MAXIMAL PLURISUBHARMONIC MODELS

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ABSTRACT. An analytic pair of dimension \( n \) and center \( V \) is a pair \( (V, M) \) where \( M \) is a complex manifold of (complex) dimension \( n \) and \( V \subset M \) is a closed totally real analytic submanifold of dimension \( n \). To an analytic pair \( (V, M) \) we associate the class \( \mathcal{U}(V, M) \) of the functions \( u : M \to [0, \pi/4] \) which are plurisubharmonic in \( M \) and such that \( u(p) = 0 \) for each \( p \in V \). If \( \mathcal{U}(V, M) \) admits a maximal function \( u \), the triple \( (V, M, u) \) is said to be a maximal plurisubharmonic model. After defining a pseudo-metric \( E_{V, M} \) on the center \( V \) of an analytic pair \( (V, M) \) we prove (see Theorem 4.1, Theorem 5.1) that maximal plurisubharmonic models provide a natural generalization of the Monge-Ampère models introduced by Lempert and Szöke in [16].

1. INTRODUCTION

An analytic pair of dimension \( n \) is a pair \( (V, M) \) where \( M \) is a complex manifold of (complex) dimension \( n \) and \( V \subset M \) is a closed totally real analytic submanifold of dimension \( n \). The submanifold \( V \) is said to be the center of the analytic pair \( (M, V) \). We denote by \( TM, TV \subset TM \) the respective (real) tangent fibre bundles and \( J : TM \to TM \) the complex structure of \( M \).

To an analytic pair \( (V, M) \) we associate the class \( \mathcal{U}(V, M) \) of the functions

\[ u : M \to [0, \pi/4] \]

which are plurisubharmonic in \( M \) and such that \( u(p) = 0 \) for each \( p \in V \). The choice of the constant \( \pi/4 \) will be explained later.

A function \( u \in \mathcal{U}(V, M) \) is said to be maximal (for the pair \( (V, M) \)) if

\[ v(p) \leq u(p) \]

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for every $p \in M$, $v \in \mathcal{U}(V, M)$, $u$ vanishes exactly on $V$, that is $u(p) > 0$ for every $p \in M \setminus V$, and
\[ \sup_{p \in M} u(p) = \frac{\pi}{4}. \]
Clearly, a maximal element in $u \in \mathcal{U}(V, M)$ is unique, provided it exists. We say that a triple $(V, M, u)$ is a maximal plurisubharmonic model (of bounded type), for short a maximal model, if $(V, M)$ is an analytic pair and $u \in \mathcal{U}(V, M)$ a maximal function.

With a (little) abuse of language we say that an analytic pair $(V, M)$ is a bounded maximal model provided there exists a maximal function $u \in \mathcal{U}(V, M)$.

Let now $u \in \mathcal{U}(V, M)$ where $(V, M)$ is an analytic pair. For $p \in V$ and $\xi \in TV \subset TM$ the formula
\[ E_{u, M}(p, \xi) = \text{“slope of } u \text{ at } p \text{ in the direction } J\xi \text{”} \]
defines a pseudo-metric on $V$ associated to the function $u$.

Taking
\[ \sup_{u \in \mathcal{U}(V, M)} E_{u, M}(p, \xi) \]
we define a pseudo-metric $E_{V, M}$ on $V$ which depends only on the geometry of $\mathcal{U}(V, M)$. If $(V, M, u)$ is a maximal model $E_{V, M}$ actually coincides with $E_{u, M}$. (See Section 3) for the precise definitions).

We now explain the motivations of our construction.

Following [16] we recall that a Monge-Ampère model of dimension $n$ is a triple $(V, M, u)$ where
1) $(V, M)$ is an analytic pair of dimension $n$;
2) $u$ is a continuous, plurisubharmonic function such that $V = \{u = 0\}$;
3) $u$ is a smooth solution on $M \setminus V$ of the (complex) Monge-Ampère equation
\[ (dd^c u)^n = 0; \]
4) $u^2$ is smooth and strictly plurisubharmonic exhaustion function on $M$.
In such conditions $V$ is called the center of the Monge-Ampère model $(V,M,u)$. Moreover, if the function $u$ is bounded then $(V,M,u)$ is said to be of bounded type.

A holomorphic map $F : (V_1,M_1,u_1) \to (V_2,M_2,u_2)$ between Monge-Ampère models is a holomorphic map $F : M_1 \to M_2$ such that $F(V_1) \subset V_2$ and $u_1 = F \circ u_2$.

Two Monge-Ampère models $(V_1,M_1,u_1)$ and $(V_2,M_2,u_2)$ are said to be isomorphic if there exists a biholomorphic map $F : M_1 \to M_2$ such that $F(V_1) = V_2$ and $u_1 = F \circ u_2$.

The center $V$ of a Monge-Ampère model $(V,M,u)$ is a Riemannian manifold with metric $g$ given by the restriction to the tangent bundle $TV$ of the Levi form $\mathcal{L}(u^2)$ of $u^2$.

Lempert and Szöke proved in [16] that every compact Riemannian manifold $(V,g)$ is, canonically, the center of a Monge-Ampère model of bounded type $(V,M,u)$. Moreover, $(V,M,u)$ is completely determined (up to isomorphisms) by the Riemannian manifold $(V,g)$, i.e. two bounded Monge-Ampère models $(V_1,M_1,u_1)$ and $(V_2,M_2,u_2)$ are isomorphic if and only if their respective centers $(V_1,g_1)$ and $(V_2,g_2)$ are isometric Riemannian manifolds.

The canonical model is constructed as follows. Let $u$ be the length function $|\cdot| : TV \to [0, +\infty]$, associated to $g$. Identify $V$ with the zero section of $TV$, and consider, for $0 < r \leq +\infty$, the $r$-tube

$$\mathcal{T}_rV = \{ \xi \in TV \mid u(\xi) < r \}$$

with center $V$. Then, for $r > 0$ small enough, $\mathcal{T}_rV$ carries an unique complex structure such that the triple $(V, \mathcal{T}_rV, u)$ is a Monge-Ampère model and the restriction to the tangent bundle $TV$ of the form $2\mathcal{L}(u^2)$ is exactly the Riemannian metric $g$ (see [16] and [18], or [10]).

The manifold $\mathcal{T}_rV$ is called a Grauert tube of radius $r$ over the Riemannian manifold $V$. The name “Grauert tube” is due to the following theorem proved by Grauert in [9]: every real analytic manifold $V$ of dimension $n$ embeds as a maximal totally real submanifold of an $n$-dimensional complex manifold $M$ in such a way to have a basis of Stein neighbourhoods.
A Grauert tube $T_rV$ is said to be rigid if each biholomorphic automorphism $f : T_rV \to T_rV$ preserves the center $V$.

Grauert tubes, and their extension to non compact centers, are widely studied complex manifolds, especially in connection with curvature problems ([16]) and rigidity problems (see e.g. [6], [5], [11], [13] and [12]).

By the way it would be interesting to have an analogous of the canonical model starting from a center equipped with a Finsler metric.

The goal of this paper is to show that maximal models of bounded type provide a natural generalization of the (bounded) Monge-Ampère ones. The results obtained here must be considered as a preliminary exploration of the geometry of such models. Clearly, assuming no kind of regularity of $u^2$ the Riemannian geometry (of the center) should be replaced by a "pseudo-metric geometry". We claim that the pseudo-metric $E_{V,M}$ defined in this paper is the right object for our scope.

The paper is organized as follows.

