JM} \hat{Y}^{(j_s,j_t)}_{JM},$$

where the suffix $(s, t)$ represents the $(s, t)$ block in an $\hat{N} \times \hat{N}$ matrix and $s, t = 1, \ldots, T$. The coefficients of the mode expansion $a^{(s,t)}_{\mu JM}$ and $\varphi^{(s,t)}_{JM}$ are $N_s \times N_t$ matrices and the fuzzy sphere harmonics $\hat{Y}^{(j_s,j_t)}_{JM}$ is a $(2j_s+1) \times (2j_t+1)$ matrix. Therefore, $a^{(s,t)}_{\mu JM}$ and $\varphi^{(s,t)}_{JM}$ are $N_s (2j_s+1) \times N_t (2j_t+1)$ rectangular matrices. Similarly, ghosts and anti-ghosts fields are expanded by the fuzzy sphere harmonics:

$$c^{(s,t)} = \sum_{J=|j_s-j_t|}^{j_s+j_t} \sum_{M=-J}^{J} c^{(s,t)}_{JM} \hat{Y}^{(j_s,j_t)}_{JM},$$

$$b^{(s,t)} = \sum_{J=|j_s-j_t|}^{j_s+j_t} \sum_{M=-J}^{J} b^{(s,t)}_{JM} \hat{Y}^{(j_s,j_t)}_{JM},$$

where $c^{(s,t)}_{JM}$ and $b^{(s,t)}_{JM}$ are the expansion coefficients. As for structure constant, we adopt that

$$f_{ijk} = \beta \delta_{ijk}, \quad \text{other } f_{\mu\nu\rho} = 0.$$

When we expand the deformed IIB matrix model up to the forth order of the quantum fluctuations, we obtain the following action:

$$\hat{S} = -\text{Tr} \sum_{stu} \left( \frac{1}{4} [p_i, p_j]^2 - \frac{i}{3} f_{ijk} [p_i, p_j] p^k + \frac{1}{2} [p_i, a^{(s,t)}] [p_i, a^{(t,s)}] + \frac{1}{2} \varphi^{(s,t)} \Gamma^i [p_i, \varphi^{(t,s)}] 
+ [p_i, a^{(s,t)}][a^{(t,w)i}, a^{(u,s)]} - b^{(s,t)} [p_i, [a^{(t,w)i}, c^{(u,s)}]] 
+ \frac{1}{2} \varphi^{(s,t)} \Gamma^u [a^{(t,u)i}, \varphi^{(u,s)}] - \frac{1}{3} f_{ijk} [a^{(s,t)i}, a^{(t,u)j}] a^{(u,s)k} + \frac{1}{4} [a^{(s,t)}_\mu, a^{(t,u)\nu}]^2 \right).$$

(4.18)
Because of the $j_{st} = j_s - j_t$ depends only on $s - t$, we can impose the following condition on the quantum fluctuations and ghosts and anti-ghosts fields:

\[
\begin{align*}
    a^{(s+1,t+1)}_{\mu} &= a^{(s,t)}_{\mu}, & \varphi^{(s+1,t+1)} &= \varphi^{(s,t)} \\
    c^{(s+1,t+1)} &= c^{(s,t)}, & b^{(s+1,t+1)} &= b^{(s,t)}.
\end{align*}
\]

From the above condition, we obtain the condition for the mode expansion coefficients:

\[
\begin{align*}
    a^{(s,t)}_{\mu J M} &= a^{(s,t)}_{\mu J M}, & \varphi^{(s,t)}_{J M} &= \varphi^{(s,t)}_{J M} \\
    c^{(s,t)}_{J M} &= c^{(s,t)}_{J M}, & b^{(s,t)}_{J M} &= b^{(s,t)}_{J M}.
\end{align*}
\]

We can rewrite this condition as follows:

\[
\begin{align*}
    a^{(s,t)}_{\mu J M} &= a_{\mu J M j_{st}}, & \varphi^{(s,t)}_{J M} &= \varphi_{J M j_{st}} \\
    c^{(s,t)}_{J M} &= c_{J M j_{st}}, & b^{(s,t)}_{J M} &= b_{J M j_{st}}.
\end{align*}
\]

We rewrite the action (4.18) by using (4.21). For example, we consider the following term:

\[
- \text{Tr} \sum_{st} \left( \frac{1}{2} [p_i, a^{(s,t)}_{\mu}]^2 \right) = \frac{\beta^2}{2} \text{Tr} \sum_{st} a^{(s,t)}_{\mu} L_i \circ L^i \circ \delta^{\mu \nu} a^{(t,s)}_{\nu}
\]

\[
= \frac{\beta^2}{2} \sum_{st} \sum_{J_1 M_1 J_2 M_2} a^{(s,t)}_{\mu J_1 M_1} \otimes \hat{Y}^{(j_i j_j)}_{J_1 M_1} J_2 (J_2 + 1) \delta^{\mu \nu} a^{(t,s)}_{\nu J_2 M_2} \otimes \hat{Y}^{(j_i j_j)}
\]

\[
= \frac{N_0 \beta^2}{2} \sum_{st} \sum_{J=|j_s-j_t|}^{\infty} \sum_{M=-J}^{J} \sum_{M=-\tilde{M}}^{\tilde{M}} a^{(s,t)}_{\mu J M} (-1)^{M - \tilde{M}} J (J + 1) \delta^{\mu \nu} a^{(t,s)}_{\nu J M \tilde{M}},
\]

where we use the property of fuzzy spherical harmonics in the appendix C. We set that $s - t = p$ and $s = h$, so $p$ and $h$ are integers, to make the connection between super Yang-Mills theory on the $S^3$ and IIB matrix model on the $S^3$. In this way, we obtain that

\[
\frac{N_0 \beta^2}{2} \sum_{h} \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \sum_{\tilde{M}=-J}^{\tilde{M}} a_{\mu J M \tilde{M}} (-1)^{M - \tilde{M}} J (J + 1) \delta^{\mu \nu} a_{\nu J M \tilde{M}},
\]

where we set that $p/2 = \tilde{M}$ and $a^{(s,t)}_{\mu J M} = a_{\mu J M \tilde{M}}$. In a similar way, we can evaluate the other terms in the action (4.18). Then, the overall factor $\sum_{h}$ appears in front of the action. After factoring out the overall factor $\sum_{h}$, we find that the mode expansion of the deformed IIB matrix model around the background $Y_i$ is identical to the action of super Yang-Mills on the $S^3$.

In what follows, we investigate the relations between the gauge theory on the $S^3$ background and the matrix models on $S^3$. Although they are classically equivalent, the relation is more subtle at the quantum level since the matrix model compactification assumes a definite cutoff procedure. We have evaluated the effective action of the gauge theory on $S^3$ using the background field method in section 3. In this section, we evaluate an effective action $\hat{W}$ of the deformed IIB matrix model on the $S^3$ background $Y_i$:

\[
\hat{W} = -\log \int d\sigma d\varphi d\sigma d\varphi e^{-S},
\]
where $\hat{S}$ is the action (4.18). Firstly, we evaluate the effective action at the tree level:

$$
\hat{W}_{\text{tree}} = -\text{Tr} \left( \frac{1}{4} [p_i, p_j]^2 - \frac{i}{3} f_{ijk} [p^i, p^j] p^k \right)
$$

$$
= -\frac{\beta^4}{6} \text{Tr} \sum_{s=1}^{T} N_s \left( L^{[js]}_i \right)^2
$$

$$
= -\frac{\beta^4}{6} \sum_{s=1}^{T} N_s j_s \ (j_s + 1) (2j_s + 1) . \tag{4.25}
$$

We impose the conditions $2j_s + 1 = N_0 + s$ in order to connect super Yang-Mills theory on the $S^3$ and IIB matrix model on the $S^3$. We obtain

$$
\hat{W}_{\text{tree}} = -\frac{\beta^4}{24} \sum_{h=1}^{\infty} \left[ \left( N_0 + h \right)^3 - \left( N_0 + h \right) \right] , \tag{4.26}
$$

where we set that $s = h$, and $N_s = N = 1$ for simplicity. Since $h$ takes integer values, the sum over $h$ is formally divergent. We have a cutoff scale on $h$ at a number $2\Lambda = T$ which is equal to the number of $(s, t)$ blocks. When we take a large $N_0$ limit in such a way that $N_0 \gg \Lambda$, we obtain the following effective action at the tree level:

$$
\hat{W}_{\text{tree}} \xrightarrow{N_0 \to \infty} \sum_{h=1}^{2\Lambda} \left( -\frac{\beta^4}{24} N_0^3 \right) . \tag{4.27}
$$

Secondly, we calculate the effective action at the 1-loop level as follows:

$$
\hat{W}_{1-\text{loop}} \sim -\text{Tr} \sum_{st} \left( \frac{1}{P^2_i} \right)^2 F_{ij} F^{ji} \tag{4.28}
$$

$$
= \text{Tr} \sum_{st} \frac{2\beta^2}{P^2_i} \tag{4.28}
$$

$$
= \sum_{st} \sum_{j_s+j_t} \frac{2}{J(J+1)} (2J+1) .
$$

We obtain that

$$
\hat{W}_{1-\text{loop}} \sim \sum_{h=1}^{\infty} \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \frac{2}{J(J+1)} (2J+1) . \tag{4.29}
$$

We have a cutoff such that $h < 2\Lambda$, so that the maximal value of $J$ and $\tilde{M}$ are $N_0$ and $\Lambda$, respectively. We divide the summation over $J$ into two parts at the value $\Lambda$ as the following:

$$
\hat{W}_{1-\text{loop}} \sim \sum_{h=1}^{2\Lambda} \sum_{J=1/2}^{\Lambda} \frac{2}{J(J+1)} (2J+1)^2 + \sum_{h=1}^{2\Lambda} \sum_{J=\Lambda+1/2}^{N_0} \frac{2}{J(J+1)} (2J+1) (2\Lambda+1) , \tag{4.30}
$$

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where we omit a zero mode of $J$. When we take a large $N_0$ limit with $N_0 \gg \Lambda$, we obtain the effective action at the 1-loop level:

$$
\tilde{W}_{1\text{-loop}} \rightarrow \sum_{h=1}^{2\Lambda} \left( 8\Lambda + 8\Lambda \log N_0 \right).
$$

