In this work, we extend the convex body chasing problem to an adversarial setting, where a player is tasked to chase a sequence of convex bodies generated adversarially by an opponent. The Player aims to minimize the cost associated with its total movements, while the Opponent tries to maximize it. The set of feasible convex bodies is finite and known to both agents, which allows us to provide performance guarantees with max-min optimality rather than in the form of competitive ratio. Under certain assumptions, we showed the continuity of the optimal value function, based on which, we designed an algorithm that numerically generates policies with performance guarantees. Finally, the theoretical results are verified through numerical examples.

1 Introduction

Convex Bodies Chasing (CBC) problem was proposed in [1] to study the interaction between convexity and metrical task systems. Later, it was realized that many problems can be seen as a variant of chasing convex bodies, including scheduling [2], efficient covering [3], safely using machine-learned advice [4, 5], self-organizing lists [6], and the famous k-server problem [7, 8]. In a CBC problem, an online player receives a request sequence of \( T \) convex sets \( Q_1, \ldots, Q_T \) contained in a normed space \( X \) of dimension \( d \). The player starts at \( x_0 \) and at timestep \( t \) observes the set \( Q_t \) and then moves to a new point \( x_t \in Q_t \), paying a cost \( \|x_t - x_{t-1}\| \). Previous works [1, 9, 10, 11] aimed to design online algorithms that ensure the total cost exceeds the minimum possible total cost by at most a bounded factor \( \alpha \), known as the ‘competitive ratio’. The most recent work [11] has achieved a competitive ratio \( O(\sqrt{d \log T}) \), despite that the proposed algorithm chooses \( x_n \) without the knowledge of the future sets \( Q_{t+1}, \ldots, Q_T \).

In the classic CBC problem, with no restriction on mechanism that generates the convex sets, the Player needs to select a point that balances the future performance for all possible subsequent convex sets. Consequently, the competitive ratio is considered as the performance metrics for most of the previous algorithms. In many real-world scenarios, however, the the convex bodies are selected (potentially adversarially) from a known set of convex sets (e.g. dynamic Blotto game [12], etc.). With the additional information, one expects to obtain performance guarantees better than the competitive ratio.

In this work, we consider the adversarial convex bodies chasing (aCBC) problem, where a (finite) set of possible compact convex bodies are known to the Player, but the sequence of selected bodies is generated by an adversary: the Opponent. The adversarial selection of the convex bodies is further constrained over a graph. As a result, the index of the convex body selected at the current timestep has an impact on the convex bodies available at the next timestep. The Player’s movement is also constrained within its (compact) reachable set, dependent on its current position. We formulate this competitive game as a zero-sum sequential game, where the Player aims to minimizes its total cost, while the Opponent tries to maximize it.

The contribution of this work is threefold: (i) the novel formulation of the adversarial CBC game; (ii) theoretical guarantees of a Lipschitz continuous max-min value function under mild assumptions; and (iii) a numerical algorithm that provides \( \epsilon \)-suboptimal performance guarantee with respect to the max-min solution.

The rest of the paper is organized as follows: Section 2 formally presents the formulation of the adversarial convex body chasing game; Section 3 introduces the value function and provides results regarding its continuity; In Section 4, we propose a numerical algorithm that discretizes the domain and computes the optimal policies for the two Players. We further prove that the obtained policy is only \( \epsilon \) away from the optimal min-max solution; In Section 5, we demonstrate the effectiveness of the proposed algorithm through numerical examples. Finally, Section 5 concludes this work.

*Equal contribution.
2 Problem Formulation

The adversarial convex body chasing (aCBC) game is played sequentially between two agents: the Player and the Opponent. The game evolves over a subset $\mathcal{X}$ of the Euclidean space $\mathbb{R}^N$. At each time step, a convex region $Q_t \subseteq \mathcal{X}$ is first selected by the Opponent, and then the Player must choose a point $x_t \in Q_t$. The timeline of the game is presented in Figure 1. The Player tries to minimize its total cost $\sum_{t=0}^{T-1} c(x_t, x_{t+1})$ over a finite horizon $T$ for some non-negative cost function $c$, while the Opponent aims to maximize this total cost.

We use $x_t$ as the state of the Player at timestep $t$ and treat $\mathcal{X}$ as the state space of the Player. Different from the classical CBC problem, we restrict the Player’s selection of its next state to a neighborhood of its current state. More specifically, we require that $x_{t+1} \in R(x_t)$ for all $t = 0, \ldots, T - 1$, where $R(x_t)$ denotes the reachable set from the current state $x_t$.

Assumption 1. For all $x \in \mathcal{X}$, the reachable set $R(x)$ is compact.

We further assign a state $i \in \mathcal{V}$ to the Opponent, which evolves over a graph $G = (\mathcal{V}, \mathcal{E})$. Given the Opponent’s current state $i_t$, the Opponent can move to any one of the neighboring nodes $i_{t+1}$, such that $(i_t, i_{t+1}) \in \mathcal{E}$. We denote the set of all neighbors of node $i$ as $\mathcal{N}_i$. Before the game starts, we assign the Opponent a finite set of convex regions $\mathcal{Q} = \{Q^{(i)}_t\}_{t=0, i=1}^{T, |\mathcal{V}|}$. The Opponent automatically selects one convex region $Q^{(i)}_t$ by selecting its state $i_t$. In other words, instead of having the freedom to choose an arbitrary subset of $\mathcal{X}$ as in the original CBC problem, the Opponent in an aCBC game can only choose from a given set of convex bodies through selecting the next state (node) to visit. Furthermore, the Opponent’s future selections depend on its past actions due to the graph constraints on its state dynamics.

As discussed in the introduction, we make the following two assumptions on the set of convex bodies $\mathcal{Q}$ and the information structure of the game.

Assumption 2. For all $i \in \mathcal{V}$ and $t = 0, \ldots, T$, the convex region $Q^{(i)}_t$ is compact.

Assumption 3. The set of convex bodies $\mathcal{Q}$ is a common knowledge to both agents.

A toy example of the proposed aCBC game is presented in Figure 2.

Figure 2: A toy example of the aCBC game with a 4-node graph and two-dimensional Player state $\mathcal{X}$. At timestep 0, the Opponent starts on Node 2 and the Player selects a point $x_0$ within $Q^{(2)}_0$. The Opponent then moves to Node 4 and the Player selects the point $x_1 \in R(x_0) \cap Q^{(4)}_1$, inducing a cost of $c(x_0, x_1)$. The Opponent then moves to Node 3 and the Player moves to $x_2 \in R(x_1) \cap Q^{(3)}_2$ with a cost of $c(x_1, x_2)$, and the game continues.

With all the additional assumptions, one natural question is, what are the key differences between the classic CBC problem and the proposed adversarial CBC game? Note that one can simply use a fully connected graph with self-loop to remove the graph constraint. Furthermore, the reachability constraint can be removed by using the following
correspondence:
\[ \mathcal{R}(x) = \bigcup_{t \in \mathcal{T}, i \in \mathcal{V}} Q^{(i)}_t \quad \forall x \in \mathcal{X}. \]

Consequently, one sees that the graph constraint and the reachability constraint do not restrict the class of problems covered by aCBC. Instead, the major difference from the original CBC formulation is that (i) the feasible convex bodies are compact, and (ii) the set of feasible convex bodies in aCBC game is finite and is also a common information for both agents. In other words, the compactness of the convex body and the finiteness of the feasible set of convex bodies significantly reduces the Opponent’s “freedom” of selecting convex regions and consequently allows the construction of min-max solution that will be discussed in the next sections.

To ensure the existence of a min-max solution, we further make the following assumptions on the continuity of the cost function \( c \) and the reachable correspondence \( \mathcal{R} \).

**Assumption 4.** The cost function \( c : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R} \) is uniformly continuous over both \( \mathcal{X} \).

**Assumption 5.** For all \( x \in \mathcal{X} \), the reachability correspondence \( \mathcal{R} : \mathcal{X} \mapsto \mathcal{X} \) is continuous with respect to the Hausdorff distance.\(^1\)

Finally, to avoid degeneracy, we assume that any admissible sequence of convex bodies is always feasible for the Player regardless the sequence of actions selected by the agent. In other words, the Player never has to worry about the future feasibility. Consequently, the aCBC game is an optimization problem rather than a feasibility problem.

**Assumption 6.** For all timestep \( t = 0, \ldots, T-1 \), the given set of convex bodies \( \mathcal{Q} = \{ Q^{(i)}_t \}_{t=0, i=1}^{T, |\mathcal{V}|} \) satisfies:
\[
\left( \mathcal{R}(x_t) \cap Q^{(i_{t+1})}_t \right)^\circ \neq \emptyset \quad \text{for all} \ x_t \in Q^{(i_t)}_t \text{ and } i_{t+1} \in \mathcal{N}_{i_t}, \tag{1}
\]
where \( F^\circ \) denotes the interior of a set \( F \).

We consider the class of Markov policies for both agents. The Player’s (deterministic) strategy at time \( t \geq 1 \) is given by \( \pi_t(x_{t-1}, i_t) \in \mathcal{R}(x_{t-1}) \cap Q^{(i_t)}_t \), which explicitly considers the constraints due to the convex body \( Q^{(i_t)}_t \) selected by the Opponent at time \( t \) and the reachability constraint from its previous state \( x_{t-1} \). Similarly, the Opponent’s (deterministic) strategy at time \( t \geq 1 \) is given by \( \sigma_t(x_{t-1}, i_{t-1}) \in \mathcal{N}_{i_{t-1}} \), which reflects the graph constraint on the Opponent’s state. To initialize the game, at time \( t = 0 \), the Opponent first selects a node \( i_0 \in \mathcal{V} \) according to its strategy \( \sigma_0(\mathcal{Q}, \mathcal{G}) \). Here, we make the policy’s dependence on the convex body set and the graph explicitly. Then the Player selects a point \( x_0 \in Q^{(i_0)}_0 \) according to \( \pi_0(i_0) \). We denote the sequences of policies used by the Player and the Opponent as \( \pi = \{ \pi_t \}_{t=0}^T \) and \( \sigma = \{ \sigma_t \}_{t=0}^T \) respectively.

A policy pair \((\pi, \sigma)\) would induce a (deterministic) \( x \)-trajectory and consequently, a total cost \( C(\pi, \sigma) \). We are interested in subgame perfect Nash equilibrium, where at each stage of the game the Player is always minimizing its future cumulative cost-to-go while the Opponent maximizes. We denote \( C^* \) as the optimal value of the given aCBC game under a subgame perfect Nash equilibrium. In this work, we address the following problem.

**Problem 1.** Given a graph \( \mathcal{G} \), a set of convex bodies \( \mathcal{Q} = \{ Q^{(i)}_t \}_{t=0, i=1}^{T, |\mathcal{V}|} \), a cost function \( c \) and the reachability mapping \( \mathcal{R} \). Under the information structure in Assumption 3, what is the value \( C^* \) of the aCBC game, and what are the corresponding optimal policies for both agents?

### 3 Value Functions

To reflect the different information available to the two agents at their decision points, we introduce two value functions: \( V_t(x_{t-1}, i_t) \) for the optimal value function of the Player and \( U_t(x_{t-1}, i_{t-1}) \) for the Opponent at time \( t \).

The two optimal value functions will be computed through a backward induction scheme, similar to other finite horizon decision-making problems [13].

#### 3.1 Backward Induction

At the terminal timestep \( T \), the Player has knowledge on its previous state \( x_{T-1} \) and the Opponent state \( i_T \), and the Player is about to make the final move. Since there is no moves after time \( T \), the Player only needs to consider the

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\(^1\)See Definition 1 for the definition of the Hausdorff metric that measures the distance between two sets.
optimality with respect to the immediate cost. Consequently, the optimal terminal value for the Player can be computed as

$$V_T (x_{T-1}, i_T) \triangleq \inf_{x_T \in R(x_{T-1}) \cap Q^{(i_T)}_T} c(x_{T-1}, x_T).$$  \hspace{1cm} (2)$$

In words, the above value depicts the best outcome for the Player, given that its previous state is $x_{T-1}$ and the Opponent has selected $i_T$ at the terminal timestep.

