ON POINCARÉ SERIES ASSOCIATED WITH LINKS OF NORMAL SURFACE SINGULARITIES

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Abstract. Assume that $M$ is a rational homology sphere plumbed 3–manifold associated with a connected negative definite tree $T$. If we consider the topological Poincaré series ($N08$) associated with $T$, one can attach to it a counting function which expresses topological information, e.g. the Seiberg–Witten invariant of $M$ ($N11$), from the series.

In this article, we study the counting function interpreting as an alternating sum of coefficient functions associated with some Taylor expansions. The method is different from the one in $[LN14]$, it is motivated by a theorem of Szenes and Vergne $[SzV03]$ which expresses these coefficient functions in terms of Jeffrey–Kirwan residues. This will be used to prove the uniqueness of the quasipolynomiality inside a special cone associated with the topology of $M$. We also put in this context a surgery formula appeared in $[N11]$ for the counting function, and in $[BN10]$ for the Seiberg–Witten invariant of $M$.

1. Introduction

1.1. Let $M$ be a closed oriented plumbed 3–manifold associated with a connected negative definite plumbing graph $T$ with vertices $V$. $M$ can be realized as the link of a complex normal surface singularity $(X, 0)$ and it determines completely the topology of the surface $X$ at the singular point. Hence it is natural to ask how the invariants of the analytic type of $(X, 0)$ can be determined from the topological invariants of $M$ (cf. Artin–Laufer program, see $[N99]$, $[L13]$).

One of the strategy to attack these question would be to study the topological analogues of the analytic invariants and understand their connections.

A good example is the Seiberg–Witten invariant conjecture of Némethi and Nicolaescu $[NN02]$, which targets the connections of the geometric genus of the possible analytic types with the Seiberg–Witten invariant of the given topological type $M$.

Another concept, which is highly related to the Seiberg–Witten invariant conjecture, is the theory of Hilbert–Poincaré series associated with the analytic and topological types of the singularity $(X, 0)$. Series of articles of Campillo, Delgado and Gusein-Zade (cf. $[CDGZ04, CDGZ08]$) studied the Poincaré series $P(t)$ associated with a divisorial multi-index filtration of functions on $(X, 0)$. Némethi $[N08]$ generalized the formulas of this concept. Moreover, motivated by the rational function presentation of $P(t)$ associated with a rational singularity (cf. $[CDGZ04]$), he defined the topological counterpart, the multivariable topological Poincaré series $Z(t)$ which is expressed from the combinatorics of the graph $T$. He showed the identity ‘$Z = P$’ in some ‘nice’ cases and constructed examples with $Z \neq P$, showing geometric obstruction for the identity, in...
general. In the sequel we provide some details on the definition and the topological information
given by $Z$.

1.2. We assume that $M$ is a rational homology sphere, or, equivalently $\mathcal{T}$ is a tree and genera
of the cores in the plumbing are zero. Let $\tilde{X}$ be the plumbed 4–manifold associated with $\mathcal{T}$, hence
$\partial\tilde{X} = M$. Its second homology $L := H_2(\tilde{X}, \mathbb{Z})$ is freely generated by the 2–spheres $\{E_v\}_{v \in V}$, and
its second cohomology $L' := H^2(\tilde{X}, \mathbb{Z})$ by the (anti)dual classes $\{E_v^*\}_{v \in V}$; the intersection form
$I = (\_,\_)$ embeds $L$ into $L'$. Set $t^2 := (t', t')$.

Let $K \in L'$ be the canonical class, $\sigma_{\text{can}}$ the canonical $\text{spin}^c$–structure on $\tilde{X}$ with $c_1(\sigma_{\text{can}}) = −K$, and $\sigma_{\text{can}} \in \text{Spin}^c(M)$ its restriction on $M$. Set $H := H_1(M, \mathbb{Z}) = L' / L$. Then $\text{Spin}^c(M)$ is
an $H$–torsor, with action denoted by $\ast$. We denote by $\text{sw}_\sigma(M)$ the Seiberg–Witten invariant of $M$
indexed by the $\text{spin}^c$–structure $\sigma$ of $M$.

Next, consider the multivariable Taylor expansion $Z(\mathbf{t}) = \sum_{l'} p_{l'} t'^l$ at the origin of
$$ \prod_{v \in V} (1 - t^{E_v^*})^{\delta_v - 2}, $$
where we write $t'' = \prod_{v \in V} t_v^{l_v}$ for any $l' = \sum_{v \in V} l_v E_v \in L'$, and $\delta_v$ is the valency of $v$. This lives in $\mathbb{Z}[L']$, the submodule of formal power series $\mathbb{Z}[\mathbb{Z}^{1/4}]$ in variables $\{t_v^{\pm 1/4}\}_v$, where $d = \det(-I)$. It has a natural decomposition $Z(\mathbf{t}) = \sum_{h \in H} Z_h(\mathbf{t})$, where $Z_h(\mathbf{t}) = \sum_{|l'| = h} p_{l'} t'^l$ ($[l']$ denotes the class of $l'$).

The first main achievement on the topological model is the result of [NT1], which shows that the
Seiberg–Witten invariant can be expressed from a counting function of the coefficients of $Z_h$
associated with a special truncation coming from a partial ordering $'\geq'$ on $L'$ (see [NT2]). More
precisely, if $l' \in -K + \text{int}(\mathbb{Z}_{\geq 0}(E_v^*))$ then
$$ \sum_{l \in L, l \geq 0} p_{l'+l} = -\frac{(K + 2l')^2 + |V|}{8} - \text{sw}_{[-l']} \ast \sigma_{\text{can}}(M). \tag{1} $$
Notice that to guarantee the polynomiality (right hand side) of the counting function (left hand
side), $l'$ should sit in the affine cone given above.

This condition is made precise in the work of [LNT3], where the authors define the multi-
variable equivariant periodic constant $pc_{\mathcal{T}}(Z) = pc(Z_h)$ of $Z(\mathbf{t})$, extending the one-variable case
introduced in [NO09, O08]. They develop an equivariant multivariable Ehrhart theory which explains
the (quasi)polynomial behaviour of the counting function and leads to the generalization
of the periodic constant. This method constructs a (conical) chamber decomposition of the parameter
space $\mathbb{R}\langle E_v \rangle_{v \in V}$ and it associates with any chamber an Ehrhart type quasipolynomial,
which describes the counting function in an affine subcone of the given chamber. In general, the
chamber decomposition divides the Lipman cone. Hence, the disadvantage of this approach is
that the counting function is realized by different quasipolynomials inside the Lipman cone, which
associate different periodic constants as well. However, [NT1] suggests that all the quasipolynomi-
als (and periodic constants) appearing in this cone are equal and interpret the Seiberg–Witten
invariant.
It is natural to ask whether there exists a more ‘compact’ view of the counting function in the sense that it shows the uniqueness of the quasipolynomiality inside the Lipman cone.

1.3. The main goal of this article is to propose a different interpretation of the counting function of $Z_h(t) = \sum_{[l'] = h} p_{l'} t^{l'}$. More precisely, one can define a counting function $Q_h(l') = \sum_{[l'']} [l''] = h p_{l''}$ associated with the coefficients of $Z_h(t)$, which is a finite sum since $Z$ is supported on the Lipman cone (cf. (4)). In Section 3.1 we represent this function as

$$(2) \quad Q_h(l') = \sum_{\emptyset \neq I \subseteq V} (-1)^{|I|-1} \text{Coeff}(R(t_I), t_{l' I})^{−1} \left(\prod_{v \in I} (1 - t_{E_v}^{E_v})^{-2} \cdot \prod_{v \in I} t_{E_v}^{E_v} \cdot t_{l' I}^{l' I}\right),$$

where $r_{h-[l']} \in \sum_{v \in V} [0, 1) E_v$ is a lift of $h - [l']$, and $\text{Coeff}(T[R(t_I)], t_{l' I})$ is the coefficient of $t_{l' I}$ in the Taylor expansion of the rational function $R$ with variable $t_{l' I} := \prod_{v \in I} t_{l' I}^{l' I}$ for any $l' = \sum_{v \in V} l_v E_v \in L'$.

Using this interpretation, one can connect the counting function $Q_h(l')$ with Jeffrey–Kirwan residues (see Section 3.2) by a theorem of Szenes and Vergne [SzV03]. It concludes that these coefficient functions from (2) are equal with sums of Jeffrey–Kirwan residues and they are quasipolynomials on a special cone.

We would like to emphasize that this connection has two messages: it gives an explicit formula for the computation of quasipolynomials of this type. Moreover, using this theorem, in Section 3.3 we prove the uniqueness of the quasipolynomial associated with the counting function inside an affine subcone of the Lipman cone. In particular, this is a polynomial on a sufficiently sparse sublattice of $L'$, which confirms the form of Equation (1).

This approach is interesting from other point of view as well. The result of [N11, Theorem 3.2.13] deduces a recursion formula for the quasipolynomial associated with the counting function. The essence of Section 4 is to show how to use the methods of Section 3 in order to deduce the recursion first for the counting function, then for the quasipolynomial associated with it on a suitable affine cone. In particular, we discuss how the recursion behaves on the level of periodic constants and we compare that with the Braun–Némethi surgery formula for the Seiberg–Witten invariant.

The proof of the recurrence is based on a decomposition of the sum from (2) in terms of the graph $\mathcal{T}$, and on a projection property (Lemma 16) of the coefficient functions.

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2. Preliminaries

2.1. Links of normal surface singularities.

2.1.1. We consider a connected negative definite plumbing graph $\mathcal{T}$ with vertices $V$, which determines an oriented plumbed 3–manifold $M$. The graph $\mathcal{T}$ can be realized as the resolution graph of some complex normal surface singularity $(X, 0)$, and $M$ will be the link of $(X, 0)$. In the sequel, we assume that $M$ is a rational homology sphere which is equivalent to the fact that $\mathcal{T}$ is a tree and all the genus decorations are zero.
Let \( \pi : \tilde{X} \to X \) be a good resolution with dual graph \( \mathcal{T} \). Then \( \tilde{X} \) is the smooth 4–manifold with boundary \( M \) which can be obtained by plumbing disc bundles along \( \mathcal{T} \) too. \( L := H_2(\tilde{X}, \mathbb{Z}) \) is freely generated by the classes of the irreducible exceptional curves \( \{E_v\}_{v \in \mathcal{V}} \) (or the cores of the plumbing construction). \( L \) is a lattice with a nondegenerate negative definite intersection matrix \( I := \langle [E_v, E_w] \rangle_{v, w} \).

If \( L' \) denotes \( H^2(\tilde{X}, \mathbb{Z}) \), then the intersection form provides an embedding \( L \hookrightarrow L' \) with finite factor \( H \cong H_1(M, \mathbb{Z}) \cong H^2(\partial \tilde{X}, \mathbb{Z}) \); \( [l'] \) denotes the class of \( l' \). The form \( (, ) \) extends to \( L' \) (since \( L' \subset L \otimes \mathbb{Q} \)). The module \( L' \) over \( \mathbb{Z} \) is freely generated by the (anti)duals \( \{E^*_v\}_{v \in \mathcal{V}} \), where we prefer the convention \( (E^*_v, E_w) = -\delta_{vw} \) (the negative of the Kronecker symbol). The negative inverse of \( I \) has entries \( (1 - I)^{-1}_{vw} = -(E^*_v, E^*_w) \), all of them are positive. These are the entries of the vectors \( E^*_v \) in the \( \{E_v\}_{v \in \mathcal{V}} \) basis. Using the notation \( A \) for the nondegenerate positive definite matrix \( -I \), we have the relation

\[
[E^*_v]_{v \in \mathcal{V}} \cdot A = [E_v]_{v \in \mathcal{V}},
\]

which will be used frequently in the sequel.

