Perturbative Energy Shifts of the Luttinger Liquid Due to the Presence of Additional Dimensions

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Abstract

We consider a system of spinless fermions in a nearly one-dimensional cylindrical trap. We introduce four-fermion interaction terms through which particles in low-lying modes of the trap may interact with particles in radially or angularly excited states. Treating these interactions as perturbations about a purely one-dimensional Luttinger liquid, we calculate the zero-temperature energy shifts of the bosonic quanta of the Luttinger model. We perform this calculation in two ways. First we use the bosonized forms of the fermion operators. Then we introduce a simplified, physically-motivated model; in this model, the methods used immediately generalize, so that we may also calculate the energy shifts of the Tomonaga-Luttinger model. By equating the results from the two disparate models, we obtain the numerical value of the Luttinger liquid bandwidth for this system.

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1 Introduction: fermions trapped in one dimension

A low-temperature system of fermions in one dimension has the known property of bosonization \[1, 2, 3\]. The coherent fermion-hole excitations of the system behave like bosons, with a spectrum that remains exactly solvable even in the presence of certain arbitrarily strong interactions. This phenomenon is described by the Luttinger and Tomonaga-Luttinger models.

The main physical applications of the Luttinger model occur in the context of the quantum Hall effect \[4, 5\]. However, there is another situation in which we expect Luttinger-type behavior to arise quite naturally. If a large number of degenerate fermions are trapped in a potential well with a very high aspect ratio, so that the particles are effectively constrained to move along a single axis, we would expect one-dimensional behavior to dominate. We shall consider a system of this type. Since the trapping potential cannot be purely one-dimensional, the second and third dimensions will affect the energy levels of the system. We shall calculate these energy shifts, by treating the effects of the small transverse dimensions as perturbations around a purely one-dimensional model. Since the field of fermion trapping is advancing very rapidly \[6, 7, 8, 9\], such calculations should be highly relevant in the analysis of future research.

After a brief discussion of trapped fermions and the one-dimensional Luttinger model, we shall introduce a physically motivated interaction Hamiltonian in Section 2. We then calculate the perturbative energy shift due to this interaction in two different ways. In Section 3 we use bosonized forms for the fermion operators. The results of the bosonized calculation depend upon the cutoff of the model. We may determine the physically meaningful value of the cutoff by comparing the bosonized result with the result of our second calculation (Section 4), which uses an entirely different, simplified model of the one-dimensional Fermi gas. Finally, in Section 5 we extend our results to systems with additional interactions and discuss the significance of our findings. Our main result is expression (55), which describes the energy shifts of the bosonic modes.

We shall consider a system with a large number of fermions trapped in a cylindrical potential. In the cylindrical coordinates, \((z, \theta, \rho)\), the wave function is periodic in \(z\) with period \(L\) and has a Dirichlet boundary condition at \(\rho = b \ll L\). The boundary condition for \(\rho\) corresponds to a trap with hard walls, but our general treatment will also be applicable to other forms for the trapping potential. The eigenfunctions of this trap are (setting \(\hbar = 1\))

\[
\psi_{k,\ell,n_{\rho}}(z,\theta,\rho) = e^{\frac{ikz}{L}} e^{\frac{i\ell\theta}{2\pi}} \frac{J_{\ell} \left( j_{\ell,n_{\rho}}(\rho) / b \right)}{\sqrt{L} \sqrt{2\pi} b \sqrt{\frac{\sqrt{J_{\ell+1}(j_{\ell,n_{\rho}})} J_{\ell-1}(j_{\ell,n_{\rho}})}{2}}},
\]

where \(k = \frac{2\pi r}{L}\) for some integer \(r\), \(\ell\) is an integer, and \(n_{\rho}\) is a positive integer, which represents the number of radial nodes in the wave function. The number \(j_{\ell,n_{\rho}}\) is the \(n_{\rho}\)-th
zero of the Bessel function $J_\ell$. The energies of these modes are

$$E_{k,\ell,n_\rho} = \frac{1}{2m} \left( k^2 + \left( \frac{j_{\ell,n_\rho}}{b} \right)^2 \right),$$

(2)

so there are energy gaps $\Delta_{\ell,n_\rho} = \frac{(j_{\ell,n_\rho})^2 - (j_{0,0})^2}{2mb^2}$ between the $n_\rho = \ell = 0$ states and the higher excited states.

In this paper, we are interested in the situation where the Fermi level is low enough so that only the $n_\rho = \ell = 0$ states are occupied, and the excited states of the trap can enter only virtually. This requires that the transverse size $b$ of the trap be of the order of the mean spacing between adjacent particles or smaller. (and most of our calculations will also require that the temperature of the system be vanishing). When no radially or angularly excited states are occupied, $k$ becomes the only relevant quantum number, and the system becomes effectively one-dimensional. This reduction in dimensionality allows us to make use of the peculiar theory of one-dimensional degenerate fermions, with its property of bosonization.

The phenomenon of bosonization is expressed in the Luttinger model, in which the noninteracting (spinless), one-dimensional fermionic Hamiltonian is approximated by

$$H_0 = v_F \sum_k \left[ (k - k_F) a^\dagger_k a_k + (-k - k_F) b^\dagger_k b_k \right],$$

(3)

where $a$ ($b$) operators correspond to right- (left-)moving fermions, and $v_F$ is the Fermi velocity. The linearization of the energy spectrum comes from the fact that the energy required to excite a particle near the Fermi surface is $(k_F + q)^2 - k_F^2 = v_Fq + \frac{q^2}{2m}$; if the excitation momentum $q$ is small compared to the Fermi momentum, then we may neglect the $q^2$ term. In the Luttinger model, the further approximation is made that this Hamiltonian is valid for all momenta $k$, but that states corresponding to $a_k$ and $b_{-k}$ for $k < 0$ are filled [10, 11]. The introduction of fictional negative-energy states, which lie far from the Fermi surface, should have little effect on the physics at low temperatures.

With these approximations, the Hamiltonian may be expressed in the form

$$H_0 = \frac{2\pi v_F}{L} \sum_{q>0} \left[ \rho_+(q) \rho_+(-q) + \rho_-(q) \rho_-(-q) \right],$$

(4)

where the $\rho_\pm$ operators

$$\rho_+(q) = \sum_k a^\dagger_{k+q} a_k$$

(5)

$$\rho_-(q) = \sum_k b^\dagger_{k+q} b_k$$

(6)

obey the Bose commutation relations $[\rho_+(q), \rho_+(q')] = [\rho_-(q), \rho_-(-q')] = \delta_{qq'} \frac{QL}{2\pi}$ and $[\rho_+(q), \rho_-(q')] = 0$. The bosonic quanta created and annihilated by the $\rho_\pm$ operators have
energies $\omega_q = v_F |q|$. It is also possible to express the fermion field operators in terms of the bosonic $\rho_\pm$ operators. We shall make use of this representation in Section 3.

The Tomonaga-Luttinger model is formed by adding scattering effects \[12\] through the interaction Hamiltonian

\[ H_1 = \frac{1}{2L} \sum_q \{ 2g_2(q) \rho_+(q) \rho_-(-q) + g_4(q) [ \rho_+(q) \rho_+(-q) + \rho_-(-q) \rho_-(-q) ] \}. \tag{7} \]

The $g_2$ and $g_4$ are the Fourier transforms of the interaction potentials. This interaction (like the similar interactions to be introduced in Section 2) describes forward scattering processes between fermions, with momentum exchange $q$. The Tomonaga-Luttinger model is exactly solvable, since $H_0 + H_1$ remains bilinear in the $\rho_\pm$. We shall return to this interaction in Section 5, but first we shall consider interactions between the Luttinger liquid bosons and fermions in excited states of the trap.

## 2 Interaction Hamiltonians

The Luttinger model should be a good description of a system of fermions in a narrow, axially symmetric trap, in which no radially or angularly excited states are occupied. We shall add to this system additional two-fermion interaction terms. These new interactions will be similar in structure to $H_1$ (when $H_1$ is expanded in terms of fermion operators). These interactions must conserve linear momentum along and angular momentum around the axis of the trap, as well as fermion number. One such interaction term is

\[ H' = \sum_{k,k',q} C_{k,k',q} c_{k+q}^\dagger d_k (a_{k'}^\dagger a_{k'} + b_{k'}^\dagger b_{k'}). \tag{8} \]

The operators $c_k$ and $d_k$ correspond to arbitrary fermion states, subject to the condition that at least one of them corresponds to an excited state of the trap (i.e. is not an $a_k$ or a $b_k$). This ensures that we shall be studying new effects. The effective coupling $C_{k,k',q}$ will depend upon the radial and angular quantum numbers corresponding to $c_k$ and $d_k$ (in part because $C_{k,k',q}$ includes integrals over the radial and angular eigenmodes of the trap). Angular momentum conservation demands that $c_k$ and $d_k$ annihlitate states with the same angular momentum.

