Exact One–Loop Thermal Free Energies of Solitons

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I show how to compute the exact one–loop thermal correction to the free energy of a soliton. The method uses the effective potential as an auxiliary step to ensure that the soliton is quantized around the appropriate vacuum. The exact result is then computed using scattering theory techniques, and includes all orders in the derivative expansion. It can be efficiently combined with a calculation of the exact quantum correction to yield the full free energy to one loop. I demonstrate this technique with explicit computations in $\phi^4$ models, obtaining the free energy for a kink in 1 + 1 dimensions and a domain wall in 2 + 1 dimensions.

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INTRODUCTION

The effective potential is an essential tool in the analysis of the vacuum states of a quantum field theory. It provides a computational framework into which quantum and thermal corrections can be efficiently incorporated, allowing for analysis of phenomena including spontaneous symmetry breaking and restoration. A key characteristic of these problems is that they can be analyzed in terms of constant background fields. For background fields that are not constant, the effective potential is still useful as an approximate calculation, in which the contribution of the higher derivative terms in the effective action have been ignored. Once the length scale over which the field varies is no longer large compared to the Compton wavelength of the fluctuating fields, however, this approximation is no longer justified. In typical soliton problems, these two length scales are expected to be comparable. In this case, it is therefore desirable to replace the derivative expansion with an exact one–loop computation.

For quantum corrections, such a procedure has been demonstrated in detail in \[5\], which implements ideas originating in \[4\]. In this case, it is essential to precisely implement the counterterms, which cancel the divergences of the quantum fluctuations. To obtain a meaningful finite result, these counterterms must enforce definite renormalization conditions fixed in perturbation theory.

Thermal corrections are finite in the ultraviolet, since the contribution of a mode with energy $\omega$ typically will fall like $e^{-|\omega|/T}$ for large $|\omega|$. Indeed, they must be finite, since there are no additional counterterms available beyond those that have already been fixed at $T = 0$. While the absence of divergences means that the calculation of thermal corrections doesn’t involve the subtleties of renormalization, it introduces a different problem: Thermal corrections can modify the classical vacuum expectation value of the fields. While quantum corrections could in principle do so as well, there is a generally a counterterm available that can cancel this effect. Standard renormalization schemes specify that this counterterm be chosen so that the classical vacuum expectation value of the field is unchanged, which is represented diagrammatically by the vanishing of the tadpole graph. Such counterterms are $T$-independent, however, so they cannot cancel any further finite shifts in the vacuum caused by thermal corrections.

Since analysis of the vacuum only requires constant fields, the thermal effective potential is well-equipped to analyze it. If a soliton exists at $T = 0$, then we expect that for small enough $T$, there should exist a similar soliton, built around the thermally modified vacuum state. The original $T = 0$ soliton configuration will now have infinitely higher free energy, since it approaches a value of the field that is not a minimum of the effective potential.

Once we have constructed the thermally corrected soliton, the thermal effective potential gives an approximation to its free energy. But if we want the exact one–loop thermal free energy, accurate to all orders in the derivative expansion, we must instead sum the full contributions from all the small oscillation modes around the thermal soliton. At the same time, we must not double-count the approximation to this quantity already included by shifting the vacuum. We will see that the phase shift formalism of \[6\] provides an efficient mechanism for organizing this calculation.

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This approach should be applicable for a wide class of field theory solutions, allowing for efficient numerical calculation. Since we do not use any high-temperature expansions, we can continuously track the soliton starting from \( T = 0 \). To illustrate the calculation in a concrete way, we will choose a simple example, where much of the computation can be done analytically. We will consider the kink soliton in 1 + 1 dimensions, where the small oscillations potential is of the exactly solvable reflectionless Pöschl-Teller form. We will also consider the corresponding domain wall in 2 + 1 dimensions. Similar calculations have been considered in this model in [5]. However, that work did not address questions related to the vacuum shift.

