Uniform Capacitated Facility Location Problems with Penalties/Outliers

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Abstract

In this paper, we present a framework to design approximation algorithms for capacitated facility location problems with penalties/outliers using LP-rounding. Primal-dual technique, which has been particularly successful in dealing with outliers and penalties, has not been very successful in dealing with capacities. For example, despite unbounded integrality gap for facility location problem with outliers (FLPO), Charikar et al. [20] were able to get around the hardness by guessing the maximum facility opening cost in the optimal and provide a primal-dual solution for the problem. On the other hand, no primal-dual solution has been able to break the hardness of capacitated facility location problem (CFLP). LP-rounding techniques had also not been very successful in dealing with capacities until a recent work by Grover et al. [10] [11]. Their constant factor approximation violating the capacities by a small factor (1 + ϵ) is promising while dealing with capacities. Though LP-rounding has not been very promising while dealing with penalties and outliers, we successfully apply it to deal with them along with capacities. That is, our results are obtained by rounding the solution to the natural LP once again exhibiting the power of LP-rounding technique. Solutions obtained by LP-rounding are easy to integrate with other LP-based algorithms.

In this paper, we apply our framework to obtain first constant factor approximations for capacitated facility location problem with outlier (CFLPO) and capacitated k-facility location problem with penalty (CkFLPP) for hard uniform capacities using LP-rounding. Our solutions incur slight violations in capacities, (1 + ϵ) for the problems without cardinality(k) constraint and (2 + ϵ) for the problems with the cardinality constraint. For the outlier variant, we also incur a small loss (1 + ϵ) in outliers. Due to the hardness of the underlying problems, the violations are inevitable. Thus we achieve the best possible by rounding the solution of natural LP for these problems. To the best of our knowledge, no results are known for CFLPO and CkFLPP. The only result known for CkFLPP uses local search.

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1 Introduction

The facility location problem is a fundamental and well-studied problem in operational research and theoretical computer science \[22, 6, 15, 2, 5, 18\]. In (uncapacitated) facility location problem (FLP), we are given a set \(C\) of \(m\) clients and a set \(F\) of \(n\) facility locations. Setting up a facility at location \(i\) incurs cost \(f_i\) (called the facility opening cost or simply the facility cost) and servicing a client \(j\) by a facility \(i\) incurs cost \(c(i, j)\) (called the service cost). We assume that the costs are metric, i.e., they satisfy the triangle inequality. The objective is to select a set \(F' \subseteq F\), so that the total cost of opening the facilities in \(F'\) and the cost of servicing all the clients by opened facilities is minimized. If the maximum number of clients a facility \(i\) can serve is bounded by \(u_i\), the problem is called capacitated facility location problem. In this paper, we present our results for uniform capacities, i.e., \(u_i = u\) for all \(i \in F\).

The above formulation of the problem is not robust towards noisy clients (few distantly located clients called outliers) that can significantly increase the cost of the solution. To address the concern, Charikar et al. \[20\] defined the problems of \(k\)-center, \(k\)-median and facility location with outliers. Though the problem of clustering with outliers had been studied in the world of practitioners, the theoretical complexity of the problems was first studied by them. In facility location problem with outliers (FLPO), we are given a bound \(t\) on the maximum number of clients that can be considered as outliers. The objective now is to identify the locations to install facilities and select at least \(m - t\) clients to be served, so that the total cost for setting up the facilities and servicing the selected clients is minimized. In another closely related variant of the problem, the facility location problem with penalties (FLPP), instead of a hard bound on the number of outliers, we are allowed to leave a client \(j \in C\) unserved by paying a penalty cost \(p_j\). The objective, then, is to identify the locations to install facilities and select the clients to be served, so that the total cost for setting up the facilities, servicing the selected clients and the penalty cost of the unserved clients is minimized. The problem of outliers becomes more severe when there is a bound on the maximum number of clients a facility can serve. In this paper, we present a framework to design approximation algorithms for capacitated facility location problems with penalties/outliers using LP-rounding and apply it to obtain approximations for some of the very fundamental problems with outliers/penalties.

We first study the uniform capacitated facility location problem with outliers (CFLPO). Charikar et al. \[20\] gave a \(3 + \epsilon\) approximation for (uncapacitated) FLPO using primal dual techniques. A PTAS for FLPO was given by Friggstad et al. \[9\] using multiswap local search for a restricted variant of the problem with uniform facility opening costs and doubling metrics. Recently, Wang et al. \[23\] gave a 2 factor approximation using a combination of primal dual and greedy schema for the general setting. To the best of our knowledge, capacitated FLPO has not been studied earlier. We present first constant factor approximation for CFLPO violating both the capacities as well as the outliers by a small factor of \((1 + \epsilon)\). The result is obtained by rounding a solution to the natural LP; the violations are inevitable as both CFLP \[22\] as well as FLPO \[20\] are known to have unbounded integrality gaps from the natural LPs. In particular, we present the following result:

\[\textbf{Theorem 1.} \] There is a polynomial time algorithm that approximates uniform capacitated facility location problem with outliers within a constant factor \((O(1/\epsilon))\) violating the capacities by a factor of at most \((1 + \epsilon)\) leaving at most \((1 + \epsilon)t\) outliers, for every fixed \(\epsilon > 0\).

We next present our result for uniform capacitated facility location problem with penalty (CFLPP). Though there has been significant amount of work on uncapacitated FLPP
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with the current best being 1.8526 due to Xu and Xu [12], the only result known for the capacitated case is due to Gupta et al. [13] using local search. In this paper, we present first constant factor approximation for the problem using LP rounding with slight violation in capacities, which is inevitable due to hardness of CFLP [22]. Though our result is weaker than the result in [13] it is interesting as it is obtained by rounding solution to natural LP. LP-rounding based solutions are interesting as they are easier to integrate with other LP-based solutions. One direct application is presented in this paper itself by extending the result to a more generalized problem in Theorem 3. In particular, we present the following result for CFLPP:

▶ **Theorem 2.** There is a polynomial time algorithm that approximates uniform capacitated facility location problem with penalties within a constant factor \((O(1/\epsilon))\) violating the capacities by a factor of at most \((1 + \epsilon)\), for every fixed \(\epsilon > 0\).

We next consider a more general problem of Capacitated \(k\)-facility location problem with penalties (C\(k\)FLPP) where-in we are given an additional bound \(k\) on the maximum number of facilities that can be opened with the cost function same as that of CFLPP. We present first constant factor approximations for the problem with and without violating the cardinality, violating the capacities a little. Our results for C\(k\)FLPP also provide same results for the well known capacitated \(k\)-median problem with penalties (C\(k\)MP) as a special case. Though, some results are known for uncapacitated \(k\)-median problem with penalties (\(k\)MP) [20, 14, 25] and uncapacitated \(k\)-facility location problem with penalties (kFLPP) [21] using primal dual schema and local search, no result has been obtained using LP-rounding. Thus, our results for C\(k\)FLPP provide first LP-rounding based approximations for the uncapacitated variants of these problems as well. To the best of our knowledge, no results are known for the capacitated variants of the problems.

▶ **Theorem 3.** There is a polynomial time algorithm that approximates uniform capacitated \(k\)-facility location problem with penalties within a constant factor \((O(1/\epsilon))\) violating the capacities by a factor of at most \((1 + \epsilon)\) and cardinality by a factor of at most 2, for every fixed \(\epsilon > 0\).

▶ **Theorem 4.** There is a polynomial time algorithm that approximates uniform capacitated \(k\)-facility location problem with penalties within a constant factor \((O(1/\epsilon))\) violating the capacities by a factor of at most \((2 + \epsilon)\), for every fixed \(\epsilon > 0\).

The hardness results of capacitated \(k\)-median [3] apply to C\(k\)FLP as well, i.e., no constant factor approximation can be obtained by rounding an LP solution, violating one of the cardinality/capacity by less than a factor of 2 without violating the other. Thus the violations in Theorems 3 and 4 are inevitable.

