CONSTRAINT ON THE GIANT PLANET PRODUCTION BY CORE ACCRETION

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ABSTRACT

The issue of giant planet formation by core accretion (CA) far from the central star is rather controversial because the growth of a massive solid core necessary for triggering the gas runaway can take longer than the lifetime of the protoplanetary disk. In this work, we assess the range of separations at which CA may operate by (1) allowing for an arbitrary (physically meaningful) rate of planetesimal accretion by the core and (2) properly taking into account the dependence of the critical mass for the gas runaway on the planetesimal accretion luminosity. This self-consistent approach distinguishes our work from similar studies in which only a specific planetesimal accretion regime was explored and/or the critical core mass was fixed at some arbitrary level. We demonstrate that the largest separation at which the gas runaway can occur within 3 Myr corresponds to the surface density of solids in the disk $\gtrsim 0.1 \text{ g cm}^{-2}$ and is 40–50 AU in the minimum mass solar nebula. This limiting separation is achieved when the planetesimal accretion proceeds at the fastest possible rate, even though the high associated accretion luminosity increases the critical core mass, delaying the onset of gas runaway. Our constraints are independent of the mass of the central star and vary only weakly with the core density and its atmospheric opacity. We also discuss various factors that can strengthen or weaken our limits on the operation of CA.

Key words: planetary systems – planets and satellites: formation – protoplanetary disks

1. INTRODUCTION

Recent discoveries of distant planetary companions (separations of at least tens of AU) around several nearby stars by direct imaging (Marois et al. 2008; Thalmann et al. 2009) have stimulated investigation of their formation mechanisms. Two contending theories of the planet formation—the core accretion (CA; Perri & Cameron 1974; Harris 1978; Mizuno 1980) and gravitational instability (Cameron 1978; Boss 1998)—differ quite dramatically in their ability to form planets at various distances from the star.

It is generally thought that CA should be capable of forming giant planets close to the star, at separations $\lesssim 10$ AU. Both theoretical modeling (Mizuno 1980; Stevenson 1982; Pollack et al. 1996) and the existence of two gas giants at semimajor axes of 5.2 AU and 9.5 AU in our own solar system (coupled with some knowledge about the evolution of the planetary orbital architecture in the early solar system) attest to this statement. At the same time it seems unlikely that giant planets can be formed by the gravitational instability in the inner parts of protoplanetary disks because of the long local cooling time of gas (Matzner & Levin 2005; Rafikov 2005, 2007).

In contrast, giant planet formation by the direct gravitational instability appears feasible at large distances from the star, beyond $\sim 70$–100 AU (Boley 2009; Rafikov 2009; Clarke 2009; but see Boss 2006 for an opposite view) where the disk cooling time is short enough to permit fragmentation of the gravitationally unstable disk (Gammie 2001). Whether CA can operate beyond several tens of AU from the star is not so clear.

The well-known problem faced by CA far from the star is that the buildup of massive refractory core (necessary for triggering the vigorous gas accretion) is thought to take very long time, longer than the 1–10 Myr lifetime of the gaseous component of protoplanetary nebula. Studies illustrating this problem are usually based on two key assumptions (Dodson-Robinson et al. 2009): first, that the gas runaway is triggered whenever a $M_{\text{crit}} \sim 10 M_{\oplus}$ core is built by planetesimal accretion, and, second, that the planetesimals accrete onto the growing core at rather moderate rates (Ida & Lin 2004; we will explain later what do we mean by that). When these conditions are imposed it is generally found that the gas runaway does not commence at $a \gtrsim 20$–30 AU prior to the nebular gas removal.

In this paper, we show that the aforementioned assumptions are too restrictive and neither of them need to be adopted in determining the feasibility of CA. First, planetesimal accretion by the growing core can (at least potentially) be much faster than what has been previously assumed. Second, the critical core mass itself strongly depends on the planetesimal accretion rate. Based on these observations we formulate in this work a novel constraint on the operation of CA in protoplanetary disks.

2. PLANETESIMAL ACCRETION

Operation of CA is intimately related to the accretion of solid material by the protoplanetary core, as discussed in more detail in Section 3. For this reason we start by reviewing the process of planetesimal accretion by growing cores.

It is well known that the rate at which a core accretes planetesimals is a strong function of planetesimal random velocities (Dones & Tremaine 1993, hereafter DT93). An important parameter for determining the dynamical state of planetesimal population is the ratio $p = R_e/R_H$ of the core radius $R_e$ to the Hill radius $R_H \equiv a(M_e/M_*)^{1/3}$, where $a$ and $M_e$ are the semimajor axis and the mass of the core, and $M_*$ is the mass of the star. At large separations from the star

$$p = \left( \frac{3M_e}{4\pi \rho_c a^3} \right)^{1/3} \approx 5 \times 10^{-4} a_{10}^{-1/3} M_{e,1}^{1/3} \rho_{c,1}^{-1/3}$$

(1)

is much less than unity, where $\rho_c$ is the core density and we define $a_{10} \equiv a/(10 \text{ AU})$, $M_{e,1} \equiv M_e/M_\odot$, $\rho_{c,1} = \rho/1 \text{ g cm}^{-3}$.

DT93 have demonstrated that whenever $p \ll 1$ there are four possible regimes of planetesimal accretion.

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1. When the planetesimal random velocity dispersion $\sigma \gtrsim \Omega R_H p^{-1/2}$ ($\Omega = \sqrt{GM_*/a^3}$ is the angular frequency at the core’s location)—a regime that we call very high dispersion—accretion is very slow because gravitational focusing is inefficient and the collision cross section is given by the geometric cross section of the core.

2. When $\Omega R_H \lesssim \sigma \lesssim \Omega R_H p^{-1/2}$ (high dispersion in the notation of DT93), the relative motions of planetesimals but is rather set by the shear in the differentially rotating disk, so that $v_{\text{rel}} \approx \Omega R_H$. Gravitational focusing is strong and saturates at a constant value whenever $\sigma \lesssim \Omega R_H$ and $M$ keeps increasing as $\sigma$ decreases simply because the thickness of the disk goes down increasing local volume density of bodies that are being accreted.

