ADVECTION-DIFFUSION EQUATION ON A HALF-LINE WITH BOUNDARY LÉVY NOISE

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Abstract. In this paper we study a one-dimensional linear advection-diffusion equation on a half-line driven by a Lévy boundary noise. The problem is motivated by the study of contaminant transport models under random sources (P. P. Wang and C. Zheng, Ground water, 43 (2005), [34]). We determine the closed form formulae for mild solutions of this equation with Dirichlet and Neumann noise and study approximations of these solutions by classical solutions obtained with the help of Wong–Zakai approximations of the driving Lévy process.

1. Introduction, classical solutions and formulation of the problem. This paper is motivated by a physically important model of a contaminant transport in a one-dimensional semi-infinite pipe with a constant flow velocity and diffusion. In the classical setting, the contaminant concentration \( u = u(t,x) \) satisfies the advection-diffusion equation

\[
\begin{aligned}
\partial_t u(t,x) &= \nu \partial_x u(t,x) - \partial_x \nu \partial_x u(t,x), \quad t > 0, \ x > 0, \\
\frac{\partial}{\partial x} u(0,x) &= 0, \\
B u(t,0) &= g(t), \quad t \geq 0.
\end{aligned}
\]

with zero initial concentration and continuously differentiable source process \( g = g(t) \), which affects the concentration of the contaminant at the boundary point \( x = 0 \). We assume that the diffusion coefficient equals 1, the flow velocity \( \nu \in \mathbb{R} \). The boundary conditions are treated in a unified way with the help of the boundary operator \( B \), namely we set

\[
\begin{aligned}
B_D u(t,0) &= \lim_{x \downarrow 0} u(t,x), \\
B_N u(t,0) &= \lim_{x \uparrow 0} \partial_x u(t,x),
\end{aligned}
\]

for the Dirichlet and Neumann problems respectively. In the Dirichlet case, the source \( g \) prescribes the concentration of the contaminant at the boundary, in the Neumann setting it determines the transfer rate through the boundary. If the input

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function \( g \in C^1_0(\mathbb{R}_+, \mathbb{R}) \), the solution (1.1) is well known in the closed form, see e.g. [9, 28].

In realistic models, the assumption that \( g \) is smooth and deterministic is too restrictive. For instance, in [34, 10, 20] the authors allow \( g \) to be a random source of contamination in an open channel flow. In particular, the contaminant can be released at random time instants, so that \( g \) can consist of a random train of delta-spikes or be a Brownian noise.

In the present paper we tackle two problems. First, we will solve equation (1.1) with a general boundary Lévy noise \( g = \tilde{Z} \), including Brownian motion or Poisson process. We will determine the mild solution of (1.1) as process with values in the fractional Sobolev space \( H^{\theta}_{p, q}(\mathbb{R}_+) \), find its explicit form as a convolution integral w.r.t. the driving Lévy process and determine its law in the large time limit. Second, we study the so-called Wong–Zakai approximations of solutions, namely we consider absolutely continuous approximations of the driving process \( Z \) and study convergence of classical solutions to the mild solution of the original equation in the non-standard \( M_1 \)-Skorokhod topology.

More details on the theory of PDEs with boundary noise can be found e.g. in [13, 12, 5, 8]. PDEs with the Lévy noise on the boundary were considered in [27], and more recently in [16] and [7]. In the deterministic case, controllability of the one-dimensional heat equation on the half line was studied in [22], whereas [2, 1, 14, 21] considered the one-dimensional heat equation on the half line with white Gaussian noise on the boundary.

Eventually, we mention the works [18, 34, 20] for applications of the mathematical model (1.1) to hydrology, [23] for a discussion on the proper choice of boundary conditions from the physical point of view, and [17] for a bibliography on transport of chemicals through soil.

2. Solutions of the advection-diffusion equation on a half line with boundary Lévy noise. On a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), let \( Z = (Z(t))_{t \geq 0} \) be a Lévy process with the characteristic function

\[
E e^{i\lambda Z(t)} = e^{e^{i\lambda t}} \Phi(\lambda), \quad \lambda \in \mathbb{R},
\]

\[
\Phi(\lambda) = -\frac{\sigma^2}{2} \lambda^2 + i a \lambda + \int_{\mathbb{R}} (e^{i\lambda z} - 1 - i\lambda z 1(|z| \leq 1)) \mu(dz),
\]

(2.1)

where the Gaussian variance, drift and the Lévy measure satisfy \( \sigma^2 \geq 0, a \in \mathbb{R}, \) and \( \mu(\{0\}) = 0, \int (z^2 + 1) \mu(dz) < \infty.\)

We solve the advection-diffusion equation

\[
\begin{align*}
\partial_t u(t, x) &= \partial_{xx} u(t, x) - \nu \partial_x u(t, x), \quad t > 0, \quad x > 0, \\
u u(0, x) &= 0, \\
Bu(t, 0) &= \tilde{Z}(t), \quad t \geq 0,
\end{align*}
\]

(2.2)

where \( B \) is given by a Dirichlet or Neumann boundary operator (1.2) or (1.3). Following [3, 14, 27], we consider (2.2) as an evolution equation in an appropriate Hilbert space \( \mathcal{H} \) and derive an integral formula for its solution. The space \( \mathcal{H} \) should satisfy two properties. First, the operator \( A_B = \partial_{xx} - \nu \partial_x \) with the domain \( \mathcal{D}(A_B) = \{ u \in \mathcal{H}; A_B u \in \mathcal{H}, \ Bu = 0 \} \) should generate a \( C_0 \)-semigroup \( (S_B(t))_{t \geq 0} \) in \( \mathcal{H} \). Second, \( \mathcal{H} \) should be rich enough to guarantee that the solution of (2.2) is càdlàg, so that we can talk about convergence in the Skorokhod topology. It turns out
that it is convenient to work in fractional Sobolev spaces $\mathcal{H} = H^\theta(\mathbb{R}_+), \theta \in \mathbb{R}$ (see Section 5 for definitions).

Define the Dirichlet map operator $D_B: \mathbb{R} \rightarrow C^2_0(\mathbb{R}_+, \mathbb{R})$ by the relation $D_B a = \phi$, where $\phi$ is a unique bounded solution of the ordinary differential equation

$$\begin{aligned}
\phi''(x) - \nu \phi'(x) = (1 + \nu)\phi(x), & \quad x > 0, \\
B\phi(0) = a.
\end{aligned}$$

A straightforward calculation yields that

$$(D_B a)(x) = ae^{-x}.$$ 

Assume for a moment that we are in the classical setting (1.1) and the input $g = \tilde{Z}$ is a smooth function, $g \in C^1(\mathbb{R})$. Consider the non-homogeneous equation

$$\begin{aligned}
\partial_t \tilde{u}(t,x) &= A\tilde{u}(t,x) dt + \left((1 + \nu)D_B g(t) - D_B \tilde{g}(t)\right)(x), \quad t > 0, x > 0, \\
\tilde{u}(0,x) &= -(D_B(g(0)))(x), \\
B\tilde{u}(t,0) &= 0.
\end{aligned} \tag{2.3}$$

We claim that $u(t,x) = \tilde{u}(t,x) + \left(D_B(g(t))\right)(x)$. Indeed, the direct substitution yields

$$\begin{aligned}
\partial_t u(t,x) &= \partial_t \tilde{u}(t,x) + \left(D_B(\tilde{g}(t))\right)(x) \\
&= \partial_{xx} \tilde{u}(t,x) - \nu \partial_x \tilde{u}(t,x) + \left((1 + \nu)D_B(g(t))\right)(x) \\
&= \partial_{xx} u(t,x) - \nu \partial_x u(t,x),
\end{aligned}$$

and the initial and boundary conditions of (1.1) are also satisfied:

$$\begin{aligned}
u u(0,x) &= \tilde{u}(0,0) + \left(D_B(g(0))\right)(x) = 0, \\
B u(t,0) &= B\tilde{u}(t,0) + \left(B D_B(g(t))\right)(0) = g(t).
\end{aligned}$$

