THE TORELLI MAP RESTRICTED TO THE HYPERELLIPTIC LOCUS

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Abstract. Let $g \geq 2$ and let the Torelli map denote the map sending a genus $g$ curve to its principally polarized Jacobian. We show that the restriction of the Torelli map to the hyperelliptic locus is an immersion in characteristic not 2. In characteristic 2, we show the Torelli map restricted to the hyperelliptic locus fails to be an immersion because it is generically inseparable; moreover, the induced map on tangent spaces has kernel of dimension $g - 2$ at every point.

1. Introduction

Let $\mathcal{H}_g$ denote the moduli stack of hyperelliptic curves of genus $g$ over $\text{Spec} \mathbb{Z}$, $\mathcal{M}_g$ denote the moduli stack of curves of genus $g$ over $\text{Spec} \mathbb{Z}$, and $\mathcal{A}_g$ denote the moduli stack of principally polarized abelian varieties of dimension $g$ over $\text{Spec} \mathbb{Z}$. For $R$ a ring, we use $\mathcal{H}_{g,R}$, $\mathcal{M}_{g,R}$, $\mathcal{A}_{g,R}$ to denote the base changes of $\mathcal{H}_g$, $\mathcal{M}_g$, and $\mathcal{A}_g$ over $\text{Spec} \mathbb{Z}$ along $\text{Spec} R \to \text{Spec} \mathbb{Z}$. Throughout, we assume $g \geq 2$.

The main goal of this note is to understand whether the composition $\mathcal{H}_g \xrightarrow{\iota_g} \mathcal{M}_g \xrightarrow{\tau_g} \mathcal{A}_g$ is an immersion, (i.e., a locally closed immersion,) for $\tau_g$, the Torelli map sending a curve to its principally polarized Jacobian. Let $\phi_g : \mathcal{H}_g \to \mathcal{A}_g$ denote this composition. We use $\tau_g$, $\mathcal{R}$ and $\phi_g$, $\mathcal{R}$ for the base changes of $\tau_g$ and $\phi_g$ along a map $\text{Spec} R \to \text{Spec} \mathbb{Z}$. To start, we recall the classical characterization of when $\tau_g$ is injective on tangent vectors. This follows from [OS79, Theorems 2.6 and 2.7] together with the converse of [OS79, Theorem 2.7], which is easy to verify directly, see Proposition 3.1.

Theorem 1.1 ([OS79] Theorems 2.6 and 2.7). Let $k$ be a field. For $g \geq 3$, and $[C] \in \mathcal{M}_{g,k}$ a field-valued point, the Torelli map $\tau_{g,k} : \mathcal{M}_{g,k} \to \mathcal{A}_{g,k}$ is injective on tangent vectors at $[C]$ if and only if $[C] \in \mathcal{M}_{g,k} - \mathcal{H}_{g,k} \subset \mathcal{M}_{g,k}$. When $g = 2$, the map $\tau_{g,k}$ is injective on tangent vectors at all points $[C] \in \mathcal{M}_{g,k}$.

In particular, Theorem 1.1 shows that $\tau_g$ is not even a monomorphism at points of $\mathcal{H}_g$ when $g \geq 3$. It is therefore natural to ask whether the restriction of $\tau_g|_{\mathcal{H}_g} : \phi_g : \mathcal{H}_g \to \mathcal{A}_g$ is a monomorphism. Our main theorem answers this question. We say a morphism of algebraic stacks is a radimmersion if it factors as the composition of a finite radicial morphism and an open immersion, see [Definition A.1].

Theorem 1.2. For $g \geq 2$, the map $\phi_g : \mathcal{H}_g \to \mathcal{A}_g$ over $\text{Spec} \mathbb{Z}$ is a radimmersion. Additionally, $\phi_2$ is an immersion and $\phi_{g,Z}[1/2] : \mathcal{H}_{g,Z}[1/2] \to \mathcal{A}_{g,Z}[1/2]$ is an immersion. However, when $g > 2$, for $k$ a field of characteristic 2, $\phi_{g,k}$ is not an immersion; instead, $\phi_{g,k}$ is generically inseparable and the induced map on tangent spaces at any point of $\mathcal{H}_{g,k}$ has kernel of dimension $g - 2$.

We carry out the proof of Theorem 1.2 at the end of §6. To paraphrase the statement, Theorem 1.1 says, loosely speaking, that there are many tangent vectors to a hyperelliptic point in $\mathcal{M}_g$ that are killed under $\tau_g$. We wish to understand whether those tangent
vectors which are killed can lie inside $\mathcal{H}_g$, or whether they correspond to deformations to non-hyperelliptic curves. The answer, provided by Theorem 1.2, is that they do all correspond to deformations to non-hyperelliptic curves when the characteristic is not 2, but this fails quite badly in characteristic 2.

**Remark 1.3.** The statement that $\phi_{g,k}$ is an immersion for a field $k$ of characteristic not 2 appears in [OS79, p. 176], though some details are omitted. Our guess is that the authors verified the map $\phi_{g,k}$ is injective on tangent spaces via explicitly calculating the Kodaira Spencer map sending a differential to a corresponding deformation, in order to understand the image of the map $T_{[C]}\mathcal{H}_{g,k} \to T_{[C]}\mathcal{M}_{g,k} \cong H^1(C, T_C)$ on tangent spaces at a point $[C] \in \mathcal{H}_{g,k}$. However, no indication is given there as to how to use this to show the map of stacks is an immersion. In this article we take a different approach, which also applies in characteristic 2; we do not see how to carry out the approach of [OS79] in characteristic 2.

**Remark 1.4.** Our initial interest in this problem was motivated by the computation of the essential dimension of level structure covers of $\mathcal{H}_g$. For $n \in \mathbb{Z}_{\geq 1}$, let $\mathcal{A}_g[n]$ denote the stack over $\mathbb{Z}[1/n]$ parameterizing pairs $(A, \mathcal{B})$ where $A$ is a principally polarized abelian variety and $\mathcal{B}$ is a symplectic basis for the $n$-torsion of $A$. Let $\mathcal{H}_g[n]$ denote the stack over $\mathbb{Z}[1/n]$ which is the pullback of $\mathcal{A}_g[n] \to \mathcal{A}_g$ along $\phi_g : \mathcal{H}_g \to \mathcal{A}_g$. As explained in [FKW19, 3.3.3-3.3.5], for $n$ prime to 2 and $k$ a field of characteristic 0, the essential dimension of the map of coarse spaces associated to $\mathcal{H}_g[n] \to \mathcal{H}_{g,k}$ is $\dim \mathcal{H}_{g,k} = 2g - 1$. On the other hand, for $n = 2, g \geq 3$, and $k$ a field of characteristic 0, the essential dimension of the map of coarse spaces associated to $\mathcal{H}_g[n] \to \mathcal{H}_{g,k}$ is only $g + 1$. Using the techniques of [FKW19], this can be related to fact that for a field $k$ with char$(k) \neq 2$, the map $\phi_{g,k} : \mathcal{H}_{g,k} \to \mathcal{A}_{g,k}$ is an immersion, and so the image of the map on tangent spaces at any point has dimension $2g - 1$, while if char$(k) = 2$, and $g \geq 2$, the image of the map on $\phi_{g,k}$ tangent spaces has dimension $g + 1 = (2g - 1) - (g - 2)$.

There are two main components to the proof of Theorem 1.2. The first component of the proof is to describe the map $\phi_g$ induces on tangent vectors. This is done by analyzing the deformation theory of hyperelliptic curves, which is possible by means of their relatively simple equations. The key tool to analyzing the induced map on tangent spaces is Proposition 3.1 which relies on a nonstandard definition of $\mathcal{H}_g$ given in [2.2]. The second component of the proof is to verify that $\phi_g$ is a radimmersion. This will imply $\phi_g$ is an immersion when it is a monomorphism, i.e., away from characteristic 2. For checking $\phi_g$ is a radimmersion, we use a valuative criterion, which roughly says that a map of stacks $f : X \to Y$ is a radimmersion when, given a map from a DVR to $Y$ with its two points factoring through $X$, the map from the DVR factors uniquely through $X$. We verify this valuative criterion using that $\phi_g$ factors as the composition of an immersion into the moduli stack of compact type curves $\mathcal{H}_g \to \mathcal{M}_g^c$, and a proper "compactified Torelli map" $\mathcal{M}_g^c \to \mathcal{A}_g$.

The outline of the paper is as follows: In §2 we recall background on the infinitesimal Torelli theorem and the moduli stack of hyperelliptic curves. In §3 we describe the map $\phi_g$ induces on tangent spaces. We then compute this map on tangent spaces when the characteristic is not 2 in §4 and when the characteristic is 2 in §5. After some preliminaries on the compactified Torelli map, we prove Theorem 1.2 in §6. Finally, in Appendix A,
we prove a valuative criterion for immersions of stacks, which is used in the proof of Theorem 1.2

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2. Background

In this section, we review relevant background notation we will need from Theorem 1.1 and also a nonstandard construction of the moduli stack of hyperelliptic curves which will be crucial to the ensuing proof.

2.1. Key inputs in the proof of the infinitesimal Torelli theorem. We next review the key inputs in the proof of Theorem 1.1 as we will rely on understanding explicitly the map on tangent spaces associated to \( \tau_s : \mathcal{M}_g \rightarrow \mathcal{A}_g \) in our ensuing analysis of the map \( \mathcal{H}_g \rightarrow \mathcal{A}_g \).

The statement regarding injectivity on points is the classical Torelli theorem \([Cor86, \text{Chapter VII, Theorem 12.1(a)}]\), see also the original proof by Torelli \([Tor13]\) and Andreotti’s beautiful proof \([And58]\). Thus, we just address the statement on tangent vectors. This too is classical, and boils down to Noether’s theorem, regarding the map (2.2) below, though is perhaps less well known.