In Section 2, for the sake of completeness, we prove some simple variations of Hopf lemma and Phragmen-Lindelöf principle for subharmonic function of one complex variable, in a form that we need in the sequel.

In Section 3 we introduce the pseudo-metrics $E_{u,M}$ and $E_{V,M}$ and describe their basic properties. It turns out that if $M$ is the unit disc $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ and $V = ] - 1,1[\]$, then the associated metric on the center $] - 1,1[\] is the restriction of the Poincaré metric on $\Delta$ (this is the reason for the constant $\pi/4$ above). Moreover, if $(V_1,M_1)$, $(V_2,M_2)$ are analytic pairs and $F : M_1 \to M_2$ is a holomorphic map which such that $F(V_1) \subset V_2$ then $F$ is a contraction for the corresponding pseudo-metrics on the centers. Thus, our theory is a Kobayashi-like pseudo-metric theory. In [8] it was proved that the class of all Finsler pseudo-metrics on the center of an analytic pair $(V,M)$ having this contraction property admits a largest element $F_{V,M}$, so that $E_{V,M} \leq F_{V,M}$. For the definition and the main properties of the metric $F_{V,M}$ we refer to [8]. It turns out that the equality $E_{V,M} =$
$F_{V,M}$ is related to the existence of “complex geodesic” for such pseudo-metrics (see Theorem 3.5).

It should be observed that the pseudo-metric $E_{V,M}$ is positively homogeneous but in general it is not symmetric, that is, for $p \in V$ and $\xi \in T_p V$ it may happen that $E_{V,M}(p, -\xi) \neq E_{V,M}(p, \xi)$.

Sections 4, Section 5 are devoted to the interplay between maximal functions for analytic pairs $(V, M)$ and solutions of the complex Monge-Ampère equation on $M \setminus V$. After proving that if $u \in \mathcal{U}(V, M)$ is a continuous exhaustion function on $M$ then $(V, M, u)$ is a maximal model if and only if $(dd^c u)^n = 0$ on $M \setminus V$ (see Theorem 4.1), in Section 5 we show that for a Monge-Ampère model $(V, M, u)$ of bounded type one has

$$E_{V,M}(p, \xi) = F_{V,M}(p, \xi) = \sqrt{2L_u^2(p, \xi)}.$$ 

i.e. the pseudo-metric $E_{V,M}(p, \xi)$ coincides with the Riemannian metric on $V$ (see Theorem 5.1).

Finally, in the last two sections we give two significant examples of generalized Monge-Ampère models maximal model which are not (unless exceptional cases) Monge-Ampère models.

In Section 6 we prove that if $\mu : \mathbb{R}^n \to [0, +\infty]$ is the Minkowsky functional associated to a bounded open convex subset of $\mathbb{R}^n$ containing the origin (not necessarily symmetric with respect to the origin) then $(\mathbb{R}^n, X_{\mu}, u)$, where

$$X_{\mu} = \{z = x + iy \in \mathbb{C}^n \mid \mu(y) < \pi/2\}$$

and $u(z) = u(x + iy) = \mu(y)$, is a maximal model. It is worthy of observing that this example easily generalizes if $\mathbb{R}^n$ is replaced by an arbitrary real Banach space where, in general, we have no analogous of the Monge-Ampère operator, while the definition of maximal plurisubharmonic bounded function is exactly the same.
In Section 7 we shall prove that if $D \subset \mathbb{R}^n$ is a bounded open convex set and $D_{\text{ell}}$ the elliptic tube over the convex set $D$ described by Lempert in [15], then $(D, D_{\text{ell}})$ is a bounded maximal model for which an explicit description of the corresponding maximal function $u$ is provided.

2. Subharmonic Functions

The purpose of this section is to prove following Theorem 2.1, which is the “Schwarz lemma” in our context.

Let us begin recalling the classical Hopf lemma in the form that we need in the sequel.

**Proposition 2.1.** Let $D \subset \mathbb{C}$ be open, $D \neq \mathbb{C}$, and $u : D \to [-\infty, 0]$ a negative subharmonic function. For $z \in D$ denote by $\delta(z)$ the distance from $z$ to $\partial D$. Let $x \in \partial D$ and assume that $\partial D$ is of class $C^2$ in a neighbourhood of $x$. Then

$$\limsup_{D \ni z \to x} \frac{u(z)}{\delta(z)} < 0.$$ 

For a proof see e.g. Proposition 12.2 of [7].

**Theorem 2.1.** Given $r, a > 0$, let

$$D = \{ z \in \mathbb{C} \mid 0 < \text{Im} \, z < r \}$$

and $u : D \to [0, a]$ a bounded subharmonic function such that

$$\lim_{D \ni z \to x} u(z) = 0$$

for each $x \in \mathbb{R}$. Then, for every $z \in D, x \in \mathbb{R}$

$$u(z) \leq \frac{a}{r} \text{Im} \, z, \quad \limsup_{y \to 0^+} \frac{u(x + iy)}{y} \leq \frac{a}{r}.$$ 

If there exist either $z_0 \in D$ such that

$$u(z_0) = \frac{a}{r} \text{Im} \, z_0$$

or $x_0 \in \mathbb{R}$ such that

$$\limsup_{y \to 0^+} \frac{u(x_0 + iy)}{y} = \frac{a}{r},$$
then
\[ u(z) = \frac{a}{r} \text{Im} z \]
for every \( z \in D \).

Proof. The function \( v : D \to \mathbb{R} \) defined by
\[ v(z) = u(z) - \frac{a}{r} \text{Im} z \]
is bounded, subharmonic on \( D \) and satisfy
\[ \limsup_{D \ni z \to \xi} v(z) \leq 0 \]
for each \( \xi \in \partial D \).

By the Phragmen-Lindelöf principle (see e.g. Proposition 4.9.45, pag. 463 of [3]) for each \( z \in D \)
\[ v(z) \leq 0, \]
that is
\[ u(z) \leq \frac{a}{r} \text{Im} z, \]
and hence, for each \( x \in \mathbb{R} \),
\[ \limsup_{y \to 0^+} \frac{u(x + iy)}{y} = \frac{a}{r}. \]
If \( u(z_0) = a/r \text{Im} z_0 \), for some \( z_0 \in D \), then \( v(z_0) = 0 \) and hence, by the maximum principle for the subharmonic functions, \( v(z) = 0 \) for each \( z \in D \). It follows that
\[ u(z) = \frac{a}{r} \text{Im} z, \]
and consequently, for each \( x \in \mathbb{R} \),
\[ \limsup_{y \to 0^+} \frac{u(x + iy)}{y} \leq \frac{a}{r}. \]
Otherwise \( v(z) < 0 \) for every \( z \in D \), that is
\[ u(z) < \frac{a}{r} \text{Im} z, \]
and for each \( x \in \mathbb{R} \), in view of Theorem (2.1),
\[ \limsup_{y \to r^-} \frac{v(x + iy)}{y} < 0, \]
namely
\[ \limsup_{y \to 0^+} \frac{u(x + iy)}{y} < \frac{a}{r}. \]
This proves the theorem. \( \rule{2cm}{0.5mm} \)

3. PSEUDO-METRICS

Let \( M \) a (connected) complex manifold of dimension \( n \).

Given \( p \in M, \xi \in T_pM \), we denote by \( \Gamma_M(p, \xi) \) the space of \( C^1 \) maps \( \gamma : [-\varepsilon, \varepsilon] \to M \), for some \( \varepsilon > 0 \), which satisfy \( \gamma(0) = p \),
\[ \gamma'(0) := d\gamma(0) \left( \frac{d}{dt} \right) = \xi. \]

For any subset \( V \subseteq M \) we denote \( \mathcal{U}(V, M) \) the class of functions
\[ u : M \to [0, \pi/4]\]
which are plurisubharmonic in \( M \) and vanishing on \( V \).

