GHZ paradoxes based on an even number of qubits

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Abstract. GHZ paradoxes are presented for all even numbers of qubits from four up. They are obtained from proofs of the Kochen-Specker (KS) theorem by showing how the assumption of noncontextuality can be justified on the basis of locality. A number of different paradoxes are presented and the nature of the entangled states involved in them is discussed. Some multiqubit proofs of the KS theorem are also presented in the form of easily understood diagrams. The implications of our results are discussed.

After Greenberger, Horne and Zeilinger [1] gave their proof of Bell’s theorem without inequalities [2], there have been many refinements and extensions of their results [3,4,5,6,7,8,9,10,11,12]. Several authors have proposed GHZ paradoxes (as we will term proofs of Bell’s theorem without inequalities or probabilities) based on qudits (i.e., d-dimensional quantum systems): Ref. [7] constructs paradoxes based on a maximally entangled state of d qudits shared among d observers; Ref. [8] presents a variety of paradoxes based on an arbitrary number of qudits and also clarifies what it means for a paradox to be genuinely multipartite and multidimensional; and Ref. [10] discusses paradoxes based on graph states of qudits. Despite this progress, it is surprising that GHZ paradoxes have not been formulated for an arbitrary even number of qubits (by contrast, Mermin [3] presented a proof for three qubits and Di-Vincenzo and Peres [6] one for five, from which a generalization to all odd numbers is easily achieved [12,10]). It is the purpose of this Letter to fill this gap by presenting GHZ paradoxes for all even numbers of qubits from four up [13]. Our paradoxes are interesting because they involve entangled states that are distinct from GHZ, W, cluster or graph states. The state $\Psi_4$ involved in our four-qubit paradox has been used earlier by Yeo and Chua [15] to discuss teleportation and dense coding, but the fact that it can be used to demonstrate...
a GHZ paradox does not seem to have been noticed. It is also interesting that
Ψ₄ is the lowest member of an infinite class of states that gives rise to GHZ
paradoxes and whose members can be expressed as sums of products of Bell
states. The second purpose of this Letter is to draw attention to a large num-
ber of multiqubit proofs of the Kochen-Specker (KS) theorem based on the
observables of the Pauli group. We will discuss the implications of our results
after we have first presented them.

Table 1

| Z | Z | Z | Z | Z | ... | Z |
|---|---|---|---|---|-----|---|
| X | X | Z | Z | Z | ... | Z |
| Z | X | X | I | I | I | ... | I |
| Z | Z | I | X | I | I | ... | I |
| I | I | I | X | X | ... | I |
| I | I | I | ... | I | X | X |
| I | I | X | I | ... | I | X |

Table 1

Left: Five mutually commuting observables of a four-qubit system, arranged hori-
zontally, with their corresponding qubits vertically aligned. Right: 2ᴺ + 1 mutually com-
muting observables of a 2ᴺ-qubit system, for N ≥ 3, in the same format as at left.

Table 1, left, shows five mutually commuting observables of a four-qubit system
arranged in the form of a 5 × 4 array, with the rows representing
the observables and the columns their corresponding qubits (X, Y, Z and I
are the Pauli and identity operators of the qubits). A noncontextual hidden
variables theory (NHVT) is required to assign the value +1 or −1 to
each of the four-qubit observables, as well as all the single qubit observ-
ables of which they are made up, in such a way that all the operator rela-
tions satisfied by the observables are mirrored in algebraic relations satisfied
by their values. This requires that the value assigned to any four-qubit ob-
serve be equal to the product of the values assigned to its single-qubit con-
stituents (for example, v(ZXXI) = v(Z₁)v(X₂)v(X₃), where subscripts
have been used on the right to indicate the qubits involved) and also that
v(ZZZZ)v(ZXXI)v(XZZZ)v(XZIX)v(IIXX) = −1 (since the product of
these observables is the negative of the identity operator). However, if one
reexpresses the left side of the last equation in terms of the values of the
single-qubit observables, one finds that each value occurs twice, implying that
the left side is +1. But this leads to the contradictory equation +1 = −1,
which shows that NHVTs are untenable and proves the KS theorem.

