ON THE BACKWARD UNIQUENESS OF THE STOCHASTIC
PRIMITIVE EQUATIONS WITH ADDITIVE NOISE

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ABSTRACT. The previous works focus on the uniqueness for the initial-value
problems of stochastic primitive equations. Uniqueness for the initial-value
problems means that if the two initial conditions are the same, then the two
solutions coincide with each other. However there is no work to answer what
will happen to the solutions if the two initial conditions are different. This
problem for the stochastic three dimensional primitive equations is addressed
by the backward uniqueness established in this article. The backward unique-
ness means that if two solutions intersect at time $t > 0$, then they are equal
everywhere on the interval $(0, t)$. In other words, given two different initial-
value conditions, the corresponding two solutions will never cross in the future.
Hence this article can be viewed as a further study of the dependence of the
solutions on the initial data.

1. Introduction. The paper is concerned with the primitive equations (PEs) in a
bounded domain with Wiener noises. To outline its content in detail, we introduce
a bounded domain $\Omega = M \times (-\frac{1}{2}, \frac{1}{2})$ with $M = (0, 1) \times (0, 1)$ and consider the
following 3D stochastic viscous PEs of Geophysical Fluid Dynamics.

\[
\begin{align*}
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla_H) \mathbf{v} + w \partial_z \mathbf{v} + f v^\perp + \nabla_H p - \Delta \mathbf{v} &= \dot{W}_1, \quad (1) \\
\partial_z p + \theta &= 0, \quad (2) \\
\nabla_H \cdot \mathbf{v} + \partial_z w &= 0, \quad (3) \\
\partial_t \theta + \mathbf{v} \cdot \nabla_H \theta + w \partial_z \theta - w - \Delta \theta &= Q + \dot{W}_2. \quad (4)
\end{align*}
\]

The unknowns for the 3D stochastic viscous PEs are the fluid velocity field $(\mathbf{v}, w) = (v_1, v_2, w) \in \mathbb{R}^3$ with $\mathbf{v} = (v_1, v_2)$ and $v^\perp = (-v_2, v_1)$ being horizontal, the temperature $\theta$ and the pressure $p$. $f = f_0(\beta + y)$ is the given Coriolis parameter, $Q$ is a
given heat source which is periodic in $x, y, z$ and is odd in $z$. In this paper we use

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the notations $\nabla_H = (\partial_x, \partial_y)$ and $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ to denote the horizontal gradient and the three dimensional Laplacian operators. Here, we take $\tilde{W}_i(t, x, y, z), i = 1, 2,$ the informal derivative for the Wiener process $W_i$ given below.

We supplement (1)-(4) with the following boundary and initial conditions:

\[ \begin{align*}
\textbf{v}, w, p, \text{ and } \theta & \text{ are periodic in } x, y, z, \\
\textbf{v} \text{ and } p & \text{ are even in } z, w \text{ and } \theta \text{ are odd in } z, \\
(\textbf{v}, \theta)|_{t=0} & = (\textbf{v}_0, \theta_0).
\end{align*} \]  

(5)

(6)

(7)

The primitive equations are the basic model used in the study of climate and weather prediction, which can be used to describe the motion of the atmosphere when the hydrostatic assumption is enforced (see [18, 23, 24] and the references therein). This model has been intensively investigated because of the interests stemmed from physics and mathematics. As far as we know, their mathematical study was initiated by Lions, Temam and Wang (see e.g. [29, 30, 31, 32]). For example, the existence of global weak solutions for the primitive equations was established in [30]. Guillén-González et al. obtained the global existence of strong solutions to the primitive equations with small initial data in [21]. The local existence of strong solutions to the primitive equations under the small depth hypothesis was established by Hu et al. in [25]. Cao and Titi developed a beautiful approach to dealing with the $L^6$-norm of the fluctuation $\tilde{v}$ of horizontal velocity and obtained the global well-posedness for the 3D viscous primitive equations in [9]. Subsequently, in [26], Kukavica and Ziane developed a different method to handle non-rectangular domains and boundary conditions with physical reality. For the global well-posedness of 3D primitive equations with partial dissipation, we refer the reader to some papers, see e.g. [5, 6, 7, 8, 10].

Along with the great successful developments of deterministic primitive equations, the random case has also been developed rapidly. In [19], Guo and Huang studied the long-time behavior of the strong solution to the stochastic primitive equations with additive stochastic forcing. When the noise is multiplicative, Gao and Sun [16] proved the global existence and uniqueness for the strong solution. Moreover, when the noise tends to zero, Gao and Sun [17] established the large deviation principle for this stochastic system. In [22], Gao and Sun studied the long-time behavior of stochastic PEs when the velocity is perturbed by an additive noise. Debussche, Glatt-Holtz, Temam and Ziane concerned the global well-posedness of strong solution when the primitive equations are driven by multiplicative stochastic forcing in [12]. Under the periodic conditions, Glatt-Holtz, Kukavica, Vicol and Ziane considered the existence of invariant measure for the 3D PEs in [20].

The uniqueness of the invariant measure and large deviations for the 3D stochastic primitive equations were obtained by Dong, Zhai and Zhang in [14, 15] under the periodic boundary conditions. Some analytical properties of weak solutions of 3D stochastic primitive equations with periodic boundary conditions were obtained in [13], in which the martingale problem associated to this model is shown to have a family of solutions satisfying the Markov property.

Our main goal of this article is to establish the backward uniqueness of the stochastic three dimensional primitive equations. This kind of uniqueness is different from uniqueness for the initial-value problems. Backward uniqueness means that given two solutions if these solutions intersect at some time $t$, they must equal everywhere before $t$. In other words, if the two solutions start with two different
initial state, then the two solutions will never cross. Therefore, this article can be viewed as a further study about the dependence of the solutions to stochastic primitive equations on the initial data. To prove our result we adopt the logarithmic energy method from [33]. The idea of the method is to try to estimate \( \log |U (t) \|^2 \) where \( U \) is the difference of the two solutions. Exactly speaking, we try to obtain some kind of estimates like

\[
\log |U(t)|^2 \geq \log |U(t_0)|^2,
\]

where \( t \) represents the present time and \( t_0 \) is the initial time. The above inequality implies that if the two solutions cross at present time \( t \), they will equal at any previous time \( t_0 \) (see Theorem 3.4). To show the inequality, the difficulty lies in the establishment of the bound of \( \frac{\| U(t) \|_2}{\| U(t_0) \|_2} \) on the finite interval, where \( \| \cdot \|_2 \) and \( \| \cdot \|_1 \) are norms in Lebesgue space \( L^2 \) and Sobolev space \( H^1 \) respectively. The a priori estimates of \( \frac{\| U(t) \|_2}{\| U(t_0) \|_2} \) heavily rely on the delicate and careful argument on the special geometric structure of stochastic primitive equations (see Theorem 3.3).

The remaining of this paper is organized as follows. Notations and some results are recalled in section 2. In section 3, the regularity of the strong solutions to stochastic PEs is improved (see Theorem 3.1 and Theorem 3.2). Then the bound of \( \frac{\| U(t) \|_2}{\| U(t) \|_2} \) is obtained in Theorem 3.3 by virtue of Theorem 3.1 and Theorem 3.2. The main result the backward uniqueness for the 3D stochastic PEs is established in Theorem 3.4. As usual, constants \( C \) may change from one line to the next, unless, we give a special declaration; we denote by \( C(a) \) a constant which depends on some parameter \( a \).

