REPRESENTATIONS OF THE KAUFFMAN SKEIN ALGEBRA I: INVARIANTS AND MIRACULOUS CANCELLATIONS

FRANCIS BONAHON AND HELEN WONG

ABSTRACT. We study finite-dimensional representations of the Kauffman skein algebra of a surface $S$. In particular, we construct invariants of such irreducible representations when the underlying parameter $q = e^{2\pi i \hbar}$ is a root of unity. The main one of these invariants is a point in the character variety consisting of group homomorphisms from the fundamental group $\pi_1(S)$ to $\text{SL}_2(\mathbb{C})$, or in a twisted version of this character variety. The proof relies on certain miraculous cancellations that occur for the quantum trace homomorphism constructed by the authors. These miraculous cancellations also play a fundamental role in subsequent work of the authors, where novel examples of representations of the skein algebra are constructed.

For an oriented surface $S$ of finite topological type and for a Lie group $G$, many areas of mathematics involve the character variety

$$\mathcal{R}_G(S) = \{\text{group homomorphisms } \pi_1(S) \to G\}/G,$$

where $G$ acts on homomorphisms by conjugation. For $G = \text{SL}_2(\mathbb{C})$, Turaev [Tu] showed that the corresponding character variety $\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$ can be quantized by the Kauffman skein algebra of the surface; see also [BuFK1, BuFK2, PrS]. In fact, if one follows the physical tradition that a quantization of a space $X$ replaces the commutative algebra of functions on $X$ by a non-commutative algebra of operators on a Hilbert space, the actual quantization of the character variety $\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$ should be a representation of the Kauffman skein algebra.

This article studies finite-dimensional representations of the skein algebra of a surface. The Kauffman skein algebra $S^A(S)$ depends on a parameter $A = e^{\pi i \hbar} \in \mathbb{C} - \{0\}$, and is defined as follows: one first considers the vector space freely generated by all isotopy classes of framed links in the thickened surface $S \times [0, 1]$, and then one takes the quotient of this space by the skein relation that

$$[K_1] = A^{-1}[K_0] + A[K_{\infty}]$$

whenever the three links $K_1$, $K_0$ and $K_{\infty} \subset S \times [0, 1]$ differ only in a little ball where they are as represented on Figure 1. The algebra multiplication is defined by superposition of skeins. See [2] for details.

Our goal is to study representations of the skein algebra, namely algebra homomorphisms $\rho: S^A(S) \to \text{End}(V)$ where $V$ is a finite-dimensional vector space over $\mathbb{C}$. See [BoW2] for an interpretation of such representations as generalizations of the Kauffman bracket invariant of framed links in $\mathbb{R}^3$. When $A$ is a root of unity, a well-known example of a finite-dimensional representation of the skein algebra

\[\text{Date: May 1, 2014.}\]
\[\text{This research was partially supported by grants DMS-0604866, DMS-1105402 and DMS-1105692 from the National Science Foundation, and by a mentoring grant from the Association for Women in Mathematics.}\]
Figure 1. A Kauffman triple

$S_A^A(S)$ arises from the Witten-Reshetikhin-Turaev topological quantum field theory associated to the fundamental representation of the quantum group $U_q(sl_2)$ [Wit, ReT, BHMV, Tu].

To analyze such representations, we need to find a way to distinguish them. We thus introduce invariants for these representations.

We focus attention on the case where $A^2$ is a root of unity, and more precisely where $A^2$ is a primitive $N$-root of unity with $N$ odd. While the restriction to roots of unity is natural, the parity condition for $N$ is forced on us by the mathematics.

The main invariant comes from Chebyshev polynomials of the first kind. The $n$-th normalized Chebyshev polynomial of the first kind is the polynomial $T_n(x)$ determined by the trigonometric identity that $2 \cos n \theta = T_n(2 \cos \theta)$. More precisely, if $A^2$ is a primitive $N$-root of unity with $N$ odd, we consider the Chebyshev polynomial $T_N$. We separate the discussion into two cases, depending on whether $A^N = +1$ or $A^N = -1$.

**Theorem 1.** Suppose that $A^2$ is a primitive $N$-root of unity with $N$ odd, and that $A^N = -1$. Then, for every finite-dimensional irreducible representation $\rho: S_A(S) \to \text{End}(V)$ of the skein algebra, there exists a unique character $r_\rho \in \mathbb{R}\text{SL}_2(\mathbb{C})(S)$ such that

$$T_N(\rho([K])) = - (\text{Tr} \ r_\rho(K)) \text{Id}_V$$

for every framed knot $K \subset S \times [0,1]$ whose projection to $S$ has no crossing and whose framing is vertical. (Here, $\text{Tr} \ r_\rho(K) \in \mathbb{C}$ denotes the trace of $r_\rho(K) \in \text{SL}_2(\mathbb{C})$.)

In particular, interpreting the elements of $\text{End}(V)$ as matrices, many “miraculous cancellations” occur in the entries of $T_N(\rho([K]))$ when one evaluates the Chebyshev polynomial $T_N$ over the endomorphism $\rho([K]) \in \text{End}(V)$. A more general version of Theorem 1 and of these miraculous cancellations, valid for all framed links in $S \times [0,1]$, is provided by Theorem 16 in §5.1 below.

In general, researchers working on the Kauffman skein algebra (or on the representation theory of the quantum group $U_q(sl_2)$) are more familiar with the normalized Chebyshev polynomials of the second kind, defined as the polynomials $S_n(x)$ such that $\sin((n + 1)\theta) = \sin \theta \ S_n(2 \cos \theta)$. The occurrence of the other Chebyshev polynomials $T_n(x)$ is here somewhat surprising.

There is a companion statement to Theorem 1 when $A^N = +1$, except that it now involves a twisted product $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}^\text{Spin}(S) = \mathcal{R}_{\text{PSL}_2(\mathbb{C})}^0(S) \times \text{Spin}(S)$ of a component of the character variety $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S)$ with the set $\text{Spin}(S)$ of isotopy classes of spin structures on $S$. This twisted character variety $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}^\text{Spin}(S)$ is more natural than would appear at first glance; for instance, the monodromy of a hyperbolic metric on $S \times (0,1)$ determines an element of $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}^\text{Spin}(S)$. See §5.1 for details.

---

1However, see [FrG] for an earlier occurrence, as well as [HAP] for a related but different context.
The definition is designed so that, for every twisted character $r \in \mathcal{R}^{\text{Spin}}_{\text{PSL}_2(\mathbb{C})}(S)$ and for every framed knot $K \subset S \times [0, 1]$, there is a well-defined trace $\text{Tr} r(K) \in \mathbb{C}$.

**Theorem 2.** Suppose that $A^2$ is a primitive $N$–root of unity with $N$ odd, and that $A^N = +1$. Then, for every finite-dimensional irreducible representation $\rho: \mathcal{S}^A(S) \to \text{End}(V)$ of the skein algebra, there exists a unique twisted character $r_\rho \in \mathcal{R}^{\text{Spin}}_{\text{PSL}_2(\mathbb{C})}(S)$ such that

$$T_N(\rho([K])) = -(\text{Tr} r_\rho(K)) \text{Id}_V$$

for every framed knot $K \subset S \times [0, 1]$ whose projection to $S$ has no crossing and with vertical framing.

Again, a more general version of Theorem 2 for all framed links in $S \times [0, 1]$ is provided by Theorem 10 in [BoW4].

For a finite-dimensional irreducible representation $\rho: \mathcal{S}^A(S) \to \text{End}(V)$, Theorems 1 and 2 both associate to this quantum object a point in the character variety $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\text{Spin}}_{\text{PSL}_2(\mathbb{C})}(S)$, which in particular is a classical (= non-quantum) geometric object. We call $r_\rho \in \mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\text{Spin}}_{\text{PSL}_2(\mathbb{C})}(S)$ the classical shadow of the representation $\rho: \mathcal{S}^A(S) \to \text{End}(V)$.

For instance, when $A^N = -1$, we can consider the finite-dimensional representation $\rho_{\text{WRT}}: \mathcal{S}^A(S) \to \text{End}(V)$ provided by the SO(3) version of the Witten-Reshetikhin-Turaev topological quantum field theory [ReT, BMV, Tu]. An easy variation [BoW2] of the arguments of [Rob] shows that this representation $\rho_{\text{WRT}}$ is irreducible. In [BoW3], we show that its classical shadow in $\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$ is the character corresponding to the trivial homomorphism $\pi_1(S) \to \{\text{Id}\} \subset \text{SL}_2(\mathbb{C})$.

The key to the proof of Theorems 1 and 2 is that, for every knot $K \subset S \times [0, 1]$ with no crossing and with vertical framing, the evaluation $T_N([K])$ of the Chebyshev polynomial $T_N$ at the element $[K] \in \mathcal{S}^A(S)$ is central in $\mathcal{S}^A(S)$.

When the surface $S$ is non-compact, there are central elements that are easier to identify, and provide more invariants. Let $P_k$ be a small simple loop that goes around the $k$–th puncture in $S$. Consider $P_k$ as a knot in $S \times [0, 1]$, and endow it with the vertical framing. It is immediate that $[P_k] \in \mathcal{S}^A(S)$ is central in $\mathcal{S}^A(S)$, and the following result easily follows.

**Proposition 3.** For every finite-dimensional irreducible representation $\rho: \mathcal{S}^A(S) \to \text{End}(V)$, there exists a number $p_k \in \mathbb{C}$ such that $\rho([P_k]) = p_k \text{Id}_V$.

Suppose in addition that $A^2$ is a primitive $N$–root of unity with $N$ odd, and that $r_\rho \in \mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\text{Spin}}_{\text{PSL}_2(\mathbb{C})}(S)$ is the classical shadow associated to $\rho$ by Theorems 1 or 2. Then, $T_N(p_k) = -\text{Tr} r_\rho(P_k)$.

The numbers $p_1$, $p_2$, ..., $p_s$ thus associated to the representation $\rho$ are the **puncture invariants** of $\rho$. The second part of Proposition 3 shows that, up to finitely many choices, these puncture invariants are essentially determined by the classical shadow of $\rho$.

Thus, we have extracted two types of invariants from a finite-dimensional irreducible representation $\rho: \mathcal{S}^A(S) \to \text{End}(V)$ of the skein algebra: the classical shadow of $\rho$ in $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\text{Spin}}_{\text{PSL}_2(\mathbb{C})}(S)$, according to whether $A^N = -1$ or $+1$; the puncture invariants $p_k \in \mathbb{C}$.

The articles [BoW3, BoW2], which are the natural continuation of this one (see also the expository article [BoW4]), provide a converse statement. More precisely,
suppose that $A^2$ is a primitive $N$–root of unity with $N$ odd and that we are given the following data: a character $r \in \mathcal{R}_{\text{SL}_2(C)}(S)$ if $A^N = -1$, or a twisted character $r \in \mathcal{R}_{\text{PSL}_2(C)}(S)$ if $A^N = +1$; a number $p_k \in \mathbb{C}$ for each puncture of $S$ such that $T_N(p_k) = -Tr(r(P_k))$, where $P_k$ is a small loop going around the puncture endowed with the vertical framing. Then \cite{BoW} provide a finite-dimensional irreducible representation $\rho: S^A(S) \to \text{End}(V)$ whose classical shadow is equal to $r$ and whose puncture invariants are equal to the $p_k$.

The construction of irreducible representations in \cite{BoW} uses the quantum trace homomorphism $\text{Tr}_\lambda^\omega$ constructed in \cite{BoW1}, and defined when the surface $S$ has at least one puncture. (The article \cite{BoW} uses punctured surfaces as a tool to construct representations of the skein algebras of closed surfaces). For such a punctured surface $S$ and for $\omega = A^{-\frac{1}{2}}$, the quantum trace homomorphism $\text{Tr}_\lambda^\omega: S^A(S) \to \mathcal{T}^\omega(\lambda)$ embeds the skein algebra $S^A(S)$ in an incarnation $\mathcal{T}^\omega(\lambda)$ of the quantum Teichmüller space of Chekhov, Fock \cite{Fo, ChF}, and Kasaharv \cite{Kash}, associated to an ideal triangulation $\lambda$ of $S$. In the special case where $A = 1$, the quantum trace $\text{Tr}_\lambda^{\pm 1}([K])$ is just the Laurent polynomial expressing, for $r \in \mathcal{R}_{\text{PSL}_2(C)}(S)$, the trace $Tr(r(K)$ in terms of suitable square roots $z_i = \sqrt{x_i}$ of the shear-bend coordinates of $r$ with respect to $\lambda$ (if these exist); a similar property holds when $A = -1$, connecting $\text{Tr}_\lambda^{\pm 1}([K])$ to the classical trace $\text{Tr}(r(K)$ for $r \in \mathcal{R}_{\text{SL}_2(C)}(S)$. See \cite{BoW} §11.

The algebraic structure of the quantum Teichmüller space $\mathcal{T}^\omega(\lambda)$ is relatively simple, and its finite-dimensional irreducible representations are easily classified \cite{BoL}. Composing these representations $\mathcal{T}^\omega(\lambda) \to \text{End}(V)$ with the quantum trace homomorphism $\text{Tr}_\lambda^\omega: S^A(S) \to \mathcal{T}^\omega(\lambda)$, one obtains many representations $S^A(S) \to \text{End}(V)$.

This motivates the second part of the present article (in addition to the fact that this second part is used to prove the results of the first part). When constructing representations of the skein algebra from representations of the quantum Teichmüller space, the challenge is to control the classical shadow of the representations so constructed. Recall that this classical shadow is defined using the Chebyshev polynomial $T_N$. It turns out that the quantum trace homomorphism is extremely well behaved with respect to $T_N$. This is expressed in the following result, which provides more miraculous cancellations.

**Theorem 4.** Suppose that $A^2$ is a primitive $N$–root of unity with $N$ odd, and set $\varepsilon = A^N = \pm 1$ and $i = \sqrt{-1} \in \{\pm 1, \pm i\}$. Then, for every knot $K \subset S \times [0,1]$ whose projection to $S$ has no crossing and with vertical framing, the element $\text{Tr}_\lambda^\omega(T_N([K])) \in \mathcal{T}^\omega(\lambda)$ is obtained from the classical trace polynomial $\text{Tr}_\lambda(T_N([K])) \in \mathcal{T}^\omega(\lambda)$ (corresponding to the case where $A$ is replaced by $\varepsilon = \pm 1$) by replacing each generator $Z_i$ by $Z_i^N$.

Since $\text{Tr}_\lambda^\omega$ is an algebra homomorphism, $\text{Tr}_\lambda^\omega(T_N([K]))$ is also equal to the evaluation $T_N(\text{Tr}_\lambda^\omega([K]))$ of the Chebyshev polynomial $T_N$ over the quantum trace $\text{Tr}_\lambda^\omega([K])$.

When $A$ is not a root of unity, $\text{Tr}_\lambda^\omega(T_N([K]))$ is a Laurent polynomial in non-commuting variables $Z_i$ and, as $N$ tends to $\infty$, the number of its terms grows as $cN^k$ for appropriate constants $c, k \geq 0$. A consequence of Theorem 4 is that, if we specialize $A$ to an appropriate root of unity, most of these terms disappear and we
are just left with a constant number of monomials, all of which involve only \(N\)-th powers of the generators \(Z_i\).

To some extent, the miraculous cancellations of Theorem 4 are even more surprising than those of Theorems 1 and 2, which at least are conceptually explained by the fact that certain elements of \(S^A(S)\) are central. We use special cases of Theorem 4 to prove Theorems 1 and 2. Our proof of Theorem 4 essentially relies on brute force, and is quite unsatisfactory in this regard. It seems that this result is quite likely to be grounded in deeper facts about the representation theory of the quantum group \(U_q(\mathfrak{sl}_2)\).

1. Constants

The article involves various fractional powers of a non-zero complex number \(q = e^{\frac{2\pi i}{N}} \in \mathbb{C}\). Systematically writing these constants as powers of \(q\) would lead to somewhat cumbersome notation, such as \(S^{q^{-\frac{1}{2}}}(S)\) or \(T^{q^{\frac{1}{4}}N^2}(\lambda)\). It is more convenient (and consistent with the weight of history) to introduce a few more constants related to \(q\). While we will regularly remind the reader of the definition of these constants, we collect these definitions here so that this section can be used as an easy reference if needed.

Again, \(q \in \mathbb{C} - \{0\}\) is a non-zero complex number. We choose successive square roots \(A = \sqrt{q^{-1}}\) and \(\omega = \sqrt{A^{-1}} = q^{\frac{1}{4}}\).

For most of the article, \(q\) is assumed to be a root of unity, with various restrictions. The strongest restriction used, under which all properties hold, is that \(q\) is a primitive \(N\)-root of unity with \(N\) odd. This is sometimes eased to the weaker assumption that \(q^2 = A^{-4}\) is a primitive \(N\)-root of unity, with no parity condition on \(N\).

Under any of the above hypotheses, we will set \(\varepsilon = A^{N^2}\) and \(\iota = \omega^{N^2}\). Note that \(\varepsilon = \pm 1\) if \(A^{2N} = 1\), and that \(\iota^2 = \varepsilon^{-1}\).

