GEOMETRIC REPRESENTATION OF CLASSES OF CONCAVE FUNCTIONS AND DUALITY

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Abstract. Using a natural representation of a \(1/s\)-concave function on \(\mathbb{R}^d\) as a convex set in \(\mathbb{R}^{d+1}\), we derive a simple formula for the integral of its \(s\)-polar. This leads to convexity properties of the integral of the \(s\)-polar function with respect to the center of polarity. In particular, we prove that the reciprocal of the integral of the polar function of a log-concave function is log-concave as a function of the center of polarity. Also, we define the Santaló regions for \(s\)-concave and log-concave functions and generalize the Santaló inequality for them in the case the origin is not the Santaló point.

1. Introduction

Log-concave and \(s\)-concave functions provide a natural extension of the theory of convex bodies. Starting with functional version of the famous Blaschke–Santaló inequality [3, 4, 8, 22, 32], much research has been devoted to the study of such functions in recent years. This has led to e.g., functional analogs of the floating body [33], John ellipsoids [1, 28] and Löwner ellipsoids [2, 34]. More examples can be found in e.g., [19, 20, 24, 46]).

Motivated by the setting of convex bodies, we investigate in this paper the properties of log-concave and \(1/s\)-concave functions related to duality transforms associated with the corresponding class of functions.

1.1. Background. For \(s > 0\), a non-negative function \(f\) on \(\mathbb{R}^d\) is called \(1/s\)-concave if the function \(f^{1/s}\) is concave on its support. It is well known that for a positive integer \(s\), a \(1/s\)-concave function on \(\mathbb{R}^d\) is the marginal of the indicator function of a convex set in \(\mathbb{R}^{d+s}\). A non-negative function on \(\mathbb{R}^d\) is called log-concave if its logarithm is concave on its support. It is also well known that any log-concave function is the local uniform limit of certain \(1/s\)-concave functions, as \(s\) tends to infinity, e.g., [14, Section 2.2]. This observation has been useful in many instances in the setting of log-concave functions, e.g., [3, 16, 30], as it allows to pass results from \(1/s\)-concave functions to log-concave functions.

The concept of duality is a cornerstone of both, geometry in general, and asymptotic geometric analysis in particular. In convex analysis, the concept of duality is tightly connected with the notion of a polar set. Recall that the polar of a set \(K\) in \(\mathbb{R}^d\) is the set \(K^\circ\) given by

\[
K^\circ = \{y \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}.
\]

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Natural generalizations of this definition to the setting of classes of concave functions are as follows. For any $s > 0$, the $s$-polar transform, introduced in [3], is defined by

$$
\mathcal{L}_s f(y) = \inf_{\{x : f(x) > 0\}} \frac{(1 - \langle x, y \rangle)^s}{f(x)},
$$

where $(a)_+ = \max\{a, 0\}$. The polar (or log-conjugate) of a non-negative function $f$ on $\mathbb{R}^d$ is defined by

$$
\mathcal{L}_\infty f(y) = \inf_{\{x : f(x) > 0\}} e^{-\langle x, y \rangle} f(x).
$$

As shown in [5, Theorem 1], the $s$-polar transform is essentially the only order reversing involution on the class of upper semi-continuous $1/s$-concave functions containing the origin in the interior of support. Likewise, as shown in [4, Corollary 12], $\mathcal{L}_\infty$ is essentially the only order reversing involution on the class of upper semi-continuous log-concave functions. See also [13] for a similar characterization of the polar transformation in the setting of convex bodies.

Alexandrov [7] noticed that the reciprocal of volume of the polar of a convex body $K$ in $\mathbb{R}^d$ as a function of the center of polarity is a $1/d$-concave function on the interior of $K$. More formally, denote the shift of a body $K$ by a vector $z$ by

$$\text{Shift} [K, z] = K - z.
$$

Then Alexandrov’s results says that for a convex body $K \subset \mathbb{R}^d$, the function

$$z \mapsto (\text{vol}_d (\text{Shift} [K, z]))^{-1/d}
$$

is concave on the interior of $K$, where $\text{vol}_d$ is the standard $d$-dimensional volume.

In [28] (see also [42]), the authors suggested a certain way of representing a $1/s$-concave function on $\mathbb{R}^d$ as a convex set in $\mathbb{R}^{d+1}$. They called it $s$-lifting and defined it as follows

$$(s)\text{Lift} f = \left\{(x, \xi) \in \mathbb{R}^{d+1} : x \in \text{cl supp } f, \ |\xi| \leq (f(x))^{1/s}\right\},
$$

where $\text{cl supp } f$ denotes the closure of the support of $f$. We note that the $s$-lifting allows us to use tools and results for convex sets in the setting of $1/s$-concave functions. The key observation of the current paper is the following simple lemma.

**Lemma 1.1.** Let $f : \mathbb{R}^d \to [0, \infty)$. Then

$$(s)\text{Lift} f \circ (s)\text{Lift} f = (s)\text{Lift} (\mathcal{L}_s f).
$$

This representation allows to express the integral of an $s$-polar function via a simple formula, even for non-integer $s$. We elaborate on this in Section 2.

In summary, the primary objective of this paper is to demonstrate the feasibility of leveraging tools from classical convexity alongside the examination of various classes of concave/convex functions simultaneously. Our aim is to illustrate that it is possible to derive classical results in convexity while also generating new outcomes within the functional context through specific limit arguments. However, it is important to exercise caution when employing these arguments, as, in some cases, duality, similar to what occurs in Lemma 1.1, may be lost during the process. For further insights into such instances, we direct interested readers to [29].
1.2. The main theorems. For a function \( f \) on \( \mathbb{R}^d \), we denote its shift by a vector \( z \) by
\[
\text{Shift}[f, z](x) = f(x - z), \quad x \in \mathbb{R}^d.
\]
The barycenter of an integrable function \( f \) on \( \mathbb{R}^d \) is defined as
\[
\frac{\int_{\mathbb{R}^d} x f(x) \, dx}{\int_{\mathbb{R}^d} f(x) \, dx},
\]
if the quotient exists. Any log-concave function of finite integral has the barycenter \([14, \text{Lemma 2.2.1}].\)

A main result of this paper is the following generalization of Alexandrov’s theorem.

**Theorem 1.** Let \( s \in (0, \infty) \) and \( f : \mathbb{R}^d \to [0, \infty) \) be an upper semi-continuous, \(1/s\)-concave function with finite integral. Then we have

1. The function \( z \mapsto \int_{\mathbb{R}^d} \mathcal{L}_s(\text{Shift}[f, z]) \) is convex on the interior of the support of \( f \). It attains the minimum at point \( \tilde{z} \) such that the origin is the barycenter of \( \mathcal{L}_s(\text{Shift}[f, \tilde{z}]) \).
2. The function \( z \mapsto \left( \int_{\mathbb{R}^d} \mathcal{L}_s(\text{Shift}[f, z]) \right)^{-\frac{1}{s+1}} \) is concave on the interior of the support of \( f \).

Note that Theorem 1 yields Alexandrov’s result, since the indicator function of a convex body is \( s \)-concave for any positive \( s \).

Using a limit argument, we obtain the following version of Alexandrov’s theorem for log-concave functions.

**Theorem 2.** Let \( f : \mathbb{R}^d \to [0, \infty) \) be an upper semi-continuous log-concave function with finite integral. Then the function
\[
z \mapsto \int_{\mathbb{R}^d} \mathcal{L}_\infty(\text{Shift}[f, z])
\]
is convex and its reciprocal is log-concave on the interior of support of \( f \).

We also provide a simple direct proof of Theorem 2 in Subsection 3.1.

