Robust Linear Estimation with Non-parametric Uncertainty:  
Average and Worst-case Performance (Full Version)  

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Abstract

In this paper, two types of linear estimators are considered for three related estimation problems involving set-theoretic uncertainty pertaining to $H_2$ and $H_\infty$ balls of frequency-responses. The problems at stake correspond to robust $H_2$ and $H_\infty$ estimation in the face of non-parametric “channel-model’ uncertainty and to a nominal $H_\infty$ estimation problem. The estimators considered here are defined by minimizing the worst-case squared estimation error over the “uncertainty set” and by minimizing an average cost under the constraint that the worst-case error of any admissible estimator does not exceed a prescribed value. The main point is to explore the derivation of estimators which may be viewed as less conservative alternatives to minimax estimators, or in other words, that allow for trade-offs between worst-case performance and better performance over “large” subsets of the uncertainty set. The “average costs” over $H_2$—signal balls are obtained as limits of averages over sets of finite impulse responses, as their length grows unbounded. The estimator design problems for the two types of estimators and the three problems addressed here are recast as semi-definite programming problems (SDPs, for short). These SDPs are solved in the case of simple examples to illustrate the potential of the “average cost/worst-case constraint” estimators to mitigate the inherent conservatism of the minimax estimators.

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Notation

- \( \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{Z}^+, \mathbb{R}^+ \), and \( \mathbb{R}^+ \) stand, respectively, for the sets of integers, real, complex, positive integer, positive and non-negative real numbers.

- \( S_0^{p \times m} \) – the set of all doubly-infinite sequences of matrices in \( \mathbb{R}^{p \times m} \), i.e.,

  \[
  S_0^{p \times m} = \{ F^s = \{ F_k : k \in \mathbb{Z} \} : F_k \in \mathbb{R}^{p \times m} \}.
  \]

- \( S_c^{p \times m} \) – the set of all causal sequences, i.e., \( S_c^{p \times m} = \{ F^s \in S_0^{p \times m} : \forall k < 0, \ F_k = 0 \} \).

- \( r_c^{p \times m} \) – the set of all impulse responses of causal, stable and finite-dimensional systems, i.e.,

  \[
  r_c^{p \times m} = \{ F^s \in S_c^{p \times m} : F_0 = D, \ \forall k \geq 1, \ F_k = C_p A_p^{k-1} B_p \ \text{for some} \ A_p \in \mathbb{R}^{n \times n}, \ B_p \in \mathbb{R}^{n \times m}, C_p \in \mathbb{R}^{p \times n} \ \text{and} \ D \in \mathbb{R}^{p \times m} \ \text{with} \ \rho(A_p) < 1 \}.
  \]

- \( \mathcal{R}_c^{p \times m} \) – the set of frequency-responses corresponding to \( r_c^{p \times m} \), i.e., \( \mathcal{R}_c^{p \times m} = \mathcal{F}(r_c^{p \times m}) \) where for \( F^s \in r_c^{p \times m} \), \( \mathcal{F}(F^s) \) denotes the Fourier transform of \( F^s \), i.e., \( \mathcal{F}(e^{j\theta}) = \sum_{k=0}^{\infty} F_k e^{-j\theta k} \).

- \( \mathcal{R}^{p \times m} \) – the set of all sums involving matrix functions in \( \mathcal{R}_c^{p \times m} \) and the conjugate transposes thereof, i.e., \( \mathcal{R}^{p \times m} = \{ G = E_i + E_i^* : E_i \in \mathcal{R}_c^{p \times m}, i = 1, 2 \} \), where \( E_i(e^{j\theta}) = E(e^{j\theta})^* = E(e^{-j\theta})^T \).

- \( \mathcal{R}_h^{p \times m} \) – matrix functions which are Hermitian on the unit circle, i.e.,

  \[
  \mathcal{R}_h^{p \times m} = \{ G = E + E^* : E \in \mathcal{R}_c^{p \times m} \}.
  \]

- \( \mathcal{R}_0^{p \times m} \) – the subset of \( \mathcal{R}^{p \times m} \) containing only matrix functions which are positive semidefinite on the unit circle, i.e.,

  \[
  \mathcal{R}_0^{p \times m} = \{ G \in \mathcal{R}_h^{p \times m} : \forall \theta \in [0, 2\pi], \ G(e^{j\theta}) \geq 0 \}.
  \]

- \( \{ F \}_{ca} \) – causal part of \( F \in \mathcal{R}^{p \times m} \), i.e., for

  \[
  F(e^{j\theta}) = \sum_{k=-\infty}^{\infty} F_k e^{-j\theta k}, \ \{ F \}_{ca} = \sum_{k=0}^{\infty} F_k e^{-j\theta k}.
  \]

- \( M^*, \ \rho(M) \) – the conjugate transpose and the spectral radius of the matrix \( M \).

- \( \langle F, G \rangle \triangleq (2\pi)^{-1} \int_{0}^{2\pi} \text{tr}\{F(e^{j\theta})^* G(e^{j\theta})\} d\theta, \| F \| ^2 = \langle F, F \rangle, \ F \in \mathcal{R}^{p \times m}, G \in \mathcal{R}^{p \times m}; \)

  \( \triangleq \) denotes equality by definition.

- \( \langle A, B \rangle_f \triangleq \text{tr}(A^* B) \).

- \( \{ M \}_{ij}, M^T \) – \( ij \)-th entry and transpose of the matrix \( M \).
• $\text{tr}(A)$, $\|A\|$, $\|A\|_F$, $\lambda_{\text{min}}(M)$ and $\lambda_{\text{max}}(M)$ denote the trace, spectral norm and Frobenius norm of $A \in \mathbb{C}^{m \times p}$ and the maximum and minimum eigenvalues of the Hermitian matrix $M$.

• $\|x\|_E$ denotes the euclidean norm of $x \in \mathbb{C}^n$.

• diag$(A_1, \ldots, A_n)$ denotes a block diagonal matrix where $A_i$ is the $i$–th diagonal block.

• $A \otimes B$ denotes Kronecker product of the matrices $A$ and $B$, i.e.,

$$A \otimes B = \begin{bmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{bmatrix}$$

where $\{a_{ij}\}$ are the entries of the $m \times n$ matrix $A$.

• $\text{rvec}(F)$ denotes the column matrix $\{f_{11} \cdots f_{1m} \cdots f_{p1} \cdots f_{pm}\}^T$ where $\{f_{ij}\}$ are entries of the $p \times m$ matrix $F$.

• diag$(\{M_k\})$ denotes a block diagonal matrix where $M_k$ is the $k$–th diagonal block.
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1 Introduction

Decision problems involving set-theoretic uncertainty on problem-data lead naturally to (robust) decision procedures based on minimizing, with respect to the admissible procedures, the worst-case value of a given risk function over the problem data set involved – for example, to select an estimator (decision procedure) for signals corrupted by noise when several signal models (problem data) are envisaged, it is natural to look for the estimator that minimizes the worst-case of the mean-square estimation error (risk function) over all signal models considered. Indeed, such minimax procedures have been widely considered in control and estimation problems (see, for example, [1] – [13]). However, it has been acknowledged that minimax procedures tend to be “overly conservative” in the sense that (roughly speaking) “point-wise performance” is compromised over much of the problem-data set in order to attain relatively small values of the loss function at the “most unfavourable” region of that set.

Attempts to overcome this drawback have been made on the basis of the so-called regret function associated to a given risk function – namely, the difference between the risk function at a given pair (decision-rule, problem-data point) and its minimal value (over all decision rules) at the same problem-data point. In these works, robust decision procedures are sought which minimize the worst-case value of the regret function (or approximation thereof) over the problem-data set (as done in [14] – [16] in connection with estimation problems).

In spite of being an attractive concept, minimax regret problems are often difficult to solve, specially in problems involving dynamic models, requiring either approximations of the regret function or restrictive assumptions on the problem set-up (see [17], for a brief discussion of this point). Thus, the motivation arises for pursuing other ways of obtaining less conservative robust decision procedures (vis-à-vis a minimax procedure) which handle more directly the trade off between worst-case and “point-wise” performance. In this paper, this theme is explored in connection with three closely-related discrete-time, linear-estimator design problems, namely, linear mean-squared error (MSE, for short) estimation involving non-parametric “channel” model uncertainty ($H_2$-balls of channel frequency-responses), nominal $H_\infty$ estimation and robust $H_\infty$ estimation with $H_\infty$ model uncertainty. In the case of nominal $H_\infty$ estimation, the reduction of conservatism is connected with $H_2$-balls of signals instead of “channel” frequency responses; whereas in the case of robust $H_\infty$ estimation it pertains to both signals and channel models.

As the central issue here is to obtain, in a computationally-efficient way, minimax and other potentially less conservative robust estimators, for each of these problems the corresponding minimax estimator – design problem is cast as a semi-definite programming problem (SDP, for short). Average cost functionals are introduced for the MSE over $H_2$-balls of “channel” frequency-responses and for the (deterministic) estimation-error magnitudes over $H_2$-balls of (information and noise) signals – these cost functionals are derived as limits of averages taken over finite impulse-responses (FIRs, for short) of a given length, as their length grows unlimited. On the basis of the solutions to the minimax problems, classes of admissible
linear estimators are defined to be used in the formulation of “average cost/worst-case constraint” (a/w, for short) problems, – i.e., minimization with respect to admissible estimators of an average cost functional under the constraint that the worst-case estimation error magnitude (over the uncertain model or signal set) does not exceed a prescribed value. Invoking conditions obtained in the conversion of the minimax problems into SDPs, these a/w problems are also converted into SDPs. Simple numerical examples are then presented to illustrate the “conservatism-reduction” potential of the resulting estimators vis-à-vis the point-wise performance (over the uncertain sets) of the associated minimax estimators.

It should be noted that robust estimation problems with $H_2$ (MSE) and $H_\infty$ estimation criteria have attracted considerable attention (e.g., [5], [6], [8], [13], [22], [23], [25], [26], [29]–[31]) mainly in connection with parametric uncertainty. The nominal $H_\infty$ estimation problem has also been successfully tackled (see, [19]–[21], [24] and its references).

These topics are revisited here mainly as set-ups for exploring robust alternatives to minimax estimators. However, the robust estimation problems addressed here involve non-parametric uncertainty (weighted $H_2$ and $H_\infty$ balls of frequency-responses). In addition, in the case of $H_\infty$ filtering criteria considered here (as explained in greater detail below), the radiiuses of the “information” and noise signal balls are independently specified rather than being included (in a potentially conservative way) in a single ball of larger radius. Finally, it is noted that the introduction of average costs over function balls in $H_2$, the corresponding robust estimator-design problems and their conversion to SDPs as well as the role of these estimators as less conservative alternatives to minimax procedures have not been contemplated in the existing literature.

This paper is organized as follows. In Section 2, a linear estimation set-up is presented and the estimation problems treated here are explicitly formulated. In Section 3, average performance criteria are introduced in connection with $H_2$ (MSE) robust estimation involving sets of possible “channel” models (frequency-responses) and with $H_\infty$ nominal estimation involving $H_2$—balls of signals. In Section 4, minimax problems for robust $H_2$ estimation and for nominal and robust $H_\infty$ estimation are posed and converted into SDPs. In Section 5, average cost/worst-case constraint problems are posed and also converted into SDPs. In Section 6, the possible trade-offs between worst-case and point-wise performance attainable with the a/w estimators are illustrated in simple examples. Unless otherwise stated, proofs are to be found in the Appendix.

2 Background and Problem Formulation

Consider the signal processing set-up of Figure 1 where the exogenous signals $y$ and $v$

pass through causal and stable, discrete-time, linear, multivariable filters $H$ (channel), $H_1$ (“reference” filter, usually set to the identity or the $k$th–step delay) and $G$ (estimator) with frequency-responses $H \in \mathcal{R}_c^{m_w \times m_y}$, $H_1 \in \mathcal{R}_c^{m_w \times m_y}$ and $G \in \mathcal{R}_c^{m_e \times m_v}$.
2.1 $H_2$ Estimation

In the first problem addressed in this paper, $y$ and $v$ are taken to be independent, wide-sense stationary, discrete-time random processes with zero mean and power spectral densities $\Gamma_y \in \mathbb{R}^{m_y \times m_y}$ and $\Gamma_v \in \mathbb{R}^{m_v \times m_v}$. For a given set-up $(H_I, H, \Gamma_y, \Gamma_v)$, the (nominal) performance of the linear estimator defined by $G$ is measured by the steady-state expected value of $e(t)^T e(t)$, where $e(t)$ denotes the estimation error signal, and is given by

$$J(G; H) \triangleq \langle (H_I - GH)\Gamma_y, (H_I - GH) \rangle + \langle G\Gamma_v, G \rangle.$$  \hfill (2.1)

The major aim here is to introduce estimators which achieve better "point-wise" performance (than that of minimax estimators) over "favourable" subsets of $S_H$ at the expense of a moderate increase in the resulting worst-case MSE (over that of a minimax estimator). To this effect, estimation problems are posed in which a cost-functional is minimized under the constraint that $\tilde{\mathcal{J}}(G; \mathcal{S}_H)$ does not exceed a prescribed value, i.e.,

$$\min_{G \in S_G} c(G) \quad \text{subject to} \quad \tilde{\mathcal{J}}(G; \mathcal{S}_H) \leq (1 + \alpha) \bar{\mathcal{J}},$$  \hfill (2.2)

where $\bar{\mathcal{J}} = \inf\{\tilde{\mathcal{J}}(G; \mathcal{S}_H) : G \in S_G\}$, $\alpha > 0$ and $c(\cdot)$ is a cost functional taking into account other properties of the MSE function $\mathcal{J}(G; \cdot) : \mathcal{S}_H \to \mathbb{R}$ other than its supremum.

The set $\mathcal{S}_H$ considered here is defined as a weighted $H_2$-ball centered on the nominal, frequency-response $H_0$, i.e.,

$$\mathcal{S}_H \triangleq \left\{ H \in \mathbb{R}^{m_v \times m_y} : \|H - H_0\|_2^2 \leq \gamma^2 \right\},$$  \hfill (2.3)

where the weighting function $W \in \mathbb{R}^{m_y \times m_y}$ is such that $W^{-1} \in \mathbb{R}^{m_y \times m_y}$. 

---

Figure 1: Estimation set-up.
Remark 2.1. $\mathcal{H}_\infty$–uncertainty on the channel frequency-response can also be cast (albeit conservatively) as $\mathcal{S}_H$ above with an appropriate choice of the set-up data. Indeed, given

$$\hat{\mathcal{S}}_H = \left\{ H \in \mathcal{R}_c^{m_y \times m_y} : \| (H - H_0) \hat{W} \|_\infty \leq \hat{\gamma} \right\},$$

where $\hat{W}$ and $\hat{W}^{-1} \in \mathcal{R}_c^{m_y \times m_y}$, note that $H\phi_y = H_0\phi_y + (H - H_0)\hat{W}(\hat{W}^{-1}\phi_y)$ and, hence,

$$\forall H \in \hat{\mathcal{S}}_H, \forall \alpha \in [0, 2\pi], \left\| (H - H_0) \phi_y \right\|_F^2 \leq \hat{\gamma}^2 \left\| \hat{W}^{-1} \phi_y \right\|_F^2 = \hat{\gamma}^2 |\phi_y (e^{j\alpha})|_F^2,$$

where $\phi_y(e^{j\alpha})^*\phi_y(e^{j\alpha}) = \text{tr} \left\{ [\hat{W}^{-1}\phi_y] (e^{j\alpha})^*[\hat{W}^{-1}\phi_y] (e^{j\alpha}) \right\}.$

Thus, whenever the function in the right-hand side of the last equation does not have zeros on $\{e^{j\alpha} : \alpha \in [0, 2\pi]\}$, $\phi_y(e^{j\alpha})$ can be taken to be a spectral factor and, hence, $\forall H \in \hat{\alpha} \in [0, 2\pi]$

$$\left\| (H - H_0) \phi_y \phi_y^{-1} (e^{j\alpha}) \right\|_F^2 \Rightarrow \left\| (H - H_0) \phi_y \phi_y^{-1} (e^{j\alpha}) \right\|_2^2.$$

Thus, given $\hat{W}$ and $\hat{\gamma}$, taking a spectral factorization

$$W(e^{j\alpha})W(e^{j\alpha})^* = [\phi_y \phi_y^{-1} (e^{j\alpha})][\phi_y \phi_y^{-1} (e^{j\alpha})]^*$$

and making $\gamma = \hat{\gamma}$, it follows that $\hat{\mathcal{S}}_H \subset \mathcal{S}_H$.

For $\mathcal{S}_H$ as in (2.3), in addition to the minimax estimator defined by

$$\text{Prob. 1: } \min_{G \in \mathcal{S}_G} \mathcal{J}(G; \mathcal{S}_H),$$

where $\mathcal{S}_G \subset \mathcal{R}_c^{m_y \times m_y}$, another robust estimator will be considered which is defined as in (2.2) above with $c(G)$ defined as a limit of the average values of $\mathcal{J}(G; \cdot)$ over classes of FIRs of increasing length – the latter is derived in Subsection 3.1.

### 2.2 Nominal $\mathcal{H}_\infty$–Filtering

The second estimation problem considered here is a nominal, “$\mathcal{H}_\infty$–filtering” problem in which the class of admissible estimators corresponds to linear systems with a prescribed maximum state-space dimension - $\mathcal{H}_\infty$ filtering problems have been widely considered (see, for example, [19] – [25] and references therein) and motivation for such problems vis-à-vis $\mathcal{H}_2$ filtering is briefly discussed in [25].

More specifically, consider the block-diagram of Figure 1 and, for $y \in \mathcal{R}_c^{m_y}$ and $v \in \mathcal{R}_c^{m_v}$, let

$$e(z; G, H_0) = H_1y - (GH_0y + Gv) = (H_{10} - GH_{01})z,$$

where $H_0$ is the “nominal” frequency-response, $H_{10} = [H_1 : 0_{m_x \times m_y}], H_{01} = [H_0 : I_{m_v}]$ and $z^T = [y^T : v^T]$.

Let $S_\alpha = \{ \alpha \in \mathcal{R}_c^{m_{\alpha}} : \| W_{\alpha} \|_2 \leq \gamma_\alpha \}, \quad \alpha = y, v$, and the weighting function, $W_\alpha \in \mathcal{R}_c^{m_{\alpha} \times m_{\alpha}}$ be such that $W_\alpha^{-1} \in \mathcal{R}_c^{m_{\alpha} \times m_{\alpha}}$. The “$\mathcal{H}_\infty$” filtering criterion $\mathcal{J}_\infty$ considered here is then defined as

$$\mathcal{J}_\infty(G; H_0) = \sup \{ \| e(z; G, H_0) \|_2 : z^T = [y^T : v^T], y \in S_y, v \in S_v \}.$$
To simplify a little the derivation to follow, let \( \bar{y} = W_y y, \bar{v} = W_v y, W_z = \text{diag}(W_v, W_v), \bar{z} = W_z z \) and \( \bar{e}(\bar{z}; G, H) = (H_{1y} - GH_{oz})\bar{z} \), where \( H_{1y} \triangleq H_{1o} W_z^{-1} = [H_1 W_y^{-1} : 0_{m_x \times m_v}] \) and \( H_{oz} \triangleq H_{0z} W_z^{-1} = [H_0 W_y^{-1} : W_v^{-1}] \). Then, for \( \bar{S}_\alpha = \{ \bar{\alpha} \in \mathcal{R}^m_{\alpha} : \|\bar{\alpha}\|_2 \leq \gamma_\alpha \} \), \( J_\infty \) can be rewritten as

\[
J_\infty(G; H_0) = \sup\{\|\bar{e}(\bar{z}; G, H_0)\|_2^2 : \bar{z}^T = [\bar{y}^T : \bar{v}^T], \bar{y} \in \bar{S}_y, \bar{v} \in \bar{S}_v\}.
\]

**Remark 2.2.** Note that the \( \mathcal{H}_\infty \)-norm of the error system \((H_{1y} - GH_{oz})\) is given by

\[
\sup\{\|\bar{e}(\bar{z}; G, H)\|_2^2 : \bar{z} \in \mathcal{R}^m_{\alpha} \times m_v, \|\bar{z}\|_2 \leq 1\}. \tag{2.4}
\]

Thus, the criterion \( J_\infty \) introduced here differs from the usual \( \mathcal{H}_\infty \) in that the \( \mathcal{H}_2 \) norms of the “information” \((y)\) and noise \((v)\) signals are independently bounded, rather than having their squared sum subject to a single upper bound – the alternative pursued here appears to be more natural in the signal processing setup of Figure 7.

For a given \( H \in \mathcal{R}^m_{c \times m_y} \) and \( S^*_G \subset \mathcal{R}^m_{c \times m_v} \) the nominal “\( \mathcal{H}_\infty \)-estimation” problem is then posed as follows

\[
\text{Prob. 2:} \quad \min_{G \in S^*_G} J_\infty(G; H). \tag{2.5}
\]

### 2.3 Robust \( \mathcal{H}_\infty \) Estimation

The third estimator-design problem tackled here is a robust “\( \mathcal{H}_\infty \)-estimation” problem in which the class of possible channel models is defined by weighted \( \mathcal{H}_\infty \)-balls of frequency responses. More specifically, let a nominal model \( H_0 \in \mathcal{R}^m_{c \times m_y} \) and the class \( S_{H_\infty} \) be given where

\[
S_{H_\infty} \triangleq \{ H \in \mathcal{R}^m_{c \times m_y} : \|H - H_0\|_H \leq \gamma_H \}, \quad W_H \in \mathcal{R}^m_{c \times m_y} \quad \text{is such that} \quad W_H^{-1} \in \mathcal{R}^m_{c \times m_y}. \]

The worst-case “\( \mathcal{H}_\infty \)-performance” of a given estimator \( G \) over \( S_{H_\infty} \) is given by

\[
\sup\{J_\infty(G; H) : H \in S_{H_\infty}\}, \quad \text{or, equivalently,}
\]

\[
\sup\left\{ \|\bar{e}(\bar{z}; G, H)\|_2^2 : \bar{z}^T = [\bar{y}^T : \bar{v}^T], \bar{y} \in \bar{S}_y, \bar{v} \in \bar{S}_v, H \in S_{H_\infty}\right\}. \tag{2.4}
\]

Rewriting for \( H_s \triangleq (H - H_0) \), \( \bar{e}(\bar{z}; G, H) = (H_{1y} - GH_{oz})\bar{z} - GH_y W_y^{-1} \bar{y} \) and defining \( w \triangleq H_s W_y^{-1} \bar{y}, \bar{z}_a = [\bar{z}^T : w^T], \) it follows that \( \bar{e}(\bar{z}; G, H) = e(z_a; G) \triangleq (H_{1a} - GH_{oz})z_a \), where \( H_{1a} \triangleq [H_1 W_y^{-1} : 0_{m_x \times m_v}], \) and \( H_{oz} \triangleq [H_0 W_y^{-1} : W_v^{-1} : I_{m_v}] \), so that (2.4) can be rewritten as

\[
\sup\left\{ \|e(z_a; G)\|_2^2 : z_a^T = [\bar{y}^T : \bar{v}^T : w], \bar{v} \in \bar{S}_v, (\bar{y}, w) \text{ is such that } \bar{y} \in \bar{S}_y \right\}. \tag{2.5}
\]

To obtain a more tractable optimization problem, the worst-case \( \mathcal{H}_\infty \)-performance index given by (2.5) will be replaced by an upper bound. To this effect, note that \( w = H_s W_y^{-1} \bar{y} = (H_s W_y)H_s W_y^{-1} \bar{y} \), where

\[
W_{hy} \triangleq (W_y W_h^{-1})^{-1}, \quad \text{and, hence, for any pair } (\bar{y}, w) \text{ as in (2.5), } \bar{y} \in \bar{S}_y \text{ and } \|w\|_2^2 \leq \gamma_H^2 W_{hy} \bar{y} \| y \|^2. \]

This
observation leads to the following upper bound on \( \sup \{ J_\infty(G; H) : H \in S_{\infty} \} \):

\[
J_\infty^a(G) \triangleq \sup \left\{ \| e(z_a; G) \|_2^2 : z_a^T = \left[ \bar{y}^T ; \bar{y}^T \right], \bar{v} \in \mathcal{S}_v, \bar{y} \in \mathcal{S}_y, ||w||_2^2 \leq \gamma^2 \| W_{ny} \bar{y} \|_2^2 \right\}
\]

and, for a given class \( S_G^a \subset \mathcal{R}^{m_e \times m_v} \) of admissible estimators, the corresponding robust estimator design problem is posed as

\[
\text{Prob. 3: } \min_{G \in S_G^a} J_\infty^a(G).
\]

### 2.4 Average Cost/Worst-Case Constraint Problem

As Prob. 1–3 are minimax problems, an alternative formulation for each of these problems is sought which allow for obtaining trade-offs between worst-case and point-wise performance in specific estimator-design exercises (leading to Prob. 3–6, respectively). To this effect, average criteria (say, \( \eta_{av} \) and \( \eta_{w} \)) associated to \( J(G; \cdot) : S_h \to \mathbb{R} \) and \( \| e(\cdot; G, H) \|_2^2 : \mathcal{S}_y \times \mathcal{S}_v \to \mathbb{R} \) are derived in the next section. Then, on the basis of approximate solutions to the minimax problems, linear classes of frequency-responses for admissible estimators are defined and average cost/worst-case constraint ("a/w", for short) estimation problems are posed. More specifically, let \( \bar{J}_a \) denote the optimal value of Prob. 1, i.e., \( \bar{J}_a = \inf \{ J(G; S_h) : G \in S_G \} \) and let \( G^o_1 \) denote an approximate solution of Prob. 1, i.e., \( G^o_1 \in S_G \) and \( \bar{J}(G^o_1; S_H) = (1 + \varepsilon) \bar{J}_a \) for a "small" \( \varepsilon > 0 \). For a minimal realization (\( A^o_1, B^o_1, C^o_1, D^o_1 \)) of \( G^o_1 \) let

\[
S_{G_n} = \{ G = D + CY^o_1B^o_1; C \in \mathbb{R}^{m_e \times n_1}, D \in \mathbb{R}^{m_e \times m_v} \},
\]

where \( Y^o_1(e^{j\phi}) = (e^{j\phi}I - A^o_1)^{-1}, A^o_1 \in \mathbb{R}^{n_1 \times n_1} \) (note that \( G^o_1 \in S_{G_n} \)).

A robust estimation problem is then posed as follows:

\[
\text{Prob. 4: } \min_{G \in S_{G_n}} \eta(G) \quad \text{subject to } \bar{J}_\infty(G) \leq (1 + \alpha) \bar{J}_a,
\]

where \( \alpha > \varepsilon \) — note that in Prob. 4 the worst-case performance of an estimator given by \( G \) (i.e., \( \bar{J}_\infty(G) \)) is allowed to be bigger than the minimum one (\( \bar{J}_a \)) by at the most \( \alpha \bar{J}_a \).

Similar considerations are brought to bear on Prob. 2 and 3 thereby leading to corresponding "a/w" estimation problems (Prob. 5 and 6). In Section 4 Prob. 4–6 are also converted into SDPs.

### 3 Average Performance Criteria

#### 3.1 Average MSE over \( H_2 \)-Balls of Frequency-Responses

In this case, the uncertain model class is \( S_h \) and the estimation criteria is \( J(G; H) \).

To simplify the derivation to follow, a simple change-of-variable is introduced replacing \( H \) by \( X \), where \( X \triangleq (H - H_o)W \). Thus, \( J(G; H) = J_X(G; X) \) and

\[
\bar{J}(G; S_H) = \bar{J}_X(G; S_X),
\]

(3.1)
where
\[ J_X(G; X) = \langle (X_0(G) - GX), Y_1 \rangle + \langle GY, G \rangle, \tag{3.2} \]
\[ X_0(G) \triangleq (H_1 - GH_0)W, \quad J_X(G; S_X) \triangleq \sup \{ J_X(G; X) : X \in S_X \}, \tag{3.3} \]
\[ S_X \triangleq \left\{ X : \mathcal{R}_{\mathbb{C}}^{m_x \times m_y} : \|X\|_2 \leq \gamma \right\} \quad \text{and} \quad \Gamma_{y_1} = W^{-1} \Gamma_y (W^{-1})^*. \tag{3.4} \]

To derive an average performance criterion to assess a given G over S_X (or, equivalently over S_H) a family of subspaces \( S_X^N \subseteq \mathcal{R}_{\mathbb{C}}^{m_x \times m_y} \) is considered which is such that for any \( X \in \mathcal{R}_{\mathbb{C}}^{m_x \times m_y}, \)
\[ \inf \left\{ \|X - \hat{X}\|_2 : \hat{X} \in S_X^N \right\} \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty, \quad \text{namely,} \]
\[ S_X^N \triangleq \left\{ X_N(\beta) \in \mathcal{R}_{\mathbb{C}}^{m_x \times m_y} : X_N(\beta) = \beta \bar{Y}_X^N, \beta \in \mathbb{R}^{m_x \times (n_x^N + m_y)} \right\}, \]
where \( \bar{Y}_X \triangleq \begin{bmatrix} Y_X^N & B_N \\ I_{m_y} \end{bmatrix}, \quad Y_X^N(e^{j\alpha}) \triangleq (e^{j\alpha}I - A_N)^{-1}, \rho(A_N) < 1, \quad (A_N, B_N) \text{ is controllable and} \]
\( A_N \in \mathbb{R}^{n_x \times n_x}. \)

**Remark 3.1.** One such family of subspaces corresponds to \( m_x \times m_y \) finite impulse responses (FIRs) of length \( N \). This family will be exploited in the derivation to follow.

Let \( \theta = \text{rvec}(\beta) \) and (with a slight abuse of notation) write \( X_N(\beta) \) as \( X_N(\theta) \). Let
\[ S^N_\theta \triangleq \left\{ \theta \in \mathbb{R}^{m_\theta} : X_N(\theta) \in S_X \right\}, \quad \text{where} \quad m_\theta \triangleq m_x (n_x^N + m_y). \] The average MSE attained with G over \( S_X^N \cap S_X^N \) (with respect to a uniform distribution on \( S^N_\theta \)) is given by
\[ \eta_N(G) = \left\{ \int_{S^N_\theta} J_X(G; X_N(\theta))d\theta \right\} \nu_N^{-1}, \quad \text{where} \quad \nu_N \triangleq \int_{S^N_\theta} d\theta \quad \text{and} \quad d\theta \quad \text{stands for} \quad d\theta_1 \ldots d\theta_{m_\theta}. \]

Note that \( S^N_\theta \) can be written as
\[ S^N_\theta = \left\{ \theta \in \mathbb{R}^{m_\theta} : \left\langle \beta P^N_\theta, \beta \right\rangle_F \leq \gamma^2 \right\}, \]
where \( P^N_\theta \triangleq (1/2\pi) \int_0^{2\pi} Y_X^N(e^{j\alpha})^*Y_X^N(e^{j\alpha})d\alpha \), or equivalently,
\[ S^N_\theta = \left\{ \theta \in \mathbb{R}^{m_\theta} : \theta^* P^N_\theta \theta \leq \gamma^2 \right\}, \quad \text{where} \quad P^N_\theta \triangleq I \otimes (P^N_\theta)^T. \]

Note also that (writing \( \Gamma_{y_1} = \phi_{y_1}^* \phi_{y_1} \)) \( GX_N(\theta) \phi_{y_1} = F_N(G)\theta \) so that it follows from (3.2) that
\[ J_X(G; X_N(\theta)) = J(G; H_0) - 2 \langle Z_y(G), F_N(G) \theta \rangle + \langle F_N(G) \theta, F_N(G) \theta \rangle, \]
where \( Z_y(G) \triangleq \text{rvec}(X_0(G) \phi_{y_1}), \quad F_N(G) \triangleq G \otimes (Y_X^N \phi_{y_1})^T. \)
As a result, \( \eta_N(G) \) can be written as
\[
\eta_N(G) = \left\{ J(G; H_0) \int_{S^N \theta} \eta N d\theta - 2 \int_{S^N \theta} \xi_J^N(G) \theta d\theta + \int_{S^N \theta} \theta^T P_J^N(G) \theta d\theta \right\} \nu_N^{-1},
\]
where \( \xi_J^N(G)^T = (1/2\pi) \int_0^{2\pi} \{ Z_\theta(G)^* F_N(G) \} (e^{i\alpha}) d\alpha \) and \( P_J^N(G) \triangleq (1/2\pi) \int_0^{2\pi} \{ F_N(G)^* F_N(G) \} (e^{i\alpha}) d\alpha \).

It then follows from the fact that \( S^N \theta \) is symmetric with respect to the origin that \( \int_{S^N \theta} \xi_J^N(G)^T \theta d\theta = 0 \) so that
\[
\eta_N(G) = J(G; H_0) + \nu_N^{-1} \int_{S^N \theta} \theta^T P_J^N(G) \theta d\theta.
\]

Consider the following proposition (a proof is presented in the Appendix).

Proposition 3.1. Let \( n \triangleq m^N_\theta \) and \( \tilde{P}_J^N(G) \triangleq (P_J^N)^{-1/2} P_J^N(G)(P_J^N)^{-1/2} \):

(a) \( \nu_N = \left| \det \left( P_J^N \right) \right|^{-1/2} \prod_{k=1}^n \tilde{P}_J^N(G)_{kk} \left| \det \left( P_J^N \right) \right|^{-1/2} \times \prod_{i=k}^K \int_0^\pi \sin(\alpha_i)^{n-1-i} d\alpha_i, \ K \geq k, \ K \leq n-2, \)

(b) \[
\int_{S^N \theta} \theta^T P_J^N(G) \theta d\theta = \left\{ \sum_{k=1}^n \tilde{P}_J^N(G)_{kk} \right\} \left| \det \left( P_J^N \right) \right|^{-1/2} \times \prod_{i=k}^K \int_0^\pi \cos(\alpha_i)^2 \sin(\alpha_i)^{n-2} d\alpha_i, \prod_{k=2n-2}^{2n} \right\}.
\]

(c) \( \eta_N(G) = J(G; H_0) + \frac{\gamma^2}{n+2} \sum_{k=1}^n \tilde{P}_J^N(G)_{kk} \).

