Rouquier’s Cocovering Theorem and Well-generated Triangulated Categories

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Abstract

We study cocoverings of triangulated categories, in the sense of Rouquier, and prove that for any regular cardinal \( \alpha \) the condition of \( \alpha \)-compactness, in the sense of Neeman, is local with respect to such cocoverings. This was established for ordinary compactness by Rouquier. Our result yields a new technique for proving that a given triangulated category is well-generated. As an application we describe the \( \alpha \)-compact objects in the unbounded derived category of a quasi-compact semi-separated scheme.

1. Introduction

Let \( T \) be a triangulated category with coproducts, and recall that an object \( Y \) of \( T \) is compact if the functor \( T(Y, -) \) commutes with coproducts. When \( T = D(X) \) is the unbounded derived category of quasi-coherent sheaves on a reasonable scheme \( X \), the condition of compactness in \( T \) is local: given an open cover \( \{ U_1, \ldots, U_n \} \) of \( X \), an object \( F \) is compact in \( D(X) \) if and only if \( F|_{U_i} \) is compact in \( D(U_i) \) for \( 1 \leq i \leq n \). For arbitrary \( T \), Rouquier introduces in [Rou08, §5] a suitable generalisation: he defines a cocovering of \( T \) to be a special family of Bousfield subcategories \( F = \{ I_1, \ldots, I_n \} \) (the precise definition is recalled below). The analogue of restriction to \( U_i \) is then passage to the quotient \( T \rightarrow T/I_i \), and under some natural hypotheses on \( F \), compactness in \( T \) is local: an object \( Y \) is compact in \( T \) if and only if the image of \( Y \) is compact in \( T/I_i \) for \( 1 \leq i \leq n \).

This article concerns the large cardinal generalisation. Let \( \alpha \) be a regular cardinal, that is, \( \alpha \) is an infinite cardinal which is not the sum of fewer than \( \alpha \) cardinals, all smaller than \( \alpha \). In his book [Nee01] Neeman associates to \( \alpha \) a class \( T^{\alpha} \subseteq T \) of \( \alpha \)-compact objects. The definition is not so easily stated, but in typical examples, say the homotopy category of spectra or the derived category of an associative ring, the condition of \( \alpha \)-compactness is very natural; see Section 4. In particular, the \( \aleph_0 \)-compact objects are precisely the compact objects. Our main theorem says, among other things, that the condition of \( \alpha \)-compactness is local: given a cocovering \( F \) of \( T \) as above, satisfying some natural hypotheses on \( F \), compactness in \( T \) is local: an object \( Y \) is compact in \( T \) if and only if the image of \( Y \) is compact in \( T/I_i \) for \( 1 \leq i \leq n \).

In order to give the precise statements, we need some notation: recall that a localising subcategory \( S \) of \( T \) is a triangulated subcategory closed under small coproducts, and \( S \) is Bousfield if the inclusion \( S \rightarrow T \) has a right adjoint. Given a class \( S \) of objects in \( T \), we write \( \langle S \rangle \) for the smallest localising subcategory of \( T \) containing \( S \). Let \( \alpha \) be a regular cardinal. If \( T^{\alpha} \) is essentially small and \( \langle T^{\alpha} \rangle = T \), then \( T \) is said to be \( \alpha \)-compactly generated, and \( T \) is called well-generated if it is \( \alpha \)-compactly generated for some regular cardinal \( \alpha \). If \( T \) is \( \alpha \)-compactly generated, then a localising subcategory \( S \subseteq T \) is \( \alpha \)-compactly generated in \( T \) if there is a set \( S \subseteq T^{\alpha} \) such that \( S = \langle S \rangle \). In this case \( S \) is \( \alpha \)-compactly generated, and \( S^{\alpha} = S \cap T^{\alpha} \) (see Theorem 5).

Two Bousfield subcategories \( I_1, I_2 \) of \( T \) are said to intersect properly if, for every pair \( I \in I_1 \) and \( J \in I_2 \), any morphism \( I \rightarrow J \) or \( J \rightarrow I \) factors through an object of \( I_1 \cap I_2 \) [Rou08, (5.2.3)].
Therefore, a cocovering of $T$ is a finite family of Bousfield subcategories $\mathcal{F} = \{I_1, \ldots, I_n\}$ of $T$ which are pairwise properly intersecting, such that $\bigcap_{i=1}^n I_i = 0$; see [Rou08, (5.3.3)]. The $\alpha = \aleph_0$ case of the following theorem is the aforementioned result of Rouquier, namely [Rou08, Theorem 5.15].

**Theorem 1.** Let $T$ be a triangulated category with coproducts and $\alpha$ a regular cardinal. Suppose that $\mathcal{F} = \{I_1, \ldots, I_n\}$ is a cocovering of $T$ with the following properties:

1. $T/I$ is $\alpha$-compactly generated for every $I \in \mathcal{F}$.
2. For every $I \in \mathcal{F}$ and nonempty subset $\mathcal{F}' \subseteq \mathcal{F} \setminus \{I\}$ the essential image of the composite

   \[ \bigcap_{I' \in \mathcal{F}'} I' \xrightarrow{\text{inc}} T \xrightarrow{\text{can}} T/I \]

   is $\alpha$-compactly generated in $T/I$.

Then $T$ is $\alpha$-compactly generated, and an object $X \in T$ is $\alpha$-compact if and only if the image of $X$ is $\alpha$-compact in $T/I$ for every $I \in \mathcal{F}$. Let $S$ be a Bousfield subcategory of $T$ intersecting properly with each $I \in \mathcal{F}$, such that:

3. $S/(S \cap I)$ is $\alpha$-compactly generated in $T/I$ for every $I \in \mathcal{F}$.
4. For every $I \in \mathcal{F}$ and nonempty subset $\mathcal{F}' \subseteq \mathcal{F} \setminus \{I\}$ the essential image of the composite

   \[ S \cap \bigcap_{I' \in \mathcal{F}'} I' \xrightarrow{\text{inc}} T \xrightarrow{\text{can}} T/I \]

   is $\alpha$-compactly generated in $T/I$.

Then $S$ is $\alpha$-compactly generated in $T$.

To return to the geometric example: if $T = \text{D}(X)$ and we are given an open cover as above, then for each $1 \leq i \leq n$ denote by $I_i = \text{D}_{X \setminus U_i}(X)$ the full subcategory of $\text{D}(X)$ consisting of complexes with cohomology supported on $X \setminus U_i$. There is a canonical equivalence $T/I_i \cong \text{D}(U_i)$, the quotient functor $T \rightarrow T/I_i$ corresponds to restriction, and the family $\mathcal{F} = \{I_1, \ldots, I_n\}$ is a cocovering of $\text{D}(X)$ satisfying the hypotheses (1), (2) of the theorem for $\alpha = \aleph_0$ [Rou08, §6.2]. For this choice of $T$ and $\mathcal{F}$ the hypotheses are very natural, and easily verified; for the full elaboration, see Section 5.

Applying the theorem (recall that, since $\alpha = \aleph_0$, this is just Rouquier’s [Rou08, Theorem 5.15]) one obtains a proof of the fact, due originally to Neeman [Nee96], that the compact objects in $\text{D}(X)$ are precisely the perfect complexes; see [Rou08, Theorem 6.8]. Using the $\alpha > \aleph_0$ case of the theorem we obtain in Section 5 a description of the $\alpha$-compact objects in $\text{D}(X)$.

We have another application in mind, which will appear in the forthcoming [Mur08]. Let $A$ be an associative ring with identity, $K(\text{Proj} A)$ and $K(\text{Flat} A)$ the homotopy categories of projective and flat left $A$-modules, respectively. A complex of left $A$-modules $F$ is pure acyclic if it is acyclic, and $N \otimes_A F$ is acyclic for every right $A$-module $N$. Let $K_{\text{pac}}(\text{Flat} A)$ denote the full subcategory of pure acyclic complexes in $K(\text{Flat} A)$. This is a triangulated subcategory, and Neeman proves in [Nee08] that the composite

\[ K(\text{Proj} A) \xrightarrow{\text{inc}} K(\text{Flat} A) \xrightarrow{\text{can}} K(\text{Flat} A)/K_{\text{pac}}(\text{Flat} A) \tag{1} \]

is an equivalence. Now let $X$ be a quasi-compact semi-separated scheme. Unless $X$ is affine, projective quasi-coherent sheaves on $X$ are rare, and the homotopy category of projective quasi-coherent sheaves on $X$ is often the zero category. In this case, the equivalence (1) suggests a suitable replacement. Let $K(\text{Flat} X)$ be the homotopy category of flat quasi-coherent sheaves on $X$, and denote by $K_{\text{pac}}(\text{Flat} X)$ the full subcategory of acyclic complexes $\mathcal{F}$ with the property that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}$ is acyclic for every quasi-coherent sheaf $\mathcal{A}$. Define

\[ N(\text{Flat} X) := K(\text{Flat} X)/K_{\text{pac}}(\text{Flat} X), \]
and let \( \{U_1, \ldots, U_n\} \) be an affine open cover of \( X \), with say \( U_i \cong \text{Spec}(A_i) \) for \( 1 \leq i \leq n \). We show in [Mur08] that there is a cocovering of \( N(\text{Flat } X) \) by Bousfield subcategories \( \{N_{X \setminus U_i}(\text{Flat } X)\}_{1 \leq i \leq n} \), where \( N_{X \setminus U_i}(\text{Flat } X) \) is the kernel of a natural restriction functor \( N(\text{Flat } X) \longrightarrow N(\text{Flat } U_i) \). Moreover, there are canonical equivalences

\[
N(\text{Flat } X)/N_{X \setminus U_i}(\text{Flat } X) \cong N(\text{Flat } U_i) \cong K(\text{Proj } A_i).
\]

Neeman proves in \( \text{loc. cit.} \) that \( K(\text{Proj } A_i) \) is \( \aleph_1 \)-compactly generated, and even compactly generated when \( A_i \) is coherent. In [Mur08] we combine Neeman’s results with Theorem 1 to see that the global category \( N(\text{Flat } X) \) is \( \aleph_1 \)-compactly generated, and compactly generated when \( X \) is noetherian.