Finally, we calculate the effective action at the 2-loop level due to planar diagrams. We describe the detailed calculations of the 2-loop effective action in appendix D. The result is

$$
\tilde{W}_{2\text{-loop}} \sim \frac{36}{\beta^4 N_0} \sum_{stu} \sum_{J_1=|j_s-j_t|} \sum_{J_2=|j_t-j_u|} \sum_{J_3=|j_u-j_s|} \left( \frac{(2J_1+1)(2J_2+1)(2J_3+1)}{J_1(J_1+1)J_2(J_2+1)J_3(J_3+1)} \left( \frac{J_1}{\tilde{M}_1} \frac{J_2}{\tilde{M}_2} \frac{J_3}{\tilde{M}_3} \right)^2 \right),
$$

where $\tilde{M}_1 = j_s - j_t$, $\tilde{M}_2 = j_t - j_u$ and $\tilde{M}_3 = j_u - j_s$. We can further rewrite it as

$$
\tilde{W}_{2\text{-loop}} \sim \frac{36}{\beta^4 N_0} \sum_{h=1}^{2\Lambda} \sum_{J_1=1/2}^{\Lambda} \sum_{J_2=1/2}^{\Lambda} \sum_{J_3=1/2}^{\Lambda} \left( \frac{(2J_1+1)(2J_2+1)(2J_3+1)}{J_1(J_1+1)J_2(J_2+1)J_3(J_3+1)} \left( \frac{J_1}{\tilde{M}_1} \frac{J_2}{\tilde{M}_2} \frac{J_3}{\tilde{M}_3} \right)^2 \right),
$$

where we set that $p/2 = \tilde{M}_1$, $q/2 = \tilde{M}_2$ and $(-p-q)/2 = \tilde{M}_3$, and omit a zero mode of $J_1$, $J_2$ and $J_3$. We have the cutoff scale $2\Lambda$ on $h$, and divide the summation over $J_1$, $J_2$ and $J_3$ into two sections at $\Lambda$ as following:

$$
\tilde{W}_{2\text{-loop}} \sim \frac{36}{\beta^4 N_0} \sum_{h=1}^{2\Lambda} \sum_{J_1=1/2}^{\Lambda} \sum_{J_2=1/2}^{\Lambda} \sum_{J_3=1/2}^{\Lambda} \left( \frac{(2J_1+1)(2J_2+1)(2J_3+1)}{J_1(J_1+1)J_2(J_2+1)J_3(J_3+1)} \left( \frac{J_1}{\Lambda} \frac{J_2}{\tilde{M}_2} \frac{J_3}{\tilde{M}_3} \right)^2 \right) + \frac{3 \cdot 36}{\beta^4 N_0} \sum_{h=1}^{2\Lambda} \sum_{J_1=\Lambda+1/2}^{\Lambda} \sum_{J_2=\Lambda+1/2}^{\Lambda} \sum_{J_3=\Lambda+1/2}^{\Lambda} \left( \frac{(2J_1+1)(2J_2+1)(2J_3+1)}{J_1(J_1+1)J_2(J_2+1)J_3(J_3+1)} \left( \frac{J_1}{\Lambda} \frac{J_2}{\Lambda} \frac{J_3}{\tilde{M}_3} \right)^2 \right) + \frac{3 \cdot 36}{\beta^4 N_0} \sum_{h=1}^{2\Lambda} \sum_{J_1=\Lambda+1/2}^{\Lambda} \sum_{J_2=\Lambda+1/2}^{\Lambda} \sum_{J_3=\Lambda+1/2}^{\Lambda} \left( \frac{(2J_1+1)(2J_2+1)(2J_3+1)}{J_1(J_1+1)J_2(J_2+1)J_3(J_3+1)} \left( \frac{J_1}{\Lambda} \frac{J_2}{\Lambda} \frac{J_3}{\Lambda} \right)^2 \right).
$$
When we take the large $N_0$ limit with $N_0 \gg \Lambda$, we find that the first term of (4.34) is logarithmically infinite while the others are finite. We describe the detailed calculations of (4.34) in appendix D.9. In this way, we obtain the effective action at the 2-loop level:

$$\hat{W}_{2\text{-loop}} \to \sum_{h=1}^{2\Lambda} \left( \frac{576\pi^2}{\beta^4 N_0} \frac{1}{N_0 \log \Lambda} \right),$$

(4.35)

We summarize the effective action of the deformed IIB matrix model on $S^3$ in a matrix model compactification procedure up to 2-loop level:

$$\hat{w} \equiv \hat{W} / \sum_{h} \to \frac{\beta^4}{24} N_0^3 + 8\Lambda + 8\Lambda \log N_0 + \frac{576\pi^2}{\beta^4 N_0} \frac{1}{N_0 \log \Lambda},$$

(4.36)

where we have factored out the overall factor $\sum_{h}$ in front of the effective action.

We make the comparison between the effective action (3.26) of the gauge theory on $S^3$ and the effective action (4.36) of a matrix model compactified on $S^3$. Tr over matrices corresponds to the integration over the volume as follows:

$$\frac{1}{N_0} \text{Tr} \to \int d\Omega.$$

(4.37)

Therefore, we obtain the following relation by comparing the tree level action:

$$\alpha^4 = \beta^4 N_0,$$

(4.38)

as they are classically equivalent. We note here that the tree level and the 1-loop contributions are highly divergent just like the gauge theory on $S^3$ case in the preceding section. In fact they are very sensitive to the cutoff procedure. However we find the identical 2-loop contribution as it is only logarithmically divergent. We also expect that the higher loop contributions are finite since the 3-dimensional gauge theory is super renormalizable. With our identification of the inverse coupling constant $\alpha^4 = \beta^4 N_0$, we conclude again that the effective action of the deformed IIB matrix model on the $S^3$ is stable against the quantum corrections as it is dominated by the tree level contribution.

5 Conclusions and discussions

In this paper, we have shown that the $S^3$ background is one of nontrivial solutions of a deformed IIB matrix model. We have evaluated the effective actions of a deformed IIB matrix model on the $S^3$ background up to the 2-loop level in a two different cutoff procedure. We have found that the effective action of the deformed IIB matrix model on the $S^3$ background is stable on the condition that the coupling constant is $O(1)$. Since we have evaluated only planar diagrams, our investigation is valid in the large $N$ limit of $U(N)$ gauge theory on $S^3$.

In section 3, we have used the three derivatives $i \partial^{(1)}_i$ on the $S^3$ to evaluate the effective action on the $S^3$ background. In [20], the authors pointed out that the bosonic parts $A_\mu$ of IIB matrix model can be interpreted as derivatives on a curved space. They claim that IIB matrix model represents generic curved spaces in their interpretation. We believe that their claim is clarified by our concrete investigations on $S^3$ especially at the quantum level.
In [21], the $S^3$ is realized by three matrices $Y_i$ which are the vacuum configuration of a matrix model. We have also investigated the effective actions on the $S^3$ background $Y_i$ in this generalized matrix model compactification. In both cases, we find that the highly divergent contributions at the tree and 1-loop level are sensitive to the UV cutoff. However the 2-loop level contributions are universal since they are only logarithmically divergent. We expect that the higher loop contributions are insensitive to the UV cutoff since 3-dimensional gauge theory is super renormalizable. We can thus conclude that the effective action of the deformed IIB matrix model on the $S^3$ is stable against the quantum corrections as it is dominated by the tree level contribution.

We recall here that we have obtained the identical conclusions for the $S^2$ case. We thus believe that the 2- and 3-dimensional spheres are classical objects in IIB matrix model since the tree level effective action dominates. We can in turn conclude that they are not the solutions of the IIB matrix model without a Myers term.

We have investigated the IIB matrix model in a matrix model compactification procedure by imposing the periodicity on the blocks of the matrices. We believe it is also interesting to investigate the same background without such a condition. Such a case appears to correspond to the quenched matrix model in the flat background case. It is also interesting to investigate higher dimensional spaces such as $S^4$ or $S^3 \times R$ as they are physically and quantum mechanically more interesting.

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**A Spherical harmonics on $S^3$**

In this appendix, we summarize the spherical harmonics on the $S^3$. We adopt the following representation of the spherical harmonics on the $S^3$ [17]:

$$Y^\frac{n}{m \tilde{m}}_n (\Omega) = \sqrt{n + 1} e^{im\phi + i\tilde{m}\tilde{\phi}} d^\frac{n}{2} (m + \tilde{m}), \frac{1}{2} (m - \tilde{m}) (2\theta), \quad (A.1)$$

where

$$d^J_{M, \tilde{M}} (2\theta) = \frac{[(J + M)!(J - M)!]}{[(J + M)!(J - M)!]}^{1/2} \times (\cos \theta)^{M + \tilde{M}} (\sin \theta)^{M - \tilde{M}} P^\frac{M - \tilde{M}, M + \tilde{M}}{J - M} (\cos 2\theta), \quad (A.2)$$

$J = n/2 = 0, 1/2, 1, \cdots$, $M = (m + \tilde{m})/2 = -J, -J + 1, \cdots, J - 1, J$ and $\tilde{M} = (m - \tilde{m})/2 = -J, -J + 1, \cdots, J - 1, J$. Additionally, we make use of the Rodrigues formulas for the Jacobi polynomial:

$$P^\frac{M - \tilde{M}, M + \tilde{M}}{J - M} (\cos 2\theta) = \frac{(-1)^{J - M}}{2^{J - M} (J + \tilde{M})!} (1 - \cos 2\theta)^{-M + \tilde{M}} (1 + \cos 2\theta)^{-M - \tilde{M}}$$

$$\times \frac{d^{J - M}}{d \cos 2\theta^{J - \tilde{M}}} \left[ (1 - \cos 2\theta)^{J - \tilde{M}} (1 + \cos 2\theta)^{J + \tilde{M}} \right]. \quad (A.3)$$
$Y^{n}_{m\bar{m}}$ satisfies the following equations:

\[ J^{2}_{1} Y^{n}_{m\bar{m}}(\Omega) = \frac{n}{2} \left( \frac{n}{2} + 1 \right) Y^{n}_{m\bar{m}}(\Omega), \]
\[ J^{2}_{2} Y^{n}_{m\bar{m}}(\Omega) = \frac{n}{2} \left( \frac{n}{2} + 1 \right) Y^{n}_{m\bar{m}}(\Omega), \]
\[ J_{13} Y^{n}_{m\bar{m}}(\Omega) = \frac{1}{2} (m + \bar{m}) Y^{n}_{m\bar{m}}(\Omega), \]
\[ J_{23} Y^{n}_{m\bar{m}}(\Omega) = \frac{1}{2} (m - \bar{m}) Y^{n}_{m\bar{m}}(\Omega), \]  

(A.4)

where $J_{1i}$ and $J_{2i}$ are the generators of SU(2) algebra. $Y^{n}_{m\bar{m}}$ are normalized as follows:

\[ \int d\Omega Y^{n}_{m \bar{m}}(\Omega) Y^{n}_{m \bar{m}}(\Omega) = (-1)^{-\bar{m}1} \delta_{n1n2} \delta_{m1-m2} \delta_{\bar{m}1-\bar{m}2}, \]  

(A.5)

The complex conjugate of $Y^{n}_{m\bar{m}}$ is that

\[ Y^{n*}_{m\bar{m}}(\Omega) = (-1)^{\bar{m}} Y^{-n}_{-m-\bar{m}}(\Omega). \]  

(A.6)

The integrals of the product of three spherical harmonics [18, 19] can be evaluated as

\[ \int d\Omega Y^{n1}_{m1\bar{m}1}(\Omega) Y^{n2}_{m2\bar{m}2}(\Omega) Y^{n3}_{m3\bar{m}3}(\Omega) = (-1)^{(n1+n2+n3)/2} \sqrt{(n1+1)(n2+1)(n3+1)} \]
\[ \times \left( \begin{array}{c} n1/2 \\ n2/2 \\ n3/2 \end{array} \right) \left( \begin{array}{c} n1/2 \\ n2/2 \\ n3/2 \end{array} \right) \left( \begin{array}{ccc} M1 & M2 & M3 \\ \bar{M}1 & \bar{M}2 & \bar{M}3 \end{array} \right), \]  

(A.7)

where $(\cdots)$ represents the 3-j symbol of Wigner [22, 23].

