Fair and efficient contribution valuation for vertical federated learning

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January 11, 2022

Abstract

Federated learning is a popular technology for training machine learning models on distributed data sources without sharing data. Vertical federated learning or feature-based federated learning applies to the cases that different data sources share the same sample ID space but differ in feature space. To ensure the data owners’ long-term engagement, it is critical to objectively assess the contribution from each data source and recompense them accordingly. The Shapley value (SV) is a provably fair contribution valuation metric originated from cooperative game theory. However, computing the SV requires extensively retraining the model on each subset of data sources, which causes prohibitively high communication costs in federated learning. We propose a contribution valuation metric called vertical federated Shapley value (VerFedSV) based on SV. We show that VerFedSV not only satisfies many desirable properties for fairness but is also efficient to compute, and can be adapted to both synchronous and asynchronous vertical federated learning algorithms. Both theoretical analysis and extensive experimental results verify the fairness, efficiency, and adaptability of VerFedSV.

1 Introduction

The efficient development of artificial intelligence methods requires collecting enormous volumes of training data in order to create powerful and robust machine learning models. However, in many industrial scenarios, training data is siloed across multiple companies, and data sharing is often not possible because of regulatory limits on data protection [22]. Federated learning (FL) is an emerging machine learning framework where a central server and multiple data owners, i.e., clients, collaboratively train a machine learning model without sharing data [24, 36, 18].

FL can be further be classified into two main categories: horizontal federated learning (HFL) and vertical federated learning (VFL) [36]. HFL refers to the scenario in which various clients’ data sets share the same feature space but have separate sample IDs, and VFL refers to the scenario in which data sets owned by different clients share the identical sample IDs but have distinctive features.

The effectiveness of FL depends on the active participation of motivated clients. Because fair cooperation and reward can motivate clients’ participation, it is vital to understand how to fairly and effectively evaluate clients’ contributions.

Shapley value [28] is a classical metric derived from cooperative game theory that is used to appropriately assess participants’ contributions. The Shapley value of a participant is defined as the expected marginal contribution of the participant over all possible subsets of the other participants. The Shapley value is the only metric that satisfies all four requirements of Shapley’s fairness criteria:
balance, symmetry, zero element, and additivity [8]; these basic fairness requirements are described in Section 4. Although the Shapley value has many desirable characteristics, its evaluation in the FL context requires repeatedly training and evaluating a machine learning model on all possible subsets of clients. The corresponding communication and computational costs are exponential, and thus prohibitive in practice [29, 34, 9].

Variants of the Shapley value make equitable data owner assessment feasible in FL. For HFL, Wang et al. [35] proposed a contribution valuation metric called federated Shapley value (FedSV). The key idea is to calculate Shapley values for clients in each round of training and then to report the total of the Shapley values for all clients as the final results. Computation of the FedSV does not need model retraining and retains some, but not all, of Shapley’s fairness criteria. Fan et al. [9] further improved the fairness of this approach by leveraging low-rank matrix factorization techniques.

Relative to HFL, adapting the Shapley value to VFL faces another challenge because of the stronger model dependence in the vertical context. More precisely, the Shapley value computation requires us to form the model produced by all the different subsets of clients. This requirement is easy to satisfy under HFL because the global model is defined as the additive aggregation of the local models and thus we only need to aggregate local models from different subsets of clients [35]. In the vertical context, however, the global model is the concatenation of local models that are not shared with the server and so simply concatenating local models from different subsets of participants does not work.

This work concentrates on developing efficient and equitable methods for evaluating clients’ contributions in VFL systems. We propose a contribution valuation metric called the vertical federated Shapley value (VerFedSV), which elaborates the idea proposed by Wang et al. [35] so that the clients’ contributions are computed at multiple time points during the training process. We resolve the model concatenation problem by carefully utilizing clients’ embeddings at different time-stamps. See Section 4 for a detailed discussion. We demonstrate that our design retains many desirable fairness properties and can be efficiently implemented without retraining.

Additionally, VFL algorithms can be divided into two categories: synchronous [12, 39, 23], where periodic synchronization among clients are required, and asynchronous [16, 13, 5], where clients are allowed to conduct local computations asynchronously. We show that VerFedSV applies to both synchronous and asynchronous VFL environments. Under the synchronous setting, we show that FedSV can be computed by leveraging some matrix-completion techniques. Although there are many similarities between synchronous and asynchronous VFL environments, contribution valuation under asynchronous VFL environments is a bit more complicated because the contribution of a client not only depends on the relevance of their data to the training task, but also depends on their local computational resource. We demonstrate that our design can reflect the strength of clients’ local computational resources under the asynchronous setting.

Our contributions can be summarized as follows:

1. Our proposed vertical federated Shapley value (VerFedSV) for vertical federated learning (Definition 3) satisfies desirable properties for fairness (Theorem 1).

2. Under the synchronous vertical federated learning environment, we show that VerFedSV can be computed by solving low-rank matrix completion problems for embedding matrices, which are proven to be approximately low rank (Proposition 1). We also give approximation guarantee on VerFedSV given the tolerance for matrix completion (Proposition 2).

3. Under the asynchronous vertical federated learning environment, we show that VerFedSV can be directly computed and can indeed reflect the strength of clients’ local computational resources (Proposition 3).
4. We show that the computational complexity of VerFedSV can be further reduced by applying Monte-Carlo sampling methods (section 7).

2 Related work

This section reviews existing contribution valuation strategies in machine learning and federated learning. Limited by space, here we restrict our discussion to Shapley-value-based valuation strategies. Shapley value [28] has had extensive influence in economics [14]. Dubey [8] showed that Shapley value is the unique measure that satisfies the four fundamental requirements of fairness proposed by Shapley [28]. See Section 4 for a more detailed introduction on Shapley value.

Ghorbani and Zou [11] proposed the Shapley value-based metric for quantifying data contributions in the context of machine learning. They introduced a data Shapley value metric for quantifying the contribution of a single data point to a learning task. They noted that directly computing the data Shapley value requires exponential-time complexity and suggested two heuristic approximation approaches to increase efficiency. Jia et al. [17] presented various efficient techniques for estimating the data Shapley value, including approaches based on group testing and compressed sensing. Ghorbani et al. [10] extended the notion of data Shapley value from a fixed training data set to arbitrary data distribution, called distributional Shapley value. Theoretically, they proved that their proposed measure is stable as similar distributions yield similar value functions. Kwon et al. [21] then developed analytic expressions for distributional Shapley value for several machine learning tasks, including linear regression and binary classification.