**EFFECTIVE POTENTIAL**

We will consider a \( \phi^4 \) model in 1 + 1 dimensions, with Lagrangian density

\[
\mathcal{L} = \frac{m^2}{2\lambda} \left( (\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{4}(\phi^2 - 1)^2 \right) + C(\phi^2 - 1)
\]

where \( \phi \) is a real scalar field and \( m \) is its classical mass. Here \( C \) is a counterterm, which depends on the cutoff but not on \( T \). At \( T = 0 \), the classical potential is

\[
V_c(\phi) = \frac{m^4}{8\lambda}(\phi^2 - 1)^2.
\]

and the model has a classical soliton solution

\[
\phi_{\text{kink}}(x) = \tanh \frac{mx}{2}
\]

which interpolates between the equivalent classical vacua \( \phi = \pm 1 \). Once we allow the temperature to be nonzero, we have to take care to ensure that we continue to consider solitons built around the correct vacua, which we will find using the quantum and thermal effective potentials.

First, we consider the quantum effective potential, which we compute at \( T = 0 \). We use the standard method [1] to compute the effective potential and fix the counterterms. We choose a renormalization scheme in which we hold the location of the minimum of the effective potential fixed. This condition is equivalent to demanding that the tadpole graph vanish, and fixes \( C \) uniquely. Once we have defined the model in this way, the counterterm is fixed and cannot be changed when we consider \( T \neq 0 \). We also add an overall constant, independent of \( \phi \) and \( T \), so that the value of the potential is zero at its minimum. This calculation yields the quantum effective potential

\[
V_q(\phi) = \frac{1}{8\pi} \left[ U(\phi) \left( 1 - \log \frac{U(\phi)}{m^2} \right) - m^2 \right]
\]

where the small oscillations potential is

\[
U(\phi) = \frac{m^2}{2}(3\phi^2 - 1) = \frac{\lambda}{m^2} V_c''(\phi).
\]

Following [2], the thermal effective potential is given by

\[
V_t(\phi) = \frac{T}{\pi} \int_0^\infty \log(1 - e^{-\sqrt{q^2 + U(\phi)/T}}) dq
\]

where \( T \) is the temperature.

The total effective potential to one loop is then given by the sum

\[
V(\phi) = V_c(\phi) + V_q(\phi) + V_t(\phi).
\]

At \( T = 0 \), we have defined the theory so that the minima of this potential are at \( \phi = \pm 1 \). But as \( T \) increases from zero, we cannot prevent the minima from shifting. For \( T \) large enough, the minimum will move to \( \phi = 0 \) and the symmetry will be restored. At that point, the soliton disappears. For \( T \) nonzero but well below this value, however, we will have a modified soliton. Our goal is to compute the difference between its free energy and the free energy of the trivial vacuum.
We can find the new minimum $\phi_0(T)$ simply by minimizing eq. (7), which is easily done numerically. We must now build the soliton around this new vacuum. To accomplish this, we define a modified classical potential

$$\tilde{V}_c(\phi) = V_c(\phi) + A(T)(\phi^2 - \phi_0(T)^2) + B(T)$$

where we have introduced artificial, finite counterterms $A(T)$ and $B(T)$. By choosing

$$A(T) = -\frac{V'_c(\phi_0(T))}{2\phi_0(T)} = -\frac{m^4}{4\lambda}(\phi_0(T)^2 - 1)$$
$$B(T) = -V_c(\phi_0(T)) = -\frac{m^4}{8\lambda}(\phi_0(T)^2 - 1)^2$$

we force $\tilde{V}_c(\phi)$ to have its minimum at $\phi_0(T)$, the same value of $\phi$ as the full effective potential. Since we will only be interested in the difference between the free energy of the soliton configuration and the free energy of the trivial vacuum, we are free to add an overall constant independent of $\phi$, even one that depends on $T$. We have used this freedom to force the value of the potential at its minimum to be zero.