The solution to the problem (2.3) is found with the help of the convolution formula (Duhamel’s principle). Let $S_B$ be the $C_0$-semigroup of the operator $A_B = \partial_{xx} - \nu \partial_x$ on the domain $\mathcal{D}(A_B)$. Then $\tilde{u}$ is found explicitly as

$$\begin{aligned}
\tilde{u}(t) &= -S_B(t)D_B(g(0)) + \int_0^t S_B(t-s) \left((1 + \nu)D_B g(t) - D_B \tilde{g}(t)\right) ds. \tag{2.4}
\end{aligned}$$

Using the $C_0$-continuity of $S_B$ we note that

$$\frac{d}{ds} S_B(t-s) = -A_B S_B(t-s),$$

so that the integration by parts gives

$$\int_0^t S_B(t-s)D_B(\tilde{g}(s)) ds = S_B(t-s)D_B(g(s)) \bigg|_0^t - \int_0^t A_B S_B(t-s) D_B(g(s)) ds.$$ 

Together with (2.4) this gives

$$u(t,x) = \int_0^t \left((1 + \nu)\text{Id} - A_B\right) S_B(t-s) D_B(g(s))(x) ds. \tag{2.5}$$

The formula (2.5) allows us to work with the following definition.
Definition 2.1. We call the process

\[ u(t, x) := \int_0^t \left( (1 + \nu) \text{Id} - A_B \right) (S_B(t - s)D_B)(x) \, dZ(s) \]

a mild solution of (2.2) in the state space \( \mathcal{H} \).

The latter definition presupposes that the integral on the right hand side exists. The construction of an integral of Hilbert-valued deterministic integrand w.r.t. a Lévy process is standard, see e.g. [11, 29].

The semigroup \( S_B \) has a well-known explicit representation in terms of the Green function of the heat equation, see [9, 28]:

\[ S_B(t)f(x) = \int_0^\infty f(y)\Lambda_B(x, y, t) \, dy, \quad f \in C_0(\mathbb{R}_+), \]

where

\[
\begin{align*}
\Lambda_D(x, y, t) &= \frac{1}{2\sqrt{\pi t}} e^{-\frac{(y-x)^2}{4t}} - \frac{1}{2\sqrt{\pi t}} \left( e^{-\frac{(x+y)^2}{4t}} - e^{-\frac{(x-y)^2}{4t}} \right), \\
\Lambda_N(x, y, t) &= \frac{1}{2\sqrt{\pi t}} e^{-\frac{(y-x)^2}{4t}} \left( e^{-\frac{(x+y)^2}{4t}} + e^{-\frac{(x-y)^2}{4t}} \right) + \frac{\nu}{2} e^{-\nu y} \text{erfc} \left( \frac{x + y - \nu t}{2\sqrt{t}} \right), \\
\text{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt.
\end{align*}
\]

Hence straightforward integration allows us to simplify

\[
\begin{align*}
G_D(x, t) &:= \left( (1 + \nu) \text{Id} - A_D \right) S_D(t)e^{-x} = \frac{x}{2\sqrt{\pi t^{3/2}}} e^{-\frac{(x-\nu t)^2}{4t}}, \\
G_N(x, t) &:= \left( (1 + \nu) \text{Id} - A_N \right) S_N(t)e^{-x} = \frac{1}{\sqrt{\pi t}} e^{-\frac{(x-\nu t)^2}{4t}} + \frac{\nu}{2} e^{-\nu x} \text{erfc} \left( \frac{x - \nu t}{2\sqrt{t}} \right),
\end{align*}
\]

which yields the closed form solution for \( u \).

The first result of this paper is formulated in the following Theorem which essentially determines the appropriate Hilbert space \( \mathcal{H} = H^\theta(\mathbb{R}_+) \).

Theorem 2.2. (D) The equation (2.2) with a Dirichlet boundary condition has a mild solution in \( H^\theta(\mathbb{R}_+) \) for \( \theta < -\frac{3}{2} \) which has the explicit form

\[ u(t, x) = \int_0^t \frac{x}{2\sqrt{\pi (t-s)^{3/2}}} e^{-\frac{(x-\nu(t-s))^2}{4(t-s)}} \, dZ(s). \]

(N) The equation (2.2) with a Neumann boundary condition has a mild solution in \( H^\theta(\mathbb{R}_+) \) for \( \theta < -\frac{1}{2} \) which has the explicit form

\[ u(t, x) = \int_0^t \left( \frac{1}{\sqrt{\pi (t-s)}} e^{-\frac{(x-\nu(t-s))^2}{4(t-s)}} + \frac{\nu}{2} \text{erfc} \left( \frac{x - \nu(t-s)}{2\sqrt{t-s}} \right) \right) \, dZ(s). \]

In all cases the mild solution is unique and the paths \( t \mapsto u(t, \cdot) \) are càdlàg in \( H^\theta(\mathbb{R}_+) \) a.s. Moreover, for any \( x > 0 \), the paths \( t \mapsto u(t, x) \) are continuous in \( \mathbb{R} \).

The proof of the Theorem is given in Section 5. Sample paths of solutions \( u \) driven by an \( \alpha \)-stable Lévy subordinator and a symmetric \( \alpha \)-stable Lévy process are presented in Fig. 1 and Fig. 2. Note that negative jumps of the noise may cause negative values of the solution. This explains why Lévy subordinators should be used to model contaminant concentrations.
3. Limiting probability distribution of the contaminant concentration.

The explicit form of the solution allows us to calculate the stationary contaminant distribution in the large time limit.

To determine the limiting distribution of $u$ in the stationary regime, we consider the equation (2.2) on the time interval $[-\tau, 0]$, $\tau > 0$, driven by a shifted Lévy process $Z_\tau = (Z(t-\tau))_{t \geq \tau}$. Let $u_\tau = u_\tau(t, x)$, $t \in [-\tau, 0]$ be its solution. For $x > 0$, we consider the limit in law

$$u(x) = \lim_{-\tau \to -\infty} u_\tau(0, x) = \lim_{-\tau \to -\infty} \int_{-\tau}^0 G(-s, x) \, dZ_\tau(s) \overset{d}{=} \int_0^\infty G(s, x) \, dZ(s),$$

provided the integral on the r.h.s. exists. Recalling (2.1) we find the Fourier transform of $u(x)$ explicitly as

$$\mathbb{E}e^{i\lambda u(x)} = \exp\left(\int_0^\infty \Phi(s) \, G(s, x) \lambda \, ds\right), \quad \lambda \in \mathbb{R},$$

provided the integral in the exponent exists.
In physically meaningful models, the process $Z$ does not take negative values, i.e. is a Lévy subordinator with the Laplace transform

$$E e^{-\lambda Z(t)} = e^{\Psi(\lambda)}, \quad \lambda \geq 0,$$

$$\Psi(\lambda) = -b \lambda + \int_0^\infty (e^{-\lambda z} - 1) \mu(dz),$$

with $b \geq 0$ and the jump measure satisfying $\mu(\{0\}) = 0$, $\int_0^\infty (z \wedge 1) \mu(dz) < \infty$. In this case, $u(t, x) \geq 0$ a.s. and its Laplace transform is

$$E e^{-\lambda u(x)} = \exp \left( \int_0^\infty \Psi \left( G(s, x) \lambda \right) ds \right), \quad \lambda \geq 0.$$

It is instructive to calculate the limiting law in the following particular case.

