Entanglement measure and dynamics of multiqubit systems: non-Markovian versus Markovian and generalized monogamy relations

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Abstract. In this paper, we first present a simple measure for multiqubit entanglement based on the strategy of bipartite cuts and the measure of negativity. Then, we establish generalized monogamy inequalities and associated partition-dependent residual entanglement (PRE) accounting for arbitrary partitions of a multiqubit system. By virtue of the defined quantities, we investigate the entanglement dynamics of a system of $N$ qubits, either in the Greenberger–Horne–Zeilinger (GHZ)-type state or in the W state, interacting with $N$ independent reservoirs in both Markovian and non-Markovian regimes. We observe entanglement revivals of qubits at instantaneous points of disappearance or after a finite interval of abrupt vanishing due to the memory effect of non-Markovian reservoirs. We also follow the whole entanglement evolution in terms of the PRE to demonstrate the process of transition between the bipartite entanglement of all possible bipartitions and the multipartite entanglement. In particular, we show that the change in time of entanglement formats differs qualitatively for the GHZ-type and W states.

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1. Introduction

Entanglement is a key concept in quantum physics and has been increasingly realized as an indispensable resource to perform various intriguing global tasks in quantum computing and quantum information processing [1]. In relation to its practical application, the investigation of entanglement dynamics is of paramount importance, since a real quantum system unavoidably interacts with its surroundings undergoing consequent decoherence and entanglement degradation. However, a full understanding of entanglement dynamics relies on the availability of entanglement measures that can reflect the time variation of the system of interest, including the change of entanglement formats during the process of evolution. So far, only bipartite entanglement has been well understood and quantified [2, 3], partially due to the feasible comparison with the ready-made maximally entangled sample. Multipartite entanglement, however, still remains under extensive exploration since there is no such fixed model with which every other state can be compared. Therefore, in order to study multipartite entanglement dynamics, one should first figure out some efficient entanglement measures.

In quantifying multipartite entanglement, one strategy is to generalize the bipartite entanglement measure to the multipartite case [4, 5]. However, the calculation for the mixed state of a multipartite system requires an optimization process, which is very difficult to solve exactly and can only be determined purely algebraically in the regime where the mixing is moderate [4]. Another strategy one usually adopts in analyzing multipartite entanglement is to deal with bipartite cuts (bipartitions) by dividing the system into two arbitrary blocks, with some particles constituting one block and all the other particles the other block [6]–[12]. For each such cut one can pick up the entanglement between the two blocks via a bipartite entanglement measure. However, different cuts may lead to different values of entanglement and the number of possible bipartite cuts increases rapidly with the number of particles in the system. Furthermore, the entanglement status of one or some bipartitions cannot describe the global feature of a multipartite system. Therefore, an efficient and calculable entanglement measure that can reflect the global information of a multipartite system is required. In this work, we shall present a simple entanglement measure to quantify a multiqubit entangled state based on the idea of bipartitions.
The cognition of various properties of multipartite entanglement is an efficient way for its quantification and application. One of the most fundamental properties of multipartite quantum systems discovered recently is that entanglement is monogamous: if two particles are maximally entangled, neither of them can be entangled, in any way, with a third one [13]. Mathematically, in the case of three qubits \( A, B \) and \( C \) in pure state, Coffman, Kundu and Wootters (CKW) established a monogamy inequality as \( C_{AB}^2 \geq C_{AB}^2 + C_{AC}^2 \), with \( C_{AB} \) (\( C_{AC} \)) being the concurrence [2] of qubit pair \( AB \) (\( AC \)) and \( C_{A|BC} \) that of qubit \( A \) with qubits \( BC \) as a whole. What is more relevant to multipartite entanglement measure is the difference between the two sides of the inequality, indicating the existence of any genuine multipartite entanglement that cannot be accounted for by pairwise entanglements inside the multipartite system. Actually, the so-called residual entanglement \( \tau_{ABC} = C_{A|BC}^2 - C_{AB}^2 - C_{AC}^2 \) [13] can serve as a measure for tripartite entanglement. Recently, the CKW inequality has been generalized to the \( N \)-qubit case by Osborne and Verstraete in [14]. The entanglement monogamy inequality and associated residual entanglement have also been constructed in the context of high-dimension systems [15], continuous variable systems [16] and in terms of other entanglement measures [17, 18]. For an arbitrary bipartition of a multipartite system, however, the bound on the entanglement between individual particles and even between the particles’ sub-blocks across the bipartition is still an open question and of fundamental importance from both mathematical and dynamical perspectives. From this point of view, one should proceed from the single-party bipartition and pairwise entanglements of the usual entanglement monogamy relation to arbitrary partitions and take the entanglement of particles’ sub-blocks into account. Our other purpose in this paper is thus to establish generalized monogamy inequalities applicable to all possible bipartitions via multi-party cuts involving the entanglement between qubits’ sub-blocks as well as between qubit pairs.

Depending on the initial conditions of the system of interest and environment, the decoherence may terminate entanglement in an infinite or a finite time. The finite-time disentanglement, also termed entanglement sudden death (ESD), has resulted in several studies both in theory [19] and in experiment [20]. Fortunately, as has been shown recently, the entanglement of two noninteracting qubits embedded either in separated non-Markovian environments [21, 22] or in a common one [23] can revive after a period of sudden death. However, to our best of our knowledge, the entanglement dynamics of multiqubit systems in non-Markovian environments has not been investigated yet. In this paper, we study the memory effect of non-Markovian environments to see how it changes the picture of multipartite entanglement dynamics in comparison with Markovian environments. Generally, the initial entanglement may spread over all possible degrees of freedom of the entire system involving separate environments that can become entangled during the dynamical process. Thus, a proper study of entanglement dynamics should take into account possible newly built entanglement formats as well as the transition between them. In [24], the authors studied a decoherence model where two entangled cavity photons locally interacted with independent reservoirs and showed that the ESD of entangled cavity photons necessarily triggers a delay of creation of entanglement (also called entanglement sudden birth (ESB)) of the associated reservoirs. In this connection, it is worth generalizing the bipartite entanglement dynamics to the multipartite case by taking the whole system, including environments, into account. By doing so we can closely follow not only the entanglement-changing process of the interested multiqubit subsystems but also the constrained distribution of entanglement between different sub-blocks in the
light of our generalized monogamy inequalities. Furthermore, the transition between bipartite entanglements of all possible bipartitions and the corresponding multipartite entanglement can be demonstrated by virtue of residual entanglement defined through our generalized monogamy inequalities. The decoherence model we use is a system of \( N \) qubits that are prepared at \( t = 0 \) in the GHZ-type state [25] or the W [26] state and for \( t > 0 \) interact with \( N \) independent reservoirs. Due to their inherent differences, we shall show that the change in time of entanglement formats differs qualitatively for the two classes of states under consideration.

