Abstract

In this paper, we give an extension of the classical $L^p$-theorem of Fourier integral operators into the product space $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \cdots \times \mathbb{R}^{N_n}$. We require the phase functions satisfying the crucial non-degeneracy condition, on each coordinate subspace $\mathbb{R}^{N_i}$ for $i = 1, 2, \ldots, n$. In the other hand, by classifying on their symbols, we prove that the Fourier integral operators are bounded on $L^p(\mathbb{R}^N)$, whenever

$$\left| \frac{1}{2} - \frac{1}{p} \right| \leq \frac{-m}{N-n}$$

where $m$ is the order of symbols and $N = N_1 + N_2 + \cdots + N_n > n$.

1 Introduction

The $L^p$-theorem of Fourier integral operators defined on $\mathbb{R}^n$ was first established in 1991, by Seeger, Sogge and Stein in [1]. In this paper, we give an extension of the above result into product spaces.

Such product theorem was first obtained by R. Fefferman and Stein in [2], for translation invariant singular integrals. It was later refined by Nagel, Ricci and Stein in [9], which gives a classification of the corresponding symbols and kernels by Fourier transforms. A recent result for non-translation invariant singular integrals was proved in [7], whereas the class of operators can be viewed either as singular integrals of non-convolution type, or as pseudo differential operators whose symbols satisfy certain characteristic properties. Our $L^p$-estimate of Fourier integral operators actually gives a generalization of the above, by considering their symbols belonging to the same symbol class investigated thereby. Indeed, by singular integral realization, the kernel of Fourier integral operators depends on the variety of phase functions, and has singularity appeared not necessarily restricted to dimension zero. A further background of Fourier integral operators can be found in the work of Sogge [4], Peral [6] and Treves [8].
2 Statement of Main Result

Let \( N = N_1 + N_2 + \cdots + N_n \) be the dimension of a product space, whereas each \( N_i \) is the homogeneous dimension of the \( i \)-th subspace, for \( i = 1, 2, \ldots, n \). We write

\[
x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \cdots \times \mathbb{R}^{N_n} = \mathbb{R}^N
\]  

and \( \xi = (\xi^1, \xi^2, \ldots, \xi^n) \) denotes its dual variable in the frequency space. Define the inner product

\[
x \cdot \xi = x^1 \cdot \xi^1 + x^2 \cdot \xi^2 + \cdots + x^n \cdot \xi^n.
\]  

Let \( f \in S \) be a Schwartz function. A Fourier integral operator defined on the product space is

\[
\left( Ff \right)(x) = \int_{\mathbb{R}^N} e^{2\pi i \Phi(x, \xi)} \sigma(x, \xi) \hat{f}(\xi) d\xi
\]  

where \( \sigma(x, \xi) \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N) \) has a compact support in \( x \).

The phase function \( \Phi \) is real, in the form of

\[
\Phi(x, \xi) = \Phi_1(x^1, \xi^1) + \Phi_2(x^2, \xi^2) + \cdots + \Phi_n(x^n, \xi^n)
\]  

and smooth away from coordinate subspaces \( \xi^i = 0, \ i = 1, 2, \ldots, n \). Each \( \Phi_i \) is homogeneous of degree one in \( \xi^i \), and satisfies the non-degeneracy condition

\[
det \left( \frac{\partial^2 \Phi_i}{\partial x^j \partial \xi^k} \right)(x, \xi^i) \neq 0
\]  

for \( \xi^i \neq 0 \) and every \( i = 1, 2, \ldots, n \) on the support of \( \sigma(x, \xi) \).

By singular integral realization, we have

\[
\left( Ff \right)(x) = \int_{\mathbb{R}^N} f(y) \Omega(x, y) dy
\]  

where

\[
\Omega(x, y) = \int_{\mathbb{R}^N} e^{2\pi i \left( \Phi(x, \xi) - y \cdot \xi \right)} \sigma(x, \xi) d\xi.
\]  

By the localization principal of oscillatory integrals, \( \Omega \) in (2.7) has singularity appeared at

\[
\nabla_\xi \left( \Phi(x, \xi) - y \cdot \xi \right) = 0
\]  

for some \( \xi \). See chapter VIII of [3]. At each \( x \), we consider the variety

\[
\Sigma_x = \{ y : y = \nabla_\xi \Phi(x, \xi) \text{ for some } \xi \}
\]  

which is the locus of the singularity of \( y \rightarrow \Omega(x, y) \). By definition of \( \Phi \) in (2.4)-(2.5), \( \nabla_\xi \Phi_i \) is homogeneous of degree zero in \( \xi^i \), for \( i = 1, 2, \ldots, n \). The projection of \( \Sigma_x \) on each coordinate subspace \( \mathbb{R}^{N_i} \) has dimension at most equal to \( N_i - 1 \).

We next recall the symbol class first investigated in [7].
Definition of $S_p$: Let $0 \leq p < 1$. A symbol $\sigma(x, \xi)$ belongs to the symbol class $S_p^m$ if it satisfies the differential inequality

$$
\left| \frac{\partial^n}{\partial \xi^a} \frac{\partial^\beta}{\partial x^\beta} \sigma(x, \xi) \right| \leq A_{\alpha, \beta} \prod_{i=1}^{n} \left( \frac{1}{1 + |\xi|^a} \right)^{|\alpha|} (1 + |\xi|)^{m+p|\beta|} \quad (2.10)
$$

for every multi-index $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n)$ and $\beta$.

Let $m \leq 0$. When $p = 0$ in (2.10), $\sigma$ is essentially a product symbol as was introduced in [9]. When $0 < p < 1$, differential inequality (2.10) has a mixture of homogeneities. In general, $S_p^m$ forms a sub-class of the exotic class $S_{\rho, \rho'}^0$, see chapter VII of [3]. In the other hand, $\sigma \in S_p^0$ satisfies a variant of Marcinkiewicz theorem:

$$
\left| \left( \xi^a \frac{\partial^\beta}{\partial \xi^\beta} \right)^\alpha \sigma(x, \xi) \right| \leq A_{\alpha} \quad (2.11)
$$

for every $\alpha$ uniformly in $x$.

In the special case of $N_1 = N_2 = \cdots = N_n = 1$, the non-degeneracy condition (2.5) imply that $\Phi(x, \xi) \approx x \cdot \xi$ under appropriate local diffeomorphisms of $\mathbb{R}$. Then $F$ defined in (2.3)-(2.5) is essentially a pseudo differential operator. It is proved in [7] that pseudo differential operators with their symbols belonging to $S_p^m$ are $L^p$-continuous for $1 < p < \infty$.

Our main result is the following:

Theorem 1 Let $\sigma \in S_p^m$. Fourier integral operator $F$ defined in (2.3)-(2.5) with $-(N-n)/2 < m \leq 0$, initially defined on $S$, extends to a bounded operator on $L^p\left( \mathbb{R}^N \right)$, whenever

$$
\left| \frac{1}{2} - \frac{1}{p} \right| \geq \frac{-m}{N - n} \quad (2.12)
$$

for $N - n > 0$. When $N - n = 0$, the $L^p$-estimate holds for $1 < p < \infty$ provided that $m \leq 0$.

