Supersymmetric Defect Expansion in CFT from AdS Supertubes

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Abstract: The AdS/dCFT correspondence is used to show that a planar $q$-dimensional superconformal CFT defect expands, under the addition of electric charge and angular momentum, to a supersymmetric higher-dimensional defect of geometry $\mathbb{R}^q \times C$, where $C$ is an arbitrary curve. The dual string theory process is the expansion of D-branes and fundamental strings into a supertube in an AdS background.

Keywords: D-branes, Supersymmetry and Duality, AdS/CFT Correspondence, Brane Dynamics in Gauge Theories.
1. Introduction

The AdS/dCFT correspondence [1] is an interesting extension of the AdS/CFT correspondence [2] in which additional structure is added on both sides. On the string theory side a brane that extends to the AdS boundary is added. On the CFT side we have a defect where the brane meets the AdS boundary. A necessary condition for the defect to preserve (part of) the conformal invariance of the bulk CFT is that the induced metric on the brane include an AdS factor; we will see, however, that this is not sufficient.

In type IIB string theory, an AdS/dCFT correspondence can be ‘derived’ for each orthonormal $q$-dimensional intersection of $N$ D3-branes and one D$p$-brane, which we shall denote as $(q|\text{D}3 \perp \text{D}p)$. For example, consider the $1/4$-supersymmetric $(2|\text{D}3 \perp \text{D}5)$ intersection represented by the array

\begin{align}
\text{D3:} & \quad 1 \ 2 \ 3 \ \ldots \ \ldots \\
\text{D5:} & \quad \ldots \ 2 \ 3 \ 4 \ 5 \ 6 \ \ldots 
\end{align}

(1.1)

This is the case originally introduced in [1]. If $g_s \ll 1$ and $g_s N \gg 1$, where $g_s$ is the string coupling constant, then the perturbative description of this system in the decoupling limit is in terms of an $AdS_4 \times S^2$ D5-brane probe in the $AdS_5 \times S^5$ near-horizon geometry of the D3-branes; the probe intersects the $AdS$ boundary at an $\mathbb{R}^2$-defect. The $AdS_4$-factor arises from the 023-directions together with the radial coordinate in the 456-space, whereas the two angular coordinates in this space yield the $S^2$-factor. On the other hand, if $g_s N \ll 1$ then the perturbative description of the system is that of the $\mathcal{N} = 4$ super Yang-Mills (SYM)
theory in the presence of an $\mathbb{R}^2$-defect. The AdS/dCFT correspondence states that these two descriptions are actually equivalent throughout the entire parameter space. The degrees of freedom on the defect arise from the open strings connecting the D5-brane to the N D3-branes \[3, 4\], and are the holographic duals of the open-string degrees of freedom on the probe D5-brane \[5\].

More generally, every 1/4-supersymmetric ($q|D3 \perp Dp$) intersection leads to an analogous duality involving a probe $AdS_{q+2} \times S^{p-q-2}$ Dp-brane in $AdS_5 \times S^5$ and an $\mathbb{R}^q$-defect in the SYM theory. Since each probe Dp-brane preserves eight Poincaré and eight special conformal supersymmetries \[6\] and (part of) the conformal symmetry in $AdS_5 \times S^5$, so does the dual defect in the gauge theory.

In the IIB Minkowski vacuum, the addition of angular momentum to a collection of orthogonal Dp-branes and fundamental strings causes them to expand to a D($p + 2$)-brane supertube \[7\], which we henceforth refer to as a D($p + 2$)-tube. This is a 1/4-supersymmetric tubular D($p + 2$)-brane of geometry $\mathbb{R}^{p+1} \times C$, where the cross-section $C$ is a completely arbitrary curve\(^1\) in $\mathbb{R}^{8-p}$ \[8\]; the initial strings and D$p$-branes appear on the D($p + 2$)-tube as dissolved charges represented by electric and magnetic Born-Infeld (BI) fields. Although originally found as a solution of the Dirac-Born-Infeld (DBI) action, the supertube is an exact solution (to all orders in $\alpha'$) of classical open-string theory \[9\]. One crucial feature of the D($p + 2$)-tube is that it preserves the same supersymmetries as a collection of strings and D$p$-branes; in particular, there is no trace of a condition associated to the presence of the D($p+2$)-brane \[7\]. This paper is based on the observation that the supersymmetric expansion to a D($p + 2$)-tube just described also takes place in the presence of (suitably oriented) D3-branes, in particular in their $AdS_5 \times S^5$ background. The holographically dual process in the CFT is a supersymmetric defect expansion.

Consider for concreteness the 1/8-supersymmetric intersection

\[
\begin{align*}
D3: & \quad 1 \ 2 \ 3 \ \cdots \\
D3: & \quad \ \cdots \ 3 \ 4 \ 5 \ \cdots \\
F1: & \quad \ \cdots \ 6 \ \cdots
\end{align*}
\]

(1.2)

This can be viewed as a (1|D3 $\perp$ D3) intersection to which fundamental strings have been added. If the D3-branes in the first line are replaced by an $AdS_5 \times S^5$ geometry then the D3-brane in the second line becomes an $AdS_3 \times S^1$ probe that intersects the boundary of $AdS_5 \times S^5$ on a line-defect (along the 3-direction), on which the endpoints of the strings appear as an electric charge density. The addition of angular momentum in the 12-plane to this probe D3/F1 system causes it to expand to a D5-tube with geometry $\mathbb{R}^4 \times C$, so the array (1.2) is replaced by

\[
\begin{align*}
D3: & \quad 1 \ 2 \ 3 \ \cdots \\
D5\text{-tube}: & \quad \bullet \ \bullet \ 3 \ 4 \ 5 \ 6 \ \cdots
\end{align*}
\]

(1.3)

\(^1\) Although the expansion always leads to a closed cross-section because of charge conservation, 1/4-supersymmetry allows it to be an ‘open’ curve without boundary (which must therefore extend to infinity).
Figure 1: Here we have depicted three examples of D5-tubes intersecting D3-branes. (It should be understood that the D5-tubes continue through to the other side of the D3-branes.) The cross-sections of the tubes may be open (C' and C'') or closed (C). The numbers of D3-branes $N$ and $N'$ (here with $N \geq N'$) on either side of a D5-tube may differ in the case that the cross-section is a straight line (C'). In this case the boundaries of the D3-branes deform the D5-tube and source a magnetic BI field $B$ on it. If the D3-branes are replaced by an $AdS_5 \times S^5$ background then the D5-tubes intersect the boundary of $AdS_5$ (at $r \to \infty$) on a tubular defect with cross-section equal to that of the corresponding tube. The gauge group of the SYM theory is $SU(N)$ on one side of the planar defect associated to the tube with linear cross-section $C'$, and $SU(N')$ on the other.