Song et al. [29] introduced the concept of data Shapley value to HFL, which he named the contribution index (CI). They pointed out that directly calculating CI requires retraining the model exponentially, which is costly in federated learning. They offered two gradient-based techniques to approximate the contribution index as a solution. Wang et al. [35] alternatively solved this problem by proposing a new measure for HFL, called federated Shapley value, which can be determined from local model updates in each training iteration. Federated Shapley value does not need model retraining and preserves some but not all of the favorable qualities of the traditional Shapley value. Fan et al. [9] further improved the fairness of this approach.

Wang et al. [34] and more recently Han et al. [15] extended the notion of data Shapley value to VFL. As we described above, the need to retrain the model for different subsets of participants is a bottleneck of Shapley value computation in federated learning. This problem was solved by introducing model-independent utility functions, where model-independent means the contribution of a client does not depend on the performance of the final model, and thus does not require retraining of the model [34, 15]. In particular, Wang et al. [34] suggested using the situational importance (SI) [1], which computes the difference between the embeddings with true feature and expected feature. However, computing SI requires knowing the expectation of each feature and can be impractical under VFL. Han et al. [15] suggested to use the conditional mutual information (CMI) [4], which computes the tightness between the label and features. However, computing CMI requires every client to access the labels, which may also be impractical under VFL. Besides the above shortcomings, the model-independent utility function itself may cause some fairness issues when the VFL is conducted asynchronously. As we discussed in Section 1 under the asynchronous setting, the contribution of a client not only relates to the quality of the local dataset but also depends on the power of local computational resources. Model-independent utility functions can not fully reflect a client’s contribution in this case.
3 Vertical federated learning

In this section, we revisit the VFL model. Consider the following standard VFL scenario: \( M \) clients and a single server collaborate to train a machine learning model on \( N \) data samples \( \{(x_i \in \mathbb{R}^d, y_i \in \mathbb{N}) \}_{i=1}^N \), where \( x_i \) is the feature vector and \( y_i \) is the label. Let \([N] = \{1, \ldots, N\}\) denote the set of all sample indices. Every feature vector \( x_i \) is distributed across \( M \) clients \( \{x^m_i \in \mathbb{R}^{d^m} : m \in [M]\} \), where \( d^m \) is the feature dimension for client \( m \) such that \( \sum_{m=1}^M d^m = d \) and \([M] = \{1, \ldots, M\}\) denotes the set of all clients. The local data set for any client \( m \) is \( D^m = \{x^m_i : i \in [N]\} \).

The server maintains all the labels \( D^s = \{y_i : i \in [N]\} \). The collaborative training problem can be formulated as

\[
\min_{\theta_1, \ldots, \theta_M} \mathcal{L}(\theta_1, \ldots, \theta_M; D^1, \ldots, D^M, D^s) := \frac{1}{N} \sum_{i=1}^N \ell(\theta_1, \ldots, \theta_M; \{x_i, y_i\}),
\]

(1)

where \( \theta_m \in \mathbb{R}^{d^m} \) denotes the training parameters of the \( m \)th client, \( \ell(\cdot) \) denotes the loss function, and \( \lambda \) is the hyper-parameter. For a wide range of models such as linear and logistic regression, and support vector machines, the loss function has the form

\[
\ell(\theta_1, \ldots, \theta_M; \{x_i, y_i\}) := f(h_i; y_i) \text{ where } h_i = \sum_{m=1}^M h^m_i, \quad h^m_i = \langle x^m_i, \theta_m \rangle,
\]

(2)

and \( f(\cdot; y) \) is a differentiable function for any \( y \). For each client \( m \), the term \( h^m_i \) can be viewed as client’s embedding of the local data point \( x^m_i \) and the local model \( \theta_m \). To preserve the privacy, clients are not allowed to share their local data set \( D^m \) or local model \( \theta_m \) with other clients or with the server. Instead, clients can share only their local embeddings \( \{h^m_i | i \in [N]\} \) to the server for training.

3.1 Assumptions and preliminaries

Communication constraint The communication between the server and the clients is widely recognised as the primary bottleneck in federated learning. Each cycle of communication consists of an upload and a download. We follow Dinh et al. [7], and assume that download times are negligible in comparison to upload times. This assumption is reasonable because downlinks typically occur over large bandwidth channels, and the power of the edge server is much more than the transmission power of the client. To describe the computational constraint on the uplink for each client, we make the following assumption.

Assumption 1. Let \( \Delta t > 0 \) be the unit time for any communication round. Each client \( m \in [M] \) can upload at most \( \tau_m > 0 \) local embeddings during each time interval \( \Delta t \).

Structure of loss function We introduce several special classes of functions that are considered in our paper.

Definition 1. Consider a differentiable function \( f : \mathbb{R}^n \to \mathbb{R} \) and any points \( x, y \in \mathbb{R}^n \). We have the following definitions:

- \( f \) is convex if \( f \) satisfies

\[
f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle;
\]


• \( f \) is \( \mu \)-strongly convex for some \( \mu > 0 \) if \( f \) satisfies
  \[
  f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \| y - x \|^2;
  \]

• \( f \) is \( G \)-Lipschitz for some \( G > 0 \) if \( f \) satisfies
  \[
  |f(x) - f(y)| \leq G \| x - y \|_2;
  \]

• \( f \) is \( L \)-smooth for some \( L > 0 \) if \( f \) satisfies
  \[
  \| \nabla f(x) - \nabla f(y) \|_2 \leq L \| x - y \|_2.
  \]

**Net and covering number** The \( \epsilon \)-net and covering number are widely used tools in high-dimensional probability [33]. We formally introduce the definitions.

**Definition 2.** A subset \( N \subset \mathbb{R}^n \) is called an \( \epsilon \)-net of a compact subset \( K \subseteq \mathbb{R}^n \) if for any \( x \in K \), there exists \( y \in N \) such that \( \| x - y \|_2 \leq \epsilon \).

The smallest cardinality of an \( \epsilon \)-net of \( K \) is called the covering number of \( K \) and is denoted by \( N(K, \epsilon) \).

### 4 Vertical federated Shapley value

Fairness is one of the most desirable properties for a contribution valuation metric [11, 26]. Under the FL setting, a contribution valuation metric should reflect how much each data owner contributes to the performance of the final model. Formally, we assume a black-box utility function \( U : 2^{[M]} \to \mathbb{R} \) such that for any subset of clients \( S \subseteq [M] \), the function \( U(S) \) returns a utility score of the model collaboratively trained by clients in \( S \), such as the performance of the model. Let \( v : [M] \to \mathbb{R} \) be the evaluation metric associated with the utility function \( U \). Given the utility function \( U \), the Shapley fairness criteria [28] has four fundamental requirements for any metric \( v \).

