EQUIDISTRIBUTION OF ZEROS OF RANDOM POLYNOMIALS IN $\mathbb{C}^m$

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ABSTRACT. We study an equidistribution problem for zeros of random polynomials represented with respect to bases whose elements are so-called $Z$-asymptotically Chebyshev polynomials (which might not be orthonormal) in $\mathbb{C}^m$. We use certain results obtained in a very recent work of Bayraktar, Bloom and Levenberg and have an equidistribution result in a more general probabilistic setting than what the paper of Bayraktar, Bloom and Levenberg considers even though the bases they use are more general than $Z$-asymptotically Chebyshev polynomials. Our equidistribution result is based on the expected distribution and the variance estimate of random zero currents associated with the zero sets of polynomials. This equidistribution result of general nature shows that equidistribution result turns out to be true without the random coefficients $a_j$ being i.i.d. (independent and identically distributed), which also means that there is no need to use any probability distribution function associated with these random coefficients.

1. INTRODUCTION AND BACKGROUND

The statistical issues of zero sets of random functions of several variables, such as random polynomials of multivariable real and complex variables, have piqued the interest of many researchers. It is impossible to give all the references here since there is an extensive literature in this subject. For this reason we will be a little short for explaining what has been done so far. Many results regarding both Gaussian and non-Gaussian cases and historical advancements of this polynomial theory may be found, for example, in [Bay17b, BL15, BS07, BL05, BD18, ROJ, SHSM, HN08] (and references therein). For example, in [BL15], the authors work with the complex random variables that have bounded distribution functions on the whole complex plane $\mathbb{C}$ and outside of a very large disk with radius $\rho$, its integral with respect to the two dimensional Lebesgue measure has an upper bound depending on $\rho$, the latter condition is called the tail-end estimate. Long before these advances, as is commonly known, the works of Polya and Bloch, Littlewood-Offord, Kac, Hammersley, and Erdős-Turan were the first efforts on the distribution of roots of random algebraic equations in single real variable, and the interested reader can go to the articles [Bl P, Kac43, LO43, HAM56, ET50].

As another interesting direction, there is an expanding physics literature dealing with the equidistribution and probabilistic problems concerned with the zeros of complex random polynomials. See, for example, [FH, Hann, NV98] for foundational research in this area.

As the most general setting so far, the equidistribution, expected distribution and variance of zero currents of integration of random holomorphic sections with different probabilistic settings (including Gaussian and non-Gaussian types) are studied in [BCM, Bay16, Shif, SZ99]. The initial and pioneering work ([SZ99]) in this setting belongs to Shiffman and Zelditch. In this paper, we will have a variance estimation and the techniques are based on the papers of [SZ99] and [Shif]. In [Bay16], [Gnyz], [Shif], [SZ08], [SZ10], the reader can find variance estimations in different probabilistic setups in this general setting of holomorphic line bundles over Kahler manifolds. As for the asymptotic expansion of variance, Shiffman has the most recent result. In his paper [Shif1],

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the asymptotic series expansion of variance (for codimension 1) in Gaussian setting is derived. These problems are heavily related to the asymptotics of Bergman kernel.

The pluricomplex Green function of a non-pluripolar compact set \( K \subset \mathbb{C}^m \) is defined as follows

\[
V_K(z) := \sup\{u(z) : u|_K \leq 0, \ u \in \mathcal{L}(\mathbb{C}^m)\},
\]

where \( \mathcal{L}(\mathbb{C}^m) \) represents the Lelong class consisting of all functions \( u \) plurisubharmonic on \( \mathbb{C}^m \) such that \( u(\zeta) - \ln |\zeta| \) is bounded from above near infinity. The upper semicontinuous regularization of \( V_K(z) \) is the following

\[
V^*_K(z) := \limsup_{\zeta \to z} V_K(\zeta).
\]

As is well-known, \( V^*_K(z) \in \mathcal{L}(\mathbb{C}^m) \) (precisely if \( K \) is non-pluripolar, see Corollary 5.2.2 of [Kl]). For more detail about the pluricomplex Green function, [Kl] may be useful.