As explained in the introduction, an element \( \bar{u} \in \mathcal{U}(V, M) \) is said to be maximal if \( u(p) \leq \bar{u}(p) \) for every \( u \in \mathcal{U}(V, M) \) and \( p \in M \). Clearly, a maximal element in \( \mathcal{U}(V, M) \) is unique, provided it exists, and a maximal element exists in \( \mathcal{U}(V, M) \) if and only if
\[ \sup_{u \in \mathcal{U}(V, M)} u = \bar{u}. \]
Assume now that \((V, M)\) is an analytic pair. For \( p \in V, \xi \in T_pV \subset T_pM, u \in \mathcal{U}(V, M) \) we set
\[ E_{u, M}(p, \xi) = \inf_{\gamma \in \Gamma(p, \xi)} \limsup_{t \to 0^+} \frac{u(\gamma(t))}{t} \]
and
\[ E_{V, M}(p, \xi) = \sup_{u \in \mathcal{U}(V, M)} E_{u, M}(p, \xi). \]
Clearly, if \( \bar{u} \in \mathcal{U}(V, M) \) admits a maximal element \( \bar{u} \), one has
\[ E_{\bar{u}, M}(p, \xi) = E_{V, M}(p, \xi). \]
Moreover, \( E_{u, M} \) and \( E_{V, M} \) are positively homogeneous functions on \( TV \), i.e.
\[ E_{u, M}(p, t\xi) = tE_{u, M}(p, \xi), \]
for $t > 0$. Observe that, in general, $E_{u,M}$, $E_{V,M}$ are not symmetric with respect to $\xi$.

Assuming a few of regularity on $u$ the definition of $E_{u,M}(p, \xi)$ simplifies:

**Lemma 3.1.** Let $u \in \mathcal{U}(V, M)$ and $p \in V$. If $u$ is Lipschitz in a neighbourhood of $p$ then for every $\xi \in T_p V$ and $\gamma \in \Gamma_M(p, J_\xi)$

$$E_{u,M}(p, \xi) = \limsup_{t \to 0^+} \frac{u(\gamma(t))}{t}.$$ 

**Proof.** It is sufficient to prove that for arbitrary $\gamma_1, \gamma_2 \in \Gamma_M(p, J_\xi)$ it results

$$\limsup_{t \to 0^+} \frac{u(\gamma_1(t))}{t} = \limsup_{t \to 0^+} \frac{u(\gamma_2(t))}{t}.$$

Let $\gamma_1, \gamma_2 \in \Gamma_M(p, J_\xi)$ and $z_1, \ldots, z_n$ local complex coordinates near $x$. Then, for $t > 0$ sufficiently small, we have

$$u(\gamma_1(t)) \leq u(\gamma_2(t)) + (u(\gamma_1(t)) - u(\gamma_2(t)))$$

$$\leq u(\gamma_2(t)) + C |z(\gamma_1(t)) - z(\gamma_2(t))| \leq u(\gamma_2(t)) + o(t),$$

and consequently

$$\limsup_{t \to 0^+} \frac{u(\gamma_1(t))}{t} \leq \limsup_{t \to 0^+} \frac{u(\gamma_2(t))}{t}.$$ 

Interchanging $\gamma_1$ and $\gamma_2$ we get the opposite inequality. //

**Theorem 3.1.** Let

$$M = \{ z \in \mathbb{C} \mid |\text{Im}z| < \pi/4 \}.$$ 

Then the function $u(z) = |\text{Im}z|$ belongs to $\mathcal{U}(\mathbb{R}, M)$ and is maximal. Moreover,

$$E_{\mathbb{R}, M}(x, \xi) = |\xi|$$

for every $x \in \mathbb{R}$, $\xi \in \mathbb{R} = T_x \mathbb{R}$. 

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**Proof.** Maximality is a consequence of Theorem (2.1). Since $u$ is Lipschitz, Lemma (3.1) then implies 

$$E_{R,M}(x,\xi) = \limsup_{t \to 0^+} \frac{|\text{Im}(x + it\xi)|}{t} = |\xi|.$$ 

\[
\]

**Theorem 3.2.** Let $\Delta$ be the (open) unit disc in $\mathbb{C}$, $I$ the interval $]-1,1[$. The function 

$$u(z) = |\text{Im}(\text{arctanh}(z))|$$

is maximal in $\mathcal{U}(I,\Delta)$. Moreover, for every $x \in I$ and $\xi \in \mathbb{R} = T_x I$ one has 

$$E_{I,\Delta}(x,\xi) = \frac{|\xi|}{1-x^2}.$$ 

**Proof.** We observe that the function 

$$f(z) = \text{arctanh}(z) = \frac{1}{2} \log \frac{1+z}{1-z}$$

is a biholomorphism between $\Delta$ and 

$$M = \{z \in \mathbb{C} \mid |\text{Im}(z)| < \pi/4\}$$

$f(I) = \mathbb{R}$ and 

$$f'(z) = \frac{1}{1-z^2}.$$ 

The statement is then an immediate consequence of Theorem (2.1). 

The quantities $E_{V,M}$ decrease by holomorphic maps:

**Theorem 3.3.** Let $(V,M)$, $(W,N)$ be analytic pairs and $f : M \to N$ a holomorphic map such that $f(V) \subset W$. Then 

$$E_{W,N}(f(p),df(p)(\xi)) \leq E_{V,M}(p,\xi).$$

for every $p \in V$, $\xi \in T_p V$. 

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Proof. We may assume that $E_{V,M}(p, \xi) < +\infty$. Let $\varepsilon > 0$ be fixed and $u \in \mathcal{U}(W,N)$. Then $u \circ f \in \mathcal{U}(V,M)$ and by definition of $E_{u,M}$ there exists $\gamma \in \Gamma_M(p,J\xi)$ such that

$$
\limsup_{t \to 0^+} \frac{u \circ f(\gamma(t))}{t} < E_{u \circ f,M}(p, \xi) + \varepsilon \leq E_{V,M}(p, \xi) + \varepsilon.
$$

Then $u \circ f \in \Gamma_N(f(p), Jd f(p)(\xi))$ and consequently

$$
E_{u,N}(f(p), d f(p)(\xi)) \leq \limsup_{t \to 0^+} \frac{u(f \circ \gamma(t))}{t} = \limsup_{t \to 0^+} \frac{u \circ f(\gamma(t))}{t} < E_{V,M}(p, \xi) + \varepsilon.
$$

Since $\varepsilon > 0$ is arbitrary we get

$$
E_{u,N}(f(p), d f(p)(\xi)) \leq E_{V,M}(p, \xi).
$$

We obtain the desired inequality taking the supremum over $\mathcal{U}(W,N)$. II

Consider now the unit disc $\Delta$ and recall that for $p \in V, \xi \in T_p V$ one defines

$$
F_{V,M}(p, \xi) = \inf\left\{ a > 0 \mid \exists f \in \text{Hol}(\Delta, M), f([-1, 1[ \subset M, f(0) = p, f'(0) = a^{-1}\xi \right\}.
$$

(1)

(cf. [3]) If $f \in \text{Hol}(\Delta, M)$ satisfies $f([-1, 1[ \subset M, f(0) = p, f'(0) = a^{-1}\xi$ then, in view of Theorems 3.3, 3.2, we have

$$
E_{V,M}(p, \xi) \leq E_{[-1,1[|\Delta}(0, a) = a;
$$

taking the infimum of $a$ over all maps $f \in \text{Hol}(\Delta, M)$ we get:

**Theorem 3.4.** Let $V \subset M$ be an analytic pair. Then

$$
E_{V,M}(p, \xi) \leq F_{V,M}(p, \xi).
$$

for every $p \in V, \xi \in T_p V$.