1. **Symmetry.** For any two clients \( i, j \in [M] \), if for any subset of clients \( S \subseteq [M] \setminus \{i, j\} \), \( U(S \cup \{i\}) = U(S \cup \{j\}) \), then \( v(i) = v(j) \).

2. **Zero element.** For any client \( i \in [N] \), if for any subset of clients \( S \subseteq [M] \setminus \{i\} \), \( U(S \cup \{i\}) = U(S) \), then \( v(i) = 0 \).

3. **Additivity.** If the utility function \( U \) can be expressed as the sum of separate utility functions, namely \( U = U_1 + U_2 \) for some \( U_1, U_2 : 2^I \to \mathbb{R} \), then for any client \( i \in [M] \), \( v(i) = v_1(i) + v_2(i) \), where \( v_1 \) and \( v_2 \) are the evaluation metrics associated with the utility functions \( U_1 \) and \( U_2 \), respectively.

4. **Balance.** \( v([M]) = \sum_{i \in [M]} v(i) \).

Note that, although balance is a necessary condition in many economic contests because it ensures payment are fully distributed to all clients, it is irrelevant in the context of our paper because we are only concerned with the relative contributions of clients. It has been shown that if the contribution valuation metric \( v \) satisfies symmetry, zero element, and additivity, then \( v \) must have the form

\[
  v(i) = c \sum_{S \subseteq [N] \setminus \{i\}} \frac{1}{\binom{N-1}{|S|-1}} [U(S \cup \{i\}) - U(S)]
\]
for some positive constant $c$ [3]. However, the contribution valuation metric in [3] is impractical in the federated learning context because the evaluation of the utility function $U$ requires retraining models [11, 35]. To make a fair evaluation of data owners computationally practical in federated learning, Wang et al. [35] recently proposed the federated Shapley value (FedSV) for HFL. The key idea is to compute the Shapley values for clients periodically during the training and then report the summation over all the periods as the final results. We adopt this idea to the VFL context.

Suppose we pre-determine $T$ time-stamps for contribution valuation and use $[T] = \{1, \ldots, T\}$ to denote the set of all time-stamps. At each time $t \in [T]$, we define the utility function $U_t : 2^{|M|} \to \mathbb{R}$ such that for any subset of clients $S \subseteq [M]$, the utility $U_t(S)$ denotes the decrease in loss made by clients in $S$ during the time period $[t-1, t]$, i.e.,

$$U_t(S) = \frac{1}{N} \sum_{i=1}^{N} f \left( \frac{M}{\sum_{m=1}^{M} (h_i^m)^{(t-1)}; y_i \right) - \frac{1}{N} \sum_{i=1}^{N} f \left( \sum_{m \in S} (h_i^m)^{(t)} + \sum_{m \notin S} (h_i^m)^{(t-1)}; y_i \right),$$

where $(h_i^m)^{(t)} = (x_i^m, \theta_i^m)$ denotes the local embedding of data point $x_i^m$ by client $m$ at time $t$. The idea behind this definition is that if client $m$ participates in the training during the period $[t-1, t]$, then we use their latest embedding $(h_i^m)^{(t)}$ for evaluation, otherwise, we use their previous embedding $(h_i^m)^{(t-1)}$. We can thus formally define the vertical federated Shapley value.

**Definition 3.** Given $T$ predetermined contribution valuation time points, the vertical federated Shapley value (VerFedSV) for any client $m \in [M]$ is

$$s_m = \frac{1}{MT} \sum_{t=1}^{T} \sum_{S \subseteq [M] \setminus \{m\}} \frac{1}{|S|} [U_t(S \cup \{m\}) - U_t(S)].$$

Computing VerFedSV does not require retraining the model. Next, we will demonstrate the fairness of VerFedSV. Since Shapley value is the unique measure satisfying the above requirements for fairness, we cannot expect any practical contribution valuation metric in VFL to exactly satisfy the above requirements. Our following proposition describes the fairness properties satisfied by the VerFedSV.

**Theorem 1** (Fairness Guarantee). Let $U : 2^{|M|} \to \mathbb{R}$ be the average utility function for the whole training process:

$$U(S) = \frac{1}{T} \sum_{t=1}^{T} U_t(S) \quad \forall S \subseteq [M].$$

The VerFedSV (Definition 3) satisfies the following requirements for fairness:

- **Symmetry.** For any two clients $i, j \in [M]$, if for any subset of clients $S \subseteq [M] \setminus \{i, j\}$, $U(S \cup \{i\}) = U(S \cup \{j\})$, then $s_i = s_j$; and

- **Zero element.** For any client $i \in [M]$, if for any subset of clients $S \subseteq [M] \setminus \{i\}$, $U(S \cup \{i\}) = U(S)$, then $s_i = 0$.

- **Periodic additivity.** If in each round, the utility function $U_t$ can be expressed as the sum of separate utility functions, i.e., $U_t = U^1_t + U^2_t$ for some $U^1_t, U^2_t : 2^{|M|} \to \mathbb{R}$, then for any client $m \in [M]$,

$$s_m = s^1_m + s^2_m,$$

where $s^i$ denotes the VerFedSV computed with respect to the utility functions $\{U^i_t : t \in [T]\}$.
Proof. We notice that the VerFedSV can be expressed as

$$s_m = \frac{1}{MT} \sum_{t=1}^{T} \left( \sum_{M \subseteq \{m\}} \frac{1}{(M-1)} \right) \left[ U_t(S \cup \{m\}) - U_t(S) \right]$$

$$= \frac{1}{M} \sum_{M \subseteq \{m\}} \left( \frac{1}{(M-1)} \right) \left[ \frac{1}{T} \sum_{t=1}^{T} U_t(S \cup \{m\}) - \frac{1}{T} \sum_{t=1}^{T} U_t(S) \right]$$

$$= \frac{1}{M} \sum_{M \subseteq \{m\}} \left( \frac{1}{(M-1)} \right) [U(S \cup \{m\}) - U(S)],$$

which matches the expression for classical Shapley value. The result then follows.

