Bernoulli measure on strings, and Thompson-Higman monoids

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May 3, 2010

Abstract

The Bernoulli measure on strings is used to define height functions for the dense \( R \)- and \( L \)-orders of the Thompson-Higman monoids \( M_{k,1} \). The measure can also be used to characterize the \( D \)-relation of certain submonoids of \( M_{k,1} \). The computational complexity of computing the Bernoulli measure of certain sets, and in particular, of computing the \( R \)- and \( L \)-height of an element of \( M_{k,1} \) is investigated.

1 Introduction

Since their introduction in the 1960s [15, 14, 16] the groups of Richard J. Thompson have become well known for their remarkable properties. They were generalized by Graham Higman [12] to a class of groups \( G_{k,i} (k \geq 2, k > i \geq 1) \) with similar properties: They are finitely presented infinite simple groups, containing all finite groups. The Thompson-Higman groups have a large literature; see the list of references in [6, 3]. The groups \( G_{k,i} \) can be generalized in a straightforward way to monoids \( M_{k,i} \) which also have remarkable properties: They are finitely generated with word problem decidable in polynomial time, they are congruence-simple, and they contain all finite monoids [3], [2], [1].

The Green relations of \( M_{k,1} \) were characterized in [2], [1]: The \( J \)- and the \( D \)-relations do not have much structure, as \( M_{k,1} \) is \( J \)-0-simple and has exactly \( k-1 \) non-zero \( D \)-classes. The \( R \)- and \( L \)-orders are complicated: They are dense, and are characterized by image sets and partitions, in analogy with the monoid of all partial functions on an infinite set. We will define \( M_{k,1} \) and describe its Green relations in more detail below. Here we will only talk about \( M_{k,1} \) \((k \geq 2)\), and ignore \( M_{k,i} \) for \( i \neq 1 \).

The aim of this paper is to introduce height functions for the \( R \)- and \( L \)-orders of \( M_{k,1} \), which is not obvious for dense orders. It turns out that the Bernoulli measure on the free monoid \( A^* \) (over a finite alphabet \( A \) with \( |A| = k \)) plays a crucial role for defining \( R \)- and \( L \)-height functions in \( M_{k,1} \). The Bernoulli measure enables us also to characterize the \( D \)-relation of certain submonoids of \( M_{k,1} \), namely submonoids of transformations that map fixed-length words to fixed-length words. We also determine the computational complexity of computing the Bernoulli measure associated with an element of \( M_{k,1} \). The appendix contains a proof that \( M_{k,1} \) is congruence-simple (the proof of this fact in [3] was incomplete).

1.1 Definition of the Thompson-Higman groups and monoids

In order to make this paper (mostly) self-contained we start with definitions and notations. We will define \( M_{k,1} \) by a partial action on \( A^* \), the free monoid over \( A \) (consisting of all finite sequences of elements of \( A \)), where \( A \) is an alphabet of size \( k \). We call the elements of \( A^* \) words, and we also include the empty word \( \varepsilon \). We denote the length of \( w \in A^* \) by \( |w| \). The concatenation of \( u, v \in A^* \) is denoted by \( uv \) or by \( u \cdot v \); more generally, the concatenation of \( B, C \subseteq A^* \) is \( BC = \{ uv : u \in B, v \in C \} \). A word \( u \in A^* \) is a prefix of \( v \in A^* \) iff \( uz = v \) for some \( z \in A^* \), and we write \( u \text{ pref } v \). We say that \( u \) and \( v \) are prefix-comparable if \( u \text{ pref } v \) or \( v \text{ pref } u \). A prefix code is a subset \( C \subseteq A^* \) whose elements are
two-by-two prefix-incomparable. A prefix code is maximal iff it is not strictly contained in any other prefix code.

A subset \( R \subseteq A^* \) is called a right ideal iff \( RA^* \subseteq R \). We call \( R \) an essential right ideal iff \( R \) intersects every right ideal of \( A^* \). More generally, for right ideals \( R' \subseteq R \subseteq A^* \), \( R' \) is essential in \( R \) iff \( R' \) intersects all right ideals included in \( R \).

A right ideal \( R \) is generated by a set \( C \subseteq A^* \) iff \( R \) is the intersection of all right ideals that contain \( C \); equivalently, \( R = CA^* \). One proves easily that a right ideal \( R \) has a unique minimal (under inclusion) generating set, and that this minimal generating set is a prefix code. The prefix code that generates \( R \) is maximal iff \( R \) is an essential right ideal. In this paper we only use right ideals that are finitely generated.

For a partial function \( f : A^* \to A^* \) we denote the domain by \( \text{Dom}(f) \) and the image by \( \text{Im}(f) \). A function \( \varphi : R_1 \to A^* \) is a right ideal homomorphism of \( A^* \) iff \( \text{Dom}(\varphi) = R_1 \) is a right ideal and for all \( x_1 \in R_1 \) and all \( w \in A^* \): \( \varphi(x_1 w) = \varphi(x_1) w \). It follows that \( \text{Im}(\varphi) \) is a right ideal, and if \( R_1 \) is finitely generated (as a right ideal) then \( \text{Im}(\varphi) \) is finitely generated. We write the action of partial functions on the left of the argument; equivalently, functions are composed from right to left.

A right ideal homomorphism \( \varphi : R_1 \to R_2 \) is uniquely determined by its restriction \( P_1 \to S_2 \), where \( P_1 \) is the prefix code that generates \( R_1 \) as a right ideal, and \( S_2 \) is a set (not necessarily a prefix code) that generates \( R_2 \) as a right ideal. This finite total surjective function \( P_1 \to S_2 \) is called the table of \( \varphi \). The finite prefix code \( P_1 \) is called the domain code of \( \varphi \) and is denoted by \( \text{domC}(\varphi) \). When \( S_2 \) is a prefix code it will be denoted by \( \text{ImC}(\varphi) \) and called image code.

A right ideal homomorphism \( \Phi : R'_1 \to A^* \) is called an essentially equal restriction of a right ideal homomorphism \( \varphi : R_1 \to A^* \) (or, equivalently, \( \varphi \) is an essentially equal extension of \( \Phi \)) iff \( R'_1 \) is essential in \( R_1 \), and for all \( x'_1 \in R'_1 \): \( \varphi(x'_1) = \Phi(x) \). The multiplication in \( M_{k,1} \) (and in \( G_{k,1} \)) depends on the following facts: (1) Every homomorphism \( \varphi \) between finitely generated right ideals of \( A^* \) has a unique maximal essentially equal extension (Prop. 1.2(2) in [3]); this extension is denoted by \( \text{max}(\varphi) \). (2) Every right ideal homomorphism \( \varphi \) has an essentially equal restriction \( \varphi' \) whose table \( P' \to Q' \) is such that both \( P' \) and \( Q' \) are prefix codes (remark after Prop. 1.2 in [3]).

We are now ready to define the Higman-Thompson monoid \( M_{k,1} \): As a set, \( M_{k,1} \) consists of all homomorphisms (between finitely generated right ideals of \( A^* \)) that have been maximally essentially equally extended. In other words, as a set,

\[
M_{k,1} = \{ \text{max}(\varphi) : \varphi \text{ is a homomorphism between finitely generated right ideals of } A^* \}
\]

The multiplication is composition followed by maximal essentially equal extension (which is unique). This multiplication is associative (Prop. 1.4 in [3]). Thus we have a partial action of \( M_{k,1} \) on \( A^* \) (since the multiplication is not just composition, but needs to be followed by maximal essential extension).

The Higman-Thompson monoid \( M_{k,1} \) also has a true action by partial functions on \( A^\omega \) (the Cantor space, consisting of all \( \omega \)-sequences over \( A \)). The action of \( \varphi \in M_{k,1} \) on \( z \in A^\omega \) is defined by \( \varphi(z) = yw \) if \( z \) can be written as \( z = xw \) for some \( x \in \text{domC}(\varphi) \), where \( y = \varphi(x) \); \( \varphi(z) \) is undefined if \( z \) has no prefix in \( \text{domC}(\varphi) \).

For a right ideal \( R \subseteq A^* \) generated by a prefix code \( P \) we call \( PA^\omega \) the set of ends of \( R \), denoted by \( \text{ends}(R) \). We call two right ideals \( R_1, R_2 \) essentially equal iff \( \text{ends}(R_1) = \text{ends}(R_2) \), and we denote this by \( R_1 =_{\text{ess}} R_2 \).

### 1.2 The Bernoulli measure

For a fixed alphabet \( A \) with \( |A| = k \), and any set \( X \subseteq A^* \) we define the Bernoulli measure of \( X \) by

\[
\mu(X) = \sum_{x \in X} k^{-|x|}.
\]

For the empty set the measure is 0. When \( X \) is infinite, \( \mu(X) \) can be any positive real number or \( +\infty \). For a non-empty finite set \( X \), \( \mu(X) \) it is a strictly positive \( k \)-ary rational number.
By definition, a \textit{k-ary rational number} is a number of the form \(a/k^n\), where \(a \in \mathbb{Z}\) and \(n \in \mathbb{N}\); equivalently, it is a rational number that has a finite representation in base \(k\). The ring of \(k\)-ary rational numbers is denoted by \(\mathbb{Z}[1/k]\). For any \(r = a/k^n \in \mathbb{Z}[1/k]\) we say that \(a/k^n\) is \(k\)-reduced if \(a\) is not divisible by \(k\); in that case we denote the numerator \(a\) by \(\text{num}(r)\).

Obviously, the definition of \(\mu\) amounts to viewing each word in \(A^*\) as a sequence of independent \(k\)-ary Bernoulli trials in which each choice \(a \in A\) has the same probability, namely \(1/k\).

If \(X\) is a prefix code (finite or infinite) then \(\mu(X) \leq 1\), with equality iff \(X\) is maximal. This is the \textit{Kraft inequality} (or equality). Hence \(\varphi \in M_{k,1}\) is total (on the Cantor space \(A^\omega\)) iff \(\mu(\text{dom}C(\varphi)) = 1\); \(\varphi\) is surjective (onto \(A^\omega\)) iff \(\mu(\text{im}C(\varphi)) = 1\).

\section{Height functions for the \(R\)- and \(L\)-orders of \(M_{k,1}\)}

In a finite monoid, the \(R\)-height of an element \(x\) is defined to be the length of a longest ascending \(R\)-chain from a minimal \(R\)-class to \(x\) (and similarly for the \(L\)-height). For infinite monoids there may be no good way to define an \(R\)- or \(L\)-height function at all, especially if the \(R\)- and \(L\)-orders are dense (as is the case for \(M_{k,1}\), by Section 4 in \([2]\)). The main result of this section is that for \(M_{k,1}\) we can nevertheless define an \(R\)-height and an \(L\)-height. For a pre-order, in general, we define height functions as follows:

\textbf{Definition 2.1} \(A\) height function for a pre-order \((M, \preceq)\) is any function \(h : M \to \mathbb{R}\) such that for all \(x, y \in M\): if \(y \preceq x\) then \(h(y) \leq h(x)\), and if \(y < x\) then \(h(y) < h(x)\) (where, \(y < x\) means \(“y \preceq x\) and \(x \neq y\)”). It follows that \(h(x) = h(y)\) when \(x \equiv y\) (where \(x \equiv y\) means \(“x \preceq y\) and \(y \preceq x\)”).

The definition of the \(R\)- and \(L\)-height functions will be guided by the characterizations of the \(R\)- and \(L\)-orders in \(M_{k,1}\), given in \([2]\). Regarding \(\leq_R\) we have for all \(\psi, \varphi \in M_{k,1}\):

\begin{itemize}
  \item \(\psi \leq_R \varphi\) iff \(\text{ends}(\text{im}(\psi)) \subseteq \text{ends}(\text{im}(\varphi))\) iff
  \item every right ideal of \(A^*\) that intersects \(\text{im}(\psi)\) also intersects \(\text{im}(\varphi)\).
\end{itemize}

The characterization of \(\leq_L\) is more complicated and involves right-congruences. We first need some definitions. For any right ideal homomorphism \(\varphi\), let \(\text{part}(\varphi)\) be the right congruence on \(\text{Dom}(\varphi)\) defined by \((x_1, x_2) \in \text{part}(\varphi)\) if \(\varphi(x_1) = \varphi(x_2)\). This definition of \(\text{part}(\varphi)\) can be extended to \(\text{ends}(\text{Dom}(\varphi))\): for \(v_1, v_2 \in A^\omega\) we have \((v_1, v_2) \in \text{part}(\varphi)\) iff \(\varphi(v_1) = \varphi(v_2)\); this is iff there exist \(w \in A^\omega\) and \(x_1, x_2 \in \text{Dom}(\varphi)\) such that \((x_1, x_2) \in \text{part}(\varphi)\) and \(v_1 = x_1w, v_2 = x_2w\).
Two right congruences \( \simeq_1 \) (on \( P_1 A^* \)) and \( \simeq_2 \) (on \( P_2 A^* \)) are called essentially equal iff \( \text{ends}(P_1 A^*) = \text{ends}(P_2 A^*) \) and the extension of \( \simeq_1 \) to \( \text{ends}(P_1 A^*) \) is equal to the extension of \( \simeq_2 \) to \( \text{ends}(P_1 A^*) \). We denote this by \( \simeq_1 = \text{ess} \simeq_2 \). We say that \( \simeq_1 \) is an essentially equal restriction of \( \simeq_2 \) (and equivalently, \( \simeq_2 \) is an essentially equal restriction of \( \simeq_1 \)) iff \( P_2 A^* \subseteq P_1 A^* \) and \( \simeq_1 = \text{ess} \simeq_2 \).

Let \( \simeq_{\text{domC} (\varphi)} \) be the restriction of the right congruence \( \text{part} (\varphi) \) to the finite prefix code \( \text{domC} (\varphi) \). We would like \( \text{part} (\varphi) \) to be determined in a simple way by \( \simeq_{\text{domC} (\varphi)} \) as follows. Let \( P \subset A^* \) be a finite prefix code and let \( \simeq_P \) be an equivalence relation on \( P \); we call a right congruence \( \simeq \) on \( PA^* \) a prefix code congruence (determined by \( \simeq_P \)) iff for all \( p_1, p_2 \in P \):

- if \( p_1 \simeq_P p_2 \) then for all \( w \in A^* \), \( p_1 w \simeq_P p_2 w \);
- for all \( x, y \in A^* \): if \( p_1 \not\equiv_P p_2 \) or if \( x \neq y \) then \( p_1 x \not\equiv_P p_2 y \).

In other words, \( \simeq \) is a prefix code congruence on \( PA^* \) iff \( \simeq \) is the smallest (i.e., finest) right congruence that agrees with the restriction of \( \simeq \) to \( P \).

Although \( \text{part} (\varphi) \) is always a right congruence, it is not always a prefix congruence. In [2] it was proved that \( \text{part} (\varphi) \) is a prefix congruence (determined by \( \simeq_{\text{domC} (\varphi)} \)) iff \( \varphi (\text{domC} (\varphi)) \) is a prefix code. In that case \( \varphi (\text{domC} (\varphi)) \) is denoted by \( \text{imC} (\varphi) \).

Let \( \simeq \) be a prefix code congruence on \( PA^* \) determined by \( \simeq_P \). If \( C \subseteq P \) is a class of \( \simeq_P \) then a class-wise replacement step consists of replacing \( C \) by the set of classes \( \{ C_1, \ldots, C_k \} \) and replacing \( P \) by \( Q = (P - C) \cup C_1 \cup \ldots \cup C_k \), where \( A = \{ a_1, \ldots, a_k \} \). In [2] it was proved that the resulting equivalence relation \( \simeq_Q \) also determines a prefix code congruence (on \( QA^* \)) which is essentially equal to the congruence determined by \( \simeq_P \). Hence, if we apply a finite sequence of class-wise replacement steps or inverses of replacement steps to a prefix code congruence \( \simeq \), we obtain a prefix code congruence that is essentially equal to \( \simeq \). Conversely, it was proved in [2] that if two prefix code congruences \( \simeq_1 \) and \( \simeq_2 \) are essentially equal then each one is obtained from the other one by a finite number of class-wise replacement steps and their inverses.

A prefix code congruence \( \simeq \) is called maximal iff \( \simeq \) is maximal with respect to \( \subseteq \text{ess} \). It is easy to see that inverse class-wise replacements form a terminating and confluent rewriting system. Hence, every prefix code congruence \( \simeq \) is \( \subseteq \text{ess} \)-contained in a unique maximal prefix code congruence, which we denote by \( \text{max}(\simeq) \).

We can now state the characterization of the \( L \)-order of \( M_{k,1} \) given in [2]. For \( \varphi, \psi \in M_{k,1} \),

- \( \psi \leq_L \varphi \) iff
- every class of \( \text{ends} (\text{part} (\psi)) \) is a union of classes of \( \text{ends} (\text{part} (\varphi)) \) iff
- every class of \( \text{part} (\psi) \) is a union of classes of \( \text{max}(\text{part} (\varphi)) \).

### 2.1 A height function for the \( R \)-order

We just saw that the \( R \)-order in \( M_{k,1} \) is determined by the inclusion relation between the sets \( \text{ends}(\text{imC} (\varphi)) \). This suggests the following definition of an \( R \)-height function for \( M_{k,1} \).

**Definition 2.2** Let \( \varphi \in M_{k,1} \) be described by a table \( P \to Q \) where \( P, Q \subset A^* \) are finite prefix codes, i.e., \( P = \text{domC} (\varphi) \) and \( Q = \text{imC} (\varphi) \). Then the \( R \)-height of \( \varphi \), denoted by \( \text{height}_R (\varphi) \), is defined by

\[
\text{height}_R (\varphi) = \mu (\text{imC} (\varphi)).
\]

We will prove below (Prop. 2.5) that \( \text{height}_R \) depends only on the \( R \)-class of \( \varphi \) and that it is indeed a height function for \( (M_{k,1}, \leq_R) \). In particular, it does not depend on the particular table \( P \to Q \) used to represent \( \varphi \).