For timesteps $t \in \{1, \ldots, T-1\}$, the Player needs to optimize its selection of a new state $x_t$ to minimize both the immediate cost and the future cost. Furthermore, the Player also has to consider the fact that the Opponent will observe its move and then best-respond with $i_{t+1}$ to maximize the future cumulative costs. Consequently, the optimal value for the Player is computed as

$$V_t (x_{t-1}, i_t) \triangleq \inf_{x_t \in R(x_{t-1}) \cap Q^{(i_t)}_t} \left\{ c(x_{t-1}, x_t) + \sup_{i_{t+1} \in N_i} V_{t+1} (x_t, i_{t+1}) \right\}. \hspace{1cm} (3)$$

Finally, for the initial Player state selection at $t = 0$, there is no reachability constraint. As a result, the value function only depends on the initial Opponent state $i_0$, while the rest of the optimization is similar to the value function in (3), and we have

$$V_0 (i_0) \triangleq \inf_{x_0 \in Q^{(i_0)}_0} \sup_{i_1 \in N_{i_0}} V_1 (x_0, i_1). \hspace{1cm} (4)$$

The Opponent’s value function is constructed similar to that of the Player. The only difference comes from the information structure. Namely, the Opponent makes decisions based on the previous Player location $x_{t-1}$ and its own previous state $i_{t-1}$. Formally, the Opponent’s value is defined as

$$U_T (x_{T-1}, i_{T-1}) \triangleq \sup_{i_T \in N_{i_{T-1}}} \left\{ \inf_{x_T \in R(x_{T-1}) \cap Q^{(i_T)}_T} c(x_{T-1}, x_T) \right\}, \hspace{1cm} (5)$$

$$U_t (x_{t-1}, i_{t-1}) \triangleq \sup_{i_t \in N_{i_{t-1}}} \left\{ \inf_{x_t \in R(x_{t-1}) \cap Q^{(i_t)}_t} \left\{ c(x_{t-1}, x_t) + U_{t+1} (x_t, i_t) \right\} \right\}, \hspace{1cm} (6)$$

$$U_0 (G, Q) \triangleq \sup_{i_0 \in V} \inf_{x_0 \in Q^{(i_0)}_0} U_1 (x_0, i_0). \hspace{1cm} (7)$$

**Remark 1.** The Opponent’s value $U_0$ is equivalent to the game value $C^*$ of an aCBC game.

**Remark 2.** For value functions $V_t (x_{t-1}, i_t)$ at $t = 1, \ldots, T$ with the double arguments on Player and Opponent states respectively, we implicitly assume $x_{t-1} \in Q^{(i_{t-1})}_{t-1}$, where $i_{t-1} \in N_{i_{t-1}}$. Similarly, we assume that $i_{t-1} \in V$ for $U_t (x_{t-1}, i_{t-1})$ and $i_{t} \in N_{i_{t-1}}$ for $V_t (x_{t-1}, i_{t})$. Otherwise, the value function may not be well-defined, as the intersection of reachable sets and the new convex set could be empty. So is the set of neighboring nodes. This implicit assumption can also be viewed as a natural result of the game dynamics, since any valid trajectory up to time $t$ will automatically satisfy these assumptions.

### 3.2 Uniform Continuity of the Value Functions

The first question one may ask regarding the value functions is whether the supremum and the infimum can be attained. The following lemma on the continuity of the value functions gives an affirmative answer.

**Lemma 1.** For all $t \in \{1, \ldots, T\}$, $i_{t-1} \in V$ and $i_t \in N_{i_{t-1}}$, the optimal value functions $V_t (x_{t-1}, i_t)$ and $U_t (x_{t-1}, i_{t-1})$ are both uniformly continuous with respect to $x_{t-1} \in Q^{(i_{t-1})}_{t-1}$.

**Proof.** See Appendix A for details.

**Remark 3.** Note that at $t = 0$, neither of the value functions have the $x$-argument. Consequently, we do not consider the $x$-continuity of $V_0$ and $U_0$.

**Corollary 2.** The extrema of the optimal value function can be attained and are finite.

**Proof.** Since the intersection between reachable sets and the convex regions is compact, this corollary is a direct consequence of Lemma 1.
Due to Corollary 2, we can replace the infimum and supremum in the value functions with minimum and maximum. The resulting optimal value functions of the Player are given as

\[ V_T (x_{T-1}, i_T) = \min_{x_T \in R(x_{T-1}) \cap Q_T^{(i_T)}} c(x_{T-1}, x_T), \]  

\[ V_t (x_{t-1}, i_t) = \min_{x_t \in R(x_{t-1}) \cap Q_t^{(i_t)}} \{ c(x_{t-1}, x_t) + \max_{i_{t+1} \in \mathcal{N}_t} V_{t+1} (x_t, i_{t+1}) \}, \quad \forall t \in \{1, \ldots, T-1\}, \]  

\[ V_0 (i_0) = \min_{x_0 \in \mathcal{Q}_0^{(i_0)}} \max_{i_1 \in \mathcal{N}_0} V_1 (x_0, i_1), \]  

Similarly, the value functions of the Opponent are as follows.

\[ U_T (x_{T-1}, i_{T-1}) = \max_{i_T \in \mathcal{N}_{T-1}} \begin{cases} \min_{x_T \in R(x_{T-1}) \cap Q_T^{(i_T)}} c(x_{T-1}, x_T) \end{cases}, \]  

\[ U_t (x_{t-1}, i_{t-1}) = \max_{i_t \in \mathcal{N}_{t-1}} \begin{cases} \min_{x_t \in R(x_{t-1}) \cap Q_t^{(i_t)}} \{ c(x_{t-1}, x_t) + U_{t+1} (x_t, i_t) \} \end{cases}, \quad \forall t \in \{1, \ldots, T-1\}, \]  

\[ U_0 (\mathcal{G}, \mathcal{Q}) = \max_{i_0 \in \mathcal{V}} \min_{x_0 \in \mathcal{Q}_0^{(i_0)}} U_1 (x_0, i_0). \]

### 3.3 Relationship between the Two Value Functions

The next natural question is, how are the two value functions \( V \) and the \( U \) related to each other? The answer to this question is given in the following two lemmas.

**Lemma 3.** For all \( t \in \{1, \ldots, T\} \), the Opponent value is related to the Player value via

\[ U_t (x_{t-1}, i_t) = \max_{i_t \in \mathcal{N}_{t-1}} V_t (x_{t-1}, i_t). \]

Furthermore, when \( t = 0 \),

\[ U_0 (\mathcal{G}, \mathcal{Q}) = \max_{i_0 \in \mathcal{V}} V_0 (i_0). \]

Similarly, for all \( t \in \{1, \ldots, T-1\} \), the Player value is related to the Opponent value through

\[ V_t (x_{t-1}, i_t) = \min_{x_t \in R(x_{t-1}) \cap Q_t^{(i_t)}} \{ c(x_{t-1}, x_t) + U_{t+1} (x_t, i_t) \}. \]

Furthermore, when \( t = 0 \),

\[ V_0 (i_0) = \min_{x_0 \in \mathcal{Q}_0^{(i_0)}} U_1 (x_0, i_0). \]

**Proof.** See appendix B.

### 3.4 Optimal Policies

From the construction of the optimal value function, one can easily see the optimal policy for the aCBC game. The optimal Player policy is defined as

\[ \pi_T (x_{T-1}, i_T) \in \arg\min_{x_T \in R(x_{T-1}) \cap Q_T^{(i_T)}} c(x_{T-1}, x_T), \]  

\[ \pi_t (x_{t-1}, i_t) \in \arg\min_{x_t \in R(x_{t-1}) \cap Q_t^{(i_t)}} \{ c(x_{t-1}, x_t) + U_{t+1} (x_t, i_t) \}, \quad \forall t = 1, \ldots, T-1, \]  

\[ \pi_0 (i_0) \in \arg\min_{x_0 \in \mathcal{Q}_0^{(i_0)}} U_1 (x_0, i_0). \]

Similarly, the optimal Opponent policy is defined as

\[ \sigma_T (x_{T-1}, i_{T-1}) \in \arg\max_{i_T \in \mathcal{N}_{T-1}} V_T (x_{T-1}, i_T), \quad \forall t = 1, \ldots, T, \]  

\[ \sigma_0 (\mathcal{G}, \mathcal{Q}) \in \arg\max_{i_0 \in \mathcal{V}} V_0 (i_0). \]
3.5 Lipschitz Continuity of the Value Functions

In previous sections, we have developed the formula for the optimal value functions and the optimal policies. However, the value functions have a continuous x-argument, which makes the storing and optimizing the value functions challenging. One natural approach is to discretize the domain of x using a mesh and then represent the value functions with their function values on the vertices of the mesh. In Section 4, we will develop an algorithm that employ this discretization idea. However, to provide performance bounds for numerical algorithm that operates on the discretized value functions, one needs to ensure that the optimal value functions are well-behaved. In this work, we want to ensure that the value functions are Lipschitz continuous. Aside from performance bounds, the Lipschitz continuity result allows us to understand what discretization resolution δ_{X,t} is needed at each timestep to obtain policies with a desired ϵ-suboptimality guarantee.

To ensure the Lipschitz continuity, we requires the following strengthened assumptions.

**Assumption 7.** The cost function c : X × X → R is Lipschitz continuous in X × X with respect to the Manhattan distance. Formally,

\[ |c(x, y) - c(x', y')| \leq L_c \cdot (\|x - x'\| + \|y - y'\|) \quad \forall x, x', y, y' \in X, \]

where L_c denotes the Lipschitz constant.

**Assumption 8.** For all x ∈ X and Q_t^{(i)} ∈ Q such that R(x) ∩ Q_t^{(i)} ≠ ∅, the intersection correspondence Θ_t^{(i)}(x) = R(x) ∩ Q_t^{(i)} is Lipschitz continuous under the Hausdorff distance with a Lipschitz constant of L_Θ. Formally,

\[ \text{dist}_H(Θ_t^{(i)}(x), Θ(x')) \leq L_Θ \cdot \|x - x'\| \quad \forall x, x' ∈ X, \ i ∈ V \text{ and } t ∈ T. \]

**Remark 4.** If R(x) is sufficiently large so that R(x) ∩ Q_t^{(i)} = Q_t^{(i)} for all valid x ∈ X, then L_Θ = 0.

**Lemma 4.** Under the strengthened Assumptions 7 and 8, we have that for all t ∈ {1, ..., T}, i_{t-1} ∈ V and i_t ∈ N_{i_{t-1}}, the optimal value functions V_t (x_{t-1}, i_t) and U_t (x_{t-1}, i_{t-1}) are both L_{v,t}-Lipschitz, where Lipschitz constant is given by

\[ L_{v,t} = L_c \cdot \sum_{k=1}^{T-t+1} (1 + L_Θ)^k \] (15)

**Proof.** See Appendix C for details. ⊓⊔

4 Algorithmic Solution

In this section, we propose to solve the aCBC problem numerically through discretization. We will provide error-bounds on the approximation and provide performance guarantee for the solutions found through discretization.

From Lemma 4, we observe that the Lipschitz constant L_{v,t} decreases monotonically as timestep t approaches the horizon T. Naturally, finer discretization resolution is preferred at the beginning of the game to ensure low approximation error. Consequently, we allow different resolutions at different timesteps. We use δ_{X,t} to denote the discretization size of X at timestep t, and we denote the set of vertices on the mesh as X̂_t = {x̂_k^t}^M_t_{k=1}. To ensure that the x-optimization domain in (8) is properly discretized, with a discretization size δ_{X,t}, we require

\[ \min_{x_t ∈ X_t} \|x_t - x_t\| ≤ δ_{X,t} \quad \forall x_t ∈ X \] for t = 0, ..., T, (16a)

\[ \min_{x_0 ∈ X_0 ∩ Q_0^{(i_0)}} \|x_0 - x_0\| ≤ δ_{X,0} \quad \forall x_0 ∈ Q_0^{(i_0)} \text{ for all } i_0 ∈ V, \] (16b)

\[ \min_{x_t ∈ X_t ∩ R(x_{t-1}) ∩ Q_t^{(i_t)}} \|x_t - x_t\| ≤ δ_{X,t} \quad \forall x_t ∈ R(x_{t-1}) \bigcap Q_t^{(i_t)} \text{ for } t = 1, ..., T, \] (16c)

where x_{t-1} ∈ X̂_{t-1} such that x_{t-1} ∈ Q_t^{(i_{t-1})} and i_t ∈ N_{i_{t-1}}.

