A topological framework for signed permutations

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Abstract
In this paper we present a topological framework for studying signed permutations and their reversal distance. As a result we can give an alternative approach and interpretation of the Hannenhalli-Pevzner formula for the reversal distance of signed permutations. Our approach utilizes the Poincaré dual, upon which reversals act in a particular way and obsoletes the notion of “padding” of the signed permutations. To this end we construct a bijection between signed permutations and an equivalence class of particular fatgraphs, called π-maps, and analyze the action of reversals on the latter. We show that reversals act via either slicing, gluing or half-flipping of external vertices, which implies that any reversal changes the topological genus by at most one. Finally we revisit the Hannenhalli-Pevzner formula employing orientable and non-orientable, irreducible, π-maps.

Keywords: signed permutation, reversal distance, fatgraph, Poincaré dual, π-map.

1. Introduction

In a seminal paper Hannenhalli and Pevzner [1] give a polynomial time algorithm as well as an explicit formula to compute the reversal distance of a signed permutation $b_n$. Employing the framework of breakpoint graphs
they express the reversal distance as

\[
d(b_n) = \begin{cases} 
  b - c + h + 1 & \text{if } h \geq 3, h \equiv 1 \mod 2, \text{ all } h \text{ hurdles are super-hurdles}, \\
  b - c + h & \text{otherwise}
\end{cases}
\]

\hspace{1cm} (1)

where \( b \) is the number of breakpoints, \( c \) is the number of cycles and \( h \) is the number of hurdles in the breakpoint graph of the signed permutation \([1]\).

The algorithm was implemented with time complexity \( O(n^4) \) \((O(n^5) \text{ with padding})\) in \([1]\) and improved later by \([3]\) to \( O(n^2) \) \((O(n^3) \text{ with padding})\). A linear time algorithm for finding the reversal distance of signed permutation is given by \([6]\). Subsequent work on analyzing this question for unsigned permutations however, has not yet succeeded, and only approximations can be found \([4, 7, 8, 9]\).

In this paper we present a topological framework for studying signed permutations, representing an alternative to the breakpoint graph. We believe that this framework can be adapted for studying the transposition distance of unsigned permutations. To understand this problem one has to study the action of transpositions on orientable cell-complexes. Accordingly, the reversal distance of signed and the transposition distance of unsigned permutations become closely related problems. The former can be described by reversals acting on cell-complexes of non-orientable surfaces and the latter by transpositions acting on cell-complexes of orientable surfaces.

Specifically, we construct from a signed permutation an equivalence class of particular fatgraphs \([10]\), called \( \pi \)-maps. While several features of \( \pi \)-maps can be found also in breakpoint graphs, like for instance oriented cycles of external vertices, the key difference lies in considering the Poincaré dual, which comes natural within the topological framework.

\( \pi \)-maps offer a combinatorial interpretation for the action of reversals on signed permutations. The combinatorial interpretation has several implications: first there is no need to introduce any “padding” \([1]\), i.e. inflating the underlying signed permutation into a specific one, having the same reversal distance. In \([1]\) it is stipulated that padding can be avoided, but all constructions are based on padded configurations, i.e. are not directly applied to the original, signed permutation. Secondly the proofs become very intuitive, a consequence of passing to the Poincaré dual in which reversals have only a “local” effect that has concrete, combinatorial interpretation. To be explicit, the action of reversals does not relocate the sectors around the center vertex:
its does only affect their orientations.

Cell-complexes of non-orientable surfaces are typically studied using the orientational double-cover \([10]\). The double-cover allows to mimic the permutation framework of the orientable cell complexes and fits therefore into the notions of half-edges and fatgraphs \([10]\). We shall, however, adopt here a different point of view: we base our definition of fatgraph on the notion of sectors, which is a pair of half-edges, together with an orientation. While this allows to reduce everything to permutations of sectors, one has to give up the fact that the fixed-point free involution algebraically relates vertices and boundary components, see Section 2.

The paper is organized as follows: first we recall in Section 2 some basic facts on signed permutations and on fatgraphs. In Section 3 we construct from a signed permutation a \(\pi\)-map. This association is, however, not unique. This gives rise to a bijection between signed permutations and certain equivalence classes of \(\pi\)-maps. We then proceed analyzing basic properties of \(\pi\)-maps.

In Section 4 we characterize irreducibility and components of \(\pi\)-maps. We study orientable and non-orientable components, characterized by the combinatorial \(\sigma\)-crossing.

We then analyze in Section 5 the action of reversals on \(\pi\)-maps and show that reversals act by either slicing, gluing or half-flipping of vertices. As a result the action of reversals is Lipschitz continuous with respect to topological genus and implies that the genus of the \(\pi\)-map is a sharp lower bound for the reversal distance.

In Section 6 we revisit a result of \([1]\) concerning the successive breakdown of non-orientable components. As a consequence topological genus is a sharp lower bound for the reversal distance.

In Section 7 we collect some facts about the action of reversals on a set of orientable components.

Finally, we reformulate in Section 8 the treatment of hurdles of \([1]\) into the language of \(\pi\)-maps and give an interpretation of the Hannenhalli-Pevzner formula, eq. \([11]\). Namely, two of the three terms express just the topological genus of the underlying \(\pi\)-map.

2. Signed permutations and fatgraphs

Let \(S_n\) denote the symmetric group over \([n]\). A permutation is a one-to-one mapping \(x : [n] \rightarrow [n]\) and represented as an \(n\)-tuple \(x = [x_1, \ldots, x_n]\),
where \( x_i = x(i) \). Furthermore, let \( \varepsilon = [\varepsilon_1, \ldots, \varepsilon_n] \in \{-1, +1\}^n \), the \( n \)-tuple of "signs".

A signed permutation is a pair \( b_n = [\varepsilon, x] = [\varepsilon_1x_1, \ldots, \varepsilon_nx_n] \) and we denote the set of signed permutations by \( \mathfrak{B}_n \). Clearly we have \( |\mathfrak{B}_n| = 2^n n! \) and \( \mathfrak{B}_n \) carries a natural structure of a group via

\[
[\varepsilon, x] \cdot [\varepsilon', y] = [\varepsilon, \varepsilon' x_i x \cdot y], \quad \text{where } [\varepsilon]_i = [\varepsilon_i x(i)].
\]

That is, there is an additional action on \( \varepsilon_y \) when commuting it with \( x \), given by the \( x \)-permutation of the coordinates. A reversal \( \rho_{i,j} \) is the particular signed permutation:

\[
\rho_{i,j} = [\varepsilon_{i,j}, (1, \ldots, i-1, j, j-1, \ldots, i, i+1, \ldots)]
\]

where

\[
(\varepsilon_{i,j})_h = \begin{cases} 
-1 & \text{for } i \leq h \leq j \\
+1 & \text{otherwise}
\end{cases}
\]

Accordingly, a reversal \( \rho_{i,j} \) acts (via right-multiplication) on \( B_n \) as follows:

\[
[\varepsilon_1x_1, \ldots, \varepsilon_i x_i, \ldots, \varepsilon_j x_j, \ldots, \varepsilon_n x_n] \cdot \rho_{i,j} = [\varepsilon_1x_1, \ldots, -\varepsilon_j x_j, \ldots, -\varepsilon_i x_i, \ldots, \varepsilon_n x_n].
\]

\( \rho_{i,j} \) transforms the subsequence \((\varepsilon_i x_i, \ldots, \varepsilon_j x_j)\) into \((-\varepsilon_j x_j, \ldots, -\varepsilon_i x_i)\) by inverting order and signs within the interval \([i, j]\), for instance

\[
[-5, +1, -3, +2, +4, +6] \cdot \rho_{3,4} = [-5, +1, -2, +3, +4, +6] \\
[-5, +1, -2, +3, +4, +6] \cdot \rho_{3,3} = [-5, +1, +2, +3, +4, +6].
\]

A sector \( x = (\lambda_x, \omega_x) \) is a pair consisting of a label \( \lambda_x \in \{1, \ldots, 2n\} \) and an orientation denoted by \( \omega_x \in \{+, -\} \). We may depict \( x \) as a labeled, oriented wedge, composed by an in- and out-half edge. We denote counterclockwise and clockwise orientations by \( \omega_x = + \) and \( \omega_x = - \), respectively. By abuse of notation we shall also write a sector alternatively as \( \pm \lambda_x \). By abuse of notion, we shall refer to an counterclockwise oriented sector \( x = (\lambda_x, +) \) as \( \lambda_x \) and a clockwise oriented sector \( x = (\lambda_x, -) \) as \( -\lambda_x \).

**Definition 1.** A fatgraph is a triple \( G = (H_{2n}, \sigma, \gamma) \) where \( H_{2n} = \{1, \ldots, 2n\} \) is a set of labeled sectors and \( \sigma, \gamma \) are permutations of sectors such that to
Figure 1: Signed permutations and the action of reversals. The signed permutation $[-5, +1, -3, +2, +4, +6]$ is transformed into the identity via the actions of the four reversals $\rho_{2,3}, \rho_{3,3}, \rho_{1,5}$ and $\rho_{1,4}$.

Each pair $(x, \sigma(x))$ there exists a unique $(y, \sigma(y))$ such that

$$
\begin{array}{c}
\sigma \quad \sigma(x) \\
\gamma \quad \gamma \\
\sigma(y) \quad y
\end{array}
\quad \text{or} \quad
\begin{array}{c}
\sigma \quad \sigma(x) \\
\gamma \quad \gamma \\
\sigma(y) \quad y
\end{array}
$$

The directions of the $\gamma$-verticals are implied by the orientations of pairs of sectors $(x, \sigma(x))$ and $(y, \sigma(y))$ and we shall refer to the above diagrams as untwisted and twisted ribbons, respectively. The genus of a fatgraph $G$ is the genus of its underlying topological quotient space, obtained by identifying the $\gamma$-sides of the ribbons.

The cycles of the permutation $\sigma$ are called vertices, $v$, i.e. a vertex is a cycle of sectors. As in the case of orientable fatgraphs [10], we follow the convention that vertex-cycles are traversed counterclockwise. Note that $\sigma$ and $\gamma$ are unsigned permutations of sectors.

A cycle $\gamma_1 = (s_1^1, \ldots, s_k^1)$ of $\gamma$ can be depicted as to connect the out-half edge of $s_i^1$ with the in-half edge of $s_{i+1}^1$. By construction this does not imply that successive sectors have equal orientations, since traversing twisted ribbons changes the latter. A cycle of $\gamma$ is called a boundary component. A fatgraph is called unicellular if $\gamma$ is an unique cycle.

A ribbon can be denoted by $((x, \sigma(x)), (y, \sigma(y))$. For untwisted ribbons we have $\omega_x = \omega_{\sigma(y)}$ and $\omega_{\sigma(x)} = \omega_y$, while for twisted ribbons $\omega_x = -\omega_y$ and $\omega_{\sigma(x)} = -\omega_{\sigma(y)}$ holds. Ribbons with mono- and bi-directional verticals are
Figure 2: (A) a fatgraph with vertices \((1, 8, -3, 5), (6), (2, -4, 7)\) and boundary components \((1, 2, -3, -4, 5, 6), (7, 8)\). For the pair of sectors, \((-3, 5)\), there is the corresponding pair, \((2, -4)\), such that \(-3 = \gamma(2)\) and \(5 = \gamma(4)\), forming a twisted ribbon. There are in addition the three untwisted ribbons: \(((1, 8), (7, 2)), ((5, 1), (6, 6))\) and \(((8, -3), (-4, 7))\). (B) flipping \((2, -4, 7)\) to \((-2, -7, 4)\): the flipping changes the twist property of ribbons connecting \((2, -4, 7)\) to other vertices.

called \(m\)- and \(b\)-ribbons, respectively. For \(m\)-ribbons we have \(\omega_x = -\omega_{\sigma(x)}\) (as well as \(\omega_y = -\omega_{\sigma(y)}\)) and for \(b\)-ribbons \(\omega_x = \omega_{\sigma(x)}\) (as well as \(\omega_y = \omega_{\sigma(y)}\)).

