Non Abelian Toda models and the Constrained KP hierarchies

I. Cabrera-Carnero, J.F. Gomes, E.P. Gueuvoghlanian, G.M. Sotkov and A.H. Zimerman

Instituto de Física Teórica - IFT/UNESP
Rua Pamplona 145
01405-900, São Paulo - SP, Brazil

ABSTRACT

A general construction of affine Non Abelian Toda models in terms of gauged two loop WZNW model is discussed. Its connection to non relativistic models corresponding to constrained KP hierarchies is established in terms of time evolution associated to positive and negative grading of the Lie algebra.

\footnotetext[1]{talk given at 7th International Wigner Symposium, College Park, Maryland, August 2001}
\footnotetext[2]{cabrera@ift.unesp.br}
\footnotetext[3]{jfg@ift.unesp.br}
\footnotetext[4]{gueuvogh@ift.unesp.br}
\footnotetext[5]{sotkov@ift.unesp.br}
\footnotetext[6]{zimerman@ift.unesp.br}
1 Introduction

The abelian affine Toda field theories provide a large class of relativistic invariant integrable models in two dimensions associated to an affine Lie algebra $\hat{G}$ (loop algebra) admitting solitons solutions. The Toda fields are defined to parametrize a finite dimensional abelian manifold (Cartan subalgebra of $\hat{G}$) and their solitonic character (in the imaginary coupling regime) is a consequence of the infinite dimensional Lie algebraic structure responsible for the multivacua configuration leading to a nontrivial topological structure.

A more general class of affine Toda models is obtained by introducing a non abelian structure to the abelian manifold (Cartan subalgebra of $\hat{G}$) parametrized by the Toda fields. A systematic manner in classifying the Toda models is in terms of a grading operator $Q$ that decomposes the affine Lie algebra $\hat{G} = \oplus G_i$, where the graded subspaces are defined by $[Q, G_i] = iG_i$. The Toda fields are defined to parametrize the zero grade subspace $G_0 \subset G$.

The simplest model is obtained when $G = SL(2)$, is decomposed according to the homogeneous gradation leading to the Lund-Regge model. For $SL(r+1)$, a general construction is discussed in ref. [3] leading to actions corresponding to Lund-Regge interacting with abelian Toda models. In this note we discuss a systematic construction of non abelian Toda models associated to $G_0 = SL(2)^p \otimes U(1)^{rank G - p}$. The simplest non trivial generalization of the Lund-Regge model is obtained $G_0 = SL(2) \otimes SL(2) \otimes U(1) \subset SL(4)$. The action, equations of motion and zero curvature representation are explicitly constructed. The class of relativistic invariant non abelian Toda models share a common algebraic structure with the constrained KP type models [4]. These are non relativistic systems whose multi time equation of motion generate an integrable hierarchy associated to positive grading. It was shown in [5] that the Lund-Regge and the non linear Schroedinger models correspond to the same hierarchy when time evolution is also assign to negative grade generators. The problem of constructing integrable hierarchies associated to positive and negative gradings was further investigated and generalized to $SL(r+1)$ in [6] where it was shown to be connected to the Riemann-Hilbert problem. Here, several examples for $SL(2), SL(3)$ and $SL(4)$ are discussed in detail. In particular the soliton solutions within positive and negative hierarchies are shown to be related.

2 Construction of the Model

The generic NA Toda models are classified according to a $G_0 \subset G$ embedding induced by the grading operator $Q$ decomposing an finite or infinite dimensional Lie algebra $G = \oplus_i G_i$ where $[Q, G_i] = iG_i$ and $[G_i, G_j] \subset G_{i+j}$. A group element $g$ can then be written in terms of the Gauss decomposition as

$$g = NBM$$

(2.1)

where $N = \exp G_{<}, B = \exp G_0$ and $M = \exp G_{>}$. The physical fields lie in the zero grade subgroup $B$ and the models we seek correspond to the coset $H_- \backslash G / H_+$, for $H_\pm$ generated by positive/negative grade operators.

For consistency with the hamiltonian reduction formalism, the phase space of the $G$-invariant WZNW model is reduced by specifying the constant generators $\epsilon_\pm$ of grade $\pm 1$. In
order to derive an action for $B$, invariant under

$$g \rightarrow g' = \alpha_- g \alpha_+,$$  \hspace{1cm} (2.2)

where $\alpha_\pm (z, \bar{z})$ lie in the positive/negative grade subgroup we have to introduce a set of auxiliary gauge fields $A \in G_<$ and $\bar{A} \in G_>$ transforming as

$$A \rightarrow A' = \alpha_- A \alpha_-^{-1} + \alpha_- \partial \alpha_-^{-1}, \quad \bar{A} \rightarrow \bar{A}' = \alpha_+^{-1} \bar{A} \alpha_+ + \partial \alpha_+^{-1} \alpha_+,$$  \hspace{1cm} (2.3)

where $z = \frac{1}{2}(t + x)$, $\bar{z} = \frac{1}{2}(t - x)$. The resulting action is the $G/H(= H_\perp G/H_\perp)$ gauged WZNW

$$S_{G/H}(g, A, \bar{A}) = S_{WZNW}(g) - \frac{k}{4\pi} \int d^2 x Tr \left( A(\bar{g} g^{-1} - \epsilon_+) + \bar{A}(g^{-1} \partial g - \epsilon_-) + A g \bar{A} g^{-1} \right).$$

Since the action $S_{G/H}$ is $H$-invariant, we may choose $\alpha_- = N^{-1}$ and $\alpha_+ = M^{-1}$. From the orthogonality of the graded subspaces, i.e. $Tr(G_i G_j) = 0$, $i + j \neq 0$, we find

$$S_{G/H}(g, A, \bar{A}) = S_{G/H}(B, A', \bar{A}') = S_{WZNW}(B) - \frac{k}{4\pi} \int d^2 x Tr [-A' \epsilon_+ - \bar{A}' \epsilon_- + A' B \bar{A}' B^{-1}],$$  \hspace{1cm} (2.4)

where

$$S_{WZNW} = -\frac{k}{4\pi} \int d^2 x Tr (g^{-1} \partial g g^{-1} \partial g) + \frac{k}{24\pi} \int_D \epsilon^{ijk} Tr (g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_k g) d^3 x,$$  \hspace{1cm} (2.5)

and the topological term denotes a surface integral over a ball $D$ identified as space-time.