In the above criteria, we first require that the mesh has the required resolution over the whole domain X at all timesteps as in (16a). We further require that the resolution within the intersections of reachable sets and the convex body is fine enough as in (16b) and (16c). An example of the discretization is presented in Figure 3. The black vertices
are constructed to discretize the domain $\mathcal{X}$ according to (16a). One can see, for this discretization scheme, there is no vertex in $Q^{(1)}_0$. Consequently, the two blue points are added to $\hat{x}_0$ to satisfy (16b). Meanwhile, although the intersection $\mathcal{R}(\hat{x}_0) \cap Q^{(2)}_1$ has a vertex within, the mesh does not have a fine enough resolution within the intersection. As a result, two red points are added to satisfy (16c).

![Figure 3: An example of the discretization scheme.](image)

With the discretization of $\mathcal{X}_t$, the Player restricts its action selection $x_t$ to the vertices $\hat{x}_t$, which leads to a decrease in its performance. We consider the worst case scenario, where the Opponent knows the mesh used by the Player. The resulted value functions then takes the vertices of the mesh $\hat{x}_t$ as the optimization domain for the Player. We denote the discretized value functions as $\hat{V}$ and $\hat{U}$ for the Player and the Opponent respectively. For example, the Player’s value function at timestep $1 \leq t \leq T - 1$ is given by

$$\hat{V}_t (\hat{x}_{t-1}, i_t) = \min_{\hat{x}_t \in \hat{x}_t \cap \mathcal{R}(\hat{x}_{t-1}) \cap Q^{(i_t)}_t} \left\{ c (\hat{x}_{t-1}, \hat{x}_t) + \max_{i_{t+1} \in \hat{x}_{t+1}} \hat{V}_{t+1} (\hat{x}_t, i_{t+1}) \right\} . \tag{17}$$

For the detailed definition of discretized value functions, see Appendix D.1.

With the optimization domain being $\hat{x}_t \cap \mathcal{R}(\hat{x}_{t-1}) \cap Q^{(i_t)}_t$, one needs to ensure that the resolution in the intersection is fine enough to provide performance guarantee, and thus we require the mesh to satisfy (16b) and (16c). Note that if one simply enforces (16a), it is possible that the optimization domain $\hat{x}_t \cap \mathcal{R}(\hat{x}_{t-1}) \cap Q^{(i_t)}_t$ in (17) is empty, even if we have assumed $\mathcal{R}(\hat{x}_{t-1}) \cap Q^{(i_t)}_t \neq \emptyset$. Furthermore, it is also possible that the mesh under only (16a) does not have fine enough resolution within the intersection set.

We denote the discretized ‘optimal’ policies induced from the discretized value functions as $\hat{\pi}$ and $\hat{\sigma}$ for the Player and the Opponent respectively. The discretized policies are defined leveraging the argmax and argmin operators similar to those in (13) and (14). For example, the Player’s discretized optimal policy at timestep $1 \leq t \leq T - 1$ is defined as

$$\hat{\pi}(\hat{x}_{t-1}, i_t) \in \arg\min_{\hat{x}_t \in \hat{x}_t \cap \mathcal{R}(\hat{x}_{t-1}) \cap Q^{(i_t)}_t} \left\{ c (\hat{x}_{t-1}, \hat{x}_t) + \hat{U}_{t+1} (\hat{x}_t, i_{t+1}) \right\} .$$

For the detailed definition of discretized optimal policies, see Appendix D.2.

**Remark 5.** The discretized value $\hat{V}$ corresponds to the optimal worst-case performance of the Player using the discretization scheme $\hat{x}$, since we implicitly assumed in the discrete value propagation rules that the Opponent has perfect knowledge on the discretization scheme used by the Player and it exploits this knowledge through the maximization.

In other words, if both the Player and Opponent applies $\hat{\pi}$ and $\hat{\sigma}$, then the game value $\hat{U}_0(G, Q)$ is realized. On the other hand, if the Opponent unilaterally deviates and applies a policy other than $\hat{\sigma}$, then a value less than or equal to $\hat{U}_0(G, Q)$ will incur, which is favorable to the Player. As an special case, the original optimal policy $\sigma$ is also a deviation in this case. However, it is also noteworthy that $U_0(G, Q) \geq \hat{U}_0(G, Q)$ as we will show in the next subsection, which means that the ‘equilibrium’ induced by discretizing Player’s domain has, as expected, a lower Player performance than the original equilibrium.

In the rest of this section, we will provide error bounds for the discretization, and then we will present an algorithm that solves aCBC games through discretization.
4.1 Error Bounds for Discretization

Based on the Lipschitz continuity of the value functions, one expects to have good approximation of the original value function with a fine mesh. However, the discretized value at timestep $t$ is computed based on the discretized value at $t+1$ as shown in (17). Consequently, the approximation error propagates over time, and it is relatively unclear how fine a discretization is needed to provide a given desired performance guarantee. We answer the above question with the following theorem.

**Theorem 5.** Given the discretization sizes over time $\{\delta_{X,t}\}_{t=0}^T$ that satisfies (16a) to (16c), the difference between discretized value function and the original value function is bounded for all $t \in \{1, \ldots, T\}$ and $\hat{x}_{t-1} \in \hat{X}_{t-1} \cap Q_{t-1}^{(i_{t-1})}$,

\[
U_t (\hat{x}_{t-1}, i_{t-1}) \leq \hat{U}_t (\hat{x}_{t-1}, i_{t-1}) \leq U_t (\hat{x}_{t-1}, i_{t-1}) + \sum_{\tau=t}^{T-1} (L_c + L_v,_{\tau+1})\delta_{X,\tau} + L_c \cdot \delta_{X,T} \quad \forall i_{t-1} \in \mathcal{V},
\]

(18)

\[
V_t (\hat{x}_{t-1}, i_t) \leq \hat{V}_t (\hat{x}_{t-1}, i_t) \leq V_t (\hat{x}_{t-1}, i_t) + \sum_{\tau=t}^{T-1} (L_c + L_v,_{\tau+1})\delta_{X,\tau} + L_c \cdot \delta_{X,T} \quad \forall i_t \in \mathcal{N}_{t-1}.
\]

(19)

**Proof.** See Appendix D.3.

As a direct consequence of the Theorem 5, we have the following corollary regarding the game value after discretization.

**Corollary 6.** The game value after discretization $\hat{U}_0 (\mathcal{V}, \mathcal{Q})$ exceeds the original game value $U_0 (\mathcal{V}, \mathcal{Q})$ by at most $\mathcal{E}(\delta_X) = \sum_{\tau=1}^{T} (L_c + L_v,_{\tau+1})\delta_{X,\tau} + L_v,_{0} \cdot \delta_{X,0} + L_c \cdot \delta_{X,T}$.

**Proof.** See Appendix D.3.

**Remark 6.** Corollary 6 states that the discretization introduces a drop in Player’s performance, assuming the Opponent properly counteracts. However, the worst-case performance under discretization does not deviate much from the worst-case performance induced from the original value function.

Corollary 6 implies, with a discretization scheme $\delta_X = \{\delta_{X,t}\}_{t=0}^T$, the worst case performance of the Player only decreases $\mathcal{E}(\delta_X)$ comparing with the optimal performance computed from the original value function. Furthermore, the performance drop diminishes as the discretization sizes $\delta_{X,t}$ approaches zero at all timestep.

Supposing a desired performance bound $\mathcal{E}$ is given, one easy way to construct the discretization is to follow the following discretization sizes

\[
\delta_{X,0} = \frac{\mathcal{E}}{(T+1)L_v,_{1}}, \quad \delta_{X,T} = \frac{\mathcal{E}}{(T+1)L_c}, \quad \text{and} \quad \delta_{X,t} = \frac{\mathcal{E}}{(T+1)(L_c + L_v,_{t+1})} \quad \text{for} \ t = 1, \ldots, T - 1.
\]

(20)

The following algorithms present the procedure for computing the discretized optimal values.

**Algorithm 1:** Solve Discretized Value Function

**Inputs:** An aCBC instance $(\mathcal{G}, \mathcal{Q}, c, \mathcal{R}, T)$, desired suboptimality $\mathcal{E}$;

1. Compute the Lipschitz constants $L_c$ and $L_\Theta$;
2. Compute the discretization scheme $\delta_X$ via (20);
3. Construct meshes $\hat{X}$ according to the computed $\delta_X$;
4. Compute the discretized optimal value functions under $\hat{X}$ according to (17);
5. **Return** Discretized value functions $\hat{V}$ and $\hat{U}$

5 Conclusion

In this work, we extended the convex body chasing problem to an adversarial setting, where a Player chases a sequence of convex bodies assigned adversarially by an Opponent. We showed that under the assumption that the set of convex bodies is finite and known to both agents, max-min optimal policies can be obtained, which have a stronger performance guarantee than those in the classical CBC literature with competitive ratio. We presented the formula to
compute the optimal value and proved its continuity under certain assumptions. An algorithm was then proposed to first discretize the domain and then numerically solve the value function. We further provided performance guarantees for the policies constructed within the discretized domain. Through numerical examples, we verified the theoretical results. Future work will further exploit the convexity and examine geometry-based approaches for policy generation. We also want to utilize this framework to introduce cost structure to adversarial resource allocation problems, such as the dynamic Defender-Attacker Blotto Games [12].
Appendix A  Proof of Lemma 1

Recall the following definition of the Player’s value function. Note that the optimization domain depends on $x_{t-1}$. Consequently, to show that $V_t$ is continuous with respect to $x_{t-1}$, we need to first ensure that the optimization domain does not vary much if $x_{t-1}$ moves slightly.

$$V_t(x_{t-1}, i_t) = \min_{x_t \in \mathcal{R}(x_{t-1}) \cap Q_t^{(i_t)}} \left\{ c(x_{t-1}, x_t) + \max_{i_{t+1} \in \mathcal{N}_{it}} V_{t+1}(x_t, i_{t+1}) \right\}$$

To properly define how much an optimization domain moves, we leverage the Hausdorff distance to measure the distance between sets.

**Definition 1.** Consider two sets $A$ and $B$ in a metric space $(X, \text{dist})$, the Hausdorff distance between the two sets is defined as

$$\text{dist}_H(A, B) = \max \left\{ \sup_{y \in A} \inf_{y' \in B} \text{dist}(y, y'), \sup_{y' \in B} \inf_{y \in A} \text{dist}(y, y') \right\}.$$

The following lemma provides us the desired continuity property of the optimization domain with respect to the Hausdorff distance.

**Lemma 7.** For all $t \in \{1, \ldots, T\}$, $i_t-1 \in \mathcal{V}$ and $i_t \in \mathcal{N}_{i_t-1}$, the correspondence $\Theta_{t-1}(\cdot) = \mathcal{R}(\cdot) \cap Q_t^{(i_t)}$ is continuous on $Q_{i_t-1}^{(i_t)}$ with respect to the Hausdorff distance.

**Proof.** Under Assumptions 5 and 6, this lemma is a direct result of Lemma 8 presented in the Appendix A.1. \(\square\)

**Lemma 1.** For all $t \in \{1, \ldots, T\}$, $i_{t-1} \in \mathcal{V}$ and $i_t \in \mathcal{N}_{i_{t-1}}$, the optimal value functions $V_t(x_{t-1}, i_t)$ and $U_t(x_{t-1}, i_{t-1})$ are both uniformly continuous with respect to $x_{t-1} \in Q_{i_{t-1}}^{(i_{t-1})}$.

**Proof.** Due to Remark 3, we only make continuity argument for the value functions $V_t$ and $U_t$ starting at $t = 1$. For convenience, we denote the set $\mathcal{R}(x_{t-1}) \cap Q_t^{(i_t)}$ as $\Theta_t(x_{t-1})$ for all $t \in \{1, \ldots, T\}$, where $\Theta_t$ is a correspondence from $Q_{i_{t-1}}^{(i_{t-1})}$ to $Q_t^{(i_t)}$.

We first prove the uniform continuity of Player’s value function by induction.