The theorem which follows characterizes the “complex geodesic” for the pseudo-metrics $E_{V,M}$.\textsuperscript{11}
Theorem 3.5. Let $(V, M)$ be an analytic pair. Let $S = \{ |\text{Im}(z)| < \pi/4 \}$ and $f : S \to M$ be a holomorphic map such that $f(\mathbb{R}) \subset V$. Then for a function $u \in \mathcal{U}(V, M)$ the following conditions are equivalent:

i) for every $z \in S$
$$u(f(z)) = |\text{Im}(z)|;$$

ii) for every $x \in \mathbb{R}$, $\xi \in \mathbb{R} = T_x \mathbb{R}$
$$E_{u,M}(f(x), df(x)(\xi)) = |\xi|;$$

iii) there is $x_0 \in \mathbb{R}$ such that
$$E_{u,M}(f(x_0), df(x_0)(\xi)) = |\xi|.$$

for every $\xi \in \mathbb{R} = T_x \mathbb{R}$.

Moreover, if such conditions are fulfilled, for every $x \in \mathbb{R}$, $\xi \in \mathbb{R}$ the following identities hold
$$E_{u,M}(f(x), df(x)(\xi)) = E_{V,M}(f(x), df(x)(\xi)) = F_{V,M}(f(x), df(x)(\xi)) = |\xi|.$$

Proof. The implications i) $\implies$ ii), ii) $\implies$ iii) are evident and that iii) $\implies$ i) follows immediately from Theorem 2.1. In order to prove the last equality it is sufficient to observe that by definition
$$E_{u,M}(f(x), df(x)(\xi)) \leq E_{V,M}(f(x), df(x)(\xi));$$

moreover, by Theorem 3.4
$$E_{V,M}(f(x), df(x)(\xi)) \leq F_{V,M}(f(x), df(x)(\xi)),$$

and by the properties of $F_{V,M}$ (cf. [8])
$$F_{V,M}(f(x), df(x)(\xi)) \leq F_{\mathbb{R},S}(x, \xi) = |\xi|.$$

Then if ii) holds
$$|\xi| = E_{u,M}(f(x), df(x)(\xi)).$$

II

The holomorphic maps $f : S \to M$ which satisfy $f(\mathbb{R}) \subset V$ and the conditions of Theorem 3.5 are called $E_u-$complex geodesic. The $E-$complex geodesic are a useful tool to give sufficient conditions in order to state maximality of plurisubharmonic functions in $\mathcal{U}(W, N)$. 

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Theorem 3.6. Let \((V, M)\) an analytic pair and \(u \in \mathcal{U}(V, M)\). Suppose that for every \(q \in M \setminus V\) there exists an \(E_u\)-complex geodesic \(f : S \to M\) such that \(q \in f(S)\). Then \(u\) is maximal.

Proof. Let \(w \in \mathcal{U}(V, M)\). We have to prove that \(w(q) \leq u(q)\) for every \(q \in M\) so let \(q \in M\). If \(q \in V\) then \(w(q) = u(q) = 0\) and in such a case the thesis is evident, so we assume that \(q \in M \setminus V\). Let \(S = \{|\Im(z)| < \pi/4\}\) and \(f : S \to M\) una \(E\)-complex geodesic such that \(f(z_0) = q, z_0 \in S\). In view of Theorem 3.5 we have \(u(q) = |\Im(z_0)|\). Observe now that \(u(f(z))\) is subharmonic in \(S\) and \(0 \leq u(f(z)) \leq \pi/4\), so, in view of Theorem 2.1 we have \(w(f(z)) \leq |\Im(z)|\) for every \(z \in S\). In particular

\[ v(q) = v(f(z_0)) \leq |\Im(z_0)| = u(q). \]

and from this it follows that \(u\) is maximal, \(q\) being arbitrary. \(\square\)

4. Complex Monge-Ampère Equation

The theorem which follows put in evidence the relationship between maximal functions for analytic pairs \((V, M)\) and solutions of the complex Monge-Ampère equation on \(M \setminus V\). For the main results about existence, unicity and maximum principle for solutions of the complex Monge-Ampère equation we refer to \[2\].

Theorem 4.1. Let \((V, M)\) be an analytic pair and \(u \in \mathcal{U}(V, M)\). Then

i) if \(u\) is continuous and maximal, \((dd^c u)^n = 0\) on \(M \setminus V\);

ii) if \(u\) is an exhaustion function such that \((dd^c u)^n = 0\) on \(M \setminus V\) and \(u(p) = 0\) if and only if \(p \in V\) then \(u\) is maximal.

In particular, if \(u\) is a continuous exhaustion function and \(u(p) = 0\) if and only if \(p \in V\) then \(u\) is maximal if and only if \((dd^c u)^n = 0\) on \(M \setminus V\).
Proof. Assume that $u$ is continuous and maximal and let us show that actually $u$ is a solution of $(dd^cu)^n = 0$ on $M \setminus V$.

Let $p \in M \setminus V$ and $U \subset M \setminus V$ be a relatively compact neighbourhood of $p$. Let $w : M \to [0, \pi/4]$ be the function defined by

$$w(p) = \begin{cases} u(p) & p \in M \setminus U, \\
v(p) & p \in U, \end{cases}$$

where $v$ is the solution of the problem

$$\begin{cases} (dd^c v)^n = 0 & \text{in } U, \\
v = u & \text{in } \partial U. \end{cases}$$

The function $v$ is characterized by

$$v(p) = \sup \{ w(p) \}$$

where the supremum is taken over the set of functions $w$ which are plurisubharmonic in $U$, continuous on $\overline{U}$ and satisfying $w \leq u$ on $\partial U$. The function $w$ belongs to $\mathcal{U}(V, M)$ and, by construction, $w \geq u$. Since $u$ is maximal then $u = w$; in particular $u$ is a solution of Monge-Ampère in a neighbourhood of $p \in M \setminus V$. Thus $u$ is a solution of $(dd^cu)^n = 0$ on $V \setminus M$, $p \in M \setminus V$ being arbitrary.