In the second line the underlined direction indicates the orientation of the strings dissolved within the tube, whilst the remaining directions are those of the D3-branes also dissolved within the tube. The bullets indicate that the cross-section $C$ of the D5-tube is restricted here to be a curve in the 12-plane (we will consider the general case in Section 2). Since the D5-tube preserves the same supersymmetries as the collection of strings and D3-branes from which it originates, the intersection (1.3) is 1/8-supersymmetric. If the D3-branes are now replaced by the $AdS_5 \times S^5$ background then the resulting probe D5-tube ends at the $AdS_5$ boundary on a defect of geometry $\mathbb{R} \times C$ (see Figure 1). From the viewpoint of the $\mathcal{N} = 4$ SYM
theory we conclude that the addition of electric charge and angular momentum to a line-defect causes it to expand to a defect of one dimension higher with geometry $\mathbb{R} \times \mathbb{C}$ while leaving unbroken four of the eight Poincaré supersymmetries preserved by the flat defect; conformal (super)symmetry is of course completely broken for non-linear $C$. Analogous arguments show that a similar expansion occurs for the defects in columns I and II of Table 1, with the exception of those involving D7-branes.

An $\mathbb{R}^q$-defect can thus be regarded as an expanded $\mathbb{R}^{q-1}$-defect in the limit in which $C$ becomes a straight line; if in addition the electric and magnetic BI fields on the dual D-tube are set to zero then conformal (super)symmetry is restored and the resulting $\mathbb{R}^q$-defect reduces to one of the defects in Table 1. Similarly, an $\mathbb{R}^{q-1}$-defect may be viewed as a collapsed $q$-dimensional defect in the limit in which $C$ reduces to a point and the BI fields are set to zero. Thus all defects associated to the D3/$Dp$ intersections in column I may be considered in a unified way; the same applies to those defects in column II. We will see in Section 5 how defects in columns I and II are related by considering configurations with multiple defects.

The $\mathbb{R}^2$-defect is special because, being of codimension one, it can separate two different CFTs defined on either side of it; in particular, the ranks of their gauge groups may differ. In Section 4 we investigate whether this is also possible for the non-planar ($\mathbb{R} \times \mathbb{C}$)-defects uncovered here.

### 2. Anti-de Sitter Supertubes

In this section we will explicitly verify that a D5-tube embedded in a D3-brane background with the orientation

\[
\text{D3: } 1 \ 2 \ 3 \ \cdots \ \cdots \\
\text{D5-tube: } \bullet \bullet \ 3 \ 4 \ 5 \ \overline{6} \ \bullet \bullet
\]  

indeed preserves a fraction of supersymmetry, as stated in the Introduction; the calculation for any other $Dp$-tube is completely analogous. Note that (2.1) generalizes (1.3) in that the cross-section of the D5-tube is now a completely arbitrary curve in the $\mathbb{R}^5$-space corresponding to the 12789-directions; this is the D5-tube to which the D3/F1 system in (1.2) expands for a general angular momentum two-form in this $\mathbb{R}^5$-space.

The non-zero fields of the supergravity solution sourced by D3-branes take the form

\[
ds^2 = H^{-1/2} ds^2 (E^{(1,3)}) + H^{1/2} ds^2 (E^6), \\
F_5 = \omega + \ast \omega, \quad \omega = \text{vol} (E^{(1,3)}) \wedge dH^{-1}, \quad e^\phi = g_s,
\]

where $H$ is a harmonic function on $E^6$. If $H = 1$ this background reduces to flat space; if instead $H = 1 + R^4/r^4$, where $R^4 = g_s N$ and $r$ is the radial coordinate in $E^6$, then it describes

### Table 1: 1/4-supersymmetric D3/$Dp$ intersections.

| I | II |
|---|---|
| (0|D3 $\perp$ D1) | (0|D3 $\perp$ D5) |
| (1|D3 $\perp$ D3) | (1|D3 $\perp$ D7) |
| (2|D3 $\perp$ D5) | |
| (3|D3 $\perp$ D7) | |

The last entry in column I is not really associated to a defect because the intersection has codimension zero inside the D3-brane; rather, adding a probe D7-brane is associated to introducing dynamical ‘quarks’ (or, more precisely, matter in the fundamental representation of the gauge group) in the gauge theory [10].

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$N$ D3-branes at $r = 0$. The near-horizon limit is effectively obtained by removing the ‘1’ in the harmonic function, in which case the geometry becomes $AdS_5 \times S^5$. For the moment we need not choose a specific form for $H$ because most of the results we will obtain do not depend on it.

In view of (2.1) we denote the coordinates in $E^{(1,3)}$ as $x = (x^0, x^3)$ and $X = (X^a)$, where $a = 1, 2$, and for those in $E^6$ we use $y = (y^m)$, where $m = 4, 5, 6$, and $Y = (Y^7, Y^8, Y^9)$. Since the D5-tube extends along the 03456-directions, we identify $x$ and $y$ with worldvolume coordinates. The additional worldvolume coordinate $\sigma$ parametrizes the cross-section $\mathcal{C}$, which is specified by

$$X = X(\sigma), \quad Y = Y(\sigma). \quad (2.3)$$

The BI field strength on the D5-tube takes the form

$$F = \mathcal{E} \, dx^0 \wedge dy^6 + \mathcal{B} \, dy^6 \wedge d\sigma. \quad (2.4)$$

The electric and magnetic components $\mathcal{E}$ and $\mathcal{B}$ correspond, respectively, to fundamental strings along the 6-direction and D3-branes along the 345-directions dissolved in the D5-brane. The Poynting momentum-density generated by the crossed electric and magnetic fields is responsible for the stability of the tube with an arbitrary cross-section [7, 8].