3. When $\Omega R_H p^{1/2} \lesssim \sigma \lesssim \Omega R_H$ (intermediate dispersion) one finds that $v_{\text{rel}}$ is no longer determined by the random motions of planetesimals but is rather set by the shear in the differentially rotating disk, so that $v_{\text{rel}} \sim \Omega R_H$. Gravitational focusing is strong and saturates at a constant value whenever $\sigma \lesssim \Omega R_H$ and $M$ keeps increasing as $\sigma$ decreases simply because the thickness of the disk goes down increasing local volume density of bodies that are being accreted.

4. Finally, when $\sigma \lesssim \Omega R_H p^{1/2}$ (very low dispersion) disk becomes so thin that a core can accrete the whole vertical column of material doomed for collision with it. Given that focusing is at its maximum this situation corresponds to the most efficient accretion regime.

The rate of accretion in the very low dispersion case is independent of the planetesimal velocity dispersion $\sigma$ (as long as $\sigma \lesssim \Omega R_H p^{1/2}$) and is given by (DT93)

$$\dot{M} = \dot{M}_{\text{max}} \approx (6.47 \pm 0.02)\Omega p^{1/2} \Sigma R_H^2, \quad (2)$$

$$\approx 5.1 G^{1/2} M^{2/3} \rho_c^{-1/6}, \quad (3)$$

where $\Sigma_s$ is the surface density of solids. In this study we parameterize $\Sigma_s$ as

$$\Sigma_s(a) = 1 \Sigma_1 a^{-\alpha_1} \text{g cm}^{-2}, \quad (4)$$

where $\Sigma_1 = \Sigma_s/(1 \text{ g cm}^{-2})$ and $\alpha$ is a constant (minimum mass solar nebula (MMSN) corresponds to $\alpha \approx 3/2$ and $\Sigma_1 \approx 1$). Note that $\dot{M}_{\text{max}}$ is independent of $M_*$. 

For comparison, it is often assumed that protoplanetary cores predominantly accrete planetesimals with the dispersion of random velocities $\sigma = \Omega R_H$, which corresponds to the transition between the so-called shear- and dispersion-dominated dynamical regimes (e.g., Dodson-Robinson et al. 2009). This is what we called a “moderate” accretion rate before (i.e., right at the border between the intermediate and high dispersion regimes of planetesimal accretion). At this transition DT93 find

$$\dot{M}_t \approx (6-10)\rho \Sigma_s \Omega R_H^2, \quad (5)$$

and one immediately sees that $\dot{M}_t \ll \dot{M}_{\text{max}}$ since $p \ll 1$. 

Previous studies of CA have been limited to considering planetesimal accretion only in the high-velocity regime ($\sigma \gtrsim \Omega R_H$) or at the boundary with the intermediate velocity regime (i.e., for $\sigma \approx \Omega R_H$). While large (tens to hundreds km in size) planetesimals indeed likely accrete in these dynamical regimes at rather slow rates, it is quite possible that most of the mass growth of the core is not due to accretion of these large bodies. According to Rafikov (2004a) and Goldreich et al. (2004), as the core becomes massive enough it dynamically excites large planetesimals in its vicinity since gas drag are not effective for massive bodies. This leads to fragmentation (rather than growth) of planetesimals when they collide with each other. Resulting fragmentation cascade converts significant fraction of the solid mass into small mass debris. These fragments finally reach small enough sizes that they are dynamically “cooled” by gas drag to low velocities, which makes it possible for them to be accreted at very high rate compatible with $\dot{M}_{\text{max}}$ (Rafikov 2004a, 2004b) in the very low dispersion regime. Recent coagulation calculations done by Kenyon & Bromley (2009) confirm this general picture, lending support to the possibility of maximally efficient accretion in the very low dispersion regime.

In this work, we are interested in obtaining a robust upper limit on the distance from the central star at which CA can operate. To this effect we will consider all possible modes of planetesimal accretion and actually determine the accretion regime facilitating the onset of gas runway within the limited amount of time: on the one hand high $M$ allows to build the core of a given mass faster, but on the other hand high $M$ also implies larger critical mass (see Section 4), which takes longer to build.

We will discover in Section 5.2 that fastest route to gas runaway lies through the most efficient accretion at the rate $\sim \dot{M}_{\text{max}}$ and for this reason we parameterize the planetesimal accretion rate as

$$\dot{M} = \chi \dot{M}_{\text{max}} \approx 6.47 \rho p^{1/2} \Sigma \Omega \left(\frac{M_c}{M_*}\right)^{2/3}, \quad \approx 2 \times 10^{-5} M_{\odot} \text{yr}^{-1} \chi a^{-\alpha} \Sigma_1 \rho_c^{-1/6} M_c^{2/3}, \quad (6)$$

($M_{c,10} = M_c/10 M_{\oplus}$) where the parameter $\chi$ accounts for the deviation of $\dot{M}$ from $\dot{M}_{\text{max}}$. In general, $\chi$ may be a function of $M_*$ and $a$, and it is expected that $\chi \lesssim 1$ (although in some situations $\chi \gtrsim 1$ may possible; see Section 6). But we will show in Section 4 that slow accretion (i.e., $\chi \ll 1$) results in smaller distance from the star at which planets could form by CA than in the case $\chi \sim 1$. It should also be emphasized that our concentration on the highest $\dot{M}$ accretion regime is not only for the sake of the argument—as mentioned before this regime may actually occur quite naturally in the course of the core buildup.