A flip of a vertex \(v\) is obtained by reversing the cyclic ordering of the sectors incident on \(v\) and changing their respective orientations. This is tantamount to replacing any untwisted and twisted ribbon, that is incident to another vertex by a twisted and untwisted ribbon, respectively. Furthermore loops remain unchanged. Flipping does not affect the underlying topological quotient space, see Fig. 2(B). Indeed, by the fundamental structure theorem of surfaces \([11]\), the latter depends only on the relative directions of the sides of ribbons which are unaffected by flipping. In Fig. 2 we illustrate the concept of fatgraphs and vertex-flips.

**Definition 2.** Two fatgraphs \(G_1 = (H^1_{2n}, \sigma^1, \gamma^1)\) and \(G_2 = (H^2_{2n}, \sigma^2, \gamma^2)\) are isomorphic if there is an bijection, mapping \(H^1_{2n}, \sigma^1\) and \(\gamma^1\) into \(H^2_{2n}, \sigma^2\) and \(\gamma^2\), respectively.

By construction, this bijection preserves the cyclic order of the sectors around the vertices as well as the order of the sectors along the boundary component and maps ribbons into ribbons. There are many additional ways to define isomorphisms of fatgraphs, see \([10]\). **Definition 2** is tailored to facilitate the identification of components with irreducible fatgraphs, see Section \([1]\).
A fatgraph, $G$, represents a cell-complex of a surface $F(G)$: the topological quotient space $F(G)$ is obtained by identifying the sides of the $G$-ribbons using the simplicial homeomorphism, see [10]. Accordingly the genus, $g$, of $G$ is the topological genus of $F(G)$, i.e.

$$2 - g - b = v - e,$$

where $b, e, v$ are the numbers of boundary components, ribbons and vertex-cycles of $G$. We shall write $G_{n,g}$ if we wish to emphasize that $G$ is a fatgraph having $2n$ sectors and genus $g$.

**Lemma 1. (Poincaré dual)** Let $G_{n,g} = (H_{2n}, \sigma, \gamma)$ be a fatgraph, then we have:

(a) $G_{n,g}^* = (H_{2n}, \gamma, \sigma)$ is a fatgraph and $(G_{n,g}^*)^* = G_{n,g}$,

(b) if $G_{n,g} = (H_{2n}, \sigma, \gamma)$ is orientable then $G_{n,g}^* = G_{n,g}$.

**Proof.** Switching $\gamma$ and $\sigma$ for untwisted ribbons results in

$$\begin{array}{cc}
\begin{array}{c}
x \\
\gamma \\
\sigma(y)
\end{array}
\xrightarrow{\sigma}
\begin{array}{c}
\sigma(x) \\
\gamma \\
y
\end{array}
&
\begin{array}{c}
x \\
\gamma \\
\gamma(y)
\end{array}
\xleftarrow{\sigma}
\begin{array}{c}
\sigma(x) \\
\gamma \\
y
\end{array}
\end{array}$$

Depending on the directions of the verticals we derive exactly one of the following ribbons

$$\begin{array}{cc}
\begin{array}{cc}
\gamma(y) & x \\
\gamma & \\
y & \gamma(x)
\end{array}
\xrightarrow{\sigma}
\begin{array}{cc}
\gamma(y) & x \\
\gamma & \\
y & \gamma(x)
\end{array}
&
\begin{array}{cc}
\gamma(y) & x \\
\gamma & \\
y & \gamma(x)
\end{array}
\leftarrow
\begin{array}{cc}
\gamma(y) & x \\
\gamma & \\
y & \gamma(x)
\end{array}
\end{array}$$

We analogously analyze the effect of switching $\gamma$ and $\sigma$ for twisted ribbons:

$$\begin{array}{cc}
\begin{array}{cc}
\sigma(y) & x \\
\gamma & \\
y & \sigma(x)
\end{array}
\xleftarrow{\sigma}
\begin{array}{cc}
\sigma(y) & x \\
\gamma & \\
y & \sigma(x)
\end{array}
&
\begin{array}{cc}
\sigma(y) & x \\
\gamma & \\
y & \sigma(x)
\end{array}
\xrightarrow{\sigma}
\begin{array}{cc}
\sigma(y) & x \\
\gamma & \\
y & \sigma(x)
\end{array}
\end{array}$$

where the verticals have an unique direction induced by the orientations of the sectors. Depending on these directions we derive exactly one of the
following ribbons

\[
\begin{array}{cccc}
  x & \sigma & y \\
  \gamma(x) & \sigma & \gamma(y) \\
  y & \sigma & x \\
  \gamma(y) & \sigma & \gamma(x) \\
  x & \sigma & y \\
  \gamma(x) & \sigma & \gamma(y) \\
  y & \sigma & x \\
  \gamma(y) & \sigma & \gamma(x) \\
\end{array}
\]

Accordingly, each $G_{n,g}$-ribbon is mapped uniquely into a ribbon in $G_{n,g}^*$ and $G_{n,g}^* = (H, \gamma, \sigma)$ is a fatgraph. By construction this dualization preserves topological genus.

It remains to prove the second assertion. Suppose $G_{n,g} = (H_{2n}, \sigma, \gamma)$ is orientable. Then all sectors are $\oplus$ and as a result we have only untwisted ribbons of the following form

\[
\begin{array}{c}
  x \sigma \rightarrow \sigma(x) = \gamma(y) \\
  \gamma \downarrow \sigma \downarrow \gamma \\
  \sigma(y) = \gamma(x) \sigma \leftarrow y \\
\end{array}
\]

Now switching $\sigma$ and $\gamma$ we derive

\[
\begin{array}{c}
  x \sigma \rightarrow \gamma(y) = \sigma(x) \\
  \gamma \downarrow \sigma \downarrow \gamma \\
  \gamma(x) = \sigma(y) \sigma \leftarrow y \\
\end{array}
\]

Accordingly, each $G_{n,g}$-ribbon is mapped identically to a $G_{n,g}^*$-ribbon. Furthermore, if $G_{n,g} = (H_{2n}, \sigma, \gamma)$ is orientable, then $G_{n,g}^* = (H_{2n}, \gamma, \sigma)$ is.

The Poincaré dual maps untwisted ribbons into $b$-ribbons and twisted ribbons into $m$-ribbons. In general, the dual does not preserve twisted or untwisted ribbons and in particular, the dual of a fatgraph having an unique vertex-cycle is a fatgraph with an unique boundary component. We illustrate dualization in Fig. 4.

3. $\pi$-maps

In this section, we shall give a bijection between signed permutations and certain equivalence classes of unicellular fatgraphs. This correspondence is the cornerstone of the paper.
Figure 3: Dualization: mapping ribbons into ribbons. Assume that we have \( b = \gamma(a) \) and \( d = \gamma(c) \). (A) \( d = \sigma(a) \) and \( b = \sigma(c) \), (B) \( c = \sigma(a) \) and \( b = \sigma(d) \), (C) \( d = \sigma(a) \) and \( c = \sigma(b) \), (D) \( c = \sigma(a) \) and \( d = \sigma(b) \). Let \( \overline{\sigma} = \gamma \) and \( \overline{\gamma} = \sigma \), then \( \overline{\sigma}(a) = \gamma(a) = b \) and \( \overline{\gamma}(c) = \gamma(c) = d \). (A) \( d = \overline{\gamma}(a) = \sigma(a) \) \( b = \overline{\gamma}(c) = \sigma(c) \), (B) \( c = \overline{\gamma}(a) = \sigma(a) \) and \( b = \overline{\gamma}(d) = \sigma(d) \), (C) \( d = \overline{\gamma}(a) = \sigma(a) \) and \( c = \overline{\gamma}(b) = \sigma(b) \), (D) \( c = \overline{\gamma}(a) = \sigma(a) \) and \( d = \overline{\gamma}(b) = \sigma(b) \).

Let \( z_{2k+1} = (2k + 1, \omega_{2k+1}) \), i.e. \( z_{2k+1} \) has label \( 2k + 1 \) but arbitrary orientation.

**Definition 3. (\( \pi \)-map)** A unicellular fatgraph, \( \mathbb{P}_{n,g} = (H_{2n+2}, \gamma, \sigma) \), is a \( \pi \)-map if it contains a vertex of the form

\[
V^* = (z_{2n+1}, z_{2n-1}, \ldots, z_3, z_1).
\]

\( V^* \) is called the center. A \( \pi \)-map, \( \mathbb{P}_{n,g} \), is called *reduced* if it does not contain any vertices of degree one.

In Fig. 3 we illustrate the concept of a \( \pi \)-map.

**Definition 4. (Equivalence)** Two \( \pi \)-maps, \( \mathbb{P}_{n,g}^1, \mathbb{P}_{n,g}^2 \), with

\[
\gamma^1 = (x^1_{2n+2}, x^1_{2n+1}, \ldots, x^1_2, x^1_1), \quad \gamma^2 = (x^2_{2n+2}, x^2_{2n+1}, \ldots, x^2_2, x^2_1)
\]

are equivalent, \( \mathbb{P}_{n,g}^1 \sim \mathbb{P}_{n,g}^2 \), iff

- \( \mathbb{P}_{n,g}^1 \) and \( \mathbb{P}_{n,g}^2 \) have the same center,
Figure 4: (A) a fatgraph with \( \sigma = (1, 8, -3, 5)(6)(2, -4, 7) \) and \( \gamma = (1, 2, -3, -4, 5, 6)(7, 8) \). (B) the dual of (A) obtained by exchanging \( \sigma \) and \( \gamma \).

In general the dualization does not the preserve twist property: the untwisted \( m \)-ribbon \(((8, -3), (7, 8))\), contained in (A), dualizes into the twisted \( b \)-ribbon \(((−3, −4), (7, 8))\).

\[ \lambda_{x_{2k}} = \lambda_{x_{2\mu(k)}} \] for some permutation \( \mu \),

- the external \( \mathbb{P}^1_{n,g} \)-vertices can be transformed into \( \mathbb{P}^2_{n,g} \)-vertices via flipping.

Let \( [\mathbb{P}_{n,g}] \) denote the equivalence class of \( \mathbb{P}_{n,g} \) and \( \mathcal{P}_{n,g} \) be the set of equivalence classes.

Clearly, any two equivalent \( \pi \)-maps have the same topological genus. The equivalence class is obtained by permutation of the even labels of the external vertices and by flipping them.

