Information versus stability in an anti-de Sitter black hole

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Information entropies associated with the energy density in position and momentum spaces are build for an anti-de Sitter (AdS) black hole. These quantities, that satisfy an entropic uncertainty relation, vary with the temperature. The higher is the black hole temperature, the greater/smaller is the information encoded respectively in the position/momentum distributions of energy. On the other hand, as it is well know, AdS black holes are subject to the Hawking Page phase transition. The amplitude for dominance of the black hole phase over the thermal AdS phase increases with the temperature. So, as the system becomes more stable, there is a change in the way that information is stored. In particular: information stored in the spatial energy density increases while information stored in the energy density in momentum space decreases.

I. INTRODUCTION

It has been proposed by Gleiser and Sowinski [1] that the stability of compact objects can be characterized by a quantity, called configuration entropy, introduced by Gleiser and Stamatopoulos in refs.[2, 3]. The smaller is this quantity, the more stable is the system. In the recent years, many different examples appeared in the literature where there is a similar relation between the variation of the configurational entropy and the stability was shown to appear in as diverse systems as compact astrophysical objects [12] and holographic AdS/QCD models [5–10]. Many other recent applications of configuration entropy are found in the literature, as for example [11–27].

The basis for the definition of the configuration entropy[1] is the information entropy of Shannon [28] that has the form

\[ -\sum_{n} p_{n} \log p_{n} , \]  

where \( p_{n} \) is the probability distribution of a discrete variable. For the configuration entropy, one considers the continuous version

\[ f = -\int d^{3}r \rho(\vec{r}) \log \rho(\vec{r}) , \]  

where \( \rho(\vec{r}) = |v(\vec{r})|^{2} \) is a normalized function \( \int d^{3}r \rho(\vec{r}) = 1 \), called modal fraction. It is important to note the discrete case of eq. (1) is positive definite since the probabilities satisfies \( p_{n} \leq 1 \). The same rule does not apply to the continuous case, that involves densities. So, one should not take eq. (2) as an absolute measure of the information content, but rather consider the variations of \( f \) as representing variation in the information content.

The configuration entropy, as defined in [1], is the momentum space version of \( f \)

\[ \tilde{f} = -\int d^{3}k \tilde{\rho}(\vec{k}) \log \tilde{\rho}(\vec{k}) , \]

where \( \tilde{\rho}(\vec{k}) = |\tilde{v}(\vec{k})|^{2} \) is the momentum space modal fraction, with

\[ \tilde{v}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^{3}\vec{x} v(\vec{x}) \exp(-i\vec{k} \cdot \vec{r}) . \]  

Information entropies like \( f \) and \( \tilde{f} \), based in conjugate variables in the sense of eq. (4) satisfy the so called entropic uncertainty relations [29], that, for this 3-dimensional case takes the form:

\[ f + \tilde{f} \geq 3(1 + \log \pi) . \]  

So, one could guess that a variation of the configuration entropy, defined in momentum space, \( \tilde{f} \) should be associated with a variation of the conjugate quantity, \( f \), defined in position space. The purpose of this letter is to go one step ahead and investigate the relation between momentum and position entropies and their relation with stability.

The idea is to use the anti-de Sitter (AdS) black hole as an example, motivated by the fact that for this physical system one finds a simple way to characterize what one means by stability. That is: stability of the black hole is related to the amplitude for the dominance of the black hole phase over the thermal AdS phase, as we will see in section II.

A discussion of the configuration entropy for an anti-de Sitter black hole already appeared in ref. [17]. However, there two completely new aspects in the present letter. One is a technical point. That is: in ref. [17] the energy density was obtained by using a particular regularization in order to get rid of a surface contribution to the mass. This regularization process, as we will discuss in section III, is not unique. So, the definition for the energy density was ambiguous. We will present in section III an alternative non ambiguous way of introducing the energy density.

The second point is that, in contrast to ref. [17], here we will investigate not only the configuration entropy in momentum space \( \tilde{f} \) but also the corresponding dual entropy \( f \) in position space. We will see that these two quantities, that are subject to the inequality (5), vary with the black hole temperature. However, the sum \( f + \tilde{f} \)
is constant for the black hole case. So that, if these two quantities represent the information content in momentum and position spaces, respectively, such a result indicates that the total information is conserved.

There is a non trivial point regarding the interpretation of the entropies that we will calculated using the black hole energy densities. Information is associated with the degree of unpredictability of the result of an observation. In the Shannon entropy of eq. (1) the factors $p_i$ are probabilities. As a simple illustration of the relation between the information content and the degree of unpredictability, one can consider the particular case of the Shannon entropy when there is just one possible result for an observation. In this situation, there is just one value of $i$, with $p_i = 1$ and the entropy vanishes. In simple words: when there is no unpredictability the information entropy vanishes.