The supersymmetries preserved by a D5-brane are those generated by Killing spinors $\epsilon$ of the background that satisfy (see, for example, [11])

$$\Gamma \epsilon = \epsilon, \quad (2.5)$$

where $\Gamma$ is the matrix appearing in the kappa-symmetry transformations of the D5-brane worldvolume fermions [12, 13]. For the case of interest here we have

$$\Delta \Gamma = \mathcal{B} \Gamma_{0345} I - \gamma \Gamma_{345} (\Gamma_{06} K - \mathcal{E}), \quad (2.6)$$

where $\{\Gamma_0, \ldots, \Gamma_9\}$ are ten flat-space constant Dirac matrices, $K$ and $I$ are operators that act on $SO(1, 9)$ chiral complex spinors as

$$K \psi = \psi^*, \quad I \psi = -i \psi, \quad (2.7)$$

and $\Delta = \sqrt{-\det(g + F)}$ is the DBI determinant, which in the present case is given by

$$\Delta = \sqrt{\mathcal{B}^2 + (1 - \mathcal{E}^2)(|\partial_\sigma X|^2 + \mathcal{H}|\partial_\sigma Y|^2)}. \quad (2.8)$$

We have also introduced

$$\gamma = \sum_{i=1}^{2} \Gamma_i \partial_\sigma X^i + H^{1/2} \sum_{i=7}^{9} \Gamma_i \partial_\sigma Y^i. \quad (2.9)$$

For generic $H$ the Killing spinors of the background (2.2) take the form

$$\epsilon = H^{-1/8} \epsilon_0, \quad (2.10)$$
where $\epsilon_0$ is a constant spinor subject to the constraint
\begin{equation}
\Gamma_{0123} I \epsilon_0 = \epsilon_0
\end{equation}
(2.11)
associated to the presence of D3-branes along the 123-directions. This means that the background is invariant under sixteen real supersymmetries. This is enhanced to thirty-two supersymmetries in two particular cases. If $H = 1$ then the background is just the Minkowski vacuum and $\epsilon_0$ is unconstrained. If $H = R^4/r^4$ then the background is $AdS_5 \times S^5$ and there are sixteen special conformal supersymmetries in addition to the Poincaré supersymmetries generated by $\epsilon_0$. We will not need the specific form of the conformal supersymmetries in this section.

Equation (2.5) is satisfied for arbitrary cross-section if
\begin{equation}
|\mathcal{E}| = 1,
\end{equation}
(2.12)
$B$ is a constant-sign function on the D5-brane worldvolume, and $\epsilon_0$ satisfies the additional constraints
\begin{equation}
\Gamma_{06} K \epsilon_0 = sgn(\mathcal{E}) \epsilon_0, \quad \Gamma_{0345} I \epsilon_0 = sgn(B) \epsilon_0.
\end{equation}
(2.13)
These are respectively associated to the presence of string charge in the 6-direction and D3-brane charge in the 345-directions, as expected from the charges carried by the D5-tube. Since these constraints are compatible with each other and with (2.2), the D5-tube preserves four of the sixteen supersymmetries of the D3-brane background$^3$. Note that, as anticipated, there is no projection corresponding to the presence of the D5-brane. Although supersymmetry allows $B$ to depend on all the worldvolume coordinates, the Bianchi identity $dF = 0$ restricts it to depend only on $y^6$ and $\sigma$, and the equations of motion for $F$ further disallow the dependence on $y^6$. Hence we conclude that $B$ may be a constant-sign but otherwise arbitrary function of $\sigma$. We would like to emphasize that if $B \neq 0$ then $|\mathcal{E}| = 1$ is a subcritical electric field, by which we mean that the DBI Lagrangian is real and nonzero. As anticipated, the BPS equations (that is, the conditions on $E$ and $B$) and the projections (2.13) we have derived for preservation of supersymmetry are independent of the form of $H$.

The supersymmetry preserved by the D5-tube is enhanced if the cross-section is a straight line in the 12-plane and $B$ is constant: in this case the D5-tube preserves eight supersymmetries for any (subcritical) constant value of $\mathcal{E}$, because the matrix $\Gamma$ becomes a constant matrix that commutes with that in equation (2.11).

3. Features of Expanded Defects

We will now analyze in more detail the features of the expanded defects in the $\mathcal{N} = 4$ SYM theory uncovered here, so we set $H = R^4/r^4$. Again we concentrate on the D5-tube for concreteness since the extension to other D$p$-tubes is straightforward.

$^3$If $H = 1$ the D5-tube preserves sixteen of the thirty-two Minkowski supercharges; the case of $AdS_5 \times S^5$ is discussed below.

$^4$We have verified that all the remaining D5-brane equations of motion are satisfied under these circumstances.
The probe D5-tube is embedded in $\text{AdS}_5 \times S^5$ in a way determined by the cross-section $C$. The two particular cases in which $C$ lies entirely along the 12-directions, which we refer to as ‘expansion along $\text{AdS}_5$’, or along the 789-directions, which we refer to as ‘expansion along $S^5$’, are especially interesting.

### 3.1 Expansion along $\text{AdS}_5$

In this case $Y$ is constant and the D5-tube intersects the boundary of $\text{AdS}_5$ on a defect of geometry $\mathbb{R} \times C$, where $C$ is a curve in the 12-plane. The induced metric on the D5-tube is

$$ds^2 = \frac{y^2 + Y^2}{R^2} (-dx_0^2 + d\sigma^2 + dx_3^2) + \frac{R^2}{y^2 + Y^2} dy^2 + \frac{R^2 y^2}{y^2 + Y^2} ds^2 (S^2), \quad (3.1)$$

where $y = |y|$, $Y = |Y|$ and we have set $|\partial_\mu X|^2 = 1$ without loss of generality by assuming an affine parametrization.