We said that we only use tables \( P \to Q \) where \( Q \) is a prefix code. Let us briefly investigate what happens when \( Q \) is not a prefix code.
**Proposition 2.3** Let $\Phi \in M_{k,1}$ be described by a table $P \rightarrow Q$, where $Q$ is not a prefix code, and let $\varphi$ be any essentially equal extension or restriction of $\Phi$ such that $\text{im} C(\varphi)$ is a prefix code. Then, $\mu(Q) > \mu(\text{im} C(\varphi))$.

**Proof.** When $Q$ is not a prefix code then there exists a prefix code $Q_0 \subset Q$ such that $QA^* = Q_0A^*$. In fact, $Q_0 = Q - QA^*$, so $Q_0$ is uniquely determined by $Q$. Since $Q_0A^* = QA^*$, and since $Q_0$ and $\text{im} C(\varphi)$ are prefix codes, we have (by Prop. 1.3): $\mu(Q_0) = \mu(\text{im} C(\varphi))$. Moreover, $\mu(Q) = \mu(Q_0) + \mu(Q - Q_0) > \mu(Q_0)$. □

The following example illustrates Prop. 2.3.

**Example.** Let $A = \{a, b\}$ and let $\varphi \in M_{2,1}$ be given by the following tables, all describing the same element of $M_{2,1}$:

$$
\phi_1 = \begin{array}{c|cc|c}
  & a & b \\
 a & \text{aa} & \text{aa} \\
 a & \text{aa} & \text{aa}
\end{array}, \quad \phi_2 = \begin{array}{c|cc|c}
  & a & b \\
 a & \text{aa} & \text{aa} \\
 a & \text{aa} & \text{aa}
\end{array}, \\
\phi_3 = \begin{array}{c|cc|c}
  & a & b \\
 a & \text{aaa} & \text{aaa} \\
 a & \text{aaa} & \text{aaa}
\end{array}, \quad \text{and} \quad \phi_4 = \begin{array}{c|cc|c}
  & a & b \\
 a & \text{aaa} & \text{aaa} \\
 a & \text{aaa} & \text{aaa}
\end{array}.
$$

The measure of the set $\{a, aa, aaa\}$ in the table of $\phi_1$ is $\mu(\{a, aa, aaa\}) = \frac{7}{5}$. By essentially equal restriction we obtain $\phi_2$ and the measure of the image set of the table of $\phi_2$ is $\mu(\{aa, ab, aaa\}) = \frac{5}{8}$. Next we get the table $\phi_3$ with measure $\mu(\{aaa, aab, ab, aa\}) = \frac{3}{4}$. Finally, another restriction step yields $\phi_4$, for which $\phi_4(\text{dom} C(\phi_1)) = \text{im} C(\phi_4)$ is a prefix code. Thus we finally obtain the measure $\mu(\text{im} C(\phi_4)) = \mu(\{aaa, aab, ab, aa\}) = \frac{1}{2}$; so, $\mu(\text{im} C(\varphi)) = \frac{1}{2}$. □

As a consequence we have:

**Proposition 2.4** For a right-ideal homomorphism $\varphi$ the following are equivalent:

1. $\text{part}(\varphi)$ is a prefix congruence;
2. $\varphi(\text{dom} C(\varphi))$ is a prefix code;
3. $\mu(\varphi(\text{dom} C(\varphi))) = \min \{\mu(Q) : P \rightarrow Q \text{ is a table that represents } \varphi\}$.

**Proof.** The equivalence of (1) and (2) was proved in [2]. The equivalence of (2) and (3) is given by Prop. 2.3 combined with Prop. 1.3 □

The term “$R$-height” is justified by the following.

**Proposition 2.5** For all $\varphi \in M_{k,1}$, $\mu(\text{im} C(\varphi))$ depends only on the $R$-class of $\varphi$.

The function $\text{height}_R : \varphi \in M_{k,1} \mapsto \mu(\text{im} C(\varphi)) \in \mathbb{Z}[\frac{1}{k}] \cap [0, 1]$ is a height function for the pre-order $(M_{k,1}, \leq_R)$ (according to Definition 2.7).

**Proof.** The fact that $\mu(\text{im} C(\varphi))$ depends only on the $R$-class of $\varphi$ follows immediately from the characterization of the $R$-order mentioned at the beginning of section 2 (Theorem 2.1 in [2]), and from Prop. 1.3 above.

Suppose $\psi$ and $\varphi$ are represented by tables $P_\psi \rightarrow Q_\psi$, respectively $P_\varphi \rightarrow Q_\varphi$. We only prove that $\varphi \geq_R \psi$ if $\text{height}_R(\varphi) \geq \text{height}_R(\psi)$ (under the assumption that $\psi$ and $\varphi$ are $R$-comparable); the other two case have a similar proof. By Theorem 2.1 in [2], $\varphi \geq_R \psi$ if $\text{ends}(Q_\psi A^*) \subseteq \text{ends}(Q_\varphi A^*)$. After an essentially equal restriction of $\psi$ and $\varphi$, if necessary, the latter holds if $Q_\psi \subseteq Q_\varphi$. This holds if $\mu(Q_\psi) < \mu(Q_\varphi)$, under the assumption that $Q_\psi$ and $Q_\varphi$ are comparable under inclusion. □

The following Lemma and Proposition show that the height function $\text{height}_R$ is onto $\mathbb{Z}[\frac{1}{k}] \cap [0, 1]$, and that for any chain of numbers in $\mathbb{Z}[\frac{1}{k}] \cap [0, 1]$ there are corresponding $<_R$-chains.
Lemma 2.6
(1) For every $h \in \mathbb{Z}[\frac{1}{k}] \cap [0,1]$ there exists a finite prefix code $P_h \subset A^*$ with $\mu(P_h) = h$.
(2) For all $g, h \in \mathbb{Z}[\frac{1}{k}]$ with $0 \leq g < h \leq 1$ the finite prefix codes $P_g, P_h$ constructed in (1) satisfy $P_h \subset P_g A^*$, and $P_g A^* \neq \text{ess} P_h A^*$.

Hence the set \{ $P_h A^* : h \in \mathbb{Z}[\frac{1}{k}] \cap [0,1]$ \} is a chain of right ideals, no two of which are essentially equal, and with $\mu(P_h) = h$.

Proof. (1) As before, $A = \{a_1, \ldots, a_k\}$. When $h = 0$ we pick $P_0 = \emptyset$, and when $h = 1$ we pick $P_1 = \{e\}$ (where $e$ is the empty string). We assume next that $0 < h < 1$. Then $h$ has a unique finite base-k expansion of the form $h = d_1 \ldots d_i \ldots d_k$ where $d_n \neq 0$, and $d_i \in \{0,1,\ldots,k-1\}$ for $i = 1, \ldots, n$. With $h$ we associate the following prefix code:

\[ P_h = \{a_1, \ldots, a_{d_1}\} \cup a_{d_1+1} \{a_1, \ldots, a_{d_2}\} \cup a_{d_1+1}a_{d_2+1} \{a_1, \ldots, a_{d_3}\} \cup \ldots \]
\[ \ldots \cup a_{d_1+1} \ldots a_{d_{i-1}+1} \{a_1, \ldots, a_{d_i}\} \cup a_{d_1+1} \ldots a_{d_{i-1}+1} a_{d_i+1} \{a_1, \ldots, a_{d_{i+1}}\} \cup \ldots \]
\[ \ldots \cup a_{d_1+1} \ldots a_{d_{n-1}+1} \{a_1, \ldots, a_{d_n}\} \].

Here, the set \{ $a_1, \ldots, a_{d_i}\} is empty if $d_i = 0$ (1 $\leq i \leq n$). It is easy to verify that

$\mu(a_{d_1+1} \ldots a_{d_{i-1}+1} \{a_1, \ldots, a_{d_i}\}) = k^{-i} d_i$,

hence $h = \mu(P_h)$. Note also that $|P_h| = \sum_{i=1}^{n} d_i$.

For example, for $k = 5$ and $h = 0.0031042$ (in base 5) we have

$P_h = a_1 a_1 \{a_1, a_2, a_3\} \cup a_1 a_1 a_4 \{a_1\} \cup a_1 a_1 a_4 a_2 a_1 \{a_1, a_2, a_3, a_4\} \cup a_1 a_4 a_2 a_1 a_5 \{a_1, a_2\}$.

(2) The above construction of $P_h$ from $h$ has the following property: For any k-ary rationals $h, g$ with $0 < h < g \leq 1$ we have $P_h \subset P_g A^*$. This is proved next.

When $g = 1$ this is clear (since then $P_g$ is a maximal prefix code, and $P_h$ is not maximal). When $h < g < 1$ then $g$ has a base-k expansion of the form $g = 0.d_1 \ldots d_{i-1} b_i \ldots b_m$ with $d_i < b_i$ (for some $i$ with $1 \leq i \leq \min\{m, n\}$), where $b_1, \ldots, b_m$ are base-k digits. Then

$P_g = \{a_1, \ldots, a_{d_1}\} \cup a_{d_1+1} \{a_1, \ldots, a_{d_2}\} \cup a_{d_1+1}a_{d_2+1} \{a_1, \ldots, a_{d_3}\} \cup \ldots \]
\[ \ldots \cup a_{d_1+1} \ldots a_{d_{j-1}+1} \{a_1, \ldots, a_{d_j}\} \cup a_{d_1+1} \ldots a_{d_{j-1}+1} a_{d_j+1} \{a_1, \ldots, a_{d_{j+1}}\} \cup \ldots \]
\[ \ldots \cup a_{d_1+1} \ldots a_{d_{m-1}+1} a_{b_m+1} \{a_1, \ldots, a_{b_m}\} \].

Since $d_i < b_i$ we have

$\{a_{d_1+1} \ldots a_{d_{i-1}+1} \{a_1, \ldots, a_{d_i}\} \subset a_{d_1+1} \ldots a_{d_{i-1}+1} \{a_1, \ldots, a_{d_i}, \ldots, a_{b_i}\}$.

And we have for all $j$ ($i \leq j \leq n$):

$\{a_{d_1+1} \ldots a_{d_{i-1}+1} a_{d_i+1} \ldots a_{d_{j-1}+1} \{a_1, \ldots, a_{d_j}\} \subset a_{d_1+1} \ldots a_{d_{i-1}+1} \{a_1, \ldots, a_{d_i}\}$

indeed, $a_{d_{i+1}} \in \{a_1, \ldots, a_{b_i}\}$ (because $d_i + 1 \leq b_i$), and $a_{d_{i+1}} \ldots a_{d_{j-1}+1} \{a_1, \ldots, a_{d_j}\} \subset A^*$. Thus, $P_h \subset P_g A^*$, hence $P_h A^* \subset P_g A^*$. Also, $P_h A^* \neq \text{ess} P_g A^*$ since $\mu(P_h A^*) = h \neq g = \mu(P_g A^*)$.

Notation: For a subset $X \subset A^*$, $id_X$ denotes the partial identity function. In other words, for any $w \in A^*$, $id_X(w) = w$ if $w \in X$; and $id_X(w)$ is undefined if $w \notin X$.

Proposition 2.7
(1) For every $h \in \mathbb{Z}[\frac{1}{k}] \cap [0,1]$ there exists $\varphi_h \in M_k, 1$ such that $\text{height}_R(\varphi_h) = h$.
Moreover, $\varphi$ can be chosen to be of the form $id_{PA^*}$ for some finite prefix code $P$ with $\mu(P) = h$.

(2) For all $g, h \in \mathbb{Z}[\frac{1}{k}]$ with $0 \leq g < h \leq 1$ the elements $\varphi_g, \varphi_h \in M_k, 1$ constructed in (1) satisfy $\varphi_g <_R \varphi_h$.

Hence \{ $\varphi_h : h \in \mathbb{Z}[\frac{1}{k}] \cap [0,1]$ \} forms a dense $<_R$-chain of elements of $M_k, 1$ with $\text{height}_R(\varphi_h) = h$. 

6
2.2 A height function for the \( L \)-order

The construction of an \( L \)-height (Definitions 2.9 and 2.12 below) requires a few preliminary definitions and facts.

Let \( \simeq \) be a partition of a set \( S \subseteq A^* \). For each \( x \in S \) the \( \simeq \)-class containing \( x \) is denoted by \([x]\). A set of representatives of \( \simeq \) is, by definition, a set consisting of exactly one element from each \( \simeq \)-class. A set of minimum-length representatives of \( \simeq \) is a set \( R \) of representatives such that each \( r \in R \) has minimum length in \([r]\). A set of maximum-length representatives is defined in a similar way.

**Lemma 2.8** For a right ideal homomorphism \( \varphi \) let \( P = \text{dom}\mathcal{C}(\varphi) \) and assume that \( \text{part}(\varphi) \) is a prefix code congruence. Then we have:

1. Let \( n \) be at least as large as the length of the longest word in \( \text{im}\mathcal{C}(\varphi) \). There exists an essential restriction \( \varphi_0 \) of \( \varphi \) such that all the words in \( \text{im}\mathcal{C}(\varphi_0) \) have the same length, i.e., \( \text{im}\mathcal{C}(\varphi_0) \subseteq A^n \).
2. There is an essential restriction \( \varphi_1 \) of \( \varphi \) such that the minimum-length representatives of the partition \( \simeq_{\text{dom}\mathcal{C}(\varphi_1)} \) all have the same length, and \( \text{part}(\varphi_1) \) is a prefix code congruence.

Similarly, there is an essential restriction \( \varphi_2 \) of \( \varphi \) such that all the maximum-length representatives of \( \simeq_{\text{dom}\mathcal{C}(\varphi_2)} \) have the same length, and \( \text{part}(\varphi_2) \) is a prefix code congruence.

**Proof.** (1) Let \( P_i \) be a class of \( \varphi \) in \( \text{dom}\mathcal{C}(\varphi) \), and let \( \varphi(P_i) = y_i \in \text{im}\mathcal{C}(\varphi) \); hence, \( \varphi^{-1}(y_i) = P_i \). We consider the essential restriction that replaces the set of table entries \( P_i \times \{y_i\} = \{(x, y_i) : x \in P_i\} \) by the set \( \bigcup_{j=1}^k P_i a_j \times \{y_i a_j\} = \{(x a_1, y_i a_1) : x \in P_i\} \cup \ldots \cup \{(x a_k, y_i a_k) : x \in P_i\} \). By such a replacement we can make the shortest element of \( \text{im}\mathcal{C}(\varphi) \) longer. By repeating this, we can give the same length to all elements of \( \text{im}\mathcal{C}(\varphi) \).

(2) Similarly, a class-wise replacement step can be used to make all the elements of \( P_i \) longer; in particular, minimum- (or maximum-) length elements can be made longer. By repeating this on the class that has the shortest among the minimum- (or maximum-) length elements over all classes, we can give the same length to all the minimum-length (or maximum-length) representatives. Since only class-wise replacements are used, \( \text{part}(\varphi_1) \) remains a prefix code congruence.

**Remark.** Any right ideal homomorphism \( \varphi \) can be essentially equally restricted to a right ideal homomorphism \( \Phi \) such that all words in \( \text{dom}\mathcal{C}(\Phi) \) have the same length. However, in general \( \text{part}(\Phi) \) will no longer necessarily be a prefix code congruence.

For any finite prefix code \( P \subseteq A^* \), a complementary prefix code of \( P \) is a finite prefix code \( Q \subseteq A^* \) such that \( \text{ends}(P A^*) \cap \text{ends}(Q A^*) = \emptyset \) and \( \text{ends}(P A^*) \cup \text{ends}(Q A^*) = A^\omega \). This was introduced in Definition 3.29 in [2]. By Lemma 3.30 in [2], every finite prefix code \( P \) has a complementary prefix code (which is empty if \( P \) is a maximal prefix code).

We now start the construction of an \( L \)-height function; this is more subtle than the \( R \)-height since now we have to measure how fine a partition is rather than just how large a set is. Intuitively, elements \( \varphi \in M_{k,1} \) that are higher in the \( L \)-order have “smaller” and “more” classes in \( \text{part}(\varphi) \); here we should treat the complement of \( \text{dom}\mathcal{C}(\varphi) \) like a (virtual) class too (called the “undefined class”). All classes of \( \text{part}(\varphi) \) are finite (of size \( \leq |\text{dom}\mathcal{C}(\varphi)| \)). The highest elements in the \( L \)-order (i.e., the \( L \)-class of the identity and the injective total maps of \( M_{k,1} \)) only have singleton classes. This suggests that the singleton classes should be given a largeness of zero, and that the concept of “collisions” of a function...
is relevant for measuring the largeness of the classes. A total injective function has no collisions. In a non-injective function \( f \), a collision is any pair \((x_1, x_2)\) such that \( x_1 \neq x_2 \) and \( f(x_1) = f(x_2) \). The concept of collision is commonly used in Algorithms and Data Structures. Thus, the first idea is to say that a class \( f^{-1}(y) \) of \( f \) has \(|f^{-1}(y)| - 1 \) collisions, if \( y \in \text{Im}(f) \); the subtraction of 1 is justified by the fact that one element by itself is not a collision (collisions only start with the second element in a class). Also, any \( x \) for which \( f(x) \) is undefined will be treated as a collision all by itself; thus, the undefined class \( C_{\emptyset} \) has \(|C_{\emptyset}| \) collisions. Moreover, for \( M_{k,1} \) we need to use the measure \( \mu \) rather than cardinality.

This motivates the following:

**Definition 2.9** Let \( \simeq \) be a prefix code congruence on a right ideal \( PA^* \), where \( P \) is a finite prefix code. Let \( \simeq_P \) be the restriction of \( \simeq \) to \( P \). Let \( \{P_1, \ldots, P_n\} \) be the classes of \( \simeq_P \), and let \( P_{\emptyset} \) be a complementary prefix code of \( P \) in \( A^* \).

