Section sigma models coupled to symplectic duality bundles on Lorentzian four-manifolds

C. I. Lazaroiu\textsuperscript{1} and C. S. Shahbazi\textsuperscript{2}

\textsuperscript{1} Center for Geometry and Physics, Institute for Basic Science, Pohang 790-784, Republic of Korea, E-mail: calin@ibs.re.kr
\textsuperscript{2} Department of Mathematics, University of Hamburg, Germany, E-mail: carlos.shahbazi@uni-hamburg.de

Abstract: We give the global mathematical formulation of a class of generalized four-dimensional theories of gravity coupled to scalar matter and to Abelian gauge fields. In such theories, the scalar fields are described by a section of a surjective pseudo-Riemannian submersion $\pi$ over space-time, whose total space carries a Lorentzian metric making the fibers into totally-geodesic connected Riemannian submanifolds. In particular, $\pi$ is a fiber bundle endowed with a complete Ehresmann connection whose transport acts through isometries between the fibers. In turn, the Abelian gauge fields are “twisted” by a flat symplectic vector bundle defined over the total space of $\pi$. This vector bundle is endowed with a vertical taming which locally encodes the gauge couplings and theta angles of the theory and gives rise to the notion of twisted self-duality, of crucial importance to construct the theory. When the Ehresmann connection of $\pi$ is integrable, we show that our theories are locally equivalent to ordinary Einstein-Scalar-Maxwell theories and hence provide a global non-trivial extension of the universal bosonic sector of four-dimensional supergravity. In this case, we show using a special trivializing atlas of $\pi$ that global solutions of such models can be interpreted as classical “locally-geometric” U-folds. In the non-integrable case, our theories differ locally from ordinary Einstein-Scalar-Maxwell theories and may provide a geometric description of classical U-folds which are “locally non-geometric”.

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1. Introduction

The construction of four-dimensional supergravity theories usually found in the physics literature (see, for example, [1,2,3,4]) is local in the sense that it is carried out ignoring the topology of the space-time manifold and without specifying the precise global description of the configuration space or the global mathematical structures required to define it. Such constructions are discussed traditionally only on a contractible subset $U$ of space-time, which guarantees that any fiber bundle defined on $U$ is trivial and
hence that any section of such a bundle can be identified with a map from $U$ into the fiber. Accordingly, the physics literature traditionally treats all classical fields as functions defined on $U$ and valued in some target space, which is either a vector space or (for the scalar fields) a manifold $\mathcal{M}$ endowed with a Riemannian metric $\mathcal{G}$. It is often also tacitly assumed that $\mathcal{M}$ is contractible, which implies that the duality structure $[5,6]$ of the Abelian gauge theory coupled to the scalar fields is described by a trivial flat symplectic vector bundle $S$ defined on $\mathcal{M}$, whose data can then be encoded by a symplectic vector space $S^0$ (the fiber of $S$) $[2,3]$. Due to this assumption, the gauge field strengths and their Lagrangian conjugates are usually treated as two-forms defined on $U$ and valued in $S^0$. It is not unambiguously clear how such local constructions can be made into complete, mathematically rigorous, global definitions of classical theories of matter coupled to gravity when $\mathcal{M}$ and $\mathcal{M}$ are not contractible. To fully define such theories, one must decide how to interpret globally various local formulas and differential operators. Such global interpretations are generally non-unique in the sense that they depend on choices of auxiliary geometric data which are not visible in the usual local formulation $[5,6,7]$.

In this paper, we consider the issue of finding a general global mathematical formulation of the universal bosonic sector of four-dimensional supergravity theories, which consists of gravity coupled to scalars and to Abelian gauge fields. We find, among other results, that the traditional local formulas $[1,2,4,8,9,10]$ can be interpreted globally in a manner which provides a mathematical definition of "classical locally-geometric U-folds" as global solutions of the equations of motion of the resulting globally-defined classical theory. Furthermore, the global theory that we obtain represents a non-trivial extension of the standard bosonic sector of ungauged four-dimensional supergravity. The supersymmetrization of the theories introduced in this paper is currently an open problem involving several objects and structures of mathematical interest, such as spinor bundles and Lipschitz structures, Special-Kähler and Quaternionic-Kähler manifolds or exceptional Lie groups, all interacting in a delicate equilibrium dictated by supersymmetry.

In our construction, the scalar map of the sigma model is first promoted to a section of a Kaluza-Klein space defined over space-time and endowed with a vertical scalar potential $\Phi$, leading to a section sigma model for the scalar fields. The latter is described by a Lagrangian density defined on the space of sections of $\pi$. In the particular case when $\Phi$ vanishes, the solutions of the equations of motion of the section sigma model are the pseudoharmonic sections studied by C. M. Wood $[11,12,13,14,15,16]$. The section sigma model is then coupled to Abelian gauge fields governed by a non-trivial duality structure, thereby extending the construction of $[5]$ to this more general setting. The globally-defined theory obtained in this manner will be called generalized Einstein-Section Maxwell (GESM) theory.

By definition, a Kaluza-Klein space over a Lorentzian manifold $(M,g)$ is a surjective submersion $\pi: E \rightarrow M$ whose total space $E$ is endowed with a Lorentzian metric $h$ (known as a Kaluza-Klein metric) such that $\pi$ is a pseudo-Riemannian submersion $[17]$ from $(E,h)$ to $(M,g)$, whose fibers are totally geodesic Riemannian connected submanifolds of $(E,h)$. Such spaces were studied in the literature on the mathematical foundations of Kaluza-Klein theory (see, for example, $[18,19,20,21]$) and their Riemannian counterpart played an important role in the construction of non-trivial examples of Einstein metrics $[22]$ and in the study of metrics of positive sectional curvature $[23]$. The orthogonal complement of the vertical distribution of $\pi$ with respect to a Kaluza-Klein metric $h$ is a horizontal distribution $H$ which gives a complete Ehresmann connection for $\pi$. Since the fibers are totally geodesic, the Ehresmann transport $T$ defined by $H$ proceeds through isometries between the fibers. As a consequence, all fibers can be identified with some model Riemannian manifold $(M,\mathcal{G})$ and the holonomy group $\mathcal{G}$ of the Ehresmann connection is a subgroup of the isometry group $\text{Iso}(M,\mathcal{G})$ of the fiber, thereby being a finite-dimensional Lie group. This implies that $\pi$ is a fiber bundle with structure group $\mathcal{G}$ (a fiber $\mathcal{G}$-bundle in the sense of $[24,25]$) and hence is associated to a principal $\mathcal{G}$-bundle $\Pi$ (known as the holonomy bundle) through the isometric action of $\mathcal{G}$ on $M$. Moreover, the Ehresmann connection $H$ is induced by a principal connection $\theta$ defined on $\Pi$. In fact, the Kaluza-Klein metric $h$ is uniquely determined by $H$, $g$ and $\mathcal{G}$ or, equivalently, by $\theta$, $g$ and $\mathcal{G}$ together with an embedding $G \hookrightarrow \text{Iso}(M,\mathcal{G})$. We say that the Kaluza-Klein space is integrable if the distribution $H$ is Frobenius integrable, which happens if and only if the principal connection $\theta$ is flat. A vertical scalar potential on $\pi$ is a smooth real-valued map $\Phi$ defined on the total space $E$ of $\pi$ such that the restrictions of $\Phi$ to the fibers of $\pi$ are related by the Ehresmann transport and hence can be identified with a smooth real-valued function $\Phi$ defined on $M$, which is invariant under the holonomy group $\mathcal{G}$.

When $\pi$ is a topologically trivial fiber bundle and $H$ is a trivial Ehresmann connection, the sections of $\pi$ can be identified with the graphs of smooth functions from $M$ to $\mathcal{M}$ and the section sigma model reduces to
the ordinary scalar sigma model defined by the “scalar structure” \((\mathcal{M}, \mathcal{G}, \Phi)\) [5,6]. This reduction always happens when \(M\) is contractible and \(H\) is integrable, since in that case \(\pi\) is necessarily topologically trivial and \(H\) is necessarily a trivial Ehresmann connection. In particular, scalar sigma models with integrable Kaluza-Klein space are locally indistinguishable from ordinary scalar sigma models. When \(\pi\) is topologically trivial but \(H\) is not integrable, the section sigma model reduces to a modified nonlinear sigma model for maps from \(M\) to \(\mathcal{M}\), which, to our knowledge, has not been considered before. As a consequence, section sigma models with non-integrable Kaluza-Klein space are locally distinct from ordinary sigma models.

As mentioned above, section sigma models can be coupled to Abelian gauge fields, leading to the construction of generalized Einstein-Section-Maxwell (GESM) theories. The most general coupling of the ordinary scalar sigma model to Abelian gauge fields can be described globally [5] using a so-called “electromagnetic structure” defined on \(\mathcal{M}\). In our situation, this is promoted to a “horizontally constant” electromagnetic structure defined on the total space \(E\) of the Kaluza-Klein space. By definition, this is a tamed flat symplectic vector bundle \(\mathcal{S}\) defined over \(E\) such that the taming is invariant under the lift of the Ehresmann transport of \(\pi\) along the flat connection of \(\mathcal{S}\).

The fiber bundle \(\pi\) admits special trivializing atlases supported on convex covers \((U_\alpha)_{\alpha \in I}\) of \(M\). The local trivialization maps of such an atlas are constructed using the Ehresmann transport along geodesics lying inside \(U_\alpha\) and ending at some reference point chosen in each \(U_\alpha\). The behavior of the section sigma model with respect to such atlases depends on whether the Ehresmann connection \(H\) is integrable or not:

A. When the Kaluza-Klein space is integrable, any special trivializing atlas allows one to identify the restriction of \((\pi, H)\) to \(U_\alpha\) with the topological trivial fiber bundle \((E_\alpha^0 \overset{def}{=} U_\alpha \times \mathcal{M}, \pi_\alpha^0 \overset{def}{=} \pi|_{E_\alpha^0})\) defined over \(U_\alpha\), endowed with the trivial Ehresmann connection. In this case, the restriction of the section sigma model to \(U_\alpha\) identifies with the ordinary sigma model of maps from \(U_\alpha\) to \(\mathcal{M}\). This implies that a global solution of the section sigma model can be obtained by patching local solutions of the ordinary scalar sigma model using symmetries of the equations of motion of the latter and thus it can be interpreted as a classical counterpart of a U-fold. Similar statements apply after coupling to Abelian gauge fields. Thus:

*When the Kaluza-Klein space is integrable, global solutions of the GESM theory correspond to classical locally-geometric U-folds glued from local solutions of the ordinary sigma model coupled to Abelian gauge fields (which may have a non-trivial duality structure) using symmetries of the equations of motion of the latter.*

This can be taken as a rigorous definition of classical locally-geometric U-folds and may provide a global geometric description of the classical limit of certain string theory U-folds in four dimensions.

