Principal Series Representations of Direct Limit Groups

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9 February 2004

Abstract

We combine the geometric realization of principal series representations of \([28]\) with the Bott–Borel–Weil Theorem for direct limits of compact groups found in \([22]\), obtaining limits of principal series representations for direct limits of real reductive Lie groups. We introduce the notion of weakly parabolic direct limits and relate it to the conditions that the limit representations are norm–preserving representations on a Banach space or unitary representations on a Hilbert space. We specialize the results to diagonal embedding direct limit groups. Finally we discuss the possibilities of extending the results to limits of tempered series other than the principal series.

1 Introduction

Harmonic analysis on a real reductive Lie group \(G\) depends on several series of representations, one for each conjugacy class of Cartan subgroups of \(G\). See \([11]\), \([12]\), \([13]\) and \([14]\) for the case where \(G\) is Harish–Chandra class, \([28]\), \([16]\) and \([17]\) for the general case. The simplest of these series is the principal series. It consists of representations constructed from representations of compact Lie groups, characters on real vector groups, and the induced representation construction. The other series are somewhat more delicate, replacing Élie Cartan’s theory of representations of compact Lie groups by Harish–Chandra’s theory of discrete series representations of real reductive Lie groups.

This paper is the first step in a program to extend the construction, analysis and geometry of those series of representations from the finite–dimensional setting to a nontrivial but well behaved family of infinite–dimensional Lie groups, the direct limits of real reductive Lie groups. Here we consider the case of the principal series. The case of the discrete series, and then the general case, will be considered separately in \([32]\) and \([33]\).

The classical Bott–Borel–Weil Theorem \([6]\) realizes representations of compact Lie groups as cohomology spaces of holomorphic vector bundles over complex flag manifolds. It since has been extended to direct limits of compact Lie groups and direct limits of complex Lie groups, both in the analytic category \([22]\) and in the algebraic category \([7]\). With some technical adjustment, the results of \([22]\) replace Cartan’s theory of representations of compact Lie groups for construction of...
Section 2 recalls our class of finite–dimensional real reductive Lie groups and the standard construction of their not–necessarily–unitary principal series representations. Section 3 recalls the geometric realization of those representations on partially holomorphic cohomologies of vector bundles over closed orbits in complex flag manifolds. In Section 4 we discuss alignment questions for minimal parabolic subgroups. The alignment is needed in order to define limit principal series representations of our direct limit groups. In effect, this is the first technical issue, and it addresses the question of whether \( G = \lim G_i \) can have a meaningful direct limit of principal series representations. For that we need the connecting maps \( \phi_{j,i} : G_i \to G_j \) of the direct system to respect the ingredients of the principal series recipe. Initially that must be done for the components \( M_i, A_i \) and \( N_i \) of minimal parabolic subgroups \( P_i = M_i A_i N_i \subset G_i \). That alignment on components is not quite automatic, but it holds (possibly after passing to a cofinal subsystem — which yields the same limit group) for the most interesting cases, the diagonal embedding direct limit groups of Section 9. See Proposition 9.12. Next, it must be done on the level of representations of the \( M_i \). That, of course, is automatic for spherical principal series representations, but more generally we use an appropriate extension of Cartan’s highest weight theory. Thus we obtain representations of \( G \) that are direct limits of principal series representations of the \( G_i \).

The second issue is to construct good geometric realizations of these “principal series” representations of the limit groups \( G \). This is the heart of the paper. The method of [29], illustrated in [29] Section 1, gives natural partially holomorphic realizations of principal series representations \( \pi_i \) of \( G_i \). That involves a certain extension of the classical Bott–Borel–Weil Theorem [3] which we need for the groups \( M_i \). In order to pass to the limit, we construct and study the appropriate limit flag manifolds, limit of closed orbits, limits of holomorphic arc components, and limit sheaves, in Sections 5 and 6. This is done in such a way that the limit Bott–Borel–Weil Theorem of [22] applies over the holomorphic arc components of the closed orbits. That defines the geometric setting for the representations in question. In order to see that the cohomology of the limit sheaf is the limit of the cohomologies, we prove a Mittag–Leffler condition at the end of Section 6. Thus we have the possibility of obtaining good geometric realizations of limit principal series representations of \( G \) directly on cohomology spaces.

We actually construct the geometric realizations in Section 7. Theorem 7.1 is the 0–cohomology result in the style of the Borel–Weil Theorem, and Theorem 7.2 is the higher cohomology result in the style of the Bott–Borel–Weil Theorem. For the latter it is essential to have the cohomologies all occur in the same degree. That is the third technical issue, and we reduce it to the same question for \( M = \lim M_i \), where it was settled in [22].

The fourth issue is whether these principal series representations of \( G \) are norm–preserving Banach space representations, or even unitary representations, of \( G \). That is settled in Theorem 8.5–8. There the key idea is that of weakly parabolic direct system.

It is very important to have a large number of interesting examples. For that we consider diagonal embedding direct limits of classical real simple Lie groups. We examine their behavior relative to the various general notions studied earlier and see that our constructions work very well for these interesting direct limit groups. This is done in Section 9. These diagonal embedding direct limits have been studied extensively in the context of locally finite Lie algebras. That is a rapidly developing area; see [18] and the references there. A locally finite Lie algebra of countable dimension can be represented as a direct limit \( \lim \{ g, d\phi_{n,m} \}_{m,n \in \mathbb{Z}^+} \) of finite dimensional Lie algebras, and the diagonal embedding direct limits are essentially just those where the group level maps \( \phi_{n,m} \) are polynomials of degree 1.
Finally, in Section 10 we discuss the place of the principal series in our program for constructing limit representations corresponding to all tempered series, and indicate some of the problems to be settled in [32] and [33].

The notions of parabolic and weakly parabolic direct systems developed from a conversation with Andrew Sinton.

2 Principal Series for General Reductive Groups

Let $G$ be a reductive real Lie group. In other words, its Lie algebra $\mathfrak{g}$ is reductive in the sense that it is the direct sum of a semisimple Lie algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ and an abelian ideal $\mathfrak{z}$ which is the center of $\mathfrak{g}$. As usual, $\mathfrak{g}_c$ denotes the complexification of $\mathfrak{g}$, so $\mathfrak{g}_c = \mathfrak{g}' \oplus \mathfrak{z}$, direct sum of the respective complexifications of $\mathfrak{g}'$ and $\mathfrak{z}$. We assume that $G$ satisfies the conditions of [28]:

\begin{equation}
\text{if } g \in G \text{ then } Ad(g) \text{ is an inner automorphism of } \mathfrak{g}_c, \text{ and }
\end{equation}

$G$ has a closed normal abelian subgroup $Z$ such that

\begin{equation}
\begin{aligned}
Z & \text{ centralizes the identity component } G^0 \text{ of } G, \\
ZG^0 & \text{ has finite index in } G, \text{ and } \\
Z \cap Z_G^0 & \text{ is co-compact in the center } Z_{G^0} \text{ of } G^0.
\end{aligned}
\end{equation}

These are the conditions, inherited by Levi components of cuspidal parabolic subgroups of $G$, that lead to a nice Plancherel formula. See [28], [16], and [17]. The famous Harish-Chandra class is the case where the semisimple component $(G^0)' := [G^0, G^0]$ of $G^0$ has finite center and the component group $G/G^0$ is finite.

Condition (2.1) says that the standard tempered representation construction yields representations that have an infinitesimal character. It can be formulated: $Ad(G) \subset Int(\mathfrak{g}_c)$.

Note that the kernel of $Ad : G \to Ad(G)$ is the centralizer $Z_G(G^0)$ of the identity component and that the image $Ad(G)$ is a closed subgroup of the complex semisimple group $Int(\mathfrak{g}_c)$ with only finitely many topological components. Thus $Ad(G)$ has maximal compact subgroups, as usual for semisimple linear groups, and every maximal compact subgroup of $Ad(G)$ is of the form $K/Z_G(G^0)$ for some closed subgroup $K \subset G$.

Given a maximal compact subgroup $K/Z_G(G^0)$ of $Ad(G)$, it is known [28] Lemma 4.1.1 that $K$ is the fixed point set of a unique involutive automorphism $\theta$ of $G$. These automorphisms $\theta$ are called Cartan involutions of $G$, and they are lifts of the Cartan involutions of the linear group $Ad(G)$. The groups $K$ are the maximal compactly embedded subgroups of $G$.

One also knows [28] Lemma 4.1.2 that $K \cap G^0$ is the identity component $K^0$ of $K$, that $K$ meets every topological component of $G$, that any two Cartan involutions of $G$ are conjugate by an element of $Ad(G^0)$, and that every Cartan subgroup of $G$ is stable under some Cartan involution. Here we use the usual definition: Cartan subgroup of $G$ means the centralizer of a Cartan subalgebra of $\mathfrak{g}$.

Fix a Cartan involution $\theta$ of $G$ and the corresponding maximal compactly embedded subgroup $K = G^0$ of $G$. Denote

\begin{equation}
\mathfrak{a} : \text{ maximal abelian subspace of } \{ \xi \in \mathfrak{g} \mid \theta(\xi) = -\xi \}.
\end{equation}

If $\xi \in \mathfrak{a}$ then $ad(\xi)$ is a semisimple linear transformation of $\mathfrak{g}$ with all eigenvalues real. Now, as usual, $\mathfrak{g}$ is the direct sum of the joint eigenspaces (= restricted root spaces)

\begin{equation}
\mathfrak{g}^\gamma = \{ \eta \in \mathfrak{g} \mid [\xi, \eta] = \gamma(\xi)\eta \text{ for every } \xi \in \mathfrak{a} \} \text{ where } \gamma \text{ ranges over } \mathfrak{a}^*.
\end{equation}
The \(a\)-root system of \(\mathfrak{g}\) is \(\Sigma(\mathfrak{g}, a) = \{\gamma \in (a^* \setminus \{0\}) \mid \gamma^\vee \neq 0\}\). Fix \nabla = \Sigma(\mathfrak{g}, a)^+. \(\Sigma(\mathfrak{g}, a)^+\): positive \(a\)-root system of \(\mathfrak{g}\).

Any two such systems are conjugate by the normalizer of \(a\) in \(K\).

Then \(\Sigma(\mathfrak{g}, a)^+\) specifies a nilpotent subalgebra and a nilpotent subgroup, by

\[
\mathfrak{n} = \sum_{\gamma \in \Sigma(\mathfrak{g}, a)^+} \mathfrak{g}^{-\gamma} \subseteq \mathfrak{g} \quad \text{and } N \text{ is the analytic subgroup of } G \text{ for } \mathfrak{n}.
\]

The corresponding minimal parabolic subalgebra \(\mathfrak{p} \subseteq \mathfrak{g}\) and minimal parabolic subgroup \(P \subset G\) are given by

\[
\mathfrak{p} \text{ is the normalizer of } \mathfrak{n} \text{ in } \mathfrak{g} \quad \text{and} \quad P \text{ is the normalizer of } N \text{ in } G.
\]

Now denote

\[
A : \text{analytic subgroup of } G \text{ for } a \quad \text{and} \quad M : \text{centralizer of } A \text{ in } K.
\]

Then

\[
\mathfrak{p} = \mathfrak{m} + a + \mathfrak{n} \quad \text{and} \quad P = MAN \text{ with } MA = M \times A
\]

and the corresponding Iwasawa decompositions are

\[
\mathfrak{g} = \mathfrak{k} + a + \mathfrak{n} \quad \text{and} \quad G = KAN.
\]

Both \(M\) and \(MA\) have the properties \(2.11\) and \(2.2\). Also, \(M\) is compact modulo \(Z_M(M^0)\), the centralizer of \(M^0\) in \(M\). We write \(\sim\) for unitary dual. If \(\xi \in Z_M(M^0)\), we write \((\tilde{M}^0)_{\xi}\) for the classes \([\eta] \in \tilde{M}^0\) such that \(\eta^0 | Z_M^0\) is a multiple of \(\xi\), and we write \((Z_M(M^0))_{\xi}\) for the classes \([\chi] \in Z_M(M^0)\) such that \(\chi | Z_M^0\) is a multiple of \(\xi\).

The extension of Cartan's highest weight theory appropriate for \(M\) is

**Proposition 2.11** (Compare \(23\) Proposition 1.1.3.)

1. \(M = Z_M(M^0)M^0\).
2. Every irreducible representation of \(M\) is finite dimensional.
3. If \([\eta] \in \tilde{M}\) there exist unique \(\xi \in Z_M(M^0)\) \(\in (Z_M(M^0))_{\xi}\) and \([\eta^0] \in (\tilde{M}^0)_{\xi}\) such that 
   \(\eta = [\chi \otimes \eta^0]\).
4. Let \(\mathfrak{t}\) be a Cartan subalgebra of \(\mathfrak{m}\), \(\Sigma(\mathfrak{m}_c, t_c)^+\) a positive \(t_c\)-root system on \(\mathfrak{m}_c\), and \(T^0 = \exp(\mathfrak{t})\), so \(\Lambda^+_m = \{\nu \in i\mathfrak{t}^* \mid e^\nu \text{ is well defined on } T^0 \text{ and } \langle \nu, \gamma \rangle \geq 0 \text{ for all } \gamma \in \Sigma(\mathfrak{m}_c, t_c)^+\}\) is the set of dominant integral weights for \(M^0\). Then there is a bijection \(\nu \leftrightarrow [\eta_{\nu}]\) of \(\Lambda^+_m\) onto \(M^0\) given by; \(\nu\) is the highest weight of \(\eta_{\nu}\). Furthermore, \([\eta_{\nu}] \in (\tilde{M}^0)_{\xi}\) where \(\xi = e^\nu | Z_M^0\).
5. \(M = TM^0\) where \(T\) is the Cartan subgroup \(\{m \in M \mid Ad(m)\mu = \mu, \text{ every } \mu \in \mathfrak{t}\}\) of \(M\) that corresponds to the Cartan subalgebra \(\mathfrak{t}\) of \(\mathfrak{m}\).

Define \(\mathfrak{h} = \mathfrak{t} + a\). It is a maximally split Cartan subalgebra of \(\mathfrak{g}\), and any two such Cartan subalgebras are \(Ad(G^0)\)-conjugate. The positive root systems \(\Sigma(\mathfrak{g}, a)^+\) and \(\Sigma(\mathfrak{m}_c, t_c)^+\) determine a positive \(\mathfrak{h}_c\)-root system \(\Sigma(\mathfrak{g}_c, h_c)^+\) for \(\mathfrak{g}_c\) as follows. A root \(\gamma \in \Sigma(\mathfrak{g}_c, h_c)^+\) is positive if it is nonzero and positive on \(a\), or if it is zero on \(a\) and positive on \(t_c\). In other words,

\[
\Sigma(\mathfrak{g}, a)^+ = \{\gamma |_a \mid \gamma \in \Sigma(\mathfrak{g}_c, h_c)^+ \text{ and } \gamma |_a \neq 0\}\] and
\[
\Sigma(\mathfrak{m}_c, t_c)^+ = \{\gamma |_t \mid \gamma \in \Sigma(\mathfrak{g}_c, h_c)^+ \text{ and } \gamma |_a = 0\}.
\]
Now let

\[(2.13) \quad [\eta_{\chi,\nu}] = [\chi \otimes \eta_{\nu}] \in \widehat{M} \text{ and } \sigma \in a^*_c.\]

That is equivalent to the datum

\[(2.14) \quad \mathcal{F}(G, [\pi_{\chi,\nu,\sigma}] = \mathcal{F}(G, \pi_{\chi,\nu,\sigma}) \in \mathcal{P} \] defined by \(\eta_{\chi,\nu,\sigma}(man) = e^{\sigma(a)}\eta_{\chi,\nu}(m)\) for \(m \in M, a \in A\) and \(n \in N\).

Here \(e^a(\exp(\xi))\) means \(e^{\sigma(\xi)}\) for \(\xi \in \mathfrak{a}\). In other words \(e^a(a)\) means \(e^{\sigma(\log a)}\). Also, we will write \(V_{\chi,\nu,\sigma}\) for the representation space of \(\pi_{\chi,\nu,\sigma}\) although as a vector space it is independent of \(\sigma\).

The corresponding principal series representation of \(G\) is

\[(2.15) \quad \pi_{\chi,\nu,\sigma} = \text{Ind}^G_P(\eta_{\chi,\nu,\sigma}), \text{ induced representation.}\]

Here one must be careful about the category in which one takes the induced representation. For example, if \(\mathcal{F}\) is a smoothness class of functions such as \(C^k, 0 \leq k \leq \infty, C^\infty_c\) (test functions), \(C^{-\infty}\) (distributions), \(C^\infty\) (analytic), or \(C^{-\omega}\) (hyperfunctions), one can take \(\pi_{\chi,\nu,\sigma}\) to be the natural representation (by translation of the variable) of \(G\) on

\[(2.16) \quad \mathcal{F}(G, [V_{\chi,\nu,\sigma}] : \text{all } f \in \mathcal{F}(G, [\pi_{\chi,\nu,\sigma}]) \text{ with } f(gman) = e^{-\sigma(a)}\eta_{\chi,\nu}(m)^{-1}(f(g))\]

for all \(g \in G, m \in M, a \in A\), and \(n \in N\). The representation is always given by the formula \(\pi_{\chi,\nu,\sigma}(g)(f(g')) = f(g^{-1}g')\).

One can also consider the analog of (2.10) using \(K\)-finite functions. Those functions are \(C^\infty\), and the representations spaces of the resulting \(K\)-finite induced representations are the common underlying Harish-Chandra modules for the representation spaces of the various smoothness classes (2.10) of induced representations.