Let \( k \) be a field and let \( [C] \in \mathcal{M}_{g,k} \) be the point corresponding to the curve \( C \). We’d like to understand whether the map

\[
T_{[C]} \mathcal{M}_{g,k} \rightarrow T_{[\tau_g(C)]} \mathcal{A}_{g,k}
\]

is injective, for \( T_{[C]} \mathcal{M}_{g,k} \) denoting the tangent space to \( \mathcal{M}_{g,k} \) at \( [C] \). By deformation theory, \( T_{[C]} \mathcal{M}_{g,k} \simeq H^1(C, T_C) \). For \( V \) a vector space, define \( \text{Sym}^2 V \) as the kernel of \( V \rightarrow \wedge^2 V \), where \( \wedge^2 V = V / \text{Span}(v \otimes v : v \in V) \). (Note that \( \text{Sym}^2 V \simeq \text{Sym}^2 V \) in characteristic \( p \neq 2 \), but differs in characteristic 2. Here \( \text{Sym}^2 V \) denotes the natural quotient of \( V \otimes V \) by the span of \( v \otimes w - w \otimes v \) for \( v, w \in V \).) Further,

\[
T_{[\tau_g(C)]} \mathcal{A}_{g,k} = \ker(H^1(\tau_g(C), \mathcal{O}_{\tau_g(C)}) \otimes H^1(\tau_g(C), \mathcal{O}_{\tau_g(C)})) \hookrightarrow \wedge^2 H^1(\tau_g(C), \mathcal{O}_{\tau_g(C)}))
\]

as described in the proof of \([OS79, \text{Theorem 2.6}]\). Therefore, we wish to understand whether the natural map

\[
H^1(C, T_C) \rightarrow \text{Sym}^2 H^1(C, \mathcal{O}_C),
\]

induced by \( T_{[C]} \tau_g : T_{[C]} \mathcal{M}_{g,k} \rightarrow T_{[\tau_g(C)]} \mathcal{A}_{g,k} \), is injective.

Applying Serre duality, since \( H^1(C, T_C) \) is dual to \( H^0(C, \omega_C^\otimes 2) \), it is equivalent to understand surjectivity of the corresponding map

\[
H^0(C, \omega_C^\otimes 2) \hookrightarrow \text{Sym}^2 H^0(C, \omega_C).
\]
This duality, valid even in characteristic 2, uses that, for $V$ a finite dimensional vector space, $\text{Sym}_2 V^\vee$ can be viewed as the second graded piece of the algebra $\text{Sym}^* V^\vee$ with its divided power structure that is naturally dual to $\text{Sym}^* V$. The map (2.2) is explicitly the map given by multiplying two sections, as stated in [OS79, Theorem 2.6]. By Noether’s theorem [SD73, Theorem 2.10] the map (2.2) is surjective when $C$ is not hyperelliptic and fails to be surjective when $C$ is hyperelliptic. See §4.1 and §5.2 for an explicit description of the image of (2.2) in the hyperelliptic case.

2.2. Definition and properties of $\mathcal{H}_g$. There are different definitions of $\mathcal{H}_g$, the moduli stack of hyperelliptic curves of genus $g$, in the literature. For the purposes of this paper, we will be especially concerned with the more delicate case when 2 is not invertible on the base, so let us now expand a bit on the definition of $\mathcal{H}_g$ over $\text{Spec} \mathbb{Z}$ we employ. We assume $g \geq 2$. We will give a definition, and then observe that, $\mathcal{H}_g$ is realized as a closed substack of $\mathcal{M}_g$. So, in particular, it will follow that $\mathcal{H}_g$ is algebraic. Loosely speaking, we will define $\mathcal{H}_g$ as is the “$g\frac{1}{2}$ locus” (the locus with a degree 2 line bundle and a 2 dimensional space of global sections) in $\mathcal{M}_g$.

To make this more precise, we first define a stack $\mathcal{H}_g'$ which will be a $\mathbb{G}_m$ gerbe over $\mathcal{H}_g$. A $B$-valued point of $\mathcal{H}_g'$ is a tuple $(C, \pi, L, \eta, D)$ where $\pi : C \to B$ is a relative curve, $L$ is a degree 2 invertible sheaf on $C$, $D$ is a locally free sheaf of rank 2 on $B$, and $\eta : \pi_* L \to D$, is a surjective map. (Note that by cohomology and base change, $\eta$ is actually an isomorphism.) An isomorphism between $(C, \pi, L, \eta, D)$ and $(D, \sigma, M, \psi, R)$ is given by $(\rho, \chi, \mu)$ where $\rho : C \to D$ is an isomorphism so that $\rho \circ \sigma = \pi$, and $\chi : L \to \rho^* M$, $\mu : D \to R$ are isomorphisms such that $\psi \circ \pi_* \chi = \mu \circ \eta$ (where $\psi : \sigma_* M \to R$ is identified with the map $\pi_* \rho^* M \to R$ using the identification $\pi_* \rho^* M \simeq \pi_* \rho_*^{-1} M = \sigma_* M$ via the adjunction between $\rho^*$ and $\rho_*$). This finishes the construction $\mathcal{H}_g'$.

We next define $\mathcal{H}_g$. Observe that $\mathcal{H}_g'$ does not parameterize hyperelliptic curves because $\mathcal{H}_g'$ has “unwanted” automorphisms given by rescaling the line bundle $L$. To deal with this, let $I\mathcal{H}_g'$ denote the inertia stack of $\mathcal{H}_g'$ and let $G \subset I\mathcal{H}_g'$ denote the subgroup stack which is isomorphic to $\mathbb{G}_m$ over $\mathcal{H}_g'$ corresponding to automorphisms which act as the identity on the relative curve $C$ and scale $L$. Then, define $\mathcal{H}_g$ as the rigidification $\mathcal{H}_g'//G$ (see [AOV08, Theorem A.1], this essentially “removes” the automorphisms corresponding to $G$).

As an alternative construction of $\mathcal{H}_g$, one can form the $GL_2$ cover $\tilde{\mathcal{H}}_g$ of $\mathcal{H}_g'$ obtained by additionally specifying a basis of $H^0(C, L)$. Then, $\tilde{\mathcal{H}}_g$ is in fact the Hurwitz stack of flat degree 2 maps from smooth proper genus $g$ curves with geometrically connected fibers to $\mathbb{P}^1$. The quotient $[\tilde{\mathcal{H}}_g / \text{PGL}_2]$, with $\text{PGL}_2$ acting by automorphisms of the target $\mathbb{P}^1$ is again identified with $\mathcal{H}_g$.

Remark 2.1. Using the above definition, one can check $\mathcal{H}_g$ is smooth of relative dimension $2g - 1$ over $\text{Spec} \mathbb{Z}$ and the natural map $\mathcal{H}_g \to \mathcal{M}_g$ is a closed immersion. We now sketch these verifications.

To check that $\mathcal{H}_g \to \mathcal{M}_g$ is a closed immersion, one can first verify that the map induces isomorphisms on isotropy groups at each point, and so is a representable monomorphism. Because proper monomorphisms are closed immersions, it suffices to check $\mathcal{H}_g \to \mathcal{M}_g$ is proper. For this verification, we can use the valuative criterion for properness. The essential input in this verification is that any degree 2 line bundle on a curve of positive
genus has at most a 2-dimensional space of global sections (plus cohomology and base change) and the line bundle defining the hyperelliptic map to \( \mathbb{P}^1 \) is unique when \( g \geq 2 \).

Further, it is not difficult to check that \( \mathcal{H}_g \) with our definition is smooth. Using that one can deform hyperelliptic curves in positive characteristic to characteristic 0 via their explicit equations, it suffices to check the fibers over points of \( \text{Spec } \mathbb{Z} \) are smooth of relative dimension \( 2g - 1 \). One way to see this is via deformation theory: It suffices to check the \( \text{PGL}_2 \) cover \( \mathcal{H}_g \) of \( \mathcal{H}_g \) described above is smooth. Geometric points of \( \mathcal{H}_g \) correspond to degree two finite flat covers \( f : C \to \mathbb{P}^1 \) for \( C \) a smooth proper connected curve. The obstructions to deforming \( f \) are given by \( \text{Ext}^2_C(\Omega_f, \mathcal{O}_C) \simeq H^1(C, T_f) \). However, \( H^1(C, T_f) \) vanishes because \( T_f \) is supported on a finite set of points, using that \( f \) is generically étale since \( g > 0 \).

To compute \( \dim \mathcal{H}_g \), the tangent space of \( \mathcal{H}_g \) at any point \( f : C \to \mathbb{P}^1 \) is given by \( \text{Ext}^1_C(\Omega_f, \mathcal{O}_C) \simeq H^0(C, T_f) \), which has dimension \( 2g + 2 \) by Riemann Hurwitz. Therefore, \( \dim \mathcal{H}_g = \dim \mathcal{H}_g - \dim \text{PGL}_2 = (2g + 2) - 3 = 2g - 1 \).

3. The General Setup for Checking Injectivity on Tangent Vectors

Let \( k \) be a field and let \( C \) be a hyperelliptic curve over \( k \). To understand whether the map \( \phi_\mathfrak{g} : \mathcal{H}_g \to \mathcal{A}_g \) is injective on tangent vectors, we want to understand the composition

\[
T_{[C]} \mathcal{H}_g \xrightarrow{T_{[C]t_{\mathfrak{g}^k}} T_{[C]} \mathcal{M}_g \xrightarrow{T_{[C]}t_{\mathfrak{g}^k}} T_{[\tau_\mathfrak{g}(C)]}\mathcal{A}_g.\]

We have already explicitly described \( T_{[C]}t_{\mathfrak{g}^k} \) by identifying it as dual to (2.2) (see also \S 4.1 and \S 5.2 below for explicit descriptions of (2.2) in terms of differentials) so we next want to understand the image of \( T_{[C]t_{\mathfrak{g}^k}} \). Following [ACGT11, Chapter 21, \S 5-\S 6] we can identify \( T_{[C]t_{\mathfrak{g}^k}} \) as follows.

