Parametric resonances and resonant delocalisation in quasi-phase matched photon-pair generation and quantum frequency conversion

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Abstract
The existing widely-accepted theory of photon-pair generation via spontaneous down-conversion (SPDC) in nonlinear optical crystals and waveguides is incomplete, as it fails to account for the important physical phenomenon of parametric resonances. We demonstrate that exponential gain of classical fields in the regime of parametric resonance corresponds to resonant delocalisation in the Glauber–Fock model of quantum SPDC. We propose a quantitative measure of localisation of Floquet eigen-modes as an analogue of classical gain to identify regimes of resonant delocalisation. Using this method, we are able to reconstruct the classical ‘Arnold tongues’ map of domains of instabilities for SPDC. We also predict novel regimes of resonant delocalisation in the two-level model describing quantum frequency conversion processes.

1. Introduction
Parametric resonances are a well-known instability mechanism triggered by periodical modulation of a system parameter. A classical example is an oscillator with periodically modulated eigenfrequency, whose dynamics is governed by the renowned Mathieu equation [1]. The important distinct feature of parametric resonances is the existence of multiple frequency ranges of instability, even when the modulation is purely harmonic. Furthermore, positions and bandwidths of the instability regions change with the modulation amplitude. Parametric resonances govern a wide range of physical phenomena, including pattern formation in liquids on a vibrating substrate [2], periodically forced reaction-diffusion systems [3], Bose–Einstein condensates with modulated interactions [4, 5], multi-mode lasers [6], and unstable vibrations of London’s Millennium bridge [7]. In nonlinear optical systems, similar parametric instabilities arise from spatial modulation of dispersive [8], dissipative [9, 10] or nonlinear [11, 12] properties of the medium.

In a different context, modulation of nonlinearity along the path of interacting optical waves is a well-known technique for effective compensation of their momentum mismatch known as quasi-phase matching (QPM) [13]. In particular, periodic alternation of the sign of $\chi^2$ nonlinearity has become a widely recognised technique for efficient second harmonic generation in bulk crystals and waveguides [14–17]. Later QPM has also been adapted for optical parametric oscillation, parametric amplification [18, 19] and spontaneous parametric down conversion (SPDC) processes [20–22]. The latter form the basis of one of the most promising and robust schemes of generation of correlated photon pairs for applications in quantum computing, metrology, and development of heralded single photon sources [23–25]. Despite apparent similarities, the relationship between parametric instabilities and QPM-driven parametric processes has not been fully explored.

In this work we demonstrate that the conventional quantum-mechanical treatment of photon-pair generation by SPDC fails to capture parametric resonances, in contrast with the classical model. We reveal the intrinsic connection between parametric resonances and the phenomenon of resonant delocalisation in...
Glauber–Fock lattices, and obtain the quantum SPDC analogue of the classical 'Arnold tongues' picture of resonance domains. We furthermore explore this connection to predict novel regimes of resonant delocalisation and Rabi oscillations in the two-level model, describing sum- and difference-frequency generation. For clarity, we focus our discussion on the case of nonlinear interactions in one dimension in a material with modulated $\chi^2$ nonlinearity, as shown in figure 1. The archetypal example of this is a single-mode waveguide periodically poled to achieve QPM, however our analysis can be straightforwardly extended to any $\chi^2$ or $\chi^3$ material exhibiting periodic modulation of its nonlinearity.

2. Photon pair generation and two-photon state function in SPDC

In a $\chi^2$-driven SPDC process, a higher energy photon (pump, $\omega_p$) from a bright source is spontaneously converted into a pair of lower energy photons (signal and idler, $\omega_s + \omega_i = \omega_p$), see figure 1. In a waveguide, all photons propagate in the same direction and the important parameter which ultimately determines the properties of the generated signal-idler pairs, for example their joint frequency structure, is the momentum (propagation constant) mismatch of the interacting waveguide modes [26]:

$$\Delta \beta(\omega_s, \omega_i) = \beta_p(\omega_s + \omega_i) - \beta_i(\omega_i) - \beta_s(\omega_i).$$