This modified classical potential is therefore identical to the zero-temperature potential, except that the artificial counterterm has shifted the mass of the small oscillations to

$$\tilde{m} = \sqrt{m^2 + \frac{2\lambda}{m^2}A(T)}$$

reflecting the vacuum shift caused by the thermal fluctuations. The classical potential is then

$$\tilde{V}_c(\phi) = \frac{\tilde{m}^4}{8\lambda\phi_0(T)^2}(\phi^2 - \phi_0(T)^2)^2$$

and the kink solution is

$$\tilde{\phi}_{kink}(x) = \phi_0(T) \tanh \frac{\tilde{m}x}{2}.$$  

Since it is built around the correct vacuum, we will be able to use this soliton for calculations of the quantum and thermal corrections. At the end of the calculation, we will use its close relation to the original potential to subtract the artificial counterterms back out again, yielding the result in the original model.

**EXACT QUANTUM CORRECTIONS**

By considering the modified potential $\tilde{V}_c(\phi)$, we can now find the quantum corrections computed around the correct vacuum. The artificial counterterms $A(T)$ and $B(T)$ summarize the effect of the thermal contribution in shifting the vacuum. Later, when we compute the full thermal correction, we will subtract the contributions of the artificial counterterms and replace them by the full thermal corrections. The quantum calculation for $\tilde{V}_c(\phi)$ now proceeds as a direct application of the techniques developed in [3, 6]. We will outline this calculation, in the process introducing quantities that will be useful for the thermal calculation as well.

We consider the small oscillations around the soliton, which are given by the solutions to the Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + \tilde{U}(\tilde{\phi}(x))\right)\psi_k(x) = (k^2 + \tilde{m}^2)\psi_k(x)$$

where

$$\tilde{U}(\tilde{\phi}) = \frac{\lambda}{\tilde{m}^2}\tilde{V}''(\tilde{\phi})$$

is the modified small oscillations potential. For the kink, we have

$$\tilde{U}(\tilde{\phi}_{kink}(x)) = \frac{\tilde{m}^2}{2\phi_0(T)^2}(3\tilde{\phi}_{kink}(x)^2 - \phi_0(T)^2) = \tilde{m}^2 \left( -\frac{3}{2} \text{sech}^2 \frac{\tilde{m}x}{2} + 1 \right).$$
The quantum correction to the energy in the modified theory is given formally by the sum in the shifts in the zero-point energies of the small oscillation modes,

$$\tilde{\mathcal{E}}_q[\phi(x)] \sim \frac{1}{2} \left( \sum_j |\tilde{E}_j| - \sum_j |\tilde{E}_j^0| \right) + \tilde{\mathcal{E}}_{ct}[\phi(x)]$$  

(16)

where the modes have energies $\tilde{E}_j$ in the background $\tilde{\phi}(x)$ and energies $\tilde{E}_j^0$ in the trivial vacuum. $\tilde{\mathcal{E}}_{ct}$ is the contribution of the cutoff-dependent counterterm. To express this quantity more precisely, we will work in the continuum. Then the sum over modes is replaced by a sum over bound states plus an integral over scattering states, weighted by the difference in the density of states between the free and interacting cases. For a general soliton background with spherical symmetry, the spectrum can be decomposed in a partial wave representation into channels $\ell$ with degeneracy $D_\ell$. In our one-dimensional example, these partial waves are just the symmetric and antisymmetric channels. In each partial wave, the difference in the density of states between the free and interacting systems is related to the scattering phase shift by

$$\rho_\ell(k) - \rho_\ell^0(k) = \frac{1}{\pi} \frac{d\delta_\ell(k)}{dk}$$  

(17)

and there are $n_\ell$ bound states, where

$$\delta_\ell(0) = \pi n_\ell$$  

(18)

which is modified to

$$\delta_+(0) = \pi (n_+ - \frac{1}{2})$$  

(19)

in the case of the symmetric channel in one dimension. In this case, there is a “half-bound” threshold state in the free background, $\psi = 1$. As the name indicates, such states enter the sum over bound states suppressed by a factor of $2 \frac{1}{2}$. For the case of a charged particle, we must sum the phase shift for both signs of the energy corresponding to each value of the momentum $k$.