2. The Kauffman skein algebra

Let \(S\) be an oriented surface (without boundary) with finite topological type. Namely, \(S\) is obtained by removing finitely many points from a compact oriented surface \(\bar{S}\). We consider framed links in the thickened surface \(\bar{S} \times [0, 1]\), namely unoriented 1-dimensional submanifolds \(K \subset \bar{S} \times [0, 1]\) endowed with a continuous choice of a vector transverse to \(K\) at each point of \(K\). A framed knot is a connected framed link.

The following definition provides a convenient way to describe a framing, in particular when representing a link by a picture. If the projection of \(K \subset S \times [0, 1]\) to \(S\) is an immersion, the vertical framing for \(K\) is the framing that everywhere is parallel to the \([0, 1]\) factor and points towards 1.

The framed link algebra \(\mathcal{K}(S)\) is the vector space (over \(\mathbb{C}\), say) freely generated by the isotopy classes of all framed links \(K \subset S \times [0, 1]\). This vector space \(\mathcal{K}(S)\) can be endowed with a multiplication, where the product of \(K_1\) and \(K_2\) is defined by the framed link \(K_1K_2 \subset S \times [0, 1]\) that is the union of \(K_1\) rescaled in \(S \times [0, \frac{1}{2}]\) and \(K_2\) rescaled in \(S \times [\frac{1}{2}, 1]\). In other words, the product \(K_1K_2\) is defined by superposition of the framed links \(K_1\) and \(K_2\). This superposition operation is compatible with isotopies, and therefore provides a well-defined algebra structure on \(\mathcal{K}(S)\).
Three framed links $K_1$, $K_0$ and $K_\infty$ in $S \times [0,1]$ form a **Kauffman triple** if the only place where they differ is above a small disk in $S$, where they are as represented in Figure 1 (as seen from above) and where the framing is vertical and pointing upwards (namely the framing is parallel to the $[0,1]$ factor and points towards 1).

For $A \in \mathbb{C} - \{0\}$, the **Kauffman skein algebra** $S^A(S)$ is the quotient of the framed link algebra $\mathcal{X}(S)$ by the linear subspace generated by all elements $K_1 - A^{-1}K_0 - AK_\infty$ as $(K_1, K_0, K_\infty)$ ranges over all Kauffman triples. The superposition operation descends to a multiplication in $S^A(S)$, endowing $S^A(S)$ with the structure of an algebra. The class $[S]$ of the empty link is an identity element in $S^A(S)$.

An element $[K] \in S^A(S)$, represented by a framed link $K \subset S \times [0,1]$, is a "skein" in $S$. The construction is defined to ensure that the **skein relation**

$$[K_1] = A^{-1}[K_0] + A[K_\infty]$$

holds in $S^A(S)$ for every Kauffman triple $(K_1, K_0, K_\infty)$.

3. **Central elements of the skein algebra**

3.1. **Central elements coming from the punctures.** The simplest central elements of the skein algebra $S^A(S)$ come from the punctures of $S$, if any.

**Proposition 5.** If $P_k$ is a small embedded loop in $S$ going once around the $k$–th puncture. Considering $P_k$ as a knot in $S \times [0,1]$ endowed with the vertical framing, the skein $[P_k]$ is central in $S^A(S)$, for every value of $A \in \mathbb{C} - \{0\}$.

These are somewhat obvious elements of the center of $S^A(S)$. When $A$ is a root of unity, this center contains other elements which are much less evident, defined using Chebyshev polynomials.

3.2. **Chebyshev polynomials.** Chebyshev polynomials of the first and second kind play a prominent rôle in this article. Both types are polynomials $P_n(x)$ that satisfy the recurrence relation

$$P_n(x) = xP_{n-1}(x) - P_{n-2}(x).$$

The (normalized) **Chebyshev polynomials of the first kind** are the polynomials $T_n(x)$ defined by this recurrence relation and by the initial conditions that $T_0(x) = 2$ and $T_1(x) = x$. The (normalized) **Chebyshev polynomials of the second kind** $S_n(x)$ are similarly defined by the above recurrence relation and by the same initial condition $S_1(x) = x$, but differ in the other initial condition $S_0(x) = 1$.

For instance, $T_2(x) = x^2 - 2$, $T_3(x) = x^3 - 3x$ and $T_4(x) = x^4 - 4x^2 + 2$, while $S_2(x) = x^2 - 1$, $S_3(x) = x^3 - 2x$ and $S_4(x) = x^4 - 3x^2 + 1$.

One reason why the occurrence of Chebyshev polynomials in this work is not completely unexpected is that both types are closely related to the trace function in $\text{SL}_2(\mathbb{C})$, through the following classical properties.

**Lemma 6.** For any $M \in \text{SL}_2(\mathbb{C})$,

1. $\text{Tr } M^n = T_n(\text{Tr } M)$;
2. If $\rho_n : \text{SL}_2(\mathbb{C}) \to \text{GL}_{n+1}(\mathbb{C})$ is the unique $(n + 1)$–dimensional irreducible representation of $\text{SL}_2(\mathbb{C})$, then $\text{Tr } \rho_n(M) = S_n(\text{Tr } M)$. [\square]

Applying Lemma 6 to the matrix $M = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ yields the more classical relations that $\cos n\theta = \frac{1}{2}T_n(2\cos \theta)$ and $\sin n\theta = \sin \theta S_{n-1}(2\cos \theta)$.

For future reference, we note the following two elementary properties.
Lemma 7. For \( n \geq 2 \),
\[
T_n(x) = S_n(x) - S_{n-2}(x).
\]
Proof. The difference \( T_n(x) - S_n(x) + S_{n-2}(x) \) satisfies the linear recurrence relation
\[
P_n(x) = xP_{n-1}(x) - P_{n-2}(x),
\]
and is equal to 0 for \( n = 2 \) and \( n = 3 \). \(\square\)

Lemma 8.

1. If \( x = a + a^{-1} \), then \( T_n(x) = a^n + a^{-n} \);
2. If \( y = b + b^{-1} \), the set of solutions to the equation \( T_n(x) = y \) consists of the numbers \( x = a + a^{-1} \) as \( a \) ranges over all \( n \)-roots of \( b \).

Proof. For a matrix \( M \in \text{SL}_2(\mathbb{C}) \), the data of its trace \( x \) is equivalent to the data of its spectrum \( \{a, a^{-1}\} \). The first property is then an immediate consequence of the fact that \( \text{Tr} \ M^n = T_n(\text{Tr} \ M) \). The second property then follows. \(\square\)

3.3. Threading a Chebyshev polynomial along a framed link. Chebyshev polynomials of the first kind surprisingly provide central elements of the skein algebra \( S^A(S) \).

We first introduce some notation. Consider a polynomial
\[
P(x) = \sum_{i=0}^{n} a_i x^i.
\]
We can then associate to each framed knot \( K \) in \( S \times [0,1] \) the linear combination
\[
[K^P] = \sum_{i=1}^{n} a_i [K^{(i)}] \in S^A(S)
\]
where, for each \( i \), \( K^{(i)} \) is the framed link obtained by taking \( i \) parallel copies of \( K \) in the direction indicated by the framing.

More generally, if \( K \) is a framed link with components \( K_1, K_2, \ldots, K_l \),
\[
[K^P] = \sum_{1 \leq i_1, i_2, \ldots, i_l \leq n} a_{i_1} a_{i_2} \cdots a_{i_l} [K_1^{(i_1)} \cup K_2^{(i_2)} \cup \cdots \cup K_l^{(i_l)}].
\]
We will say that the element \( [K^P] \in S^A(S) \) is obtained by threading the polynomial \( P \) along the framed link \( K \).

When \( K \subset S \times [0,1] \) projects to a simple closed curve in \( S \) and is endowed with the vertical framing, \( [K^P] \) is equal to the evaluation \( P([K]) \in S^A(S) \) of the polynomial \( P \) at the skein \( [K] \), for the algebra structure of \( S^A(S) \). More generally, if \( K \), with components \( K_1, K_2, \ldots, K_l \), projects to an embedded submanifold of \( S \) and is endowed with the vertical framing, then \( [K^P] = P([K_1])P([K_2]) \cdots P([K_l]) \).

Beware that this is in general false for a knot or link whose projection to \( S \) admits crossings, or whose framing is different from the vertical framing.

Theorem 9. If \( A^2 \) is a primitive \( N \)-root of unity, threading the Chebyshev polynomial \( T_N \) along a framed link produces a central element of the skein algebra, namely \( [K^{TN}] \) is central in \( S^A(S) \) for every framed link \( K \subset S \times [0,1] \).

Theorem 9 holds with no parity condition on \( N \).

The key to the proof of Theorem 9 is the following special case in the once-punctured torus \( T \), represented in Figure 2 as a square with its corners removed and with opposite sides identified.
Lemma 10. In the once-punctured torus $T$, let $L_0$ and $L_\infty$ be the two curves represented in Figure 2 and consider these curves as framed knots with vertical framing in $T \times [0, 1]$. If $A^2$ is a primitive $N$-root of unity, then

$$[L_0^{T_N}]|L_\infty] = [L_\infty]|L_0^{T_N}]$$

in the skein algebra $S^A(T)$.

![Figure 2. The curves $L_0$, $L_\infty$, $L_1$, and $L_{-1}$ in the once-punctured torus](image)

There is an elementary proof of Lemma 10 borrowed from the proof of the Product-to-Sum Formula of \cite{FrG} for the non-punctured torus. See also the (much less elementary) combination of \cite{BuP}, §2 and \cite{HaP} Lemma 2. We provide another proof in \cite{21} where it is proved at the same time as Lemma 12 below.

Proof of Theorem 9, assuming Lemma 10: We need to show that $[K^{T_N}]|L] = [L]|K^{T_N}]$ for any two framed links $K$ and $L$.

Let $K_1 \in S \times [0, 1]$ be isotopic to $K$, and contained in a small neighborhood of $S \times \{0\}$ so that $[K_1^{T_N} \cup L] = [K^{T_N}]|L]$ in $S^A(S)$ (where $[K^{T_N} \cup L]$ denotes the element of $S^A(S)$ obtained by threading the Chebyshev polynomial $T_N$ along the components of $K_1$, and leaving $L$ untouched).

By progressively changing the undercrossings of $K$ with $L$ to overcrossings, we construct a sequence of framed links $K_1$, $K_2$, ..., $K_n$ such that each $K_{i+1}$ is obtained from $K_i$ by an isotopy crossing $L$ exactly once, and such that $K_n$ can be isotoped without crossing $L$ into a small neighborhood of $S \times \{1\}$. In particular, $[K_i^{T_N} \cup L] = [L]|K^{T_N}]$ in $S^A(S)$. It therefore suffices to show that $[K_{i+1}^{T_N} \cup L] = [K_i^{T_N} \cup L]$ for every $i$.

We now consider an embedded thickened once-punctured torus which “swallows” the crossing between $K_i$ and $K_{i+1}$ in question and “follows” the two curves otherwise. We then apply Lemma 10 to exchange the swallowed undercrossing to an overcrossing. More precisely, by construction of $K_{i+1}$ from $K_i$ there is an embedding $\varphi: T \times [0, 1] \to S \times [0, 1]$ for which, with the notation of Lemma 10 the intersections $K_i \cap \varphi(T \times [0, 1])$, $L \cap \varphi(T \times [0, 1])$ and $K_{i+1} \cap \varphi(T \times [0, 1])$ respectively correspond to the framed knots $L_\infty \times \{\frac{1}{2}\}$, $L_0 \times \{\frac{1}{2}\}$ and $L_\infty \times \{\frac{1}{3}\}$ in $T \times [0, 1]$, after isotopy in $T \times [0, 1]$. Applying Lemma 10 and composing with the homomorphism $S^A(T) \to S^A(S)$ induced by $\varphi$, shows that $[K_i^{T_N} \cup L] = [K_{i+1}^{T_N} \cup L]$.

By iteration, this concludes the proof.

4. The Chebyshev Homomorphism

Theorem 11. If $A^2$ is a primitive $N$-root of unity and if $\varepsilon = A^{N^2}$, there is a unique algebra homomorphism $T^A: S^A(S) \to S^A(S)$ which, for every framed link $K$ in $S \times [0, 1]$, associates $[K^{T_N}] \in S^A(S)$ to the skein $[K] \in S^A(S)$. 
A key step in the proof of Theorem 11 is the following computation.

**Lemma 12.** Suppose that $A^4$ is a primitive $N$–root of unity. In the once-punctured torus $T$, let $L_0$, $L_\infty$, $L_1$ and $L_{-1}$ be the curves represented in Figure 2. Considering these curves as knots in $T \times [0,1]$ and endowing them with the vertical framing,

$$[L_0^{T_N}] [L_\infty^{T_N}] = A^{-N^2} [L_1^{T_N}] + A^{N^2} [L_{-1}^{T_N}]$$

in the skein algebra $S^A(T)$.

We postpone the proof of Lemma 12 to §7.9. This result is very reminiscent of the Product-to-Sum Formula of [FrG], which holds in the unpunctured torus but for all values of $A$. Unlike that formula, Lemma 12 only holds when $A^4$ is an $N$–root of unity. It would probably be more satisfying to find a simpler proof of Lemma 12 perhaps similar to or using the arguments of [FrG].

![Figure 3. The 1–submanifolds $L_1$, $L_0$, $L_\infty$ and $L_{-1}$ in the twice-punctured plane](image)

The proof of Theorem 11 also uses the following similar computation.

**Lemma 13.** Suppose that $A^4$ is a primitive $N$–root of unity. For the twice-punctured plane $U$, let $L_1$, $L_0$, $L_\infty$ and $L_{-1}$ be the links in $U \times [0,1]$ represented in Figure 3 endowed with the vertical framing. Then,

$$[L_1^{T_N}] = A^{-N^2} [L_0^{T_N}] + A^{N^2} [L_\infty^{T_N}]$$

and

$$[L_{-1}^{T_N}] = A^{N^2} [L_0^{T_N}] + A^{-N^2} [L_\infty^{T_N}]$$

in the skein algebra $S^A(U)$.

Again, we postpone the proof of Lemma 13 to §7.9.

**Proof of Theorem 11 (assuming Lemmas 12 and 13).** We have to check that the Chebyshev threads $[K^{T_N}] \in S^A(S)$ satisfy the skein relation with $A$ replaced by $\varepsilon$.

Let the framed links $K_1$, $K_0$, $K_\infty \subset S \times [0,1]$ form a Kauffman triple, as in Figure 1. Let $\tilde{K}_1$, $\tilde{K}_0$, $\tilde{K}_\infty$ denote the union of the (one or two) components of $K_1$, $K_0$, $K_\infty$, respectively, that appear in Figure 1 (There may be additional components that do not appear on the picture.) Considering the way the strands represented connect outside of the picture, we distinguish three cases, according to the number of components of $\tilde{K}_1$, $\tilde{K}_0$ and $\tilde{K}_\infty$. One easily sees that exactly two of these three links are connected, while the remaining one has two components.

If $\tilde{K}_1$ is disconnected, there exists an embedding $\varphi: T \times [0,1] \to S \times [0,1]$ of the thickened punctured torus $T \times [0,1]$ such that, using the notation of Lemma 12, $\varphi^{-1}(K_1)$ is isotopic to the union of $L_0 \times \{\frac{1}{2}\}$ and of $L_\infty \times \{\frac{1}{2}\}$ in $T \times [0,1]$, while $\varphi^{-1}(K_0)$ and $\varphi^{-1}(K_\infty)$ are respectively isotopic to $L_1 \times \{\frac{1}{2}\}$ and $L_{-1} \times \{\frac{1}{2}\}$. Lemma 12 shows that

$$[\tilde{K}_1^{T_N}] = A^{-N^2} [\tilde{K}_0^{T_N}] + A^{N^2} [\tilde{K}_\infty^{T_N}] = \varepsilon^{-1} [\tilde{K}_0^{T_N}] + \varepsilon [\tilde{K}_\infty^{T_N}]$$
in $S^A(\varphi(T \times [0, 1]))$. Since $K_1, K_0, K_\infty$ coincide outside of $\varphi(T \times [0, 1])$, it follows that

$$[K_1^{T_N}] = \varepsilon^{-1}[K_0^{T_N}] + \varepsilon[K_\infty^{T_N}]$$

in $S^A(S)$.

The other two cases are similar, but using Lemma [13] this time. If $\tilde{K}_\infty$ is disconnected, then there exists an embedding $\varphi: U \times [0, 1] \to S \times [0, 1]$ of the thickened twice-punctured plane $U \times [0, 1]$ such that, using the notation of Lemma [13] $\varphi^{-1}(K_1) = L_1$, $\varphi^{-1}(K_0) = L_0$ and $\varphi^{-1}(K_\infty) = L_\infty$. Applying the first identity of Lemma [13] shows that

$$[\tilde{K}_1^{T_N}] = \varepsilon^{-1}[\hat{K}_0^{T_N}] + \varepsilon[\hat{K}_\infty^{T_N}]$$

in this case as well, so that $[K_1^{T_N}] = \varepsilon^{-1}[K_0^{T_N}] + \varepsilon[K_\infty^{T_N}]$ in $S^A(S)$.