2.1.2. For any \( l_1, l_2 \in L \otimes \mathbb{Q} \) one writes \( l_1 \geq l_2 \) if \( l_1 - l_2 = \sum_{v \in \mathcal{V}} r_v E_v \) with all \( r_v \geq 0 \). Denote by \( \mathcal{S}' \) the Lipman cone \( \{l' \in L' : (l', E_v) \leq 0 \ \text{for all} \ v \} \). It is generated over \( \mathbb{Z}_{\geq 0} \) by the elements \( E^*_v \). Since all the entries of \( E^*_v \) are strictly positive, for any fixed \( a \in L' \) one has:

\[
\{l' \in \mathcal{S}' : l' \not\in a\}
\]

is finite.

For any \( J \subseteq \mathcal{V} \) let \( \delta_v, J \) be the valency of the vertex \( v \) in the complete subgraph \( \mathcal{T}_J \) with vertices \( J \) of \( \mathcal{T} \). We will use shorter notation \( \delta_v = \delta_v, \mathcal{V} \). We also use the notation \( \mathcal{E} = \{v \in \mathcal{V} : \delta_v = 1\} \) for the subset of end vertices of \( \mathcal{T} \).

For more details on the links and resolution graphs of normal surface singularities see e.g. [N99, N05, N07].

2.1.3. *Spin\(^c\)–structures.* Let \( \tilde{\sigma}_{\text{can}} \) be the canonical *spin\(^c\)–structure* on \( \tilde{X} \), its first Chern class \( c_1(\tilde{\sigma}_{\text{can}}) = -K \in L' \), where \( K \) is the canonical element in \( L' \) defined by the adjunction formulas

\[
(K + E_v, E_v) + 2 = 0 \quad \text{for all} \ v \in \mathcal{V}
\]

(cf. [GS99, p. 415]). The set of *spin\(^c\)–structures* \( \text{Spin}^c(\tilde{X}) \) of \( \tilde{X} \) is an \( L' \)-torsor; if we denote the \( L' \)-action by \( l' \ast \tilde{\sigma} \), then \( \text{cl}(l' \ast \tilde{\sigma}) = c_1(\tilde{\sigma}) + 2l' \). Furthermore, all the *spin\(^c\)–structures* of \( M \) are obtained by restrictions from \( \tilde{X} \). \( \text{Spin}^c(M) \) is an \( H \)-torsor, compatible with the restriction and the projection \( L' \to H \). The canonical *spin\(^c\)–structure* \( \sigma_{\text{can}} \) of \( M \) is the restriction of the canonical *spin\(^c\)–structure* \( \tilde{\sigma}_{\text{can}} \) of \( \tilde{X} \).

2.1.4. *Distinguished representatives.* For any class \( h \in H = L'/L \) we consider two distinguished representatives: the unique element \( r_h \in L' \) characterized by \( r_h = \sum_v [0, 1] E_v \) and \( [r_h] = h \), and the unique minimal element \( s_h \) of \( \{l' \in L' : [l'] = h\} \cap \mathcal{S}' \) (cf. [N05, 5.4]). One has \( s_h \geq r_h \), and in general \( s_h \neq r_h \) (for an example see [N07, 4.5]). Both representatives appear from different perspectives in the ‘normalization’ term of the Seiberg–Witten invariants (e.g. [BN10] and [N05]). In Section 4.3 we will discuss their behaviour under certain projections.
2.2.2. Seiberg-Witten invariants and the Poincaré series. For any closed 3-manifold $M$ one can associate the Seiberg–Witten invariant $\text{sw}(M) : \text{Spin}^c(M) \to \mathbb{Q}$, $\sigma \mapsto \text{sw}_\sigma$ (cf. [Lim00, Nic04]). In general, it is difficult to compute the invariant using its very definition. Therefore, in the last several years several combinatorial expressions were established regarding the Seiberg–Witten invariants.

For rational homology spheres, one direction was opened by the result of Nicolaescu [Nic04] showing that $\text{sw}(M)$ is equal with the Reidemeister–Turaev torsion normalized by the Casson–Walker invariant. In the case when $M$ is a negative definite plumbed rational homology sphere, combinatorial formula for Casson–Walker invariant in terms of the plumbing graph can be found in Lescop [Les96]. The Reidemeister–Turaev torsion is determined by Némethi and Nicolaescu [NN02] using Dedekind–Fourier sums, which is still hard to determine in most of the cases.

Another direction is a combinatorial interpretation for $\text{sw}(M)$ from the Poincaré series $Z(t)$, which uses qualitative properties of the coefficients. We define the counting function of the coefficients of $Z_h(t) = \sum_{[l'] = h} p_{l'} t^{l'}$ by

$$l' \mapsto Q_h(l') := \sum_{l' \in V, [l'] = h} p_{l'}.$$ 

Notice that the above sum is finite by (4). The interpretation is given by the following theorem.

**Theorem 1** (Némethi [N11]). Fix some $l' \in -K + \text{int}(S')$. Then

$$Q_{[l']}(l') = -\frac{(K + 2l')^2 + |V|}{8} - \text{sw}_{-[l'] \sigma_{can}}(M).$$
If we write \( l' = l + r_h \) with \( l \in L \) and \( h = |l'| \), The right hand side of (7) can be seen as a multivariable quadratic polynomial on \( L \) with constant term
\[
\frac{(K + 2r_h)^2 + |V|}{8} = \text{sw}_{-h^{\ast}\sigma_m}(M),
\]
which we call the \( r_h \)-normalized Seiberg–Witten invariant of \( M \) associated with the class \( h \in H \). Similarly, one can also define the \( s_h \)-normalized Seiberg–Witten invariant by writing \( l' = l + s_h \) for some \( l \in L \) and \( h = |l'| \).

Remark 2. It is important to emphasize that the identity (7) was motivated by a similar analytic identity which targets the geometric genus of the complex normal surface singularity \((X, 0)\) whose link is \( M \) (cf. [N12, O08]).

2.2.3. Periodic constants and the Ehrhart theoretical approach. Once we have the identification given by (7), it is interesting to understand how one recovers this information at the level of series (or rational function) in order to obtain closed formulas for the Seiberg–Witten invariants of \( M \).

In the sequel we briefly recall the main ideas from the case of one-variable series introduced by Némethi and Okuma in [NO09, O08], and then we state the definition of the multivariable equivariant periodic constant specialized to the series \( Z(t) \).

Let \( S(t) = \sum_{l \geq 0} c_l t^l \in \mathbb{Z}[[t]] \) be a formal power series with one variable and assume that for some \( p \in \mathbb{Z}_{>0} \) the counting function \( Q_p(n) := \sum_{i=0}^{m-1} c_i \) is a polynomial in \( n \). Then the constant term \( Q_p(0) \) is independent of \( p \) and it is called the periodic constant \( \text{pc}(S) \) of the series \( S \). One can also understand the meaning of the periodic constant on the level of series: if \( S(t) \) is the Hilbert series associated with a graded algebra/vector space \( A = \oplus_{i \geq 0} A_i \) (i.e. \( c_i = \dim A_i \)), and the series \( S \) admits a Hilbert quasipolynomial \( Q(l) \) (that is, \( c_l = Q(l) \) for \( l \gg 0 \)), then the periodic constant of the ‘regularized series’ \( S_{\text{reg}}(t) := \sum_l Q(l) t^l \) is zero. Hence the periodic constant of \( S(t) \) measures exactly the difference between \( S(t) \) and \( S_{\text{reg}}(t) \), that is \( \text{pc}(S) = (S - S_{\text{reg}})(1) \) collecting all the anomalies of the starting elements of \( S \). When \( S(t) \) is a rational function of the form \( B(t)/A(t) \) with \( A(t) = \prod_i (1 - t^{a_i}) \) one has a decomposition \( S(t) = P(t) + \frac{D(t)}{A(t)} \), where \( P \) and \( D \) are polynomials and \( D(t)/A(t) \) has negative degree. In this case, one can prove that the \( \text{pc}(S) \) can be interpreted and calculated by \( P(1) \). For more details on this decomposition we refer to [BN10] 7.0.2 and we give a similar equivariant interpretation in Section 3.3.

For the multivariable case we consider the settings of Sections 2.1 and 2.2.1. We take a closed real cone \( \mathcal{K} \subset L' \oplus \mathbb{R} \) whose affine closure has positive dimension and assume that there is a sublattice \( \tilde{L} \subset L' \) of finite index and \( l'_0 \in \mathcal{K} \) such that \( Q_h(l'_0) \) is a quasipolynomial for any \( \tilde{L} \cap (l'_0 + \mathcal{K}) \). Then we say that \( Z(t) \) admits a multivariable equivariant periodic constant in \( \mathcal{K} \) and it is defined by
\[
\text{pc}^\mathcal{K}_h(Z) := Q_h(0) \quad \text{for any } h \in H.
\]
We will omit \( \mathcal{K} \) from the notations when it is clear from the context.
Remark 3. The definition does not depend on the choice of the sublattice $\tilde{L}$, which corresponds to the choice of $p$ in the one-variable case. This is responsible for the name ‘periodic’ in the definition.

In order to see the quasipolynomial behaviour of $Q$ in some special chambers, László and Némethi [LN14] developes a theory which associates with the denominator of $f(t)$ a (non-convex) polytope situated in the lattice $L'$. Taking into account the representation of $H$, one can consider the multivariable equivariant Ehrhart piecewise quasipolynomial associated with these data. This construction also determines a conical chamber decomposition of the parameter space $L' \otimes \mathbb{R}$. Then the numerator of $f$ specializes an affine cone inside any chamber where the counting function can be represented using the Ehrhart quasipolynomial. If the chosen chamber belongs to the Lipman cone, the theory also shows that the first three top degree coefficients carry topological information, including the Seiberg–Witten invariants of the manifold $M$. More details about these construction can be found in [LN14, Sections 3 and 4].

On one hand, the chamber decomposition given by the above theory is too fine in the sense that the Lipman cone can be divided into several chambers and different quasipolynomials are attached to the counting function. On the other hand, (7) suggests that they are the same. In the next section we will give a different interpretation of the counting function which reflects to the uniqueness of the quasipolynomial using a theorem of Szenes and Vergne [SzV03].

3. Residues and the uniqueness of the quasipolynomial

3.1. Counting function as coefficients of Taylor expansions. We express the counting function as alternating sum of coefficient functions of Taylor expansions. Using this presentation one can associate quasipolynomials to the counting function in a standard manner, moreover it will also help us in their study.