Let \( C \) be a hyperelliptic curve as above and \( \mathcal{L} \) the unique invertible sheaf giving rise to a hyperelliptic map \( C \to \mathbb{P}^1 \). Let

\[
\mu_0 : H^0(C, \mathcal{L}) \otimes H^0(C, \omega_C \otimes \mathcal{L}^\vee) \to H^0(C, \omega_C)
\]
denote the multiplication map. Then, as in [ACGT11, Chapter 21, (6.1)], there is a canonical map

\[
\mu_1 : \ker \mu_0 \to H^0(C, \omega_C^{\otimes 2})
\]
which explicitly sends \( r \otimes s \mapsto dr \cdot s \).

**Proposition 3.1.** For \( C \) a hyperelliptic curve, the composition

\[
T_{[C]} \mathcal{H}_g \xrightarrow{T_{[C]t_{\mathfrak{g}^k}}} T_{[C]} \mathcal{M}_g \xrightarrow{T_{[C]}t_{\mathfrak{g}^k}} T_{[\tau_\mathfrak{g}(C)]}\mathcal{A}_g
\]
is dual to the composition

\[
H^0(C, \omega_C^{\otimes 2}) / \text{im } \mu_1 \xrightarrow{(T_{[C]t_{\mathfrak{g}^k}})^\vee} H^0(C, \omega_C^{\otimes 2}) \xrightarrow{(T_{[C]}t_{\mathfrak{g}^k})^\vee} \text{Sym}^2 H^0(C, \omega_C).
\]

Hence, the dimension of the kernel of (3.1) agrees with the dimension of the cokernel of (3.2). In particular, (3.1) is injective if and only if (3.2) is surjective. Further, \( \dim \text{im } \mu_1 = g - 2 \) and \( \dim \text{im} (T_{[C]}t_{\mathfrak{g}^k})^\vee = 2g - 1 \).
Proof. For \( C \) a smooth projective geometrically connected curve and \( L \) an invertible sheaf on \( C \), there is a rank 2 locally free sheaf, \( \Sigma_L \) on \( C \), as defined in [ACG11, p. 804] such that \( H^1(C, \Sigma_L) \) parameterizes first order deformations of the pair \((C, L)\) [ACG11, Chapter 21, Proposition 5.15]. Further, as described in [ACG11 Chapter 21, (5.24)], there is a natural map \( \mu : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^\vee) \to H^0(C, \omega_C \otimes \Sigma_L^\vee) \). The key property of \( \mu \) is that if \( \alpha \in H^1(C, \Sigma_L) \) corresponds to a first order deformation \((\mathcal{E}, \mathcal{L})\) of \((C, L)\), then all sections of \( L \) extend to sections of \( \mathcal{L} \) (meaning that, if \( \pi : \mathcal{E} \to \text{Spec} k[\epsilon]/(\epsilon^2) \) is the structure map, \( \pi_* \mathcal{L} \) is locally free) if and only if \( \alpha \in (\text{im } \mu) \perp \) [ACG11 Chapter 21, Proposition 5.26]. Here \((\text{im } \mu) \perp \) denotes the orthogonal subspace to \( \text{im } \mu \) under the Serre duality pairing \( H^0(C, \omega_C \otimes \Sigma_L^\vee) \otimes H^1(C, \Sigma_L) \to k \).

Suppose \( C \) is a hyperelliptic curve with corresponding line bundle \( L \) defining the hyperelliptic map \( C \to \mathbb{P}^1 \). Given \( \beta \in H^1(C, T_C) \), we obtain a deformation of \( C \), corresponding to a curve \( \pi : \mathcal{E} \to \text{Spec} k[\epsilon]/(\epsilon^2) \). Our main goal is to show that \( \beta \) corresponds to a hyperelliptic curve precisely when \( \beta \in (\text{im } \mu) \perp \). By definition, if \([\mathcal{E}]\) is hyperelliptic, there is an invertible sheaf \( \mathcal{L} \) on \( \mathcal{E} \) with \( \pi_* \mathcal{L} \) locally free of rank 2. As described above, using [ACG11 Chapter 21, Proposition 5.15], we obtain that the pair \((\mathcal{E}, \mathcal{L})\) corresponds to some \( \alpha \in H^1(C, \Sigma_L) \) with \( \alpha \in (\text{im } \mu) \perp \).

We next show that if \( \beta \) corresponds to a hyperelliptic curve, then \( \beta \in (\text{im } \mu) \perp \). In other words, we will show \( T_{[C]} \mathcal{H}_{g,k} \subset (\text{im } \mu) \perp \). By [ACG11 Chapter 21, Proposition 5.15], there is a natural map of sheaves \( \Sigma_L \to T_L \) inducing the map \( \sigma : H^1(C, \Sigma_L) \to H^1(C, T_L) \) which sends the deformation \([\mathcal{E}, \mathcal{L}]\) to the deformation \([\mathcal{E}]\). In particular, \( \sigma(\alpha) = \beta \).

Let \( (T_{[C]} \tau_{g,k})^\vee : H^0(C, \omega_C^{\otimes 2}) \to H^0(C, \omega_C \otimes \Sigma_L^\vee) \) denote the map which is Serre dual to \( \sigma \). By definition of \( \mu_1 \) (see [ACG11 Chapter 21, (6.1)]) we have a commutative diagram

\[
\begin{array}{ccc}
H^0(C, L) \otimes H^0(C, \omega_C \otimes L^\vee) & \xrightarrow{\mu} & H^0(C, \omega_C \otimes \Sigma_L^\vee) \\
\uparrow & & \uparrow \sigma^\vee \\
\ker \mu_0 & \xrightarrow{\mu_1} & H^0(C, \omega_C^{\otimes 2}).
\end{array}
\]

For any \( \alpha \in \text{im } \mu \), we have \( \langle \alpha, \alpha \rangle = 0 \), where \( \langle \cdot, \cdot \rangle \) denotes the pairing from Serre duality. Therefore, for \( b \in \text{im } \mu_1 \), commutativity of (3.3) implies \( \langle \sigma^\vee(b), \alpha \rangle = 0 \). Functoriality of Serre duality then implies 0 = \( \langle \sigma^\vee(b), \alpha \rangle = \langle b, \sigma(\alpha) \rangle = \langle b, \beta \rangle \), and hence \( \beta \in (\text{im } \mu_1) \perp \).

We claim that \( \dim \ker \mu_0 = g - 2, \dim \text{im } (T_{[C]} \tau_{g,k})^\vee = 2g - 1 \), and \( \mu_1 \) is injective. We now explain why verifying these three claims finishes the proof. These claims imply \( \dim \text{im } \mu_1 = g - 2 \). We know \( T_{[C]} \mathcal{H}_{g,k} \) is \( 2g - 1 \) dimensional (as \( \mathcal{H}_{g,k} \) is smooth of dimension \( 2g - 1 \) [Remark 2.1]) and \( T_{[C]} \mathcal{H}_{g,k} \subset (\text{im } \mu_1) \perp \) inside the \( 3g - 3 \) dimensional vector space \( H^0(C, T_C) \), as shown above. Since \( \dim \text{im } \mu_1 = g - 2 \), by dimensional considerations, the containment \( T_{[C]} \mathcal{H}_{g,k} \subset (\text{im } \mu_1) \perp \) must be an equality. Therefore, \( T_{[C]} \mathcal{H}_{g,k} \) is dual to the surjection \( H^0(C, \omega_C^{\otimes 2}) \to H^0(C, \omega_C^{\otimes 2}) / \text{im } \mu_1 \).

It remains to check \( \dim \ker \mu_0 = g - 2, \dim \text{im } (T_{[C]} \tau_{g,k})^\vee = 2g - 1 \), and \( \mu_1 \) is injective. First, let us check \( \dim \ker \mu_0 = g - 2 \). Because \( C \) is hyperelliptic, we have \( \omega_C \otimes L^\vee \simeq L^{\otimes (g-1)} \). In particular, \( \omega_C \otimes L^\vee \simeq L^{\otimes (g-2)} \) and so \( \mu_0 \) is given by the map \( H^0(C, L) \otimes H^0(C, L^{\otimes (g-2)}) \to H^0(C, L^{\otimes (g-1)}) \). Letting \( \{1, x\} \) denote a basis for \( H^0(C, L) \), we see
\{1, x, \ldots, x^i\} is a basis for \(H^0(C, L^{\otimes i})\). Applying this when \(i = g - 2\) and \(i = g - 1\), we find \(\mu_0\) is surjective, so \(\dim \ker \mu_0 = g - 2\). Explicitly, we see \(\ker \mu_0\) is generated by \(\{1 \otimes x^j - x \otimes x^{j-1}\}_{1 \leq j \leq g - 2}\).

Next, we verify \(\dim \text{im}(T_{[C]}T_{g,k})^\vee = 2g - 1\). As in the previous paragraph, we may choose a basis \(\{1, x\}\) for \(H^0(C, L)\) so that \(1, x, \ldots, x^{g-1}\) is a basis for \(H^0(C, \omega_C)\). Then, the image of the multiplication map \((T_{[C]}T_{g,k})^\vee\) is spanned by \(1, x, \ldots, x^{2g-2}\), which has dimension \(2g - 1\).