2. Multiqubit entanglement measure and generalized monogamy inequalities

2.1. Multiqubit entanglement measure

The initial idea to quantify entanglement was connected with its usefulness in terms of communication [27, 28]. However, one can apply an axiomatic point of view, by allowing any function of state to be a measure, provided it satisfies some postulates. The most important postulate for entanglement measure is monotonicity [28], from which one can derive another axiom, i.e., entanglement vanishes for separable states. The above two axioms are essentially the only ones that should be necessarily required from entanglement measures [29]. Here, we generalize a simple multiqubit entanglement quantifier based on the idea of bipartition and the measure negativity (which is two times the absolute value of the sum of the negative eigenvalues of the corresponding partially transposed matrix of a state \( \rho \)) [3]. For an arbitrary \( N \)-qubit state \( \rho_{s_1 s_2 ... s_N} \), a multiqubit entanglement measure can be formulated as

\[
\mathcal{N} = \frac{N}{2} \sum_{k=1}^{N/2} \mathcal{N}_{k|N-k}(\rho_{s_1 s_2 ... s_N}),
\]

where \( N \) is assumed even, otherwise \( \frac{N}{2} \) should be replaced by \( \frac{N-1}{2} \), and \( \mathcal{N}_{k|N-k}(\rho_{s_1 s_2 ... s_N}) \) is the entanglement in terms of negativity between two blocks of a bipartition \( k|N-k \) of the state \( \rho_{s_1 s_2 ... s_N} \). The defined entanglement measure \( \mathcal{N} \) obviously satisfies the axiom of monotonicity since it is only a linear average of negativity. Also, \( \mathcal{N} \) vanishes when all the terms \( \mathcal{N}_{k|N-k} \) become zero, implying the \( N \)-qubit system is fully separable. In other words, \( \mathcal{N} \) can reflect the global entanglement of a multiqubit system in the sense that it accounts for all the possible bipartite entanglements. Our entanglement measure is not a general one for multipartite entanglement, but it is sufficient for our purpose to study the dynamics of \( N \)-qubit GHZ-type [25] and W-type [26] states.

2.2. Generalized monogamy inequalities and residual entanglement

For an \( N \)-qubit system \( s_1, s_2, ..., s_N \), an arbitrary bipartition \( k|N-k \) divides the system into two blocks labeled by \( s_1 \ldots s_k | s_{k+1} \ldots s_N \) and each of the blocks can be further cut into two more sub-blocks labeled by \( s_1 \ldots s_m | s_{m+1} \ldots s_k \) and \( s_{k+1} \ldots s_n | s_{n+1} \ldots s_N \). The bound on the entanglement between sub-blocks and qubits belonging to different blocks of the first bipartition is an open question. Here, instead of giving a general answer to this question, we shall put forward two partition-dependent entanglement monogamy-like inequalities (3) and (4) in terms of negativity \( \mathcal{N} \) aiming at the entanglement of some typical multiqubit states, such as GHZ-type...
and W-type states with rank-2 reduced matrix density, as follows:

\[ N^2_{S_1|S_2|S_3|S_4} \leq N^2_{S_1|S_2|S_4} + N^2_{S_3|S_4} \]

\[ \geq N^2_{S_1|S_2|S_4} + N^2_{S_3|S_4} + N^2_{S_1|S_2|S_3|S_4} + N^2_{S_3|S_4} \]

\[ \geq \sum_{i=1}^{N} \sum_{j=k+1}^{N} N^2_{S_i,S_j}, \]

where \( N_{S_i|S_j|S_k|S_{N-k}} \) denotes negativity associated with the bipartition \( S_i \ldots S_j | S_k \ldots S_m \) and \( N_{S_i|S_j} \) that of the pair of qubits \( S_i \) and \( S_j \). Remarkably, the above monogamy inequalities hold for any partitions characterized by \( m, k \), and \( n \) with \( 1 \leq m \leq k \leq n \leq N \). These reduce to the bound that derived in [14] for the particular case with \( 1 = m = k = n = N \).

Obviously, inequalities (3) and (4) put general constraints on the sum of entanglement between any two qubits’ sub-blocks across an arbitrary bipartition as well as on the sum of entanglement between any two individual qubits across the bipartition. Namely, inequality (3) can be interpreted as follows. If a block of \( k \) qubits \( S_1 \ldots S_k \) has some amount of entanglement with a block of the remaining \( N-k \) qubits \( S_{k+1} \ldots S_N \), then this amount sets the upper bound for the total entanglement between any two sub-blocks taken from blocks \( S_1 \ldots S_k \) and \( S_{k+1} \ldots S_N \), respectively. The sum of entanglement between two such sub-blocks in turn, as revealed by (4), bounds the total pairwise entanglement of individual qubits belonging to different blocks. In deriving inequalities (2) and (3), we have used the monogamy inequality in terms of negativity for the three-qubit state, \( N_{ABC}^2 \geq N_{AB}^2 + N_{AC}^2 \) [17], and single-qubit reduction for the qubits’ block with rank-2 reduced state. After repeatedly applying (3) to each term of its right-hand side until the situation when each sub-block consists of only one qubit, we obtain nothing else but the last inequality (4).

Based on inequalities (3) and (4), we can define the following partition-dependent residual entanglements (PREs):

\[ \Pi_{q_1 \ldots q_m | q_{m+1} \ldots q_k | q_{k+1} \ldots q_{N}} = N^2_{q_1 \ldots q_m | q_{m+1} \ldots q_k | q_{k+1} \ldots q_{N}} - N^2_{q_1 \ldots q_m | q_{k+1} \ldots q_{N}} - N^2_{q_{m+1} \ldots q_k | q_{k+1} \ldots q_{N}} - N^2_{q_{m+1} \ldots q_k | q_{k+1} \ldots q_{N}} \]

(5)

and

\[ \Pi'_{q_1 \ldots q_k | q_{k+1} \ldots q_{N}} = N^2_{q_1 \ldots q_k | q_{k+1} \ldots q_{N}} - \sum_{i=1}^{k} \sum_{j=k+1}^{N} N^2_{q_i,q_j} \]

(6)

