Deriving Sorting Algorithms

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Abstract

This paper shows how 3 well-known sorting algorithms can be derived by similar sequences of transformation steps from a common specification. Each derivation uses an auxiliary algorithm based on insertion into an intermediate structure. The proofs given involve both inductive and coinductive reasoning, which are here expressed in the same program calculation framework, based on unicity properties.
Abstract. This paper proposes new derivations of three well-known sorting algorithms, in their functional formulation. The approach we use is based on three main ingredients: first, the algorithms are derived from a simpler algorithm, i.e. the specification is already a solution to the problem (in this sense our derivations are program transformations). Secondly, a mixture of inductive and coinductive arguments are used in a uniform, algebraic style in our reasoning. Finally, the approach uses structural invariants so as to strengthen the equational reasoning with logical arguments that cannot be captured in the algebraic framework.

1 Introduction

This paper presents new derivations of three well-known sorting algorithms, in the functional setting. Our approach can be summarized as follows:

1. It is based on program transformation in the sense that we depart from a specification that is already a (not very efficient) algorithm for solving the problem. Traditional derivations of sorting algorithms (building on the work of Burstall and Darlington) formalize the “is sorted” property on lists. Instead, we take the insertion sort algorithm to be a specification of sorting, and derive, by sequences of correct steps, more efficient algorithms from it.

2. The algorithms that we derive follow the derive and conquer strategy and as such are not structurally recursive on their arguments. For this reason a combination of inductive and coinductive reasoning must be used. We adhere here to the equational style of reasoning usually known to functional programmers as program calculation, which relies on uniqueness properties of certain recursion patterns. Although the proofs are independent of this choice, we find that this allows for greater uniformity between the inductive and coinductive arguments.

3. In two of our three derivations, the equational reasoning must be strengthened by using invariants on certain intermediate data-structures, since some of the equalities one needs to prove are not universal for a given data-type. For instance, it is not true that the in-order traversal of any binary tree produces a sorted list. This is however true for trees produced in a certain way. As far as we know there is very little work on program calculation strengthened with invariants.

4. The algorithms are derived as hylomorphisms, i.e. as explicit compositions of a recursive function with a co-recursive one, with an intermediate data-structure of a tree type, which can be deforested to produce the standard formulation of the algorithms. Sorting algorithms have been defined as hylomorphims elsewhere [1].

The paper is structured as follows: Section 2 reviews standard material on sorting in the functional setting, including the algorithms that will be considered in the main sections of
the paper. Section 3 contains background material on program calculation, based on unicity (or universal) properties of recursion pattern operators. Section 4 then introduces two generic algorithms for sorting, based on insertion into an intermediate structure of a container type (in a leftwards and rightwards fashion respectively). Sections 5, 6, and 7 present the derivations of merge sort, quicksort, and heapsort, which are based on instantiations of the generic algorithms. Finally we conclude the paper in Section 8.

2 Sorting Homomorphisms and Divide-and-conquer Algorithms

Consider a very simple algorithm for sorting a list, usually known under the name of insertion sort. We give it here written in Haskell.

\[
\text{isort} \ [
\] = 
\text{isort} \ (x:xs) = \text{insert} \ x \ (\text{isort} \ xs)
\]

where insert inserts an element in a sorted list. This is certainly a natural way of sorting a list in a traditional functional language: since the list structurally consists of a head element \(x\) and a tail sublist \(xs\), it is natural to recursively sort \(xs\) and then combine this sorted list with \(x\). This pattern of recursion can be captured by the \text{foldr} operator, resulting in the following definition where explicit recursion has been removed.

\[
\text{isort} = \text{foldr} \ \text{insert} \ [\]
\]

Actually, any sorting function is a list homomorphism \[3,6\], which means that if the initial unsorted list is split at any point and the two resulting sublists are recursively sorted, there exists a binary operator \(\odot\) that can combine the two results to give the final sorted list.

\[
\text{isort} \ (l_1 \cup l_2) = (\text{isort} \ l_1) \odot (\text{isort} \ l_2)
\]

This \(\odot\) operator is of course the (linear time) function of type \([a] \rightarrow [a] \rightarrow [a]\) that merges two sorted lists:

\[
\begin{align*}
\text{merge} \ [\] \ l &= l \\
\text{merge} \ l \ [\] &= l \\
\text{merge} \ (h1:t1) \ (h2:t2) &= \\
| \ (h1 \leq h2) &= h1:(\text{merge} \ t1 \ (h2:t2)) \\
| \ \text{otherwise} &= h2:(\text{merge} \ (h1:t1) \ t2)
\end{align*}
\]

The operator \(\odot\) is associative with the empty list as unit, forming a monoid over lists. It is also commutative. insert can be defined in terms of \(\odot\) as follows

\[
\text{insert} \ x \ l = [x] \odot l \quad (1)
\]

Insertion sort runs in quadratic time. Most well-known efficient sorting algorithms perform recursion twice, on subsequences obtained from the input sequence, and then combine the results (for this reason they are called divide-and-conquer algorithms). As such, they do not fit the simple iteration pattern captured by \text{foldr}. In the following we describe three different divide-and-conquer algorithms.
Heapsort. The principle behind heapsort is to traverse the list to obtain, in linear time, its minimum element \( y \) and a pair of lists of approximately equal size, containing the remaining elements (function \( \text{haux} \)). The lists are then recursively sorted and merged together, and \( y \) pasted at the head of the resulting list.

\[
\text{haux} \; x \; [] = (x, [], []) \\
\text{haux} \; x \; (y:ys) = \text{let} \; (z, l, r) = \text{haux} \; y \; ys \\
\hspace{1cm} \text{in} \; \text{if} \; x < z \; \text{then} \; (x, z:r, l) \; \text{else} \; (z, x:r, l)
\]

\[
\text{hsort} \; [] = [] \\
\text{hsort} \; (s:xs) = \text{let} \; (y, l, r) = \text{haux} \; x \; xs \\
\hspace{1cm} \text{in} \; y:((\text{merge} \; (\text{hsort} \; l) \; (\text{hsort} \; r))
\]

Quicksort. The criterion for obtaining the two sublists is here to use the head of the list (the only element accessible in constant time) as a pivot used to separate the remaining elements. The two sorted results need only be concatenated (with the pivot in the middle) to give the final result.

\[
\text{qaux} \; _{} \; [] = ([], []) \\
\text{qaux} \; x \; (h:t) = \text{let} \; (l, r) = \text{qaux} \; x \; t \\
\hspace{1cm} \text{in} \; \text{if} \; h <= x \; \text{then} \; (h:l, r) \; \text{else} \; (l, h:r)
\]

\[
\text{qsort} \; [] = [] \\
\text{qsort} \; (x:xs) = \text{let} \; (l, r) = \text{qaux} \; x \; xs \\
\hspace{1cm} \text{in} \; ((\text{qsort} \; l)) \; ++ \; x:(\text{qsort} \; r)
\]

Merge Sort. This is similar to heapsort except that the minimum element is not extracted when the list is traversed. For this reason an extra base case is used.

\[
\text{maux} \; [] = ([], []) \\
\text{maux} \; (x:xs) = (x:b, a) \; \text{where} \; (a, b) = \text{maux} \; xs
\]

\[
\text{msort} \; [] = [] \\
\text{msort} \; [x] = [x] \\
\text{msort} \; xs = \text{let} \; (l, r) = \text{maux} \; xs \\
\hspace{1cm} \text{in} \; \text{merge} \; (\text{msort} \; l) \; (\text{msort} \; r)
\]

These functional versions of the algorithms may be difficult to recognize for a reader used to the imperative formulations, where the sorting is usually done in place, on indexed arrays. All three are however widely known in the functional programming community formulated as above.