B. When the Kaluza-Klein space is not integrable, the restriction of \(H\) to \(U_\alpha\) identifies with a non-integrable horizontal distribution of \(\pi_\alpha^0\). In this case, local sections of \(\pi\) defined over \(U_\alpha\) are the graphs of smooth maps into the fiber \(\mathcal{M}\) and local solutions can be interpreted as solutions of the modified non-linear sigma model for maps mentioned above. Hence:

*When the Kaluza-Klein space is not integrable, global solutions of the GESM theory can be glued from local solutions of the modified sigma model coupled to Abelian gauge fields (which may have a non-trivial duality structure).*

Together with the results of [5], point A. above implies that GESM theories with integrable Kaluza-Klein space are locally indistinguishable from usual Einstein-Scalar-Maxwell theories and hence provide admissible global interpretations of the local formulas governing the universal bosonic sector of four-dimensional supergravity. On the other hand, point B. above implies that GESM theories with non-integrable Kaluza-Klein space are locally distinct from ordinary ESM theories.
The paper is organized as follows. Section 2 discusses Kaluza-Klein spaces, their special trivializing atlases and their classification in the integrable case. The same section discusses vertical scalar potentials and bundles of scalar structures as well as the classification of the latter in the integrable case. Section 3 discusses section sigma models and the U-fold interpretation of their global solutions in the integrable case. Section 4 discusses bundles of scalar-electromagnetic structures, which are Kaluza-Klein spaces endowed with “horizontally-constant” data describing the scalar potential and electromagnetic bundle needed to couple the section sigma model to Abelian gauge fields. Section 5 gives the global formulation of a generalized Einstein-Section-Maxwell theory. In the integrable case, we show that special trivializing atlases of the underlying Kaluza-Klein space allow one to interpret global solutions of such models as classical locally-geometric U-folds. This section contains the main result of the paper, namely Theorem 5.1, which proves in precise terms the local equivalence between GESM theories and ordinary ESM theories. Section 6 illustrates section sigma models with a simple example, showing how in a special case they recover the celebrated Scherk-Schwarz construction. Finally, Section 7 contains our conclusions and some directions for further research. Appendix A contains technical material on pseudo-Riemannian submersions and Kaluza-Klein spaces. Appendix B shows that, in the non-integrable case, a section sigma model reduces locally to a modified sigma model for maps, which we describe explicitly using adapted local coordinates. In addition, the same Appendix gives local expressions in adapted coordinates for some key objects used in the formulation of GESM theories.

1.1. Notations and conventions. All manifolds considered are smooth, connected, Hausdorff and paracompact (hence also second countable) while all fiber bundles considered are smooth. All submersions considered are assumed to be surjective and to have connected fibers. Given vector bundles \( \mathcal{S} \) and \( \mathcal{S}' \) over some manifold \( M \), we denote by \( \text{Hom}(\mathcal{S}, \mathcal{S}') \), \( \text{Isom}(\mathcal{S}, \mathcal{S}') \) the bundles of morphisms and isomorphisms from \( \mathcal{S} \) to \( \mathcal{S}' \) and by \( \text{Hom}(\mathcal{S}, \mathcal{S}') \overset{\text{def}}{=} \Gamma(M, \text{Hom}(\mathcal{S}, \mathcal{S}')) \), \( \text{Isom}(\mathcal{S}, \mathcal{S}') \overset{\text{def}}{=} \Gamma(M, \text{Isom}(\mathcal{S}, \mathcal{S}')) \) the sets of smooth sections of these bundles. When \( \mathcal{S}' = \mathcal{S} \), we set \( \text{End}(\mathcal{S}) \overset{\text{def}}{=} \text{Hom}(\mathcal{S}, \mathcal{S}) \), \( \text{Aut}(\mathcal{S}) \overset{\text{def}}{=} \text{Isom}(\mathcal{S}, \mathcal{S}) \) and \( \text{End}(\mathcal{S}) \overset{\text{def}}{=} \text{Hom}(\mathcal{S}, \mathcal{S}) \), \( \text{Aut}(\mathcal{S}) \overset{\text{def}}{=} \text{Isom}(\mathcal{S}, \mathcal{S}) \). Given a smooth map \( f : M_1 \to M_2 \) and a vector bundle \( \mathcal{S} \) on \( M_2 \), we denote the \( f \)-pull-back of \( \mathcal{S} \) to \( M_1 \) by \( \mathcal{S}^f \). Given a section \( s \in \Gamma(M_2, \mathcal{S}) \), we denote its \( f \)-pullback by \( s^f \in \Gamma(M_1, \mathcal{S}^f) \). Given \( T \in \text{Hom}(\mathcal{S}, \mathcal{S}') \), where \( \mathcal{S}, \mathcal{S}' \) are vector bundles defined on \( M_2 \), we denote the \( f \)-pullback of \( T \) by \( T^f \in \text{Hom}(\mathcal{S}^f, (\mathcal{S}')^f) \). Let \( \text{Met}_{p,q}(W) \) denote the set of non-degenerate symmetric pairings of signature \( (p,q) \) on a vector bundle \( W \) of rank \( \text{rk} W = p + q \). Let \( \text{Met}(W) \overset{\text{def}}{=} \sqcup_{p,q \geq 0, p+q = \text{rk} W} \text{Met}_{p,q}(W) \) denote the set of all non-degenerate metrics on \( W \). When \( W = TM \) is the tangent bundle of a manifold \( M \), we set \( \text{Met}_{p,q}(M) \overset{\text{def}}{=} \text{Met}_{p,q}(TM) \) and \( \text{Met}(M) \overset{\text{def}}{=} \text{Met}(TM) \). By definition, a Lorentzian manifold is a pseudo-Riemannian manifold of “mostly plus” signature. Given a manifold \( M \), let \( \mathcal{P}(M) \) denote the set of paths (piecewise-smooth curves) \( \gamma : [0,1] \to M \). We sometimes use various notations and conventions introduced in [5].

Remark 1.1. Throughout the paper, we use the mathematical concept of “Kaluza-Klein space”, which is well-established in the mathematics and mathematical physics literature (see, for example, [18,19,20,21,22]). Such spaces were initially defined and studied in the context of mathematical foundations of Kaluza-Klein theories, which arise by reducing a higher-dimensional theory of gravity on such a manifold. In the present paper, such spaces are used merely as convenient auxiliary mathematical data which parameterize a GESM theory, despite the historical context in which they were introduced initially. Throughout the paper, no reduction of any putative higher dimensional theory on such spaces is ever suggested or performed.

2. Kaluza-Klein spaces, vertical potentials and bundles of scalar structures

2.1. Lorentzian submersions and Kaluza-Klein metrics. Let \( (M, g) \) be a connected four-dimensional Lorentzian manifold. Let \( E \) be a connected manifold of dimension \( \text{dim} E = n + 4 \), where \( n > 0 \). Recall that a smooth map \( \pi : E \to M \) is called a surjective submersion if \( \pi \) is surjective and the linear map \( \partial_\pi \pi : T_e E \to T_{\pi(e)} M \) is surjective for all \( e \in E \). We shall always assume that the fibers of \( \pi \) are connected. \(^1\)

\(^1\) Albeit without gauging of any putative continuous isometry of the scalar manifold. In fact, our construction does not assume existence of any continuous isometries.
Let $\pi : E \to M$ be a surjective submersion with connected fibers. The vertical distribution of $\pi$ is the rank $n$ distribution $V \overset{\text{def}}{=} \ker(d\pi) \subset TE$ defined on $E$, which is Frobenius integrable and integrates to the foliation whose leaves are the fibers of $\pi$. A Lorentzian metric $h$ on $E$ is called $\pi$-positive if its restriction to each fiber of $\pi$ is positive-definite. In this case, the $h$-orthogonal complement $H := H(h)$ of $V$ inside $TE$ is a distribution of rank four called the horizontal distribution of $\pi$ determined by $h$. Let $hv$ and $hV$ denote the metrics induced by $h$ on $V$ and $H$, which we call the vertical and horizontal metrics induced by $h$. For any $e \in E$, the map $d\pi_e$ restricts to a linear bijection from $H_e$ to $T_{\pi(e)} M$. Hence the restriction $d\pi|_H$ of $d\pi$ to $H$ induces a based isomorphism of vector bundles $(d\pi)_H : H \overset{\sim}{\to} (TM)^\pi$.

**Definition 2.1.** Let $h$ be a $\pi$-positive Lorentzian metric on $E$. The surjective submersion $\pi : E \to M$ is called a Lorentzian submersion from $(E,h)$ to $(M,g)$ if the bundle isomorphism $(d\pi)_H$ is an isometry from $(H,h_H)$ to $((TM)^\pi, g^\pi)$.

When $\pi$ is a Lorentzian submersion from $(E,h)$ to $(M,g)$, the pair $(H,h_H)$ is a pseudo-Euclidean distribution of signature $(3,1)$ defined on $E$ while $(V,h_V)$ is a Euclidean distribution. In particular, $\pi$ is a surjective pseudo-Riemannian submersion in the sense of [17].

**Definition 2.2.** A Kaluza-Klein metric for the surjective submersion $\pi : E \to M$ relative to the Lorentzian metric $g \in \text{Met}_3(M)$ is a $\pi$-positive Lorentzian metric $h$ on $E$ which makes $\pi$ into a Lorentzian submersion from $(E,h)$ to $(M,g)$.

More information about pseudo-Riemannian submersions and Kaluza-Klein metrics can be found in Appendix A.

Any horizontal distribution $H$ defines a horizontal lift of vector fields, which takes $Q \in \chi(M)$ into the unique horizontal vector field $\tilde{Q} \in \Gamma(E,H)$ satisfying $d\pi(\tilde{Q}) = Q$. Moreover, any vector field $X \in \chi(E)$ decomposes uniquely as $X = X_H \oplus X_V$, where $X_H \in \Gamma(E,H)$ and $X_V \in \Gamma(E,V)$. The curvature $\mathcal{F} \in \Omega^2(E,V)$ of $H$ is defined as [25]:

$$\mathcal{F}(X,Y) \overset{\text{def}}{=} [X_H,Y_H]_V, \quad \forall X,Y \in \chi(E).$$

Its restriction to $H$ gives a section $\mathcal{F}_H \in \Gamma(E, \wedge^2 H^\ast \otimes V)$ which satisfies:

$$\mathcal{F}_H(X,Y) = \mathcal{F}(X,Y) = [X,Y]_V, \quad \forall X,Y \in \Gamma(E,H)$$

and hence coincides up to a constant factor with the restriction to $H$ of O’Neill’s second fundamental tensor $A$ [26] of the pseudo-Riemannian submersion $\pi$:

$$A_X Y = -A_Y X = \frac{1}{2} \mathcal{F}_H(X,Y), \quad \forall X,Y \in \Gamma(E,H).$$

A basic property of O’Neill’s tensor is that it vanishes if and only if $\mathcal{F}_H$ does. Hence either of $A$ or $\mathcal{F}_H$ describe the obstruction to Frobenius integrability of $H$.

The distribution $H$ is called complete if the flow $T$ defined by horizontal lifts of vector fields is globally-defined, which amounts to the condition that any curve in $M$ lifts to a horizontal curve in $E$ through any point lying in the fiber above its source. When $H$ is complete, it follows from a result of [27] that $\pi$ is a fiber bundle, though its structure group need not be a finite-dimensional Lie group. In this case, $H$ is an Ehresmann connection for $\pi$ and $T$ is called its Ehresmann transport. Reference [28] shows that a sufficient condition for $H$ to be complete and for the structure group to be a finite-dimensional Lie group is that the fibers of $\pi$ be geodesically complete and totally geodesically connected submanifolds of $(E,h)$. In this case the Ehresmann transport $T_\gamma$ is isometric, which means that $T_\gamma$ is an isometry from $E_{\pi(0)}$ to $E_{\pi(1)}$ for any path $\gamma \in \mathcal{P}(M)$. Hence the structure group is isomorphic to the Riemannian isometry group of the fiber of $\pi$ over any given point in $M$.