Let $Z$ be an $\alpha$-stable subordinator with $\Psi(\lambda) = -c \lambda^\alpha$, $c > 0$ being the scale parameter and $\alpha \in (0, 1)$ the stability index. Then

$$E e^{-\lambda u(x)} = \exp \left( -c \lambda^\alpha \int_0^\infty G(s, x)^\alpha ds \right), \quad \lambda \geq 0.$$

In other words, the limiting concentration $u(x)$ at the location $x > 0$ has a spectrally positive $\alpha$-stable distribution with the scale

$$c(x) = c \int_0^\infty G(s, x)^\alpha ds. \quad (3.1)$$

The straightforward integration allows to determine the limiting scale $c_D(x)$ in case of the Dirichlet boundary noise as

$$c_D(x) = \begin{cases} 
  c \cdot \frac{2^{1-\alpha}}{\pi^{\alpha/2}} |\nu|^{2\alpha-2} \cdot e^{-\frac{x^\alpha (1-\nu)}{2}} K_{2\alpha-2} \left( \frac{\alpha x |\nu|}{2} \right), & \nu \neq 0, \quad \alpha \in (0, 1), \\
  c \cdot \Gamma \left( \frac{3\alpha - \nu}{2} \right) \cdot \frac{\alpha^{2-3\alpha}}{2^{2(1-\alpha)} \pi^{\alpha/2}} \cdot x^{2(1-\alpha)}, & \nu = 0, \quad \alpha \in (2/3, 1), \\
  +\infty, & \nu = 0, \quad \alpha \in (0, 2/3].
\end{cases}$$

where $K_\nu$ is the modified Bessel function of the second kind. Taking into account the asymptotic expansion

$$K_\nu(x) = \sqrt{\frac{\pi}{2}} x^{-1/2} e^{-x} (1 + \mathcal{O}(|x|^{-1})), \quad |x| \to \infty,$$
we get that for large values of $x$ and $\alpha \in (0, 1)$
\[
c_D(x) \approx \begin{cases} 
  c \cdot \frac{2^{1-\alpha} \pi^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} \nu^{\frac{3}{2}(\alpha-1)} \cdot e^{-\nu x^\alpha} x^{-\frac{1-\alpha}{\alpha}}, & \nu > 0, \\
  c \cdot \frac{2^{1-\alpha} \pi^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} |\nu|^{\frac{3}{2}(\alpha-1)} \cdot x^{-\frac{1-\alpha}{\alpha}}, & \nu < 0,
\end{cases}
\]
see Fig. 3 (left).

In the Neumann case, it is clear that $c_N(x) = +\infty$ for $\nu \geq 0$. For $\nu < 0$, the result of numerical integration is presented in Fig. 3 (right).

It is interesting to note that the integral (3.1) diverges for $\nu \neq 0$ and $\alpha \in (0, 2/3]$ in the Dirichlet case. The same critical value $\alpha = 2/3$ was discovered in [25] in the analysis of limiting distributions of Lévy driven transport dynamics.

4. **Approximations by classical solutions of the Wong–Zakai type.** From the point of view of applications, the boundary noise $\tilde{Z}$ in (2.2) is an idealization of a very fast continuous injection process taking place at the opening of the pipe. A natural question about the convergence of approximations to the solution $u$ arises.

Approximations of irregular trajectories of random processes by smooth paths are well known in the literature under the name of Wong–Zakai approximations [36, 37]. In particular for dynamical systems driven by Brownian motion there is a number of results in both finite and infinite dimensional settings which state that the approximations converge in the uniform topology to the solution of the Stratonovich equation, see, e.g. [6, 31, 33].

Commonly used examples of absolutely continuous approximations of a Lévy process $Z$ are polygonal approximations,
\[
Z^n(t) = Z\left(\frac{k-1}{n}\right) + n\left(Z\left(\frac{k}{n}\right) - Z\left(\frac{k-1}{n}\right)\right)\left(t - \frac{k}{n}\right), \quad k \geq 1, \quad n \geq 1, \quad (4.1)
\]
red noise approximations
\[
Z^n(t) = \int_0^t \left(1 - e^{-n(t-s)}\right) dZ(s), \quad n \geq 1, \quad (4.2)
\]
or short memory averaging
\[
Z^n(t) = n \int_{(t-n^{-1})\wedge 0}^t Z(s) \, ds, \quad n \geq 1. \quad (4.3)
\]
A common feature of these approximations is that they approximate a continuous process $Z$ (i.e. a Brownian motion with drift) in the uniform topology. If $Z$ has jumps, these jumps are approximated continuously and in a monotonous way. Such type of approximations can be very well described with the help of the so-called $M_1$-Skorokhod topology.

Let $V$ be a separable Banach space with the norm $\| \cdot \|$. In this paper, we will mainly deal with $V = \mathbb{R}$ for approximations of the Lévy process $Z$ and $V = H^\theta(\mathbb{R})$ for approximations of solutions of the equation (2.2). For a fixed time $T > 0$, the space of $V$-valued càdlàg functions is denoted by $D([0, T], V)$. Each $f \in D([0, T], V)$ may have at most countably many discontinuities.

For two elements $v_1, v_2 \in V$ we define a segment $[v_1, v_2]$ as a straight line between $v_1$ and $v_2$:
\[
[v_1, v_2] := \{ v \in V : v = \alpha v_1 + (1-\alpha)v_2 \text{ for } \alpha \in [0, 1] \}.
\]
In order to define the so-called (strong) $M_1$ metric on $D([0,T];V)$, we define for each $f \in D([0,T], V)$ the extended graph of $f$ by

$$\Gamma(f) := \{(t, v) \in [0,T] \times V : v \in [f(t), f(t)]\},$$

where $f(0-) := f(0)$. A total order relation on $\Gamma(f)$ is given by

$$(t_1, v_1) \leq (t_2, v_2) \iff \begin{cases} t_1 < t_2 & \text{or} \\ t_1 = t_2 \text{ and } \|f_1(t_1) - v_1\| \leq \|f_1(t_1) - v_2\|. \end{cases}$$

A parametric representation of the extended graph of $f$ is a continuous, non-decreasing, surjective function

$$(r, u): [0,1] \to \Gamma(f), \quad (r, u)(0) = (0, f(0)), \quad (r, u)(1) = (T, f(T)).$$

Let $\Pi(f)$ denote the set of all parametric representations of $f$.

For $f_1, f_2 \in D([0,T], V)$, we define

$$d_M(f_1, f_2) := \inf \left\{ |r_1 - r_2|_\infty \vee \|u_1 - u_2\|_\infty : (r_i, u_i) \in \Pi(f_i), i = 1, 2 \right\}.$$  

The mapping $d_M$ is called strong $M_1$ metric on $D([0,T], V)$. This topology was introduced by Skorokhod in his seminal paper [30]. The extensive analysis of $M_1$-topology in the finite dimensional setting can be found in [35]. For a generalization to Banach and Hilbert spaces, see [24].

**Remark 4.1.** The approximations $Z^n$ defined in (4.1), (4.2) and (4.3) are absolutely continuous and converge to $Z$ a.s. in the (strong) $M_1$ topology in $D([0,T], \mathbb{R})$.

The second main result of this paper is the following theorem

**Theorem 4.2.** Let $T > 0$ and let $Z^n \to Z$ in probability in $D([0,T], \mathbb{R}; d_M)$ as $n \to \infty$, and let $Z_n$, $n \geq 1$, be absolutely continuous. Then the classical solutions $u^n$ driven by $Z^n$ converge to $u$ determined in Theorem 2.2 in probability in $D([0,T], H^\theta(\mathbb{R}_+); d_M)$ as $n \to \infty$.

Finally we note that away of the boundary $x = 0$, the solution $(t,x) \mapsto u(t,x)$ is a smooth function. Thus the following theorem holds.

**Theorem 4.3.** Let $T > 0$ and let $Z^n \to Z$ in probability in $D([0,T], \mathbb{R}; d_M)$ as $n \to \infty$, and let $Z^n$, $n \geq 1$, be absolutely continuous. Then for any $x > 0$

$$\sup_{t \in [0,T]} |u^n(t,x) - u(t,x)| \to 0$$

in probability, as $n \to \infty$.

5. **Proof of the Theorem 2.2.** To show the existence of a mild solution we first have to determine a suitable Hilbert space $H$, so that the operator $A_B = \partial_{xx} - \nu \partial_x$ with the boundary condition $B$ generates a $C_0$-semigroup.

