Locating and Identifying Codes in Circulant Networks

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Abstract

A set $S$ of vertices of a graph $G$ is a dominating set of $G$ if every vertex $u$ of $G$ is either in $S$ or it has a neighbour in $S$. In other words $S$ is dominating if the sets $S \cap N[u]$ where $u \in V(G)$ and $N[u]$ denotes the closed neighbourhood of $u$ in $G$, are all nonempty. A set $S \subseteq V(G)$ is called a locating code in $G$, if the sets $S \cap N[u]$ where $u \in V(G) \setminus S$ are all nonempty and distinct. A set $S \subseteq V(G)$ is called an identifying code in $G$, if the sets $S \cap N[u]$ where $u \in V(G)$ are all nonempty and distinct. We study locating and identifying codes in the circulant networks $C_n(1,3)$. For an integer $n \geq 7$, the graph $C_n(1,3)$ has vertex set $\mathbb{Z}_n$ and edges $xy$ where $x, y \in \mathbb{Z}_n$ and $|x - y| \in \{1, 3\}$. We prove that a smallest locating code in $C_n(1,3)$ has size $\lceil n/3 \rceil + c$, where $c \in \{0, 1\}$, and a smallest identifying code in $C_n(1,3)$ has size $\lceil 4n/11 \rceil + c'$, where $c' \in \{0, 1\}$.

Keywords: Domination, locating code, locating-dominating set, identifying code, circulant network

1. Introduction

All graphs considered in this paper are simple, without multiple edges or loops. Given a graph $G = (V,E)$, for any vertex $u \in V$, we denote the neighbourhood of $u$ in $G$ by $N_G(u) = \{x \in V : ux \in E\}$. By the closed neighbourhood of $u \in V$, we mean the set $N_G[u] = N_G(u) \cup \{u\}$. When the graph $G$ is clear from the context, we omit the subscripts in this notation. Given a subset $S \subseteq V$, the shadow of a vertex $u \in V$ on $S$ is defined to be the set $S_u = N[u] \cap S$. The set $S$ is a dominating set of $G$ if every $u \in V$ has a nonempty shadow on $S$. The set $S$ is said to be an identifying code, if it is dominating, and distinct vertices $u,v \in V$ have distinct shadows on $S$. The smallest size of an identifying code in a graph $G$ (if one exists) is called the identifying number of $G$ and is denoted by $\gamma^{ID}(G)$. The set $S$ is said to be a locating-dominating set or a locating code, if it is dominating, and distinct vertices $u,v \in V \setminus S$ have distinct shadows on $S$. The smallest size of a locating code in a graph $G$ is called the locating number of $G$ and is denoted by $\gamma^{LOC}(G)$. Locating codes

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were first introduced in [21], motivated by nuclear power plant safety. Vertices of a locating-dominating set \( S \) correspond to safeguards that are able to locate an intruder corresponding to a vertex in \( V - S \). Identifying codes were first introduced in more general form in [16]. Karpovsky et al. study \( r \)-identifying codes in specific topologies of interest in distributed computing for diagnosis of faulty units in multi-processor networks. In the definition of \( r \)-identifying codes and \( r \)-locating-dominating sets the neighbourhood \( N[u] \) is replaced by the set \( N^r[u] = \{ x \in V : d(u, x) \leq r \} \) for a constant \( r \geq 1 \), where \( d(u, x) \) is the graph distance between vertices \( u \) and \( x \). The \( r \)-identifying and \( r \)-locating codes correspond to the identifying and locating codes in the \( r \)th power \( G^r \) of \( G \). Locating and identifying codes have received a great deal of attention from researchers [19, 3, 22, 10, 5, 2, 23]. In particular, locating and identifying codes in special classes of networks have been studied. Examples of such articles include locating codes in trees [15, 12], locating codes in infinite grids [14], locating codes in series-parallel networks [6], locating codes in the infinite triangular grid [13], identifying codes in the infinite hexagonal grid [7], identifying codes in cages [17], identifying codes in binary Hamming spaces [9], identifying and locating codes in geometric networks [18].