### 2.2. Lorentzian Kaluza-Klein spaces.

**Definition 2.3.** A Lorentzian Kaluza-Klein space over $(M,g)$ is a Lorentzian submersion $\pi : (E,h) \to (M,g)$ such that $(E,h)$ is connected and such that the fibers of $\pi$ are geodesically complete and totally geodesic connected submanifolds of $(E,h)$. The Kaluza-Klein space is called integrable if the horizontal distribution $H(h) \subset TE$ defined by $h$ is Frobenius integrable.
Consider a Lorentzian Kaluza-Klein space \( \pi : (E, h) \rightarrow (M, g) \) with horizontal distribution \( H := H(h) \), whose Ehresmann transport we denote by \( T := T(h) \). Let \( m_0 \) be fixed point of \( M \) and let \( M \overset{\text{def}}{=} E_{m_0} = \pi^{-1}\{m_0\} \) and \( \mathcal{G} \overset{\text{def}}{=} h|_{E_{m_0}} \). As mentioned above, the results of [28] imply that \( H \) is a complete Ehresmann connection, that \( \pi \) is a fiber bundle and that the transport \( T \) is isometric. Hence the restriction \( h_m \overset{\text{def}}{=} h|_{E_m} \) of \( h \) to the fibers of \( E \) is uniquely determined by \( \mathcal{G} \) and by \( T \). Therefore, the metric \( h \) of a Kaluza-Klein space is uniquely determined by \( g, \mathcal{G} \) and \( H \). When \( g \) and \( \mathcal{G} \) are fixed, we thus have a bijection between Ehresmann connections \( H \) for \( \pi \) and Lorentzian metrics \( h \) on \( E \) such that \( \pi : (E, h) \rightarrow (M, g) \) is a Kaluza-Klein space and such that \( h_{m_0} = \mathcal{G} \). Let:

\[
G \overset{\text{def}}{=} \{ T_\gamma, \ | \ \gamma \in \mathcal{P}(M), \ \gamma(0) = \gamma(1) = m_0 \} \subseteq \text{Iso}(M, \mathcal{G})
\]

be the Ehresmann holonomy group at \( m_0 \). For any \( m \in M \), consider the set:

\[
\Pi_m \overset{\text{def}}{=} \{ T_\gamma | \ \gamma \in \mathcal{P}(M) : \gamma(0) = m_0 \& \gamma(1) = m \},
\]

and let \( \Pi \overset{\text{def}}{=} \bigcup_{m \in M} \Pi_m \) be endowed with the obvious projection to \( M \). Then \( \Pi \) is a principal \( G \)-bundle (known as the holonomy bundle of \( H \) relative to \( m_0 \)) under the obvious right action of \( G \). Moreover, the Ehresmann connection \( H \) induces a principal connection \( \theta \) on \( \Pi \). Conversely, let \( \rho \) be the isometric action of \( G \) on \( M \) given by the inclusion \( G \subseteq \text{Iso}(M, \mathcal{G}) \). Then the associated bundle construction associates to a principal bundle \( \Pi \) with principal connection \( \theta \) the fiber bundle \( E_\Pi \overset{\text{def}}{=} \Pi \times_\rho M \) with induced Ehresmann connection \( H_\theta \). The two correspondences described above give mutually quasi-inverse functors between the groupoid of Kaluza-Klein spaces over \( (M, g) \) which have typical fiber \( (M, \mathcal{G}) \) and are endowed with a horizontal distribution \( H \) with holonomy \( G \subseteq \text{Iso}(M, \mathcal{G}) \) and the groupoid of principal \( G \)-bundles \( \Pi \) defined over \( M \) and endowed with a principal connection \( \theta \) together with an embedding \( G \subseteq \text{Iso}(M, \mathcal{G}) \) up to conjugation. Hence:

**Proposition 2.1.** Let \( g \) be a fixed Lorentzian metric on \( M \). Then isomorphism classes of Kaluza-Klein spaces over \( (M, g) \) with typical fiber \( (M, \mathcal{G}) \) and horizontal distribution \( H \) having Ehresmann holonomy contained in \( \text{Iso}(M, \mathcal{G}) \) are in bijection with isomorphism classes of principal \( G \)-bundles \( \Pi \) defined over \( M \) and endowed with a principal connection \( \theta \) whose holonomy is embedded in \( \text{Iso}(M, \mathcal{G}) \). Moreover, \( H \) is integrable if and only if \( \theta \) is flat.

Let \((M, \mathcal{G})\) be a Riemannian manifold. Let \( E^0 \overset{\text{def}}{=} M \times M \) and let \( \pi^0 : M \times M \rightarrow M \) and \( p^0 : M \times M \rightarrow M \) denote the canonical projections. Notice that \( \pi^0 \) is the trivial fiber bundle over \( M \) with fiber \( M \). Let \( V^0 = (TM)^{p^0} \) denote the vertical distribution of \( \pi^0 \), which is endowed with the pull-back metric \( \mathcal{G}^{p^0} \).

**Definition 2.4.** A topologically trivial Kaluza-Klein space over \((M, g)\) with fiber \((M, \mathcal{G})\) is a Kaluza-Klein space of the form \( \pi^0 : (E^0, h^0) \rightarrow (M, g) \), whose vertical metric is the pull-back metric \( h^{0 \mathcal{G}} \overset{\text{def}}{=} \mathcal{G}^{p^0} \). The metric \( h^0 = h(g, \mathcal{G}, H) \) of such a space is uniquely determined by \( g, \mathcal{G} \) and by the horizontal distribution \( H \), whose parallel transport \( T \) must preserve \( \mathcal{G}^{p^0} \). For any \( \gamma \in \mathcal{P}(M) \) and all \( p \in M \), we define:

\[
T_\gamma(p) = (\gamma(1), T_\gamma(p)),
\]

where \( T_\gamma \in \text{Iso}(M, \mathcal{G}) \) satisfy \( \hat{T}_{\gamma_1 \gamma_2} = \hat{T}_{\gamma_1} \circ \hat{T}_{\gamma_2} \).

**Definition 2.5.** The product Kaluza-Klein space over \((M, g)\) with fiber \((M, \mathcal{G})\) is the topologically trivial Kaluza-Klein space \( \pi^0 : (E^0, h^0) \rightarrow (M, g) \) with fiber \((M, \mathcal{G})\) whose horizontal distribution is the trivial integrable Ehresmann connection \( H^0_{\text{triv}} \overset{\text{def}}{=} (TM)^{p^0} \), with induced horizontal metric \( h^0_{\text{triv}} \).

For any \( p \in M \), the horizontal lift through \((\gamma(0), p) \in E^0_{\gamma(0)} \) of any path \( \gamma \in \mathcal{P}(M) \) defined by the trivial Ehresmann connection \( H^0 \) is the path given by \( \gamma_{p, \text{triv}}(s) = (\gamma(s), p) \) for all \( s \in [0, 1] \). Hence the Ehresmann transport \( T^0_\gamma \) defined by \( H^0 \) is given by \( T^0_\gamma = \text{id}_M \). This implies that the Kaluza-Klein metric \( h^0_{\text{triv}} = h_{g, \mathcal{G}, H^0_{\text{triv}}} \) equals the product metric \( g \times \mathcal{G} \).
As explained above, integrable Kaluza-Klein spaces correspond to flat principal $G$-bundles $(\Pi, \theta)$ defined over $M$ together with an embedding $G \subseteq \text{Iso}(\mathcal{M}, \mathcal{G})$. It is well-known that flat principal $G$-bundles are classified up to isomorphism by their holonomy morphism $\text{Hol}_{P, \theta} : \pi_1(M) \to G$, considered up to the conjugation action of $G$. Hence the set of isomorphism classes of integrable Kaluza-Klein spaces over $(M, g)$ with fiber $(\mathcal{M}, \mathcal{G})$ and Ehresmann holonomy contained in $\text{Iso}(\mathcal{M}, \mathcal{G})$ is in bijection with the character variety:

$$\mathcal{M}_{\text{Iso}(\mathcal{M}, \mathcal{G})}(M) \overset{\text{def}}{=} \text{Hom}(\pi_1(M), \text{Iso}(\mathcal{M}, \mathcal{G}))/\text{Iso}(\mathcal{M}, \mathcal{G}) .$$

When the space-time $M$ is simply-connected, any integrable Kaluza-Klein space over $(M, g)$ is isomorphic with a product space (see below). In that case, we necessarily have $G = 1$ (the trivial group) and $\Pi$ is gauge-equivalent with the trivial Ehresmann connection.

### 2.3. Vertical scalar potentials and scalar bundles

Let $\pi : (E, h) \to (M, g)$ be a Kaluza-Klein space and $T$ be its Ehresmann transport.

**Definition 2.6.** A vertical scalar potential for $\pi$ is a smooth $T$-invariant real-valued function $\Phi \in \mathcal{C}^\infty(E, \mathbb{R})$ defined on the total space $E$ of $\pi$.

The $T$-invariance of $\Phi$ means that the restrictions $\Phi_m \overset{\text{def}}{=} \Phi|_{E_m} \in \mathcal{C}^\infty(E_m, \mathbb{R})$ to the fibers of $E$ satisfy:

$$\Phi_\gamma(1) \circ T_\gamma = \Phi_{\gamma(0)}, \quad \forall \gamma \in \mathcal{P}(M) ,$$

a condition which is equivalent with the requirement that $\Phi$ is annihilated by any horizontal vector field defined on $E$:

$$X(\Phi) = 0, \quad \forall X \in \Gamma(E, H). \quad (2.2)$$

This implies that all fiber restrictions $\Phi_m$ ($m \in M$) can be recovered from $\Phi_{m_0}$, where $m_0$ is any fixed point of $M$. In particular, the isomorphism type of the scalar structure\(^2\) $(E_m, h_m, \Phi_m)$ is independent of $m$. The holonomy group $G_m \subset \text{Iso}(E_m, h_m)$ of $\pi$ at any point $m \in M$ preserves $\Phi_m$:

$$G_m \subset \text{Iso}(E_m, h_m, \Phi_m) \overset{\text{def}}{=} \{ \varphi \in \text{Iso}(E_m, h_m) | \Phi_m \circ \varphi = \Phi_m \} .$$

Relation (2.2) amounts to $d(\Phi)(X) = 0$ for all $X \in \Gamma(X, H)$, i.e. $d(\Phi) \circ P_H = 0$. Since $P_V + P_H = \text{id}_{TE}$, this gives $d(\Phi) = d(\Phi) \circ P_V$, which shows that $d(\Phi)$ can be viewed as an element of $\Gamma(E, V^*)$. Since $H$ and $V$ are $h$-orthogonal, this implies that the gradient of $\Phi$ is a vertical vector field:

$$\text{grad}_h \Phi \in \Gamma(E, V) .$$

**Definition 2.7.** A scalar bundle over $(M, g)$ is a pair $(\pi : (E, h) \to (M, g), \Phi)$, where $\pi$ is a Kaluza-Klein space and $\Phi$ is a vertical scalar potential for $\pi$. The scalar bundle is called integrable if the Kaluza-Klein space $\pi$ is integrable.

The isomorphism type of the scalar structures $(E_m, h_m, \Phi_m)$ (which, as explained above, does not depend on the point $m \in M$) is called the type of the scalar bundle and will be generally denoted by $(\mathcal{M}, \mathcal{G}, \Phi)$. The classification of integrable Kaluza-Klein spaces immediately implies the following.

