Waveguide bandgap N-qubit array with a tunable transparency resonance

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(Dated: September 22, 2020)

We study a single photon transmission through 1D N-qubit chain. The qubits are supposed to be identical with equal distance between neighbors. We express the transfer matrix of N-qubit chain in terms of Chebyshev polynomials, which allows us to obtain simple expressions for the transmission and reflection amplitudes for arbitrarily large N. If the distance between neighbor qubits is equal to half wavelength, the transmission spectrum exhibits a flat bandgap structure with very steep walls. We show that for odd N the tuning of the excitation frequency of a central qubit gives rise to the appearance within a bandgap of a narrow resonance with a full transmission. The position of the resonance and its width can be controlled by the frequency of a central qubit. We show that the formation of the bandgap and of the transmission resonance is conditioned by the overlapping of the widths of individual qubits which results from the strong coupling between qubits and waveguide photons.

I. INTRODUCTION

One-dimensional (1D) waveguide-quantum electrodynamics (QED) systems are promising candidates for quantum information processing [1]. One important implementation is waveguide QED, where a quantum multi-qubit system coupled to a common waveguide interacts coherently with the continuum of modes of a waveguide instead of a cavity [2, 3].

The multiple qubit system obtains an infinite range photon mediated effective interaction which can be tuned with the inter-qubit distance. Furthermore, this system exhibits collective excitations with lifetimes ranging from extremely sub- to superradiant relative to the radiative lifetime of the individual qubits [4, 5, 6, 7].

Among the qubit family, the chain which consists of N identical equally spaced qubits is the most simple one. There are many papers where the photon transmission through this structure has been analytically treated, however, here we refer only to the papers [8, 9] where the analytical treatment is similar to the one used in this paper.

In the present paper we propose a matrix formalism for the calculation of a transmission amplitude in 1D open waveguide filled with a number of equally spaced identical qubits. The formalism is based on the decomposition of the transfer matrix for N-qubit chain in terms of Chebyshev polynomials. This method has originally been developed for the description of light scattering in periodic stratified media [10, 11].

In Section II we express transfer matrix for equally spaced N identical qubits in terms of Chebyshev polynomials, which allows a concise expression for transmission and reflection amplitudes. We show that if the distance L between neighbor qubits is equal to a half wavelength at the qubit frequency Ω, the transmission spectrum exhibits a bandgap with steep walls. The width of the bandgap weakly depends on the qubit number N, whereas the suppression of transmitted signal at the bandgap boundaries scales as $N^{-2}$. The existence of this bandgap is not surprising because it is a common feature of many periodic structures [12–18]. In principle, bandgaps appear at any nonzero value of the qubit phase $k_0 L = \Omega L/v_g$, where $v_g$ is the group velocity [10]. However, the value $k_0 L = \pi$ has a unique property, which is studied in Section III.

In Section III we consider the N-odd qubit chain where N − 1 qubits have the same excitation frequency $\Omega$, whereas the excitation frequency $\Omega_0$ of a central qubit is different. In general, if we add new energy $\Omega_0$ to the system it can give rise to new resonance, as was experimentally shown in [12]. However, we show that for our structure there exists inside the gap a frequency where the reflection amplitude equals exactly zero. Therefore, within the bandgap a narrow resonance line appears which provides a full transmission. The position of this resonance and its width depend on the ratio $\Omega_0/\Omega$. Therefore, in our case the resonance line which gives rise to a full transmission, moves within the bandgap with its position and the width being controlled by the excitation frequency of the central qubit.

We show that both the bandgap and the narrow resonance result from strong photon-qubit coupling when the width of individual qubits become overlapped. From the quantum point of view they represent superradiant and subradiant states, respectively.

At the end of this section we consider how the radiation losses to the environment influence the photon transmission trough this structure.

The basic properties of Chebyshev polynomials, which are used in the main text are given in the Appendix.