While it is often not possible to achieve direct phase matching $\Delta \beta = 0$ for a desired combination of frequencies and waveguide modes, the QPM technique relies on modulation of the interaction strength along the waveguide to effectively compensate a non-zero mismatch. Treating the pump field classically, and neglecting pump depletion (only a few photons from the bright light pump are converted into signal-idler pairs), the spontaneous creation of signal-idler photon pairs can be described by the following interaction Hamiltonian [22, 27]:

$$\hat{H}_I = \gamma(\eta) [e^{-i \beta_p \eta} \hat{a}_s \hat{a}_i + e^{i \beta_i \eta} \hat{a}_s \hat{a}_i],$$

where $\eta = \kappa z$ is the dimensionless propagation distance related to the modulation period $L = 2\pi / \kappa$, $\gamma(\eta) = \sqrt{P_0} \gamma(\eta) / \kappa$ is the effective interaction which encapsulates the modulated waveguide nonlinearity $\gamma(\eta + 2\pi) = \gamma(\eta)$ and the pump power $P_0, R = \Delta \beta / \kappa$ is the ratio between the momentum mismatch and the reciprocal modulation period. Hence, setting vacuum state $|\text{vac}\rangle$ as the initial condition at $\eta = 0$, the state vector is given by:

$$|\psi(\eta)\rangle = \exp \left[ -i \int_0^\eta \hat{H}_I(\eta') d\eta' \right] |\text{vac}\rangle.$$ 

The next commonly-used step is to apply a perturbation expansion of the exponential term in the above expression, assuming a weak interaction [28–30]:

$$|\psi(\eta)\rangle \approx \left[ 1 - i \int_0^\eta \hat{H}_I(\eta') d\eta' + \ldots \right] |\text{vac}\rangle,$$

which naturally leads to the decomposition of the state into single- and multiple photon-pair terms: $|\psi(\eta)\rangle \approx |\text{vac}\rangle + |\psi_2\rangle + |\psi_4\rangle + \ldots$. In particular, from equation (4) the two-photon state is obtained:

$$|\psi_2(\eta)\rangle = \left\{ -i \int_0^\eta \gamma(\eta') e^{-i \beta_p \eta'} d\eta' \right\} \hat{a}_s \hat{a}_i |\text{vac}\rangle.$$
Expanding the $2\pi$-periodic interaction function in the Fourier series:

$$\tilde{\gamma}(\eta) = \sum_{m} \tilde{\gamma}_{m} e^{i m \eta},$$

(6)

it is easy to see that the two-photon function amplitude grows linearly with propagation distance if $R$ is integer. In other words, photon pair generation occurs when the momentum mismatch $\Delta \beta$ coincides with the reciprocal period of $m$th harmonic of the nonlinearity modulation function $\gamma(\eta)$.

We emphasise, that for the case of simple harmonic modulation:

$$\tilde{\gamma}(\eta) = \tilde{\gamma}_0 \cos(\eta),$$

(7)

according to the well-known in literature result for the two-photon function in equation (5), the growth of the two-photon state amplitude is only observed when $R = \pm 1$, i.e. when $\Delta \beta = \pm \kappa$. Furthermore, this result does not depend on the amplitude of the modulation $\tilde{\gamma}_0$.

3. Parametric resonances in classical parametric amplification

Let us now consider the classical analogue of the SPDC process, i.e. the process of parametric amplification. Under the same assumption of undepleted pump as used in derivation of the Hamiltonian in equation (2), the interacting (weak) signal and idler field amplitudes $A_{s}$, evolve along the waveguide length according to [31]:

$$\frac{dA_{s}}{d\eta} = i \hat{\gamma}(\eta) e^{-i R \eta} A_{s}^{*}, \quad \frac{dA_{i}}{d\eta} = i \tilde{\gamma}(\eta) e^{-i R \eta} A_{i}^{*}.$$

(8)

Making the substitution $X = [A_{s} + A_{i}] \exp(i R \eta/2)$, and using the simple harmonic modulation in equation (7), the above system can be cast into a Mathieu-type oscillator equation:

$$\frac{i}{2} \frac{dX}{d\eta} + \frac{R}{2} X + \tilde{\gamma}_0 \cos(\eta) X^{*} = 0.$$

(9)

