SUPPORTS OF WEIGHTED EQUILIBRIUM MEASURES:
COMPLETE CHARACTERIZATION

MUHAMMED ALİ ALAN AND NİHAT GÖKHAN GÖĞÜŞ

Abstract. In this paper, we prove that a compact set \( K \subset \mathbb{C}^n \) is the support of a weighted equilibrium measure if and only it is not pluripolar at each of its points extending a result of Saff and Totik to higher dimensions. Thus, we characterize the supports weighted equilibrium measures completely. Our proof is a new proof even in one dimension.

1. Introduction and Background

The supports of weighted extremal measures \( S_w \), are important in pluripotential theory, approximation theory, complex geometry, and they are loosely related to parabolic manifolds [AS11].

Once we know the support of the weighted extremal measure, the weighted extremal function, \( V_{K,Q} \), can be determined by solving the homogenous complex Monge-Ampère equation in the bounded components of the complement with boundary value \( Q \). Furthermore, \( V_{K,Q} = Q \) on the support \( S_w \) quasi everywhere.

Another advantage of determining the supports of weighted extremal measures is as follows: The weighted extremal function of \( K \) with respect to \( Q \) and the weighted extremal function of the support \( S_w \) with respect to the weight \( Q|_{S_w} \) are equal. Thus, determining the support of weighted extremal measures makes approximating the weighted capacities very efficient (see [RRR10].)

Some applications in weighted approximation are as follows. By Theorem 2.12 of Appendix B of [ST97], a weighted polynomial attains its essential supremum on the support \( S_w \). In order to make a weighted approximation of a continuous function \( f \) on \( K \), \( f \) must vanish outside of \( K \). Namely, if \( f \) is continuous on \( K \) and there is a sequence of weighted polynomials \( w^dP_d \) converging uniformally to \( f \) on \( K \), then \( f \equiv 0 \) on \( S_w \) (see [ST97, Cal07].)

Since the weighted extremal function \( V^*_{K,Q} \) is locally bounded, the weighted extremal measure \( (dd^c V^*_{K,Q})^n \) does not put mass on pluripolar sets, i.e., \( \text{supp}(dd^c V^*_{K,Q})^n \) is not pluripolar at each of its points; i.e., for all \( z \in K \)
and all $r > 0$, $B(z, r) \cap K$ is not pluripolar. It is natural to ask the converse. Namely, if $K$ is a compact set which is not pluripolar at each of its points, then does there exist an admissible weight $Q$ on $K$ such that $\text{supp}(dd^cV_{K,Q})^n = K$?

The following theorem gives the converse in $\mathbb{C}$, which characterizes the supports of weighted extremal measures in $\mathbb{C}$.

**Theorem 1.1.** [ST97, Theorem IV.1.1] If $K$ is a compact subset of $\mathbb{C}$ which is not pluripolar at each of its points, then there exists an admissible weight on $K$ such that $\text{supp}(\Delta V_{K,Q}) = K$.

Unfortunately, the proof of the theorem uses logarithmic potentials which is not available in $\mathbb{C}^n$. Branker and the first author investigated the supports of weighted extremal measures (see [Bra04, Ala]). In this paper, we obtain the same theorem [1.1] in $\mathbb{C}^n$ as our main result.

First we recall few facts from weighted and unweighted (pluri-)potential theory. Standard references are [Ran95] for unweighted potential theory, [Kli91] for unweighted pluripotential theory, [ST97] for weighted potential theory, and Appendix B in the same book by Thomas Bloom for weighted pluripotential theory.

Let $K$ be a closed subset of $\mathbb{C}^n$. An admissible weight function on $K$ is a lower semicontinuous function $Q : K \to (-\infty, \infty]$ such that

i) $\{ z \in K \mid Q(z) < \infty \}$ is not pluripolar.

ii) If $K$ is unbounded, then $Q(z) - \log |z| \to \infty$ as $|z| \to \infty$, $z \in K$.