▶ **Corollary 5.** There is a polynomial time algorithm that approximates uniform capacitated \(k\)-median problem with penalties within a constant factor \((O(1/\epsilon))\) violating the capacities by a factor of at most \((1 + \epsilon)\) and cardinality by a factor of at most 2, for every fixed \(\epsilon > 0\).

▶ **Corollary 6.** There is a polynomial time algorithm that approximates uniform capacitated \(k\)-median problem with penalties within a constant factor \((O(1/\epsilon))\) violating the capacities by a factor of at most \((2 + \epsilon)\), for every fixed \(\epsilon > 0\).

▶ **Corollary 7.** There is a polynomial time algorithm that approximates uncapacitated \(k\)-facility location problem with penalties within a constant factor \((O(1/\epsilon))\), for every fixed \(\epsilon > 0\).

Table I shows the known hardness results for some of the special cases of our problems and Table 2 summarises the results of this paper.
Table 1 Known hardness results. They apply to the generalisations considered in this paper.

| Problem     | Hardness | Capacity Violation | Outlier Violation | Cardinality Violation | Reference |
|-------------|----------|--------------------|-------------------|------------------------|-----------|
| FLPO        | Unbdd IG | NA                 | Nil               | NA                     | [20]      |
| CFLP        | Unbdd IG | Nil                | NA                | NA                     | [22]      |
| kMO         | Unbdd IG | NA                 | Nil               | Nil                    | [20]      |
| CkM/CkFLP  | Unbdd IG | < 2                | NA                | Nil                    | [20]      |
| CkM/CkFLP  | Unbdd IG | Nil                | < 2               | Nil                    | [20]      |

Table 2 Our Results

| Problem     | Approximation Ratio | Capacity Violation | Outlier Violation | Cardinality Violation |
|-------------|---------------------|--------------------|-------------------|------------------------|
| CFLPO       | $O(1/\epsilon)$     | $(1 + \epsilon)$  | $(1 + \epsilon)$ | NA                     |
| CFLPP       | $O(1/\epsilon)$     | $(1 + \epsilon)$  | NA                | NA                     |
| CkM/CkFLPP | $O(1/\epsilon)$     | $(1 + \epsilon)$  | NA                | 2                      |
| CkM/CkFLPP | $O(1/\epsilon)$     | $(2 + \epsilon)$  | NA                | Nil                    |
| kFLPP       | $O(1/\epsilon)$     | NA                 | NA                | Nil                    |

1.1 Related Work

Charikar et al. [20] were the first to define the problems of $k$-center, facility location and $k$-median with penalties/outliers in the theory world. Uncapacitated FLPO is known to have an unbounded integrality gap with natural LP [20]. Charikar et al. [20] get around the hardness by guessing the most expensive facility opened by the optimal and gave a $(3 + \epsilon)$-factor approximation using primal dual technique. Friggstad et al. [9] gave a PTAS using multiswap local search on a restricted variant of the problem with uniform facility opening costs and doubling metrics.

For uncapacitated FLPP, a 3-factor approximation using primal dual techniques was given by Charikar et al. [20] which was subsequently improved to 2 by Jain et al. [16] using dual-fitting and greedy approach. Wang et al. [23] also gave a 2-factor approximation using a combination of primal-dual and greedy technique. Later Xu and Xu [26] gave a $2 + 2/\epsilon$ using LP rounding. The factor was improved to 1.8526 by the same authors in [12] using a combination of primal-dual schema and local search. For linear penalties, using LP-rounding, Li et al. [19] gave a 1.5148-factor which was subsequently improved to 1.488 by Qiu et al. [21].

Uncapacitated $k$-median problem with outlier($k$MO) was also shown to have an unbounded integrality gap with natural LP [20]. Charikar et al. [20] gave a $4(1 + 1/\epsilon)$-approximate solution using primal-dual technique with $(1 + \epsilon)$-factor violation in outliers. Friggstad et al. [9] used local search techniques to obtain a $(3 + \epsilon)$-factor approximation with $(1 + \epsilon)$-factor violation in outliers. The first true constant factor approximation was given by Chen [4] using a combination of primal-dual and local search. Their approximation factor is large. The current best known result for the problem is due to Krishnaswamy et al. [17] who gave a $(7.081 + \epsilon)$-factor approximation using iterative rounding framework and strengthened LP.

For $k$-Median with penalties, Charikar et al. gave a 4-factor approximation in [20] which was later improved to $(3 + \epsilon)$ by Hajiaghayi et al. [14] using local search technique. For uniform penalties, Wu et al. [24] gave a $(1 + \sqrt{3} + \epsilon)$-approximation algorithm via pseudopolynomial. For $k$-facility location problem with linear penalties, Wang et al. [21] used local search schema to obtain $(2 + \sqrt{3} + \epsilon)$-approximation.
In capacitated world, the only result known is for CFLPP \cite{13} by Gupta et al. They give 3 and 5 factor approximation for uniform and non-uniform capacities respectively using local search techniques. A 25-factor approximation for Capacitated $k$-center with outliers is given by Cygan and Kociumaka in \cite{8}. To the best of our knowledge, no result is known for CFLPO and C$k$LPP.

### 1.2 Our Techniques

Most of the work dealing with outliers/penalty uses primal-dual technique or a combination of the primal-dual/dual-fitting with greedy/local search schema. Since primal-dual technique has not been able to handle capacities, these works cannot be extended to the capacitated variant of these problems. Local search alone does not perform well for outliers even for the uncapacitated variants. We use LP-rounding to obtain our desired claims. In particular, we extend the work in \cite{10, 11} to penalty and outliers.

The proposed framework works in two/three steps. In the first step we identify the set of clients that serve as outliers (/pay penalty) in our solution. In the second step, we call upon the solution to the underlying problem without outliers(/penalty). In some cases, we are able to directly plug-in the solution of the underlying problem eliminating the need for the third step. However, in some other cases, we modify the solution of the underlying problem to open the facilities integrally and solve the transportation problem with outliers (/penalty) to obtain the integral assignments finally in the third step. Note that the LP for the transportation problem with outliers (/penalty) is TUM(totally unimodular) and hence provides an integral optimal solution.

In step 2 of CFLPP (and C$k$LPP with cardinality violation), we raise the assignment of the remaining clients to 1 and the openings accordingly so that the solution so obtained is a feasible solution for the LPs of CFLP (/C$k$LFP). In this case, we are able to directly plug-in the solution of CFLP (/C$k$LFP) eliminating the need for the third step. For CFLPO (and C$k$LPP without cardinality violation), in our reduced problem after step 1, servicing clients to full extent leads to large outlier/cardinality violation. Thus, we cannot simply plug-in the solutions of CFLP (/C$k$LFP). Hence, in step 2, we modify the solution of the underlying problem to open the facilities integrally preserving the extent to which clients are serviced from step 1.

### 1.3 Organisation of the paper

In Section 2 we present our algorithm for CFLPO followed by CFLPP in Section 3. In Section 4 we present our results for capacitated $k$-facility location problem with penalties.

### 2 Capacitated Facility Location Problem with Outsiders

CFLPO can be formulated as the following integer program (IP):

Minimize $\text{CostCFLPO}(x, y, z) = \sum_{j \in C} \sum_{i \in F} c(i, j)x_{ij} + \sum_{i \in F} f_i y_i$

subject to

\begin{align*}
\sum_{i \in F} x_{ij} + z_j & \geq 1 \quad \forall j \in C \quad (1) \\
\sum_{j \in C} x_{ij} & \leq u_i y_i \quad \forall i \in F \quad (2) \\
 x_{ij} & \leq y_i \quad \forall i \in F, j \in C \quad (3) \\
\sum_{j \in C} z_j & \leq t \quad (4) \\
z_j, y_i, x_{ij}, & \in \{0, 1\} \quad (5)
\end{align*}
where variable \( y_i \) denotes whether facility \( i \) is open or not, \( z_j \) indicates whether client \( j \) is an outlier and, \( x_{ij} \) indicates if client \( j \) is served by facility \( i \) or not. Constraints 1 ensure that the extent to which a client is served and the extent, to which it is an outlier, sum to 1. Constraints 2 make sure that the total demand assigned to a facility is no more than its capacity. Constraints 3 ensure that a client is assigned only to an open facility and Constraint 4 ensures that total number of unserved clients is at most \( t \). LP-Relaxation of the problem is obtained by allowing the variables \( z_j, y_i, x_{ij} \in [0, 1] \). Call it \( LP_{CFLPO} \).