Accretion luminosity corresponding to $\chi = 1$ given by Equation 6 is

$$L = \frac{G M_1 \dot{M}}{R_c} \approx 8.2 \chi \Sigma_s G^{3/2} M_c^{4/3} \rho_c^{1/6} \approx 8 \times 10^{-6} L_\odot \chi a^{-\alpha} \Sigma_1 \rho_c^{1/6} M_c^{4/3} M_{c,10}^{4/3}. \quad (7)$$

According to Equation (6), accretion at the rate $\dot{M}_{\text{max}}$ results in the core radius growing linearly with time.\(^{3}\) Time needed for the core mass to reach a predetermined value $M_c$ by accretion at the rate $\chi \dot{M}_{\text{max}}$ is (see Equation (2))

$$\tau(M_c) \approx 0.6 \rho_c^{1/6} M_{c,10}^{1/3} \chi \Sigma_G G^{-1/2} \approx 0.3 \text{Myr} a^{-\alpha} \rho_c^{1/6} M_{c,10}^{1/3} \chi \Sigma_1. \quad (8)$$

\(^3\) One can easily see that accretion at the rate $\dot{M}_t$ (see Equation (5)) also leads to $M_c^{1/3} \propto \tau$, although the coefficient of this relation is considerably smaller than in the case of $\dot{M} = \dot{M}_{\text{max}}$ and has different scaling with $a$. 

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2 For simplicity here we do not differentiate between the horizontal and vertical velocity dispersion.
It follows from this formula that a core can grow to $10 \, M_\oplus$ in 3 Myr by accretion at the rate $M_{\text{max}}$ even at $\sim 50 \, \text{AU}$ in the MMSN. Previous results which did not account for the factors leading to the fast accretion (planetaryesimal fragmentation and gas drag on debris) predict much smaller distance ($\lesssim 20 \, \text{AU}$) at which such core would be able to form on that timescale.

Having said all that, in the next section we show that the formation of the $10 \, M_\oplus$ core is not a prerequisite for gas runaway and that the $M_{\text{crit}}$ is a function of accretion rate.

3. PROTOPLANETARY ATMOSPHERE

Numerical calculations (Mizuno 1980; Pollack et al. 1996; Ikoma et al. 2000) and analytical theory (Stevenson 1982) show that gas runaway commences when the core mass $M_{\text{crit}}$ is so large that the mass of its atmosphere is comparable to the core mass itself,

$$M_{\text{atm}}(M_{\text{crit}}) = \eta M_{\text{crit}},$$

i.e., when the gaseous component of the protoplanet becomes self-gravitating. Here $\eta$ is a parameter of order unity; its exact value may depend on the accretion history of the core (Ikoma et al. 2000). Condition (9) should be regarded as an equation for $M_{\text{crit}}$ which can be easily solved as long as the dependence of $M_{\text{atm}}$ on $M_{\text{crit}}$ and other parameters of the problem is specified. Thus, calculation of $M_{\text{crit}}$ involves understanding properties of planetary atmospheres and calculation of $M_{\text{atm}}$ in particular.

It has been known since the works of Mizuno (1980) and Stevenson (1982) that the critical core mass is a rather weak function of the density $\rho_0$ and temperature $T_0$ of the surrounding nebula (see Rafikov 2006 and Section 5.3 for the discussion of the conditions under which this behavior is expected). This has led to the wide acceptance of the CA idea since the interiors of the solar system giant planets are believed to harbor $\sim 10 \, M_\oplus$ cores despite their different separations from the Sun. Unfortunately, this observation of the $M_{\text{crit}}$ invariance has shadowed the fact first noticed by Stevenson (1982) that $M_{\text{crit}}$ is a strong function of the planetesimal accretion luminosity $L_{\text{cr}}$, or, equivalently, planetesimal accretion rate $M_{\text{cr}}$. This fact has been subsequently confirmed both numerically (Ikoma et al. 2000; Hori & Ikoma 2010) and analytically (Rafikov 2006).

Stevenson’s (1982) analytical results strictly apply only to cores possessing radiative atmospheres with constant opacity. Rafikov (2006, hereafter R06) has studied more general types of atmospheres and has shown that they segregate into two general classes depending on the relationship between $L$ and a critical luminosity $L_{\text{cr}}$, defined as the value of $L$ at which the atmosphere becomes convective all the way to the planetary Bondi radius $R_B \equiv (G M_\star) / \alpha^2_0$ (here $c_0$ is the sound speed of the nebular gas around the core). Whenever $L \gtrsim L_{\text{cr}}$, the intense energy release near the core surface makes protoplanetary atmosphere and the nebular gas in the whole Hill sphere of the core convectively unstable. Such atmospheres have high entropy and rather low mass relative to the mass of the core. In the opposite case, when $L \lesssim L_{\text{cr}}$, protoplanetary atmosphere is separated from the nebular gas by a roughly isothermal shell of gas in which energy is transported radiatively and gas entropy decreases from the nebular value to a much smaller value characteristic for the atmosphere. Density in this shell increases roughly exponentially toward the planet which makes $M_{\text{atm}}$ much higher (for a given core mass) than in the high luminosity case.

It was demonstrated in R06 that far from the star protoplanetary atmospheres are virtually always characterized by $L \ll L_{\text{cr}}$, even if planetesimal accretion proceeds at the maximally efficient rate $M_{\text{max}}$. For that reason we will consider only the low luminosity atmospheres in this work. The total atmospheric mass $M_{\text{atm}}$ in the low luminosity case was computed in R06 by making simplifying assumptions about the conditions in the deep layers of the planetary atmosphere: either a polytropic model with constant polytropic index (mimicking the fully convective interior) or a fully radiative atmosphere with a simple power-law parameterization of the opacity dependence on gas pressure and temperature. Under these assumptions a significant fraction of the atmospheric mass resides in the outer layer, near the inner boundary of the roughly isothermal external radiative zone, in agreement with calculations by Mizuno (1980). It can then be shown that (R06)

$$M_{\text{atm}} \approx \zeta \left(\frac{G M_\star \mu}{k} \right)^\frac{4}{\kappa_0 L},$$

where $k_B$ is the Boltzmann constant, $\kappa_0$ is the gas opacity in the outer radiative layer of the atmosphere, and $\zeta$ is a dimensionless factor, which can be computed exactly for a given density distribution inside the atmosphere. It should be noted here that $\kappa_0$ is essentially the same as the opacity of the nebular gas surrounding the core, which is determined by the nebular conditions. This is because opacity is predominantly due to dust and depends only on the gas temperature, which is roughly the same both in the outer radiative zone of the atmosphere and in the surrounding nebular gas. Formula (10) explicitly demonstrates how $M_{\text{atm}}$ depends on important parameters like $M_{\text{cr}}$ and $\kappa_0$ and on accretion history, to which $L$ is sensitive.