II. TRANSFER MATRIX FOR N IDENTICAL QUBITS

Our approach is based on the transfer matrix for a single photon scattering on two-level atom or qubit which

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is written as follows [19]:

$$ T_1 = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 1 + i\alpha & i\alpha \\ -i\alpha & 1 - i\alpha \end{pmatrix} = \begin{pmatrix} e^{-i\theta}(1 + i\alpha) & e^{-i\theta}i\alpha \\ -e^{i\theta}i\alpha & e^{i\theta}(1 - i\alpha) \end{pmatrix} \quad (1) $$

where $\alpha = \Gamma/(\omega - \Omega)$, $\Omega$ is the qubit excitation frequency, $\Gamma$ is the rate of spontaneous emission of excited qubit to a waveguide. It is proportional to the photon-qubit interaction.

We assume that $\Omega$ is much larger than the cutoff frequency of the waveguide, then the dispersion relation of photons at near resonant frequency, $\omega \approx \Omega$, can be taken as linear: $\omega = \nu_g k$, where $\nu_g$ is the group velocity.

The definition (1) is slightly different from that in [19] by the inclusion of the phase matrix for the first qubit:

$$ \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \quad (2) $$

where $\theta = kL = \omega L/\nu_g$, $L$ is the distance between neighbor qubits in multi qubit system. The inclusion of the phase matrix [22] for the first qubit in the array is made solely for convenience of calculations. It does not influence the absolute values of reflection or transmission amplitudes.

We define the transfer matrix, $T^{(n)}$ for $n$-th qubit in the chain in such a way that it relates incoming, $a_{n-1}$ and outgoing, $b_{n-1}$ wave amplitudes before the qubit to outgoing, $a_n$ and incoming, $b_n$ wave amplitudes behind the qubit.

$$ \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} \rightarrow \begin{pmatrix} a_n \\ b_n \end{pmatrix} $$

$$ \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} T_{11}^{(n)} & T_{12}^{(n)} \\ T_{21}^{(n)} & T_{22}^{(n)} \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \quad (3) $$

where $a_0 = 1$, $b_0 = r$ is the reflection amplitude.

Transfer matrix for $N$ identical qubits is calculated as the $N$-th power of the matrix $T_1$:

$$ T_N = (T_1)^N \quad (4) $$

so that

$$ \begin{pmatrix} 1 \\ r \end{pmatrix} = \begin{pmatrix} (T_N)_{11} & (T_N)_{12} \\ (T_N)_{21} & (T_N)_{22} \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} \quad (5) $$

with $r$ and $t$ being the reflection and transmission amplitudes for $N$-qubit chain, respectively. Below, we use the engineering notations for these quantities: $r \equiv S_{11}$, $t \equiv S_{21}$.

From (3) the transmission and reflection amplitudes are as follows:

$$ S_{21} = \frac{1}{(T_N)_{11}} \quad (6) $$

$$ S_{11} = \frac{(T_N)_{21}}{(T_N)_{11}} \quad (7) $$

Next, we use the Abeles theorem [11] (see Eq. A4 in appendix), which allows us to write matrix $T_N$ in terms of Chebyshev polynomials of second kind:

$$ T_N = U_{N-1}(y)T_1 - U_{N-2}(y)I \quad (8) $$

where

$$ y = \frac{1}{2}Sp(T_1) = \cos \theta + \alpha \sin \theta \quad (9) $$

From (8) and (11) we obtain

$$ (T_N)_{11} = U_{N-1}(y)(T_1)_{11} - U_{N-2}(y) \quad (10) $$

$$ = U_{N-1}(y)e^{-i\theta}(1 + i\alpha) - U_{N-2}(y) $$

$$ |(T_N)_{11}|^2 = U_{N-1}^2(y)(1 + \alpha^2) + U_{N-2}^2(y) $$

$$ -2yU_{N-1}(y)U_{N-2}(y) = 1 + \alpha^2U_{N-1}^2(y) $$

$$ (T_N)_{21} = -i\alpha e^{i\theta}U_{N-1}(y) \quad (12) $$

On deriving the expression (11) we use the definition (9) of $y$ and the identity (A3) from appendix.