Let us first suppose that $Y = 0$. Although in this case the metric is that of $\text{AdS}_4 \times S^2$, this does not imply that part of the conformal isometries of $\text{AdS}_5$ are preserved, because the submanifold occupied by the brane is not mapped into itself by these isometries unless $C$ is a straight line. From the viewpoint of the gauge theory it is clear that conformal symmetry (as well as translational and rotational symmetry in the 12-plane) must be broken if $C$ is not a straight line since its size (or radius of curvature) introduces a scale. Similarly, conformal supersymmetry of $\text{AdS}_5 \times S^5$ is completely broken if the cross-section of the D5-tube is not a straight line, as we show in the Appendix. It follows that the corresponding tubular defect with generic $C$ breaks all the conformal supersymmetries of the $\mathcal{N} = 4$ SYM theory; of course, it preserves four of the sixteen Poincaré supersymmetries, as we saw in the previous section.

It is thus clear that a necessary condition for preservation of conformal symmetry is that the cross-section of the D5-tube be a straight line, in which case the submanifold occupied by the D5-tube is invariant under the conformal isometries of $\text{AdS}_5$. However, this condition is not sufficient because the BI field strength on the worldvolume of the D5-tube is not invariant unless $\mathcal{E}$ and $\mathcal{B}$ are set to zero (recall that precisely for a linear cross-section in the 12-plane this can be done while preserving Poincaré supersymmetry). We thus conclude that (part of) the conformal symmetries of $\text{AdS}_5$ are preserved by the D5-tube if and only if its cross-section is a straight line and $\mathcal{E} = \mathcal{B} = 0$; we show in the Appendix that the same applies for preservation of conformal supersymmetry. Under these circumstances the D5-tube reduces to an ordinary D5-brane, the intersection (2.1) becomes a $(2|D3 \perp D5)$ intersection, and the corresponding $\mathbb{R}^2$-defect preserves (part of) the conformal invariance of the $\mathcal{N} = 4$ SYM theory.

If $Y \neq 0$ then the metric (3.1) is that of $\text{AdS}_4 \times S^2$ for $y \gg Y$ but deviates from it in the interior of $\text{AdS}_5$. This means that the theory on the defect is not conformal even if $C$ is a straight line and $\mathcal{E} = \mathcal{B} = 0$. In this case conformal invariance is broken because for non-zero $Y$ the D3-D5 open strings, and hence the corresponding degrees of freedom on the defect, are non-zero.

5Technically this means that its Lie derivative along the Killing vector fields that generate these isometries is non-zero.
massive\(^6\). The fact that the embedding becomes conformal for large \(y\) reflects the fact that these masses become negligible in the ultra-violet limit.

### 3.2 Expansion along \(S^5\)
Here we shall directly concentrate on the case in which \(C\) is a straight line. Since \(X\) is constant the induced metric on the D5-tube is

\[
ds^2 = \frac{r^2}{R^2} (-dx_0^2 + dx_3^2) + \frac{R^2}{r^2} ds^2 + R^2 ds^2(S^3).
\]

Although the submanifold occupied by the D5-tube is indeed invariant under part of the conformal symmetries of \(AdS\), as suggested by the \(AdS_3 \times S^3\) form of the induced metric, these symmetries are again broken by the BI field strength. Note that setting the BI fields to zero is now incompatible with supersymmetry, since a pure D5-brane embedded as an \(AdS_3 \times S^3\) submanifold is not supersymmetric\(^7\) [6], as could be expected from the fact that it would seem to originate from a \((1|D3 \perp D5)\) intersection. It is precisely the non-zero worldvolume electromagnetic fields which make supersymmetry possible in this case. The dual defect is a supersymmetric non-conformal line-defect.

### 4. D3-branes Ending on D5-tubes
The \(\mathbb{R}^2\)-defect dual to the \((2|D3 \perp D5)\) intersection permits the CFTs on either side of it to be different; the case of interest here is when the ranks of the gauge groups, \(N\) and \(N'\), differ. In the brane construction, this means that some D3-branes actually end on the D5-brane rather than intersect it, so that the difference in the numbers of D3-branes on either side is \(\delta N \equiv N - N' \neq 0\). This imbalance has two effects on the D5-brane probe [1] (see Figure 1). Firstly, it induces \(\delta N\) units of BI magnetic flux through the two-sphere in the 456-space, because the two-dimensional boundary of a D3-brane acts as a magnetic monopole charge in the five-dimensional worldspace of the D5-brane. Secondly, the D5-brane worldspace is deformed close to the boundaries of the D3-branes due to the imbalance in the forces exerted by their tensions.

Given the existence of supersymmetric (that is, stable) tubular defects with arbitrary cross-section, the question of whether the gauge group can also jump across these defects while preserving supersymmetry arises. To answer this we will look for the corresponding dual D5-tubes in \(AdS_5 \times S^5\). We will argue that in the probe approximation used here these do not exist, and then discuss the meaning of this result and the limitations of the approximation.

The D5-tubes we seek should be oriented as indicated by (1.3), so we set \(Y = 0\), but the cross-section specified before by \(X(\sigma)\) should now be allowed to vary in the 456-directions to

\(^6\)Note that this breaking of conformal invariance is explicit and supersymmetry-preserving, as opposed to the non-supersymmetry-preserving breaking by vacuum expectation values discussed in [6].

\(^7\)Technically this manifests itself in the fact that the matrix \(\Gamma\) anti-commutes with that in equation (2.2) if the cross-section lies in the 789-space and \(E = B = 0\).
describe the deformation due to the D3-brane tension imbalance; in other words, we should now have \( X = X(\sigma, y) \). We must also generalize the form of the BI field strength (2.4) to incorporate the magnetic flux sourced by the D3-brane boundaries. Therefore we must set

\[
F = \mathcal{E} \, dx^0 \wedge dy^6 + \mathcal{B} \, dy^b \wedge d\sigma + \frac{1}{2} \epsilon_{mnp} B^m \, dy^n \wedge dy^p
\]  

(4.1)

and require

\[
\delta N \equiv \frac{1}{2\pi} \int_{S^2} \vec{B} \cdot d\vec{S} \neq 0,
\]

(4.2)

where \( \vec{B} = (B^m) \) and \( S^2 \) is a two-sphere in the 456-space. Recall that \( a, b, \ldots = 1, 2 \) and \( m, n, \ldots = 4, 5, 6 \). Since we will need to distinguish between the 45-directions and the 6-direction because the symmetry between them is broken by \( \mathcal{E} \) and \( \mathcal{B} \), we will further set \( y = (y^m) = (y^l, y^6) \) with \( i, j = 4, 5 \). Note that in principle we should expect \( \mathcal{B} \) and \( B^m \) to depend on \( \sigma \) and \( y^m \), of course subject to the conditions

\[
\partial_m B^m = 0, \\
\partial_\sigma B_4 = \partial_5 \mathcal{B}, \\
\partial_\sigma B_5 = -\partial_4 \mathcal{B}, \\
\partial_\sigma B_6 = 0
\]

(4.3) - (4.6)

implied by the Bianchi identity \( dF = 0 \).