2. Preliminaries. For \( 1 \leq p \leq \infty \), let \( L^p(\mathbb{O}) \) be the usual Lebesgue spaces with the norm \( \| \cdot \|_p \), when \( p = 2 \), we denote by \( \langle \cdot, \cdot \rangle \) the inner product in \( L^2(\mathbb{O}) \). For a positive integer \( m \), we denote by \( (H^{m,p}(\mathbb{O}), \| \cdot \|_{m,p}) \) the usual Sobolev spaces, see [1]). When \( p = 2 \), we denote by \( (H^m(\mathbb{O}), \| \cdot \|_m) \) with inner product \( \langle \cdot, \cdot \rangle_{H^m} \).

In the following we will need terminology to describe periodic boundary conditions. Let \( m \in \mathbb{N} \). Then we say that a smooth function \( f : \mathbb{O} \rightarrow \mathbb{R} \) is periodic of order \( m \) on \( \partial \mathbb{O} \) if

\[
\frac{\partial^\alpha f}{\partial x^\alpha}(0, y, z) = \frac{\partial^\alpha f}{\partial x^\alpha}(1, y, z) \quad \text{and} \quad \frac{\partial^\alpha f}{\partial y^\alpha}(x, 0, z) = \frac{\partial^\alpha f}{\partial y^\alpha}(x, 1, z),
\]

\[
\frac{\partial^\alpha f}{\partial z^\alpha}(x, y, -\frac{1}{2}) = \frac{\partial^\alpha f}{\partial z^\alpha}(x, y, \frac{1}{2})
\]

for all \( \alpha = 0, \ldots, m \). We assume space periodicity with period \( \mathbb{O} \), meaning that all functions are taken to satisfy:

\[
f(x, y, z, t) = f(x + 1, y, z, t) = f(x, y + 1, z, t) = f(x, y, z + 1, t),
\]

when extended to \( \mathbb{R}^3 \). Define

\[
\mathcal{H}^m_{per}(\mathbb{O}) := \{ f \in H^m(\mathbb{O}) | \text{f is periodic of order } m - 1 \text{ on } \partial \mathbb{O} \}
\]

\[
\text{and satisfy } \int_{\mathbb{O}} f d\mathbb{O} = 0. \tag{9}
\]

We introduce our working space in the following

\[
V_1 = \left\{ \mathbf{v} \in (\mathcal{H}^1_{per}(\mathbb{O}))^2 | \mathbf{v} \text{ even in } z, \int_{-\frac{1}{2}}^{\frac{1}{2}} \nabla_H \cdot \mathbf{v} dz = 0 \right\},
\]

\[
V_2 = \left\{ \theta \in \mathcal{H}^1_{per}(\mathbb{O}) | \theta \text{ odd in } z \right\},
\]
We define the inner product of $H^1$ Wiener processes in $W^2_1$. We denote by $\mathbb{H}^{-m}$ and $\mathbb{H}^{-m}$ the dual space of $\mathbb{H}^m$ and $\mathbb{H}^m$ respectively. $W_1$ and $W_2$ are standard Wiener processes in $\mathbb{H}$ and $\mathbb{H}$ defined below and have the form of

$$W_j(t) := \sum_{n=1}^{\infty} \lambda_n^{1/2} e_{i,j} B_i(t), j = 1, 2,$$

where $((B_i(t))_{t \in \mathbb{R}})_{i \in \mathbb{N}}$ be a sequence of one-dimensional, independent, identically distributed Brownian motions defined on the complete probability space $(\Omega, \mathcal{F}, P)$, $(e_{i,1})_{i \in \mathbb{N}}$ and $(e_{i,2})_{i \in \mathbb{N}}$ are orthonormal basis in $\mathbb{H}$ and $\mathbb{H}$ respectively. $(\lambda_i)_{i \in \mathbb{N}}$ is a convergent sequence of positive numbers which ensure $W_1$ and $W_2$ are standard Wiener processes in $\mathbb{H}$ and $\mathbb{H}$ respectively. Let $U_i := (v_i, \theta_i)$ be the horizontal velocity and temperature with $i = 1, 2, 3$. We equip $V := \mathbb{H}^1 \times \mathbb{H}^1$ with the inner product

$$\langle U_1, U_2 \rangle_V := \int_{\Omega} (\nabla v_1 \cdot \nabla v_2 + \nabla \theta_1 \cdot \nabla \theta_2) d\Omega,$$

where $\nabla = (\partial_x, \partial_y, \partial_z)$. Subsequently, the norm of $V$ is defined by $\|U_i\|_1 = \langle U_i, U_i \rangle_V^{1/2}$. We define the inner product of $H := (L^2(\Omega))^3$ by

$$\langle U_1, U_2 \rangle_H := \langle v_1, v_2 \rangle + \langle \theta_1, \theta_2 \rangle = \int_{\Omega} (v_1 \cdot v_2 + \theta_1 \cdot \theta_2) d\Omega.$$

Define $a : V \times V \to \mathbb{R}$ bilinear, continuous, coercive

$$a(U_1, U_2) = \langle U_1, U_2 \rangle_V, \tag{11}$$

$b : V \times V \times \mathbb{H}^2 \to \mathbb{R}$ trilinear:

$$b(U_1, U_2, U_3) = \langle v_1 \cdot \nabla_H v_2, v_3 \rangle + \langle w(v_1) \partial_z v_2, v_3 \rangle + \langle v_1 \cdot \nabla_H \theta_2, \theta_3 \rangle + \langle w(v_1) \partial_z \theta_2, \theta_3 \rangle, \tag{12}$$

where $w(f) = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \nabla_H \cdot f(x, y, z') dz'$, $f \in \mathbb{R}^2$. Obviously, we have $b(U_1, U_2, U_3) = -b(U_1, U_3, U_2)$ and $b(U_1, U_2, U_2) = 0$. Define $e : V \times V \to \mathbb{R}$ bilinear, continuous:

$$e(U_1, U_2) = \langle v_1^*, v_2 \rangle + \langle \theta_1, w(v_2) \rangle - \langle \theta_2, w(v_1) \rangle. \tag{13}$$

Obviously, $e(U_1, U_1) = 0$. Denote by $V'$ the dual space of $V$. In the following, we introduce the operators $A, B$ and $E$.

$A$ is a linear continuous from $V$ into $V'$, given by

$$\langle AU_1, U_2 \rangle = a(U_1, U_2), \quad \forall U_1, U_2 \in V.$$

$B$ is a bilinear continuous from $V \times V$ into $\mathbb{H}^{-2}$, defined by

