Toroidal solitons in 3 + 1 dimensional integrable theories

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Abstract

We analyze the integrability properties of models defined on the symmetric space $SU(2)/U(1)$ in 3 + 1 dimensions, using a recently proposed approach for integrable theories in any dimension. We point out the key ingredients for a theory to possess an infinite number of local conservation laws, and discuss classes of models with such property. We propose a 3+1-dimensional, relativistic invariant field theory possessing a toroidal soliton solution carrying a unit of topological charge given by the Hopf map. Construction of the action is guided by the requirement that the energy of static configuration should be scale invariant. The solution is constructed exactly. The model possesses an infinite number of local conserved currents. The method is also applied to the Skyrme-Faddeev model, and integrable submodels are proposed.
1 Introduction

In this paper we consider scalar field theories in $3 + 1$ dimensions, defined on $S^2$, or equivalently on the symmetric space $SU(2)/U(1)$. One of the motivations to study such theories is that some of them present topological solitons. The requirement of finite energy for static configurations imposes, in general, that the fields should be constant at spatial infinity. Consequently, for such purpose space can be taken to be $S^3$, and the solutions define a mapping $S^3 \rightarrow S^2$. The topological charges carried by the solitons are then determined by a Hopf map. That differs from the case of magnetic monopoles, for instance, where the charges are winding numbers of the map $S^2 \rightarrow S^2$. The Hopf index is given by the linking number of the pre images of a given pair of points of $S^2$.

As a consequence, the solitons tend to have string like configurations, and those with charge unity to have a toroidal shape.

Conventional wisdom of two dimensional soliton physics holds that the existence of solitons is linked to the notion of integrability. The reasoning is that the high degree of symmetries underlying the infinite set of conserved quantities accounts for the conspiracy among the degrees of freedom, necessary for the appearance of solitons. We give indications here that also in the setup of higher dimensional integrable models the solitons appear in theories with infinite number of conserved quantities.

We analyze the integrability properties of those scalar theories using the approach of [1], which generalizes the concept of zero curvature in two dimensions to theories defined in a space-time of any dimension. Those ideas are reviewed in section 2. In section 3 we define the models we are interested in by presenting their zero curvature representation. Then we discuss the conditions the model has to satisfy to contain an infinite number of local conserved currents. In section 4 we study some examples of such integrable theories. One of the important ingredients to have soliton solutions, is that the energy should be stable under scaling of the space variables (Derrick’s theorem). In section 4.1 we introduce a model where the energy for static configurations is invariant under such scalings. We then construct the exact solution for a soliton carrying one unity of topological charge. In section 4.2 we discuss the integrability of the Skyrme-Faddeev model [6], and propose a submodel of it which possesses an infinite number of conserved currents.
2 Integrability in any dimension

As we said, we shall analyze the integrability properties of the models considered in this paper using the approach of [1]. The main idea there is to generalize the zero curvature condition in two dimensions guided by the fact that it embodies conservation laws. Indeed, the flatness condition for the Lax operators \( A_\mu \) implies that its path ordered integral is path independent, as long as the end point are kept fixed. For a closed path that leads to a Gauss type law and so, conserved quantities. Therefore, the central idea in [1] to bring such concepts to higher dimensions, is to introduce quantities integrated over hypersurfaces and to find the conditions for them to be independent of deformations of the hypersurfaces which keep their boundaries fixed. Such an approach certainly leads to conservation laws in a manner very similar to the two dimensional case. However, the main problem of that is how to introduce non-linear zero curvatures keeping things as local as possible. The way out is to introduce connections to allow for parallel transport.

The zero curvature obtained in [1] is in general non local but there are interesting conditions under which it becomes local. The structures underlying those conditions involve a Lie algebra \( \mathcal{G} \) and a representation \( R \) of it. Then one introduces the non-semisimple Lie algebra \( \mathcal{G}_R \) as

\[
[T_a, T_b] = f_{ab}^c T_c \\
[T_a, P_i] = P_j R_{ji} (T_a) \\
[P_i, P_j] = 0 \quad (2.1)
\]

where \( T_a \) constitute a basis of \( \mathcal{G} \) and \( P_i \) a basis for the abelian ideal \( P \) (representation space). The fact that \( R \) is a matrix representation, i.e.

\[
[R (T_a), R (T_b)] = R ([T_a, T_b]) \quad (2.2)
\]

follows from Jacobi identities.

