On transitivity dynamics of topological semiflows

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Abstract
Let \( T \times X \rightarrow X, (t,x) \mapsto tx \), be a topological semiflow on a topological space \( X \) with phase semigroup \( T \). We introduce and discuss in this paper various transitivity dynamics of \((T, X)\).

Keywords: Semiflow; transitivity; sensitivity; thick stability

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1. Introduction
Let \( T \) be a topological semigroup with a neutral element \( e \) and \( X \) a non-singleton topological space. We consider a semiflow \( T \times X \rightarrow X, (t,x) \mapsto tx \), denoted by \((T, X)\), which satisfies conditions:

1. \((t, x) \mapsto tx\) is jointly continuous;
2. \(ex = x \forall x \in X\); and
3. \(t(sx) = (ts)x \forall s, t \in T, x \in X\).

Here \( T \) is called the phase semigroup and \( X \) the phase space. Moreover,

a). when \( T \) is a group, we will call \((T, X)\) a flow with phase group \( T \);

b). if each \( t \in T \) is surjective, then \((T, X)\) will be said to be surjective;

c). if each \( t \in T \) is bijective, then \((T, X)\) will be called invertible and \( \langle T \rangle \) stands for the smallest group of self-homeomorphisms of \( X \) containing \( T \).

Clearly a flow is invertible and an invertible semiflow is certainly surjective. Let \( TA = \bigcup_{t \in T} tA \) for all \( A \subseteq X \). If \((T, X)\) is surjective, then \( TX = X \); however, the converse need not be true. For example, let \( X = \{a, b\} \) a discrete space, \( f: a \mapsto b \mapsto b \) and \( g: b \mapsto a \mapsto a \); and let \( T = \langle f, g \rangle \), be the semigroup generated by \( f \) and \( g \). Then \( TX = X \) but \( f(X) = \{b\} \neq X \).

Definition 1.1. \((T, X)\) is called topologically transitive (TT) if for all non-empty open sets \( U, V \) in \( X \), the set

\[ N_T(U, V) = \{t \in T \mid U \cap t^{-1}V \neq \emptyset\} \]

is non-empty. Clearly, \( N_T(U, V) = \{t \in T \mid tU \cap V \neq \emptyset\} \). See Theorem 2.1 for two equivalent descriptions of TT.
Definition 1.2. \((T, X)\) is called point-transitive (PT) if the set

\[
\text{Tran}(T, X) = \{ x \in X | T x \text{ is dense in } X \}
\]

is non-empty. Every point of Tran\((T, X)\) is a transitive point of \((T, X)\).

Both of TT and PT are the most important and basic transitivity of semiflows. If the phase space \(X\) is not separable, then TT \(\Rightarrow\) PT in general even for \(T = \mathbb{Z}\) (cf. [7, Example 4.17]). However, if \(X\) is a Polish space, then \((T, X)\) is TT iff \(\text{Tran}(T, X)\) is a dense \(G_\delta\) set of \(X\) (cf. [18] and [5, Basic fact 1]).

Let \((T, X)\) be a flow with phase group \(T\). Then by \(\text{cls}_X Tsx = \text{cls}_X T x\) for all \(s \in T\) and \(x \in X\), \(\text{Tran}(T, X)\) is invariant so if \((T, X)\) is PT, \(\text{Tran}(T, X)\) is dense in \(X\) and \((T, X)\) is thus TT.

However, PT \(\Rightarrow\) TT in general if \((T, X)\) is not a flow even for \(X\) is a compact metric space. Let us consider a simple counterexample as follows.

Example 1.3. Let \(T = (\mathbb{Q}_+, +)\) the nonnegative rational numbers semigroup and \(X = \mathbb{R}_+ \cup \{ +\infty \}\) with the usual one-point compactification topology. Define

\[
T \times X \to X, \quad (t, x) \mapsto t + x.
\]

Clearly, \(\text{Tran}(T, X) = \{0\}\) so \((T, X)\) is PT. But \((T, X)\) is not TT, for \(\text{Tran}(T, X)\) is not a dense \(G_\delta\)-subset of \(X\). In fact, for open subsets \(U = (10, 12)\) and \(V = (5, 6)\) of \(X\), we have \(N_T(U, V) = \emptyset\).

As mentioned before, if \((T, X)\) is PT with \(\text{Tran}(T, X)\) dense in \(X\) then \((T, X)\) is TT. But this sufficient condition is not a necessary one for TT with non-Polish phase space. In this note, we will consider the following question:

**When does \(\text{Tran}(T, X) \not= \emptyset\) imply TT for a semiflow \((T, X)\) with \(T\) not a group?**

We will also consider other kinds of important transitivity dynamics and their relationships with TT and PT in this paper.

In particular, in §4 we shall consider “universally transitive” semiflow with compact Hausdorff phase space and show that a universally transitive PT semiflow is minimal distal (cf. Definition 4.2.2 and Theorem 4.16).

In §6 we shall consider “syndetically transitive” semiflow and show that a non-minimal syndetically transitive semiflow is “syndetically sensitive” (cf. Definition 6.1 and Theorem 6.12). This will generalizes some results of Glasner and Weiss on \(E\)-systems.

2. Prolongation and nonwandering points

Let \((T, X)\) be a semiflow on a topological space \(X\) and \(A \subseteq X\). We say \(A\) is invariant if \(T x \subseteq A\) for all \(x \in A\); i.e., \(T A \subseteq A\); and \(A\) is called negatively- or \(T^{-1}\)-invariant if \(T^{-1}x \subseteq A\) for all \(x \in A\) and \(t \in T\); i.e., \(T^{-1}A \subseteq A\). By \(\text{Int}_T A\) we will denote the interior of \(A\) relative to \(X\).

In this section we will consider basic properties of TT. We can easily verify the following two elementary sufficient and necessary conditions for TT of semiflow.

**Theorem 2.1.** Let \((T, X)\) be a semiflow on a topological space \(X\) with phase semigroup \(T\). Then the following conditions are pairwise equivalent.

1. \((T, X)\) is TT.
2. \((T, X)\) is syndetically transitive.
3. \((T, X)\) is point-transitive.

(2) Every invariant set with non-empty interior is dense in $X$.

(3) Every non-empty, open, and $T^{-1}$-invariant set is dense in $X$.

**Proof.** Condition (1) $\Rightarrow$ (2). To be contrary assume there is an invariant set $X_0$ with $\text{Int}_X X_0 \neq \emptyset$ such that $\text{cls}_X X_0 \neq X$. Let $U = \text{Int}_X X_0$ and set $V = X \setminus \text{cls}_X X_0$; then $U \neq \emptyset$ and $V \neq \emptyset$ such that $N_T(U, V) = \emptyset$, a contradiction to TT.

Condition (2) $\Rightarrow$ (1). Let $U, V$ be two non-empty open subsets of $X$. Since $e \in T$ so that $U \subseteq TU$, $TU$ is an invariant set with non-empty interior in $X$. Hence $TU \cap V \neq \emptyset$ so that $tU \cap V \neq \emptyset$ for some $t \in T$. This implies that $U \cap T^{-1}V \neq \emptyset$ for some $t \in T$. Thus $(T, X)$ is TT.

Condition (3) $\Rightarrow$ (1). Given non-empty open sets $U, V$ in $X$, since $T^{-1}V$ is $T^{-1}$-invariant open non-empty, $T^{-1}V$ is dense in $X$ so $U \cap T^{-1}V \neq \emptyset$ for some $t \in T$. Thus $(T, X)$ is TT.