$$\langle B(U_1, U_2), U_3 \rangle = b(U_1, U_2, U_3), \quad \forall U_1, U_2 \in V, \quad \forall U_3 \in \mathbb{H}^2 \times \mathbb{H}^2.$$
$E$ is a linear continuous from $V$ into $V'$ satisfying
\[ \langle EU_1, U_2 \rangle = e(U_1, U_2), \quad \forall U_1, U_2 \in V. \]
Define the linear operator $A_1 : \mathbb{H}^1 \to \mathbb{H}^{-1}$, as follows:
\[ \langle A_1 v_1, v_2 \rangle = \langle \nabla v_1, \nabla v_2 \rangle, \forall v_1, v_2 \in \mathbb{H}^1. \]
Define the linear operator $A_2 : \mathbb{H}^1 \to \mathbb{H}^{-1}$, as follows:
\[ \langle A_2 \theta_1, \theta_2 \rangle = \langle \nabla \theta_1, \nabla \theta_2 \rangle, \forall \theta_1, \theta_2 \in \mathbb{H}^1. \]
Since the operator $A_i, i = 1, 2$ is positive selfadjoint with compact resolvent, by the classical spectral theorems there exists a sequence $\{\alpha_{i,j}\}_{j \in \mathbb{N}}$ of eigenvalues of $A_i$ such that
\[ 0 < \alpha_{i,1} \leq \alpha_{i,2} \leq \cdots, \quad \alpha_{i,j} \to \infty \]
corresponding to the eigenvectors $e_{i,j}$. Assume
\[ \sum_{j=1}^{\infty} \lambda_j \alpha_{i,j}^3 < \infty. \quad (14) \]
For arbitrary constant $T > 0$, $i = 1, 2$ and $j \in \mathbb{N}$, we define
\[ z_i(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \int_0^t e^{-A_i(t-s)} e_{i,n} dB_n(s), \quad t \in [0, T] \]
and
\[ z_i(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \int_0^t e^{-A_i(t-s)} e_{i,n} dB_n(s), \quad t \in [0, T]. \]
Obviously,
\[ z_1(w) \in C([0, T]; \mathbb{H}^3) \text{ and } z_2(w) \in C([0, T]; \mathbb{H}^1), \quad \mathbb{P} - \text{a.e. } \omega \in \Omega. \]
For $k \in \mathbb{N}$ and $k > j$, in view of an infinite dimensional version of Burkholder-Davis-Gundy type of inequality for stochastic convolutions (see Theorem 1.2.6 in [27] and references therein), we have
\[ E \sup_{t \in [0, T]} \| A_i^{3/2}(z_i^1 - z_i^k) \|_{L^2}^2 \]
\[ \leq C T \sum_{n=j+1}^{\infty} \lambda_n \alpha_{i,n}^3 \to 0, \quad \text{as } j \to \infty. \]
Therefore
\[ z_1(w) \in C([0, T]; \mathbb{H}^3), \text{ and } z_2(w) \in C([0, T]; \mathbb{H}^1), \quad \mathbb{P} - \text{a.e. } \omega \in \Omega. \quad (15) \]
Obviously, $(z_1(t))_{t \in [0, T]}$ is the unique solution which is periodic in $x, y, z$ and even in $z$ to the following initial condition problem defined on $\Omega$
\[ d z_1 - \Delta z_1 dt = dW_1, \quad z_1|_{t=0} = 0. \]
Similarly, $(z_2(t))_{t \in [0, T]}$ is the unique solution which is periodic in $x, y, z$ and odd in $z$ to the following initial condition problem defined on $\Omega$
\[ d z_2 - \Delta z_2 dt = dW_2, \quad z_2|_{t=0} = 0. \]
To simplify the notations we set
\[ \int_{\Omega} := \int_{\Omega} d\bar{\Omega}. \]
Definition 2.1. Given $T > 0$, we say a continuous $\mathcal{V}$-valued $(\mathcal{F}_t)$ = $(\sigma(W^H_t(s), s \in [0, t]), j = 1, 2)$ adapted random field $(U(\cdot, t))_{t \in [0, T]} = (\nu(\cdot, t), \theta(\cdot, t))_{t \in [0, T]}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a strong solution to problem (1)-(7) if the following two conditions hold:

1. For $(v_0, \theta_0) \in \mathbb{H} \times \mathbb{H}^1$, we have $U \in C([0, T]; \mathbb{H} \times \mathbb{H}^1) \cap L^2([0, T]; \mathbb{H} \times \mathbb{H}^1)$ a.s.

2. The integral relation

\[
\int_0^t \nu(t) \cdot \phi_1 - \int_0^t ds \int_{\Omega} \left\{ \left[ (\nu \cdot \nabla) \phi_1 + w(\nu) \partial_z \phi_1 \right] v - \left[ (f \cdot \nu) \cdot \phi_1 + \left( \int_{-\frac{1}{2} t}^z \theta \cdot dz' \right) \nabla \cdot \phi_1 \right] \right\} + \int_0^t ds \int_{\Omega} \nu \cdot A_1 \phi_1 = \int_{\Omega} \nu_0 \cdot \phi_1 + \int_{\Omega} W_1(t, w) \cdot \phi_1,
\]

\[
\int_0^t \theta(t) \phi_2 - \int_0^t ds \int_{\Omega} \left\{ \left[ (\nu \cdot \nabla) \phi_2 + w(\nu) \partial_z \phi_2 \right] \theta - \theta A_2 \phi_2 \right\} = \int_{\Omega} \theta_0 \phi_2 + \int_{\Omega} W_2(t, w) \cdot \phi_2,
\]

hold a.s. for all $t \in [0, T]$ and $\phi = (\phi_1, \phi_2) \in D(A_1) \times D(A_2)$.

Denote by $u = \nu - z_1$ and $\vartheta = \theta - z_2$. Consider the primitive equations driven by a stochastic forcing in a Cartesian system. It is easy to see that if $(u, \vartheta)$ is the unique strong solution of the following system (16)-(21), then equivalently $(\nu, \theta)$ is the unique strong solution of (1)-(7).

\[
\begin{align*}
\partial_t u - \Delta u + [(u + z_1) \cdot \nabla_H](u + z_1) + w(u + z_1) \partial_z (u + z_1) & + f(u + z_1) + \nabla_H p_s - \int_{-\frac{1}{2}}^z \nabla_H \vartheta \cdot dz' = 0; \\
\partial_t \vartheta - \Delta \vartheta + [(u + z_1) \cdot \nabla_H](\vartheta + z_2) & + w(u + z_1) \partial_z (\vartheta + z_2) - w(u + z_1) = 0;
\end{align*}
\]

(16)\hspace{1cm} (17)

\[u, \text{ and } \vartheta \text{ are periodic in } x, y, z;\]

\[u \text{ is even in } z, \vartheta \text{ is odd in } z;\]

(18)\hspace{1cm} (19)

\[\int_{-1}^0 \nabla_H \cdot udz = 0;\]

(20)

\[\left. (u_{\lambda_{t=0}}, \vartheta_{\lambda_{t=0}}) \right|_{t=0} = (v_0, \theta_0).\]

(21)

Definition 2.2. Let $z_j, j = 1, 2$, are defined above, $(v_0, \theta_0) \in \mathbb{H} \times \mathbb{H}^1$ and $T$ be a fixed positive time. For $P - a.e. \omega \in \Omega$, $(u, \vartheta)$ is called a strong solution of the system (16)-(21) on the time interval $[0, T]$ if it satisfies (16)-(17) in the weak sense such that

\[u \in C([0, T]; \mathbb{H}) \cap L^2([0, T]; \mathbb{H}^2),\]

\[\vartheta \in C([0, T]; \mathbb{H}^1) \cap L^2([0, T]; \mathbb{H}^2).\]

In our paper, we will frequently use the following inequalities, so we state them in the following. For their proof, one can refer to [9].
Lemma 2.3. Let $v \in (H^2(\Omega))^2$, $\mu \in (H^1(\Omega))^2$ and $\nu \in (L^2(\Omega))^2$. Then, there exists a positive constant $c$ independent of $v$, $\mu$ and $\nu$ such that

$$\left| \langle \left( \int_0^1 \nabla_H \cdot v(t; \theta, \phi, \xi') d\xi' \right) \mu, \nu \rangle \right| \leq c|\nabla_H \cdot v|_{\mathcal{F}} \left| |\nabla_H \mu|_{\mathcal{F}} + |\Delta_H \mu|_{\mathcal{F}} \right| |\nu|_{\mathcal{F}}.$$  

Lemma 2.4. Let $v \in (H^1(\Omega))^2$, $\mu \in (H^1(\Omega))^2$ and $\nu \in (H^1(\Omega))^2$. Then, there exists a positive constant $c$ independent of $v$, $\mu$ and $\nu$ such that

$$\left| \langle \left( \int_0^1 \nabla_H \cdot v(t; \theta, \phi, \xi') d\xi' \right) \mu, \nu \rangle \right| \leq c|\nabla_H \cdot v|_{\mathcal{F}} \left| |\nabla_H \mu|_{\mathcal{F}} + |\nabla_H \nu|_{\mathcal{F}} \right|.$$  

In [19], one of the authors of this article obtained the global well-posedness for the stochastic 3D primitive equations.