The amount of collision of \( \simeq \) in \( P_{\emptyset} \) is \( \mu(P_{\emptyset}) \). For a class \( P_i \) of \( \simeq_P \) \((1 \leq i \leq n)\), let \( m_i \) be any chosen minimum-length element in \( P_i \). The amount of collision of \( \simeq \) in \( P_i \) is \( \mu(P_i - \{m_i\}) \). The total amount of collision of the prefix code congruence \( \simeq \) is

\[
\text{coll}(\simeq) = \mu(P_{\emptyset}) + \sum_{i=1}^n \mu(P_i - \{m_i\}).
\]

Since \( P_{\emptyset} \cup P_1 \cup \ldots \cup P_n \) is a maximal prefix code, \( \mu(P_{\emptyset}) + \sum_{i=1}^n \mu(P_i) = 1 \); hence we also have

\[
\text{coll}(\simeq) = 1 - \sum_{i=1}^n \mu(m_i).
\]

Accordingly, the amount of non-collision of the prefix code congruence \( \simeq \) is defined by

\[
\text{noncoll}(\simeq) = \sum_{i=1}^n \mu(m_i).
\]

Further justifications of this definition:

The motivation for removing an element \( m_i \) from the class \( P_i \) when we measure the collisions is that one element by itself creates no collision; only subsequent additions of elements to a class cause collisions. We choose to remove the most probable (i.e., shortest) element from each class, and let all the other elements in the class account for the collisions. At the end of this subsection there is a discussion of other possible definitional choices.

The value of \( \text{coll}(\simeq) \) depends only on \( \simeq \). Indeed, first, it is easy to see that \( \text{coll}(\simeq) \) does not depend on the choice of a particular minimum-length word \( m_i \) in \( P_i \), since \( \text{coll}(\simeq) \) depends only on the lengths of words. Second, we easily show that \( \text{coll}(\simeq) \) does not depend on the choice of \( P_{\emptyset} \), since all complementary prefix codes of \( P \) in \( A^* \) have the same ends (namely \( A^\omega - \text{ends}(PA^*) \)), hence the same measure. The values of \( \text{coll}(\simeq) \) are \( k \)-ary rational numbers that range from 0 (for the identity congruence) to 1 (for the empty congruence, on an empty domain).

**Lemma 2.10** If \( \simeq' \) and \( \simeq \) are prefix code congruences and \( \simeq' =_{\text{ess}} \simeq \) then \( \text{coll}(\simeq') = \text{coll}(\simeq) \).

**Proof.** If we apply a class-wise replacement step \( C \rightarrow \{Ca_1, \ldots, Ca_k\} \) to \( \simeq \), where \( C \) is a class of \( \simeq \), the resulting prefix code congruence \( \simeq' \) satisfies:

\[
\text{coll}(\simeq') = \text{coll}(\simeq) - \mu(C - \{m\}) + \mu(Ca_1 - \{ma_1\}) + \ldots + \mu(Ca_k - \{ma_k\}),
\]

where \( m \) is a minimum-length element of \( C \). For any set \( S \) we have \( \mu(S) = \mu(Sa_1) + \ldots + \mu(Sa_k) \), since \( \mu(Sa_i) = \frac{1}{k} \mu(S) \); it follows that \( \text{coll}(\simeq') = \text{coll}(\simeq) \).

In a similar way one proves that an inverse class-wise replacement step preserves the amount of collision. Hence, iteration of replacement steps and inverse replacement steps preserves the amount of collision. \( \square \)

More generally we have the following (note the order reversal, since finer congruences have fewer collisions):
Lemma 2.11 Suppose $\simeq_1$ and $\simeq_2$ are prefix code congruences that are comparable in the order $\leq_{\text{ends}}$. Then we have $\simeq_1 \lessdot_{\text{ends}} \simeq_2$, or $\simeq_1 =_{\text{ess}} \simeq_2$, or $\simeq_1 \gtrdot_{\text{ends}} \simeq_2$, according as $\text{coll}(\simeq_1) > \text{coll}(\simeq_2)$, or $\text{coll}(\simeq_1) = \text{coll}(\simeq_2)$, or $\text{coll}(\simeq_1) < \text{coll}(\simeq_2)$.

Proof. Suppose $\simeq_1 \gtrdot_{\text{ends}} \simeq_2$. Then, by Lemma 2.10 we can essentially equally restrict $\simeq_1$ and $\simeq_2$ so that in the resulting prefix code congruences (which we still call $\simeq_1$ and $\simeq_2$) we have: Every class of $\simeq_2$ is a union of classes of $\simeq_1$.

Suppose $Q$ is a class of $\simeq_2$ in $\text{domC}(\simeq_2)$, and suppose $P_1, \ldots, P_n$ are the classes of $\simeq_1$ in $\text{domC}(\simeq_1)$ such that $Q = P_1 \cup \ldots \cup P_n$, $2 \leq n$. Then the amount of collision in $\simeq_2$ for $Q$ is $\mu(Q - m)$ (where $m$ is a shortest element of $Q$). The amount of collision in $\simeq_1$ for $P_1, \ldots, P_n$ (with shortest element in $P_i$ denoted by $m_i$) is

$$
\mu(P_1 - m_1) + \ldots + \mu(P_n - m_n) = \mu(P_1) + \ldots + \mu(P_n) - \mu(m_1) - \ldots - \mu(m_n)
$$

= $\mu(Q) - \mu(m_1) - \ldots - \mu(m_n) < \mu(Q) - \mu(m)$. 

The last “$<$” is due to the fact that $\mu(m)$ is equal to one of the numbers $\mu(m_1), \ldots, \mu(m_n)$, since $Q = P_1 \cup \ldots \cup P_n$ and $n \geq 2$. We conclude that

$$
\mu(P_1 - m_1) + \ldots + \mu(P_n - m_n) < \mu(Q - m).
$$

In other words, coarser classes have larger amounts of collision.

Moreover, if $C$ is a class of $\simeq_1$ that does not intersect the domain of $\simeq_2$, then $C$ is in the undefined class of $\simeq_2$, hence $\mu(C)$ will be counted in the amount collision in $\simeq_2$ (but only $\mu(C - m)$ will be counted in $\simeq_1$). So, here again, the amount of collision in $\simeq_2$ is larger. So, $\text{coll}(\simeq_1) < \text{coll}(\simeq_2)$.

In a similar way we can prove that $\simeq_1 \lessdot_{\text{ends}} \simeq_2$ implies $\text{coll}(\simeq_1) > \text{coll}(\simeq_2)$. And the proof that $\simeq_1 =_{\text{ess}} \simeq_2$ implies $\text{coll}(\simeq_1) = \text{coll}(\simeq_2)$ was already given in Lemma 2.10.

For the converse: Suppose we have $\text{coll}(\simeq_1) > \text{coll}(\simeq_2)$ and suppose that $\simeq_1$ and $\simeq_2$ are comparable for the $\leq_{\text{ends}}$-order. This leaves only the three possibilities: $\simeq_1 \lessdot_{\text{ends}}, =_{\text{ess}}$, and $\simeq_1 \gtrdot_{\text{ends}}$. But we already proved that $=_{\text{ess}}$ and $\gtrdot_{\text{ends}}$ would contradict $\text{coll}(\simeq_1) > \text{coll}(\simeq_2)$. So we have $\simeq_1 \lessdot_{\text{ends}} \simeq_2$. □

Definition 2.12 For any element of $M_{k,1}$ represented by a right ideal homomorphism $\varphi$ we define the $\mathcal{L}$-height by $\text{height}_{\mathcal{L}}(\varphi) = 1 - \text{coll}(\text{part}(\varphi))$ (i.e., the amount of non-collision). Hence,

$$
\text{height}_{\mathcal{L}}(\varphi) = \sum_{i=1}^{n} \mu(m_i),
$$

where $m_i$ is a shortest representative of the class $P_i$ of $\text{part}(\varphi)$ (denoting the classes of $\text{part}(\varphi)$ in $\text{domC}(\varphi)$ by $P_1, \ldots, P_n$).

By the characterization of the $\mathcal{L}$-order of $M_{k,1}$ and by Lemma 2.10 above, $\text{height}_{\mathcal{L}}(\varphi)$ depends only on $\varphi$ as an element of $M_{k,1}$ and not on the right ideal homomorphism chosen. Lemma 2.11 implies that $\text{height}_{\mathcal{L}}(\cdot)$ is indeed a height function for $\leq_{\mathcal{L}}$, i.e., that we have:

Proposition 2.13 Suppose $\varphi, \psi \in M_{k,1}$ are comparable in the $\mathcal{L}$-order. Then we have $\varphi \gtrdot_{\mathcal{L}} \psi$, or $\varphi \equiv_{\mathcal{L}} \psi$, or $\varphi \lessdot_{\mathcal{L}} \psi$, according as $\text{height}_{\mathcal{L}}(\varphi) > \text{height}_{\mathcal{L}}(\psi)$, or $\text{height}_{\mathcal{L}}(\varphi) = \text{height}_{\mathcal{L}}(\psi)$, or $\text{height}_{\mathcal{L}}(\varphi) < \text{height}_{\mathcal{L}}(\psi)$. □

Proposition 2.14 (1) For every $h \in \mathbb{Z}[\frac{1}{k}] \cap [0,1]$ there exists $\varphi_h \in M_{k,1}$ such that $\text{height}_{\mathcal{L}}(\varphi_h) = h$.

(2) For all $g, h \in \mathbb{Z}[\frac{1}{k}]$ with $0 \leq g < h \leq 1$ the elements $\varphi_g, \varphi_h \in M_{k,1}$ constructed in (1) satisfy $\varphi_g \lessdot_{\mathcal{L}} \varphi_h$. The set $\\{\varphi_h : h \in \mathbb{Z}[\frac{1}{k}] \cap [0,1]\}$ forms a dense $\lessdot_{\mathcal{L}}$-chain of elements of $M_{k,1}$ with $\text{height}_{\mathcal{L}}(\varphi_h) = h$.

Proof. This is proved in the same way as Prop. 2.7. □
Variants of the definition of an \( \mathcal{L} \)-height function:

We chose the definition \( \text{height}_{\mathcal{L}}(\varphi) = \sum_{i=1}^{n} \mu(m_i) \) where each word \( m_i \) is a minimum-length representative of a \( \text{part}(\varphi) \)-class in \( \text{domC}(\varphi) \). For the remainder of this subsection we will call this function \( \text{height}_{\mathcal{L}}^{\text{min}}(.) \). If in the definition of \( \mathcal{L} \)-height we replace minimum-length by maximum-length representatives we obtain a function \( \text{height}_{\mathcal{L}}^{\text{max}}(.) \) that is also an \( \mathcal{L} \)-height function (according to Def. 2.1). For all \( \varphi \in M_{k,1} \) we obviously have \( \text{height}_{\mathcal{L}}^{\text{max}}(\varphi) \leq \text{height}_{\mathcal{L}}^{\text{min}}(\varphi) \). For idempotents the following relation holds between the \( \mathcal{R} \)-height function and the two \( \mathcal{L} \)-height functions.

**Proposition 2.15** For any idempotent \( \eta = \eta^2 \in M_{k,1} \),

\[
\text{height}_{\mathcal{L}}^{\text{max}}(\eta) \leq \text{height}_{\mathcal{R}}(\eta) \leq \text{height}_{\mathcal{L}}^{\text{min}}(\eta).
\]

If \( \eta = \eta^2 \in \text{Inv}_{k,1} \) then \( \text{height}_{\mathcal{L}}^{\text{max}}(\eta) = \text{height}_{\mathcal{R}}(\eta) = \text{height}_{\mathcal{L}}^{\text{min}}(\eta) \).

**Proof.** For an idempotent, the elements of \( \text{imC}(\eta) \) form a set of representatives of the \( \text{part}(\eta) \)-classes in \( \text{domC}(\eta) \), assuming that \( \eta \) has been restricted so as to make \( \text{part}(\eta) \) a prefix code congruence. Hence, the lengths of the elements of \( \text{imC}(\eta) \) are between the lengths of the minimum-length representatives and the maximum-length representatives. The inequalities follow.

When the idempotent \( \eta \) is injective, the congruence classes of \( \text{part}(\eta) \) are singletons, so the minimum-length and the maximum-length representatives of a class are the same. \( \Box \)

An \( \mathcal{L} \)-height function could also be defined by using the average of the lengths in each block of \( \text{part}(\varphi) \):

\[
\text{height}_{\mathcal{L}}^{\text{ave}}(\varphi) = \sum_{y \in \text{imC}(\varphi)} \mu(\text{ave}(\varphi^{-1}(y))) \, ,
\]

where for any finite set \( S \subseteq A^* \) we define \( \text{ave}(S) = \frac{1}{|S|} \sum_{x \in S} |x| \). This is indeed an \( \mathcal{L} \)-height function, as a consequence of the fact that for two disjoint finite sets \( S_1, S_2 \) we have

\[
\min\{\text{ave}(S_1), \text{ave}(S_2)\} \leq \text{ave}(S_1 \cup S_2) \leq \max\{\text{ave}(S_1), \text{ave}(S_2)\}.
\]

The same reasoning works with the average replaced by the median.

### 2.3 A connection between the \( \mathcal{R} \) - and \( \mathcal{L} \)-heights and the \( \mathcal{D} \)-relation

Interestingly, \( \text{height}_{\mathcal{R}}(\varphi) \) determines the \( \mathcal{D} \)-class of \( \varphi \); similarly, \( \text{height}_{\mathcal{L}}(\varphi) \) determines the \( \mathcal{D} \)-class. First of all, obviously,

\[
\text{height}_{\mathcal{L}}(\varphi) = 0 \text{ iff } \varphi = 0 \text{ iff } \text{height}_{\mathcal{R}}(\varphi) = 0.
\]

When \( \varphi \neq 0 \) it was proved (Theorem 2.5 in [3]) that the \( \mathcal{D} \)-class of \( \varphi \) is

\[
D(\varphi) = \{ \psi \in M_{k,1} : |\text{imC}(\psi)| \equiv |\text{imC}(\varphi)| \mod k - 1 \} .
\]

The number \( i \in \{1, \ldots, k-1\} \) such that \( i \equiv |\text{imC}(\varphi)| \mod k - 1 \) is called the index of the \( \mathcal{D} \)-class of \( \varphi \) (when \( \varphi \neq 0 \)). In this paper, integers modulo \( k - 1 \) will be picked in the range \( \{1, \ldots, k-1\} \).

Recall that for a \( k \)-ary rational number \( r = a/k^n \), where \( k \) does not divide \( a \), the numerator \( a \) is denoted by \( \text{num}(r) \).

**Proposition 2.16** For every \( \varphi \in M_{k,1} \) the \( \equiv_{\mathcal{D}} \)-class of \( \varphi \) is uniquely determined by \( \text{height}_{\mathcal{R}}(\varphi) \) (and similarly, by \( \text{height}_{\mathcal{L}}(\varphi) \)). More precisely, when \( \varphi \neq 0 \) we have the following formulas.

1. The \( \mathcal{D} \)-class index \( i \in \{1, \ldots, k-1\} \) of \( \varphi \) is determined by
   \[
   i \equiv \text{num}(\text{height}_{\mathcal{L}}(\varphi)) \equiv \text{num}(\text{height}_{\mathcal{R}}(\varphi)) \mod k - 1 .
   \]

2. When base-\( k \) representations of \( \text{height}_{\mathcal{L}}(\varphi) \) and \( \text{height}_{\mathcal{R}}(\varphi) \) are given we have:
   - If \( \text{height}_{\mathcal{L}}(\varphi) = 0.d_1 \ldots d_m \) or \( \text{height}_{\mathcal{R}}(\varphi) = 0.d'_1 \ldots d'_n \) we have
     \[
     i \equiv d_1 + \ldots + d_m \mod k - 1 , \quad \text{or}
     i \equiv d'_1 + \ldots + d'_n \mod k - 1 .
     \]
   - If \( \text{height}_{\mathcal{L}}(\varphi) = 1 \) or \( \text{height}_{\mathcal{R}}(\varphi) = 1 \) then \( i = 1 \).
Proof. Part (2) immediately follows from part (1), since $k \equiv 1 \mod k - 1$. Let us prove part (1).

$\mathcal{R}$-height formula: We have $\text{height}_{\mathcal{R}}(\varphi) = \mu(\text{imC}(\varphi))$, and we can write $|\text{imC}(\varphi)| = i + j \cdot (k - 1)$, for some integers $i, j$ such that $1 \leq i \leq k - 1$ and $j \geq 0$. Moreover, $\mu(\text{imC}(\varphi)) = (i + j \cdot (k - 1)) \cdot k^{-N}$, for some integer $N > 0$. Note that for the $\mod k - 1$ value of the numerator of a $k$-ary rational number, it does not matter whether the numerator is divisible by $k$ (since $k \equiv 1 \mod k - 1$). Hence, $|\text{imC}(\varphi)| \equiv i \equiv \text{num}(\mu(\text{imC}(\varphi))) \mod k - 1$.

$\mathcal{L}$-height formula: We have $\text{height}_{\mathcal{L}}(\varphi) = \sum_{i=1}^{n} \mu(m_i)$, where $n = |\text{imC}(\varphi)|$, and $\{m_i : i = 1, \ldots, n\}$ is the set of minimum-length representatives of the $\text{part}(\varphi)$ classes in $\text{domC}(\varphi)$. By Lemma 2.18 we can assume that all $m_i$ have the same length, say $\ell$. Then, $\text{height}_{\mathcal{L}}(\varphi) = n \cdot k^{-\ell}$. Again, for the $\mod k - 1$ value of the numerator it does not matter whether the numerator is divisible by $k$. Hence, $|\text{imC}(\varphi)| = n \equiv \text{num(\text{height}_{\mathcal{L}}(\varphi)) \mod k - 1}$. □

Proposition 2.17 (Independence of the $\mathcal{R}$- and $\mathcal{L}$-heights in $M_{k,1}$ and in $\text{Inv}_{k,1}$).

