Two-dimensional topological superconducting phases emerged from d-wave superconductors in proximity to antiferromagnets

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Abstract – Motivated by the recent observations of nodeless superconductivity in the monolayer CuO\textsubscript{2} grown on the Bi\textsubscript{2}Sr\textsubscript{2}CaCu\textsubscript{2}O\textsubscript{8+δ} substrates, we study the two-dimensional superconducting (SC) phases described by the two-dimensional t-J model in proximity to an antiferromagnetic (AF) insulator. We found that i) the nodal d-wave SC state can be driven via a continuous transition into a nodeless d-wave pairing state by the proximity-induced AF field. ii) The energetically favorable pairing states in the strong field regime have extended s-wave symmetry and can be nodal or nodeless. iii) Between the pure d-wave and s-wave paired phases, there emerge two topologically distinct SC phases with (s + id) symmetry, \textit{i.e.}, the weak and strong pairing phases, and the weak pairing phase is found to be a $Z_2$ topological superconductor protected by valley symmetry, exhibiting robust gapless nonchiral edge modes. These findings strongly suggest that the high-$T_c$ superconductors in proximity to antiferromagnets can realize fully gapped symmetry-protected topological SC.

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Introduction. – Despite the intensive research in the past 30 years, the field of high-$T_c$ superconductivity (SC) in the cuprates [1–4] continues to generate surprising and challenging issues. Up to now all the high-$T_c$ superconducting copper oxides have layered structures, and the superconducting layers are sandwiched by insulating charge reservoir layers. It is usually believed that the CuO\textsubscript{2} layers are antiferromagnetic (AF) Mott insulators. The modulation of charge carriers in the CuO\textsubscript{2} planes is realized through the substitution of chemical elements in the reservoir. In hole-doped cuprates, it has been established that SC around the optimal doping has a $d$-wave pairing with gap nodes along the zone diagonals [5–7].

Recently, Zhong \textit{et al.} reported that a monolayer CuO\textsubscript{2} is successfully grown on the optimally doped Bi\textsubscript{2}Sr\textsubscript{2}CaCu\textsubscript{2}O\textsubscript{8+δ} (Bi-2212) substrates via molecular beam epitaxy [8], enabling a direct probe of the CuO\textsubscript{2} plane by scanning tunneling microscopy. Their results are interesting and important. Unlike the sandwiched CuO\textsubscript{2} layers in the bulk Bi-2212, the overall electronic spectral density on the monolayer films is characterized by a large ($\sim$ 2 eV) Mott-Hubbard-like gap [8]. In the low-energy regime, however, two distinct and spatially separated energy gaps are observed on the films: the V-shaped gap is similar to the gap observed on the BiO layer, and the U-shaped gap is identified by its superconducting nature [8]. Such an U-shaped superconducting gap is in striking contrast with the nodal gap in the $d_{x^2-y^2}$-wave pairing symmetry [5–7]. Therefore, the observed superconductivity in the monolayer CuO\textsubscript{2} on optimally doped Bi-2212 substrates raises a challenge whether a new nodeless SC appears at the interface between nearly optimally doped Bi-2212 and the CuO\textsubscript{2} AF insulating layer (see fig. 1(a)).

Motivated by this recent experiment, we examine the possible superconducting phases derived from the standard $t$-$J$ model with nearest-neighbor singlet pairings in proximity to an antiferromagnetic insulating layer. In the absence of the AF layer, the superconducting state of the $t$-$J$ model has a purely $d$-wave pairing phase with four nodes for doping up to $\delta = 0.25$. The proximity effect induced by the AF layer is modeled by applying an external AF field. With the parameters being relevant for the cuprates, $J/t = 0.3$, $t'/t = 0.2$ and $0.01 < \delta < 0.16$, our findings are summarized in the phase diagram in terms
of hole doping $\delta$ and the AF field $m_z$, shown in fig. 1(b). From weak to moderate $m_z$, we found a continuous transition from the nodal $d$-wave to a nodeless $d$-wave SC with reduced doping or increased $m_z$. However, the strong AF field drives the $d$-wave pairing unstable and the extended $s$-wave pairing more favored energetically, leading to a fully gapped $s$-wave phase and a continuous transition to the nodal $s$-wave pairing at higher doping.

In the intermediate region of $m_z$, the pure $d$-wave and $s$-wave pairings are degenerate in energy such that the SC states with mixed $(s + id)$ pairing emerge as the energetically most favorite. These mixed pairing states preserve valley and mirror symmetries. Remarkably, we found that there exist two topologically distinct gapped SC states, termed as $(s + id)_s$ and $(s + id)_w$ corresponding to the strong and weak pairing SC phases, respectively [9]. The $(s + id)_w$ phase is identified as a $Z_2$ topological SC protected by the valley symmetry and supports robust gapless nonchiral edge modes. Our findings strongly suggest that the high-$T_C$ copper-oxide superconductors in proximity to AF insulating phases can not only produce fully gapped SC states of various pairing symmetries, but also potentially realize topological valley SC.