**B Two-loop effective action on background $i \partial^{(1)}$**

In this appendix, we evaluate the effective action at the 2-loop level on the background: $i \partial^{(1)}$. We can evaluate the effective action $W$ of the deformed IIB matrix model on the $S^3$ as follows:

\[ W = - \log \int da d\varphi dc db e^{-\tilde{S}} \]
\[ = W_{\text{tree}} + W_{1-\text{loop}} + W_{2-\text{loop}}, \]  

(B.1)

where

\[ W_{2-\text{loop}} = \left\langle \exp \left\{ \text{Tr} \left[ \left[ p_i, a_\mu \right] [a^i, a^\mu] - b \left[ p_i, [a^i, c] \right] + \frac{1}{2} \varphi \Gamma^\mu [a_\mu, \varphi] - \frac{1}{3} f_{ijk} [a^i, a^j] a^k + \frac{1}{4} [a_\mu, a_\nu]^2 \right] \right\} \right\rangle_{1PI}, \]  

(B.2)

and $\langle \cdots \rangle_{1PI}$ represents that we sum only the 1PI (1-Particle-Irreducible) diagrams. There are five 1PI diagrams to evaluate which are illustrated in Fig. 2. The diagrams (a), (b) and (c) represent the contributions from gauge fields, and (c) involves the Myers type interaction. The diagrams (d) and (e) represent the contributions from ghost and fermion fields respectively.
B.1 Bosonic propagators

From the quadratic terms for the gauge fields $a_\mu$, we can read out the propagators of gauge boson modes $a_{\mu n}^n$.

\[
\text{Tr} \left( \frac{1}{2} a_\mu P_i^2 \delta^{\mu \nu} a_\nu \right) 
\rightarrow \int d\Omega \left( \frac{1}{2} \sum_{n_1 m_1 \tilde{m}_1} a_{\mu n_1 m_1} Y_{n_1 m_1} (\Omega) \frac{\alpha^2}{4} n_2 (n_2 + 2) \delta^{\mu \nu} a_{\nu m_2 \tilde{m}_2} Y_{n_2 \tilde{m}_2} (\Omega) \right)
\]

\[
= \frac{1}{2} \sum_{n_1 m_1 \tilde{m}_1} a_{\mu n_1 m_1} \alpha^2 \sum_{n_2 \tilde{m}_2} \frac{n_2 (n_2 + 2)}{4} \delta^{\mu \nu} (-1)^{\tilde{m}_1} \delta_{n_1 m_2} \delta_{\tilde{m}_1 \tilde{m}_2} a_{\nu m_2 \tilde{m}_2}.
\]

(B.3)

In the first step, we have taken the semi-classical limit and substituted the quantum fluctuations which are expanded by the spherical harmonics on the $S^3$:

\[
a_\mu = \sum_{nm\tilde{m}} a_{\mu nm\tilde{m}} Y_{nm\tilde{m}} (\Omega).
\]

Therefore,

\[
\langle a_{\mu n_1 m_1} a_{\nu n_2 \tilde{m}_2} \rangle = \frac{4}{\alpha^2} \frac{(-1)^{\tilde{m}_1}}{n_1 (n_1 + 2)} \delta_{\mu \nu} \delta_{n_1 m_2} \delta_{\tilde{m}_1 \tilde{m}_2}.
\]

(B.5)

The propagators of gauge boson fields become

\[
\langle a_\mu a_\nu \rangle = \frac{4}{\alpha^2} \sum_{nm\tilde{m}} \frac{(-1)^{\tilde{m}}}{n (n + 2)} \delta_{\mu \nu} Y_{n \tilde{m}}^n (\Omega_1) Y_{-\tilde{m}}^{-n} (\Omega_2).
\]

(B.6)

In the same way, we can read off the propagators of ghost fields.

\[
\langle c_{m_1 \tilde{m}_1} b_{m_2 \tilde{m}_2} \rangle = \frac{4}{\alpha^2} \frac{(-1)^{\tilde{m}_1}}{n_1 (n_1 + 2)} \delta_{n_1 m_2} \delta_{\tilde{m}_1 \tilde{m}_2},
\]

\[
\langle c b \rangle = \frac{4}{\alpha^2} \sum_{n m \tilde{m}} \frac{(-1)^{\tilde{m}}}{n (n + 2)} Y_{n \tilde{m}}^n (\Omega_1) Y_{-\tilde{m}}^{-n} (\Omega_2).
\]

(B.7)

We have introduced the quantum fluctuations which are expanded by the spherical harmonics on the $S^3$ as follows:

\[
c = \sum_{nm\tilde{m}} c_{nm\tilde{m}} Y_{nm\tilde{m}} (\Omega),
\]

\[
b = \sum_{nm\tilde{m}} b_{nm\tilde{m}} Y_{nm\tilde{m}} (\Omega).
\]

(B.8)
B.2 Contribution from four-point gauge boson vertex (a)

We evaluate the 1PI diagram involving a 4-point gauge boson vertex.

\[
V_4 = \frac{1}{4} \text{Tr} [a_\mu, a_\nu]^2
= \frac{1}{2} \text{Tr} (a_\mu a_\nu a_\mu a_\nu - a_\mu a_\mu a_\nu a_\nu).
\] (B.9)

We can calculate \(V_4\) using the Wick contraction.

\[
V_4 \rightarrow \left[ \frac{1}{2} (10 + 10) - \frac{1}{2} (10^2 + 10) \right] \int d\Omega_1 \left( \frac{4}{\alpha^2} \right)^2 \sum_{n_1m_1\bar{m}_1} \sum_{n_2m_2\bar{m}_2} \frac{(-1)^{\bar{m}_1+\bar{m}_2}}{n_1(n_1 + 2)n_2(n_2 + 2)}
\times Y_{m_1\bar{m}_1}^{n_1} (\Omega_1) Y_{-m_1-\bar{m}_1}^{-n_1} (\Omega_1) Y_{m_2\bar{m}_2}^{n_2} (\Omega_1) Y_{-m_2-\bar{m}_2}^{-n_2} (\Omega_1)
= -45 \int d\Omega_1 \int d\Omega_2 \left( \frac{4}{\alpha^2} \right)^2 \sum_{n_1m_1\bar{m}_1} \sum_{n_2m_2\bar{m}_2} \frac{(-1)^{\bar{m}_1+\bar{m}_2}}{n_1(n_1 + 2)n_2(n_2 + 2)}
\times Y_{m_1\bar{m}_1}^{n_1} (\Omega_1) Y_{-m_1-\bar{m}_1}^{-n_1} (\Omega_2) Y_{m_2\bar{m}_2}^{n_2} (\Omega_1) Y_{-m_2-\bar{m}_2}^{-n_2} (\Omega_2) \delta (\Omega_1 - \Omega_2).
\] (B.10)

Here we can insert the complete set as follows:

\[
\sum_{nm\bar{m}} (-1)^{\bar{m}} Y_{nm\bar{m}}^{n} (\Omega_1) Y_{-m-\bar{m}}^{-n} (\Omega_2) = \delta (\Omega_1 - \Omega_2).
\] (B.11)

Therefore, we can get

\[
V_4 \rightarrow -45 \sum_{123} \Psi_{123}^* (\Omega_2) \frac{1}{P^2 Q^2} \Psi_{123} (\Omega_1),
\] (B.12)

where \(P, Q\) and \(R\) are defined as follows:

\[
P_i Y_{m_i\bar{m}_i}^{n_i} (\Omega) \equiv p_i Y_{m_i\bar{m}_i}^{n_i} (\Omega),
Q_i Y_{m_2\bar{m}_2}^{n_2} (\Omega) \equiv p_i Y_{m_2\bar{m}_2}^{n_2} (\Omega),
R_i Y_{m_3\bar{m}_3}^{n_3} (\Omega) \equiv p_i Y_{m_3\bar{m}_3}^{n_3} (\Omega).
\] (B.13)

We have introduced the wave functions such that

\[
\Psi_{123} (\Omega) \equiv \int d\Omega Y_{m_1\bar{m}_1}^{n_1} (\Omega) Y_{m_2\bar{m}_2}^{n_2} (\Omega) Y_{m_3\bar{m}_3}^{n_3} (\Omega).
\] (B.14)

\(\sum_{123}\) denotes \(\sum_{n_1m_1\bar{m}_1} \sum_{n_2m_2\bar{m}_2} \sum_{n_3m_3\bar{m}_3}\).