For any function \( f \) on \( \mathbb{R}^d \), define
\[
f_s(x) = \left( \frac{1 + \log f(x)}{s} \right)_+. \quad (1.1)
\]
Clearly, for a log-concave function \( f \), the function \( f_s \) is \( s \)-concave and \( f_s \to f \) pointwise on \( \mathbb{R}^d \) as \( s \to +\infty \). Then, to carry out the limit argument, we use the following technical observation, which, surprisingly, seems to be new.

**Theorem 3.** Let \( f : \mathbb{R}^d \to [0, \infty) \) be an upper semi-continuous, log-concave function with finite integral, containing the origin in the interior of its support. Denote by \( A \) the set of all points in \( \mathbb{R}^d \) that are not in the boundary of the support of \( \mathcal{L}_\infty f \). Then, as \( s \to \infty \),
\[
\mathcal{L}_s f_s \left( \frac{x}{s} \right) \to \mathcal{L}_\infty f(x)
\]
locally uniformly on \( A \).
A weaker version of this result is [3, Lemma 3.3].

One of the most important results in convex geometry is the Blaschke–Santaló inequality, see, e.g., [25, 47]. It says that there is a unique $z_0$ in $\text{int}(K)$, the interior of $K$, for which $\text{vol}_d((\text{Shift}[K, z]))^\circ$ is minimal and then

$$\text{vol}_d(K) \cdot \text{vol}_d((\text{Shift}[K, z]))^\circ \leq \left(\text{vol}_d B^d\right)^2.$$  

Equality holds if and only if $K$ is an ellipsoid. Here, and throughout the paper, $B^d$ denotes the $d$-dimensional Euclidean unit ball. Meyer and Pajor [38] proved a more general form of this inequality, which we state now:

Let $K$ be a convex body in $\mathbb{R}^d$. Let $H$ be an affine hyperplane with half-spaces $H_+$ and $H_-$, such that $\text{vol}_d(H_+ \cap K) = \lambda \text{vol}_d(K)$. Then there exists $z \in H \cap \text{int}(K)$ such that

$$\text{vol}_d(K) \cdot \text{vol}_d((\text{Shift}[K, z]))^\circ \leq \left(\text{vol}_d B^d\right)^2 \frac{4\lambda (1 - \lambda)}{\kappa_{d+1}}.$$  \hfill (1.2)

We note here that $B^d$ is the only self-polar set in $\mathbb{R}^d$. Let $| \cdot |$ be the Euclidean norm and put

$$\hat{h}(x) = \begin{cases} \left[1 - |x|^2\right]^{1/2}, & \text{if } x \in B^d \\ 0, & \text{otherwise.} \end{cases}$$

It is not hard to see that $\hat{h}^s$ is self-$s$-polar, that is,

$$L_s\left(\hat{h}^s\right) = \hat{h}^s,$$

and thus this function plays the role of the unit ball in the class of $1/s$-concave functions. Similarly, the standard Gaussian density $e^{-|x|^2/2}$ is self-polar in the class of log-concave functions. The Blaschke–Santaló inequality for log-concave functions was obtained in e.g., [3, 8, 32]. The Blaschke–Santaló inequality for a more general “duality relation” including the $s$-polar transform was obtained in [22]. In [31], Lehec generalized (1.2) to log-concave functions.

Using Lehec’s approach, we prove the following extension of (1.2) to the class of $1/s$-concave functions.

**Theorem 4.** Let $f : \mathbb{R}^d \to [0, \infty)$ be a $1/s$-concave function with finite integral. Let $H$ be an affine hyperplane with half-spaces $H_+$ and $H_-$ and such that $\lambda \int_{\mathbb{R}^d} f = \int_{H_+} f$ for some $\lambda \in (0, 1)$. Then there exists $z \in H$ such that

$$\int_{\mathbb{R}^d} f \int_{\mathbb{R}^d} L_s(\text{Shift}[f, z]) \leq \frac{(s)\kappa_{d+1}^2}{\kappa_{d+1}^2},$$  \hfill (1.3)

where $(s)\kappa_{d+1} = \int_{\mathbb{R}^d} \hat{h}^s$. Again, this theorem implies both the analogous result (1.2) for convex sets and Lehec’s result [31] for log-concave functions. A kind of reverse Santaló inequality, that can be considered a counterpart to Theorem 4, was proved in [39] for the case $d = 1$. See also [23] for another proof of this result.

Finally, generalizing the definition of Meyer and Werner [40], we define and list several properties of the Santaló $s$-region of a non-negative function with bounded support. This region is essentially the set of points $z$ such that the integral of the $s$-polar transform of $\text{Shift}[f, s]$ is bounded by some positive constant. We give formal definitions and discuss possible definitions of Santaló $s$-function in Section 5.
The rest of the paper is organized as follows. In 2, we define the \( s \)-lifting of a function, study its properties related to duality, and prove Theorem 1. In Section 3, we recall several definitions of convex analysis and prove Theorem 3. We also show that Theorem 2 is a consequence of Theorems 1 and 3. Next, we prove Theorem 4 in Section 4. Finally, we introduce and study the Santaló \( s \)-region in Section 5.

**Notation.** We use \( \langle x, y \rangle \) to denote the standard inner product of vectors \( x \) and \( y \) of \( \mathbb{R}^d \). We write \( \mathbb{R}^d \subset \mathbb{R}^{d+1} \), when we consider \( \mathbb{R}^d \) as the subspace of \( \mathbb{R}^{d+1} \) of vectors with zero last coordinates. We say that a set \( K \subset \mathbb{R}^{d+1} \) is \( d \)-symmetric if \( K \) is symmetric with respect to \( \mathbb{R}^d \subset \mathbb{R}^{d+1} \). The closure of a set \( K \subset \mathbb{R}^d \) is denoted by \( \text{cl} K \). The support function of a convex body \( K \) at \( y \neq 0 \) is defined by

\[
h_K(y) = \sup_{x \in K} \langle x, y \rangle.
\]

The convex hull of a set \( K \) is denoted by \( \text{conv} K \). The support of a non-negative function \( f \) on \( \mathbb{R}^d \) is the set

\[
\text{supp } f = \{ x \in \mathbb{R}^d : f(x) > 0 \}.
\]

We will refer to an integrable lower semi-continuous function of finite integral as a proper function. We will integrate over domains in \( \mathbb{R}^d \) and \( \mathbb{R}^{d+1} \) and use \( \lambda_{d+1} \) to denote the standard Lebesgue measure on \( \mathbb{R}^{d+1} \) and \( \sigma \) to denote the uniform measure on the unit sphere \( S^d \subset \mathbb{R}^{d+1} \).

### 2. The \( s \)-lifting and its properties

The notion of \( s \)-lifting of a function was first introduced in [28]. We recall its definition and prove Theorem 1 in the current section.

Let \( f : \mathbb{R}^d \to [0, \infty) \) be a function and \( s > 0 \). The \( s \)-lifting of \( f \) is a \( d \)-symmetric set in \( \mathbb{R}^{d+1} \) defined by

\[
^{(s)}\text{Lift } f = \left\{ (x, \xi) \in \mathbb{R}^{d+1} : x \in \text{cl supp } f, \ |\xi| \leq (f(x))^{1/s} \right\}.
\]  \hspace{1cm} (2.1)

Clearly, the \( s \)-lifting of a \( 1/s \)-concave function is a convex set. Thus, the \( s \)-lifting gives a nice way of representing a \( 1/s \)-concave function on \( \mathbb{R}^d \) as a convex set in \( \mathbb{R}^{d+1} \). In fact, it is represented as a \( d \)-symmetric convex set. There are other representations of \( 1/s \)-concave function as convex sets. One of them is mentioned in (5.2). The advantages of the representation (2.1) are

- it holds for non-integer \( s \)
- one can investigate the properties of the \( s \)-lifting instead of studying \( 1/s \)-concave functions directly.