The proof of Theorem 1 is by induction on the size \( n = |F| \) of the cocovering. The real content is in the initial step of the induction, which we separate into its own section. The proof of the theorem is completed in Section 3. Our basic reference for triangulated categories is [Nee01], whose notation we follow with one exception: given a class \( \mathcal{C} \) of objects in \( T \), we write

\[
\mathcal{C}^\perp = \{Y \in T \mid \text{Hom}_T(\Sigma^n X, Y) = 0 \text{ for all } X \in \mathcal{C} \text{ and } n \in \mathbb{Z}\},
\]

\[
^\perp \mathcal{C} = \{X \in T \mid \text{Hom}_T(X, \Sigma^n Y) = 0 \text{ for all } Y \in \mathcal{C} \text{ and } n \in \mathbb{Z}\}
\]

for the orthogonals, which are triangulated subcategories of \( T \). For further information on the theory of well-generated triangulated categories, the reader is referred to [Nee05, Kra07]. In this article, all triangulated categories have “small Homs”.

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2. Initial step of the induction

Throughout, \( T \) is a triangulated category with coproducts. Two Bousfield subcategories \( \mathcal{I}_1, \mathcal{I}_2 \) of \( T \) are orthogonal if \( \mathcal{I}_1 \subseteq \mathcal{I}_2^\perp \) and \( \mathcal{I}_2 \subseteq \mathcal{I}_1^\perp \). In this situation the composite \( \mathcal{I}_a \longrightarrow T \longrightarrow T/\mathcal{I}_b \) is fully faithful for \( \{a, b\} = \{1, 2\} \) and \( \mathcal{I}_a \) may be identified with a Bousfield subcategory of \( T/\mathcal{I}_b \). Let us state the \( n = 2 \) case of the Theorem 1 as a proposition:

**Proposition 2.** Let \( \mathcal{I}_1, \mathcal{I}_2 \) be orthogonal Bousfield subcategories of \( T \), and suppose that for some regular cardinal \( \alpha \), we have:

1. \( T/\mathcal{I}_a \) is \( \alpha \)-compactly generated for \( a \in \{1, 2\} \),
2. \( \mathcal{I}_a \) is \( \alpha \)-compactly generated in \( T/\mathcal{I}_b \) for \( \{a, b\} = \{1, 2\} \).

Then \( T \) is \( \alpha \)-compactly generated, and an object \( X \in T \) is \( \alpha \)-compact if and only if the image of \( X \) is \( \alpha \)-compact in both \( T/\mathcal{I}_1 \) and \( T/\mathcal{I}_2 \). Let \( S \) be a Bousfield subcategory of \( T \) intersecting properly with \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \), and suppose that:

1. \( S/(S \cap \mathcal{I}_a) \) is \( \alpha \)-compactly generated in \( T/\mathcal{I}_a \) for \( a \in \{1, 2\} \),
2. \( S \cap \mathcal{I}_a \) is \( \alpha \)-compactly generated in \( T/\mathcal{I}_b \) for \( \{a, b\} = \{1, 2\} \).

Then \( S \) is \( \alpha \)-compactly generated in \( T \).

We develop the proof as a series of lemmas. Since the \( \alpha = \aleph_0 \) is handled by [Rou08, Proposition 5.14], we assume that \( \alpha > \aleph_0 \). Throughout this section the notation of the proposition is in force. For \( a \in \{1, 2\} \) we write \( i_{a*} : \mathcal{I}_a \longrightarrow T \) for the inclusion, \( j^*_a \) for the right adjoint of \( i_{a*} \), \( j^*_a : T \longrightarrow T/\mathcal{I}_a \) for the quotient functor and \( j_{a*} \) for the right adjoint of \( j^*_a \).
To prove that $\mathcal{T}$ is $\alpha$-compactly generated we need to produce, in the language of [Nee01, Ch.8], an $\alpha$-perfect set of $\alpha$-small objects, which generates. The condition of $\alpha$-smallness is very simple: an object $X \in \mathcal{T}$ is $\alpha$-\textit{small} if for every family $\{Y_i\}_{i \in I}$ of objects of $\mathcal{T}$, any morphism

$$X \to \bigoplus_{i \in I} Y_i$$

factors through a subcoproduct $\bigoplus_{i \in I} Y_i$ for some subset $J \subseteq I$ of cardinality $|J| < \alpha$. An object $X$ is $\aleph_0$-small if and only if $T(X, -)$ commutes with coproducts, and in this case one says that $X$ is \textit{compact}. We refer the reader to [Nee01, Ch.3] for the definition of $\alpha$-perfect classes, and restrict ourselves here to one trivial fact: any triangulated subcategory of $\mathcal{T}$ is an $\aleph_0$-perfect class.

By hypothesis the quotients $\mathcal{T}/\mathcal{I}_a$ are $\alpha$-compactly generated, and $\mathcal{I}_b$ is $\alpha$-compactly generated in $\mathcal{T}/\mathcal{I}_a$ for $\{a, b\} = \{1, 2\}$. Hence these categories all possess $\alpha$-perfect classes of $\alpha$-small objects which generate. The strategy employed by Rouquier [Rou08] in the $\alpha = \aleph_0$ case is to take generating sets $\mathcal{E}$ and $\mathcal{E}'$ for $\mathcal{I}_2$, $\mathcal{T}/\mathcal{I}_2$ respectively, use a gluing argument to lift $\mathcal{E}'$ to class $\mathcal{E}''$ of compact objects in $\mathcal{T}$, and take the union $\mathcal{E} \cup \mathcal{E}''$. This is a generating set of compact objects for $\mathcal{T}$.

In the $\alpha > \aleph_0$ case we take a different approach, in which it seems easier to manage the perfection condition (which is trivial for $\alpha = \aleph_0$). To proceed, we first recall how to rephrase the condition on our generating set in terms of a property of a certain exact functor between abelian categories.

A triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ is said to be $\alpha$-\textit{localising} if the coproduct of fewer than $\alpha$ objects of $\mathcal{S}$ lies in $\mathcal{S}$. For example, the class $\mathcal{T}^\alpha$ of $\alpha$-compact objects is an $\alpha$-localising triangulated subcategory of $\mathcal{T}$. Given an $\alpha$-localising subcategory $\mathcal{S}$ of $\mathcal{T}$ we denote by $\text{Add}_\alpha(\mathcal{S}^{\text{op}}, \mathcal{A}b)$ the abelian category of all functors $\mathcal{S}^{\text{op}} \to \mathcal{A}b$ which preserve products of fewer than $\alpha$ objects, where $\mathcal{A}b$ is the category of abelian groups. There is a canonical homological functor

$$\mathcal{T} \to \text{Add}_\alpha(\mathcal{S}^{\text{op}}, \mathcal{A}b), \quad X \mapsto \mathcal{T}(-, X)|_\mathcal{S}. $$

Let $\mathcal{T} \to A(\mathcal{T})$ be Freyd’s universal homological functor, where $A(\mathcal{T})$ is the abelianisation of $\mathcal{T}$ [Nee01, Ch. 5]. From the universal property of this construction, we deduce an exact functor

$$\pi : A(\mathcal{T}) \to \text{Add}_\alpha(\mathcal{S}^{\text{op}}, \mathcal{A}b),$$

and Neeman proves in [Nee01, Theorem 1.8] that $\mathcal{S}$ is an $\alpha$-perfect class of $\alpha$-small objects precisely when $\pi$ preserves coproducts. Here, then, is the strategy of our proof: in Definition 3 below we take the obvious candidate for a generating set $\mathcal{S} = \mathcal{T}^{|\alpha|}$ of $\mathcal{T}$. We have to prove two things: firstly, that this is an $\alpha$-perfect class of $\alpha$-small objects, and secondly, that it generates. The second condition is easily verified, and for the first we just need to prove that $\pi$ preserves coproducts. The cocovering $\{\mathcal{I}_1, \mathcal{I}_2\}$ of $\mathcal{T}$ leads to a pair of localisations of $\text{Add}_\alpha(\mathcal{S}^{\text{op}}, \mathcal{A}b)$, which we may think of as a “cover” of this abelian category. Checking that $\pi$ preserves coproducts then becomes a “local” problem with respect to this cover. The local pieces in the cover correspond to the quotients $\mathcal{T}/\mathcal{I}_a$, and we can use the fact that these categories are $\alpha$-compactly generated to complete the proof.

**DEFINITION 3.** We define a full subcategory of $\mathcal{T}$ by

$$\mathcal{T}^{|\alpha|} = \{X \in \mathcal{T} \mid j_a^*(X) \in (\mathcal{T}/\mathcal{I}_a)^\alpha \text{ for } a \in \{1, 2\}\}.$$ 

**LEMMA 4.** $\mathcal{T}^{|\alpha|}$ is an $\alpha$-localising subcategory of $\mathcal{T}$.

**Proof.** Follows from the fact that $j_a^*$ preserve coproducts, and $(\mathcal{T}/\mathcal{I}_a)\alpha$ is $\alpha$-localising.

Let us recall the statement of the Neeman-Ravenel-Thomason localisation theorem.

**THEOREM 5.** Let $\mathcal{R}$ be a triangulated category with coproducts which is $\alpha$-compactly generated, and let $\mathcal{S} \subseteq \mathcal{R}$ be a localising subcategory $\alpha$-compactly generated in $\mathcal{R}$. Then $\mathcal{S}$ is $\alpha$-compactly
generated, and $S^\alpha = R^\alpha \cap S$. The canonical functor $R \to R/S$ preserves $\alpha$-compactness and the induced functor

$$R^\alpha / S^\alpha \to (R/S)^\alpha$$

is an equivalence (recall that $\alpha > \aleph_0$).

**Proof.** See [Nee01, Theorem 4.4.9].

The full subcategory of $\alpha$-compact objects in $I_a$ is denoted by $I_a^\alpha$. One needs to be careful to distinguish between objects $X \in I_a$ which are $\alpha$-compact in $I_a$, and those that are $\alpha$-compact in the larger category $T$. At this point, we do not know that these classes are the same. It follows from hypotheses (1) and (2) of Proposition 2, and Theorem 5, that $I_a^\alpha \subseteq I_a$ is precisely the class of objects $X \in I_a$ with the property that $j_b^\alpha(X) \in (T/I_b)^\alpha$, where $\{a, b\} = \{1, 2\}$. Moreover, $I_a = \langle I_a^\alpha \rangle$.