2.1 Step 1: Identifying the outliers

We first identify the set of clients that will be treated as outliers in our solution. Let \( \rho^* = < x^*, y^*, z^* > \) denote the optimal solution of \( LP_{CFLPO} \) and \( LP_{opt} \) denote the cost of \( \rho^* \). Let \( t \geq 2 \) be a fixed parameter. Let \( J \in C \) be such that \( z^*_j \geq 1 - \frac{1}{t} \), we treat such clients as outliers in our solution. Let \( C_0 \) be the set of these clients and \( C_r \) be the set of rest of the clients. Let \( \tilde{\rho} = < \tilde{x}, \tilde{y}, \tilde{z} > \) be the solution so obtained. That is, \( \tilde{z}_j = 1 \) and \( \tilde{x}_{ij} = 0 \) \( \forall i \in F \) for all \( j \in C_0 \).

Facility openings and the assignments of the remaining clients remain the same. Note that \( \sum_{j\in C} \tilde{z}_j \leq \left( \frac{r^*}{r^* + \ell} \right) \sum_{j\in C} z^*_j \leq \left( \frac{1}{r^* + \ell} \right) t \). Also, Cost\(CFLPO(\tilde{x}, \tilde{y}, \tilde{z}) \leq \) Cost\(CFLPO(x^*, y^*, z^*) \).

2.2 Step 2: Integrally Open Solution

We obtain an integrally open solution slightly modifying the algorithm for CFLP of \[10, 11\] on the reduced set \( C_r \) of clients. Note that we cannot plug in the solution of \[10, 11\] as the clients in our reduced instance are not fully served. We preserve the extent to which the clients in \( C_r \) are served while opening the facilities integrally. Note that \( \forall j' \in C_r, \sum_{i \in F} \tilde{x}_{ij} > 1/\ell \).

We first sparsify the problem instance by removing some clients from the client set \( C_r \) and distributing their demands (the extent up to which the client is served in \( \tilde{\rho} \)) to the remaining clients. This is done using clustering technique in the same manner as done in \[10, 11\] (see Figure 1): for \( j \in C_r \), let \( \hat{C}_j \) denote the average connection cost of \( j \) in \( \hat{\rho} \), i.e., \( \hat{C}_j = \left( \sum_{i \in F} \hat{x}_{ij} c(i, j) \right) / \left( \sum_{i \in F} \hat{x}_{ij} \right) \). Further, let \( ball(j) \) be the set of facilities within a distance \( t\hat{C}_j \) of \( j \), i.e., \( ball(j) = \{ i \in F; c(i, j) \leq t\hat{C}_j \} \). Then, \( size(\hat{y}, ball(j)) \geq \left( 1 - \frac{1}{t} \right) \sum_{i \in F} \hat{x}_{ij} \geq \frac{1}{t} \) where \( size(\hat{y}, ball(j)) = \sum_{i \in ball(j)} \hat{y}_i \) denotes the total extent up to which facilities in \( ball(j) \) are opened under \( \hat{\rho} \). Clients in \( C_r \) are considered in the increasing order of their radii \( \{t\hat{C}_j\} \).

For a client \( j \) at hand, remove all the clients \( k : c(j, k) \leq 2t \max\{\hat{C}_j, \hat{C}_k\} \) and repeat the process with the remaining clients. Let \( C' \) be the set of clients remaining after all the clients in \( C_r \) have been considered. Clusters, of facilities, are formed around the clients in \( C' \) by assigning a facility to the cluster of the nearest client in \( C' \), i.e., if, for \( j' \in C', N_{j'} \) denotes the cluster centered at \( j' \) then a facility \( i \) belongs to \( N_{j'} \) if and only if \( j' \) is the closest client in \( C' \) to \( i \). Note that \( ball(j') \subseteq N_{j'} \). The clients in \( C' \) are called the cluster centers. Any two cluster centers \( j', k' \) in \( C' \) satisfy the separation property: \( c(j', k') > 2t \max\{\hat{C}_j, \hat{C}_k\} \).

Demand of the removed clients are distributed strategically to various cluster centers: let \( j \in C_r \) be a client that was removed while sparsifying the problem instance, \( j' \in C' \) and let \( A_{j'}(j, N_{j'}) \) be the extent up to which \( j \) was served by the facilities in \( N_{j'} \). Then \( A_{j'}(j, N_{j'}) \) extent of \( j \) is moved to \( j' \). This can be done incurring \( 2(\ell + 1) \)-factor loss in cost \[10, 11\].

For \( j' \in C' \), let \( d_{j'} \) denote the demand accumulated at \( j' \), i.e., \( d_{j'} = \sum_{j \in C_r} A_{j'}(j, N_{j'}) \). A cluster is said to be sparse if \( d_{j'} \leq u \) otherwise it is called dense. Let \( C_{S} = \{ j' \in C' ; d_{j'} \leq u \} \) and \( C_{D} = C' \setminus C_{S} \).

For \( j' \in C_{S} \), we open a cheapest facility in \( ball(j') \) at a loss of factor \( 2\ell \) in the facility opening cost and transfer all the assignments in the cluster onto it at a loss of constant factor in the service cost \[11\]. Since \( d_{j'} \leq u \) there is no violation in the capacity.
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(a) (b) (c)

Figure 1 (a) The balls around the clients in $C_r$. (b) Let $\ell = 4$. Reduced set of clients $C' = \{j_1, j_4\}$, partition of $F': \mathcal{N}_{j_1} = \{i_1, i_2, i_5\}, \mathcal{N}_{j_4} = \{i_3, i_4\}$ and assignment by LP solution. (c) Partitioning of demand: $d_{j_1} = \sum_{j \in C_r} (\hat{x}_{i_1j} + \hat{x}_{i_2j} + \hat{x}_{i_5j}), d_{j_4} = \sum_{j \in C_r} (\hat{x}_{i_3j} + \hat{x}_{i_4j})$.

To handle dense clusters, for every $j' \in C_D$, we solve the LP: Minimize $\text{Cost}_{CF}(w) = \sum_{i \in \mathcal{N}_{j'}} (f_i + wc(i, j')) w_i$ s.t. $\sum_{i \in \mathcal{N}_{j'}} w_i \geq d_j'$ and $w_i \in [0, 1]$. It can be shown that $w_i = \sum_{j \in C_r} \hat{x}_{ij}$ is a feasible solution with cost at most $\sum_{i \in \mathcal{N}_{j'}} f_i \hat{y}_i$ and $\sum_{i \in \mathcal{N}_{j'}} \sum_{j \in C_r} \hat{x}_{ij}[c(i, j) + 4 \hat{C}_j]$. An almost integral solution $w'$ is obtained by arranging the fractionally opened facilities in $w$ in non-decreasing order of $f_i + c(i, j')u$ and greedily transferring the openings $w$ without increasing the cost of the solution.

The demand $d_j'$ is distributed to the opened facilities by assigning $u \cdot w'_i$ demand to the $i$th facility. If the opening of fractional facility is $\leq \epsilon$, we transfer the demand to an integrally opened facility at $(1 + \epsilon)$-factor loss in capacity and service cost. Else, i.e., if the opening of fractional facility is greater than $\epsilon$, we open the facility at constant factor $(1/\epsilon)$ loss in facility opening cost.