We are only interested in interactions that will renormalize the bosonic energies. The simplest way for bosonized operators to arise from from an interaction Hamiltonian such as $H'$ is for $C_{k,k',q}$ to be effectively independent of $k$ and $k'$. Then the $a_{k'}^\dagger$ and $a_k$ operators combine to form a boson operator, as in \[5\], and the $b_{k'}^\dagger$ and $b_k$ combine similarly. This leads to an interaction of the form

\[ H_2 = \frac{1}{2L} \sum_{q \neq 0} g(q) \left( \sum_k c_{k+q}^\dagger d_k \right) [\rho_+(q) + \rho_-(q)] + \text{h.c.}. \tag{9} \]
We shall calculate the energy shifts of the low-lying bosonic states to second order in \( H_2 \). (The first-order shifts obviously vanish.) If both \( c_{k-q} \) and \( d_k \) correspond to excited states of the trap, then the operator \( H_2 \) preserves the number of excited fermions. Hence, \( H_2 \) has zero matrix element between low-lying (Luttinger liquid) states and states with excited fermions. So we only get nonvanishing contributions from terms where one of \( c_k \) and \( d_k \) corresponds to an excited fermion state, while the other corresponds to a low-lying mode. Moreover, the excited mode must have angular momentum \( \ell = 0 \) about the axis of the trap.

Let us fix \( q > 0 \). We shall calculate the energy shift for the \( \rho_+(q) \) quanta. In order for these quanta to be well-behaved Luttinger liquid phonons, we must have \( \omega_q \ll \epsilon_F \equiv \frac{1}{2m}k_F^2 \), or \( q \ll k_F \). We may write the interactions affecting the \( \rho_+(q) \) quanta as

\[
H_{2+q} = \frac{1}{L} g(q) \left\{ \sum_k (c_{k-q}^\dagger d_k + d_{k-q}^\dagger c_k) \rho_+(q) + \left[ \sum_k (c_{k+q}^\dagger d_k + d_{k+q}^\dagger c_k) \right] \rho_+(-q) \right\},
\]

where \( d_k \) annihilates an excited fermion and \( c_k \) annihilates a low-lying fermion. \([\text{If } c_k \text{ annihilates a right-moving fermion, additional terms are actually be required in order make } H_{2+q} \text{ hermitian. However, these terms do not involve } \rho_+, \text{ so their matrix elements are smaller by a factor of } O(L^{-1/2}) \text{ and may be neglected.}]\) We shall denote by \( \Delta \) the energy gap \( \Delta_{0,n_\rho} \) for the particular \( \ell = 0 \) states corresponding to the operators \( d_k \).

3 Energy shift calculated with bosonized operators

We shall calculate the perturbative energy shift arising from \( (10) \) in two different models. In this section, we use the bosonized field operators of the Luttinger model to perform the calculation. It is possible to obtain a closed-form expression for the energy shift in two different situations—when the temperature vanishes or when the energy gap is very large. (In each of these cases, the “transverse temperature,” which is the effective temperature for excitations of the trap’s radial and angular modes, is zero.) In these two situations, we find that the leading contributions to the the infinite sum over intermediate states may be resummed. This resummation allows us to find a simple expression for the energy shift.

We shall restrict our attention to the case in which \( c_k \) is \( a_k \) (rather than \( b_k \)). In this case, the matrix element which we must calculate is \( \langle \{m_+\}; K | H_{2+q} | \{n_+\} \rangle \). The state \( |\{n_+\} \rangle \) is a number state of the Luttinger model; the \( \{n_+\} \) are the occupation numbers of the \( \rho_+ \) quanta. The state \( |\{m_+\}; K \rangle \) has differing occupation numbers \( \{m_+\} \), as well as an excited state fermion with momentum \( K \).

The matrix element is simply

\[
\langle \{m_+\}; K | H_{2+q} | \{n_+\} \rangle = \frac{1}{L} g(q) \langle \{m_+\}; K | \sum_k \left[ d_{k-q}^\dagger a_k \rho_+(q) + d_{k+q}^\dagger a_k \rho_+(-q) \right] |\{n_+\} \rangle
\]

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where the occupation numbers \( \{n'_+\} \) and \( \{n''_+\} \) are the same as the \( \{n_+\} \), except that 
\( n'_+ = n_+ + 1 \) and \( n''_+ = n_+ - 1 \). As \( L \to \infty \), \( \{n'_+\} \) and \( \{n''_+\} \) differ from \( \{n_+\} \) only on a set of measure zero, and we may replace the matrix elements \( \langle \{m'_+\}|a_k|\{n'_+\} \rangle \) and \( \langle \{m''_+\}|a_k|\{n''_+\} \rangle \) appearing in (11) by \( \langle \{m'_+\}|a_k|\{n_+\} \rangle \) and \( \langle \{m''_+\}|a_k|\{n_+\} \rangle \), 
where \( m'_+ = m_+ - 1 \) and \( m''_+ = m_+ + 1 \). We shall eventually be summing over all possible sets of occupation numbers \( \{m_+\} \), so we may ultimately replace both matrix elements in (11) with \( \langle \{m_+\}|a_k|\{n_+\} \rangle \). So we see that the crucial quantity to calculate is \( \langle \{m_+\}|a_k|\{n_+\} \rangle \), for an arbitrary set of occupation numbers \( \{m_+\} \). We shall perform this calculation by taking a Fourier transform, 
\[
\langle \{m_+\}|a_k|\{n_+\} \rangle = \frac{1}{\sqrt{L}} \int_0^L dx e^{-ikx} \langle \{m_+\} | \psi_+(x) | \{n_+\} \rangle,
\]
(12)
where \( \psi_+ \) is the field operator for the right-moving fermions (while \( \psi_- \) is the left-moving 
fermions’ field operator). So we must now turn our attention to the operators \( \psi_{\pm} \).

The bosonized form of the fermion field operator is [13] [14] [15] [16] [17] 
\[
\psi_{\pm}(x) = \frac{1}{\sqrt{2\pi\alpha}} U_{\pm} \exp[\pm ik_F x \mp i\phi(x) + i\theta(x)].
\]
(13)
In the pure Luttinger model, this is only an operator identity when we take the limit 
\( \alpha \to 0 \) [13]. However, we shall interpret \( \alpha \) as a finite cutoff, so that \( v_F\alpha^{-1} \) is the bandwidth
of the Luttinger liquid [2] [14] [19] [20] [21]. The finiteness of \( \alpha \) corresponds to a deviation of the energy spectrum from the Luttinger form far from the Fermi surface. We shall discuss which values of \( \alpha \) are physically meaningful in Sections 4 and 5. The \( U_{\pm} \) operators decrease the total particle number on their corresponding branches by unity; these operators are necessary because the bosonic fields conserve the total fermion number.

The bosonic quantities \( \phi \) (the boson field) and \( \theta \) (the integral of the momentum conjugate to \( \phi \)) are given by 
\[
- i\phi = -\frac{\pi}{L} \sum_{p \neq 0} \frac{1}{p} e^{-|p|/2} e^{-ipx} \left[ \rho_+(p) + \rho_-(p) \right] + i\pi \frac{N_+ - N_-}{L} x
\]
(14)
\[
i\theta = -\frac{\pi}{L} \sum_{p \neq 0} \frac{1}{p} e^{-|p|/2} e^{-ipx} \left[ \rho_+(p) - \rho_-(p) \right] + i\pi \frac{N_+ - N_-}{L} x,
\]
(15)
where \( N_+ \) and \( N_- \) are the excess numbers of right- and left-moving fermions relative to the ground state, respectively. So the argument of the exponential in \( \psi_{\pm} \) is
\[
\pm ik_F x \mp i\phi + i\theta = \pm ik_F x \mp \frac{2\pi}{L} \sum_{p \neq 0} \frac{1}{p} e^{-|p|/2} e^{-ipx} \rho_\pm(p) \pm \frac{2\pi i N_\pm}{L} x.
\]
(16)
For states sufficiently close to the ground state, we may neglect the $N_\pm$ terms; this will be equivalent to neglecting an $\mathcal{O}(L^{-1})$ term in the argument of a momentum-conserving $\delta$-function. So we have

$$\pm i k_F x \mp i \phi + i \theta = \pm i k_F x \mp \frac{2\pi}{L} \sum_{p>0} \frac{1}{p} e^{-\alpha p/2} \left[ e^{i p x} \rho_+ (p) - e^{-i p x} \rho_- (-p) \right]. \quad (17)$$

Since $[\rho_+(p), \rho_- (p')] = 0$ if $p \neq \pm p'$, the elements of the sum in (17) all commute with one-another. This allows us to write $\psi_\pm$ in the product form

$$\psi_\pm (x) = \frac{1}{\sqrt{2\pi \alpha}} U_\pm e^{\pm i k_F x} \prod_{p>0} \exp \left\{ \pm \left( \frac{2\pi}{pL} \right) e^{-\alpha p/2} \left[ e^{i p x} \rho_+ (-p) - e^{-i p x} \rho_- (p) \right] \right\}. \quad (18)$$

So the study of the matrix elements of $\psi_\pm$ between different Luttinger liquid states leads naturally to the study of harmonic oscillator matrix elements of the form $\langle m | e^{\lambda A^\dagger - \lambda^* A} | n \rangle$, where $A^\dagger$ and $A$ are raising and lowering operators. In our case, since $\sqrt{2\pi/(pL)} \rho_\pm$ is a canonically normalized ladder operator,

$$\lambda = \mp \sqrt{\frac{2\pi}{pL}} e^{\mp i p x} e^{-\alpha p/2}. \quad (19)$$

It is important to note that $\lambda$ is $\mathcal{O}(L^{-1/2})$.