3 Recursion Patterns, Unicity, and Hylomorphisms

We direct the reader to [7] for an extensive introduction to the field of program calculation, and include here only the basic notions needed for expressing the proofs included in the paper.
data BTree a = Empty | Node a (BTree a) (BTree a)
type Heap = BTree
data LTree a = Leaf (Maybe a) | Branch (LTree a) (LTree a)

unfoldBTree :: (b -> (Either (a,b,b) b)) -> b -> BTree a
unfoldBTree g x = case (g x) of
  Right () -> Empty
  Left (y,l,r) -> Node y (unfoldBTree g l) (unfoldBTree g r)

foldBTree :: (a -> b -> b -> b) -> b -> BTree a -> b
foldBTree f e Empty = e
foldBTree f e (Node x l r) = f x (foldBTree f e l) (foldBTree f e r)

unfoldLTree :: (b -> (Either (b,b) Maybe a)) -> b -> LTree a
unfoldLTree g x = case (g x) of
  Right y -> Leaf y
  Left (l,r) -> Branch (unfoldLTree g l) (unfoldLTree g r)

foldLTree :: (b->b->b) -> ((Maybe a)->b) -> LTree a -> b
foldLTree f e (Leaf x) = e x
foldLTree f e (Branch l r) = f (foldLTree f e l) (foldLTree f e r)

### Table 1. Types and recursion patterns for binary trees

The fold recursion pattern can be generalized for any regular type; in the context of the algebraic theory of data-types folds are *datatype-generic* (in the sense that they are parameterized by the base functor of the type), and usually called *catamorphisms*. The result of a fold on a node of some tree data-type is a combination of the results of recursively processing each subtree (and the contents of the node, if not empty).

The dual notion is the *unfold* (also called *anamorphism*): a function that constructs (possibly infinite) trees in the most natural way, in the sense that the subtrees of a node are recursively constructed by unfolding.

In the present paper we will need to work with two flavours of binary trees: leaf-labelled (for merge sort) and node-labelled trees (for the remaining algorithms). These types, and the corresponding recursion patterns, are defined in Table 1.

In principle, a fold is a recursive function whose domain is a type defined as a least fixpoint (an initial algebra), and an unfold is a recursive function whose codomain is defined as a greatest fixpoint (a final coalgebra). However, in lazy languages such as Haskell, least and greatest fixpoints coincide, and are simply called recursive types.

At an abstract level, folds (as well as other structured forms of recursion, such as primitive recursion) enjoy an initiality property among the algebras of the base functor of the domain type. In concrete terms, this makes possible the use of *induction* as a proof technique. Dually,
unfolds are final coalgebras; techniques for reasoning about unfolds include fixpoint induction and coinduction [5].

Unicity. The program calculation approach is based on the use of initiality and finality directly as an equational proof principle. Both properties can be formulated in the same framework, as universal or unicity properties. In this paper we generally adhere to the equational style for proofs, but often resort to induction for the sake of simplicity (in particular when none of the sides of the equality one wants to prove is directly expressed using a recursion pattern, applying a unicity property may require substantial manipulation of the expressions). See [4] for a study of program calculation carried out purely by using fusion, including an adequate treatment of strictness conditions.

We give below the unicity properties that we shall require in the rest of the paper, for the foldr, unfoldLTree, and unfoldBTree operators. A weaker fusion law for foldr is also shown, which is easily derived from unicity.

\[ f = \text{foldr } g e \quad \iff \quad \{ \text{unicity-foldr} \} \]
\[
\begin{align*}
& f \; [] = e \\
& \text{for all } x, xs, \\
& \quad f \; (x : xs) = g \; x \; (f \; xs)
\end{align*}
\]

\[ h \circ \text{foldr } g e = \text{foldr } g' e' \quad \iff \quad \{ \text{foldr-fusion} \} \]
\[
\begin{align*}
& h \; e = e' \\
& h \; (g \; x) = (g' \; x) \circ h
\end{align*}
\]

\[ f = \text{unfoldLTree } g \quad \iff \quad \{ \text{unicity-unfoldLTree} \} \]
\[
\begin{align*}
\text{for all } x, \\
& f \; x = \text{case } (g \; x) \text{ of} \\
& \quad \text{Right } y \rightarrow \text{Leaf } y \\
& \quad \text{Left } (l, r) \rightarrow \text{Branch } (f \; l) \; (f \; r)
\end{align*}
\]

\[ f = \text{unfoldBTree } g \quad \iff \quad \{ \text{unicity-unfoldBTree} \} \]
\[
\begin{align*}
\text{for all } x, \\
& f \; x = \text{case } (g \; x) \text{ of} \\
& \quad \text{Right } () \rightarrow \text{Empty} \\
& \quad \text{Left } (y, l, r) \rightarrow \text{Node } y \; (f \; l) \; (f \; r)
\end{align*}
\]

Hylomorphisms. The composition of a fold over a regular type \( T \) with an unfold of that type is a recursive function whose recursion tree is shaped in the same way as \( T \). Such a definition can be deforested [10], i.e. the construction of the intermediate data-structures can be eliminated, yielding a direct recursive definition. As an example, the definition \( h = (\text{foldLTree } f \; e) \circ (\text{unfoldLTree } g) \) can be deforested to give:

\[ h \; x = \text{case } (g \; x) \text{ of} \\
\quad \text{Right } y \rightarrow e \; y \\
\quad \text{Left } (l, r) \rightarrow f \; (h \; l) \; (h \; r) \]

This corresponds to a new generic recursion pattern, called a hylomorphism. Hylomorphisms do not possess a unicity property, but they are still useful for reasoning about programs, using the properties of their fold and unfold components. In particular, hylomorphisms are useful for capturing the structure of functions that are not directly defined by structured recursion or co-recursion, as is the case of the divide-and-conquer sorting algorithms: the unfold component takes the unsorted list and constructs a tree; the fold iterates over this structure to produce the sorted list. The sorting algorithms introduced in the previous section were studied as hylomorphisms in [1]. In the present paper we use this hylomorphic structure to calculate these algorithms from a common specification.
4 Sorting by Insertion

In the rest of the paper we will repeatedly apply the following principles. Consider a type constructor $C$ and the following functions:

$$
\begin{align*}
\text{istC} : a \to C \ a \to C \\
\text{C2list} : C \ a \to [a]
\end{align*}
$$

The idea is that $C \ a$ is a container type for elements of type $a$ (typically a tree-shaped type); $\text{istC}$ inserts an element in a container to give a new container; and $\text{C2list}$ converts a container into a sorted list of type $a$.

A generic sorting algorithm can then be defined, with a container acting as intermediate data-structure. The idea is that elements are inserted one by one by folding over the list; a sorted list is then obtained using $\text{C2list}$. $\varepsilon :: C \ a$ is an appropriate “empty value”.

$$
isortC = \text{C2list} \circ (\text{foldr istC} \ \varepsilon)
$$

It is easy to see that the algorithm is correct if the intermediate data-structure contains exactly the same elements as the initial list, and $\text{C2list}$ somehow produces a sorted list from the elements in the intermediate structure. This can be formalized by constructing a proof of equivalence to insertion sort, which gives necessary conditions for the algorithm to be correct.

$$
\begin{align*}
\text{C2list} \circ (\text{foldr istC} \ \varepsilon) &= \text{foldr insert} \ [ ] \\
\Leftrightarrow & \{ \text{unicity-foldr} \} \\
& \{ \text{C2list} (\text{foldr istC} \ \varepsilon \ [ ]) = [ ] \} \\
& \{ \text{C2list} (\text{foldr istC} \ \varepsilon \ (x : xs)) = \text{insert} \ x \ (\text{C2list} (\text{foldr istC} \ \varepsilon \ xs)) \} \\
\Leftrightarrow & \{ \text{def. foldr} \} \\
& \{ \text{C2list} \ \varepsilon = [ ] \} \\
& \{ \text{C2list} (\text{istC} \ x \ (\text{foldr istC} \ \varepsilon \ xs)) = \text{insert} \ x \ (\text{C2list} (\text{foldr istC} \ \varepsilon \ xs)) \}
\end{align*}
$$

Alternatively one can use fusion, which leads to stronger conditions:

$$
\begin{align*}
\text{C2list} \circ (\text{foldr istC} \ \varepsilon) &= \text{foldr insert} \ [ ] \\
\Leftarrow & \{ \text{foldr fusion} \} \\
& \{ \text{C2list} \ \varepsilon = [ ] \} \\
& \{ \text{C2list} \circ (\text{istC} \ x) = (\text{insert} \ x) \circ \text{C2list} \}
\end{align*}
$$

Thus for each concrete container type it is sufficient to prove equation $\Box$ and one of $\Box$ or $\Box$ to establish that the corresponding function $\text{sortC}$ is indeed a sorting algorithm:

$$
\begin{align*}
\text{C2list} \ \varepsilon &= [ ] \\
\text{C2list} \circ (\text{istC} \ x) &= (\text{insert} \ x) \circ \text{C2list} \\
\text{C2list} (\text{istC} \ x \ (\text{foldr istC} \ \varepsilon \ xs)) &= \text{insert} \ x \ (\text{C2list} (\text{foldr istC} \ \varepsilon \ xs))
\end{align*}
$$