For $h \in H$ we have considered the counting function $Q_h(x') = \sum_{l' \in L', \langle l' \rangle = h} p_{l'}$, $x' \in S'$ associated with the Taylor expansion $T[f(t)] = \sum_{l' \in L'} p_{l'} t^{l'}$ of the rational fraction $f(t) = \prod_{v \in \mathcal{V}} (1 - t E_v)^{s_v - 2}$ at $t = 0$. We can write

$$Q_h(x') = \sum_{l' \in \mathcal{P}(x') \cap L'} p_{l'} = \sum_{l' \in \mathcal{P}(x') \cap (h + L)} p_{l'}$$

as a counting function supported on the semi-open, bounded, concave polytope $\mathcal{P}(x') := S_{R} \setminus (x' + R_{\geq 0}(E_v)_{v \in \mathcal{V}})$, where $S_{R} = R_{\geq 0}(E_v^*)_{v \in \mathcal{V}}$. The decomposition of $\mathcal{K}(x')$ into semi-open convex polytopes $\mathcal{P}_I(x') := \{\sum_{v \in \mathcal{V}} y_v E_v \in S_{R} \mid y_v < x'_v , \forall v \in I\}$, where $x' = \sum_{v \in \mathcal{V}} x'_v E_v$ and $\emptyset \neq I \subseteq \mathcal{V}$, expressed in terms of characteristic functions as $\mathcal{P}(x') = \sum_{\emptyset \neq I \subseteq \mathcal{V}} (-1)^{|I|-1} \mathcal{P}_I(x')$, yields the following decomposition of the counting function

$$Q_h(x') = \sum_{\emptyset \neq I \subseteq \mathcal{V}} (-1)^{|I|-1} \sum_{l' \in \mathcal{P}_I(x') \cap (h + L)} p_{l'}.$$
Look at the restriction of $Q_h$ to a coset $g+L \subset L'$: for $x' \in (g+L) \cap S'$ we have $l' \in P_{\mathcal{I}}(x') \cap (h+L)$ exactly when

\[ l'_v \in x'_v + (r_{h-g})_v + \mathbb{Z}_{\leq 0}, \quad \text{for all } v \in \mathcal{I}, \]

where $r_{h-g} = \sum_{v \in V} (r_{h-g})_v E_v$ is the unique lift of $h-g \in L'/L$ in $\sum_{v \in V} [0,1] E_v$.

For any non-empty subset $\mathcal{I} \subseteq V$ let $V_\mathcal{I} = \mathbb{R}(E_i)_{i \in \mathcal{I}}$ and denote $\pi_{\mathcal{I}} : V \to V_\mathcal{I}$ the projection along $V \setminus V_\mathcal{I}$. For any $y = \sum_{v \in V} y_v E_v \in L'$ introduce notations $t^{(y)}_I = \pi_{\mathcal{I}}(t^{(y)}) = \prod_{v \in \mathcal{I}} t^{(y)}_v$. Then (9) can be reformulated as

\[ (9) \quad \pi_{\mathcal{I}}(l'_v) + z_v = \pi_{\mathcal{I}}(x'_v + r_{h-g} - \sum_{v \in V} E_v) \quad \text{for some } z_v \in \mathbb{Z}_{\geq 0}(E_i)_{i \in \mathcal{I}}, \]

hence $\sum_{v \in P_{\mathcal{I}}(x') \cap (h+L)} p_{l'_v}$ is the coefficient of $t_{x'_v + r_{h-g} - \sum_{v \in V} E_v}$ in the Taylor expansion of $\frac{f(t)}{\prod_{v \in \mathcal{I}} 1 - t_{l'_v}^{E_v}}$ at $t_\mathcal{I} = 0$, that is

\[ \sum_{v \in P_{\mathcal{I}}(x') \cap (h+L)} p_{l'_v} = \text{Coeff} \left( T \left[ \frac{f(t)}{\prod_{v \in \mathcal{I}} 1 - t_v^{E_v}} \right], t_{x'_v + r_{h-g} - \sum_{v \in V} E_v} \right) = \text{Coeff} \left( t_{x'_v + r_{h-g}} T \left[ \prod_{v \in \mathcal{I}} \frac{t_v^{E_v}}{1 - t_v^{E_v}} \right], t_{x'_v} \right). \]

Finally, with short notation $F_{\mathcal{I}}(t) := T \left[ \prod_{v \in \mathcal{I}} \frac{t_v^{E_v}}{1 - t_v^{E_v}} \right]$ we get that for any $x' \in (g+L) \cap S'$

\[ Q_h(x') = \sum_{\emptyset \neq \mathcal{I} \subseteq V} (-1)^{|\mathcal{I}|} \text{Coeff} \left( t_{x'_v + r_{h-g}} F_{\mathcal{I}}(t), t_{x'_v} \right). \]

It is more convenient to define the function $Q : L' \to \mathbb{R}$,

\[ (10) \quad Q(x') := \sum_{\emptyset \neq \mathcal{I} \subseteq V} (-1)^{|\mathcal{I}|} \text{Coeff}(F_{\mathcal{I}}(t), t_{x'_v}^{E_{x'}}), \]

because for any $h \in H$ the function $Q_h$ can be recovered from relations

\[ (11) \quad Q_{[x']}(x') = Q(x') \quad \text{and} \quad Q_{[x']}(x' - q) = Q(x') \]

for all $q \in L' \cap \sum_{v \in V} [0,1] E_v$. In particular, we have $Q_{[x']}(x') = Q_{[x']}(x' - q)$ for all $q \in L' \cap \sum_{v \in V} [0,1] E_v$. (cf. [LN14, 4.3.15]). Therefore, in the sequel we will study the quasipolynomial behaviour of the function $Q$.

3.2. The Szenes–Vergne formula. In this section we recall the notion of Jeffrey–Kirwan residue and the theorem of Szenes and Vergne about quasipolynomiality of coefficient functions, following [SZV03] and [BV99].

Let $V$ be an $r$-dimensional real vector space with a rank $r$ lattice $\Gamma \subset V$. We consider a finite collection of non-zero vectors $\Psi \subset \Gamma$ (elements of $\Psi$ are not necessarily distinct). Assume that all elements of $\Psi$ lie in an open half-space of $V^*$. Denote $\mathcal{A}(\Psi)$ the set of all bases $\sigma \subset \Psi$ of $V$. Let $\Gamma^* = \{ p \in V^*_\mathbb{C} \mid e^{\langle \alpha, p \rangle} = 1, \forall \alpha \in \Gamma \}$ be the $2\pi\sqrt{-1}$-multiple of the dual lattice of $\Gamma$. A big chamber is a connected component of $V \setminus \bigcup_{\sigma \in \mathcal{A}(\Psi)} \partial \mathbb{R}_{\geq 0}(\sigma)$, where $\partial \mathbb{R}_{\geq 0}(\sigma) = \{ v = \sum_{\beta \in \sigma} v_{\beta} \beta \mid v_{\beta} \geq 0, \forall \beta \text{ s.t. } v_{\beta} \neq 0 \}$ is the boundary of the closed cone $\mathbb{R}_{\geq 0}(\sigma)$. For a big chamber $\epsilon$ we define the set of bases $\mathcal{A}(\Psi, \epsilon) = \{ \sigma \in \mathcal{A}(\Psi) \mid \epsilon \subset \mathbb{R}_{\geq 0}(\sigma) \}$. 

3.2.1. **Total residue.** Consider elements of \( V \) as linear functions on \( V^* \). Denote by \( \mathbb{C}[V^*] \) the polynomial ring on \( V^* \) (which can be identified by the symmetric algebra of \( V \)). Let \( R_\Psi \) be the complex vector space spanned by fractions of form \( \psi = \frac{\theta}{\prod_{k+1}^L \beta_k} \), where \( \theta \in \mathbb{C}[V^*] \), \( \beta_k \in \Psi \) and \( R \in \mathbb{Z}_{>0} \). For any \( \sigma \in \mathfrak{B}(\Psi) \) the fraction \( \frac{1}{\prod_{k+1}^L \beta_k} \) is called simple. Let \( S_\Psi \) be the subspace of \( R_\Psi \) spanned by simple fractions.

**Theorem 4 ([BV99]).** For \( v \in V^* \) and \( \psi \in R_\Psi \) let \( \partial_v \psi(z) := \left. \frac{d}{dz} \psi(z + \epsilon v) \right|_{\epsilon=0} \) be the differentiation in direction \( v \). Then there is a direct sum decomposition

\[
R_\Psi = \left( \sum_{v \in V^*} \partial_v R_\Psi \right) \oplus S_\Psi.
\]

Thus, the projection to the second component \( \text{Tres}_\Psi : R_\Psi \to S_\Psi \) is a well defined map, which is called the total residue.

A grading on \( R_\Psi \) is defined as follows. If \( \theta \in \mathbb{C}[V^*] \) is a homogeneous polynomial of degree \( N \) then \( \frac{\theta}{\prod_{k+1}^L \beta_k} \in R_\Psi \) has homogeneous degree \( N - R \). The homogeneous degree \( d \) part of \( R_\Psi \) is denoted by \( R_\Psi[d] \) and one has decomposition \( R_\Psi = \bigoplus_{d \in \mathbb{Z}} R_\Psi[d] \).

**Remark 5.** Since \( S_\Psi \subset R_\Psi[-r] \) the total residue \( \text{Tres}_\Psi \) vanishes on \( R_\Psi[d] \) unless \( d = -r \).

Denote \( \mathbb{C}[[V^*]] \) the formal power series on \( V^* \) and let \( \hat{R}_\Psi \) be the complex vector space spanned by fractions of form \( \psi = \frac{\theta}{\prod_{k+1}^L \beta_k} \), where \( \theta \in \mathbb{C}[[V^*]] \) and \( \beta_k \in \Psi \). There is a filtration by degree on \( \mathbb{C}[[V^*]] \): for \( \Theta = \sum_{l \geq 0} \Theta_l \) with \( \Theta_l \in \mathbb{C}[V^*] \) homogeneous of degree \( l \) we say that \( \Theta \in \mathbb{C}[[V^*]]_{\geq d} \) if \( \Theta_l = 0 \) for all \( l < d \). It induces a filtration \( \hat{R}_\Psi = \sum_{d \in \mathbb{Z}} \hat{R}_\Psi[\geq d] \) compatible with the grading on \( R_\Psi \): we have \( \frac{\psi}{\prod_{k+1}^L \beta_k} \in \hat{R}_\Psi[\geq d] \) if \( \Theta \in \mathbb{C}[[V^*]]_{\geq d - R} \). The total residue can be extended to \( \hat{R}_\Psi \) as follows. Fix a \( d > -r \) and write \( \varphi \in \hat{R}_\Psi \) uniquely as \( \varphi = \varphi_{\text{poly}} + \varphi_{\text{ps}} \) with \( \varphi_{\text{poly}} \in \oplus_{k \leq d} R_\Psi[k] \) and \( \varphi_{\text{ps}} \in \hat{R}_\Psi[\geq d] \), then define \( \text{Tres}_\Psi(\varphi) := \text{Tres}_\Psi(\varphi_{\text{poly}}) \). Note that it does not depend on the choice of \( d \) by Remark 5.