**Proof of Lemma 1.1.** Let \( y \in \mathbb{R}^d \) and \( \tau \in \mathbb{R} \). Then \( (y, \tau) \in \left( ^{(s)}\text{Lift } f \right)^{\circ} \) if and only if the inequality

\[
\langle x, y \rangle + t\tau \leq 1
\]

holds for all \( x \in \text{supp } f \) and \( t \in \mathbb{R} \) such that \( (x, t) \in ^{(s)}\text{Lift } f \). By the symmetry of the \( s \)-lifting, we conclude that \( (y, \tau) \in \left( ^{(s)}\text{Lift } f \right)^{\circ} \) if and only if

\[
\langle x, y \rangle + |\tau| f^{1/s}(x) \leq 1
\]
for any \( x \in \text{supp} \ f \). Thus, \( \langle x, y \rangle \leq 1 \) and

\[
|\tau|^s \leq \frac{(1 - \langle x, y \rangle)^s}{f(x)}.
\]

Taking the infimum, we see that \((y, \tau) \in \left( (s) \text{Lift} f \right)^{\circ} \) if and only if

\[
|\tau|^s \leq \inf_{x \in \text{supp} \ f} \frac{(1 - \langle x, y \rangle)^s}{f(x)} = \mathcal{L}_s f(y).
\]

That is, \((y, \tau) \in \left( (s) \text{Lift} f \right)^{\circ} \) if and only if \((y, \tau)\) belongs to the \(s\)-lifting of \( \mathcal{L}_s f \). \( \square \)

Let \( C \subset \mathbb{R}^{d+1} \) be a \( d \)-symmetric Borel set. The \textit{s-volume} of \( C \) is defined by

\[
(s) \mu(C) = \int_{\mathbb{R}^d} \left[ \frac{1}{2} \text{length} \ (C \cap \ell_s) \right]^s \, dx.
\]

By Fubini's theorem, we have

\[
\int_{\mathbb{R}^d} f = (s) \mu \left( (s) \text{Lift} f \right) = \frac{s}{2} \int_{(s) \text{Lift} f} |\langle e_{d+1}, x \rangle|^{s-1} \, d\lambda_{d+1}(x). \tag{2.2}
\]

**Lemma 2.1.** Fix \( s > 0 \). Let \( K \) be a \( d \)-symmetric convex body in \( \mathbb{R}^{d+1} \). The functional

\[
z \mapsto \frac{s}{2(d+s)} \int_{S^d} \frac{|\langle e_{d+1}, u \rangle|^{s-1}}{h_{\text{shift}[K,z]}(u)^{d+s}} \, d\sigma(u)
\]

is convex on the interior of \( K \). Moreover, if \( K \) is the \( s \)-lifting of a \( 1/s \)-concave function \( f \) and \( z \in \mathbb{R}^d \), then the value of this functional at \( z \) is equal to

\[
\int_{\mathbb{R}^d} \mathcal{L}_s(\text{Shift} [f, z]) .
\]

**Proof.** Since \( h_{\text{shift}[K,z]}(u) = h_K(u) - \langle z, u \rangle \), one sees that \( (h_{\text{shift}[K,z]}(u))^{d+s} \) is a convex function of \( z \) for a fixed \( u \in S^d \) and any \( s > 0 \). The convexity of the functional follows immediately.

Assume now that \( z \in \mathbb{R}^d \) and put \( K = (s) \text{Lift} f \) for a \( 1/s \)-concave function \( f \). By Lemma 1.1 and with equation (2.2), we get

\[
\int_{\mathbb{R}^d} \mathcal{L}_s(\text{Shift} [f, z]) = (s) \mu(\text{Shift} [K, z]) = \frac{s}{2} \int_{(s) \text{Lift} f} |\langle e_{d+1}, x \rangle|^{s-1} \, d\lambda_{d+1}(x) .
\]

Using spherical coordinates gives

\[
\int_{\mathbb{R}^d} \mathcal{L}_s(\text{Shift} [f, z]) = \frac{s}{2} \int_{(s) \text{Lift} f} |\langle e_{d+1}, ru \rangle|^{s-1} r^d \, dr \, d\sigma(u) =
\]

\[
\frac{s}{2} \int_{S^d} |\langle e_{d+1}, u \rangle|^{s-1} \int_{0}^{1} r^{d+s-1} \, dr \, d\sigma(u).
\]

That is,

\[
\int_{\mathbb{R}^d} \mathcal{L}_s(\text{Shift} [f, z]) = \frac{s}{2(d+s)} \int_{S^d} \frac{|\langle e_{d+1}, u \rangle|^{s-1}}{(h_{\text{shift}[K,z]}(u))^{d+s}} \, d\sigma(u) . \tag{2.3}
\]

This completes the proof. \( \square \)
Proof of Theorem 1. Denote $K_z = \text{Lift} (\text{Shift} [f, z])$, $K_z^o = \text{Lift} \mathcal{L}_s (\text{Shift} [f, z])$, and
\[\Phi(z) = \int_{\mathbb{R}^d} \mathcal{L}_s (\text{Shift} [f, z]).\]

Lemma 2.1 implies that $\Phi$ is convex on the interior of the support of $f$.

We now address the second assertion of the theorem. Denote $\Psi(z) = \Phi(z)^{-\frac{1}{d+s}}$, and let $\nu_z$ be a measure on $S^d$ with density given by
\[d\nu_z(u) = \frac{s}{2(d+s)} \frac{|\langle e_{d+1}, u \rangle|^{s-1}}{(h_{K_z}(u))^{d+s+1}} d\sigma(u).\]

The directional derivative of $h_{K_z}(u)$ in the direction of the $i$-th standard basis vector $e_i = -u[i]$, where $a[i]$ stands for the $i$-th coordinate of $a$. Differentiating (2.3), we obtain that
\[\Phi'_i(z) = (d+s) \frac{s}{2(d+s)} \int_{S^d} \frac{|\langle e_{d+1}, u \rangle|^{s-1} u[i]}{(h_{K_z}(u))^{d+s+1}} d\sigma(u) = (d+s) \int_{S^d} \frac{u[i]}{h_{K_z}(u)} d\nu_z(u),\]
and
\[\Phi''_{ij}(z) = (d+s)(d+s+1) \int_{S^d} \frac{u[i]u[j]}{(h_{K_z}(u))^{d+s+2}} d\nu_z(u).\]

On the other hand,
\[\Phi''_{ij}(z) = \frac{1}{d+s} \left( \frac{d+s+1}{d+s} \right) \Phi(z)^{-\frac{1}{d+s}} - 2 \Phi'_i(z) \Phi'_j(z) - \frac{1}{d+s} \Phi(z)^{-\frac{1}{d+s}} \Phi''_{ij}(z) = -\frac{1}{d+s} \Phi(z)^{-\frac{1}{d+s} - 2} \left( \Phi(z) \Phi''_{ij}(z) - \frac{d+s+1}{d+s} \Phi'_i(z) \Phi'_j(z) \right).\]

Therefore, $\Psi$ is concave on its support if and only if the matrix $A_z$ given by
\[A_z[i, j] = \Phi(z) \Phi''_{ij}(z) - \frac{d+s+1}{d+s} \Phi'_i(z) \Phi'_j(z)\]
is positive semi-definite at every point of the interior of the support of $\Psi$. Using the formulas for the partial derivatives of $\Phi$ and (2.3), we get
\[\frac{A_z[i, j]}{(d+s)(d+s+1)} = \int_{S^d} d\nu_z(u) \cdot \int_{S^d} \frac{u[i]u[j]}{(h_{K_z}(u))^{d+s+2}} d\nu_z(u) - \int_{S^d} \frac{u[i]}{h_{K_z}(u)} d\nu_z(u) \int_{S^d} \frac{u[j]}{h_{K_z}(u)} d\nu_z(u).\]

That is, $A_z$ is a covariance matrix, and hence it is positive semi-definite.

