A mean field theory for the spin ladder system

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Abstract

In the present paper, we propose a mean field approach for spin ladders based upon the Jordan-Wigner transformation along an elaborately ordered path. We show on the mean field level that ladders with even number legs open a energy gap in their low energy excitation with a magnitude close to the corresponding experimental values, whereas the low energy excitation of the odd-number-leg ladders are gapless. It supports the validity of our approach. We then calculate the gap size and the excitation spectra of 2-leg-ladder system. Our result is in good agreement with both the experimental data and the numerical results.

Key words: spin ladder, Jordan Wigner transformation

Category: Ca2

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I. INTRODUCTION

The study on low dimensional Heisenberg anti-ferromagnetic model is one of the most active research field in condensed matter physics. Haldane [2] conjectured that for integer-spin one dimensional anti-Ferromagnetic chain a energy gap exists in the low energy excitation spectrum, but for half integer-spin case the excitation spectrum is gapless. The spin 1/2 anti-Ferromagnetic chain can be solved exactly by Bathe ansatz [3]. The excitation spectrum is found to be gapless. The measurement on the realistic ladder material such as $SrCu_2O_3$ (two-leg), $Sr_2Cu_3O_5$ (three-leg), or $(VO)_2P_2O_7$ (two-leg) [10] shows that the spin excitation gap is opened in the ladders with even numbers of legs, while for ladders with odd numbers of legs, no gap is found in spin excitation. This conclusion is predicted by the early numerical calculation [14] and was explained qualitatively by Khveshchenko [8]. In Khveshchenko’s explanation a topological term appears in the effective Hamiltonian of the long wavelength dynamics in odd-leg ladder and is absent in even-leg ladder. Recently, G.Sierra [1] has mapped the ladder problem onto a effective one dimensional non-linear Sigma model. An extra topological term appears in odd-leg system and is absent in even-ladder system. This difference between the odd and even-leg ladders is essentially an extension for the difference between the half integer and integer Heisenberg chains.

We also have the experience from the 1-D quantum Heisenberg spin chain, that it is not so trivial to incorporate the subtle physics of the topological term in a mean field approach. For two-leg ladder, the existence of energy gap in spin excitation for nonzero inter chain coupling $J'$ is confirmed by various method such as Lanczos, quantum Monte Carlo [14,15], renormalization group [5], variational method [3], strong coupling expansion [7], spin liquid mean field approach [3], as well as the mean field approach based on the Jordan-Wigner transformation [16]. And for $J=J'$ ($J$ is the exchange coupling along the chains) which is the physical parameter of the real ladder compounds, the energy gap obtained by numerical calculation is $0.5J$ which is very close to the experimental value. The spin wave excitation spectra has also been obtained by the numerical calculation which shows the
minimum of the spectra is located at the wave number $\pi$ [14,15]. This was also confirmed by the recent neutron scattering experiment [11].

A mean field treatment of the two-leg Heisenberg ladder with the application of Jordan-Wigner (J-W) transformation has been proposed by M.Azzouz and et al [16]. In their approach the Jordan-Wigner transformation is introduced to map the spin 1/2 system to the spinless Fermion system. The gap is obtained in their mean field approach which is about 0.7J under the case of J=J’ which does not fit well with the experimental value. The excitation spectra is also calculated in their approach, which contents a minimum at the wave number $\pi$. But the shape of the spectra is not consistent with the numerical result which predicts the maximum of the spectra locating between the 0 and $\pi/2$ [15].

In the present paper, we propose a mean field approach also based on the J-W transformation [17]. Our mean field approach is quite different from the approach used in paper [16]. When performing J-W transformation in the ladder system, one must put the sites in a queue. Then the spin operator can be expressed as $S_i^+ = c_i^{+} e^{i\pi\Phi(i)}$, where $\Phi(i)$ is nothing but the summation of number of the sites from $-\infty$ to the (i-1)th site in the particularly ordered queue which are occupied by spinless Fermions. The difference of particularly elaborated queues used in our approach and that in paper [16] are shown in Fig.1.(a) and (b). In our approach the sites in the odd number rungs are labeled from upward to downward but from downward to upward in the even number rungs (shown as Fig.1(a)), whereas in paper [16] all rungs are labeled from upward to downward (shown as Fig.2(b)).