In \((3 + 1)\) dimensions, which is the case of interest here, one then introduces a connection \( A_\mu \) belonging to \( \mathcal{G} \) and a rank 3 antisymmetric tensor \( B_{\mu\nu\rho} \) belonging to \( P \), i.e.

\[
A_\mu = A_\mu^a T_a \quad , \quad B_{\mu\nu\rho} = B_{\mu\nu\rho}^i P_i \quad (2.3)
\]

Then the local zero curvature conditions are given by

\[
D_\lambda B_{\mu\nu\rho} - D_\mu B_{\nu\rho\lambda} + D_\nu B_{\rho\lambda\mu} - D_\rho B_{\lambda\mu\nu} = 0 \quad (2.4)
\]
and
\[ F_{\mu\nu} \equiv [\partial_{\mu} + A_{\mu}, \partial_{\nu} + A_{\nu}] = 0 \quad (2.5) \]
where we have introduced the covariant derivative
\[ D_{\mu} \cdot \equiv \partial_{\mu} \cdot + [A_{\mu}, \cdot] \quad (2.6) \]
Introducing the dual of \( B_{\mu\nu\rho} \) as
\[ \tilde{B}^\mu \equiv \frac{1}{3!} \varepsilon^{\mu\nu\rho\lambda} B_{\nu\rho\lambda} \quad (2.7) \]
one can write (2.4) as
\[ D_\mu \tilde{B}^\mu = 0 \quad (2.8) \]
The relations (2.3) and (2.8) constitute the local generalization to higher dimensions of the zero curvature condition in two dimensions. They lead to local conservation laws. Indeed, since the connection \( A_{\mu} \) is flat it can be written as
\[ A_{\mu} = -\partial_{\mu} W W^{-1} \quad (2.9) \]
and consequently (2.8) implies that the currents
\[ J_\mu \equiv W^{-1} \tilde{B}^\mu W \quad (2.10) \]
are conserved:
\[ \partial_\mu J^\mu = 0 \quad (2.11) \]
The zero curvature conditions (2.4) and (2.3) are invariant under the gauge transformations
\[ A_{\mu} \rightarrow g A_{\mu} g^{-1} - \partial_{\mu} g g^{-1} \]
\[ B_{\mu\nu\rho} \rightarrow g B_{\mu\nu\rho} g^{-1} \quad (2.12) \]
and
\[ A_{\mu} \rightarrow A_{\mu} \]
\[ B_{\mu\nu\rho} \rightarrow B_{\mu\nu\rho} + D_\mu \alpha_{\nu\rho} + D_\nu \alpha_{\rho\mu} + D_\rho \alpha_{\mu\nu} \quad (2.13) \]
In (2.12) \( g \) is an element of the group obtained by exponentiating the Lie algebra \( \mathcal{G} \). The transformations (2.13) are symmetries of (2.4) and (2.7) as a consequence of the fact that the connection \( A_{\mu} \) is flat, i.e. \([D_{\mu}, D_{\nu}] = 0\). In addition, the parameters \( \alpha_{\mu\nu} \) take values in the abelian ideal \( P \).
The currents (2.10) are invariant under the transformations (2.12), while under (2.13) they transform as

$$J_\mu \rightarrow J_\mu + \varepsilon_{\mu\nu\rho\lambda} \partial^\nu \left( W^{-1} \alpha^\rho W \right)$$ (2.14)

The transformations (2.12) and (2.13) do not commute and their algebra is isomorphic to the non-semisimple algebra $\mathcal{G}_R$ introduced in (2.1).

3 Integrable models on SU(2)/U(1)

The models we shall be considering involve scalar fields living on the two dimensional sphere $S^2$, and we will denote them as $n = (n_1, n_2, n_3)$, with $n^2 = 1$. Alternatively, one can use the stereographic projection of $S^2$ and work with two unconstrained scalar fields, which we shall choose to constitute a complex scalar field $u$ related to $n$ by

$$n = \frac{1}{1 + |u|^2} \begin{pmatrix} u + u^* , -i(u - u^*) , |u|^2 - 1 \end{pmatrix}$$ (3.1)

