d-wave superconductivity in Hubbard model on the square lattice perturbed by weak 3D uniaxial anisotropy

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The Hubbard model on a square lattice is one of the most studied condensed-matter quantum problems. Here we find evidence that for intermediate $U/4t$ values and a hole-concentration range $x \in (x_c, x_*)$ the ground state of the Hubbard model on the square lattice perturbed by weak three-dimensional (3D) uniaxial anisotropy has long-range $d$-wave superconducting order. Here $t$ is the effective nearest-neighbor transfer integral and $U$ the effective on-site repulsion. The lower critical concentration $x_c$ involves the Ginzburg number $G_i$ and is approximately given by $x_c \approx G_i + x_0 \approx 0.05$. Here $x_0 < G_i$ is a small critical hole concentration that marks a sharp quantum phase transition from a Mott-Hubbard insulator with long-range antiferromagnetic order for $x < x_0$ to an Anderson insulator with short-range incommensurate spiral spin order for $x \in (x_0, x_c)$. The value of the critical hole concentration $x_c$ depends on $U/4t$ and is given by $x_c \approx 0.27$ for $U/4t \approx 1.525$. The long-range $d$-wave superconducting order emerges below a critical temperature $T_c$ for a hole concentration range centered at $x_{op} = (x_c + x_*)/2 \approx 0.16$. It results from the effects of the residual interactions of the charge $c$ and spin-neutral two-spinon $s$ fermions of Ref.14, as a by-product of the short-range spin correlations. The spin subsystem provides through such interactions the energy needed for the effective pairing coupling between the $c$ fermions of the virtual-electron pair configurations.

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I. INTRODUCTION

The Hubbard model on a square lattice\textsuperscript{1} is one of the most studied condensed-matter quantum problems. However, it has no exact solution and many open questions about its properties remain unsolved. Besides being a many-electron problem with physical interest in its own right, there is some consensus that it is the simplest toy model for describing the effects of electronics correlations in the cuprate superconductors\textsuperscript{2-10} and their Mott-Hubbard insulators parent compounds.\textsuperscript{11,12} In addition, the model can be experimentally realized with unprecedented precision in systems of ultra-cold fermionic atoms on an optical lattice. One may expect very detailed experimental results over a wide range of parameters to be available.\textsuperscript{13} The pairing of fermions lies at the heart of both superconductivity studied in this paper and superfluidity. Recent studies of a system of $^6$Li ultra-cold atoms observed a zero-temperature quantum phase transition from a fully paired state to a partially polarized normal state.\textsuperscript{14} Hence, our studies are of interest for both cuprate superconductors and systems of ultra-cold fermionic atoms on an optical lattice.

The virtual-electron pairing mechanism studied in this paper is consistent with the evidence provided in Refs.\textsuperscript{15,16} that unconventional superconductivity is in different classes of systems such as cuprate superconductors, heavy-fermion superconductors, and iron arsenides mediated by magnetic fluctuations. Our investigations have as starting point the square-lattice quantum liquid of Ref.1, which refers to the Hubbard model in the one- and two-electron subspace. It is spanned by the ground state and the excited states generated by application onto it of one- and two-electron operators. For such a square-lattice quantum liquid only the charge $c$ fermions and spin-neutral two-spinon $s$ fermions play an active role. The $c$ and $s$ fermion description and the related general rotated-electron description are consistent with the global $SO(3) \times SO(3) \times U(1) = [SO(4) \times U(1)]/Z_2$ symmetry found recently in Ref.\textsuperscript{17} for the Hubbard model on any bipartite lattice and thus on the square lattice. Such a global symmetry is an extension of the $SO(4)$ symmetry known to occur for the model on such lattices.\textsuperscript{18}

An extended presentation of the results of this manuscript, including their application to the study of the unusual properties of the hole-doped cuprate superconductors, can be found in Ref.\textsuperscript{19}

The Hubbard model on a square lattice with torus periodic boundary conditions, spacing $a$, $N_a^2 = [N_a]^2$ sites, lattice edge length $L = N_a a$, and $N_a \gg 1$ even reads,

$$\hat{H} = -t \sum_\langle \vec{r}_j, \vec{r}_f \rangle \sum_{\sigma = \uparrow, \downarrow} \left[ c_{\vec{r}_j, \sigma}^\dagger c_{\vec{r}_f, \sigma} + h.c. \right] + U \left[ N_a^2 - \hat{Q} \right]/2; \quad \hat{Q} = \sum_{j=1}^{N_a^2} n_{\vec{r}_j, \sigma} \left( 1 - n_{\vec{r}_j, -\sigma} \right).$$

(1)

Here $n_{\vec{r}_j, \sigma} = c_{\vec{r}_j, \sigma}^\dagger c_{\vec{r}_j, \sigma}$ and $-\sigma = \uparrow$ (and $-\sigma = \downarrow$) for $\sigma = \downarrow$ (and $\sigma = \uparrow$) counts the number of electron singly occupied sites. Consistently, the operator $\hat{Q} = [\hat{N} - \hat{Q}]/2$ where $\hat{N} = \sum_\sigma \hat{N}_\sigma$ and $\hat{N}_\sigma = \sum_{j=1}^{N_a^2} n_{\vec{r}_j, \sigma}$ counts that of electron
doubly occupied sites. We denote the $\eta$-spin (and spin) value of the energy eigenstates by $S_\eta$ (and $S_s$) and the corresponding projection by $S^\eta = -[N^2 - N]/2$ (and $S^s = -[N_f - N_i]/2$). We focus our attention on initial ground states with hole concentration $x = [N^2 - N]/N^2 \geq 0$ and spin density $m = [N_f - N_i]/N^2 = 0$ and their excited states belonging to the one- and two-electron subspace.

As found in Ref.\textsuperscript{1}, one can perform an extended Jordan-Wigner transformation\textsuperscript{20,21} that maps the $s1$ bond particles onto $s1$ fermions with operators $\hat{f}^\dagger_{\vec{r}_j,s1}$ and $\hat{f}_{\vec{r}_j,s1}$ whose real-space coordinates $\vec{r}_j$ where $j = 1, ..., N^2_{a_{11}}$ are those of the square $s1$ effective lattice with $N^2_{a_{11}} \approx (1-x)N^2_{a_{12}}/2$ sites. Such operators are related to the corresponding $s1$ bond-particle operators $\hat{g}^\dagger_{\vec{r}_j,s1}$ and $\hat{g}_{\vec{r}_j,s1}$, respectively, defined in Ref.\textsuperscript{22} as follows,

$$f^\dagger_{\vec{r}_j,s1} = e^{i\phi_{\vec{r}_j,s1}} \hat{g}^\dagger_{\vec{r}_j,s1}; \quad f_{\vec{r}_j,s1} = e^{-i\phi_{\vec{r}_j,s1}} \hat{g}_{\vec{r}_j,s1},$$

$$\phi_{\vec{r}_j,s1} = \sum_{j' \neq j} f^\dagger_{\vec{r}_{j'},s1} f_{\vec{r}_{j'},s1} \phi_{\vec{r}_{j'},s1} = \Phi_0 \sum_{j' \neq j} n_{\vec{r}_{j'},s1} \delta(\vec{r}_{j'} - \vec{r}_j) \vec{e}_x,$$

(2)

Here the phase $\phi_{\vec{r}_{j'},s1}$ is defined in the range $\phi_{\vec{r}_{j'},s1} \in (0, 2\pi)$ and $x_{j1}$ and $x_{j2}$ (and $x_{j'1}$ and $x_{j'2}$) are the Cartesian components of the vector $\vec{r}_j$ (and $\vec{r}_{j'}$). The phase $\phi_{\vec{r}_j,s1}$ is that created by a gauge field whose fictitious magnetic field and corresponding effective vector potential read,

$$\vec{B}_{s1}(\vec{r}_j) = \nabla \times \vec{A}_{s1}(\vec{r}_j) = \Phi_0 \sum_{j' \neq j} n_{\vec{r}_{j'},s1} \delta(\vec{r}_{j'} - \vec{r}_j) \vec{e}_x,$$

$$\vec{A}_{s1}(\vec{r}_j) = \Phi_0 \sum_{j' \neq j} \frac{n_{\vec{r}_{j',s1}}}{|\vec{r}_j - \vec{r}_{j'}|} \vec{e}_{\phi_{\vec{r}_{j'},s1} + \pi/2},$$

(3)

respectively. In these expressions $\vec{e}_x$ is a unit vector perpendicular to the plane, $\Phi_0 = 1$ in our units, and $n_{\vec{r}_{j',s1}} = f^\dagger_{\vec{r}_{j'},s1} f_{\vec{r}_{j'},s1}$ is the $s1$ fermion local density operator. In equation (3) and the remaining of this paper we denote by,

$$\vec{e}_\phi = \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix},$$

(4)

an in-plane unit vector whose direction is defined by the angle $\phi$. The effective potential (3) generates long-range interactions between the $s1$ fermions. In addition, each spin-neutral two-spinon $s1$ fermion has in average a flux tube of one flux quantum attached to it.

The components $q_{jx_1}$ and $q_{jx_2}$ of the discrete momenta $\vec{q}_j$ of the $s1$ fermions are the eigenvalues of the two corresponding $s1$ translation generators in the presence of the fictitious magnetic field $\vec{B}_{s1}$ of Eq. (3). The $s1$ fermion operators are defined in and act onto subspaces spanned by mutually neutral states.\textsuperscript{18} Therefore, such operators commute and the two components $q_{jx_1}$ and $q_{jx_2}$ can be simultaneously specified. A property that plays a central role in the studies of this paper and Ref.\textsuperscript{22} is that for vanishing spin density $m = 0$ ground states at finite hole concentrations $x > 0$ the $s1$ fermion momentum band is full and for one-electron and two-electron excited states displays a single hole and none or two holes, respectively. The $s1 - s1$ fermion interactions associated with the effective vector potential of Eq. (3) are stronger than those that arise between the emerging $s1$ fermions and pre-existing $c$ fermions. In spite of that, the former do not lead to $s1 - s1$ fermion inelastic scattering. The obvious reason is that due to phase-space restrictions associated with the exclusion principle and energy and momentum conservation requirements there are no available momentum values in the $s1$ band for excited-state occupancy configurations.

According to Mermin-Wagner-Berezinskii Theorem\textsuperscript{24-27}, in 2D quantum systems destruction of the fluctuations of long-range orders occurs at finite temperatures. To search for long-range superconducting orders at finite temperature we add a small three-dimensional (3D) uniaxial anisotropy perturbation to the square-lattice Hamiltonian (1),

$$\hat{H}_{3D} = \hat{H} + \hat{H}_\perp; \quad \hat{H}_\perp = -t_\perp \sum_{(\vec{r}_{j},\vec{r}_{j'};\perp)} \sum_{\sigma = \uparrow, \downarrow} [c^\dagger_{\vec{r}_{j},\sigma} c_{\vec{r}_{j'},\sigma} + h.c.]; \quad M = \frac{1}{2t_\perp}.$$  

(5)

Here the sum $\sum_{(\vec{r}_{j},\vec{r}_{j'};\perp)}$ runs over first-neighboring sites on nearest-neighboring planes, $t_\perp \ll t$ is a small transfer integral associated with electron hopping between such planes, and the mass $M$ is given in units of lattice spacing and Planck constant one. For the Hamiltonian $\hat{H}_{3D}$ of Eq. (5) $N$ denotes the expectation value of the number of electrons per plane. In addition to weak 3D uniaxial anisotropy, in this paper we find evidence that for the hole concentration range $x \in (x_c, x_*)$ the effects of the cuprates intrinsic disorder are very weak and can be accounted for by an oversimplified scheme involving a single effective suppression coefficient $\gamma_d$. Here $x_c \approx 0.05$ is a critical hole
concentration introduced below and \( x_* \) is the critical hole concentration above which there is no short-range spin order at zero temperature.

In this paper we consider three completely different types of anisotropy: (i) The 3D uniaxial anisotropy associated with the small Hamiltonian term \( \hat{H}_L \) of Eq. (1); (ii) The in-plane anisotropy as measured by the values of the Fermi-velocity anisotropy coefficient \( \eta_{\text{3D}} = \max |V_{3D}/V_{4D}| \) and Fermi-energy anisotropy coefficient \( \eta_0 = |\Delta|/W_c \) introduced in Ref.\(^{1,2}\) and given in Section II - It refers to the anisotropy of the Fermi line associated with the dependence of its finite energy width on the Fermi angle \( \phi \) given in Eq. (A10) of Appendix A; (iii) Finally, the in-plane superfluid-density anisotropy, which refers to the in-plane anisotropy of the superfluid density of some cuprate superconductors. These three types of anisotropy are in the following in general called 3D uniaxial anisotropy, anisotropy, and superfluid-density anisotropy, respectively.

The effects of the 3D uniaxial anisotropy perturbation \( \hat{H}_L \) on the square-lattice Hamiltonian \( \hat{H}_4 \) can for very small anisotropy parameter \( \varepsilon^2 = m^*_c/M \ll 1 \) be ignored for most properties. Here \( m^*_c \) is the c fermion mass of expression (A11) of Appendix A and \( M = 1/2t_L > 1/2t \) the effective mass given in Eq. (4) associated with electron hopping between nearest-neighboring planes. The c fermion mass becomes infinite in the limit \( U/4t \to 0 \). Hence in spite of \( t_L/t \ll 1 \) for the original Hamiltonian \( \hat{H}_{3D} \) of Eq. (3), which is written in terms of electron operators, for small \( U/4t < u_0 \approx 1.3 \) the 3D uniaxial anisotropy parameter reaches values \( \varepsilon^2 = m^*_c/M > 1 \) larger than one upon decreasing \( U/4t \). Indeed, rather than the bare electronic transfer integral \( t \) and corresponding mass \( m^\infty = 1/2t \), the relevant parameter for that quantum liquid is the c fermion mass \( m^*_c = 1/(2r_c, t) \). That follows from the occupancy configurations of the c fermions generating the charge degrees of freedom of states close to the energy eigenstates of \( \hat{H}_{3D} \). The parameter \( r_c = m^\infty_c/m^*_c \) in the c fermion mass expression is the mass ratio given in Eq. (A1) of Appendix A. According to that equation, its limiting values are \( r_c \to 0 \) and \( r_c \to 1 \) for \( U/4t \to 0 \) and \( U/4t \to \infty \), respectively. Our study refers to the range \( U/4t > u_0 \) for which approximately \( r_c \in (0.74, 1.00) \). For it that \( t_L/t \ll 1 \) for the original Hamiltonian \( \hat{H}_{3D} \) of Eq. (3) implies as well that \( \varepsilon^2 = m^*_c/M \ll 1 \). Here \( u_0 \approx 0.13 \) is the \( U/4t \) value at which the important energy scale \( \Delta_0 \) of Eq. (A11) of Appendix A reaches its maximum magnitude.

The studies of this paper focus on the Hamiltonian \( \hat{H}_{3D} \) of Eq. (5) in the one- and two-electron subspace. For hole concentrations \( x \in (x_c, x_*) \) we call virtual-electron pair quantum liquid (VEP quantum liquid) that quantum problem under the weak suppression effects considered below. The main effect of the small perturbation \( \hat{H}_L \) of the VEP quantum liquid is for approximately \( U/4t > u_0 \) the emergence of the 3D uniaxial anisotropy parameter \( \varepsilon^2 = m^*_c/M \ll 1 \) in the expressions of some physical quantities sensitive to the thermal and quantum fluctuations.\(^{10}\) Specific examples are the critical hole concentration \( x_c \approx G + x_0 \), the phases \( \theta \) of the virtual-electron pairs considered below, and related quantities. Here \( G \) is the Ginzburg number\(^{10}\) and \( x_0 < G \) is a small critical hole concentration that marks a sharp quantum phase transition from a Mott-Hubbard insulator with long-range antiferromagnetic order for \( x < x_0 \) to a Anderson insulator with short-range incommensurate spiral spin order for \( x \in (x_0, x_c) \). As discussed in Appendix B, the strong intrinsic-disorder hole-trapping effects present in the hole-doped cuprate superconductors for \( x \in (0, x_c) \) render \( x_0 \) a critical hole concentration. Moreover, such effects, which are active mainly for hole concentrations below \( x_c \), shift the critical hole concentration from \( x_c \approx G + x_0 \) to \( x_c \approx G + x_0 \). This is their only effect on the physics for \( x \in (x_c, x_*) \).

Concerning the short-range spin order occurring for \( x \in (x_0, x_*) \), for both \( x \in (x_0, x_c) \) and \( 0 < (x - x_c) \ll 1 \) it is an incommensurate spiral spin order as that of the square-lattice quantum liquid of Ref.\(^{2}\) for small \( x \). Further details on that order for instance for \( x \) near the optimal hole concentration \( x_{op} = (x_c + x_*)/2 \) and \( U/4t \in (u_{op}, u_1) \) remains an interesting open problem. We find in this paper that for both \( 0 < (x - x_c) \ll 1 \) and \( 0 < (x_* - x) \ll 1 \) the quantum fluctuations are large, so that the VEP quantum-liquid ground state refers to a liquid. In turn, it is found that at and near \( x \approx x_{op} \) such fluctuations are smaller. Hence the ground state is expected to be intermediate between a liquid, where such fluctuations are large, and a crystal, where they are small. The results of Ref.\(^{23}\) seem to indicate that the VEP quantum-liquid physics is for the latter range of \( x \) values controlled by a quantum critical point.

The possibility of using \( U/4t \) as a tuning parameter plays a central role in our scheme. Indeed, the change in the \( U/4t \) ratio is strongly linked to a change to the electron-lattice coupling, since increasing \( U/4t \) also involves increasing the strength of the periodic potential. In contrast to large \( U/4t \) values for which the electron-lattice coupling strength further increases and the important energy scale \( \Delta_0 \) of Eq. (A11) of Appendix A becomes small, for intermediate \( U/4t \in (u_0, u_1) \) values that strength remains much smaller, hence freeing the correlated VEP quantum liquid from its rigid link to the underlying lattice. Intermediate \( U/4t \) values then refer to relatively high effective correlation strength without an increase in the effective strength of the periodic potential. The magnitude of the energy scale \( \Delta_0 \) provides a measure of such an correlation strength effectiveness, vanishing both for \( U/4t \to 0 \) and \( U/4t \to \infty \) and being largest for \( U/4t \in (u_0, u_1) \). Such a correlation strength effectiveness is probably behind the exotic physics emerging for both intermediate \( U/4t \in (u_0, u_1) \) values and hole concentrations at and near \( x_{op} = (x_c + x_*)/2 \), which as mentioned above is according to the studies of Ref.\(^{23}\) controlled by a quantum critical point.

Hence often our results focus on the intermediate-\( U/4t \) range \( U/4t \in (u_0, u_1) \) and hole concentrations \( x \in (x_c, x_*) \).
As discussed in Ref.\textsuperscript{14}, the value of the parameter $\varepsilon^2 = m_s^2/M \ll 1$ is set so that $x_c \approx \text{Gi} + x_0 \approx 0.05$. In turn, that of the zero-temperature upper hole-concentration $x_\ast$ is insensitive to $\varepsilon^2$ and rather depends on $U/4t$. Alike for the square-lattice quantum liquid of Ref.\textsuperscript{14}, it changes smoothly from $x_\ast \approx 0.23$ for $u_0 \approx 1.3$ to $x_\ast \approx 0.28$ for $u_0 \approx 1.6$. Specifically, the value $x_\ast = 0.27$ appropriate to the families of hole-doped cuprates considered in this paper corresponds to $U/4t \approx u_\ast = 1.525$. Provided that $\varepsilon^2 = m_s^2/M \ll 1$, it is expected that the momentum occupancy configurations of the $c$ and $s$ fermions that generate the energy eigenstates of the $\varepsilon^2 = 0$ square-lattice quantum liquid of Ref.\textsuperscript{14} are for $x \in (x_c, x_\ast)$ close to energy eigenstates of the VEP quantum liquid. It then follows that the interactions of such objects remain being residual.

The phases of the phase-coherent virtual-electron pair configurations studied in later sections have the form $\theta = \theta_0 + \theta_1$. Their fluctuations play an important role in the VEP quantum liquid physics. At zero temperature the fluctuations of the phases $\theta_0$ and $\theta_1$ become large for $x \to x_c$ and $x \to x_\ast$, respectively. Expression in terms of the rotated-electron operators of the effective microscopic Hamiltonian describing the quantum fluctuations of the VEP quantum liquid leads to a problem with basic similarities to that considered in Ref.\textsuperscript{14}. Fortunately, the very involved problem of the derivation of an effective action for the phases $\theta$ can be mapped onto a corresponding problem already investigated in that reference. That is achieved on replacing in the Hamiltonian (1) of Ref.\textsuperscript{14} electron operators by rotated electron operators. The transformation that relates rotated electrons to electrons is unitary, so that the effective action for the phases $\theta$ considered in this paper is valid for approximately $U/4t > u_0$ and specifically at $U/4t \approx u_\ast = 1.525$. Expression of the quantum problem that describes the fluctuations of the VEP effective microscopic Hamiltonian describing the quantum fluctuations of the VEP is for the five representative systems very small, $\gamma_d = 1$ physics remains unchanged under these small effects.

Indeed, strong evidence is found below that for intermediate $U/4t$ values, vanishing spin density $m = 0$, and hole concentrations in the range $x \in (x_c, x_\ast)$ the ground state of the VEP quantum liquid has a $d$-wave long-range superconducting order, coexisting with its short-range spin order. Superconductivity emerges below a $d$ dependent critical temperature $T_c$ and results from the effects of the residual $c$ - $s$ fermion interactions, as a by-product of the short-range spin correlations. The spin subsystem provides through such interactions the energy needed for the effective pairing coupling between the $c$ fermions of the virtual-electron pair configurations introduced in the following. The suppression effects due to intrinsic disorder or superfluid density anisotropy slightly lessen the magnitude of $T_c$ and related physical quantities. Otherwise the $\gamma_d = 1$ physics remains unchanged under these small effects.

Our scheme is used in Refs.\textsuperscript{12,23} in the further understanding of the mechanisms behind the unusual properties of hole-doped cuprates. For the five representative systems of the hole-doped cuprates for which $x_c \approx 0.05$ and $x_\ast \approx 0.27$: YBa$_2$Cu$_3$O$_{6+\delta}$ (YBCO 123), Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ (Bi 2212), HgBa$_2$CuO$_{4+\delta}$ (Hg 1201), Tl$_2$Ba$_2$CuO$_{6+\delta}$ (Tl 2201), and La$_{2-x}$Sr$_x$CuO$_4$ (LSCO). In Section III-D it is found that for $\gamma_d = 1$ some of the VEP quantum liquid expressions are valid provided that $x_c/x_\ast < 1/4$. The critical hole concentration values $x_c \approx 0.05$ and $x_\ast \approx 0.27$ are compatible with such a requirement. One finds $x_c - x_\ast \approx \text{Gi} \approx 0.026$ for the four representative hole-doped cuprates other than LSCO and $(x_c - x_\ast) \approx \text{Gi} \approx 0.037$ for LSCO. The relation $\text{Gi} = G/\varepsilon^2 = G[M/m_s^2]$ then implies that our scheme is not valid for the $\varepsilon^2 = m_s^2/M = 0$ 2D square-lattice quantum liquid studied in Ref.\textsuperscript{14}. Indeed, such a problem corresponds to vanishing values of $\varepsilon^2 = m_s^2/M$, such that $1/\varepsilon^2 = M/m_s^2 \to \infty$ and thus $x_c/x_\ast > 1/4$. Fortunately, one finds that $G$ is for the five representative systems very small, $G \approx 10^{-5} - 10^{-4}$. This allows that the ratio $\varepsilon^2 = m_s^2/M \approx 10^{-4} - 10^{-2}$ is smaller than $\text{Gi}$. Interestingly, our results provide evidence that the zero-temperature hole-concentration width $(x_\ast - x_c)$ of the superconducting dome decreases upon decreasing the 3D uniaxial anisotropy. This is consistent with the weak 3D uniaxial anisotropy effects associated with the small Hamiltonian term $\hat{H}_L$ of Eq.\textsuperscript{10} being needed for the occurrence of superconductivity in the quantum liquid under investigation in this paper.

Our preliminary studies on the five representative systems focus mainly onto the hole concentration range $x \in (x_c, x_\ast)$ where $x_c \approx 0.05$ and $x_\ast \approx 0.27$ at $U/4t \approx u_\ast = 1.525$ for which the strong hole trapped effects caused by intrinsic disorder discussed in Appendix B are not active. For that $x$ range the suppression effects originated by intrinsic disorder or in-plane superfluid density anisotropy are very weak for the four representative hole-doped cuprates other than the random alloy LSCO. Indeed, the minimum magnitude $\gamma_d^\text{min}$ reached at $x_\text{op} = (x_c + x_\ast) / 2 \approx 0.16$ of the corresponding suppression coefficient $\gamma_d = T_c/T_c[\text{op} = 0]$ is found in Ref.\textsuperscript{14} to belong to the range $\gamma_d^\text{min} \in (0.94, 0.98)$, whereas $\gamma_d = 1$ both for $0 < (x_c - x) \ll 1$ and $0 < (x_\ast - x) \ll 1$. Our investigations and those of Ref.\textsuperscript{23} provide evidence that for $x \in (x_c, x_\ast)$ the interplay of the electronic correlations described in this paper by the $c$ - $s$ fermion residual interactions with the weak effects of the 3D uniaxial anisotropy perturbation $\hat{H}_L$ of Eq.\textsuperscript{10} being behind the unusual universal properties of the hole-doped cuprate superconductors\textsuperscript{12,23}.

For the VEP quantum liquid referring to the range $x \in (x_c, x_\ast)$ the $x$ dependence of the fluctuations of the phases $\theta = \theta_0 + \theta_1$ is for intermediate $U/4t$ values found to imply naturally a dome-like dependence on $x$ for the critical temperature $T_c$. It reaches its maximum magnitude $T_c^\text{max}$ at the above-mentioned optimal hole concentration $x_\text{op} = (x_c + x_\ast) / 2 \approx 0.16$ for $x_c = 0.05$ and $x_\ast = 0.27$. The energy parameter $\Delta_0$ appearing in the critical temperature and other energy scales expressions plays an important role in the square-lattice quantum liquid\textsuperscript{14}. It vanishes both in
the limits $U/4t \to 0$ and $U/4t \to \infty$ and goes through a maximum magnitude $\max\{\Delta_0\} \approx t/\pi$ at $U/4t = u_0 \approx 1.302$. For instance, it is found in Ref.\textsuperscript{26,27} to control the dependence of $T_{c}^{\text{max}}$ on pressure $P$. Under application of $P$ it first increases until it reaches a maximum magnitude at a pressure value $P = P_0$. A generalized formula valid for arbitrary magnitudes of the critical hole concentrations $x_c$ and $x_s$ and pressure $P \in (0, P_0)$ is introduced. At $P = 0$ it becomes the empirical formula $(1 - T_c/T_{c}^{\text{max}}) = 82.64(x - 0.16)^2$ found by many authors to apply to several families of hole-doped cuprates for the range $x \in (x_c, x_s)$.\textsuperscript{28,29} Importantly, an universal ratio $T_{c}^{\text{max}}(P_0)/T_{c}^{\text{max}}(0) \approx 1.26$ is predicted for hole-doped cuprates with $P = 0$ and $T = 0$ critical hole concentrations $x_c \approx 0.05$ and $x_s \approx 0.27$. The validity of such a theoretical prediction is confirmed for two systems with completely different values of $T_{c}^{\text{max}}(0)$.\textsuperscript{30-33}

Moreover, the low-temperature incommensurate peaks in the inelastic neutron scattering of LSCO observed at momenta $\delta \mathbf{P} = [\pi \pm 2\pi x, \pi]$ and $\delta \mathbf{P} = [\pi, \pi \pm 2\pi x]$ for the hole concentration range $x \in (0,0.12)$.\textsuperscript{31,32,34} follow within the VEP quantum liquid physics from a suitable generalization of the $s1$ fermion microscopic processes found in Ref.\textsuperscript{26} to lead at $x = 0$ to the sharpest features in the neutron scattering of LCO. The incommensurate character of the LSCO peaks results from the contraction of the $s1$ momentum band boundary line upon increasing the hole concentration $x$.

The VEP quantum liquid is consistent with the coexisting two-gap scenario.\textsuperscript{23,33,35,36} A dome-like superconducting energy scale $2|\Omega| = 4k_BT_c/(1-x_cT_c/x_sT_{c}^{\text{max}})$ and pseudogap $2|\Delta| = (1-x/x_s)2\Delta_0$, over the whole dome $x \in (x_c, x_s)$. The energy parameters $2|\Delta|$ and $|\Delta|$ are the maximum magnitudes of the spinon pairing energy $2|\Delta_s(q_{B1}^2)| = 2|\Delta||\cos 2\phi|$ and corresponding one-electron gap $|\Delta||\cos 2\phi|$, respectively. Here $q_{B1}^2$ is a $s1$ band boundary line momentum and $d = \pm 1$ the doublonicity. It is reached at the values $\phi = 0, \pi/2$ of the Fermi angle $\phi$ of Eq. (A6) of Appendix A. The energy scale $2|\Omega|$ is the maximum magnitude reached at $\phi = \pi/4$ of the superconducting virtual-electron pairing energy $2|\Omega_1(q_{B1}^2)| = 2|\Omega||\sin 2\phi|$ introduced in this paper. Both the magnitudes and $x$ dependences of the energy scales $2|\Delta|$ and $2|\Omega|$ and of the superfluid density predicted by the VEP quantum liquid scheme agree with the corresponding experimental results of the five representative cuprate superconductors. That for $x = 0.04 < x_c$ only the $d$-wave gap $2\Delta = |\Delta||\cos 2\phi|$ is observed in the experiments of Ref.\textsuperscript{26} is not inconsistent with our two-gap picture. Indeed, $2|\Omega|$ emerges for $x > x_c$.

The paper is organized as follows. The energy scales of the quantum liquid and other basic physical quantities are discussed in Section II. In Section III evidence is found that the ground state of the VEP quantum liquid has for intermediate $U/4t$ values and hole concentrations in the range $x \in (x_c, x_s)$ a $d$-wave long-range superconducting order coexisting with its short-range spin order. A corresponding effective superconductivity theory is then introduced. The results presented in Section IV include the study of the VEP quantum liquid general energy functional, dependence on the hole concentration $x$ of the superfluid density, and role of the $c$ - $s1$ fermion residual interactions in the $c$ fermion strong effective coupling. Finally, Section V contains the concluding remarks.

II. ENERGY SCALES, CROSSOVER HOLE CONCENTRATIONS, AND FERMION MOMENTUM

In this section and in Appendix A some of the results of Refs.\textsuperscript{1,22} needed for our studies are shortly reviewed. That as considered in Ref.\textsuperscript{26} the angle between the $c$ Fermi hole momentum and $s1$ boundary line momentum of the $c$ fermion and $s1$ fermion hole, respectively, created upon an one-electron addition excitation is exactly $\pi/2$ is a good approximation for the study of some properties. However, having in view the study of the one-electron scattering properties fulfilled in Ref.\textsuperscript{26}, in this paper we account for the small deviations of that angle from $\pi/2$.

Concerning our short review of the results of Refs.\textsuperscript{1,22}, there occurs for the square-lattice quantum liquid a sharp quantum phase transition such that the $m = 0$ ground state has a long-range antiferromagnetic order at $x = 0$ and a short-range spiral-incommensurate spin order with strong antiferromagnetic correlations for $0 < x < 1$. As a result of it, the maximum magnitude $2|\Delta|$ of the $s1$ fermion spinon-pairing energy has a singular behavior at $x = 0$. Indeed, one has that $2\Delta_{|x=0} = \mu^0$ for $x = 0$ and $m = 0$ whereas $\lim_{x \to 0} 2|\Delta| = 2\Delta_0$. Here $\mu^0$ is the chemical potential $\mu$ in the limit $\lim_{x \to 0} \mu$ whose behaviors are given in Eq. (A3) of Appendix A. It is one half the Mott-Hubbard gap, consistently with for $x = 0$ and $m = 0$ the chemical potential belonging to the range $\mu \in (-\mu^0, \mu^0)$. In spite of $2\mu^0$ being the charge Mott-Hubbard gap, it also refers to the spin degrees of freedom.\textsuperscript{2} The energy scale $2\Delta_{|x=0} = \mu^0$ is the excitation energy below which the long-range antiferromagnetic order survives for $x = 0$, $m = 0$, and zero temperature $T = 0$. In turn, the energy scale $\lim_{x \to 0} 2|\Delta| = 2\Delta_0$ plays a key role in the $x > 0$ and $x > x_0$ physics of the square-lattice quantum liquid of Ref.\textsuperscript{26} and the VEP quantum liquid, respectively. The small hole concentration $x_0 < x_c$ is given below. According to the analysis of Appendix B, the hole-trapping effects caused for hole concentrations below $x_c$ by the cuprates intrinsic disorder are strong and shift that sharp quantum phase transition from $x = 0$ to $x = x_0$.

The approximate limiting behaviors of the energy scale $2\Delta_0$ are provided in Eq. (A1) of Appendix A. As a function of $u = U/4t$, it is such that $\partial \Delta_0(u)/\partial u = 0$ at $u = U/4t = u_0$. This is consistent with it reaching a maximum magnitude $t/\pi$ at that $U/4t$ value. It can be expressed in terms of the $s1$ fermion energy dispersion bandwidth $W_{s1}^0$
Here $\epsilon_s(q)$ is the s1 fermion energy dispersion derived for the square-lattice quantum liquid in Ref.\textsuperscript{30} whose generalized expression is for the VEP quantum liquid introduced below and $q_{d1}$ is the s1 boundary line nodal momentum defined in that reference.

For the Hubbard model on the square lattice at zero temperature $T = 0$ a small concentration $x$ of holes prevents the occurrence of long-range antiferromagnetic order. At vanishing spin density $m = 0$ and both for small finite temperature $T > 0$ and zero hole concentration $x = 0$ and for vanishing or small finite temperature $T > 0$ and small finite hole concentration $0 < x \ll 1$ the system is driven into a renormalized classical regime where the $T = 0$ and $x = 0$ long-range antiferromagnetic order is replaced by a quasi-long-range spiral-incommensurate spin order as that studied in Ref.\textsuperscript{30} for simpler spin systems. For hole concentrations obeying the inequality $0 < x < x_c$ the system is in a short-range spin ordered phase whose order parameter is the maximum magnitude of the $x = 0$ long-range antiferromagnetic order is replaced by a quasi-long-range spiral-incommensurate spin order as that

\begin{align}
\Delta_0 &= r_s 4W_{s1}^0 = 4W_{s1}^0 e^{-\lambda_s}; \quad W_{s1}^0 = \lim_{x \to 0} W_{s1} = W_{s1}|_{x=0}, \\
W_{s1} &= \left[ \epsilon_{s1}(q_{d1}) - \epsilon_{s1}(0) \right]; \quad \lambda_s = |\ln(\Delta_0/4W_{s1}^0)|. \quad (6)
\end{align}

as,

At vanishing spin density $m = 0$ and both for small finite temperature $T > 0$ and zero hole concentration $x = 0$ and for vanishing or small finite temperature $T > 0$ and small finite hole concentration $0 < x \ll 1$ the system is driven into a renormalized classical regime where the $T = 0$ and $x = 0$ long-range antiferromagnetic order is replaced by a quasi-long-range spiral-incommensurate spin order as that studied in Ref.\textsuperscript{30} for simpler spin systems. For hole concentrations obeying the inequality $0 < x < x_c$ the system is in a short-range spin ordered phase whose order parameter is the maximum magnitude of the $s_1$ fermion spinon-pairing energy $2|\Delta| = 2\Delta_0 (1 - x/x_c)$. Following the analysis of Appendix B, for the quantum problem considered here such an order occurs for a smaller range $x \in (x_0, x_c)$. In either case the above expression is valid approximately for the range $U/4t \in (u_s, u_1)$ where $u_s > u_1$ is the $U/4t$ value at which $x_c = 1/\pi \approx 0.32$. For approximately $U/4t > u_1$ and thus including at $U/4t \approx u_s$ one has that $r_s < r_c < 2r_s$ and the approximate expression $r_s \approx e^{-4t u_s/U}$ of Eq. (A10) of Appendix A is not valid. Often our results refer to the smaller range $U/4t \in (u_s, u_1)$ for which the relation $r_c \approx 2r_s \approx 2e^{-4t u_s/U}$ approximately holds.

The s1 fermion mass ratios in Fermi-liquid theory\textsuperscript{40} the $U/4t$-dependent spin ratio $r_s = \Delta_0/4W_{s1}^0$ and charge mass ratio $r_c = m_c^\infty/m_c^*$, both given in Eq. (A1) of Appendix A, control the effects of the electronic correlations in many physical quantities. These ratios play an important role in the physics of both the square-lattice quantum liquid of Ref.\textsuperscript{30} and VEP quantum liquid. In contrast to a Fermi liquid, here the mass $m_c^\infty = \lim_{U/4t \to \infty} m_c^* = 1/2t$, which refers to the limit of infinite on-site interaction, plays the role of bare mass. In turn, the VEP quantum liquid mass ratio $1/e^2 = M/m_c^*$ involving the mass $M$ of Eq. (5) and the $c$ fermion mass $m_c^*$ of expression (A11) of Appendix A controls the effects of the weak 3D uniaxial anisotropy.

For $U/4t > 0$ the $s_1$ fermion dispersion acquires a finite energy bandwidth. Both the square-lattice quantum liquid of Ref.\textsuperscript{30} and the VEP quantum liquid are defined in the one- and two-electron subspace. For finite $U/4t$ values the $c$ and $s_1$ fermions describe the charge and spin degrees of freedom, respectively, of the occupancy configurations of the rotated electrons that generate the states that span that subspace. In it only the $c$ and $s_1$ fermions play an active role. The c momentum band has the same shape and area $(2\pi/L)^2 N_d^2$ as the first Brillouin zone. The $s_1$ momentum band is exotic and such that its momentum area and space are subspace dependent. For ground states corresponding to hole concentrations $x \in (x_c, x_s)$ and spin density $m = 0$ the $c$ fermions have for the square-lattice quantum liquid of Ref.\textsuperscript{30} an isotropic hole like $c$ Fermi line. The hole momenta $q_{d1}^{ph}$ belonging to that line, defined in Eq. (10), are centered at the momentum $-\pi$. Out of the $c$ band $N_d^2$ discrete hole momentum values $q_{d1}^j$, where $j = 1, ..., N_d^2$, $N_d = 2S_c$ are filled and $N_h = [N_d^2 - 2S_c]$ unfilled and correspond to $c$ fermions and $c$ fermion holes, respectively. For the ground state one has that $N_c = 2S_c = N = (1 - x) N_d^2$ and the hole c Fermi line encloses a momentum area $[N_d^2 - N_c] [2\pi/L]^2 = [N_d^2 - 2S_c] [2\pi/L]^2 = 4\pi^2 x$.

For ground states corresponding to vanishing spin density $m = 0$ the $s_1$ band is full. Hence the number of holes in the $s_1$ band vanishes, $N_s^0 = 0$, and the number of $s_1$ fermions reads $N_{s1} = N/2 = (1 - x) N_d^2/2$. The $s_1$ momentum band is particle like so that its boundary line is centered at $q = 0$ and encloses a momentum area $2\pi^2 (1 - x)$. The momenta $q_{d1}^{s1}$ of the $s_1$ band boundary line are given in the following. For the states that span the one- and two-electron subspace the number of holes in the $s_1$ momentum band is given by $N_{s1}^h = 1$ and $N_{s1}^b = 0$ or $N_{s1}^b = 2$, respectively.

For each $[N_\uparrow, N_\downarrow] = [N/2, N/2 - 1]$-electron ground state there is a corresponding zero-spin-density and $[N_\uparrow, N_\downarrow] = [N/2, N/2]$-electron ground state with one more spin-down electron. We say that such a ground state is the $m = 0$ generating ground state of both the $[N/2, N/2 - 1]$-electron ground state and its $[N/2, N/2 - 1]$-electron excited states
of small momentum and low energy. The reduced one-electron subspace is spanned by a \([N/2, N/2 - 1]\)-electron ground state and its \([N/2, N/2 - 1]\)-electron excited states of small momentum and low energy. The doublet \(d = \pm 1\) labels the s1 fermions whose momentum occupancy configurations generate the states that span that subspace. The s1 band momenta \(\vec{q}\) of the reduced one-electron subspace s1 band are related to the s1 band momenta \(\vec{q}_0\) of the corresponding \(m = 0\) generating ground state as 
\[
\vec{q} = A^d_{1s1} \vec{q}_0; \quad \lim_{x \to 0} A^d_{1s1} = I.
\]

Here we neglect terms of order \(1/N^2\), which vanish in the thermodynamic limit. \(I\) denotes the 2 \(\times\) 2 unit matrix and for \(x \in (x_c, x_*)\) and momenta \(\vec{q}\) at or near the s1 boundary line one has that \(A^d_{1s1} = A^d_F\). The \(F\) rotation matrix \(A^d_F\) is given below.