Action (2.4) describes the non singular Toda models among which we find the Conformal and the Affine abelian Toda models where $Q = \sum_{i=1}^{r} \frac{2\beta_i H_i}{a_i^2}$, $\epsilon_\pm = \sum_{i=1}^{r} \mu_\pm i E_{\pm \alpha_i}$ and $Q = h d + \sum_{i=1}^{r} \frac{2\lambda_i H_i^{(0)}}{a_i^2}$, $\epsilon_\pm = \sum_{i=1}^{r} \mu_\pm i E_{\pm \alpha_i}^{(0)} + \mu_0 E_{\pm \psi}^{(\pm 1)}$ respectively, $\psi$ denotes the highest root, $\lambda_i$ are the fundamental weights and $h = 1 + \sum_{i=1}^{r} \psi \cdot \lambda_i$ is the coxeter number of $G$.

Performing the integration over the auxiliary fields $A$ and $\bar{A}$, the functional integral

$$Z_{\pm} = \intDA \exp (F_{\pm}),$$  \hspace{1cm} (2.6)

where

$$F_{\pm} = -\frac{k}{2\pi} \int \left( Tr(A - B \epsilon_- B^{-1}) B(A - B^{-1} \epsilon_+ B) B^{-1} \right) d^2 x$$  \hspace{1cm} (2.7)

yields the effective action

$$S = S_{WZNW}(B) + \frac{k}{2\pi} \int Tr \left( \epsilon_+ B \epsilon_- B^{-1} \right) d^2 x$$  \hspace{1cm} (2.8)

The action (2.8) describes integrable perturbations of the $G_0$-WZNW model. Those perturbations are classified in terms of the possible constant grade $\pm 1$ operators $\epsilon_\pm$. 

2
More interesting cases arises in connection with non abelian embeddings $\mathcal{G}_0 \subset \mathcal{G}$. In particular, if we supress $p$ alternate fundamental weights from $Q$, the zero grade subspace $\mathcal{G}_0$, acquires a nonabelian structure $\mathfrak{sl}(2)^{\nu} \otimes u(1)^{\text{rank}\mathcal{G}-p}$. Let us consider for instance $Q = h'd + \sum_{i \neq 1,3}^{r} \frac{2\lambda_{i} \cdot H}{\alpha_{i}^{2}}$, where $h' = 0$ or $h' \neq 0$ corresponding to the Conformal or Affine nonabelian (NA) Toda respectively. The absence of $\lambda_{al}, al = 1,3, \cdots$ in $Q$ prevents the contribution of the simple root step operator $E_{\alpha_{al}}^{(0)}$ in constructing the grade one operator $\epsilon_{+}$. It in fact, allows for reducing the phase space even further. This fact can be understood by enforcing the nonlocal constraint $J_{Y^{l}H} = \tilde{J}_{Y^{l}H} = 0$ where $Y^{l}, l = 1, \cdots t \leq p$ is such that $[Y^{l} \cdot H, \epsilon_{\pm}] = 0$ and $J = g^{-1} \partial g$ and $\tilde{J} = -\partial gg^{-1}$. Those generators of $\mathcal{G}_0$ commuting with $\epsilon_{\pm}$ define a subalgebra $\mathcal{G}_{0}' \subset \mathcal{G}_0$. Such subsidiary constraint is incorporated into the action by requiring symmetry under

$$g \longrightarrow g' = \alpha_{0}g\alpha'_{0}$$

(2.9)

where we shall consider $\alpha'_{0} = \alpha_{0}(z, \bar{z}) \in \mathcal{G}_{0}'$, i.e., axial symmetry (the vector gauging is obtained by choosing $\alpha'_{0} = \alpha_{0}^{-1}(z, \bar{z}) \in \mathcal{G}_{0}'$, see for instance [1]). Auxiliary gauge fields $A_{0} = \sum_{l=1}^{t} \alpha_{0}Y^{l} \cdot H$ and $\tilde{A}_{0} = \sum_{l=1}^{t} \alpha_{0}Y^{l} \cdot \tilde{H} \in \mathcal{G}_{0}'$ transforming as

$$A_{0} \longrightarrow A'_{0} = A_{0} - \alpha_{0}^{-1}\partial \alpha_{0}, \quad \tilde{A}_{0} \longrightarrow \tilde{A}'_{0} = \tilde{A}_{0} - \bar{\partial} \alpha'_{0}(\alpha'_{0})^{-1}.$$  

are introduced to construct an invariant action under transformations (2.3)

$$S(B, A_{0}, \tilde{A}_{0}) = S(g_{1}^{0}, A'_{0}, \tilde{A}'_{0}) = S_{WZW}(B) + \frac{k}{2\pi} \int Tr \left( \epsilon_{+}B\epsilon_{-}B^{-1} \right) d^{2}x$$

$$- \frac{k}{2\pi} \int Tr \left( A_{0}\bar{\partial}BB^{-1} + \tilde{A}_{0}B^{-1}\partial B + A_{0}B\tilde{A}_{0}B^{-1} + A_{0}\tilde{A}_{0} \right) d^{2}x$$