We perform the J-W transformation with such a specially ordered queue. The advantages of our approach are as the following. First of all if we replace the phase of the hopping term by its average value, we can easily obtain the mean field Hamiltonian for n-leg ladders in which the effective hopping terms of the spinless Fermions have alternative signs along both the chain and rung directions. This mean field Hamiltonian is similar with the one used in paper [16] for 2-leg case. But in their approach a further assumption, i.e. adding a $\pi$-flux to each plaquette is needed to obtain such kind of mean field Hamiltonian, whereas in our approach the mean field Hamiltonian is obtained directly by replacing the phase factor in
the hopping terms with their average values. We can then easily show that the spin gap is only opened in even-leg ladders. In certain degree, it supports the validity of our way of Jordan-Wigner transformation construction.

Secondly we can treat the phase factor in the hopping terms more carefully by introduce a self consistent procedure for 2-leg ladder. We find that both the gap magnitude and the spin excitation spectra are in good agreement with the numerical and experimental results in the case of \( J' = J \) which is of the physical parameter for the real ladder compounds. Actually the energy gap is found to be \( 0.46J \) in the case \( J' = J \) which is very close to the experimental result \( 0.47 \pm 0.2J \). The spin wave excitation spectra obtained by our mean field calculation is also consistent with the numerical result with the minimum at the wave number \( \pi \) and the maximum at the wave number \( 0.356\pi \). Compared with the approach used in [16], our mean field approach is much better in the case of \( J' > 0.5J \). In the weak coupling regime our approach works not so good, the energy gap is even persisted in the case of \( J' = 0 \). In spite of the unsatisfactory aspect in the weak coupling regime, our approach is still valuable because it works very well in the intermediate and strong coupling regime, which has the experimental correspondence.

In section II, we calculate the spin excitation gap of the spin ladders with various numbers of legs and show that in the mean field level the energy gap only exists in ladder system with even number legs. In section III, we propose a more carefully treatment of the 2-legs spin ladder. The spin excitation gap, as well as the excitation spectra is calculated. Finally, we make the conclusion remarks in Sec.IV.
II. Mean field treatment for the spin gap in \( n \)-leg ladders

We begin with the \( 2M \)-leg anti-ferromagnetic Heisenberg ladder Hamiltonian:

\[
H = J' \sum_{i}^{2M-1} \sum_{p=1}^{2M} \vec{S}_{i,p} \cdot \vec{S}_{i,p+1} + J \sum_{i}^{2M} \sum_{p=1}^{2M} \vec{S}_{i,p} \cdot \vec{S}_{i+1,2M+1-p}
\]

\[
= J' \sum_{i}^{2M-1} \sum_{p=1}^{2M} S_{i,p}^z \cdot S_{i,p+1}^z + J \sum_{i}^{2M} \sum_{p=1}^{2M} S_{i,p}^z \cdot S_{i+1,2M+1-p}^z +
\]

\[
\frac{J'}{2} \sum_{i}^{2M-1} \sum_{p=1}^{2M} S_{i,p}^+ \cdot S_{i,p+1}^- + \frac{J}{2} \sum_{i}^{2M} \sum_{p=1}^{2M} S_{i,p}^+ \cdot S_{i+1,2M+1-p}^- + H.C.
\]

(1)

In the above Hamiltonian, ‘\( i \)’ represents the site position along the chains and ‘\( p \)’ represents the \( 2M \) sites of different chains coupled by the inter-chain coupling constant \( J' \). As shown in Fig.1, \( p \) is labeled from upward to downward at the even sites while downward to upward at the odd sites. This is different from what used in the paper of Azzouz et.al. [16]. Then we introduce the generalized J-W transformation:

\[
S_{p,i}^+ = c_{p,i}^+ e^{i\pi \sum_{n=-\infty}^{i-1} \sum_{l=1}^{2M} c_{i,n}^+ c_{l,n} + i\pi \sum_{l=1}^{p+1} c_{i,l}^+ c_{l,i}}
\]