The sphere $S^2$ can be mapped in a one-to-one manner into the symmetric space $SU(2)/U(1)$ and we shall explore that fact to construct the local zero curvature conditions (2.4) and (2.5). The $U(1)$ is the subgroup invariant under the involutive automorphism of $SU(2)$

$$\sigma(T_3) = T_3 \quad \sigma(T_\pm) = -T_\pm$$ (3.2)

where $T_3, T_\pm$ are the generators of $SU(2)$ satisfying

$$[T_3, T_\pm] = \pm T_\pm, \quad [T_+, T_-] = 2T_3$$ (3.3)

The automorphism (3.2) is inner and given by

$$\sigma(T) \equiv e^{i\pi T_3} T e^{-i\pi T_3}$$ (3.4)

The elements of $SU(2)/U(1)$ can be parametrized by the variable $x(g) \equiv g\sigma(g)^{-1}$, $g \in SU(2)$, since $x(g) = x(gk)$ with $k \in U(1)$. In addition one has that $\sigma(x) = x^{-1}$.

1 The sphere $S^2$ is diffeomorphic to $SO(3)/SO(2)$, and $SO(3) \sim SU(2)/Z_2$. The elements of the subgroup $U(1)$ in $SU(2)/U(1)$ are given by $g(\theta) = \exp(i\theta T_3)$. Therefore, $g(0) = 1$ and $g(2\pi) = -1$, and such $U(1)$ has twice as many elements as $SO(2)$. In fact, $SO(2) \sim U(1)/Z_2$, and so the points of $S^2$ and $SU(2)/U(1)$ are in one-to-one correspondence.

2 Actually, these are the generators of $SL(2)$. The generators of $SU(2)$ are $T_i, i = 1, 2, 3$ with $T_\pm = \frac{1}{2}(T_1 \pm iT_2)$.
We then choose the group element $W$ in (2.9) to be of the form of the variable $x(g)$, i.e.

$$W \equiv e^{iuT_+} e^{\varphi T_3} e^{iu^* T_-}$$

where $\varphi = \ln(1 + |u|^2)$. In the defining (spinor) representation of $SU(2)$ one has

$$R^{(1/2)}(W) = \frac{1}{\sqrt{1 + |u|^2}} \begin{pmatrix} 1 & iu \\ iu^* & 1 \end{pmatrix}$$

Indeed, one can check that $\sigma(W) = W^{-1}$, using in (3.4) that

$$R^{(1/2)}(e^{i\pi T_3}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

We then introduce the potentials

$$A_\mu = -\partial_\mu W W^{-1} = \frac{1}{1 + |u|^2} (-i\partial_\mu u T_+ - i\partial_\mu u^* T_- + (u\partial_\mu u^* - u^* \partial_\mu u) T_3)$$

$$\tilde{B}_\mu = \frac{1}{1 + |u|^2} \left( K_\mu P_1^{(1)} - K_\mu^* P_{-1}^{(1)} \right)$$

where $K_\mu$ is a functional of the fields $u$ and $u^*$ and their derivatives. In addition, $P_{m \pm 1}$ stand for the states of eigenvalues $\pm 1$ of $T_3$ in the triplet representation of $SU(2)$. According, to (2.1) they are generators of the abelian subalgebra $P$ of $G_R$. Here we give the commutation relations for any spin-$j$ representation ($m = -j, -j + 1, \ldots, j - 1, j$)

$$[T_3, P_m^{(j)}] = m P_m^{(j)}$$

$$[T_\pm, P_m^{(j)}] = \sqrt{j(j+1) - m(m \pm 1)} P_{m \pm 1}^{(j)}$$

$$[P_m^{(j)}, P_{m'}^{(j')}]} = 0$$

Obviously, the zero curvature relation (2.3) is trivially satisfied because we have chosen $A_\mu$ of the pure gauge form. So, it does not impose any condition on the fields $u$ and $u^*$.

Requiring, that

$$\text{Im} (K_\mu \partial_\mu u^*) = 0$$

one obtains that the zero curvature condition (2.8) implies that

$$\left( 1 + |u|^2 \right) \partial^\mu K_\mu - 2 u^* K_\mu \partial^\mu u = 0$$
together with its complex conjugate equation.