**Model and theory.** We start to consider the square lattice $t$-$J$ model for the $d$-wave superconductor in proximity to an AF insulating layer (fig. 1(a)),

$$H = -t \sum_{\mathbf{r}, \eta, \sigma} c^\dagger_{\mathbf{r}, \sigma} c_{\mathbf{r}+\eta, \sigma} + t' \sum_{\mathbf{r}, \gamma, \sigma} c^\dagger_{\mathbf{r}, \sigma} c_{\mathbf{r}+\gamma, \sigma} + \frac{J}{2} \sum_{\mathbf{r}, \eta} (S^x_{\mathbf{r}} S^x_{\mathbf{r}+\eta} - \frac{1}{4} m_{\mathbf{r}} m_{\mathbf{r}+\eta}) - \mu_0 \sum_{\mathbf{r}, \sigma} c^\dagger_{\mathbf{r}, \sigma} c_{\mathbf{r}, \sigma} + m_z \sum_{\mathbf{r}, \sigma} \sigma (-1)^{\mathbf{r}} c^\dagger_{\mathbf{r}, \sigma} c_{\mathbf{r}, \sigma},$$

(1)

where $t$ and $t'$ are the nearest-neighbor (NN) and next-nearest-neighbor (NNN) hoppings, $\eta$ and $\gamma$ denote the corresponding vectors, $J$ is the NN Heisenberg exchange interaction, and the proximate AF insulating layer has an in-plane staggered field $m_z$ along the $y$-direction. The square lattice is bipartitioned into a checkerboard A-B sublattice. Although the AF ordering inevitably breaks the time-reversal symmetry $T = i \sigma_y K$, each unit cell shows zero net magnetic field, and the product of the time reversal and one unit lattice translation $\tau$ (equivalent to switching A-B sub-lattices) $\tilde{T} \equiv i \sigma_y \tau_z K$ is respected, where $\tau_z$ denotes the Pauli matrix acting upon the sublattice spinor space with the eigen-spinor of $\tau_z = \pm 1$ living on A/B sublattice. The site-centered mirror symmetry with respect to the $y = 0$ plane $M_y = i \sigma_y$ is respected, while the mirror reflection regarding $x = 0$ or $z = 0$ plane is preserved only by combining the unit lattice translation $M_{x/z} \equiv i \sigma_x/z \tau_z$, which is the bond-centered mirror reflection symmetry. Thanks to the singlet pairing nature, shifting the AF field from in-plane to out-of-plane only adapts the mirror symmetries analysis, while most of our results still hold.

In this paper, we fix the parameters to be relevant for the cuprates: $J/t = 0.3, t'/t = 0.2$ and $0.01 < \delta < 0.16$. The local constraint of no double occupancy $\sum_{\mathbf{r}, \sigma} c^\dagger_{\mathbf{r}, \sigma} c_{\mathbf{r}, \sigma} \leq 1$ has to be imposed. Writing the electron operators in terms of a fermionic spinon and a bosonic holon $c_{\mathbf{r}, \sigma} = b^\dagger_{\mathbf{r}, \sigma} f_{\mathbf{r}, \sigma}$, the constraint changes into an equality $b^\dagger_{\mathbf{r}} b_{\mathbf{r}} + \sum_{\mathbf{r}} f^\dagger_{\mathbf{r}} f_{\mathbf{r}} = 1$, which can be enforced using a Lagrangian multiplier $\lambda$. The doping concentration is given by $\delta = \langle b^\dagger_{\mathbf{r}} b_{\mathbf{r}} \rangle$ and we have $\sum_{\mathbf{r}} \langle f^\dagger_{\mathbf{r}} f_{\mathbf{r}} \rangle = 1 - \delta$.

In the mean-field (MF) approach to the SC state [4,10,11], the holons condense, i.e., $b^\dagger_{\mathbf{r}}$ and $b_{\mathbf{r}}$ are replaced by their expectation value $\sqrt{\delta}$. The superexchange interaction in eq. (1) can be decoupled in the paramagnetic valence bond and spin-singlet pairing channels by introducing the order parameters

$$\chi \equiv \frac{J}{4} \langle f^\dagger_{\mathbf{r}} f_{\mathbf{r}+\eta} + f^\dagger_{\mathbf{r}} f_{\mathbf{r}+\eta} \rangle,$$

$$\Delta_0 \equiv \frac{J}{4} \langle f_{\mathbf{r}+\eta} f_{\mathbf{r}} - f_{\mathbf{r}} f_{\mathbf{r}+\eta} \rangle.$$