Given positive integers \( n \) and \( d_1, \ldots, d_k < n/2 \), we define the circulant graph \( C_n(d_1, \ldots, d_k) \) to have vertex set \( \mathbb{Z}_n = \{ 0, 1, \ldots, n-1 \} \), in which two vertices \( x, y \) are adjacent if and only if \( |x - y| \in \{ d_1, \ldots, d_k \} \). For positive integers \( d_1, \ldots, d_k \), the infinite circulant graph \( C_\infty(d_1, \ldots, d_k) \) is defined on the vertex set \( \mathbb{Z} \) with edges \( xy \) such that \( |x - y| \in \{ d_1, \ldots, d_k \} \). The density of \( S \subseteq \mathbb{Z} \) in \( \mathbb{Z} \) is defined by

\[
\rho(S) = \limsup N \frac{|S \cap [-N, N]|}{2N + 1}.
\]

Identifying and locating codes of the circulant graphs \( C_n(1, 2, \ldots, r) \) are studied in [11, 19, 24, 20, 4, 8] as \( r \)-locating and \( r \)-identifying codes of cycles. The values of \( \gamma_{\text{LOC}}(C_n(1, 2)) \) are established in [4]: for \( n \geq 6 \),

\[
[n/3] \leq \gamma_{\text{LOC}}(C_n(1, 2)) \leq [n/3] + 1.
\]

The values of \( \gamma_{\text{ID}}(C_n(1, 2)) \) are established in [20]: for \( n \geq 8 \),

\[
[n/2] \leq \gamma_{\text{ID}}(C_n(1, 2)) \leq [n/2] + 2.
\]

Motivated by these results, we study locating and identifying codes of the circulant graphs \( C_n(1, 3) \). We prove for \( n \geq 9 \),

\[
[n/3] \leq \gamma_{\text{LOC}}(C_n(1, 3)) \leq [n/3] + 1,
\]

and

\[
[4n/11] \leq \gamma_{\text{ID}}(C_n(1, 3)) \leq [4n/11] + 1.
\]

We also prove that the least density of a locating (resp. identifying) code in \( C_\infty(1, 3) \) is 1/3 (resp. 4/11).
2. General lower bounds

Recall that for a graph $G = (V, E)$ and a dominating set $S \subset V$, by the shadow of a vertex $u \in V$ on $S$ we mean the set $S_u = S \cap N[u]$. The profile of $u \in V$ to be the $d_G(u) + 1$–tuple $\pi(u)$ with entries $|S_x|$ where $x \in N[u]$, in ascending order. The share of a vertex $u \in S$ in $S$ is defined by $\gamma(u; S) = \sum_{x \in N[u]} \frac{1}{|S_x|}$. When the set $S$ is clear from the context, we refer to $\gamma(u; S)$ simply as the share of $u$ and we denote it by $\gamma(u)$. The following lemma, proved by a simple double-counting argument, is a powerful tool in obtaining lower bounds on (various flavors of) domination numbers.

Lemma 2.1. [23] Let $G$ be a graph of order $n$ and let $S$ be a dominating set of $G$. Then $\sum_{u \in S} \gamma(u) = n$.

The above lemma yields the following lower bounds on the size of locating and identifying codes in a general graph.

Proposition 2.2. [21] For a graph $G$ of order $n$ and maximum degree $\Delta$ we have $\gamma^{\text{LOC}}(G) \geq \frac{2n}{\Delta + 3}$.

Proposition 2.3. [16] For a graph $G$ of order $n$ and maximum degree $\Delta$ we have $\gamma^{ID}(G) \geq \frac{2n}{\Delta + 2}$.

3. Locating number of $C_n(1, 3)$

From Proposition 2.2 it follows that if $G$ is a 4–regular graph of order $n$ then $\gamma^{\text{LOC}}(G) \geq 2n/7$. In this section we obtain a better lower bound for the locating number of the circulant graphs $C_n(1, 3)$, and we show that this bound is asymptotically tight. Let $n \geq 13$, and let $S$ be a locating code in the graph $C_n(1, 3)$. A vertex $u \in S$ is said to be heavy if $\gamma(u) \geq 3$.

Lemma 3.1. Let $u \in S$ be a heavy vertex. Then $\pi(u)$ is either $(1, 1, 2, 2, 3)$ or $(1, 1, 2, 3, 4)$. Moreover, we may assign to each heavy vertex $u \in S$, a vertex $u' \in S$, called the mate of $u$, such that $\gamma(u) + \gamma(u') \leq 6$. Moreover, distinct heavy vertices have distinct mates.