Finally, if $\tilde{K}_0$ is disconnected, we use an embedding $\varphi: U \times [0, 1] \to S \times [0, 1]$ of the thickened twice-punctured plane $U \times [0, 1]$ such that $\varphi^{-1}(K_1) = L_{-1}$, $\varphi^{-1}(K_0) = L_\infty$ and $\varphi^{-1}(K_\infty) = L_0$. The second identity of Lemma [13] then provides the relation sought.

Therefore, $[K_1^{T_N}] = \varepsilon^{-1}[K_0^{T_N}] + \varepsilon[K_\infty^{T_N}]$ in $S^A(S)$ for every Kauffman triple $K_1, K_0, K_\infty$. This proves that the threading map $K \mapsto [K^{T_N}] \in S^A(S)$ induces a linear map $T^A: S^\varepsilon(S) \to S^A(S)$. It is immediate that $T^A$ is an algebra homomorphism.

\[ \square \]

5. Invariants of finite-dimensional irreducible representations

Let $\rho: S^A(S) \to \text{End}(V)$ be a finite-dimensional irreducible representation of the skein algebra $S^A(S)$. To avoid degenerate cases (see our discussion of the trivial homomorphism $S^A(S) \to \mathbb{C}$ before Proposition [14] later on) we always require that, for a representation $\rho$ of $S^A(S)$, there exists at least one non-empty framed link $K$ such that $\rho([K]) \neq 0 \in \text{End}(V)$.

We want to construct invariants of the irreducible representation $\rho$.

To take advantage of Theorems [9] and [11] we are going to need that both $A^4$ is a primitive $N$–root of unity and $A^{2N} = 1$. This is equivalent to the property that $A^2$ is a primitive $N$–root of unity with $N$ odd. In particular, $\varepsilon = A^{N^2} = A^N = \pm 1$.

5.1. The classical shadow. Composing the representation $\rho: S^A(S) \to \text{End}(V)$ with the Chebyshev homomorphism $T^A: S^\varepsilon(S) \to S^A(S)$ of Theorem [11] we obtain a homomorphism $S^\varepsilon(S) \to \text{End}(V)$. By Theorem [9] this homomorphism factors through the center of $S^\varepsilon(S)$ and therefore, by irreducibility of $\rho$ and by Schur’s lemma, there exists an algebra homomorphism

$$\kappa_\rho: S^\varepsilon(S) \to \mathbb{C}$$

such that

$$\rho \circ T^A([K]) = \rho([K^{T_N}]) = \kappa_\rho([K])\text{Id}_V$$

for every skein $[K] \in S^A(S)$.

Because $\varepsilon = \pm 1$, such a homomorphism $\kappa_\rho: S^\varepsilon(S) \to \mathbb{C}$ has a geometric interpretation. More precisely, consider the character variety

$$\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S) = \{\text{group homomorphisms } r: \pi_1(S) \to \text{SL}_2(\mathbb{C})\} / \text{SL}_2(\mathbb{C})$$

where $\text{SL}_2(\mathbb{C})$ acts on homomorphisms by conjugation, and where the double bar $\|$ indicates that the quotient is taken in the sense of geometric invariant theory [MePK] in algebraic geometry. For a group homomorphism $r: \pi_1(S) \to \text{SL}_2(\mathbb{C})$
and a closed curve $K \subset S \times [0, 1]$, the trace $\text{Tr} r(K) \in \mathbb{C}$ depends only on the class of $r$ in the character variety $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S)$. An observation of Doug Bullock, Charlie Frohman, Jozef Przytycki and Adam Sikora \cite{Bu1, Bu2, BuFK, BuFK2, PrS} then shows that this defines an algebra homomorphism

$$\text{Tr}_r : S^{-1}(S) \to \mathbb{C}$$

by the property that

$$\text{Tr}_r([K]) = -\text{Tr} r(K)$$

for every (connected) framed knot $K \subset S \times [0, 1]$. Note that $\text{Tr}_r([K])$ is independent of the framing of $K$.

There exists a twisted version of this for $S^{+1}(S)$. Let $\text{Spin}(S)$ denote the space of isometry classes of spin structures on $S$. Given a spin structure $\sigma \in \text{Spin}(S)$, John Barrett \cite{Ba} constructs an algebra homomorphism $B_\sigma : S^+1(S) \to S^{-1}(S)$, which to a skein $[K] \in S^+1(S)$ associates $(-1)^{k+\sigma(k)}[K] \in S^{-1}(S)$, where $k$ is the number of components of the link $K$ and where $\sigma(K) \in \mathbb{Z}_2$ is the monodromy of the framing of $K$ with respect to the spin structure $\sigma$. A pair $r = (\hat{r}, \sigma)$, consisting of a group homomorphism $\hat{r} : \pi_1(S) \to \text{SL}_2(\mathbb{C})$ and a spin structure $\sigma \in \text{Spin}(S)$, then defines a trace map

$$\text{Tr}_r = \text{Tr}_\hat{r} \circ B_\sigma : S^+1(S) \to \mathbb{C},$$

which to a connected skein $[K] \in S^+1(S)$ associates

$$\text{Tr}_r([K]) = (-1)^{\sigma(k)} \text{Tr} \hat{r}(K).$$

This trace map $\text{Tr}_r$ is unchanged under certain modifications of the pair $(\hat{r}, \sigma)$. Indeed, two spin structures in $\text{Spin}(S)$ differ by an obstruction in $H^1(S; \mathbb{Z}_2)$; this defines an action of $H^1(S; \mathbb{Z}_2)$ on $\text{Spin}(S)$. The cohomology group $H^1(S; \mathbb{Z}_2)$ also acts on the character variety $\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$ by the property that, for a homomorphism $\hat{r} : \pi_1(S) \to \text{SL}_2(\mathbb{C})$ and $\alpha \in H^1(S; \mathbb{Z}_2)$, the character $\alpha \hat{r} \in \mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$ is represented by

$$\alpha \hat{r}(\gamma) : \pi_1(S) \to \text{SL}_2(\mathbb{C})$$

$$\gamma \mapsto (-1)^{\alpha(\gamma)} \hat{r}(\gamma).$$

Hence the trace map $\text{Tr}_r : S^+1(S) \to \mathbb{C}$ depends only on the image of $r = (\hat{r}, \sigma)$ in the quotient

$$\mathcal{R}^{\text{Spin}}_{\text{PSL}_2(\mathbb{C})}(S) = (\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S) \times \text{Spin}(S)) / H^1(S; \mathbb{Z}_2).$$

To explain the appearance of the group $\text{PSL}_2(\mathbb{C})$ in the notation, consider the character variety

$$\mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S) = \{\text{group homomorphisms } r : \pi_1(S) \to \text{PSL}_2(\mathbb{C})\} / \text{PSL}_2(\mathbb{C})$$

and the subset $\mathcal{R}^0_{\text{PSL}_2(\mathbb{C})}(S) = \mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S) / H^1(S; \mathbb{Z}_2)$ of $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S)$ consisting of those homomorphisms $\pi_1(S) \to \text{PSL}_2(\mathbb{C})$ that lift to $\text{SL}_2(\mathbb{C})$. Bill Goldman \cite{Go} showed that $\mathcal{R}^0_{\text{PSL}_2(\mathbb{C})}(S)$ is equal to the whole character variety $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S)$ when $S$ is non-compact, and is equal to one of the two components of $\mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S)$ when $S$ is compact. Then, $\mathcal{R}^{\text{Spin}}_{\text{PSL}_2(\mathbb{C})}(S)$ can be seen as a twisted product of $\mathcal{R}^0_{\text{PSL}_2(\mathbb{C})}(S)$ with $\text{Spin}(S)$. In particular, both $\mathcal{R}_{\text{SL}_2(\mathbb{C})}(S)$ and $\mathcal{R}^{\text{Spin}}_{\text{PSL}_2(\mathbb{C})}(S)$ are coverings of $\mathcal{R}^0_{\text{PSL}_2(\mathbb{C})}(S)$ with fiber $\text{Spin}(S) \cong H^1(S; \mathbb{Z}_2)$. Choosing a spin structure $\sigma \in \text{Spin}(S)$
Proof. Let Lemma 15.

Conversely, all algebra homomorphisms \( \kappa : S^\varepsilon(S) \to \mathbb{C} \) but one are obtained in this way, via a trace map. The exceptional homomorphism is the trivial homomorphism, defined by the property that \( \kappa([\varnothing]) = 1 \) and \( \kappa([K]) = 0 \) for every nonempty framed link \( K \) in \( S \times [0, 1] \).

**Proposition 14.** Every non-trivial homomorphism \( \kappa : S^\varepsilon(S) \to \mathbb{C} \) is equal to the trace homomorphism \( \text{Tr}_r \) associated to a unique character \( r \in \mathcal{R}_{\text{SL}_2(\mathbb{C})}(S) \) if \( \varepsilon = -1 \), or a twisted character \( r \in \mathcal{R}_{\text{PSL}_2(\mathbb{C})}(S) \) if \( \varepsilon = +1 \).

**Proof.** The case \( \varepsilon = -1 \) is a result of Heinz Helling [He, Prop. 1]. The case \( \varepsilon = +1 \) easily follows by using the Barrett isomorphism \( B_r : S^{+1}(S) \to S^{-1}(S) \) provided by a spin structure \( \sigma \in \text{Spin}(S) \).

Let us now return to our analysis of a finite-dimensional irreducible representation \( \rho : S^A(S) \to \text{End}(V) \), and to its associated homomorphism \( \kappa_\rho : S^\varepsilon(S) \to \mathbb{C} \) defined by the property that \( \rho([K^T_n]) = \kappa_\rho([K]) \text{Id}_V \) for every skein \([K] \in S^A(S)\).

**Lemma 15.** The homomorphism \( \kappa_\rho : S^\varepsilon(S) \to \mathbb{C} \) is non-trivial.

**Proof.** Let \( O \) be the boundary of a small disk in \( S \times \{\frac{1}{2}\} \), endowed with the vertical framing.

The skein \([O]\) is clearly in the center of \( S^A(S) \). By irreducibility of \( \rho \) and by Schur’s lemma, there consequently exists a number \( o \in \mathbb{C} \) such that \( \rho([O]) = o \text{Id}_V \). A classical consequence of the invariance of skeins under isotopy (and in particular under the second Reidemeister move) is that \([O]^{}[K] + (A^2 + A^{-2})[K] = 0 \) in \( S^A(S) \) for every skein \([K] \in S^A(S)\) represented by a non-empty link \( K \subset S \times [0, 1] \); see for instance [Lic1] Lemmas 3.2 and 3.3. As a consequence, \((o + A^2 + A^{-2})\rho([K]) = 0 \) for every non-empty skein \([K] \in S^A(S)\). Applying this to a non-empty skein \([K] \in S^A(S)\) such that \( \rho([K]) \neq 0 \), which exists by our convention for representations of \( S^A(S) \) (see the beginning of [8]), this proves that \( o = -(A^2 + A^{-2}) \), and therefore that

\[
\rho([O]) = -(A^2 + A^{-2}) \text{Id}_V.
\]

Then, because \( O \) has no crossing and is endowed with the vertical framing,

\[
\rho([O^T_n]) = \rho(T_N([O])) = T_N(\rho([O])) = T_N((-A^2 - A^{-2}) \text{Id}_V)
\]

\[
= T_N((-A^2 - A^{-2}) \text{Id}_V = ((-A^2)^N + (-A^{-2})^N) \text{Id}_V
\]

\[
= -2 \text{Id}_V
\]

where the second equality comes from the fact that \( \rho \) is an algebra homomorphism, while the fifth identity is a consequence of Lemma [8], and the last equality uses the fact that \( A^{2N} = 1 \) and \( N \) is odd. In particular, \( \kappa_\rho([O]) = -2 \neq 0 \), and \( \kappa_\rho \) is non-trivial.

Combining the definition of the algebra homomorphism \( \kappa_\rho : S^\varepsilon(S) \to \mathbb{C} \) with Proposition [14] and Lemma [15] provides the following statement.

**Theorem 16.** Suppose that \( A^2 \) is a primitive \( N \)–root of unity with \( N \) odd, so that \( \varepsilon = A^N \) is equal to \( \pm 1 \). Then, for every finite-dimensional irreducible representation \( \rho : S^A(S) \to \text{End}(V) \) of the skein algebra \( S^A(S) \), there exists a unique character
$r_\rho \in \mathcal{R}_{S\text{L}_2(\mathbb{C})}(S)$ if $A^N = -1$, or a unique twisted character $r_\rho \in \mathcal{R}_{S\text{P\text{L}}_2(\mathbb{C})}(S)$ if $A^N = +1$, such that

$$
\rho([K^{TN}]) = (\text{Tr}_{r_\rho}([K])) \text{Id}_V
$$

for every skein $[K] \in \mathcal{S}^A(S)$. \hfill $\Box$

In particular, Theorem 16 implies Theorems 1 and 2 stated in the introduction since, for a knot $K \subset S \times [0, 1]$ with no crossing and with vertical framing, $T_N(\rho([K])) = \rho([K^{TN}])$ and $\text{Tr}_{r_\rho}([K]) = -\text{Tr}_{r_\rho}(K)$.

We call the character $r_\rho \in \mathcal{R}_{S\text{L}_2(\mathbb{C})}(S)$ or $\mathcal{R}_{S\text{P\text{L}}_2(\mathbb{C})}(S)$, associated to $\kappa_\rho$ by Theorem 16, the classical shadow of the irreducible representation $\rho: \mathcal{S}^A(S) \rightarrow \text{End}(V)$.

5.2. Puncture invariants. The central elements associated to the punctures of $S$ provide similar invariants for the finite-dimensional irreducible representation $\rho: \pi_1(S) \rightarrow \text{End}(V)$. More precisely, let $P_k \subset S \times \{\frac{1}{2}\} \subset S \times [0, 1]$ be a small simple loop going around the $k$-th puncture, endowed with the vertical framing. Proposition 5 shows that the corresponding skein $[P_k] \in \mathcal{S}^A(S)$ is central. Therefore, by irreducibility of $\rho$, there exists a number $p_k \in \mathbb{C}$ such that

$$
\rho([P_k]) = p_k \text{Id}_V.
$$

This number $p_k \in \mathbb{C}$ is the $k$-th puncture invariant of the finite-dimensional irreducible representation $\rho$.

These numbers $p_k$ are clearly constrained by the classical shadow $r_\rho \in \mathcal{R}_{S\text{L}_2(\mathbb{C})}(S)$ or $\mathcal{R}_{S\text{P\text{L}}_2(\mathbb{C})}(S)$ of $\rho$, in terms of the Chebyshev polynomial $T_N$. Indeed:

Lemma 17.

$$
T_N(p_k) = -\text{Tr}_{r_\rho}(P_k).
$$

Proof. By definition of the homomorphism $\text{Tr}_{r_\rho}: \mathcal{S}^c(S) \rightarrow \mathbb{C}$ and by Theorem 16

$$
(\text{Tr}_{r_\rho}(P_k)) \text{Id}_V = -\text{Tr}_{r_\rho}([P_k]) \text{Id}_V = -\rho([P_k^{TN}]) = T_N(p_k) \text{Id}_V
$$

where the third equality uses the fact that $P_k$ has no crossings and has vertical framing, and where the fourth equality holds because $\rho$ is an algebra homomorphism. \hfill $\Box$

In particular, Lemma 8 shows that, once the classical shadow $r_\rho$ is given, there are at most $N$ possibilities for each puncture invariant $p_k$.

6. The Quantum Teichmüller Space and the Quantum Trace Homomorphism

In [BoW3, BoW4], we prove a converse to Theorem 16 and Lemma 17 by showing that every character $r \in \mathcal{R}_{S\text{L}_2(\mathbb{C})}(S)$ or $\mathcal{R}_{S\text{P\text{L}}_2(\mathbb{C})}(S)$ and any set of puncture weights $p_k$ with $T_N(p_k) = \text{Tr}_{r_\rho}([P_k])$ can be realized as the classical shadow and the puncture invariants of a finite-dimensional irreducible representation $\rho: \mathcal{S}^A(S) \rightarrow \text{End}(V)$.

The proof of this result relies on the quantum Teichmüller space of [FG, ChF2] and on the quantum trace homomorphism constructed in [BoW3]. A crucial step in the argument is a compatibility property between the quantum trace homomorphism and the Chebyshev homomorphism, appearing here as Theorem 20. This
section is devoted to the precise statement of this result. This property is of interest by itself, because of the surprising “miraculous cancellations” that it exhibits. A special case of Theorem 20 also provides a key step in the proof of Lemmas 10, 12 and 13 which we had temporarily postponed for this very reason.

6.1. The Chekhov-Fock algebra. The Chekhov-Fock algebra is the avatar of the quantum Teichmüller space associated to an ideal triangulation of the surface $S$. If $S$ is obtained from a compact surface $\bar{S}$ by removing finitely many points $v_1, v_2, \ldots, v_s$, an ideal triangulation of $S$ is a triangulation $\lambda$ of $\bar{S}$ whose vertex set is exactly $\{v_1, v_2, \ldots, v_s\}$.