More precisely, let $S(\mathbb{R})$ be the Schwartz space of rapidly decreasing functions and let $S'(\mathbb{R})$ be its dual space. Let $L^2(\mathbb{R})$ be the Hilbert space of equivalence classes of square-integrable functions $f: \mathbb{R} \to \mathbb{C}$ with scalar product $\langle f, g \rangle_2 = \int_{\mathbb{R}} f(x)\overline{g(x)} \, dx$ and the associated norm $\|f\|_2^2 := \int_{\mathbb{R}} |f(x)|^2 \, dx$. On $S(\mathbb{R})$ or $S'(\mathbb{R})$ respectively we define the Fourier transform $\mathcal{F}$ such that for $\varphi \in S(\mathbb{R})$

$$(\mathcal{F}\varphi)(\xi) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi} \varphi(x) \, dx.$$
and for $T \in \mathcal{S}'(\mathbb{R})$, $\mathcal{F}T$ is the functional on $\mathcal{S}'(\mathbb{R})$, such that $\mathcal{F}T(\varphi) = T(\mathcal{F}\varphi)$ for every $\varphi \in \mathcal{S}(\mathbb{R})$.

For $\theta \in \mathbb{R}$, $H^\theta(\mathbb{R})$ denotes the fractional Sobolev space, namely a separable Hilbert space

$$H^\theta(\mathbb{R}) := \{ f \in \mathcal{S}'(\mathbb{R}) : (1 + \xi^2)^{\theta/2}(\mathcal{F}f)(\xi) \in L^2(\mathbb{R}) \}$$

with the norm

$$\|f\|_{\theta, 2} := \left\| (1 + \xi^2)^{\theta/2}(\mathcal{F}f)(\xi) \right\|_2,$$

see e.g. [15]. In the present paper we will work in the restriction of $H^\theta(\mathbb{R})$ to $\mathbb{R}_+$, denoted by $H^\theta(\mathbb{R}_+)$. We equip this space with the norm

$$\|g\|_{H^\theta(\mathbb{R}_+)} := \inf_{\xi_+ \in g} \|\xi_+\|_{\theta, 2}.$$

To define the associated scalar product in $H^\theta(\mathbb{R}_+)$, note that for every $f \in H^\theta(\mathbb{R}_+)$ there is a unique extension $\text{ext}\ f$ to $\mathbb{R}$, such that $\|f\|_{H^\theta(\mathbb{R}_+)} = \|\text{ext}\ f\|_{\theta, 2}$ and such that the relation

$$\langle f, g \rangle_{H^\theta(\mathbb{R}_+)} := \langle \text{ext}\ f, \text{ext}\ g \rangle_{\theta, 2}$$

defines a scalar product on $H^\theta(\mathbb{R}_+)$, see the Appendix for details. Completeness and separability of $H^\theta(\mathbb{R}_+)$ then follow from the completeness and separability of $H^\theta(\mathbb{R})$.

Furthermore, we write $\mathcal{D}(\mathbb{R}_+)$ for the space of infinitely differentiable functions $f : \mathbb{R}_+ \to \mathbb{C}$ with compact support in $(0, \infty)$. We need this space to give meaning to the boundary condition of operator $A$. First note that if $Au \in H^\theta(\mathbb{R}_+)$ then $u \in H^{\theta+2}(\mathbb{R}_+)$ (see the following Lemma 5.1 and its proof). Since the Sobolev spaces are spaces of equivalence classes of functions, the meaning of the boundary conditions $u(0) = 0$ and $\partial_x u(0) = 0$ for $u \in H^{\theta+2}(\mathbb{R}_+) \leq H^{\theta+2}(\mathbb{R}_+)$ may not be obvious, whereas for $u \in \mathcal{D}(\mathbb{R}_+)$ these conditions are well defined. So in what follows, the expressions $u(0) = 0$ and $\partial_x u(0) = 0$ will be understood in the sense of closures of $\mathcal{D}(\mathbb{R}_+)$ in $H^{\theta+2}(\mathbb{R}_+)$. This relies on the important fact that if $\theta < \frac{1}{2}$, then $\mathcal{D}(\mathbb{R}_+)$ is dense in $H^\theta(\mathbb{R}_+)$, see the appendix for details.

Lemma 5.1. (i) For $\theta < -\frac{3}{2}$, the operator $A = A_D$ with domain $\mathcal{D}(A) = \{ u \in H^\theta(\mathbb{R}_+) : Au \in H^\theta(\mathbb{R}_+), u(0) = 0 \}$ generates a $C_0$-semigroup in $H^\theta(\mathbb{R}_+)$. (ii) For $\theta < -\frac{1}{2}$, the operator $A = A_N$ with domain $\mathcal{D}(A) = \{ u \in H^\theta(\mathbb{R}_+) : Au \in H^\theta(\mathbb{R}_+), \partial_x u(0) = 0 \}$ generates a $C_0$-semigroup in $H^\theta(\mathbb{R}_+)$. 

Proof. In the following we write $\mathcal{H}$ for $H^\theta(\mathbb{R}_+)$. We use the Hille–Yosida Theorem for contractive $C_0$-semigroups (see [26], Theorem 3.1, p. 8). First we show, that $\mathcal{D}(A) = H^{\theta+2}(\mathbb{R}_+)$ and therefore $A$ is dense in $\mathcal{H}$. For the Laplace operator $\Delta_D$ on $\mathbb{R}_+$ with the Dirichlet boundary condition we have $\mathcal{D}(\Delta_D) = H^{\theta+2}(\mathbb{R}_+) \leq H^{\theta+2}(\mathbb{R}_+)$ for $\theta < -\frac{3}{2}$. Now note, that

$$\|Au\|_{\mathcal{H}} < \infty \Rightarrow \|\Delta_D u\|_{\mathcal{H}} < \infty$$

and on the other hand

$$\|u\|_{H^{\theta+2}(\mathbb{R}_+)} < \infty \Rightarrow \|Au\|_{\mathcal{H}} < \infty$$

and so,

$$\mathcal{D}(\Delta_D) \subseteq \mathcal{D}(A) \subseteq H^{\theta+2}(\mathbb{R}_+) = \mathcal{D}(\Delta_D).$$
Now, we need to take a look at the resolvent set \( \rho(A) \) and show that \( (0, \infty) \subseteq \rho(A) \) and for all \( \lambda > 0 \)
\[
\| (\lambda \text{Id} - A)^{-1} \|_{L(H^\theta(\mathbb{R}^+), D(A))} \leq \frac{1}{\lambda}.
\]

Let \( f \in H \). We define
\[
h := \mathcal{F}^{-1}\left( \frac{\mathcal{F}(f)}{\lambda + \xi^2 + i\nu \xi} \right)
\]
and \( g := h|_{\mathbb{R}^+} \). Then, because of the properties of the Fourier transform,
\[
g = (\lambda \text{Id} - A)^{-1} f.
\]
Furthermore
\[
\| g \|_H^2 \leq \| h \|_{\theta, 2}^2 = \int_{-\infty}^{\infty} (1 + \xi^2)^\theta \frac{|\mathcal{F}(f)|^2}{\lambda + \xi^2 + i\nu \xi} \, d\xi
\leq \frac{1}{\lambda^2} \int_{-\infty}^{\infty} (1 + \xi^2)^\theta |\mathcal{F}(f)|^2 \, d\xi
= \frac{1}{\lambda^2} \| f \|_{\theta, 2}^2 = \frac{1}{\lambda^2} \| f \|_H^2.
\]

Noting that
\[
\frac{1}{\lambda^2} \int_{-\infty}^{\infty} (1 + \xi^2)^\theta |\mathcal{F}(f)|^2 \, d\xi < \infty \quad \Rightarrow \quad \int_{-\infty}^{\infty} (1 + \xi^2)^{\theta+2} \frac{|\mathcal{F}(f)|^2}{\lambda + \xi^2 + i\nu \xi} \, d\xi < \infty,
\]
we also get \( g \in H^{\theta+2}(\mathbb{R}^+) = D(A) \) and thus
\[
\| (\lambda \text{Id} - A)^{-1} \|_{L(H^\theta(\mathbb{R}^+), D(A))} \leq \sup_{\| f \|_H \leq 1} \frac{1}{\lambda} \| f \|_H = \frac{1}{\lambda}.
\]