Intuitively, a full understanding of entanglement for a multiqubit system should involve the exploration of various formats of quantum correlation therein, including both bipartite and multipartite entanglements with respect to all the possible partitions of the entire system. Anyway, it seems that a single value cannot completely describe the actual entanglement. The entanglement monogamy inequalities (3), (4) and the derived PREs (5), (6) cover all the possible bipartitions for an \( N \)-qubit system. Depending on the concrete partitions of the system in terms of \( k, m \), and \( n \), PREs (5) and (6) can reflect essential multi-way entanglements with different levels and different formats. In other words, the defined PREs have two distinct advantages: the introduction of entanglements between sub-blocks and the allowance for arbitrary partitions of the system. More precisely, \( \Pi_{q_1 \ldots q_k | q_{k+1} \ldots q_{N}} \) (\( \Pi'_{q_1 \ldots q_k | q_{k+1} \ldots q_{N}} \)) measures the amount of residual entanglement between two blocks \( \{q_1,q_2 \ldots q_k\} \) and \( \{q_{k+1}q_{k+2} \ldots q_{N}\} \) that cannot be accounted for by the bipartite entanglements between separate sub-blocks (qubits).
In order to show the applicability of our inequalities and PREs, we present two examples. The first one is an $N$-qubit GHZ-type state \cite{25} of the form
\begin{equation}
|G\rangle_{q_1 q_2 \ldots q_N} = (\alpha |00\ldots0\rangle + \beta |11\ldots1\rangle)_{q_1 q_2 \ldots q_N},
\end{equation}
with $|\alpha|^2 + |\beta|^2 = 1$. It is easy to check that
\begin{equation}
\Pi'_{q_1 \ldots q_{N-2} q_{N-1} q_N} = \Pi'_{q_1 \ldots q_{N-1} q_{N}} = \mathcal{N}^2_{q_1 \ldots q_{N-1} q_N} = 4\alpha^2 \beta^2.
\end{equation}
This means that the $N$-qubit GHZ-type state is a genuine multipartite entangled state containing neither bipartite entanglement between sub-blocks nor pairwise entanglement between individual qubits. The second example is an $N$-qubit W state \cite{26}
\begin{equation}
|W\rangle_{q_1 q_2 \ldots q_N} = \frac{1}{\sqrt{N}}(|10\ldots0\rangle + |01\ldots0\rangle + \cdots + |00\ldots1\rangle)_{q_1 q_2 \ldots q_N}.
\end{equation}
For this state, we have
\begin{equation}
\Pi_{q_1 \ldots q_{N-2} q_{N-1} q_N} = \frac{4k(N-k)}{N^2} - \frac{1}{N^2} \left[\sqrt{(N-k)^2 + 4m(n-k) - (N-k)}\right]^2
- \frac{1}{N^2} \left[\sqrt{(N-k)^2 + 4(k-m)(n-k) - (N-k)}\right]^2
- \frac{1}{N^2} \left[\sqrt{(N-k)^2 + 4(k-m)(N-n) - (N-k)}\right]^2.
\end{equation}
and
\begin{equation}
\Pi'_{q_1 \ldots q_{N-1} q_N} = \frac{4k(N-k)}{N^2} - \frac{k(N-k)}{N^2} \left[\sqrt{4 + (N-2)^2 - (N-2)}\right]^2
= \frac{2k(N-k)}{N^2} (N-2) \left[\sqrt{4 + (N-2)^2 - (N-2)}\right].
\end{equation}
As seen from equations (10) and (11), both $\Pi_{q_1 \ldots q_{N-2} q_{N-1} q_N}$ and $\Pi'_{q_1 \ldots q_{N-1} q_N}$ are closely related to the bipartition in terms of $k$, while the former depends on further partitions in terms of $m$ and $n$. This confirms the fact that a single value cannot assess the total entanglement of a multiqubit system as different partitions give different amounts and formats of entanglement. Therefore, a comprehensive analysis of the whole system is compulsory for a full understanding of multiqubit entanglement. In this connection, our defined PREs have unique usefulness when one deals with qubits’ blocks and sub-blocks in studying the entanglement dynamics of a multiqubit system. In the next section, by virtue of PREs, we study the entanglement dynamics of an $N$-qubit system which is prepared at time $t = 0$ in either the GHZ-type or the W state and then at time $t > 0$ interacts locally with $N$ independent multi-mode reservoirs.

3. Entanglement dynamics of multiqubit systems

3.1. Physical model

Consider a realistic quantum network of $N$ remote nodes, each of which contains a matter qubit (e.g. a two-level atom, a spin-half particle and so on) in contact with a multimode reservoir.
Since there are no interactions at all between the nodes, the whole network can be described via the sum of $N$ independent qubit–reservoir Hamiltonians of the form ($\hbar = 1$)

$$\hat{H} = \omega_q \hat{\sigma}_z \hat{\sigma}_- + \sum_j \omega_j \hat{a}_j^\dagger \hat{a}_j + \sum_j (g_j \hat{\sigma}_z \hat{a}_j + g_j^* \hat{\sigma}_- \hat{a}_j^\dagger),$$

(12)

where $\hat{a}_j^\dagger$ ($\hat{a}_j$) is the creation (annihilation) operator of mode $j$ of the reservoir, $\hat{\sigma}_z = |1\rangle \langle 0|$, $\hat{\sigma}_- = |0\rangle \langle 1|$, $\omega_q$ ($\omega_j$) is the transition frequency of the qubit (the reservoir mode $j$) and $g_j$ measures the strength of coupling between the qubit and the reservoir mode $j$. The overall dynamics can simply be obtained from the evolution of the individual qubit–reservoir subsystem. Let each reservoir be initially in the vacuum state $\{|0\rangle\rangle_r = \prod_{k=1}^N |0_k\rangle_r$. Then, the state $|0\rangle\rangle_r$ does not evolve in time, while the state $|\phi(0)\rangle = |1\rangle \otimes |\bar{0}\rangle_r$ will evolve to

$$|\phi(t)\rangle = C_0(t)|1\rangle \otimes |\bar{0}\rangle_r + \sum_j C_j(t)|0\rangle\rangle_r,$$

(13)

with $|1\rangle\rangle_r$ the reservoir state containing only one excitation in the $j$th mode. The equations of motion for the probability amplitudes in the interaction picture take the form

$$\dot{C}_0(t) = -i \sum_j g_j e^{-i(\omega_j - \omega_q)t} C_j(t),$$

(14)

$$\dot{C}_j(t) = -i g_j^* e^{i(\omega_j - \omega_q)t} C_0(t).$$

(15)

Integrating equation (15) with the initial condition $C_j(0) = 0$ and inserting its solution into equation (14), one obtains an integro-differential equation for the amplitude $C_0(t)$,

$$\dot{C}_0(t) = -\int_0^t \sum_j |g_j|^2 e^{-i(\omega_j - \omega_q)(t-t')} C_0(t') \, dt'.$$

(16)

The sum $\sum_j |g_j|^2 e^{-i(\omega_j - \omega_q)(t-t')}$ in the above equation is recognized as nothing else but the reservoir correlation function $\langle f(t-t') = \langle |0\rangle |A(t)|A^\dagger(t')|\bar{0}\rangle_r$, where $A(t) = \sum_k g_k a_k e^{-i(\omega_k - \omega_q)t}$. In the limit of a large number of modes, the summation over the reservoir modes can be changed to an integration $\int d\omega J(\omega) e^{-i(\omega - \omega_q)(t-t')}$, where $J(\omega) = |g(\omega)|^2 S(\omega)$ is referred to as the reservoir effective spectral density with $S(\omega)$ a density of states introduced such that $S(\omega) \, d\omega$ yields the number of modes with frequencies lying in the interval $[\omega, \omega + d\omega]$. Then, equation (16) becomes

$$\dot{C}_0(t) = -\int_0^t \int d\omega J(\omega) e^{-i(\omega - \omega_q)(t-t')} C_0(t') \, dt',$$

(17)

whose solution depends on the reservoir structure characterized by $J(\omega)$.

