Slow Diffeomorphisms of a Manifold with $\mathbb{T}^2$ Action

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Abstract
The uniform norm of the differential of the $n$-th iteration of a diffeomorphism is called the growth sequence of the diffeomorphism. In this paper we show that there is no lower universal growth bound for volume preserving diffeomorphisms on manifolds with an effective $\mathbb{T}^2$ action by constructing a set of volume-preserving diffeomorphisms with arbitrarily slow growth.

1 Introduction

Let $M$ be a smooth compact connected manifold. Let $f$ be a diffeomorphism of the manifold $M$. Define $\Gamma_n : \text{Diff}(M) \to \mathbb{R}$, the growth sequence, as

$$\Gamma_n(f) = \max \left\{ \max_{x \in M} \|df^n_x\|, \max_{x \in M} \|df^{-n}_x\| \right\}, \quad n \in \mathbb{N}.$$  

Here $f^n$ is the $n$-th iteration of $f$ and $\|df_x\|$ is the operator norm of the differential, calculated with respect to a Riemannian metric on $M$. We write $a_n \geq b_n$, when $a_n$ and $b_n$ are two positive sequences, and there exists $c > 0$ such that $a_n \geq cb_n$ for all $n \in \mathbb{N}$. Two sequences $a_n$ and $b_n$ are called equivalent if $a_n \geq b_n$ and $b_n \geq a_n$. Under this definition, the equivalence class of the growth sequence is an invariant of $f$ under conjugations in $\text{Diff}(M)$. It is called the growth type of $f$.

The growth type of a diffeomorphism is a basic dynamical invariant (see [KH]). The behavior of the growth sequence of different categories of diffeomorphisms is an interesting

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*This paper is the author’s M.Sc. thesis, being carried out under the supervision of Prof. Leonid Polterovich, at Tel-Aviv university.
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This topic was first brought up by D'Ambra and Gromov [DAG]. In this paper we show that there is no lower universal growth bound for volume preserving diffeomorphisms of manifolds with an effective $\mathbb{T}^2$ action. An action is called effective if the only element of the group that defines the identity diffeomorphism is the identity element.

**Theorem 1.0.1 (Main Theorem).** Let $M$ be a smooth compact connected oriented manifold with an effective $\mathbb{T}^2$ action, $\phi: \mathbb{T}^2 \times M \to M$. Let $\Psi$ be a positive, unbounded increasing, function on $\mathbb{R}_+$ such that $\Psi(x) = o(x)$ for $x \to \infty$. Then there exists a volume preserving diffeomorphism, $f$, of $M$, such that

$$0 < \limsup_{n \to \infty} \frac{\Gamma_n(f)}{\Psi(n)} < \infty. \tag{1.1}$$

We will refer to such an $f$ as to a slow diffeomorphism.

In works of Polterovich and Sikorav (see [P1, PSi]) it has been found that there are lower growth bounds for Hamiltonian diffeomorphisms of symplectic manifolds, $M$, with $\pi_2(M) = 0$. A Hamiltonian diffeomorphism always has fixed points and vanishing flux (The definition of the flux is given in Section 3).

In the case of symplectic, but non-Hamiltonian, diffeomorphisms (i.e. with non-vanishing flux) Polterovich prove the existence of lower growth bounds if the diffeomorphism has a fixed point with some special property.

In recent works of Polterovich (see [P2]) and Borichev (see [B]) it has been found that there are no lower growth bounds ("continuous spectrum") in the case of symplectic diffeomorphisms without fixed points. They gave examples of sequences of diffeomorphisms on the two dimensional torus with arbitrarily slow growth.

In the case of smooth category Polterovich and Sodin [PSO] show that there are no growth bounds:

**Theorem 1.0.2.** Let $\{u(n)\}$ be a sequence of positive real numbers which goes to infinity as $n \to +\infty$. Then there exists a diffeomorphism $f \in Diff_0(M) \setminus \{1\}$ with a fixed point so that

$$\liminf_{n \to \infty} \frac{\Gamma_n(f)}{u(n)} < \infty.$$
Let us emphasize that the diffeomorphisms constructed in this theorem are dissipative, that is to say, they do not preserve any smooth volume form.

In this work we tackle the open problem: what happens in volume-preserving category?

Main theorem 1.0.1 shows that in this case, on manifolds with an effective $\mathbb{T}^2$ action, there is no universal lower growth bound for volume preserving diffeomorphisms.