Under these circumstances the kappa-symmetry matrix takes the form

\[
\Delta \Gamma = \mathcal{B} \Gamma_{0345} I + H^{-1} B^m \partial_{[m} X^1 \partial_{\sigma]} X^2 \Gamma_{0123} I \\
+ \left[ \Gamma_{345} \left( \gamma + H^{-1/2} \partial_i X^1 \partial_{\sigma} X^2 \Gamma_{i12} \right) I + H^{-1/2} B_6 \Gamma_{3} \gamma K I \right] (\Gamma_{06} K - \mathcal{E}) \\
+ \Gamma_{03} K I \left( \mathcal{B} B_6 + \Gamma_{1245} \partial_{[6} X^1 \partial_{\sigma]} X^2 \right) + H^{-1/2} \left[ \mathcal{B} \partial_a X^a \Gamma_{ai} \Gamma_{0345} I + B_i \partial_\sigma X^{[a} \Gamma_{i} \Gamma_{0123} I \right] \\
+ H^{-1} \left[ \frac{1}{6} \Gamma_{0123} \gamma K I \partial_{[m} \partial_{n} X^1 \partial_{p]} X^2 + \Gamma_{0123} I \left( \mathcal{B} - \frac{1}{3} \mathcal{E} \Gamma_0 \gamma \right) \partial_{[4} X^1 \partial_{5]} X^2 \\
+ \frac{1}{3} \Gamma_{03a} K I \delta_{bc} \Gamma_{[m} \partial_{n} X^a \partial_{p]} X^b \partial_\sigma X^c - \frac{1}{3} \mathcal{E} \Gamma_{3a} I \delta_{bc} \partial_{[i} X^a \partial_{j]} X^b \partial_\sigma X^c \right],
\]

(4.7)

where

\[
\gamma = \partial_\sigma X^a \Gamma_a.
\]

(4.8)

It follows that equation (2.5) is satisfied if we impose the projections (2.11) and (2.13), as well as the BPS conditions (2.12) and

\[
B_i \partial_\sigma X^a = -B \epsilon^{ab} \partial_i X^b, \\
\mathcal{B} B_6 = \partial_{[6} X^1 \partial_{\sigma]} X^2, \\
0 = \partial_{[m} X^1 \partial_{n]} X^2,
\]

(4.9) - (4.11)
in which case the DBI determinant reduces to
\[ \Delta = \left| \mathcal{B} + H^{-1} B^m \partial_{[m} X^1 \partial_{\sigma]} X^2 \right|. \] (4.12)

Note the non-linearity of the BPS equations, as well as the asymmetry between the 45- and the 6-directions.

The projections above are expected given the charges carried by the D5-tube. In particular, we need to impose condition (2.11) even if \( H = 1 \), that is, even if there are no background D3-branes, because these should arise here as a ‘spike’ excitation on the D5-tube described by \( X(\sigma, y) \).

It is well-known that the rank of the gauge group may change while preserving supersymmetry across an \( \mathbb{R}^2 \)-defect on which \( \mathcal{E} = \mathcal{B} = 0 \) [1]. Let us therefore first show that this is also possible with non-zero BI fields. By rotational invariance in the 12-plane, we assume without loss of generality that the linear cross-section is aligned with the 2-axis, that is, we set
\[ X^2 = \sigma, \quad X^1 = X(y). \] (4.13)

The BPS equations then simplify to
\[ B_i = \mathcal{B} \partial_i X, \quad \mathcal{B} B_6 = \partial_6 X. \] (4.14)

Under these circumstances the equations of motion for the BI gauge field on the D5-brane in the background (2.2) reduce to the conditions that \( \mathcal{B} \) is constant and that
\[ \partial_4^2 X + \partial_5^2 X + \mathcal{B}^{-2} \partial_6^2 X = 0. \] (4.15)

The latter equation coincides with the divergence-free condition (4.3) for \( B^m \) upon using (4.14); since \( \mathcal{B} \) is constant, it is solved by any harmonic function of \( \tilde{y} \), where \( \tilde{y}^i = y^i \) and \( \tilde{y}^6 = B^6 y^6 \). The choice \( X = \delta N / 4\pi |\tilde{y}| \) satisfies (4.2), and represents \( \delta N \) open D3-branes ending on the D5-tube at \( y = 0 \).

We now turn to non-linear cross-sections. We will show that the cross-section cannot be non-linear unless \( \delta N = 0 \). It will suffice to prove this for a circular cross-section since any curve is approximated locally by a circle, and the condition for preservation of supersymmetry (2.5) is local. Let us therefore set
\[ X^1 = R(y) \sin \sigma, \quad X^2 = R(y) \cos \sigma. \] (4.16)

The BPS equations (4.9)–(4.11) then reduce to
\[ R B_i = \mathcal{B} \partial_i R, \quad \mathcal{B} B_6 = R \partial_6 R. \] (4.17)

Since \( R \) is \( \sigma \)-independent, these equations together with (4.4)–(4.6) imply that \( B^m \) cannot depend on \( \sigma \), and also that \( \mathcal{B} \) may only depend on \( y^6 \). The D5-brane equations of motion then force \( R \) to take the form
\[ R(y) = f(y^4, y^5) |\mathcal{B}(y^6)|^{1/2}, \] (4.18)
where \( f \) and \( B \) must satisfy

\[
(\partial_4^2 + \partial_5^2) \log f = \lambda f^2, \quad \partial_6^2 \log B = -2\lambda B,
\]

for some undetermined constant \( \lambda \). We seek a solution for which \( R \) tends to a non-zero constant far away from the D3-brane boundaries, which is at \( y \to \infty \) if we assume (without loss of generality) that the D3-branes end on the D5-tube at \( y = 0 \). Since this means that the left-hand sides of (4.19) vanish asymptotically, we must set \( \lambda = 0 \). If we additionally restrict ourselves to solutions invariant under rotations in the 45-plane, then

\[
f = a (y_4^2 + y_5^2)^b, \quad B = ce^{dy^6},
\]

for constants \( a, b, c, d \). The requirement that \( R \) tends to a non-zero constant implies that \( c = d = 0 \), in which case \( R \) is everywhere constant and \( B^m = 0 \). This is therefore a particular case of the general D5-tube with an arbitrary but \( y \)-independent cross-section and \( \delta N = 0 \).