However, it is not obvious that the simplifying assumptions about the atmospheric properties used in deriving Formula (10) are justified given the complexity of the physical effects encountered deep in the atmosphere: self-gravity gradually becomes important as $M_{\text{cr}}$ approaches $M_{\text{crit}}$, grain sublimation changes opacity in a non-trivial fashion, the equation of state may vary with depth because of molecular dissociation, an atmosphere may have both radiative and convective regions at the same time. These complications likely do not affect the qualitative conclusions reached in R06, but should be important for quantitative comparisons.

On the other hand, numerical calculations of CA which include the aforementioned physical effects (Ikoma et al. 2000) are typically limited to exploring the dependence of $M_{\text{crit}}$ on only a limited set of input parameters such as $\kappa_0$ and $M_{\text{cr}}$. The planetesimal accretion rate $M_{\text{cr}}$ is often taken to be constant (or is given by a particular prescription) through the calculation and this assumption significantly limits the direct applicability of these numerical results to realistic situations (since typically $M_{\text{cr}}$ increases as $M_{\text{cr}}$ grows), including our present study.

To apply the existing numerical results to cases where the CA history is non-trivial ($M \neq \text{const}$) while at the same time being able to use a simple Formula (10) motivated by Stevenson (1982) and R06 in our analysis, we have resorted to the following approach. First, we still use Formula (10) to calculate $M_{\text{atm}}$ but now we do not assume the coefficient $\zeta$ to be constant as various physical effects in the deep atmosphere introduce additional variations of the density profile not captured by the analytical theory of R06. Second, using Equation (9) we calculate $M_{\text{crit}}$ under the assumption $M = \text{const}$ used in Ikoma et al. (2000). With this information in hand we can calibrate the dependence of

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4 This is different from the case of an atmosphere with constant opacity considered by Stevenson (1982).
\( \zeta \) on various physical parameters of the problem by comparing our results with those of Ikoma et al. (2000). Finally, after \( \zeta \) has been calibrated for a particular accretion history \( M = \text{const} \), Equation (10) can be applied for a more general situation, e.g., for the \( M(M_\ast) \) dependence used in this work.

This approach should work well as long as the state of the atmosphere is fully determined by the current value of \( M \), and is independent of the full accretion history. For this to be the case the thermal timescale of the atmosphere must be short compared to the planetesimal accretion timescale, and this has been verified in R06. This gives us confidence that the outlined method should be robust and justifies its application to the problem at hand.

Since \( \zeta \) is predominantly affected by the changes of the thermodynamical state of material in the atmosphere (grain sublimation, molecular dissociation, variations of the equation of state depend on gas temperature and density) one should expect \( \zeta \) to depend most strongly on \( M_\ast \) and \( L \)—variables that determine \( T \) and \( \rho \) deep in the envelope. Ikoma et al. (2000) have shown that in their case \( M_{\text{crit}} \) scales with \( M \) and \( \kappa_0 \) roughly as power laws. This motivates us to look for the dependence of \( \zeta \) on \( M_\ast \) and \( L \) also in the power-law form. The details of the calibration procedure are presented in the Appendix where we show that

\[
\zeta = \zeta_0 M_\ast^\delta, \quad \delta \approx 1.2,
\]

\[
\zeta_0 \approx 4 \times 10^{-32} \text{g}^{-3} \approx 82 M_{\odot}^{-3}
\]

provide a reasonably good fit to the numerical results of Ikoma et al. (2000). Note that \( \zeta \) has been found to depend on \( L \) only weakly, and can be considered a function of \( M_\ast \) only. In deriving this result for \( \zeta \) we have assumed that the coefficient \( \eta \) appearing in the relation (9) is independent of the planetesimal accretion history of the core and is thus the same in Ikoma et al.’s work as well as in our case.

### 4. CRITICAL CORE MASS

Results obtained in the previous section allow us to compute \( M_{\text{crit}} \) for \( M \) given by Equation (6): substituting \( M_{\text{diss}} \) in the form (10) with \( \zeta \) given by Equation (11) into the condition of the onset of CA (Equation (9)) and using Equation (7) for the core’s accretion luminosity, we find

\[
M_{\text{crit}} \approx \left[ \frac{8.2}{\zeta_0} \left( \frac{k_B}{\mu} \right)^4 \left( \frac{\Sigma \rho_0}{\sigma G^{5/2}} \right)^{3/(5+3\delta)} \right] \left[ \frac{\chi \Sigma_1 \kappa_1 \rho_{0.1}}{\mu_2^3} \right]^{0.35} \approx 40 M_{\odot} \chi_0^{1/2} \rho_0^{-0.35},
\]

Equation (12) shows among other things that \( M_{\text{crit}} \) is indeed a function of the planetesimal accretion regime: according to Equation (6) we may think of the free parameter \( \chi \) as a direct measure of planetesimal \( M_\ast \) and \( M_{\text{crit}} \) scales roughly as \( \chi^{0.35} \). Just for illustration, if (as has been done in Dodson-Robinson et al. 2009) planetesimal random velocities were kept at \( \sigma \sim \Omega_{\ast H} \) (rather than being essentially zero as assumed in deriving Equation (6)), Equation (6) could still be used but with \( \chi(a) \approx [p(a)]^{1/2} \ll 1 \) (DT93; R06). At \( a = 40 \) AU where

\[
p \approx 10^{-4}
\]

one would then find \( M_{\text{crit}} \) smaller by a factor of five compared to what Equation (12) predicts for \( \chi = 1 \) at the same location; see Figure 1.