In the following, we consider the structured reservoir as the electromagnetic field inside a lossy cavity. In this case, the fundamental mode $\omega_c$ supported by the cavity displays a Lorentzian broadening due to the nonperfect reflectivity of the cavity mirrors. The effective spectral density of the intracavity field can be modeled as

$$J(\omega) = \frac{R^2}{\pi} \frac{\Gamma}{(\omega - \omega_c)^2 + \Gamma^2}.$$  

(18)

In equation (18), $R$ is related to the strength of the qubit–cavity coupling and $\Gamma$ is the half-width at half-maximum of the intracavity field spectrum profile. The cavity correlation function then is $f(t-t') = R^2 \exp[-\Gamma(t-t')]$, allowing one to interpret $1/\Gamma = \tau_c$ as the time scale for the
cavity. As for the qubit, its time scale can be shown to be $\tau_q = \Gamma / 2R^2$. Explicitly in terms of $\tau_c$ and $\tau_q$, the exact solution of equation (17) with $J(\omega)$ given by equation (18) can be obtained as

$$
C_0(t) = e^{-(1-i\Delta)t/2} \left[ \cosh \left( \frac{\tau}{2} \sqrt{(1-i\Delta)^2 - \frac{2\tau_c}{\tau_q}} \right) + \frac{1-i\Delta}{\sqrt{(1-i\Delta)^2 - \frac{2\tau_c}{\tau_q}}} \sinh \left( \frac{\tau}{2} \sqrt{(1-i\Delta)^2 - \frac{2\tau_c}{\tau_q}} \right) \right],
$$

(19)

where $\tau = \Gamma t$ and $\Delta = \delta / \Gamma$ with $\delta = \omega_q - \omega_c$ the detuning of the qubit frequency from the fundamental cavity mode frequency. Solution (19) is rigorous since its derivation is nonperturbative and keeps track of the reservoir memory. The behavior of $C_0(t)$ is qualitatively distinguished in two regimes: the weak coupling regime $2\tau_c/\tau_q < 1$ (i.e. $R < \Gamma/2$), which corresponds to Markovian dynamics, and the strong coupling regime $2\tau_c/\tau_q \geq 1$ (i.e. $R \geq \Gamma/2$), which exhibits non-Markovian dynamics.

In this paper, making use of the rigorous solution (19) we shall be able to explore different dynamical behavior of certain types of multiqubit states in Markovian as well as in non-Markovian regimes. In terms of the cavity’s normalized collective state with only one excited mode $|\tilde{t}_r\rangle = C(t)^{-1} \sum_s C_j(t)|1_s\rangle_r$, equation (13) can be rewritten as

$$
|\phi(t)\rangle = C_0(t)|1_0\rangle_r + C(t)|0\rangle_r|\tilde{t}_r\rangle,
$$

(20)

where $C(t) = \sqrt{1 - |C_0(t)|^2}$. Note that each cavity can be either in the vacuum or in a state with only one excited mode so it can also be regarded as a qubit. Thus, the network as a whole is a system of $2N$ qubits ($N$ matter qubits plus $N$ reservoir qubits) and we can apply our monogamy inequalities and PREs to such a $2N$-qubit system. In what follows, however, we refer to matter qubits simply as qubits and to reservoir qubits simply as reservoirs.

### 3.2. GHZ-type initial states of $N$ qubits

We first deal with $N$ qubits $s_1, s_2, \ldots, s_N$ prepared at time $t = 0$ in the GHZ-type state (7) with each qubit $s_i$ interacting independently with an empty local reservoir $r_i$. For simplicity, we suppose the interactions between qubits and reservoirs are resonant (i.e. $\delta = 0$) throughout the paper. The initial state of the total system

$$
|\Phi(0)\rangle_{s_1\ldots s_N r_1\ldots r_N} = (\alpha|0\rangle^{\otimes N} + \beta|1\rangle^{\otimes N})_{s_1\ldots s_N} \otimes |\bar{0}\rangle^{\otimes N}_{r_1\ldots r_N}
$$

(21)

evolves at time $t > 0$ to

$$
|\Phi(t)\rangle_{s_1\ldots s_N r_1\ldots r_N} = \alpha|0\rangle_{s_1\ldots s_N} |\bar{0}\rangle^{\otimes N}_{r_1\ldots r_N} + \beta \prod_{i=1}^{N} \{C_0(t)|1\rangle_{s_i r_i} + C(t)|0\rangle_{r_i} \}
$$

(22)

For all possible bipartitions $k|N - k$ within the $N$ qubit–reservoir pairs, we shall, on the same footing, study the entanglements between sub-blocks across the bipartitions, that is $s_1 \ldots s_k |s_{k+1} \ldots s_N$ for the qubits, $r_1 \ldots r_k |r_{k+1} \ldots r_N$ for the reservoirs and $s_1 \ldots s_k |r_{k+1} \ldots r_N$, $s_{k+1} \ldots s_N |r_1 \ldots r_k$ for the qubit–reservoir cuts as well as the PRE $\Pi_{s_1 r_1 \ldots s_k r_{k-1} s_{k+1} \ldots s_N r_N}$. To this end, we should first obtain the reduced matrices of the corresponding subsystems and then calculate their negativities. In the following, we derive the

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analytical expressions for the reduced states and the corresponding entanglement of each of the concerned subsystems.  

The reduced density matrix of the qubits’ subsystem $\rho_{s_1 \ldots s_k}(t)$ is obtained by tracing $\rho_{s_1 \ldots s_k r_1 \ldots r_N}(t) \equiv |\Phi(t)\rangle\langle\Phi(t)|$ over the reservoirs’ degrees of freedom, which can be expressed as

$$
\rho_{s_1 \ldots s_k}(t) = \mu_0|0\rangle\langle0| \otimes \sum_{k=1}^{N} \mu_k \mathcal{P}_S\left([|0\rangle\langle0|]^{\otimes(N-k)} \otimes (|1\rangle\langle1|)^{\otimes k}\right) + \alpha \beta^* C_0^N(t)(|0\rangle\langle1|)^{\otimes N}
$$

$$
+ \beta^* \alpha C_0^N(t)(|1\rangle\langle0|)^{\otimes N},
\tag{23}
$$

where $\mathcal{P}_S$ accounts for all possible permutations of the state of $N$ qubits, $\mu_0 = |\alpha|^2 + |\beta|^2 |C_0(t)|^{2k} |C(t)|^{2(N-k)}$ and $\mu_k = |\beta|^2 |C_0(t)|^{2k} |C(t)|^{2(N-k)}$. For the bipartition $k|N-k$, the minimal eigenvalue of the states’ partial transposition can be calculated as

$$
\Lambda_k^S(t) = \frac{1}{2}\left[(\mu_k + \mu_{N-k}) - \sqrt{(\mu_k - \mu_{N-k})^2 + 4|\alpha \beta|^2|C_0(t)|^{2N}}\right].
\tag{24}
$$