For the continuum case, in place of the discrete probability, one uses the modal fractions $\rho(\bar{x})$ and $\rho(\bar{q})$ in the definitions of the entropies in eqs. (2) and (3) respectively. If it happens that the modal fractions can be interpreted as probability densities, then the entropies $f$ and $\bar{f}$ can be interpreted as representing the information content in position and momentum spaces, respectively. In this letter we will build up modal fractions as the normalized square of the energy densities of the AdS black hole, in position and momentum spaces. These quantities are not probability densities if one makes an observation of the energy. The energy distribution is completely known, once the density is given. So, regarding energy distribution there is no unpredictability. However, one can think about an “experiment” where one searches for the position of a particle and the probability density of finding the particle at a given position is the normalized square of the spatial energy density. In a similar way, for the momentum case one can consider a process of determining the momentum of a particle that has a probability density in momentum space equal to the modal fraction obtained from the energy density in momentum space. It is in this sense that we will interpret $f$ and $\bar{f}$ as representing information content.

This letter is organized as follows. In section II we briefly review the Hawking Page transition in an AdS black hole. In section II we present the new approach for finding the energy density of the AdS black hole. Then in section IV we present the results for the entropies and some final conclusions.

II. HAWKING PAGE TRANSITION

A very interesting description of an anti-de Sitter (AdS) black hole was found by Hawking and Page [30] (see also [31] for enlightening discussions). The point of view, following semi-classical arguments, is that the black hole is a physical system consisting of a superposition of two different geometries with the same asymptotic boundaries. Both are solutions of the vacuum Einstein equation with a negative cosmological constant. One is the (thermal) Euclidean AdS space

$$ds^2 = \left(1 + \frac{r^2}{\bar{b}^2}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{\bar{b}^2}} + r^2d\Omega^2_{(2)}, \quad (6)$$

with a boundary, at $r \to \infty$, that is the product of a spatial $S^2$ sphere with a temporal $S^3$ circle. The time coordinate is periodic: $t \sim t + \beta$.

The other geometry is the AdS-Schwarzschild black hole space

$$ds^2 = \left(1 + \frac{r^2}{\bar{b}^2} - \frac{2MG_N}{r}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{\bar{b}^2}} - \frac{2MG_N}{r} + r^2d\Omega^2, \quad (7)$$

where $M$ is the black hole mass and $G_N$ is the Newton constant. In this case the temporal periodicity is $t \sim t + \beta$. The black hole space is the region $r > r_h$, were $r_h$, for our purposes, is the largest root of:

$$1 + \frac{r^2}{\bar{b}^2} - \frac{2MG_N}{r} = 0. \quad (8)$$

In order to avoid a conical singularity, the temporal period must be

$$\beta = \frac{4\pi b^2 r_h}{3r_h^2 + b^2}. \quad (9)$$

The black hole temperature is $T = 1/\beta$.

The Einstein gravity actions for the geometries (6) and (7) are respectively

$$I_{AdS} = \frac{3}{2b^2G_N} \int_0^\beta d\bar{t} \int_0^R r^2dr, \quad (10)$$

$$I_{BH} = \frac{3}{2b^2G_N} \int_0^\beta dt \int_{r_h}^R r^2dr, \quad (11)$$

where $R$ is a large (regulator) radius. In order that the actions have the same asymptotic geometry at $r \to \infty$, one imposes that the temporal circles have equal length:

$$\sqrt{\left(1 + \frac{R^2}{\bar{b}^2} - \frac{2MG_N}{R}\right)} \ dt = \sqrt{\left(1 + \frac{R^2}{\bar{b}^2}\right)} \ d\bar{t}. \quad (12)$$

This implies that at large $R$:

$$\beta \approx \beta \sqrt{1 - \frac{r_h^3 + b^2r_h}{2R^3}}. \quad (13)$$

The regularized action $I$, that represents the effect of the presence of the black hole, with respect to the AdS space without the black hole, reads

$$I = I_{BH} - I_{AdS}, \quad (14)$$

and is finite in the $R \to \infty$ limit.

The black hole solutions considered here are restricted to $b/\sqrt{3} < r_h$. The action $I$ is positive for $r_h < b$, when
the AdS space is dominant and is negative for \( r_h > b \), 
when the black hole space become dominant. The amplitude 
for finding the black hole geometry is approximately 
governed by the factor \( \exp(-I) \). Since the action \( I \) 
decreases monotonically with \( r_h \), the black hole becomes 
more and more stable against the Hawking Page transition 
as \( r_h \) and, correspondingly the temperature \( T \), 
increase.