Theorem 2.5. For $Q \in L^2(\Omega)$, $(v_0, \theta_0) \in \mathbb{H}^1 \times H^1$. There exists a unique strong solution $(v, \theta)$ to (1)-(7) or equivalently there exists a unique strong solution $(u, \vartheta)$ to (16)-(21).

In order to study the backward uniqueness of 3D stochastic primitive equations, we need to improve the regularity of the strong solution. The following Lemma, a special case of a general result of Lions and Magenes [28], will help us to achieve our goal.

Lemma 2.6. Let $V, H, V'$ be three Hilbert spaces such that $V \subset H \subset V'$, where $H$ and $V'$ are the dual spaces of $H$ and $V$ respectively. Suppose $u \in L^2(0, T; V)$ and $u' \in L^2(0, T; V')$. Then $u$ is almost everywhere equal to a function continuous from $[0, T]$ into $H$.

3. Backward uniqueness for the strong solutions of the three dimensional primitive equations.

Theorem 3.1. Given $(v_0, \theta_0) \in \mathbb{H}^1 \times H^1$ and $(\partial_z v_0, \partial_z \theta_0) \in \mathbb{H}^1 \times H^1$, let $(u, \vartheta)$ be the unique strong solution to (16)-(21) then $u_{zz} \in C([0, T]; \mathbb{H}^1) \cap L^2([0, T]; H^2)$ and $\vartheta_z \in C([0, T]; H^1) \cap L^2([0, T]; H^2)$.

Proof. Taking the derivative of (16) with respect to $z$ yields

$$\begin{align*}
\partial_t u_z - \Delta u_z + ([u + z_1] \cdot \nabla_H)(u_z + \partial_z z_1) + ([u_z + \partial_z z_1] \cdot \nabla_H)(u + z_1) \\
-(u_z + \partial_z z_1) \nabla_H \cdot (u + z_1) + w(u + z_1)(u_{zz} + \partial_z z_1) \\
+ fk \times (u_z + \partial_z z_1) - \nabla H \vartheta = 0. \tag{22}
\end{align*}$$

Multiplying $-\Delta u_z$ and integrating over $\Omega$, we have

$$\begin{align*}
\frac{1}{2} \partial_t |u_z|^2 + |\Delta u_z|^2 & = \\
-(\langle (u + z_1) \cdot \nabla_H \rangle (u_z + \partial_z z_1), -\Delta u_z angle \\
-(\langle (u_z + \partial_z z_1) \cdot \nabla_H \rangle (u + z_1), -\Delta u_z angle \\
+(\langle u_z + \partial_z z_1 \rangle \nabla_H \cdot (u + z_1), -\Delta u_z angle \\
-w(u + z_1) \langle u_{zz} + \partial_z z_1 \rangle, -\Delta u_z angle \\
-\langle fk \times (u_z + \partial_z z_1), -\Delta u_z \rangle + \langle \nabla H \vartheta, -\Delta u_z \rangle = \sum_{k=1}^{n} I^k. \tag{23}
\end{align*}$$
By the Hölder inequality, the interpolation inequality and Young’s inequality we have
\[
I_1 \leq |\Delta u_z|_2 |\nabla_H (u_z + \partial_z z_1)|_2 |u + z_1|_6
\leq C |\Delta u_z|_2 |\nabla_H (u_z + \partial_z z_1)|_2^{1/2} |\Delta (u_z + \partial_z z_1)|_2^{1/2} |u + z_1|_6
\leq \varepsilon |\Delta u_z|_2^2 + C |z_1|_2^2 + C |\nabla_H (u_z + \partial_z z_1)|_2^2 |u + z_1|_6^2
\leq \varepsilon |\Delta u_z|_2^2 + C |z_1|_2^2 + C (|u|_2^2 + |z_1|_2^2) (|u|_4^2 + |z_1|_4^2).
\]

In view of the Hölder inequality, the interpolation inequality, Sobolev imbedding theorem and Young’s inequality we obtain
\[
I_2 \leq |\Delta u_z|_2 |\nabla_H (u + z_1)|_4 |u_z + \partial_z z_1|_4
\leq C |\Delta u_z|_2 |\nabla_H (u + z_1)|_2^{1/4} |u + z_1|_2^{3/4} |u_z + \partial_z z_1|_2^{1/4} |u_z + \partial_z z_1|_1^{3/4}
\leq C |\Delta u_z|_2 |u + z_1|_2^{1/4} |u_z + \partial_z z_1|_1^{3/4}
\leq \varepsilon |\Delta u_z|_2^2 + C |u + z_1|_2^2 |u_z + \partial_z z_1|_1^2.
\]

Similarly,
\[
I_3 \leq |\Delta u_z|_2 |\nabla_H (u + z_1)|_3 |u_z + \partial_z z_1|_6
\leq |\Delta u_z|_2 |u + z_1|_2 |u_z + \partial_z z_1|_1
\leq \varepsilon |\Delta u_z|_2^2 + C |u + z_1|_2^2 |u_z + \partial_z z_1|_1^2.
\]

By Lemma 2.3 and Young’s inequality we have
\[
I_4 \leq |\Delta u_z|_2 |u_{zzz} + \partial_{zz} z_1|_2^{1/2} |\nabla_H u_{zzz} + \nabla_H \partial_{zz} z_1|_2^{1/2}
\times |u + z_1|_2^{1/2} |u + z_1|_2^{1/2}
\leq \varepsilon |\Delta u_z|_2^2 + \varepsilon |\nabla_H u_{zzz} + \nabla_H \partial_{zzz} z_1|_2^2
\leq C |u_{zzz} + \partial_{zz} z_1|_2^2 |u + z_1|_2^2 |u + z_1|_2^2
\leq \varepsilon |\Delta u_z|_2^2 + C |z_1|_2^3 + C |u_{zzz}|_2^2 |u + z_1|_2^2 |u + z_1|_2^2
\leq C |z_1|_2^3 |u + z_1|_2^2 |u + z_1|_2^2.
\]

Thanks to the Hölder inequality and Young’s inequality we obtain
\[
I_5 + I_6 \leq \varepsilon |\Delta u_z|_2^2 + C |u + z_1|_1^2 + C |\theta|_1^2.
\]

Based on (23), estimates from $I_1$ to $I_6$, the Gronwall inequality and the Theorem 2.5 we conclude that
\[
\begin{equation}
\begin{aligned}
\text{u}_z \in L^\infty([0, T]; \mathbb{H}^1) \cap L^2([0, T]; \mathbb{H}^2).
\end{aligned}
\end{equation}
\]