B.3 Contribution from three-point gauge boson vertex (b)

We evaluate the 1PI diagram involving 3-point gauge boson vertices. We can express the contribution corresponding the diagram (b) as follows:

\[
V_3 = \frac{1}{2} (\text{Tr} [p_i, a_\mu] [a^i, a^\mu])^2.
\] (B.15)
We can express the result as a compact form:

\[ V_3 \longrightarrow \frac{9}{2} \sum_{123} \Psi^*_{123} (\Omega_2) \frac{2P^2 - P \cdot Q - P \cdot R}{P^2Q^2R^2} \Psi_{123} (\Omega_1). \]  

(B.16)

We use the following relation:

\[ P \cdot Q \Psi_{123} (\Omega) = \frac{R^2 - P^2 - Q^2}{2} \Psi_{123} (\Omega), \]  

(B.17)

and the momentum conserved relation: \( P + Q + R = 0 \). Therefore, we can simplify the result as

\[ V_3 \longrightarrow \frac{27}{3} \sum_{123} \Psi^*_{123} (\Omega_2) \frac{1}{P^2Q^2} \Psi_{123} (\Omega_1). \]  

(B.18)

We note the relation that

\[ \sum_{m\tilde{m}} (-1)^{\tilde{m}} Y_{m\tilde{m}}^n (\Omega) P_i Y_{m-\tilde{m}}^n (\Omega) = - \sum_{m\tilde{m}} (-1)^{\tilde{m}} (P_i Y_{m\tilde{m}}^n (\Omega)) Y_{m-\tilde{m}}^n (\Omega). \]  

(B.19)

### B.4 Contribution from the Myers type interaction (c)

The diagram involving the Myers type interactions is represented as follows:

\[ V_M = -\frac{1}{18} (\text{Tr} f_{ijk}[a^i, a^j]a^k)^2. \]  

(B.20)

We get the following result

\[ V_M \longrightarrow 4\alpha^2 \sum_{123} \Psi^*_{123} (\Omega_2) \frac{1}{P^2Q^2R^2} \Psi_{123} (\Omega_1), \]  

(B.21)

where we have used the relation: \( f_{ijk} f^{ijk} = 6\alpha^2 \).

### B.5 Contribution from the ghost interaction (d)

We evaluate the contribution from the ghost interactions.

\[ V_{gh} = \frac{1}{2} \left( \text{Tr} [p_i, b][a^i, c] \right)^2. \]  

(B.22)

The result is

\[ V_{gh} \longrightarrow -\frac{1}{2} \sum_{123} \Psi^*_{123} (\Omega_2) \frac{1}{P^2Q^2} \Psi_{123} (\Omega_1). \]  

(B.23)
B.6 Fermion propagator

We can read off the fermion propagator from the quadratic terms in the action (3.11):

\[ \text{Tr} \left( -\frac{1}{2} \bar{\varphi} \Gamma^i P_i \varphi \right) \]

\[ \longrightarrow \int d\Omega \left( -\frac{1}{2} \sum_{n_1 m_1 \tilde{m}_1} \sum_{n_2 m_2 \tilde{m}_2} \bar{\varphi}_{n_1 \tilde{m}_1} \Gamma^i P_i \varphi_{n_2 \tilde{m}_2} \right) \]

\[ = -\frac{1}{2} \sum_{n_1 m_1 \tilde{m}_1} \sum_{n_2 m_2 \tilde{m}_2} \bar{\varphi}_{n_1 \tilde{m}_1} \Gamma^i P_i \delta_{n_1 n_2} \delta_{m_1 m_2} \delta_{\tilde{m}_1 \tilde{m}_2} \varphi_{n_2 \tilde{m}_2}, \quad (B.24) \]

where we have expanded the quantum fluctuations by the spherical harmonics on the \( S^3 \) as follows:

\[ \varphi = \sum_{nm} \varphi_{nm} Y_{nm} (\Omega). \quad (B.25) \]

Therefore,

\[ \langle \varphi_{n_1 \tilde{m}_1} \bar{\varphi}_{n_2 \tilde{m}_2} \rangle = -\frac{1}{\Gamma^i P_i} \delta_{n_1 n_2} \delta_{m_1 m_2} \delta_{\tilde{m}_1 \tilde{m}_2}. \quad (B.26) \]

The fermion propagator is

\[ \langle \varphi \bar{\varphi} \rangle = \sum_{nmn} \left( -\frac{1}{\Gamma^i P_i} \right) (-1)^{\tilde{m}} Y_{nm}^* (\Omega_1) Y_{n}^{\tilde{m}} (\Omega_2). \quad (B.27) \]

We can further expand the fermion propagator in powers of \( 1/P^2 \) as follows:

\[ -\frac{1}{\Gamma^i P_i} = -\frac{1}{P^2 + \frac{i}{2} F_{ij} \Gamma^j} \Gamma^k P_k \]

\[ = -\frac{1}{P^2} \Gamma^i P_i + \frac{i}{2} \left( \frac{1}{P^2} \right)^2 F_{ij} \Gamma^j \Gamma^k P_k + \frac{1}{4} \left( \frac{1}{P^2} \right)^3 F_{ij} \Gamma^j F_{kl} \Gamma^k P_a + \ldots. \quad (B.28) \]

The second term of (B.28) is that

\[ \frac{i}{2} \left( \frac{1}{P^2} \right)^2 F_{ij} \Gamma^i \Gamma^j \Gamma^k P_k = \frac{i}{2} \left( \frac{1}{P^2} \right)^2 (f_{ijkl} \Gamma^l P_i - 2i \Gamma \cdot P + \ldots) \]

\[ = \frac{i}{2} \left( \frac{1}{P^2} \right)^2 f_{ijkl} \Gamma^l P_i + \left( \frac{1}{P^2} \right)^2 \Gamma \cdot P + \ldots. \quad (B.29) \]

Here we have used the formula as follows:

\[ \Gamma^i \Gamma^j \Gamma^k = \Gamma^{ijk} + \delta^{ij} \Gamma^k - \delta^{ik} \Gamma^j + \delta^{jk} \Gamma^i. \quad (B.30) \]

While the third term of (B.28) is

\[ \frac{1}{4} \left( \frac{1}{P^2} \right)^3 F_{ij} \Gamma^j F_{kl} \Gamma^k \Gamma^a P_a = \frac{1}{4} \left( \frac{1}{P^2} \right)^3 (-4 \Gamma \cdot PP^2 + \ldots) \]

\[ = -\left( \frac{1}{P^2} \right)^2 \Gamma \cdot P + \ldots. \quad (B.31) \]
Here we have used the formula as follows:
\[
\Gamma^{ijkl a} = \Gamma^{ijkl a} - \Gamma^{aijkl} + \delta^{ik} \Gamma^{jka} - \delta^{ja} \Gamma^{ikl} + \delta^{ja} \Gamma^{ika} + \delta^{ia} \Gamma^{ikl} \\
- \delta^{ka} \Gamma^{ijl} + \delta^{la} \Gamma^{ijk} + \delta^{ik} \delta^{ja} \Gamma^{a} + \delta^{ja} \delta^{ik} \Gamma^{a} - \delta^{ja} \delta^{ik} \Gamma^{k} \\
+ \delta^{ja} \delta^{ik} \Gamma^{l} - \delta^{ia} \delta^{il} \Gamma^{k} + \delta^{ka} \delta^{il} \Gamma^{j} - \delta^{ka} \delta^{il} \Gamma^{i} - \delta^{ja} \delta^{jk} \Gamma^{i} + \delta^{ja} \delta^{jk} \Gamma^{i}.
\]
(B.32)

In this way, we obtain
\[
- \frac{1}{\Gamma^i P_i} = - \frac{1}{P^2} \Gamma^i P_i + \frac{i}{2} \left( \frac{1}{P^2} \right)^2 f_{ijkl} \Gamma^{ijkl} P^j P_l + O \left( \left( \frac{1}{P^2} \right)^3 \right).
\]
(B.33)

### B.7 Contribution from the fermion interaction (e)

Finally, we evaluate the diagram involving fermion interactions.

\[
V_F = \frac{1}{8} \left( \mathrm{Tr} \bar{\phi} \Gamma^\mu[a_\mu, \varphi] \right)^2 \\
= \frac{1}{2} \left( \mathrm{Tr} \bar{\varphi} \Gamma^\mu a_\mu \varphi \right)^2.
\]
(B.34)

We perform the Wick contractions and evaluate it in the semi-classical limit:

\[
V_F \rightarrow \frac{1}{2} \int d\Omega_1 \int d\Omega_2 \\
\times \mathrm{tr} \sum_{n_1 m_1 \bar{m}_1} \left( \frac{1}{P^2} \Gamma^i P_i - \frac{i}{2} \left( \frac{1}{P^2} \right)^2 f_{ijkl} \Gamma^{ijkl} P^j P_l + \cdots \right) \left( \frac{1 + \Gamma_{11}}{2} \right) \Gamma^\mu \\
\times (-1)^{\bar{m}_1} Y^{n_1}_{n_1 \bar{m}_1} (\Omega_1) Y^{n_1}_{-n_1 \bar{m}_1} (\Omega_2) \\
\times \sum_{n_2 m_2 \bar{m}_2} \frac{4}{\alpha^2 n_2 (n_2 + 2)} \delta_{\mu \nu} Y^{n_2}_{n_2 \bar{m}_2} (\Omega_1) Y^{n_2}_{-n_2 \bar{m}_2} (\Omega_2) \\
\times \sum_{n_3 m_3 \bar{m}_3} \left( -\frac{1}{P^2} \Gamma^a P_a + \frac{i}{2} \left( \frac{1}{P^2} \right)^2 f_{abc} \Gamma^{abcd} P^c P_d + \cdots \right) \left( \frac{1 + \Gamma_{11}}{2} \right) \Gamma^\nu \\
\times (-1)^{\bar{m}_3} Y^{n_3}_{n_3 \bar{m}_3} (\Omega_1) Y^{n_3}_{-n_3 \bar{m}_3} (\Omega_2),
\]
(B.35)

where \( \mathrm{tr} \) represents the trace over gamma matrices. Firstly, we evaluate the leading term in the 1/P^2 expansion. The trace of products of gamma matrices in the leading term is as evaluated follows:

\[
\mathrm{tr} \Gamma^\mu \Gamma^\nu \Gamma_\mu \left( \frac{1 + \Gamma_{11}}{2} \right) = -8 \mathrm{tr} \Gamma^\mu \Gamma_\mu \left( \frac{1 + \Gamma_{11}}{2} \right) = -128 \delta^{ia}.
\]
(B.36)

The trace of products of gamma matrices in the next leading term is evaluated as follows:

\[
\mathrm{tr} \Gamma^\mu \Gamma^\nu \Gamma^{abcd} \Gamma_\mu \left( \frac{1 + \Gamma_{11}}{2} \right) = -8 \mathrm{tr} \Gamma^\nu \Gamma^{abcd} \left( \frac{1 + \Gamma_{11}}{2} \right) = 0.
\]
(B.37)
The trace of products of gamma matrices in the next leading term is evaluated as follows:
\[
\text{tr} \Gamma^{ijl} \Gamma^{\mu} \Gamma^{abd} \Gamma_{\mu} \left( \frac{1 + \Gamma_{11}}{2} \right) = 4 \text{tr} \Gamma^{ijl} \Gamma^{\mu} \Gamma^{abd} \Gamma_{\mu} \left( \frac{1 + \Gamma_{11}}{2} \right) = -64 \left( \delta^{ia} \delta^{jd} \delta^{lb} - \delta^{ia} \delta^{jd} \delta^{lb} + \delta^{ia} \delta^{ib} \delta^{ld} + \delta^{ia} \delta^{ib} \delta^{jd} - \delta^{ja} \delta^{id} \delta^{lb} \right).
\] (B.38)

In this way, we obtain the following result
\[
V_F \sim \sum_{123} \Psi^*_{123} (\Omega_2) \left[ -64 \frac{P \cdot Q}{P^2 Q^2 R^2} + 32 \alpha^2 \frac{P^2 Q^2}{(P^2 Q^2)^2 R^2} \right] \Psi_{123} (\Omega_1).
\] (B.39)

**B.8 Two-loop effective action**

We summarize the 2-loop effective action on the \(S^3\) as follows:
\[
W_{2\text{-loop}} \sim V_4 + V_3 + V_M + V_{gh} + V_F = 36 \alpha^2 \sum_{123} \Psi^*_{123} (\Omega_2) \frac{1}{P^2 Q^2 R^2} \Psi_{123} (\Omega_1) = \frac{2304}{\alpha^4} \sum_{n_1n_2n_3} \frac{(n_1 + 1)(n_2 + 1)(n_3 + 1)}{n_1(n_1 + 2)n_2(n_2 + 2)n_3(n_3 + 2)}.
\] (B.40)

Here we have used the formula (A.7), and evaluated the summations over \(m_1, m_2, m_3, \tilde{m}_1, \tilde{m}_2\) and \(\tilde{m}_3\).