(2.10)

Such residual gauge symmetry allows us to eliminate extra fields associated to $Y^{l} \cdot H, l = 1, \cdots, t \leq p$. Notice that the physical fields $g_{1}^{0}$ lie in the coset $\mathcal{G}_{0}/\mathcal{G}_{0}' = (\mathfrak{sl}(2)^{\nu} \otimes u(1)^{\text{rank}\mathcal{G}'-p})/u(1)^{t}$ and are classified according to the gradation $Q$. It therefore follows that $S(B, A_{0}, \tilde{A}_{0}) = S(g_{1}^{0}, A'_{0}, \tilde{A}'_{0}).$

In [8] a detailed study of the gauged WZNW construction for finite dimensional Lie algebras leading to Conformal NA Toda models was presented. For an infinite dimensional Kac-Moody algebra $\hat{\mathcal{G}}$

$$[T_{m}^{a}, T_{n}^{b}] = f^{abc}T_{m+n}^{c} + \hat{c}m\delta_{m+n}\delta^{ab}$$

$$[\hat{d}, T_{n}^{a}] = nT_{n}^{a}; \quad [\hat{c}, T_{n}^{a}] = [\hat{c}, \hat{d}] = 0$$

(2.11)

the NA Toda models are associated to gradations of the type $Q(h') = h'd + \sum_{i \neq 1,3}^{r} \frac{2\lambda_{i} \cdot H^{(0)}}{\alpha_{i}^{2}}$, where $h'$ is chosen such that the gradation, $Q(h')$, acting on infinite dimensional Lie algebra $\hat{\mathcal{G}}$ ensures that the zero grade subgroup $\mathcal{G}_{0}$ coincides with its counterpart obtained with $Q(h') = \hat{\mathcal{G}}$.\footnote{For the Kac-Moody case we are suppresing the index (0) in defining the Cartan subalgebra of $\hat{\mathcal{G}}$.}
0) acting on the Lie algebra $\mathcal{G}$ of finite dimension apart from two commuting generators $\hat{c}$ and $\hat{d}$. Since they commute with $\mathcal{G}_0$, the kinetic part decouples such that the conformal and the affine singular NA-Toda models differ only by the potential term characterized by $\epsilon_{\pm}$.

The integration over the auxiliary gauge fields $A$ and $\tilde{A}$ requires explicit parametrization of $B$.

$$B = \exp\left(\sum_{l=1}^{p} \chi_{al}E_{al}^{(0)}\right) \exp\left(\sum_{l=1}^{t} R^{l} \sum_{i=1}^{r} y_{i}^{l} H_{i}(0) + \Phi(H) + \nu \hat{c} + \eta \hat{d}\right) \exp\left(\sum_{l=1}^{p} \psi_{al}E_{al}^{(0)}\right)$$  \hspace{1cm} (2.12)

where $\Phi(H) = \sum_{j=1}^{r} \sum_{i=1}^{t} \varphi_{i} X_{j}^{l} H_{j}(0)$, where $\sum_{j=1}^{r} y_{i}^{l} X_{j}^{l} = 0, i = 1, \ldots, r - t, l = 1, \ldots, t$. After gauging away the nonlocal fields $R^{l}$, the factor group element becomes

$$g_{0}^{f} = \exp\left(\sum_{l=1}^{p} \chi_{al}E_{al}^{(0)}\right) \exp(\Phi) \exp(\sum_{l=1}^{p} \psi_{al}E_{al}^{(0)})$$ \hspace{1cm} (2.13)

where $\chi_{al} = \hat{\chi}_{al}e^{\frac{1}{2} \sum_{s=1}^{t}(Y_{s \cdot \alpha - al})R^{s}}$, $\psi_{al} = \hat{\psi}_{al}e^{\frac{1}{2} \sum_{s=1}^{t}(Y_{s \cdot \alpha - al})R^{s}}$. We therefore get for the zero grade component

$$F_{0} = -\frac{k}{2\pi} \int Tr \left( A_{0} \partial g_{0}^{f} (g_{0}^{f})^{-1} + \hat{A}_{0} (g_{0}^{f})^{-1} \partial g_{0}^{f} + A_{0} g_{0}^{f} \hat{A}_{0} (g_{0}^{f})^{-1} + A_{0} \hat{A}_{0}\right) d^{2}x$$

$$= -\frac{k}{2\pi} \int \left(\sum_{l=1}^{t} a_{0l} \hat{a}_{0l} 2(Y^{l} \cdot Y^{l}) \Delta_{v^{l} l^{l}}\right)$$

$$- \sum_{l=1}^{t} 2 \left(\frac{\alpha_{al} Y^{l}}{2} \frac{Y^{l} \cdot \alpha_{al}}{2Y^{l} \cdot Y^{l}} \right) (a_{0l} \hat{a}_{0l} \partial \chi_{al} + a_{0l} \chi_{al} \partial \hat{a}_{0l}) e^{\Phi(\alpha_{al})} ) d^{2}x$$ \hspace{1cm} (2.14)

where $\Delta_{v^{l} l^{l}} = 1 + \sum_{l=1}^{t} \frac{2 \left(\frac{\alpha_{al} Y^{l}}{2} \frac{Y^{l} \cdot \alpha_{al}}{2Y^{l} \cdot Y^{l}} \right) \psi_{al} \chi_{al} e^{\Phi(\alpha_{al})}}{\frac{Y^{l} \cdot \alpha_{al}}{2Y^{l} \cdot Y^{l}}}$ and $[\Phi(H), E_{al}^{(0)}] = \Phi(\alpha_{al}) E_{al}^{(0)}$. In matrix form $F_{0}$ reads,