Following [3], we write the quantum correction as

$$\tilde{\mathcal{E}}_q[\phi(x)] = \frac{1}{2} \sum_j (\tilde{\omega}_j - \tilde{m}) + \int_0^\infty \frac{dk}{2\pi} \tilde{\omega}(k) \frac{d}{dk} \delta(k) - \frac{\tilde{m}}{4} + \tilde{\mathcal{E}}_{ct}[\phi(x)]$$  

(20)

where $\tilde{\omega}(k) = \sqrt{k^2 + \tilde{m}^2}$, the $\tilde{\omega}_j$ are the bound state energies, and $\tilde{\delta}(k) = \tilde{\delta}_-(k) + \tilde{\delta}_+(k) = \frac{1}{2\pi} \log \det \tilde{S}(k)$ is the total phase shift in the potential of eq. (13). The $\frac{\tilde{m}}{4}$ term comes from the energy of the “half-bound” state in the free case. Since the phase shift falls like $1/k$ at large $k$, the integral in eq. (20) is logarithmically divergent, and therefore should be considered as a function of a regulator, such as the dimension of spacetime [3]. The counterterm contribution also depends on the same regulator. Using Levinson’s theorem we have

$$\tilde{\mathcal{E}}_q[\phi(x)] = \frac{1}{2} \sum_j (\tilde{\omega}_j - \tilde{m}) + \int_0^\infty \frac{dk}{2\pi} (\tilde{\omega}(k) - \tilde{m}) \frac{d}{dk} \tilde{\delta}(k) + \tilde{\mathcal{E}}_{ct}[\phi(x)]$$  

(21)

and then we implement the counterterm contribution by expressing it in terms of the first Born approximation to the phase shift,

$$\tilde{\delta}_1(k) = -\frac{1}{2k} \int_{-\infty}^\infty \left( \tilde{U}(\phi(x)) - \tilde{m}^2 \right) dx$$  

(22)

giving

$$\tilde{\mathcal{E}}_q[\phi(x)] = \frac{1}{2} \sum_j (\tilde{\omega}_j - \tilde{m}) + \int_0^\infty \frac{dk}{2\pi} (\tilde{\omega}(k) - \tilde{m}) \frac{d}{dk} \left( \tilde{\delta}(k) - \tilde{\delta}_1(k) \right).$$  

(23)

The Born term depends on the potential only through its integral over $x$, as it must since that is also how the counterterm depends on the potential.
In general the phase shifts and bound states can be efficiently computed numerically. Eq. (13), however, is of the exactly solvable Pöschl-Teller reflectionless potential form. We have

$$\tilde{\delta}(k) = 2 \arctan \frac{\tilde{m}}{k} + 2 \arctan \frac{\tilde{m}}{2k} \quad \text{and} \quad \tilde{\delta}_1(k) = \frac{3\tilde{m}}{k}$$

where the bound states are

$$\tilde{\omega}_0 = 0 \quad \text{and} \quad \tilde{\omega}_1 = \frac{\tilde{m}}{2}$$

in addition to a “half-bound” threshold state at $\tilde{\omega} = \tilde{m}$. Plugging these data into eq. (23) gives

$$\tilde{\mathcal{E}}_q[\phi_{\text{kink}}(x)] = \tilde{m} \left( \frac{1}{4\sqrt{3}} - \frac{3}{2\pi} \right)$$

for the quantum correction to the modified potential.

**EXACT THERMAL CORRECTIONS**

The one–loop thermal correction to the free energy in a background $\phi_0$ is given formally by a sum over the thermal occupation number of the small oscillation modes,

$$\mathcal{E}_t[\phi(x)] \sim -T \left( \sum_{E_j} \log \left( \sum_{n=0}^{\infty} e^{-n|E_j|/T} \right) - \sum_{E_j'} \log \left( \sum_{n=0}^{\infty} e^{-n|E_j'|/T} \right) \right)$$

where we have taken the difference between the interacting and free backgrounds. The soliton has a zero mode, corresponding to a translation rather than an excitation. It should be omitted from this sum and handled by collective quantization, representing the motion of the soliton as a point particle. We will compute the classical energy plus the corresponding to a translation rather than an excitation. It should be omitted from this sum and handled by collective where we have used Levinson’s theorem in the second line.