Condition (1) $\Rightarrow$ (3). Let $U$ be a non-empty open $T^{-1}$-invariant subset of $X$. If $V = X - \text{cls}_X U$ were not empty, then $N_T(V, U) = \emptyset$ and this contradicts TT. Thus $U$ must be dense in $X$.

This proves Theorem 2.1. $\square$

So, if $X$ contains a non-dense open $T$- or $T^{-1}$-invariant non-empty set, then $(T, X)$ is not TT. Moreover, if $(T, X)$ is TT with $\text{Int}_X Tx \neq \emptyset$, then $x \in \text{Tran}(T, X)$.

In (2) $\Rightarrow$ (1) of Theorem 2.1, the neutral element $e$ plays a role. By Theorem 2.1 we can obtain the following well-known sufficient and necessary condition.

**Corollary 2.2.** Let $(T, X)$ be a flow; then $(T, X)$ is TT if and only if every invariant open non-empty subset is dense in $X$.

The following notion is a kind of generalized orbit closure of a semiflow $(T, X)$ on a topological space $X$.

**Definition 2.3.** Let $(T, X)$ be a semiflow with $T$ a locally compact non-compact topological semigroup. Let $\mathcal{N}_e$ be the collection of compact neighborhoods of $e$ in $T$ and $x \in X$.

1. We say $y \in X$ is in the prolongation of $x$, denoted by $y \in \Omega_T(x)$, if for all neighborhoods $U$ of $x$ and $V$ of $y$ in $X$ and all $K \in \mathcal{N}_e$ in $T$, one can find some $x' \in U$ and $t \in T \setminus K$ such that $tx' \in V$.

2. If $x \in \Omega_T(x)$, write $x \in \Omega_T(T, X)$, then $x$ is called a nonwandering point of $(T, X)$. If $\Omega(T, X) = X$, then $(T, X)$ itself is called a nonwandering semiflow.

Clearly,

- Both $\Omega_T(x)$ and $\Omega_T(T, X)$ are closed subsets of $X$; moreover, they are invariant whenever $T$ is a group or an abelian semigroup.

If $x$ is a nonwandering point of $(T, X)$, then for all neighborhood $U$ of $x$, there is some $t \in T$ outside any compact subset of $T$ such that $U \cap tU \neq \emptyset$.

**Lemma 2.4.** Let $(T, X)$ be a semiflow with $T$ a locally compact non-compact semigroup and with $X$ a locally compact connected Hausdorff space. Then $(T, X)$ is TT if and only if the prolongation of every point of $X$ is $X$, i.e., $\Omega_T(x) = X \forall x \in X$.

**Proof.** Let $(T, X)$ be TT and $x \in X$. Let $y \in X$ and let $U_x$ and $V_y$ be neighborhoods of $x$ and $y$ in $X$, respectively. Let $K$ be a compact neighborhood of $e$ in $T$.

Since $X$ is locally compact, we can assume $U_x$ is compact. Since $X$ is connected, we may suppose $V_y$ is open but not closed. Then as $KU_y$ is compact, it follows that $V_y \setminus KU_y \neq \emptyset$ so by
TT we can conclude that \( N_T(U, V) \cap (T \setminus K) \neq \emptyset \). Thus, there are \( t \in T \setminus K \) and \( x' \in U_s \) with \( tx' \in V_r \). So \( y \in \Omega_T(x) \).

Conversely, assume \( \Omega_T(x) = X \) for all \( x \in X \) and let \( U, V \) be two non-empty open subsets of \( X \). Take some \( x \in U \) and then by \( \Omega_T(x) \cap V \neq \emptyset \) it follows that \( N_T(U, V) \neq \emptyset \).

Thus, under the same situation of Lemma 2.4, if \((T, X)\) is TT, then every point of \( X \) is nonwandering. In particular, we can obtain the following.

**Corollary 2.5.** Let \((T, X)\) be a semiflow with \( T \) a locally compact non-compact semigroup and with \( X \) a topological manifold. If \((T, X)\) is TT, then \((T, X)\) is nonwandering.

In view of the question we are concerned with, the following corollary of Lemma 2.4 is somewhat of interest.

**Corollary 2.6.** Let \((T, X)\) be PT with \( T \) a locally compact non-compact semigroup and with \( X \) a locally compact connected Hausdorff space. Then \((T, X)\) is TT iff \( \Omega_T(x) \cap \text{Tran}(T, X) \neq \emptyset \) for all \( x \in X \).

**Proof.** The necessity follows at once from Lemma 2.4. To prove the sufficiency, let \( D(x) \) be the invariant closed subset of \((T, X)\) defined as follows: \( y \in D(x) \) iff there are nets \( \{x_n\} \) in \( X \) and \( \{t_n\} \) in \( T \) with \( x_n \rightarrow x \) and \( t_nx_n \rightarrow y \). Then \( \Omega_T(x) \subseteq D(x) \) and so \( D(x) \cap \text{Tran}(T, X) \neq \emptyset \) for all \( x \in X \). Thus \( D(x) = X \) for all \( x \in X \). Let \( U, V \) be two non-empty open subsets of \( X \). Take some \( x \in U \) and then by \( D(x) \cap V \neq \emptyset \) it follows that \( N_T(U, V) \neq \emptyset \).

**Remark 2.7.** The prolongation \( \Omega_T(x) \) relies on the topology of \( T \). For example, let \((t, x) \mapsto tx \) be a classical \( C^0 \)-flow on a manifold \( X \) with phase group \( \mathbb{R} \). If \( \mathbb{R} \) is under the discrete topology, then every point of \( X \) is nonwandering so that this flow is nonwandering. Of course, there are nonwandering classical \( C^0 \)-flows if \( \mathbb{R} \) is under the usual topology.

3. Pre-recurrent transitive points of semiflows

Let \((T, X)\) be a semiflow on a topological space \( X \) with phase semigroup \( T \). We first introduce a notion which only works for semiflows with \( T \) not groups. By \( \text{cls}_X A \) we will denote the closure of a set \( A \) in \( X \).

**Definition 3.1.** A point \( x \in X \) is called **pre-recurrent** for \((T, X)\) if \( x \in \text{cls}_X Tx \) for every \( s \in T \). We could also define **pointwise pre-recurrent**.

**Definition 3.2.** An \( x \in X \) is called a **minimal point** of \((T, X)\) if \( \text{cls}_X Tx \) is a minimal subset of \((T, X)\); that is, \( \text{cls}_X Ty = \text{cls}_X Tx \) for all \( y \in \text{cls}_X Tx \).

The point \( x = 0 \) in Example 1.3 is transitive but not pre-recurrent. By definitions the following lemma is evident and so we will omit its proof here.

**Lemma 3.3.** Each minimal point is pre-recurrent for \((T, X)\).

**Definition 3.4.**

1. A subset \( A \) of \( T \) is called **thick** if for all compact subset \( K \) of \( T \), one can find an element \( t \in T \) such that \( Kt \subseteq A \).
2. A subset \( S \) of \( T \) is called **syndetic** if there is a compact subset \( K \) of \( T \) such that \( Kt \cap S \neq \emptyset \) for all \( t \in T \).
A subset of $T$ is syndetic iff it intersects non-voidly every thick set of $T$ (cf., e.g., [3, Lemma 2.5]).