**Proposition 1.** There exists a bijection \( \varphi_n \) between the set of signed permutations, \( \mathcal{B}_{n} \), and equivalence classes of \( \pi \)-maps, \( \mathcal{P}_{n,g} \):

\[ \varphi_n : \mathcal{B}_{n} \rightarrow \mathcal{P}_{n}, \quad b_n \mapsto [\mathbb{P}_{n,g}] \]

**Proof.** Given a signed permutation, \( b_n = (\varepsilon_1y_1, \ldots, \varepsilon_ny_n) \), we associate to \( \varepsilon_py_i \) the sector \( x_{2i+1} \), where \( \lambda_{x_{2i+1}} = 2y_i + 1 \), and \( \omega_{x_{2i+1}} = + \) if \( \varepsilon_i = 1 \) and \( - \), otherwise. Let \( x_1 = (1, +) \) be an additional, + -sector, and let

\[ \gamma^* = (z_{2n+1}, z_{2n-1}, \ldots, z_5, z_3, z_1 = x_1) \]

with \( z_{2k+1} = (2k + 1, \omega_{x_{2k+1}}) \).

**Claim.** Given \( b_n \) and \( x_{2k+1} \) constructed as above, there exists a set of fatgraphs \( \mathcal{G}_{n,g} \) such that
Figure 5: $\pi$-maps: here $v^* = (9, 7, 5, 3, 1)$ and $(2)(4)(6)(8)(10)$ are the external vertices.

(a) $\gamma^*$ is a $\gamma$-cycle in $G_{n,g}$ and $\sigma = (x_{2n+2}, x_{2n+1}, \ldots, x_4, x_3, x_2, x_1)$,
(b) any two of these, $G_{n,g}$ and $G'_{n,g}$ differ by choosing a labeling of the even sectors and an orientation of each boundary component over even sectors.

We shall interpret $\gamma^* = (z_{2n+1}, z_{2n-1}, \ldots, z_5, z_3, z_1 = x_1)$ as a boundary component of length $n$ that traverses all sectors $z_{2n+1}, \ldots, z_1$.

We make the Ansatz

$$\sigma = (x_{2n+2}, x_{2n+1}, \ldots, x_4, x_3, x_2, x_1),$$

where $\lambda_{x_2i} = 2\mu(i)$ and $\mu \in S_n$, i.e. the even sectors are arbitrarily labeled with even numbers. Note that at this point the orientations of the even sectors are not determined, yet. Furthermore, $\gamma^*$ can be expressed in terms of the odd sectors $x_{2k-1}$. Then we have $\gamma^* = (x_{2\tau(i)-1})$, where $x_{2\tau(i)-1} = z_{2i-1}$.

We proceed by producing the orientations of the even sectors $x_{2i}$ as well as the boundary component $\gamma$, containing the cycle $\gamma^*$. These are constructed using the fact that we have to generate ribbons. This defines $\gamma$ via $\gamma(x_{2i}) = x_{2j}$ as follows:

$$\begin{align*}
\begin{cases}
  x_{2i+1} & \overset{\sigma}{\longrightarrow} x_{2i} \\
  x_{2j-1} & \overset{\sigma}{\longrightarrow} x_{2j}
\end{cases} \quad \text{or} \quad
\begin{cases}
  x_{2i+1} & \overset{\sigma}{\longrightarrow} x_{2i} \\
  x_{2j} & \overset{\sigma}{\longrightarrow} x_{2j+1}
\end{cases}
\end{align*}$$

(2)

Accordingly the ribbon structure induces $\gamma$ as a collection of boundary components over even sectors and the unique cycle of odd sectors, $\gamma^*$. The only
choice is that of selecting the orientation of one sector, \( x_{2j} \), for each respective boundary component over even sectors, \( \gamma_r \). The given orientations of the odd sectors naturally determine whether we have an untwisted or a twisted ribbon and eq. (2) defines \( \gamma \) on even-indexed sectors, such that \( \sigma \) and \( \gamma \) produce ribbons.

This construction is unique up to choosing an orientation in each \( \gamma \)-cycle, \( \gamma_r \), over even sectors. Eq. (2) shows that this induces orientations for all sectors (and the directions of the respective ribbon sides) contained in \( \gamma_r \). Accordingly, to a signed permutation corresponds the set of fatgraphs

\[
G_{n,g} = (H_{2n+2}, \sigma, \gamma),
\]

such that any two of them differ by choosing a labeling of the even sectors and an orientation of each boundary component except \( \gamma^* \). This proves the Claim.

We next consider the dual \( P_{n,g} = G^*_{n,g} \). By construction, in \( P_{n,g} \), \( \gamma^* \) becomes the center

\[
v^* = (z_{2n+1}, z_{2n-1}, \ldots, z_5, z_3, z_1 = x_1)
\]

and \( P_{n,g} \) is, by construction, unicellular having boundary component \( \sigma \). Note that in any \( \pi \)-map, \( v^* \) contains the odd sectors labeled in descending order, the only difference consists in their orientations. Thus we have a well-defined mapping

\[
\varphi_n : \mathcal{B}_n \longrightarrow P_{n,g}, \quad \varphi_n(b_n) = [P_{n,g}].
\]

We proceed by constructing \( \varphi_n^{-1} \): given an equivalence class of \( \pi \)-maps \([P_{n,g}]\), we choose a representant, \( P_{n,g} \) and dualize. That is we choose \( \mu \) and the orientations of the cycles over even sectors. This produces the fatgraph \( G_{n,g} \) that has a boundary component cycle \( \gamma^* = (z_{2n+1}, z_{2n-1}, \ldots, z_5, z_3, z_1) \) traversing all odd sectors and the vertex

\[
\sigma = (x_{2n+2}, x_{2n+1}, \ldots, x_4, x_3, x_2, x_1).
\]

To recover the signed permutation we only need partial information: the sequence

\[
(x_{2n+1}, x_{2n-1}, \ldots, x_5, x_3)
\]

is obviously independent of the choice of the representant. This induces an unique signed permutation where the sign of \( \varepsilon_i y_i \) equals the orientation of
Figure 6: From the signed permutation $[-5, +1, -3, +2, +4]$ to its $\pi$-map: first we compute $x_{11} = 9$, $x_9 = 5$, $x_7 = -7$, $x_5 = 3$, $x_3 = -11$ and $x_1 = 1$. Then we set $\sigma = (x_{12} = -12, x_{11} = 9, \ldots, x_3 = -11, x_2 = 2, x_1 = 1)$, where $2\mu(i) = \lambda_{x_2i}$, $\mu \in S_n$ and $x_2i = \sigma(x_{2i+1})$. Furthermore, $\gamma^* = (-11, 9, -7, 5, 3, 1)$. This produces the fatgraph $G_5$, (A). In order to recover the signed permutation, we extract the sequence $x_{11} = 9$, $x_9 = 5$, $x_7 = -7$, $x_5 = 3$, $x_3 = -11$. Then the signed permutation is $[-5, +1, -3, +2, +4]$ obtained via $y_i = (\lambda_{x_{2i+1}} - 1)/2$, for $1 \leq i \leq 5$. In (B) we depict the $\pi$-map of (A) after dualization.

the sector $x_{2i+1}$ and $y_i = (\lambda_{x_{2i+1}} - 1)/2$. □

In Fig. 6 we give an example to show how to construct a $\pi$-map from a signed permutation and back.

We next collect some facts about $\pi$-maps.

**Lemma 2.** Let $b_n$ be a signed permutation and $[P_{n,g}] = \varphi_n(b_n)$, and $G^*_{n,g} = P_{n,g}$. Then

(a) any $P_{n,g}$-ribbon is incident to $v^*$,
(b) a $P_{n,g}$-ribbon is a $m$-ribbon if and only if it is induced by a twisted $G_{n,g}$-ribbon.

**Proof.** (a) immediately follows, since $P_{n,g}$ is the dual of $G_{n,g}$ induced by $b_n$.
and we have:

\[
\begin{array}{c}
x_{2i-1} \xrightarrow{\sigma} x_{2i} \\
\gamma^* \quad \gamma \\
\downarrow \quad \downarrow \\
x_{2j-1} \xrightarrow{\sigma} x_{2j-2}
\end{array}
\quad \leftrightarrow \quad
\begin{array}{c}
x_{2i-1} \xrightarrow{\gamma^*} x_{2j-1} \\
\gamma \quad \gamma \\
\downarrow \quad \downarrow \\
x_{2i} \xrightarrow{\sigma} x_{2j-2}
\end{array}
\]

The case of twisted ribbons is analogous.

Ad (b): this follows from Lemma 1. Switching \(\gamma\) and \(\sigma\) for twisted \(G_{n,g}\)-ribbons (induced by the sign-change of the \(\gamma^*\)-sectors, \(x_{2i-1}, x_{2j-1}\)) means

\[
\begin{array}{c}
x_{2i-1} \xrightarrow{\sigma} x_{2i} \\
\gamma \\
\downarrow \\
x_{2j} \xrightarrow{\sigma} x_{2j-1}
\end{array}
\quad \leftrightarrow \quad
\begin{array}{c}
x_{2i-1} \xrightarrow{\gamma} x_{2i} \\
\gamma \\
\downarrow \\
x_{2j} \xrightarrow{\gamma} x_{2j-1}
\end{array}
\]

The latter diagram is equivalent to exactly one of the following ribbons, depending on the directions of the diagonals,

\[
\begin{array}{c}
x_{2i-1} \xrightarrow{\sigma} x_{2j-1} \\
\gamma \quad \gamma \\
\downarrow \quad \downarrow \\
x_{2i} \xrightarrow{\sigma} x_{2j}
\end{array}
\quad \begin{array}{c}
x_{2i-1} \xrightarrow{\sigma} x_{2j-1} \\
\gamma \quad \gamma \\
\downarrow \quad \downarrow \\
x_{2j} \xrightarrow{\sigma} x_{2i}
\end{array}
\quad \begin{array}{c}
x_{2j-1} \xrightarrow{\sigma} x_{2i-1} \\
\gamma \quad \gamma \\
\downarrow \quad \downarrow \\
x_{2j} \xrightarrow{\sigma} x_{2i}
\end{array}
\quad \begin{array}{c}
x_{2j-1} \xrightarrow{\sigma} x_{2i-1} \\
\gamma \quad \gamma \\
\downarrow \quad \downarrow \\
x_{2i} \xrightarrow{\sigma} x_{2j}
\end{array}
\]

We call a \(\pi\)-map, \(\mathbb{P}_{n,g}\), orientable if it induces an orientable surface, \(F(\mathbb{P}_{n,g})\), and non-orientable, otherwise. Note that the notion of orientability here is different from the notion of oriented cycles in break-point graphs \([1]\). In fact, a permutation with oriented cycles in a breakpoint graph corresponds one to one to a non-orientable \(\pi\)-map.

**Lemma 3. (Non-orientability)** Let \(\mathbb{P}_{n,g}\) be a reduced \(\pi\)-map. Then the following assertions are equivalent:

(a) \(\mathbb{P}_{n,g}\) is non-orientable,
(b) \(\mathbb{P}_{n,g}\) contains a \(m\)-ribbon.
(c) \(\mathbb{P}_{n,g}\) contains an external vertex incident to both: a twisted as well as an untwisted ribbon.

**Proof.** (a) \(\Rightarrow\) (b): suppose \(\mathbb{P}_n\) contains no \(m\)-ribbon. Then \(\mathbb{P}_n\) represents a 2-dimensional cell-complex with exclusively complementary edge pairs. By
the structure theorem of surfaces its underlying topological quotient space is an orientable surface, whence (a) ⇒ (b).