5 Computation of VerFedSV under synchronous setting

This section shows how to equip VerFedSV with synchronous VFL algorithms. Almost all the synchronous VFL algorithms \cite{11, 14, 23} share the same framework that in each round, every client uploads local embeddings for the same batch of data points to the server, downloads gradient information from the server, and then do local updates. The main difference among these algorithms is the scheme of local updates. Since the VerFedSV is computed by the server and is independent of the clients’ local updates, for clearness, we consider the vanilla stochastic gradient descent algorithm for VFL (a.k.a. FedSGD \cite{23}). All the results can be easily generalized to more complicated synchronous algorithms.

5.1 Algorithm

We show the sketch of one training round of the FedSGD algorithm bellow. Note that according to the Assumption \ref{assumption1}, the batch size can not be bigger than

$$\tau := \min\{\tau_m \mid m \in [M]\}.$$  

In each iteration $t$, the FedSGD algorithm executes the following steps:

1. Server selects a mini-batch $B^{(t)} \subset [N]$ of data points with $|B^{(t)}| = \tau$;
2. Each client $m \in [M]$ computes local embeddings

$$(h^m_i)^{(t)} = \langle x^m_i, \theta^{(t)}_m \rangle \mid i \in B^{(t)};$$

where $\theta^{(t)}_m$ is the client’s current model, and sends them to the server;
3. Server computes gradient information

$$\left\{ g^{(t)}_i := \frac{\partial f(h^m_i; y_i)}{\partial h^m_i} \mid i \in B^{(t)}; h^m_i = \sum_{m=1}^{M} (h^m_i)^{(t)} \right\}$$

and sends it to every client $m \in [M]$;
4. Each client $m \in [M]$ updates the local model via

$$\theta^{(t+1)}_m = \theta^{(t)}_m - \eta^{(t)}_m \left[ B^{(t)} \right] \sum_{i \in B^{(t)}} g^{(t)}_i x^m_i,$$

where $\eta^{(t)}$ is the learning rate.
5.2 Computation of VerFedSV

Next, we show how to compute VerFedSV with the FedSGD algorithm. Under the synchronous setting, we can just set the time points for contribution valuation to be the ends of training rounds. In order to compute the VerFedSV, the key challenge is that at each round, we know only the local embeddings for the current mini-batch $B^{(t)}$, i.e.,

$$H^{(t)} = \{(h^m_i)^{(t)} | m \in [M], i \in B^{(t)}\},$$

where computing VerFedSV requires all the local embedding in each round, i.e.,

$$\hat{H}^{(t)} = \{(h^m_i)^{(t)} | m \in [M], i \in N\};$$

see Definition 3. Therefore, we want to obtain some reasonable approximations of the missing local embeddings.

Embedding matrix For each client $m$, we define the embedding matrix $H^m \in \mathbb{R}^{T \times N}$ where each element $(t, i)$ is defined as

$$H^m_{t,i} = (h^m_i)^{(t)}.$$

However we can only make partial observations $\{H^m_{t,i} : t \in [T], i \in B^{(t)}\}$. We notice that the embedding matrices can be decomposed as

$$H^m = \Theta_m X_m \quad \text{where} \quad \Theta_m = \begin{bmatrix} - (\theta_m^1)^T & \cdots & - (\theta_m^T)^T \end{bmatrix} \in \mathbb{R}^{T \times d_m} \quad \text{and} \quad X_m = \begin{bmatrix} x^m_1 & \cdots & x^m_N \end{bmatrix} \in \mathbb{R}^{d_m \times N}.$$

Note that when $d_m < \min\{T, N\}$, the embedding matrix $H^m$ is low-rank because

$$\text{rank}(H^m) \leq \min\{\text{rank}(\Theta_m), \text{rank}(X_m)\} \leq d_m.$$

When $d_m \geq \min\{T, N\}$, we observe that the model matrix $\Theta_m$ is approximately low-rank due to the similarity of local model between successive rounds, and the data matrix $X_m$ can also be approximately low-rank due to the similarity between local data points. Our following proposition theoretically formalizes this observation. Before stating the result, we first give a formal definition of approximately low-rank, proposed by Udell and Townsend [32].

**Definition 4 (ε-rank, [32, Def. 2.1]).** Let $X \in \mathbb{R}^{m \times n}$ be a matrix and $\epsilon > 0$ a tolerance. The $\epsilon$-rank of $X$ is given by

$$\text{rank}_\epsilon(X) = \min\{\text{rank}(Z) : Z \in \mathbb{R}^{m \times n}, \|Z - X\|_{\infty} \leq \epsilon\},$$

where $\|\cdot\|_{\infty}$ is the absolute maximum matrix entry. Thus, $k = \text{rank}_\epsilon(X)$ is the smallest integer for which $X$ can be approximated by a rank $k$ matrix, up to an accuracy of $\epsilon$.

The following proposition characterizes the approximate rank of the embedding matrix.

**Proposition 1 (Rank of the embedding matrix).** Assume that the function $f(\cdot; y)$ is $L$-smooth for any label $y$, and the local data sets are normalized, i.e., $\|x^m_i\| = 1$ for all $m \in [M]$ and $i \in [N]$, and the learning rate is defined by $\eta^{(t)} := \frac{1}{t}$. Then for any $\epsilon > 0$ and any $m \in [M],$

$$\text{rank}_\epsilon(H^m) \leq \min \left\{ d_m, \left[ \frac{L \log(T)}{\epsilon} \right], \mathcal{N}\left(\mathcal{D}_m^m, \frac{\epsilon}{\gamma m}\right) \right\} \quad (5)$$

where $\mathcal{N}(\cdot, \cdot)$ is the covering number (Definition 3).
Proof. It is evident that \( \text{rank}_\epsilon(\mathcal{H}^m) \leq \text{rank}(\mathcal{H}^m) \leq d_m \) for all \( m \in [M] \). So we only need to prove the remaining two upper bounds. First, we consider the difference between successive rows of the embedding matrix \( \mathcal{H}^m \). For any \( t \in [T-1] \) and \( i \in [N] \),

\[
|\mathcal{H}_{t,i}^m - \mathcal{H}_{t+1,i}^m| = \| (\theta_{m}^{(t)} - \theta_{m}^{(t+1)}) \|
\]

we can obtain an upper bound on the \( \text{rank}_\epsilon(\mathcal{H}^m) \) by

\[
\text{rank}_\epsilon(\mathcal{H}^m) \leq \left[ \frac{1}{\epsilon} \sum_{t=1}^{T-1} \| \mathcal{H}^m[t,:] - \mathcal{H}^m[t+1,:]\|_{\max} \right] \leq \frac{L}{\epsilon} \sum_{t=1}^{T-1} \| \mathcal{H}^m[t,:] - \mathcal{H}^m[t+1,:]\|_{\max} \leq \left[ \frac{L \log(T)}{\epsilon} \right].
\]