The Fermi-velocity anisotropy coefficient \(\eta_\Delta = \max V_{B_{s1}}^\Delta / V_{Fc}\), Fermi-energy anisotropy coefficient \(\eta_0 = |\Delta|/W^h\), and hole concentrations \(x_\Delta\) and \(x_0\) at which \(\eta_\Delta = 1\) and \(\eta_0 = 1\), respectively, play an important role. For approximately \(U/4t \in (u_0, u_1)\) such quantities read
\[
\eta_\Delta = \max r_\Delta \approx \sqrt{\frac{x_0}{\pi}} \gamma_0; \quad r_\Delta = \frac{V_{B_{s1}}^\Delta}{V_{Fc}}; \quad \eta_0 = \frac{|\Delta|}{W^h} \approx \sqrt{\frac{2x_\Delta}{\pi}} \left(\frac{1}{x - x_*}\right); \quad x_\Delta = \frac{1}{2\pi} \left(\frac{\Delta_0}{4F_c t}\right)^2;
\]
\[
x_0 = \frac{\Delta_0}{t} \left(\frac{x_*}{(2rc^2 + \Delta_0/t)}\right) \approx \frac{\Delta_0}{t} \left(\frac{1}{2\pi^2 x_*}\right); \quad x_{c1} = \frac{1}{8}.
\]

Here \(\Delta_0\) is the energy parameter of Eq. 4, \(r_c \approx 2r_c \approx 2e^{-4tu_0/U}\) where \(u_0 \approx 1.302\), and \(W^h \approx 2\pi/m^*_c\) is the unfilled \(c\) fermion sea energy bandwidth. Following the discussions of Appendix B, the hole trapping effects caused by strong intrinsic disorder render \(x_0\) a critical hole concentration. It marks a sharp transition between zero-temperature states with long-range antiferromagnetic order and short-range spin order. In Ref. x_{c1} was defined as the hole concentration at which the inequality \(\eta_\Delta = 2x_0\) is satisfied. For \(U/4t \in (u_0, u_1)\) that formula leads to \(x_{c1} \approx 1/8\). Since \(x_{c1}\) is a mere crossover hole concentration in this paper we use the magnitude \(x_{c1} = 1/8\), valid for the interaction range \(U/4t \in (u_0, u_1)\). The absolute value \(V_{B_{s1}}^\Delta\) of the s1 fermion velocity \(V_{B_{s1}}^\Delta(\vec{q})\) of Eq. (A12) of Appendix A at the s1 boundary line and that of the \(c\) fermion velocity at hole momenta belonging to the \(c\) Fermi line also appearing in the expressions of Eq. (8) read,
\[
V_{B_{s1}}^\Delta \equiv V_{B_{s1}}^\Delta(\vec{q}_{B_{s1}}) = \frac{|\Delta|}{\sqrt{2}} \sin 2\phi; \quad V_{Fc} = \frac{q^h_{Fc}}{m^*_c} \approx \frac{\sqrt{\pi} 2}{m^*_c},
\]
respectively. The dependence on the Fermi angle \(\phi \in (0, 2\pi)\) defined in Eq. (A6) of Appendix A of the absolute value \(V_{B_{s1}}^\Delta\) of the s1 fermion velocity confirms the anisotropic character of the s1 boundary line. In turn, that for hole concentrations \(x \in (x_c, x_*)\) the \(c\) Fermi line is isotropic follows from the independence of the angle \(\phi\) of the \(c\) fermion velocity.

For \(x \in (x_c, x_*)\) and \(U/4t\) values approximately in the range \(U/4t \in (u_0, u_1)\) the Fermi-line level of anisotropy is controlled by the interplay of the s1 boundary line anisotropy and \(c\) Fermi line isotropy. The Fermi energy has the form \(E_F = \mu + \delta E_F\) where \(\mu \approx \mu^0 + W^h, \mu^0\) is given in Eq. (A1) of Appendix B, and below it is found that the square-lattice quantum liquid expression \(\delta E_F \approx |\Delta| |\cos 2\phi|\) acquires an extra term due to the superconducting fluctuations of the VEP quantum liquid studied in this paper.

According to the criteria of Ref. x_{c1} for the approximate range \(U/4t \in (u_0, u_1)\) of intermediate \(U/4t\) values the Fermi line is strongly anisotropic for hole concentrations below \(x_{c1} = 1/8\), has some Fermi-energy anisotropy yet the Fermi velocity is nearly isotropic for the hole concentration range \(x \in (x_c, x_{c2})\), and is nearly isotropic for the range \(x \in (x_{c2, c1})\). Here the hole concentration \(x_{c2} = (2\gamma_0 + 1/[3\gamma_0 + 1]) x_* \) where \(\gamma_0 = (1 - x_0/x_*)\) is introduced below in Section III-E. The Fermi line is hole and particle like for \(x < x_{h}\) and \(x > x_{h}\), respectively. The value of the hole concentration \(x_{h} \geq x_{c2}\) is not accurately known. Likely it belongs to the range \(x_{h} \in (x_{c2}, x_*)\). Provided that \(x_{h} < x_{c1}\), the angle \(\phi_{AN}\) of Eq. (A6) of Appendix A vanishes for hole concentrations \(x \leq x_{h}\) and is small for \(x \in (x_{1h}, x_{c1})\). Hence for hole concentrations in the range \(x \in (x_{h}, x_*)\) we consider only corrections up to first order in \(\phi_{AN}\), so that max \{\(\delta E_F\)\} \(\approx |\Delta|\) and \(\min \{V_{B_{s1}}^\Delta\} \approx |\phi_{AN} \sqrt{2} | |\Delta|\).

The Fermi momentum \(\vec{k}_F = \vec{k}_F + \vec{\pi}\) given in Eq. (A6) of Appendix A where \(\vec{\pi} = [\pi, \pi]\) can be expressed in terms of the corresponding c hole Fermi momenta \(\vec{q}_{Fc}^d\) and s1 boundary-line momentum \(\vec{q}_{B_{s1}}^d\) as follows.
\[
\vec{k}_F^h = \vec{q}_{Fc}^h - \vec{q}_{B_{s1}}^d = k^h_{Fc}(\phi) \vec{e}_\phi; \quad \vec{q}_{Fc}^d = \vec{q}_{Fc}^d + \vec{\pi},
\]
\[
\vec{q}_{B_{s1}}^d = \vec{q}_{Fc}^d(\vec{\phi}_c^d) \vec{e}_{\phi_d}; \quad \vec{q}_{B_{s1}}^d = q_{B_{s1}}(\vec{\phi}_c^d) \vec{e}_{\phi_{s1}^d}, \quad d = \pm 1.
\]
These general expressions are for one-electron excited states valid for \( x \in (x_c, x_s) \). For \( x \in (x_{c1}, x_{c2}) \) the specific general dependences on the Fermi angle \( \phi \) of the angles \( \phi^d_\ell \in (0, 2\pi) \) and \( \phi^s_\ell \in (0, 2\pi) \) are,

\[
\phi^d_\ell = [\phi - d_\pi \pi / 2 + \phi^d_\ell]; \quad \phi^s_\ell = [\phi + \pi + \phi^d_\ell], \quad d = \pm 1. \tag{11}
\]

In the studies of Ref.\textsuperscript{11} it is considered that for the Fermi velocity isotropic \( x \) range \( x \in (x_{c1}, x_{c2}) \) the c Fermi hole momentum \( \vec{q}_F^c \) and s1 boundary line momentum \( \vec{q}_B^{s1} \) of the c Fermi and s1 fermion hole, respectively, created upon one-electron addition excitation are perpendicular, so that \( \vec{q}_F^c \cdot \vec{q}_B^{s1} = 0 \). Since \( \pi \pi / 2 \) is the angle between the momentum-space directions of \( \vec{q}_F^c \) and \( \vec{q}_B^{s1} \), this is equivalent to considering that \( \pi \pi / 2 = \pi / 2 \) in the \( \phi^c_\ell \) expression of Eq. (11). Hence, in spite of the c momentum band remaining unaltered upon such an excitation, for states belonging to the reduced one-electron subspace there are two alternative c Fermi momenta. Those are associated with the two doublet values \( d = \pm 1 \), respectively. Their expression is given in Eq. (11). Specifically, the doublet \( d = \pm 1 \) of a c fermion created or annihilated under an one-electron excitation equals that of the corresponding s1 fermion hole created under the same excitation.

The concept of doublet remains valid when the angle \( \pi \pi / 2 \) between the momentum-space directions of \( \vec{q}_F^c \) and \( \vec{q}_B^{s1} \) appearing in the \( \phi^c_\ell \) expression of Eq. (11) is \( \phi \) dependent, provided that the following integrals involving the corresponding function \( \pi \pi = \pi \pi (\phi) \) and the c Fermi line hole momentum and s1 boundary line momentum absolute values \( q_F^c (\phi) \) and \( q_B^{s1}(\phi) \), respectively, vanish,

\[
\int_0^{\pi / 4} d \phi q_F^c (\phi) q_B^{s1} (\phi) \cos \left( \frac{\pi \pi (\phi)}{2} \right) = \int_0^{\pi / 2} d \phi q_F^c (\phi) q_B^{s1} (\phi) \cos \left( \frac{\pi \pi (\phi)}{2} \right) = 0. \tag{12}
\]

This ensures that the Fermi line centered at \(-\pi\) encloses the correct momentum area,

\[
\int_0^{2\pi} d \phi \pi \left[ k_F^c (\phi) \right]^2 = \int_0^{2\pi} d \phi \frac{\pi}{2} \left( [q_F^c (\phi)]^2 + [q_B^{s1} (\phi)]^2 + 2q_F^c (\phi) q_B^{s1} (\phi) \cos \left( \frac{\pi \pi (\phi)}{2} \right) \right) = x 4\pi^2 + (1-x) 2\pi^2 = (1+x) 2\pi^2. \tag{13}
\]

Here \( x = 4\pi^2 \) and \( (1-x) 2\pi^2 \) are the momentum areas enclosed by the c Fermi line and s1 boundary line when centered at \(-\pi\) and \( \vec{0} = [0, 0] \), respectively. In this paper we consider the general case for which the angle \( \pi \pi / 2 = \pi \pi (\phi) / 2 \) is for \( x \in (x_{c1}, x_{c2}) \) a periodic function of period \( \pi / 2 \). In addition, for \( \phi \in (0, \pi / 4) \) it obeys the equality \( \pi \pi (\phi) / 2 = \pi \pi (\pi / 2 - \phi) / 2 \) where \( \pi / 2 - \phi \in (\pi / 4, \pi / 2) \). It follows that for \( x \in (x_{c1}, x_{c2}) \) the one-electron F angle \( \phi^F (\phi) \) appearing in Eq. (11) reads,

\[
\phi^d_\ell (\phi) = d \arctan \left( \frac{q_F^c (\phi) \sin \left( \frac{\pi \pi (\phi)}{2} \right)}{q_B^{s1} (\phi) + q_F^c (\phi) \cos \left( \frac{\pi \pi (\phi)}{2} \right)} \right), \quad d = \pm 1. \tag{14}
\]

It can have two values, \( \phi^F_\ell \) and \( \phi^F_\ell \). The two corresponding rotations refer to the doublet \( d = -1 \) and \( d = +1 \), respectively. For \( \pi \pi / 2 = \pi / 2 \) this general expression recovers that found in Ref.\textsuperscript{11}. The F angle \( \phi^F (\phi) \) is associated with the one-electron F rotation matrix \( A^F \) such that,

\[
q_B^{s1} = A^F q_B^{s1}; \quad A^F = \begin{bmatrix} \cos \phi^F & -\sin \phi^F \\ \sin \phi^F & \cos \phi^F \end{bmatrix}, \quad d = \pm 1. \tag{15}
\]

Alike in Ref.\textsuperscript{11}, \( q_B^{s1} \) denotes the auxiliary momentum of the s1 boundary momentum \( \vec{q}_B^{s1} \) by \( \vec{q}_B^{s1} \). For \( x \in (x_{c1}, x_{c2}) \) and \( \pi \pi \) at or near the s1 boundary line the matrix \( A^F \) of Eq. (11) is orthogonal and equals the F rotation matrix \( A^F \) given here.

If follows from the above expressions that for the hole-concentration range \( x \in (x_{c1}, x_{c2}) \) the absolute value \( k_F^c (\phi) \) of the hole Fermi momentum given in Eq. (10) and Eq. (A6) of Appendix A reads,

\[
k_F^c (\phi) = \sqrt{[q_F^c (\phi)]^2 + [q_B^{s1} (\phi)]^2 + 2q_F^c (\phi) q_B^{s1} (\phi) \cos \left( \frac{\pi \pi (\phi)}{2} \right)}. \tag{16}
\]

Again, for \( \pi \pi / 2 = \pi / 2 \) this recovers the corresponding expression of Ref.\textsuperscript{11}. The hole Fermi momentum \( \vec{k}_F^h \) of Eq. (10) for any value of the Fermi angle \( \phi \) and the nodal Fermi momentum \( \vec{k}_F^N = \vec{k}_F^{N, -\pi} - \pi \) for \( \phi = \pi / 4 \) can be expressed as follows,

\[
\vec{k}_F^h (\phi) = -\frac{k_F^h (\phi)}{q_B^{s1} (\phi)} q_B^{s1} (\phi) \quad \vec{k}_F^N = -\frac{k_F^N}{q_B^{s1}} q_B^{s1}; \quad q_B^{s1} (\phi) = q_B^{s1} (\pi / 2 - \phi). \tag{17}
\]
Here $\tilde{k}_p^F(\phi)$ and $\tilde{k}_N^F$ are centered at $-\pi$ and $0$, respectively, and $k_p^N$ is the absolute value of the vector $\tilde{k}_N^F$. Note that the auxiliary $s1$ boundary momentum $\tilde{q}_{B11}$ of Eq. (15) has been constructed to inherently pointing in the same direction as $\tilde{k}_p^F$. This justifies the validity of the equalities (17), which involve the auxiliary $s1$ boundary-line momentum rather than the corresponding $s1$ boundary-line momentum.

The ratios $r_s$ and $r_c$ of Eq. (A11) of Appendix A, which for $U/4t \in \{u_0, u_1\}$ are related as $r_c = \pi x_s \approx 2r_s$, control the effects of electronic correlations in many quantities. For instance, for such a $U/4t$ range they control the range width $\max(\pi c_s/2)-\min(\pi c_s/2) = r_s$ of the angle $\pi c_s/2$. Indeed, $\pi c_s/2 \in [1-x_A]/2, [1+x_A]/2)$. Here $x_A \approx x_s/2 = 0.153$ is a crossover hole concentration slightly larger than $x_c = 1/8 = 0.125$. According to the studies of Ref. [8] it marks the emergence of a scale invariance in the VEP quantum liquid, which dominates the physics in a hole concentration range $x \in (x_A, x_{c2})$. The smallest and largest $\pi c_s/2$ magnitudes are reached for hole Fermi momenta pointing in the nodal and anti-nodal directions,

$$\pi c_s(\pi/4)/2 = [1-x_A]/2; \quad \pi c_s(0)/2 = \pi c_s(\pi/2) = [1+x_A]/2; \quad x_A \approx x_s/2,$$

respectively. For $U/4t \approx 1.525$ this gives $\pi c_s(\pi/4)/2 \approx 0.43 \pi$ and $\pi c_s(0)/2 = \pi c_s(\pi/2) \approx 0.57 \pi$. Hence $\pi c_s/2 \in (0.43 \pi, 0.57 \pi)$ has indeed values near $\pi/2$. Both this and that its average value is $\pi/2$ justifies the approximation of Ref. [8] that it reads $\pi/2$.

Except for the angle $\pi c_s/2 = \pi/2$ being replaced by a $\phi$ dependent angle $\pi c_s/2 \in [(1-x_A)/2, [1+x_A]/2, \pi/2)$, the expressions and physics reported in Ref. [8] remain valid. Moreover, the $c$ fermion energy dispersion $\epsilon_c(q^B)$ provided in Eq. (A10) of Appendix A is not changed by the VEP quantum liquid superconducting fluctuations studied in this paper. In contrast, the $s1$ fermion energy dispersion $\epsilon_{s1}(q^B)$ is, as found in the following.

### III. Long-range $d$-Wave Superconducting Order

Here evidence is provided that the ground state of the VEP quantum liquid is superconducting for zero spin density $m = 0$, hole concentrations in the range $x \in (x_c, x_s)$, and intermediate $U/4t$ values. We recall that for approximately $U/4t > u_0$ the VEP quantum liquid refers to the Hamiltonian $H_{3D}$ of Eq. (5) in the one- and two-electron subspace for $t_\perp/t \ll 1$ plus the very small suppression effects considered below. For such a $U/4t$ range of values, that the inequality $t_\perp/t \ll 1$ holds assures that the anisotropy parameter $\varepsilon^2 = m_t^c/M$ is very small as well.

We start by considering the ground state of the $\gamma_d = 1$ and $t_\perp/t = 0$ square-lattice quantum liquid of Ref. [8] at vanishing spin density $m = 0$. We find that a necessary condition for it being that of a $d$-wave superconductor is fulfilled for finite hole concentrations below $x_s$. Such a result is valid for the VEP quantum liquid as well provided that the only significant effects of the weak 3D uniaxial anisotropy and suppression effects are the emergence of the small anisotropic parameter $\varepsilon^2 = m_t^c/M$ and parameter $\gamma_d \approx 1$ in the expression of several physical quantities associated with quantum and thermal fluctuations. For the five representative systems the parameter $\varepsilon^2 = m_t^c/M$ is found in Ref. [8] to belong to the range $\varepsilon^2 \in (3 \times 10^{-4}, 1 \times 10^{-2})$. For such a small values of that parameter and $\gamma_d \approx 1$ the energy eigenstates of the $\gamma_d = 1$ and $t_\perp/t = 0$ square-lattice quantum liquid studied in Ref. [8] are expected to be a good approximation for those of the VEP quantum liquid.

Consistently with the Mermin-Wagner-Berezinskii Theorem [24-27], the introduction of the weak 3D uniaxial anisotropy prevents the destruction of the fluctuations of long-range orders at finite temperature. For $t_\perp/t \ll 1$ very small and approximately $U/4t > u_0$ we find indeed strong evidence of virtual-electron-pair phase coherence below some critical temperature $T_c$. (The concept of a virtual electron pair is introduced below.) At zero temperature such a virtual-electron-pair phase coherence occurs for the hole-concentration range $x \in (x_c, x_s)$. Here the lower critical concentration $x_c \approx 0.153$. The higher critical concentration $x_c \approx 0.153$, where $x_c$ is fully determined by the anisotropic parameter $\varepsilon^2 = m_t^c/M \approx 1$. The suppression effects of the hole-trapping effects of Appendix B. The magnitude of the proportionality constant $G$ is found in Ref. [8] to be in the range $G \in (1 \times 10^{-5}, 3 \times 10^{-4})$ for the five representative systems. Hence it is so small that $G \approx 10^{-2}$ in spite of $m_t^c/M \approx 1$ being much smaller than $G$. The weak 3D uniaxial anisotropy allows considering the pairing energy within a small coherent volume. Such an elementary volume controls the magnitude of parameters associated with the thermal and quantum fluctuations of the square-lattice quantum liquid perturbed bery weak 3D uniaxial anisotropy. The suppression effects slightly lessen the $T_c$ magnitude, which is linear in $\gamma_d \approx 1$.

The physical picture that emerges from our study is that superconductivity arises in the VEP quantum liquid as a by-product of the short-range spin correlations. We then construct a consistent scheme concerning the $d$-wave pairing mechanism, phase-coherent-pair superconducting order, and corresponding order parameter, which follows from the properties of that quantum liquid. Our results are inconclusive on whether the ground state of the 2D Hubbard model on the square lattice is superconducting. They seem to indicate that some small 3D uniaxial anisotropy is needed for the emergence of superconductivity.
A. A necessary condition for the ground state at zero spin density being that of a \(d\)-wave superconductor

Let us consider the \(t_\perp/t = 0\) and \(\gamma_d = 1\) Hubbard model on the square lattice. The one- and two-electron subspace is spanned by states whose deviation \(\delta N_{h}^{s1}\) in the number of \(c\) band holes and number \(\delta N_{s1}^{h}\) of \(s1\) band holes read

\[
\delta N_{c}^{h} = -\delta N = 0, \mp 1, \pm 2; \quad \delta N_{s1}^{h} = \pm(\delta N_\uparrow - \delta N_\downarrow) + 2L_{s, \mp 1/2} + 2N_{s2} = L_{s, -1/2} + L_{s, +1/2} + 2N_{s2} = 0, 1, 2. \tag{19}\]

Here \(\delta N = [\delta N_\uparrow + \delta N_\downarrow]\) is the deviation in the number of electrons, \(\delta N_\uparrow\) and \(\delta N_\downarrow\) those in the number of spin-projection \(\uparrow\) and \(\downarrow\) electrons, respectively, \(N_{s2}\) the number of the excited-state \(s2\) fermions, and \(L_{s, \pm 1/2}\) that of independent spinons of spin projection \(\pm 1/2\).

For finite hole concentrations below \(x_s\) and intermediate \(U/4t\) values the use of the energy functional introduced in Ref.\(^1\) reveals that excited states with \(s1\) band hole numbers \(N_{s1}^h = 0\) and \(N_{s1}^h = 1, 2\) and involving addition to or removal from the \(c\) Fermi line of \(c\) fermions refer to a gapless excitation branch and have an energy gap, respectively. For the latter excited states the energy gap vanishes only if the auxiliary momentum of the \(s1\) band hole \(\left(N_{s1}^{h} = 1\right)\) or both \(s1\) holes \((N_{s1}^{h} = 2)\) points in the nodal directions. Use of Eq. \(\tag{19}\) then reveals that addition or removal of (i) one electron and (ii) two electrons with the same spin projection to or from the hole-like Fermi line whose hole fermion momenta are given in Eq. \(\tag{10}\) involves creation of (i) one and (ii) two holes, respectively, in the \(s1\) momentum band. Except for the excitation-momentum nodal directions these excited states have a finite energy gap. In contrast, note that addition or removal of two electrons of opposite spin projection to or from that Fermi line leads to a final excited state whose \(s1\) band is full alike that of the initial ground state. Indeed such excitations correspond to \(N_{s2} = L_{s, \mp 1/2} = 0\) in Eq. \(\tag{19}\), so that \(N_{s2} = 0\) for \(\delta N_{s} = \delta N_{j} = \pm 1\). Hence the latter processes refer to a gapless branch of two-electron excitations. Specifically, for excitations whose \(s1\) holes are created at the \(s1\) boundary line the energies of such processes read to first order in the \(c\) and \(s1\) hole momentum distribution-function deviations,

\[
\delta E = \delta E_F(\phi) = |\Delta| \cos 2\phi; \quad \delta N_{\sigma} = \pm 1, \quad \delta N_{-\sigma} = 0, \\
\delta E = \delta E_F(\phi) + \delta E_F(\phi') = |\Delta| \left[ |\cos 2\phi| + |\cos 2\phi'| \right]; \quad \delta N_{\sigma} = \pm 2, \quad \delta N_{-\sigma} = 0, \\
\delta E = 0; \quad \delta N_{\uparrow} = \delta N_{\downarrow} = \pm 1. \tag{20}\]

Such general spectra refer to vanishing spin densities \(m = 0\) and approximately \(U/4t \in (u_{0}, u_{0})\). (For the square-lattice quantum liquid of Ref.\(^1\) and the VEP quantum liquid they refer to \(x \in (0, x_s)\) and \(x \in (x_{c}, x_s)\), respectively.) In the expressions provided in Eq. \(\tag{20}\), \(\delta E_F = |\Delta_{s1}(\vec{q}_{B_{s1}}^{d})|\) is the anisotropic Fermi-energy term and \(\left|\Delta_{s1}(\vec{q}_{B_{s1}}^{d})\right|\) is the \(s1\) fermion pairing energy per spinon given in Eq. \(\tag{A12}\) of Appendix A. The Fermi energy reads \(E_F = \mu + \delta E_F\) where \(\mu\) is the zero-temperature chemical potential. The anisotropic Fermi energy \(\delta E_F\) vanishes for \(x > x_s\) since then \(\max\{\delta E_F\} = |\Delta| = 0\). It follows that for the hole concentration range \(x \in (x_{c}, 1)\) the ground state is a disordered state without short-range spin order. Consistently, for that range of \(x\) values the \(s1\) fermion spinon-pairing energy \(2|\Delta_{s1}(\vec{q}_{B_{s1}}^{d})|\) vanishes for all momentum values.

The numbers of \(s1\) fermions and \(s1\) fermion holes in the \(s1\) momentum band equal those of occupied and unoccupied sites, respectively, in the \(s1\) effective lattice. Hence in the following we discuss the structure of the different energy spectra \(\delta E\) of Eq. \(\tag{20}\) in terms of the numbers of sites, occupied sites, and unoccupied sites of that lattice. Those equal the corresponding numbers of discrete momentum values, filled discrete momentum values, and unfilled discrete momentum values, respectively, of the \(s1\) fermion band. That the energy \(\delta E\) of Eq. \(\tag{20}\) vanishes for excitations involving creation of two electrons of opposite spin projection and except for the nodal directions is for \(x \in (x_{c}, x_s)\) finite and given by \(\delta E = |\Delta E_F(\phi) + \delta E_F(\phi')|\) for those involving creation of two electrons with the same spin projection is not a trivial result. (The additional term to \(\delta E_F = |\Delta_{s1}(\vec{q}_{B_{s1}}^{d})|\) found below in Section IV-A as a result of the superconducting fluctuations does not change the basic property that \(\delta E = 0\) and \(\delta E > 0\) for excitations involving creation of two electrons of opposite spin projection and the same spin projection, respectively.) That behavior follows from the number \(N_{s_{a1}}^{2}\) of sites of the \(s1\) effective lattice being for the Hubbard model on the square lattice a subspace-dependent functional.\(^1\) For the one- and two-electron subspace referring both to the square-lattice quantum liquid of Ref.\(^1\) and the VEP quantum liquid, the expressions of the number deviations \(\delta N_{a_{1}}^{2}\) and \(\delta N_{s1}\) of sites and occupied sites, respectively, of the \(s1\) effective lattice read,

\[
\delta N_{a_{1}}^{2} = \delta N_\uparrow + L_{s, -1/2}; \quad \delta N_{s1} = \delta N_\downarrow - L_{s, -1/2} - 2N_{s2}. \tag{21}\]

Such expressions are consistent with that of \(N_{s1}^{h} = [\delta N_{a_{1}}^{2} - \delta N_{s1}]\), given in Eq. \(\tag{19}\). For the deviation numbers and numbers \(\delta N_{\uparrow} = \delta N_{\downarrow} = \pm 1\) and \(L_{s, \pm 1/2} = N_{s2} = 0\) of the \(\delta N = \pm 2\) excited states for which the two added or removed electrons are in a spin-singlet configuration one finds from the use of the expressions provided in Eqs. \(\tag{19}\) and \(\tag{21}\) that the corresponding deviations in the numbers of sites and occupied sites of the \(s1\) effective lattice read \(\delta N_{a_{1}}^{2} = \pm 1\) and \(\delta N_{s1} = \pm 1\), respectively. Therefore, the deviation in the number of unoccupied sites vanishes,
\[ \delta N^h = [\delta N^2_{a_{s1}} - \delta N_{s1}] = 0. \] Indeed, under such excitations the creation or annihilation of one \( s1 \) fermion is exactly cancelled by an increase or decrease, respectively, in the number of sites of the \( s1 \) effective lattice. As a result the number of unoccupied sites of the corresponding excited states remains being zero, as for the initial ground state. Use of the energy functional introduced in Ref. and given below in Section IV-A then leads to the gapless spectrum \( \delta E = 0 \) of Eq. (20).

In turn, for the deviation numbers and numbers \( \delta N_1 = \pm 2, \delta N_2 = 0, L_{s1, \pm 1/2} = 2 \) and \( L_{s2, \pm 1/2} = N_{s2} = 0 \) or \( \delta N_1 = \pm 2, \delta N_1 = 0, L_{s1, \pm 1/2} = 2 \) and \( L_{s2, \pm 1/2} = N_{s2} = 0 \) of the \( \delta N = \pm 2 \) excited states for which the two added or removed electrons are in a spin-triplet configuration one finds from the use of the expressions provided in Eqs. (19) and (21) that the corresponding deviations in the numbers of sites and occupied sites of the \( s1 \) effective lattice read \( \delta N_{a_{s1}} = [\pm 1 + 1] \) and \( \delta N_{s1} = [\pm 1 - 1] \), respectively. It then follows that the deviation in the number of unoccupied sites are given by \( \delta N^{h}_{s1} = [\delta N^2_{a_{s1}} - \delta N_{s1}] = 2 \). In contrast to the above excitations involving a spin-singlet electron pair, the creation or annihilation of one \( s1 \) fermion is not cancelled under the latter excitations by an increase or decrease, respectively, in the number of sites of the \( s1 \) effective lattice. As a result, its number of unoccupied sites increases from zero to two for the excited states. Use of the general energy functional of Ref. then leads to the gapped spectrum given in Eq. (20) for these excitations. (Use of the modified VEP quantum liquid functional of Section IV-A leads to a gapped spectrum including an additional term, which as mentioned above does not change the physics discussed here.)

A similar analysis can be carried out for the \( \delta N = \pm 1 \) electron and \( \delta N = 0 \) spin excitations. Their energy spectrum is in general gapped. The \( d \)-wave-like structure of the one- and two-electron spectra of the Hubbard model on the square lattice given in Eq. (20) follows from the momentum dependence of the \( s1 \) fermion dispersion studied in Ref. Whether the general energy spectrum (20) is gapless or displays a gap is fully controlled by the interplay between the deviations in the numbers of sites and occupied sites, respectively, of the \( s1 \) effective lattice. That lattice is exotic in that its number of sites is subspace dependent. The spectrum (20) refers to a gapless branch of excitations whenever the corresponding creation or annihilation of \( s1 \) fermions is exactly cancelled by an increase or decrease, respectively, in the number of sites of the \( s1 \) effective lattice so that its number of unoccupied sites remains being zero as for the initial ground state. Above we gave the example of creation or annihilation of a spin-singlet electron pair. Such a canceling occurs for creation or annihilation of any finite number of such spin-singlet electron pairs.

In turn, for one-electron excitations and creation or annihilation of spin-triplet electron pairs the spectrum of Eq. (20) is in general gapped except for some momentum directions. The \( d \)-wave-like structure of that spectrum is a necessary condition for the ground state of the model being for vanishing spin density \( m = 0 \), finite hole concentrations \( x \in (x_c, x_s) \), and thus finite \( 2|\Delta| \) that of a \( d \)-wave superconductor. However, it is not a sufficient condition for the occurrence of coherent pairing needed for the macroscopic condensate. Below we find strong evidence that for intermediate \( U/4t \) values, vanishing spin density \( m = 0 \), and the hole concentration range \( x \in (x_c, x_s) \) the sufficient condition of phase coherence is met by the VEP quantum liquid. At zero temperature and both for hole concentrations in the ranges \( x \in (x_0, x_c) \) and \( x \in (x_c, x_s) \) there is short-range spin order. For \( x \in (x_0, x_c) \) pairing correlations occur that due to strong phase fluctuations do not lead to coherent pairing and superconductivity.

Finally, we emphasize that the number deviation and number expressions provided in Eqs. (19) and (21) also hold for \( D = 1 \) spatial dimensions with \( N^2_{a_{s1}} \) replaced by \( N_{s1} \). However, for the 1D Hubbard model there is no short-range spin order for any range of \( x \) values so that \( |\Delta| = 0 \) in Eq. (20) and \( \delta E = 0 \) for all one- and two-electron excitations under consideration. Consistently, the ground state of the 1D model is not superconducting. Such an analysis reveals that a necessary condition for the occurrence of a superconductivity order in the VEP quantum liquid is the occurrence of short-range spin order associated with the finite energy parameter \( 2|\Delta| > 0 \). That strongly suggests that the occurrence of superconductivity in such a system is a by-product of the short-range spin correlations associated with the energy scale \( 2|\Delta| \). However, our results are inconclusive on whether for the Hubbard model on the square lattice the phase coherence needed for the occurrence of long-range superconducting order does occur.

### B. Short-range spin order, the pseudogap energy scale, and the pseudogap temperature \( T^* \)

Here we provide some basic information about the energy scales associated with the short-range spin order, which for vanishing spin density \( m = 0 \) and hole concentrations in the range \( x \in (x_0, x_s) \) refers to finite temperatures below a pseudogap temperature \( T^* \). Indeed, the VEP quantum liquid scheme accounts for the hole trapping effects reported in Appendix B, so that the short-range spin order occurs for the range \( x \in (x_0, x_s) \) rather than \( x \in (0, x_s) \) for the square-lattice quantum liquid of Ref. Evidence is provided in Ref. that for the hole concentration range \( x \in (x_c, x_s) \) the suppression effects due to intrinsic disorder or superfluid density anisotropy are for the four representative systems other than LSCO very small. In contrast, the LSCO cation-randomness effects introduced in Ref. are not small. Fortunately, they are merely accounted for by multiplying the energy scale \( \Delta_0 \) and related magnitudes by a factor of 1/2, leaving the pseudogap temperature \( T^* \) considered in the following unaltered. Its magnitude remains unaltered as
well under the suppression effects. The results of this paper and Ref. 22 confirm that our oversimplified description of the effects intrinsic disorder or superfluid density anisotropy and LSCO randomness in terms of suppression effects and cation-randomness effects, respectively, leads to agreement between theory and experiments on the five representative systems.

At zero temperature the energy order parameter $2|\Delta|$ of the short-range spin correlations is the maximum magnitude of the $s_1$ fermion spinon-pairing energy of Eq. (13) of Appendix A. Upon increasing $x$ the residual $c$ - $s_1$ fermion interactions tend to suppress the short-range spin correlations. For approximately $U/4t \in (u_0, u_\pi)$ and hole concentrations $x \in (x_0, x_\pi)$ such an effect leads to the linear decreasing $2|\Delta| \approx (1 - x/x_\pi)2\Delta_0$ of the zero-temperature order parameter of the corresponding short-range spin order. That energy parameter plays the role of pseudogap energy scale, as it controls the magnitude of the pseudogap temperature $T^*$ above which there is no short-range spin order, associated with $s_1$ fermion spin pairing. However, there are several definitions of $T^*$ corresponding to effects of the pseudogap associated with $2|\Delta|$ on different physical quantities. Indeed, such effects may appear at different temperatures.

For the square-lattice quantum liquid of Ref. 21, for which the short-range spin order occurs for $0 < x < x_\pi$, we identify in the limit $x \to 0$ the temperature $T^*$ with the temperature $T_\pi$ of Ref. 21. In that limit it is related to the energy parameter $\Delta_0 = \lim_{x\to 0} |\Delta|$ as $T_\pi \approx \Delta_0/k_B$. For the quantum problem considered here and approximately $U/4t \in (u_0, u_\pi)$ the corresponding expression valid for finite hole concentrations in the range $x \in (x_0, x_\pi)$ is then,

$$T^* \approx \left(1 - \frac{x}{x_\pi}\right) \frac{\Delta_0}{k_B}.$$

This is actually the maximum magnitude that $T^*$ can achieve. Its magnitudes as obtained from different properties should obey the inequality $T^* \leq (1 - x/x_\pi)|\Delta_0/k_B|$. Hence the zero-temperature magnitude of the short-range spin order parameter $2|\Delta| \approx (1 - x/x_\pi)2\Delta_0$ controls the range of the pseudogap temperature $T^* \approx 2|\Delta|/2k_B$. For temperatures above $T^*$ the system is at zero spin density $n = 0$ and for finite hole concentrations in the range $x \in (x_0, x_\pi)$ driven into a spin disordered state without short-range spin order.

C. Selected Hamiltonian terms: The VEP quantum-liquid microscopic Hamiltonian

For the hole concentration ranges $x \in (0, x_c)$ and $x \in (x_c, x_\pi)$ the quantum problem considered in this paper is qualitatively different. For $x \in (0, x_c)$ it refers to that reported in Appendix B, for which the effects of intrinsic disorder are strong. For $x \in (x_c, x_\pi)$ the VEP quantum-liquid suppression effects are found in Ref. 22 to be very small. That under the LSCO cation-randomness effects introduced in that section the energy scale $\Delta_0$ is lessened by a factor two is behind the hole concentration $x_0$ of Eq. (8) being different for the parameters appropriate to LSCO and the remaining four representative systems: It reads $x_0 \approx 0.013$ and $x_0 \approx 0.024$, respectively.

Expression of the $t_{i+}/t \ll 1$ Hamiltonian $H_{3D}$ of Eq. (3) in terms of rotated-electron creation and annihilation operators leads to an infinite number of terms. Although in terms of electron operators the Hamiltonian has only on-site interactions, in terms of rotated electron operators there emerge effective interactions involving rotated electrons on different sites. Only a small number of such Hamiltonian terms are relevant to the physics of the present quantum problem. However, the appropriate selection of the latter Hamiltonian terms is a problem of huge complexity.

Here we use as criterion for that selection the general strongly correlated microscopic Hamiltonian that, according to the analysis of Ref. 21, almost certainly underlies the essential physics of the representative hole-doped cuprates. For $U/4t > u_0$ this leads to a simplified effective Hamiltonian with the same general form as that given in Eq. (1) of Ref. 21, but with the electron creation and annihilation operators replaced by corresponding rotated electron operators. Its kinetic-energy terms include those generated from the operator $\hat{H}_{\perp}$ given in Eq. (5). In Section IV-A we introduce the general energy functional corresponding to such a microscopic Hamiltonian. The small hole concentration denoted by $x_0$ in Ref. 21, which reads $x_0 \approx 0.01$ for LSCO, is identified here with the hole concentration $x_0$ of Eq. (8). For the VEP quantum liquid $x_0$ is the critical hole-concentration at which there occurs a sharp quantum phase transition from the Mott-Hubbard insulator with long-range antiferromagnetic order to a short-range incommensurate-spiral spin ordered state. That in Ref. 21 it is found to read $x_0 \approx 0.013$ for LSCO and as given in Eq. (8) is proportional to $\Delta_0/t$ is consistent with the results of Ref. 21.

The selected Hamiltonian terms of our microscopic Hamiltonian are basically the same as those of the microscopic Hamiltonian (1) of Ref. 21, with the electron operators replaced by rotated-electron operators and without the term containing the parameter $U_p$. Indeed, after expression of the rotated-electron operators in terms of $c$ and $s_1$ fermion operators the one- and two-electron subspace with no rotated-electron double occupancy is well-defined: The generators of the states that span such a subspace have simple expressions in terms of $c$ and $s_1$ fermion operators. As a result there is no need of introducing artificial Hamiltonian terms to impose the lack of double occupancy. In our case that is achieved by defining the microscopic Hamiltonian in the one- and two-electron subspace. Indeed, the
corresponding VEP quantum liquid is defined in that subspace. Below we express the terms of the microscopic Hamiltonian that control the thermal and quantum fluctuations in terms of $c$ and $s_1$ fermion operators. The kinetic-energy terms of the VEP quantum-liquid microscopic Hamiltonian involve both the $U/4t$-dependent $c$ fermion mass $m^*_c$ and the much larger mass $M$ of Eq. (4).

The advantage of the rotated-electron operator description of Ref. is that it has been constructed to inherently single and double rotated-electron occupancies being good quantum numbers for $U/4t > 0$. Hence for such operators the lack of rotated-electron double occupancy is exact for the range $U/4t > u_0$ of the VEP quantum liquid. The transformation that relates rotated electrons to electrons is unitary, so that the effective action for the phases $\theta$ considered in this paper is valid for approximately $U/4t > u_0$ and specifically at $U/4t \approx u_0 = 1.525$. This is important in view of the results of Refs. 19,23. Such results provide evidence that within the description of the properties of several classes of hole-doped cuprates with $x_c \approx 0.05$ and $x_s \approx 0.27$ by the VEP quantum liquid the appropriate value of $U/4t$ is not large and reads $U/4t \approx u_0 = 1.525 \in (u_0, u_1)$. This includes the five representative systems. The intermediate value $U/4t \approx 1.525$ is also that found in Ref. 24 to be appropriated for the description of the spin-wave spectrum of the $x = 0$ parent compound La$_2$CuO$_4$ (LCO) 12.

Most of the VEP quantum-liquid microscopic Hamiltonian terms refer to in-plane processes. The very small $t_{1\perp}$-dependent terms generated from the operator $\hat{H}_1$ given in Eq. (5) control the magnitude of the hole concentration $x_c$ and thus have effects on the virtual-electron pair phases $\theta_{j,0}$ and $\theta_{j,1}$ introduced below. In the superconducting phase considered in the following such terms allow the Josephson tunneling through nearest-neighboring planes of a very small density of vanishing-energy spin-singlet electron pairs. For each square-lattice plane such excitations correspond to the gapless excited states of the general spectrum (20), which refer to creation or annihilation of spin-singlet electron pairs. The small average numbers of such pairs that leave and arrive to a given square-lattice plane are identical. The main role of that tunneling is to allow for thermal and quantum fluctuations associated with in-plane long-range superconducting order at finite temperatures.

The general energy functional introduced in Section IV-A corresponds to the VEP quantum-liquid microscopic Hamiltonian, yet it has an in-plane character. It includes implicitly the needed 3D uniaxial anisotropy effects. Indeed, part of such effects are effectively described by the use of a suitable mean-field approximation for the square-lattice physics. Specifically and alike in the microscopic Hamiltonian of Ref. 24, in some of the selected Hamiltonian terms studied below a complex gap function $\Delta_{j,j',\nu}$ replaces the corresponding pairing operator. Indeed within mean-field theory finite-temperature long-range orders may occur in the 2D system. Although this is excluded by the exact Mermin-Wagner-Berezinskii Theorem 24,25, mean-field theory refers to an additional effective way to indirectly account for the effects of the very small spin-singlet electron pair Josephson tunneling through nearest-neighboring planes. In turn, in the pseudogap state considered below the small $t_{1\perp}$-dependent terms are behind very small one-electron transfer between first-neighboring planes. For each square-lattice plane such excitations correspond to excited states of the general spectrum (20) with a finite energy gap. Those refer to creation or annihilation of a single electron. As discussed in Ref. 19, such excitations are behind the energy gap of the normal-state conductivity in the direction perpendicular to the planes.