(2)

in which \( c \) is the spinless Fermion operator. The summation in the phase factor is nothing but the number of the occupied sites before the \( i \)th site along the particular queue shown in Fig.1(a). Then the quantum spin 1/2 Hamiltonian can be mapped onto a spinless Fermion Hamiltonian as:

\[
H = J' \sum_{i,p=1}^{2M-1} \left( \frac{1}{2} - c_{i,p}^+ c_{i,p} \right) \cdot \left( \frac{1}{2} - c_{i,p+1}^+ c_{i,p+1} \right) + J \sum_{i,p=1}^{2M} \left( \frac{1}{2} - c_{i,p}^+ c_{i,p} \right) \cdot \left( \frac{1}{2} - c_{i+1,2M+1-p}^+ c_{i+1,2M+1-p} \right) +
\]

\[
\frac{J'}{2} \sum_{i,p=1}^{2M+1} \left( c_{i,p}^+ c_{i,p+1} + H.C. \right) + \frac{J}{2} \sum_{i,p}^{2M} \left( c_{i,p}^+ c_{i+1,2M+1-p} e^{-i\hat{\Phi}(p)} + H.C. \right)
\]

(3)

where

\[
\hat{\Phi}(p) = \pi \sum_{l=p+1}^{2m} \left( n_{i,l} + n_{i+1,2M+1-l} \right)
\]

In our mean field approach, we replace \( n_{i,l} \) by \( < n_{i,l} > \). For the present study, we further assume that the finite magnetization in this system is not possible because of the strong
quantum fluctuation. This is reasonable for systems with its leg number much less then the site number of each chain. Then we have \( \langle S_{i,l}^z \rangle = < 1/2 - c_{i,l}^+c_{i,l} > = 0 \) which implies \( < n_{i,l} > = 0.5 \). Consequently, the phase factor in eq.(3) can be approximated by: \( \Phi(p) = \pi(2M - p) \). Moreover we decouple the four fermion interaction term in the above Hamiltonian by the Hatree-Fock approximation. Finally the mean field Hamiltonian of the spinless Fermions has the expression:

\[
H = \sum_{i,p} \left( c_{i,p}^+ c_{i+1,p} + H.C. \right) + \sum_{i,p} \left( c_{i,p}^+ c_{i+1,2M+1-p}(-1)^{p+1} + H.C. \right)
\]

If we introduce a Fourier transformation for the site indices, we have:

\[
H = a \sum_{k,p} \left( c_{k,p}^+ c_{k+1,p} + H.C. \right) + \frac{J}{2} \sum_{k,p} (i\gamma_k (-1)^{p} c_{k,p}^+ c_{k+1,2M+1-p} + H.C.)
\]

where \( a = \frac{J'}{2} \gamma_k = -J \sin(k) \). The Hamiltonian then can be written in a form \( H = \sum_k C_k^+ h(k) C_k \) with \( h(k) \):

\[
h(k) = \begin{pmatrix}
0 & a & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & i\gamma \\
 a & 0 & a & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & -i\gamma \\
0 & a & 0 & a & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & i\gamma \\
\ldots & \ldots & a & 0 & a & \ldots & \ldots & \ldots & \ldots & \ldots & i\gamma \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & -i\gamma & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & a \\
\ldots & \ldots & \ldots & \ldots & \ldots & a & 0 & a & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & -i\gamma & \ldots & \ldots & a & 0 & a & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & i\gamma & \ldots & \ldots & a & 0 & a & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & a & 0 & a & \ldots & \ldots & \ldots & a \\
\end{pmatrix}
\]

The above matrix contains \( 2M \) eigenvalues for a given wave number \( k \) corresponding to the \( 2M \) individual bands separated by gaps. It can be proved straightforwardly that for a
given wave number $k$, half of the eigenvalues are less than zero and other half of them are
great than zero. Furthermore we can also prove that zero is not an eigenvalue of the above
matrix for any non zero $J'$. We will prove the two statements in the Appendix. This result
shows clearly that half of the energy bands of the spinless Fermions is below zero energy
and another half is above zero. The energy gap between them is nonzero because zero is not
the eigenvalue of the above matrix for nonzero $J'$ and arbitrary $k$. Assuming there are no
self-magnetization in one dimensional system, only the lower half of the states is occupied
by spinless Fermions in ground state. Therefore such spinless Fermion system is very similar
with the traditional insulator in which the valence band is fully occupied and the conductive
band is fully empty in ground state. So a nonzero minimum energy is needed to excite
the system from the ground state which indicates a spin excitation energy gap. For spin
ladders with odd number legs there exist odd numbers of energy band. Since only half of the
states is occupied in the ground state there must exist at least one band which is partially
occupied. This picture is very similar with the traditional conductor in which exists at least
one partial occupied band (conduction band). Then the low energy excitations are gapless
for odd-number-leg ladders.