(b) ⇒ (a): suppose \( P_n \) contains a \( m \)-ribbon, \( e \). Since any \( P_n \)-ribbon is incident to \( v^* \) we have a single vertex containing two subsequent sectors with different orientations. Flipping external vertices does not change the fact that \( e \) is an \( m \)-ribbon and \( v^* \) cannot be flipped, by construction. This implies that the quotient space of \( P_n \) is a connected sum of projective planes and as such non-orientable.

(a) ⇒ (c): follows by transposition.

(c) ⇒ (a): the existence of such an external vertex implies that \( v^* \) contains two subsequent sectors with different orientations and hence a \( m \)-ribbon. We then employ (b) ⇒ (a).

Removing all vertices incident to exactly one ribbon and subsequent relabeling induces a projection map \( P \mapsto P^\rho \) from \( \pi \)-maps to reduced \( \pi \)-maps. By Euler’s characteristic equation this projection preserves topological genus.

**Lemma 4.** Let \( b_n \) be a signed permutation and \( [P_{n,g}] = \varphi_n(b_n) \). Then \( b_n \) and the signed permutation \( \varphi^{-1}( [P_{n,g}] ) \) have equal reversal distance.

**Proof.** For an external vertex, that is incident to only one ribbon in \( P_n \), we have by Lemma 2 (a) the following alternative

\[
\begin{align*}
x_{2i+1} \xrightarrow{\gamma^*} x_{2j+1} & \quad \text{or} \quad x_{2i+1} \xrightarrow{\gamma^*} x_{2j+1} \\
x_{2i} & \quad \ x_{2i} \\
x_{2j} & \quad \ x_{2j}
\end{align*}
\]

where the orientations of \( x_{2i+1}, x_{2j+1} \) are equal, both being either \( + \) or \( - \). Without loss of generality, we may assume that \( x_{2i}, x_{2j} \) is \( + \), i.e. we have

\[
\lambda_{x_{2i+1}} + 2 = \lambda_{x_{2j+1}}.
\]

As a result we have, depending on the orientations of \( x_{2i+1}, x_{2j+1} \), being \( + \) or \( - \), either two successive numbers \( (k - 1), k \) or \( -(k - 1), k \), respectively, in \( \varphi_n^{-1}([P_{n,g}]) \). Removing the sector \( x_{2i} \) and replacing the sectors \( x_{2i+1}, x_{2j+1} \) by \( x_{2i+1} \) is equivalent to replacing \( \varphi_n^{-1}([P_{n,g}]) \) by the signed permutation \( b_{n-1} \), obtained by removing \( k \) or \( -(k - 1) \), respectively and by setting \( \pm j \mapsto \pm(j - 1) \) for any \( j > k \); see Fig. 7. Clearly, \( b_n \) and \( b_{n-1} \) have the same reversal distance, whence the lemma.
Figure 7: (A) the $\pi$-map induced by the signed permutation $[-2, -1, 3]$. (B) removing the vertex of degree one, $(-6)$, results in a $\pi$-map corresponding to the signed permutation $[-1, 2]$. This is equivalent to replacing the segment $[-2, -1]$ by $[-1]$. Both $[-2, -1, 3]$ and $[-1, 2]$ have reversal distance 1.

**Corollary 1.** The signed permutation corresponding to the equivalence class of $\pi$-maps having exclusively external vertices of degree one, has reversal distance zero.

### 4. Irreducibility and components

Let $\mathbb{P}_{n,g} = (H_{2n+2}, \sigma, \gamma)$ be a $\pi$-map with center $v^* = (z_{2n+1}, z_{2n-1}, \ldots, z_1)$ where $\lambda_{2k+1} = 2k + 1$. By construction, $v^*$ contains all odd $\mathbb{P}_{n,g}$-sectors.

By abuse of notation we shall identify the $v^*$-sectors with their labels, when their respective orientation is not of relevance. Let $i$ and $j$ be two $v^*$-sectors where $i <_{\sigma} j$, we set

$$[i, j]_\sigma = \{ t \mid t = 2k + 1, \ i \leq_{\sigma} t \leq_{\sigma} j \}$$

$$[i, j]_\gamma = \{ x \mid x = 2k + 1, \ i \leq_{\gamma} x \leq_{\gamma} j \lor j \leq_{\gamma} x \leq_{\gamma} i \}.$$

Note that in both intervals, $[a, b]_\sigma$ and $[a, b]_\gamma$, all sectors are odd. For any $[i, j]_\sigma$ the construction of Proposition 1 possibly produces a $\pi$-map to which
we refer to as $\mathbb{P}_{n,g}^{i,j}$. An interval $[i, j]_\sigma$ inducing the $\pi$-map $\mathbb{P}_{n,g}^{i,j}$ is called minimal, if there exists no sector $i < \sigma k < \sigma j$ such that both $[i, k]_\sigma$ and $[k, j]_\sigma$ induce $\pi$-maps, respectively.

Let $I_{P_{n,g}}$ denote the set of intervals $[i, j]$ inducing $\pi$-maps, $\mathbb{P}_{n,g}^{i,j}$. Then $I_{P_{n,g}}$ becomes via inclusion, i.e.

$$[a, b]_\sigma \subset [c, d]_\sigma \iff c < \sigma a < \sigma b < \sigma d,$$

a partially ordered set. By construction $[1, 2n + 1]_\sigma$ is the unique maximal element of $(I_{P_{n,g}}, \subset)$.

**Definition 5. (Irreducibility)** A $\pi$-map, $\mathbb{P}_{n,g}$, is called irreducible if $I_{P_{n,g}} = \{[1, 2n + 1]\}$.

**Lemma 5.** Suppose $[i, j]_\sigma$ induces a $\pi$-map, then we have

$$[i, j]_\sigma = [i, j]_\gamma.$$

**Proof.** Let $t \in [i, j]_\sigma$, since $[i, j]_\sigma$ induces the $\pi$-map, $\mathbb{P}_{n,g}^{i,j}$, $t$ is necessarily in its boundary component, i.e. $t \in [i, j]_\gamma$ and $[i, j]_\sigma \subseteq [i, j]_\gamma$. $[i, j]_\gamma \subseteq [i, j]_\sigma$ follows analogously. $\square$

**Definition 6. ($\mathbb{P}_n$-component)** Let $\mathbb{P}_{n,g}$ be a $\pi$-map with minimal interval $[i, j]_\sigma$. An interval, $[a_k, b_k]_\sigma \subsetneq [i, j]_\sigma$, is gap if it induces a $\pi$-map and is of maximal length. Suppose

$$v^* = (\ldots, i, \ldots, a_k, a_k', \ldots, b_k', b_k, \ldots, j \ldots),$$

then the disjoint union of intervals

$$C = [i, j]_\sigma \setminus \bigcup_k [a'_k, b'_k]_\sigma$$

is called a $\mathbb{P}_n$-component. A component $C = [i, i + 2]_\sigma$ is called trivial.

We illustrate the concept of a component in Fig. 8.

Let $C$ be a $\mathbb{P}_{n,g}$-component. Collapsing all $C$-gaps and subsequent re-labelling of the sectors we derive:

**Lemma 6.** Let $\mathbb{P}_{n,g}$ be a $\pi$-map, then the following assertions hold

(a) any $\mathbb{P}_{n,g}$-component is isomorphic to an irreducible $\pi$-map,

(b) any $\pi$-map can be uniquely decomposed into a set of components.
A component is orientable if it is nontrivial and its associated, irreducible \( \pi \)-map is orientable, i.e. by Lemma 3, it contains only \( b \)-ribbons.

As for the relation between two distinct component \( C_1, C_2 \) we observe

**Lemma 7.** Let \( C_1 = \bigcup_j [c^{(1)}_j, d^{(1)}_j]_\sigma \) and \( C_2 = \bigcup_j [c^{(2)}_j, d^{(2)}_j]_\sigma \) be two components. Then

\[
\exists 1 \leq j < n - 1; \quad d^{(1)}_j \leq c^{(2)}_1 \quad \implies \quad C_2 \subset [d^{(1)}_j, c^{(1)}_{j+1}]_\sigma.
\]

That is, two components are either subsequent around \( v^* \) or one is contained in a gap of the other.

**Proof.** By construction, the intervals of two components only intersect on their boundaries. Suppose \( d^{(1)}_j \leq c^{(2)}_1 \) for some \( 1 \leq j < n - 1 \). By assumption \([d^{(1)}_j, c^{(1)}_{j+1}]_\sigma\) is a gap and as such it induces a maximal \( \pi \)-map. Since \( c^{(2)}_1 \in [d^{(1)}_j, c^{(1)}_{j+1}]_\sigma \) this implies \( C_2 \subset [d^{(1)}_j, c^{(1)}_{j+1}]_\sigma \).

By Lemma 2 any \( P_{n,g} \)-ribbon, \( t \), is incident to \( v^* \) i.e. it is determined by its pair of incident, odd sectors \( (t, \sigma(t)) \).

**Definition 7.** (\( \sigma \)-crossing) Let \( v_1, v_2 \) be two external vertices and let

\[
[v] = \{r \mid r \text{ is a ribbon that is incident to } v\}.
\]
Figure 9: $\sigma$-crossing: $v_1$ and $v_2$ are $\sigma$ crossing since $-9 < \sigma - 3 \leq \sigma - 3 < \sigma 11$, where ribbon $(-9, -3) \in [v_2]$ and ribbon $(-3, 11) \in [v_1]$. $v_1$ and $v_3$ are not $\sigma$-crossing.

Then $v_1, v_2$ are $\sigma$-crossing if and only if there exist four ribbons $r_1, r_3 \in [v_1]$, $r_2, r_4 \in [v_2]$ such that $t_{r_1} <_\sigma t_{r_2} \leq t_{r_3} <_\sigma t_{r_4}$.

Let (●) be the following property: for any for two external vertices $v_1, v_2$ there exists a sequence $(v_1 = w_1, w_2, \ldots, w_{k-1}, w_k = v_2)$ such that $w_i, w_{i+1}$ are $\sigma$-crossing, see Fig. 9.

Lemma 8. Let $\mathbb{P}_{n,g}$ be a $\pi$-map, then the following assertions are equivalent:
(a) $\mathbb{P}_{n,g}$ is irreducible,
(b) $\mathbb{P}_{n,g}$ satisfies (●).

Proof. (a) $\Rightarrow$ (b): for an arbitrary external vertex, $v$, let $S(v)$ be the set of $v^*$-sectors associated to ribbons that are either incident to $v$ or incident to vertices crossing $v$. The sets $S(v)$ are partially ordered via

$$S(v) \leq S(v') \iff [\min S(v), \max S(v)]_\sigma \subset [\min S(v'), \max S(v')]_\sigma.$$ 

Let $S(v_0)$ be a $\leq$-minimal element. Then $S(v_0)$ is an interval $[i, k]_\sigma$ such that any external vertex incident to a ribbon attached to $[i, k]_\sigma$ has all its incident ribbons attached to $[i, k]_\sigma$. This means $[i, k]_\sigma$ forms an induced $\pi$-map. Since $\mathbb{P}_{n,g}$ is irreducible we have $[i, k]_\sigma = [1, 2n + 1]_\sigma$, whence for any two external vertices (●) holds.