To conclude, we show \(\mu_1\) is injective. Recall \(\{1 \otimes x^j - x \otimes x^{j-1}\}_{1 \leq j \leq g - 2}\) spans \(\ker \mu_0\), as shown above, and recall that \(\mu_1\) is given by \(r \otimes s \mapsto dr \otimes s \) [ACG11, Chapter 21, (6.6)], (alternatively, see [ACG11, p. 813-p. 814]). Therefore, \(\text{im} \mu_1\) is spanned by \(dx \cdot x^{j-1}\) for \(1 \leq j \leq g - 2\), which are independent elements of \(H^0(C, \omega_C \otimes L^{\otimes(g-1)}) \simeq H^0(C, \omega_C^{\otimes 2})\). Hence, \(\mu_1\) is injective.

### 4. Hyperelliptic curves in characteristic not 2

The key to analyzing the map induced by \(\phi_{g,k}\) on tangent spaces for \(\text{char}(k) \neq 2\) is \textbf{Lemma 4.1} below. Let \(k\) be an algebraically closed field of characteristic \(p\) (allowing \(p = 0\)) with \(p \neq 2\). Before proving injectivity, we set up some notation.

#### 4.1. Hyperelliptic differentials in characteristic not 2

Every hyperelliptic curve \(C\) over an algebraically closed field \(k\) can be expressed as the proper integral model of the affine curve \(y^2 = f\) for \(f \in k[x]\) a polynomial of degree \(2g + 1\) with no repeated roots. We can choose a basis of differentials for \(C\) of the form
\[
\frac{dx}{y}, \frac{x \, dx}{y}, \ldots, \frac{x^{g-1} \, dx}{y}. 
\]
where here \(x\) and \(y\) are viewed as rational functions and \(dx\) is viewed as a rational section of \(H^0(C, \omega_C)\).

In the above basis, the multiplication map \((2.2)\) above (which is dual to \(T_{[C]}\mathcal{M}_{g,k} \rightarrow T_{[\mathcal{T}_g(C)]}\mathcal{M}_{g,k}\)) has image
\[
\frac{(dx)^2}{y^2}, \ldots, \frac{x^{2g-2} (dx)^2}{y^2}.
\]
Written another way, the basis is \(\frac{1}{2} (dx)^2, \ldots, \frac{x^{2g-2} (dx)^2}{y^2}\). This is just what one obtains by multiplying together pairs of functions from the above described basis \((4.1)\). In particular, the image is a \(2g - 1\) dimensional subspace of the \(3g - 3\) dimensional vector space \(H^0(C, \omega_C^{\otimes 2})\).

#### 4.2. Computing the tangent map in characteristic not 2

The following was essentially already shown in the end of the proof of \textbf{Proposition 3.1} albeit with slightly different notation.

\textbf{Lemma 4.1.} For \(C\) a hyperelliptic curve over an algebraically closed field \(k\) of characteristic \(p \neq 2\), the composition \((3.2)\) is surjective.

\textbf{Proof.} We have an explicit understanding of the image of \((2.2)\) from \textbf{4.1}. If we can also explicitly describe \(\text{im} \mu_1\), then we will be able to determine surjectivity of the composite
map (5.2). To start, let’s describe \( \mu_0 \), using the notation from \( \S 4.1 \) Letting \( \{1, x\} \) denote a basis of \( \mathcal{L} \), we find that \( \{ \frac{dx}{y}, \frac{x\,dx}{y}, \ldots, \frac{x^{g-2}\,dx}{y} \} \) is a basis for \( H^0(C, \omega_C \otimes \mathcal{L}^\vee) \). Then, it follows that

\[
(4.3) \quad x \otimes \frac{dx}{y} - 1 \otimes \frac{x\,dx}{y}, x \otimes \frac{x\,dx}{y} - 1 \otimes \frac{x^2\,dx}{y}, \ldots, x \otimes \frac{x^g\,dx}{y} - 1 \otimes \frac{x^{g-1}\,dx}{y}
\]

is a basis for \( \ker \mu_0 \).

The map \( \mu_1 \) is given explicitly by sending \( \alpha \otimes \beta \mapsto d\alpha \cdot \beta \) \([ACG11, \text{Chapter 21, (6.6)}]\). Therefore, applying \( \mu_1 \) to (4.3), we find \( \text{im } \mu_1 \) is generated by

\[
(4.4) \quad \frac{(dx)^2}{y}, \frac{x^2(dx)^2}{y}, \ldots, \frac{x^{g-1}(dx)^2}{y}.
\]

On the other hand, the image of the map \( H^0(C, \omega_C^2) \leftarrow \text{Sym}^2 H^0(C, \omega_C) \) from (2.2) is given in (4.1). Since together, the union of the \( 2g-1 \) elements of (4.1) and the \( g-2 \) elements of (4.4) span \( H^0(C, \omega_C^2) \), it follows that the composite map (5.2) is surjective. \( \square \)

5. Hyperelliptic curves in characteristic 2

As in \( \S 4 \), to check injectivity of \( \phi_X \) on tangent vectors, we may assume our base field \( k \) is algebraically closed. To conclude the proof of Theorem 1.2, we only need prove Lemma 5.1 below. We now set up notation for the proof. The key difference in characteristic 2 is that hyperelliptic curves cannot be described in terms of an equation of the form \( y^2 = f \), for \( f \in k[x] \), of degree more than 1, as any such curve would be singular at the roots of \( \frac{df}{dx} \). We now describe a general form for hyperelliptic curves in characteristic 2.

5.1. Equations for hyperelliptic curves in characteristic 2. Over an algebraically closed field of characteristic 2, every hyperelliptic curve can be written as the projective completion of a curve of the form \( y^2 - y = f \), for \( f \in k(x) \), see \([EPI3, \text{Notation 1.1}] \). This can be seen by starting with an equation of the form \( ay^2 + byz + cz^2 = 0 \) with \( a \in k, b \in k[x], c \in k[x] \) and changing variables appropriately. If the genus of the curve is \( g \), then \( b \) has degree at most \( g + 1 \) and \( c \) of degree at most \( 2g + 2 \). Upon the above change of variables, applying a partial fraction decomposition and changing variables again, it follows that a general such curve can be written in the form

\[
(5.1) \quad f = a_0 + \frac{a_1}{x-a_1} + \cdots + \frac{a_g}{x-a_g}
\]

with \( a_0, \ldots, a_g, a_1, \ldots, a_g \in k \), see \([EPI3, \text{Notation 1.1}] \). As a side remark, a hyperelliptic curve can be written in this form precisely if it is ordinary, though we will not need this fact.

5.2. Hyperelliptic differentials in characteristic 2. Now let’s proceed to explicitly write down the differentials, as in the characteristic 0 case.

Let \( k \) be a field of characteristic 2 and let \( C \) be a hyperelliptic curve given as the projective model of the curve defined by the equation \( y^2 - y = f \), for \( f = a_0 + \frac{a_1}{x-a_1} + \cdots + \frac{a_g}{x-a_g} \in k(x) \). As described in \( \S 5.1 \), \( \pi : C \to \mathbb{P}^1 \) is ramified at the preimages of the points \( a_1, \ldots, a_g \) and \( \infty \) (where \( \infty \) is the point in the complement of \( \text{Spec } k[x] \cong \mathbb{A}^1 \subset \mathbb{P}^1 \)).
By the assumption that $f$ is as in (5.1), there are $g + 1$ ramification points of $\pi$, and the differential $dx$ is ramified to order 2 at $\pi^{-1}(V(x - a_1)), \ldots, \pi^{-1}(V(x - a_g))$. In other words, the relative sheaf of differentials $\Omega_{\pi}$ is a skyscraper sheaf at $\pi^{-1}(V(x - a_i))$ with degree 2 at $\pi^{-1}(V(x - a_i))$, for $1 \leq i \leq g$. Since $\deg dx = 2g - 2$, we conclude that $dx$ has a pole of order 2 at the unique point over $\infty \in \mathbb{P}^1$. Further, the function $x - a_i$ vanishes to order 2 at the preimage of $\pi^{-1}(V(x - a_i))$ and therefore has a pole of order 2 at $\pi^{-1}(\infty)$. It follows that the functions

$$\omega_1 := \frac{dx}{x - a_1}, \ldots, \omega_g := \frac{dx}{x - a_g}$$

form a basis of $H^0(C, \omega_C)$.

In the above basis, the map (2.2)

$$H^0(C, \omega_C^{\otimes 2}) \leftarrow \text{Sym}^2 H^0(C, \omega_C).$$

has image spanned by elements of the form

$$(5.2) \quad \frac{(dx)^2}{(x - a_i)(x - a_j)}$$

Of course, these elements will not be independent, but they will necessarily span a $2g - 1$ dimensional subspace of $H^0(C, \omega_C^{\otimes 2})$ by Proposition 3.1.

5.3. Computing the map on tangent spaces in characteristic 2.

**Lemma 5.1.** For $C$ a hyperelliptic curve of genus $g \geq 2$ over an algebraically closed field $k$ of characteristic 2, the composition (3.2) has cokernel of dimension $g - 2$.

**Proof.** For $C$ any hyperelliptic curve, with notation as in Proposition 3.1, $\dim \text{im} \mu_1 = g - 2$ and $\dim \text{im}(\widetilde{T}_C^* T_k)^\vee = 2g - 1$. It follows that the composition (3.2) has image of dimension at least $(2g - 1) - (g - 2) = g + 1$. Equivalently, (3.2) has cokernel of dimension at most $g - 2$, since $\dim \mathcal{H}_{g, \mathbb{F}_2} = 2g - 1 = (g + 1) + (g - 2)$. Therefore, by upper semicontinuity of cokernel dimension, in order to show (3.2) has cokernel of dimension $g - 2$ at all $[C] \in \mathcal{H}_{g, \mathbb{F}_2}$ over a field of characteristic 2, it suffices to show this holds over a general such $[C] \in \mathcal{H}_{g, \mathbb{F}_2}$.