Moreover, we calculate the gap size of the 2,4,6,8, and 10-leg spin ladder, the results are
shown in Fig.2 together with the experimental value. Although our approach is quite rough,
the result is in good agreement with the experimental value $[10,11,12,13]$.

The spin excitation spectra can also be obtained from the above mean field approach, the
spin wave dispersion is $\sqrt{(J')^2 + J^2 \sin k^2}$ for the two-leg case. It has two energy minimum,
one is at 0 and another at $\pi$. The shape of the spectra is not consistent with the numerical
result which has the spectra minimum located only at wave number $\pi$ and the maximum is
near 0.356$\pi$. This is due to that the treatment for the phase term of eq.(3) is too rough.
In the next section we will propose a more careful treatment of the phase factor in eq.(3).
We can then obtain a much more improved spin excitation spectra, which is very close to
the numerical results.
II. The mean field theory of 2-leg ladder

For the two-leg case, we can introduce two bipartite lattice labeled as $\alpha$ and $\beta$. Following the same procedure shown in the above section, the Hamiltonian for spinless Fermions becomes:

$$H = J' \sum_i \left( \frac{1}{2} - \alpha_i^+ \alpha_i \right) \left( \frac{1}{2} - \beta_i^+ \beta_i \right) + J \sum_i \left( \frac{1}{2} - \alpha_i^+ \alpha_i \right) \left( \frac{1}{2} - \beta_{i+1}^+ \beta_{i+1} \right) + J \sum_i \left( \frac{1}{2} - \alpha_{i+1}^+ \alpha_{i+1} \right) \left( \frac{1}{2} - \beta_i^+ \beta_i \right) + J' \sum_i \left( \alpha_i^+ \beta_i + h.c. \right) + J \sum_i \left( \alpha_i^+ \beta_{i+1} e^{i\pi(\beta_i^+ \beta_{i+1} + \alpha_i^+ \alpha_{i+1})} + h.c. \right) + \frac{J}{2} \sum_i \left( \beta_i^+ \alpha_{i+1} + h.c. \right)$$

In our mean field approach, different with the simple treatment used in the previous section, we first replace the phase factor in (6) by it’s average value:

$$\langle e^{i\pi(\beta_i^+ \beta_{i+1} + \alpha_i^+ \alpha_{i+1})} \rangle = \langle (1 - 2\beta_i^+ \beta_i)(1 - 2\alpha_{i+1}^+ \alpha_{i+1}) \rangle = -4\chi_1 \chi_2$$

where we define:

$$\chi_1 = \langle \beta_i^+ \alpha_{i+1} \rangle \quad \chi_2 = \langle \alpha_{i+1}^+ \beta_i \rangle \quad \chi_0 = \langle \beta_i^+ \alpha_i \rangle$$

Then the Fermion-Fermion interacting term $\left( \frac{1}{2} - \alpha_i^+ \alpha_i \right) \left( \frac{1}{2} - \beta_{i+1}^+ \beta_{i+1} \right)$ can be factorized as:

$$\left( \frac{1}{2} - \alpha_i^+ \alpha_i \right) \left( \frac{1}{2} - \beta_{i+1}^+ \beta_{i+1} \right) = \frac{1}{4} - \chi_0 \beta_i^+ \alpha_i - \chi_0^+ \alpha_i^+ \beta_i + \chi_0 \chi_0^+$$

We decouple the other two interacting term in the same manner and obtain the following mean field Hamiltonian of spinless Fermions.

$$H_{MF} = \sum_k \gamma_k \alpha_k^+ \beta_k + h.c. \quad \text{(7)}$$

where:

$$\gamma_k = \left( \frac{J'}{2} - J' \chi_0 \right) + \left( \frac{J}{2} - 2J |\chi_2|^2 - J \chi_1 - J \chi_2 \cos(k) \right) +$$
\[ i \sin(k) \left( J\chi_2 - J\chi_1 - 2J|\chi_2|^2 - \frac{J}{2} \right) \]

Then the above Hamiltonian can be diagonalized as:

\[ H_{MF} = \sum_k E_k \left( \tilde{\alpha}_k^+ \tilde{\alpha}_k - \tilde{\beta}_k^+ \tilde{\beta}_k \right) \]  