**Proposition 2.2.** Integrable scalar bundles defined over $(M, g)$ having type $(\mathcal{M}, \mathcal{G}, \Phi)$ are classified up to isomorphism by the points of the character variety:

$$\mathcal{M}_{\text{Iso}(\mathcal{M}, \mathcal{G}, \Phi)}(M) \overset{\text{def}}{=} \text{Hom}(\pi_1(M), \text{Iso}(\mathcal{M}, \mathcal{G}, \Phi))/\text{Iso}(\mathcal{M}, \mathcal{G}, \Phi) .$$

\(^2\) As defined in reference [5].
2.4. Special trivializing atlases for a Kaluza-Klein space and for a scalar bundle. Recall that an open subset $U$ of a pseudo-Riemannian manifold is called geodesically convex [17] if it is a normal neighborhood for each of its points. If $U$ is convex, then for any two points $p, q \in U$ there exists a unique geodesic segment which is contained in $U$ and which connects $p$ and $q$. Any point of a pseudo-Riemannian manifold has a basis of geodesically convex neighborhoods (see [17, p. 129]). A convex cover of a pseudo-Riemannian manifold is an open and geodesically convex set which has the property that any nontrivial intersection of two of its elements is geodesically convex. Given any open cover $\mathcal{U}$ of a pseudo-Riemannian manifold, there exists a convex cover $\mathcal{U}$ such that any element of $\mathcal{U}$ is contained in some element of $\mathcal{U}$ [17, Lemma 5.10]. In particular, any pseudo-Riemannian manifold admits convex covers.

Let $\pi : (E, h) \to (M, g)$ be a Kaluza-Klein space. Let $m_0 \in M$ be a fixed point and set $\mathcal{M} \overset{\text{def}}{=} E_{m_0}$ and $\mathcal{G} \overset{\text{def}}{=} h_{m_0}$. Let $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ be a convex cover for $(M, g)$, where the indexing set is chosen such that $0 \notin I$.

Fix points $m_\alpha \in U_\alpha$ and paths $\alpha^\alpha \in \mathcal{P}(M)$ such that $\alpha^\alpha(0) = m_\alpha$ and $\alpha^\alpha(1) = m_0$. For any $m \in U_\alpha$, let $\gamma^\alpha : [0, 1] \to U_\alpha$ be the unique smooth geodesic contained in $U_\alpha$ such that $\gamma^\alpha(0) = m = \gamma^\alpha(1) = m_\alpha$. For any $\alpha \in I$, let $g_\alpha \overset{\text{def}}{=} g|_{U_\alpha}$, $E_\alpha \overset{\text{def}}{=} E|_{U_\alpha}$, $\pi_\alpha \overset{\text{def}}{=} \pi|_{U_\alpha}$. Let $E_0 \overset{\text{def}}{=} U_\alpha \times M$ and $\pi_0 : E_0 \to U_\alpha$, $p_\alpha : E_\alpha \to M$ be the projections on the first and second factors. Then $\pi_0 : E_0 \overset{\text{def}}{=} U_\alpha \times M \to U_\alpha$ is the trivial fiber bundle over $U_\alpha$ with fiber $M$. Define diffeomorphisms $q_\alpha : E_\alpha \to U_\alpha \times M$ through:

$$q_\alpha(e) = (\pi(e), q_\alpha(e)), \quad \forall e \in E_\alpha,$$

where $\tilde{q}_\alpha : E_\alpha \to M$ is given by the following differentiable surjective map:

$$\tilde{q}_\alpha(e) \overset{\text{def}}{=} p_\alpha^0 \circ (T_{\lambda^\alpha} \circ T_{\gamma^\alpha}^{-}(e)) = p_\alpha^0 \circ T_{\lambda^\alpha \circ \gamma^\alpha(e)}(e) \in M, \quad \forall e \in E_\alpha.$$

Here we have used the identification $\mathcal{M} \overset{\text{def}}{=} E_{m_0}$ and the fact that:

$$(T_{\lambda^\alpha} \circ T_{\gamma^\alpha}^{-})(e) \in \{m_0\} \times E_{m_0}, \quad \forall e \in E_\alpha.$$  

The restriction $\tilde{q}_\alpha(m) \overset{\text{def}}{=} \tilde{q}_\alpha|_{E_m} : E_m \to \mathcal{M}$ to the fiber at $m \in U_\alpha$ is an isometry from $(E_m, h_m)$ to $(\mathcal{M}, \mathcal{G})$ which is given explicitly by:

$$\tilde{q}_\alpha(m)(e_m) = p_\alpha^0 \circ T_{\lambda^\alpha \circ \gamma^\alpha(e_m)}(m \times e_m), \quad \forall e_m \in E_m.$$  

The maps $q_\alpha$ are diffeomorphisms from $E_\alpha$ to $E_\alpha^0 = U_\alpha \times M$ which fit into the following commutative diagram:

$$\begin{array}{ccc}
E_\alpha & \xrightarrow{q_\alpha} & E_\alpha^0 \\
\pi_\alpha \downarrow & & \downarrow \pi_0 \\
U_\alpha & \xrightarrow{\text{id}_{U_\alpha}} & U_\alpha
\end{array}$$

Hence $(U_\alpha, q_\alpha)$ is a trivializing atlas for the fiber bundle $\pi$, called the special trivializing atlas determined by the convex cover $(U_\alpha)_{\alpha \in I}$, by the reference point $m_0$ and by the choices of points $m_\alpha \in U_\alpha$ and of paths $\alpha^\alpha$ from $m_\alpha$ to $m_0$.

Let $h_\alpha \overset{\text{def}}{=} h|_{E_\alpha}$. Since the Ehresmann transport $T$ is isometric, the vertical metric $h_\alpha|_V$ agrees with $\mathcal{G}^{D_\alpha}$ through the diffeomorphism $q_\alpha : E_\alpha \to U_\alpha \times M$. Hence $h_\alpha$ corresponds through $q_\alpha$ to a Kaluza-Klein metric $h_\alpha^0$ on the trivial bundle $\pi_\alpha^0 : E_\alpha \to U_\alpha$. The latter is the Kaluza-Klein metric determined by $g|_{U_\alpha}$, $\mathcal{G}$ and the distribution $H_\alpha \overset{\text{def}}{=} \pi_\alpha^0(H_{E_\alpha})$, where $H_{E_\alpha} \overset{\text{def}}{=} H|_{E_\alpha}$. The diffeomorphism $q_\alpha$ is an isometry from $(E_\alpha, h_\alpha)$ to $(E_\alpha^0, h_\alpha^0)$ which makes diagram (2.5) into an isomorphism of Kaluza-Klein spaces from the Kaluza-Klein space $\pi_\alpha : (E_\alpha, h_\alpha) \to (U_\alpha, g_{U_\alpha})$ to the Kaluza-Klein space $\pi_\alpha^0 : (E_\alpha^0, h_\alpha^0) \to (U_\alpha, g_{U_\alpha})$. Notice that the second of these need not be a product Kaluza-Klein space, since $h_\alpha^0$ may differ from the product metric $g_{U_\alpha} \times \mathcal{G}$.

Since $(M, g)$ admits convex covers, any Kaluza-Klein space admits special trivializing atlases. In particular, any Kaluza-Klein space is locally isomorphic with a topologically trivial Kaluza-Klein space (which need not be a product Kaluza-Klein space!).
For any $\alpha, \beta \in I$ such that $U_{\alpha\beta} \defeq U_\alpha \cap U_\beta \neq \emptyset$, we have:

$$(q_\beta \circ q_\alpha^{-1})(m, p) = (m, g_{\alpha\beta}(m)(p)), \quad \forall m \in U_{\alpha\beta}, \quad \forall p \in \mathcal{M},$$

where the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to \text{Iso}(\mathcal{M}, \mathcal{G})$ are given by:

$$g_{\alpha\beta}(m) \defeq \hat{q}_\alpha(m) \circ \hat{q}_\alpha^{-1}(m) \in \text{Iso}(\mathcal{M}, \mathcal{G}), \quad \forall m \in U_{\alpha\beta}.$$  \hfill (2.6)

Using (2.4), this gives:

$$g_{\alpha\beta}(m) = T_{\lambda^\beta} \circ T_{\gamma_m^\beta} \circ T_{\gamma_m^{-1}} \circ T_{\lambda^\alpha}^{-1} = T_{c_m^{\alpha\beta}} \in \text{Iso}(\mathcal{M}, \mathcal{G}), \quad \forall m \in U_{\alpha\beta}.$$  \hfill (2.7)

Here:

$$c_m^{\alpha\beta} \defeq \lambda^\beta \circ \gamma_m^\beta \circ (\gamma_m^\alpha)^{-1} \circ (\lambda^\alpha)^{-1},$$

is the closed path starting and ending at $m_0$ and passing through the point $m \in U_{\alpha\beta}$ which is shown in Figure 2.1.

**Fig. 2.1.** The transition functions of a special trivializing atlas are determined by closed paths based at the reference point $m_0 \in M$ and passing through $m \in U_{\alpha\beta}$.

Let $\Phi$ be a vertical potential for $\pi$. In this case, a special trivializing atlas for the Kaluza-Klein space $\pi$ is also called a special trivializing atlas for the scalar bundle $(\pi, \Phi)$. Let $\Phi_\alpha \defeq \Phi|_{E_\alpha}$ and set $\Phi \defeq \Phi_{m_0}$. Since $\Phi$ is $T$-invariant, the definition (2.4) of $\hat{q}_\alpha$ implies:

$$\Phi_\alpha = \Phi \circ \hat{q}_\alpha.$$  \hfill (2.8)

This gives $\Phi \circ g_{\alpha\beta} = \Phi$, i.e. $g_{\alpha\beta} \in \text{Iso}(\mathcal{M}, \mathcal{G}, \Phi)$.

When the Kaluza-Klein space $\pi$ is integrable, the Ehresmann transport depends only on the homotopy class of curves in $M$. In this case, $T_{c_m^{\alpha\beta}}$ is independent of the point $m \in U_\alpha \cap U_\beta$ since $U_\alpha \cap U_\beta$ is path connected and hence the homotopy class of $c_m^{\alpha\beta}$ does not depend on $m$. Moreover, integrability of $H$ implies that $H_0^\pi$ coincide with the trivial Ehresmann connections $H^{\text{triv}}$ of $\pi_0$. In this case, $\pi_0 : (E_0^{\alpha}, h_0) \to (U_\alpha, g_\alpha)$ is a product Kaluza-Klein space and $g_\alpha$ is an isometry from $(E_\alpha, h_0)$ to $(U_\alpha \times M, g_\alpha \times \mathcal{G})$. Thus:

**Proposition 2.3.** Let $\pi : (E, h) \to (M, g)$ be an integrable Kaluza-Klein space. Then the local trivializing maps $g_\alpha$ of a special trivializing atlas consist on isometries from $(E_\alpha, h_\alpha)$ to $(U_\alpha \times M, g_\alpha \times \mathcal{G})$ and the transition functions $g_{\alpha\beta}$ defined by such an atlas are constant on $U_\alpha \cap U_\beta$. In particular, $\pi$ is locally isomorphic with a product Kaluza-Klein space.

In this case, relation (2.6) gives:

$$\hat{q}_\beta|_{E_{\alpha\beta}} = g_{\alpha\beta} \circ \hat{q}_\alpha|_{E_\alpha}, \quad \forall \alpha, \beta \in I,$$  \hfill (2.9)

where $E_{\alpha\beta} \defeq E|_{\pi^{-1}(U_{\alpha\beta})}$ and we define $g_{\alpha\beta} \defeq \text{id}_\mathcal{M}$ when $U_{\alpha\beta} = \emptyset$.

