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GLOBAL MONODROMY MODULO 5 OF THE QUINTIC-MIRROR FAMILY

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Abstract

The quintic-mirror family is a well-known one-parameter family of Calabi–Yau threefolds. A complete description of the global monodromy group of this family is not yet known. In this paper, we give a presentation of the global monodromy group in the general linear group of degree 4 over the ring of integers modulo 5.

1. Introduction

The quintic-mirror family \((W_b)_{b \in U} \to \mathbb{P}^1\) is a family, whose restriction \(f: (W_b)_{b \in U} \to U \) on \(U := \mathbb{P}^1 - \{0, 1, \infty\}\) is a smooth projective family of Calabi–Yau manifolds. Fix \(b \in U\) and let \(\langle , \rangle\) be the anti-symmetric bilinear form on \(H^3(W_b, \mathbb{Z})\) defined by the cup product. The global monodromy group \(\Gamma\) is the image of the representation \(\pi_1(U, b) \to \text{Aut}(H^3(W_b, \mathbb{Z}), \langle , \rangle)\) corresponding to the local system \(R^3f_*\mathbb{Z}\) with the fiber \(H^3(W_b, \mathbb{Z})\) over \(b\). When we take a symplectic basis, we can identify \(\text{Aut}(H^3(W_b, \mathbb{Z}), \langle , \rangle)\) with \(\text{Sp}(4, \mathbb{Z})\).

In this paper, we are concerned with a description of \(\Gamma\). Matrix presentations of the generators of \(\Gamma\) are well studied and it is also known that \(\Gamma\) is Zariski dense in \(\text{Sp}(4, \mathbb{Z})\) (e.g. [1], [3]). However, it is not known whether the index of \(\Gamma\) in \(\text{Sp}(4, \mathbb{Z})\) is finite or not (e.g. [2]). A direct approach for this problem is to describe \(\Gamma\) explicitly.

In the main theorem of this paper, we give a presentation of \(\Gamma\) in \(\text{GL}(4, \mathbb{Z}/5\mathbb{Z})\), which is a small attempt toward a description of \(\Gamma\).

On the other hand, Chen, Yang and Yui find a congruence subgroup \(\Gamma(5, 5)\) of \(\text{Sp}(4, \mathbb{Z})\) of finite index, which contains \(\Gamma\) in [2]. Combining their result and our main theorem, we can construct a smaller congruence subgroup \(\tilde{\Gamma}(5, 5)\) of \(\text{Sp}(4, \mathbb{Z})\) of finite index, which contains \(\Gamma\). However this result is merely the fact that \(\tilde{\Gamma}(5, 5)\) contains \(\Gamma\). After all, the index of \(\Gamma\) in \(\text{Sp}(4, \mathbb{Z})\) is still unknown.

2. The quintic-mirror family

The quintic-mirror family was constructed by Greene and Plesser. We review the construction of the quintic-mirror family after [4].

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A finite group $G$, which is abstractly isomorphic to $(\mathbb{Z}/5\mathbb{Z})^3$, acts on $Q$ as follows.

Let $\mu_5$: the multiplicative group of the 5-th root of 1, $G = (\mu_5)^5/\{\alpha_1, \ldots, \alpha_5\} \in (\mu_5)^5 \mid \alpha_1 = \cdots = \alpha_5\}$.

$G = \{(\alpha_1, \ldots, \alpha_5) \in G \mid \alpha_1 \cdots \alpha_5 = 1\}$.

$G \times Q \rightarrow Q_{\psi}, \quad ((\alpha_1, \ldots, \alpha_5), (x_1, \ldots, x_5)) \mapsto (\alpha_1 x_1, \ldots, \alpha_5 x_5)$.