Conversely, suppose that $u$ is an exhaustion function for $M$ and a solution of the Monge-Ampère equation on $V \setminus M$. In particular, $V$ is a compact submanifold of $M$. Let $w$ be an arbitrary function of $\mathcal{U}(V, M)$. We have to prove that $w(p) \leq u(p)$ for every $p \in M$. This is certainly true if $p \in M$ since then $w(p) = u(p) = 0$. Thus we assume that $p \in M \setminus V$. By hypothesis $u(p) > 0$. Let $\varepsilon > 0$ be such that $u(p) < \frac{\pi}{4} - \varepsilon$,

$$D = \{ q \in V \mid 0 < u(q) < \frac{\pi}{4} - \varepsilon \};$$

since $u$ is an exhaustion function $D$ is relatively compact. Let $F_1$ be the subset of the boundary $\partial D$ of $D$ where $u$ takes the value $\frac{\pi}{4} - \varepsilon$

$$u_\varepsilon = \frac{\pi/4}{\pi/4 - \varepsilon} u.$$

Then $\partial D = M \cup F_1$ and we are going to show that $u_\varepsilon \geq w$ on $\partial D = M \cup F_1$. Indeed, if $q \in V$ then $w(q) = 0 = u_\varepsilon(q)$ and if
\( q \in F_1 \) then \( w(q) < \pi/4 = u_{\varepsilon}(q) \). Since \( w \) is plurisubharmonic we have \((dd^c v)^n \geq 0 = (dd^c u_{\varepsilon})^n\) on \( D \), whence \( w(q) \leq u_{\varepsilon}(q) \) for every \( q \in D \). In particular \( w(p) \leq u_{\varepsilon}(p) \) for every \( \varepsilon > 0 \), hence \( w(p) \leq u(p) \). Since \( p \in M \) is arbitrary \( u \geq w \) on \( M \). //

5. THE SMOOTH CASE

Let \( M \) be a complex manifold and \( u \) a \( C^2 \) function on \( M \). Let \( p \in M, \xi \in T_pM \) and \( f \) be a germ at \( 0 \in \mathbb{C} \) of a holomorphic map with values in \( M \) such that \( f(0) = p, f'(0) = \xi \). Then the complex number

\[
\mathcal{L}_u(p, \xi) = \frac{\partial^2(u \circ f)(0)}{\partial z \partial \overline{z}},
\]

depends only on \( u, p, \xi \) and it is nothing but that the Levi form of \( u \) at \( p \) evaluated at \( \xi \).

**Proposition 5.1.** Let \((V, M)\) be an analytic pair and \( u \in \mathcal{U}(V, M) \). Assume that \( u^2 \) is \( C^2 \) around \( V \). Then, for every \( p \in V \) and \( \xi \in T_pV \), we have the following equality

\[
E_{u, M}(p, \xi) = \sqrt{2 \mathcal{L}_{u^2}(p, \xi)}.
\]

**Proof:** Let \( p \in V, \xi \in T_pV \) and \( f \) be a holomorphic map with values in \( M \), defined in a neighbourhood \( U \) of the origin \( 0 \in \mathbb{C} \) and such that \( f(0) = p, f'(0) = \xi \). Since \( u^2 \) is of class \( C^2 \) in a neighbourhood of \( V \) it is locally Lipschitz in a neighbourhood of \( V \). Then Lemma 3.1 implies

\[
E_{u, M}(p, \xi) = \limsup_{y \to 0^+} \frac{u \circ f(\text{iy})}{\xi}.
\]

Now we set

\[
g(y) = (u \circ f(\text{iy}))^2,
\]

and observe that, since \( u \circ f \) vanishes on \( U \cap \mathbb{R} \), one has

\[
2 \mathcal{L}_{u^2}(p, \xi) = 2 \frac{\partial^2(u \circ f)^2(0)}{\partial z \partial \overline{z}} = \frac{1}{2} \Delta(u \circ f)^2(0) = \frac{1}{2} g''(0),
\]

where

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.
\]
On the other hand, since \( g(y) \) is \( C^2 \), non negative and vanishing on \( y = 0 \), from the elementary identity

\[
\sqrt{\frac{1}{2}g''(0)} = \lim_{y \to 0} \frac{\sqrt{g(y)}}{y}
\]

we get

\[
\sqrt{2\mathcal{L}_{u^2}(p,v)} = \sqrt{\frac{1}{2}g''(0)} = \lim_{y \to 0^+} \frac{u \circ f(iy)}{y} = E_{u,M}(p, \xi).
\]

This proves the proposition. \( \square \)

**Theorem 5.1.** Let \((V, M)\) be an analytic pair and \( u \in \mathcal{U}(V, M) \). Assume that \((V, M, u)\) is a Monge-Ampère model. Then, for every \( p \in V, \xi \in T_pV \)

\[
E_{V,M}(p, \xi) = F_{V,M}(p, \xi) = \sqrt{2\mathcal{L}_{u^2}(p, \xi)}.
\]

In particular, \((V, M, u)\) is a maximal model.

**Proof.** We have that \( V = \{u = 0\} \), \((dd^cu)^n = 0\) on \( M \setminus V \) and \( u^2 \) is a smooth, strictly plurisubharmonic exhaustion function for \( M \).

Let \( p \in V, \xi \in T_pV \). Without loss of generality we may assume

\[
\sqrt{2\mathcal{L}_{u^2}(p, \xi)} = 1.
\]

In view of Theorem 4.1 the function \( u \in \mathcal{U}(V, M) \) is maximal i.e.

\[
E_{V,M}(p, \xi) = E_{u,M}(p, \xi),
\]

so, by Proposition 5.1

\[
E_{u,M}(p, \xi) = \sqrt{2\mathcal{L}_{u^2}(p, \xi)} = 1.
\]

Let us denote \( g \) the Riemannian metric induced on \( V \) by the restriction to \( TV \) of the Levi form \( dd^c(u^2) \) (cf. [17]). Since \( u \) is an exhaustion function for \( M \), \((V, g)\) is a compact Riemannian manifold, so there is a geodesic \( \gamma: \mathbb{R} \to M \) such that \( \gamma(0) = p, \gamma'(0) = \xi \) and

\[
g(\gamma(x), \gamma'(x), \gamma'(x)) = 1
\]
for every \( x \in \mathbb{R} \).
In view of the results proved in \cite{17} (cf. also Theorem 3.1 of \cite{16}), if \( S = \{ |\text{Im} z| < \pi/4 \} \), the map \( f : S \to M \) defined by
\[
f(z) = f(x + iy) = d\gamma(x)(y)
\]
is holomorphic. By construction \( f(\mathbb{R}) \subset M \) and, moreover,
\[
1 = E_{u,M}(p, \xi) = E_{u,M}(f(0), f'(0)).
\]

Theorem \[ \ref{3.5} \] now implies that \( f \) is an \( E \)-complex geodesic and, since \( p = f(0), \xi = f'(0) \), one has
\[
F_{V,M}(p, \xi) = E_{V,M}(p, \xi) = E_{u,M}(p, \xi) = \sqrt{2L_{u^2}(p, \xi)} = 1.
\]

\[
//
\]

6. **Convex Homogeneous Real Functions**

For every point \( z \in \mathbb{C}^n \) we set \( z = x + iy, x, y \in \mathbb{R}^n, x = \text{Re} z, y = \text{Im} z \).

Let \( \mu : \mathbb{R}^n \to [0, +\infty[ \) be a positively homogeneous convex function such that \( \mu(x) = 0 \) if and only if \( x = 0 \); \( \mu \) is the Minkowsky functional associated to a bounded open convex subset of \( \mathbb{R}^n \). Observe that we do not require the property \( \mu(-x) = \mu(x) \) for \( x \in \mathbb{R}^n \).

Let
\[
X_\mu = \{ z \in \mathbb{C}^n | \mu(\text{Im}(z)) < \pi/4 \}
\]
and \( u_\mu : X_\mu \to [0, \pi/4[ \) be the function defined by
\[
u_\mu(z) = \mu(\text{Im}(z))
\]
for every \( z \in X_\mu \).