**Remark 2.1.** Suppose that the integrable Kaluza-Klein space $\pi$ is endowed with a vertical potential $\Phi$ whose restriction to $\mathcal{M}$ we denote by $\Phi$. Then the constant transition functions in a special trivializing atlas satisfy $g_{\alpha\beta} \in \text{Iso}(\mathcal{M}, \mathcal{G}, \Phi)$.
3. Section sigma models

Given a Kaluza-Klein space $\pi : (E, h) \to (M, g)$, let $P_V : TE \to V$ and $P_H : TE \to H$ denote the corresponding $h$-orthogonal projectors and $T$ denote the Ehresmann transport of $H \overset{\text{def}}{=} H(h)$. Let $\nabla^v \overset{\text{def}}{=} P_V \circ \nabla$ denote the connection induced on $V$ by the Levi-Civita connection $\nabla$ of $(E, h)$. Let $\Phi \in \mathcal{C}^\infty(E, \mathbb{R})$ be a vertical scalar potential for $\pi$, so that $(\pi, \Phi)$ is a bundle of scalar structures.

3.1. The scalar section sigma model defined by $\pi$ and $\Phi$.

**Definition 3.1.** The vertical Lagrange density of $\pi$ is the map $e^v_\Phi : \Gamma(\pi) \to \mathcal{C}^\infty(M, \mathbb{R})$ defined, for every $s \in \Gamma(\pi)$, as follows:

$$e^v_\Phi(g, h, s) \overset{\text{def}}{=} \frac{1}{2} \text{Tr}_g s^\ast(h_V) + \Phi^s,$$

where $s^\ast(h_V)$ is the vertical first fundamental form of $s$ (see Appendix A) and $\Phi^s = \Phi \circ s \in \mathcal{C}^\infty(M, \mathbb{R})$. Let:

$$e^v(g, h, s) \overset{\text{def}}{=} e^v_\Phi(g, h, s) = \frac{1}{2} \text{Tr}_g s^\ast(h_V).$$

**Definition 3.2.** The section sigma model defined by $\pi$ and $\Phi$ has action functional $S_{sc, \Omega} : \text{Met}_{3+1}(E) \times \Gamma(\pi) \to \mathbb{R}$ given by:

$$S_{sc, \Omega}[g, h, s] = -\int_U \nu_M(g) e^v_\Phi(g, h, s)$$

for any relatively-compact subset $U \subset M$.

Let $s \in \Gamma(\pi)$ be a section. The differential $ds : TM \to TE$ of $s$ is an unbased morphism of vector bundles equivalent to a section $ds \in \Omega^1(M, TE^s)$ which for simplicity we denote by the same symbol. We define the vertical differential $d^v s \overset{\text{def}}{=} P^g_V \circ ds \in \Omega^1(M, V^s)$, where $P^g_V$ denotes the vertical projection of the pull-back bundle (see Appendix A). The Levi-Civita connection on $(M, g)$ and the $s$-pull-back of the connection $\nabla^v$ on $V$ induce a connection on $T^s M \otimes V^s$, which for simplicity we denote again by $\nabla^v$.

**Definition 3.3.** The vertical tension field of $s \in \Gamma(\pi)$ is defined through:

$$\tau^v(g, h, s) \overset{\text{def}}{=} \text{Tr}_g \nabla^v d^v s \in \Gamma(M, V^s).$$

**Remark 1.** The ordinary Lagrange density and tension field of $s \in \Gamma(\pi)$ are defined similarly to their vertical counterparts except that there is no vertical projection involved. More precisely, we define:

$$\epsilon_\Phi(g, h, s) \overset{\text{def}}{=} \frac{1}{2} \text{Tr}_g s^\ast(h) + \Phi^s,$$

as well as:

$$\tau(g, h, s) \overset{\text{def}}{=} \text{Tr}_g \nabla ds \in \Gamma(M, TE^s),$$

where $\nabla$ denotes the connection on $T^s M \otimes TE^s$ induced by the Levi-Civita connection on $(M, g)$ and the $s$-pullback of the Levi-Civita connection on $(E, h)$. For simplicity we will sometimes drop the explicit dependence on $g$ and $h$ in $e_\Phi(g, h, s)$, $\tau(g, h, s)$ etc.

The following is an easy adaptation of a result due to [11].

**Proposition 3.1.** The vertical density $e^v(s)$ differs from $\epsilon(s)$ by a constant and we have $\tau^v(s) = P_V \circ \tau(s)$. Moreover, the critical points of (3.1) with respect to $s \in \Gamma(\pi)$ are solutions of the equation:

$$\tau^v(s) = - \langle \text{grad}_h \Phi \rangle^s.$$

(3.2)

**Remark 2.** When $\Phi = 0$, equation (3.2) becomes the vertical pseudoharmonic equation:

$$\tau^v(s) = 0$$

and its solutions are called pseudoharmonic sections of $\pi$.

Notice that (3.1) is extremized only with respect to vertical variations of $s$, since $s$ is subject to the section constraint $\pi \circ s = \text{id}_M$. As a consequence, a section which is pseudo-harmonic as an unconstrained map from $(M, g)$ to $(E, h)$ is a pseudo-harmonic section, but not every pseudo-harmonic section is pseudo-harmonic as an unconstrained map from $(M, g)$ to $(E, h)$.
3.2. The sheaves of configurations and solutions. The local character of the model allows us to define two sheaves of sets on $M$, namely:

- The sheaf of configurations $\text{Conf}_\pi$, which coincides with the sheaf of local smooth sections of $\pi$.
- The sheaf of solutions $\text{Sol}_{\pi,\Phi}$, defined as the sub-sheaf of $\text{Conf}_\pi$ whose set of sections $\text{Sol}_{\pi,\Phi}(U)$ over an open subset $U \subset M$ consist of pseudo-harmonic sections of the restricted Kaluza-Klein space $\pi_U : (E_U, h_U) \to (U, g_U)$ endowed with the vertical potential $\Phi|_{E_U}$, where $E_U \overset{\text{def}}{=} \pi^{-1}(U)$, $h_U \overset{\text{def}}{=} h|_{E_U}$, $\pi_U \overset{\text{def}}{=} \pi|_{E_U}$, and $g_U \overset{\text{def}}{=} g|_U$.

3.3. A modified sigma model for maps. Let $\pi^0 : (E^0, h^0) \to (M, g)$ be a topologically-trivial Kaluza-Klein space over $(M, g)$ endowed with a vertical scalar potential $\Phi \in C^\infty(E^0, \mathbb{R})$. Let $H^0$ be the horizontal distribution determined by $h^0$. Let graph : $C^\infty(M, M) \to \Gamma(\pi^0)$ be the bijective map given by:

$$\text{graph}(\varphi)(m) \overset{\text{def}}{=} (m, \varphi(m)) \quad \forall m \in M,$$

whose inverse is the map ungraph : $\Gamma(\pi^0) \to \Gamma(\pi^0)$ given by:

$$\text{ungraph}(s^0) \overset{\text{def}}{=} \rho^0 \circ s^0.$$

Using this correspondence, the scalar section sigma model defined by $\pi^0 : (E^0, h^0) \to (M, g)$ together with the vertical potential $\Phi \in C^\infty(E^0, \mathbb{R})$ can be viewed as a generalization of the ordinary scalar sigma model of maps from $(M, g)$ to $(\mathcal{M}, \mathcal{G}, \Phi)$.

Definition 3.4. The modified scalar sigma model determined by $(M, g)$, $(\mathcal{M}, \mathcal{G})$ and $H^0$ is defined by the action:

$$S_{H^0, \Phi, U^0}[\varphi] = S_{\text{sc}, \pi^0, \Phi, U^0}[\text{graph}(\varphi)].$$

for any relatively compact open set $U^0 \subset M$, where $S_{\text{sc}, \pi^0, \Phi, U^0}$ is the action of the section sigma model of the topologically trivial Kaluza-Klein space $\pi^0 : (E^0, h^0) \to (M, g)$ with fiber $(\mathcal{M}, \mathcal{G})$ and horizontal distribution $H^0$.

Definition 3.5. Let $H^0$ be any horizontal distribution for the trivial bundle $\pi^0 : E^0 \to M$. A map $\varphi : (M, g) \to (\mathcal{M}, \mathcal{G})$ is called $H^0$-pseudoharmonic if the graph $s^0$ of $\varphi$ is a pseudo-harmonic section of the topologically trivial Kaluza-Klein space $\pi^0 : (E^0, h^0) \to (M, g)$ with fiber $(\mathcal{M}, \mathcal{G})$, where $h^0$ is the metric on $E^0$ determined by $H^0, \mathcal{G}$ and $g$.

It is clear that the solutions of the equations of motion of the modified scalar sigma model are $H^0$-pseudoharmonic maps. The modified scalar sigma model defined reduces to the ordinary sigma model when $H^0$ is the trivial Ehresmann connection of $\pi^0$, as we explain next.

Remark 3.3. Let $\pi^0 : (E^0, h^0) \overset{\text{def}}{=} M \times \mathcal{M}, g^0 \overset{\text{def}}{=} g \times \mathcal{G}) \to (M, g)$ be the product Kaluza-Klein space with fiber $(\mathcal{M}, \mathcal{G})$ defined over $(M, g)$ and horizontal distribution $H^0_{\text{triv}}$. For any $s \in \Gamma(\pi^0)$, we have $s^*(h) = \varphi^*(\mathcal{G})$ and $d^v s = d\varphi^0$, where $\varphi = \text{ungraph}(s) = \rho^0 \circ s$. Thus $\epsilon(s) = \epsilon(\varphi)$ and $\tau(s) = \tau(\varphi)$. Moreover, we have $\Phi = \Phi \circ \rho^0$ for some $\Phi \in C^\infty(\mathcal{M}, \mathbb{R})$. Hence the section sigma model action (3.1) reduces to the action of an ordinary sigma model of maps from $(M, g)$ to $(\mathcal{M}, \mathcal{G})$, while the equations of motion (3.2) reduce to those of an ordinary sigma model. Setting $\Phi = 0$, we conclude that a map $\varphi : (M, g) \to (\mathcal{M}, \mathcal{G})$ is $H^0_{\text{triv}}$-pseudoharmonic if and only if it is pseudo-harmonic.