When we take the quotient of the hypersurface $Q_{\psi}$ by $G$, canonical singularities appear. For $\psi \in \mathbb{C} \subset \mathbb{P}^1$, it is known that there is a simultaneous minimal desingularization of these singularities, and we have the one-parameter family $(W_{\psi})_{\psi \in \mathbb{P}^1}$ whose fibres are listed as follows:

- When $\psi$ belongs to $\mu_5 \subset \mathbb{C} \subset \mathbb{P}^1$, $W_{\psi}$ has one ordinary double point.
- $W_\infty$ is a normal crossing divisor in the total space.
- The other fibres of $(W_{\psi})_{\psi \in \mathbb{P}^1}$ are smooth with Hodge numbers $h^{p, q} = 1$ for $p + q = 3$, $p, q \geq 0$.

By the action of

$\alpha \in \mu_5, \quad (x_1, \ldots, x_5) \mapsto (x_1, \ldots, x_4, \alpha^{-1} x_5)$,

we have the isomorphism from the fibre over $\psi$ to the fibre over $\alpha \psi$. Let $\lambda = \psi^5$ and let

$$(W_{\lambda})_{\lambda \in \mathbb{P}^1} \cong ((W_{\psi})_{\psi \in \mathbb{P}^1})/\mu_5 \xrightarrow{(\lambda \text{-plane})} (\psi \text{-plane})/\mu_5.$$

This family $(W_{\lambda})_{\lambda \in \mathbb{P}^1}$ is the so-called quintic-mirror family. (For more details of the above construction, see e.g. [4], [5].)

3. Monodromy

Let $b \in \mathbb{P}^1 - \{0, 1, \infty\}$ on the $\lambda$-plane. In [1], Candelas, de la Ossa, Green and Parks constructed a symplectic basis \{A^1, A^2, B_1, B_2\} of $H_3(W_b, \mathbb{Z})$ and calculated the monodromies around $\lambda = 0, 1, \infty$ on the period integrals of a holomorphic 3-form on this basis. By the relation in [5, Appendix C] between the symplectic basis \{B_1, B_2, A_1, A_2\} of $H^3(W_b, \mathbb{Z})$, which is defined to be the dual basis of \{B_1, B_2, A_1, A_2\}, and the period
integrals, we have the matrix representations of the local monodromies for the basis \( \{ \beta^1, \beta^2, \alpha_1, \alpha_2 \} \). We recall their results.

Matrix representations \( A, T, T_\infty \) of local monodromies around \( 0, 1, \infty \) for the basis \( \{ \beta^1, \beta^2, \alpha_1, \alpha_2 \} \) are as follows:

\[
A = \begin{pmatrix}
11 & 8 & -5 & 0 \\
5 & -4 & -3 & 1 \\
20 & 15 & -9 & 0 \\
5 & -5 & -3 & 1
\end{pmatrix},
\quad
T = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\quad
T_\infty = \begin{pmatrix}
-9 & -3 & 5 & 0 \\
0 & 1 & 0 & 0 \\
-20 & -5 & 11 & 0 \\
-15 & 5 & 8 & 1
\end{pmatrix}.
\]

In particular, the above \( A \) and \( T \) are the inverse matrices of the matrices \( A \) and \( T \) in the lists of [1], respectively.

Let \( \langle , \rangle \) be the anti-symmetric bilinear form on \( H^3(W_b, \mathbb{Z}) \) defined by the cup product. The global monodromy \( \Gamma \) is \( \text{Im}(\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \to \text{Aut}(H^3(W_b, \mathbb{Z}), \langle , \rangle) \).

When we take \( \{ \beta^1, \beta^2, \alpha_1, \alpha_2 \} \) as the basis of \( H^3(W_b, \mathbb{Z}) \), \( \text{Aut}(H^3(W_b, \mathbb{Z}), \langle , \rangle) \) is identified with \( \text{Sp}(4, \mathbb{Z}) \), and \( \Gamma \) is the subgroup of \( \text{Sp}(4, \mathbb{Z}) \) which is generated by \( A \) and \( T \).