Similarly, by tracing $\rho_{s_1 \ldots s_k r_1 \ldots r_N}(t)$ over the qubits’ degrees of freedom, the reduced density matrix of the reservoirs $\rho_{r_1 \ldots r_k}(t)$ can be expressed as

$$
\rho_{r_1 \ldots r_k}(t) = v_0(|0\rangle\langle0|) \otimes \sum_{k=1}^{N} v_k \mathcal{P}_R\left([|0\rangle\langle0|]^{\otimes(N-k)} \otimes (|1\rangle\langle1|)^{\otimes k}\right) + \alpha \beta^* C_0^N(t)(|0\rangle\langle1|)^{\otimes N}
$$

$$
+ \beta^* \alpha C_0^N(t)(|1\rangle\langle0|)^{\otimes N},
\tag{25}
$$

where $\mathcal{P}_R$ accounts for all possible permutations of the state of $N$ reservoirs, $v_0 = |\alpha|^2 + |\beta|^2 |C_0(t)|^{2k} |C(t)|^{2k}$ and $v_k = |\beta|^2 |C_0(t)|^{2k} |C(t)|^{2k}$. For the bipartition $k|N-k$, the minimal eigenvalue of the states’ partial transposition is

$$
\Lambda_k^R(t) = \frac{1}{2}\left[(v_k + v_{N-k}) - \sqrt{(v_k - v_{N-k})^2 + 4|\alpha \beta|^2|C(t)|^{2N}}\right].
\tag{26}
$$

As for the subsystem of $k$ qubits $s_1 \ldots s_k$ plus $N-k$ reservoirs $r_{k+1} \ldots r_N$, its reduced density matrix $\rho_{s_1 \ldots s_k r_{k+1} \ldots r_N}(t)$ can be obtained by tracing $\rho_{s_1 \ldots s_k r_{k+1} \ldots r_N}(t)$ over the other $N-k$ qubits and $k$ reservoirs as

$$
\rho_{s_1 \ldots s_k r_{k+1} \ldots r_N}(t) = |\alpha|^2 (|0\rangle\langle0|) \otimes \sum_{k=0}^{N} v_k \mathcal{F}_k \mathcal{P}_{SR}\left([|0\rangle\langle0|]^{\otimes(N-k)} \otimes (|1\rangle\langle1|)^{\otimes k}\right)
$$

$$
+ \alpha \beta^* C_0^k(t) C^{*(N-k)}(t)(|0\rangle\langle0|)^{\otimes N} + \beta^* \alpha C_0^{N-k}(t) C^k(t)(|1\rangle\langle1|)^{\otimes N},
\tag{27}
$$

where $\mathcal{F}_k$ denotes bit-flip operations on the first $k$ particles, $\mathcal{P}_{SR}$ accounts for all possible permutations of the state of $k$ qubits plus $N-k$ reservoirs and $v_k = |\beta|^2 |C_0(t)|^{2k} |C(t)|^{2k}$. For the bipartition $k|N-k$, we obtain the minimal eigenvalue of the states’ partial transposition as

$$
\Lambda_k^{SR}(t) = \frac{1}{2}\left[(v_0 + v_N) - \sqrt{(v_0 - v_N)^2 + 4|\alpha \beta|^2|C_0(t)|^{2k}|C(t)|^{2(N-k)}}\right].
\tag{28}
$$

With the same procedure, we obtain the reduced density matrix $\rho_{r_1 \ldots r_k s_{k+1} \ldots s_N}(t)$ for the subsystem of $k$ reservoirs $r_1 \ldots r_k$ plus $N-k$ qubits $s_{k+1} \ldots s_N$ as

$$
\rho_{r_1 \ldots r_k s_{k+1} \ldots s_N}(t) = |\alpha|^2 (|0\rangle\langle0|) \otimes \sum_{k=0}^{N} v_k \mathcal{F}_k \mathcal{P}_{SR}\left([|0\rangle\langle0|]^{\otimes(N-k)} \otimes (|1\rangle\langle1|)^{\otimes k}\right)
$$

$$
+ \alpha \beta^* C_0^{*(N-k)}(t) C^k(t)(|0\rangle\langle0|)^{\otimes N} + \beta^* \alpha C_0^{N-k}(t) C^k(t)(|1\rangle\langle1|)^{\otimes N},
\tag{29}
$$

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Figure 1. The entanglement $\mathcal{N}_{s_1s_2s_3s_4} \equiv \mathcal{N}^s$ of qubits in GHZ-type state (7) as functions of dimensionless time $Rt$ and parameter $|\alpha|^2$ in (a) non-Markovian and (b) Markovian reservoirs, respectively. Correspondingly, (c) and (d) are for the entanglement $\mathcal{N}_{r_1r_2r_3r_4} \equiv \mathcal{N}^R$ for non-Markovian and Markovian reservoirs, respectively. For non-Markovian reservoirs we choose $\Gamma_1 = 0.1R$, whereas $\Gamma_1 = 5R$ is used for Markovian reservoirs.

Now we are in a position to study the entanglement dynamics of both the qubits’ subsystem and reservoirs by virtue of our defined multiqubit entanglement measure $\mathcal{N}$, equation (1). Without loss of generality, let us consider a system consisting of four qubits $s_1, s_2, s_3, s_4$ in the GHZ-type (7) state interacting with four independent reservoirs $r_1, r_2, r_3, r_4$. In figure 1, we plot the evolution of entanglement $\mathcal{N}_{s_1s_2s_3s_4}$ of qubits and $\mathcal{N}_{r_1r_2r_3r_4}$ of the corresponding reservoirs. Figures 1(a) and (b) are for the qubits in the non-Markovian and Markovian regimes, respectively. They clearly show that in the Markovian regime (figure 1(b)) ESD occurs for $|\alpha|^2 < 1/2$ but for $|\alpha|^2 \geq 1/2$ the qubits’ entanglement decays only asymptotically. Such behavior exhibits strong contrast with the non-Markovian regime (figure 1(a)) in which the qubits’ entanglement oscillates while decaying to zero for $|\alpha|^2 \geq 1/2$, but for $|\alpha|^2 < 1/2$ zero- and finite-entanglement domains appear alternately. This means that the qubits’ entanglement revives after a period of complete death. The entanglement revival phenomenon that originates from the memory effect of the reservoirs proves very useful since it prolongs the usage time of entanglement during its evolution. The information flow from qubits to reservoirs will give rise to the creation of entanglement of the reservoirs. This can be seen from figures 1(c) and (d), where the reservoirs’ entanglement evolution is shown. A notable feature is that when $|\alpha|^2 \geq 1/2$ the reservoirs’ entanglement can arise right from $t = 0$, but when $|\alpha|^2 < 1/2$ it is zero.

with the minimal eigenvalue of the states’ partial transposition

$$\Lambda^R_k(t) = \frac{1}{2} \left[ (v_0 + v_N) - \sqrt{(v_0 - v_N)^2 + 4|\alpha\beta|^2 |C_0(t)|^2 |C(t)|^2} \right]. \quad (30)$$
for an initial period of time and then suddenly appears, i.e. ESB occurs. The reservoirs’ ESB is shown always happening under the same condition for the qubits’ ESD. In the non-Markovian regime, figure 1(c), the reservoirs’ entanglement evolution is oscillatory. For $|\alpha|^2 < 1/2$, it can abruptly vanish for a while and then revives again and such a birth-followed-by-death phenomenon can occur a number of times. In contrast, as shown in figure 1(d), the reservoirs’ entanglement only grows monotonically without any oscillations in the Markovian regime.