Before introduction in Section IV of the general energy functional corresponding to the VEP quantum-liquid microscopic Hamiltonian, in the following we express the creation and annihilation rotated-electron operators of the terms of such a Hamiltonian that control the thermal and quantum fluctuations in terms of $c$ and $s_1$ fermion operators. Analysis of the form of the obtained effective Hamiltonian terms reveals that the spin bonds and the charge-2e sector of Ref. 24 correspond to the spin-singlet two-spinon $s_1$ fermions and $c$ fermion pairs, respectively. For the hole concentration range $x \in (x_c, x_s)$ of the VEP quantum liquid both the effects of 3D uniaxial anisotropy and intrinsic disorder are small. Hence for such a $x$ range the interactions of these objects are within our description residual. This results from the residual character of such interactions within the starting square-lattice quantum liquid of Ref. 24. That property greatly simplifies the derivation of the one-electron scattering rate carried out in Ref. 24.

The $d$-wave-like structure of the Fermi-line one- and two-electron energy spectrum given in Eq. (20) is only a necessary condition for at zero spin density $m = 0$ and finite hole concentrations in the range $x \in (x_0, x_s)$ for which $|\Delta| > 0$ the ground state of the present quantum problem to be that of a $d$-wave superconductor. That energy spectrum refers to $\varepsilon^2 = m^*_c/M \rightarrow 0$. It also applies here provided that the 3D anisotropic parameter $\varepsilon^2 = m^*_c/M$ is very small and $\gamma_d \approx 1$. In the remaining of this section we access the $x$ dependence of the $T = 0$ parameters associated with the VEP quantum-liquid thermal and quantum fluctuations for the approximate range $U/4t \in (u_0, u_\pi)$. Indeed some of our expressions are not valid for $U/4t > u_{\pi}$.

In the following evidence from the study of pairing phase fluctuations is found that for zero spin density $m = 0$ and hole concentrations in the range $x \in (x_c, x_\pi)$ the ground state of the VEP quantum liquid has phase coherence, which is associated with a long-range superconducting order. While our results refer to $U/4t \in (u_0, u_\pi)$, this may hold for $U/4t > u_s$ as well. Our results also indicate that for vanishing spin density $m = 0$ and finite hole concentrations in the range $x \in (x_0, x_s)$ there is short-range spin order and pairing correlations yet the lack of phase coherence prevents long-range superconducting order. Finally, it is expected that alike for the system of Ref. 24, for $m = 0$ and $x \in (0, x_0)$
monopole-antimonopole pairs of the type considered in that reference unbind and proliferate, leading to long-range antiferromagnetic order, consistently with the analysis of Appendix B.

1. Hamiltonian terms that control the quantum fluctuations of the phases associated with competing orders

Out of the selected terms of the VEP quantum-liquid microscopic Hamiltonian, here we consider those that control the quantum fluctuations of the phases associated with the competing orders. It is assumed that the $c$ and $s1$ fermion occupancy configurations that generate the energy eigenstates of the $t_{1} / t = 0$ and $\gamma_{d} = 1$ problem generate states close to energy eigenstates for the VEP quantum liquid at very small $t_{1} / t \ll 1$ values and $\gamma_{d} \approx 1$. Such terms are given by,

$$\hat{H}_{\text{bonds}} = \sum_{j=1}^{N/2} \sum_{j' \in \{j-\text{const}\}} \Delta_{j'j''} [\hat{c}_{j',\uparrow} \hat{c}_{j'',\downarrow} - \hat{c}_{j',\downarrow} \hat{c}_{j'',\uparrow}] + (\text{h.c.}),$$

(23)

consistently with the maximum number of in-plane independent bonds being $N/2$. Indeed, for hole concentrations below $x_{s}$ and zero spin density $m = 0$ the ground states are spin-singlet states. For $x < x_{0}$ and $x \in (x_{0}, x_{s})$ such states have long-range and short range spin order, respectively. They contain $N_{s1} = N/2$ two-spinon $s1$ bond particles. The two spinons of such spin-neutral objects describe the spin degrees of freedom of rotated electrons that singly occupy sites in the ground-state configurations. Therefore, in the one- and two-electron subspace without rotated-electron doubly occupancy where such ground states are contained there is an energetic preference for the formation of spin-singlet rotated-electron bonds.

The summation $\sum_{j'j''(j-\text{const})}$ on the right-hand side of Eq. (23) is over all in-plane $N/2$ spin-singlet rotated-electron bonds considered in Ref.22. Alike in Ref.22, to arrive to the Hamiltonian terms given here we used the usual mean-field approximation within which the complex gap function $\Delta_{j'j''}$ has replaced the corresponding pairing operator. The weak effects of the very small 3D uniaxial anisotropy perturbation $\hat{H}_{\perp}$ of Eq. (5) occur on the quantum liquid in-plane Hamiltonian terms (23) through the phase of the complex gap function $\Delta_{j'j''}$, as discussed below. Each spin-singlet rotated-electron bond is centered at a real-space point $|\vec{r}_{j'} + \vec{r}_{j''}|/2$, near that of coordinate $\vec{r}_{j} = |\vec{r}_{j'} + \vec{r}_{j''}|/2 - l \langle a_{s}/2 \rangle \vec{e}_{x}$. Here the two-site bond indices $l \pm 1$ and $d = 1, 2$ are those used within the notation of Ref.22 discussed below and $a_{s} = a/\sqrt{1-x}$ is the spacing of the spin effective lattice. (The bond index $d = 1, 2$ is unrelated to the duality $d = \pm 1$ considered in Section II and Appendix A.)

The rotated-electron creation and annihilation operators of the Hamiltonian terms (23) act onto the one- and two-electron subspace with zero rotated-electron double occupancy. For finite hole concentrations in the range $x \in (x_{0}, x_{s})$ such terms refer to energy scales below and around the energy $\Delta_{0}$ of Eq. (9). The complex gap function is defined on the two-site bonds in the spin effective lattice studied in Ref.22. The energy scale $2\Delta_{0}$ equals the absolute maximum excitation energy below which the short-range spin order survives. Therefore, $\Delta_{0}$ is the absolute maximum magnitude of the pairing energy per spinon of the two-site and two-spinon bond associated with the complex function $\Delta_{j'j''}$. The amplitude of $\Delta_{j'j''}$ is for finite hole concentrations in the range $x \in (x_{0}, x_{s})$ frozen below the energy $\Delta_{0}/\sqrt{2}$. Here $1/\sqrt{2}$ is a suitable normalization factor, which is absorbed in the expressions of the two-site and two-spinon operators of the $s1$ bond-particle operators considered below. What remains are the fluctuations of the phases $\theta_{j'j''}$ of rotated-electron pairs. The complex gap function reads,

$$\Delta_{j'j''} = e^{i\theta_{j'j''}} \left( \frac{-1}{\sqrt{2}} \Delta_{0} \right) x_{0} = \left| \frac{\Delta_{j'j''}}{t} \right| C_{0}; \quad C_{0} = \left( \frac{\sqrt{2} x_{s} \Delta_{0} / t}{2 \gamma_{s}^{2} \pi^{2}} \right), \quad 0 < x < x_{s},$$

$$\Delta_{j'j''} = e^{i\theta_{j'j''}} \left( \frac{-1}{\sqrt{2}} \mu^{0} \right) \mu^{0} / 2, \quad 0 < x < x_{s}.$$

(24)

The hole concentration $x_{0}$ of Eq. (8) is here expressed in terms of $|\Delta_{j'j''}| = \Delta_{0}/\sqrt{2}$. It corresponds to the hole concentration also called $x_{0}$ in Ref.22. The studies of that reference estimated it to be proportional to $|\Delta_{j'j''}|/t$ and such that $x_{0} = |\Delta_{j'j''}|/t C_{0} \ll 1$. Its expressions given in Eqs. (8) and (24) are consistent with such an estimation.

Following the $d$-wave character of the spinon pairing of the $s1$ bond particles, one has in the expressions of Eq. (24) that $d = 1$ and $d = 2$ for the families of spin-singlet two-site bonds whose primary bonds considered in Ref.22 are horizontal and vertical, respectively. The different magnitudes $\Delta_{0}$ and $\mu^{0}/\sqrt{2}$ that the energy scale $|\Delta_{j'j''}|$ has for finite hole concentrations in the ranges $x \in (x_{0}, x_{s})$ and $x \in (0, x_{0})$, respectively, are due to the sharp quantum phase transition occurring at $x = x_{0}$. (We recall that $\mu^{0}$ is the energy scale given in Eq. (A3) of Appendix A.)

Usually one restricts the in-plane summations $\sum_{j'j''}$ of Eq. (23) to nearest neighboring sites. In turn, the terms given in that equation involve contributions from all possible in-plane bonds whose two rotated electrons are located
at arbitrarily distant sites. In order to capture the physics of the quantum problem studied here one must start by taking into account all such contributions. After some algebra one then arrives to an effective Hamiltonian, which is equivalent to restricting the in-plane summations $\sum_{j,j'\sigma}$ of Eq. (23) to nearest neighboring sites.

The importance of the selected Hamiltonian terms (23) is that they contain the phases $\theta_{j,j''}$ whose fluctuations control the physics of the VEP quantum liquid. It is useful to express (23) in terms of c fermion operators and two-site and two-spinon bond operators. This is straightforwardly achieved by direct use of the expressions of the rotated-electron creation and annihilation operators in terms of c fermion operators and two-site and two-spinon bond operators, with the result,

$$\Delta_{j,j''} = \sum_{j,j',\sigma} \hat{c}^\dagger_{j,\sigma} \hat{c}_{j',\sigma} - \sum_{j,j',\sigma} \hat{c}^\dagger_{j',\sigma} \hat{c}_{j,\sigma}$$

(25)

Here $\Delta_{j,j''}$ is the two-site bond centre $\bar{r}_j$ of the complex gap function $\Delta$ appearing in Eq. (25) can be written as,

$$\hat{r}^\dagger_{j,\sigma} \hat{r}_{j''} = e^{i\theta_{j,j''}}$$

(27)

Here $\theta_{j,j''}$ is the overall centre-of-mass phase. In turn, $\theta_{j,j''}$ is the part of $\theta_{j,j''}$, which cannot be reduced to arbitrary configurations of site phases and regulates the relative motion of a pair. Hence the phases $\theta_{j,j''}$ are related to the internal pairing degrees of freedom. Within charge excitations the phases $\theta_{j,j''}$ are associated with the c fermion pair of Eq. (24). For one-electron and spin excitations they are associated with the virtual-electron pair defined below.

Use of expression (24) in Eq. (25) leads to,

$$\Delta_{j,j''} = \sum_{j,j',\sigma} \hat{c}^\dagger_{j,\sigma} \hat{c}_{j',\sigma} - \sum_{j,j',\sigma} \hat{c}^\dagger_{j',\sigma} \hat{c}_{j,\sigma}$$

(28)

Hence one finds from use of this expression in Eq. (26),

$$\hat{H}^{bonds} = \sum_{j=1}^{N_{s1}} \sum_{j'',\{j''\text{constant}\}} e^{i\theta_{j,j''}} \Delta_0 \hat{f}_{j,j''}^\dagger \hat{f}_{j,j''} + \text{h.c.}$$

(29)

The c effective lattice is identical to the original lattice. For the particular set of rotated-electron pairs and corresponding c fermion pairs of real-space coordinates $\bar{r}_j$ and $\bar{r}_{j''}$ associated with the same point of coordinate $\bar{r}_j = [\bar{r}_j + \bar{r}_{j''}] / 2 - \bar{r}_{d,l}$ the phases $\theta_{j,j''}$ of Eq. (24) are nearly equal. The reason is that such a set of pairs interact
with the two-site and two-spinon bonds of the same $s1$ fermion of real-space coordinate $\vec{r}_j$ whose operator is given in Eq. (26). Thus we introduce the following notation for the phase factors,

$$e^{i\theta_j} = e^{i\theta_{j,0}} e^{i\theta_{j,1}}. \quad (30)$$

The phases $\theta_{j,0}$ and $\theta_{j,1}$ are identical to the phases $\theta_{j,j',0}$ and $\theta_{j,j',1}$, respectively, of the primary $c$ fermion pairs associated with the four primary two-site bonds referring to nearest-neighboring sites of the in-plane spin effective lattice whose centre of mass is near the real-space point of coordinate $\vec{r}_j = [\vec{r}_{j'} + \vec{r}_{j''}] / 2 - \vec{r}_{d,4}$. 

In order to describe the physics of the quantum liquid in terms of $c$ and $s1$ fermions, in Appendix C it is shown that the Hamiltonian terms (29) are approximately equivalent to,

$$\hat{H}_{c \text{bonds}}^{bonds} = \sum_{j=1}^{N_{s1}} \sum_{(j',j'')} \frac{e^{i\theta_{j,p}}}{|h_0|} \Delta_0 f_{j',j'}^{\dagger} f_{j',j''}^{\dagger} f_{j,j',s1}^{\dagger} + \text{(h.c.)}. \quad (31)$$

Here the summation $\sum_{(j',j'')} \text{runs over} c$ fermion pairs associated with rotated-electron pairs on singly occupied sites. In the spin effective lattice the centre $[\vec{r}_{j'} + \vec{r}_{j''}] / 2$ of the corresponding in-plane nearest-neighboring real-space coordinates $\vec{r}_{j'}$ and $\vec{r}_{j''}$ of the spins of such rotated electrons is near the real-space coordinate $\vec{r}_j = [\vec{r}_{j'} + \vec{r}_{j''}] / 2 - l [a_s / 2] \vec{e}_x$. In the present $N_a^2 \rightarrow \infty$ thermodynamic limit it coincides with it. Moreover,

$$e^{i\theta_{j,p}} = e^{i\theta_{j,p}(\vec{r}_j)} = e^{i[\theta_{j,0} - \phi_{j,s1}^0]} = e^{i[\theta_{j,0} + \theta_{j,1} - \phi_{j,s1}^0]}; \quad \phi_{j,s1}^0 = \sum_{j' \neq j} \phi_{j',j,s1}. \quad (32)$$

Here $\phi_{j',j,s1}$ are the phases defined in Eq. (2). It follows from the derivation of Appendix C that the primary two-site bonds, i.e. those whose length $\xi_0 = a_s / \sqrt{1 - x}$ is minimum, have most of the spectral weight of a $s1$ bond particle and corresponding $s1$ fermion. The coefficient $1/|h_0|$ of expression (31) compensates the weight associated with bonds of larger length.

The phases $\theta_{c,p} = \theta_{c,p}(\vec{r}_j)$ appearing in the Hamiltonian terms (31) are associated with the rotated-electron pair and corresponding $c$ fermion pair whose centre of mass $\vec{r}_j = [\vec{r}_{j'} + \vec{r}_{j''}] / 2 - l [a_s / 2] \vec{e}_x$ of the $s1$ fermion of operator $f_{j,j,s1}^{\dagger}$. Such Hamiltonian terms describe the interaction of the $c$ fermion pair with the $s1$ fermion. That pair feels the latter object through such a phase. Hence the real-space coordinate $\vec{r}_j$ in the argument of the phases $\theta_{c,p} = \theta_{c,p}(\vec{r}_j)$ corresponds both approximately to the centre of mass of the $c$ fermion pair and to the real-space coordinate of the $s1$ fermion that the two $c$ fermions interact with. For the construction of an effective action for these important phases the contributions from the Hamiltonian terms involving nearest-neighboring sites at $\vec{r}_{j'}$ and $\vec{r}_{j''}$ are sufficient. In turn, concerning some other aspects of the virtual-electron pairing mechanism the contributions from rotated-electron pairs and corresponding $c$ fermion pairs at larger distances cannot be ignored.

Since $\theta_{c,p} = [\theta_{j} - \phi_{j,s1}^0]$, the $c$ fermion pairs of Eq. (31) feel the $s1$ fermion effective vector potential through the phases $\phi_{j,s1}^0$. Those are obtained from the operator phase $\phi_{j,s1}$ also given in that equation by replacing $f_{j,j,s1}^{\dagger} f_{j,j,s1}^{\dagger}$ by its average $\langle f_{j,j,s1}^{\dagger} f_{j,j,s1} \rangle ≈ 1$. Indeed, for the $s1$ fermion occupancies of the states that span the one- and two-electron subspace there are none, one, or two unoccupied sites in the $s1$ effective lattice. In addition, the total number of such sites $N_{a,s}^2 \approx N / 2 = (1 - x)N_a^2 / 2$ is of the order of $N_a^2$. Therefore, within the present thermodynamic limit such a replacement is a good approximation. Furthermore, for the states that span such a subspace the fluctuations of the phases $\phi_{j,s1}^0$ are very small. Hence the fluctuations of the phases $\theta_{c,p}$ of Eq. (32) are fully controlled by those of the phases $\theta_{j,0}$ and $\theta_{j,1}$ of Eq. (30).

The effective Hamiltonian terms (31) are consistent with considering only the Hamiltonian terms,

$$\hat{H}_{c \text{bonds}}^{bonds} = \sum_{j=1}^{N/2} \sum_{(j',j'')} \Delta_{j,j'} \left[ \phi_{j,j'}^{\dagger} \phi_{j,j''}^{\dagger} - \phi_{j,j'}^{\dagger} \phi_{j,j''}^{\dagger} \right] + \text{(h.c.)}. \quad (33)$$

Those correspond to the spin-singlet rotated-electron pairs of Eq. (29) involving nearest-neighboring sites. Indeed, the spin-singlet rotated-electron pairs of Eq. (33) involve the same sites as the $c$ fermion pairs of Eq. (31).

2. The fluctuations of the phases $\theta_{j,0}$ and $\theta_{j,1}$

A next step is the construction of an effective action for the phases $\theta_{c,p}$. The exact calculation of some of the physical quantities involved in that derivation is an involved open problem. Nevertheless, one can construct by means of several
approximations an effective action whose key features faithfully reflect general properties of the VEP quantum-liquid microscopic Hamiltonian. To follow the fluctuations in $\theta_{j,0}$ and $\theta_{j,1}$, we should integrate the $c$ and $s1$ fermions in the expression of Eq. (33) about a suitable saddle point. Such a procedure can as well be fulfilled by integrating the rotated-electrons in the expression of Eq. (34). Those are however extremely complex problems.

Fortunately, since the VEP quantum-liquid microscopic Hamiltonian has the same general form as the microscopic Hamiltonian (1) of Ref. 4, the corresponding effective action for the phases $\theta_{cp}$ has also basic similarities to that constructed in that reference. Given the nearly equivalence between the two actions, we omit here the details that are common to both investigations. Those can be found in Refs. 4, 44. The main results are summarized in the following. They are used and further developed in Section III-D and following sections. Alike in the problem of Refs. 4, 44, one arrives to an effective continuum Lagrangian. In spite of being a simplification of the selected rotated-electron Hamiltonian terms, such a Lagrangian is expected to faithfully reflect the general properties of the microscopic problem under consideration.

First one finds that our phases $\theta_{j,0}$ and $\theta_{j,1}$ correspond to the phases $\theta_{CM}^{j,j''}$ and $\theta_{CM}^{j,j'''}$, respectively, of Refs. 4, 44, where the site indices $j'j''$ are called $jk$. Except that the rotated electrons play here the role played by the electrons in that reference, the physics is very similar. For instance, the charge–2e sector of Ref. 4 refers to the charge–2e $c$ fermion pairs. Indeed, upon expression of the rotated-electron operators in terms of $c$ fermion and two-site and two-spinon bond operators one finds that the Cooper pairs of that reference are mapped onto the virtual-electron pairs considered below. The charge degrees of freedom of such virtual-electron pairs refer to the $c$ fermion pairs. Within charge excitations the latter pairs behave independently of the $s1$ fermion that contains the two-spinon spin-singlet configuration of each virtual-electron pair.

Second it is found that the fluctuations in the phases $\theta_{j,0}$ and $\theta_{j,1}$ proliferate in different regions of the phase diagram. For finite hole concentrations in the range $x \in (x_0, x_*)$ and vanishing temperature $T = 0$ the fluctuations of the phases $\theta_{j,0}$ increase for $x \to x_*$ and remain large for finite hole concentrations in the range $x \in (x_0, x_*)$. Those of $\theta_{j,1}$ increase for $x \to x_*$. It follows that the fluctuations of the phases $\theta_{j}$ increase both for $x \to x_*$ and $x \to x_*$. Otherwise, the zero-temperature fluctuations of the phases $\theta_{j,0}$ and $\theta_{j,1}$ remain small for finite hole concentrations in the ranges $x \in (x_c, x_*)$ and $x \in (x_0, x_*)$, respectively. Here $x_*$ is the critical hole concentration of Eq. (42) of Appendix A introduced in Ref. 4. Above it there is no short-range spin order at zero temperature. As further discussed below, the critical hole concentration $x_c$, that emerges from such studies is directly related to the Ginzburg number and critical hole concentration $x_0, x_* \approx \text{Gi} + x_0$. The expression of GI suitable to the present quantum problem is provided below in Section III-D. That relation of $x_c$ to the Ginzburg number and $x_0$ is valid for very small values of the 3D uniaxial anisotropy parameter $\varepsilon^2 = \gamma_d / M$ and $\gamma_d \approx 1$. For those $x_c \approx 0.05$ and we find below that $\text{GI} \approx 0.037$ for LSCO and $\text{GI} \approx 0.026$ for the remaining four representative hole-doped cuprates.

Third one finds that at zero temperature and hole concentrations in the range $x \in (x_c, x_*)$ and for temperatures $T$ lower than a critical temperature $T_c < T^*$ and a smaller temperature-dependent hole concentration range centered at the optimal hole concentration $x = x_{op} \approx (x_0 + x_*)/2$ the phases $\theta_{j}$ and thus $\theta_{cp}$ all line up into a phase-coherent superconductor. For these $x$ and $T$ ranges both the $c$ fermion kinetic energy and the energy order parameter $2|\Delta|$ of the short-range spin correlations are finite. Combination of the results on the fluctuations of the phases $\theta_{j,0}$ and $\theta_{j,1}$ with those of Section III-A on the general $d$-wave spectrum then provides strong evidence that for the VEP quantum liquid there is a $d$-wave long-range superconducting order for such hole concentrations and temperature ranges. In that phase the fluctuations of both the phases $\theta_{j,0}$ and $\theta_{j,1}$ are small. This ensures that the average phase factors $\langle e^{i\theta_{j,0}} \rangle \neq 0$ and $\langle e^{i\theta_{j,1}} \rangle \neq 0$ are non vanishing. It then follows that the average phase factor $\langle e^{i\theta_{cp}} \rangle \neq 0$ is also non vanishing. This is consistent with the phases $\theta_{cp}$ all lining up into a phase-coherent superconductor.

Fourth it is found that for finite hole concentrations in the range $x \in (x_0, x_*)$ at zero temperature and a temperature-dependent hole concentration range for temperatures belonging to the range $T \in (T_c, T^*)$ where $T_c = 0$ for $x < x_c$ and $T_c > 0$ for $x \in (x_c, x_*)$ and the pseudogap temperature $T^*$ is approximately given by Eq. (22) the fluctuations of the phases $\theta_{j,1}$ are small yet those of the phases $\theta_{j,0}$ are large. As a result, one finds that the average phase factor $\langle e^{i\theta_{j,0}} \rangle = 0$ vanishes and the average phase factor $\langle e^{i\theta_{j,1}} \rangle \neq 0$ remains non vanishing. This then implies that the average phase factor $\langle e^{i\theta_{cp}} \rangle = 0$ vanishes. In the corresponding pseudogap state there is no phase coherence and thus no long-range superconducting order. Nonetheless there remain strong pairing correlations. Indeed, virtual-electron pairs remain existing in such a normal state yet their phases $\theta_{cp}$ do not line up. Consistently, the virtual electron pair – VEP quantum liquid refers both to the normal and superconducting states. In addition the short-range spin order prevails and infinite vorticity loops proliferate. Since the fluctuations of the phases $\theta_{j,1}$ remain small, the monopole-antimonopole pairs of the type considered in Ref. 4 remain bound and the vorticity is conserved in the low-energy limit.

Finally one finds that for the hole concentration range $x \in (0, x_0)$ at zero temperature $T = 0$ and a small temperature dependent hole concentration range for finite temperatures the fluctuations of the phases $\theta_{j,0}$ and $\theta_{j,1}$ are large. It follows that the average phase factors $\langle e^{i\theta_{j,0}} \rangle = 0$, $\langle e^{i\theta_{j,1}} \rangle = 0$, and $\langle e^{i\theta_{cp}} \rangle = 0$ all vanish. As a result vorticity is not properly conserved. Then the above monopole-antimonopole pairs unbind and proliferate. At $T = 0$ this is a
Mott-Hubbard insulator quantum phase. The corresponding ground state has long-range antiferromagnetic order.

Alike in the problem of Ref\textsuperscript{4}, a sharp distinction is drawn between the microscopic properties of the quantum liquids referring to hole concentrations in the range \(x \in (0, x_0)\) and finite hole concentrations \(x \in (x_0, x_*)\), respectively, in terms of the global symmetry of the effective action for the phases \(\theta_j\). For the latter hole concentrations it is a global \(U(1)\) symmetry. It is commonly understood that Hamiltonian symmetries by themselves are not sufficient to prove that a particular symmetry is broken in the ground state. However, for vanishing spin density \(m = 0\) and finite hole concentrations in the range \(x \in (x_0, x_*)\) the global symmetry of that action is related to the hidden global \(U(1)\) symmetry of the Hubbard model on the square lattice found recently\textsuperscript{15}. The latter \(U(1)\) symmetry is contained in the model global \(SO(3) \times SO(3) \times U(1) = [SO(4) \times U(1)]/\mathbb{Z}_2\) symmetry, beyond its previously well-known global \(SO(4)\) symmetry\textsuperscript{6}. Indeed the representations of the model hidden global \(U(1)\) symmetry correspond to the occupancy configurations of the \(c\) fermions in their \(c\) momentum band\textsuperscript{15,16}.

Consistently, for the present quantum problem in the normal state at zero temperature and finite hole concentrations in the range \(x \in (x_0, x_*)\) and in the superconducting state at zero temperature and hole concentrations in the range \(x \in (x_*, x_*)\) the \(c\) fermions configurations are organized in terms of zero-momentum \(c\) fermion pairs. Those play a key role in the non-coherent and coherent virtual-electron pairing, respectively, of these two states. In turn, at \(m = 0\) and \(x = 0\) the \(c\) band is for the absolute ground state of the Hubbard model on a square lattice full so that there is a single occupancy configuration. Consistently, the symmetry of the effective action for the phases \(\theta_j\) is instead a local compact gauge symmetry. Here that holds for \(x \in (0, x_0)\) as well.

Our results are inconclusive on whether the ground state of the \(t_\perp/t = 0\) square-lattice Hubbard model\textsuperscript{11} is superconducting for intermediate \(U/4t\) values and hole concentrations in the range \(x \in (x_*, x_*)\). If it is, the zero-temperature lower hole concentration \(x_c\) must have an expression different from that found here, \(x_c \approx G + x_0\). Indeed, this expression gives \(x_c \to \infty\) for \(t_\perp/t \to 0\) and thus \(M/m_c^* \to \infty\) for \(U/4t > 0\). However the expression \(x_c \approx G + x_0\) applies provided that \(x_c\) remains small. It indicates though that the superconducting dome hole-concentration width \((x_* - x_c)\) decreases upon decreasing \(t_\perp/t\) in the regime where \((x_c - x_0) = G = G[M/m_c^*] + x_0\) remains small.

For small hole concentrations obeying the inequality \(0 < (x - x_0) \ll 1\) the motion of the \(c\) fermion pairs and the associated kinetic energy play the role of symmetry-breaking Higgs terms. The presence of such terms supresses free monopoles and is behind the replacement at \(T = 0\) of the long-range antiferromagnetic order for \(x \in (0, x_0)\) by the short-range incommensurate-spiral spin order for \(0 < (x - x_0) \ll 1\). Indeed, independently of the occurrence of phase coherence virtual-electron pairs and corresponding \(c\) fermion pairs prevail for \(x > x_0\) and only for \(x \in (0, x_0)\) are replaced by the monopoles and antimonopoles. The Higgs terms associated with the motion of the \(c\) fermion pairs leaves behind the global \(U(1)\) symmetry of the short-range spin ordered phase. This is alike in the related problem of Ref\textsuperscript{4}, where the motion of the \(c\) fermion pairs corresponds to that of the \(-2e\) charges.

D. Relation of virtual-electron pairing to spinon and \(c\) fermion pairings and the superconducting order parameter

In this and following sections further evidence is provided that for intermediate (and probably large) \(U/4t\) values a ground-state long-range superconducting order occurs in the VEP quantum liquid at zero spin density \(m = 0\) and hole concentrations in the range \(x \in (x_*, x_*)\), as a by-product of the short-range spin order.

1. Small suppression effects and relation of virtual-electron pairing to spinon and \(c\) fermion pairings

For the hole-concentration range \(x \in (x_*, x_*)\) at zero temperature and a smaller temperature dependent \(x\) range centered at \(x = x_{op} \approx (x_* + x_*)/2\) for finite temperatures below the critical temperature \(T_c\) given below the fluctuations of the phases \(\theta_{j,0}\) and \(\theta_{j,1}\) remain small. Hence the amplitudes,

\[
g = \vert\langle e^{i\theta_j}\rangle\vert = \vert\langle e^{i\theta_{j,0} + i\theta_{j,1}}\rangle\vert = g_0 g_1; \quad g_0 = \vert\langle e^{i\theta_{j,0}}\rangle\vert; \quad g_1 = \vert\langle e^{i\theta_{j,1}}\rangle\vert,
\]

(34)

remain finite. Such amplitudes play an important role in the physics of the VEP quantum liquid.

According to our scheme, the effects of the cuprates intrinsic disorder or superfluid density anisotropy\textsuperscript{46,47}, called here suppression effects, are behind the lessening of the experimental critical temperature relative to its magnitude predicted by the \(\gamma_d = 1\) VEP quantum liquid scheme. Within our oversimplified description of such effects, they are
accounted for by a single suppression coefficient, for
\[
\gamma_d = \left(1 - \alpha_c \frac{4\tilde{g}}{\gamma_c}\right) = \frac{T_c}{T_c|\alpha_c=0} ; \quad \gamma_c = \left(1 - \frac{x_c}{x}\right) ; \quad \gamma_d^{min} = \gamma_d|x=x_{op} = (1 - \alpha_d) \in (\gamma_c, 1) ,
\]
\[
\alpha_d = \left(1 - \gamma_d^{min}\right) \in \left(0, \frac{x_c}{x}\right) \text{ for } x_c/x \in \left(0, \frac{1}{4}\right) ; \quad \alpha_d \in \left(\frac{x_c}{x}, \frac{x_c - \frac{1}{4} x}{x}\right) \text{ for } x_c/x \in \left(\frac{1}{4}, \frac{1}{2}\right) . \quad (35)
\]
Here \(\tilde{g}\) denotes the amplitude \(g\) of Eq. 31 at \(T = 0\), whose maximum magnitude \(\tilde{g}_{\max}^{\gamma} = \gamma_c/4\) is found below to be reached at \(x = x_{op} = (x_c + x)/2\), \(T_c|\alpha_c=0\) denotes the critical temperature in the absence of suppression effects introduced in the following, and \(T_c\) is its actual magnitude as obtained from experiments.

For \(\tilde{g} \to 0\) the very strong phase fluctuations are the main mechanism for depressing \(T_c\), so that \(\gamma_d \to 1\). This is found below to occur both for \(0 < (x - x_c) \ll 1\) and \(0 < (x_c - x) \ll 1\). In turn, \(\gamma_d\) reaches its minimum magnitude \(\gamma_d^{min} = (1 - \alpha_d)\) at the optimal hole concentration \(x \approx x_{op} = (x_c + x)/2\), when such fluctuations are smallest.

For the five representative systems \(x_c/x_s < 1/4\) so that the strength of their suppression effects is measured by the magnitude of \(\alpha_d \in (0, x_c/x_s)\) and thus of \(\gamma_d^{min} \in (\gamma_c, 1)\). The \(\alpha_d = (1 - \gamma_d^{min})\) ranges given in Eq. (35) are justified below.

The validity of our scheme requires that the experimental \(T_{c,\max}^{\alpha_c=0}\) is smaller than \(T_{c,\max}^{\alpha_c=0}\) where \(\gamma_c \approx 0.81\) and thus \(x_c/x_s \approx 0.19 < 1/4\) for \(x_c = 0.05\) and \(x_s = 0.27\). In Ref.14 it is found that \(\gamma_d^{min} \in (0.94, 0.98)\) and thus \(\alpha_d \in (0.02, 0.06)\) for the four representative systems other than LSCO. The corresponding weakness of the suppression effects is consistent with the experimental results of Refs. 31, 32, 33. In turn, it is found that \(\gamma_d^{min} = 0.82\) and thus \(\alpha_d = 0.18\) for LSCO. The inequalities \(\gamma_c T_{c,\max}^{\alpha_c=0} < T_{c,\max} < T_{c,\max}^{\alpha_c=0}\), \(\gamma_d^{min} > \gamma_c\), and \(\alpha_d < x_c/x_s\) for \(x_c/x_s \approx 0.19 < 1/4\) are then met for all representative systems.

For temperatures below the pseudogap temperature \(T^*\) the short-range spin order parameter \(2|\Delta|\) is identified with \(2|\Delta| = g_1 2\Delta_0\). For a large range of finite temperatures below \(T^*\) such that \((T^* - T) > 0\) is not too small the amplitude \(g_1 = |\langle e^{i\theta_{j,1}}\rangle|\) is expected to be approximately given by \(g_1 \approx \tilde{g}_1\) where \(\tilde{g}_1\) is its zero-temperature value. It follows that provided that \((T^* - T) > 0\) is not too small the energy scale \(2|\Delta|\) is for the range \(U/4t \in (u_0, u_{\pi})\) for which \(x_s \in (0.23, 0.32)\) approximately given by,
\[
2|\Delta| = g_1 2\Delta_0 \approx \tilde{g}_1 2\Delta_0 \approx 2\Delta_0 \left(1 - \frac{x}{x_s}\right) . \quad (36)
\]

The results of Ref.11 about the dependence of the energy scale \(|\Delta|\) on the hole concentration \(x\) then indicate that for the square-lattice quantum liquid the amplitude \(\tilde{g}_1\) reads \(\tilde{g}_1 \approx (1 - x/x_s)\) for small hole concentrations \(0 < x < 1\). This is expected to hold for the VEP quantum liquid for \(0 < (x - x_0) \ll 1\). Interestingly, this expression has for the whole range \(x \in (x_0, x_s)\) the correct behavior \(\tilde{g}_1 < 1\), which is imposed by the inequality \(g_1 = |\langle e^{i\theta_{j,1}}\rangle| \leq 1\). Indeed its maximum magnitude reached for \((x - x_0) \to 0\) reads \(\tilde{g}_1 = \gamma_0 = (1 - x_0/x_s) < 1\). Provided that \(U/4t \in (u_0, u_{\pi})\) and thus the expression \(2|\Delta| \approx 2\Delta_0 (1 - x/x_s)\) is valid, we then consider that \(\tilde{g}_1 \approx (1 - x/x_s)\) for hole concentrations in the range \(x \in (x_0, x_s)\).

Concerning properties such as the fluctuations of the phases \(\theta_{j,0}\) and \(\theta_{j,1}\), the Hamiltonian terms (33) involving spin-singlet rotated-electron pairs at nearest-neighboring sites contain the relevant information. In turn, in order to capture the extra information needed for the further study of the coherent virtual-electron pairing and clarification of its relation to fermion pairing and s1 fermion spinon pairing one must take into account the more general Hamiltonian terms provided in Eq. 23. When expressed in terms of c fermion operators and two-site and two-spinon bond operators these Hamiltonian terms are of the form given in Eq. 29.

The amplitude fluctuations near a real-space point can be neglected. Nevertheless the \(x\) dependence of the average over the whole system \(|\langle e^{i\theta_{j,j'}}\rangle|\) of the quantity \(e^{i\theta_{j,j'}}\Delta_0\) appearing in the Hamiltonian terms provided in Eq. 29 plays an important role. It follows from Eqs. 34 and 35 that \(|\langle e^{i\theta_{j,j'}}\rangle|\) \(\Delta_0\) can at zero temperature be written as \(\tilde{g}_0|\Delta| = \tilde{g}|\Delta_0|\). Here \(\tilde{g}_0\) and \(\tilde{g}\) denote the \(T = 0\) values of the amplitudes \(g_0\) and \(g\), respectively. The physical meaning of such terms is that the two-site and two-spinon bonds of the s1 fermion of Eq. 26 provide through the residual interactions with the c fermion pairs the energy needed for effective pairing coupling between the c fermions. The source of such an energy transfer is the energy scale \(|\Delta|\), which is one half the order parameter associated with the short-range spin correlations. Hence within the VEP quantum-liquid scheme the short-range spin correlations provide the energy needed for the c fermion effective coupling involved in the pairing mechanism behind the long-range superconducting order. And this is consistent with the short-range spin order parameter \(2|\Delta|\) having the form \(2|\Delta| = g_1 2\Delta_0 \approx \tilde{g}_1 2\Delta_0\), as given in Eq. 36.

The expression in Eq. 29 of the VEP quantum-liquid effective microscopic Hamiltonian terms of Eq. 28 in terms of the local c fermion operators and the two-site and two-spinon bond operators contains important physical information. Indeed it reveals that the set of c fermion pairs whose centre of mass \(\tilde{r}_j + \tilde{r}_j'\) nearly coincides with the real-space coordinate \(\tilde{r}_j = [\tilde{r}_j' + \tilde{r}_j'']/2 - \tilde{r}_{d,1}\approx [\tilde{r}_j' + \tilde{r}_j'']/2\) of a given local s1 fermion interact with the
two-site and two-spinon bonds contained in that s1 fermion, as given in Eq. (20). An important concept is then that of a local virtual-electron pair of real-space coordinate \( \vec{r}_j \). Such an object has the same real-space coordinate as a given local s1 fermion. Indeed there is a local virtual-electron pair for each local s1 fermion. In addition to that s1 fermion, a local virtual-electron pair of real-space coordinate \( \vec{r}_j \) involves a superposition of quantum configurations of all \( c \) fermion pairs whose centre of mass \( [\vec{r}_{j'} + \vec{r}_{j''}] / 2 \) nearly coincides with its real-space coordinate \( \vec{r}_j = [\vec{r}_{j'} + \vec{r}_{j''}] / 2 - \vec{r}_{dd} \approx [\vec{r}_{j'} + \vec{r}_{j''}] / 2 \). Each of its configurations includes the charge \(-2e\) of a \( c \) fermion pair of centre of mass \( [\vec{r}_{j'} + \vec{r}_{j''}] / 2 \) and the spin-singlet configuration of two spin-1/2 spinons of the local s1 fermion of real-space coordinate \( \vec{r}_j = [\vec{r}_{j'} + \vec{r}_{j''}] / 2 - \vec{r}_{dd} \approx [\vec{r}_{j'} + \vec{r}_{j''}] / 2 \). In each of these configurations the local s1 fermion of real-space coordinate \( \vec{r}_j = [\vec{r}_{j'} + \vec{r}_{j''}] / 2 - \vec{r}_{dd} \approx [\vec{r}_{j'} + \vec{r}_{j''}] / 2 \) interacts with the \( c \) fermions of a pair of centre of mass \( [\vec{r}_{j'} + \vec{r}_{j''}] / 2 \). As discussed below, the \( c \) fermion effective coupling results from the energy supplied to each pair by the s1 fermion under consideration.

In the pseudogap state whose \( x \) and \( T \) range was mentioned above the amplitude \( g_1 \) vanishes. However, there remain \( c \) fermion pairing correlations associated with the amplitude \( g_1 \), which remains finite. Indeed, the fluctuations of the phases \( \theta_1 \) remain small. Therefore, the concepts of a virtual-electron pair and a VEP quantum liquid hold for the pseudogap state as well, yet for it such a pairing has no phase coherence.