(b) $\Rightarrow$ (a): suppose $\mathbb{P}_{n,g}$ is not irreducible. Then there exists some interval $[i, k]_\sigma \neq [1, 2n + 1]_\sigma$ such that $\mathbb{P}^{i,k}_{n,g}$ is a $\pi$-map. Let $v_1$ be an external vertex
that is incident to a ribbon attached to \([i, k]_{\sigma}\) and let \(v_2\) be an external vertex, incident to a ribbon not attached to \([i, k]_{\sigma}\).

By Lemma 5, since \(P_{n,k}\) is a \(\pi\)-map, we have \([i, j]_{\sigma} = [i, j]_{\gamma}\) and any vertex incident to a \([i, k]_{\sigma}\)-ribbon has all its incident ribbons attached to \([i, k]_{\sigma}\). Consequently, for \(v_1\) and \(v_2\) a sequence \((v_1 = w_1, w_2, \ldots, w_{k-1}, w_k = v_2)\) of mutually crossing vertices does not exist. This means that (ii) implies that \(P_{n,g}\) is irreducible.

5. Reversals

In this section we study the action of reversals on \(\pi\)-maps. Suppose the signed permutation

\[b_n = (\epsilon_1 y_1, \ldots, \epsilon_1 y_i, \ldots, \epsilon_j y_j, \ldots, \epsilon_n y_n),\]

is acted upon by the reversal \(\rho_{i,j}\). The action produces the signed permutation

\[b_n \cdot \rho_{i,j} = \bar{b}_n = (\epsilon_1 y_1, \ldots, -\epsilon_j y_j, \ldots, -\epsilon_i y_i, \ldots, \epsilon_n y_n).\]

\(b_n\) and \(\bar{b}_n\) induce by Proposition 1 the equivalence classes \([P_n]\) and \([\bar{P}_n]\). That is we have the diagram

\[
\begin{array}{c}
\downarrow \phi_n \\
\rho_{i,j} \\
\uparrow \phi_n \\
\bar{b}_n \rightarrow [\bar{P}_n] \\
\rho_{i,j} \\
\downarrow \\
b_n \rightarrow[P_n]
\end{array}
\]

Accordingly, we have a natural reversal action on equivalence classes of \(\pi\)-maps induced by the reversal-right multiplication in the group of signed permutations and making the above diagram commutative.

In order to describe the action of reversals on \([P_n]\) combinatorially, we reconsider the relation between sectors and ribbons in fatgraphs. On the one hand a ribbon is a diagram

\[
\begin{array}{c}
\sigma \downarrow x \quad \sigma \uparrow \sigma(x) \\
\gamma \quad \gamma \\
\sigma(y) \downarrow y \quad \sigma(y) \uparrow \sigma(y)
\end{array}
\]

or

\[
\begin{array}{c}
\sigma \downarrow x \quad \sigma \uparrow \sigma(x) \\
\gamma \quad \gamma \\
\sigma(y) \downarrow y \quad \sigma(y) \uparrow \sigma(y)
\end{array}
\]

where the directions of the verticals labeled by \(\gamma\) are implied by the orientations of \(x, \sigma(x), y, \sigma(y)\) determined by these four sectors. On the other hand,
a sector $\sigma(x)$ is determined by the pair of its incident ribbons

\[
\begin{array}{c}
\sigma & \sigma(x) \\
\gamma & \gamma & \gamma & \gamma \\
\sigma^2(y) & \sigma(y) & \sigma(y) & \sigma(y)
\end{array}
\]

depicted here, w.l.o.g. as being untwisted. Furthermore, any even vertex can be described by the sequence of its incident ribbons $(e_1, \ldots, e_k)$, such that $e_i = (\sigma^{-1}(t), t)$ and $e_{i+1} = (t, \sigma(t))$, for $1 \leq i \leq k$.

Let $\mathbb{P}_n$ be a $\pi$-map with boundary component

\[
\gamma = (x_{2n+2}, \ldots, x_{2j+2}, x_{2j+1}, \ldots, x_{2i+1}, x_{2i}, x_{2i-1}, \ldots, x_1).
\]

**Scenario 1 (gluing):** suppose the sectors $x_{2j+2}$ and $x_{2i}$ are located at the two distinct vertices $v_1$ and $v_2$. Without changing the equivalence class of $\mathbb{P}_n$, we can, (by means of flipping $v_2$ if necessary, which has no effect on the $v^*$-sectors) assume that $v_1$ and $v_2$ are given by

\[
v_1 = ((\lambda_{x_{2j+2}}, +), h_1^{(1)}, \ldots, h_{m_1}^{(1)}),
\]
\[
v_2 = ((\lambda_{x_{2i}}, -), h_1^{(2)}, \ldots, h_{m_2}^{(2)}).
\]

We represent $v_1$ and $v_2$ by the sequences of ribbons

\[
v_1 = ((h_1^{(1)}, x_{2j+2}), (x_{2j+2}, h_1^{(1)}), e_3^{(1)}, \ldots, e_{m_1}^{(1)}),
\]
\[
v_2 = ((h_2^{(2)}, x_{2i}), (x_{2i}, h_1^{(2)}), e_3^{(2)}, \ldots, e_{m_2}^{(2)}).
\]

Next we set

\[
\tilde{v} = ((h_1^{(1)}, x_{2j+2}), (x_{2i}, h_1^{(2)}), e_3^{(2)}, \ldots, e_{m_2}^{(2)}, h_1^{(2)}, x_{2i}), (x_{2j+2}, h_1^{(1)}), e_3^{(1)}, \ldots, e_{m_1}^{(1)})
\]

and obtain by replacing $v_1$ and $v_2$ by $\tilde{v}$ the fatgraph $\mathbb{P}_{\tilde{v}}$, see Fig. 10.

**Scenario 2 (slicing):** suppose $x_{2j+2}$ and $x_{2i}$ are located at $v$ and that $x_{2j+2}$ and $x_{2i}$ have different orientations. Without changing the equivalence
class of $\mathbb{P}_n$ we may assume that

$$v = (\lambda_{x_{2j+2}}, +), h_{1}^{(2)}, \ldots, h_{m_2}^{(2)}, (\lambda_{x_{2i}}, -), h_{1}^{(1)}, \ldots, h_{m_1}^{(1)}).$$

We next express $v$ as the sequence of ribbons

$$v = ((h_{m_1}^{(1)}, x_{2j+2}), (x_{2j+2}, h_{1}^{(1)}), e_{3}^{(1)}, \ldots, e_{m_1}^{(1)}). (h_{m_2}^{(2)}, x_{2i}), (x_{2i}, h_{1}^{(1)}), e_{3}^{(1)}, \ldots, e_{m_1}^{(1)}).$$

Let

$$\tilde{v}_1 = ((h_{m_1}^{(1)}, x_{2j+2}), (x_{2i}, h_{1}^{(1)}), e_{3}^{(1)}, \ldots, e_{m_1}^{(1)}), \quad \tilde{v}_2 = ((h_{m_2}^{(2)}, x_{2i}), (x_{2j+2}, h_{1}^{(2)}), e_{3}^{(2)}, \ldots, e_{m_2}^{(2)}).$$

Replacing $v$ by $\tilde{v}_1$ and $\tilde{v}_2$ in $\mathbb{P}_n$, we obtain a fatgraph $\tilde{\mathbb{P}}_n$ having the same ribbons as $\mathbb{P}_n$, see Fig. 11.

**Scenario 3 (half-flipping):** suppose $x_{2j+2}$ and $x_{2i}$ are located at $v$ and
suppose furthermore that \( x_{2j+2} \) and \( x_{2i} \) have the same orientation, i.e.,

\[
v = ((\lambda x_{2j+2}, +), h^{(1)}_1, \ldots, h^{(1)}_{m_1}, (\lambda x_{2i}, +), h^{(2)}_1, \ldots, h^{(2)}_{m_2}).
\]

We represent \( v \) as

\[
v = ((h^{(2)}_{m_2}, x_{2j+2}), (x_{2j+2}, h^{(1)}_1), e^{(1)}_3, \ldots, e^{(1)}_{m_1}, (h^{(1)}_{m_1}, x_{2i}), (x_{2i}, h^{(2)}_1), e^{(2)}_3, \ldots, e^{(2)}_{m_2}).
\]

Next we set

\[
\tilde{v} = ((h^{(2)}_{m_2}, x_{2j+2}), (x_{2i}, h^{(1)}_1), \tilde{e}^{(1)}_m, \ldots, \tilde{e}^{(1)}_{m_1}, (h^{(1)}_{m_1}, x_{2j+2}), (x_{2i}, h^{(2)}_1), \tilde{e}^{(2)}_3, \ldots, \tilde{e}^{(2)}_{m_2}).
\]

Replacing \( v \) by \( \tilde{v} \) we obtain the fatgraph \( \tilde{P}_n \), see Fig. 12.

\[\text{Lemma 9.}\] Let \( P_n \) be a \( \pi \)-map satisfying eq. (4) and having the boundary component

\[
\gamma = (x_{2n+2}, \ldots, x_{2j+2}, x_{2j+1}, \ldots, x_{2i+1}, x_{2i}, x_{2i-1}, \ldots, x_1).
\]

Then \( \tilde{P}_n \) is a \( \pi \)-map having the boundary component

\[
\tilde{\gamma} = (x_{2n+2}, \ldots, \tilde{x}_{2j+2}, -x_{2i+1}, \ldots, -x_{2i+2}, \ldots, -x_{2j+1}, \tilde{x}_{2i}, \ldots, x_1),
\]

where \( -x_k = (\lambda x_k, -\omega x_k) \) denotes the sector \( x_k \) having reversed orientation.

\[\text{Proof.}\] By construction \( \tilde{P}_n \) is a fatgraph having a center vertex \( \tilde{v}^* \). As for
\(\tilde{\mathbb{P}}_n\)-boundary components, we consider the sector \(x_{2n+2}\). The \(\tilde{\mathbb{P}}_n\)-boundary component starting at \(x_{2n+2}\) visits the same sectors and ribbons as in \(\mathbb{P}_n\) before arriving at \(\tilde{x}_{2j+2}\). \(\tilde{x}_{2j+2}\) is the sector formed by the pair of ribbons 

\[
(h_{m_1}^{(1)}, x_{2j+2}) = e_1^{(1)} \quad (x_{2i}, h_{1}^{(2)}) = e_2^{(2)} \quad \text{(gluing)}
\]

\[
(h_{m_1}^{(1)}, x_{2j+2}) = e_1^{(1)} \quad (x_{2i}, h_{1}^{(1)}) = e_2^{(1)} \quad \text{(slicing)}
\]

\[
(h_{m_2}^{(2)}, x_{2j+2}) = e_1^{(1)} \quad (x_{2i}, h_{1}^{(1)}) = e_2^{(2)} \quad \text{(half-flipping)}
\]

In all three cases, the next sector of the tour is traversing \(x_{2i+1}\) in reverse orientation, i.e. we have \(-x_{2i+1} = (\lambda_{x_{2i+1}}, -\omega_{x_{2i+1}})\). We continue now traversing the \(\tilde{\mathbb{P}}_n\)-boundary component as the \(\mathbb{P}_n\)-boundary component, \(\gamma\), in reverse order via \(-x_{2i+2}, \ldots, -x_{2j+1}\). We then arrive at the new sector \(\tilde{x}_{2i}\), given by the pair of ribbons 

\[
(h_{m_2}^{(2)}, x_{2i}) = e_1^{(2)} \quad (x_{2j+2}, h_{1}^{(2)}) = e_2^{(1)} \quad \text{(gluing)}
\]

\[
(h_{m_1}^{(1)}, x_{2i}) = e_1^{(1)} \quad (x_{2j+2}, h_{1}^{(1)}) = e_2^{(2)} \quad \text{(slicing)}
\]

\[
(h_{1}^{(1)}, x_{2j+2}) = e_1^{(1)} \quad (x_{2i}, h_{1}^{(2)}) = e_2^{(2)} \quad \text{(half-flipping)}
\]

The next sectors traversed by the \(\tilde{\mathbb{P}}_n\)-boundary component are 

\(x_{2i-1} = x_{2i-1}, x_{2i-2}, \ldots, x_1\)

and thus traversed as in the \(\mathbb{P}_n\)-boundary component \(\gamma\). Accordingly, \(\tilde{\mathbb{P}}_n\) has the unique boundary component 

\[
\tilde{\gamma} = (x_{2n+2}, \ldots, \tilde{x}_{2j+2}, -x_{2i+1}, \ldots, -x_{2i+2}, -x_{2j+1}, \tilde{x}_{2i}, \ldots, x_1)
\]

and the lemma follows. \(\square\)

Inspecting the effect of Lemma 9 on the underlying signed permutation, we shall categorize the action of reversals as to either glue, splice or half-flip.