We develop our $L^p$-estimate in the framework of Littlewood-Paley projections, whereas the frequency space is decomposed into the product of Dyadic balls. For each ball lies in the coordinate subspace, we construct a second Dyadic decomposition, in the same spirit of the estimation given implicitly in [10]. A well illustrated proof for $n = 1$ can be found in chapter IX of [3]. In compare of that, for $n > 1$ we have obtained a decaying estimate on the kernel of every partial sum operators. This eventually makes the breakthrough in our analysis. Lastly, the estimate (2.12) is sharp, in the sense of the following: Suppose $n = 1$. Let $a(x) \in C_\infty^\infty\left( \mathbb{R}^N \right)$ not vanishing on $|x| = 1$ and $\eta(\xi) \sim (1 + |\xi|)^m$. Fourier integral operator $F$, with phase function $\Phi(x, \xi) = x \cdot \xi \pm |\xi|$ and symbol $\sigma(x, \xi) = a(x)\eta(\xi)$, is not bounded on $L^p\left( \mathbb{R}^N \right)$ for $|1/2 - 1/p| > m/(1 - N)$. Regarding details can be found in 6.13, chapter IX of [3].

Abbreviations:

- We write $\Phi_\xi = \nabla_\xi \Phi$ and $\Phi_{\xi^i} = \nabla_\xi \Phi_i$ for $i = 1, 2, \ldots, n$.

- Unless otherwise indicated, we write $\int = \int_{\mathbb{R}^N}$ and $L^p = L^p\left( \mathbb{R}^N \right)$. 
We write $H^1 = H^1\left(\mathbb{R}^N\right)$ to be the $H^1$-Hardy space, and $\text{BMO}$ denotes the space of bounded mean oscillations.

We write $x^i = \left(x^i_j, (x^i_j)^\dagger\right) \in \mathbb{R}^1 \times \mathbb{R}^{N-1}$ and $\xi^i = \left(\xi^j_j, (\xi^j_j)^\dagger\right) \in \mathbb{R}^1 \times \mathbb{R}^{N-1}, j = 1, 2, \ldots, N_i$ for every $i = 1, 2, \ldots, n$.

We always write $A$ as a generic, positive constant with a subindex indicating its dependence.

### 3 $L^2$-Estimate

By Plancherel’s theorem, our estimate is reduced to a similar assertion for the operator

$$ (Sf)(x) = \int e^{2\pi i \Phi(x, \xi)} \sigma(x, \xi) f(\xi) d\xi. \quad (3.1) $$

Let $c$ be a small, positive constant. We define the narrow cone inside the frequency space in the following way: whenever $\xi$ and $\eta$ belong to the cone and $|\eta| \leq |\xi|$, the component of $\eta$ which is perpendicular to $\xi$, denoted by $\eta^\perp$, satisfies $|\eta^\perp| \leq c|\xi|$. It is clear that any given $S$ can be written as a finite sum of operators where $\sigma(x, \xi)$ is supported on the narrow cone, by using a suitable smooth partition of unity.

We claim that

$$ \left| \nabla_x (\Phi(x, \xi) - \Phi(x, \eta)) \right| \geq \text{const} |\xi - \eta| \quad (3.2) $$

whenever $\xi$ and $\eta$ belong to the same narrow cone. For the result in (3.2), we refer to the estimates given in section 3, chapter IX of [3].

Let $\phi \in C_c^\infty(\mathbb{R}^N)$, such that $\phi \equiv 1$ when $|\xi| \leq 1$ and $\phi \equiv 0$ when $|\xi| > 2$. Define

$$ \psi(\xi) = \phi(\xi) - \phi(2\xi) \quad (3.3) $$

so that $\psi$ is supported on a spherical shell $\frac{1}{2} \leq |\xi| \leq 2$. Let $j$ be a positive integer. We write

$$ \psi_j(\xi) = \phi\left(2^{-j}\xi\right) - \phi\left(2^{-j+1}\xi\right). \quad (3.4) $$

Consider the partial sum operator $S_j$ defined by

$$ (S_j f)(x) = \int e^{2\pi i \Phi(x, \xi)} \psi_j(\xi) \sigma(x, \xi) f(\xi) d\xi. \quad (3.5) $$

It has a dual operator

$$ (S_j^* f)(x) = \int e^{-2\pi i \Phi(y, \xi)} \psi_j(\xi) \overline{\sigma(y, \xi)} f(x) dy. \quad (3.6) $$

We have

$$ (S_i S_j^* f)(x) = \int e^{2\pi i \Phi(x, \xi)} \psi_i(\xi) \sigma(x, \xi) (S_j^* f)(\xi) d\xi $$

$$ = \int e^{2\pi i \Phi(x, \xi)} \psi_i(\xi) \sigma(x, \xi) \left\{ \int e^{-2\pi i \Phi(\eta, \xi)} \psi_j(\xi) \overline{\sigma}(y, \xi) f(y) dy \right\} d\xi \quad (3.7) $$

$$ = \int f(y) \left\{ \int e^{2\pi i (\xi(\phi(\xi), \xi) - \Phi(y, \eta))} \psi_i(\xi) \psi_j(\xi) \sigma(x, \xi) \overline{\sigma}(y, \xi) d\xi \right\} dy. $$
Denote $\Omega^\flat_{ij}(x, y)$ to be the kernel of $S_iS_j^*$ so that

$$
(S_iS_j^* f)(x) = \int f(y)\Omega^\flat_{ij}(x, y)dy
$$

(3. 8)

and

$$
\Omega^\flat_{ij}(x, y) = \int e^{2\pi i(\Phi(x,y) - \Phi(\xi, \eta))}\psi_i(\xi)\psi_j(\eta)\sigma(x, \xi, \eta) d\xi.
$$

(3. 9)

Observe that by definition of $\psi_j$ in (3. 4), we have $\Omega^\flat_{ij} = 0$ whenever $|i - j| \geq 2$. In the other hand, direct computation shows

$$
(S_i^*S_j f)(\xi) = \int f(\eta)\Omega_i^\flat(\xi, \eta)d\eta
$$

(3. 10)

where

$$
\Omega_i^\flat(\xi, \eta) = \int e^{2\pi i(\Phi(x, \eta) - \Phi(\xi, \eta))}\psi_i(\xi)\psi_j(\eta)\sigma(x, \eta)dx.
$$

(3. 11)

Since $\sigma(x, \xi)$ has $x$-compact support, $\Omega_i^\flat$ is bounded indeed. Moreover, integrating by parts with respect to $x$ implies $\Omega_i^\flat(\xi, \eta)$ equals

$$
\Bigg|2\pi V_x(\Phi(x, \eta) - \Phi(x, \xi))\Bigg|^{-2N} \int e^{2\pi i(\Phi(x, \eta) - \Phi(\xi, \eta))}\psi_i(\xi)\psi_j(\eta)\sigma(x, \eta)\Delta_x^N(\sigma(x, \eta)d\xi)(\xi)dx
$$