**C Fuzzy sphere harmonics**

In this appendix, we summarize the fuzzy sphere harmonics based on the work [21]. The fuzzy sphere harmonics is the eigen function on a set of fuzzy \(S^2\) with different radii. Let us consider a set of linear maps \(\mathcal{M}_{jj'}\) from a \((2j + 1)\)-dimensional complex vector space \(V_j\) to a \((2j' + 1)\)-dimensional complex vector space \(V_{j'}\), where \(j\) and \(j'\) are non-negative half-integers. \(\mathcal{M}_{jj'}\) is \([(2j + 1) \times (2j' + 1)]\)-dimensional complex vector space. They constructed a basis of \(\mathcal{M}_{jj'}\) to use a basis of the spin \(j\) and \(j'\) representations of \(SU(2)\) as a basis of \(V_j\) and \(V_{j'}\):
\[
|jr\rangle |j'r\rangle ,
\] (C.1)
where \(r = -j, -j + 1, \cdots, j - 1, j\) and \(r' = -j', -j' + 1, \cdots, j' - 1, j'\). Then, an arbitrary element of \(\mathcal{M}_{jj'}\): \(M\) is represented by
\[
M = \sum_{rr'} M_{rr'} |jr\rangle |j'r\rangle .
\] (C.2)

They defined linear maps from \(\mathcal{M}_{jj'}\) to \(\mathcal{M}_{jj'}\) by its operation on the basis as follows:
\[
L_i \circ |jr\rangle |j'r\rangle = L_i |jr\rangle |j'r\rangle - |jr\rangle |j'r\rangle L_i,
\] (C.3)
where $L_i$ is a representation matrix [4.6] for a $\tilde{N}$-dimensional representation of $SU(2)$. The matrix element $M_{rr'}$ is transformed under linear maps from $M_{j'j}$ to $M_{j''j''}$:

$$(L_i \circ M)_{rr'} = \left(L_i^{[j]}\right)_{rp} M_{pr'} - M_{rr'} \left(L_i^{[j']\dagger}\right)_{r'p'},$$

(C.4)

where $L_i^{[j]}$ is the $(2j + 1) \times (2j + 1)$ representation matrix of for the spin $j$ representation of $SU(2)$. Additionally, the following identity holds:

$$(L_i \circ L_j \circ -L_j \circ L_i) |jr\rangle|j'r'\rangle = i\epsilon_{ijr} L^k \circ |jr\rangle|j'r'\rangle.$$  

(C.5)

The fuzzy sphere harmonics is defined as a basis of $M_{j'j}$:

$$\hat{Y}_{JM}^{(jj')} = \sum_{rr'} (-1)^{3j-r+2J-M-(\zeta+\zeta')/2} \sqrt{N_0 (2J+1)} \begin{pmatrix} j & j' \\ -r & m \end{pmatrix} |jr\rangle|j'r'\rangle,$$

(C.6)

where $J = |j-j'|+1, \cdots, j+j'-1, j+j'$ and $M = -J,-J+1, \cdots, J-1, J$. Furthermore, they have introduced a positive integer $N_0$ as the following parameter:

$$2j+1 = N_0 + \zeta, \quad 2j' + 1 = N_0 + \zeta'.$$

(C.7)

Therefore, $J = |\zeta - \zeta'|/2, |\zeta - \zeta'|/2+1, \cdots, (\zeta + \zeta')/2+N_0-2, (\zeta + \zeta')/2+N_0-1$. $N_0$ plays a role of an ultraviolet cutoff scale for the angular momentum. Since $\hat{Y}_{JM}^{(jj')}$ is the basis of the spin $J$ irreducible representation of $SU(2)$, the following equations hold

$$L_i \circ L_i \circ \hat{Y}_{JM}^{(jj')} = J(J+1) \hat{Y}_{JM}^{(jj')},$$

$$L_3 \circ \hat{Y}_{JM}^{(jj')} = M \hat{Y}_{JM}^{(jj')}.$$  

(C.8)

$\hat{Y}_{JM}^{(jj')}$ is normalized as follows:

$$\frac{1}{N_0} \text{Tr} \hat{Y}_{J_1M_1}^{(jj')} \hat{Y}_{J_2M_2}^{(jj')} = (-1)^{M_1-(j-j')} \delta_{J_1J_2} \delta_{M_1-M_2},$$

(C.9)

where $\text{Tr}$ stands for a trace over $(2j+1) \times (2j+1)$ matrices. The hermitian conjugate of $\hat{Y}_{JM}^{(jj')}$ is defined as

$$\hat{Y}_{JM}^{(jj')\dagger} = (-1)^{M-(j-j')} \hat{Y}_{J-M}^{(j'j)}.$$  

(C.10)

The product of three fuzzy spherical harmonics can be evaluated as

$$\frac{1}{N_0} \text{Tr} \hat{Y}_{J_1M_1}^{(jj')} \hat{Y}_{J_2M_2}^{(j'j'')} \hat{Y}_{J_3M_3}^{(j''j)} = (-1)^{-j+j'+J_1+J_2+J_3-\zeta/2-\zeta'/2-\zeta''} \sqrt{N_0 (2J_1+1) (2J_2+1) (2J_3+1)}$$

$$\times \begin{pmatrix} J_1 & J_2 & J_3 \\ M_1 & M_2 & M_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3 \\ j & j & j' \end{pmatrix},$$

(C.11)

where $(\cdots)$ and $\{ \cdots \}$ represent the 3-$j$ and 6-$j$ symbol of Wigner, respectively [22, 23]. In the large $N_0$ limit, we can obtain that

$$\frac{1}{N_0} \text{Tr} \hat{Y}_{J_1M_1}^{(jj')} \hat{Y}_{J_2M_2}^{(j'j'')} \hat{Y}_{J_3M_3}^{(j''j)} \longrightarrow (-1)^{2J_2-2J_3-\tilde{M}_1} \sqrt{(2J_1+1) (2J_2+1) (2J_3+1)}$$

$$\times \begin{pmatrix} J_1 & J_2 & J_3 \\ M_1 & M_2 & M_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3 \\ \tilde{M}_1 & \tilde{M}_2 & \tilde{M}_3 \end{pmatrix},$$

(C.12)

where $\tilde{M}_1 = j-j'$, $\tilde{M}_2 = j' - j''$ and $\tilde{M}_3 = j'' - j$. 

24
D Two-loop effective action on background $Y_i$

In this appendix, we calculate the effective action at the 2-loop level on the background $Y_i$. We can investigate the effective action $\hat{W}$ of the deformed IIB matrix model on the background $Y_i$ as follows:

$$\hat{W} = -\log \int da d\varphi dc db e^{-S} = \hat{W}_{\text{tree}} + \hat{W}_{1-\text{loop}} + \hat{W}_{2-\text{loop}},$$

(D.1)

where

$$\hat{W}_{2-\text{loop}} = \left\langle \exp \left\{ \begin{array}{l} \text{Tr} \sum_{stu} \left[ \mathcal{P}_{i} a^{(s,t)}_{\mu} a^{(t,u)}_{\nu} - b^{(s,t)} p_i, a^{(t,u)}_{\nu} c^{(u,s)} \right] \\ + \frac{1}{2} \mathcal{P}^{(s,t)} \Gamma^{(t,u)} a^{(t,u)}_{\nu} c^{(u,s)} - \frac{1}{3} f_{ijk} a^{(s,t)}_{\nu} a^{(t,u)}_{\mu} a^{(u,s)}_{\lambda} + \frac{1}{4} \left[ a^{(s,t)}_{\mu}, a^{(t,u)}_{\nu} \right]^2 \end{array} \right\} \right\rangle_{1\text{PI}}$$

(D.2)

and $\langle \cdots \rangle_{1\text{PI}}$ implies that we sum only the 1PI diagrams. The evaluation procedure parallels to that of appendix A. There are precisely identical five 1PI diagrams to evaluate which are illustrated in Fig. 2.