We can partially normalize \( A \) and \( T \) simultaneously as follows.

**Lemma.** There exists \( P \in \text{GL}(4, \mathbb{Q}) \) such that

\[
p^{-1}A^{-1}P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 \\
0 & 0 & 1 & -1 \\
5 & 5 & 5 & -4
\end{pmatrix},
\quad
p^{-1}T^{-1}P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

**Proof.** We take \( P = \begin{pmatrix}
5 & -3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
10 & -5 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \). The assertion follows.

\[\square\]

4. Main result

Let \( \Gamma' = \{ p^{-1}XP \in \text{GL}(4, \mathbb{Z}) \mid X \in \Gamma \} \), and let \( \rho: \text{GL}(4, \mathbb{Z}) \to \text{GL}(4, \mathbb{Z}/5\mathbb{Z}) \) be the natural projection. Define \( \tilde{\Gamma} = \rho(\Gamma') \). We will study \( \tilde{\Gamma} \).

Let \( \tilde{A} = \rho(p^{-1}A^{-1}P), \tilde{T} = \rho(p^{-1}T^{-1}P) \in \text{GL}(4, \mathbb{Z}/5\mathbb{Z}) \). By a simple calculation, we obtain

\[
\tilde{A}^n = \begin{pmatrix}
1 & n & 3n(n + 4) & n(n + 1)(4n + 1) \\
0 & 1 & n & 2n(n + 1) \\
0 & 0 & 1 & 4n \\
0 & 0 & 0 & 1
\end{pmatrix} \in \text{GL}(4, \mathbb{Z}/5\mathbb{Z}).
\]
Let $\hat{\Gamma}$ be
\[
\left\{ \begin{pmatrix}
1 & n & 3n^2 + 2n & a \\
0 & 1 & n & b \\
0 & 0 & 1 & c \\
0 & 0 & 0 & 1
\end{pmatrix} \in \text{GL}(4, \mathbb{Z}/5\mathbb{Z}) \bigg| n, a, b, c \in \mathbb{Z}/5\mathbb{Z} \right\}.
\]

$\hat{\Gamma}$ is a subgroup of $\text{GL}(4, \mathbb{Z}/5\mathbb{Z})$ which contains $\tilde{A}$ and $\tilde{T}$. The following Theorem and Corollary are the main results of this paper.

**Theorem.** $\hat{\Gamma} = \tilde{\Gamma}$.

**Proof.** $\tilde{\Gamma} \subset \hat{\Gamma}$ follows from what we just mentioned. So we shall prove the converse inclusion.

From the presentations of elements of $\tilde{\Gamma}$, we see that $\tilde{\Gamma}$ is generated by $\tilde{A}, \tilde{T}, E_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Therefore, it is enough to show $E_1$ and $E_2$ belong to $\tilde{\Gamma}$. In fact, we have
\[
E_2 = \tilde{A} \tilde{T} \tilde{A}^4 \tilde{T}^4,
\]
and
\[
E_1 = (E_2^2 \tilde{A}^2 \tilde{T}^4 \tilde{A}^3 \tilde{T})^4.
\]

Hence $E_1, E_2 \in \tilde{\Gamma}$. \qed

**Corollary.** Let $X \in \Gamma$. Then the characteristic polynomial of $X$ is
\[
x^4 + (5m + 1)x^3 + (5n + 1)x^2 + (5m + 1)x + 1,
\]
where $m, n$ are some integers. In particular, if $X$ is not the unit matrix and the order of $X$ is finite, then the order of $X$ is 5 and the eigenvalues of $X$ are $\exp(2\pi i/5)$, $\exp(4\pi i/5)$, $\exp(6\pi i/5)$, $\exp(8\pi i/5)$.