A compact set \( K \) in \( \mathbb{C}^m \) is regular if \( V_K \equiv 0 \) on \( K \) (and therefore \( V_K \) is continuous on \( \mathbb{C}^m \)). The compact sets we consider in this paper will be assumed to be regular.

We are going to use the notation \( \|f\|_D := \sup \{ |f(z)| : z \in D \} \) for a function \( f : D \to \mathbb{C} \). Let \( \mathbb{N}^n \) be the collection of all \( n \)-dimensional vectors with non-negative integer coordinates. For \( k = (k_1, \ldots, k_\nu, \ldots, k_n) \in \mathbb{N}^n \) and \( z = (z_1, \ldots, z_n) \in \mathbb{C}^m \), let \( z^{k(j)} = z_1^{k_1(j)} \ldots z_n^{k_n(j)} \), \( j \in \mathbb{N} \) and \( |k(j)| := k_1(j) + \ldots + k_n(j) \) be the degree of the monomial \( e_j(z) = z^{k(j)} \). We consider the enumeration \( \{k(j)\}_{j \in \mathbb{N}} \) of the set \( \mathbb{N}^n \) such that \( |k(j)| \leq |k(j+1)| \) and on each set \( \{ |k(j)| = n \} \) the enumeration coincides with the lexicographic order. We will write \( s(j) := |k(j)| \). The number of multiindices of degree at most \( n \) is \( d_n := C_m^n = \dim (\mathcal{P}_n) \), where \( \mathcal{P}_n \) is the vector space of polynomials on \( \mathbb{C}^m \) of degree at most \( n \).

The standard \((m-1)\)-simplex will be taken into consideration

\[
\Delta := \left\{ \theta = (\theta_\nu) \in \mathbb{R}^n : \theta_\nu \geq 0, \ \nu = 1, \ldots, n; \ \sum_{\nu=1}^n \theta_\nu = 1 \right\},
\]

and its interior (with respect to the relative topology on the hyperplane containing \( \Delta \))

\[
\Delta^* := \{ \theta = (\theta_\nu) \in \Delta : \theta_\nu > 0, \ \nu = 1, \ldots, n \}.
\]

For \( \theta \in \Delta \) we denote by \( C_\theta \) the set of all infinite sequences \( N \subset \mathbb{N} \) such that \( \frac{k(j)}{s(j)} \to \theta \).

Leja raised the problem as to whether there is usual limit for transfinite diameter in several complex variables ([L]). Zakharyuta in his seminal work [Za1] solved this problem affirmatively for an arbitrary compact set \( K \subset \mathbb{C}^n \) by introducing the following what is called directional Chebyshev constants

\[
\tau(K, \theta) := \limsup_{j \to \infty} \tau_j := \sup_{L \in \mathcal{L}_\theta} \limsup_{j \to \infty} \tau_j, \ \theta \in \Delta,
\]

\[
\tau_j = \tau_j(K) := (M_j)^{1/s(j)}, \ j \in \mathbb{N},
\]

where

\[
M_j := \inf \left\{ |p|_K : p = e_j + \sum_{l=1}^{j-1} e_l \right\}, \ j \in \mathbb{N}
\]

The constants \( M_j \) are known as the least uniform deviation of monic polynomials from the identical zero on compact set \( K \). A polynomial which attains its infimum in (1.4) is called a Chebyshev polynomial. In the context of the theory of best approximation in Banach spaces ([Ah], section 8), this kind of polynomials always exist, but the uniqueness is not ensured.
Let \( P(k(j)) := \{ t(z) = e_j(z) + \sum_{l<j} c_l e_l(z) : c_l \in \mathbb{C} \} \). Next definition is due to Bloom ([BI01]) (see also [BBL]).

**Definition 1.1.** Let \( K \subset \mathbb{C}^m \) be compact and \( \theta \in \Delta^0 \) be given. A sequence of polynomials \( \{ t_j \}_{j \in \mathbb{N}} \), where \( \mathbb{N} \subset \mathbb{N} \), is said to be \( \theta \)-asymptotically Chebyshev if

- For every \( j \in \mathbb{N} \), there is one \( m \)-tuple \( k(j) \) such that \( t_j \in P(k(j)) \),
- \( s(j) = |k(j)| \to \infty \) and \( N \subset \mathbb{C}_\theta \),
- \( \| t_j \|_{1/|k(j)|} \to \tau(K, \theta) \) when \( j \to \infty \).