Banach space representations, in particular unitary representations, are more delicate. We have to discuss this because we will have to keep track of how they behave in a direct limit process. The modular function of \(P\) is \(\Delta_P(man) = e^{-2\rho_{\mathfrak{a}}(a)}\), where \(2\rho_{\mathfrak{a}}(\xi)\) is the trace of \(ad(\xi)|_n\) for \(\xi \in \mathfrak{a}\), because we defined \(\mathfrak{n}\) to be the sum of the negative \(\mathfrak{a}\)-root spaces. \(G\) is unimodular, so \(\Delta_G/\Delta_P(man) = \Delta_P^{-1}(man) = e^{2\rho_{\mathfrak{a}}(a)}\). Let \(\xi\) be a norm–preserving representation of \(P\) on a Banach space \(V\), and let \(1 \leq p \leq \infty\). If \(f \in C_c(G, P : \xi \otimes \Delta_P^{-1/p}) = C_c(G, P : \xi \otimes e^{(2/p)\rho_{\mathfrak{a}}})\) then \(\|f\|_{V_p} \in C_c(G, P; \Delta_P^{-1/p})\), so the global norm

\[(2.17) \quad \|f\|_p = \left(\int_{G/P} \|f(gP)\|^p_{V_p} \, d\mu_{G/P}(gP)\right)^{1/p} \quad \text{for } p < \infty,\]

\[
\|f\|_p = \text{ess sup}_{G/P} \|f(gP)\|_{V_p} \quad \text{for } p = \infty,
\]

is well defined and invariant under the left translation action of \(G\). Denote

\[L_p(G, P : \xi \otimes e^{(2/p)\rho_{\mathfrak{a}}}) : \text{Banach space completion of } \left(C_c(G, P : \xi \otimes e^{(2/p)\rho_{\mathfrak{a}}}), \|\cdot\|_p\right).\]

Each \(\pi_{\xi \otimes e^{(2/p)\rho_{\mathfrak{a}}}}(g)\) extends by continuity from \(C_c(G, P : \xi \otimes e^{(2/p)\rho_{\mathfrak{a}}})\) to a norm–preserving operator on \(L_p(G, P : \xi \otimes e^{(2/p)\rho_{\mathfrak{a}}})\) and that defines a norm–preserving Banach representation of \(G\) on \(L_p(G, P, \xi \otimes e^{(2/p)\rho_{\mathfrak{a}}})\). If \(\xi\) is unitary then the global inner product

\[(2.18) \quad \langle f, f' \rangle = \int_{G/P} \langle f(gH), f'(gH) \rangle_{V_p} \, d\mu_{G/P}(gP) \text{ for } f, f' \in C_c(G, P : \xi \otimes e^{p_{\mathfrak{a}}}).\]

is \(G\)-invariant, \(L_2(G, P : \xi \otimes e^{p_{\mathfrak{a}}})\) is the Hilbert space completion of \((C_c(G, P : \xi \otimes e^{p_{\mathfrak{a}}}), \langle \cdot, \cdot \rangle)\), and \(\pi_{\xi \otimes e^{p_{\mathfrak{a}}}}\) is a unitary representation of \(G\). Those unitary representations form the unitary principal series of \(G\). We translate the discussion to our terminology (2.15) for principal series representations as follows.
Proposition 2.19 The principal series representation $\pi_{\chi, \nu, \sigma}$ extends by continuity from a representation of $G$ on $C_c(G, P : V_{\chi, \nu, \sigma})$ to a norm-preserving representation on $L_p(G, P : V_{\chi, \nu, \sigma})$ if and only if $\sigma \in \mathfrak{a}^* + \frac{2}{\beta} \rho_{\theta, a}$. In particular it extends by continuity to a unitary representation of $G$ on $L_2(G, P : V_{\chi, \nu, \sigma})$ if and only if $\sigma \in \mathfrak{a}^* + \rho_{\theta, a}$.

Remark 2.20 The restriction $\pi_{\chi, \nu, \sigma}|_K = \text{Ind}_{M}^{G}(\eta_{\chi, \nu})$, independent of $\sigma$. Decompose $\sigma = \sigma' + \sigma''$ where $\sigma' \in \mathfrak{a}^* + \rho_{\theta, a}$ and $\sigma'' \in \mathfrak{a}$. Then $\pi_{\chi, \nu, \sigma}|_K = \pi_{\chi, \nu, \sigma'}|_K$, restriction of a unitary representation. In other words, we may always view the underlying Harish–Chandra module of $\pi_{\chi, \nu, \sigma}$ as a pre-Hilbert space. This will be important when we look at direct limit groups.

3 Geometric Form of Principal Series Representations

Let $G_c$ be a connected reductive complex Lie group for which $G$ is a real form. In other words there is a homomorphism $\varphi : G \to G_c$ with discrete kernel such that $d\varphi(g)$ is a real form of $\mathfrak{g}_c$.

If $Q$ is a parabolic subgroup of $G_c$, then we can view the complex flag manifold $Z = G_c/Q$ as the set of all $G_c$-conjugates of $Q$, say $Z \ni z \mapsto Q_z \subset G_c$, because $Q$ is its own normalizer in $G_c$. Now we can view $Z$ as the set of all $\text{Int}(\mathfrak{g}_c)$-conjugates of $\mathfrak{q}$ by $Z \ni z \mapsto z \in \mathfrak{g}_c$.

The condition $\mathfrak{a}^* + \rho_{\theta, a}$ ensures that $G$ acts on $Z$ through $\varphi$ and conjugation. In other words $G$ acts on $Z$ through its adjoint action on $\mathfrak{g}_c$. Thus

\[(3.1) \quad G \text{ acts on } Z \text{ by } g(z) = z' \text{ where } q_zv = \text{Ad}(g)(q_z).\]

This will be important when we construct direct limits of complex flag manifolds.

Let $\Psi \subset \Sigma(\mathfrak{m}_c, t_c)^+$ be any set of simple roots. That defines a parabolic subalgebra $\mathfrak{r} = \mathfrak{i}_c + \mathfrak{n}_m$ in $\mathfrak{m}_c$, with nilradical $\mathfrak{n}_m$ and Levi component $\mathfrak{l}_c$, where the reductive algebra $\mathfrak{i}_c$ contains $\mathfrak{t}_c$ and has simple root system $\Psi$. The corresponding parabolic subgroup of $M_c$ is $J_c N_m$, and its $\varphi^{-1}$–image is a real form $J$ of $J_c$. Note that $J = TJ^0$ where $T$ is the Cartan subgroup of $M$ corresponding to $\mathfrak{t}$.

Conversely to (2.12) we extend roots of $\mathfrak{m}_c$ to roots of $\mathfrak{m}_c + \mathfrak{a}_c$ by zero on $\mathfrak{a}_c$ and obtain $\Sigma(\mathfrak{m}_c, t_c)^+ = \Sigma(\mathfrak{m}_c + \mathfrak{a}_c, h_c)^+ = \Sigma(\mathfrak{g}_c, h_c)^+$. Every $\psi \in \Psi$ remains simple for $\Sigma(\mathfrak{g}_c, h_c)^+$. Thus $\Psi$ also defines a parabolic subalgebra $\mathfrak{q} = \mathfrak{i}_c + \mathfrak{u}$ in $\mathfrak{g}_c$, with nilradical $\mathfrak{u}$ and Levi component $\mathfrak{l}_c$, where the reductive algebra $\mathfrak{i}_c$ contains $\mathfrak{h}_c$ and has simple root system $\Psi$. The corresponding parabolic subgroup of $G_c$ is $Q = L_cU$, and its $\varphi^{-1}$–image is a real form $L$ of $L_c$. Note that $L = JA = AJ = ATJ^0 = HJ^0$ where $H = T \times A$ is the Cartan subgroup of $G$ corresponding to $\mathfrak{h}$.

As before, $Z$ is the complex flag manifold $G_c/Q$. Let $z_0 = 1Q \in Z$. Then the closed $G$-orbit in $Z$ is $F := G(z_0) = K(z_0)$. We will realize principal series representations on partially holomorphic vector bundles over $F$.

Define $S = M(z_0)$. Note that $M$ acts on $Z$ as a compact group. The basic properties of $S$, from [28] Chapter 1, are: (1) $S = M_c(z_0)$, so $S$ is a complex flag manifold, $S = M_c/R$ where $R = \{m \in M_c \mid m(z_0) = z_0\}$ is a parabolic subgroup of $M_c/R$ with Lie algebra $\mathfrak{r}$ as described above, (2) $S \cong M/J$ with $J$ as described above, and (3) If $g, g' \in G$ and $gS$ meets $g'S$ then $gS = g'S$; and $P = \{g \in G \mid gS = S\}$. In fact, in the notation of [28] the $gS$ are the holomorphic arc components of $F$. Thus we have

Proposition 3.2 Define $\beta : F \to G/P = \{gS \mid g \in G\}$ by $\beta(gz_0) = gS$. Then $\beta : F \to G/P$ is a well defined $C^\infty$ fiber bundle with structure group $P$. The fiber over $gP$ is $gS$, which is maximal among complex submanifolds of $Z$ that are contained in $F$. 

6
Note that $T = Z_M(M^0)T^0$. For $Z_M(M^0)$ centralizes $t$, thus is contained in $T$, and if $t \in T$ then $\text{Ad}(t)|_{M^0}$ is an inner automorphism of $M^0$ that fixes every $\mu \in t$, thus given by $\text{Ad}(t')|_{M^0}$ for some $t' \in T^0$, so $tT^0 \subset Z_M(M^0)T^0$. As $T \subset J$ and $J \cap M^0 = J^0$ we have

**Lemma 3.3** (Compare [28, Proposition 1.1.3].)
1. $J = Z_M(M^0)J^0$.
2. If $[\xi] \in \tilde{J}$ there exist unique $\xi \in \tilde{Z}_M$, $[\chi] \in (Z_M(M^0))_{\xi}$ and $[\zeta^0] \in (\tilde{M})_{\xi}$ such that $[\xi] = [\chi \otimes \zeta^0]$.
3. Let $\Lambda^+_1 = \{ \nu \in \text{it}^* | e^\nu$ is well defined on $T^0 \}$ and $\langle \nu, \gamma \rangle \geq 0$ for all $\gamma \in \Sigma(j_c, t_c)^+$, the set of dominant integral weights for $J^0$. Then there is a bijection $\nu \mapsto [\zeta^0]$ of $\Lambda^+_1$ onto $\tilde{J}$ given by: $\nu$ is the highest weight of $\zeta^0$. Furthermore, $[\zeta^0] \in (\tilde{J})_{\xi}$ where $\xi = e^\nu|_{Z_M}$.

The set of $m$–nonsingular dominant integral weights for $J^0$ is

$$\langle \Lambda^+_1 \rangle : = \{ \nu \in \Lambda^+_1 | \langle \nu + \rho, \gamma \rangle \neq 0 \} \quad \text{for all} \quad \gamma \in \Sigma(j_c, t_c)^+ \}$$

where $\rho_{m,t}$ is half the sum of the roots in $\Sigma(m, t)$. If $\nu \in (\Lambda^+_1)$ there is a unique Weyl group element $w \in W(m, t)$ such that

$$\Delta := w(\nu + \rho_{m,t}) - \rho_{m,t} \in \Lambda^+_m.$$ 

We write $q(\nu)$ for the length $\ell(w)$ of that Weyl group element.

Let $\nu \in (\Lambda^+_1)^\prime$. Let $\zeta(\nu)$ denote the irreducible representation of $J^0$ with highest weight $\nu$ as in Lemma 3.3. Denote $\xi = e(\nu)|_{Z_M}$ and choose $[\chi] \in Z_M(M^0)_{\xi}$. Then $[\zeta(\nu)] = [\chi \otimes \zeta^0]$ is a well defined element of $\tilde{J}$. Let $E_{\chi, \nu}$ denote the representation space. Let $\sigma \in a_{\chi}^\ast$. The isotropy subgroup of $G$ at $z_0$ is $JAN$, and the representation $\zeta(\chi, \nu, \sigma)(jan) = e^\sigma(a)\zeta(\chi, \nu, \sigma)(b)$ of $JAN$ defines

$$E_{\chi, \nu, \sigma} \rightarrow F : G$$-homogeneous vector bundle with fiber $E_{\chi, \nu, \sigma}$ over $z_0$

where $E_{\chi, \nu, \sigma}$ is the representation space $E_{\chi, \nu}$ of $\zeta(\chi, \nu, \sigma)$. Note that $E_{\chi, \nu, \sigma}|_{gS} \rightarrow gS$ is holomorphic, for every fiber $gS$ of $F \rightarrow G/P$. Initially one is tempted to define the corresponding sheaf as

$$\mathcal{O}_n(E_{\chi, \nu, \sigma}) \rightarrow F : \text{germs of } C^\infty \text{ functions } h : G \rightarrow E_{\chi, \nu, \sigma} \text{ such that}$$

(i) $h(gjan) = \zeta(\chi, \nu, \sigma)(jan)^{-1}(h(g))$ for $g \in G$ and $jan \in JAN$

(ii) $h(g; \xi) + d\zeta(\chi, \nu, \sigma)(\xi)h(g) = 0$ for $g \in G$ and $\xi \in (\ mortal + \ n)_{\chi}$

but that causes a number of technical problems, and it is better to use hyperfunctions as in [25] and [26] to ensure that the differentials in the cohomology of $\mathcal{O}_n(E_{\chi, \nu, \sigma}) \rightarrow F$ have closed range. The correct definition is

$$\mathcal{O}_n(E_{\chi, \nu, \sigma}) \rightarrow F : \text{germs of } C^{-\omega} \text{ functions } h : G \rightarrow E_{\chi, \nu, \sigma} \text{ such that}$$

(i) $h(gjan) = \zeta(\chi, \nu, \sigma)(jan)^{-1}(h(g))$ for $g \in G$ and $jan \in JAN$

(ii) $h(g; \xi) + d\zeta(\chi, \nu, \sigma)(\xi)h(g) = 0$ for $g \in G$ and $\xi \in (\ mortal + \ n)_{\chi}$.

Apply the Bott–Borel–Weil Theorem to each $E_{\chi, \nu, \sigma}|_{gS} \rightarrow gS$. By elliptic regularity, use of hyperfunction coefficients results in the same cohomology as use of smooth coefficients. The result is

**Proposition 3.8** (Compare [28, Theorem 1.2.19].) If $\nu \not\in (\Lambda^+_1)^\prime$ then $H^q(F; \mathcal{O}(E_{\chi, \nu, \sigma})) = 0$ for every integer $q$. If $\nu \in (\Lambda^+_1)^\prime$, then $H^q(F; \mathcal{O}(E_{\chi, \nu, \sigma})) = 0$ for $q \neq q(\nu)$, and the natural action of $G$ on $H^q(\nu)(F; \mathcal{O}(E_{\chi, \nu, \sigma}))$ is infinitesimally equivalent (same underlying Harish–Chandra module) to the principal series representation $\pi_{\chi, \nu, \sigma}$.
4 Principal Series for Direct Limit Groups

Consider a countable strict direct system \( \{ G_i, \phi_{k,i} \}_{i,k \in I} \) of reductive Lie groups. Thus \( I \) is a countable partially ordered set. If \( i, k \in I \) there exists \( \gamma \in I \) with \( i \leq \gamma \) and \( k \leq \gamma \). Each \( G_i \) is a reductive Lie group. If \( i \leq k \) then \( \phi_{k,i} : G_i \to G_k \) is a continuous group homomorphism.

Then we have the direct limit group \( G = \lim_{\to} G_i \), with direct limit topology, and the \( \phi_{k,i} \) specify continuous group homomorphisms \( \phi_i : G_i \to G_k \). The strictness condition is that the homomorphisms \( \phi_i \) are homeomorphisms onto their images. So we may in fact view the \( \phi_{k,i} \) as inclusions and view \( G \) as the union of the \( G_i \), and then the original topology on each \( G_i \) is the subspace topology. In particular \( G_i \) sits in \( G \) as a closed (thus regularly embedded) submanifold.

Countability of \( I \) has two important consequences. First, it guarantees the existence of a \( C^\omega \) (real analytic) Lie group structure on \( G \). See [20], [21], [22] and [9]. Second, it guarantees that \( I \) either is finite or has a cofinal subset order–isomorphic to the positive integers. Whenever it is convenient we will replace \( I \) by that subset; this change in the defining direct system \( \{ G_i, \phi_{k,i} \}_{i,k \in I} \) has no effect on the direct limit group \( G = \lim_{\to} G_i \).

We always assume that every \( G_i \) satisfies [21] and [22].

We have the corresponding strict direct system \( \{ g_i, d\phi_{k,i} \}_{i,k \in I} \) of reductive Lie algebras, the direct limit algebra \( g = \lim_{\to} g_i \) with the direct limit topology, and injective homomorphisms \( d\phi_i : g_i \to g \) that are \( C^\omega \) diffeomorphisms onto their images. We also have the exponential map \( \exp : g \to G \), direct limit of the \( \exp : g_i \to G_i \). The \( C^\omega \) Lie group structure on the limit group \( G \) is specified by the condition that \( \exp : g \to G \) is a \( C^\omega \) diffeomorphism from a neighborhood on 0 in \( g \) onto a neighborhood of 1 in \( G \). Again see [20], [21], [22] and [9].

Consider a compatible family of representations \( \{ \pi_i, W_i, \psi_{k,i} \}_{i,k \in I} \) of \( \{ G_i, \phi_{k,i} \} \). Thus \( W_i \) is a locally convex topological vector space (usually Hilbert or Fréchet). \( W_i, \psi_{k,i} \) is a strict direct system, \( \pi_i \) is a continuous representation of \( G_i \) on \( W_i \), and

\[
(4.1) \quad \text{if } i \leq k, g_i \in G_i \text{ and } w_i \in W_i \text{ then } \pi_k(\phi_{k,i}(g_i))(\psi_{k,i}(w_i)) = \psi_{k,i}(\pi_i(g_i)(w_i))
\]

That of course results in continuous injective linear maps \( \psi_i : W_i \to W \) with closed image, where \( W = \lim_{\to} W_k \). We have the direct limit representation \( \pi = \lim_{\to} \pi_i \) of \( G \) on \( W \) given by

\[
(4.2) \quad \text{if } g = \phi_i(g_i) \in G \text{ and } w = \psi_i(w_i) \in W \text{ then } \pi(g)w = \psi_i(\pi_i(g_i)(w_i)).
\]

We now examine the situation where the \( \pi_i \) are principal series representations of the \( G_i \). For that we need direct limits of minimal parabolic subgroups.

As mentioned above we may assume \( I = \{ 1, 2, 3, \ldots \} \) with the usual order. Then we recursively construct Cartan involutions \( \theta_i \) of \( g_i \) such that if \( i \leq k \) then \( \theta_i|_{d\phi_{k,i}(g_i)} = \theta_i \), in other words \( d\phi_{k,i}(\mathfrak{t}_i) = \mathfrak{t}_k \cap d\phi_{k,i}(\mathfrak{g}_i) \). We know that \( \theta_i \) extends uniquely to \( G_i \) in such a way that its fixed point set \( K_i \) has Lie algebra \( \mathfrak{t}_i \), contains the kernel of the adjoint representation of \( G_i \), and meets every component of \( G_i \). Thus \( K_i \) is the \( G_i \)-normalizer of \( K_i^0 = K_i \cap G_i^0 \). Because of components, however, we must explicitly assume that

\[
(4.3) \quad \text{if } i \leq k \text{ then } \phi_{k,i}(K_i) \subset K_k, \text{ so we have } K = \lim_{\to} K_i.
\]

While it is tempting to try to get around the assumption \( 4.3 \) by assuming that the \( G_i \) are connected, we would still meet the same problem with the groups \( M_i \) indicated below.