Moreover, in certain situations, the slow diffeomorphisms appearing in the main theorem have features similar to those of Hamiltonian diffeomorphisms: they have fixed points and their flux vanishes.

**Theorem 1.0.3.** There exist volume-preserving slow diffeomorphisms with vanishing flux and fixed points on the manifold $S^1 \times S^2$.

### 1.1 Organization of the Work

In Section 2 we construct slow diffeomorphisms of a manifold with an effective $\mathbb{T}^2$ action and thus prove Theorem 1.0.1. Farther, we give examples of slow diffeomorphisms of manifold $S^1 \times S^2$.

In Section 3 we define the flux homomorphism and prove Theorem 1.0.3.
2 Proof of Main Theorem

2.1 Topological preliminaries

Let $M$ be a connected manifold of dimension $n$ with an effective $\mathbb{T}^2$ action, $\phi: \mathbb{T}^2 \times M \to M$. Denote $\phi(g, x) = \phi_g(x) = gx$. Consider the torus as the group $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

We say that the action is free at $x \in M$, if the map $\mathbb{T}^2 \to M, g \mapsto \phi(g, x)$ is an embedding. We write $M_e$ as the set of elements of $M$ where the action is free. Then, by [CGK, Corollary B.48], when the action is effective, $M_e$ is open and dense.

Let $x_0 \in M_e$, then the orbit $V = \mathbb{T}^2 x_0$ is diffeomorphic to $\mathbb{T}^2$. Then there exists a neighborhood, $W$, of $x_0$ such that $\phi(\cdot, x): \mathbb{T}^2 \to M$ is an embedding for all $x$ in the neighborhood. Let $D$ be an open disc in $W$ of dimension $n - 2$ such that $D$ is transversal to $V$. Finally, denoted by $U$ the orbit of the disc under the torus action, then $U = \mathbb{T}^2 D \cong \mathbb{T}^2 \times D$.

Let $u = (u_1, \ldots, u_{n-2})$ where $\sum_{i=1}^{n-2} u_i^2 < 1$, be the coordinates of the disc $D$. Let $(\varphi_1, \varphi_2)$ where $\varphi_1, \varphi_2 \in S^1 = \mathbb{R}/\mathbb{Z}$ be coordinates of the torus. Accordingly, the triple $(\varphi_1, \varphi_2, u)$ represents coordinates in $U$.

Let $\tilde{B}$ and $B$ be two sets satisfying the following:

- $\tilde{B} \subset B \subset D$
- $B$ is a compact set
- $\tilde{B}$ is an open set.

We define a smooth function $A: D \to \mathbb{R}$ such that supp$(A) \subseteq B$ and $A|_{\tilde{B}} = 1$.

2.2 Constructing the Diffeomorphism

We define the diffeomorphism on $M \setminus U$ and on $U$ separately. On the set $U$ we construct a slow diffeomorphism using the function from Borichev’s theorem [B] and the function $A: D \to \mathbb{R}$.

For a function $F: S^1 \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, we consider the Weyl sum

$$W(N, x, \alpha) = \sum_{k=0}^{N-1} F(x + k\alpha).$$
Theorem 2.2.1 (Borichev). Let \( \Psi \) be a positive, unbounded increasing, function on \( \mathbb{R}_+ \) such that \( \Psi(x) = o(x) \) for \( x \to \infty \). Then, there exists a real-analytic and 1-periodic function, \( F: \mathbb{R} \to \mathbb{R} \), and \( \alpha \in \mathbb{R} \) such that \( \int_0^1 F(x)dx = 0 \) and

\[
0 < \limsup_{N \to \infty} \max_{0 \leq x < 1} \frac{W'(N, x, \alpha)}{\Psi(N)} < \infty. \tag{2.1}
\]

Let \( F \) be the function and \( \alpha \) be the constant from Borichev’s theorem. Then, we define \( f_1: U \to U \) as the following diffeomorphism:

\[
f_1(x) = (\varphi_1 + \alpha, \varphi_2 + A(u) \cdot F(\varphi_1), u),
\]

where we use the coordinates \((\varphi_1, \varphi_2, u) = x \in U \) previously mentioned.

On the set \( M \setminus U \) we define \( f_2: M \setminus U \to M \setminus U \) as the action of the element \((\alpha, 0) \in \mathbb{T}^2\) on \( M \setminus U \): \( f_2(x) = \phi(\alpha, 0)x \). Indeed, \( U \) is the orbit of a set, hence, \( f_2 \) is onto \( M \setminus U \).