Note that together, equations $\Box$ and $\Box$ mean that $\text{C2list}$ is a homomorphism between the structures $(C \ a, \text{istC}, \varepsilon)$ and $([a], \text{insert}, [ ])$. Observe that the above algorithm constructs the intermediate structure by inserting the elements from right to left. A tail-recursive version of $\text{sortC}$ can be derived by a standard transformation based on fusion $[2]$. This will construct the intermediate structure in a rightwards fashion. We start by writing a specification for this function $\text{sortC}_t$. 
\[\text{isortLT} = \text{LT2list} \circ \text{buildLT} \]
\[\text{buildLT} = \text{foldr isLT} (\text{Leaf Nothing})\]
\[\text{LT2list} = \text{foldLT} (\circ) t\]
\[\text{where } t \text{ Nothing} = [\]
\[t \text{ (Just } x\text{)} = [x]\]
\[\text{isLT} x \text{ (Leaf Nothing)} = \text{Leaf} \text{ (Just } x\text{)}\]
\[\text{isLT} x \text{ (Leaf(Just } y\text{))} = \text{Branch} \text{ (Leaf (Just } x\text{)) (Leaf(Just } y\text{))}\]
\[\text{isLT} x \text{ (Branch } l r\text{)} = \text{Branch} \text{ (isLT } x r\text{) } l\]

| Table 2. Sorting by insertion in a leaf tree |
|----|----|
| \text{isortC}_t : [a] \to C \to a \to [a] | \text{isortC}_t l y = (\text{isort } l) \circ (\text{C2list } y) |

The tail-recursive function uses an extra accumulator argument of the chosen container type. In the call \text{isortC}_t l y, l is the list that remains to be sorted, and the accumulator y contains elements already inserted in the container. The right-hand side of the equality states how the final result can be obtained using insertion sort and the conversion of y to a list.

The following definition satisfies the specification (proof is given in Appendix A.1).

\[\text{isortC}_t = \text{foldr istC'} \circ \text{C2list}\]
\[\text{where } \text{istC'} x f y = f \text{ (istC } x y\text{)}\]

Then \text{isortC}_t l \varepsilon = \text{iSort } l holds as an immediate consequence of the specification and eq. (3) above. An alternative version of this can be defined, which separates the tail-recursive construction of the intermediate structure from its conversion to a sorted list (note \text{ap} f = f \varepsilon):

\[\text{isortC'} = \text{C2list} \circ \text{ap} \circ (\text{foldr} \text{ istC'} \circ \text{id})\]  \hfill (6)

It is straightforward to establish that \text{isortC'} l \varepsilon = \text{iSortC}_t l \varepsilon, thus \text{isortC'} = \text{iSort}.

In the next sections, the container type and its empty value, together with the functions \text{istC} and \text{C2list}, will be instantiated to produce three different insertion-based algorithms, using schemes 2 and 6.

Each algorithm will be proved correct by calculating eqs. 3 and 4 or 5 above. The next step will be to transform each algorithm into a hylomorphism that can then be deforested, resulting in a well-known sorting algorithm. For this, it will suffice to transform the function that constructs the intermediate tree into co-recursive form.

5 A Derivation of Merge Sort

Our first concrete sorting algorithm based on insertion into an intermediate structure uses leaf-labelled binary trees. This is given in Table 2. We remark that to cover the case of the empty list, a Maybe type is used in the leaves of the trees.

\[\text{Proposition 1. } \text{isortLT is a sorting algorithm.}\]
Proof. We instantiate eqs. (3) and (4). Note that the empty value here is \( \varepsilon = \text{Leaf Nothing} \).

\[
\begin{align*}
\text{LT2list (Leaf Nothing)} & = [] \\
\text{LT2list } \circ (\text{istLT } x) & = (\text{insert } x) \circ \text{LT2list}
\end{align*}
\]

The first equality is true by definition; the second can be proved by induction, or alternatively using fusion. The latter proof is given in Appendix A.2. Together these equations establish that \( \text{LT2list} \) is a homomorphism between the structures \((\text{LTree } a, \text{istLT}, \text{Leaf Nothing})\) and \(([a], \text{insert}, [])\).

It is also easy to see that the intermediate tree is balanced: the difference between the heights of the subtrees of a node is never greater than one, since subtrees are swapped at each insertion step. Note that the insertion function \( \text{istLT} \) was carefully designed with efficiency in mind, which grants execution in time \( O(N \log N) \); other solutions would still lead to sorting algorithms, albeit less efficient.

**Proposition 2.** *The trees constructed by buildLT are balanced.*

**Proof.** It can be proved by induction on the structure of the argument list that either the subtrees of the constructed tree have the same height, or the height of the left subtree is greater than the height of the right subtree by one unit. The function \( \text{istLT} \) preserves this invariant.

The next transformation step applies to the function that constructs the intermediate tree. An alternative way of constructing a balanced tree is by unfolding: the initial list is traversed and its elements placed alternately in two subsequences, which are then used as arguments to recursively construct the subtrees. Note that the sequences will have approximately the same length. For singular and empty lists, leaves are returned.

\[
\text{unfoldmsort} = \text{unfoldLTree } g \\
\text{where } g [ ] = \text{Right Nothing} \\
g \ [x] = \text{Right } (\text{Just } x) \\
g \ [x:s] = \text{Left } (\text{maux } x:s) \\
\text{maux } [ ] = ([], []) \\
\text{maux } (h : t) = (h : b, a) \text{ where } (a, b) = \text{maux } t
\]

**Proposition 3.** *The above function constructs the same intermediate trees as those obtained by folding over the argument list:*

\[
\text{foldr istLT (Leaf Nothing)} = \text{unfoldmsort}
\]

**Proof.** We use the unicity property of leaf-tree unfolds:
\[
\text{isortH} = \text{H2list} \circ \text{buildH} \\
\text{buildH} = \text{foldr altH Empty} \\
\text{H2list} = \text{foldr aux []} \\
\text{where aux x l r} = x : (l \odot r) \\
\text{altH x Empty} = \text{Node x Empty Empty} \\
\text{altH x (Node y l r)} \mid x < y = \text{Node x (altH y r) l} \\
\text{otherwise} = \text{Node y (altBST x r) l}
\]

Table 3. Sorting by insertion in a heap

\[
\text{foldr altLT (Leaf Nothing)} = \text{unfoldLTTree g} \\
\equiv \{ \text{unicity-unfoldLTTree} \} \\
\text{for all x,} \\
(\text{foldr altLT (Leaf Nothing)) x} = \text{case (g x) of} \\
\text{Right y} \rightarrow \text{Leaf y} \\
\text{Left (l, r)} \rightarrow \text{Branch (f l) (f r)} \\
\equiv \{ \text{by cases} \} \\
\begin{cases} 
\text{Leaf Nothing} = \text{Leaf Nothing} & \text{if x = [ ]} \\
\text{Leaf(Just h)} = \text{Leaf(Just h)} & \text{if x = [h]} \\
\text{altLT h_1 (foldr altLT (Leaf Nothing)) (h_2 : t))} & \text{if x = h_1 : h_2 : t} \\
= \text{Branch (foldr altLT (Leaf Nothing) l) (foldr altLT (Leaf Nothing) r)} & \text{where (l, r) = maux (h_1 : h_2 : t)} \\
\end{cases}
\]

And the last equality can be easily proved by induction on the structure of \( t \).

Substituting this in the definition of \( \text{isortLT} \) yields a hylomorphism that is of course still equivalent to insertion sort. It is immediate to see that this can be deforested, and the result is merge sort:

\[
\text{LT2list} \circ \text{unfoldmsort} = \text{msort}
\]

6 A Derivation of Heapsort

In the \textit{heapsort} algorithm, one computes the minimum of the list prior to the recursive calls. This will determine that each node of the intermediate structure (the recursion tree) this minimum for some tree; it is thus a binary \textit{node-labelled} tree.

We repeat the program taken for the derivation of the merge sort: we design a function that inserts a single element in the intermediate tree (\text{altH}), iterate this function over the initial list (\text{buildH}) and then provide a function that recovers the ordered list from the tree (\text{H2list}). These functions are shown in Table 3.

\textbf{Proposition 4.} \textit{isortH} is a sorting algorithm.

\textit{Proof.} We instantiate eqs. \text{(3)} and \text{(5)}. We set \( \epsilon = \text{Empty} \), and thus eq. \text{(3)} results directly from the definition. For eq. \text{(5)}, we need to prove that for every list \( l \),

\[
\text{insert x (H2list (buildH l))} = \text{H2list (altH x (buildH l))}
\]
In order to prove this, we rely on the fact that trees generated by buildH are always heaps, i.e. the root element is the least of the tree. The complete derivation is presented in appendix B (Propositions 8 and 1).