(3. 12)

for every $N \geq 1$. From differential inequality (2. 10), together with (3. 2), we have

$$
\bigg|\Omega_i^\flat(\xi, \eta)\bigg| \leq A_N \left(\frac{|\xi|^p}{|\xi - \eta|} + \frac{|\eta|^p}{|\xi - \eta|}\right)^{2N}
$$

(3. 13)

where $2^{i-1} < |\xi| < 2^{i+1}$ and $2^{j-1} < |\eta| < 2^{j+1}$. It is easy to observe that

$$
\bigg|\Omega_i^\flat(\xi, \eta)\bigg| \leq A_N 2^{2N\max(i, j)} \times 2^{-2N\max(i, j)}
$$

(3. 14)

$$
\leq A_N 2^{-N(1-p)Ni} \times 2^{-N(1-p)Nj}, \quad 0 \leq \rho < 1
$$

for every $N \geq 1$. Since $\Omega_i^\flat$ has its $\xi$-support inside $2^{i-1} < |\xi| < 2^{i+1}$, by taking $N$ to be sufficiently large in (3. 14), we shall have

$$
\bigg\|\big(S_i^*S_j f\big)\bigg\|_{L_2} \leq A_\rho 2^{-(1-p)i} \times 2^{-(1-p)j} \|f\|_{L_2}.
$$

(3. 15)

Lastly, we need to show that each $S_j$ is bounded uniformly in $j$. Observe that each $\psi_j(\xi)\sigma(x, \xi)$ is supported in the shell $2^{j-1} < |\xi| < 2^{j+1}$. By letting

$$
\zeta_j(x, \xi) = \psi_j(2^p \xi) \sigma(2^{-p}x, 2^p \xi),
$$

(3. 16)

we have $\zeta_j \in S_{0,0}$ and has compact support in $\xi$ for every $j$. Consider

$$
\big(S_j f\big)(x) = \int e^{2\pi i\Phi(x, \xi)}\zeta_j(x, \xi)f(\xi)d\xi
$$

(3. 17)
and its dual operator
\[ (\widetilde{S}_j f)(x) = \int e^{-2\pi i \Phi(x, \xi)} \mathcal{S}_j(x, \xi) f(x) \, dx. \] (3.18)

We have
\[ (\widetilde{S}_j \widetilde{S}_j f)(\xi) = \int f(\eta) \Omega_j(\xi, \eta) \, d\eta \] (3.19)
where
\[ \Omega_j(\xi, \eta) = \int e^{2\pi i (\Phi(x, \eta) - \Phi(x, \xi))} \mathcal{S}_j(x, \eta) \mathcal{S}_j(x, \xi) \, dx \] (3.20)
is bounded for every \( j \). It is clear that \( \widetilde{S}_j \widetilde{S}_j \) is bounded on the \( L^2 \)-spaces, and so does \( \widetilde{S}_j \), uniformly in \( j \). Next, let \( \tau_j \) be the scaling operator
\[ (\tau_j f)(x) = f(2^{\rho j} x). \] (3.21)

It is easy to verify that
\[ S_j = \tau_j \widetilde{S}_j \tau_j^{-1}. \] Since
\[ \|\tau_j f\|_{L^2} = 2^{\rho N j} \|f\|_{L^2} \quad \text{and} \quad \|\tau_j^{-1} f\|_{L^2} = 2^{-\rho N j} \|f\|_{L^2}, \] (3.22)
the \( L^2 \)-boundedness of \( \widetilde{S}_j \) implies that \( S_j \) is a bounded operator on the \( L^2 \)-spaces.

At this point, by writing
\[ S = \sum_{j=0}^{\infty} S_j = \sum_{\text{odd}} S_j + \sum_{\text{even}} S_j, \] (3.23)
we obtain a result of almost orthogonality:
\[ \|S_j\| \leq A_\rho, \quad \text{for every } j, \]
\[ \|S_j^* S_j\| \leq A_\rho 2^{-(1-\rho)j} \times 2^{-(1-\rho)j}, \] (3.24)
\[ \|S_j S_j^*\| = 0, \quad \text{whenever } i, j \in \text{odd or } i, j \in \text{even}. \]

By Cotlar-Stein lemma in chapter VII of [3], we have
\[ \|S f\|_{L^2} \leq A_\rho \|f\|_{L^2} \] (3.25)
which implies \( F \) is \( L^2 \)-continuous.

## 4 Littlewood-Paley Projections

In this section, we introduce the Littlewood-Paley projections developed in section 3 of [7]. Let \( t_i, i = 1, 2, \ldots, n \) be positive integers and write \( q = 1/\rho \). Define the non-isotropic dilations
\[ \mathbf{t}_i \xi = \left( 2^{-q t_i} \xi^1, \ldots, 2^{-q t_i} \xi^{i-1}, 2^{-t_i} \xi^i, 2^{-q t_i} \xi^{i+1}, \ldots, 2^{-q t_i} \xi^n \right) \] (4.1)
for every \( i = 1, 2, \ldots, n \).
Let $\psi \in C^\infty_\alpha (\mathbb{R}^N)$ in (3.3). For each $n$-tuple $t = (t_1, t_2, \ldots, t_n)$, we write

$$\delta_t(\xi) = \prod_{i=1}^n \psi(t_i \xi).$$

(4.2)

The partial sum operator $\Delta_t$ is defined by

$$(\Delta_t f)(\xi) = \delta_t(\xi) \hat{f}(\xi).$$

(4.3)

For $0 < \rho < 1$, the support of $\delta_t(\xi)$ lies inside the intersection of $n$ elliptical shells, with different homogeneities of given dilations. For $\rho = 0$, the support of $\delta_t(\xi)$ lies inside the Dyadic rectangle $\{\xi \in \mathbb{R}^N : 2^{l-1} \leq |\xi_i| \leq 2^{l+1}, i = 1, 2, \ldots, n\}$. For each $t$, consider the hypothesis set below:

(H) There exists at least one $i \in \{1, 2, \ldots, n\}$ such that

$$t_i > \frac{1}{\alpha - 1} (2 + 2 \log_2 n).$$

(4.4)

We recall the combinatorial result proved in [7]:

Lemma 4.1 Let $t$ satisfying hypothesis (H) and $0 < \rho < 1$. Set $I \cup J = \{1, 2, \ldots, n\}$ such that

$$\frac{t_i + 2 + \log_2 n}{\alpha} < t_{i_2} < q t_i - (2 + \log_2 n) \quad \text{for all } i_1, i_2 \in I,$n

and

$$qt_j - (2 + \log_2 n) \leq t_i \quad \text{for some } i \in I \text{ and all } j \in J.$$n

(4.5)

Then, the support of $\delta_t(\xi)$ is non-empty if and only if

$$|\xi_i| \sim 2^{l_i} \quad \text{for every } i \in I,$n

$$|\xi_j| \lesssim 2^{l_j} \quad \text{and} \quad |\xi_i| \sim 2^{l_i} \sim 2^{\alpha l_i} \quad \text{for every } j \in J \text{ and some } i \in I.$$n

(4.6)

A direct outcome from Lemma 4.1 is the following:

Lemma 4.2 Let $\sigma \in S^0_\alpha$. Suppose $t$ satisfying hypothesis (H), we have

$$\left| \prod_{i=1}^n \left( \frac{\partial}{\partial \xi^i} \right)^{\alpha_i} \delta_t(\xi) \sigma(x, \xi) \right| \leq A_\alpha \prod_{i=1}^n 2^{-t_i |\alpha_i|}$$

for every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

(4.8)

Proof: Certainly, for every $t$ satisfying (H), $\delta_t(\xi)$ by definition satisfies differential inequality (4.8). Recall $\sigma \in S^0_\alpha$ is admitted in differential inequality (2.10). Therefore, at each $i$-th component, we have

$$\left| \left( \frac{\partial}{\partial \xi^i} \right)^{\alpha_i} \sigma(x, \xi) \right| \leq \left( \frac{1}{1 + |\xi_i| + |\xi|^\rho} \right)^{|\alpha_i|}$$

for $i = 1, 2, \ldots, n$.