Now \( d\phi_{k,i} \) maps the \((-1)\)-eigenspace of \( \theta_i \) into the \((-1)\)-eigenspace of \( \theta_k \), so we can recursively construct maximal abelian subspaces \( a_i \subset \{ \xi \in g_i \mid \theta_i(\xi) = -\xi \} \) as in [22] such that \( d\phi_{k,i}(a_i) \subset a_k \) for \( i \leq k \). Then the corresponding analytic subgroups satisfy

\[
(4.4) \quad \text{if } i \leq k \text{ then } \phi_{k,i}(A_i) \subset A_k, \text{ so we have } A = \lim_{\to} A_i.
\]
Note \(d\phi_{k,i}(a_i) = a_i \cap d\phi_{k,i}(g_i)\). This allows us to recursively construct a sequence of elements \(\zeta_i \in a_i^*\) such that \(\langle \zeta_i, a_i \rangle \neq 0\) for all \(a_i \in \Sigma(g_i, a_i)\) and \(d\phi_{k,i}(\zeta_k) = \zeta_i\) for \(i \leq k\). Taking roots where that inner product is positive we have positive root systems \(\Sigma(g_i, a_i)^+\) such that \(d\phi_{k,i}\) maps every negative restricted root space \(g_i^{-\zeta_i}\) into \(n_k := \sum_{\beta \in \Sigma(g_i, a_i)^+} g_k^{-\beta_k}\). Again, the corresponding analytic subgroups satisfy

\[
\text{if } i \leq k \text{ then } \phi_{k,i}(N_i) \subset N_k, \text{ so we have } N = \lim_{\rightarrow} N_i.
\]

Essentially as before, let \(M_i\) denote the centralizer \(Z_{K_i}(A_i)\) of \(A_i\) in \(K_i\). In general the behavior of the \(M_i\) (or even their identity components and Lie algebras) under the \(\phi_{k,i}\) is unclear. Thus we explicitly assume that

\[
\text{if } i \leq k \text{ then } \phi_{k,i}(M_i) \subset M_k, \text{ so we have } M = \lim_{\rightarrow} M_i.
\]

Now we put all this together. Under the assumptions (4.3) and (4.6) we have

\[
\text{Iwasawa decompositions } G_i = K_i A_i N_i \text{ and minimal parabolics } P_i = M_i A_i N_i
\]

such that \(\phi_{k,i}\) maps \(K_i \rightarrow K_k, M_i \rightarrow M_k, A_i \rightarrow A_k\) and \(N_i \rightarrow N_k\).

In particular,

\[
\text{we have an Iwasawa decomposition } G = KAN
\]

and if \(i \leq k\) then \(\phi_{k,i}(P_i) \subset P_k\) so

\[
\text{we have the limit minimal parabolic } P = \lim_{\rightarrow} P_i = MAN.
\]

Here \(G = KAN\) and \(P = MAN\) mean

\[
\text{(i) } (k,a,n) \mapsto \text{kan is a } C^\infty \text{ diffeomorphism of } K \times A \times N \text{ onto } G,
\]

\[
\text{(ii) } (m,a,n) \mapsto \text{man is a } C^\infty \text{ diffeomorphism of } M \times A \times N \text{ onto } P.
\]

**Example 4.11** The diagonal embedding direct limit groups described in [22, Section 5], and their extension to noncompact real forms, all satisfy (4.3) and (4.6), leading to the limits and decompositions \(G = KAN\) and \(P = \lim_{\rightarrow} P_i = MAN\) of (4.8) and (4.9). For example, let \(\{r_n\}\) and \(\{s_n\}\) be sequences of integers \(\geq 0\) where \(1 \leq n < \infty\) and \(r_n + s_n \geq 1\). Fix \(k_1 > 1\), and recursively define \(k_{n+1} = r_n k_n + s_n\), define \(G_n = SL(k_n; \mathbb{R})\) and \(d\phi_{n+1,n} : G_n \rightarrow G_{n+1}\) by \(d\phi_{n+1,n}(g) = \text{diag}(g, \ldots, g; 1, \ldots, 1)\) with \(r_n\) of \(g\)'s and \(s_n\) of 1's. Here \(K_n\) is the special orthogonal group \(SO(k_n)\), \(A_n\) consists of the diagonal matrices of determinant 1 with positive diagonal entries in \(G_n\), \(M_n\) consists of the diagonal matrices determinant 1 with diagonal entries \(\pm 1\) in \(G_n\), and \(N_n\) consists of the lower triangular matrices in \(G_n\) with all diagonal entries equal to 1. The limit groups depend on the choice of sequences \(\{r_n\}\) and \(\{s_n\}\), and it is quite nontrivial to see when pairs of sequences lead to isomorphic limits.

In order to discuss representations of \(M\) we need direct systems of Cartan subalgebras and appropriate root orders. With \(I = \{1, 2, \ldots\}\) we recursively construct

\[
\text{Cartan subalgebras } t_i \text{ in } m_i, \text{ } d\phi_{k,i} t_i \subset t_k, \text{ and positive systems } \Sigma(m_i, t_i)^+\]

\[
\text{such that } d\phi_{k,i} \left( \sum_{\alpha \in \Sigma(m_i, t_i)^+} m_i^\alpha \right) \subset \sum_{\beta \in \Sigma(m_k, t_k)^+} m_k^\beta \text{ for } i \leq k.
\]

Then \(t = \lim_{\rightarrow} t_i\) is a Cartan subalgebra of \(m\), the root system \(\Sigma(m_i, t_i) = \lim_{\rightarrow} \Sigma(m_i, t_i)\) (inverse limit), and the positive system \(\Sigma(m_i, t_i)^+ = \lim_{\rightarrow} \Sigma(m_i, t_i)^+\) is well defined. The
Cartan subalgebra \( t \) of \( m \) defines a Cartan subgroup \( T = \{ m \in M \mid \text{Ad}(m) \xi = \xi \text{ for all } \xi \in t \} \), and \( T^0 = T \cap M^0 \) is the corresponding Cartan subgroup of \( M^0 \).

Each \( M_i/M^0 \) is discrete, so \( \lim M_i \) is connected, closed and open in \( M \). Thus \( \lim M_0 = M^0 \) and \( M/M^0 \) is discrete. The same considerations hold for \( G, K \) and \( T \). Also, since each \( M_0 = T \cdot T^0 \) we have \( M = TM^0 \). Here note \( T^0 = T \cap M^0 \). In the special case where each \( \phi_{k,i} : Z_{M_i}(M^0) \to Z_{M_k}(M^0) \) we have \( Z_{M}(M^0) = \lim Z_{M_i}(M^0) \) and \( M = Z_{M}(M^0)M^0 \).

A linear functional \( \nu \in t_\nu^* \) is called integral (or \( m \)-integral) if \( e^\nu \) is a well defined homomorphism \( T^0 \to \mathbb{C}^\times \), in other words if the pull-backs \( \nu_k = \phi_{k,i}^* (\nu) \in t_{\nu_k,x}^* \) are integral. Here note \( \nu = \lim \nu_k \). The functional \( \nu \) is called dominant (or \( m \)-dominant) if \( (\nu, a) \geq 0 \) for every \( a \in \Sigma(m, t_{\nu}^*) \), in other words if \( \nu_k \) is \( m_i \)-dominant for each \( i \). We use these notions for a small variation on the Mackey little-group method.

**Proposition 4.13** Let \( \nu \in t_\nu^* \) be a dominant integral linear functional. It determines an irreducible unitary representation \( \eta_{\nu} \) of \( M^0 \) as follows. Let \( \eta_{\nu_k, \nu} \) denote the irreducible unitary representation of \( M^0 \) with lowest weight \( -\nu = \phi_k^* (-\nu) \). Choose a unit lowest weight vector \( v_{\nu_k, \nu} \) in the representation space \( V_{\nu_k, \nu} \) of \( \eta_{\nu_k, \nu} \). For \( k \leq i \) extend the Lie algebra homomorphism \( d\phi_{k,i} : g_k \to g_i \) as usual to an enveloping algebra homomorphism \( U(g_k) \to U(g_i) \), which we also denote \( d\phi_{k,i} \), and define \( \psi_{k,i} : V_{\nu_k, \nu} \to V_{\nu_k, \nu} \) by \( \psi_{k,i}(d\eta_{\nu_k, \nu}(\Xi_{\nu_k, \nu}))(v_{\nu_k, \nu}) = d\eta_{\nu_k, \nu}(d\phi_{k,i}(\Xi_{\nu_k, \nu}))(v_{\nu_k, \nu}) \) for \( \Xi_{\nu_k, \nu} \in U(m_i) \).

(1) \( \{ \eta_{\nu_k, \nu}, V_{\nu_k, \nu}, \psi_{k,i} \} \) is a compatible system of irreducible representations of \( \{ M_0^0, \phi_{k,i} \} \), so \( \eta_{\nu} = \lim \eta_{\nu_k} \) is a well defined irreducible unitary representation of \( M^0 \), with representation space \( V_{\nu} = \lim V_{\nu_k, \nu} \). Further, \( \eta_{\nu} \) is a lowest weight representation with lowest weight \( -\nu \), and \( v_{\nu} := \lim v_{\nu_k, \nu} \) is a lowest weight unit vector.

(2) If \( m \in M \) then \( \eta_{\nu} \circ \text{Ad}(m)^{-1} \) is unitarily equivalent to \( \eta_{\nu} \).

(3) Denote \( M_{\nu} \): equivalence classes of irreducible unitary representations \( \eta \) of \( M \) such that \( \eta|_{M^0} \) weakly contains \( \eta_{\nu} \) in the sense that the kernel of \( d\eta \) on the enveloping algebra \( U(m) \) is contained in the kernel of \( d\eta_{\nu} \). Then \( M_{\nu} = \{ \eta \in \tilde{M} \mid [\eta]_{M^0} \) is a multiple of \( \eta_{\nu} \} \).

(4) Let \( \eta_{\chi, \nu} \in M_{\nu} \). Let \( V_{\chi, \nu} \) denote its representation space. Choose a subspace \( V_{\nu} \subset V_{\chi, \nu} \) on which \( M^0 \) acts by \( \eta_{\nu} \), let \( v \to v' \) denote the intertwining map of \( V_{\nu} \) onto \( V_{\nu} \) and let \( v_{\nu} \) be the image of the lowest weight unit vector \( v_{\nu} \) of \( \eta_{\nu} \). Then the image of \( V_{\nu} \) in \( V_{\nu} \) is \( d\eta_{\nu}(U(m))(v_{\nu}) \), and \( V_{\nu} = \lim d\eta_{\nu}(U(m))(v_{\nu}) \).

(5) In the special case where \( M = Z_{M}(M^0)M^0 \), the set \( M_{\nu} \) consists of all \( \{ \chi \otimes \eta_{\nu} \} \) such that \( \chi \in (Z_{M}(M^0))_{\xi} \) where \( \xi = e^{-\nu}|_{Z_{M}\nu} \).

**Note.** In general we write the elements of \( M_{\nu} \) as \( [\eta_{\chi, \nu}] \) where \( \chi \) is just a parameter. In the case of Statement (5) the parameter \( \chi \) is interpreted as an element of \( (Z_{M}(M^0))_{\xi} \).

**Proof.** Statement (1) is satisfied by construction.

For Statement (2) let \( m \in M \) and \( \eta_{\nu} = \eta_{\nu} \circ \text{Ad}(m)^{-1} \). We view \( M \) as the union of the \( M_i \). Then \( m \) belongs to some \( M_i \), hence to \( M_i \) for \( i \geq \delta \). Altering \( m \) by an element of \( M_0^0 \) we may assume that \( m \in Z_{M_i}(M_0^0) \), so \( \text{Ad}(m)^* (v_0) = \nu_0 \), and thus \( \eta_{\nu_0, \nu}(v_0) = \) some multiple \( c_0 v_0 \) of \( v_0 \). Apply the enveloping algebra now to see that \( v \to c_0 v \) intertwines \( \eta_{\nu_0, \nu} \) with \( \eta_{\nu_0, \nu} \). The point here is that we may replace \( c_0 \) by any other modular scalar, for example by \( 1 \). Now \( v \to v \) intertwines \( \eta_{\nu_0, \nu} \) with \( \eta_{\nu_0, \nu} \), and thus intertwines \( \eta_{\nu_0, \nu} \) with \( \eta_{\nu_0, \nu} \).

For Statement (3) let \( K \) denote the kernel of \( d\eta \) on \( U(m) \) and let \( K_{\nu} \) denote the kernel of \( d\eta_{\nu} \). If \( \eta|_{M^0} \) is a multiple of \( \eta_{\nu} \) then \( K = K_{\nu} \). Now let \( K \subset K_{\nu} \). Then the associative algebra \( U(m)/K_{\nu} \) is a quotient of \( U(m)/K \). Remember that \( \eta_{\nu} \) is irreducible. Since \( M^0 \) is connected and generated by \( \exp(m) \) now \( \eta_{\nu} \) is equivalent to a quotient representation of \( \eta|_{M^0} \). By unitarity now \( \eta_{\nu} \) is equivalent to a subrepresentation of \( \eta|_{M^0} \). Let \( w \) be a cyclic unit vector for that
irreducible subrepresentation and let \( W \) be a set of representatives of \( M \) modulo \( M^0 \). Then the representation space of \( \eta \) is generated by the \( \eta(M)(\eta(x)w), x \in W \). By Statement (2), the action of \( M^0 \) on the closed span of \( \eta(M)(\eta(x)w) \) is equivalent to \( \eta_\nu \). Thus \( \eta|_{M^0} \) is a multiple of \( \eta_\nu \).

Statements (4) and (5) follow from (1) and (3).

Fix \( [\eta_{X,\nu}] \in \hat{M}_\nu \) as in Proposition \ref{prop:4.13}. In the notation of Proposition \ref{prop:4.13} identify \( V_\nu \) with its image \( V'_\nu = d\eta_\nu(\mathcal{U}(m))(v'_\nu) \) in \( \chi,\nu \) and identify the lowest weight vector \( v'_\nu \) of \( \eta_\nu \) with its image \( v'_\nu \) in \( \chi,\nu \). Let \( V_{i,\chi,\nu} \) denote the closed span of \( \eta_{X,\nu}(M_i)(v'_\nu) \) and let \( \eta_{i,\chi,\nu} \) denote the representation of \( M_i \) on \( V_{i,\chi,\nu} \). Unwinding the definitions one sees that

\[
\eta_{X,\nu} = \lim \eta_{i,\chi,\nu}.
\]

Now let

\[
\sigma \in \mathfrak{a}^*_c \text{ and } \sigma_i = \phi^*_i(\sigma) \in (\mathfrak{a}_i)^*.
\]

As in Section \ref{sec:2} that is equivalent to the data

\[
\eta_{X,\nu,\sigma} \in \hat{P} : \eta_{X,\nu,\sigma}(man) = e^\sigma(a)\eta_{X,\nu}(m) \text{ for } m \in M, a \in A \text{ and } n \in N, \text{ and } \\
\eta_{i,\chi,\nu,\sigma} \in \hat{P}_i : \eta_{i,\chi,\nu,\sigma}(man) = e^{\sigma_i}(a)\eta_{i,\chi,\nu}(m) \text{ for } m \in M_i, a \in A_i \text{ and } n \in N_i.
\]

Again as in Section \ref{sec:2} we write \( V_{X,\nu,\sigma} \) for the representation space of \( \eta_{X,\nu,\sigma} \): as a vector space it just \( V_{\chi,\nu} \). Similarly we write \( V_{i,\chi,\nu,\sigma} \) for the representation space of \( \eta_{i,\chi,\nu,\sigma} \).

The principal series representation of \( G \) defined by \( [\eta_{X,\nu}] \in \hat{M}_\nu \) and \( \sigma \in \mathfrak{a}_c^* \) is

\[
\pi_{X,\nu,\sigma} = \text{Ind}^G_P(\eta_{X,\nu,\sigma}), \text{ induced representation.}
\]

This representation is always given by the formula \( \pi_{X,\nu,\sigma}(g)(f(g')) = f(g^{-1}g') \). Of course we also have the principal series representations \( \pi_{i,\chi,\nu,\sigma} = \text{Ind}^{G_i}_{P_i}(\eta_{i,\chi,\nu,\sigma}) \) of \( G_i \).

The principal series representations \( \pi_{X,\nu,\sigma} \) of (4.17) has representation space that consists of an appropriate class of functions \( f : G \to V_{X,\nu,\sigma} \) such that \( f(gman) = e^{-\sigma(a)}\eta_{X,\nu}(m)^{-1}.f(g) \) for \( g \in G \) and \( man \in MAN = P \). Here recall that \( V_{X,\nu,\sigma} \) is the representation space of \( \eta_{X,\nu,\sigma} \). View the representation space \( V_{i,\chi,\nu,\sigma} \) of \( \eta_{i,\chi,\nu,\sigma} \) as the closed \( M_i \)-invariant subspace of \( V_{X,\nu,\sigma} \) generated by \( \eta_{X,\nu,\sigma}(M_i)(v'_\nu) \). Then the representation space of \( \pi_{i,\chi,\nu,\sigma} \) is the subspace of the representation space of \( \pi_{X,\nu,\sigma} \), given by \( f(G_i) \subset V_{i,\chi,\nu,\sigma} \). Since \( G \) is the union of the \( G_i \) and \( V_{X,\nu,\sigma} \) is the union of the \( V_{i,\chi,\nu,\sigma} \) we have proved

**Proposition 4.18** The principal series representations of a countable strict direct limit are just the direct limits of principal series representations. Specifically, \( \pi_{X,\nu,\sigma} = \lim \pi_{i,\chi,\nu,\sigma} \).