Throughout the rest of the article, we will assume that $S$ admits such an ideal triangulation $\lambda$, which is equivalent to the property that $S$ has at least one puncture and that its Euler characteristic is negative.

Let $e_1, e_2, \ldots, e_n$ denote the edges of $\lambda$. Let $a_{ij} \in \{0, 1, 2\}$ be the number of times an end of the edge $e_j$ immediately succeeds an end of $e_i$ when going counterclockwise around a puncture of $S$, and set $\sigma_{ij} = a_{ij} - a_{ji} \in \{-2,-1,0,1,2\}$. The Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$ of $\lambda$ is the algebra defined by generators $Z_1, Z_2, \ldots, Z_n$ associated to the edges $e_1, e_2, \ldots, e_n$ of $\lambda$, and by the relations

$$Z_i Z_j = \omega^{2\sigma_{ij}} Z_j Z_i.$$

(The actual Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$ that is at the basis of the quantum Teichmüller space uses the constant $q = \omega^4$ instead of $\omega$. The generators $Z_i$ of $\mathcal{T}^\omega(\lambda)$ appearing here are designed to model square roots of the original generators of $\mathcal{T}^\omega(\lambda)$.)

An element of the Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$ is a linear combination of monomials $Z_{i_1}^{n_1} Z_{i_2}^{n_2} \ldots Z_{i_l}^{n_l}$ in the generators $Z_i$, with $n_1, n_2, \ldots, n_l \in \mathbb{Z}$. Because of the skew-commutativity relation $Z_i Z_j = \omega^{2\sigma_{ij}} Z_j Z_i$, the order of the variables in such a monomial does matter. It is convenient to use the following symmetrization trick.

The Weyl quantum ordering for $Z_{i_1}^{n_1} Z_{i_2}^{n_2} \ldots Z_{i_l}^{n_l}$ is the monomial

$$[Z_{i_1}^{n_1} Z_{i_2}^{n_2} \ldots Z_{i_l}^{n_l}] = \omega^\sum_{u < v} n_u n_v \sigma_{uv} Z_{i_u}^{n_u} Z_{i_v}^{n_v} \ldots Z_{i_l}^{n_l} Z_{i_1}^{n_1} Z_{i_2}^{n_2} \ldots Z_{i_l}^{n_l}.$$

The formula is specially designed so that $[Z_{i_1}^{n_1} Z_{i_2}^{n_2} \ldots Z_{i_l}^{n_l}]$ is invariant under any permutation of the $Z_{i_u}$.

Note that the Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$ depends only on $\omega^2$, and that there is a canonical isomorphism $\mathcal{T}^\omega(\lambda) \cong \mathcal{T}^{-\omega}(\lambda)$. However, the Weyl quantum ordering does depend on $\omega$, and justifies the emphasis on $\omega$ (as opposed to $\omega^2$) in the notation for $\mathcal{T}^\omega(\lambda)$.

6.2. The Frobenius homomorphism. The following homomorphism plays a fundamental role in the classification of finite-dimensional irreducible representations of the quantum Teichmüller space $\text{BoL}$.

**Proposition 18.** If $\iota = \omega^{-2}$, there is a unique algebra homomorphism

$$F^\omega : \mathcal{T}^\omega(\lambda) \to \mathcal{T}^\omega(\lambda)$$

which maps each generator $Z_i \in \mathcal{T}^\omega(\lambda)$ to $Z_i^N \in \mathcal{T}^\omega(\lambda)$, where in the first instance $Z_i \in \mathcal{T}^\omega(\lambda)$ denotes the generator associated to the $i$-th edge $e_i$ of $\lambda$, whereas the second time $Z_i \in \mathcal{T}^\omega(\lambda)$ denotes the generator of $\mathcal{T}^\omega(\lambda)$ associated to the same edge $e_i$. 

Proof. If $Z_iZ_j = \omega^{2\sigma_{ij}}Z_jZ_i$ for generators $Z_i$, $Z_j$ of $\mathcal{T}^\omega(\lambda)$, then $Z_i^N Z_j^N = \omega^{2\sigma_{ij}N^2} Z_j^N Z_i^N = \omega^{2\sigma_{ij}N} Z_j^N Z_i^N$. □

Borrowing the terminology from [FoG], the algebra homomorphism $F^\omega : \mathcal{T}^\omega(\lambda) \to \mathcal{T}^\omega(\lambda)$ is the Frobenius homomorphism of $\mathcal{T}^\omega(\lambda)$. When $\omega^N = (-1)^{N+1}$, this Frobenius is remarkably well-behaved with respect to coordinate changes in the quantum Teichmüller space; see [ChF, BoL, FoG]. However, we won’t need this property here, although it somewhat foreshadows the compatibility property of Theorem 20.

### 6.3. The quantum trace homomorphism

Another key ingredient is the quantum trace homomorphism $\text{Tr}^\omega : S^A(S) \to \mathcal{T}^\omega(\lambda)$ constructed in [BoW]. The computations of [W] will (unfortunately) require familiarity with the details of this embedding of the skein algebra $S^A(S)$ into the Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$. We therefore need to briefly summarize the main technical features of its construction.

In particular, we need to consider situations where $S$ is a punctured surface with boundary, obtained by removing finitely many points $v_1, v_2, \ldots, v_s$ from a compact connected oriented surface $\bar{S}$ with boundary $\partial \bar{S}$. We require that each component of $\partial S$ contains at least one puncture $v_k$, that there is at least one puncture, and that the Euler characteristic of $S$ is less than the number of components of $\partial S$. In particular, the boundary $\partial S$ consists of disjoint open intervals. These topological restrictions are equivalent to the existence of an ideal triangulation $\lambda$ for $S$. The Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$ is then defined as before, with skew-commuting generators corresponding to the edges of $\lambda$ (including the components of $\partial S$).

To define the skein algebra for such a punctured surface $S$ with boundary, we take a framed link $K$ in $S \times [0, 1]$ to be a 1–dimensional framed submanifold $K \subset S \times [0, 1]$ such that:

1. $\partial K = K \cap (\partial S \times [0, 1])$ consists of finitely many points in $(\partial S) \times [0, 1]$;
2. at every point of $\partial K$, the framing is vertical, namely parallel to the $[0, 1]$ factor and pointing in the direction of 1;
3. for every component $e$ of $\partial S$, the points of $\partial K$ that are in $e \times [0, 1]$ sit at different elevations, namely have different $[0, 1]$–coordinates.

Isotopies of framed links are required to respect all three conditions above. The third condition is particular important for the gluing construction described below.

The skein algebra $S^A(S)$ is then defined as before, first considering the vector space $\mathcal{X}(S)$ freely generated by all isotopy classes of framed links, and then taking its quotient by the skein relation. The multiplication of the algebra structure is provided by the usual superposition operation.

The extension to surfaces with boundary enables us to glue surfaces and skeins. Given two surfaces $S_1$ and $S_2$ and two boundary components $e_1 \subset \partial S_1$ and $e_2 \subset \partial S_2$, we can glue $S_1$ and $S_2$ by identifying $e_1$ and $e_2$ to obtain a new oriented surface $S$. There is a unique way to perform this gluing so that the orientations of $S_1$ and $S_2$ match to give an orientation of $S$. We allow the “self-gluing” case, where the surfaces $S_1$ and $S_2$ are equal as long as the boundary components $e_1$ and $e_2$ are distinct. If we are given an ideal triangulation $\lambda_1$ of $S_1$ and an ideal triangulation $\lambda_2$ of $S_2$, these two triangulations fit together to give an ideal triangulation $\lambda$ of the glued surface $S$.

Now, suppose in addition that we are given skeins $[K_1] \in S^A(S_1)$ and $[K_2] \in S^A(S_2)$, represented by framed links $K_1$ and $K_2$ such that $K_1 \cap (e_1 \times [0, 1])$ and
for the identification given by the gluing, and if the points of $K_1 \cap (e_1 \times [0, 1])$ (and of $K_2 \cap (e_2 \times [0, 1])$) sit at different elevations, the framed link $K$ is now uniquely determined up to isotopy. Also, this operation is well behaved with respect to the skein relations, so that $K$ represents a well-defined element $[K] \in S^A(S)$. We say that $[K] \in S^A(S)$ is obtained by gluing the two skeins $[K_1] \in S^A(S_1)$ and $[K_2] \in S^A(S_2)$.

The gluing operation is also well-behaved with respect to Chekhov-Fock algebras. Indeed, if $S$ is obtained by gluing two surfaces $S_1$ and $S_2$ along boundary components $e_1 \subset \partial S_1$ and $e_2 \subset \partial S_2$, and if the ideal triangulations $\lambda_1$ of $S_1$ and $\lambda_2$ of $S_2$ are glued to provide an ideal triangulation $\lambda$ of $S$, there is a canonical embedding $\Phi: \mathcal{I}^\omega(\lambda) \to \mathcal{I}^\omega(\lambda_1) \otimes \mathcal{I}^\omega(\lambda_2)$ defined as follows. If $Z_e$ is a generator of $\mathcal{I}^\omega(\lambda)$ associated to an edge $e$ of $\lambda$:

1. $\Phi(Z_e) = Z_{e_1} \otimes Z_{e_2}$ if $e$ is the edge coming from the boundary components $e_1 \subset \partial S_1$ and $e_2 \subset \partial S_2$ that are glued together, and if $Z_{e_1} \in \mathcal{I}^\omega(\lambda_1)$ and $Z_{e_2} \in \mathcal{I}^\omega(\lambda_2)$ are the generators associated to these edges $e_1$, $e_2$ of $\lambda_1$, $\lambda_2$;
2. $\Phi(Z_e) = Z_{e'_1} \otimes 1$ if $e$ corresponds to an edge $e'_1$ of $\lambda_1$ that is not glued to an edge of $\lambda_2$;
3. $\Phi(Z_e) = 1 \otimes Z_{e'_2}$ if $e$ corresponds to an edge $e'_2$ of $\lambda_2$ that is not glued to an edge of $\lambda_1$.

There is a similar embedding $\mathcal{I}^\omega(\lambda) \to \mathcal{I}^\omega(\lambda_1)$ in the case of self-gluing, when the surface $S$ and the ideal triangulation $\lambda$ are obtained by gluing two distinct boundary components of a surface $S_1$ and an ideal triangulation $\lambda_1$ of $S_1$.

Finally, a state for a skein $[K] \in S^A(S)$ is the assignment $s: \partial K \to \{+, -\}$ of a sign $\pm$ to each point of $\partial K$. Let $S^A(S)$ be the algebra consisting of linear combinations of stated skeins, namely of skeins endowed with states. In particular, $S^A(S) = S^A(S)$ when $S$ has empty boundary.

In the case where $K \in S^A(S)$ is obtained by gluing the two skeins $K_1 \in S^A(S_1)$ and $K_2 \in S^A(S_2)$, the states $s: \partial K \to \{+, -, \}$, $s_1: \partial K_1 \to \{+, -, \}$, $s_2: \partial K_2 \to \{+, -\}$ are compatible if $s_1$ and $s_2$ coincide on $\partial K_1 \cap (e_1 \times [0, 1]) = \partial K_2 \cap (e_2 \times [0, 1])$ for the identification given by the gluing, and if $s$ coincides with the restrictions of $s_1$ and $s_2$ on $\partial K \subset \partial K_1 \cup \partial K_2$.

**Theorem 19 (BoW).** For $A = \omega^{-2}$, there is a unique family of injective algebra homomorphisms

$$Tr^\omega_\lambda: S^A(S) \to \mathcal{I}^\omega(\lambda),$$

defined for each punctured surface $S$ with boundary and for each ideal triangulation $\lambda$ of $S$, such that:

1. (State Sum Property) If the surface $S$ is obtained by gluing $S_1$ to $S_2$, if the ideal triangulation $\lambda_1$ of $S_1$ and $\lambda_2$ of $S_2$, and if the skeins $[K_1] \in S^A(S_1)$ and $[K_2] \in S^A(S_2)$ are glued together to give $[K] \in S^A(S)$, then

$$Tr^\omega_\lambda([K, s]) = \sum_{\text{compatible } s_1, s_2} Tr^\omega_\lambda([K_1, s_1]) \otimes Tr^\omega_\lambda([K_2, s_2])$$

where the sum is over all states $s_1: \partial K_1 \to \{+, -\}$ and $s_2: \partial K_2 \to \{+, -\}$ that are compatible with $s: \partial K \to \{+, -\}$ and with each other. Similarly
if the surface $S$, the ideal triangulation $\lambda$ of $S$, and the skein $[K] \in S^A(S)$ are obtained by gluing the surface $S_1$, the ideal triangulation $\lambda_1$ of $S_1$, and the skein $[K_1] \in S^A(S_1)$, respectively, to themselves, then

$$\text{Tr}_\lambda^\omega([K, s]) = \sum_{\text{compatible } s_1} \text{Tr}_\lambda^\omega([K_1, s_1]).$$

(2) (Elementary Cases) When $S$ is a triangle and $K$ projects to a single arc embedded in $S$, with vertical framing, then

(a) in the case of Figure 4(a), where $\varepsilon_1, \varepsilon_2 = \pm$ are the signs associated by the state $s$ to the end points of $K$,

$$\text{Tr}_\lambda^\omega([K, s]) = \begin{cases} 0 & \text{if } \varepsilon_1 = -\text{ and } \varepsilon_2 = + \\ [Z_1^{\varepsilon_1} Z_2^{\varepsilon_2}] & \text{if } \varepsilon_1 \neq -\text{ or } \varepsilon_2 \neq + \end{cases}$$

where $Z_1$ and $Z_2$ are the generators of $T^\omega(\lambda)$ associated to the sides $e_1$ and $e_2$ of $S$ indicated, and where $[Z_1^{\varepsilon_1} Z_2^{\varepsilon_2}] = \omega^{-\varepsilon_1 \varepsilon_2} Z_1^{\varepsilon_1} Z_2^{\varepsilon_2} = \omega^{\varepsilon_1 \varepsilon_2} Z_2^{\varepsilon_2} Z_1^{\varepsilon_1}$ (identifying the sign $\varepsilon = \pm$ to the exponent $\varepsilon = \pm 1$) is the Weyl quantum ordering defined in §6.1.

(b) in the case of Figure 4(b), where the end point of $K$ marked by $\varepsilon_1$ is higher in $\partial S \times [0, 1]$ than the point marked by $\varepsilon_2$,

$$\text{Tr}_\lambda^\omega([K, s]) = \begin{cases} 0 & \text{if } \varepsilon_1 = \varepsilon_2 \\ -\omega^5 & \text{if } \varepsilon_1 = +\text{ and } \varepsilon_2 = - \\ \omega^{-1} & \text{if } \varepsilon_1 = -\text{ and } \varepsilon_2 = + \end{cases}$$

Figure 4. Elementary skeins in the triangle

When $A = \pm 1$ and $S$ has no boundary, $\text{Tr}_\lambda^\omega([K])$ is just the Laurent polynomial which, for an element $r \in R^\text{spin}_{PSL(2)\mathbb{C}}(S)$ or $R_{SL(2)\mathbb{C}}(S)$, expresses the trace $\text{Tr}_r([K])$ in terms of (the square roots of) the shear coordinates of $r$ with respect to $\lambda$. See [BoW1].

When gluing skeins, it is crucial to keep track of the ordering of boundary points by their elevation. Figure 4(b) illustrates a convenient method to describe this ordering in pictures representing a link $K$ in $S \times [0, 1]$. For a component $d$ of $\partial S$, we first choose an orientation of $d$ and indicate it by an arrow. We then modify $K$ by an isotopy (respecting the elevations of boundary points) so that the ordering of the points of $\partial K \cap (d \times [0, 1])$ by increasing elevation exactly corresponds to the ordering of their projections to $S$ for the chosen orientation of $d$; in this way, the ordering of these points becomes immediately clear on the picture. In addition, we will always assume in pictures that the framing is vertical, namely pointing towards the reader.
6.4. Compatibility between the Chebyshev, quantum trace and Frobenius homomorphisms. Theorem 20 below shows that the Chebyshev homomorphism $T^A: S^ε(S) → S^A(S)$, the Frobenius homomorphism $F^A: T^\lambda(\lambda) → T^\omega(\lambda)$, and the quantum trace homomorphisms $Tr^\omega_A: S^A(S) → T^\omega(\lambda)$ and $Tr^\lambda_A: S^\epsilon(S) → T^\lambda(\lambda)$ are remarkably compatible with each other. Theorem 4 in the introduction is an immediate consequence of this statement.

**Theorem 20.** Let $S$ be a punctured surface with no boundary, and let $\lambda$ be an ideal triangulation of $S$. Then, if $A^4$ is a primitive $N$–root of unity, $A = \omega^{-2}$, $\epsilon = AN^2$ and $i = \omega N^2$, the diagram

$$
\begin{array}{ccc}
S^A(S) & \xrightarrow{Tr^\omega_A} & T^\omega(\lambda) \\
\downarrow{F^A} & & \downarrow{F^\omega} \\
S^\epsilon(S) & \xrightarrow{Tr^\lambda_A} & T^\lambda(\lambda)
\end{array}
$$

is commutative. Namely, for every skein $[K] \in S^\epsilon(S)$, the quantum trace $Tr^\omega_A([K^T\lambda])$ of $[K^T\lambda]$ is obtained from the classical trace polynomial $Tr^\lambda_A([K])$ by replacing each generator $Z_i ∈ T^\lambda(\lambda)$ by $Z_i^N ∈ T^\omega(\lambda)$.