Therefore, the absolute values of the transmission and reflection amplitudes can be written in the following form:

$$ |S_{21}|^2 = \frac{1}{1 + \alpha^2U_{N-1}^2(y)} \quad (13) $$

$$ |S_{11}|^2 = \frac{\alpha^2U_{N-1}^2(y)}{1 + \alpha^2U_{N-1}^2(y)} \quad (14) $$

The energy conservation clearly follows from (13) and (14)

$$ |S_{11}|^2 + |S_{21}|^2 = 1 \quad (15) $$

If the distance between neighbor qubits is small compared to the wavelength ($\theta \ll 1$), then $y = 1, U_{N-1}(1) = N$ and we obtain for $S_{21}$:

$$ |S_{21}|^2 = \frac{(\omega - \Omega)^2}{(\omega - \Omega)^2 + (N\Gamma)^2} \quad (16) $$
This is well known phenomena of a superradiance: closely spaced $N$ identical qubits decay $N$ times faster than a single qubit [20].

From the definition of $U_n(y)$ (see (A1) in appendix) it follows that for $|y| > 1$ the Chebyshev polynomial $U_n(y)$ can take sufficiently large values which give rise to a strong suppression of the transmission.

The formation of the bandgap begins when the number of qubits $N$ exceeds the value $\sqrt{\Omega/\Gamma}$ [21]. For real atoms in nanofiber this value is on the order of $10^4$ [21]. However, for solid state qubits in the regime of strong coupling ($\Gamma/\Omega \approx 0.1$) the quantity $\sqrt{\Omega/\Gamma} \approx 3$. Therefore, for this system the formation of the bandgap starts with several qubits in the chain.

The plots of transmission amplitude (13) for five identical qubits together with the frequency dependence of $y$ are shown in Fig. 1. All qubits are assumed to satisfy the Bragg condition: $\theta_0 = \frac{k_0 L}{v_g} \pi = \pi$. We see that in vicinity of $\theta \approx \pi$ there exists a broad bandgap where the transmission is strongly suppressed, and within the band $y < -1$.

For subsequent calculations it is convenient to write $\theta$ as

$$\theta = \frac{\omega}{v_g} L = k_0 L \left(1 + \frac{\Delta}{\Omega}\right)$$

where $k_0 = \frac{\Omega}{v_g}$, $\Delta = \omega - \Omega$.

We take $k_0 L = \pi$ and assume $\Delta \ll \Omega$. Thus we obtain for $y$

$$y = -\cos \left(\frac{\pi \Delta}{\Omega}\right) - \frac{\Gamma}{\Delta} \sin \left(\frac{\pi \Delta}{\Omega}\right) \approx b + a \left(\frac{\Delta}{\Omega}\right)^2$$

where

$$b = -1 - \frac{\Gamma \pi}{\Omega}; \quad a = \frac{\pi^2}{2} \left(1 + \frac{\Gamma \pi}{3\Omega}\right)$$

Next, we assume that the walls of the band correspond to $y = y_b = -1$. Equating (13) to the boundary value $y_b = -1$ we obtain for half bandwidth

$$\frac{\Delta}{\Omega} = \frac{1}{\pi} \sqrt{\frac{2G\pi}{1 + G\pi^2}}$$

where $G = \Gamma/\Omega$, $\Delta_b = \omega_b - \Omega$.

It should be noted that the estimation (20) does not depend on the number of qubits. However, a weak dependence of the bandgap width on $N$ does exist, while the steepness of the bandgap walls depends on $N$ much more strongly. This is seen in Fig. 2 where we compare two bandgaps, for $N = 5$ and $N = 11$.

Next we estimate the value of transmission amplitude at the boundary of the band, where $y \approx -1$, $U_{N-1}(-1) = (-1)^{N-1} N$ [15].