Finally, differentiating (2.3) again, we get that the directional derivative $\Phi'_h$ of $\Phi$ in direction $h \in \mathbb{R}^d$ is
\[\Phi'_h(z) = \frac{s}{2} \int_{S^d} \frac{|\langle e_{d+1}, u \rangle|^{s-1}}{(h_{K_z}(u))^{d+s+1}} \langle h, u \rangle d\sigma(u).\]

Reversing the chain of identities in the proof of Lemma 2.1, one gets
\[\frac{s}{2} \int_{K_z^o} |\langle e_{d+1}, ru \rangle|^{s-1} \langle h, ru \rangle r^d dr d\sigma(u) = \frac{s}{2} \int_{K_z^o} \frac{|\langle e_{d+1}, x \rangle|^{s-1}}{(h_{K_z^o}(x))^{d+s+1}} \langle h, x \rangle d\lambda_{d+1}(x).\]
Since $h \in \mathbb{R}^d$ and by the definition of $K^\circ$, the latter is
\[
\int_{\mathbb{R}^d} \langle h, y \rangle \mathcal{L}_s(\text{Shift } [f, z]) (y) \, dy.
\]
By convexity, all directional derivatives of $\Phi$ vanish at the argmin. Consequently, the above calculations show that at the argmin $\tilde{z}$, the identity
\[
\int_{\mathbb{R}^d} y \mathcal{L}_s(\text{Shift } [f, \tilde{z}]) (y) \, dy = 0
\]
holds. Thus, the origin is the barycenter of $\mathcal{L}_s(\text{Shift } [f, \tilde{z}])$.

This finishes the proof of Theorem 1. 

\[
\square
\]

As was pointed to us by the anonymous referee, using the measure representation of the integral from Lemma 2.1, combined with the appropriate version of Minkowski inequality for means yield the inequality of Theorem 1. Let us sketch the idea.

In the notation used in the proof of Theorem 1,
\[
\Psi(z) = \|h_K - \langle z, \cdot \rangle\|_{L_{-(d+s)}(\tilde{\nu}_s)},
\]
where $\tilde{\nu}_s$ is a measure on $S^d$ with density given by
\[
d\tilde{\nu}_s(u) = \frac{s}{2(d+s)} |\langle e_{d+1}, u \rangle|^{s-1} d\sigma(u).
\]
Using the Minkowski inequality for means, one gets:
\[
\Psi(\lambda z_1 + (1 - \lambda) z_2) = \|\lambda (h_K - \langle z_1, \cdot \rangle) + (1 - \lambda) (h_K - \langle z_2, \cdot \rangle)\|_{L_{-(d+s)}(\tilde{\nu}_s)} \geq \lambda \|h_K - \langle z_1, \cdot \rangle\|_{L_{-(d+s)}(\tilde{\nu}_s)} + (1 - \lambda) \|h_K - \langle z_2, \cdot \rangle\|_{L_{-(d+s)}(\tilde{\nu}_s)} = \lambda \Psi(z_1) + (1 - \lambda) \Psi(z_2).
\]

3. LOG-CONCAVE FUNCTIONS

In this section, we study the properties of log-concave functions related to the polar transform and prove Theorems 3 and 2.

3.1. Consequences of Theorem 3. Before proving Theorem 3, we derive several results from it including Theorem 2.

Proof of Theorem 2. Recall that
\[
\lim_{s \to +\infty} \left( \lambda a^{\frac{1}{d+s}} + (1 - \lambda) b^{\frac{1}{d+s}} \right)^{d+s} = a^\lambda b^{1-\lambda}
\]
for any positive real numbers $a$ and $b$.

Let now $f$ be a proper log-concave function containing the origin in the interior of its support. Then, to prove Theorem 2, it suffices to show that
\[
\int_{\mathbb{R}^d} \mathcal{L}_\infty f = \lim_{s \to +\infty} s^d \int_{\mathbb{R}^d} \mathcal{L}_s f_s,
\]
where $f_s$ is as in (1.1). Indeed, the convexity of $z \mapsto \int_{\mathbb{R}^d} \mathcal{L}_\infty(\text{Shift } [f, z])$ follows immediately from assertion (1) of Theorem 1 and (3.2). The log-concavity of
\[
z \mapsto \frac{1}{\int_{\mathbb{R}^d} \mathcal{L}_\infty(\text{Shift } [f, z])}
\]
follows from assertion (2) of Theorem 1 and identity (3.1).
Identity (3.2) follows from Theorem 3 and [3, Lemma 3.2]. We use the following
simplified version of this lemma.

Lemma 3.1. [3] Let \( \{f_i\}_{i=1}^{\infty} \) be a sequence of log-concave functions converging to a log-
concave function \( f \) of finite integral on a dense subset \( A \subset \mathbb{R}^d \). Then \( \int_{\mathbb{R}^d} f_n \to \int_{\mathbb{R}^d} f \).

This completes the proof of Theorem 2. \( \square \)

Also, there is a direct proof of Theorem 2.

Direct proof of Theorem 2. Denote \( \Phi(z) = \int_{\mathbb{R}^d} L_{\infty}(\text{Shift } [f, z]) \), and let \( f = e^{-\psi} \).
One has
\[
\Phi(z) = \int_{\mathbb{R}^d} e^{-L\psi(y)-(z,y)} \, dy.
\]
By the Hölder inequality,
\[
\Phi((1-\lambda)z_1 + \lambda z_2) = \int_{\mathbb{R}^d} e^{-(1-\lambda)(L\psi(y)-(z_1,y))-\lambda(L\psi(y)-(z_2,y))} \, dy
\leq \left( \int_{\mathbb{R}^d} e^{-L\psi(y)-(z_1,y)} \, dy \right)^{1-\lambda} \left( \int_{\mathbb{R}^d} e^{-L\psi(y)-(z_2,y)} \, dy \right)^{\lambda} = \Phi(z_1)^{1-\lambda}\Phi(z_2)^{\lambda}.
\]
So, the function \( \Phi \) is log-convex. Consequently, it is convex and the desired func-
tional, which is reciprocal of \( \Phi \), is log-concave.

\( \square \)

The next corollary is another consequence of Theorem 3.

Corollary 3.1. Let \( f : \mathbb{R}^d \to [0, \infty) \) be a proper log-concave function containing the
origin in the interior of its support. Then
\[
\int f \cdot \int L_{\infty} f = \lim_{s \to +\infty} s^d \int f_s \cdot \int L_s f_s.
\]

3.2. Proof of Theorem 3. To prove Theorem 3, we recall several definitions and
facts of convex analysis which can be found in e.g., [45].

We start with the definition of the classical convex conjugate transform or Legendre
transform \( L \) defined for functions \( \varphi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\} \) by
\[
(L \varphi)(y) = \sup_{x \in \mathbb{R}^d} \{ \langle x, y \rangle - \varphi(x) \}.
\]
Thus, for \( f = e^{-\psi} : \mathbb{R}^d \to [0, \infty) \),
\[
L_{\infty} f(y) = e^{-(L \psi)(y)}.
\]
A vector \( p \) is said to be a subgradient of a convex function \( \psi \) on \( \mathbb{R}^d \) at the point \( x \) if
\[
\psi(y) \geq \psi(x) + \langle p, y - x \rangle
\]
for all \( y \in \mathbb{R}^d \). The set of all subgradients of \( \psi \) at \( x \) is called the subdifferential of \( \psi \) at \( x \) and is denoted by \( \partial \psi(x) \).
The effective domain \( \text{dom} \psi \) of a convex function \( \psi \) on \( \mathbb{R}^d \) is the set
\[
\text{dom} \psi = \{ x : \psi(x) < +\infty \}.
\]
The epigraph of a convex function $\psi$ on $\mathbb{R}^d$ is the set
\[ \{(x, \xi) : x \in \text{dom} \psi, \xi \in \mathbb{R}, \xi \geq \psi(x)\} . \]

In the remainder of the paper we work with convex functions that have non-empty effective domain and closed epigraph.