In terms of $l \equiv m - n$, the general formula for the matrix element in question is

$$\langle m | e^{\lambda A^\dagger - \lambda^* A} | n \rangle = \frac{1}{|l|!} \left( \frac{m!}{n!} \right)^{\frac{1}{2} \text{sgn}(l)} \lambda^{(|l|+l)/2} (-\lambda^*)^{(|l|-l)/2} \times \left\{ e^{\lambda^2/2} F \left[ \max(m,n) + 1; |l| + 1; -|\lambda|^2 \right] \right\}, \quad (20)$$

where $F(a; b; z)$ is a confluent hypergeometric function. This formula is derived in Appendix A.

The formulas above pertain to both $\psi_+$ and $\psi_-$, but we shall henceforth restrict our attention to $\psi_+$ only. We shall now proceed to calculate the energy shift for the case in which the temperature is effectively zero when compared to the transverse energy scale of the trap. The fact that the temperature vanishes allows us to make some substantial simplifications of the mathematics. We shall show explicitly that these simplifications are not valid (and would give rise to unphysical results) when the “transverse temperature” is nonzero.

It may initially appear that the bracketed factors in (20) may be neglected, since both $e^{\lambda^2/2}$ and $F[\max(m,n) + 1; |l| + 1; -|\lambda|^2]$ are power series in $|\lambda|^2$, and $|\lambda|^2$ is $\mathcal{O}(L^{-1})$. Each power series is dominated by its first term in the large $L$ limit, and for both series, that term is unity. However, these terms may not actually be neglected in the $L \to \infty$ limit, because there are an infinite number of such terms, corresponding to all the allowed
momenta. Since the density of states in momentum space is proportional to $L$, the infinite product does not approach unity.

Let us denote the infinite product in question by $\mathcal{P}$; that is, $\mathcal{P}$ is defined to be

$$
\mathcal{P} = \prod_{p>0} \exp \left[ \left( \frac{\pi}{pL} \right) e^{-\alpha p} \right] F \left[ \max (m_{+p}, n_{+p}) + 1; |m_{+p} - n_{+p}| + 1; -\left( \frac{2\pi}{pL} \right) e^{-\alpha p} \right].
$$

(21)

When we calculate the energy shift, we sum up terms consisting of a matrix element squared divided by an energy difference. If the characteristic energy differences are $O(L)$, then the sum of the matrix elements squared must also be $O(L)$ if we are to obtain a nonvanishing contribution. This is only possible if the occupation numbers of a macroscopic number of states are $O(L)$. This is a natural situation at finite temperature; however, at $T = 0$, the occupation numbers must be small, so that the excitation energy of the system is not extensive.

So at $T = 0$ we only get a nonvanishing contribution when the difference in energies between $|\{m_{+p}\}|$ and $|\{n_{+p}\}|$ grows more slowly than $L$. That is, almost all of the $m_{+p}$ must be equal to the $n_{+p}$. We may then separate out those values of $p$ at which $m_{+p}$ and $n_{+p}$ differ and get

$$
\mathcal{P} = \prod_{p,l_{+p} \neq 0} \left\{ \exp \left[ \left( \frac{\pi}{pL} \right) e^{-\alpha p} \right] F \left[ \max (m_{+p}, n_{+p}) + 1; |l_{+p}| + 1; -\left( \frac{2\pi}{pL} \right) e^{-\alpha p} \right] \right\}
\times \prod_{p,l_{+p} = 0} \left\{ \exp \left[ \left( \frac{\pi}{pL} \right) e^{-\alpha p} \right] F \left[ n_{+p} + 1; 1; -\left( \frac{2\pi}{pL} \right) e^{-\alpha p} \right] \right\}.
$$

(22)

The number of terms in the first product must grow more slowly than $L$. Since each of these terms has the form $1 + O(L^{-1})$ as $L \to \infty$, the entire product is one in that limit.

The second product in (22) is more complicated. Using the Kummer transformation formula $F(a; b; z) = e^z F(b - a; b; -z)$ and the fact that $F(-n; 1; z)$ is just the Laguerre polynomial $L_n(z)$ \cite{2}, we may rewrite $\mathcal{P}$ as

$$
\mathcal{P} = \prod_{p,l_{+p} = 0} \left\{ \exp \left[ -\left( \frac{\pi}{pL} \right) e^{-\alpha p} \right] L_{n_{+p}} \left[ \left( \frac{2\pi}{pL} \right) e^{-\alpha p} \right] \right\}
$$

(23)

We may evaluate this explicitly at $T = 0$, where $n_{+p} = 0$ for almost all $p$. The product of the Laguerre polynomials is one, and all that remain are the exponentials. Each momentum $p$ represented in the product has the form $p(r) = 2\pi r/L$, for some positive integer $r$, so we have

$$
\mathcal{P}(T = 0) = \exp \left[ -\sum_{r,l_{+p}(r)=0} \frac{e^{-2\pi or/L}}{2r} \right].
$$

(24)

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Since \( l_+p = 0 \) for almost all \( p \), we may extend the sum to include all positive integers \( r \). Then the sum is just the Taylor series for \( \frac{1}{2} \log \left( 1 - e^{-2\pi \alpha/L} \right) \), so, to leading order in \( L^{-1} \), we have

\[
\mathcal{P}(T = 0) = \sqrt{\frac{2\pi \alpha}{L}}. \tag{25}
\]

The factor of \( \sqrt{\frac{\alpha}{L}} \) will be necessary in order to make our final result finite.

If we were to assume that the only terms which contributed had \( m_+p = n_+p \) for almost all \( p \) even when \( T > 0 \), we would obtain the expression

\[
\mathcal{P} = \sqrt{\frac{2\pi \alpha}{L}} \prod_{p,l_+p=0} \left( \frac{2\pi}{pL} \right)^{l_+p} \left( e^{-\alpha p} \right)^{l_+p}. \tag{26}
\]

We shall use this expression to demonstrate explicitly that our assumption is inconsistent unless \( T \) vanishes.

We may now turn our attention to remaining factors in the matrix elements—those arising from the unbracketed terms in (20). We shall divide the matrix elements of this operator into groups. Each matrix element \( \langle \{m_+p\} | \psi_+ | \{n_+p\} \rangle \) is labeled by an integer \( j \geq 0 \), where \( j \) is the number of quanta by which \( \{m_+p\} \) and \( \{n_+p\} \) differ. That is,

\[
j = \sum_{p>0} |m_+p - n_+p|. \tag{27}
\]

According to our earlier argument, at \( T = 0 \), the relevant values of \( j \) are smaller than \( \mathcal{O}(L) \), and we shall for the moment assume that this is also the case at finite \( T \) (although we shall eventually show that this assumption is inconsistent).

Each of the \( j \) quanta by which \( \{m_+p\} \) and \( \{n_+p\} \) differ corresponds to a momentum \( p_i > 0 \), \( i \in 1, 2, \ldots, j \). Since \( j \) is smaller than \( \mathcal{O}(L) \), in the continuum limit all the \( p_i \) are distinct, except on a set of measure zero. So we shall only need to use (20) in the case where \( |l| \leq 1 \). Keeping this in mind, we may write the matrix elements of \( \psi_+ \) as

\[
\langle \{m_+p\} | \psi_+ (x) | \{n_+p\} \rangle = \mathcal{P} \frac{e^{ik_Fx}}{\sqrt{2\pi\alpha}} \prod_{i=1}^{j} \left( \frac{2\pi}{p_iL} \right)^{i} \left( e^{-\alpha p_i} \right)^{i} \sqrt{n_+p_i + 1} \sqrt{n_+p_i + 1}. \tag{28}
\]

The upper term applies if \( m_+p_i > n_+p_i \) (i.e. if \( m_+p_i = n_+p_i + 1 \)), and the lower term otherwise. We note that this matrix element is \( \mathcal{O}(L^{-j/2}) \). Such a matrix element can only give rise to a nonvanishing energy shift if it is multiplied by an appropriate positive power of \( L \).