Multiplying (22) by $\eta \in \mathbb{H}^1$, integrating with respect to space variable yields
\[
\begin{align}
(\partial_t A_1^{1/2} u_z, \eta) &= (\partial_t u_z, A_1^{1/2} \eta)
= (\Delta u_z, A_1^{1/2} \eta) - ((u + z_1) \cdot \nabla H) (u_z + \partial_z z_1, A_1^{1/2} \eta)
- \langle((u_z + \partial_z z_1) \cdot \nabla_H (u + z_1), A_1^{1/2} \eta)
+ \langle(u_z + \partial_z z_1) \nabla_H (u + z_1, A_1^{1/2} \eta) - \langle w(u + z_1)(u_{zzz} + \partial_{zz} z_1), A_1^{1/2} \eta)
- (f k \times (u_z + \partial_z z_1) - \nabla_H \theta, A_1^{1/2} \eta).
\end{align}
\]
By the Hölder inequality, Lemma 2.3 and Sobolev imbedding theorem we have
\[
\|\partial_t A_1^{1/2} u_z\|_{\mathbb{H}^{-1}} \leq C\|u_z\|_2 + C|\nabla_H (u_z + \partial_z z_1)\|_3 |u + z_1|_6 \\
+ C\|u_z + \partial_z z_1\|_4 |\nabla_H (u + z_1)|_4 \\
+ C\|u_{zz} + \partial_{zz} z_1\|_{1/2} \|u_z + \partial_z z_1\|_{1/2} \|u + z_1\|_{1/2} \\
+C\|u_z + \partial_z z_1\|_2 + C|\nabla_H \partial z_2|_2
\]
which implies that
\[
\partial_t A_1^{1/2} u_z \in L^2([0, T]; \mathbb{H}^{-1}).
\]
Therefore in view of Lemma 2.6, we have
\[
u_z \in C([0, T]; \mathbb{H}^1).
\]
Taking the derivative of (17) with respective to \(z\) yields
\[
\partial_t \partial_z - \Delta \partial_z + [(u_z + \partial_z z_1) \cdot \nabla_H] (\partial_z + z_2) \\
+ [(u + z_1) \cdot \nabla_H] (\partial_z + \partial_z z_2) - (\nabla_H \cdot u + \nabla_H \cdot z_1) (\partial_z + \partial_z z_2) \\
+ w(u + z_1)(\partial_z + \partial_z z_2) + \nabla_H \cdot u + \nabla_H \cdot z_1 = 0.
\]
Multiplying \(-\Delta \partial_z\) and integrating over \(\Omega\) yields,
\[
\frac{1}{2} \partial_t \|\partial_z\|_2^2 + |\Delta \partial_z|_2^2 = \\
- \langle (u_z + \partial_z z_1) \cdot \nabla_H \rangle (\partial_z + z_2), -\Delta \partial_z \rangle \\
- \langle (u + z_1) \cdot \nabla_H \rangle (\partial_z + \partial_z z_2), -\Delta \partial_z \rangle \\
+ \langle (\nabla_H \cdot u + \nabla_H \cdot z_1) (\partial_z + \partial_z z_2), -\Delta \partial_z \rangle \\
- \langle w(u + z_1)(\partial_z + \partial_z z_2), -\Delta \partial_z \rangle \\
- \langle (\nabla_H \cdot u + \nabla_H \cdot z_1), -\Delta \partial_z \rangle = \Sigma_{k=1}^5 J_k.
\]
By the Hölder inequality, the Sobolev imbedding theorem and Young’s inequality we have
\[
J_1 \leq \varepsilon |\Delta \partial_z|_2^2 + C|\nabla_H (\partial + z_2)|_2^2 |u_z + \partial_z z_1|_4^2 \\
\leq \varepsilon |\Delta \partial_z|_2^2 + C|\partial + z_2|_2^2 |u_z + \partial_z z_1|_1^2.
\]
In view of the Sobolev imbedding theorem, the Hölder inequality and Young’s inequality we obtain
\[
J_2 \leq |u + z_1|_\infty \|\partial_z + \partial_z z_2\|_1 |\Delta \partial_z|_2 \\
\leq \varepsilon |\Delta \partial_z|_2 + C|\partial + z_2|_2^2 \|\partial_z + \partial_z z_2\|_1^2.
\]
Taking a similar argument as in \(J_1\) yields,
\[
J_3 \leq \varepsilon |\Delta \partial_z|_2^2 + C\|\partial_z + \partial_z z_2\|_2^2 |u + z_1|_1^2.
\]
According to Lemma 2.3 and Young’s inequality,
\[ J_4 \leq |\Delta \vartheta_z|^2 |\nabla_H (\vartheta_{zz} + \partial_z z_2)|^2 |\vartheta_{zz} + \partial_z z_2|_2^2 |u + z_1|_1^2 |u + z_1|_1^2 \]
\[ \leq \varepsilon |\Delta \vartheta_z|^2 + C \|z_2\|_2^2 + C |\vartheta_{zz} + \partial_z z_2|_2^2 |u + z_1|_1^2 |u + z_1|_1^2 \]
\[ \leq \varepsilon |\Delta \vartheta_z|^2 + C \|z_2\|_2^2 + C \|\vartheta + \partial_z z_2\|_2^2 |u + z_1|_1^2 |u + z_1|_1^2 . \]

By Young’s inequality, we obtain
\[ J_5 \leq \varepsilon |\Delta \vartheta_z|^2 + C \|u\|^2 + C \|z_1\|^2 . \]
By (29) and estimates from $J_1$ to $J_5$ we obtain,
\[ \partial_t |\vartheta_z|^2 + |\Delta \vartheta_z|^2 \leq C \|\vartheta + z_2\|^2 |u + \partial_z z_1|^2 \]
\[ + C (1 + \|u + z_1\|_1^2) \|u + z_1\|_1^2 \|\vartheta + \partial_z z_2\|^2 \]
\[ + C \|u\|^2 + C \|z_1\|^2 . \]

Based on (30), the Gronwall inequality and the Theorem 2.5 we conclude that
\[ \vartheta_z \in L^\infty([0, T]; H^1) \cap L^2([0, T]; H^2) . \]

Multiplying (29) by $\xi \in H^1$, integrating with respect to space variable yields
\[ \langle \partial_t A_2^{1/2} \vartheta_z, \xi \rangle = \langle \partial_t \vartheta_z, A_2^{1/2} \xi \rangle \]
\[ = \langle \Delta \vartheta_z, A_2^{1/2} \xi \rangle - \langle ([u + \vartheta_z] \cdot \nabla_H) (\vartheta + z_2), A_2^{1/2} \xi \rangle \]
\[ - \langle ([u + z_1] \cdot \nabla_H) (\vartheta_z + \vartheta_z z_2), A_2^{1/2} \xi \rangle \]
\[ + \langle (\nabla_H \cdot u + \nabla_H \cdot z_1) (\vartheta_z + \vartheta_z z_2), A_2^{1/2} \xi \rangle \]
\[ - \langle w(u + z_1)(\vartheta_z + \vartheta_z z_2), A_2^{1/2} \xi \rangle \]
\[ - \langle \nabla_H \cdot u + \nabla_H \cdot z_1, A_2^{1/2} \xi \rangle . \]

By the Hölder inequality and the Sobolev inequality, we obtain
\[ \|\partial_t A_2^{1/2} \vartheta_z\|_{H^{-1}} \leq C |\Delta \vartheta|^2 + C \|u + \partial_z z_1\| \|\vartheta + z_2\|_2 \]
\[ + C \|u + z_1\|_\infty \|\vartheta_z + \partial_z z_2\|_1 \]
\[ + C \|u + z_1\|_2 \|\vartheta_z + \partial_z z_2\|_1 + 1 . \]

Hence, by (24), (32) and Theorem 2.5, we infer
\[ \partial_t A_2^{1/2} \vartheta_z \in L^2([0, T]; H^{-1}) , \]
which combined with (32) and Lemma 2.6 implies
\[ \vartheta_z \in C([0, T]; H^1) . \]

Finally, the conclusions of this theorem follows from (24), (28), (32) and (35).

**Theorem 3.2.** Given $(v_0, \theta_0) \in H^2 \times H^2$, let $(u, \vartheta) \in C([0, T]; H^2 \times H^2) \cap L^2([0, T]; H^3 \times H^3)$.