Lemma 10. Let \(\rho_{i,j}\) be a reversal acting on the \(\pi\)-map \([\mathbb{P}_{n,g}]\) and let \(b_n = \varphi_n^{-1}(\mathbb{P}_{n,g})\). Then

\[
[\mathbb{P}_{n,g}] \cdot \rho_{i,j} = [\tilde{\mathbb{P}}_{n,g}]^\prime.
\]

That is \(\tilde{\mathbb{P}}_{n,g} \sim \mathbb{P}_{n,g}^\prime\), where \(\tilde{\mathbb{P}}_{n,g}^\prime\) is obtained by either gluing, splicing or
half-flipping. Furthermore we have $|g - g'| \leq 1$ and
\[ d(b_n) \geq g. \]

**Proof.** The reversal $\rho_{i,j}$ determines uniquely the pair of even sectors $(x_{2i}, x_{2j+2})$ in $\mathbb{P}_{n,g}$. If the two sectors belong to two distinct $\mathbb{P}_{n,g}$-vertices, by Proposition 1, the orientations of $x_{2i}, x_{2j+2}$ can be chosen as to satisfy eq. (4). Otherwise, $x_{2i}, x_{2j+2}$ have either distinct or equal orientations. This corresponds to the three scenarios: gluing, slicing and half-flipping.

By Lemma 9 any of these generates the $\pi$-map $\tilde{\mathbb{P}}_{n,g'}$ having the boundary component $\tilde{\gamma}$, respectively. By Proposition 1 a $\pi$-map with boundary component $\tilde{\gamma}$ induces an equivalence class that corresponds to the signed permutation $b_n \rho_{i,j}$. Consequently we have
\[ \varphi_n^{-1}([\tilde{\mathbb{P}}_{n,g}]) = b_n \rho_{i,j}. \]

Since $\varphi_n$ is a bijection between the set of signed permutations and equivalence classes of $\pi$-maps we derive
\[ \tilde{\mathbb{P}}_{n,g'} \sim \mathbb{P}_{n,g'}. \]

Euler’s characteristic equation immediately implies $|g - g'| \leq 1$, whence any reversal decreases the genus of the underlying $\pi$-map by at most one, i.e.
\[ d(b_n) \geq g. \]

\[ \square \]

6. Non-orientable components

By Lemma 9 any $\pi$-map $\mathbb{P}_{n,g}$ can be uniquely decomposed into components, $C_i$, having genus $g_i$. Each of these is isomorphic to an irreducible $\pi$-map and we have $\sum_i g_i = g$.

In the following we shall show that any non-orientable component, $C_i$, of $\mathbb{P}_{n,g}$, or equivalently any non-orientable, irreducible $\pi$-map can be spliced into $\varphi_n^{-1}([\text{id}])$ using $g_i$ reversals. This implies in particular the sharpness of the lower bound on the reversal distance given by Lemma 10.

The following result is due to [1]. The proof given here is based on the characterization of components via $\sigma$-crossings.
In the following we present Theorem 4 of [1] employing the \( \pi \)-map framework. This theorem plays a key role of computing the reversal distance of signed permutations. An irreducible, non-orientable \( \pi \)-map corresponds to an oriented component in the breakpoint graph [1].

**Lemma 11.** Let \( \mathbb{P}_{n,g} \) be an irreducible, non-orientable \( \pi \)-map of genus \( g \), then its associated, signed permutation, \( b_n = \varphi_n^{-1}(\mathbb{P}_{n,g}) \), has reversal distance \( g \).

**Proof.** Since \( C \) is non-orientable there exists some \( m \)-ribbon \( e \). We shall show

**Claim 0.** There exists some \( m \)-ribbon \( e \) in \( C \) such that splicing \( e \) decomposes \( C \) into exclusively non-orientable components. In particular, \( C \) can be successively spliced into trivial components.

By Lemma 3 there exists some \( m \)-ribbon, \( e \) in \( C \). Suppose splicing \( e \) decomposes \( C \) into the components \( C_{e1}, \ldots, C_{ek} \). If \( C_{e1} \) is orientable, then it contains exclusively \( b \)-ribbons \( y_{e1} \). Since \( C_1 = \{ y \in C \mid y \in C_{e1} \} \) is, due to the presence of \( e \), not a component in \( C \), not all of them can be \( b \)-ribbons. That is, there exists some \( b \)-ribbon, \( e_{e1} \), that was originally a \( m \)-ribbon in \( C \). Since \( e_{e1} \in C_{e1} \) and \( C_{e1} \) is orientable, \( e_{e1} \) is untwisted. Since splicing does not change untwisted edges into twisted ones, we can conclude that \( e_1 \) is untwisted.

**Claim 1.** Splicing \( e_1 \) produces from \( C \) a non-orientable component \( C_{e1}^* \).

By Lemma 3 \( C \) contains an external vertex, \( v \), incident to a twisted and an untwisted ribbon, respectively. We shall show that also \( v^{e1} \), the vertex obtained by slicing \( e_1 \), has this property in \( C_{e1}^* \). Since \( e_1 \) is untwisted, we have to assure that splicing \( e_1 \) does not eliminate the only untwisted \( v \)-ribbon.

Clearly, if \( v \) is not incident \( e_1 \), then \( v^{e1} \) is still incident to a twisted and an untwisted ribbon.

Otherwise, we observe that there exists at least one additional untwisted ribbon incident to \( v \). Indeed, splicing \( e \) produces by assumption the orientable component \( C_{e1}^* \), containing \( e_1 \). Suppose now \( e_1 \) were the only untwisted ribbon incident to \( v \). Then the orientability of \( C_{e1}^* \) guarantees that (a) \( v \) cannot be incident to any twisted ribbon, since slicing \( e \) does preserve twisted and untwisted ribbons, (b) \( C_{e1}^* \) contains only the ribbon \( e_{e1} \). (a) and (b) imply that \( C_{e1}^* \) is trivial, which is impossible and Claim 1 follows.

We next show that \( C_{e1}^* \) is quite “large”: let \( R_e \) and \( R_{e1} \) denote the sets of ribbons derived from \( C \) by splicing \( e \) and \( e_1 \), respectively. We call two
Figure 13: (A) A non-orientable component, $C$, with $m$-ribbons $e = ((-3,17),(-4,18))$ and $e_1 = ((-7,19),(-8,20))$. (B) Slicing $e$ in $C$ induces $C^e_1$ and $C^e_2$, both of which being orientable components. The ribbon $e_1$ is contained in $C^e_2$. (C) Slicing $e_1$ instead of $e$ in $C$ generates the component $C^{e_1}$. $C^{e_1}$ is non-orientable and contains all ribbons that are not associated to $C^e_1$-ribbons.

ribbons $y^e$ and $y^{e_1}$ associated if they are induced by the same $C$-ribbon, $y$.

$$\xymatrix{ & y \ar[dl]_{y^e} \ar[dr]^{y^{e_1}} & }$$

**Claim 2.** $C^{e_1}$ contains all ribbons $y^{e_1}$ that are not associated to $C^e_1$-ribbons.

By assumption, splicing $e$ produces the components $C^e_1, \ldots, C^e_k$. Let $V$ denote the set of external $C$-vertices that are, after slicing $e$, contained in $C^e_2, \ldots, C^e_k$. By Lemma[8] components are characterized by $\sigma$-crossing, whence slicing $e_1^e \in C^e_1$ does not affect any of the components $C^e_2, \ldots, C^e_k$. The key point here is that w.r.t. $\sigma$-crossing, $e_1$ and $e^e_1$ have the same effect since they differ only by $e_1$ being a $m$-ribbon and $e^e_1$ being a $b$-ribbon, see Fig. 13. Therefore slicing $e_1$ in $C$ does not affect the $\sigma$-crossing property of $V$-vertices. By construction, $e$ connects all $V$-vertices which proves that $C^{e_1}$ contains all ribbons $y^{e_1}$ that are not associated to $C^e_1$-ribbons.

Claim 1 and Claim 2 show that splicing $e_1$ instead of $e$ generates the non-orientable component $C^{e_1}$, together with a set of ribbons, $z^{e_1}$, that are associated to $C^e_1$-ribbons. By assumption $C^e_1$ is an orientable component and since $e_1$ is spliced the number of these $z^{e_1}$ is strictly smaller than the number of those contained in $C^e_1$. Accordingly, Claim 0 follows by induction on the
number of ribbons contained in orientable components.

By Lemma 10 each such splicing reduces the genus by one eventually into a \( \pi \)-map containing only external vertices of degree one. The lemma follows then from Corollary 1.

As a result we now have

**Theorem 1.** Let \( b_n \) be a signed permutation, then we have

\[
d(b_n) \geq g,
\]

i.e. the topological genus is a sharp bound for the reversal distance.

7. Orientable components

By Lemma 11 irreducible, non-orientable \( \pi \)-maps of genus \( g \) have reversal distance \( g \). Thus it remains to analyze orientable components or equivalently, orientable, irreducible \( \pi \)-maps. In difference to non-orientable components, that could be treated individually, orientable components acted upon by reversals have to be considered as an ensemble. This is a result of Lemma 7, i.e. these components are either concatenated or nested and the action of reversals affects entire chains of them.

We shall begin by showing that half-flipping transforms an orientable component into an non-orientable component.

**Lemma 12.** Suppose \( P_n \) is a \( \pi \)-map and \( C \) is a non-trivial, orientable \( P_n \)-component having genus \( g \). Let \( e \) be a b-ribbon in \( C \) having the two even sectors, \( x_{2j+2}, x_{2i} \) and let \( \tilde{P}_n \) be the \( \pi \)-map obtained by half-flipping the vertex incident to \( e \) w.r.t. \( x_{2i} \) and \( x_{2j+2} \). Then \( \tilde{C} \) is a non-orientable \( \tilde{P}_n \)-component having genus \( g \).