Proof. By symmetry, we may assume $u = 0$. Note that if there is at least one $x \in N[0]$ with $|S_x| = 1$, then $\gamma(0) \leq 1 + 4/2 = 3$. Thus there is $x \in N(0)$ such that $S_0 = S_x = \{0\}$. Since $N(0) = \{-3, 1, 1, 3\}$, we may assume without loss of generality that $x \in \{-1, -3\}$.

Case 1: $x = -1$. Since $S_0 = S_{-1} = \{0\}$, we have $[-4, 3] \cap S = \{0\}$. Thus we must have $-6 \in S$ since otherwise, $S_{-3} = \{0\}$, which contradicts the locating
property of \( S \). Similarly, we must have \( 4 \in S \) since otherwise, \( S_1 = \{0\} \). We must also have \( 6 \in S \) since otherwise, \( S_1 = S_3 = \{0, 4\} \). We now have \( |S_0| = |S_{-1}| = 1, |S_{-3}| = |S_1| = 2, \) and \( |S_3| = 3 \), giving \( \pi(0) = (1, 1, 2, 2, 3) \). Moreover, we must have \( 5 \in S \) since otherwise, \( S_2 = \emptyset \).

Case 2: \( x = -3 \). Since \( S_0 = S_{-3} = \{0\} \), we have \( \{-6, -4, -3, -2, -1, 1, 3\} \cap S = \emptyset \). Thus we must have \( 2 \in S \) since otherwise, \( S_{-1} = \{0\} \). Since 2 is a common neighbour of \(-1, 1, \) and 3, we must have \( 4 \in S \), and since 4 is a common neighbour of 1 and 3, we must have \( 6 \in S \). We now have \( |S_0| = |S_{-1}| = 1, |S_{-1}| = 2, |S_1| = 3, \) and \( |S_3| = 4 \), giving \( \pi(0) = (1, 1, 2, 3, 4) \).

In case 1, we assign 4 as the mate of 0. We have \( |S_1|, |S_4|, |S_7| \geq 2 \) and \( |S_3|, |S_5| \geq 3 \), thus \( \gamma(4) \leq 13/6 \). We see that \( \gamma(0) + \gamma(4) \leq 11/2 < 6 \). In case 2, we assign 2 as the mate of 0. We have \( |S_2| \geq 1, |S_{-1}| = 2, |S_1|, |S_5| \geq 3, \) and \( |S_3| = 4 \), thus \( \gamma(2) \leq 29/12 \). We see that \( \gamma(0) + \gamma(2) \leq 11/2 < 6 \).

Note that in case 1, the vertices 1, 2, 3 between 0 and its mate 4 are not in \( S \), and in case 2, the vertex 1 between 0 and its mate 2 is not in \( S \). Since in case 1, \( 5 \in S \) and in case 2, \( 4 \in S \), we see that in either case, \( u' \) cannot also be the mate of \( u' + 4 \). Also since \( 5 \in S \) in case 1, we see that in this case, \( u' \) cannot also be the mate of \( u' + 2 \). It remains to show that in case 2, \( u' = 2 \) is not also the mate of 4. This is true since all neighbours of 4 have a shadow of size at least 2, hence 4 is not a heavy vertex.

\[ \square \]

**Theorem 3.2.** For every \( n \geq 13 \), \( \gamma_{\text{LOC}}(C_n(1, 3)) \geq n/3 \).

**Proof.** Let \( S \) be a locating code in \( C_n(1, 3) \). Lemma 3.1 gives a unique mate \( u' \) for every heavy vertex \( u \), such that \( \gamma(u) + \gamma(u') \leq 6 \). On the other hand, for every other vertex \( v \in S \) we have \( \gamma(v) \leq 3 \). Thus the total share of vertices of \( S \) is at most \( 3|S| \). The result now follows from Lemma 2.1 \( \square \)

Note that the proof of Lemma 3.1 works also for the graph \( C_\infty(1, 3) \). On the other hand, the neighbours of each vertex \( u \in \mathbb{Z} \) are within short numeric distances of \( u \) (at most 3). These allow us to prove a lower bound of 1/3 on the density of any locating set in \( C_\infty(1, 3) \).