**Proof.** Taking inner product of (16) with $A_1^2 u$ in $H$ yields
\[ \frac{1}{2} \partial_t |A_1^2 u|^2 + |A_1^{3/2} u|^2 = - \int_\Omega ([u + Z_1] \cdot \nabla_H) (u + Z_1) \cdot A_1^2 u \]
\[ - \int_\Omega w(u + z_1) \partial_z (u + z_1) \cdot A_1^2 u - \int_\Omega f(u + z_1) \cdot A_1^2 u \]
\[ - \int_\Omega \nabla H p_s \cdot A_1^2 u + \int_\Omega \left( \int_0^z \nabla_H \vartheta dz' \right) \cdot A_1^2 u = \Sigma_{k=1}^5 J_k . \]
By integration by parts and Young's inequality, we have
\[ J_3 = \int_{\Omega} f(u + z_1) \cdot A_1^2 u = \int_{\Omega} f(z_1) \cdot A_1^2 u \leq \varepsilon |A_1^{3/2} u|^2 + C \|z_1\|^2 \]
and
\[ J_4 = \int_{\Omega} \nabla_H p_s \cdot A_1^2 u = 0. \]
Similarly
\[ J_5 = \int_{\Omega} \left( \int_{-1}^1 \nabla_H \partial_t z' \right) \cdot A_1^2 u \leq \varepsilon |A_1^{3/2} u|^2 + C \|\theta\|^2. \]
Since by integration by parts,
\[ J_1 + J_2 \]
\[ = \int_{\Omega} (u + z_1) \cdot \nabla_H (A_1(u + z_1)) \cdot A_1 u + \int_{\Omega} w(u + z_1) \partial_z (A_1(u + z_1)) \cdot A_1 u \]
\[ + \int_{\Omega} \left( \nabla(u + z_1) : \nabla_H \nabla(u + z_1) \right) \cdot A_1 u \]
\[ + \int_{\Omega} \left( \nabla w(u + z_1) : \partial_z \nabla(u + z_1) \right) \cdot A_1 u \]
\[ + \int_{\Omega} A_1(u + z_1) \cdot \nabla_H (u + z_1) A_1 u + \int_{\Omega} A_1 w(u + z_1) \partial_z (u + z_1) A_1 u. \quad (37) \]
In the following, we will estimate the terms on the right hand side of (37) respectively. Obviously, by integration by parts we have
\[ \int_{\Omega} u \cdot \nabla_H (A_1 u) \cdot (A_1 u) + \int_{\Omega} w(u) \partial_z (A_1 u) \cdot (A_1 u) = 0. \quad (38) \]
In view of the Hölder inequality, the Sobolev inequality and Young’s inequality we obtain
\[ \int_{\Omega} z_1 \cdot \nabla_H A_1(u + z_1) \cdot A_1 u + \int_{\Omega} w(z_1) \partial_z A_1(u + z_1) \cdot A_1 u \]
\[ \leq |z_1|_{\infty} |A_1^{3/2}(u + z_1)|_2 |A_1^2 u|_2 + |z_1|_3 |A_1^{3/2}(u + z_1)|_2 |A_1^2 u|_2 \]
\[ \leq \varepsilon |A_1^{3/2} u|^2 + C |z_1|_3^2 \|u\|_2 + C |z_1|_3 \|u\|_2^2. \quad (39) \]
Thanks to the Hölder inequality, the Sobolev inequality and Young’s inequality we have
\[ \int_{\Omega} u \cdot (\nabla_H A_1 z_1) \cdot A_1 u + \int_{\Omega} w(u) \partial_z (A_1 z_1) \cdot A_1 u \]
\[ \leq C \|u\|_{\infty} \|A_1^2 u\|_2 \|z_1\|_3 + |z_1|_3^2 \|A_1^3/2 u\|_2^2 \|A_1^{3/2} u\|_2^2 \|A_1^2 u\|_2 \]
\[ \leq \varepsilon |A_1^{3/2} u|^2 + C |z_1|_3 \|u\|_2^2 + C |z_1|_3^2 \|u\|_2^2 + C \|u\|_2^2. \quad (40) \]
By the aid of the Hölder inequality, the interpolation inequality and Young’s inequality we have
\[ \int_{\Omega} \nabla(u + z_1) \cdot \nabla_H \nabla(u + z_1) \cdot A_1 u \]
\[ \leq |A_1 u|_2 \|\nabla_H \nabla(u + z_1)\|_4 \|\nabla(u + z_1)\|_4 \]
\[ \leq |A_1 u|_2 |A_1(u + z_1)|_2^{1/4} |A_1^{3/2}(u + z_1)|_2^{3/4} \|\nabla(u + z_1)\|_2^{1/4} \|u + z_1\|_2^{3/4}. \]
\[
\begin{align*}
&\leq \varepsilon|A_{1/2}^{1/2}(u + z_1)|_{2/5}^2 + C|A_{1/2}^{3/2}|A_{1/2}^{1/2}||\nabla(u + z_1)|_{2/5}^2\|u + z_1|_{6/5}^6
\\&\leq \varepsilon|A_{1/2}^{3/2}|u_{1/2}^2 + C|z_1|_{2/5}^2 + C|A_{1/2}^{3/2}|u_{1/2}^2\|u + z_1|_{2/5}^2
\\&\leq \varepsilon|A_{1/2}^{3/2}|u_{1/2}^2 + C|z_1|_{1/2}^3 + C\|u + z_1|_{1/2}^3 + C|A_{1/2}^{3/2}|u_{1/2}^2\|u + z_1|_{1/2}^2.
\end{align*}
\]

Thanks to Lemma 2.4 and Young's inequality we obtain
\[
\int_{\Omega} \left( \nabla w(u + z_1) : \partial_z \nabla(u + z_1) \right) \cdot A_1 u \leq |A_1(u + z_1)|_{2/3}^2 |\nabla H(u + z_1)|_{1/4}^4 A_1 \leq C|A_1(u + z_1)|_{2}^2 |\nabla H(u + z_1)|_{1}^4 A_1 \leq \varepsilon|A_{1/2}^{3/2}|u_{1/2}^2 + C|z_1|_{1/2}^2 \|u + z_1|_{1/2}^6 + C|z_1|_{1/2}^2 \|u + z_1|_{1/2}^2.
\]

By the Hölder inequality, the interpolation inequality and Young's inequality we have
\[
\int_{\Omega} A_1(u + z_1) \cdot \nabla H(u + z_1) \cdot A_1 u \leq |A_1(u + z_1)|_{1/4} A_1 \leq C|A_1(u + z_1)|_{2}^2 |\nabla H(u + z_1)|_{1}^4 A_1 \leq \varepsilon|A_{1/2}^{3/2}|u_{1/2}^2 + C|z_1|_{1/2}^2 \|u + z_1|_{1/2}^6 + C|z_1|_{1/2}^2 \|u + z_1|_{1/2}^2.
\]

In view of Lemma 2.4 and Young's inequality we obtain
\[
\int_{\Omega} A_1 w(u + z_1) \cdot \partial_z w(u + z_1)(A_1 u) \leq C|A_{1/2}^{3/2}|u_{1/2}^2 |A_{1/2}^{3/2}|u_{1/2}^2 \|\nabla(u + z_1)|_{1/2}^2 |A_1(u + z_1)|_{1/2}^2
\\\leq \varepsilon|A_{1/2}^{3/2}|u_{1/2}^2 + C|z_1|_{1/2}^2 \|u + z_1|_{1/2}^2 + C|z_1|_{1/2}^2 + C\|u + z_1|_{1/2}^6.
\]

From (37)-(44) we obtain the estimates of $J_1 + J_2$ that
\[
J_1 + J_2 \leq C(|z_1|_{1/2}^4 + 1)(|u|_{2}^2 + 1)
\]
\[
+ C(|z_1|_{1/2}^2 + |A_{1/2}^{3/2}|u_{1/2}^2)(|u|_{2}^2 + |z_1|_{1/2}^2 + 1)(|u|_{2}^2 + |z_1|_{1/2}^2 + 1)
\]
\[
+ C|z_1|_{1/2}^2(|u|_{1}^2 + |\nabla z_1|_{1})(|u|_{2}^2 + |\nabla z_1|_{2})
\]