**Lemma 6.** There is an inclusion $I_a^\alpha \cup I_2^\alpha \subseteq T^{[\alpha]}$.

**Proof.** If $X \in I_a^\alpha$ then by (2), $j_b^\alpha(X) \in (T/I_b)^\alpha$. Since $j_b^\alpha(X) = 0$, it follows that $X \in T^{[\alpha]}$.

**Lemma 7.** Given $X \in T^{[\alpha]}$ and $Y \in I_a$ for $a \in \{1, 2\}$, any morphism $f : X \to Y$ in $T$ factors as

$$X \to I \to Y$$

for some $I \in I_a^\alpha$.

**Proof.** Let $b \in \{1, 2\}$ be such that $b \neq a$. By hypothesis (2) there is a set $Q \subseteq (T/I_b)^\alpha \cap I_a$ such that $\langle Q \rangle = I_a$. By [Nee01, Theorem 4.3.3] the morphism $j_b^\alpha f : j_b^\alpha X \to j_b^\alpha Y$ factors in $T/I_b$ as

$$j_b^\alpha X \to N \to j_b^\alpha Y$$

for some $N \in \langle Q \rangle^\alpha = I_a^\alpha$. Since $I := j_{bs}N$ belongs to $I_a^\alpha$, the composite

$$X \xrightarrow{\text{can}} j_{bs}j_b^\alpha X \to j_{bs}N \to j_{bs}j_b^\alpha Y \cong Y$$

provides the desired factorisation of $f$.

We use several facts about proper intersection of subcategories developed by Rouquier [Rou08, §5]. For the reader’s convenience, the necessary facts are recalled here in Appendix B. For example, since $I_1, I_2$ are properly intersecting the Verdier sum operation is commutative: $I_1 \star I_2 = I_2 \star I_1$. It follows that $I_1 \star I_2$ is a Bousfield subcategory of $T$ and, following the notation of [Rou08, §5], we write $i_{1,2} : I_1 \star I_2 \to T$ for the inclusion, $i_{1,2}^\alpha$ for its right adjoint, $j_{1,2}^\alpha : T \to T/(I_1 \star I_2)$ for the quotient, and $j_{1,2,\alpha}$ for its right adjoint. Note that in loc.cit. Rouquier writes $\langle I_1 \cup I_2 \rangle_\infty$ for $I_1 \star I_2$, to reflect the fact that this is the smallest triangulated subcategory of $T$ containing $I_1 \cup I_2$. For $\{a, b\} = \{1, 2\}$ the quotient $j_{a,\alpha}^\star$ induces a functor $j_{a,\alpha}^\star : T/I_a \to T/(I_1 \star I_2)$ fitting into a sequence

$$0 \to I_b \to T/I_a \to T/(I_1 \star I_2) \to 0$$

which is *exact*, in the sense that $I_b \to T/I_a$ is fully faithful and $j_{a,\alpha}^\star$ is, up to natural equivalence, the Verdier quotient of $T/I_a$ by $I_b$. We write $j_{a,\alpha}^\star$ for the right adjoint of $j_{a,\alpha}^\star$.

**Lemma 8.** Given $a \in \{1, 2\}$ and $Y \in (T/I_a)^\alpha$, there is $X \in T^{[\alpha]}$ such that $j_a^\alpha(X) \cong Y$.

**Proof.** We use the argument given in the proof of [Rou08, Proposition 5.14]. Let $b \in \{1, 2\}$ be such that $\{a, b\} = \{1, 2\}$. From hypotheses (1), (2) and Theorem 5 we deduce that the quotient functor $j_{b,\alpha}^\star$ preserves $\alpha$-compactness. Hence, if we set $D_a = Y$, then the object $D_{a,\alpha} := j_{a,\alpha}^\star D_a$ is $\alpha$-compact in $T/(I_1 \star I_2)$. Also by Theorem 5, the canonical functor

$$j_{b,\alpha}^\star : (T/I_b)^\alpha \to \left(T/(I_1 \star I_2)\right)^\alpha$$

is an equivalence.
is a Verdier quotient, so we can find $D_b \in (T/I_b)^{\alpha}$ and an isomorphism $j_b^* D_b \cong D_\cup$. There are unit morphisms $\eta_1 : D_1 \rightarrow j_1 \cup D_\cup$, $\eta_2 : D_2 \rightarrow j_2 \cup D_\cup$ and we define $\delta$ to be the morphism induced out of the coproduct $j_1 \cup D_1 \oplus j_2 \cup D_2$ by $j_1(\eta_1) - j_2(\eta_2)$. If we define $X$ by extending $\delta$ to a triangle $$X \rightarrow j_1 \cup D_1 \oplus j_2 \cup D_2 \xrightarrow{\delta} j_\cup \cup D_\cup \rightarrow +,$$
then one checks that $j_b^* X \cong D_a = Y$, and that $j_b^* X \cong D_b$, so $X \in T^{\alpha}$ as required.

**Lemma 9.** $T^{\alpha}$ is essentially small.

**Proof.** For $X \in T^{\alpha}$ there is a canonical triangle [Rou08, Proposition 5.10]

$$X \rightarrow j_1 \cup j_1^* X \oplus j_2 \cup j_2^* X \rightarrow j_\cup \cup j_\cup^* X \rightarrow +.$$  \hfill (2)

By hypothesis $j_a^* X \in (T/I_a)^{\alpha}$ for $a \in \{1, 2\}$. Now, since $T/I_a$ is $\alpha$-compactly generated, $(T/I_a)^{\alpha}$ is essentially small. It follows that there is, up to isomorphism, only a “set” of possible objects $X$ in a triangle of the form (2), whence $T^{\alpha}$ is essentially small.

**Lemma 10.** For $a \in \{1, 2\}$ the canonical functor

$$j_a^* : T^{\alpha}/(T^{\alpha} \cap I_a) \rightarrow (T/I_a)^{\alpha} \quad \hfill (3)$$

is an equivalence.

**Proof.** The composite $T^{\alpha} \xrightarrow{\text{inc}} T \xrightarrow{j_a^*} T/I_a$ factors, by definition, through the inclusion $(T/I_a)^{\alpha} \rightarrow T/I_a$. The factorisation $T^{\alpha} \rightarrow (T/I_a)^{\alpha}$ vanishes on $T^{\alpha} \cap I_a$, and induces a functor (3). To verify that (3) is fully faithful, we use a standard argument. Let $s : X \rightarrow Y$ be a morphism in $T$ with cone in $I_a$ and $Y \in T^{\alpha}$. Extend to a triangle

$$X \xrightarrow{s} Y \xrightarrow{f} I \xrightarrow{+}.$$ \hfill (4)

The map $f$ factors, by Lemma 7, as $Y \rightarrow I' \rightarrow I$ with $I' \in I_a^\alpha$. From the octahedral axiom, applied to the pair of morphisms in this factorisation, we obtain objects $C, D$ and triangles

$$Y \rightarrow I' \rightarrow C \xrightarrow{+}, \quad \hfill (5)$$

$$I' \rightarrow I \rightarrow D \xrightarrow{+}, \quad \hfill (6)$$

Since $I' \in I_a^\alpha \subseteq T^{\alpha}$ we find that $C$ belongs to $T^{\alpha}$ and $D$ to $I_a$. Thus $\Sigma^{-1} C \rightarrow X$ is a morphism with domain in $T^{\alpha}$, the composite of which with $s : X \rightarrow Y$ has cone in $I_a$. It now follows easily that (3) is fully faithful. To see that it is surjective on objects, we use Lemma 8.

Fix an index $a \in \{1, 2\}$. By Lemma 10 the canonical functor $j_a^* : T^{\alpha} \rightarrow (T/I_a)^{\alpha}$ is a Verdier quotient which preserves $\alpha$-coproducts. By [Kra07, Lemma B.8] the (exact) restriction functor

$$q_a^* : \text{Add}_a((T/I_a)^{\alpha})^{\text{op}}, Ab) \rightarrow \text{Add}_a((T^{\alpha})^{\text{op}}, Ab),$$

$$q_a^*(F) = F \circ j_a^*$$

has an exact left adjoint $q_a*$. The right adjoint $q_a^*$ is fully faithful, so $q_a*$ is a Gabriel localisation of
As we will see, it is reasonable to think of the pair of localisations
\[ \text{Add}_\alpha(\{T^{[\alpha]}\}^{\text{op}}, Ab) \]

as a covering of \( \text{Add}_\alpha(\{T^{[\alpha]}\}^{\text{op}}, Ab) \). To make this precise, we show that a functor \( F \) which is sent to zero by both localisations, must already be zero.

**Lemma 11.** \( \text{Ker}(q_{1*}) \cap \text{Ker}(q_{2*}) = 0 \).

**Proof.** Fix \( a \in \{1, 2\} \). By [Kra07, Lemma B.8] a functor \( F \in \text{Add}_\alpha(\{T^{[\alpha]}\}^{\text{op}}, Ab) \) belongs to \( \text{Ker}(q_{2*}) \) if and only if for any \( C \in T^{[\alpha]} \), every morphism \( T^{[\alpha]}(-, C) \rightarrow F \) factors via \( T^{[\alpha]}(-, \gamma) : T^{[\alpha]}(-, C) \rightarrow T^{[\alpha]}(-, C') \) for some morphism \( \gamma : C \rightarrow C' \) in \( T^{[\alpha]} \) with \( j_a^\gamma \) = 0 in \( T/I_a \). From Lemma 10 we deduce that \( j_a^\gamma \) = 0 if and only if \( \gamma \) factors, in \( T^{[\alpha]} \), via an object of \( T^{[\alpha]} \cap I_a \). We conclude that \( F \) belongs to \( \text{Ker}(q_{2*}) \) if and only if every morphism \( T^{[\alpha]}(-, C) \rightarrow F \) factors via \( T^{[\alpha]}(-, I) \) for some \( I \in T^{[\alpha]} \cap I_a \).