Having selected the facilities to be opened integrally, we solve the transportation problem with outliers for uniform supplies $\hat{u} = \lceil (1 + \epsilon)u \rceil$, unit demand and the number of outliers, $\hat{\ell} = \lceil \sum_{j \in C_r} \hat{z}_j \rceil$. By choosing $\ell = O(1/\epsilon)$, we get the desired claim.

3 Capacitated Facility Location Problem with Penalties

CFLPP can be formulated as the following integer program (IP):

\[
\begin{align*}
\text{Minimize } & \quad \text{Cost}_{CFLPP}(x, y, z) = \sum_{j \in C} \sum_{i \in F} c(i, j)x_{ij} + \sum_{i \in F} f_i y_i + \sum_{j \in C} p_j z_j \\
\text{subject to } & \quad \sum_{i \in F} x_{ij} + z_j \geq 1 \quad \forall j \in C \tag{6} \\
& \quad \sum_{j \in C} x_{ij} \leq u_i y_i \quad \forall i \in F \tag{7} \\
& \quad x_{ij} \leq y_i \quad \forall i \in F, j \in C \tag{8} \\
& \quad z_j, y_i, x_{ij} \in \{0, 1\} \tag{9}
\end{align*}
\]

where variable $z_j$ denotes if client $j$ pays penalty or not. Constraints (6) ensure that the extent to which the client is served and the extent to which it pays the penalty sum to 1.

1 a solution is said to be an almost integral solution if it has at most one fractionally opened facility.
LP-Relaxation of the problem is obtained by allowing the variables \( z_j, y_i, x_{ij} \in [0, 1] \). Call it \( LP_{CFLPP} \).

### 3.1 Step 1: Identifying the clients that pay the penalty

We first identify the clients that will penalty in our solution; rest of the clients are serviced fully. Let \( \rho^* =< x^*, y^*, z^* > \) denote the optimal LP solution of \( LP_{CFLPP} \) and \( LP_{opt} \) denote the cost of solution \( \rho^* \). Let \( \ell \geq 2 \) be a fixed parameter. Let \( j \in C \) be such that \( z^*_j \geq 1/\ell \), we pay penalty for such clients in our solution incurring at most \( \ell \)-factor loss in penalty costs. Let \( C_p \) be the set of these clients and \( C_r \) be the set of rest of the clients.

For \( j \in C_r \), \( \forall i \in F : x^*_{ij} > 0 \), we raise the assignment of \( j \) on \( i \) proportionately so that \( j \) is fully serviced. Opening of \( i \) is raised accordingly so as to keep the solution feasible.

Let \( \hat{\rho} =< \hat{x}, \hat{y}, \hat{z} > \) be the solution obtained, i.e., (i) \( \forall j \in C_p, \hat{x}_{ij} = 0 \) \( \forall i \in F \) and \( \hat{z}_j = 1 \leq \ell z^*_j \) (ii) \( \forall j \in C_r, \hat{x}_{ij} = x^*_{ij} / \sum_{i \in F} x^*_{ij} \). Note that \( x^*_{ij} \leq \ell \hat{x}_{ij} \leq \ell x^*_{ij} \). (iii) \( \forall i \in F \), set \( \hat{y}_i = \min\{1, y^*_{ij} \cdot \max_{j \in C_p, z^*_j > 0}(\hat{x}_{ij}/x^*_{ij})\} \). Note that \( y^*_{ij} \leq \hat{y}_i \leq \frac{1}{\ell} y^*_{ij} \).

Note that \( \hat{\rho} \) satisfies constraint [9] as for \( j \in C_p, \hat{z}_j = 1 \) and \( \hat{x}_{ij} = 0 \), therefore \( \sum_{i \in F} \hat{x}_{ij} + \hat{z}_j = 1 \) whereas for \( j \in C_r, \hat{z}_j = 0 \) therefore \( \sum_{i \in F} \hat{x}_{ij} + \hat{z}_j = \sum_{i \in F} \hat{x}_{ij} = \sum_{i \in F} x^*_{ij} / \sum_{i \in F} x^*_{ij} = 1 \). It also satisfies constraint [10]: If \( \hat{y}_i = 1 \) then it holds trivially as \( \hat{x}_{ij} \leq 1 \) else for \( x^*_{ij} > 0 \), \( \hat{y}_i \geq \frac{\hat{x}_{ij}}{2} \cdot \hat{x}_{ij} \geq \hat{x}_{ij} \) as \( y^*_{ij} \cdot x^*_{ij} \geq 1 \). Also, capacities are violated at most by a factor of \( \frac{\ell}{\ell} \) as \( \forall i \in F, \sum_{j \in C_r} x^*_{ij} \leq \frac{\ell}{\ell} \sum_{j \in C_r} z^*_j \leq \frac{\ell}{\ell}(\sum_{i \in F} x^*_{ij} + \sum_{i \in F} \hat{y}_i) \). And, 
\[
\text{Cost}_C^{CFLPP}(\hat{x}, \hat{y}, \hat{z}) \leq \frac{\ell}{\ell}(\sum_{i \in F} x^*_{ij} + \sum_{i \in F} \hat{y}_i) + \ell \sum_{j \in C_r} \hat{y}_i \leq \text{LP}_{opt},
\]
where the last inequality follows since \( \ell \geq \frac{\ell}{\ell} \) for \( \ell \geq 2 \).

### 3.2 Step 2: Reducing to CFLP

Next, we solve an instance of capacitated facility location where-in the client set is reduced to \( C_r \) and capacities are scaled up by a factor of \( \frac{\ell}{\ell} \). < \( \hat{x}, \hat{y} > \) provides a feasible solution for the LP of CFLP with cost at most \( (\frac{\ell}{\ell})\sum_{i \in F} \hat{y}_i + \sum_{j \in C_r} c(i, j)x^*_{ij} \).

\[\textbf{Lemma 8.} \text{ Let } < \hat{x}, \hat{y} > \text{ be an } \alpha\text{-approximate solution for CFLP with } \beta\text{-factor violation in capacities. Then, } < \hat{x}, \hat{y}, \hat{z} > \text{ is a solution to CFLPP of cost within a constant factor } (\max\{\alpha(1 + \frac{1}{\ell}), \ell\}) \text{ of } LP_{opt} \text{ violating the capacities by a factor of } \beta(1 + \frac{1}{\ell}) \text{ for } \ell \geq 2.\]

\[\textbf{Proof.} \text{ (i) Cost Bound: } \text{Cost}_C^{CFLPP}(\hat{x}, \hat{y}, \hat{z}) \leq \text{Cost}_C^{CFLPP}(\hat{x}, \hat{y}) + \ell(\sum_{j \in C_r} \hat{z}_j) \leq \alpha \cdot \text{Cost}_C^{CFLPP}(\hat{x}, \hat{y}) + \ell(\sum_{j \in C_r} \hat{z}_j) \leq \alpha \cdot (\frac{\ell}{\ell})\sum_{i \in F} \hat{y}_i + \sum_{j \in C_r} c(i, j)x^*_{ij} + \ell(\sum_{j \in C_r} \hat{z}_j) \leq (\max\{\alpha(1 + \frac{1}{\ell}), \ell\}) \text{LP}_{opt}. \text{ (ii) Violation in capacities: } \beta \hat{u} = \beta(\frac{\ell}{\ell})u = \beta(1 + \frac{1}{\ell})u. \]

By choosing \( \ell : \frac{1}{\ell} \leq \epsilon \), and using the result of \[10, 11\], \( \alpha = O(1/\epsilon), \beta = (1 + \epsilon) \), we arrive at Theorem \[2\].