Note that in order to prove the correctness of this algorithm, we cannot rely on the strongest hypothesis given by eq. 4 (obtained from the use of the fusion law) as we have done for merge sort. The reason for this is that, for an arbitrary tree \( t \),

\[
\text{insert } x \ (\text{H2list } t) \neq \text{H2list } (\text{istH } x \ t).
\]

On the other hand, the weaker requisite given by eq. 5 (obtained by the use of unicity or induction) retains the information that we restrict our attention to trees constructed by buildBST, and these will satisfy the required equality.

We also note that the intermediate tree is again balanced (essentially by the same argument used for merge sort). This means that this sorting algorithm also executes in time \( O(N \lg N) \).

It remains to show that the intermediate tree can be constructed coinductively. For that, consider the following function:

\[
\begin{align*}
\text{unfoldhsort} &= \text{unfoldBT} \ 
\text{ree } g \\
\text{where } &g \ [ ] = \text{Right } () \\
&g \ (x : xs) = \text{Left } (\text{haux } x \ xs) \\
&\text{haux } x \ [ ] = (x, [], []) \\
&\text{haux } x \ (y : ys) \ |
\begin{cases}
  x < m = (x, m : b, a) \\
  \text{otherwise} = (m, x : b, a)
\end{cases}
\text{ where } (m, a, b) = \text{haux } y \ ys
\end{align*}
\]

**Proposition 5.** The above function constructs the same intermediate trees as those obtained by folding over the argument list:

\[
\text{buildH} = \text{unfoldhsort}
\]

**Proof.**

\[
\begin{align*}
\text{unfoldhsort} &= \text{buildH} \\
\Leftrightarrow \{ \text{by unicity } - \text{unfoldBT} \ 
\text{ree } g \} \\
&\{ \text{definitions} \} \\
&\text{buildH } [ ] = \text{Empty} \\
&\text{buildH } (x : xs) = \text{Node } z \ (\text{buildH } a) \ (\text{buildH } b) \\
&\quad \text{ where } (z, a, b) = \text{haux } x \ xs \\
\Leftrightarrow \{ \text{definitions} \} \\
&\text{Empty } = \text{Empty} \\
&\text{istH } x \ (\text{buildH } xs) = \text{Node } z \ (\text{buildH } a) \ (\text{buildH } b) \\
&\quad \text{ where } (z, a, b) = \text{haux } x \ xs
\end{align*}
\]

The second equality is proved by structural induction on \( xs \). For the base case \( (xs = []) \), it follows directly from evaluating the definitions. For the inductive step \( (xs = y : ys) \), let us assume that \( (z, a, b) = \text{haux } y \ ys \). The definition of haux tell us that

\[
(z', a', b') = \text{haux } x \ (y : ys) = \begin{cases}
  (x, z : b, a) & \text{if } x < z, \\
  (z, x : b, a) & \text{if } x \geq z.
\end{cases}
\]
isortBST = BST2list ∘ buildBST
buildBST = (ap Empty) ∘ bAcc
   where ap x f = f x
         bAcc = foldr aux id
               aux x f a = f (istBST x a)

BST2list = foldBTree aux []
   where aux x l r = l ++ (x : r)

istBST x (Empty)   = Node x Empty Empty
istBST x (Node y l r) | x < y = Node y (istBST x l) r
                        | otherwise = Node y l (istBST x r)

Table 4. Sorting by insertion in a binary search tree

Thus,

\[
\begin{align*}
\text{istH} & \ x \ (\text{buildH} \ (y : ys)) \\
& = \ \{\text{def. buildH}\} \\
& \text{istH} \ x \ (\text{istH} \ y \ (\text{buildH} \ ys)) \\
& = \ \{\text{induction hypotheses}\} \\
& \text{istH} \ x \ (\text{Node} \ z \ (\text{buildH} \ a) \ (\text{buildH} \ b)) \\
& = \ \{\text{def. istH}\} \\
& \begin{cases} \\
& \text{Node} \ x \ (\text{istH} \ z \ (\text{buildH} \ b)) \ (\text{buildH} \ a) \ \text{if} \ x < z, \\
& \text{Node} \ z \ (\text{istH} \ x \ (\text{buildH} \ b)) \ (\text{buildH} \ a) \ \text{if} \ x \geq z.
\end{cases} \\
& = \ \{\text{def. of } z', a', b'\} \\
& \text{Node} \ z' \ (\text{buildH} \ a') \ (\text{buildH} \ b')
\end{align*}
\]

As would be expected, the hylomorphism obtained replacing \text{buildH} by \text{unfoldhsort} can be deforested, and the result is the original \text{hsort}.

\[
\text{H2list} \circ \text{unfoldhsort} = \text{hsort}
\]

7 A Derivation of Quicksort

In the quicksort algorithm, the activity performed prior to the recursive calls is different from that in heapsort: instead of finding the minimum of the list, the head of the list is used as a pivot for splitting the tail. Again, the intermediate structure is a node-labelled binary tree. But now, its ordering properties are different — the constructed trees will be binary search trees, and it suffices to traverse these trees in-order to produce the desired sorted list.

Following the same line as in the derivation of the previous algorithms, we define an algorithm that iteratively inserts elements from a list into a binary tree and then reconstructs the list by the in-order traversal. This algorithm is given in Table 4.

Observe that this algorithm iterates on the initial list from left to right (we may think of it as using the Haskell \text{foldl} operator, but we write it as a higher-order function using \text{foldr}, to exploit the application of the rules presented earlier). This will become evident below when we replace this function by one that constructs the intermediate tree corecursively. For the correctness argument, we know that the order of traversal for the initial list is irrelevant (as shown in Section 4).
Proposition 6. isortBST is a sorting algorithm.

Proof. We instantiate eqs. 3 and 5. We set $\epsilon = \text{Empty}$, and thus Equation 3 results directly from the definition. For Equation 5, we need to prove that for every list,

\[
\text{insert } x (\text{BST2list } (\text{buildBST } l)) = \text{BST2list } (\text{istBST } x (\text{buildBST } l)).
\]

In order to prove this, we rely on the fact that trees generated by buildBST are always binary search trees. The complete derivation is presented in appendix B (Propositions 10 and 2).

To obtain the well-known quicksort algorithm, we need to replace the iterated insertion function buildBST by an unfold.

\[
\text{unfoldqsort} = \text{unfoldBTree } g
\]

where \( g \)\[\] = Right ()

\[
g (x : xs) = \text{Left } (\text{qaux } x xs)
\]

\[
\text{qaux } x \]\[\] = (x, [], [])

\[
\text{qaux } x (y : ys) | y < x = (x, y : b, a)
\]

| otherwise = (m, a, y : b)

where \((a, b) = \text{qaux } x ys\)

Proposition 7. The above function constructs the same intermediate trees as those obtained by folding over the argument list:

\[
\text{buildBST} = \text{unfoldqsort}
\]

Proof.

\[
\text{unfoldqsort} = \text{buildBST}
\]

\[
\Leftrightarrow \{\text{unicity-unfolBTree}\}
\]

\[
\text{buildBST } [] = \text{Empty}
\]

\[
\text{buildBST } (x : xs) = \text{Node } x (\text{bAcc } a \text{ Empty} ) (\text{bAcc } b \text{ Empty})
\]

\[
\text{where } (a, b) = \text{qaux } x xs
\]

\[
\Leftrightarrow \{\text{definitions}\}
\]

\[
\text{bAcc } [] \text{ Empty} = \text{Empty}
\]

\[
\text{bAcc } (x : xs) \text{ Empty} = \text{Node } x (\text{bAcc } a \text{ Empty} ) (\text{bAcc } b \text{ Empty})
\]

\[
\text{where } (a, b) = \text{qaux } x xs
\]

\[
\Leftrightarrow \{\text{simplification}\}
\]

\[
\text{Empty} = \text{Empty}
\]

\[
\text{bAcc } xs (\text{Node } x \text{ Empty} \text{ Empty} ) = \text{Node } x (\text{bAcc } a \text{ Empty} ) (\text{bAcc } b \text{ Empty})
\]

\[
\text{where } (a, b) = \text{qaux } x xs
\]