Assume now that \( F \) belongs to \( \text{Ker}(q_{1*}) \cap \text{Ker}(q_{2*}) \) and let \( x : T^{[\alpha]}(-, C) \rightarrow F \) be any morphism. By the above, this must factor as \( T^{[\alpha]}(-, C) \rightarrow T^{[\alpha]}(-, I) \rightarrow F \) for some \( I \in T^{[\alpha]} \cap I_1 \). Since \( F \) also belongs to \( \text{Ker}(q_{2*}) \), the morphism \( T^{[\alpha]}(-, I) \rightarrow F \) factors as \( T^{[\alpha]}(-, I) \rightarrow T^{[\alpha]}(-, I') \rightarrow F \) for some \( I' \in T^{[\alpha]} \cap I_2 \). But since \( I_1 \) and \( I_2 \) are orthogonal, the morphism \( T^{[\alpha]}(-, I) \rightarrow T^{[\alpha]}(-, I') \) vanishes, and we conclude that \( x = 0 \). It follows that \( F = 0 \), as claimed.

**Proposition 12.** \( T^{[\alpha]} \) is an \( \alpha \)-perfect class of \( \alpha \)-small objects in \( T \).

**Proof.** We have \( \alpha \)-localising subcategories \( T^{[\alpha]} \subseteq T \), \( (T/I_1)^{[\alpha]} \subseteq T/I_1 \) and \( (T/I_2)^{[\alpha]} \subseteq T/I_2 \) and, as discussed at the beginning of this section, there are canonical exact functors
\[
\pi : A(T) \rightarrow \text{Add}_\alpha(\{T^{[\alpha]}\}^{\text{op}}, Ab), \\
\pi_1 : A(T/I_1) \rightarrow \text{Add}_\alpha(\{(T/I_1)^{[\alpha]}\}^{\text{op}}, Ab), \\
\pi_2 : A(T/I_2) \rightarrow \text{Add}_\alpha(\{(T/I_2)^{[\alpha]}\}^{\text{op}}, Ab).
\]

We claim that for \( a \in \{1, 2\} \) the diagram
\[
\begin{array}{ccc}
A(T) & \xrightarrow{A(j_a^\ast)} & A(T/I_0) \\
\pi \downarrow & & \pi_a \downarrow \\
\text{Add}_\alpha(\{T^{[\alpha]}\}^{\text{op}}, Ab) & \xrightarrow{q_{a*}} & \text{Add}_\alpha(\{(T/I_0)^{[\alpha]}\}^{\text{op}}, Ab)
\end{array}
\]
commutes up to natural equivalence, where \( A(j_a^\ast) \) is the induced functor between the abelianisations. By the universal property of the abelianisations, it suffices to prove that the related diagram
\[
\begin{array}{ccc}
T & \xrightarrow{j_a^\ast} & T/I_0 \\
\rho \downarrow & & \rho_a \downarrow \\
\text{Add}_\alpha(\{T^{[\alpha]}\}^{\text{op}}, Ab) & \xrightarrow{q_{a*}} & \text{Add}_\alpha(\{(T/I_0)^{[\alpha]}\}^{\text{op}}, Ab)
\end{array}
\]
commutes, where \( \rho \) and \( \rho_a \) are the restricted Yoneda functors. To do this, we recycle an argument of Krause from the proof of [Kra07, Theorem 6.3]. The first thing to observe is that the composite
Here is where we use commutativity of (7). Both $q_{as} \circ \rho$ vanishes on $I_a$; one uses the description in [Kra07, Lemma B.8] of the kernel of $q_{as}$, together with Lemma 7. For $C \in T^{[a]}$ and $X \in T/I_a$ there is an adjunction isomorphism
\[ T/I_a(j_a^* C, X) \cong T(C, j_a^* X), \]
and it follows that there is a natural equivalence $q_{as}^* \circ \rho_a \cong \rho \circ j_{as}$. Composing with $q_{as}$ we obtain a natural equivalence $\rho_a \cong q_{as} \circ q_{as}^* \circ \rho_a \cong q_{as} \circ \rho \circ j_{as}$ and consequently $\rho_a \circ j_{as}^* \cong q_{as} \circ \rho \circ j_{as} \circ j_{as}'$. From the unit $\eta: 1 \to j_{as} \circ j_{as}'$ we obtain a natural transformation
\[ q_{as} \circ \rho \xrightarrow{(q_{as} \circ \rho) \eta} q_{as} \circ \rho \circ j_{as} \circ j_{as}' \cong \rho_a \circ j_{as}'. \]
This is the desired natural equivalence, because for every $X \in T$ the cone of $\eta_X: X \to j_{as}j_{as}'(X)$ is an object of $I_a$, on which $q_{as} \circ \rho$ vanishes.

Since $(T/I_a)^{\alpha}$ is an $\alpha$-perfect class of $\alpha$-small objects, we infer from [Nee01, Theorem 1.8] that $\pi_{\alpha}$ preserves coproducts. Let $\{x_\lambda\}_\lambda$ be a family of objects in $A(T)$, and let $\xi: \bigoplus x_\lambda \to \pi(\bigoplus x_\lambda)$ be the canonical morphism in $\text{Add}_{\alpha}(\{T^{[\alpha]}\}^{\text{op}}, \text{Ab})$. Extend on both sides to an exact sequence
\[ 0 \to \text{Ker}(\xi) \to \bigoplus x_\lambda \xrightarrow{\xi} \pi\left(\bigoplus x_\lambda\right) \to \text{Coker}(\xi) \to 0, \]
which maps under $q_{as}$ to an exact sequence
\[ 0 \to q_{as} \text{Ker}(\xi) \to q_{as} \bigoplus x_\lambda \xrightarrow{q_{as}\pi(\xi)} q_{as}\pi\left(\bigoplus x_\lambda\right) \to q_{as} \text{Coker}(\xi) \to 0. \]
Here is where we use commutativity of (7). Both $\pi_{\alpha}$ and $\pi_A(j_{as})$ preserve coproducts, whence $q_{as} \circ \pi \cong \pi \circ A(j_{as})$ preserves coproducts. Since $q_{as}$ preserves coproducts (it has a right adjoint), we conclude that $q_{as}(\xi)$ (is an isomorphism, and thus $q_{as} \text{Ker}(\xi)$ and $q_{as} \text{Coker}(\xi)$ both vanish. Since $a \in \{1, 2\}$ was arbitrary, it follows from Lemma 11 that $\text{Ker}(\xi) = \text{Coker}(\xi) = 0$, whence $\xi$ is an isomorphism and $\pi$ preserves coproducts. By [Nee01, Theorem 1.8], $T^{[\alpha]}$ is an $\alpha$-perfect class of $\alpha$-small objects. \[ \square \]

**Proof of Proposition 2.** First we prove that $T^{[\alpha]}$ is an $\alpha$-compact generating set$^1$ for $T$, in the sense of [Nee01, Definition 8.16]. In light of Proposition 12, it suffices to prove that if an object $x \in T$ satisfies $T(y, x) = 0$ for all $y \in T^{[\alpha]}$ then $x = 0$. Note that $I_1^{\alpha} \subseteq T^{[\alpha]}$, so $x \in (T^{[\alpha]})^\perp \subseteq (I_1^{\alpha})^\perp = I_1^\perp$, since $(I_1^{\alpha}) = I_1$. Let $t \in (T/I_1)^{[\alpha]}$ be given, and choose by Lemma 8 a $t' \in T^{[\alpha]}$ with $j_{as}(t') \cong t$. Then
\[ 0 = T(t', x) \cong T/I_1(j_{as}(t'), j_{as}(x)) \cong T/I_1(t, j_{as}(x)). \]
But $T/I_1$ is $\alpha$-compactly generated and $t$ was arbitrary, so $j_{as}(x) = 0$. Hence $x$ belongs to both $I_1$ and $I_1^\perp$, which is only possible if $x = 0$. It now follows from [Nee01, Proposition 8.4.2] that $T = (T^{[\alpha]})^\perp$, and from [Nee01, Theorem 4.4.9] that $T^{[\alpha]}$ is the smallest $\alpha$-localising subcategory of $T$ containing $T^{[\alpha]}$. Hence $T^{[\alpha]} = T^{[\alpha]}$, which settles the first statement of the theorem.

The second statement of the theorem deals with a Bousfield subcategory $S$. The intersections $S \cap I_1, S \cap I_2$ are orthogonal Bousfield subcategories of $S$. We want to apply the first part of the theorem to $S$ and this pair of subcategories. Condition (1) is certainly satisfied, since by hypothesis (3) the quotients $S/(S \cap I_1)$ are $\alpha$-compactly generated. For condition (2) we must show that $S \cap I_0$ is $\alpha$-compactly generated in $S/(S \cap I_0)$ for $\{a, b\} = \{1, 2\}$. It follows from hypothesis (4) that
\[ (S \cap I_0)^{[\alpha]} = (S \cap I_0) \cap (T/I_b)^{[\alpha]}, \]
and from hypothesis (3) that
\[ \left(S/(S \cap I_0)\right)^{[\alpha]} = \left(S/(S \cap I_0)\right) \cap (T/I_b)^{[\alpha]}, \]
$\text{Strictly speaking } T^{[\alpha]}$ is an essentially small class, not a set, but let us replace $T^{[\alpha]}$ by a representative set of objects and ignore the distinction.
This implies that the inclusion \( S \cap I_a \rightarrow S/(S \cap I_a) \) preserves \( \alpha \)-compactness, from which we deduce that the former category is \( \alpha \)-compactly generated in the latter. Now, using the first part of the theorem, we conclude that \( S \) is \( \alpha \)-compactly generated and that an object \( X \in S \) is \( \alpha \)-compact in \( S \) if and only if the image under \( S \longrightarrow S/(S \cap I_a) \) is \( \alpha \)-compact for each \( a \in \{1, 2\} \). By (10) these are precisely the \( X \in S \) that are \( \alpha \)-compact in \( T \), so \( S \) is \( \alpha \)-compactly generated in \( T \).

\[ \square \]

3. Proof of the Theorem

Let us briefly recall the setup of Theorem 1. We are given a triangulated category \( T \) with coproducts, a regular cardinal \( \alpha \), and a cocovering \( F = \{I_1, \ldots, I_n\} \) satisfying some conditions (1), (2), and we wish to prove that \( T \) is \( \alpha \)-compactly generated. Once again, since the \( \alpha = \aleph_0 \) case is handled by [Rou08, Theorem 5.15], we restrict to the case \( \alpha > \aleph_0 \). In what follows we make implicit use of the properties of proper intersection described in Appendix B, particularly Lemma 38.