**Proof.** We shall prove the first part. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the eigenvalues of $X$. Then the characteristic polynomial $p(X)$ of $X$ is
\[
x^4 - \left( \sum_{1 \leq i < j < k \leq 4} \lambda_i \lambda_j \lambda_k \right)x^3 + \left( \sum_{1 \leq i < j \leq 4} \lambda_i \lambda_j \right)x^2 - \left( \sum_{1 \leq i \leq 4} \lambda_i \right)x + 1.
\]
On the other hand, the characteristic polynomial $p(X^{-1})$ of $X^{-1}$ is

$$x^4 - \left( \sum_{1 \leq i \leq j \leq k \leq 4} \frac{1}{\lambda_i^2 \lambda_j \lambda_k} \right) x^3 + \left( \sum_{1 \leq i < j \leq 4} \frac{1}{\lambda_i \lambda_j} \right) x^2 - \left( \sum_{1 \leq i < j \leq k \leq 4} \frac{1}{\lambda_i \lambda_j \lambda_k} \right) x + 1$$

$$= x^4 - \left( \sum_{1 \leq i \leq 4} \lambda_i \right) x^3 + \left( \sum_{1 \leq i < j \leq 4} \lambda_i \lambda_j \right) x^2 - \left( \sum_{1 \leq i < j < k \leq 4} \lambda_i \lambda_j \lambda_k \right) x + 1.$$  

Since $X \in \text{Sp}(4, \mathbb{Z})$, $p(X) = p(X^{-1})$. So $p(X)$ is the form $x^4 + ax^3 + bx^2 + ax + 1$, where $a, b \in \mathbb{Z}$. It follows from the theorem that $a \equiv -4$, $b \equiv 6 \mod 5$. Hence the claim of the first part follows.

Next we shall prove the latter part. Let $\lambda$ be an eigenvalue of $X$. It follows from $p(X) = p(\tilde{X})$ and $p(X) = p(X^{-1})$ that $\tilde{\lambda}, 1/\lambda, 1/\tilde{\lambda}$ are also eigenvalues of $X$. Since the determinant of $X$ is 1, if 1 or $-1$ is an eigenvalue of $X$, its multiplicity is even. Since the order of $X$ is finite, we can express eigenvalues of $X$ by $\exp(i\theta_1), \exp(-i\theta_1), \exp(i\theta_2), \exp(-i\theta_2)$ ($0 \leq \theta_1, \theta_2 \leq \pi$). Then the characteristic polynomial of $X$ is

$$x^4 - 2(\cos \theta_1 + \cos \theta_2)x^3 + 2(\cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2) + 1)x^2 - 2(\cos \theta_1 + \cos \theta_2)x + 1.$$  

By the claim of the first part of this Corollary, we have

$$-2(\cos \theta_1 + \cos \theta_2) = 5m + 1, \quad 2(\cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2) + 1) = 5n + 1, \quad m, n \in \mathbb{Z}.$$  

By the addition theorem for cosines, we have

$$2(\cos \theta_1 + \cos \theta_2) = -5m - 1, \quad 4 \cos \theta_1 \cos \theta_2 = 5n - 1.$$  

It follows from $-4 \leq 2(\cos \theta_1 + \cos \theta_2) \leq 4$ that $m = 0$ or $-1$. If $m = -1$, then $\cos \theta_1, \cos \theta_2 = 1$ and all eigenvalues of $X$ are 1. Since the order of $X$ is finite, $X$ is the unit matrix. This contradicts the assumption that $X$ is not the unit matrix. Hence $m = 0$ and

$$\cos \theta_1 + \cos \theta_2 = -\frac{1}{2}.$$  

It follows from $-4 \leq 4 \cos \theta_1 \cos \theta_2 \leq 4$ that $n = 0$ or 1. If $n = 1$, then $\cos \theta_1 = \pm 1, \cos \theta_2 = \pm 1$. This contradicts the fact that $\cos \theta_1 + \cos \theta_2 = -1/2$. Hence $n = 0$ and

