1. Introduction

The study of fixed point results in partially ordered sets finds its root in the work of Knaster [1] and Tarski [2]. In 1955, Tarski published his work in the context of complete lattices; the result states that each monotone mapping from a complete lattice to itself has a fixed point. In [3], Abian and Brown extended the result of Knaster–Tarski to chain-complete poset with a least or largest element and showed that every order-preserving map has a fixed point.

On the contrary, several fixed-point theorems in metric spaces endowed with a partial order have been stated and studied. In 2004, Ran and Reuring (see [4]) combined successfully the Banach Contraction Principle and Knaster–Tarski fixed point. They managed to prove that every monotone mapping in a complete metric space has a fixed point provided that it satisfies contraction condition only for comparable elements. Jachymski [5] managed to prove an equivalent result of Ran and Reuring’s in a metric space endowed with a graph.

In the same vein, the authors in [6, 7] extended the result of Ran and Reuring to the case of monotone nonexpansive mappings. Their starting point was to approach the fixed point by iterative techniques and successive approximations. Recently, Espínola and Wiśnicki [8] generalized the above results in Hausdorff topological spaces endowed with partial order. The key ingredient in such a generalization is the compactness of the order intervals mixed with Knaster–Tarski fixed point. For more details, see [9, 10].

In this work, we generalize several known results in the context of topological spaces endowed with a digraph instead of partial order. For this purpose, we introduce the concept of $G$-regular monotone mapping and we give some applications in modular function spaces of the obtained results.

2. Main Results

Since the main result of this work relates topological properties to graphs, the following definition is needed. The interested reader can consult [11], for more details.

Definition 1. A directed graph or digraph $G$ is determined by a nonempty set $V(G)$ of its vertices and the set $E(G) \subseteq V(G) \times V(G)$ of its directed edges. A digraph is reflexive if each vertex has a loop. Given a digraph $G = (V, E)$,

(i) If whenever $(x, y) \in E(G) \Rightarrow (y, x) \notin E(G)$, then the digraph $G$ is called an oriented graph.
Definition 2. Let $G = (V(G), E(G))$ be a reflexive digraph and $a, b \in V(G)$.

(i) We define the $G$-intervals as follows:

$$[a, \longrightarrow] = \{x \in V(G), x \in [a]_G\},$$

$$\langle \longrightarrow, a \rangle = \{x \in V(G), a \in [x]_G\},$$

and

$$[a, b] = [a, \longrightarrow] \cap \langle \longrightarrow, b \rangle.$$  

(ii) For a subset $A$ of $V(G)$, we say that $a \in V(G)$ is a $G$-upper bound of $A$ if $a \in [x, \longrightarrow)$, $\forall x \in A$. A $G$-lower bound of $A$ if $a \in (\longrightarrow, x]$, $\forall x \in A$.

(iii) A $G$-upper bound of $A$ that belongs to $A$ is called $G$-maximal element of $A$, and a $G$-lower bound of $A$ that belongs to $A$ is called $G$-minimal element of $A$.

(iv) We say that $a \in V(G)$ is a $G$-supremum of $A$ if

$$a \text{ is a } G\text{-upper bound for every } G\text{-upper bound } b \text{ of } A, a \in (\longrightarrow, b]$$

and $a \text{ is a } G\text{-lower bound for every } G\text{-lower bound } b \text{ of } A, a \in [b, \longrightarrow)$.

The following example illustrates this last definition.

Example 1

(1) Let $V(G) = \mathbb{R}$ and $E(G) = \{(x, y) : |x| \leq |y|\}$. $G$ is a reflexive digraph. Set $A_1 = (-\infty, 0]$.

$a \in A_1$ is a $G$-upper bound if for each $x \in A_1$,

$$a \in [x, \longrightarrow) \Leftrightarrow |x| \leq |a|,$$

thus, $A_1$ has no $G$-upper bound.

$a \in A_1$ is a $G$-lower bound if for each $x \in A_1$,

$$a \in (\longrightarrow, x] \Leftrightarrow |a| \leq |x|,$$

thus, $0$ is the only $G$-lower bound of $A_1$. Moreover, since $0 \in A_1$, it is the only $G$-minimal element and it is the $G$-infimum.