We prove the second equality in a slightly strengthened formulation. For every tree \((\text{Node } x l r)\) and list \(xs\),

\[
\text{bAcc } xs (\text{Node } x l r) = \text{Node } x (\text{bAcc } a l ) (\text{bAcc } b r)
\]

\[
\text{where } (a, b) = \text{qaux } x xs
\]
By induction on the structure of $xs$. For the base case ($xs = []$), it follows directly from evaluating the definitions. For the inductive step ($xs = y : ys$), we reason by cases. If $x < y$, then

\[
\text{bAcc} (y : ys) (\text{Node} x l' r') = \text{Node} x (\text{bAcc} a' l') (\text{bAcc} b' r')
\]

where $(a', b') = \text{qaux} x (y : ys)
\]

$\Leftrightarrow \{ \text{definition, } x < y \}$

\[
\text{bAcc} ys (\text{istBST} y (\text{Node} x l' r')) = \text{Node} x (\text{bAcc} a l') (\text{bAcc} b (y : b) r')
\]

where $(a, b) = \text{qaux} x ys
\]

$\Leftrightarrow \{ \text{definition, } x < y \}$

\[
\text{bAcc} ys (\text{Node} x l' (\text{istBST} y r')) = \text{Node} x (\text{bAcc} a l') (\text{bAcc} b (\text{istBST} y r'))
\]

where $(a, b) = \text{qaux} x ys
\]

$\Leftrightarrow \{ \text{induction hypotheses} \}$

\[
\text{bAcc} ys (\text{Node} x l' (\text{istBST} y r')) = \text{bAcc} ys (\text{Node} x l' (\text{istBST} y r'))
\]

where $(a, b) = \text{qaux} x ys
\]

Similarly for the case ($x \geq y$). This concludes the proof.

We conclude with the statement that the hylomorphism obtained is, as intended, the forested version of the original quicksort algorithm.

\[
\text{BST2list} \circ \text{unfoldqsort} = \text{qsort}
\]

8 Conclusion

This paper illustrates the strengths of the “program calculation” style of reasoning, in particular the simplicity of using the unicity property of unfolds as an alternative to using coinductive principles based on bisimulations, and more generally the structural aspects of proofs. Inductive proofs are however often much simpler to carry out than using the equational style, so we are not dogmatic about the style in which proofs are presented.

Apart from the proofs of correctness which as far as we know are new, the contributions of this paper include (two versions of) a generic sorting algorithm, of which 3 concretizations are used. The role played by structural invariants in this study should also be emphasized.

Even when they are not crucial to the calculations, invariants provide a much more natural setting for conducting them. Moreover, efficiency properties of the algorithms, which we have left out of this study, can only be established using well-balancing invariants on the intermediate trees (these invariants can easily be proved by induction for both $\text{isortL}$ and $\text{isortH}$, which run in time $O(N \lg N)$).

Another application of invariants would come up in a generic programming setting: the $\text{C2list}$ functions would have a single definition for every tree type: the function would merge together the lists resulting from recursive calls with the (wrapped) contents of nodes and leaves. For each concrete intermediate type, the structural invariants would then allow us to refine the definition into the one given in this paper.

This study opens the way to a richer interplay between invariants and recursion patterns – a topic that is not explored in this paper, but is being currently investigated by the authors.

Finally, we have left completely out of the paper a study of stability of the sorting algorithms, an important property in the presence of data-types for which the order is not total. Some of the algorithms derived are stable and others are not, which means that under this premise, which invalidates commutativity of $\circ$, they are not all equivalent.
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A Proofs and Calculations

A.1

The function

\[ \text{isort}_C = \text{foldr} \ \text{ist}' \ C \text{2list} \]

where \( \text{ist}' \ x \ f \ y = f \ (\text{ist} \ x \ y) \)

satisfies the specification

\[ \text{isort}_C : [a] \rightarrow C \rightarrow [a] \]
\[ \text{isort}_C \ l \ y = (\text{isort} \ l) \circ (\text{C2list} \ y) \]

Proof. The specification can be rewritten as

\[ \text{isort}_C \ l \ y = (\text{isort} \ l) \oplus y \]

or

\[ \text{isort}_C = (\oplus) \circ \text{isort} \]

with the \( \oplus \) operator defined as

\[ s \oplus y = s \circ (\text{C2list} \ y) \]

This appeals to the use of the fusion law since \text{isort} is defined as a fold.

\[ \text{isort}_C = (\oplus) \circ \text{isort} \]
\[ \iff \{ \text{definitions} \} \]
\[ \text{foldr} \ \text{ist}' \ C \text{2list} = (\oplus) \circ (\text{foldr} \ [\ ] ) \]
\[ \iff \{ \text{foldr fusion, with } \oplus \text{ strict} \} \]
\[ \{ (\oplus) \ [\ ] = \text{C2list} \}
\[ \{ (\oplus) \circ (\text{insert} \ x) = (\text{ist}' \ x) \circ (\oplus) \}
\[ \iff \{ \eta\text{-expansion} \} \]
\[ \{ [\ ] \oplus y = \text{C2list} \ y \}
\[ \{ (\text{insert} \ x \ l) \oplus y = \text{ist}' \ x \ ((\oplus) \ l) \ y \}
\[ \iff \{ \text{def. } \oplus, \text{ properties of } \circ, \text{ def. } \text{ist}' \} \]
\[ \{ \text{C2list} \ y = \text{C2list} \ y \}
\[ \{ (\text{insert} \ x \ l) \oplus y = l \oplus (\text{ist} \ x \ y) \}
\[ \iff \{ \text{eq. (1)}, \text{ def. } \oplus \} \]
\[ [x] \circ l \circ (\text{C2list} \ y) = l \circ (\text{C2list} \ (\text{ist} \ x \ y)) \]
\[ \iff \{ \text{eq. (1)} \} \]
\[ [x] \circ l \circ (\text{C2list} \ y) = l \circ (\text{insert} \ x \ (\text{C2list} \ y)) \]
\[ \iff \{ \text{eq. (1)}, \text{ properties of } \circ \} \]
\[ [x] \circ l \circ (\text{C2list} \ y) = [x] \circ l \circ (\text{C2list} \ y) \]

A.2

We prove

\[ \text{LT2list} \circ (\text{istLT} \ x) = (\text{insert} \ x) \circ \text{LT2list} \]

first by calculation, and then using induction.
ParaLTree :: (LTree a) -> b -> (LTree a) -> b -> b -> ((Maybe a) -> b) -> LTree a -> b

ParaLTree f g (Leaf x) = g x
ParaLTree f g (Branch l r) = f l (ParaLTree f g l) r (ParaLTree f g r)

\[ h = \text{ParaLTree } f \text{ } g \]
\[ h \circ \text{ParaLTree } f \text{ } g = \text{ParaLTree } a \text{ } b \]

\[ \begin{align*}
  h \circ \text{Leaf} &= g \\
  h \text{ (Branch l r)} &= f l \text{ (h l)} \text{ } r \text{ (h r)} \\
\end{align*} \]

\[ \begin{align*}
  h \circ \text{ParaLTree } f \text{ } g &= \text{ParaLTree } a \text{ } b \\
  h \circ (\text{ParaLTree } f \text{ } g) &= \text{ParaLTree } a \text{ } b \\
\end{align*} \]

| | 
|---|---|
| h = ParaLTree f g | h ∘ (ParaLTree f g) = ParaLTree a b |
| \{unicity-ParaLTree\} | \{ParaLTree-fusion\} |
| h ∘ Leaf = g | h strict ∧ h ∘ g = b ∧ |
| for all l, r, | h(f l l' r r') = a l (h l') r (h r') |
| h (Branch l r) = f l (h l) r (h r) | |

Table 5. The list paramorphism recursion pattern and laws

**Proof by Calculation.** It is easy to see that the insertion function istLT cannot be written as a fold over trees, since it uses one of the subtrees unchanged (insertion will proceed recursively in the other subtree). This is a typical example of a situation where iteration is not sufficient: primitive recursion is required. This has been studied as the paramorphism recursion pattern [8]. The operator in Table 5 embodies this pattern for leaf-trees. The corresponding unicity property and fusion law [9] are also shown in the table.