We may find the corresponding matrix element of \( a_k \) by taking the Fourier transform \( (12) \). We have

\[
\langle \{m_+p\} | a_k | \{n_+p\} \rangle = \mathcal{P} \frac{2\pi}{\sqrt{2\pi\alpha}} \left( \frac{2\pi}{L} \right)^{j/2} \frac{1}{\sqrt{L}} L \int_0^L dx \exp \left( -ikx + ik_Fx + \sum_{i=1}^{j} \mp i p_i x \right).
\]
\[
\times \exp \left( -\frac{1}{2} \alpha \sum_{i=1}^{j} p_i \right) \prod_{i=1}^{j} \frac{1}{\sqrt{p_i}} \left\{ -\sqrt{n_{+p_i}} + 1 \right\}^\frac{1}{2}. \tag{29}
\]

For each \(i\), we have two possibilities, depending upon the relative magnitudes of \(m_{+p_i}\) and \(n_{+p_i}\). In each case, the upper (lower) sign in the exponential corresponds to the upper (lower) square root. The integral over \(x\) squared is then

\[
|\langle m_{+p} | a_k | n_{+p}\rangle|^2 = \frac{\mathcal{P}^2}{2\pi \alpha} \left( \frac{2\pi}{L} \right)^j \sqrt{L \delta_{k,k_F + \sum_{i=1}^{j} \mp p_i}} \exp \left( -\alpha \sum_{i=1}^{j} p_i \right) \prod_{i=1}^{j} \frac{1}{p_i} \left\{ n_{+p_i} + 1 \right\}^\frac{1}{2}. \tag{30}
\]

To find the perturbative energy shift due to terms of this form, we must divide the matrix element squared by the energy defect and integrate over all possible momenta. The phase-space factors arising from the integrations will cancel out the \(j\) factors of \(2\pi/L\) in \(a_k\). The energy defect is given by

\[
E_{(n_{+p})k\mp q} - E_{(n_{+p})} = \Delta + \frac{1}{2m} \left[ (k \mp q)^2 - k^2_F \right] \pm \omega_q + \sum_{i=1}^{j} \pm \omega_{p_i}. \tag{31}
\]

\(\Delta + \frac{1}{2m}(k \mp q)^2\) is the energy of the excited fermion; \(\frac{1}{2m}k^2_F\) is the energy lost by the Luttinger liquid ground state with the removal of one fermion; and \(\pm \omega_q + \sum_{i=1}^{j} \pm \omega_{p_i}\) is the energy change due to the phonons gained or lost in the intermediate state. There are \(2^j\) possible choices of signs in the sum over the quanta created or annihilated by \(a_k\).

The momentum range over which we must integrate extends from \(e^{-\gamma p_{\text{min}}} \equiv e^{-\gamma 2\pi \alpha} \) to \(+\infty\). The Euler-Mascheroni constant \(\gamma\) enters when we change from a sum to an integral, through the formula \(\lim_{m \to +\infty} \left( \sum_{i=1}^{m} \frac{1}{i} - \int_{1}^{m} \frac{dx}{x} \right) = \gamma\). We may then absorb this \(\gamma\)-dependence into the limits of integration. When we integrate all the \(p_i\) over this range, we must also divide by \(j!\). This factor arises from our overcounting of intermediate states—permuting the \(p_i\) does not give rise to a new state.

So the energy shift consists of terms of the form

\[
\Delta E_j = -\frac{\mathcal{P}^2}{2\pi \alpha} \frac{q}{2\pi L} |g(q)|^2 \left( \frac{2\pi}{L} \right)^j \sum_k \sum_{s = \pm 1} \sum_{t = 1}^{2^j} \frac{1}{j!} \left( \frac{L}{2\pi} \right)^j
\]

\[
\times \int_{\gamma}^{+\infty} \frac{\mathcal{P}^2}{2\pi \alpha} \frac{q}{2\pi L} |g(q)|^2 \left( \frac{2\pi}{L} \right)^j \sum_k \sum_{s = \pm 1} \sum_{t = 1}^{2^j} \frac{1}{j!} \left( \frac{L}{2\pi} \right)^j \frac{L \delta_{k,k_F + \sum_{i=1}^{j} \mp p_i} \left( n_{+q} + \frac{s+1}{2} \right)}{\Delta + \frac{1}{2m} [(k - s q)^2 - k^2_F] + s \omega_q + \sum_{i=1}^{j} \pm \omega_{p_i}}
\]

The sum over \(t\) runs over the \(2^j\) choices of sign (and of \(n_{+p_i} + 1\) or \(n_{+p_i}\)). The integrand in \(32\) behaves as \(p_i^{-1}\) for small values of \(p_i\) and as \(p_i^{-3}\) for large values of \(p_i\); so the integral is dominated by the region where all the \(p_i\) are small.

Near \(p_i = p_{\text{min}}, p_i\) and \(\omega_{p_i}\) are each \(O(L^{-1})\), so we may neglect \(\sum_{i=1}^{j} \mp p_i\) and \(\sum_{i=1}^{j} \pm \omega_{p_i}\) unless \(j\) is \(O(L)\) or greater. Since we have assumed that \(j\) is smaller than \(O(L)\), we may
drop $\sum_{i=1}^{j} \mp p_i$ and $\sum_{i=1}^{j} \pm \omega p_i$. We shall not drop the $\exp \left(-\alpha \sum_{i=1}^{j} p_i\right)$ term; this factor is needed to provide a cutoff for the momentum integral.

By dropping the two sums, we simplify the integral substantially. The expression becomes

$$
\Delta E_j = -\frac{\mathcal{P}^2}{4\pi^2\alpha} \sum_{s=\pm 1} \sum_{t=1}^{2^j} \frac{|g(q)|^2 q \left(n_{+q} + \frac{s+1}{2}\right)}{\Delta + \frac{1}{2m} \left[(k_F - sq)^2 - k_F^2\right] + s\omega_q} \frac{1}{n_{+p_i}} \left\{ n_{+p_i} + 1 \right\}.
$$

(33)

We may now perform the sum over $t$, to get

$$
\Delta E_j = -\frac{\mathcal{P}^2}{4\pi^2\alpha} \sum_{s=\pm 1} \sum_{t=1}^{2^j} \frac{|g(q)|^2 q \left(n_{+q} + \frac{s+1}{2}\right)}{\Delta + \frac{1}{2m} \left[(k_F - sq)^2 - k_F^2\right] + s\omega_q} \frac{1}{n_{+p_i}} \left[ f_{\text{pmin}/e^2}^{+\infty} dp_i e^{-\alpha p_i} (2n_{+p_i} + 1) \right]^t.
$$

(34)

Furthermore, we may easily sum up all the terms with different values of $j$, getting the exponential function

$$
\Delta E_+ = -\frac{\mathcal{P}^2}{4\pi^2\alpha} \sum_{s=\pm 1} \sum_{t=1}^{2^j} \frac{|g(q)|^2 q \left(n_{+q} + \frac{s+1}{2}\right)}{\Delta + \frac{1}{2m} \left[(k_F - sq)^2 - k_F^2\right] + s\omega_q} \frac{1}{n_{+p_i}} \left[ f_{\text{pmin}/e^2}^{+\infty} dp_i e^{-\alpha p_i} (2n_{+p_i} + 1) \right]^t.
$$

(35)

(The subscript “$+$” in $\Delta E_+$ indicates that this energy shift only includes the terms arising from $\psi_+$—i.e. the terms in which $c_k$ is $a_k$.) It now remains to evaluate the integral and add the contributions from the terms for which $c_k$ is $b_k$.

It is impossible to evaluate the integral for a general set of $n_{+p_i}$. However, we may simplify our expression for the energy shift substantially by separating the two terms in the integral and transforming the integral back into an infinite sum. We have

$$
\exp \left[ \int_{\text{pmin}}^{+\infty} dp_i e^{-\alpha p_i} (2n_{+p_i} + 1) \right] = \exp \left[ 2 \sum_{p_i > 0} \left( \frac{2\pi}{p_i L} \right) e^{-\alpha p_i} n_{+p_i} \right] \exp \left[ \sum_{p_i > 0} \left( \frac{2\pi}{p_i L} \right) e^{-\alpha p_i} \right].
$$

(36)

The second factor on the right-hand side of (36) contains the same sum that we encountered when we evaluated $\mathcal{P}(T = 0)$; in fact, this term is exactly $[\mathcal{P}(T = 0)]^{-2}$. So we define a new infinite product $\mathcal{D}$ as

$$
\mathcal{D} = \mathcal{P}^2 \exp \left[ \int_{\text{pmin}}^{+\infty} dp_i e^{-\alpha p_i} (2n_{+p_i} + 1) \right] = \prod_{p_i > 0} \left( \exp \left[ \left( \frac{2\pi}{p_i L} \right) e^{-\alpha p_i} n_{+p_i} \right] L_{n_{+p_i}} \left( \left( \frac{2\pi}{p_i L} \right) e^{-\alpha p_i} \right) \right)^2.
$$

(37)

$\mathcal{D}$ is an overall factor multiplying our expression for $\Delta E_+$. It is clear that $\mathcal{D} = 1$ when $T = 0$ (since almost all the $n_{+p}$ are then zero). For $T > 0$, $\mathcal{D}$ contains all the energy
shift’s temperature dependence, so it is natural to calculate its thermal average. We shall find that this average diverges for all finite temperatures; this unphysical result signals that we cannot assume that $j$ is smaller than $O(L)$ unless the temperature is strictly vanishing.