The following statement is a direct consequence of the definition of subdifferential.

**Lemma 3.2 (Geometric meaning of subdifferential).** Let $\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous, convex function with non-empty effective domain. Let $x \in \text{dom} \varphi$. If $p \in \mathbb{R}^d$ and a negative number $\xi$ are such that
\[ \langle (p, \xi), (y, \varphi(y)) \rangle \leq \langle (p, \xi), (x, \varphi(x)) \rangle \]
for all $y \in \text{dom} \psi$, then there are a subgradient $q$ of $\varphi$ at $x$ and a positive constant $\alpha$ such that
\[ (p, \xi) = \alpha (q, -1). \]

**Remark 3.1.** The assertion of Lemma 3.2 can be rephrased as: $(p, \xi)$ belongs to the normal cone to the epigraph of $\psi$ at point $(x, \psi(x))$.

The following facts on the subgradient can be found in e.g., [45, Chapter 23]. See also [18].

Let $\psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous convex function with non-empty effective domain, and let $z$ be in the interior of $\text{dom} L\psi$. Let $\varepsilon > 0$ be such that $z + \varepsilon B^d_2 \subset \text{dom} L\psi$. Then $L\psi$ is Lipschitz on $z + \varepsilon B^d_2$ and, denoting the Lipschitz constant by $C$, the following statements hold.

1. **Fact 1:** $\partial L\psi(z)$ is a non-empty compact convex set and $\partial L\psi(z) \subset CB^d_2$.
2. **Fact 2:** Let $q \in \partial L\psi(z)$. Then $z \in \partial \psi(q)$, and
\[ \psi(q) + L\psi(z) = \langle q, z \rangle \]  \hspace{1cm} (3.3)
3. **Fact 3:** Let $q \in \partial L\psi(z)$. Then, by the previous assertion,
\[ |\psi(q)| \leq |L\psi(z)| + C |z| . \]  \hspace{1cm} (3.4)
4. **Fact 4:** Let $q \in \partial L\psi(z)$. Then for all $p \in \mathbb{R}^d$,
\[ \langle z, p \rangle - \psi(p) \leq L\psi(z) . \]  \hspace{1cm} (3.5)

To prove Theorem 3, we will also need the following lemmas.

**Lemma 3.3.** Let $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function. Let $z$ be a point in the interior of $\text{supp} L_{\infty} f$. Let $C$ be the Lipschitz constant of the convex function $L\psi = -\log L_{\infty} f$ on some open neighborhood of $z$. Then for any $s > |L\psi(z)| + C |z|$,\[ L_s f_s \left( \frac{z}{L\psi(z) + s} \right) = \left( 1 + \frac{L\psi(z)}{s} \right)^{-s}. \]  \hspace{1cm} (3.6)
GEOMETRIC REPRESENTATION OF CLASSES OF CONCAVE FUNCTIONS AND DUALITY

Let \( f \) be a proper log-concave function such that the origin is in the interior of \( \text{supp} f \). Denote \( \mathcal{L} \psi = -\log \mathcal{L}\psi \). If \( z \) is in the interior of \( \text{supp} \mathcal{L}_s f_s \), then there exists \( z_s \) in the support of \( \mathcal{L}_s f_s \) such that

\[
\mathcal{L} \psi(z) = \mathcal{L} \psi(z_s) + s \quad \text{and} \quad \mathcal{L}_s f_s(z) = \left(1 + \frac{\mathcal{L} \psi(z_s)}{s}\right)^{-s}.
\]

(3.7)

**Proof.** Let \( (y, f_s^{1/s}(y)) \) be a dual point to the convex set \( \text{(s)Lift} \mathcal{L}_s f_s \) at \( (z, (\mathcal{L}_s f_s)^{1/s}(z)) \), that is

\[
\left\langle (y, f_s^{1/s}(y)), (\tilde{z}, (\mathcal{L}_s f_s)^{1/s}(\tilde{z})) \right\rangle \leq \left\langle (y, f_s^{1/s}(y)), (z, (\mathcal{L}_s f_s)^{1/s}(z)) \right\rangle = 1
\]

(3.8)

for all \( \tilde{z} \in \text{supp} \mathcal{L}_s f_s \) and

\[
\left\langle (\tilde{y}, f_s^{1/s}(\tilde{y})), (z, (\mathcal{L}_s f_s)^{1/s}(z)) \right\rangle \leq \left\langle (y, f_s^{1/s}(y)), (z, (\mathcal{L}_s f_s)^{1/s}(z)) \right\rangle = 1
\]

(3.9)

for all \( \tilde{y} \in \text{supp} f_s \).

Since \( z \) is in the interior of \( \text{supp} \mathcal{L}_s f_s \), inequality (3.8) implies that \( f_s(y) > 0 \). Note that \( y \) might belong to the boundary of the support of \( f_s \). Consider the convex function

\[
\varphi(x) = \begin{cases} -f_s^{1/s}(x), & x \in \text{cl supp} f_s \\ +\infty, & x \notin \text{cl supp} f_s. \end{cases}
\]

Using inequality (3.9) in Lemma 3.2, we conclude that

\[
\left\langle z, (\mathcal{L}_s f_s)^{1/s}(z) \right\rangle = \alpha \left(\frac{z_s}{s}, 1\right),
\]

where \( \alpha > 0 \) and \( \frac{z_s}{s} \) belongs to the subdifferential of the convex function \( -f_s^{1/s} \) at \( y \). Since \( f_s^{1/s}(y) > 0 \) and \( f_s \) is lower semi-continuous, there is an open neighborhood \( U \) of \( y \) such that for all \( \tilde{y} \) that are in \( U \) and in the boundary of \( \text{cl supp} f_s \) we have \( f_s(\tilde{y}) > 0 \). Moreover, since \( \psi(y) < s \) and \( \psi \) is upper semi-continuous, one has that \( \psi(\tilde{y}) = +\infty \) for all \( \tilde{y} \in (y + \varepsilon B^d_{\tilde{y}}) \cap (\mathbb{R}^d \setminus \text{supp} f_s) \) for some \( \varepsilon > 0 \). That is,
the function \( \varphi \) coincides with \(-1 + \frac{\psi(x)}{s}\) on some open neighborhood of \( y \). Hence, \( z_s \in \partial \psi(y) \). Therefore, by (3.3),

\[
\left\langle \left( y, f_s^{1/s}(y) \right), \left( \frac{z_s}{s}, 1 \right) \right\rangle = 1 + \frac{(z_s, y) - \psi(y)}{s} = 1 + \frac{\mathcal{L}_s \psi(z_s)}{s} < +\infty,
\]

and the identities (3.7) hold.

The following corollary is an immediate consequence of Lemma 3.4.

**Corollary 3.2.** Let \( f : \mathbb{R}^d \to [0, \infty) \) be a proper log-concave function containing the origin in it’s support. Denote \( \mathcal{L}_\psi = -\log \mathcal{L}_\infty f \) and \( M = \min_{\mathbb{R}^d} \mathcal{L}_\psi \). Then for any \( s \) such that the origin is in the interior of \( \text{supp} f_s \) and \( s + M > 0 \), one has

\[
\text{supp} \mathcal{L}_s f_s \subset \frac{1}{M + s} \text{cl } \text{supp} \mathcal{L}_\infty f.
\]

We are now ready for the proof of Theorem 3.