According to (2.10) and (2.11) one gets three conserved currents corresponding to three states of the triplet representation. They are given by

$$J^{(1)}_{\mu} = \sum_{m=-1}^{1} J^{(1,m)}_{\mu} P^{(1)}_m$$

(3.15)

with

$$J^{(1,1)}_{\mu} = \frac{\mathcal{K}_\mu + \mathcal{K}_\mu^* u^2}{(1 + |u|^2)^2}$$
$$J^{(1,0)}_{\mu} = \frac{i \sqrt{2} \left( \mathcal{K}_\mu^* u - \mathcal{K}_\mu u^* \right)}{(1 + |u|^2)^2}$$
$$J^{(1,-1)}_{\mu} = -J^{(1,1)*}_{\mu}$$

(3.16)

Notice that the condition (3.13) implies that the term in the direction of $P^{(1)}_0$ vanishes, i.e. $\left[ \partial_\mu u T_+ + \partial_\mu u^* T_-, \mathcal{K}_\mu P^{(1)}_1 - \mathcal{K}_\mu^* P^{(1)}_{-1} \right] = 0$. Therefore, the equations of motion (3.14) are determined only by the way that $\tilde{B}_\mu$ transforms under the $U(1)$ subgroup generated by $T_3$. In fact, $\tilde{B}_\mu$ contains two irreducible representations of $U(1)$ which are the singlets $P^{(1)}_{\pm 1}$ of charges $\pm 1$. That means that if we change the representation of $SU(2)$ where $\tilde{B}_\mu$ lives, we do not change the equations of motion if $\tilde{B}_\mu$ still transforms under the same two singlets of the $U(1)$ subgroup. What can happen is that the commutator of the $T_\pm$ part of $A_\mu$ does not commute with $\tilde{B}_\mu$ anymore, and then we get some additional equations which should be considered as constraints on the model. Consequently, we would be dealing with submodels of the original theory. See ref. [2] for a detailed discussion on that.

One way of implementing these ideas is as follows. Any integer spin-$j$ representation of $SU(2)$ possesses a charge zero singlet of the $U(1)$ generated by $T_3$, which is $P^{(j)}_0$. Therefore, if one considers representations of $SU(2)$ which are tensor products of these representations, one obtains several singlets of $U(1)$ transforming like $P^{(1)}_{\pm 1}$, which are given by tensor products of $P^{(j)}_{\pm 1}$ with copies of $P^{(j)}_0$. For instance one has

$$\left[ 1 \otimes T_3 + T_3 \otimes 1, P^{(j')}_0 \otimes P^{(j)}_{\pm 1} \right] = \pm \left( P^{(j')}_0 \otimes P^{(j)}_{\pm 1} \right)$$

(3.17)

Therefore, for the case of the tensor product of $n$ integer spin representations, one introduces the potentials

$$A^{(j_1,\ldots,j_n)}_{\mu} \equiv \sum_{l=0}^{n-1} (\otimes 1)^l \otimes \left( A^\pm_\mu + A^3_\mu \right) (\otimes 1)^{n-l-1} \equiv A^{(j_1,\ldots,j_n,\pm)}_\mu + A^{(j_1,\ldots,j_n,3)}_\mu$$

(3.18)

$$\tilde{B}^{(j_1,\ldots,j_n)}_{\mu} \equiv \sum_{l=0}^{n-1} \tilde{P}^{(j_1)}_0 \otimes \tilde{P}^{(j_2)}_0 \otimes \ldots \frac{\left( \mathcal{K}_\mu P^{(j_1)}_1 - \mathcal{K}_\mu^* P^{(j_1)}_{-1} \right)}{1 + |u|^2} \otimes \tilde{P}^{(j_{l+1})}_0 \ldots \otimes \tilde{P}^{(j_n)}_0$$
with $j_l$, $l = 1, 2, \ldots n$, being positive integers numbers, and where we have rescaled the zero charge singlets as
\[
\tilde{P}^{(ji)}_0 \equiv P^{(ji)}_0 / \sqrt{j_l(j_l + 1)}
\]  
(3.19)

In addition, we have denoted
\[
A^{\pm}_\mu \equiv \frac{1}{1 + |u|^2} (-i \partial_\mu u T_+ - i \partial_\mu u^* T_-)
\]

\[
A^3_\mu \equiv \frac{1}{1 + |u|^2} (u\partial_\mu u^* - u^* \partial_\mu u) T_3
\]  
(3.20)

The zero curvature condition (2.5) for these potentials is still trivially satisfied because $A^{(j_1 \ldots j_n)}_\mu$ is of the pure gauge form. The condition (2.8) can be split in two terms:
\[
\partial^\mu \tilde{B}^{(j_1 \ldots j_n)}_\mu + \left[ A^{(j_1 \ldots j_n, \pm)}, \tilde{B}^{(j_1 \ldots j_n)}_\mu \right] = - \left[ A^{(j_1 \ldots j_n, \pm)}_\mu, \tilde{B}^{(j_1 \ldots j_n)}_\mu \right]
\]  
(3.21)

The l.h.s. of such equation vanishes as a consequence of the equations of motion (3.14).