**Base case:** Consider $t = T$. The continuity of cost function $c$ and the compactness of $\Theta_{T-1}(x_{T-1})$ imply the infimum in (2) is attainable and finite. Hence,

$$V_T(x_{T-1}, i_T) = \min_{x_T \in \mathcal{R}(x_{T-1}) \cap Q_T^{(i_T)}} c(x_{T-1}, x_T).$$

From Lemma 7, we know that the correspondence $\Theta_T(\cdot) = \mathcal{R}(\cdot) \cap Q_T^{(i_T)}$ is continuous for all $i_T \in \mathcal{V}$. Together with Assumption 4 on cost function $c$, Lemma 15 implies that for all $i_{T-1} \in \mathcal{V}$ and $i_T \in \mathcal{N}_{i_{T-1}}$, Player’s value function $V_T(x_{T-1}, i_T)$ is uniformly continuous on $Q_{i_{T-1}}^{(i_{T-1})}$.

**Inductive hypothesis:** Suppose that $V_t(x_{t-1}, i_t)$ is uniformly continuous on $Q_{t-1}^{(i_{t-1})}$ for all $i_{t-1} \in \mathcal{V}$ and $i_t \in \mathcal{N}_{i_{t-1}}$.

**Induction step:** We want to show that $V_{t-1}(x_{t-2}, i_{t-1})$ is uniformly continuous on $Q_{t-2}^{(i_{t-2})}$ for all $i_{t-2} \in \mathcal{V}$ and $i_{t-1} \in \mathcal{N}_{i_{t-2}}$. Since $V_t(x_{t-1}, i_t)$ is uniformly continuous and $\mathcal{N}_{i_{t-1}}$ consists of finitely many nodes, we can replace the sup operator in (3) with a max operator. Furthermore, Lemma 16 implies that for all $i_{t-1} \in \mathcal{V}$,

$$g_t(x_{t-1}, i_{t-1}) = \sup_{i_t \in \mathcal{N}_{i_{t-1}}} V_t(x_{t-1}, i_t) = \max_{i_t \in \mathcal{N}_{i_{t-1}}} V_t(x_{t-1}, i_t)$$

is uniformly continuous on $Q_{t-1}^{(i_{t-1})}$. Together with the assumed uniform continuity of the cost function $c$, one can further conclude that for all $i_{t-2} \in \mathcal{V}$ and $i_{t-1} \in \mathcal{N}_{i_{t-2}}$, the function $c(x_{t-2}, x_{t-1}) + \max_{i_t \in \mathcal{N}_{i_{t-1}}} V_t(x_{t-1}, i_t)$ is uniformly continuous with respect to both $x_{t-1} \in Q_{i_{t-1}}^{(i_{t-1})}$ and $x_{t-2} \in Q_{i_{t-2}}^{(i_{t-2})}$. Since $\Theta_{t-1}(x_{t-2})$ is compact, the infimum in $V_{t-1}(x_{t-2}, i_{t-1})$ is also attainable and finite. Therefore,

$$V_{t-1}(x_{t-2}, i_{t-1}) = \min_{x_{t-1} \in \mathcal{R}(x_{t-2}) \cap Q_{t-1}^{(i_{t-1})}} \left\{ c(x_{t-2}, x_{t-1}) + \max_{i_t \in \mathcal{N}_{i_{t-1}}} V_t(x_{t-1}, i_t) \right\}.$$
Since the correspondence \( \Theta_{t-1} \) is also continuous on \( Q_{t-1}^{i_{t-1}} \), by Lemma 15, we conclude that \( V_{t-1} (x_{t-2}, i_{t-1}) \) is uniformly continuous on \( Q_{t-2}^{i_{t-2}} \) for all \( i_{t-2} \in \mathcal{V} \) and \( i_{t-1} \in \mathcal{N}_{i_{t-2}} \), which completes the induction step, and we have shown the uniform continuity of the Player’s value function.

The proof for the uniform continuity of the Opponent’s value function can be constructed in a similar manner and therefore, is omitted.

\[ \]

A.1 Supporting Results for Lemma 7

**Lemma 8.** Consider a compact set \( Q \subseteq \mathcal{V} \) and a compact-valued correspondence \( \Gamma : \mathcal{X} \mapsto \mathcal{Y} \) such that \( \Gamma(x) \) is compact for all \( x \in \mathcal{X} \). Suppose \( \Gamma \) is continuous, then the correspondence \( \Theta(x) = \Gamma(x) \cap Q \) is also continuous on \( \mathcal{D} = \{ x \in \mathcal{X} : (\Theta(x))^o \neq \emptyset \} \).

**Proof.** We prove the continuity of \( \Theta \) by showing its upper and lower hemicontinuity.

We start with the upper hemicontinuity. For all \( x \in \mathcal{D} \), consider an arbitrary open neighborhood \( E \) of \( \Theta(x) \). With \( B_\epsilon(y) \) denoting the \( \epsilon \)-open ball around \( y \in \mathcal{Y} \), we define

\[
(\Gamma(x))^o = \bigcup_{y \in \Gamma(x)} B_\epsilon(y),
\]

which is non-empty and open.

Consequently, the set \( E \cup (\Gamma(x))^o \) is an open neighborhood of \( \Gamma(x) \). Since \( \Gamma \) is upper hemicontinuous, there exists a neighborhood \( F \) of \( x \) such that \( \Gamma(x') \subseteq E \cup (\Gamma(x))^o = E \cup ((\Gamma(x))^o \setminus E) \) for all \( x' \in F \). Notice

\[
\Theta(x') = \Gamma(x') \cap Q \subseteq (E \cap Q) \cup (((\Gamma(x))^o \setminus E) \cap Q) \subseteq E \cup (((\Gamma(x))^o \cap Q) \setminus E).
\]

Because \( \Gamma(x) \) and \( Q \) are always compact, \( \Gamma(x) \cap Q \) is also compact. Recall \( E \) is an open neighborhood of \( \Gamma(x) \cap Q \), from Lemma 9 we know that there exists \( \delta > 0 \) such that \( (\Gamma(x) \cap Q)_\delta \subseteq E \). Furthermore, since \( \Gamma(x) \) and \( Q \) are compact sets in \( \mathbb{R}^d \) and \( \Gamma(x) \cap Q \neq \emptyset \), Lemma 10 implies that, for this \( \delta > 0 \), there exists \( \epsilon(\delta) > 0 \) such that \( \Gamma_{\epsilon(\delta)}(x) \cap Q \subseteq (\Gamma(x) \cap Q)_\delta \subseteq E \). Consequently, \( \Theta_{\epsilon(\delta)}(x) \cap Q \setminus E = \emptyset \). With this \( \epsilon(\delta) \), we have from (21) that

\[
\Theta(x') \subseteq E \cup (((\Gamma(x))^o \cap Q) \setminus E) = E.
\]

Therefore, for all \( x \in \mathcal{D} \) and any open neighborhood \( E \) of \( \Theta(x) \), there exists a neighborhood \( F \) of \( x \) such that \( \Theta(x') \subseteq E \) for all \( x' \in F \). Consequently, \( \Theta \) is upper hemicontinuous on \( \mathcal{D} \).

Next, we show the lower hemicontinuity of \( \Theta \). For all \( x \in \mathcal{D} \), consider an arbitrary open set \( E \) such that \( E \cap \Theta(x) \neq \emptyset \). Since the interior \( (\Theta(x))^o \) is nonempty and convex, we have from Lemma 11 that \( E \cap (\Theta(x))^o \neq \emptyset \). Let \( G \) be an open set such that \( G \subseteq F \cap (\Theta(x)) = F \cap Q \cap \Gamma(x) \). Obviously, \( G \cap \Gamma(x) \neq \emptyset \). Consequently, from the lower hemicontinuity of \( \Gamma \), there exists an open neighborhood \( F' \) around \( x \) such that for all \( x' \in F' \), \( F \cap \Gamma(x') \neq \emptyset \). Since \( G \subseteq Q \), such open neighborhood also ensures that \( G \cap \Gamma(x') \cap Q \neq \emptyset \). Thus, the correspondence \( \Theta \) is also lower hemicontinuous.

Finally, by Lemma 19, we have that \( \Theta \) is continuous with respect to the Hausdorff distance over \( \mathcal{D} \).

**Lemma 9.** For every compact subset \( F \) of arbitrary open set \( E \) in \( \mathbb{R}^d \), there exists a \( \delta > 0 \) such that \( F_\delta \subseteq E \), where \( F_\delta = \bigcup_{x \in F} B_\delta(x) \).

**Proof.** Since \( E \) is open, for all \( x \in F \), there exists \( \delta(x) > 0 \) such that \( B_\delta(x) \subseteq E \). As a result, we have \( \{ B_\delta(x) \} \) as an open covering of \( F \). Due to compactness of \( F \), there is a finite subcovering \( \{ B_\delta(x_i) \} \) of \( F \).

For all \( x \in F \), there exists some \( i \in I \) such that \( x \in B_{\delta(x_i)}(x_i) \). Let \( \delta = \min_{i \in I} \{ \delta(x_i) \} > 0 \). Then, for all \( y \in B_\delta(x) \), we have \( \text{dist} (y, x_i) \leq \text{dist} (y, x) + \text{dist} (x, x_i) < \delta(x_i) \)

Therefore, for all \( x \in F \), there exists some \( i \in I \) such that \( B_\delta(x) \subseteq B_\delta(x_i)(x_i) \subseteq E \). Notice \( \{ B_\delta(x) \} \) also forms an open covering of \( F \), so there exists a \( \delta > 0 \) such that

\[
F_\delta = \bigcup_{x \in F} B_\delta(x) \subseteq E.
\]
Lemma 10. Consider two compact sets $E$ and $F$ in $\mathbb{R}^d$, such that $E \cap F \neq \emptyset$. Define the $\epsilon$ neighborhood of a set $E$ as

$$E_{\epsilon} = \bigcup_{x \in E} B_{\epsilon}(x).$$

Then, given an arbitrary $\delta > 0$, there is an $\epsilon > 0$ such that

$$E_{\epsilon} \cap F \subseteq (E \cap F)_\delta.$$

Proof. We provide a proof by contradiction. Suppose this is not the case, then for a certain $\delta_0 > 0$, there exists a sequence of $x_n \in F$ such that

$$\text{dist} (x_n, E) < \frac{1}{n} \quad \text{and} \quad \text{dist} (x_n, E \cap F) \geq \delta_0.$$

Since $F$ is compact, there is a convergent subsequence $x_{n_k} \to x_\infty$, where $x_\infty \in F$. Furthermore, by continuity of distance function $\text{dist} (\cdot, E)$ on $F$, we have $\lim_{k \to \infty} \text{dist} (x_{n_k}, E) = \text{dist} (x_\infty, E) = 0$, which implies $x_\infty \in E$. Consequently, the subsequence $x_{n_k}$ converges to $x_\infty \in E \cap F$, leading to a contradiction with the assumption $\text{dist} (x_\infty, E \cap F) \geq \delta_0 > 0$. \qed

Lemma 11. Consider an arbitrary open set $E$ and a convex set $F$ such that the interior of $F$ is non-empty, i.e. $F^o \neq \emptyset$. Then, $E \cap F \neq \emptyset$ implies $E \cap F^o \neq \emptyset$.

Proof. We provide a proof by contradiction. Suppose $E \cap F \neq \emptyset$ but $E \cap F^o = \emptyset$.

Let $x \in E \cap F$. Since $E$ is an open set, there is an $\epsilon > 0$, such that $B_{\epsilon}(x) \subseteq E$. From the assumption, we also have $B_{\epsilon}(x) \cap F^o = \emptyset$.

Take an arbitrary $y \in F^o$. From the results in [14], we have for arbitrary $\lambda \in (0, 1)$, $\lambda x + (1 - \lambda)y \in F^o$. By letting $\lambda > \left(1 - \frac{\epsilon}{\|x - y\|}\right)$, we have $\|x - z\| < \epsilon$, which implies that $z \in B_{\epsilon}(x) \cap F^o$. A contradiction. \qed

A.2 Supporting Results for Lemma 1

Proposition 12. Given an continuous correspondence $\Gamma : \mathcal{X} \mapsto \mathcal{Y}$, such that $\Gamma(x) \subseteq \mathcal{Y}$ is compact for all $x \in \mathcal{X}$. Consider a uniformly continuous function $f : \mathcal{Y} \mapsto \mathbb{R}$. Then, given an arbitrary $\epsilon > 0$ and $x \in \mathcal{X}$, there exists $\delta(x, \epsilon) > 0$ such that if $x' \in \mathcal{X}$ and $\|x - x'\| < \delta(x, \epsilon)$, then for all $y' \in \Gamma(x')$, there exists $y \in \Gamma(x)$, that satisfies $|f(y) - f(y')| < \epsilon$.