(2)

In general, we assume $\Delta_x = \Delta_x + i \Delta_y$ and $\Delta_y = \Delta_y - i \Delta_d$, where $\Delta_x$ and $\Delta_d$ are the amplitudes of the NN...
spin-singlet pairing with $s_{x^2+y^2}$-symmetry and $d_{x^2-y^2}$-symmetry, respectively. In momentum space, the MF Hamiltonian can be written in terms of Nambu spinors $\Psi^\dagger_{\alpha,k} = (f^\dagger_{\uparrow,k}, f^\dagger_{\uparrow,k+Q}, f_{\downarrow,-k}, f_{\downarrow,-k-Q})$, where $Q = (\pi, \pi)$ is the AF wave vector. Writing $H_{MF} = \frac{1}{2} \sum_{k} F^\dagger_{k} H_{k} F_{k}$ in the unfolded Brillouin zone, we derive the Hamiltonian matrix as

$$H_{k} = \begin{pmatrix} m_{s} \tau^{x} + \epsilon_{k} \tau^{z} + \epsilon'_{k} - \mu & \Delta_{k} \tau^{y} \\ \Delta_{k} \tau^{y} & m_{s} \tau^{x} - \epsilon_{k} \tau^{z} - \epsilon'_{k} + \mu \end{pmatrix},$$

where $\mu \equiv \mu_{0} - \lambda$ is the renormalized chemical potential, $\epsilon_{k} = -2(\delta + \chi)(\cos k_{x} + \cos k_{y})$, $\epsilon'_{k} = 4t' \delta \cos k_{x} \cos k_{y}$, and the SC gap function is defined by

$$\Delta_{k} = 2\Delta_{d}(\cos k_{x} - \cos k_{y}) - i2\Delta_{s}(\cos k_{x} + \cos k_{y})$$

up to a global phase that can be gauge-fixed.

It should be emphasized that the MF Hamiltonian with pure singlet pairing does not break the symmetry $\hat{T}$ and mirror symmetries of the prototypical Hamiltonian, but the mixed phase of $s$-wave and $d$-wave pairings would break the symmetry $\hat{T}$ spontaneously. To make the physics more transparent, we adopt a two-step strategy to diagonalize the MF Hamiltonian: first the normal sector spanned by AF ordering is diagonalized as

$$\xi_{\pm,k} = \epsilon'_{k} - \mu \pm \sqrt{\epsilon'_{k}^{2} + m_{s}^{2}},$$

with corresponding AF quasi-particles

$$\begin{align*}
\psi^\dagger_{\uparrow, k, \sigma} &= (\cos \theta_{k}) f^\dagger_{\uparrow, k, \sigma} + \sigma (\sin \theta_{k}) f^\dagger_{\uparrow, k+Q, \sigma}, \\
\psi^\dagger_{\downarrow, k, \sigma} &= (\sin \theta_{k}) f^\dagger_{\downarrow, k, \sigma} - \sigma (\cos \theta_{k}) f^\dagger_{\downarrow, k+Q, \sigma},
\end{align*}$$

where $\theta_{k} \equiv \frac{1}{2} \tan^{-1} \frac{m_{s}}{\Delta_{d}} \in [0, \pi]$. The normal state owns two-fold Kramer’s degeneracy due to the symmetry $\hat{T}$, which is anti-unitary and $\hat{T}^{2} = -1$. Furthermore, the symmetry $\hat{T}$ acting on the slave spinor is equivalent to the time-reversal $\hat{T}$ acting on the AF quasi-particles, and similarly for the mirror symmetries:

$$\hat{T}^{-1} \left( f^\dagger_{k, \sigma} f^\dagger_{k+Q, \sigma} \right) \hat{T} \leftrightarrow \hat{T}^{-1} \left( \psi^\dagger_{\uparrow, k, \sigma} \psi^\dagger_{\downarrow, k+Q, \sigma} \right) \hat{T} = \left( \psi^\dagger_{\uparrow, k, \sigma} \psi^\dagger_{\downarrow, k+Q, \sigma} \right),$$

where $M_{x/z}^{-1} f^\dagger_{k+Q, \sigma} M_{x/z} \leftrightarrow M_{x/z}^{-1} \psi^\dagger_{\uparrow, k, \sigma} M_{x/z} \psi^\dagger_{\downarrow, k+Q, \sigma}$,

$$M_{y}^{-1} f^\dagger_{k, \sigma} M_{y} \leftrightarrow M_{y}^{-1} \psi^\dagger_{\uparrow, k, \sigma} M_{y} \psi^\dagger_{\downarrow, k+Q, \sigma}.$$\hspace{1cm}(6)