$$F_{0} = -\frac{k}{2\pi} \int \left(\hat{A}_{0} M A_{0} - \hat{A}_{0} N - \hat{N} A_{0}\right)$$

$$= -\frac{k}{2\pi} \int \left(\hat{A}_{0} - \hat{N} M^{-1} \right) \left(M A_{0} - M^{-1} N - \hat{N} M^{-1} N\right)$$ \hspace{1cm} (2.15)

where

$$A_{0} = \begin{pmatrix} a_{01} \\ a_{03} \\ \vdots \\ a_{0t} \end{pmatrix}, \quad \hat{A}_{0} = \begin{pmatrix} \hat{a}_{01} & \hat{a}_{03} & \cdots & \hat{a}_{0t} \end{pmatrix}, \quad N = \begin{pmatrix} \sum_{l=1}^{p} 2 a_{0l} Y^{l} \psi_{al} \partial \chi_{al} e^{\Phi(\alpha_{al})} \\ \sum_{l=1}^{p} 2 a_{0l} Y^{l} \psi_{al} \partial \chi_{al} e^{\Phi(\alpha_{al})} \\ \cdots \\ \sum_{l=1}^{p} 2 a_{0l} Y^{l} \psi_{al} \partial \chi_{al} e^{\Phi(\alpha_{al})} \end{pmatrix}$$

$$\hat{N} = \begin{pmatrix} \sum_{l=1}^{p} 2 a_{0l} Y^{l} \chi_{al} \partial \hat{a}_{0l} e^{\Phi(\alpha_{al})} \\ \sum_{l=1}^{p} 2 a_{0l} Y^{l} \chi_{al} \partial \hat{a}_{0l} e^{\Phi(\alpha_{al})} \\ \cdots \\ \sum_{l=1}^{p} 2 a_{0l} Y^{l} \chi_{al} \partial \hat{a}_{0l} e^{\Phi(\alpha_{al})} \end{pmatrix}$$

and

$$M = \begin{pmatrix} 2 Y^{1} \cdot Y^{1} \Delta_{11} & 2 Y^{1} \cdot Y^{2} \Delta_{12} & \cdots \\ 2 Y^{2} \cdot Y^{1} \Delta_{21} & 2 Y^{2} \cdot Y^{2} \Delta_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$ \hspace{1cm} (2.16)
The effective action is obtained by integrating over the auxiliary matrix fields $A_0, \bar{A}_0$,

$$Z_0 = \int D A_0 D \bar{A}_0 \exp(F_0) \quad (2.17)$$

The total effective action is therefore given as

$$S_{eff} = -\frac{k}{2\pi} \int \left( Tr \left( \partial \Phi(H) \bar{\partial} \Phi(H) \right) + 2 \sum_{l=1}^{p} \partial \chi_{al} \bar{\partial} \psi_{al} e^\Phi(\alpha_{al}) - \bar{N} M^{-1} N - V \right) \quad (2.18)$$

where $V = Tr (\epsilon_+ B \epsilon_- B^{-1})$.

### 2.1 The Lund-Regge Model, $A_1^{(1)}$

The simplest non abelian affine Toda model consists in taking the $A_1^{(1)}$ Kac-Moody algebra generated by $\{ E_{\pm \alpha}^{(n)}, H^{(n)} \}$, gradation $Q = d$ and $\epsilon_\pm = H^{(\pm 1)}$, such that $G_0 = SL(2)$. The factor group $G_0 / G_0^0 = SL(2) / U(1)$ is then parametrized as

$$g_{0}^f = \exp(\chi E_{-\alpha}^{(0)}) \exp(\psi E_{\alpha}^{(0)}) \quad (2.19)$$

We therefore find by direct calculation

$$F_0 = -\frac{k}{2\pi} \int \left( a_0 \bar{a}_0 2 \lambda_1^2 \Delta - \bar{a}_0 \psi \partial \chi + a_0 \bar{\partial} \psi \right) d^2x \quad (2.20)$$

where $\Delta = 1 + \psi \chi$. After integration over $a_0$ and $\bar{a}_0$ we find the total effective action of the Lund-Regge model

$$\mathcal{L}_{eff} = \frac{\partial \chi \bar{\partial} \psi}{\Delta} - (1 + 2\psi \chi) \quad (2.21)$$

and equations of motion given by [3]

$$\bar{\partial} \left( \frac{\partial \chi}{\Delta} \right) + \chi \frac{\partial \chi \bar{\partial} \psi}{\Delta^2} + 2\chi = 0,$$

$$\partial \left( \frac{\partial \psi}{\Delta} \right) + \psi \frac{\partial \chi \bar{\partial} \psi}{\Delta^2} + 2\psi = 0 \quad (2.22)$$

### 2.2 The $p = 1$, $A_2^{(1)}$ Model

Consider the model defined by the Kac-Moody algebra $A_2^{(1)}$, gradation $Q = 2 \hat{d} + 2 \frac{\lambda_2 H}{\alpha_2^2}$ and constant grade $\pm 1$ generators $\epsilon_\pm = E_{\pm \alpha_2}^{(0)} + E_{+ \alpha_2}^{(\pm 1)}$. The zero grade subspace is $SL(2) \otimes U(1)$, generated by $G_0 = \{ E_{\pm \alpha_1}^{(0)}, h_1^{(0)}, h_2^{(0)} \}$. It follows that $G_0^0 = \{ \lambda_1 \cdot H \}$, i.e. $Y^1 = \lambda_1$, $A_0 = a_0 \lambda_1 \cdot H$, $\bar{A}_0 = \bar{a}_0 \lambda_1 \cdot H$.

