N-FIBER-FULL MODULES

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ABSTRACT. Let $A$ be a Noetherian flat $K[t]$-algebra, $h$ an integer and let $N$ be a graded $K[t]$-module, we introduce and study “$N$-fiber-full up to $h$” $A$-modules. We prove that an $A$-module $M$ is $N$-fiber-full up to $h$ if and only if $\Ext^i_A(M,N)$ is flat over $K[t]$ for all $i \leq h - 1$. And we show some applications of this result extending the recent result on square-free Gröbner degenerations by Conca and Varbaro.

1. INTRODUCTION

Throughout this paper, $A$ is a Noetherian flat $K[t]$-algebra, $M$ and $N$ are finitely generated $A$-modules which are flat over $K[t]$, and all of $A$, $M$ and $N$ are graded $K[t]$-modules. At the “CIME-CIRM Course on Recent Developments in Commutative Algebra” conference in 2019, Matteo Varbaro introduced the notion of “fiber-full modules” providing a new proof of the main result of [2]. The starting point of this paper is to find some possible generalizations of this concept. We recall that $M$ is a fiber-full $A$-module if, for any $m \in N_{>0}$, the natural projection $M/t^m M \to M/tM$ induces injective maps $\Ext^i_A(M/tM,A) \to \Ext^i_A(M/t^m M,A)$ for all $i \in \mathbb{Z}$. And one of the “most important” properties of fiber-full modules is that being fiber-full is related to the flatness over $K[t]$ of every $\Ext^i_A(M,A)$ [9]. We notice that if $N$ is a graded $K[t]$-module, the induced map $\Ext^0_A(M/tM,N) \to \Ext^0_A(M/t^m M,N)$ is always injective. Motivated by this, let us introduce the “$N$-fiber-full up to $h$” modules:

Definition 1.1. Let $h$ be an integer. We say that $M$ is $N$-fiber-full up to $h$ as an $A$-module if, for any $m \in N_{>0}$, the natural projection $M/t^m M \to M/tM$ induces injective maps $\Ext^i_A(M/tM,A) \to \Ext^i_A(M/t^m M,A)$ for all $i \leq h$. An important question is: if $M$ is $N$-fiber-full up to $h$ as an $A$-module, can we obtain the flatness of some $\Ext^i_A(M,N)$? The main theorem of this paper goes in this direction.

Main Theorem(cf. Theorem 2.10). Let $h$ be an integer. $M$ is $N$-fiber-full up to $h$ as an $A$-module if and only if $\Ext^i_A(M,N)$ is flat over $K[t]$ for all $i \leq h - 1$.

To prove this theorem, the most difficult part is the proof of Lemma 2.5 below, concerning the equivalence between two properties 1) and 2). At the beginning, I thought one could prove it using a way similar to the proof of Lemma 3.5 in [9], actually the proof of implication 1 $\Rightarrow$ 2) is done in this way. But the proof of the other implication of this lemma is the hardest point of this paper: the whole section 2 is to prove Lemma 2.5 and the main theorem. In section 3, we will talk about some applications of $N$-fiber-full modules. A main consequence, as we will see, is that the notion “$N$-fiber-full up to $h$” allows us to infer interesting results whenever the special fiber $M/tM$ has “nice” properties after removing primary components of big height. For example,
Theorem (cf. Theorem 3.5). Let $S$ be the polynomial ring $K[X_1, \ldots, X_n]$ over a field $K$, let $I \subseteq S$ be a homogeneous ideal. Fixing a monomial order on $S$, we denote by $\text{in}(I)$ the initial ideal of $I$ with respect to this monomial order. If $I$ is such that $\text{in}(I)^{\text{sat}}$ is square-free, then

$$\dim_K H^i_m(S/I)_j = \dim_K H^i_m(S/\text{in}(I))_j$$

for all $i \geq 2$ and for all $j \in \mathbb{Z}$.

Equivalently,

Theorem. Let $\mathbb{P}^n$ be the $n$-dimensional projective space over a field $K$, let $X \subseteq \mathbb{P}^n$ be a projective scheme and let $I$ be a homogeneous ideal of $S = K[X_0, \ldots, X_n]$ such that $X = \text{Proj}(S/I)$. Fix a monomial order $<$ on $S$ and assume that $\text{in}(X) = \text{Proj}(S/\text{in}(I))$ is reduced, where $\text{in}(I)$ is the initial ideal of $I$ with respect to $<$. Then

$$\dim_K H^i(X, \mathcal{O}_X(j)) = \dim_K H^i(\text{in}(X), \mathcal{O}_{\text{in}(X)}(j))$$

for all $i > 0$ and for all $j \in \mathbb{Z}$.

This theorem has already been announced by Varbaro in his paper [11] (Theorem 4.4) without writing the complete proof. By a private communication, Varbaro told me that he realized later that it was not clear to him how to extend the proof given in [9] for the same reasons explained above. But we will show that we can complete the proof of a very general version of this theorem using Theorem 2.10 of this paper.

2. Definition and Properties of $N$-fiber-full Modules

In this section we prove some basic properties of $N$-fiber-full modules, in order to show the equivalence between being $N$-fiber-full up to $h$ and the flatness of modules $\text{Ext}_A^i(M, N)$ with $i \leq h - 1$.

Remark 2.1. If $A$ is a Cohen-Macaulay complete local ring, $N$ is the canonical module of $A$ and $M = A/I$ with $I$ an ideal of $A$, then $M$ is $N$-fiber-full up to $\dim A$ is equivalent to saying that $t$ is a surjective element in $M$ in the sense of [3] section 3.1.

If $A$ is a local ring and $N = A$, then $M$ is $N$-fiber-full up to $h$ for each $h \in \mathbb{N}$ is equivalent to saying that $M$ is fiber-full defined as in [9] Definition 3.8.

One implication of the main theorem is not difficult to prove, but it is very useful because we need it to prove the other one.

Theorem 2.2. Let $h$ be an integer and let $\text{Ext}_A^i(M, N)$ be flat over $K[t]$ for all $i \leq h - 1$. Then $M$ is $N$-fiber-full up to $h$ as an $A$-module.

Proof. If $\text{Ext}_A^i(M, N)$ is flat over $K[t]$ for all $i \leq h - 1$, then

$$\text{Ext}_A^i(M, N) \xrightarrow{t^{m-1}} \text{Ext}_A^i(M, N)$$

is injective for all $i \leq h - 1$ and for all $m \in \mathbb{N}_{>0}$, hence

$$\text{Ext}_A^{i-1}(M, N) \xrightarrow{t^{m-1}} \text{Ext}_A^{i-1}(M, N)$$
is injective for all \( i \leq h \) and for all \( m \in \mathbb{N}_{>0} \). The commutative diagram of \( A \)-modules with exact rows

\[
\begin{array}{ccc}
0 & \longrightarrow & M \\
& \downarrow{t^m} & \downarrow{t^{m-1}} \\
0 & \longrightarrow & M \\
& \uparrow{t^{m-1}} & \uparrow{t^m} \\
& \longrightarrow & M/tM \\
\end{array}
\]

yields the following commutative diagram of \( A \)-modules with exact rows

\[
\begin{array}{ccc}
\text{Ext}_A^{i-1}(M, N) & \longrightarrow & \text{Ext}_A^{i-1}(M, N) \\
\downarrow{t^m} & \downarrow{t^{m-1}} & \downarrow{t^m} \\
\text{Ext}_A^{i-1}(M, N) & \longrightarrow & \text{Ext}_A^i(M/tM, N) \\
\end{array}
\]

\[
\begin{array}{ccc}
& \longrightarrow & \text{Ext}_A^i(M, N) \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Ext}_A^{i-1}(M, N) & \longrightarrow & \text{Ext}_A^i(M/tM, N) \\
\text{Ext}_A^{i-1}(M, N) & \longrightarrow & \text{Ext}_A^i(M/t^mM, N) \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Ext}_A^i(M, N) \\
\end{array}
\]