Denoting the inverse temperature by $\beta$, we find that the thermal average of $\mathcal{D}$ is

$$\langle \mathcal{D} \rangle = \mathcal{N} \sum_{\{n_{+p}\}} \left\{ \exp \left[ -\beta v_F \sum_{p>0} (pn_{+p}) \right] \mathcal{D} \right\}. \quad (38)$$

The sum over the $\{n_{+p}\}$ runs over all possible sets of occupation numbers, and $\mathcal{N}$ is the necessary normalization factor

$$\mathcal{N} = \left\{ \sum_{\{n_{+p}\}} \exp \left[ -\beta v_F \sum_{p>0} (pn_{+p}) \right] \right\}^{-1} = \prod_{p>0} \left( 1 - e^{-\beta v_F p} \right). \quad (39)$$

$\langle \mathcal{D} \rangle$ may be factorized into an infinite product of sums indexed by $p$. This allows us to calculate the thermal average separately for each value of the momentum, so $\langle \mathcal{D} \rangle$ becomes

$$\langle \mathcal{D} \rangle = \mathcal{N} \prod_{p>0} \sum_{n_{+p}=0}^{\infty} \left( \exp \left[ -\beta v_F pn_{+p} \right] \exp \left[ \left( \frac{2\pi}{pL} \right) e^{-\alpha p n_{+p}} \right] L_{n_{+p}} \left[ \left( \frac{2\pi}{pL} \right) e^{-\alpha p} \right]^2 \right). \quad (40)$$

The sum is now in the form of a product of generating functions for $(L_n)^2$. That is,

$$\langle \mathcal{D} \rangle = \mathcal{N} \prod_{p>0} \sum_{n=0}^{\infty} [L_n(x)]^2 z^n, \quad (41)$$

with $z = \exp \left[ \left( \frac{2\pi}{pL} \right) e^{-\alpha p} - \beta v_F p \right]$ and $x = \left( \frac{2\pi}{pL} \right) e^{-\alpha p}$.

The sum of this generating function is known [23]. In fact, when $|z| < 1$,

$$\sum_{n=0}^{\infty} [L_n(x)]^2 z^n = \frac{1}{1-z} \exp \left( -\frac{2xz}{1-z} \right) I_0 \left( 2xz \sqrt{\frac{z}{1-z}} \right), \quad (42)$$

where $I_0$ is the modified Bessel function $I_0(u) = J_0(iu)$. The series diverges if $|z| > 1$. So $\langle \mathcal{D} \rangle$ converges only if $\exp \left[ \left( \frac{2\pi}{pL} \right) e^{-\alpha p} - \beta v_F p \right] < 1$ for all $p$. Since $\left( \frac{2\pi}{pL} \right) e^{-\alpha p} - \beta v_F p$ is a strictly decreasing function of $p$, we need only consider $p = p_{\min}$. At this value of $p$, we may neglect $e^{-\alpha p}$, so the condition for convergence becomes $\beta > (v_F p_{\min})^{-1}$. As $L \to \infty$, this can not be satisfied at any finite temperature. So the thermal average of $\mathcal{D}$ is equal to unity at $T = 0$ (when $z = 0$ for all $p$) and diverges for $T > 0$. For finite $L$ and very large $v_F$, the condition that $\beta > (v_F p_{\min})^{-1}$ gives us a quantitative estimate of how small the temperature must be in order for our approximation to be valid. Current experiments with degenerate fermions [6, 7, 8, 9] typically achieve temperatures for which $\beta^{-1} \sim \epsilon_F$. 

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In order for the $T = 0$ approximation to be valid, the temperature must be smaller by a factor of $O(N)$, so this regime is far beyond the limits of current technique.

The divergence of $\langle D \rangle$ at nonzero $T$ indicates the breakdown of our approximation. We may see this more explicitly by looking at the argument of the exponential in (35). The Taylor series $\sum_{j=0}^{\infty} \frac{x^j}{j!}$ for $e^x$ is dominated by the terms for which $j$ is $O(x)$. At $T = 0$, the argument of the exponential appearing in (35) is $\int_{-\infty}^{+\infty} p_{\min} e^{-\alpha p_i} dp_i = \log \left( \frac{L}{2\pi\alpha} \right)$, which is $O(\log L)$. Therefore, the dominant contributions to the exponent come from values of $j$ that are likewise $O(\log L)$. However, when $T > 0$, the average number of quanta present in each of the lowest-lying modes is $O(L)$ unless $\beta > (v_F p_{\min})^{-1}$. The argument of the exponential and the dominant values of $j$ are then $O(L \log L)$. This contradicts our assumption that $j$ is smaller than $O(L)$.

We have shown that when the temperature is nonzero, we must receive a substantial contribution from terms for which $j$ is $O(L)$ or larger, so the methods we have used so far do not apply to this case. However, the $T = 0$ treatment does provide us with several insights that will prove useful when we analyze the case in which $T > 0$ but the “transverse temperature” is still vanishing. We shall return to that case later; however, there are several more points about the zero-temperature situation that we must consider first.

Thus far, we have assumed that $\alpha^{-1}$ is the correct cutoff for the momentum integral appearing in (35). However, there is also another cutoff we must consider. In order to obtain a physically meaningful result, we must determine which cutoff is the relevant one.

A real Luttinger liquid has a finite bandwidth $v_F \alpha^{-1}$, because the phonon description breaks down at large values of $p_i$. The Luttinger model only describes the low-energy excitations of the fermion system. When $p_i$ becomes comparable to $k_F$, the energy $(k_F + p_i)^2 - k_F^2 / 2m$ of an excitation can no longer be approximated by $v_F p_i$ and is no longer much smaller than $\epsilon_F$, so we expect the physical cutoff $\alpha^{-1}$ to be $O(k_F)$.

The second cutoff arises from the energy denominator (31). When $p_i$ is large compared to $\sqrt{m\Delta}$, the integrand decays as $p_i^{-3}$. If there existed an intermediate range of $p_i$ values, where $m\Delta / k_F \ll p_i \ll \sqrt{m\Delta}$, the integrand would decay as $p_i^{-2}$ in this region. However, since we must have $\Delta > \frac{1}{2m} k_F^2$ in order to have an energy gap at all, such a region cannot exist. So the rapid $p_i^{-3}$ decay at large values of $p_i$ will provide a cutoff for the integral which we expect to be $O(\sqrt{m\Delta})$.

We must determine which of these two cutoff scales is smaller, since the smaller one determines the behavior of the energy shift. When $\Delta \gg \epsilon_F$, the first cutoff, near the Fermi momentum, is smaller. However, when the energy gap is comparable to the Fermi energy, $\sqrt{m\Delta}$ is itself $O(k_F)$. Without more precise values for the cutoffs, it is not clear which one matters in this situation.

We shall now estimate the cutoff $p_{\max}'$ arising from the $p_i^{-3}$ behavior of (31). To do this, we shall again make use of the fact that the integral over the $p_i$ is dominated by the region where the $p_i$ are small. The value of $p_{\max}'$ is determined by the behavior of
the integrand when not all the $p_i$ are small; at least one of the $p_i$ must be comparable to $\sqrt{m\Delta}$. However, the dominant contribution under such conditions comes from the region where only a single $p_i$ is large compared to $p_{\text{min}}$. If more than one $p_i$ is large, the contribution is much smaller. This means that we may determine $p'_{\text{max}}$ by analyzing the case in which all but one of the $p_i$ are negligible.

More precisely, we approximate $\delta_{k,k_F+\sum_{i=1}^j \pm p_i}$ by $\delta_{k,k_F-p_i}$ and $\sum_{i=1}^j \pm \omega_{p_i}$ by $\omega_{p_i}$. The choice of signs corresponds to the fact that the virtual phonon at momentum $p_i$ must be created, not annihilated, since almost all the phonon modes are empty of quanta when $T = 0$. One of the $p_i$ integrations is now nontrivial. We shall perform this integration explicitly and extract the cutoff from it. Our prescription for doing this is that

$$\int_{p_{\text{min}}}^{+\infty} dp_i \frac{1}{p_i \Delta + \frac{1}{2m}[(k_F - p_i)^2 - k_F^2] + \omega_{p_i}} \approx \frac{1}{\Delta} \log \left( \frac{p'_{\text{max}}}{p_{\text{min}}} \right). \quad (43)$$

The integral in (43) is elementary. The result is

$$\int_{p_{\text{min}}}^{+\infty} dp_i \frac{1}{p_i \Delta + \frac{1}{2m}[(k_F - p_i)^2 - k_F^2]} \left( - \frac{\alpha}{p_i} \right) \approx \log \left( \frac{L p_{\text{max}}}{2\pi} \right), \quad (44)$$

so, neglecting $e^{2\gamma}p_{\text{min}}^2$ compared to $2m\Delta$, we have that $p'_{\text{max}} = e^\gamma \sqrt{2m\Delta}$. (This is really the appropriate value of $p'_{\text{max}}$ only when $q \ll \sqrt{2m\Delta}$. A more careful calculation, which takes into account the $q$-dependence of the cutoff, is located in Appendix B.)