As described in §5.1, a general hyperelliptic curve over $k = \overline{k}$ with char($k$) = 2 can be written in the form $y^2 - f = 0$, for $f = a_0 + \frac{a_1}{x - a_1} + \cdots + \frac{a_g}{x - a_g} \in k(x)$, with elements $a_0, \ldots, a_g \in k$ and with $a_1, \ldots, a_g$ distinct. Following the strategy outlined in §3, we next determine ker $\mu_0$ and then im $\mu_1$ for such curves $C$.

We can choose $1, x - a_1$ as a basis for $H^0(C, \mathcal{L})$ identifying $\mathcal{L} \simeq \mathcal{O}_C(2, \pi^{-1}(\infty))$, for $\pi : C \rightarrow \mathbb{P}^1$ the hyperelliptic map. For $1 < i \leq g$, the differential forms $\frac{dx}{a_i - a_1} (\omega_1 - \omega_i) = \frac{dx}{(x - a_1)(x - a_i)}$ have a zero of order 2 at $\infty$, and therefore lie in $H^0(C, \omega_C \otimes \mathcal{L}^\vee)$. Because they are also independent, it follows that

$$(5.3) \quad \frac{dx}{(x - a_1)(x - a_2)}, \frac{dx}{(x - a_1)(x - a_3)}, \ldots, \frac{dx}{(x - a_1)(x - a_g)}$$
form a basis for $H^0(C, \omega_C \otimes L^\vee)$. The kernel of $\mu_0$ is then spanned by elements of the form

$$(x - a_1) \otimes r_i + 1 \otimes t_i$$

for $1 \leq i \leq g - 2$ with $r_i, t_i$ linear combinations of the elements in (5.3).

Therefore, im $\mu_1$ is generated by $d(x - a_1) \otimes r_i + d(1) \otimes t_i = dx \otimes r_i$ for $1 \leq i \leq g - 2$. In particular, the $r_i$ are some linear combination of elements appearing in (5.3). It follows that $dx \otimes r_i$ are also linear combinations of the elements appearing in (5.2). Therefore, im $\mu_1$ \subset im($H^0(C, \omega_C^\otimes 2) \leftarrow \text{Sym}^2 H^0(C, \omega_C)$). By Proposition 3.1, dim im $\mu_1 = g - 2$ and dim im($T_{C|C} \tau_g)$\textsuperscript{\vee} = $2g - 1$. Hence, the composition (5.2) has image of dimension $(2g - 1) - (g - 2) = g + 1$ and cokernel of dimension $(2g - 1) - (g + 1) = g - 2$. \hfill $\square$

6. PROOF OF THEOREM 1.2

At this point, nearly everything is in place to prove Theorem 1.2. The work of previous sections will show that $\phi_8$ is a monomorphism over Spec $\mathbb{Z}$[[1/2]], but fails to be a monomorphism when restricted to any field $k$ with char($k$) = 2. In order to show $\phi_8 : \mathcal{H}_8 \to \mathcal{A}_8$ is an immersion over Spec $\mathbb{Z}$[[1/2]], we will verify the valuative criterion for immersions (or radimmersions) in Proposition 6.3. We then use Corollary A.5 to deduce $\phi_8$ is an immersion over Spec $\mathbb{Z}$[[1/2]]. This valuative criterion loosely says that a monomorphism $f : X \to Y$ is an immersion when, given a map from a DVR to $Y$ with its two points factoring through $X$, the map from the DVR factors through $X$.

We next set up some notation. Let $\overline{\mathcal{M}}_g$ denote the Deligne-Mumford compactification of $\mathcal{M}_g$ and let $\mathcal{M}_g^c$ denote the moduli stack of stable compact type curves of genus $g$. Recall that $\mathcal{M}_g^c$ can be constructed as the open substack of $\overline{\mathcal{M}}_g$ which, loosely speaking, parameterizes curves whose dual graph of components is a tree. The geometric points of $\overline{\mathcal{M}}_g$ lie in $\mathcal{M}_g^c$ precisely when the Jacobian of the corresponding curve is an abelian variety, as follows from [BLR90, \S 9.2, Example 8]. The following lemma is well-known:

**Lemma 6.1.** The Torelli map $\phi_8 : \mathcal{H}_8 \to \mathcal{A}_8$ factors as the composition of an immersion $\mathcal{H}_8 \to \overline{\mathcal{M}}_8$ and a proper map $\tau_8^c : \mathcal{M}_8^c \to \mathcal{A}_8$.

**Proof.** It is shown in [Ale04 Corollary 5.4] that there is a compactification of $\mathcal{A}_8$, which we denote $\overline{\mathcal{A}}_8$, and a map $\tau_8 : \overline{\mathcal{M}}_8 \to \overline{\mathcal{A}}_8$ extending $\tau_8$. (In [Ale04 Corollary 5.4], it is only stated that this yields a map of coarse spaces, but the map is in fact constructed as a map of stacks.) Since $\overline{\mathcal{M}}_8$ and $\overline{\mathcal{A}}_8$ are proper, $\overline{\tau}_8$ is as well, and therefore the resulting map $\mathcal{M}_8^c = \overline{\mathcal{M}}_8 \times_{\overline{\mathcal{A}}_8} \mathcal{A}_8 \to \mathcal{A}_8$ is also proper. Finally, $\mathcal{H}_8 \to \mathcal{M}_8$ is an immersion because it is a composition of the immersions $\mathcal{H}_8 \to \mathcal{M}_8 \to \mathcal{M}_8^c$. \hfill $\square$

**Remark 6.2.** We used the rather difficult result of [Ale04 Corollary 5.4] in the proof of Lemma 6.1 but Lemma 6.1 can also be verified directly. One can extend the principal polarization on the universal Jacobian over $\mathcal{M}_8$ to that over $\mathcal{M}_8$, and then use the valuative criterion of properness and [BLR90, \S 7.4, Proposition 3] to verify properness of $\mathcal{M}_8^c \to \mathcal{A}_8$. See [Aar] for further discussion of this. For the sake of brevity, we omit this more direct proof.

We now verify $\phi_8$ satisfies the valuative criterion for radimmersions.
Proposition 6.3. The map \( \phi_g : \mathcal{H}_g \to \mathcal{A}_g \) satisfies the valuative criterion for radimmersions for traits as in Definition A.3.

Proof. Let \( T \) be a trait (the spectrum of a discrete valuation ring) with 2-commutative diagrams as in (A.1) for \( X = \mathcal{H}_g \) and \( Y = \mathcal{A}_g \). By Lemma 6.1 and the valuative criterion for properness [Ols16, Theorem 11.5.1] there is a dominant map of traits \( T' \to T \) so that the resulting map \( T' \to \mathcal{A}_g \) factors through \( \mathcal{M}_g^c \). Let \( J \to T' \) denote the family of principally polarized abelian varieties of dimension \( g \) corresponding to the map \( T' \to \mathcal{A}_g \).

The factorization \( T' \to \mathcal{M}_g^c \) yields a family of stable curves \( C \to T' \), so that the principally polarized Jacobian of the generic fiber of \( C \) agrees with the generic fiber of \( J \to T' \).

With notation for \( T, T', t, t', \eta, \eta' \) as in Definition A.3, we have diagrams

\[
\begin{array}{c}
\eta' \\
\downarrow \\
T' \\
\downarrow \\
T \\
\downarrow \\
\mathcal{H}_g \\
\downarrow \\
\mathcal{M}_g^c \\
\downarrow \\
\mathcal{M}_g \\
\downarrow \\
\mathcal{A}_g
\end{array}
\begin{array}{c}
t' \\
\downarrow \\
t \\
\downarrow \\
\mathcal{H}_g \\
\downarrow \\
\mathcal{M}_g^c \\
\downarrow \\
\mathcal{M}_g \\
\downarrow \\
\mathcal{A}_g
\end{array}
\]

The first diagram in (6.1) 2-commutes by the valuative criterion for properness. We claim the second diagram also 2-commutes. Granting this claim, observe that \( \mathcal{H}_g \to \mathcal{M}_g^c \) is an immersion, being the composition of a closed immersion \( \mathcal{H}_g \to \mathcal{M}_g \) and an open immersion \( \mathcal{M}_g \to \mathcal{M}_g^c \). Therefore, the valuative criterion for immersions [A.5] implies \( T' \to \mathcal{M}_g^c \) lifts to a map \( T' \to \mathcal{H}_g \) making the diagrams (A.2) 2-commute for \( X = \mathcal{H}_g \) and \( Y = \mathcal{A}_g \).

So, to conclude, we just need to check the second diagram in (6.1) 2-commutes. This will necessarily follow if we verify the fiber product \( \mathcal{C} \times_{\phi_g : \mathcal{H}_g} \mathcal{M}_g^c \) contains a unique geometric point for any geometric point \( \mathcal{C} \in \mathcal{H}_g \). First, we show \( \mathcal{C} \times_{\phi_g : \mathcal{H}_g} \mathcal{M}_g^c \) does not contain any geometric points mapping to \( \mathcal{M}_g^c - \mathcal{M}_g \). Indeed, the theta divisor associated to a singular compact type curve is always geometrically reducible, while that associated to a curve in \( \mathcal{M}_g \) is geometrically irreducible. Hence, no geometric points of \( \mathcal{M}_g^c - \mathcal{M}_g \) can map under \( \tau_g^c \) to \( \phi_g([C]) \). Therefore, it suffices to show \( \mathcal{C} \times_{\phi_g : \mathcal{H}_g} \mathcal{M}_g^c \) contains a unique geometric point in \( \mathcal{M}_g \), which follows from the classical Torelli theorem [Cor86, Chapter VII, Theorem 12.1].