**Theorem 3.3.** Every locating code in \( C_\infty(1, 3) \) has density at least 1/3.

**Proof.** Let \( S \) be a locating set in \( C_\infty(1, 3) \). Note that the mate of each heavy vertex found in Lemma 3.1 is within numeric distance at most 4 of that vertex. Thus for any positive integer \( N \), the set \( S' = S \cap [-N, N] \) contains at most two heavy vertices (one at each end) whose mate is not present in \( S' \). Since by Lemma 3.1, the share in \( S \) of a heavy vertex is at most \( 3 + 1/3 \), we obtain

\[
\sum_{u \in S'} \gamma(u) \leq 3|S'| + 2/3.
\]

On the other hand,

\[
\sum_{u \in S'} \gamma(u) = \sum_{u \in S'} \sum_{x \in \mathbb{Z}} \frac{1}{|S_x|} \geq \sum_{x \in [-N, N]} \sum_{u \in S_x} \frac{1}{|S_x|} = 2N + 1.
\]
The inequality appears since not all neighbours of every \( u \in S' \) are necessarily in the range \([-N, N]\). These inequalities give \( 2N + 1 \leq 3|S'| + 2/3 \), or

\[
\frac{|S \cap [-N, N]|}{2N + 1} \geq \frac{1}{3} - \frac{2}{9(2N + 1)}.
\]

We conclude that \( \rho(S) \geq 1/3 \).

In the remainder of this section, we provide constructions of locating codes in circulant graphs \( C_n(1, 3) \). From Theorem 3.2, we know that such codes have size at least \( \lceil n/3 \rceil \). We give general constructions for \( n \geq 13 \). These codes have size \( \lceil n/3 \rceil \), unless when \( n \equiv 2 \mod 3 \), where the constructed code has size \( \lceil n/3 \rceil + 1 \). We do not know whether this is best possible, but using a brute-force computer search, we verified that for \( 14 \leq n \leq 38 \), a locating code of size \( \lceil n/3 \rceil \) does not exist in this case. For \( n < 13 \), we verified using this program that

\[
\gamma_{LOC}(C_7(1, 3)) = 3, \quad \gamma_{LOC}(C_8(1, 3)) = 6, \quad \gamma_{LOC}(C_9(1, 3)) = \gamma_{LOC}(C_{10}(1, 3)) = \gamma_{LOC}(C_{11}(1, 3)) = 4, \quad \text{and} \quad \gamma_{LOC}(C_{12}(1, 3)) = 5.
\]

For a positive integer \( t \), let

\[
A_t = \{6i + j : 0 \leq i \leq t - 1 \text{ and } j \in \{0, 1\}\}.
\]

It is easy to see that for \( t \geq 3 \), the set \( A_t \) is a locating code in \( C_{6t}(1, 3) \). The sets \( A_t \) can indeed be used in constructions of locating codes for the graphs \( C_n(1, 3) \) when \( n \) is not necessarily a multiple of 6. Such constructions are presented in Table 1.

| \( n \) | A locating code for \( C_n(1, 3) \) |
|---|---|
| \( 6t + 1 \) | \( A_t \cup \{6t - 2\} \) |
| \( 6t + 2 \) | \( A_{t+1} \) |
| \( 6t + 3 \) | \( A_{t+1} \) |
| \( 6t + 4 \) | \( A_{t+1} \) |
| \( 6t + 5 \) | \( A_{t+1} \cup \{6t - 2\} \) |
| \( 6t + 6 \) | \( A_{t+1} \) |

Table 1: Constructions of locating codes for the circulant graphs \( C_n(1, 3) \). Here \( t \geq 2 \) is an integer.

We omit the proofs here. The proofs are straightforward, and all take advantage of the “local” structure of the graphs \( C_n(1, 3) \), namely the fact that each neighbourhood is contained in an interval of length 6. We present an example of these codes in Figure 1. These results are summarized in the next theorem.

**Theorem 3.4.** Let \( n \geq 9 \). Then \( \gamma_{LOC}(C_n(1, 3)) = \lceil n/3 \rceil \) if \( n \not\equiv 2 \mod 3 \), and \( \lceil n/3 \rceil \leq \gamma_{LOC}(C_n(1, 3)) \leq \lceil n/3 \rceil + 1 \) if \( n \equiv 2 \mod 3 \).