We now have, using (3.13), that
\[
\left[ A^{(j_1 \ldots j_n, \pm)}_\mu, K^{\mu} P_1^{(ji)} - K^{\mu*} P_{-1}^{(ji)} \right] = -i \sqrt{j_l(j_l + 1)} - 2 \left( K^{\mu} \partial_\mu P_2^{(ji)} - K^{\mu*} \partial_\mu u^* P_{-2}^{(ji)} \right)
\]  
(3.22)

In addition
\[
\left[ A^{(j_1 \ldots j_n, \pm)}_\mu, \tilde{P}^{(ji)}_0 \right] = -i \left( \partial_\mu u P_1^{(ji)} + \partial_\mu u^* P_{-1}^{(ji)} \right)
\]  
(3.23)

Now, let us analyze the r.h.s. of (3.21). Consider the terms containing commutators of $A^{(j_1 \ldots j_n, \pm)}_\mu$ with $\tilde{P}^{(ji)}_0$ and $\tilde{P}^{(j_m)}_0$ ($l < m$). Then one gets, using (3.23), the terms
\[
\tilde{P}^{(ji)}_0 \otimes \ldots \left( -i \right) \left( \partial_\mu u P_1^{(ji)} + \partial_\mu u^* P_{-1}^{(ji)} \right) \otimes \ldots \frac{\left( K^{\mu} P_1^{(jm)} - K^{\mu*} P_{-1}^{(jm)} \right)}{1 + |u|^2} \otimes \ldots \tilde{P}^{(jn)}_0
\]

\[
+ \tilde{P}^{(ji)}_0 \otimes \ldots \frac{\left( K^{\mu} P_1^{(ji)} - K^{\mu*} P_{-1}^{(ji)} \right)}{1 + |u|^2} \otimes \ldots \left( -i \right) \left( \partial_\mu u P_1^{(jm)} + \partial_\mu u^* P_{-1}^{(jm)} \right) \otimes \ldots \tilde{P}^{(jn)}_0
\]

Therefore, taking into account (3.13), one observes that if one imposes the constraint
\[
K^{\mu} \partial_\mu u = 0
\]  
(3.24)

those two terms cancel. The same constraint cancels the terms involving the commutator (3.22). Therefore, the r.h.s. of (3.21) vanishes.

Consequently, the zero curvature conditions (2.3) and (2.8) for the potentials (3.18) lead to the equations of motion (3.14) and the constraint (3.24). And so, they lead to the submodel defined by equations
\[
\partial^\mu K^{\mu} = 0 \quad K^{\mu} \partial_\mu u = 0
\]  
(3.25)
According to (2.10) and (2.11) one obtains the conserved current
\[ J_\mu^{(j_1\ldots j_n)} = (W^{-1} \otimes \ldots \otimes W^{-1}) \tilde{B}_\mu^{(j_1\ldots j_n)} (W \otimes \ldots \otimes W) \]
\[ \equiv \sum_{l=1}^{n} \sum_{m_1=-j_l}^{j_l} J_\mu^{(j_1\ldots j_n),(m_1\ldots m_n)} P_{m_1}^{(j_1)} \otimes \ldots \otimes P_{m_n}^{(j_n)} \]  
\( (3.26) \)

Therefore, one gets \( \prod_{l=1}^{n} (2j_l + 1) \) currents. However, since \( n \) and \( j_l \) can be any positive integer number, such submodel contains an infinity of conserved currents. All such currents are linear in \( K_\mu \) and \( K_\mu^* \), with the coefficients being functionals of \( u \) and \( u^* \).

Notice that any current of the form
\[ J_\mu = K_\mu \frac{\delta G}{\delta u} - K_\mu^* \frac{\delta G}{\delta u^*} \]  
\( (3.27) \)

with \( G \) being any functional of \( u \) and \( u^* \) only (no derivatives), are conserved as a consequence of (3.13) and (3.25). We have checked that for the case where all \( j_l \)'s are equal to 1, the currents (3.26) are of the form (3.27) \( [2] \).