The \( c \) fermion discrete momenta \( \vec{q}_j = [\vec{q}^h_j - \vec{a}] \) where \( j = 1, ..., N^2_a \) and s1 fermion discrete momenta \( \vec{q}_j \) where \( j = 1, ..., N^2_s \), are the conjugate of the real-space coordinates of the \( c \) and s1 effective lattices of such objects, respectively. Such momenta are good quantum numbers for the square-lattice quantum liquid of Ref. 1. They are close to good quantum numbers for the VEP quantum liquid at hole concentrations in the range \( x \in (x_e, x_s) \). Unlike a local virtual-electron pair has the same spatial coordinate \( \vec{r}_j \) as the corresponding local s1 fermion, a virtual-electron pair configuration of momentum \( \vec{q}_j \) involves one s1 fermion carrying that momentum. Therefore, the virtual-electron pair configuration momenta belong to the s1 band. Such a virtual-electron pair configuration consists of a zero-momentum pair of \( c \) fermions of hole momenta \( \vec{q}^h \) and \(-\vec{q}^h \), which interact with a s1 fermion of momentum \( \vec{q}_j \). Within a local virtual-electron pair the well-defined pairs with the same centre of mass \( [\vec{r}_{j'} + \vec{r}_{j''}] / 2 \) interact with the s1 fermion of real-space coordinate \( \vec{r}_j = [\vec{r}_{j'} + \vec{r}_{j''}] / 2 - \vec{r}_{dd} \approx [\vec{r}_{j'} + \vec{r}_{j''}] / 2 \). Consistently, the effective pairing coupling between \( c \) fermions of hole momenta \( \vec{q}^h \) and \(-\vec{q}^h \) results from elementary processes where such objects have interactions with a s1 fermion of well-defined momentum \( \vec{q}_j \). We emphasize that a pair of \( c \) fermions of hole momenta \( \vec{q}^h \) and \(-\vec{q}^h \) is a superposition of local \( c \) fermion pairs whose centre of mass \( [\vec{r}_{j'} + \vec{r}_{j''}] / 2 \) is different and thus interact with different s1 fermions. Therefore, in the occupancy configurations of the superconducting ground state there is no one-to-one correspondence between a coherent \( c \) fermion pair and a given s1 fermion. The occupancies of such a ground state involve superpositions of phase-coherent virtual-electron pair configurations of different momenta. The same pair of \( c \) fermions of hole momenta \( \vec{q}^h \) and \(-\vec{q}^h \) participates in different virtual-electron pair configurations where it interacts with a s1 fermion of different momentum \( \vec{q}_j \). And vice versa, each s1 fermion of momentum \( \vec{q}_j \) participates in different virtual-electron pair configurations where it interacts with different pairs of \( c \) fermions.

The short-range spin correlations provide through the \( c \)-s1 fermion interactions the energy needed for the effective coupling between the two \( c \) fermions of a pair. These zero-momentum \( c \) fermion pairs couple to charge probes independently of the s1 fermions. The superconducting macroscopic condensate refers to such pairs. In turn, the virtual-electron pair configurations are coupled to spin probes through the s1 fermions. It follows that concerning charge excitations the zero-momentum coherent \( c \) fermion pairs behave as independent objects relative to the virtual-electron pair configurations. In turn, one-electron and spin excitations break phase-coherent virtual-electron pairs. Indeed, as discussed below virtual-electron pairs exist in intermediate virtual states generated by pair breaking processes upon such excitations. A virtual electron pair configuration involves the charge \(-2e\) of its \( c \) fermion pair and the spin-singlet configuration of the two spin-1/2 spinons of its composite s1 fermion.

A virtual-electron pairing configuration involves two types of pairing: (i) The zero-momentum \( c \) fermion pairing whose effective coupling between the \( c \) fermions results from residual interactions with the s1 fermion; (ii) The spin-singlet spinon pairing of the composite two-spinon s1 fermion. As reported below in Section III-E, at zero temperature and hole concentrations in the range \( x \in (x_e, x_s) \) one may reach the pseudogap state by applying a magnetic field \( H \) aligned perpendicular to the square-lattice planes. The occupancies of the \( g_1 > 0 \) and \( g = 0 \) normal ground state involve again superpositions of virtual-electron pair configurations of different momenta \( \vec{q}_j \). However such pair configurations and corresponding \( c \) fermion pairs have no phase coherence. The same applies to the normal ground state for vanishing magnetic field \( H = 0 \) and finite hole concentrations in the range \( x \in (x_0, x_e) \).

Strong effective coupling between \( c \) fermions of a pair is defined below as that whose breaking under one-electron excitations leads to sharp spectral features in the corresponding \((\vec{k}, \omega)\)-space distribution. One then classifies the \( c \)-s1 interactions into two classes: Those that lead and do not lead to strong effective coupling, respectively. Importantly, only \( c \) fermion pairs with strong effective coupling can participate in phase-coherent virtual electron pair configurations. Indeed, \( c \) fermion strong effective coupling is a necessary but not sufficient condition for coherent pairing.
According to the analysis presented below in Section IV-C, the momentum \( \vec{q} \) of strongly coupled virtual-electron pairs may belong to different \( s1 - sc \) lines labelled by the line normal momentum absolute value \( q_{N}^{sc} = (q_{sc}^{N}, q_{sc}^{N}) \). Here \( sc \) stands for strong coupling. The \( s1 - sc \) line of the momentum of a \( s1 \) fermion contributing to the strong effective coupling of two \( c \) fermions is determined by the absolute value \( q^{h} \) of their hole momenta \( \vec{q}^{h} \) and \(-\vec{q}^{h} \). Indeed, there is an one-to-one correspondence between each \( \vec{q}_{sc}^{N} \) value labeling a given \( s1 - sc \) line and \( q^{h} = |\vec{q}^{h}| \in (q_{sc}^{N}, q_{sc}^{N}) \). The set of \( c \) fermions with the same hole momentum absolute value \( q^{h} = |\vec{q}^{h}| \in (q_{sc}^{N}, q_{sc}^{N}) \) are involved in virtual-electron pairs whose momentum \( \vec{q} \) belongs to the same \( s1 - sc \) line. Hence the \( s1 \) fermions whose momentum belongs to a given \( s1 - sc \) line interact with the \( c \) fermions whose hole momenta \( \vec{q}^{h} \) and \(-\vec{q}^{h} \) belong to an approximately circular \( s - sc \) line of radius \( q^{h} = |\vec{q}^{h}| \in (q_{sc}^{N}, q_{sc}^{N}) \) centered at \(-\vec{q} = -[\pi, \pi] \). And vice versa. \( q_{sc}^{N} \) and \( q_{sc}^{N} \) are the minimum momentum and corresponding maximum hole momentum, respectively, for which there is \( s1 \) fermion strong effective coupling. (Such properties refer only to \( c - s1 \) fermion interactions leading to \( c \) fermion strong effective coupling.)

Combination of the expression of the number of holes in the \( s1 \) band \( N_{s1}^{h} \) given in Eq. (19) with that provided in Eq. (21) of its deviation \( \delta N_{s1} \) due to a ground-state - excited-state transition gives \( N_{s1}^{h} = [\delta N - 2\delta N_{s1} - 2N_{s2}] \). Here \( N_{s2} = 0, 1 \) is the number of four-spinon \( s2 \) fermions. This is the form that the exact relation between the excited-states number of holes in the \( s1 \) band and the corresponding deviation \( \delta N \) from the ground-state electron-number value has for the one- and two-electron subspace. It implies that the matrix elements between one-electron (and two-electron) excitations and excited states with an even (and an odd) number of \( s1 \) fermion holes vanish. Such an exact selection rule applies when the initial ground state refers to vanishing spin density \( m = 0 \). It reveals that excited states with a single hole in the \( s1 \) band are generated by application onto that ground state of one-electron operators. In turn, those with none or two holes in that band are generated by application of two-electron operators. Furthermore, the transformation laws under the electron - rotated-electron transformation of the quantum objects whose occupancy configurations generate the energy eigenstates of the Hubbard model can be used to show that nearly the whole one-electron (and two-electron) spectral weight corresponds to excited states with a single \( s1 \) fermion hole (and none or two \( s1 \) fermion holes). The same is expected to hold for the VEP quantum liquid.

In conventional superconductivity the objects that pair are Fermi-liquid quasiparticles. Here virtual-electron pairing involves a pair of charge \( c \) fermions and one spin-neutral two-spinon \( s1 \) fermion. The occurrence of \( s1 \) fermion spin-singlet spinon pairing is needed for coherent virtual-electron pairing. In contrast, the former pairing can occur independently of the latter. Indeed, the spin-singlet spinon pairing is needed for coherent virtual-electron pairing. In contrast, the former pairing can occur independently of the latter. Concerning charge excitations the coherent \( c \) fermion pairs of charge \(-2e \) behave as independent entities, which carry the superfluid current. The flux quantization is \(-hc/2e \). Hence in the superconducting state in the presence of a vector potential, the \( c \) band dispersion whose expression is provided in Eqs. (A10) and (A11) of Appendix A is shifted and approximately given by,

\[
\epsilon_{c}(q^{h}) \approx \epsilon_{c}(\vec{q}^{h}) + \left( \frac{1}{2e} \vec{\nabla} \theta_{cp} - \vec{A} \right) \cdot \vec{j}_{c}(q^{h}).
\]

Here \( \theta_{cp} \) is the phase of Eq. (32) and \( \vec{j}_{c}(q^{h}) \) the \( c \) fermion current of Eq. (A16) of Appendix A. It reads \( j_{c}(q^{h}) = q_{sc}^{h}/m_{c}^{2} \) for hole momenta \( q^{h} \) at or near the \( c \) Fermi line. Its absolute value is for \( x \in (x_{e}, x_{c}) \) and \( U/4t \in (u_{c}, u_{e}) \) approximately given by \( j_{c}(q^{h}) \approx \sqrt{\pi} 2/2m_{c}^{2} \). The renormalized transport mass \( m_{c}^{2} \) is provided in Eq. (A16) of Appendix A. In the units used here it reads \( m_{c}^{2} = r_{c}/2t \).

2. Virtual-electron pairs in virtual intermediate states of one- and two-electron excitations

As discussed in Section III-A, removal of two electrons in a spin-singlet configuration leads to an excited state with no holes in the \( s1 \) band. Upon such an excitation, before the two involved \( c \) fermions recombine with one \( s1 \) fermion giving rise to the removed electron pair under consideration, the system is driven into an intermediate virtual state. In it the \( c \) fermion pair under consideration stops interacting with all \( s1 \) fermions other than that \( s1 \) fermion. Hence, in that intermediate virtual state both the \( c \) fermion pair and the \( s1 \) fermion participate in a single virtual-electron pair. In it such a pair exists as an individual object. The occurrence of that intermediate virtual state is behind the designation virtual-electron pair. This holds both for the superconducting and pseudogap states. Such a two-electron process corresponds to a gapless excitation branch. It refers to the last excitation energy \( \delta E = 0 \) given in Eq. (20).

In turn, under one-electron removal and spin excitations there emerges one and two holes in the \( s1 \) band, respectively. Also within such excitations the system is driven into an intermediate virtual-electron state. In it one \( c \) fermion pair...
and one s1 fermion contribute to a single virtual electron pair: that broken under the excitation. Such a discontinuous change is brought about by application onto the ground state of the corresponding one-electron and spin two-electron operator, respectively. Within these processes the c fermion pair ceases to interact with s1 fermions other than that under consideration. Furthermore, that s1 fermion ceases to interact with all remaining c fermion pairs. The virtual electron pair and the corresponding c fermion pair and s1 fermion spinon pair are then broken under the one-electron removal or spin excitation. Finally, under the former excitation one c fermion and one spinon are removed along with the electron and one hole emerges in the s1 band. In turn, under a spin-triplet excitation a spinon spin-flip occurs. It generates two independent spinons. Under a spin-singlet excitation the two spinons recombine with those of another s1 fermion giving rise to a spin-singlet four-spinon s2 fermion. Such spin-two-electron processes are followed by the emergence of two holes in the s1 band. In addition, the c fermion pair left over under these spin excitations restarts interacting within virtual-electron pairs with other s1 fermions. This is how one-electron and spin quantum measurements destroy the ground-state wavefunction and create a new wavefunction.

3. Critical temperature, superconducting order parameter, and coherent length

The order parameter associated with the short-range spin correlations is the maximum magnitude of the s1 fermion spinon-pairing energy $2|\Delta|$. It is expected that the absolute value $2|\Omega|$ of the superconducting order parameter is the maximum magnitude of the coherent virtual-electron-pairing energy. According to the pairing mechanism emerging from the above analysis, the short-range spin correlations supply the latter pairing energy. It is then expected that for hole concentrations in the range $x \in (x_c, x_s)$ and zero temperature both the inequality $2|\Omega| < 2|\Delta|$ and the relation $2|\Omega| \propto 2|\Delta|$ hold. The pseudogap temperature (22) is given by $T \approx \tilde{\gamma}_1 \Delta_0/k_B$. It is controlled by the zero-temperature amplitude $\tilde{g}_1$, which is finite for hole concentrations in the range $x \in (x_0, x_s)$ where there is short-range spin order. Within the present approach the superconductivity critical temperature reads $T_c \approx \gamma_d \tilde{g} \Delta_0/2k_B$, so that $T_c/\Delta_0 = \tilde{\gamma}_d \approx \tilde{g} \Delta_0/2k_B$ in Eq. 55. It is controlled by the amplitude $\tilde{g}$, which is finite for hole concentrations in the range $x \in (x_c, x_s)$. For a long-range superconducting order coexists with the short-range spin order, as its by-product.

Physically, the energy scales (i) $2|\Omega|/T = 0$ and (ii) $4k_B T_c$ are the maximum magnitude of the coherent c fermion pairing energy within pair breaking (i) at zero-temperature under spin-triplet excitations and (ii) upon increasing the temperature. The relation $2|\Delta|/T = 0 \approx 2k_B T^* c$ holds. The corresponding expected relation $2|\Omega|/T = 0 \approx 4k_B T_c$ holds both in the limits $0 < (x - x_c) \ll 1$ and $0 < (x_s - x) \ll 1$. Hence in these limits when the phase fluctuations are strong the superconducting energy scales $2|\Omega|/T = 0$ and $4k_B T_c$ have the same magnitude. In turn, at $x = x_{op} = (x_c + x_s)/2$ such fluctuations are smallest and these two energy scales have slightly different magnitudes. Due to correlations the energy scales considered in the following are not additive, so that our discussion is merely qualitative. The maximum energy that the short-range spin correlations associated with the energy scale $2|\Delta|(x_{op})$ can supply to coherent pairing is at $x = x_{op}$ approximately given by $\approx 2|\Delta|(x_{op})/T = 0/\gamma_c = \Delta_0/2$. In the absence of suppression effects the above pairing energies read $4k_B T_c(x_{op}) \approx \tilde{g}_0 2|\Delta|(x_{op})/T = 0 = \gamma_c \Delta_0/2$ and $2|\Omega|(x_{op})/T = 0 \approx \Delta_0/2$. For pair breaking under spin excitations the pairing energy $2|\Omega|(x_{op})/T = 0$ equals that delivered by the short-range spin correlations $\approx \Delta_0/2$. In contrast, in the case of $4k_B T_c(x_{op}) \approx (1 - x_c/x_s) \Delta_0/2$ only a fraction of the latter energy is supplied to pairing, the energy $\approx (x_c/x_s) \Delta_0/2$ left over being lost due to phase and thermal fluctuations. If one accounts for the suppression effects the above two related pairing energies then approximately read $4k_B T_c(x_{op}) \approx \gamma_d^\text{min} \tilde{g} \Delta_0/2|\Delta|(x_{op})/T = 0 = \gamma_d^\text{min} \gamma_c \Delta_0/2$ and $2|\Omega|(x_{op})/T = 0 \approx \gamma_d^\text{min} \Delta_0/2$. Hence the relation,

$$\frac{4k_B T_c(x_{op})}{2|\Omega|(x_{op})/T = 0} = \gamma_c; \quad x_{op} = \frac{1}{2}(x_s + x_c),$$

holds. The following expression satisfies such limiting behaviors,

$$\frac{4k_B T_c}{2|\Omega|/T = 0} \approx \frac{4k_B T_c}{\tilde{\beta}_c} = \frac{\gamma_d \tilde{g} 2\Delta_0}{\tilde{\beta}_c} = \frac{1 - x_c}{x_s} \frac{T_c}{x_s T_{c,max}}.$$

The inequalities $\alpha_d < x_c/x_s$ and $x_s/x_c < 1$ in the range of the parameter $\alpha_d = (1 - \gamma_d^\text{min})$ of Eq. (35) are justified by the physical range and inequality $2|\Omega|/T_{c,max} = (\tilde{g} 2\Delta_0, \Delta_0/2)$ and $2|\Delta|/T = 0/2|\gamma_c = \Delta_0/2 < 2|\Delta|/T = 0$, respectively, corresponding to $x = x_{op}$. The latter follows from the physical requirement that the maximum energy supplied by the short-range spin correlations being smaller than their own energy scale $2|\Delta|$. In turn, concerning the above range only a fraction $(1 - \alpha_d)\Delta_0/2$ of the available maximum energy $\Delta_0/2$ is supplied, whereas the energy $\alpha_d \Delta_0/2$ left over is lost due to the suppression effects. Our oversimplified scheme in terms of such effects is valid provided that $2|\Omega|/\Omega_{c,0} = \gamma_d \Delta_0/2 < 2|\Delta|/\Delta_{c,0}$. At $\alpha_d = x_c/x_s$ the equality $\gamma_d = \tilde{\beta}_c$ holds, so that $2|\Omega|/\Omega_{c,0} = \gamma_d 2\Delta_0$. Otherwise, $\gamma_d < \tilde{\beta}_c$ and $\tilde{g} 2\Delta_0 < 2|\Omega|/T_{c,0}$ for $\alpha_d < x_c/x_s$ and $\gamma_d^\text{min} > \gamma_c$. Below it is confirmed that
\( T_c^{\text{max}} = T_c(x_{\text{op}}) \) and 2\( |\Omega|^{\text{max}}_{T=0} = 2|\Omega|_{T=0} \). For the critical hole concentration values \( x_c \approx 0.05 \) and \( x_s \approx 0.27 \) of the five representative systems the optimal hole concentration of Eq. (38) is given by \( x_{\text{op}} \approx 0.16 \), in agreement with experiments.

It follows from the above analysis that the order parameter 2\( \Omega \) and the energy scale 4\( k_B T_c \) read,

\[
2\Omega = e^{i\theta_{\text{cp}}} 2|\Omega|; \quad \theta_{\text{cp}} = \theta_j - \phi_{j,s,1}^0; \quad \theta_j = \theta_{j,0} + \theta_{j,1}; \quad 4k_B T_c \approx \gamma_d \bar{g} 2\Delta_0. \tag{40}
\]

Here 2\( |\Omega| \) is for \( T = 0 \) provided in Eq. (39), \( \theta_{\text{cp}} = \theta_{\text{cp}}(\vec{r}_j') \) are the phases of Eq. (32), and \( \bar{g} = \bar{g}_0 \bar{g}_1 \) is the zero-temperature value of the overall amplitude given in Eq. (44).

Generalization to finite temperature \( T > 0 \) of the zero-temperature 2\( |\Omega| \) expression provided in Eq. (39) leads for a temperature-dependent \( x \) range centered at \( x = x_{\text{op}} \) to,

\[
2|\Omega| = \frac{\gamma_d \bar{g} 2\Delta_0}{\beta_c}; \quad \beta_c = \left( 1 - \frac{x_c}{x_s} \frac{4\bar{g}}{\gamma_c} \right). \tag{41}
\]

Here \( \gamma_c \) is the parameter given in Eq. (35). Note that \( \beta_c|_{T=0} = \bar{\beta}_c \) where \( \bar{\beta}_c \) is defined in Eq. (39).

The 2\( \Omega \) expression of Eq. (40) is such that 2\( \Omega \propto e^{i\theta_{\text{cp}}} \langle e^{i\theta_{\text{cp}}\gamma} \rangle 2\Delta_0 \approx e^{i\theta_{\text{cp}}\gamma} g 2\Delta_0. \) However, the amplitude fluctuations are not accounted for twice. The point is that \( \langle |e^{i\theta_{\text{cp}}\gamma}| \rangle \) refers to an average over the whole system. Hence the amplitude \( \langle |e^{i\theta_{\text{cp}}\gamma}| \rangle \) is independent of the spatial coordinate \( \vec{r}_j' \). In turn, the \( \vec{r}_j' \) dependence of the phase factor \( e^{i\theta_{\text{cp}}\gamma} \) corresponds to a small real-space region around \( \vec{r}_j' \) where the phases \( \theta_{\text{cp}} \) change little and smoothly. Then amplitude fluctuations can be neglected in that small region around \( \vec{r}_j' \) and averaging \( e^{i\theta_{\text{cp}}\gamma} \) over the local virtual-electron pairs contained in it gives \( \langle |e^{i\theta_{\text{cp}}\gamma}|_{\text{near-}j'} \rangle \approx 1 \). This justifies why such fluctuations are not accounted for twice.

That local normalization is not fulfilled when the phase fluctuations become large. Nevertheless, then \( g \to 0 \) and thus 2\( \Omega \to 0 \). Hence the expressions provided in Eqs. (40) and (41) remain valid. One then concludes that the range of validity of the expression 2\( \Omega = e^{i\theta_{\text{cp}}} 2|\Omega| \) whose \( \vec{r}_j' \) dependence occurs through the phases \( \theta_{\text{cp}} = [\theta_j - \phi_{j,s,1}^0] \) refers to a small real-space region around \( \vec{r}_j' \). We recall that the fluctuations of the phases \( \theta_{\text{cp}} \) result mostly from those of the phases \( \theta_j = [\theta_{j,0} + \theta_{j,1}] \). Indeed, the fluctuations of the phases \( \phi_{j,s,1}^0 \) are very small for the whole hole-concentration range \( x \in (x_0,x_s) \).

At \( \alpha_d = 0 \) and thus \( \gamma_d = 1 \) the superconducting energy scale 4\( k_B T_c \) of Eq. (40) equals the zero-temperature average over the whole system \( \langle |e^{i\theta_{j,s}^\prime\gamma}| \Delta_0 \rangle \) of the basic quantity \( e^{i\theta_{j,s}^\prime\gamma} \Delta_0 \) appearing in the Hamiltonian terms (29).

The expressions provided in Eqs. (39), (38), (40), and (41) confirm that the short-range spin order parameter 2\( |\Delta| \), energy scale 4\( k_B T_c \), and superconducting order parameter 2\( |\Omega| \) are closely related. This is consistent with the short-range spin order and long-range superconducting order being closely related as well. The presence of the amplitude \( g_1 = \langle |e^{i\theta_{j,s}^\prime\gamma}| \rangle \) within the overall amplitude \( g = g_0 g_1 \) of the superconducting order parameter (41) is consistent with the short-range spin correlations providing the energy needed for the coherent virtual-electron pairing. Simultaneously, such a supplying by the short-range spin correlations suppresses them through the increasing fluctuations of the phases \( \theta_{j,1} \). The latter phases refer to the internal degrees of freedom of the virtual-electron pairs.

The phases \( \theta_j,0 \) and \( \theta_j,1 \) are such that \( \bar{g}_0 = \langle |e^{i\theta_{j,0}}| \rangle_{T=0} = 0 \) and \( \bar{g}_1 = \langle |e^{i\theta_{j,1}}| \rangle_{T=0} = \gamma_c \) for \( 0 < (x - x_c) \to 0 \) and \( \bar{g}_0 = \langle |e^{i\theta_{j,0}}| \rangle_{T=0} = 1 \) and \( \bar{g}_1 = \langle |e^{i\theta_{j,1}}| \rangle_{T=0} = 0 \) for \( 0 < (x_s - x) \to 0 \). These behaviors are consistent with the critical temperature \( T_c \) being small and given by \( T_c \approx \bar{g}_0 \Delta_0/2k_B \) and \( T_c \approx \bar{g}_1 \Delta_0/2k_B \) for \( 0 < (x - x_c) \ll 1 \) and \( 0 < (x_s - x) \ll 1 \), respectively. In these limits \( \bar{g} = \bar{g}_0 \bar{g}_1 \to 1 \), so that \( \gamma_d \to 1 \), the suppression effects play no role, and the physics is controlled by the very strong phase fluctuations. Symmetry arguments associated with the physics specific to these limits for which the critical temperature \( T_c \) is controlled only by strong phase fluctuations imply that \( \bar{g}_0 \approx (x/x_s)^{z\nu} \) and \( \bar{g}_1 \approx (x_s - x)/x_s^{z\nu} \) are for \( 0 < (x - x_c) \ll 1 \) and \( 0 < (x_s - x) \ll 1 \), respectively, controlled by the same dynamical exponent \( z = 1 \) and unknown exponent \( \nu \).

Treatments involving the use of the effective action for the phases \( \theta_j \) without incorporating the Berry phase lead to \( \nu \approx 2/3 \). More detailed treatments, incorporating the latter phase, lead to different values for that exponent.\(^{25}\) Symmetry arguments suggest that \( \bar{g}_1 \approx (x_s - x)/x_s \) has the same overall exponent \( z\nu \) for hole concentrations obeying both the inequalities \( 0 < (x - x_c) \ll 1 \) and \( 0 < (x - x_0) \ll 1 \). On combining the expression \( 2\Delta_0 = g_1 2\Delta_0 \) of Eq. (36) with the square-lattice quantum-liquid results of Ref.\(^{34}\) for \( 0 < x_s \ll 1 \), which in our case hold for \( 0 < (x - x_s) \ll 1 \), we then find \( z\nu = 1 \) so that \( \nu = 1 \). Within our scheme the suppression effects lessen the critical temperature and related energy scales, yet do not affect the phases and corresponding amplitude \( g = g_0 \bar{g}_1 \). We then consistently consider that for approximately the range \( U/4t \in (u_0,u_\alpha) \) and hole concentrations \( x \in (x_0,x_s) \) the zero-temperature amplitude \( \bar{g}_1 \) is given by \( \bar{g}_1 \approx (1 - x/x_s) \). Furthermore, the expression \( \bar{g}_0 \approx (x - x_c)/(x_s - x_c) \) obeys for \( x \in (x_c,x_s) \) the inequality \( \bar{g}_0 = \langle |e^{i\theta_{j,s}^\prime\gamma}| \rangle_{T=0} \leq 1 \). It also obeys the expected boundary condition \( \bar{g}_0 \to 1 \) as \( x \to x_s \). Therefore, we consider that \( \bar{g}_0 \approx (x - x_c)/(x_s - x_c) \) for hole concentrations in the range \( x \in (x_c,x_s) \). Hence within the present approach the expressions of the amplitudes \( g_0 \) and \( g_1 \) of Eq. (41) are for intermediate interaction.
values $U/4t \in (u_0, u_*)$ and zero temperature approximately given by,
\begin{align*}
g & = |\langle e^{i\theta_j/\hbar} \rangle|_{T = 0} = \bar{g}_0 \bar{g}_1 \approx \frac{(x - x_c)}{(x_s - x_c)} \left( 1 - \frac{x}{x_s} \right), \\
g & = |\langle e^{i\theta_j/\hbar} \rangle|_{T = 0} \approx \frac{(x - x_c)}{(x_s - x_c)} \text{ for } x \in (x_c, x_*) \text{,} \\
\bar{g}_1 & = |\langle e^{i\theta_j/\hbar} \rangle|_{T = 0} \approx \left( 1 - \frac{x}{x_s} \right) \text{ for } x \in (x_0, x_*); \\
\max \{\bar{g}_1\} & = \gamma_0 = \left( 1 - \frac{x_0}{x_s} \right) \text{ for } 0 < (x - x_0) \ll 1, \\
\bar{g}_0 & = \bar{g} = 0 \text{ for } 0 \leq x \leq x_c \text{ and } x \in (x_0, x_*); \\
\gamma_0 & = \gamma_1 = \bar{g} = 0 \text{ for } x \in (0, x_0) \text{ and } x \in (x_0, 1). \quad (42)
\end{align*}

Due to the hole-trapping effects reported in Appendix B, the fluctuations of the phases $\theta, \gamma_1$ are very strong for $x \in (0, x_0)$ and the zero-temperature amplitude $\bar{g}_1 = |\langle e^{i\theta_j/\hbar} \rangle|_{T = 0}$ vanishes. For $x \in (x_0, x_*)$ the hole-trapping effects remain active yet are weaker than for $x \in (0, x_0)$. In contrast to the range $x \in (0, x_0)$, the short-range spiral incommensurate spin order of the related square-lattice quantum liquid of Ref. 1 survives for hole concentrations $x \in (x_0, x_*)$, coexisting with the Anderson insulating behavior brought about by the hole-trapping effects.

The behaviors of the zero-temperature amplitudes $\bar{g}_0$ and $\bar{g}_1$ follow from the fluctuations of the corresponding phases $\theta$ and $\theta_1$ of Eqs. 39 and 40 becoming large for $0 < (x - x_c) \ll 1$ and $0 < (x_* - x) \ll 1$, respectively. The singular behavior $\bar{g}_1 \to \gamma_0$ for $0 < (x - x_0) \to 0$ and $\bar{g}_1 = 0$ at $(x - x_0) = 0$ is due to a sharp quantum phase transition. It marks the onset of the long-range antiferromagnetic order for $x < x_0$. In turn, the singular behavior $\bar{g}_0 \to 1$ for $0 < (x_* - x) \to 0$ and $\bar{g}_0 = 0$ at $(x_* - x) = 0$ is also due to a sharp quantum phase transition marking the onset of a disordered state without short-range spin order for $x > x_*$. Due to such sharp quantum-phase transitions, the phases $\theta_1$ and $\theta_0$ have also a singular behavior at $x = x_0$ and $x = x_*$, respectively. The fluctuations of these phases are small for $0 < (x - x_0) \ll 1$ and $0 < (x_* - x) \ll 1$ and large for $x < x_0$ and $x > x_*$, respectively.

Both in the Anderson insulator phase with short-range spiral incommensurate spin order occurring in the hole concentration range $x \in (x_0, x_*)$ due to the hole-trapping effects and in the phase corresponding to the range $x \in (x_0, x_*)$ for which the short-range spin order coexists with a long-range superconducting order the symmetry of the phases $\theta = \theta_0 + \theta_1$ action is a global $U(1)$. Such a global $U(1)$ symmetry is related to the Hubbard-model global $[SO(4) \times U(1)]/Z_2$ symmetry found recently beyond its known $SO(4)$ symmetry 17.

On combining Eqs. 40 and 44 one finds the following expression for the critical temperature,
\begin{align*}
T_c & \approx \gamma_d T_c |_{\alpha = 0} \approx \bar{g} \left( 1 - \frac{4 \bar{g}}{\gamma_d} \right) \\
& \approx \frac{\Delta_0}{2KB}; \\
T_c |_{\alpha = 0} & \approx \bar{g} \frac{\Delta_0}{2KB} = \left( 1 - \frac{x}{x_s} \right) \frac{\Delta_0}{2KB}. \quad (43)
\end{align*}

The related energy scale $2|\Omega||_{T = 0}$ is then given by expression 39 with $T_c$ as provided here. Hence the energy scales $2|\Delta||_{T = 0}$ of Eq. 39 at zero temperature and $2|\Omega||_{T = 0}$ associated with the short-range spin order and long-range superconducting order read,
\begin{align*}
2|\Delta||_{T = 0} & = \bar{g}_1 2\Delta_0 \cos(x - x_0); \\
2|\Omega||_{T = 0} & \approx \frac{4KB T_c}{\beta_c} = \frac{\gamma_d \bar{g} \Delta_0}{\beta_c}, \quad (44)
\end{align*}

respectively. Analysis of these expressions reveals that the physical requirement that $2|\Omega| < 2|\Delta|$ is met for the whole hole concentration range $x \in (x_0, x_*)$ provided that the $\alpha = (1 - \gamma_d^{|\min|})$ ranges given in Eq. 39 are fulfilled.

It follows from the expressions of the energy scale $2|\Omega||_{T = 0}$ and critical temperature $T_c$ provided in Eqs. 39 and 45, respectively, that the maximum magnitudes of these quantities read,
\begin{align*}
2|\Omega||_{T = 0} & \approx \frac{\Delta_0}{2}; \\
T_c |_{\alpha = 0} & = \frac{\Delta_0}{4KB} \gamma_d \gamma_c; \\
\gamma_d^{|\min|} & = (1 - \alpha_d) \in (\gamma_c, 1). \quad (45)
\end{align*}

Those are achieved at the optimal hole concentration given in Eq. 38.

Complementarily to the complex parameter $2\Omega$ of Eq. 40, one can consider the order parameter $\phi_{c\rho} = \sqrt{n_{c\rho} / 2} e^{i\theta_{c\rho}}$. Here $n_{c\rho}$ denotes the density of paired $c$ fermions contributing to phase-coherent virtual-electron pair configurations. The $x$ dependence of $n_{c\rho}$ is studied below in Section IV. The complex parameter $\phi_{c\rho}(\vec{r}) = \sqrt{n_{c\rho} / 2} e^{i\theta_{c\rho}}$ describes the macroscopic properties of the zero-momentum $c$ fermion pairs superfluid condensate. Indeed, $|\phi_{c\rho}|^2 = n_{c\rho} / 2$ is a measure of the local superfluid density of coherent fermion pairs. The average fermion distance or length $\xi_1$ of the set of $c$ fermion pairs contributing to a local virtual-electron pair is related to the coherent length $\xi$ associated with coherent virtual-electron pairing. For the isotropic Fermi-velocity range $x \in (x_c, x_*)$ and approximately $U/4t \in (u_0, u_*)$ such lengths read,
\begin{align*}
\xi_1 & \approx \sum_{g = 0}^{\lfloor N_1 / 4 - 1 \rfloor} |2h| g |^2 \xi_g \approx \frac{hV_F}{2\gamma_d \Delta_0} \approx \frac{[2\pi x_* \sqrt{xx}]}{\gamma_d \Delta_0} a, \\
\xi & \approx \frac{\gamma_c}{4g} \xi_1; \\
\xi |_{T = 0} & \approx \frac{\gamma_c}{4g} \xi_1 \approx \frac{\gamma_c}{4} \frac{hV_F}{4KB T_c} = \frac{\gamma_c}{4g} \frac{hV_F}{g^2 \Delta_0}; \\
\xi_{\text{min}} & = \xi |_{T = 0, x = x_{op}} = \xi_1. \quad (46)
\end{align*}
In the expression of $\xi_1$ we included explicitly here the lattice spacing $a$ and the factor $\gamma_c/4g$ appearing in the $\xi$ expression assures that its minimum magnitude $\xi_{min}$ equals $\xi_1$. It is achieved at zero temperature and $x = x_{op}$. Furthermore, $\xi_2$ is the distance between the $c$ fermions of each pair contributing to the same local virtual-electron pair. For the present isotropic Fermi-velocity range the Fermi velocity $V_F$ equals approximately the $c$ fermion velocity $V_{Fc} \approx \sqrt{2}\pi/2m^*_c$ of Eq. (29). The quantity $|2h_g| = \sqrt{2h_g^2}$ is the absolute value of the coefficients controlling the ratios $\langle f^\dagger_{F'}, c | f^\dagger_{F''}, c, g \rangle / \langle f^\dagger_{F'}, c | f^\dagger_{F''}, c, h \rangle | \approx |2h_g|^2/|2h_0|$ of Eq. (C3) of Appendix C. Those are associated with a $c$ fermion pair with the same real-space coordinates $r_{F'}$ and $r_{F''}$ as a two-site bond of a given rotated-electron pair. The relation between such $c$ fermion and rotated-electron pair is given in Eq. (28). The coefficients $\theta_g$ obey the sum-rule provided in Eq. (C3) of Appendix C. They are twice those appearing in the expression of the annihilation operator of the local $s1$ fermion of a local virtual-electron pair. Such an expression is obtained from Eq. (20). The absolute value $|2h_g| = \sqrt{2h_g^2}$ decreases upon increasing the magnitude of the length $\xi_{op}$. $|2h_g|$ is largest at $g = 0$, which corresponds to $\xi_0 = a_s = a/\sqrt{1-x}$. It falls rapidly upon increasing $g$. Therefore, the average length $\xi_1$ is typically very small as compared to the penetration depth considered below in Section IV-E.

The overall amplitude $g = g_0g_1$ associated with the phases $\theta_j = \theta_{j,0} + \theta_{j,1}$ whose dome-like $x$ dependence is for zero temperature given in Eq. (42) gives a measure of the strength of the $c$ fermion effective coupling leading to the superconducting state coherent pairing. Such an effective coupling also occurs when $g_1 > 0$ and $g_0 = 0$ yet for the corresponding pseudogap state does not lead to coherent virtual-electron pairing. In either state it is due to the energy provided to the $c$ fermions of such pairs by the short-range spin correlations. This occurs through the residual interactions of the two $c$ fermions with the $s1$ fermion within each virtual-electron pair. The energy lost by the spin-subsystem to supply the $c$ fermions of a pair with the energy needed for their effective coupling weakens the short-range spin order. This occurs through the enhancement of the fluctuations of the phases $\theta_{j,1}$ associated with the amplitude $g_1$. For the $c$ fermions $g_1$ refers to their effective pair coupling. Complementarily, for the $s1$ fermions it is behind the weakening of the short-range spin correlations upon increasing $x$. Consistently, the value of $g_1$ provides a measure of the ability and power of the short-range spin correlations supplying the $c$ fermions with the energy needed for the occurrence of virtual-electron pairing.

The physical picture that emerges is that the coherent-pair superconducting order rather than competing with the short-range spin order is a by-product of it. It coexists with the latter order for the hole concentration range $x \in (x_c, x_2)$ at zero temperature and a smaller $x$ range centered at $x = x_{op}$ for finite temperatures below $T_c$. Moreover, the type of $d$-wave long-range superconducting order considered here cannot occur without the simultaneous occurrence of short-range spin correlations. As the limit $0 < (x_s - x) < 1$ is reached at zero temperature, the short-range spin correlations use up all their energy. This is consistent with the behaviors $\tilde{g}_1 \rightarrow 0, \tilde{g} = \tilde{g}_0 \tilde{g}_1 = \tilde{g}_1 \rightarrow 0$, $2|\Delta||r_{0}=0$, $2|\Omega||r_{0}=0$, and $2|\Omega||r_{0}=0/2|\Delta||r_{0}=0/2$ occurring in that limit. Indeed, $\gamma_{d} \rightarrow 1$ as $\tilde{g} \rightarrow 0$ for $0 < (x_s - x) < 1$ when the physics is controlled by strong phase fluctuations. The order parameter of the long-range superconducting order vanishes upon the vanishing of that of the short-range spin order. Such parameters vanish upon the simultaneous disappearance of the corresponding orders.

Except in the limits $0 < (x_c - x) < 1$ and $0 < (x_s - x) < 1$ when the phase fluctuations are very strong, for $x \in (x_c, x_2)$ the suppression effects lessen both the critical temperature $T_c$ of Eq. (45) and superconducting energy scale $2|\Omega||$ of Eq. (44), and enhance the coherent length $\xi$ of Eq. (46), relative to their $\alpha_d = 0$ magnitudes. In turn, the $c$ fermion energy dispersion $\varepsilon_c(q^2)$ given in Eqs. (A10) and (A11) of Appendix A, the form of the $s1$ fermion energy dispersion $\varepsilon_s(q^2)$ given below, and the magnitudes of the pseudogap temperature $T^*$ of Eq. (22) and the critical hole concentration $x_s$ of Eq. (17) remain unaltered. Although here we consider intrinsic disorder, this is consistent with the experimental results of Refs.48,49, according to which the disorder induced by Zn substitution does not lead to a clear change in the magnitude of the pseudogap temperature $T^*$.

4. The ratios controlling the effects weak 3D uniaxial anisotropy and electronic correlations

In the present limit of very weak 3D uniaxial anisotropy its effects occur mainly through the dependence of the critical hole concentration $x_c$ on the parameter $\varepsilon^2$. It follows that the amplitudes $\tilde{g}$ and $\tilde{g}_0$ of Eq. (42), whose expressions involve $x_c$, also depend on the latter parameter. For approximately $U/4t > u_0$ the critical hole concentration deviation $(x_c - x_0)$ and the critical hole concentration $x_c$ are proportional to two important ratios,

$$ (x_c - x_0) \approx G = \frac{G}{\varepsilon^2} = G \frac{M}{m^*_c}; \quad G = \frac{1}{8} \left[ \frac{T_e^{\text{max}}(K)}{\varepsilon_{\text{op}}(A)\Delta_0} \right]_x = x_{\varepsilon = x_{\text{op}}, T=0} \left( 1.964 \times 10^8 \right) \xi_{\text{min}}(A); \quad x_s = \frac{2r_s}{d} = \frac{2}{4W_1^a}; \quad U/4t > u_0. \quad (47) $$

That $(x_c - x_0)$ equals approximately the Ginzburg number $G$ is a result obtained from the study of the effective action of the phases $\theta_j$. In the $G$ expression provided here $\lambda_{ab}$ denotes the in-plane penetration depth introduced
below in Section IV-E, $T^\text{max}$ is the maximum critical temperature of Eq. (44), reached at $x = x_{op}$, and $\xi^{\text{min}}$ is the corresponding $x = x_{op}$ minimum magnitude of the coherent length given in Eq. (10). The ratio $\xi^2 = m^*_c/M$ appearing in the $(x_c - x_0)$ expression of Eq. (17) controls the effects of weak 3D uniaxial anisotropy. Here the $U/4t$ dependent mass $m^*_c$ is that of the $c$ fermion energy dispersion of Eq. (A11) of Appendix A and $M \gg m^*_c$ is the effective mass of Eq. (3) associated with electron hopping between the planes.

The critical hole concentration $x_c$ expression given in Eq. (47) is derived in Ref. 18. The spin ratio $r_s$ appearing in that expression and the $x_c$ limiting values are provided in Eqs. (A11) and (A2) of Appendix A, respectively. For the smaller range of intermediate interaction values $U/4t \in (u_0, u_1)$ one approximately has that $r_c = m^*_c/m^*_s \approx 2r_s$. Hence for such a range the upper critical hole concentration can be expressed as $x_c \approx r_c/\pi$, as found in Ref. 18. The ratio $r_s = \Delta_0/4W^0_{s1}$ and $r_c = m^*_c/m^*_s$ control the effects of electronic correlations. Here $m^*_c = \lim_{U/4t \to \infty} m_c = 1/2t$ and $\Delta_0$ and $4W^0_{s1}$ are the energy scales of Eq. (3).