3.2.2. **The Jeffrey–Kirwan residue.** Fix a volume form (or a scalar product) on \( V \). For each \( \sigma \in \mathfrak{B}(\Psi) \) it gives the volume \( \text{vol}(\sigma) \) of any parallelepiped \( \sum_{\beta \in \sigma} [0, 1] / l \sigma \). A big chamber \( c \) is also fixed. The **Jeffrey–Kirwan residue** on \( S_\Psi \) is defined by the formula

\[
\text{JK}^\sigma \left( \frac{1}{\prod_{\beta \in \sigma} \beta} \right) = \left\{ \begin{array}{ll}
\text{vol}(\sigma) & \sigma \in \mathfrak{B}(\Psi, c) \\
0 & \sigma \in \mathfrak{B}(\Psi) \setminus \mathfrak{B}(\Psi, c)
\end{array} \right.
\]

3.2.3. **The Szenes–Vergne formula.** Consider a rational function \( f(t) = \frac{\sum_{i \in I} t_i}{\prod_{i \in I} (1 - t_i)} \) with \( I \) finite subset of \( \Gamma \) and \( \beta_k \in \Psi \). It can be associated with a (meromorphic) function on \( V^* \) of form \( \varphi(z) = f(e^z) = \frac{\sum_{i \in I} e^{z_i(t_i)}}{1 - \sum_{i \in I} e^{z_i(t_i)}} = \frac{\sum_{i \in I} e^{z_i(t_i)}}{\prod_{i \in I} (1 - e^{z_i(t_i)})} \). Expand \( e^{z(t_i)} \) and \( (1 - e^{z_i(t_i)})^{-1} = -(\beta_k, z)^{-1} \sum_{l \geq 0} \gamma_k(z)^l \), where \( \gamma_k(z) = 1 + \frac{(\beta_k, z)}{2!} + \frac{(\beta_k, z)^2}{3!} + \ldots \) into power series, thus \( f(e^z) \) can be considered as an element of \( \hat{R}_\Psi \).

Let the set of poles \( \text{SP}(\varphi, \Gamma^*) \) of \( \varphi \) be the set of those \( p \in V^*_c \) such that \( \{ \beta_k \mid e^{(\beta_k, p)} = 1 \} \) spans \( V \). Note that \( \text{SP}(\varphi, \Gamma^*) \) is invariant under translation by elements of \( \Gamma^* \). Let the reduced set of poles be the quotient set \( \text{RSP}(\varphi, \Gamma^*) = \text{SP}(\varphi, \Gamma^*) / \Gamma^* = \cup_{\sigma \in \mathfrak{B}(\Psi)} (Z\sigma)^* / \Gamma^* \), where \( (Z\sigma)^* = \{ p \in \)}
\(V_\alpha^a | e^{(\alpha, p)} = 1, \forall \alpha \in \sigma\) is the \((2\pi \sqrt{-1})\)-multiple of the dual lattice of \(Z\sigma\). Note that reduced set of poles is a finite set and moreover, \(e^{(\alpha, p)}\) can be defined for any \(p \in RSP(\varphi, \Gamma^*)\) and \(\alpha \in \Gamma\), since it does not depend on the representant of \(p\).

Finally, we state a weaker version of the [SZV03], Theorem 2.3 and Lemma 2.2.

**Theorem 6.** Denote \(vol \Gamma\) the volume form on \(V\) such that parallelepiped of basis vectors of \(\Gamma\) has volume \(1\). Then for a big chamber \(c\) the function \(\lambda \mapsto \sum_{p \in RSP(\varphi, \Gamma^*)} JK^c \left( e^{(\lambda z - p)} \varphi(p - z) \right) \) is an exponential-polynomial (quasipolynomial) on \(\Gamma\), and for all \(\lambda \in \Gamma \cap \xi \in I (\xi + c)\) we have

\[
\sum_{p \in RSP(\varphi, \Gamma^*)} JK^c \left( \text{Tres}_\varphi \left( e^{(\lambda z - p)} \varphi(p - z) \right) \right)_{\text{vol} \Gamma} = \text{Coeff}(T[f(t)], t^\lambda).
\]

**Remark 7.** By the properties of the Jeffrey–Kirwan residue, the degree of the quasipolynomial is at most the number of linear terms in the denominator of \(f\) minus the dimension of \(V\).

**Remark 8.** From [SZV03], Theorem 2.3 also follows that if \(\xi \in \sum_{k=1}^R \mathbb{Z}/0, 1/\beta_k\) for all \(\xi\) in the numerator of \(f\) then the quasipolynomial in Theorem 6 vanishes for \(\lambda = 0\).

In dimension one the quasipolynomial in the above theorem can be computed as follows. Assume that \(\beta_1, \ldots, \beta_k\) are positive numbers and let \(q > 0\) be the generator of the lattice \(\Gamma\), i.e. \(\Gamma = q\mathbb{Z}\). In this case \(\Gamma^* = \mathbb{Z}^{\mathbb{Z}}\mathbb{Z}\) and \(RSP(\varphi, \Gamma^*) = 2\pi \sqrt{-1} \mathbb{Z} \sum_{k=1}^R \left\{ 0, \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_k} \right\}\) with \(\beta_k = \frac{\beta_k}{q} \in \mathbb{Z}\). Then the quasipolynomial is given by in terms of usual residue

\[
\lambda \mapsto \sum_{p \in RSP(\varphi, \Gamma^*)} \text{Res}_{z=0} \left( e^{\lambda(z - p)} \frac{\sum_{\xi \in I} c_k e^{\xi(p - z)}}{\prod_{k=1}^R (1 - e^{p \beta_k})} \right) dz
\]

(we chose the volume form on \(V\) such that \(vol(q) = q\)).

We emphasize that the above theorem gives an explicit formula how to compute this quasipolynomial, nevertheless the explicit computation may be lengthy.

### 3.3. Periodic constant in dimension 1 revisited.

Let \(f(t) = \frac{B(t)}{A(t)}\) with \(A(t) = \prod_i (1 - t^{m_i})\) polynomial and \(B(t)\) Laurent polynomial in \(t^{\frac{1}{d}}\) for some \(d \in \mathbb{Z}_{>0}\). Let \(L = \mathbb{Z}\) and \(L' = \mathbb{Z}^{\frac{1}{d}}\). The Taylor expansion \(Z(t) = T[f(t)] = \sum_{n \in L} p_n t^n\) can be decomposed into \(L' / L\)-equivariant parts: \(Z(t) = \sum_{h \in L'/L} Z_h(t)\), where \(Z_h(t) = \sum_{n' \in L', |n'|=h} p_n t^{\frac{n'}{d}}\). We can write uniquely \(f(t) = \sum_{h \in L'/L} P_h(t) + \frac{D(t)}{A(t)}\) such that \(t^{-r_h} P_h(t)\) are Laurent polynomials in \(t\) and \(D(t)/A(t)\) is rational in \(t^{\frac{1}{d}}\) with negative degree. We have the following interpretation of the equivariant periodic constant.

**Lemma 9.** \(\text{pc}(Z_h) = P_h(1)\).

**Proof.** Similarly as in Section 3.1 \(Q_h(l) = \text{Coeff}(T[f(t)] t^{-r_h}, t^l)\) is the counting function of \(Z_h(t)\) for \(l \in L\). By Theorem 6 there is a quasipolynomial \(L_h\) such that \(L_h(l) = Q_h(l)\) for sufficiently large \(l > 0\). We have to show that \(L_h(0) = P_h(1)\). By the decomposition \(f(t) = \sum_{h \in L'/L} P_h(t) + \frac{D(t)}{A(t)}\) we have that

\[
\sum_{h \in L'/L} P_h(t) + \frac{D(t)}{A(t)} = \sum_{h \in L'/L} P_h(t) + \frac{D(t)}{A(t)}
\]

and

\[
\sum_{h \in L'/L} P_h(t) + \frac{D(t)}{A(t)} = \sum_{h \in L'/L} P_h(t) + \frac{D(t)}{A(t)}
\]

Therefore, \(\text{pc}(Z_h) = P_h(1)\).
\[ \sum_{h \in L/L} P_h(t) + \frac{D(t)}{A(t)} \] for \( l \in L \) we have
\[ Q_h(l) = \sum_{k \in L/L} \text{Coeff} \left( T \left[ P_h(t) \frac{t^{1-r_h}}{1-t} \right], t^l \right) + \text{Coeff} \left( T \left[ \frac{D(t) t^{1-r_h}}{A(t)(1-t)} \right], t^l \right) \]
\[ = \text{Coeff} \left( T \left[ P_h(t) \frac{t^{1-r_h}}{1-t} \right], t^l \right) + \text{Coeff} \left( T \left[ \frac{D(t) t^{1-r_h}}{A(t)(1-t)} \right], t^l \right). \]

Denote \( L'_h \) and \( L''_h \) the quasipolynomials such that \( L'_h(l) = \text{Coeff}(T[P_h(t) \frac{t^{1-r_h}}{1-t}], t^l) \) and \( L''_h(l) = \text{Coeff}(T[D(t) \frac{t^{1-r_h}}{A(t)(1-t)}], t^l) \) for \( l \gg 0 \). Note that \( L'_h(l) \) is constant starting from a large \( l \) with value \( P_h(1) \), hence \( L'_h(l) = P_h(1) \) for all \( l \in L \). Moreover, \( L''_h(0) = 0 \) by Remark 8. Thus, \( L_h(0) = L'_h(0) + L''_h(0) = P_h(1) \).

3.4. Uniqueness theorem. Recall that the counting function \( Q(x') \) can be written as an alternating sum of coefficient functions of form \( \text{Coeff} \left( T \left[ \prod_{i \in V} (1 - t_i x'_i)^{d_h - 2} \prod_{i \in E} \frac{t_i x'_i}{1-t_i}, t^{x'} \right] \) for \( x' \in S' \). In general, the associated vector configuration \( \Psi_\mathcal{I} = \{ \pi_\mathcal{I}(E_i^\prime), E_i | l \in \mathcal{E}, i \in \mathcal{I} \} \) divides \( \text{int}(\pi_\mathcal{I}(S'_h)) \) into several big chambers. However, we will show that still there is an affine subcone \( y + \text{int}(S'_h) \) of the Lipman cone such that \( Q(x') \) is a quasipolynomial on \( y + \text{int}(S') \). In particular, the restriction of \( Q \) in the open cone \( \text{int}(S'_h) \) to a sufficiently sparse finite rank sublattice of \( L' \) is a polynomial.

**Theorem 10.** There exist unique quasipolynomials \( \mathcal{L} \) and \( \mathcal{L}_h \) on the lattice \( L' \) such that
\[ Q(x') = \mathcal{L}(x') \quad \text{and} \quad Q_h(x') = \mathcal{L}_h(x') \quad \text{for all} \ x' \in y + \text{int}(S'). \]

**Remark 11.** It is enough to prove the theorem only for \( Q \) since the part for \( Q_h \) follows by relations (11). The uniqueness of \( \mathcal{L} \) can be deduced as follows. If there is another quasipolynomial \( \mathcal{L}' \) such that \( Q(x') = \mathcal{L}'(x') \) for all \( x' \in y' + \text{int}(S') \), then the quasipolynomials \( \mathcal{L} \) and \( \mathcal{L}' \) coincide on the maximal dimensional affine cone \( y'' + \text{int}(S') \) for any \( y'' \in (y + S') \cap (y' + S') \), hence they are equal.

To prove the above theorem it is enough to show the following proposition.

**Proposition 12.** For all non-empty vertex set \( \mathcal{I} \subseteq V \) the function \( x' \mapsto \text{Coeff}(F_\mathcal{I}(t), t_\mathcal{I}^x) \) is a quasipolynomial on \( y_\mathcal{I} + \text{int}(S') \) for some \( y_\mathcal{I} \in \text{int}(S') \).

Using the following lemma, basically we reduce the proof of the proposition to the particular case when \( \mathcal{I} \) contains the set of end vertices, which is proved in Lemma 14.