Following [BBL], a sequence \( \{ t_j \}_{j \in \mathbb{N}} \) is called asymptotically Chebyshev for \( K \) if for any \( \theta \in \Delta^0 \), there is a subsequence \( N \subset \mathbb{N} \) that satisfies the above three conditions. If the sequence has also the condition that for each \( \theta \in \Delta^0 \), every sequence of \( \beta \in \mathbb{N}^m \) with \( \lim_{|\beta| \to \infty} \frac{2}{|\beta|} = \theta \), one has \( \lim \| t_j \|_{1/|\beta_G|} = \tau(K, \theta) \), then we say that \( \{ t_j \} \) is a \( Z \)-asymptotically Chebyshev sequence.

We always assume also that \( u_1(z) = t_1(z) := 1 \).

As observed in [BI01] and [BBL], the concept of a sequence of \( Z \)-asymptotically Chebyshev polynomials is a generalization of many other important polynomial types studied in the literature such as Fekete polynomials associated with an array of Fekete points in a compact set \( K \), Leja polynomials associated with a sequence of so-called Leja points in a compact set \( K \) and \( L^2(\mu) \)-minimal polynomials for a compact set \( K \), where \( \mu \) is a Bernstein-Markov measure. For more nice examples, see [BBL]. We will be also working with \( Z \)-asymptotically Chebyshev polynomials in the context of Definition 1.1, and as mentioned and investigated in [BBL], our bases in this paper do not have to be orthonormal either.

Let \( \{ t_j \} \) be a \( Z \)-asymptotically Chebyshev sequence for \( K \). Write

\[
(1.5) \quad u_j(z) := \frac{t_j}{\| t_j(z) \|_K}.
\]

In [BBL], the authors study the following Chebyshev-Bergman functions

\[
(1.6) \quad \Gamma_n(z) := \sum_{j=1}^{d_n} |u_j(z)|^2.
\]

In Proposition 2.3 of [BBL], by using a Zakharyuta-Siciak type theorem of Bloom ([BI01], Theorem 4.2) for \( Z \)-asymptotically Chebyshev sequences for compact sets in \( \mathbb{C}^m \) and a diagonalization argument, the authors prove for the sequence \( \Gamma_n \) that when a subsequence \( L \) of \( \mathbb{N} \) is given, one can find another subsequence \( L' \subset L \) and a countable dense subset of points \( \{ z_k \} \) \((k = 1, 2, \ldots) \) in \( \mathbb{C}^m \) such that the following holds

\[
\lim_{n \to \infty, n \in L'} \frac{1}{2n} \log \Gamma_n(z_k) = V_K(z_k), \quad k = 1, 2, \ldots
\]

Theorem 4.2 in [BI01] also yields that the sequence \( \{ \frac{1}{2n} \log \Gamma_n \} \) is locally uniformly bounded from above on \( \mathbb{C}^m \).

As a consequence of Proposition 2.3 in [BBL], the following lemma is proved, which will be crucial for the purposes of this paper.

**Lemma 1.2** (Corollary 2.6, [BBL]). Given a compact set \( K \subset \mathbb{C}^m \), for a sequence of \( Z \)-asymptotically Chebyshev polynomials for \( K \), one has

\[
\frac{1}{2n} \log \Gamma_n \to V_K
\]

in \( L^1_{loc}(\mathbb{C}^m) \).
\( \mathcal{D}^{p,q}(\mathbb{C}^m) \) denotes the space of test forms of bidegree \((p,q)\) on \( \mathbb{C}^m \).

Our probabilistic setup necessary in the sequel will be as follows. Following the papers such as [CM1] (see also [BCM] and references therein), we describe how we randomize the space \( \mathcal{P}_n \).