The factor group element $g_{0}^f \in G_0 / G_0^0 = (SL(2) \otimes U(1)) / U(1)$ is then parametrized as

$$g_{0}^f = \exp(\chi E_{-\alpha_1}^{(0)}) \exp(\varphi h_2^{(0)}) \exp(\psi E_{\alpha_1}^{(0)}) \quad (2.23)$$
2.3 The $p = t = 2$, $A_3^{(1)}$ Model

Consider the model defined by the Kac-Moody algebra $A_3^{(1)}$, gradation $Q = 2\hat{d} + 2\frac{\lambda_2}{\lambda_3}H$ and constant grade $\pm 1$ generators $\epsilon_\pm = E_0^{(0)} \pm E_0^{(1)} \pm \alpha_2$. The zero grade subspace is $SL(2) \otimes SL(2) \otimes U(1)$, generated by $G_0 = \{E_{\pm \alpha_1}, E_{\pm \alpha_3}, h_1, h_2, h_3\}$. It follows that $G_0^0 = \{\lambda_1 \cdot H, \lambda_3 \cdot H\}$, i.e. $Y^1 = \lambda_1, Y^2 = \lambda_3$.

The 2-singular structure is obtained by setting

$$A_0 = a_{01} \lambda_1 \cdot H + a_{03} \lambda_3 \cdot H, \quad \bar{A}_0 = \bar{a}_{01} \lambda_1 \cdot H + \bar{a}_{03} \lambda_3 \cdot H \quad (2.26)$$

The factor group element $g_0^f \in G_0^0 = (SL(2) \otimes SL(2) \otimes U(1)) \backslash U(1)^2$ is then parametrized as

$$g_0^f = \exp(\chi_1 E_{-\alpha_1}^{(0)} + \chi_3 E_{-\alpha_3}^{(0)}) \exp(\varphi h_2^{(0)}) \exp(\psi_1 E_{\alpha_1}^{(0)} + \psi_3 E_{\alpha_3}^{(0)}) \quad (2.27)$$

The total effective action is therefore given as

$$S_{eff} = -\frac{k}{2\pi} \int (\varphi \partial \varphi + \chi_3 \partial \chi_3) \mathcal{D} \bar{\psi}_1 \partial \chi_1 + \frac{1}{4} \chi_1 \bar{\psi}_3 \partial \chi_3 e^{-\varphi}$$

$$+ \Delta_i \partial \psi_i \partial \chi_3 + \frac{1}{4} \chi_3 \bar{\psi}_3 \partial \psi_i \partial \chi_1 e^{-\varphi} - V) \quad (2.28)$$

where $\Delta = \frac{1}{2}(1 + \frac{3}{4}(\psi_1 \chi_1 + \psi_3 \chi_3) e^{-\varphi} + \frac{1}{4} \psi_1 \chi_1 \psi_3 \chi_3 e^{-2\varphi})$, $\Delta_i = 1 + \frac{3}{4} \psi_i \chi_i e^{-\varphi}$, $i = 1, 3$ and $V = e^{-2\varphi} + e^{2\varphi}(1 + \psi_3 \chi_3 e^{-\varphi})(1 + \psi_1 \chi_1 e^{-\varphi})$.

The above examples describe non abelian Affine Toda Models characterized by broken conformal invariance and they all allow electrically charged soliton solutions. For the models with one singular structure, their spectra and soliton solutions where explicitly constructed in refs. [3] and [10]. The generalized multi-charged model of example in this subsection provides a generalization of the Lund-Regge model. Its systematic construction, its soliton solutions and spectra are discussed in detail in [11].
3 Zero Curvature and Equations of Motion

The equations of motion for the action (2.8) are [4]

$$\dot{\bar{a}}(B^{-1}\partial B) + [\epsilon_-, B^{-1}\epsilon_+ B] = 0, \quad \bar{a}(\bar{\partial} BB^{-1}) - [\epsilon_+, B\epsilon_- B^{-1}] = 0$$  \hspace{1cm} (3.1)

The subsidiary constraint $J_{Y^l, H(0)} = Tr(B^{-1}\partial B Y^l \cdot H(0))$ and $\bar{J}_{Y^l, H(0)} = Tr(\bar{\partial} BB^{-1} Y^l \cdot H(0)) = 0$ can be consistently imposed since $[Y^l \cdot H(0), \epsilon_\pm] = 0$, $l = 1, \ldots, t \leq p$ as can be obtained from (3.1) by taking the trace with $Y^l \cdot H(0)$. We shall be considering $Y^l \cdot H = \lambda_\alpha \cdot H$, $l = 1, \ldots, t$. In terms of parametrization (2.13), they yield the following system of equations for the nonlocal fields $R_k$

$$\sum_{l=1}^{t}(\lambda_{ak} \cdot \lambda_{al})\partial R_l - \tilde{\psi}_{al} \partial \tilde{\chi}_{al} e^{R_l - \Phi(\alpha_{al})} = 0$$  
$$\sum_{l=1}^{t}(\lambda_{ak} \cdot \lambda_{al})\tilde{\partial} R_l - \tilde{\chi}_{al} \tilde{\psi}_{al} e^{R_l - \Phi(\alpha_{al})} = 0$$  \hspace{1cm} (3.2)