Proof. For the given $\epsilon > 0$, due to the uniform continuity of $f$, there exists $\eta(\epsilon) > 0$ such that if $\|y - y'\| < \eta(\epsilon)$, we have $|f(y) - f(y')| < \epsilon$.

The continuity of the correspondence $\Gamma$ implies that for the given $x \in \mathcal{X}$ and this $\eta(\epsilon) > 0$, there exists $\delta(x, \epsilon) > 0$ such that for all $x' \in B_{\delta(x, \epsilon)}(x)$, we have

$$\sup_{y' \in \Gamma(x')} \inf_{y \in \Gamma(x)} \|y' - y\| < \eta(\epsilon).$$

Due to the compactness of $\Gamma(x)$, for a fixed $y' \in \Gamma(x')$, there exists a $y \in \Gamma(x)$ such that

$$\|y' - y\| < \eta(\epsilon). \quad (22)$$

Then, with the uniform continuity of $f$, the above equation (22) implies $|f(y) - f(y')| < \epsilon$. \qed

Lemma 13. Consider a continuous correspondence $\Gamma : \mathcal{X} \mapsto \mathcal{Y}$, such that $\Gamma(x) \subseteq \mathcal{Y}$ is compact for all $x \in \mathcal{X}$. Suppose a function $f : \mathcal{Y} \mapsto \mathbb{R}$ is uniformly continuous, then the function $\psi(x) = \min_{y \in \Gamma(x)} f(y)$ is uniformly continuous.

Proof. Consider an arbitrary $\epsilon > 0$ and a fixed $x \in \mathcal{X}$. From Proposition 12, there exists $\delta(x, \epsilon) > 0$ such that if $x' \in \mathcal{X}$ and $\|x - x'\| < \delta(x, \epsilon)$, then for all $y' \in \Gamma(x')$, there exists $y \in \Gamma(x)$ that satisfies $|f(y) - f(y')| < \epsilon$.

We will show that this specific $\delta(x, \epsilon)$ also ensures that $|\min_{y \in \Gamma(x)} f(y) - \min_{y' \in \Gamma(x')} f(y')| < \epsilon$. We consider an arbitrary $x' \in \mathcal{X}$ such that $\|x - x'\| < \delta(x, \epsilon)$ for the rest of the proof.

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The continuity of $f$ and the compactness of $\Gamma(x)$ imply that the set of minima of $f$ over the domain $(\Gamma(x) \cup \Gamma(x'))$ is non-empty. Consequently, the minimum is attained either in $\Gamma(x)$ or $\Gamma(x')$ or both.

Without loss of generality, consider the case where the minimum can be attained in $\Gamma(x')$. Formally,

$$\left( \arg\min_{y \in \Gamma(x) \cup \Gamma(x')} f(y) \right) \cap \Gamma(x') \neq \emptyset. \quad (23)$$

In this case, we have

$$\min_{y \in \Gamma(x)} f(y) \geq \min_{y' \in \Gamma(x)} f(y) = \min_{y' \in \Gamma(x')} f(y').$$

Now, consider $y^* \in \left( \arg\min_{y \in \Gamma(x) \cup \Gamma(x')} f(y) \right) \cap \Gamma(x')$, we have

$$\left| \min_{y \in \Gamma(x)} f(y) - \min_{y' \in \Gamma(x')} f(y') \right| = \min_{y \in \Gamma(x)} f(y) - \min_{y' \in \Gamma(x')} f(y') = \min_{y \in \Gamma(x)} f(y) - f(y^*).$$

With $|x-x'| < \delta(x, \epsilon)$, from Proposition 12, we can find $\hat{y} \in \Gamma(x)$ for the $y^* \in \Gamma(x')$, such that $|f(\hat{y}) - f(y^*)| < \epsilon$. Consequently, we have

$$\left| \min_{y \in \Gamma(x)} f(y) - \min_{y' \in \Gamma(x')} f(y') \right| = \min_{y \in \Gamma(x)} f(y) - f(y^*) \leq f(\hat{y}) - f(y^*) < \epsilon,$$

which implies that $\psi : \mathcal{X} \mapsto \mathbb{R}$ is continuous. Furthermore, if $\mathcal{X}$ is compact, we have that $\psi$ is uniformly continuous.

\[\square\]

**Lemma 14.** Consider a compact set $\mathcal{X} \subseteq \mathbb{R}^d$ and two continuous functions $f, g : \mathcal{X} \mapsto \mathbb{R}$. Suppose for some $\epsilon \geq 0$, we have $|f(x) - g(x)| \leq \epsilon$ for all $x \in \mathcal{X}$, then $\min_{x \in \mathcal{X}} f(x) - \min_{x \in \mathcal{X}} g(x) \leq \epsilon$.

**Proof.** Consider an arbitrary $\hat{x} \in \arg\min_{x \in \mathcal{X}} f(x)$, we have

$$\min_{x \in \mathcal{X}} g(x) - \epsilon \leq g(\hat{x}) \leq f(\hat{x}) = \min_{x \in \mathcal{X}} f(x).$$

Similarly, consider an arbitrary $\hat{x} \in \arg\min_{x \in \mathcal{X}} g(x)$

$$\min_{x \in \mathcal{X}} f(x) \leq f(\hat{x}) \leq g(\hat{x}) + \epsilon = \min_{x \in \mathcal{X}} g(x) + \epsilon.$$

In summary, we have

$$\left| \min_{x \in \mathcal{X}} f(x) - \min_{x \in \mathcal{X}} g(x) \right| \leq \epsilon.$$

\[\square\]

**Lemma 15.** Consider $\mathcal{Y} \subseteq \mathbb{R}^d$, compact subset $\mathcal{X} \subseteq \mathbb{R}^d$ and an continuous correspondence $\Gamma : \mathcal{X} \mapsto \mathcal{Y}$ under Hausdorff distance, such that for all $x \in \mathcal{X}$, $\Gamma(x) \subseteq \mathcal{Y}$ is compact. Suppose a function $f : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$ is uniformly continuous over both $\mathcal{X}$ and $\mathcal{Y}$, then the function $\phi(x) = \min_{y \in \Gamma(x)} f(x, y)$ is also continuous.

**Proof.** Since $f$ is continuous and the set $\Gamma(x)$ is always compact, the minimization in $\phi(x)$ is well-defined. Consider an arbitrary point $x \in \mathcal{X}$ and some other point $x' \in \mathcal{X}$ in its neighborhood, we have

$$|\phi(x) - \phi(x')| \leq \min_{y \in \Gamma(x)} f(x, y) - \min_{y \in \Gamma(x')} f(x, y') + \min_{y \in \Gamma(x')} f(x, y') - \min_{y' \in \Gamma(x')} f(x', y') \quad (24)$$

Since the correspondence $\Gamma$ is continuous and the function $f$ is uniformly continuous, by Lemma 13, the function $\psi(\hat{x}) = \min_{y \in \Gamma(x)} f(x, y)$ is continuous given an arbitrary $x \in \mathcal{X}$. We choose $\hat{x}$ to be the point $x$ that we have selected. Then, for an arbitrary $\epsilon > 0$, there exists a $\xi(\epsilon) > 0$, such that for all $x' \in B_{\xi(\epsilon)}(x)$, we have

$$A = \left| \min_{y \in \Gamma(x)} f(x, y) - \min_{y' \in \Gamma(x')} f(x, y') \right| = \left| \psi(x) - \psi(x') \right| < \frac{\epsilon}{2}.$$
Due to the uniform continuity of $f$, for an arbitrary $\epsilon > 0$, there exists a $\xi_2(\epsilon) > 0$, such that $||x - x'|| < \xi_2(\epsilon)$ implies $|f(x, y') - f(x', y')| < \frac{\epsilon}{4}$ for all $y' \in \mathcal{Y}$. Since $\Gamma(x') \subseteq \mathcal{Y}$ is compact, by Lemma 14, we know that the same $\xi_2(\epsilon)$ guarantees

$$B = \left| \min_{y' \in \Gamma(x')} f(x, y') - \min_{y' \in \Gamma(x')} f(x', y') \right| \leq \frac{\epsilon}{4} < \frac{\epsilon}{2}.$$  

In summary, for any point $x \in \mathcal{X}$ and $\epsilon > 0$, if $||x - x'|| < \min\{\xi_1(x, \epsilon), \xi_2(\epsilon)\}$, then we have $|\phi(x) - \phi(x')| < \epsilon$, which implies the continuity of $\phi$. Furthermore, since the domain $\mathcal{X}$ is compact, $\phi$ is then uniformly continuous.

\[\square\]

**Lemma 16.** Let $\mathcal{X} \subseteq \mathbb{R}^d$ and $N \geq 2$. Suppose $f_i : \mathcal{X} \mapsto \mathbb{R}$ is uniformly continuous for all $i \in \{1, \ldots, N\}$, then the function $f_{\max}(x) = \max_{i \in \{1, \ldots, N\}} f_i(x)$ is uniformly continuous.

**Proof.** This lemma can be easily proved using the identity $\max\{a, b\} = \frac{1}{2}((a + b) - |a - b|)$.

\[\square\]

**Appendix B  Proof of Lemma 3**

**Lemma 3.** For all $t \in \{1, \ldots, T\}$, the Opponent value is related to the Player value via

$$U_t(x_{t-1}, i_{t-1}) = \max_{i_t \in \mathcal{N}_{i_{t-1}}} V_t(x_{t-1}, i_t). \quad (10)$$

Furthermore, when $t = 0$,

$$U_0(G, \mathcal{Q}) = \max_{i_0 \in \mathcal{V}} V_0(i_0). \quad (11)$$

Similarly, for all $t \in \{1, \ldots, T - 1\}$, the Player value is related to the Opponent value through

$$V_t(x_{t-1}, i_t) = \min_{x_t \in \mathcal{R}(x_{t-1}) \cap \mathcal{Q}^{(i_t)}} \{c(x_{t-1}, x_t) + U_{t+1}(x_t, i_{t+1})\}. \quad (12)$$

Furthermore, when $t = 0$,

$$V_0(i_0) = \min_{x_0 \in \mathcal{Q}^{(i_0)}} U_1(x_0, i_0).$$

**Proof.** We first apply induction to prove that (10) holds for all $t \in \{1, \ldots, T\}$ and $i_{t-1} \in \mathcal{V}$. Notice $\mathcal{N}_t$ is a finite set for all $i \in \mathcal{V}$.

**Base Case:** At $t = T$, (8a) and (9a) directly imply that for all $i_{T-1} \in \mathcal{V}$ and $x_{T-1} \in \mathcal{Q}_T^{(i_{T-1})}$,

$$U_T(x_{T-1}, i_{T-1}) = \max_{i_T \in \mathcal{N}_{i_{T-1}}} V_T(x_{T-1}, i_T). \quad (25)$$

**Inductive Hypothesis:** Suppose at some $t \in \{2, \ldots, T\}$, the relation $U_t(x_{t-1}, i_{t-1}) = \max_{i_t \in \mathcal{N}_{i_{t-1}}} V_t(x_{t-1}, i_t)$ holds for all $i_{t-1} \in \mathcal{V}$ and $x_{t-1} \in \mathcal{Q}_{t-1}^{(i_{t-1})}$.

**Induction Step:** We want to show that $U_{t-1}(x_{t-2}, i_{t-2}) = \max_{i_{t-2} \in \mathcal{N}_{i_{t-2}}} V_{t-1}(x_{t-2}, i_{t-2})$ for all $i_{t-2} \in \mathcal{V}$ and $x_{t-2} \in \mathcal{Q}_{t-2}^{(i_{t-2})}$.