Since

\[
\text{Ext}_A^{i-1}(M, N) \xrightarrow{t^{m-1}} \text{Ext}_A^{i-1}(M, N)
\]

is injective for all \( i \leq h \), by the proof of the Five Lemma we have that

\[
\text{Ext}_A^i(M/tM, N) \longrightarrow \text{Ext}_A^i(M/t^mM, N)
\]

is injective for all \( i \leq h \).

\[ \square \]

**Remark 2.3.** The map \( M/t^kM \xrightarrow{t^{l-k}} M/t^lM \) is injective for each \( k, l \in \mathbb{N}_{>0} \) such that \( l \geq k \).

**Proof.** If \( x \in M \) is such that \( \overline{t^{l-k}x} = 0 \) in \( M/t^lM \), then there exists \( y \in M \) such that \( t^l y = t^{l-k} x \), hence \( t^{l-k}(x - t^k y) = 0 \). Since \( M \) is flat over \( K[t] \), we have \( x = t^k y \), it follows that \( \overline{x} = 0 \) in \( M/t^kM \).

\[ \square \]

**Lemma 2.4.** The following are equivalent:

1. \( M \) is N-fiber-full up to \( h \).
2. For each \( k, l \in \mathbb{N}_{>0} \) such that \( l \geq k \), the short exact sequence

\[
0 \longrightarrow M/t^kM \xrightarrow{t^{l-k}} M/t^lM \longrightarrow M/t^{l-k}M \longrightarrow 0
\]

induces a short exact sequence

\[
0 \longrightarrow \text{Ext}_A^i(M/t^{l-k}M, N) \longrightarrow \text{Ext}_A^i(M/t^lM, N) \xrightarrow{f_{k,l}^i} \text{Ext}_A^i(M/t^kM, N) \longrightarrow 0
\]

for all \( i \leq h - 1 \).
3. For each \( k, l \in \mathbb{N}_{>0} \) such that \( l \geq k \), the natural projection \( M/t^lM \longrightarrow M/t^kM \) induces injective maps \( \text{Ext}_A^i(M/t^kM, N) \longrightarrow \text{Ext}_A^i(M/t^lM, N) \) for all \( i \leq h \).

Furthermore, up to the identifications

\[
t^{l-k} \text{Ext}_A^i(M/t^lM, N) \cong \text{Ext}_A^i(M/t^kM, N) \quad \text{for each } i \leq h - 1,
\]

the map \( f_{k,l}^i \) in 2) corresponds to the surjective map

\[
\text{Ext}_A^i(M/t^lM, N) \xrightarrow{t^{l-k}} t^{l-k} \text{Ext}_A^i(M/t^lM, N)
\]

for each \( i \leq h - 1 \).
Proof. In the case $k = l$, 2) and 3) are always true, so we suppose $k < l$.

1 $\Rightarrow$ 2) We consider the short exact sequence

$$0 \rightarrow M/t^k M \xrightarrow{d} M/t^{k+1} M \rightarrow M/t M \rightarrow 0,$$

it induces a long exact sequence

$$\cdots \rightarrow \Ext^i_A(M/t^k M, N) \rightarrow \Ext^i_A(M/t M, N) \rightarrow \Ext^i_A(M/t^{k+1} M, N) \rightarrow \cdots.$$ 

Since $M$ is $N$-fiber-full up to $h$, $\Ext^i_A(M/t M, N) \rightarrow \Ext^i_A(M/t^{k+1} M, N)$ is injective for each $i \leq h$, it follows that $f^j_{k,k+1}$ is surjective for each $j \leq h - 1$.

Hence $f^j_{k,l} = f^j_{k,k+1} \circ \cdots \circ f^j_{l-1,l}$ is surjective for each $j \leq h - 1$. Therefore

$$0 \rightarrow \Ext^j_A(M/t^{-k} M, N) \rightarrow \Ext^j_A(M/t^l M, N) \xrightarrow{f^j_{k,l}} \Ext^j_A(M/t^k M, N) \rightarrow 0$$

is exact for all $j \leq h - 1$.

2 $\Rightarrow$ 3) The short exact sequence

$$0 \rightarrow M/t^{-k} M \xrightarrow{d^k} M/t^l M \rightarrow M/t^k M \rightarrow 0$$

induces a long exact sequence

$$\cdots \rightarrow \Ext^i_A(M/t^{-k} M, N) \rightarrow \Ext^{i+1}_A(M/t^k M, N) \rightarrow \Ext^{i+1}_A(M/t^l M, N) \rightarrow \cdots.$$ 

By 2) $f^i_{k,l}$ is surjective for all $i \leq h - 1$, it follows that

$$\Ext^i_A(M/t^k M, N) \rightarrow \Ext^i_A(M/t^l M, N)$$

is injective for all $j \leq h$.

3 $\Rightarrow$ 1) Take $k = 1$.

Furthermore, we observe that the endomorphism

$$M/t^l M \xrightarrow{d^l} M/t^l M$$

corresponds to the composition of maps

$$M/t^l M \xrightarrow{p} M/t^k M \xrightarrow{d^{l-k}} M/t^l M$$

where $p$ is the natural projection, therefore the following diagram

$$\begin{array}{ccc}
\Ext^i_A(M/t^l M, N) & \xrightarrow{d^{l-k}} & \Ext^i_A(M/t^l M, N) \\
& f^i_{k,l} & \downarrow \\
& \Ext^i_A(M/t^k M, N) & \\
\end{array}$$

is commutative. By 3) $\Ext^i_A(M/t^k M, N) \rightarrow \Ext^i_A(M/t^l M, N)$ is injective for all $i \leq h - 1$, and by 2) $f^i_{k,l}$ is surjective for each $i \leq h - 1$. We have that for each $i \leq h - 1$,

$$\Ext^i_A(M/t^k M, N) \cong d^{l-k} \Ext^i_A(M/t^l M, N)$$

is a submodule of $\Ext^i_A(M/t^l M, N)$, and $f^i_{k,l}$ corresponds to the surjective map

$$\Ext^i_A(M/t^l M, N) \xrightarrow{d^{l-k}} t^{l-k} \Ext^i_A(M/t^l M, N).$$
The next lemma is crucial to show Theorem 2.10. The most difficult implication 2 \(\Rightarrow\) 1) surprisingly uses Theorem 2.2.

**Lemma 2.5.** Let \(h\) be an integer. The following are equivalent:

1) \(\text{Ext}^i_A(M, N)\) is flat over \(K[t]\) for all \(i \leq h\).