### 4 Examples

The methods discussed above can be used to construct integrable models with an infinite number of conserved currents in a space-time of any dimension. The example of \( CP^1 \) in \( (2 + 1) \) was discussed in \( [1] \), and corresponds to the choice \( K_\mu \rightarrow \partial_\mu u \). Examples involving other symmetric spaces (or homogeneous spaces) were also considered in \( [2, 3, 4] \).

A particular class of models can be constructed using the quantity
\[ K_\mu = (\partial^\nu u^* \partial_\nu u) \partial_\mu u - (\partial_\nu u)^2 \partial_\mu u^* \]  
\( (4.1) \)

since it automatically satisfies
\[ K_\mu \partial^\mu u = 0 \]  
\( (4.2) \)

In addition one has that
\[ \text{Im} (K_\mu \partial^\mu u^*) = 0 \]  
\( (4.3) \)

Therefore, if \( \mathcal{F} \) is any real functional of \( u, u^* \) and their derivatives, it follows that the choice
\[ K_\mu \rightarrow \mathcal{F} (u) K_\mu \]  
\( (4.4) \)
satisfies (3.13) and (3.24), and consequently leads to a class of models defined by the equations of motion (see (3.25))

$$\partial^\mu (F(u) K_\mu) = 0$$

(4.5)

and possessing an infinite number of local conserved currents given by (3.26)-(3.27).

### 4.1 A solvable model presenting toroidal solitons

Consider the quantity

$$H_{\mu\nu} \equiv n \cdot (\partial_\mu n \times \partial_\nu n)$$

(4.6)

where $n$ are scalar fields living on $S^2$. Using (3.1) one obtains

$$H_{\mu\nu} = -\frac{2i}{(1 + |u|^2)^2} (\partial_\mu u \partial_\nu u^* - \partial_\nu u \partial_\mu u^*)$$

(4.7)

We introduce the Lagrangean

$$\mathcal{L} \equiv -\eta_0 \left( H_{\mu\nu}^2 \right)^{\frac{3}{4}} = -\eta_0 \frac{8}{3} \frac{(K_\mu \partial^\mu u^*)^{\frac{3}{2}}}{(1 + |u|^2)^3}$$

(4.8)

where $K_\mu$ is the same as in (4.1), and where $\eta_0 = \pm 1$, determines the choice of the signature of the Minkowski metric, $g_{\mu\nu} = \eta_0 \text{diag} (1, -1, -1, -1)$.

The corresponding equations of motion are

$$\partial^\mu \left( \frac{(K \partial u^*)^{-\frac{1}{2}} K_\mu}{1 + |u|^2} \right) = 0$$

(4.9)

and its complex conjugate.

This model possesses a representation in terms of the zero curvature (2.4) and (2.8), with the potentials being given by (3.18) and

$$K_\mu \to \frac{(K \partial u^*)^{-\frac{1}{2}} K_\mu}{1 + |u|^2}$$

(4.10)

Indeed, such $K_\mu$ satisfies (3.13) and (3.24) as a consequence of (1.2) and (1.3).

Consequently, the model (4.8) is integrable (or solvable) in the sense that it possesses an infinite number of conserved currents given by (3.26)-(3.27).

We are interested in constructing exact static finite energy solutions with non vanishing topological charges. The finite energy requirement imposes that the field $n$...
should be constant at spatial infinity. Therefore, for such purpose one can consider the three dimensional space as an $S^3$ where the spatial infinity is identified with the north pole. The relevant topological invariant is given by the Hopf map $S^3 \to S^2$, and the topological charge is

$$Q_h \equiv \frac{1}{4\pi^2} \int \epsilon_{ijk} H_{ij} A_k d^3x \quad ; \quad H_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{abc} n^a \partial_i n^b \partial_j n^c$$

(4.11)

One of the main difficulties of constructing such type of solutions comes from scaling instabilities in the energy [5, 6, 7]. The choice of the Lagrangean density (4.8) is made to avoid such problems. Indeed, the energy for static configurations is given by

$$E \equiv \int d^3x \Theta_{00} = 8 \frac{1}{4} \int d^3x \frac{(K_i \partial^i u^*)^2}{(1 + |u|^2)^3}$$

(4.12)

with $i = 1, 2, 3$, and $\Theta_{\mu\nu}$ being the canonical energy-momentum tensor. Under a rescaling $x^i \to \lambda x^i$, one has $K_i \to \lambda^{-3} K_i$, and so energy is scale invariant.