**Lemma 13.** (i) Let \( v \neq v'' \) be an edge of a tree \( \mathcal{T} \) with vertex set \( V \). Decompose \( \mathcal{T} \setminus \{v, v''\} \) into disjoint union of trees \( \mathcal{T}' \) and \( \mathcal{T}'' \) with vertex sets \( V' \ni v' \) and \( V'' \ni v'' \), respectively. Then for all non-empty vertex set \( \mathcal{J} \subseteq V \) the fraction
\[ (1 - t_\mathcal{J}^{E_\mathcal{J}'}) \prod_{v \in V'} (1 - t_\mathcal{J}^{E_\mathcal{J}'})^{d_v - 2} \in \mathbb{R}[t_\mathcal{J}^{E_\mathcal{J}'} : v \in V], \]

i.e. it is a Laurent polynomial.
(ii) Let \( J \subseteq V \) be a non-empty vertex set such that its associated subgraph \( T_J \) is a tree. Then
\[
\prod_{v \in V} (1 - t_J^v)_{\delta_s,v - 2} = P_J(t, J) \prod_{j \in J} (1 - t_J^v)^{\delta_{s,j} - 2},
\]
where \( P_J(t, J) \in \mathbb{R}[t_{J^*}^v : v \in V] \) is a Laurent polynomial.

(iii) There exists a positive definite rational matrix \( \tilde{A}_J \) associated with the subtree \( T_J \) such that
\[
[\pi_J(E^*_v)]_{v \in J} \cdot \tilde{A}_J = [E^*_v]_{v \in J}.
\]

Proof. Let \( d_{v',v} = \max\{d(v, v') \mid v \in V'\} \), where \( d(u,v) \) is the length of the minimal path between vertices \( u \) and \( v \) in \( T_V \). We proceed by induction on \( d_{v',v} \).

If \( d_{v',v} = 0 \) then \( V' = \{v'\} \) and \( v' \) is an end vertex of \( T \). Since \( V' \subseteq V \setminus J \), by the relation \( A_{v',v'}E^*_v - E^*_v = E_{v'} \) coming from (3) we get \( A_{v',v'}\pi_J(E^*_v) = \pi_J(E^*_v) \), and (12) becomes
\[
1 \frac{1 - t_{v'}^{E^*_v}}{1 - t_{J^*}^{E^*_v}} = 1 \frac{1 - t_{J^*}^{E^*_v}}{1 - t_{J^*}^{E^*_v}} = 1 + t_{J^*}^{E^*_v} + \ldots + t_{J^*}^{(A_{v',v'} - 1)E^*_v},
\]
which is a polynomial in \( \mathbb{R}[t_{J^*}^v : v \in V] \).

If \( d_{v',v} = 1 \) then \( T_v \) is star-shaped with node \( v' \) and we have \( \delta_{v',v'} = |V'| - 1 \). That is, if \( V' = \{v' = v_0, v_1, \ldots, v_s\} \) then \( v_1, \ldots, v_s \in E \) and \( \delta_{v_0,v'} = \delta_{v_0,v} - 1 = s \). Moreover, we have relations
\[
A_{v_0,v_i}\pi_J(E^*_v) = \pi_J(E^*_v), \quad i = 1, \ldots, s,
\]
\[
A_{v_0,v_0} \pi_J(E^*_v) - \pi_J(E^*_v) - \ldots - \pi_J(E^*_v) = \pi_J(E^*_v).
\]
Note that in general one has the following decomposition
\[
1 - t_{v_0}^{E^*_v} = \sum_{\emptyset \neq S \subseteq \{0, \ldots, s\}} (-1)^{|S|} \prod_{i \in S} (1 - t_{E^*_v}^i) = \sum_{i=0}^{s} q_i (1 - t_{E^*_v}^i),
\]
where \( q_i \in \mathbb{R}[t_{E^*_v}^i : i = 0, \ldots, s] \). Hence, together with (13) imply
\[
1 - t_{E^*_v}^{E^*_v} = q_0 (1 - t_{E^*_v}^{A_{v_0,v_0}E^*_v}) + \sum_{i=1}^{s} q_i (1 - t_{E^*_v}^{E^*_v}) = \sum_{i=0}^{s} p_i (1 - t_{E^*_v}^{E^*_v}),
\]
where \( p_0 = q_0 (1 + t_{E^*_v}^{E^*_v} + \ldots + t_{E^*_v}^{A_{v_0,v_0}E^*_v}) \) and \( p_i = q_i (-t_{E^*_v}^{E^*_v}) \), \( i = 1, \ldots, s \) are Laurent polynomials. Then (12) becomes
\[
\frac{(1 - t_{E^*_v}^{E^*_v})(1 - t_{E^*_v}^{E^*_v})_{s-1}}{1 - t_{E^*_v}^{E^*_v}} = p_0 \prod_{l=1}^{s} (1 - t_{E^*_v}^{E^*_v}) + \sum_{i=1}^{s} p_i \prod_{l \neq i}^{s} (1 - t_{E^*_v}^{E^*_v})
\]
\[
= p_0 \prod_{l=1}^{s} (1 + t_{E^*_v}^{E^*_v} + \ldots + t_{E^*_v}^{A_{v_0,v_l}E^*_v}) + \sum_{i=1}^{s} p_i \prod_{l \neq i}^{s} (1 + t_{E^*_v}^{E^*_v} + \ldots + t_{E^*_v}^{A_{v_0,v_l}E^*_v})
\]
which is a Laurent polynomial.

In general, we have decomposition \( V' = \{v_0\} \cup V'_1 \cup \ldots \cup V'_s \) such that \( T_{V' \setminus \{v_0\}} = T_{V'_1} \cup \cdots \cup T_{V'_s} \) is disjoint union of trees, and let \( v_i \in V'_i, i = 1, \ldots, s \) be the neighboring vertices of \( v_0 \) in \( T_{V'} \). We
introduce notation $\phi_{\mathcal{V}_i} = \prod_{v \in \mathcal{V}_i} (1 - t^{E^*_{jv}})_{s, \nu - 2}$. Then by (15) there are Laurent polynomials $p_i$ such that

$$(1 - t^{E^*_{jv_i}}) \prod_{v \in \mathcal{V}_i} (1 - t^{E^*_{jv}})_{s, \nu - 2} = (1 - t^{E^*_{jv_i}})(1 - t^{E^*_{jv}})^{s-1} \prod_{i=1}^s \phi_{\mathcal{V}_i}$$

$$= \left[ \sum_{i=0}^s p_i (1 - t^{E^*_{jv_i}}) \right] \cdot (1 - t^{E^*_{jv}})^{s-1} \prod_{i=1}^s \phi_{\mathcal{V}_i}$$

$$= p_0 \prod_{i=1}^s (1 - t^{E^*_{jv}}) \phi_{\mathcal{V}_i} + \sum_{i=1}^s \prod_{(t \neq i)} p_i \left( (1 - t^{E^*_{jv}}) \phi_{\mathcal{V}_i} \right) \bigg[ \prod_{(t \neq i)} (1 - t^{E^*_{jv}}) \phi_{\mathcal{V}_i} \bigg].$$

Since $d_{\mathcal{V}_i, v_i} < d_{\mathcal{V}_i, v_i}$, by induction hypothesis $(1 - t^{E^*_{jv_i}}) \phi_{\mathcal{V}_i}$ is a Laurent polynomial. Finally, we have to show that the term $(1 - t^{E^*_{jv_i}}) \phi_{\mathcal{V}_i}$ is also a Laurent polynomial. One can write $\mathcal{T}_{\mathcal{V}_i \setminus \{v_i\}} = \mathcal{T}_{W_i} \cup \ldots \cup \mathcal{T}_{W_r}$ as disjoint union of trees and let $w_k$ be the (unique) neighboring vertex of $v_i$ such that $w_k \in W_k$, $k = 1, \ldots, r$. Then

$$(16) \quad (1 - t^{E^*_{jv_i}}) \phi_{\mathcal{V}_i} = \prod_{k=1}^r (1 - t^{E^*_{jv_k}}) \phi_{W_k},$$

and $d_{W_k, w_k} < d_{\mathcal{V}_i, v_i}$, hence (16) is also a Laurent polynomial by induction. This completes the proof of (16).

The subgraph $\mathcal{T}_{\mathcal{V} \setminus \mathcal{J}}$ decomposes to several trees $\mathcal{T}_{\mathcal{J}_1}, \ldots, \mathcal{T}_{\mathcal{J}_r}$ such that $\mathcal{J}_1 \cup \ldots \cup \mathcal{J}_r = \mathcal{V} \setminus \mathcal{J}$. For all $i = 1, \ldots, r$ denote $j_i \in \mathcal{J}$ the unique vertex such that there is an edge in $\mathcal{T}$ from the vertex $j_i$ to a vertex in $\mathcal{J}$ (it is possible that $j_i = j_k$ for some $i \neq k$). Then

$$\prod_{v \in \mathcal{V}} (1 - t^{E^*_{jv}})_{s, \nu - 2} = \prod_{j \in \mathcal{J}} (1 - t^{E^*_{jv}})_{s, \nu - 2} \cdot \prod_{k=1}^r (1 - t^{E^*_{j_{k}}} \prod_{u \in \mathcal{J}_k} (1 - t^{E^*_{jv}})_{s, \nu - 2},$$

and by (16) each term $(1 - t^{E^*_{j_{k}}}) \prod_{u \in \mathcal{J}_k} (1 - t^{E^*_{jv}})_{s, \nu - 2}$ is a a Laurent polynomial.

For $\mathcal{J} = \mathcal{V}$ we choose $\tilde{A}_V = A$. Let $i \in \mathcal{J}$ be an end vertex with $j$ neighboring vertex and denote $\mathcal{J}' = \mathcal{J} \setminus \{i\}$. Then $\pi_{\mathcal{J}'}$ is the composition of $\pi_{\mathcal{J}}$ and $\pi_{\mathcal{J}_i}$, the projection along $\mathbb{R}E_i$. The later projection corresponds to removal of edge $\overline{ij}$ from $\mathcal{T}_{\mathcal{J}}$, hence by [EN85] Section 21] we get that $[\pi_{\mathcal{J}'}(E^*_{jv})]_{v \in \mathcal{J}'} = [E_v]_{v \in \mathcal{J}}$, where $(\tilde{A}_{\mathcal{J}'})_{kl} = (\tilde{A}_{\mathcal{J}})_{kl}$ for all $k, l \in \mathcal{J}'$, except $(\tilde{A}_{\mathcal{J}'})_{ii} = (\tilde{A}_{\mathcal{J}})_{ii} - 1/(\tilde{A}_{\mathcal{J}})_{ii}$. Moreover, $\tilde{A}_{\mathcal{J}'}$ is positive definite and it is associated with $\mathcal{T}_{\mathcal{J'}}$. Finally, if $\mathcal{J} \subseteq \mathcal{V}$ then we can realize $\pi_{\mathcal{J}}$ as composition of projections $\pi_{\mathcal{J}_i}$ corresponding to successive removal of edges not in $\mathcal{T}_{\mathcal{J}}$, and which is end edges of the respective intermediate trees.