2) \(\text{Ext}^i_{A/t^m A}(M/t^m M, N/t^m N)\) is flat over \(K[t]/(t^m)\) for all \(m \in \mathbb{N}_{>0}\) and for all \(i \leq h-1\).

**Proof.** 1 \(\Rightarrow\) 2): Since \(N\) is flat over \(K[t]\), there is a short exact sequence

\[ 0 \rightarrow N \overset{t^m}{\rightarrow} N \rightarrow N/t^m N \rightarrow 0. \]

Consider the induced long exact sequence of \(\text{Ext}^i_A(M, -)\):

\[ \ldots \rightarrow \text{Ext}^i_A(M, N) \overset{t^m}{\rightarrow} \text{Ext}^i_A(M, N) \rightarrow \text{Ext}^i_A(M, N/t^m N) \rightarrow \ldots \]

Since \(\text{Ext}^i_A(M, N)\) is flat over \(K[t]\) for all \(k \leq h\), \(t^m\) is an \(\text{Ext}^i_A(M, N)\)-regular element for all \(m \in \mathbb{Z}_{+}\), and so we obtain a short exact sequence

\[ 0 \rightarrow \text{Ext}^i_A(M, N) \overset{t^m}{\rightarrow} \text{Ext}^i_A(M, N) \rightarrow \text{Ext}^i_A(M, N/t^m N) \rightarrow 0 \]

for all \(i \leq h-1\). It follows that

\[ \text{Ext}^i_A(M, N/t^m N) \cong \frac{\text{Ext}^i_A(M, N)}{t^m \text{Ext}^i_A(M, N)} \]

for all \(i \leq h-1\). Furthermore, using again 1) we have that \(\text{Ext}^i_A(M, N)/t^m \text{Ext}^i_A(M, N)\) is flat over \(k[t]/(t^m)\) for \(i \leq h\) (see [3] Section 7). Therefore,

\[ \text{Ext}^i_{A/t^m A}(M/t^m M, N/t^m N) \cong \text{Ext}^i_A(M, N/t^m N) \]

is flat over \(k[t]/(t^m)\) for all \(i \leq h-1\).

1 \(\Leftarrow\) 2): We use induction on \(h \geq 0\). If \(h = 0\), we consider the long exact sequence

\[ 0 \rightarrow \text{Hom}_A(M/tM, N) \rightarrow \text{Hom}_A(M, N) \overset{t}{\rightarrow} \text{Hom}_A(M, N) \rightarrow \ldots, \]

induced by the short exact sequence \(0 \rightarrow M \overset{t^l}{\rightarrow} M \rightarrow M/t^l M \rightarrow 0\). Since \(N\) is flat over \(K[t]\), we have \(\text{Hom}_A(M/tM, N) = 0\), and it follows that the map \(\text{Hom}_A(M, N) \overset{t}{\rightarrow} \text{Hom}_A(M, N)\) is injective. Hence 1) is true.

If \(h \geq 1\), we suppose that \(\text{Ext}^i_{A/t^m A}(M/t^m M, N/t^m N)\) is flat over \(K[t]/(t^m)\) for all \(m \in \mathbb{N}_{>0}\) and for all \(i \leq h-1\). By the inductive hypothesis \(\text{Ext}^i_A(M, N)\) is flat over \(K[t]\) for all \(i \leq h-1\). Furthermore, \(M\) is \(N\)-fiber-full up to \(h\) by Theorem 2.2. We prove that \(\text{Ext}^i_A(M, N)\) is flat over \(K[t]\). By contradiction, suppose that there exists \(x \in \text{Ext}^i_A(M, N), x \neq 0\) and there exists \(k \in \mathbb{N}\) such that \(t^k x = 0\). Let \(l\) be an integer such that \(l > k\). The commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & M & \overset{t^l}{\rightarrow} & M & \rightarrow & M/t^l M & \rightarrow & 0 \\
& & \downarrow t^k & & \downarrow t^k & & \downarrow t^k & & \\
0 & \rightarrow & M & \overset{t^{l-k}}{\rightarrow} & M & \rightarrow & M/t^{l-k} M & \rightarrow & 0
\end{array}
\]
yields the following commutative diagram of $A$-modules with exact rows

$$\begin{array}{cccc}
\Ext^h_A(M, N) & \xrightarrow{\psi} & \Ext^h_A(M/t^l M, N) & \xrightarrow{\phi} & \Ext^h_A(M, N) \\
& / & f^h_{l-k,l} & / & \\
\Ext^h_A(M, N) & \xrightarrow{\eta} & \Ext^h_A(M/t^{l-k} M, N) & \xrightarrow{\zeta} & \Ext^h_A(M, N) \\
& / & t^k & / & \\
& \Ext^h_A(M, N) & \xrightarrow{d^h} & \Ext^h_A(M, N). \\
\end{array}$$

Since $t^k x = 0$, $t > k$, we have $t^l x = 0$, hence there exists $y \in \Ext^h_A(M/t^l M, N)$, $y \neq 0$ such that $\phi(y) = x$. We may suppose that $f^h_{l-k,l}(y) = 0$. Indeed, if $f^h_{l-k,l}(y) \neq 0$, since $\zeta(f^h_{l-k,l}(y)) = t^k(\phi(y)) = t^k x = 0$, there exists $z \in \Ext^h_A(M, N)$ such that $\eta(z) = f^h_{l-k,l}(y)$. Set $y' = y - \psi(z)$. We have

$$\phi(y') = \phi(y) - \phi(\psi(z)) = x$$

and

$$f^h_{l-k,l}(y') = f^h_{l-k,l}(y) - f^h_{l-k,l}(\psi(z)) = f^h_{l-k,l}(y) - \eta(z) = 0.$$

Hence, using Lemma 2.4, we may suppose that $t^k y = 0$ in $\Ext^h_A(M/t^l M, N)$. Furthermore, since $\Ext^h_A(M/t^l M, N) \cong \Ext^{h-1}_{A/t_{l}A}(M/t^l M, N/t^l N)$ (see [1] Lemma 3.1.16), there exists $[\varphi] \in \Ext^{h-1}_{A/t_{l}A}(M/t^l M, N/t^l N)$, $[\varphi] \neq 0$ such that $t^l [\varphi] = 0$.