The soliton we found is constructed using the rational map approach [8]. It has a Hopf charge $Q_H = 1$ and corresponds to a spherically symmetric hedgehog Skyrme field defined in terms of a rational map $R : S^2 \to S^2$ and a radial profile function $f(r)$ which enter as follows in the expression for the complex field $u$

$$u = \frac{1}{2} \frac{R}{|R|} \left( |R| - \frac{1}{|R|} + i \left( |R| + \frac{1}{|R|} \right) g(r) \right)$$

(4.13)

where

$$g(r) \equiv \cotan f(r)$$

(4.14)

In what follows we choose

$$R(\theta, \phi) = \tan(\theta/2) \exp(i\phi)$$

(4.15)

which can be identified via stereographic projection with a point $z$ on the sphere defined by polar coordinates $(\theta, \phi)$. With the choice of (4.13) the complex field $u$ becomes:

$$u = -\frac{e^{i\phi}}{\sin(\theta)} \left( \cos(\theta) - ig(r) \right) \quad ; \quad 1 + |u|^2 = \frac{1 + g^2(r)}{\sin^2(\theta)}$$

(4.16)

for which the Hopf charge $Q_H$ is equal to one [9].

Plugging the ansatz (4.16) back into equations of motion (4.9) we find that it is a solution of equations of motion for

$$(1 + g^2(r))/r^2 = g'(r) \quad ; \quad g(r) = \pm(r^{-1} - r)/2$$

(4.17)
Correspondingly, the soliton solutions of equations of motion are given by

\[ u_{\pm} = -\frac{e^{i\phi}}{\sin(\theta)} \left( \cos(\theta) \pm \frac{i}{2} \left( \frac{1}{r} - r \right) \right) \]  

(4.18)

According to (4.17), we can take the profile function \( f(r) \) to be

\[ f(r) = \arctan \left( \frac{2r}{r^2 - 1} \right) \]  

(4.19)

which is monotonically decreasing function with the boundary conditions \( f(0) = \pi \) and \( f(\infty) = 0 \).

We also find that the soliton energy is given by

\[ E = \int \left( H_{ij} \right)^{3/4} d^3x = \frac{(8 \times 4^3)^{3/4} 2\pi}{\sqrt{(r^4 + 2r^2 + 1)^{3/2}}} \]  

(4.20)

and since \( \int r^2 dr (r^4 + 2r^2 + 1)^{-3/2} = \pi/16 \) we obtain

\[ E = 8(2^{3/4})\pi^2 = 132.78 \]  

(4.21)

Alternatively, one can rewrite the soliton solutions (4.18) as a composite of the Hopf map:

\[ u_{\pm} = \pm i \frac{\Phi_4 \pm i\Phi_3}{\Phi_1 - i\Phi_2} \]  

(4.22)

together with the stereographic map: \( \mathbb{R}^3 \rightarrow S^3 \) of degree 1:

\[ \Phi_i = \frac{2x_i}{r^2 + 1}, i = 1, 2, 3; \Phi_4 = \frac{r^2 - 1}{r^2 + 1} \]  

(4.23)

Eq. (4.18) is reproduced when making use of the spherical representation \( x_1 + ix_2 = r \sin \theta \exp(i\phi), x_3 = r \cos \theta \). We recognize in (4.22) the soliton solution of reference [6], where expression of the same form as in (4.22) was found as a solution to the equations of motion of the model:

\[ \mathcal{L} = -\eta_0 \left( -\eta_0 \frac{1}{4} (\partial n)^2 \right)^{3/2} \]  

(4.24)

Note, that the scaling property of this model is such that it circumvents the Derrick’s theorem in the similar manner to the model defined by (4.8).

Equations of motion of (4.24) differ from that of (4.8). However, when the additional constraint

\[ (\partial u)^2 = 0 \]  

(4.25)

is imposed then equations of motion for both models take an identical and simplified form of

\[ \partial^\mu \left( h(u) \partial_\mu u \right) = 0 \]  

(4.26)
where
\[ h(u) \equiv \frac{(\partial u \partial u^*)^{\frac{1}{2}}}{1 + \mid u \mid^2} \] (4.27)

In the case of soliton solutions given by (4.16) or (4.18) the condition (4.25) is automatically satisfied, which explains why (4.18) is a common solution for both these models.