Since $A_1$ has no $G$-upper bound the sets of $G$-maximal and $G$-supremum are empty.

(2) Unlike the case of partially ordered, the $G$-supremum of set may not be unique. Indeed, we consider $A_2 = [0, 1]$, thus we have

The set $G$-upper bound of $A_2$ is $(-\infty, -1) \cup [1, \infty)$

1 is the only $G$-maximal element of $A_2$

$a$ is $G$-supremum of $A_2$ if and only if $a \in [-1, 1]$

Recall that a collection of sets $E$ has the finite intersection property (f.i.p.), if, for every family $F$ of members of $E$, the intersection of $F$ is nonempty provided the intersection of all finite subfamilies of $F$ are nonempty.

Definition 3. Let $G = (V(G), E(G))$ be a reflexive digraph; a subset $L$ of $V(G)$ is said $G$-directed if every finite subset of $L$ has a $G$-upper bound in $L$.

We then get the following generalization of the result obtained in [8] for graphs.

Lemma 1. Let $G = (V(G), E(G))$ be a reflexive digraph; if $G$ has the finite intersection property for $G$-intervals, then every $G$-directed subset $L$ of $V(G)$ has a $G$-supremum.

Proof 1. We consider the set $M = \bigcap_{x \in L} [x, \longrightarrow)$, as $L$ is $G$-directed; every finite intersection $\bigcap_{i=1}^{n} [x_i, \longrightarrow)$, where $x_1, x_2, \ldots, x_n \in L$, is nonempty since it contains every $G$-upper bound that is in $L$ of the finite subset $\{x_1, x_2, \ldots, x_n\}$ and as $G$ has the finite intersection property, and $M$ is nonempty.

Let us consider now the set $M' = \bigcap_{x \in L, y \in M} [y, x]$, then, again, every finite intersection $\bigcap_{i=1}^{n} [y_i, x_i]$ where $x_1, x_2, \ldots, x_n \in L$ and $y_1, y_2, \ldots, y_n \in M$ is nonempty (also it contains any $G$-upper bound that is in $L$ of the finite subset $\{x_1, x_2, \ldots, x_n\}$ and $G$ has the finite intersection property $M'$ which is nonempty. And, it is clear that every element of $M'$ is a $G$-supremum of $L$.

Recall that a map $T: X \longrightarrow X$ is said to be $G$-monotone if for all $x, y \in X$, whenever $y \in [x]_G$, then $T(y) \in [T(x)]_G$.

Next, we introduce the notion of $G$-regular monotone.

Definition 4. Let $X$ be a set endowed with a graph $G = (V(G), E(G))$ a map $T: X \longrightarrow X$ is said to be $G$-regular monotone; if $T$ is $G$-monotone and for every $x, y \in X$, if $x \in [y]_G$ and $y \in [x]_G$, then $T(x) = T(y)$.
The following theorem is the cornerstone of what follows.

**Theorem 1.** Let $X$ be a topological space endowed with a reflexive digraph $G = (V(G), E(G))$ such that $G$-intervals are compact, and let $T: X \rightarrow X$ be $G$-regular monotone map; if there exists $x_0 \in X$ such that $T(x_0) \in [x_0]_G$ then the set of fixed points of $T$ is not empty and has a $G$-maximal element.

**Proof.** Let $L = \{T^n(x_0) : n \in \mathbb{N}\}$. Then, $L$ is $G$-directed and for all $x \in L$, $T(x) \in L$ and $T(x) \in [x]_G$. Set now

$$\mathcal{F} = \{G - \text{directed } L \subset J, \text{ and } \forall x \in J, T(x) \in J \text{ and } T(x) \in [x]_G\},$$

then $\mathcal{F}$ is nonempty inductive set with respect to the $\subset$ order if $(I)_i \in \mathcal{F}$ is a chain in $\mathcal{F}$ then $\bigcup_{i \in I} I_i$ is an upper bound of $(I)_i \in \mathcal{F}$.

By Zorn’s lemma, there exists a maximal $G$-directed set $L_0$ such that $L \subset L_0$ and for all $x \in L_0$, $T(x) \in L_0$ and $T(x) \in [x]_G$. Since the $G$-intervals are compacts, $G$ has the finite intersection property for $G$-intervals; thus, $L_0$ has a $G$-supremum $s$.