In dealing with principal series representations one must be very careful about the category in which he takes the induced representation. Smoothness categories such as \( C^k, 0 \leq k \leq \infty \), \( C^\infty \) (test functions), \( C^{-\infty} \) (distributions), \( C^\omega \) (analytic), or \( C^{\omega} \) (hyperfunctions) are still available for principal series representations of \( G \), but anything involving integration over \( G/P \) is excluded. We will get around this problem by constructing geometric realizations that provide \( L_p \) versions of the principal series for \( G \).
5 Groups and Spaces for the Limit Principal Series

The Iwasawa decompositions \( \boxtimes \), and the Cartan subalgebras \( t_i \subset m_i \) and the positive root systems of \( \boxtimes \) define

Cartan subalgebras \( h_i = t_i \times a_i \) in \( g_i \) such that \( d\phi_{k,i} \) maps \( h_i \to h_k \)

and positive root systems \( \Sigma(g_i, c, h_i, c)^+ \) given by \( \boxtimes \)

\[
\text{such that } d\phi_{k,i} \left( \sum_{\alpha \in \Sigma(g_i, c, h_i, c)^+} \alpha \right) \subset \sum_{\beta \in \Sigma(g_k, c, h_k, c)^+} \beta \text{ for } i \leq k.
\]

Then \( h = \lim h_i \) is a Cartan subalgebra of \( g, \Sigma(g_\cdot, h_\cdot) := \lim \Sigma(g_i, c, h_i, c) \) is its root system, and \( \Sigma(g_\cdot, h_\cdot)^+ := \lim \Sigma(g_i, c, h_i, c)^+ \) is a positive subsystem. Further, we will need

\[
q_i = l_i \subset g_i, \quad \text{parabolic subalgebras such that}
\]

\[
(i) \text{ the } q_i \text{ are defined by sets } \Psi_i \text{ of } \Sigma(m_i, c, t_i, c)^+ \text{–simple roots, as in } \boxtimes
\]

\[
(ii) \text{ } d\phi_{k,i}^{-1}(q_k) = q_i.
\]

Then \( l_i := g_i \cap l_i, \quad \text{and } g_i \cap q_i = l_i + n_i, \) as in Section \( \boxtimes \)

Let \( G_{i,c} \) denote the connected simply connected Lie group with Lie algebra \( g_{i,c} \). In general \( G_i \) will not be a real form of \( G_{i,c} \) because in general \( g_i \mapsto g_{i,c} \) will not integrate to a homomorphism \( G_i \to G_{i,c} \), but at least we have the connected complex simply connected group \( G_c = \lim G_{i,c} \) with Lie algebra \( g_c = \lim g_{i,c} \).

Let \( Q_i \) be the parabolic subgroup with Lie algebra \( q_i \), and let \( Z_i \) denote the complex flag manifold \( G_{i,c}/Q_i \). Note that we would get the same \( Z_i \) if we did this construction starting with arbitrary complex Lie groups \( 'G_{i,c} \) for which the \( G_{i,c} \) are the universal covering groups, in particular if we started with any connected complex Lie group \( 'G_{i,c} \) for which \( G_i \) is a real form. For \( Z_i \) can be identified as the set of all \( \text{Int}(g_i, c) \)–conjugates of \( q_i \) in \( g_{i,c} \), with the action of \( G_i \) given by conjugation as in \( \boxtimes \).

The reason for this indirection is that in general we cannot choose a family of complex Lie groups \( 'G_{i,c} \), for which the \( G_{i,c} \) are the universal covering groups, such that the \( 'G_{i,c} \) constitute a well defined direct system of complex Lie groups and holomorphic homomorphisms \( '\phi_{k,i} \) with \( d\phi_{k,i} = d'\phi_{k,i} \).

We now recall some structural information concerning the limit groups and limit flags from \( \boxtimes \) Sections 1 & 2.

The parabolic \( Q_i = L_i/c U_i, \) semidirect product, where \( L_i/c \) and \( U_i \) are the respective complex analytic subgroups of \( G_{i,c} \) for \( l_i \subset g_i \) and \( u_i \). The direct systems \( \{ G_{i,c}, \phi_{k,i} \} \) and \( \{ q_i, d\phi_{k,i} \} \) define direct systems \( \{ L_i/c, \phi_{k,i} \} \) and \( \{ U_i, \phi_{k,i} \} \). Let \( Q = \lim Q_i, \quad L_c = \lim L_i/c, \) and \( U = \lim U_i \).

Then \( Q, L_c \) and \( U \) are closed complex analytic subgroups of \( G \), and \( Q = L_c U \) semidirect product.

We define a direct system \( \{ Z_i, \phi_{k,i} \} \) by \( Z_i = \{ Ad(g_i)q_i \mid g_i \in G_i \} \) and \( \phi_{k,i}(z_i) = z_k \), where \( z_i = Ad(g_i)(q_i) \) gives \( z_k = Ad(\phi_{k,i}(g_i))(q_k) \). Then \( Z_i, \phi_{k,i} \) is a strict direct system of complex manifolds and holomorphic maps, so the limit \( Z = \lim Z_i \) is a complex manifold and the \( \phi_{k,i}' : Z_i \to Z \) are holomorphic injections with closed image. The \( Z_i \) are complex homogeneous spaces \( G_{i,c}/Q_i \), and it follows that the limit flag manifold \( Z \) is a complex homogeneous space \( G_c(z_0) = G_c/Q \) where \( z_0 \) is the base point in \( Z \), i.e. \( \phi_i'(z_{i,0}) = z_0 \) for every \( i \). Further, the action \( G \times Z \to Z \) is holomorphic.

Let \( F_i \) denote the closed orbit \( G_i(z_{i,0}) = K_i(z_{i,0}) \) in \( Z_i \). Let \( S_i = M_i(z_{i,0}) \), complex flag manifold \( M_i/c/R_i \), where \( R_i \) is the parabolic subgroup of \( M_{i,c} \) for the set \( \Psi_i \) of simple \( (m_i, c, t_i, c) \)–roots whose extension to \( h_i, c \) defines \( q_i \) as in \( \boxtimes \). We have \( r_i = l_i/c + n_i/m \), reductive part and
nilradical, and \( i_{k,C} = \tau_i \cap i_{k,C} \) and \( n_m = \tau_i \cap n_{i,m} \). Thus \( R_i = J_{i,C}N_{i,m} \). Up to finite covering, let \( L_i \) denote the real form \( \phi_i^{-1}(L_{i,C}) \) of \( L_{i,C} \) and let \( J_i \) denote the real form \( \phi_i^{-1}(J_{i,C}) \) of \( J_{i,C} \). Then \( L_i = J_i \times A_i \).

Now go to the limit: \( F = \lim F_i \), \( L = \lim L_i \), \( S = \lim S_i \), \( R = \lim R_i \), and \( J_c = \lim J_{i,C} \). Then \( L = J \times A, G/LN = G/KAN \cong F \cong K/J \), and \( S' \cong M/J \). Further, two translates \( gS \) and \( g'S \) either coincide or are disjoint, and \( P = \{ g \in G \mid gS = S \} \). Thus we have a fibration exactly as in Proposition 4.2.

**Proposition 5.3** Define \( k : F \to G/P = \{ gS \mid g \in G \} \) by \( k(gz_0) = gS \). Then \( k : F \to G/P \) is a well defined \( C^\infty \) fiber bundle with structure group \( P \), where the fiber over \( gS \) is the complex submanifold \( gS \) of \( Z \) that is contained in \( F \).

### 6 Bundles and Sheaves for the Limit Principal Series

Retain the notation of Section 5. In order to construct a coherent family of homogeneous vector bundles \( E_{j,\chi,\nu,\sigma} \to F_i \), we start with a coherent family of representations, as in Proposition 4.13. The proof of Proposition 6.1 just below, is the same as the proof of Proposition 4.13.

**Proposition 6.1** Let \( \nu \in \mathfrak{t}_\nu^* \) be a \( j_c \)-dominant integral linear functional. It determines an irreducible unitary representation \( \zeta_\nu \) of \( J^0 \) as follows. Let \( \zeta_{i,\nu} \) denote the irreducible unitary representation of \( J^0_i \) with lowest weight \(-\nu_i = \phi_i^*(\nu)\). Choose a unit lowest weight vector \( e_{i,-\nu} \) in the representation space \( E_{i,\nu} \) of \( \zeta_{i,\nu} \). For \( i \leq k \) define \( \psi_{k,i} : E_{i,\nu} \to E_{k,\nu} \) by \( \psi_{k,i}(d\zeta_{i,\nu}(\xi)(e_{i,-\nu})) = d\zeta_{k,\nu}(d\phi_{k,i}(\xi))(e_{k,-\nu}) \) for \( \xi \) in the enveloping algebra \( U(j_i) \).

1. \( \{ \zeta_{i,\nu}, E_{i,\nu}, \psi_{k,i} \} \) is a compatible system of irreducible representations of \( \{ J^0_i, \phi_{k,i} \} \), so \( \zeta_\nu = \lim_{i \to k} \zeta_{i,\nu} \) is a well defined irreducible unitary representation of \( J^0 \), with representation space \( E_\nu = \lim_{i \to k} V_{i,\nu} \).

2. If \( j \in J \) then \( \zeta_\nu \circ \text{Ad}(j)^{-1} \) is unitarily equivalent to \( \zeta_\nu \).

3. Denote \( \tilde{J}_\nu \): equivalence classes of irreducible unitary representations \( \zeta \) of \( J \) such that \( \zeta \mid_{J^0} \) weakly contains \( \zeta_\nu \) in the sense that the kernel of \( d\zeta \) on the enveloping algebra \( U(j) \) is contained in the kernel of \( d\zeta_\nu \). Then \( \tilde{J}_\nu = \{ [\zeta] \in \tilde{J} \mid [\zeta]_{J^0} \) is a multiple of \( \zeta_\nu \} \).

4. Let \( \tilde{\zeta}_{X,\nu} \in \tilde{J}_\nu \). Let \( E_{X,\nu} \) denote its representation space. Choose a subspace \( E'_\nu \subset E_{X,\nu} \) on which \( J^0 \) acts by \( \zeta_\nu \), let \( e \mapsto e' \) denote the intertwining map of \( E_\nu \) onto \( E'_\nu \), and let \( e'_{-\nu} \) be the image of the lowest weight unit vector \( e_{-\nu} \) of \( \zeta_\nu \). Then the image of \( E_{i,\nu} \) in \( E_{X,\nu} \) is \( d\zeta_{i,\nu}(U(j_i))(e'_{-\nu}) \), and \( E'_\nu = \lim_{i \to k} d\zeta_{i,\nu}(U(j_i))(e'_{-\nu}) = d\zeta_\nu(U(j))(e'_{-\nu}) \).

5. In the special case where \( J = Z_J(J^0)J^0 \), the set \( \tilde{J}_\nu \) consists of all \([ \chi \tilde{\otimes} \zeta_\nu ] \) such that \( \chi \in (\hat{Z}_J(J^0))_\xi \) where \( \xi = e^{-\nu}|_{Z_{J^0}} \).

Note. In general we write the elements of \( \tilde{J}_\nu \) as \( [\zeta_{X,\nu}] \) where \( \chi \) is just a parameter. In the case of Statement (5) the parameter \( \chi \) is interpreted as an element of \( (\hat{Z}_J(J^0))_\xi \).

Now let

\[ [\zeta_{X,\nu}] \in \tilde{J}_\nu \text{ as in Proposition 6.1 and let } \sigma \in \mathfrak{a}_c^* . \]

As in Sections 2 and 4 that is equivalent to the datum

\[ \zeta_{X,\nu,\sigma} \in \tilde{J}_AN \text{ defined by } \zeta_{X,\nu,\sigma}(jan) = e^\sigma(a)\zeta_{X,nu}(b) \text{ for } j \in J, a \in A \text{ and } n \in N. \]

As in Sections 2 and 4 we write \( E_{X,\nu,\sigma} \) for the representation space of \( \zeta_{X,\nu,\sigma} \). Now we have

\[ E_{X,\nu,\sigma} \to F : G \text{-homogeneous vector bundle with fiber } E_{X,\nu,\sigma} \text{ over } z_0 \]
as in (3.6). If \( g \in G \) then \( E_{\chi,\nu,\sigma}|gS \to gS \) is a holomorphic vector bundle.

Note that the limit \( E_{\chi,\nu,\sigma} = \lim E_{i,\chi,\nu,\sigma} \) where \( E_{i,\chi,\nu,\sigma} \) is the subspace of \( E_{\chi,\nu,\sigma} \) generated by \( \zeta_{\chi,\nu,\sigma}(J_iA_iN_i)(e_\nu') \). Let \( E_{i,\chi,\nu,\sigma} \to F_i \) denote the associated \( G_i \)-homogeneous vector bundle.

It is holomorphic over each \( g_iS_i \). The maps

\[
\phi_{k,i} \times \psi_{k,i} : G_i \times E_{i,\chi,\nu,\sigma} \to G_k \times E_{k,\chi,\nu,\sigma}
\]

induce bundle maps \((\phi_{k,i}, \psi_{k,i}) : E_{i,\chi,\nu,\sigma} \to E_{k,\chi,\nu,\sigma}\). These bundle maps form a coherent system and give us

\[
(6.5) \quad E_{\chi,\nu,\sigma} = \lim E_{i,\chi,\nu,\sigma}.
\]

Write \( E^*_{\chi,\nu,\sigma} \) for the strong topological dual of \( E_{\chi,\nu,\sigma} \) and write \( E^*_{i,\chi,\nu,\sigma} \to F \) for the associated homogeneous vector bundle. Again, the restricted bundles \( E^*_{i,\chi,\nu,\sigma}|gS \to gS \) are holomorphic vector bundles, for every fiber \( gS \) of \( F \to G/P \), by Lemma 2.2 of [22]. By elliptic regularity for hyperfunctions, Dolbeault cohomology is the same for \( C \) of the sheaf over \( \phi \) with stalk \( U \), and also the sheaf \( O_n(E^*_{\chi,\nu,\sigma}) \to F \) corresponding to the dual bundle. These are the sheaves of germs of \( C^{-\omega} \) sections that are holomorphic over the fibers \( gS \) of \( F \to G/P \).

For simplicity of notation, we write \( O_n(E_{i,\chi,\nu,\sigma}) \to F_i \), instead of \( O_n(E_{i,\chi,\nu,\sigma}) \to F_i \), for the sheaf over \( F_i \) analogous to that of (6.6).

We recall the definition of the inverse limit sheaf \( \lim O_n(E^*_{i,\chi,\nu,\sigma}) \). First, identify \( Z_i \) with \( \phi_i(Z_i) \subset Z \), thus also identifying \( F_i \) with \( \phi_i(F_i) \subset F \), and view \( O_n(E^*_{i,\chi,\nu,\sigma}) \) as a sheaf over \( F \) with stalk \( \{0\} \) over every point \( z \notin F_i \). The open subsets of \( F_i \) are the sets \( U_i = U \cap F_i \) where \( U \) is open in \( F \). Let \( \Gamma_i(U) \) denote the abelian group of sections of \( O_n(E^*_{i,\chi,\nu,\sigma})|U_i \). The \( \Gamma_i(U) \) form a complete presheaf, corresponding to \( E^*_{i,\chi,\nu,\sigma} \). Also, the abelian group \( \Gamma(U) \) of sections of \( O_n(E^*_{\chi,\nu,\sigma})|U \) is the inverse limit, \( \Gamma(U) = \lim \Gamma_i(U) \), corresponding to the inverse system given by restriction of sections and then extension by zero. Also, the \( \Gamma(U) \) form a complete presheaf corresponding to \( O_n(E^*_{\chi,\nu,\sigma}) \). Thus, by definition,

\[
(6.7) \quad O_n(E^*_{\chi,\nu,\sigma}) = \lim_i O_n(E^*_{i,\chi,\nu,\sigma}).
\]

**Proposition 6.8** (Compare [22] Proposition 2.4) Let \( q \geq 0 \). Then there is a natural \( G \)-equivariant isomorphism from the cohomology \( H^q(F; O_n(E^*_{\chi,\nu,\sigma})) \) of the inverse limit onto the inverse limit \( H^q(F; O_n(E^*_{i,\chi,\nu,\sigma}))) \) of the cohomologies.

**Proof.** Apply [15] Chapter I, Theorem 4.5 with the global section functor \( \Gamma \) in place of \( T \) to see that our sheaf cohomologies are the derived functors of \( \Gamma \). Our neighborhood bases on \( F \) and the \( F_i \) are properly aligned, as described in the above description of the definition of the inverse limit sheaf, so that we have a base \( B \) for the topology of \( F \) such that each \( B_i := \{U_i = U \cap F_i \mid U \in B\} \) forms a base for the topology of \( F_i \). We can refine \( B \) so that the neighborhoods \( U \in B \) have the following property. If \( U \in B \) and \( g_i \in G_i \) such that \( U \cap g_iS_i \neq \emptyset \) then each \( U \cap g_iS_i \) is Stein, and \( U_i \) is the product of \( U \cap g_iS_i \) with a cell. Then, for every \( U \in B \),

(a) the inverse system \( \{\Gamma_i(U)\} \) is surjective, in other words if \( i \leq k \) and \( s_i \in \Gamma_i(U) \) then there exists \( s_k \in \Gamma_k(U) \) such that \( s_i = s_k|U_i \), and

(b) if \( q > 0 \) then \( H^q(U, O_n(E^*_{i,\chi,\nu,\sigma})|U_i) = 0 \) for all \( i \).
Let \( G \rightarrow \) be the proof that \( \lim \) to the \( \nu \) of \([28, \text{Theorem } 1.2.19]\) the action of \( \nu \) for the unitary case, and note that its proof suffices for the general case. The point there is equivalent to the principal series representation \( \pi \nu \). Let \( \eta_{i,\chi,\nu} \) denote the representation of \( M_i \) on \( V_i := H^{q-1}(S_i; \mathcal{O}_n(E^*_{i,\chi,\nu,\sigma})) \). If \( \nu_i \) is \( m_i \)-singular then \( V_i = 0 \), so \( H^{q-1}(F_i; \mathcal{O}_n(E^*_{i,\chi,\nu,\sigma})) = 0 \) and the Mittag–Leffler condition \( (6.10) \) is trivially satisfied. Now assume that \( \nu_i \) is \( m_i \)-nonsingular. From Proposition \( 6.8 \) (or see \[28, \text{Theorem } 1.2.19\]) the action of \( G_i \) on \( H^{q-1}(F_i; \mathcal{O}_n(E^*_{i,\chi,\nu,\sigma})) \) is a certain principal series representation. Those representations have finite composition series; see \[28, \text{Theorem } 4.4.4\] for the unitary case, and note that its proof suffices for the general case. The point there is that the infinitesimal equivalence and the \( K_i \)-restriction are fixed, and that forces finiteness for the composition series. Since each subspace in the filtration \( \{ H^{q-1}(F_k; \mathcal{O}_n(E^*_{k,-\chi,\nu,\sigma})) \mid k \geq i \} \) of \( H^{q-1}(F_i; \mathcal{O}_n(E^*_{i,\chi,\nu,\sigma})) \) is an \( M_i \)-submodule, there are only finitely many possible composition factors, and \( (6.10) \) is immediate. That completes the proof of Proposition \( 6.8 \) \( \square \)

7 Geometric Realization of the Limit Principal Series

In this section we establish the geometric realization of principal series representations of direct limit groups, and look at some of the consequences. In effect we combine Propositions \( 6.1 \) and \( 6.8 \) and use ideas of Bott–Borel–Weil theory from \[22\].