Let $h_1, h_2$ be any $k$-ary rational with $0 < h_1, h_2 \leq 1$ and such that $\text{num}(h_1) \equiv \text{num}(h_2) \mod k - 1$ (i.e., $h_1$ and $h_2$ determine the same non-zero $\equiv_{\mathcal{D}}$-class). Then there exists an element $\varphi \in \text{Inv}_{k,1} (\subset M_{k,1})$ such that $\text{height}_{\mathcal{L}}(\varphi) = h_1$ and $\text{height}_{\mathcal{R}}(\varphi) = h_2$.

Proof. This follows directly from Lemma 2.18 which will be proved next. □

Lemma 2.18.

(1) Let $R$ be any non-zero $\mathcal{R}$-class of $M_{k,1}$ and let $h_1$ be any $k$-ary rational with $0 < h_1 \leq 1$, such that the $\mathcal{D}$-class of $R$ coincides with the $\mathcal{D}$-class determined by $h_1$ (used as an $\mathcal{L}$-height); in other words, we assume that $\text{num(\text{height}_{\mathcal{R}}(R)))} \equiv \text{num}(h_1) \mod k - 1$. Then there exists an element $\varphi \in R \cap \text{Inv}_{k,1}$ such that $\text{height}_{\mathcal{L}}(\varphi) = h_1$.

(2) Similarly, let $L$ be any non-zero $\mathcal{L}$-class and let $h_2$ be any $k$-ary rational with $0 < h_2 \leq 1$, such that the $\mathcal{D}$-class of $L$ coincides with the $\mathcal{D}$-class determined by $h_2$ (used as an $\mathcal{R}$-height). Then there exists an element $\psi \in L \cap \text{Inv}_{k,1}$ such that $\text{height}_{\mathcal{R}}(\psi) = h_2$.

Proof. We only prove part (1), since (2) is similar. We consider the base-$k$ representation $h_1 = 0.d_1 \ldots d_n$ (or $h_1 = 1$). As in the proof of Lemma 2.6 we construct the finite prefix code $P_{h_1}$ from $h_1$, with $\mu(P_{h_1}) = h_1$ and $|P_{h_1}| = \sum_{i=1}^{n} d_i$ (or $\mu(P_{h_1}) = |P_{h_1}| = 1$ if $h_1 = 1$). We can increase the size of $P_{h_1}$ by multiples of $k - 1$, as follows: In $P_{h_1}$ we replace $a_{d_1+1} \ldots a_{d_{n-1}+1} a_{d_n} A$ by $a_{d_1+1} \ldots a_{d_{n-1}+1} a_{d_n} A$, thus obtaining a prefix code $P_{h_1}^{(1)}$ of size $|P_{h_1}| + k - 1$, with the measure remaining unchanged at $\mu(P_{h_1}^{(1)}) = h_1$ (by Lemma 1.1). This can be repeated: In $P_{h_1}^{(1)}$ we replace $a_{d_1+1} \ldots a_{d_{n-1}+1} a_{d_n} A$ by $a_{d_1+1} \ldots a_{d_{n-1}+1} a_{d_n} A$, thus obtaining a prefix code $P_{h_1}^{(2)}$ of size $|P_{h_1}| + 2 \cdot (k - 1)$, with unchanged measure $\mu(P_{h_1}^{(2)}) = h_1$. As a result, for any $j \geq 1$ we obtain a prefix code $P_{h_1}^{(j)}$ of size $|P_{h_1}| + j \cdot (k - 1)$, with unchanged measure $\mu(P_{h_1}^{(j)}) = h_1$. More precisely, inductively, $P_{h_1}^{(j)} = (P_{h_1}^{(j-1)} - \{a_{d_1} \ldots a_{d_{n-1}+1} a_{d_n}^2\}) \cup a_{d_1} \ldots a_{d_{n-1}+1} a_{d_n}^2 A$

For all $\psi$ in the $\mathcal{R}$-class $R$ the right ideals $\text{Im}(\psi)$ are essentially equal, and all are essentially equal to $QA^*$ for a fixed finite prefix code $Q$. Since $R$ and $h_1$ correspond to the same $\mathcal{D}$-class, we have $|Q| \equiv \text{num}(h_1) \mod k - 1$. We can increase the size of $Q$ by any multiple of $k - 1$ without changing the corresponding $\mathcal{R}$-class, as follows: For any $q \in Q$ we replace $Q$ by $Q^{(1)} = (Q - \{q\}) \cup QA$, of size $|Q^{(1)}| = |Q| + k - 1$, and such that $Q^{(1)} A^* = \text{ess} Q A^*$. Then we replace $Q^{(1)}$ by $Q^{(2)} = (Q^{(1)} - \{q a_k\}) \cup q a_k A$, etc. After $j$ steps we obtain a prefix code $Q^{(j)}$ of size $|Q^{(j)}| = |Q| + j \cdot (k - 1)$, such that $Q^{(j)} A^* = \text{ess} Q A^*$.

Since $R$ and $h_1$ correspond to the same $\mathcal{D}$-class, i.e., $|Q| \equiv \text{num}(h_1) = |P_{h_1}| \mod k - 1$, there exist $j$ and $j'$ such that $|P_{h_1}^{(j)}| = |Q^{(j')}|$. Let us define $\varphi$ by any bijection $P_{h_1}^{(j)} \rightarrow Q^{(j')}$.
since \( Q^{(j)} A^* =_{ess} QA^* \). Also, \( \text{height}_L(\varphi) = h_1 \) since \( \varphi \) is injective and since \( \text{domC}(\varphi) = P^{(j)}_{h_1} \) with \( \mu(P^{(j)}_{h_1}) = h_1 \). □

We saw in Prop. 2.15 that for idempotents of \( M_{k,1} \), there are relations between the \( R \)-height and the \( L \)-height.

### 3 The Green relations of \( \text{plep} M_{k,1} \) and \( \text{tlep} M_{k,1} \)

#### 3.1 The monoids \( \text{plep} M_{k,1} \) and \( \text{tlep} M_{k,1} \)

The submonoid \( \text{tlep} M_{k,1} \) of total length-equality preserving elements of \( M_{k,1} \) was introduced in [4], where it was simply called \( \text{lep} M_{k,1} \). We now add the “t” (for total) in order to distinguish it from the submonoid \( \text{plep} M_{k,1} \) of partial length-equality preserving elements of \( M_{k,1} \). As usual, partial does not rule out total, so \( \text{tlep} M_{k,1} \subseteq \text{plep} M_{k,1} \). More precisely, these submonoids of \( M_{k,1} \) are defined as follows.

\[
\text{plep} M_{k,1} = \{ \varphi \in M_{k,1} : \text{for all } x_1, x_2 \in \text{Dom}(\varphi), |x_1| = |x_2| \text{ implies } |\varphi(x_1)| = |\varphi(x_2)| \}.
\]

\[
\text{tlep} M_{k,1} = \{ \varphi \in \text{plep} M_{k,1} : \text{Dom}(\varphi) \text{ is an essential right ideal} \}.
\]

In words, \( \varphi \in M_{k,1} \) belongs to \( \text{plep} M_{k,1} \) iff \( \varphi \) transforms equal-length inputs to equal-length outputs, hence the name “length equality preserving”. Recall that \( \text{Dom}(\varphi) \) is essential iff \( \text{ends}(\text{Dom}(\varphi)) = A^\omega \), i.e., iff \( \varphi \) is total on \( A^\omega \).

One can easily prove the following characterization: \( \varphi \in M_{k,1} \) belongs to \( \text{plep} M_{k,1} \) iff there is an essentially equal restriction \( \Phi \) of \( \varphi \) such that for some \( m, n > 0 \),

\[
\text{domC}(\Phi) \subseteq A^m \text{ and } \text{imC}(\Phi) \subseteq A^n.
\]

For \( \text{tlep} M_{k,1} \) we have in addition that \( \text{domC}(\Phi) = A^m \).

An important motivation for the study of \( \text{plep} M_{k,1} \) and \( \text{tlep} M_{k,1} \) is their similarity to (partial) acyclic boolean circuits. In [4] it was proved that \( \text{tlplep} M_{k,1} \) has a generating set of the form \( \Gamma \cup \tau \) where \( \Gamma \) is finite and \( \tau = \{ \tau_{i,i+1} : i \geq 1 \} \). Each \( \tau_{i,i+1} \) is a position transposition (or “wire crossing”), defined as follows: \( \tau_{i,i+1}(uvbw) = ubaw \) for all \( u \in A^{i-1}, a, b \in A, v \in A^* \); and \( \tau_{i,i+1}(x) \) is undefined when \( |x| < i+1 \). When \( k = 2 \), the set \( \Gamma \) can be chosen to be \( \{ \text{and}, \text{or}, \text{not}, \text{fork} \} \). These are the classical circuit gates, given by the tables \( \text{and} = \{(00, 0), (01, 0), (10, 0), (11, 1)\} \), or \( \{(00, 0), (01, 1), (10, 1), (11, 1)\} \), not \( \{(0, 1), (1, 0)\} \), for \( \text{fork} = \{(0, 00), (1, 11)\} \).

It was proved in [4] that for elements in \( \text{tlep} M_{2,1} \), word-length over \( \Gamma \cup \tau \) is polynomially equivalent to circuit-size. For this reason we call generating sets of \( \text{tlep} M_{k,1} \) (or, more generally, of \( M_{k,1} \), or of \( \text{plep} M_{k,1} \)) of the form \( \Gamma \cup \tau \) (where \( \Gamma \) is finite and \( \tau \) is as above) circuit-like generating sets. The monoids \( M_{k,1} \) and \( \text{tlep} M_{k,1} \) have circuit-like generating sets, and Prop. 3.2 below will show the same for \( \text{plep} M_{k,1} \).

If a different \( \Gamma \) is used, the word-length changes only linearly, by the following general observation (whose proof is straightforward):

**Proposition 3.1** If two (possibly infinite) generating sets \( \Gamma_1 \) and \( \Gamma_2 \) for a monoid \( M \) differ only by a finite amount (i.e., their symmetric difference \( \Gamma_1 \Delta \Gamma_2 \) is finite), then the word-lengths of \( M \) over \( \Gamma_1 \), respectively \( \Gamma_2 \), are linearly related. □

**Notation:** When \( P \) is a prefix code we abbreviate the partial identity map \( \text{id}_{P^{A^*}} \) by \( \text{id}_P \). E.g., denoting the elements of the alphabet \( A \) by \( \{a_1, \ldots, a_k\} \), the partial identity \( \text{id}_{A-a_1} \) is undefined on words that start with \( a_1 \) and is the identity on all other words in \( A^* \).

**Proposition 3.2** The monoid \( \text{plep} M_{k,1} \) has a circuit-like generating set. More specifically, if \( \Gamma \cup \tau \) is any circuit-like generating set of \( \text{tlep} M_{k,1} \) then \( \Gamma \cup \tau \cup \{ \text{id}_{A-a_1} \} \) generates \( \text{plep} M_{k,1} \).
Proof. For any \( \varphi \in \text{plep} M_{k,1} \) with \( \text{dom} C(\varphi) \subseteq A^m \), we can define an element \( \psi \in \text{tlep} M_{k,1} \) by extending the domain of \( \varphi \) as follows: if \( x \in \text{dom} C(\varphi) \) then \( \psi(x) = \varphi(x) \), and if \( x \in A^m - \text{dom} C(\varphi) \) then \( \psi(x) = y_0 \), where \( y_0 \) is any fixed element chosen in \( \text{im} C(\varphi) \). Then we have: \( \varphi = \psi \circ \text{id}_{\text{dom} C(\varphi)} \).

This shows that \( \text{plep} M_{k,1} \) is generated by \( \Gamma \cup \tau \), together with the partial identities of the form \( \text{id}_P \) (where \( P \) ranges over the finite subsets of \( A^m \) for all non-negative integers \( m \)). Moreover, for \( P \subseteq A^m \) we have

\[
\text{id}_P = \prod_{s \in A^m - P} \text{id}_{A^m - \{s\}}
\]

(and this composition of partial identities is commutative). So, it will suffice to prove that each partial identity of the form \( \text{id}_{A^m - \{s\}} \) is generated by \( \Gamma \cup \{\text{id}_{A - a_1}\} \cup \tau \) for some finite subset \( \Gamma \) of \( \text{tlep} M_{k,1} \).

For each letter \( a_i \in A = \{a_1, a_2, \ldots, a_k\} \) we introduce the function \( E_{a_i} : A \to \{a_1, a_2\} \), defined by \( E_{a_i}(a_j) = a_1 \) if \( a_j \neq a_i \), and \( E_{a_i}(a_j) = a_2 \) if \( a_j = a_i \). We also introduce the function \( \text{and} : A^2 \to \{a_1, a_2\} \), defined by \( \text{and}(a_i, a_j) = a_1 \) if \( a_i = a_1 \) or \( a_j = a_1 \), and \( \text{and}(a_i, a_j) = a_2 \) if \( a_i \neq a_1 \neq a_j \).

And we define \( \text{not} : A \to \{a_1, a_2\} \) by \( \text{not}(a_1) = a_2 \), and \( \text{not}(a_i) = a_1 \) for \( a_i \in A - \{a_1\} \). Thus, the letter \( a_1 \) plays the role of the boolean value \( \text{false} \), and the other letters play the role of \( \text{true} \). We also use the function \( \text{fork} : A \to A^2 \), defined by \( \text{fork}(a) = aa \). And we use \( \text{proj}_2 : A^2 \to A \), defined by \( \text{proj}_2(a, a_j) = a_j \). Then \( \text{id}_{A^m - \{s\}} \) (for any \( s = s_1 \ldots s_m \in A^m \)) is generated by

\[\{E_{a_i} : i = 1, \ldots, k\} \cup \{\text{and}, \text{not}, \text{fork}, \text{proj}_2, \text{id}_{A - a_1}\} \cup \tau.\]

Let us show how to simulate \( \text{id}_{A^m - \{s\}} \) by a fixed sequence of elements of the above set. On input \( x_1 x_2 \ldots x_m \ x_1 x_2 \ldots x_m w \) (where \( x_i \in A \) for \( i = 1, \ldots, m \) and \( w \in A^* \)), the output should be either undefined (if \( x_1 x_2 \ldots x_m = s_1 s_2 \ldots s_m \)), or equal to the input (if \( x_1 x_2 \ldots x_m \neq s_1 s_2 \ldots s_m \)).

First, by using \( m \) copies of the fork function, together with transpositions (\( \in \tau \)), a second copy of \( x_1 x_2 \ldots x_m \) is made:

\[x_1 x_2 \ldots x_m w \mapsto x_1 x_2 \ldots x_m x_1 x_2 \ldots x_m w.
\]

Next, by using \( E_{s_1}, E_{s_2}, \ldots, E_{s_m} \) and transpositions, we implement

\[x_1 x_2 \ldots x_m x_1 x_2 \ldots x_m w \mapsto e_1 e_2 \ldots e_m x_1 x_2 \ldots x_m w,
\]

where \( e_i = a_1 \) if \( x_i \neq s_i \), and \( e_i = a_2 \) if \( x_i = s_i \) (\( 1 \leq i \leq m \)). Then, to \( e_1 e_2 \ldots e_m \) we apply \( m - 1 \) copies of and, as well as transpositions; this is followed by one application of not; in effect, we compute the \( m \)-input \( \text{and} \) of \( e_1 e_2 \ldots e_m \). This implements

\[e_1 e_2 \ldots e_m x_1 x_2 \ldots x_m w \mapsto e x_1 x_2 \ldots x_m w,
\]

where \( e = a_1 \) if \( e_1 = e_2 = \ldots = e_m = a_1 \) (i.e., if \( x_1 x_2 \ldots x_m = s_1 s_2 \ldots s_m \)), and \( e = a_2 \) otherwise. We apply \( \text{id}_{A - a_1} \) now; the operation is undefined if \( e = a_1 \), and is the identity otherwise. Finally, applying \( \text{proj}_2 \) produces the output \( x_1 x_2 \ldots x_m w \) if \( e = a_2 \) (i.e., if \( \text{id}_{A - a_1} \) was defined); the result is undefined otherwise. \( \square \)

Open problems: Are \( \text{plep} M_{k,1} \) and \( \text{tlep} M_{k,1} \) (not) finitely generated? Is \( M_{k,1} \) (not) finitely presented?

3.2 The \( \mathcal{R} \)-, \( \mathcal{L} \)-, and \( \mathcal{J} \)-relations of \( \text{tlep} M_{k,1} \) and \( \text{plep} M_{k,1} \)

We will show that the \( \mathcal{R} \)-, \( \mathcal{L} \)-, and \( \mathcal{J} \)-orders of \( \text{plep} M_{k,1} \) and \( \text{tlep} M_{k,1} \) are very similar to the ones in \( M_{k,1} \), and that \( \text{plep} M_{k,1} \) is also congruence-simple.

The monoids \( \text{plep} M_{k,1} \) and \( \text{tlep} M_{k,1} \) are regular; this is easily proved from the definition. (A semigroup \( S \) is called regular iff for every \( s \in S \) there exists \( t \in S \) such that \( sts = s \).)

Proposition 3.3 The \( \mathcal{R} \)- and \( \mathcal{L} \)-orders of \( \text{plep} M_{k,1} \) and of \( \text{tlep} M_{k,1} \) are induced by the corresponding orders of \( M_{k,1} \). In other words, for all \( \varphi_1, \varphi_2 \in \text{plep} M_{k,1} \),

\[\varphi_1 \geq_R \varphi_2 \text{ in } \text{plep} M_{k,1} \text{ iff } \varphi_1 \geq_R \varphi_2 \text{ in } M_{k,1} ;
\]
Another consequence of Prop. 3.3 is that the \( R \)- and \( L \)-orders of \( \text{plep}_{k,1} \) are also induced by the corresponding orders of \( M_{k,1} \).

**Proof.** This follows from the fact that \( \text{plep}_{k,1} \) and \( \text{tlep}_{k,1} \) are regular semigroups, and the general fact that if \( S_2 \) is a subsemigroup of a semigroup \( S_1 \) and \( S_2 \) is regular then the \( \geq_R \) and \( \geq_L \) orders of \( S_2 \) are induced from \( S_1 \). See e.g. [7] or p. 289 of [9]. \( \square \)

An immediate consequence of Prop. 3.3 is the following.