Now, for all $x \in L_0$, we have $s \in [x]_G$ and $T(s) \in [T^n(x)]_G$ and $T(x) \in [x]_G$ for all $x \in L$. Hence, $T(s)$ is a $G$-upper bound of $L_0$, that is, $T(s) \in [s]_G$. We then claim that $s \in L_0$. Indeed if $s \notin L_0$, then

$$\bar{L} = L_0 \cup \{T^n(s) : n \in \mathbb{N}\},$$

is a $G$-directed subset of $X$ that contains strictly $L_0$ (and thus $J$ as well) and such that, for all $x \in \bar{L}$, $T(x) \in \bar{L}$ and $T(x) \in [x]_G$, which is in contradiction with maximality of $L_0$.

As $s \in L_0$ and $T(s) \in L_0$ too, we have $s \in [T(s)]_G$ and as $T(s) \in [s]_G$ we get then $T(T(s)) = T(s)$, that is, $T(s)$ is a fixed point of $T$.

Notice that if $z$ is a fixed point of $T$, $z \in L_0$, and thus $s \in [z]_G$ and $T(s) \in [T(z)]_G$ as $T$ is $G$-monotone.

**Example 2.** Let $X$ the unit disc in $C$, i.e., $X = \{z \in C, \ |z| = 1\}$, endowed with the graph $G = (V(G), E(G))$, where $V(G) = X$ and $(x, y) \in E(G)$ iff $|x| \leq |y|$.

Consider $T: X \rightarrow X$ defined $T(x) = |x|$ for all $x \in X$ with the usual topology.

It is easy then to check that $G$-intervals are compact, and that $T$ is $G$-regular monotone mapping and that 1 is a fixed point for $T$ (every real positive number in $X$ is indeed a fixed point for $T$).

In the same way, we obtain a common fixed point for commuting family of $G$-monotone mappings.

**Theorem 2.** Let $X$ be a topological space endowed with a reflexive digraph $G = (V(G), E(G))$ such that $G$-intervals are compact, and let $\{T_\lambda : X \rightarrow X, \lambda \in \Lambda\}$ be a family of commuting $G$-regular monotone mappings; if there exists $x_0 \in X$ such that $T_\lambda(x_0) \in [x_0]_G$ for all $\lambda \in \Lambda$ then the set of common fixed points of the family $(T_\lambda)_{\lambda \in \Lambda}$ is nonempty and has a $G$-maximal element.

**Proof.** Let $L = \{T_{\lambda_1} \cdot T_{\lambda_2} \cdot \ldots \cdot T_{\lambda_n}(x_0) : n \in \mathbb{N} \text{ and } \lambda_1 \ldots \lambda_n \in \Lambda\}$; then, $L$ is $G$-directed and we have $T_{\lambda}(x) \in L$ and $T_{\lambda}(x) \in [x]_G$ for all $x \in L$ and $\lambda \in \Lambda$. Let

$$\mathcal{F} = \{G - \text{directed } L \subset J, \text{ and } \forall x \in J, \forall \lambda \in \Lambda, T_\lambda(x) \in J \text{ and } T_\lambda(x) \in [x]_G\},$$

then $\mathcal{F}$ is a nonempty inductive set with respect to the inclusion order. Indeed, if $(I)_i \in \mathcal{F}$ is a chain in $\mathcal{F}$, then $\bigcup_{i \in I} I_i$ is an upper bound of $(I)_i \in \mathcal{F}$. By Zorn’s lemma, there exists a maximal $G$-directed set $L_0$ such that $L \subset L_0$ and for all $x \in L_0$ and $\lambda \in \Lambda$, we get $T_\lambda(x) \in L_0$ and $T_\lambda(x) \in [x]_G$.

As $G$ has the finite intersection property for $G$-intervals, $L_0$ has a $G$-supremum $s$.

For all $x \in L_0$, we have $s \in [x]_G$ and $T_\lambda(s) \in [T_{\lambda}(x)]_G$ and $T_\lambda(x) \in [x]_G$ for all $x \in L_0$ and for all $\lambda \in \Lambda$. Hence, $T_\lambda(s)$ is a $G$-upper bound of $L_0$, for any $\lambda \in \Lambda$. That is, $T_\lambda(s) \in [s]_G$. As $G$ has the finite intersection property for $G$-intervals, $L_0$ has a $G$-supremum $s$.