**Proof of Theorem 3.** We put \( \psi = -\log f \). Then \( \mathcal{L}_\psi = -\log \mathcal{L}_\infty f \).

Corollary 3.2 shows that it suffices to consider points in the interior of the support of \( \mathcal{L}_\infty f \).

Let therefore \( x \) be a point in the interior of the support of \( \mathcal{L}_\infty f \). That is, \( x \) is an interior point of \( \text{dom} \mathcal{L}_\psi \). Let \( \varepsilon_1 \) be such that \( x + \varepsilon_1 B_2 \subset \text{supp} \mathcal{L}_\infty f \). As noted above, \( \mathcal{L}_\infty f \) is then Lipschitz on some open neighborhood in \( x + \varepsilon_1 B_2 \) with Lipschitz constant \( C \). By Lemma 3.3, we have for all sufficiently large \( s \),

\[
\left\{ \left( \frac{z}{\mathcal{L}_\psi(z) + s} \right) : z \in x + \varepsilon_1 B_2 \right\} \subset \text{supp} \mathcal{L}_s f_s.
\]

Denote by \( \ell \) the line passing through the origin and \( x \). If \( x \) and the origin coincide, then \( \ell \) is an arbitrary linear one-dimensional subspace of \( \mathbb{R}^d \). By continuity, for any \( 0 < \varepsilon_2 < 1 \) there exists \( s_0 \) such that

\[
x \in \left\{ \frac{s z}{\mathcal{L}_\psi(z) + s} : z \in \ell, \ |z - x| < \varepsilon_2 \right\}
\]

for all \( s > s_0 \). It follows that for all sufficiently large \( s \) there exists \( z_s \) satisfying

1. \( \frac{s z_s}{\mathcal{L}_\psi(z_s) + s} = x \)
2. \( z_s \to x \) as \( s \to \infty \).

This, continuity and identity (3.6) yield

\[
\lim_{s \to \infty} \mathcal{L}_s f_s \left( \frac{x}{s} \right) = \lim_{s \to \infty} \mathcal{L}_s f_s \left( \frac{z_s}{\mathcal{L}_\psi(z_s) + s} \right) = \lim_{s \to +\infty} \left( 1 + \frac{\mathcal{L}_\psi(z_s)}{s} \right)^{-s} = \mathcal{L}_\infty f (x).
\]

Since all the functions considered are continuous at any point of the interior of the support, we conclude that \( f_s \) converges locally uniformly to \( f \) on \( A \).

This finishes the proof of Theorem 3. \( \square \)
4. A Blaschke–Santaló inequality

In this section, we prove the following theorem, which implies Theorem 4. Recall also that

\[(s)\kappa_{d+1} = \int_{B_2^d} \left[1 - |x|^2\right]^{s/2} \, dx. \tag{4.1}\]

**Theorem 5.** Let \( f : \mathbb{R}^d \to [0, \infty) \) be \( 1/s \)-concave function with finite integral. Let \( H \) be an affine hyperplane with half-spaces \( H_+ \) and \( H_- \) and such that \( \lambda \int_{\mathbb{R}^d} f = \int_{H_+} f \) for some \( \lambda \in (0, 1) \). Then there exists \( z \in H \) such that for any Borel function \( g : \mathbb{R}^d \to [0, \infty) \) satisfying

\[ \forall x, y \in \mathbb{R}^d, \quad f(x + z)g(y) \leq \left(1 - \langle x, y \rangle\right)^s \]

the inequality

\[ \int_{\mathbb{R}^d} f \int_{\mathbb{R}^d} g \leq \frac{(s)\kappa_{d+1}^2}{4\lambda(1 - \lambda)}. \tag{4.2} \]

holds.

The idea of our proof mostly follows Lehec’s arguments [31] – namely, we prove it by induction on the dimension. However, in our setting the one dimensional case requires a more subtle analysis of the Lebesgue level sets of the functions than in the case of log-concave functions.

4.1. The one-dimensional case. The following lemma is a particular case of Proposition 1 of [22].

**Lemma 4.1.** Let \( \varphi_1 : [0, \infty) \to [0, \infty) \) and \( \varphi_2 : [0, \infty) \to [0, \infty) \) be two Borel functions satisfying the duality relation

\[ \text{for all } t_1, t_2 \in [0, \infty), \quad \varphi_1(t_1)\varphi_2(t_2) \leq (1 - t_1t_2)^s. \tag{4.3} \]

Then

\[ \int_{[0, \infty)} \varphi_1 \int_{[0, \infty)} \varphi_2 \leq \left(\frac{(s)\kappa_2}{2}\right)^2. \tag{4.4} \]

**Proof.** Define \( \varphi_3(t) = (1 - t^2)^s; \) for \( i = 1, 2, 3 \), define \( g_i(t) = \varphi_i(e^t)e^t \). On the one hand, \( \int_{\mathbb{R}} g_i = \int_{[0, \infty)} \varphi_i \). On the other hand, inequality (4.3) takes the following form:

\[ g_1(t_1)g_2(t_2) \leq g_3\left(\frac{t_1 + t_2}{2}\right). \]

The desired inequality follows now from the Prékopa–Leindler inequality [43], see also [24, 47]. \( \square \)

**Corollary 4.1.** Let \( f : \mathbb{R} \to [0, \infty) \) be a Borel function such that \( \int_{\mathbb{R}} f \, dt < \infty \), and let \( \int_{[0, \infty)} f = \lambda \int_{\mathbb{R}} f \) for some \( \lambda \in (0, 1) \). Then for any Borel function \( g : \mathbb{R} \to [0, \infty) \) satisfying

\[ \forall t_1, t_2 \in \mathbb{R}, \quad f(t_1)g(t_2) \leq \left(1 - \langle t_1, t_2 \rangle\right)^s \]

the inequality

\[ \int_{\mathbb{R}} f \int_{\mathbb{R}} g \leq \frac{(s)\kappa_2^2}{4\lambda(1 - \lambda)}. \tag{4.5} \]
holds.

Proof. We use Lemma 4.1 twice. First with \( \varphi_1(t) = f(t) \) and \( \varphi_2(t) = g(t) \) and then with \( \varphi_1(t) = f(-t) \) and \( \varphi_2(t) = g(-t) \). In both cases, the condition

\[
\varphi_1(t_1) \varphi(t_2) \leq (1 - t_1 t_2)^s
\]

for all \( t_1, t_2 \in [0, \infty) \) is satisfied. Therefore,

\[
\left( \frac{\kappa_2}{2} \right)^2 \geq \int_{[0, \infty)} f(t) \, dt \int_{[0, \infty)} g(t) \, dt
\]

and

\[
\left( \frac{\kappa_2}{2} \right)^2 \geq \int_{(-\infty, 0]} f(t) \, dt \int_{(-\infty, 0]} g(t) \, dt.
\]

By the assumption \( \int_{[0, \infty)} f = \lambda \int_{\mathbb{R}} f \), we get

\[
\left( \frac{\kappa_2}{2} \right)^2 \geq \lambda \int_{\mathbb{R}} f(t) \, dt \int_{[0, \infty)} g(t) \, dt
\]

and

\[
\left( \frac{\kappa_2}{2} \right)^2 \geq (1 - \lambda) \int_{\mathbb{R}} f(t) \, dt \int_{(-\infty, 0]} g(t) \, dt.
\]

Summing these inequalities, we obtain

\[
\left( \frac{\kappa_2}{2} \right)^2 \frac{1}{\lambda(1 - \lambda)} = \left( \frac{\kappa_2}{2} \right)^2 \left( \frac{1}{\lambda} + \frac{1}{(1 - \lambda)} \right) \geq \int_{\mathbb{R}} f \int_{\mathbb{R}} g.
\]

\[ \square \]

4.2. Induction on the dimension.

Proof of Theorem 5. We prove the theorem by induction on the dimension. The one-dimensional case was proved in Corollary 4.1.