Denote \( f: M \to M \) as

\[
f(x) = \begin{cases} 
  f_1(x) & x \in U \\
  f_2(x) & x \in M \setminus U.
\end{cases}
\]

It is clear that \( f \) is a diffeomorphism. We show that \( f \) satisfies the conditions in the main theorem \[1.0.1\].

Lemma 2.2.2. The diffeomorphism \( f_1 \) satisfies the inequality

\[
0 < \limsup_{n \to \infty} \frac{\Gamma_n(f)}{\Psi(n)} < \infty.
\]

on the submanifold \( U \).

Proof. Take \((\varphi_1, \varphi_2, u)\) as the coordinates on \( U \). The \( m \)-th iteration of \( f_1 \) is equal to

\[
f_1^m = (\varphi_1 + m\alpha, \varphi_2 + A(u) \cdot W(m, \varphi_1, \alpha), u).
\]
Hence,

\[
    d_x f_1^m = \begin{pmatrix}
        1 & 0 & 0 & \cdots & 0 \\
        A(u) \cdot W'(m, \varphi_1, \alpha) & 1 & \frac{\partial A}{\partial u_1} \cdot W(m, \varphi_1, \alpha) & \cdots & \frac{\partial A}{\partial u_{n-2}} \cdot W(m, \varphi_1, \alpha) \\
        0 & 0 & 1 & \cdots & \cdots \\
        \vdots & \vdots & \vdots & \ddots & \vdots \\
        0 & 0 & 0 & \cdots & 1
    \end{pmatrix}
\]

where \( u = (u_1, \ldots, u_{n-2}) \in D \).

Define a norm of an \( n \times n \) matrix \( Q = (q_{ij}) \) as

\[
    \|Q\| = \max_i \sum_j |q_{ij}|. \tag{2.2}
\]

Every other norm is equivalent to \( \|\cdot\| \), hence, by using the fact that \( A(u) \) and its derivatives are bounded functions of \( u \), it is sufficient to prove that \( F \) and \( \alpha \) satisfy the following condition:

\[
    0 < \limsup_{N \to \infty} \max_{0 \leq \varphi < 1} \frac{|W(N, \varphi, \alpha)| + |W'(N, \varphi, \alpha)|}{\Psi(N)} < \infty.
\]

The lower bound directly follows inequality (2.1). For the upper bound we use the fact that there exists \( \tilde{x} \in [0, 1] \) such that \( W(N, \tilde{x}, \alpha) = 0 \). Indeed, \( \int_0^1 W(N, x, \alpha)dx = 0 \) and \( W(N, x, \alpha) \) is continuous as a function of \( x \). Hence, using \( W(N, \varphi, \alpha) = \int_{	ilde{x}}^{\varphi} W'(N, x, \alpha)dx + W(N, \tilde{x}, \alpha) \), we get

\[
    \max_{0 \leq \varphi < 1} |W(N, \varphi, \alpha)| = \max_{0 \leq \varphi < 1} \left| \int_{	ilde{x}}^{\varphi} W'(N, x, \alpha)dx \right| \leq \max_{0 \leq \varphi < 1} |W'(N, \varphi, \alpha)|.
\]

Hence, inequality (2.1) yields

\[
    \limsup_{N \to \infty} \max_{0 \leq \varphi < 1} \frac{|W(N, \varphi, \alpha)|}{\Psi(N)} < \limsup_{N \to \infty} \max_{0 \leq \varphi < 1} \frac{|W'(N, \varphi, \alpha)|}{\Psi(N)} < \infty
\]

and we prove the upper bound in \( U \).

\[ \square \]

**Lemma 2.2.3.** Look at \( \Gamma_m(f_2) \) as a function of \( m \). Then, \( \Gamma_m(f_2) \) is bounded.
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Proof. The diffeomorphism $f_2: M \setminus U \to M \setminus U$ is the action of $(\alpha, 0) \in \mathbb{T}^2$ on $M \setminus U$ and the two dimensional torus is compact, therefore, $\|d_x f_2^n\|$ is bounded. 