We now show how the above KS proof can be converted into a GHZ para-
dox. Let Ψ₄ be the simultaneous eigenstate of the four-qubit observables of
Table 1, left, with eigenvalue +1 for the first four and −1 for the last. Suppose
a source repeatedly emits four qubits in the state Ψ₄, sending one qubit
to each of the observers A, B, C and D who are at spacelike separations from
one other. Suppose each observer randomly measures $X$, $Z$ or $I$ on his/her
qubit in each of the runs ($I$ simply means they do nothing) and that they get
together later to analyze their results, keeping only the runs in which their
measurements correspond to the single-qubit observables in one of the rows
of Table 1, left. We now show that all the single qubit observables in Table
1, left, are “elements of reality”, i.e., they have values that can be determined
without disturbing the qubits in any way. This is so because, in any run, the
value of any one of the measured observables can be deduced from the observed
values of all the others if one uses the fact that the product of all the values is
fixed (and equal to the eigenvalue of the appropriate four-qubit observable).
Locality also guarantees that this value is independent of which alternative
sets of measurements are carried out on the other qubits, or that the value
is noncontextual. The earlier KS argument then shows that it is impossible
to assign values to all the single qubit observables in such a way that all the
four-qubit observables have their required (eigen)values, and one gets a full
fledged GHZ paradox.

Table 1, right, shows a generalization of the above four-qubit proof to one
for $2N$ qubits, for $N \geq 3$. As before, the KS proof can be converted into a GHZ
paradox by making use of any joint eigenstate of the $2N$-qubit observables. It
should be stressed that all members of this chain of KS/GHZ proofs are gen-
true multipartite proofs [8] because the elimination of any columns (qubits)
and/or rows (multiqubit observables) from Table 1, right, does not leave a valid proof. We have verified this explicitly by means of a computer program
for all proofs of up to ten qubits and are confident that it holds generally, but
do not have an analytic demonstration of it.

We now discuss the structure of the joint eigenstates that occur in our
paradoxes. The state $\Psi_4$ can be expressed in terms of the Bell states $\Phi^\pm, \Psi^\pm$ as $\Psi_4 = \Phi^+_{12} \Phi^+_{34} - \Psi^+_{12} \Phi^+_{34}$, where the subscripts on the Bell states label the
qubits (which are numbered in ascending order from left to right in all our observables). State $\Psi_4$ has the property that a measurement on any two qubits in
the computational basis leaves the other two qubits in a Bell state. Briegel and
Rauusendorf [14] discussed a state with this property and Yeo and Chua [15]
used essentially the state $\Psi_4$ to discuss teleportation and dense coding in two-
qubit systems. The joint eigenstate of our six-qubit paradox, with eigenvalue
$-1$ for the last observable, is

$$
\begin{align*}
\Psi_6 &= \Phi^+_{12} (\Phi^+_{34} \Phi^+_{56} + \Psi^+_{34} \Psi^+_{56}) - \Psi^+_{12} (\Phi^+_{34} \Psi^+_{56} + \Psi^+_{34} \Phi^+_{56}) \\
&= (00)_{13} (\Phi^+_{24} \Phi^+_{56} + \Psi^+_{24} \Psi^+_{56}) - (01)_{13} (\Phi^+_{24} \Phi^+_{56} + \Psi^+_{24} \Psi^+_{56}) \\
&+ (10)_{13} (\Psi^+_{24} \Phi^+_{56} + \Phi^+_{24} \Psi^+_{56}) - (11)_{13} (\Phi^+_{24} \Phi^+_{56} + \Psi^+_{24} \Psi^+_{56})
\end{align*}
$$

The above decompositions, and other similar ones, show that a measurement
on any two qubits leaves the other four in a state similar to $\Psi_4$. State $\Psi_6$ is
closely related to the six-qubit state of Borras et al. [16] that has been used to discuss teleportation and state sharing in two- and three-qubit systems [17].

The eigenstate of our eight-qubit paradox can be expressed as a sum of eight terms, each of which is a product of four Bell states, and has the property that a measurement on any two qubits leaves the other six in a state similar to $\Psi_6$. We expect this hierarchical feature to be true of all the higher eigenstates.

We next present several multiqubit proofs of the KS theorem, which are similar to (but more involved than) the two- and three-qubit proofs that have been presented earlier [3,28]. Figure 1 shows two different proofs based on four qubits. The observables are placed within circles and mutually commuting sets of observables are joined by lines, with a thin or a thick line being used for a set whose product is $+I$ or $-I$ ($I$ is the identity operator in the space of all the qubits, and all commuting sets are of one of these two types). It is easy to verify that the diagrams prove the KS theorem by noting that they have an odd number of thick lines and that each observable lies on an even number of lines. However these KS proofs cannot be converted into GHZ paradoxes because each involves more than one set of commuting multiqubit observables, and using an eigenstate of one of the sets to set up a paradox does not allow the observables of the others to be established as elements of reality. The “kite” and “wheel” diagrams of Fig.1 are each members of an infinite chain that stretches upwards for all numbers of qubits. The higher members of the kite family involve more observables strung out along the tail of the kite. The wheel diagram for five qubits has a simple structure, with the commuting observables lying on three concentric circles and five “spokes” radiating outward from the center. The wheel diagram for six qubits also consists of circles and
Fig. 1 The four-qubit “kite” (left) and the four-qubit “wheel” (right).