For a locating code in \( C_\infty(1, 3) \) with density \( 1/3 \), one may take the code

\[
A_\infty = \{6i + j : i \in \mathbb{Z} \text{ and } j \in \{0, 1\}\}.
\]
4. Identifying number of $C_n(1, 3)$

In this section we obtain a lower bound for the identifying number of the circulant graphs $C_n(1, 3)$, and we show that this bound is asymptotically tight. We assume that $n \geq 13$ is an integer, and $S$ is an identifying code in the circulant graph $G = C_n(1, 3)$. A vertex $u \in S$ is said to be a heavy vertex, if $\gamma(u) > 11/4$. The subgraph of $G$ induced by $S$ is denoted by $\Gamma$. The connected component of $\Gamma$ containing a vertex $u \in S$ is denoted by $\Gamma_u$. By a heavy component of $\Gamma$, we mean a connected component whose vertices have average share larger than $11/4$.

**Lemma 4.1.** Let $u \in S$ be a heavy vertex. Then $\pi_u = (1, 2, 2, 2, 3)$.

**Proof.** If $\pi(u)$ contains at least two numbers greater than 2, then

$$\gamma(v) \leq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} < \frac{11}{4}. $$

This contradicts the choice of $u$ as a heavy vertex. Then $\pi(u) = (1, 2, 2, 2, a)$ for some integer $a \geq 2$. Moreover, if $a \geq 4$, then $\gamma(u) \leq 11/4$. Hence $\pi(u)$ is either $(1, 2, 2, 2, 2)$ or $(1, 2, 2, 2, 3)$.

Suppose that $\pi(u) = (1, 2, 2, 2, 2)$. We may assume $u = 0$. By the assumption on $\pi(0)$, the shadow $S_0$ has size 1 or 2. If $|S_0| = 1$, then $N[0] \cap S = \{0\}$, and $|S_x| = 2$ for all $x \in N(v)$. In particular, $S_1 = \{0, y\}$ where $y \in \{-2, 2, 4\}$. If $y = \pm 2$, then $S_{-1} = S_1$, and if $y = 4$, then $S_1 = S_3$. These both contradict the identifying property of $S$. If $|S_0| = 2$, let $S_0 = \{0, x\}$. Then $\{0, x\} \subseteq S_x$, thus $|S_x| \geq 2$. Since $|S_x| \leq 2$, this gives $S_x = S_0$, a contradiction. \qed
Note that if \( u \in S \), then \( d_\Gamma(u) = |S_u| - 1 \). On the other hand, if \( u \) is a heavy vertex, it has profile \((1, 2, 2, 2, 3)\) by the above lemma, so \(|S_u| \leq 3\). We conclude that \( d_\Gamma(u) \in \{0, 1, 2\}\). We first prove that \( d_\Gamma(u) \neq 0 \) when \( u \) is a heavy vertex.

**Lemma 4.2.** Let \( u \in S \) be a heavy vertex. Then \( u \) is not an isolated vertex in \( \Gamma \).

**Proof.** We may assume \( u = 0 \). Suppose \( d_\Gamma(0) = 0 \), namely, \([-3, -1, 1, 3] \cap S = \emptyset \). Since \( \pi(0) = (1, 2, 2, 2, 3) \), we may assume that \(|S_1| = |S_3| = 2\). Now \( 2 \not\in S \) and \( 4 \not\in S \), since otherwise, we have \( S_1 = S_3 \), a contradiction. Hence \(-2, 6 \in S \). This is a contradiction since \([-2, 0]\) is now contained in \( S_{-3}, S_{-1}, \) and \( S_1 \), while these sets are distinct and two of them have size 2.

**Lemma 4.3.** If \( u \in S \) is a heavy vertex with \( d_\Gamma(u) = 2 \), then \( \Gamma_u \) is isomorphic to \( P_2 \), the path graph of length 2.

**Proof.** Let \( N_\Gamma(u) = \{v, w\} \). Then \( d_\Gamma(v) \geq 1 \) and \( d_\Gamma(w) \geq 1 \). Since \(|S_u| = 3 \) and \( u \) is heavy, we must have \( d_\Gamma(v) = 1 \) and \( d_\Gamma(w) = 1 \). Thus none of the vertices in the set \( \{u, v, w\} \) has a neighbour outside this set in \( \Gamma \).

