Digital simulator of a random process using Markov model

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Abstract. Sources (generators) of random processes (signals) are widely used in various technology areas as software modules in computer simulations or hardware blocks in experimental works. The equipment can be built on analog or digital bases. Noise physical sources (generators) form random processes of a certain type (Gaussian noise), which properties and characteristics are set very approximately. Digital generators (sensors) of pseudo-random processes have better technical characteristics, but the range of their types is very limited. The random signal simulator should form its values with specified statistical characteristics (probability distribution and its parameters, correlation properties, etc.) and the possibility of changing them. From this point of view, the Markov model of random process (a simple Markov chain) provides good results. It is the base for a structural scheme of the simple digital algorithm to form pseudo-random numbers and the hardware simulator implementation, to research its capabilities and effectiveness.

1. Introduction

The need to simulate random signals (processes) with given statistical, correlation, spectral and structural properties arises in various fields of technology in the tasks of their research, modeling, measurement of characteristics, testing, etc. [1, 2]. Known analog and digital methods for generating random signals do not have wide versatility. The random signals generated by physical noise sensors and pseudorandom signals, the generation of which are determined by algorithms that provide wide scope to choose statistical properties are recognized.

The various algorithms for generating pseudorandom numbers with given probability distributions described in [3] do not allow the complex two-dimensional statistical properties of simulated processes to be displayed. The most easily independent pseudo-random number sensors with a uniform probability distribution are implemented.

A digital pseudorandom signal simulator based on their Markov model [5, 6] is proposed [4], which can be implemented SW-based in computer models or hardware based on microprocessors [7] or, more expediently, field programmable gate arrays (FPGAs) [8-10]. The Markov model is characterized by a large number of parameters (transition probabilities) and allows to display different two-dimensional probability simulated signals distributions quantized by level and time. It can be
formed according to a given mathematical description of a two-dimensional probability density of a simulated signal or based on statistical processing of its experimental implementation.

The analysis confirms the high versatility of the simulator and the ability to generate a variety of high-frequency pseudorandom signals.

2. Materials and methods

2.1 Markov Model

A simple Markov chain [5, 6] describes a random process in the form of a sequence of integer quantities $z_n$ (numbers of quantization levels of signal samples from the output of the analog-to-digital converter ADC) with $M = 2^m$ possible values from 1 to $M$, $m$ – binary code capacity, $n = 1, N$ – sample point number $t_n$, $N$ – sample volume.

In a simple Markov chain, the probabilities of values $z_{n+1}$ depend only on the previous value $z_{n+1}$ and don’t dependent on earlier readings. It is described by a transition probability matrix in the form of

$$
[P_\pi] = \begin{bmatrix}
P_{11} & P_{12} & \cdots & P_{1M} \\
P_{21} & P_{22} & \cdots & P_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
P_{M1} & P_{M2} & \cdots & P_{MM}
\end{bmatrix},
$$

where $P_{ij}$ – the probability of a process transitioning from a value $z_n = i$ to $z_{n+1} = j$. The probabilities $q_i$ of the initial values $z_1 = i$ are given by the following matrix

$$
[q_1] = \begin{bmatrix}
q_1 \\
q_2 \\
\vdots \\
q_M
\end{bmatrix}.
$$

The model has $M$ ($M-1$) independent parameters $P_{ij}$, which allows using it to display various properties of random processes.

The transition probability matrix should be represented graphically in a three-dimensional image.

2.2 Markov Model parameters determination

For a given two-dimensional probability distribution $w(x_1, x_2)$ of values $x_1$ and $x_2$ with average $x_{cp}$ and thresholds of uniform quantization of the ADC

$$
g_m = \begin{cases} 
-\infty & \text{when } m = 0, \\
\left(\frac{m - M}{2}\right) \cdot h + x_{cp} & \text{when } m = 1, (M - 1), \\
\infty & \text{when } m = M,
\end{cases}
$$

where $h$ – quantization step by level, joint probability distribution of values of a discrete random process is written as
\[ P(z_1 = i, z_2 = j) = \int_{x_{i-1}}^{x_i} \int_{x_{j-1}}^{x_j} w(x_1, x_2) dx_2 dx_1, \quad (4) \]