Although the above result was derived for a particular ansatz for the BI fields (4.1), this ansatz is grounded in a physical understanding of supertubes, and hence we consider it unlikely that there exists a supersymmetric solution not captured by this ansatz. Another caveat is that effects beyond the probe approximation could allow a supersymmetric non-linear cross-section provided that its radius of curvature is of the order \( N/\delta N \), consistent with the fact that in the limit \( \delta N/N \to 0 \) only linear cross-sections are supersymmetric.

5. Multiple Defects

In this section we discuss configurations with multiple defects, some of which involve defects from both columns I and II in Table 1.

A first observation is that any number of defects with geometries \( \mathbb{R}^q \times C_i \) (for fixed \( q \)), where the \( \mathbb{R}^q \)-factors are mutually-aligned and the cross-sections \( C_i \) are arbitrary, preserve 1/4 of the Poincaré supersymmetries of the \( \mathcal{N} = 4 \) SYM theory. (Three of them with \( q = 1 \) have been depicted in Figure 1.) This follows immediately from the fact that \( D_p \)-tubes with mutually-aligned axes and arbitrary cross-sections preserve 1/4-supersymmetry, as was shown in \([8]\) by exhibiting a 1/4-supersymmetric supergravity solution representing such a multi-tube configuration\(^8\). Note that this includes as a particular case parallel defect/anti-defect configurations, for which the dual system consists of a supersymmetric parallel brane/anti-brane pair \([8, 16]\).

We have shown that the line-defect of column I can be viewed as a collapsed tubular two-dimensional defect. This suggests that the line-defect could have an endpoint on a planar \( \mathbb{R}^2 \)-defect, or that it could even be smoothly attached to it. The dual configuration in \( \text{AdS}_5 \times S^5 \)

\(^8\)For circular cross-sections the supergravity solution was first found in \([14]\), and solutions in Matrix theory were constructed in \([15]\).
that realizes this possibility is described by the array

\[
\begin{align*}
\text{D3:} & \quad 1 \ 2 \ 3 \ 
\vdots \ 
\vdots \\
\text{D5:} & \quad \_ \ 2 \ 3 \ 4 \ 5 \ 6 \ 
\vdots \\
\text{D3:} & \quad 1 \ 
\vdots \ 4 \ 5 \ 
\vdots
\end{align*}
\]

As usual the D3-branes at the top provide the \( AdS_5 \times S^5 \) background. The probe D3-brane at the bottom ends on the probe D5-brane, that is, it extends only along the (say) positive values of the 1-coordinate. From the viewpoint of the SYM theory this represents a line-defect ending on an \( \mathbb{R}^2 \)-defect. Since the D3-brane can be viewed as a smooth solitonic excitation of the D5-brane [17, 18], one might expect that the line defect can be similarly viewed as a solitonic excitation of the conformal field theory on the \( \mathbb{R}^2 \)-defect as depicted in Figure 2.

Figure 2: A line-defect ending on an \( \mathbb{R}^2 \)-defect.

The CFT on the \( \mathbb{R}^2 \)-defect associated to the \( (2|D3 \perp D5) \) intersection was deduced in [5], following the earlier work of [3, 4]. Essentially, it is a linear \( \mathcal{N} = 4 \ d = 3 \) sigma-model with 4\( N \) scalar fields but with additional couplings to the pullbacks of the \( d = 4 \) SYM fields\(^9\). A vortex solution of this theory could interpolate between a region of non-zero scalar fields at the vortex core to an asymptotic region of vanishing scalar fields far away from the core. Due to the coupling to the four-dimensional background fields, non-vanishing scalar fields on the defect imply discontinuities of the Higgs fields across the defect. To take this properly into account would require going beyond the probe approximation.

Given that a line-defect can end on a planar defect, one may wonder whether a tubular defect formed by expansion of a line-defect can also end on a planar defect. This would seem

\(^9\)The same field theory, but in a lower dimension, should therefore give the dynamics of the line- and point-defects associated to the \( (1|D3 \perp D3) \) and \( (0|D3 \perp D1) \) intersections because each of them can be regarded as a collapsed \( \mathbb{R}^2 \)-defect.
to require a corresponding bulk configuration in which a D5-tube has a boundary on a D5-brane, and there is no known supersymmetric configuration of this kind; a possible resolution is that the D5-brane expands *locally* to a D7-tube, on which the D5-tube can end. This illustrates a rather general problem in any attempt to find a supertube ending on another brane, as opposed to merely intersecting it. An examination of all possibilities [19] shows that the only case in which the problem does not arise is that recently discussed [9] of a D2-tube ending on a D4-brane, and the T-dual cases in which a D*p*-tube ends on a D*(p + 2)*-brane. Note that a naive application of T-duality rules to systems with branes ending on branes can lead to incorrect results. For example, T-duality along a direction longitudinal to the D4-brane and orthogonal to the D2-brane that ends on it might seem to transform this system into one in which a D3-brane has a one-dimensional boundary (on another D3-brane), which is impossible.