3. A point $x$ is called almost periodic for $(T, X)$ if for all neighborhood $U$ of $x$, the set

$$N_T(x, U) = \{t \in T \mid tx \in U\}$$

is syndetic in $T$.

It should be noted that since here $X$ is not necessarily a regular space, an almost periodic point need not be a minimal point. Conversely, since $X$ is not necessarily compact, a minimal point need not be an almost periodic point.

**Lemma 3.5.** Every almost periodic point is pre-recurrent for $(T, X)$.

**Proof.** Let $x$ be an almost periodic point of $(T, X)$ and $U$ an arbitrary neighborhood of $x$; let $s \in T$. Since $N_T(x, U)$ is syndetic, there is some $k \in T$ with $ks \in N_T(x, U)$ so $U \cap Tsx \neq \emptyset$. Since $U$ is arbitrary, this implies $x \in \text{cls}_X Tsx$.

Next, based on pre-recurrent point we can easily obtain the following two simple criteria for TT of semiflows.

**Proposition 3.6.** If there is an $x \in \text{Tran}(T, X)$ such that $x$ is pre-recurrent, then $(T, X)$ is TT.

**Proof.** Let $x \in \text{Tran}(T, X)$ be a pre-recurrent; then for all $s \in T$, $x \in \text{cls}_X Tsx$ implies that $X = \text{cls}_X Tsx$ and so $sx \in \text{Trans}(T, X)$. Thus Tran $(T, X)$ is dense in $X$ and so $(T, X)$ is TT.

**Proposition 3.7.** Suppose $(T, X)$ is PT. Then $(T, X)$ is TT if and only if whenever (even for some) $x \in \text{Tran}(T, X)$ and for all non-empty open sets $U, V$ in $X$, there are $s, t$ in $T$ such that $sx \in U$ and $tx \in V$.

**Proof.** The sufficiency is obvious. Now for the necessity, let $x$ be a transitive point and $U, V$ non-empty open sets. By TT, there is some $t \in T$ and a non-empty open set $W \subseteq U$ with $tW \subseteq V$. Finally by PT, there is an $s \in T$ such that $sx \in W$ so $tsx \in V$.

In view of Example 1.3, the pre-recurrence in Proposition 3.6 is important. More general than Proposition 3.6, we can obtain the following result.

**Theorem 3.8.** Let $(T, X)$ be PT such that there exists an $x \in \text{Tran}(T, X)$ which is a limit of pre-recurrent points. Then $(T, X)$ is TT.

**Proof.** Let $x \in \text{Tran}(T, X)$ which is a limit of pre-recurrent points, and let $U, V$ be non-empty open subsets of $X$. Since $x$ is a transitive point, there exist a neighborhood $W$ of $x$ and $s, t \in T$ such that $sW \subseteq V$ and $tx \in U$. Hence there exists a pre-recurrent point $z \in W$ such that $tz \in U$. This implies that there is some $\tau \in T$ such that $\tau tz \in W$ so that $\tau tzU$ intersects $V$ non-voidly. Thus $(T, X)$ is TT by Definition 1.1. This proves Theorem 3.8.

**Corollary 3.9.** Let $(T, X)$ be a PT semiflow. If minimal points or almost periodic points are dense in $X$, then $(T, X)$ is TT.

**Proof.** By Lemma 3.3 and Lemma 3.5, the pre-recurrent points are dense in $X$. Then Corollary 3.9 follows from Theorem 3.8.
4. PT and universally transitive semiflows

It has already been a well-known fact that

If \( f : Y \to Y \) is a continuous surjective transformation of a compact metric space \( X \), then \( PT \Rightarrow TT \) for \( (f, Y) \) (cf. [17, Theorem 2.2.2]).

In fact, we can obtain the following generalization to this result, in which the surjective condition is important according to Example 1.3.

**Theorem 4.1.** Let \( (T, X) \) be PT and surjective with \( T \) an abelian semigroup. Then \( Tran (T, X) \) is invariant and thus \( (T, X) \) is TT.

**Proof.** Given \( x_0 \in Trans (T, X) \) and \( s \in T \), since \( cls_T s x_0 = cls_T x_0 \supseteq scls_T x_0 = X \), thus we have \( sx_0 \in Trans (T, X) \). This proves Theorem 4.1.

**Definition 4.2.** Let \( (T, X) \) be a semiflow on a topological space \( X \) with phase semigroup \( T \).

1. Let \( Aut (T, X) \) be the automorphism group of \( (T, X) \); i.e., \( Aut (T, X) \) is the group of all self-homeomorphisms \( a \) of \( X \) such that \( at = ta \) for all \( t \in T \).
2. If \( Aut (T, X) x = X \) for some \( x \in X \) (so for all \( x \in X \)), then \( (T, X) \) is called universally transitive (UT) (or algebraically transitive in [15]).

**Note.** If \( (T, X) \) is UT, then \( Aut (T, X) \) is referred to as transitive on \( X \) (cf. [2, Theorem 2.13]).

Using UT condition instead of the one that each \( t \in T \) is surjective, we can obtain the following corollary.

**Corollary 4.3.** If \( (T, X) \) is PT and UT with \( T \) an abelian semigroup, then \( Tran (T, X) \) is invariant and thus \( (T, X) \) is TT.

**Proof.** By Theorem 4.1, it is sufficient to show \( tX = X \) for all \( t \in T \). In fact, since \( atX = taX = tX \) for all \( t \in T \) and \( a \in Aut (T, X) \), hence \( atx \in tX \) for all \( a \in Aut (T, X) \). So by UT, \( tX = X \).

We notice here that TT + PT \( \Rightarrow \) UT even for flows with compact metric phase spaces. Let’s see such a simple example as follows.

**Example 4.4.** Let \( X = \mathbb{R} \cup \{\infty\} \) be the one-point compactification of \( \mathbb{R} \) with the usual topology and let \( T = (\mathbb{R}, +) \); define \( \pi : T \times X \to X \), \((t, x) \mapsto t + x\), which is of course a flow. Then \( (T, X) \) is TT and PT with \( Tran (T, X) = \mathbb{R} \) but \( Aut (T, X) = \{\infty\} \neq X \). Thus \( (T, X) \) is not UT.

Under UT condition, if our phase space \( X \) is compact Hausdorff or compact metric, then we can gain more. First, let’s recall a classical theorem of Gottschalk.

**Gottschalk’s theorem** (cf. [15, Theorem 7]). Let \( (T, X) \) be a PT flow on a compact metric space with \( T \) an abelian group. Then \( (T, X) \) is UT iff \( (T, X) \) is equicontinuous.

Next we will generalize Gottschalk’s theorem for semiflows. For this, we need to introduce some concepts and lemmas for self-closeness of this note.

**Definition 4.5.** Let \( (T, X) \) be a semiflow on a compact Hausdorff space \( X \) with the compatible symmetric uniform structure \( \mathbb{U}_X \).
1. We say \((T, X)\) is distal if given \(x, y \in X\) with \(x \neq y\), there is an \(\alpha \in \mathcal{U}_X\) such that \((tx, ty) \notin \alpha\) for all \(t \in T\). Thus if \((T, X)\) is distal, then for two different initial points \(x, y \in X\), their orbits \(Tx\) and \(Ty\) are synchronously far away.