\( \Box \)

6. Proof of Theorem 4.2. Similarly to the convergence in the uniform topology and in the standard Skorokhod metric \( J_1 \), convergence of a sequence of functions in the metric \( d_M \) can be described by quantifying the oscillation of the functions. For \( v, v_1, v_2 \in V \) the distance from \( v \) to the segment \( \| v_1, v_2 \| \) is defined by
\[
M(v_1, v, v_2) := \inf_{\alpha \in [0, 1]} \| v - (\alpha v_1 + (1 - \alpha) v_2) \|.
\]

Define for \( f \in D([0, T]; V) \) and \( \delta > 0 \) the oscillation function by
\[
M(f; \delta) := \sup \left\{ M\left( f(t_1), f(t), f(t_2) \right) : 0 \leq t_1 < t < t_2 \leq T \text{ and } t_2 - t_1 \leq \delta \right\}.
\]
Let \( T > 0 \), and let \( \{Z^n\}_{n \geq 1} \) be a sequence of absolutely continuous functions, such that \( Z_n \rightarrow Z \) a.s. in \( M_1 \)-topology on \( [0, T] \). Since
\[
\lim_{A \rightarrow \infty} \mathbb{P}\left( \sup_{t \in [0, T]} |\Delta Z(t)| > A \right) = 0
\]
from now on we assume that the jumps of \( Z \) are bounded by some constant \( A > 0 \). Furthermore due to the \( M_1 \)-convergence we can assume that for \( n \) large enough
\[
\sup_{s \in [0, T]} |Z^n(s)| \leq \sup_{s \in [0, T]} |Z(s)| + 1. \quad \text{(6.1)}
\]
Let \( u^n \) be classical continuous solutions to (1.1). From [24], Theorem 3.2, we know it is sufficient to show that

(i) for every \( t \in [0, T] \) we have \( \| u^n(t, \cdot) - u(t, \cdot) \|_{H^\theta(\mathbb{R}^+)} \xrightarrow{\mathbb{P}} 0, \, n \rightarrow \infty, \) and
(ii) for every $\varepsilon > 0$ the oscillation function $M(u^n, \delta)$ obeys

$$\lim_{\delta \to 0} \lim_{n \to \infty} P(M(u^n, \delta) \geq \varepsilon) = 0.$$ 

1. Neumann case ($\theta < -\frac{1}{2}$):

First note that we can extend the solutions $u(t, \cdot)$ and $u^n(t, \cdot)$ to $\mathbb{R}$, simply by extending $G_N(s, x)$ defined in (2.7) to a function $G_N : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in the following way:

$$G_N(x, s) := \frac{1}{\sqrt{\pi s}} e^{-\frac{(x-s)^2}{4s}} + \frac{\nu}{2} \text{erfc} \left( \frac{|x| - \nu s}{2\sqrt{s}} \right)$$

and

$$F_G(t - s)(\xi) = 2 e^{-\frac{(\xi^2 + i\nu \xi)(t-s)}{4t}} + \frac{\nu}{2} \frac{\text{erfc} \left( \frac{1}{2} \left( \frac{t-s}{\sqrt{t}} \right) \right)}{\sqrt{t}}\xi.$$ 

If we can show that (i) and (ii) hold in $H^q(\mathbb{R})$ for these explicit extensions, then the result follows easily for $H^q(\mathbb{R}_+)$.

Linearity of the integral allows us to split both $u$ and $u^n$ into two parts and consider each separately. So, for (i), we see that for every $t \in [0, T]$

$$\|u^n(t, \cdot) - u(t, \cdot)\|_{H^q(\mathbb{R}_+)} \leq \|u^n_1(t, \cdot) - u_1(t, \cdot)\|_{H^q(\mathbb{R}_+)} + \|u^n_2(t, \cdot) - u_2(t, \cdot)\|_{H^q(\mathbb{R}_+)}.$$ 

where

$$u_1(t, x) = \int_0^t \frac{1}{\sqrt{\pi s}} e^{-\frac{(x-s)^2}{4s}} dZ(s),$$

$$u_2(t, x) = \int_0^t \frac{\nu}{2} \text{erfc} \left( \frac{x - \nu(t-s)}{2\sqrt{t-s}} \right) dZ(s),$$

and $u^n_1$ and $u^n_2$ are defined analogously.

To estimate the convergence $u^n_1 \to u_1$, we integrate by parts to obtain

$$\|u^n_1(t, \cdot) - u_1(t, \cdot)\|_{H^q(\mathbb{R}_+)}^2 \leq 4 \int_{-\infty}^{\infty} (1 + \xi^2)^\theta \int_0^t e^{-\frac{(\xi^2 + i\nu \xi)(t-s)}{4}} d(Z^n(s) - Z(s))^2 d\xi$$

$$\leq 4 \int_{-\infty}^{\infty} (1 + \xi^2)^\theta |Z^n(t) - Z(t) - Z^n(0)|^2$$

$$+ \int_0^t e^{-\frac{(\xi^2 + i\nu \xi)(t-s)}{4}} (\xi^2 + i\nu \xi)(Z^n(s) - Z(s)) d\xi$$

$$\leq 8 \left( |Z^n(t) - Z(t)|^2 + |Z^n(0)|^2 \right) \int_{-\infty}^{\infty} (1 + \xi^2)^\theta d\xi$$

$$+ 8 \int_{-\infty}^{\infty} (1 + \xi^2)^\theta \int_0^t e^{-\frac{(\xi^2 + i\nu \xi)(t-s)}{4}} (\xi^2 + i\nu \xi)(Z^n(s) - Z(s)) d\xi$$

Since $\{Z^n\}_{n \geq 1}$ converge to $Z$ in $M_1$ on $[0, T]$ and $Z$ is stochastically continuous, it follows that $Z^n(t) \to Z(t)$ for any $t \in [0, T]$ in probability, so that the first summand vanishes in probability as $n \to \infty$.