The line-defect of column I in Table 1 can also end on the point-defect associated to the (0|D3 ⊥ D5) intersection of column II (see Figure 3). The corresponding brane configuration is

\[
\begin{align*}
\text{D3:} & \quad 1 \ 2 \ 3 \ - \ - \ - \ - \\
\text{D5:} & \quad - \ - \ 4 \ 5 \ 6 \ 7 \ 8 \ - \\
\text{D3:} & \quad - \ - \ 3 \ 4 \ 5 \ - \ - \ - ,
\end{align*}
\]  

(5.2)

where the probe D3-brane at the bottom ends on the probe D5-brane. If strings along the 9-direction and angular momentum in the 12-plane are added, then both probes can expand supersymmetrically yielding a configuration in which a D5-tube ends on a D7-tube:

\[
\begin{align*}
\text{D3:} & \quad 1 \ 2 \ 3 \ - \ - \ - \ - \\
\text{D7-tube:} & \quad \bullet \ \bullet \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \\
\text{D5-tube:} & \quad \bullet \ \bullet \ 3 \ 4 \ 5 \ - \ - \ 9 ,
\end{align*}
\]  

(5.3)

Both tubes share a common cross-section \( C \) in the 12-plane. From the viewpoint of the CFT theory, a tubular \( \mathbb{R} \times C \)-defect with axis along the 3-direction ends on a one-dimensional \( C \)-defect (see Figure 3). In the limit in which \( C \) becomes a straight line, the \( C \)-defect reduces to the line-defect associated to the second intersection in column II of Table 1, and the configuration above becomes an \( \mathbb{R}^2 \)-defect with a boundary line-defect. For a line in the (say)
2-direction this corresponds to the array

\[
\begin{align*}
\text{D3:} & \quad 1 \ 2 \ 3 \ - \ - \ - \\
\text{D7:} & \quad - \ 2 \ - 4 \ 5 \ 6 \ 7 \ 8 \ 9 \\
\text{D5:} & \quad - \ 2 \ 3 \ 4 \ 5 \ - \ - \ - 9 .
\end{align*}
\]

(A.4)

\[5.4\]

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A. Special Conformal Supersymmetries

If the cross-section of the D5-tube is a line in the 12-plane (at \(Y = 0\)) and \(E = B = 0\) then eight of the sixteen special conformal supersymmetries of \(AdS_5 \times S^5\) are preserved (together with eight of the sixteen Poincaré supersymmetries) \([6]\). We show here that these conditions are not only sufficient but also necessary for preservation of conformal supersymmetry, as stated in Section 3.1.

It is convenient to set \(R = 1\) and to use spherical coordinates \(\{r, \theta_\alpha\} (\alpha = 1, \ldots, 5)\) for the \(\mathbb{R}^6\)-space transverse to the D3-branes such that \(Y = 0\) corresponds to \(\theta_1 = \theta_2 = \theta_3 = \pi/2\). The D5-tube then wraps the \(S^2\) within the \(S^5\) parametrized by \(\theta_4, \theta_5\), and its embedding in \(AdS_5\) is specified by \(X(\sigma)\). In these coordinates the BI field strength (2.4) and the DBI determinant take the form

\[
F = \mathcal{E} \cos \theta_4 \, dx_0 \wedge dr - \mathcal{E} r \sin \theta_4 \, dx_0 \wedge d\theta_4 + \mathcal{B} \cos \theta_4 \, dr \wedge d\sigma - \mathcal{B} r \sin \theta_4 \, d\theta_4 \wedge d\sigma ,
\]

(A.1)

\[
\Delta = r^2 \sin \theta_4 \sqrt{\mathcal{B}^2 + (1 - \mathcal{E}^2)|\partial_\sigma X|^2} ,
\]

(A.2)

whereas the kappa-symmetry matrix is

\[
\Delta \Gamma = r^2 \sin \theta_4 \Gamma_{\theta_4} I \left\{ [\Gamma_0 \theta_4 K + \mathcal{E} (\cos \theta_4 \Gamma_0 + \sin \theta_4 \Gamma_r)] \gamma - \mathcal{B} \Gamma_0 (\cos \theta_4 \Gamma_0 + \sin \theta_4 \Gamma_r) \right\} .
\]

(A.3)

Here \(\gamma\) is as defined in (4.8) and \(\{\Gamma_0, \ldots, \Gamma_3, \Gamma_r, \Gamma_{\theta_1}, \ldots, \Gamma_{\theta_5}\}\) are ten tangent-space constant Dirac matrices.

Following \([6]\) closely we write the Killing spinors for \(AdS_5 \times S^5\) as

\[
\epsilon = r^{1/2} h(\theta_\alpha) \eta_1 + \left[ -r^{-1/2} \Gamma_r + r^{1/2} (x_0 \Gamma_0 + X^a \Gamma_a + x_3 \Gamma_3) \right] h(\theta_\alpha) \eta_2 ,
\]

(A.4)

where \(\eta_1\) and \(\eta_2\) are constant sixteen-component complex spinors of negative and positive chirality respectively, subject to the additional constraints

\[
\Gamma_{0123} I \eta_1 = \eta_1 , \quad \Gamma_{0123} I \eta_2 = -\eta_2 ,
\]

(A.5)
and the matrix

\[ h(\theta_\alpha) = \exp \left( \frac{\theta_1}{2} \Gamma_{r \theta_1} \right) \exp \left( \frac{\theta_2}{2} \Gamma_{\theta_1 \theta_2} \right) \exp \left( \frac{\theta_3}{2} \Gamma_{\theta_2 \theta_3} \right) \exp \left( \frac{\theta_4}{2} \Gamma_{\theta_3 \theta_4} \right) \exp \left( \frac{\theta_5}{2} \Gamma_{\theta_4 \theta_5} \right) \quad (A.6) \]

accounts for the tangent-space rotation induced by the change from Cartesian to spherical coordinates in \( \mathbb{R}^6 \). If \( \eta_2 = 0 \) then the remaining eight independent complex Killing spinors are the Poincaré spinors \((2.10)\). The Killing spinors \( \epsilon \) with \( \eta_2 \neq 0 \) (but \( \eta_1 \) not necessarily zero) correspond to special conformal supersymmetries, as indicated by their dependence on the coordinates along the D3-branes.