2. \((T, X)\) is called equicontinuous in case given \(\varepsilon \in \mathcal{U}_X\), there is some \(\delta \in \mathcal{U}_X\) such that if \((x, y) \in \delta\) then \((tx, ty) \in \varepsilon\) for all \(t \in T\).

3. We say \(x \in X\) is an equicontinuous point of \((T, X)\), denoted \(x \in \text{Equi}(T, X)\), if given \(\varepsilon \in \mathcal{U}_X\), there is \(\delta \in \mathcal{U}_X\) such that \((tx, ty) \in \varepsilon\) whenever \((x, y) \in \delta\).

**Lemma 4.6** (cf. [3, Lemma 1.6]). If \(\text{Equi}(T, X) = X\) with \(X\) a compact Hausdorff space, then \((T, X)\) is equicontinuous.

**Lemma 4.7** (cf. [18, Lemma 3.3] and [6, Lemma 4.1]). If \((T, X)\) is a \(TT\) semiflow with \(X\) a compact Hausdorff space, then \(\text{Equi}(T, X) \subseteq \text{Tran}(T, X)\).

It is a well-known basic fact that

An equicontinuous flow is minimal if and only if it is PT (cf., e.g. [2, p. 37]).

But this is not the case in semiflow situation. Let’s see a simple example.

**Example 4.8.** Let \(X = \{a, b\}\) and \(f : a \mapsto b \mapsto b\); then the cascade \((f, X)\), which induces a \(\mathbb{Z}_+\)-action, is equicontinuous and PT with \(\text{Tran}(f, X) = \{a\}\), but it is not minimal.

However if we consider TT instead of PT, then by Lemma 4.7 we can obtain the following.

**Lemma 4.9.** Let \((T, X)\) be an equicontinuous semiflow on a compact Hausdorff space; then \((T, X)\) is TT if and only if it is minimal.

**Proof.** If \((T, X)\) is minimal, then it is obviously TT. Conversely, if it is TT, then by Lemma 4.7, \(X = \text{Equi}(T, X) \subseteq \text{Tran}(T, X)\) so \(\text{cls}_X Tx = X\) for all \(x \in X\) and thus \((T, X)\) is minimal.

Recall that if \(\text{Equi}(T, X)\) is dense in \(X\), then \((T, X)\) is called almost equicontinuous (cf. [1, 12]). It should be noticed that if we relax “equicontinuous” by “almost equicontinuous”, then the above statement is false even for flows as will be shown by the following example.

**Example 4.10.** We now construct a non-minimal cascade \((f, X)\) with \(\text{Equi}(f, X) = \text{Tran}(f, X)\) dense. Let \(X\) be the compact metric space with

\[
X = [0, 1] \cup \left\{2^{-2^n} \mid n = 0, 1, 2, \ldots\right\} \cup \left\{2^{-1/2^n} \mid n = 1, 2, \ldots\right\}
\]

and \(f : x \mapsto x^2\). Clearly, \((f, X)\) is TT and PT such that \(\text{Equi}(f, X) = \text{Tran}(f, X) = X \setminus [0, 1]\) is dense in \(X\). But \((f, X)\) is not minimal as a flow.

Of course, if we relax “equicontinuous” by “almost equicontinuous” and meanwhile we strengthen “TT” by “ST” (cf. Definition 6.1), then the statement of Lemma 4.9 still holds by Theorem 6.12 in §6.

**Lemma 4.11** (cf. [3, Corollary 3.3]). If \((T, X)\) is a minimal semiflow on a compact Hausdorff space \(X\) with \(T\) abelian, then \((T, X)\) is surjective.

**Lemma 4.12** (cf. [3, Theorem 1.13]). Let \((T, X)\) be a semiflow on a compact Hausdorff space \(X\) with phase semigroup \(T\). Then:
(1) If \((T, X)\) is distal, then it is invertible.
(2) If \((T, X)\) is equicontinuous surjective, then it is distal.
(3) If \((T, X)\) is invertible equicontinuous, then \((T, X)\) is an equicontinuous flow.

**Lemma 4.13** ([16, 2]). Let \(\langle \varphi_n : X \to Y \rangle_{n=1}^\infty\) be a sequence of continuous functions on a Baire space \(X\) to a metric space \(Y\), which converges pointwise to a function \(\varphi : X \to Y\). Let \(E\) be the set of all \(x \in X\) such that \(\varphi_n \to \varphi\) uniformly at \(x\). Then \(E\) is a residual subset of \(X\).

Theorem 4.14 below gives us a sufficient and necessary condition for equicontinuity of TT semiflow with abelian phase semigroup in terms of UT. However, it should be mentioned that (1) of Theorem 4.14 in the important special case that \((T, X)\) is a flow is originally due to Fort 1949 [9] (also cf. [16, Theorem 9.36] and [2, Theorem 2.13]).

**Theorem 4.14.** Let \((T, X)\) be a semiflow on a compact metric space \(X\) with phase semigroup \(T\). Then the following two statements hold.

1. If \((T, X)\) is UT, then \((T, X)\) is equicontinuous invertible.
2. If \((T, X)\) is TT equicontinuous with \(T\) abelian, then \((T, X)\) is UT.

**Note 1.** In view of Example 4.8, PT + Equicontinuous + Abelian phase semigroup \(\not\Rightarrow\) UT, in general semiflow setting.

**Note 2.** In fact, it is easy to verify that (2) of Theorem 4.14 also holds on compact Hausdorff phase space \(X\) by using Ellis semigroup.

**Proof.** (1) Let \((T, X)\) be UT. Without loss of generality, we can regard \(T\) as a subset of \(C(X, X)\) the continuous self-maps of \(X\), provided with the topology of uniform convergence.

Let \(\langle t_n \rangle\) be any sequence in \(T\). Choose a point \(x_0 \in X\). Some subsequence \(\langle t_i x_0 \rangle\) of \(\langle t_n x_0 \rangle\) converges, for \(X\) is a compact metric space. Then we may suppose \(\lim_{n \to \infty} t_i x_0 = y_0\). If \(x \in X\) and if \(a \in \text{Aut} (T, X)\) such that \(ax_0 = x\), then \(ay_0 = \lim_{i \to \infty} at_i x_0 = \lim_{i \to \infty} t_i ax_0 = \lim_{i \to \infty} t_i x\). Hence the sequence \(\langle t_i \rangle\) converges pointwise to some function \(\varphi : X \to X\). Since \(X\) is a Baire space, then by Lemma 4.13 there exists some point \(x_1 \in X\) such that \(t_i \to \varphi\) uniformly at \(x_1\). Since \(t_1 a = a t_1 \forall a \in \text{Aut} (T, X)\), hence \(\varphi a = a \varphi \forall a \in \text{Aut} (T, X)\). Then by UT of \((T, X)\), we can see that \(\varphi\) is continuous on \(X\). In fact, we need to show that \(t_i \to \varphi\) uniformly on \(X\). For this, let \(d\) be a compatible metric on \(X\) and let \(\varepsilon > 0\). Let \(a \in \text{Aut} (T, X)\) with \(ax_1 = x\) and let \(\delta > 0\) such that if \(d (w, z) < \delta\), then \(d (aw, az) < \varepsilon\). Let \(V_{\delta}\) be a neighborhood of \(x_1\) and \(i_0\) a positive integer such that if \(i \geq i_0\) and \(w \in V_{\delta}\), then \(d (t_i w, \varphi w) < \delta\). Hence \(d (t_i a w, a \varphi w) = d (at_i w, a \varphi w) < \varepsilon\). Thus if \(\varepsilon \in U = a [V_{\delta}]\), then \(d (t_i y, \varphi y) < \varepsilon\) for all \(i \geq i_0\). Therefore, \(t_i \to \varphi\) uniformly on \(X\).