When $0 < \rho < 1$, by Lemma 4.1, we either have $|\xi_i| \sim 2^{l_i}$ or otherwise there exists $\xi^i$ such that $|\xi^i| \sim 2^{l_i} \sim 2^{\alpha_i l_i}$, whenever $\xi \in \text{supp} \delta_t(\xi)$. When $\rho = 0$, we have $|\xi| \sim 2^{l_i}$ for every $i = 1, 2, \ldots, n$. □
Suppose $\xi \in \text{supp} \delta_t(\xi)$ with $t$ satisfying (H). By Lemma 4.1, if the support is non-empty, we have $|\xi|^i \sim 2^i$ for every $i \in I$, and $|\xi| \leq 2^{i/p}$ for every $j \in J$ and some $i \in I$, with respect to (4. 5)-(4. 6). As an necessity, we have $t_i \geq t_i$ for all other $i \neq t$ and therefore $|\xi| \sim |\xi|^i \sim 2^i$.

Consider the $n$-tuple $s = (s_1, s_2, \ldots, s_n)$ whereas each $s_i, i = 1, 2, \ldots, n$ is a positive integer. Let $\psi_i \in C_c^\infty(\mathbb{R}^N_i)$ be defined as same as (3. 3), on each coordinate subspace $\mathbb{R}^N_i, i = 1, 2, \ldots, n$. We set

$$
\delta_{t,s}(\xi) = \delta_t(\xi) \prod_{i=1}^n \psi_i^j \left(2^{-s_i} \xi^i\right) \tag{4. 10}
$$

where its support is either empty, or lies inside the Dyadic rectangle $\left\{2^{s_i-1} \leq |\xi^i| \leq 2^{s_i+1}\right\}$ for every $s_i \leq t_i, i = 1, 2, \ldots, n$.

L$^p$-Estimate for $N - n = 0$

We now turn to an $L^p$-estimate of Fourier integral operators $\mathcal{F}$ defined in (2. 3)-(2. 5), in the special case of $N_1 = N_2 = \cdots = N_n = 1$. Since $\Phi_i$ is homogeneous of degree one in $\xi^i$, we shall have

$$
\Phi_i(x^i, \xi^i) = \Phi_i(x^i, \pm 1)|\xi^i| \tag{4. 11}
$$

for every $i = 1, 2, \ldots, n$. The non-degeneracy condition (2. 5) implies $d\Phi_i/dx^i\left(x^i, \pm 1\right) \neq 0$. Under appropriate smooth partition of unity, Fourier integral operator $\mathcal{F}$ can be written as

$$
\sum_{\lambda} \left(T^\lambda_\sigma f\right)\left(\Phi_1(x^1, \pm 1), \Phi_2(x^2, \pm 1), \ldots, \Phi_n(x^n, \pm 1)\right) + \left(\mathcal{E}f\right)(x) \tag{4. 12}
$$

where each $T^\lambda_\sigma$ in (4. 12) is a pseudo differential operator with symbol $\sigma \in S^0_\rho$, supported in the quadrant

$$
\left\{\lambda \cdot \xi : |\xi^i| > 1, i = 1, 2, \ldots, n\right\} \tag{4. 13}
$$

with $\lambda = (\text{sign} \xi^1, \text{sign} \xi^2, \ldots, \text{sign} \xi^n)$.

The $L^p$-theorem obtained in [7] implies that $T^\lambda_\sigma$ is bounded on $L^p$ for $1 < p < \infty$, for every $\lambda$ among the permutations. In the other hand, the error term $\mathcal{E}f$ in (4. 12) has its symbol supported in

$$
\left\{\lambda \cdot \xi : |\xi^i| \leq 1, i \in I \text{ and } |\xi^j| > 1, j \in J\right\} \tag{4. 14}
$$

for every $I \cup J = \{1, 2, \ldots, n\}$ with $|I| \geq 1$. On each of these regions, we can take the local diffeomorphisms

$$
x^i \mapsto \Phi_j\left(x^j, \pm 1\right), \quad j \in J. \tag{4. 15}
$$

The resulting operator can be estimated in Littlewood-Paley projections, with partial sum operators $\Lambda_k$ defined in (4. 3), on the subspace $\bigoplus_{k \in J} \mathbb{R}^N_k$. By carrying out the estimation in analogue to section 5 of [7], together with that $\sigma(x, \xi)$ has a compact support in $x$, we conclude that $\mathcal{E}$ is bounded on $L^p$ for $1 < p < \infty$.

5 Region of Influence

In order to obtain our desired $L^p$-estimate, the main point is to show that the operator $\mathcal{F}$ defined in (2. 3)-(2. 5), with $\sigma \in S^m_\rho$ for $m = -(N - n)/2$, is bounded from $H^1$ to $L^1$. 

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Let \( B_\delta \subset \mathbb{R}^N \) be the ball centered on \( x = 0 \) with radius \( \delta > 0 \). We consider \( a \) as an \( \mathcal{H}^1 \)-atom in \( \mathbb{R}^N \), such that

\[
\begin{align*}
\text{supp } a &\subset B_\delta, \\
|a(x)| &\leq |B_\delta|^{-1} \quad \text{for almost everywhere } x, \\
\int a(x) dx & = 0.
\end{align*}
\]  

(5.1)

As was classified in [11], if \( f \in \mathcal{H}^1 \), then \( f \) can be written as \( \sum \lambda_k a_k \), where each \( a_k \) is an \( \mathcal{H}^1 \)-atom associated to the ball \( B_{\delta_k} \) and \( \sum |\lambda_k| < \text{const} \). Therefore, it is suffice to show that

\[
\int |\mathcal{F}a(x)| dx \lesssim \|a\|_{L^1}, \quad a \in \mathcal{H}^1
\]  

(5.2)

for \( \sigma \in S^m_p \) with \( m = -(N - n)/2 \).