To analyse dynamics of this ordinary differential equation (ODE) with periodically varying coefficients, it is convenient to consider the corresponding Floquet operator, which maps the field over one period: $[X(\eta + 2 \pi), X^{*}(\eta + 2 \pi)]^{T} = \hat{F} \cdot [X(\eta), X^{*}(\eta)]^{T}$. The operator $\hat{F}$ can be constructed numerically by integrating equation (9) with two orthogonal initial conditions. Spectral properties of $\hat{F}$ determine stability of the system in equation (9):

$$\hat{F} \cdot \psi^{(n)} = \lambda_{n} \psi^{(n)}.$$

(10)

An eigenvalue $\lambda_{n}$, with a positive real part corresponds to exponential gain in signal/idler fields. In figure 2(a) the corresponding gain regions are indicated on the plane of parameters ($R, \tilde{\gamma}_0$), and have the typical ‘Arnold tongues’ structure known for solutions to the Mathieu equation and seen in other systems exhibiting parametric resonance [1]. For a fixed interaction strength $\tilde{\gamma}_0$, the system is unstable within multiple regions of $R$. These regions emerge from the set of points $R = m, m = \pm 1, \pm 3, \pm 5, ...$ on $\tilde{\gamma}_0 = 0$ axis, expanding and shifting as $\tilde{\gamma}_0$ increases. In figure 2(b) the maximal gain, i.e. real part of eigenvalues $\lambda_{n}$ as a function of interaction strength is plotted for the first three ‘tongues’ ($m = 1, 3, 5$). It scales as $\sim \tilde{\gamma}_0^{n}$, consistent with Arnold’s scaling law [32, 33].

4. Parametric resonances and resonant delocalisation in SPDC

The analysis above reveals a fundamental inconsistency between classical theory and the approximation commonly used in the quantum-mechanical treatment of QPM down-conversion shown in equation (4). It is easy to see that with the interaction Hamiltonian defined in equation (2), equations for $\hat{a}_{s}$ and $\hat{a}_{i}$ operators in the Heisenberg picture have similar structure to equations (8). Therefore one should expect to observe growth of signal and idler photon pair numbers in the parameter regions where the classical model predicts parametric amplification. However, neither the existence of higher order resonances ($m = \pm 3, \pm 5, ...$), nor the resonance bandwidth and position dependencies on the modulation strength are reflected in the two-photon function amplitude in equation (5) with the simple harmonic modulation in equation (7). The inclusion of higher-order expansion terms in equation (4) does not restore any of these well-known parametric resonance features. Apparently, the widely adapted perturbation expansion procedure in equation (4) fails to capture the important physical aspects of SPDC processes, and needs to be reconsidered.

To develop an analogue of the classical Floquet analysis for the SPDC process, we adapt the Fock basis of signal-idler photon pairs $|\psi_{m}^{\pm} = |nm\rangle e^{-i \Delta \beta \eta}|, \text{where} |nm\rangle = (\hat{a}_{s}^{\dagger} \hat{a}_{i}^{\dagger})^{n}|\text{vac}\rangle$. This allows the signal and idler creation and annihilation operators to be absorbed into outer products. Hence, the interaction Hamiltonian in equation (2) becomes:
and the evolution of the state vector \( |\psi\rangle = \sum_n U_n |\psi_n\rangle \) is governed by the set of ODEs with periodic coefficients:

\[
-\frac{dU_n}{dt} = \gamma(n) [nU_{n-1} + (n + 1) U_{n+1}] - nRU_n
\]

The corresponding Floquet operator can be obtained by taking the product of a semi-infinite set of linearly independent solutions of the above system integrated over one modulation period: \( \hat{F} = \phi_1 \otimes \phi_2 \). This was done numerically with the help of the automated Adams/Backward Differential Formula ODE integrator available in the ODEPACK library \[34\]. Evolution of an arbitrary initial state is then obtained by repeated translations with \( \hat{F} \).

Unlike its classical counterpart in equation (8), the system in equation (12) preserves the norm \( \sum_n U_n^2 \), and therefore cannot have exponentially growing solutions. In figures 3(a)–(c) the evolution of the state vector is illustrated for the case of simple harmonic modulation \( \gamma(t) \) in equation (7), with \( R = 1, 3, 5 \), respectively, and initial vacuum state \( |\psi\rangle(0) = |\text{vac}\rangle \). Two qualitatively different types of evolution are observed for \( R = 1, 3 \) and \( R = 2 \) cases. In \( R = 2 \) case (no parametric resonance in classical system), figure 3(b), a partial beating between the vacuum and higher order terms is observed. In contrast, in \( R = 1, 3 \) cases (parametric resonances), figures 3(a) and (c), the system gradually evolves into the pairwise-correlated thermal state. The total number of signal and idler photons \( \langle \psi | \hat{n}_1 | \psi \rangle \) grows in this process, which corresponds to the exponential explosion of the classical field intensities. The characteristic length scales of resonant coupling dynamics in \( R = 1 \) and \( R = 3 \) cases are different by three orders of magnitude, which is in agreement with the scaling law of parametric resonances, see figure 2(b).