### 4 Capacitated \( k \)-Facility Location Problem with penalties

\( CKFLPP \) can be formulated as the following integer program (IP):

\[\begin{align*}
\text{max} & \quad \sum_{i \in F} \sum_{j \in C} c(i,j) x_{ij} \\
\text{subject to} & \quad \sum_{j \in C} x_{ij} = 1, \quad \forall i \in F \\
& \quad \sum_{i \in F} x_{ij} \leq \beta, \quad \forall j \in C \\
& \quad x_{ij} \in \{0,1\}, \quad \forall i \in F, \forall j \in C
\end{align*}\]

\[\text{where } C = \{1, 2, \ldots, K\} \]

\[2\text{ Wlog we assume that } \sum_{i \in F} x_{ij} \leq 1.\]
Minimize \( \text{Cost}_{CkFLPP}(x, y, z) = \sum_{i \in C} \sum_{j \in F} c(i, j)x_{ij} + \sum_{i \in F} f_i y_i + \sum_{j \in C} p_j z_j \)

subject to

\[
\begin{align*}
\sum_{i \in F} x_{ij} + z_j & \geq 1 \quad \forall j \in C & (10) \\
\sum_{j \in C} x_{ij} & \leq u_i \quad \forall i \in F & (11) \\
\sum_{i \in F} y_i & \leq k & (12) \\
x_{ij} & \leq y_i \quad \forall i \in F, j \in C & (13) \\
z_j, y_i, x_{ij} & \in \{0, 1\} & (14)
\end{align*}
\]

Constraint (12) ensures that at most \( k \) facilities are opened in a feasible solution. LP-Relaxation of the problem is obtained by allowing the variables \( z_j, y_i, x_{ij} \in [0, 1] \). Call it \( LP_{CkFLPP} \).

**Proof Of Theorem 3**: Using \( O(1/\epsilon) \) factor approximation for CkFLLP with \((1 + \epsilon)\) violation in capacities and \( 2/(1 + \epsilon) \) factor loss in cardinality, the approach of Section 5 leads to Theorem 3.

We next present a result without violating the cardinality. The capacity violation increases slightly to \((2 + \epsilon)\) in the process. Let \( \rho^* = < x^*, y^*, z^* > \) denote the optimal \( LP_{CkFLPP} \) solution and \( LP_{opt} \) denote the cost of solution \( \rho^* \). Let \( \ell \geq 4 \) be a fixed parameter.

**Step 1: Identifying the set of clients that pay the penalty**: We identify the set of clients that pay the penalty in our solution in the same manner as was done in Section 3.1. Let \( \hat{\rho} \) be solution so obtained. Clients in \( C_r \) are then served to an extent \( \geq (1 - \frac{1}{\ell}) \), i.e., \( \sum_{i \in F} \hat{x}_{ij} \geq (1 - \frac{1}{\ell}) \). Unlike CFLLP, raising the assignments of clients in \( C_r \) to \( 1 \) leads to \( \ell \) factor loss in cardinality. Hence we can not directly plug in the solution of CkFLLP as the approach relies on the fact that clients are served to full extent to guarantee sufficient opening within a cluster.

**Step 2: Obtaining an integrally open solution**: We modify the constant factor approximation for CkFLLP in [10][11] that violates the capacities by a factor of \((2 + \epsilon)\) without violating the cardinality to obtain an integrally open solution. The extent to which clients are served is preserved from Step 1.

**Step 3: Obtaining an integral solution**: Integral assignments are obtained by solving an instance of transportation problem with penalties (TPP) with scaled up capacities.

Step 2 works in two phases: In phase I, the problem instance is sparsified using clustering techniques in the same manner as explained in Section 2.2. Let \( \hat{C}_j \) denote the average connection cost of a client \( j \in C \), in \( \hat{\rho} \), i.e., \( \hat{C}_j = \frac{\sum_{i \in F} \hat{x}_{ij} c(i, j)}{\sum_{i \in F} \hat{x}_{ij}} \). In this case, \( \text{size}(\hat{y}, \text{ball}(j)) \geq (1 - \frac{1}{\ell})(\sum_{i \in F} \hat{x}_{ij}) \geq (1 - \frac{1}{\ell})^2 \). Sparse and dense clusters are defined in the same manner as in Section 2.2. For phase II, we present a brief sketch of the approach, omitting some of the details that are similar to the algorithm in [10][11].

1. For a dense cluster, sufficient facilities are opened in it so that its entire demand can be assigned to the opened facilities within the cluster; we call such clusters as self-sufficient.
2. For a sparse cluster, we are not able to guarantee this. That is, we may not be able to open even one facility in a sparse cluster. We try to send the unmet demand of such a cluster to a nearby cluster in which we guarantee that its demand can be assigned to the opened facilities within the claimed bounds. To achieve the goal, a forest of routing trees is defined. The edges costs are non-increasing as we move up the tree.
3. In order to ensure that demand of every sparse cluster is assigned to facilities that are not too far from it, we make groups, called Meta-Clusters, of \( \ell/2 \) clusters and write an auxiliary LP (ALP) to open sufficient number of facilities within each Meta-Cluster so that all except at most \( u \) units of its demand is served within the Meta-Cluster itself.

\[ A \text{ careful analysis shows that the cardinality loss in [10][11] is actually } 2/(1 + \epsilon) \]
ALP is defined such that the opened facilities are well spread out amongst the clusters of a Meta-Cluster (we make sure that at most 1 (sparse) cluster has no facility opened in it and demand of a dense cluster is satisfied within the cluster itself.)

4. An iterative rounding algorithm of [10] [11] is then used to obtain a pseudo-integral solution that has at most two fractionally opened facilities, for the auxiliary LP.

5. Pseudo integral solution is converted to fully integral solution by opening the facility with larger opening to full extent.

We start with describing the routing tree which is used to form the Meta-Clusters. This is followed by describing the auxiliary LP (ALP) to open the facilities integrally.

4.1 Constructing the Routing Trees

We define a graph \( G = (V, E) \) on set \( C' \) of cluster centers. For \( j' \in C_S \), let \( \eta(j') \) be the nearest other cluster center in \( C' \), i.e., \( \eta(j') = k'(\neq j') \in C' : k' \in C' \Rightarrow c(j', k') \leq c(j', k'' \in C', \eta(j') = j' \). The graph \( G \) consists of directed edges \((j', \eta(j'))\). Note that each connected component of the graph is a tree except a 2-cycle at the root, we delete any one of the two edges from the cycle arbitrarily. The resulting graph is a forest.

Following Lemmas will be helpful in providing a feasible solution of bounded cost for the auxiliary LP. For a client \( j' \in C_S \), Lemma 9 bounds the cost of serving major part \( (1 - \frac{1}{\ell}) \) extent of \( d_{ij} \). Let a client \( j' \) is served to an extent of \( \gamma \) (in \( \beta \)) within its cluster \( N_{j'} \) and to an extent of \( \delta \) outside. If \( \gamma \geq (1 - 1/\ell) \) then Lemma 10 bounds the cost of serving \( \gamma \) extent of \( d_{ij} \) within \( N_{j'} \). Otherwise it bounds the cost of serving \( \gamma \) extent of \( d_{ij} \) within \( N_{j'} \) and sending \((1 - 1/\ell) - \gamma \) extent of it to the nearest cluster center. Proof of Lemma 10 uses Lemma 9. Note that \( \gamma + \delta \geq (1 - 1/\ell) \).