D.1 Bosonic propagators

From the quadratic terms for the gauge fields $a^{(s,t)}_{\mu}$ in the action (4.18), we can read out the propagators of gauge boson modes $a^{(s,t)}_{\mu JM}$:

$$\text{Tr} \sum_{st} \left( \frac{1}{2} \mathcal{P}_{i} a^{(s,t)}_{\mu} P_{i} \delta^{(t,s)}_{\nu} a^{(t,s)}_{\nu} \right)$$

$$= \text{Tr} \sum_{st} \left( \frac{1}{2} \sum_{J_{1}M_{1}} \sum_{J_{2}M_{2}} a^{(s,t)}_{\mu J_{1}M_{1}} \mathcal{A}_{\mu J_{2}M_{2}} \delta^{(t,s)}_{\nu} a^{(t,s)}_{\nu} \delta_{J_{1},J_{2}} \beta^{2} J_{2} \left( J_{2} + 1 \right) \delta_{\mu \nu} \hat{Y}_{J_{1}J_{2}} \right)$$

(D.3)

We expand the quantum fluctuations by the fuzzy spherical harmonics as follows:

$$a^{(s,t)}_{\mu} = \sum_{JM} a^{(s,t)}_{\mu JM} \mathcal{Y}_{JM} \hat{Y}_{JM}.$$

(D.4)

Therefore,

$$\left\langle a^{(s,t)}_{\mu J_{1}M_{1}} a^{(t,s)}_{\nu J_{2}M_{2}} \right\rangle = \frac{1}{\beta^{2} N_{0}} \frac{(-1)^{M_{1}-(J_{s}-j_{t})}}{J_{1} \left(J_{1} + 1 \right)} \delta_{\mu \nu} \delta_{J_{1}J_{2}} \delta_{J_{M_{1}J_{M_{2}}}}.$$

(D.5)

The propagators of gauge boson fields become

$$\left\langle a^{(s,t)}_{\mu} a^{(t,s)}_{\nu} \right\rangle = \frac{1}{\beta^{2} N_{0}} \sum_{JM} \frac{(-1)^{M-(J_{s}-j_{t})}}{J \left(J + 1 \right)} \delta_{\mu \nu} \mathcal{Y}_{JM} \hat{Y}_{JM}.$$

(D.6)
In the same way, we can read off the propagators of ghost fields:

\[
\langle c_{J_1 M_1}^{(s,t)} b_{J_2 M_2}^{(t,s)} \rangle = \frac{1}{\beta^2 N_0} \frac{(-1)^{M_1 - (j_s - j_t)}}{J_1 (J_1 + 1)} \delta_{J_1 J_2} \delta_{M_1 - M_2},
\]

\[
\langle c^{(s,t)} b^{(t,s)} \rangle = \frac{1}{\beta^2 N_0} \sum_{J M} \frac{(-1)^{M - (j_s - j_t)}}{J (J + 1)} \hat{\gamma}_{J M}^{(j_s j_t)} \hat{\gamma}_{J M}^{(j_t j_s)}.
\] (D.7)

We expand the ghost fields by the fuzzy spherical harmonics as follows:

\[
c^{(s,t)} = \sum_{J M} c_{J M}^{(s,t)} \otimes \hat{\gamma}_{J M}^{(j_s j_t)},
\]

\[
b^{(s,t)} = \sum_{J M} b_{J M}^{(s,t)} \otimes \hat{\gamma}_{J M}^{(j_s j_t)}.
\] (D.8)

### D.2 Contribution from four-point gauge boson vertex (a)

We evaluate the 1PI diagram involving a 4-point gauge boson vertex:

\[
\hat{V}_4 = \frac{1}{4} \text{Tr} \sum_{stu} \left[ a_{\mu}^{(s,t)} a_{\nu}^{(t,u)} \right]^2 = \frac{1}{2} \text{Tr} \sum_{stuw} \left( a_{\mu}^{(s,t)} a_{\nu}^{(t,u)} a_{\rho}^{(u,v)} a_{\sigma}^{(v,w)} - a_{\mu}^{(s,t)} a_{\rho}^{(t,u)} a_{\sigma}^{(u,v)} a_{\nu}^{(v,w)} \right),
\] (D.9)

We can calculate \( \hat{V}_4 \) by performing the Wick contraction.

\[
\hat{V}_4 = \left[ \frac{1}{2} (10 + 10) - \frac{1}{2} (10^2 + 10) \right] \text{Tr} \sum_{stuw} \sum_{J_1 M_1} \sum_{J_2 M_2} \left( \frac{1}{\beta^2 N_0} \right)^2 \frac{(-1)^{M_1 + M_2 - (j_s - j_t) - (j_u - j_v)}}{J_1 (J_1 + 1) J_2 (J_2 + 1)} \times \hat{\gamma}_{J_1 M_1}^{(j_s j_t)} \hat{\gamma}_{J_1 M_1}^{(j_t j_u)} \hat{\gamma}_{J_2 M_2}^{(j_u j_v)} \hat{\gamma}_{J_2 M_2}^{(j_v j_s)}
\]

\[
= -45 \text{Tr} \sum_{stuw} \sum_{pq} \sum_{J_1 M_1} \sum_{J_2 M_2} \left( \frac{1}{\beta^2 N_0} \right)^2 \frac{(-1)^{M_1 + M_2 - (j_s - j_t) - (j_u - j_v)}}{J_1 (J_1 + 1) J_2 (J_2 + 1)} \times \hat{\gamma}_{J_1 M_1}^{(j_s j_t)} \hat{\gamma}_{J_1 M_1}^{(j_t j_p)} \hat{\gamma}_{J_2 M_2}^{(j_p j_q)} \hat{\gamma}_{J_2 M_2}^{(j_q j_s)} \delta_{pu} \delta_{qs}.
\] (D.10)

Here we have inserted the complete set as follows:

\[
\frac{1}{N_0} \text{Tr} \sum_{J_3 M_3} (-1)^{M_3 - (j_p - j_q)} \hat{\gamma}_{J_3 M_3}^{(j_p j_q)} \hat{\gamma}_{J_3 M_3}^{(j_q j_p)} = \delta_{pu} \delta_{qs}.
\] (D.11)

Therefore, we can get

\[
\hat{V}_4 = \frac{-45}{N_0} \sum_{123} \hat{\Psi}^{\dagger}_{123} \frac{1}{P^2 Q^2} \hat{\Psi}_{123},
\] (D.12)

where \( \hat{P} \), \( \hat{Q} \) and \( \hat{R} \) are defined as follows:

\[
\hat{P}_{J_1 M_1} \hat{\gamma}_{J_1 M_1}^{(j_s j_t)} \equiv \left[ p_i, \hat{\gamma}_{J_1 M_1}^{(j_s j_t)} \right],
\]

\[
\hat{Q}_{J_2 M_2} \hat{\gamma}_{J_2 M_2}^{(j_s j_t)} \equiv \left[ q_i, \hat{\gamma}_{J_2 M_2}^{(j_s j_t)} \right],
\]

\[
\hat{R}_{J_3 M_3} \hat{\gamma}_{J_3 M_3}^{(j_s j_t)} \equiv \left[ r_i, \hat{\gamma}_{J_3 M_3}^{(j_s j_t)} \right].
\] (D.13)
We have introduced the following wave function:

\[
\hat{\Psi}_{123} \equiv \frac{1}{N_0} \text{Tr} \sum_{stu} \hat{\chi}^{(j_{s}j_{t})}_{J_{1}M_{1}} \hat{\chi}^{(j_{t}j_{u})}_{J_{2}M_{2}} \hat{\chi}^{(j_{u}j_{s})}_{J_{3}M_{3}}.
\]  

(D.14)

\[\sum_{123}\] denotes \[\sum_{J_{1}M_{1}} \sum_{J_{2}M_{2}} \sum_{J_{3}M_{3}}\].

D.3 Contribution from three-point gauge boson vertex (b)

We evaluate the 1PI diagram involving 3-point gauge boson vertices. We can express the contribution corresponding to the diagram (b) as follows:

\[
\hat{V}_{3} = \frac{1}{2} \left( \text{Tr} \sum_{stu} [p_{i}, a_{\mu}^{(s,t)}] \left[ a_{\mu}^{(t,u)} , a_{\mu}^{(u,s)} \right] \right)^2.
\]  

(D.15)

We can express the result as a following compact form:

\[
\hat{V}_{3} = \frac{9}{2N_0} \sum_{123} \hat{\Psi}_{123}^{\dagger} 2 \hat{\mathcal{P}}^{2} - \hat{\mathcal{P}} \cdot \hat{\mathcal{Q}} - \hat{\mathcal{P}} \cdot \hat{\mathcal{R}} \hat{\Psi}_{123}.
\]  

(D.16)

We use the following relation:

\[
\hat{P} \cdot \hat{Q} \hat{\Psi}_{123} = \frac{\hat{R}^{2} - \hat{P}^{2} - \hat{Q}^{2}}{2} \hat{\Psi}_{123},
\]  

(D.17)

and the momentum conserved relation: \[\hat{P} + \hat{Q} + \hat{R} = 0\]. Therefore, we can simplify the result as

\[
\hat{V}_{3} = \frac{27}{3N_0} \sum_{123} \hat{\Psi}_{123}^{\dagger} \frac{1}{\hat{P}^{2}Q^{2}R^{2}} \hat{\Psi}_{123}.
\]  

(D.18)

We note the following relation

\[
\sum_{M} (-1)^{M-(j_{s}-j_{t})} \hat{\chi}^{(j_{s}j_{t})}_{J_{M}} \hat{P}_{J} \hat{\chi}^{(j_{t}j_{s})}_{J-M} = - \sum_{M} (-1)^{M-(j_{s}-j)} \left( \hat{P}_{J} \hat{\chi}^{(j_{s}j_{t})}_{J_{M}} \right) \hat{\chi}^{(j_{t}j_{s})}_{J-M}.
\]  

(D.19)

D.4 Contribution from the Myers type interaction (c)

The diagram involving the Myers type interactions is represented as follows:

\[
\hat{V}_{M} = -\frac{1}{18} \left( \text{Tr} \sum_{stu} f_{ijk} [a_{\mu}^{(s,t)i}, a_{\mu}^{(t,u)j}] a_{\mu}^{(u,s)k} \right)^2.
\]  

(D.20)

We can evaluate it as

\[
\hat{V}_{M} = \frac{4\beta^{2}}{N_0} \sum_{123} \hat{\Psi}_{123}^{\dagger} \frac{1}{P^{2}Q^{2}R^{2}} \hat{\Psi}_{123},
\]  

(D.21)

where we have used the relation: \[f_{ijk}f^{ijk} = 6\beta^{2}\].