This shows that every sequence \(\langle t_n \rangle\) in \(T\) has a uniformly convergent subsequence. Whence \(T\) is relatively compact in \(C(X, X)\). Therefore \(T\) is equicontinuous on \(X\) by the Ascoli-Arzelà theorem. Finally by UT, each \(t \in T\) is a surjection of \(X\) as in the proof of Corollary 4.3. Thus \((T, X)\) is invertible by Lemma 4.12. So \((T, X)\) is equicontinuous surjective.

(2) Let \((T, X)\) be equicontinuous TT with \(T\) an abelian semigroup. Lemma 4.9 follows that \((T, X)\) is minimal and further by Lemma 4.11, \((T, X)\) is equicontinuous surjective. Then by Lemma 4.12, \((T, X)\) is equicontinuous invertible. Further by Lemma 4.12, it follows that \((T, X)\) is an equicontinuous PT flow with \(T\) an abelian group. Gottschalk’s theorem follows that \((T, X)\) is UT. Since \(\text{Aut} (T, X) \not\subseteq \text{Aut} (T, X)\), thus \((T, X)\) is UT.

This thus concludes Theorem 4.14. \(\square\)
Since PT implies TT in flows, hence we now can generalize Gottschalk’s theorem from flows to semiflows as follows:

**Corollary 4.15.** Let \( (T, X) \) be a TT semiflow on a compact metric space \( X \) with \( T \) an abelian semigroup. Then \((T, X)\) is UT iff it is equicontinuous.

We note that the metric on \( X \) has played an important role in the proof of (1) of Theorem 4.14. However, if \( X \) is only a compact Hausdorff space non-metrizable, then what can we say?

Let \((T, X)\) be a minimal UT semiflow with compact Hausdorff phase space \( X \). Then given \( x, y \in X \), there is an \( a \in \text{Aut}(T, X) \) such that \( y = a(x) \). This implies that \( (x, y) \) is almost periodic. Indeed, if \( t_n x \to x' \) (i.e. \( t_n(x, y) \to (x', a(x')) \)) then there is \( s_n x' \to x \) (so \( s_n(x', a(x')) \to (x, y) \)) for some net \( (s_n) \) in \( T \) since \((T, X)\) is minimal. Thus \((T, X)\) is distal.

For non-minimal case, we can obtain the following, whose proof may be simplified by using Ellis’ semigroup [8, pp. 15–22].

**Theorem 4.16.** Let \((T, X)\) be a (resp. PT) semiflow on a compact Hausdorff space \( X \) with phase semigroup \( T \). If \((T, X)\) is UT, then \((T, X)\) is (resp. minimal) distal.

**Proof.** For simplicity, write \( H = \text{Aut}(T, X) \). Then \( H x = X \) for all \( x \in X \) by UT. To be contrary, assume \((T, X)\) is not distal; then there are two points \( y, w \in X \) with \( y \neq w \) such that there is a net \( (t_n) \) in \( T \) with \( \lim_n t_n y = \lim_n t_n w = z \), for \( X \) is compact.

Let \( X^X \) be the compact Hausdorff space of all functions, continuous or not, from \( X \) to itself with the pointwise convergence topology. Let \( E \) be the closure of \( T \) in \( X^X \), where we identify each \( t \in T \) with the transition map \( x \mapsto tx \) of \( X \) to \( X \) associated with \((T, X)\). Then \( E_{\gamma,w} \), defined by \( E_{\gamma,w} = \{ p \in E \mid p(y) = p(w) \} \), is a non-empty semigroup with the composition of maps such that \( E_{\gamma,w} \) is compact Hausdorff and for all \( q \in E, R_q: p \mapsto pq \) is continuous under the pointwise topology. Whence there is an element \( u \in E_{\gamma,w} \) with \( u^2 = u \) (cf. [8, Lemma 2.9]). Clearly, \( hu = uh \forall h \in H \). Now let \( x \in X \). Then \( Hx = Hux \). Hence there exists \( h \in H \) with \( hx = ux \). Then \( hx = uhx = u^2 x = ux = hx \) implies \( ux = x \). Thus \( u \in E_{\gamma,w} \) is the identity so that \( y = w \) a contradiction.

This shows that \((T, X)\) is distal if it is UT. Because a distal point is almost periodic, \((T, X)\) is minimal if it is PT and UT. Thus proves Theorem 4.16. 

Then by Theorem 4.16 combining with Furstenberg [10], we can easily obtain the following consequence.

**Corollary 4.17.** If \((T, X)\) is a UT semiflow on a compact Hausdorff space \( X \), then it admits an invariant Borel probability measure.

Therefore by Theorem 4.16, it follows that Ellis’ ‘two circle’ minimal set [8, Example 5.29] is not UT; for otherwise, it would be distal.

In addition, since a minimal semiflow is TT, hence PT + UT = TT on compact Hausdorff phase spaces by Theorem 4.16.

5. Almost right C-semigroup actions

We will first introduce a kind of phase semigroup, which includes the two important special cases: \( T = (\mathbb{R}, +) \) and \( T = (\mathbb{Z}_+, +) \) equipped respectively with the usual topologies.

Recall that a topological semigroup \( T \) is called a right C-semigroup [18] if \( T \setminus Ts \) is relatively compact in \( T \) for all \( s \in T \). In particular, \( T \setminus Ts \), for \( s \in T \), is a finite set if \( T \) is a discrete right C-semigroup semigroup like \( T = \mathbb{Z}_+ \).
Definition 5.1 (cf. [4]). A topological semigroup $T$ is called an *almost right C-semigroup* if and only if for $t \in T \mid \text{cls}_T(T \setminus Tt)$ is compact in $T$ is dense in $T$.

Clearly each topological group is an almost right C-semigroup. See [4, Examples 0.5] for some examples of almost right C-semigroups which are not right C-semigroups.

**Theorem 5.2.** Assume $(T, X)$ is a PT semiflow satisfying the following two conditions:

(a) $\text{Int}_T T x_0 = \emptyset$ for some $x_0 \in \text{Tran}(T, X)$ and 

(b) $T$ is an almost right C-semigroup.

Then $\text{Tran}(T, X)$ is dense in $X$ and hence $(T, X)$ is $TT$.

*Proof.* Let $x_0 \in \text{Tran}(T, X)$ and let $U$ be any non-empty open subset of $X$. We simply write $G$ for the dense set $\{ t \in T \mid \text{cls}_T(T \setminus Tt)$ is compact in $T \}$. Then $\text{cls}_X G x_0 = \text{cls}_X T x_0 = X$.