From the definition in (10) we have

$$U_{t-1}(x_{t-2}, i_{t-2}) = \max_{i_{t-2} \in \mathcal{N}_{i_{t-2}}} \left\{ \min_{x_{t-1} \in \mathcal{R}(x_{t-2}) \cap \mathcal{Q}_{t-1}^{(i_{t-1})}} \{c(x_{t-2}, x_{t-1}) + U_t(x_{t-1}, i_{t-1})\} \right\}$$

Replace $U_t(x_{t-1}, i_{t-1})$ in the above equation with the assumed relation in the inductive hypothesis, we have

$$U_{t-1}(x_{t-2}, i_{t-2}) = \max_{i_{t-1} \in \mathcal{N}_{i_{t-2}}} \left\{ \min_{x_{t-1} \in \mathcal{R}(x_{t-2}) \cap \mathcal{Q}_{t-1}^{(i_{t-1})}} \{c(x_{t-2}, x_{t-1}) + \max_{i_t \in \mathcal{N}_{i_{t-1}}} V_t(x_{t-1}, i_t)\} \right\}$$

$$= \max_{i_{t-1} \in \mathcal{N}_{i_{t-2}}} V_{t-1}(x_{t-2}, i_{t-1}).$$
As the selections of \( i_{t-2} \in V \) and the \( x_{t-2} \) are arbitrary, we conclude that for all \( i_{t-2} \in V \) and \( x_{t-2} \in Q_{i_{t-2}}^{(i_{t-2})} \), \( U_{i_{t-2}}(x_{t-2}, i_{t-2}) = \max_{i_{t-1} \in N_{i_{t-2}}} V_{i_{t-1}}(x_{t-2}, i_{t-1}) \), which completes the induction step.

Due to the different argument presented in \( V_0 \) and \( U_0 \), we discuss \( U_0 \)'s relation to \( V_0 \) as a special case.

Based on the induction analysis, we have \( U_1(x_0, i_0) = \max_{i_1 \in N_0} V_1(x_0, i_1) \) for all \( i_0 \in V \) and \( x_0 \in Q_0^{(i_0)} \). Then (8c) implies \( V_0(i_0) = \min_{x_0 \in Q_0^{(i_0)}} U_1(x_0, i_0) \), which then implies that \( U_0(V, Q) = \max_{i_0 \in V} V_0(i_0) \).

The results relating the Opponent’s value to the Player’s value can be shown in a similar manner.

\[ \square \]

**Appendix C  Proof of Lemma 4**

**Lemma 4.** Under the strengthened Assumptions 7 and 8, we have that for all \( t \in \{1, \ldots, T\} \), \( i_{t-1} \in V \) and \( i_t \in N_{i_{t-1}} \), the optimal value functions \( V_t(x_{t-1}, i_t) \) and \( U_t(x_{t-1}, i_{t-1}) \) are both \( L_{v,t} \)-Lipschitz, where Lipschitz constant is given by

\[
L_{v,t} = L_c \cdot \sum_{k=1}^{T-t+1} (1 + L_\Theta)^{k} \tag{15}
\]

**Proof.** We will only prove the above result for \( V_t \) through induction, since the case for \( U_t \) can be easily obtained from the relations between \( V_t \) and \( U_t \) using Lemma 17.

**Base case:** Consider \( t = T \) and two distinct arbitrary points \( x_{T-1}, x'_{T-1} \in Q_{T-1}^{(i_{T-1})} \). From the assumptions, we know that for all \( i_T \in N_{i_{T-1}} \), \( \Theta(x_{T-1}) = R(x_{T-1}) \cap Q_{T-1}^{(i_T)} \neq \emptyset \), likewise for \( x'_{T-1} \). Then, we have

\[
|V_T(x_{T-1}, i_T) - V_T(x'_{T-1}, i_T)| = \min_{x_T \in \Theta(x_{T-1})} c(x_{T-1}, x_T) - \min_{x'_T \in \Theta(x'_{T-1})} c(x'_{T-1}, x'_T) \leq \min_{x_T \in \Theta(x_{T-1})} c(x_{T-1}, x_T) - \min_{x'_T \in \Theta(x'_{T-1})} c(x'_{T-1}, x'_T).
\]

Since \( c \) is Lipschitz continuous, we have \( |c(x_{T-1}, x_T) - c(x'_{T-1}, x'_T)| \leq L_c \|x_{T-1} - x'_T\| \) for all \( x_T \). Consequently, with \( \Theta(x_{T-1}) \) being compact, Lemma 14 implies \( A \leq L_c \|x_{T-1} - x'_T\| \). From Lemma 18, we have \( B \leq L_\Theta L_c \|x_{T-1} - x'_T\| \). Consequently, we have

\[
|V_T(x_{T-1}, i_T) - V_T(x'_{T-1}, i_T)| \leq L_c (1 + L_\Theta) \|x_{T-1} - x'_T\| = L_{v,T} \|x_{T-1} - x'_T\|
\]

**Inductive hypothesis:** Suppose for all \( i_{t+1}, V_{t+1}(x_t, i_{t+1}) \) is \( L_{v,t+1} \)-Lipschitz with respect to \( x_t \).

**Induction:** Recall the definition of the Player's value function shown in equation (8b)

\[
V_t(x_{t-1}, i_t) = \min_{x_t \in \Theta(x_{t-1})} \left\{ c(x_{t-1}, x_t) + \max_{i_{t+1} \in N_{i_t}} V_{i_{t+1}}(x_t, i_{t+1}) \right\}.
\]

From the inductive hypothesis and Lemma 17, we have that \( U_{t+1}(x_t, i_t) = \max_{i_{t+1} \in N_{i_t}} \{ V_{t+1}(x_t, i_{t+1}) \} \) is also \( L_{v,t+1} \)-Lipschitz. Combining with \( c \) being \( L_c \) Lipschitz and Lemma 18, we have that \( V_t(x_{t-1}, i_t) \) has a Lipschitz constant of \( (L_c + L_{v,t+1})(1 + L_\Theta) \). Plug in the definition of \( L_{v,t+1} \), we have

\[
L_{v,t} = (L_c + L_{v,t+1})(1 + L_\Theta) = (L_c + L_c \sum_{k=1}^{T-t} (1 + L_\Theta)^k)(1 + L_\Theta) = L_c \sum_{k=1}^{T-t+1} (1 + L_\Theta)^k,
\]

which completes the induction. \( \square \)
C.1 Supporting Results for Lemma 4

Lemma 17. Let $\mathcal{X} \subseteq \mathbb{R}^d$ and $N \geq 2$, suppose $f_i : \mathcal{X} \to \mathbb{R}$ is $L_i$-Lipschitz continuous for all $i \in \{1, \ldots, N\}$, then the function $f_{\max}(x) = \max_{i \in \{1, \ldots, N\}} f_i(x)$ is also $L$-Lipschitz continuous, where $L = \max_i L_i$.

Proof. We will only prove the case where $N = 2$. To extend to $N > 2$, one can use an inductive argument. Fix two arbitrary points $x, x' \in \mathcal{X}$, we have

$$f_1(x') \leq f_1(x) + L_1 \|x - x'\| \leq f_{\max}(x) + L_1 \|x - x'\|$$

and

$$f_2(x') \leq f_2(x) + L_2 \|x - x'\| \leq f_{\max}(x) + L_2 \|x - x'\|,$$

which implies

$$f_{\max}(x') \leq f_{\max}(x) + L \|x - x'\|.$$

By symmetry, we also have $f_{\max}(x) \leq f_{\max}(x') + L \|x - x'\|$, and consequently is the function $f_{\max}$ $L$-Lipschitz continuous.

Lemma 18. Consider a Lipschitz continuous function $f : \mathcal{Y} \to \mathbb{R}$ with a Lipschitz constant $L_f$ and a compact-valued correspondence $\Gamma : \mathcal{X} \to \mathcal{Y}$ that is Lipschitz under the Hausdorff distance with a Lipschitz constant $L_\Gamma$. Then, the real-valued function $\psi(x) = \min_{y \in \Gamma(x)} f(y)$ is also Lipschitz continuous with a Lipschitz constant of $L_\psi = L_f \cdot L_\Gamma$.

Proof. Consider two arbitrary points $x, x' \in \mathcal{X}$ and the difference between the value of the $\psi$ function,

$$|\psi(x) - \psi(x')| = \left| \min_{y \in \Gamma(x)} f(y) - \min_{y' \in \Gamma(x')} f(y') \right|.
$$

Similar to the proof for Lemma 13, without loss of generality, we assume that there exists a $y^* \in \mathcal{Y}$ such that

$$y^* \in \left( \arg\min_{y \in \Gamma(x)} f(y) \right) \cap \Gamma(x).$$

Then, we have

$$\min_{y \in \Gamma(x)} f(y) = \min_{y \in \Gamma(x)} f(y) \leq \min_{y' \in \Gamma(x')} f(y'),$$

which implies

$$|\psi(x) - \psi(x')| = \min_{y' \in \Gamma(x')} f(y') - \min_{y \in \Gamma(x)} f(y) = \min_{y \in \Gamma(x)} f(y) - f(y^*),$$

where $y^* \in \Gamma(x)$ as defined in (26). From the Lipschitz continuity of the correspondence $\Gamma$ and the definition of the Hausdorff distance, there exists $\bar{y} \in \Gamma(x')$, such that $\|\bar{y} - y^*\| \leq L_\Gamma \|x - x'\|$. Consequently, we have

$$|\psi(x) - \psi(x')| \leq \max_{y' \in \Gamma(x')} f(y') - f(y^*) \leq L_f \|\bar{y} - y^*\| \leq L_f L_\Gamma \|x - x'\|,$$

App. D Theoretical Results on Discretization

D.1 Definition of Discretized Value Functions

The following are the propagation rules for the discretized Player value functions. Notice that the discretized value function depends on the discretized domain $\{\mathcal{X}_t\}_{t=0}^{T-1}$. Furthermore, the discretized value function is only defined on the mesh vertices on $\{\mathcal{X}_t\}_{t=0}^{T-1}$ at each timestep.

$$\hat{V}_T(\hat{x}_T, i_T) = \min_{\hat{x}_T \in \mathcal{X}_T \cap \mathcal{R}(\hat{x}_T, i_T) \cap \mathcal{Q}^{(r)}_T} c(\hat{x}_T, i_T),$$

$$\hat{V}_t(\hat{x}_t, i_t) = \min_{\hat{x}_t \in \mathcal{X}_t \cap \mathcal{R}(\hat{x}_t, i_t) \cap \mathcal{Q}^{(r)}_t} \left\{ c(\hat{x}_t, i_t) + \max_{i_{t+1} \in \mathcal{N}_i} \hat{V}_{t+1}(\hat{x}_{t+1}, i_{t+1}) \right\}, \forall t = 1, \ldots, T - 1,$$

$$\hat{V}_0(i_0) = \min_{\hat{x}_0 \in \mathcal{X}_0 \cap \mathcal{Q}^{(r)}_0} \max_{i_1 \in \mathcal{N}_0} \hat{V}_1(\hat{x}_0, i_1).$$
The discretized Opponent value function has the following propagation rule, similar to the Player’s.

\[
\hat{U}_T (\hat{x}_{T-1}, i_{T-1}) = \max_{i_t \in \mathcal{N}_{i_{T-1}}} \left\{ \min_{\hat{x}_t \in \hat{x}_T \cap \mathcal{R}(\hat{x}_{T-1}) \cap Q_T^{(i_T)}} c (\hat{x}_{T-1}, \hat{x}_T) \right\}, \tag{29a}
\]

\[
\hat{U}_t (\hat{x}_{t-1}, i_{t-1}) = \max_{i_t \in \mathcal{N}_{i_{t-1}}} \left\{ \min_{\hat{x}_t \in \hat{x}_t \cap \mathcal{R}(\hat{x}_{t-1}) \cap Q_t^{(i_t)}} \left\{ c (\hat{x}_{t-1}, \hat{x}_t) + \hat{U}_{t+1} (\hat{x}_t, i_t) \right\} \right\}, \quad \forall \ t = 1, \ldots, T - 1,
\]

\[
\hat{U}_0 (\mathcal{V}, Q) = \max_{i_0 \in \mathcal{V}} \min_{\hat{x}_0 \in \hat{x}_0 \cap Q_0^{(i_0)}} \hat{U}_1 (\hat{x}_0, i_0). \tag{29b}
\]