With Equation (12) we also show for our chosen \( \dot{M} \) behavior that \( M_{\text{crit}} \) varies with the distance from the central object: for \( \chi = \text{const} \) the critical mass scales as \( \approx a^{-0.53} \) in the MMSN \((\alpha = 3/2)\). This makes \( M_{\text{crit}} \) at 10 AU twice as large as \( M_{\text{crit}} \) at 40 AU, everything else being equal. Thus, taking \( M_{\text{crit}} \) to be independent of the regime of planetesimal accretion and distance from the central object is generally not justified.

### 5. LIMIT ON CORE ACCRETION

For the gas runaway to happen before the nebular gas dispersal the growth time of the core with mass \( M_{\text{crit}} \) must be less than the lifetime of the protoplanetary nebula \( \tau_{\text{neb}} \), i.e., \( \tau(M_{\text{crit}}) < \tau_{\text{neb}} \). Plugging in our result (12) into Equation (8) this constraint can be rephrased in terms of the lower limit on the surface density of solids needed for gas runaway to occur before \( \tau_{\text{neb}} \):

\[
\chi \Sigma_1 > \Sigma_{\text{lim}},
\]

\[
\Sigma_{\text{lim}} \approx 0.74 \left[ \frac{\rho_0^{(2+\delta)/2} (k_B/\mu)^4 \kappa_0}{\zeta_0 \rho^1 G^{(10+3\delta)/2} \sigma} \right] \approx 0.1 \text{ g cm}^{-2} \left( \frac{\tau_{\text{neb}}}{3 \text{ Myr}} \right)^{-1.13} \frac{p_{c,1}^{0.21} \kappa_0^{0.13}}{\mu_2^{0.33}}.
\]

This inequality represents the main result of this work: a robust lower limit on the planetesimal surface density at which CA is capable of producing giant planets within a protoplanetary
nebula lifetime. Note that $\Sigma_{\text{lim}}$ is a sensitive function of the nebula lifetime $\tau_{\text{neb}}$, while it depends rather weakly on the bulk density of the core $\rho_c$, atmospheric dust opacity $\kappa_0$ (this is addressed in Section 5.3), and the mean molecular weight of the atmospheric gas $\mu$.

Now we can also determine the limiting core mass $M_{\text{lim}}$ defined as the critical core mass for $\chi = \Sigma_{\text{lim}}$, i.e., at the very extreme of the region where CA can still occur within $\tau_{\text{neb}}$. It is found by substituting (14) into (12):

$$M_{\text{lim}} = \left[ \frac{4.9}{\zeta_0} \left( \frac{k_B}{\mu} \right)^4 \frac{\rho_c^{1/3} \kappa_0}{\sigma G^3 \tau_{\text{neb}}} \right]^{3/(4+3\delta)} \approx 15 M_\odot \left( \frac{\tau_{\text{neb}}}{3 \text{Myr}} \right)^{-0.4} \left( \frac{\rho_c^{13/4} \kappa_0^{0.4}}{\mu_2^{1.58}} \right). \quad (15)$$

By construction $M_{\text{lim}}$ is independent of the surface density profile in the nebula. This mass is to be compared with the isolation mass $M_{\text{iso}}$ (the core mass at which it has accreted all solid mass within its feeding zone)—an annulus centered on the core’s orbit and having a full width equal to $\xi RH$:

$$M_{\text{iso}} = \frac{(2\pi \xi \Sigma_1 a_{g,1}^2)^{1/2}}{M_\odot^{1/2}}, \quad \approx 2 \left( \frac{\xi}{5} \right)^{3/2} \frac{\Sigma_1^{3/2}}{a_{g,1}^{3/4}} M_\odot \quad (16)$$

(an MMSN-like density profile was used in making numerical estimate). At 44 AU one finds $M_{\text{iso}} \approx 6 M_\odot$, which is smaller than $M_{\text{lim}}$. However, a modest radial displacement of the core due to some type of migration can easily expose it to additional fresh material allowing $M_c$ to reach $M_{\text{lim}}$ (Alibert et al. 2005). Alternatively, increasing $\Sigma_c$ (boosting up $\Sigma_1$) by a factor of two makes $M_{\text{iso}} \approx M_{\text{lim}}$ at 44 AU.

Quite interestingly, the value of $M_{\text{lim}}$ is not too far from $10 M_\odot$ commonly accepted as the core mass throughout the whole protoplanetary disk. This coincidence is accidental since our estimate of $M_{\text{lim}}$ was derived self-consistently rather than postulated in an ad hoc fashion.

### 5.1. Limiting Distance for Core Accretion

Given a constraint (14) and having a particular model of the radial distribution of $\Sigma$, one can determine the maximal radial extent $a_{\text{lim}}$ of the region in the protoplanetary disk in which CA can produce giant planets within the nebula lifetime $\tau_{\text{neb}}$. Since $\Sigma_c$ is expected to be a decreasing function of $a$ this would only be possible for $a < a_{\text{lim}}$. Using our power-law parameterization (4) of $\Sigma_c$ we find that in a MMSN-like disk with $\alpha = 3/2$

$$a_{\text{lim}}^{\text{MMSN}} \approx 44 \text{ AU} \left( \frac{\Sigma_c}{\rho_c^{14/3} \kappa_0^{0.09}} \right)^{1/2} \left( \frac{\tau_{\text{neb}}}{3 \text{Myr}} \right)^{0.75}. \quad (17)$$

Accretion at $M = M_{\text{max}}$ corresponds to $\chi = 1$ in this formula. The limiting distance found in Equation (17) is similar to previous estimates (e.g., Dodson-Robinson et al. 2009) obtained for less vigorous planetesimal accretion (i.e., for smaller $M$) and fixed $M_{\text{crit}} = 10 M_\odot$ but this is just a coincidence.