As in the quantum case, we go to the continuum and represent the density of states in terms of the phase shift. Decomposing the scattering into channels $\ell$ with degeneracy $D_\ell$, we have

$$\mathcal{E}_t[\phi(x)] = T \sum_\ell D_\ell \left( \sum_j \log \left( 1 - e^{-|\omega_{j,\ell}|/T} \right) + \int_0^\infty \frac{dk}{\pi} \log \left( 1 - e^{-\omega(k)/T} \right) \frac{d\delta_j(k)}{dk} \right)$$

$$= T \sum_\ell D_\ell \left( \sum_j \log \left( \frac{1 - e^{-|\omega_{j,\ell}|/T}}{1 - e^{-m/T}} \right) - \log \left( 1 - e^{-m/T} \right) + \int_0^\infty \frac{dk}{\pi} \log \left( \frac{1 - e^{-\omega(k)/T}}{1 - e^{-m/T}} \right) \frac{d\delta_j(k)}{dk} \right)$$

where we have used Levinson’s theorem in the second line. The prime indicates that the zero mode has been omitted, which also accounts for the mismatch with Levinson’s theorem resulting in the second term in the second line. In our one–dimensional example, this result becomes

$$\mathcal{E}_t[\phi(x)] = T \left( \sum_j \log \left( \frac{1 - e^{-|\omega_{j}|/T}}{1 - e^{-m/T}} \right) - \log \left( 1 - e^{-m/T} \right) + \int_0^\infty \frac{dk}{\pi} \log \left( \frac{1 - e^{-\omega(k)/T}}{1 - e^{-m/T}} \right) \frac{d\delta_j(k)}{dk} \right)$$

1. For fermions, where the background field is fixed externally, there is no vacuum shift. Thus we can compute the fermion free energy directly as $\mathcal{E}_t'[\phi(x)] = -T \sum_\ell D_\ell \left( \sum_j \log \left( 1 + e^{-|\omega_{j,\ell}|/T} \right) + \int_0^\infty \frac{dk}{\pi} \log \left( 1 + e^{-\omega(k)/T} \right) \frac{d\delta_j(k)}{dk} \right)$. This expression gives the thermal correction to the fermion free energy in a classical background. It is similar to the calculation used in [3] for the thermal correction to the charge.
which is, as expected, a finite integral.

Next we must modify this calculation to be able to consistently combine it with our manipulations of the $T = 0$ classical and quantum contributions. There we introduced an artificial counterterm as a placeholder for the thermal shift in the vacuum. First we introduce these same terms into the thermal problem, so that we write eq. (29) in terms of the modified potential $\tilde{U}(\phi_{\text{kink}}(x))$ built around the true vacuum $\phi_0(T)$. Next we must cancel out all the artificial terms we have added. We will accomplish this by imposing a consistency condition: We know that $\phi_0(T)$ is the local minimum of the full effective potential. By construction, it is a minimum of the modified classical potential, and remained a minimum when the modified quantum corrections were added, because we arranged to cancel the quantum tadpole. Thus if we cancel the modified thermal tadpole with a counterterm of the same form, we can be sure that we have compensated for all the previous manipulations. As with the quantum case, subtracting the contribution from the first Born expansion implements this prescription. It depends only on the background field through the quantity $\tilde{\omega}_j^2$, so it is indeed of the right form to implement this subtraction. Thus if we define

$$\tilde{\mathcal{E}}_t[\phi(x)] = T \left( \sum_j \log \frac{1 - e^{-\tilde{\omega}_j/T}}{1 - e^{-\tilde{m}/T}} - \log \left( 1 - e^{-\tilde{m}/T} \right) + \int_0^\infty \frac{dk}{\pi} \log \left( \frac{1 - e^{-\tilde{\omega}(k)/T}}{1 - e^{-\tilde{m}/T}} \right) \frac{d}{dk} \left( \tilde{\delta}(k) - \tilde{\delta}_1(k) \right) \right)$$