Lemma 9. \[ \sum_{j' \in C'}\sum_{i \in N_{j'}} d_{ij'} \sum_{i \in \mathcal{X}} c(i, j') \hat{x}_{ij'} \leq 3 \sum_{j' \in C_S} \sum_{i \in \mathcal{X}} c(i, j') \hat{x}_{ij} = 3 \text{Cost}_{CkFLPP}(\bar{p}) \]

Lemma 10. \[ \sum_{j' \in C_S} d_{ij'} (\sum_{i \in N_{j'}} c(i, j') \hat{x}_{ij'} + c(j', \eta(j')) (\frac{\ell - 1}{\ell} - \min\{\frac{\ell - 1}{\ell}, \sum_{i \in N_{j'}} \hat{x}_{ij'}\})) \leq 6 \cdot \text{Cost}_{CkFLPP}(\bar{p}). \]

Proof. The second term of LHS is

\[
\sum_{j' \in C_S} \sum_{i \in \mathcal{X}} c(i, j') \hat{x}_{ij'} - \sum_{i \in N_{j'}} \hat{x}_{ij'} = \sum_{j' \in C_S} \sum_{i \in \mathcal{X}} c(i, j') \hat{x}_{ij'} (\sum_{i \in N_{j'}} \hat{x}_{ij'}) - \sum_{i \in N_{j'}} \hat{x}_{ij'}
\]

The in-degree of a node in the above routing tree is unbounded which can lead to arbitrarily large capacity violations. We convert these trees into binary trees using the standard procedure, also explained in [10] [11]. (Refer to Appendix A.1 for figures.) The binary routing trees have the following properties: (i) There is at most one dense cluster and it must be the root of the tree, (ii) the in-degree of root is at most 1 and, (iii) edge costs decrease as we move up from the leaves to the root. Let \( \psi(j') \) be the parent of cluster \( j' \) in the newly formed binary trees. Then, \( c(j', \psi(j')) \leq 2c(j', \eta(j')) \) and therefore,

\[
\frac{\ell - 1}{\ell} - \min\{\frac{\ell - 1}{\ell}, \sum_{i \in N_{j'}} \hat{x}_{ij'}\} = 0 \text{ if } \min\{\frac{\ell - 1}{\ell}, \sum_{i \in N_{j'}} \hat{x}_{ij'}\} = (\frac{\ell - 1}{\ell}) \text{ and it is } \leq (\sum_{i \in \mathcal{X}} \hat{x}_{ij'} - \sum_{i \in N_{j'}} \hat{x}_{ij'}) \text{ otherwise, where the inequality follows as } \sum_{i \in \mathcal{X}} \hat{x}_{ij'} \geq (1 - 1/\ell). \text{ Thus the claim follows in either case.} \]
Claim 11. \( \sum_{j' \in C_{r'}} d_{j'} \left( \sum_{i \in N_{j'}} c(i, j') \hat{x}_{ij'} + c(j', \psi(j')) \left( \frac{t_{ij}}{r} - \min \left\{ \frac{t_{ij}}{r}, \sum_{i \in N_{j'}} \hat{x}_{ij'} \right\} \right) \right) \leq 12 \cdot \text{CostCkFLPP}(\hat{\rho}) \)

4.2 Constructing the Meta-Clusters and writing the ALP

Recall that each cluster has at least \((1 - 1/\ell)^2\) opening in it and hence \(q(1 - 1/\ell)^2 \geq q - 1\) for \(q \leq \ell/2\). Thus, we make Meta-Clusters consisting of \(\ell/2\) clusters. Meta-Clusters are formed in the same way as is done in [10, 11]: for every binary routing tree \(T\), make meta-clusters by processing nodes of the tree \(T\) greedily from the root node. Starting from the root node, extend the Meta-Cluster by adding a node that is connected to the Meta-Cluster by a cheapest edge. We grow the Meta-Cluster until either we have included \(\ell/2\) nodes/clusters or there are no more nodes to add (i.e. we have reached the leaves of the tree). Remove the nodes of the Meta-Cluster and repeat the above process if there are more nodes in the tree (the tree could have gotten disconnected after removing the nodes included in the Meta-Cluster). Each Meta-Cluster is a binary rooted tree in itself. The construction imposes a natural rooted tree structure on the Meta-Clusters also. Some Meta-Clusters towards the leaves may have less than \(\ell/2\) clusters. (see Appendix A.2 for the detailed algorithm of construction of Meta-Clusters.)

Let \(G_r\) be a Meta-Cluster (whose binary rooted tree is) rooted at cluster \(r\). Let \(\delta_r\) (0 or 1) be the number of dense clusters and \(\sigma_r (\leq \ell/2)\) be the number of sparse clusters in \(G_r\) and \(\mathcal{H}(G_r)\) be the subgraph of \(T\) induced by the nodes in \(G_r\). We will open \(\alpha_r = \max \{0, \sigma_r - 1\}\) facilities in the sparse clusters of \(G_r\) and \([d_{j'}/u]\) facilities in a dense cluster centered at \(j'\), if any. For \(j' \in C_S\), we would like to open a facility in \(N_{j'}\) only if it is no farther than \(\psi(j')\) from \(j'\). Thus, we define \(\tau(j') = \{i \in N_{j'} : c(i, j') \leq c(j', \psi(j'))\}\) if \(j' \in C_S\) and \(\tau(j') = N_{j'}\) if \(j' \in C_D\). Let \(w_i\) denote whether facility \(i\) is opened in the solution or not. Thus we arrive at the following auxiliary LP:

\[
\text{CostALP}(w) = \sum_{j' \in C_S} d_{j'} \left( \sum_{i \in N_{j'}} c(i, j') w_i + c(j', \psi(j')) \left( \frac{t_{ij}}{r} - \min \left\{ \frac{t_{ij}}{r}, \sum_{i \in N_{j'}} \hat{x}_{ij'} \right\} \right) \right) + u \sum_{j' \in C_D} \sum_{i \in N_{j'}} c(i, j') w_i + \sum_{i \in F} f_i w_i
\]

s.t.

\[
\begin{align*}
\sum_{i \in \tau(j')} w_i & \leq 1 & \forall j' \in C_S & (15) \\
\sum_{i \in \tau(j')} w_i & \geq \lfloor d_{j'}/u \rfloor & \forall j' \in C_D & (16) \\
\sum_{j' \in G_r \cap C_S} \sum_{i \in \tau(j')} w_i & \geq \alpha_r & \forall r : G_r \text{ is a MC} & (17) \\
\sum_{i \in F} w_i & \leq k & (18) \\
0 & \leq w_i \leq 1 & \forall i \in F & (19)
\end{align*}
\]

Lemma 12. A feasible solution \(w'\) to above LP can be obtained such that \(\text{CostALP}(w') \leq (2\ell + 14) \cdot \text{CostCkFLPP}(\hat{\rho})\).

Proof. For all \(i \in F\), let \(l_i = \sum_{j \in C_{r'}} \hat{x}_{ij'}.\) For \(j' \in C_D\), \(i \in \tau(j')\), set \(w'_i = \frac{1}{d_{j'}/u} \hat{x}_{ij'}\). For \(j' \in C_S\), set \(w'_i = \hat{x}_{ij'}\) for \(i \in \tau(j')\) and \(w'_i = 0\) for \(i \in N_{j'} \setminus \tau(j')\). It is easy to see that the solution is feasible (see Appendix A.3 for detailed proof) It is easy to bound the service cost for dense clusters and the facility opening cost. The first term of the cost function that corresponds to the sparse clusters is bounded by \(\leq 12 \cdot \text{CostCkFLPP}(\hat{\rho})\) by claim (11). (see Appendix A.3 for detailed proof)

A pseudo-integral solution \(\tilde{w}\) of the ALP of bounded cost (\(\text{CostALP}(\tilde{w})\) bounded by \((2\ell + 14) \cdot \text{CostCkFLPP}(\hat{\rho})\)) is obtained by using the iterative rounding algorithm [10, 11]: compute an extreme point solution \(w^{(o)}\) for the original ALP. LP for the next iteration is obtained as follows: remove the integral variables from the LHS and subtract the sum of
values, in \( w^{(6)} \), of the corresponding variables from the RHS of each constraint. If RHS of any constraint becomes zero, remove the constraint. The process is repeated until either all the variables are integral or all of them are fractional. If all the variables are integral we are done, otherwise we use the properties of extreme point solution to claim that the number of non-zero (and hence fractional) variables are at most two.

We open the facility with larger fractional opening and close the other fractionally opened facility to obtain an integrally open solution. Lemma [13] bounds the cost of the new solution so obtained. Lemmas [14] and [15] bound the violation in capacities and the assignment costs within the Meta-Clusters. And, Lemma [16] bounds the violation in capacities and the assignment costs across the Meta-Clusters.