**Theorem 6.1.** Let \( X_\mu, u_\mu \) be as above. Then \( u_\mu \in U(\mathbb{R}^n, X_\mu) \) is maximal and for every \( x \in \mathbb{R}^n, \xi \in \mathbb{R}^n = T_x\mathbb{R}^n \) the following identity holds true
\[
E_{\mathbb{R}^n, X_\mu}(x, \xi) = E_{u_\mu, X_\mu}(x, \xi) = \mu(\xi).
\]
Proof. Assume first that \( \mu \) is of class \( C^2 \). Then, since the function \( u_\mu (x + iy) \) does not depend on \( x \) it follows that

\[
\frac{\partial^2 u_\mu (x + iy)}{\partial z_i \partial \bar{z}_j} = \frac{1}{4} \frac{\partial^2 \mu (y)}{\partial y_i \partial y_j};
\]

it follows that \( u_\mu \) is plurisubharmonic, since \( \mu \) is convex. If \( \mu \) is only continuous the same conclusion is obtained approximating \( \mu \) by smooth functions.

In order to show the maximality of \( u_\mu \), we have to prove that, if \( w \in \mathcal{U (\mathbb{R}^n, X_\mu)} \), then \( w(z) \leq u_\mu (z) \) for every \( z \in X_\mu \).

This is obviously true if \( \text{Im} z = 0 \) for then \( w(z) = 0 \leq u_\mu (z) \), so let \( \text{Im} z \neq 0 \). Define on

\[ S^+ = \{ \zeta \in \mathbb{C} \mid 0 < \text{Im} \zeta < \pi/4 \} \]

the function \( f : S^+ \to X_\mu \) setting, for every \( \zeta \in \mathbb{C} \)

\[ f(\zeta) = \text{Re} z + \zeta \mu (\text{Im} z)^{-1} \text{Im} z. \]

Then, by construction

\[ u_\mu (f(\zeta)) = \mu ((\text{Im} \zeta) \mu (\text{Im} z)^{-1} \text{Im} z) = \text{Im} \zeta. \]

The function \( w \circ f : S^+ \to [0, \pi/4] \) is subharmonic and satisfies

\[ \lim_{S \ni z \to x} \sup w \circ f(z) = 0 \]

for every \( x \in \mathbb{R} \), so, in view of Theorem 2.1, we have

\[ w(f(\zeta)) \leq \text{Im} \zeta = u_\mu (f(\zeta)). \]

for every \( \zeta \in S^+ \). In particular, for \( \zeta_0 = i \mu (\text{Im} z) \) we get \( f(\zeta_0) = z \) and consequently

\[ v(z) = v(f(\zeta_0))) \leq u_\mu (f(\zeta_0))) = u_\mu (z). \]

This proves that \( u_\mu \) is maximal.

Now we observe that, since \( \mu \) is a convex function, \( u_\mu \) is Lipschitz; then, by Lemma 3.1 we have

\[ E_{\mathbb{R}^n, X_\mu}(x, \xi) = E_{u_\mu, X_\mu}(x, \xi) = \lim_{t \to 0^+} \frac{u_\mu (x + ity)}{t} = \lim_{t \to 0^+} \frac{\mu (ty)}{t} = \mu (y), \]

for every \( x \in \mathbb{R}^n, \xi \in \mathbb{R}^n = T_x \mathbb{R}^n \).

This proves completely the theorem. //
Remark 6.1. It would be interesting to provide a characterization of the models \((\mathbb{R}^n, X_\mu, u_\mu)\) as done by Abate e Patrizio in [1], where \(\mu\) is assumed to be in \(C^\infty(\mathbb{R}^n \setminus \{0\})\) and symmetric i.e. \(\mu(-x) = \mu(x)\) for every \(x \in \mathbb{R}^n\).

7. The elliptic tube of Lempert

The following construction is due to Lempert [15].

Given a segment \(I \subset \mathbb{R}^n \subset \mathbb{C}^n\) of positive length we denote 
\(L(I) \subset \mathbb{C}^n\) the unique complex line which contains \(I\). We assume that \(I\) is a relatively open interval in the real straight line of \(L(I)\) containing it. Let \(\tilde{I} \subset L(I)\) be the relatively open disc in \(L(I)\) whose diameter is \(I\).

Let now \(D \subset \mathbb{R}^n\) be a convex domain. The elliptic tube over \(D\) is defined by

\[D^{\text{ell}} = \bigcup_I \{\tilde{I} \mid I \subset D, I \text{ segment}\} \]

The main result of this section consists of finding the maximal function \(u \in \mathcal{U}(D, D^{\text{ell}})\) and the explicit computation of \(E_{D, D^{\text{ell}}}\) when \(D\) is a bounded convex domain in \(\mathbb{R}^n\).

If \(z = x + iy \in \mathbb{C}^n, x \in D\), consider the functional of Minkowski of \(D\) centered at \(x\) and evaluated at \(y\)

\[p(z) = p_D(z) = \inf\{t > 0 \mid x + t^{-1}y \in D\}\]

It is easy to check that a point \(z = x + iy \in \mathbb{C}^n\) belongs to \(z \in D^{\text{ell}}\)
if and only if \(x = \text{Re} z \in D\) and \(p(z)p(\overline{z}) < 1\).

We have the following

Theorem 7.1. Let \(D \subset \mathbb{R}^n\) be a bounded convex domain and \(p(z)\) the Minkowski functional. Let \(u : D^{\text{ell}} \to [0, \pi/4]\) be defined by

\[u(z) = u_D(z) = \frac{\arctan(p(z)) + \arctan(p(\overline{z}))}{2}\]

Then

1) \(u\) is locally Lipschitz and \(u(z) = u(x + iy) = 0\) if and only if \(y = 0\);
2) \(u\) is plurisubharmonic on \(D^{\text{ell}}\);
3) \(u \in \mathcal{U}(D, D^{\text{ell}})\) is maximal;
4) \((dd^c u)^n = 0\) on \(D^{\text{ell}} \setminus D\).
Moreover, if \(z = x + iy\) with \(x \in D\), \(y \in \mathbb{R}^n = T_xD\), then
\[
E_{u,D^e}(x,y) = E_{D,D^e}(x,y) = F_{D,D^e}(x,y) = \frac{p(z) + p(\overline{z})}{2}.
\]
Finally, if \(\partial D\) is of class \(C^2\) then also \(p\) and \(u\) are of class \(C^2\) on \(D^{\text{ell}} \setminus D\) and for each \(i, j = 1, \ldots, n\)
\[
(3) \quad \frac{\partial^2 u(z)}{\partial z_i \partial \overline{z}_j} = \frac{1}{4} \left( \frac{\partial^2 p(z)}{\partial y_i \partial y_j} + \frac{\partial^2 p(\overline{z})}{\partial y_i \partial y_j} \right).
\]
Proof. The statement 1) is a consequence of convexity and boundedness of \(D\).
In order to prove 2) and 3) define for \(z \in D\)
\[
\tilde{u}(z) = \sup \left\{ v(z) \mid v \in \mathcal{U}(D, D^e) \right\};
\]
it is then sufficient to show that
\[
u(z) = \tilde{u}(z)
\]
for every \(z \in D^e\). If \(z \in D\) the equality is evident, so we assume that \(z = x + iy \in D^e\) with \(y \neq 0\). Set
\[
t_1 = p(z)^{-1},
\]
\[
t_2 = p(\overline{z})^{-1},
\]
\[
x_1 = x + t_1 y,
\]
\[
x_2 = x - t_2 y.
\]
Then \(x_1, x_2 \in \partial D\) and the segment with endpoints \(x_1\) and \(x_2\) is contained in \(D\); we easily derive
\[
x = \frac{t_2}{t_1 + t_2} x_1 + \frac{t_1}{t_1 + t_2} x_2,
\]
\[
y = \frac{1}{t_1 + t_2} x_1 - \frac{1}{t_1 + t_2} x_2,
\]
namely
\[
z = \frac{t_2 + i}{t_1 + t_2} x_1 + \frac{t_1 - i}{t_1 + t_2} x_2.
\]
Let \(\Delta\) be the unit disc in \(\mathbb{C}\) and \(f : \Delta \to \mathbb{C}^n\) defined by \(\zeta \in \Delta\)
\[
f(\zeta) = \frac{1 - \zeta}{2} x_1 + \frac{1 + \zeta}{2} x_2.
\]
Since $f$ sends $]-1,1[$ in $D$ then $f(\Delta) \subset D^\text{ell}$ ([15]). Setting