then the full free energy of the true kink system is given by

$$\mathcal{F}[\phi(x)] = \tilde{\mathcal{E}}_c[\phi(x)] + \tilde{\mathcal{E}}_t[\phi(x)] + \tilde{\mathcal{E}}_q[\phi(x)]$$

where we have defined the classical contribution

$$\tilde{\mathcal{E}}_c[\phi(x)] = \int \tilde{V}_c(\phi(x)) dx$$

which gives

$$\tilde{\mathcal{E}}_c[\phi_{\text{kink}}(x)] = \frac{\tilde{m}^3}{3\lambda}$$

for the kink. We have thus made the unique choice such that the free energy is minimized at the true vacuum $\phi(x) = \phi_0(T)$, and the value of the energy at this point is zero.

**DOMAIN WALL**

It is straightforward to extend this formalism to the case of a domain wall. As an example, we will consider the same $\phi^4$ theory in $2 + 1$ dimensions. This model was also considered in \[8, 9\]. We now have the same kink solution, where the field is independent of the second spatial coordinate. Since there are no new divergences, we can continue to use the same renormalization scheme. In this case, the quantum effective potential is

$$V_q(\phi) = \frac{1}{12\pi} \left( -U(\phi) \sqrt{U(\phi)} + \frac{3}{2} mU(\phi) - \frac{1}{2} m^3 \right)$$

and the thermal effective potential is given by

$$V_t(\phi) = \frac{T}{2\pi} \int_0^\infty q \log(1 - e^{-\sqrt{q^2 + U(\phi)/T}}) dq.$$  

The method of \[10\] gives the full quantum correction to the energy per unit transverse length as

$$\frac{\tilde{\mathcal{E}}_q[\phi(x)]}{L} = -\frac{1}{4\pi} \left( \frac{1}{2} \sum_j \omega_j^2 \log \frac{\tilde{\omega}_j^2}{m^2} + \int_0^\infty \frac{dk}{2\pi} \tilde{\omega}(k)^2 \log \frac{\tilde{\omega}(k)^2}{m^2} \frac{d}{dk} \left( \tilde{\delta}(k) - \tilde{\delta}_1(k) \right) \right).$$

where the scattering data $\tilde{\delta}(k)$, $\tilde{\delta}_1(k)$, and $\tilde{\omega}_j$ are unchanged from the case of one space dimension, since the small oscillation wavefunctions can be separated into

$$\psi_{k,p}(x, y) = \psi_k(x) e^{ipy}$$
where $p$ is the momentum in the transverse direction. Plugging in these data yields

$$\frac{\tilde{E}_q[\phi_{\text{kink}}(x)]}{L} = \frac{3\tilde{m}^2}{16\pi} (\text{arccoth}(2) - 2)$$

(38)

for the quantum correction to the domain wall. The analog of eq. (30) is now the thermal free energy per unit transverse length,

$$\frac{\tilde{E}_t[\phi(x)]}{L} = T \int_{-\infty}^\infty \frac{dp}{2\pi} \left[ \sum_j' \log \left( \frac{1 - e^{-\tilde{\omega}_j(p)/T}}{1 - e^{-\tilde{m}(p)/T}} \right) - \log \left( 1 - e^{-\tilde{m}(p)/T} \right) \right]
+ \int_0^\infty \frac{dk}{\pi} \log \left( \frac{1 - e^{-\tilde{\omega}(k,p)/T}}{1 - e^{-\tilde{m}(p)/T}} \right) \frac{d}{dk} \left( \tilde{\delta}(k) - \tilde{\delta}_1(k) \right)$$

(39)

where $m(p) = \sqrt{m^2 + p^2}$, $\omega(k,p) = \sqrt{k^2 + m(p)^2}$ and $\omega_j(p) = \sqrt{\omega_j^2 + p^2}$.