**Lemma 13.** Cost of converting pseudo-integral solution \( \bar{w} \) into an integral solution \( \tilde{w} \) is bounded by 3 \cdot \text{CostALP}(\bar{w})

**Proof.** Let \( i_1, i_2 \) be the two fractionally opened facilities in \( \bar{w} \). WLOG let us assume \( \bar{w}_{i_1} \geq \bar{w}_{i_2} \). Since \( \bar{w}_{i_1} + \bar{w}_{i_2} = 1 \) we must have, \( \bar{w}_{i_1} \geq 1/2 \). Also by our choice of rounding \( \tilde{w}_{i_1} = 1 \) and \( \tilde{w}_{i_2} = 0 \).

1. **When both** \( i_1 \) and \( i_2 \) **belong to same sparse cluster** \( j' \):

   Adding the following three inequalities we get the desired claim:
   
   \[
   \begin{align*}
   &d_{j'}[c(i_1, j')\bar{w}_{i_1} + c(i_2, j')\bar{w}_{i_2}] \leq 2 \cdot d_{j'}[c(i_1, j')\tilde{w}_{i_1} + c(i_2, j')\tilde{w}_{i_2}] \\
   &\left(\frac{\ell - 1}{\ell} - \min\left\{\frac{\ell - 1}{\ell}, \tilde{w}_{i_1}\right\}\right) = 0 \leq \left(\frac{\ell - 1}{\ell} - \min\left\{\frac{\ell - 1}{\ell}, \tilde{w}_{i_1}\right\}\right)
   \end{align*}
   \]

2. **When both** \( i_1 \) and \( i_2 \) **belong to same dense cluster** \( j' \):

   \[
   \begin{align*}
   &\bar{u}[c(i_1, j')\bar{w}_{i_1} + c(i_2, j')\bar{w}_{i_2}] \leq 2u[c(i_1, j')\tilde{w}_{i_1} + c(i_2, j')\tilde{w}_{i_2}].
   \end{align*}
   \]

3. **When** \( i_1, i_2 \) **belong to two sparse clusters** \( j', j'' \) **respectively of a group** \( G_r \):

   \[
   \begin{align*}
   &d_{j'}[c(i_1, j')\bar{w}_{i_1} + d_{j''}[c(j'', \psi(j''))(\frac{\ell - 1}{\ell} - \bar{w}_{i_2})] \\
   &\leq 2 \cdot d_{j'}[c(i_1, j')\tilde{w}_{i_1} + d_{j''}[c(j'', \psi(j''))(\frac{\ell - 1}{\ell} - \tilde{w}_{i_2})] \\
   &\leq 2 \cdot d_{j'}[c(i_1, j')\tilde{w}_{i_1} + c(j', \psi(j')(\frac{\ell - 1}{\ell} - \min\left\{\frac{\ell - 1}{\ell}, \tilde{w}_{i_1}\right\})] + 3 \cdot d_{j''}[c(j'', \psi(j''))(\frac{\ell - 1}{\ell} - \min\left\{\frac{\ell - 1}{\ell}, \tilde{w}_{i_1}\right\}) \\
   &\leq d_{j'}[c(i_1, j')\tilde{w}_{i_1} + c(j', \psi(j')(\frac{\ell - 1}{\ell} - \min\left\{\frac{\ell - 1}{\ell}, \tilde{w}_{i_1}\right\})] + 3 \cdot d_{j''}[c(j'', \psi(j''))(\frac{\ell - 1}{\ell} - \min\left\{\frac{\ell - 1}{\ell}, \tilde{w}_{i_1}\right\})] \\
   &\left(\frac{\ell - 1}{\ell} - \min\left\{\frac{\ell - 1}{\ell}, \tilde{w}_{i_1}\right\}\right).
   \end{align*}
   \]

**Lemma 14.** Consider a Meta-Cluster \( G_r \). Suppose the capacities are scaled up by a factor of 2 for facilities in the sparse clusters and by a factor of 3 for facilities in the dense clusters. Then, (i) Let \( \beta_r = \left[\frac{d_{\text{root}}}{u}\right] + \max\{0, \sigma - 1\} \), where \( j_{\text{root}} \) is the center of the dense root cluster (if any) in \( G_r \). Then, at least \( \beta_r \) facilities are opened in \( G_r \). (ii) The dense cluster in \( G_r \) (if any) is self-sufficient, i.e., its demand can be completely assigned within the cluster itself at a loss of at most factor 2 in cost. (iii) There is at most one cluster with no facility opened in it and it is a sparse cluster. (iv) Any (cluster) center responsible for the unserved demand of \( j' \in G_r \) is an ancestor of \( j' \) in \( H(G_r) \). (v) At most \( u \) units of demand in \( G_r \) remain un-assigned and it must be in the root cluster of \( G_r \). Such a Meta-Cluster cannot be a root Meta-Cluster.

**Proof.** Refer Appendix A.4
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Lemma 15. Total distance traveled by demand $d_{j'}$ of $j' \neq r \in G_r$ to reach the center of the cluster in which it is served is bounded by $(\frac{1}{\ell-1})d_{j'}c(j', \psi(j')) \sum_{i \in F} \hat{x}_{ij'}$

Proof. Demand $d_{j'}$ of such a $j'$ is assigned to $\psi(j')$ and $d_{j'}c(j', \psi(j')) = (\frac{1}{\ell-1})(\frac{1}{\ell})d_{j'}c(j', \psi(j')) \leq (\frac{1}{\ell-1})d_{j'}c(j', \psi(j')) \sum_{i \in F} \hat{x}_{ij'}$ where the last inequality follows because $\sum_{i \in F} \hat{x}_{ij'} \geq (1 - \frac{1}{\ell})$

Lemma 16. [10, 11] Consider a Meta-Cluster $G_r$. Suppose the capacities are scaled up by a factor of $2 + \frac{8}{\ell-2}$, $\ell \geq 4$ for facilities in the sparse clusters and by a factor of 3 for facilities in the dense clusters. Then, (i) The demand of $G_r$ and the demand coming onto $G_r$ from the children Meta-Clusters can be assigned to the facilities opened in $G_r$. (ii) Total distance traveled by demand $d_{j'}$ of $j' \in G_r$ to reach the centers of the clusters in which it is served is bounded by $O(\ell)d_{j'}c(j', \psi(j')) \sum_{i \in F} \hat{x}_{ij'}$.

Proof. Refer Appendix [A.5]

Lemma 17. The cost of assigning the demands collected at the centers to the facilities opened in their respective clusters is bounded by $O(\ell)\text{CostCkFLPP}(\hat{\rho})$.

Proof. Refer Appendix [A.6]

Once we have selected the facilities to be opened integrally, in step 3, we solve the transportation problem with penalties for uniform supplies $\hat{u} = [3u]$ and unit demand to obtain integral assignments. Choosing $\ell \geq 4$, such that $2 + \frac{8}{\ell-2} = 3$, $\Rightarrow \ell = 10$. Hence, by choosing $\ell \geq 10$ and $O(1/\epsilon)$ we get the desired result. Using the ideas of [10, 11], the violations in capacities can be reduced to $(2 + \epsilon)$-factor.

5 Conclusion

In this paper, we presented a framework for designing approximation algorithms for capacitated facility location problems with outliers/penalty and applied it to obtain first constant factor approximations for some very fundamental problems like CFLPO and CkMPP. Our solutions incur a slight violations in capacities, $(1 + \epsilon)$ for the problems without cardinality constraint and $(2 + \epsilon)$ for the problems with the cardinality constraint. For the outlier variant, we also incur a small loss $(1 + \epsilon)$ in outliers. Due to the hardness of the underlying problems, the violations are inevitable. Thus we achieve the best possible by rounding the solution of natural LP for these problems. The results of CkFLPP should be extendable to another closely related problem, Capacitated Knapsack Median with Penalties. Using the approach of Aardal et al. [1] in step 2, we will be able to get rid of capacity violation for CkFLPP when the facility opening costs are uniform maintaining the cardinality violation at 2.

As a by-product of our results, we also provide first approximation result for the uncapacitated $k$ facility location problem with penalties. We also provide first approximation results for $k$MP and FLPO using LP-rounding.