Physically, this suggests the original time-reversal $\hat{T}$ and $M_{x/z}$ restore for the AF quasi-particles without combining the sublattice transformations, which can be understood as that AF quasi-particles simply lose sight of the sublattice. The inter-band pairing between these two species of AF quasi-particles is found to be absent due to the NN singlet pairing nature. Hence the MF Hamiltonian is decoupled into $H_{MF} = \frac{1}{4} \sum_{k, \alpha = \pm} \xi_{\alpha,k} \Psi^\dagger_{\alpha,k} H_{\alpha,k} \Psi_{\alpha,k}$, where

$$H_{\pm,k} = \xi_{\pm,k} \rho_{2} \pm (\Delta_{k} \rho_{+} + \Delta_{k} \rho_{-}),$$

with the Nambu spinor $\Psi^\dagger_{\pm,k} \equiv \left( \psi^\dagger_{\uparrow,k+Q}, \psi^\dagger_{\downarrow,-k} \right)$, $\psi_{\pm,k} = \psi_{\pm,k+Q}$ and $\psi_{\pm,-k}$. Thus, the energy spectrum appears pairwise as $\pm E_{\pm}(k)$,

$$E_{\pm}(k) = \sqrt{\xi_{\pm}^{2}(k) + |\Delta_{k}|^{2}},$$

The SC ground-state energy density is given by

$$\varepsilon_{g} = \frac{1}{2N} \sum_{k} \left[ E_{+}(k) + E_{-}(k) \right] + \frac{J}{8} (1 - \delta) - \mu \delta$$

$$+ \frac{8}{J} \left( \kappa^{2} + \Delta_{s}^{2} + \Delta_{d}^{2} \right) + \mu_{0} (\delta - 1).$$

Then the saddle point equations are obtained by minimizing the ground-state energy $\partial \varepsilon_{g}/\partial (\chi, \Delta_{s}, \Delta_{d}, \mu) = 0$, from which the MF parameters $(\chi, \Delta_{s}, \Delta_{d}, \mu)$ are determined self-consistently.

**Phase diagram of superconducting phases.** – The complete phase diagram in the $m_{s}$-$\delta$ plane (fig. 1(b)) is obtained by minimizing the ground-state energy and considering the nodeness of the quasi-particle spectrum. As is shown in fig. 2, when the AF field $m_{s}$ is relatively weak, the $d_{x^2-y^2}$-wave pairing symmetry dominates which concurs with the consensus. However, a strong proximate AF field drives the energetically favorite pairing symmetry from $d_{x^2-y^2}$-wave to $s_{x^2+y^2}$-wave continuously through a mixed pairing regime of $s_{x^2+y^2}+id_{x^2-y^2}$. Further, each region is bisected into two different phases.

In the $d_{x^2-y^2}$-wave phase, the nodes of the lower Bogoliubov quasi-particle spectrum $E_{-}(k)$ appear when the zero lines of $\xi_{-}(k) = 0$ intercepts that of $\Delta_{k} = 0$, i.e., satisfying the condition $a \cos^{2}2k_{x} + b \cos 2k_{y} + c = 0$, where $a = (t' \delta)^2$, $b = 2(t' \delta)^2 - \mu' \delta - 2(t' + \gamma) \kappa$, and $c = (t' \delta - \mu)^2 - 2(t' + \gamma)^2 - m_{s}^{2}/4$. According to this criterion, the $d_{x^2-y^2}$-wave region is divided into the nodal phase for moderate doping or a weak AF field, and a nodeless phase for underdoping or relatively strong AF field.
Quite similarly, the $s_{x^2+y^2}$-wave phase shows nodes in the spectrum when $\cos 2K_{xy} = -\frac{m_x^2 + m_y^2}{2t^2} = 1$ is satisfied, and is therefore divided into nodal and nodeless phases as well. The mixed pairing $s_{x^2+y^2} + id_{x^2-y^2}$ also consists of two fully gapped phases $(s + id)_w$ and $(s + id)_s$ separated by a critical line on which the spectrum exhibits four nodes at $X_{\pm} \equiv (\pi/2, \pm \pi/2)$ and their inversion partners.

In fig. 1(b), the critical line of the nodal and nodeless $d_{x^2-y^2}$-wave SC (marked by red line) connects the critical line dividing the $(s + id)_w$ and $(s + id)_s$ phases (green line), which further joins the critical line separating the nodal and nodeless $s_{x^2+y^2}$-wave phases (blue line). This is guaranteed by the hidden topological nature. In fact, as we shall show later, the nodal $d_{x^2-y^2}$, $(s + id)_w$ and nodal $s_{x^2+y^2}$ SC phases can be classified as the weak pairing and topologically nontrivial, while the nodeless-$d_{x^2-y^2}$, $(s + id)_s$ and nodeless-$s_{x^2+y^2}$ as the strong pairing and topologically trivial. So the joint critical lines penetrating the phase diagram are essentially the phase transition from weak pairing to strong pairing regardless of pairing symmetry.