Suppose \( \delta > 1 \). We have an a priori estimate of

\[
\int |\mathcal{F}a(x)| dx \lesssim \|F a\|_{L^2} \leq A_\rho \|a\|_{L^2}.
\]  

(5.3)

The first inequality in (5. 3) follows that \( \mathcal{F} \) has a compact support in \( x \), whereas the second inequality is proved in section 2. Since \( a \) is an \( \mathcal{H}^1 \)-atom, we must have

\[
\|a\|_{L^2} \leq |B_\delta|^{-\frac{1}{2}} \leq \text{const}
\]  

(5.4)

for \( \delta \geq 1 \). In the other hand, when \( \delta < 1 \), the pseudo-local property of \( \mathcal{F} \) eliminated us to consider so-called the region of influence with respect to \( B_\delta \), denoted by \( B_{\delta^*} \), which is essentially the set \( \{ \mathbf{x} : \text{dist} ( \Sigma_{\mathbf{c}} B_{\delta} ) \leq \delta \} \). Since \( \Sigma_{\mathbf{c}} \) defined in (2. 9) could be very singular, and has dimension at most equal to \( N - n \), we shall take

\[
|B_{\delta^*}| \lesssim \delta^d
\]  

(5.5)

for \( \delta \) is small. By Schwartz inequality, we have

\[
\int_{B_{\delta^*}} |\mathcal{F}a(x)| dx \leq |B_{\delta^*}|^{\frac{1}{2}} \|\mathcal{F}a\|_{L^2} \lesssim \delta^d \|\mathcal{F}a\|_{L^2}.
\]  

(5.6)

Write

\[
(\mathcal{F}f)(\mathbf{x}) = \int e^{2\pi i \Phi(\mathbf{x}, \mathbf{\xi})} \sigma(\mathbf{x}, \mathbf{\xi}) \left( 1 + |\mathbf{\xi}|^2 \right)^{-\frac{N-m}{2}} \hat{f}(\mathbf{\xi}) d\mathbf{\xi}
\]  

(5.7)

where the Fourier multiplier operator \( T \) defined by \( \hat{Tf}(\mathbf{\xi}) = \left( 1 + |\mathbf{\xi}|^2 \right)^{-\frac{N-m}{2}} \hat{f}(\mathbf{\xi}) \) is a singular integral of convolution type. Its kernel satisfies a size estimate of

\[
|\Omega(\mathbf{x})| \lesssim |\mathbf{x}|^{-N-m}.
\]  

(5.8)

By Hardy-Littlewood-Sobolev inequality, we have \( \|\mathcal{F}a\|_{L^2} \lesssim \|a\|_{L^p} \) if

\[
\frac{1}{p} = \frac{1}{2} + \frac{N-n}{2N}
\]  

(5.9)
of which the symbol \( \sigma(x, \xi) \) has order \( m = -(N - n)/2 \). Now, the \( H^1 \)-atom \( a \) in (5.1) implies that \( \|a\|_{L^p} \leq |B|^{-1/p} \) for \( 1 < p < \infty \). By taking the largest possible bound, we have

\[
\int_{B^*_\delta} |F_a(x)| \, dx \leq \delta^2 \delta^{N(1 + \frac{1}{p})} < \text{const.} \quad (5.10)
\]

**Construction of \( B^*_\delta \):**

Set

\[
\{I \cup J = \{1, 2, \ldots, n\}
\]

such that \( N_i > 1 \) for every \( i \in I \) and \( N_j = 1 \) for every \( j \in J \). From now on, we assume \( |I| \geq 1 \). Let \( S^{N_i-1} \) be the unit sphere in the coordinate subspace \( \mathbb{R}^{N_i} \), for \( i \in I \). We consider an equally distributed set of points on \( S^{N_i-1} \) with grid length equal to \( 2^{-s_i}/2 \). Denote the collection of such points by \( \{ \xi_{s_i}^{(j)} \}_{s_i} \). It is clear that there are at most a constant multiple of \( 2^{(N_i-1)/2} \) elements in \( \{ \xi_{s_i}^{(j)} \}_{s_i} \) for every \( s_i \geq 1 \).

Define the rectangles by

\[
R_{s_i}^{(j)} = \left\{ x^j : \left| \Phi(x^j, \xi_{s_i}^{(j)}) \right| \leq \text{const} \, 2^{-s_i/2}, \, \left| \tau_{s_i}^{(j)} \Phi(x^j, \xi_{s_i}^{(j)}) \right| \leq \text{const} \, 2^{-s_i} \right\} \quad (5.11)
\]

for every \( i \in I \), where \( \tau_{s_i}^{(j)} \) is the orthogonal projection in the direction of \( \xi_{s_i}^{(j)} \). The region of influence \( B^*_\delta \) is defined by

\[
B^*_\delta = \bigcup_{i \in I} \left( \bigcup_{2^{-s_i} \leq \delta} \bigcup_{s_i} R_{s_i}^{(j)} \right). \quad (5.12)
\]

The set \( B^*_\delta \) in (5.12) satisfies the norm estimate (5.5) for \( \delta < 1 \), provided that \( |I| \leq n \). For regarding details, please see 4.3, chapter IX of [3].

Let \( \Gamma_{s_i}^{(j)} \) denotes the cone whose central direction is \( \xi_{s_i}^{(j)} \), such that

\[
\Gamma_{s_i}^{(j)} = \left\{ \xi^j : \left| \frac{\xi^j}{|\xi^j|} - \xi_{s_i}^{(j)} \right| \leq 2 \times 2^{-s_i/2} \right\}, \quad i \in I. \quad (5.13)
\]

Recall from 4.4, chapter IX of [3]. We can construct an associated partition of unity \( \chi_{s_i}^{(j)} \), with homogeneity zero in \( \xi^j \) and supported in \( \Gamma_{s_i}^{(j)} \). Moreover, we have

\[
\sum_{s_i} \chi_{s_i}^{(j)}(\xi^j) = 1 \quad \text{for } \xi^j \neq 0 \text{ and all } s_i \geq 1 \quad (5.14)
\]

such that

\[
\left| \left( \frac{\partial}{\partial \xi^j} \right)^\alpha \chi_{s_i}^{(j)}(\xi^j) \right| \leq A_{\alpha, i} 2^{\alpha |s_i|/2} |\xi^j|^{-|\alpha|}. \quad (5.15)
\]

Lastly, we define simultaneously

\[
\chi_{s}^{(j)}(\xi) = \prod_{i \in I} \chi_{s_i}^{(j)}(\xi^j) \prod_{j \in J} \left( 1 - \psi^j(\xi^j) \right), \quad (5.16)
\]

\[
\hat{c} \chi_{s_i}^{(j)}(\xi) = \prod_{i \in I} \chi_{s_i}^{(j)}(\xi^j) \left( 1 - \prod_{j \in J} \left( 1 - \psi^j(\xi^j) \right) \right). \quad (5.17)
\]
6 Estimation on Kernel

The L\textsuperscript{p} estimate will be carried out in analogue to the estimation in chapter IX of [3]. We aim to prove that

\[
\int_{B_0} |\mathcal{F}a(x)| \, dx \lesssim \|a\|_{L^1},
\]

whenever \(a\) is an \(H^1\)-atom.