At the boundary

$$\alpha_b = \frac{\Gamma}{\Delta_b} = \frac{\Gamma}{\Omega} \frac{\Omega}{\Delta_b} \approx G\pi \sqrt{\frac{1 + G\pi}{2G\pi}}$$

The substitution of (21) in (13) leads to the following estimation for the amplitude of transmission at the boundary of the band:

$$\left|S_{21}\right|^2 = \frac{1}{1 + N^2 G\pi \left(1 + \frac{G\pi}{3}\right)}$$

It is worthwhile to note that if we disregarded the frequency dependence of $\theta$ and simply put $\theta = n\pi$, where $n$ is integer (as was done in [10]), the essential interference effects which are contained in the second term in (9) would be lost. In this case we would obtain a broadened Lorentzian [16] for the transmission amplitude. As the example, we compare in Fig. 3 the transmission amplitudes for five identical qubits calculated from exact expression (13), where the frequency dependence of the phase $\theta$ is accounted for, with the one calculated from (13), where the frequency dependence of the phase is ignored. The bandgap (black curve is obtained for $k_0 L = \pi$, whereas the broad Lorentzian- type (red) curve is obtained for $kL = \pi$, where the frequency dependence of the phase is disregarded.

It is known that for $N$- qubit chain there are $N - 1$ transmission peaks, where the reflection is zero [4]. This
III. TUNABLE RESONANCE WITHIN THE BANDGAP

Here we consider the chain with odd number of qubits, \( N = 2n + 1 \), where \( n \) is a positive integer. All qubits in the chain are supposed to be identical except for the central qubit whose excitation frequency \( \Omega_0 \) differs from that of the other qubits in the chain. We show that in this case there exists within a stopband a narrow resonance which provides a full transmission. The position and the width of the resonance depend on the ratio \( \Omega_0/\Omega \) of the frequency of the central qubit, \( \Omega_0 \) to the frequency of other qubits, \( \Omega \).

A. Basic expressions

The transfer matrix for the qubit structure described above is of the form:

\[
T_{2n+1} = T_1^n T_0 T_1^n
\]  

(23)

where \( T_1 \) is given in (1), and

\[
T_0 = \begin{pmatrix}
 e^{-i\theta} & 0 \\
 0 & e^{i\theta}
\end{pmatrix} \begin{pmatrix}
 1 + i\alpha_0 & i\alpha_0 \\
 -i\alpha_0 & 1 - i\alpha_0
\end{pmatrix}
\]  

(24)

where \( \alpha_0 = \Gamma/(\omega - \Omega_0) \), \( \Omega_0 \) is the excitation frequency of the central qubit.

The application of Abeles theorem to the matrix \( T_1^n \) gives

\[
T_{2n+1} = U_{n-1}^2(y)T_1^n T_0 T_1^n - U_{n-1}(y)U_{n-2}(y)[T_1 T_0]_+ + U_{n-2}(y)T_0
\]  

(25)

where

\[
[T_0 T_1]_+ = T_0 T_1 + T_1 T_0
\]

Next we define the quantity which characterizes the difference between the central and other qubits in the chain:

\[
\delta = \alpha_0 - \alpha = \alpha_0 \alpha \eta
\]  

(26)

where \( \eta = (\Omega_0 - \Omega)/\Gamma \).

Therefore, the matrix \( T_0 \) can be written as

\[
T_0 = T_1 + i\delta R
\]  

(27)

where

\[
R = \begin{pmatrix}
 e^{-i\theta} & e^{-i\theta} \\
 -e^{i\theta} & -e^{i\theta}
\end{pmatrix}
\]  

(28)

If \( \delta = 0 \) the expression (28) reduces to (5). Therefore, we can rewrite (25) as follows:

\[
T_{2n+1} = T_N + i\delta X
\]  

(29)

where \( T_N \) is given in (5) and

\[
X = U_{n-1}^2(y)T_1 R T_1^n - U_{n-1}(y)U_{n-2}(y)[T_1 R]_+ + U_{n-2}(y)R
\]  