Then for the transition probabilities we get

\[ P_{ij} = \frac{P(z_1 = i, z_2 = j)}{P(z_1 = i)} = \frac{\int_{x_{i-1}}^{x_i} \int_{x_{j-1}}^{x_j} w(x_1, x_2) dx_2 dx_1}{\int_{x_{i-1}}^{x_i} \int_{-\infty}^{\infty} w(x_1, x_2) dx_2 dx_1}, \quad (5) \]

and for the probabilities of the initial values, respectively,

\[ P(z_1 = i) = \int_{x_{i-1}}^{x_i} \int_{-\infty}^{\infty} w(x_1, x_2) dx_2 dx_1. \quad (6) \]

Expressions (5) and (6) allow us to construct a Markov model of a random process with an arbitrary known two-dimensional probability density \( w(x_1, x_2) \). An example is the Gaussian random process [5], for which three-dimensional matrix diagrams \( P_{ij} \) at \( M=32 \) (ADC capacity \( m = 5 \)) with different correlation coefficients \( r \) are shown in figure 1. As can be seen, the value \( r \) significantly changes the parameters of the model and they also reflect changes in the average value and variance.

In practice, \( m = 8÷12 \) and, accordingly, the matrix has a dimension from 256x256 to 4096x4096, which allows the signal samples to be generated quite accurately.

![Figure 1. Three-dimensional diagrams of transition probability matrices.](image)

Based on the transition probability matrix \( P_{ij} \), a matrix of a two-dimensional probability distribution function

\[ F_{ij} = \sum_{m=1}^{j} P_{im}, \quad (7) \]

with which a signal is simulated is formed.
2.3 Measurement of Markov model parameters

If there is an experimental implementation of a quantized ADC random process \( z_n = 1, N \), then the transition probabilities can be estimated by calculating the transition numbers \( l_{ij} \) of the values \( z_n \) from \( z_{n+1} = 1 \) to \( z_n = j \) for all \( n = 2, N \). Then the statistical estimate \( \tilde{P}_{ij} \) of the transition probabilities \( P_{ij} \) is written as

\[
\tilde{P}_{ij} = \frac{l_{ij}}{\sum_{k=1}^{M} l_{ik}}.
\]  

(8)

With an unlimited increase \( N \) for a stationary random process, estimates \( \tilde{P}_{ij} \) (8) tend to \( P_{ij} \). To estimate \( \tilde{q}_i \) the probabilities \( q_i \) (2) we obtain

\[
\tilde{q}_i = \frac{\sum_{k=1}^{M} l_{ik}}{N - 1}.
\]  

(9)

Statistical estimation of the transition probability matrix \( P_{ij} \) of simple Markov chain (7) reflects the probabilistic properties of the observed quantized random process and the relationship of neighboring samples. If in any \( n \)-th row of a matrix of transition count \( l_{ij} = 0, j = 1, M \), then we need to add to all \( l_{ij} \) a constant, for example 1.

Based on \( \tilde{P}_{ij} \) the matrix estimate of the two-dimensional probability distribution function is calculated

\[
\tilde{F}_{ij} = \sum_{m=1}^{j} \tilde{P}_{im}.
\]  

(10)

2.4 Simulator schematic structure

Simulator schematic structure [4] is shown in figure 2. The clock pulse generator (CPG) generates control pulses with a frequency \( f_T \) that determines the frequency of formation of the signal samples. For each of them, a pseudorandom-number generator (PRNG) generates \( k \)-bit binary code \( A_i \). The composition of the simulator in figure 2 includes a storage unit (SU), a register (RG) and a digital filter (DF).
Pseudorandom binary sequence with a uniform probability distribution are generated by a hardware sensor based on shift registers with feedbacks, for example, an M-sequence generator or a software shaper [3, 11]. In particular, with a shift register length of 61 bits, the period of the M-sequence [11] is $2^{61}-1=2,306 \cdot 10^{18}$, which is quite enough for the formation of realizations of a random signal of long duration.