The function $w = e^{-Q}$ is also used equivalently in the terminology. Especially, the notation $w$ is used more often in weighted approximation (see [Blo09, BL03, ST97]).

The **weighted Siciak-Zahariuta extremal function** of $K$ with respect to $Q$ is defined as

$$V_{K,Q}(z) := \sup \{ u(z) \mid u \in L, u \leq Q \text{ on } K \}.$$  \hfill (1.1)

Recall that $L$ is the Lelong class:

$$L := \{ u \mid u \text{ is plurisubharmonic on } \mathbb{C}^n, u(z) \leq \log^+ |z| + C_u \}. $$ \hfill (1.2)

If $Q = 0$, then $V_{K,0}$ is called the **(unweighted) Siciak-Zahariuta extremal function** of $K$ and $V_K$ denotes it.

A compact set $K$ is called **regular** if $V_K$ is continuous. If $K \cap \overline{B(z, r)}$ is regular for all $z \in K$ and $r > 0$, the set $K$ is called **locally regular**. Here we use the notation $B(z_0, r)$ for the open ball of radius $r$ and center $z_0$. 
It is well known that the upper semicontinuous regularization of \( V_{K,Q} \) is plurisubharmonic and in \( L^+ \) where
\[
L^+ := \{ u \in L | \log^+ |z| + C_u \leq u(z) \}.
\]
Recall that the upper semicontinuous regularization of a function \( v \) is defined by
\[
v^*(z) := \limsup_{w \to z} v(w).
\]

A subset \( P \subset \mathbb{C}^n \) is called pluripolar if \( E \subset \{ z \in \mathbb{C}^n | u(z) = -\infty \} \) for some plurisubharmonic function \( u \). If a property holds everywhere except on a pluripolar set we will say that the property holds quasi everywhere. It is a well-known fact that \( V_{K,Q} = V_{K,Q}^* \) quasi everywhere. See [Kli91].

Let \( S_w \) denotes the support of the \((dd^c V_{K,Q})^n\), where \((dd^c u)^n\) is the Monge-Ampère measure of \( u \). The following lemma is very useful to determine the supports of Monge-Ampère measures.

**Lemma 1.2.** [ST97, Appendix B, Theorem 1.3] Let \( S^*_w := \{ z \in \mathbb{C}^n | V_{K,Q}^*(z) \geq Q(z) \} \). Then we have \( S_w \subset S^*_w \).

**Theorem 1.3.** [Dem92, Proposition 11.9] Let \( u, v \) be locally bounded plurisubharmonic functions on \( \Omega \). Then we have the following inequality
\[
(dd^c \max\{u, v\})^n \geq \chi_{\{u \geq v\}}(dd^c u)^n + \chi_{\{u < v\}}(dd^c v)^n.
\]
Here \( \chi_A \) is the characteristic function of \( A \). The inequality \((1.3)\) will be called the Demailly inequality.

**Proposition 1.4.** [Sic81, Proposition 2.13] If \( K \) is locally regular and \( Q \) is continuous, then \( V_{K,Q} \) is continuous.

2. Characterization of the Supports

**Proposition 2.1.** Let \( K \) be a non-pluripolar compact set in \( \mathbb{C}^n \) and let \( u \) be a continuous plurisubharmonic function in Lelong class. If \( Q \) is the weight on \( K \) defined by \( Q := u|_K \), then we have \( V_{K,Q} = u |_K \).

**Proof.** Because \( u \) itself is a competitor in the envelope defining \( V_{K,Q} \), we have \( u \leq V_{K,Q} \) on \( \mathbb{C}^n \); and \( V_{K,Q} \leq Q = u \) on \( K \). Thus \( V_{K,Q} = u \) on \( K \). \( \square \)

Note that \( u = V_{K,Q}^* \) quasi everywhere on \( K \); i.e., we have \( u = V_{K,Q}^* \) on \( K \setminus P \) where \( P \) is a pluripolar set. The following theorem is our main result which gives the complete characterization of supports of weighted extremal measures.