The proof of Theorem 20 will occupy all of the next section. As should already be apparent from the statement, it involves a large number of “miraculous cancellations”.

7. Quantum traces of Chebyshev skeins

7.1. Algebraic preliminaries. For a parameter $a \in \mathbb{C} - \{0\}$, we will make use of two types of quantum integers,

$$
(n)_a = \frac{a^n - 1}{a - 1} = a^{n-1} + a^{n-2} + a^{n-3} + \cdots + 1,
$$

and

$$
[n]_a = \frac{a^n - a^{-n}}{a - a^{-1}} = a^{n-1} + a^{-n-3} + a^{-n-5} + \cdots + a^{-(n-1)} = a^{-(n-1)}(n)_a.
$$

In particular, the first type of quantum integers will occur in the quantum factorials

$$
(n)_a! = (n)_a (n - 1)_a \cdots (2)_a (1)_a,
$$

and in the quantum binomial coefficients

$$
\binom{n}{p}_a = \frac{(n)_a (n - 1)_a \cdots (n - p + 2)_a (n - p + 1)_a}{(p)_a (p - 1)_a \cdots (2)_a (1)_a} = \frac{(n)_a!}{(p)_a! (n - p)_a!}.
$$

Although this definition of the quantum binomial coefficient $\binom{n}{p}_a$ requires that $a^k \neq 1$ for every $k \leq \min\{p, n - p\}$, see Lemmas 21 or 22 below to make sense of it in all cases.

Quantum binomial coefficients naturally arise in the following classical formula (see for instance Kass §IV.2).

**Lemma 21** (Quantum Binomial Formula). If the quantities $X$ and $Y$ are such that $YX = aXY$, then

$$
(X + Y)^n = \sum_{p=0}^{n} \binom{n}{p}_a X^{n-p} Y^p.
$$

□
We will encounter quantum binomial coefficients in a more combinatorial setting, where they occur as generating functions. Let \( P \) be a linearly ordered set of \( n \) points. A state for \( P \) is a map \( s: P \to \{+, -\} \) assigning a sign \( s(x) = + \) or \( - \) to each point \( x \in P \). An inversion for \( s \) is a pair \((x, x')\) of two points \( x, x' \in P \) such that \( x < x' \) for the ordering of \( P \), but such that \( s(x) > s(x') \) (namely \( s(x) = + \) and \( s(x') = - \)). Let \( \iota(s) \) denote the number of inversions of \( s \), and let \( |s| = \#\{x \in P; s(x) = +\} \) be the number of \(+\) signs in \( s \).

**Lemma 22.**

\[
\binom{n}{P/a} = \sum_{|s|=p} a^{\iota(s)}
\]

where the sum is over all states \( s: \{1, 2, \ldots, n\} \to \{+, -\} \) with \(|s| = p\).

**Proof.** This essentially is a rephrasing of the Quantum Binomial Formula. Indeed, the expansion of \((X + Y)^n\) is the sum of all monomials \( Z_1Z_2\ldots Z_n \) where each \( Z_i \) is equal to \( X \) or \( Y \). Such a monomial \( Z_1Z_2\ldots Z_n \) can be described by a state \( s: \{1, 2, \ldots, n\} \to \{+, -\} \), defined by the property that \( s(i) = - \) if \( Z_i = X \) and \( s(i) = + \) when \( Z_i = Y \). Then \( Z_1Z_2\ldots Z_n = a^{\iota(s)}X^{|s|}Y^{n-|s|}\) after reordering the terms. The result now follows from the Quantum Binomial Formula of Lemma 21 by considering the coefficient of \( X^{n-p}Y^p \). \( \square \)

7.2. **Quantum traces in the biangle.** In practice, computing the quantum trace of Theorem 19 is made easier by including biangles among allowable surfaces, namely infinite strips diffeomorphic to \([0, 1] \times \mathbb{R}\).

![Elementary skeins in the biangle](image)

**Figure 5.** Elementary skeins in the biangle

**Proposition 23** (Proposition 13 and Lemma 14 of [BoW1]). For \( A = \omega^{-2} \), there is a unique family of algebra homomorphisms

\[ \text{Tr}_B^\omega: \mathcal{S}_\omega^A(B) \to \mathbb{C}, \]

defined for all oriented biangles \( B \), such that:

1. **(State Sum Property)** if the biangle \( B \) is obtained by gluing together two distinct biangles \( B_1 \) and \( B_2 \), and if \([K_1] \in \mathcal{S}^A(B_1)\) and \([K_2] \in \mathcal{S}^A(B_2)\) are glued together to give \([K] \in \mathcal{S}^A(B)\), then

\[ \text{Tr}_B^\omega([K, s]) = \sum_{\text{compatible } s_1, s_2} \text{Tr}_{B_1}^\omega([K_1, s_1]) \text{Tr}_{B_2}^\omega([K_2, s_2]), \]

where the sum is over all states \( s_1: \partial K_1 \to \{+, -\} \) and \( s_2: \partial K_2 \to \{+, -\} \) that are compatible with \( s: \partial K \to \{+, -\} \) and with each other;
(2) (Elementary Cases) if the biangle $B$ is represented by a vertical strip in the plane as in Figure 5 and if $K$ projects to a single arc embedded in $B$, then

(a) in the case of Figure 5(a), where $\varepsilon_1$, $\varepsilon_2 = \pm$ are the signs associated by the state $s$ to the end points of $K$,

\[
\text{Tr}_{B}^{\omega}([K, s]) = \begin{cases} 1 & \text{if } \varepsilon_1 = \varepsilon_2 \\ 0 & \text{if } \varepsilon_1 \neq \varepsilon_2; \end{cases}
\]

(b) in the case of Figure 5(b),

\[
\text{Tr}_{B}^{\omega}([K, s]) = \begin{cases} 0 & \text{if } \varepsilon_1 = \varepsilon_2 \\ -\omega^{-5} & \text{if } \varepsilon_1 = + \text{ and } \varepsilon_2 = - \\ \omega^{-1} & \text{if } \varepsilon_1 = - \text{ and } \varepsilon_2 = +. \end{cases}
\]

(c) in the case of Figure 5(c),

\[
\text{Tr}_{B}^{\omega}([K, s]) = \begin{cases} 0 & \text{if } \varepsilon_1 = \varepsilon_2 \\ \omega & \text{if } \varepsilon_1 = + \text{ and } \varepsilon_2 = - \\ -\omega^{-5} & \text{if } \varepsilon_1 = - \text{ and } \varepsilon_2 = +. \end{cases}
\]

\[\square\]

Note that Figure 5 uses the picture conventions indicated at the end of Section 3. In particular, framings are everywhere vertical and, in Figures 5(b) and (c), the end point labelled by $\varepsilon_1$ sits at a higher elevation than the point labelled by $\varepsilon_2$.

The quantum trace $\text{Tr}_{B}^{\omega}$ of Proposition 23 is just a version of the classical Kauffman bracket for tangles.

The relationship between the quantum traces of Proposition 23 and Theorem 19 is the following. If we glue one side of a biangle $B_1$ to one side of a triangle $T_2$, the resulting surface is a triangle $T$. Each side of $T$ is naturally associated to a side of $T_2$, so that there is a natural isomorphism $\mathcal{T}^{\omega}(T) \cong \mathcal{T}^{\omega}(T_2)$ between their associated Chekhov-Fock algebras.

The following is an immediate consequence of the construction of the quantum trace homomorphism in [BoW 6] (which uses biangles as well as triangles).

**Proposition 24.** If the triangle $T$ is obtained by gluing a biangle $B_1$ to a triangle $T_2$, and if $[K_1] \in S^A(B_1)$ and $[K_2] \in S^A(T_2)$ are glued together to give $[K] \in S^A(T)$, then for $A = \omega^{-2}$

\[
\text{Tr}_{T}^{\omega}([K, s]) = \sum_{\text{compatible } s_1, s_2} \text{Tr}_{B_1}^{\omega}([K_1, s_1]) \text{Tr}_{T_2}^{\omega}([K_2, s_2]) \in \mathcal{T}^{\omega}(T_2) \cong \mathcal{T}^{\omega}(T),
\]

where the sum is over all states $s_1: \partial K_1 \to \{+, -\}$ and $s_2: \partial K_2 \to \{+, -\}$ that are compatible with $s: \partial K \to \{+, -\}$ and with each other.

\[\square\]

Note that, because a triangle $T$ has a unique triangulation $\lambda$, we write $\text{Tr}_{T}^{\omega}$ and $\mathcal{T}^{\omega}(T)$ instead of $\text{Tr}_{\lambda}^{\omega}$ and $\mathcal{T}^{\omega}(\lambda)$, by a slight abuse of notation.

**7.3. Jones-Wenzl idempotents.** The Jones-Wenzl idempotents are special elements $JW_n \in S^A(B)$ of the skein algebra of the biangle $B = [0, 1] \times \mathbb{R}$. The $n$-th Jones Wenzl idempotent $JW_n$ is a linear combination of skeins, each of which meets each of the two components of $\partial B \times [0, 1]$ in $n$ points.

It is convenient (and traditional) to represent $JW_n$ by a box with $n$ strands emerging on each side, as on the left-hand side of Figure 6. In this case, it is important to remember that the box does not represent a single link, but a linear
combination of links. When several boxes appear on a picture, multiple sums need to be taken.

\[
\begin{align*}
&= + [n-1]_{A^2} [n]_{A^2} - [n-1]_{A^2} [n-1]_{A^2} \\
&= + [n-1]_{A^2} [n]_{A^2} - [n-1]_{A^2} [n-1]_{A^2} \\
&= + [n-1]_{A^2} [n]_{A^2} - [n-1]_{A^2} [n-1]_{A^2}
\end{align*}
\]

**Figure 6.** The Jones-Wenzl recurrence relation

The Jones-Wenzl idempotents are then inductively defined by the property that \( JW_1 \) is the skein consisting of a single unknotted arc with vertical framing, as in Figure 5(a), and by the recurrence relation indicated in Figure 6. In particular, \( JW_n \) is defined as long as \( [k]_{A^2} \neq 0 \), or equivalently \( A^{4k} \neq 1 \), for every \( k \leq n \).

The book chapter [Lic1, Chap. 13] or the survey article [Lic2] are convenient references for Jones-Wenzl idempotents. We will use the relations in \( S^A(B) \) summarized in Figures 7, 8 and 9.

\[
\begin{align*}
\text{Figure 7.} \quad &\text{The Jones-Wenzl idempotent property} \\
\text{Figure 8.} \quad &\text{Gluing a U-turn to a Jones-Wenzl idempotent} \\
\text{Figure 9.} \quad &\text{What makes Jones-Wenzl idempotents convenient for us is the following property.}
\end{align*}
\]

If we glue the two boundary components of the biangle \( B \) together, we obtain a cylinder \( C \).

**Lemma 25.** Let the cylinder \( C \cong S^1 \times \mathbb{R} \) be obtained by gluing together the two sides of the biangle \( B = [0,1] \times \mathbb{R} \). Then, the element of \( S^A(C) \) obtained by applying this gluing to the Jones-Wenzl idempotent \( JW_n \in S^A(B) \) is equal to the evaluation \( S_n([K]) \) of the Chebyshev polynomial \( S_n \) at the skein \([K] \in S^A(C)\) represented by an embedded loop \( K \subset S \times \{1/2\} \) going around the cylinder, endowed with the vertical framing. \( \square \)
See for instance [Lic1, §13] or [Lic2, p. 715] for a proof.

We will use Lemma 25 to compute the quantum traces $\text{Tr}^\omega_K([K^S_N])$ and $\text{Tr}^\omega_K([K^T_N])$ of the elements $[K^S_N]$, $[K^T_N] \in \mathcal{S}^4(K)$ respectively obtained by threading the Chebyshev polynomials $S_N$ and $T_N$ along a framed link $K$. A particularly useful feature is the idempotent property of Figure 7, which will enable us to localize the computations.

Jones-Wenzl idempotents however have the drawback that not all of them are defined when $A$ is a root of unity (and in particular that $JW_N$ is undefined when $A^{4N} = 1$, which is precisely that case we are interested in). Consequently, we will temporary assume that $A$ is generic, namely is not a root of unity, to ensure that all Jones-Wenzl idempotents are defined. We will then let $A$ tend to an appropriate root of unity, and analyze what happens in the limit to complete our computation.

7.4. Quantum traces of Jones-Wenzl idempotents in the bangle. As indicated above, we assume for a while that $A$ is generic. Let us fix an integer $N \geq 1$.

Let $s$ be a state for the Jones-Wenzl idempotent $JW_N \in \mathcal{S}^4(B)$. This means that we are assigning the same state $s$ to each of the links that appear in the expression of $JW_N$. It is here important that all these links have the same boundary, which we denote by $\partial JW_N$.

Let $e_1$ and $e_2$ be the two boundary components of the bangle $B$, oriented in a parallel way and so that $e_1$ is the component on the left. The state $s$ for $JW_N$ consists of a state $s_1$ on $e_1 \cap \partial JW_N$ and a state $s_2$ on $e_2 \cap \partial JW_N$ (where $e_i \cap \partial JW_N$ is shorthand notation for $(e_i \times [0,1]) \cap \partial JW_N$). By convention, we order the points of $e_i \cap \partial JW_N$ from bottom to top, in increasing order vertically upwards along the $[0,1]$–direction. As in Figure 7, let $|s_i|$ and $\iota(s_i)$ respectively denote the number of $+$ signs and the number of inversions of $s_i$.

**Proposition 26.** For a generic $A \in \mathbb{C} - \{0\}$,

\[
\text{Tr}_B(JW_N, s) = 0 \quad \text{if } |s_1| \neq |s_2|, \quad \text{and}
\]

\[
\text{Tr}_B(JW_N, s) = \frac{A^{2\iota(s_1)A^{2\iota(s_2)}}}{N^\lambda} \quad \text{if } |s_1| = |s_2| = p.
\]

**Proof:** The first property is an easy consequence of the definition of the quantum trace of Proposition 23, see [BoW1, Lemma 21].

For the second property, we first reduce our computation to the case where there is no inversion.

Suppose that $s_1$ has at least one inversion, corresponding to two points $x$, $x' \in e_1 \cap \partial JW_N$ such that $x < x'$ for the ordering of $e_1 \cap \partial JW_N$, $s(x) = +$ and $s(x') = -$. Without loss of generality, we can assume that $x$ and $x'$ are next to each other for the order on $e_1 \cap \partial JW_N$. Let the state $s'$ be obtained from $s$ by exchanging $s(x)$ and $s(x')$. In particular, $\iota(s') = \iota(s_1) - 1$.

Glue to $B$ another bangle $B_1$ along $e_1$, and glue to $[JW_N] \in \mathcal{S}^4(B)$ along the points $x$ and $x'$ a U-turn $K_1$ as in Figure 8. By the identity described by Figure 8, the resulting element of $\mathcal{S}^4(B \cup B_1)$ is trivial. Combining the State Sum Property of Proposition 23 with the values of the quantum trace for the skein of Figure 8(c),

\[0 = \omega \text{Tr}_B(JW_N, s') - \omega^5 \text{Tr}_B(JW_N, s)\]

and therefore $\text{Tr}_B(JW_N, s) = A^2 \text{Tr}_B(JW_N, s')$ since $A = \omega^{-2}$. 

The same formula holds when \( s' \) is obtained from \( s \) by removing an inversion of consecutive points in \( e_2 \). This proves that

\[
\text{Tr}_B(JW_N, s) = A^{2(s_1)} A^{2(s_2)} \text{Tr}_B(JW_N, s_0)
\]

where \( s_0 \) is the state that has no inversion, and has \( p \) signs + on \( e_1 \cap \partial JW_N \) and on \( e_2 \cap \partial JW_N \).

It therefore suffices to show that \( C(N, p) = \text{Tr}_B(JW_N, s_0) \) is equal to \( \left( \frac{N}{p} \right)_{A^4}^{-1} \).

We will prove this by induction on the number \( N - p \) of minus signs in the states \( s_1 \) and \( s_2 \).

Using the recurrence relation of Figure 5 to expand \( [JW_N] \in S^4(B) \) as a linear combination of skeins, the only skein in this expansion that does not contain a U-turn as in Figure 5(b) is the one consisting of \( N \) parallel strands as in Figure 5(a), and the coefficient of this term in the expansion is equal to 1. It follows that \( C(N, N) = 1 \), which proves the initial step of the induction, when \( N - p = 0 \).