**Proof.** Non-triviality of \( C \) implies that \( v \) has at least two incident ribbons and as in Scenario 3 we write \( v \) as

\[
v = \left( \lambda_{
abla x_{2j+2}}, +, h^{(1)}_1, \ldots, h^{(1)}_{m_1}, \lambda_{\nabla x_{2i}}, +, h^{(2)}_1, \ldots, h^{(2)}_{m_2} \right)_{x_{2j+2}, x_{2i}}
\]

\[
= \left( \lambda_{\nabla h^{(2)}_{m_2}, x_{2j+2}}, \lambda_{\nabla x_{2j+2}}, h^{(1)}_1, e_3^{(1)}, \ldots, e_{m_1}^{(1)}, \lambda_{\nabla h^{(1)}_{m_1}, x_{2i}}, \lambda_{\nabla x_{2i}}, h^{(2)}_1, e_3^{(2)}, \ldots, e_{m_2}^{(2)} \right)_{e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_2^{(2)}}.
\]
Since $C$ is orientable, all ribbons incident to $v$ are untwisted. By half-flipping, $v$ becomes the vertex $\tilde{v}$, 
\[
\tilde{v} = \left( (h^{(2)}_{m_2}, x_{2j+2}), (x_{2i}, h^{(1)}_{m_1}), e^{(1)}_1, \ldots, e^{(1)}_3, \left( h^{(1)}_{1}, x_{2j+2} \right), (x_{2i}, h^{(2)}_{1}), e^{(2)}_3, \ldots, e^{(2)}_{m_2} \right),
\]
where the ribbon $\bar{e}$ is obtained by twisting $e$. By Lemma 8, a component is characterized by $(\bullet)$, whence $\tilde{C}$ is a $\tilde{\mathbb{P}}_n$-component containing the external vertex $\tilde{v}$. Furthermore $\tilde{v}$ is incident to both: twisted and untwisted ribbons, respectively. By Lemma 3, $\tilde{C}$ is non-orientable. Euler’s characteristic equation implies that $\tilde{C}$ has genus $g$. \[\square\]

We now proceed by formalizing the partial order of orientable components implied by Lemma 8. Let $\mathfrak{O}_{\mathbb{P}_n}$ denote the set of orientable components. By Lemma 7, we have the partial order $C \vartriangleleft C' \iff C \neq C'$ and $C$ is nested in $C'$. We shall add to $(\mathfrak{O}_{\mathbb{P}_n}, \sqsubseteq)$ the element $\ast$, which contains any other orientable component. If we consider $(\mathfrak{O}_{\mathbb{P}_n} \setminus \{C\}, \sqsubseteq)$, we say $C$ is deleted from $(\mathfrak{O}_{\mathbb{P}_n}, \sqsubseteq)$. Let $C_1, C_2 \in \mathfrak{O}_{\mathbb{P}_n}$ and suppose $C_1, C_2 \in \mathfrak{O}_{\mathbb{P}_n}$ are not nested. Then we shall, w.l.o.g., assume 
\[
\min_{\sigma} C_1 <_{\sigma} \max_{\sigma} C_1 <_{\sigma} \min_{\sigma} C_2.
\]
We set $C_{1,2}$ to be the smallest orientable component containing $C_1, C_2$ and 
\[
[C_1, C_2] = \{ C_j \mid C_1 \sqsubseteq C_j \sqsubseteq C_2 \}.
\]
Let $C \in \mathfrak{O}_{\mathbb{P}_n}$ such that $C_1 \sqsubseteq C, C_2 \sqsubseteq C$. Then $C$ separates $C_1$ and $C_2$ if $C_1$ and $C_2$ are contained in distinct $C$-gaps, see Fig. 14 (A) and (B). We write $C_1 \ast C_2$ if and only if 
\[
\max_{\sigma} C_1 <_{\sigma} \max_{\sigma} C_2.
\]

**Lemma 13.** Let $\mathbb{P}_n$ be a $\pi$-map with $C_1, C_2 \in \mathfrak{O}_{\mathbb{P}_n}$ and $x_{2j+2}, x_{2i}$ be two even sectors contained in $C_1$ and $C_2$, respectively. Then gluing $x_{2j+2}$ and $x_{2i}$ generates a new, non-orientable component $C_*$, obtained by merging the
Figure 14: Suppose $C_1 \subset C_2 \subset C_3$ and $C_1, C_2 \sqsubset C_3 = C_{1,2}$. (A): $C_3$ does not separate $C_1$ and $C_2$, while in (B) $C_3$ does. (C): gluing $C_1$ and $C_2$ in (A) does not merge $C_3$ into the non-orientable component $C_\ast$. (D): gluing $C_1$ and $C_2$ in (B) merges $C_3$ into the non-orientable component $C_\ast$.

following set of orientable $\mathbb{P}_n$-components

$$M = \begin{cases} [C_1, C_{1,2}] \cup [C_2, C_{1,2}] \cup \{C_{1,2}\} & \text{if } C_{1,2} \text{ separates } C_1 \text{ and } C_2 \\ [C_1, C_{1,2}] \cup [C_2, C_{1,2}] & \text{otherwise.} \end{cases} \quad (6)$$

Proof. Let $C_1, C_2 \in \mathcal{O}_{\mathbb{P}_n}$ and let $v_1$ be the $C_1$-vertex containing $x_{2j+2}$ and $v_2$ the $C_2$-vertex containing $x_{2i}$. By Lemma 9 gluing merges $v_1$ and $v_2$ into $v_\ast$, without changing the $\sigma$-crossings of any other vertices. Consequently, Lemma 9 and Lemma 8 imply that all $C_1$- and $C_2$-vertices merge in $\tilde{\mathbb{P}}_n$ into one component, $C_\ast$. By Lemma 9 gluing produces a pair of sectors $\tilde{x}_{2j+2}, \tilde{x}_{2i}$, that are contained in a single, external vertex and that have different orientations. Thus, by Lemma 9 $C_\ast$, is a non-orientable component.

We consider $C_3 \in [C_1, C_{1,2}]$ such that $C_1 \sqsubset C_3$. Since $C_1 \sqsubset C_3$ there are two ribbons $(a, \sigma(a)), (b, \sigma(b))$ incident to some $C_3$-vertex $v_3$ such that

$$\sigma(a) \prec_\sigma d \prec_\sigma \sigma(d) \prec_\sigma b, \quad (7)$$
where \( d, \sigma(d) \) are the odd sectors of an arbitrary \( C_1 \)-ribbon. \( C_2 \) is by definition of \([C_1, C_{1.2}]\) not nested in \( C_3 \). Thus \( C_2 \) lies, counterclockwise around \( v^* \), to the right of \( C_3 \). As a result we have

\[
\sigma(a) <_{\sigma} \min_{\sigma} C_1 <_{\sigma} \max_{\sigma} C_1 <_{\sigma} b <_{\sigma} \min_{\sigma} C_2.
\]

Gluing \( v_1, v_2 \) w.r.t. the sectors \( x_{2j+2} \) and \( x_{2i} \) generates the vertex \( v_* \) and a \( \sigma \)-crossing of \( v_* \) and \( v_3 \). Therefore \( C_3 \) is merged into \( C_* \), see Fig. 14 (D).

The case of \( C_{1.2} \) separating \( C_1 \) and \( C_2 \) is analogous: then there exists a \( C_{1.2} \)-vertex, \( w \), together with two incident ribbons \((a, \sigma(a)), (b, \sigma(b))\) such that eq. (8) holds. Accordingly, gluing \( v_1, v_2 \) w.r.t. \( x_{2j+2} \) and \( x_{2i} \) generates a \( \sigma \)-crossing of \( v_* \) and \( w \), see Fig. 14 (C). The case of \([C_2, C_{1.2}]\) is argued analogously, whence the lemma.

Lemma 13 suggests to glue two vertices contained in minimal \( O_P n \)-elements, \( C_1, C_2 \). This collapses at least the entire chains \([C_1, C_{1.2}]\) and \([C_2, C_{1.2}]\), respectively, as well as possibly \( C_{1.2} \) if \( C_{1.2} \) separates \( C_1, C_2 \).

8. The reversal distance

This section is the reformulation of Hannenhalli and Pevzner’s treatment of hurdles [1] into the topological framework. To relate our approach to breakpoint graphs, we note that an orientable component in a \( \pi \)-map corresponds to a component without oriented cycles in the breakpoint graph. Furthermore, gluing two hurdles in a \( \pi \)-map corresponds to the merging two hurdles in the breakpoint graph [1].

**Definition 8.** [1] A hurdle is either

- a minimal \( O_{P_n} \)-element, i.e. an interval \( v^*-[i, k]_{\sigma} \) inducing an irreducible \( \pi \)-map, or
- the maximum element in \((O_{P_n}, \sqsubset)\) which does not separate any pair of leaves.

A super-hurdle is a \((O_{P_n}, \sqsubset)\)-hurdle, whose deletion creates a \((O_{P_n} \setminus \{C\}, \sqsubset)\)-hurdle.

A reversal is called safe if it reduces \((g + h)\) by one, i.e., either \( g \) decreases by one and \( h \) persists, or \( g \) increases by one and \( h \) decreases by two.

Then we have
Figure 15: Hurdles: (A) a $\pi$-map with 6 orientable components. $C_1$ and $C_3$ are minimal and $C_6$ is a maximal hurdle. $C_1$ and $C_3$ are in addition super-hurdles, removing $C_1$ and $C_3$ renders $C_2$ and $C_4$ as hurdle, respectively and neither, $C_2$ or $C_4$ were originally a hurdle. $C_6$ is not a super-hurdle, since $C_5$ separates $C_1$ and $C_3$. (B) $C_6$ is here a super-hurdle, since $C_5$ does not separate $C_1$ and $C_3$ and becomes the maximum hurdle upon removal of $C_6$.

Lemma 14. Let $\mathbb{P}_n$ be a $\pi$-map containing the two hurdles $H_1, H_2$ such that $x_{2j+2} \in H_1$ and $x_{2i} \in H_2$ are two even sectors. Then gluing w.r.t. $x_{2j+2}$ and $x_{2i}$ is safe if there exist two hurdles $U_1$ and $U_2$ such that

$$H_1 \triangleleft U_1 \triangleleft H_2 \triangleleft U_2 \text{ or } U_1 \triangleleft H_1 \triangleleft U_2 \triangleleft H_2.$$ 

Proof. Suppose that gluing the sectors $x_{2j+2}$ and $x_{2i}$ generates the $\pi$-map, $\mathbb{P}_n$ and the $\mathfrak{D}_{\mathbb{P}_n}$-hurdle, $C$. We shall distinguish the scenarios of (a) $C$ being minimal in $\mathfrak{D}_{\mathbb{P}_n}$, or (b) $C$ being not minimal.

Ad (a): since $C$ is generated by the gluing of $H_1, H_2, C$ cannot have been minimal in $\mathfrak{D}_{\mathbb{P}_n}$. Furthermore we have $H_1 \sqsubset C$ and $H_2 \sqsubset C$. By Lemma 13, $[H_1, C]$ and $[H_2, C]$ collapse, merging into the non-orientable component, $C_*$. In case of $H_1 \triangleleft U_1 \triangleleft H_2 \triangleleft U_2$, we have $U_1 \sqsubset C$ and by construction $U_1$ does not merge into $C_*$. This means that $C$ cannot be minimal in $\mathfrak{D}_{\mathbb{P}_n}$, contradiction, see Fig. 16 (a). In case of $U_1 \triangleleft H_1 \triangleleft U_2 \triangleleft H_2$ we argue analogously.