Equation (17) shows that as $\Sigma_c$ (or $\Sigma_1$) increases, the extent of CA-capable part of the protoplanetary disk also grows. But $\Sigma_c$ cannot be increased without limit; at some point the gaseous component of the protoplanetary nebula would become self-gravitating, and, depending on its cooling time (Gammie 2001), would either fragment or evolve quasi-viscously while maintaining the marginally gravitationally unstable state (Rafikov 2009; Clarke 2009). Thus, the most extreme value of $a_{\text{lim}}$ can be obtained by taking $\Sigma_{\text{iso}}(a) = f_{\text{iso}}\Sigma_{\text{lim}}(a)$ in Equation (14), where $f_{\text{iso}}$ is the dust-to-gas ratio, which we take to be $10^{-2}$, and $\Sigma_{\text{iso}}(a)$ is the gas surface density at which the disk is marginally gravitationally unstable, which happens when the Toomre $Q \equiv \Omega_c a / \kappa G \Sigma_{\text{iso}}$ is of order unity (Safronov 1960; Toomre 1964). Assuming that the disk is heated by a central star with luminosity $1 L_\odot$ at normal incidence (i.e., neglecting complications related to flaring geometry, Chiang & Goldreich 1997), we find that

$$f_{\text{iso}} \Sigma_{\text{iso}}(a) \approx 20 a_1^{-7/4} \text{ g cm}^{-2}. \quad (18)$$

Using this density profile in Equation (14) we find

$$a_{\text{lim}}^{Q=1} \approx 200 \text{ AU} \left( \frac{\chi}{0.57} \right)^{1/6} \left( \frac{\tau_{\text{neb}}}{3 \text{Myr}} \right). \quad (19)$$

In practice, this limit will hardly apply to real systems because self-gravitating disks with $Q \sim 1$ would not persist for the lifetime of the nebula, as we already mentioned.

### 5.2. Sensitivity to Planetesimal Accretion Efficiency

Through our calculations we have retained in all formulae the parameter $\chi$ defined in Equation (6), which characterizes the efficiency of planetesimal accretion. This allows us to see how the variation of planetesimal $M$ with respect to $M_{\text{max}}$ affects the planet formation by the CA. From Equation (14) we see that smaller $\chi$ (corresponding to less efficient accretion) results in higher $\Sigma_c$ and smaller $a_{\text{lim}}$—see Equations (17), (19).

As we emphasized in Section 3, this result is not trivial since less efficient accretion implies lower planetesimal luminosity, bigger $M_{\text{lim}}$ for the same $M_c$, and lower $M_{\text{crit}}$, making gas runaway easier to get going. However, it turns out that the competing effect of being able to grow the massive core faster at higher $M$ is more important for the gas runaway to commence within the limited amount of time. For this reason the largest extent of the region in which CA can operate in time $\tau_{\text{neb}}$ is reached when the core is able to accrete at the highest possible rate, namely at $M_{\text{max}}$ (we discuss whether it is potentially possible to exceed $M_{\text{max}}$ in Section 6.2). This makes our numerical estimates in Equations (17), (19) very robust.

Just for illustration let us also consider a situation in which the core accretes planetesimals at the rate $\dot{M}_p$ given by Equation (5), and determine $a_{\text{lim}}$ in this case. Comparing expressions (2) and (5), one can easily see that we can do this by simply setting $\chi = p^{1/2} (p(a)$ is defined by Equation (1)) as appropriate for planetesimal accretion at the rate $\dot{M}_p$ either in Equation (14) or in Equation (17) and determining $a_{\text{lim}}$ from the resulting expression:

$$a_{\text{lim}}^{\dot{M}_p} \approx 5 \text{ AU} \left( \frac{\Sigma_1^{0.5}}{\rho_c^{0.14} \kappa_0^{0.09}} \right)^{1/2} \left( \frac{\tau_{\text{neb}}}{3 \text{Myr}} \right)^{0.57}. \quad (20)$$

where we assumed MMSN-like disk properties to provide direct comparison with $a_{\text{lim}}^{\text{MMSN}}$ given by Equation (17). If, as we did in deriving Equation (19), we assume that the disk is marginally gravitationally unstable we would find $a_{\text{lim}}^{Q=1,\dot{M}_p} \approx 20$ AU for

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6 See also the discussion after Equation (12).
and metal-free (i.e., containing only H and He) atmospheres and found values of $M_{\text{crit}}$ lower by up to an order of magnitude (as low as $2 \, M_0$ in the metal-free case for $M = 10^{-6} \, M_0$ yr$^{-1}$) than in the dusty case. It would certainly be interesting to repeat calculations done in Sections 4, 5, and the Appendix for the case of dust-free atmosphere to see how this extreme reduction of opacity would extend the radial range available for CA.

In practice, we cannot do such calculation at present since it is not possible to calibrate $M_{\text{atm}}$ against the results of Hori & Ikoma (2010) as we did in Equations (10), (11), and the Appendix. This is because in the absence of dust $\kappa$ is a function not only of gas temperature but also of gas density. As demonstrated in R06, in this case $M_{\text{crit}}$ is no longer independent of the ambient temperature $T_0$ and density $\rho_0$ of the nebula as the original analyses of Mizuno (1980) and Stevenson (1982) suggest. Instead, one finds that $M_{\text{crit}} \propto (\rho_0/\rho_0)^{q/(1+q)}$, where $q \equiv \ln \kappa/\ln \rho$ (R06). Obviously, $M_{\text{crit}}$ then also depends on $\rho_0$ and $T_0$, and knowledge of this dependence is very important for obtaining $\Sigma_{\text{lim}}$ and $a_{\text{lim}}$ in the dust-free case. Unfortunately, we do not possess this knowledge from first principles as opacity calculations are rather complicated, and in any case we cannot currently calibrate $M_{\text{crit}}$ as functions of $M_\ast$, $T_0$, and $\rho_0$ against numerical results because the calculations of Hori & Ikoma (2010) were done for a single value of the planetary semimajor axis (meaning fixed values of $T_0$ and $\rho_0$), while $M_\ast$ was varied. The scaling of $M_{\text{atm}}$ and $M_{\text{crit}}$ with $T_0$ and $\rho_0$ and its implication for the possibility of CA thus remain worthwhile issues for future investigation.