**Corollary 3.4** Every \( \mathcal{H} \)-class of \( \text{plep}_{k,1} \) (or of \( \text{tlep}_{k,1} \)) has the form \( H \cap \text{plep}_{k,1} \) (respectively \( H \cap \text{tlep}_{k,1} \)), where \( H \) is an \( \mathcal{H} \)-class of \( M_{k,1} \). \( \square \)

Another consequence of Prop. 3.3 is that the \( R \)-height and \( L \)-height functions that we defined for \( M_{k,1} \) also work for \( \text{plep}_{k,1} \) and \( \text{tlep}_{k,1} \).

Additional facts about the \( R \)- and \( L \)-orders of \( \text{tlep}_{k,1} \) and \( \text{plep}_{k,1} \):

Every \( R \)-class of \( M_{k,1} \) intersects \( \text{plep}_{k,1} \), and every non-zero \( R \)-class of \( M_{k,1} \) intersects \( \text{tlep}_{k,1} \). In particular, for any \( \varphi \in M_{k,1} \) with table \( \varphi : P \to Q \) we have \( \varphi \equiv_R \text{id}_Q \in \text{plep}_{k,1} \). To find an idempotent of \( \text{tlep}_{k,1} \) in every non-zero \( R \)-class of \( M_{k,1} \) we can just extend \( \text{id}_Q \) to a total function (by taking a complementary prefix code of \( Q \) and mapping it to any element of \( Q \)).

Not every \( L \)-class of \( M_{k,1} \) contains an element of \( \text{plep}_{k,1} \). For example, if some class of \( \text{part} \) contains words of different lengths then the \( L \)-class of \( \varphi \) does not intersect \( \text{plep}_{k,1} \).

**Proposition 3.5** The monoid \( \text{plep}_{k,1} \) is 0-\( \mathcal{J} \)-simple (i.e., it consists of 0 and one non-zero \( \mathcal{J} \)-class), and it is congruence-simple (i.e., there are only two congruences in \( \text{plep}_{k,1} \), the equality relation, and the one-class congruence). The monoid \( \text{tlep}_{k,1} \) is \( \mathcal{J} \)-simple.

**Proof.** It was proved in [3] (Prop. 2.2) that \( M_{k,1} \) is 0-\( \mathcal{J} \)-simple. Congruence-simplicity of \( M_{k,1} \) was proved (incompletely) in Theorem 2.3 in [3]; a complete proof appears in the proof of Prop. 5.1 in the Appendix of the present paper.

For 0-\( \mathcal{J} \)-simplicity and congruence-simplicity of \( \text{plep}_{k,1} \) we observe that the proofs for \( M_{k,1} \) also apply for \( \text{plep}_{k,1} \) since the multipliers used in those proofs belong to \( \text{plep}_{k,1} \).

Similarly, the proof of \( \mathcal{J} \)-simplicity of \( \text{tot}_{k,1} \) (in Prop. 2.2 in [3]) also works for \( \text{tlep}_{k,1} \). \( \square \)

**Question:** Are \( \text{tot}_{k,1} \) and \( \text{tlep}_{k,1} \) congruence-simple for all (or some) \( k \geq 2 \) ?

### 3.3 The \( \mathcal{D} \)-relation of \( \text{tlep}_{k,1} \) and \( \text{plep}_{k,1} \)

This subsection gives another unexpected application of the Bernoulli measure \( \mu \), namely a simple characterization of the \( \mathcal{D} \)-relation of \( \text{plep}_{k,1} \) and of \( \text{tlep}_{k,1} \). Recall that for a \( k \)-ary rational number \( a/k^n \) with \( a \) not divisible by \( k \) we denote the numerator \( a \) by \( \text{num}(r) \).

**Theorem 3.6 (\( \mathcal{D} \)-relation of \( \text{plep}_{k,1} \) and \( \text{tlep}_{k,1} \)).** For any non-zero \( \varphi_1, \varphi_2 \in \text{plep}_{k,1} \),

\[
\varphi_1 \equiv_{\mathcal{D}(\text{plep}_M)} \varphi_2 \iff \text{num}(\mu(\text{imC}(\varphi_1))) = \text{num}(\mu(\text{imC}(\varphi_2))).
\]

The same holds for \( \text{tlep}_{k,1} \).

**Proof.** \([\Rightarrow]\) If \( \varphi_1 \equiv_R \varphi_2 \) then (by the characterization of the \( R \)-order), \( \text{Im}(\varphi_1) = \text{ess} \text{Im}(\varphi_2) \), hence \( \mu(\text{imC}(\varphi_1)) = \mu(\text{imC}(\varphi_2)) \) (by Prop. 1.3 and Prop. 3.3). Thus \( \text{num}(\mu(\text{imC}(\varphi_1))) = \text{num}(\mu(\text{imC}(\varphi_2))). \)

Suppose \( \varphi_1 \equiv_L \varphi_2 \). Then after essential restrictions (if necessary), and since \( \varphi_1, \varphi_2 \in \text{plep}_{k,1} \), we have by the characterization of the \( L \)-order: \( \text{domC}(\varphi_1) = \text{domC}(\varphi_2) \subseteq A^m \) (for some \( m > 0 \)), and \( \text{part}(\varphi_1) = \text{part}(\varphi_2) \). Hence, \( |\text{imC}(\varphi_1)| = |\text{imC}(\varphi_2)| = |\text{part}_{\text{domC}}(\varphi_1)| = |\text{part}_{\text{domC}}(\varphi_2)| \), where \( \text{part}_{\text{domC}}(\varphi_1) \) denotes the restriction of \( \text{part}(\varphi_1) \) to \( \text{domC}(\varphi_1) = \text{domC}(\varphi_2) \), and \( |\text{part}_{\text{domC}}(\varphi_1)| \) denotes...
the number of classes of the partition on $\text{domC}(\varphi_1) = \text{domC}(\varphi_2)$. Also, $\varphi_1, \varphi_2 \in \text{plep} M_{k,1}$ implies that $\text{imC}(\varphi_1) \subseteq A^{n_1}$ and $\text{imC}(\varphi_2) \subseteq A^{n_2}$, for some $n_1, n_2 > 0$. It follows that $\mu(\text{imC}(\varphi_1))$ and $\mu(\text{imC}(\varphi_2))$ are of the form

$$\mu(\text{imC}(\varphi_1)) = |\text{imC}(\varphi_1)| \times k^{-n_1}, \quad \text{and} \quad \mu(\text{imC}(\varphi_2)) = |\text{imC}(\varphi_2)| \times k^{-n_2}.$$ 

Hence, since $|\text{imC}(\varphi_1)| = |\text{imC}(\varphi_2)|$ we have

$$\mu(\text{imC}(\varphi_1)) \times k^{n_2} = \mu(\text{imC}(\varphi_2)) \times k^{n_1}.$$ 

After removing powers of $k$, we obtain the $k$-reduced numerators, hence

$$\text{num}(\mu(\text{imC}(\varphi_1))) = \text{num}(\mu(\text{imC}(\varphi_2))).$$ 

We proved that both $\equiv_R$ and $\equiv_L$ preserve $\text{num}(\mu(\text{imC}(\varphi)))$, hence $\equiv_D$ preserves $\text{num}(\mu(\text{imC}(\varphi)))$.

The reasoning works in the same way for $\text{tlep} M_{k,1}$.

Let $\varphi_1, \varphi_2 \in \text{plep} M_{k,1}$ be represented by maps $\varphi_1 : P_1 \to Q_1$ and $\varphi_2 : P_2 \to Q_2$, where $P_1, Q_1, P_2, Q_2$ are finite prefix codes with $Q_1 \subseteq A^{n_1}$ and $Q_2 \subseteq A^{n_2}$, and $\mu(Q_1) = \mu(Q_2)$. By Lemma 4.1 in [2], $\varphi_1 \equiv_R \text{id}_Q$, and $\varphi_2 \equiv_R \text{id}_Q$, so we only need to prove that $\text{id}_Q \equiv_D \text{plep} M \text{id}_Q$.

We have $\mu(Q_1) = |Q_1| \times k^{-n_1}$ and $\mu(Q_2) = |Q_2| \times k^{-n_2}$. Moreover, the assumption is that $\mu(Q_1) = N \times k^{-j_1}$ and $\mu(Q_2) = N \times k^{-j_2}$, for some common numerator $N > 0$ and some $j_1, j_2 \geq 0$, such that $N$ is not divisible by $k$. Hence, $|Q_1| = N \times k^{j_1}$ and $|Q_2| = N \times k^{j_2}$ for some $i_1, i_2 \geq 0$.

Suppose that, for example, $i_1 \geq i_2$. We can essentially restrict $\text{id}_Q$ to $\text{id}_{Q_2}$ where $Q_2' = Q_2 A^{i_1-i_2} \subseteq A^{n_2+i_1-i_2}$. Now we have

$$Q_2' = |Q_2| \times k^{i_1-i_2} = N \times k^{i_2}.$$ 

So there exists a bijection $\beta : Q_1 \to Q_2'$. Since all words in $Q_1$ have the same length, and all words in $Q_2'$ have the same length, we have $\beta \in \text{plep} M_{k,1}$. Of course, $\text{id}_{Q_2}$ and $\text{id}_{Q_2'}$ represent the same element of $\text{plep} M_{k,1}$. Now we have $\beta \circ \text{id}_Q(\cdot) \circ \beta^{-1} = \text{id}_{Q_2'}(\cdot)$. Hence, $\text{id}_Q \equiv_D \text{plep} M \text{id}_{Q_2'}$, since $\text{id}_Q \equiv_L \beta \circ \text{id}_Q, \equiv_R \beta \circ \text{id}_Q \circ \beta^{-1} (= \text{id}_{Q_2'})$. So, $\text{id}_Q \equiv_D \text{id}_{Q_2'}$ in $\text{plep} M_{k,1}$.

In case $\varphi_1, \varphi_2 \in \text{tlep} M_{k,1}$ the same reasoning works, except that we replace $\text{id}_Q$ and $\text{id}_{Q_2}$ (which are not total) by $\eta_Q : A^{n_1} \to Q_1$ and $\eta_{Q_2} : A^{n_2+i_1-i_2} \to Q_2'$, defined as follows: For $q_1 \in Q_1$ we let $\eta_Q(q_1) = q_1$, and for $x \in A^{n_1} - Q_1$ we let $\eta_Q(x) = q_{0,1}$ (where $q_{0,1}$ is a fixed element, chosen arbitrarily in $Q_1$). Note that the definition of $\eta_Q$ depends on $Q_1$, $q_{0,1}$, and $A^{n_1} - Q_1$ (however, $n_1$ is determined by $Q_1$ since $Q_1 \subseteq A^{n_1}$, so $A^{n_1} - Q_1$ is determined by $Q_1$). Similarly, for $q \in Q_2$, $\eta_{Q_2}(q) = q$, and for $x \in A^{n_2+i_1-i_2} - Q_2'$, $\eta_{Q_2}(x) = q_{0,2}$ (where $q_{0,2}$ is a fixed element, chosen arbitrarily in $Q_2'$). Then $\varphi_1 \equiv_R Q_1$ and $\varphi_2 \equiv_R Q_2'$.

As above, let $\beta : Q_1 \to Q_2'$ be a bijection; we assume in addition that $\beta(q_{0,1}) = q_{0,2}$. We define $B : A^{n_1} \to Q_2'$ by $B(q_{1}) = \beta(q_1)$ for all $q_1 \in Q_1$; and $B(q) = q_{0,2}$ when $q \in A^{n_1} - Q_1$. We define $B' : A^{n_2+i_1-i_2} \to Q_1$ by $B'(q_2) = \beta^{-1}(q_2)$ for $q_2 \in Q_2'$; and $B'(q) = q_{0,1} (= \beta^{-1}(q_{0,2}))$ when $q \in A^{n_2+i_1-i_2} - Q_2'$. Obviously, $\eta_{Q_1}, \eta_{Q_2}', B, B' \in \text{tlep} M_{k,1}$. It is then straightforward to check that $B' \circ B \circ \eta_Q = B \circ \eta_Q$, and $B \circ B' \circ \eta_{Q_2} = B \circ B'$.

Hence, $\eta_{Q_1} \equiv_L B \circ \eta_{Q_1} \equiv_R B \circ \eta_{Q_2}' \equiv L B \circ \eta_{Q_2}'$. So, $\eta_{Q_1} \equiv_D \eta_{Q_2}'$ in $\text{tlep} M_{k,1}$.

Proposition 3.7. For any positive integer $i$ not divisible by $k$ there exists $\varphi \in \text{tlep} M_{k,1}$ such that $i = \text{num}(\mu(\text{imC}(\varphi)))$.

Proof. For any $i > 0$ there exists a fixed-length prefix code $Q \subseteq A^n$ (for some $n > \log_k i$), with $|Q| = i$. So we have $\mu(Q) = i/k^n$. Hence, if $i$ is not divisible by $k$ and if we take $\varphi = \text{id}_Q$ we obtain the result for $\text{plep} M_{k,1}$. To get the result for $\text{tlep} M_{k,1}$ we extend $\text{id}_Q$ to a total function (by taking a complementary prefix code of $Q$ and mapping it to any element of $Q$).
Theorem 3.6 and Prop. 3.7 give a one-to-one correspondence between the non-zero \( D \)-classes of \( \text{plep} M_{k,1} \) (and of \( \text{tlep} M_{k,1} \)) and the positive integers that are not divisible by \( k \).

So the \( D \)-relation of \( \text{plep} M_{k,1} \) is **not** induced by the \( D \)-relation of \( M_{k,1} \) (since \( M_{k,1} \) has only \( k - 1 \) non-zero \( D \)-classes). In other words (since the \( R \)- and \( L \)-orders of \( \text{plep} M_{k,1} \) are induced by \( M_{k,1} \)), there are \( \psi, \varphi \in \text{plep} M_{k,1} \) such that \( R_{M_{k,1}}(\psi) \cap L_{M_{k,1}}(\varphi) \neq \emptyset \), but \( R_{\text{plep} M_{k,1}}(\psi) \cap L_{\text{plep} M_{k,1}}(\varphi) = \emptyset \). (Notation: \( R_M(x) \) and \( L_M(x) \) denote the \( R \)- respectively \( L \)-class of \( x \) in a monoid \( M \)).

It is also interesting a look at an example. When \( k = 2 \) and \( A = \{a, b\} \), \( M_{2,1} \) has just one non-zero \( D \)-class, so in \( M_{2,1} \) we have \( 1 \equiv_D \text{id}_{\{a,a,b\}} \). Obviously, \( 1 \) and \( \text{id}_{\{a,a,b\}} \) belong to \( \text{plep} M_{k,1} \). We have \( \mu(\text{id}_{\{a,a,b\}}) = 1 \), so the \( k \)-ary numerator is \( 1 \). On the other hand, \( \mu(\text{im}\text{C}(\text{id}_{\{a,a,b\}})) = \mu(\{a,a,b\}) = \frac{3}{2} \), so the \( k \)-ary numerator is \( 3 \). Hence, \( 1 \not\equiv_D \text{id}_{\{a,a,b\}} \) in \( \text{plep} M_{k,1} \).

**Definition 3.8** For \( \varphi \in \text{plep} M_{k,1} \) (or \( \text{tlep} M_{k,1} \)) with \( \varphi \neq 0 \), the positive integer \( \text{num}(\mu(\text{im}\text{C}(\varphi))) \) is called the index of the \( D \)-class of \( \varphi \) in \( \text{plep} M_{k,1} \) (or \( \text{tlep} M_{k,1} \)).

The indices range over all the positive integers that are not divisible by \( k \). Moreover, as we saw, the index determines one non-zero \( D \)-class of \( \text{plep} M_{k,1} \) (or \( \text{tlep} M_{k,1} \)) uniquely, and vice versa.

Although the characterization of \( \equiv_D \) of \( \text{plep} M_{k,1} \) in Theorem 3.6 is simple, it is hard to picture what the number \( \text{num}(\mu(\text{im}\text{C}(\varphi))) \) means. The following gives perhaps a better insight.

**Proposition 3.9** For any non-zero \( \varphi_1, \varphi_2 \in \text{plep} M_{k,1} \) the following are equivalent:

1. \( \text{num}(\mu(\text{im}\text{C}(\varphi_1))) = \text{num}(\mu(\text{im}\text{C}(\varphi_2))) \)
2. there are essential class-wise restrictions \( \Phi_1, \Phi_2 \) of \( \varphi_1 \), respectively \( \varphi_2 \), such that for some \( n \geq 1 \), \( \text{im}\text{C}(\Phi_1) \subseteq A^n \), \( \text{im}\text{C}(\Phi_2) \subseteq A^n \), and \( |\text{im}\text{C}(\Phi_1)| = |\text{im}\text{C}(\Phi_2)| \).

**Proof.** We prove \([(1) \Rightarrow (2)] \) (the converse is obvious). Let \( \varphi_1, \varphi_2 \in \text{plep} M_{k,1} \) be represented by maps \( \varphi_1 : P_1 \to Q_1 \) and \( \varphi_2 : P_2 \to Q_2 \), where \( P_1, Q_1, P_2, Q_2 \) are finite prefix codes with \( Q_1 \subseteq A^{n_1} \) and \( Q_2 \subseteq A^{n_2} \), and \( \text{num}(\mu(Q_1)) = \text{num}(\mu(Q_2)) \). Moreover, by assumption, \( \mu(Q_1) = N \times k^{-j_1} \) and \( \mu(Q_2) = N \times k^{-j_2} \), for a common numerator \( N > 0 \) and some \( j_1, j_2 \geq 0 \), such that \( N \) is not divisible by \( k \). Hence, \( |Q_1| = N \times k^{i_1} \) and \( |Q_2| = N \times k^{i_2} \) for some \( i_1 \geq i_2 \).