It would be interesting to obtain similar results for the outlier variants of Capacitated $k$-Facility Location and/or Capacitated $k$-Median. Three hard bounds viz. capacities, cardinality and the outliers, make the problems very challenging. Due to the hardness of CkM, we must violate at least one of cardinality and capacities by a factor of at least 2 and by the hardness of CkMO, we must violate at least one of cardinality and outlier constraint. The challenge is to keep the violations low (some small constant). In particular, it would be interesting to see a result for CkMO violating only cardinality and/or a result for CkMO violating any two constraints up to small constants or violating all three by $(1 + \epsilon)$. 
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A CkFLPP

A.1 Routing Tree

Refer figure 2.

A.2 Construction of Meta-Clusters

Refer Algorithm 1. At any point of time, the set $\mathcal{N}$ contains the set of nodes in $\mathcal{T}$, not yet grouped; in line number 4, we either start with the root of $\mathcal{T}$ or a topmost node in $\mathcal{T}$ which has not yet been grouped.

A.3 Proof of Lemma 12

**Proof.** For all $i \in \mathcal{F}$, let $l_i = \sum_{j \in \mathcal{C}} \hat{x}_{ij}$. We define a feasible solution to the $LP_{ALP}$ as follows: let $j' \in \mathcal{C}_D$, $i \in \tau(j')$, set $w_i^j = \frac{1}{u} [d_{j'}/u] = \frac{1}{u} \lfloor d_{j'}/u \rfloor \leq l_i \leq \hat{y}_i$. For $j' \in \mathcal{C}_S$, we set $w_i^j = \hat{x}_{ij'}$ for $i \in \tau(j')$ and $w_i^j = 0$ for $i \in \mathcal{N}_{j'} \setminus \tau(j')$. We will next show that the solution is feasible.
Next, let $x \in N$. Then for a Meta-Cluster $G_k$, we have $\sum_{j \in C} w_j = \sum_{j \in N'} d_{ij}^r$, where $\sum_{j \in N'} d_{ij} = \sum_{j \in C} d_{ij}$. Therefore, $\sum_{j \in C} w_j = \sum_{j \in N'} d_{ij}$. This completes the proof.

**Algorithm 1** Meta-cluster Formation

1. **Meta-cluster**($T$)
   1. $N \leftarrow$ set of nodes in $T$.
   2. while there are non-grouped nodes in $N$ do
   3. Pick a topmost non-grouped node, say $k$ of $N$: form a new MC, $G_k$.
   4. when $G_k$ has fewer than $\ell/2$ nodes do
   5. if $N = \emptyset$ then break and stop.
   6. Let $j = \arg\min_{u \in N} \{ c(u, v) : (u, v) \in T, v \in G_k \}$, set $G_k = G_k \cup \{ j \}$. $N \leftarrow N \setminus \{ j \}$.
   7. end while
   8. end while

1. For $j' \in C$, $\sum_{i \in \partial(j')} w_i = \sum_{i \in N}_{j'} x_{ij} \leq 1$.
2. Next, let $j' \in D$, then $\sum_{i \in \partial(j')} w_i = \sum_{i \in N}_{j'} \frac{\ell}{w_{ij}} = \sum_{i \in N}_{j'} l_i = d_{j'}$.
3. For a Meta-Cluster $G_r$, we have $\sum_{j' \in \partial(j')} w_i = \sum_{j' \in C_{\partial}} \sum_{i \in \partial(j')} x_{ij} \geq \sum_{j' \in C_{\partial}} \sum_{i \in \partial(j')} \bar{x}_{ij} \geq \sum_{i \in C_{\partial}} \sum_{j \in C_{\partial}} \bar{x}_{ij} c(i, j) + 2(\ell + 1) \cdot CostCkFLPP(\tilde{\beta})$. Summing over all $j' \in C_D$ we get $\sum_{j' \in C_D} \sum_{j \in C_{\partial}} \bar{x}_{ij} c(i, j) + 2(\ell + 1) \cdot CostCkFLPP(\tilde{\beta}) \leq 12 \cdot CostCkFLPP(\tilde{\beta})$ by Claim (11).
4. Now consider the part of objective function for $C_S$. For $j' \in C_S$, we have $\sum_{i \in \partial(j')} c(i, j') w'_i = \sum_{i \in N}_{j'} c(i, j') \left( \frac{\ell}{w_{ij}} - \min \left\{ \frac{\ell}{w_{ij}}, \frac{\ell}{w_{ij}} \right\} \right) = \sum_{i \in N}_{j'} \sum_{j \in C_{\partial}} \bar{x}_{ij} c(i, j) + 2(\ell + 1) \cdot CostCkFLPP(\tilde{\beta})$. Thus, the solution $w'$ is feasible and $CostCkFLPP(w')$.

Theorem 1: For $\ell = 2, \ldots, M$, the algorithm computes a solution $w$ with $\sum_{i \in N} w_i = 1$, $\sum_{i \in C} w_i = \sum_{j \in C} d_{ij}$, and

$$\sum_{i \in E} f_i w_i \leq \frac{9}{2} \cdot \sum_{j' \in C_D} d_{j'} \left( \sum_{i \in \partial(j')} \bar{x}_{ij} c(i, j') \left( \frac{\ell}{w_{ij}} - \min \left\{ \frac{\ell}{w_{ij}}, \frac{\ell}{w_{ij}} \right\} \right) \right) + u \sum_{j' \in C_D} \sum_{i \in N}_{j'} c(i, j') w_i + \sum_{i \in F} f_i w_i \leq (2\ell + 14) \cdot CostCkFLPP(\tilde{\beta}).$$
A.4 Proof of Lemma 14

(i) follows from the constraints 16 and 17 of the ALP. (ii) follows as there is at least \( \lfloor d_j/u \rfloor \) opening in each dense cluster and hence \( d_j \) demand can be served from them at a loss of factor 2 in capacity and service cost. (iii) follows from constraint 17 and the definition of \( \alpha_r \). (iv, v) hold because for \( j' \neq r \), the unmet demand \( d_j \) is assigned to \( \psi(j') \in G_r \) at a loss of factor 2 in capacities and for \( j' = r \), if \( G_r \) is a root Meta-Cluster \( d_j \) is assigned to \( \psi(j') \in G_r \) at a loss of factor 2 in capacities else \( d_j \) is not assigned within \( G_r \). It is rather sent to the parent Meta-Cluster of \( G_r \).

A.5 Proof of Lemma 16

(i) \[ \frac{\text{Total demand to be served in } G_r}{\text{Total opening in } G_r} \leq \frac{\left( \left\lfloor \frac{d_j}{u} \right\rfloor + 1 + \sigma_r \right) u + \left( \frac{4}{\ell} + 1 \right) u}{\beta_r u} \leq \frac{(\beta_r + 2) u + (\frac{4}{\ell} + 1) u}{\beta_r u} \leq \frac{2(\beta_r + 2) u}{\beta_r u} = 2 + \frac{4}{\beta_r} = 2 + \frac{8}{(\ell/2 - 1)} \] where the last equality follows as \( \beta_r \geq (\delta_r + \sigma_r - 1) = (\ell/2 - 1) \) for a non-leaf Meta-Cluster

(ii) Let \( j' = r \) and \( d_j \) remains unassigned within \( G_r \) in Lemma 14 then \( d_j \) is distributed in the nodes of the parent Meta-Cluster utilising their remaining capacities. The claim follows as the edges in a parent Meta-Cluster are cheaper than \( c(j', \psi(j')) \) and there are \( O(\ell) \) edges in the parent Meta-Cluster.

A.6 Proof of Lemma 17

The main observation here is that the cost of assigning \( \delta \) extent of \( d_j \) to a facility \( i \in \tau(k') \) from the center \( k' \in C_S \) is bounded by \( \delta d_j c(j', \psi(j')) \). This is because \( i \in \tau(k') \implies c(i, k') \leq c(k', \psi(k')) \) which is further \( \leq c(j', \psi(j')) \).