**Nodal $d$-wave SC and its phase transitions.** The nodal $d$-wave SC in the presence of AF field shows a pair of inequivalent nodes in the first quadrant of the unfolded Brillouin zone as a result of the band folding. The locations of the nodes are denoted by $K_{\pm} \equiv (K_+, K_{\pm})$ ($0 < K_+ < \pi/2$ and $K_- = \pi - K_+ > \pi/2$). The nodes located on the other quadrants of the Brillouin zone are related to $K_{\pm}$ by mirror reflection or inversion. This underlying topology is encoded in the low-energy Bogoliubov quasi-particles in the vicinity of these nodes. To examine the low-energy effective Hamiltonian, we expand the MF Hamiltonian around the nodal points $K_{\pm}$. In the AF quasi-particle basis, $\Psi^{\dagger}_{\pm}(\mathbf{k})$ is fully gapped and frozen in the low-energy limit, therefore, the BdG effective Hamiltonian for $\Psi^{\dagger}_{\pm}(\mathbf{k})$ is obtained

$$H_{\text{eff}}(K_{\pm} + \mathbf{q}) = \pm v_1 q_x \rho_x + v_1 q_y \rho_y \equiv \tilde{h}_{\pm}(\mathbf{q}) \cdot \tilde{\rho}, \quad (10)$$

where $v_{\pm} \equiv v_x \pm v_y$ and the two characteristic velocities are $v_1 = 2\Delta_q \sin K_+$ and $v_1 = -2\delta' \sin 2K_+ + 4(\delta + \kappa)^2 \sin 2K_+ / \sqrt{m_x^2 + 16(\delta + \kappa)^2 \cos^2 K_+}$. Actually this matrix is similar to that describing a pair of two-dimensional Weyl fermions with opposite chirality around $K_{\pm}$, and the pseudo-magnetic field $\tilde{h}_{\pm}(\mathbf{q})$ exhibits anti-vortex/vortex topological texture with nodes being the vortex cores. This underlying topology entails robust Andreev bound states on the edges with momentum residing between the projection of nodes [12]. For convenience, eq. (10) only shows the two valleys in the first quadrant of Brillouin zone while leaving their mirror partners on $\pm(K_{\pm}, -K_0)$ and time-reversal partners on $-K_{\pm}$ behind. The whole system preserves the mirror symmetries and emergent $T$ symmetry. In fact, the nontrivial topology of the nodal $d$-wave SC is protected by the emergent $T$ symmetry, which forbids the mass term proportional to the matrix $\rho_y$ and confines $\tilde{h}_{\pm}(\mathbf{q})$ to lie in-plane. So the anti-vortex/vortex structure is guaranteed and cannot be destroyed by arbitrary weak perturbations.

Actually there are two distinct ways to gap out the nodes by bestowing a mass term upon the Weyl fermion-like quasi-particles. The first one is to gradually tune the positions of a pair of nodes to merge so that the coupling of quasi-particles generates a mass for each other, driving them into a massive Dirac fermion. This is the only way allowed by the emergent $T$ symmetry. From the perspective of $\tilde{h}_{\pm}(\mathbf{q})$, the vortex and anti-vortex annihilate with each other, killing the nontrivial topology. This process can be achieved by increasing the AF field or decreasing the doping concentration, and the weak pairing nodal $d$-wave SC thus changes into the nodeless $d$-wave SC through a continuous phase transition [13,14]. The evolution of the Bogoliubov quasi-particle spectrum through this transition is shown in fig. 3. The critical point is characterized by a highly anisotropic Bogoliubov dispersion: along the nodal line the dispersion is quadratic nonrelativistic, while when perpendicular to the nodal line it is the linear Dirac dispersion.

The other way of gapping out the nodes is to directly introduce a mass term upon the pair of Weyl fermion-like quasi-particles. This can be achieved in the process from the nodal $d$-wave SC to the $(s + id)_w$ SC with the emergent additional extended s-wave pairing component. The corresponding low-energy effective Hamiltonian eq. (10) is 37004-p4
then changed into
\[ H_{\text{eff}}(K_{\pm} + q) = \pm v_3 q_y \rho_z + v_1 q_x \rho_x \mp (4 \Delta_x \cos K_{\pm}) \rho_y \equiv \tilde{n}_{\pm}(q) \cdot \tilde{\rho}. \] (11)

The corresponding dispersion evolves as shown in fig. 3. It is important to point out that the mass term introduced via \( s_{x^2+y^2} \) pairing does not suppress the topology of the nodal \( d \)-wave SC. At the cost of breaking the time-reversal symmetry \( T \), the \( s_{x^2+y^2} \) pairing contributes to the out-of-plane component of \( \tilde{n}_{\pm}(k) \), driving the antivortex/vortex texture into a meron/anti-meron instead, forming a skyrmion. The skyrmion is approximately localized in the valleys, and is complete only in the low-energy limit, where it loses sight of the Brillouin zone and is absolutely isolated from its mirror partner (the anti-skyrmion living on the other valley). Consequently, the \( (s + id) \) SC in essence realizes the topological crystalline SC. Formally, the topological Chern number can be calculated and the weak to strong pairing transition of \( (s + id) \) SC should be expected.