27
D.5 Contribution from the ghost interaction (d)

We evaluate the contribution from the ghost interactions.

\[ \hat{V}_{gh} = \frac{1}{2} \left( \text{Tr} \sum_{stu} [\beta_i, b^{(s,t)}] \left[ a^{(t,u)}i, c^{(u,s)} \right] \right)^2. \]  \hspace{1cm} (D.22)

The result is

\[ \hat{V}_{gh} = -\frac{1}{2N_0} \sum_{123} \hat{\Psi}_{123} \hat{P}_2 \hat{Q}^2 \hat{\Psi}_{123}. \]  \hspace{1cm} (D.23)

D.6 Fermion propagator

We can read off the fermion propagator from the quadratic term of \( \varphi \) in the action (4.18):

\[
\begin{align*}
\text{Tr} \sum_{st} \left( -\frac{1}{2} \bar{\varphi}^{(s,t)} \Gamma^i P_i \varphi^{(t,s)} \right) \\
= \text{Tr} \sum_{st} \left( -\frac{1}{2} \sum_{J_1M_1 J_2M_2} \bar{\varphi}^{(s,t)} J_1M_1 \otimes \hat{Y}_{J_1M_1}^{(j_s, j_t)} \Gamma^i P_i \varphi^{(t,s)} J_2M_2 \otimes \hat{Y}_{J_2M_2}^{(j_t, j_s)} \right) \\
= -\frac{1}{2} \sum_{J_1M_1 J_2M_2} \bar{\varphi}^{(s,t)} J_1M_1 \Gamma^i P_i N_0 \delta_{J_1J_2} \delta_{M_1M_2} \varphi^{(t,s)} J_2M_2, \hspace{1cm} (D.24)
\end{align*}
\]

where we expanded the quantum fluctuations by the fuzzy spherical harmonics as follows:

\[ \varphi^{(s,t)} = \sum_{JM} \varphi_{JM}^{(s,t)} \otimes \hat{Y}_{JM}^{(s,t)}. \]  \hspace{1cm} (D.25)

We thus find

\[ \langle \varphi_{J_1M_1}^{(s,t)} \varphi_{J_2M_2}^{(t,s)} \rangle = -\frac{1}{N_0 \Gamma^i P_i} \delta_{J_1J_2} \delta_{M_1M_2}. \]  \hspace{1cm} (D.26)

The fermion propagator is

\[ \langle \varphi^{(s,t)} \varphi^{(t,s)} \rangle = \sum_{JM} \left( -\frac{1}{N_0 \Gamma^i P_i} \right) (-1)^{M-(j_s-j_t)} \hat{Y}_{JM}^{(j_s, j_t)} \hat{Y}_{JM}^{(j_t, j_s)}. \]  \hspace{1cm} (D.27)

We can further expand the fermion propagator in powers of \( 1/P^2 \) as in appendix A.6.

\[ -\frac{1}{\Gamma^i P_i} = -\frac{1}{P^2} \Gamma^i P_i + i \frac{1}{2} \left( \frac{1}{P^2} \right)^2 f_{ijk} \Gamma^{ijkl} P^k P_l + \mathcal{O} \left( \left( \frac{1}{P^2} \right)^3 \right). \]  \hspace{1cm} (D.28)

D.7 Contribution from the fermion interaction (e)

Finally, we evaluate the diagram involving fermion interactions.

\[ \hat{V}_F = \frac{1}{8} \left( \text{Tr} \sum_{stu} \bar{\varphi}^{(s,t)} \Gamma_{\mu} \left[ a^{(t,u)}_\mu, \varphi^{(u,s)} \right] \right)^2 \hspace{1cm} (D.29) \]

\[ = \frac{1}{2} \left( \text{Tr} \sum_{stu} \bar{\varphi}^{(s,t)} \Gamma_{\mu} a^{(t,u)}_\mu \varphi^{(u,s)} \right)^2. \]
We can perform the Wick contractions:

\[
\hat{V}_F = \frac{1}{2} \text{Tr} \sum_{stu} \text{Tr} \sum_{pqr} \\
\times \sum_{J_1 M_1} \frac{1}{N_0} \left( \frac{1}{P^2} \rho_i P_i - \frac{i}{2} \left( \frac{1}{P^2} \right)^2 f_{ijk} \Gamma^{ijkl} P_i + \cdots \right) \left( \frac{1 + \Gamma_{11}}{2} \right) \Gamma^\mu \\
\times (-1)^{M_1 - (j_i - j_r)} \hat{\Psi}^{(j_i j_r)} J_1 M_1 \hat{\Psi}^{(j_r j_i)} J_1 - M_1 \\
\times \frac{1}{\beta^2 N_0} \sum_{J_2 M_2} \frac{(-1)^{M_2 - (j_r - j_q)}}{J_2 (J_2 + 1)} \delta_{\mu \nu} \hat{Y}^{(j_r j_q)} J_2 M_2 \hat{Y}^{(j_q j_r)} J_2 - M_2 \\
\times \sum_{J_3 M_3} \frac{1}{N_0} \left( -\frac{1}{P^2} \Gamma^\alpha a + \frac{i}{2} \left( \frac{1}{P^2} \right)^2 f_{abc} \Gamma^{abcd} P_d + \cdots \right) \left( 1 + \Gamma_{11} \right) \Gamma^\nu \\
\times (-1)^{M_3 - (j_q - j_p)} \hat{\Psi}^{(j_q j_p)} J_3 M_3 \hat{\Psi}^{(j_p j_q)} J_3 - M_3.
\]

We can evaluate the traces of products of gamma matrices as in appendix A.7. We obtain the following result:

\[
\hat{V}_F \sim \sum_{123} \frac{1}{N_0} \hat{\Psi}^\dagger_{123} \left[ -64 \frac{\hat{P} \cdot \hat{Q}}{P^2 \hat{Q}^2 R^2} + 32 \beta^2 \frac{\hat{P}^2 \hat{Q}^2}{(P^2)^2 (\hat{Q}^2)^2 R^2} \right] \hat{\Psi}_{123}.
\]

(D.31)

**D.8 Two-loop effective action**

We summarize the 2-loop effective action on the background \( Y_i \) as follows:

\[
\hat{W}_{2\text{-loop}} \sim \hat{V}_4 + \hat{V}_3 + \hat{V}_M + \hat{V}_{gh} + \hat{V}_F \\
= \frac{36 \beta^2}{N_0} \sum_{123} \hat{\Psi}^\dagger_{123} \frac{1}{P^2 \hat{Q}^2 R^2} \hat{\Psi}_{123} \\
\rightarrow \frac{36 \beta^4 N_0}{\beta^4 N_0} \sum_{stu} \sum_{J_1 J_2 J_3} \frac{(2J_1 + 1)(2J_2 + 1)(2J_3 + 1)}{J_1 (J_1 + 1) J_2 (J_2 + 1) J_3 (J_3 + 1)} \left( \begin{array}{ccc} J_1 & J_2 & J_3 \\ M_1 & M_2 & M_3 \end{array} \right)^2.
\]

(D.32)

Here we have used the formula (C.12), and performed the summations over \( M_1, M_2 \) and \( M_3 \).

**D.9 Two-loop effective action at large \( N_0 \) limit**

We evaluate the 2-loop effective action in such a large \( N_0 \) limit that \( N_0 \gg \Lambda \). We impose cutoff scale 2\( \Lambda \) on \( h \), and separate the summation over \( J_1, J_2 \) and \( J_3 \) into two parts at a cutoff scale
\[ \Lambda \text{ as (4.34). The first term of (4.34) is calculated by using the result of (3.24):} \]

\[
\tilde{W}_{2-\text{loop}}^{(1)} \equiv \frac{36}{\beta^4 N_0} \sum_{h=1}^{2\Lambda} \sum_{J_1=1/2}^{\Lambda} \sum_{J_2=1/2}^{J_1} \sum_{J_3=1/2}^{J_2} \sum_{J_4=1/2}^{J_3} \sum_{J_5=-J_4}^{J_3} \frac{(2J_1 + 1)(2J_2 + 1)(2J_3 + 1)}{J_1(J_1 + 1)J_2(J_2 + 1)J_3(J_3 + 1)} \left( \begin{array}{ccc}
J_1 & J_2 & J_3 \\
\tilde{M}_1 & \tilde{M}_2 & \tilde{M}_3
\end{array} \right)^2
\]

\[
\tilde{W}_{2-\text{loop}}^{(1)} = \frac{36}{\beta^4 N_0} \sum_{h=1}^{2\Lambda} \sum_{J_1=1/2}^{\Lambda} \sum_{J_2=1/2}^{J_1} \sum_{J_3=1/2}^{J_2} \sum_{J_4=1/2}^{J_3} \sum_{J_5=-J_4}^{J_3} \frac{(2J_1 + 1)(2J_2 + 1)(2J_3 + 1)}{J_1(J_1 + 1)J_2(J_2 + 1)J_3(J_3 + 1)}.
\]

(D.33)

where we have evaluated the summations over \(\tilde{M}_1, \tilde{M}_2\) and \(\tilde{M}_3\). We can assume that \(J_1, J_2, J_3 \gg 1\):

\[
\tilde{W}_{2-\text{loop}}^{(1)} \xrightarrow{N_0 \to \Lambda} \sum_{h} \left( \frac{576\pi^2}{\beta^4 N_0} \log \Lambda \right).
\]

(D.35)

The second term in (4.34) is calculated as follows:

\[
\tilde{W}_{2-\text{loop}}^{(2)} \equiv \frac{3 \cdot 36}{\beta^4 N_0} \sum_{h=1}^{2\Lambda} \sum_{J_1=1/2}^{\Lambda} \sum_{J_2=1/2}^{J_1} \sum_{J_3=1/2}^{J_2} \sum_{J_4=1/2}^{J_3} \sum_{J_5=-J_4}^{J_3} \frac{(2J_1 + 1)(2J_2 + 1)(2J_3 + 1)}{J_1(J_1 + 1)J_2(J_2 + 1)J_3(J_3 + 1)} \left( \begin{array}{ccc}
J_1 & J_2 & J_3 \\
\Lambda & \tilde{M}_2 & \tilde{M}_3
\end{array} \right)^2
\]

\[
\tilde{W}_{2-\text{loop}}^{(2)} \approx \frac{3 \cdot 36}{\beta^4 N_0} \sum_{h=1}^{2\Lambda} \sum_{J_1=\Lambda+1/2}^{\Lambda} \sum_{J_2=\Lambda+1/2}^{J_1} \sum_{J_3=\Lambda+1/2}^{J_2} \sum_{J_4=\Lambda+1/2}^{J_3} \sum_{J_5=-J_4}^{J_3} \frac{(2J_1 + 1)(2J_2 + 1)(2J_3 + 1)}{J_1(J_1 + 1)J_2(J_2 + 1)J_3(J_3 + 1)} \left( \begin{array}{ccc}
J_1 & J_2 & J_3 \\
0 & \tilde{M}_2 & \tilde{M}_3
\end{array} \right)^2
\]

\[
\tilde{W}_{2-\text{loop}}^{(2)} = \frac{3 \cdot 36}{\beta^4 N_0} \sum_{h=1}^{2\Lambda} \sum_{J_1=\Lambda+1/2}^{\Lambda} \sum_{J_2=\Lambda+1/2}^{J_1} \sum_{J_3=\Lambda+1/2}^{J_2} \sum_{J_4=\Lambda+1/2}^{J_3} \sum_{J_5=-J_4}^{J_3} \frac{(2J_1 + 1)(2J_2 + 1)(2J_3 + 1)}{J_1(J_1 + 1)J_2(J_2 + 1)J_3(J_3 + 1)} \frac{1}{2J_1 + 1}.
\]