The function \( \text{istLT} \ x \) can now be written as the following paramorphism of leaf trees

\[ \text{istLT } x = \text{ParaLTree } f \text{ } g \]
\[ \text{where } g \text{ Nothing} = \text{Leaf(Just } x) \]
\[ g \text{ (Just } y) = \text{Branch (Leaf(Just } x)) \text{ (Leaf(Just } y)) \]
\[ f \text{ l l' r r'} = \text{Branch r' l} \]

We use the following strategy: we apply fusion to prove the left-hand side of the equality equivalent to a new paramorphism; subsequently we prove by unicity that the right-hand side of the equality is also equivalent to this paramorphism.

\[ \text{LT2list} \circ (\text{istLT } x) = \text{ParaLTree } a \text{ } b \]
\[ \Leftrightarrow \{ \text{def. of istLT } x \text{ as a paramorphism}\} \]
\[ \text{LT2list} \circ (\text{ParaLTree } f \text{ } g) = \text{ParaLTree } a \text{ } b \]
\[ \Leftrightarrow \{ \text{ParaLTree-fusion, with LT2list strict} \} \]
\[ \{ \text{LT2list } g = b \} \]
\[ \{ \text{LT2list}(f \text{ l l' r r'}) = a l \text{ (LT2list } l') \text{ } r \text{ (LT2list } r') \} \]
\[ \Leftrightarrow \{ \eta\text{-expansion, def. } f, g \} \]
\[ \{ \text{LT2list (Leaf(Just } x)) = b \text{ Nothing} \} \]
\[ \{ \text{LT2list (Branch (Leaf(Just } x)) \text{ (Leaf(Just } y))} = b \text{ (Just } y) \} \]
\[ \{ \text{LT2list (Branch r' l)} = a l \text{ (LT2list } l') \text{ } r \text{ (LT2list } r') \} \]
\[ \Leftrightarrow \{ \text{def. LT2list} \} \]
\[ \{ [x] = b \text{ Nothing} \} \]
\[ \{ [x] \odot [y] = b \text{ (Just } y) \} \]
\[ \{ \text{LT2list } r' \odot \text{ (LT2list } l) = a l \text{ (LT2list } l') \text{ } r \text{ (LT2list } r') \} \]

We are thus led to define

\[ b \text{ Nothing} = [x] \]
\[ b \text{ (Just } y) = [x] \odot [y] \]
\[ a l l'' r r'' = r'' \odot \text{ (LT2list } l) \]
It remains to prove \((\text{insert } x) \circ \text{LT2list} = \text{paraTree } a \ b\). Again we proceed by using fusion; the trick is now to write the fold \(\text{LT2list}\) as a paramorphism (this is always possible since it is a particular case).

\[
(\text{insert } x) \circ \text{LT2list} = \text{paraTree } a \ b
\]

\[
\iff \quad \{\text{unicity-paraTree, with } (\text{insert } x) \text{ strict}\}
\]

\[
(\text{insert } x) \circ \text{LT2list} \circ \text{Leaf} = b
\]

\[
(\text{insert } x) \ (\text{LT2list} \ (\text{Branch } l \ r)) = a \ l \ (\text{insert } x \ (\text{LT2list } l)) \ r \ (\text{insert } x \ (\text{LT2list } r))
\]

\[
\iff \quad \{\text{def. of LT2list}\}
\]

\[
(\text{insert } x) \circ g = b
\]

\[
\text{insert } x \ (f \ l \ (\text{LT2list } l) \ r \ (\text{LT2list } r)) = a \ l \ (\text{insert } x \ (\text{LT2list } l)) \ r \ (\text{insert } x \ (\text{LT2list } r))
\]

where

\[
g \text{ Nothing} = [ ]
\]

\[
g \ (\text{Just } y) = [y]
\]

\[
f \ l \ l' \ r \ r' = l' \circ r'
\]

\[
\iff \quad \{\eta\text{-expansion, def. of } f, g, a, b\}
\]

\[
(\text{insert } x \ [ ] = [x]
\]

\[
(\text{insert } x \ [y] = [x] \circ [y]
\]

\[
(\text{insert } x \ (\text{LT2list } l \circ \text{LT2list } r) = (\text{insert } x \ (\text{LT2list } r)) \circ (\text{LT2list } l)
\]

\[
\iff \quad \{1\text{ and properties of } \circ\}
\]

\[
[x] = [x]
\]

\[
[x] \circ [y] = [x] \circ [y]
\]

\[
[x] \circ (\text{LT2list } l) \circ (\text{LT2list } r) = [x] \circ (\text{LT2list } l) \circ (\text{LT2list } r)
\]

\[
\]

Proof by Induction.

1. \(c = \text{Leaf Nothing}\)

\[
\text{LT2list} \ (\text{istLT } x \ (\text{Leaf Nothing})) = \text{insert } x \ (\text{LT2list } (\text{Leaf Nothing}))
\]

\[
\iff \quad \{\text{def. istLT, LT2list}\}
\]

\[
\text{LT2list} \ (\text{Leaf } (\text{Just } x)) = \text{insert } x \ [ ]
\]

\[
\iff \quad \{\text{def. LT2list, insert}\}
\]

\[
[x] = [x]
\]

2. \(c = \text{Leaf (Just } y\))

\[
\text{LT2list} \ (\text{istLT } x \ (\text{Leaf (Just } y))\)) = \text{insert } x \ (\text{LT2list } (\text{Leaf (Just } y)))
\]

\[
\iff \quad \{\text{def. istLT, LT2list}\}
\]

\[
\text{LT2list} \ (\text{Branch } (\text{Leaf } (\text{Just } x)) \ (\text{Leaf } (\text{Just } y))) = \text{insert } x \ [y]
\]

\[
\iff \quad \{\text{def. LT2list, Spec. theorem}\}
\]

\[
(\text{LT2list } (\text{Leaf } (\text{Just } x))) \circ (\text{LT2list } (\text{Leaf } (\text{Just } y))) = (\text{wrap } x) \circ [y]
\]

\[
\iff \quad \{\text{def. LT2list, wrap}\}
\]

\[
[x] \circ [y] = [x] \circ [y]
\]
3. $c = \text{Branch } l \; r$

\[
\begin{align*}
\text{LT}2\text{list} \; (\text{istLT } x \; (\text{Branch } l \; r)) &= \text{insert } x \; (\text{LT}2\text{list} \; (\text{Branch } l \; r)) \\
\iff & \quad \{ \text{def. istLT, LT}2\text{list} \} \\
\text{LT}2\text{list} \; (\text{Branch } (\text{istLT } x \; r) \; l) &= \text{insert } x \; ((\text{LT}2\text{list } l) \odot (\text{LT}2\text{list } r)) \\
\iff & \quad \{ \text{def. LT}2\text{list, Spec. theorem} \} \\
(\text{LT}2\text{list} \; (\text{istLT } x \; r)) \odot (\text{LT}2\text{list } l) &= (\text{wrap } x) \odot ((\text{LT}2\text{list } l) \odot (\text{LT}2\text{list } r)) \\
\iff & \quad \{ \text{induction, commut. } \odot \} \\
(\text{insert } x \; (\text{LT}2\text{list } r)) \odot (\text{LT}2\text{list } l) &= (\text{wrap } x) \odot ((\text{LT}2\text{list } r) \odot (\text{LT}2\text{list } l)) \\
\iff & \quad \{ \text{Spec. theorem, assoc. } \odot \} \\
(\text{wrap } x) \odot (\text{LT}2\text{list } r) \odot (\text{LT}2\text{list } l) &= (\text{wrap } x) \odot (\text{LT}2\text{list } r) \odot (\text{LT}2\text{list } l)
\end{align*}
\]
B Tree Invariants

In order to prove certain equalities, it is convenient to introduce a notion of invariant that captures properties satisfied by the intermediate structures. These invariants are defined structurally on the data types.

For every predicate \( p : A \to \text{Bool} \), we consider the following inductive predicates:

\[
\begin{align*}
\text{(AllL } p \text{ [])} & \quad \forall x, xs. (p x) \land \text{(AllL } p \text{ xs)} \Rightarrow \text{(AllL } p \text{ (x : xs)}) \\
\text{(AllT } p \text{ Empty)} & \quad \forall x, l, r. (p x) \land \text{(AllT } p \text{ l)} \land \text{(AllT } p \text{ r)} \Rightarrow \text{(AllT } p \text{ (Node x l r)}) \\
\text{(BST Empty)} & \quad \forall x, l, r. \text{(AllT } (< x) \text{ l)} \land \text{(AllT } (\geq x) \text{ r)} \land \text{(BST } l) \land \text{(BST } r) \Rightarrow \text{(BST } \text{ (Node x l r)}) \\
\text{(HEAP Empty)} & \quad \forall x, l, r. \text{(AllT } (\geq x) \text{ l)} \land \text{(AllT } (\geq x) \text{ r)} \land \text{(HEAP } l) \land \text{(HEAP } r) \Rightarrow \text{(HEAP } \text{ (Node x l r)})
\end{align*}
\]

Let us start stating some simple properties concerning lists and trees.