Recall that $\mathcal{T}_\mathcal{I}$ is the subgraph of $\mathcal{T}$ with vertex set $\mathcal{I}$. Let $\mathcal{J} \subseteq \mathcal{V}$ be such that $\mathcal{T}_{\mathcal{J}}$ is the minimal subtree of $\mathcal{T}$ containing $\mathcal{T}_{\mathcal{J}}$ as subgraph. Note that $\mathcal{I} \subseteq \mathcal{J}$ and $\mathcal{I}$ contains all the end vertices of the tree $\mathcal{T}_{\mathcal{J}}$. We write the projection $\pi_{\mathcal{J}}$ as composition of projections $\pi_{\mathcal{T}} : \mathcal{V} \to \mathcal{V}_{\mathcal{J}}$ along $\mathcal{V}_{\mathcal{V} \setminus \mathcal{J}}$ and $\mathcal{V}_{\mathcal{J}} \to \mathcal{V}_{\mathcal{J}}$ along $\mathcal{V}_{\mathcal{J} \setminus \mathcal{J}}$. We introduce notation $\tilde{E}_\mathcal{V} = \pi_{\mathcal{J}}(E^*_\mathcal{V})$, $\tilde{A}_\mathcal{V} = \pi_{\mathcal{J}}(\mathcal{V})$, and $\tilde{L} = \pi_{\mathcal{J}}(L)$. By Lemma [3.3.3] we have $[\tilde{E}_\mathcal{V}]_{v \in \mathcal{J}} = \tilde{A}_\mathcal{V}[E_{\mathcal{J}}]_{v \in \mathcal{J}}$, where $\tilde{A}_\mathcal{V}$ is a rational positive definite matrix associated with $\mathcal{T}_{\mathcal{J}}$. Then one can prove the following.
Lemma 14. Let $\tilde{P}(t) \in R[t^{\pm \tilde{E}^*_j} : v \in \tilde{V}]$ be a Laurent polynomial. If $\mathcal{I} \subseteq \mathcal{J}$ such that the set of end vertices $\mathcal{E}_\mathcal{J}$ of $T_\mathcal{J}$ is contained in $\mathcal{I}$ then there exists $\tilde{y}_\mathcal{J} \in \mathbb{R}_{>0}(\tilde{E}^*_j)_{j \in \mathcal{J}}$ such that

$$\tilde{x} \mapsto \text{Coeff} \left( \prod_{j \in \mathcal{J}} (1 - t_{\tilde{E}^*_j}^\mathcal{J}) \prod_{i \in \mathcal{I}} \frac{t_{\tilde{E}^*_i}}{1 - t_{\tilde{E}^*_i}} \right)$$

is a quasipolynomial on $y_\mathcal{J} + \mathbb{R}_{>0}(\tilde{E}^*_j)_{j \in \mathcal{J}} \cap \tilde{L}$.

Proof. Recall that $t_{\tilde{E}^*_j} = t^{\pi_j(\tilde{E}^*_j)}$. Denote $\Psi_\mathcal{I} = \{\pi_j(\tilde{E}^*_j), E_i \mid j \in \mathcal{E}_\mathcal{J}, i \in \mathcal{I}\}$ the set of vectors which appears in the denominator of $\tilde{P}(t_\mathcal{J}) \prod_{j \in \mathcal{J}} (1 - t_{\tilde{E}^*_j}^\mathcal{J}) \prod_{i \in \mathcal{I}} \frac{t_{\tilde{E}^*_i}}{1 - t_{\tilde{E}^*_i}}$. By Theorem 6 it is enough to show that $\mathbb{R}_{>0}(\pi_j(\tilde{E}^*_j))_{j \in \mathcal{J}}$ is not split into several big chambers, that is for any $\sigma \in \mathcal{B}(\Psi_\mathcal{I})$ we have either $\mathbb{R}_{>0}(\pi_j(\tilde{E}^*_j))_{j \in \mathcal{J}} \subset \mathbb{R}_{>0}(\sigma)$ or $\mathbb{R}_{>0}(\pi_j(\tilde{E}^*_j))_{j \in \mathcal{J}} \cap \mathbb{R}_{>0}(\sigma) = \emptyset$. Equivalently, no facet of $\mathbb{R}_{>0}(\sigma)$ cuts into $\mathbb{R}_{>0}(\pi_j(\tilde{E}^*_j))_{j \in \mathcal{J}}$, that is

$$\mathbb{R}_{>0}(\pi_j(\tilde{E}^*_j), \ldots, \pi_j(\tilde{E}^*_i), E_{i_{t+1}}, \ldots, E_{i_{|\mathcal{I}|-1}}) \cap \mathbb{R}_{>0}(\pi_j(\tilde{E}^*_j))_{j \in \mathcal{J}} = \emptyset,$$

for all $j_1, \ldots, j_t \in \mathcal{E}_\mathcal{J}$ and $i_{t+1}, \ldots, i_{|\mathcal{I}|-1} \in \mathcal{I}$. Suppose that (17) does not hold. Then there are $j_1, \ldots, j_t \in \mathcal{E}_\mathcal{J}$ and $i_{t+1}, \ldots, i_{|\mathcal{I}|-1} \in \mathcal{I}$ such that

$$\mathbb{R}_{>0}(\pi_j(\tilde{E}^*_j), \ldots, \pi_j(\tilde{E}^*_i), E_{i_{t+1}}, \ldots, E_{i_{|\mathcal{I}|-1}}) \cap \mathbb{R}_{>0}(\pi_j(\tilde{E}^*_j))_{j \in \mathcal{J}} \neq \emptyset.$$

Denote $\mathcal{J}' = \mathcal{J} \setminus \{j_1, \ldots, j_t\}$. Let $\pi'$ be the projection along $\mathbb{R}(\tilde{E}^*_j, \ldots, \tilde{E}^*_i)$ to $\mathbb{R}(E_j)_{j \in \mathcal{J}'}$ and note that $\pi'$ restricted to $V_{\mathcal{J}'}$ is the identity map. Hence

$$\mathbb{R}_{>0}(\pi'(E_{i_{t+1}}), \ldots, \pi'(E_{i_{|\mathcal{I}|-1}})) + V_{\mathcal{J}' \cap \mathbb{R}_{>0}(\pi_j(\tilde{E}^*_j))_{j \in \mathcal{J}'}} \neq \emptyset.$$

Denote $i' \in \mathcal{J}$ the unique neighboring vertex for the end vertex $i \in \mathcal{E}_\mathcal{J}$. Then for all $i \in \{j_1, \ldots, j_t\}$ the relation $\tilde{A}_i \tilde{E}^*_i - \tilde{E}^*_i = E_i$ implies that $\pi'(E_i) = -\pi'(\tilde{E}^*_i)$, hence

$$\pi'(E_i) = \begin{cases} -\pi'(\tilde{E}^*_i) & \text{if } i \in \{j_1, \ldots, j_t\}, \\ E_i & \text{if } i \in \mathcal{J}' \end{cases}.$$

Relabeling in such a way that $\{j_1, \ldots, j_t\} \cap \{i_{t+1}, \ldots, i_{|\mathcal{I}|-1}\} = \{i_{t+1}, \ldots, i_s\}$ the relation (19) becomes

$$\mathbb{R}_{>0}(-\pi'(\tilde{E}^*_j), \ldots, -\pi'(\tilde{E}^*_i), E_{i_{t+1}}, \ldots, E_{i_{|\mathcal{I}|-1}}) \cap \mathbb{R}_{>0}(\pi'(\tilde{E}^*_j))_{j \in \mathcal{J}'} \neq \emptyset.$$
Note that \( \text{int}(\pi_{\mathcal{J}}(S'_h)) = \mathbb{R}_{>0}(\tilde{E}'_{j})_{j \in \mathcal{J}} \). By Lemma \[\text{L}4\]

\[ (21) \quad \tilde{x} \mapsto \text{Coeff} \left( T \left[ \prod_{j \in \mathcal{J}} (1 - t_{x,j}^{E'} \delta_{x,j} - 2), t_{\tilde{x}}^{\tilde{t}_{x}} \right] \right) = \text{Coeff} \left( T \left[ \prod_{v \in \mathcal{V}} (1 - t_{v}^{E} \delta_{v} - 2), t_{\tilde{x}}^{\tilde{t}_{x}} \right] \right) \]

is a quasipolynomial on \( \tilde{y}_{\mathcal{J}} + \text{int}(\pi_{\mathcal{J}}(S')) \) for some \( \tilde{y}_{\mathcal{J}} \in \text{int}(\pi_{\mathcal{J}}(S')) \). Composing \( \pi_{\mathcal{J}} \) with the map \[\text{L}21\] we get that \( x' \mapsto \text{Coeff}(F_{x'}(t), t_{x'}^{\tilde{t}_{x}'}) \) is also a quasipolynomial on \( y_{\mathcal{J}} + \text{int}(S') \) for some \( y_{\mathcal{J}} \in \pi_{\mathcal{J}}^{-1}(\tilde{y}_{\mathcal{J}}) \cap \text{int}(S') \).

4. Surgery (recursion) formula for the counting function

In this section we present a recursion formula for the counting function using the interpretation of Section \[\text{S}3.1\] It is given in terms of a surgery on the graph \( \mathcal{T} \). The idea is based on special decomposition and projection properties of the coefficient functions from \[\text{S10}\]. In particular, we project the recurrence to the level of periodic constants too, and compare with a surgery formula of Braun and Némethi \[\text{BN10}\] proved for the Seiberg–Witten invariants. For sake of completeness, we recall first the formula from \[\text{BN10}\].

4.1. The Braun–Némethi surgery formula. Although the formula is true for any vertex of \( \mathcal{T} \), we restrict our attention to an end–vertex \( u \in \mathcal{E} \). Denote the plumbed 4– and 3–manifold associated with \( \mathcal{T} \setminus u \) by \( \tilde{X}_{u} \) and \( M_{u} \) respectively.

For any \( h \in H \), the \( \text{spin}^c \)–structure \( \sigma = h \ast \sigma_{\text{can}} \in \text{Spin}^c(M) \) can be extended uniquely to \( \tilde{\sigma} \in \text{Spin}^c(\tilde{X}) \) such that \( \tilde{\sigma} = r_{h} \ast \tilde{\sigma}_{\text{can}} \). We consider the projection \( \pi^{(u)} : L' \rightarrow L'_{\mathcal{T}\setminus u} \), where \( \pi^{(u)}(E_{v}^*) = E_{v}^* \) if \( v \in \mathcal{V}(\mathcal{T} \setminus u) \) and \( \pi^{(u)}(E_{u}^*) = 0 \). Since the canonical \( \text{spin}^c \)–structure of \( \tilde{X} \) projects to the canonical \( \text{spin}^c \)–structure of \( \tilde{X}_{u} \), \( \tilde{\sigma} \) projects to \( \tilde{\sigma}_{u} := \pi^{(u)}(r_{h}) \ast \tilde{\sigma}_{\text{can},u} \), whose restriction to the boundary \( M_{u} \) is \( \sigma_{u} = [\pi^{(u)}(r_{h})] \ast \sigma_{\text{can},u} \). Then the main result of \[\text{BN10}\] is the following theorem.

**Theorem 15.** (Braun–Némethi surgery formula)

\[
\text{sw}_{-h \ast \sigma_{\text{can}}}(M) = \frac{(K_{\mathcal{T}} + 2r_{h})^{2} + |\mathcal{V}|}{8} = \text{sw}_{[-\pi^{(u)}(r_{h})] \ast \sigma_{\text{can},u}}(M_{u}) + \frac{(K_{\mathcal{T}\setminus u} + 2\pi^{(u)}(r_{h}))^{2} + |\mathcal{V} \setminus u|}{8} - \text{pc}(Z'_{h})
\]

where \( Z'_{h} \) is the one–variable series \( Z_{h}|_{t_{v}=1,v \neq u} \).