Combining the estimates from $J_1$ to $J_5$ and (36) as well as the Gronwall inequality yields
\[
u \in L^\infty([0, T]; H^2) \cap L^2([0, T]; H^3).
\]

Following the same steps as in (25) and (26) with minor revisions, we can further prove
\[
u \in C([0, T]; H^2) \cap L^2([0, T]; H^3).
\]
Taking inner product of (17) with $A_2^2\theta$ in $H$ yields,
\[
\frac{1}{2} \partial_t |A_2\theta|^2 + |A_2^{3/2}\theta|^2 = -\int_{\Omega} \left[ (u + z_1) \cdot \nabla H \right] (\theta + z_2) A_2^2 \theta
\]
\[
- \int_{\Omega} w(u + z_1) \partial_x (\theta + z_2) A_2^2 \theta + \int_{\Omega} w(u + z_1) A_2^2 \theta
\]
\[
=: K_1 + K_2 + K_3. \tag{46}
\]
Taking a similar argument as in (37) yields,
\[
K_1 + K_2
\]
\[
= -\int_{\Omega} [A_2(u + z_1) \cdot \nabla H(\theta + z_2) A_2 \theta - \int_{\Omega} w(A_2(u + z_1)) \partial_x (\theta + z_2) A_2 \theta
\]
\[- \int_{\Omega} [(u + z_1) \cdot \nabla H] A_2 \theta + \int_{\Omega} w(u + z_1) \partial_x (A_2 \theta + A_2 z_2) A_2 \theta
\]
\[+ \int_{\Omega} \nabla (u + z_1) \nabla (\theta + z_2) A_2 \theta + \int_{\Omega} \nabla w(u + z_1) \partial_x (\nabla \theta + \nabla z_2) A_2 \theta. \tag{47}
\]
In the following we will estimate each term on the right hand side of (47) respectively. By integration by parts,
\[
- \int_{\Omega} u \cdot \nabla H A_2 \theta A_2 \theta - \int_{\Omega} w(u) \partial_x A_2 \theta A_2 \theta = 0. \tag{48}
\]
In view of the Hölder inequality, the Lemma 2.3 and Young’s inequality we have
\[
- \int_{\Omega} u \cdot \nabla H A_2 z_2 A_2 \theta - \int_{\Omega} w(u) \partial_x A_2 z_2 A_2 \theta
\]
\[- \int_{\Omega} z_1 \cdot \nabla H A_2 (\theta + z_2) A_2 \theta - \int_{\Omega} w(z_1) \partial_x (A_2 \theta + A_2 z_2) A_2 \theta
\]
\[
\leq C\|u\|_\infty \|z_2\|_3 \|A_2 \theta\|_2 + C \|z_2\|_3 \|A_2^{3/2} \theta\|_2^{1/2} \|u\|_1 \|u\|_2^{1/2}
\]
\[+ C \|z_1\|_3 (\|A_2^{3/2} \theta\|_2 + \|z_2\|_3) \|A_2 \theta\|_2
\]
\[
\leq \varepsilon \|A_2^{3/2} \theta\|_2^2 + C \|u\|_2^2 + C (\|z_1\|_3^4 + \|z_2\|_3^4 + 1) (\|A_2 \theta\|_2^2 + 1). \tag{49}
\]
By aid of the Hölder inequality and Young’s inequality we deduce,
\[
- \int_{\Omega} A_2(u + z_1) \cdot \nabla H(\theta + z_2) A_2 \theta
\]
\[
\leq |\nabla H(\theta + z_2)|_\infty \|A_2(u + z_1)\|_2 \|A_2 \theta\|_2
\]
\[
\leq \varepsilon \|A_2^{3/2} \theta\|_2^2 + C \|z_2\|_3^2 + C (\|u\|_2^2 + \|z_1\|_3^2) \|A_2 \theta\|_2^2. \tag{50}
\]
By virtue of Lemma 2.3 and Young’s inequality, we obtain
\[
- \int_{\Omega} w(A_2 u + A_2 z_1) \partial_x (\theta + z_2) A_2 \theta
\]
\[
\leq C \|A_2 u + A_2 z_1\|_2 \|u + z_1\|_3^{1/2} \|\theta + z_2\|_1^{1/2} \|A_2 \theta\|_2 + \|z_2\|_2 \|A_2^{3/2} \theta\|_2
\]
\[
\leq \varepsilon \|A_2^{3/2} \theta\|_2^2 + C \|u + z_1\|_2^2 \|u + z_1\|_2^2 + \|z_2\|_3^2
\]
\[+ C \|\theta + z_2\|_2^2 \|\theta + z_2\|_2^2. \tag{51}
\]
With the aid of the Hölder inequality, the interpolation inequality and Young’s inequality we have

$$
\int_I \nabla(u + z_1) \nabla(\vartheta + z_2) A_2 \vartheta
\leq |A_2 \vartheta|^2 \int |\nabla(u + z_1)|^4 |\nabla^2(\vartheta + z_2)|^4 + C|A_2 \vartheta||u + z_1|^{3/2} \| \vartheta + z_2 \|^4 |A_2^{3/2}(\vartheta + z_2)|^{3/4}
\leq \varepsilon |A_2^{3/2}\vartheta|^2 + C|z_2|^2 + C\| \vartheta + z_2 \|^2 + C|A_2 \vartheta|^2 \|u + z_1\|^2.
$$

(52)

Taking a similar argument as in (42) yields,

$$
\int_I \nabla w(u + z_1) \partial_2(\nabla \vartheta + \nabla z_2) A_2 \vartheta
\leq |A_2 \vartheta|^2 \| \vartheta + z_2 \|^2 |A_2^{3/2}(\vartheta + z_2)| \| u + z_1 \|^2 \| u + z_1 \|^2
\leq \varepsilon |A_2^{3/2}\vartheta|^2 + C|z_2|^2 + C|A_2 \vartheta|^2
+C(|A_2 \vartheta|^2 + \| z_2 \|^2)|u + z_1|^2 \| u + z_1 \|^2.
$$

(53)

In view of Young’s inequality we get

$$
K_3 \leq \varepsilon |A_2^{3/2}\vartheta|^2 + C\| u + z_1 \|^2.
$$

(54)

With the aid of (46)-(54), we obtain

$$
\frac{1}{2} \partial_t |A_2 \vartheta|^2 + |A_2^{3/2}\vartheta|^2
\leq C(1 + \| z_1 \|^3 + \| z_2 \|^3 + \| u \|^2 + \| \vartheta \|^2)(|A_2 \vartheta|^2 + 1)
+C(\| u \|^2 + \| z_1 \|^2)(\| u \|^2 + \| z_1 \|^2)(|A_2 \vartheta|^2 + \| z_2 \|^2 + 1).
$$

By (15), (45) and the Gronwall inequality, we have

$$
\vartheta \in L^\infty([0, T]; \mathbb{H}^2) \cap L^2([0, T]; \mathbb{H}^3).
$$

Following the same steps as in (33)-(35) with minor revisions we can further prove

$$
\vartheta \in C([0, T]; \mathbb{H}^2) \cap L^2([0, T]; \mathbb{H}^3).
$$

(55)

Then conclusions of Theorem 3.2 follows from (45) and (55).

\[ \square \]

**Theorem 3.3.** Let $G \in L^2([0, T]; \mathbb{H}^1 \times \mathbb{H}^1), U_0 \in \mathbb{H}^1 \times \mathbb{H}^1$. Assume $U$ be the solution to the linear primitive equations:

$$
U'(t) + AU(t) + EU(t) = G,
U(0) = U_0.
$$

For all time such that $U(t) \neq 0$, define

$$
g(t) = \frac{\langle (A + E)U(t), U(t) \rangle}{\|U(t)\|^2},
$$

Then

$$
g'(t) \leq \frac{\|G(t)\|^2}{\|U(t)\|^2}.
$$
Proof. Taking a similar argument as in Theorem 2.5, we can show that $U$ satisfies $U \in C([0, T]; \mathbb{H}^1 \times \mathbf{H}^1) \cap L^2([0, T]; \mathbb{H}^2 \times \mathbf{H}^2)$.