Since equation \((2.5)\) must hold at all points of the D5-tube, terms with different functional dependencies on the worldvolume coordinates must vanish independently. The term with \( r^{3/2} \) dependence in \((2.5)\) reduces to

\[ \Gamma_{3\theta_5} I \left\{ [-\Gamma_{0 \theta_4} K + \mathcal{E} (\cos \theta_4 \Gamma_4 - \sin \theta_4 \Gamma_r)] \gamma - \mathcal{B} \Gamma_0 (\cos \theta_4 \Gamma_4 - \sin \theta_4 \Gamma_r) \right\} h(\theta_\alpha) \eta_2 = \sqrt{B^2 + (1 - \mathcal{E}^2) |\partial_\sigma X|^2} h(\theta_\alpha) \eta_2 . \quad (A.7) \]

The term with \( x_0 \) dependence similarly reduces to

\[ \Gamma_{3\theta_5} I \left\{ [-\Gamma_{0 \theta_4} K + \mathcal{E} (\cos \theta_4 \Gamma_4 + \sin \theta_4 \Gamma_r)] \gamma + \mathcal{B} \Gamma_0 (\cos \theta_4 \Gamma_4 + \sin \theta_4 \Gamma_r) \right\} h(\theta_\alpha) \eta_2 = \sqrt{B^2 + (1 - \mathcal{E}^2) |\partial_\sigma X|^2} h(\theta_\alpha) \eta_2 . \quad (A.8) \]

Subtracting the two equations above, we see that

\[ \left[ \mathcal{E} \sin \theta_4 \Gamma_r \gamma + \mathcal{B} \cos \theta_4 \Gamma_0 \theta_4 \right] h(\theta_\alpha) \eta_2 = 0 . \quad (A.9) \]

Using the commutation properties of the gamma matrices to bring \( h(\theta_\alpha) \) through to the left, we find that

\[ \left[ \exp \left( \frac{\theta_5}{2} \Gamma_{\theta_4 \theta_5} \right) \Gamma_{\theta_4} \gamma \sin \theta_4 \mathcal{E} - \exp \left( -\frac{\theta_5}{2} \Gamma_{\theta_4 \theta_5} \right) \Gamma_{\theta_5} \cos \theta_4 \mathcal{B} \right] \eta_2 = 0 . \quad (A.10) \]

Since the two terms must vanish independently for all \( \theta_4, \theta_5 \) and the matrices in each term are invertible, we see that this equation can be satisfied with non-zero \( \eta_2 \) only if \( \mathcal{E} = \mathcal{B} = 0 \). If we substitute this condition back into \((A.8)\) we find that

\[ \Gamma_{03r \theta_4 \theta_5} K I \frac{\gamma}{|\partial_\sigma X|^2} \tilde{\eta} = \tilde{\eta} , \quad (A.11) \]

where \( \tilde{\eta} = h(\theta_\alpha) \eta_2 \). Fixing \( \theta_\alpha \) and demanding that this equation holds for all \( \sigma \) for some non-zero \( \eta_2 \) (note that \( h(\theta_\alpha) \) is invertible) requires that the matrix \( \gamma / |\partial_\sigma X| \) is constant, which in turn implies that \( \partial_\sigma X^a \) are constants, that is, that the cross-section is a straight line.
References

[1] A. Karch and L. Randall, Open and Closed String Interpretation of SUSY CFT’s on Branes with Boundaries, J. High Energy Phys. 06 (2001) 063, hep-th/0105132.

[2] J. M. Maldacena, The Large N Limit of Superconformal Field Theories and Supergravity, Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200.

[3] S. Sethi, The Matrix Formulation of Type IIB Five-Branes, Nucl. Phys. B 523 (1998) 158, hep-th/9710005.

[4] A. Kapustin and S. Sethi, The Higgs Branch of Impurity Theories, Adv. Theor. Math. Phys. 2 (1998) 571, hep-th/9804027.

[5] O. DeWolfe, D. Z. Freedman and H. Ooguri, Holography and Defect Conformal Field Theories, Phys. Rev. D 66 (2002) 025009, hep-th/0111135.

[6] K. Skenderis and M. Taylor, Branes in AdS and pp-wave Spacetimes, J. High Energy Phys. 06 (2002) 025, hep-th/0204054.

[7] D. Mateos and P. K. Townsend, Supertubes, Phys. Rev. Lett. 87 (2001) 011602, hep-th/0103030.

[8] D. Mateos, S. Ng and P. K. Townsend, Tachyons, Supertubes and Brane/Anti-Brane Systems, J. High Energy Phys. 03 (2002) 016, hep-th/0112054.

[9] M. Kruczenski, R. C. Myers, A. W. Peet and D. J. Winters, Aspects of Supertubes, J. High Energy Phys. 05 (2002) 017, hep-th/0204103.

[10] A. Karch and E. Katz, Adding Flavor to AdS/CFT, J. High Energy Phys. 06 (2002) 043, hep-th/0205236.

[11] E. Bergshoeff, R. Kallosh, T. Ortín and G. Papadopoulos, Kappa-symmetry, supersymmetry and intersecting branes, Nucl. Phys. B 502 (1997) 149, hep-th/9705040.

[12] M. Cederwall, A. von Gussich, B. E. W. Nilsson, P. Sundell and A. Westerberg, The Dirichlet Super-p-Branes in Ten-Dimensional Type IIA and IIB Supergravity, Nucl. Phys. B 490 (1997) 179, hep-th/9611159.

[13] E. Bergshoeff and P. K. Townsend, Super D-branes, Nucl. Phys. B 490 (1997) 145, hep-th/9611173.

[14] R. Emparan, D. Mateos and P. K. Townsend, Supergravity Supertubes, J. High Energy Phys. 07 (2001) 011, hep-th/0106012.

[15] D. Bak and K. Lee, Noncommutative Supersymmetric Tubes, Phys. Lett. B 509 (2001) 168, hep-th/0103148.

[16] D. Bak and A. Karch, Supersymmetric Brane-Antibrane Configurations, Nucl. Phys. B 626 (2002) 165, hep-th/0110039.

[17] C. G. Callan and J. M. Maldacena, Brane Dynamics From the Born-Infeld Action, Nucl. Phys. B 513 (1998) 198, hep-th/9708147.

[18] G. W. Gibbons, Born-Infeld Particles and Dirichlet p-branes, Nucl. Phys. B 514 (1998) 603, hep-th/9709027.

[19] D. Mateos and S. Ng, Supertubes Ending on D4-branes, unpublished, 2001; to appear in S. Ng, Aspects of Branes in String Theory, PhD thesis, University of Cambridge, 2002.