**Theorem 2.2.** Let \( K \) be a compact set in \( \mathbb{C}^n \) which is not pluripolar at each of its points; i.e., for all \( z \in K \) and all \( r > 0 \), \( B(z, r) \cap K \) is not pluripolar. There exists a continuous weight \( Q \) on \( K \) so that \( K = \text{supp}(dd^c V_{K,Q}^*)^n \).
Proof. Since \( K \) is compact, \( K \subset K_r \) for some \( r > 0 \), where \( K_r := B(z, r) \). Let \( Q_r \) be the weight on \( K_r \) defined by \( Q_r := \frac{1}{\sqrt{2\pi r}} |z|^2 \). By Example 3.7 of [Ala], we have \( \text{supp}(dd^c V_{K_r, Q_r})^n = K_r \).

We define \( Q|_K := u = V_{K_r, Q_r} \). By proposition 2.1 we have \( V_{K_r, Q_r} = u \) quasi everywhere on \( K \). By Demailly’s inequality we have

\[
(dd^c V_{K_r, Q_r})^n \geq \chi \{ V_{K_r, Q_r} \geq V_{K_r, Q_r} \}(dd^c V_{K_r, Q_r})^n + \chi \{ V_{K_r, Q_r} > V_{K_r, Q_r} \}(dd^c V_{K_r, Q_r})^n.
\]

Due to the facts that the set \( \{ V_{K_r, Q_r} > V_{K_r, Q_r} \} \cap K \) is pluripolar, and that \( V_{K_r, Q_r} \) is locally bounded, we have \( (dd^c V_{K_r, Q_r})^n \) vanishes on \( \{ V_{K_r, Q_r} > V_{K_r, Q_r} \} \cap K \). Therefore, we have \( (dd^c V_{K_r, Q_r})^n \geq (dd^c V_{K_r, Q_r})^n \) quasi everywhere on \( K \). Namely, for any non-pluripolar (Borel) subset \( E \) of \( K \), we have

\[ (dd^c V_{K_r, Q_r})^n(E) \geq (dd^c V_{K_r, Q_r})^n(E) > 0. \]

For any \( z \in K \) for every \( r > 0 \), we have \( (dd^c V_{K_r, Q_r})^n(K \cap B(z, r)) > 0 \). Therefore \( z \in \text{supp}(dd^c V_{K_r, Q_r})^n \). □

Corollary 2.3. Let \( K \) be a locally regular compact subset of \( \mathbb{C}^n \). Then there exists a continuous weight \( Q \) on \( K \) such that \( K = \text{supp}(dd^c V_{K_r, Q_r})^n \) and \( Q = V_{K_r, Q} \) on \( K \).

Proof. We define \( K_r \) and \( Q_r \) as in the proof of above theorem. By above theorem we have \( K = \text{supp}(dd^c V_{K_r, Q_r})^n \). By Proposition 2.4 we have \( V_{K_r, Q} \) is continuous, thus \( V_{K_r, Q} \leq Q \) on \( K \). By combining these with Lemma 2.2 we have \( Q = V_{K_r, Q} \) on \( K \). □

As a corollary, we obtain the following unexpected result.

Corollary 2.4. There exists a continuous plurisubharmonic function \( u \in L^+ \), such that \( \text{supp}(dd^c u)^n = \partial \Delta^n \), where \( \Delta^n \) is the polydisc in \( \mathbb{C}^n \).

Open Problem 2.5. A compact set \( K \subset \mathbb{C}^n \) is locally regular if and only if it is the support of the Monge-Ampère measure of a continuous function in \( L^+ \).

Remark 2.6. Note that the above open problem might be a step to understand the measures which are Monge-Ampère measures of continuous plurisubharmonic function.

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(Muhammed Ali Alan) **Syracuse University, Syracuse, NY, 13244 USA**

*E-mail address: malan@syr.edu*

(Nihat Gökhan Göğüş) **Sabancı University, Orhanlı, Tuzla 34956, Istanbul, TURKEY. E-mail: nggogus@sabanciuniv.edu**