Using Lemmas 2.2.2 and 2.2.3 we get that the diffeomorphism $f$ satisfies the inequality

$$0 < \limsup_{n \to \infty} \frac{\Gamma_n(f)}{\Psi(n)} < \infty.$$ 

The manifold $M$ is oriented, hence, there exists a volume form on $M$. Define the volume form $\omega$ on $M$ as follows. Let $\Omega_0 = d\varphi_1 \wedge d\varphi_2 \wedge du_1 \wedge \ldots \wedge du_{n-2}$ on $U$. Let $\Omega$ be any extension of $\Omega_0$ to the entire manifold. Let $\mu$ be a Haar measure on the torus. We average the pullbacks of $\Omega$ by the diffeomorphism $f$, where $g \in \mathbb{T}^2$ and set

$$\omega = \int_{\mathbb{T}^2} g^* \Omega d\mu(g).$$

Lemma 2.2.4. The diffeomorphism $f$ preserves the volume form $\omega$.

Proof. The action preserves $\omega$, for every $g \in \mathbb{T}^2$ $g^* \omega = \omega$. The diffeomorphism $f_2$ is the action of the element $(\alpha, 0) \in \mathbb{T}^2$ on $M$, hence, $f$ preserves $\omega$ on $M \setminus U$.

On $U$ $\det(d_x f_1) = 1$. Hence $f$ preserves the volume form $\omega|_U = \Omega_0 = d\varphi_1 \wedge d\varphi_2 \wedge du_1 \wedge \ldots \wedge du_{n-2}$ on $U$. Therefore, $f$ preserves the volume form $\omega$ on $M$ as required. 

2.3 Examples of Manifolds with Slow Diffeomorphisms

An example of manifolds satisfying the conditions of the main theorem are the spheres, $S^n$, where $n \geq 3$.

Another example is $S^1 \times S^2$. We will construct two diffeomorphisms that satisfies inequality (1.1) on $S^1 \times S^2$.

Consider the unit sphere $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Define the polar coordinates $(\rho, \Theta)$ in the $(x, y)$-plane: $x = \rho \cos \Theta$ and $y = \rho \sin \Theta$. Put $\theta = \frac{\Theta}{\pi}$. Then $\{(\theta, z) : \theta \in [0, 1), z \in (-1, 1)\}$ can be taken as coordinates of the sphere without the poles.

Let $R_\alpha$ be the rotation of $S^2$ around the $z$-axis by angle $2\pi \alpha \in S^1 = \mathbb{R}/2\pi \mathbb{Z}$:

$$R_\alpha(w) = (\theta + \alpha, z)$$
where $w = (\theta, z)$.

**Example 2.3.1.** Consider the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and define a torus action on $S^1 \times S^2$ as follows

$$\phi_{(\varphi_1, \varphi_2)}(\lambda, w) = (\lambda + \varphi_1, R_{\varphi_2}(w))$$

where $(\varphi_1, \varphi_2) \in \mathbb{T}^2$, $w \in S^2$ and $\lambda \in S^1 = \mathbb{R}/\mathbb{Z}$.

Let us construct the set $U$ from Section 2.1. The action $\phi$ is free on the element $(0, w_0) \in S^1 \times S^2$, where $w_0 = (0, 0) \in S^2$. The set $D = \{(0, w) : w = (0, z), z \in (-1, 1)\} \subset S^1 \times S^2$ is a one dimensional disc. Then $U = \mathbb{T}^2 D$ and we get $U = S^1 \times (S^2 \setminus \{a_1, a_2\})$, where $a_1$ and $a_2$ are the poles of the sphere.

Let $\Psi$ be a positive, unbounded increasing, function on $\mathbb{R}_+$ such that $\Psi(x) = o(x)$ for $x \to \infty$. Let $F$ be the function from Borichev’s theorem and $\alpha$ be the constant from Borichev’s theorem. In this case the slow diffeomorphism of $S^1 \times S^2$, satisfying inequality (1.1), is

$$f(\lambda, w) = \begin{cases} f_1(\lambda, w) = (\lambda + \alpha, R_{A(z)}F(\lambda)(w)) & (\lambda, w) \in U, \ w = (\theta, z) \\ f_2(\lambda, w) = (\lambda + \alpha, w) & (\lambda, w) \in S^1 \times \{a_1, a_2\}. \end{cases}$$

Notice the diffeomorphism does not have a fixed point.

**Example 2.3.2.** Define the torus action $\tilde{\phi} : \mathbb{T}^2 \times (S^1 \times S^2) \to S^1 \times S^2$ as follows

$$\tilde{\phi}_{(\varphi_1, \varphi_2)}(\lambda, w) = (\lambda + \varphi_2, R_{\varphi_1}(w))$$

where $(\varphi_1, \varphi_2) \in \mathbb{T}^2$, $w \in S^2$ and $\lambda \in S^1$.