A direct computation shows that

\[
\left(\mathcal{F}\Delta_t f\right)(x) = \int f(y)\Omega_t(x, y) \, dy
\]

where

\[
\Omega_t(x, y) = \int e^{2\pi i (\Phi(x, \xi) - y \cdot \xi)} \delta_t(\xi)\sigma(x, \xi) \, d\xi.
\]

Observe that by definition of \(\delta_t(\xi)\) in (4. 2), we have

\[
\Omega = \sum_t \Omega_t + \Omega_o
\]

where \(t\) in the summation satisfies hypothesis \((H)\) in (4. 4). Observe that \(\Omega_o \in C^\infty_o \left(\mathbb{R}^N\right)\).

Let \(j\) be a positive integer such that \(2^{j-1} \leq |\xi| \leq 2^{j+1}\) for \(\xi \in \text{supp} \delta_t(\xi)\). By Lemma 4.1, we must have \(t_i \leq j\) for every \(i = 1, 2, \ldots, n\). In particular, we write

\[
\Omega_t = \Omega_j
\]

for \(t_1 = t_2 = \cdots = t_n = j\).

The main objective in this section is proving the following estimates of the kernel \(\Omega_t\):

**Lemma 6.1** Suppose \(\sigma \in S_b^m\) with \(m = -(N - n)/2\). We have

\[
\int |\Omega_t(x, y)| \, dx \lesssim \prod_{i=1}^n 2^{(t_i-j)(N_i-1)/2} \quad \text{for } y \in \mathbb{R}^N,
\]

\[
\int |\Omega_j(x, y) - \Omega_j(x, z)| \, dx \lesssim 2^j |y - z| \quad \text{for } y, z \in \mathbb{R}^N
\]

and

\[
\int_{B_0} |\Omega_j(x, y)| \, dx \lesssim \frac{2^{-j}}{\delta} \quad \text{for } y \in B_0
\]

whenever \(2^j > \delta^{-1}\).

**Remark 6.1** By assumption, there is at least one \(i \in \{1, 2, \ldots, n\}\) such that \(N_i \geq 2\). We thus have \(m = -(N - n)/2 \leq -1/2\). In fact, suppose \(\Phi(x, \xi) = x \cdot \xi\). Then, \(\mathcal{F}\) defined in (2. 3) with its symbol \(\sigma \in S_b^{\varepsilon}\) for any \(\varepsilon > 0\), satisfies the weak type-(1,1) estimate.
Proof: First, we write
\[ \Omega_t = \sum_{s,v} \left( \Omega_{ts}^v + \varepsilon \Omega_{ts}^v \right) \]  \hspace{1cm} (6.8)
where
\[ \Omega_{ts}^v(x, y) = \int e^{2\pi i \left( \Phi(x, \xi) - y \cdot \xi \right)} \lambda_{s}^{v} (\xi) \delta_{ts}(\xi) \sigma(x, \xi) d\xi \]  \hspace{1cm} (6.9)
and
\[ \varepsilon \Omega_{ts}^v(x, y) = \int e^{2\pi i \left( \Phi(x, \xi) - y \cdot \xi \right)} \varepsilon \lambda_{s}^{v} (\xi) \delta_{ts}(\xi) \sigma(x, \xi) d\xi \]  \hspace{1cm} (6.10)
with \( \delta_{ts}(\xi) \) defined in (4.10) and \( \lambda_{s}^{v}, \varepsilon \lambda_{s}^{v} \) defined respectively in (5.16)-(5.17).

In each \( i \)-th subspace for \( i \in I \), we can choose a new framework under some appropriate linear transformation, such that the first coordinate coincides with the direction of \( \xi_{j}^{v} \), whenever \( \xi \in \text{supp} \lambda_{s}^{v}(\xi) \delta_{ts}(\xi) \). It should be clear that differential inequality (2.10) remains to be true.

With a bit omitted on notations, we then write
\[ \Phi_i \left( x^i, \xi^i \right) - y^i \cdot \xi^i = \left( \Phi_{\xi_i} \left( x^i, \xi^i \right) - y^i \right) \cdot \xi^i + \left( \Phi_i \left( x^i, \xi^i \right) - \Phi_{\xi_i} \left( x, \xi_i \right) \cdot \xi^i \right) \]  \hspace{1cm} (6.11)
where \( \xi_i = \xi_i / \| \xi_i \| \) is in the direction of \( \xi_{i}^{v} \), for every \( i \in I \). Recall \( \Phi_{\xi_i} \) is homogeneous of degree zero in \( \xi_i \). Define
\[ \varphi_i \left( x^i, \xi^i \right) = \Phi_i \left( x^i, \xi^i \right) - \Phi_{\xi_i} \left( x, \xi_i \right) \cdot \xi^i, \quad i \in I. \]  \hspace{1cm} (6.12)

Let \( \xi^i = \left( \tau^i, \eta^i \right) \in \mathbb{R} \times \mathbb{R}^{N_i-1} \) for \( i \in I \) such that
\[ \tau^i = \xi_1^i \in \mathbb{R}, \quad \eta^i = \left( \xi_1^i \right)^\top \in \mathbb{R}^{N_i-1}. \]  \hspace{1cm} (6.13)

By carrying out the estimation given in 4.5, chapter IX of [3], we have
\[ \left\| \left( \begin{array}{l} \frac{\partial}{\partial T^i} \\ \frac{\partial}{\partial \xi^i} \end{array} \right) \right\|_{N} \varphi_i \left( x^i, \xi^i \right) \leq A_N 2^{-N\xi_i}, \]  \hspace{1cm} (6.14)
\[ \left\| \left( \begin{array}{l} \nabla \eta^i \\ \nabla \xi^i \end{array} \right) \right\|_{N} \varphi_i \left( x^i, \xi^i \right) \leq A_N 2^{-N\xi_i/2} \]
for every \( N \geq 1 \).

Let
\[ \omega_{ts}^v(x, \xi) = \prod_{j \in U} e^{2\pi i \varphi_j(x^j, \xi^j)} \lambda_j^v(\xi) \delta_{ts}(\xi) \sigma(x, \xi). \]  \hspace{1cm} (6.15)

We rewrite \( \Omega_{ts}^v \) as
\[ \int \left\{ \prod_{j \in V} e^{2\pi i \left( \Phi_j(x^j, \xi^j) - y^j \cdot \xi^j \right)} \right\} \left\{ \prod_{j \in U} e^{2\pi i \left( \Phi_{\xi_j}(x^j, \xi^j) - y^j \right) \cdot \xi^j} \right\} \omega_{ts}^v(x, \xi) d\xi. \]  \hspace{1cm} (6.16)

Let \( \sigma \in \mathbb{S}^{m}_p \) and has an order of \( m = -(N - n)/2 \). If \( |\xi| \sim 2^j \), we have
\[ \left| \omega_{ts}^v(x, \xi) \right| \leq \left( \frac{1}{1 + |\xi|} \right)^{\frac{N-m}{2}} \sim 2^{-j(N-n)/2}. \]  \hspace{1cm} (6.17)
The rest of estimation will be accomplished in several steps.