(30)
In this case, the transmission amplitude can be written as follows

$$|S_{21}|^2 = \frac{1}{|(T_N)_{11} + i\delta(X)_{11}|^2} \quad (31)$$

In order to transform (30) in more convenient form we introduce commutator of two matrices, \( R \) and \( T_1 \):

$$[R,T_1] = RT_1 - T_1R = 2ie^{i\theta} \sin \theta \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} \quad (32)$$

and rewrite (30) as follows:

$$X = [2yU_{n-1}^2 - 2U_{n-1}U_{n-2}] T_1R + U_{n-1}^2T_1[RT_1]_- - U_{n-1}U_{n-2}[RT_1]_- - (U_{n-1}^2 - U_{n-2}^2)R \quad (33)$$

With the aid of recurrence relation (A2) and properties of Chebyshev polynomials (A7), (A8) we write (30) in the final form;

$$X = U_{n-2}T_1R + U_{n-1}^2T_1[RT_1]_- - U_{n-1}U_{n-2}[RT_1]_- - U_{N-3}R \quad (34)$$

B. Frequency of the resonance inside the bandgap

In what follows we need only two matrix elements \( X_{11} \) and \( X_{21} \).

$$X_{11} = U_{N-2} \left[ e^{-2i\theta}(1 + i\alpha) - i\alpha \right]$$

$$-U_{n-1}^22\alpha e^{i\theta} \sin \theta - U_{N-3}e^{-i\theta} \quad (35)$$

$$X_{21} = U_{N-2} \left[ -i\alpha - e^{2i\theta}(1 - i\alpha) \right]$$

$$+U_{n-1}^22ie^{i\theta} e^{2i\theta} \sin \theta e^{2i\theta} \quad (36)$$

$$-U_{n-1}U_{n-2}2ie^{i\theta} \sin \theta + U_{N-3}e^{i\theta}$$

We show below that within a bandgap there exists a frequency where the reflection amplitude is exactly equal to zero.

For the numerator of reflection amplitude (7) we have

$$(T_{2n+1})_{21} = -i\alpha e^{i\theta}U_{n-1}(y) + i\delta(X)_{21} \quad (37)$$

Inside the bandgap \( \theta \approx \pi, \alpha \gg 1 \). In these approximations the quantities \( X_{11} \) and \( X_{21} \) can be approximated as

$$X_{11} = U_{N-2} + U_{N-3}; X_{21} = -(U_{N-2} + U_{N-3}) \quad (38)$$

For (37) within the bandgap we thus obtain

$$(T_{2n+1})_{21} \approx i\alpha U_{N-1}(y) - i\delta(U_{N-2} + U_{N-3}) \quad (39)$$

Making use of (20) we can obtain from (39) the equation which defines the frequency of the resonance within the bandgap

$$U_{N-1}(y) = \alpha_0\eta(U_{N-2}(y) + U_{N-3}(y)) = 0 \quad (40)$$

From (41) we obtain the expression for the resonance frequency, \( \omega_p \) which is valid for \( \Delta \ll \Omega \).

$$\omega_p = \Omega_0 + (\Omega_0 - \Omega) \frac{U_{N-2}(b) + U_{N-3}(b)}{U_{N-1}(b)} \quad (41)$$

where \( b \) is defined in (19).

From (41) we see that the resonance frequency moves to the walls of the band as the frequency of the central qubit deviates from \( \Omega \). The corresponding picture is shown in Fig (4) for five qubits with \( \Gamma/\Omega = 0.1 \).

Two dimensional distribution of transmission amplitude for five qubits against the frequency, \( \omega \) and the ratio \( \Omega_3/\Omega \) is shown in Fig (7). The resonance line is seen as a narrow colored stripline in the middle of the bandgap.
The investigation of transmission amplitude $S_{21}$ near resonance leads to the following expression for the spectrum of transmission line (the details of derivation are given in the appendix):\[ S_{21}(\omega) = \frac{\omega - \Omega}{\omega - \Omega + i\Gamma B_N(y_p)(\omega - \omega_p)} \quad (44) \]
where the quantity $B_N(y_p)$ is given in (39):\[ B_N(y_p) = \frac{2\omega_p\Delta}{\omega_p} \left( U'_{N-1} - \alpha_0(\omega_p)\eta \left[U'_{N-2} + U'_{N-3}\right]\right) + \frac{U'_{N-1}}{\Delta_{\omega_0}} \quad (45) \]