4.2 The Skyrme-Faddeev model

The Skyrme-Faddeev (SF) model is defined by the Lagrangean
\[ \mathcal{L} = m^2 (\partial n)^2 - \eta_0 \frac{1}{e^2} H_{\mu \nu}^2 + \lambda \left( n^2 - 1 \right) \] (4.28)
where \( H_{\mu \nu} \) was defined in (4.6), \( \eta_0 = \pm 1 \) determines the signature of the Minkowski metric (see (4.8)), and \( \lambda \) is a Lagrange multiplier.

The corresponding equations of motion, in terms of the complex field \( u \) introduced in (3.1), are
\[ \left( 1 + \mid u \mid^2 \right) \partial^\mu L_\mu - 2 u^* \left( L^\mu \partial_\mu u \right) = 0 \] (4.29)
and its complex conjugate, where
\[ L_\mu \equiv m^2 \partial_\mu u - \eta_0 \frac{4}{e^2} \frac{K_\mu}{(1 + \mid u \mid^2)^2} \] (4.30)
and \( K_\mu \) is defined in (4.1).

One of the main properties of such theory is that the two terms compensate the scaling instabilities in the energy that each term would present if considered separate. Indeed, the energy for static configurations is given by
\[ E \equiv \int d^3x \Theta_{00} = E_1 + E_2 \] (4.31)
with
\[ E_1 \equiv 4m^2 \int d^3x \frac{\mid \nabla u \mid^2}{(1 + \mid u \mid^2)^2} \]
\[ E_2 \equiv \frac{32}{e^2} \int d^3x \frac{(\nabla u_1)^2 (\nabla u_2)^2 (1 - \cos^2 \gamma)}{(1 + \mid u \mid^2)^4} \] (4.32)
where \( \gamma \) is the angle between the vectors \( \nabla u_1 \) and \( \nabla u_2 \), and so \( E_1 \) and \( E_2 \) are positive definite.
By rescaling the space variables as \( x_i \to \lambda x_i \), one has

\[
E(\lambda) = \lambda E_1 + \frac{1}{\lambda} E_2 \tag{4.33}
\]

Expanding around \( \lambda = 1 \) \((\lambda - 1 \sim \varepsilon)\)

\[
E(\lambda) = (E_1 + E_2) + (E_1 - E_2) \varepsilon + E_2 \varepsilon^2 + O(\varepsilon^3) \tag{4.34}
\]

one observes that it is necessary to have

\[
E_1 = E_2 \tag{4.35}
\]

to get stable configurations.

However, no solution with non trivial topological charge has been explicitly found for this model. The existence of soliton solutions has been corroborated by numerical calculations using variational methods to find configurations which minimize the energy \([7, 9]\).

We now discuss the integrability properties of the Skyrme-Faddeev model. Notice that

\[
\text{Im} \left( L_{\mu} \partial^\mu u^* \right) = 0 \tag{4.36}
\]

Therefore, if one makes the correspondence

\[
K_\mu \to L_\mu \tag{4.37}
\]

one notices that (3.13) is satisfied, and the Skyrme-Faddeev model admits a zero curvature representation with the potentials being given by (3.8) and (3.9).

The conserved currents are given by (3.16), and they correspond to the three Noether currents associated to the invariance of the model under the \(O(3)\) symmetry. The Skyrme-Faddeev model, however, does not admit an infinite number of conserved currents, obtainable through the procedures described in section 3, because the condition (3.24) is not satisfied, since

\[
L_\mu \partial^\mu u = m^2 (\partial u)^2 \tag{4.38}
\]

However, the submodel obtained by imposing the constraint (4.25), does possess an infinite number of conserved currents given by (3.26) (with the replacement (1.37)). Then the equations of motion (4.29) become

\[
\partial^\mu \left( f(u) \partial_\mu u \right) = 0 \tag{4.39}
\]

\(^3\)One could also impose \(m^2 = 0\) to obtain an integrable submodel. However, the static solutions of such theory would present, in \(3+1\) dimensions, scaling instabilities in the energy. To avoid such instabilities one would have to consider such submodel in \(4+1\) dimensions.
where
\[ f(u) \equiv m^2 - \eta_0 \frac{4}{e^2} \frac{\partial_\mu u \partial^\mu u^*}{(1 + |u|^2)^2} \quad (4.40) \]

Since such submodel possesses an infinite number of local conserved currents, we believe that it is easier to be solved than the full Skyrme-Faddeev model. We are now investigating the solitons it may have.

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