4.2. Recursion for the counting function and its quasipolynomial. To obtain a recursion for the quasipolynomial and the periodic constant we use partial fraction decomposition. Fix an end–vertex \( u \in \mathcal{E} \) and let \( u' \in \mathcal{V} \) be its unique neighboring vertex. Denote \( \mathcal{V}' = \mathcal{V} \setminus u \) and \( \mathcal{E}' = \mathcal{E}' \cup \mathcal{E} \setminus u \). We group non-empty subsets of \( \mathcal{V} \) as follows:

- subsets \( \mathcal{I} \) such that \( u \in \mathcal{I} \) and \( \mathcal{I} \setminus u \neq \emptyset \),
- subsets \( \mathcal{I}' \) such that \( u \notin \mathcal{I}' \),
- the subset \( \{u\} \).
We use notation $I' = I \setminus u$ for any subset $I$ belonging to the first group. We decompose the function $Q$ given in (10) associated with the tree $T$ as

\begin{equation}
Q^T(x) = \sum_{u \in \mathcal{I} \subseteq \mathcal{V}} (-1)^{|I'|} \text{Coeff}(F_I, t^2_I) + \sum_{\emptyset \neq I' \subseteq \mathcal{V}} (-1)^{|I'| - 1} \text{Coeff}(F_{I'}, t^2_{I'}) + \text{Coeff}(F_{\{u\}}, t^2_u)
\end{equation}

($t_u = t_{\{u\}}$), and recall that $F_{I'} = T \left[ \prod_{v \in \mathcal{V}} (1 - t^2_{E^+_I}) \prod_{v \notin \mathcal{I}} \frac{t^2_{E^+_I}}{1-t^2_{E^+_I}} \right]$. In general, if $z = x + y$ then we have the following partial fraction decomposition

\[
\frac{1}{(1-t^2)(1-t^y)} = \frac{1}{(1-t^2)(1-t^2)} + \frac{1}{(1-t^2)(1-t^2)} - \frac{1}{(1-t^2)}.
\]

In particular, for $x = A_{uu} \pi_2(E^+_u)$, $y = -E_u$ and $z = \pi_2(E^+_u)$ it yields

\[
\frac{t^2_{E^+_u}}{(1-t^2_{E^+_u})(1-t^2_{E^+_u}^v)} = \left(\sum_{k=0}^{A_{uu}-1} \frac{t^{kE^+_u}}{(1-t^2_{E^+_u})(1-t^2_{E^+_u})} \left(1-t^{kE^+_u}I\right)\right) = \frac{1}{1-t^2_{E^+_u}} - \sum_{k=0}^{A_{uu}-1} \frac{t^{kE^+_u}}{1-t^2_{E^+_u}}.
\]

If we write $F_{I'} = T \left[ \frac{t^2_{E^+_u}}{(1-t^2_{E^+_u})(1-t^2_{E^+_u}^v)} \varphi_{I'} \right]$ with $\varphi_{I'} = \prod_{v \notin \mathcal{I}} (1-t^2_{E^+_I}) \prod_{v \notin \mathcal{I}} \frac{t^2_{E^+_I}}{1-t^2_{E^+_I}}$, then the above relation yields

\[
\text{Coeff}(F_{I'}, t^2_I) = \text{Coeff}(T \left[ \frac{t^2_{E^+_u} \varphi_{I'}}{(1-t^2_{E^+_I})(1-t^2_{E^+_u}^v)} \right], t^2_I) = \text{Coeff}(T \left[ \frac{t^2_{E^+_u} \sum_{k=0}^{A_{uu}-1} t^{kE^+_u}}{(1-t^2_{E^+_u})(1-t^2_{E^+_u}^v)} \varphi_{I'} \right], t^2_I)
\]

(23)

\[
- \text{Coeff}(T \left[ \frac{\varphi_{I'}}{(1-t^2_{E^+_I})(1-t^2_{E^+_u}^v)} \right], t^2_I) + \text{Coeff}(T \left[ \frac{\sum_{k=0}^{A_{uu}-1} t^{kE^+_u}}{1-t^2_{E^+_u}^v} \varphi_{I'} \right], t^2_I).
\]

We will apply the following lemma to each summand on the right hand side of the above relation.

**Lemma 16.** Let $V$ be an $r$-dimensional real vector space and $L' \subset V$ be a rank $r$ lattice. Let $\Psi = \{\gamma_1, \ldots, \gamma_n\} \subset L'$ be a collection of non-zero vectors which span $V$ and lie in an open half-space of $V$. We assume that $\gamma_1 \notin \mathbb{R}_{\geq 0}\{\gamma_2, \ldots, \gamma_n\}$ and there is a basis of $L'$ such that $\gamma_1$ is a basis element. Let $W \subset V$ such that $V = \mathbb{R}\gamma_1 \otimes W$ and denote $\pi': V \rightarrow W$ the projection along $\mathbb{R}\gamma_1$. Then

\begin{equation}
\text{Coeff}(T \left[ \frac{\sum_{\eta \in \mathcal{N}} b^\eta t^{\pi'(\eta)}}{\prod_{\eta=1}^{\mathcal{N}} (1-t^{\pi'(\eta)})} \right], t^{\pi'(\lambda)}) = \text{Coeff}(T \left[ \frac{\sum_{\eta \in \mathcal{N}} b^\eta t^{\pi'(\eta)}}{\prod_{\eta=2}^{\mathcal{N}} (1-t^{\pi'(\eta)})} \right], t^{\pi'(\lambda)})
\end{equation}

for all $\lambda \in L' \cap \left( \bigcap_{\eta \in \mathcal{N}} (\eta + \mathbb{R}_{\geq 0}(\Psi) \setminus \mathbb{R}_{\geq 0}(\Psi')) \right)$, where $\Psi' = \Psi \setminus \{\gamma_1\}$ and $\mathcal{N} \subset L'$ is finite.

**Proof.** First, remark that for any $\lambda \in \mathbb{R}_{\geq 0}\{\gamma_1, \ldots, \gamma_n\} \setminus \mathbb{R}_{\geq 0}\{\gamma_2, \ldots, \gamma_n\}$ such that $\lambda \in \mu + \mathbb{R}_{\geq 0}\gamma_1$ we have that $\lambda = \mu + \mathbb{R}_{\geq 0}\gamma_1$. Indeed, suppose that $\lambda - \mu = -q\gamma_1$ with $q \geq 0$. There is a simplicial subcone of $\mathbb{R}_{\geq 0}\{\gamma_2, \ldots, \gamma_n\}$ which contains $\mu$. We assume that this cone is $\mathbb{R}_{\geq 0}\{\gamma_2, \ldots, \gamma_{r+1}\}$. The cone $\mathbb{R}_{\geq 0}\{\gamma_1, \ldots, \gamma_{r+1}\}$ can be written as union of $r$-dimensional simplicial cone which intersect each other only at their boundaries as follows.
Let $\mathbb{R}_0(\gamma_2, \ldots, \hat{\gamma}_i, \ldots, \gamma_{r+1})$, $i = 1, \ldots, k$ be the facets of $\mathbb{R}_0(\gamma_2, \ldots, \gamma_{r+1})$ such that their supporting hyperplane separates $\gamma_2, \ldots, \gamma_{r+1}$ from $\gamma_1$. Then $\mathbb{R}_0(\gamma_1, \ldots, \gamma_{r+1})$ is the union of cones $\mathbb{R}_0(\gamma_2, \ldots, \gamma_{r+1})$ and $\mathbb{R}_0(\gamma_1, \gamma_2, \ldots, \hat{\gamma}_i, \ldots, \gamma_{r+1})$, $i = 1, \ldots, k$, moreover they intersect only in their faces. Finally, $\lambda$ lies in one of $\mathbb{R}_0(\gamma_1, \hat{\gamma}_i, \ldots, \gamma_{r+1})$, $i \in \{1, \ldots, k\}$, thus by assumption $q \geq 0$ and convexity $\mu = \lambda + q\gamma_1$ is also in same cone, which contradicts $\mu \in \mathbb{R}_0(\gamma_2, \ldots, \gamma_{r+1})$.

Since $\Psi$ is an open half-space of $V$ and $\gamma_1 \not\in \mathbb{R}_0(\Psi')$ the right hand side of (24) is well-defined. It is enough to prove the lemma for $1 \in \mathbb{R}$. Let $\lambda \in L' \cap \mathbb{R}_0(\Psi') \setminus \mathbb{R}_0(\Psi')$. By the beginning remark $(\lambda + R\gamma_1) \cap \mathbb{Z}_0(\Psi') = (\lambda + R\gamma_1) \cap \mathbb{Z}_0(\Psi') = \{\lambda - q\gamma_1 \mid q = q_1, \ldots, q_k \in \mathbb{Q}_0\}$ is finite, since $\Psi$ is an open half-space of $V$. Furthermore, $q_1, \ldots, q_k \in \mathbb{Z}_0$ since $\gamma_1$ is a basis element of $L'$. Finally, we compute

$$\text{Coeff} \left( \left[ \prod_{i=2}^{\infty} (1 - t^{\varphi_i}) \right]^{T}, t^{\varphi_i} \right) = \sum_{\mu \in (\lambda + R\gamma_1) \cap \mathbb{Z}_0(\Psi')} \text{Coeff} \left( \left[ \prod_{i=2}^{\infty} (1 - t^{\varphi_i}) \right]^{T}, t^{\varphi_i} \right).$$

Remark 17. The description of the precise set of lattice points where the above lemma holds can be cumbersome, nevertheless it always contains a maximal dimensional affine subcone of $L' \cap \mathbb{R}_0(\Psi) \setminus \mathbb{R}_0(\Psi')$, which will be sufficient for us.

Let $\Psi'_I = \{ \pi_I(E_i^*) \mid j \in J', i \in J' \}$. First, projecting along $E_u$ for $\pi_I(x) \in L'$ in a suitable affine subcone of $L' \cap \mathbb{R}_0(\Psi') \setminus \mathbb{R}_0(\Psi'_I)$ given by the above lemma we get

$$\text{Coeff} \left( \left[ \prod_{i \in J'} (1 - t^{E_i}) \right]^{T}, t^{E_i} \right) = \text{Coeff} \left( \left[ \prod_{i \in J'} (1 - t^{E_i}) \right]^{T}, t^{E_i} \right).$$

Secondly, denote $\pi^{(u)}_I : V_J \to V_I$, the projection along $\pi_I(E_u^*)$ and use short notation $\pi^{(u)} = \pi^{(u)}_I$ for the projection along $E_u^*$. Note that we have commutation relation $\pi^{(u)}_I \pi_I = \pi_I \pi^{(u)}$, moreover $\pi^{(u)}_I(E_i) = E_i$ for $i \neq u$. Projecting along $\pi^{(u)}_I(E_u^*)$ for $\pi_I(x) \in L'$ in a suitable affine subcone of $L' \cap \mathbb{R}_0(\Psi'_I) \setminus \mathbb{R}_0(\Psi'_I)$ we get

$$\text{Coeff} \left( \left[ \prod_{i \in J'} (1 - t^{E_i}) \right]^{T}, t^{E_i} \right) = \text{Coeff} \left( \left[ \prod_{i \in J'} (1 - t^{E_i}) \right]^{T}, t^{E_i} \right).$$

the coefficients which appear in the $Q$ function associated with the tree $T \setminus u$:

$$Q^{T \setminus u}(\pi^{(u)}_I(x)) = \sum_{0 \neq \varphi' \leq \varphi''} (-1)^{|\varphi''| - 1} \text{Coeff} \left( \left[ \prod_{i \in J''} (1 - t^{\varphi''}) \right]^{T}, t^{\varphi''} \right).$$
Thirdly, for $\pi_T(x) \notin \bigcup_{k=0}^{d_{\pi_T}} k\pi_T(E_u^*) + \mathbb{R}_{\geq 0}(\Psi_T^*) \cap L'$ we have
\begin{equation}
\text{Coeff} \left( T \left[ \frac{\sum_{k=0}^{d_{\pi_T}} E_u^*}{1 - t_{E_u^*}}, \varphi_T \right] , t_T^x \right) = 0.
\end{equation}

Finally, we need a maximal dimensional affine subcone of the Lipman cone where relations (26), (27) hold simultaneously. A suitable affine subcone of a maximal dimensional subcone $c$ with $\pi_T(c) \subset \mathbb{R}_{\geq 0}(E_u^*) \setminus \mathbb{R}_{\geq 0}(\Psi_T^*)$ and $\pi_T(c) \subset \mathbb{R}_{\geq 0}(\pi_T(E_u^*), \Psi_T^*) \setminus \mathbb{R}_{\geq 0}(\Psi_T^*)$ will have this property.