$$\cos \theta_1 \cos \theta_2 = -\frac{1}{4}.$$  

Combining these two equations, we have

$$\cos^2 \theta_1 + \frac{1}{2} \cos \theta_1 - \frac{1}{4} = 0.$$
When we solve this equation for \( \cos \theta_1 \),
\[
\cos \theta_1 = \frac{-1 \pm \sqrt{5}}{4}, \quad \sin \theta_1 = \frac{\sqrt{10 \pm 2\sqrt{5}}}{4},
\]
\[
\cos \theta_2 = \frac{-1 \mp \sqrt{5}}{4}, \quad \sin \theta_2 = \frac{\sqrt{10 \mp 2\sqrt{5}}}{4}.
\]

Then we can verify easily that \((\exp(i\theta_1))^5\) and \((\exp(i\theta_2))^5 = 1\). Hence \((\theta_1, \theta_2) = (2\pi/5, 4\pi/5)\) or \((4\pi/5, 2\pi/5)\).

5. A relation to the other result

In this section, we shall compare the main result of this paper with the result of Chen, Yang and Yui. In [2], they find the congruence subgroup \(\Gamma'(5, 5)\) which contains the global monodromy \(\Gamma\). Combining their result and our theorem, we can find a smaller group which contains \(\Gamma\).

The congruence subgroup \(\Gamma(5, 5)\) is defined by
\[
\Gamma(5, 5) = \left\{ X \in \text{Sp}(4, \mathbb{Z}) \left| \gamma \equiv \begin{pmatrix}
1 & * & * & * \\
0 & 1 & * & * \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \mod 5 \right. \right\}.
\]

\(\Gamma(5, 5)\) contains the principal congruence group \(\Gamma(5) = \text{Ker}(\text{Sp}(4, \mathbb{Z}) \to \text{Sp}(4, \mathbb{Z}/5\mathbb{Z}))\) as a normal subgroup of finite index.

Let \(X \in \Gamma(5, 5)\) and express \(X\) by
\[
\begin{pmatrix}
5x_{11} + 1 \\
5x_{21} \\
5x_{31} \\
5x_{41}
\end{pmatrix}
\begin{pmatrix}
x_{12} \\
x_{22} + 1 \\
x_{32} \\
x_{42}
\end{pmatrix}
\begin{pmatrix}
x_{13} \\
x_{23} \\
x_{33} + 1 \\
x_{43}
\end{pmatrix}
\begin{pmatrix}
x_{14} \\
x_{24} \\
x_{34} \\
x_{44} + 1
\end{pmatrix}, \quad x_{ij} \in \mathbb{Z} \quad (1 \leq i, j \leq 4).
\]

Then we have
\[
\text{GL}(4, \mathbb{Z}) \ni P^{-1}X P = \begin{pmatrix}
1 & -9x_{31} & -x_{12} + 3x_{32} & -x_{14} + 3x_{34} \\
0 & 1 & -2x_{12} & -2x_{14} \\
0 & 0 & 1 & x_{24} \\
0 & 0 & 0 & 1
\end{pmatrix} \mod 5.
\]

By the main theorem, if \(X \in \Gamma\), then \(\rho(P^{-1}X P) \in \Gamma\) and
\[
-9x_{31} \equiv n, \quad -2x_{12} \equiv n, \quad -x_{12} + 3x_{32} \equiv 3n^2 + 2n \quad \text{(mod 5)}.
\]
where \( n \) is some integer. From a simple calculation, the above equation is equivalent to

\[
x_{31} \equiv 3x_{12}, \quad x_{32} \equiv 4x_{12}^2 + 4x_{12} \pmod{5}.
\]

So we define

\[
\tilde{\Gamma}(5, 5) = \left\{ \left( \begin{array}{cccc}
5x_{11} + 1 & x_{12} & x_{13} & x_{14} \\
5x_{21} & 5x_{22} + 1 & x_{23} & x_{24} \\
5x_{31} & 5x_{32} & 5x_{33} + 1 & 5x_{34} \\
5x_{41} & 5x_{42} & x_{43} & 5x_{44} + 1 \\
\end{array} \right) \in \text{Sp}(4, \mathbb{Z}) \mid \left( \begin{array}{c}
x_{31} \equiv 3x_{12}, \\
x_{32} \equiv 4x_{12}^2 + 4x_{12} \\
\end{array} \right) \pmod{5} \right\}.
\]

Then we have the following Corollary.