**Lemma 4.4.** Every heavy component of \( \Gamma \) is isomorphic to \( P_2 \).

**Proof.** Suppose that \( \Gamma \) has a heavy component with at least 4 vertices. Then this component has at least one heavy vertex \( u \). By Lemma 4.3, we have \( d_\Gamma(u) = 1 \). Let \( N_\Gamma(u) = \{v\} \). The vertex \( v \) is called the mate of \( u \). Since \( u \) is heavy and \( \Gamma_u \) has order at least 4, we have \( d_\Gamma(v) = 2 \). Let \( N_\Gamma(v) = \{w, u\} \). Since \( \Gamma \) has order at least 4, we have \( d_\Gamma(w) \geq 2 \), thus \(|S_w| \geq 3 \). This shows that \( \gamma(v) \leq \frac{1}{1} + \frac{2}{2} + \frac{3}{3} = \frac{8}{3} \), which in turn gives \( \gamma(u) + \gamma(v) \leq 11/2 \). On the other hand, since \(|S_u| = 3 \) and \(|S_w| \geq 3 \), \( w \) is not heavy. Therefore, \( v \) is not the mate of two heavy vertices. Averaging \( \gamma(x) \) over all \( x \in V(\Gamma_u) \), we see that each heavy vertex and its mate contribute \( 11/2 \) together. Since every other vertex has share less than \( 11/4 \), the average share of the vertices of \( \Gamma_u \) is at most \( 11/4 \). This contradicts the assumption that \( \Gamma_u \) is heavy.

**Lemma 4.5.** Every heavy component of \( \Gamma \) is isomorphic to a path of length 2, all whose vertices are heavy. Moreover, the vertices of this component are of the form \( \{u - 1, u, u + 3\} \) or \( \{u - 3, u, u + 1\} \) for some \( u \in \mathbb{Z}_n \).

**Proof.** Consider a heavy component of \( \Gamma \) and let \( W \) denote its vertex set. Without loss of generality, we may assume that \( 0 \in W \), thus we may refer to this component as \( \Gamma_0 \). By Lemma 4.4, we know that \(|W| = 3 \). By symmetries of \( C_\infty(1, 3) \), we may assume that \( W \) is one of the sets \( \{-3, 0, 3\}, \{-1, 0, 1\}, \{-1, 0, 2\}, \) or \( \{-1, 0, 3\} \). In what follows, we show that the first three of these choices yield to a contradiction.

If \( W = \{-3, 0, 3\} \), then \( \{\pm 1, \pm 2, \pm 4, \pm 6\} \cap S = \emptyset \). This gives \( S_{-1} = S_1 = \{0\} \), a contradiction.

If \( W = \{-1, 0, 1\} \), then \( \{\pm 2, \pm 3, \pm 4\} \cap S = \emptyset \). If none of the vertices \(-5, 5\) is in \( S \), we have \( S_{-2} = S_2 = \{-1, 1\} \), a contradiction. Therefore, at least one of \(-5 \) and \( 5 \), say \( 5 \), is in \( S \). This gives \( |S_{-2}|, |S_4| \geq 2, |S_0| = |S_2| = 3, \) and \( |S_1| = 2 \).
hence $\gamma(1) \leq 13/6$. We now obtain \( \frac{1}{3} (\gamma(-1) + \gamma(0) + \gamma(1)) \leq 47/18 < 11/4 \). This contradicts the choice of $\Gamma_0$.

If $W = \{-1, 0, 2\}$, then $6 \in S$ since otherwise, $S_1 = S_3 = \{0, 2\}$. This gives $\gamma(2) \leq 13/6$, which yields a contradiction similarly to the previous case.