**Proof of Theorem 1.** The proof is by induction on the number \( n \geq 2 \) of elements in the cocover \( F \) (to be clear, the induction includes the second statement of the theorem, about \( S \)). The \( n = 2 \) case is given by Proposition 2, and for the inductive step the argument is identical to the inductive step in the proof of [Rou08, Theorem 5.15]. For the reader’s convenience, let us repeat the argument here. Assume that \( n > 2 \) and set

\[ I_\cap = I_2 \cap \cdots \cap I_n. \]

Then \( \{I_1, I_\cap\} \) is an orthogonal pair of Bousfield subcategories of \( T \). By hypothesis \( T/I_1 \) is \( \alpha \)-compactly generated, and \( I_\cap \) is \( \alpha \)-compactly generated in \( T/I_1 \), so in order to apply the \( n = 2 \) case of the Theorem to the pair \( I_1, I_\cap \) it remains to check that

(i) \( T/I_\cap \) is \( \alpha \)-compactly generated, and

(ii) \( I_1 \) is \( \alpha \)-compactly generated in \( T/I_\cap \).

Set \( T = T/I_\cap \) and for \( I \in F \) define \( I = I/(I \cap I_\cap) \). This is a Bousfield subcategory of \( T \), and \( \{I_2, \ldots, I_n\} \) is a cocovering of \( T \) (Lemma 38). Moreover:

- For \( I \in F \setminus \{I_1\} \) the category \( T/I \cong T/I_\cap \) is \( \alpha \)-compactly generated.
- For \( I \in F \setminus \{I_1\} \) and a nonempty subset \( F' \subseteq F \setminus \{I, I_1\} \) the image of the canonical functor

\[ \bigcap_{I' \in F'} \overline{T}^{\text{inc}} \xrightarrow{\text{can}} T/I \cong T/I_\cap \]

is just the essential image of the composite

\[ \bigcap_{I' \in F'} T^{\text{inc}} \xrightarrow{\text{can}} T/I, \]

which is, by hypothesis, \( \alpha \)-compactly generated in \( T/I \).

From the inductive hypothesis, we deduce that \( T \) is \( \alpha \)-compactly generated, and that \( X \in T \) is \( \alpha \)-compact if and only if the images of \( X \) in \( T/I \cong T/I_\cap \) are \( \alpha \)-compact for each \( I \in F \setminus \{I_1\} \). This verifies condition (i) above, and it remains to check (ii).

Identify \( I_1 \) as a subcategory of \( T \) via the embedding \( I_1 \rightarrow T \rightarrow T/I_\cap \). Then \( I_1 \) is a Bousfield subcategory, properly intersecting \( T \) for \( I \in F \setminus \{I_1\} \). Moreover:

- For \( I \in F \setminus \{I_1\} \) the subcategory \( I_1/(I_1 \cap I) \) of \( T/I \) is identified, under the equivalence \( T/I \cong T/I_\cap \), with \( I_1/(I_1 \cap I) \), which is \( \alpha \)-compactly generated in \( T/I \) by hypothesis.

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- For every $\mathcal{I} \in \mathcal{F} \setminus \{\mathcal{I}_1\}$ and nonempty subset $\mathcal{F}' \subseteq \mathcal{F} \setminus \{\mathcal{I}, \mathcal{I}_1\}$ the image of
  \[ \mathcal{I}_1 \cap \bigcap_{\mathcal{I}' \in \mathcal{F}'} \mathcal{T}' \xrightarrow{\text{inc}} \mathcal{T} \xrightarrow{\text{can}} \mathcal{T} / \mathcal{I} \cong \mathcal{T} / \mathcal{I} \]
  is just the essential image of the composite
  \[ \mathcal{I}_1 \cap \bigcap_{\mathcal{I}' \in \mathcal{F}'} \mathcal{T}' \xrightarrow{\text{inc}} \mathcal{T} \xrightarrow{\text{can}} \mathcal{T} / \mathcal{I} \]

  which is, by hypothesis, $\alpha$-compactly generated in $\mathcal{T} / \mathcal{I}$.

From the inductive hypothesis (with $\mathcal{S} = \mathcal{I}_1$) we conclude that $\mathcal{I}_1$ is $\alpha$-compactly generated in $\mathcal{T}$. Having now established both $(i)$ and $(ii)$ above, we deduce from the $n = 2$ case of the Theorem that $\mathcal{T}$ is $\alpha$-compactly generated, and that $X \in \mathcal{T}$ is $\alpha$-compact if and only if $X$ is $\alpha$-compact in both $\mathcal{T} / \mathcal{I}_1$ and $\mathcal{T}$. But the image of $X$ in $\mathcal{T}$ is $\alpha$-compact if and only if the images of $X$ in $\mathcal{T} / \mathcal{I} \cong \mathcal{T} / \mathcal{I}$ are $\alpha$-compact for $\mathcal{I} \in \mathcal{F} \setminus \{\mathcal{I}_1\}$, which gives the desired criterion for $\alpha$-compactness in $\mathcal{T}$.

To complete the inductive step, it remains to treat the second statement: we are given a Bousfield subcategory $\mathcal{S}$ properly intersecting every $\mathcal{I} \in \mathcal{F}$, satisfying conditions (3), (4). By hypothesis, then, $\mathcal{S}/(\mathcal{S} \cap \mathcal{I}_\alpha)$ and $\mathcal{I}_\alpha \cap \mathcal{S}$ are $\alpha$-compactly generated in $\mathcal{T}/\mathcal{I}_1$, and to apply the $n = 2$ case of the Theorem to $\mathcal{S}$ and the cocover $\{\mathcal{I}_1, \mathcal{I}_\alpha\}$ it remains to check that

$(i)'$ $\mathcal{S}/(\mathcal{S} \cap \mathcal{I}_\alpha)$ is $\alpha$-compactly generated in $\mathcal{T}$, and

$(ii)'$ $\mathcal{I}_\alpha \cap \mathcal{S}$ is $\alpha$-compactly generated in $\mathcal{T}$.

Set $\overline{\mathcal{S}} = \mathcal{S}/(\mathcal{S} \cap \mathcal{I}_\alpha)$. This is a Bousfield subcategory of $\mathcal{T}$ properly intersecting every element of the cocovering $\{\overline{\mathcal{T}}_2, \ldots, \overline{\mathcal{T}}_n\}$ of $\mathcal{T}$. Moreover:

- For $\mathcal{I} \in \mathcal{F} \setminus \{\mathcal{I}_1\}$ the subcategory $\overline{\mathcal{S}}/(\overline{\mathcal{S}} \cap \overline{\mathcal{T}})$ of $\mathcal{T} / \mathcal{I}$ is identified under the equivalence $\mathcal{T} / \mathcal{I} \cong \mathcal{T} / \mathcal{I}$ with the subcategory $\mathcal{S}/(\mathcal{S} \cap \mathcal{I})$, which is $\alpha$-compactly generated in $\mathcal{T} / \mathcal{I}$ by hypothesis.

- For every $\mathcal{I} \in \mathcal{F} \setminus \{\mathcal{I}_1\}$ and nonempty subset $\mathcal{F}' \subseteq \mathcal{F} \setminus \{\mathcal{I}, \mathcal{I}_1\}$ the image of
  \[ \overline{\mathcal{S}} \cap \bigcap_{\mathcal{I}' \in \mathcal{F}'} \overline{\mathcal{T}} \xrightarrow{\text{inc}} \overline{\mathcal{T}} \xrightarrow{\text{can}} \overline{\mathcal{T}} / \mathcal{I} \cong \overline{\mathcal{T}} / \mathcal{I} \]
  is just the essential image of the composite
  \[ \mathcal{S} \cap \bigcap_{\mathcal{I}' \in \mathcal{F}'} \mathcal{T}' \xrightarrow{\text{inc}} \mathcal{T} \xrightarrow{\text{can}} \mathcal{T} / \mathcal{I}, \]

  which is, by hypothesis, $\alpha$-compactly generated in $\mathcal{T} / \mathcal{I}$.

From the inductive hypothesis, we conclude that $\overline{\mathcal{S}}$ is $\alpha$-compactly generated in $\overline{\mathcal{T}}$, which is $(ii)'$ above. A similar argument verifies $(i)'$, and from the $n = 2$ case of the Theorem we conclude that $\mathcal{S}$ is $\alpha$-compactly generated in $\mathcal{T}$. This completes the inductive step, and thus the proof. \hfill $\Box$

**Corollary 13.** In the situation of Theorem 1, for any regular cardinal $\beta \geq \alpha$ an object $X \in \mathcal{T}$ is $\beta$-compact if and only if the image of $X$ is $\beta$-compact in $\mathcal{T} / \mathcal{I}$ for every $\mathcal{I} \in \mathcal{F}$.

**Proof.** If a triangulated category $\mathcal{Q}$ is $\alpha$-compactly generated, or a subcategory $\mathcal{S}$ is $\alpha$-compactly generated in some larger triangulated category, then the same is true for any regular cardinal $\beta \geq \alpha$. Hence, if the cocover $\mathcal{F}$ satisfies the hypotheses of Theorem 1 for $\alpha$, it satisfies the same conditions for $\beta \geq \alpha$, whence the claim. \hfill $\Box$

**4. Derived Categories of Rings**

In the next section we obtain a characterisation of the $\alpha$-compact objects in the derived category of a scheme. We will use a reduction to the affine case, so in this section we prepare the ground with a
review of some facts about the derived category of a ring. Throughout, a ring is a (not necessarily commutative) associative ring with identity, and all modules are left modules. Given a ring $R$ we denote by $\mathcal{D}(R)$ the unbounded derived category of $R$-modules. If $\alpha$ is a regular cardinal, then $\mathcal{D}(R)^\alpha$ denotes the full subcategory of $\alpha$-compact objects in $\mathcal{D}(R)$.