(D.36)

where we can assume that \(J_1 \sim N_0\) and \(N_0 \gg \Lambda\) in the 3-\(j\) symbol, and perform the summations.
over $\tilde{M}_2$ and $\tilde{M}_3$. Additionally, we can assume that $J_1, J_2, J_3 \gg 1$:

$$
\bar{W}_{2\text{-loop}}^{(2)} \sim \frac{3 \cdot 36}{\beta^4 N_0} \sum_{h=1}^{2\Lambda} \sum_{J_1=\Lambda+1/2}^{N_0} \sum_{J_2=\Lambda+1/2}^{N_0} \sum_{J_3=\Lambda+1/2}^{N_0} \left(2\Lambda\right) \left(\frac{2J_1}{J_1^2} \frac{2J_2}{J_2^2} \frac{2J_3}{J_3^2}\right) \frac{1}{2J_1} \\
\sim \frac{13824\Lambda}{\beta^4 N_0} \sum_h \int_{2\Lambda+1}^{2N_0} dn_1 \int_{\Lambda}^{2\Lambda} dn_2 \int_{\Lambda}^{2\Lambda} dn_3 \left(\frac{1}{n_1^2 n_2 n_3}\right),
$$

where we have set that $J_1 = n_1/2$, $J_2 = n_2/2$ and $J_3 = n_3/2$. Therefore, we can estimate it as

$$
\bar{W}_{2\text{-loop}}^{(2)} \underset{N_0 \to \Lambda}{\sim} \sum_h \left(\text{const}\right),
$$

The third term in (4.34) is calculated as follows:

$$
\bar{W}_{2\text{-loop}}^{(3)} \equiv \frac{3 \cdot 36}{\beta^4 N_0} \sum_{h=1}^{2\Lambda} \sum_{J_1=\Lambda+1/2}^{N_0} \sum_{J_2=\Lambda+1/2}^{N_0} \sum_{J_3=\Lambda+1/2}^{N_0} \sum_{J_3=\Lambda+1/2}^{N_0} \left(2\Lambda\right)^2 \frac{1}{J_1 (J_1 + 1) J_2 (J_2 + 1) J_3 (J_3 + 1)} \left(\frac{J_1}{\Lambda} \frac{J_2}{\Lambda} \frac{J_3}{M_3}\right)^2
$$

$$
\sim \frac{3 \cdot 36}{\beta^4 N_0} \sum_{h=1}^{2\Lambda} \sum_{J_1=\Lambda+1/2}^{N_0} \sum_{J_2=\Lambda+1/2}^{N_0} \sum_{J_3=\Lambda+1/2}^{N_0} \sum_{J_3=\Lambda+1/2}^{N_0} \left(2\Lambda\right)^2 \frac{1}{J_1 (J_1 + 1) J_2 (J_2 + 1) J_3 (J_3 + 1)} \left(\frac{J_1}{\Lambda} \frac{J_2}{\Lambda} \frac{J_3}{M_3}\right)^2,
$$

where we can assume the conditions that $J_1, J_2 \sim N_0$ and $N_0 \gg \Lambda$ in the 3-\text{j}-symbol. Additionally, we can assume the condition that $J_1, J_2, J_3 \gg 1$ and $J_1, J_2 \gg J_3$:

$$
\bar{W}_{2\text{-loop}}^{(3)} \sim \frac{3 \cdot 36}{\beta^4 N_0} \sum_{h=1}^{2\Lambda} \sum_{J_1=\Lambda+1/2}^{N_0} \sum_{J_2=\Lambda+1/2}^{N_0} \sum_{J_3=\Lambda+1/2}^{N_0} \sum_{J_3=\Lambda+1/2}^{N_0} \left(2\Lambda\right)^2 \left(\frac{2J_1}{J_1^2} \frac{2J_2}{J_2^2} \frac{2J_3}{J_3^2}\right) \\
\times \frac{-1}{2J_2 + 1} \delta_{J_1-J_2+J_3} \left(\frac{(J_1 - J_2 + J_3)!}{(J_1 + J_2 + J_3)!}\right)^2, \\
$$

where we have used the following relation for the squares of 3-\text{j}-symbol in the semi-classical limit [23]:

$$
\sum_{M_3} \left(\frac{J_1}{0} \frac{J_2}{0} \frac{J_3}{M_3}\right)^2 \sim \frac{1}{2J_2 + 1} \left[D_{0,-J_1+J_2}^J \left(0; \frac{\pi}{2}, 0\right)\right]^2.
$$

The Wigner $D$-functions are given by

$$
D_{0M}^J \left(0; \frac{\pi}{2}, 0\right) = (-1)^{J-M} \delta_{J-M,2n} \frac{\sqrt{(J-M)!(J+M)!}}{2^J \left(\frac{J+M}{2}\right)!\left(\frac{J-M}{2}\right)!},
$$

The Wigner $D$-functions are given by
where \( \delta_{J-M,2n} \) implies that \( J-M \) must be even numbers. After making use of Stirling’s formula, we obtain
\[
\hat{W}^{(3)}_{2\text{-loop}} \sim 3 \cdot 36 \sum_{h=1}^{2N_0} \sum_{J_1=\Lambda+1/2}^{N_0} \sum_{J_2=\Lambda+1/2}^{N_0} \sum_{J_3=\Lambda+1/2}^{N_0} (2\Lambda)^2 \frac{(2J_1)(2J_2)(2J_3)}{J_1^2 J_2^2 J_3^2} \times \frac{(-1)^{J_1-J_2+J_3}}{2J_2 + 1} \frac{2}{\delta_{J_1-J_2+J_3,2n}} \frac{1}{2\pi \sqrt{(J_1-J_2+J_3)(-J_1+J_2+J_3)}}
\]
\[
\sim \frac{110592\Lambda^2}{\pi^3 N_0 \sum_h 2N_0} \int_{2\Lambda+1}^{2N_0} dn_1 \int_{2\Lambda+1}^{2N_0} dn_2 \int_1^{2\Lambda} dn_3 \frac{1}{n_1 n_2 n_3} \frac{1}{\sqrt{n_3^2 - (n_1 - n_2)^2}}
\]
(D.43)

where we have set that \( J_1 = n_1/2, J_2 = n_2/2 \) and \( J_3 = n_3/2 \). Therefore, we can estimate it as
\[
\hat{W}^{(3)}_{2\text{-loop}} \xrightarrow{N_0 \to \infty \ N_0 \gg \Lambda} \sum_h (\text{const})
\]
(D.44)

The forth term in (4.34) is calculated as follows:
\[
\hat{W}^{(4)}_{2\text{-loop}} = \sum_{h=1}^{2N_0} \sum_{J_1=\Lambda+1/2}^{N_0} \sum_{J_2=\Lambda+1/2}^{N_0} \sum_{J_3=\Lambda+1/2}^{N_0} \frac{36}{\beta^4 N_0} (2\Lambda + 1)^2 \frac{(2J_1+1)(2J_2+1)(2J_3+1)}{J_1(J_1+1)J_2(J_2+1)J_3(J_3+1)} \left( \begin{array}{ccc} J_1 & J_2 & J_3 \\ \Lambda & \Lambda & \Lambda \end{array} \right)^2
\]
\[
\sim \sum_{h=1}^{2N_0} \sum_{J_1=\Lambda+1/2}^{N_0} \sum_{J_2=\Lambda+1/2}^{N_0} \sum_{J_3=\Lambda+1/2}^{N_0} \frac{36}{\beta^4 N_0} (2\Lambda + 1)^2 \frac{(2J_1+1)(2J_2+1)(2J_3+1)}{J_1(J_1+1)J_2(J_2+1)J_3(J_3+1)} \left( \begin{array}{ccc} J_1 & J_2 & J_3 \\ 0 & 0 & 0 \end{array} \right)^2
\]
(D.45)

where we assume the conditions that \( J_1, J_2, J_3 \sim N_0 \) and \( N_0 \gg \Lambda \) in the 3-\( j \) symbol. Additionally, we assume the condition that \( J_1, J_2, J_3 \gg 1 \):
\[
\hat{W}^{(4)}_{2\text{-loop}} \sim \sum_{h=1}^{2N_0} \sum_{J_1=\Lambda+1/2}^{N_0} \sum_{J_2=\Lambda+1/2}^{N_0} \sum_{J_3=\Lambda+1/2}^{N_0} \frac{36}{\beta^4 N_0} \frac{(2\Lambda)^2}{J_1^2 J_2^2 J_3^2} \frac{1}{\pi \sqrt{-J_1^4 - J_2^4 - J_3^4 + 2J_1^2 J_2^2 + 2J_2^2 J_3^2 + 2J_3^2 J_1^2}}
\]
(D.46)

where we can make use of the following relation for the squares of 3-\( j \) symbol in the semiclassical limit [23]:
\[
4\pi \left( \begin{array}{ccc} J_1 & J_2 & J_3 \\ M_1 & M_2 & M_3 \end{array} \right)^2 \sim \delta_{M_1+M_2+M_3,0} \sqrt{A^2 + \frac{1}{4} (J_1^2 M_2 M_3 + J_2^2 M_3 M_1 + J_3^2 M_1 M_2)}
\]
(D.47)

and
\[
16A^2 = -J_1^4 - J_2^4 - J_3^4 + 2J_1^2 J_2^2 + 2J_2^2 J_3^2 + 2J_3^2 J_1^2.
\]
(D.48)
We obtain
\[ \hat{W}^{(4)}_{2\text{-loop}} \sim \frac{36864\Lambda^2}{\pi \beta^4 N_0} \sum_h \int_{2\Lambda+1}^{2N_0} dn_1 \int_{2\Lambda+1}^{2N_0} dn_2 \int_{2\Lambda+1}^{2N_0} dn_3 \]
\[ \times \frac{1}{n_1 n_2 n_3} \sqrt{-n_1^4 - n_2^4 - n_3^4 + 2n_1^2 n_2^2 + 2n_2^2 n_3^2 + 2n_3^2 n_1^2} \]
where we have set that \( J_1 = n_1/2 \), \( J_2 = n_2/2 \) and \( J_3 = n_3/2 \). Therefore, we can estimate it as
\[ \hat{W}^{(4)}_{2\text{-loop}} \rightarrow \sum_h (\text{const}) \quad \text{(D.50)} \]

We conclude that the 2-loop effective action is given as follows in such a large \( N_0 \) limit that \( N_0 \gg \Lambda \):
\[ \hat{W}^{(4)}_{2\text{-loop}} \rightarrow \sum_h \left( \frac{576\pi^2}{\beta^4 N_0} \log \Lambda \right) \quad \text{(D.51)} \]

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