Next, we consider the difference between any two columns of the embedding matrix \( \mathcal{H}^m \). For any \( t \in [T] \) and \( i,j \in [N] \), we have

\[
|\mathcal{H}_{t,i}^m - \mathcal{H}_{t,j}^m| = \| (\theta_{m}^{(t)} - \theta_{m}^{(t)} - \theta_{j}^{(t)} + \theta_{j}^{(t)}) \|
\]

It follows that

\[
\| \mathcal{H}^m[:,i] - \mathcal{H}^m[:,j]\|_{\max} \leq \max_{t \in [T]} \| \theta_{m}^{(t)} \| \cdot \| x_{i}^{m} - x_{j}^{m} \|.
\]

Let \( \gamma^m = \max_{t \in [T]} \| \theta_{m}^{(t)} \| \) and \( \mathcal{N} \) be an \( \frac{\epsilon}{\gamma^m} \)-net for \( \{ x_{i}^{m} : i \in [N] \} \). We can thus conclude that

\[
\text{rank}_\epsilon(\mathcal{H}^m) \leq |\mathcal{N}|.
\]

By definition of the covering number, it follows that

\[
\text{rank}_\epsilon(\mathcal{H}^m) \leq \mathcal{N}(\gamma^m) \cdot \frac{\epsilon}{\gamma^m},
\]

\[\square\]

**Low-rank matrix completion** Proposition 1 shows that the embedding matrix is approximately low rank. For any client \( m \in [M] \), we propose the following factorization-based low-rank matrix completion problem to complete the embedding matrix \( \mathcal{H}^m \):

\[
\text{minimize}_{W \in \mathbb{R}^T \times r, H \in \mathbb{R}^N \times r} \sum_{t=1}^{T} \sum_{i \in B(t)} (\mathcal{H}_{t,i}^m - w_i^T h_i)^2 + \lambda(\| W \|^2_F + \| H \|^2_F),
\]

where \( r \) is a user-specified rank parameter, \( \lambda \) is a positive regularization parameter, \( \| \cdot \|_F \) is the Frobenius norm, and \( w_i \) and \( h_i \), respectively, are the \( t \)-th and \( i \)-th row vectors of the matrices \( W \) and \( H \). The rank parameter \( r \) can be determined via Proposition 1. The low-rank matrix completion model (6) was first used in completing the information for recommender systems [20]. Its effectiveness has been extensively studied both theoretically and empirically [19, 30]. We can therefore adopt well-established matrix-completion methods [37, 6] for solving (6).
Approximation guarantee The following proposition shows that if we can obtain an accurate factorization, i.e.,
\[ \mathcal{H}^m \approx W^m(H^m)^\top \]
via solving (6), then we can also guarantee a good approximation to the true VerFedSV.

**Proposition 2** (Approximation guarantee). Define

\[ \varepsilon := \frac{1}{M} \sum_{m=1}^{M} \| \mathcal{H}^m - W^m(H^m)^\top \|_{\text{max}}. \]

For any client \( m \in [M] \), let \( \hat{s}_m \) denote the VerFedSV computed with \( \{W^m, H^m : m \in [M]\} \), i.e.,

\[ \hat{s}_m = \frac{1}{MT} \sum_{t=1}^{T} \sum_{S \subseteq [M] \setminus \{m\}} \frac{1}{(M-1)} [\hat{U}_t(S \cup \{m\}) - \hat{U}_t(S)], \]

with

\[ \hat{U}_t(S) = \frac{1}{N} \sum_{i=1}^{N} f \left( \sum_{m=1}^{M} (w_{t-1,i}^m)\mathcal{H}_t^m; y_i \right) - \frac{1}{N} \sum_{i=1}^{N} f \left( \sum_{m \in S} (w_{t}^m)\mathcal{H}_t^m + \sum_{m \notin S} (w_{t-1,i}^m)\mathcal{H}_t^m; y_i \right). \]

If the function \( f(\cdot; y) \) is \( G \)-Lipschitz for any label \( y \), then

\[ |\hat{s}_m - s_m| \leq 2G\varepsilon \]

for all client \( m \in [M] \).

**Proof.** Define \( U, \hat{U} : 2^{[M]} \to \mathbb{R} \) by

\[ U(S) := \frac{1}{T} \sum_{t=1}^{T} U_t(S) \quad \text{and} \quad \hat{U}(S) := \frac{1}{T} \sum_{t=1}^{T} \hat{U}_t(S). \]

For any \( S \subset [M] \), we know that

\[ |U(S) - \hat{U}(S)| \leq \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \left| f \left( \sum_{m=1}^{M} \mathcal{H}_{t-1,i}^m; y_i \right) - f \left( \sum_{m=1}^{M} (w_{t-1}^m)\mathcal{H}_{t}^m; y_i \right) \right| \]

\[ + \frac{1}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \left| f \left( \sum_{m \in S} \mathcal{H}_{t,i}^m + \sum_{m \notin S} \mathcal{H}_{t-1,i}^m; y_i \right) - f \left( \sum_{m \in S} (w_{t}^m)\mathcal{H}_{t}^m + \sum_{m \notin S} (w_{t-1,i}^m)\mathcal{H}_{t}^m; y_i \right) \right| \]

\[ \leq \frac{G}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{m=1}^{M} \left| \mathcal{H}_{t-1,i}^m - \sum_{m=1}^{M} (w_{t-1}^m)\mathcal{H}_{t}^m \right| \]

\[ + \frac{G}{NT} \sum_{t=1}^{T} \sum_{i=1}^{N} \left| \sum_{m \in S} \mathcal{H}_{t,i}^m + \sum_{m \notin S} \mathcal{H}_{t-1,i}^m \right| - \left( \sum_{m \in S} (w_{t}^m)\mathcal{H}_{t}^m + \sum_{m \notin S} (w_{t-1,i}^m)\mathcal{H}_{t}^m \right) \]

\[ \leq GM\varepsilon. \]

Then we can obtain a bound on \( |s_m - \hat{s}_m| \) by

\[ |s_m - \hat{s}_m| \leq \frac{1}{M} \sum_{S \subseteq [M] \setminus \{m\}} \frac{1}{(M-1)} \left| U(S \cup \{m\}) - \hat{U}(S \cup \{m\}) \right| + \left| U(S) - \hat{U}(S) \right| \]

\[ \leq 2G\varepsilon. \]

\[ \square \]
6 Computation of VerFedSV under asynchronous setting

Algorithms using synchronous computation are inefficient when applied to real-world VFL tasks, partly, when clients’ computational resources are unbalanced. In this section, we show how to equip VerFedSV with asynchronous VFL algorithms.