Since $M$ is $N$-fiber-full up to $h$, the natural map

$$\Ext^h_A(M/t^l M, N) \longrightarrow \Ext^h_A(M/t^m M, N)$$

is injective for each $m \geq l$ by Lemma 2.4. Hence $[\varphi] \in \Ext^{h-1}_{A/t_{m}A}(M/t^m M, N/t^m N)$ for each $m \geq l$ and $t^l [\varphi] = 0$ in $\Ext^{h-1}_{A/t_{m}A}(M/t^m M, N/t^m N)$ for each $m \geq l$. Now it is enough to find a positive integer $m \geq l$ such that $[\varphi] \not\in t^{m-k}\Ext^{h-1}_{A/t_{m}A}(M/t^m M, N/t^m N)$. Let us take an $A$-free resolution $F_\bullet$ of $M$, and let $(G_\bullet, \partial_\bullet)$ be the complex

$$\Hom_A(F_\bullet, N),$$

so that $\Ext^i_A(M, N)$ is the $i$-th cohomology module of $G_\bullet$. Since $M$ and $A$ are flat over $K[t]$, $F_\bullet/t^m F_\bullet$ is an $A/t^m A$-free resolution of $M/t^m M$ for all $m \in \mathbb{N}_{>0}$. Let $(\overline{G_\bullet}, \partial_\bullet)$ denote the complex

$$\Hom_{A/t^m A}(F_\bullet/t^m F_\bullet, N/t^m N),$$

so that $\Ext^i_{A/t^m A}(M/t^m M, N/t^m N)$ is the $i$-th cohomology module of $\overline{G_\bullet}$ and $\pi^\bullet$ the natural map of complexes from $G_\bullet$ to $\overline{G_\bullet}$. Since $F_{h-1}$ is free and finitely generated, there exists $\beta_{h-1} \in \mathbb{N}$ such that $F_{h-1} = A^{\beta_{h-1}}$, hence

$$G^{h-1} = \Hom_A(F_{h-1}, N) = \Hom_A(A^{\beta_{h-1}}, N) \cong N^{\beta_{h-1}}$$

and

$$\overline{G^{h-1}} = \Hom_{A/t^m A}(F_{h-1}/t^m F_{h-1}, N/t^m N) \cong (N/t^m N)^{\beta_{h-1}}.$$
therefore \( \pi_{h^{-1}} : G^{h^{-1}} \rightarrow \overline{G^{h^{-1}}} \) is surjective. It follows that there exists \( \delta \in G^{h^{-1}} \) such that \( \pi_{h^{-1}}(\delta) = \varphi \).

\[
G^{h^{-1}} \xrightarrow{\partial^{h^{-1}}} G^{h} \xrightarrow{\partial^{h}} G^{h+1} \\
\downarrow \pi^{h^{-1}} \downarrow \pi^{h} \\
\overline{G^{h^{-1}}} \xrightarrow{\overline{\partial^{h^{-1}}}} \overline{G^{h}}
\]

Set \( w = \partial^{h^{-1}}(\delta) \in G^{h} \). Since \( \overline{\partial^{h^{-1}}}(\varphi) = 0 \), then

\[
\pi^{h}(w) = 0 \in \overline{G^{h}} = \text{Hom}_{A/tmA}(F_{h}/t^{m}F_{h}, N/t^{m}N) \\
\cong \text{Hom}_{A}(F_{h}, N)/t^{m}\text{Hom}_{A}(F_{h}, N).
\]

So there exists \( w' \in G^{h} \) such that \( w = t^{m}w' \). Hence \( w \in t^{m}G^{h} \) for all \( m \gg 0 \). Using Krull’s intersection theorem we have that \( \partial^{h^{-1}}(\delta) = w = 0 \), thus \( [\delta] \in \text{Ext}^{h^{-1}}_{A}(M, N) \). Since \( [\varphi] \neq 0 \), \( \varphi \not\in \text{Im}(\partial^{h^{-2}}) \). If \( \varphi = \pi^{h^{-1}}(\delta) \in t^{m-k}\ker(\overline{\partial^{h^{-1}}}) \),

\[
\delta \in t^{m-k}\ker(\partial^{h^{-1}}) + t^{m}G^{h^{-1}} \subseteq t^{m-k}G^{h^{-1}}.
\]

But \( \delta \neq 0 \), so \( \varphi \not\in t^{m-k}\ker(\overline{\partial^{h^{-1}}}) \) for all \( m \gg 0 \) by Krull’s intersection theorem. Therefore, there exists \( m \) such that

\[
t^{k}[\varphi] = 0
\]

and

\[
[\varphi] \not\in t^{m-k}\text{Ext}^{h^{-1}}_{A}(M/t^{m}M, N/t^{m}N),
\]

but this contradicts the fact that \( \text{Ext}^{h^{-1}}_{A}(M/t^{m}M, N/t^{m}N) \) is flat over \( K[t]/(t^{m}) \).

\[
\square
\]

**Notations 2.6.** We introduce some notations which are useful in the following results.

- \( A_{m} = A/t^{m}A \) for each \( m \in \mathbb{Z}_{+} \).
- \( M_{m} = M/t^{m}M \) for each \( m \in \mathbb{Z}_{+} \).
- \( N_{m} = N/t^{m}N \) for each \( m \in \mathbb{Z}_{+} \).
- \( \iota_{j} : t^{j+1}M_{m} \rightarrow t^{j}M_{m} \) the natural inclusion for each \( m \in \mathbb{Z}_{+} \) and for each \( j \in \mathbb{N} \).
- \( \mu_{j} : t^{j}M_{m} \rightarrow t^{m-1}M_{m} \) the multiplication by \( t^{m-1-j} \) for each \( m \in \mathbb{Z}_{+} \) and for each \( j \in \mathbb{N} \), \( j \leq m-1 \).
- \( E_{m}^{i}(-) \) the contravariant functor \( \text{Ext}^{i}_{A}(M, N_{m}) \) for each \( i \in \mathbb{N} \).

**Remark 2.7.** Suppose that \( k \in \mathbb{Z}_{+} \). Since \( t^{m} \) is an \( A \)- and \( N \)-regular element for all \( m \in \mathbb{Z}_{+} \) and \( t^{m}M_{k} = 0 \) for all \( m \geq k \), we have that

\[
\text{Ext}^{i+1}_{A}(M_{k}, N) \cong \text{Ext}^{i}_{A/tmA}(M_{k}, N/t^{m}N) = E_{m}^{i}(M_{k})
\]

for all \( i \in \mathbb{N} \) and for all \( m \geq k \) by Lemma 3.1.16 in [1]. In particular,

\[
\text{Ext}^{i+1}_{A}(M_{k}, N) \cong E_{k}^{i}(M_{k})
\]

for all \( i \in \mathbb{N} \). Therefore,

\[
E_{k}^{i}(M_{k}) \cong \text{Ext}^{i+1}_{A}(M_{k}, N) \cong E_{m}^{i}(M_{k})
\]

for all \( i \in \mathbb{N} \) and for all \( k \leq m \).
Remark 2.8. The following map

\[ M_i \rightarrow t^{m-j}M_m \]

is an isomorphism for all \( j \leq m \).

**Proof.** If \( j = m \), it is trivial. We suppose that \( j < m \). It is clear that the above map is surjective. If \( t^{m-j}m \in t^m M \), then there exists \( n \in M \) such that \( t^{m-j}(m - t^n n) = t^{m-j}m - t^m n = 0 \). Since \( t^{m-j} \) is an \( M \)-regular element for all \( j < m \), \( m = t^n n \in t^j M \). It follows that the above map is injective. \( \square \)

Remark 2.9. Let \( h \) be an integer and let \( j \leq m - 1 \). If \( M \) is \( N \)-fiber-full up to \( h \), then the following short exact sequence

\[ 0 \rightarrow t^{j+1}M_m \xrightarrow{\iota_j} t^j M_m \xrightarrow{\mu_j} t^{m-1}M_m \rightarrow 0 \]

yields a short exact sequence

\[ 0 \rightarrow E^i_m(t^{m-1}M_m) \xrightarrow{E^i_m(\mu_j)} E^i_m(t^j M_m) \xrightarrow{E^i_m(\iota_j)} E^i_m(t^{j+1}M_m) \rightarrow 0 \]

for all \( i \leq h - 2 \).