**Weak to strong \( s + id \) pairing SC.** — The \( (s + id) \) SC phase is divided into two fully gapped \( (s + id) \) and \( (s + id) \) SC by a critical line. The critical line is characterized by the nodes on the crossing point of the \( \mu \) and \( A > 0 \) nodal lines, i.e., \( X_{\pm} = (\tilde{\gamma}_1, \pm \tilde{\gamma}_2) \). Near the critical point the low-energy effective Hamiltonian can be obtained by expanding the MF Hamiltonian for \( \Psi^{-\dagger}(k) \) around the valleys \( X_{\pm} \) leading order, because \( \Psi_{\pm}(k) \) is frozen in the low-energy limit. Then, we have

\[
\tilde{H}_{\text{eff}}(X_{\pm} + q) \equiv \tilde{n}_{\pm}(q) \cdot \tilde{\rho} \\
= (-\mu' - A q_{x}^2 - A' q_{y}^2) \rho_{z} + 2 q_{x} \Delta_{d} \rho_{x} + 2 q_{y} \Delta_{\rho_{y}}. \] (12)

where \( \mu' = \mu + m_{s} \), \( A = 2(\delta \phi + \chi)^{2}/m_{s} - \delta \theta \), \( A' = \delta \phi \), \( q_{x} = q_{s} + q_{y} \) and \( q_{y} = q_{s} - q_{y} \). Within our phase diagram \( A > 0 \) and \( A' > 0 \), the effective chemical potential \( \mu' \) controls the phase transition: \( \mu' < 0 \) accounts for the \( (s + id) \) SC phase, while \( \mu' > 0 \) represents the \( (s + id) \) SC phase. \( \mu' \) gives rise to the critical point. Note that in the phenomenological sense, eq. (12) resembles two copies of the effective Hamiltonian of weak pairing \( (p_{x} \pm ip_{y}) \) SC discussed by Read and Green [9], analogous to the time-reversal invariant topological superconductor [15], where \( (p_{x} + ip_{y})_{11} \) and \( (p_{x} - ip_{y})_{11} \) are related by \( T \). However, \( \tilde{H}_{\text{eff}}(X_{+} + q) \) and \( \tilde{H}_{\text{eff}}(X_{-} + q) \) living on the two valleys \( X_{\pm} \) are related by mirror symmetries \( M_{x/y} \) instead of the symmetry \( T \). Nevertheless, our discussion of its topology goes quite parallel.

In the fully gapped \( s + id \) pairing states, since both \( \Delta_{x,y} \neq 0 \), the pairing gap function \( \Delta(k) \) is complex in eq. (12), where \( \tilde{n}(k) \) pins down Anderson's pseudospin [16], giving rise to the SC ground-state wave function as

\[ |\Omega\rangle \propto \exp \left( \sum_{k} g_{k} b_{k} |b_{-k}\rangle \right) |FS\rangle, \] (13)

where \( g_{k} = (1 + \tilde{n}_{z})/(\tilde{n}_{x} - i \tilde{n}_{y}) \) with \( \tilde{n} = \tilde{n}_{x} - i \tilde{n}_{y} \). Note that the SC ground state is written in the hole representation, because the normal state Fermi sea in the low-energy theory is given by the hole band \( \xi_{+}(k) \), leading to the hole pocket Fermi surfaces around \( X_{\pm} \) for \( \mu' < 0 \). It is then straightforward to show that

\[ g_{X_{\pm}} \propto \frac{1}{(\Delta_{s} q_{x}^{2} + i \Delta_{d} q_{y}^{2})^{\frac{1}{2}}} \] (14)

which signifies singularity on \( X_{\pm} \) for the \( (s + id) \) pairing phase, resulting in a modulated long tail of the pairing wave function in real space. Actually, the function \( g_{X_{\pm}}(k) \) defines a map from the \( k \)-space torus \( T^{2} \) to a sphere \( S^{2} \) parametrized by the corresponding pseudo-spinor. Such a map is classified by the homotopy group \( \pi_{2}(S^{2}) \), and the singular vortex structure of \( g_{X_{\pm}} \) is identical to a topological monopole charge [9]. As shown schematically in fig. 4(a), the singular vortices residing on the two valleys \( X_{\pm} \) are mirror partners in the \( (s + id) \) phase, so that they carry monopoles with opposite topological charges \( Q_{X_{\pm}} = \pm \). In contrast, in the \( (s + id) \) phase with \( \mu' > 0 \), the pairing wave function \( g_{X_{\pm}} \) is analytic at \( X_{\pm} \) and the topological charge is zero, indicating that the strong pairing \( (s + id) \) phase corresponds to a topologically trivial phase.