To estimate the second summand, we apply the Hölder inequality:
\[
\left| \int_0^t e^{-(\xi^2 + iv\xi)(t-s)}(\xi^2 + iv\xi)(Z^n(s) - Z(s)) \, ds \right|
\]
\[
\leq (\xi^2 + |\nu| |\xi|) \int_0^t e^{-\xi^2(t-s)}|Z^n(s) - Z(s)| \, ds
\]
\[
\leq (\xi^2 + |\nu| |\xi|) \varepsilon^{-2/p} \Delta \left( \int_0^t \xi e^{-\xi^2(t-s)} \, ds \right)^{1/p} \left( \int_0^t |Z^n(s) - Z(s)|^q \, ds \right)^{1/q}
\]
and choose \( p > 1 \) such that \( 1 - \frac{1}{p} < \frac{-2|\theta|-1}{4} \) to get
\[
\int_{-\infty}^{\infty} (1 + \xi^2)^q (\xi^2 + |\nu| |\xi|)^{-2/p} \left(1 - e^{-\xi^2 t}\right)^2 \, d\xi =: C(\theta, p, \nu, t) < \infty.
\]
Finally we note that the estimate (6.1) and the boundedness of jumps of \( Z \) imply that for any \( q > 1 \) and \( n \) large enough there are \( C_1, C_2 > 0 \) such that
\[
E \int_0^t |Z^n(s) - Z(s)|^q \, ds \leq C_1 + C_2 E \sup_{s \in [0, T]} |Z(t)|^q < \infty.
\]
Hence, the dominated convergence theorem yields
\[
E \int_0^t |Z^n(s) - Z(s)|^q \, ds \to 0.
\]
To estimate the difference \( \|u_2(t, \cdot) - u_2(t, \cdot)^{H^p} (x) \|_{H^p} \), we integrate by parts again. Note that for \( 0 \leq s < t \leq T \) and \( x \in \mathbb{R}, \frac{x-\nu(t-s)}{2\sqrt{(t-s)}} \geq -\frac{\nu}{2} \sqrt{T} \). Since \( x \to \text{erfc}(|x|) \) is integrable, so is \( x \to \text{erfc} \left( \frac{|x|-\nu(t-s)}{2\sqrt{(t-s)}} \right) \), and we can estimate the Fourier transform in the following way:
\[
\sup_{\xi \in \mathbb{R}} \left| \mathcal{F} \left\{ \text{erfc} \left( \frac{|\cdot| - \nu(t-s)}{2\sqrt{(t-s)}} \right) \right\}(\xi) \right| \leq \frac{1}{\sqrt{2\pi}} \| \text{erfc} \left( \frac{|\cdot| - \nu(t-s)}{2\sqrt{(t-s)}} \right) \|_{L_1(\mathbb{R})} \leq C(T),
\]
where \( C(T) \) is a constant that only depends on \( T \). It follows, that
\[
\left| \left[ (Z^n(s) - Z(s)) \mathcal{F} \left( \frac{|x| - \nu(t-s)}{2\sqrt{(t-s)}} \right) \right]_{s=t} \right|^2 \leq 2 \left( |Z^n(t) - Z(t)|^2 + |Z^n(0)|^2 \right) (C(T))^2.
\]
Furthermore, for the derivative of the error function we see:
\[
\frac{d}{ds} \text{erfc} \left( \frac{|x| - \nu(t-s)}{2\sqrt{(t-s)}} \right) = -\frac{1}{\sqrt{\pi}} \left( \frac{\nu}{2\sqrt{(t-s)}} + \frac{|x|}{2(t-s)^{3/2}} \right) e^{-\frac{(|x| - \nu(t-s))^2}{4(t-s)}}.
\]
Obviously, \( x \to e^{-\frac{(|x| - \nu(t-s))^2}{4(t-s)}} \) is integrable for every \( s \in [0, t] \), so we can estimate
\[
\sup_{\xi \in \mathbb{R}} \left| \mathcal{F} \left( \frac{\nu}{2\sqrt{\pi(t-s)}} e^{-\frac{(|x| - \nu(t-s))^2}{4(t-s)}} \right) \right| \leq \frac{C_1}{\sqrt{t-s}}.
\]
for some constant $C_1$, that does not depend on $\xi$ or $s$. For the second term we make a simple substitution to see
\[
\int_0^\infty \frac{x - \nu(t-s)}{2(t-s)^2} e^{-\frac{(x-\nu(t-s))^2}{4(t-s)^2}} \, dx = \int_{-\infty}^{\infty} \frac{2y}{\sqrt{t-s}} e^{y^2} \, dy = \int_{-\infty}^{\infty} \frac{2|y|}{\sqrt{t-s}} e^{y^2} \, dy \leq \frac{C_2}{\sqrt{t-s}},
\]
with $C_2$ being another constant. Eventually this yields for some $C_3 > 0$
\[
\|u^n_t(t, \cdot) - u^n_t(t, \cdot)\|_{H^p(\mathbb{R}_+)}^2 \leq 8 \left( |Z^n(t) - Z(t)|^2 + |Z^n(0)|^2 \right) (C(T))^2 \int_{-\infty}^{\infty} (1 + \xi^2)^\theta \, d\xi
\]
\[
+ 8 \int_{-\infty}^{\infty} (1 + \xi^2)^\theta \, d\xi \cdot \left( \int_0^t \frac{C_3}{\sqrt{(t-s)}} |Z^n(s) - Z(s)| \, ds \right)^2.
\]
Since $\int_0^t (t-s)^{-p/2} \, ds < \infty$ for any $1 < p < 2$ we can use the Hölder inequality to get the anticipated convergence.

Now, we turn to condition (ii). We will only look at the first summand of $G_N$ here. The term containing $u^n_t$ can be treated similarly. For any $0 \leq t_1 \leq t \leq t_2 \leq T$, $|t_2 - t_1| \leq \delta$ and any $\alpha \in [0, 1]$ we estimate
\[
|M(u^n_t(t), u^n_t(t), u^n_t(t_2))|^2
\]
\[
\leq \int_{-\infty}^{\infty} (1 + \xi^2)^\theta \left| \mathcal{F}(u^n(t) - \alpha u^n(t_1) - (1 - \alpha)u^n(t_2)) \right|^2 \, d\xi,
\]
where
\[
|\mathcal{F}(u^n(t) - \alpha u^n(t_1) - (1 - \alpha)u^n(t_2))|
\]
\[
= \left| \int_0^t e^{-(\xi^2 + i\nu\xi)(t-s)} \, dZ^n(s) \right|
\]
\[
- \alpha \int_0^{t_1} e^{-(\xi^2 + i\nu\xi)(t_1-s)} \, dZ^n(s) - (1 - \alpha) \int_0^{t_2} e^{-(\xi^2 + i\nu\xi)(t_2-s)} \, dZ^n(s)|
\]
\[
\leq |Z^n(t) - \alpha Z^n(t_1) - (1 - \alpha)Z^n(t_2)|
\]
\[
+ \alpha \int_0^{t_1} \left| e^{-(\xi^2 + i\nu\xi)(t-s)} - e^{-(\xi^2 + i\nu\xi)(t_1-s)} \right| (\xi^2 + i\nu\xi) Z^n(s) \, ds
\]
\[
+ (1 - \alpha) \int_0^{t_2} \left| e^{-(\xi^2 + i\nu\xi)(t-s)} - e^{-(\xi^2 + i\nu\xi)(t_2-s)} \right| (\xi^2 + i\nu\xi) Z^n(s) \, ds
\]
\[
+ \alpha \int_0^{t_1} e^{-(\xi^2 + i\nu\xi)(t-s)} (\xi^2 + i\nu\xi) Z^n(s) \, ds
\]
\[
+ (1 - \alpha) \int_0^{t_2} e^{-(\xi^2 + i\nu\xi)(t_2-s)} (\xi^2 + i\nu\xi) Z^n(s) \, ds
\]
\[
= |Z^n(t) - \alpha Z^n(t_1) - (1 - \alpha)Z^n(t_2)| + \alpha I_1 + (1 - \alpha) I_2 + \alpha I_3 + (1 - \alpha) I_4.
\]
Because of the $M_1$-convergence of $Z^n$, for any $\varepsilon > 0$ there is $\alpha$, such that for $n \to \infty$
\[
|Z^n(t) - \alpha Z^n(t_1) - (1 - \alpha)Z^n(t_2)| < \varepsilon.
\]
We estimate the first integral as
\[ I_1 \leq \sup_{s \in [0,1]} \left| Z^\prime(t) \cdot (\xi^2 + |\nu||\xi|) \right| \int_0^1 \left| e^{-(\xi^2 + i\nu\xi)(t_1 - s)} - e^{-(\xi^2 + i\nu\xi)(t - s)} \right| ds, \]
and
\[ \int_0^{t_1} \left| e^{-(\xi^2 + i\nu\xi)(t_1 - s)} - e^{-(\xi^2 + i\nu\xi)(t - s)} \right| ds \]
\[ = \int_0^{t_1} e^{-\xi^2 r} \left| e^{i\nu\xi r} - e^{i\nu\xi(t - t_1 + r)} + e^{i\nu\xi(t - t_1 + r)} - e^{\xi^2(t - t_1)} e^{i\nu\xi(t - t_1 + r)} \right| dr \]
\[ \leq \int_0^{t_1} e^{-\xi^2 r} \left| 1 - e^{i\nu\xi(t - t_1)} \right| dr + \int_0^{t_1} e^{-\xi^2 r} \left| 1 - e^{\xi^2(t - t_1)} \right| dr \]
\[ \leq (|\nu\xi\delta| \wedge 2) \int_0^{t_1} e^{-\xi^2 r} dr + (1 - e^{-\xi^2}) \int_0^{t_1} e^{-\xi^2 r} dr \]
\[ = (|\nu\xi\delta| \wedge 2) \xi^{-2}(1 - e^{-\xi^2 t_1}) + (1 - e^{-\xi^2})\xi^{-2}(1 - e^{-\xi^2 t_1}). \]

Finally
\[ \int_{-\infty}^{\infty} (1 + \xi^2)^\theta (\xi^2 + |\nu||\xi|)^2 \xi^{-2} (|\nu\xi\delta| \wedge 2)^2(1 - e^{-\xi^2 t_1})^2 d\xi \]
\[ = \nu^2 \delta^2 \int_{|\xi| \leq 2/|\nu|\delta} (1 + \xi^2)^\theta (\xi^2 + |\nu||\xi|)^2 \xi^{-4} (1 - e^{-\xi^2 t_1})^2 d\xi \]
\[ + 4 \int_{|\xi| > 2/|\nu|\delta} (1 + \xi^2)^\theta (\xi^2 + |\nu||\xi|)^2 \xi^{-4} d\xi \leq C\delta^{2|\theta|-1} \to 0, \quad \delta \to 0. \]

The terms $I_2, I_3, I_4$ are estimated analogously.