**Proof.** It is easy to check that \( 0 \rightarrow t^{j+1}M_m \xrightarrow{\iota_j} t^j M_m \xrightarrow{\mu_j} t^{m-1}M_m \rightarrow 0 \) is a short exact sequence and, using the identifications in Remark 2.8, the following diagram is commutative

\[ \begin{array}{cccccc}
0 & \xrightarrow{\iota_j} & t^j M_m & \xrightarrow{\mu_j} & t^{m-1}M_m & \rightarrow 0 \\
0 & \xrightarrow{\iota_j} & t^j M_m & \xrightarrow{\mu_j} & t^{m-1}M_m & \rightarrow 0
\end{array} \]

Applying \( E^i_m(-) \) on the short exact sequence and the above diagram we obtain a long exact sequence

\[ \ldots \rightarrow E^i_m(t^{m-1}M_m) \xrightarrow{E^i_m(\mu_j)} E^i_m(t^j M_m) \xrightarrow{E^i_m(\iota_j)} E^i_m(t^{j+1}M_m) \rightarrow E^{i+1}_m(t^{m-1}M_m) \xrightarrow{E^{i+1}_m(\mu_j)} E^{i+1}_m(t^j M_m) \rightarrow \ldots \]

and a commutative diagram

\[ \begin{array}{ccc}
E^i_m(t^{m-1}M_m) & \xrightarrow{E^i_m(\mu_j)} & E^i_m(t^j M_m) \\
\downarrow{\cong} & & \downarrow{\cong} \\
E^i_m(M_1) & \xrightarrow{\cong} & E^i_m(M_{m-j})
\end{array} \]

for all \( i \in \mathbb{N} \). Since \( M \) is \( N \)-fiber-full up to \( h \), \( \text{Ext}^j_A(M/tM, N) \rightarrow \text{Ext}^j_A(M/t^m M, N) \) is injective for all \( j \leq h \), and so \( E^i_m(M_1) \rightarrow E^i_m(M_{m-j}) \) is injective for all \( i \leq h - 1 \) by Remark 2.7. Hence \( E^i_m(\mu_j) \) is injective for all \( i \leq h - 1 \) by the above diagram. It follows that

\[ 0 \rightarrow E^i_m(t^{m-1}M_m) \xrightarrow{E^i_m(\mu_j)} E^i_m(t^j M_m) \xrightarrow{E^i_m(\iota_j)} E^i_m(t^{j+1}M_m) \rightarrow 0 \]

is exact for all \( i \leq h - 2 \). \( \square \)
\textbf{Theorem 2.10.} Let \( h \) be an integer. \( M \) is \( N \)-fiber-full up to \( h \) as an \( A \)-module if and only if \( \text{Ext}^i_A(M, N) \) is flat over \( K[t] \) for all \( i \leq h - 1 \).

\textit{Proof.} \( \iff \) See Theorem \[2.9\].

\( \Rightarrow \) Suppose \( i \leq h - 2 \). By Lemma \[2.6\], it is enough to show that \( E^i_m(M_m) \) is flat over \( K[t]/(t^m) \) for all \( m \in \mathbb{Z}_+ \). We show this by induction on \( m \geq 1 \):

If \( m = 1 \), then \( E^i_1(M_1) \) is flat since \( K[t]/t \) is a field.

Now assume \( m \geq 2 \) and assume that \( E^i_{m-1}(M_{m-1}) \) is flat over \( K[t]/(t^{m-1}) \).

Since the ideal \((t^{m-1})/(t^m)\) is nilpotent, using the local criterion for flatness (see \[3\] Theorem \[22.3\]) we have that \( E^i_m(M_m) \) is flat over \( K[t]/(t^m) \) if and only if the following two conditions holds:

i) \( E^i_m(M_m)/t^{m-1}E^i_m(M_m) \) is flat over \( K[t]/(t^{m-1}) \), and

ii) the natural multiplication map

\[ \theta : (t^{m-1})/\left\langle t^m \right\rangle \otimes_{K[t]/(t^m)} E^i_m(M_m) \longrightarrow E^i_m(M_m) \]

is injective, that is, \( 0 : E^i_m(M_m) t^{m-1} = 0 \).

For each \( j \leq m - 2 \), we denote by \( \nu^j \) the composition of natural inclusions

\[ \nu^j = t_j \circ \ldots \circ t_{m-2} : t^{m-1}M_m \longrightarrow t^j M_m. \]

Since \( \mu_j : t^j M_m \longrightarrow t^{m-1}M_m \) is the multiplication by \( t^{m-1-j} \) and \( \nu^j \) is the natural inclusion, \( \nu^j \circ \mu_j : t^j M_m \longrightarrow t^j M_m \) is the multiplication by \( t^{m-1-j} \). Furthermore, since \( E^i_m(t_k) \) is surjective for all \( k \in \mathbb{N} \) by Remark \[2.9\] and since \( E^i_m(\cdot) \) is a functor, we have that \( E^i_m(\nu^j) \) is surjective for all \( j \leq m - 2 \).

Therefore, since \( E^i_m(\cdot) \) is a \( A_m \)-linear functor, we have

\[ \text{Im}(E^i_m(\mu_j)) = \text{Im}(E^i_m(\nu^j \circ E^i_m(\mu_j))) = t^{m-1-j}E^i_m(t^j M_m) \]

for all \( j \leq m - 2 \). Using again Remark \[2.9\],

\[ \text{Ker}(E^i_m(t_j)) = \text{Im}(E^i_m(\mu_j)) = t^{m-1-j}E^i_m(t^j M_m), \]

and so

\[ E^i_m(t^{j+1} M_m) \cong \frac{E^i_m(t^j M_m)}{t^{m-1-j}E^i_m(t^j M_m)} \]

for all \( j \leq m - 2 \).

Since \( m \geq 2 \), we can plug in \( j = 0 \) and we get

\[ E^i_m(M_m) / t^{m-1}E^i_m(M_m) \cong E^i_m(tM_m). \]

Using Remark \[2.8\] and Remark \[2.7\],

\[ E^i_m(tM_m) \cong E^i_m(M_m-1) \cong E^i_{m-1}(M_{m-1}) \]

is flat over \( K[t]/(t^{m-1}) \) by the inductive hypothesis, and it follows that \( E^i_m(M_m)/t^{m-1}E^i_m(M_m) \) is flat over \( K[t]/(t^{m-1}) \). So the condition i) is proved.

Before proving the condition ii), we show first that \( \text{Ker}(E^i_m(\nu^j)) = tE^i_m(t^j M_m) \) by induction on \( j \leq m - 2 \).