We first look at a limit construction for principal series representations in the geometric style of the limit Borel–Weil Theorem, where there is no problem of cohomology degree.

**Theorem 7.1** Let \( \nu \in t^*_c \) be an \( \underline{1} \)-dominant integral linear functional. Let \( \zeta_{\chi,\nu} \in \widehat{J}_\nu \) as in Proposition \( 6.1 \). Let \( \zeta_{\chi,\nu} = \zeta_{\chi,\nu}^* \), the dual (contragredient) of \( \zeta_{\chi,\nu} \). For each \( i \) suppose that \( \nu_i := \phi^*_i(\nu) \) is \( m_i \)-dominant. Let \( \sigma \in a^*_c \). Let \( \sigma^* \) denote its complex conjugate, using conjugation of \( a^*_c \) over \( a^* \). Then the natural action of \( G \) on \( H^0(F; \mathcal{O}_n(E^*_{\chi,\nu,\sigma})) \) is infinitesimally equivalent to the principal series representation \( \pi_{\chi,\nu,\sigma} \), and its dual is infinitesimally equivalent to the principal series representation \( \pi_{\chi,\nu,\sigma}^* \).

**Proof.** Apply Proposition \( 6.8 \) to each \( E_{i,\chi,\nu,\sigma} \rightarrow F_i \). Since \( \nu \) is dominant, \( \nu_i \in (\Lambda^+_1)' \), \( q(\nu_i) = 0 \), and \( \nu_i = \nu_i \). Thus the natural action of \( G_i \) on \( H^0(F_i; \mathcal{O}_n(E_{i,\chi,\nu,\sigma})) \) is the principal series representation \( \pi_{i,\chi,\nu,\sigma} \).

Note \( \zeta_{\chi,\nu}^* = \zeta_{\chi,\nu} \) for some index \( \chi^* \) and \( \chi^* \) is in fact the dual of \( \chi \) when we are in the situation \( J = Z_f(J^0) \) of Proposition \( 6.1 \). Also, \( e^{\sigma^*} \) is the dual of \( e^\sigma \). Thus the bundles \( E_{\chi,\nu,\sigma} \) and \( E_{\chi,\nu,\sigma}^* \) are dual, at least at the \( K \)-finite level. Now \( \pi_{\chi,\nu,\sigma} \) and \( \pi_{\chi,\nu,\sigma}^* \) are dual, so the natural action of \( G \) on \( H^0(F; \mathcal{O}_n(E^*_{\chi,\nu,\sigma})) \) is \( \pi_{\chi,\nu,\sigma} \), and the natural action of \( G \) on \( H^0(F; \mathcal{O}_n(E^*_{\chi,\nu,\sigma})) \) is \( \pi_{\chi,\nu,\sigma}^* \).

Similarly \( \pi_{i,\chi,\nu,\sigma} \) and \( \pi_{i,\chi,\nu,\sigma}^* \) are dual, so the natural action of \( G_i \) on \( H^0(F_i; \mathcal{O}_n(E^*_{i,\chi,\nu,\sigma})) \) is \( \pi_{i,\chi,\nu,\sigma} \). Now Proposition \( 6.8 \) says that \( \pi_{\chi,\nu,\sigma} = \lim \pi_{i,\chi,\nu,\sigma} \), and thus also \( \pi_{\chi,\nu,\sigma} = \lim \pi_{i,\chi,\nu,\sigma}^* \). \( \square \)
In order to extend Theorem 7.1 to higher cohomology we face the same problem as in [22]. We have to find conditions under which the cohomology degrees

\[ q_i = q(\nu_i) := \{ \gamma_i \in \Sigma(m_i, t_i) : (\nu_i + \rho_{i,m,t}, \gamma_i) < 0 \} \]

remain constant as \( i \) increases indefinitely. So we recall some definitions from [22, Section 4].

Suppose that \( \nu_i + \rho_{i,m,t} \) is nonsingular. Then there is a unique element \( w_i \) in the Weyl group \( W(m_i, t_i) \) that carries \( \nu_i + \rho_{i,m,t} \) to a dominant weight, and \( q_i = q(w_i) \) is the length \( \ell(w_i) \).

The Weyl group \( W = W(m, t) \) is defined to be the group of all \( w \), where \( w \) is an automorphism of \( m \) such that (i) \( w(t) = t \) and (ii) for some index \( i_0 \) if \( i \geq i_0 \) then \( w(d\phi_i(m_i)) = d\phi_i(m_i) \) and \( w|_{d\phi_i(m_i)} \) is an inner automorphism of \( m_i \).

Our hypothesis amounts to a choice of Borel subalgebra \( b = \lim_i b_i \) of \( m \) such that \( t_i \subset b_i \subset t_i \) and \( d\phi_{k,i}(b_i) \subset b_k \), where \( r_t = d\phi_i^{-1}(t) \). This choice determines the finite Weyl group \( W_F = W_F(m, b, t) \) consisting of all \( w \) in \( W \) such that \( w(b) \subset b \) has finite codimension in \( b \).

We define this codimension to be the length \( \ell(w) \).

Let \( w \in W(m, t) \). Then we have the classically defined lengths \( \ell(w_i) \) relative to the positive root systems \( \Sigma(m_i, t_i) \). If \( w \in W_F(m, b, t) \) then there is an index \( i_0 \), which in general depends on \( w \), such that \( \ell(w_i) = \ell(w_k) \) for \( k \geq i \geq i_0 \), and this common length is \( \ell(w) \).

A linear functional \( \nu \in \mathfrak{t}_+^* \) is classically cohomologically finite if there exist \( w \in W_F(m, b, t) \) and \( i_0 \) as above, and an integral linear functional \( \bar{\nu} \in \mathfrak{t}_+^* \), with the following property. If \( i \geq i_0 \) then \( d\phi_i^*(\bar{\nu}) \) is dominant relative to the positive root system \( \Sigma(m_i, t_i) \), and \( d\phi_i^*(\bar{\nu}) = w_i(\nu_i + \rho_{i,m,t} - \rho_{i,m,t}) \). A linear functional \( \nu \in \mathfrak{t}_+^* \) is cohomologically finite of degree \( q_0 \), if whenever \( i \) is sufficiently large, say \( i \geq i_0 \), (i) \( \nu_i + \rho_{i,m,t} \) is nonsingular and (ii) \( q_i = q_0 \) constant in \( i \). If \( \nu \) is classically cohomologically finite by means of \( w \in W_F \) then it is cohomologically finite of degree \( \ell(w) \). By contrast, there are cases where \( \nu \) is cohomologically finite of degree \( q > 0 \) while \( W_F = \{1\} \), so \( \nu \) is not classically cohomologically finite.

Drawing on [22, Theorem 4.6] we now have a limit construction for principal series representations in the geometric style of the Bott–Borel–Weil theorem, as follows.

**Theorem 7.2** Let \( \nu \in \mathfrak{t}_+^* \) be a \( \chi \)-dominant integral linear functional. Let \( \sigma \in a^*_+ \).

1. If \( \nu \) is not cohomologically finite then every \( H^q(F; \mathcal{O}_n(\mathcal{E}_{\nu,\sigma}^*)) = 0 \).

2. Assume that \( \nu \) is cohomologically finite of degree \( q_\nu \). Then

   (a) \( H^q(F; \mathcal{O}_n(\mathcal{E}_{\nu,\sigma}^*)) = 0 \) for \( q \neq q_\nu \), and

   (b) the natural action of \( G \) on \( H^q(F; \mathcal{O}_n(\mathcal{E}_{\nu,\sigma}^*)) \) is infinitesimally equivalent to a principal series representation of the form \( \pi_{\nu,\sigma} = \lim_\pi_{\nu,\sigma} \), and its dual is infinitesimally equivalent to a principal series representation of the form \( \pi_{\nu,\sigma} = \lim_\pi_{\nu,\sigma} \).

3. If further \( \nu \) is classically cohomologically finite, say by means of \( w \in W_F \), then \( q_\nu = \ell(w) \) and in (2) we may take \( \mu = \bar{\nu} \), defined by \( \mu_i = w_i(\nu_i + \rho_{i,m,t} - \rho_{i,m,t}) \) for \( i \) sufficiently large.

**Proof.** Suppose that \( \nu \) is not cohomologically finite. Fix an integer \( p \geq 0 \). If \( \nu_i + \rho_{i,m,t} \) is singular then \( H^p(S_1; \mathcal{O}_n(\mathcal{E}_{\nu,\sigma}^*|S_1)) = 0 \). If \( \nu_i + \rho_{i,m,t} \) is nonsingular, then \( H^p(S_1; \mathcal{O}_n(\mathcal{E}_{\nu,\sigma}^*|S_1)) = 0 \) unless \( q(\nu_i) = p \). The \( q(\nu_i) \) are increasing in \( i \). Since \( \nu \) is not cohomologically finite, the \( q(\nu_i) \) are unbounded. Thus, \( H^p(S_1; \mathcal{O}_n(\mathcal{E}_{\nu,\sigma}^*|S_1)) \) becomes 0 and stays 0 as \( i \) increases. Let \( \eta_i^* \) denote the representation of \( M_i \) on \( H^p(S_1; \mathcal{O}_n(\mathcal{E}_{\nu,\sigma}^*|S_1)) \), and let \( \eta^* \) denote the representation of \( M \) on \( H^p(S_1; \mathcal{O}_n(\mathcal{E}_{\nu,\sigma}^*|S_1)) \). We have just seen that \( H^p(S_1; \mathcal{O}_n(\mathcal{E}_{\nu,\sigma}^*|S_1)) = \lim_\eta^* H^p(S_1; \mathcal{O}_n(\mathcal{E}_{\nu,\sigma}^*|S_1)) = 0 \), so the representation space of \( \eta^* \) is 0, and thus the representation space \( H^p(F; \mathcal{O}_n(\mathcal{E}_{\nu,\sigma}^*)) \) of \( \text{Ind}^G_{M \times \mathbb{C}} (\eta \otimes e^\sigma) \) is zero. That proves assertion (1).
Let \( k \) \( p \) Unitarity, \( L \). The first is studied extensively in \([22]\), and we now turn to the second.

8 Unitarity, \( L^p \) Boundedness, and Related Questions

According to Proposition \([2.11]\) the \( L_p \) condition for \( \pi_{i,\chi,\nu,\sigma} \) is \( \sigma_i \in i\mathfrak{a}_i^* + \frac{2}{p} \rho_{i,a} \). So the \( L_\infty \) condition is transparent: \( \sigma_i \in i\mathfrak{a}_i^* \) for all \( i \) if and only if \( \sigma \in i\mathfrak{a}^* \). Now we set that case aside and suppose \( 1 \leq p < \infty \).

**Lemma 8.1** Let \( 1 \leq p < \infty \). If \( k \geq i \) view \( d\phi_{k,i} : \mathfrak{g}_i \to \mathfrak{g}_k \) as an inclusion \( \mathfrak{g}_i \hookrightarrow \mathfrak{g}_k \). Then the \( \pi_{i,\chi,\nu,\sigma} \) satisfy the \( L_p \) condition for all \( i \geq i_0 \) if and only if (i) \( \sigma_{i_0} \in i\mathfrak{a}_{i_0}^* + \frac{2}{p} \rho_{i_0,a} \) and (ii) if \( k \geq i \geq i_0 \) then \( \rho_{k,a} \| a_i \| = \rho_{i,a} \). In that case \( \rho_a := \lim_{i \to \infty} \rho_{i,a} \in \mathfrak{a}^* \) is well defined.

**Proof.** The \( \pi_{i,\chi,\nu,\sigma} \) satisfy the \( L_p \) condition for all \( i \geq i_0 \) if and only if \( \sigma_i \in i\mathfrak{a}_i^* + \frac{2}{p} \rho_{i,a} \) for \( i \geq i_0 \), in other words \( Re \sigma_i = \frac{2}{p} \rho_{i,a} \). If (i) and (ii) hold, it is obvious that the \( \pi_{i,\chi,\nu,\sigma} \) satisfy the \( L_p \) condition for all \( i \) sufficiently large, say \( i \geq i_0 \). Conversely, suppose that \( k \geq i \geq i_0 \) and that \( \pi_{k,\chi,\nu,\sigma} \) satisfies the \( L_p \) condition for \( \ell = k, i, i_0 \). Then \( \frac{2}{p} \rho_{i,a} = \frac{2}{p} \rho_{k,a} \| a_i \| \), in other words \( \rho_{i,a} = \rho_{k,a} \| a_i \| \), in addition to \( \sigma_{i_0} \in i\mathfrak{a}_{i_0}^* + \frac{2}{p} \rho_{i_0,a} \). The last assertion follows. \( \square \)

Recall the structure theory for real parabolic subalgebras. Let \( \Psi_i \) denote the set of simple roots of \( \Sigma(\mathfrak{g}_i, \mathfrak{a}_i)^+ \). The \( G_i^0 \)-conjugacy classes of (real) parabolic subalgebras of \( \mathfrak{g}_i \) are in one to one correspondence \( \Phi_i \leftrightarrow \mathfrak{p}_{i,\Phi} \) with the subsets \( \Phi_i \subset \Psi_i \) by

\[
\mathfrak{p}_{i,\Phi} = \mathfrak{m}_{i,\Phi} + \mathfrak{a}_{i,\Phi} + \mathfrak{n}_{i,\Phi}
\]

(8.2)

\[
\mathfrak{a}_{i,\Phi} = \{ \xi \in \mathfrak{a}_i \ | \ \psi_i(\xi) = 0 \text{ for all } \psi_i \in \Phi_i \},
\]

\[
\mathfrak{m}_{i,\Phi} = \theta(\mathfrak{m}_{i,\Phi}) \text{ and } \mathfrak{m}_{i,\Phi} + \mathfrak{a}_{i,\Phi} \text{ is the centralizer of } \mathfrak{a}_{i,\Phi} \text{ in } \mathfrak{g}_i,
\]

\[
\mathfrak{n}_{i,\Phi} \text{ is the sum of the negative } \mathfrak{a}_i \text{-root spaces not in } \mathfrak{m}_{i,\Phi}.
\]

Here \( \mathfrak{n}_{i,\Phi} \) is the nilradical, \( \mathfrak{m}_{i,\Phi} + \mathfrak{a}_{i,\Phi} \) is the Levi component (reductive part), and \( \mathfrak{p}_{i,\Phi} \) is the normalizer of \( \mathfrak{n}_{i,\Phi} \) in \( \mathfrak{g}_i \). \( \Phi_i \) is the simple root system for \( \mathfrak{m}_{i,\Phi} + \mathfrak{a}_{i,\Phi} \). The minimal parabolic is the case \( \Phi_i = \emptyset \). The derived algebra \( \mathfrak{m}'_{i,\Phi} = [\mathfrak{m}_{i,\Phi}, \mathfrak{m}_{i,\Phi}] \) is a maximal semisimple subalgebra of \( \mathfrak{p}_{i,\Phi} \), and we refer to it as the semisimple component of \( \mathfrak{p}_{i,\Phi} \).

If \( \gamma \in \Sigma(\mathfrak{g}_i, \mathfrak{a}_i) \) we write \( \text{mult}(\gamma) \) for the multiplicity of \( \gamma \) as an \( \mathfrak{a}_i \)-root, in other words for the dimension \( \dim \mathfrak{g}_{i,\gamma} \) of the root space. Thus \( \rho_{\mathfrak{g},\mathfrak{a},\Phi} = \sum_{\gamma \in \Sigma(\mathfrak{g}_i, \mathfrak{a}_i)} \text{mult}(\gamma) \gamma \).

The following lemma is standard in the context on non-restricted roots, but we have not been able to find it in the literature, so we give a proof for the convenience of the reader.

**Lemma 8.3** If \( \psi \in \Psi_i \) then \( \frac{2(\rho_{\mathfrak{g},\mathfrak{a},\Phi}, \psi)}{\langle \psi, \psi \rangle} = \text{mult}(\psi) + 2 \text{mult}(2\psi) \).

**Proof.** Let \( \psi \in \Psi_i \) denote the Weyl group reflection for the simple restricted root \( \psi \). Then \( \psi \Sigma(\mathfrak{g}_i, \mathfrak{a}_i)^+ = \Sigma(\mathfrak{g}_i, \mathfrak{a}_i)^+ \setminus S(\psi) \) where \( S(\psi) \) is \( \{ \psi \} \) if \( 2\psi \) is not a restricted root, \( \{ \psi, 2\psi \} \)
if $2\psi$ is a restricted root. Now compute

$$2(\rho_{i,g,a} - \text{mult}(\psi)\psi - \text{mult}(2\psi)2\psi) = 2w_\psi(\rho_{i,g,a})$$

$$= \sum_{\gamma \in \Sigma(g,\alpha_i)} w_\psi(\gamma)$$

$$= \sum_{\gamma \in \Sigma(g,\alpha_i)} \left( \gamma - \frac{2(\gamma,\psi)}{\langle \psi,\psi \rangle} \right)$$

$$= 2\rho_{i,g,a} - \frac{2(\rho_{i,g,a},\psi)}{\langle \psi,\psi \rangle}.$$ 

Thus $\text{mult}(\psi) + 2\text{mult}(2\psi) = \frac{2(\rho_{i,g,a},\psi)}{\langle \psi,\psi \rangle}$, as asserted. \qed

Now we are ready to look at Condition (ii) of Lemma 8.4.

**Proposition 8.4** Let $g_i \subset g_k$, real semisimple Lie algebras. Choose a Cartan involution $\theta$ of $g_k$ that preserves $g_i$, let $a_i$ be a maximal abelian subspace of $\{ \xi \in g_i \mid \theta(\xi) = -\xi \}$, and enlarge $a_i$ to a maximal abelian subspace $a_k$ of $\{ \xi \in g_k \mid \theta(\xi) = -\xi \}$. Suppose that $a_i = a_i \oplus a_{k,i}$ where $a_{k,i}$ centralizes $g_i$, in other words that $g_i \oplus a_{k,i}$ is a subalgebra of $g_k$. Then following conditions are equivalent.