EXAMPLES AND CONCLUSIONS

We illustrate these results with some simple numerical calculations. Figure 1 shows the full effective potential in $1 + 1$ dimensions and $2 + 1$ dimensions, with both plotted as a function of $\phi$ at fixed $T$.

![Figure 1](image)

**FIG. 1:** Left: Real part of the full effective potential $V(\phi)/m^2$ for the $1 + 1$ dimensional model with $T = 0.7m$ and $\lambda = 0.2m^2$, as a function of $\phi$. Right: Real part of the full effective potential $V(\phi)/m^3$ for the $2 + 1$ dimensional model with $T = 2.4m$ and $\lambda = 0.2m$, as a function of $\phi$. In both cases, the zero–temperature mass $m$ sets the scale of units. Away from the minimum, the effective potential can become complex, which gives a cusp in the plot.

Figure 2 shows the full free energy for the soliton and domain wall. In both cases, the plot shows the free energy of the extended object with the free energy of the trivial background subtracted from it, considered as a function of $T$, with the renormalized zero–temperature mass and coupling held fixed. The free energy cost increases as the temperature increases, as we would expect since the soliton and domain wall are low–entropy objects.

This technique should be of general use in soliton problems at non-zero temperature. Since we have not relied on the high-temperature expansion, the calculation allows us to track the soliton continuously as we increase the temperature from zero all the way up to the point where the soliton melts away. In models with conserved charges [11], one could use this approach to compare the free energy of a soliton with a state carrying the same charge built on top of the trivial vacuum. It could also be used to study phase transitions and find percolation temperatures in statistical systems. One might also use the free energy to estimate rates of soliton formation after phase transitions. Finally, we have seen that this approach extends naturally to strings and domain walls using the techniques developed in [10].

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FIG. 2: Left: Full free energy $F[\phi(x)]/m$ for the kink soliton in $1+1$ dimensions as a function of $T/m$, with $\lambda = 0.2m^2$. Right: Full free energy per unit transverse length $F[\phi(x)]/mL$ in $2+1$ dimensions as a function of $T/m$ with $\lambda = 0.2m$. In both cases, the zero–temperature mass $m$ sets the scale of units.

91ER40662.

[1] S. Coleman, Aspects of Symmetry (Cambridge University Press, 1985).
[2] L. Dolan and R. Jackiw, Phys. Rev D9 (1974) 3320; J. Kapusta, Finite-temperature field theory (Cambridge University Press, 1989).
[3] E. Farhi, N. Graham, P. Haagensen and R. L. Jaffe, Phys. Lett. B427 (1998) 334; E. Farhi, N. Graham, R. L. Jaffe, and H. Weigel, Nucl. Phys. B585 (2000) 443; Nucl. Phys. B595 (2001) 536.
[4] J. Schwinger, Phys. Rev. 94 (1954) 1362; J. Baacke, Z. Phys. C55 (1992) 402.
[5] C. A. de Carvalho, [arXiv:hep-ph/0110103](http://arxiv.org/abs/hep-ph/0110103).
[6] N. Graham and R. L. Jaffe, Phys. Lett. B435 (1998) 145; N. Graham and R.L. Jaffe, Nucl. Phys. B544 (1999) 432; N. Graham and R.L. Jaffe, Nucl. Phys. B549 (1999) 516.
[7] G. V. Dunne and K. Rao, Phys. Rev. D64, 025003 (2001); A. J. Niemi and G. W. Semenoff, Phys. Lett. B135 (1984) 121.
[8] C. A. de Carvalho, J. M. Cornwall and A. J. da Silva, Phys. Rev. D64 (2001) 025021.
[9] M. B. Einhorn and D. R. T. Jones, Nucl. Phys. B398 (1993) 611.
[10] N. Graham, R. L. Jaffe, M. Quandt, and H. Weigel, Phys. Rev. Lett. 87 (2001) 131601.
[11] A. Kusenko, Phys. Lett. B404 (1997) 285; M. Laine and M. E. Shaposhnikov, Nucl. Phys. B532 (1998) 376; N. Graham, Phys. Lett. B513 (2001) 112.