$$\zeta_0 = \frac{t_1 - t_2}{t_1 + t_2} + i \frac{2}{t_1 + t_2};$$

the inequality $p(z)p(\overline{z}) < 1$ implies $\zeta_0 \in \Delta$; moreover $f(\zeta_0) = z$ so

$$\text{arctanh}(\zeta_0) = \frac{1}{2} \log \frac{1 + \zeta_0}{1 - \zeta_0} = \frac{1}{2} \log \frac{t_1 - i}{t_2 + i}$$

$$= \frac{1}{2} \log \left( \frac{t_1 t_2 - 1}{t_2^2 + 1} - i \frac{t_1 + t_2}{t_2^2 + 1} \right)$$

whence

$$|\text{Im}(\text{arctanh}(\zeta_0))| = \frac{1}{2} \left| \text{arg} \left( \frac{t_1 t_2 - 1}{t_2^2 + 1} - i \frac{t_1 + t_2}{t_2^2 + 1} \right) \right| = \frac{1}{2} \text{arctan} \frac{t_1 + t_2}{t_1 t_2 - 1}.$$

Since $t_1 = p(z)^{-1}, t_2 = p(\overline{z})^{-1}$ we deduce

$$|\text{Im}(\text{arctanh}(\zeta_0))| = \frac{1}{2} \text{arctan} \frac{p(z) + p(\overline{z})}{1 - p(z)p(\overline{z})}.$$

Finally, putting

$$p(z) = \tan(\text{arctan}(p(z))$$

$$p(\overline{z}) = \tan(\text{arctan}(p(\overline{z}))$$

into the elementary formula

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)},$$

we get

$$|\text{Im}(\text{arctanh}(\zeta_0))| = \frac{\text{arctan}(p(z)) + \text{arctan}(p(\overline{z}))}{2} = u(z).$$

Now let $w \in \mathcal{U}(D, D^\text{ell})$. Then $w \circ f : \Delta \to [0, \pi/4[$ is subharmonic and vanishing on $]-1,1[$. In view of Theorem 3.2 for every $\zeta \in \Delta$ we have

$$w \circ f(\zeta) \leq \left| \text{Im}(\text{arctanh}(\zeta)) \right|.$$
in particular
\[ v(z) = v \circ f(\zeta_0) \leq |\text{Im}(\text{arctanh}(\zeta_0))| = u(z) \]
since \( w \) is arbitrary
\[ \tilde{u}(z) \leq u(z). \]
As for the opposite inequality we observe that there is a holomorphic map \( g : D^{\text{ell}} \to \Delta \) satisfying \( g(D) \subset ]-1, 1[ \), \( g(f(\zeta)) = \zeta \) for every \( \zeta \in \Delta \) (cf. [15]). It follows
\[ |\text{Im}(\text{arctanh}(g))| \in \mathcal{H}(D, D^e), \]
which implies
\[ \tilde{u}(z) \geq |\text{Im}(\text{arctanh}(g(z)))| = |\text{Im}(\text{arctanh}(\zeta_0)))| = u(z). \]
since \( g(z) = g(f(\zeta_0)) = \zeta_0 \).
4) is an immediate consequence of continuity of \( u \) in view of Theorem [4.1].
In order to prove the last equality let \( x \in D, y \in \mathbb{R}^n = T_xD \).
Since \( u \) is locally Lipschitz, by Lemma [3.1] we have
\[
E_{u,D^e}(x,y) = E_{D,D^e}(x,y) = \lim_{t \to 0^+} \frac{\arctan(tp(z)) + \arctan(tp(\bar{z}))}{2t} = \frac{p(z) + p(\bar{z})}{2},
\]
where \( z = x + iy \). Let \( S = \{ |\text{Im}(z)| < \pi/4 \} \). We are going to show that the map \( h : S \to D^{\text{ell}} \) defined by
\[ h(\eta) = f(\tanh(\eta)). \]
is an \( E \)-complex geodesic.
Define \( k : D^{\text{ell}} \to S \) by
\[ k(z) = \text{arctanh}(g(z)). \]
In view of the identity \( g \circ f(\zeta) = \zeta \) we get \( k \circ h(\eta) = \eta \) for every \( \eta \in S \); it follows
\[ |\xi| = E_{R,S}(x,\xi) \leq E_{D,D^e}(h(x),dh(x)(\xi)) \leq E_{R,S}(k \circ h(x),d(k \circ h)(x)(\xi)) = E_{R,S}(x,\xi) = |\xi| \]
for every \( x \in \mathbb{R}, v \in \mathbb{R} = T_x\mathbb{R} \), and consequently
\[ E_{D,D^{\text{ell}}}(h(x),dh(x)(\xi)) = |\xi|. \]
Theorem 3.5 now implies that \( h \) is a \( E \)-complex geodesic. Moreover, again in view of Theorem 3.5 we have

\[
E_{D,D^\text{ell}}(x,v) = E_{D,D^\text{ell}}(h(k(x)), dh(k(x))(dk(x)(v))) = F_{D,D^\text{ell}}(h(k(x)), dh(k(x))(dk(x)(v))) = F_{D,D^\text{ell}}(x,v).
\]

Assume now that \( \partial D \) is smooth of class \( C^2 \). Then we claim that the function \( p(z) \) (and hence \( u(z) \)) is smooth of class \( C^2 \) on \( D^\text{ell} \setminus D \) and for \( i, j = 1, \ldots, n \),

(A) \[
\frac{\partial p}{\partial x_i} = p \frac{\partial p}{\partial y_i},
\]

(B) \[
\frac{\partial^2 p}{\partial x_i \partial y_j} = p \frac{\partial^2 p}{\partial y_i \partial y_j} + \frac{\partial p}{\partial y_i} \frac{\partial p}{\partial y_j},
\]

(C) \[
\frac{\partial^2 p}{\partial x_i \partial x_j} = p^2 \frac{\partial^2 p}{\partial y_i \partial y_j} + 2p \frac{\partial p}{\partial y_i} \frac{\partial p}{\partial y_j}.
\]

Assuming for granted such relations we have

\[
\frac{\partial^2 \arctan p}{\partial z_i \partial \bar{z}_j} = (1 + p^2)^{-2} \left[ (1 + p^2) \frac{\partial^2 p}{\partial z_i \partial \bar{z}_j} - 2p \frac{\partial p}{\partial z_i} \frac{\partial p}{\partial \bar{z}_j} \right].
\]