**Lemma 18.** There is a maximal dimensional open polyhedral cone $c$ in the Lipman cone $S$ such that for all $I \subseteq V$ with $u \in I$ we have $\pi_T(c) \cap \mathbb{R}_{\geq 0}(\Psi_T^*) = \emptyset$.

**Proof.** If $|V| = 2$ then we can choose $c = \text{int}(S)$, thus we may assume that $|V| > 2$. Denote $E_i' = E_i$ if $i \neq u'$ and $E_{u'} = E_u^* + E_{u'}$. Then $[E_i']_{i \in V'} A' = [E_i^*]_{i \in V'}$, where $A'$ is the positive definite matrix associated with $T \setminus u$. Therefore, $\mathbb{R}_{\geq 0}(\Psi_T^* \setminus \{E_{u'}\}) \subset \mathbb{R}_{\geq 0}(E_u^* \setminus v \in V')$, but they are not equal since $|V| > 2$ implies $E_{u'} \neq E_u^*$.

Let $\mathcal{J} := I \cup u'$, $\mathcal{J}' = I' \cup u'$ and let $\pi_{\mathcal{J}} : \mathbb{R}(E_i' \setminus i \in V') \rightarrow \mathbb{R}(E_i' \setminus i \in \mathcal{J}')$ be the projection along $\mathbb{R}(E_i' \setminus i \notin \mathcal{J}')$ It maps $\Psi_T^* \setminus \{E_{u'}\}$ into $\mathbb{R}_{\geq 0}(E_i' \setminus i \notin \mathcal{J}')$. Moreover, the restriction of $\pi_{\mathcal{J}}$ (the projection along $\mathbb{R}(E_i \setminus i \notin \mathcal{J})$ to $\mathbb{R}(E_i \setminus i \notin \mathcal{J})$) to $\mathbb{R}(E_i' \setminus i \notin \mathcal{J}')$ is the same as $\pi_{\mathcal{J}}$, followed by the isomorphism $E_i' \rightarrow E_i$, for $i \neq u, u'$ and $E_{u'} \rightarrow \pi_T(E_u^*) + \pi_T(E_{u'})$, thus $\mathbb{R}_{\geq 0}(\Psi_T^* \setminus \{E_{u'}\})$ is in $\mathbb{R}_{\geq 0}(\pi_T(E_u^*) \setminus \pi_T(E_{u'}) \setminus E_i \setminus i \notin \mathcal{J}' \setminus u')$, but they are not equal.

If $u' \in I$ then $\pi_T(E_{u'}) = E_{u'}$ and the hyperplane $\mathbb{R}(\pi_T(E_u^*) \setminus E_{u'}) \setminus E_i \setminus i \notin \mathcal{J}' \setminus u')$ separates $\pi_T(E_u^*), E_{u'}$ from $E_{u'}$, hence $c_\mathcal{J} := \pi_T(S) \setminus \mathbb{R}_{\geq 0}(\Psi_T^*) \neq \emptyset$ is a maximal dimensional subcone of $\pi_T(S)$.

If $u' \notin I$ then $\pi_T(E_{u'}) = 0$, hence $\mathbb{R}_{\geq 0}(\Psi_T^*)$ is a proper closed subset of $\mathbb{R}_{\geq 0}(\pi_T(E_u^*) \setminus E_i \setminus i \notin \mathcal{J}')$ not containing $\pi_T(E_u^*)$, therefore $c_\mathcal{J} := \pi_T(S) \setminus \mathbb{R}_{\geq 0}(\Psi_T^*)$ is a maximal dimensional subcone of $\pi_T(S)$.

Finally, $c := \bigcap_{u \in I \subseteq V} \pi_T^{-1}(c_\mathcal{J}) \cap \text{int}(S)$ is an open maximal dimensional subcone of $S$ satisfying the required properties. \qed

Summarizing the above computations, relations (22), (23), (24), (25) and (27) imply the following recursion for $Q$ functions
\begin{equation}
Q^T(x) = Q^{T \setminus u}(\pi(u)(x)) + \text{Coeff}(F_{\{u\}}, t_u^x)
\end{equation}
on a suitable maximal dimensional affine subcone $c$ of the Lipman cone.

Denote $\mathcal{L}^T$ (resp. $\mathcal{L}_u^T$) the quasipolynomial on $L'$ such that $\mathcal{L}^T(x) = Q^T(x)$ (resp. $\mathcal{L}_u^T(x) = Q_u^T(x)$) on $y + S'$ for some $y \in S'$. Such a unique quasipolynomial exists by Theorem 6 and Proposition 12. Similarly to the relations (11) we can also recover $\mathcal{L}_u^T$ from $\mathcal{L}^T : \mathcal{L}_{u'}(x') = \mathcal{L}^T(x')$ and $\mathcal{L}_{u'}(x' - q) = \mathcal{L}_{u'}(x')$ for any $q \in L' \cap \sum_{v \in V \setminus \{0\}} E_v$.

Denote $\mathcal{L}^u$ the quasipolynomial such that $\mathcal{L}^u(y') = \text{Coeff}(F_{\{u\}}(t_u^x), t_{u'}^{y'})$ for $y' \gg 0$. Then (28) implies the relation
\begin{equation}
\mathcal{L}^T(x) = \mathcal{L}^{T \setminus u}(\pi(u)(x)) + \mathcal{L}^u(x_u)
\end{equation}of quasipolynomials on $L'$. 

The next lemma shows that

Denote \( P(t_u)(1 - t_u^{E_u})^{-2} \). Thus, \( L^u \) is the quasipolynomial associated with the coefficient function of the Taylor expansion of \( P(t_u)(1 - t_u^{E_u})^{-2}(1 - t_u)^{-1} \), hence \( L^u \) has degree two by Remark \( \ref{remark:degree} \). Moreover, by \( \ref{lemma:degree} \) we get that \( \mathcal{L}^T \) is also a quasipolynomial of degree two.

4.3. Recursion for the periodic constants. Let \( x' \in L' \) such that \( [x'] = h \) for a fixed \( h \in H \).

4.3.1. \( r_h \)-normalization. We represent \( x' = \overline{x} + r_h \) for some \( \overline{x} \in L \). Then relations \( \ref{formula:telescoping} \) and the formula \( \ref{formula:recursion} \) imply that

\[
\mathcal{L}_h(x') = \mathcal{L}_h^T(\overline{x}) = \mathcal{L}_h^{T \setminus u}(\overline{x}+r_h)(\pi(u)(\overline{x}+r_h)) + \mathcal{L}_h^u((\overline{x}+r_h)_u).
\]

One can happen that \( [\pi(u)(\overline{x}+r_h)] \) varies in \( L_1^{T \setminus u} / L_T \setminus u \), hence the formula chooses different quasipolynomials on \( T \setminus u \). However, from periodic constant point of view it is enough to look at a sparse sublattice \( \overline{L} \) of \( L \) such that \( [\pi(u)(\overline{x}+r_h)] \) is constant and equals with \( [\pi(u)(r_h)] \).

Therefore, by \( \ref{corollary:recurrence} \) we get

\[
\text{pc}_h(Z^T) = \mathcal{L}_h^T(0) = \mathcal{L}_h^{T \setminus u}(\pi(u)(r_h)) + \text{pc}(Z^h).
\]

The problem is that in general \( \tilde{r}_h := \pi(u)(r_h) \) is not in \( \sum_{v \in V'[0,1]}E_v \), hence \( \mathcal{L}_h^{T \setminus u}(\tilde{r}_h) \) is not equal with \( \text{pc}_{\tilde{r}_h}(Z^{T \setminus u}) \). This behaviour of the recurrence on the periodic constant level is in accordance with the Braun–Némethi formula (Theorem \( \ref{theorem:braun-nemethi} \), since for \( \tilde{r} = r_h * \sigma_{can} \) we have \( \tilde{s}_u = \tilde{r}_h * \sigma_{can,u} \). In special cases (e.g. \( h = 0 \) or \( r_h = s_h \)) we get purely a recursion of periodic constants.

4.3.2. \( s_h \)-normalization. In this case, we write \( x' = \overline{x} + s_h \) for some \( \overline{x} \in L \). One can also modify the definition of the periodic constant associated with \( Z \) and introduce \( s_h \)-normalized periodic constant by

\[
\overline{\text{pc}_h}(Z) := \mathcal{L}_h(s_h),
\]

where \( L_h \) is the quasipolynomial on \( h + L \) associated with \( Z \). Then by the same argument as in the above section we get

\[
\mathcal{L}_h^T(s_h) = \mathcal{L}_h^{T \setminus u}(\pi(u)(s_h)) + \mathcal{L}_h^u((s_h)_u).
\]

The next lemma shows that \( s_h \) is projected under \( \pi(u) \) into a representative of the same type.

**Lemma 20.** \( \pi(u)(s_h) = \sigma_{[\pi(u)(s_h)]} \) in \( L_1^{T \setminus u} \).

**Proof.** Denote \( [\pi(u)(s_h)] \) by \( \overline{s}_h \). By the definition of \( \pi(u) \) one has \( \pi(u)(s_h) \in S_1^{T \setminus u} \), the Lipman cone in \( L_1^{T \setminus u} \), therefore the unique representative associated with \( h \) can be written as \( s_h = \pi(u)(s_h) - l \), with \( l \in L_1^{T \setminus u} \subset L \) and \( l \geq 0 \). We set \( s := s_h - l \) and we show that \( s \in S' \), i.e. \( (s, E_v) \leq 0 \) for all \( v \). This would imply that \( l = 0 \) by the minimality of \( s_h \), hence \( \pi(u)(s_h) = s_h \).

Notice that \( s_h \in S_1^{T \setminus u} \) is equivalent with \( (\pi(u)(s_h) - l, E_v) \leq 0 \) for all \( v \neq u \). Moreover \( (s_h, E_v) = (\pi(u)(s_h), E_v) \) for all \( v \neq u \), hence \( (s, E_v) \leq 0 \) for all \( v \neq u \). On the other hand, \( (l, E_u) \geq 0 \) since the \( E_u \)-coefficient of \( l \) is 0. Hence \( (s, E_u) = (s_h, E_u) - (l, E_u) \leq 0 \) by \( s_h \in S' \).
Therefore, (30) can be interpreted as a recursion of $s_h$-normalized periodic constants

$$
\text{pc}_h(Z^T) = \text{pc}_{[\tau(s_h)]}(Z^T)^u + \text{pc}(t^{-(s_h)}u)Z^u,
$$

where $Z^u$ is the one-variable series $Z^T|_{t^v=1,v\neq u}$.

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