**D.2 Definition of Discretized Policies**

The discretized optimal Player policies are defined as

\[
\hat{\pi}_T (\hat{x}_{T-1}, i_T) \in \argmin_{\hat{x}_T \in \hat{x}_T \cap \mathcal{R}(\hat{x}_{T-1}) \cap Q_T^{(i_T)}} c (\hat{x}_{T-1}, \hat{x}_T), \tag{30a}
\]

\[
\hat{\pi}_t (\hat{x}_{t-1}, i_t) \in \argmin_{\hat{x}_t \in \hat{x}_t \cap \mathcal{R}(\hat{x}_{t-1}) \cap Q_t^{(i_t)}} \left\{ c (\hat{x}_{t-1}, \hat{x}_t) + \hat{U}_{t+1} (\hat{x}_t, i_t) \right\}, \quad \forall \ t = 1, \ldots, T - 1, \tag{30b}
\]

\[
\hat{\pi}_0 (i_0) \in \argmin_{\hat{x}_0 \in \hat{x}_0 \cap Q_0^{(i_0)}} \hat{U}_1 (\hat{x}_0, i_0). \tag{30c}
\]

Similarly, the optimal Opponent policy is defined as

\[
\hat{\sigma}_T (\hat{x}_{t-1}, i_{t-1}) \in \argmax_{i_t \in \mathcal{N}_{i_{t-1}}} \hat{V}_t (\hat{x}_{t-1}, i_t), \quad \forall \ t = 1, \ldots, T, \tag{31a}
\]

\[
\hat{\sigma}_0 (\mathcal{G}, Q) \in \argmax_{i_0 \in \mathcal{V}} \hat{V}_0 (i_0). \tag{31b}
\]

**D.3 Error Bounds for Discretization**

**Theorem 5.** Given the discretization sizes over time \( \{\delta x_t\}_{t=0}^T \) that satisfies (16a) to (16c), the difference between discretized value function and the original value function is bounded for all \( t \in \{1, \ldots, T\} \) and \( \hat{x}_{t-1} \in \hat{x}_{t-1} \cap Q_{t-1}^{(i_{t-1})} \).

\[
U_t (\hat{x}_{t-1}, i_{t-1}) \leq \hat{U}_t (\hat{x}_{t-1}, i_{t-1}) \leq U_t (\hat{x}_{t-1}, i_{t-1}) + \sum_{\tau = t}^{T-1} (L_c + L_{c,T}) \delta x_{\tau} \leq \mathcal{V}, \quad \forall \ i_{t-1} \in \mathcal{V}, \tag{18}
\]

\[
V_t (\hat{x}_{t-1}, i_{t}) \leq \hat{V}_t (\hat{x}_{t-1}, i_{t}) \leq V_t (\hat{x}_{t-1}, i_{t}) + \sum_{\tau = t}^{T-1} (L_c + L_{c,T}) \delta x_{\tau} \leq \mathcal{V}, \quad \forall \ i_{t} \in \mathcal{N}_{i_{t-1}}. \tag{19}
\]

**Proof.** We provide a proof through induction.

**Base case:** Consider the terminal timestep \( T \) and an arbitrary \( i_{T-1} \). Fix a \( \hat{x}_{T-1} \in \hat{x}_{T-1} \cap Q_{T-1}^{(i_{T-1})} \). Clearly, we have for all \( i_T \in \mathcal{N}_{i_{T-1}}, \)

\[
\hat{V}_T (\hat{x}_{T-1}, i_T) = \min_{\hat{x}_T \in \hat{x}_T \cap \mathcal{R}(\hat{x}_{T-1}) \cap Q_T^{(i_T)}} c (\hat{x}_{T-1}, \hat{x}_T) \geq \min_{\hat{x}_T \in \mathcal{R}(\hat{x}_{T-1}) \cap Q_T^{(i_T)}} c (\hat{x}_{T-1}, \hat{x}_T) = V_T (\hat{x}_{T-1}, i_T).
\]

From the Lipschitz continuity of cost \( c \) and with the discretization size \( \delta x_{\tau} \) defined in (16c), we have that for arbitrary \( x_T \in \mathcal{R}(x_{T-1}) \cap Q_T^{(i_T)} \), there is a vertex that approximates the function value at \( x_T \) well. Formally,

\[
\min_{\hat{x}_T \in \hat{x}_T \cap \mathcal{R}(\hat{x}_{T-1}) \cap Q_T^{(i_T)}} |c (\hat{x}_{T-1}, x_T) - c (\hat{x}_{T-1}, \hat{x}_T)| \leq L_c \cdot \delta x_{\tau}.
\]
Let $\mathbf{x}_T^* \in \text{argmin}_{\mathbf{x}_T \in \mathcal{R}(\mathbf{x}_{T-1}) \cap Q_T^{(i_T)}} c(\mathbf{x}_T-1, \mathbf{x}_T)$ be the original optimal selection of $\mathbf{x}_T$, then there exists an

$$\hat{\mathbf{x}}_T^* \in \text{argmin}_{\mathbf{x}_T \in \mathcal{R}(\mathbf{x}_{T-1}) \cap Q_T^{(i_T)}} \|\mathbf{x}_T^* - \hat{\mathbf{x}}_T^*\|$$

such that $|c(\mathbf{x}_T-1, \mathbf{x}_T^*) - c(\mathbf{x}_T-1, \hat{\mathbf{x}}_T^*)| \leq L_c \cdot \delta_{X,T}$. Consequently, we have

$$V_T(\hat{\mathbf{x}}_T-1, i_T) = \min_{\mathbf{x}_T \in \mathcal{R}(\mathbf{x}_{T-1}) \cap Q_T^{(i_T)}} c(\mathbf{x}_T-1, \mathbf{x}_T) = c(\mathbf{x}_T-1, \hat{\mathbf{x}}_T^*)$$

$$\geq c(\mathbf{x}_T-1, \hat{\mathbf{x}}_T^*) - L_c \cdot \delta_{X,T}$$

$$\geq \min_{\mathbf{x}_T \in \hat{x}_T \cap \mathcal{R}(\mathbf{x}_{T-1}) \cap Q_T^{(i_T)}} c(\mathbf{x}_T-1, \hat{\mathbf{x}}_T^*) - L_c \cdot \delta_{X,T}$$

$$= V_T(\hat{\mathbf{x}}_T-1, i_T) - L_c \cdot \delta_{X,T}.$$

Based on the relationship between the Player and the Opponent value function, we have

$$\hat{U}_T(\hat{\mathbf{x}}_T-1, i_T-1) = \max_{i_T \in N_{i_T-1}} \hat{V}_T(\hat{\mathbf{x}}_T-1, i_T) \leq \max_{i_T \in N_{i_T-1}} V_T(\hat{\mathbf{x}}_T-1, i_T) + L_c \cdot \delta_{X,T}$$

$$= U_T(\hat{\mathbf{x}}_T-1, i_T-1) + L_c \cdot \delta_{X,T}.$$

Similarly, we also have

$$\hat{U}_T(\hat{\mathbf{x}}_T-1, i_T-1) \geq U_T(\hat{\mathbf{x}}_T-1, i_T-1).$$

In summary, we have for all $\hat{\mathbf{x}}_{T-1} \in \hat{\mathcal{X}}_{T-1} \cap Q_T^{(i_{T-1})}$ and $i_T \in \mathcal{Y}$

$$U_T(\hat{\mathbf{x}}_{T-1}, i_T-1) \leq \hat{U}_T(\hat{\mathbf{x}}_{T-1}, i_T-1) \leq U_T(\hat{\mathbf{x}}_{T-1}, i_T-1) + L_c \cdot \delta_{X,T},$$

$$V_T(\hat{\mathbf{x}}_{T-1}, i_T) \leq \hat{V}_T(\hat{\mathbf{x}}_{T-1}, i_T) \leq V_T(\hat{\mathbf{x}}_{T-1}, i_T) + L_c \cdot \delta_{X,T}.$$

**Inductive hypothesis:** Suppose at some timestep $t \in \{2, \ldots, T\}$, we have for all $\hat{\mathbf{x}}_{t-1} \in \hat{\mathcal{X}}_{t-1} \cap Q_T^{(i_{t-1})}$ that

$$U_t(\hat{\mathbf{x}}_{t-1}, i_{t-1}) \leq U_t(\hat{\mathbf{x}}_{t-1}, i_{t-1}) + \sum_{\tau = t}^{T-1} (L_c + L_{v, \tau+1}) \delta_{X, \tau} + L_c \cdot \delta_{X,T} \quad \forall i_{t-1} \in \mathcal{Y},$$

$$V_t(\hat{\mathbf{x}}_{t-1}, i_t) \leq V_t(\hat{\mathbf{x}}_{t-1}, i_t) + \sum_{\tau = t}^{T-1} (L_c + L_{v, \tau+1}) \delta_{X, \tau} + L_c \cdot \delta_{X,T} \quad \forall i_t \in \mathcal{N}_{i_{t-1}}.$$

For notation simplicity, we denote

$$\hat{\epsilon}_t = \sum_{\tau = t}^{T-1} (L_c + L_{v, \tau+1}) \delta_{X, \tau} + L_c \cdot \delta_{X,T}$$

**Induction:** Consider the timestep $t - 1$ and fix a discretization point $\mathbf{x}_{t-2} \in \hat{\mathcal{X}}_{t-2} \cap Q_T^{(i_{t-2})}$. From the definition of $\hat{V}_{t-1}$, we have

$$\hat{V}_{t-1}(\hat{\mathbf{x}}_{t-2}, i_{t-1}) = \min_{\mathbf{x}_{t-1} \in \hat{\mathcal{X}}_{t-1} \cap \mathcal{R}(\mathbf{x}_{t-2}) \cap Q_T^{(i_{t-1})}} \left\{ c(\hat{\mathbf{x}}_{t-2}, \mathbf{x}_{t-1}) + \hat{U}_t(\hat{\mathbf{x}}_{t-1}, i_{t-1}) \right\}$$

$$\leq \min_{\mathbf{x}_{t-1} \in \hat{\mathcal{X}}_{t-1} \cap \mathcal{R}(\mathbf{x}_{t-2}) \cap Q_T^{(i_{t-1})}} \left\{ c(\hat{\mathbf{x}}_{t-2}, \hat{\mathbf{x}}_{t-1}) + U_t(\hat{\mathbf{x}}_{t-1}, i_{t-1}) \right\} + \hat{\epsilon}_t$$

$$\leq \min_{\mathbf{x}_{t-1} \in \mathcal{R}(\mathbf{x}_{t-2}) \cap Q_T^{(i_{t-1})}} \left\{ c(\hat{\mathbf{x}}_{t-2}, \mathbf{x}_{t-1}) + U_t(\mathbf{x}_{t-1}, i_{t-1}) \right\} + L_c \delta_{X,t-1} + L_{v,t} \delta_{X,t-1} + \hat{\epsilon}_t$$

$$= V_{t-1}(\mathbf{x}_{t-2}, i_{t-1}) + \sum_{\tau = t-1}^{T-1} (L_c + L_{v, \tau+1}) \delta_{X, \tau} + L_c \cdot \delta_{X,T}$$

$$= V_{t-1}(\hat{\mathbf{x}}_{t-2}, i_{t-1}) + \hat{\epsilon}_{t-1},$$

where the first inequality is a result from the inductive hypothesis. The second inequality comes from the Lipschitz continuity of the value function $U$ and the cost $c$, as well as the fine resolution of the mesh $\hat{\mathcal{X}}_{t-1}$. 