1. The restriction $\rho_{k,g,a}|_{a_i} = \rho_{i,g,a}$.
2. $(g_i + m_k) \oplus a_{k,i}$ is the centralizer of $a_{k,i}$ in $g_k$.
3. Modulo $m_k$, the algebra $g_i$ is the semisimple component of a real parabolic subalgebra of $g_k$ that contains $a_k$.

**Proof.** Assume (3). Then there is a subset $\Phi \subset \Psi$ such that, modulo $m_k$, $g_i$ is the semisimple component $s$ of $p_{k,\Phi}$. In particular $\Phi$ is the simple root system for $\Sigma(g_i \oplus a_{k,i}, a_k)^+$, so $\Sigma(g_i \oplus a_{k,i}, a_k) = \Sigma(s \oplus a_{k,i}, a_k)$, and the multiplicities $\text{mult}_s(\gamma) = \text{mult}_g(\gamma)$ for every root $\gamma \in \Sigma(s \oplus a_{k,i}, a_k)$. Thus $\rho_{i,g,a} = \rho_{s,a_i}$. But Lemma 8.3 shows that $\langle \rho_{k,g,a}, \varphi \rangle = \langle \rho_{s,a_i}, \varphi \rangle$ for every $\varphi \in \Phi$, so $\rho_{k,g,a}|_{a_i} = \rho_{s,a_i}$. That proves (1).

Assume (1). Denote $t = (g_i + m_k) \oplus a_{k,i}$. We have not yet proved that $t$ is an algebra, but we do have $\rho_{t,a_k} := \frac{1}{2} \sum_{\gamma \in \Sigma(t,a_k)} \text{dim}(\gamma \cap g_i^\perp)\gamma$, $\rho_{t,a_k} = \rho_{g_i \oplus a_{k,i}, a_k}$ by definition of $t$.

Let $\mathfrak{z}$ denote the centralizer of $a_{k,i}$ in $g_k$. Then $\rho_{k,g,a}|_{a_i} = \rho_{\mathfrak{z},a_i}|_{a_i}$. Using assumption (1) now $\rho_{\mathfrak{z},a_k} = \rho_{s,a_i}$. By construction of $t$ and of $\mathfrak{z}$, if $\gamma \in \Sigma(t,a_k)^+$ then $\gamma \in \Sigma(\mathfrak{z},a_k)^+$ and its multiplicities satisfy $\text{mult}_s(\gamma) \geq \text{mult}_g(\gamma)$. As $\rho_{\mathfrak{z},a_k} = \rho_{s,a_i}$ now $\sum_{\gamma \in \Sigma(t,a_k)^+} \text{mult}_s(\gamma)\gamma = 0$. Take inner product with $\rho_{\mathfrak{z},a_i}|_{a_i}$. Since each $\langle \rho_{\mathfrak{z},a_i}|_{a_i}, \gamma \rangle > 0$ and each $\text{mult}_s(\gamma) \geq \text{mult}_g(\gamma)$ it follows that $\text{mult}_t(\gamma) = \text{mult}_g(\gamma)$. That proves $t = \mathfrak{z}$, which is the assertion of (2).

Assume (2). Then $\mathfrak{z} = (g_i + m_k) \oplus a_{k,i}$ is the reductive component of a parabolic subalgebra of $g_k$.

The corresponding semisimple component is $[\mathfrak{z},\mathfrak{z}] = [g_i + m_k, g_i + m_k]$. If $\gamma \in \Sigma(g_i \oplus a_{k,i}, a_k)$ then $(g_i \oplus a_{k,i})^\perp = g_i^\perp$, so $[m_k, g_i] \subset g_i$. Now $[\mathfrak{z},\mathfrak{z}] = g_i + [m_k, m_k]$. That proves (3), completing the proof of the Proposition. \qed

**Corollary 8.5** Let $g_i$ be the semisimple component of a real parabolic subalgebra of $g_k$ that contains $a_k$. Then the restriction $\rho_{k,g,a}|_{a_i} = \rho_{i,g,a}$.

**Definition 8.6** The strict direct system $\{G_i, \Phi_{k,i}\}$ of reductive Lie groups is weakly parabolic if for every pair $k \geq i$ the subalgebra $d\Phi_{k,i}(g'_i) \to g'_k$ satisfies the conditions of Proposition 8.4, where $g'_i$ denotes the derived algebra of $g'_i$. It is parabolic if for every pair $k \geq i$ the subalgebra $d\Phi_{k,i}(g'_i)$ is the semisimple component of a real parabolic subalgebra of $g_k$.

**Remark 8.7** The condition that $\{G_i, \Phi_{k,i}\}$ be weakly parabolic is slightly less restrictive than the corresponding condition (7.1) of “coherent root orderings” in [22]. The context and applications are different, but the core idea is similar.
Now we come to the main result of this section:

**Theorem 8.8** Suppose that the direct system $\{G_i, \phi_k, i\}$ is weakly parabolic. Let $\nu \in U^*$ be a $\chi_\nu$-dominant integral linear functional that is cohomologically finite of degree $q_\nu$. Let $\sigma \in a^*_\nu$. Recall the principal series representation $\pi_{\nu, \sigma} = \lim_{p \rightarrow \infty} \pi_{\nu, \sigma}^p$ of $G$ on $H^{q_\nu}(F; \mathcal{O}_n(E^{*}_{\chi, \nu, \sigma})$ and its dual $\pi_{\nu, \sigma}^* = \lim_{p \rightarrow \infty} \pi_{\nu, \sigma}^p$. Suppose that $\sigma \in i a^*_\nu + \frac{2}{p} \rho_{i, \nu}$, where $1 \leq p \leq \infty$. Then $\pi_{\nu, \sigma}$ is infinitesimally equivalent to a Banach space representation on $\lim_{p \rightarrow \infty} L_2(G_i, P_i : V_{i, \chi, \nu, \sigma})$. In particular, if $\sigma_i \in i a^*_\nu + \rho_{i, \nu}$ then $\pi_{\nu, \sigma}$ is infinitesimally equivalent to a unitary representation on $\lim_{p \rightarrow \infty} L_2(G_i, P_i : V_{i, \chi, \nu, \sigma})$. □

**Proof.** Combine Theorem 7.2 with Lemma 8.1 Proposition 8.4 and Definition 8.6

In Theorem 8.8 it would be better to derive the $L_p$ norm directly from the limit bundle $E_{\chi, \nu, \sigma} \rightarrow F$. We do this by using a partially holomorphic cohomology space, as in [28]. The fibers $gS_i$ of $F_i \rightarrow G_i / P_i$ are compact, so any cohomology class $c_{gS_i} \in H^q(gS_i, \mathcal{O}_n(E^{*}_{\chi, \nu, \sigma} | gS_i))$ is represented by a harmonic $E_{\chi, \nu, \sigma} | gS_i$-valued $(0, q)$-form $\omega_{gS_i}$, and $\omega_{gS_i}$ has a well-defined $L_p$ norm $||\omega_{gS_i}||_p = (\int_{M} ||\omega_{gS_i}(m)||^p \text{d}m)^{1/p}$. We now look at the Banach spaces

$$B^q_{i, p}(F_i; E_{i, \chi, \nu, \sigma}) : \text{measurable } E_{i, \chi, \nu, \sigma} \text{-valued } (0, q) \text{-forms } \omega \text{ on } F_i \text{ such that}$$

$$||\omega||_p = \int_{M} (||\omega_{gS_i}||_p)^p \text{d}k < \infty$$

where

$$(8.9) \quad ||\omega||_p = \left( \int_{K_i / M_i} (||\omega_{gS_i}||_p)^p \text{d}k \right)^{1/p} = \left( \int_{K_i / J_i} (||\omega(kJ_i)||^p) \text{d}k \right)^{1/p}$$

For $p = 2$ the norm is given by the inner product

$$(8.10) \quad \langle \omega, \omega' \rangle = \int_{K_i / M_i} \left( \int_{M_i / J_i} \omega(kmz_0) \wedge \# \omega(kmz_0) \text{d}m \right) \text{d}k.$$

There $\omega$ is the $E^{*}_{\chi, \nu, \sigma}$-valued $(s, s - q)$-form, $s = \dim S_i$, which along $kS_i$ is the Hodge–Kodaira orthogonal of $\omega$, and $\wedge$ is exterior product followed by pairing of $E_{i, \chi, \nu, \sigma}$ with $E^{*}_{i, \chi, \nu, \sigma}$. That gives us a Hilbert space

$$H^q_{i, 2}(F_i; E_{i, \chi, \nu, \sigma}) = (B^q_{i, 2}(F_i; E_{i, \chi, \nu, \sigma}), \langle \cdot, \cdot \rangle).$$

Now, if we stay with a cofinal weakly parabolic subsystem of $\{G_i, \phi_k, i\}$ as in Theorem 8.8 we have Banach space representations

$$(8.13) \quad \pi_{\chi, \nu, \sigma}^* \text{ on } B^q_{i, p}(F_i; E^{*}_{\chi, \nu, \sigma}) = \lim_{p \rightarrow \infty} B^q_{i, p}(F_i; E_{i, \chi, \nu, \sigma})$$

and $\pi_{\chi, \nu, \sigma}^* \text{ on } B^q_{i, p}(F_i; E^{*}_{\chi, \nu, \sigma}) = \lim_{p \rightarrow \infty} B^q_{i, p}(F_i; E_{i, \chi, \nu, \sigma})$

where $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p < \infty$. In the case $p = 2$ we have unitary representations

$$(8.14) \quad \pi_{\chi, \nu, \sigma}^* \text{ on } H^q_{i, 2}(F_i; E^{*}_{\chi, \nu, \sigma}) = \lim_{p \rightarrow \infty} H^q_{i, 2}(F_i; E_{i, \chi, \nu, \sigma})$$

and $\pi_{\chi, \nu, \sigma}^* \text{ on } H^q_{i, 2}(F_i; E^{*}_{\chi, \nu, \sigma}) = \lim_{p \rightarrow \infty} H^q_{i, 2}(F_i; E_{i, \chi, \nu, \sigma})$.

Here note that $\lim_{p \rightarrow \infty}$ and $\lim_{p \rightarrow \infty}$ are the same in the Hilbert space category.
9 Diagonal Embedding Direct Limits

In this section we study an important class of direct limit groups that includes those obtained
from weakly parabolic direct systems. These diagonal embedding direct limits were introduced
on the complex Lie algebra level (see, for example, [2], [3], [4], [5], [35] and [36]). This topic
is now plays a central role in the theory of locally finite Lie algebras. The idea was somewhat extended and applied on both the compact and the complex group level in [22, Section 5], and
that is our starting point.

**Linear Groups.** We consider limits of real, complex and quaternionic special linear groups.

Fix sequences \( r = \{ r_n \}_{n \geq 1}, s = \{ s_n \}_{n \geq 1} \) and \( t = \{ t_n \}_{n \geq 1} \) of non–negative integers with all \( r_n + s_n > 0 \). Start with \( d_0 > 0 \) and recursively define \( d_{n+1} = d_n (r_{n+1} + s_{n+1}) + t_{n+1} \). Let \( \mathbb{F} \) be one of \( \mathbb{R} \) (real), \( \mathbb{C} \) (complex) or \( \mathbb{H} \) (quaternions) and define \( G_n = SL(d_n; \mathbb{F}) \). Let \( \delta \) denote the outer automorphism of \( G_n \) given by

\[
\delta(g) = J(g^t)^{-1}J^{-1} \quad \text{where} \quad J = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.
\]

(The point of \( J \) here is that it, as defined, preserves the standard positive root system.) Then
we have strict direct systems \( \{ G_m, \phi_{n,m} \}_{n \geq m \geq 0} \) given by

\[
\phi_{n+1,m} : G_n \to G_{n+1} \quad \text{by} \quad \phi_{n+1,n}(g) = \text{diag}\{ g, \ldots, g; \delta(g), \ldots, \delta(g); 1, \ldots, 1 \}
\]

with \( r_{n+1} \) blocks \( g \), with \( s_{n+1} \) blocks \( \delta(g) \), and with \( t_{n+1} \) entries 1. That gives us

\[
G = SL_{r,s,t}(\infty; \mathbb{F}) := \lim_{\longrightarrow} \{ G_m, \phi_{n,m} \} \quad \text{for the given} \ r, s \ \text{and} \ t.
\]

Thus we have \( SL_{r,s,t}(\infty; \mathbb{R}), SL_{r,s,t}(\infty; \mathbb{C}) \) and \( SL_{r,s,t}(\infty; \mathbb{H}) \). Of course the situation is exactly the same to construct infinite general linear groups \( GL_{r,s,t}(\infty; \mathbb{R}) \) and \( GL_{r,s,t}(\infty; \mathbb{C}) \).

**Unitary Groups.** We consider limits of real, complex and quaternionic unitary groups.

Here \( SU(p, q; \mathbb{R}) \) denotes the special orthogonal group \( SO(p, q) \) for a nondegenerate
form \( (p, q) \), \( SU(p, q; \mathbb{C}) \) denotes the usual complex special unitary \( SU(p, q) \) for a
nondegenerate hermitian form \( (p, q) \), and \( SU(p, q; \mathbb{H}) \) is the quaternionic special
unitary group for a nondegenerate hermitian form \( (p, q) \). In each case we write the
form as \( b(z, w) = \sum_{1 \leq i \leq p} z_i \overline{w_i} - \sum_{1 \leq i \leq q} \overline{z_{p+i}} w_{p+i} \), reflecting the fact that we view \( \mathbb{F}^{p+q} \) as a
right vector space over \( \mathbb{F} \) so that linear transformations act on the left.

Fix sequences \( r = \{ r_n \}_{n \geq 1}, s = \{ s_n \}_{n \geq 1} \), plus two new sequences \( t' = \{ t'_n \}_{n \geq 1} \) and \( t'' = \{ t''_n \}_{n \geq 1} \), all of non–negative integers with each \( r_n + s_n > 0 \) and \( d_{n+1} = d'_n (r_{n+1} + s_{n+1}) + t'_{n+1} \) and denote \( d_{n+1} = d'_n + d''_{n+1} \). Let \( G_n \) be the real special unitary group \( SU(d'_n, d''_{n+1}; \mathbb{F}) \) over \( \mathbb{F} \). If \( \mathbb{F} = \mathbb{R} \), or if \( \mathbb{F} = \mathbb{C} \) and \( d_n \) is odd, then \( G_n, \mathbb{C} \) has no outer automorphism, and we denote \( \delta = 1 \in \text{Aut}(G_n) \). Otherwise (except when \( \mathbb{F} = \mathbb{R} \) and \( d_n = 8 \)) \( G_n, \mathbb{C} \) has outer automorphism
group generated modulo inner automorphisms by \( \delta_0 := Ad\left( \frac{1}{2} \right) \) if \( \mathbb{F} = \mathbb{R} \), by \( \delta_0 : g \mapsto g^{-1} \)
if \( \mathbb{F} = \mathbb{C} \), and we choose \( \delta \in \delta_0 \text{Int}(G_n) \) that preserves the standard positive root system. Then we have \( \phi_{n+1,n} : SU(d'_n, d''_{n+1}; \mathbb{F}) \to SU(d'_n, d''_{n+1}; \mathbb{F}) \) given by

\[
\phi_{n+1,n}(g) = \text{diag}\{ 1, \ldots, 1; g, \ldots, g; \delta(g), \ldots, \delta(g); 1, \ldots, 1 \}
\]

with \( t'_{n+1} \) entries 1, \( r_{n+1} \) blocks \( g, s_{n+1} \) blocks \( \delta(g) \), and finally \( t''_{n+1} \) entries 1, where all \( s_n = 0 \) in the case \( \mathbb{F} = \mathbb{H} \). Now \( \lim_{\longrightarrow} \{ G_m, \phi_{n,m} \} \) defines a strict direct system \( \{ G_m, \phi_{n,m} \} \). Let \( d' = \lim d'_n \) and \( d' = \lim d''_{n+1} \); both usually are \( \infty \) but of course it can happen that one is finite, even zero. In any case we have

\[
G = SU_{t', r, s, t''}(d', d'', \mathbb{F}) := \lim_{\longrightarrow} \{ G_m, \phi_{n,m} \} \quad \text{for the given} \ t', r, s \ \text{and} \ t''.
\]
Thus we have the groups $SO_{\nu', r, s, t'}(d', d'')$, $SU_{\nu', r, s, t'}(d', d'')$ and $Sp_{\nu', r, s, t'}(d', d'')$.

The same process gives us real limit orthogonal groups $O_{\nu', r, s, t'}(d', d'')$ and the complex limit unitary groups $U_{\nu', r, s, t'}(d', d')$. In the $O_{\nu', r, s, t'}(d', d'')$ case, $\text{(2.1)}$ requires that each $d_n$ should be odd. $s_n = 0$.

**Symplectic Groups.** We consider limits of real and complex symplectic groups. Fix sequences $\mathbf{r} = \{r_n\}_{n \geq 1}$ and $\mathbf{t} = \{t_n\}_{n \geq 1}$ with all $r_n > 0$. Start with $d_0 > 0$ and recursively define $d_{n+1} = d_n r_{n+1} + t_{n+1}$. Our convention is that $Sp(n; \mathbb{F})$ is the automorphism group of $\mathbb{F}^n$ with a nondegenerate antisymmetric bilinear form; that forces $\mathbb{F}$ to be $\mathbb{R}$ or $\mathbb{C}$. Let $G_n = Sp(d_n; \mathbb{F})$, either the real symplectic group $Sp(d_n; \mathbb{R})$ or the complex symplectic group $Sp(n; \mathbb{C})$. Then we have strict direct systems $\{G_m, \phi_{m,n}\}_{m \geq n \geq 0}$ with

$$\phi_{n+1,n} : G_n \to G_{n+1} \text{ by } \phi_{n+1,n}(g) = \text{diag}(g, \ldots , g; 1, \ldots, 1)$$

with $r_{n+1}$ blocks $g$ and with $2t_{n+1}$ entries 1. That gives us

$$G = Sp_{\mathbf{r}, \mathbf{t}}(\infty; \mathbb{F}) := \lim_{n \to \infty} \{G_m, \phi_{m,n}\} \text{ for the given } \mathbf{r} \text{ and } \mathbf{t}.$$

**Complex Orthogonal Groups.** Now consider the complex special orthogonal groups $G_n = SO(d_n; \mathbb{C})$. The formula $\text{(13)}$ defines maps $\phi_{n+1,n} : SO(d_n; \mathbb{C}) \to SO(d_{n+1}; \mathbb{C})$ so it defines a strict direct system $\{G_m, \phi_{m,n}\}$. Then we have

$$G = SO_{\mathbf{r}, \mathbf{s}, \mathbf{t}}(\infty; \mathbb{C}) := \lim_{n \to \infty} \{G_m, \phi_{m,n}\} \text{ for the given } \mathbf{r}, \mathbf{s} \text{, and } \mathbf{t}.$$

The same process gives us complex limit orthogonal groups $O_{\mathbf{r}, \mathbf{s}, \mathbf{t}}(\infty; \mathbb{C})$; as before, here $\text{(2.1)}$ requires that each $d_n$ be odd.