Our scheme is valid for very small values of the 3D uniaxial anisotropy mass ratio $m^*_c/M$. That the relation $(x_c - x_0) \approx G_i$ holds is fully consistent with the related results of Ref. 18. The studies of Ref. 18 reveal that a critical hole concentration $x_c \approx 0.05$ corresponds to $\xi^2 = m^*_c/M \approx 10^{-4}$ for LSCO and $\xi^2 = m^*_c/M \approx 10^{-3} - 10^{-2}$ for the other four systems. This follows from $G$ being very small and given by $G \approx 10^{-5}$ and $G \approx 10^{-4}$ for LSCO and the other four systems, respectively. The smallness of $G$ allows that $G_i \approx (x_c - x_0) \approx 0.02$ is small in spite of $1/\xi^2 = M/m^*_c$ being large and $(x_c - x_0) \propto 1/\xi^2 = M/m^*_c$. Hence the value $x_c \approx 0.05$ is set by choosing an appropriate small magnitude for the 3D uniaxial anisotropy parameter $\xi^2 = m^*_c/M$. In turn, the value $x_c \approx 0.27$ is set by choosing $U/4t \approx 1.525$. Often one considers the range $U/4t \in (u_0, u_e)$ for which according to Eq. (A2) of Appendix A the $x_c$ values belong to the domain $x_e \in (2e^{-1/\pi}, 1/\pi) \approx (0.23, 0.32)$. Some of our expressions refer to the smaller range $U/4t \in (u_0, u_1)$ of intermediate interaction values for which $r_c \approx 2r_s \approx \pi x_s \approx 2e^{-4}u_0/\pi$ and $x_c \approx 0.23, 0.28$.

Our results reveal that for constant values of $U/4t$ the superconducting-dome hole-concentration width $(x_c - x_0)$ decreases upon further decreasing the ratio mass $m^*_c/M$. That could mean that such a width vanishes in the 2D limit $m^*_c/M \to 0$ so that the ground state of the Hubbard model on the square lattice is not superconducting. However, note that our results do not apply to the 2D limit $\xi^2 = m^*_c/M \to 0$. Then $G_i$ becomes large, so that the expression $x_c \approx (G_i + x_0)$ obtained from the effective action of the phases $\theta_j$ may not be valid.

5. The rate equations for suppression of $2|\Delta|$ and variation of $4k_B T_c$

The dependence on the hole concentration $x$ of the zero-temperature short-range spin order parameter $2|\Delta|(x)|_{T=0}$ of Eq. (30) and critical temperature $T_c$ provided in Eq. (43) is described by the two rate differential equations under suitable and physical boundary conditions given in the following. Those are valid for vanishing spin density $m = 0$ and finite hole concentrations in the ranges $x \in (x_0, x_s)$ and $x \in (x_c, x_s)$, respectively.

The rate equation for suppression of the order parameter of the short-range spin correlations $2|\Delta|(x)|_{T=0}$ upon increasing the value of $x$ and its boundary condition are for the approximate range $U/4t \in (u_0, u_e)$ and $x \in (x_0, x_s)$ given by,

$$\frac{\partial 2|\Delta|(x)|_{T=0}}{\partial x} = -\frac{\Delta_0}{r_s} \theta(x_s - x); \quad 2|\Delta|(x_0)|_{T=0} = \gamma_0 2\Delta_0; \quad \gamma_0 = \left(1 - \frac{x_0}{x_s}\right).$$

Here $\theta(z)$ is a theta function such that $\theta(z) = 1$ for $z > 0$ and $\theta(z) = 0$ for $z \leq 0$. In turn, the rate equation for variation of the superconducting energy scale $4k_B T_c$ upon increasing $x$ and the corresponding boundary condition read,

$$\frac{\partial 4k_B T_c(x)}{\partial x} = \left(\frac{2\gamma_d - 1}{\gamma_c} \right) \pi \frac{r_s}{r_c} \left[2|\Delta|(x) - 2|\Delta|(x_{op})\right]|_{T=0} \theta(x_s - x); \quad 4k_B T_c(x_c) = 0.$$
E. Zero-temperature magnetic-field $c$ fermion diamagnetic orbital coupling and $s^1$ fermion Zeeman coupling

It is useful for the study of the pseudogap state to consider a uniform magnetic field $H$ aligned perpendicular to the square-lattice plane, to suppress superconductivity. The usual type II superconductor field $H_{c1}$ is very small for the VEP quantum liquid at intermediate $U/4t \in (u_0, u_\tau)$ values, so that here we consider that $H_{c1} \approx 0$. Our study refers to such a $U/4t$ range.

It is well known that for the type-II superconductors the Pauli field $H_p$, for which the Zeeman splitting spin alignment starts to be profitable is given by $H_p \approx k_B T_c/\mu_B g$. Here $\mu_B$ is the Bohr magneton and $g \approx 2$. For conventional type-II superconductors such a field is found to lie above the $H_{c2}$ line. In that case the Pauli pair breaking will become at zero temperature an important issue at high magnetic fields $H > H_p > H_{c2}$. In contrast, we find in the following that here $H_{c2} > H_p$. Important properties are then that:

The diamagnetic orbital coupling of the magnetic field is to the charge $c$ fermions. It occurs through the $c$ band hole momenta, $\vec{q} - e\vec{A}$, where $\vec{A}$ is the vector potential associated with the magnetic field $\vec{H} = \nabla \times \vec{A}$. Hence diamagnetic pair breaking refers here to the zero-momentum charge $c$ fermion pairing.

The Zeeman coupling of the magnetic field is to the spin-singlet two-spinon $s^1$ fermions. Hence Pauli pair breaking refers here to the spin-singlet spinon pairing of the $s^1$ fermions.

Since virtual-electron pairing involves both zero-momentum charge $c$ fermion pairing and spin-singlet spinon pairing of the $s^1$ fermions, diamagnetic orbital coupling and Zeeman coupling are direct probes of the two types of pairing involved in the virtual-electron pairing.

That the energy needed for coherent $c$ fermion pairing is supplied by the short-range spin correlations associated with $s^1$ fermion spinon pairing implies an interplay between Pauli spinon pair breaking and diamagnetic $c$ fermion pair breaking. Below a critical field $H_0$ the energy supplied to the $c$ fermions by the spin correlations associated with $s^1$ fermion spinon pairing is at zero temperature enough to prevent that diamagnetic $c$ fermion pair breaking stops phase coherence. For $H < H_0 \approx k_B T_c/\mu_B g$ there are no significant effects on the spin-singlet spinon pairing from Zeeman coupling to the $s^1$ fermions and only diamagnetic orbital coupling to the $c$ fermions plays an active role in the physics. In turn, for $H > H_0$ the $s^1$ fermion spinon pairing starts to be affected by the Zeeman coupling. This lessens the energy supplied by the spin sub-system to the $c$ fermion strong effective coupling. As a result, the effects of diamagnetic $c$ fermion pair breaking become stronger for $H > H_0$ and the critical field $H_0 > H_{c2}$ above which there is no $c$ fermion pairing phase coherence is at zero temperature proportional to $H_p$. However, for $H \in (H_{p}, H_0)$ the diamagnetic orbital coupling to the $c$ fermions remains much stronger than the Zeeman coupling to the $s^1$ fermions. Within our scheme the critical field $H_0$ is that above which both the diamagnetic orbital coupling to the $c$ fermions and the Zeeman coupling to the $s^1$ fermions play an active role. It is approximately given by $H_0 \approx 2H_p$. Since $H_p \approx k_B T_c/\mu_B g$ where $g \approx 2$ one then finds that at zero temperature $H_0 \approx k_B T_c/\mu_B$.

Hence within our picture, at zero temperature the fluctuations of the phases $\theta_1$ remain small and long-range superconducting order prevails provided that the uniform magnetic field $H$ is below a critical field $H_0 \approx k_B T_c/\mu_B$. Above that critical field, diamagnetic $c$ fermion pair breaking prevents phase coherence yet there remain $c$ fermion pairing correlations.

At zero temperature such a critical field expression approximately reads,

$$H_0 \approx k_B T_c/\mu_B \approx \gamma d \frac{(x - x_c)}{(x_s - x_c)} \left(1 - \frac{x}{x_s}\right) \frac{\Delta_0}{2\mu_B}, \quad x \in (x_c, x_s); \quad H_{0}^\text{max} = \gamma d \min \gamma_c \frac{\Delta_0}{8\mu_B}, \quad x = x_{op}. \quad (50)$$

Here the hole concentration $x_{op}$ is given in Eq. (38). Interestingly, the expressions of the energy scale $|2\Omega|_{T=0}$ given in Eq. (39), critical temperature $T_c$ in Eq. (43) and magnetic field $H_0$ in Eq. (50) have the dome-like $x$ dependence observed in the five representative hole-doped cuprates. Within our scheme such a type of $x$ dependence follows directly from that of the overall zero-temperature amplitude $\tilde{g} = \bar{g}_0 \bar{g}_1$ of Eq. (42). The $x$ dependence of such an amplitude is fully controlled by the interplay of the $x$ dependences of the fluctuations of the phases $\theta_{j,0}$ and $\theta_{j,1}$ such that $\theta_j = \theta_{j,0} + \theta_{j,1}$, as discussed above in Section III-D.

At zero temperature, magnetic field $H$ in the range $H \in (H_0, H_{c2})$, and finite hole concentrations in a $H$-dependent range below a hole concentration $x = x_{c}$ the phases $\theta_{j,0}$ have large fluctuations. This refers to the pseudogap state for which both the diamagnetic orbital coupling to the $c$ fermions and the Zeeman coupling to the $s^1$ fermions play an active role. In it the phase-factor averages $(e^{i\theta_0})$ and $(e^{i\theta_2})$ and the amplitudes $g_0$ and $g$ vanish. However, the fluctuations of the phases $\theta_{j,1}$ remain small. As a result, the amplitude $g_1 = |(e^{i\theta_1})|$ remains finite and the short-range spin order prevails in the pseudogap state. The same applies to the pseudogap state reached at zero temperature, magnetic field in the range $H \in (H_0, H^*)$, and a $H$-dependent hole concentration range above $x = x_{c}$ and below $x = x_s$. Here $H_{c2}$ and $H^*$ are the fields above which the phases $\theta_{j,1}$ have large fluctuations, so that the amplitude $g_1 = |(e^{i\theta_1})|$ vanishes and there is no short-range spin order.
While the phase transition occurring at $H = H_0$ results mostly from $c$ fermion diamagnetic pair breaking, the disappearance of short-range spin order occurring at the $H_{c2}$ and $H^*$ lines results both from $c$ fermion diamagnetic and Zeeman spin-singlet spinon $s1$ fermion pairs breaking. Consistently, at zero temperature the fields $H = H_{c2}$ for $x \in (x_0, x_{c2})$ and $H = H^*$ for $x \in (x_{c2}, x_*)$ are those at which Zeeman spin-singlet spinon $s1$ fermion pair breaking fully suppresses the short-range spin order and thus the diamagnetic $c$ fermion pair breaking fully suppresses the corresponding incoherent $c$ fermion pairing correlations.

For hole concentrations in the range $x \in (x_0, x_{c1})$ we use the method of Ref.~24 to derive the $x$ dependence of the magnitude $H_{c2}$ of the uniform magnetic field $H$ aligned perpendicular to the square-lattice plane. This involves expanding the quadratic terms of the continuum Lagrangian representing the charge sector associated with the $c$ fermion pairs. The quadratic terms reflect the centre of mass motion of such pairs. They are expanded in charge $\pm 2e$ Landau levels resulting from the diamagnetic orbital coupling of the magnetic field to the charge $c$ fermions. In turn, we did not derive an expression valid for hole concentrations in the range $x \in (x_{c1}, x_{c2})$. For that $x$ range we could only access an inequality obeyed by $H_{c2}$ and the hole concentration $x_{c2}$, respectively. In turn, the $x$ dependence of the upper field $H^*$ is a simpler problem. Alike $H_0 \approx k_B T_c/\mu_B$ is proportional to the superconducting critical temperature $T_c$ for $x \in (x_{c2}, x_*)$ it is proportional to the pseudogap temperature $T^*$ of Eq.~(22). For the interaction range $U/4t \in (u_0, u_x)$ our results for the fields $H_{c2}$ and $H^*$ are,

$$H_{c2} \approx \mathcal{F} = \frac{(x - x_0) \Delta_0}{(x \gamma_0) \delta \mu_B}, \quad x \in (x_0, x_{c1}); \quad \frac{\tilde{g}_1(x)}{g_1(x_{c1})} \mathcal{F} \leq H_{c2} \leq \mathcal{F}, \quad x \in (x_{c1}, x_{c2}),$$

$$H^* \approx \frac{k_B T_c}{\mu_B} \approx \frac{\Delta_0}{\mu_B}, \quad x \in (x_{c2}, x_*) .$$

Here the amplitude $\tilde{g}_1 = \tilde{g}_1(x)$ is given in Eq.~(12). We call $H_{c2}^*$ the field magnitude reached at the hole $x = x_{c2}$ at which $H_{c2} = H^*$. Hence $H_{c2} = H_{c2}(x_{c2}) = H^*(x_{c2})$. The inequalities obeyed by $H_{c2}$ and $x_{c2}$ are then given by,

$$0 \leq H_{c2}^* \leq \frac{\gamma_0 \Delta_0}{(3\gamma_0 + 1)\mu_B}; \quad x_{c2}^\text{min} \leq x_{c2} \leq x_*; \quad x_{c2}^\text{min} = \frac{(2\gamma_0 + 1)}{(3\gamma_0 + 1)} x_* ,$$

respectively. Here $H_{c2}^* = \gamma_0 \Delta_0/[3(\gamma_0 + 1)\mu_B]$ and $H_{c2}^* = 0$ for $x = x_{c2}^\text{min}$ and $x = x_{c2}^\text{max} = x_*$, respectively, and $\gamma_0$ is the parameter given in Eq.~(15). It is expected that $H_{c2}^*$ is closer to $\gamma_0 \Delta_0/[3(\gamma_0 + 1)\mu_B]$ than to zero, so that $x_{c2} \approx x_{c2}^\text{min}$ and $H_{c2} \approx \mathcal{F}$ both for $x \in (x_0, x_{c1})$ and $x \in (x_{c1}, x_{c2})$.

That the expression $H^* \approx \tilde{g}_1[\Delta_0/\mu_B]$ involves the zero-temperature and zero-field value of the amplitude $g_1 = |\langle \psi_{B_1} \rangle|$ is consistent with it marking the full suppression of the short-range spin correlations. Indeed, for $H > H^*$ and hole concentrations in the range $x \in (x_{c2}, x_*)$ the system is driven into a disordered state without short-range spin order. In turn, for hole concentrations $x \in (x_0, x_{c1})$ the field $H_{c2}$ grows linearly with $x$. If such a behavior is a good approximation for $x \in (x_{c1}, x_{c2})$ as well, for $x_1 = 0.27$ one finds $x_{c2} \approx x_{c2}^\text{min} \approx 0.20$ for the five representative systems. For $x \in (x_1, x_{c2})$ the actual $H_{c2}$ $x$ dependence may slightly deviate to below the linear-$x$ behavior. If so, the hole concentration $x_{c2} \approx 0.20$ may increase to $\approx 0.21 - 0.22$.

The pseudogap state occurring for fields below $H_{c2}$ and $H^*$ and above $H_0$ for a $H$-dependent hole-concentration range centered at $x = x_{c2}$ is a quantum vortex liquid. It shows strong vortex fluctuations and enhanced diamagnetism. Along the line $H = H_0$, this quantum vortex liquid freezes into a solid. Then any small deviation $-\delta H < 0$ of the field $H = [H_0 - \delta H]$ near $H_0$ leads to superconductivity. For magnetic fields above the field $H^*$ corresponding to the range $x \in (x_{c2}, x_*)$ the system is driven into a disordered spin state. It is similar to that reached at zero field and zero temperature for hole concentrations larger than $x_*$. Upon lowering $H$ from above $H^*$, the field $H^*$ marks the onset of the short-range spin order. Therefore, it marks a crossover rather than a sharp transition. The same applies to the pseudogap temperature $T^*$ of Eq.~(22) for vanishing magnetic field $H = 0$ and finite hole concentrations in the range $x \in (x_0, x_{c1})$. Within a mean-field treatment, the upper field $H_{c2}$ refers to a crossover as well. For finite fields $H$ and vanishing temperature the true quantum transition to the superconducting phase takes place at the line $H_0 = H_0(x)$. Along it the quantum vortex liquid freezes into a solid. Whether beyond mean-field theory the line associated with $H_{c2}(x)$ marks a sharp quantum phase transition and the onset of a new order for fields $H > H_{c2}$ and finite hole concentrations in the range $x \in (x_0, x_{c2})$ remains an open question.

IV. THE QUANTUM-LIQUID ENERGY FUNCTIONAL, VIRTUAL-ELECTRON STRONG EFFECTIVE COUPLING PAIRING MECHANISM, AND SUPERFLUID DENSITY

Here we study the general energy functional associated with the VEP quantum-liquid microscopic Hamiltonian considered in Section III-C. It refers to that Hamiltonian in normal order relative to the $m = 0$ initial ground state.
The 2D energy functional considered here accounts for the weak 3D uniaxial anisotropy effects through the dependence of the critical hole concentration $x_c$ and pairing phases on the parameter $\varepsilon^2 = m^*_s/M \ll 1$.

Such a functional is obtained from that constructed in Ref. 1 on introducing in it the effects of the coherent virtual-electron pairing phase fluctuations, accounting for the small suppression effects. For $x \in (x_c, x_*)$ this refers to the occurrence in the VEP quantum liquid of a long-range superconducting order. That program involves the evaluation of the general superconducting virtual-electron pairing energy, which emerges in the spectrum of such an energy functional. The derivation of such a pairing energy requires the study of the virtual-electron strong effective coupling pairing mechanism. This involves as well investigations on the energy range of the $c - s_1$ fermion interactions behind it and signatures of virtual-electron pairs in the one-electron removal spectral-weight distribution. In addition, in this section we study the hole-concentration dependence of the zero-temperature superfluid density.

### A. The quantum-liquid energy functional

The phase and amplitude of the complex gap function $\Delta_M$ appearing in the VEP quantum-liquid Hamiltonian terms [23] show up in the superconducting order parameter $\Omega$ of Eq. [10]. The absolute value of this order parameter is the maximum magnitude of the superconducting virtual-electron pairing energy $2|\Omega_1(\mathbf{q})|$ evaluated below in Section IV-B for $s_1$ band momenta $\mathbf{q}_i \approx \mathbf{q}_{i|s_1}$ at or near the $s_1$ boundary line and in Section IV-F for momenta belonging to the general coherent $s_1 - sc$ lines introduced in that section. (Here the index $sc$ stands for strong coupling.) For temperatures below $T_c$ and a $T$ dependent $x$ range centered at $x = x_{op}$, which is largest and given by $x \in (x_c, x_*)$ at $T = 0$, the superconducting pairing energy per electron $|\Omega_1(\mathbf{q})|$ derived in these sections emerges in the $s_1$ fermion energy dispersion $\epsilon_1(\mathbf{q})$. This dispersion appears in the general energy functional introduced in the following. That functional involves the ground-state normal-ordered $c$ and $s_1$ fermion hole momentum distribution function deviations,$^1$

$$\delta N^h_s(\mathbf{q}_j) = [N^h_s(\mathbf{q}_j) - N^h_{s^0}(\mathbf{q}_j)], \quad \delta N^h_1(\mathbf{q}_j) = [N^h_1(\mathbf{q}_j) - N^h_{1^0}(\mathbf{q}_j)].$$  

Here $N^h_{s^0}(\mathbf{q}_j)$ and $N^h_{1^0}(\mathbf{q}_j)$ are the corresponding initial-ground-state values. According to the results of Ref. 1, the $c$ and $s_1$ fermion discrete hole momenta $\mathbf{q}^0_s$ and momenta $\mathbf{q}^0_1$, respectively, are good quantum numbers for the $\varepsilon^2 = 0$, $\alpha_d = 0$, and $x_0 = 0$ square-lattice quantum liquid of Ref. 1. Under the very weak 3D uniaxial anisotropy effects and suppression effects they are for $x \in (x_c, x_*)$ close to good quantum numbers of the VEP quantum liquid. This is in contrast to the range $x \in (0, x_c)$, in which the strong hole tapping effects reported in Appendix B change qualitatively the square-lattice quantum liquid physics of Ref. 1.

It follows that alike for that quantum liquid, for the the VEP quantum liquid considered here the hole momentum distribution functions $N^h_s(\mathbf{q}_j)$ and $N^h_1(\mathbf{q}_j)$ approximately read 1 and 0 for unfilled and filled, respectively, discrete-hole-momentum values $\mathbf{q}^0_s$ or discrete-momentum values $\mathbf{q}_j$. Furthermore, the first-order energy terms of the energy functional given in the following correspond to the dominant contributions. The $c - s_1$ fermion interactions vanish or are extremely small and can be ignored, whereas the $s_1 - s_1$ fermion interactions do not lead to inelastic scattering.$^223$

In turn, the effects of the $c - s_1$ fermion interactions of interest for the studies of this paper are implicitly accounted for in the pairing energy involved in the $s_1$ band energy dispersion $\epsilon_1(\mathbf{q})$ appearing in the first-order terms of the energy functional. This justifies why its second-order terms in the deviations [53] are not studied here. An approximate expression of the matrix element $\lim_{\mathbf{q} \rightarrow 0} W_{c,s_1}(\mathbf{q}^0_s, \mathbf{q}, \mathbf{p})$ of the $c - s_1$ fermion effective interaction between the initial and final states needed for the derivation of the one-electron inverse lifetime and related to the second-order forward-scattering $f_{c,s_1}(\mathbf{q}^0_s, \mathbf{q}, \mathbf{p})$ function is for transfer momentum $\mathbf{p} \rightarrow 0$ evaluated in Ref. 22.

For the interaction range $U/4t \in (u_0, u_c)$, hole concentrations $x \in (x_c, x_*)$, and vanishing spin density $m = 0$ the VEP quantum liquid general energy functional is to first order in the deviations [53] given by,

$$\delta E = - \sum_{\mathbf{q}^0_s} \epsilon_s(\mathbf{q}^0_s)\delta N^h_s(\mathbf{q}^0_s) - \sum_{\mathbf{q}} \epsilon_1(\mathbf{q})\delta N^h_1(\mathbf{q}).$$  

The energy dispersion $\epsilon_s(\mathbf{q}^0_s)$ appearing here is given in Eqs. [A10] and [A11] of Appendix A. The dispersion $\epsilon_1(\mathbf{q}_j)$ reads,

$$\epsilon_1(\mathbf{q}_j) = -\sqrt{[\epsilon^0_1(\mathbf{q}_j)]^2 + [\Delta_{v-el}(\mathbf{q}_j)]^2}; \quad [\Delta_{v-el}(\mathbf{q}_j)] = [\Delta_{s_1}(\mathbf{q}_j)] + [\Omega_{s_1}(\mathbf{q}_j)].$$  

Here the energy dispersion $\epsilon^0_1(\mathbf{q}_j)$ and pairing energy per spinon $[\Delta_{s_1}(\mathbf{q}_j)]$ are given in Eq. [A12] of Appendix A, $[\Delta_{v-el}(\mathbf{q}_j)]$ is the virtual-electron pairing energy per electron, and $[\Omega_{s_1}(\mathbf{q}_j)]$ is its coherent part introduced below in Sections IV-B and IV-F. The short-range spin order and its order parameter $2|\Delta|$ refer to the range $x \in (x_0, x_*)$. 


However, due to the effects of the Anderson insulator behavior coexisting with that order for \( x \in (x_0, x_c) \), the \( s_1 \) fermion energy dispersion \( \epsilon_{s_1}(\tilde{q}) \) of Eq. (55) and the corresponding momentum-dependent energies \( |\Delta_{s_1}(\tilde{q})| \) and \( |\Omega_{s_1}(\tilde{q})| \) are well defined only for the range \( x \in (x_c, x_s) \).

As discussed in Section III-D, removal of two electrons in a spin-singlet configuration leads to an excited state with no holes in the \( s_1 \) band. Hence it follows from the form of the energy functional (55) that no virtual-electron pair is broken under that excitation. Instead, such a pair becomes the removed electron pair. In turn, one-electron and spin excitations lead to the emergence of one and two holes in the \( s_1 \) band, respectively. Thus their energy spectrum involves a finite pairing energy associated with virtual-electron pair breaking.

We recall that the virtual-electron pairs carry the same momentum \( \tilde{q} \) as the corresponding \( s_1 \) fermion. Indeed, their two \( c \) fermions have hole momentum \( \tilde{q}^h \) and \( -\tilde{q}^h \), respectively. Therefore, the \( c \) fermion pair net momentum contribution vanishes. This justifies why the energy scale \( |\Delta_{v-el}(\tilde{q})| \) given in Eq. (55) appears in the \( \tilde{q} \) dependent energy dispersion \( \epsilon_{s_1}(\tilde{q}) = -\sqrt{|\epsilon_{s_1}^d(\tilde{q})|^2 + |\Delta_{v-el}(\tilde{q})|^2} \). Indeed, that it and thus \( |\Omega_{s_1}(\tilde{q})| \) appear in such a dispersion follows from the virtual-electron pair associated with these pairing energies per electron carrying momentum \( \tilde{q} \). As discussed in Section III-D, such virtual-electron pairs of momentum \( \tilde{q} \) exist as individual objects only in intermediate states of one-electron and spin excitations. Nevertheless, the virtual-electron pair energy is related to the spectrum of one-electron excitations and has physical significance, as confirmed below in Section IV-D. It reads,

\[
E_{v-el}(\tilde{q}) = 2|\epsilon_{c}(\pm \tilde{q}^h)| + |\epsilon_{s_1}(\tilde{q})|.
\]

Note that such an energy is not in general a pairing energy and thus is different from the virtual-electron pairing energy per electron \( |\Delta_{v-el}(\tilde{q})| \) of Eq. (55). It is found below that for momenta \( \tilde{q} \approx \tilde{q}_{B1s}^d \) at or near the \( s_1 \) boundary line coherent pairing is associated with \( c \) fermions of hole momenta \( \tilde{q}^h \) and \( -\tilde{q}^h \) near the \( c \) Fermi line. In this case \( E_{v-el}(\tilde{q}_{B1s}^d) \approx |\Delta_{v-el}(\tilde{q}_{B1s}^d)| \), so that the virtual-electron-pair energy becomes the virtual-electron pairing energy per electron. At zero temperature the energy \( |\Omega_{s_1}(\tilde{q})| \) is finite for \( s_1 \) band momenta \( \tilde{q} \) corresponding to \( s_1 \) fermions that mediate coherent \( c \) fermion pairing. These momenta belong to the range \( \tilde{q} \in Q_{cp}^{s_1} \) defined in Section IV-F. It refers to a set of coherent \( s_1 \) \( - \) \( sc \) lines centered at zero \( s_1 \) band momentum. Moreover, \( s_1 \) fermions whose momentum belongs to a given coherent \( s_1 \) \( - \) \( sc \) line only interact with \( c \) fermions whose hole momenta belong to a related nearly circular coherent \( c \) \( - \) \( sc \) line centered at the \( c \) band momentum \( -\pi = -[\pi, \pi] \). There is a one-to-one correspondence between such \( s_1 \) \( - \) \( sc \) lines and \( c \) \( - \) \( sc \) lines.

The non-coherent \( |\Delta_{s_1}(\tilde{q})| \) and coherent \( |\Omega_{s_1}(\tilde{q})| \) parts of the virtual-electron pairing energy \( |\Delta_{v-el}(\tilde{q})| \) of Eq. (55) are associated with \( s_1 \) fermion spin-singlet spinon pairing and coherent zero-momentum \( c \) fermion pairing, respectively. That they appear in the \( s_1 \) fermion spectrum and their maximum magnitudes \( |\Delta| \) and \( |\Omega| = \gamma_d |\Omega_0| |\Delta| \) are closely related and as confirmed in Ref. 15 correspond to the superconducting- and normal-state maximum one-electron gap \( |\Delta| \) and superconducting-state low-temperature magnetic resonance energy \( 2|\Omega| \), respectively, is consistent with the interplay between magnetic fluctuations and unconventional superconductivity discussed in Ref. 16.

The corresponding general excitation momentum functional is linear in the deviations (53) and reads,

\[
\delta \tilde{P} = \delta \epsilon_{s_1}^d + \sum_{\tilde{q}^h} \delta \Omega_{s_1}(\tilde{q}^h) - \sum_{\tilde{q}} \delta \Omega_{s_1}(\tilde{q}) = \delta \epsilon_{s_1}^d + \sum_{\tilde{q}^h} \delta \Omega_{s_1}(\tilde{q}^h) - \sum_{\tilde{q}} \delta \Omega_{s_1}(\tilde{q}) .
\]

Here \( \delta \epsilon_{s_1}^d \) is the subspace-dependent small momentum deviation considered in Ref. 15.

Except for the superconducting pairing energy per electron \( |\Omega_{s_1}(\tilde{q})| \) appearing in the expression of the dispersion \( \epsilon_{s_1}(\tilde{q}) \) of Eq. (55), the first-order terms of the energy functional (54) are those derived in Ref. 15 for approximately the range \( U/4t > u_0 \approx 1.302 \). Indeed and as found in that reference, for \( U/4t < u_0 \) the energy scale \( 2\Delta_0 \) whose limiting behaviors are given in Eq. (A13) of Appendix A becomes very small. It follows that the amplitude fluctuations of the order parameter of the short-range spin correlations cannot be ignored. As discussed in Ref. 15, for small \( U/t \) values the \( s_1 \) fermion energy dispersion \( \epsilon_{s_1}(\tilde{q}) \) is then not expected to have the form given in Eq. (55).

### B. \( d \)-wave virtual-electron pairing mechanism near the Fermi line

Here we evaluate the superconducting pairing energy per electron \( |\Omega_{s_1}(\tilde{q})| \) in the \( s_1 \) band energy spectrum of Eq. (55) for momenta \( \tilde{q} \approx \tilde{q}_{B1s}^d \) at or near the \( s_1 \) boundary line. As given in Eq. (A13) of Appendix A, the order parameter \( 2|\Delta| \) associated with the short-range spin correlations is the maximum magnitude of the \( s_1 \) fermion spinon-pairing energy \( 2|\Delta_{s_1}(\tilde{q})| \). Similarly, the absolute value \( 2|\Omega||\tau = 0 \) of the order parameter associated with phase-coherent-pair superconducting order is the maximum magnitude of the superconducting virtual-electron pairing energy . The maximum magnitude of the \( s_1 \) fermion spinon-pairing energy \( 2|\Delta_{s_1}(\tilde{q})| \) corresponds according to Eq. (A13) of Appendix A to \( s_1 \) band momentum values \( \tilde{q} = \tilde{q}_{B1s}^d \). Those belong to the \( s_1 \) boundary line and their corresponding
auxiliary momentum values \( \vec{q}_0 = \vec{q}_{AN}^d \) given in Eq. (47) of that Appendix point in the anti-nodal directions. (The equality \( \vec{q}_{BS1}^{AN} = \vec{q}_{BS1}^d \) holds for ground states at vanishing spin density \( m = 0 \) and their two-electron excited states. In turn, one has that \( \vec{q}_{BS1}^{AN} \neq \vec{q}_{BS1}^d \) for their one-electron excited states.)

The VEP quantum-liquid \( d \)-wave long-range superconducting order occurring at zero temperature for the \( x \) range \( x \leq 0 \) \((x - x_c) \ll 1 \) is a by-product of the short-range spin correlations. For small \( 0 < (x - x_c) \ll 1 \) this shows up in the relation of the zero temperature energy scale \( \Omega_{s1}(\vec{q}_{BS1}^d) \) to the \( s1 \) fermion pairing energy per spinon associated with such short-range correlations. Indeed for \( 0 < (x - x_c) \ll 1 \) the energy scale \( |\Omega_{s1}(\vec{q})| \) is the deviation of the \( s1 \) fermion pairing energy per spinon \( \Delta_{s1}(\vec{q}_{BS1}^d - \vec{q}_{0}) \) from its value \( \Delta_{s1}(\vec{q}_{BS1}^d) \) at \( \vec{q}_{0} = 0 \). Here \( \vec{q}_{0} = |g_0/\sqrt{2}| \vec{e}_{\phi_{AN}^s} \). (The physical meaning of the vector \( \vec{q}_{0} \) is discussed below.) Expanding \( \Delta_{s1}(\vec{q}_{BS1}^d - \vec{q}_{0}) \) gives,

\[
|\Delta_{s1}(\vec{q}_{BS1}^d - \vec{q}_{0})| \approx |\Delta_{s1}(\vec{q}_{BS1}^d)| + \vec{V}_{s1}^d(\vec{q}_{BS1}^d) \cdot \vec{q}_{0} = |\Delta_{s1}(\vec{q}_{BS1}^d)| + |\Omega_{s1}(\vec{q}_{BS1}^d)|, \quad T = 0.
\] (58)

Hence for \( 0 < (x - x_c) \ll 1 \) the zero-temperature superconducting virtual-electron pairing energy per electron reads,

\[
|\Omega_{s1}(\vec{q}_{BS1}^d)| = |\vec{V}_{s1}^d(\vec{q}_{BS1}^d)| \cdot \vec{q}_{0} = g_0 \sqrt{2} \vec{V}_{s1}^d = |\Omega| |x - 0| \sin 2\phi, \quad T = 0.
\] (59)

Here the \( s1 \) fermion velocity \( \vec{V}_{s1}^d(\vec{q}_{BS1}^d) \) is given in Eq. (A15) of Appendix A, \( \vec{V}_{s1}^d \) denotes its absolute value of Eq. (39), and \(|\Omega|\big|_{T=0}\) is the superconducting energy scale of Eq. (39).

The expansion of Eq. (58) refers to small \( 0 < (x - x_c) \ll 1 \) and thus to small absolute values \( g_0 \sqrt{2} = |(x - x_c)/(x - x_c)|/|\sqrt{2} | \) of the vector \( \vec{q}_{0} = |g_0/\sqrt{2}| \vec{e}_{\phi_{AN}^s} \). As mentioned above, that for such a range of \( x \) values near and above the critical hole concentration \( x_c \), the coherent virtual-electron pairing energy per electron \( |\Omega_{s1}(\vec{q}_{BS1}^d)| \) associated with the phase-coherent-pair superconducting order is the energy deviation \( \vec{V}_{s1}^d(\vec{q}_{BS1}^d) \cdot \vec{q}_{0} \) of the expansion \( 58 \) is consistent with such an order being a by-product of the short-range spin correlations. Indeed, the latter are associated with the \( s1 \) fermion pairing energy per spinon \( |\Delta_{s1}(\vec{q}_{BS1}^d)| \), which corresponds to the zeroth-order term of that expansion.

For larger hole concentration values in the range \( x \leq 0 \) \((x - x_c) \ll 1 \) the expansion of Eq. (58) is not valid anymore. However, within our scheme the expression for the zero-temperature coherent virtual-electron pairing energy per electron \( |\Omega_{s1}(\vec{q}_{BS1}^d)| \approx \gamma_d g_0 \sqrt{2} \vec{V}_{s1}^d \) of Eq. (59) multiplied by the suppression coefficient \( \gamma_d \) of Eq. (55) is. Indeed, that equation refers to the limit \( 0 < (x - x_c) \ll 1 \) for which \( \gamma_d = 1 \). For virtual-electron pairs of momentum \( \vec{q} \approx \vec{q}_{BS1}^d \) the full virtual-electron pairing energy per electron has the following form,

\[
|\Delta_{s1}(\vec{q}_{BS1}^d)| = |\Delta_{s1}(\vec{q}_{BS1}^d)| + |\Omega_{s1}(\vec{q}_{BS1}^d)| \approx |\Delta||\cos 2\phi| + |\Omega||\sin 2\phi|.
\] (60)

For small \( 0 < (x - x_c) \ll 1 \) this energy reads \( |\Delta_{s1}(\vec{q}_{BS1}^d) - \vec{q}_{0})| \), as given in Eq. (58). The form of the pairing energy \( \Omega_{s1}(\vec{q}_{BS1}^d) \) is consistent with virtual-electron-pair pairing having a \( d \)-wave character and involving both spin-singlet spinon \( s1 \) fermion pairing and \( c \) fermion pairing. For the normal-state ground states at finite magnetic field \( H \) considered in Section III-E the virtual-electron pairs exist yet loose phase coherence. Then the virtual-electron pairing energy per electron equals for virtual-electron pairs of momentum \( \vec{q} \approx \vec{q}_{BS1}^d \) the spin-singlet spinon \( s1 \) fermion pairing energy for the same momentum, \( |\Delta_{s1}(\vec{q}_{BS1}^d)| \).

It follows from the above analysis that for hole concentrations in the range \( x \leq 0 \) \((x - x_c) \ll 1 \) the zero-temperature absolute value \( 2|\Omega| T=0 \) defined by Eqs. (48) and (49) of the order parameter introduced in Eq. (10) associated with the phase-coherent-pair superconducting order reads \( 2|\Omega| |T=0 = 2|\Omega_{s1}(\vec{q}_{BS1}^d)| \). It corresponds to the maximum magnitude of the coherent virtual-electron pairing energy. Such a maximum magnitude refers to virtual-electron pairs of momentum at the \( s1 \) boundary line and whose auxiliary momentum points in the nodal directions. We recall that the maximum magnitude of the zero-temperature spin-singlet spinon \( s1 \) fermion pairing energy \( 2|\Delta_{s1}(\vec{q}_{BS1}^d)| \) is \( 2|\Delta_{s1}(\vec{q}_{BS1}^d, AN) \approx 2|\Delta_{s1}(\vec{q}_{BS1}^d)| \) corresponds to momenta belonging to the \( s1 \) boundary line and whose auxiliary momenta \( \vec{q}_{AN}^{d} \) point instead in the anti-nodal directions.

For hole concentrations in the range \( x \leq 0 \) \((x - x_c) \ll 1 \) the momenta \( \vec{q} \approx \vec{q}_{BS1}^d \) at or near the \( s1 \) boundary line, and temperatures \( T \) smaller than \( T_c \) the coherent part \( |\Omega_{s1}(\vec{q})| \) of the pairing energy per electron reads,

\[
|\Omega_{s1}(\vec{q})| = |\Omega_{s1}(-\vec{q})| = |\Omega_{s1}(\vec{q})| = |\gamma_d \vec{V}_{s1}^d(\vec{q}) \cdot \vec{q}_{0} = \gamma_d g_0 \sqrt{2} \vec{V}_{s1}^d(\vec{q})| = g_0 \vec{e}_{\phi_{AN}^s} \approx \vec{q}_{BS1}^d.
\] (61)

It involves the general amplitude \( g_0 \) rather than its \( T = 0 \) magnitude \( \bar{g}_0 = (x - x_c)/(x + x_c - x_c) \) of Eq. (42).

The \( c - s1 \) fermion residual interactions are behind the effective pairing coupling of the two \( c \) fermions of hole momentum \( \vec{q}^h \) and \(-\vec{q}^h \) such that \( \vec{q}^h \approx \vec{q}_{BS1}^d \) is at or near the \( c \) Fermi line. Such \( c \) fermion pairs are associated with the pairing energy per electron \( |\Omega_{c1}(\vec{q})| \approx \gamma_d \vec{V}_{c1}^d(\vec{q}) \cdot \bar{g}_0 \) of phase-coherent virtual-electron pair configurations of momentum \( \vec{q} \approx \vec{q}_{BS1}^d \) at or near the \( s1 \) boundary line. The short-range spin correlations provide the energy \( |\Omega_{s1}(\vec{q})| \)
through the interactions of the s1 fermion of momentum \( \vec{q} \approx \vec{q}_{\text{Be1}} \), energy \( \epsilon_{s1}(\vec{q}_{\text{Be1}}) \approx -|\Delta_{s1}(\vec{q}_{\text{Be1}})| \), and velocity \( \vec{V}_{s1}^{\Delta}(\vec{q}) \approx -\nabla_{\vec{q}}|\Delta_{s1}(\vec{q})| \) with the two c fermions.

The vector \( \vec{g}_{s0} \) plays the role of coupling constant of the c fermion-pair degrees of freedom to the s1 fermion of velocity \( \vec{V}_{s1}^{\Delta}(\vec{q}) \). It is through it that the pairing energy \( |\Omega_{s1}(\vec{q})| \approx \gamma_{d} \vec{V}_{s1}^{\Delta}(\vec{q}) \cdot \vec{g}_{s0} \) needed for the c fermion effective coupling and corresponding coherent virtual-electron pairing is supplied to the two c fermions. The absolute value \( g_{0}/\sqrt{2} \) of the vector \( \vec{g}_{s0} \) involves the amplitude \( g_{0} = g_{/1}/g_{1} \) of Eq. (34). It measures the relative strength of the coherent virtual-electron pairing and s1 fermion spinor pairing. Indeed, the magnitudes of the amplitudes \( g \) and \( g_{1} \) also given in Eq. (34) provide a measure of the strengths of the coherent virtual-electron pairing and s1 fermion spinor pairing, respectively.