To obtain an explicit expression for \( \eta_{av}(G) = \lim_{N \to \infty} \eta_N(G) \), \( S^N_x \) is taken to be the set of frequency-responses corresponding to FIRs of length \( N \) (i.e., impulse responses \( \{ F_k : k = 0, 1, \ldots \} \) such that \( \forall k > N, F_k = 0 \)), with state-space realizations \( (A^N_x, B^N_x, C, D) \) given by \( A^N_x = \text{diag} \left( A^N_c, \ldots, A^N_c \right) \),
\[
B^N_x = \text{diag} \left( b^N_c, \ldots, b^N_c \right) \in \mathbb{R}^{(N-1)m_y \times m_y}, \ A^N_c = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \ b^N_c = e_i(N), \ n^N_x = Nm_y,
\]
\[
A^N_x \in \mathbb{R}^{N_x \times N_x}, \ C = [F_1 e_1(m) \cdots F_N e_1(m) : \cdots : F_1 e_m(\pi) \cdots F_N e_m(\pi)] \in \mathbb{R}^{m_y \times N}, \ D = F_0 \in \mathbb{R}^{m_y \times m_y} \text{ and } n = m^N_\theta = m_x m_y (N+1).
\]

The desired limit is presented in the following proposition.

Proposition 3.2. The average MSE criterion over \( S^N \) defined by \( \eta_{av}(G) \triangleq \lim_{N \to \infty} \eta_N(G) \) is given by
\[
\eta_{av}(G) = J(G; H_0) + \left( \gamma^2 / (m_x m_y) \right) \left( G \otimes \phi_{y1}^T, G \otimes \phi_{y1}^T \right),
\]
where \( S^N \) is given by \( \{2, 3\} \) and \( \phi_{y1}, \phi_{y1}^* = W^{-1} \Gamma_y(W^{-1})^* \). \( \nabla \)
Remark 3.2. The “average” criterion \( \eta_{av}(G) \) consists of the nominal MSE \( \langle J(G; H_0) \rangle \) supplemented by a “channel-output noise” term reflecting the “effect” of the channel model perturbations on the input signal’s power spectral density \( \Gamma_y = \phi_y \phi_y^* \) – indeed, in the SISO case, the additional term \( \gamma^2(\phi_{y_1}; \phi_{y_2}) \) is exactly like the observation noise term of \( J(G; H_0) \)  (i.e., \( \langle G \Gamma_y, G \rangle \) with \( \Gamma_y \) replaced by \( \gamma^2 W^{-1} \Gamma_y(W^{-1})^* \). ∇

3.2 Average Estimation Error Over \( \mathcal{H}_2 \) Signal Balls

In this case, the estimation error magnitude for a pair of signals \( (\tilde{y}, \tilde{v}) \) is given by \( \| \tilde{e}(\tilde{z}; G, H) \|_2^2 \) where \( \tilde{z}^T = [\tilde{y}^T \tilde{v}^T] \) and the corresponding average criterion is to be defined with respect to the set \( \tilde{S}_\alpha = \tilde{S}_y \times \tilde{S}_v \), where \( \tilde{S}_\alpha, \alpha = \tilde{y}, \tilde{v} \) is as defined in Section 2. To this effect, consider FIR subsets of \( \tilde{S}_\alpha \), namely,

\[
\{ \alpha_\alpha(\beta_\alpha) = Y_{\alpha N}^{\alpha} \beta_\alpha, \beta_\alpha \in S_\alpha \}, \quad \text{where} \quad S_N^{\alpha} \triangleq \{ \beta_\alpha \in \mathbb{R}^{n_{\alpha N} + m_{\alpha}} : \| \alpha_\alpha(\beta_\alpha) \|_2 \leq \gamma_\alpha \},
\]

\[
Y_{\alpha N}^{\alpha} \triangleq [C_{\alpha N} Y_{\alpha N}^{\alpha} : I_{m_{\alpha}}], \quad Y_{\alpha N}^{\alpha} e(i\tilde{\phi}) = (e^{i\tilde{\phi}} I - A_{\alpha N})^{-1}, \quad A_{\alpha N} = \tilde{A}_{\alpha N}^T \quad \text{and} \quad C_{\alpha N} = \tilde{B}_{\alpha N}^T, \quad n_{\alpha N} = N m_{\alpha},
\]

where \( \tilde{A}_{\alpha N} \) and \( \tilde{B}_{\alpha N} \) are given by \( A_{\alpha N} \) and \( B_{\alpha N} \) above with \( m_y \) replaced by \( m_{\alpha} \).

The average value of the squared, estimation error over these signal sets is given by

\[
\eta_{av}^a(G; H) \triangleq \left\{ \int_{S_N^v} \int_{S_N^y} \| \tilde{e}(\tilde{z}_N(\beta_y, \beta_v); G, H) \|_2^2 d\beta_v d\beta_y \right\} (\mu_{av}^a)^{-1},
\]

where \( \tilde{z}_N(\beta_y, \beta_v)^T \triangleq [\tilde{y}_N(\beta_y)^T : \tilde{v}_N(\beta_v)^T] \), \( \mu_{av}^a \triangleq \int_{S_N^v} \int_{S_N^y} d\beta_v d\beta_y \), and \( d\beta_{\alpha} \triangleq d\beta_1, \ldots, d\beta_{n_{\alpha N} + m_{\alpha}} \) or, equivalently,

\[
\mu_{av}^a \eta_{av}^a(G; H) = \int_{S_N^y} \int_{S_N^v} \left( H_{ty} - G H_{oz} \right) \begin{bmatrix} \tilde{y}_N(\beta_y) \\ \tilde{v}_N(\beta_v) \end{bmatrix} \|_{2}^2 d\beta_v d\beta_y \quad \iff
\]

\[
\mu_{av}^a \eta_{av}^a(G; H) = \int_{S_N^y} \int_{S_N^v} \left( H_{ty} - G H_{oz} \right) \begin{bmatrix} Y_{\alpha N}^{av} 0 \\ 0 Y_{\alpha N}^{av} \end{bmatrix} \begin{bmatrix} \beta_y \\ \beta_v \end{bmatrix} \|_{2}^2 d\beta_v d\beta_y \quad \iff
\]

\[
\mu_{av}^a \eta_{av}^a(G; H) = \int_{S_N^y} \int_{S_N^v} [\beta_y^T : \beta_v^T] \Gamma_{av}^e \begin{bmatrix} \beta_y \\ \beta_v \end{bmatrix} d\beta_v d\beta_y,
\]

where \( \Gamma_{av}^e \triangleq (1/2\pi) \int_0^{2\pi} F_{GY} e(i\phi)^* F_{GY} (e^{i\phi}) d\phi \) and \( F_{GY} \triangleq (H_{ty} - G H_{oz}) \text{diag}(Y_{\alpha N}^{av}, Y_{\alpha N}^{av}) \).

Thus, for \( \Gamma_{av}^e \triangleq \begin{bmatrix} \Gamma_{av}^{ey} \\ \Gamma_{av}^{ey} \end{bmatrix} \),

\[
\mu_{av}^a \eta_{av}^a(G; H) = \int_{S_N^y} \int_{S_N^v} (\beta_y^T \Gamma_{av}^{ey} \beta_y + 2\beta_y^T \Gamma_{av}^{ey} \beta_v + \beta_v^T \Gamma_{av}^{ey} \beta_v) d\beta_v d\beta_y \quad \iff
\]

\[
\mu_{av}^a \eta_{av}^a(G; H) = \mu_{av}^a \int_{S_N^v} \beta_y^T \Gamma_{av}^{ey} \beta_y + \mu_{av}^a \int_{S_N^v} \beta_v^T \Gamma_{av}^{ey} \beta_v d\beta_v.
\]
where \( \mu^a_{\alpha N} \triangleq \int_{S^u_N} d\beta_\alpha \) (note that \( \int_{S^u_N} \int_{S^e_N} 2\beta_y \Gamma e_{N\nu} \beta_y d\beta_y = 0 \) since \( \beta_v \in S^v u \iff -\beta_v \in S^v u \)).

Thus, as \( \mu^a y = \mu^a u \mu^a e \), \( \eta^a_{\alpha N}(G, H) = \eta^a_{\nu N}(G; H) + \eta^a_{\omega N}(G; H) \), where \( \eta^a_{\nu N} = (\mu^a y N)^{-1} \int_{S^u_N} 2\beta_y \Gamma e_{N\nu} \beta_y d\beta_y \) and \( \eta^a_{\omega N} = (\mu^a e N)^{-1} \int_{S^e_N} \beta_y \Gamma e_{N\nu} \beta_y d\beta_y \).

As a result, pursuing the path that led to \( \eta \) leads to the following proposition.

**Proposition 3.3.**

\[
\eta^a_{\alpha N} = \left( \frac{\gamma^2}{m_{\alpha N} + 2} \right) (N + 1) \langle \bar{A}_\alpha(G; H), \bar{A}_\alpha(G; H) \rangle,
\]

where \( m_{\alpha N} = (N + 1)m_\alpha \), \( \bar{A}_y(G; H) = (H_1 - GH)W_\nu^{-1} \) and \( \bar{A}_v(G; H) = GW_\nu^{-1} \).

Moreover, defining \( \eta^a(G; H) = \lim_{N \to \infty} \eta^a_{\alpha N}(G; H) \), it follows that

\[
\eta^a(G; H) = (\gamma^2/m_y) \langle \bar{A}_y(G; H), \bar{A}_y(G; H) \rangle + (\gamma^2/m_v) \langle \bar{A}_v(G; H), \bar{A}_v(G; H) \rangle.
\]

\[\nabla\]

**Remark 3.3.** Note that the limit process yielding the “average” criterion \( \eta^a(G; H) \) naturally led to the estimation error due to the deterministic signals in (the weighted \( H_2 \)-balls) \( S_y \) and \( S_v \) being represented as the estimation MSE due to stochastic signals with power spectral densities \( (\gamma^2/m_y)(W^*W_y)^{-1} \) and \( (\gamma^2/m_v)(W^*W_v)^{-1} \).

\[\nabla\]

### 3.3 Average Criterion for the Robust \( H_\infty \) Problem

The results in Subsections 3.1 and 3.2 are now combined in a simple way to yield a cost-functional that takes into account the average estimation error over \( H_2 \) signals balls and the “channel-model” set

\[
S_{H_\infty} = \{ H \in \mathbb{R}_{c}^{m_y \times m_y} : \| (H - H_0)W \|_\infty \leq \gamma H \}
\]

introduced in Subsection 2.3. This is done, in line with the derivation of \( \eta^a(\cdot) \) and Remark 3.3, by viewing the signal balls \( S_y \) and \( S_v \) as “formally equivalent” (for the purpose of defining an average criterion) to stochastic signals with power spectral densities \( \Gamma^a y = \phi^a y (\phi^a y)^* \) and \( \Gamma^a v = \phi^a v (\phi^a v)^* \), where \( \phi^a y = (\gamma y / \sqrt{m_y})W_\nu^{-1} \) and \( \phi^a v = (\gamma v / \sqrt{m_v})W_\nu^{-1} \), and by (conservatively) taking into account the set \( S_{H_{\infty}} \) by means of a \( H_2 \)-ball of frequency-responses, as described in Remark 2.1, namely,

\[
\bar{S}_{H_\infty} = \{ H \in \mathbb{R}_{c}^{m_y \times m_y} : \| (H - H_0)\bar{W} \|_2 \leq \gamma H \},
\]

where \( \bar{W} = \phi^a y \phi^{-1}_{y} \) and \( \phi^a y \) is a spectral factor of

\[
\phi^a y (e^{j\alpha}) = \text{tr}\{(W^{-1}_H \phi^a y (e^{j\alpha})^*[W^{-1}_H \phi^a y (e^{j\alpha})]\}
\]
Then, replacing $\gamma$, $\Gamma_y$, $\Gamma_v$, and $\phi_y$ respectively by $\gamma_H$, $\Gamma_y$, $\Gamma_v$, and $\tilde{\phi}_y = \tilde{W}^{-1}\phi_y$ in the expression of $\eta_{av}(G)$ leads to (since $\tilde{\phi}_y = \phi_y W I_m$)

$$
\eta^b(G; S_{H\infty}) = \eta^a(G; H_0) + \left(\gamma_H^2/m_y m_v\right) \left(G \otimes (\phi_y W I_m), G \otimes (\phi_y W I_m)_y\right)
$$
or, equivalently,

$$
\eta^b(G; S_{H\infty}) = \eta^a(G; H_0) + \left(\gamma_H^2/m_v\right) \left(G \phi_y W, G \phi_y W\right).
$$

Note that $\eta^b(\cdot)$ consists of the average criterion for the nominal $H_{\infty}$ problem plus an additive term which takes into account the weighting function $W_H$ (by means of $\phi_y W$) and the $H_{\infty}$-uncertainty radius $\gamma_H$.

4 Minimax Estimators

In this section, the minimax problems $Prob. 1 - 3$ are recast as SDPs.

4.1 Minimax $H_2$ Estimators with $H_2$ Model Uncertainty

The first problem to be considered in this section is formulated as follows:

$$
Prob. 1 : \min_{G \in S_{\mathcal{G}}} \bar{J}(G; S_{H}),
$$

where $S_{H} = \{ H \in \Re_{c \times m_y} : \| (H - H_0) W \|_2 \leq \gamma \}$ and $S_{\mathcal{G}}$ is a subset of $\Re_{c \times m_y}$, or, equivalently (cf. (3.1) – (3.4))

$$
Prob. 1 : \min_{G \in S_{\mathcal{G}}} \bar{J}_X(G; S_{X}),
$$

where $\bar{J}_X$ and $S_{X}$ are defined by (3.1) – (3.4).

The major aim of this Subsection is to recast $Prob. 1$ as an SDP. This was also carried out in [18], but the SDP introduced here is simpler than the one previously obtained as one of the LMIs involved in the latter was eliminated. This, together with the fact that the simplified conditions are a part of the average cost/worst-case constraint problem below, provides motivation for presenting the modified SDP here.

Proceeding as in [18], the first step is to introduce the Lagrangian and dual functionals

$$
Lag(X, \lambda; G) = J_X(G; X) - \lambda \left\{ \| X \|_2^2 - \gamma^2 \right\}
$$

and

$$
\varphi_D(\lambda; G) \triangleq \sup \left\{ Lag(X, \lambda; G) : X \in \Re_{c \times m_y} \right\},
$$

so that Theorem 2 of [12] can be invoked to yield

$$
\bar{J}_X(G; S_{X}) = \inf \left\{ \varphi_D(\lambda; G) : \lambda > 0 \right\}.
$$
To facilitate the derivation to follow, \( \mathcal{J}_X(G;X) \) and \( \text{Lag}(X,\lambda;G) \) are rewritten as

\[
\mathcal{J}_X(G;X) = \| X_0(G)F_y - GXF_y + GF_v \|^2_2
\]

and

\[
\text{Lag}(X,\lambda;G) = \lambda \gamma^2 - L_a(X,\lambda;G),
\]

where

\[
L_a(X,\lambda;G) = \left\langle \begin{bmatrix} \lambda I & 0 \\ 0 & -I \end{bmatrix} \right| \begin{bmatrix} XF_w \\ GXF_y \end{bmatrix} - A_o(G) \right| \begin{bmatrix} XF_w \\ GXF_y \end{bmatrix} - A_0(G) \right>,
\]

\[
F_y \triangleq \begin{bmatrix} W^{-1} \phi_y & 0_{m_y \times m_y} \end{bmatrix}, \quad F_v \triangleq \begin{bmatrix} 0_{m_y \times m_y} & \phi_v \end{bmatrix}, \quad F_w \triangleq \begin{bmatrix} I_{m_y} & 0_{m_y \times m_y} \end{bmatrix} \text{ and } A_o(G) \triangleq \begin{bmatrix} 0 \\ X_0(G)F_y + GF_v \end{bmatrix}, \text{ or, equivalently, for } Z = \text{rvec}(X)
\]

\[
L_a(Z,\lambda;G) = \left\langle M(\lambda)FZ - \mathcal{X}_0(G), FZ - \mathcal{X}_0(G) \rightangle,
\]

where \( M(\lambda) = \text{diag}(\lambda I_{m_x} - I_{m_x}) \otimes I_{(m_y + m_y)}, \ F = \begin{bmatrix} I_{m_y} \otimes F_w^T \\ G \otimes F_v^T \end{bmatrix} \) and \( \mathcal{X}_0(G) = \text{rvec}(A_0(G)) \).

Note that it follows from (4.2), (4.4) and (4.5) that

\[
\varphi_\Delta(\lambda;G) = \lambda \gamma^2 - \inf \left\{ L_a(Z,\lambda;G) : Z \in \mathcal{R}^{m_y \times m_y}_c \right\}.
\]

To proceed towards the conversion of Prob. 1 into an SDP, the range of \( \lambda \) in (4.3) is restricted to a set of values \( \mathcal{S}_\lambda \), say) over which the inf of \( L_a(\cdot) \) (see (4.6)) can be recast as the maximum of a linear functional under a matrix inequality constraint (in the light of Lemma A1, [17]). This is stated in the following proposition.

**Proposition 4.1.** (a) \( \tilde{J}(G;\mathcal{S}_\lambda) = \inf \left\{ \varphi_\Delta(\lambda;G) : \lambda \in \mathcal{S}_\lambda \right\}, \) where

\[
\mathcal{S}_\lambda \triangleq \{ \lambda > 0 : \forall \phi \in [0,2\pi], \left\{ F^*M(\lambda)F \right\}e^{i\phi} > 0 \}.
\]

(b) For \( \lambda \in \mathcal{S}_\lambda \),

\[
\inf \{ L_a(Z,\lambda;G) : Z \in \mathcal{R}^{m_y \times m_y}_c \} = \sup \{ x_o^T P x_o : P = P^T \text{ and } Q_{LQ}(P;\Sigma_a, M(\lambda)) > 0 \},
\]

where \( Q_{LQ}(P;\Sigma_a, M(\lambda)) = Q_\mathcal{J}(P;A,B) + S(\Sigma_a, M(\lambda)), \) \( \Sigma_a = (A_a, [B_b : b], C_a, [D_d : d]) \) is a realization of

\[
F = \begin{bmatrix} F_w \\ G \end{bmatrix}, \quad A = \begin{bmatrix} A_a & b_a \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_a \\ 0 \end{bmatrix}, \quad x_o = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \rho(A_a) < 1, \quad R = \begin{bmatrix} C_a & d_a & D_d \end{bmatrix},
\]

\[
Q_{\mathcal{J}}(P;A,B) \triangleq \begin{bmatrix} A^T PA - P & A^T PB \\ B^T PA & B^T PB \end{bmatrix} \text{ and } S(\Sigma_a, M) \triangleq R^T MR.
\]

(c) \( \lambda \in \mathcal{S}_\lambda \) if and only if there exists \( P = P^T \) such that \( Q_{LQ}(P;\Sigma_a; M(\lambda)) > 0. \) \( \nabla \)
In the light of (4.2) – (4.6) and Proposition 4.1, \( \tilde{J}_X(G;S_X) \) can be written as
\[
\tilde{J}_X(G;S_X) = \inf \left\{ \lambda \gamma^2 - \inf \left\{ L_a(Z, \lambda, G) : Z \in \mathcal{R}_{c}^{m_u \times m_y} \right\} : \lambda \in S_X \right\} \quad \Leftrightarrow \\
\tilde{J}_X(G;S_X) = \inf \left\{ \lambda \gamma^2 - \sup \left\{ x_o^T P x_o : P = P^T, \quad Q_{LQ}(P; \Sigma_a, M(\lambda)) > 0 \right\} : \lambda \in S_X \right\} \quad \Leftrightarrow \\
\tilde{J}_X(G;S_X) = \inf \left\{ \lambda \gamma^2 + \inf \left\{ -x_o^T P x_o : P = P^T, \quad Q_{LQ}(P; \Sigma_a, M(\lambda)) > 0 \right\} : \lambda \in S_X \right\} \quad \Leftrightarrow \\
\tilde{J}_X(G;S_X) = \inf \left\{ \lambda \gamma^2 + x_o^T (-P)x_o : \lambda > 0, \quad P = P^T, \quad Q_{LQ}(P; \Sigma_a, M(\lambda)) > 0 \right\}.
\] (4.7)

The nonlinear term \( S(\Sigma_a, M(\lambda)) \) in the matrix inequality above can be eliminated on the basis of the Schur complement formula. To this effect, note that
\[
\begin{bmatrix}
F & -X_o(G)
\end{bmatrix} = \begin{bmatrix}
I_{m_u} \otimes F_w^T & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
0 \\
I
\end{bmatrix} F_X(G),
\]
where \( F_X(G) \triangleq \begin{bmatrix} G \otimes F_y^T & -rvec(X_o(G)F_y + GF_y) \end{bmatrix} \), so that, for a minimal realization
\[
\left( A_a, \begin{bmatrix} B_a & b_a \end{bmatrix}, \widehat{C}_a, \begin{bmatrix} \widehat{D}_a \mid \widehat{d}_a \end{bmatrix} \right) \text{ of } F_X(G), \quad C_a = \begin{bmatrix} 0 \\
I \end{bmatrix} \widehat{C}_a, \quad D_a = \begin{bmatrix} I_{m_u} \otimes F_w^T \\
\widehat{D}_a \end{bmatrix}, \quad \begin{bmatrix} 0 \\
\widehat{d}_a \end{bmatrix},
\]
\[
R = \begin{bmatrix} R_1 & R_2 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0_{m_u \times (m_y + m_u)} \\
0_{m_u \times (m_y + m_u) \times 1}
\end{bmatrix} : I_{m_u} \otimes F_w^T, \quad R_2 = \begin{bmatrix} \widehat{C}_a \mid \widehat{d}_a \mid \widehat{D}_a \end{bmatrix} \text{ and, hence, } S(\Sigma_a, M(\lambda)) = \lambda R_1^T R_1 - R_2^T R_2.
\]
As a result,
\[
Q_{LQ}(P; \Sigma_a, M(\lambda)) > 0 \quad \Leftrightarrow \quad \begin{bmatrix}
Q_{\beta}(P; A, B) + \lambda R_1^T R_1 & R_2^T \\
R_2 & I
\end{bmatrix} > 0
\]
so that (4.7) can be rewritten (\( Q = -P \)) as
\[
\tilde{J}_X(G;S_X) = \inf \left\{ \lambda \gamma^2 + x_o^T Q x_o : \lambda > 0, \quad Q = Q^T, \quad Q_{\beta}(Q; \Sigma_a, M(\lambda)) < 0 \right\},
\] (4.8)
where
\[
Q_{\beta}(Q; \Sigma_a, M(\lambda)) = \begin{bmatrix}
Q_{\beta}(Q; A, B) - \lambda R_1^T R_1 & R_2^T \\
R_2 & -I
\end{bmatrix}.
\] (4.9)

Confining estimators’ frequency-responses to a finite-dimensional subspace of \( \mathcal{R}_{c}^{m_u \times m_u} \) (see Remark 4.2), Prob. 1 can be converted to an SDP on the basis of (4.8). Indeed, let \( \mathcal{S}_G \) be defined as
\[
\mathcal{S}_G = \{ G = D + C Y_G B_G : C \in \mathbb{R}^{m_u \times n_G}, \quad D \in \mathbb{R}^{m_u \times m_u} \},
\] (4.10)
where \( Y_G(e^{j\phi}) = (e^{j\phi} I - A_G)^{-1}, \quad A_G \in \mathbb{R}^{n_G \times n_G}, \rho(A_G) < 1, \quad B_G \in \mathbb{R}^{n_G \times m_u}, \quad (A_G, B_G) \) controllable, or equivalently,
\[
\mathcal{S}_G = \left\{ G(\beta) = \beta Y_G^a : C = \begin{bmatrix} C & D \end{bmatrix} \in \mathbb{R}^{m_u \times (n_G + m_u)} \right\},
\]
where \( Y_G^a = \begin{bmatrix} Y_G B_G \\
I_{m_u}
\end{bmatrix} \). In this case, as \( X_o(G)F_y - GF_y = (H_I - GH_o)WF_y - GF_y, \)
\[
F_x(G) = \left[ (\beta \otimes I_{(m_y + m_x)}) (Y^a_G \otimes F_y^T) \right] \colon X_{1w} + \left( \beta \otimes I_{(m_y + m_x)} \right) X_{0w}, \quad \text{where } X_{1w} = -\text{rvec}(H_1WF_y),
\]
\[
X_{0w} = \text{rvec}\{ Y^a_G(H_{0w} - F_y) \} \quad \text{and} \quad H_{0w} \triangleq H_0WF_y, \text{or, equivalently, } F_x(G) = A_r(\beta)F^o_x, \text{ where}
\]
\[
A_r(\beta) \triangleq \left[ \beta \otimes I : I \right] \quad \text{and} \quad F^o_x = \begin{bmatrix} Y^a_G \otimes F_y^T : X_{0w} \\ 0 : X_{1w} \end{bmatrix}
\]
so that, letting \( (A_a, [B_a : b_a], C_{x'}, [D_x : d_x]) \) denote a minimal realization of \( F^o_x \),
\[
R_x(\beta) = \begin{bmatrix} \hat{C}_a : \hat{d}_a : \hat{D}_a \end{bmatrix} = A_r(\beta) \begin{bmatrix} C_{x'} : d_x : D_x \end{bmatrix}.
\]

\( \text{Prob. 1 can then be recast as an SDP as stated in the following proposition (it follows immediately from } \text{(4.8) } \text{-- } \text{(4.12)}. \)

**Proposition 4.2.** \( \text{Prob. 1 can be recast as the following SDP:} \)
\[
\min_{\beta, P=P^T, \lambda > 0} \lambda \gamma^2 + x_o^T P x_o \quad \text{subject to} \quad Q_{jX}(P, \lambda, \beta) < 0,
\]
where \( Q_{jX}(P, \lambda, \beta) \triangleq \begin{bmatrix} Q_j(P; A, B) - \lambda R^T_1 R_1 & R_2(\beta) \\ R_2(\beta) & -I \end{bmatrix}, \quad R_2(\beta) \text{ is an affine function of } \beta, \)
\( R_1 = [0 : 0 : I_{m_y} \otimes F_{1w}^T] \) and \( (A_a, [B_a : b_a], C_{x'}, [D_x : d_x]) \) is a minimal realization of \( F^o_x \) (given by \( \text{(4.11)}. \))

\( \nabla \)

**Remark 4.1.** Once \( \beta_o \) is obtained from a solution \( (\beta_o, P_o, \lambda_o) \) to the problem posed in Proposition \( \text{4.2, } \) the corresponding minimax estimator has frequency-response given by \( G(\beta_o) = \beta_o Y^a_G. \)

**Remark 4.2.** Taking \( \mathcal{S}_G \) to be a linear subspace of \( \mathcal{R}_c^{m_x \times m_y} \) is instrumental to the recasting of Prob. 1 as a SDP. It would be quite natural and conceptually somewhat preferable to take \( \mathcal{S}_G \) to be the subset of \( \mathcal{R}_c^{m_x \times m_y} \) corresponding to state-space realizations of a prescribed maximum dimension. This was indeed done in connection with robust estimator design problems involving the MSE and \( H_\infty \) criteria and parametric uncertainty classes (e.g., [6]) without precluding their recasting as SDPs – this is also the case with the nominal and robust \( H_\infty \) estimation problems tackled in Subsections \( \text{4.2 } \text{and } \text{4.3 } \) below. However, in the case of the MSE criterion and non-parametric, \( H_2 \)-ball, channel-model uncertainty addressed here, the approach pursued in the subsequent sections to achieve the desired conversions into SDPs (hinging upon the so-called Elimination Lemma) would not seem to be applicable beyond the SISO case.

Further justification for taking \( \mathcal{S}_G \) as in \( \text{(4.10)}. \) comes from the fact that any frequency-response in \( \mathcal{R}_c^{m_x \times m_y} \) can be approximated (as close as desired) with respect to the \( H_2 \)-norm in classes of FIRs (of sufficiently large length) and as mentioned above, such a class can be cast in the form of \( \text{(4.10)}. \) In addition, computing the optimal (over the whole of \( \mathcal{R}_c^{m_x \times m_y} \)) nominal MSE estimator (say, \( G_o \) with minimal real-
ization \((A_o, B_o, C_o, D_o)\) leads to a “problem-specific” and well-motivated class of admissible estimators (say, \(S_{G}^{\text{nom}}\) as in (4.10) with \(A_G = A_o, B_G = B_o\), as done in the example presented in Subsection 6.2)

Finally, it is noted that lower bounds on \(\mu_c(S_X) = \inf \{ \tilde{I}(G; S_X) : G \in \mathcal{R}_{c}^{m_x \times m_y} \} \) can be obtained as optimal values of SDPs (see [18]) so that, in any given estimation exercise, upper bounds can be obtained on the increase of the minimax MSE brought about by confining estimators to \(S_G\) instead of optimizing over the whole of \(\mathcal{R}_{c}^{m_x \times m_y}\) — in fact, as illustrated in [18], the class \(S_{G}^{\text{nom}}\) mentioned above may lead to optimal minimax performance which is quite close to \(\mu_c(S_X)\).

\[ \nabla \]

4.2 Nominal “\(H_\infty\)–Estimation”

Let \(H \in \mathcal{R}_{c}^{m_x \times m_y}, S_{G}^{o} \subset \mathcal{R}_{c}^{m_x \times m_y}\) and consider the “\(H_\infty\)–filtering” problem

\[
\text{Prob. 2 : } \min_{G \in S_{G}^{o}} \tilde{J}_\infty(G; H).
\]

The major aim of this section is to show that, for a given \(n_G\) and

\[ S_{G}^{o} = \{ G \in \mathcal{R}_{c}^{m_x \times m_y} : G \text{ has a realization } (A_G, B_G, C_G, D_G) \text{ with } A_G \in \mathbb{R}_{c}^{n_G \times n_G}, \rho(A_G) < 1 \}, \]

a solution to Prob. 2 can be obtained on the basis of SDPs.

To this effect and proceeding along the lines which led to Proposition 4.1, let a Lagrangian and dual functional be given by

\[
\text{Lag}_\infty(\tilde{z}, \sigma; G, H) = \| \tilde{z} \|_2^2 - \sigma_y(\| \tilde{y} \|_2^2 - \gamma_y^2) - \sigma_v(\| \tilde{v} \|_2^2 - \gamma_v^2)
\]

and \(\varphi_{D_\infty}(\sigma; G, H) \triangleq \sup \{ \text{Lag}_\infty(\tilde{z}, \sigma; G, H) : \tilde{z} \in \mathcal{R}_{c}^{m_x} \}\), where \(\sigma = (\sigma_y, \sigma_v)\) and \(m_z = m_y + m_v\).

It then follows from Theorem 2 in [12] that

\[
\tilde{J}_\infty(G; H) = \varphi_{D_\infty}(G; H) \triangleq \inf \{ \varphi_{D_\infty}(\sigma; G, H) : \sigma > 0 \}.
\]

Note now that

\[
\text{Lag}_\infty(\tilde{z}, \sigma; G, H) = \langle (H_{1y} - GH_{0})\tilde{z}, (H_{1y} - GH_{0})\tilde{z} \rangle - \langle M_{\sigma} \tilde{z}, \tilde{z} \rangle + \sigma_y \gamma_y^2 + \sigma_v \gamma_v^2,
\]

where \(H_{1y}\) and \(H_0\) are defined in Subsection 2.2 and \(M_{\sigma} = \text{diag}(\sigma_y I_{m_x}, \sigma_v I_{m_v})\). Thus, the dual functional \(\varphi_{D_\infty}\) can be written as

\[
\varphi_{D_\infty}(\sigma; G, H) = \sigma_y \gamma_y^2 + \sigma_v \gamma_v^2 - \hat{\varphi}_{D_\infty}(\sigma; G, H),
\]

where \(\hat{\varphi}_{D_\infty}(\sigma; G, H) = \inf \{ \langle (M_{\sigma} - F_{G}^* F_{G})\tilde{z}, \tilde{z} \rangle : \tilde{z} \in \mathcal{R}_{c}^{m_y} \}\) and \(F_{G} \triangleq H_{1y} - GH_{0}\).

To proceed, consider the following proposition.
Proposition 4.4. \( \hat{\varphi}_{D_\infty}(G;H) \) can be written as
\[
\hat{\varphi}_{D_\infty}(G;H) = \inf \{ \sigma_y^2 + \sigma_v^2 : \sigma_y > 0, \sigma_v > 0 \quad \text{and} \quad \forall \phi \in [0, 2\pi], \quad (M_\sigma - F_G^e F_G)(e^{j\phi}) > 0 \}. \quad \nabla
\]

In the light of Proposition 4.3, the next proposition is an immediate consequence of the so-called (discrete-time) bounded-real lemma ([27]).