It is seen from expression (44) that the lineshape of resonance inside the band is not the Lorentzian function: its width depends on the frequency. The full width at half maximum (FWHM) of the resonance we find from the equation\[ |S_{21}(\omega_{\pm})|^2 = \frac{1}{2} \quad (46) \]

We thus obtain:\[ \omega_+ = \frac{\Omega - \Gamma B_N \omega_p}{1 - \Gamma B_N}, \quad \omega_- = \frac{\Omega + \Gamma B_N \omega_p}{1 + \Gamma B_N} \quad (47) \]

Finally, the FWHM of the resonance line is as follows:\[ \Delta \omega = \omega_+ - \omega_- = \frac{2\Gamma B_N(\Omega - \omega_p)}{1 - \Gamma^2 B_N^2} \quad (48) \]

It follows from (48) that the width of resonance line depends on its position within the bandgap. The closer the resonance line to the wall of the bandgap, the greater its width. The plots of the width of the resonance line and the corresponding quality factor, $Q = \omega_p/\Delta \omega$ against the frequency of central qubit are shown for five-qubit structure in Fig.7 and Fig.8 respectively. The solid (black) curves are calculated from exact expression (31), the dashed (red) curves are calculated from the estimation (45).

In fact, the resonance line inside the bandgap corresponds to a subradiant state. Its decay rate is just the width of resonance line. In Fig.9 we compare the decay rate for these subradiant states with that of individual qubits for several $N$-odd qubit chains. We see from Fig.9 that as the number of qubits is increased, the resonance width is strongly reduced.

It is interesting to look at the process of a formation of the bandgap and that of the resonance line. Consider the dependence of the transmission curve on the rate of single qubit spontaneous emission, $\Gamma$ into a waveguide. This rate is the measure of the qubit-photon interaction. If all qubits are identical and the qubit-photon interaction is weak, the transmission amplitude looks like the one shown by dashed (black) curve in Fig.10 panel (a). As the qubit-photon interaction is increased, the transmission curve starts to deform, the width of the bandgap

\[ |(T_{2n+1})_{11}|^2 = |(T_N)_{11} + i\delta(X)_{11}|^2 \]

\[ = |(T_N)_{11}|^2 + \delta^2|(X)_{11}|^2 + \delta \left[(X)_{11} (T_N)_{11}^* - (X)_{11}^* (T_N)_{11}\right] \quad (42) \]

Making use of (11) and the approximation (38) for (X)_{11}, we obtain from (42):\[ |(T_{2n+1})_{11}|^2 = 1 + \alpha^2[U_{n-1} - \alpha_0 \eta (U_{N-2} + U_{N-3})]^2 \quad (43) \]

The quantity in square brackets in the right hand side of (43) is just the one in the left hand side of (10), which defines the position of resonance inside the band. As is seen from (43) at the point of resonance a full transmission is observed.
FIG. 7: The width of the resonance line against the frequency of a central qubit. \( N = 5, G = 0.1 \). Solid (black) plot is calculated from exact expression (31), dashed (red) plot is calculated from the estimation (48).

FIG. 8: The quality factor of the resonance line against the frequency of a central qubit. \( N = 5, G = 0.1 \). Solid (black) plot is calculated from exact expression (31), dashed (red) plot is calculated from the estimation (48).

becomes larger, and finally it transforms to the structure with a bandgap as is shown in the panel (e) of Fig. 10. This is a direct manifestation of the formation of a superradiant state which results from the overlapping of the qubit individual widths [22].