**Corollary.**

(i) \( \tilde{\Gamma}(5, 5) \) is a subgroup of \( \Gamma(5, 5) \).

(ii) \( \Gamma \subset \tilde{\Gamma}(5, 5) \not\subset \Gamma(5, 5) \).

(iii) \( \tilde{\Gamma}(5, 5) \) is a congruence subgroup of \( \text{Sp}(4, \mathbb{Z}) \) of finite index.

**Proof.** Let \( \rho' : \Gamma(5, 5) \to \text{GL}(4, \mathbb{Z}), X \mapsto P^{-1}XP \) and let \( \pi = \rho \circ \rho' : \Gamma(5, 5) \to \text{GL}(4, \mathbb{Z}/5\mathbb{Z}) \). \( \tilde{\Gamma}(5, 5) = \pi^{-1}(\tilde{\Gamma}) \) follows from what we just mentioned. Since \( \pi \) is a group homomorphism, \( \pi^{-1}(\tilde{\Gamma}) \) is a subgroup of \( \Gamma(5, 5) \). Hence the claim of (i) follows.

We can verify easily that \( A \) and \( T \) belong to \( \tilde{\Gamma}(5, 5) \). Therefore \( \tilde{\Gamma}(5, 5) \) contains \( \Gamma \).

We shall show that \( \tilde{\Gamma}(5, 5) \) is a proper subgroup of \( \Gamma(5, 5) \). We take \( X = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
5 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right) \).

Then \( X \in \Gamma(5) \subset \Gamma(5, 5) \) and \( X \notin \tilde{\Gamma}(5, 5) \). Hence the claim of (ii) follows.

Finally, we shall show the claim of (iii). \( \tilde{\Gamma}(5, 5) \) contains the principal congruence subgroup \( \Gamma(25) = \text{Ker}(\text{Sp}(4, \mathbb{Z}) \to \text{Sp}(4, \mathbb{Z}/25\mathbb{Z})) \) as a normal subgroup. Hence we obtain \( |\tilde{\Gamma}(5, 5) : \text{Sp}(4, \mathbb{Z})| < |\Gamma(25) : \text{Sp}(4, \mathbb{Z})| = |\text{Sp}(4, \mathbb{Z}/25\mathbb{Z})| < \infty \).

**QUESTION.** There are other 13 mirror families of Calabi–Yau threefolds with \( h^{2,1} = 1 \) as discussed in [2]. Is it possible to find smaller subgroups in those 13 cases as well?

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**References**

[1] P. Candelas, C. de la Ossa, P.S. Green, and L. Parks: A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory, Nuclear Phys. B 359 (1991), 21–74.
[2] Y.-H. Chen, Y. Yang and N. Yui: Monodromy of Picard–Fuchs differential equations for Calabi–Yau threefolds, J. Reine Angew. Math. 616 (2008), 167–203.

[3] P. Deligne: Local behavior of Hodge structures at infinity; in Mirror Symmetry, II, AMS/IP Stud. Adv. Math. I, Amer. Math. Soc., Providence, RI., 1997, 683–699.

[4] D.R. Morrison: Picard–Fuchs equations and mirror maps for hypersurfaces; in Essays on Mirror Manifolds, Int. Press, Hong Kong, 1992, 241–264.

[5] D.R. Morrison: Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians, J. Amer. Math. Soc. 6 (1993), 223–247.

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