**Proposition 4.4.**

\[
\mathcal{J}_\infty(G,H) = \inf \{ \sigma_y^2 + \sigma_v^2 : \sigma_y > 0, \sigma_v > 0, P = P^T > 0 \quad \text{and} \quad Q_{BR}(P; \Sigma_{FG}, M_\sigma) < 0 \}
\]

where
\[
Q_{BR}(P; \Sigma_{FG}, M_\sigma) \triangleq \begin{bmatrix}
A_{FG}^T P A_{FG} - P & A_{FG}^T P B_{FG} \\
B_{FG}^T P A_{FG} & B_{FG}^T P B_{FG}
\end{bmatrix} + \begin{bmatrix}
C_{FG}^T \\
D_{FG}
\end{bmatrix} \begin{bmatrix}
C_{FG} & D_{FG}
\end{bmatrix} - \begin{bmatrix}
0 & 0 \\
0 & M_\sigma
\end{bmatrix}
\]

and \( \Sigma_{FG} = (A_{FG}, B_{FG}, C_{FG}, D_{FG}) \) is a realization of \( F_G \). \quad \nabla

As a result, for such \( S^0 \) and in the light of Proposition 4.4, Prob. 2 can be stated as

**Prob. 2a:**

\[
\min_{\theta, \sigma \in \mathbb{R}^n, \sigma > 0, \sigma > 0} \quad \sigma_y^2 + \sigma_v^2 \quad \text{subject to} \quad Q_{BR}(P, \Sigma_{FG}(\theta), M_\sigma) < 0,
\]

where \( \theta = \begin{bmatrix} A_G & B_G \\ C_G & D_G \end{bmatrix} \).

To convert Prob. 2a into a SDP, the approach to the \( \mathcal{H}_\infty \) control problem pursued in [27] is followed here. Its first step is to separate the estimator parameter \( \theta \) from \( (P, \sigma) \) in the constraint above so that the so-called Elimination Lemma (see [27]) can be invoked. This is done rewriting the condition “\( Q_{BR}(\cdot) < 0 \)” and exploiting the fact that \( A_{FG} = A_o + A_L(\theta), B_{FG} = B_o + B_L(\theta), C_{FG} = C_o + C_L(\theta), D_{FG} = D_o + D_L(\theta) \), where \( A_L, B_L, C_L, D_L \) are linear functions of \( \theta \) and \( (A_o, B_o, C_o, D_o) \) are given matrices. This leads to the following proposition.

**Proposition 4.5.** (a) \( Q_{BR}(P, \Sigma_{FG}(\theta), M(\sigma)) < 0 \) \iff \( \psi(P, \sigma, \theta) > 0 \),

where
\[
\psi(P, \sigma, \theta) \triangleq \begin{bmatrix}
P^{-1} & A_{FG} & B_{FG} & 0_{n_{FG} \times m_e} \\
A_{FG}^T & P & 0_{n_{FG} \times m_y} & C_{FG}^T \\
B_{FG}^T & 0_{m_y \times n_{FG}} & M_\sigma & D_{FG}^T \\
0_{m_e \times n_{FG}} & C_{FG} & D_{FG} & I_{m_e}
\end{bmatrix}.
\]

(b) \( \psi(P, \sigma, \theta) = \psi_o(P, \sigma) + T_a^T \theta T_b + (T_a^T \theta T_b)^T > 0 \),
Proposition 4.6. (a) The optimal value \( \mathcal{J}_\infty^o(\mathbf{H}) \) of Prob. 2 is given by

\[
\mathcal{J}_\infty^o(\mathbf{H}) = \inf \{ \sigma_y \gamma_y^2 + \sigma_v \gamma_v^2 : \sigma_y > 0, \sigma_v > 0, \mathbf{P} = \mathbf{P}^T > 0, \mathbf{P} \in \mathbb{R}^{(n_G + n_x) \times (n_G + n_x)} \}
\]

are such that \( \psi(\sigma, \mathbf{P}, \theta) > 0 \}

is also given by

\[
\mathcal{J}_\infty^o(\mathbf{H}) = \inf \{ \sigma_y \gamma_y^2 + \sigma_v \gamma_v^2 : \sigma_y > 0, \sigma_v > 0, \mathbf{P} = \mathbf{P}^T > 0 \text{ are such that } (4.13) \text{ holds.} \}
\]

(b) If \( \mathbf{P}^o \) and \( \sigma^o \) are such that \( (4.13) \) holds there exists \( \theta \) such that

\[
\psi_o^o(\mathbf{P}^o, \sigma^o) + \mathbf{T}_a^T \theta \mathbf{T}_b + (\mathbf{T}_a^T \theta \mathbf{T}_b)^T > 0 \quad (4.14)
\]

and \( \rho(\mathbf{A}_G) < 1 \).

Moreover, if \( \sigma_y \gamma_y^2 + \sigma_v \gamma_v^2 = \mathcal{J}_\infty^o(\mathbf{H}) + \varepsilon \) then any \( \theta \) for which holds is such that

\[
\mathcal{J}_\infty(\mathbf{G}(\theta); \mathbf{H}) \leq \mathcal{J}_\infty^o(\mathbf{H}) + \varepsilon.
\]

\[\nabla\]

Remark 4.3. In words, an approximate solution, say \( \hat{\mathbf{G}} \), to Prob. 2 can be obtained on the basis of an approximate solution \((\mathbf{P}^o, \sigma^o)\) of Prob. 2b in the following way:

find a solution \( \hat{\theta} \) of the LMI given in (4.14) (which is guaranteed to exist) and take \( \hat{\mathbf{G}} \) with realization

\[
(\hat{\mathbf{A}}_G, \hat{\mathbf{B}}_G, \hat{\mathbf{C}}_G, \hat{\mathbf{D}}_G), \text{ where } \hat{\theta} = \begin{bmatrix} \hat{\mathbf{A}}_G & \hat{\mathbf{B}}_G \\ \hat{\mathbf{C}}_G & \hat{\mathbf{D}}_G \end{bmatrix}.
\]

\[\nabla\]
Note that whereas, for a given pair \((P^0, \sigma^0)\) which satisfies (4.13), (4.14) is a LMI on \(\theta\) (with non-empty solution set), the constraint (4.13) of Prob. 2c is non-linear on \(P\) as it is affine on \((P, P^{-1})\). However, exploiting the “zero-structure” of \(W_a\) and \(W_b\) as well as a specific parametrization of \(P\) (as done in [27]), (4.13) can be converted into two LMIs (on the “free” parameters of \(P\)), as is now stated in detail.

To this effect, \(P\) is written as
\[
\begin{bmatrix}
S & N \\
N^T & X
\end{bmatrix}
\]
and
\[
P^{-1} = 
\begin{bmatrix}
R & M \\
M^T & Z
\end{bmatrix},
\]
where the dimensions of \(S, X, R\) and \(Z\) are equal, i.e., \(n_G\) is taken to be equal to \(n_{az} + n_{ty}\) (see Appendix). Then, as \(R = (S - NX^{-1}N^T)^{-1}\), \(P\) is parametrized by \(S, X\) and \(R\) (rather than \(S, X\) and \(N\)), \(S = S^T > 0, R = R^T > 0, X = X^T > 0\) with \(S \geq R^{-1} \iff \begin{bmatrix} S & I \\ I & R \end{bmatrix} \geq 0\), \(N\) is given by \(NX^{-1}N^T = S - R^{-1}\). Condition (4.13) on \(P\) is then converted into LMIs on \(S\) and \(R\), as it is now precisely stated:

**Proposition 4.7.** Let \(P = \begin{bmatrix} S & N \\ N^T & X \end{bmatrix} \in \mathbb{R}^{n_{az} \times n_{az}}\) and \(X = X^T \in \mathbb{R}^{n_G \times n_G}\), \(n_{az} = n_{ty} + n_{az}\), \(n_G = n_{az}\) and write
\[
P^{-1} = \begin{bmatrix} R & M \\ M^T & Z \end{bmatrix},
\]
(a) Let \((A_{ty}, B_{ty}, C_{ty}, D_{ty})\) and \((A_{az}, B_{az}, C_{az}, D_{az})\) be minimal realizations of \(H_{ty}\) and \(H_{az}\).

\[
W_a^T \psi_a(P, \sigma)W_a > 0 \iff P > 0 \quad \text{and} \quad Q_a(R) \triangleq \begin{bmatrix}
R - A_{az}RA_{az}^T & B_{az} \\
B_{az}^T & M_{az}
\end{bmatrix} > 0, \tag{4.15}
\]
where \(A_{az} \triangleq \text{diag}(A_{ty}, A_{az})\) and \(B_{az} \triangleq \begin{bmatrix} B_{ty} \\ B_{az} \end{bmatrix}\).

(b) \(W_b^T \psi_b(P, \sigma)W_b > 0 \iff P > 0 \quad \text{and} \quad Q_b(S, \sigma) > 0,
\]
where \(Q_b(S) = Q_{b1}(S, \sigma) - E_{bo}^T E_{bo}, \quad Q_{b1}(S, \sigma) = \text{diag}(S, \sigma \gamma_y I_{m_y}) + \sigma \gamma_y E_{bo}^T E_{bo} - \psi_{b2}^T S \psi_{b2}, \quad E_{bo}, \quad E_{b} \quad \text{and} \quad \psi_{b}\) are given in the Appendix.

Combining Propositions (4.6) and (4.7) lead to the conversion of Prob. 2 into two SDPs, as follows.

**Proposition 4.8.** (a) The optimal value of Prob. 2 equals the optimal value of

\[
\begin{align*}
\text{Prob. 2d:} \quad & \min \limits_{\sigma > 0, S = S^T > 0, R = R^T > 0} \quad \sigma \gamma_y^2 + \sigma \gamma_v^2 \quad \text{subject to} \quad Q_a(R, \sigma) > 0, \quad Q_b(S, \sigma) > 0 \\
& \quad \begin{bmatrix} S & I \\ I & R \end{bmatrix} \geq 0.
\end{align*}
\]

(b) If \((\sigma^0, S_o, R_o)\) is a feasible solution of Prob. 2d and \(\sigma^0 \gamma_y^2 + \sigma^0 \gamma_v^2 = J_{\infty}(H) + \varepsilon\), then, for any \(X = X^T > 0\) and any unitary matrix \(V\), defining \(P^o = \begin{bmatrix} S_o & Q_{sr} VX^{1/2} \\ (Q_{sr} VX^{1/2})^T & X \end{bmatrix}\), where \(Q_{sr} = (S_o - R_o^{-1})^{1/2}\), it follows that \(P^o > 0\) and there exists \(\theta\) such that \(\psi_a(P^o, \sigma^0) + T^T a \theta T_a + (T_a \theta T_b)^T > 0\) and \(\rho(A_G) < 1\). Moreover, for any such \(\theta\), \(J_{\infty}(G(\theta), H) \leq J_{\infty}(H) + \varepsilon\). \(\nabla\)
Remark 4.4. An approximate solution  $G_M$ to Prob. 2 can be obtained on the basis of Proposition 4.8 in the following way: solve Prob. 2d to get $(S_o, R_o, \sigma_o^\circ, \sigma_o^\circ)$; for $P_o$ as defined in Proposition 4.8(b) obtain a solution $\theta_o$ to the LMI above involving $\psi_o(P_o, \sigma_o)$; obtain a realization $(A_o, B_o, C_o, D_o)$ for $G_M$, where $\theta_o = \begin{bmatrix} A_o & B_o \\ C_o & D_o \end{bmatrix}$.

\[ \nabla \]

4.3 Minimax $H_\infty$ Estimation for $H_\infty$–Balls of Uncertain Models

Prob. 3 (see Subsection 2.3) is now approached exactly along the lines pursued in connection with Prob. 2, with the class of admissible estimators given by

\[
S^\circ_G \triangleq \{ G \text{ has a realization } (A_G, B_G, C_G, D_G) : 
\begin{align*}
A_G &\in \mathbb{R}^{n_G \times m_y}, \\
C_G &\in \mathbb{R}^{m_e \times n_G}, \\
D_G &\in \mathbb{R}^{m_e \times m_w}, \\
\rho(A_G) &< 1 \},
\end{align*}
\]

The first step of this approach is to give a characterization of $J^\circ_G$ involving a matrix inequality constraint.

To this effect, recall that $e(z_o; G) = F_G z_o$, where $z_o \triangleq \begin{bmatrix} y^T & \bar{v}^T & \bar{w}^T \end{bmatrix}$, $F_G \triangleq H_{1a} - G H_{0a}$, and consider the Lagrangian and dual functionals $\text{Lag}^\circ_G(\cdot)$ and $\varphi^\circ_G(\cdot)$

\[
\text{Lag}^\circ_G(z_o; \sigma, G) = \|e(z_o; G)\|_2^2 - \sigma_y(\|\bar{y}\|_2^2 - \gamma_y^2) - \sigma_v(\|\bar{v}\|_2^2 - \gamma_v^2) - \sigma_w(\|\bar{w}\|_2^2 - \gamma_w^2), \\
\varphi^\circ_G(\sigma; G) = \sup \{ \text{Lag}^\circ_G(z_o; \sigma, G) : z_o \in \Re^{m_y+2m_w} \},
\]

where $\sigma = (\sigma_y, \sigma_v, \sigma_w)$.

It then follows from Theorem 2 in [12] that

\[
J^\circ_G = \varphi^\circ_G(G) \triangleq \inf \{ \varphi^\circ_G(\sigma; G) : \sigma = (\sigma_y, \sigma_v, \sigma_w), \sigma_y > 0, \sigma_v > 0, \sigma_w > 0 \}. \]

Note now that

\[
\text{Lag}^\circ_G(z_o; \sigma, G) = \langle F_G z_o, F_G z_o \rangle - \langle M^\circ_G z_o, z_o \rangle + \sigma_w \gamma_w^2 \langle W_{H_y} \bar{y}, W_{H_y} \bar{y} \rangle + \sigma_v \gamma_v^2
\]

or, equivalently,

\[
\text{Lag}(z_o; \sigma, G) = \langle (F_G W_{H_y} F_G - M^\circ_G) z_o, z_o \rangle + \sigma_y \gamma_y^2 + \sigma_v \gamma_v^2,
\]

where

\[
M^\circ_G \triangleq \text{diag}(\sigma_y I_{m_y}, \sigma_v I_{m_v}, \sigma_w I_{m_w}), \quad F_G \triangleq \left[ \begin{array}{c} \sigma_w^{1/2} W^\circ_{H_y} \\
W_{H_y} \end{array} \right], \quad W^\circ_{H_y} \triangleq \left[ \begin{array}{c} W_{H_y} \\
0_{m_y \times m_w} \end{array} \right].
\]

Proceeding as in Subsection 4.2 it follows that

\[
J^\circ_G = \inf \{ \sigma_y \gamma_y^2 + \sigma_v \gamma_v^2 : \sigma_y > 0, \sigma_v > 0, \sigma_w > 0 \text{ and } \forall \phi \in [0, 2\pi], F_G (e^{j\phi})^T F_G (e^{j\phi}) - M^\circ_G < 0 \}.
\]

(4.16)
Noting now that
\[ F_{GW}(e^{j\phi})^*F_{GW}(e^{j\phi}) - M_o^a < 0 \iff I - (M_o^a)^{-1/2}F_{GW}(e^{j\phi})^*F_{GW}(e^{j\phi})(M_o^a)^{-1/2} > 0, \]
it follows that
\[ \forall \phi \in [0, 2\pi], \ F_{GW}(e^{j\phi})^*F_{GW}(e^{j\phi}) - M_o^a < 0 \iff \|F_{GW}(M_o^a)^{-1/2}\|_\infty < 1 \]
or, equivalently, taking a realization \[ \Sigma_{GW} = (A_{GW}, B_{GW}, C_{GW}, D_{GW}) \] of \[ F_{GW}, \ \rho(A_{GW}) < 1, \]
and invoking the discrete-time, bounded-real lemma (as done in the nominal \[ \mathcal{H}_\infty \] derivation above)
\[ \exists P = P^T > 0 \] such that \[ Q_{br}(P; \Sigma_{GW}, M_o^a) < 0. \]

Thus,
\[ J_\infty^0(G) = \inf \{ \sigma_y^2 + \sigma_w^2 : \sigma = (\sigma_y, \sigma_v, \sigma_w), \sigma_y > 0, \sigma_v > 0, \sigma_w > 0, P = P^T > 0 \]
and \[ Q_{br}(P; \Sigma_{GW}, M_o^a) < 0 \}. \] (4.17)

Now, as shown in Section 4.2
\[ Q_{br}(P; \Sigma_{GW}, M_o^a) < 0 \iff \psi_a(P, \sigma, \theta) > 0, \]
where \[ \theta = \begin{bmatrix} A_G & B_G \\ C_G & D_G \end{bmatrix}, \] \( (A_G, B_G, C_G, D_G) \) is a realization of \( G \) and
\[
\psi_a(P, \sigma, \theta) = \begin{bmatrix}
P^{-1} & A_{GW} & B_{GW} & 0 \\ A_{GW}^T & P & 0 & C_{GW}^T \\ B_{GW} & 0 & M_o^a & D_{GW} \\ 0 & C_{GW} & D_{GW} & I
\end{bmatrix}
\]
or, equivalently, separating the estimator parameter \( \theta \) from \( (P, \sigma) \),
\[ Q_{br}(P; \Sigma_{GW}, M_o^a) < 0 \iff \psi_o^0(P, \sigma) + T_1^T \theta T_2 + T_2^T \theta^T T_1 > 0, \] (4.18)
where
\[
\psi_o^0(P, \sigma) = \begin{bmatrix}
P^{-1} & A_a & B_a & 0 \\ A_a^T & P & 0 & C_a^T \\ B_a^T & 0 & M_o^a & D_a^T \\ 0 & C_a & D_a & I
\end{bmatrix}
\]
where \( A_{GW} = A_a + A_L(\theta), B_{GW} = B_a + B_L(\theta), C_{GW} = C_a + C_L(\theta), D_{GW} = D_a + D_L(\theta), A_L(\cdot), B_L(\cdot), C_L(\cdot) \) and \( D_L(\cdot) \) are linear functions (of \( \theta \)) given in the Appendix,
\[ T_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{oa} & 0 & D_{oa}^* & I & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \]
\( \hat{C}_{\infty} = [0_{m_x \times n_{\text{w}_x}} : C_{\infty}] \) and \( \hat{D}_{\infty} = [D_{1\infty} D_{w}^{-1} : D_{w_{\infty}}^{-1}] \), \( A_{\infty} = \text{diag}(A_{a_1}, 0_{n_G \times n_G}) \), \( n_G \) is the dimension of \( A_{a_1}, A_{a_2}, B_{\text{a}}, C_{\text{a}} \) and \( D_{\text{a}} \) are also given in the Appendix.

For \( \psi_{a_{\infty}}(P, \sigma) \), \( T_1 \) and \( T_2 \) so defined, it follows from (4.17) and (4.18) that

\[
J_{\infty}^a(G(\theta)) = \inf \{ \sigma_y y + \sigma_v \gamma^2 : \sigma_y > 0, \sigma_v > 0, \sigma_w > 0, P = P^T > 0 \text{ and } \psi_{a_{\infty}}(P, \sigma) + T_1^T \theta T_2 + T_2^T \theta^T T_1 > 0 \}.
\]

Bringing in matrices \( W_1 \) and \( W_2 \) whose columns respectively constitute bases for the null spaces of \( T_1 \) and \( T_2 \) together with the corresponding constraints

\[
W_1^T \psi_{a_{\infty}}(P, \sigma) W_1 > 0 \quad \text{and} \quad W_2^T \psi_{a_{\infty}}(P, \sigma) W_2 > 0
\]

and invoking the Elimination Lemma, a statement entirely similar to Proposition 4.6 can be seen to hold as follows.

**Proposition 4.9.** Let

\[
S_G^a \triangleq \{ G \text{ has a realization } (A_G, B_G, C_G, D_G) : A_G \in \mathbb{R}^{n_G \times n_G}, B_G \in \mathbb{R}^{n_G \times n_{\text{w}_G}}, C_G \in \mathbb{R}^{m_x \times n_G}, D_G \in \mathbb{R}^{m_x \times m_v}, \rho(A_G) < 1 \},
\]

and \( J_{\infty}^a \triangleq \inf \{ J_{\infty}^a(G) : G \in S_G^a \} \). Then (a) \( \forall G \in S_G^a \)

\[
J_{\infty}^a(G) \triangleq \inf \{ \sigma_y y + \sigma_v \gamma^2 : \sigma_y > 0, \sigma_v > 0, \sigma_w > 0, P = P^T > 0, \theta \in \mathbb{R}^n, \psi_{a_{\infty}}(P, \sigma) + T_1^T \theta T_2 + T_2^T \theta^T T_1 > 0 \},
\]

where \( \theta = \begin{bmatrix} A_G & B_G \\ C_G & D_G \end{bmatrix} \).

(b) \( J_{\infty}^a \) is given by

\[
J_{\infty}^a = \inf \{ \sigma_y y + \sigma_v \gamma^2 : \sigma_y > 0, \sigma_v > 0, \sigma_w > 0, P = P^T > 0 \text{ and } (4.19) \text{ holds} \}.
\]

Moreover, if \( P^T \) and \( \sigma^o = (\sigma_y^o, \sigma_v^o, \sigma_w^o) \) satisfy the constraints above and \( \sigma_y^o y + \sigma_v^o \gamma^2 = J_{\infty}^a + \varepsilon \), then there exists \( \theta \) such that

\[
\psi_{a_{\infty}}(P^o, \sigma^o) + T_1^T \theta T_2 + T_2^T \theta^T T_1 > 0
\]

and, for any such \( \theta \), \( J_{\infty}^a(G(\theta)) \leq J_{\infty}^a + \varepsilon \).

\( \nabla \)

Proceeding along the lines that led to Propositions 4.7 and 4.8, (4.19) is converted into LMIs on \( R \) and \( S \), where \( P = \begin{bmatrix} S & N \\ N^T & X \end{bmatrix} \) and \( P^{-1} = \begin{bmatrix} R & M^T \\ M & Z \end{bmatrix} \), as it is now stated.
Proposition 4.10. (a) $W_1^T \phi_a(P, \sigma) W_1 > 0 \iff \sigma_w > 0$, $P > 0$, and $Q_1(R, \sigma) > 0$, where
\[
Q_1(R, \sigma) \triangleq \begin{bmatrix} R & B_{a1}^t \\ (B_{a1}^t)^T & D_{a1}^t \end{bmatrix} \begin{bmatrix} C_{a1}^W \\ 0 \end{bmatrix} = \begin{bmatrix} A_{a1}^t \\ 0 \end{bmatrix},
\]
$C_{a1}^W = [\gamma_H C_{WH} : 0_{m_y \times n_{AHs}} : 0_{m_y \times n_{Aoa}}]$ and $D_{a1}^W = [\gamma_H D_{WHy} : 0_{m_y \times m_w}]$, $M_\sigma = \text{diag}(\sigma_y I_{m_y}, \sigma_v I_{m_w})$, $A_{a1}$ and $B_{a1}$ are given in the Appendix.

(b) $W_2^T \phi_a(P, \sigma) W_2 > 0 \iff P > 0$ and $Q_2(S, \sigma) > 0$, where
\[
Q_2(S, \sigma) \triangleq \tilde{Q}_2(S, \sigma) - E_o^T E_a - E_s^T S E_a,
\]
\[
\tilde{Q}_2(S, \sigma) \triangleq \begin{bmatrix} S & 0 \\ 0 & M_\sigma \end{bmatrix} + \sigma_w \begin{bmatrix} \tilde{C}_{oa}^T \\ \tilde{D}_{oa}^T \end{bmatrix} \begin{bmatrix} \tilde{C}_{oa} \tilde{D}_{aoa} \end{bmatrix} - \sigma_w \begin{bmatrix} (C_{a1}^W)^T \\ (D_{a1}^W)^T \end{bmatrix} \begin{bmatrix} C_{a1}^W \\ D_{a1}^W \end{bmatrix},
\]
$E_a$ and $E_s$ are given in the Appendix.

Finally, combining Proposition 4.9 and Proposition 4.10 leads to the counterpart of Proposition 4.8 which is now stated (the proofs of “$P^* > 0$” and “$\rho(A_a) < 1$” follow exactly the same argument invoked in the proof of Proposition 4.8).

Proposition 4.11. (a) The optimal value of $\mathcal{J}_{00}^a$ of Prob. 3 equals to the optimal value of the following problem
\[
\text{Prob. 3a} \quad \min_{\sigma_y > 0, \sigma_v > 0, \sigma_w > 0} \sigma_y \gamma_y^2 + \sigma_v \gamma_v^2 \\
S = S^T > 0, R = R^T > 0
\]
subject to $Q_1(R, \sigma) > 0$, $Q_2(S, \sigma) > 0$, $[S \ I \ I] > 0$, where $Q_1(\cdot)$ is affine on $(R, \sigma_y, \sigma_v)$ and $Q_2(\cdot)$ is affine (see Proposition 4.10).

(b) If $(\sigma_y^0, \sigma_v^0, \sigma_w^0, S^0, R^0)$ is a feasible solution of Prob. 3a and $\sigma_y^0 \gamma_y^2 + \sigma_v^0 \gamma_v^2 = \mathcal{J}_{00}^a + \varepsilon$, then for any $X_a = X_a^T > 0$ and any unitary matrix $V$, defining $P^0 = \begin{bmatrix} S^T & Q_{SR}^a \ 
Q_{SR}^a \ V X_a^{1/2} \\ \ 
\end{bmatrix} \begin{bmatrix} X_a \\ V X_a^{1/2} \end{bmatrix}$, where $Q_{SR}^a = [S^0 - (R^0)^{-1}]^{1/2}$, it follows that $P^0 > 0$ and there exists $\theta$ such that
\[
\psi_a^0(P^0, \sigma^0) + T_1^T \theta T_2 + T_2^T \theta^T T_1 > 0
\]
and $\rho(A_a) < 1$. Moreover, for any such $\theta$, $\mathcal{J}_{00}^a(G(\theta)) \leq \mathcal{J}_{00}^a + \varepsilon$.

Remark 4.5. Note that $Q_1(R, \sigma)$ is not affine on $\sigma_w$. However, if a value is assigned to $\sigma_w$ so that it is no longer a decision variable of Prob. 3a, the resulting problem is a SDP on the remaining ones $(S, R, \sigma_y, \sigma_v)$. Thus, a natural way of tackling Prob. 3a is by means of a line search with respect to $\sigma_w$ at each step of which a SDP is solved (if $W_{HY}$ is constant “$Q_1(R, \sigma) > 0$” can be rewritten as a LMI).
Note, in addition, that for a given solution \((\mathbf{P}^*, \sigma^*_y, \sigma^*_w)\) of Prob. 3a, \(\theta\) is obtained from the LMI in Proposition 4.11(b).

\[\n\]

## 5 Robust Estimation Based on an Average Criteria

The major aim of this section is to formulate estimation problems (Prob. 4 – 6 below) based on the average cost-functionals \(\eta_{av}, \eta^a\) and \(\eta^b\) and convert them into SDPs, with the purpose of enabling trade-offs to be achieved between worst-case and “pointwise” performance. Prob. 4 – 6 can be respectively cast as

\[
\begin{align*}
\min_{\mathbf{G} \in \mathcal{S}_{G}^{i}} \eta^i(\mathbf{G}) & \quad \text{subject to} \quad J^i(\mathbf{G}) \leq (1 + \alpha)J^i_0, \quad \alpha > 0
\end{align*}
\]

where \((\eta^i, J^i) = (\eta_{av}, J_X), (\eta^2, J^2) = (\eta^a, J_\infty), (\eta^3, J^3) = (\eta^b, J_\infty^a), \) and \(J^i, i = 1, 2, 3\) are respectively the optimal values of the minimax problems Prob. 1 – 3. As the main point here is to generate “less conservative” alternatives to the corresponding minimax estimators, in each case \(\mathcal{S}_{G}^{i}\) is taken to be a linear class

\[
\mathcal{S}_{G}^{i} \triangleq \left\{ \mathbf{G}(\beta) = \beta \mathbf{Y}_a : \beta = \left[ \mathbf{C}_G \mathbf{D}_G \right], \mathbf{C}_G \in \mathbb{R}^{m_x \times n^i_G}, \mathbf{D}_G \in \mathbb{R}^{n^i_G \times m_y} \right\},
\]

where \(\mathbf{Y}_a = \left[ \mathbf{Y}_a^i \mathbf{Y}_a^b \right] \mathbf{I}_{m_y} \mathbf{I}_{m_y} \mathbf{I}_{m_y}, \mathbf{Y}_a^i(\epsilon) = (\epsilon \mathbf{I} - \mathbf{A}_G^i)^{-1}\) and \((\mathbf{A}_G^i, \mathbf{B}_G^i, \mathbf{C}_G^i, \mathbf{D}_G^i)\) is a minimal realization of an \(\epsilon\)-approximate solution \(\mathbf{G}^i\) to the associated minimax problem \((\mathbf{G}^i \in \text{feasible solution of Prob. } i \text{ and } J^i(\mathbf{G}) \leq J^i_0 + \epsilon)\) and \(\alpha > \epsilon\). The three derivations to convert Prob. 4 – 6 into SDPs are entirely analogous and go as follows: (a) the results of Section \(\mathcal{I}\) for problems 1 – 3 provide the constraints that impose upperbounds on worst-case performance of any admissible estimator in each linear class \(\mathcal{S}_{G}^{i}; \) (b) then each average, quadratic cost functional \(\eta^i\) is converted into a linear one with the introduction of additional LMI constraints.

With respect to (b), note that the cost-functionals \(\eta_{av}, \eta^a\) and \(\eta^b\) can be written as

\[
\begin{align*}
\eta_{av}(\mathbf{G}) &= C(\mathbf{G}; \mathbf{\Gamma}_y, \mathbf{\Gamma}_v) + \left(\gamma^2/(m_y m_v)\right) \left(\mathbf{G} \otimes \phi^T_{y_1} \mathbf{G} \otimes \phi^T_{y_1}\right)
\eta^a(\mathbf{G}) &= C(\mathbf{G}; \mathbf{\Gamma}_y^a, \mathbf{\Gamma}_v^a)
\eta^b(\mathbf{G}) &= C(\mathbf{G}; \mathbf{\Gamma}_y^b, \mathbf{\Gamma}_v^b),
\end{align*}
\]

where \(C(\mathbf{G}; \hat{\mathbf{\Gamma}}_y, \hat{\mathbf{\Gamma}}_v) \triangleq \left(\mathbf{H}_1 - \mathbf{G} \mathbf{H}_0\right) \hat{\mathbf{\Gamma}}_y + \left(\mathbf{H}_1 - \mathbf{G} \mathbf{H}_0\right) \hat{\mathbf{\Gamma}}_v, \mathbf{\Gamma}_\alpha = (\gamma^2/\alpha)^{-1}(\mathbf{W}_\alpha \mathbf{W}_\alpha)^{-1}, \alpha = y, v \) and \(\mathbf{\Gamma}_v^a = \mathbf{\Gamma}_v^b + \left(\gamma^2/m_v\right) \phi^T_{y_1} \mathbf{G} \mathbf{\phi}^T_{y_1} \mathbf{I}_{m_v} \).

As a result, the replacement of \(\eta_{av}, \eta^a\) and \(\eta^b\) by linear cost-functionals will be based on the following equalities: for \(\mathbf{G}(\beta) \in \mathcal{S}_{G}^{i};\)

\[
\begin{align*}
C(\mathbf{G}(\beta); \mathbf{\Gamma}_y, \mathbf{\Gamma}_v) &= \inf \left\{ \text{tr}(\mathbf{P}_c) : \mathbf{P}_c = \mathbf{P}_c^T \text{ and } Q_{\beta q}(\mathbf{P}_c, [m^T_\alpha : \beta]; Q_{c}) \geq 0 \right\}, \quad (5.1)
\end{align*}
\]

\[
\begin{align*}
\langle \mathbf{G}(\beta) \otimes \phi^T_{y_1}, \mathbf{G}(\beta) \otimes \phi^T_{y_1} \rangle &= \inf \left\{ \text{tr}(\mathbf{R}_c) : \mathbf{R}_c = \mathbf{R}_c^T \text{ and } Q_{\beta q}(\mathbf{R}_c, \beta : \mathbf{I}; Q_{Gc}) \geq 0 \right\}, \quad (5.2)
\end{align*}
\]
where \( Q_{\mathcal{J}}(P, M; Q) \triangleq \left[ \begin{array}{c} P \\ Q^{1/2} M^T \\ I \end{array} \right] \),

\[
Q_c \triangleq (1/2\pi) \int_0^{2\pi} \left\{ \left[ \begin{array}{ccc} H_i & \hat{\Gamma}_y & \left[ \begin{array}{c} 0 \\ Y_a \end{array} \right] \end{array} \right] \left[ \begin{array}{c} H_i \\ -Y_a H_o \end{array} \right]^* + \left[ \begin{array}{c} 0 \\ Y_a \end{array} \right] \left[ \begin{array}{c} 0 \\ Y_a \end{array} \right]^* \right\} (e^{i\phi}) d\phi,
\]

\[
Q_{Gc} \triangleq (1/2\pi) \int_0^{2\pi} \left( Y_a^i \otimes \phi_y^T \right) \left( Y_a^i \otimes \phi_y^T \right)^* (e^{i\phi}) d\phi.
\]

### 5.1 MSE Estimation With \( \mathcal{H}_2 \) Model Uncertainty

A robust estimation problem is now considered in which the cost functional \( \eta_{av} \) (cf. Proposition 3.2) is minimized with respect to \( G \) under the constraint that the worst-case MSE of a given estimator does not exceed a prescribed value, i.e.,

\[
\text{Prob. 4:} \quad \min_{G \in \mathcal{S}_G^1} \eta_{av}(G) \quad \text{subject to} \quad \tilde{J}_X(G : \mathcal{S}_X) \leq (1 + \alpha) \tilde{J}_o^1,
\]

where \( \tilde{J}_o^1 \equiv \inf \{ \tilde{J}_X(G; \mathcal{S}_X) : G \in \mathcal{S}_G \} \).

To convert Prob. 4 into an SDP the constraint above is recast in the light of (4.8) and (4.9) as follows.