Suppose that, for example, \( i_1 \geq i_2 \). We can essentially restrict \( \varphi_2 : P_2 \to Q_2 \) to \( \varphi'_2 : P'_2 \to Q'_2 \) where \( Q'_2 = Q_2 A^{i_1-i_2} \subseteq A^{n_2+i_1-i_2} \). Now we have \( |Q'_2| = |Q_2| \times k^{i_1-i_2} = N \times k^{i_2} \times k^{i_1-i_2} = |Q_1| \). \( \square \)

### 3.4 The maximal subgroups of \( \text{plep} M_{k,1} \) and \( \text{tlep} M_{k,1} \)

It is well known in semigroup theory that all the maximal subgroups in the same \( D \)-class are isomorphic, and that the maximal subgroups are exactly the \( H \)-classes that contain an idempotent. So, to find all the maximal subgroups (up to isomorphism) we only need to find one idempotent (and its \( H \)-class) in every \( D \)-class.

In [5] we defined the subgroup \( \text{lp} G_{k,1} \) of length-preserving elements of the Thompson-Higman group \( G_{k,1} \); one motivation for studying \( \text{lp} G_{k,1} \) is that \( G_{k,1} \) is a Zappa-Szep product of \( \text{lp} G_{k,1} \) and \( F_{k,1} \) (proved in [5]). More generally, for the Higman group \( G_{k,m} \) we can define the subgroup

\[
\text{lp} G_{k,m} = \{ \varphi \in G_{k,m} : (\forall x \in \text{Dom}(\varphi)) |\varphi(x)| = |x| \}.
\]

Note that when \( n \equiv m \mod k - 1 \) then \( \text{lp} G_{k,n} \simeq \text{lp} G_{k,m} \). This is proved in the same way as Prop. 3.1 in [1] (which shows that \( n \equiv m \mod k - 1 \) implies \( M_{k,n} \simeq M_{k,m} \)).

By Prop. 3.1 and Theorem 2.1 in [1] (and their proofs) we have:
Lemma 3.10  Let $P \subset A^*$ be any finite prefix code such that $m \equiv |P| \mod k - 1$, let $id_P$ be the partial identity on $P A^*$, and let

$$G(id_P) = \{ \varphi \in M_k, \text{ Dom}(\varphi) =_{\text{ess}} \text{Im}(\varphi) =_{\text{ess}} P A^*, \text{ and } \varphi \text{ is injective} \}.$$  

Then $G(id_P)$ is isomorphic to the Higman group $G_{k,m}$. \qed 

Recall that in this paper, the integers modulo $k - 1$ are taken in the range $\{1, \ldots, k - 1\}$.

Proposition 3.11  

(1) The group of units of both $plepM_{k,1}$ and $tlepM_{k,1}$ is $lpG_{k,1}$.

(2) The maximal subgroups of $plepM_{k,1}$ (and of $tlepM_{k,1}$) are isomorphic to the groups $lpG_{k,m}$ (for $1 \leq m \leq k - 1$). More precisely, for any positive integer $i$ not divisible by $k$, all the maximal subgroups of the $D$-class with index $i$ are isomorphic to $lpG_{k, i \mod k - 1}$.

Proof. (1) By Corollary 3.4, every $H$-class of $plepM_{k,1}$ is of the form $H \cap plepM_{k,1}$, where $H$ is any $H$-class in $M_{k,1}$. The group of units of $plepM_{k,1}$ is the $H$-class of $1$ in $plepM_{k,1}$, and the $H$-class of $1$ in $M_{k,1}$ is $G_{k,1}$ (by Prop. 2.1 in [3]). Hence, the group of units of $plepM_{k,1}$ is $G_{k,1} \cap plepM_{k,1} = G_{k,1} \cap tlepM_{k,1} = lpG_{k,1}$.

Since the $L$-class of $1$ in $M_{k,1}$ contains only elements with domain essentially equal to $A^*$, the group of units of $plepM_{k,1}$ is in $tlepM_{k,1}$. Hence, the groups of units of $tlepM_{k,1}$ is equal to the group of units of $plepM_{k,1}$. This proves part (1) of the Theorem.

Proof of (2) for $plepM_{k,1}$:

Let $D_i$ be the $D$-class of $plepM_{k,1}$ with index $i$. We choose any $n > 0$ such that $i < k^n$, and any prefix code $Q \subset A^n$ such that $|Q| = i$. Then the partial identity $id_Q$ is an idempotent in $D_i$. The $H$-class of $id_Q$ in $M_{k,1}$ consists of the elements $\varphi \in M_{k,1}$ such that $\varphi \equiv_R id_Q$ (i.e., $\text{Im}(\varphi) =_{\text{ess}} QA^*$), and $\varphi \equiv_L id_Q$ (i.e., $\text{Dom}(\varphi) =_{\text{ess}} QA^*$, and $\varphi$ is injective). Hence, the $H$-class of $id_Q$ in $M_{k,1}$ is

$$G(id_Q) = \{ \varphi \in M_{k,1} : \text{ Dom}(\varphi) =_{\text{ess}} \text{Im}(\varphi) =_{\text{ess}} QA^*, \text{ and } \varphi \text{ is injective} \} ,$$

and by Lemma 3.10 this is a group isomorphic to $G_{k,m}$. By Corollary 3.4 the $H$-class of $id_Q$ in $plepM_{k,1}$ is $G(id_Q) \cap plepM_{k,1}$, hence it is isomorphic to $lpG_{k,m}$. This proves (2) for $plepM_{k,1}$.

Proof of (2) for $tlepM_{k,1}$:

Let $Q \subset A^n$ be as in the proof of (2) for $plepM_{k,1}$, and let $q_0$ be a fixed element, arbitrarily chosen in $Q$. Consider the idempotent $\eta_{Q,q_0} \in tlepM_{k,1}$ defined by $\eta_{Q,q_0}(q) = q$ for all $q \in Q$, and $\eta_{Q,q_0}(x) = q_0$ for all $x \in A^n - Q$. The $H$-class of $\eta_{Q,q_0}$ in $tlepM_{k,1}$ consists of the elements $\varphi \in M_{k,1}$ such that we have $\varphi \equiv_R \eta_{Q,q_0}$ (i.e., $\text{Im}(\varphi) =_{\text{ess}} QA^*$), and we have $\varphi \equiv_L \eta_{Q,q_0}$ (i.e., $\text{Dom}(\varphi) =_{\text{ess}} A^*$, and part($\varphi$) is essentially equivalent to the partition $\{ \{ q \} : q \in Q \} \cup \{ A^n - Q \}$ of $A^n$). So the $H$-class of $\eta_{Q,q_0}$ in $M_{k,1}$ is

$$G(\eta_{Q,q_0}) = \{ \varphi \in M_{k,1} : \text{ Dom}(\varphi) =_{\text{ess}} A^*, \text{ Im}(\varphi) =_{\text{ess}} QA^*, \text{ and } \varphi \text{ is injective on } QA^* \cap \text{Dom}(\varphi), \varphi(A^n - Q) = \{ \varphi|_{q_0} \} \} .$$

Then $G(\eta_{Q,q_0})$ is a maximal subgroup of $M_{k,1}$ with identity $\eta_{Q,q_0}$ (since the $H$-class of an idempotent is a maximal subgroup).

We saw that for $\varphi \in G(\eta_{Q,q_0})$, the restriction $\varphi : QA^* \cap \text{Dom}(\varphi) \to QA^* \cap \text{Im}(\varphi)$ is injective. Hence, the inverse of $\varphi \in G(\eta_{Q,q_0})$ is $\varphi' = \varphi^{-1} \circ \eta_{Q,q_0}(\cdot) \in G(\eta_{Q,q_0})$; indeed, one easily verifies that $\varphi' = \varphi$. So $G(\eta_{Q,q_0})$ is isomorphic to $G(id_Q)$, and $G(\eta_{Q,q_0}) \cap tlepM_{k,1}$ is isomorphic to $G(id_Q) \cap tlepM_{k,1}$. 

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An isomorphism can be defined by $\iota : \varphi \in G(\eta_{Q,q_0}) \mapsto \varphi_Q \in G(id_Q)$, where $\varphi_Q$ is the the restriction $QA^* \cap \text{Dom}(\varphi) \to QA^* \cap \text{Im}(\varphi)$ of $\varphi$ as above. Bijectiveness and the homomorphism property of the map $\iota$ follow easily from the definition of $G(\eta_{Q,q_0})$ and $G(id_Q)$.

The same isomorphism $\iota$ shows that $G(\eta_{Q,q_0}) \cap \text{tlep}M_{k,1}$ is isomorphic to $G(id_Q) \cap \text{plep}M_{k,1}$, i.e., to $i\rho G_{k,m}$. This proves the Claim.

By Corollary 3.4 $G(\eta_{Q,q_0}) \cap \text{tlep}M_{k,1}$ is the $H$-class of $\eta_{Q,q_0}$ in $\text{tlep}M_{k,1}$. This proves (2) for $\text{tlep}M_{k,1}$. □

4 Complexity of computing the Bernoulli measure

We consider the problem of computing the numbers $\mu(\text{imC}(\varphi)) = \text{\text{height}}_R(\varphi)$, $\mu(\text{domC}(\varphi))$, and the amount of collision $\text{coll}(\varphi) = 1 - \text{\text{height}}_L(\varphi)$. Here we assume that the input $\varphi \in M_{k,1}$ is given by a word over a generating set of $M_{k,1}$; we can consider either a finite generating set $\Gamma$, or a circuit-like generating set $\Gamma \cup \tau$. These numbers belong to $[0,1] \cap \mathbb{Z}[\frac{1}{k}]$, and we want to express them in base $k$, i.e., in the form $0$, or $1$, or $0.d_1\ldots d_n$ with $d_n \neq 0$, where $d_1,\ldots,d_n \in \{0,1,\ldots,k-1\}$ are base-$k$ digits. Before we compute these numbers we need some preliminary algorithms.

4.1 Complexity for inputs over a finite generating set

**Lemma 4.1** The following computational problems can be solved in deterministic polynomial time.

**Input:** $y \in A^*$ and $\varphi \in M_{k,1}$, the latter given by a word over a finite generating set $\Gamma$.

**Output 1:** The $k$-ary rational number $\mu(\varphi^{-1}(y))$ expressed in base $k$.

**Output 2:** The $k$-ary rational number $\mu(m_y)$ expressed in base $k$, where $m_y$ is any minimum-length element of the class $\varphi^{-1}(y)$ of $\text{part}(\varphi)$.

The output in both cases will be a finite string over the alphabet $\{\cdot,0,1,\ldots,k-1\}$, where “$\cdot$” is the base-$k$ dot.

**Proof.** We use the following result from Corollary 4.15 in [3]: There is a deterministic algorithm which on input $(\varphi,y)$ constructs an acyclic DFA (deterministic finite automaton) $A_y$ with a single accept state, that accepts the language $\varphi^{-1}(y) \subseteq A^*$. The time complexity of this algorithm is a polynomial in $|y| + |\varphi|$. (where $|\varphi|$ denotes the word-length of $\varphi$ over $\Gamma$).

We saw that the $\text{part}(\varphi)$-classes are finite, so the language $\varphi^{-1}(y)$ is finite; but its cardinality can grow exponentially with $|y| + |\varphi|$. On the other hand, since $A_y$ can be constructed deterministically in polynomial time, $A_y$ has only polynomially many states and edges. The underlying directed graph of $A_y$ is acyclic, and it has only one source (namely the start state $q_0$) and one sink (namely the accept state $q_{\text{acc}}$). By definition, a source is a vertex without incoming edges, and a sink is a vertex without outgoing edges; a finite acyclic directed graph always has at least one source and at least one sink.

Let us compute **Output 2** first: A breadth-first search is performed in the directed graph of $A_y$, starting at the start state $q_0$ and ending when the accept state is found. This easily yields the length of a shortest path from the start state to the accept state; and this length is $|m_y|$. **Output 2** is $\mu(m_y) = 0.0|m_y|-11$ (i.e., after the dot in the base-$k$ representation there are $|m_y| - 1$ digits “0” and one digit “1”).

To compute **Output 1**, a more elaborate algorithm is needed. For every state $q$ of $A_y$ we will compute $\mu(L_q)$, where $L_q \subseteq A^*$ is the set of labels of all the paths in $A_y$ from the start state $q_0$ to $q$. In other words, $L_q$ is the language that would be accepted by $A_y$ if $q$ (instead of $q_{\text{acc}}$) were the accept state. We call $\mu(L_q)$ the measure of the state $q$, and denote it by $\mu(q)$. Since $A_y$ is acyclic, $L_q$ is finite for every state $q$. It follows from these definitions that $\mu(q_0) = 1$ (since $L_{q_0} = \{\varepsilon\}$), and $L_{q_{\text{acc}}} = \varphi^{-1}(y)$, so $\mu(q_{\text{acc}}) = \mu(\varphi^{-1}(y))$. Hence, $\mu(q_{\text{acc}})$ is the desired **Output 1**.
Theorem 4.2 On input \( \varphi \in M_{k,1} \), given by a word over a finite generating set \( \Gamma \), the following numbers (expressed in base \( k \)) can be computed by a deterministic polynomial-time algorithm: \( \mu(\text{imC}(\varphi)) \) (i.e., \( \text{height}_{\mathbb{R}}(\varphi) \)), \( \mu(\text{domC}(\varphi)) \), and \( \text{coll}(\varphi) \) (i.e., \( 1 - \text{height}_{\mathcal{L}}(\varphi) \)).

Proof. (1) For \( \mu(\text{imC}(\varphi)) \) we use the following result from Corollary 4.11 in [3]. There is a deterministic polynomial-time algorithm which on input \( \varphi \) (expressed over \( \Gamma \)), outputs \( \text{imC}(\varphi) \) explicitly as a list of words.

Next, for a given word \( w \in A^* \) the measure \( \mu(w) \) can be immediately computed: \( \mu(w) = 0.0|w|^{-1}1 \), where \( 0|w|^{-1} \) denotes a sequence of \( |w| - 1 \) zeros. Thus, we obtain \( \mu(\text{imC}(\varphi)) = \sum_{y \in \text{imC}(\varphi)} \mu(y) \) deterministically in polynomial-time.

(2) By part (1) of this proof and from Output 1 of Lemma [4,1] we can compute the sequence \( (\mu(\varphi^{-1}(y)) : y \in \text{imC}(\varphi)) \) explicitly in deterministic polynomial time. Hence we find \( \mu(\text{domC}(\varphi)) = \sum_{y \in \text{imC}(\varphi)} \mu(\varphi^{-1}(y)) \) in polynomial time.

(3) For \( \text{coll}(\varphi) \) we use part (1) of this proof and Output 2 of Lemma [4,1] to compute \( \text{coll}(\varphi) = 1 - \sum_{y \in \text{imC}(\varphi)} \mu(m_y) \). \( \square \)

For a \( k \)-ary rational number \( 0.d_1 \ldots d_n \) (with \( d_n \neq 0 \)) written in base \( k \), the \( k \)-reduced numerator is \( \text{num}(0.d_1 \ldots d_n) = d_1 \ldots d_n \); the latter may have leading zeros (if \( d_1 = 0 \), etc.) that will be dropped. Hence Theorem 4.2 implies the following.

Corollary 4.3 If \( \varphi \in M_{k,1} \) is given by a word over a finite generating set \( \Gamma \), the following integers (expressed in base \( k \)) can be computed by a deterministic polynomial-time algorithm: \( \text{num}(\mu(\text{imC}(\varphi))) \), \( \text{num}(\mu(\text{domC}(\varphi))) \), and \( \text{num}(\mu(\text{coll}(\varphi))) \). \( \square \)

4.2 New complexity classes for the Bernoulli measure and for counting

Measures are similar to counting, so along the same lines as the counting complexity classes we can define measure classes.
Definition 4.4  For any complexity class $C$ of decision problems we introduce the measure class $\mu \bullet C$ consisting of all functions of the form

$$f_R : v \in B^* \mapsto \mu(\{w \in A^* : (v,w) \in R\}) \in [0,1] \cap \mathbb{Z}[\frac{1}{k}],$$

where $R$ ranges over all predicates $R \subseteq B^* \times A^*$ (for any finite alphabets $A,B$ with $|A| = k$), with the following properties:

- The predicate $R$ is polynomially balanced; by definition, this means that there exists a polynomial $p(.)$ such that for all $(v,w) \in R$, $|w| \leq p(|v|)$.
- The membership problem of $R$ (i.e., the question, “given $(v,w) \in B^* \times A^*$ is $(v,w) \in R$ ?”) is in the complexity class $C$.
- For every $v \in B^*$ the set $(v)R = \{w \in A^* : (v,w) \in R\}$ is a finite prefix code. (Finiteness of $(v)R$ already follows from polynomial balancedness of $R$.)

Compare this with the well-known counting class $# \bullet C$ consisting of all functions of the form

$$f_R : v \in B^* \mapsto |\{w \in A^* : (v,w) \in R\}| \in \mathbb{N},$$

where $R$ has the same properties as in Definition 4.4 except that $(v)R$ is just a finite set (not required to be a prefix code).

The counting class $# \bullet P$ (commonly just denoted $#P$) is called Valiant’s class [18], and many well-known problems are in that class. An example of a function in $\mu \bullet P$ is given by the following Proposition. Here we will count every transposition $\tau_{i-1,i} \in \tau$ as having length $|\tau_{i-1,i}| = i$. The alphabet $A \cup \{\cdot\}$ will be used for the base-$k$ representation of $k$-ary rationals, where $k = |A|$ and where the letters $a_1,a_2,\ldots,a_k$ represent the $k$-ary digits $0,1,\ldots,k-1$.

Proposition 4.5  The following function belongs to $\mu \bullet P$:

$$\varphi \in M_{k,1} \mapsto \mu(\text{dom}(\varphi)) \in [0,1] \cap \mathbb{Z}[\frac{1}{k}],$$

where $\varphi$ is given by a word over $\Gamma \cup \tau$, and $\mu(\text{dom}(\varphi))$ is expressed by a finite string over the alphabet $A \cup \{\cdot\}$.