1. Define the differential operators

\[
L^i = I - 2^{2i} \left( \frac{\partial}{\partial \tau^i} \right)^2 - 2^{\nu_i} \Delta_{\eta^i}, \quad i \in I, \tag{6. 18}
\]

\[
L^j = I - 2^{2s_j} \Delta_{\xi_j}, \quad j \in J.
\]

Let \( \xi \in \text{supp} \chi^\nu_s(\xi) \delta_{t,s}(\xi) \). Recall estimates (6. 14)-(6. 17) and (5. 15), together with Lemma 4.1 and Lemma 4.2. We have

\[
\left| (L^i)^M (L^j)^N (\omega^\nu_s(x, \xi)) \right| \leq A_{MN} 2^{-j(N-n)/2} \tag{6. 19}
\]

for every \( i \in I, j \in J \) and every \( M \geq 1, N \geq 1 \).

Turning to the other side,

\[
(L^i)^N e^{2\pi i (\Phi_i(x^i, \xi^i) - y^i)} e^{i \xi^i} = \left\{ 1 + 4\pi^2 2^{2s_i} \left| (\Phi_i(x^i, \xi^i) - y^i) \right|^2 + 4\pi^2 2^{s_i} \left| (\Phi_i(x^i, \xi^i) - y^i) \right|^2 \right\}^N e^{2\pi i (\Phi_i(x^i, \xi^i) - y^i)} e^{i \xi^i} \tag{6. 20}
\]

for every \( i \in I \). Recall that each \( \Phi_i \) satisfies the non-degeneracy condition (2. 5). We therefore can make a change of variables

\[
x^i \rightarrow \Phi_i(x^i, \xi^i) \tag{6. 21}
\]

whose Jacobian is bounded from below.

In the other hand, we have

\[
(L^j)^M e^{2\pi i (\Phi_j(x^j, \xi_j) - y^j)} = \left\{ 1 + 4\pi^2 2^{2s_j} \left| (\Phi_j(x^j, \xi_j) - y^j) \right|^2 \right\}^M e^{2\pi i (\Phi_j(x^j, \xi_j) - y^j)} \tag{6. 22}
\]

for every \( j \in J \). In the support of \( \chi^\nu_s(\xi) \) as defined in (5. 16), the non-degeneracy condition (2. 5) implies the local diffeomorphisms

\[
x^j \rightarrow \Phi_j(x^j, \pm 1) \tag{6. 23}
\]

for every \( j \in J \), as in (4. 15).

By definitions in (5. 13)-(5. 16), the support of \( \chi^\nu_s(\xi) \delta_{t,s}(\xi) \) has a size bounded by

\[
\prod_{i \in I} 2^{s_i} 2^{(N_i-1)/2} \times \prod_{j \in J} 2^{s_j} \tag{6. 23}
\]

with \( s_i \leq t_i \) for every \( i = 1, 2, \ldots, n \).
By putting all together the estimates (6.19)-(6.23), with $N$ and $M$ sufficiently large, we have
\[
\int |\Omega_{t,s}^{\nu}(x,y)| \, dx \leq \prod_{i=1}^{2^{s_i/2}(s_i-j)(N_i-1)/2} \int_{\mathbb{R}^{N_i}} \left(1 + 2^{s_i/2} \left| \left(x^i - y^i\right)_1 \right| + 2^{s_i/2} \left| \left(x^i - y^i\right)_1^+ \right| \right)^{-2N} \, dx^i
\times \prod_{j \in J} 2^{N|s_j|} \int_{\mathbb{R}^{N_j}} \left(1 + 2^{s_j/2} \left| \left(x^j - y^j\right) \right| \right)^{-2M} \, dx^j
\leq 2^{-j(N-n)/2}.
\tag{6.24}
\]

Next, we claim that $c\Omega_{t,\text{s}}^{\nu}$ in (6.10) satisfies the same estimate as above. The argument follows the same paragraphs in the end of section 3: By definition of $c\chi_{s}^{\nu}$ in (5.17), the integrant of (6.10) has support bounded at least in one coordinate of the subspace $\bigoplus_{j \in J} \mathbb{R}^{N_j}$. On its complement subspace, we can carry out the exactly same estimation as before. Since $\sigma(x, \xi)$ has compact support in $x$, integration in the extra dimensions within a compact support in the frequency space has no impact on the size of $c\Omega_{t,\text{s}}^{\nu}$ other than a multiple of constant. Therefore, the estimate in (6.24) is also valid for $c\Omega_{t,\text{s}}^{\nu}$.

Recall from the construction of $B_{\nu}^{s}$ in section 5, there are at most a constant multiple of $\prod_{i=1}^{2^{s_i(N_i-1)/2}}$ elements in the collection of $\nu$. We have
\[
\int |\Omega_{t}(x,y)| \, dx \leq \sum_{s} \sum_{\nu} \int \left(|\Omega_{t,s}^{\nu}(x,y)| + |c\Omega_{t,s}^{\nu}(x,y)|\right) \, dx
\leq \prod_{i=1}^{n} \left\{ \sum_{s_i \leq l_i} 2^{s_i-j}(N_i-1)/2 \right\}
\leq \prod_{i=1}^{n} 2^{l_i-j}(N_i-1)/2
\tag{6.25}
\]
uniformly in $y$.

2. Suppose $t_1 = t_2 = \cdots = t_n = j$. We have $|\xi^i| \sim 2^i$ for every $i = 1, 2, \ldots, n$, whenever $\xi \in \delta_t(\xi)$ by Lemma 4.1. In this case, we can assume $t = s$ since the support of $\delta_{t,s}(\xi)$ by definition is nonempty only if
\[
|t-s| = \sum_{i=1}^{n} |t_i - s_i| < \text{const}.
\]

Observe that a differentiation with respect to $y$ in (6.16) gives a factor bounded by $2^i$. By carrying out the same estimation in step 1, we obtain
\[
\int |\nabla_y \Omega_{j}(x,y)| \, dx \leq 2^i.
\tag{6.26}
\]
Therefore, for every $y, z \in \mathbb{R}^N$, we have
\[
\int |\Omega_{j}(x,y) - \Omega_{j}(x,z)| \, dx \leq 2^i|y - z|.
\tag{6.27}
\]
3. Suppose $t_1 = t_2 = \cdots = t_n = t$. Let $k$ be a positive integer such that $2^{k-1} \leq \delta \leq 2^{k+1}$. For $x \in \mathcal{C}B^*_\delta$, by definition in (5. 11)-(5. 12), there exists at least one $x^i$ for some $i \in I$, such that
\[ |\pi^i_0 \Phi^i_0 (x^i, \xi^i_0)| \geq \text{const} \, 2^{-k} \quad \text{and} \quad |\Phi^i_0 (x^i, \xi^i_0)| \geq \text{const} \, 2^{-k/2}. \quad (6. 28)\]