If \( j = m - 2 \), then \( t^{m-2} = t_{m-2} \) and we have shown above that

\[ \text{Ker}(E^i_m(t_{m-2})) = tE^i_m(t^{m-2}M_m). \]
Hence $\ker(E_m^i(t^{m-2})) = tE_m^i(t^{m-2}M_m)$. If $j < m - 2$, since $E_m^i(t^j) = E_m^i(t_{m-2}) \circ \cdots \circ E_m^i(t_j)$, we have

$$E_m^i(t_{m-3}) \circ \cdots \circ E_m^i(t_j) (tx) \in tE_m^i(t^{m-2}M_m) = \ker(E_m^i(t_{m-2}))$$

for each $x \in E_m^i(t^j M_m)$. It follows that $tE_m^i(t^j M_m) \subseteq \ker(E_m^i(t^j))$. On the other hand, if $u \in \ker(E_m^i(t^j))$, then

$$E_m^i(t_j)(u) \in \ker(E_m^i(t^{j+1})) = tE_m^i(t^{j+1}M_m)$$

by the inductive hypothesis, and so there exists $v \in E_m^i(t^{j+1}M_m)$ such that $E_m^i(t_j)(u) = tv$. Since $E_m^i(t_j)$ is surjective by Remark 2.9 there exists $w \in E_m^i(t^j M_m)$ such that $E_m^i(t_j)(w) = v$. Hence

$$u - tw \in \ker(E_m^i(t_j)) = t^{m-1-j}E_m^i(t^j M_m).$$

It follows that $u \in tE_m^i(t^j M_m) + t^{m-1-j}E_m^i(t^j M_m) = tE_m^i(t^j M_m)$. Therefore, $tE_m^i(t^j M_m) = \ker(E_m^i(t^j))$ for all $j \leq m - 2$.

In particular, we have $tE_m^i(t M_m) = \ker(E_m^i(t^0))$.

Now we prove the condition ii). Since

$$E_m^i(\mu_0) \circ E_m^i(t^0) = E_m^i(\mu_0 \circ t^0) : E_m^i(M_m) \rightarrow E_m^i(M_m)$$

is the multiplication by $t^{m-1}$ and since $E_m^i(\mu_0)$ is injective by Remark 2.9 we have

$$0 : E_m^i(M_m) t^{m-1} = \ker(E_m^i(\mu_0) \circ E_m^i(t^0)) = \ker(E_m^i(t^0)) = tE_m^i(M_m).$$

$\square$

3. Applications

In this section, we study some applications of $N$-fiber-full modules.

**Notations 3.1.** Let $R = K[X_1, \ldots, X_n]$ be the polynomial ring and fix $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ a weight vector. Notice that for each $f \in R$ there exists a unique (finite) subset of the set of monomials of $R$, denoted by $\text{Supp}(f)$, such that

$$f = \sum_{\mu \in \text{Supp}(f)} a_\mu \mu \quad \text{with} \quad a_\mu \in K \setminus \{0\}.$$ 

If $\mu = X_1^{u_1} \cdots X_n^{u_n}$, then we set $w(\mu) = w_1u_1 + \cdots + w_nu_n$. If $f = \sum_{\mu \in \text{Supp}(f)} a_\mu \mu \in R$, $f \neq 0$, we set

$$w(f) = \max\{w(\mu) : \mu \in \text{Supp}(f)\},$$

$$\text{init}_w(f) = \sum_{\mu \in \text{Supp}(f)} a_\mu \mu,$$

and we call

$$\text{hom}_w(f) = \sum_{\mu \in \text{Supp}(f)} a_\mu \mu t^{w(f) - w(\mu)} \in R[t]$$

the $w$-homogenization of $f$.

Given an ideal $I \subseteq R$, $\text{in}_w(I)$ denotes the ideal of $R$ generated by $\text{init}_w(f)$ with $f \in I$, and $\text{hom}_w(I)$ denotes the ideal of $R[t]$ generated by $\text{hom}_w(f)$ with $f \in I$. 
Given a monomial order < on \( R = K[X_1, \ldots, X_n] \) and given an ideal \( I \subseteq R \), there exists a weight vector \( w = (w_1, \ldots, w_n) \in (\mathbb{N}_{>0})^n \) such that \( \text{in}_<(I) = \text{in}_w(I) \) (see [9] Proposition 3.4).

The following corollary is a generalization of Corollary 3.3 in [9].

**Corollary 3.2.** Let \( I \subseteq R = K[X_1, \ldots, X_n] \) be an ideal, \( w = (w_1, \ldots, w_n) \in \mathbb{N}^n \) a weight vector and suppose that \( S = P/\text{hom}_w(I) \) is \( N \)-fiber-full up to \( h \) as a \( P \)-module, where \( P = R[t], N \) is finitely generated and flat over \( K[t] \). Then \( \text{Ext}^i_P(S, N) \) is a flat \( K[t] \)-module for \( i \leq h - 1 \) by the previous theorem. So, if furthermore \( I \) is homogeneous, we have

\[
\dim_K(\text{Ext}^i_R(R/I, N/tN)_j) = \dim_K(\text{Ext}^i_R(R/\text{in}_w(I), N/tN)_j)
\]

for all \( i \leq h - 2 \) and for all \( j \in \mathbb{Z} \).

In particular, if \( N = P \), then

\[
\dim_K(H^i_m(R/I)_j) = \dim_K(H^i_m(R/\text{in}_w(I))_j)
\]

for all \( i \geq n - h + 2 \) and for all \( j \in \mathbb{Z} \).

**Proof.** We observe that \( \text{Ext}^i_P(S, N) \) is a flat \( K[t] \)-module for \( i \leq h - 1 \) follows directly from Theorem 2.10. Let us give a graded structure to \( R = K[X_1, \ldots, X_n] \) by putting \( \deg X_i = g_i \) for each \( i \in \{1, \ldots, n\} \), where \( g = (g_1, \ldots, g_n) \) is a vector of positive integers. Suppose that \( I \) is a \( g \)-homogeneous ideal, and note that \( \text{hom}_w(I) \) is homogeneous with respect to the bi-graded structure on \( P = R[t] \) given by \( \deg(X_i) = (g_i, w_i) \) and \( \deg(t) = (0,1) \). So \( S = P/\text{hom}_w(I) \) and \( \text{Ext}^i_P(S, N) \) are finitely generated bi-graded \( P \)-modules, and it follows that

\[
\text{Ext}^i_P(S, N)_{(j,k)} = \bigoplus_{k \in \mathbb{Z}} \text{Ext}^i_P(S, N)_{(j,k)}
\]

is a finitely generated graded (with respect to the standard grading) \( K[t] \)-module for all \( j \in \mathbb{Z} \). By Remark 3.7 and Remark 3.8 in [9], we have that for each \( i, j \in \mathbb{Z} \):

\[
\text{Ext}^i_P(S, N)_{(j,k)} \cong K[t]^{a_{i,j}} \oplus (\bigoplus_{k \in \mathbb{N}_{>0}} (K[t]/(t^k))^{b_{i,j,k}})
\]

for some natural numbers \( a_{i,j} \) and \( b_{i,j,k} \). Set \( b_{i,j} = \sum_{k \in \mathbb{N}_{>0}} b_{i,j,k} \). Repeating a discussion similar to Theorem 3.1 in [9] we obtain

\[
\dim_K(\text{Ext}^i_R(R/I, N/tN)_j) = a_{i,j}
\]

and

\[
\dim_K(\text{Ext}^i_R(R/\text{in}_w(I), N/tN)_j) = a_{i,j} + b_{i,j} + b_{i+1,j}
\]

for every \( i, j \in \mathbb{Z} \). Since \( \text{Ext}^i_P(S, N) \) is a flat \( K[t] \)-module for \( i \leq h - 1 \), \( b_{i,j} = 0 \) for all \( i \leq h - 1 \) and so \( b_{i+1,j} = 0 \) for all \( i \leq h - 2 \). Hence for all \( i \leq h - 2 \) and for all \( j \in \mathbb{Z} \)

\[
\dim_K(\text{Ext}^i_R(R/I, N/tN)_j) = a_{i,j} = \dim_K(\text{Ext}^i_R(R/\text{in}_w(I), N/tN)_j)
\]

If \( N = P \), then

\[
\dim_K(H^i_m(R/I)_j) = \dim_K(H^i_m(R/\text{in}_w(I))_j)
\]

for all \( i \geq n - h + 2 \) and for all \( j \in \mathbb{Z} \) by the local duality theorem for graded modules (see [11] Theorem 3.6.19). 