Corollary 6.4. The map \( \phi_g : \mathcal{H}_g \to \mathcal{A}_g \) is a radimmersion (as defined in Definition A.1).

Proof. This will follow from Proposition 6.3 and Theorem A.4 once we verify \( \phi_g \) is representable and induces a universal homeomorphism on isotropy groups at every point of \( \mathcal{H}_g \). We first check that \( \phi_g \) induces isomorphisms on isotropy groups at every point of \( \mathcal{H}_g \). Observe that \( \phi_g \) induces a bijection on geometric points of isotropy groups at every point of \( \mathcal{H}_g \) by the Torelli theorem [Cor86, Chapter VII, Theorem 12.1(b)]. Since both \( \mathcal{H}_g \) and \( \mathcal{A}_g \) are Deligne-Mumford stacks, their isotropy groups at any geometric point are constant group schemes. Therefore, \( \phi_g \) induces an isomorphism on isotropy groups at geometric points of \( \mathcal{H}_g \). As a consequence, \( \phi_g \) is representable, as follows by applying [Con07, Theorem 2.2.5] to the pullback of \( \phi_g \) along a schematic cover of \( \mathcal{A}_g \).
Combining Corollary 6.4 with Proposition 3.1, Lemma 4.1, and Lemma 5.1, we now prove Theorem 1.2.

**Proof of Theorem 1.2.** We know $\phi_g$ is a radimmersion by Corollary 6.4. Next, we show $\phi_g$ is an immersion over $\text{Spec} \mathbb{Z}[1/2]$ or when $g = 2$. Using Theorem A.4 and Corollary A.5, we only need to check $\phi_g$ is a monomorphism over $\text{Spec} \mathbb{Z}[1/2]$ or when $g = 2$. Equivalently, we just need to verify $\phi_g$ is injective on points and tangent vectors [Gro67, 17.2.6].

It follows from the classical Torelli theorem [Cor86, Chapter VII, Theorem 12.1(a)] that the map $\phi_g : \mathcal{H}_g \to \mathcal{A}_g$ is injective on points. Since smooth hyperelliptic curves $C$ in over a field $k$ of any characteristic can be deformed to smooth hyperelliptic curves of characteristic 0, (as can be seen by using explicit equations,) we obtain an identification of the kernel of $T[C]\phi_g : T[C]\mathcal{H}_g \to T[C]\mathcal{A}_g$ with $\ker (T[C]\phi_{g,k})$. Note that $T[C]\phi_{g,k}$ is given as the composition (3.1). The vanishing of $\ker T[C]\phi_{g,k}$ over $\text{Spec} \mathbb{Z}[1/2]$ follows from combining Proposition 3.1 with Lemma 4.1. Therefore, $\phi_g$ is injective on tangent vectors over $\text{Spec} \mathbb{Z}[1/2]$. In the case $g = 2$ we find $\phi_g$ is a monomorphism by combining the above with Lemma 5.1.

To conclude the proof, we just need to check that the restriction of $\phi_g$ to a field of characteristic 2 induces a map on tangent spaces with kernel of dimension $g - 2$. Again using the identification $\ker T[C]\phi_g = \ker T[C]\phi_{g,k}$ mentioned above, this follows from combining Proposition 3.1 with Lemma 5.1. □

**Appendix A. The valuative criterion for locally closed immersions**

A.1. **Statement of the criterion.** In this section we state a valuative criterion for immersions of algebraic stacks in Corollary A.5. The result is a generalization of [Moc14, Chapter 1, Corollary 2.13] used in proving Theorem 1.2. Many of the ideas are present in [Moc14, Chapter 1, §2.4], though nontrivial care has to be taken to deal with algebraic stacks in place of schemes.

We begin by introducing definitions to state the valuative criterion for radimmersions, which will imply the analogous valuative criterion for immersions. Recall that a morphism of schemes $X \to Y$ is radial if for every field $K$, $X(\text{Spec} K) \to Y(\text{Spec} K)$ is injective, or equivalently each geometric fiber has at most one geometric point. A morphism $f : X \to Y$ of algebraic stacks is an immersion (or locally closed immersion) if it factors as a composition $X \to U \to Y$ where $U \to Y$ is an open immersion and $X \to U$ is a closed immersion.

**Definition A.1.** A morphism of algebraic stacks $X \to Y$ is a radimmersion if it factors as a composition $X \to U \to Y$ where $U$ is an algebraic stack with $U \to Y$ an open immersion and $X \to U$ a finite radicial map.

We note in particular that radimmersions are representable by schemes.

**Remark A.2.** In the context of maps of algebraic stacks, being radial is not equivalent to being injective on geometric points as it is for maps of schemes. For example, for $k$ a field, $\text{Spec} k \to [\text{Spec} k/ (\mathbb{Z}/2\mathbb{Z})]$ is bijective on geometric points but is not radial because after pulling back to a schematic cover of the target, the resulting map is not radial. This distinction will play a significant role in what follows.
For the next definition recall that a \textit{trait} is a scheme of the form \text{Spec} \, R for \, R a discrete valuation ring.

\textbf{Definition A.3.} Let \( f : X \to Y \) be a map of algebraic stacks. We say \( f \) \textit{satisfies the valuative criterion for radimmersions} if the following property holds: Let \( T \) be the spectrum of a valuation ring with generic point \( \eta \) and closed point \( t \), and let \( g : T \to Y \) be any map. Suppose we have 2-commutative diagrams

\[
\begin{array}{ccc}
\eta & \xrightarrow{j_\eta} & X \\
\downarrow{i_\eta} & & \downarrow{f} \\
T & \xrightarrow{g} & Y \\
\end{array}
\quad \quad \begin{array}{ccc}
t & \xrightarrow{j_t} & X \\
\downarrow{i_t} & & \downarrow{f} \\
T & \xrightarrow{g} & Y \\
\end{array}
\]

(A.1)

with 2-morphisms \( \gamma_\eta : f \circ j_\eta \simeq g \circ i_\eta \) and \( \gamma_t : f \circ j_t \simeq g \circ i_t \) witnessing 2-commutativity of the diagrams. Then, there exists a spectrum of a valuation ring \( T' \) with closed point \( t' \) and generic point \( \eta' \) with a specified dominant map \( T' \to T \) such that there is a unique morphism \( h : T' \to X \) making the diagrams

\[
\begin{array}{ccc}
\eta' & \xrightarrow{j_{\eta'}} & X \\
\downarrow{h'} & & \downarrow{f} \\
T' & \xrightarrow{g} & Y \\
\end{array}
\quad \quad \begin{array}{ccc}
t' & \xrightarrow{j_{t'}} & X \\
\downarrow{h'} & & \downarrow{f} \\
T' & \xrightarrow{g} & Y \\
\end{array}
\]

(A.2)

2-commute compatibly with the above choices of \( \gamma_\eta \) and \( \gamma_t \) (as for dotted arrows in \cite[Tag 0CLA]{Sta}).

We say \( f \) \textit{satisfies the valuative criterion for radimmersions with} \( T = T' \) if for every spectrum of a valuation ring \( T \) and diagrams (A.1), there exists a map \( h : T \to X \) such that (A.2) holds with \( T' = T \) and the map \( T' \to T \) being the identity map.

We say \( f \) \textit{satisfies the valuative criterion for radimmersions for traits} if \( f \) satisfies the valuative criterion for radimmersions for all traits \( T \) and \( T' \).

We say \( f \) \textit{satisfies the valuative criterion for radimmersions for traits with} \( T = T' \) if \( f \) satisfies the valuative criterion for radimmersions with \( T = T' \) for all traits \( T \).

We can now state the valuative criterion for radimmersions. We note that in the case that \( X \) and \( Y \) are finite type schemes over a noetherian base \( S \), the first two conditions of Theorem A.4 were shown to be equivalent in \cite[Chapter 1, Theorem 2.12]{Moc14}. Recall that for \( S \) a scheme and \( x : S \to X \) a point of an algebraic stack, the isotropy group at \( x \) is by definition the algebraic space \( \text{Isom}_X(x, x) \).

\textbf{Theorem A.4} (Valuative criterion for radimmersions). Let \( f : X \to Y \) be a representable finite type quasi-separated morphism of algebraic stacks with \( Y \) locally noetherian. Then, the following are equivalent:

1. \( f \) is a radimmersion
2. \( f \) induces a universal homeomorphism on isotropy groups at every geometric point of \( X \), and \( f \) satisfies the valuative criterion for radimmersions for traits with \( T = T' \)
3. \( f \) induces a universal homeomorphism on isotropy groups at every geometric point of \( X \), and \( f \) satisfies the valuative criterion for radimmersions for traits.
The proof is given at the end of this section. Before giving the proof, we deduce the following valuative criterion for locally closed immersions, which generalizes [Moc14, Corollary 2.13].

**Corollary A.5** (Valuative criterion for locally closed immersions). Let \( f : X \to Y \) be a finite type quasi-separated monomorphism of algebraic stacks with \( Y \) locally noetherian. Then, the following are equivalent:

1. \( f \) is an immersion
2. \( f \) satisfies the valuative criterion for radimmersions for traits with \( T = T' \)
3. \( f \) satisfies the valuative criterion for radimmersions for traits.

**Proof.** Recall by definition that a map is a monomorphism if it is representable (i.e., representable by algebraic spaces) and is fppf locally a monomorphism. Further, by [Sta, Tag 04ZZ] monomorphisms must induce isomorphisms on isotropy groups at every point, and so in particular, the map on isotropy groups a universal homeomorphism.