For the general case, suppose the property proved for every \( N', p' \) with \( N' - p' < N - p \). Consider the identity in \( S^4(B) \) represented in Figure 5 or [Lic] (see [Lic], p. 137) or [Lic2, p. 714] for a proof. Endow both sides with the state that has no inversion, and has \( p \) signs + on each side of \( B \). If we use the State Sum Property to compute the quantum trace of the left hand side of the equation of Figure 5, only two terms have non-trivial contributions, and this relation gives

\[
[N + 1]_{A^4} C(N - 1, p) = -[N]_{A^4} \left( (-\omega^5) C(N, p + 1) \omega^{-1} + \omega A^{2p} A^{2p} C(N, p)(-\omega^{-5}) \right).
\]

Remembering that \( [N]_{A^2} = A^{-2(N-1)}(N)_{A^4} \) and \( A = \omega^{-2} \), and using the induction hypothesis,

\[
C(N, p) = A^{-4(p+1)} \frac{(N+1)_{A^4}}{(N)_{A^4}^4} C(N, p) - A^{-4(p+1)} C(N, p + 1)
\]

\[
= A^{-4(p+1)} \frac{(N+1)_{A^4}^4}{(N)_{A^4}^4} \left( (p)_{A^4} (p-1)_{A^4} \cdots (1)_{A^4} \right) \frac{(N-1)_{A^4} (N-2)_{A^4} \cdots (N-p-1)_{A^4}}{(N)_{A^4} (N-1)_{A^4} \cdots (N-p)_{A^4}}
\]

\[
= A^{-4(p+1)} \frac{(N+1)_{A^4}^4}{(N-p)_{A^4}^4} \left( (p+1)_{A^4} (p)_{A^4} \cdots (1)_{A^4} \right) \frac{(N-1)_{A^4} (N-2)_{A^4} \cdots (N-p)_{A^4}}{(N)_{A^4} (N-1)_{A^4} \cdots (N-p+1)_{A^4}}
\]

\[
= \left( \frac{N}{p} \right)_{A^4}^{-1}.
\]

The last step uses the property that \( (N+1)_{A^4}^4 - (p+1)_{A^4}^4 = A^{4(p+1)} (N-p)_{A^4}^4 \), which is an immediate consequence of the definition of the quantum integers \( (k)_{A^4} \).

This concludes the proof by induction that \( C(N, p) = \left( \frac{N}{p} \right)_{A^4}^{-1} \), and therefore that

\[
\text{Tr}_B(JW_N, s) = A^{2(s_1)} A^{2(s_2)} \text{Tr}_B(JW_N, s_0) = A^{2(s_1)} A^{2(s_2)} \left( \frac{N}{p} \right)_{A^4}^{-1}
\]

when \( |s_1| = |s_2| = p \).
7.5. Quantum traces of Jones-Wenzl skeins in the triangle. We now consider a Jones-Wenzl idempotent $JW_N$ in a triangle $T$, as represented in Figure 10(a).

Proposition 27. In the triangle $T$, let $JW_N \in S^A(T)$ be the Jones-Wenzl idempotent represented in Figure 10(a). Let $e_1$ and $e_2$ be the two sides of $T$ indicated in Figure 10(a), and let $Z_1$ and $Z_2$ denote the corresponding generators of $T^\omega(T)$. Let $s$ be a state for $JW_N$, consisting of a state $s_1$ on $e_1 \cap JW_N$ and a state $s_2$ on $e_2 \cap JW_N$, let $\iota(s_1)$ and $\iota(s_2)$ be the respective numbers of inversions of $s_1$ and $s_2$, and set $p_1 = |s_1|$ and $p_2 = |s_2|$.

Then $\text{Tr}_T^\omega([JW_N, s]) = 0$ if $p_2 > p_1$, and otherwise

$$\text{Tr}_T^\omega([JW_N, s]) = A^{2(\iota_1)} A^{2(\iota_2)} \frac{(N-p_2)_{A^4}(p_1)_{A^4}}{(N)_{A^4}(p_1-p_2)_{A^4}} A^{-(p_1-p_2)(N-p_1+p_2)} [Z_1^{2p_1-N} Z_2^{2p_2-N}]$$

where $[Z_1^{2p_1-N} Z_2^{2p_2-N}]$ denotes the Weyl quantum ordering for the monomial $Z_1^{2p_1-N} Z_2^{2p_2-N}$, as defined in 6.1.

**Proof.** The same U-turn trick as in the proof of Proposition 26 reduces the computation to the case where there are no inversions. Therefore, we henceforth assume that $s_1$ and $s_2$ have no inversion.

Split the triangle $T$ into a biangle $B_1$ and a triangle $T_2$ as in Figure 10(b), and let $e_0$ be their common edge $B_1 \cap T_2$. In particular, the Jones-Wenzl idempotent $JW_N$ in $T$ splits into a Jones-Wenzl idempotent $JW_N^1$ in the biangle $B_1$ and into a family $K_2$ of parallel strands in the triangle $T_2$. Applying the State Sum Property of Proposition 24

$$\text{Tr}_T^\omega(JW_N, s) = \sum_{s_0} \text{Tr}_{B_1}^\omega(JW_N, s_1 \cup s_0) \text{Tr}_{T_2}^\omega(K_2, s_2 \cup s_0)$$

where the sum is over all states $s_0$ for $e_0 \cap JW_N$.

By Proposition 26, a state $s_0$ with a non-trivial contribution to the above sum is such that $|s_0| = p_1$, and in this case

$$\text{Tr}_{B_1}^\omega(JW_N, s_1 \cup s_0) = A^{2(\iota_0)} \left( \frac{N}{p_1} \right)^{-1}_{A^4}.$$

We now consider the terms coming from the triangle $T_2$. Let $s_j(i) \in \{-, +\}$ denote the sign assigned by the state $s_j$ to the $i$-th point of $e_j \cap JW_n$. In particular, since $s_2$ has no inversion, $s_2(i) = +$ if and only if $i > N - p_2$. Therefore, by Case 2(a) of Theorem 19 if $\text{Tr}_{T_2}^\omega(K_2, s_2 \cup s_0) \neq 0$ then necessarily $s_0(i) = +$ for every $i > N - p_2$. In addition, identifying the sign ± to the number ±1 in the exponents, this contribution is then equal to

$$\text{Tr}_{T_2}^\omega(K_2, s_2 \cup s_0) = [Z_1^{s_0(1)} Z_2^{-1}] [Z_1^{s_0(2)} Z_2^{-1}] \cdots [Z_1^{s_0(N-p_2)} Z_2^{-1}] [Z_1 Z_2]^{p_2}.$$
Using the property that
\[ [Z_1 Z_2^{-1}] [Z_1^{-1} Z_2^{-1}] = \omega^{-4} [Z_1^{-1} Z_2^{-1}] [Z_1 Z_2^{-1}] = A^2 [Z_1^{-1} Z_2^{-1}] [Z_1 Z_2^{-1}], \]
the terms in this contribution can be reordered as
\[ \text{Tr}_\omega^i (K_2, s_2 \cup s_0) = A^{\alpha(s_0)} [Z_1^{-1} Z_2^{-1}]^{N-|s_0|} [Z_1 Z_2^{-1}]^{|s_0|-p_2} [Z_1 Z_2]^{p_2} \]

Combining this with Lemma \ref{lemma22}, we obtain
\[ \text{Tr}_\omega^i (JW_N, s) = \sum_{s_0(i) = + \text{ if } i > N - p_2} A^{\alpha(s_0)} \binom{N}{p_1}^{-1} \left[ Z_1^{-1} Z_2^{-1} \right]^{N-p_1} [Z_1 Z_2^{-1}]^{p_1-p_2} [Z_1 Z_2]^{p_2} \]
\[ = \binom{N}{p_1-p_2} A_{A^4} \binom{N}{p_1}^{-1} [Z_1^{-1} Z_2^{-1}]^{N-p_1} [Z_1 Z_2^{-1}]^{p_1-p_2} [Z_1 Z_2]^{p_2} \]
\[ = \binom{N}{p_1-p_2} A_{A^4} \binom{N}{p_1-p_2}^{-1} \left[ Z_1^{-2p_1-N} Z_2^{-2p_2-N} \right] \]

after a final grouping of terms.

This concludes the proof of Proposition \ref{proposition27} \hfill \Box

7.6. Evaluation of Chebyshev threads of the second kind. Let \([K] \in S^A(S)\) be a skein in a surface \(S\) without boundary, and let \(\lambda\) be an ideal triangulation of \(S\). We will restrict attention to the case where \(K \subset S \times [0,1]\) is a \(\lambda\)–simple knot, in the sense that it projects to a simple closed curve in \(S\) that meets each triangle of \(\lambda\) in at most one arc, and that its framing is everywhere vertical.

This \(\lambda\)–simple condition is not essential, but it will greatly simplify our exposition since we will not have to worry about the order in which we take terms in the contribution of each triangle. It will also be sufficient to restrict attention to this case, because \(\lambda\)–simple skeins generate \(S^A(S)\) as an algebra \([Bu]3\).

Arbitrarily pick an orientation for \(K\). Let \(e_1, e_2, \ldots, e_k, e_{k+1} = e_1\) denote, in this order, the edges of \(\lambda\) that are crossed by the projection of \(K\) to \(S\). Similarly, let \(T_1, T_2, \ldots, T_k\) be the triangles of \(\lambda\) that are crossed by \(K\), in such a way that \(K\) crosses \(T_i\) between \(e_{i-1}\) and \(e_i\). When \(K\) crosses \(e_i\), the orientations of \(K\) and \(S\) determine a left and a right endpoint for \(e_i\), and there are four possible configurations according to whether \(e_{i-1}\) and \(e_{i+1}\) are respectively adjacent to the left or right endpoint of \(e_i\). We will say that \(K\) crosses \(e_i\) in a left-left, left-right, right-left or right-right pattern accordingly. For instance \(K\) crosses \(e_i\) in a left-right pattern if \(e_{i-1}\) is adjacent to the left endpoint of \(e_i\), and \(e_{i+1}\) is adjacent to its right end point.

Let \([K^S_N] \in S^A(S)\) be obtained by threading the Chebyshev polynomial of the second kind \(S^N_N\) along \(K\).

**Proposition 28.** Let \([K] \in S^A(S)\) be a \(\lambda\)–simple skein in the surface \(S\). Then, for a generic \(A\),
\[ \text{Tr}_\lambda^i ([K^S_N]) = \sum_{p_1, p_2, \ldots, p_k} a_0 b_2 \ldots b_k [Z_1^{2p_1-N} Z_2^{2p_2-N} \ldots Z_k^{2p_k-N}] \]
where the sum is over all integers $p_i$ with $0 \leq p_i \leq N$, where

$$a_0 = \prod_{i=1}^{k} \frac{A^{(p_i-p_{i+1})^2}}{(|p_i - p_{i+1}|)A^4}$$

and where

$$b_i = \begin{cases} 1 & \text{if } p_{i-1} \geq p_i \geq p_{i+1} \text{ when } K \text{ crosses } e_i \text{ in a left-left pattern,} \\ 0 & \text{otherwise} \end{cases}$$

$$b_i = \begin{cases} 1 & \text{if } p_{i-1} \leq p_i \leq p_{i+1} \text{ when } K \text{ crosses } e_i \text{ in a right-right pattern,} \\ 0 & \text{otherwise} \end{cases}$$

$$b_i = \begin{cases} A^{2Np_1(N-p_i)A^4} & \text{if } p_{i-1} \geq p_i \leq p_{i+1} \text{ when } K \text{ crosses } e_i \text{ in a left-right pattern,} \\ 0 & \text{otherwise} \end{cases}$$

$$b_i = \begin{cases} A^{-2Np_1(N-p_i)A^4} & \text{if } p_{i-1} \leq p_i \geq p_{i+1} \text{ when } K \text{ crosses } e_i \text{ in a right-left pattern.} \\ 0 & \text{otherwise} \end{cases}$$

Proof. We will use Jones-Wenzl idempotents.

By Lemma 25, the element $[K^{S_N}] \in S^A(S)$ can be obtained by threading the Jones-Wenzl idempotent $JW_N$ along $K$ (using a thickened annulus with core $K$ embedded in $S \times [0,1]$). Using the idempotent property of Figure 7, we can even put a Jones-Wenzl idempotent above each triangle $T_j$ of the ideal triangulation $\lambda$.

Consequently, for each triangle $T_j$, replace the arc $K \cap (T_j \times [0,1])$ by a Jones-Wenzl idempotent $JW_{N}^{(j)}$ as in Figure 10(a). Our discussion above shows that $[K^{S_N}]$ is equal to the element of $S^A(S)$ obtained by gluing the $JW_{N}^{(i)}$ together.

We can then apply the State Sum Property, which gives

$$\text{Tr}_{A}^{N}([K^{S_N}]) = \sum_{s_1, s_2, \ldots, s_k} \prod_{i=1}^{k} \text{Tr}_{T_i}^{N}(JW_{N}^{(i)}, s_{i-1} \cup s_i)$$

where the $s_i$ range over all states for $e_i \cap K^{S_N}$. Letting $p_i = |s_i|$ be the number of + signs in $s_i$, Proposition 27 computes the contribution of each family of states $s_1, s_2, \ldots, s_k$ as

$$\prod_{i=1}^{k} \text{Tr}_{T_i}^{N}(JW_{N}^{(i)}, s_{i-1} \cup s_i) = c_1 \ldots c_k \ A^{4s(s_1)} \ldots A^{4s(s_k)} \ Z_1^{2p_1-N} \ldots Z_k^{2p_k-N}$$

where

$$c_i = \begin{cases} 0 & \text{if } p_{i-1} < p_i \\ \frac{(N-p_i)A^4(p_i-1)A^4}{(N)A^4(p_{i-1}-p_i)A^4} A^{(p_{i-1}-p_i)^2+N(p_i-p_{i-1})} & \text{if } p_{i-1} \geq p_i \end{cases}$$

when $e_{i-1}$ and $e_i$ are adjacent to the left, and

$$c_i = \begin{cases} 0 & \text{if } p_{i-1} > p_i \\ \frac{(N-p_i)A^4(p_i-1)A^4}{(N)A^4(p_{i-1}-p_i)A^4} A^{(p_{i-1}-p_i)^2+N(p_i-p_{i-1})} & \text{if } p_{i-1} \leq p_i \end{cases}$$

when $e_{i-1}$ and $e_i$ are adjacent to the right. Note that we are here using the property that $K$ is $\lambda$–simple, namely that $K$ crosses each triangle at most once, so that the contributions of the various $i = 1, \ldots, k$ commute. In particular, the quantum
ordering \( [Z_1^{2p_1-N} \ldots Z_k^{2p_k-N}] \) is equal to the product of the quantum orderings of these \( k \) contributions.

If we fix the numbers \( p_1, p_2, \ldots, p_k \) and sum over all states \( s_i \) with \( |s_i| = p_i \), Lemma \(^22\) shows that

\[
\sum_{|s_i|=p_i} A^{4k(s_1)} A^{4k(s_2)} \ldots A^{4k(s_k)} = \left( \frac{N}{p_1} \right) A^4 \left( \frac{N}{p_2} \right) A^4 \ldots \left( \frac{N}{p_k} \right) A^4.
\]

Therefore, the contribution of the states \( s_i \) with \( |s_i| = p_i \) is equal to

\[
e_1 \ldots e_k \left( \frac{N}{p_1} \right) A^4 \ldots \left( \frac{N}{p_k} \right) A^4 [Z_1^{2p_1-N} \ldots Z_k^{2p_k-N}].
\]

This product is often 0. When it is not, many of the quantum factorials involved in the coefficients \( e_i \) and \( \left( \frac{N}{p_i} \right) A^4 \) cancel out. For instance, all terms \( (N)_{A_4}! \) disappear. The remaining terms are then easily grouped as in the statement of Proposition \(^28\) \( \□ \)

7.7. Evaluation of Chebyshev threads of the first kind. We now turn to Chebyshev polynomials of the first kind \( T_N \). Remembering from Lemma \(^7\) that \( T_N = S_N - S_{N-2} \), we now want to evaluate

\[
\text{Tr}_\lambda^\omega([K^{T_N}]) = \text{Tr}_\lambda^\omega([K^{S_N}]) - \text{Tr}_\lambda^\omega([K^{S_{N-2}}])
\]

for a \( \lambda \)-simple skein \([K] \in S^\lambda(S)\).

For this, it is convenient to rephrase the formula of Proposition \(^28\) by putting more emphasis on the powers \( n_i = 2p_i - N \) of the generators \( Z_i \). Note that \(-N \leq n_i \leq N\), and that \( n_i \) has the same parity as \( N \).