From another perspective, since the pseudo-spinor is polarized by the unit vector \( \tilde{n}(k) \), the \( S^{2} \) of the topological map can be alternatively spanned by \( \tilde{n} \), which naturally entails the topological invariant Chern number \( C = \frac{1}{4\pi} \int_{BZ} d^{2}k \tilde{n} \cdot (\partial_{k_{x}} \tilde{n} \times \partial_{k_{y}} \tilde{n}) \) that characterizes the map. Physically, the Chern number is the integral of the Berry curvature which counts the total Berry flux. Since the Berry curvature is sharply peaked around the valley points \( X_{\pm} \), the total Berry flux can be approximately attributed as the sum of the Berry flux carried by each valley \( C = C_{X_{+}} + C_{X_{-}} \), in which \( C_{X_{\pm}} \) converges fast to a quantized value in the low-energy limit. By treating eq. (12) in infinite large space [17], our calculation shows that

\[ C_{X_{\pm}} = \frac{1}{4\pi} \int_{\infty} d^{2}q \tilde{n}_{X_{\pm}} \cdot (\partial_{k_{x}} \tilde{n}_{X_{\pm}} \times \partial_{k_{y}} \tilde{n}_{X_{\pm}}) \]

\[ = \begin{cases} 
\pm 1, & \mu' < 0, \\
0, & \mu' > 0.
\end{cases} \] (15)

Although the total Chern number \( C = 0 \) in both phases, it is meaningful to introduce a valley Chern number [18,19], \( C_{v} = C_{X_{+}} - C_{X_{-}} \) that describes the two topologically distinct phases and classifies the topological phases by a \( Z \) valley index. Note that the second-order terms in eq. (12) are essential in removing the marginality, giving rise to an integer instead of a half-integer for \( C_{X_{\pm}} \) [20]. So the topological nontrivial phase and the trivial one are distinguishable. To conclude, in the low-energy limit we lose sight of the Brillouin zone living in two inequivalent valleys \( X_{\pm} \), and we have two copies of the nontrivial \( (p_{x} \pm ip_{y}) \)-like topological SC that are related by the mirror symmetry \( M_{x/y} \) instead of the time-reversal symmetry \( T \) for the
(s + id)\(w\) phase. This weak pairing SC is characterized by a nonzero topological valley Chern number \(C_v\) (fig. 4(a)). In contrast, the \((s + id)_s\) SC is a trivial strong pairing phase.

There’s one thing remained to be addressed, i.e., the protecting symmetry of the \((s + id)_w\) SC phase. Since the two inequivalent valleys form a valley-spinor, we denote the Pauli matrices acting on this spinor as \(\gamma_{\alpha=x,y,z}\), with the valley on \(X_\pm\) being the eigen-spinor of \(\gamma_2 = \pm 1\), respectively. In this way, the low-energy effective Hamiltonian eq. (12) can be rewritten as

\[
\tilde{H}_{\text{eff}}(\mathbf{q}) = (-p' - Bq^2)\rho_z + 2q_x(\Delta_d\rho_x + \Delta_s\rho_y) \\
+ (-B'q_x\rho_z + \Delta_s\rho_y - \Delta_d\rho_x)2m_0\gamma_2,
\]

where \(\mathcal{V} = A + A'\) and \(B' = A - A'\). The valley symmetry \(\mathcal{V}\) of \(\gamma_2\) is preserved in the low-energy limit, and the mirror reflection amounts to \(\gamma_2\) that flips the two valleys. As long as \(\mathcal{V}\) is present, the valley-spinor is conserved and all possible inter-valley couplings are forbidden, as a result the valley Chern number \(C_v\) is well defined and cannot be changed without gap closing. Namely, in the presence of the valley symmetry \(\mathcal{V}\), all mass terms for \((s + id)_w\) are forbidden, so there is no way of adiabatically connecting the \((s + id)_w\) and \((s + id)_s\) SC phases, evidencing their topological distinction [21]. Analogous to the quantum spin Hall Hamiltonian [22] and the \(Z_2\) time-reversal invariant topological SC protected by the time-reversal symmetry [15], the \((s + id)_w\) SC phase is a topological valley SC that nevertheless goes beyond the conventional ten-fold way classification [23–25].