Therefore, using [Theorem A.4] it suffices to show that a monomorphism \( f : X \to Y \) is an immersion if and only if it is a radimmersion. Certainly immersions are monomorphisms and radimmersions. So, we just need to check a radimmersion which is a monomorphism is an immersion. Both immersions and radimmersions are representable by schemes by definition, and so it suffices to check a radimmersion monomorphism of schemes is an immersion. Further, by factoring \( f : X \to Y \) as a composition of a finite morphism and an open immersion, it suffices to check a finite monomorphism is a closed immersion. This is shown in [Gro67, 18.12.6]. □

### A.2. Remarks and Examples

We next make some comments on the valuative criterion for radimmersions and give some examples and non-examples.

**Remark A.6.** One can similarly state and prove a version of the valuative criterion for radimmersions [Theorem A.4] and the valuative criterion for locally closed immersions [Corollary A.5] where one removes the noetherian hypotheses on \( Y \) at the cost of assuming \( f \) is finitely presented (instead of just of finite type) and working with all valuation rings (instead of just discrete valuation rings). The proof is essentially the same, where one replaces the references to the noetherian valuative criteria for properness and separatedness for discrete valuation rings with references to valuative criteria for properness and separatedness for general valuation rings.

**Example A.7.** As we have seen in [Proposition 6.3] the restricted Torelli map \( \phi_{g} : \mathcal{H}_{g} \to \mathcal{A}_{g} \) satisfies the valuative criterion for radimmersions. Another example of a map of algebraic stacks which can be seen to be an immersion using the valuative criterion is the map from the moduli stack of smooth plane curves of degree \( d \) for \( d \geq 4 \) to \( \mathcal{M}_{g} \). Here, the moduli stack of plane curves can be defined by taking the open in the Hilbert scheme of plane curves corresponding to smooth plane curves, and quotienting by the PGL\(_{3}\) action. See [LSTX19, Remark 5.4] for some more details. We note that we do not know how to see either of these maps are immersions without the valuative criterion.

**Example A.8.** Radimmersions of algebraic stacks do not always induce isomorphisms on isotropy groups. For example, for \( k \) a field of characteristic \( p \), the map \( \text{Spec} k \to \text{Spec} k / \mu_{p} \) is a radimmersion that does not induce an isomorphism on isotropy groups. More generally, we can replace \( \mu_{p} \) with any group scheme with a single geometric point over \( \text{Spec} k \) in the above example.
Example A.9. We now give an example of a map which satisfies the valuative criterion for radimmersions for traits but which is not a radimmersion. For $k$ a field, consider the representable map $f : \text{Spec } k \to [\text{Spec } k/ (\mathbb{Z}/2\mathbb{Z})]$. This satisfies the valuative criterion for radimmersions because a $\mathbb{Z}/2\mathbb{Z}$ torsor over a trait which is trivial over the generic fiber is necessarily trivial, using normality of the trait. Nevertheless, $f$ is not a radimmersion because the fiber of $f$ over $\text{Spec } k$ has two geometric points. In particular, $f$ induces the map $\{\text{id}\} \to \mathbb{Z}/2\mathbb{Z}$ on isotropy groups, and so is not a universal homeomorphism on isotropy groups and hence does not satisfy Theorem A.4(3). More generally, one can replace $\mathbb{Z}/2\mathbb{Z}$ in the above example with any nontrivial constant group scheme.

Example A.10. In addition to Example A.9, another example of a map which satisfies the valuative criterion for radimmersions for traits but which is not a radimmersion is the Torelli map $\tau_g : M_g \to A_g$ when $g \geq 3$. This can be verified using the same method as in the proof of Proposition 6.3. Of course, $\tau_g$ does not induce an isomorphism on isotropy groups because a generic genus $g$ curve for $g \geq 3$ has only the trivial automorphism, while all principally polarized abelian varieties have $\times[-1]$ as a nontrivial automorphism.

Example A.11. An example of a map which is bijective on geometric points but which does not satisfy the valuative criterion for radimmersions for traits is $[\text{Spec } \mathbb{R}/ (\mathbb{Z}/2\mathbb{Z})] \to \text{Spec } \mathbb{R}$. This fails to satisfy the valuative criterion because one can map the generic point of a trait to a trivial $\mathbb{Z}/2\mathbb{Z}$ torsor and the closed point to a nontrivial $\mathbb{Z}/2\mathbb{Z}$ torsor, and there will be no maps from the trait extending these. Indeed, any $\mathbb{Z}/2\mathbb{Z}$ torsor over a trait which is generically trivial is trivial.

The above examples raise the following question:

Question A.12. Is there a simple characterization of maps of algebraic stacks $f : X \to Y$ which satisfy the valuative criterion for radimmersions?

Note that Question A.12 is not answered by Theorem A.4 because we do not assume that $f$ is representable and a universal homeomorphism on isotropy groups.

A.3. Proving the valuative criterion. Before proving Theorem A.4 at the end of this section, we establish a number of preliminary lemmas. One of the main obstructions we face, not encountered in the schematic version from [Moc14, Chapter 1, §2.4], is to verify that $f$ is representable by schemes. This is verified using Zariski’s main theorem in Lemma A.15 after we show $f$ is separated. We next verify that $f$ satisfying Theorem A.4(3) have geometric fibers with at most one geometric point.

Lemma A.13. Suppose $f : X \to Y$ is a finite type representable morphism of algebraic stacks, inducing a universal homeomorphism on isotropy groups at each geometric point of $X$ and satisfying the valuative criterion for radimmersions for traits. Then each geometric fiber of $f$ has at most 1 geometric point.

Proof. Begin with a geometric point $y : \text{Spec } k \to Y$. Suppose $\alpha : \text{Spec } k \to X$ and $\beta : \text{Spec } k \to X$ are two geometric points of $X$ with 2-morphisms $f \circ \alpha \simeq y \simeq f \circ \beta$. Because the map induced by $f$ is a universal homeomorphism on isotropy groups, if $\alpha$ and $\beta$ map to 2-isomorphic points of $X$, they must map to the same point of $X_y := \text{Spec } k \times_{y, X} f X$. (In general, this property may fail when $f$ is not a universal homeomorphism on isotropy
groups, such as in the case of \( f : \text{Spec} k \to [\text{Spec} k/ (\mathbb{Z}/2\mathbb{Z})] \). Therefore, in order to show \( X_y \) has at most one geometric point, it suffices to exhibit a 2-morphism \( \alpha \simeq \beta \).

By the finite type hypothesis \( \alpha \) and \( \beta \) both factor though closed points of \( X_y \). On the other hand, taking \( T = \text{Spec} k[[x]] \) in Definition A.3 we may choose diagrams (A.1) sending the generic point of \( T \) to the image of \( \alpha \) via the inclusion \( k \hookrightarrow k((x)) \) and the closed point of \( T \) to the image of \( \beta \). Therefore, by the valuative criterion, we obtain a map \( T' \to X \) sending the generic point of \( T' \) to \( \alpha \) and the closed point to \( \beta \). Hence \( \beta \) lies in the closure of \( \alpha \). Because \( \alpha \) and \( \beta \) are both closed geometric points, we find \( \beta \simeq \alpha \). Therefore, \( X \) has at most one geometric point over \( y \) and hence \( X_y \) has at most one geometric point by the preceding paragraph.

\[ \Box \]

**Lemma A.14.** Suppose \( f : X \to Y \) is a finite type quasi-separated representable morphism of algebraic stacks with \( Y \) locally noetherian, and suppose \( f \) satisfies Theorem A.4(3). Then \( f \) is separated.

**Proof.** By [Sta, Tag 0E80] to show \( f \) is separated, it suffices to verify the uniqueness part of the valuative criterion for discrete valuation rings. So, suppose we are given some dominant map of traits \( T' \to T \) two maps \( h_1 : T' \to X \) and \( h_2 : T' \to X \) (in place of \( h \)) making the first diagram in (A.2) 2-commute. We claim the second diagram in (A.2) also 2-commutes. First, observe that by Lemma A.13 for any geometric point \( y : \text{Spec} k \to Y \), the fiber of \( f \), \( X_y := X \times_{Y, y} \text{Spec} k \), is 0-dimensional and quasi-separated. Therefore, \( X_y \) is a scheme by [Ols16, Theorem 6.4.1]. So, to show the maps \( \alpha_1 : t' \to T' \xrightarrow{h_1} X_y \) and \( \alpha_2 : t' \to T' \xrightarrow{h_2} X_y \) agree, it suffices to show their images map to the same geometric point. This follows from Lemma A.13 because \( f \circ \alpha_1 \simeq f \circ \alpha_2 \) by 2-commutativity of the right diagram of (A.2). Hence, it follows that both diagrams in (A.2) commute, and so \( h_1 \) agrees with \( h_2 \) by the uniqueness aspect of the valuative criterion for radimersions for traits.

We now deduce that morphisms of algebraic spaces satisfying the valuative criterion for radimersions for traits are representable by schemes.

**Lemma A.15.** Suppose \( f : X \to Y \) is a finite type quasi-separated morphism of algebraic spaces with \( Y \) a locally noetherian scheme, and suppose \( f \) satisfies the valuative criterion for radimersions for traits. Then \( X \) is in fact a scheme.

**Proof.** By Lemma A.13 \( f \) is radicial, hence quasi-finite. By Lemma A.14 \( f \) is separated. Hence, by [Sta, Tag 082J], (a variant of Zariski’s main theorem,) \( f \) is quasi-affine, and therefore \( f \) is a representable by schemes. Therefore, \( X \) is a scheme.

We next state a lemma with the goal of establishing (1) \( \Rightarrow \) (2) in Theorem A.4.

**Lemma A.16.** If \( f : X \to Y \) is a radimmersion of schemes with \( Y \) a locally noetherian scheme, then \( f \) satisfies the valuative criterion for traits with \( T = T' \).