Assume now that the theorem is true in dimension \( d - 1 \). Let \( b_+ = \int_{H_+} x f(x) \, dx \) and \( b_- = \int_{H_-} x f(x) \, dx \), that is, \( b_\pm \) is a scalar multiple of the barycenter of the restriction of the measure with density \( f \) on \( H_\pm \), respectively. Since \( f \) is not concentrated on \( H \), the point \( b_\pm \) belongs to the interior of \( H_\pm \), and similarly for \( b_- \). Hence the line passing through \( b_\pm \) and \( b_- \) intersects \( H \) at one point, which we call \( z \). We will show that \( z \) satisfies (4.5), for all functions \( g \) that satisfy the assumption.

Clearly, replacing \( f \) by \( \text{Shift} [f, -z] \) and \( H \) by \( H - z \), we can assume that \( z = 0 \). Let \( g \) be such that

\[
\forall x, y \in \mathbb{R}^d, \quad f(x)g(y) \leq (1 - \langle x, y \rangle)_+^s.
\]

(4.6)

Let \( e_1, \ldots, e_d \) be an orthonormal basis of \( \mathbb{R}^d \) such that \( H = e_1^\perp \) and \( \langle b_+, e_d \rangle > 0 \). Let \( v = b_+/\langle b_+, e_d \rangle \) and let \( A \) be the linear operator on \( \mathbb{R}^d \) that maps \( e_d \) to \( v \) and \( e_i \) to itself, for \( i = 1 \ldots d - 1 \). Let \( B = (A^{-1})^t \). Define

\[
F_+ : H \rightarrow \mathbb{R}_+ \quad \text{by} \quad y_1 \mapsto \int_{\mathbb{R}_+} f(y_1 + tv) \, dt
\]
The function \( F \) the barycenter of \( \tau \) attains the minimum at the origin, applying Theorem 4.6.

Returning to (4.6), we have

\[
\langle y_1 + t_1v, B y_2 + t_2 e_d \rangle = \langle y_1, y_2 \rangle + t_1 t_2 \text{ for all } t_1, t_2 \in \mathbb{R} \text{ and } y_1, y_2 \in H.
\]

Let \( t_1, t_2 > 0 \) and assume that \( \langle y_1, y_2 \rangle < 1 \). Set \( c = \sqrt{1 - \langle y_1, y_2 \rangle} \). Then, using (4.8) with \( \tau_i = t_i/c \), we get

\[
f(y_1 + cv \cdot \tau_1)g(B y_2 + ce_d \cdot \tau_2) \leq (1 - \tau_1 \tau_2)^s.
\]

By Lemma 4.1, we get that

\[
\frac{1}{c^2s} \int_{[0, \infty)} f(y_1 + cv \cdot \tau_1) d\tau_1 \int_{[0, \infty)} g(B y_2 + ce_d \cdot \tau_2) d\tau_2 \leq \left( \frac{(s) \kappa_2}{2} \right)^2.
\]

Returning to \( t_1 \) and \( t_2 \) in the integrals, one has

\[
F_+(y_1) G_+(y_2) \leq \left( \frac{(s) \kappa_2}{2} \right)^2 (1 - \langle y_1, y_2 \rangle)^{s+1}
\]

for all \( y_1, y_2 \) in \( H \) satisfying \( \langle y_1, y_2 \rangle < 1 \).

If \( \langle y_1, y_2 \rangle \geq 1 \), inequality (4.8) implies that \( F_+(y_1) G_+(y_2) = 0 \). Thus,

\[
F_+(y_1) G_+(y_2) \leq \left( \frac{(s) \kappa_2}{2} \right)^2 (1 - \langle y_1, y_2 \rangle)^{s+1}.
\]

The function \( F_+ \) is a function with finite integral on the \((d - 1)\)-dimensional space \( H \), and it is \( \frac{1}{1+s} \)-concave by the Borell–Brascamp–Lieb inequality [15]. Thus, by induction assumption, there exists \( v \in H \) such that

\[
\int_H F_+ \int_H \mathcal{L}_{s+1}(\text{Shift} [F_+, v]) \leq (s+1) \kappa_d.
\]

Such a \( v \) can be found in any hyperplane inside \( H \) bisecting \( F_+ \). Since the origin is the barycenter of \( F_+ \), we see that the integral

\[
\int_H \mathcal{L}_{s+1}(\text{Shift} [\mathcal{L}_{s+1} F_+, q])
\]

attains the minimum at the origin, applying Theorem 1 to \( \mathcal{L}_{s+1} F_+ \). Using again the induction assumption, we can assume that \( v = 0 \) in the previous inequality.
Inequality (4.9) yields that $G_+$ is pointwise less or equal to $\left(\frac{(s)K_2}{2}\right)^2 L_{s+1}(F_+)$. Hence, we get
\[
\int_H F_+ \int_H G_+ \leq \left(\frac{(s)K_2}{2}\right)^2 \int_H F_+ \int_H L_{s+1}(F_+) \leq \left(\frac{(s)K_2}{2}\right)^2 \left(\frac{(s+1)\kappa_d}{2}\right)^2.
\]
However, by direct computation,
\[
(s)\kappa_{d+1} = (s)\mu(B^{d+1}) = \pi^{d/2} \frac{\Gamma(s/2 + 1)}{\Gamma(s/2 + d/2 + 1)}
\]
where $\Gamma(\cdot)$ is the Euler Gamma function, and thus
\[
\left(\frac{(s)K_2}{2}\right)^2 \left(\frac{(s+1)\kappa_d}{2}\right)^2 = \left(\frac{(s)\kappa_{d+1}}{2}\right)^2.
\]
Hence, also using (4.7),
\[
\frac{(s)\kappa_{d+1}}{2} \geq \int_H F_+ \int_H G_+ = \lambda \int_{\mathbb{R}^d} f \int_{H_+} g(Bx) \, dx = \lambda \int_{\mathbb{R}^d} f \int_{H_+} g.
\]
Similarly,
\[
\frac{(s)\kappa_{d+1}}{2} \geq (1 - \lambda) \int_{\mathbb{R}^d} f \int_{H_-} g.
\]
Therefore,
\[
\int_{\mathbb{R}^d} f \int_{\mathbb{R}^d} g = \int_{\mathbb{R}^d} f \left(\int_{H_+} g + \int_{H_-} g\right) \leq \left(\frac{(s)\kappa_{d+1}}{2}\right)^2 \left(\frac{1}{\lambda} + \frac{1}{1 - \lambda}\right) = \left(\frac{(s)\kappa_{d+1}}{2}\right)^2 \frac{1}{\lambda(1 - \lambda)}.
\]
This completes the proof of Theorem 5. \qed

5. Santaló s-regions

In this section, we will define Santaló regions for functions and discuss possible approaches to define Santaló functions.

For a convex body $K$, its Santaló region $S(K, t)$ with parameter $t$ was introduced and studied in [40]. It is defined as
\[
S(K, t) = \left\{ x \in K : \operatorname{vol}_d K \cdot \operatorname{vol}_d \text{Shift}[K, x] \leq t \left(\operatorname{vol}_d B^d\right)^2 \right\}. \tag{5.1}
\]
Note that the Santaló region approximates the initial set as $t \to \infty$. More importantly, it was shown in [40, Theorem 10] that the affine surface area of the initial convex body can be computed as a limit as $t \to \infty$ of the volume difference of the convex body and its Santaló regions.