**Proposition 5.1.** \( G \in \mathcal{S}_G^1 \) satisfies the constraint “\( \tilde{J}_X(G; \mathcal{X}) \leq (1 + \alpha) \tilde{J}_o^1 \)” in Prob. 4 if

(i) \( \exists \lambda > 0 \) and \( P = P^T > 0 \) such that \( (i) \lambda \gamma^2 + x_o^T P x_o \leq (1 + \alpha) \tilde{J}_o^1 \) and \( (ii) \) \( Q_{Ja}(P, \Sigma_a(G), M(\lambda)) < 0 \). Moreover, the optimal value of Prob. 4 equals the optimal value of the following problem

\[
\min_{G \in \mathcal{S}_G^1, P = P^T > 0, \lambda > 0} \eta_{av}(G) \quad \text{subject to } (i) \text{ and } (ii),
\]

where \( Q_{Ja}(\cdot) \) is given by (4.9).

Thus, noting that \( Q_{Ja}(P, \Sigma_a(G(\beta)), M(\lambda)) < 0 \) if and only if \( Q_{Ja}(P, \lambda, \beta) < 0 \), it follows from equations (5.1) and (5.2) (replacing \( (\hat{\Gamma}_y, \hat{\Gamma}_v, Y_a^i) \) by \( (\Gamma_y, \Gamma_v, Y_a^i) \)) and Proposition 5.1 that Prob. 4 can be converted into a SDP as stated in the following proposition.

**Proposition 5.2.** Prob. 4 can be recast as

\[
\min_{\beta, P, P = P^T, P = P^T, \lambda > 0} \text{tr}(P_{\mathcal{J}}) + \text{tr}(P_{\mathcal{J}})
\]

subject to

\[
Q_{Ja}(P_{\mathcal{J}}, [m_e : \beta]; Q_{oe}) \geq 0, \quad Q_{Ja}(P_{\mathcal{J}}, [\beta \otimes I]; Q_{Ga}) \geq 0,
\]

\[
\lambda \gamma^2 + x_o^T P x_o \leq (1 + \alpha) \tilde{J}_o^1, \quad \text{and} \quad Q_{Ja}(P, \lambda, \beta) < 0,
\]

where \( x_o \) and \( Q_{Ja}(P, \lambda, \beta) \) are as in Proposition 4.2. \( Q_{oe} \) and \( Q_{Ga} \) are defined as \( Q_c \) and \( Q_{Gc} \) (in (5.3), (5.4)) replacing \( (\hat{\Gamma}_y, \hat{\Gamma}_v, Y_a^i) \) by \( (\Gamma_y, \Gamma_v, Y_a^i) \).

\[ \nabla \]
5.2 A Nominal $\mathcal{H}_\infty$ Estimator Based on an Average Cost

In this subsection, a linear filter is sought in a class of admissible ones which minimizes the average cost $\eta^a(\cdot; H_o)$ (for a given “nominal” $H_o$) under the constraint that its worst-case estimation error over $S_y \times S_v$ does not exceed a prescribed value.

To this effect, consider the optimization problem

$$\text{Prob. 5: } \min_{G \in \mathcal{S}_G} \eta^0(G; H_o) \quad \text{subject to } J_\infty(G; H_o) \leq (1 + \alpha)J_o^2,$$

where $J_o^2 = \inf \{ J_\infty(G; H_o) : G \in \mathcal{S}_G \}$.

In the light of (4.4), Prob. 5 can be recast as

$$\min_{\beta > 0, \sigma_y > 0, \sigma_v > 0, P = P^T > 0} \eta^0(G(\beta); H) \quad \text{subject to } \begin{cases} \sigma_y \gamma_y^2 + \sigma_v \gamma_v^2 \leq (1 + \alpha)J_o^2, \\ Q_{BR}(P; \Sigma^\circ_{FG}(G(\beta)), M_\sigma) < 0, \end{cases} \quad (5.5)$$

where $\Sigma^\circ_{FG}(G(\beta)) = (A_{FG}^o, B_{FG}^o, C_{FG}(\beta), D_{FG}(\beta))$ is a realization of $F_G(\beta) = H_y - \beta Y_G^2 H_{zz}$, obtained from a minimal realization $(A_{FG}^o, B_{FG}^o, C_{FG}, D_{FG})$ of $[-Y_G^2, H_{zz}]$, and $[C_{FG}(\beta) : D_{FG}(\beta)] = [I_{m_\sigma} : \beta] [C_{FG} : D_{FG}]$.

The second constraint in (5.5) can be rewritten as $\hat{Q}_a(P, \sigma, \beta) > 0$

where $\hat{Q}_a(P, \sigma, \beta) \triangleq \begin{bmatrix} \hat{Q}_{a1}(P, \sigma) & C_{FG}(\beta)^T \\ C_{FG}(\beta) & D_{FG}(\beta)^T \end{bmatrix}$

$$\hat{Q}_{a1}(P, \sigma) = \begin{bmatrix} P & 0 \\ 0 & M_\sigma \end{bmatrix} - \begin{bmatrix} A_{FG} & B_{FG} \\ B_{FG}^T & A_{FG}^o \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & M_\sigma \end{bmatrix} \begin{bmatrix} A_{FG}^o & B_{FG}^o \\ B_{FG} & A_{FG} \end{bmatrix}.$$

It then follows from equation (5.1) (replacing $(\Gamma_y, \Gamma_v, Y_a)$ by $(\Gamma_y^o, \Gamma_v^o, Y_a^2)$) and (5.4) – (5.5) that Prob. 5 can be recast as an SPD as follows.

**Proposition 5.3.** Prob. 5 can be recast as

$$\min_{\beta, \sigma_y > 0, \sigma_v > 0, P = P^T > 0, R_a = R_a^T} \text{tr}[R_a] \quad \text{subject to } \begin{cases} Q_{Jq}(R_a, [I_{m_\sigma} : \beta]; \Gamma_a) \geq 0, \\ \sigma_y \gamma_y^2 + \sigma_v \gamma_v^2 \leq (1 + \alpha)J_o^2, \\ \text{and } \hat{Q}_a(P, \sigma, \beta) > 0, \end{cases}$$

where $\Gamma_a$ is defined as $Q_c$ (in (5.3)) replacing $(\Gamma_y, \Gamma_v, Y_a)$ by $(\Gamma_y^o, \Gamma_v^o, Y_a^2)$. $\nabla$

**Remark 5.1.** The frequency-response of the “average-cost” $\mathcal{H}_\infty$ estimator is obtained from a solution $(\beta_o, P_o, R_o, \sigma^o_y, \sigma^o_v)$ of the SDP introduced in Proposition 5.3 and is given by $G(\beta_o) = \beta_o Y_a^2$. $\nabla$
5.3 A Robust $\mathcal{H}_\infty$ Estimator Based on an Average Cost

In this subsection, a robust estimator is introduced for the set-up of Subsection 4.3 with the aim of enabling trade-offs to be achieved between worst-case and “point-wise” performance over the $\mathcal{H}_\infty$—balls of channel frequency-responses and the $\mathcal{H}_2$—balls of exogenous signals.

More specifically, consider the following optimization problem

$$
\text{Prob. 6: } \min_{\mathbf{G} \in S^3_{\infty}} \eta^b(\mathbf{G}) \text{ subject to } J^a(\mathbf{G}) \leq (1 + \alpha)J^a_o,
$$

where $J^a_o = \inf\{J^a(\mathbf{G}) : \mathbf{G} \in S^3_{\infty}\}$.

It then follows from (4.17) (in the same way Prob. 5 was recast as (5.5)) that Prob. 6 can be stated as

$$
\min_{\sigma_y > 0, \sigma_w > 0, \eta^b(\mathbf{G}; \mathbf{H}_o) \text{ subject to } \sigma_y^2 + \sigma_w^2 \leq (1 + \alpha)J^a, \quad Q_{BR}(\mathbf{P}, \Sigma^b_{\mathbf{G}_W}(\sigma_w, \beta), \mathbf{M}_o^a) < 0, \quad (5.6)
$$

where $\beta = \left[ \mathbf{C}_G : \mathbf{D}_G \right]$, $\Sigma^b_{\mathbf{G}_W}(\sigma_w, \beta) = (\mathbf{A}^b_{\mathbf{G}_W}, \mathbf{B}^b_{\mathbf{G}_W}, \mathbf{C}_{\mathbf{G}_W}(\sigma_w, \beta), \mathbf{D}_{\mathbf{G}_W}(\sigma_w, \beta))$ is a realization of $\mathbf{F}_{\mathbf{G}_W}(\beta)$ (see Subsection 4.3) obtained from a minimal realization $(\mathbf{A}^b_{\mathbf{G}_W}, \mathbf{B}^b_{\mathbf{G}_W}, \mathbf{C}_{\mathbf{G}_W}, \mathbf{D}_{\mathbf{G}_W})$ of $[(\mathbf{W}^{a}_{\mathbf{H}})^T : \mathbf{H}_{e}^a : (\mathbf{Y}^{a}_{\mathbf{H}})^T]^T$ letting $[\mathbf{T}_y^T : \mathbf{T}_e(\beta)^T]^T \equiv \text{diag}(\mathbf{I}_{m_y}, \mathbf{I}_{m_e} : \beta)[\mathbf{C}_{\mathbf{G}_W} : \mathbf{D}_{\mathbf{G}_W}, \mathbf{M}_o^a$ as in Subsection 4.3 and $[\mathbf{C}_{\mathbf{G}_W}(\sigma_w, \beta) : \mathbf{D}_{\mathbf{G}_W}(\sigma_w, \beta)] = \text{diag}(\sigma_w^{1/2} \gamma^y, \mathbf{I}_{m_y}, \mathbf{I}_{m_e} | \mathbf{T}_y^T : \mathbf{T}_e(\beta)^T]^T$.

It can then be shown that

$$
Q_{BR}(\mathbf{P}, \Sigma^b_{\mathbf{G}_W}(\sigma_w, \beta), \mathbf{M}_o^a) < 0 \Leftrightarrow \hat{Q}_b(\mathbf{P}, \sigma, \beta) > 0,
$$

where

$$
\hat{Q}_b(\mathbf{P}, \sigma, \beta) \equiv \begin{bmatrix} \hat{Q}_{s1}(\mathbf{P}, \sigma) - \sigma_w^2 \gamma^y (\mathbf{T}_y)^T \mathbf{T}_y & \mathbf{T}_e(\beta)^T \\ \mathbf{T}_e(\beta)^T & \mathbf{I}_{m_e} \end{bmatrix},
$$

$$
\hat{Q}_{s1}(\mathbf{P}, \sigma) \equiv \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{M}_o^a \end{bmatrix} - \begin{bmatrix} \mathbf{A}^b_{\mathbf{G}_W} \mathbf{T}_e(\beta) \\ \mathbf{B}^b_{\mathbf{G}_W} \mathbf{T}_e(\beta) \end{bmatrix} \mathbf{P} \begin{bmatrix} \mathbf{A}^b_{\mathbf{G}_W} & \mathbf{B}^b_{\mathbf{G}_W} \end{bmatrix}.
$$

In the light of (5.6), (5.7) and equation (5.1) (replacing $(\hat{\mathbf{F}}_y, \hat{\mathbf{F}}_v, \mathbf{Y}_{\alpha}^i)$ by $(\mathbf{F}_y, \hat{\mathbf{F}}_v, \mathbf{Y}_{\alpha}^a)$), Prob. 6 can be recast as a SDP as it is now stated in detail.

**Proposition 5.4.** Prob. 6 can be recast as the following SDP

$$
\min_{\sigma_y > 0, \sigma_w > 0, \sigma_v > 0, \beta, \mathbf{P} = \mathbf{P}^T, \mathbf{R} = \mathbf{R}^T} \text{tr}(\mathbf{R})
$$

subject to $Q_{s2}(\mathbf{R}, [\mathbf{I}_{m_e} : \beta], \Gamma^b_\eta) \geq 0$, $\sigma_y^2 + \sigma_w^2 + \sigma_v^2 \leq (1 + \alpha)J^a_o$, and $\hat{Q}_b(\mathbf{P}, \sigma, \beta) > 0$,

where $\hat{Q}_b(\mathbf{P}, \sigma, \beta)$ is given by (5.8), $\sigma = (\sigma_y, \sigma_v, \sigma_w)$, and $\Gamma^b_\eta$ is defined as $Q_c$ (in (5.3)) replacing $(\hat{\mathbf{F}}_y, \hat{\mathbf{F}}_v, \mathbf{Y}_{\alpha}^i)$ by $(\mathbf{F}_y, \hat{\mathbf{F}}_v, \mathbf{Y}_{\alpha}^a)$. \(\nabla\)
Remark 5.2. In exactly the same way described in Remark 5.1, the frequency-response of the “average-cost” estimators for the robust $H_{\infty}$ problem defined by Prob. 6 is obtained from a solution $(\beta_o, P_o, R_o, \sigma_y^o, \sigma_v^o, \sigma_w^o)$ of the SDP introduced in Proposition 5.4 as $G(\beta_o) = \beta_o Y^3$.

6 Comparing Robust Estimators

In this section, the average cost/worst-case constraint estimators introduced in Section 5 are compared with the corresponding minimax ones. The main issues addressed here are first discussed in connection with robust $H_2$ estimation.

6.1 Robust $H_2$ Estimators

The assessment of a linear MSE estimator given by $G \in \mathcal{R}^{m_e \times m_v}$ in connection with a model set $S_H$ may be carried out in terms of various features of (functionals computed on) the “MSE function” $J(G; \cdot) : S_H \rightarrow \mathbb{R}$. As pointed out in Section 1, the supremum of $J(G; \cdot)$ over $S_H$ (denoted by $\bar{J}(G; S_H)$), i.e., the worst-case MSE over $S_H$, has been extensively used as the main assessment feature in the case of set-theoretic, model uncertainty, leading to minimax estimators (denoted below by $G_M$). To mitigate the conservatism of such estimators other features may be considered together with $\bar{J}(G; S_H)$ such as the “average ($L_1$-norm)” of $J(G; \cdot)$ over $S_H$ or the nominal MSE, $J(G; H_0)$, which lead to the estimator defined by Prob. 4 above with $\eta(G) = \eta_{av}(G)$ – this estimator will be denoted by $G_{av}$.

When comparing this estimator with the minimax estimator the questions naturally arise as to whether the improvement brought about at the expense of the increase in the worst-case MSE is significant (at some points of $S_H$) or whether $G_{av}$ outperforms $G_M$ on a “sizable part” of $S_H$.

In the first case (range of “point-wise improvement”), a suitable additional feature of $J(G; \cdot)$ (for $G = G_{av}$) could be defined by

$$\eta_{PW}(G) \triangleq \sup \{ J(G_M; H) - J(G; H) : H \in S_H \},$$

or, in relative terms, by

$$\eta_{RW}(G) = \inf \{ J(G; H)/J(G_M; H) : H \in S_H \}. $$

Remark 6.1. It should be noted that for a given $G \in S_G$, $\eta_{PW}(G)$ can also be characterized as the optimal value of an SDP along the lines which led to Proposition 4.1 and equation (4.7). 

In the second case (“relative size” of the subset of $S_H$ over which point-wise performance was improved), the “improvement set” $S_I$ could be defined as $S_I(G; G_M) = \{ H \in S_H : J(G; H) < J(G_M; H) \}$ or, equivalently,

$$S_I(G; G_M) = \{ X \in S_X : \delta_J(X; G) < 0 \},$$

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where $\delta_j(X; G) \triangleq J_X(G; X) - J_X(G_M; X)$, i.e., $\delta_j(X; G) = \delta_{j_o} - 2\delta_r(X; G) + \delta_q(X; G)$, $\delta_{j_o} = \mathcal{J}(G; H_0) - \mathcal{J}(G_M; H_0)$, $\delta_i(X; G) = \ell_X(X; G) - \ell_X(X; G_M)$, $\delta_q(X; G) = q_X(X; G) - q_X(X; G_M)$, $\ell_X(X; G) = \langle GXG_{y_1}, X_0(G) \rangle$ and $q_X(X; G) = \langle GXG_{y_1}, GX \rangle$.

Due to the difficulty in comparing the “volume” (in $\mathcal{H}_2$) of $S_j(G; G_M)$ with that of $S_X$ (the “projections” of $S_j(G; G_M)$ along radial line segments in $S_X$ are considered in the search for conditions which indicate that $G$ improves on $G_M$ over a “sizeable” part of $S_X$). To this effect, consider radial line segments of model perturbations $X$ (the nominal model corresponds to $X = 0$) $S_{\beta}(\bar{X}) = \{X = \beta\bar{X} : \beta \in [-1, 1]\}$, where $\bar{X} \in \mathcal{R}_{c_m \times m_y}$ is such that $\|\bar{X}\|_2 = 1$ and $\delta_i(\bar{X}) \leq 0$ – note that no loss of generality is incurred by the condition “$\delta_i(\bar{X}) \leq 0$” since $S_{\beta}(\bar{X})$ and $S_{\beta}(-\bar{X})$ define the same line segment. Along one such segment the projection of $S_j(\cdot)$ (say, $S_{j\beta}(\bar{X}; G)$) is given by

$$S_{j\beta}(\bar{X}; G) = \{\beta \in [-1, 1] : \delta_{j_o} + 2\beta|\delta_r(\bar{X}; G)| + \beta^2|\delta_q(\bar{X}; G)| < 0\}$$

so that, if the length (Lebesgue measure) $\mu_j(\bar{X}; G)$ of $S_{j\beta}(\bar{X}; G)$ is greater than $\alpha \in [0, 2]$, it can be said that “$G$ improves on $G_M$ on a fraction of the segment defined by $\bar{X}$ which is larger than $\alpha/2$”.

If this holds for any radial linear segment, it can be said that $G$ improves on $G_M$ in a “radial” sense on “more than $\alpha/2$” of the perturbed model set.

A lower bound on how much $G$ improves on $G_M$ in this sense is provided by the next proposition.

**Proposition 6.1.** Let $\delta_{j \delta} \triangleq \gamma\|\{(G^*X_o(G) - G_{M^*}X_o(G_M))^{\Gamma_{y_1}}\}_{\ell_2}\|_2$ and $\delta_{q \delta} \triangleq \lambda_\infty(G^*G - G_{M^*}G_m)$ $\|\phi_{y_1}\|_{\infty, 2}^2$, $\Gamma_{y_1} = \phi_{y_1}^*\phi_{y_1}$, where for $E(e^{j\phi}) = E_a(e^{j\phi}) + E_a(e^{j\phi})^*$ with $E_a \in \mathcal{R}_{c_m \times m_y}$, $\lambda_\infty(E) \triangleq \sup\{\lambda_{\max}(E(e^{j\phi})) : \phi \in [0, 2\pi]\}$ and $\lambda_{\max}(P)$ denotes the maximum eigenvalue of the Hermitian matrix $P$. Let $\nu_{c} \triangleq (1/2)|\delta_{j_o}|/\delta_{j \delta}$, $\nu_{c} \triangleq |\delta_{j_o}|/\delta_{q \delta}$, $\nu_{\beta} \triangleq (1/2)\nu_{c} \sqrt{\left\{(\delta_{i}/\delta_{q \delta})^2 + \nu_{c}\right\}^{1/2}}$ and define $\mu_{\gamma}(G) = \min\{2, 1 + \nu_{\gamma}, 2 + \nu_{\gamma}, 2\nu_{c}\}$. Let $G$ be such that $\delta_{j_o}(G) < 0$. Then $\forall \bar{X} \in \mathcal{R}_{c_m \times m_y}$ such that $\|\bar{X}\|_2 = \gamma^2$, $\mu_{\gamma}(\bar{X}; G) \geq \mu_{\gamma}(G)$.

In the next section, three numerical examples will be presented in which a given robust estimator $G$ will be assessed on the basis of its worst-case performance ($\mathcal{J}_X(G)$), “range of point-wise improvement” ($\eta_{\text{rw}}(G)$ or $\eta_{\text{rw}}(G)$) and of estimates of the “relative size” of the “improvement set” $S_j(G; G_M)$. The first two performance indexes will be computed with SDPs whereas the third feature will be (conservatively) assessed by means of Proposition 6.1 and by estimates of the Lebesgue measure of $S_j(G; G_M) \cap S_X^\gamma$ obtained from pseudo-random samples.

### 6.2 Numerical Examples

A simple SISO numerical example is now presented to illustrate the potential of the estimator given by $G_{av}$ (obtained on the basis of *Prob. 4*) in the search of trade-offs between worst-case and “point-wise” performance over the perturbed model set. The comparison between this estimator and the minimax estimator given by
\( G_M \) (obtained on the basis of Prob. 1) will be based on the estimate \( \mu^0_i(G_{av})/2 \) of the “fraction of the model set” upon which \( G_{av} \) improves on \( G_M \) and on the largest (over the “channel” model set) relative decrease of the estimator error brought about by \( G_{av} \), i.e., \( 1 - \eta_{RM}(G_{av}) \), where

\[
\eta_{RM}(G_{av}) = \inf \{ \mathcal{J}(G_{av}; H)/\mathcal{J}(G_M; H) : H \in S_H \}.
\]

Additionally, for randomly-generated samples of SISO, FIR model perturbations \( X \) of a pre-specified length \( L \) (say \( \{X_1, \ldots, X_N \} \), with \( X \) uniformly distributed on \( \{\theta \in \mathbb{R}^{L+1} : \|\theta\|_{\infty} \leq \gamma \} \)), the following statistics where computed \( i_{fN} = i_{\chi}/N_x \), where \( i_{\chi} \) is the number of instances \( X_i \) such that \( \mathcal{J}(G_{av}; H_0 + X_i) < \mathcal{J}(G_M; H_0 + X_i) \), and \( \min \{\mathcal{J}(G_{av}; H_0 + X_i)/\mathcal{J}(G_M; H_0 + X_i) : i = 1, \ldots, N \} \) - note that \( i_{fN} \) is a consistent estimator for the ratio between the “volumes” of the “improvement set” (say \( v_{XL} \)) and that of \( S_{\chi} \) (say \( v_{XL} \)) when \( X \) is confined to the set of FIRs of length \( L \).

The possible point-wise improvements were to be obtained at the expenses of a pre-specified, maximum allowed increase in the worst-case MSE vis-à-vis that attained with the minimax estimator.

To generate the numerical results, the following sequence of computations is required for a given set-up specified by \( \{\Gamma_y, \Gamma_v, H_0, H_1, W, \gamma\} \):

\( (i) \) Compute (an approximation to) the solution \( G_o \) (with minimal realization \( (A_o, B_o, C_o, D_o) \)) of the nominal MSE problem \( \min_{G \in \mathbb{R}^{m_x \times m_y}} \mathcal{J}(G; H_0) \) - note that \( G_o = \{H_1 \Gamma_y H_0^* (\psi_o)^{-1}\} A_o \psi_o^{-1} \) where \( \psi_o \) is a spectral factor of \( \Gamma_v + H_0 \Gamma_y H_0^* = \psi_o \psi_o^* \).

\( (ii) \) Compute the frequency-response \( G_m \) of the minimax MSE estimator in the class \( S_G = S_G^{nom} \triangleq \{G(\beta) = \beta Y_o^a : \beta \in \mathbb{R}^{m_x \times (n_o + m_w)} \} \), where \( Y_o^a = \begin{bmatrix} Y_o B_o \\ I_{n_w} \end{bmatrix} \) and \( Y_o (e^{j\phi}) = (e^{j\phi} I - A_o)^{-1} \), and the corresponding worst-case error \( \mathcal{J}_{\chi}(S_G) \), solving the SDP of Proposition 4.2.

\( (iii) \) Choose an upper bound for worst-case performance \( \eta_o = (1 + \alpha) \mathcal{J}_{\chi}(S_G) \), \( \alpha \in (0, 1) \) and obtain \( G_{av} \) solving the SDP of Proposition 5.2.

\( (iv) \) Compute \( \mu^o_i(G_{av}) \) as defined in Proposition 6.1.

\( (v) \) Compute \( \eta_{RW}(G_{av}) = \inf \{\mathcal{J}(G_{av}; H) - \lambda \mathcal{J}(G_M; H) : H \in S_H, \lambda \in \mathbb{R}_+ \} \), i.e.,

\[
\eta_{RW}(G_{av}) = \inf \{\lambda > 0 : f_{RW}(\lambda) \leq 0 \}, \quad f_{RW}(\lambda) = \inf \{\mathcal{J}(G_{av}; H) - \lambda \mathcal{J}(G_M; H) : H \in S_H \}
\]

by means of a line search with respect to \( \lambda \) with \( f_{RW}(\lambda) \) (for a given value of \( \lambda \)) computed by means of a SDP.

To ensure that tight confidence intervals for \( v_{IL}/v_{XL} \) can be constructed from a sample \( \{X_1, \ldots, X_N \} \), a lower bound on \( N_x \) is enforced to ensure that a double-sided confidence interval of length 2\( \varepsilon \) around \( i_{fN} \) has confidence level of \( (1 - \delta) \), namely, \( N_x \geq \frac{1}{2\varepsilon^2} \log(2/\delta) \) \cite{28} – thus, with \( \varepsilon = 10^{-2} \) and \( \delta = 10^{-2} \), \( N_x \geq 16.505 \).

A simple, SISO numerical experiment was carried out with the following data:

\[
m_y = m_v = m_e = 1, \quad \Gamma_y = \sigma_y^2, \quad \Gamma_v = \sigma_v^2, \quad \sigma_v = 0.1 \sigma_y, \quad \sigma_y = 5, \quad H_1 = 1, \quad H_0(e^{j\phi}) = \sum_{k=0}^{5} M_k e^{-j\phi k},
\]
\[
\begin{bmatrix}
  M_0 & M_1 & \cdots & M_5
\end{bmatrix} = 2 \times \begin{bmatrix}
  1.0 & -1.3963 & 0.9638 & -0.8713 & 0.5593 & -0.1389
\end{bmatrix}, \ W = 1, \alpha = 0.15 \ (\text{see } (\text{iii}) \text{ above}), \gamma = 0.3\|H_o\|_2.
\]

The results obtained were as follows:
\[
\delta_{\mathcal{J}^2} = -6.7966, \ \mu_o^*(G_{av}) = 1.4684 \ (\mu_i^*(G_M) = 1.4703) \text{ indicating that } G_{av} \text{ improved on } G_M \text{ over at least } 0.7342\% \text{ of the perturbed model set } \mathcal{S}_H; \ \eta_{RW}(G_{av}) \approx 0.2500 \text{ showing that at certain points of the perturbed model set } G_{av} \text{ yields a MSE value which is close to a 75\% reduction of the one obtained with } G_M \text{ – the maximum point-wise improvement observed in the sample in the case of } L = 6, \text{ achieved at a frequency-response } H_o \text{ (say) corresponds to } \mathcal{J}(G_{av}; H_o) = 3.0202 \text{ and } \mathcal{J}(G_M; H_o) = 12.5158.
\]

The role of \( \mu_i^o \) as a conservative estimate of the relative size of the “improvement set” is borne out by the values of \( i_fN \) obtained in three Monte Carlo experiments, each one with \( N = 65,000 \) samples with FIR model perturbations of length \( L = 6, 9 \) and \( 13 \). The obtained values were as follows: for \( L = 6, i_fN = 0.9673 \) for \( L = 9, i_fN = 0.9822 \) and for \( L = 13, i_fN = 0.9923 \).

Summing up, for a 15\% increase in the worst-case MSE over its minimum value, the robust estimator given by \( G_{av} \) is guaranteed in this example to improve on \( G_M \) over at least “0.73 of the perturbed model set \( \mathcal{S}_H \),” bringing about a decrease in the MSE error (vis-à-vis that of the minimax estimator) by up to 75\% in some points of \( \mathcal{S}_H \) – in fact, the “relative size” of the improvement set may be expected to be substantially bigger than 0.73 as Monte Carlo results with FIR model perturbations of length 6, 9 and 13 led to lower confidence bounds (with 0.99 confidence level) for this ratio respectively greater than 0.94, 0.97 and 0.98 (i.e., the “improvement set” in each these cases is at least 94\%, 97\% and 98\% of the uncertain model set).

The numerical results for this SISO example are summarized in Tables (recall that \( \mathcal{J}, \eta^o \) and \( \eta_{RW} \) are the worst-case, average smallest relative MSE on \( \mathcal{S}_H \))

| \( G_{av} \) | \( \mathcal{J}(G) \) | \( \eta^o(G) \) | \( \mathcal{J}(G; H_o) \) | \( \eta_{RW}(G) \) |
|---|---|---|---|---|
| \( G_M \) | 26.8604 | 12.6040 | 10.1015 | \( \approx 0.25 \) |

Table 1: Performance on \( \mathcal{S}_H \).

Table 2 exhibits the relative frequencies of the improvement set \( i_fN \) in three experiments involving 65,000 samples of FIR perturbations of length \( L_{FIR} = 6, 9 \) and \( 13 \), respectively

| \( L_{FIR} \) | 6 | 9 | 13 |
|---|---|---|---|
| \( i_fN \) | 0.9673 | 0.9822 | 0.9923 |

Table 2: Relative frequency of the improvement set.

In addition, similar numerical experiments were performed with two, \( 2 \times 2 \) MIMO examples. The first one involves a FIR of length 4 as nominal channel model (\( H_o \)), a “coloured” signal \( y \) (filtered white-noise), with \( \phi_y \) as the frequency-response of the shaping filter (\( H_o \) and \( \phi_y \) are presented in the Appendix), white
observation noise with covariance matrix given by $\sigma_v^2 I$ and signal-to-noise ratio (at the channel output) $\sigma_v / \| H_0 \phi_y \|_2 = 0.2$, perturbations radius $\gamma = 0.2 \times \| H_0 \phi_y \|_2$ and average MSE minimization allowing for 0.1 increase on worst-case MSE above its minimum value. The numerical examples for the first MIMO example are presented in Tables 3 and 4.

| $G_{av}$ | $\mathcal{J}(G)$ | $\eta^a(G)$ | $\mathcal{J}(G; H_0)$ | $\bar{\eta}_{RW}(G)$ |
|----------|-----------------|-------------|------------------------|------------------------|
| $G_M$    | 3.8090          | 1.961       | 1.8027                 | $\approx 0.45$        |
|          | 3.4628          | 2.5947      | 2.5289                 | 1                      |

Table 3: MIMO example 1: Performance on $\mathcal{S}_H$.

The relative frequencies of the improvement set in three experiments involving FIR perturbations of length $L_{\text{FIR}} = 4, 8$ and 10 are displayed in Table 3.

| $L_{\text{FIR}}$ | 4  | 8  | 10 |
|------------------|----|----|----|
| $f_i/N$          | 0.9987 | 0.9991 | 0.9999 |

Table 4: MIMO example 1: Relative frequency of the improvement set.

Analogously to what happened in the case of the SISO example presented above, with small allowed increase on worst-case performance (10% in this example) it is possible to achieve significantly better “pointwise” performance with $G_{av}$, including reduction up to 55% on the MSE of the minimum estimator at specific channel models and very high frequency of improvement over $G_M$ on Monte-Carlo experiments.

To further illustrate the “pointwise” improvements brought about by the a/w estimator, a plot is presented below which exhibits the variation of the error estimation criterion attained by the minimax and the a/w estimators over a path in the “uncertain” model set (with FIRs of length 4). This path goes from the most favourable to the most unfavourable model for the a/w estimator, passing through the nominal model.

![Figure 2: Pointwise performance of $G_{av}$ and $G_M$.](image-url)
It can be seen that, at the most unfavourable model for the a/w estimator (corresponding to the value of 21 in the horizontal axis), its steady-state MSE exceeds that of the minimax estimator by about 10%; whereas at the most favourable model (value of 1 in the horizontal axis) its steady-state MSE is about 0.54 of the MSE of the minimax estimator (the nominal model corresponds to the value 11 in the horizontal axis). In addition, in about “90% of the path” the pointwise performance of $G_{av}$ is superior to that of $G_M$.

In the second MIMO example, $H_0$ is given by a fifth-order state-space model with non-zero eigenvalues (see the Appendix), $\gamma = 0.3 \times \|H_0 \phi_y\|_2$ and the remaining data is as in the first MIMO example. The numerical results obtained are given in Tables 5 and 6.

| $G_{av}$ | $J(G)$ | $\eta^p(G)$ | $J(G;H_0)$ | $\eta_{RW}(G)$ |
|----------|--------|-------------|-------------|----------------|
| $G_M$    | 3.2680 | 1.8904      | 1.5001      | $\approx 0.6$  |

Table 5: MIMO example 2: Performance on $S_H$.

| $L_{\text{FIR}}$ | 4 | 8 | 10 |
|------------------|---|---|----|
| $\ell_{/N}$      | 0.9104 | 0.9632 | 0.9747 |

Table 6: MIMO example 2: Relative frequency of the improvement set.

In this case, with worst-case MSE only 8% larger than that of $G_M$, $G_{av}$ achieves MSE reduction of up to 40% at specific points of the channel model set and improves on $G_M$ over most of the Monte-Carlo samples examined.

### 6.3 Nominal $H_\infty$ Estimation

In this subsection, possible ways of comparing the performance of the linear estimators defined by Prob. 2 and Prob. 5 (with frequency-response $G_M$ and $G$, respectively) are discussed along the lines pursued in Subsection 6.1.

In this case, the largest improvement in pointwise performance (by $G$ over $G_M$) is defined by

$$\eta^\infty_p(G) = \sup \{ \langle \Gamma_0 \bar{z}, \bar{z} \rangle : \bar{z} \in S_\delta \}, \quad \text{where} \quad \Gamma_\delta = \Gamma_{e0} - \Gamma_{e1},$$

$$\Gamma_{e0} = H_{xz}(G_M)^*H_{xz}(G_M), \quad \Gamma_{e1} = H_{xz}(G)^*H_{xz}(G), \quad H_{xz} = H_{y} - GH_{xz} \quad \text{and} \quad S_\delta = \left\{ \bar{z} \in R_{m_y + m_v} : \bar{z} = \begin{bmatrix} \bar{y}^T & \bar{v}^T \end{bmatrix}, \|\bar{y}\|_2 \leq \gamma_y, \|\bar{v}\|_2 \leq \gamma_v \right\}.$$
Proposition 6.2. For $\Gamma = E_r + E^*_r$, where $E_r \in \mathcal{R}^{m \times m}$, let
\[ \lambda_\infty (\Gamma) \triangleq \sup \{ \tilde{\lambda}(E_r(e^{j\phi})) : \phi \in [0, 2\pi] \}, \] where $\tilde{\lambda}(R)$ is the maximum eigenvalue of the Hermitian matrix $R$. Then,
(a) $\eta_\infty^G(G) \geq \lambda_\infty(M, I_{\delta} M_\gamma)$, where $M_\gamma = \text{diag} \left( \gamma_x I_{m_y}, \gamma_x I_{m_y} \right)$.
(b) If $\mu > 0$ is such that $\lambda_\infty(M_\gamma (\mu \Gamma_0 - \Gamma_{\alpha 1}) M_\gamma) \geq 0$, $\eta_\infty^G(G) \leq \mu$.