As a whole from the above descriptions, the entanglement of reservoirs exhibits complementarity to the entanglement variation of qubits due to the transfer of information initially possessed by qubits to reservoirs. Although in the non-Markovian regime the information can be attained reciprocally, the back actions of information are finite. Thus, the entanglement of qubits would eventually decay to zero, being completely transferred into reservoirs. However, we note that the entanglement of corresponding reservoirs alone cannot account for all the entanglement loss of qubits in the intermediate stage of evolution. For example, as shown in figure 2, for small $|\alpha|^2$ there exist time windows within which the sum $N_s^1s_2^2s_3^3s_4^4 + N_r^1r_2^2r_3^3r_4^4$ is zero, i.e. the entanglement of both qubits and reservoirs simultaneously vanish. To understand such simultaneous blank entanglement windows, a more detailed exploration for the process of entanglement evolution is necessary.

With respect to the bipartition $k|N-k$, the condition for ESD of the qubits is $\Lambda^S_k(t) = 0$ and that for ESB of the corresponding reservoirs is $\Lambda^R_k(t) = 0$. A noteworthy property, which can be checked from equations (24) and (26), is the $k$-independence of the time at which qubits’ ESD (reservoirs’ ESB) occurs, implying simultaneous vanishing (arising) of the qubits’ (reservoirs’) entanglement with respect to all the possible bipartitions $k|N-k$. Also, as we have shown above, ESD and ESB occur under the same condition $|\beta/\alpha| > 1$ ($|\alpha|^2 < 1/2$), i.e. ESD necessarily triggers ESB and vice versa. Interestingly, ESB may occur before, simultaneously or even after ESD if $|\beta/\alpha| < \sqrt{2^N}$, $|\beta/\alpha| = \sqrt{2^N}$ or $|\beta/\alpha| > \sqrt{2^N}$, respectively. Existence of the above-mentioned simultaneous blank entanglement windows is now understood: this is the situation when ESB occurs after ESD. However, there is still another unclarified question: where has entanglement gone inside the blank entanglement windows? There should be other formats of entanglement besides the considered bipartite one. We shall make this clear by virtue of our generalized monogamy inequalities and the PREs established in subsection 2.1.
Let us consider the case when ESB occurs after ESD. Thus, for concreteness, we set $\alpha = \sqrt{\frac{1}{34}}$ and $\beta = \sqrt{\frac{33}{34}}$. In figure 3, for the same system of four qubits interacting with four independent reservoirs, we plot the time evolution of $N_{s_1}^2$, $N_{s_1}^2$, $\Pi_{s_1}^2$, $\Pi_{s_1}^2$, $N_{s_1}^2$, $N_{s_1}^2$, $\Pi_{s_1}^2$, $\Pi_{s_1}^2$, which correspond to $k = 1$, and $N_{s_1}^2$, $N_{s_1}^2$, $\Pi_{s_1}^2$, $\Pi_{s_1}^2$, which correspond to $k = 2$, in both Markovian and non-Markovian regimes. Here, although we do not plot the entanglement between remote qubits and reservoirs (i.e. $N_{s_1}^2$, $N_{s_1}^2$, $N_{s_1}^2$, $N_{s_1}^2$), these can also vanish in the simultaneous absence of qubits’ and reservoirs’ entanglement. From figure 3(a) for the Markovian reservoir, one observes that, although precise evolution of the concerned quantities is affected by $k$, there is a feature that is independent of $k$: all the squared negativities vanish simultaneously at some moment $t_1$ and then become simultaneously again greater than zero at another later moment $t_2$. In other words, within the time interval $[t_1, t_2]$ all the possible entanglements between remote sub-blocks are absent identically. This, however, does not mean that the overall system loses all its entanglement. By looking at the evolution of the PRE with $k = 1$ (i.e. $\Pi_{s_1}^2$) and that with $k = 2$ (i.e. $\Pi_{s_1}^2$), we recognize that both of them first increase and then reach a maximum value of $4|\alpha\beta|^2$ at the moment $t_1$ when all the negativities vanish. Interestingly enough, the maximum value of the PREs remains invariant during the time from $t_1$ to $t_2$, but decreases after the moment $t_2$ when the negativities start to increase from zero. Such behavior of the negativities and the PREs can be interpreted by the time-dependent change of entanglement formats. At $t = 0$ the PREs are zero and the total entanglement amount is $4|\alpha\beta|^2 = N_{s_1}^2 (0)$, which is solely of bipartite format. As time

**Figure 3.** Time evolution of $N_{s_1}^2$ (red solid line), $N_{s_1}^2$ (black solid line) and $\Pi_{s_1}^2$ (green solid line) corresponding to bipartition of $k = 1$, and that of $N_{s_1}^2$ (red dotted line), $N_{s_1}^2$ (black dotted line) and $\Pi_{s_1}^2$ (green dotted line) for $k = 2$ in (a) Markovian ($\Gamma = 5R$) and (b) non-Markovian ($\Gamma = 0.1R$) cases. The four qubits $s_1, s_2, s_3, s_4$ are initially in the GHZ-type state with $\alpha = \sqrt{\frac{1}{34}}$ and $\beta = \sqrt{\frac{33}{34}}$. 

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passes the PREs become greater than zero, implying a partial transfer of entanglement from bipartite to multipartite formats. Since in the time interval $t_2 - t_1$ the maximum value of the PREs is exactly $4|\alpha \beta|^2$ and all the negativities vanish, the phenomenon can be explained by a full transfer of all the bipartite entanglement formats to the multipartite ones. Only for time later than $t_2$ are the multipartite entanglement formats gradually transferred back to the bipartite ones, which are the eventual format of entanglement in the long-time limit possessed solely by the reservoir subsystem. As for the non-Markovian case, more blank entanglement windows appear such as $[t_1, t_2]$, $[t_3, t_4]$, $[t_5, t_6]$ and $[t_7, t_8]$, as seen from figure 3(b). The strong coupling strength in this regime favors virtual energy exchange between the qubits and the reservoirs, leading to oscillatory behavior of all the quantities concerned. Therefore, the change of entanglement formats from bipartite to multipartite and vice versa happens more frequently.