3.4. U-fold interpretation of globally-defined solutions in the integrable case. Let $(\pi : (E, h) \to (M, g), \Phi)$ be an integrable bundle of scalar data of type $(\mathcal{M}, \mathcal{G}, \Phi)$. Consider a special trivializing atlas of $\pi$ defined by the convex cover $(U_\alpha)_{\alpha \in I}$ of $(M, g)$. We freely use the notations introduced in Subsection 2.4. Since $\pi$ is integrable, the trivializing maps defined in (2.3) give isometries $q_\alpha : (E_\alpha, h_\alpha) \cong (U_\alpha \times \mathcal{M}, g_\alpha \times \mathcal{G})$. For any pair of indices $\alpha, \beta \in I$ such that $U_{\alpha \beta} \overset{\text{def}}{=} U_\alpha \cap U_\beta$ is non-empty, the composition $q_{\alpha \beta} \overset{\text{def}}{=} q_\beta \\ q_\alpha^{-1} : U_{\alpha \beta} \times \mathcal{M} \to U_{\alpha \beta} \times \mathcal{M}$ has the form $q_{\alpha \beta}(m, p) = (m, g_{\alpha \beta}(p))$, where:

$$g_{\alpha \beta} \in \text{Iso}(\mathcal{M}, \mathcal{G}, \Phi).$$
We remind the reader that \( \Phi \overset{\text{def}}{=} \Phi_{m_0} = \Phi|_{E_{m_0}} \). Setting \( g_{\alpha \beta} = \text{id}_M \) for \( U_{\alpha \beta} = \emptyset \), the collection \( (g_{\alpha \beta})_{\alpha, \beta \in I} \) satisfies the cocycle condition:

\[
g_{\beta \delta} g_{\alpha \beta} = g_{\alpha \delta}, \quad \forall \alpha, \beta, \delta \in I. \tag{3.3}
\]

For any section \( s \in \Gamma(\pi) \), the restriction \( s_\alpha \overset{\text{def}}{=} s|_{U_\alpha} \) corresponds through \( q_\alpha \) to the graph \( \text{graph}(\varphi^\alpha) \in \Gamma(\alpha^0) \) of a uniquely-defined smooth map \( \varphi^\alpha \in C^\infty(U_\alpha, M) \):

\[
s_\alpha = q_\alpha^{-1} \circ \text{graph}(\varphi^\alpha) \quad \text{i.e.} \quad q_\alpha(s_\alpha(m)) = (m, \varphi^\alpha(m)), \quad \forall m \in U_\alpha. \tag{3.4}
\]

Composing the first relation from the left with \( p_\alpha \) gives:

\[
\varphi^\alpha = \hat{q}_\alpha \circ s_\alpha.
\]

Using relation (2.9), this implies:

\[
\varphi^\beta(m) = g_{\alpha \beta} \varphi^\alpha(m) \quad \forall m \in U_{\alpha \beta}, \tag{3.5}
\]

where juxtaposition in the right hand side denotes the tautological action of the group \( \text{Iso}(M, G, \Phi) \) on \( M \). Conversely, any family of smooth maps \( \{\varphi^\alpha \in C^\infty(U_\alpha, M)\}_{\alpha \in I} \) satisfying (3.5) defines a smooth section \( s \in \Gamma(\pi) \) whose restrictions to \( U_\alpha \) are given by (3.4).

From the previous discussion, the equation of motion (3.2) for \( s \) is equivalent with the condition that each \( \varphi^\alpha \) satisfies the equation of motion of the ordinary sigma model defined by the scalar data \( (M, G, \Phi) \) on the space-time \( (U_\alpha, q_\alpha) \):

\[
\tau^e(h, s) = -(\text{grad} \Phi)^e \Leftrightarrow \tau(g, \varphi^\alpha) = -(\text{grad} \Phi)^\varphi^\alpha \quad \forall \alpha \in I. \tag{3.6}
\]

Thus global solutions \( s \) of the equations of motion (3.2) are glued from local solutions \( \varphi^\alpha \in C^\infty(U_\alpha, M) \) of the equations of motion of the ordinary sigma model using the \( \text{Iso}(M, G, \Phi) \)-valued constant transition functions \( g_{\alpha \beta} \) which satisfy the cocycle condition (3.3). This realizes the ideology of classical\(^3\) U-folds, namely gluing local solutions through symmetries of the equations of motion.

### 3.5. Sheaf-theoretical description.

The observations above have the following sheaf-theoretical description. Let \( \text{Conf}^3_M \overset{\text{def}}{=} \text{Conf}_M|_{U_{\alpha \beta}} \) and \( \text{Sol}^3_M \overset{\text{def}}{=} \text{Sol}^3_M|_{U_{\alpha \beta}} \). The sections \( g_{\alpha \beta} : \text{Conf}^3_M|_{U_{\alpha \beta}} \to \text{Conf}^3_M|_{U_{\alpha \beta}} \) are the isomorphism of sheaves defined through:

\[
g_{\alpha \beta}(s) \overset{\text{def}}{=} g_{\alpha \beta} s, \quad \forall U \subset U_{\alpha \beta}, \quad \forall s \in \text{Conf}^3_M(U). \tag{3.7}
\]

Since \( g_{\alpha \beta} \) acts by symmetries of the equations of motion of the ordinary sigma model, this restricts to an isomorphism of sheaves of sets from \( \text{Sol}^3_M|_{U_{\alpha \beta}} \) to \( \text{Sol}^3_M|_{U_{\alpha \beta}} \). These isomorphisms of sheaves satisfy the cocycle conditions:

\[
g_{\beta \gamma} g_{\alpha \beta} = g_{\alpha \gamma}, \quad \forall \alpha, \beta, \gamma \in I.
\]

The sheaves \( \text{Conf}^3_M \) and \( \text{Sol}^3_M \) defined on \( U_{\alpha \beta} \) glue using these isomorphisms to sheaves \( \text{Conf}_M \) and \( \text{Sol}_M \) defined on \( M \) which satisfy \( \text{Conf}_M|_{U_{\alpha \beta}} \simeq \text{Conf}^3_M \) and \( \text{Sol}_M|_{U_{\alpha \beta}} \simeq \text{Sol}^3_M \).

The discussion above shows that the trivialization maps \( q_\alpha \) of \( \pi \) induce isomorphisms of sheaves:

\[
\text{Conf}_\pi \simeq \text{Conf}_M, \quad \text{Sol}_{\pi, \Phi} \simeq \text{Sol}^3_M.
\]

which present \( \text{Conf}_M \) and \( \text{Sol}_M \) respectively as the sheaves of configurations and solutions of the section sigma model defined by the scalar structure \( (\pi, \Phi) \). These isomorphisms of sheaves encode the U-fold interpretation of the section sigma model defined by \( (\pi, \Phi) \).

\(^3\) As opposed to string-theoretical.
3.6. Classical scalar locally-geometric U-folds. Let \((M, \mathcal{G}, \Phi)\) be a scalar structure. The previous discussion motivates the following mathematically rigorous definition:

**Definition 3.6.** A classical scalar locally-geometric U-fold of type \((M, \mathcal{G}, \Phi)\) is a smooth global solution \(s \in \Gamma(\pi)\) of the equations of motion (3.2) of the section sigma model defined by an integrable bundle of scalar data \((\pi : (E, h) \to (M, g), \Phi)\) having type \((M, \mathcal{G}, \Phi)\).

As explained above, any such object can be constructed by gluing local solutions of the ordinary sigma model with target \((M, \mathcal{G})\) using the \(\text{Iso}(M, \mathcal{G}, \Phi)\)-valued transition functions of \(\pi\). Moreover, the discussion above shows that a section sigma model based on an integrable Kaluza-Klein space is locally indistinguishable from an ordinary sigma model. Thus section sigma models based on integrable Kaluza-Klein spaces provide allowed globalizations of the local formulas used in the sigma model literature, in the sense that they are locally indistinguishable from the latter. When the space-time \(M\) is not simply-connected, the number of inequivalent global formulations of this type is in general continuously infinite, since so is the character variety of \(\pi_1(M)\) for the group \(\text{Iso}(M, \mathcal{G}, \Phi)\). This illustrates the highly ambiguous character of the local formulation of theories involving sigma models (such as supergravity theories coupled to scalar matter in four dimensions). Clearly such local formulations are far from sufficient when one tries to specify the theory uniquely on a non-contractible space-time.

4. Scalar-electromagnetic bundles

Let \(\pi : (E, h) \to (M, g)\) be a Kaluza-Klein space with Ehresmann transport \(T\) associated to the horizontal distribution \(H \subset TE\). Let \(\Delta = (S, \omega, D)\) be a flat symplectic vector bundle defined over \(E\) with symplectic structure \(\omega\) and symplectic connection \(D\). Given a point \(m \in M\), let \((S_m, D_m, \omega_m)\) be the restriction of \((S, \omega, D)\) to the fiber \(E_m\). This is a flat symplectic vector bundle defined on the Riemannian manifold \((E_m, h_m)\) and hence a duality structure as defined in [5]. For any path \(\Gamma \in \mathcal{P}(E)\) in the total space of \(E\), let \(U_\Gamma : S_{\Gamma(0)} \to S_{\Gamma(1)}\) be the parallel transport defined by \(D\) along \(\Gamma\). Since \(D\) is a symplectic connection, \(U_\Gamma\) is a symplectomorphism between the symplectic vector spaces \((S_{\Gamma(0)}, \omega_{\Gamma(0)})\) and \((S_{\Gamma(1)}, \omega_{\Gamma(1)})\). For any path \(\gamma \in \mathcal{P}(M)\), let \(\gamma_e \in \mathcal{P}(E)\) denote its horizontal lift starting at the point \(e \in E_{\gamma(0)}\).

**Definition 4.1.** The extended horizontal transport along a path \(\gamma \in \mathcal{P}(M)\) is the unbased isomorphism of vector bundles \(T_\gamma : S_{\gamma(0)} \to S_{\gamma(1)}\) defined through:

\[T_\gamma(e) \overset{\text{def}}{=} U_{\gamma_e} : S_e \to S_{\gamma(e)}, \quad \forall e \in E_{\gamma(0)},\]

which linearizes the Ehresmann transport \(T_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}\) along \(\gamma\).

Clearly \(T_\gamma\) is an isomorphism of flat symplectic vector bundles:

\[T_\gamma : (S_{\gamma(0)}, D_{\gamma(0)}, \omega_{\gamma(0)}) \xrightarrow{\sim} (S_{\gamma(1)}, D_{\gamma(1)}, \omega_{\gamma(1)}),\]

which lifts the isometry \(T_\gamma : (E_{\gamma(0)}, h_{\gamma(0)}) \to (E_{\gamma(1)}, h_{\gamma(1)})\).

**Definition 4.2.** Let \(\pi : (E, h) \to (M, g)\) be a Kaluza-Klein space. A duality bundle \(\Delta\) is a flat symplectic vector bundle \(\Delta = (S, \omega, D)\) over \(E\). Let \(\Delta_1\) and \(\Delta_2\) be duality bundles. A morphism of duality bundles from \(\Delta_1\) to \(\Delta_2\) is a bundle morphism of the underlying flat symplectic vector bundles.

Let \(\Delta = (S, D, \omega)\) be a duality bundle over \(\pi : E \to M\) and let \(m_0 \in M\) be a fixed point in \(M\). Since \(M\) is path-connected, it follows that all fiber restrictions \((S_m, D_m, \omega_m)\), \(m \in M\), can be recovered by extended horizontal transport from the flat symplectic vector bundle \((S_{m_0}, D_{m_0}, \omega_{m_0})\) over \(E_{m_0}\). The flat vector bundle \((S_m, D_m, \omega_m)\) is a duality structure as defined in [5]. In particular, the isomorphism class of the duality structures \(\Delta_m \overset{\text{def}}{=} (S_m, D_m, \omega_m)\) is independent of \(m\) and is called the type of \(\Delta\). We will drop the subscript and write \(\Delta \overset{\text{def}}{=} (S, D, \omega)\) for the type of \(\Delta\). Notice that \(\Delta\) can be viewed as a bundle whose fibers are the duality structures \(\Delta_m\), endowed with the complete Ehresmann connection given by the extended horizontal transport \(T\). Such objects defined over \((M, g)\) form a category when equipped with the obvious notion of (based) morphism.
4.1. Vertical tamings and scalar-electromagnetic bundles. Let $\Delta$ be a duality bundle over a Kaluza-Klein space $\pi : (E, h) \rightarrow (M, g)$. Recall that a taming $J \in \text{Aut}(\mathcal{S}, \omega)$ is an automorphism of the symplectic vector bundle $(\mathcal{S}, \omega)$ satisfying [29]:

$$J^2 = -\text{Id}_\mathcal{S}, \quad \omega(J e, e) > 0, \quad \forall e \in \Gamma(E, \mathcal{S}).$$

Tamings always exist. Given a taming $J$ of $(\mathcal{S}, \omega)$ and a point $m \in M$, we denote by $J_m \overset{\text{def}}{=} J|_{E_m}$ the taming on $(\mathcal{S}_m, \omega_m)$ induced by the restriction of $J$ to the fiber $E_m$ of $\pi$.