which can be solved in terms of new variables $\psi_{al} = \tilde{\psi}_{al} e^\frac{1}{2} R_l$, $\chi_{al} = \tilde{\chi}_{al} e^\frac{1}{2} R_l$. Since $\lambda_j = K_{jl}^{-1} \alpha_l$, we rewrite (3.2) in matrix form as

$$\sum_{l=1}^{t} \left( K_{aj,al}^{-1} + \frac{1}{2} \psi_{al} \chi_{al} e^{-\Phi(\alpha_{al})} \delta_{aj,al} \right) \partial R_l = \psi_{aj} \partial \chi_{aj} e^{-\Phi(\alpha_{aj})},$$
$$\sum_{l=1}^{t} \left( K_{aj,al}^{-1} + \frac{1}{2} \psi_{al} \chi_{al} e^{-\Phi(\alpha_{al})} \delta_{aj,al} \right) \tilde{\partial} R_l = \chi_{aj} \tilde{\psi}_{aj} e^{-\Phi(\alpha_{aj})}, \quad j = 1, \ldots, t$$  \hspace{1cm} (3.3)

In matrix form, eqns. (3.3) can be rewritten as

$$M_{ij} \partial R_{aj} = N_i, \quad M_{ij} \bar{\partial} R_{aj} = \bar{N}_{tr}^i$$  \hspace{1cm} (3.4)

where $M, N$ and $\bar{N}$ are given in (2.16). The solution is of the form

$$\partial R_{ai} = \left( M^{-1} \right)_{ij} (N)_j, \quad \bar{\partial} R_{ai} = \left( M^{-1} \right)_{ij} \left( \bar{N}^t \right)_j$$

Alternatively, (3.1) admits a zero curvature representation

$$\partial \bar{A} - \bar{\partial} A - [A, \bar{A}] = 0$$  \hspace{1cm} (3.5)

where

$$A = -B\epsilon_- B^{-1}, \quad \bar{A} = \epsilon_+ + \bar{\partial} BB^{-1}$$  \hspace{1cm} (3.6)

Whenever the constraints (3.3) are incorporated into $A$ and $\bar{A}$ in (3.6), equations (3.3) yield the equations of motion of the NA singular Toda models, which coincide with those derived from the general action (2.18).
4 Multi-time Evolution Equations

In this section we review the construction of the cKP hierarchy in terms of the decomposition of zero grade algebra $G_0 = \text{Im} (\text{ad} \epsilon_+) \oplus \text{Ker} (\text{ad} \epsilon_+)$. The time evolution equations are given in terms of the zero curvature representation,

$$\partial_{t_N} A_x - \partial_x A_{t_N} - [A_{t_N}, A_x] = 0, \quad N > 0$$  (4.7)

where $A_x = A_0 + \epsilon_+, A_0 \in G_0$, $A_0 \in \text{Im} (\text{ad} \epsilon_+)$ and

$$A_{t_N} = D_N^{(N)} + D_N^{(N-1)} + \cdots D_N^{(0)}, \quad D_N^{(k)} \in G_k$$  (4.8)

where the upper index denotes the grading, i.e. $D_N^{(j)} \in G_j$. Equation (4.7) can be decomposed according to the grading into the system of algebraic equations

$$[D_N^{(N)}, \epsilon_] = 0$$

$$-\partial_x D_N^{(N)} - [D_N^{(N)}, A_0] - [D_N^{(N-1)}, \epsilon_] = 0$$

$$-\partial_x D_N^{(N-1)} - [D_N^{(N-1)}, A_0] - [D_N^{(N-2)}, \epsilon_] = 0$$

$$\vdots = \vdots$$

$$-\partial_x D_1^{(1)} - [D_1^{(1)}, A_0] - [D_0^{(0)}, \epsilon_] = 0$$

$$\partial_{t_N} A_0 - \partial_x D_0^{(0)} - [D_0^{(0)}, A_0] = 0$$  (4.9)

which can be solved by taking $D_N^{(N)} \in \text{Ker} (\text{ad} \epsilon_+)$. Substituting the result in the second eqn. (4.3) we determine $D_N^{(N)}$ and the component of $D_N^{(N-1)}$ lying in the $\text{Im} (\text{ad} \epsilon_+)$. Substituting in the third eqn. (4.3) we determine the component of $D_N^{(N-1)}$ lying in the $\text{Ker} (\text{ad} \epsilon_+)$ and the component of $D_N^{(N-2)}$ lying in the $\text{Im} (\text{ad} \epsilon_+)$. We solve for $D_N$ by repeating the argument $N$ times until we reach the last eqn. (4.9) from where we determine the component of $D_0^{(0)}$ lying in the $\text{Ker} (\text{ad} \epsilon_+)$ together with the $t_N$-evolution equations for the fields $A_0 \in \text{Im} (\text{ad} \epsilon_+)$. 

As examples of positive grade time evolution let us consider the first nontrivial case $t_N = t_2$.