6.1 Algorithm

We follow the vertical asynchronous federated learning (VAFL) algorithm proposed by Chen et al. [5], where the algorithm allows each client to run stochastic gradient algorithms without coordination with other clients. We sketch the training process of the VAFL algorithm.

- **Server** maintains the latest embeddings \( \{ h^m_i \mid i \in [N], m \in [M] \} \) and waits a message from an active client \( m \). The message contains either an embedding update or a gradient query:
  1. **update**: Client \( m \) sends the embeddings \( \{ \hat{h}^m_i \mid i \in B_m \} \) to the server. In which case, the server then updates its latest embeddings by \( h_i^m = \hat{h}^m_i \) for all \( i \in B_m \);
  2. **query**: client \( m \) requests the partial gradient with respect to batch \( B_m \). In which case, the server then sends the partial gradient
     \[
     \left\{ g_i := \frac{\partial f(h_i; y_i)}{\partial h_i} \mid i \in B_m, h_i = \sum_{m=1}^{M} h^m_i \right\}
     \] (7)
     back to client \( m \).

- **Client** \( m \) executes the following steps:
  1. Randomly selects a batch \( B_m \subset [N] \) with \( |B_m| = \tau_m \) (Assumption [1]) and computes local embeddings
     \[
     \{ \hat{h}^m_i := \langle \theta^m, x_i \rangle \mid i \in B_m \};
     \]
  2. Uploads embeddings \( \{ \hat{h}^m_i \mid i \in B_m \} \) to the server;
  3. Query gradient from server and update local model as
     \[
     \theta_m \leftarrow \theta_m - \frac{\eta_m}{|B_m|} \sum_{i \in B_m} g_i x_i^m,
     \]
     where \( \eta_m \) is the local learning rate and \( g_i \) is defined in (7).

6.2 Computation of VerFedSV

Next, we show how to compute VerFedSV with the VAFL algorithm. Under the asynchronous setting, there is no definition of training rounds from the perspective of server. According to Definition [3] we can pre-determine \( T \) time-stamps for contribution valuation. At any contribution valuation time \( t \in [T] \), the server keeps two sets of embeddings for valuation, i.e.,

\[
H^{(t-1)} = \{ (h^m_i)^{(t-1)} \mid m \in [M], i \in [N] \} \quad \text{(embeddings at time point } t-1)\]
\[
H^{(t)} = \{ (h^m_i)^{(t)} \mid m \in [M], i \in [N] \} \quad \text{(embeddings at time point } t)\]

Note that we initialize the server’s embeddings with
\[
(h^m_i)^{(0)} = 0, \forall m \in [M], \forall i \in [N],
\]
which is true if all the local models \( \{ \theta_m : m \in [M] \} \) are initialized with 0. Then we can compute VerFedSV according to equation (4) and Definition 3.

A careful reader may ask why we do not need matrix completion in this scenario? A short answer to this question is that under the asynchronous setting, the contribution of a client is related to both the quality of the local dataset, and the power of local computational resources, which is reflected by the parameter \( \tau_m \) (Assumption 1). More precisely, at any contribution valuation time point \( t \in [T] \), for any client \( m \in [M] \) there exist \( B \subset [N] \) such that only the embeddings corresponding to \( B \) are updated, i.e.,

\[
(h_i^m(t)) 
eq (h_i^m(t-1)) \forall i \in B \quad \text{and} \quad (h_i^m(t)) = (h_i^m(t-1)) \forall i \not\in B,
\]

where the size of \( B \) is proportional to \( \tau_m \). Then according to equation (4), for any \( S \subset [M] \setminus \{ m \} \), we have

\[
U_t(S \cup \{ m \}) - U_t(S) = \frac{1}{N} \sum_{i \in B} \left[ f \left( (h_i^m(t-1) + (h_i^{-m})^m(t), y_i) \right) - f \left( (h_i^m(t)) + (h_i^{-m}(t), y_i) \right) \right],
\]

with

\[
(h_i^{-m}(t)) := \sum_{k \in S} (h_k^i(t)) + \sum_{k \not\in S \cup \{ m \}} (h_k^i(t-1)).
\]

Therefore, we can see that the contribution of client \( m \) is proportional to \( \tau_m \), which is indeed an important feature of asynchronous VFL algorithms. Thus, if we do complete the embedding matrix as under the synchronous setting, then we will lose this important feature, which is unfair for the clients with more powerful local computational resource.

**Motivation for more updates** The above discussion also suggests that VerFedSV can motivate clients to communicate more with the server in the asynchronous setting, i.e., entirely using their local computational resource. Formally, the following proposition shows that if two clients with the same local datasets but different communication frequencies, they will receive different valuations. The one who communicates more with the server will receive a higher valuation.

**Proposition 3** (Harder work leads to more rewards). We consider a simple case where we have two clients with identical local datasets i.e.,

\[
x_1^i = x_2^i = x_i \quad \forall i \in [N].
\]

The two clients have different communication frequencies in the sense that there exist \( \rho \in [0, 1] \), such that every time client 1 sends local embeddings to server, client 2 will send local embeddings with probability \( \rho \). Suppose that the loss function

\[
g(\theta) := \frac{1}{N} \sum_{i=1}^{N} f(\langle \theta, x_i \rangle, y_i)
\]

is \( \mu \)-strongly convex and \( \theta^* \) is the global minimum point, \( \theta_1 \) and \( \theta_2 \) are initialized at 0, and we stop training when we reach the optimum point. Then it follows that

\[
E[\theta^*_1] = \frac{1}{1 + \rho} \theta^* \quad \text{and} \quad E[\theta^*_2] = \frac{\rho}{1 + \rho} \theta^*.
\]

Furthermore, if we only do one round of contribution valuation when the training ends,

\[
E[s_1] \geq E[s_2] + \mu \left( \frac{1 - \rho}{1 + \rho} \right)^2 \| \theta^* \|^2,
\]

where \( s_1, s_2 \) are the VerFedSV for client 1 and 2 respectively.
Proof. We know that $\theta_1^*$ and $\theta_2^*$ are the optimal variables for the following convex optimization problem

$$\min_{\theta_1, \theta_2} \frac{1}{N} \sum_{i=1}^{N} f((\theta_1, x_i) + (\theta_2, x_i); y_i) = g(\theta_1 + \theta_2).$$