We will say that a sequence \( n_1, n_2, \ldots, n_k \) is admissible if each of the corresponding \( p_i = \frac{N+n_i}{2} \) contributes a non-trivial term to the formula of Proposition \(^28\) namely if \(-N \leq n_i \leq N\) for every \( i \), each \( n_i \) has the same parity as \( N \), and if

\[
n_{i-1} \geq n_i \text{ when } e_{i-1} \text{ and } e_i \text{ are adjacent to the left, and } n_{i-1} \leq n_i \text{ when } e_{i-1} \text{ and } e_i \text{ are adjacent to the right}.
\]

Then Proposition \(^28\) can be rephrased as

**Proposition 29.** For a generic \( A \),

\[
\text{Tr}_\lambda^\omega([K^{S_N}]) = \sum_{\text{admissible } n_1,n_2,\ldots,n_k} a_0 b_1 b_2 \ldots b_k [Z_1^{n_1} Z_2^{n_2} \ldots Z_k^{n_k}]
\]

where

\[
a_0 = \prod_{i=1}^k \frac{A^\left( -n_i + n_{i+1} \right)^2}{\left( n_i - n_{i+1} \right)^2 A_4!}
\]

and where

\[
b_i = 1 \text{ when } K \text{ crosses } e_i \text{ in a left-left or right-right pattern,}
\]

\[
b_i = A^{N(N+n_i)} \frac{(N-n_i)^2}{4} A_4! \text{ when } K \text{ crosses } e_i \text{ in a left-right pattern, and}
\]

\[
b_i = A^{-N(N+n_i)} \frac{(N+n_i)^2}{4} A_4! \text{ when } K \text{ crosses } e_i \text{ in a right-left pattern.} \quad \Box
\]
An almost identical formula holds for $\text{Tr}_A^\omega([K^{S_{N-2}}])$, except that the admissible sequences $n_1, n_2, \ldots, n_k$ are further constrained by the condition that $-N + 2 \leq n_i \leq N - 2$ for every $i$. Because of the parity condition, this is equivalent to $-N < n_i < N$.

More precisely,

$$\text{Tr}_A^\omega([K^{S_{N-2}}]) = \sum_{\text{admissible } n_1, n_2, \ldots, n_k} a_0 b_1' b_2' \cdots b_k' [Z_1^{n_1} Z_2^{n_2} \cdots Z_k^{n_k}]$$

where $a_0$ is defined as in Proposition 29 and

$\quad b_i' = 1$ when $K$ crosses $e_i$ in a left-left or right-right pattern,

$\quad b_i' = A^{4(N-2)(N-2+n_i)} A^4 \left( \frac{\ell - n_i}{2} \right)^{A_4}$ when $K$ crosses $e_i$ in a left-right pattern,

$\quad b_i' = A^{-(N-2)(N-2+n_i)} A^4 \left( \frac{\ell - n_i}{2} \right)^{A_4}$ when $K$ crosses $e_i$ in a right-left pattern.

So far, our computations assumed that $A^4$ was a primitive $N$–root of unity. However, because $(N)_A^4 = 0$ in this case, we have to be careful in the definition of the quantities considered, and make sure that we never attempt to divide by 0. We will make sense of these properties by a limiting process.

**Lemma 30.** Let $n_1, n_2, \ldots, n_k$ be an admissible sequence such that $-N < n_i < N$ for every $i$. Then, the respective contributions

$$a_0 b_1 b_2 \cdots b_k [Z_1^{n_1} Z_2^{n_2} \cdots Z_k^{n_k}]$$

and

$$a_0 b_1' b_2' \cdots b_k' [Z_1^{n_1} Z_2^{n_2} \cdots Z_k^{n_k}]$$

of this admissible sequence to $\text{Tr}_A^\omega([K^{S_{N}}])$ and $\text{Tr}_A^\omega([K^{S_{N-2}}])$ have the same limit as $A^4$ tends to a primitive $N$–root of unity.

**Proof.** Because of the assumption that $-N < n_i < N$, all quantum integers involved in these contributions are different from $(N)_A^4$ and we do not really have to worry about taking limits here. By continuity, choosing $A^4$ to be a primitive $N$–root of unity will be sufficient. We therefore need to show that

$$b_1 b_2 \cdots b_k = b_1' b_2' \cdots b_k'$$

under this hypothesis on $A$.

Let us compare the coefficients $b_i$ and $b_i'$. When $K$ crosses $e_i$ in a left-left or right-right patterns, we of course have that $b_i = b_i' = 1$.

When $K$ crosses $e_i$ in a left-right pattern,

$$b_i = A^{4(N+n_i)} \left( \frac{\ell - n_i}{2} \right)^{A_4} = A^{4N+2n_i-4} A^{N-2(N-2+n_i)} \left( \frac{\ell - n_i}{2} \right)^{A_4} \left( \frac{\ell - n_i}{2} - 1 \right)^{A_4},$$

$$= A^{4N+2n_i-4} \left( \frac{\ell - n_i}{2} \right)^{A_4} A^4 \left( \frac{\ell - n_i}{2} - 1 \right)^{A_4} = A^{4N+2n_i-4} A^{2N-2n_i-4} A^{2N+2n_i-1} - b_i' = -A^{2N-4} b_i',$n

using the fact that $A^{4N} = 1$ for the last equality.

When $K$ crosses $e_i$ in a right-left pattern, a similar computation gives

$$b_i = -A^{-(2N-4)} b_i'.$$
Therefore, as we compute the product of the $\omega$ unity, all the terms of $\text{Tr}_n(\omega)$ and Lemma 31.

Let $n$ tend to 0 are the terms $(\pm b_i)$. In the expression of $\text{Tr}_n(\omega)$, when $d$ indices modulo $k$, $n$ see one more right-left pattern than left-right patterns. It follows that this maximal sequence of consecutive $n_i = \pm N$ is admissible, so $a_{i-1}$ and $a_i$ are necessarily adjacent to the right of $K$, whereas $c_{i+1}$ are adjacent to the left. Therefore, if we examine how $K$ crosses $c_i$ when $i_1 \leq i \leq i_2$, we see one more right-left pattern than left-right patterns. It follows that this maximal sequence of consecutive $n_i = \pm N$ contributes one more $(N)_{A^4}$ to the numerator than to the denominator of the product of the corresponding $b_i$.

Similarly, a maximal sequence of consecutive $n_i = -N$ contributes one more $(N)_{A^4}$ to the numerator than to the denominator of the product of the corresponding $b_i$.

Because of our assumption that there exists at least one $n_j \neq \pm N$, the number of $i$ where $n_i$ switches from $n_i = \pm N$ to $n_{i+1} = \mp N$ is strictly less than the total number of maximal sequences of consecutive $n_i = +N$ and of maximal sequences of consecutive $n_i = -N$. It follows that there is at least one more $(N)_{A^4}$ in the numerator of $a_{0}b_{1}b_{2}\ldots b_{k}$ than in the denominator. This term therefore converges to 0 as $A^4$ tends to a primitive $N$–root of unity. □

Lemma 32. Let $n_1, n_2, \ldots, n_k$ be an admissible sequence where each $n_i$ is equal to $\pm N$. Then the contribution

$$a_0b_1b_2\ldots b_k [Z_1^{n_1} Z_2^{n_2} \ldots Z_k^{n_k}]$$
of this admissible sequence to \( \text{Tr}^\omega_\chi([K^{TN}]) \) is equal to \([Z_1^{n_1} Z_2^{n_2} \cdots Z_k^{n_k}]\).

Proof. The proof of Lemma 31 shows that, in this case, we have exactly as many quantum factorials \((N)_A!\) in the numerator as in the denominator of \(a_0 b_1 b_2 \cdots b_k\). These quantum factorials therefore cancel out.

All remaining quantum factorials are equal to \((0)_A! = 1\). We therefore only have to worry about the powers of \(A\) that occur in \(a_0 b_1 b_2 \cdots b_k\).

Going back to the definition of the constants \(a_0\) and \(b_i\) in Proposition 29, one obtains that
\[
a_0 b_1 b_2 \cdots b_k = A^{2N^2(\alpha + \beta - \beta')}
\]
where
\[
\alpha = \frac{1}{2} \sum_{i=1}^k \left( \frac{n_{i+1} - n_i}{2N} \right)^2 = \frac{1}{2} \# \{ i; n_i \neq n_{i+1} \}
\]
\[
\beta = \sum_{\text{left-right pattern at } e_i} \frac{N+n_i}{N} = \# \{ i; n_i = +N \text{ and left-right pattern at } e_i \}
\]
and
\[
\beta' = \sum_{\text{right-left pattern at } e_i} \frac{N+n_i}{N} = \# \{ i; n_i = +N \text{ and right-left pattern at } e_i \}.
\]

If all \(n_i\) are equal to each other, then \(\alpha = 0\) and \(\beta = \beta'\), so that \(\alpha + \beta - \beta' = 0\).

Otherwise, \(\alpha\) is equal to the number of intervals \(I = \{ u, u+1, \ldots, v-1, v \} \) in the index set (counting indices modulo \(k\)) where \(n_i = +N\) for every \(i \in I\) while \(n_{u-1} = n_{v+1} = -N\). For such an interval \(I\), the \(i \in I\) contribute a total of \(-1\) to \(\beta - \beta'\). Since there are \(\alpha\) such intervals \(I\), we conclude that \(\alpha + \beta - \beta' = 0\) in this case as well.

This proves that \(a_0 b_1 b_2 \cdots b_k = 1\) in all cases. □

If we combine Proposition 29 and Lemmas 30, 31 and 32 we now have the following computation.

Proposition 33. Suppose that \(A^4\) is a primitive \(N\)–root of unity. Then, for every \(\lambda\)–simple skein \([K] \in \mathcal{S} \Lambda(S)\),
\[
\text{Tr}^\omega_\chi([K^{TN}]) = \sum_{\text{admissible } n_1, n_2, \ldots, n_k} [Z_1^{n_1} Z_2^{n_2} \cdots Z_k^{n_k}]
\]
where the sum is over all admissible sequences \(n_1, n_2, \ldots, n_k\) with all \(n_i = \pm N\), and where the \(Z_i\) and the notion of admissibility are defined as in §7.6 and at the beginning of this §7.7. □

Note the dramatic difference between the number of terms in the expression of \(\text{Tr}^\omega_\chi([K^{TN}])\) provided by Proposition 33 and that in the formula for generic \(A\) given in Proposition 29. Indeed, the number of monomials in the formula of Proposition 33 is independent of \(N\). On the other hand, the number of terms in the expression of Proposition 29 is a polynomial in \(N\) of degree \(k\) (it is the Ehrhart polynomial of a certain \(k\)–dimensional polytope determined by the admissibility conditions).
7.8. Proof of Theorem [20] Assuming that \(A^4\) is a primitive \(N\)-root of unity, we want to prove that the diagram

\[
\begin{array}{c}
S^A(S) \xrightarrow{\text{Tr}_\omega} \mathcal{T}(\lambda) \\
S^\epsilon(S) \xrightarrow{\text{Tr}_\epsilon} \mathcal{T}(\lambda)
\end{array}
\]

is commutative.

By [Bu3], the skein algebra \(S^\epsilon(S)\) is generated as an algebra by \(\lambda\)-simple skeins \([K]\). Since the maps involved are algebra homomorphisms, it therefore suffices to show that

\[
\text{Tr}_\epsilon(\mathcal{T}^A([K])) = \mathcal{F}_\omega(\text{Tr}_\epsilon([K]))
\]

for every \(\lambda\)-simple skein \([K] \in S^\epsilon(S)\).

The quantum trace \(\text{Tr}_\epsilon(\mathcal{T}^A([K])) = \text{Tr}_\epsilon([K^{TN}])\) is computed by Proposition 33. With the notation of that statement,

\[
\text{Tr}_\epsilon([K^{TN}]) = \sum_{\text{admissible } n_1, n_2, \ldots, n_k} [Z_1^{n_1} Z_2^{n_2} \ldots Z_k^{n_k}]
\]

Similarly, by Theorem 19 or Proposition 33

\[
\text{Tr}_\epsilon([K]) = \sum_{\text{admissible } \epsilon_1, \epsilon_2, \ldots, \epsilon_k} [Z_1^{\epsilon_1} Z_2^{\epsilon_2} \ldots Z_k^{\epsilon_k}].
\]

Because \(t = \omega^{N^2}\), \(\mathcal{F}_\omega([Z_1^{\epsilon_1} Z_2^{\epsilon_2} \ldots Z_k^{\epsilon_k}]) = [Z_1^{N \epsilon_1} Z_2^{N \epsilon_2} \ldots Z_k^{N \epsilon_k}]\) by definition of the Weyl quantum ordering. Therefore, \(\mathcal{F}_\omega(\text{Tr}_\epsilon([K]))\) is equal to \(\text{Tr}_\epsilon([K^{TN}]) = \text{Tr}_\epsilon(\mathcal{T}^A([K]))\) by comparison of the above formulas. □

7.9. The cases of the once-punctured torus and the twice-punctured plane. At this point, there is a gap that needs to be filled in. Indeed, our arguments so far assumed that the Chebyshev homomorphism \(\mathcal{T}^A: S^\epsilon(S) \to \mathcal{S}^A(S)\) is a well-defined algebra homomorphism. This property relied on Lemmas 12 and 13 whose proofs we temporarily postponed. We now prove these two statements, using the computations of §7.7. At the same time, we will prove Lemma 10 which was used in Theorem 9 to show that Chebyshev threads \([K^{TN}]\) are central in \(\mathcal{S}^A(S)\).

![Figure 11](image.png)

**Figure 11.** The curves \(L_0, L_\infty, L_1\) and \(L_{-1}\) in the once-punctured torus, with an ideal triangulation \(\lambda\)

We begin with Lemma [10] which we repeat for the convenience of the reader.
Lemma 34. In the once-punctured torus \( T \), let \( L_0 \) and \( L_\infty \) be the two curves represented in Figure [11] (or Figure [2]), and consider these curves as framed knots with vertical framing in \( T \times [0,1] \). If \( A^2 \) is a primitive \( N \)-root of unity, then

\[
[L_0^{T_N}] [L_\infty] = [L_\infty] [L_0^{T_N}]
\]

in \( S^A(T) \).

Proof. Consider the ideal triangulation \( \lambda \) indicated in Figure [11] with edges \( e_0, e_1 \) and \( e_\infty \). The trace map \( \text{Tr}_\lambda^\omega \colon S^A(S) \rightarrow \mathcal{T}^\omega(\lambda) \) is an injective algebra homomorphism \([\text{BoW}] \) Prop. 29], so that it suffices to check that

\[
\text{Tr}_\lambda^\omega ([L_0^{T_N}]) \text{Tr}_\lambda^\omega ([L_\infty]) = \text{Tr}_\lambda^\omega ([L_\infty]) \text{Tr}_\lambda^\omega ([L_0^{T_N}]).
\]

First consider the case where \( N \) is odd. Then, \( A^4 \) is also a primitive \( N \)-root of unity, and we can use the computations of \([7.7]\). Let \( Z_0, Z_1, Z_\infty \) be the generators of \( \mathcal{T}^\omega(\lambda) \) respectively associated to the edges \( e_0, e_1, e_\infty \) of \( \lambda \). Proposition [33] then shows that the quantum trace \( \text{Tr}_\lambda^\omega ([L_0^{T_N}]) \) is a Laurent polynomial in the variables \( Z_i^2 \) and \( Z_i^N \).

Note that, when \( i \neq j \), the two ends of the edge \( e_j \) are adjacent to the two ends of \( e_j \), and that \( Z_iZ_j = \omega^{\pm 4}Z_jZ_i \). In particular, \( Z_i^N \) commutes with \( Z_j \) since \( \omega^{AN} = A^{-2N} = 1 \). As a consequence, \( \text{Tr}_\lambda^\omega ([L_0^{T_N}]) \) is central in \( \mathcal{T}^\omega(\lambda) \), and in particular commutes with \( \text{Tr}_\lambda^\omega ([L_\infty]) \). This proves the desired property when \( N \) is odd.

When \( N \) is even, \( A^4 \) is a primitive \( \frac{N}{2} \)-root of unity. Proposition [33] now shows that \( \text{Tr}_\lambda^\omega ([L_0^{T_N/2}]) = T_N^\omega (\text{Tr}_0^\omega ([L_0])) \) is a linear combination of monomials \( Z_1^{n_1}Z_\infty^{n_2} \) with \( n_1, n_2 \in \{-\frac{N}{2}, \frac{N}{2}\} \). By definition of Chebyshev polynomials, \( T_N = T_2 \circ T_2 \) (use for instance Lemma [11]) and \( T_2(x) = x^2 - 2 \). Therefore

\[
\text{Tr}_\lambda^\omega ([L_0^{T_N}]) = T_N (\text{Tr}_0^\omega ([L_0])) = T_2 \left( T_2 (\text{Tr}_0^\omega ([L_0])) \right) = \left( \text{Tr}_0^\omega ([L_0^{T_N/2}]) \right)^2 - 2
\]

is a linear combination of monomials \( Z_1^{m_1}Z_\infty^{m_2} \) with \( m_1, m_2 \in \{-N, 0, +N\} \). As a consequence, \( \text{Tr}_\lambda^\omega ([L_0^{T_N}]) \) is again central in \( \mathcal{T}^\omega(\lambda) \), which concludes the proof as before.

We now address Lemma [12]

Lemma 35. Suppose that \( A^4 \) is a primitive \( N \)-root of unity. In the once-punctured torus \( T \), let \( L_0, L_\infty, L_1 \) and \( L_{-1} \) be the curves represented in Figure [11] (or Figure [2]). Considering these curves as knots in \( T \times [0,1] \) and endowing them with the vertical framing,

\[
[L_0^{T_N}] [L_\infty] = A^{-N^2} [L_1^{T_N}] + A^{N^2} [L_{-1}^{T_N}]
\]

in the skein algebra \( S^A(T) \).