3.3. W-type initial states of $N$ qubits

We next deal with the entanglement dynamics using the same model of decoherence but with the $N$-qubit system prepared in the W state (9). Thus the total qubits–reservoir state at $t = 0$,

$$|\Psi(0)\rangle_{s_1...s_Nr_1...r_N} = \frac{1}{\sqrt{N}}(|10...00\rangle + |01...00\rangle + ... + |00...01\rangle)_{s_1...s_N} \otimes (\underline{00}...\underline{00})_{r_1...r_N},$$

(31)

evolves at $t > 0$ to

$$|\Psi(t)\rangle_{s_1...s_Nr_1...r_N} = \frac{1}{\sqrt{N}}[(C_0(t)|1\underline{0}) + C(t)|1\underline{1})]_{s_1r_1} |0\underline{0}\rangle_{s_2r_2} ... |0\underline{0}\rangle_{s_Nr_N} + |0\underline{0}\rangle_{s_1r_1} (C_0(t)|1\underline{0}) + C(t)|1\underline{1})]_{s_Nr_N}.$$

(32)

The reduced density matrix of qubits $\rho_{s_1...s_N}(t)$ and of the corresponding reservoirs $\rho_{r_1...r_N}(t)$ read

$$\rho_{s_1...s_N}(t) = \frac{1}{N} \left[ N C^2(t)|00...00\rangle_{s_1...s_N} \langle 00...00| + C_0^2(t)|10...00\rangle_{s_1...s_N} \langle 10...00| 
+ |01...00\rangle_{s_1...s_N} \langle 01...00| + ... + |00...01\rangle_{s_1...s_N} \langle 00...01| 
+ C^2_0(t)|11...00\rangle_{s_1...s_N} \langle 11...00| + ... + |00...10\rangle_{s_1...s_N} \langle 00...10| + h.c. \right]$$

(33)

and

$$\rho_{r_1...r_N}(t) = \frac{1}{N} \left[ C_0^2(t)(|\underline{10}...\underline{00}\rangle_{r_1...r_N} \langle \underline{10}...\underline{00}| + |\underline{01}...\underline{00}\rangle_{r_1...r_N} \langle \underline{01}...\underline{00}| + ... 
+ |\underline{00}...\underline{01}\rangle_{r_1...r_N} \langle \underline{00}...\underline{01}| + N C_0^2(t)|\underline{00}...\underline{00}\rangle_{r_1...r_N} \langle \underline{00}...\underline{00}| 
+ C^2(t)(|\underline{10}...\underline{00}\rangle_{r_1...r_N} \langle \underline{10}...\underline{00}| + ... + |\underline{00}...\underline{10}\rangle_{r_1...r_N} \langle \underline{00}...\underline{10}| + h.c. \right].$$

(34)

For the bipartition $k |N - k$, the minimal eigenvalues of the partial transposes of $\rho_{s_1...s_N}(t)$ and $\rho_{r_1...r_N}(t)$ are

$$\Lambda_k^*(t) = \frac{1}{2} \left( |C(t)|^2 - \sqrt{|C(t)|^4 + \frac{4k(N-k)}{N^2}|C_0(t)|^4} \right).$$

(35)
Figure 4. Time evolution of $N_{s1s2s3s4}^2$ (red solid line), $N_{r1r2r3r4}^2$ (black solid line) and $\Pi_{s1s2s3s4}$ (green solid line) corresponding to bipartition of $k = 1$, and that of $N_{s1s2s3s4}^2$ (red dotted line), $N_{r1r2r3r4}^2$ (black dotted line) and $\Pi_{s1s2s3s4}$ (green dotted line) for $k = 2$ in (a) Markovian ($\Gamma = 5R$) and (b) non-Markovian ($\Gamma = 0.1R$) cases for the qubits initially in the W state.

\[
\Lambda_k^R(t) = \frac{1}{2} \left( |C_0(t)|^2 - \sqrt{|C_0(t)|^4 + \frac{4k(N-k)}{N^2}|C(t)|^4} \right),
\]
respectively. Obviously, both $\Lambda_k^S(t)$ and $\Lambda_k^R(t)$ in equations (35) and (36) are negative for $0 < t < \infty$, implying that the entanglement of the qubit subsystem decays asymptotically in parallel with an asymptotic growth of the entanglement of the reservoir subsystem from $t = 0$. In other words, the qubits in the W state and the corresponding reservoirs do not exhibit ESD and ESB at all. Consider again for simplicity the system of four qubits plus four reservoirs. The overall dynamical scenario in the case of the W state is represented in figure 4 for both Markovian and non-Markovian cases. Of interest here again is the fact that the entanglement format changes in time. As recognized from the evolution of the PREs $\Pi_{s1s2s3s4}$ and $\Pi_{r1r2r3r4}$, they also increase from zero towards a maximum value and then vanish asymptotically. Note that the maxima of the PREs are higher for larger $k$, but they are reached simultaneously regardless of the concrete value of $k$. It can be verified that the crossing point of all the bipartite entanglements between any two sub-blocks across a definite bipartition (i.e. $N_{s1s2s3s4}^2, N_{r1r2r3r4}^2, N_{s1s2s4}^2, N_{r1r2r4}^2, N_{s1s3s4}^2, N_{r1r3r4}^2$ for $k = 1$ and $N_{s1s2s3s4}^2, N_{r1r2r3r4}^2, N_{s1s2s4}^2, N_{s1s3s4}^2, N_{r1r2r4}^2, N_{r1r3r4}^2$ for $k = 2$) corresponds to the minimal values of their sums, which are reached exactly at the moment the PREs become maximal. Physically, this manifests a pronounced transition between the bipartite and multipartite entanglement formats involving different partitions of the system in the time evolution. Remarkably, all the time (i.e., for $0 < t < \infty$) the system exhibits coexistence of both bipartite and multipartite entanglement formats. This is in clear contrast with the case of the GHZ-type
state for which there is a time interval during which only multipartite entanglement formats survive. Continuous exchange of entanglement formats also occurs in the non-Markovian case but with oscillatory character, as evident from figure 4(b).