A complex of $R$-modules $P$ is called $K$-projective if, for every acyclic complex $X$ of $R$-modules, the complex of abelian groups $\text{Hom}_R(P, X)$ is acyclic [Spa88]. For example, any bounded above complex of projective $R$-modules is $K$-projective. The $K$-projective resolution of a complex of $R$-modules $M$ is a quasi-isomorphism $P \to M$, where $P$ is $K$-projective. In this case, $P$ is the unique (up to homotopy equivalence) $K$-projective complex isomorphic to $M$ in $\mathcal{D}(R)$.

**Theorem 14** (Neeman). Let $R$ be a ring. The derived category $\mathcal{D}(R)$ is compactly generated, and given a regular cardinal $\alpha > \aleph_0$ a complex of $R$-modules is $\alpha$-compact in $\mathcal{D}(R)$ if and only if it is quasi-isomorphic to a $K$-projective complex of free $R$-modules of rank $< \alpha$.

**Remark 15.** Since we are allowing noncommutative rings $R$, one has to be a bit careful about the meaning of “rank”; we direct the reader to [Nee08, (5.2)].

**Proof.** Part of this criterion is stated without proof in [Nee01], and the full statement can be deduced from the more general results of [Nee08, §7]. To be precise: taking $K$-projective resolutions defines a fully faithful functor $\mathcal{D}(R) \to K(\text{Proj} R)$, and we identify $\mathcal{D}(R)$ as a subcategory of $K(\text{Proj} R)$ via this embedding. Neeman proves in [Nee08] that $K(\text{Proj} R)$ is $\aleph_1$-compactly generated. Since $\mathcal{D}(R)$ is a localising subcategory of $K(\text{Proj} R)$ generated by $R$, which is compact in $K(\text{Proj} R)$, it follows from Theorem 5 that for any regular cardinal $\alpha > \aleph_0$ we have

$$\mathcal{D}(R)^\alpha = K(\text{Proj} R)^\alpha \cap \mathcal{D}(R),$$

that is, a complex $M$ of $R$-modules is $\alpha$-compact in $\mathcal{D}(R)$ if and only if the $K$-projective resolution $P$ of $M$ is $\alpha$-compact in $K(\text{Proj} R)$. But by [Nee08, Proposition 7.4] and [Nee08, Proposition 7.5], $P$ is $\alpha$-compact in $K(\text{Proj} R)$ if and only if it is homotopy-equivalent to a complex of free $R$-modules of rank $< \alpha$.

**Remark 16.** In the context of the theorem, it is natural to ask if the condition of $K$-projectivity can be dropped, that is: are the $\alpha$-compact objects in $\mathcal{D}(R)$ precisely the complexes quasi-isomorphic to a complex of free modules of rank $< \alpha$? We will see in Theorem 19 that this is true provided that $R$ is either left noetherian or has cardinality $< \alpha$. In general, however, the answer is negative: we construct a counterexample in Appendix A, consisting of a ring $B$ and a complex of free $B$-modules of rank $< \alpha$ that is not $\alpha$-compact in $\mathcal{D}(B)$.

**Definition 17.** Let $R$ be a ring and $\alpha$ an infinite cardinal. An $R$-module $M$ is said to be $\alpha$-generated if it can be generated as an $R$-module by a subset of cardinality $\alpha$. We say that $M$ is sub-$\alpha$-generated if it can be generated by a subset of cardinality $< \alpha$.

We include a proof of the following standard fact:

**Lemma 18.** Let $R$ be a ring, $\alpha$ an infinite cardinal, and suppose that $R$ is either left noetherian, or has cardinality $\leq \alpha$. Then if an $R$-module $M$ is $\alpha$-generated, every submodule of $M$ is $\alpha$-generated.

**Proof.** Let $\kappa$ be a fixed regular cardinal with $\kappa > \alpha$. We prove the following statement by transfinite induction: if $x$ is an ordinal $< \kappa$, and $M$ is any $R$-module generated by a set of cardinality $|x|$, then every submodule of $M$ is generated by a set of cardinality $< \kappa$. Call this statement $B(x)$. Taking $x = \alpha$ and $\kappa$ to be the successor cardinal of $\alpha$ (which is regular) gives the result.

Successor ordinals: if $x$ is an ordinal $< \kappa$ then either $|x| = |x^+|$, in which case the statement $B(x^+)$ is just $B(x)$ and the inductive step is trivial, or these two cardinals are distinct, in which
case $x$ is a finite cardinal. If $x$ is finite, then since $R$ is either left noetherian or has cardinality $\leq \alpha$, it is straightforward to verify that $B(x)$ holds.

Limit ordinals: assume that $x$ is a limit ordinal $< \kappa$, and that $B(\beta)$ holds for all $\beta < x$. We may assume that $x$ is a cardinal, since otherwise the inductive step is trivial. Let $M$ be an $R$-module generated by a set of cardinality $x$, say by $\{m_t \mid t < x\}$, and for $t < x$ let $M_{< t}$ be the submodule of $M$ generated by the set $\{m_s \mid s < t\}$. This is a generating set of cardinality $< x$, so by the inductive hypothesis for any submodule $N$ of $M$, the intersection $N \cap M_{< t}$ can be generated by a set $\lambda_t$ of cardinality $< \kappa$. From the equality

\[
N = N \cap M = N \cap \sum_{t < x} M_{< t} = \sum_{t < x} (N \cap M_{< t})
\]

we deduce that $N$ can be generated by the union $\lambda = \cup_{t < x} \lambda_t$, which is of cardinality $|\lambda| \leq \sum_{t < x} |\lambda_t| < \kappa$

because $\kappa$ is regular. \hfill $\square$

The following argument was kindly explained to the author by Neeman:

**Theorem 19 (Neeman).** Let $R$ be a ring and $\alpha > \aleph_0$ a regular cardinal. Suppose that $R$ is either left noetherian, or has cardinality $< \alpha$. Then for a complex $F$ of $R$-modules the following are equivalent:

(i) $F$ is $\alpha$-compact in $D(R)$.

(ii) $F$ is isomorphic, in $D(R)$, to a complex of free $R$-modules of rank $< \alpha$.

(iii) $H^i(F)$ is a sub-$\alpha$-generated $R$-module for all $i \in \mathbb{Z}$.

**Proof.** The implication (i) $\Rightarrow$ (ii) is a consequence of Theorem 14, while (ii) $\Rightarrow$ (iii) follows from Lemma 18, so it remains to prove that (iii) $\Rightarrow$ (i). Let $D_\alpha(R)$ denote the full subcategory of $D(R)$ consisting of complexes with sub-$\alpha$-generated cohomology. Using Lemma 18 this is easily checked to be an $\alpha$-localising triangulated subcategory. We already know that $D(R)^\alpha \subseteq D_\alpha(R)$, and we want to prove the reverse inclusion. Let us begin with a technical observation.

Given a complex $F$ in $D_\alpha(R)$, we claim that it is possible to construct a morphism of complexes $\phi : F \to F'$ with $F' \in D_\alpha(R)$ such that the mapping cone of $\phi$ is $\alpha$-compact and $\phi$ is a ghost, that is, the induced maps $H^i(\phi) : H^i(F) \to H^i(F')$ are zero for all $i \in \mathbb{Z}$.

For each $i \in \mathbb{Z}$ there exists a surjective map $P_i^i \to H^i(F)$ from some free $R$-module $P_i^i$ of rank $< \alpha$, and this lifts to a morphism of complexes $g_i : \Sigma^{-i}P_i^i \to F$. The coproduct $P = \bigoplus_{i \in \mathbb{Z}} \Sigma^{-i}P_i^i$ is $\alpha$-compact in $D(R)$, and the sum $g : P \to F$ of the $g_i$’s is a morphism of complexes surjective on cohomology. Extending to a triangle

\[
P \xrightarrow{g} F \xrightarrow{\phi} F' \xrightarrow{} \Sigma P
\]

we observe that $\phi$ is a ghost with $\alpha$-compact mapping cone, and $F'$ has sub-$\alpha$-generated cohomology. If we begin with a fixed $F = F_0$ in $D_\alpha(R)$ and iterate this process to construct a sequence

\[
F_0 \xrightarrow{\phi_0} F_1 \xrightarrow{\phi_1} F_2 \xrightarrow{\phi_2} \cdots
\]

de g of ghost maps $\phi_i$ in $D_\alpha(R)$ with $\alpha$-compact cones, then the homotopy colimit of this sequence in $D(R)$ is acyclic (that is, zero). Since $D_\alpha(R)$ is closed under countable coproducts, this also calculates the homotopy colimit in $D_\alpha(R)$, and thus in the quotient $D_\alpha(R)/D(R)^\alpha$. But in this quotient each $\phi_i$ is an isomorphism (as it has zero cone), so in this case the homotopy colimit is equal to $F$. Together, these observations imply that $F \in D(R)^\alpha$, as claimed. \hfill $\square$
Remark 20. We learn from the theorem that when \( \alpha > |R| \), the \( \alpha \)-compacts in \( D(R) \) can be characterised by the “size” of their cohomology. In fact, this is a general phenomenon in well-generated triangulated categories, as explained by Krause in [Kra01, Theorem B] and [Kra02, Theorem C].

5. Derived Categories of Schemes

In this section we apply Theorem 1 to the derived category of quasi-coherent sheaves on a scheme. We refer the reader to [Lip09, Nee96, AIL97, AJPV08] for background on unbounded derived categories of schemes. A scheme is semi-separated if it admits a covering by affine open subsets \( \{V_i\}_{i \in I} \) with all pairwise intersections \( V_i \cap V_j \) affine; see [AJPV08, TT90]. Separated schemes are semi-separated, and semi-separated schemes are quasi-separated. Given a scheme \( X \) we denote by \( D(X) \) the unbounded derived category of quasi-coherent sheaves on \( X \). If \( X \) is quasi-compact and semi-separated, then \( D(X) \) is equivalent to the full subcategory of complexes with quasi-coherent cohomology in the derived category of sheaves of \( \mathcal{O}_X \)-modules [BN93].