Therefore, if $\Gamma_0$ is heavy, then $W = \{-1, 0, 3\}$, up to symmetries of $C_n(1, 3)$. It remains to prove that all vertices in $W$ are heavy. With $W = \{-1, 0, 3\}$ we obtain $\{-4, -3, -2, 1, 2, 4, 6\} \cap S = \emptyset$. Moreover, $-5 \notin S$ since otherwise, $|S_{-4}|, |S_{-2}|, |S_{-1}|, |S_0| \leq 2$ and $|S_5| = 3$ which give $\gamma(-1) \leq 7/3$. This is a contradiction with the choice of $\Gamma_0$. Similarly, $5 \notin S$ and $7 \notin S$ since otherwise, $\gamma(3) \leq 7/3$. On the other hand, $-6 \in S$ since otherwise, $S_{-3} = S_1 = \{0\}$, $-7 \in S$ since otherwise, $S_{-4} = S_{-2} = \{-1\}$, and $9 \in S$ since otherwise, $S_4 = S_6 = \{3\}$. We obtain $\pi(-1) = \pi(0) = \pi(3) = (1, 2, 2, 2, 3)$. \qed

**Theorem 4.6.** For every $n \geq 13$ we have $\gamma^{iv}(C_n(1, 3)) \geq 4n/11$.

**Proof.** Let $S$ be an identifying code in $C_n(1, 3)$, where $n \geq 13$. We assign to each heavy component of $\Gamma$, a unique subset of $S$ referred to as the mate of that component, such that the average share of vertices in a heavy component and its mate together is at most $11/4$.

Let $\Gamma_u$ be a heavy component of $\Gamma$. By the proof of Lemma 4.5, and by symmetry, we may assume that $u = 0$ and $V(\Gamma_0) = W = \{-1, 0, 3\}$. Then $S \cap \{-7, 10\} = \{-7, -6, -5, -1, 0, 3, 8, 9, 10\}$. This is by the proof of Lemma 4.5 and that if $8 \notin S$, then $S_7 = \emptyset$. Also if $10 \notin S$, then $S_5 = S_7 = \{8\}$.

If $\{11, 12, 13\} \cap S \neq \emptyset$, the mate of $\Gamma_0$ is defined to be the set $W' = \{8, 9\}$. If $11 \in S$, then $\gamma(8) \leq 5/2$ and $\gamma(9) \leq 2$. If $12 \in S$, then $\gamma(8) \leq 31/12$ and $\gamma(9) \leq 9/4$. If $13 \in S$, then $\gamma(8) \leq 17/6$ and $\gamma(9) \leq 13/6$. In either of these cases we have

$$\frac{1}{5} (\gamma(-1) + \gamma(0) + \gamma(3) + \gamma(8) + \gamma(9)) \leq \frac{11}{4}.$$  

If $\{11, 12, 13\} \cap S = \emptyset$, we see that $\Gamma_0 \cong P_2$ and by the proof of Lemma 4.5 we have $\frac{1}{2} (\gamma(8) + \gamma(9) + \gamma(10)) \leq \frac{47}{18}$. In this case we assign $W'' = \{8, 9, 10\}$ as the mate of $\Gamma_0$, and we have

$$\frac{1}{6} (\gamma(-1) + \gamma(0) + \gamma(3) + \gamma(8) + \gamma(9) + \gamma(10)) \leq \frac{1}{2} \left( \frac{17}{6} + \frac{47}{18} \right) < \frac{11}{4}.$$  

The mates defined above do not contain any mates assigned in the proof of Lemma 4.4 since those are adjacent to a heavy vertex. On the other hand, since there are four vertices $4, 5, 6, 7$ not in $S$, between $W$ and each of $W'$ and $W''$ we see that $W''$ does not overlap with any mate assigned to other heavy components. Moreover, if $14 \notin S$, then $S_7 = S_{11} = \{8, 10\}$, a contradiction. Thus $W''$ does not overlap any other mates (the four vertex gap is not present after $W''$).

We conclude that the average share of the vertices of $S$ is at most $11/4$, which by Lemma 2.1 gives $|S| \geq 4n/11$. \qed

Similarly to the proof of Theorem 3.3 we may prove the following theorem.
Theorem 4.7. Every identifying code in $C_\infty(1, 3)$ has density at least $4/11$.