Lemma 1. For every values \( x, y \) and lists \( l_1, l_2 \), we have:

1. \( x < y \Rightarrow \text{insert } x \ (l_1 ++ [y] ++ l_2) = (\text{insert } x \ l_1) ++ [y] ++ l_2 \)
2. \( \text{(AllL } (\leq x) \text{ l)} \Rightarrow \text{insert } x \ (l_1 ++ l_2) = l_1 ++ (\text{insert } x \ l_2) \)
3. \( (\forall x. p x \Rightarrow q x) \Rightarrow \text{AllL } p \ l_1 \Rightarrow \text{AllL } q \ l_1 \)
4. \( \text{(AllL } p \ (l_1 ++ l_2)) \Leftrightarrow (\text{AllL } p \ l_1) \land (\text{AllL } p \ l_2) \)
5. \( \text{(AllL } (< x) \text{ l)} \Rightarrow \text{insert } x \ l_1 = x : l_1 \)

Proof. Simple induction on \( l_1 \).

Lemma 2. For every tree \( t \) and value \( x \),

1. \( \text{(AllT } p \ t) \Rightarrow \text{(AllL } p \ (\text{BST2list } t)) \)
2. \( \text{(AllT } p \ t) \Rightarrow \text{(AllL } p \ (\text{H2list } t)) \)
3. \( (p x) \land \text{(AllT } p \ t) \Rightarrow \text{(AllT } p \ (\text{istBST } x \ t)) \)
4. \( (p x) \land \text{(AllT } p \ t) \Rightarrow \text{(AllT } p \ (\text{istH } x \ t)) \)

Proof. Induction on \( t \).

We are now able to prove the required properties. For heapsort, we explore the fact that the intermediate structure is a heap (its root keeps the least element).

For the heapsort algorithm, we explore the fact that the intermediate tree is a heap.

Proposition 8.

\[
\text{(HEAP } t) \Rightarrow \text{insert } x \ (\text{H2list } t) = \text{H2list } (\text{istH } t)
\]
Proof. By induction on the structure of $t$. The base case follows immediately from the definitions. For the induction step we have:

\[
\begin{align*}
\text{insert } x \ (\text{H2list} \ (\text{Node } y \ l \ r)) \\
= & \quad \{\text{def. H2list}\} \\
\text{insert } x \ (y : (\text{H2list} \ l) \circ (\text{H2list} \ r)) \\
= & \quad \{\text{def. insert}\} \\
& \quad \begin{cases}
  x : y : ((\text{H2list} \ l) \circ (\text{H2list} \ r)) & \text{if } x < y, \\
  y : (\text{insert } x \ ((\text{H2list} \ l) \circ (\text{H2list} \ r))) & \text{if } x \geq y,
\end{cases} \\
= & \quad \{\text{lemma 1 (5)}\} \\
& \quad \begin{cases}
  x : (\text{insert } y \ ((\text{H2list} \ l) \circ (\text{H2list} \ r))) & \text{if } x < y, \\
  y : (\text{insert } x \ ((\text{H2list} \ l) \circ (\text{H2list} \ r))) & \text{if } x \geq y,
\end{cases} \\
= & \quad \{\text{commutativity and associativity of } \circ\} \\
& \quad \begin{cases}
  x : (((y \circ (\text{H2list} \ r)) \circ (\text{H2list} \ l)) & \text{if } x < y, \\
  y : (((x \circ (\text{H2list} \ r)) \circ (\text{H2list} \ l)) & \text{if } x \geq y,
\end{cases} \\
= & \quad \{\text{induction hypotheses}\} \\
& \quad \begin{cases}
  x : ((\text{H2list} \ (\text{istH } y \ l)) \circ (\text{H2list} \ r)) & \text{if } x < y, \\
  y : ((\text{H2list} \ (\text{istH } x \ l)) \circ (\text{H2list} \ r)) & \text{if } x \geq y,
\end{cases} \\
= & \quad \{\text{def. H2list}\} \\
& \quad \begin{cases}
  \text{H2list} \ (\text{Node } x \ (\text{istH } y \ l \ r)) & \text{if } x < y, \\
  \text{H2list} \ (\text{Node } y \ (\text{istH } x \ l \ r)) & \text{if } x \geq y,
\end{cases} \\
= & \quad \{\text{def. istH}\} \\
& \quad \text{H2list} \ (\text{istH } x \ (\text{Node } y \ l \ r))
\end{align*}
\]

To prove that the intermediate tree is actually a heap, we prove that insertion of elements preserves the invariant.

**Proposition 9.** For every value $x$ and tree $t$,

\[
(\text{HEAP } t) \Rightarrow (\text{HEAP } (\text{istH } x \ t)).
\]

Proof. Induction on $t$. The base case follows immediately from the definitions. For the induction step we have:

\[
\begin{align*}
(\text{HEAP } (\text{istH } x \ (\text{Node } y \ l \ r))) \\
\Leftrightarrow & \quad \{\text{def. istH}\} \\
& \quad \begin{cases}
  (\text{HEAP } (\text{Node } x \ (\text{istH } y \ r \ l))) & \text{if } x < y, \\
  (\text{HEAP } (\text{Node } y \ (\text{istH } x \ r \ l))) & \text{if } x \geq y,
\end{cases}
\end{align*}
\]

In fact, when $x < y$ we have:

\[
\begin{align*}
& \begin{cases}
  (\forall l \geq x \ (\text{istH } y \ l)) & \text{by lemma 2 (2) and (HEAP } (\text{Node } y \ l \ r)) \\
  (\forall l \geq x \ r) & \text{by (HEAP } (\text{Node } y \ l \ r)) \\
  (\text{HEAP } (\text{istH } x \ r)) & \text{by induction hypotheses and (HEAP } (\text{Node } y \ l \ r)) \\
  (\text{HEAP } l) & \text{by (HEAP } (\text{Node } y \ l \ r))
\end{cases}
\end{align*}
\]

We reason similarly when $x \geq y$. 
And now, the required result follows directly by induction.

**Corollary 1.** For every list \( l \),

\[
(\text{HEAP } \text{buildH } l)
\]

**Proof.** Simple induction on \( l \).

For the quicksort algorithm, we explore the fact that the intermediate tree is a binary search tree.

**Proposition 10.** For every value \( x \) and tree \( t \),

\[
(\text{BST } t) \Rightarrow \text{insert } x (\text{BST2list } t) = \text{BST2list } (\text{istBST } t)
\]

**Proof.** By induction on the structure of \( t \). The base case follows immediately from the definitions. For the induction step we have:

\[
\text{insert } x (\text{BST2list } (\text{Node } y l r))
\]

\[
= \left\{ \text{def. BST2list} \right\}
\]

\[
\text{insert } x ((\text{BST2list } l) ++[y] ++(\text{BST2list } r))
\]

\[
= \left\{ \text{lemma } 1(1,2), 2(1) \text{ and hypotheses } (\text{BST } (\text{Node } y l r)) \right\}
\]

\[
\begin{cases}
\left( \text{insert } x (\text{BST2list } l) ++[y] ++(\text{BST2list } r) \right) & \text{if } x < y,
\left( \text{BST2list } l ++[y] ++(\text{insert } x (\text{BST2list } r)) \right) & \text{if } x \geq y,
\end{cases}
\]

\[
= \left\{ \text{induction hypotheses} \right\}
\]

\[
\begin{cases}
\left( \text{BST2list } (\text{istBST } x l) ++[y] ++(\text{BST2list } r) \right) & \text{if } x < y,
\left( \text{BST2list } l ++[y] ++(\text{BST2list } (\text{istBST } x r)) \right) & \text{if } x \geq y,
\end{cases}
\]

\[
= \left\{ \text{def. BST2list} \right\}
\]

\[
\begin{cases}
\text{BST2list } (\text{Node } y (\text{istBST } x l) r) & \text{if } x < y,
\text{BST2list } (\text{Node } y l (\text{istBST } x r)) & \text{if } x \geq y.
\end{cases}
\]

Again, we note that the insertion function preserves the invariant.