Let \( X \) be a quasi-compact semi-separated scheme, and \( \{U_1, \ldots, U_n\} \) a cover of \( X \) by affine open subsets. For \( 1 \leq i \leq n \) set \( Z_i = X \setminus U_i \) and denote by \( D_{Z_i}(X) \) the full subcategory of \( D(X) \) consisting of complexes with cohomology supported on \( Z_i \). The inclusion \( D_{Z_i}(X) \hookrightarrow D(X) \) and restriction \( (-)|_{U_i} \) fit into a sequence of functors

\[
0 \longrightarrow D_{Z_i}(X) \xrightarrow{\text{inc}} D(X) \xrightarrow{(-)|_{U_i}} D(U_i) \longrightarrow 0
\]

which is exact, in the sense that \( (-)|_{U_i} \) induces an equivalence \( D(X)/D_{Z_i}(X) \cong D(U_i) \). In fact this is a localisation sequence: the right adjoint of \( D_{Z_i}(X) \hookrightarrow D(X) \) is Grothendieck’s local cohomology functor \( R\Gamma_{Z_i}(-) \), and the right adjoint of \( (-)|_{U_i} \) is the derived direct image \( \mathbb{R}f_* \), where \( f : U_i \longrightarrow X \) is the inclusion. In particular, each \( D_{Z_i}(X) \) is a Bousfield subcategory of \( D(X) \).

The family \( \mathcal{F} = \{D_{Z_1}(X), \ldots, D_{Z_n}(X)\} \) is a covering of \( D(X) \), and Rouquier proves in [Rou08, §6.2] that this covering satisfies the hypotheses (1), (2) of Theorem 1 for the cardinal \( \alpha = \aleph_0 \). Let us examine the content of these hypotheses, and sketch Rouquier’s argument in each case:

1. requires that \( D(X)/D_{Z_i}(X) \cong D(U_i) \) be compactly generated for \( 1 \leq i \leq n \). But \( U_i \cong \text{Spec}(A_i) \) is affine, and thus \( D(U_i) \cong D(A_i) \) is known to be compactly generated.

2. requires, given an index \( 1 \leq j \leq n \) and nonempty subset \( I \subseteq \{1, \ldots, n\} \) not containing \( j \), that the essential image of the composite

\[
\bigcap_{i \in I} D_{Z_i}(X) \xrightarrow{\text{inc}} D(X) \xrightarrow{(-)|_{U_j}} D(U_j)
\]

be compactly generated in \( D(U_j) \). But this image is just \( D_Z(U_j) \), where \( Z \) is the complement of \( U_j \cap \bigcup_{i \in I} U_i \) in \( U_j \). Since \( U_j \) is affine one can generate this subcategory by a Koszul complex, which is compact in \( D(U_j) \); see [Rou08, Proposition 6.6] or [BN93, Proposition 6.1].

In particular, Rouquier proves that if \( Z \) is a closed subset of \( X \) with quasi-compact complement \( U \), then \( D_Z(X) \) is compactly generated in \( D(X) \). Since the restriction functor \( (-)|_U : D(X) \longrightarrow D(U) \) factors as \( D(X) \longrightarrow D(X)/D_Z(X) \cong D(U) \) we infer from [Nee01, Theorem 4.4.9] that the functor \( (-)|_U \) preserves \( \alpha \)-compactness for any regular cardinal \( \alpha \).

Let us record the following special case of Corollary 13, with \( \mathcal{T} = D(X) \) and \( \mathcal{F} \) as above.

Proposition 21. Let \( X \) be a quasi-compact semi-separated scheme and \( \alpha \) a regular cardinal. Given a cover \( \{U_1, \ldots, U_n\} \) of \( X \) by affine open subsets, a complex \( \mathcal{F} \) of quasi-coherent sheaves on \( X \) is \( \alpha \)-compact in \( D(X) \) if and only if \( \mathcal{F}|_{U_i} \) is \( \alpha \)-compact in \( D(U_i) \) for \( 1 \leq i \leq n \).
The proposition reduces the problem of understanding the \( \alpha \)-compacts in \( \mathbf{D}(X) \) to the problem of understanding the \( \alpha \)-compacts in the derived category of a ring, which was settled in Section 4.

**Proposition 22.** Let \( X \) be a quasi-compact semi-separated scheme, and \( \alpha > \aleph_0 \) a regular cardinal. A complex \( \mathcal{F} \) of quasi-coherent sheaves on \( X \) is \( \alpha \)-compact in \( \mathbf{D}(X) \) if and only if, for every \( x \in X \), there is an affine open neighborhood \( U \) of \( x \) such that \( \Gamma(U, \mathcal{F}) \) is quasi-isomorphic as a complex of \( A = \Gamma(U, \mathcal{O}_X) \)-modules to a \( K \)-projective complex of free \( A \)-modules of rank \( < \alpha \).

**Proof.** Using Proposition 21 we reduce to the case of affine \( X \), which is Theorem 14.

Let \( \mathbf{D}(X)^\alpha \) denote the full subcategory of \( \alpha \)-compact objects in \( \mathbf{D}(X) \). Subcategories of \( \mathbf{D}(X) \) are typically defined by imposing conditions on homology, so it is comforting to have such a description of \( \mathbf{D}(X)^\alpha \). We begin with some definitions.

**Definition 23.** Let \( \alpha \) be an infinite cardinal. A quasi-coherent sheaf \( \mathcal{F} \) on a scheme \( X \) is said to be locally \( \alpha \)-generated if for every \( x \in X \), there exists an open neighborhood \( U \) of \( x \) together with an epimorphism \( \bigoplus_{j \in J} \mathcal{O}_X|_U \rightarrow \mathcal{F}|_U \), for some index set \( J \) of cardinality \( \alpha \). If for each \( x \in X \) we can arrange for the set \( J \) to be of cardinality \( < \alpha \), then \( \mathcal{F} \) is locally sub-\( \alpha \)-generated.

**Lemma 24.** Let \( R \) be a commutative ring, \( \alpha \) an infinite cardinal and \( F \) an \( R \)-module. Then the following are equivalent:

(i) \( F \) is an \( \alpha \)-generated \( R \)-module.

(ii) The complex \( \mathcal{F} \) of quasi-coherent sheaves on \( \text{Spec}(R) \) associated to \( F \) is locally \( \alpha \)-generated.

**Proof.** (i) \( \Rightarrow \) (ii) is clear. For (ii) \( \Rightarrow \) (i), set \( X = \text{Spec}(R) \) and suppose that \( \mathcal{F} \) is locally \( \alpha \)-generated. We may find generators \( f_1, \ldots, f_r \) of the unit ideal of \( R \), with the property that on each \( U_i = D(f_i) \) there is an epimorphism \( \bigoplus_{j \in J_i} \mathcal{O}_X|_{U_i} \rightarrow \mathcal{F}|_{U_i} \) for some set \( J_i \) of cardinality \( \alpha \). That is, \( F[f_i^{-1}] \) can be generated as an \( R[f_i^{-1}] \)-module by a subset \( \{a_j/f_i^n\}_{j \in J_i} \) of cardinality \( |J_i| = \alpha \). Form a set \( J \) consisting of the union, over each \( 1 \leq i \leq r \), of the set of numerators \( \{a_j\}_{j \in J_i} \). This set has cardinality \( \alpha \) and generates \( F \) as an \( R \)-module.

Finally, we arrive at a characterisation of \( \alpha \)-compactness in terms of cohomology sheaves.

**Corollary 25.** Let \( X \) be a quasi-compact semi-separated scheme and \( \alpha > \aleph_0 \) a regular cardinal. Suppose that \( X \) is either

(a) noetherian, or

(b) admits a cover by open affines \( \{U_i\}_{1 \leq i \leq n} \) with \( \Gamma(U_i, \mathcal{O}_X) \) of cardinality \( < \alpha \) for \( 1 \leq i \leq n \).

Then a complex \( \mathcal{F} \) of quasi-coherent sheaves on \( X \) is \( \alpha \)-compact in \( \mathbf{D}(X) \) if and only if \( H^i(\mathcal{F}) \) is locally sub-\( \alpha \)-generated for every \( i \in \mathbb{Z} \).

**Proof.** Under either hypothesis on \( X \) there is an affine open cover \( \{U_1, \ldots, U_n\} \) of \( X \) such that the conclusion of Theorem 19 applies to each of the rings \( \Gamma(U_i, \mathcal{O}_X) \). By Lemma 24, a quasi-coherent sheaf \( \mathcal{G} \) on \( X \) is locally sub-\( \alpha \)-generated if and only if \( \Gamma(U_i, \mathcal{G}) \) is a sub-\( \alpha \)-generated \( \Gamma(U_i, \mathcal{O}_X) \)-module for \( 1 \leq i \leq n \), so the result follows from Proposition 21.

**Appendix A. Counterexample**

Let \( R \) be a ring and \( \alpha > \aleph_0 \) a regular cardinal. We proved in Theorem 19 that when \( R \) is either left noetherian or has cardinality \( < \alpha \), the \( \alpha \)-compact objects in \( \mathbf{D}(R) \) are the objects isomorphic to a complex of free \( R \)-modules of rank \( < \alpha \). In this appendix we show that this characterisation of the \( \alpha \)-compact objects cannot hold for arbitrary rings, by constructing for any given regular cardinal...
\( \alpha > \aleph_0 \) a commutative local ring \( B \) and a complex of free \( B \)-modules of rank one which is not \( \alpha \)-compact. Needless to say, \( B \) is non-noetherian and has cardinality \( \geq \alpha \).