Let \( K \subset \mathbb{C}^m \) be compact. Let \( \{u_j\}_{j=1}^{d_n} \) be a basis for \( \mathcal{P}_n \) consisting of \( Z \)-asymptotically Chebyshev polynomials for \( K \) such that the polynomials \( u_j \) are as in (1.5). Then, for any polynomial \( F \in \mathcal{P}_n \) of degree \( n \), we have

\[
F(z) = \sum_{l=1}^{d_n} a_l u_l(z) := \langle a, u(z) \rangle \in \mathcal{P}_n,
\]

where \( a = (a_1, \ldots, a_{d_n}) \in \mathbb{C}^{d_n} \) and \( u(z) = (u_1(z), \ldots, u_{d_n}(z)) \in \mathcal{P}_n^{d_n} \). We identify the space \( \mathcal{P}_n \) with \( \mathbb{C}^{d_n} \) and furnish it with a probability measure \( \mu_n \) satisfying the moment condition below:

There exist a constant \( \alpha \geq 2 \) and for every \( n \geq 1 \) constants \( C_n > 0 \) such that

\[
\int_{\mathbb{C}^{d_n}} |\log |\langle a, v \rangle||^\alpha d\mu_n(a) \leq C_n
\]

for every \( v \in \mathbb{C}^{d_n} \) with \( ||v|| = 1 \).

Many other widely used probability measures verify this moment condition (1.8), such as Gaussian, Fubini-Study, locally moderate measures etc., we refer the interested reader to [BCM] for a more detailed exposition of these cases.

The zero set of \( F \) is denoted by \( Z_F \), that is, \( Z_F := \{ z \in \mathbb{C}^m : F(z) = 0 \} \). Associated with this zero set, we consider the random current of integration over the zero set \( Z_F \), in symbols \( [Z_F] \), defined as follows, given a test form \( \phi \in \mathcal{D}^{m-1,m-1}(\mathbb{C}^m) \)

\[
\langle [Z_F], \phi \rangle := \int_{\{F(z) = 0\}} \phi.
\]

The expectation and the variance of the random current \( [Z_F] \) are defined by

\[
\mathbb{E}\langle [Z_F], \phi \rangle := \int_{\mathcal{P}_n} \langle [Z_F], \phi \rangle d\mu_n(F)
\]

\[
\text{Var}\langle [Z_F], \phi \rangle := \mathbb{E}\langle [Z_F], \phi \rangle^2 - (\mathbb{E}\langle [Z_F], \phi \rangle)^2,
\]

where \( \phi \in \mathcal{D}^{m-1,m-1}(\mathbb{C}^m) \) and \( \mu_n \) is the probability measure on \( \mathbb{C}^m \) coming from the identification of \( \mathcal{P}_n \). Variance and expectation can be regarded as current-valued random variables as well.

Note that the moment condition (1.8) is slightly different than the one given in [BCM] and [CM1] (see, e.g., p.3, assumption (B) in [BCM]) in order to guarantee that the variance of random currents of integration of zero sets of polynomials (see section 4) is well-defined. As is well-known, we have the Poincaré- Lelong formula

\[
[Z_F] = dd^c \log |Z_F|,
\]

where \( dd^c = \frac{i}{2} \partial \overline{\partial} \) is used throughout the paper. In the sequel, we study the random currents of integration by normalizing them with the degree of the polynomial, namely, given \( F_n \in \mathcal{P}_n \) of degree \( n \) having a representation in (1.7), \( \hat{Z}_{F_n} := \frac{1}{n}[Z_{F_n}] \).

We remark that for simplicity we work with test forms throughout the paper but our results are of course true for continuous forms with compact support in \( \mathbb{C}^m \) by density of test forms in continuous forms with compact support.
2. Equidistribution Result

2.1. Expected Distribution of zeros. As before, let a compact set $K \subset \mathbb{C}^m$ be given. For a random polynomial $F_n \in \mathcal{P}_n$ for $K$ as in (1.7), we have

**Lemma 2.1.** Assume that $C_n = o(n^\alpha)$. Then the following holds true

$$
E[Z_{F_n}] \to dd^c V_K
$$

in the weak* topology of currents as the degree $n \to \infty$.