(8)

in which:

\[ E_k = \sqrt{\left( \frac{J'}{2} - J'\chi_0 \right) + \left( \frac{J}{2} - 2J|\chi_2|^2 - J\chi_1 - J\chi_2 \cos(k) \right)^2 + \sin^2(k) \left( J\chi_2 - J\chi_1 - 2J|\chi_2|^2 - \frac{J}{2} \right)^2} \]

The three parameters \( \chi_1, \chi_2 \) and \( \chi_0 \) is determined selfconsistantly. The gap size obtained within the present approach is shown in Fig.3 compared with the numerical results. Our results fits quite well to the numerical results in the parameter regime \( J'/J > 0.5 \). For the case of \( J' = J \), the three parameters are found to be \( \chi_1 = -0.188J, \chi_2 = 0.237J \) and \( \chi_0 = 0.3867J \), and the gap is found to be 0.46J which is very close to the experimental value \( (0.47 \pm 0.2)J \) [11]. In strong coupling limit \( (J' >> J) \) our result fit well with the result obtained by strong coupling expansion [7], which shows \( \Delta/J' \rightarrow 1 \) when \( J'/J \rightarrow \infty \). But in the regime \( J'/J < 0.5 \) our results deviated from the numerical results, and a nonzero gap persists even at the case \( J'=0 \). So our mean field approach is valid only in the intermediate and strong coupling regime. In the weak coupling regime our mean field picture breaks due to the over estimation of the inter-chain interaction. Since the phase factor in the hopping term in equation(6) is replaced by its average value, it makes the hopping term within one chain being strongly modified by the motion of spinless Fermions in the other chain, which is not valid for the weak coupling regime. We believe this approach is valid when the inter-chain coupling is in the order of unit, but is not valid for the weak coupling case.

Another advantage of the present mean field approach is that in the case of \( J' = J \) it gives the same spin wave dispersion predicted by the numerical calculation as shown in Fig.4. The minimum of the spectra is at the wave number \( \pi \), and the maximum is at the wave number \( 0.356\pi \). This result is in good agreement with the numerical result which has
a minimum at $\pi$ and maximum at $0.3\pi$. We can also calculate the two-magnon continuum from our mean field theory. The two magnon continuum is proportional to

$$\int dt < S^z(q, t)S^z(-q, 0) > e^{-i\omega t}$$

which can be transformed into the density-density correlation of the spinless Fermions by a Jordan-Wigner transformation. Then the two magnon excitation can be viewed as particle-hole excitation of the spinless Fermions. The spectra of the two magnon excitation with several specific $q$ number is shown in Fig.5. And the bottom(top) of the two magnon continuum is just the minimum(maximum) energy of hole-particle excitation of the spinless Fermions for a given wave number. The result is shown by dashed line(bottom) and dotted line(top) in Fig.4 which fits the numerical result quite well. Compared to the numerical method such as DMRG, quantum Monte Carlo and Lanczos method, our mean field theory based on the Jordan-Wigner Transformation gives a more transparent understanding of the gap formation in even-number-leg spin ladders and the low energy spin excitation.

The spin susceptibility is also obtained by introduce a magnetic field in the original spinless Fermion Hamiltonian, this term acts like the chemical potential:

$$H = \sum_k \gamma_k \alpha_k^+ \beta_k + h.c. - \frac{1}{N} \sum_k (\alpha_k^+ \alpha_k + \beta_k^+ \beta_k)h$$

The magnetization $m$ then has the expression as:

$$m = \frac{1}{N} \sum_i (1 - < \alpha_i^+ \alpha_i > - < \beta_i^+ \beta_i >)$$

And the spin susceptibility could be derived as:

$$\chi_s = \frac{\partial m}{\partial h}$$

We calculate the spin susceptibility in the case of $J' = J$ in a wide range of temperature, the result is shown in Fig.6. Our result explains the temperature behavior of the spin susceptibility quite well, and it is again in good agreement with the numerical results which gives a maximum at $T = 0.8J$. 