4. Conclusion

In summary, we have presented a simple multiqubit entanglement quantifier $\mathcal{N}$, equation (1), based on the idea of bipartitions and the measure of negativity. A particular advantage of $\mathcal{N}$ is its ability to detect the global separability of a multiqubit system, namely, it reaches zero exactly when a multiqubit system becomes fully separable. We have established generalized monogamy relations and associated residual entanglements in terms of (the squares of) negativities by taking into account arbitrary bipartitions of a multipartite system. This generalization is natural and necessary when one aims at accounting for entanglements due to arbitrary partitions in a multipartite system. The previously known entanglement monogamy relations [14, 17] treated only the one-party cut, i.e. bipartite entanglements between one party and the remaining ones. In contrast, in establishing our generalized monogamy relations multi-party cuts were taken into account and thus entanglements between two sub-blocks as well as between two individual qubits can be assessed consistently. Note that, as it should be, our inequalities (3) and (4) reduce to that derived in [17] when $k = 1$. The physical meaning of our inequalities is that for an $N$-qubit system the entanglement between two qubit blocks specified by a bipartite cut $k|N - k$ is the upper bound for the sum of entanglement between two qubit sub-blocks across the cut, which is in turn the upper bound for the sum of entanglement between two individual qubits also across the cut. We have also studied the entanglement dynamics of a system of $N$ qubits which are prepared at $t = 0$ either in the GHZ-type state or in the W state and interact at $t > 0$ with $N$ independent Markovian or non-Markovian reservoirs. By using the defined $\mathcal{N}$, we have calculated the global dynamics of the qubits’ subsystem and of corresponding reservoirs. In comparison with Markovian reservoirs, the memory effect of non-Markovian reservoirs can rebirth the qubits’ dead entanglement. Furthermore, thanks to the generalized monogamy relations and associated residual entanglement, we have been able to demonstrate in detail a whole entanglement evolution process with respect to the transition between different entanglement formats. We have observed qualitatively different entanglement dynamics between the GHZ-type state and the W one. For the GHZ-type state, when the ESD of the qubits occurs before the ESB of the reservoirs the following scenario was found: bipartite and multipartite entanglements coexist prior to the moment of the ESD and after the moment of the ESB, but during the time period within the two moments only multipartite entanglements survive. This means that in this period only multipartite entanglement accounts for the total correlations of the whole system, including qubits and reservoirs. In contrast, the W state behaves quite distinctly: the coexistence of bipartite and multipartite entanglements exists all the time, although multipartite entanglements dominate around the moment of crossing between bipartite entanglements of qubits’ and reservoirs’ subsystems. These studies would shed some light on understanding multiqubit entanglement from both mathematical and dynamical points of view. As a final remark, we would like to note that the entanglement measure and generalized monogamy relations established in this paper are quite universal in the sense that they can be used to study more general multipartite entangled (pure and mixed) states under different initial conditions ($T = 0$ and $T > 0$) of the reservoirs. These issues deserve further separate research.
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References

[1] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[2] Wootters W K 1998 Phys. Rev. Lett. 80 2245
[3] Vidal G and Werner R F 2002 Phys. Rev. A 65 032314
[4] Carvalho A R R, Minter F and Buchleitner A 2004 Phys. Rev. Lett. 93 230501
[5] Minter F, Kuś M and Buchleitner A 2005 Phys. Rev. Lett. 95 260502
[6] Dür W and Cirac J I 2000 Phys. Rev. A 61 042314
[7] Aolita L, Chaves R, Cavalcanti D, Acín A and Davidovich L 2008 Phys. Rev. Lett. 100 080501
[8] Simon C and Kempe J 2002 Phys. Rev. A 65 052327
[9] Dür W and Beigel H J 2004 Phys. Rev. Lett. 92 180403
[10] Hein M, Dür W and Briegel H J 2005 Phys. Rev. A 71 032350
[11] Man Z X, Xia Y J and An N B 2008 Phys. Rev. A 78 064301
[12] An N B and Kim J 2009 Phys. Rev. A 79 022303
[13] Coffman V, Kundu J and Wootters W K 2000 Phys. Rev. A 61 052306
[14] Osborne T J and Verstraete F 2006 Phys. Rev. Lett. 96 220503
[15] Kim J S, Das A and Sanders C 2009 Phys. Rev. A 79 012329
Yu C S and Song H S 2005 Phys. Rev. A 71 042331
Ou Y C 2007 Phys. Rev. A 75 034305
[16] Hiroshima T, Adesso G and Illuminati F 2007 Phys. Rev. Lett. 98 050503
[17] Ou Y C and Fan H 2007 Phys. Rev. A 75 062308
[18] Huang J H and Zhu Y S 2008 Phys. Rev. A 78 012325
Chi D P et al 2008 J. Math. Phys. 49 112102
[19] Yu T and Eberly J H 2004 Phys. Rev. Lett. 93 140404
Yu T and Eberly J H 2006 Phys. Rev. Lett. 97 140403
Eberly J H and Yu T 2007 Science 316 555
Yu T and Eberly J H 2009 Science 323 598
Dodd P J and Halliwell J J 2004 Phys. Rev. A 69 052105
Santos M F, Milman P, Davidovich L and Zagury M 2006 Phys. Rev. A 73 040305
Yönaç M, Yu T and Eberly J H 2006 J. Phys. B: At. Mol. Opt. Phys. 39 S621
Yönaç M, Yu T and Eberly J H 2007 J. Phys. B: At. Mol. Opt. Phys. 40 S45
Sainz I and Björk G 2007 Phys. Rev. A 76 042313
Man Z X, Xia Y J and An N B 2008 J. Phys. B: At. Mol. Opt. Phys. 41 085503
Man Z X, Xia Y J and An N B 2008 J. Phys. B: At. Mol. Opt. Phys. 41 155501
Ficek Z and Tanašt R 2006 Phys. Rev. A 74 024304
Zyczkowski K, Horodecki P, Horodecki M and Horodecki R 2001 Phys. Rev. A 65 012101
Yu T and Eberly J H 2006 Opt. Commun. 264 393
Liu R F and Chen C C 2006 Phys. Rev. A 74 024102
Cui H T, Li K and Yi X X 2007 Phys. Lett. A 365 44
Yu T and Eberly J H 2007 Quantum Inform. Comput. 7 459
[20] Almeida M P et al 2007 Science 316 579
    Salles A et al 2008 Phys. Rev. A 78 022322
    Laurat J, Choi K S, Deng H, Chou C W and Kimble H J 2007 Phys. Rev. Lett. 99 180504
[21] Bellomo B, Franco R Lo and Compagno G 2007 Phys. Rev. Lett. 99 160502
[22] Bellomo B, Franco R Lo and Compagno G 2008 Phys. Rev. A 77 032342
[23] Mazzola L, Maniscalco S, Piilo J, Suominen K A and Garraway B M 2009 Phys. Rev. A 79 042302
[24] López C E, Romero G, Lastra F, Solano E and Retamal J C 2008 Phys. Rev. Lett. 101 080503
[25] Greenberger D M, Horne M A and Zeilinger A 1989 Bell’s Theorem, Quantum Theory, and Conceptions of the Universe (Dordrecht: Kluwer)
[26] Dür W, Vidal G and Cirac J I 2000 Phys. Rev. A 62 062314
[27] Bennett C H et al 1996 Phys. Rev. Lett. 76 722
[28] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Phys. Rev. A 54 3824
[29] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Rev. Mod. Phys. 81 865