**Proposition 11.** For every value \( x \) and tree \( t \),

\[
(\text{BST } t) \Rightarrow (\text{BST } (\text{istBST } x t))).
\]

**Proof.** Induction on \( t \). The base case follows immediately from the definitions. For the induction step we have:

\[
(\text{BST } (\text{istBST } x (\text{Node } y l r)))
\]

\[
\iff \left\{ \text{def. istBST} \right\}
\]

\[
\begin{cases}
(\text{BST } (\text{Node } y (\text{istBST } x l) r)) & \text{if } x < y,
(\text{BST } (\text{Node } y l (\text{istBST } x r))) & \text{if } x \geq y,
\end{cases}
\]

In fact, when \( x < y \) we have:

\[
\begin{cases}
(\text{AllT } (\text{< } y) \text{ (istBST } x l)) & \text{by lemma } 2(2)
(\text{AllT } (\text{\geq } y) r) & \text{by } (\text{BST } (\text{Node } y l r))
(\text{BST } (\text{istBST } x l)) & \text{by induction hypotheses and } (\text{BST } (\text{Node } y l r))
(\text{BST } r) & \text{by } (\text{BST } (\text{Node } y l r))
\end{cases}
\]

We reason similarly when \( x \geq y \).
And the required result follows directly by induction.

**Corollary 2.** For every list \(l\),

\[
(BST (buildBST l))
\]

*Proof.* Simple induction on \(l\).
C Alternative Derivation

In this appendix we present a slight variation on the strategy for deriving the sorting algorithms. This variation clarifies the role of the invariants on intermediate structures in the correctness argument of these algorithms.

When we compare the proof effort required to establish the correctness of the “sorting by insertion” algorithms, we note that there is significant difference between isortLT and the other two algorithms (isortH and isortBST). As explained in the main text, this is because the correctness for the last two algorithms depend on properties of the intermediate structure. However, we can explain that difference at a more abstract level — one might argue that isortLT is closer to the specification of a generic insertion sort presented at Section 4. To illustrate this point, let us recall the definition of these algorithms (we omit the definitions not relevant for this discussion):

\[
\begin{align*}
isortLT &= \text{LT2list} \circ \text{buildLT} \\
isortH &= \text{H2list} \circ \text{buildH} \\
isortBST &= \text{BST2list} \circ \text{buildBST} \\
\text{LT2list} &= \text{foldLTree} (\circ) t \\
&\quad \text{where } t \text{ Nothing} = [ ] \\
&\quad t (\text{Just } x) = [x] \\
\text{H2list} &= \text{foldr aux} [ ] \\
&\quad \text{where } aux \ x \ l \ r = x : (l \circ r) \\
\text{BST2list} &= \text{foldBTree aux} [ ] \\
&\quad \text{where } aux \ x \ l \ r = l ++ (x : r)
\end{align*}
\]

We observe that LT2list uses only \( \circ \) to construct (non trivial) lists. On the other side, H2list and BST2list make use of other functions (namely \((:)\) and \((++))\). That distinction makes the later two sensible to the ordering attributes of the intermediate tree.

Let us make one step back and define the following variants of isortH and isortBST algorithms:

\[
\begin{align*}
isortH' &= \text{BT2list} \circ \text{buildH} \\
isortBST' &= \text{BT2list} \circ \text{buildBST} \\
\text{BT2list} &= \text{foldr aux} [ ] \\
&\quad \text{where } aux \ x \ l \ r = [x] \circ (l \circ r)
\end{align*}
\]

Now, the conversion of binary trees into lists (BT2list) does not assume any ordering constraints on these trees. In fact, BT2list and LT2list should be read as two instances of the same polytypic function.

It is interesting to verify that, for these modified functions, the correctness argument is essentially the same as for isortLT.

**Proposition 12.** isortH' and isortBST' are sort algorithms.

*Proof.* We instantiate eqs. (3) and (4) for both functions. We set \( \epsilon = \text{Empty} \), and thus eq. (3) results directly from the definition. For eq. (5), we need to prove that for every binary tree \( t \)
and value $x$,

$$(\text{BT2list} \circ \text{istH } x) \ t = ((\text{insert } x) \circ \text{BT2list}) \ t$$

$$(\text{BT2list} \circ \text{istBST } x) \ t = ((\text{insert } x) \circ \text{BT2list}) \ t$$

These are proved by induction on the structure of $t$. We show the proof of the first one (the second is similar). The base case is trivial. For the induction step we have:

$$\text{BT2list(istH } x \ (\text{Node } y \ l \ r)) =$$

{def. \text{istH}}

\begin{align*}
& ((\text{BT2list}(\text{Node } x \ (\text{istH } y \ r) \ l)) \text{ if } x < y, \\
& (\text{BT2list}(\text{Node } y \ (\text{istH } x \ r) \ l)) \text{ if } x \geq y,
\end{align*}

= {def. \text{BT2list}}

\begin{align*}
& [x] \odot ((\text{BT2list (istH } y \ r) \odot \text{BT2list } l)) \text{ if } x < y, \\
& [y] \odot ((\text{BT2list (istH } x \ r) \odot \text{BT2list } l)) \text{ if } x \geq y,
\end{align*}

= {induction hypotheses}

\begin{align*}
& [x] \odot (([y] \odot (\text{BT2list } r)) \odot \text{BT2list } l) \text{ if } x < y, \\
& [y] \odot (([x] \odot (\text{BT2list } r)) \odot \text{BT2list } l)
\end{align*}

= {comutativity and associativity of $\odot$}

$$[x] \odot ([y] \odot ((\text{BT2list } l) \odot (\text{BT2list } r)))$$

= {def. \text{BT2list}}

$$[x] \odot (\text{BT2list (Node } y \ l \ r))$$

In order to refine $\text{isortH'}$ and $\text{isortBST'}$ to heap sort and quicksort, we should now proceed in two independent paths:

– to show that the construction of the intermediate tree can be performed co-inductively (i.e. $\text{buildH}$ and $\text{buildBST}$ are equal to $\text{unfoldhsort}$ and $\text{unfoldqsort}$ respectively);

– to show that the tree conversion into the resultant list can be simplified to their standard formulation (i.e. $\text{BT2list}$ can be replaced by $\text{H2list}$ for the heapsort and by $\text{BST2list}$ for the quicksort).

The first point was performed in the main text (c.f. Propositions 5 and 7). The second is the one that should consider the ordering properties induced by the building process for each case — more precisely, one proves:

$$\text{BT2list} \circ \text{buildH} = \text{H2list} \circ \text{buildH}$$

$$\text{BT2list} \circ \text{buildBST} = \text{BST2list} \circ \text{buildBST}$$

As in appendix B it is convenient to make explicit the structural invariants possessed by the intermediate structures in each case. That is,

$$\text{HEAP } t \implies \text{BT2list } t = \text{H2list } t$$

$$\text{BST } t \implies \text{BT2list } t = \text{BST2list } t$$

The proof require a simple lemma relating $\odot$ with ordering predicates.
Lemma 3. For every

1. \((\text{AIL} (x \leq l_1)) \implies [x] \odot l_1 = x : l_1\)
2. \((\text{AIL} (x > l_1)) \implies l_1 \odot (x : l_2) = l_1 ++ (x : l_2)\)
3. \((\text{AIL} p l_1) \land (\text{AIL} p l_2) \implies (\text{AIL} p (l_1 \odot l_2))\)

Proof. The first two are proved by simple induction on the structure of \(l_1\). The third by mutual induction on \(l_1\) and \(l_2\).

Now, the required properties follow by simple induction. The base case is, in both cases, trivial. For the induction step, we have for \(H2list\):

\[
\begin{align*}
\text{BT2list} (\text{Node } x l r) &= \{\text{def. BT2list}\} \\
&= [x] \odot ((\text{BT2list} l) \odot (\text{BT2list} r)) \\
&= \{\text{induction hypotheses}\} \\
&= [x] \odot ((\text{H2list} l) \odot (\text{H2list} r)) \\
&= \{\text{def. of HEAP and lemma } 3(1,3)\} \\
&= x : ((\text{H2list} l) \odot (\text{H2list} r)) \\
&= \{\text{def. H2list}\} \\
&= \text{H2list} (\text{Node } x l r)
\end{align*}
\]

and for \(BST2list\):

\[
\begin{align*}
\text{BT2list} (\text{Node } x l r) &= \{\text{def. BT2list}\} \\
&= [x] \odot ((\text{BT2list} l) \odot (\text{BT2list} r)) \\
&= \{\text{induction hypotheses}\} \\
&= [x] \odot ((\text{BST2list} l) \odot (\text{BST2list} r)) \\
&= \{\text{commutativity and associativity of } \odot\} \\
&= ((\text{BST2list} l) \odot ([x] \odot (\text{BST2list} r))) \\
&= \{\text{def. of BST and lemma } 3(1,2)\} \\
&= (\text{BST2list} l) \odot (x : (\text{BST2list} r)) \\
&= \{\text{def. BST2list}\} \\
&= \text{BST2list} (\text{Node } x l r)
\end{align*}
\]