**Proof.** First, by using the relation (1.6), form the following unit vectors in $\alpha$

$$
\lambda(z) := \left( \frac{u_1(z)}{\sqrt{\Gamma_n(z)}}, \ldots, \frac{u_d(z)}{\sqrt{\Gamma_n(z)}} \right).
$$

Observe that

$$
\frac{1}{n} \log |F_n(z)| = \frac{1}{n} \log |\langle a, \lambda(z) \rangle| + \frac{1}{2n} \log \Gamma_n(z),
$$

here $a = (a_1, \ldots, a_d_n) \in \mathbb{C}^d_n$. Let us take a test form $\phi \in \mathcal{D}^{1,1}(\mathbb{C}^m)$. We have, by definition of expectation, the Poincare-Lelong formula (1.10), the identification of $\mathcal{P}_n$ with $\mathbb{C}^d_n$ and Fubini-Tonelli’s theorem,

$$
\frac{1}{n} E[\langle Z_{F_n}, \phi \rangle] = \int_{\mathbb{C}^d_n} \left( \frac{1}{2n} dd^c \log \Gamma_{n,m} \phi \right) d\mu_n(a) + \frac{1}{n} \int_{\mathbb{C}^m} \log |\langle a, \lambda(z) \rangle| d\mu_n(a) dd^c \phi(z).
$$

By our moment condition (1.8) and Hölder’s inequality, the second term in (2.4) can be estimated from above as follows

$$
\frac{1}{n} \int_{\mathbb{C}^m} \log |\langle a, \lambda(z) \rangle| d\mu_n(a) dd^c \phi(z) \leq \frac{C_n^{1/\alpha}}{n} D_\phi,
$$

where $D_\phi$ is some finite constant depending on the form $\phi$ having a compact support in $\mathbb{C}^m$, to be specific here, it can be taken to be the sum of the supremum norms of the coefficients of the form $dd^c \phi$. When we pass to the limit as $n \to \infty$ in (2.4), the second term goes to zero owing to the inequality (2.5) and our hypothesis $C_n = o(n^\alpha)$. Therefore the first term converges to $dd^c V_K$ in the weak* topology by Lemma [1.2], which concludes the proof. \[\square\]

**Remark 2.2.** $|E[\langle Z_{F_n}, \phi \rangle]|$ is bounded for any $\phi \in \mathcal{D}^{1,1}(\mathbb{C}^m)$ because $\left\{ \frac{1}{2n} \log \Gamma_n \right\}$ is locally uniformly bounded from above on $\mathbb{C}^m$ ([BBL], page 6) and $\phi$ has a compact support in $\mathbb{C}^m$ so, in the expression (2.4), the first term is bounded from above and the second integral has, as seen from the proof, already a bound from above (and also from below), which all in all gives the boundedness of $|E[\langle Z_{F_n}, \phi \rangle]|$.

Observe also that the exponent $\alpha$ in Lemma 2.1 does not have to be bigger than or equal to 2, here the condition $\alpha \geq 1$ works as well, however in the next section, we shall need $\alpha$ to satisfy $\alpha \geq 2$.

2.2. Variance Estimate. We establish a variance estimate of a random zero current of integration over its zero set. Before starting, with the help of Remark 2.2, in the definition of variance (1.9), it will be enough to find an upper bound for $E[\langle Z_{F_n}, \phi \rangle]^2$. We follow the way used in, for instance, [SZ99] (see also [Shif]).

**Theorem 2.3.** Assume that the probability space $(\mathcal{P}_n, \mu_n)$ verifies the moment condition (1.8). Then for any form $\phi \in \mathcal{D}^{1,1}(\mathbb{C}^m)$, the following variance estimate of the random current of integration $[Z_{F_n}]$ holds

$$
\text{Var}(\langle Z_{F_n}, \phi \rangle) \leq D_\phi^2 + 2D_\phi \frac{C_n^{1/\alpha}}{n} + D_\phi^2 (C_n)^{\frac{\alpha}{2}} \frac{1}{n^2},
$$
where $D_\phi$ is a constant depending on the test form $\phi$.