2. Dirichlet case ($\theta < -\frac{3}{2}$): In the Dirichlet case the Fourier transform of $G_D$ has the explicit form
\[ \mathcal{F}G_D(t - s)(\xi) = -(\nu - 2i\xi)e^{-(\xi^2 + i\nu\xi)(t - s)}. \]

Obviously, the only difference to the first summand in the Neumann case is the factor $-(\nu - 2i\xi)$. But since in this case $\theta < -\frac{3}{2}$, we only have to note, that the term
\[ ||(\nu - 2i\xi)||^2(1 + \xi^2)^\theta \]
plays the same role as $(1 + \xi^2)^\theta$ for $\theta < -\frac{1}{2}$ in the Neumann case. Consequently, the proof in the Dirichlet case essentially repeats the steps of the Neumann case.

7. Proof of the Theorem 4.3. We consider the case of Neumann boundary conditions.

It is easy to see that for $x > 0$ and $t > 0$ the function
\[ G_N(t, x) = \frac{1}{\sqrt{\pi t}} e^{-\frac{(x - \nu t)^2}{4t}} + \frac{\nu}{2} \text{erfc} \left( \frac{x - \nu t}{2\sqrt{t}} \right), \quad t > 0, \]
\[ G_N(0, x) = \lim_{t \downarrow 0} G_N(t, x) = 0, \]
is absolutely continuous, and its time derivative equals
\[ \frac{d}{dt} G_N(t, x) = G_N(t, x) = \frac{x^2 - t(2 + \nu^2)}{4\sqrt{\pi tb^{5/2}}} e^{-\frac{(x - \nu t)^2}{4t}} + \frac{\nu}{\sqrt{\pi}} \left( \frac{\nu}{4\sqrt{t}} + \frac{x}{4t^{3/2}} \right) e^{-\frac{(x - \nu t)^2}{4t}} \]
\[ = \frac{x^2 + \nu tx - 2t}{4\sqrt{\pi t^{5/2}}} e^{-\frac{(x - \nu t)^2}{4t}}. \]
we have

\[ u(t, x) = \int_0^t G_N(t - s, x) \, dZ(s) = \int_0^t \dot{G}_N(t - s, x) Z(s) \, ds, \]

Thus

\[ u^n(t, x) = \int_0^t G_N(t - s, x) \, dZ^n(s) = Z^n(0) + \int_0^t \dot{G}_N(t - s, x) Z^n(s) \, ds. \]

Thus

\[
\sup_{t \in [0, T]} |u(t, x) - u^n(t, x)| \leq |Z^n(0)| + \sup_{t \in [0, T]} \int_0^t |\dot{G}_N(t - s, x)| \cdot |Z(s) - Z^n(s)| \, ds \\
\leq |Z^n(0)| + M(x) \int_0^T |Z(s) - Z^n(s)| \, ds,
\]

which converges to 0 in probability due to convergence \( Z^n(t) \to Z(t) \) in \( M_1 \) in probability.

8. Appendix. Sobolev spaces have been studied in numerous books, e.g. in [19, 15, 32]. However, a lot of the results presented there are either valid for \( \theta > 0 \), or they are proved in a much broader generality and the proofs rely heavily on the more complex theory of function spaces. The aim of this appendix is to direct proofs specifically for the properties of \( H^\theta(\mathbb{R}_+) \) that we needed for this article.

For our main result (Theorem 2.2) we used that \( H^\theta(\mathbb{R}_+) \) is a separable Hilbert space. For \( \theta > 0 \) this result follows, e.g., from [15], p. 73, Proposition 3.39. In [32], Theorem 4.5.5, an explicit extension operator is given for a more general class of function spaces. In the following lemma we will prove the the minimality of the norm of the extension directly.

**Lemma 8.1.** Let \( \theta \in \mathbb{R} \).

(i) For every \( f \in H^\theta(\mathbb{R}_+) \), there is a unique extension \( \text{ext} \) to \( \mathbb{R} \) such that

\[ \| f \|_{H^\theta(\mathbb{R}_+)} = \| \text{ext} f \|_{\theta, 2}. \]

(ii) The operator \( \text{ext} : H^\theta(\mathbb{R}_+) \to H^\theta(\mathbb{R}) \) is bounded and linear.

**Proof.** We start by showing the existence of the extension for fixed \( f \in H^\theta(\mathbb{R}_+) \). Let \( E \) be the subset of \( H^\theta(\mathbb{R}) \), containing all extensions of \( f \) to \( \mathbb{R} \). Let

\[ \delta := \| f \|_{H^\theta(\mathbb{R}_+)} = \inf_{g \in E} \| g \|_{\theta, 2}. \]

For \( g, h \in E \) we have by the parallelogram law that

\[ \| g - h \|_{\theta, 2}^2 = 2\| g \|_{\theta, 2}^2 + 2\| h \|_{\theta, 2}^2 - 4 \| \frac{g + h}{2} \|_{\theta, 2}^2. \]

Since for every \( \varphi \in \mathcal{D}(\mathbb{R}_+) \)

\[ \left( \frac{g + h}{2} \right)(\varphi) = \frac{1}{2} (g(\varphi) + h(\varphi)) = \frac{1}{2} (f(\varphi) + f(\varphi)) = f(\varphi), \]

\[ \frac{g + h}{2} \in E \quad \text{and} \quad \| \frac{g + h}{2} \|_{\theta, 2} \geq \delta^2. \]

So we get

\[ \| g - h \|_{\theta, 2}^2 \leq 2\| g \|_{\theta, 2}^2 + 2\| h \|_{\theta, 2}^2 - 4\delta^2. \] (8.1)

Now, there is a sequence \( (g_n)_{n \in \mathbb{N}} \) in \( E \) with \( \lim_{n \to \infty} \| g_n \|_{\theta, 2} = \delta \). Then by (8.1) we have

\[ \| g_n - g_m \|_{\theta, 2}^2 \leq 2\| g_n \|_{\theta, 2}^2 + 2\| g_m \|_{\theta, 2}^2 - 4\delta^2 \to 0, \quad n, m \to \infty. \]
That means, \((g_n)\) is a Cauchy sequence and since \(H^\theta(\mathbb{R})\) is complete, there is \(\tilde{f} \in H^\theta(\mathbb{R})\) with \(\tilde{f} = \lim_{n \to \infty} g_n\) in \(H^\theta(\mathbb{R})\). Since \(H^\theta(\mathbb{R})\) is a subspace of \(S'(\mathbb{R})\), this especially implies, that for \(\varphi \in D(\mathbb{R}_+)\),

\[
\tilde{f}(\varphi) = \lim_{n \to \infty} g_n(\varphi) = \lim_{n \to \infty} f(\varphi) = f(\varphi).
\]

Thus \(\tilde{f} \in E\) and, due to the continuity of the norm, \(\|\tilde{f}\|_{\theta,2} = \lim_{n \to \infty} \|g_n\|_{\theta,2} = \delta\). The uniqueness (in the sense of equivalent classes) follows easily from (8.1). Let \(g \in E\) be another element with \(\|g\|_{\theta,2} = \delta\). Then

\[
\|g - \tilde{f}\|_{\theta,2}^2 \leq 2\|g\|_{\theta,2}^2 + 2\|\tilde{f}\|_{\theta,2}^2 - 4\delta^2 = 0.
\]

For (ii) we only have to prove the linearity of the operator. Boundedness directly follows from the definition.