As confirmed in Section IV-D, breaking under one-electron excitations of the virtual-electron pairs considered here refers to removal or addition of electrons from and to the Fermi line. Therefore, virtual-electrons of s1 band momentum at or near the s1 boundary line are within the electron representation at or near the Fermi line, respectively. Consistently, the superconducting fluctuations lead to an additional contribution \( |\Omega_{s1}(\vec{q}_{\text{Be1}})\rangle \approx |\Omega|\sin 2\phi \) to the anisotropic term \( \delta E_{F} = |\Delta_{s1}(\vec{q}_{\text{Be1}}^{\text{d}})| \) of the Fermi energy introduced in Ref. A, which appears in the spectra given in Eq. (20). Hence for the present VEP quantum liquid the Fermi energy reads,

\[
\delta E_{F} = \mu + \delta E_{F}(\phi); \quad \mu \approx \mu^{0} + W_{c}^{\text{h}},
\]

\[
\delta E_{F}(\phi) = -\epsilon_{s1}(\vec{q}_{\text{Be1}}^{\text{d}}) = |\Delta_{s1-c\text{-el}}(\vec{q}_{\text{Be1}}^{\text{d}})| = |\Delta_{s1}(\vec{q}_{\text{Be1}}^{\text{d}})| + |\Omega_{s1}(\vec{q}_{\text{Be1}}^{\text{d}})| \approx |\Delta| \cos 2\phi + |\Omega| \sin 2\phi,
\]

where the superconducting energy scale \( |\Omega| \) is provided in Eq. (41) and \( W_{c}^{\text{h}} \) is the ground-state c fermion-hole energy bandwidth given in Eq. (A11) of Appendix A. For intermediate \( U/4t \) values approximately in the range \( U/4t \in (u_{0}, u_{1}) \) the \( W_{c}^{\text{h}} \) expression provided in that equation is a very good approximation for hole concentrations \( x \in (x_{c}, x_{1}) \) and a reasonably good approximation for the range \( x \in (x_{1}, x_{s}) \). Rather than the square-lattice quantum-liquid expression \( \mu \approx \mu^{0} + W_{c}^{\text{h}} \), the expression \( \mu \approx \mu^{0} + W_{c}^{\text{h}} \) given here for the chemical potential refers to the VEP quantum liquid. In it \( \mu^{0} = \mu^{0} + \delta \mu \) where \( \mu^{0} = \lim_{x_{c} \to 0} \mu \) is one half the Mott-Hubbard gap of Eq. (A3) of Appendix A and the shift \( \delta \mu \) generated by the hole trapping effects is given in Eq. (51) of Appendix B.

As discussed in Ref. A, in the Fermi energy expression \( E_{F} = \mu + \delta E_{F}(\phi) \) the chemical potential \( \mu \) arises from the isotropic c fermion energy dispersion and \( \delta E_{F} = |\epsilon_{s1}(\vec{q}_{\text{Be1}}^{\text{d}})| = |\Delta_{s1-c\text{-el}}(\vec{q}_{\text{Be1}}^{\text{d}})| \) stems from the anisotropic s1 fermion energy dispersion. The extra term \( |\Omega_{s1}(\vec{q}_{\text{Be1}}^{\text{d}})| \approx |\Omega| \sin 2\phi \) appearing for \( x \in (x_{c}, x_{s}) \) in the expression of the anisotropic Fermi energy term \( \delta E_{F} \) given in Eq. (62) does not change the physics discussed in Section III-A. In turn, for \( x \in (x_{0}, x_{c}) \) and \( T = 0 \) the equality \( \delta E_{F} = |\Delta_{s1-c\text{-el}}(\vec{q}_{\text{Be1}}^{\text{d}})| = |\Delta_{s1}(\vec{q}_{\text{Be1}}^{\text{d}})| \) holds. The anisotropic Fermi energy term \( \delta E_{F} \) plays an important role in the VEP quantum-liquid physics. Its maximum magnitude \( \max \{ \delta E_{F} \} = |\Delta| \) determines and equals that of the anti-nodal one-electron gap \( \delta E_{F}^{\text{AN}} = \max \{ \delta E_{F} \} = |\Delta| \). From the use of the critical-temperature expressions given in Eq. (42) and (43) and \( \delta E_{F}(\phi) \) expression provided in Eq. (62), \( \delta E_{F}^{\text{AN}} \) can be expressed as a function of \( T_{c}/T_{c}^{\max} \) as,

\[
\delta E_{F}^{\text{AN}} = \max \{ \delta E_{F} \} = |\Delta|(x) = |\Delta|(x_{op})(1 - \sgn(x - x_{op})\sqrt{1 - T_{c}(x)/T_{c}^{\max}}); \quad |\Delta|(x_{op}) = \gamma_{c} \frac{\Delta_{0}}{2}.
\]

C. The general virtual-electron strong effective coupling pairing mechanism

According to our approach, c fermion strong effective coupling is that whose breaking upon one-electron excitations leads to sharp features in the one-electron spectral function. It is a necessary but not sufficient condition for the occurrence of coherent pairing in the initial ground state. The absolute value \( q^{h} = |\vec{q}^{h}| \) of the hole momenta \( \vec{q}^{h} \) and \(-\vec{q}^{h}\) of two c fermions with strong effective coupling belongs to a well-defined range \( q^{h} \in (q_{c}^{h}, q_{c}^{h}) \). Here \( q_{c}^{h} \) is the maximum hole momentum given below for which there is c fermion strong effective coupling. We denote by \( Q_{cc}^{h} \) and \( Q_{cc}^{h} \) the corresponding c and s1 band momentum domains for which the c and s1 fermions, respectively, participate in virtual-electron pairs with strong effective coupling. \( Q_{cc}^{h} \) and \( Q_{cc}^{h} \) correspond to a set of \( c - sc \) and \( s1 - sc \) lines, respectively. As further discussed below, within the present scheme:

i) Only strong effective coupling, i.e. that whose breaking upon one-electron excitations leads to sharp features in the one-electron spectral function, can lead to coherent virtual-electron pairing.

ii) For each c band hole momentum absolute value \( q^{h} \) in a well-defined range \( q^{h} \in (q_{c}^{h}, q_{c}^{h}) \) there is a c band approximately circular \( c - sc \) line of radius \( q^{h} \) centered at \(-\vec{q}^{h}\). Such a range refers to the c fermions of hole momenta \( q^{h} \) and \(-\vec{q}^{h}\) whose effective coupling is strong. There is no one-to-one correspondence between the c fermion pairs and the s1 fermions of strongly coupled virtual-electron pairs. As justified in this section, the strong effective coupling of c fermions of hole momenta \( q^{h} \) and \(-\vec{q}^{h}\) at or near a \( c - sc \) line of radius \( q^{h} \) results from interactions with s1 fermions.
of momentum $\vec{q}$ at or near a corresponding $s_1 - sc$ line centered at the $s_1$ band zero momentum. The four nodal momenta belonging to the latter line have the same absolute value $q^N_{arc} = q^N_{arc}(q^h)$, which is uniquely determined by the radius magnitude $q^h$ of the corresponding $c - sc$ line. Here $q^N_{arc} \in (q^N_{arc}, q^N_{arcB})$ where $q^N_{arc} = q^N_{ec}$ and $q^N_{arc} = q^N_{bs}$ correspond to $q^h = q^h_{ec}$ and $q^h = q^h_{bs}$, respectively. For $q^N_{arc} < q^N_{bs}$ and thus $q^h > q^h_{sc}$, the $s_1 - sc$ line is constituted by four disconnected line arcs centered at zero momentum and crossing perpendicularly the four nodal lines, respectively, whose auxiliary momenta refer to Fermi angles $\phi = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$. The angular width of these four line arcs is a decreasing function of $q^h$, which vanishes at $q^h = q^h_{sc}$. As found below, in that limit these four line arcs reduce to four nodal momenta whose absolute value is $q^N_{ec}$. The function $q^N_{arc} = q^N_{arc}(q^h)$ and the magnitudes of $q^h_{ec}$ and $q^h_{bs}$ are given above. That the number of $s_1$ fermions that mediate the strong effective coupling of $c$ fermion pairs of hole momenta $q^h_{ec}$ and $-q^h_{ec}$ decreases upon increasing $q^h$, vanishing for $q^h > q^h_{ec}$, is consistent with for the range $q^h \in (q^h_{sc}, q^h_{ec})$ the strength of such effective coupling being a decreasing function of the $c$ fermion absolute energy, $|\epsilon_c(q^h)| = (|q^h|^2 - |q^h_{sc}|^2)/2m^*_c$.

In turn, weak effective pairing coupling between $c$ fermions is that whose breaking upon one-electron excitations leads to broad flat features. Weak effective pairing coupling never contributes to coherent virtual-electron pairing. Strong effective pairing coupling may or may not lead to such a coherent pairing.

In the following we justify and supplement the information reported in point (ii). The ground-state $ec$-pairing line is the $c - sc$ line of largest radius $q^h_{ec}$. It separates in the $c$ momentum band the $c$ fermions with strong effective pairing coupling from those with only weak effective pairing coupling, respectively. (The designation $ec$-pairing line follows from it separating $c$ fermions with two different types of effective coupling $[ec]$.) The hole momentum domain $Q^c_{ec}$ refers to the set of $c - sc$ lines. It is limited by the $c$ Fermi line and $ec$-pairing line, respectively. As confirmed below in Section IV-E, the energy scale,

$$W_{ec} = \frac{|q^h_{ec}|^2 - |q^h_{sc}|^2}{2m^*_c} = \frac{4\Delta_0}{\gamma_c},$$

is the maximum magnitude of both the energy bandwidth of the superconducting-ground-state sea of $c$ fermions contributing to coherent pairing and the energy bandwidth corresponding to the hole momentum domain $Q^c_{ec}$ considered here.

Alike the smaller $c - sc$ lines enclosed by it, for the interaction range $U/4t \in (u_0, u_1)$ the $ec$-pairing line is nearly circular for hole concentrations $x \in (x_c, x_a)$. It is a fairly good approximation to consider that it remains circular for $x \in (x_c, x_a)$. The energy range $|\epsilon_c(q^h)| = |\epsilon_c(-q^h)| \in (0, W_{ec})$ of $c$ fermions having strong effective coupling in the ground state corresponds to the uniquely defined hole momentum range $q^h \in (q^h_{sc}, q^h_{ec})$ reported in (ii). For these $c$ fermions, the hole momenta $q^h_{ec}$ belonging to the $ec$-pairing line are those of largest absolute value $q^h_{ec}$. For the interaction range $U/4t \in (u_0, u_1)$ and $x \in (x_c, x_a)$ it is approximately given by,

$$q^h_{ec} = \sqrt{1 + \frac{W_{ec}}{|q^h_{sc}|^2 - q^h_{sc}}} q^h_{ec} = \sqrt{1 + \frac{\Delta_0}{x(x_a - x_c)\pi t}} q^h_{ec}.$$  

An interesting property is that for $x \in (x_c, x_a)$ the $c$ band hole momentum area corresponding to $c$ fermions with strong effective pairing coupling in the ground state is independent of $x$ and given by,

$$S^c_{ec} = \pi(|q^h_{ec}|^2 - |q^h_{sc}|^2) = \frac{4\Delta_0}{(x_a - x_c)t}.$$  

It refers to the hole momentum area of the domain $Q^c_{ec}$. Its expression involves only the superconducting dome hole concentration width $(x_a - x_c)$ and the basic energy scales $\Delta_0$ and $t$.

In Appendix D it is found that for $U/4t \in (u_0, u_1)$ the $ec$-pairing line bandwidth $|\epsilon_c(q^h)|$ of $c$ fermions of momenta $q^h$ and $-q^h$ having strong effective pairing coupling due to residual interactions with $s_1$ fermions of momentum $\vec{q} \in Q^{s_1}_{ec}$ obeys the inequality,

$$|\epsilon_c(q^h)| \leq W_{ec} \left(1 - \frac{|\Delta s_1(\vec{q})|}{|\Delta|}\right) \approx W_{ec} (1 - |\cos 2\phi|); \quad |\Delta s_1(\vec{q})| \approx |\Delta| \cos 2\phi \text{ for } \vec{q} \in Q^{s_1}_{ec}.$$  

For $x \in (x_c, x_a)$ the inequality $W_{ec} < W_P^c$ holds. (Here $W_P^c$ is the energy bandwidth of the ground-state $c$ momentum band filled by $c$ fermions.) Hence only $s_1$ fermions whose auxiliary momenta point in the nodal directions have interactions with strongly coupled $c$ fermions whose energy $|\epsilon_c(q^h)|$ belongs to the whole range $|\epsilon_c(q^h)| \in (0, W_{ec})$.

The inequality (67) is obeyed by the following hole momentum range,

$$q^h \in (q^h_{sc}, q^h_{ec}); \quad q^h_a = a_q q^h_{sc}; \quad a_q = \sqrt{1 + (a_q^2 - 1) \left(1 - \frac{|\Delta s_1(\vec{q})|}{|\Delta|}\right)} \approx \sqrt{1 + (a_q^2 - 1)(1 - |\cos 2\phi|)},$$  

(68)
Equivalent to that inequality are the following inequalities obeyed by the magnitude \(|\Delta s_1(\vec{q})|\) of the \(s_1\) fermion pairing energy per spinon and Fermi angle \(\phi\) defined in Eq. (A9),

\[
|\Delta s_1(\vec{q})| \leq \Delta \left( 1 - \frac{|\epsilon_c(\vec{q}^h)|}{W_{ec}} \right); \quad |\cos 2\phi| \leq \left( 1 - \frac{|\epsilon_c(\vec{q}^h)|}{W_{ec}} \right),
\]

respectively. The second inequality given here involving the Fermi angle \(\phi\) is in turn equivalent to restricting the range \(\phi \in (0, \pi/2)\) to an arc of angular width \(2\phi_{arc}\) centered at \(\phi = \pi/4\),

\[
\phi \in (\pi/4 - \phi_{arc}, \pi/4 + \phi_{arc}); \quad \phi_{arc} = \frac{1}{2} \arcsin \left( \frac{|q_{ec}^h| - |q^h|}{q_{ec}^h - |q^h|} \right) = \frac{1}{2} \arcsin \left( \frac{\epsilon_c(\vec{q}^h)}{W_{ec}} \right); \quad (0, \pi/4) \quad \cdots \quad (70).
\]

Here \(\epsilon_c(\vec{q}^h)\) is the following \(c\) fermion energy dispersion,

\[
\epsilon_c(\vec{q}^h) = W_{ec} + \epsilon_c(\vec{q}^h); \quad 0 \leq \epsilon_c(\vec{q}^h) \leq W_{ec}.
\]

It is the energy of a \(c\) fermion of hole momentum \(\vec{q}^h\) measured from the energy level of the \(ec\)-pairing line. The energy \(\epsilon_c(\vec{q}^h)\) is largest at the \(c\) Fermi line and vanishes at the \(ec\)-pairing line,

\[
\epsilon_c(\vec{q}_{ec}) = 0; \quad \epsilon_c(\vec{q}_{ec}^d) = W_{ec}.
\]

The shape of the \(ec\)-pairing line is for \(x \in (x_c, x_s)\) fully determined by the form of the \(c\) band energy dispersion \(\epsilon_c(\vec{q}^h)\) as follows,

\[
\vec{q}_{ec}^d \in ec \text{ – pairing line } \iff \epsilon_c(\vec{q}_{ec}^d) = 0.
\]

For simplicity we limit our analysis to the quadrant referring to the Fermi angle range \(\phi \in (0, \pi/2)\) and corresponding \(s_1-sc\) line arc whose angular range is that given in Eq. (74), \(\phi \in (\pi/4 - \phi_{arc}, \pi/4 + \phi_{arc})\). To each \(c\) band approximately circular \(c-sc\) line of radius \(q^h\) centered at \(-\vec{q}\) corresponds such a \(s_1\) band \(s_1-sc\) line arc whose momenta \(\vec{q}_{arc}^d\) are approximately given by,

\[
\vec{q}_{arc}^d(\phi) = A_{s_1}^d(\phi) \vec{q}_{arc}(\phi); \quad \vec{q}_{arc}(\phi) \approx \frac{q_{bs}^N}{q_{bs}^N - q_{bs}^N}; \quad q_{arc}(\phi) = q_{bs}(\pi/2 - \phi); \quad \phi \in (\pi/4 - \phi_{arc}, \pi/4 + \phi_{arc}.)
\]

Here \(q_{arc}(\phi) \approx [q_{arc}(\phi)/q_{bs}(\phi)] q_{bs}(\phi)\) is the absolute value of the auxiliary momentum \(\vec{q}_{arc}(\phi), A_{s_1}^d\) is the matrix of Eq. (74), and the \(s_1-sc\) line arc nodal momentum absolute value \(q_{arc}^N = q_{bs}^N(q_{bs}^N)\) is given below. As stated in (ii), for each \(c\) band \(c-sc\) line of radius \(q^h\) in the range \(q^h \in (q_{ec}^h, q_{ec}^h)\) there is for \(\phi \in (0, \pi/2)\) exactly one of such \(s_1\) band \(s_1-sc\) line arcs. The line arc limiting angular widths \(2\phi_{arc} = 0\) and \(2\phi_{arc} = \pi/2\) refer to \(q^h = q_{ec}^h\) and \(q^h = q_{ec}^h\), respectively. The angle range \(\phi \in (\pi/4 - \phi_{arc}, \pi/4 + \phi_{arc})\) of each \(s_1-sc\) line arc runs symmetrical around \(\phi = \pi/4\). The physical importance of the \(s_1-sc\) line arcs follows from the corresponding sharp spectral features line arcs studied below, which emerge in the one-electron removal weight distribution and are observed in experiments on the cuprates.

We note that for \(\phi \in (0, \pi/2)\) the \(s_1\) band auxiliary momenta belong to the quadrant for which their two components are negative. For instance, for momenta near the \(s_1\) boundary line and \(x \in (x_c, x_s)\) this follows from the relation of the \(s_1\) band momentum angle \(\phi_s = \phi + \pi\) of Eq. (11) to the Fermi angle \(\phi\). Indeed the range \(\phi_s \in (\pi, 3\pi/2)\) refers to \(\phi \in (0, \pi/2)\). Consistently with Eq. (73), the \(s_1\) boundary line is for \(\phi \in (0, \pi/2)\) the \(s_1-sc\) line arc whose nodal momentum absolute value is given by \(q_{arc}^N = q_{bs}^N\). It corresponds to the approximately circular \(c\) Fermi line whose radius reads \(q^h = q_{ec}^h\). One confirms below that \(q_{arc}^N = q_{bs}^N\) for \(q^h = q_{ec}^h\). Hence the auxiliary momentum angle \(\phi_s = \phi + \pi\) can be generalized to all \(s_1-sc\) line arcs. Each of such line arcs then refers to an auxiliary momentum angle range \(\phi_s \in (5\pi/4 - \phi_{arc}, 5\pi/4 + \phi_{arc})\) corresponding to \(\phi \in (\pi/4 - \phi_{arc}, \pi/4 + \phi_{arc})\).

As mentioned above, there is one-to-one correspondence between the \(c\) fermion pairing lines and corresponding \(s_1\) fermions participate in virtual-electron-pair pairs with strong effective coupling. Concerning general \(c-s_1\) fermion interactions, there are in average two \(c\) fermions for each \(s_1\) fermion. However, only part of these \(c\) fermions and \(s_1\) fermions participate in strongly coupled virtual-electron pairs. There are four classes of strongly coupled virtual-electron pairs associated with the four disconnected \(s_1-sc\) line arcs, respectively. For any strongly coupled virtual-electron pair configuration whose auxiliary momentum Cartesian components are \([q_{arc} \cos \phi_s, q_{arc} \sin \phi_s]\) where \(\phi_s = \phi + \pi\) there are in the other three disconnected \(s_1-sc\) line arcs three strongly coupled virtual-electron pair configurations, respectively, with the same energy and momentum absolute value \(q_{arc}\). The \(s_1\) band auxiliary momenta of the four strongly coupled virtual electron pair configurations under consideration have Cartesian components \(\pm [q_{arc} \cos \phi_s, q_{arc} \sin \phi_s]\) and \(\pm [-q_{arc} \sin \phi_s, q_{arc} \cos \phi_s]\). The set of four \(s_1-sc\) line arcs that such four momenta belong to are associated
with four sub-domains $Q_{ec}^{±,+s1}$ and $Q_{ec}^{−,−s1}$, respectively, of the s1 band momentum domain $Q_{ec}^{N}$ of strongly coupled virtual-electron pair configurations.

The point is that the strongly coupled virtual electron pair configurations whose auxiliary momentum has Cartesian components $±[q_{arc} \cos \phi_{s1}, q_{arc} \sin \phi_{s1}]$ couple to different $c$ fermion pairs than those of Cartesian components $±[-q_{arc} \sin \phi_{s1}, q_{arc} \cos \phi_{s1}]$. Hence one can divide the c band domain $Q_{ec}^{N}$ into two sub-domains that in average have half of the $c$ fermion pairs each. We call them $Q_{ec}^{+}$ and $Q_{ec}^{-}$, respectively. The strong effective coupling of $c$ fermions of hole momenta belonging to $Q_{ec}^{+}$ and $Q_{ec}^{-}$ results from interactions with s1 fermions whose momenta belong to $Q_{ec}^{±,+s1}$ and $Q_{ec}^{±,−s1}$, respectively. Furthermore, symmetry arguments concerning the relationship of the two strongly coupled virtual-electron pairs with the same energy, momentum absolute value, and momentum direction but different senses suggest that the number of s1 fermions of each of the two s1 band sub-domains $Q_{ec}^{+,+s1}$ and $Q_{ec}^{−,−s1}$ is one half that of $c$ fermions in the $c$ band domain $Q_{cc}^{N}$ (and $Q_{cc}^{-}$). Therefore, for each of the four sub-domains $Q_{ec}^{+,+s1}$ and $Q_{ec}^{−,−s1}$ the s1 band momentum area $S_{s1}^{N}$ corresponding to s1 fermions of virtual-electron pairs with strong effective coupling in the ground state is for $x \in (x_c, x_*)$ one fourth that of Eq. [66].

$$S_{s1}^{N} = \frac{\Delta_{0}}{(x_s - x_c) t}. \quad (75)$$

It follows that the momentum area of the whole domain $Q_{ec}^{N}$ equals that of $Q_{ec}^{c}$ and reads $4S_{s1}^{N} = 4\Delta_{0}/(x_s - x_c) t$. It refers to the four sets of concentric $s1 - sc$ line arcs centered at $0$. The present results are valid for $U/4t \in (u_0, u_1)$. The simultaneous validity of the equalities [65] for the whole c band domain $Q_{cc}^{N}$ and [76] for each of the four sub-domains of $Q_{ec}^{N}$ requires the hole concentration to obey the inequality $x < x_{ec}^{max} = 1 - [2\Delta_{0}/(\pi^2(x_s - x_c) t)]$. Hence $x_s$ must be smaller than or equal to $x_{ec}^{max}$. This requires that $\Delta_{0}/t < [\pi^2/2](1 - x_s)/(x_s - x_c)$. For $x_s = 0.5$ and $x_*= 0.27$ this gives approximately $\Delta_{0}/t < 0.79$. The fulfillment of this inequality assures that for $x \in (x_c, x_*)$ the total numbers of $c$ fermions and s1 fermions are larger than those of $c$ fermions and s1 fermions, respectively, participating in strongly coupled virtual-electron pairs. For $U/4t = 1.525$ and the t magnitude $t \approx 195$ meV found in Ref.12 to be appropriate to the four representative hole doped cuprates other than LSCO for which $x = 0.5$ and $x_*= 0.27$ one finds $\Delta_{0} \approx 84$ meV, so that $\Delta_{0}/t \approx 0.28$. In turn, the LSCO cation-randomness effects considered in that section lessen the $\Delta_{0}$ magnitude to $\Delta_{0} \approx 42$ meV, so that $\Delta_{0}/t \approx 0.14$ for that random alloy.

That in average the number of s1 fermions whose momenta $\vec{q}$ belong to the domain $Q_{ec}^{N}$ is larger than that of $c$ fermion pairs of hole momenta $\vec{q}^{h}$ and $\vec{q}^{b}$ belonging to $Q_{sc}^{N}$, is consistent with the dependences on $q^h$ of the s1–sc and c–sc lines length. The c–sc line length $\approx 2\pi q^h$ increases upon increasing $q^h$. In contrast, that of the s1–sc line,

$$L_{s1}(q_{arc}^{N}) = 4l_{s1}(q_{arc}^{N}) \equiv 2(1 - \sin 2\phi_{arc}) \sqrt{[q_{B1}^{AN}]^2 - [q_{B1}^{N}]^2} = 2\frac{q_{B1}^{AN} - q_{B1}^{N}}{q_{B1}^{AN} - q_{B1}^{N}} \sqrt{[q_{B1}^{AN}]^2 - [q_{B1}^{N}]^2}, \quad (76)$$

decreases. Here $l_{s1}(q_{arc}^{N})$ is the approximate length of each of the four $s1 - sc$ line arcs, $q_{B1}^{N}$ is the minimum absolute value of the nodal $s1 - sc$-line-arc momenta $q_{sc}^{N}$ and their auxiliary momenta $q_{s1}^{N} = q_{arc}(\pi/4)$ of Eq. [74], and $q_{B1}^{AN}$ and $q_{B1}^{N}$ are the absolute values of Eq. [E1] of Appendix E of the nodal and anti-nodal s1 boundary-line momenta $q_{sc}^{N}$ and $q_{sc}^{AN}$, respectively. According to the definition of Ref.12, their corresponding auxiliary momenta $q_{s1}^{N}$ and $q_{s1}^{AN}$, respectively, have for the quadrant for which $\phi \in (0, \pi/2)$ so that $\phi_{s1} \in (\pi, 3\pi/2)$ and thus $q_{s1} \leq 0$ and $q_{0x2} \leq 0$ the Cartesian components given in Eq. [AS] of Appendix A. The x dependence of $q_{B1}^{AN}$ and $q_{B1}^{N}$ is discussed in Appendix E.

Hence the number of s1 fermions available to supply the energy needed for $c$ fermion strong effective pairing coupling decreases upon increasing the $c$–sc line radius $q^h$. For $q^h$ equal or close to $q_{B1}^{N}$, there are plenty s1 fermions to supply the energy needed for the $c$ fermion strong effective coupling. In the opposite limit of $q^h$ tending to $q_{B1}^{N}$ the c–sc line length $2\pi q_{B1}^{h}$ is maximum. However, the length $L_{s1}(q_{arc}^{N})$ given in Eq. [76] tends to zero, as each s1–sc line arc becomes a single discrete momentum value. Therefore, then only two sets of two s1 fermions whose auxiliary momenta have Cartesian components $\pm[q_{arc}^{N}/\sqrt{2}, q_{arc}^{N}/\sqrt{2}]$ and $\pm[-q_{arc}^{N}/\sqrt{2}, q_{arc}^{N}/\sqrt{2}]$ are available to supply the energy needed for the ground-state strong effective coupling of c fermions of hole momenta belonging to $Q_{ec}^{+}$ and $Q_{ec}^{-}$, respectively. Consistently, there is no strong effective pairing coupling for $q^h > q_{ec}^{N}$ and $q_{arc} < q_{ec}^{N}$.

The momentum area $S_{sc}^{N}$ of Eq. [77] is approximately given by,

$$S_{sc}^{N} = \int_{q_{ec}^{N}}^{q_{B1}^{N}} dq_{arc} l_{s1}(q_{arc}^{N}) \approx \langle q_{B1}^{N} - q_{ec}^{N} \rangle \sqrt{[q_{B1}^{AN}]^2 - [q_{B1}^{N}]^2}. \quad (77)$$

Here $l_{s1}(q_{arc}^{N})$ is the approximate length of each s1–sc line arc given in Eq. [76]. From the use of the two $S_{sc}^{N}$
expressions provided in Eqs. (75) and (77), respectively, one straightforwardly arrives to,

\[
q^N_{ec} \approx q^N_{Bn1} - \frac{\Delta_0}{(x_s - x_c) \sqrt{|q^{AN}_{Bn1}|^2 - |q^{N}_{Bn1}|^2 t}}.
\]

(78)

On combining this expression with the auxiliary \(s1 - sc\)-line momentum expression of Eq. (74) one confirms that,

\[
\begin{align*}
q_{arc}^{N}(\phi) &= q_{Bn1}^{N}(\phi) \quad \text{for } q_{arc}^{N} = q^{N}_{Bn1}, \quad \phi \in (0, \pi/2), \\
&= q_{ec}^{N}(\phi) \quad \text{for } q_{arc}^{N} = q^{N}_{ec}, \quad \phi = \pi/4.
\end{align*}
\]

(79)

Here \(q_{ec}^{N}\) is the auxiliary momentum of the \(s1 - sc\)-line momentum \(q_{ec}^{dN} = A_{ec}^{d} q_{ec}^{N}\) of smallest absolute value (78).

The absolute value \(q_{arc}^{N} \in (q_{ec}^{N}, q_{Bn1}^{N})\) of the nodal momentum belonging to a given \(s1 - sc\) line arc is obtained by replacing in expression (78) the energy scale \(\Delta_0\) by the energy spectrum \(|\Delta_{ec}(\tilde{q}^h)|\) defined in Eq. (D1) of Appendix D. As given in Eq. (D2) of that Appendix, its magnitude varies from \(|\Delta_{ec}(\tilde{q}^h)| = 0\) for \(\tilde{q}^h = q_{ec}^h\) to \(|\Delta_{ec}(\tilde{q}^h)| = \Delta_0\) for \(\tilde{q}^h = q_{ec}^h\). This is consistent with \(c\) fermions with strong effective coupling and hole momenta at or near the \(c\) Fermi line interacting with \(s1\) fermions of momenta at or near the \(s1\) boundary line. In turn, \(c\) fermions with strong effective coupling and hole momenta at or near the \(ec\) pairing line interact with the \(s1\) fermion whose nodal momentum has absolute value \(q_{dec}\) given in Eq. (78). In the limit \(q_{arc}^{N} \to q_{ec}^{N}\) the \(s1 - sc\) line arc of minimum absolute nodal momentum \(q_{arc}^{N}\) has vanishing length and for \(\phi \in (0, \pi/2)\) and thus \(\phi_{n1} \in (\pi, 3\pi/2)\) reduces to a single discrete nodal momentum \(q_{ec}^{dN} = A_{ec}^{d} q_{ec}^{N}\).

Replacement in expression (78) of the energy scale \(\Delta_0\) by the energy spectrum \(|\Delta_{ec}(\tilde{q}^h)|\) then leads to,

\[
q_{arc}^{N} \approx q_{Bn1}^{N} - \frac{|\Delta_{ec}(\tilde{q}^h)|}{(x_s - x_c) \sqrt{|q^{AN}_{Bn1}|^2 - |q^{N}_{Bn1}|^2 t}} = q_{Bn1}^{N} - \frac{|\tilde{q}^h|^2 - |q_{ec}^h|^2}{4 \sqrt{|q^{AN}_{Bn1}|^2 - |q^{N}_{Bn1}|^2 t}} \pi; \quad \tilde{q}^h \in (q_{ec}^{h1}, q_{ec}^{h2}).
\]

(80)

These expressions are equivalent to,

\[
q_{arc}^{N} \approx q_{Bn1}^{N} - (1 - \sin 2\phi_{arc})(q_{Bn1}^{N} - q_{ec}^{N}).
\]

(81)

The angle \(\phi_{arc}\) of Eq. (80) can be expressed in terms of the nodal momentum \(q_{arc}^{N}\) of Eq. (80) as follows,

\[
\phi_{arc} = \frac{1}{2} \arcsin \left( \frac{q_{arc}^{N} - q_{ec}^{N}}{q_{Bn1}^{N} - q_{ec}^{N}} \right) \in (0, \pi/4).
\]

(82)

As given in Eq. (81), the absolute value \(q_{arc}^{N}\) provided in Eq. (80) of the nodal momentum belonging to the \(s1 - sc\) line arc can be expressed as a function of the angle \(\phi_{arc}\).

Consistently with the limiting behaviors reported in Eq. (79), the \(s1 - sc\) line arc considered here becomes for \(q_{arc}^{N} = q_{ec}^{N}\) and thus \(\tilde{q}^h = q_{ec}^{h}\) a single discrete nodal momentum \(q_{ec}^{dN}\) whose absolute value is given in Eq. (78). Hence \(q_{arc}^{d} = q_{ec}^{dN}\). In the opposite limit reached at \(q_{arc}^{N} = q_{Bn1}^{N}\) and then \(q_{arc}^{N} = q_{ec}^{N}\) the \(s1 - sc\) line arc becomes the \(s1\) boundary line of angular range \(\phi \in (0, \pi/2)\), so that \(q_{arc}^{d} = q_{Bn1}^{N}\). Between these two limits the angular range \(\phi \in (\pi/4 - \phi_{arc}, \pi/4 + \phi_{arc})\) of the \(s1 - sc\) line arc increases from a single angle \(\phi = \pi/4\) for \(q_{arc}^{N} = q_{ec}^{N}\) and \(q_{arc}^{d} = q_{ec}^{dN}\) to a maximum angular width \(\phi \in (0, \pi/2)\) for \(q_{arc}^{N} = q_{Bn1}^{N}\) and \(q_{arc}^{d} = q_{Bn1}^{N}\). This corresponds to auxiliary momentum angular ranges increasing from a single angle \(\phi_{n1} = 5\pi/4\) to a maximum angular width \(\phi_{n1} \in (\pi, 3\pi/2)\), respectively.

The inequality (77) is equivalent to restricting the energy dispersions \(\epsilon^{ce}(\tilde{q}^{h})\) of Eq. (71) and \(\epsilon^{ce}(\tilde{q}^{h})\) to the following ranges,

\[
\epsilon^{ce}(\tilde{q}^{h}) \in (V^{eff}_{ce}(\tilde{q}), W_{ec}); \quad \epsilon^{ce}(\tilde{q}^{h}) \in (-[W_{ec} - V^{eff}_{ce}(\tilde{q})], 0),
\]

(83)

respectively. Here the energy \([W_{ec} - V^{eff}_{ce}(\tilde{q})]\) is positive or vanishing. For a \(c\) fermion of a \(c\) strongly coupled pair whose energy \(\epsilon^{ce}(\tilde{q}^{h})\) is measured from the \(ce\)-pairing line the energy scale,

\[
V^{eff}_{ce}(\tilde{q}) = \frac{W_{ec}}{|\Delta|} |\Delta_{s1}(\tilde{q})|; \quad V^{eff}_{ce}(\tilde{q}) \in (0, W_{ec}),
\]

(84)

with limiting values,

\[
V^{eff}_{ce}(q^{N}_{dec}) = V^{eff}_{ce}(q_{Bn1}^{N}) = 0 \quad V^{eff}_{ce}(q_{Bn1}^{AN d}) = W_{ec},
\]

plays the role of an effective potential energy.
Since the momentum $\vec{q}$ of a strongly coupled virtual-electron pair configuration equals that of the corresponding s1 fermion, $\vec{q}$ is at or near a s1 – sc line arc. It follows from Eq. (67) that the energy $E_{v-el}(\vec{q})$ given in Eq. (54) of the corresponding virtual-electron pair obeys the inequality,

$$E_{v-el}(\vec{q}) \leq E_1(\phi) + |\varepsilon_s(\vec{q}_{darc})|; \quad E_1(\phi) = 2W_{ec}(1 - |\cos 2\phi|).$$

(85)

Here $q_{darc}^d \pm$ is a s1 – sc-line arc momentum $q_{darc}$ pointing in the directions defined by the angle $\phi = \pi/4 \pm \phi_{arc}$. Hence breaking virtual-electron pairs of momentum $\vec{q} = \vec{q}_{darc}$ belonging to a s1 – sc line arc under one-electron excitations leads to sharp spectral features provided that their energy obeys the inequalities of Eq. (85). Moreover, only such virtual-electron pairs may have phase coherence. Note that the Fermi angle range $\phi \in (\pi/4 - \phi_{arc}, \pi/4 + \phi_{arc})$ of a s1 – sc line arc has been constructed to inherently the inequalities of Eqs. (55) and (56) and the energy ranges of Eq. (83) being obeyed.

For $U/4t \in (u_0, u_1)$ and $x \in (x_A, x_c)$ where $x_A \approx x_c/2$ the maximum magnitude of the energy $|\varepsilon_s(\vec{q}_{darc})|$ is often below the resolution of experiments on hole doped cuprates. In that case the inequality (85) approximately reads,

$$E_{v-el}(\vec{q}_{darc}) \leq E_1(\phi) = 2W_{ec}(1 - |\cos 2\phi|).$$

(86)

D. Virtual-electron pairing breaking under electron removal: Signatures of the virtual-electron pairs in the one-electron spectral-weight distribution

Taking into account the ranges of Eqs. (55), (55), and (55) and using the general energy functional of Eq. (54), one finds that the maximum energy of the one-electron removal processes that lead to sharp spectral features is reached as follows: Upon breaking a virtual-electron pair, a c fermion pair whose c fermions have hole momenta $q^{h'}$ and $-q^{h'}$, at or near the c – sc line corresponding to the energy $|\varepsilon_c(\pm q^{h'})| = W_{ec}(1 - |\cos 2\phi|)$ is broken. Simultaneously, a spinon pair of a s1 fermion whose momentum $\vec{q} = \vec{q}_{darc}$ is at or near the s1 – sc line arc whose nodal momentum absolute value $q_{darc}^N$ is given by Eq. (80) with $q^h = q^{h'} \equiv |q^{h'}|$ is broken as well. The maximum energy condition imposes that such a s1 – sc line arc momentum corresponds to a maximum $\pi/4 - \phi_{arc}$ or maximum $\pi/4 + \phi_{arc}$ Fermi angle $\phi$ magnitude. Furthermore, one of the two involved c fermions recombines with one spinon within the removed electron. The spinon left behind then leads to the emergence of one s1 band hole, whose momentum $q = q_{darc}^d$ is that of the broken s1 fermion. Finally, the second c fermion goes over to the c line Fermi line. The resulting energy spectrum is straightforwardly obtained by use of the hole-momentum-distribution-function deviations,

$$\delta N_c^h(\vec{q}^{h}) = [\delta q^{h}, -\vec{q}^{h}, + \delta q^{h}, \vec{q}^{h}, - \delta q^h, \vec{q}_p^{h}] ; \quad \delta N_c^h(\vec{q}) = \delta_q, q_{darc},$$

(87)

in the energy functional of Eq. (54). It is given by the energy scale $E_1(\phi)$ of Eqs. (55) and (55). Such an energy scale appears on the right-hand side of the inequality $E_{v-el}(\vec{q}_{darc}) \leq E_1(\phi)$ also provided in the former equation. It equals the maximum magnitude of the virtual-electron pair energy $E_{v-el}(\vec{q}_{darc})$ of strongly coupled virtual-electron pairs of momentum $\vec{q} = \vec{q}_{darc}$. It follows that the one-electron spectrum $E_1(\phi)$ corresponds to a line in the $(\phi, \omega)$ plane associated with the boundary separating the one-electron sharp features from broad incoherent features. Such a boundary refers to the equality $E_{v-el}(\vec{q}_{darc}^d \pm) \approx E_1(\phi)$ associated with virtual-electron pairs whose s1 – sc-line arc momentum has $\phi$ limiting magnitudes given by $\pi/4 \pm \phi_{arc}$. In turn, the above inequality $E_{v-el}(\vec{q}_{darc}^d \pm) \leq E_1(\phi)$ of Eq. (67) corresponds to a $(\phi, \omega)$ plane domain bounded by it and associated with virtual-electron-removal excitations that lead to sharp spectral features provided that their effective coupling is strong. This implies that their energy obeys such an inequality. It corresponds to a $(\phi, \omega)$ plane domain constituted by a set of sharp one-electron removal spectral features lines.

There is an one-to-one correspondence between each sharp one-electron removal spectral feature line arc (1-el-sharp-feature line arc) and a virtual-electron-pair s1 band s1 – sc line arc. Specifically, for $x \in (x_A, x_c)$ and $U/4t \in (u_0, u_1)$ the energy scale $|\varepsilon_s(\vec{q}_{darc})|$ is small and below the experimental resolution, so that the energy and hole momentum of the 1-el-sharp-feature line arcs read,

$$E(\vec{k}^h) \approx \frac{4}{\pi} \left( \frac{q_{N}}{B_{s1}} - q_{darc}^N \right) \sqrt{\left[ q_{N}^{A_{s1}} \right]^2 - \left[ q_{N}^{B_{s1}} \right]^2} = E_{v-el}(2\phi_{arc}) = 2W_{ec}(1 - \sin 2\phi_{arc}),$$

$$\vec{k}^h = \vec{q}_{Fc}^d - \vec{q}_{darc}^d, \quad \phi \in (\pi/4 - \phi_{arc}, \pi/4 + \phi_{arc}); \quad \vec{k}^N = \vec{k}^h N - \vec{p} = -\frac{k_{N}}{q_{N}^{B_{s1}} - q_{darc}},$$

(88)

respectively. Importantly, the energy scale $E(\vec{k}) \approx E(\vec{q}_{darc})$ exactly equals the energy $E_{v-el}$ of the strongly coupled virtual-electron pair broken under the one-electron removal excitation. Except for the small energy scale $|\varepsilon_s(\vec{q}_{darc})|$
ignored here, strongly coupled virtual-electron pairs associated with the same 1-el-sharp-feature line arc have the same energy. Indeed, $E_{v-el}(q_{arc}^N)$ depends only on the nodal momentum absolute value $q_{arc}$ of the corresponding s1 – sc line arc. It may alternatively be expressed as a function of the angular width $2\phi_{arc}$ of the line arc under consideration.