4.1 AKNS Hierarchy

The $A_1^{(1)}$ AKNS hierarchy is defined by

$$A_0 = q E_\alpha^{(0)} + r E_{-\alpha}^{(0)}, \quad \epsilon_+ = H^{(1)} = \mu \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$$  (4.10)

where $\mu$ denote the spectral parameter. The equations of motion for $t_N = t_2$ in (4.9) are given by the non linear Schroedinger equation,

$$\partial_{t_2} q + \partial_x^2 q - 2rq^2 = 0$$

$$\partial_{t_2} r - \partial_x^2 r + 2r^2q = 0$$  (4.11)
4.2 Yajima-Oikawa Hierarchy

The $A_2^{(1)}$ hierarchy is defined by the following algebraic structure

$$A_0 = qE^{(0)}_{\alpha_1} + rE^{(0)}_{\alpha_2} + Uh^{(0)}_2, \quad \epsilon_+ = E^{(0)}_{\alpha_2} + E^{(1)}_{\alpha_2}$$

(4.12)

and the equations of motion obtained from (4.9) with $t_N = t_2$ are the Yajima-Oikawa equations,

$$\partial_{t_2} q - \partial^2 q = q \partial_x U + qU^2 + rq^2 = 0$$
$$\partial_{t_2} r + \partial^2 r = r \partial_x U - rU^2 - r^2 q = 0$$
$$\partial_{t_2} U + \partial_x (rq) = 0$$

(4.13)

4.3 The $p = t = 2$ $A_3^{(1)}$ Hierarchy

Let $G = SL(4)$ hierarchy defined by

$$A_0 = q_1 E^{(0)}_{\alpha_1} + q_3 E^{(0)}_{\alpha_3} + r_1 E^{(0)}_{\alpha_1} + r_3 E^{(0)}_{\alpha_3} + Uh^{(0)}_2, \quad \epsilon_+ = E^{(0)}_{\alpha_2} + E^{(1)}_{\alpha_2}$$

(4.14)

The $t_2$ time evolution is then written as

$$\partial_{t_2} q_1 - \partial^2 q_1 + q_1 \partial_x U + q_1U^2 + r_1 q_1^2 + r_3 q_1 q_3 = 0$$
$$\partial_{t_2} r_1 + \partial^2 r_1 + r_1 \partial_x U - r_1 U^2 - r_1 q_1^2 + r_3 r_1 q_3 = 0$$
$$\partial_{t_2} q_3 + \partial^2 q_3 - q_3 \partial_x U - q_3U^2 - r_3 q_3^2 - r_1 q_1 q_3 = 0$$
$$\partial_{t_2} r_3 - \partial^2 r_3 - r_3 \partial_x U + r_3 U^2 + r_3 q_3^2 + r_3 r_1 q_1 = 0$$
$$\partial_{t_2} U + \partial_x (q_1 r_1) - \partial_x (r_3 q_3) = 0$$

(4.15)

5 Negative grade Evolution Equations

The relativistic invariant models constructed in section 2 share the same algebraic structure of the Constrained KP (cKP) models discussed in ref. [4]. As explained in the examples, the Lund-Regge and its $SL(3)$ generalization belong to the non linear Schroedinger and the Yajima-Oikawa hierarchies respectively. The $SL(4)$ example of subsections (2.3) and (4.3) also correspond to the same hierarchy. Their field transformation is given by

$$A_0 = \partial BB^{-1}$$

(5.1)

and its soliton solution are related. In fact the Leznov-Saveliev equations (3.1) correspond to zero curvature equation for $t_{-1} = z$. It was shown in [3] that the general negative “time” hierarchy is given by

$$\partial_{t_{-N}} A_z - \partial_z A_{t_{-N}} - [A_{t_{-N}}, A_z] = 0, \quad N > 0$$

(5.2)

where

$$A_{t_{-N}} = D^{(-N)}_N + D^{(-N+1)}_N + \cdots + D^{(-1)}_N, \quad D^{(-k)}_N \in G_{-k}$$

(5.3)

For $N = 1$, a general solution is given in closed form by

$$D^{(-1)}_1 = -B\epsilon B^{-1}$$

(5.4)
6 One Soliton Solutions

In this section we discuss the relation between the one soliton solutions within the same hierarchy. Let us first consider the $Sl(2)$ case of the Lund-Regge equations of motion (2.22) with solution given by

\[ \psi = \frac{be^{\frac{ab}{2} \bar{\gamma} t + \bar{\gamma} x}}{1 + \Gamma e^{\frac{1}{2}(\frac{1}{\bar{\gamma} + \frac{1}{\bar{\gamma} - \bar{\gamma}})(\bar{\gamma} - x)}}, \quad \chi = \frac{ae^{\frac{1}{2} - \bar{\gamma} t}}{1 + \Gamma e^{\frac{1}{2}(\frac{1}{\bar{\gamma} + \frac{1}{\bar{\gamma} - \bar{\gamma}})(\bar{\gamma} - x)}}}, \]

where $\Gamma = \frac{ab}{(\bar{\gamma} - \bar{\gamma})}$, $z = t - 1$, $\bar{\varepsilon} = x$

The one soliton solution of the nonlinear Schrödinger equation (1.11) is given by

\[ r = -\frac{a\gamma_1 e^{-\gamma_1^2 t_2 - \gamma_1 x}}{1 + \frac{ab\gamma_1}{(\bar{\gamma} - \bar{\gamma})} e^{(\gamma_1^2 - \bar{\gamma}) t_2 + (\bar{\gamma} - \gamma_1) x}}, \quad q = \frac{b\gamma_2 e^{-\gamma_2^2 t_2 + \gamma_2 x}}{1 + \frac{ab\gamma_2}{(\bar{\gamma} - \bar{\gamma})} e^{(\gamma_2^2 - \bar{\gamma}) t_2 + (\bar{\gamma} - \gamma_2) x}} \]

The solutions agree when we parametrize $q$ and $r$ in terms of $\psi$ and $\chi$ defined as in (5.1), i.e.

\[ q = \frac{\partial \psi}{\Delta} e^R, \quad r = \bar{\partial} \chi e^{-R} \]

and $\frac{1}{\bar{\gamma}_i} \to \gamma_i^2 t_2$, $i = 1, 2$, $\bar{\varepsilon} \to x$.