Instead of the PRNG, we can use a random number generator based on a physical noise source (thermal, for example) and $k$-bit quantizer. In this case, the simulator will generate random signals, but the possibility of conducting repeated statistical experiments will disappear.

Values $F_{ij}$ and their estimates with growth $j$ vary from $F_{i0}=0$ to $F_{iM}=1$, and this area is divided into $V=2^k$ intervals. Their lower bounds $R_d$ may be represented as binary $k$-bit code $A_i = d_{k-1}d_{k-2}..d_0$,

$$R_d = \frac{1}{V}d, \quad (11)$$

$d$ – decimal equivalent of code $d_{k-1}d_{k-2}..d_0$, $k \geq m$, $d=0, V-1$ and $R_0=1$.

For each previous value $i$ of signal reference for all possible codes $d_{k-1}d_{k-2}..d_0$ we find the values $j = j_d$ next reference according to inequality

$$R_d \leq R_d < R_{d+1}. \quad (12)$$

The resulting arrays $j_d$ are recorded in a storage unit (SU) in the form of pages with cell addresses $d_{k-1}d_{k-2}..d_0$ (lower order bits of the SU address $A_i$). Pages are addressed in binary code $i$ (high order bits of the address $A_i$).

At the start of the simulator, the first CPG pulse forms the first code $A_1$. The initial state of the RG register (may be any) determines the first sample $i$ ($m$-bit binary code $A_2$) of the simulated signal. It sets the SU page (high order bits of the address), and the code $A_i$ selects a memory location containing the next sample $j$ of the simulated signal. The binary code $j$ appears at the output of the memory. By the next clock pulse, the code $j$ is written to the RG, becoming the previous sample, and the new PRNG code selects the next signal sample. Then the procedure is repeated and the random signal samples are sent to a digital filter, which generates the required frequency and correlation properties of the simulated random process and transfers them to the digital output $z$ of the simulator. If necessary, they can be submitted to a digital-to-analog converter to obtain an analog signal $s(t)$.

**Figure 2.** Simulator Schematic Structure.
It is advisable to select the $m$ bit depth of the signal sample code in the interval $m = 6 \div 10$, and code $d_{N-2}d_{N-3}...d_0$ of the PRNG $k = 8 \div 16$, as a result, the bit width of the address bus of the memory unit is $m + N = 14 \div 26$, that is, the capacity of the memory unit will be no more than 64MB.

### 2.5 Gaussian signals simulation

Statistical modeling of a digital simulator was carried out. Figure 3 presents the results of simulating a normal random process with zero mean and dispersion $\sigma^2$ at a $M = 32$ correlation coefficient $r = 0.4$, with the transition probability matrix shown in figure 1b, and with quantization levels (3) at step

$$h = \frac{10 \cdot \sigma}{M}.$$  

(13)

Figure 3a shows a probability distribution diagram $F_{ij}$ (7); figure 3b shows a temporary implementation of a sequence of discrete samples $z_n$; its histogram coincides with the initial normal probability distribution. Figure 3c shows the dependence of the correlation coefficient $R_k$ on the magnitude of the offset of the sample of samples $k$, the dotted line shows the theoretical dependence $r^k$. As we can see, the model well reflects the two-dimensional properties of the Gaussian random process.

![Figure 3](image)

**Figure 3.** The results of the simulation algorithm.

#### 2.6 Non-Gaussian Random Process Simulation

Consider a simulation of a random process with a two-dimensional probability density of the form

$$w(x, y) = \frac{1}{2} \sin(x + y) \text{ when } 0 \leq x, y \leq \pi/2,$$

(14)

its diagram is presented in figure 4a. Figure 4b shows a three-dimensional diagram of the matrix of the joint probability distribution $P(i, j)$ (4) at $M = 32$, and figure 4c shows the transition probability matrices $P_{ij}$ (6). It is easy to verify that the probabilistic properties of the random process (14) significantly differ from the normal one.
Figure 4. Markov model characteristics.