In what follows, we suppose furthermore that \( A = \bigoplus_{i \in \mathbb{N}} A_i \) is positively graded with \( A_0 = K \) and \( t \in A_1 \), and \( M \) and \( N \) are graded \( A \)-modules.
Notation 3.3. Let $m$ the homogeneous maximal ideal of $A$, $d = \dim A$ and let

$$I = \bigcap_{i=1}^{s} q_i$$

be a homogeneous ideal of $A$, where $q_i$ are the primary components of $I$. For each integer $h$, we set

$$I^{\leq h} = \bigcap_{i=1}^{\dim(A/q_i) \geq d-h} q_i.$$

Notice that,

- $f^{\leq d-1} = I^{\text{sat}},$
- $H^i_m(A/I) \cong H^i_m(A/I^{\leq h})$ for all $i \geq d - h$.

**Proposition 3.4.** If $A$ is a $d$-dimensional Cohen-Macaulay ring, $N = \omega_A$ is the canonical module of $A$, $I$ is a homogeneous ideal of $A$ and $A/(I,t)^{\leq h}$ is cohomologically full (see Definition 1.1 in [3]) with $h$ an integer, then $A/I$ is $N$-fiber-full up to $h$.

**Proof.** First, supposing $m \in \mathbb{N}_{>0}$ and using Theorem 2.3 in [10] we observe that

$$((I,t^m)^{\leq h} = \{g \in A| \dim \left( A/(I,t^m) : g \right) < d - h \}$$

$$= \{g \in A| \dim A - \dim \left( A/(I,t^m) : g \right) > h \}$$

in particular,

$$((I,t)^{\leq h} = \{g \in A| \dim \left( I, t : g \right) > h \}.$$

If $g \in ((I,t^m)^{\leq h}$, then $\dim \left( I, t^m : g \right) > h$. Since $((I,t^m) \subseteq (I,t)$, we have

$$((I,t^m) : g \subseteq (I,t) : g,$$

it follows that

$$\dim \left( I, t : g \right) > \dim \left( I, t^m : g \right) > h.$$

Hence $((I,t^m)^{\leq h} \subseteq (I,t)^{\leq h}$.

We recall that $f \in \sqrt{(I,t)^{\leq h}}$ if and only if there exists $N \in \mathbb{N}$ such that $f^N \in (I,t)^{\leq h}$, if and only if there exists $N \in \mathbb{N}$ such that $\dim \left( I, t : f^N \right) > h$. Let $g_1, \ldots, g_l$ be the minimal generators of $(I,t) : f^N$. We have $g_i f^N \in (I,t)$ for all $i \in \{1, \ldots, l\}$, hence

$$g_i^m f^m = (g_i f^N)^m \in (I,t)^m \subseteq (I,t^m)$$

for all $i \in \{1, \ldots, l\}$. It follows that

$$\left( g_1^m, \ldots, g_l^m \right) \subseteq (I,t^m) : f^{Nm}.$$

Since

$$\dim \left( g_i^m, \ldots, g_l^m \right) = \dim \left( I, t : f^N \right) > h,$$

we have

$$\dim \left( I, t^m : f^{Nm} \right) \geq \dim \left( g_i^m, \ldots, g_l^m \right) > h,$$

therefore $f \in \sqrt{(I,t^m)^{\leq h}}$. Thus $\sqrt{(I,t)^{\leq h}} = \sqrt{(I,t^m)^{\leq h}}$.

Now for each $m \in \mathbb{N}_{>0}$, we set $m_m = m/(I,t^m)$. Since $A/(I,t)^{\leq h}$ is cohomologically full, the natural map

$$H^j_{m_m}(A/(I,t^m)^{\leq h}) \rightarrow H^j_{m_1}(A/(I,t)^{\leq h})$$
is surjective for all $j$. In general, for all $j$ we have $H^1_{m_i}(A/(I,t)) \cong H^i_{m_i}(A/(I,t))$ and $H^2_{m_i}(A/(I,t^m)) \cong H^i_{m_i}(A/(I,t^m))$, hence the natural map

$$H^i_{m_i}(A/(I,t^m)) \rightarrow H^i_{m_i}(A/(I,t))$$

is surjective for all $j \geq d - h$. Since $A$ is *complete, by the local duality theorem for graded modules (see [1] Theorem 3.6.19), $\text{Ext}^i_A(A/(I,t), N) \rightarrow \text{Ext}^i_A(A/(I,t^m), N)$ is injective for all $i \leq h$. \hfill \Box

Let $R$ be the polynomial ring $K[X_1, \ldots, X_n]$ over a field $K$ and let $I \subseteq R$ be a homogeneous ideal. In the paper of Conca and Varbaro [2] they obtained the following result:

if $\text{in}(I)$ is a square-free monomial ideal for some term order, then

$$\dim_K H^i_{m}(R/I)_j = \dim_K H^i_{m}(R/\text{in}(I))_j$$

They actually showed that

$$\dim_K H^i_{m}(R/I)_j = \dim_K H^i_{m}(R/\text{in}(I))_j$$

for all $i, j$ if $R/\text{in}(I)$ is cohomologically full. If $\text{in}(I)$ is a square-free monomial ideal then $R/\text{in}(I)$ is cohomologically full (see [6] Theorem 1).

Using the notion of $N$-fiber-full we can prove the following theorem:

**Theorem 3.5.** Let $I \subseteq R = K[X_1, \ldots, X_n]$ be a homogeneous ideal, $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ a weight vector and $P = R[t]$. If $R/\text{in}_w(I)^{\leq h}$ is cohomologically full, then

$$\dim_K (H^i_{m}(R/I)_j) = \dim_K (H^i_{m}(R/\text{in}_w(I))_j)$$

for all $i > n - h$ and for all $j \in \mathbb{Z}$.

In particular, fixing a monomial order $<$ on $R$,

- if $R/\text{in}_w(I)^{\leq h}$ is cohomologically full, then $\dim_K (H^i_{m}(R/I)_j) = \dim_K (H^i_{m}(R/\text{in}_w(I))_j)$ for all $i > n - h$ and for all $j \in \mathbb{Z}$;

- if $\text{in}_w(I)^{\leq h}$ is square-free, then $\dim_K (H^i_{m}(R/I)_j) = \dim_K (H^i_{m}(R/\text{in}_w(I))_j)$ for all $i > n - h$ and for all $j \in \mathbb{Z}$.