**Proof.** We can factor \( f \) as a composition \( X \to U \to Y \) for \( X \to U \) a finite radicial morphism and \( U \to Y \) an open immersion. Suppose we have commutative diagrams as in (A.1). Because both \( t \) and \( \eta \) factor through \( U \subset Y \), we may replace \( Y \) by \( U \). Then, we may assume the map \( f \) is finite, and in particular proper. By the valuative criterion for properness, a morphism \( h : T' = T \to Y \) exists making the first diagram in (A.2) commute. The second
diagram in (A.2) then also commutes since the map \( t \to Y \) is uniquely determined by the composition \( t \to X \xrightarrow{f} Y \) by Lemma A.13.

We now bootstrap the preceding lemma to morphisms of algebraic stacks.

**Corollary A.17.** If \( f : X \to Y \) is a radimmersion of algebraic stacks with \( Y \) locally noetherian, then \( f \) satisfies the valuative criterion for radimmersions for traits with \( T = T' \).

**Proof.** Suppose we are given 2-commuting diagrams as in (A.1) with \( T \) and \( T' \) traits. Because \( X \to Y \) is a radimmersion, the fiber product \( X \times_Y T \) is a scheme and the resulting map \( X \times_Y T \to T \) is a radimmersion. By the universal property of fiber products, we obtain 2-commuting diagrams

\[
\begin{array}{ccc}
\eta & \longrightarrow & X \times_Y T \\
\downarrow & & \downarrow \\
T & \longrightarrow & Y
\end{array}
\quad
\begin{array}{ccc}
t & \longrightarrow & X \times_Y T \\
\downarrow & & \downarrow \\
T & \longrightarrow & Y
\end{array}
\]

Hence, by applying Lemma A.16 to the left squares in the above diagrams, we obtain a unique lift \( h' : T \to X \times_Y T \) making the squares in (A.3) 2-commute. This implies there is a unique lift \( h : T \to X \) making (A.2) 2-commute as in the valuative criterion for radimmersions for traits with \( T = T' \). Specifically, \( h \) is the composition of \( h' \) with the projection \( X \times_Y T \to X \).

For the implication \((1) \implies (2)\) we will also need the following verification that the map on isotropy groups is a universal homeomorphism.

**Lemma A.18.** A radimmersion \( f : X \to Y \) of algebraic stacks induces a universal homeomorphism on isotropy groups at each geometric point of \( X \).

**Proof.** We can factor \( f : X \to Y \) as \( X \to U \to Y \) where \( X \to U \) is finite radicial and \( U \to Y \) is an open immersion. An open immersion induces an isomorphism on isotropy groups at every point of the source, so it suffices to prove the lemma in the case that \( f \) is finite radicial. Choose a geometric point \( x : \text{Spec} k \to X \) and let \( y : \text{Spec} k \to X \to Y \) denote the composition. Let \( X_y := X \times f, Y, \text{Spec} k \). The map \( x : \text{Spec} k \to X \) and the 2-morphism \( x \circ f \simeq y \) induce a map \( z : \text{Spec} k \to X_y \). Then, we have the following diagram, where all squares are cartesian

\[
\begin{array}{ccc}
\text{isom}_X(x, x) & \longrightarrow & \text{Spec} k \\
\downarrow & & \downarrow \quad \quad \downarrow z \\
\text{isom}_Y(y, y) & \longrightarrow & X_y \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow y \\
\text{Spec} k & \longrightarrow & X \\
\downarrow x \quad \quad \downarrow f \\
\text{Spec} k & \longrightarrow & Y
\end{array}
\]

Because \( f \) is a finite radicial map, \( g : X_y \to \text{Spec} k \) is also finite radicial. Additionally, \( g \) is surjective because \( z \) is a section of \( g \). So, \( g \) is finite radicial and surjective, therefore a universal homeomorphism [Gro67, 18.12.11]. Since \( z : \text{Spec} k \to X_y \) is a section of \( g \), it is also a universal homeomorphism. Therefore, the map \( \text{isom}_X(x, x) \to \text{isom}_Y(y, y) \), which is the base change of \( z \) along \( h \), is also a universal homeomorphism. 

\[\square\]
The following lemma will be useful for reducing the implication \((3) \implies (1)\) to the case that \(Y\) is an algebraic space.

**Lemma A.19.** Suppose \(\alpha : \mathcal{X} \to \mathcal{Y}\) is a finite type quasi-separated representable morphism of algebraic stacks with \(Y\) locally noetherian and suppose \(\alpha\) satisfies \(\text{[Theorem A.4]}\). Then, for any scheme \(Z\), the base change map \(\mathcal{X} \times_{\mathcal{Y}} Z \to Z\) also satisfies the valuative criterion for radimmersions for traits.

**Proof.** Given any map \(T \to Z\) from a trait \(T\), making \(\text{(A.1)}\) commute with \(X = \mathcal{X} \times_{\mathcal{Y}} Z\) and \(Y = Z\), we wish to show there is a unique dominant map of traits \(T' \to T\) and \(h : T' \to \mathcal{X} \times_{\mathcal{Y}} Z\) making the resulting diagrams in \(\text{(A.2)}\) commute. Since \(\mathcal{X} \to \mathcal{Y}\) satisfies the valuative criterion for radimmersions for traits, we obtain a map \(T' \to \mathcal{X}\) making the diagrams in \(\text{(A.2)}\) associated to the map \(\alpha : \mathcal{X} \to \mathcal{Y}\) 2-commute. Since we are also given a map \(T' \to Z\), we obtain the desired map \(h : T' \to \mathcal{X} \times_{\mathcal{Y}} Z\) making \(\text{(A.2)}\) commute. We only need verify uniqueness of the map \(h\). There is a unique geometric point \(q\) of \(\mathcal{X} \times_{\mathcal{Y}} Z\) over the image of \(\eta'\) in \(Z\) by \(\text{[Lemma A.13]}\). This implies that any such map \(h\) must send \(\eta' \mapsto q\). From the valuative criterion for separatedness, to show the map \(h\) is unique, it suffices to show \(\mathcal{X} \times_{\mathcal{Y}} Z \to Z\) is separated. This holds by \(\text{[Lemma A.14]}\) because \(\mathcal{X} \times_{\mathcal{Y}} Z \to Z\) is the base change of the separated map \(\mathcal{X} \to \mathcal{Y}\). \(\square\)

Combining the above, we now prove \(\text{[Theorem A.4]}\).

**Proof of Theorem A.4.** \((1)\) implies \((2)\) by \(\text{[Corollary A.17]}\) and \(\text{[Lemma A.18]}\). Also, \((2)\) implies \((3)\) by definition. So it only remains to check \((3)\) implies \((1)\).

In order to verify \(f : X \to Y\) is a radimmersion, we may do so smooth locally on \(Y\). Therefore, for \(U \to Y\) a smooth cover, the map \(U \times_Y X \to U\) again satisfies \((3)\) by \(\text{[Lemma A.19]}\). Hence, we may assume that \(Y\) is a scheme. By representability of \(f\), \(X\) is an algebraic space. By \(\text{[Lemma A.15]}\), we find that \(X\) is also a scheme. Since \(Y\) is locally noetherian, and \(f\) is finite type, \(f\) is in fact finitely presented.

Next, we reduce to the case that \(Y\) is a strictly Henselian local ring whose closed point lies in the image of \(f\) and \(f\) is finite radicial. Indeed, for any \(x \in X\), suppose we know \(f : X \to Y\) is a finite radicial map after base change to the strict henselization at \(f(x) \in Y\). Note the strict henselization is again noetherian by \(\text{[Gro67, 18.8.8(iv)]}\). Then, since \(f\) is finitely presented, by spreading out for finite morphisms \(\text{[Gro66, 8.10.5(x)]}\), the map \(f\) is finite after base change to some étale neighborhood of \(f(x) \in Y\). Let \(U\) be the union over the images of all such neighborhoods in \(Y\). By fppf descent for finite morphisms \(\text{[Sta, Tag 02LA]}\), it follows that \(f : X \to U\) is finite. Further, \(X \to U\) is radicial because it is so on all fibers over points of \(U\). Therefore, we find that \(f\) factors through \(U\), with \(U \to Y\) an open immersion, and \(X \to U\) finite radicial.

To conclude the proof, we verify that \(f : X \to Y\) is a finite radicial map in the case \(Y\) is a strictly Henselian scheme and the closed point of \(Y\) lies in the image of \(f\). From \(\text{[Lemma A.13]}\), \(f\) is radicial, hence quasi-finite. From \(\text{[Lemma A.14]}\), \(f\) is separated. So, by Zariski’s main theorem \(\text{[Gro67, 18.12.13]}\), we find \(f\) can be factored as \(X \to Z \to Y\) with \(\alpha : X \to Z\) an open immersion and \(\beta : Z \to Y\) finite. By \(\text{[Gro67, 18.5.11(a)]}\), \(Z\) is a disjoint union \(C_1 \sqcup \cdots \sqcup C_r\) with \(C_i = \text{Spec } R_i\), for \(R_i\) a local ring. We next show that \(r = 1\). Chose a point \(x \in X\) with \(f(x)\) mapping to the closed point of \(Y\). We may assume \(x \in C_1\). Choose some \(w \in X\) lying in \(C_i\) for some \(i, 1 \leq i \leq r\). We will show that \(i = 1\). Since \(Y\) is noetherian, we may construct a trait \(T\) whose closed point maps to \(f(x)\) and whose generic point maps to \(f(w)\) \(\text{[Sta, Tag 054F]}\). Hence, by the valuative criterion for
radimmersions for traits, we can find an extension $T' \to T$ of traits whose closed point maps to $x$ and whose generic point maps to $w$. Therefore, $x$ is in the closure of $w$ and $i = 1$. We conclude that we $\alpha$ factors through $C_1$, so we may take $\alpha$ to be an isomorphism, as $\text{im } \alpha$ contains the closed point of $C_1$, and hence $f$ is finite radicial.

\[\square\]

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