**Definition 4.4.** A taming $J$ of $(\mathcal{S}, \omega, D)$ is called vertical if it is $T$-invariant, which means that it satisfies:

$$T_\gamma \circ J_\gamma(0) = J_\gamma(1) \circ T_\gamma, \quad \forall \gamma \in \mathcal{P}(M).$$

It is clear that $J$ is vertical if and only if it satisfies:

$$D_X \circ J = J \circ D_X, \quad \forall X \in \Gamma(E, H).$$

In this case, $T_\gamma$ is an isomorphism of tamed flat symplectic vector bundles:

$$T_\gamma : (\mathcal{S}_\gamma(0), \omega_\gamma(0), D_\gamma(0), J_\gamma(0)) \overset{\sim}{\rightarrow} (\mathcal{S}_\gamma(1), \omega_\gamma(1), D_\gamma(1), J_\gamma(1)),$$

which covers the isometry $T_\gamma : (E_\gamma(0), h_\gamma(0)) \rightarrow (E_\gamma(1), h_\gamma(1))$, i.e. the following diagram commutes:

$$
\begin{array}{ccc}
(S_\gamma(0), \omega_\gamma(0), D_\gamma(0), J_\gamma(0)) & \xrightarrow{T_\gamma} & (S_\gamma(1), \omega_\gamma(1), D_\gamma(1), J_\gamma(1)) \\
\downarrow{\pi_\gamma(0)} & & \downarrow{\pi_\gamma(1)} \\
(E_\gamma(0), h_\gamma(0)) & \xrightarrow{T_\gamma} & (E_\gamma(1), h_\gamma(1))
\end{array}
$$

**Definition 4.5.** Let $\pi : (E, h) \rightarrow (M, g)$ be a Kaluza-Klein space. An electromagnetic bundle $\Xi$ is a duality bundle $\Delta$ over $E$ equipped with a vertical taming $J$. We write $\Xi \overset{\text{def}}{=} (\Delta, J) = (\mathcal{S}, \omega, D, J)$. Let $\Xi_1$ and $\Xi_2$ be two electromagnetic bundles. A morphism of electromagnetic bundles $f : \Xi_1 \rightarrow \Xi_2$ from $\Xi_1$ to $\Xi_2$ is a morphism of the underlying duality structures which satisfies $J_2 \circ f = f \circ J_1$.

**Definition 4.6.** A scalar-electromagnetic bundle defined over $(M, g)$ is a triple $\mathcal{D} = (\pi : (E, h) \rightarrow (M, g), \Phi, \Xi)$ consisting of a Kaluza-Klein space $\pi : (E, h) \rightarrow (M, g)$, a vertical potential $\Phi$ and an electromagnetic bundle $\Xi$ defined over the total space $E$ of $\pi$. The scalar-electromagnetic bundle $\mathcal{D}$ is called integrable if $\pi : (E, h) \rightarrow (M, g)$ is an integrable Kaluza-Klein space.

Let $\mathcal{D} = (\pi : (E, h) \rightarrow (M, g), \Phi, \Xi)$ be a scalar-electromagnetic bundle, which for simplicity (and when no confusion can arise) we will denote by $\mathcal{D} = (\pi, \Phi, \Xi)$. Since $M$ is path-connected, it follows that all fiber restrictions $(\pi, \Phi, \Xi)|_{E_m} \overset{\text{def}}{=} (E_m, h_m, \Phi_m, \mathcal{S}_m, \omega_m, D_m, J_m)$ can be recovered by extended horizontal transport from the ’value’ of $\mathcal{D}$ at a fixed point $m_0 \in M$. In particular, each $D_m \overset{\text{def}}{=} (E_m, h_m, \Phi_m, \mathcal{S}_m, \omega_m, D_m, J_m)$ is a scalar-electromagnetic structure in the sense of [5]. The isomorphism class of $D_m$ is independent of $m$ and is called the type of the scalar-electromagnetic bundle $\mathcal{D}$.

Hence we will drop the subscript and write $\mathcal{D} \overset{\text{def}}{=} (M, \mathcal{G}, \Phi, \mathcal{S}, \omega, D, J)$. Notice that $\mathcal{D}$ can be viewed as a bundle whose fibers are the scalar-electromagnetic structures $D_m$, endowed with the complete Ehresmann connection given by the extended horizontal transport $T$. Such objects defined over $(M, g)$ form a category when equipped with the obvious notion of (based) morphism.

**Definition 4.7.** The extended holonomy group of $\mathcal{D}$ at the point $m \in M$ in $\text{Aut}(D_m)$ defined through:

$$G_m \overset{\text{def}}{=} \{T_\gamma | \gamma \in \mathcal{P}(M), \gamma(0) = \gamma(1) = m\} \subset \text{Aut}(D_m).$$

**Remark 4.1.** The holonomy groups associated to $\mathcal{D}$ at different points in $M$ are related by conjugation inside $\text{Aut}(\mathcal{D})$ and hence isomorphic. Therefore, we can speak of the holonomy group $G$ of $\mathcal{D}$ without further reference to base points.
4.2. Topologically trivial scalar-electromagnetic bundles. Let \( \pi^0 : (E^0, h^0) \to (M, g) \) be a topologically trivial Kaluza-Klein space over \((M, g)\) with fiber \((M, \mathcal{G})\), where \(E^0 = M \times M\) and \(\pi^0\) is the projection in the first factor. Let \(p^0\) be the projection of \(E^0\) on the second factor. Let \(H \subset TE^0\) be the horizontal distribution determined by \(h\) and let \(\mathcal{D} = (M, \mathcal{G}, \Phi, S, \omega, D, J)\) be a scalar-electromagnetic structure. There exists a unique flat connection \(D^0\) on the pulled-back bundle \(S^0\) which satisfies the following conditions for all \(e \in E^0\):

\[
\begin{align*}
D^0_0(\sigma^0) &= 0, \quad \forall \sigma \in \Gamma(M, S), \quad \forall x \in H_x(h), \\
D^0_x &= D^0_x, \quad \forall x \in V_e,
\end{align*}
\]

where \(D^0\) is the pullback of the connection \(D\) through \(p^0\). Let \(S^0 \overset{\text{def}}{=} (\mathcal{S}^0, \omega^0, D^0, J^0)\) be an electromagnetic bundle defined over \(E^0\) and \(\Phi^0 \overset{\text{def}}{=} \Phi^0 \overset{\text{def}}{=} \Phi \circ p^0\) is a vertical potential for \(\pi^0\). A section \(\sigma \in \Gamma(E^0, S^0)\) satisfies \(D^0_X \sigma = 0\) for all \(X \in \Gamma(E^0, H)\) if and only if there exists \(\sigma \in \Gamma(M, S)\) such that \(\sigma = \sigma^0\). It is easy to see that \(\sigma\) is invariant under the extended horizontal transport \(T^0\) associated to \(H^0\) and \(D^0\) if and only if \(\sigma\) is invariant under the action of the subgroup:

\[
\mathcal{G}^h \overset{\text{def}}{=} \{ \mathcal{T}^h \xi \in \mathcal{P}(M) \} \subset \text{Isom}(M, \mathcal{G}),
\]

where \(\mathcal{T}^h\) was defined in (2.1).

**Definition 4.7.** The triple \(D^0 = (\pi^0 : (E^0, h^0) \to (M, g), \Phi^0, S^0)\) is called a topologically trivial scalar-electromagnetic bundle of type \(D\) defined over \((M, g)\).

**Definition 4.8.** A topologically-trivial scalar-electromagnetic bundle \(D^0\) whose underlying Kaluza-Klein space \(\pi^0 : (E^0, h^0) \to (M, g)\) is the product Kaluza-Klein bundle (where \(h^0 = g \times \mathcal{G}\)) is called a metrically trivial scalar-electromagnetic bundle defined over \((M, g)\).

For a metrically-trivial scalar-electromagnetic bundle, we have \(\mathcal{G} = \text{id}_M\). Hence a section \(\sigma \in \Gamma(E^0, S^0)\) is \(T^0\)-invariant if and only if \(\sigma = \sigma^0\) for some section \(\sigma \in \Gamma(M, S)\) of \(S\).

4.3. Special trivializing atlases for scalar-electromagnetic bundles. Consider a scalar-electromagnetic bundle \(\mathcal{D} = (\pi, \Phi, \mathcal{S})\). Let \((U_\alpha)_{\alpha \in I}\) be a convex cover of \(M\) with \(0 \not\in I\). Fix points \(m_0 \in M\), \(m_\alpha \in U_\alpha\) as well as paths \(\lambda^\alpha\) from \(m_\alpha\) to \(m_0\) as in Subsection 2.4, whose notations we will use freely. Let \(\mathcal{D}_\alpha \overset{\text{def}}{=} (\mathcal{M}, \mathcal{G}, \Phi, S, \omega, D, J)\) be the type of a scalar-electromagnetic bundle \(\mathcal{D}\) and let \(\mathcal{D}_\alpha \overset{\text{def}}{=} (\pi_\alpha, \Phi_\alpha, S_\alpha)\) denote its restriction to \(E_\alpha\). Then \(\mathcal{D}_\alpha \overset{\text{def}}{=} (\pi_\alpha, \Phi_\alpha, \mathcal{S}_\alpha)\) is a scalar-electromagnetic bundle defined over \((E_\alpha, h_\alpha)\), with Kaluza-Klein metric given by \(h_\alpha \overset{\text{def}}{=} h|_{U_\alpha}\). Let \(\mathcal{D}_\alpha^0 = (\pi_\alpha^0, \Phi_\alpha^0, S_\alpha^0)\) be the topologically trivial scalar-electromagnetic bundle of type \(\mathcal{D}\) defined over \(E_\alpha^0\), with Kaluza-Klein metric \(h_\alpha^0\) making:

\[
q_\alpha : E_\alpha \to U_\alpha \times M,
\]

into an isometry (see Subsection 2.4). We have \(S_\alpha^0 = S^0\), \(D_\alpha^0 = D^0\), \(J_\alpha^0 = J^0\), \(\omega_\alpha^0 = \omega^0\).

The extended horizontal transport \(\mathcal{T}\) along geodesics inside \(U_\alpha\) can be used to define unbiased isomorphisms of electromagnetic structures \(q_\alpha : (\mathcal{S}_\alpha, \omega_\alpha, D_\alpha, J_\alpha) \to (S_\alpha^0, \omega_\alpha^0, D_\alpha^0, J_\alpha^0)\) which linearize the diffeomorphisms \(q_\alpha : E_\alpha \to E_\alpha^0\) defined in equation (2.3). For any \(e \in E_\alpha\), the linear isomorphism \(q_{\alpha, e} : \mathcal{S}_\alpha \to S_\alpha^0(e) = S_\alpha \circ q_{\alpha}(e)\) is defined through:

\[
q_{\alpha, e} = \mathcal{T}_{\lambda^\alpha \circ q_{\alpha}(e)}(e).
\]
This gives commutative diagrams:

\[
\begin{array}{ccc}
\mathcal{S}_\alpha & \xrightarrow{q_\alpha} & \mathcal{S}^0 \\
\downarrow & & \downarrow \\
E_\alpha & \xrightarrow{\pi_\alpha} & E^0 \\
\pi_\alpha & & \pi^0 \\
U_\alpha & \xrightarrow{\text{id}_{U_\alpha}} & U_\alpha \\
\end{array}
\]

which extend the diagrams (2.5). In particular, \( q_\alpha \) are determined by \( q_\alpha \). It is easy to see that \( q_\alpha \) is an isomorphism of electromagnetic structures from \((\mathcal{S}_\alpha, \omega_\alpha, D_\alpha, J_\alpha)\) to \((\mathcal{S}^0_\alpha, \omega^0_\alpha, D^0_\alpha, J^0_\alpha)\). Hence \( q_\alpha \) can be viewed as an isomorphism of scalar-electromagnetic structures from \(D_\alpha\) to \(D^0_\alpha\). By definition, the family \((U_\alpha, q_\alpha)_{\alpha \in I}\) is the special trivializing atlas of the scalar-electromagnetic bundle \(D\) defined by the convex cover \((U_\alpha)_{\alpha \in I}\) and by the data \(m_0\) and \((m^\alpha, X^\alpha)_{\alpha \in I}\). Since any scalar-electromagnetic bundle admits such an atlas, it follows that any scalar-electromagnetic bundle is locally isomorphic with a topologically trivial scalar-electromagnetic bundle.