Let \( k \) be a field and \( \beta \) an infinite cardinal, and \( I = \{ x_i \}_{i \in \beta} \) a set of variables indexed by \( \beta \). We denote by \( k[[I]] \) the ring of formal power series in the set of variables \( I \). More precisely, let \( \mathbb{N}^{(I)} \) be the set of all functions \( \gamma : I \rightarrow \mathbb{N} \) with finite support, and define

\[
k[[I]] := \{ \text{functions } f : \mathbb{N}^{(I)} \rightarrow k \},
\]

with the usual addition \((f + g)(\gamma) = f(\gamma) + g(\gamma)\) and product \((f \cdot g)(\gamma) = \sum_{\alpha + \beta = \gamma} f(\alpha)g(\beta)\). Then \( k[[I]] \) is a commutative local ring, with maximal ideal given by the ideal of power series with zero constant term. We say that a monomial in the set of variables \( I \) is pure if it is of the form \( x_i^n \) for some \( i \in \beta \) and \( n \geq 0 \), with \( x_0^0 \) understood to be the identity in \( k[[I]] \). Let \( a \) denote the ideal of power series in which the coefficient of every pure monomial is zero (e.g. \( x_ix_j \) for \( i \neq j \)) and define

\[B' := k[[I]]/a.\]

Each residue class of \( B' \) contains a unique power series \( f \) in which only pure monomials have nonzero coefficients, and such \( f \) can be written as a formal sum

\[f = \lambda \cdot 1 + \sum_{i \in \beta} \sum_{n \geq 1} f_{i,n} \cdot x_i^n, \quad \lambda, f_{i,n} \in k.\]  

(11)

We say that a power series \( f \) involves a variable \( x_i \) if the coefficient of \( x_i^n \) in \( f \) is nonzero for some \( n \geq 1 \). Finally, we may define the desired ring \( B \) as a subring of \( B' \).

**Definition 26.** Given a field \( k \) and an infinite cardinal \( \beta \), we define a commutative \( k \)-algebra \( B \) by

\[B := \{ f \in B' | f \text{ involves only finitely many variables in } I \}.\]

Concretely, this is the ring of formal power series of the form (11) in the set of variables \( \{ x_i \}_{i \in \beta} \), with each power series involving only a finite number of variables. Power series are multiplied according to the relations \( x_i^n x_j^m = x_i^{n+m} \) and \( x_ix_j = 0 \) for \( i \neq j \). This is a commutative local ring, with maximal ideal \( m_B \) given by the set of power series with zero constant term, and residue field \( k = B/m_B \).

**Remark 27.** The following properties of \( B \) are immediate:

- Given a nonempty subset \( J \subseteq \beta \), the ideal \((x_j)_{j \in J} \) in \( B \) consists precisely of those power series \( f \) with zero constant term, which do not involve any variable \( x_i \) with \( i \in \beta \setminus J \).
- Given \( i \in \beta \), the kernel of \( B \xrightarrow{x_i} B \) is the ideal \((x_j)_{j \in \beta \setminus \{i\}} \).
- For any nonempty subset \( J \subseteq \beta \) there is an internal direct sum \((x_j)_{j \in J} \cong \bigoplus_{j \in J} B(x_j) \).

We will need the following consequence of Theorem 14.

**Lemma 28.** Let \( A \) be a commutative local ring with residue field \( k \), and \( \alpha > \aleph_0 \) a regular cardinal. If \( M \) is an \( A \)-module belonging to \( \textbf{D}(A)^{\alpha} \) then \( \text{rank}_k \text{Tor}_n(M, k) < \alpha \) for all \( n \geq 0 \).

**Proof.** If \( M \) belongs to \( \textbf{D}(A)^{\alpha} \) then by Theorem 14 it admits a \( \textbf{K} \)-projective resolution by a complex \( P \) of free \( A \)-modules of rank \( < \alpha \). Hence \( P \otimes_A k = M \otimes_A k \) is a complex of \( k \)-vector spaces of dimension \( < \alpha \), and the claim follows.

The key pathology of the ring \( B \) becomes apparent in the next lemma.

**Lemma 29.** Given \( i \in \beta \) and the corresponding ideal \((x_i) \) in \( B \), we have

\[
\text{rank}_k \text{Tor}_n((x_i), k) = \begin{cases} 1 & n = 0, \\ \beta & n > 0. \end{cases}
\]

Consequently, if \( \alpha > \aleph_0 \) is a regular cardinal such that \( \alpha \leq \beta \), then \((x_i) \) does not belong to \( \textbf{D}(B)^{\alpha} \).
Definition 33. Two triangulated subcategories \( \mathcal{A}, \mathcal{B} \) of \( \mathcal{T} \) are said to intersect properly if there is an equality of subcategories \( \mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A} \).

In the next lemma we verify that this definition of proper intersection agrees with the one given by Rouquier [Rou08, (5.2.3)] for Bousfield subcategories.
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Lemma 34. Let \( \mathcal{A}, \mathcal{B} \) be triangulated subcategories of \( T \). The following are equivalent:

(i) \( \mathcal{A} \) and \( \mathcal{B} \) intersect properly, that is, \( \mathcal{A} \ast \mathcal{B} = \mathcal{B} \ast \mathcal{A} \).

(ii) Given \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \), every morphism \( A \to B \) and every morphism \( B \to A \) factors through an object of \( \mathcal{A} \cap \mathcal{B} \).

Proof. (i) \( \Rightarrow \) (ii) Let \( f : A \to B \) be given, with \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \), and extend to a triangle

\[
A \xrightarrow{f} B \xrightarrow{} X \xrightarrow{} \Sigma A.
\]

Since \( X \) belongs to \( \mathcal{B} \ast \mathcal{A} = \mathcal{A} \ast \mathcal{B} \), there is a triangle \( A' \to X \to B' \to \Sigma A' \) with \( A' \in \mathcal{A} \) and \( B' \in \mathcal{B} \). Applying the octahedral axiom to the pair \( B \to X, X \to B' \) we obtain an object \( D \) and triangles

\[
B \to B' \to D \xrightarrow{\gamma} \Sigma B, \\
A \xrightarrow{\delta} \Sigma^{-1} D \to A' \to \Sigma A,
\]

such that \( \gamma \circ \Sigma \delta = \Sigma f \). From the first triangle we deduce that \( D \in \mathcal{B} \), and from the second triangle we conclude that \( D \in \mathcal{A} \), whence \( f \) factors via \( \mathcal{A} \cap \mathcal{B} \). The factorisation argument for a morphism \( B \to A \) is dual. (ii) \( \Rightarrow \) (i) Let \( X \in \mathcal{A} \ast \mathcal{B} \) be given, so that there is a triangle

\[
A \to X \to B \xrightarrow{s} \Sigma A.
\]

By hypothesis \( s \) factors as \( B \to D \to \Sigma A \) for some \( D \in \mathcal{A} \cap \mathcal{B} \), and from the octahedral axiom applied to the pair of morphisms in this factorisation of \( s \), we conclude that \( X \in \mathcal{B} \ast \mathcal{A} \). This shows that \( \mathcal{A} \ast \mathcal{B} \subseteq \mathcal{B} \ast \mathcal{A} \), and the reverse inclusion follows similarly.

Lemma 35. Let \( \mathcal{A}, \mathcal{B} \) be properly intersecting triangulated subcategories of \( T \). Then \( \mathcal{A} \ast \mathcal{B} \) is a triangulated subcategory of \( T \).

Proof. It suffices to prove that \( \mathcal{A} \ast \mathcal{B} \) is closed under mapping cones. Let \( f : X \to Y \) be a morphism in \( T \) with \( X, Y \in \mathcal{A} \ast \mathcal{B} \), and fix triangles

\[
X \xrightarrow{f} Y \to C \to \Sigma X, \\
A_X \to X \to B_X \to \Sigma A_X, \\
A_Y \to Y \to B_Y \to \Sigma A_Y,
\]

where \( A_X, A_Y \in \mathcal{A} \) and \( B_X, B_Y \in \mathcal{B} \). Applying the octahedral axiom to the pair \( A_Y \to Y, Y \to C \) yields an object \( D \) and triangles

\[
A_Y \to C \to D \to \Sigma A_Y, \tag{12} \\
B_Y \to D \to \Sigma X \to \Sigma B_Y. \tag{13}
\]

Using the proper intersection property, we infer from (13) that

\[
D \in \mathcal{B} \ast (\mathcal{A} \ast \mathcal{B}) = \mathcal{B} \ast (\mathcal{B} \ast \mathcal{A}) = (\mathcal{B} \ast \mathcal{B}) \ast \mathcal{A} = \mathcal{B} \ast \mathcal{A} = \mathcal{A} \ast \mathcal{B},
\]

whence by (12) we have

\[
C \in \mathcal{A} \ast (\mathcal{A} \ast \mathcal{B}) = (\mathcal{A} \ast \mathcal{A}) \ast \mathcal{B} = \mathcal{A} \ast \mathcal{B}.
\]

Hence \( \mathcal{A} \ast \mathcal{B} \) is closed under mapping cones, and therefore triangulated.

Remark 36. Let \( \mathcal{A}, \mathcal{B} \) be properly intersecting triangulated subcategories of \( T \). Then \( \mathcal{A} \ast \mathcal{B} \) is clearly the smallest triangulated subcategory of \( T \) containing \( \mathcal{A} \cup \mathcal{B} \). Notice that if \( \mathcal{A} \) and \( \mathcal{B} \) are localising, then so is \( \mathcal{A} \ast \mathcal{B} \).

We will need the following results from [Rou08].
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**Lemma 37.** Let $A, B$ be properly intersecting Bousfield subcategories of $T$. Then $A \cap B$ and $A \ast B$ are Bousfield subcategories of $T$.

**Proof.** See [Rou08, Lemma 5.8].

**Lemma 38.** Let $\mathcal{F}$ be a finite family of Bousfield subcategories of $T$, any two of which intersect properly. Given a subset $\mathcal{F}' \subseteq \mathcal{F}$, the intersection $\bigcap_{I \in \mathcal{F}} I$ is a Bousfield subcategory of $T$ intersecting properly with any subcategory in $\mathcal{F}$. Given $I_1, I_2, I \in \mathcal{F}$, the quotients $I_1/(I_1 \cap I)$ and $I_2/(I_2 \cap I)$ are properly intersecting Bousfield subcategories of $T/I$.

**Proof.** See [Rou08, Lemma 5.9].

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