Note that different 1-el-sharp-feature line arcs have different energy magnitudes $E(q_{arc}^N) = E_{v-el}(q_{arc}^N)$. Hence they refer to broken strongly coupled virtual-electron-pairs of different energy. It follows that the energy $E(q_{arc}^N) = E_{v-el}(q_{arc}^N)$ of each 1-el-sharp-feature line arc provides a direct experimental signature of the corresponding broken strongly coupled virtual-electron-pairs. Its energy absolutely equals that of such pairs. The $E_{v-el}(2\phi_{arc})$ expression provided in Eq. (88) refers to the energy $2\pi_c(\vec{q}^h_{arc})$ expressed in terms of the corresponding s1 – sc-line arc angular width $2\phi_{arc}$. The coefficient $(1 - \sin 2\phi_{arc})$ equals the factor $(1 - |\cos 2\phi|)$ of the $E_1$ expression given in Eq. (82) for $2\phi = (\pi/2 \pm 2\phi_{arc})$ and $2\phi_{arc} \in (0, \pi/2)$.

The 1-el-sharp-feature line arc excitation hole momentum $\vec{k}^h = \vec{k} + \vec{\pi}$ also given in Eq. (88) is centered at momentum $-\vec{\pi}$. In that expression an expression for the corresponding momentum $\vec{k}$ valid only for the nodal direction is also provided. Each of the four arcs of the 1-el-sharp-feature line arc has exactly the same angular width $2\phi_{arc}$ as the corresponding s1 – sc line four arcs. Moreover, the Fermi angle $\phi$ range provided in Eq. (88) of each 1-el-sharp-feature line arc has been constructed to inherently the inequalities of Eqs. (67), (85), and (86) being obeyed. Its energy absolutely equals that of such pairs. The $E_{v-el}(2\phi_{arc})$ expression provided in Eq. (88) refers to the energy $2\pi_c(\vec{q}^h_{arc})$ expressed in terms of the corresponding s1 – sc-line arc angular width $2\phi_{arc}$. The coefficient $(1 - \sin 2\phi_{arc})$ equals the factor $(1 - |\cos 2\phi|)$ of the $E_1$ expression given in Eq. (82) for $2\phi = (\pi/2 \pm 2\phi_{arc})$ and $2\phi_{arc} \in (0, \pi/2)$.

Concerning charge excitations and currents associated with the phases $\theta_{cp}$ the zero-momentum $c$ fermion pairs participating in phase-coherent virtual-electron-pair configurations behave independently of the corresponding s1 fermions of momentum $\vec{q}$. Hence within such phenomena the phases $\theta_{cp}$ are associated with the zero-momentum $c$ fermion pairs of the phase-coherent-virtual-electron-pair superconducting state macroscopic condensate. In the following we call them coherent $c$ fermion pairs. For such a state the energy cost of a phase twist is for small $0 < (x - x_c) \ll 1$ and zero temperature approximately given by $\rho_{cp}(0)\vec{\nabla}\theta_{cp}^2/2$. Here $\rho_{cp}(T)$ is the superfluid density at temperature $T$ of coherent fermion pairs. For $0 < (x - x_c) \ll 1$ the fluctuations of the phases $\theta_{j,0}$ and thus of the $c$ fermion-pair phases $\theta_{cp}$ are strong. Hence the critical temperature $T_c$ is governed by such fluctuations. As a result for $0 < (x - x_c) \ll 1$ the transition taking place from the superconducting state to the quantum vortex liquid is for the quasi-2D VEP quantum liquid a Berezinskii-Kosterlitz-Thouless like transition. One finds according to the properties of such a transition that under the suppression effects the following relation holds,

$$
\frac{8}{\pi} k_B T_c = \frac{n_{cp}(T_c)}{m^*_c} = \rho_{cp}(T_c).
$$

(89)
Here \( n_{cp}(T_c) \) is the density \( \lim_{T \to T_c} n_{cp}(T) \) of coherently paired \( c \) fermions for \( 0 < (T_c - T) \ll 1 \). (For \( 0 < (T - T_c) \ll 1 \) such a density vanishes.) \( n_{cp}(T_c)/2 \) is the corresponding density of the macroscopic condensate coherent \( c \) fermion pairs. In units of lattice spacing \( a \) one, the \( c \) fermion mass can be written in terms of the coefficient \( r_c \) of Eq. (A1) of Appendix A as \( m_c^* = h^2/2r_c a^2 \). Here \( h \) is the Planck constant. (Often we write such a mass in units of both \( a \) and \( h \) one, so that it reads \( m_c^* = 1/2r_c a^2 \).) From the \( T_c \) expression given in Eq. (42) one finds \( T_c = \tilde{\gamma}_0 \Delta_0/2k_B \rightarrow 0 \) as \( (x - x_c) \rightarrow 0 \). Indeed, in that limit the suppression coefficient of Eq. (35) reads \( \gamma_d = 1 \). For \( 0 < (x - x_c) \ll 1 \) the use of the relations given in Eq. (60) then leads to,

\[
\rho_{cp}(0) \approx \tilde{\gamma}_0 \frac{4\Delta_0}{\pi h^2}.
\]

Here \( \tilde{\gamma}_0 \) is the amplitude given in Eq. (42). So that the relations \( r_c \approx 2r_s \approx 2e^{-4t_{uo}/U} \) and \( x_s = 2r_s/\pi \approx r_c/\pi \approx (2/\pi)e^{-4t_{uo}/U} \) apply, expression (90) and most expressions given below are derived for the approximate range \( U/4t \in (u_0, u_1) \). It follows from Eq. (90) that for \( 0 < (x - x_c) \ll 1 \) the superconducting-ground-state \( c \) band momentum area that corresponds to the \( c \) fermions contributing to coherent pairing is,

\[
S_{cp}^c = (x - x_c) \frac{4\pi^2}{r_c(r_c - \pi x_c)} \frac{2\Delta_0}{t}.
\]

The hole-momentum domain \( Q_{cp}^c \) of the \( c \) fermions that contribute to coherent pairing refers to a set of \( c - sc \) lines whose radius \( q^h \) belongs to the range \( q^h \in (q^h_{cp}, q^h_{s}) \). Here \( q^h_{cp} \) is the radius of the zero-temperature coherent unpaired \( c \) fermion line whose hole momenta are denoted by \( q^h_{s} \). It is the coherent \( c - sc \) line of largest radius \( q^h_{cp} \). For \( 0 < (x - x_c) \ll 1 \) it is nearly circular and separates in the \( c \) momentum band the \( c \) fermions participating in coherent pairing from those that do not participate. The hole-momentum domain \( Q_{cp}^c \) is bounded by the \( c \) Fermi line and coherent unpaired \( c \) fermion line, respectively.

For \( 0 < (x - x_c) \ll 1 \) the coherent unpaired \( c \) fermion line radius \( q^h_{cp} \) approximately reads,

\[
q^h_{cp} \approx \sqrt{1 + \frac{W_{cp}}{[q^h_{Fc}]^2 r_c t}} q^h_{Fc} \approx \sqrt{1 + \frac{W_{cp}}{4\pi^2 r_c t}} q^h_{Fc}.
\]

For \( U/4t \in (u_0, u_1) \) this expression is a very good approximation for the range \( x \in (x_c, x_{c1}) \) and a reasonably good approximation for the range \( x \in (x_{c1}, x_s) \).

The energy bandwidth \( W_{cp} \approx [(q^h_{cp})^2 - (q^h_{Fc})^2]/2m_c^* \) of the corresponding superconducting-ground-state sea of \( c \) fermions contributing to coherent pairing reads,

\[
W_{cp} = \tilde{\gamma}_0 8\Delta_0; \quad W_{ec} \equiv \max \ W_{cp} = \frac{4\Delta_0}{\gamma_c}.
\]

This confirms the validity of the maximum energy bandwidth expression provided in Eq. (64). The parameter \( \gamma_c \) appearing here is given in Eq. (38). The expression of \( W_{cp} \) provided in this equation is a good approximation for a range of hole concentrations \( x \in (x_c, x_{cp}) \) for which \( W_{cp} < W_{ec} \). The maximum magnitude \( W_{ec} \) of the energy bandwidth \( W_{cp} \) is reached at a hole concentration \( x = x_{cp} \) given below. Only for \( x \) below \( x_{cp} \) there is formation of coherent \( c \) fermion pairing upon increasing \( x \). Such a formation occurs while the short-range spin correlations are strong enough to supply the energy needed for it. For the hole concentration range \( x \in (x_{cp}, x_s) \) the energy bandwidth \( W_{cp} \) is independent of \( x \) and reads \( W_{cp} = W_{ec} \). Furthermore, it vanishes for \( x > x_s \). The singular behavior occurring at \( x = x_s \) marks a sharp quantum phase transition to a disordered state without short-range spin order and thus without long-range superconducting order.

For the approximate range \( U/4t \in (u_0, u_1) \) generalization of the zero-temperature superfluid density expression (60) found above for \( 0 < (x - x_c) \ll 1 \) to the hole-concentration range \( x \in (x_c, x_s) \) for which there is long-range superconducting order leads to,

\[
\rho_{cp} \approx \left[ \tilde{\gamma}_0 \Theta(x_{cp} - x) + \frac{1}{2\gamma_c} \theta(x - x_c) \right] \frac{4\Delta_0}{\pi h^2}; \quad x_{cp} = \frac{x_s}{2} + x_c.
\]

Here the theta function \( \Theta(z) \) is such that \( \Theta(z) = 1 \) for \( z \geq 0 \) and \( \Theta(z) = 0 \) for \( z < 0 \). The superfluid density \( \rho_{cp} \) vanishes both for \( x < x_c \) and \( x > x_s \). Hence it has a singular behavior at \( x = x_s \). It marks the above mentioned sharp quantum phase transition to a disordered state without short-range spin order and thus without long-range superconducting order for \( x > x_s \).
Why does the superfluid density \( \rho_{\text{sf}} \) not depend of the suppression coefficient \( \gamma_d \)? Since there are no suppression effects both for \( 0 < (x-x_c) \ll 1 \) and \( 0 < (x_s-x) \ll 1 \) when \( \gamma_d = 1 \) and such effects are strongest at \( x = x_{\text{op}} \), to answer such a question it is useful to consider the ratio,

\[
\frac{2|\Omega|}{2|\Delta|}_{T=0,x=x_{\text{op}}} = \frac{\gamma_d^{\min}}{2\gamma_c} = \frac{(1-\alpha_d)}{2\gamma_c}.
\]

The dependence on the suppression coefficient \( \gamma_d^{\min} = (1-\alpha_d) \) of the ratio \( \frac{2|\Omega|}{2|\Delta|}_{T=0,x=x_{\text{op}}} \) justifies physically why the zero-temperature superfluid density \( \rho_{\text{sf}} \) remains unaltered under the suppression effects and given by Eqs. (94). If the ratio \( \frac{2|\Omega|}{2|\Delta|}_{T=0,x=x_{\text{op}}} \) was independent of \( \gamma_d^{\min} \), the superfluid density would be suppressed. In that case one would expect that \( \rho_{\text{sf}} \propto \gamma_d^{\min} \) at \( x = x_{\text{op}} \). However, while the energy available to the short-range correlations \( 2|\Delta|_{T=0} \) to supply pair formation remains unaltered, the energy cost \( 2|\Omega|_{T=0} \) of that pair formation decreases by a factor given exactly by the suppression coefficient \( \gamma_d^{\min} \). This effect cancels the decreasing of \( \rho_{\text{sf}} \), whose magnitude remains independent of \( \gamma_d^{\min} \). Such an analysis is straightforwardly generalized to the whole range \( x \in (x_c, x_s) \).

It follows from expression (94) that the corresponding superconducting-ground-state energy bandwidth of the sea of \( c \) fermions whose pairs contribute to the superfluid density is for hole concentrations in the range \( x \in (x_c, x_s) \), vanishing spin density \( m = 0 \), and zero temperature given by,

\[
W_{cp} = \left[ g_0 \Theta(x_{cp} - x) + \frac{1}{2\gamma_c} \theta(x-x_{cp}) \right] 8\Delta_0.
\]

The in-plane penetration depth \( \lambda_{ab} \) can be expressed in terms of the superfluid density of Eq. (94) as follows,

\[
\lambda_{ab} = \frac{1}{(-\epsilon)} \sqrt{\frac{d_\parallel}{\mu_0 \rho_{cp}}} ; \quad \frac{\lambda^2_{ab}}{d_\parallel} = \frac{\mu_0 e^2}{\hbar} \rho_{cp}.
\]

Here \( d_\parallel \) is the plane separation of the system described by the quasi-2D Hamiltonian (1), \(-\epsilon\) denotes the electronic charge, and \( \mu_0 \) is the vacuum permeability. As confirmed in Ref.22, for the parameter values appropriate to the five representative hole-doped cuprate superconductors the penetration depth \( \lambda_{ab} \) is much larger than the coherence length \( \xi \) of Eq. (16).

Finally, the \( x \) dependence of the zero-temperature superfluid density \( \rho_{cp} \) of Eq. (94) can for approximately \( U/4t \in (u_0, u_1) \) be obtained from solution of the following rate differential equation of superfluid-density increase upon increasing the hole concentration \( x \), under suitable physical boundary conditions,

\[
\frac{\partial \rho_{cp}}{\partial x} = \frac{W_{ec}}{2\hbar^2 r_s} \theta(x_{cp} - x) ; \quad \rho_{cp}|_{x_{cp} < x < x_s} = \rho_{cp}|_{x = x_{cp}} = \frac{W_{ec}}{\pi \hbar^2} ; \quad \rho_{cp}|_{x = x_s} = \rho_{cp}|_{x = x_s} = 0.
\]

F. General superconducting virtual-electron pairing energy

Here we generalize the expression of the superconducting virtual-electron pairing energy per electron \( |\Omega_{s1}(q)| \) given in Eq. (61) for momenta \( q \approx q_{Da}^d \) at or near the \( s1 \) boundary line to momenta \( q = q_{arc}^d \) belonging to any other phase-coherent virtual-electron pair configuration \( s1 - sc \) line arc, as defined below. The whole set of such lines generates the momentum range \( q \in Q_{cp}^{s1} \) of the phase-coherent virtual-electron pair configurations. The general function \( |\Omega_{s1}(q)| \) given here is that involved in the \( s1 \) band energy dispersion \( \epsilon_{s1}(q) \) expression of Eq. (55). That dispersion appears in the first-order terms of the general energy functional provided in Eq. (54).

The general coherent virtual-electron pairing energy per electron \( |\Omega_{s1}(q)| \) refers to pairs of \( c \) fermions whose hole momenta belong to any \( c - sc \) line between the \( c \) Fermi line and the coherent unpaired \( c \) fermion line. The radius \( q^h \) of such a \( c - sc \) line belongs to the range \( q^h \in (q_{Fc}^h, q_{cp}^h) \). Only for it is the coherent virtual-electron pairing energy per electron introduced here finite. The energy bandwidth \( W_{cp} \) of such \( c \) fermions provided in Eq. (99) plays the same role for the coherent unpaired \( c \) fermion line as the energy bandwidth \( W_{ec} \) given in Eq. (64) for the \( ec \)-pairing line. As found below, for \( x \in (x_{cp}, x_s) \) these two lines are the same line.

For the range of hole concentrations \( x \in (x_c, x_s) \) the shape of the coherent unpaired \( c \) fermion line is at zero temperature defined by the following relation,

\[
q_{cp}^h \in \text{coherent unpaired } c \text{ line } \iff \epsilon_{cp}(q_{cp}^h) = 0.
\]

Here the \( c \) fermion energy dispersion \( \epsilon_{cp}(q^h) \),

\[
\epsilon_{cp}(q^h) = W_{cp} + \epsilon_c(q^h) ; \quad 0 \leq \epsilon_{cp}(q^h) \leq W_{cp} ; \quad \epsilon_{cp}(q_{cp}^h) = 0 ; \quad \epsilon_{cp}(q_{Fc}^h) = W_{cp},
\]

(100)
has the vanishing energy level at the coherent unpaired c fermion line. For $U/4t \in (u_0, u_1)$ and the hole-concentration range $x \in (x_c, x_s)$ the coherent unpaired c fermion line radius is approximately given by Eq. (92).

The corresponding s1 band momentum domain $Q^N_{ec}$ of the phase-coherent virtual electron pair configurations refers to the set of coherent s1 – sc line arcs associated with coherent c – sc lines contained in the c band hole-momentum domain $Q^N_{cp}$. For instance, for the s1 – sc line arc corresponding to the Fermi angle range $\phi \in (0, \pi/2)$ whose s1 band quadrant has auxiliary momentum angle range $\phi_{s1} \in (\pi, 3\pi/2)$ and components $q_{0s1} \leq 0$ and $q_{0sc} \leq 0$ it is limited by the s1 boundary line and the coherent s1 – sc line arc with minimum nodal momentum absolute value $q^N_{cp}$. The latter is given by expression (80) at $q^h = q_{sc}$. This leads to,

$$q^N_{cp} \approx q^N_{B1} = \frac{|q_{sc}^h|^2 - |q_{sc}^c|^2}{4\sqrt{|q_{B1}^{hN}|^2 - |q_{B1}^{cN}|^2}} \pi = q^N_{B1} - \frac{\tilde{g}_0 2\Delta_0}{r_c \sqrt{|q_{B1}^{hN}|^2 - |q_{B1}^{cN}|^2}} \pi , \quad (101)$$

for $x \in (x_c, x_{cp})$ and $q^N_{cp} = q^N_{ec}$ for $x \in (x_{cp}, x_s)$ where $q^N_{ec}$ is provided in Eq. (85). The s1 boundary momenta $q^N_{B1}$ and $q^N_{sc}$ appearing here are given in Eq. (E3) of Appendix E.

For the hole concentration range $x \in (x_c, x_{cp})$ the coherent s1 – sc line arc angle width $2\phi_{arc}$ belongs to the range $2\phi_{arc} \in (\pi/2 - 2\phi_{arc}^N, \pi/2 + 2\phi_{arc}^N)$. Here $\phi_{arc}^N$ is the angle $\phi_{arc}$ of Eq. (82) at $q^N_{arc} = q^N_{cp}$ and thus $q^h = q_{sc}$. It reads,

$$\phi_{arc}^N = 1/2 \arcsin \left( \frac{q^N_{cp} - q^N_{ec}}{q^N_{B1} - q^N_{ec}} \right) = 1/2 \arcsin \left( \frac{2(x_{cp} - x)}{x_s} \right) \in (0, \pi/4) ; \quad x \in (x_c, x_{cp}) ,$$

$$= 0 ; \quad x \in (x_{cp}, x_s) . \quad (102)$$

Its limiting values are $\phi_{arc}^N = \pi/4$ at $x = x_c$ and $\phi_{arc}^N = 0$ at $x = x_{cp}$. The c fermions of hole momenta $q^h$ and $-q^h$ at or near the c Fermi line participate in phase-coherent virtual-electron-pair configurations of momentum $\vec{q} \approx q^N_{B1}$ at or near the s1 boundary line. The strong effective coupling of such c fermions results from residual interactions with the corresponding s1 fermions of momentum $\vec{q} \approx q^N_{B1}$ at or near the s1 boundary line. In turn, in the opposite limit of c fermion hole momenta $q^h$ and $-q^h$ belonging to the coherent unpaired c fermion line such a residual interactions are with s1 fermions of momentum $\vec{q}$ belonging to the coherent s1 – sc line arc whose nodal momentum has minimum absolute value $q^N_{sc}$. For the hole concentration range $x \in (x_c, x_{cp})$ that line has the minimum angular range $\phi \in (\pi/4 - \phi_{arc}^N, \pi/4 + \phi_{arc}^N)$ of the coherent s1 – sc line arcs and its nodal momentum has absolute value $q^N_{arc}$ given in Eq. (104). In turn, for $x \in (x_{cp}, x_s)$ the angular width $2\phi_{arc}$ vanishes and $q^N_{arc} = q^d_{arc}$ equals the absolute-value nodal momentum $q^d_{arc}$ given in Eq. (75). For the latter $x$ range the coherent s1 – sc line arc of minimum nodal momentum absolute value $q^N_{arc}$ then reduces to the discrete nodal momentum $\vec{q} = q^d_{arc}$. Hence for $x \in (x_{cp}, x_s)$ the coherent unpaired c fermion line is the cc-pairing c fermion line. For intermediate $q^h$ and $-q^h$ hole momenta at or near the coherent c – sc lines between the c Fermi line and the coherent unpaired c fermion line the strong effective coupling associated with coherent pairing is due to residual interactions of the c fermions under consideration with s1 fermions of momentum at or near the corresponding coherent s1 – sc line arcs whose nodal momentum absolute value belongs to the range $q^N_{arc} \in (q^d_{arc}, q^N_{B1})$.

Based on the results of Sections IV-C and IV-E, we are now ready to introduce the superconducting pairing energy spectrum $|\Omega_{s1}(\vec{q})|$ contributing to the s1 band energy dispersion $\epsilon_{s1}(\vec{q})$ expression (55). For a virtual-electron pair of momentum $\vec{q} = q^d_{arc}$ belonging to a given coherent s1 – sc line arc are the general superconducting virtual-electron pairing energy per electron reads,

$$|\Omega_{s1}(q^d_{arc})| = \Theta \left( \epsilon_{cp}(q^h) - V^{eff}_{cp}(q^d_{arc}) \right) \Theta \left( W_{cp} - \epsilon_{cp}(q^h) \right) \gamma_d 90 \sqrt{2} V_{s1}^{d} \frac{2\Delta_0(q^d_{arc}) \epsilon_{cp}(q^h) + \epsilon_{cp}(q^h)}{2W_{cp}} . \quad (103)$$

Here $q^h$ is the radius of the corresponding c – sc line. It is uniquely related to the nodal momentum absolute value $q^d_{arc}$ of the coherent s1 – sc line arc under consideration by the function $q^N_{arc} = q^N_{arc}(q^h)$ defined in Eq. (80). Moreover, since coherent pairing requires strong effective coupling, the energy range of Eq. (67) together with the expression $\epsilon_{cp}(q^h) = W_{cp} + \epsilon_c(q^h)$ of Eq. (100) imply that the energy scale $V^{eff}_{cp}(\vec{q})$ appearing in this pairing energy expression reads,

$$V^{eff}_{cp}(q^d_{arc}) = 0 ; \quad W_{cp} < W_{ec} \left( 1 - \frac{\Delta_{s1}(q^d_{arc})}{|\Delta|} \right)$$

$$= W_{cp} - W_{ec} \left( 1 - \frac{\Delta_{s1}(q^d_{arc})}{|\Delta|} \right) ; \quad W_{cp} > W_{ec} \left( 1 - \frac{\Delta_{s1}(q^d_{arc})}{|\Delta|} \right) . \quad (104)$$
The pairing energy given in Eq. (103) is expressed in terms of the energy $\epsilon_{cp}(\vec{q}^h) + \epsilon_{cp}(\vec{-q}^h)$ of the $c$ fermion pairs of hole momentum at or near the coherent $c-sc$ line of radius $q^h$. Those interact with the $s1$ fermion of momentum $\vec{q} = \vec{q}_{arc}^d$ at or near the corresponding coherent $s1-sc$ line arc. This occurs within the phase-coherent virtual-electron pair configuration of the same momentum. The use of the function $q^N_{arc} = q^N_{arc}(q^h)$ of Eq. (90) leads to the following equivalent yet simpler expression for the general pairing energy per electron defined by Eqs. (103) and (111),

$$|\Omega_{s1}(\vec{q}_{arc})| = \gamma_d g_0 \sqrt{2} V_{s1}^\Delta(\vec{q}_{arc}) \left( \frac{q^N_{arc} - q^N_{cp}}{q^N_{B1} - q^N_{cp}} \right) \approx \gamma_d g \left( \frac{q^N_{arc} - q^N_{cp}}{q^N_{B1} - q^N_{cp}} \right) \Delta_0 |\sin 2\phi|.$$  

(105)

Here,

$$V_{s1}^\Delta(\vec{q}) = |\nabla_{\vec{q}} \Delta_{s1}(\vec{q})| = \frac{|\Delta|}{\sqrt{2}} G_{s1}(\vec{q}), \quad \vec{q} = \vec{q}_{arc}^d; \quad G_{s1}(\vec{q}_{arc}) \approx |\sin 2\phi|,$$

(106)

where the function $G_{s1}(\vec{q})$ is defined in Ref. 36. The value $G_{s1}(\vec{q}_{arc}) \approx |\sin 2\phi|$ is a good approximation for $s1$ band momenta belonging to coherent $s1-sc$ line arcs.

The energy $|\Omega_{s1}(\vec{q}_{arc})|$ has its maximum magnitude for momenta $\vec{q}_{arc}^d \approx \vec{q}_{B1}^d$, when it involves $c$ fermions of hole momenta at or near the $c$ Fermi line such that $\epsilon_{cp}(\pm \vec{q}^h) = W_{cp}$. The strong effective coupling of such fermions results from interactions with $s1$ fermions of momenta at or near the $s1$ boundary line. It vanishes in the opposite limit referring to $c$ fermions of hole momenta at or near the coherent unpaired $c$ fermion line for which $\epsilon_{cp}(\pm \vec{q}^h) = 0$.

Both the superconducting pairing energy expressions (103) and (105) have physical significance only for $s1$ band momenta $\vec{q} \approx \vec{q}_{arc}^d$ at or near a coherent $s1-sc$ line arc. Interestingly, the energy $|\Omega_{s1}(\vec{q}_{arc})|$ has a much simpler expression (105) in terms of the momenta of the virtual-electron pair coherent $s1-sc$ line arc than as given in Eq. (103). Indeed, the coherent $s1-sc$ line arcs have been constructed to inherently the inequalities (103) being obeyed.

On varying the virtual-electron pair coherent $s1-sc$ line arc nodal momentum absolute value within the range $q^N_{arc} \in (q^N_{cp}, q^N_{B1})$ and thus that of the $c$ fermions holes momenta within $q^h \in (q^h_{B1}, q^h_{cp})$ the pairing energy $2|\Omega_{s1}(\vec{q}_{arc})|$ varies within the corresponding range $2|\Omega_{s1}(\vec{q}_{arc})| \in (0, 2|\Omega||\sin 2\phi|)$. It has the following limiting behaviors,

$$\max 2|\Omega_{s1}(\vec{q}_{arc})| = 2|\Omega_{s1}(q_{B1})||_{T=0} = 2|\Omega||_{T=0}; \quad 2|\Omega_{s1}(q_{cp})| = 0; \quad 2|\Omega_{s1}(q_{B1})| \approx 2|\Omega||\sin 2\phi|.$$  

(107)

V. CONCLUDING REMARKS

In conventional superconductivity of isotropic 3D many-electron problems the objects that pair are the quasiparticles of the Fermi-liquid theory. The low-energy eigenstates are described by occupancy configurations of such quasiparticles so that their momenta are good quantum numbers and their interactions are residual. That property simplifies enormously the description of the many-electron physics. In turn, in the VEP quantum liquid the charge $c$ fermions and spin-neutral two-spinon $s1$ fermions are for $x \in (x_c, x_s)$ the constituents of the phase-coherent virtual-electron pair configurations. The momentum occupancies of such quantum objects generate the energy eigenstates of the Hubbard model on the square lattice in the one- and two-electron subspace. For the VEP quantum liquid studied in this paper such $c$ and $s1$ band momenta are close to good quantum numbers. The $c-s1$ fermion interactions play a central role in the pairing mechanism of that quantum liquid, consistently with the evidence that unconventional superconductivity is in different classes of systems mediated by magnetic fluctuations.

Our studies consider the fluctuations of the phases associated with the long-range antiferromagnetic, short-range spin, and long-range superconducting orders occurring at zero temperature for $x \in (0, x_0)$, $x \in (x_0, x_s)$, and $x \in (x_c, x_s)$, respectively. For $x \in (x_0, x_c)$ the short-range spin order coexists with Anderson insulating behavior brought about by intrinsic disorder. For hole concentrations in the range $x \in (x_c, x_s)$ the short-range spin order and long-range superconducting order coexist. Fortunately, for that $x$ range the effects of intrinsic disorder or superfluid-density anisotropy are very weak. This justifies the success of our oversimplified description of such effects in terms of a single suppression coefficient $\gamma_{pin}^n = (1 - \alpha_d)$. The results of this paper focus on the physics associated with the corresponding hole concentration range $x \in (x_c, x_s)$. (The effects of intrinsic disorder are much stronger for $x \in (0, x_c)$, as shortly discussed in Appendix B.)

The exotic $d$-wave superconducting phase of the VEP quantum liquid involves virtual-electron pairs whose phases are coherent. Those are constituted by one zero-momentum $c$ fermion pair and one spin-singlet two-spinon $s1$ fermion of momentum $\vec{q} \in Q_{sc}^1$. Hence a virtual electron pair involves the two charges $-e$ of its $c$ fermions and the spin-singlet configuration of the two spin-1/2 spinons of its composite $s1$ fermion. The phases of the virtual-electron pairs read $\theta = \theta_0 + \theta_1$. Here $\theta_0$ are overall centre-of-mass phases and the phases $\theta_1$ are related to the internal pairing degrees of freedom. The corresponding macroscopic condensate of zero-momentum $c$ fermion pairs is associated with the phase
coherence occurring for \( x \in (x_c, x_s) \). The fluctuations of the phases \( \theta_0 \) and \( \theta_1 \) become large for \( 0 < (x - x_c) \ll 1 \) and \( 0 < (x - x_s) \ll 1 \), respectively. For the intermediate interaction range \( U/4t \in (u_0, u_1) \) the critical hole concentrations are given by \( x_c \approx 0.05 \) and \( x_s \in (0.23, 0.28) \). For it such fluctuations are controlled by the weak 3D uniaxial anisotropy effects and electronic correlations through the mass ratios \( 1/\varepsilon^2 = M/m_c^* \) and \( r_c = m_c^*/m_s^* \approx 2r_s \approx 2e^{-4t_u/\Omega} \), respectively. The direct relation to the problem investigated in Ref.\(^a\) has simplified the study of the fluctuations of the phases \( \theta_0 \) and \( \theta_1 \).

The virtual-electron pairing energy is supplied to the zero-momentum \( c \) fermion pair by the short-range spin correlations associated with the spin-singlet spinon pairing. This occurs through the \( c - s1 \) fermion interactions within each virtual-electron pair. The interrelated spinon and \( c \) fermion pairings involved in the exotic coherent virtual-electron pairing are associated with the coexisting two-gap scenario consistent with the unusual properties of the representative hole doped cuprates\(^\text{15,17,22}\): A superconducting energy scale \( 2\Omega \) and pseudogap \( 2|\Delta| \), over the whole dome \( x \in (x_c, x_s) \), as illustrated in Fig. 1 of Ref.\(^a\). The energy parameter \( 2|\Delta| \) is the maximum magnitude reached at \( \phi = 0, \pi/2 \) of the spinon pairing energy \( 2|\Delta_{s1}(\overrightarrow{q}_{|\Delta|})| \approx 2|\Delta| \cos 2\phi \) appearing in Eq. (102). The energy scale \( 2\Omega \) is the maximum magnitude reached at \( \phi = \pi/4 \) of the superconducting virtual-electron pairing energy \( 2|\Omega_{s1}(\overrightarrow{q}_{\Omega_{s1}})| \approx 2\Omega \sin 2\phi \) of Eq. (107). The latter pairing energy is associated with the effective coupling of the corresponding zero-momentum coherent \( c \) fermion pairs. Such zero-momentum \( c \) fermion pairs couple to charge probes independently of the \( s1 \) fermions, which couple to spin probes. It follows that concerning charge excitations the zero-momentum coherent \( c \) fermion pairs behave as independent objects relative to the virtual-electron pairs. In turn, the virtual-electron pairs exist in virtual intermediate states generated by one-electron and spin excitations, which break such pairs.

The pseudogap state occurs both for hole concentrations \( x \in (x_0, x_c) \) at temperatures below the pseudogap temperature \( T^* \) and for a temperature-dependent hole concentration range centered at \( x = x_{op} \) for finite temperatures in the range \( T \in (T_c, T^*) \). For such a state the \( s1 \) fermion spinon pairing energy remains finite and there are \( c \) fermion pairing correlations, yet there is no phase coherence. Normal ground states may be reached by applying a magnetic field \( H \) aligned perpendicular to the planes. Our results are inconclusive on whether for \( H = 0 \) the ground state of the \( \varepsilon^2 = m_c^*/M = 0 \) Hubbard model on the square lattice is superconducting. They seem to indicate that some small 3D uniaxial anisotropy is needed for the emergence of superconductivity. Indeed, at constant \( U/4t \) values the hole-concentration width \( (x_s - x_c) \) of the superconducting dome decreases upon decreasing the small 3D uniaxial anisotropy parameter \( \varepsilon^2 = m_s^*/M \).

Concerning previous related studies on the large-\( U \) Hubbard model and \( t - J \) model and Heisenberg model on a square lattice involving for instance Jordan-Wigner transformations\(^\text{21}\) or the slave particle formalisms\(^\text{32,33}\), here the single occupancy constraint is naturally implemented for all \( U/4t \) finite values. Indeed, the spin-1/2 spinons refer to the rotated electrons of the singly occupied sites of the energy eigenstates configurations\(^\text{12,22}\). In the one- and two-electron subspace where the VEP quantum liquid is defined only the charge \( c \) fermions and spin-neutral two-spinon \( s1 \) fermions play an active role. The main difference relative to the above related schemes is that for them the spinless fermions arise from individual spin-1/2 spins or spinons. In contrast, according to the studies of Ref.\(^a\) the \( s1 \) fermions of the representation used in the studies of this paper emerge from an extended Jordan-Wigner transformation performed on spin-neutral two-spinon composite \( s1 \) bond particles. A property of physical importance for the unusual scattering properties of the VEP quantum liquid investigated in Ref.\(^a\) is that the \( s1 \) bond is full for \( m = 0 \) ground states and contains a single hole for one-electron excitations.

When expressed in terms of the rotated-electron operators, the VEP quantum liquid microscopic Hamiltonian has basic similarities to that considered in Ref.\(^a\) in terms of electron operators. The advantage of our scheme is that it applies to intermediate \( U/4t \) values. Indeed, the results of Refs.\(^\text{22,23}\) strongly suggest that the physics of several classes of hole-doped cuprates with critical hole concentrations \( x_c \approx 0.05 \) and \( x_s \approx 0.27\) corresponds to \( U/4t \approx u_s = 1.525 \in (u_0, u_1) \). The effective superconductivity approach for such cuprates introduced in this paper is consistent with its physics being closely related to the doping of a Mott insulator\(^\text{23}\). There are previous studies where spin-charge interactions also lead to superconducting effective pair coupling\(^\text{15,34-37}\). Our VEP quantum liquid acts on the one- and two-electron subspaces. For it rotated-electron double occupancy is exactly zero. This implies that the mechanisms behind its long-range superconducting order are different from those of the \( SO(5) \) theory\(^\text{38}\).

Finally, in Refs.\(^\text{19,23}\) evidence is provided that the use of the theoretical VEP quantum liquid scheme introduced in this paper leads to quantitative agreement with experiments concerning the universal properties of the cuprate superconductors.

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Appendix A: Basic information on the c and s1 fermion description

In this Appendix some basic information on the c and s1 fermion description used in the studies of Refs.1,2,22 is provided. The critical hole concentration $x_s$ expression given in Eq. (18) refers to the interaction range $U/4t > u_0 \approx 1.302$ and involves the spin ratio $r_s$. That ratio can be defined for the whole range of $U/4t$ values and increases smoothly upon increasing $U/4t$. The charge and spin ratios are for the approximate range $U/4t \in (u_0, u_1)$ given by $r_c \approx 2r_s \approx 2e^{-4tu_0/U}$. Here $u_1 \approx 1.600$. For $U/4t > 0$ such ratios have the following limiting behaviors,

$$
\begin{align*}
    r_c &= \frac{m_c^\infty}{m_c^s}; & r_s &= \frac{\Delta_0}{4W_0} = e^{-\lambda_s}, \\
    r_c &= r_s = 0, & U/4t &\to 0; & r_s &\approx e^{-\pi\sqrt{4t/U}}, & U/4t &\ll 1, \\
    r_c &\approx 2r_s \approx 2e^{-1} \approx 0.736, & U/4t &\approx u_0 \approx 1.302, \\
    r_c &\approx 2r_s \approx 2e^{-4tu_0/U}, & U/4t &\in (u_0, u_1), \\
    r_c &= r_s = 1, & U/4t &\to \infty. 
\end{align*}
$$

(A1)

For the range $U/4t \in (u_0, u_1)$ the critical hole concentration $x_s$ is given by $x_s \approx 2r_s/\pi$. For $U/4t \geq u_0$ its limiting magnitudes are,

$$
\begin{align*}
    x_s &\approx \frac{2}{\pi} e^{-4tu_0/U}, & U/4t &\in (u_0, u_1), \\
    &\approx 0.23, & U/4t &= u_0 \approx 1.302, \\
    &\approx 0.27, & U/4t &\approx 1.525, \\
    &\approx 0.28, & U/4t &= u_1 \approx 1.600, \\
    &= \frac{1}{\pi}, & U/4t &= u_\pi > u_1, \\
    &\leq \frac{2}{\pi} \approx 0.64, & U/4t &\to \infty. 
\end{align*}
$$

(A2)

An important energy scale is the the $x = 0$ and $m = 0$ Mott-Hubbard gap $2\mu^0$ given by $\Delta_{0}$,

$$
\begin{align*}
    2\mu^0 &\approx 64t e^{-\pi\sqrt{\pi}}, & U/4t &\ll 1, \\
    &\approx \frac{4e^{1/4}}{\pi} \sqrt{1 + (U/4t - u_0)}, & U/4t &\in (u_0, u_1), \\
    &\approx [U - 8t]; & U/4t &\gg 1. 
\end{align*}
$$

(A3)

Another energy scale that plays a key role in our studies is that of Eq. (5), $\lim_{x\to 0} |\Delta| = \Delta_0$. It has the following approximate limiting behaviors,

$$
\begin{align*}
    \Delta_0 &\approx 16t e^{-\pi\sqrt{4t/U}}, & U/4t &\ll 1, \\
    &= \max \{\Delta_0\} \approx t/\pi, & U/4t &= u_0, \\
    &\approx e^{(1 - 4tu_0/U)}(t/\pi) \left[1 - (U/4t - u_0) e^{-u_2 - u_0/4t} \ln(u_0)\right], & U/4t &\in (u_0, u_1), \\
    &\approx \pi [4t]^2/U, & U/4t &\gg 19. 
\end{align*}
$$

(A4)

where $u_0 \approx 1.3$, $u_* = 1.525$, and $u_1 \approx 1.6$. The $U/4t$ dependence reported here for approximately $U/4t \in (u_0, u_1)$ corresponds to an interpolation function used to connect the following limiting behaviors valid for $0 \leq (U/4t - u_0)/(u_1 - u_0) \ll 1$ and $U/4t \approx u_* = 1.525$,

$$
\begin{align*}
    \Delta_0 &\approx e^{(1 - 4tu_0/U)} \frac{t}{\pi} \left(2 - \frac{U}{4t u_0}\right), & 0 \leq (U/4t - u_0)/(u_1 - u_0) &\ll 1, \\
    &\approx e^{(1 - 4tu_0/U)} \frac{t}{\pi} \left[1 - \left(\frac{u_1 - u}{U/4t}\right)\right], & U/4t &\approx u_* = 1.525, 
\end{align*}
$$

(A5)

respectively.
Let us consider a Brillouin zone centered at the momentum \(-\vec{\pi}\). The hole Fermi momentum \(\vec{k}^h_F\) is related to the Fermi momentum \(\vec{k}_F\) as follows,

\[
\vec{k}^h_F = \vec{k}_F + \vec{\pi} = k^h_F(\phi) \vec{e}_\phi; \quad \phi = \arctan \left( \frac{k^h_Fx_2}{k^h_Fx_1} \right),
\]

\[
\phi \in \left( \phi_{AN}; \frac{\pi}{2} - \phi_{AN} \right); \quad \phi_{AN} = 0, x \leq x_h.
\] (A6)

Here the expression of the Fermi angle \(\phi\), which defines the direction of the hole Fermi momentum \(\vec{k}^h_F\), also holds for \(\phi \in (0, 2\pi)\). The angle \(\phi_{AN}\) appearing in Eq. (A6) is small for the hole concentration range \(x \in (x_h, x_s)\) for which the Fermi line is particle-like. The precise magnitude of the hole concentration \(x_h > x_c\) remains unknown. The studies of Ref. consider that it belongs to the range \(x_h \in (x_c, x_s)\). The hole Fermi momentum \(\vec{k}^h_F\) can be expressed in terms of the \(c\) Fermi line hole momenta and \(s\) boundary momenta as given in Eq. (11). For \(x \in (x_{c1}, x_{c2})\) the corresponding angles \(\phi^d_c\) and \(\phi^d_s\) are provided in Eq. (17).

For one-electron excited states, the \(s1\) band momentum,

\[
q_{BS1} = [A^{'d}_{11}]^{-1} q_{BS1} = q_{BS1}(\phi) \vec{e}_{\phi+\pi}.
\] (A7)

is called auxiliary momentum of the \(s1\) boundary-line momentum \(q_{BS1}^{d}\). Both for \(x < x_{c1}\) and \(x > x_{c2}\) the form of the matrix \(A^{'d}_{11}\) is not known. For approximately \(x \in (x_{c1}, x_{c2})\) it reads \(A^{'d}_{11} \approx A^{'d}_{F}\) where the \(F\) rotation matrix \(A^{'d}_{F}\) is given in Eq. (15).