Now, as $s \in L_0$, then, for each $\lambda \in \Lambda$, $T_\lambda(s) \in L_0$ and $s \in [T_\lambda(s)]_G$; hence, $T_\mu(T_\lambda(s)) = T_\mu(s)$, $\forall \lambda, \mu \in \Lambda$. By commutativity, $T_\mu(s)$ is a common fixed point for the family of mappings $(T_\lambda)_{\lambda \in \Lambda}$.

Finally, if $z$ is a common fixed point of the family of mappings $(T_\lambda)_{\lambda \in \Lambda}$ then $z \in L$ thus $s \in [z]_G$ and $T_\mu(s) \in [T_\mu(z)]_G$, for all $\mu \in \Lambda$. Then, $T_\lambda(s)$ is $G$-maximal element of the set of common fixed points of $(T_\lambda)_{\lambda \in \Lambda}$.

### 3. Application to Modular Function Spaces

For the sake of completeness, we begin by recalling some definitions and properties of modular function spaces that we used later. For more details, see [12].

Let $\Omega$ be a nonempty set and $\mathcal{P}$ a nontrivial $\delta$-ring of subsets of $\Omega$, and let $\Sigma$ be the smallest $\sigma$-algebra of subsets of $\Omega$ such that $\Sigma$ contains $\mathcal{P}$ such that $E \cap A \in \mathcal{P}$ for every $E \in \mathcal{P}$ and $A \in \Sigma$; $K_\lambda(\Omega)$, where $K_\lambda \in \mathcal{P}$, for all $\lambda$. $\mathcal{E}$ is the linear space of $\mathcal{P}$-simple functions; $\mathcal{M}_\infty$ is the set of measurable functions. We denote by $\lambda_\lambda$ the characteristic function of $A$, where $A \in \Omega$.

**Definition 5.** An even convex function $\rho: \mathcal{M}_\infty \rightarrow [0, +\infty]$ is called regular convex function pseudomodular if

(i) $\rho(0) = 0$.

(ii) $\rho$ is monotone, i.e., if for $f, g \in \mathcal{M}_\infty$, $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$, then $\rho(f) \leq \rho(g)$. 
(iii) $\rho$ is orthogonally subadditive, i.e., $\rho(f,A,B) = \rho(f,A) + \rho(f,B)$, whenever $A, B \in \Sigma$ and $A \cap B = \emptyset$ and $f \in \mathcal{M}$.

(iv) $\rho$ has the Fatou property, i.e., if $\{(f_n(\omega))\}_{n \geq 1}$ $\rho(\omega)$, for all $\omega \in \Omega$, $f \in \mathcal{M}$, then $\rho(f_n \rho f)$.

(v) $\rho$ is order continuous in $\mathcal{M}$, i.e., $g_n \in \mathcal{M}$ and $|g_n| \to 0$ implies $\rho(g_n) \to 0$.

Let $\rho$ be a regular convex function pseudomodular; we then introduce these notions:

(i) A set $A \in \Sigma$ is said $\rho$-null, if $\rho(g, 1_A) = 0$, $\forall g \in \mathcal{E}$.

(ii) A property (P) is said to hold $\rho$ almost everywhere if the exceptional set is $\rho$-null.

(iii) We will identify pair of measurable sets whose symmetric difference is $\rho$-null, as well as pair of measurable function differing only on a $\rho$-null set.

(iv) $\mathcal{M}(\Omega, \Sigma, \rho, \rho)$ is the set of all nonzero regular convex function pseudomodulars on $\Omega$.

Definition 6. Let $\rho \in \mathcal{E}$:

(i) We say that $(f_n)_n \in L_\rho$ $\rho$-converges to $f$, and write $f_n \to f (\rho)$, if $\rho(f_n - f) \to 0$, and a sequence $(f_n)_n \in L_\rho$ is $\rho$-Cauchy if $\rho(f_n - f_m) \to 0$ as $(n, m) \to \infty$.