Given any $s \in G$ with $s \neq e$, we will show that $U \cap \text{cls}_X T s x_0 \neq \emptyset$. To be contrary, assume that $U \cap \text{cls}_X T s x_0 = \emptyset$. Set $K = \text{cls}_T(T \setminus T s)$, which is compact in $T$ by condition (b). Since $T = K \cup T s$ and so

$$X = \text{cls}_X T x_0 = \text{cls}_X(K x_0 \cup T s x_0) = K x_0 \cup \text{cls}_X T s x_0,$$

hence $U \subseteq K x_0$, which contradicts condition (a). Thus $\text{cls}_X T s x_0 = X$ for all $s \in G$ and then $G x_0 \subseteq \text{Tran}(T, X)$. This proves Theorem 5.2. \hfill $\square$

We notice here that condition (a) is very important for the consequences of Theorem 5.2 and Corollary 5.3 below, which implies that $(T, X)$ has no isolated orbit if $T$ is a group. Otherwise, Example 4.8 is a counterexample.

**Corollary 5.3.** Let $(T, X)$ be a PT equicontinuous semiflow with $X$ a compact Hausdorff space such that:

(a) $\text{Int}_T T x_0 = \emptyset$ for some $x_0 \in \text{Tran}(T, X)$ and 

(b) $T$ is an abelian almost right C-semigroup.

Then $(T, X)$ is UT, minimal, and distal.

*Proof.* By Theorem 5.2, $(T, X)$ is TT equicontinuous with $T$ abelian. So $(T, X)$ is UT by Note 2 to Theorem 4.14. Then by Theorem 4.16, $(T, X)$ is minimal distal. \hfill $\square$

Clearly a discrete semigroup is a right C-semigroup if and only if it is an almost right C-semigroup. The second part of the following Proposition 5.4 is just [18, (1) of Proposition 3.2].

**Proposition 5.4.** Let $(T, X)$ be a PT semiflow, where $X$ has no isolated point. If $T$ is a right C-semigroup under the discrete topology, then $\text{Tran}(T, X)$ is dense in $X$ and hence $(T, X)$ is $TT$.

*Proof.* In the above proof of Theorem 5.2, $K = T \setminus T s$ is finite and $U \subseteq K x_0$ implies that $X$ contains isolated points. This contradiction completes the proof. \hfill $\square$

**Corollary 5.5.** Let $(T, X)$ be a PT semiflow with $T$ a countable semigroup, where $X$ has no isolated point. If $T$ is an almost right C-semigroup, then $\text{Tran}(T, X)$ is dense in $X$ and hence $(T, X)$ is $TT$.

*Proof.* Since $T$ is a countable semigroup, then $\text{Int}_T T x = \emptyset$, for all $x \in X$ for $X$ has no isolated points. Then Corollary 5.5 follows from Theorem 5.2. \hfill $\square$
6. Syndetic transitivity and syndetic sensitivity

We will consider another kind of transitivity, which is important for chaos of semiflows (see, e.g., [5] and Theorem 6.12).

**Definition 6.1.** \((T, X)\) is called **syndetically transitive** (ST) if \(N_f(U, V)\) is syndetic in \(T\) for all non-empty open sets \(U, V\) in \(X\).

Note that ST is TT, but the converse is false. For example, \(\mathbb{R} \times X \to X, \,(t, x) \mapsto t + x\), where \(X = \mathbb{R} \cup \{\infty\}\) is the one-point compactification as in Example 4.4, is TT but not ST.

On a Polish space \((T, X)\) is TT if and only if \(\text{Tran}(T, X)\) is a residual subset of \(X\) (cf. [18, (2) of Proposition 3.2] and [5]). However, this is not the case for semiflows with non-separable phase spaces (cf. [7, Example 4.17]).

In fact, in general, \(\text{ST} \Rightarrow \text{PT}\), for flows on compact non-separable Hausdorff spaces. Let’s construct such an example as follows:

**Example 6.2.** Let \(X = Y^T\) be the space of all functions \(f : T \to Y\), continuous or not, equipped with the pointwise convergence topology, where \(Y\) is a compact Hausdorff space and \(T\) is an infinite discrete group. Then \(X\) is a compact Hausdorff space. Given \(t \in T\) and \(f \in X\), define \(f^t : T \to Y\) by \(\tau \mapsto f(\tau t)\). We now define the flow on \(X\) with the phase group \(T\) as follows: \(T \times X \to X\) by \((t, f) \mapsto f^t\).

1. First we can assert that \((T, X)\) is ST. Indeed, for all non-empty open subsets \(U, V\) of \(X\) and \(\tau_1, \ldots, \tau_n, s_1, \ldots, s_n\) in \(T\), let

   \[U = \{[\tau_1, \ldots, \tau_n], U\} = \{f \in X \mid f(\tau_i) \in U, i = 1, \ldots, n\}\]

   \[V = \{[s_1, \ldots, s_n], V\}.\]

   Let

   \[T_0 = \{s_i^{-1}\tau_j \mid i = 1, \ldots, n; j = 1, \ldots, n\} \cup \{s_j^{-1}s_i \mid i = 1, \ldots, n; j = 1, \ldots, n\}\]

   and

   \[T_1 = T \setminus T_0.\]

   Since \(T\) is an infinite discrete group and \(T_0\) is finite, it is easy to check that \(T_1\) is syndetic in \(T\).

   (In fact, if \(K = \{e\} \cup t_0T_0^{-1}\) for some \(t_0 \in T_1\), then \(Kt \cap T_1 \neq \emptyset\) for all \(t \in T = T_0 \cup T_1\) so \(T_1\) is syndetic in \(T\).) Thus for all \(t \in T_1\),

   \[{s_1t, \ldots, s_nt} \cap \{\tau_1, \ldots, \tau_n, s_1, \ldots, s_n\} = \emptyset.\]

   Now choose \(f \in X\) such that \(f(\tau_i) \in U\) and \(f(s_it) \in V\) for \(1 \leq i \leq n\). Thus \(f \in U\) and \(f^t \in V\) so that \(N_f(U, V) \supseteq T_1\). Thus

   \(\bullet\) \((T, X)\) is ST.

2. Now we choose \(Y\) a non-separable space (so \(Y\) has no countable dense subset) and let \(T\) be a countable infinite discrete group. Then \(X\) has no countable dense subset. Because \(X\) is not separable and \(T\) is countable, it follows that:

   \(\bullet\) \((T, X)\) is not PT: i.e., \(\text{Tran}(T, X) = \emptyset.\)

This completes the construction of our Example 6.2.
In view of Example 6.2 we now ask two questions:

1. If \((T, X)\) is a TT and pointwise almost periodic flow/semiflow on a compact non-separable Hausdorff space, is it PT (or equivalently minimal)?

2. If \((T, X)\) is minimal and ST with \(X\) a locally compact, non-compact, Hausdorff space, is \((T, X)\) pointwise almost periodic?

Let \(T\) be an infinite discrete group. Recall that \((T, X)\) is called strongly mixing if \(N_T(U, V)\) is co-finite for all non-empty open sets \(U, V\) in \(X\). By the arguments in Example 6.2, \((T, X)\) is strongly mixing based on any compact Hausdorff space \(Y\).

**Lemma 6.3** ([5, Lemma 2.3]). If \((T, X)\) is TT with dense almost periodic points, then it is ST.

Thus by Corollary 3.9 together with Lemma 6.3, we can easily obtain the following.