A simple SISO numerical example is now presented to illustrate the possible trade-off between worst-case and “pointwise” performance enabled by an average cost/worst-case constraint estimator. The nominal model in this case is $(1/2)H_0$, where $H_0$ is as in Subsection 6.1. The remaining elements of the estimation set-up are $H_t = 1, \gamma_y = 5, \gamma_v = 0.5, m_v = 1, m_x = m_a = 1, W_y = 1$, and $W_v = 1$. The estimators obtained from the approximate solution of Prob. 2 ($G_M$, see Proposition 4.3 and the subsequent “Remark”) and Prob. 5 ($G_{av}$, see Proposition 5.3 and Remark 5.1) give rise, respectively, to worst-case, squared estimation errors smaller than 18.3903 and 19.7594 (an increase of less than 10% in the minimum value of the worst-case performance index). Using Proposition 6.2 and approximately computing (taking a grid on $[0, 2\pi]$) $\lambda_\infty(M_\gamma I_{\delta} M_\gamma)$ and $\lambda_\infty(M_\gamma (\mu \Gamma_0 - \Gamma_{\alpha 1}) M_\gamma)$ for several values of $\mu$, the following results were obtained: $\eta_\infty^G(G_{av}) \geq 8.5866$ and $\eta_\infty^G(G_{av}) \leq 0.15$ – in words, at some points in the disturbance set $S_x$, $G_{av}$ diminishes the squared estimation error attained by $G_0$ by at least 8.6 (in the range of $(0, 18.3903)$ and brings it down to 15% of its value.

In addition, Monte Carlo experiments involving $\tilde{y} \in \tilde{S}_q$ and $\tilde{v} \in \tilde{S}_v$ defined by FIRs of prescribed length (say, $L$) were carried out to estimate the “relative size” (Lebesgue measure) of the set of signal pairs on which $G_{av}$ yields a small squared estimation error than $G_M$. Such an estimate is obtained from a pseudo-random sample of $N_s = 30,000$ pairs $\tilde{z}_k^{\top} = (\tilde{y}_k^{\top}, \tilde{v}_k^{\top})$ computing the relative frequency $\frac{1}{N_s} \sum_{i=1}^{N_s} \mathbb{I}_z (\tilde{z}_k; G_n)$, where $\mathbb{I}_z (\tilde{z}_k; G_n) = 1$, if $\| e(\tilde{z}_k; G_n) \|_2 < \| e(\tilde{z}_k; G_M) \|_2$ and $\mathbb{I}_z (\tilde{z}_k; G_n) = 0$ otherwise. The sample $\{ \tilde{z}_k : k = 1, \ldots, N_s \}$, in turn, is obtained from $N_s$ independent samples of FIRs of length $L$, $F_{\alpha} = (F_{\alpha 1}, \ldots, F_{\alpha (L+1)})$, $\alpha = y, v$, with $F_{\alpha}$ uniformly distributed on $\{ F \in \mathbb{R}^{L+1} : \| F \|_2 \leq \gamma_\alpha \}$.

The relative frequency obtained in an experiment with FIRs of length 30 was 0.7114 so that with the sample size equal to 30,000, a confidence interval for the desired probability was obtained with lower limit equal to 0.70 and confidence level equal to 0.99 (cf. Subsection 6.1). In other similar experiments, results were obtained to the effect that the corresponding probabilities grow with the length of the FIRs involved.

Summing up, this simple example illustrates the possibility of improving on $G_M$ over a “large portion” of the disturbance set, with significant absolute and relative decreases in the squared estimation error at some points of the set, if a relatively small increase is allowed in the worst-case performance index over its achievable minimum. Similar numerical results were obtained in the case of robust $H_\infty$ estimation.
7 Concluding Remarks

In this paper, three basic linear estimation problems involving set-theoretical uncertainty were revisited with the major aim of designing estimators which may be viewed as alternatives to minimax estimators. The problems addressed were robust $H_2$ and $H_\infty$ estimation in the face of non-parametric “channel-model” uncertainty ($H_2$ and $H_\infty$ balls of frequency-responses) and a nominal Hinf problem (in this case, set-theoretical uncertainty pertains to $H_2$ balls of “information” and noise signals). To provide trade-offs between worst-case and “pointwise” performance over the uncertainty set, in each case, average criteria on $H_2$ balls were derived as limits of averages over sets of FIRs of a given length (say, $L$) as $L$ grows unbounded. Linear estimation design problems were then formulated as minimization of an average cost under the constraint that worst-case performance of any admissible estimator does not exceed a prescribed value. The corresponding minimax and average cost/worst-case constraint problems were all recast as SDPs. A brief discussion was presented on how to compare such estimators considering the “size” of the part of the uncertainty set on which a “constrained-average” estimator improves on the corresponding minimax estimator, as well as on how much absolute or relative improvement is brought about by the former at some points of the uncertainty set. The SDPs involved were solved in the case of simple examples and the numerical results obtained indicate the potential of this approach to provide attractive alternatives to minimax estimators.
8 References

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9 Appendix

Notation for Problem Data

For a string of symbols \( s \), a state-space realization \( (A_s, B_s, C_s, D_s) \) is denoted by \( \Sigma_s \).

Estimation Set-up and \( \mathcal{H}_2 \) Problem

\[ H_1, \Gamma_y = \phi_y \phi_y^*, \Gamma_v = \phi_v \phi_v^* \] (Subsection 2.1).

\( H_n : \) nominal “channel” frequency-response with minimal realization (mr, for short) \( \Sigma_{H_0} \).

\( W : \) weighting function for \( \mathcal{H}_2 \)–uncertainty (eq. \( 2.3 \)).

Nominal \( \mathcal{H}_\infty \) Problem

\( W_\alpha, \alpha = y, v : \) weighting function for the \( \mathcal{H}_2 \)–signal ball \( S_\alpha \) with mr \( \Sigma_{W_\alpha} \).

\( H_{ty} \triangleq [H_tW_y^{-1} : 0_{m_x \times m_v}] \) and \( H_{oa} \triangleq [H_oW_y^{-1} : W_v^{-1}] \) with mrs \( \Sigma_{ty} \) and \( \Sigma_{oa} \).

Robust \( \mathcal{H}_\infty \) Problem

\( H_{tn} = [H_{ty} : 0_{m_x \times m_v}] \) and \( H_{on} = [H_{oa} : I_{m_y}] \) with mrs \( \Sigma_{Htn} \) and \( \Sigma_{on} \).

\( W_{Ht} : \) weighting function for \( \mathcal{H}_\infty \)–uncertainty.

\( W_{Hy} \triangleq (W_yW_{Ht})^{-1} \) with mr \( \Sigma_{W_{Hy}}, W_{Hy} = W_{Hy}[I_{m_y} : 0_{m_y \times m_v} : 0_{m_y \times m_v}] \).

Matrix definitions for Proposition 4.5 and 4.7

Given \( \Sigma_{ty}, \Sigma_{oa} \) as in as in the definitions for the Nominal \( \mathcal{H}_\infty \) Problem above above,

\[ A_{ty} \in \mathbb{R}^{n_{ty} \times n_{ty}}, A_{oa} \in \mathbb{R}^{n_{oa} \times n_{oa}}, n_G = n_{ty} + n_{oa}; A_o = \text{diag}(A_{ty}, A_{oa}, 0_{n_G \times n_G}) \).

\[ b_o = [b_{ty} : b_{oa} : 0_{m_x \times n_{oa}}], m_x = m_y + m_v \).

\[ c = [C_{ty} : 0_{m_x \times (n_{oa} + n_G)}], C_{oa} = [0_{m_v \times n_{ty}} : C_{oa}], \hat{C}_o = [C_{ty} : 0_{m_x \times n_{oa}}] \).

\[ D_{ty} = D_{ty}([I_{m_y} : 0_{m_y \times m_v}]^T) \).

\[ e_b = -D_{W_v}[C_x : D_{Ht}D_{W_y}^{-1}], e_{bo} = [\hat{c}_o : D_{ty}] \).

\[ \psi_b \triangleq [A_{oa} : B_{ax1}] + B_{ax2}e_b, A_{ax} = \text{diag}(A_{ty}, A_{ax}), B_{ax} = [B_{ax1} : B_{ax2}] \triangleq [B_{ty}^T : B_{oa}^T]^T, \]

\[ B_{ax1} = B_{ax}([I_{m_y} : 0_{m_y \times m_v}]^T) \).

Matrix Definitions for Subsection 4.3

Given \( \Sigma_{oa}, \Sigma_{Htn} \) and \( \Sigma_{W_{Hy}} \) as in the definitions for the Robust \( \mathcal{H}_\infty \) Problem above,

\[ A_{a1} = \text{diag}(A_{ty}, A_{Htn}, A_{oa}), n_G \] denotes the dimension of \( A_{a1} \).

\[ b_a = [b_{a1} : 0_{(m_y + 2m_v) \times n_{oa}}], B_{a1} = [B_{Hy}^T : B_{Htn}^T : B_{oa}^T], B_{a} = B_{Hy}[I_{m_y} : 0_{m_y \times 2m_v}] \).

\[ B_{a1} = B_{a1}([I_{m_x} : 0_{m_x \times m_y}]^T) \).

\[ C_a = \text{diag}(\sigma_{m}^{1/2}I_{m_y}, I_{m_x})([\hat{C}_{Wa} : 0_{(m_y + m_x) \times (n_{oa} + n_G^2)}, \hat{C}_{Wa} = \text{diag}(\gamma_{H}C_{W^H}, C_{Htn}) \)].
Thus, where $\bar{MSE}$ and the response is now briefly reviewed for both finite time and as time goes to infinity. To this effect, note first that the same letter), the response from zero initial conditions would be given by $G$ so that $E_z = \{0\}$, then

$\Rightarrow \parallel F_k \parallel_2 < \infty$ (non-zero initial conditions would fade-out with time due to asymptotic stability) so that $E\{\parallel e(K)\parallel_E^2\} = \text{tr}\left\{\sum_{k=0}^{\infty} F_{ak} E[\hat{z}(K-k)\hat{z}(K-l)] F_{ak}^T\right\} \Rightarrow E\{\parallel e(K)\parallel_E^2\} = \text{tr}\left\{\sum_{k=0}^{\infty} F_{ak} F_{ak}^T\right\}$

Thus, denoting by $F_e$ the frequency-response from $\hat{z} = \begin{bmatrix} \hat{y}^T & \hat{v}^T \end{bmatrix}^T$ to $e$, i.e., $e = (H_a - GH)\phi_y \hat{y} + G\Phi_v \hat{v} = F_e \hat{z}$ (by a slight abuse of notation both signal sequences and their Fourier transforms are denoted by the same letter), the response from zero initial conditions would be given by $e(K) = \sum_{k=0}^{K} F_{eK} \hat{z}(K-k)$, where $F_e(e^{j\phi}) = \sum_{k=0}^{\infty} F_{ak} e^{-jk\phi}$ (non-zero initial conditions would fade-out with time due to asymptotic stability) so that $E\{\parallel e(K)\parallel_E^2\} = \sum_{k=0}^{\infty} F_{ak} E[\hat{z}(K-k)\hat{z}(K-l)] F_{ak}^T \Rightarrow E\{\parallel e(K)\parallel_E^2\} = \text{tr}\left\{\sum_{k=0}^{\infty} F_{ak} F_{ak}^T\right\}

Thus, $E_{\bar{z}} = [C_{a1} : D_{a1}]$, and $E_e = [A_{a1} : B_{a1}].$

The connection between the mean-squared estimation error and the $H_2$-norm of the “error system” frequency-response is now briefly reviewed for both finite time and as time goes to infinity. To this effect, note first that the assumptions on $y$ and $v$ amount to taking them to be filtered “versions” of independent, zero-mean sequences $\hat{y}$ and $\hat{v}$ such that $\forall k, \ell, E[\hat{y}(k)\hat{y}(\ell)^T] = \delta_{kk} I_{m_y}, E[\hat{v}(k)\hat{v}(\ell)^T] = \delta_{k\ell} I_{m_v}$, where $\delta_{kk} = 1$ if $k \neq \ell$, $\delta_{kk} = 1$.

If stationarity assumptions are relaxed to the effect that $\forall k, \ell, E[\hat{z}(k)\hat{z}(\ell)] = \delta_{k\ell} Q_s(k)$ and $\forall k, \lambda_{\max}(Q_s(k)) \leq \mu_s$, then $\forall K, E\{\parallel e(K)\parallel_E^2\} \leq \mu_s \parallel F_e\parallel_2^2$ i.e., even in this case $\parallel F_e\parallel_2^2$ yields a uniform upper bound on the MS estimation error.

To consider sample means of the squared estimation error, note that

$\frac{1}{N+1} \sum_{K=0}^{N} \parallel e(K)\parallel_E^2 = \text{tr}\left\{F_{eN} R_{K,N} F_{eN}^T\right\},$}

where $F_{eN} = [F_{e0} \cdots F_{eN}]$ and $R_{K,N} \triangleq \text{diag}(R_K,0_{N-K,N-K})$, $R_K \triangleq \begin{bmatrix} \hat{z}(K)^T \\ \vdots \\ \hat{z}(0)^T \end{bmatrix}$.

Thus,

$\frac{1}{N+1} \sum_{K=0}^{N} \parallel e(K)\parallel_E^2 = \text{tr}\left\{F_{eN} R_{K,N} F_{eN}^T\right\},$ 

(R1)

where $R_N \triangleq \frac{1}{N+1} \sum_{K=0}^{N} R_{K,N}$ is a symmetric matrix of sample correlations with $(p, p + q)$ blocks given by $\forall p = 0, 1, \ldots, N, \forall q = 0, 1, \ldots, N - p, \{R_{N}\}_{p,q} = \frac{1}{N+1} \sum_{K=q}^{N-p} \hat{z}(K)\hat{z}(K-q)^T$.

It follows from (R1) that

$\forall N, \frac{1}{N+1} \sum_{K=0}^{N} \parallel e(K)\parallel_E^2 \leq \lambda_{\max}(R_N) \parallel F_e\parallel_2^2
so that \( \| F_e \|_2^2 \) also yields a uniform upper bound in the sample-averages of the squared estimation error which is based on correlation estimates.

MIMO Example Data

Data for the First MIMO Example:

Data for the Second MIMO Example:

\[ A_{\phi_y} = \begin{bmatrix} 0.6 & 0.4 & 0.2 & 0.3 & 0.7 & 0.5 \\ 0.0 & 0.1 & 0.2 & 0.1 & 0.3 & 0.4 \end{bmatrix}, \quad A_{\phi_y} = \text{diag}(\lambda_{\lambda_{\phi_y}}), \quad V_{\phi_y} = \begin{bmatrix} 1.0 & 1.3 & 1.0 & 1.4 & 1.0 & 1.5 \\ 0.0 & 0.0 & 0.0 & 0.2 & 0.0 & 1.9 \end{bmatrix}, \]

\[ A_{\phi_y} = V_{\phi_y} * A_{\phi_y} * [V_{\phi_y}]^{-1}, \quad B_{\phi_y} = \begin{bmatrix} 0.6 & 0.4 & 0.2 & 0.3 & 0.7 & 0.5 \\ 0.0 & 0.1 & 0.2 & 0.1 & 0.3 & 0.4 \end{bmatrix}, \quad B_{\phi_y} = V_{\phi_y} B_{\phi_y}, \]

\[ C_{\phi_y} = [0.9635 -0.8713 0.5593 -0.4389 0.4015], \quad C_{\phi_y} = \begin{bmatrix} \phi_x \phi_y \end{bmatrix}, \quad D_{\phi_y} = [1.0 0.3] \]

\( \phi_y \) is as in the first MIMO example.

Proof Outlines

Proof of Proposition 3.1: To compute \( \nu_N = \int_{S^N_{\theta}} d\theta \) and \( \nu_{\gamma_N} \triangleq \int_{S^N_{\theta}} \theta^T P_{\gamma_N}(\theta) d\theta \), define

\[ \hat{\theta} = \left( P_{\gamma}^{1/2} \right) \theta \]

and rely on polar coordinates to obtain

\[ \nu_N = \det \left( P_{\gamma}^{1/2} \right)^n \frac{1}{n} \left\{ \int_0^\pi [\sin(\alpha)]^{n-2} d\alpha \right\} \prod_{k=2}^{n-2} (2\pi), \]

\[ \nu_{\gamma_N} = \left\{ \sum_{i=1}^n [\hat{P}_{\gamma}^{1/2}]_{ii} \right\} \cdot \left\{ \int_0^\pi [\sin(\alpha)]^{n-2} d\alpha \right\} \prod_{k=2}^{n-2} (2\pi), \]

where \( \prod_{k=2}^{n-2} \triangleq \sum_{k=2}^{n-2} \int_0^\pi [\sin(\alpha)]^{n-2} d\alpha \).

Proof of Proposition 3.2: It turns out that \( P_{\gamma} = I \) and, hence, \( P_{\gamma}^{1/2} = I \) and \( P_{\gamma}^{1/2}(\theta) = P_{\gamma}(\theta) \). As

\[ F_N(\theta) = (G \otimes \phi_{y_1})(I_{m_y} \otimes (Y_{x_1})^T), \quad [I_{m_y} \otimes (Y_{x_1})^T][I \otimes (Y_{x_1})^T] = I_{m_y} \otimes [(Y_{x_1})^T (Y_{x_1})^T] \]

and \( (Y_{x_1})^T (Y_{x_1})^T = (N + 1)I_{m_y} \), it follows that \( \text{tr} \left\{ \hat{P}_{\gamma}(\theta) \right\} = (N + 1) \left\{ G \otimes \phi_{y_1}^T, G \otimes \phi_{y_1}^T \right\} \). Thus (in the
light of Proposition III.1) \( \eta_N^a(G) = J(G; H_o) + \left\{ \frac{\gamma^2(N+1)}{m_\alpha m_y(N + 1) + 2} \right\} \langle G \otimes \phi_y^T, G \otimes \phi_y^T \rangle \) from which Proposition 3.2 follows. ■

Proof of Proposition 3.3: Note that \( \eta_{\alpha N}^a(G; H) = (\mu_{\alpha N}^a)^{-1} \mu_{\alpha N}^r(G; H) \), where \( \mu_{\alpha N}^r = \int_{\mathcal{S}_{H_\infty}} \beta_\alpha^\top \Gamma_{\alpha \beta} \beta_\alpha d\beta_\alpha \), so that \( \mu_{\alpha N}^a \) and \( \mu_{\alpha N}^r \) are formally identical to \( \nu_N \) and \( \nu_{\mathcal{T}N} \) above and, hence, the derivation of \( \eta_N(\cdot) \) applies mutatis mutandis to \( \eta_{\alpha N}^a \) – note that in this case the “nominal term” is zero and it is not necessary to rely on Kronnecker products. ■

Derivation of \( \eta^b(G; \mathcal{S}_{H_\infty}) \) (Subsection 3.3): First, the set \( \bar{\mathcal{S}}_{H_\infty} \supset \mathcal{S}_{H_\infty} \), is introduced where

\[
\bar{\mathcal{S}}_{H_\infty} = \{ H \in \mathcal{R}_e^{m_x \times m_y} : \| (H - H_0) \mathcal{W} \|_2 \leq \gamma_h \}, \quad \mathcal{W} = \phi_y^T \phi_y^{-1}
\]

and \( \phi_{yW} \) is a spectral factor of \( \eta_{\mathcal{T}W} \). Then, replacing \( \gamma, \Gamma_y, \Gamma_v, \) and \( \phi_{y\gamma} \) respectively by \( \gamma_h, \Gamma_y^a, \Gamma_v^a, \) and \( \tilde{\phi}_{y\gamma} = W^{-1} \phi_y^a \) in the expression of \( \eta_{ax}(G) \) leads to (since \( \tilde{\phi}_{y\gamma} = \phi_{yW} I_{m_y} \))

\[
\eta^b(G; \mathcal{S}_{H_\infty}) = \eta^a(G; H_0) + (\gamma_h^2/m_y m_v) \langle G \otimes (\phi_{yW} I_{m_y}), G \otimes (\phi_{yW} I_{m_y}) \rangle
\]
or, equivalently,

\[
\eta^b(G; \mathcal{S}_{H_\infty}) = \eta^a(G; H_0) + (\gamma_h^2/m_v) \langle G \phi_{yW}^T, G \phi_{yW} \rangle
\]

Proof of Proposition 4.1:

(a) Letting \( \lambda_o \triangleq \| F_{y} \|_2^2 \), where \( F_y = G \otimes F_y^T \), and noting that \( \lambda \in \mathcal{S}_\alpha \iff \lambda > \lambda_o \), the proof consists of the following intermediate statements:

1. \( \forall \lambda \leq \lambda_o, \varphi_\mathcal{D}(\lambda; D) = +\infty \).

2. \( \inf \{ \varphi_\mathcal{D}(\lambda; G) : \lambda \geq \lambda_o \} = \inf \{ \varphi_\mathcal{D}(\lambda; G) : \lambda \in \mathcal{S}_\lambda \} \).

(b) It is a direct consequence of Lemma A1 ([13]).

(c) The proof hinges on invoking the (so-called) discrete-time, bounded-real lemma ([27]) and using elementary congruence transformations (row and column permutations), say \( T \) and \( T^T \), to convert the condition “\( \lambda \in \mathcal{S}_\lambda \)” into the positive-definiteness of a diagonal block of \( TQ_{LQ}(P; \Sigma_a, M(\lambda))T^T \). ■

Proof of Proposition 4.5: Writing

\[
-Q_{BR}(P, \Sigma_{FG}(\theta), M(\sigma)) = \begin{bmatrix} P & 0 \\ 0 & M_a \end{bmatrix} - \begin{bmatrix} A_{FG}^T & C_{FG}^T \\ B_{FG}^T & D_{FG}^T \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{FG} & C_{FG} \\ B_{FG} & D_{FG} \end{bmatrix}
\]

and invoking the Schur complement formula, the condition “\( Q_{BR}(\cdot) < 0 \)” can be recast as “\( \tilde{\psi}(P, \sigma, 0) > 0 \)” where \( \tilde{\psi}(P, \sigma, 0) = \text{diag}(P, M_a, P^{-1}, I_{m_a}) + \tilde{\psi}_a(\sigma) + \tilde{\psi}_a(\sigma)^T \), \( \tilde{\psi}_a(\sigma) = \begin{bmatrix} 0 \\ A_{FG} & C_{FG} \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{FG} & C_{FG} \end{bmatrix} \).

The proof is concluded by bringing elementary congruence transformations to bear on “\( \tilde{\psi}(P, \sigma, \theta) \)” and writing

\[
A_{o} = \text{diag}(A_{az}, 0_{n_G \times n_G}), \quad B_{o} = \begin{bmatrix} B_{az} \\ 0_{n_G \times m_y} \end{bmatrix}, \quad C_{o} = \begin{bmatrix} C_{ly} : 0_{m_a \times (n_{az} + n_G)} \end{bmatrix}, \quad D_{o} = D_{Ly} + D_{L(\theta)},
\]

\[
A_{L}(\theta) = \begin{bmatrix} 0_{n_{az} \times n_G} & 0_{n_{az} \times n_G} \\ B_G C_{az} & A_G \end{bmatrix}, \quad B_{L}(\theta) = \begin{bmatrix} 0_{n_{az} \times m_y} \\ B_G D_{az} \end{bmatrix}, \quad C_{L}(\theta) = \begin{bmatrix} -D_G C_{az} : -C_G \end{bmatrix}, \quad D_{L}(\theta) = -D_G D_{az}.
\]

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Proof of Proposition 4.6: (a) follows directly from, Propositions 4.4 and 4.5, (i) and (ii).

(b) follows from Proposition 4.5, (i) and (ii) with \( \rho(A_G) < 1 \) being a consequence of Proposition 4.5 and the fact that \( Q_{BB}(P^\sigma, \Sigma_{FG}(\theta), M(\sigma^\sigma)) < 0 \) \( \Rightarrow \rho(A_G) < 1 \) (as, in this case, \( A_{FG}^T P^\sigma A_{FG} - P^\sigma < 0 \) with \( P^\sigma > 0 \)) since \( A_{GF} \) is taken to be lower triangular with \( A_G \) as one of its diagonal blocks.

Proof of Proposition 4.7: (a) As \( W_a^T \) and \( W_a^T \psi_\alpha(P)W_a \) are given by

\[
W_a^T = \begin{bmatrix}
1 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad W_a^T \psi_\alpha(P)W_a = \begin{bmatrix}
R & A_{az} & 0 & B_{az} \\
A_{az} & S & N & 0 \\
0 & N^T & X & 0 \\
B_{az} & 0 & 0 & M_\sigma
\end{bmatrix}.
\]

Pre and post-multiplying \( W_a^T \psi_\alpha(P)W_a \) by \( I_x \) and \( I_x \) (say), where \( I_x \) is a column permutation matrix, leads to \( W_a^T \psi_\alpha(P)W_a > 0 \) \( \iff \) (invoking the Schur complement formula)

\[
\begin{bmatrix}
S & N \\
N^T & X
\end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix}
R & B_{az} \\
B_{az}^T & M_\sigma
\end{bmatrix} - \begin{bmatrix}
A_{az} & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
S & N \\
N^T & X
\end{bmatrix}^{-1} \begin{bmatrix}
A_{az} & 0 \\
0 & 0
\end{bmatrix} > 0
\]

Moreover, since \( \begin{bmatrix}
S & N \\
N^T & X
\end{bmatrix}^{-1} = \begin{bmatrix} R & M \end{bmatrix} \), the last LMI above can be rewritten as

\[
\begin{bmatrix}
R & B_{az} \\
B_{az}^T & M_\sigma
\end{bmatrix} - \begin{bmatrix} I_{12} \end{bmatrix} A_{az} R A_{az}^T I_{12} > 0 \quad \iff \quad \begin{bmatrix}
R - A_{az} R A_{az}^T B_{az} \\
B_{az}^T & M_\sigma
\end{bmatrix} > 0.
\]

(b) Bringing a similar argument to bear on \( W_b^T \psi_\alpha(P, \sigma)W_b \) provides a proof of Proposition IV.5(b) with

\[
W_b^T = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \text{where} \quad T_{63} = -D_{az2}^{-1} C_{az}, \quad T_{65} = -D_{az2}^{-1} D_{oz1}, \quad D_{oz1} = D_{Ho} D_{Wg}^{-1} D_{oz2} = D_{Wg}^{-1}
\]

and \( C_{az} \) is as above.

Proof of Proposition 4.8: Proposition 4.8(a) is a direct consequence of Propositions 4.6 and 4.7 and the fact that \( S \geq R^{-1} \) \( \iff \begin{bmatrix} S & I \end{bmatrix} R \end{bmatrix} \geq 0 \).

(b) The fact that \( P^\sigma > 0 \) follows from the following logical sequence based on Schur Complements: as \( X > 0, P^\sigma > 0 \) \( \iff (S_o - Q_{sr} X V^{1/2} X^{-1} V^T Q_{sr}) > 0 \) \( \iff (S_o - Q_{sr}^2) > 0 \) \( \iff (S_o - (S_o - R_o^{-1})) > 0 \) \( \iff R_o^{-1} > 0 \) \( \iff R_o > 0 \). The statements about \( \theta \) at the end of Proposition 4.8(b) follow from Proposition 4.6(b) and Proposition 4.7.

Equations (5.1) and (5.2): (5.1) and (5.2) follow from the equalities (for \( Q = Q^T \geq 0 \))

\[
\text{tr}(M Q M^T) = \inf \{ \text{tr}(P) : P = P^T, P \geq M Q M^T \} = \inf \left\{ \text{tr}(P) : P = P^T, \left[ \begin{array}{c}
P \\ Q_{1/2} M^{T} I \\
\hline
\end{array} \right] \geq 0 \right\}.
\]

The definition of \( \langle \cdot, \cdot \rangle \), the parametrizations \( G(\beta) = \beta Y_i, i = 1, 2, 3 \), and the identity \( M_1 M_2 \otimes M_3 = [M_1 \otimes I][M_2 \otimes M_3]^T \).

Proof of Proposition 5.1: The first part is an immediate consequence of (4.8). The second part follows from Proposition A.1 below.
Proofs

**Upper bounds on** $\tilde{J}_\infty(G; H)$ **and** $\tilde{J}_\infty^a(G)$: Notice first that “$\bar{y} \in S_y$ and $\bar{v} \in S_v$” $\Leftrightarrow$ “$\|\bar{y}\|^2 - \gamma_y^2 \leq 0$ and $\|\bar{v}\|^2 - \gamma_v^2 \leq 0$.”

Thus, $\forall \bar{y} \in S_y$, $\forall \bar{v} \in S_v$, $\forall \sigma_y > 0$, $\forall \sigma_v > 0$,

$Lag_\infty(\bar{z}, \sigma; G, H) \geq \|\bar{e}(\bar{z}; G, H)\|^2_2 \Rightarrow$

$$\tilde{J}_\infty(G; H) \leq \sup\{Lag_\infty(\bar{z}, \sigma; G, H_0): \bar{y} \in S_y, \bar{v} \in S_v\} \leq \sup\{Lag_\infty(\bar{z}, \sigma; G, H_0): \bar{z} \in \mathcal{R}^m_x\} \Rightarrow$$

$\forall \sigma_y > 0$, $\forall \sigma_v > 0$, $\tilde{J}_\infty(G; H_0) \leq \varphi^a_{\infty}(\sigma; G, H_0) \Rightarrow$

$$\tilde{J}_\infty^a(G) \leq J_\infty(G; H_0) \overset{\triangle}{=} \inf\{\varphi^a_{\infty}(\sigma; G, H_0): \sigma_y > 0, \sigma_v > 0\}.$$

In an entirely similar way, it can be shown that

$\forall \sigma_y > 0$, $\forall \sigma_v > 0$, $\forall \sigma_w > 0$, $\tilde{J}_\infty^a(G) \leq \varphi^a_{\infty}(\sigma; G) \Rightarrow$

$$\tilde{J}_\infty^a(G) \leq J_\infty(G) \overset{\triangle}{=} \inf\{\varphi^a_{\infty}(\sigma; G): \sigma = (\sigma_y, \sigma_v, \sigma_w), \sigma_y > 0, \sigma_v > 0, \sigma_w > 0\}.$$

$\blacksquare$

**Proof of Proposition 3.1** (a) Note first that

$$\nu_N = \int_{S_\theta}^{\mathbb{R}^n} d\theta = \int_{S_\theta}^{\mathbb{R}^n} \left| \det \left( \left( P_\theta^N \right)^{-1/2} \right) \right| d\theta = \left| \det \left( \left( P_\theta^N \right)^{-1/2} \right) \right| \int_{S_\theta}^{\mathbb{R}^n} d\theta, \quad (A.1)$$

where $S_\theta^N = \left\{ \hat{\theta} \in \mathbb{R}^n : ||\hat{\theta}||^2_2 \leq \gamma^2 \right\}$ (i.e., $\hat{\theta} = \left( P_\theta^N \right)^{1/2} \theta$). Note also that for $n > 2$

$$\int_{S_\theta}^{\mathbb{R}^n} d\theta = \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \int_0^{r_{n-1}} \cdots \int_0^{r_1} \prod_{i=1}^{n-2} \{\sin(\alpha_i)\}^{n-1-i} dr_1 \cdots dr_{n-2} \int_0^{r_{n-2}} \cdots \int_0^{r_1} \{\sin(\alpha_i)\}^{n-1-i} \int_0^{2\pi} \int_0^{r_{n-1}} \cdots \int_0^{r_1} \{\sin(\alpha_i)\} \int_0^{2\pi} \int_0^{r_{n-1}} \cdots \int_0^{r_1} \{\sin(\alpha_i)\}^{n-1-i} \alpha_i \}^{1/2} \quad (A.2).$$

Proposition 3.1(a) follows immediately from (A.1) and (A.2).