Our observables-based KS proofs (or KS systems, for short) can be used to obtain “parity proofs” of the KS theorem based on sets of projectors and bases. A “parity proof” (in a state space of dimension $\geq 4$) consists of a set of projectors, of possibly different ranks, that forms an odd number of bases (defined as a set of projectors that sums to the identity) in such a way that each projector occurs in an even number of bases. Such a configuration proves the KS theorem because it is impossible for a NHVT to assign the value 0 or 1 to each of the projectors in such a way that the sum of the values in every basis is unity. The first parity proof, discovered by Cabello et al [18], involved 18 rank-1 projectors and 9 bases in a state space of four dimensions. Since then thousands of other parity proofs have been discovered in four and eight dimensions [19,20,28]. We show here how a large number of parity proofs can be obtained in $2^N$ dimensions, for $N \geq 4$.

To obtain all the parity proofs associated with a KS system, one begins by constructing the simultaneous eigenstates of all the sets of commuting observables in it. Instead of the eigenstates we will talk of the associated projectors, which are rank-1 for a commuting set that is complete and of higher rank for one that is not. The projectors form two types of bases that we will term “pure” and “hybrid”. The pure bases arise from the commuting sets of observables whereas the hybrid ones consist of mixtures of projectors from different pure bases. We will term the union of the pure and hybrid bases the “basis table” of the KS system. A remarkable feature of all the KS systems we have studied is that their basis tables are “saturated”, i.e., they contain all the...
orthogonalities of the projectors in the system. The basis table is thus completely equivalent to the “Kochen-Specker diagram” of the system (a graph in which the projectors are represented as vertices and edges join vertices that correspond to orthogonal projectors). Deletion of sets of bases from a basis table leads to all the parity proofs associated with a KS system.

The parity proofs associated with our KS diagrams have many features of interest, of which we will mention only a few. Consider the kite diagram of Fig.1. The most compact parity proof contained in it involves 24 projectors and 9 bases, which we can denote by the symbol 24-9. However we can also use the more informative symbol $12^2 4^2 - 4^4 4^1 1_8$, in which the first half shows the number of projectors of each rank (with the rank indicated as a superscript and the number of occurrences of the projector among the bases as a subscript) and the second half shows the number of bases of each size (with the size indicated as a subscript). The kite diagram actually has 33 different types of parity proofs in it, involving all odd numbers of bases from 9 to 17 [27]. Each of these proofs has a large number of replicas under symmetry, causing the total number of distinct proofs in this system to be 33152. A similar story can be told about our other diagrams. We should add that the only parity proofs we ever consider are ones that are critical, i.e., ones that fail if even a single basis is dropped from them. An interesting numerical curiosity of many (but not all) of the KS systems we have looked at is that they contain a total of $2^H$ distinct proofs (with replicas included), where $H$ is half the number of hybrid bases associated with the system. We have found systems with $H = 9, 10$ [28], 12 (Table 1, left), 20 (Fig.1, right), 15 [19] and 11, 13, 16, 17 [27]. An interesting feature of some KS systems we have looked at is that they allow very compact parity proofs to be constructed in spaces of high dimension. For example, we have found a proof in 128 dimensions (the dimension of a seven-qubit system) involving 36 projectors and 9 bases and having the detailed symbol $24^8 12^4 - 4^4 4^2 1_16$ [27].

Our results demonstrate that KS proofs, whether in the observables form advocated by Mermin [3] or the parity proof form advocated by Peres and others [18,19,20], are not rare occurrences but can be assembled in the thousands out of the observables of the Pauli group. A subset of these proofs (namely, those consisting of only a single set of commuting multiqubit observables) can also be converted into GHZ paradoxes. This rich store of proofs is worth exploring further for its foundational and practical interest. On the foundational side, it provides new GHZ paradoxes and KS proofs based on small numbers of qubits, which are still the most accessible systems. The experimental realization of these proofs is not trivial because they call for measurements of sets of commuting multiqubit observables in a single context, but this task is not beyond the scope of current or foreseeable technology. Noise and imperfections will prevent the realization of the ideal scenarios we have discussed here, but our “all or nothing” proofs can be converted into Bell inequalities that can be tested experimentally. Our parity proofs are of interest because, following
Cabello [21], they can be turned into inequalities that can be used to rule out noncontextuality, and they could also find application in quantum key distribution [22], quantum error correction [23,24], random number generation [25] and parity oblivious transfer [26].
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