Since

\[
\frac{\partial}{\partial z_i} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} + \frac{1}{i} \frac{\partial}{\partial y_i} \right),
\]

\[
\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - \frac{1}{i} \frac{\partial}{\partial y_j} \right),
\]

then, using (A) we obtain

\[
\frac{\partial p}{\partial z_i} = \frac{1}{2} (p - i) \frac{\partial p}{\partial y_i},
\]

\[
\frac{\partial p}{\partial \bar{z}_j} = \frac{1}{2} (p + i) \frac{\partial p}{\partial y_j},
\]

and hence

\[
\frac{\partial p}{\partial z_i} \frac{\partial p}{\partial \bar{z}_j} = \frac{1}{4} \left( 1 + p^2 \right) \frac{\partial p}{\partial y_i} \frac{\partial p}{\partial y_j}.
\]
Recalling that
\[ \frac{\partial^2}{\partial z_i \partial \bar{z}_j} = \frac{1}{4} \left[ \left( \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial y_i \partial y_j} \right) + \frac{1}{i} \left( \frac{\partial^2}{\partial x_i \partial y_j} - \frac{\partial^2}{\partial x_j \partial y_i} \right) \right], \]

from (5) and (6) we obtain
\[ \frac{\partial^2 p}{\partial z_i \partial \bar{z}_j} = \frac{1}{4} \left( 1 + p^2 \right) \frac{\partial p}{\partial y_i} \frac{\partial p}{\partial y_j} + \frac{p \partial p \partial p}{2 \partial y_i \partial y_j}, \]
and hence, from (5) and (6), we obtain
\[ \frac{\partial^2 p}{\partial z_i \partial \bar{z}_j} = \frac{1}{4} \left( 1 + p^2 \right) \frac{\partial^2 p}{\partial y_i \partial y_j} + \frac{p \partial p \partial p}{2 \partial y_i \partial y_j}. \]

Inserting (8) and (9) in (7) we easily obtain
\[ \frac{\partial^2 \arctan p}{\partial z_i \partial \bar{z}_j} = \frac{1}{4} \frac{\partial^2 p}{\partial y_i \partial y_j}, \]
and this easily implies (3).

It remains to prove that the function \( p(z) \) is of class \( C^2 \) on \( D^{\text{ell}} \setminus D \) and the equations (4), (5) and (6) hold.

Let \( \mu \in C^2(\mathbb{R}^n) \) be a global defining function for \( D \), that is \( x \in D \) if, and only if, \( \mu(x) < 0 \) and \( (D_1 \mu(x), \ldots, D_n \mu(x)) \neq 0 \) if \( x \in \partial D \). Here \( D_i \mu(x) \) is the derivative of \( \mu(x) \) with respect to the variable \( x_i \).

Let \( U : D \times (\mathbb{R}^n \setminus \{0\}) \times [0, +\infty[ \) defined by
\[ U(x, y, p) = \mu \left( x + p^{-1}y \right). \]

We denote the derivatives \( \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial y_i} \) and \( \frac{\partial F}{\partial p} \) respectively as \( F_{x_i}, F_{y_i} \) and \( F_p \) (and similarly for the higher order derivatives).

Since \( D \) is convex then
\[ U_p(x, y, p) = -p^{-2} \sum_{\alpha=1}^{n} y_{\alpha} D_{\alpha} \mu(x + p^{-1}y) \neq 0 \]
when \( x + p^{-1}y \in \partial D \) that is if \( F_p(x, y, p) = 0 \). Setting \( z = x + iy \) the function \( p(z) \) is characterized by the condition
\[ U(x, y, p(z)) = 0. \]
By the Dini implicit function theorem, $p(z)$ is of class $C^2$ on $U : D \times (\mathbb{R}^n \setminus \{0\})$.

Taking the derivatives in (11) we obtain

\begin{align*}
(12) & \quad U_{x_i} + U_p p_{x_i} = 0, \\
(13) & \quad U_{y_i} + U_p p_{y_i} = 0.
\end{align*}

From (10) we obtain

\begin{align*}
(14) & \quad U_{x_i} = D_i \mu, \\
(15) & \quad U_{y_i} = p^{-1} D_i \mu,
\end{align*}

and hence

$$p_{x_i} = -U_p^{-1} U_{x_i} = -p U_p^{-1} U_{y_i} = p p_{y_i},$$

which proves (4).

Differentiating (12) with respect to $x_j$ we compute

$$U_{x_i x_j} + U_{x_i p} p_{x_j} + U_{x_j p} p_{x_i} + U_{pp} p_{x_i p} p_{x_j} + U_p p_{x_i x_j} = 0,$$

obtaining

\begin{align*}
(16) & \quad p_{x_i x_j} = -U_p^{-1} \left( U_{x_i x_j} + U_{x_i p} p_{x_j} + U_{x_j p} p_{x_i} + U_{pp} p_{x_i p} p_{x_j} \right).
\end{align*}

Similarly we have

\begin{align*}
(17) & \quad p_{x_i y_j} = -U_p^{-1} \left( U_{x_i y_j} + U_{x_i p} p_{y_j} + U_{y_j p} p_{x_i} + U_{pp} p_{x_i p} p_{y_j} \right)
\end{align*}

and

\begin{align*}
(18) & \quad p_{y_i y_j} = -U_p^{-1} \left( U_{y_i y_j} + U_{y_i p} p_{y_j} + U_{y_j p} p_{y_i} + U_{pp} p_{y_i p} p_{y_j} \right).
\end{align*}

But from (12) and (13) we have

\begin{align*}
U_{x_i x_j} & = D_i D_j \mu, \\
U_{x_i y_j} & = p^{-1} D_i D_j \mu, \\
U_{y_i y_j} & = p^{-2} D_i D_j \mu, \\
U_{x_i p} & = p^{-2} \sum_{\alpha=1}^{n} y_{\alpha} D_{\alpha} D_i \mu, \\
U_{y_i p} & = -p^{-2} D_i \mu - p^{-3} \sum_{\alpha=1}^{n} y_{\alpha} D_{\alpha} D_i \mu,
\end{align*}
and hence
\[ U_{x_ix_j} = p^2 U_{y_iy_j}, \]
\[ U_{x_iy_j} = p U_{y_iy_j}, \]
\[ U_{x_ip} = p U_{y_ip} + U_{y_i}. \]

Inserting such values in (17) we obtain
\[ p_{x_iy_j} = -U_p^{-1} (U_{x_iy_j} + U_{x_ip}p_{y_j} + U_{y_jp}p_{x_i} + U_{pp}p_{x_ip}y_j) \]
\[ = -U_p^{-1} (pU_{y_ip} + (pU_{y_i}p + U_{y_i})p_{y_j} + pU_{y_jp}p_{y_i} + pU_{pp}p_{y_ip}y_j) \]
\[ = -pU_p^{-1} (U_{y_iy_j} + U_{y_ip}p_{y_j} + U_{y_jp}p_{y_i} + U_{pp}p_{y_ip}y_j) - U_p^{-1} U_{y_jp_{y_j}} \]
\[ = pp_{y_ip} + p_{y_jp_{y_j}}, \]
and this proves (5).

Finally, from (18)
\[ p_{x_iy_j} = -U_p^{-1} (U_{x_iy_j} + U_{x_ip}p_{x_j} + U_{x_jp}p_{x_i} + U_{pp}p_{x_ip}x_j) \]
\[ = -U_p^{-1} \left( p^2 U_{y_iy_j} + p(pU_{y_ip} + U_{y_j})p_{y_j} \right. \]
\[ + p(pU_{y_jp} + U_{y_j})p_{y_i} + p^2 U_{pp}p_{y_ip}y_j \]
\[ = -p^2 U_p^{-1} (U_{y_iy_j} + U_{y_ip}p_{y_j} + U_{y_jp}p_{y_i} + U_{pp}p_{y_ip}y_j) \]
\[ -pU_p^{-1} U_{y_jp_{y_j}} - pU_p^{-1} U_{y_ip_{y_j}} \]
\[ = p^2 p_{y_ip} + 2 pp_{y_ip}y_j, \]

obtaining hence (6).

The proof of the theorem is completed. \( \square \)

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