In the remainder of this section, we provide constructions of identifying codes in circulant graphs $C_n(1, 3)$. From Theorem 4.6 we know that such codes have size at least $\lceil 4n/11 \rceil$. We give general constructions for $n \geq 11$. These codes have size $\lceil 4n/11 \rceil$, unless when $n \equiv 8 \mod 11$, where the constructed code has size $\lceil 4n/11 \rceil + 1$. We do not know whether this is best possible, but using a brute-force computer search, we verified that for $n = 19, 30, 41$, an identifying code of size $\lceil 4n/11 \rceil$ does not exist. For $n < 13$, we verified using this program that $\gamma^\text{id}(C_7(1, 3)) = \gamma^\text{id}(C_9(1, 3)) = \gamma^\text{id}(C_{10}(1, 3)) = 4$, and $\gamma^\text{id}(C_n(1, 3)) = 6$.

For a nonnegative integer $t$, let

$$B_t = \{11i + j : 0 \leq i \leq t - 1 \text{ and } j \in \{0, 4, 5, 6\}\}.$$ 

In particular, $B_0 = \emptyset$. It is easy to see that $B_t$ is indeed an identifying code in $C_{11t}(1, 3)$. The sets $B_t$ can indeed be used in constructions of identifying codes for the graphs $C_n(1, 3)$ when $n$ is not necessarily a multiple of 11. Such constructions are presented in Table 2.

| $n$      | An identifying code for $C_n(1, 3)$ |
|----------|------------------------------------|
| $11t$    | $B_t$                              |
| $11t + 1$| $B_t \cup \{11t - 4\}$            |
| $11t + 2$| $B_{t-1} \cup \{11t - 11, 11t - 10, 11t - 5, 11t - 4, 11t - 1\}$ |
| $11t + 3$| $B_t \cup \{11t, 11t + 1\}$      |
| $11t + 4$| $B_t \cup \{11t, 11t + 1\}$      |
| $11t + 5$| $B_{t-1} \cup \{11t - 11, 11t - 10, 11t - 5, 11t - 4, 11t + 1, 11t + 2\}$ |
| $11t + 6$| $B_t \cup \{11t, 11t + 1, 11t + 4\}$ |
| $11t + 7$| $B_t \cup \{11t, 11t + 1, 11t + 4\}$ |
| $11t + 8$| $B_t \cup \{11t, 11t + 1, 11t + 6, 11t + 7\}$ |
| $11t + 9$| $B_t \cup \{11t, 11t + 1, 11t + 6, 11t + 7\}$ |
| $11t + 10$| $B_t \cup \{11t, 11t + 1, 11t + 6, 11t + 7\}$ |

Table 2: Constructions of identifying codes for the circulant graphs $C_n(1, 3)$. Here $t$ is a positive integer.

We omit the proofs here. The proof are straight-forward, and all take advantage of the “local” structure of the graphs $C_n(1, 3)$, namely the fact that each neighbourhood is contained in an interval of length 6. We present an example of these codes in Figure 2. These results are summarized in the next theorem.

Theorem 4.8. Let $n \geq 9$. Then $\gamma^\text{id}(C_n(1, 3)) = \lceil 4n/11 \rceil$ if $n \not\equiv 8 \mod 11$, and $\lceil 4n/11 \rceil \leq \gamma^\text{id}(C_n(1, 3)) \leq \lceil 4n/11 \rceil + 1$ if $n \equiv 8 \mod 11$.

For an identifying code in $C_\infty(1, 3)$ with density $4/11$, one may take the code

$$B_\infty = \{11i + j : i \in \mathbb{Z} \text{ and } j \in \{0, 4, 5, 6\}\}.$$
Figure 2: A minimum identifying code of $C_{14}(1,3)$. Vertices in the code are in black. The number next to a vertex is its label. The set next to each vertex is its shadow on this code.

5. Concluding remarks

Determining locating and identifying numbers of general circulant graphs remain open. In particular the circulant graphs $C_n(1,d)$ with $d \geq 4$ are of interest. For larger values of $d$, proofs similar to those presented in this paper get too complicated, so a new approach seems necessary. We close this article by two problems involving the only graphs $C_n(1,3)$ whose exact locating/identifying number is not settled here.

Problem 5.1. Show that if $n \geq 13$ and $n \equiv 2 \mod 6$, the circulant graph $C_n(1,3)$ does not admit a locating code of size $\lceil n/3 \rceil$.

Problem 5.2. Show that if $n \equiv 8 \mod 11$, the circulant graph $C_n(1,3)$ does not admit an identifying code of size $\lceil 4n/11 \rceil$.

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