Proof. We consider the predicate $R = \{((\varphi,x) \in (\Gamma \cup \tau)^* \times A^* : x \in \text{dom}(\varphi))\}$. By Prop. 5.5(1) in [2], the membership problem of this predicate (called the domain code membership problem) is in $P$. The proof of Prop. 5.5(1) in [2] also shows that the predicate is polynomially balanced; in fact, for any $x \in \text{dom}(\varphi)$ we have $|x| \leq c \cdot |\varphi|_{\Gamma \cup \tau}$ (for some constant $c$), so the predicate is linearly balanced. And $\text{dom}(\varphi)$ is of course a prefix code.

In order to represent $\Gamma \cup \{\tau\}$ by a finite alphabet we will express every transposition $\tau_{i-1,i} \in \tau$ by $t^i$. So $\Gamma \cup \tau$ is represented by the finite alphabet $\Gamma \cup \{t\}$.  

The Bernoulli measure is closely related to counting, and we would like to explore the connection between the measure complexity classes $\mu \bullet C$ and the counting complexity classes $# \bullet C$.

We have to overcome a syntactic obstacle, namely the fact that the functions in counting classes output natural integers, whereas the functions in measure classes output rational numbers in the interval $[0,1]$. We therefore introduce output reductions that consist of moving the base-$k$ dot; these are a special case of polynomial-time output reductions. We will define them next and we will show that $\mu \bullet C = # \bullet C$, where overlining indicates closure under polynomial-time dot-shift reduction.

Definition 4.6  (1) Let $f_1 : B^* \rightarrow A_1^*$ and $f_2 : B^* \rightarrow A_2^*$ be two total functions. A polynomial-time output reduction from $f_1$ to $f_2$ is a polynomial-time computable total function $\rho : A_2^* \rightarrow A_1^*$ such that $f_1(.) = \rho \circ f_2(.)$.

(2) Suppose $A_1 = A_2 = A \cup \{\cdot\} = \{a_1,\ldots,a_k,\cdot\}$, where $A \cup \{\cdot\}$ is used for the base-$k$ representation. Suppose that $\text{Im}(f_1) \cup \text{Im}(f_2) \subseteq A^* \cup \{\cdot\} A^* \cup A^*$, i.e., the output strings of $f_1$ and $f_2$ contain at most one dot. A polynomial-time dot-shift reduction is a polynomial-time output reduction $\rho$ from $f_1$ to $f_2$
such that for all \( z \in A^*\{\}A^* \cup A^* \) we have: \( z \) and \( \rho(z) \) are identical except possibly for occurrences of \( a_1 \) at the left end (corresponding to leading 0's), occurrences of \( a_1 \) at the right end (corresponding to trailing 0's), and the position (including presence or absence) of the dot. Equivalently, if \( z \) and \( \rho(z) \) are viewed as rationals in in base-k representation, they differ only by a multiplicative factor \( k^n \) for some integer \( n \) (positive or negative or 0).

The defining property of the polynomial-time dot-shift reduction can also be expressed as follows: If in both \( z \) and \( \rho(z) \) one deletes the dot and all occurrences of \( a_1 \) at the right end and the left end, the same string is obtained from \( z \) and \( \rho(z) \).

The closure of a set of functions \( \mathcal{F} \) under polynomial-time dot-shift reduction is the set

\[
\overline{\mathcal{F}} = \{ \rho \circ f(.) : f \in \mathcal{F} \text{ and } \rho \text{ is a polynomial-time dot-shift reduction} \}.
\]

The following Theorem is stated abstractly for a complexity class \( \mathcal{C} \) with certain properties. But we are mainly thinking of the classes \( \mathcal{P} \), \( \mathcal{NP} \) and \( \text{coNP} \). Each one of these classes is closed under intersection with languages in \( \mathcal{P} \) (i.e., \( L \in \mathcal{C} \) and \( L \subseteq \mathcal{P} \) implies \( L \cap L \subseteq \mathcal{C} \)), and is closed under polynomial-time disjunctive reduction. Disjunctive polynomial-time reductions are defined in [10].

**Theorem 4.7** Let \( \mathcal{C} \) be any complexity class of decision problems that is closed under intersection with languages in \( \mathcal{P} \), and closed under disjunctive polynomial-time reduction. Then

\[
\overline{\mu \bullet \mathcal{C}} = \# \bullet \mathcal{C} ,
\]

where overlining indicates closure under polynomial-time dot-shift reduction.

**Proof. (1)** To prove \( \mu \bullet \mathcal{C} \subseteq \overline{\# \bullet \mathcal{C}} \), consider any \( f \in \mu \bullet \mathcal{C} \). So there is a predicate \( R \subseteq B^* \times A^* \) such that \( R \in \mathcal{C} \), \( R \) is polynomially balanced, \( (v)R = \{ w \in A^* : (v, w) \in R \} \) is a finite prefix code (for every \( v \in B^* \)), and \( f(v) = \mu((v)R) \) (for every \( v \in B^* \)). Let \( p(.) \) be the balancing polynomial of \( R \). From \( R \) we construct a new predicate \( R' \subseteq B^* \times A^* \) defined by

\[
(v, z) \in R' \text{ iff } |z| = p(|v|), \text{ and there exists a prefix } w \text{ of } z \text{ such that } (v, w) \in R .
\]

Hence for all \( v \in B^* \), \( (v)R' \subseteq A^{p(|v|)} \); it follows that \((v)R'\) is a fixed-length prefix code; it follows also that \( p(.) \) is a balancing polynomial for \( R' \).

The membership problem of \( R' \) is in \( \mathcal{C} \). Indeed, given \((v,z)\), the relation \(|z| = p(|v|)| \) can be checked in deterministic polynomial time. Since \( z \) has only linearly many prefixes, checking whether some prefix \( x \) of \( z \) satisfies \((v,w) \in R \) leads to at most \(|z| \) membership tests in \( R \); this problem belongs to \( \mathcal{C} \) since \( \mathcal{C} \) is closed under disjunctive polynomial-time reduction.

Since \((v)R'\) is a fixed-length prefix code (of length \( p(|v|) \)) we have: \( \mu((v)R') = |(v)R'| \cdot k^{p(|v|)} \).

The right ideals \((v)R' \cdot A^* \) and \((v)R \cdot A^* \) are essentially equal since every \( z \in (v)R' \) has a prefix in \((v)R \) and every \( w \in (v)R \) is the prefix of an element of \((v)R' \); the latter follows from the fact that the elements of \((v)R' \) have length \( p(|v|) \) whereas all elements of \((v)R \) have length \( \leq p(|v|) \). Since both \((v)R \) and \((v)R' \) are prefix codes it follows now (by Prop. [13]) that \( \mu((v)R') = \mu((v)R) \).

Hence, \( \mu((v)R) = |(v)R'| \cdot k^{-p(|v|)} \), and thus \( f \) is obtained by a polynomial-time dot-shift reduction from the function \( v \in B^* \mapsto |(v)R| \). The latter function belongs to \( \# \bullet \mathcal{C} \).

In summary, the main idea in proof (1) is to transform each prefix code \((v)R \) into a fixed-length prefix code \((v)R' \), while preserving the measure; for fixed-length prefix codes there is a simple relation between cardinality and measure.

**Proof. (2)** To prove \( \# \bullet \mathcal{C} \subseteq \overline{\mu \bullet \mathcal{C}} \) consider any \( f \in \# \bullet \mathcal{C} \). So there is a predicate \( R \subseteq B^* \times A^* \) such that \( R \in \mathcal{C} \), \( R \) is polynomially balanced, and for all \( v \in B^* : f(v) = |(v)R| \). Note that here, \((v)R \) is not necessarily a prefix code.

Before constructing a new predicate from \( R \) we introduce an injective encoding homomorphism \( c : A^* \to A^* \), defined for all \( a_i \in A = \{ a_1, a_2, \ldots, a_k \} \) by \( a_i \mapsto a_i a_2 \). Then \(|c(w)| = 2 \cdot |w| \) for
all $w \in A^*$. By injectiveness, $|c((v)R)| = |(v)R|$ for all $v \in B^*$. Now we define a new predicate $R' \subseteq B^* \times A^*$ by

$$(v, z) \in R' \iff |z| = 2 \cdot p(|v|),$$

where $p(.)$ is the balancing polynomial of $R$. The role of $a_1^*$ is to pad $c(w)$ with trailing zeros in order to make $z$ have length $2 \cdot p(|v|)$. Then $(v)R'$ is a fixed-length prefix code with $(v)R' \subseteq A^2p(|v|)$, and $2 \cdot p(.)$ is a balancing polynomial for $R'$. The membership problem of $R'$ is in $C$, for similar reasons as in the proof of (1). Also, for the same reason as in (1), $\mu((v)R') = |(v)R'| \cdot k^{-2p(|v|)}$, for all $v \in B^*$.

Note that for $(v, z) \in R'$ there exists exactly one $w \in (v)R$ such that $z = c(w) a_1^2p(|v|)-2|w|$, because $c(w)$ ends with the letter $a_2$. Hence, $|(v)R'| = |(v)R|$.

Now we have $|(v)R| = |(v)R'| = \mu((v)R') \cdot k^{p(|v|)}$, so $|(v)R|$ can be computed from $\mu((v)R')$ by a dot-shift. This yields a polynomial-time dot-shift reduction from the function $f$ to a problem in $\mu \circ C$.

Remarks. (1) We have some flexibility in the way we formulate the assumptions on $C$ in Theorem 4.7 above. Instead of closure under disjunctive reduction we could assume that $C$ is closed under right-concatenation with free monoids; this means that $R \in C$ implies $RA^* \in C$. Here we assume that any binary predicate $R \subseteq B^* \times A^*$ is represented by the language $\{(x, y) \in B^*\times A^* : (x, y) \in R\}$, where $\$ is a letter that does not belong to $A \cup B$.

(2) The proof of Theorem 4.7 above shows more than what we stated: The equality $\# \circ C = \# \circ C$ is effective, in the sense since that given a predicate $R$ that represents a function $f$ in $\mu \circ C$, one easily finds a predicate $R'$ that represents a dot-shift of $f$ that belongs to $\# \circ C$, and vice versa.

As a consequence of Theorem 4.7 and Prop. 4.5, we have:

**Corollary 4.8** The function problem $\varphi \in M_{k,1} \mapsto \mu(\text{dom} C(\varphi))$ is $\# \circ P^\text{-complete}$ (when elements of $M_{k,1}$ are given by words over $\Gamma \cup \tau$).

**Proof.** It was proved at the end of Section 6.2 in [1] that the function problem $C \mapsto |\text{Dom}(C)|$ (where $C$ ranges over partial acyclic circuits) is $\# \circ P^\text{-complete}$. Hence the problem is also $\# \circ P^\text{-complete}$ with respect to polynomial-time parsimonious reductions and dot-shift reductions. And it follows from Prop. 4.5 and Theorem 4.7 that the problem is in $\# \circ P (= \mu \circ P)$. □

### 4.3 Complexity for inputs over a circuit-like generating set $\Gamma \cup \tau$

We saw that computing $\mu(\text{dom} C(\varphi))$ is $\# \circ P^\text{-complete}$ when $\varphi \in M_{k,1}$ is given by a word over $\Gamma \cup \tau$. We will show now that computing $\mu(\text{im} C(\varphi))$ is $\# \circ \text{NP}$-complete, and that computing the amount of collision $\text{coll}(\varphi)$ is $\# \circ \text{coNP}$-complete. It is known from [17] (see also [11]) that $\# \circ \text{NP} \subseteq \# \circ \text{coNP}$.

Recall that, in the definition of the word-length $|\varphi|_{\Gamma \cup \tau}$ we use $|\gamma| = 1$ for $\gamma \in \Gamma$ and $|\tau_{i-1,i}| = i$.

**Lemma 4.9** For all $\varphi \in M_{k,1}$ and all $y \in \text{im} \varphi$ there exists $x \in \varphi^{-1}(y)$ such that

$$|x| \leq |y| + c_{\Gamma} \cdot |\varphi|_{\Gamma \cup \tau},$$

where $c_{\Gamma} = \max\{\ell(\gamma) : \gamma \in \Gamma\}$.

**Proof.** Let $\varphi = \alpha_N \ldots \alpha_1$, where $\alpha_N, \ldots, \alpha_1 \in \Gamma \cup \tau$. When a generator $\gamma \in \Gamma$ is applied to an argument, the output is at most $\ell(\gamma)$ letters shorter than the argument (where $\ell(\gamma)$ denotes the length of the longest word in the table of $\gamma$). Hence, the length decrease $|x| - |\varphi(x)|$ is at most $c_{\Gamma} \cdot |\varphi|_{\Gamma \cup \tau}$. Hence we have $|y| \geq |x| - c_{\Gamma} \cdot |\varphi|_{\Gamma \cup \tau}$. □

**Theorem 4.10** The following problem is $\# \circ \text{NP}$-complete.

**Input:** $\varphi \in M_{k,1}$, given by a word over $\Gamma \cup \tau$.

**Output:** The $k$-ary fraction $\mu(\text{im} C(\varphi))$, written in base $k$.  

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Proof. We first show that this problem is $\# \bullet \text{NP}$-hard (or, equivalently, $\# \bullet \text{NP}$-hard), by reducing the image-size problem to it. The latter problem was proved to be $\# \bullet \text{NP}$-complete in Section 6.2 of [1]; it is specified as follows.

Input: A partial circuit $C$ with $m$ input wires and $n$ output wires; $m, n$ are part of the input.

Output: The integer $\vert \text{Im}(C) \vert$, written in base $k$.

To obtain a reduction we view a partial circuit $C$ as an element (call it $\varphi_C$) of $M_{k,1}$, written as a word over $\Gamma \cup \tau$. Indeed, any partial function between finite prefix codes (in this case, subsets of $A^m$ and $A^n$) is an element of $M_{k,1}$. For the representation of $C$ by $\varphi_C$ we have $\vert \text{Im}(C) \vert = \vert \text{Im}(\varphi_C) \cap A^n \vert$.

By knowing $n$ and finding $\mu(\text{Im}(\varphi_C))$ (by $\mu(\text{Im}(\varphi_C) \cap A^n)$), we can solve the image size problem, by computing $\vert \text{Im}(C) \vert = k^n \cdot \mu(\text{Im}(\varphi_C))$. This is a polynomial-time dot-shift reduction.

Proof that that the problem is in $\# \bullet \text{NP}$: We consider the predicate $R \subseteq A^* \times (\Gamma \cup \tau)^*$ defined by

$$(y, \varphi) \in R \iff (\exists x \in A^*)[y = \varphi(x) \land \vert y \vert = c_\Gamma \cdot \vert \varphi \vert],$$

where $c_\Gamma = \max\{\ell(\gamma) : \gamma \in \Gamma \}$. The quantified variable $x$ in $(\exists x \in A^*)$ has polynomially bounded length. Indeed, if $\vert y \vert = c_\Gamma \cdot \vert \varphi \vert$ and $y \in \text{Im}(\varphi)$ then there exists $x \in A^*$ such that $y = \varphi(x)$ and (by Lemma [2]), $|x| \leq |y| + |\varphi| \cdot \max\{\ell(\gamma) : \gamma \in \Gamma \}$. Also, $y = \varphi(x)$ can be verified in polynomial time when $x, y$ and $\varphi$ are given. So, the membership problem of the predicate $R$ is in NP.

Hence the function $\varphi \mapsto \{y : (y, \varphi) \in R\}$ is in $\# \bullet \text{NP}$; here, $\varphi$ is represented by a word over $\Gamma \cup \tau$ and the integer $\{y : (y, \varphi) \in R\}$ is written in base $k$.

By Theorem 4.5(2) in [3], the length $\ell(\varphi)$ of the longest word in $\text{Im}(\varphi) \cup \text{dom}(\varphi)$ is at most $c_\Gamma \cdot \vert \varphi \vert$. Hence, if $n = c_\Gamma \cdot \vert \varphi \vert$ then the right ideal $\text{Im}(\varphi) \cap A^n A^*$ is essential in $\text{Im}(\varphi)$. Hence, $\text{Im}(\varphi) \cap A^n$ is equal to $\text{Im}(\Phi)$ for some essentially equal restriction $\Phi$ of $\varphi$. Moreover, since $\text{Im}(\Phi) \subseteq A^n$, $\text{Im}(\Phi)$ is a prefix code. By the definition of the predicate $R$ we have $\{y : (y, \varphi) \in R\} = |\text{Im}(\Phi)|$; thus, $|\text{Im}(\Phi)|$ is computable in $\# \bullet \text{NP}$.

Finally, from $|\text{Im}(\Phi)|$ we can compute $\mu(\text{Im}(\varphi)) = \mu(\text{Im}(\Phi))$ in deterministic polynomial time, by computing $\mu(\text{Im}(\Phi)) = k^{-n} \cdot |\text{Im}(\Phi)|$. This is a polynomial-time dot-shift reduction, hence $\mu(\text{Im}(\Phi))$ is computable in polynomial time from $|\text{Im}(\Phi)|$, which itself is computable in $\# \bullet \text{NP}$. $\square$

For the next theorem we will need a lemma.

Lemma 4.11 Let $B(x, y)$ be any boolean formula $B(x, y)$ where $x$ and $y$ are strings of boolean variables with $|x| = m$ and $|y| = n$. Suppose that there exists $y \in \{0, 1\}^n$ such that $(\forall x)[B(x, y) = 0]$; i.e., $B(x, y)$ does not have property (2) below. Then there exists a boolean formula $\beta(X, y)$ with $|X| = m + 1$ and $|y| = n$, such that:

1. $\{y : (\forall x)[B(x, y) = 1]\} = \{y : (\forall X)[\beta(X, y) = 1]\}$;
2. the sentence $(\forall y)(\exists X)[\beta(X, y) = 1]$ is true;
3. $\beta$ can be constructed from $B$ in deterministic polynomial time.