Among the \(s1\) boundary-line momenta \(q_{BS1}^{d}\), it is useful to consider those whose corresponding auxiliary \(s1\) boundary-line momenta \(q_{BS1}^{AN}\) point in the nodal and anti-nodal directions. The nodal and anti-nodal \(s1\) boundary-line momenta \(q_{BS1}^{d AN}\) and \(q_{BS1}^{d AN}\), respectively, are defined as those whose corresponding auxiliary momenta \(q_{BS1}^{AN}\) and \(q_{BS1}^{AN}\) have for instance for the quadrant such that \(q_{0x1} \leq 0\) and \(q_{2x2} \leq 0\) the following Cartesian components,

\[
q_{BS1}^N = \left[ \begin{array}{c} q_{BS1}^N/\sqrt{2} \\ q_{BS1}^N/\sqrt{2} \end{array} \right]; \quad q_{BS1}^{AN} = \left[ \begin{array}{c} q_{BS1}^{AN} \\ 0 \end{array} \right]; \quad \left[ \begin{array}{c} 0 \\ q_{BS1}^{AN} \end{array} \right].
\] (A8)

Here \(q_{BS1}^N\) and \(q_{BS1}^{AN}\) are the absolute values of both the auxiliary momenta \(q_{BS1}^{AN}\) and \(q_{BS1}^{AN}\) and corresponding momenta \(q_{BS1}^{d AN}\) and \(q_{BS1}^{d AN}\), respectively.

For \(m = 0\) ground states and their two-electron excited states the \(s1\) boundary-line momentum reads \(q_{BS1}^{d} = q_{BS1}(\phi) \vec{e}_{\phi+\pi}\) for \(x \in (x_c, x_s)\). Thus it equals the auxiliary momentum. For the sake of generality, we often use the notation \(q_{BS1}^{d}\) for the \(s1\) boundary-line momenta of the \(s1\) fermion occupancy configurations of these states. For such states the hole \(c\) Fermi momentum is independent of the duality \(d\) introduced in Ref. It is given by,

\[
\vec{q}^h_c = \vec{q}^h_F + \vec{\pi} = q^h_F(\phi) \vec{e}_\phi; \quad \phi_c = \phi.
\] (A9)

The \(c\) and \(s1\) energy dispersions appearing in the general energy functional of Eq. (54) depend on the Cartesian components of the \(c\) band hole momentum and \(s1\) band momentum through the elementary functions \(\epsilon_c(q)\) and \(\epsilon_{s1}(q)\), respectively. Those are known in some limits. The \(c\) fermion energy dispersion \(\epsilon_c(q^h)\) is for \(U/4t > 0\), \(m = 0\), and \(x \in (x_c, x_s)\) given by,

\[
\epsilon_c(q^h) = \epsilon^0_c(q^h) = \sum_{i=1,2} \left[ \epsilon_c(q^h_i) - \epsilon_c(q^h_{Fx1}) \right].
\] (A10)

For \(x \in (x_c, x_s)\) and \(U/4t \geq u_0\) the \(c\) Fermi line is approximately circular and the \(c\) fermion energy dispersion \(\epsilon^0_c(q^h)\) and the chemical potential \(\mu\) are given by,

\[
\epsilon_c(q^h) \approx -\frac{|q^h|^2 - |q^h_{Fx1}|^2}{2m^c}; \quad |\epsilon_c(q^h)| < W^c_{x=x_s}; \quad W^c_{x} \approx \frac{2x\pi}{m^c}; \quad \mu \approx \mu_0 + W^c_{x}.
\] (A11)

Provided that \(\mu_0\) is replaced by the energy scale \(\mu^0\) given in Eq. (11) of Appendix B, the \(\mu\) expression given here is a good approximation for the VEP quantum liquid at \(x \in (x_c, x_s)\) and \(U/4t \in (u_0, u_1)\).

The \(s1\) fermion energy dispersion \(\epsilon_{s1}(\vec{q})\) and pairing energy per spinon \(|\Delta_{s1}(\vec{q})|\) appearing in Eq. (55) are given by,

\[
\epsilon_{s1}(\vec{q}) = \epsilon^0_{s1}(\vec{q}); \quad |\Delta_{s1}(\vec{q})| = |\Delta^s_{s1}(\vec{q})|.
\] (A12)

Here \(\vec{q}^0\) obeys Eq. (7), \(\epsilon^0_{s1}(\vec{q}^0) = \sum_{i=1,2} \left[ \epsilon_{s1}(q_{0x1}) - \epsilon_{s1}(q_{BS1x}) \right]\), \(|\Delta^s_{s1}(\vec{q}^0)| = \gamma_0 \Delta^s_{s1} F^s_{s1}(\vec{q}^0), F^s_{s1}(\vec{q}^0) = \left[ |\epsilon_{s1}(q_{0x1}) - \epsilon_{s1}(q_{0x2})|/W_{s1} \right] \), and the energy bandwidth \(W_{s1}\) decreases for increasing values of \(U/4t\).
The elementary function \( e_{s1}(q) \) is such that \( \Delta_{s1}(\vec{q}_{BS1}^{d}) = 0 \), \( |e_{s1}(\vec{q}_{BS1}^{d}) - e_{s1}(0)| = W_{s1} \), and \( F_{\phi_{c}}^{d}(\vec{q}_{BS1}) \approx \cos 2\phi \). Hence at \( T = 0 \) the maximum magnitude \( 2|\Delta| \) of the spinon pairing energy \( 2|\Delta_{s1}(\vec{q})| \) is reached at \( \vec{q} = \vec{q}_{BS1}^{d} \).

\[
2|\Delta||_{T=0} = \tilde{g}_{1}2\Delta_{0} = 2|\Delta_{s1}(\vec{q}_{BS1}^{d})|; \quad x \in (x_{c}, x_{s}) \; , \; m = 0.
\]  

(13)

The shapes of the \( c \) Fermi line and \( s1 \) boundary line are fully determined by the form of the auxiliary energy dispersions \( \epsilon_{c}^{0}(\vec{q}^{b}) \) and \( \epsilon_{s1}^{0}(\vec{q}) \) given in Eqs. (A10) and (A12), respectively, as follows,

\[
\begin{align*}
\vec{q}_{BS1}^{d} & \in \text{hole } c \; \text{Fermi line} \iff \epsilon_{c}^{0}(\vec{q}_{FC}^{d}) = 0, \\
\vec{q}_{BS1}^{d} & \in \text{s1 boundary line} \iff \epsilon_{s1}^{0}(\vec{q}_{BS1}) = 0.
\end{align*}
\]  

(14)

The \( c \) and \( s1 \) fermion group velocities derived from the energy dispersions of Eqs. (A5), (A10), (A11), and (A12) read,

\[
\begin{align*}
\vec{V}_{c}(\vec{q}^{b}) = & -\vec{\nabla}_{\vec{q}^{b}} \epsilon_{c}(\vec{q}^{b}) \; ; \; \vec{V}_{s1}^{0}(\vec{q}) = -\vec{\nabla}_{\vec{q}} \epsilon_{s1}(\vec{q}) ;
\end{align*}
\]  

(15)

We call their unit vectors \( \vec{e}_{\phi_{s1}}(\vec{q}) \), \( \vec{e}_{\phi_{s1}}^{0}(\vec{q}) \), \( \vec{e}_{\phi_{c}}(\vec{q}) \), and \( \vec{e}_{\phi_{c}}^{0}(\vec{q}) \), respectively.

Finally, the \( c \) fermion elementary charge current is given by,

\[
\begin{align*}
\vec{J}_{c}(\vec{q}^{b}) = & -e \alpha \vec{V}_{c}(\vec{q}^{b}) \; ; \; \vec{J}_{c}^{0}(\vec{q}_{FC}) = -e \frac{\vec{q}_{FC}}{m_{c}^{*}} \vec{e}_{\phi_{c}+\pi} \; ; \; \alpha \equiv \frac{m_{c}^{*}}{m_{c}^{0}} \approx \frac{1}{v_{c}^{0}} , \quad U/4t > u_{0} \approx 1.302.
\end{align*}
\]  

(16)

Here \( \vec{V}_{c}(\vec{q}^{b}) \) is the \( c \) fermion velocity of Eq. (A15), \( m_{c}^{*} \) a renormalized transport mass, and \( \phi_{c} = \phi \).

**Appendix B: The hole-trapping effects**

In this Appendix the strong effects of intrinsic disorder on the square-lattice quantum-liquid physics of Ref. in the hole concentration range \( x \in (0, x_{c}) \) are briefly discussed. In contrast to the range \( x \in (x_{c}, x_{s}) \) referring to the problem studied in this paper, the effects reported here change such a physics qualitatively.

As discussed in the following, intrinsic disorder is strongest in the range \( x \in (0, x_{0}) \). It has qualitatively different effects in that range and in the range \( x \in (x_{0}, x_{c}) \), respectively. Here \( x_{0} = x_{s}/\left[1 + (2\pi x_{s})^{2}(t/\Delta_{0})\right] \) is the hole concentration of Eq. (5) at which the Fermi-energy anisotropy coefficient \( \gamma_{0} = |\Delta|/W_{c}^{h} \) given in that equation reads \( \gamma_{0} = 1 \) for the square-lattice quantum liquid. It is given by \( x_{0} \approx 0.013 \) and \( x_{0} \approx 0.024 \) for \( x_{s} = 0.27 \), \( t = 295 \) meV, and the magnitudes \( \Delta_{0} = 42 \) meV and \( \Delta_{0} = 84 \) meV found in Ref. for LSCO and the remaining representative systems, respectively. For the square-lattice quantum liquid of Ref. there occurs a quantum phase transition from a Mott-Hubbard insulator with long-range antiferromagnetic order at the hole concentration \( x = 0 \) to a short-range incommensurate-spiral spin ordered state for \( 0 < x < \frac{1}{12} \). An energy scale that could be used to characterize such a transition is the \( c \) fermion unfilled sea bandwidth \( W_{c}^{h} \) of Eq. (A11) of Appendix A. It is associated with the hole kinetic energy. For the square-lattice quantum liquid it vanishes at \( x = 0 \) and is finite for finite hole concentrations.

Within the effective and simplified description of the problem considered here, for hole concentrations in the range \( x \in (0, x_{c}) \) the holes are trapped in the vicinity of randomly distributed impurities, so that their kinetic energy vanishes and the \( c \) fermion energy dispersion \( W_{c}^{h} \) is not well defined. Nevertheless, the critical hole concentration \( x_{0} \) of the Néel-state - spin-glass-state sharp quantum phase transition is that at which the ratio \( |\Delta|/W_{c}^{h} \) of the corresponding square-lattice quantum liquid is given by one. Such a ratio is for the Fermi-energy anisotropy coefficient \( \gamma_{0} = |\Delta|/W_{c}^{h} \) of Eq. (5) introduced in Ref. In turn, the absolute value \( |\epsilon_{s1}(\vec{q})| \) of the \( s1 \) fermion energy dispersion of Eq. (A11) of Appendix A changes from \( |\epsilon_{s1}(\vec{q}_{BS1})| = 0 \) for \( s1 \) boundary-line auxiliary momenta pointing in the nodal directions to \( |\epsilon_{s1}(\vec{q}_{BS1}^{d})| = |\Delta| \) for such momenta pointing in the anti-nodal directions. Therefore, the energy parameter \( |\Delta| \) gives a measure of the \( s1 \) boundary-line anisotropy.

Consistently with \( |\Delta| \) providing a measure of the \( s1 \) boundary-line anisotropy, for the square-lattice quantum liquid at hole concentrations below \( x_{0} \) for which \( |\Delta| > W_{c}^{h} \) the corresponding anisotropy effects dominate over those of the isotropic \( c \) Fermi line. It follows from Eq. (10) that the Fermi line involves contributions from both that line and the \( s1 \) boundary line. Hence for the square-lattice quantum liquid the Fermi line is strongly anisotropic for hole concentrations below \( x_{0} \). This weakens the hole kinetic-energy effects associated with the energy scale \( W_{c}^{h} \). The hole-trapping effects profit from such weakening of the hole kinetic-energy effects: They give rise to the extension of the long-range antiferromagnetic order from \( x = 0 \) to \( x \in (0, x_{0}) \) for the VEP quantum liquid. As a result, the critical concentrations \( x_{0} \) and \( x_{c} \) are those at which the energy scales \( |\Delta| \) and \( W_{c}^{h} \) emerge for the latter quantum problem, being ill defined for hole concentrations below \( x_{0} \) and below \( x_{c} \), respectively. In contrast, for the square-lattice quantum liquid they are well defined for the whole range \( x \in (0, x_{c}) \).
and randomness are larger for that compound. The Fermi-velocity anisotropy is then considered small at a smaller
quantum phase transition between the Mott-Hubbard insulator with long-range antiferromagnetic order and
that at which $\Delta \approx 0.285 \pm t$. It is considered that the Fermi-velocity anisotropy is small when $\eta_\Delta < 2x_0 \approx 0.05$. Consistently, the crossover hole concentration $x_{c1} = 1/8$ approximately is that at which $\eta_\Delta \approx 2x_0$ for $U/4t \in (u_0, u_1)$. The magnitudes given here refer both to the square-lattice quantum liquid and the four representative systems other than LSCO.

The hole-trapping effects lead to two related phenomena: (i) the extension of the long-range antiferromagnetic
order of the Mott-Hubbard insulator from $x = 0$ to the range $x \in (0, x_0)$; (ii) The coexistence for $x \in (x_0, x_c)$ of the
short-range spin order emerging at $x = x_0$ with Anderson insulating behavior. The latter behavior is brought about by
intrinsic disorder. For hole concentrations in the range $x \in (x_0, x_c)$ such a disorder and associate hole-trapping effects
are not strong enough to remove the short-range spin order. Finally, the only effect of the hole-trapping effects on
the Fermi velocity is nearly isotropic. The Fermi line is nearly isotropic for the hole concentration range

| $x$ | 0.05 | 0.11 | 1/8 | 0.15 | 0.16 | 0.21 | 0.23 | 0.24 | 0.27 |
|-----|------|------|-----|------|------|------|------|------|------|
| $\eta_\Delta$ | 0.12 | 0.06 | 0.05 | 0.04 | 0.03 | 0.02 | 0.01 | 0.01 | 0    |
| $\eta_0$  | 0.44 | 0.14 | 0.11 | 0.08 | 0.07 | 0.03 | 0.02 | 0.01 | 0    |

TABLE I: Magnitudes of the Fermi-velocity anisotropy coefficient $\eta_\Delta$ and Fermi-energy anisotropy coefficient $\eta_0$ of Eq. (5) for several hole concentrations $x$, $U/4t \approx 1.525$, and thus $\Delta \approx 0.285 \pm t$. It is considered that the Fermi-velocity anisotropy is small when $\eta_\Delta < 2x_0 \approx 0.05$. Consistently, the crossover hole concentration $x_{c1} = 1/8$ approximately is that at which $\eta_\Delta \approx 2x_0$ for $U/4t \in (u_0, u_1)$. The magnitudes given here refer both to the square-lattice quantum liquid and the four representative systems other than LSCO.

The dependence (and independence) on the Fermi angle $\phi$ of the s1 fermion velocity $V_{B_{s1}}^\Delta$ of Eq. (9) (and c fermion velocity $V_{F_c}$ also given in that equation) implies an anisotropic (and isotropic) character for the s1 boundary line (and c Fermi line.) The Fermi angle $\phi$ given in Eq. (A6) of Appendix A defines the direction of the hole Fermi momentum $\mathbf{k}_F^h$ also provided in that equation. According to the analysis of Ref. $^\Delta$, the Fermi line anisotropy is measured both by the Fermi-velocity anisotropy coefficient $\eta_\Delta = \max \{\eta_\Delta, \eta_0\}$ and Fermi-energy anisotropy coefficient $\eta_0 = \max \{\delta E_F\}/W_h$. For the square-lattice and VEP quantum liquids, those are well defined for the ranges $x \in (0, x_*)$ and $x \in (x_c, x_*)$, respectively. The velocity $V_{B_{s1}}^\Delta$ and anisotropic part of the Fermi energy $\delta E_F$ of Eq. (20) achieve their maximum magnitudes for s1 band auxiliary momenta pointing in the nodal and antinodal directions, respectively. Hence the coefficients $\eta_\Delta$ and $\eta_0$ contain complementary yet different information. For approximately $U/4t \approx (u_0, u_1)$ they are given in Eq. (5). The hole concentration $x_0$ is defined in that equation as that at which $\eta_\Delta = 1$ for the square-lattice quantum liquid. For such an intermediate $U/4t$ range it is considered that the Fermi-velocity anisotropy is small for hole concentrations larger than $x_{c1} = 1/8$.

For $U/4t \approx 1.525$ the expressions provided in Eq. (5) lead to $x_0 \approx 0.024$ for the square-lattice quantum liquid for which $\Delta_0/t \approx 0.285$ and four representative systems other than LSCO and to $x_0 \approx 0.013$ for LSCO. Indeed, according to the analysis of Ref. $^\Delta$, $\Delta_0/t \approx 0.142$ for the latter system. The magnitudes of the coefficients $\eta_\Delta$ and $\eta_0$ are for $U/4t \approx 1.525$ and several values of $x$ given in Table I for these four representative systems and in Table II for LSCO. As given in the latter table, the anisotropy coefficients are smaller for LSCO. However, intrinsic disorder and randomness are larger for that compound. The Fermi-velocity anisotropy is then considered small at a smaller $\eta_\Delta = 2x_0 \approx 0.03$ value than that $\eta_\Delta = 2x_0 \approx 0.05$ of the other four representative systems. Indeed, due to the LSCO larger intrinsic disorder and randomness, an anisotropy associated with $\eta_\Delta \approx 0.05$ has more impact on the physics than for the remaining four representative cuprates. However, in both cases the hole concentration at which $\eta_\Delta = 2x_0 \approx 0.03$ and $\eta_\Delta = 2x_0 \approx 0.05$, respectively, is approximately $x_{c1} = 1/8$.

As confirmed by the data of the Tables I and II, the coefficient $\eta_0$ decreases upon increasing $x$ slower than $\eta_\Delta$. For the VEP (and square-lattice) quantum liquid the Fermi-line anisotropy is strongest at $x = x_*$ (and for $x \to 0$), when $\eta_\Delta$ and $\eta_0$ are largest (and $\eta_\Delta, \eta_0 \to \infty$). Such an anisotropy vanishes for $x \to x_*$, when $\eta_\Delta, \eta_0 \to 0$ and the Fermi line becomes fully isotropic. Following the $x$ dependence of $\eta_\Delta$ and $\eta_0$, it is considered that the Fermi line is anisotropic for hole concentrations in the range $x \in (x_c, x_{c1})$. For the $x$ range $x \in (x_{c1}, x_{c2})$ it has some Fermi-energy anisotropy, yet the Fermi velocity is nearly isotropic. The Fermi line is nearly isotropic for the hole concentration range $x \in (x_{c2}, x_*)$. That for $x \to x_*$, the Fermi line becomes as isotropic as that of an isotropic Fermi liquid is consistent with the physics tending to that of a Fermi liquid as that critical hole concentration is approached.

For both the $x$ range $x \in (0, x_c)$ and hole concentrations obeying the inequality $0 < (x - x_c) \ll 1$ the hole-trapping effects change the chemical potential $\mu \approx \mu_0 + W_h$ of Eq. (A11) of Appendix A. For the range $x \in (0, x_c)$ it is pinned and given approximately by $\mu \approx \mu_0$. In turn, for $0 < (x - x_c) \ll 1$ it is shifted by $-\delta \mu \approx -\nu_0 2\pi/m_0^2 = -4\pi r_c x t$. The emergence of its dependence on $x$ is then shifted from the hole concentration $x = 0$ to $x = x_c$. The suppression of
the chemical potential shift is for the hole concentration range $x \in (0, x_c)$ due to its pinning by the intrinsic-disorder impurity potential. For $U/4t \geq u_0 \approx 1.302$ it is approximately given by,

$$\mu \approx \mu_0 + W_c^h : \mu_0 = \mu_0^0 - \delta \mu^0 : \delta \mu^0 = 4\pi r_c x_c t, \quad x \in (x_c, x_*) .$$

Here the second chemical potential expression refers to the hole-concentration range $x \in (x_c, x_*)$ mostly considered in the studies of this paper. For it the only modification of the chemical potential $\mu \approx \mu_0 + W_c^h$ of Eq. (A11) of Appendix A is the replacement of the $x$ independent energy term $\mu_0$ by $\mu_0^0 = [\mu_0^0 - 4\pi r_c x_c t]$, the $x$ dependent term $W_c^h$ remaining unaltered. In turn, the first $\mu$ expression of Eq. (B1) is consistent with for hole concentrations below $x_c$, the energy scale $W_c^h$ being ill defined for the VEP quantum liquid. For the particular case of LSCO, in addition to suppressing the magnitude of the energy scale $\Delta_0$ by a factor two, the cation-random effects considered in Ref. 19 cause a further pinning of the chemical potential up to approximately $x \approx 0.14$ and lessen the magnitude of $W_c^h$. Hence the formula (B1) does not apply to that random alloy.

**Appendix C: Hamiltonian terms that control the fluctuations of the $c$ fermion-pair phases**

Here it is shown that the Hamiltonian terms (23) are approximately equivalent to those given in Eq. (31). In order to reach that goal we start by expressing (23) in terms of the phases $\theta_j$,

$$\hat{H}_{\text{eff}} = \sum_{j=1}^{N_s} e^{i\theta_j} \sum_{j', j''}[j-\text{const}] \Delta_0 \langle f_{\bar{r}_{j''}, c}^\dagger f_{\bar{r}_{j''}, c} \rangle \hat{b}_{\bar{r}_{j''}, s1, d, l, g}^\dagger + (\text{h.c.}) .$$

Long-range superconducting order implies the occurrence of coherent $c$ fermion pairing, so that $\langle f_{\bar{r}_{j''}, c}^\dagger f_{\bar{r}_{j''}, c} \rangle$ is finite. We thus use a mean-field approximation for which the Hamiltonian (23) is replaced by,

$$\hat{H}_{\text{eff}} = \sum_{j=1}^{N_s} e^{i\theta_j} \sum_{j', j''}[j-\text{const}] \Delta_0 \langle f_{\bar{r}_{j''}, c}^\dagger f_{\bar{r}_{j''}, c} \rangle \hat{b}_{\bar{r}_{j''}, s1, d, l, g}^\dagger + (\text{h.c.}) .$$

According to Eq. (24), local $c$ fermion pairs with real-space coordinates $\bar{r}_{j''}$ and $\bar{r}_{j''}$ equal to those of a given two-site bond refer to the same local rotated-electron pair. Hence within our approach for pairs with the same centre of mass $\bar{r}_{j''} = [\bar{r}_{j'''} + \bar{r}_{j'''}]/2 \approx [\bar{r}_{j'''} + \bar{r}_{j'''}]/2 - \bar{r}_{d, l, g}^0$ and belonging to the same family and thus having the same values for the indices $d$ and $l$ (where $d = 1, 2$ is not the doublet) the following relation holds,

$$\frac{\langle f_{\bar{r}_{j''}, c}^\dagger f_{\bar{r}_{j''}, c} \rangle}{|\langle f_{\bar{r}_{j''}, c}^\dagger f_{\bar{r}_{j''}, c} \rangle|} \approx \frac{2\hbar_g^*}{|2\hbar_0|} \sum_{g=0}^{[N_s/4]-1} |2\hbar_g|^2 = 1 .$$

Here the index $g$ was added to specify the two-site link type. Hence the expectation values amplitudes are controlled by the coefficient $h_g^*$ of the corresponding spin two-site bond in the $s1$ fermion operator defined in Eq. (20). Such a relation follows from the two $c$ fermions and the corresponding two site bond stemming from the same spin-singlet rotated-electron pair. The expectation value amplitudes decrease upon increasing the distance between the two $c$ fermions of a pair. As discussed in Section III-C, the physics behind the relation (C3) is that the generator of the spin degrees of freedom of the overall occupancy configuration generated by each of the $N/2$ spin-singlet rotated-electron pair operators of the $j$ summation of Eq. (20) is a $s1$ bond-particle operator.
Indeed, the use of the relation (C8) and s1 bond-particle operator expression (26) in the effective Hamiltonian of Eq. (C2) allows after some straightforward algebra to express it in terms of a s1 bond-particle operator as follows,

$$\hat{H}_{\text{eff}}^{\text{bonds}} = \sum_{j=1}^{N_s} \sum_{(j',j'')} e^{i\theta_{j,j''}} \frac{\Delta_0}{4||\vec{q}||} (f^\dagger_{j',c} f^\dagger_{j'',c} b^\dagger_{\vec{r}_{j,j''1}} + \text{h.c.}) .$$  \hspace{1cm} (C4)

The use of the transformation (2) and replacement of $\phi_{j,s1}$ by the phase $\phi_{j,s1}^0$ of Eq. (C2) allows expressing (C4) in terms of the s1 fermion operator $f^\dagger_{\vec{r}_{j,j''1}}$ as follows,

$$\hat{H}_{\text{eff}}^{\text{bonds}} = \sum_{j=1}^{N_s} \sum_{(j',j'')} e^{i\theta_{j,j''}} \frac{\Delta_0}{4||\vec{q}||} (f^\dagger_{j',c} f^\dagger_{j'',c} f^\dagger_{\vec{r}_{j,j''1}} + \text{h.c.}) .$$ \hspace{1cm} (C5)

Finally, note that (C5) is the effective Hamiltonian obtained by replacing in Eq. (31) the operator $f^\dagger_{\vec{r}_{j',c}} f^\dagger_{\vec{r}_{j'',c}}$ by $\langle f^\dagger_{\vec{r}_{j',c}} f^\dagger_{\vec{r}_{j'',c}} \rangle$. This is alike for the effective Hamiltonians (C1) and (C2). Since (29) and (C1) are the same Hamiltonian terms, one concludes that those are approximately equivalent to (31).

**Appendix D: The energy range of the c - s1 fermion interactions behind c fermion strong effective coupling**

Here we derive the c fermion energy range of the c - s1 fermion interactions within a virtual-electron pair configuration. Our results refer to the interactions that lead to c fermion strong effective coupling. That range is expressed as a function of the c fermion hole momenta $\pm \vec{q}^h$ and s1 fermion momentum $\vec{q}$. We start by considering the square-lattice quantum liquid perturbed by weak 3D uniaxial anisotropy without hole-trapping effects. For it there is short-range spin order for $0 < x < x_c$ above which there is long-range superconducting order reads $x_c \approx G_i$ rather than $x_c \approx G_i + x_0$. Thereafter, we extrapolate our analysis to the range $x \in (x_c, x_*)$ of the VEP quantum liquid.

The absolute maximum magnitude of the ratio $|\Omega_{s1}(\vec{q}_{\text{arc}})|/\Delta_0$ where $|\Omega_{s1}(\vec{q}_{\text{arc}})|$ is the pairing energy per electron of Eq. (105) gives for $\gamma_d = 1$ the maximum rate concerning the energy that the short-range spin correlations can supply to c fermion strong effective coupling through the c - s1 fermion interactions within a virtual-electron-pair configuration. Indeed, in expression (105) the suppression coefficient $\gamma_d$ accounts for the part of that energy used in phase-coherent pairing. For $\gamma_d = 1$ such a ratio reaches its absolute maximum magnitude $|\Omega_{s1}(\vec{q}_{\text{arc}})|/\Delta_0 = \gamma_c/4$ at zero temperature, $\vec{q}_{\text{arc}} = \vec{q}_{\text{arc}}$ and $x = x_{\text{op}}$. Importantly, as confirmed by Eq. (64) the ratio $\Delta_0/W_{cc} = \gamma_c/4$ exactly equals that maximum magnitude.

In the absence of hole trapping effects there is c fermion strong effective coupling for $0 < x < x_c$, yet it leads to phase-coherent pairing only for $x > x_c$. The above rate ratio $|\Omega_{s1}(\vec{q}_{\text{arc}})|/\Delta_0 = \Delta_0/W_{cc} = \gamma_c/4$ plays an important role in the c - s1 fermion interactions. Indeed the c fermion energy range of these interactions contributing to strong effective coupling is for $0 < x \ll 1$ controlled by a function given by,

$$|\Delta_{cc}(\vec{q}^h)| = \frac{\Delta_0}{W_{cc}} |\epsilon_c(\vec{q}^h)| ; \quad \epsilon_c(\vec{q}^h) = -W_{cc} \frac{|\Delta_{cc}(\vec{q}^h)|}{\Delta_0} .$$ \hspace{1cm} (D1)

It is such that,

$$|\Delta_{cc}(\vec{q}^h)| = |\Delta_{cc}(-\vec{q}^h)| \in (0, \Delta_0) ; \quad |\Delta_{cc}(\vec{q}_{F_C}^h)| = 0 ; \quad |\Delta_{cc}(\vec{q}_{F_C}^h)| = \Delta_0 .$$ \hspace{1cm} (D2)

The momentum $\vec{q}_{F_C}^h$ appearing here belongs to the zero-temperature cc-pairing line defined in Eq. (73). We recall that the energy scale $W_{cc}$ of Eq. (71) is both the maximum energy bandwidth corresponding to the hole momentum domain $Q_{cc}$ of c fermions with strong effective coupling and the maximum magnitude of the energy bandwidth of the superconducting-ground-state sea of c fermions contributing to phase-coherent pairing. Alike the maximum pairing energy $2|\Omega_{s1}(\vec{q}_{\text{arc}})|$ is smaller than the energy $2\Delta_0$ associated with the short-range spin correlations, also the energy $|\Delta_{cc}(\vec{q}^h)| + |\Delta_{cc}(-\vec{q}^h)|$ is smaller than the c fermion energy $|\epsilon_c(\vec{q}^h)| + |\epsilon_c(-\vec{q}^h)|$. Indeed, for $\gamma_d = 1$ the maximum magnitude of the ratio $2|\Omega_{s1}(\vec{q}_{\text{arc}})|/2\Delta_0$ exactly equals $||\Delta_{cc}(\vec{q}^h)| + |\Delta_{cc}(-\vec{q}^h)||/|\epsilon_c(\vec{q}^h)| + |\epsilon_c(-\vec{q}^h)||$.

In the present $0 < x \ll 1$ limit the energy scale $2\Delta_0$ of Eq. (A1) of Appendix A is both the maximum magnitude of the s1 fermion spinon-pairing energy and the maximum magnitude of the c fermion energy scale $|\Delta_{cc}(\vec{q}^h)| + |\Delta_{cc}(-\vec{q}^h)|$ of Eq. (D2). For $0 < x \ll 1$ it controls the energy bandwidth of the following ranges,

$$2|\Omega_{s1}(\vec{q})| \in (0, 2\Delta_0) ; \quad |\Delta_{cc}(\vec{q}^h)| + |\Delta_{cc}(-\vec{q}^h)| \in (0, 2\Delta_0) .$$ \hspace{1cm} (D3)
Consistently with the ranges $[D3]$ and the above analysis, the energy scale $2\Delta_0$ is for $0 < x \ll 1$ the maximum energy of the short-range spin correlations. It is the source of the energy supplied through the $c$ - $s1$ fermion interactions within each virtual-electron pair configuration, to ensure $c$ fermion strong effective coupling. The occurrence of such a strong effective coupling is a necessary condition for virtual-electron pairing phase coherence. This role of the energy scale $2\Delta_0$ is consistent with: i) $2\Delta_0 = \lim_{\Delta \to 0} 2|\Delta|$ being the maximum magnitude of the $s1$ fermion spinon pairing energy associated with the short-range spin correlations; ii) the presence of $\Delta_0$ in the Hamiltonian terms of Eq. $[20]$, which control the elementary processes associated the $c$ - $s1$ fermion interactions.

Only at zero temperature and for $x \to 0$ all phases $\theta_{j,1}$ line up. Only then $g_1 = |\langle e^{i\theta_{j,1}} \rangle | \approx 1$ and the pseudogap energy $2|\Delta| = g_1 2\Delta_0$ of Eq. $[30]$ reaches its maximum magnitude $2\Delta_0$. Hence for finite hole concentrations and/or temperatures the pseudogap energy $2|\Delta| = g_1 2\Delta_0$ gives the suppressed magnitude of the energy scale $2\Delta_0$ due to the fluctuations of the phases $\theta_{j,1}$. That pseudogap is the order parameter of the short-range spin correlations. As a result of $2\Delta_0$ being the maximum energy of the short-range spin correlations, at small hole concentrations $0 < x \ll 1$ the overall quantity $|\Delta_{ec}(\hat{q}^h) + |\Delta_{ec}(-\hat{q}^h)| + 2|\Delta_{s1}(\hat{q})| \leq 2\Delta_0$. The ranges provided in Eq. $[D3]$ correspond to two limiting cases of such an overall inequality: i) The range $2|\Delta_{s1}(\hat{q})| \in (0, 2\Delta_0)$ holds for $c$ band hole momenta $\pm \hat{q}^h$ at the $c$ Fermi line, so that $|\Delta_{ec}(\hat{q}^h)| = |\Delta_{ec}(-\hat{q}^h)| = 0$; ii) The range $|\Delta_{ec}(\hat{q}^h)| + |\Delta_{ec}(-\hat{q}^h)| \in (0, 2\Delta_0)$ holds for $s1$ band momenta $\hat{q}$ pointing in the nodal directions, so that $2|\Delta_{s1}(\hat{q})| = 0$.

Hence for $0 < x \ll 1$ the residual interactions of $c$ fermions with general hole momenta $\pm \hat{q}^h$ and $-\hat{q}^h$ and energy $\epsilon_c(\hat{q}^h) = \epsilon_c(-\hat{q}^h) = -W_{ec} |\Delta_{ec}(\hat{q}^h)|/\Delta_0$ with a $s1$ fermion of general momentum $\hat{q}$ and spinon-pairing energy $2|\Delta_{s1}(\hat{q})|$ contribute to the strong effective pairing coupling of the former two objects provided that,

$$0 \leq |\Delta_{ec}(\hat{q}^h)| + |\Delta_{ec}(-\hat{q}^h)| + 2|\Delta_{s1}(\hat{q})| \leq 2\Delta_0. \tag{D4}$$

This inequality is equivalent to,

$$|\epsilon_c(\hat{q}^h)| \leq W_{ec} \left( 1 - \frac{|\Delta_{s1}(\hat{q})|}{\Delta_0} \right). \tag{D5}$$

Generalization of the $0 < x \ll 1$ inequality $[D3]$ to finite hole concentrations below $x_s$ and $U/4t \in (u_0, u_{\pi})$ involves the replacement of the energy parameter $\Delta_0 \sim \lim_{x \to 0} |\Delta|$ by the general spin energy scale $|\Delta|$ in the ratio $|\Delta_{s1}(\hat{q})|/\Delta_0$. For finite hole concentrations below $x_s$ it then becomes the inequality provided in Eq. $[67]$. Due to the hole trapping effects discussed in Appendix B, which are strongest for $x \in (0, x_0)$, for the VEP quantum liquid the latter inequality is valid for $x \in (x_0, x_s)$ rather than for $x \in (0, x_s)$.

Strong effective coupling is required for the occurrence of phase-coherent virtual-electron pairing. However, strong effective coupling also occurs in the pseudogap state. That $2|\Omega| = 0$ does not affect though the validity of the inequalities provided in Eqs. $[67]$ and $[D5]$. It defines the $c$ fermion energy range of the $c$ - $s1$ fermion interactions that lead to strong effective coupling independently on whether it is associated with phase-coherent virtual-electron pairing or not.

Appendix E: The $s1$ boundary line nodal and anti-nodal momenta

The controlled approximations used to derive the following expressions of the $s1$ boundary-line absolute-value momenta $\varrho_{B,s1}^N$ and $\varrho_{B,s1}^{AN}$ rely on the change of the $s1$ boundary line shape and size of the square-lattice quantum liquid of Ref. $[2]$ from a square of edge magnitude $\sqrt{2}\pi$ and length $\sqrt{2\pi}$ for $x \to 0$ to a circle of radius $\sqrt{(1-x)2\pi}$ and length $(1-x)2\pi$ for $x \to 1$. For $U/4t \in (u_0, u_1)$ the latter shape is a good approximation for $x > x_{c3}$ where $x_{c3} \approx (x_{c1} + x_s) \approx 0.40$ is defined below and $x_{c1} = 1/8$. The obtained approximate expressions read,

$$\varrho_{B,s1}^N \approx \pi \frac{\pi}{\sqrt{2}} (1-x), \quad x \in (x_{c1}, x_{c1}),$$
$$\approx \frac{7\pi}{\sqrt{2}} = \sqrt{(1-x_{c3})2\pi}, \quad x \in (x_{1}, x_{c3}),$$
$$\approx \sqrt{(1-x)2\pi}, \quad x \in (x_{c3}, 1);$$
$$\varrho_{B,s1}^{AN} \approx \pi \left[ 1 - \frac{1}{2\pi} \tanh \left( \sqrt{\frac{\pi}{x_{c1}}} \right) \right], \quad x \in (x_{c}, x_{c1}),$$
$$\approx C_2 \sqrt{(1-x)2\pi}, \quad x \in (x_{c1}, x_{c3}),$$
$$\approx \sqrt{(1-x)2\pi}, \quad x \in (x_{c3}, 1), \tag{E1}$$
respectively. Here,

\[
C_1 = C_{A|x=x_c} = \frac{8}{\sqrt{3}} \left[ 1 - \frac{1}{2 \pi} \tanh(\sqrt{\pi}) \right] ; \quad C_2 = \frac{C_1}{4} \left( \frac{x_3 - x}{x_3 - x_{c1}} \right),
\]

\[
x_{c3} = 1 - \frac{49 \pi}{256} \approx (x_{c1} + x_s) \approx 0.40 ; \quad x_3 = x_{c3} + \frac{(x_{c3} - x_{c1})}{C_1}\frac{\sqrt{\pi}}{4} - 1.
\]  

(E2)

The coefficient \(C_A = q_{B,s}^{AN}/\sqrt{2}q_{B,s}^N\) has then the following approximate limiting behaviors,

\[
C_A = q_{B,s}^{AN}/\sqrt{2}q_{B,s}^N \approx \frac{1}{1 - x} \left( \frac{x_3 - x}{x_3 - x_{c1}} \right) \sqrt{1 - x}, \quad x \in (x_c, x_{c1}),
\]

\[
\approx \frac{C_1}{\sqrt{2}}, \quad x \in (x_{c3}, 1).
\]  

(E3)

As discussed in Ref. for approximately \(U/4t > u_0\) and hole concentrations in the range \(x \in (0, x_{c1})\) the shape of the \(s_1\) boundary line of the square-lattice quantum liquid is independent of \(U/4t\). This justifies the independence of \(U/4t\) for that range of the \(q_{B,s}^N\) and \(q_{B,s}^{AN}\) expressions given in Eq. (E21) for \(x \in (x_c, x_{c1})\). Indeed, the results of that reference apply to the latter range. (The behavior \(q_{B,s}^{AN} \approx \pi - 2 \pi x \) reported in Ref. is valid for \(x < 1\) rather than for \(x < x_c\); note also that for \(U/4t > u_0\) the expression \(q_{B,s}^{AN} \approx \pi(1 - [1/2\pi] \tanh(\sqrt{8\pi} x))\) of Eq. (E21) applies; In turn, the crossover hole concentration magnitude \(x_{c1} = 1/8\) refers only to the range \(U/4t \in (u_0, u_1)\), for which that expression reads \(q_{B,s}^{AN} \approx \pi(1 - [1/2\pi] \tanh(\sqrt{8\pi} x))\), as given in Eq. (E21).)

For the square-lattice quantum liquid of Ref. the coefficient \(C_A = q_{B,s}^{AN}/\sqrt{2}q_{B,s}^N\) of Eq. (E3) exactly reads \(C_A = 1\) in the limit \(x \to 0\). In the range \(x \in (0, x_{c1})\) for that liquid and \(x \in (x_c, x_{c1})\) for the VEP quantum liquid the overall decreasing of \(q_{B,s}^{AN}\) and \(q_{B,s}^N\) is very similar, so that \(C_A = C_1 \approx 0.97\) at \(x = x_{c1} = 1/8\). At that value the magnitude of \(q_{B,s}^N\) equals the radius \(\sqrt{(1 - x_{c3})/2\pi}\) of the nearly circular \(s_1\) boundary line at a larger hole concentration \(x = x_{c3} \approx (x_{c1} + x_s)\). In the range \(x \in (x_{c1}, x_{c3})\) the \(q_{B,s}^{AN}\) magnitude decreases very little. This justifies why within our approximation \(q_{B,s}^{AN} \approx \pi/\sqrt{2}(1 - x_c) = \sqrt{(1 - x_{c3})/2\pi}\) for \(x \in (x_{c1}, x_{c3})\), as given in Eq. (E21). Since for \(x > x_{c3}\) it is a reasonable good approximation to consider that the \(s_1\) boundary line is a circle of radius \(\sqrt{(1 - x_{c3})/2\pi}\), it is assumed that \(q_{B,s}^{AN} \approx C_2 \sqrt{(1 - x)^{2\pi}}\) for the range \(x \in (x_{c1}, x_{c3}), C_2\) reaching the value \(C_2 = 1\) at \(x = x_{c3}\). For the latter range the magnitudes of both \(q_{B,s}^{AN}\) and \(C_2\) are decreasing functions of \(x\). While the exact form of the decreasing function \(C_2 = C_2(x)\) remains an open problem, that \(C_2 \approx \sqrt{\pi(1 - x_{c1})/2}\) and \(C_2 = 1\) for \(x = x_{c1}\) and \(x = x_{c3}\), respectively, is expected to be a good approximation. As given in Eq. (E22), here we assume that the coefficient \(C_2\) decreases linearly as \(C_2 \propto (x_3 - x)\) where \(x_3\) is provided in that equation. That is the simplest curve connecting the above \(C_2\) magnitudes at \(x = x_{c1}\) and \(x = x_{c3}\), respectively.

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