Since $\theta^*$ is the unique minimizer for $g(\theta)$, it follows that $\theta_1^* + \theta_2^* = \theta^*$. Denote $d_1(t)$ as the update for $\theta_1$ at the $t$th iteration and $d_2(t)$ as the update for $\theta_2$ at the $t$th iteration. By the construction, we know that

$$E \left[ \sum_{t=1}^{\infty} (d_1(t) + d_2(t)) \right] = \theta^*$$

and

$$E \left[ \sum_{t=1}^{\infty} d_2(t) \right] = \frac{\rho}{1 + \rho} \theta^*.$$

Therefore

$$E[\theta_1] = E \left[ \sum_{t=1}^{\infty} d_1(t) \right] = \frac{1}{1 + \rho} \theta^*$$

and

$$E[\theta_2] = E \left[ \sum_{t=1}^{\infty} d_2(t) \right] = \frac{\rho}{1 + \rho} \theta^*.$$

Now we consider VerFedSV. By the definition, we have

$$E[s_1 - s_2] = 2 \left[ g \left( \frac{\rho}{1 + \rho} \theta^* \right) - g \left( \frac{1}{1 + \rho} \theta^* \right) \right].$$

Define $h : [0, 1] \rightarrow \mathbb{R}$ as $h(\lambda) = g(\lambda \theta^*)$. Since $g$ is $\mu$-strongly convex and $\theta^*$ is the unique minimizer, it follows that $h$ is $\mu ||\theta^*||^2$-strongly convex and is monotonically non-increasing on $[0, 1]$. Thus, we can conclude that

$$E[s_1] \geq E[s_2] + 2 \left[ h \left( \frac{\rho}{1 + \rho} \right) - h \left( \frac{1}{1 + \rho} \right) \right] \geq E[s_2] + \mu \left( \frac{1 - \rho}{1 + \rho} \right)^2 ||\theta^*||^2.$$

$\square$

7 Estimating the VerFedSV

From Definition 3 we can see that the computational complexity for VerFedSV is exponential in the number of clients $N$. The same situation appears in the computation of the classical Shapley value. How to efficiently estimate the Shapley value has been studied extensively [11, 17] and those methods can be adopted for the computation of VerFedSV. Here we describe the well known Monte-Carlo sampling method [25, 11]. We can rewrite the definition of VerFedSV into an equivalent formulation using expectation:

$$s_m = \mathbb{E}_{\pi \sim \Pi([N])} \frac{1}{T} \sum_{t=1}^{T} [U_t(\pi(m) \cup \{m\}) - U_t(\pi(m))],$$

where $\Pi([N])$ is the uniform distribution over all $N!$ permutations of the set of clients $[N]$ and $\pi(m)$ is the set of clients preceding client $m$ in permutation $\pi$. With this formulation, we can use the Monte-Carlo method to get an approximation of VerFedSV. More precisely, we can randomly sample $K$ permutations $\pi_1, \ldots, \pi_K$ and get an approximation to VerFedSV $s_m$ by

$$s_m = \frac{1}{K T} \sum_{k=1}^{K} \sum_{t=1}^{T} [U_t(\pi_k(m) \cup \{m\}) - U_t(\pi_k(m))].$$

By applying the Hoeffding’s inequality, it can be shown [17] that if

$$K \geq \frac{2R^2N}{\epsilon} \log \left( \frac{2N}{\delta} \right)$$

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for some $\epsilon > 0$ and $\delta \in (0, 1)$, where

$$R := \max_{S \subseteq [N]} \left[ \frac{1}{T} \sum_{t=1}^{T} U_t(S) \right] - \min_{S \subseteq [N]} \left[ \frac{1}{T} \sum_{t=1}^{T} U_t(S) \right],$$

then we have

$$\mathbb{P}(|\hat{s}_m - s_m| \leq \epsilon) \geq 1 - \delta.$$

8 Experiments

We conduct experiments on real-world datasets to validate the fairness, efficiency, and adaptability of VerFedSV. Section 8.1 introduces the datasets and some general settings for our experiments. Section 8.2 and Section 8.3 show respectively the performance of VerFedSV when adapted with synchronous and asynchronous VFL algorithms.

8.1 Data sets and settings

**Adult** For the Adult dataset [38], the task is to predict whether a person’s income exceeds 50K/yr based on census data. The number of data points is $N = 48842$, the number of features is $d = 123$ and the number of classes is 2. We separate the features across 3 clients for Adult dataset.

**Web** For the Web dataset [27], the task is to classify whether a web page belongs to a category or not based on keyword attributes. The number of data points is $N = 119742$, the number of features is $d = 300$ and the number of classes is 2. We separate the features across 15 clients for Web dataset.

**Covtype** For the Covtype dataset [3], the task is to predict forest cover type based on cartographic variables. The number of data points is $N = 581012$, the number of features is $d = 54$ and the number of classes is 7. We separate the features across 9 clients for Covtype dataset.

**Model** For all datasets, we set the model to be multinomial logistic regression. Under both synchronous and asynchronous settings, we can achieve 85% test accuracy on Adult dataset, 94% test accuracy on Web dataset, and 72% test accuracy on Covtype dataset.

**Implementation** We implement the VFL algorithms and the corresponding VerFedSV computation schemes described in sections 5 and 6 in the Julia language [2]. The implementation of both synchronous and asynchronous VFL algorithms is publicly available at [https://github.com/ZhenanFanUBC/VerFedLogistic.jl](https://github.com/ZhenanFanUBC/VerFedLogistic.jl). The matrix completion problem (6) is solved by the Julia package [LowRankModels.jl](https://github.com/UniversalityLabs/LowRankModels.jl) [31]. The VerFedSV is computed exactly when the number of clients $M \leq 10$, and approximately using Monte Carlo when the number of clients $M > 10$. All the experiments are conducted on a Linux server with 32 CPUs and 64 GB memory.