If $y \in B^*_\delta$, under appropriate linear transformations as above, we have
\[ 2^i \left| \left( \Phi^i_0 (x^i, \xi^i_0) - y \right) \right| + 2^{i/2} \left| \left( \Phi^i_0 (x^i, \xi^i_0) - y \right) \right|^* \geq 2^{i-k} \quad (6. 29)\]
for some $i \in I$, provided that $\text{const}$ is sufficiently large. By inserting estimate (6. 29) into (6. 20), and carrying out the same estimation in step 1, we obtain
\[ \int_{B^*_\delta} |\Omega_j (x, y)| \, dx \leq \frac{2^{-j}}{\delta}, \quad y \in B^*_\delta. \quad (6. 30)\]

In order to show that (6. 1), it is suffice to consider
\[ \sum_i \int |a(y)| \left\{ \int_{B^*_\delta} |\Omega_j (x, y)| \, dx \right\} \, dy. \quad (6. 31)\]

However, from (6. 5) in Lemma 6.1, the summation over all $t$ in (6. 31) will be dominated by
\[ \int |a(y)| \left\{ \sum_{j=1}^{\infty} \int_{B^*_\delta} |\Omega_j (x, y)| \, dx \right\} \, dy. \quad (6. 32)\]

For $2^i \leq \delta^{-1}$, we write
\[ \int a(y) \Omega_j (x, y) \, dy = \int a(y) \left( \Omega_j (x, y) - \Omega_j (x, z) \right) \, dy \quad (6. 33)\]
as a fact of $\int a(y) \, dy = 0$. By (6. 6) in Lemma 6.1, we have
\[ \int |a(y)| \left\{ \int |\Omega_j (x, y) - \Omega_j (x, 0)| \, dx \right\} \, dy \leq ||a||_{L^1} \, 2^j \delta \quad (6. 34)\]
provided that $a$ is supported on $B^*_\delta$. By summing over all such $j$ s, we have
\[ \left( \sum_{2^j \leq \delta^{-1}} 2^j \right) \delta \leq \text{const}. \quad (6. 35)\]

For $2^j > \delta^{-1}$, by (6. 7) in Lemma 6.1, we have
\[ \int |a(y)| \left\{ \int_{B^*_\delta} |\Omega_j (x, y)| \, dx \right\} \, dy \leq ||a||_{L^1} \frac{2^{-j}}{\delta} \quad (6. 36)\]
for $2^j > \delta^{-1}$. By summing over all such $j$ s, we have
\[ \left( \sum_{2^j > \delta^{-1}} 2^{-j} \right) \delta^{-1} \leq \text{const}. \quad (6. 37)\]
At this point, (6.1) is proved. Together with (5.10), we have \( \|F_a\|_{L^1} \leq \|a\|_{L^1} \) for \( a \in H^1 \).

Let \( F^* \) be the adjoint operator of \( F \). We have
\[
(F^* f)(x) = \int f(y) \Omega^*(x, y) dy
\]
and
\[
= \int f(y) \left( \int e^{2\pi i (\Phi(y, \xi) - \xi \cdot x)} \eta(y, \xi) d\xi \right) dy.
\]

Observe that its kernel \( \Omega^* \) has \( x \) and \( y \) reversed in the role of \( \Omega \) in (2.7). As discussed in 4.8, chapter IX of [3], we redefine the rectangles in (5.11) to be
\[
R^\nu_{s_i} = \left\{ x^j : |x^j - \Phi^j (0, s^\nu_{s_i})| \leq \text{const} \ 2^{-s_i/2}, \ |\pi^\nu_i (x^j - \Phi^j (0, s^\nu_{s_i}))| \leq \text{const} \ 2^{-s_i} \right\}, \quad i \in I
\]
where \( \pi^\nu_i \) is the orthogonal projection in the direction of \( s^\nu_{s_i} \). The region of influence \( B^\nu_{s_i} \) will be defined as same as (5.12), but in terms of \( R^\nu_{s_i} \) in (6.39). By carrying out the same estimation developed in this section on \( \Omega^* \), we shall have (6.1) also valid for \( F^* \). In the other hand, the estimation given by 3.1.4, chapter IX of [3] implies
\[
\frac{1}{q} = \frac{1}{2} + \frac{m}{N}.
\]

Let \( m = -(N - n)/2 \). By duality and (5.3), we have (5.10) also valid for \( F^* \). All together, we have \( \|F^* a\|_{L^1} \leq \|a\|_{L^1} \) for \( a \in H^1 \). The duality between \( H^1 \) and \( BMO \), as first classified in [12], then implies simultaneously that \( \|F f\|_{BMO} \leq \|f\|_{L^\infty} \) and \( \|F^* f\|_{BMO} \leq \|f\|_{L^\infty} \) for \( f \in L^2 \cap L^\infty \).

\section{L^p-Estimate}

We now start to conclude our \( L^p \)-estimate, by applying the complex interpolation theorem set out in 5.2, chapter IV of [3]. Consider the analytic family of operators \( F_s \) in the strip \( 0 < \text{Re}(s) \leq 1 \), by
\[
F_s(x) = e^{(s - \vartheta)^2} \int e^{2\pi i \Phi(x, \xi)} \sigma(x, \xi) \left( 1 + |\xi|^2 \right)^{\nu(\xi)} \hat{f}(\xi) d\xi
\]
where
\[
\vartheta(s) = -m - \frac{s(N - n)}{2}, \quad \vartheta = -\frac{2m}{N - n}.
\]

Let \( s = u + it \). There are only finitely many derivatives of the symbol involved in our previous estimation. Therefore, the corresponding derivatives have at most a polynomial growth in \( t \), whereas the factor \( e^{(s - \vartheta)^2} \) decays rapidly as \( |t| \to \infty \).

When \( \text{Re}(s) = 0 \), \( \sigma(x, \xi) \left( 1 + |\xi|^2 \right)^{\nu(\xi)} \) has an order of zero. As a result of section 3, we have
\[
\|F_{it} f\|_{L^2} \leq A_p \|f\|_{L^2}, \quad -\infty < t < \infty.
\]

When \( \text{Re}(s) = 1 \), \( \sigma(x, \xi) \left( 1 + |\xi|^2 \right)^{\nu(\xi)} \) has an order of \( -(N - n)/2 \). As a result of section 6, we have
\[
\|F_{1+it} f\|_{BMO} \leq \|f\|_{L^\infty}, \quad -\infty < t < \infty, \quad f \in L^2 \cap L^\infty.
\]
By the desired complex interpolation, we obtain
\[ \| F_\delta f \|_{L^p} \leq A_{\delta, \rho} \| f \|_{L^p} \] (7.5)
where \( \delta = 1 - 2/p > 0 \). Observe that
\[ F_\delta = F \] (7.6)
for which \( 1/2 - 1/p = -m/(N - n) \).

In the other hand, \( F^* \) is bounded on \( L^2 \) since \( S^* \) is bounded on \( L^2 \) as proved in section 3. Together with \( \| F^* f \|_{\text{BMO}} \leq \| f \|_{L^\infty} \) at \( m = -(N - n)/2 \), We have \( F^* \) satisfying the same estimates as above. By duality, \( F \) is bounded on \( L^p \left( \mathbb{R}^N \right) \), whenever
\[ \left| \frac{1}{2} - \frac{1}{p} \right| \leq \frac{-m}{N - n} \] (7.7)

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