**Proof.** Pick any $\phi \in \mathcal{D}^{m-1,m-1}(\mathbb{C}^m)$. By the representation (1.7) for $F_n$ and the Poincare-Lelong formula (1.10), we first write

\begin{equation}
(2.7) \quad \mathbb{E}([Z_{F_n}], \phi)^2 = \frac{1}{n^2} \int_{\mathcal{P}_n} \int_{\mathbb{C}^m} \int_{\mathbb{C}^m} \log \langle a, u^{(n)}(z) \rangle \log \langle a, u^{(n)}(w) \rangle dd^c\phi(z)dd^c\phi(w)d\mu_n(F_n),
\end{equation}

where $u^{(n)}(z) := (u_1(z), \ldots, u_{dn}(z))$. By the relation (2.3), the integrand in (2.7) becomes

\begin{equation}
(2.8) \quad \frac{1}{4n^2} \log \Gamma_n(z) \log \Gamma_n(w) + \frac{1}{2n^2} \log \Gamma_n(z) \log \langle a, \lambda^{(n)}(w) \rangle + \frac{1}{2n^2} \log \Gamma_n(w) \log \langle a, \lambda^{(n)}(z) \rangle + \frac{1}{n^2} \log \langle a, \lambda^{(n)}(z) \rangle \log \langle a, \lambda^{(n)}(w) \rangle.
\end{equation}

Let us begin with the first term in (2.8). Call the triple integral of the first term $A_1$. To get a bound for $A_1$, it will suffice to estimate the integral below

\begin{equation}
(2.10) \quad \int_{\mathbb{C}^m} \frac{1}{2n} \log \Gamma_n(z)dd^c\phi(z).
\end{equation}

To do this, first by the inequality $\lim \sup_{n \to \infty} \frac{1}{2n} \log \Gamma_n(z) \leq V_K(z)$ on $z \in \mathbb{C}^m$ (the relation (2.5) at page 6 of [BBL]), we have

\begin{equation}
(2.11) \quad \int_{\mathbb{C}^m} \frac{1}{2n} \log \Gamma_n(z)dd^c\phi(z) \leq \int_{\mathbb{C}^m} V_K(z)dd^c(z) \leq D_\phi,
\end{equation}

here $D_\phi$ is, as above, a constant depending only on $\phi$ since $V_K(z)$ is locally uniformly bounded from above on $\mathbb{C}^m$ and $\phi$ does have a compact support in $\mathbb{C}^m$. Hence we have $A_1 \leq D_\phi^2$ since $\mu_n$ is a probability measure.

The second and the third terms in (2.8) will have the same estimations, so we treat just one of them and denote each of them by $A_2$. In view of (2.11), our moment condition (1.8), H"older’s inequality, the identification of $\mathcal{P}_n$ with $\mathbb{C}^{dn}$ and Fubini-Tonelli’s theorem, we get the following inequality

\begin{equation}
(2.12) \quad 2A_2 \leq 2 \int_{\mathcal{P}_n} \int_{\mathbb{C}^m} \int_{\mathbb{C}^m} \frac{1}{2n} \log \langle a, \lambda^{(n)}(w) \rangle dd^c\phi(z)dd^c\phi(w)d\mu_n(F_n) \leq 2D_\phi \frac{C_n^{1/\alpha}}{n^\beta}.
\end{equation}

Finally we will get an upper bound for the last term, which we write $A_3$ for, that is, the term in (2.9). What we do is to apply H"older’s inequality twice for the appropriate exponents. Then, by the identification of $\mathcal{P}_n$, Fubini-Tonelli’s theorem and H"older’s inequality with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, where $\alpha \geq 2$ is the exponent determined for the moment condition (1.8),