8.2 Synchronous algorithm

In this set of experiments, we adapt VerFedSV with synchronous VFL algorithms. First, we show that the embedding matrices are indeed approximately low-rank; see Proposition [1]. Besides, we show that VerFedSV satisfies the fairness property under synchronous setting; see Theorem [1]. More precisely, we want to show that clients with similar features should get similar valuations and clients with randomly generated features should get low valuations. We train the model for $T = 500$ rounds for all the datasets, which is also the number of contribution valuation time points (Definition [3]).
In this experiment, we numerically verify that the embedding matrices are approximately rank as described by Proposition 1. Because we want to access the full embedding matrix, we set the batch size equal to the number of data points for all clients. The number of clients for the Adult, Web and Covtype datasets is, respectively, \( M = 3, 15, 9 \). Since computing the \( \epsilon \)-rank (Definition 4) is NP-hard \cite{lu2021verfedsv}, we instead compute the rank of its truncated singular value decomposition as an approximation. More precisely, given a matrix \( X \in \mathbb{R}^{m \times n} \), let \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \) be its ordered singular values, where \( p = \min\{m, n\} \). Then given \( \epsilon \in (0, 1) \), we define its approximated \( \epsilon \)-rank as

\[
\hat{\text{rank}}_\epsilon(X) = \max\{r \in [1, p] \mid \sigma_r \geq \epsilon \cdot \sigma_1\}.
\]

We show a bar plot of the approximated \( \epsilon \)-rank of the embedding matrices in Figure 1 where \( \epsilon = 10^{-3} \), the x-axis represents the approximated \( \epsilon \)-rank and the y-axis represents number of clients. The plot shows that the embeddings matrices have low approximated \( \epsilon \)-ranks for all the datasets.

**Impact of feature heterogeneity** In this experiment, we show that clients with similar features will receive similar valuations under the synchronous setting. Besides the original clients, for each data set, we add 5 more clients whose features are identical to client 1 but with different level of perturbations. More precisely, for new client \( i \in \{1, \ldots, 5\} \), we add white Gaussian noise to \((i-1)10\%\) percent of his local features, which we denote as the feature heterogeneity. Then we measure the relative difference between the original client 1 and the new clients, i.e., for any new client \( i \in \{1, \ldots, 5\} \),

\[
\text{diff}_i := \frac{|s - s_i|}{s},
\]

where \( s \) is the VerFedSV for the original client 1 and \( s_i \) is the VerFedSV for the new client \( i \). We show a plot of relative VerFedSV difference vs feature heterogeneity in Figure 2 where the number of clients for the Adult, Web and Covtype datasets is, respectively, \( M \in \{8, 20, 14\} \). From the plot, we can see that the relative VerFedSV difference is proportional to the feature heterogeneity. Besides, when the feature heterogeneity is equal to 0, i.e., two clients have identical features, the relative VerFedSV difference is exactly 0 for Adult dataset, and is nearly 0 for Web dataset and Covtype dataset, where the inexactness is due to the Monte Carlo sampling.
VerFedSV for random feature In this experiment, we show that clients with randomly generated features will receive low evaluations under synchronous setting. Besides the regular clients, for each data set, we add 5 more clients whose features are randomly generated according to different distributions. More precisely, for new client $i \in \{1, \ldots, 5\}$, the features are generated from Gaussian distribution with mean equal to $i$ and variance equal to $i^2$. We show in the Table 1 the percentage of clients’ VerFedSVs in the total sum of VerFedSVs, where the number of clients for the Adult, Web and Covtype datasets is, respectively, $M = 8, 20, 14$. As we can see from the table, regardless of the distributions, clients with randomly generated features receive much lower valuations than the regular clients for all the datasets.

8.3 Asynchronous algorithm

In this set of experiments, we adapt VerFedSV with asynchronous VFL algorithms. We show that VerFedSV not only satisfies the fairness property under asynchronous setting; see Proposition 1 but also can reflect how hard clients are working; see Proposition 3. More specifically, during the training, we let clients to communicate with server at different frequencies. For all the datasets, we asynchronously train the model for 5 seconds and do the valuation every 0.01 seconds, i.e., there are $T = 500$ contribution valuation time points (Definition 3).

Impact of communication frequency In this experiment, we show that clients with identical features but different communication frequencies will receive different valuations. Furthermore, clients with higher communication frequency will receive higher valuation. Besides the original clients, for each data set, we add 5 more clients whose features are identical to client 1 but with different level of
(a) Adult dataset.  
(b) Web dataset.  
(c) Covtype dataset.

Figure 3: VerFedSVs for clients with different communication frequencies.

|                      | Adult | Web  | Covtype |
|----------------------|-------|------|---------|
| all regular clients  | 97.91 | 98.39| 98.57   |
| artificial client 1  | 0.93  | 0.26 | 0.35    |
| artificial client 2  | 0.06  | 0.31 | 0.21    |
| artificial client 3  | 0.33  | 0.36 | 0.22    |
| artificial client 4  | 0.43  | 0.40 | 0.25    |
| artificial client 5  | 0.35  | 0.28 | 0.39    |

Table 2: Percentage of clients’ VerFedSVs in the total sum of VerFedSVs

communication frequencies. More precisely, for new client $i \in \{1, \ldots, 5\}$, he will communicate with the server every $0.01i$ seconds, i.e., new client 1 has the highest communication frequency and new client 5 has the lowest. We show a plot in Figure 3 of the percentage of new clients’ VerFedSVs in the total sum of VerFedSVs, where the number of clients for the Adult, Web and Covtype datasets is, respectively, $M = 8, 20, 14$. From the plot, we can see that the percentage of VerFedSV is proportional to the communication frequency.

**VerFedSV for random feature** In this experiment, we show that clients with randomly generated features will receive low evaluations regardless of their communication frequencies. Besides the regular clients, for each data set, we add 5 more clients whose features are randomly generated according to standard Gaussian distribution. They have communication frequencies such new client $i \in \{1, \ldots, 5\}$ will communicate with the server every $0.01i$ seconds. We show in the Table 2 the percentage of clients’ VerFedSVs in the total sum of VerFedSVs, where the number of clients for the Adult, Web and Covtype datasets is, respectively, $M = 8, 20, 14$. As we can see from the table, regardless of the communication frequencies, clients with randomly generated features receive much lower valuations than the regular clients for all the datasets.

9 Conclusion and insights

In this work, we propose a contribution valuation metric called vertical federated Shapley value (VerFedSV) for vertical federated learning (VFL). We demonstrate both theoretically and empirically that VerFedSV satisfies many desirable properties for fairness, and is quite adaptable such that it can
be applied under both synchronous and asynchronous VFL environments.

There is an interesting insight from the experiments that deserve future research. We notice that when we keep adding clients with identical features, their overall VerFedSV increases. This may suggest clients to cheat by constructing new clients with identical features so that they can receive new rewards in the end. The issue can be resolved under the synchronous setting, for example, by checking the similarity between uploaded embeddings from clients. However, how to resolve this issue under the asynchronous setting has no simple solution.

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