We can still get a qualitative idea of how $a_{\text{lim}}$ changes in the dust-free case by setting opacity in Equation (17) at the very low level consistent with pure gas opacity, e.g., $\kappa = 10^{-10}$ cm$^2$ g$^{-1}$. We then find $a_{\text{lim}}^{\text{MMSS}} \approx 80$ AU compared to 44 AU that Equation (17) predicts for $\kappa = 0.1$ cm$^2$ g$^{-1}$. Thus, opacity reduction due to sedimentation and coagulation of dust grains in the protoplanetary atmosphere may help in extending the range of distances in which the CA is possible.

6. DISCUSSION

Despite the robustness of our arguments it is not inconceivable that some additional factors can weaken them and make giant planet formation by CA possible even beyond the limits represented by Equations (14) and (17). Alternatively, it is quite possible that some of the assumptions used in deriving these results are too extreme and one can get even better constraints by focusing on less dramatic assumptions. Below we review factors that can work one way or another.

6.1. Extending CA to Larger Radii

One possible way to facilitate CA and increase $a_{\text{lim}}$ is to consider the possibility of planetesimal accretion at rates exceeding $M_{\text{max}}$. This is very difficult (since there are many factors that tend to reduce $M$ compared to $M_{\text{max}}$, see Section 6.2) but may be possible if, e.g., one takes into account the increase of planetesimal capture cross section by the core caused by its extended, dense atmosphere. This effect has been previously investigated by Inaba & Ikoma (2003) who demonstrated that an increase of $M$ by a factor of $\sim 10$ compared to the value computed without atmosphere is possible. According to Equation (17), such an enhancement of $M$ (incorporated by increasing $\chi$) would boost $a_{\text{lim}}^{\text{MMSS}}$ by a factor of $\sim 4$.

Our present calculations assume that the core is accreting planetesimals continuously until the protoplanetary nebula
reach the maximum rate $1024 \text{ g}$ and would require on the order of $10^{-100}$ Myr to grow a dynamically dominant core would need to have mass of order $\frac{M}{\Omega}$ compared to $44 \text{ AU}$ estimated in Equation (17). For example, dissipation—this is important at large $a$ since massive core requires long time to be built. But one might wonder if building smaller core in shorter time and then cutting off subsequent planetesimal accretion (and energy release at the core surface, which supports atmosphere against going unstable) completely may still lead to the gas runaway and potentially extend it to larger semimajor axes. Such an accretion scenario has been adopted by, e.g., Pollack et al. (1996), Ikoma et al. (2000), and Hubickyj et al. (2005). The problem in this case is that even if $M = 0$ it still takes long time for the atmosphere around the core to grow to the mass comparable to $M_c$. INE0 show that this process occurs on the thermal timescale of the atmosphere and typically takes millions of years.

A similar problem is also encountered in a scenario where the core grows rapidly by planetesimal accretion in the inner regions of the protoplanetary disk and then gets scattered out to large radii by some massive perturber. One might expect that after the orbit of the scattered core Circularizes by dynamical friction the core would gradually accrete massive atmosphere and undergo gas runaway at some point. Given that both the orbit Circularization and envelope accretion are likely to take a long time it is not at all obvious whether the gas runaway could be achieved in this scenario within several Myr.

### 6.2. Limiting CA to Smaller Radii

There are many factors that can potentially reduce $a_{\text{lim}}$ compared to $44 \text{ AU}$ estimated in Equation (17). For example, there are several reasons why it may not be possible for $M$ to reach the maximum rate $M_{\text{max}}$.

First, the growing core may clear out a gap in the planetesimal disk around its orbit, thus significantly reducing $M$ (Tanaka & Ida 1997; Rafikov 2001, 2003a). In our previous calculations we implicitly assumed this not to happen, e.g., because of the core migration through the disk, which allows fresh planetesimal material to be constantly supplied for CA (Alibert et al. 2005).

Second, as we mentioned in Section 2, a known pathway to $M \approx M_{\text{max}}$ is via the growth of the core to the size at which it starts dominating dynamical evolution of nearby planetesimals and triggers their efficient collisional fragmentation (Rafikov 2004a). However, there is a strong implicit assumption in this scenario—that the core can reach this critical size within the nebula lifetime. Rafikov (2003b) has shown that at $a \sim 30–40 \text{ AU}$ a dynamically dominant core would need to have mass of order $10^{24} \text{ g}$ and would require on the order of $10–100 \text{ Myr}$ to grow in the MMSN. This timescale is apparently in conflict with the typical dissipation times of protoplanetary disks. Thus, one may need to either require a more massive disk at these radii or find other pathways for accretion at $M_{\text{max}}$. Formation of massive solid bodies by direct gravitational instability facilitated by various streaming instabilities (Johansen et al. 2009) may be quite relevant for the latter option. Regardless of this (arguably rather serious) issue our estimate (17) still remains a useful upper limit on $a_{\text{lim}}$.