Affine surface area was first introduced by Blaschke [11] for dimensions 2 and 3 and for smooth enough convex bodies as an integral over the boundary $\partial K$ of a power of the Gauss curvature $\kappa_K$,
\[
as(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} \, d\mu(x).
\]
Integration is with respect to the usual surface area measure μ on ∂K. It was successively extend within the last decades. Aside from the afore mentioned successful approach using the Santaló region, there are other successful approaches via the (convex) floating body resp. the illumination body in [48, 52] or in e.g., [36, 27], and they all coincide. Such extensions are desirable as the affine surface area is one of the most powerful tools in convex and differential geometry. It proved to be fundamental in the solution of the affine Plateau problem by Trudinger and Wang [50, 51], in the theory of valuations where the affine and centro-affine surface areas have been characterized by Ludwig and Reitzner [35] and Haberl and Parapatits [26] as unique valuations satisfying certain invariance properties. Affine surface area appears naturally in the approximation of general convex bodies by polytopes, e.g., [12, 44, 49]. Furthermore, there are connections to e.g., PDEs and ODEs and concentration of volume (e.g., [21, 37]), information theory (e.g., [6, 16, 41, 53]) and in a spherical and hyperbolic setting [9, 10].

It would be extremely interesting to develop an analogue of affine surface area in the functional setting. There are several approaches already how to define it for log-concave functions [6, 16, 17, 33], but those do not all coincide. We think that an approach via a Santaló function will help not only to clarify this point, but also can reveal new properties and inequalities related to log-concave functions and their integrals. Towards this goal, we define Santaló functions for 1/s-concave functions in the next subsection.

First we define, following (5.1), the Santaló s-region for a fixed positive s by \( S_{\text{reg}}(f, s, t) \) of a non-negative Borel function \( f \) on \( \mathbb{R}^d \) by

\[
S_{\text{reg}}(f, s, t) = \left\{ x \in \co \text{supp} f : \int_{\mathbb{R}^d} f \cdot \int_{\mathbb{R}^d} \mathcal{L}_s (\text{Shift}[f, x]) \leq t \left( \kappa_d \right)^{1/(s-1)} \right\},
\]

where \( \co A \) denotes the convex hull of a set \( A \). In the limit case \( s = \infty \) we define the Santaló \( \infty \)-region by

\[
S_{\text{reg}}(f, \infty, t) = \left\{ x \in \co \text{supp} f : \int_{\mathbb{R}^d} f \cdot \int_{\mathbb{R}^d} \mathcal{L}_\infty (\text{Shift}[f, x]) \leq t \cdot (2\pi)^{d/2} \right\}.
\]

We summarize the properties of the Santaló regions in the next lemmas. They follow from Lemmas 1.1 and 2.1 and Theorems 2-5.

**Lemma 5.1.** Let \( s \) be a fixed positive real number. Let \( f \) be a non-negative function on \( \mathbb{R}^d \) such that \( \mathcal{L}_s \mathcal{L}_s f \) has positive integral. Then \( S_{\text{reg}}(f, s, t) \)

1. is non-empty if \( t \geq 1 \), and has non-empty interior if \( t > 1 \),
2. is a convex set if it is non-empty,
3. is strictly convex if it has non-empty interior,
4. has \( C^\infty \)-smooth boundary if it has non-empty interior.

**Lemma 5.2.** Let \( f : \mathbb{R}^d \to [0, \infty) \) be a proper log-concave function. If \( S_{\text{reg}}(f, \infty, t) \) is non-empty, then

\[
S_{\text{reg}}(f, s, t) \to S_{\text{reg}}(f, \infty, t)
\]

in the Hausdorff metric as \( s \to \infty \).
5.1. Marginals of convex sets. Let $s$ be the reciprocal of a positive integer. Then, as we already discussed, any $1/s$-concave function on $\mathbb{R}^d$ is the marginal of a convex set in $\mathbb{R}^{d+s}$. In particular, one can lift a $1/s$-concave function $f : \mathbb{R}^d \rightarrow [0, \infty)$ into $\mathbb{R}^{d+s}$ as follows [3],

$$K_s(f) = \left\{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^s : x \in \text{cl supp } f, \ |y| \leq \left( \frac{f(x)}{\text{vol}_s B^s} \right)^{1/s} \right\}. \quad (5.2)$$

Clearly,

$$\text{vol}_{d+s} K_s(f) = \int_{\text{supp } f} \text{vol}_s B^s \left( \frac{f(x)}{\text{vol}_s B^s} \right) \, dx = \int_{\mathbb{R}^d} f. \quad (5.3)$$

From [3, Lemma 3.1] follows that for any $z$ in the interior of the support of $f$, one has

$$\text{vol}_{d+s} \text{Shift} [K_s(f), z] = (\text{vol}_s B^s)^2 \int_{\mathbb{R}^d} L_s(\text{Shift} [f, z]),$$

and therefore,

$$\int_{\mathbb{R}^d} f \int_{\mathbb{R}^s} L_s(\text{Shift} [f, z]) = \frac{\text{vol}_{d+s} K_s(f) \text{vol}_{d+s} \text{Shift} [K_s(f), z]}{(\text{vol}_s B^s)^2}. \quad (5.4)$$

On the other hand, $\text{vol}_{d+s} B^{d+s} = (s)^{K_{d+1}} \text{vol}_s B^s$.

We then define the Santaló $m$-function $S_m(f, s, t)$ of a $1/s$-concave function $f$ with integer $s$ to be such that

$$K_s(S_m(f, s, t)) = S(K_s(f), t). \quad (5.5)$$

It follows that the Santaló $m$-function $S_m(f, s, t)$ of a $1/s$-concave function $f$ is $1/s$-concave. Moreover, the following identity holds.

**Proposition 5.1.** Let $s \in \mathbb{N}$, and let $f : \mathbb{R}^d \rightarrow [0, \infty)$ be a $1/s$-concave function of finite integral. Then

$$\lim_{t \rightarrow \infty} \frac{\int_{\mathbb{R}^d} f - \int_{\mathbb{R}^d} S_m(f, s, t)}{t^{\frac{1}{s(d+s+1)}}} = \frac{s}{2} \left( \frac{\text{vol}_s B^s}{\text{vol}_{d+s} B^{d+s}} \right)^{2} \int_{\mathbb{R}^d} \left| \text{Hess} \left( f^{\frac{1}{s}} \right) \right| \frac{1}{t^{\frac{1}{s(d+s+1)}}} f^{\left( \frac{s-1}{s(d+s+1)} \right)}(u),$$

where $\text{Hess} \left( f^{\frac{1}{s}} \right)$ is the Hessian of $f^{\frac{1}{s}}$.

**Proof.** The proof follows immediately from Theorem 10 of [40] and Proposition 6 of [6]. \qed

It remains to find a reasonable definition of a Santaló function for a log-concave function. The natural first approach $\lim_{t \rightarrow \infty} S_m(f, s, t)$ does not lead to anything meaningful. Lemma 2.1 and Theorem 1 provide another possible approach.

For $s > 0$, we define the set

$$\left\{ y \in (s) \text{Lift } f : \frac{s}{2(d+s)} \int_{S^d} |\langle e_{d+1}, u \rangle|^{s-1} d\sigma(u) \cdot \int_{\mathbb{R}^d} f \leq t \left( (s)^K_{d+1} \right)^2 \right\}.$$
Lemma 2.1 shows that \( (s)S_p(f, s, t) \) is a convex \( d \)-symmetric set, and hence, it is the \( s \)-lifting of a \( 1/s \)-concave function which we denote \( S_p(f, s, t) \). The function \( S_p(f, s, t) \) seems a good candidate for a Santaló function and we investigate this in a forthcoming paper.

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