Proof. We will write $x = (x_1, \ldots, x_m)$, and $X = (x, x_{m+1})$. Assuming $B(x, y)$ does not have property (2) already, we define $\beta(x, x_{m+1}, y)$ by

$$\beta(x, 0, y) = B(x, y), \text{ and}$$

$$\beta(x, 1, y) = 1.$$

Then for every $y$ we have $(\forall x)[\beta(x, 1, y) = 1$, hence $(\exists x)[\beta(x, x_{m+1}, y) = 1]$; so property (2) holds. The definition of $\beta(x, x_{m+1}, y)$ can also be written as

$$\beta(x, x_{m+1}, y) = x_{m+1} \lor B(x, y),$$

which provides an expression for $\beta(x, x_{m+1}, y)$ from an expression for $B(x, y)$ in deterministic linear time. This proves property (3). Let us prove of property (1).

If $y$ satisfies $(\forall x)[B(x, y) = 1]$ then $(\forall x)[\beta(x, 0, y) = 1]$ and $(\forall x)[\beta(x, 1, y) = 1]$. Hence $y$ satisfies $(\forall(x, x_{m+1}))[\beta(x, x_{m+1}, y) = 1]$.
If $y$ satisfies $(\forall(x,x_{m+1}))[\beta(x,x_{m+1},y) = 1]$ then when $x_{m+1} = 0$ we have $(\forall x)[\beta(x,0,y) = 1]$. Hence, since $\beta(x,0,y) = B(x,y)$, $y$ satisfies $(\forall x)[B(x,y) = 1]$. □

**Notation.** For a boolean formula $B(x,y)$ we let

$$N_{B,1} = \{y : (\forall x)[B(x,y) = 1]\},$$
$$N_{B,0} = \{y : (\forall x)[B(x,y) = 0]\}.$$ 

Hence, $N_{B,0} = 0$ iff $B(x,y)$ satisfies $(\forall y)(\exists x)[B(x,y) = 1]$ (i.e., property (2) in Lemma 4.11).

**Theorem 4.12** The following problem is $\# \cdot \coNP$-complete.

**Input:** $\varphi \in M_{k,1}$, given by a word over $\Gamma \cup \tau$.

**Output:** The $k$-ary fraction $\text{coll}(\varphi)$, written in base $k$.

**Proof.** (1) Let us first prove that the function $\varphi \mapsto \text{noncoll}(\varphi)$ belongs to $\mu \bullet \coNP$ ($\subseteq \# \cdot \coNP$).

Recall that $\text{coll}(\varphi) = 1 - \text{noncoll}(\varphi)$ and $\text{noncoll}(\varphi) = \sum_{i=1}^{n} \mu(m_i)$ where $n = |\text{imC}(\varphi)|$ and $\{m_i : i = 1, \ldots, n\}$ consists of minimum-length representative of all the $\text{part}(\varphi)$-classes in $\text{domC}(\varphi)$ (according to Def. 2.24); this assumes that $\text{imC}(\varphi)$ is a prefix code. By Theorem 4.12 in [2] and its proof, every word in $\text{domC}(\varphi)$ has length $\leq c_{\tau} \cdot |\varphi|^{|\Gamma \cup \tau|}$, where $c_{\tau} = \max\{|z| : z \in \text{domC}(\gamma) \cup \text{imC}(\gamma), \gamma \in \Gamma\}$; i.e., $c_{\tau}$ is the length of the longest words occurring in the tables of the elements of $\Gamma$. In this proof we will abbreviate $c_{\tau}$ by $c$ and $|\varphi|^{|\Gamma \cup \tau|}$ by $|\varphi|$.

When we choose a minimum-length representatives $m$ in a class $C$ we will choose $m$ to be the first element in the dictionary order (among the minimum-length elements in $C$). This does not affect the value of $\text{noncoll}(\varphi)$, which depends only on lengths. The dictionary order in $A^*$ is defined as follows for all $x, y \in A^*$ (assuming $A$ is ordered as $a_1 < \ldots < a_k$): $x \leq_{\text{dict}} y$ iff

- $x$ is a prefix of $y$, or
- there is a common prefix $p$ of $x$ and $y$ and there are $a_i, a_j \in A$ such that
  - $p a_i$ is a prefix of $x$ and $p a_j$ is a prefix of $y$, and $a_i < a_j$.

Since $x$ has only linearly many prefixes, the property $x \leq_{\text{dict}} y$ can be checked in deterministic polynomial time when $x$ and $y$ are given.

By Lemma 2.8(2), $\varphi$ has an essential restriction $\Phi$ that is such that $\text{imC}(\Phi)$ is a prefix code and all the minimum-length representatives $m_i$ of the $\text{part}(\Phi)$-classes have (the same) length $c \cdot |\varphi|$. By Lemma 2.10, $\text{noncoll}(\varphi) = \text{noncoll}(\Phi)$.

A set of minimum-length representatives of the $\text{part}(\Phi)$-classes in $\text{domC}(\Phi)$ is the same as a set of representatives of those $\text{part}(\varphi)$-classes $C$ such that $C$ contains some element(s) of length $c \cdot |\varphi|$, and $C$ contains no shorter elements. Therefore,

$$\text{noncoll}(\Phi) = \mu\{m \in A^{c \cdot |\varphi|} \cap \text{dom}(\varphi) : (\forall x \in A^{<c \cdot |\varphi|})[\varphi(x) \neq \varphi(m)] \text{ and } (\forall x \in A^{c \cdot |\varphi|})[x <_{\text{dict}} m \Rightarrow \varphi(x) \neq \varphi(m)]\}.$$ 

Here, $(\forall x \in A^{<c \cdot |\varphi|})[\varphi(x) \neq \varphi(m)]$ expresses that $m$ has minimum length in its $\text{part}(\varphi)$-class; and $(\forall x \in A^{c \cdot |\varphi|})[x <_{\text{dict}} m \Rightarrow \varphi(x) \neq \varphi(m)]$ expresses that $m$ comes first in the dictionary order in its $\text{part}(\varphi)$-class (which implies that we have picked only one representative in this class).

The above formula for $\text{noncoll}(\Phi)$ implies that the function $\text{noncoll}(\cdot)$ belongs to $\mu \bullet \coNP$. Indeed, the restriction that $m \in A^{<c \cdot |\varphi|}$ implies polynomial balancedness. The property $[m \in \text{dom}(\varphi)]$ is in $\text{P}$ (by Prop. 5.5 in [2]). The condition $[\varphi(x) \neq \varphi(m)]$ is in $\text{P}$, since $\varphi$ can be evaluated on two given words in deterministic polynomial time (by the proof of Prop. 5.5 in [2]); hence the condition $(\forall x \in A^{<c \cdot |\varphi|})[\varphi(x) \neq \varphi(m)]$ is in $\text{coNP}$. Similarly, $[x <_{\text{dict}} m \Rightarrow \varphi(x) \neq \varphi(m)]$ is in $\text{P}$, hence the condition $(\forall x \in A^{c \cdot |\varphi|})[x <_{\text{dict}} m \Rightarrow \varphi(x) \neq \varphi(m)]$ is in $\text{coNP}$.

(2) Let us prove now that the function $\text{noncoll}(\cdot)$ is $\# \cdot \coNP$-hard, by reducing the problem $\# \cdot \Pi^P_1 \text{Sat}$ to the problem of computing $\text{noncoll}(\cdot)$. The problem $\# \cdot \Pi^P_1 \text{Sat}$ is also denoted by $\#\forall\text{Sat}$ and called “Counting forall-satisfiability”; it is specified as follows.
Input: A boolean formula $B(x, y)$ where $x$ and $y$ are strings of variables with $|x| = m$, $|y| = n$ (with $m$ and $n$ part of the input).

Output: The binary representation of the integer $\{y \in \{0,1\}^n : (\forall x \in \{0,1\}^m)[B(x, y) = 1]\}$.

The problem $\#\forall$Sat is $\# \cdot \coNP$-complete (see [8]), and remains $\# \cdot \coNP$-complete when we restrict to the case when $n = m$; we assume from now on that $n = m$. For a reduction we map any instance $B(x, y)$ of $\#\forall$Sat to the element $\varphi_B \in M_{2,1}$, defined as follows:

$$\varphi_B(0xz) = B(x, z) \cdot x \quad \text{for all } x \in \{0,1\}^n \text{ and } z \in \{0,1\}^n ;$$

$$\varphi_B(1xw) = 0x \quad \text{for all } x \in \{0,1\}^n \text{ and } w \in \{0,1\}^{n+1} .$$

So, $\text{domC}(\varphi_B) = 0 \{0,1\}^{2n} \cup 1 \{0,1\}^{2n+1}$, and $0 \{0,1\}^n \subseteq \text{imC}(\varphi_B) \subseteq \{0,1\}^{n+1}$. More precisely, $\text{imC}(\varphi_B) = 0 \{0,1\}^n \cup 1 \{x \in \{0,1\}^n : (\exists z)[B(z, x) = 1]\}$. By Lemma 4.11 we can assume that for every $x \in \{0,1\}^n$ there exists $z$ such that $B(x, z) = 1$; by the Lemma, this does not change the cardinality $\{y : (\forall x)[B(x, y) = 1]\}$. That assumption implies that $\text{imC}(\varphi_B) = \{0,1\}^{n+1}$.

By the definition of $\varphi_B$, the classes of $\text{part}(\varphi_B)$ are of the form $\varphi_B^{-1}(0x)$ for all $x \in \{0,1\}^n$, and of the form $\varphi_B^{-1}(1x)$ for all $x \in \{0,1\}^n$ such that $(\exists z)[B(z, x) = 1]$: however, this holds for all $x \in \{0,1\}^n$, since by Lemma 4.11 we assume that for all $x \in \{0,1\}^n$ we have $(\exists z)[B(x, z) = 1]$. In a class $\varphi_B^{-1}(1x)$ all elements (and hence the shortest element) have length $2n + 1$; and there are $2n$ such classes (as $x$ ranges over $\{0,1\}^n$). The shortest element in the class $\varphi_B^{-1}(0y)$ has length $2n + 2$ if $(\forall z)[B(x, z) = 1]$; the number of such classes is $N_{B,1} = |\{y : (\forall x)[B(x, y) = 1]\}|$. In a class $\varphi_B^{-1}(0y)$, the shortest element has length $2n + 1$ if $(\exists z)[B(x, z) = 0]$; the number of such classes is $2^n - N_{B,1}$. Thus for $\text{noncoll}(\varphi_B)$ we have the formula

$$\text{noncoll}(\varphi_B) = 2^{-2(n+1)} \cdot 2^n + 2^{-2(n+2)} \cdot N_{B,1} + 2^{-2(n+1)} \cdot (2^n - N_{B,1}) = 2^n - 2^{-2(n+2)} \cdot N_{B,1} .$$

It follows that, in binary representation, $N_{B,1}$ can be computed in deterministic polynomial time from $\text{noncoll}(\varphi_B)$ via the formula

$$N_{B,1} = 2^{n+2} - \text{noncoll}(\varphi_B) \cdot 2^{2n+2} ,$$

which reduces the problem $\# \cdot \Pi^p_1$Sat (i.e., the computation of $N_{B,1}$) to the problem of computing $\text{noncoll}(\varphi_B)$.

\[\square\]

5 Appendix

The following theorem was stated in [3] (Theorem 2.3), but the proof was incomplete. We give a complete proof here.

**Theorem 5.1** The monoids $M_{k,1}$ and $\text{Inv}_{k,1}$ are congruence-simple for all $k \geq 2$.

**Proof.** Let $\equiv$ be any congruence on $M_{k,1}$ that is not the equality relation. We will show that then the whole monoid is congruent to the empty map 0. We will make use of 0-$J$-simplicity.

Case 0: Assume that $\Phi \equiv 0$ for some element $\Phi \neq 0$ of $M_{k,1}$. Then for all $\alpha, \beta \in M_{k,1}$ we have obviously $\alpha \Phi \beta \equiv 0$. Moreover, by 0-$J$-simplicity of $M_{k,1}$ we have $M_{k,1} = \{\alpha \Phi \beta : \alpha, \beta \in M_{k,1}\}$ since $\Phi \neq 0$. Hence in this case all elements of $M_{k,1}$ are congruent to 0.

For the remainder we suppose that $\varphi \equiv \psi$ and $\varphi \neq \psi$, for some elements $\varphi, \psi$ of $M_{k,1} - \{0\}$.

Case 1: $\text{Dom}(\varphi) \neq \text{ess Dom}(\psi)$.

Then there exists $x_0 \in A^*$ such that $x_0 A^* \subseteq \text{Dom}(\varphi)$, but $\text{Dom}(\psi) \cap x_0 A^* = \emptyset$; or, vice versa, there exists $x_0 \in A^*$ such that $x_0 A^* \subseteq \text{Dom}(\psi)$, but $\text{Dom}(\varphi) \cap x_0 A^* = \emptyset$. Let us assume the former. Letting $\beta = (x_0 \mapsto x_0)$, we have $\varphi \beta(.) = (x_0 \mapsto \varphi(x_0))$. We also have $\psi \beta(.) = 0$, since $x_0 A^* \cap \text{Dom}(\psi) = \emptyset$. 

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So, $\varphi \beta \equiv \psi \beta = 0$, but $\varphi \beta \neq 0$. Hence case 0, applied to $\Phi = \varphi \beta$, implies that the entire monoid $M_{k,1}$ is congruent to 0.

Case 2.1: $\text{Im}(\varphi) \neq \text{ess Im}(\psi)$ and $\text{Dom}(\varphi) = \text{ess Dom}(\psi)$.

Then there exists $y_0 \in A^*$ such that $y_0 A^* \subseteq \text{Im}(\varphi)$, but $\text{Im}(\psi) \cap y_0 A^* = \emptyset$; or, vice versa, $y_0 A^* \subseteq \text{Im}(\psi)$, but $\text{Im}(\varphi) \cap y_0 A^* = \emptyset$. Let us assume the former. Let $x_0 \in A^*$ be such that $y_0 = \varphi(x_0)$. Then $(y_0 \mapsto y_0) \circ \varphi \circ (x_0 \mapsto x_0) = (x_0 \mapsto y_0)$.

On the other hand, $(y_0 \mapsto y_0) \circ \psi \circ (x_0 \mapsto x_0) = 0$. Indeed, if $x_0 A^* \cap \text{Dom}(\psi) = \emptyset$ then for all $w \in A^*$: $\psi \circ (x_0 \mapsto x_0)(x_0 w) = \psi(x_0 w) = \emptyset$. And if $x_0 A^* \cap \text{Dom}(\psi) \neq \emptyset$ then for those $w \in A^*$ such that $x_0 w \in \text{Dom}(\psi)$ we have $(y_0 \mapsto y_0) \circ \psi \circ (x_0 \mapsto x_0)(x_0 w) = (y_0 \mapsto y_0)(\psi(x_0 w)) = \emptyset$, since $\text{Im}(\psi) \cap y_0 A^* = \emptyset$. Now case 0 applies to $0 \neq \Phi = (y_0 \mapsto y_0) \circ \varphi \circ (x_0 \mapsto x_0) \equiv 0$; hence all elements of $M_{k,1}$ are congruent to 0.

Case 2.2: $\text{Im}(\varphi) = \text{ess Im}(\psi)$ and $\text{Dom}(\varphi) = \text{ess Dom}(\psi)$.

Then (after restricting), $\text{domC}(\varphi) = \text{domC}(\psi)$, and there exist $x_0 \in \text{domC}(\varphi) = \text{domC}(\psi)$ and $y_0 \in \text{imC}(\varphi)$, $y_1 \in \text{imC}(\psi)$ such that $\varphi(x_0) = y_0 = y_1 = \psi(x_0)$. We have two sub-cases.

Case 2.2.1: $y_0$ and $y_1$ are not prefix-comparable.

Then $(y_0 \mapsto y_0) \circ \varphi \circ (x_0 \mapsto x_0) = (x_0 \mapsto y_0)$. On the other hand, $(y_0 \mapsto y_0) \circ \psi \circ (x_0 \mapsto x_0)(x_0 w) = (y_0 \mapsto y_0)(y_1 w) = \emptyset$ for all $w \in A^*$ (since $y_0$ and $y_1$ are not prefix-comparable). So $(y_0 \mapsto y_0) \circ \psi \circ (x_0 \mapsto x_0) = 0$. Hence case 0 applies to $0 \neq \Phi = (y_0 \mapsto y_0) \circ \varphi \circ (x_0 \mapsto x_0) \equiv 0$.

Case 2.2.2: $y_1$ is a prefix of $y_0$. (The case where $y_0$ is a prefix of $y_1$ is similar; and since $y_0 \neq y_1$, there is no other case.)

Then $y_1 = y_0 v_1$ for some $v_1 \in A^*$, and $y_1 A^* \subseteq y_0 A^*$, so $\text{imC}(\psi) \cap y_0 A^*$ contains some string $y_2$ besides $y_1$. Indeed, the right ideal $\text{imC}(\psi) A^* \cap y_0 A^*$ is essential in $y_0 A^*$ because $\text{Im}(\varphi) = \text{ess Im}(\psi)$.

So, $y_2 = y_0 v_2$ for some $v_2 \in A^*$. Hence, $(y_2 \mapsto y_2) \circ \varphi \circ (x_0 \mapsto x_0)(x_0 v_2) = (y_2 \mapsto y_2)(y_0 v_2) = y_2$.

On the other hand, $y_1$ and $y_2$ are not prefix-comparable, since both belong to $\text{imC}(\psi)$, which is a prefix code. Hence, $(y_2 \mapsto y_2) \circ \psi \circ (x_0 \mapsto x_0)(x_0 w) = (y_2 \mapsto y_2)(y_1 w) = \emptyset$, since $y_2$ and $y_1$ are not prefix-comparable. Thus, case 0 applies to $0 \neq \Phi = (y_2 \mapsto y_2) \circ \varphi \circ (x_0 \mapsto x_0) \equiv 0$.

The same proof works for $\text{Inv}_{k,1}$ since all the multipliers used in the proof (of the form $(u \mapsto v)$ for some $u, v \in A^*$) belong to $\text{Inv}_{k,1}$.

\hfill $\Box$

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