**Robust gapless edge states of \((s + id)_w\) SC.**

To demonstrate that the topological \((s + id)_w\) phase indeed supports robust gapless edge modes, we perform an exact diagonalization to the model with \((1,1)\) open edges (fig. 4(b)). In the cylinder geometry, the momentum \(k_1\) along the edge remains a good quantum number, and the valley symmetry is preserved upon projection onto the boundary. Within the surface Brillouin zone, the two valleys are located at \(k_1 = 0, \pi\) respectively. For the singlet pairings, the Bogoliubov excitations are spin polarized. The dispersions in the surface Brillouin zone are shown in fig. 4(c) and (d) for spin-up and -down Bogoliubov quasi-particles, respectively, which are related by the particle-hole transformation \(Z = i\sigma_z\rho_y\). The spin index in our BdG basis is chosen along the \(y\)-direction and the whole spectrum satisfies the particle-hole symmetry \(Z^{-1}E_kZ = -E_{-k}\).

Having a closer look at the spin-up excitations, for instance, in the vicinity of each valley there is one gapless chiral edge mode, and the edge modes associated with the two valleys propagate in opposite directions, so the whole system is nonchiral analogous to the quantum valley Hall effect. This pattern concurs with the Chern number for each valley, because the edges being the interface of the nontrivial bulk and the trivial vacuum should close the gap and the number of edge modes is supposed to amount to the mismatch of the Chern number between the bulk valley and the vacuum. Since the Chern number for the valley on \(X_\pm\) is \(\pm 1\), there are supposed to be counter-propagating chiral edge modes associated with the two valleys. However, it is worth noting that on the same edge the edge currents contributed by the two valleys differ slightly by their velocities, so that the net velocity is nonzero and can be detected experimentally. This is the consequence of the noncentrosymmetry of the lattice edges (fig. 4(b)). In contrast, we do not observe any edge modes on \((0,0)\) or \((0,1)\) edges, because the valleys would collapse onto the same momentum on the surface Brillouin zone and break the valley symmetry. Neither is there any sign of gapless edge modes in the strong-pairing \((s + id)_s\) phase, evidencing its trivial topology distinct from \((s + id)_w\).

**Discussion and conclusion.** Now that we have elaborated on the weak pairing topological nature of the nodal \(d\)-wave SC and the \((s+id)_w\) SC as well as their phase transitions into strong pairing trivial SC phases, it should be mentioned that the properties of the nodal \(s_{x^2+y^2}\)-wave SC are similar to the nodal \(d_{x^2-y^2}\)-wave SC and the scenarios of its phase transition either into nodeless \(s_{x^2+y^2}\)-wave phase or into the \((s+id)_s\) phase are parallel to that of the \(d_{x^2-y^2}\)-wave SC. To summarize the phase diagram in fig. 1(b), the nodal \(d_{x^2-y^2}\)-wave, \((s + id)_w\) and nodal
$s_{x_2+y_2}$-wave are weak pairing and topologically nontrivial. Among them the nodal phases are topological nodal superconductors whose nodes carry the topological number and entail edge modes on edges residing between the projection of nodes, and the fully gapped $(s+id)_w$ realizes a topological valley SC. On the other hand, the nodeless $d_{x^2-y^2}$-wave and $(s+id)_s$ and the nodeless $s_{x_2+y_2}$-wave are all fully gapped strong pairing topologically trivial SC, between which we find no gap closing phase transition. The phase transition from weak to strong pairing phases necessarily experiences a critical point described by the effective Hamiltonian eq. (12).

In summary, we have shown that when a nodal $d$-wave SC is proximately coupled to an AF insulator, the nature of the SC can be remarkably changed into nodeless $d$-wave, nodeless and nodal $s$-wave, and valley-symmetry–protected $Z_2$ topological $(s+id)_w$ SC phases, depending on the dopant concentration and the proximity-induced AF field. These findings are supported by careful studies of the SC phases described by the two-dimensional $t$-$J$ model. The presence of a AF field is crucial, and its existence can be justified in the CuO$_2$ monolayer on the substrates of optimally doped cuprates [8]. Our theoretical calculations suggest that a possible candidate for the nodeless superconductivity observed in the CuO$_2$ monolayer on the optimally doped Bi-2212 substrates may be the valley-symmetry–protected topological SC, consistent with the large overall Mott gap found in the scanning tunneling spectrum. However, so far it is not clear how strong the AF field in the CuO$_2$ insulating layer can be reached experimentally. Our findings certainly broaden the scope of the investigations on high-$T_c$ cuprates to include the possibility of topological valley superconductivity. In order to confirm such novel nodeless SCs in the cuprates, further new experiments are desirable to verify our predictions.

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