(b) Let $\hat{\theta} = \left( P_\theta^N \right)^{1/2} \theta$. Thus,

$$\int_{S_\theta}^{\mathbb{R}^n} \hat{\theta}^T P_\sigma^N(G) \theta d\theta = \int_{S_\theta}^{\mathbb{R}^n} \hat{\theta}^T \left( \left( P_\theta^N \right)^{-1/2} P_\sigma^N(G) \left( P_\theta^N \right)^{-1/2} \right) \hat{\theta} \left| \det \left( \left( P_\theta^N \right)^{-1/2} \right) \right| d\theta \quad \Leftrightarrow$$

$$\int_{S_\theta}^{\mathbb{R}^n} \hat{\theta}^T P_\sigma^N(G) \theta d\theta = \int_{\hat{S}_\theta}^{\mathbb{R}^n} \hat{\theta}^T \left( \hat{P}_\sigma^N(G) \hat{\theta} \right) \left| \det \left( \left( P_\theta^N \right)^{-1/2} \right) \right|^{-1/2},$$
where $\hat{P}_N^N(G) = (P_{\theta}^N)^{-1/2} P_{\theta}^N(G)(P_{\theta}^N)^{-1/2}$, so that

$$\int_{S_{\theta}}^{N} \theta^T P_{\theta}^N(G) \theta d\theta = \left\{ \sum_{i=1}^n \sum_j \int_{S_{\theta}}^{N} [\hat{P}_N^N(G)]_{ij} \hat{\theta}_i \hat{\theta}_j d\hat{\theta}_1 \ldots d\hat{\theta}_n \right\} \left| \det(P_{\theta}^N) \right|^{-1/2}$$

so that (since $\int_{S_{\theta}}^{N} \hat{\theta}_i \hat{\theta}_j d\hat{\theta}_1 \ldots d\hat{\theta}_n = 0$ for $i \neq j$)

$$\int_{S_{\theta}}^{N} \theta^T P_{\theta}^N(G) \theta d\theta = \left\{ \sum_{i=1}^n [\hat{P}_N^N(G)]_{ii} \int_{S_{\theta}}^{N} \hat{\theta}_i^2 d\hat{\theta}_1 \ldots d\hat{\theta}_n \right\} \left| \det(P_{\theta}^N) \right|^{-1/2}. \quad (A.3)$$

Note now that

$$\int_{S_{\theta}}^{N} \hat{\theta}_i^2 d\hat{\theta}_1 \ldots d\hat{\theta}_n = \int_{S_{\theta}}^{N} \hat{\theta}_i^2 d\hat{\theta}_1 \ldots d\hat{\theta}_n. \quad (A.4)$$

Moreover,

$$\int_{S_{\theta}}^{N} \hat{\theta}_i^2 d\hat{\theta}_1 \ldots \hat{\theta}_n = \int_0^{2\pi} \int_0^\pi \ldots \int_0^\pi \int_{r=0}^{\gamma} \{ r \cos(\alpha_i) \}^2 r^{n-1} \prod_{k=1}^{n-2} [\sin(\alpha_k)]^{n-1-k} dr d\alpha_1 \ldots d\alpha_{n-1},$$

or, equivalently,

$$\int_{S_{\theta}}^{N} \hat{\theta}_i^2 d\hat{\theta}_1 \ldots \hat{\theta}_n = \left\{ \int_0^{\gamma} r^2 r^{n-1} dr \right\} \left\{ \int_0^\pi [\cos(\alpha_1)]^2 [\sin(\alpha_1)]^{n-2} d\alpha_1 \right\} \prod_{k=2}^{2n-2} (2\pi), \quad (A.5)$$

where $\prod_{k=2}^{2n-2} \triangleq \prod_{k=2}^{n-1} \int_0^\pi [\sin(\alpha_k)]^{n-1-k} d\alpha_k$.

It then follows from $[A.3] - [A.5]$ that

$$\int_{S_{\theta}}^{N} \theta^T P_{\theta}^N(G) \theta d\theta = \left\{ \sum_{i=1}^n [\hat{P}_N^N(G)]_{ii} \right\} \left| \det(P_{\theta}^N) \right|^{-1/2} \left( \frac{\gamma^{n+2}}{n+2} \right) \times \left\{ \int_0^\pi [\cos(\alpha_1)]^2 [\sin(\alpha_1)]^{n-2} d\alpha_1 \right\} \prod_{k=2}^{2n-2} (2\pi). \quad (A.6)$$

(c) In the light of Proposition 3.1(a) and (A.6),

$$\nu_N^{-1} \int_{S_{\theta}}^{N} \theta^T P_{\theta}^N(G) \theta d\theta = \gamma^2 \left( \frac{n}{n+2} \right) \left\{ \sum_{i=1}^n [\hat{P}_N^N(G)]_{ii} \right\} \times \left\{ \int_0^\pi [\cos(\alpha_1)]^2 [\sin(\alpha_1)]^{n-2} d\alpha_1 \right\} \left\{ \int_0^\pi [\sin(\alpha_1)]^{n-2} d\alpha_1 \right\}^{-1}. \quad (A.7)$$

Note now that

$$\int_0^\pi [\sin(\alpha_1)]^{n-2} d\alpha_1 = (1/n) \int_0^\pi [\sin(\alpha_1)]^{n-2} d\alpha_1. \quad (A.8)$$

It then follows from (A.7) and (A.8) that

$$\nu_N^{-1} \int_{S_{\theta}}^{N} \theta^T P_{\theta}^N(G) \theta d\theta = \frac{\gamma^2}{n+2} \left\{ \sum_{i=1}^n [\hat{P}_N^N(G)]_{ii} \right\}.$$
Thus,
\[ \eta_N(G) = \mathcal{J}(G; H_0) + \frac{\gamma^2}{(n+2)} \left\{ \sum_{i=1}^{n} \hat{P}_i^N(G) \right\}, \]

**Proof of equation (A.8):** Note first that
\[
\int_0^\pi [\sin(\alpha)]^n-2[\cos(\alpha)]^2d\alpha = \int_0^\pi [\sin(\alpha)]^n-2d\alpha - \int_0^\pi [\sin(\alpha)]^n d\alpha
\]
and that (integrating by parts yields to)
\[
\int_0^\pi [\sin(\alpha)]^n d\alpha = \int_0^\pi [\sin(\alpha)]^{n-1} \sin(\alpha)d\alpha = [\sin(\alpha)]^{n-1}(-\cos(\alpha))]^\pi_0
\]
\[
- \int_0^\pi (n-1)[\sin(\alpha)]^{n-2} \cos(\alpha)[-\cos(\alpha)]d\alpha
\]
\[\Leftrightarrow \int_0^\pi [\sin(\alpha)]^n d\alpha = -(n-1) \int_0^\pi [\sin(\alpha)]^{n-2}[-\cos^2(\alpha)]d\alpha
\]
\[\Leftrightarrow \int_0^\pi [\sin(\alpha)]^n d\alpha = (n-1) \int_0^\pi [\sin(\alpha)]^{n-2}[\cos(\alpha)]^2d\alpha.
\]
Thus,
\[
\int_0^\pi [\sin(\alpha)]^n-2[\cos(\alpha)]^2d\alpha = \int_0^\pi [\sin(\alpha)]^n-2d\alpha - (n-1) \int_0^\pi [\sin(\alpha)]^{n-2} d\alpha
\]
\[\Leftrightarrow n \int_0^\pi [\sin(\alpha)]^n-2[\cos(\alpha)]^2d\alpha = \int_0^\pi [\sin(\alpha)]^n-2 d\alpha
\]
\[\Leftrightarrow \int_0^\pi [\sin(\alpha)]^n-2[\cos(\alpha)]^2d\alpha = (1/n) \int_0^\pi [\sin(\alpha)]^n-2 d\alpha.
\]

**Proof of Proposition [3.2]:** Recall that
\[ P^N_\theta = I \otimes \left( P^N_\theta \right)^T = [I \otimes (P^N_\theta)^{1/2}][I \otimes (P^N_\theta)^{1/2}], \]
where \( P^N_\theta = (1/2\pi) \int_0^{2\pi} Y^N_x(e^{ja})Y^N_x(e^{ja})^{*}d\alpha. \) Note now that (for the family \( S^N_x \) of FIRs of length \( N \))
\[ Y^N_x(e^{ja}) = (e^{ja}I - A^N_{jc})^{-1} = \text{diag} \left( (e^{ja}I - A^N_{jc})^{-1}, \ldots, (e^{ja}I - A^N_{jc})^{-1} \right) \]
\[ Y^N_x(e^{ja})B^N_c = \text{diag} \left( (e^{ja}I - A^N_{jc})^{-1} b^N_1, \ldots, (e^{ja}I - A^N_{jc})^{-1} b^N_c \right) \in \mathbb{C}^{m_y \times (N-1) \times m_y}. \]
Now, let \( Z^N_x(e^{ja}) \triangleq (e^{ja}I - A^N_{jc})^{-1} b^N_c \) and note that

**Auxiliary Proposition 1:** \( Z^N_x(e^{ja}) = [1 e^{-ja} e^{-j2a} \ldots e^{-j(N-1)a}]^T. \)
As a result $\hat{Y}_N^x = \begin{bmatrix} \text{diag}(Z_N, \ldots, Z_N) \\ I_{m_y} \end{bmatrix}$ and $(\hat{Y}_N^x)^* = [\text{diag}(Z_N^*, \ldots, Z_N^*)^T : I_{m_y}]$ so that

$$P_N^N = \frac{1}{2\pi} \int_0^{2\pi} \begin{bmatrix} \text{diag}(Z_N Z_N^*, \ldots, Z_N Z_N^*) : \text{diag}(Z_N, \ldots, Z_N) \\ \text{diag}(Z_N^*, \ldots, Z_N^*) : I_{m_y} \end{bmatrix}^T (e^{j\alpha}) d\alpha,$$

$$P_N^N = \begin{bmatrix} \text{diag}(I_{N-1}, \ldots, I_{N-1}) & 0 \\ 0 & I_{m_y} \end{bmatrix} = I.$$

Thus, in this case, $P_N^N = I$ and, hence, $\hat{P}_N^N(G) = P_N^N(G)$.

As for $\hat{P}_N^N(G) \triangleq \frac{1}{2\pi} \int_0^{2\pi} \{F_N(G)^* F_N(G)\} (e^{j\alpha}) d\alpha$, note first that $F_N(G) = (G \otimes \phi_{y_1}^T) (I_{m_y} \otimes (\hat{Y}_x^N)^T)$ so that

$$F_N(G)^* F_N(G) = (I_{m_y} \otimes (\hat{Y}_x^N)^T)^* Q_y(G) (I_{m_y} \otimes (\hat{Y}_x^N)^T),$$

where $Q_y(G) \triangleq (G \otimes \phi_{y_1}^T)^* (G \otimes \phi_{y_1}^T)$. Thus,

$$\text{tr} \{\hat{P}_N^N(G)\} = \text{tr} \{P_N^N(G)\} = \frac{1}{2\pi} \int_0^{2\pi} \text{tr} \{Q_y(G) [(I_{m_y} \otimes (\hat{Y}_x^N)^T) (I_{m_y} \otimes (\hat{Y}_x^N)^T)^*]\} (e^{j\alpha}) d\alpha.$$

Note now that $[I_{m_y} \otimes (\hat{Y}_x^N)^T]^* = I \otimes [(\hat{Y}_x^N)^T]^* = I \otimes (\hat{Y}_x^N)^c$, (and since $I \otimes (M_1 M_2)^T = (I \otimes M_2^T)(I \otimes M_1^T)$)

$$[I_{m_y} \otimes (\hat{Y}_x^N)^T][I \otimes (\hat{Y}_x^N)^c] = I_{m_y} \otimes [(\hat{Y}_x^N)^T (\hat{Y}_x^N)^c],$$

and $(\hat{Y}_x^N)^c (\hat{Y}_x^N)^c = (N + 1) I_{m_y}$.

As a result, $\text{tr} \{\hat{P}_N^N(G)\} = (1/2\pi) \int_0^{2\pi} \text{tr} \{Q_y(G) (N + 1) I_{m_y m_y}\} (e^{j\alpha}) d\alpha$

$$\Leftrightarrow \text{tr} \{\hat{P}_N^N(G)\} = (N + 1)(1/2\pi) \int_0^{2\pi} \text{tr} \{(G \otimes \phi_{y_1}^T)^* (G \otimes \phi_{y_1}^T)\} (e^{j\alpha}) d\alpha$$

$$\Leftrightarrow \text{tr} \{\hat{P}_N^N(G)\} = (N + 1) \langle G \otimes \phi_{y_1}^T, G \otimes \phi_{y_1}^T \rangle.$$

It then follows that

$$\lim_{N \to \infty} \eta_N(G) = J(G; H_0) + \lim_{N \to \infty} \left\{ \frac{\gamma^2 (N + 1)}{m_y m_y (N + 1) + 2} \right\} \langle G \otimes \phi_{y_1}^T, G \otimes \phi_{y_1}^T \rangle \Rightarrow$$

$$\lim_{N \to \infty} \eta_N(G) = J(G; H_0) + \frac{\gamma^2}{m_y m_y} \langle G \otimes \phi_{y_1}^T, G \otimes \phi_{y_1}^T \rangle.$$
Proof of Proposition 4.1 (a) Let $\lambda_o \triangleq \|F_{Gy}\|_\infty^2$ and note that

$$\lambda \in S_\lambda \iff \lambda > \lambda_o.$$  \hfill (A.9)

Note also that if $\lambda < \lambda_o$, \text{inf}\{L_a(Z, \lambda; G) : Z \in \mathcal{R}^{m_wm_y}_c\} = -\infty$ and hence $\varphi_D(\lambda, G) = +\infty$. Thus, it follows from 4.3 that

$$\bar{J}(G; S_X) = \text{inf}\{\varphi_D(\lambda; G) : \lambda \geq \lambda_o\}.$$  \hfill (A.10)

Now, $\forall \lambda > \lambda_o, \forall Z \in \mathcal{R}^{m_wm_y}_c$, \text{inf}\{L_a(X, \lambda; G) : X \in \mathcal{R}^{m_wm_y}_c\} \geq \text{inf}\{L_a(X, \lambda_o; G) : X \in \mathcal{R}^{m_wm_y}_c\}$.

Thus, it follows from the fact that

$$\varphi_D(\lambda; G) = \lambda_o^2 + (\lambda - \lambda_o)^2 \geq \text{inf}\{L_a(X, \lambda_o; G) : X \in \mathcal{R}^{m_wm_y}_c\}$$

that

$$\varphi_D(\lambda; G) \leq \lambda_o^2 + (\lambda - \lambda_o)^2 - \text{inf}\{L_a(X, \lambda; G) : X \in \mathcal{R}^{m_wm_y}_c\} \iff \varphi_D(\lambda; G) \leq \varphi_D(\lambda_o; G) + (\lambda - \lambda_o)^2.$$  \hfill (A.11)

Thus, \text{inf}\{\varphi_D(\lambda; G) : \lambda > \lambda_o\} \leq \varphi_D(\lambda_o; G) which implies that

$$\text{inf}\{\varphi_D(\lambda; G) : \lambda \geq \lambda_o\} = \text{inf}\{\varphi_D(\lambda; G) : \lambda > \lambda_o\}.$$  \hfill (A.11)

Combining (A.9) - (A.11) leads to

$$\bar{J}(G; S_X) = \text{inf}\{\varphi_D(\lambda; G) : \lambda \in S_\lambda\}.$$  \hfill (A.11)

(b) It directly follows from Lemma A.1 ([17]) that $\forall \lambda \in S_\lambda$, \text{inf}\{L_a(Z, \lambda; G) : Z \in \mathcal{R}^{m_wm_y}_c\} equals the optimal value of \text{inf}\{x_0^TPx_0 : P \in S_p(\Sigma_a, \lambda)\}$, where $S_p(\Sigma_a, \lambda) = \{P = P^T : Q_{LQ}(P; \Sigma_a, M(\lambda)) \geq 0\}$. The proof is concluded by noting that $\forall \lambda \in S_\lambda$, $S_p(\Sigma_a, \lambda) = \{P = P^T : Q_{LQ}(P; \Sigma_a, M(\lambda)) > 0\}$ is non-empty (see Proposition 4.1(c)) and that $S_p(\Sigma_a, \lambda)$ is dense in $S_p(\Sigma_a, \lambda)$ (since $Q_{LQ}(\cdot; \Sigma_a, M(\lambda))$ is affine).

(c) Note first that (for $F_{IW} \triangleq I_{m_y} \otimes F_w^T$, $F_{Gy} \triangleq G \otimes F_y^T$, $F = \begin{bmatrix} F_{IW} \\ F_{Gy} \end{bmatrix}$)

$$F^*M(\lambda)F = [F_{IW}^* \ F_{Gy}^*] \text{diag}(\lambda I, -I) \begin{bmatrix} F_{IW} \\ F_{Gy} \end{bmatrix}.$$  

Noting that $F_{IW} = \text{diag}(F_w^T)$, so that $F_{IW}^* F_{IW} = \text{diag}(F_w F_w^T)$, it follows from $F_w = [I_{m_y} : 0_{m_y \times m_y}]$ that $F_w F_w^T = I_{m_y}$ and, hence, $F_{IW}^* F_{IW} = \text{diag}(I_{m_y}) = I_{m_w m_y}$.

As a result, $F^*M(\lambda)F = \lambda I_{m_w m_y} - F_{Gy}^* F_{Gy}$. Thus, $\forall \phi \in [0, 2\pi]$,

$$(F^*M(\lambda)F)(e^{i\theta}) > 0 \iff \|\lambda^{-1/2} F_{Gy}\|_\infty < 1.$$
Now, let \((A_{Gy}, B_{Gy}, C_{Gy}, D_{Gy})\) denote a realization of \(F_{Gy}\), \(\rho(A_{Gy}) < 1\). It then follows from the discrete-time bounded-real lemma ([27]) that \(\|\lambda^{-1/2}F_{Gy}\|_{\infty} < 1 \iff \exists X = X^T < 0\) such that

\[
\begin{bmatrix}
A^T_{Gy} \\
B^T_{Gy}
\end{bmatrix}
X[A_{Gy} B_{Gy}] - \begin{bmatrix}
X & 0 \\
0 & 0
\end{bmatrix}
- \lambda^{-1}
\begin{bmatrix}
C^T_{Gy} \\
D^T_{Gy}
\end{bmatrix}
[C_{Gy} D_{Gy}] + \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} > 0. \quad (A.12)
\]

Consider now \(Q_{LQ}(P; \Sigma_a, M(\lambda))\) given by

\[
Q_{LQ}(P; \Sigma_a, M(\lambda)) = Q_{\mathcal{F}}(P; \Sigma_a) + S(\Sigma_a; M(\lambda)),
\]

where \(Q_{\mathcal{F}}(P; \Sigma_a) \triangleq \begin{bmatrix} A^T & 0 \\ B^T & P[A B] \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}, \quad S(\Sigma_a; M(\lambda)) = R^T M(\lambda) R,
\]

\[
A = \begin{bmatrix} A_a & B_a \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_a \\ 0 \end{bmatrix}, \quad R = [C_a \; d_a \; D_a] \quad \text{and} \quad \Sigma_a = (A_a, [B_a : b_a], C_a, [D_a : d_a]) \quad \text{is a realization of} \quad |F: -\mathcal{X}_a(G)|, \quad \rho(A_a) < 1.
\]

Note that

\[
Q_{\mathcal{F}}(P; \Sigma_a) = \begin{bmatrix} A^T_a & 0 \\ b^T_a & 0 \\ B^T_a & 0 \end{bmatrix} P_1 \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} A_a & b_a & B_a \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{12} & P_{22} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow
\]

\[
Q_{\mathcal{F}}(P; \Sigma_a) = \begin{bmatrix} A^T_a \\ b^T_a \\ B^T_a \end{bmatrix} P_{11} [A_a b_a B_a] - \begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{12} & P_{22} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (A.13)
\]

Note also that (writing \(R = \begin{bmatrix} R_{a1} \\ R_{a2} \end{bmatrix}, \quad R_{ai} = [C_{ai} \; d_{ai} \; D_{ai}], i = 1, 2\))

\[
S(\Sigma_a; M(\lambda)) = [R^T_{a1} \; R^T_{a2}] \text{diag}(\lambda I, -I) \begin{bmatrix} R_{a1} \\ R_{a2} \end{bmatrix} = \lambda R^T_{a1} R_{a1} - R^T_{a2} R_{a2},
\]

where \(R_{a1} = [I_m \; m_y] R\) and \(R_{a2} = [0 \; I_m \; m_y] R \Rightarrow
\]

\[
S(\Sigma_a; M(\lambda)) = \lambda \begin{bmatrix} C^T_{a1} \\ D^T_{a1} \end{bmatrix} [C_{a1} \; d_{a1} \; D_{a1}] - \begin{bmatrix} C^T_{a2} \\ D^T_{a2} \end{bmatrix} [C_{a2} \; d_{a2} \; D_{a2}]. \quad (A.14)
\]

Permuting rows and columns of \(Q_{\mathcal{F}}(\cdot)\) and \(S(\cdot)\) (i.e., applying congruence transformations) yields \(Q_{LQ}(P; \Sigma_a) =
\]

\[
\begin{bmatrix} A^T_a \\ B^T_a \\ b^T_a \end{bmatrix} P_{11} \begin{bmatrix} A_a & B_a & b_a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S(\Sigma_a; M(\lambda)) = \lambda \begin{bmatrix} C^T_{a1} \\ D^T_{a1} \end{bmatrix} [C_{a1} \; D_{a1} \; D_{a1}] - \begin{bmatrix} C^T_{a2} \\ D^T_{a2} \end{bmatrix} [C_{a2} \; D_{a2} \; D_{a2}].
\]

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Thus, the condition \( Q_{LQ}(P; \Sigma_a, M(\lambda)) > 0 \) is equivalent to

\[
\tilde{Q}_{\gamma}(P; \Sigma_a) + \tilde{S}(\Sigma_a; M(\lambda)) > 0
\]

(A.15)

so that the top diagonal block in (A.15) is also non-negative definite, \textit{i.e.,}

\[
\begin{bmatrix}
A_a^T \\
B_a^T 
\end{bmatrix}
P_{11} [A_a B_a] -
\begin{bmatrix}
P_{11} & 0 \\
0 & 0 
\end{bmatrix}
+ \lambda \begin{bmatrix}
C_{a1}^T \\
D_{a1}^T 
\end{bmatrix}
[C_{a1} D_{a1}] -
\begin{bmatrix}
C_{a2}^T \\
D_{a2}^T 
\end{bmatrix}
[C_{a2} D_{a2}] > 0.
\] \hspace{1cm} (A.16)

Note now that \( [F : -\mathcal{X}_0(G)] = \begin{bmatrix}
I_m \otimes F_W^T & 0 \\
0 & -\mathcal{X}_0(G)
\end{bmatrix} \), where \( \mathcal{X}_0(G) = \text{rvec}\{X_0(G)F_y + GF_y\} \).

Thus, taking a realization \( (A_{a2}, [B_a : b_a], C_{a2}, [D_a : d_a]) \) of \( [F_{Gy} : -\mathcal{X}_0(G)] \) it follows that

\[
A_a = A_{a2}, \quad [B_a : b_a] = [B_{a2} : b_{a2}], \quad C_a = \begin{bmatrix}
0 \\
\ldots \\
C_{a2}
\end{bmatrix}, \quad D_a = \begin{bmatrix}
0 \\
\ldots \\
D_{a2}
\end{bmatrix}, \quad d_a = \ldots \] so that \( C_{a1} = 0 \) and

\[
D_{a1} = (I_m \otimes F_W^T).
\]

Thus, \[
\begin{bmatrix}
C_{a1}^T \\
D_{a1}^T
\end{bmatrix}
[C_{a1} D_{a1}] = \begin{bmatrix}
0 & 0 \\
0 & T_{a1} D_{a1}
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & I
\end{bmatrix}
\]

from which (A.16) is rewritten as

\[
\tilde{Q}_{\gamma1}(P_{11}; \lambda) \triangleq \begin{bmatrix}
A_{a2}^T \\
B_{a2}^T
\end{bmatrix}
P_{11} [A_{a2} B_{a2}] -
\begin{bmatrix}
P_{11} & 0 \\
0 & 0 
\end{bmatrix}
+ \lambda \begin{bmatrix}
0 & 0 \\
0 & 0 
\end{bmatrix}
- \begin{bmatrix}
C_{a2}^T \\
D_{a2}^T
\end{bmatrix}
[C_{a2} D_{a2}] > 0.
\] \hspace{1cm} (A.17)

Note that, in the light of (A.17), there exists \( P_{\gamma1} = P_{\gamma1}^T > 0 \) such that \( A_{a2}^T P_{11} A_{a2} - P_{11} - C_{a2}^T C_{a2} = P_{\gamma1} \), so that as \( \rho(A_{a2}) < 1 \), \( P_{11} = \sum_{k=0}^{\infty} (A_{a2})^k (P_{\gamma1} + C_{a2}^T C_{a2}) A_{a2} = P_{11}^T < 0 \).

Finally, noting that \( F_{Gy} = [F_{Gy} : -\mathcal{X}_0(G)] \begin{bmatrix}
I \\
0
\end{bmatrix} \), a realization of \( F_{Gy} \) is given by \( (A_{Gy}, B_{Gy}, C_{Gy}, D_{Gy}) \),

where \( A_{Gy} = A_{a2}, \quad B_{Gy} = B_{a2}, \quad C_{Gy} = C_{a2} \) and \( D_{Gy} = D_{a2} \). Then, for \( \lambda > 0 \) (A.17) is equivalent to

\[
\begin{bmatrix}
A_{Gy}^T \\
B_{Gy}^T
\end{bmatrix} \begin{bmatrix}
\lambda^{-1} P_{11} \\
0
\end{bmatrix} [A_{Gy} B_{Gy}] -
\begin{bmatrix}
\lambda^{-1} P_{11} & 0 \\
0 & 0 
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & I
\end{bmatrix}
- \lambda^{-1} \begin{bmatrix}
C_{Gy}^T \\
D_{Gy}^T
\end{bmatrix}
[C_{Gy} D_{Gy}] > 0.
\]

Thus, for \( \lambda > 0 \) and \( P = \begin{bmatrix}
P_{11} & P_{12} \\
P_{12} & P_{22}
\end{bmatrix} \) such that \( Q_{LQ}(P; \Sigma_a, M(\lambda)) > 0 \), \( \exists X = X^T < 0 \) \( (X \triangleq \lambda^{-1} P_{11}) \) such that (A.12) holds (in which case \( \|\lambda^{-1} F_{Gy}\|_\infty < 1 \)) or, equivalently, \( \lambda \in \mathcal{S}_\lambda \).

To show the converse, let \( \lambda \in \mathcal{S}_\lambda \) and write

\[
\tilde{Q}_{LQ}(P; \Sigma_a, M(\lambda)) = \tilde{Q}_{\gamma}(P; \Sigma_a) + \lambda R_1^T R_1 - R_2^T R_2,
\]

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where $\mathbf{R}_1 \triangleq \begin{bmatrix} C_{a_1} & D_{a_1} & d_{a_1} \end{bmatrix}$ and $\mathbf{R}_2 \triangleq \begin{bmatrix} C_{a_2} & D_{a_2} & d_{a_1} \end{bmatrix}$ or, equivalently,

$$ Q_{LQ}(P; \Sigma_n, M(\lambda)) = \text{diag}(Q_{d1}(P_{11}, \lambda), -P_{22}) - \begin{bmatrix} 0 & P_{12} \\ P_{12}^T & 0 \end{bmatrix} + T_{LQ} + T_{LQ}^T, \quad (A.18) $$

where

$$ T_{LQ} = \begin{bmatrix} 0 \\ \vdots \\ P_{11} [A_2 B_2 : 0] + (1/2) \begin{bmatrix} 0 \\ \vdots \\ 0 : b_{a_2} \end{bmatrix} \end{bmatrix} $$

$$ = \begin{bmatrix} 0 \\ \vdots \\ C_{a_2} D_{a_2} : 0 \end{bmatrix} + (1/2) \begin{bmatrix} 0 \\ \vdots \\ 0 : d_{a_2} \end{bmatrix} $$

so that

$$ T_{LQ} + T_{LQ}^T = \begin{bmatrix} 0 & e_1(P_{11}) \\ e_1(P_{11})^T & b_{a_2}^T P_{11} b_{a_2} \end{bmatrix} - \begin{bmatrix} 0 & e_2 \\ e_2^T & d_{a_2}^T d_{a_2} \end{bmatrix}, \quad (A.19) $$

where $e_1(P_{11}) = b_{a_2}^T P_{11} [A_2 B_2]$ and $e_2 = d_{a_2}^T [C_{a_2} D_{a_2}]$.

As $\lambda \in \mathcal{S}_\lambda$, it follows from (A.17) and the discrete-time bounded real lemma that $\exists P_{11}^o$ such that $\bar{Q}_{d1}(P_{11}^o; \lambda) > 0$. Take $P_{12}^o = e_1(P_{11}) - e_2$, $P_{22}^o$ such that $\bar{q}_{d2}(P_{22}^o) \triangleq -P_{22} + b_{a_2}^T P_{11} b_{a_2} - d_{a_2}^T d_{a_2} > 0$ and $P^o = \begin{bmatrix} P_{11}^o \\ P_{12}^o \\ (P_{12}^o)^T \\ P_{22}^o \end{bmatrix}$. It then follows from (A.18) and (A.19) that

$$ Q_{LQ}(P^o; \Sigma_n, M(\lambda)) = \text{diag}(\bar{Q}_{d1}(P_{11}^o), \bar{q}_{d2}(P_{22}^o)) > 0 \Rightarrow Q_{LQ}(P^o; \Sigma_n, M(\lambda)) > 0. $$

Proof of Proposition 4.3: Consider first the following sets

$$ \mathcal{S}_{s_1} \triangleq \{ \sigma = (\sigma_y, \sigma_v) : \sigma_y > 0, \sigma_v > 0 \text{ and } \forall \phi \in [0, 2\pi], (M_\sigma - F_G^* F_G)(e^{j\phi}) \geq 0 \} $$

and

$$ \mathcal{S}_{s_2} \triangleq \{ \sigma = (\sigma_y, \sigma_v) : \sigma_y > 0, \sigma_v > 0 \text{ and } \forall \phi \in [0, 2\pi], (M_\sigma - F_G^* F_G)(e^{j\phi}) > 0 \} $$

and note that $\forall \sigma \in \mathcal{S}_{s_1}$, $\bar{\varphi}_{D_{\infty}}(\sigma; G, H) = 0 \Rightarrow \varphi_{D_{\infty}}(\sigma; G, H) = \sigma_y^2 + \sigma_v^2$ whereas for other values of $\sigma > 0$, $\varphi_{D_{\infty}}(\sigma; G, H) = -\infty \Rightarrow \varphi_{D_{\infty}}(\sigma; G, H) = +\infty$. Hence, $\varphi_{D_{\infty}}(G; H) = \inf\{ \varphi_{D_{\infty}}(\sigma; G, H) = \sigma_y^2 + \sigma_v^2 : \sigma \in \mathcal{S}_{s_1} \}$.