**The Remaining Classical Series.** There is one other series of real classical groups, the groups $SO^*(2n)$, real form of $SO(2n; \mathbb{C})$ with maximal compact subgroup $U(n)$. The usual definition is

$$SO^*(2n) = \{ g \in U(n, n) \mid g \text{ preserves } (x,y) = \sum_{\alpha=1}^{n} (x_\alpha y_{n+\alpha} + x_{n+\alpha} y_\alpha) \text{ on } \mathbb{C}^{2n} \}.$$ 

It will be more convenient for us to use the alternate formulation of $\text{(30)}$ Section 8], which is

$$SO^*(2n) = \{ g \in SL(n; \mathbb{H}) \mid b(gx, gy) = b(x, y) \text{ for all } x, y \in \mathbb{H}^n \}$$

where $b$ is the skew–hermitian form on $\mathbb{H}^n$ given by $b(x, y) = \sum_{\alpha=1}^{n} \bar{x}_\alpha y_\alpha$. For then $\text{(61)}$ defines an outer automorphism $\delta$ of each $SO^*(2n)$. Now fix sequences $\mathbf{r} = \{r_n\}_{n \geq 1}$, $\mathbf{s} = \{s_n\}_{n \geq 1}$ and $\mathbf{t} = \{t_n\}_{n \geq 1}$ of non–negative integers with $r_0 > 0$ and all $r_n + s_n > 0$. Start with $d_0 > 0$ and recursively define $d_{n+1} = d_n (r_{n+1} + s_{n+1}) + t_{n+1}$. Define $G_n = SO^*(2d_n)$. Then we have strict direct systems $\{G_m, \phi_{m,n}\}_{n \geq m \geq 0}$ given by

$$\phi_{n+1,n} : G_n \to G_{n+1} \text{ by } \phi_{n+1,n}(g) = \text{diag}(g, \ldots , g; \delta(g), \ldots, \delta(g); 1, \ldots, 1)$$

with $r_{n+1}$ blocks $g$, with $s_{n+1}$ blocks $\delta(g)$, and with $t_{n+1}$ entries 1. That gives us

$$G = SO^*_{\mathbf{r}, \mathbf{s}, \mathbf{t}}(\infty) := \lim_{n \to \infty} \{G_m, \phi_{m,n}\} \text{ for the given } \mathbf{r}, \mathbf{s} \text{ and } \mathbf{t}.$$

We refer to the direct limit groups $\text{(2.8), (2.4), (2.7), (2.9)}$, and $\text{(2.11)}$ as diagonal embedding direct limit groups and to the associated direct systems as diagonal embedding direct systems. In the unitary symplectic case of $\text{(15)}$ we made the convention that we have the sequence $\mathbf{s}$ but each $s_n = 0$. We say that a diagonal embedding direct limit group and the associated diagonal embedding direct system are of classical type if $r_n + s_n = 1$ for all $n$ sufficiently large.

Now we collect some basic properties of diagonal embedding direct limit groups.
Proposition 9.12 Let $G = \lim \{G_m, \phi_{n,m}\}_{n \geq m \geq 0}$ be a diagonal embedding direct limit group. Then conditions (2.1) and (2.2) always hold, and if the $G_n$ are not (special) unitary groups over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ then (1.7) holds. If the $G_n$ are (special) unitary groups over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, then (1.7) holds for a cofinal subsystem (which of course yields the same limit group $G$) of $\{G_m, \phi_{n,m}\}$. In any case, if $\{G_m, \phi_{n,m}\}$ is weakly parabolic then it is of classical type.

Proof. Conditions (2.1) and (2.2) are clear: the $G_n$ are semisimple Lie groups, connected except possible for the case of orthogonal groups where we have explicitly ensured (2.1). Now we look at (1.7).

We first consider the special linear groups $G_n = SL(d_n; \mathbb{F})$. Fix a basis $\mathcal{B}$ of $\mathbb{F}^{d_n}$. Relative to $\mathcal{B}$, $A_n$ will consist of the diagonal real matrices in $G_n$ that have all entries $> 0$, $M_n$ will consist of the diagonal matrices in $G_n$ that have all entries of absolute value $1$, and $N_n$ will consist of all lower triangular matrices in $G_n$ that have all diagonal entries $= 1$. It is immediate that $\phi_{n+1,n}$ maps $A_n$ into $A_{n+1}$, maps $M_n$ into $M_{n+1}$, and maps $N_n$ into $N_{n+1}$. That is (1.7).

Now consider the symplectic groups $G_n = Sp(d_n; \mathbb{F})$. The standard basis $\{e_i\}$ of $\mathbb{F}^{2d_n}$, in which the antisymmetric bilinear form $b_n$ that defines $G_n$ has matrix $(-I, 0)$, specifies a new basis $\mathcal{B} = \{v_1, \ldots, v_{d_n}; v_1', \ldots, v_{d_n}'\}$ by $v_i = \sqrt{2} (e_i + e_{d_n+i})$ and $v'_i = \sqrt{2} (e_i - e_{d_n+i})$. Relative to $\mathcal{B}$, the group $A_n$ will consist of the diagonal real matrices in $G_n$ with all entries $> 0$, in other words all diag$(a_1, \ldots, a_{d_n}, a_1^{-1}, \ldots, a_{d_n}^{-1})$ with each $a_i > 0$. Then, as above, $M_n$ will consist of the diagonal matrices in $G_n$ that have all entries of absolute value $1$, and $N_n$ will consist of all lower triangular matrices in $G_n$ that have all diagonal entries $= 1$, so $\phi_{n+1,n}$ maps $A_n$ into $A_{n+1}$, maps $M_n$ into $M_{n+1}$, and maps $N_n$ into $N_{n+1}$. That is (1.7).

Next consider the complex special orthogonal groups $G_n = SO(d_n; \mathbb{C})$. Let $m_n = [d_n/2]$, let $\{e_i\}$ be a basis of $\mathbb{C}^{d_n}$ in which the symmetric bilinear form $b_n$ that defines $G_n$ has matrix $I$. Define $v_i = \sqrt{2}(e_i + e_{m_n+i})$ and $v'_i = \sqrt{2}(e_i - e_{m_n+i})$ for $1 \leq i \leq m_n$. Consider the basis $\mathcal{B}$ of $\mathbb{C}^{d_n}$ given by $\{v_1, \ldots, v_{m_n}; v_1', \ldots, v_{m_n}'\}$ if $d_n$ is even (hence $= 2m_n$), by $\{v_1, \ldots, v_{m_n}; v_1', \ldots, v_{m_n}'; e_{d_n}\}$ if $d_n$ is odd (hence $= 2m_n + 1$). Relative to $\mathcal{B}$, if $d_n$ is even then $A_n$ will consist of all matrices diag$(a_1, \ldots, a_{m_n}, a_1^{-1}, \ldots, a_{m_n}^{-1})$ with each $a_i > 0$, and if $d_n$ is odd it will consist of all diag$(a_1, \ldots, a_{m_n}, a_1^{-1}, \ldots, a_{m_n}^{-1}, 1)$. Then $M_n$ will consist of the diagonal matrices in $G_n$ that have all entries of absolute value $1$, and $N_n$ will consist of all lower triangular matrices in $G_n$ that have all diagonal entries $= 1$, so $\phi_{n+1,n}$ maps $A_n$ into $A_{n+1}$, maps $M_n$ into $M_{n+1}$, and maps $N_n$ into $N_{n+1}$. That is (1.7).

We now consider the groups $G_n = SO^+(2d_n)$, essentially as above. Let $U_n$ be the underlying right vector space over $\mathbb{H}$ on which $G_n$ acts. Let $\{e_i\}$ be a basis of $U_n = \mathbb{H}^{d_n}$ in which the skew–hermitian form $b_n$ that defines $G_n$ is given by $b_n(z,w) = \sum_{1 \leq i \leq d_n} w\overline{z}_i$. Define $v_i = \frac{1}{\sqrt{2}}(e_{2i-1} + e_{2i})$ and $v'_i = \frac{1}{\sqrt{2}}(e_{2i-1} + e_{2i}k)$ for $1 \leq i \leq m_n$ where $m_n = [d_n/2]$. Let $\mathcal{B} = \{v_1, \ldots, v_{m_n}; v_1', \ldots, v_{m_n}'\}$ if $d_n$ is even, i.e. $d_n = 2m_n$, and let $\mathcal{B} = \{v_1, \ldots, v_{m_n}; v_1', \ldots, v_{m_n}'; e_{d_n}\}$ if $d_n$ is odd, i.e. $d_n = 2m_n + 1$. Then $V_n = \text{Span}\{v_i\}$ and $V'_n = \text{Span}\{v'_i\}$ are maximal totally $b_n$–isotropic subspaces of $U_n$, paired by $b_n(v_i, v'_i) = \delta_{ij}$. If $d_n$ is even then $U_n = V_n + V'_n$, and if $d_n$ is odd then $U_n = V_n + V'_n + W_n$ where $W_n = \text{Span}\{e_{d_n}\} = (V_n + V'_n)^\perp$ relative to $b_n$. In the basis $\mathcal{B}$ the groups $A_n$, $M_n$ and $N_n$ are given as in the case of the complex special orthogonal groups, so $\phi_{n+1,n}$ maps $A_n$ into $A_{n+1}$, maps $M_n$ into $M_{n+1}$, and maps $N_n$ into $N_{n+1}$. That gives us (1.7).

Finally we come to the case $G_n = SU(d_n; d'_n; \mathbb{F})$ of the real orthogonal, complex unitary and unitary symplectic (quaternion unitary) groups. Let $U_n$ be the underlying right vector space, over $\mathbb{F}$ on which $G_n$ acts. Then $G_n$ is essentially the group of automorphisms of $(U_n, b_n)$ where $b_n$ is the nondegenerate $\mathbb{F}$–hermitian form on $U_n$ that defines $G_n$. Let $\{e_i\}$ be a basis of $U_n$.
in which $b_n$ has matrix $\begin{pmatrix} d_n & 0 \\ 0 & 1 \end{pmatrix}$. Let $m_n = \min(d'_n, d''_n)$, the real rank $\dim \sigma_n$ of $G_n$. Define $v_i = \frac{1}{\sqrt{2}}(e_i + e_{m_n + i})$ and $v'_i = \frac{1}{\sqrt{2}}(e_i - e_{m_n + i})$ for $1 \leq i \leq m_n$. Let $r_n = d_n - 2m_n$, and let $\{w_1, \ldots, w_{r_n}\}$ denote the ordered set of those $e_i$ not involved in the $v_j$. Then we have the basis $B = \{v_1, \ldots, v_{m_n}; w_1, \ldots, w_{r_n}; v'_{m_n}; \ldots, v'_{1}\}$ of $U_n$. On the subspace level, $V_n = \text{Span}\{v_i\}$ and $V'_n = \text{Span}\{v'_i\}$ are maximal totally $b_n$-isotropic subspaces of $U_n$, paired by $b_n(v_i, v'_j) = \delta_{ij}$, and $U_n = V_n + V'_n + V_n$ where $W_n = \text{Span}\{w_i\} = (V_n + V'_n)^{\perp}$ relative to $b_n$. Note that $W_n$ is zero if $d'_n = d''_n$, positive definite if $d'_n > d''_n$, negative definite if $d'_n < d''_n$.

We choose $A_n$ to consist of all linear transformations of $U_n$ with matrix, relative to $B$, of the form $\text{diag}\{a_1, \ldots, a_{m_n}; 1, \ldots, 1; a_{m_n}^{-1}, \ldots, a_1^{-1}\}$ with $a_i$ all real and positive. Then $M_n$ consists of all linear transformations $m \in G_n$ such that

$$m(v'_i) = m_i v'_i, \quad m(W_n) = W_n \quad \text{and} \quad m(v_i) = m_i v_i \quad \text{where} \quad m_i \in \mathbb{F} \quad \text{with} \quad |m_i| = 1$$

Thus $\phi_{n+1,n}(A_n) \subset A_{n+1}$.

The description of $N_n$ is a little more complicated. Let $V_n = \{V_n,1, \ldots, V_{n,m_n}\}$ be the maximal isotropic flag in $V_n$ given by $V_{n,j} = \text{Span}\{v_1, \ldots, v_j\}$. In almost every case we may take the minimal parabolic subgroup $P_n$ of $G_n$ to be the $G_n$-stabilizer of $V_n$. That done, let $P_{n,j}$ denote the maximal real parabolic subgroup of $G_n$ that is the stabilizer of $V_{n,j}$. Then the nilradicals of these parabolics satisfy $n_n = \sum_{1 \leq j \leq m_n} n_{n,j}$. The point of this is that we know the $n_{n,j}$ in a convenient form. Let $X_{n,j} = \text{Span}\{v_1, \ldots, v_j\}$ so that $V_n = V_{n,j} \oplus X_{n,j}$. Define $V'_{n,j} = \text{Span}\{v'_1, \ldots, v'_j\}$ and $X'_{n,j} = \text{Span}\{v'_{j+1}, \ldots, v'_{m_n}\}$ so that $V'_n = V'_{n,j} \oplus X'_{n,j}$. Denote $W_{n,j} = X_{n,j} \oplus W_n \oplus X'_{n,j}$ so that $U_n = V_{n,j} + W_{n,j} + V'_{n,j}$. According to \cite{[34]} Lemma 3.4], the nilradical $n_{n,j}$ of $p_{n,j}$ is the sum of its two subspaces

$$p_{n,j}^2 = \{\xi \in g_n \mid \xi(V_{n,j}) \subseteq V_{n,j}, \xi(W_{n,j}) = 0, \xi(V'_{n,j}) = 0\}$$

and

$$p_{n,j}^1 = \{\xi \in g_n \mid \xi(V_{n,j}) \subseteq W_{n,j}, \xi(W_{n,j}) \subseteq V_{n,j}, \xi(V'_{n,j}) = 0\}$$

while the reductive component consists of those $\xi$ in $g_n$ that stabilize each of $V'_{n,j}, W_{n,j}$ and $V_{n,j}$. Thus, relative to the basis $B$, the elements of $n_{n,j}$ have block form $\begin{pmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix}$ along $U_n = V'_{n,j} + W_{n,j} + V_{n,j}$. Summing over $j$, the elements of $n_n$ are precisely those elements of $p_n$ whose matrix relative to $B$ has block form $\begin{pmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix}$ along $U_n = V'_{n,j} + W_n + V_n$ where $\ell$ and $\ell'$ are lower triangular with zeroes on their diagonals. In the case $G_n = SO(d'_n, d''_n)$ one must be a bit more careful and take some orientation into account, as in \cite{[34]}, but the result is the same. Thus $\phi_{n+1,n}(N_n) \subset N_{n+1}$.

It certainly cannot be automatic that $\phi_{n+1,n}(M_n) \subset M_{n+1}$. Define a difference $\mu_n = \max(d'_n, d''_n) - \min(d'_n, d''_n)$. Then $\mu_n = \dim W_n$, so $M_{n,0} \cong SU(\mu_n; \mathbb{F})$ and $M_{n+1,0} \cong SU(\mu_{n+1}; \mathbb{F})$, and thus $\phi_{n+1,n}(M_n) \not\subseteq M_{n+1}$ whenever $\mu_n > \mu_{n+1}$.

Now we pin this down. The map $\phi_{n+1,n} : G_n \rightarrow G_{n+1}$ is implemented by a unitary injection $k_n : (U_{n+1}, b_{n+1}) \rightarrow (U_{n+1}, \pm b_{n+1})$. We have set things up so that, possibly after interchanging the $v_i$ and the $v'_i$ in $U_{n+1}$, $k_n(V_{n+1}) \subseteq V_{n+1,j}$ and $k_n(V'_{n+1}) \subseteq V'_{n+1,j}$, for $1 \leq j \leq m_n$. We used that to prove that $\phi_{n+1,n}$ maps $A_n$ into $A_{n+1}$ and $N_n$ into $N_{n+1}$. But $\phi_{n+1,n}(M_n) \subset M_{n+1}$ if and only if we can make the choices of $V_{n+1}$ and $V'_{n+1}$ so that $k_n(W_n) \rightarrow W_{n+1}$. That is possible if and only if $\mu_n \leq \mu_{n+1}$.

If $\mu_n \leq \mu_{n+1}$ for infinitely many indices $n$, then we have a cofinal subsystem of $\{G_m, \phi_{n,m}\}$ in which $\mu_n \leq \mu_{n+1}$ for all $n$, and thus $\phi_{n+1,n}(M_n) \subset M_{n+1}$. If $\mu_n \leq \mu_{n+1}$ for only finitely many indices $n$, then we have an index $n_0$ such that $\mu_n > \mu_{n+1}$ for all $n \geq n_0$. That is impossible. Thus $\phi_{n,m}$ is always valid for a cofinal subsystem of $\{G_m, \phi_{n,m}\}$.
Suppose that \( \{G_m, \phi_n, m\} \) is weakly parabolic. View the \( \phi_{n+1, n} \) as inclusions \( G_n \hookrightarrow G_{n+1} \). Then \( a_{n+1} = a_n + a_{n+1, n} \) as in Proposition \( \ref{S.12} \) and \( (a_n + m_n) \oplus a_{n+1, n} \) is the centralizer of \( a_{n+1, n} \) in \( G_{n+1} \). In particular, \( \Sigma(g_n \oplus g_{n+1}, \phi_{n+1}) \subset \Sigma(g_{n+1}, a_{n+1, n}) \). Thus, if \( \gamma \in \Sigma (g_n, a_n) \) there is a unique \( \gamma_{n+1} \in \Sigma(g_{n+1}, a_{n+1}) \) such that \( \gamma_{n+1} | a_n = \gamma_n \). But if \( r_{n+1} + s_{n+1} \geq 2 \) then at least two distinct elements of \( \Sigma (g_{n+1}, a_{n+1}) \) restrict to \( \gamma_n \). Thus \( r_{n+1} + s_{n+1} = 1 \).