Proof. As usual, set \( \varepsilon = A^{N^2} \) and \( \iota = \omega^{N^2} \). By injectivity of the trace homomorphism \( \text{Tr}_\lambda^\omega \colon S^A(S) \rightarrow \mathcal{T}^\omega(\lambda) \), it again suffices to check that

\[
\text{Tr}_\lambda^\omega ([L_0^{T_N}]) \text{Tr}_\lambda^\omega ([L_\infty]) = \varepsilon^{-1} \text{Tr}_\lambda^\omega ([L_1^{T_N}]) + \varepsilon \text{Tr}_\lambda^\omega ([L_{-1}^{T_N}])
\]

in \( \mathcal{T}^\omega(\lambda) \), for the ideal triangulation \( \lambda \) indicated in Figure [11].

For this, we need to compute these \( \text{Tr}_\lambda^\omega ([L_0^{T_N}]) \in S^A(S) \), and we will compare them to \( \text{Tr}_\lambda^\omega ([L_1^{T_N}]) \in S^A(S) \) through the Frobenius homomorphism \( \mathbf{F}^\omega \colon \mathcal{T}^\omega(\lambda) \rightarrow \mathcal{T}^\omega(\lambda) \). Note that we cannot use here the full force of Theorem [20] because we are...
trying to prove one of the steps in the proof of that statement, namely that threading Chebyshev polynomials \( T_N \) provides an algebra homomorphism \( T^\omega : S^*(S) \to S^A(S) \). However, we can rely on the computations of \( F^\omega \).

The framed knots \( L_0, L_\infty \) and \( L_1 \) are \( \lambda \)-simple. So we can use Proposition 33, which gives

\[
\text{Tr}^\omega_\lambda ([L_0^{TN}]) = \sum_{n_1, n_2 = 0}^{n_1, n_2} \text{[Z}^{n_1}Z_1^{n_2}] = [\text{Z}_1^{-N}Z_1^{-N}] + [\text{Z}_\infty^{-N}Z_1^{-N}] + [\text{Z}_\infty^{N}Z_1^{N}].
\]

The definition of the quantum trace in Theorem 13 also shows that

\[
\text{Tr}^\omega_\lambda ([L_0]) = [\text{Z}_1^{-1}] + [\text{Z}_\infty^{-1}] + [\text{Z}_\infty Z_1]
\]

in \( T^\omega(\lambda) \). Comparing the two formulas, we see that \( \text{Tr}^\omega_\lambda ([L_0^{TN}]) = F^\omega(\text{Tr}^\omega_\lambda ([L_0])) \).

Since \( L_\infty \) and \( L_1 \) are also \( \lambda \)-simple, the same argument shows that \( \text{Tr}^\omega_\lambda ([L_\infty^{TN}]) = F^\omega(\text{Tr}^\omega_\lambda ([L_\infty])) \) and \( \text{Tr}^\omega_\lambda ([L_1^{TN}]) = F^\omega(\text{Tr}^\omega_\lambda ([L_1])) \).

Unfortunately, \( L_{-1} \) is not \( \lambda \)-simple, so that the computation of \( \text{Tr}^\omega_\lambda ([L_{-1}^{TN}]) \) requires more care. The arguments will actually end up being the same, and the only new complication in this case is that the notation is clumsier.

Let \( T_1 \) and \( T_2 \) denote the two triangle faces of \( \lambda \). The torus \( T \) is obtained by gluing these two triangles along their sides. As indicated in 46.3 there is a canonical embedding \( \mathcal{T}^\omega(\lambda) \to \mathcal{T}^\omega(T_1) \otimes \mathcal{T}^\omega(T_2) \) that sends the generator \( Z_i \) associated to an edge of \( \lambda \) to the tensor product \( Z_{i,1} \otimes Z_{i,2} \in \mathcal{T}^\omega(T_1) \otimes \mathcal{T}^\omega(T_2) \) of the generators of \( \mathcal{T}^\omega(T_1) \) and \( \mathcal{T}^\omega(T_2) \) corresponding to the same edge.

We can arrange \( L_{-1} \) in \( T \times [0, 1] \) so that, above each triangle \( T_j \), the component of \( L_{-1} \cap (T_j \times [0, 1]) \) that is closest to the upper right corner of the square in Figure 11 sits at a higher elevation than the second component. This is consistent with the orientation of the edges of \( \lambda \) indicated by the arrows in Figure 11.

Then, the arguments used in the proof of Proposition 33 carry over to this case, except that we have to keep track of the order of the contributions of each components of a \( K_{-1} \cap (U_j \times [0, 1]) \). In our case, this gives

\[
\text{Tr}^\omega_\lambda ([L_{-1}^{TN}]) = \sum_{n_1 \leq n_2 \leq n_3 \geq n_4 \geq n_1} \text{[Z}_1^{n_1}Z_1^{n_2}][Z_0^{n_1}Z_1^{n_2}][Z_1^{-N}Z_0^{-N}] + [Z_1^{-N}Z_0^{-N}][Z_1^{N}Z_0^{N}] + [Z_1^{-N}Z_0^{-N}][Z_1^{N}Z_0^{N}]
\]

where the order of the terms is dictated by the respective elevations of the components of each \( K_{-1} \cap (T_j \times [0, 1]) \). Listing all 6 possibilities for \((n_1, n_2, n_3, n_4)\), reordering the terms in each monomial and remembering that \( Z_i \) corresponds to \( Z_{i,1} \otimes Z_{i,2} \) for the embedding \( \mathcal{T}^\omega(\lambda) \to \mathcal{T}^\omega(T_1) \otimes \mathcal{T}^\omega(T_2) \),

\[
\text{Tr}^\omega_\lambda ([L_{-1}^{TN}]) = [Z_1^{-N}Z_0^{-N}][Z_1^{-2N}Z_0^{-N}] + [Z_1^{-N}Z_0^{-N}][L_1^{-N}Z_0^{-N}] + (\omega_4^{-N} + \omega_4^{-2N})[Z_1^{-N}Z_0^{-N}]
\]

in \( \mathcal{T}^\omega(\lambda) \).
particular, the choice of elevations indicated in Figure 12 guarantees that there are
Proof.

Applying the algebra homomorphism $F$ on both sides, and using the fact that $Tr^i(\omega(\lambda)) = F^i(\lambda) = \epsilon^{-i}Tr^i(\lambda)$ for $i = 0, \infty, 1$ and $-1$, we conclude that $Tr^i(\omega(\lambda))(L_0) = F^i(\lambda)$ in this case as well, since $\epsilon = \omega^{N^2}$.

Note that $[L_0][L_\infty] = \epsilon^{-1}[L_1] + \epsilon[L_1]$ in $S^\epsilon(S)$, by the skein relation. Because $Tr^i: S^\epsilon(S) \rightarrow \mathcal{T}^\epsilon(\lambda)$ is an algebra homomorphism, it follows that

Applying the algebra homomorphism $F^i$ on both sides, and using the fact that $Tr^i(\omega(\lambda)) = F^i(\lambda)$ for $i = 0, \infty, 1$ and $-1$, we conclude that $Tr^i(\omega(\lambda))(L_T^N) = F^i(\lambda)$ in this case as well, since $\epsilon = \omega^{N^2}$.

This concludes the proof, since the algebra homomorphism $Tr^i: S^\lambda(S) \rightarrow \mathcal{T}^\epsilon(\lambda)$ is injective.

Figure 12. The 1–submanifolds $L_1$, $L_0$ and $L_\infty$ in the twice-punctured plane, with an ideal triangulation $\lambda$

The proof of Lemma 13 is very similar.

Lemma 36. Suppose that $A^2$ is a primitive $N$–root of unity with $N$ odd. In the twice-punctured plane $U$, let $L_1$, $L_{-1}$, $L_0$ and $L_\infty$ be the 1–submanifolds represented in Figure 12 (or Figure 3). Considering these submanifolds as links in $U \times [0,1]$ and endowing them with the vertical framing,

\[
[L_1^N] = A^{-N^2}[L_0^N] + A^{N^2}[L_\infty^N]
\]

and

\[
[L_{-1}^N] = A^{N^2}[L_0^N] + A^{-N^2}[L_\infty^N]
\]

in the skein algebra $S^A(U)$.

Proof. Consider the ideal triangulation $\lambda$ of $U$, with edges $e_1$, $e_2$, $e_3$ and faces $T_1$, $T_2$, represented in Figure 12. As usual, let $Z_i = Z_i^1 \otimes Z_i^2$ denote the generator of $\mathcal{T}^\epsilon(\lambda) \subset \mathcal{T}^\epsilon(T_1) \otimes \mathcal{T}^\epsilon(T_2)$ associated to $e_i$.

As for Lemma 35, we can use the arguments of the proof of Proposition 33. In particular, the choice of elevations indicated in Figure 12 guarantees that there are
no non-trivial (namely different from 1) contributions from the bangle neighborhoods of the edges of $\lambda$. The corresponding computation then gives

$$\text{Tr}_\lambda^\omega([L_1^{T_N}]) = \sum_{n_1 \geq n_2 \geq n_3 \leq n_4 \leq n_1} [Z_{2,1}^{n_1} Z_{3,1}^{n_2}] [Z_{2,1}^{n_3} Z_{3,1}^{n_4}] \otimes [Z_{1,2}^{n_5} Z_{2,2}^{n_6}] [Z_{3,2}^{n_7} Z_{2,3}^{n_8}],$$

$$= \omega^{-2N^2} [Z_1^{-N} Z_2^{-2N} Z_3^{-N}] + \omega^{2N^2} [Z_1^{-N} Z_3^{-N}] + \omega^{-2N^2} [Z_1^{-N} Z_3^{-N}]$$

$$+ \omega^{-2N^2} [Z_1^{-N} Z_3^{-N}] + \omega^{2N^2} [Z_1^{N} Z_3^{N}] + \omega^{-2N^2} [Z_1^{N} Z_3^{N}].$$

$$\text{Tr}_\lambda^\omega([L_0^{T_N}]) = [Z_1^{-N} Z_3^{-N}] + [Z_1^{N} Z_3^{N}]$$

and

$$\text{Tr}_\lambda^\omega([L_{\infty}^{T_N}]) = ([Z_1^{-N} Z_2^{-N}] + [Z_1^{N} Z_2^{N}]) ([Z_2^{-N} Z_3^{-N}] + [Z_2^{N} Z_3^{N}])$$

$$= [Z_1^{-N} Z_2^{-2N} Z_3^{-N}] + [Z_1^{-N} Z_3^{-N}] + [Z_1^{N} Z_3^{N}] + [Z_1^{N} Z_2^{2N} Z_3^{N}].$$

Remembering that $A = \omega^{-2}$, we immediately see that

$$\text{Tr}_\lambda^\omega([L_1^{T_N}]) = A^{-N^2} \text{Tr}_\lambda^\omega([L_0^{T_N}]) + A^{N^2} \text{Tr}_\lambda^\omega([L_{\infty}^{T_N}]).$$

Since the quantum trace homomorphism $\text{Tr}_\lambda^\omega : S^A(U) \to \mathcal{J}^\omega(\lambda)$ is injective, this proves the first statement of the lemma.

The second statement is proved by the same method. \hfill $\square$

This takes care of the proofs that we had postponed up to this point.

References

[Ba] John W. Barrett, *Skein spaces and spin structures*, Math. Proc. Cambridge Philos. Soc. 126 (1999), 267–275.

[BHMV] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, Topology 34 (1995), 883–927.

[BoL] Francis Bonahon, Xiaobo Liu, *Representations of the quantum Teichmüller space and invariants of surface diffeomorphisms*, Geom. Topol. 11 (2007), 889–938.

[BoWi] Francis Bonahon, Helen Wong, *Quantum traces for representations of surface groups in SL_2(C)*, Geom. Topol. 15 (2011), 1569–1615.

[BoWi2] Francis Bonahon, Helen Wong, *Kauffman brackets, character varieties and triangulations of surfaces*, in *Topology and Geometry in Dimension Three: Triangulations, Invariants, and Geometric Structures* (W. Li, L. Bartolini, J. Johnson, F. Luo, R. Myers, J. H. Rubinstein eds.), Contemporary Mathematics 560, American Math. Society, 2011.

[BoWi3] Francis Bonahon, Helen Wong, *Representations of the Kauffman skein algebra II: punctured surfaces*, preprint, arXiv:1206.1639.

[BoWi4] Francis Bonahon, Helen Wong, *Representations of the Kauffman skein algebra III: closed surfaces and naturality*, in preparation.

[BoWi5] Francis Bonahon, Helen Wong, *The Witten-Reshetikhin-Turaev representation of the Kauffman skein algebra*, preprint, arXiv:1309.0921.

[Bu] Doug Bullock, *Estimating a skein module with SL_2(C) characters*, Proc. Amer. Math. Soc. 125 (1997), 1835–1839.

[Bu2] Doug Bullock, *Rings of SL_2(C)–characters and the Kauffman bracket skein module*, Comment. Math. Helv. 72 (1997), 521–542.

[Bu3] Doug Bullock, *A finite set of generators for the Kauffman bracket skein algebra*, Math. Z. 231 (1999), 91–101.

[BuFK] Doug Bullock, Charles Frohman, Joanna Kania-Bartoszyńska, *Understanding the Kauffman bracket skein module*, J. Knot Theory Ramifications 8 (1999), 265–277.

[BuFK2] Doug Bullock, Charles Frohman, Joanna Kania-Bartoszyńska, *The Kauffman bracket skein as an algebra of observables*, Proc. Amer. Math. Soc. 130 (2002), 2479–2485.

[BuP] Doug Bullock, Józef H. Przytycki, *Multiplicative structure of Kauffman bracket skein module quantizations*, Proc. Amer. Math. Soc. 128 (2000), 923–931.
[ChF₁] Leonid O. Chekhov, Vladimir V. Fock, Quantum Teichmüller spaces, Theor. Math. Phys. 120 (1999), 1245–1259.
[ChF₂] Leonid O. Chekhov, Vladimir V. Fock, Observables in 3D gravity and geodesic algebras, in: Quantum groups and integrable systems (Prague, 2000), Czechoslovak J. Phys. 50 (2000), 1201–1208.
[Fo] Vladimir V. Fock, Dual Teichmüller spaces, unpublished preprint, 1997, arXiv:math/9702018.
[FoG] Vladimir V. Fock, Alexander B. Goncharov, Cluster ensembles, quantization and the dilogarithm, Ann. Sci. Éc. Norm. Supér. 42 (2009), 865–930.
[FrG] Charles Frohman, Răzvan Gelca, Skein modules and the noncommutative torus, Trans. Amer. Math. Soc. 352 (2000), 4877–4888.
[Go] William M. Goldman, Topological components of spaces of representations, Invent. Math. 93 (1988), 557–607.
[HaP] Miloslav Havlíček, Severin Poňta, On the classification of irreducible finite-dimensional representations of $U_q(\mathfrak{so}_3)$ algebra, J. Math. Phys. 42 (2001), 472–500.
[He] Heinz Helling, Diskrete Untergruppen von $SL_2(\mathbb{R})$, Invent. Math. 17 (1972), 217–229.
[Kash] Rinat Kashaev, Quantization of Teichmüller spaces and the quantum dilogarithm, Lett. Math. Phys. 43 (1998), 105–115.
[Kass] Christian Kassel, Quantum groups, Graduate Texts in Mathematics vol. 155, Springer-Verlag, New York, 1995.
[Lic₁] W. B. Raymond Lickorish, An introduction to knot theory, Graduate Texts in Math. 175, Springer-Verlag, 1997.
[Lic₂] W. B. Raymond Lickorish, Quantum invariants of 3–manifolds, in: Handbook of geometric topology, 707–734, North-Holland, Amsterdam, 2002.
[MuFK] David Mumford, John Fogarty, Frances Kirwan, Geometric invariant theory. Third edition, Ergebnisse der Mathematik und ihrer Grenzgebiete 34, Springer-Verlag, 1994.
[PrS] Jozef H. Przytycki, Adam S. Sikora, On skein algebras and $SL_2(\mathbb{C})$–character varieties, Topology 39 (2000), 115–148.
[ReT] Nicolai Y. Reshetikhin and Vladimir G. Turaev, Invariants of 3–manifolds via link polynomials and quantum groups, Invent. Math. 103 (1991), 547–597.
[Rob] Justin Roberts, Irreducibility of some quantum representations of mapping class groups, Knots in Hellas ’98, Vol. 3 (Delphi), J. Knot Theory Ramifications 10 (2001), 763–767.
[Tu₁] Vladimir Turaev, Skein quantization of Poisson algebras of loops on surfaces, Ann. Sci. École Norm. Sup. 24 (1991), 635–704.
[Tu₂] Vladimir G. Turaev. Quantum invariants of knots and 3-manifolds, de Gruyter Studies in Mathematics Vol. 18, Walter de Gruyter & Co., Berlin, 1994.
[Wit] Edward Witten, Quantum Field Theory and the Jones Polynomial, Commun. Math. Phys. 121 (1989), 351–399.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES CA 90089-2532, U.S.A.
E-mail address: fbonahon@math.usc.edu

DEPARTMENT OF MATHEMATICS, CARLETON COLLEGE, NORTHFIELD MN 55057, U.S.A.
E-mail address: hwong@carleton.edu