**Proof.** We claim

$$R/\text{in}_w(I)^{\leq h} \cong P/(\text{hom}_w(I), t)^{\leq h+1}.$$

Indeed, if $\text{in}_w(I) = \bigcap_{i=1}^s q_i$ is a primary decomposition of $\text{in}_w(I)$, by definition

$$\text{in}_w(I)^{\leq h} = \bigcap_{\dim(R/q_i) \geq n-h} q_i.$$

Since $q_i \subseteq R, x, t$ is a $P$-regular sequence for each $x \in R$ such that $x \neq 0$ and $x$ is not invertible, we have that $(q_i, t)$ are the primary components of the ideal $(\text{in}_w(I), t)$.
and

\[
\begin{align*}
(\text{in}_w(I)^{\leq h}, t) &= \left( \bigcap_{i=1}^s (q_i) \right) \cap (t) \\
&= \bigcap_{i=1}^s (q_i, t) \\
&= \bigcup_{i=1}^s (q_i, t) \\
&= (\text{in}_w(I), t)^{\leq h+1}.
\end{align*}
\]

By Proposition 3.5 in [9], we have

\[
(\text{hom}_w(I), t) = (\text{in}_w(I), t).
\]

Hence

\[
R/\text{in}_w(I)^{\leq h} \cong P/(\text{in}_w(I)^{\leq h}, t) \cong P/(\text{in}_w(I), t)^{\leq h+1} \cong P/(\text{hom}_w(I), t)^{\leq h+1}.
\]

Since \( R/\text{in}_w(I)^{\leq h} \) is cohomologically full, by our claim and by Proposition 3.4 we have that \( P/\text{hom}_w(I) \) is \( P \)-fiber-full up to \( h + 1 \). Therefore

\[
\dim_K(H^i_m(R/I)_j) = \dim_K(H^i_m(R/\text{in}_w(I))_j)
\]

for all \( i > n - h \) and for all \( j \in \mathbb{Z} \) by Corollary 3.2.

In particular, given a monomial order \( < \) on \( R \), there exists a weight vector \( w = (w_1, \ldots, w_n) \in (\mathbb{N}_{>0})^n \) such that \( \text{in}_w(I) = \text{in}_{<}(I) \), and this yields the statement. \( \square \)

**Remark 3.6.** In the paper of Dao, De Stefani and Ma [3] they proved the following result (see [3] Lemma 3.7):

Let \( R \) be the polynomial ring \( K[X_1, \ldots, X_n] \) over a field \( K \) and let \( J \subseteq R \) be a homogeneous ideal. If \( R/J \) is a cohomologically full ring, then \( R/J \) satisfies Serre’s condition \( (S_1) \), that is, \( \text{Min}(J) = \text{Ass}(R/J) \).

Hence, it can happen that \( R/J \) is not cohomologically full only because it has some embedded primes. But using the notion \( J^{\leq h} \), sometimes we can remove the embedded primes of \( R/J \) and make \( R/J^{\leq h} \) be a cohomologically full ring.

**Example 3.7.** Let \( I \) be a monomial ideal of the polynomial ring \( R = K[x_1, \ldots, x_n] \) and let \( \mu_1, \ldots, \mu_s \) be the minimal monomial generators of \( I \) such that the following condition holds: if there exists \( \mu \in \{ \mu_1, \ldots, \mu_s \} \) and there exists an integer \( t \geq 2 \) such that \( x_k^t | \mu \) with \( k \in \{1, \ldots, n\} \), then there exists \( g \in I^{\text{sat}} \) such that \( g \sqrt{\mu} \). Then we have \( I^{\text{sat}} = I^{\leq n-1} \) is square-free. Hence \( R/I^{\text{sat}} \) is cohomologically full.

A simple example is the following: If \( R = K[x, y, z] \) and \( J = (x^2y, xy^2, xyz) \), then we have \( \text{Ass}(R/J) = \{(x, y, z), (x, y)\} \) and \( \text{Min}(J) = \{(x, y)\} \), hence \( R/J \) is not a cohomologically full ring. We observe \( \text{Ass}(R/J^{\text{sat}}) = \{(x, y)\} = \text{Min}(J^{\text{sat}}) \). Since \( J^{\text{sat}} \) is square-free, \( R/J^{\text{sat}} \) is cohomologically full.

**Remark 3.8.** We observe that in the situation of Corollary 3.2, if \( N = K[t] \) then

\[
\dim_K(\text{Ext}^i_R(R/I, N/tN)_j) = \dim_K(\text{Ext}^i_R(R/I, K)_j) = \beta_{i,j}(R/I)
\]
is the \((i, j)\)-th Betti number of \(R/I\) and 
\[
\dim_K(\Ext^i_R(R/\in_w(I), N/tN)_j) = \beta_{i,j}(R/\in_w(I))
\]
is the \((i, j)\)-th Betti number of \(R/\in_w(I)\), hence 
\[
\beta_{i,j}(R/I) = \beta_{i,j}(R/\in_w(I))
\]
for all \(i \leq h - 2\) and for all \(j \in \mathbb{Z}\). However in general, \(\beta_{i,j}(R/I) \neq \beta_{i,j}(R/\in_w(I))\) even if \(\in_w(I)\) is a square-free monomial ideal. Hence \(R\) is fiber-full does not imply \(R\) is \(K[t]\)-fiber-full.

In practice, if \(A\) is Cohen-Macaulay and \(A/(I, t)\) is cohomologically full, then the natural map \(\Ext^i_A(A/(I, t), \omega_A) \longrightarrow \Ext^i_A(A/J, \omega_A)\) is injective for all \(i\), where \(J \subseteq (I, t)\) and \(\sqrt{J} = \sqrt{(I, t)}\). Hence, if \((I, t)\) is a square-free monomial ideal, then the natural map \(\Ext^i_A(A/(I, t), N) \longrightarrow \Ext^i_A(A/J, N)\) is injective if \(N = \omega_A\), however it is not true if \(N = K[t]\).

**Example 3.9.** Consider the following graph \(G\):

```
   1
  /|
 / |
/  |
5 - 2
/|
/  \
4 - 3
```

We obtain the binomial edge ideal of \(G\) (see [3]):
\[
J_G = (x_1y_2 - x_2y_1, x_2y_3 - x_3y_2, x_3y_4 - x_4y_3, x_4y_5 - x_5y_4, x_1y_5 - x_5y_1),
\]
and using Macaulay2 [4] we compute the initial ideal of \(J_G\):
\[
in_<(J_G) = (x_2y_1, x_3y_1, x_3y_2, x_4y_2, x_4y_3, x_1x_2x_5y_3,
\]
\[
x_5y_4, x_4y_1y_5, x_1x_4y_2y_5, x_3y_1y_4y_5),
\]
where \(<\) is the lexicographic order on \(K[x_1, x_2, \ldots, x_5, y_1, y_2, \ldots, y_5]\) induced by 
\[
x_1 > x_2 > \cdots > x_5 > y_1 > y_2 > \cdots > y_5.
\]

Therefore, \(\in_<(J_G)\) is a square-free monomial ideal, \(\beta_0(J_G) = 5\) and \(\beta_0(\in_<(J_G)) = 10 \neq \beta_0(J_G)\).

Hence, in the situation of Corollary 3.2, it would be very interesting to understand when \(S = P/\hom_w(I)\) is \(K[t]\)-fiber-full, a condition that would guarantee that the graded Betti numbers are preserved going from \(I\) to \(\in_w(I)\).

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