Figure 5a shows the implementation of a random process obtained by simulation \( z_n \), figure 5b shows a histogram of the one-dimensional probability distribution \( P_i \) of sample values \( z_n = i \) (the theoretical dependence is shown with a dashed line), and figure 5c shows normalized correlation function \( R_k \). Figure 6 shows the probabilistic characteristics of the simulation results: figure 6a shows a statistical estimate of the joint probability distribution \( P(i, j) \), and figure 6b shows the transition probabilities \( P_{ij} \).

Figure 5. Simulation results.

Figure 6. Probabilistic simulation results characteristics.
As can be seen, again the estimates of the characteristics of the simulated non-Gaussian random process coincide with the theoretical ones. Similar results were obtained for other two-dimensional probability distributions as well.

3. Results and discussion

As a result of experimental implementation of a radio signal with wideband frequency modulation (WFM) [12] with a volume of \( L = 2.5 \times 10^5 \) samples at the ADC sampling rate \( f_D = 4 f_0 \) (\( f_0 \) – carrier frequency) with number of quantization levels \( M=64 \) \( (m=6) \) an estimate of the matrix of transition count \( l_{ij} \) is obtained, the three-dimensional diagram of which is shown in figure 7a.

![Figure 7a](image1)

**Figure 7a.** Signal simulation using WFM.

Figure 7b shows the estimate of the three-dimensional diagram of the probability distribution function \( \tilde{F}_{ij} \) (10) obtained from the experimental matrix of transition count, and figure 7c shows transition count matrix formed by a signal simulator.

Thus, the considered simulator of experimentally observed random signals based on the Markov model allows us to correctly simulate their probabilistic properties. At a selected sampling frequency \( f_D \) (at the Nyquist frequency boundary [12]) for a narrow-band signal, correlation properties may be distorted; to display them, it is necessary to increase the sampling frequencies several times.

4. Conclusion

A digital random signal simulator based on the Markov model provides an accurate representation of their two-dimensional statistical and correlation properties. It is shown that modeling of two-dimensional probability distributions of Gaussian, non-Gaussian and arbitrary experimental random processes is provided. The simulator allows a simple software and hardware implementation based on microprocessors or FPGAs with a minimum number of operations independent of the simulated process. When using FPGAs, the next sample is generated in one clock interval.

A signal model can be formed theoretically from a given two-dimensional probability density or from an experimental sample of signal samples. It is recorded in a storage unit and can be quickly changed during operation of the simulator.

References

[1] Kleinen J 1978 *Statistical methods in simulation modeling* vol 2 (Moscow: Statistics)
[2] Kelton V and Low A 2004 *Simulation modeling. Classic CS* (St. Petersburg: Peter, BHV Publishing Group)
[3] Litvinenko V P and Chernoyarov O V 2017 *Modeling of random processes: training manual*
[4] Glushkov A N, Kalinin M Y, Litvinenko V P and Litvinenko Y V 2019 Digital simulator of random signals Patent Russia no 2690780

[5] Wentzel E S and Ovcharov L A 2000 Probability Theory and Its Engineering Applications (Moscow: Higher School)

[6] Kazakov V A 1973 Introduction to the theory of Markov processes and some radio engineering problems (Moscow: Soviet Radio)

[7] Krug P G 2001 Digital signal processing processors: a training manual (Moscow: Publishing House MEI)

[8] Xilinx (2009a) Spartan-6 Family Overview vol DS160 (V1.0) Xilinx Inc. Available at: https://www.xilinx.com/support/documentation/data_sheets/ds160.pdf

[9] Xilinx (2010a) 7 Series Overview vol DS150 (V1.0) Xilinx Inc. Available at: https://www.xilinx.com/support/documentation/data_sheets/ds180_7Series_Overview.pdf

[10] Cyclone IV Device Handbook vol 1. Altera Corporation. Available at: https://www.intel.ru/content/dam/www/programmable/us/en/pdfs/literature/hb/cyclone-iv/eyiv-5v1.pdf

[11] Varakin L E 1985 Communication systems with noise-like signals (Moscow: Radio and Communications)

[12] Sklar B 2001 Digital Communications. Fundamentals and Applications (New Jersey: Prentice Hall)