(ii) A set $B \subseteq L_\rho$ is called $\rho$-closed, if for any sequence $(f_n)_n \in B$, $f_n \to f (\rho)$ implies $f \in B$.

(iii) A set $B \subseteq L_\rho$ is called $\rho$-bounded, if its diameter $d_{\rho}(B) = \sup\{\rho(f - g)(\rho), g \in B\}$ is finite.

(iv) A set $B \subseteq L_\rho$ is called $\rho$-compact, if for any sequence $(f_n)_n \in L_\rho$ there exists a subsequence $(f_{k_n})_n$ and $f \in B$ such that $(f_{k_n})_n$ $\rho$-converges to $f$.

(v) A set $B \subseteq L_\rho$ is called $\rho$-a.e.-compact, if for any sequence $(f_n)_n \in L_\rho$ there exists a subsequence $(f_{k_n})_n$ and $f \in B$ such that $(f_{k_n})_n$ a.e.-converges to $f$.

(vi) A set $B \subseteq L_\rho$ is called $\rho$-a.e.-closed, if for any sequence $(f_n)_n \in B$, $f_n \to f$ a.e. implies $f \in B$.

Definition 7. Let $\rho \in \mathcal{E}$.

The modular function space is the vector space $L_\rho(\Omega, \Sigma)$ or briefly $L_\rho$, defined by

$L_\rho = \{f \in \mathcal{M}; \lim_{t \to 0} \rho(\lambda, f) = 0\}.$

The map $\|f\|_\rho: L_\rho \to [0, \infty)$ is defined by

$\|f\|_\rho = \inf\{\alpha > 0; \rho\left(\frac{f}{\alpha}\right) \leq 1\},$

which is called norm of Luxembourg on $L_\rho$.

The following properties play a prominent role in the study of modular function spaces.

Definition 8. Let $\rho \in \mathcal{E}$:

(i) We say that $\rho$ has the $\Delta_2$-property, if $\rho(2f_n) \to 0$ whenever $\rho(f_n) \to 0$, $(f_n)_n \in L_\rho$.

(ii) We say that $\rho$ has the $\Delta_2$-type condition, if there exists $k \in [0, +\infty)$ such that $\rho(2f) \leq k \rho(f)$, for any $f \in L_\rho$.

We need the following definition of the growth function.

Definition 9 (see [12]). Let $\rho$ be a function modular; the function $\omega_\rho: [0, +\infty) \to [0, +\infty)$ is defined by

$\omega_\rho(t) = \sup\{\rho(f) : f \in L_\rho \text{ and } 0 < \rho(f) < \infty\},$

which is called the growth of $\rho$.

The growth function has the following properties.

Proposition 1 (see [12]). Let $\rho \in \mathcal{E}$ that has the $\Delta_2$-type condition, and $\omega_\rho$ its growth function; then,

(i) $\omega_\rho(t) < \infty, \forall t \in [0, +\infty]$.

(ii) $\omega_\rho: [0, +\infty) \to [0, +\infty]$ is convex, and strictly increasing, if it is also continuous.

(iii) $\omega_\rho(\alpha, \beta) \leq \omega_\rho(\alpha \omega_\rho, \beta)$, for all $\alpha, \beta \in [0, +\infty)$.

(iv) $\omega_\rho^{-1}(\alpha \omega_\rho^{-1}(\beta) \leq \omega_\rho^{-1}(\alpha \beta)$, for all $\alpha, \beta \in [0, +\infty)$, where $\omega_\rho^{-1}$ is the inverse function of $\omega_\rho$.

(v) $f \omega_\rho \leq (1/\omega_\rho^{-1}(1/\rho(f)))$, for any $f \in L_\rho$.

Proofs of following theorems could be found in [12].

Theorem 3. Let $\rho \in \mathcal{E}$:

(i) $(L_\rho, \|\cdot\|_\rho)$ is a complete normed space, and $L_\rho$ is $\rho$-complete.

(ii) $f_n \rho |f| \to 0 \iff \rho(\alpha f_n) \to 0$ for every $\alpha > 0$.

(iii) If $\rho(f_n) \to 0$ there exists $(f_n)_k$ subsequence of $(f_n)_n$, such that $f_n_k \to f$, $\rho$-a.e.