If one qubit has a different frequency, \( \Omega_0 \), and the qubit-photon interaction is weak, then the transmission curve represents two dips, at \( \Omega \) and \( \Omega_0 \) as is shown by solid (red) curve in the panel (a) in Fig. 10. Between these two dips there is a broad region where transmission equals unity. As the qubit-photon interaction is increased, the region between two dips (panel (a)) undergoes transformation which finally results in a single narrow resonance where the transmission is equal to unity (panel (e) in Fig. 10). The reason for this transformation is the same as above: the photon-qubit interaction leads to the overlapping of the qubit individual widths.

Finally, we consider how the transmission is influenced by the radiative losses to the environment (out of the waveguide). It is common to model these losses by the modification of the qubit frequency with a small shift to lower half of the complex plane: \( \Omega \rightarrow \Omega - i\gamma \), where \( \gamma \) is the radiation rate to the environment. The influence of radiative losses is shown in Fig. 11. The calculation is made for five-qubit chain, for \( k_0L = \pi \), \( \Gamma/\Omega = 0.1 \), and five gradually increasing values of \( \gamma \). We see (the panel (a) in Fig. 11) that the width of the band-gap near its bottom and the position of transmission peak are practically not sensitive to the \( \gamma \). However, the line-shape of the transmission peak (the panel (b) in Fig. 11) changes drastically: as \( \gamma \) increases the height of the peak decreases, and its width increases.
FIG. 10: The process of the peak formation for five-qubit system, $\Omega_3/\Omega = 1.2$. (a) $G = 0.001$, (b) $G = 0.005$, (c) $G = 0.01$, (d) $G = 0.05$, (e) $G = 0.1$.

FIG. 11: (a) Frequency dependence of the photon transmission through five-qubit chain for different values of the radiative losses. (b) Lineshape of the transparency peak inside the band-gap for different values of the radiative losses.

IV. CONCLUSION

We have analyzed the signal transmission through a 1D chain of equally spaced $N$ qubits using the decomposition of a transfer matrix in terms of Chebyshev polynomials. We find that if the phase factor at the qubit frequency is equal to integer multiple of $\pi$, the bandgap is developed in the spectrum of transmitted signal.

Detailed analysis have been performed for the structure where $N - 1$ qubits were identical in their frequency, $\Omega$, but the frequency of a central qubit, $\Omega_0$ was different. In this case, a narrow resonance line appeared within the bandgap with its position and the width being dependent on the ratio $\Omega_0/\Omega$.

We showed that the appearance of the bandgap and the narrow resonance resulted from the strong photon-qubit interaction when the widths of individual qubits were overlapped.

Our results may be applied not only to superconducting qubits [7, 8], but to many other quantum emitters, interacting with any kind of quasi-1D photonic structures or circuits.

Acknowledgments

Ya. S. G. thanks A. Sultanov for fruitful discussions. The work is supported by the Ministry of Education and Science of Russian Federation under the project FSUN-2020-0004.

Appendix A: Basic properties of Chebyshev polynomials of second kind, $U_n(y)$

1. Definition

$$U_{N-1}(y) = \begin{cases} \frac{\sin N\Lambda}{\sin \Lambda}, & \Lambda = \arccos(y), \quad |y| \leq 1 \\
(\text{sign}(y))^{N-1} \frac{\sin N\Lambda}{\sin \Lambda}, & \Lambda = \ln(|y| + \sqrt{|y|^2 - 1}), \quad |y| > 1 \end{cases}$$ (A1)

2. Recurrence relation

$$U_N = 2yU_{N-1} - U_{N-2} \quad (A2)$$

3. Roots

Chebyshev polynomial $U_{N-1}(y)$ has $N - 1$ roots $y_m$ where

$$y_m = \cos \left( \frac{\pi m}{N} \right) \quad (A3)$$

with $m = 1, 2, ..., N - 1$.

4. Abeles theorem [11].

The $n$-th power $M^n$ of any unimodular (determinant is equal to 1) matrix $M$ can be expressed in terms of Chebyshev polynomials of second kind as follows:

$$M^n = U_{n-1}(y)M - U_{n-2}(y)I \quad (A4)$$
where $y = \frac{1}{2} S \rho(M)$ and $I$ is the identity matrix.