Although this estimate is rather inexact, it shows the general character of the cutoff. At $\Delta = \epsilon_F$, the cutoff takes on its minimum value of $p'_{\text{max}} \approx 1.78k_F$. However, we have not yet determined whether this is ever the relevant cutoff. We shall defer the final resolution of this question until Section 4. In the meantime, we shall make the replacement

$$\int_{p_{\text{min}}}^{+\infty} dp_i \frac{e^{-\alpha p_i}}{p_i} \rightarrow \log \left( \frac{L p_{\text{max}}}{2\pi} \right), \quad (45)$$

where in all cases, $p_{\text{max}}$ is $O(k_F)$. Using the prescription (45) to evaluate the integral and inserting our expression for $P(T = 0)$, we get

$$\Delta E_+ \approx \frac{1}{4\pi^2} \sum_{s=\pm 1} \frac{|g(q)|^2 q p_{\text{max}} \left( n + q + \frac{s+1}{2} \right)}{\Delta + \frac{1}{2m}[(k_F - sq)^2 - k_F^2] + s\omega_q}. \quad (46)$$

The denominator, $\Delta + \frac{1}{2m}[(k_F - sq)^2 - k_F^2] + s\omega_q = \Delta + \frac{q^2}{2m}$ is independent of $s$, so we may trivially perform the sum over $s$. Thus far, we have restricted our attention to the case for which $c_k$ corresponds to a right-moving fermion. The terms in which $c_k$ corresponds to a left-moving particle also contributes an equal amount, so we have

$$\Delta E \approx \frac{|g(q)|^2 q p_{\text{max}}}{\pi^2 (\Delta + \frac{q^2}{2m})} \left( n + q + \frac{1}{2} \right). \quad (47)$$
is the energy shift for a single bosonic mode, with momentum $+q$. To obtain the total shift for the system, we must add in the similar renormalization of the left-moving mode with momentum $-q$ and integrate over $q$. Since the density of momentum states is proportional to $L$, the total shift is extensive, as it must be. We must also sum over all the possible $n_\rho$ values to which $d_k$ may correspond; this will give an infinite sum of terms of the form (47), in which $\Delta$ takes on all the values $\Delta_{0,n_\rho}$, $n_\rho > 0$.

We shall now turn our attention to the $T > 0$ case. The crucial simplification that allowed us to evaluate the $T = 0$ energy shift in closed form was the fact that the energy defects were effectively independent of the intermediate states. A remarkable cancellation then occurred between the prefactor $P$ and the exponential that arose when we summed over all all intermediate states. A similar cancellation will occur here.

When $T > 0$, we may not necessarily neglect $\sum_{i=1}^j \mp p_i$ and $\sum_{i=1}^j \pm \omega_{p_i}$, because $j$ is not generally small. The presence of these two sums gives rise to a nontrivial energy denominator. However, if $\Delta$ is formally very large, so that any other energy may be neglected compared to it, the energy defect again becomes independent of the $p_i$. As in the previous case, the temperature is negligible when compared with the transverse energy scale; that is, the “transverse temperature” is effectively vanishing. With this simplification, it will again be possible to obtain an analytical expression for the energy shift.

If we again assemble the various terms that compose the energy shift (now assuming that $\Delta$ is very large), we have

$$\Delta E_+ = - \sum_{s=\pm 1} \frac{|g(q)|^2 q (n_{+q} + \frac{s+1}{2})}{2\pi} \sum_{\{m_+\}} \frac{|\{m_+\}| \prod_{p>0} e^{\lambda_{+p} A_{+p}^\dagger - \lambda_{+p}^* A_{+p}} |\{n_+\}|^2}{\Delta},$$

where $\lambda_{+p} = -\sqrt{2\pi/(pL)} e^{-\alpha p/2}$, and $A_{+p}^\dagger$ and $A_{+p}$ are the canonical ladder operators corresponding to the $\rho_+ (\pm p)$ quanta.

Because the energy defect in (48) is independent of the $\{m_+\}$, we may rewrite the energy shift as

$$\Delta E_+ = - \sum_{s=\pm 1} \frac{|g(q)|^2 q (n_{+q} + \frac{s+1}{2})}{4\pi^2 \epsilon \Delta} \sum_{\{m_+\}} \left\{ \prod_{p>0} \left( e^{\lambda_{+p} A_{+p}^\dagger - \lambda_{+p}^* A_{+p}} \right) \right\} \left| \sum_{\{m_+\}} |\{m_+\}\rangle \langle\{m_+\}| \prod_{p>0} \left( e^{\lambda_{+p} A_{+p}^\dagger - \lambda_{+p}^* A_{+p}} \right) \right| |\{n_+\}\rangle \right\}.$$  

(49)

The operator in brackets is just the identity, so we may drop it. Furthermore, since $\lambda_{+p} A_{+p}^\dagger - \lambda_{+p}^* A_{+p}$ is antihermitian, $\prod_{p>0} \left( e^{\lambda_{+p} A_{+p}^\dagger - \lambda_{+p}^* A_{+p}} \right) = \prod_{p>0} \left( e^{\lambda_{+p} A_{+p}^\dagger - \lambda_{+p}^* A_{+p}} \right)^{-1}$, and the entire matrix element expression is unity. [We now see that the remarkable simplicity of the $T = 0$ result is a consequence of the fact that when the energy denominator
is constant, the sum over intermediate states may be evaluated simply using the closure
relation. However, the detailed analysis that led up to (46) was necessary for us to justify
taking the energy denominator to be constant.]

This \( T > 0 \) result shows that, when \( \Delta \) is very large, we recover a similar expression
to the one we obtained in a different extreme limit, that of zero temperature. The total
energy shift is

\[
\Delta E = -\frac{|g(q)|^2 q}{\pi^2 \alpha \Delta} \left(n + \frac{1}{2}\right).
\]

(50)

This expression is actually identical to the large \( \Delta \) limit of (47). In that limit, \( \frac{q^2}{2m} \) may be
neglected compared to \( \Delta \), and the correct cutoff \( p_{\text{max}} \) is just \( \alpha^{-1} \), because \( p'_{\text{max}} \)
grows as \( \mathcal{O}(\sqrt{m\Delta}) \). However, we still do not know to what actual value \( \alpha \) corresponds. We shall
now turn our attention to a different model, which will allow us to answer that question.

## 4 Energy shift for a more physical model

In this section, we present an alternative model—one which is useful when \( T = 0 \) and
\( \Delta \) is very large. Using this model, we shall again calculate the energy shift. From our
calculation we may extract the physically relevant value of \( \alpha^{-1} \), the cutoff arising from
the breakdown of the phonon description. Moreover, we shall find in Section 5 that our
calculation requires only slight modification in the presence of strong interactions between
the low-lying fermion states.

The Luttinger model does not accurately represent the fermions lying deep within the
Fermi sea, so it is not necessary to use a bosonized form for the matrix elements of \( c_k \)
when \( k \not\approx k_F \). Since a basic assumption of the Luttinger liquid theory is that the presence
or absence of additional fermions far from the Fermi surface should have minimal effect
on the low-energy excitations, we shall simply assume that the action of \( c_k \) \( (k \not\approx k_F) \) on
a Luttinger liquid state is to create a hole in the Fermi sea, without affecting the phonon
configuration in any way. The only exception to this prescription is that when the fermion
operators are combined in the form (44), as they are in \( H_2 \), we shall used the bosonized
expressions.

Our approximation will require that \( \Delta \gg \epsilon_F \). This condition ensures that the energy
shift is not dominated by the terms with the smallest energy defects; instead, the defects
are dominated by \( \Delta \) and depend only weakly on the energy of the fermion annihilated by
\( c_k \). So the entire Fermi sea will contribute to the energy shift, and the region near the
Fermi surface (where our approximation for \( c_k \) is invalid) will make a comparatively small
contribution. The condition that \( T = 0 \) ensures that the occupied states are precisely
those with \( |k| < k_F \).

Now we must identify the states between which we shall be calculating matrix elements.
As before, we shall have the state \( \{|n_p\rangle \} \) on the right. We shall find the energy shift of
this state due to interactions with the states \( \{|n'_p\}; k \not\equiv q; k\rangle \). These states have bosonic
occupation numbers \( \{n'_{+p}\} \) which are the same as \( \{n_{+p}\} \), except that \( n'_{+q} = n_{+q} \pm 1 \). These states also have an excited fermion with momentum \( k \mp q \) and a hole in the Fermi sea with momentum \( -k \).