Since $\langle EU(t), U(t) \rangle = 0$, we have $g(t) = \frac{\langle (A + E)U(t), U(t) \rangle}{|U(t)|_2^2} = \frac{\langle AU(t), U(t) \rangle}{|U(t)|_2^2}$. Here, as $|U(t)|_2$ is continuous on $[0, T]$, $g$ is defined on the open set of $(0, T)$ which satisfies $|U(t)|_2 > 0$. The following calculations are carried out formally. To give a rigorous proof, we should perform the following calculations on the Galerkin approximation sequence and then extend the result to the infinite dimensional case by Aubin-Lions Lemma. Taking derivative of $g$ with respect to time $t$ yields,

$$g'(t) = \frac{2\langle AU'(t), U(t) \rangle}{|U(t)|_2^2} - \frac{2\langle AU(t), U'(t), U(t) \rangle}{|U(t)|_2^2}.$$

Since by integration by parts,

$$\langle AU'(t), U(t) \rangle = \langle G - AU(t) - EU(t), AU(t) \rangle = \langle G - AU(t), AU(t) \rangle = \langle G, AU(t) \rangle - \langle AU(t), AU(t) \rangle$$

and

$$\langle AU(t), U(t) \rangle \langle U'(t), U(t) \rangle = \langle AU(t), U(t) \rangle \langle G - AU(t) - EU(t), U(t) \rangle = \langle AU(t), U(t) \rangle \langle G - AU(t), U(t) \rangle,$$

we have

$$g'(t) = \frac{2\langle G, AU(t) \rangle - 2\langle AU(t), AU(t) \rangle}{|U(t)|_2^2} - \frac{2\langle AU(t), U(t) \rangle \langle G - AU(t), U(t) \rangle}{|U(t)|_2^2}$$

$$= \frac{2\langle G, AU(t) \rangle - 2\langle AU(t), AU(t) \rangle}{|U(t)|_2^2} + \frac{2\langle AU(t), U(t) \rangle^2 - 2\langle G, AU(t) \rangle \langle AU(t), U(t) \rangle}{|U(t)|_2^2}$$

$$= \frac{2\langle G, AU(t) \rangle - 2\langle AU(t), AU(t) \rangle}{|U(t)|_2^2} + \frac{2\langle AU(t), U(t) \rangle^2 - \frac{1}{2} \langle G, U(t) \rangle^2}{|U(t)|_2^2} - 2 \frac{1}{4} \frac{\langle G, U(t) \rangle^2}{|U(t)|_2^2}$$

$$\leq \frac{2\langle G, AU(t) \rangle - 2\langle AU(t), AU(t) \rangle}{|U(t)|_2^2} + \frac{2|AU(t) - \frac{1}{2}G|^2}{|U(t)|_2^2} + \frac{1}{2} \frac{|G|^2}{|U(t)|_2^2}$$

$$\leq \frac{|G|^2}{|U(t)|_2^2}.$$

\[ \Box \]

**Theorem 3.4.** Given $(v_1(0), \theta_1(0))$ and $(v_2(0), \theta_2(0)) \in \mathbb{H}^1 \times \mathbf{H}^1$, let $(v_1, \theta_1)$ and $(v_2, \theta_2)$ be the two corresponding strong solutions to stochastic primitive equations.
with (14) holds. If we assume \( v_1(t) = v_2(t) \) and \( \theta_1(t) = \theta_2(t) \) for some \( t > 0 \), a.s., then
\[
\mathbb{P}(v_1(s) = v_2(s), \theta_1(s) = \theta_2(s), \forall s \in [0, t]) = 1.
\]

**Proof.** Let \( v = v_1 - v_2, \theta = \theta_1 - \theta_2, p_1 - p_2 = p \). Then \( (v, p, \theta) \) satisfies
\[\begin{align*}
\partial_t v - \Delta v + v \cdot \nabla_H v_1 + v_2 \cdot \nabla_H v + w(v)\partial_z v_1 + w(v_2)\partial_z v + f v + \nabla_H p - \int_{-z}^{z} \nabla_H \theta dz' = 0,
\end{align*}\]
\[\begin{align*}
\nabla_H \cdot v + \partial_z w(v) = 0,
\end{align*}\]
\[\begin{align*}
\partial_t \theta - \Delta \theta + v \cdot \nabla_H \theta_1 + v_2 \cdot \nabla_H \theta + w(v)\partial_z \theta_1 + w(v_2)\partial_z \theta - w(v) = 0,
\end{align*}\]
where \( p = p_s - \int_{-z}^{z} \theta dz' \). For \( t > 0 \), define
\[\tilde{G}U = B(U, U_1) + B(U_2, U),\]
where \( U_i = (v_i, \theta_i), i = 1, 2 \). Then, rewriting the above equations in the abstract form
\[U' + AU + EU + \tilde{G}U = 0.\]
Taking inner product of (56) and (58) in \( \mathbb{H} \) and \( \mathbb{H} \) with \( v \) and \( \theta \) respectively and adding the two equations yields,
\[
\frac{1}{2} \partial_t (|v|^2 + |\theta|^2) + (|\nabla v|^2 + |\nabla \theta|^2)
\]
\[= -(v \cdot \nabla_H v_1 + w(v)\partial_z v_1, v) - (v \cdot \nabla_H \theta_1 + w(v)\partial_z \theta_1, \theta) := \langle \tilde{G}U, U \rangle.\]

Since in view of Theorem 3.3, (60) and (61), we have
\[g'(t) \leq C\langle |U|^2_{\mathbb{H}}, U_1 \rangle \leq C\|U\|_{\mathbb{H}}^2 \]
by the Gronwall inequality, we have
\[g(t) \leq g(t_0) e^{C \int_{t_0}^t \|U_1\|_{\mathbb{H}}^2 ds} \leq C.\]
In view of (61), then we try to estimates \( \log |U(t)|^2 \)
\[\frac{d \log |U(t)|^2}{dt} = -2(\langle |\nabla v|^2 + |\nabla \theta|^2 \rangle - 2\langle G(t)U, U \rangle)
\]
\[\geq -2g(t) - 2\frac{\|U(t)\|_1 \|U_1(t)\|_3}{|U(t)|^2}
\]
\[\geq -2g(t) - 2g^2(t) \|U_1(t)\|_3
\]
\[\geq -C - C\|U_1(t)\|_3.\]

Since \( U_1 \in L^2([0, T]; \mathbb{H}^3 \times \mathbb{H}^3) \), for arbitrary \( t_0 \in [0, t] \) integrating (62) over \([t_0, t]\) yields
\[\log |U(t)|^2 \geq C(t - t_0) + \log |U(t_0)|^2, \quad \forall t \in [t_0, t].\]
Hence, from (63) we see that if \( v_1(t) = v_2(t) \) and \( \theta_1(t) = \theta_2(t) \) for some \( t > 0 \) a.s., which is equivalent to \( U(t) = 0 \) a.s., then \( U(t_0) = 0 \) a.s.. Then the result of this theorem follows. \[\square\]

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