\begin{align*}
A_3 & \leq \int_{\mathbb{C}^m} \int_{\mathbb{C}^m} dd^c\phi(z)dd^c\phi(w) \frac{1}{n^2} \int_{\mathbb{C}^{4n}} \log \langle a, \lambda^{(n)}(z) \rangle \log \langle a, \lambda^{(n)}(w) \rangle dd^c\phi(z)dd^c\phi(w)d\mu_n(a) \\
& \leq \int_{\mathbb{C}^m} \int_{\mathbb{C}^m} dd^c\phi(z)dd^c\phi(w) \frac{1}{n^2} \left\{ \int_{\mathbb{C}^{4n}} \log \langle a, \lambda^{(n)}(z) \rangle d\mu_n(a) \right\}^{\frac{\alpha}{2}} \left\{ \int_{\mathbb{C}^{4n}} \log \langle a, \lambda^{(n)}(z) \rangle d\mu_n(a) \right\}^{\frac{\beta}{2}} \\
& \leq \int_{\mathbb{C}^m} \int_{\mathbb{C}^m} dd^c\phi(z)dd^c\phi(w) \frac{1}{n^2} \left( C_n \right)^{\frac{1}{\alpha}} \left\{ \int_{\mathbb{C}^{4n}} \log \langle a, \lambda^{(n)}(z) \rangle d\mu_n(a) \right\}^{\frac{\alpha}{2}} \frac{1}{n^2} \left( C_n \right)^{\frac{1}{\beta}} \frac{1}{n^2} \left( C_n \right)^{\frac{1}{\beta}}.
\end{align*}

We need to apply H"older’s inequality to the innermost integral one more time and as we mentioned, the condition that $\alpha \geq 2$ (and so $\alpha \geq 2 \geq \beta$) is important here and it will enable us to reuse H"older’s inequality, therefore we get

\begin{equation}
(2.13) \quad A_3 \leq \int_{\mathbb{C}^m} \int_{\mathbb{C}^m} dd^c\phi(z)dd^c\phi(w) \frac{1}{n^2} \left( C_n \right)^{\frac{\alpha}{2}} \leq \frac{1}{n^2} D_\phi^2 \left( C_n \right)^{\frac{\alpha}{2}}.
\end{equation}
Altogether, by (2.11), (2.12) and (2.13), we have
\begin{equation}
\text{Var}(\widehat{Z}_{F_n}, \phi) \leq D_\phi^2 + 2D_\phi \frac{C_n^{1/\alpha}}{n} + D_\phi^2 (C_n) \frac{1}{n^2},
\end{equation}
which completes the variance estimate of the random current of integration $\widehat{Z}_{F_n}$. □

Now the equidistribution result in codimension 1 will be proved.

**Theorem 2.4.** Under the same condition as in Theorem 2.3, if $\sum_{n=1}^{\infty} \frac{C_n^{1/\alpha}}{n} < \infty$, then for $\mu_n$-almost every sequence $\{F_n\}$,
\begin{equation}
\text{Var}(\widehat{Z}_{F_n}, \phi) \rightarrow dd^c V_K
\end{equation}
in the weak* topology of currents as $n \rightarrow \infty$.

**Proof.** Let $\phi \in D^{m-1,m-1}(\mathbb{C}^m)$. Following [SZ99], we consider the random variables
\begin{equation}
W_n(F_n) := (\text{Var}(\widehat{Z}_{F_n}, \phi)) \geq 0.
\end{equation}

First, by the alternative definition of variance,
\begin{equation}
\int \mathcal{P}_n W_n(F_n)d\mu_n(F_n) = \text{Var}(\widehat{Z}_{F_n}, \phi).
\end{equation}

By assumption and Theorem 2.3 one has
\begin{equation}
\int \mathcal{P}_n \sum_{n=1}^{\infty} W_n(F_n)d\mu_n(F_n) = \sum_{n=1}^{\infty} \int \mathcal{P}_n W_n d\mu_n(F_n) = \sum_{n=1}^{\infty} \text{Var}(\widehat{Z}_{F_n}, \phi) < +\infty,
\end{equation}
This yields, for $\mu_n$-almost surely, that $W_n \rightarrow 0$, namely
\[\langle \widehat{Z}_{F_n}, \phi \rangle - \mathbb{E}\langle \widehat{Z}_{F_n}, \phi \rangle \rightarrow 0\]
$\mu_n$-almost surely. This then gives, by Lemma 2.1, for $\mu_n$-almost all $F_n \in \mathcal{P}_n$,
\[\widehat{Z}_{F_n} \rightarrow dd^c V_K\]
in the weak* topology of currents, which finishes the proof. □

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