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Similar to the base case, one can show that
\[
\hat{V}_{t-1}(\hat{x}_{t-2},i_{t-1}) = \min_{\hat{x}_{t-1} \in \hat{X}_{t-1} \cap R(\hat{x}_{t-2}) \cap Q_{t-1}^{(i_{t-1})}} \left\{ c(\hat{x}_{t-2},\hat{x}_{t-1}) + \hat{U}_t(\hat{x}_{t-1},i_{t-1}) \right\} \\
\geq \min_{\hat{x}_{t-1} \in \hat{X}_{t-1} \cap R(\hat{x}_{t-2}) \cap Q_{t-1}^{(i_{t-1})}} \left\{ c(\hat{x}_{t-2},\hat{x}_{t-1}) + U_t(\hat{x}_{t-1},i_{t-1}) \right\} \\
\geq \min_{\hat{x}_{t-1} \in \hat{X}_{t-1} \cap R(\hat{x}_{t-2}) \cap Q_{t-1}^{(i_{t-1})}} \left\{ c(\hat{x}_{t-2},x_{t-1}) + U_t(x_{t-1},i_{t-1}) \right\} \\
= V_{t-1}(\hat{x}_{t-2},i_{t-1}).
\]

Together we have for all $\hat{x}_{t-2} \in \hat{X}_{t-2} \cap Q_{t-2}^{(i_{t-2})}$,
\[
V_{t-1}(\hat{x}_{t-2},i_{t-1}) \leq \hat{V}_{t-1}(\hat{x}_{t-2},i_{t-1}) \leq V_{t-1}(\hat{x}_{t-2},i_{t-1}) + \sum_{\tau=t-1}^{T-1} (L_c + L_{v,\tau+1})\delta_{X,\tau} + L_c \cdot \delta_{X,T}.
\]
Based on the relationship between the Player and the Opponent value function and through a similar argument used in the base case, we have for all $\hat{x}_{t-2} \in \hat{X}_{t-2} \cap Q_{t-2}^{(i_{t-2})}$,
\[
U_{t-1}(\hat{x}_{t-2},i_{t-1}) \leq \hat{U}_{t-1}(\hat{x}_{t-2},i_{t-1}) \leq U_{t-1}(\hat{x}_{t-2},i_{t-1}) + \sum_{\tau=t-1}^{T-1} (L_c + L_{v,\tau+1})\delta_{X,\tau} + L_c \cdot \delta_{X,T} \quad \forall \ x_{t-2} \in V,
\]
which completes the induction. \(\square\)

**Corollary 6.** The game value after discretization $\hat{U}_0(V, Q)$ exceeds the original game value $U_0(V, Q)$ by at most $E(\delta_X) = \sum_{\tau=1}^{T-1} (L_c + L_{v,\tau+1})\delta_{X,\tau} + L_{v,1} \cdot \delta_{X,0} + L_c \cdot \delta_{X,T}$.

**Proof.** From the expression of $V_0(i_0)$ in Lemma 3, we have the discretized player value at timestep 0 as
\[
\hat{V}_0(i_0) = \min_{\hat{x}_0 \in \hat{X}_0 \cap Q_0^{(i_0)}} \hat{U}_1(\hat{x}_0,i_0).
\]
From Theorem 5, we have for all $\hat{x}_0 \in \hat{X}_0 \cap Q_0^{(i_0)}$,
\[
U_1(\hat{x}_0,i_0) \leq \hat{U}_1(\hat{x}_0,i_0) \leq U_1(\hat{x}_0,i_0) + \sum_{\tau=1}^{T-1} (L_c + L_{v,\tau+1})\delta_{X,\tau} + L_c \cdot \delta_{X,T}.
\]
Then, we have
\[
\hat{V}_0(i_0) = \min_{\hat{x}_0 \in \hat{X}_0 \cap Q_0^{(i_0)}} \hat{U}_1(\hat{x}_0,i_0) \geq \min_{\hat{x}_0 \in \hat{X}_0 \cap Q_0^{(i_0)}} U_1(\hat{x}_0,i_0) \\
\geq \min_{x_0 \in Q_0^{(i_0)}} U_1(x_0,i_0) = V_0(i_0).
\]
Furthermore,
\[
\hat{V}_0(i_0) = \min_{\hat{x}_0 \in \hat{X}_0 \cap Q_0^{(i_0)}} \hat{U}_1(\hat{x}_0,i_0) \\
\leq \min_{\hat{x}_0 \in \hat{X}_0 \cap Q_0^{(i_0)}} U_1(\hat{x}_0,i_0) + \sum_{\tau=1}^{T-1} (L_c + L_{v,\tau+1})\delta_{X,\tau} + L_c \cdot \delta_{X,T} \\
\leq \min_{x_0 \in Q_0^{(i_0)}} U_1(x_0,i_0) + L_{v,1} \cdot \delta_{X,0} + \sum_{\tau=1}^{T-1} (L_c + L_{v,\tau+1})\delta_{X,\tau} + L_c \cdot \delta_{X,T} \\
= V_0(i_0) + L_{v,1} \cdot \delta_{X,0} + \sum_{\tau=1}^{T-1} (L_c + L_{v,\tau+1})\delta_{X,\tau} + L_c \cdot \delta_{X,T}.
\]
Therefore, we have for all \( i_0 \in \mathcal{V} \),
\[
V_0(i_0) \leq \tilde{V}_0(i_0) \leq V_0(i_0) + L_{v, 1} \cdot \delta_{X, 0} + \sum_{\tau = 1}^{T - 1} (L_c + L_{v, \tau + 1}) \delta_{X, \tau} + L_c \cdot \delta_{X, T}.
\]

Finally, we relate the discretized Opponent value to that of the Player via
\[
\hat{U}_0(\mathcal{V}, \mathcal{Q}) = \max_{i_0 \in \mathcal{V}} \tilde{V}_0(i_0). \tag{36}
\]

One can then easily arrive at
\[
U_0(\mathcal{V}, \mathcal{Q}) \leq \hat{U}_0(\mathcal{V}, \mathcal{Q}) \leq U_0(\mathcal{V}, \mathcal{Q}) + L_{v, 1} \cdot \delta_{X, 0} + \sum_{\tau = 1}^{T - 1} (L_c + L_{v, \tau + 1}) \delta_{X, \tau} + L_c \cdot \delta_{X, T}. \tag{37}
\]

\[\square\]

**Appendix E  Continuity of Compact-Valued Correspondences**

**Definition 2.** Consider two metric spaces \( \mathcal{X} \) and \( \mathcal{Y} \). A correspondence \( \Gamma : \mathcal{X} \rightrightarrows \mathcal{Y} \) is said to be upper hemicontinuous at the point \( \mathbf{x} \) if for any open neighbourhood \( E \) of \( \Gamma(\mathbf{x}) \), there exists a neighborhood \( F \) of \( \mathbf{x} \) such that for all \( \mathbf{x}' \in F \) we have \( \Gamma(\mathbf{x}') \subseteq E \).

**Definition 3.** Consider two metric spaces \( \mathcal{X} \) and \( \mathcal{Y} \). A correspondence \( \Gamma : \mathcal{X} \rightrightarrows \mathcal{Y} \) is said to be lower hemicontinuous at the point \( \mathbf{x} \) if for any open set \( E \) such that \( \Gamma(\mathbf{x}) \cap E \neq \emptyset \), there exists a neighborhood \( F \) of \( \mathbf{x} \) such that for all \( \mathbf{x}' \in F \) we have \( \Gamma(\mathbf{x}') \cap E \neq \emptyset \).

**Lemma 19.** Consider two metric spaces \( \mathcal{X} \) and \( \mathcal{Y} \). A compact valued correspondence \( \Gamma : \mathcal{X} \rightrightarrows \mathcal{Y} \) is continuous with respect to the Hausdorff distance if and only if it is both upper and lower hemicontinuous.

**Proof.** For the sufficiency, we want to show that if \( \Gamma \) is both upper and lower hemicontinuous, then given a fixed \( \mathbf{x} \in \mathcal{X} \) and an \( \epsilon > 0 \), there exists a \( \delta(\mathbf{x}) > 0 \), such that for all \( \mathbf{x}' \in B_{\delta(x)}(x) \), we have
\[
\text{dist}_H(\Gamma(\mathbf{x}), \Gamma(\mathbf{x}')) = \max \left\{ \sup_{y \in \Gamma(\mathbf{x}')} \inf_{y' \in \Gamma(\mathbf{x})} \text{dist}(y, y'), \sup_{y \in \Gamma(\mathbf{x})} \inf_{y' \in \Gamma(\mathbf{x}')} \text{dist}(y, y') \right\} < \epsilon \tag{38}
\]

We start with the first component in the max operator. Since the correspondence is upper hemicontinuous, for a given \( \epsilon \), we can find a \( \delta_U \), such that for all \( \mathbf{x}' \in B_{\delta_U(x)}(x) \), we have
\[
\Gamma(\mathbf{x}') \subseteq \bigcup_{y \in \Gamma(\mathbf{x})} B_\epsilon(y), \tag{39}
\]
which implies that
\[
\sup_{y' \in \Gamma(\mathbf{x}')} \inf_{y \in \Gamma(\mathbf{x})} \text{dist}(y, y') \leq \epsilon.
\]

Next, we work on the second component in (38). Since the correspondence is compact-valued, the \( \epsilon \)-neighborhood around \( \Gamma(\mathbf{x}) \) can be expressed as a finite subcover. More specifically,
\[
(\Gamma(\mathbf{x}'))_\epsilon = \bigcup_{y \in \Gamma(\mathbf{x})} B_\epsilon(y) = \bigcup_{k=1}^{K} B_\epsilon(y_k),
\]
where \( y_k \in \Gamma(\mathbf{x}) \). Since \( \Gamma \) is lower hemicontinuous, for each \( y_k \), there exists a \( \delta_{L, k}(\mathbf{x}) \) such that, for all \( \mathbf{x}' \in B_{\delta_{L, k}(x)}(x) \), we have
\[
\Gamma(\mathbf{x}') \cap B_\epsilon(y_k) \neq \emptyset. \tag{40}
\]
Take \( \delta_L(\mathbf{x}) = \min_k \delta_{L, k}(\mathbf{x}) \). Then for all \( \mathbf{x}' \in B_{\delta_L(x)}(x) \), \( \Gamma(\mathbf{x}') \cap B_\epsilon(y_k) \neq \emptyset \) for all \( k \).

Then, given an arbitrary \( y \in \Gamma(\mathbf{x}) \), we have \( y \in B_\epsilon(y_{k^*}) \) for some \( k^* \). Since \( \Gamma(\mathbf{x}') \cap B_\epsilon(y_{k^*}) \neq \emptyset \), there exists a \( \hat{y} \in \Gamma(\mathbf{x}') \) such that \( \text{dist}(y_{k^*}, \hat{y}) < \frac{\epsilon}{2} \). Consequently, this \( \hat{y} \in \Gamma(\mathbf{x}') \) satisfies
\[
\text{dist}(y, \hat{y}) \leq \text{dist}(y, y_{k^*}) + \text{dist}(y_{k^*}, \hat{y}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{41}
\]
Since the selection of $y$ is arbitrary, the above implies

$$\sup_{y \in \Gamma(x)} \inf_{y' \in \Gamma(x')} \text{dist}(y, y') \leq \epsilon.$$ 

In summary, if one take $\delta(x) = \min\{\delta_U(x), \delta_L(x)\}$, then for all $x' \in B_{\delta(x)}(x)$, we have

$$\text{dist}_H(\Gamma(x), \Gamma(x')) < \epsilon.$$ 

For the necessity, we need to show that if $\Gamma$ is continuous under the Hausdorff distance, it is both upper and lower hemicontinuous. We start with upper hemicontinuity. Fix an arbitrary $x \in X$ and consider an arbitrary open set $E$ such that $\Gamma(x) \subseteq E$. Since $\Gamma(x)$ is compact, we can find an $\epsilon > 0$ such that $\Gamma(x) \subseteq (\Gamma(x))_\epsilon \subseteq E$. Since $\Gamma$ is continuous with respect to the Hausdorff distance, we can find a $\delta(x)$ for this specific $\epsilon$, such that for all $x' \in B_{\delta(x)}(x)$, $\text{dist}_H(\Gamma(x), \Gamma(x')) < \epsilon$, which implies $\Gamma(x') \subseteq (\Gamma(x))_\epsilon \subseteq E$, and thus we have upper hemicontinuity.

For the lower hemicontinuity, fix an arbitrary $x \in X$ and consider an arbitrary open set $E$ such that $\Gamma(x) \cap E \neq \emptyset$. Let $y^* \in \Gamma(x) \cap E \neq \emptyset$. Since $E$ is open, there exists an $\epsilon > 0$, such that $B_\epsilon(y^*) \subseteq E$. Since $\Gamma$ is continuous with respect to the Hausdorff distance, we can find a $\delta(x)$ for this specific $\epsilon$, such that for all $x' \in B_{\delta(x)}(x)$, $\text{dist}_H(\Gamma(x), \Gamma(x')) < \epsilon$, which implies that for all $y \in \Gamma(x)$, there exists a $y' \in \Gamma(x')$ such that $\text{dist}(y, y') < \epsilon$. Consequently, for all $x' \in B_{\delta(x)}(x)$, there exists a $y' \in \Gamma(x')$ such that $y' \in B_\epsilon(y^*) \subseteq E$, which implies $\Gamma(x') \cap E \neq \emptyset$, and thus we have lower hemicontinuity.
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