The proof is concluded by noting that $\mathcal{S}_{s_2} \subset \mathcal{S}_{s_1}$ is dense in $\mathcal{S}_{s_1}$ – indeed, if $(\sigma_y^o, \sigma_v^o) \in \mathcal{S}_{s_1}$, $\forall \varepsilon > 0$,

$$ \sigma^o \triangleq (\sigma_y^o + \varepsilon, \sigma_v^o + \varepsilon) \in \mathcal{S}_{s_2} \text{ as } (M_\sigma - F_G^* F_G)(e^{j\phi}) = (M_\sigma - F_G^* F_G)(e^{j\phi}) + \varepsilon I > 0. $$


Proof of Proposition 4.4: Note that \( \forall \phi \in [0, 2\pi], \ (M_\sigma - F_G^* F_G)(e^{i\phi}) > 0 \iff \forall \phi \in [0, 2\pi], \ I - M_\sigma^{1/2} F_G^* F_G M_\sigma^{-1/2}(e^{i\phi}) > 0 \iff \|F_G M_\sigma^{1/2}\|_\infty < 1. \)

As, in the light of the so-called (discrete-time) bounded-real lemma ([27]), for a realization \((A_{FG}, B_{FG}, C_{FG}, D_{FG})\) of \(F_G, \rho(A_{FG}) < 1, \|F_G M_\sigma^{-1/2}\|_\infty < 1 \iff \exists X = X^T > 0, \) such that

\[
\begin{bmatrix}
A_{FG}^T X A_{FG} - X & A_{FG}^T X B_{FG} M_\sigma^{-1/2} \\
M_\sigma^{-1/2} B_{FG}^T X A_{FG} & M_\sigma^{-1/2} B_{FG}^T X B_{FG} M_\sigma^{-1/2}
\end{bmatrix}
+ \begin{bmatrix}
C_{FG}^T \\
M_\sigma^{-1/2} D_{FG}^T
\end{bmatrix}
\begin{bmatrix}
C_{FG} & D_{FG} M_\sigma^{-1/2}
\end{bmatrix}
- \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} < 0,
\]

it follows that the condition on \((M_\sigma - F_G^* F_G)\) in Proposition 4.3 is equivalent to the matrix inequality above. Moreover, if its left-hand side is pre and post-multiplied by \(\text{diag}(I_{n_{FG}}, M_{1/2})\), it is converted into

\[\exists X = X^T > 0 \text{ such that } Q_{BR}(X; \Sigma_a, M_{\sigma}) < 0\]

where

\[
Q_{BR}(P; \Sigma_a, M_{\sigma}) \triangleq \begin{bmatrix}
A_{FG}^T P A_{FG} - P & A_{FG}^T P B_{FG} \\
B_{FG}^T P A_{FG} & B_{FG}^T P B_{FG}
\end{bmatrix}
+ \begin{bmatrix}
C_{FG}^T \\
C_{FG}^T
\end{bmatrix}
\begin{bmatrix}
C_{FG} & D_{FG} \\
D_{FG}
\end{bmatrix}
- \begin{bmatrix}
0 & 0 \\
0 & M_{\sigma}
\end{bmatrix}.
\]

Thus, it follows from Proposition 4.3 that \(J_\infty(G; H)\) can be rewritten as stated above. \(\square\)

Proof of Proposition 4.5(a): Note first that \(Q_{BR}(P, \Sigma_a(\theta), M_\sigma) < 0 \iff \)

\[
\begin{bmatrix}
P & 0 \\
0 & M_\sigma
\end{bmatrix}
- \begin{bmatrix}
A_{FG}^T \\
B_{FG}^T
\end{bmatrix}
P \begin{bmatrix}
A_{FG} \\
B_{FG}
\end{bmatrix}
+ \begin{bmatrix}
C_{FG}^T \\
D_{FG}^T
\end{bmatrix}
\begin{bmatrix}
C_{FG} & D_{FG} 0_{n_{FG} \times m_e}
\end{bmatrix} > 0 \iff
\]

\[
\begin{bmatrix}
P & 0 & C_{FG}^T \\
0 & M_\sigma & D_{FG}^T \\
C_{FG} & D_{FG} & I_{m_e}
\end{bmatrix}
- \begin{bmatrix}
A_{FG}^T \\
B_{FG}^T \\
0_{m_e \times n_{FG}}
\end{bmatrix}
P \begin{bmatrix}
A_{FG} \\
B_{FG} 0_{n_{FG} \times m_e}
\end{bmatrix} > 0 \iff
\]

\[
\begin{bmatrix}
P & 0 & C_{FG}^T \\
0 & M_\sigma & D_{FG}^T \\
C_{FG} & D_{FG} & I_{m_e}
\end{bmatrix}
- \begin{bmatrix}
A_{FG}^T \\
B_{FG}^T \\
0_{m_e \times n_{FG}}
\end{bmatrix}
P \begin{bmatrix}
A_{FG} \\
B_{FG} 0_{n_{FG} \times m_e}
\end{bmatrix} > 0 \iff
\]

\[\iff \text{(pre and post-multiplying the matrix above by } I_p^T \text{ and } I_p \text{ where, } I_p \text{ is the column-block permutation}}\]
Note first that, as 
\[
\psi(P, \sigma, \theta) > 0,
\]
where 
\[
\theta = \begin{bmatrix}
A_G & B_G \\
C_G & D_G
\end{bmatrix},
\]
\( (A_G, B_G, C_G, D_G) \) is a realization of \( G \) and

\[
\psi(P, \sigma, \theta) \triangleq \begin{bmatrix}
P^{-1} & \quad A_{FG} & \quad B_{FG} & 0_{n_{FG} \times m_e} \\
A_{FG}^T & \quad P & \quad 0_{n_{FG} \times m_y} & \quad C_{FG}^T \\
B_{FG}^T & \quad 0_{m_y \times n_{FG}} & \quad M_\sigma & \quad D_{FG}^T \\
0_{m_e \times n_{FG}} & \quad C_{FG} & \quad D_{FG} & \quad I_{m_e}
\end{bmatrix}
\]

Thus,

\[
Q_{BH}(P, \Sigma_0(\theta), M_\sigma) < 0 \iff \psi(P, \sigma, \theta) > 0.
\] (A.20)

**Proof of Proposition 4.5(b):** Note first that, as \( F_G = H_{ty} - GH_{oz} \), a realization \((A_{FG}, B_{FG}, C_{FG}, D_{FG})\) of \( F_G \) is obtained from realizations \((A_{ty}, B_{ty}, C_{ty}, D_{ty})\) of \( H_{ty} \) and \((A_{GO}, B_{GO}, C_{GO}, D_{GO})\) of \( F_{GO} \triangleq GH_{oz} \) as

\[
A_{FG} = \begin{bmatrix} A_{ty} & 0 \\ 0 & A_{GO} \end{bmatrix}, \quad B_{FG} = \begin{bmatrix} B_{ty} \\ B_{GO} \end{bmatrix}, \quad C_{FG} = \begin{bmatrix} C_{ty} \vdash -C_{GO} \end{bmatrix}, \quad D_{FG} = D_{ty} - D_{GO}.
\]

In turn, a realization of \( F_{GO} \) is obtained from realizations of \( G \) and \( H_{oz} \) as

\[
A_{GO} = \begin{bmatrix} A_{oz} & 0_{n_{oz} \times n_G} \\ B_G C_{oz} & A_G \end{bmatrix}, \quad B_{GO} = \begin{bmatrix} B_{oz} \\ B_G D_{oz} \end{bmatrix}, \quad C_{GO} = \begin{bmatrix} [D_G C_{oz} \vdash C_G] \end{bmatrix}, \quad D_{GO} = D_G D_{oz}
\]

so that

\[
A_{FG} = \begin{bmatrix} A_{ty} & 0_{n_{ty} \times n_G} & 0_{n_{ty} \times n_{FG}} \\ 0 & A_{oz} & 0_{n_{oz} \times n_{FG}} \\ 0 & B_G C_{oz} & A_G \end{bmatrix}, \quad B_{FG} = \begin{bmatrix} B_{ty} \\ B_{oz} \\ B_G D_{oz} \end{bmatrix}, \quad C_{FG} = \begin{bmatrix} C_{ty} \vdash -D_G C_{oz} \vdash -C_G \end{bmatrix},
\]

\[
D_{FG} = D_{ty} - D_G D_{oz}, \quad \text{or, equivalently,}
\]

\[
A_{FG} = \begin{bmatrix} A_{oz} & 0_{n_{oz} \times n_G} \\ B_G C_{az} & A_G \end{bmatrix}, \quad B_{FG} = \begin{bmatrix} B_{oz} \\ B_G D_{oz} \end{bmatrix}, \quad \text{where} \quad A_{oz} \triangleq \text{diag}(A_{ty}, A_{oz}), \quad B_{oz} = \begin{bmatrix} B_{ty} \\ B_{oz} \end{bmatrix}
\]

and \( C_{oz} = [0_{m_y \times n_{ty}} \vdash C_{oz}] \), so that

\[
A_{FG} = A_o + A_L(\theta), \quad B_{FG} = B_o + B_L(\theta), \quad C_{FG} = C_o + C_L(\theta), \quad D_{FG} = D_{ty} + D_L(\theta),
\]

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where \( A_o = \text{diag}(A_{az}, 0_{n_G \times n_G}), \) \( B_o = \begin{bmatrix} B_{az} \\ 0_{n_G \times m_{vy}} \end{bmatrix}, \) \( C_o = [C_{ty} : 0_{m_e \times (n_{ax} + n_G)}], \) \( D_o = D_{ty}, \)
\[
A_L(\theta) = \begin{bmatrix} 0_{n_{az} \times n_{az}} & 0_{n_{az} \times n_G} \\ B_G C_{az} & A_G \end{bmatrix}, \quad B_L(\theta) = \begin{bmatrix} 0_{n_{az} \times m_{vy}} \\ B_G D_{az} \end{bmatrix}, \quad C_L(\theta) = [-D_G C_{az} : -C_G], \quad D_L(\theta) = -D_G D_{oz}.
\]

As a result, \( \psi(P, \theta) = \psi_o(P) + T_a^T \theta T_b + T_a^T \theta^T T_a, \)

where
\[
\psi_o(P) = \begin{bmatrix} P^{-1} & A_o & B_o & 0_{n_P \times m_e} \\ A_o^T & P & 0_{n_{FG} \times m_{vy}} & C_o^T \\ B_o & 0_{m_{vy} \times n_{FG}} & M_\sigma & D_{ty}^T \\ 0_{n_e \times n_{FG}} & C_o & D_{ty} & I_{m_e} \end{bmatrix}
\]

and
\[
T_a^T \theta T_b = \begin{bmatrix} 0_{n_{az} \times n_{az}} & 0_{n_{az} \times n_G} & 0_{n_{az} \times n_{az}} & 0_{n_{az} \times m_{vy}} & 0_{n_{az} \times m_{vy}} & 0_{n_P \times m_e} \\ 0_{n_G \times n_{az}} & 0_{n_G \times n_G} & 0_{n_G \times n_{az}} & 0_{n_G \times m_{vy}} & 0_{n_G \times m_{vy}} & 0_{n_G \times m_e} \\ 0_{n_G \times n_{az}} & 0_{n_G \times n_G} & 0_{n_G \times n_{az}} & 0_{n_G \times m_{vy}} & 0_{n_G \times m_{vy}} & 0_{n_G \times m_e} \\ 0_{m_{vy} \times n_{az}} & 0_{m_{vy} \times n_G} & 0_{m_{vy} \times n_{az}} & 0_{m_{vy} \times m_{vy}} & 0_{m_{vy} \times m_{vy}} & 0_{m_{vy} \times m_e} \\ 0_{m_e \times n_{az}} & 0_{m_e \times n_G} & -D_G C_{az} & -C_G & -D_G D_{az} & 0_{m_e \times m_e} \\ -D_G C_{az} & -C_G & -D_G D_{az} & 0_{m_e \times m_e} & 0_{m_e \times m_e} & 0_{m_e \times m_e} \end{bmatrix}
\]

\[
\iff T_a^T \theta T_b = \begin{bmatrix} 0_{n_{az} \times n_{G}} & 0_{n_{ax} \times m_e} \\ I_{n_G} & 0_{n_G \times m_e} \\ 0_{n_{ax} \times n_G} & 0_{n_{ax} \times m_e} \\ 0 & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_G C_{az} & A_G & B_G D_{oz} \\ B_G C_{az} & A_G & B_G D_{oz} \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \]

\[
T_a^T \theta T_b = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.
\]

Thus, with \( \theta = \begin{bmatrix} A_G & B_G \\ C_G & D_G \end{bmatrix}, \) \( T_a \) and \( T_b \) are given by
\[
T_a = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad T_b = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & C_{az} & 0 & D_{oz} & 0 \end{bmatrix}.
\]
Proof of Proposition 4.6(a): It follows from the fact that (in the light of Proposition 4.5(a)) the constraint “$Q_{BR}(\cdot) < 0$” in Prob. 2a can be replaced by “$\psi(P, \sigma, \theta)$”.

Proof of Proposition 4.6(b): The first part follows directly from Proposition 4.5(b), (i) and (ii) above. The fact that “$\rho(A_G) < 1$” follows from (4.14) $\Rightarrow$ “$Q_{BR}(P^*, \Sigma_{FG}(\theta^*), M(\sigma^*)) < 0$” (due to Proposition 4.6) $\Rightarrow$ $\rho(A_{FG}) < 1$ $\Rightarrow$ $\rho(A_G) < 1$ (see the definition of $A_{FG}$ above). The last part follows from Proposition 4.4.

Proof of Proposition 4.7: To apply the Elimination Lemma on the condition

$$
\psi_o(P, \sigma) + T_a^T \theta T_b + T_b^T \theta T_a > 0
$$

(A.22)

it is necessary to obtain matrices (say $W_a$ and $W_b$) whose columns form bases form the null spaces of $T_a$ and $T_b$.

To this effect, note that

$$
\text{Ker}(T_a) \triangleq \{v = [v_1^T \ldots v_6^T]^T : T_a v = 0\} = \{v = [v_1^T \ldots v_6^T]^T : v_2 = 0, v_6 = 0\}
$$

so that $W_a$ is given by

$$
W_a = \begin{bmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

and a necessary condition for (A.22) to hold is that

$$
W_a^T \psi_o(P) W_a > 0.
$$

(A.23)

To partition $\psi_o(P)$ in conformity with (A.21) let

$$
P = \begin{bmatrix} S & N \\ N & X \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} R & M \\ M^T & Z \end{bmatrix}, \text{ where } S = S^T \text{ and } R = R^T \text{ are } n_{az} \times n_{az} \text{ matrices and } X = X^T \text{ and } Z = Z^T \text{ are } n_G \times n_G \text{ matrices (} n_G \geq n_{az} \text{). Thus, } \psi_o(P) \text{ can be rewritten as}
$$
\[
\psi_0(P) = \begin{bmatrix}
R & M & A_{az} & 0 & B_{az} & 0 \\
M^T & Z & 0 & 0 & 0 & 0 \\
A_{az}^T & S & N & 0 & \hat{C}_0^T & 0 \\
0 & 0 & N^T & X & 0 & 0 \\
B_{az}^T & 0 & 0 & M_\sigma & D_{ty} & 0 \\
0 & 0 & \hat{C}_0 & 0 & D_{ty} & I_{m_e}
\end{bmatrix}, \text{ where } \hat{C}_0 = [C_{ty} : 0_{m_e \times n_{ax}}].
\]

As a result,
\[
W_a^T \psi_0(P) W_a = W_a^T \begin{bmatrix}
R & A_{az} & 0 & B_{az} \\
M^T & 0 & 0 & 0 \\
A_{az}^T & S & N & 0 \\
0 & N^T & X & 0 \\
B_{az}^T & 0 & 0 & M_\sigma \\
0 & \hat{C}_0 & 0 & D_{ty}
\end{bmatrix} = \begin{bmatrix}
R & A_{az} & 0 & B_{az} \\
A_{az}^T & S & N & 0 \\
0 & N^T & X & 0 \\
B_{az}^T & 0 & 0 & M_\sigma
\end{bmatrix}.
\]

Moreover, pre and post-multiplying \(W_a^T \psi_0(P) W_a\) by \(I_c^T\) and \(I_c\), where \(I_c = \begin{bmatrix} I & 0 & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
0 & I & 0 & 0
\end{bmatrix}\) (column permutation matrix),
\[
W_a^T \psi_0(P) W_a > 0 \iff I_c^T W_a^T \psi_0(P) W_a I_c > 0 \iff
\]
\[
\begin{bmatrix}
R & B_{az} & A_{az} & 0 \\
B_{az}^T & M_\sigma & 0 & 0 \\
A_{az}^T & 0 & S & N \\
0 & 0 & N^T & X
\end{bmatrix} > 0 \iff \text{ (invoking the Schur complement formula)}
\]
\[
\begin{bmatrix}
S & N \\
N^T & X
\end{bmatrix} > 0 \text{ and } \begin{bmatrix}
R & B_{az} \\
B_{az}^T & M_\sigma
\end{bmatrix} - \begin{bmatrix}
A_{az} & 0 \\
0 & S & N^{-1}
\end{bmatrix} \begin{bmatrix}
A_{az}^T & 0 \\
0 & N^T & X
\end{bmatrix} > 0
\]

Moreover, since \(P^{-1} = \begin{bmatrix} R & M \\
M^T & Z
\end{bmatrix}\), the last LMI above can be rewritten as
\[
\begin{bmatrix}
R & B_{az} \\
B_{az}^T & M_\sigma
\end{bmatrix} - \begin{bmatrix}
I & 0 \\
0 & A_{az} R A_{az}^T I_0
\end{bmatrix} > 0 \iff \begin{bmatrix}
R - A_{az} R A_{az}^T & B_{az} \\
B_{az}^T & M_\sigma
\end{bmatrix} > 0. \quad (A.24)
\]

It has thus been established that \((A.24)\) is a necessary condition for \((A.22)\) to hold. To obtain the corresponding necessary condition pertaining to \(T_b\), note that \(D_{oz} = [D_{oz1} : D_{oz2}]\), where \(D_{oz1} = D_{Ho} D_{wy}^{-1}\) and \(D_{oz2} = D_{wy}^{-1}\).
so that \( T_b = \begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{az} & 0 & D_{oz1} & D_{oz2} & 0 & 0 \end{bmatrix} \) and

\[
\text{Ker}(T) = \{ w = [w_1^T \ldots w_7^T]^T : w_4 = 0 \text{ and } C_{az} w_3 + D_{oz1} w_5 + D_{oz2} w_6 = 0 \} \iff
\text{Ker}(T) = \{ w = [w_1^T \ldots w_7^T]^T : w_4 = 0 \text{ and } w_6 = -D_{oz2}^{-1}(C_{az} w_3 + D_{oz1} w_5) \}.
\]

Thus, the columns of

\[
W_b = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & T_{63} & T_{65} & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \quad \text{where } T_{63} = -D_{oz2}^{-1} C_{az} \text{ and } T_{65} = -D_{oz2}^{-1} D_{oz1}.
\]

form a basis for \( \text{Ker}(T_b) \) (\( W_b^T = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & T_{63}^T & 0 \\ 0 & 0 & 0 & 0 & I & T_{65}^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \)).

Again, it follows from the Elimination Lemma that a necessary condition for (A.22) is that

\[
W_b^T \psi_o(P) W_b > 0. \quad \text{(A.25)}
\]

To write (A.25) explicitly, \( \psi_o(P) \) is partitioned in conformity with \( W_b, i.e., \) with

\[
B_{az} = [B_{az1} : B_{az2}] \text{ and } D_{iy} = [D_{iy1} : D_{iy2}], \text{ then}
\]

\[
\psi_o(P) = \begin{bmatrix}
R & M & A_{az} & 0 & B_{az1} & B_{az2} & 0 \\
M^T & Z & 0 & 0 & 0 & 0 & 0 \\
A_{az}^T & 0 & S & N & 0 & 0 & C_o^T \\
0 & 0 & N^T & X & 0 & 0 & 0 \\
B_{az1}^T & 0 & 0 & 0 & M_{xy} & 0 & D_{iy1}^T \\
B_{az2}^T & 0 & 0 & 0 & 0 & M_{xy} & D_{iy2}^T \\
0 & 0 & C_o & 0 & D_{iy1} & D_{iy2} & I_{me} 
\end{bmatrix}.
\]
Thus,

$$\psi_i(P)W_b = \begin{bmatrix} R & M & A_{az} + B_{axz}T_{63} & B_{ax1} + B_{ax2}T_{65} & 0 \\ M^T & Z & 0 & 0 & 0 \\ A_{az}^T & 0 & S & 0 & C_{o}^T \\ 0 & 0 & N^T & 0 & 0 \\ B_{ax1}^T & 0 & 0 & M_{sy} & D_{1y1}^T \\ B_{ax2}^T & 0 & M_{sv}T_{63} & M_{sv}T_{65} & D_{1y2}^T \\ 0 & 0 & C_{o} + D_{1y2}T_{63} & D_{1y1} + D_{1y2}T_{65} & I_{me} \end{bmatrix}$$

and, hence,

$$\psi_b \triangleq W_b^T\psi_i(P)W_b = \begin{bmatrix} R & M & \psi_{b13} & \psi_{b14} & \psi_{b15} \\ M^T & Z & 0 & 0 & 0 \\ \psi_{b13}^T & 0 & (S + M_{3w}) & \psi_{b34} & \psi_{b35} \\ \psi_{b14}^T & 0 & \psi_{b34}^T & (M_{sy} + M_{5w}) & \psi_{b45} \\ \psi_{b15}^T & 0 & \psi_{b35}^T & \psi_{b45}^T & I_{me} \end{bmatrix},$$

where

$$\psi_{b13} = A_{az} + B_{axz}T_{63}, \quad \psi_{b14} = B_{ax1} + B_{ax2}T_{65}, \quad \psi_{b15} = 0,$$

$$\psi_{b34} = T_{63}^TM_{sv}T_{65}, \quad \psi_{b35} = C_{o}^T + T_{63}^TD_{1y2}^T, \quad \psi_{b45} = D_{1y1}^T + T_{65}^TD_{1y2},$$

$$M_{3w} = T_{63}^TM_{sv}T_{63}, \quad M_{5w} = T_{65}^TM_{sv}T_{65}.$$

Pre and post-multiplying $\psi_b$ by $I_b^T$ and $I_b$, where the column permutations matrix $I_b$ is given by

$$I_b = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

it follows that $\psi_b > 0 \iff I_b^T\psi_b I_b > 0 \iff$

$$\begin{bmatrix} (S + M_{3w}) & \psi_{b34} & \psi_{b35} & \psi_{b13}^T & 0 \\ \psi_{b34}^T & (M_{sy} + M_{5w}) & \psi_{b45} & \psi_{b14}^T & 0 \\ \psi_{b35}^T & \psi_{b45}^T & I_{me} & \psi_{b15}^T & 0 \\ \psi_{b13} & \psi_{b14} & \psi_{b15} & R & M \\ 0 & 0 & 0 & M^T & Z \end{bmatrix} > 0$$

$\iff$ (in the light of the Schur complement formula) $\begin{bmatrix} R & M \\ M^T & Z \end{bmatrix} > 0$ and

$$\begin{bmatrix} (S + M_{3w}) & \psi_{b34} & \psi_{b35} \\ \psi_{b34}^T & (M_{sy} + M_{5w}) & \psi_{b45} \\ \psi_{b35}^T & \psi_{b45}^T & I_{me} \end{bmatrix} - \begin{bmatrix} \psi_{b13}^T & 0 \\ \psi_{b14}^T & 0 \\ \psi_{b15}^T & 0 \end{bmatrix} \begin{bmatrix} R & M \\ M^T & Z \end{bmatrix}^{-1} \begin{bmatrix} \psi_{b13} & \psi_{b14} & \psi_{b15} \\ 0 & 0 & 0 \end{bmatrix} > 0$$
Moreover, since  
\[
\begin{bmatrix}
R & M \\
M^T & Z
\end{bmatrix}^{-1} = P = \begin{bmatrix}
S & N \\
N^T & X
\end{bmatrix},
\]
the last LMI above can be rewritten as
\[
\begin{bmatrix}
(S + M_{yw}) & \psi_{y34} & \psi_{y35} \\
\psi_{y34}^T & (M_{sy} + M_{yw}) & \psi_{y45} \\
\psi_{y35}^T & \psi_{y45}^T & I_{m_y}
\end{bmatrix} - \begin{bmatrix}
\psi_{b13}^T \\
\psi_{b14}^T \\
\psi_{b15}^T
\end{bmatrix} S \begin{bmatrix}
\psi_{b13}^T \\
\psi_{b14}^T \\
\psi_{b15}^T
\end{bmatrix} > 0. \quad (A.26)
\]

To complete the proof note that \( \psi_{b15} = 0 \) so that the last LMI holds if and only if
\[
\begin{bmatrix}
S & 0 \\
0 & M_{sy}
\end{bmatrix} + \begin{bmatrix}
M_{sw}(\sigma) & \psi_{b34} \\
\psi_{b34}^T & M_{sw}(\sigma)
\end{bmatrix} - \begin{bmatrix}
\psi_{b35} \\
\psi_{b45}
\end{bmatrix} \begin{bmatrix}
\psi_{b35}^T \\
\psi_{b45}^T
\end{bmatrix} - \begin{bmatrix}
\psi_{b13} \\
\psi_{b14}
\end{bmatrix} S \begin{bmatrix}
\psi_{b13} \\
\psi_{b14}
\end{bmatrix} > 0
\]
or, equivalently,
\[
\begin{bmatrix}
S & 0 \\
0 & \sigma_y I_{m_y}
\end{bmatrix} + \sigma_y E_b^T E_b - \psi_b^T S \psi_b - E_{bw}^T E_{bw} > 0,
\]
where
\[
E_{bw} = \begin{bmatrix}
T_{63} & : & T_{65}
\end{bmatrix} = -D^{-1}_{by} \begin{bmatrix}
C_{ax} & : & D_{ax1}
\end{bmatrix} = -D_{by} \begin{bmatrix}
C_{ax} & : & D_{b}^{-1}_{by}
\end{bmatrix},
\]
\[
E_{bw} = \begin{bmatrix}
\hat{C}_a & : & D_{by1}
\end{bmatrix} \quad \text{(since \( D_{by2} = 0 \))}
\]
and \( \psi_b = \begin{bmatrix}
A_{ax} & : & B_{ax1}
\end{bmatrix} + B_{ax2} E_b. \)

Thus, \( \hat{Q}_b(S, \sigma) > 0 \iff Q_{b}(S, \sigma) > 0, \)
where \( Q_{b}(S, \sigma) = Q_{b1}(S, \sigma) - E_{bw}^T E_{bw}, \hat{Q}_{b1}(S, \sigma) = \text{diag}(S, \sigma_y I_{m_y}) + \sigma_y E_b^T E_b - \psi_b^T S \psi_b. \)

**Proof of Proposition 4.8**: It has been established that
\[
\psi(P, \sigma, \theta) > 0 \iff \psi_{b}(P, \sigma) + T_a^T \theta T_b + T_b^T \theta^T T_a > 0,
\]
Thus, it follows from Proposition 4.7 and the Elimination Lemma that if \( Q_a(R_a) > 0 \) and \( Q_b(S_b) > 0 \), then for any \( P = \begin{bmatrix}
S_o & \mathbf{N} \\
\mathbf{N}^T & X
\end{bmatrix} > 0 \) with \( P^{-1} = \begin{bmatrix}
R_o & M \\
M^T & Z
\end{bmatrix} \) there exists \( \theta \) such that
\[
\psi_{b}(P) + T_a^T \theta T_b + T_b^T \theta^T T_a > 0.
\]
To get \( P \) as above from a given pair \( (S_o, R_o) \) as above, note that
(with \( \dim(X) = \dim(S_o) \))
\[
R_o = (S_o - \mathbf{N} \mathbf{X}^{-1} \mathbf{N}^T)^{-1} \iff S_o - \mathbf{N} \mathbf{X}^{-1} \mathbf{N}^T = R_o^{-1} \iff S_o - R_o^{-1} = \mathbf{N} \mathbf{X}^{-1} \mathbf{N}^T
\]
so that with \( S_o - R_o^{-1} \geq 0 \) (\( \iff \begin{bmatrix}
S_o & \mathbf{I} \\
\mathbf{I} & R_o
\end{bmatrix} \geq 0 \)) one might take \( Q_{SR} = (S_o - R_o^{-1})^{1/2}, \quad X = X^T > 0 \)
and put \( \mathbf{N} \mathbf{X}^{-1/2} = Q_{SR} V \iff \mathbf{N} = Q_{SR} V X^{1/2}, \) for any unitary \( V. \)

The fact that \( P^* > 0 \) follows from the following logical sequence based on Schur Complements: as \( X > 0, \)
\[
P^* > 0 \iff (S_o - Q_{SR} V X^{1/2} \mathbf{X}^{-1/2} V^T Q_{SR}) > 0 \iff (S_o - Q_{SR}^2) > 0 \iff
\]

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As a result in the light of the Proposition 4.6, Prob. 2 can be recast as Prob. 2d.

With respect to Proposition 4.8(b), note that it follows from Proposition 4.4 and 4.5 that

\[
\mathcal{J}_\infty(G(\theta; H)) = \inf \left\{ \sigma_y \gamma_y^2 + \sigma_v \gamma_v^2 : \sigma_y > 0, \sigma_v > 0, \mathbf{P} = \mathbf{P}^T > 0 \right. \\
\left. \quad \text{and } \psi_\alpha(\mathbf{P}, \sigma) + T_a^T \theta T_b + T_b^T \theta^T T_a > 0 \right\}.
\]

Therefore, for \(\mathbf{P}^o, \sigma^o\) and \(\theta\) as above, \(\mathcal{J}_\infty(G(\theta; H)) \leq \sigma_y^o \gamma_y^2 + \sigma_v^o \gamma_v^2 = \mathcal{J}_\infty^o(H) + \varepsilon\) (the proof that \(\rho(A_G) < 1\) is as in Proposition 4.6(b)).

**Proof of equation (4.20):** Proceeding as in the **Proof of Proposition 4.3**, consider first the following sets

\[
S_{\sigma^o}(G) \triangleq \{ \sigma = (\sigma_y, \sigma_v, \sigma_w) : \sigma_y > 0, \sigma_v > 0, \sigma_w > 0 \}
\]

and \(\forall \phi \in [0, 2\pi], (F_{G^*} G_{G^*} + \sigma_w \gamma_H^2 \Gamma_{H^*}) (e^{j\phi}) \leq 0\)

where \(G_{H^*} \triangleq \text{diag}(W_{H^*} W_{H^*}^*, 0, 0)\), and

\[
S_{\sigma^o}(G) \triangleq \{ \sigma = (\sigma_y, \sigma_v, \sigma_w) : \sigma_y > 0, \sigma_v > 0, \sigma_w > 0 \}
\]

and \(\forall \phi \in [0, 2\pi], (F_{G^*} G_{G^*} + \sigma_w \gamma_H^2 \Gamma_{H^*}^a) (e^{j\phi}) < 0\)

and note that \(\forall \sigma \in S_{\sigma^o}(G)\), \(\varphi_{D^o}(\sigma; G) = \sigma_y \gamma_y^2 + \sigma_v \gamma_v^2\), whereas for the other values of \(\sigma \geq 0\), \(\varphi_{D^o}(\sigma; G) = +\infty\) – hence \(\varphi_{D^o}(\sigma; G) = \inf\{\sigma_y \gamma_y^2 + \sigma_v \gamma_v^2 : \sigma \in S_{\sigma^o}(G)\}\).

The proof is concluded by noting that \(S_{\sigma^o}(G) (\subset S_{\sigma^o}(G))\) is dense in \(S_{\sigma^o}(G)\) – indeed, if \(\sigma^o \triangleq (\sigma_y^o, \sigma_v^o, \sigma_w^o) \in S_{\sigma^o}, \forall \varepsilon > 0, \sigma^o \triangleq (\sigma_y^o + 2\varepsilon \gamma_H^2 ||W_{H^*}||_2^2, \sigma_v^o + \varepsilon, \sigma_w^o + \varepsilon) \in S_{\sigma^o}\) as

\[
(F_{G^*} G_{G^*} + \sigma_w \gamma_H^2 \Gamma_{H^*}^a - M_{\sigma^o}^a + \text{diag}(\varepsilon \gamma_H^2 ||W_{H^*}||_2^2 ||W_{H^*}||_2^2, \varepsilon ||W_{H^*}||_2^2, -\varepsilon ||W_{H^*}||_2^2, -\varepsilon ||W_{H^*}||_2^2)) (e^{j\phi}) < 0.
\]

**Proof of equation (4.18):** Note that

\[
F_{GW} = \begin{bmatrix}
\sigma_w \gamma_H^2 ||W_{H^*}||_2^2 \\
F_{G_{G^*}}
\end{bmatrix} = H_{W^*}^a - \begin{bmatrix}
0 \\
I
\end{bmatrix} G_{H^*} \text{ and } H_{W^*} = \begin{bmatrix}
\sigma_w \gamma_H^2 ||W_{H^*}||_2^2 \\
F_{G_{G^*}}
\end{bmatrix},
\]

so that introducing (minimal) realizations \(\Sigma_{W^*} \text{ of } H_{W^*}^a, \Sigma_{G^*} \text{ of } H_{G^*}\), \(\Sigma_{GW} \) is obtained as follows:

\[
A_{GW} = \begin{bmatrix}
A_{W^*} & 0 \\
0 & A_{G^*}
\end{bmatrix}, \quad B_{GW} = \begin{bmatrix}
B_{W^*} \\
B_{G^*}
\end{bmatrix}, \quad C_{GW} = \begin{bmatrix}
C_{W^*} & -C_{G^*}
\end{bmatrix}, \quad D_{GW} = D_{W^*} - D_{G^*}.
\]

where \(\Sigma_{W^*} = (A_{W^*}, B_{W^*}, C_{W^*}, D_{W^*})\) and \(\Sigma_{G^*} = (A_{G^*}, B_{G^*}, C_{G^*}, D_{G^*})\) are given by
As a result, \( A_a = \text{diag}(A_{a_1},0_{n_G \times n_G}) \), \( A_{a_1} = \text{diag}(A_{Wa},A_{oa}) \),

\[
A_{\text{Goa}} = \begin{bmatrix} A_{oa} & 0_{n_{oa} \times n_G} \\ B_G C_{oa} & A_G \end{bmatrix}, \quad B_{\text{Goa}} = \begin{bmatrix} B_{oa} \\ B_G D_{oa} \end{bmatrix}, \quad C_{\text{Goa}} = \begin{bmatrix} 0 \\ I_{m_y} \end{bmatrix}, \quad D_{\text{Goa}} = \begin{bmatrix} 0 \\ I_{m_e} \end{bmatrix}, \quad D_G D_{oa}^	op.
\]

\[
A_{Wa} = \begin{bmatrix} A_{W_{Hy}} & 0 \\ 0 & A_{HHa} \end{bmatrix}, \quad B_{Wa} = \begin{bmatrix} B^\sigma_{W_{Hy}} \\ B_{HHa} \end{bmatrix}, \quad C_{Wa} = \begin{bmatrix} C^\sigma_{W_{Hy}} \\ C_{HHa} \end{bmatrix}, \quad D_{Wa} = \begin{bmatrix} D^\sigma \\ D_{HHa} \end{bmatrix}.
\]

\[
C^\sigma_{Wa} = \begin{bmatrix} \sigma_w^\gamma H C_{W_{Hy}} & 0_{m_y \times n_A_{HHa}} \\ 0_{n_{A_{W_{Hy}}} \times n_A_{HHa}} & 0_{m_y \times n_A_{HHa}} \end{bmatrix}, \quad C^e_{Wa} = \begin{bmatrix} 0_{m_y \times n_A_{W_{Hy}}} \\ C_{HHa} \end{bmatrix}, \quad D^\sigma_{Wa} = \sigma_w^{1/2} H D^a_{W_{Hy}}.
\]

\[
D^e_{Wa} = D_{HHa} - \Sigma_{Wa} \quad \text{and} \quad \Sigma_{Goa} \quad \text{depend on the problem data} \quad (H_0,H_1,W_y,W_v,W_H) \quad \text{through the realizations of} \quad H_{oa} = \begin{bmatrix} H_0 W_{y^{-1}} & W_v^{-1} \end{bmatrix}, \quad H_{ta} = H_1 W_{y^{-1}}, \quad \text{so that} \quad D^e_{W_{Hy}} = \begin{bmatrix} 0_{m_y \times m_v} & 0_{m_y \times m_v} \end{bmatrix},
\]

and \( W^a_{Hy} = W_{Hy} \begin{bmatrix} I_{m_y} & 0_{m_y \times m_v} & 0_{m_y \times m_v} \end{bmatrix}, \quad \text{i.e.,} \quad C_L(\theta) = - \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix} \theta, \quad D_a = D_{Wa} \).