The matrix elements we shall need are then

\[
\langle \{n'_{+p}\}; k - sq; k | \rho_+ (sq) d_{k - sq}^\dagger c_k | \{n_{+p}\} \rangle = \sqrt{\frac{qL}{2\pi}} \left( n_{+q} + \frac{s + 1}{2} \right) \theta(k_F - |k|),
\]

for \( s = 1 \). The step function ensures that the matrix elements vanish when \( |k| > k_F \). The energy defects are

\[
E_{\{n'_{+p}\}; k - sq, k} - E_{\{n_{+p}\}} = \Delta + \frac{1}{2m} [(k - sq)^2 - k^2] + s\omega_q,
\]

so the energy shift is

\[
\Delta E = -\frac{1}{L^2} |g(q)|^2 \sum_{k=-k_F}^{k_F} \sum_{s=\pm 1} |\langle \{n'_{+p}\}; k - sq; k | \rho_+ (sq) d_{k - sq}^\dagger c_k | \{n_{+p}\} \rangle|^2 \quad \Delta = \frac{1}{2m} [(k - sq)^2 - k^2] + s\omega_q
\]

Converting the sum over \( k \) into an integral, we have

\[
\Delta E = -\frac{1}{L^2} |g(q)|^2 \left( \frac{qL}{2\pi} \right) \sum_{s=\pm 1} \left( n_{+q} + \frac{s + 1}{2} \right) \left( \frac{L}{2\pi} \right) \int_{-k_F}^{k_F} dk \left( \frac{1}{\Delta + \frac{q^2}{2m} + s\omega_q} - s\frac{q}{m} \right)
\]

As in Section 3, we must sum over all values of \( q \) to get an extensive energy shift.
under these circumstances. In fact, (47) and (55) agree in their region of common validity exactly if \( p_{\text{max}} = k_F \). Since \( k_F \) is substantially smaller than any possible value of \( p'_{\text{max}} \) (for large \( \Delta \)), we must conclude that the relevant cutoff is \( p_{\text{max}} = \alpha^{-1} \). The value of \( \alpha^{-1} \) is a property solely of the Luttinger liquid, and \( \alpha^{-1} \) is thus independent of \( \Delta \). Therefore, \( \alpha^{-1} \) is always equal to the Fermi momentum when \( T = 0 \). We then see that \( \alpha^{-1} < p'_{\text{max}} \) for all values of \( \Delta \), so \( \alpha^{-1} \) is always the most relevant cutoff, and (55) is the general expression for the \( T = 0 \) energy shift. (We shall discuss the possible relationship between \( \alpha \) and the temperature in Section 5.)

A simple estimate of the magnitude of any correction to (55) due to \( p'_{\text{max}} \) is that the fractional error should be approximately \( O\left(\frac{k_F}{p'_{\text{max}}}\right) \). Since we found a minimum value of \( p'_{\text{max}} \) of only 1.78\( k_F \), this error could be significant when the energy gap is small. This must be kept in mind when this formula is applied to systems for which \( \Delta \) is \( O\left(\epsilon_F\right) \).

5 Extensions and discussion

The model outlined in Section 4 may be applied to systems with more general interactions. In particular, our techniques may be applied to the interacting Tomonaga-Luttinger model. The Tomonaga-Luttinger Hamiltonian \( H_0 + H_1 \) is diagonalized by the Bogoliubov-transformed operators [24]

\[
\rho'_\pm(q) = \frac{1}{2} (X + X^{-1}) \rho_\pm(q) + \frac{1}{2} (X - X^{-1}) \rho_\mp(q),
\]

(56)

where

\[
X = \left( v_F + \frac{g_4(q)}{2\pi} + \frac{g_2(q)}{2\pi} \right)^{1/4} \left( v_F + \frac{g_4(q)}{2\pi} - \frac{g_2(q)}{2\pi} \right). \quad (57)
\]

The \( \rho'_\pm \) satisfy the same commutation relations as \( \rho_\pm \). To determine the effects of the additional interaction \( H_2 \), we only need relate \( \rho_+(q) + \rho_-(q) \) to \( \rho'_+(q) + \rho'_-(q) \). From (58), it is obvious that

\[
\rho_+(q) + \rho_-(q) = \left( v_F + \frac{g_4(q)}{2\pi} - \frac{g_2(q)}{2\pi} \right)^{1/4} \left[ \rho'_+(q) + \rho'_-(q) \right], \quad (58)
\]

so the calculation of \( \Delta E \) requires only two modifications. We must use the Tomonaga-Luttinger frequency

\[
\omega'_q = |q| \left[ \left( v_F + \frac{g_4}{2\pi} \right)^2 - \left( \frac{g_2}{2\pi} \right)^2 \right]^{1/2}, \quad (59)
\]

in place of \( \omega_q \), and we must multiply the whole expression for the energy shift by \( |X|^{-2} \). This complicates the expressions somewhat; however, if we neglect terms of \( O(\omega_q) \) com-
pared to $\Delta$ (since $q \ll k_F$), we simply have

$$\Delta E = -\frac{|g(q)|^2 q k_F}{\pi^2 \Delta} \left| v_F + \frac{g_4(q)}{2\pi} - \frac{g_2(q)}{2\pi} \right|^{1/2} \left( n_{+q} + \frac{1}{2} \right). \quad (60)$$

It is slightly more difficult to perform this calculation using the bosonized operators. The mixing of the right-moving and left-moving phonons in $\psi_{\pm}$ adds to the complexity of the calculations [2, 17, 25, 26]. Performing the calculation, we find that the matrix elements factorize into $\rho_+^\prime$ terms and $\rho_-^\prime$ terms. The cancellations that appeared in Section 3 at $T = 0$ now occur separately for the two types of terms. The end result is the same as (60), provided we set $p_{\max} = \alpha^{-1} = k_F$ and again neglect the $O(\omega_q)$ terms.

Our main result is (55), the expressions for the energy shift. It is interesting to note that (55) remains regular—without a pole or a branch point—at $\Delta = \epsilon_F$. When $\Delta_{0,1} = \epsilon_F$, we are in a slightly different regime from the one we have considered, because some $n_\rho = 0$, $\ell = \pm 1$ states of the trap will be occupied. However, we can still draw some conclusions from the behavior of the energy shift. Although the ground state changes qualitatively at this point, as fermions “spill over” out of the Luttinger liquid into the radially excited state, the phonon energies remain finite. Even if the decay of a phonon into a $n_\rho = 0$, $\ell = 0$ hole and a $n_\rho = 1$, $\ell = 0$ fermion is allowed energetically, the phonons appear to remain metastable against this decay channel. This represents an interesting “softening” of the expected phase transition at this point. However, in order to understand this situation properly, other interactions than (10), including those involving the occupied $\ell = \pm 1$ states, must also be considered.

Since the cutoff parameter $\alpha$ is related to the renormalization group flow for the Luttinger liquid [21], the identification of $k_F$ with $\alpha^{-1}$ has interesting implications. In the Luttinger model, $\alpha$ is effectively a free parameter; its value needs to be specified on the basis of physical considerations external to the model. Our results at $T = 0$ suggest that $v_F k_F = 2 \epsilon_F$ is the maximum physically meaningful bandwidth for the system. This order of magnitude for the cutoff has a clear physical basis, as discussed in Section 3. Since the Luttinger model (with spin) is known to be equivalent to several other condensed matter models [21, 27, 28, 29, 30, 31] and field theory models [15, 32], the specification of a particular value of $\alpha$ may have interesting implications for the related cutoffs of these analogous systems.

Although we have identified $\alpha^{-1} = k_F$ as the correct physical cutoff at $T = 0$, our results do not tell us anything about the renormalization group flow for nonzero temperatures. Moreover, although (50) is valid for finite temperatures, it is only meaningful in an extremely singular limit; all the temperature-dependence of the expression has been removed by making $\Delta$ very large. The temperature-independent expression (50) is consistent with a nontrivial $T$-dependence of the energy shift, because $\Delta E$ is actually formally vanishing in this limit, since it is proportional to the ratio $[|g(q)|^2 \alpha^{-1} q]/\Delta$. So the questions of the finite-temperature energy shift and the nature of the renormalization group flow for $T > 0$ remain open.
We have used several complementary techniques to evaluate the energy shift. The bosonized method used in Section 3 is valid in a wider range of situations; however, we needed the calculations of Section 4 to determine the unknown constant $p_{\text{max}}$ appearing the bosonized result. We found a frequency shift that is always $O[|g(q)|^2 q k_F / \Delta]$, even for nonzero temperatures or in the presence of strong Tomonaga-Luttinger-type interactions. When experimenters are able to build fermionic traps with sufficiently high aspect ratios to observe Luttinger liquid behavior and to trap fermions at sufficiently low temperatures, formulas such as these should become readily testable.