The system in equation (12), also known as the Glauber–Fock lattice \[35\], is equivalent to a semi-infinite 1D Bloch lattice of coupled detuned oscillators in which the \( n \)th oscillator has eigen-frequency \( nR \) and the coupling

\[
H_1 = \gamma(\eta) \sum_n \left[ (n + 1)|\psi_{n+1}\rangle \langle \psi_n| + n|\psi_{n-1}\rangle \langle \psi_n| \right]
\]
is inhomogeneous and periodically-varying. The modulation of coupling enables effective cross-talk between the detuned oscillators, leading to the so-called resonant delocalisation [35]. The phenomenon is known for the $R = \pm 1$ case [35, 36], however the present theory in [35] fails to predict higher order parametric resonances $R = \pm 3, \pm 5,...$.

While gain is replaced by resonant delocalisation, the spectrum $\lambda_v$ of $\hat{F}$ no longer carries any information about such resonances. Instead, the structure of its eigen-modes $b^{(n)}$ needs to be analysed. For this purpose, we introduce a measure of localisation of Floquet eigen-modes, similar to the so-called inverse participation ratio used e.g. for studies of Anderson localisation in lattices [37]. While in the SPDC process coupling to the vacuum state plays a crucial role, we define the localisation parameter as $\mathcal{P} = \sum_n |\nu^{(n)}|^2$, where $\nu^{(n)}$ is the first (vacuum) component of the $n$th normalised eigen-mode. In the limit of weak interaction, the eigen-modes of $\hat{F}$ converge to Fock states, i.e. the $n$th eigen-mode is localised on the respective lattice site. It is easy to see that in this limit the localisation parameter tends to its maximal value $\mathcal{P} \to 1$. In the opposite limit of strong interaction, we expect all eigen-modes to be equally spread across the lattice, so that $(|\nu^{(n)}|^2) \sim 1/N \forall n$, where $N$ is the size of the truncated Glauber--Fock lattice. In this limit the localisation parameter tends to its minimal value $\mathcal{P} \to 1/N$. In figure 3(d) we plot $\mathcal{P}$ for the system in equation (12) as function of the modulation parameter $R$ and interaction strength $\gamma_0$. We observe several distinct regions of low $\mathcal{P}$, which form the well-known classical picture of ‘Arnold tongues’, see figure 2(a), and correspond to the resonant delocalisation regime. Remarkably, our analysis predicts higher order resonances, in full correspondence with the classical model.

The perturbation solution in equations (4), (5) is recovered by assuming the hierarchy of smallness of Fock state amplitudes: $|U_0| \gg |U_1| \gg |U_2|,...$. In this regime the system in equation (12) becomes:

\[
\begin{align*}
-\frac{i}{\hbar} (dU_0/d\eta) &= 0, \\
-\frac{i}{\hbar} (dU_1/d\eta) &= -RU_1 + \tilde{\gamma}(\eta) U_0, \\
-\frac{i}{\hbar} (dU_2/d\eta) &= -2RU_2 + 2\tilde{\gamma}(\eta) U_1,
\end{align*}
\]

By solving the above system recursively, dynamics of each multi-photon state is governed by a simple driven oscillator-type equation. Here, the solution for $U_{n-1}$ from the previous step serves as an effective external driving force in the equation for $U_n$. In other words, in this perturbation expansion procedure parametric resonances are replaced by standard resonances. It is easy to see that the resonance condition is the same for all $U_n$. In particular, for simple harmonic modulation of $\gamma(\eta)$ the above system has only $R = \pm 1$ resonance. Solving equations (13) for $U_1$, the two-photon function in equation (5) is restored.