III. ENERGY DENSITY FOR THE ADS BLACK HOLE

The mass of the AdS black hole is obtained from [31]
\[
M = \frac{\partial I}{\partial \beta}, \tag{15}
\]
with the action \( I \) given by eqs. (10), (11) and (14).

The idea that we will follow, in order to find an energy 
density, is to find a way of writing the total mass of eq. 
(15) as a spatial integral:
\[
M = \int u(r) d^3r. \tag{16}
\]
This means that we have to commute the derivative with 
respect to \( \beta \) with the spatial integration, contained in 
the action integral \( I \). This is not a trivial task since 
there is a subtle point in equation (15). The limit of the 
radial integration of the black hole action depends on the 
temperature. This happens because the black hole space is 
defined as the region outside the horizon, as can be 
seen in eq. (11) and the horizon position depends on the 
temperature, as show in eq. (9). So, the derivative with 
respect to \( \beta (= 1/T) \) affects not only the temporal periods 
but also the spatial part of the action integral. In ref. 
[17], a proposal to define an energy density was presented.

But the approach used there was to use the standard 
radial coordinate, as in eq. (11), where the dependence on 
the temperature appears in the lower integration limit. 
Then, the derivative with respect to \( \beta \) lead to a surface 
term to the mass. This surface term comes from the 
derivative of the radial integration limit and in ref. [17] it 
was located at the horizon position. This term would give 
a singular contribution to the entropy, as a consequence of 
the singular localization in the radial coordinate. In 
order to fix this problem, it was proposed in [17] that 
the surface term should be replaced by a volume density 
that leads to the same contribution to the mass. This 
was interpreted as a regularization procedure. However, 
it was not possible to find a unique definition for the 
volume density. So, a consistent definition for the energy 
density was lacking.

Here we will propose a different approach where one 
does not need to introduce any arbitrary term in the density. 
We simply perform a change of the radial variable 
to \( x = r/r_h \) that moves the dependence on the temperature 
from the lower radial integration limit to the upper 
integration limit. The actions take the form
\[
I_{\text{AdS}} = \frac{3}{2b^2G_N} \bar{\beta} r_h^3 \int_0^{R/r_h} x^2 dx, \tag{17}
\]
\[
I_{\text{BH}} = \frac{3}{2b^2G_N} \bar{\beta} r_h^3 \int_1^{R/r_h} x^2 dx. \tag{18}
\]

Differentiating \( I = I_{\text{BH}} - I_{\text{AdS}} \) with respect to \( \beta \), using 
relations (9) and (13) and then taking the limit \( R \to \infty \), 
one finds the mass of the black hole expressed as 
just a volume integral. Surface terms appear when one 
differentiates with respect to the upper integration limit 
\( x = R/r_h \). However, in contrast to what happens in the 
approach of ref. [17], they vanish when one subtracts 
the contributions from the black hole and thermal AdS 
actions and then take the \( R \to \infty \) limit.

One finds two different constant densities: \( u_1(x) \) in the 
region \( 0 \leq x \leq 1 \) and \( u_2(x) \) for the region \( 1 \leq x \leq R/r_h \). 
In other words, the mass can be written as
\[
M = 4\pi \int_0^{R/r_h} x^2 u(x) dx, \tag{19}
\]
with
\[
u(x) = \begin{cases} 
\frac{3}{8\pi b^2G_N} \left( \frac{6r_h^2 + 4r_h b^2}{3r_h^2 - b^2} \right), & (0 \leq x \leq 1), \\
- \frac{3r_h^2}{8\pi b^2G_N} \left( \frac{3r_h^2 + 2r_h b^2 + r_h b^4}{R^3(3r_h^2 - b^2)} \right), & (1 \leq x \leq R/r_h). 
\end{cases} \tag{20}
\]
Note that the energy density is proportional to \( 1/R^3 \). 
In the \( R \to \infty \) limit the density goes to zero, but the con-
tribution to the mass is finite since the volume increases 
with \( R^3 \). The density in the region \( r > r_h \) outside the 
horizon, will not contribute to the information entropies.