Ad (b): $C$ becomes the unique maximal element in $\mathfrak{D}_{\mathbb{P}_n}$. We first observe that, by Lemma 13 in case of $H_1 \sqsubset C$ and $H_2 \not\sqsubset C$, $C$ necessarily merges
Figure 16: Safe reversals when gluing $H_1$ and $H_2$: an orientable component which is not a hurdle (white) and a hurdle (black). (a): $C$ being minimal in $O$, (b1): $C$ being maximal and $H_1, H_2 \subset C$ and (b2): $C$ being maximal and $H_1, H_2 \not\subset C$.

into $C_*$. Accordingly, we have the alternative:

$$H_1, H_2 \subset C \quad \text{or} \quad H_1, H_2 \not\subset C.$$  

In case of $H_1, H_2 \subset C$, $C$ does not separate $H_1, H_2$, as it would vanish, otherwise. In case of $H_1, H_2 \not\subset C$, $H_{1,2}$ necessarily separates $H_1, H_2$, since otherwise, $H_{1,2}$ remains and $H_{1,2} \not\subset C$, whence $C$ is not the unique maximum.

(b1): in case of $H_1, H_2 \subset C$ and $C$ does not separate $H_1$ and $H_2$. Since $C$ is not a hurdle in $O$, $C$ necessarily separates two orientable components. As a result, there exists a hurdle $N \neq H_1, H_2$ such that $C$ separates $N$ and $H_1, H_2$. In particular $C$ separates $N$ and $U_1$, where $H_1 \vartriangleleft U_1 \vartriangleleft H_2$. The two hurdles $N$ and $U_1$ persist, respectively, when gluing $H_1, H_2$, whence $C$ separates $N$ and $U_1$ in $O$. This implies that $C$ is not a hurdle in $O$, contradiction.

(b2): in case of $H_1, H_2 \not\subset C$ we can conclude that either $U_1$ or $U_2$ are not nested in $C$. Thus $C$ cannot be the unique, maximal element in $O$, contradiction.

Corollary 2. Let $P_{n,g}$ be a $\pi$-map and $h$ be the number of hurdles in $P_{n,g}$. Then in case of $h \neq 3$, there exists always a reversal that acts safely on $P_n$.

Proof. We label the hurdles such that $H_1 \vartriangleleft H_2 \vartriangleleft \cdots \vartriangleleft H_h$ holds.

For $h = 1$, we half-flip, which preserves $g$ and reduces $h$ by one. Therefore, any half-flip is in this scenario safe.

For $h = 2$, we glue $H_1, H_2$. If $H_1, H_2$ are both minimal, then there exists no unique maximal, non-separating hurdle. Thus $H_{1,2}$ is the unique maximal component which separates $H_1, H_2$ and consequently vanishes by the gluing. Hence $g$ increases by one and $h$ decreases by two. If $H_2$ is the maximal
hurdle, then we have a unique chain and gluing merges the latter, whence \( g \) increases by one and \( h \) decreases by two. Thus, gluing \( H_1 \) and \( H_2 \) is a safe reversal.

For \( h > 3 \), we glue \( H_1 \) and \( H_{1+[h/2]} \), where

\[
H_1 \triangleleft H_2 \triangleleft H_{1+[h/2]} \triangleleft H_h.
\]

By Lemma 10 and Lemma 14 \( g \) increases by one and \( h \) decreases by two, i.e. it is a safe reversal. By successively removing such pairs of hurdles, we can reduce the situation to \( h = 1 \) or \( h = 0 \).

Lemma 15. \( \text{[1]} \) Suppose \( \mathbb{P}_n \) is a \( \pi \)-map and \( \tilde{\mathbb{P}}_n \) is obtained by gluing the two hurdles \( H_1 \) and \( H_2 \). Then, any super-hurdle \( U \neq H_1, H_2 \) contained in \( (\Omega_{\mathbb{P}_n}, \square) \), is also a super-hurdle of \( (\Omega_{\tilde{\mathbb{P}}_n}, \square) \).

Proof. Since \( U \) is a hurdle, by Lemma 13 gluing \( H_1 \) and \( H_2 \) does not merge \( U \) into \( C_* \), whence \( U \in (\Omega_{\tilde{\mathbb{P}}_n}, \square) \). Since \( U \) is a super-hurdle, deleting \( U \) from \( (\Omega_{\mathbb{P}_n}, \square) \) creates a new hurdle, \( U' \).

The key point is to show that \( U' \in (\Omega_{\mathbb{P}_n}, \square) \) and that \( U' \) is a hurdle in \( (\Omega_{\tilde{\mathbb{P}}_n}, \square) \). To establish \( U' \in (\Omega_{\mathbb{P}_n}, \square) \) is a hurdle we consider \( U' \in (\Omega_{\mathbb{P}_n} \setminus \{U\}, \square) \) and distinguish two cases:

In case of \( U' \) being minimal, we conclude that \( U \) is minimal in \( (\Omega_{\mathbb{P}_n}, \square) \). Thus \( U' \) is not contained in \( [H_1, H_{1,2}] \cup [H_2, H_{1,2}] \cup \{H_{1,2}\} \), i.e. \( U' \) remains to be a minimal hurdle in \( (\Omega_{\mathbb{P}_n}, \square) \).

If \( U' \) is the unique maximal hurdle in \( (\Omega_{\mathbb{P}_n} \setminus \{U\}, \square) \), it does not separate any pair of hurdles. We now inspect the effect of gluing \( H_1 \) and \( H_2 \). Since \( U' \) does not separate any pair of hurdles, \( U' \) is not contained in \( [H_1, H_{1,2}] \cup [H_2, H_{1,2}] \cup \{H_{1,2}\} \) and accordingly not merged into \( C_* \). Furthermore, \( U' \) remains to be the unique maximal hurdle in \( (\Omega_{\mathbb{P}_n}, \square) \).

In both cases \( U \) remains to be a super-hurdle in \( (\Omega_{\tilde{\mathbb{P}}_n}, \square) \) and the lemma follows. \( \square \)

By Corollary 2, there exists always a safe reversal except in the case \( h = 3 \).

If not all of them are super-hurdles, then half-flipping one non super-hurdle reduces the situation to \( h = 2 \). Thus it remains to consider the case of all three hurdles being super-hurdles.

Lemma 16. \( \text{[1]} \) Let \( \mathbb{P}_n \) be a \( \pi \)-map with exactly three hurdles, all of which being super-hurdles. Then there exists no safe reversal.
Proof. A safe reversal means to either (1) half-flip, which keeps $g$ and decreases $h$ by one, or to (2) glue in which case $g$ increases by one and $h$ decreases by two.

In case of (1), since all three hurdles are super-hurdles, half-flipping is not safe as it neither reduces the number hurdles nor the topological genus. Thus it suffices to consider (2).

In case of (2), suppose we have the three super-hurdles denoted via $H_1 < H_2 < H_3$ and that deleting $H_i$ creates the hurdle $U_i$, $1 \leq i \leq 3$. It follows from

$$\max_{\sigma} H_1 <_{\sigma} \max_{\sigma} H_2 <_{\sigma} \max_{\sigma} H_3,$$

that $H_1$ and $H_2$ are necessarily minimal.

We distinguish two scenarios:

Scenario 1. suppose $H_3$ is minimal.

Then $\mathcal{D}_{F_n}$ has three chains $H_i, U_i, \ldots K_i$, $1 \leq i \leq 3$, where the $K_i$ are not necessarily all different. Furthermore, each chain has length at least two. Assume that we glue the two minimum hurdles, say $H_1$ and $H_2$ (to glue $H_1$ and $H_3$, $H_2$ and $H_3$ are argued similar). Then the two chains $H_1, U_1, \ldots K_1$ and $H_2, U_2, \ldots K_2$ merge into $C_*$ either including $H_{1,2}$ or not. We distinguish three cases:

(a) $H_{1,2} = K_1 = K_2$ remains as a minimal element. Then $H_{1,2}$ is a hurdle and $g + h$ changes into $(g + 1) + (h + 1) - 2 = g + h$,

(b1) $H_{1,2}$ is contracted and $K_3$ is not contracted. Then $K_3$ becomes the unique maximal hurdle (as it does not separate any two leaves) and $g + h$ remains constant,

(b2) $H_{1,2} = K_3$ is contracted. Then $K_1 = K_2 = K_3$ and $K_3$ separates $H_1$ and $H_2$. Since $H_3$ is a super-hurdle, deleting $H_3$ creates the hurdle, $U_3$, which implies $U_3 \neq K_3$. Accordingly, gluing of $H_1, H_2$ transforms the chain $H_3, U_3, \ldots K_3$ into a chain having a maximal element not equal to $H_3$. The latter is a hurdle, whence $g + h$ remains constant.

Scenario 2. suppose $H_3$ is maximal.

Then there are the two chains

$$H_1, U_1, \ldots, U_3, H_3, \quad H_2, U_2, \ldots, U_3, H_3,$$

where $U_1, U_2 \neq U_3$ and $H_3$, as a super-hurdle, does not separate $H_1$ and $H_2$. 

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Then gluing $H_1$ and $H_2$ is not safe as it preserves $H_3$ and produces an unique chain, creating a new, minimal hurdle. Using Lemma 8 we observe that gluing $H_1$ and $H_3$ produces an unique chain together with a new, maximal hurdle. Therefore, the number of hurdles decreases only by one, whence it is not safe.

Accordingly, we have shown that there exists no safe reversal in a $\pi$-map which contains only three hurdles, all of which are super-hurdles.

**Theorem 2.** Let $b_n$ be a signed permutation with associated class of $\pi$-maps $[\mathbb{P}_{n,g}]$, having genus $g$ and $h$ hurdles. Then we have

$$d(b_n) = \begin{cases} g + h + 1 & \text{if } h \neq 1 \text{ is odd and all } h \text{ hurdles are super-hurdles,} \\ g + h & \text{otherwise} \end{cases}$$

**Proof.** If $h$ is even, we label the hurdles $H_1 < H_2 < \cdots < H_{2k}$. In case of $k = 1$, we have $h = 2$ and gluing $H_1$ and $H_2$ is safe. In case of $k > 1$, we glue $H_1$ and $H_{k+1}$, which is safe by Lemma 14. Iterating this $h/2$ times we obtain a $\pi$-map with genus $g' = g + h/2$ without any hurdles, i.e., a $\pi$-map in which each component is non-orientable. All these non-orientable components can be reduced to trivial ones via $g'$ slicings. Accordingly, the total number of reversals, i.e., $d(b_n)$, is $h/2 + g' = h/2 + g + h/2 = g + h$.

If $h$ is odd and $\mathbb{P}_{n,g}$ contains at least one hurdle, which is not a super-hurdle, then we apply a half-flip, deriving a $\pi$-map of genus $g$ having $(h - 1)$ hurdles. This reduces this case to the case of $h$ being even and the total number of reversals is $1 + (g + h - 1) = g + h$.

Finally, suppose $h$ is odd and all hurdles are super-hurdles, $H_1 < H_2 < \cdots < H_{2k+1}$. By Lemma 14 we have safe reversals and can reduce the situation to a scenario of exactly three super-hurdles. By Lemma 16 there exists no safe reversal, then. Gluing any pair of these three creates a new hurdle and reduces the scenario to that of $h = 2$. The number of reversal in this case is $\frac{h-1}{2} + (g + \frac{h-1}{2}) + 1 + 1 = g + h + 1$.

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