To show linearity we define \(M := \{f \in H^\theta(\mathbb{R}) : f|_{\mathbb{R}_+} = 0\}\) and its orthogonal complement \(M^\perp = \{g \in H^\theta(\mathbb{R}) : \langle g, f \rangle_{\theta,2} = 0 \text{ for all } f \in M\}\). First we show, that for every \(h \in H^\theta(\mathbb{R}_+)\), \(\text{ext}(h) \in M^\perp\).

Indeed, it is enough to show, that \(\langle h, m \rangle_{\theta,2} = 0\) for all \(m \in M\) with \(\|m\|_{\theta,2} = 1\). Note that, since \(m|_{\mathbb{R}_+} = 0\), \(\text{ext}(h) - \alpha m\) is another extension of \(h\) for any \(\alpha \in \mathbb{C}\). Due to its definition, the norm of \(\text{ext}(h)\) is minimal in comparison to every other extension and therefore

\[
\langle \text{ext}(h), \text{ext}(h) \rangle_{\theta,2} \leq \langle \text{ext}(h) - \alpha m, \text{ext}(h) - \alpha m \rangle_{\theta,2}
\]

\[
= \langle \text{ext}(h), \text{ext}(h) \rangle_{\theta,2} + |\alpha|^2 \langle m, m \rangle_{\theta,2} - \bar{\alpha} \langle \text{ext}(h), m \rangle_{\theta,2} - \alpha \langle m, \text{ext}(h) \rangle_{\theta,2}.
\]

Choosing \(\alpha = \langle \text{ext}(h), m \rangle_{\theta,2}\) we get

\[
0 \leq |\alpha|^2 - \bar{\alpha} \alpha - \alpha \bar{\alpha} = -|\alpha|^2 = -|\langle \text{ext}(h), m \rangle_{\theta,2}|^2
\]

and hence

\[
\langle \text{ext}(h), m \rangle_{\theta,2} = 0. \tag{8.2}
\]

Let now \(f, g \in H^\theta(\mathbb{R}_+)\) and \(\lambda \in \mathbb{C}\). We have to show that

\[
\|\text{ext}(f + \lambda g) - \text{ext}(f) - \lambda \text{ext}(g)\|_{\theta,2} = 0.
\]

It is easy to see, that \(\langle \text{ext}(f) + \lambda \text{ext}(g) \rangle|_{\mathbb{R}_+} = f + \lambda g\) and thus

\[
m := \text{ext}(f + \lambda g) - \text{ext}(f) - \lambda \text{ext}(g) \in M.
\]

By (8.2) we have

\[
0 = \langle \text{ext}(f + \lambda g), m \rangle_{\theta,2} - \langle \text{ext}(f), m \rangle_{\theta,2} - \lambda \langle \text{ext}(g), m \rangle_{\theta,2}
\]

\[
= \langle m, m \rangle_{\theta,2}
\]

\[
= \|\text{ext}(f + \lambda g) - \text{ext}(f) - \lambda \text{ext}(g)\|_{\theta,2}^2.
\]

\[
\square
\]

Lemma 8.1 and the properties of \(H^\theta(\mathbb{R})\) imply that \(H^\theta(\mathbb{R}_+)\) is a complete, separable Hilbert space with

\[
\|f\|_{H^\theta(\mathbb{R}_+)}^2 = \langle f, f \rangle_{H^\theta(\mathbb{R}_+)} = \langle \text{ext} f, \text{ext} f \rangle_{\theta,2}.
\]

Eventually we show denseness of test functions in \(H^\theta(\mathbb{R}_+)\) for negative \(\theta\). The idea of the proof follows the argument of Lemma 1.11.1 in [19].

**Lemma 8.2.** If \(\theta < \frac{1}{2}\), then \(\mathcal{D}(\mathbb{R}_+)\) is dense in \(H^\theta(\mathbb{R}_+)\).
Proof. Denote by $D_0(\mathbb{R})$ the subspace of all $\varphi \in D(\mathbb{R})$ with $\varphi = 0$ in a neighbourhood of 0, that is:

$$D_0(\mathbb{R}) := \{\varphi \in D(\mathbb{R}) : \exists r > 0 \text{ such that } \forall x \in B_r(0) \; \varphi(x) = 0\}.$$ 

Obviously it is enough to show that $D_0(\mathbb{R}) \subset H^\theta(\mathbb{R})$ is dense if $\theta < \frac{1}{2}$, because then for any $h \in H^\theta(\mathbb{R}_+)$ there is a sequence $\{d_n\}_{n \in \mathbb{N}} \subset D_0(\mathbb{R})$ that approximates the extension of $h$ in $H^\theta(\mathbb{R})$. Since for every $n \in \mathbb{N}$, $d_n = 0$ in a neighbourhood of zero, the restriction $d_n|_{\mathbb{R}_+}$ is in $D(\mathbb{R}_+)$ and $\{d_n|_{\mathbb{R}_+}\}_{n \in \mathbb{N}}$ approximates $h$ in $H^\theta(\mathbb{R}_+)$. Let $N : H^\theta(\mathbb{R}) \to \mathbb{C}$ be a continuous linear functional. A consequence of the Hahn-Banach Theorem states, that we only have to show, that if $N$ vanishes on $D_0(\mathbb{R})$, it also vanishes on the whole space $H^\theta(\mathbb{R})$ (see [4], Corollary 1.8 and Remark 5). According to the Riesz representation theorem there is a unique element $h_N \in H^\theta(\mathbb{R})$, such that for every $u \in H^\theta(\mathbb{R})$

$$N(u) = \langle u, h_N \rangle_{H^\theta(\mathbb{R})} = \int_{\mathbb{R}} (1 + |\xi|^2)^{\theta} \overline{F(h_N)(\xi)} F(u)(\xi) \, d\xi.$$ 

Let now $N(u) = 0$ for all $u \in D_0(\mathbb{R})$. We can interpret the function $f(\xi) = F(1 + |\xi|^2)^{\theta} \overline{F(h_N)(\xi)}$ as an element of $S'(\mathbb{R})$ and get by the definition of $F$ on $S'(\mathbb{R})$ that

$$(f, u) = (1 + |\xi|^2)^{\theta} \overline{F(h_N), F u} = N(u) = 0 \quad \forall u \in D_0(\mathbb{R}).$$

This, however, means that supp $f = \{0\}$, from which follows (see [15], Theorem 2.31), that

$$f = \sum_{j \leq m} c_j D^j \delta_0$$

for some $m \in \mathbb{N}_0$ and $c_j \in \mathbb{C}$, $j = 0, \ldots, m$, where $\delta_0$ is the $\delta$-distribution.

Hence, we know

$$(1 + |\xi|^2)^{\theta} \overline{F(h_N)(\xi)} = (1 + |\xi|^2)^{-\frac{\theta}{2}} (F^{-1} f)(\xi) = (1 + |\xi|^2)^{-\frac{\theta}{2}} \sum_{j=0}^{m} c_j (-i)^j \xi^j F^{-1} \delta_0,$$

and since $h_N \in H^\theta(\mathbb{R})$, $(1 + |\xi|^2)^{\frac{\theta}{2}} \overline{F(h_N)(\xi)} \in L^2(\mathbb{R})$, so that

$$\int_{\mathbb{R}} \sum_{j=0}^{m} |c_j (-i)^j \xi^j|^2 \frac{1}{(1 + |\xi|^2)^{\theta}} \, d\xi < \infty.$$

For $\theta < \frac{1}{2}$ this is only possible, if $c_j = 0$ for $j = 0, \ldots, m$, and so $h_N \equiv 0$ and therefore $N \equiv 0$ on $H^\theta(\mathbb{R})$. \hfill \square

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