5. Identities

$$U_{2n+1} = 2U_n(yU_n - U_{n-1}) \quad \text{(A5)}$$

$$U_{2n-1} = 2U_{n-1}(yU_{n-1} - U_{n-2}) \quad \text{(A6)}$$

$$U_{n+m} = U_nU_m - U_{n-1}U_{m-1} \quad \text{(A7)}$$

$$U_{n-1}(U_n - U_{n-2}) = U_{2n-1} \quad \text{(A8)}$$

$$U_{N-1}(y) + U_{N-2}(y) - 2yU_{N-1}(y)U_{N-2}(y) = 1 \quad \text{(A9)}$$

6. First several polynomials

$$U_0(y) = 0; \quad U_0(y) = 1 \quad \text{(A10)}$$

$$U_1(y) = 2y \quad \text{(A11)}$$

$$U_2(y) = 4y^2 - 1 \quad \text{(A12)}$$

$$U_3(y) = 8y^3 - 4y \quad \text{(A13)}$$

$$U_4(y) = 16y^4 - 12y^2 + 1 \quad \text{(A14)}$$

7. Specific values

$$U_n(\pm 1) = (\pm)^n(n + 1) \quad \text{(A15)}$$

$$U_{2n}(0) = (-1)^n; \quad U_{2n+1}(0) = 0 \quad \text{(A16)}$$

Appendix B: Derivation of the resonance lineshape

The expression [13] allows us to write the transmission amplitude in the following form

$$S_{21}(y) = \frac{1}{1 + i\alpha A_N(y)} \quad \text{(B1)}$$

where

$$A_N(y) = U_{N-1}(y) - \alpha_0(\omega)\eta(U_{N-2}(y) + U_{N-3}(y)) \quad \text{(B2)}$$

where

$$\alpha_0(\omega) = \frac{\Gamma}{\omega_p - \omega} \approx \alpha_0(\omega_p) - \alpha_0(\omega_p) \frac{\delta \omega}{\Delta_{p,0}} \quad \text{(B3)}$$

$$\alpha_0(\omega_p) = \Gamma/\Delta_{p,0}, \quad \Delta_{p,0} = \omega_p - \omega_0, \quad \delta \omega = \omega - \omega_p.$$ Let at the point of resonance where $A_N(y) = 0$

$$y \equiv y_p = b + a \left(\frac{\Delta_p}{\Omega}\right)^2 \quad \text{(B4)}$$

Near resonance the transmission amplitude can be written as follows:

$$S_{21}(y_p + \delta y) = \frac{1}{1 + i\alpha A_N(y_p + \delta y)} \quad \text{(B5)}$$

where

$$\delta y = y - y_p \approx \frac{2\alpha \Delta_p}{\Omega^2} \delta \omega \quad \text{(B6)}$$

with $\delta \omega = \omega - \omega_p.$

Then, we obtain

$$A_N(y_p + \delta y) \approx B_N(y_p) \delta \omega \quad \text{(B7)}$$

where

$$B_N(y_p) = 2\frac{\alpha \Delta_p}{\Omega^2} \left(U_{N-1}' - \alpha_0(\omega_p)\eta \left[U_{N-2}' + U_{N-3}'\right]\right)$$

$$+ \frac{\alpha_0(\omega_p)}{\Delta_{p,0}} \eta \left(U_{N-2} + U_{N-3}\right) \quad \text{(B8)}$$

In expression [B8] the Chebyshev polynomials and their derivatives are taken at the point $y = y_p.$

The equation [B10] allows us to rewrite the last term in [B8], so that we finally obtain for $B_N(y_p):$

$$B_N(y_p) = 2\frac{\alpha \Delta_p}{\Omega^2} \left(U_{N-1}' - \alpha_0(\omega_p)\eta \left[U_{N-2}' + U_{N-3}'\right]\right)$$

$$+ \frac{U_{N-1}'}{\Delta_{p,0}} \quad \text{(B9)}$$

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