(iv) If $f_n \to f$ a.e., then $\rho(f) \leq \liminf_{n \to \infty} \rho(f_n)$ (the Fatou property).

(v) If $\rho$ has the $\Delta_2$-property and $\rho(\alpha f_n) \to 0$ for $\alpha > 0$, then $\rho(f) \to 0$.

A modular $\rho$ is said $\sigma$-finite if there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that, for every $n \in \mathbb{N}$, $0 < \rho(K_n) < \infty$ and $\Omega = \cup \cup_{n=1}^\infty K_n$.

Let $d: L_\rho \times L_\rho \to [0, +\infty)$ defined by

$d(f, g) = \sum_{k=1}^\infty \frac{1}{2^k} \rho\left(\frac{|f - g|}{1 + |f - g|}ight)^k.$

Theorem 4. Let $\rho \in \mathcal{E}$ if $\rho$ is $\sigma$-finite and has the $\Delta_2$-type condition; then, for any $f, g \in L_\rho$,

(1) $d(f, g) = 0 \iff f = g$ $\rho$-a.e.

(2) $d(f, g) = d(g, f)$.
(3) \( d(f, g) \leq (\omega(2)/2) (d(f, h)) + d(h, g) \)

and if \((f_n)\) is a sequence in \(L_p\) that is \(\rho\)-a.e. convergent to \(f\), then \(\lim_{n \to \infty} d(f_n, f) = 0\).

Moreover, if \(\lim_{n \to \infty} d(f_n, f) = 0\), then there exists a subsequence \((f_{n_k})\) that converges \(\rho\)-a.e. to \(f\).

Here is the first application of our main result to modular function spaces.

**Theorem 5.** Let \(\rho \in \mathfrak{R}\), that has the \(\Delta_2\)-property and \(G\) a digraph on \(L_p\) such that \(G\)-intervals are \(\rho\)-compact. Let \(T_\lambda: L_p \to L_p, \lambda \in \Lambda\) be a family of commuting \(G\)-regular monotone mappings; if there exists \(f_0 \in L_p\) such that \(T_\lambda(f_0) \in [f_0]_{L_p}\), \(\forall \lambda \in \Lambda\), then the set of common fixed points of the family \(\{T_\lambda\}_{\lambda \in \Lambda}\) is not empty and has a \(G\)-maximal element.

**Proof.** As \(\rho\) has the \(\Delta_2\)-property, then the \(\rho\)-convergence is equivalent to convergence in the Banach space \((L_p, \|\cdot\|_p)\), which means that every \(\rho\)-compact subset of \(L_p\) is a compact in \((L_p, \|\cdot\|_p)\), we can then apply Theorem 2.

Requiring more conditions on the function modular \(\rho\) one can suppose that \(G\)-intervals are only \(\rho\)-a.e. compact.

**Theorem 6.** Let \(\rho \in \mathfrak{R}\), a \(\sigma\)-finite function modular, that has the \(\Delta_2\)-property and \(G\) a digraph on \(L_p\) such that \(G\)-intervals are \(\rho\)-a.e.-compact. Let \(T_\lambda: L_p \to L_p, \lambda \in \Lambda\) be a family of commuting \(G\)-regular monotone mappings; if there exists \(f_0 \in L_p\) such that \(T_\lambda(f_0) \in [f_0]_{L_p}\), \(\forall \lambda \in \Lambda\), then the set of common fixed points of the family \(\{T_\lambda\}_{\lambda \in \Lambda}\) is not empty and has a \(G\)-maximal element.

**Proof.** Indeed, Theorem 4 states that \((L_p, d)\) is a b-metric space, and then sequential compactness is equivalent to compactness (the usual argument, which proves that fact for metric spaces, still holds in b-metric spaces). Now, if a subset \(K\) of \(L_p\) is \(\rho\)-a.e.-compact, then from every sequence of elements of \(K\) one can extract a subsequence that converges \(\rho\)-a.e.; then, by Theorem 4, one can extract a subsequence that converges in the \(G\)-metric space \((L_p, d)\). That is, \(K\) is sequentially compact in \((L_p, d)\) and thus compact, which implies that \(G\)-intervals are compact for the topology of the b-metric space \((L_p, d)\), then using Theorem 2 we get the result.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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