For the $Sl(3)$ case the solution of the relativistic model was given in ref. [3] as

\[ e^{\frac{2}{\Delta} R} = \frac{1 + \frac{ab\rho_1 \rho_2}{(1 - \gamma_1)(1 - \gamma_2)}}{1 + \frac{ab\rho_1 \rho_2 \gamma_1^2}{(1 - \gamma_1)(1 - \gamma_2)}}, \quad e^{\varphi} = \frac{1 + \frac{ab\rho_1 \rho_2 \gamma_1^2}{(1 - \gamma_1)(1 - \gamma_2)}}{1 + \frac{ab\rho_1 \rho_2 \gamma_1^2}{(1 - \gamma_1)(1 - \gamma_2)} (A + ab\rho_1 \rho_2 \gamma_1^2)^{1/2}} 
\]

\[ \chi = \frac{A\rho_1}{(A + ab\rho_1 \rho_2)^{1/4} (A + ab\rho_1 \rho_2 \gamma_1^2)^{3/4}}, \quad \psi = \frac{Ab\rho_2}{(A + ab\rho_1 \rho_2)^{1/4} (A + ab\rho_1 \rho_2 \gamma_1^2)^{3/4}} \]

where

\[ \rho_1 = e^{-\frac{1}{\bar{\gamma}_1 + \bar{\gamma}_1}}, \quad \rho_2 = e^{-\frac{1}{\bar{\gamma}_2 + \bar{\gamma}_2}}, \quad \gamma_1, 2 = \gamma_1 / \gamma_2, \quad A = (1 - \gamma_1)(1 - \gamma_2) \]

It therefore follows from (5.1) that

\[ q = \frac{\partial \psi}{\Delta} e^\frac{1}{\Delta} R - \varphi, \quad r = \left( \bar{\partial} \chi - \chi \bar{\partial} \varphi - \frac{1}{4} \chi \frac{\partial \psi}{\Delta} e^{-\varphi} \right) e^\frac{1}{\Delta} R, \quad U = \bar{\partial} \varphi + \frac{\chi}{\Delta} \frac{\partial \psi}{\Delta} e^{-\varphi} \]

where $\Delta = 1 + \frac{1}{4} \psi \chi e^{-\varphi}$. 

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Substituting (6.4) in (6.6) we find that the relativistic solution agrees with the solution of the Yajima-Oikawa model given in [12]

\[
q = - \frac{b \gamma_1 e^{\gamma_2^2 t_2 - x \gamma_2}}{1 + \frac{ab \gamma_2 \gamma_1 \epsilon^{(\gamma_1 - \gamma_2)^2 t_2 (\gamma_1^2 - \gamma_2^2)}}{(\gamma_1 + \gamma_2)(\gamma_1 - \gamma_2)^2}}, \quad r = \frac{a \gamma_1 e^{-\gamma_1^2 t_2 + x \gamma_1}}{1 + \frac{ab \gamma_2 \gamma_1 \epsilon^{(\gamma_1 - \gamma_2)^2 t_2 (\gamma_1^2 - \gamma_2^2)}}{(\gamma_1 + \gamma_2)(\gamma_1 - \gamma_2)^2}},
\]

\[
U = - \partial_x \ln \left( \frac{1 + \frac{ab \gamma_2 \gamma_1 \epsilon^{(\gamma_1 - \gamma_2)^2 t_2 (\gamma_1^2 - \gamma_2^2)}}{(\gamma_1 + \gamma_2)(\gamma_1 - \gamma_2)^2}}{1 + \frac{ab \gamma_2 \gamma_1 \epsilon^{(\gamma_1 - \gamma_2)^2 t_2 (\gamma_1^2 - \gamma_2^2)}}{(\gamma_1 + \gamma_2)(\gamma_1 - \gamma_2)^2}} \right)
\]

(6.7)

after the substitution \( \frac{z}{\gamma_i} \rightarrow \gamma_i^2 t_2, \ i = 1, 2, \ \bar{z} \rightarrow x \).

7 Concluding Remarks

The relation between certain relativistic non abelian Toda models constructed in this notes as the negative flows of the constrained KP (generalized non linear Schroedinger, Yajima-Oikawa, etc) non relativistic models is shown to generalize the well known relation between the sine-Gordon and mKdV or the Lund-Regge and non linear Schroedinger models. The common algebraic structure is the key to propose negative grade hierarchies containing the relativistic equations as the first negative flows. An interesting question that naturally arises concerns the relationship between the conserved charges. In order to illustrate that let us consider for instance the Yajima-Oikawa model where the use of eqns. of motion (4.13) yields

\[
\partial_{t_2} \left( U^2 + rq \right) = \partial_x \left( r \partial_x q - q \partial_x r - 2rqU \right)
\]

(7.1)

and hence the first hamiltonian density

\[
H_{non \ rel} = rq + U^2
\]

(7.2)

Substituting (5.6) in (7.1) we find that the hamiltonian density of the \( SL(3), p = 1 \) model described in subsection II.B,

\[
H_{rel} = \frac{\bar{\partial} \psi \bar{\partial} \chi}{\Delta} e^{-\varphi} + (\bar{\partial} \varphi)^2, \quad \Delta = 1 + \frac{3}{4} \psi \chi e^{-\varphi}
\]

(7.3)

is also conserved due to eqns. of motion (2.25), i.e.

\[
\partial_{t_{-1}} H_{rel} = \bar{\partial} \left( \psi \chi e^{-\varphi} \right)
\]

(7.4)

with the identification \( t_{-1} = z, x = \bar{z} \). Similar relations can be found for higher grade equations within the same hierarchy.

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