Among other factors hindering the onset of gas runaway and reducing $a_{\text{lim}}$ we should mention the possibility of high opacity in the protoplanetary atmosphere. It was suggested in Section 5.3 that $\kappa$ may be very low because of the dust sedimentation and growth. However, infalling planetesimals which feed CA likely get partly disrupted in the atmosphere, leaving behind a large amount of refractory material. This may actually increase $\kappa$ compared to the value of $0.1 \text{ cm}^2 \text{ g}^{-1}$ assumed in Equation (17). Nevertheless, given the weak sensitivity of $a_{\text{lim}}$ to $\kappa$ the potential increase of atmospheric opacity is unlikely to have a huge effect on $a_{\text{lim}}$.

### 7. SUMMARY

We studied the formation of giant planets by CA at different locations in the protoplanetary disk with the goal of determining the range of radii where CA is feasible within the several Myr lifetime of the protoplanetary disk. We demonstrate that this range is determined by two key factors:

1. The high planetesimal accretion rate is necessary to build the solid core as rapidly as possible at large separations from the star.
2. Intense energy release at the core surface caused by planetesimal accretion increases the critical core mass and delays the gas runaway.

The first factor turns out to be more important and the largest distance at which CA can happen, around $40–50 \text{ AU}$, is obtained when $M$ is at its highest possible value $M_{\text{max}}$ corresponding to accretion of dynamically cold planetesimals. The core mass corresponding to this case is around $15 M_\oplus$, likely compatible with the isolation mass at this distance.

Our approach is quite different from other similar studies which often assume that (1) accretion proceeds at much slower rate $M_a$ (see Equation (5)) corresponding to accretion of planetesimals moving with random velocities at the level of $\Omega R_a$, and/or (2) the gas runaway commences after the core has reached a fixed mass of around $10 M_\oplus$ irrespective of the planetesimal accretion rate or the location in the disk. Relaxing these two arbitrary assumptions we are able to obtain a significantly more robust and self-consistent limit on the CA operation which can be represented as a lower bound on the solid surface density ($\gtrsim 0.1 \text{ g cm}^{-2}$) or an upper bound on the size of the region where the gas runaway can get going within several Myr timescale. These limits are insensitive to the mass of the central star and depend only weakly on the opacity in the core atmosphere. Our predictions are relevant for interpreting the results of current and future direct imaging surveys (Marois et al. 2008) designed to uncover and characterize the population of giant planets at large separations from their parent stars.

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### APPENDIX

#### CALIBRATION OF $\zeta$

INE0 investigated the dependence of $M_{\text{crit}}$ on $\kappa_0$ and $M$ assuming the latter to be constant. Their numerical calculations of CA include processes such as the dissociation and ionization of hydrogen, realistic gas opacities with the inclusion of the dust grain contribution, and self-gravity of the atmosphere. These are the crucial ingredients missing in the analytical models of Stevenson (1982) and R06. Thus, it is expected that calculations of INE0 should be more realistic than the aforementioned analytical studies.

We calibrate the coefficient $\zeta$ entering the expression for $M_{\text{atm}}$ against the results of INE0 who found that for $\kappa_0 \gtrsim$...
and it follows from their Table 6 that the best-fit power-law exponents $q'$ and $s$ satisfy $q' = 0.27$–0.29 and $s = 0.24$–0.29. Motivated by these scalings we first assume power-law dependence in the form
\[ \zeta = \zeta_0 \eta M_{\odot} L^{-\omega}. \] (A2)

The explicit dependence of $\zeta$ on parameter $\eta$ is motivated by the CA condition (9) and the assumed independence of $\eta$ on the planetesimal accretion history (i.e., the same value of $\eta$ applies in our case of $M \propto M_c^2$, see Equation (6), as in the $M = \text{const}$ case studied by INE0). Plugging this expression into (10) and using $L = GM_c M / R_c$ and Equation (9), we find
\[ M_{\text{crit}} = \left[ \frac{(k_B / \mu)^4}{\zeta_0 \sigma G^3 \omega} \left( \frac{4 \pi \rho_c}{3} \right)^{(1 + \omega)/3} \right]^{1/3} \kappa_0 M_\odot M^4, \] (A3)

where
\[ \tilde{q} = \frac{3(1 + \omega)}{7 + 3 \delta - 2 \omega}, \quad \tilde{s} = \frac{3}{7 + 3 \delta - 2 \omega}, \] (A4)

play the role of $q'$ and $s$ in INE0 case. It follows from (A4) that $\tilde{q} / \tilde{s} = 1 + \omega$ while the INE0’s results for these indices give $1 \lesssim q' / s \lesssim 1.2$. This constrains $\omega \lesssim 0.2$, resulting in a very weak dependence of $\zeta$ on $L$. Based on this observation and given the approximate nature of our calibration procedure we decided to neglect the dependence of $\zeta$ on $L$ altogether and to consider $\zeta$ in the form (11), i.e., scaling with $M_\odot$ only. Repeating our calculation we find instead of (A3) the following expression for $M_{\text{crit}}$:
\[ M_{\text{crit}} = \left[ \frac{(k_B / \mu)^4}{\zeta_0 \sigma G^3 \omega} \left( \frac{4 \pi \rho_c}{3} \right)^{1/3} \kappa_0 M_{\odot} \tilde{q} \right]^{1/3} \tilde{s} = \tilde{q} = \frac{3}{7 + 3 \delta}. \] (A5)

Choosing $\delta = 1.2$, as stated in Equation (11), we find $\tilde{s} = \tilde{q} \approx 0.28$ in good agreement with the $q'$ and $s$ values found in INE0.

To determine $\zeta_0$ we use the fact that according to Equation (A1) $M_{\text{crit}} \approx 7 M_\odot$ for $\kappa_0 = 1 \text{ cm}^2 \text{ g}^{-1}$ and $M = 10^{-7} M_\odot \text{ yr}^{-1}$. This uniquely determines the value of $\zeta_0$ in Equation (11).

Note that in carrying out this calibration we implicitly used the fact that $M_{\text{crit}}$ is independent of the ambient conditions in the nebula, which is true if opacity is independent of gas density (R06). This is an important point since the INE0 calculations have been done without varying the external conditions. More on this issue can be found in Section 5.3.

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