Define the set $U$ as in Example 2.3.1. The slow diffeomorphism in this case is

$$\tilde{f}(\lambda, w) = \begin{cases} \tilde{f}_1(\lambda, w) = (\lambda + A(z)F(\theta), R_\alpha(w)) & (\lambda, w) \in U, \ w = (\theta, z) \\ \tilde{f}_2(\lambda, w) = (\lambda, w) & (\lambda, w) \in S^1 \times \{a_1, a_2\}. \end{cases}$$

Notice $\tilde{f}$ has fixed points at $S^1 \times \{a_1, a_2\}$. 


3 Growth and Flux

Let \((M, \omega)\) be a closed manifold of dimension \(n\) with a volume form, \(\omega\). Let \(\text{Diff}_0(M, \omega)\) be the group of volume preserving diffeomorphisms isotopic to the identity. First, let us define the flux homomorphism on \(\pi_1(\text{Diff}_0(M, \omega))\).

For any \((n - 1)\)-cycle \(C\) and any \(\{f_t\} \in \pi_1(\text{Diff}_0(M, \omega))\) we define an \(n\)-cycle \(\Phi(C) = \bigcup_t f_t(C)\) in \(M\).

Define the flux homomorphism \(\text{flux}_\omega: \pi_1(\text{Diff}_0(M, \omega)) \to H^{n-1}(M, \mathbb{R})\) as follows. Choose a loop \(\{f_t\}\) representing element \(\alpha \in \pi_1(\text{Diff}_0(M, \omega))\). Put

\[
(\text{flux}_\omega([\{f_t\}]), [C]) = (\omega, \Phi(C)) \quad \text{for all } [C] \in H_{n-1}(M, \mathbb{R}).
\]

One can show that this definition does not depend on choice of a loop \(\{f_t\}\) representing \(\alpha\) and the cycle \(C\) representing the homology class.

The image \(\Gamma = \overline{\text{flux}_\omega(\pi_1(\text{Diff}_0(M, \omega)))} \subset H^{n-1}(M, \mathbb{R})\) is called the flux group.

The notion of flux can be extended to diffeomorphisms \(f \in \text{Diff}_0(M, \omega)\). Define

\[
\text{flux}_\omega: \text{Diff}_0(M, \omega) \to H^{n-1}(M, \mathbb{R})/\Gamma
\]

as follows. Choose any path \(\{f_t\}\) of volume preserving diffeomorphisms with \(f_0 = \text{id}\) and \(f_1 = f\). Put \(\Phi(C) = \bigcup_t f_t(C)\). Then \(\text{flux}_\omega\) is given by

\[
(\text{flux}_\omega(f), [C]) = (\omega, \Phi(C)) \quad \text{for all } [C] \in H_{n-1}(M, \mathbb{R}).
\]

Using the fact that \(f_t\) is volume preserving for every \(t\), \((\omega, \Phi(C))\) does not depend on the choice of the element \(C \in [C]\).

For \(f \in \text{Diff}_0(M, \omega)\) we can choose different paths, \(\{f_t\}\) and \(\{g_t\}\), with \(f_0 = g_0 = \text{id}\) and \(f_1 = g_1 = f\). However, the difference between the fluxes of these paths connecting \(\text{id}\) to \(f\) belongs to \(\Gamma\) and thus the flux is well defined.
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Let us return to the example of $(S^1 \times S^2, \omega)$, when $\omega$ is the volume form constructed in Section 2.2 after normalization, $\int_{S^1 \times S^2} \omega = 1$. We calculate the flux of the slow diffeomorphisms $f$ and $\tilde{f}$. First, let us calculate the flux group, $\Gamma$, of this manifold. Make the following identifications:

$$H_2(S^1 \times S^2, \mathbb{Z}) = \mathbb{Z}, \quad H_3(S^1 \times S^2, \mathbb{Z}) = \mathbb{Z}, \quad H^2(S^1 \times S^2, \mathbb{Z}) = \mathbb{Z} \subset \mathbb{R} = H^2(S^1 \times S^2, \mathbb{R}).$$