Thus, partitioning \( P \) and \( P^{-1} \) conformally with \( A_a \), \( i.e., \quad P = \begin{bmatrix} S & N \\ N^T & X \end{bmatrix} \) and

\[
P^{-1} = \begin{bmatrix} R & M \\ M^T & Z \end{bmatrix},
\]

\[
\psi^o_a(P,\sigma) = \begin{bmatrix} R & M & A_{a_1} & 0 & B^\ast & 0 & 0 & 0 \\ M^T & Z & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{a_1}^T & 0 & S & N & 0 & 0 & (C^\sigma_{a_1})^T & (C^e_{a_1})^T \\ 0 & 0 & N^T & X & 0 & 0 & 0 & 0 \\ (B^\ast_{a_1})^T & 0 & 0 & 0 & M^\ast & 0 & (D^\sigma_{a_2})^T & (D^e_{a_2})^T \\ 0 & 0 & 0 & 0 & M_{\sigma} & 0 & 0 & 0 \\ 0 & 0 & C^\ast_{a_1} & 0 & D^\ast & 0 & m_y \times m_v & I_{m_y} \\ 0 & 0 & C^e_{a_1} & 0 & D_{e} & 0 & m_y \times m_v & 0 & I_{m_e} \end{bmatrix}
\]

\[= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & m_y \times m_v & 0 & I_{m_e} \\ 0 & 0 & 0 & 0 & 0 & 0 & m_y \times m_v & 0 & I_{m_e} \\ 0 & 0 & 0 & 0 & 0 & 0 & m_y \times m_v & 0 & I_{m_e} \\ 0 & 0 & 0 & 0 & 0 & 0 & m_y \times m_v & 0 & I_{m_e} \\ 0 & 0 & 0 & 0 & 0 & 0 & m_y \times m_v & 0 & I_{m_e} \\ 0 & 0 & 0 & 0 & 0 & 0 & m_y \times m_v & 0 & I_{m_e} \end{bmatrix}
\]

- note that passing from the 4–block expression for \( \psi^o_a(P,\sigma) \) to the 8–block one above \( C_a \) and \( D_a \) were written as

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It is first necessary to obtain $W_1$ and $W_2$. To this effect, note that

$$\text{Ker}(T_1) = \left\{ v^T = [v_1^T \ldots v_8^T] : T_1 v = 0 \right\} = \{ v : v_2 = 0 \text{ and } v_8 = 0 \}.$$ 

**Proof of Proposition 4.10** It is first necessary to obtain $W_1$ and $W_2$. To this effect, note that
As a result,

\[ W_1 = \begin{bmatrix}
I & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

is such that its columns form a basis for the null space of \( T_1 \).

Thus,

\[
W_1^T \psi_a(P, \sigma) W_1 = \begin{bmatrix}
R & A_{a_1} & 0 & B^z_{a_1} & 0 & 0 \\
A^T_{a_1} & S & N & N 0 & 0 & (C^\sigma)^T_{a_1} \\
0 & N^T & X & 0 & 0 & 0 \\
(B^z_{a_1})^T & 0 & 0 & M^z_{\sigma} & 0 & (D^\sigma)^T_{a_1z} \\
0 & 0 & 0 & 0 & M^w_{\sigma} & 0 \\
0 & C^\sigma_{a_1} & 0 & D^\sigma_{a_1z} & 0 & I_{m_y}
\end{bmatrix},
\]

so that \( W_1^T \psi_a(P, \sigma) W_1 > 0 \iff \sigma_w > 0 \) and

\[
\begin{bmatrix}
R & A_{a_1} & 0 & B^z_{a_1} & 0 \\
A^T_{a_1} & S & N & 0 & (C^\sigma)^T_{a_1} \\
0 & N^T & X & 0 & 0 \\
(B^z_{a_1})^T & 0 & 0 & M^z_{\sigma} & (D^\sigma)^T_{a_1z} \\
0 & C^\sigma_{a_1} & 0 & D^\sigma_{a_1z} & I_{m_y}
\end{bmatrix} > 0
\]

\( \iff \) (exchanging rows and columns) \( \sigma_w > 0 \) and

\[
\begin{bmatrix}
R & B^z_{a_1} & 0 & A_{a_1} & 0 \\
(B^z_{a_1})^T & M^z_{\sigma} & (D^\sigma)^T_{a_1z} & 0 & 0 \\
0 & D^\sigma_{a_1z} & I_{m_y} & C^\sigma_{a_1} & 0 \\
A^T_{a_1} & 0 & (C^\sigma)^T_{a_1} & S & N \\
0 & 0 & 0 & N^T & X
\end{bmatrix} > 0
\]

\( \iff \sigma_w > 0 \) and

\[
\psi_{a_1}(R, \sigma) - \begin{bmatrix}
A_{a_1} & 0 \\
0 & 0 \\
C^\sigma_{a_1} & 0
\end{bmatrix} \begin{bmatrix}
R & M \\
M^T & Z
\end{bmatrix} \begin{bmatrix}
A^T_{a_1} & 0 & (C^\sigma)^T_{a_1}
\end{bmatrix} > 0
\]
\( \Leftrightarrow \sigma_w > 0 \) and
\[
\psi_{a1}(R, \sigma) - \begin{bmatrix} A_{a1} \\ 0 \\ C_{a1}^\sigma \end{bmatrix} R \begin{bmatrix} A_{a1}^T & 0 & (C_{a1}^\sigma)^T \end{bmatrix} > 0,
\]
where
\[
\psi_{a1}(R, \sigma) = \begin{bmatrix} R & B_{a1} & 0 \\ (B_{a1}^*)^T & M_{a1}^\sigma & (D_{a1}^*)^T \\ 0 & D_{a1}^\sigma & I_{m_y} \end{bmatrix}, \quad C_{a1}^\sigma = \begin{bmatrix} \sigma_{w}^{1/2} \gamma H C_{WHy} : 0_{m_y \times n_{AHH}} : 0_{m_y \times n_{Aoa}} \end{bmatrix},
\]
\[
C_{a1}^e = \begin{bmatrix} 0_{m_e \times n_{AWHy}} : C_{H} : 0_{m_e \times n_{Aoa}} \end{bmatrix}, \quad D_{a1}^e = \begin{bmatrix} \sigma_{w}^{1/2} \gamma H D_{WHy} : 0_{m_y \times n_v} \end{bmatrix}
\]
and
\[
D_{a1}^e = \begin{bmatrix} D_{HH} D_{Wy}^{-1} : 0_{m_e \times n_v} \end{bmatrix}.
\]

Finally, pre and post-multiplying the last matrix inequality above by diag(\(I, I, \sigma_w^{1/2} I\)) it follows that
\[
W_1^T \psi_{a}(P, \sigma) W_1 > 0 \Leftrightarrow \sigma_w > 0, \ P > 0 \text{ and } Q_1(R, \sigma) > 0.
\]

Similarly,
\[
\text{Ker}(T_2) = \left\{ v^T = [v_1^T \ldots v_8^T] : v_4 = 0 \text{ and } \hat{C}_{oa} v_3 + D_{oa}^z v_5 + v_6 = 0 \right\}.
\]

Thus,
\[
\text{Ker}(T_2) = \left\{ v^T = [v_1^T \ldots v_8^T] : v_4 = 0 \text{ and } v_6 = T_{63} v_3 + T_{65} v_5 \right\},
\]
where \( T_{63} = -\hat{C}_{oa} \) and \( T_{65} = -D_{oa}^z \). \( D_{oa}^z = [D_{HH} D_{Wy}^{-1} : D_{Wy}^{-1}] \).

Thus,
\[
W_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_{63} & T_{65} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad W_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.
\]
As a result,

\[
W_2^T \psi^a(P, \sigma) W_2 = \begin{bmatrix}
R & M & \psi_{a13} & \psi_{a14} & 0 & 0 \\
M^T & Z & 0 & 0 & 0 & 0 \\
\psi_{a13}^T & 0 & S + M_{\sigma w3} & \psi_{a34} & \psi_{a35} & \psi_{a36} \\
\psi_{a14}^T & 0 & M_{\sigma}^T + M_{\sigma w5} & \psi_{a45} & \psi_{a46} & 0 \\
0 & 0 & \psi_{a35}^T & \psi_{a45} & I_{m_y} & 0 \\
0 & 0 & \psi_{a36}^T & \psi_{a46} & 0 & I_{m_v}
\end{bmatrix},
\]

where

\[
\psi_{a13} = A_{a1}, \quad \psi_{a14} = B^a_{a1},
\]

\[
\psi_{a34} = T_6^T M_{w}^T T_6, \quad \psi_{a35} = (C_{a1})^T, \quad \psi_{a36} = (C_{a1})^T, \quad \psi_{a45} = (D_{a})^T, \quad \psi_{a46} = (D_{a})^T,
\]

\[
M_{\sigma w3} = T_6^T M_{\sigma}^T T_6, \quad M_{\sigma w5} = T_6^T M_{w}^T T_6, \quad M_{\sigma} = \text{diag}(\sigma, I, \sigma, I), \quad M_{w} = \sigma I_{m_v}.
\]

Pre and post-multiplying \(W_2^T \psi^a(P, \sigma) W_2\) by \(I_{a2}^T\) and \(I_{a2}\), where \(I_{a2}\) is given by

\[
I_{a2} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix},
\]

it follows that \(W_2^T \psi^a(P, \sigma) W_2 > 0 \iff I_{a2}^T W_2^T \psi^a(P, \sigma) W_2 I_{a2} > 0\)

\[
\psi_{Wa}(S, \sigma) - \begin{bmatrix}
\psi_{a13}^T & \psi_{a14}^T & 0 & 0 \\
\psi_{a13} & \psi_{a14} & 0 & 0 \\
0 & 0 & R & M \\
0 & 0 & M^T & Z
\end{bmatrix} > 0 \iff \begin{bmatrix}
R & M \\
M^T & Z
\end{bmatrix} > 0 \quad \text{and}
\]

\[
\psi_{Wa}(S, \sigma) - \begin{bmatrix}
\psi_{a13}^T & \psi_{a14}^T & S \begin{bmatrix}
\psi_{a13} & \psi_{a14} & 0 & 0
\end{bmatrix} > 0, \text{ where}
\]

\[
\psi_{Wa}(S, \sigma) = \begin{bmatrix}
\psi_{a13}^T & \psi_{a14}^T & \psi_{a35} & \psi_{a36} \\
\psi_{a13} & \psi_{a14} & \psi_{a35} & \psi_{a36} \\
0 & 0 & I_{m_y} & 0 \\
0 & 0 & I_{m_v}
\end{bmatrix} \iff \begin{bmatrix}
R & M \\
M^T & Z
\end{bmatrix} > 0 \quad \text{and}
\]

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or, equivalently, \[ W_2^T \psi_a^0(S, \sigma) W_2 > 0 \Leftrightarrow \psi_{aS}(S, \sigma) - \begin{bmatrix} \hat{\psi}_{a13}^T \\ \hat{\psi}_{a14}^T \end{bmatrix} \begin{bmatrix} \sigma_w \gamma_H^{-2} I_{m_y} & 0 \\ 0 & I_{m_e} \end{bmatrix} \begin{bmatrix} \hat{C}_{WH}^T \\ \hat{D}_{az}^T \end{bmatrix} M_{WI}(\sigma_w) \begin{bmatrix} \hat{C}_{WH}^T \\ \hat{D}_{az}^T \end{bmatrix} > 0, \] (A.30)

where \( \psi_{aS}(S, \sigma) \triangleq \psi_{a1}(S, \sigma) - \begin{bmatrix} \psi_{a13}^T \\ \psi_{a14}^T \end{bmatrix} \begin{bmatrix} \sigma_w \gamma_H^{-2} I_{m_y} & 0 \\ 0 & I_{m_e} \end{bmatrix} \begin{bmatrix} \hat{C}_{WH}^T \\ \hat{D}_{az}^T \end{bmatrix} M_{WI}(\sigma_w) \begin{bmatrix} \hat{C}_{WH}^T \\ \hat{D}_{az}^T \end{bmatrix} > 0, \) (A.31) or, equivalently, \[ \psi_{aS}(S, \sigma) \triangleq \psi_{a1}(S, \sigma) - \begin{bmatrix} A_{a1}^T \\ B_{a1}^T \end{bmatrix} \begin{bmatrix} A_{a1} & B_{a1} \end{bmatrix}. \] (A.32)

Thus, the condition \( W_2^T \psi_a^0(P, \sigma) W_2 > 0 \) is equivalent to the LMI (A.30) on the variables \( S, \sigma_y, \sigma_v, \text{ and } \sigma_w. \) The proof is concluded by noting that \( \begin{bmatrix} T_{63} \\ T_{65} \end{bmatrix} = \begin{bmatrix} C_{ao} & D_{oa}^T \end{bmatrix}, M_w^w = \sigma_w I, E_w = \begin{bmatrix} \psi_{a13} \\ \psi_{a14} \end{bmatrix} (E_s \text{ and } E_o \text{ as in the beginning of the Appendix}) \) and

\[
\sigma_w \begin{bmatrix} (C_{a1}^w)^T \\ (D_{az}^w)^T \end{bmatrix} \begin{bmatrix} C_{a1}^w & D_{az}^w \end{bmatrix} + E_o^T E_o. \]

\[ \blacksquare \]
Proof of equation (5.7): Note first that

\[ F_{GW} = \begin{bmatrix} \sigma_w^{1/2} \gamma_H W^a_H & \sigma_w^{1/2} \gamma_H W^a_T \\ F_{Ga} & \end{bmatrix} = \begin{bmatrix} \sigma_w^{1/2} \gamma_H I_{m_y} & 0 \\ 0 & I_{m_e} \end{bmatrix} \begin{bmatrix} I_{m_y} & 0 & 0 \\ 0 & I_{m_e} & \beta \end{bmatrix} \begin{bmatrix} W^a_H \\ H_{Ga} \\ -Y_e^* H_{oa} \end{bmatrix} \Rightarrow \]

\[ [C_{GW}(\sigma_w, \beta) : D_{GW}(\sigma_w g)] = \begin{bmatrix} \sigma_w^{1/2} \gamma_H I_{m_y} & 0 \\ 0 & I_{m_e} \end{bmatrix} \begin{bmatrix} I_{m_y} & 0 \\ 0 & I_{m_e} & \beta \end{bmatrix} \begin{bmatrix} T^y_{GW} \\ T^e_{GW}(\beta) \end{bmatrix}, \]

where \( T^y_{GW} \triangleq \begin{bmatrix} I_{m_y} & 0_{m_y \times m_e} & 0_{m_y \times (n_c^e m_y)} \end{bmatrix} \begin{bmatrix} \hat{C}_{GW} : \hat{D}_{GW} \end{bmatrix} \)

and

\[ T^e_{GW} \triangleq \begin{bmatrix} 0_{m_e \times m_y} & I_{m_y} & \beta \end{bmatrix} \begin{bmatrix} \hat{C}_{GW} : \hat{D}_{GW} \end{bmatrix}. \]

As a result,

\[ Q_{CD} \triangleq \begin{bmatrix} C_{GW}(\cdot)^T & D_{GW}(\cdot)^T \end{bmatrix} \begin{bmatrix} C_{GW}(\cdot) & D_{GW}(\cdot) \end{bmatrix} = \sigma_w^2 \gamma_H^{2}(T^y_{GW})^T(T^y_{GW}) + (T^e_{GW}(\beta))^T(T^e_{GW}(\beta)). \]

Note now that \( Q_{BR}(P; \Sigma^g_{GW}(\sigma_w, \beta), M_e^g) < 0 \iff \hat{Q}_{b1}(P; \sigma) - Q_{CD} > 0 \iff \hat{Q}_{b1}(P; \sigma) - \sigma_w^2 \gamma_H^{2}(T^y_{GW})^T(T^y_{GW}) + (T^e_{GW}(\beta))^T(T^e_{GW}(\beta)) > 0 \iff \hat{Q}_{a}(P; \sigma, \beta) > 0. \]

Rewriting the second constraint of (5.5): Note that \( Q_{BR}(P; \Sigma^g_{a}(G(\beta)), M_e^g) < 0 \iff \hat{Q}_{a}(P; \sigma, \beta) > 0. \]

Proof of equations (5.1) and (5.2): Note first that

\[ C \left( G(\beta); \hat{\Gamma}_y, \hat{\Gamma}_v \right) = \left\{ \begin{bmatrix} I_{m_e} : \beta \end{bmatrix} F^y_c \hat{\Gamma}_y, \begin{bmatrix} I_{m_e} : \beta \end{bmatrix} F^y_c \hat{\Gamma}_v \right\} + \left\{ \begin{bmatrix} I_{m_e} : \beta \end{bmatrix} F^y_c \hat{\Gamma}_y, \begin{bmatrix} I_{m_e} : \beta \end{bmatrix} F^y_c \hat{\Gamma}_v \right\}, \]

where \( F^y_c \triangleq \begin{bmatrix} H_i \\ -Y_e^* H_o \end{bmatrix} \) and \( F^v_c \triangleq \begin{bmatrix} 0 \\ Y_e^* \end{bmatrix} \).

Thus,

\[ C \left( G(\beta); \hat{\Gamma}_y, \hat{\Gamma}_v \right) = \left\{ \begin{bmatrix} I_{m_e} : \beta \end{bmatrix}, \begin{bmatrix} I_{m_e} : \beta \end{bmatrix} (F^y_c \hat{\Gamma}_y (F^y_c)^* + F^v_c \hat{\Gamma}_v (F^v_c)^*) \right\}, \iff \]

\[ C \left( G(\beta); \hat{\Gamma}_y, \hat{\Gamma}_v \right) = \text{tr} \left\{ \begin{bmatrix} I_{m_e} : \beta \end{bmatrix} Q_c \begin{bmatrix} I_{m_e} : \beta \end{bmatrix}^T \right\}. \]

As a result, \( C \left( G(\beta); \hat{\Gamma}_y, \hat{\Gamma}_v \right) = \inf \left\{ \text{tr}(P) : P = P^T, P \geq \begin{bmatrix} I_{m_e} : \beta \end{bmatrix} Q_c \begin{bmatrix} I_{m_e} : \beta \end{bmatrix}^T \right\}. \)
The proof of (5.1) is concluded by noting that
\[
\mathbf{P} \geq [I_m : \beta]Q_c[I_m : \beta]^T \iff \mathbf{P} - \left\{ \left( [I_m : \beta]Q_c^{1/2} \right) \left( [I_m : \beta]Q_c^{1/2} \right)^T \right\} \geq 0
\]
\[\iff \text{(in the light of the so-called Schur complement formula)} \quad Q_{J\gamma}(\mathbf{P}, [I_m : \beta]; Q_c) \geq 0. \]
Equation (5.2) is proved in exactly the same way noting at the beginning that
\[
\left\langle \left( \mathbf{G}(\beta) \otimes \phi_y^T , \mathbf{G}(\beta) \otimes \phi_y^T \right) \right\rangle = \left\langle \left( \mathbf{I} \otimes \phi_y^T \right) (\mathbf{G}(\beta) \otimes \phi_y^T) , \left( \mathbf{I} \otimes \phi_y^T \right) (\mathbf{G}(\beta) \otimes \phi_y^T)^T \right\rangle \iff
\]
\[= \left\langle \left( \mathbf{I} \otimes \phi_y^T \right) \left( \mathbf{Y}^i \otimes \phi_y^T \right) , \left( \mathbf{Y}^i \otimes \phi_y^T \right)^T \right\rangle \iff
\]
\[= \text{tr} \left\{ \left( \mathbf{I} \otimes \phi_y^T \right) \mathbf{Q}_{G_x} \left( \mathbf{I} \otimes \phi_y^T \right)^T \right\} .
\]

Proof of Proposition [5.1]: Proposition [5.1] follows directly from Proposition [4.2] and the following auxiliary proposition.

Proposition A.1: Let \( f : \mathcal{R}^{m \times m} \rightarrow \mathbb{R} \) be \( \mathcal{H}_2 \)-continuous,
\[
S_{pr} \triangleq \{ \mathbf{G} \in S_{\mathcal{G}} : \exists (\lambda, \mathbf{P}), \lambda > 0, \mathbf{P} = \mathbf{P}^T (i) \text{ and } (ii) \text{ hold} \},
\]
where (i) \( \lambda \gamma^2 + x_o^T \mathbf{P} x_o \leq \eta_j \) and (ii) \( Q_{J\gamma}(\mathbf{P}; \Sigma_u, M(\lambda)) < 0 \), and
\[
S_{eq} \triangleq \{ \mathbf{G} \in S^1_\mathcal{G} : \mathcal{J}(\mathbf{G}; S_\mathcal{X}) \leq \eta_j \}, \quad \eta_j \triangleq (1 + \alpha)\mathcal{J}_o, \quad \alpha > \varepsilon \text{ and }
\]
\[\mathcal{J}_o = \inf \{ \mathcal{J}(\mathbf{G}; S_\mathcal{X}) : \mathbf{G} \in S_{\mathcal{G}} \} .
\]

Then, \( S_{pr} \) is non-empty, \( S_{pr} \subset S_{eq} \) and
\[
\inf \{ f(\mathbf{G}) : \mathbf{G} \in S_{eq} \} = \inf \{ f(\mathbf{G}) : \mathbf{G} \in S_{pr} \} .
\]

Proof of Proposition A.1: Let \( S_{in} \triangleq \{ \mathbf{G} \in S_{\mathcal{G}} : \mathcal{J}(\mathbf{G}; S_\mathcal{X}) < \eta_j \} \) and note that as \( \alpha > \varepsilon \), \( S_{in} \) is non-empty. Now, consider the following auxiliary propositions.

Auxiliary Proposition 2: If \( \mathbf{G} \in S_{pr} \), then \( \mathbf{G} \in S_{eq} \) (i.e., \( S_{pr} \subset S_{eq} \)). If \( \mathbf{G} \in S_{in} \), then \( \mathbf{G} \in S_{pr} \) (i.e., \( S_{in} \subset S_{pr} \)).

Auxiliary Proposition 3: For any \( \mathbf{G} \in S_{eq} \) there exists \( \{ \mathbf{G}_k \} \subset S_{in} \) such that \( \mathbf{G}_k \overset{\mathcal{H}_2}{\longrightarrow} \mathbf{G} \).

It follows from Auxiliary Proposition 2 that \( S_{in} \subset S_{pr} \subset S_{eq} \) (so that \( S_{pr} \) is non-empty) and, hence,
\[
\hat{f}_{eq} \triangleq \inf \{ f(\mathbf{G}) : \mathbf{G} \in S_{eq} \} \leq \hat{f}_{pr} \triangleq \inf \{ f(\mathbf{G}) : \mathbf{G} \in S_{pr} \} \leq \hat{f}_{in} \triangleq \inf \{ f(\mathbf{G}) : \mathbf{G} \in S_{in} \} . \tag{A.33}
\]

Now, let \( \delta > 0 \) and take \( \mathbf{G}_\delta \in S_{eq} \) such that \( f(\mathbf{G}_\delta) \leq \hat{f}_{eq} + \delta \). In the light of Auxiliary Proposition 3, \( \exists \{ \mathbf{G}_{\delta_k} \} \subset S_{in} \) such that \( \mathbf{G}_{\delta_k} \overset{\mathcal{H}_2}{\longrightarrow} \mathbf{G}_\delta \) and, hence (since \( f \) in \( \mathcal{H}_2 \)-continuous), \( f(\mathbf{G}_{\delta_k}) \to f(\mathbf{G}_\delta) \).
Thus, as ∀k ∈ \mathbb{Z}_+, f_{in} \leq f(G_{\delta k}), f_{in} \leq f(G_{\delta}) = f_{eq} + \delta. As this holds for any δ > 0, it has been established that ∀δ > 0 \ f_{in} \leq f_{eq} + \delta. Therefore, f_{in} \leq f_{eq}.

On the other hand, in the light of (A.33), f_{in} \geq f_{eq} so that f_{in} = f_{pr} = f_{eq}. **Auxiliary Proposition 2** follows directly from equation (L.8) and **Auxiliary Proposition 3** from the fact that \mathcal{S}_G is convex and \mathcal{J}_\chi(,;\mathcal{S}_G) is convex. ■

Recasting **Prob. 5** as (5.5) and **Prob. 6** as (5.7): These are the counterparts of Proposition 5.1 for Probs. 5 and 6 and can be proved using the argument (mutatis mutandis) invoked in its proof. ■

**Auxiliary Proposition 4:** Let \( \hat{X} \in \mathcal{S}_X \) be such that \( \|\hat{X}\|_2 = \gamma \) and \( \delta_\hat{X}(\hat{X}) \leq 0 \). Let \( p(\beta; \hat{X}) \triangleq \delta_\beta(\beta \hat{X}) = \delta_{\beta,0} + 2\beta|\delta_i(\hat{X})| + \beta^2 \delta_q(\hat{X}), S_{\beta,0}(\hat{X}; G) \triangleq \{ \beta \in [-1, 1]: p(\beta; \hat{X}) < 0 \} \) and let \( \mu_i(\hat{X}; G) \) denote the length of \( S_{\beta,0}(\hat{X}; G) \).

Let \( G \) and \( G_M \) be such that \( \delta_{\beta,0} \triangleq \mathcal{J}_\chi(G; 0) - \mathcal{J}_\chi(G_M; 0) < 0 \). Then,

(i) If \( \delta_q(\hat{X}) \leq 0, \ \forall \varepsilon > 0, \ \mu_i(\hat{X}; G) \geq 1 + \min(1, \nu_\varepsilon(\hat{X})), \) where \( \nu_\varepsilon(\hat{X}) = (1/2) \frac{|\delta_{\beta,0}|}{\varepsilon + |\delta_i(\hat{X})|}. \)

(ii) Let \( \hat{X} \) be such that \( \delta_q(\hat{X}) > 0 \) and let \( C(\hat{X}) \triangleq \frac{|\delta_{\beta,0}|}{|\delta_q(\hat{X})|}, \ K(\hat{X}) \triangleq \frac{|\delta_\beta(\hat{X})|}{|\delta_q(\hat{X})|}, \)

\[
\beta_{r_1} \triangleq -K(\hat{X}) - \{ K(\hat{X})^2 + C(\hat{X}) \}^{1/2} \text{ and } \beta_{r_2} \triangleq -K(\hat{X}) + \{ K(\hat{X})^2 + C(\hat{X}) \}^{1/2}.
\]

(iia) If either \( \beta_{r_1} < -1 \) or \( \beta_{r_2} > 1, \ \mu_i(\hat{X}; G) \geq 1 + \min(1, \nu_\beta(\hat{X})), \) where \( \nu_\beta(\hat{X}) \triangleq (1/2) \frac{C(\hat{X})}{\{ K(\hat{X})^2 + C(\hat{X}) \}^{1/2}}. \)

(iib) If \( |\beta_{r_1}, \beta_{r_2}| \in [-1, 1], \ \mu_i(\hat{X}; G) = 2\{ K(\hat{X})^2 + C(\hat{X}) \}^{1/2} \).

\[\nabla\]

**Proof:** (i) If \( \delta_q(\hat{X}) \leq 0 \implies p(\beta, \hat{X}) < 0 \ \forall \beta \in [-1, 0]. \) For \( \beta \in [0, 1], \ p(\beta, \hat{X}) \leq \delta_{\beta,0} + 2\beta|\delta_i(\hat{X})| \leq \delta_{\beta,0} + 2\beta(\varepsilon + |\delta_i(\hat{X})|) \) for any \( \varepsilon > 0 \) so that for any \( \beta \in [0, 1], \ p(\beta, \hat{X}) \leq 0 \) whenever, for any \( \varepsilon > 0, \)

\[
\delta_{\beta,0} + 2\beta(\varepsilon + |\delta_i(\hat{X})|) \iff \beta \leq (1/2) \frac{|\delta_{\beta,0}|}{\varepsilon + |\delta_i(\hat{X})|}.
\]

Thus \( S_{\beta,0}(\hat{X}; G) \supset [-1, 0] \cap \{0, \nu_\varepsilon(\hat{X})\} \) for any \( \varepsilon > 0 \implies \mu_i(\hat{X}; G) \geq 1 + \min(1, \nu_\varepsilon(\hat{X})), \) for any \( \varepsilon > 0 \).

(ii) Noting that for \( \Delta x > 0 (x + \Delta x)^{1/2} = x^{1/2} + \frac{1}{2x^{1/2}} \Delta x \) for some \( \hat{x} \in (x, x + \Delta x) \), there exists \( \hat{x} \in (K(\hat{X})^2, K(\hat{X})^2 + C(\hat{X})) \) such that \( \beta_{r_1} = -2K(\hat{X}) - \Delta \beta(\hat{x}), \beta_{r_2} = \Delta \beta(\hat{x}), \) where \( \Delta \beta(\hat{x}) = \frac{1}{2x^{1/2}} C(\hat{X}). \)

If \( \beta_{r_1} < -1, \ S_{\beta,0}(\hat{X}; G) = [-1, \Delta \beta(\hat{x})] \). If \( \beta_{r_2} > 1, \ as \beta_{r_2} < -\Delta \beta(\hat{x}), \ S_{\beta,0}(\hat{X}; G) \supset [-\Delta \beta(\hat{x}), 1]. \) In both cases,

\[
\mu_i(\hat{X}; G) \geq 1 + \Delta \beta(\hat{x}) \geq 1 + \frac{C(\hat{X})}{2\{ K(\hat{X})^2 + C(\hat{X}) \}^{1/2}}.
\]
Proposition 6.1

\( \forall \) \( \beta_{r_1}, \beta_{r_2} \) \( \Rightarrow \) \( \mu_j(\hat{X}; G) = 2\{K(\hat{X})^2 + C(\hat{X})\}^{1/2} \).

Proof of Proposition 6.1: Proposition 6.1 is a straightforward consequence of Auxiliary Proposition 4. To see why this is so, note first that \( \forall \hat{X} \in S_X, |\delta_i(\hat{X})| < \delta_i, |\delta_i(\hat{X})| < \delta_i, \)

(a) \( \forall \hat{X} \in S_X, \nu_c(\hat{X}) \geq (1/2)|\delta_{ja}| \) \( \Rightarrow \) \( \sup\{\nu_c(\hat{X}) : \varepsilon > 0\} \geq \nu_a. \)

(b) \( C(\hat{X}) \geq \nu_c \) and \( 2\{K(\hat{X})^2 + C(\hat{X})\}^{1/2} \geq 2C(\hat{X})^{1/2} \geq 2\nu_c^{1/2} \).

(c) \( \hat{\nu}_a(\hat{X}) = (1/2) \frac{|\delta_{ja}|}{\{\|\delta_i(\hat{X})\|^2 K(\hat{X})^2 + |\delta_i(\hat{X})||\delta_{ja}|\}^{1/2}} = (1/2) \frac{\delta_{ja}}{|\delta_i(\hat{X})|^2 + |\delta_i(\hat{X})||\delta_{ja}|} \)

\( \Rightarrow \) \( \hat{\nu}_a(\hat{X}) = (1/2) \frac{1}{\{\|\delta_i(\hat{X})\|^2 + \nu_c(\|\delta_i(\hat{X})\|)\}^{1/2}} \)

\( \Rightarrow \) \( \hat{\nu}_a(\hat{X}) \geq (1/2) \nu_c \frac{1}{\{\|\delta_i(\hat{X})\|^2 + \nu_c\}^{1/2}} \geq \nu_a. \)

In the light of (a), the lower bound on \( \mu_j(\hat{X}; G) \) given in (i) leads to

\( \mu_j(\hat{X}) \geq \min(2, 1 + \nu_a) \).

Similarly in the light of (c) and (b) the lower bounds on \( \mu_j(\hat{X}; G) \) given by (ii.a) and (ii.b) give rise, respectively, to

\( \mu_j(\hat{X}; G) \geq \min(2, 1 + \nu_a) \) and \( \mu_j(\hat{X}; G) \geq 2\nu_c^{1/2} \).

Thus, \( \forall \hat{X} \in S_X, \mu_j(\hat{X}; G) \geq \min\{\min(2, 1 + \nu_a), \min(2, 1 + \nu_a), 2\nu_c^{1/2}\} \)

\( \Rightarrow \) \( \forall \hat{X} \in S_X, \mu_j(\hat{X}; G) \geq \min\{2, 1 + \nu_a, 1 + \nu_a, 2\nu_c^{1/2}\} \).

Proof of Proposition 6.2: (a) Note first that

\( \eta_p^\infty(G) = \sup\{\langle \Gamma_{a\gamma} z_a, z_a \rangle : z_a \in S_{z_a}\} \),

where \( \Gamma_{a\gamma} = M_{a\gamma} \Gamma_{a\gamma} M_{a\gamma} \), and

\( S_{z_a} = \{z_a = \begin{bmatrix} y_a^T & v_a^T \end{bmatrix} : y_a \in \mathcal{R}^{m_v}, v_a \in \mathcal{R}^{m_v}, \|y_a\|_2 \leq 1, \|v_a\|_2 \leq 1\} \).

Note also that whenever \( \|z_a\|_2 \leq 1, z_a \in S_{z_a} \) (Since \( 1 \geq \|z_a\|_2 \geq \max\{\|y_a\|_2, \|v_a\|_2\} \)). Thus, \( \eta_p^\infty \geq \sup\{\langle \Gamma_{a\gamma} z_a, z_a \rangle : \|z_a\|_2 \leq 1\} = \lambda_\infty(\Gamma_{a\gamma}) \).
(b) Note first that

\[ \eta^\omega_R(G) \leq \mu \iff \inf \left\{ \langle \Gamma_{e_1} \tilde{z}, \tilde{z} \rangle - \mu \langle \Gamma_{e_0} \tilde{z}, \tilde{z} \rangle : \tilde{z} \in \mathcal{S}_z \right\} \leq 0 \]

\[ \iff \sup \left\{ \langle \Gamma_\mu z_a, z_a \rangle : z_a \in \mathcal{S}_{z_a} \right\} \geq 0, \]

where \( \Gamma_\mu \triangleq M_\gamma (\mu \Gamma_{e_0} - \Gamma_{e_1}) M_\gamma \). But

\[ \sup \left\{ \langle \Gamma_\mu z_a, z_a \rangle : z_a \in \mathcal{S}_{z_a} \right\} \geq \sup \left\{ \langle \Gamma_\mu z_a, z_a \rangle : \|z_a\|_2 \leq 1 \right\} = \bar{\lambda}_\infty(\Gamma_\mu), \]

so that if \( \bar{\lambda}_\infty(\Gamma_\mu) \geq 0 \), \( \eta^\omega_R \leq \mu \).

Hence, \( \eta^\omega_R(G) \leq \inf \{ \mu > 0 : \bar{\lambda}_\infty(\Gamma_\mu) \geq 0 \} \). \[ \blacksquare \]