**Acknowledgments**

The author is grateful to K. Huang for many helpful discussions.

**A Appendix: Harmonic oscillator matrix elements**

To obtain the general formula (20), we first apply the Baker-Campbell-Hausdorff formula to the operator $e^{\lambda A^\dagger - \lambda^* A}$. Since $[\lambda^* A, \lambda A^\dagger] = -|\lambda|^2$ commutes with both $A$ and $A^\dagger$, we have

$$e^{\lambda A^\dagger - \lambda^* A} = e^{-\lambda^* A} e^{\lambda A^\dagger} e^{\lambda^2 / 2}. \quad (61)$$

This reduces the problem to the determination of $\langle n| e^{-\lambda^* A} e^{\lambda A^\dagger} | m \rangle$.

There are three cases: $m = n$, $m > n$, and $m < n$. The first case is the simplest. If $m = n$, each factor of $A$ from $e^{-\lambda^* A}$ must be paired with a factor of $A^\dagger$ from $e^{\lambda A^\dagger}$; otherwise, the term does not contribute. So we get

$$\langle m| e^{-\lambda^* A} e^{\lambda A^\dagger} | m \rangle = \langle m| 1 + (-\lambda^* A) (\lambda A^\dagger) + \frac{1}{(2!)^2} (-\lambda^* A)^2 (\lambda A^\dagger)^2 + \cdots | m \rangle$$

$$= \sum_{i=0}^{\infty} (-1)^i |\lambda|^{2i} \frac{1}{(i!)^2} \frac{(m + i)!}{m!}$$

$$= F(m + 1; 1; -|\lambda|^2). \quad (62)$$

For the case of $m > n$, there must be $l$ more factors of $A^\dagger$ than factors of $A$ for a term to contribute. This immediately leads to the series

$$\langle m| e^{-\lambda^* A} e^{\lambda A^\dagger} | n \rangle = \langle m| \frac{1}{l!} (\lambda A^\dagger)^l + \frac{1}{(l + 1)!} (-\lambda^* A) (\lambda A^\dagger)^{l+1}$$

$$+ \cdots + \frac{1}{2! (l + 2)!} (-\lambda^* A)^2 (\lambda A^\dagger)^{l+2} + \cdots | n \rangle$$

$$= \lambda^l \sum_{i=0}^{\infty} \frac{1}{l! (l + i)!} (-1)^i |\lambda|^{2i} \frac{(m + i)!}{m!} \sqrt{\frac{m!}{n!}}.$$
In the third case, \( l \) is negative. The calculation proceeds along essentially the same lines as for the \( l > 0 \) case. There must be \(|l|\) more factors of \( A \) than \( A^\dagger \) for a term to be nonzero, so we get

\[
\langle m \mid e^{-\lambda^* A}e^{\lambda A^\dagger} \mid n \rangle = \langle m \mid \frac{1}{|l|!} (-\lambda^* A)^{|l|} + \frac{1}{(|l| + 1)!} (-\lambda^* A)^{|l|+1} (\lambda A^\dagger) + \frac{1}{2! (|l| + 2)!} (-\lambda^* A)^{|l|+2} (\lambda A^\dagger)^2 + \cdots |n \rangle
\]

\[
= (-\lambda^*)^{|l|} \frac{1}{|l|!} \sum_{i=0}^{\infty} \frac{|l|!}{(|l| + i)!} (-1)^i |\lambda|^{2i} \frac{(n + i)!}{n!} \sqrt{n!} \lambda A^\dagger \]

\[
= (-\lambda^*)^{|l|} \frac{1}{|l|!} \sqrt{n!} m! F \left( n + 1; |l| + 1; -|\lambda|^2 \right). \quad \text{(64)}
\]

Combining equations (14), (15), (16), and (17), we get (20).

**B Appendix: Momentum dependence of \( p'_{\text{max}} \)**

We shall now examine how the cutoff \( p'_{\text{max}} \) depends upon \( q \). We do this by modifying the prescription (13) for \( p'_{\text{max}} \) to

\[
\int_{p'_{\text{min}}}^{+\infty} \frac{dp_i}{p_i} \Delta + \frac{1}{2m} \left[(k_F - s\bar{q} - p_i)^2 - k_F^2 \right] + s\omega_q + \omega_{p_i} \approx \frac{1}{\Delta + \frac{q^2}{2m}} \log \left( \frac{p'_{\text{max}}}{p_{\text{min}}} \right). \quad \text{(65)}
\]

To do this, we use the integration formula

\[
\int_{p_{\text{min}}}^{+\infty} \frac{dp_i}{ap_i + bp_i^2 + cp_i^2} = \frac{1}{a} \log \left( \frac{e^\gamma}{p_{\text{min}}} \sqrt{\frac{a + be^{-\gamma}p_{\text{min}} + ce^{-2\gamma}p_{\text{min}}^2}{c}} \right) \quad \text{(66)}
\]

\[
+ \frac{b}{2a \sqrt{-b^2 + 4ac}} \left[ i \log \left( \frac{2ic}{\sqrt{-b^2 + 4ac}} \right) - i \log \left( \frac{-2ic}{\sqrt{-b^2 + 4ac}} \right) \right] - i \log \left( \frac{\sqrt{-b^2 + 4ac} - ib - 2ice^{-\gamma}p_{\text{min}}}{\sqrt{-b^2 + 4ac}} \right) \right] - i \log \left( \frac{\sqrt{-b^2 + 4ac} + ib + 2ice^{-\gamma}p_{\text{min}}}{\sqrt{-b^2 + 4ac}} \right) \right].
\]

In this case, we have \( a = \Delta + \frac{q^2}{2m}, b = \frac{s\bar{q}}{m}, \) and \( c = \frac{1}{2m} \). We shall define \( x \) to be the frequently-appearing ratio \( x \equiv -\frac{b}{\sqrt{-b^2 + 4ac}} = -s \frac{q}{\sqrt{2m}\Delta} \). We shall find that \( p'_{\text{max}} \) depends upon \( q \) only through \( x \).

To simplify the expression (66), we first note that since \( p_{\text{min}} \) is small compared to any other momentum scale, we may approximate \( a + be^{-\gamma}p_{\text{min}} + ce^{-2\gamma}p_{\text{min}}^2 \approx a \) and \( ib + 2ice^{-\gamma}p_{\text{min}} \approx ib \). This reduces the first term on the right-hand side to \( \frac{1}{a} \log \left( \frac{e^\gamma}{p_{\text{min}}} \right) \sqrt{\frac{a}{c}} \).
We now turn our attention to the complex terms. Since these terms appear in conjugate pairs, we need only calculate their real parts. Each term has an overall factor of $i$ times a logarithm, so we shall only need the logarithms’ imaginary parts. Because $\sqrt{-b^2 + 4ac}$ is purely imaginary, we may simplify the second and third terms on the right-hand side of (66) to

$$-\frac{x}{2a}\left[i \log\left(\frac{2ic}{\sqrt{-b^2 + 4ac}}\right) - i \log\left(\frac{-2ic}{\sqrt{-b^2 + 4ac}}\right)\right] = -\frac{x}{2a} \left[i \log i - i \log(-i)\right]$$

$$= -\frac{x}{2a} \left[i \left(\frac{\pi}{2}\right) - i \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi x}{2a}. \quad (67)$$

We may evaluate the last two terms of (66) similarly, getting

$$-\frac{x}{2a} \left[\log(1 + ix) - i \log(1 - ix)\right] = -\frac{x}{2a} \left[i \tan^{-1} x - i \tan^{-1}(-x)\right]$$

$$= \frac{x}{a} \tan^{-1} x. \quad (68)$$

So the entire expression becomes

$$\int_{p_{\text{min}}}^{p_{\text{max}}} \frac{dp_i}{ap_i + bp_i^2 + cp_i^3} = \frac{1}{a} \log\left(\frac{e^\gamma \sqrt{2m\Delta + q^2}}{2\pi}\right) + \frac{1}{a} \left[x \left(\tan^{-1} x + \frac{\pi}{2}\right)\right]. \quad (69)$$

We may absorb the second term on the right-hand side of (69) into the logarithm by exponentiating it. We then identify the cutoff as

$$p'_{\text{max}} = e^\gamma \sqrt{2m\Delta \sqrt{1 + x^2} \exp \left[x \left(\tan^{-1} x + \frac{\pi}{2}\right)\right]} \quad (70)$$

This expression depends upon $s$. However, only the value of $p'_{\text{max}}$ averaged over $s = \pm 1$ is ultimately relevant, so we may modify our expression for $p'_{\text{max}}$ to become

$$p'_{\text{max}} = e^\gamma \sqrt{2m\Delta \sqrt{1 + x^2} \exp \left(x \tan^{-1} x\right) \cosh \left(\frac{\pi x}{2}\right)}. \quad (71)$$

For fixed $\Delta$, this takes on its minimum value in the $q = 0$ case considered previously.

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