**Proposition 9.13** Let \( G = \lim\{G_m, \phi_n, m\}_{n \geq 0} \) be a diagonal embedding direct limit group. Then the following conditions are equivalent.

1. \( G = \lim\{G_m, \phi_n, m\}_{n \geq 0} \) is of classical type, in other words \( r_n + s_n = 1 \) for \( n \geq n_0 \).
2. The root system \( \Sigma (g, a) = \lim \Sigma (g_n, a_n) \) is countable.
3. \( \Sigma (g, a) = \bigcup_{n \geq 0} \Sigma (g_n, a_n) \).
4. \( g \) is restricted–root–reductive in the sense that \( g = (m + a) + \sum_{\gamma \in \Sigma (g, a)} g^\gamma \).

**Proof.** Let \( G \) be of classical type. View the \( \phi_n \) as inclusions \( G_n \hookrightarrow G \) as inclusions. Let \( \Psi_n \) denote the simple system of \( \Sigma (g_n, a_n) \). If \( n \geq m \geq n_0 \) then Proposition \( \ref{S.12} \) shows how \( \Psi_m \subset \Psi_n \) when we extend the elements of \( \Psi_m \) by zero on \( a_{n,m} \). Thus \( \Sigma (g, a) = \bigcup_{n \geq 0} \Sigma (g_n, a_n) \), which is countable, and if \( \gamma \in \Sigma (g, a) \) then \( \phi_n (\gamma) \) is a well defined subspace of the root space for \( \gamma \). We have just seen that (1) implies (2), (3), and (4). On the other hand, (4) implies (3) and (3) implies (2), at a glance. Thus we need only prove that (2) implies (1).

Suppose that \( G = \lim\{G_m, \phi_n, m\}_{n \geq 0} \) is not of classical type. Then we can pass to a cofinal subsystem in which every \( r_n + s_n \geq 2 \). That done, every root \( \gamma \in \Sigma (g_n, a_{n}^+) \) is the restriction of at least 2 roots in \( \Sigma (g_{n+1}, a_{n+1}) \), thus this is the restriction of at least 2\( \rho_0 \) roots in \( \Sigma (g, a) \). In particular \( \Sigma (g, a) \) is not countable. Thus (2) implies (1). \( \square \)

Recall the notion of Satake diagram. We use the Cartan subalgebra \( h_n = t_n + a_n \) of \( g_n \) and the positive root system \( \Sigma (g_n, C, h_n, C) \) as in \( \ref{S.12} \). Write \( \Psi (g_n, C, h_n, C) \) for the corresponding simple \( h_n, C \)-root system, and write \( \Psi (g_n, a_n) \) for the simple \( a_n \)-root system corresponding to \( \Sigma (g_n, a_n) \). Every \( \psi \in \Psi (g_n, a_n) \) is of the form \( \bar{\psi} | a_n \) for some \( \bar{\psi} \in \Psi (g_n, C, h_n, C) \). More or less conversely, if \( \bar{\psi} \in \Psi (g_n, C, h_n, C) \) then either \( \bar{\psi} | a_n = 0 \) or \( \bar{\psi} | a_n \in \Psi (g_n, a_n) \). The Satake diagram describes the restriction process. Start with the Dynkin diagram \( D_n \) of \( g_n \) whose vertices are the elements of \( \Psi (g_n, C, h_n, C) \). If there are two root lengths this is indicated by arrows rather than darkening the vertices for short roots. Now darken those \( \bar{\psi} \in \Psi (g_n, C, h_n, C) \) such that \( \bar{\psi} | a_n = 0 \). It can happen that two (but never more than two) distinct elements \( \bar{\psi}, \bar{\psi}' \in \Psi (g_n, C, h_n, C) \) have the same \( a_n \)-restriction. In that case, join them by an arrow. The result is the Satake diagram of \( g_n \). The white vertices and vertex pairs corresponding to simple \( a_n \)-roots of \( g_n \). The black vertices correspond to simple \( t_n, C \)-roots of \( m_n, C \). See \( \ref{S.2} \), pp. 90–93 or \( \ref{S.13} \), pp. 32–33, for Araki’s list of Satake diagrams.

We use the Satake diagrams to see just which \( G = \lim\{G_m, \phi_n, m\} \) of classical type are weakly parabolic. The description \( \ref{S.2} \) of real parabolic subalgebras gives us

**Lemma 9.14** The semisimple components of real parabolic subalgebras of \( g_{n+1} \) are characterized up to \( \text{Int}(G_{n+1}) \)-conjugacy by their Satake diagrams, and those Satake diagrams are obtained from the Satake diagram of \( g_{n+1} \) by deleting (i) an arbitrary set of white vertices, and then (ii) all white vertices joined by arrows (meaning the same restriction to \( a_{n+1} \)) to vertices deleted in (i).

The black vertices (restriction 0 to \( a_{n+1} \)) remain because they represent the simple roots of \( m_{n+1} \), which is contained in every real parabolic subalgebra that contains \( a_{n+1} \).

Let \( G = \lim\{G_m, \phi_n, m\} \) be a diagonal embedding direct limit group of classical type. From Araki’s list of Satake diagrams one sees that the possible inclusions \( \phi_{n+1, n} : g_n' \hookrightarrow g_{n+1} \) of weakly parabolic type are given, modulo \( m_{n+1} \), by
(9.15a) $SL(d_n; \mathbb{F}) \leftrightarrow SL(d_{n+1}; \mathbb{F})$ by $\phi_{n+1,n}(g) = \text{diag}\{g \text{ or } \delta(g), 1, \ldots, 1\}, d_{n+1} > d_n$.

(9.15b) $SO(d_n', d_n'') \leftrightarrow SO(d_n' + u_n, d_n'' + u_n)$ by $\phi_{n+1,n}(g) = \text{diag}\{g \text{ or } \delta(g), 1, \ldots, 1\}, u_n > 0$.

(9.15c) $SO(d_n; \mathbb{C}) \leftrightarrow SO(d_n + 2u_n; \mathbb{C})$ by $\phi_{n+1,n}(g) = \text{diag}\{g \text{ or } \delta(g), 1, \ldots, 1\}, u_n > 0$.

(9.15d) $SU(d_n', d_n'') \leftrightarrow SU(d_n' + u_n, d_n'' + u_n)$ by $\phi_{n+1,n}(g) = \text{diag}\{g \text{ or } \delta(g), 1, \ldots, 1\}, u_n > 0$.

(9.15e) $Sp(d_n', d_n'') \leftrightarrow Sp(d_n' + u_n, d_n'' + u_n)$ by $\phi_{n+1,n}(g) = \text{diag}\{g, 1, \ldots, 1\}, u_n > 0$.

(9.15f) $Sp(d_n; \mathbb{F}) \leftrightarrow Sp(d_{n+1}; \mathbb{F})$ by $\phi_{n+1,n}(g) = \text{diag}\{g, 1, \ldots, 1\}, d_{n+1} > d_n$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

(9.15g) $SO^*(2d_n) \leftrightarrow SU^*(2d_n + 4u_n)$ by $\phi_{n+1,n}(g) = \text{diag}\{g \text{ or } \delta(g), 1, \ldots, 1\}, u_n > 0$.

In order to pin things down we make use of the fact that $G = \varprojlim G_m, \phi_{m,n}$ is determined by any cofinal subsequence of indices. Denote

$0 = \{0, 0, 0, \ldots\}$, $1 = \{1, 1, 1, \ldots\}$ and $2 = \{2, 2, 2, \ldots\}$.

Consider, for example, the case of (9.15a). Suppose first that there are only finitely many indices $n$ for which $\phi_{n+1,n}(g) = \text{diag}\{g, 1, \ldots, 1\}$. Pass to the subsequence starting just after the last $\phi_{n+1,n}$ that involves $\delta$.

That done, we interpolate and arrive at the same limit with each $\phi_{n+1,n}(g) = \left( \begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix} \right)$. Now suppose that there are infinitely many indices $n$ for which $\phi_{n+1,n}(g) = \text{diag}\{g, 1, \ldots, 1\}$. Pass to the cofinal subsequence obtained by deleting the $G_n$ for which $\phi_{n+1,n}(g) = \text{diag}\{g, 1, \ldots, 1\}$, so now every $\phi_{n+1,n}(g)$ is of the form $g \mapsto \text{diag}\{\delta(g), 1, \ldots, 1\}$.

If $t_{n+1} > 1$ for an infinite number of $t_{n+1}$ then, recursively, we take the smallest index $n$ for which $t_{n+1} > 1$, insert $t_{n+1} - 1$ steps $g \mapsto \left( \begin{smallmatrix} \delta(g) & 0 \\ 0 & 1 \end{smallmatrix} \right)$ between $G_n$ and $G_{n+1}$, and proceed to insert steps $g \mapsto (\delta(g) 0 \ 0)$ at the next $t_{n+1} - 1$ possible places. Then we arrive at the same limit with each $\phi_{n+1,n}(g) = \left( \begin{smallmatrix} \delta(g) & 0 \\ 0 & 1 \end{smallmatrix} \right)$. If $t_{n+1} > 1$ for only finitely many $n$ we just pass to the subsequence starting just after the last $\phi_{n+1,n}$ involving a $t_{n+1}$ that is $> 1$. Thus $G = SL_{1,0,1}(\infty, \mathbb{F}) = \varprojlim SL(n + 1; \mathbb{F})$ in the first case, with $\phi_{n+1,n}(g) = \left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right)$, and $G = SL_{0,1,1}(\infty, \mathbb{F}) = \varprojlim SL(n + 1; \mathbb{F})$ in the second case, with $\phi_{n+1,n}(g) = \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)$.

Similar considerations hold in the other six cases of (9.15). The final result is

**Proposition 9.16** The weakly parabolic diagonal embedding direct limit groups $G = \{G_m, \phi_{m,n}\}$ of classical type, with $G_m$ noncompact and simple for $m$ large, are given, up to isomorphism, by one of the following.

(9.17a) $SL_{1,0,1}(\infty, \mathbb{F})$ with $g \mapsto \left( \begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix} \right)$ or $SL_{0,1,1}(\infty, \mathbb{F})$ with $g \mapsto \left( \begin{smallmatrix} \delta(g) & 0 \\ 0 & 1 \end{smallmatrix} \right)$; here we may take $G_m$ to be $SL(m + 1; \mathbb{F})$.

(9.17b) $SO_{1,0,1}(\infty, \mathbb{C})$ with $g \mapsto \left( \begin{smallmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)$ or $SO_{0,1,1}(\infty, \mathbb{C})$ with $g \mapsto \left( \begin{smallmatrix} 1 & 0 & 0 \\ 0 & \delta(g) & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)$; here we may take $G_m$ to be an $SO(d_1' + m, d_2'' + m)$ where $d_1', d_2'' \geq 1$.

(9.17c) $SO_{1,2}(\infty, \mathbb{C})$ with $g \mapsto \left( \begin{smallmatrix} g & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix} \right)$ or $SO_{0,1,2}(\infty, \mathbb{C})$ with $g \mapsto \left( \begin{smallmatrix} \delta(g) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix} \right)$; here we may take $G_m$ to be $SO(2m + 1; \mathbb{C})$ (type $B$) or $SO(2m; \mathbb{C})$ (type $D$).

(9.17d) $SU_{1,0,1}(\infty, \mathbb{C})$ with $g \mapsto \left( \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix} \right)$ or $SU_{0,1,1}(\infty, \mathbb{C})$ with $g \mapsto \left( \begin{smallmatrix} 1 & 0 & 0 \\ 0 & \delta(g) & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)$; here we may take $G_m$ to be an $SU(d_1' + m, d_2'' + m)$ where $d_1', d_2'' \geq 1$.

(9.17e) $Sp_{1,0,1}(\infty, \mathbb{C})$ with $g \mapsto \left( \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix} \right)$; here we may take $G_m$ to be an $Sp(d_1' + m, d_2'' + m)$ where $d_1', d_2'' \geq 1$.

(9.17f) $Sp_{1,2}(\infty, \mathbb{C})$ with $g \mapsto \left( \begin{smallmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$; here we may take $G_m$ to be $Sp(m; \mathbb{F})$.

(9.17g) $SO_{1,0,1}(\infty, \mathbb{C})$ with $g \mapsto \left( \begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)$ or $SO_{0,1,1}(\infty, \mathbb{C})$ with $g \mapsto \left( \begin{smallmatrix} \delta(g) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix} \right)$, (quaternionic matrices); here we may take $G_m$ to be $SO^*(2m)$. 

25
10 The Other Tempered Series

The finite dimensional real reductive Lie groups $G$ that satisfy \( \text{Proposition 2.11} \) and \( \text{Proposition 2.22} \) have a series of unitary representations for each conjugacy class of Cartan subgroups. These are the “tempered” representations, the ones that occur in the decomposition of $L_2(G)$ under the left translation action of $G$. The principal series is the tempered series corresponding to the conjugacy class of a maximally noncompact Cartan subgroup, but of course there are others. If $G$ has a Cartan subgroup with compact image under the adjoint representation, then the corresponding series is the discrete series. In general these series are constructed by combining the ideas of the discrete series and the principal series. See \( \text{[11], [12], [13] and [14]} \) for the case where $G$ is Harish–Chandra class, \( \text{[28], [16] and [17]} \) for the general case. We now recall a few relevant facts from these papers in order to indicate the corresponding extension of our principal series results.

Fix a Cartan involution $\theta$ of $G$ and let $K$ denote its fixed point set. As usual, $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$, decomposition into $(\pm 1)$–eigenspaces of $\theta$. Every $G^0$–conjugacy class of Cartan subgroup contains a $\theta$–stable Cartan. Fix a $\theta$–stable Cartan subgroup $H$ of $G$. Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ where $\mathfrak{t} = \mathfrak{k} \cap \mathfrak{t}$ and $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{s}$. Here $H = T \times A$ where $T = H \cap K$ and $A = \exp(\mathfrak{a})$. Earlier we had only considered the case where $\mathfrak{a}$ is maximal abelian in $\mathfrak{s}$; here the situation is more general. The centralizer of $A$ in $G$ has form $M \times A$ where $\theta(M) = M$. In our earlier discussions $M$ was compact modulo $Z_G(G^0)$ (relatively compact), but here the situation is more general. In any case, $M$ satisfies \( \text{Proposition 2.21} \) and \( \text{Proposition 2.22} \), and $T$ is relatively compact, so $M$ has relative discrete series representations. In the principal series setting these will be all the irreducible representations of $M$ and will necessarily be finite dimensional, but here the situation is more general.

We have the $\mathfrak{a}$–root system $\Sigma(\mathfrak{g}, \mathfrak{a}) := \{\alpha | \alpha \in \Sigma(\mathfrak{g}, \mathfrak{h})$ and $\alpha|_{\mathfrak{a}} \neq 0\}$. Fix a positive subsystem $\Sigma(\mathfrak{g}, \mathfrak{a})^+$ and define $n = \sum_{\beta \in \Sigma(\mathfrak{g}, \mathfrak{a})^+} \mathfrak{g}^{-\beta}$. Then $\mathfrak{p} := \mathfrak{m} + \mathfrak{a} + n$ is a particular kind of (real) parabolic subalgebra of $\mathfrak{g}$, distinguished by the fact that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{m}$. Those are the cuspidal parabolic subalgebras of $\mathfrak{g}$. Let $N = \exp(n)$. It is the analytic subgroup of $G$ with Lie algebra $\mathfrak{n}$, and $P = MAN$ is the parabolic subgroup of $G$ with Lie algebra $\mathfrak{p}$. Those are the cuspidal parabolic subgroups of $G$.

We use the cuspidal parabolic subgroup $P = MAN$ to describe the $H$–series representations of $G$. The analog of Proposition \( \text{Proposition 2.11} \) is the parameterization of the relative discrete series of $M$. Fix a positive $\mathfrak{t}$–root system $\Sigma(\mathfrak{m}, \mathfrak{t})^+$ on $\mathfrak{m}_+$. Let $\nu \in i^+$ such that $e^{\nu - \rho_{\mathfrak{m}, \mathfrak{t}}}$ is well defined on $T^0$ and $(\nu, \alpha) \neq 0$ for all $\alpha \in \Sigma(\mathfrak{m}, \mathfrak{t})$. Then $M^0$ has a unique unitary equivalence class of relative discrete series representations, \( [\eta^0_\nu] \), with Harish–Chandra parameter $\nu$. Here $\nu$ is the infinitesimal character of $[\eta^0_\nu]$; if $\eta^0_\nu$ has a highest weight, that weight is $\nu - \rho_{\mathfrak{m}, \mathfrak{t}}$. Set $M^1 = Z_M(M^0)M^0$. Then the relative discrete series classes of $M^1$ are the $[\eta^1_{\chi, \nu}] = [\chi \otimes \eta^0_\nu]$ where $[\chi] \subset Z_M(M^0)^*_\xi$ with $\xi = e^{\nu - \rho_{\mathfrak{m}, \mathfrak{t}}} \cdot Z_{\mathfrak{m}_0}$. The relative discrete series classes of $M$ are the $[\eta_{\chi, \nu}]$ where $\eta_{\chi, \nu} = \text{Ind}_{M^1}^M(\eta^1_{\chi, \nu})$. Let $\sigma \in \mathfrak{a}^*_+$.

Now we return to our strict direct system \{ $G_i, \phi_{k,i}$ \} of reductive Lie groups. Fix a Cartan subgroup $H = \lim H_i$ of $G = \lim G_i$. We consider limit representations $\pi = \lim \pi_i$ of $G$, where, for each $i$, $\pi_i$ is an $H_i$–series representation of $G_i$.

Here there are several problems. First, we need the discrete series analog of \( \text{Proposition 2.22} \) in order to construct the $M$–component of any $\lim \pi_i$. That falls into two parts. The first is to realize
the discrete series representations of the $M_i$ on some appropriate cohomology spaces, such as spaces of $L^2$ harmonic forms. This is done, for example, in [23], [24] and [28]. The second is to make sure that these representations all appear on cohomologies of the same degree, and to line them up properly so that one can take limits. This was done in [19] for holomorphic discrete series; there the cohomology degree is 0, the alignment is done using the universal enveloping algebra description of highest weight representations, and the result is analysed by use of [8]. It was done in [10] for other discrete series of certain diagonal embedding direct limit groups $Sp(p, \infty)$ and $SO(2p, \infty)$ of classical type using Zuckerman derived functor modules $A_q(\lambda)$ for the cohomologies. We address these matters in some generality in [32].

Second, we need an analog of the considerations of Section 8. This is not so difficult, but one has to be careful. We address this matter in [33].

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