**Corollary 6.4.** If \((T, X)\) is PT with dense almost periodic points, then \((T, X)\) is ST.

**Standing assumption 6.5.** In the remainder of this section, let \(X\) be a uniform Hausdorff space with a symmetric uniform structure \(\mathcal{U}_X\). For \(x \in X, A \subset X\) and \(\epsilon \in \mathcal{U}_X\), we write

\[\epsilon[x] = \{ y \in X \mid (x, y) \in \epsilon \} \quad \text{and} \quad \epsilon[A] = \bigcup_{x \in A} \epsilon[x].\]

Given \((T, X)\) and \(\epsilon, \delta \in \mathcal{U}_X\), the “\((\epsilon, \delta)\)-stable-time set” at a point \(x \in X\) is defined as follows:

\[T_{\epsilon, \delta}(x) = \{ t \in T \mid t(\delta[x]) \subseteq \epsilon[x] \}.\]

Next we will consider a simple application of ST in chaos. For this, we first need to introduce some notions.

**Definition 6.6** (cf. [18, 19, 5, 20]). Let \((T, X)\) be a semiflow on \((X, \mathcal{U}_X)\) with phase semigroup \(T\). Then:

1. \((T, X)\) is called sensitive if there exists an \(\epsilon \in \mathcal{U}_X\) such that for all \(x \in X\) and all \(\delta \in \mathcal{U}_X\), there is an \(x'\) with \((x, x') \in \delta\) and \(t(x, x') \notin \epsilon\) for some \(t \in T\); that is, \(T_{\epsilon, \delta}(x) \neq T\).

2. \((T, X)\) is called syndetically sensitive if there exists an \(\epsilon \in \mathcal{U}_X\) such that for all \(x \in X\) and \(\delta \in \mathcal{U}_X\), \(T^\delta_{\epsilon}(x)\) is not thick in \(T\).

3. \((T, X)\) is said to be pointwise thickly stable if given \(\epsilon \in \mathcal{U}_X\) and \(x \in X\), one can find a \(\delta \in \mathcal{U}_X\) such that \(T^\delta_{\epsilon}(x)\) is thick in \(T\).

4. \((T, X)\) is called pointwise equicontinuous if given \(\epsilon \in \mathcal{U}_X\) and \(x \in X\), one can find a \(\delta \in \mathcal{U}_X\) such that \(T^\delta_{\epsilon}(x) = T\); see 3 of Definition 4.5.

5. \((T, X)\) is called uniformly almost periodic if given \(\epsilon \in \mathcal{U}_X\), there is a syndetic subset \(A\) of \(T\) such that \(Ax \subseteq \epsilon[x]\) for all \(x \in X\).

Clearly, syndetically sensitive \(\Rightarrow\) sensitive, and equicontinuous \(\Rightarrow\) thickly stable. Moreover, by the classical “Lebesgue covering lemma”, we can easily obtain the following uniformity:

**Lemma 6.7.** If \((T, X)\) is pointwise thickly stable with \(X\) a compact Hausdorff space, then for every \(\epsilon \in \mathcal{U}_X\) there exists a \(\delta \in \mathcal{U}_X\) such that \(T^\delta_{\epsilon}(x)\) is thick in \(T\) for all \(x \in X\).

In addition, we will need another known result.
Lemma 6.8 ([6]). If \((T, X)\) is a semiflow on a compact Hausdorff space, then it is uniformly almost periodic if and only if it is equicontinuous surjective.

By definitions, pointwise equicontinuous (resp. thickly stable) is much more stronger than ‘not to be sensitive’ (resp. ‘not to be syndetically sensitive’). But we shall show that they are equivalent under ‘ST’ condition.

The following lemma is essentially contained in Furstenberg’s argument of [11, p. 74–75] for measure-preserving cascades. For the self-closeness, we will prove it here.

**Lemma 6.9.** Let \(T \times (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu), (t, x) \mapsto tx\) be a measure-preserving semiflow of a probability space \((X, \mathcal{B}, \mu)\), where \(T\) is a discrete countable semigroup with \(e \in T\). If \(A \in \mathcal{B}\) with \(\mu(A) > 0\), then \(\{t \in T; \mu(t^{-1}A \cap A) > 0\}\) is syndetic in \(T\).

**Proof.** First of all, since \(\mu(X) = 1\) and \(\mu(A) > 0\), we note that there exists a finite subset \(K\) of \(T\) such that

\[
\mu \left( \bigcup_{t \in K} t^{-1}A \right) > \mu \left( \bigcup_{t \in T} t^{-1}A \right) - \mu(A).
\]

Then for all \(s \in T\),

\[
\mu \left( \bigcup_{t \in K} t^{-1}A \right) = \mu \left( \bigcup_{t \in K} t^{-1}A \right) > \mu \left( \bigcup_{t \in T} t^{-1}A \right) - \mu(A).
\]

This implies that \(\mu \left( \bigcup_{t \in K} t^{-1}A \cap A \right) > 0\) for all \(s \in T\); for otherwise, \(e \notin K\) so that

\[
\mu \left( \bigcup_{t \in K} t^{-1}A \right) \geq \mu(A) + \mu \left( \bigcup_{t \in K} t^{-1}A \right) > \mu \left( \bigcup_{t \in T} t^{-1}A \right),
\]

a contradiction. Thus, for all \(s \in T\), there is some \(t \in K\) with \(\mu(t^{-1}A \cap A) > 0\). This implies that \(\{t \in T; \mu(t^{-1}A \cap A) > 0\}\) is syndetic in \(T\) by 2 of Definition 3.4.

The proof of Lemma 6.9 is thus completed. 

Let \(f: X \to X\) be a continuous transformation on a compact metric space \(X\). Recall that \((f, X)\) is called an \(E\)-system if it is TT and admits an invariant Borel probability measure with full support. Clearly an \(E\)-semiflow is surjective. Moreover, it is known that a non-minimal \(E\)-system \((f, X)\) is sensitive [13, Theorem 1.3] and \(N_f(U, U)\) is syndetic in \(\mathbb{Z}_n\) for all non-empty open set \(U\) in \(X\) [14].

**Definition 6.10.** \((T, X)\) is called an \(E\)-semiflow if it is TT and for all non-empty open subset \(U\) of \(X\) there is an invariant Borel probability measure \(\mu\) of \((T, X)\) with \(\mu(U) > 0\).

Clearly, an \(E\)-system of Glasner and Weiss can induce an \(E\)-semiflow with phase semigroup \(\mathbb{Z}_n\). However, since \(X\) need not be second countable, so an \(E\)-system in the sense of Definition 6.10 is not necessarily an \(E\)-system in the sense of Glasner and Weiss.

**Proposition 6.11.** Let \((T, X)\) be an \(E\)-semiflow with \(T\) a countable discrete semigroup. Then \((T, X)\) is ST.

**Proof.** Let \(U, V\) be non-empty open subsets of \(X\). By TT, let \(\tau \in T\) with \(\tau \cap \tau^{-1}V \neq \emptyset\).

Then by Lemma 6.9, it follows that \(N_\tau(A, A)\) is syndetic in \(T\). Since \(\tau N_\tau(A, A)\) is syndetic in \(T\) and \(\tau N_\tau(A, A) \subseteq N_\tau(U, V)\), thus \((T, X)\) is ST. 

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Therefore Theorem 6.12 below is a generalization and strengthening of the theorem of Glasner and Weiss on E-systems.

The following result has already been proved by Miller and Money [19, Theorem 4.4] with X a compact metric space and by Wang and Zhong [20, Theorem 4.3] with X a compact Hausdorff space, using different approaches.

**Theorem 6.12.** If \((T,X)\) is a non-minimal ST semiflow with X a uniform Hausdorff space, then it is syndetically sensitive.