For each $\gamma \in \pi_1(Diff_0(M, \omega))$ and $[a] \in H_2(S^1 \times S^2, \mathbb{Z})$ we have

$$(\text{flux}_\omega(\gamma), a) = (\omega, \Phi(a)).$$

Here $\Phi$ is a functional from $\mathbb{Z}$ to $\mathbb{Z}$ and the value of $\omega$ on the generator of $H_3(S^1 \times S^2, \mathbb{Z})$ equals 1. Hence, $(\omega, \Phi a) \in \mathbb{Z}$ and $\text{flux}(\gamma) \in H^2(S^1 \times S^2, \mathbb{Z})$ which implies that $\Gamma \subset \mathbb{Z}$. On the other hand, let us look at the following loop $\gamma \in \pi_1(Diff_0(M, \omega))$ :

$$\gamma(\lambda, w)(t) = (\lambda + t, w)$$

where $\lambda \in S^1$, $w \in S^2$ and $t \in [0, 1]$. Fix $\lambda_0 \in S^1$. Let $[C] = [\lambda_0 \times S^2]$ be the generator of $H_2(S^1 \times S^2, \mathbb{Z})$. Then $\Phi(C) = S^1 \times S^2$. Therefore,

$$(\text{flux}(\gamma), [C]) = (\omega, \Phi(C)) = 1,$$

and we conclude that $\mathbb{Z} \subset \Gamma$. Hence $\Gamma = \mathbb{Z}$.

Let us calculate the flux of the diffeomorphism from Example 2.3.1. The set $U = S^1 \times (S^2 \setminus \{a_1, a_2\})$ is the set constructed in Example 2.3.1. As before $C$ stands for $\lambda_0 \times S^2$. Then

$$(\omega, \Phi(C)) = (\omega, \Phi(C'))$$

where $C' = C \cap U = C \setminus \lambda_0 \times \{a_1, a_2\}$, and $a_1, a_2$ are the poles of $S^2$. Indeed, the dimension of $\Phi(\lambda_0 \times \{a_1, a_2\})$ is one, hence $(\omega, \Phi(\lambda_0 \times \{a_1, a_2\})) = 0$.

Let $f_t(\lambda, w) = (\lambda + t \alpha, R_{tA(\omega)F(\lambda)}(w))$, $t \in [0, 1]$, where $w = (\theta, z) \in S^2$, be a path of volume preserving diffeomorphisms on $U$ with $f_0 = \text{id}$ and $f_1 = f$. Then,

$$f_t(\{\lambda_0 \times (S^2 \setminus \{a_1, a_2\})\}) = \{(\lambda_0 + t\alpha) \times (S^2 \setminus \{a_1, a_2\})\}$$
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and

$$\Phi(\lambda_0 \times (S^2 \setminus \{a_1, a_2\})) = ([\lambda_0, \lambda_0 + \alpha] \times (S^2 \setminus \{a_1, a_2\})).$$

Thus,

$$(\text{flux}(\{f_t\}), [C]) = (\omega, \Phi(C')) = \alpha \pmod{1}.$$ 

Hence, $\text{flux}(\{f\})([C]) = \alpha \pmod{1}.$

Now, let us calculate the flux of $\tilde{f}$ from Example 2.3.2. Let $\tilde{f}_t(\lambda, w) = (\lambda + tA(z)F(\theta), R_{\alpha}(w)),$ $t \in [0, 1],$ where $w = (\theta, z) \in S^2,$ be a path of volume preserving diffeomorphisms with $\tilde{f}_0 = \text{id}$ and $\tilde{f}_1 = \tilde{f}.$ Then,

$$\tilde{f}_t(\{\lambda_0 \times (S^2 \setminus \{a_1, a_2\})\}) = \{(\lambda, \theta, z) \in S^1 \times S^1 \times (-1; 1) | \lambda = \lambda_0 + tA(z)F(\theta)\}$$

and $\Phi(\lambda_0 \times (S^2 \setminus \{a_1, a_2\}))$ is a domain bounded by hypersurfaces $\{\lambda = \lambda_0\}$ and $\{\lambda = \lambda_0 + A(z)F(\theta)\}.$

Thus,

$$(\text{flux}(\{\tilde{f}_t\}), [C]) = (\omega, \Phi(C')) = \int_{-1}^{1} \int_{0}^{1} A(z)F(\theta)d\theta dz \pmod{1}$$

where $C = \lambda_0 \times S^2.$

Now, from Borichev’s theorem, $\int_{0}^{1} F(\theta)d\theta = 0.$ Hence, $\text{flux}(\{\tilde{f}\})([C]) \equiv 0 \pmod{1}.$

Since $H_1(S^2 \times S^1, \mathbb{Z}) = \mathbb{Z}$ we get that $\text{flux}(\{f_t\}) = 0 \pmod{1}.$

In conclusion, in Example 2.3.2 we have a diffeomorphism with fixed points and vanishing flux. That proves Theorem 1.0.3.
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