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IV. The Concluding Remarks

In this paper we propose a mean field approach for spin ladders based on the Jordan-Wigner transformation along an elaborately chosen path defined in this paper. We show that in the mean field level that spin gap is opened only in the even-number-leg-ladders and vanishes in the odd-number-leg-ladders. It gives a very simple picture of the formation and vanishing of the spin gap in the above mentioned two type of spin ladders. The spin ladders with even number legs formed a 'insulator' like band for spinless Fermions, whereas in odd-number-leg ladders the band structure of the spinless Fermions is 'metal' like.

Then we take a more careful study of the 2-leg ladder. Particular for the $J = J'$ case the magnitude of the gap found in our approach is in good agreement with both the numerical result and the experimental result. Further the spin excitation spectra and the uniform susceptibility are also calculated based on our mean field treatment. The dispersion relation of the spin excitation spectra obtained by our mean field theory is very similar with the numerical result, which has its maximum locating between 0 and $\pi$. The uniform susceptibility is also consistent with the numerical results, which predicts a maximum at $T = 0.8J$. 
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Appendix

In the Appendix we prove that matrix $h(k)$ in section II has the two following properties: (i) If $\lambda$ is a eigenvalue of the matrix, $-\lambda$ is also a eigenvalue of it. (ii) Zero can not be a eigenvalue of the matrix with any nonzero $J'$.

First we divide the Hermite matrix $h(k)$ into its real and imaginary part $h(k) = A + iB$, in which the matrix $A,B$ read as:

\[
A_{ij} = a\delta_{i,j+1} + a\delta_{i,j-1} \quad B_{ij} = (-1)^{i+1}\delta_{i,2M+1-j}\gamma
\]

One can easily find that the matrix $A$ and $B$ satisfies some relations:

(II) \[A^T = A \quad B^T = -B \quad B^{-1} = \gamma^{-2}B \quad B^{-1}AB = -A\]

in which $K_{ij} = \delta_{ij}(-1)^i$ and $K_{il}K_{lj} = \delta_{ij}$

Based on the above equations we can prove the two properties straightforwardly. For (i) if we have $h(k)x = \lambda x$, we can multiply the matrix $K$ to the both side of the above equation: $K \cdot h(k) \cdot Kx = \lambda Kx$, then we have $h(k)(Kx) = -\lambda(Kx)$.

For property (ii), we can prove it as follows. First if $B=0$ the conclusion is obviously true because the determinant of matrix $A$ is nonzero for $J' \neq 0$ and this makes the equation $Ax = 0$ can not be satisfied unless $x=0$. Next when $B \neq 0$ we have:

\[(A+iB)x = 0 \text{ or equivalently } B^{-1}Ax = -ix\]

For matrix $C = B^{-1}A$, we have

\[C^+ = [B^{-1}A]^+ = \gamma^{-2}A^+B^+ = -\gamma^{-2}AB = -(B^{-1})(\gamma^{-2}B)AB = (B^{-1})A = C\]

Then the matrix $C$ is Hermite, and it can not has a imaginary eigenvalue, so the equation $(A+iB) \cdot x = 0$ can not be satisfied for nonzero $x$.

In the above proof we used the relation (II). Then from the above paragraph we prove the two properties used in section II.
FIGURE CAPTIONS

Fig. 1: To perform the Jordan-Wigner transformation the sites are put in a particular queue in our present study(a) and in paper [16](b).

Fig. 2: The spin gap for n-leg spin ladders calculated in our mean field approach which is compared by the experimental value. [11,13].

Fig. 3: The solid line shows the spin gap obtained by the self consistent procedure for 2-leg ladder as a function of inter chain coupling $J'$. The dashed line is the result of strong coupling expansion [7]. The squares shows the numerical result of paper [14]. The inset shows the results with $J'/J$ less than 0.8.

Fig. 4: The solid line is the dispersion of the spinless Fermions for 2-leg ladder calculated in our mean field approach. The dashed(dotted) line is the bottom(top) of the two magnon continuum.

Fig. 5: The spectra of two-magnon excitation with $q = 0.1\pi$ (dotted line), $0.5\pi$(solid line),$\pi$(dashed line).

Fig. 6: The spin susceptibility of 2-leg ladder in the case of $J = J'$. 
Fig. 1.
The number of legs

0.0
0.1
0.2
0.3
0.4
0.5

The spin gap

Experimental value
Calculated in this paper

Fig. 2
Fig. 3
Fig.4
The susceptibility $\chi(T)$

Fig.6