**Proof.** First of all, by non-minimality of \((T,X)\) there are \(\eta \in \mathcal{U}_X\), \(q \in X\) and a closed invariant subset \(\Lambda \) of \((T,X)\) such that \(\eta[q] \cap \eta[\Lambda] = \emptyset\). Assume \((T,X)\) is not syndetically sensitive; then for all \(\varepsilon \ll \eta\), there are \(x \in X\) and \(\delta \in \mathcal{U}_X\) such that \(T^\varepsilon_{\delta}(x)\) is a thick subset of \(T\).

Since \((T,X)\) is ST, \(N_T(\delta[x], \eta[q])\) is syndetic and so there exists a compact subset \(K\) of \(T\) such that \(K \cap N_T(\delta[x], \eta[q]) \neq \emptyset\). That is to say, for any \(t \in T\), there is some \(y \in \delta[x]\) such that \(Kty \cap \eta[q] \neq \emptyset\).

Let \(p \in \Lambda\) and since \(\Lambda\) is invariant for \((T,X)\), there exists some \(\gamma \in \mathcal{U}_X\) with \(\gamma \ll \eta\) such that \(K[\gamma[p]] \subset \varepsilon[\Lambda]\).

Let \(F\) be the collection of non-empty compact subsets of \(T\). Then since \(T^\varepsilon_{\delta}(x)\) is a thick subset of \(T\) and \(K\) is compact in \(T\), so for any \(F \in \mathcal{F}\), there is some \(t_F\) so that \((KF)t_F \subset T^\varepsilon_{\delta}(x)\). Clearly, \(T_0 = \bigcup_{F \in \mathcal{F}} F t_F\) is a thick subset of \(T\).

Moreover, since \(N_T(\delta[x], \gamma[p])\) is syndetic in \(T\) by ST, \(T_0 \cap N_T(\delta[x], \gamma[p]) \neq \emptyset\). Now having chosen \(\tau \in T_0 \cap N_T(\delta[x], \gamma[p])\), there exists some point \(y_0 \in \delta[x]\) such that \(\tau y_0 \in \gamma[p]\) so that \(K\tau y_0 \subset \varepsilon[\Lambda]\). By the choice of \(\tau\) and the definition of \(T_0\), it follows that \(K \tau \cap N_T(\delta[x], \eta[q]) \neq \emptyset\).

Therefore, \((T,X)\) must be syndetically sensitive. □

Recall that if \(T\) is a topological semigroup and if \(A\) is a thick subset of \(T\), then \(As\) is thick in \(T\) for all \(s \in T\). Now based on Theorem 6.12, we can conclude the following dichotomy theorem for any ST semiflow on uniform spaces.

**Corollary 6.13.** Let \((T,X)\) be an ST semiflow with phase semigroup \(T\). Then it is either syndetically sensitive or minimal thickly stable.

**Proof.** Clearly if \((T,X)\) is syndetically sensitive, then it is never thickly stable. Next assume \((T,X)\) is not syndetically sensitive; and then by Theorem 6.12, it follows that \((T,X)\) is minimal.

Now since \((T,X)\) is not syndetically sensitive, then for any \(\varepsilon \in \mathcal{U}_X\), there are some \(x \in X\) and \(\delta \in \mathcal{U}_X\) such that \(T^\varepsilon_{\delta}(x)\) is thick in \(T\). Let \(y \in X\) be arbitrary. Since \((T,X)\) is minimal, we can find some \(\alpha \in \mathcal{U}_X\) and \(s \in T\) such that \(s(\alpha[y]) \subseteq \delta[x]\). So \(T^\varepsilon_{\delta}(y) \supseteq T^\varepsilon_{\delta}(y)s\) is thick in \(T\). Since \(\varepsilon\) and \(y\) both are arbitrary, thus \((T,X)\) is thickly stable. □

In fact, by Theorem 6.12, we can obtain the following

**Corollary 6.14.** Let \((T,X)\) be ST with \(T\) an almost right C-semigroup. Then it is either sensitive or minimal equicontinuous.

**Proof.** Assume \((T,X)\) is not sensitive. So it is not syndetically sensitive. By Theorem 6.12, \((T,X)\) is minimal. Since \((T,X)\) is not sensitive, then for any \(\varepsilon \in \mathcal{U}_X\), we can find some \(x \in X\) and \(\alpha \in \mathcal{U}_X\) such that \(T^\varepsilon_{\delta}(x) = T\). Given any \(y \in X\), since \((T,X)\) is minimal, there are some \(\delta' \in \mathcal{U}_X\) and some \(\tau \in \{t \in T \mid \text{cls}(T \setminus Tt)\}\) is compact in \(T\) such that \(\tau(\delta'[y]) \subseteq \alpha[x]\). In addition, since \(T_0 \coloneqq T \setminus T\tau \)}
is relatively compact by Definition 5.1, there is a $\delta \in \mathcal{V}_X$ with $\delta < \delta'$ such that $t(\delta[x]) \subseteq e[ty]$ for all $t \in T_0$. Thus $T^t_{\epsilon, \delta}(y) = T$. This shows that $(T, X)$ is equicontinuous.

Finally, if $(T, X)$ is sensitive, it is never minimal equicontinuous. Thus we have concluded Corollary 6.14. \[\square\]

If $(T, X)$ is TT with dense almost periodic points, then it is called an $M$-semiflow ([13, 5]). By Lemma 6.3, $M$-semiflow is ST. Thus Corollary 6.14 generalizes [12, Theorem 1.41] that is for $M$-flow on a compact metric space $X$ and [18, Main result] that is for $M$-semiflow on a Polish space with $T$ a right $C$-semigroup by using different approaches.

The following corollary has already been observed in [5, Corollary 2.7]; but its proof presented in [5] is insufficient. Here we prove it using Theorem 6.12.

**Corollary 6.15.** Let $(T, X)$ be ST with $X$ a compact Hausdorff space and with $T$ an abelian semifrom. Then $(T, X)$ is either sensitive or minimal uniformly almost periodic.

**Proof.** If $(T, X)$ is sensitive, then it is not equicontinuous and so not uniformly almost periodic by Lemma 6.8.

Now assume $(T, X)$ is not sensitive. Then by Theorem 6.12, $(T, X)$ is minimal; otherwise it is syndetically sensitive. Moreover, for all $\epsilon \in \mathcal{V}_X$, we can find some $x \in X$ and $\delta \in \mathcal{V}_X$ such that $T^t_{\epsilon, \delta}(x) = T$. We will show that $(T, X)$ is uniformly almost periodic.

Let $\epsilon \in \mathcal{V}_X$ be any given; and take an $\eta \in \mathcal{V}_X$ with “$\eta \ll \epsilon$” and then we can choose $x_0 \in X$ and $\delta \in \mathcal{V}_X$ such that $T^t_{\epsilon, \delta}(x_0) = T$. So $t(\delta[x_0]) \subseteq \eta[tx_0]$ for all $t \in T$. Let $A = N_T(x_0, \delta[x_0])$, which is syndetic in $T$. Now for any $a \in A$ and $s \in T$, we have $ax_0 \in \delta[x_0]$ and then it follows that $(sx_0, ax_0) = (sx_0, ax_0) \in \eta$. Since $TX_0$ is dense in $X$, hence $(z, tz) \in e$ for all $z \in X$ and $t \in A$. Thus $(T, X)$ is uniformly almost periodic This proves Corollary 6.15. \[\square\]

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