5. Resonant delocalisation in sum- and difference-frequency generation

We emphasise that resonant delocalisation is a generic mechanism which can be observed in a wide range of classical and quantum coupled oscillator-type systems with periodically modulated parameters. It is instructive to consider another type of three-wave mixing process, difference- and sum-frequency generation, whereby an idler wave (or photon) is injected together with a pump into the waveguide, producing signal at $\omega_2 = \omega_1 \mp \omega_p$.
In the context of quantum optics such processes are also known as quantum frequency conversion [38]. In the undepleted pump approximation, both classical (idler wave) and quantum (idler photon) models of this process are similar to the dynamics of a two level system:

\[
\frac{dA_s}{d\eta} = i\gamma_1(\eta)e^{-i\phi_0}A_i, \quad \frac{dA_i}{d\eta} = i\gamma_1^*(\eta)e^{i\phi_0}A_s.
\]  

(14)

In the difference- (sum-) frequency generation case the initial condition is set to \(A_s(0) = 1, A_i(0) = 0\) \((A_s(0) = 0, A_i(0) = 1)\). Unlike the model in equation (8), there can be no exponential gain in the above system. Instead, by tuning the model parameters, one can observe a resonant beating between signal and idler, as illustrated in figures 4(a)–(c). The observed complete Rabi oscillation in the \(R = 1\) case is well understood. Here, one of the exponents in \(\gamma(\eta) \sim \cos(\eta) = 0.5(e^{i\eta} + e^{-i\eta})\) modulation cancels the phase-mismatch exponents, thus enabling efficient coupling. However, this simple logic fails to explain similar oscillations in the \(R = 3\) case. In figures 4(d)–(f) the structure of the corresponding Floquet eigen-modes is illustrated (for clarity, only one of the two conjugate modes is shown). In the \(R = 1\) and \(R = 3\) cases, both signal and idler components of the eigen-mode retain large amplitudes throughout the modulation period. In contrast, in \(R = 2\) case one component of the eigen-mode has a much lower amplitude than the other component, therefore signal and idler are practically de-coupled. Adapting the definition of the localisation parameter \(P\) for this case through the idler component of eigen-modes, we reveal the ‘Arnold tongue’-like structure of resonant delocalisation regions in the space of parameters \((R, \gamma)\), see figure 4(g). In full analogy to parametric resonances, the effective strength of the \(R = 3\) ‘resonance’ is weaker than \(R = 1\), and the complete frequency conversion is observed over a larger number of modulation cycles, see figures 4(a) and (c).

6. Summary

It is well-known that the couplings between bright optical fields in structures with periodically modulated nonlinearity exhibit parametric resonances. We have shown that photon pair generation via SPDC in a waveguide with simple-harmonic modulation of \(\chi_2\) nonlinearity can also be observed within multiple domains in the parameter space of modulation strength and period, in agreement with the regimes of instability of the classical Mathieu equations (‘Arnold’s tongues’). The widely accepted theory of SPDC based on the perturbative derivation of the so-called two-photon function, equations (4), (5), fails to predict such resonances.

We have demonstrated that parametric resonances in SPDC correspond to resonant delocalisation in the Glauber–Fock model. Unlike classical parametric amplification, such resonant delocalisation is not reflected in the spectrum of the corresponding Floquet operator. Instead, the structure of the Floquet eigenmodes must be analysed. By introducing the corresponding localisation parameter \(P\), we have recovered multiple domains of...
photon-pair generation. However, the localisation parameter $\mathcal{P}$ gives no information about the strength of such resonances, unlike the exponential gain parameter calculated for non-Hermitian models.

Our method helps to predict the phenomenon of resonant delocalisation in the generic class of coupled oscillator-type models. In particular, we have explored novel regimes of resonant delocalisation in the QPM photon frequency-conversion process. The established analogy between parametric resonances and resonant delocalisation in parametric down-conversion processes brings a fresh insight into such seemingly unrelated dynamical mechanisms, and can help in developing better tools for their analysis. Furthermore, we note the possibility of harnessing these resonances for previously unexplored phase matching. For example, converting emission from atomic transitions to telecommunications wavelengths typically requires very short poling periods with commensurately tight fabrication tolerances [39]; exploiting a higher-order parametric resonance would lengthen the poling period required and relax the fabrication requirements. However, as higher-order resonances are weaker, see figure 2(b), a careful consideration of the balance between the overall length of the nonlinear medium and attainable pump powers will be required in development of such applications.

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