Now we return to coordinate \( r \) and define the corre-
sponding mass/energy density:
\[
v(r) = u(x)/r_h. \tag{21}
\]
The modal fraction that we need, in order to define the 
information entropy in position space is
\[
\rho(r) = \frac{|v(r)|^2}{4\pi \int_0^{r_h} v(r)^2 r^2 dr}. \tag{22}
\]
In order to find the momentum space version we now 
define an energy density in momentum space:
\[
\hat{v}(k) = \frac{1}{(2\pi)^{3/2}} \int v(r) \exp(-ik \cdot r) d^3r. \tag{23}
\]
After taking the limit \( R \to \infty \) one finds
\[
\hat{v}(k) = \frac{1}{(2\pi)^{3/2}} \frac{3}{2b^2G_N} \left( \frac{6r_h^5 + 4r_h^3 b^2}{3r_h^2 - b^2} \right) \times \left( \frac{\sin(kr_h)}{k^3} - \frac{r_h \cos(kr_h)}{k^2} \right), \tag{24}
\]
that comes only from contributions of the density inside the 
horizon.
The modal fraction in momentum space is defined as
\[
\hat{\rho}(k) = \frac{|\hat{\psi}(k)|^2}{4\pi \int_0^\infty |\hat{\psi}(k)|^2 k^2 dk} = \frac{1}{4\pi (\pi r_h^3/6)} \left( \frac{\sin(kr_h)}{k^3} - \frac{r_h \cos(kr_h)}{k^2} \right)^2
\]  
(24)

The information entropies in position and momentum spaces are given by eqs. (2) and (3), respectively, with spherical symmetry
\[
f = -\int d^3r \rho(r) \log \rho(r),
\]
\[
f' = -\int d^3k \hat{\rho}(k) \log \hat{\rho}(k).
\]  
(25)

Using eq. (20) one gets:
\[
f = \log \left(4\pi r_h^3/3\right).
\]

It is important to remark that both \( f \) and \( f' \) receive contributions from the energy density (20) only from the region inside the horizon: \( r < r_h \). This means that the changes in information, as represented by the variations of \( f \) and \( f' \), come only from inside the black hole, as one would expect from physical grounds. There is a non trivial fact to be noticed: the black hole geometry (7) is defined only for \( r \geq r_h \) but the subtraction of the thermal AdS background (6) leads to the appearance of an energy density inside the horizon. And this is precisely the density that contributes to the information entropies.

**IV. RESULTS AND CONCLUSIONS**

We show in figure 1 plots of the entropies \( f \), \( f' \) and their sum: \( f + f' \) as a function of the horizon radius divided by the AdS radius \( r_h/b \). One notes that as the horizon radius, and correspondingly the temperature, increase, the entropy in position space \( f \) increases while the momentum space entropy \( f' \) has the opposite behavior. They satisfies the constraint given by the uncertainty relation in eq. (5) and actually behave in a trivial way: the sum is constant.

So, as the black hole temperature increases, and the black hole increases the stability against a Hawking Page transition, information encoded in the spatial distribution of energy increases while information in the momentum distribution decreases. And the total information associated with the quantities \( f \) and \( f' \) remains constant.

It is not clear, at this moment, if such a behavior is particular of the AdS black hole or could be more general. Previous studies of configuration entropy, like [1, 12, 15] indicate that the momentum space entropy decreases with stability, but for the spatial entropy, and their sum, we are not aware of any similar study. This could be an interesting topic for future investigation.

Regarding stability, the black hole states are dominant for temperatures above the critical one, that corresponds to \( r_h/b = 1 \). The semi-classical argument of Hawking and Page is that the relative probability of a given configuration is proportional to the exponential of (minus) the corresponding action. So the probability of finding the black hole is governed by the factor \( \exp (-I) \), where \( I \) is the difference between the black hole action and the thermal action. The larger the ratio \( r_h/b \), the larger the difference between the actions. So that, the black hole dominance increases smoothly with \( r_h/b \). This is consistent with the decrease in the configuration entropy \( f \) found here, that is shown in figure 1. There is no particular signature of the Hawking-Page transition point \( r_h/b = 1 \) from the point of view of the configuration entropy. This can be explained by the fact that the variation in the difference between the black hole and thermal AdS actions with the temperature is smooth.

It is interesting to note that if one looks at the logarithms of the entropies, one finds for \( f' \) a simple scaling relation. We plot \( \log(f(r_h/b)) \) in figure 2. There is an approximate linear behavior of the form: \( \log(f) = 2.20 - 0.31(r_h/b) \). For the logarithm of the position entropy there is no such linear fit.

**FIG. 1.** Information Entropies for the AdS black hole as a function of \( r_h/b \). Blue line: momentum space entropy. Orange line: position space case: \( f \). Green line: sum of the entropies.

**FIG. 2.** Logarithm of the momentum space entropy.
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