For any \(\alpha, \beta \in I\) such that \(U_{\alpha,\beta} \neq \emptyset\) and any \(m \in U_{\alpha,\beta}\), the restriction of the unbased isomorphism \(q_{\alpha,\beta} \circ q_{\alpha}^{-1} : \mathcal{S}^0_{\alpha}|_{E_{\alpha,\beta}} \to \mathcal{S}^0_{\beta}|_{E_{\alpha,\beta}}\) to the fiber \(\{m\} \times M\) can be identified with the unbased automorphism \(f_{\alpha,\beta}(m) \in \text{Aut}^{ub}(\mathcal{S})\) of \(\mathcal{S}\) given by:

\[
f_{\alpha,\beta}(m) = T_{c_{\alpha,\beta}}^m,
\]

where \(c_{\alpha,\beta}^m\) is the closed path defined in (2.7). This unbased automorphism of \(\mathcal{S}\) linearizes the isometry \(g_{\alpha,\beta}(m) \in \text{Iso}(M, \mathcal{G}, \Phi)\) of \(M\) given by the transition function \(g_{\alpha,\beta}\). Since \(J\) and \(\omega\) and the vertical connection induced by \(D\) are \(T\)-invariant, we have \(f_{\alpha,\beta}(m) \in \text{Aut}(D) = \text{Aut}^{ub}_{M, \mathcal{G}, \Phi}(\mathcal{S}, D, \omega, J)\). This gives maps \(f_{\alpha,\beta} : U_{\alpha,\beta} \to \text{Aut}(D)\), which we call the extended transition functions of the special trivializing atlas \((U_\alpha, q_\alpha)_{\alpha \in I}\). Notice that \(g_{\alpha,\beta}\) are uniquely determined by \(f_{\alpha,\beta}\). In addition, notice that \(f_{\alpha,\beta}\) satisfy the cocycle condition:

\[
f_{\beta,\gamma}f_{\alpha,\beta} = f_{\alpha,\gamma},
\]

which implies the cocycle condition (3.3) for \(g_{\alpha,\beta}\).

4.4. The fundamental bundle form and field of an electromagnetic bundle. Consider a scalar-electromagnetic bundle \(D = (\pi, \Phi, \Xi)\) with associated electromagnetic bundle \(\Xi = (\mathcal{S}, \omega, D, J)\) and Kaluza-Klein space \(\pi : (E, h) \to (M, g)\). Let:

\[
D^{ad} : \Gamma(E, \text{End}(\mathcal{S})) \to \Omega^1(E, \text{End}(\mathcal{S}))
\]

be the connection induced by \(D\) on the endomorphism bundle \(\text{End}(\mathcal{S})\) of \(\mathcal{S}\).

**Definition 4.9.** The fundamental bundle form \(\Theta\) of \(D\) is the \(\text{End}(\mathcal{S})\)-valued one-form defined on \(E\) as follows:

\[
\Theta \overset{\text{def}}{=} D^{ad}J \in \Omega^1(E, \text{End}(\mathcal{S})).
\]

The fact \(J\) is vertical together with the fact that the decomposition \(TE = H \oplus V\) is \(h\)-orthogonal implies that we have:

\[
\Theta \in \Gamma(E, V^* \otimes \text{End}(\mathcal{S})).
\]

**Definition 4.10.** The fundamental bundle field \(\Psi\) of \(D\) is the \(\text{End}(\mathcal{S})\)-valued vector field defined on \(E\) as follows:

\[
\Psi \overset{\text{def}}{=} (\sharp h \otimes \text{Id}_{\text{End}(\mathcal{S})}) \circ D^{ad}J \in \Gamma(E, V \otimes \text{End}(\mathcal{S})).
\]
We denote by $\text{End}(S_m) = \text{End}(\mathcal{S})|_{E_m}$ the restriction of $\text{End}(\mathcal{S})$ to $E_m$, which becomes the endomorphism bundle of the vector bundle $S_m$. We denote by $\Theta_m \overset{\text{def}}{=} \Theta|_{E_m}$ the restriction of $\Theta$ to $E_m$, which is a section of $V^*|_{E_m} \otimes \text{End}(S_m)$, namely:

$$\Theta_m \in \Gamma(E_m, V^*|_{E_m} \otimes \text{End}(S_m)).$$

Likewise, we define $\Psi_m \overset{\text{def}}{=} \Psi|_{E_m}$, which is a section of $V|_{E_m} \otimes \text{End}(S_m)$:

$$\Psi_m \in \Gamma(E_m, V|_{E_m} \otimes \text{End}(S_m)).$$

Since $V \subset TE$ is the vertical integrable distribution integrated by the fibers of $\pi: (E, h) \rightarrow (M, g)$ (which by assumption are connected), we have $V|_{E_m} \simeq TE_m$ and we obtain:

$$\Theta_m = \Omega^1(E_m, \text{End}(S_m)), \quad \Psi_m \in \mathcal{X}(E_m, \text{End}(S_m)).$$

The extended horizontal transport $T_\gamma$ along paths $\gamma \in \mathcal{P}(M)$ induces various isomorphisms of spaces of sections of the appropriate vector bundles. For simplicity we denote all these isomorphisms by the same symbol. For instance, $T_\gamma$ induces the following isomorphism:

$$T_\gamma: \Omega^1(E_{\gamma(0)}, \text{End}(S_{\gamma(0)})) \xrightarrow{\simeq} \Omega^1(E_{\gamma(1)}, \text{End}(S_{\gamma(1)})),$$

whose explicit action on homogeneous elements $\alpha \otimes \Sigma \in \Omega^1(E_{\gamma(0)}, \text{End}(S_{\gamma(0)}))$, with $\alpha \in \Omega^1(E_{\gamma(0)})$ and $\Sigma \in \Gamma(E_{\gamma(0)}, \text{End}(S_{\gamma(0)}))$ is given by:

$$T_\gamma \cdot (\alpha \otimes \Sigma) = (T^{-1}_\gamma)^* \alpha \otimes (T_\gamma \cdot \Sigma).$$

Here $T_\gamma \cdot \Sigma \in \Gamma(E_{\gamma(1)}, \text{End}(S_{\gamma(1)}))$. The explicit evaluation of $T_\gamma \cdot \Sigma: S_{\gamma(1)} \rightarrow S_{\gamma(1)}$ on sections of $S_{\gamma(1)}$ takes the form:

$$(T_\gamma \cdot \Sigma)(\xi) = T_\gamma \cdot \Sigma(T^{-1}_\gamma \cdot \xi) = T_\gamma \cdot \Sigma(T^{-1}_\gamma \circ \xi \circ T_\gamma) = T_\gamma \circ \Sigma \circ T^{-1}_\gamma \circ \xi, \quad \xi \in \Gamma(E_{\gamma(1)}, S_{\gamma(1)}).$$

As a general rule, and for simplicity in the notation, we will denote the action induced by extended horizontal transport $T_\gamma$ on a given module of sections by "$\cdot$", whereas we will denote by "$\circ$" the action induced by $T_\gamma$ as an unbased automorphism of the given bundles. The explicit form of the action represented by "$\circ$" will depend on the particular details of the module of sections which is acted upon and should be clear from the context. Likewise, the explicit form of the action represented by "$\cdot$" will depend on the bundles involved. The reader is referred to appendix D of [5] and to remark 4.2 for detail about the explicit form of the various actions induced by $T_\gamma$.

Let us fix $m_0 \in M$. The fact that the restriction $\Psi_{m_0}$ of the fundamental bundle field $\Psi$ to $E_{m_0}$ is a vector field over $E_{m_0}$, taking values on $\text{End}(S_{m_0})$ together with the fact that the extended horizontal transport $T_\gamma$ preserves the flat symplectic connection on $\mathcal{S}$ and covers isometries, implies:

$$T_\gamma \cdot \Theta_{m_0} = \Theta_m, \quad \forall m \in M, \quad (4.4)$$

$$T_\gamma \cdot \Psi_{m_0} = \Psi_m, \quad \forall m \in M, \quad (4.5)$$

where $\gamma \in \mathcal{P}(M)$ is a path in $M$ with initial point $\gamma(0) = m_0$ and final point $\gamma(1) = m$. We conclude that the isomorphism type of the restriction of the fundamental bundle form $\Theta$ and the fundamental bundle field $\Psi$ to $E_m$ does not depend on the point and hence will be denoted by $\Theta$ and $\Psi$, respectively.

Remark 4.2. Equation (4.4) follows from the following computation:

$$T_\gamma \cdot \Theta_{m_0} = T_\gamma \cdot D_{m_0}^{\text{ad}}(T^{-1}_\gamma \cdot T_\gamma \cdot J_{m_0}) = T_\gamma \cdot D_{m_0}^{\text{ad}}(T^{-1}_\gamma \cdot J_m) = D_m J_m = \Theta_m,$$

where we have used the fact that $T_\gamma$ preserves $D_{m_0}$ and $D_m$, i.e.:

$$D_m^{\text{ad}}(\xi) = T_\gamma \cdot D_{m_0}^{\text{ad}}(T^{-1}_\gamma \cdot \xi) = T_\gamma \cdot D_{m_0}^{\text{ad}}(T^{-1}_\gamma \circ \xi \circ T_\gamma), \quad \forall \xi \in \Gamma(E_m, S_m), \quad (4.6)$$

Unfortunately, we cannot use the same notation as in [5] since bold letters have in this article a different meaning.

Equation (4.5) can be proved similarly.
as well as the relations:

\[(T_\gamma \cdot J_{m_0})(\xi) = T_\gamma \cdot J_{m_0}(T_\gamma^{-1} \cdot \xi) = T_\gamma \cdot (J_{m_0} \circ T_\gamma^{-1} \circ \xi \circ T_\gamma)\]

\[= T_\gamma \circ (J_{m_0} \circ T_\gamma^{-1} \circ \xi \circ T_\gamma) \circ T_\gamma^{-1} = T_\gamma \circ J_{m_0} \circ T_\gamma^{-1} \circ \xi = J_m(\xi), \quad (4.7)\]

which hold for all \(\xi \in \Gamma(E_m, S_m)\). Notice that the \(T_\gamma\)-action symbols “\(\cdot\)” and “\(\circ\)” in equations (4.6) and (4.7) have a different meaning depending on the step and the sections involved. In equation (4.6) we have:

\[D_{m_0}^{ad}(T_\gamma^{-1} \circ \xi \circ T_\gamma) \in \Omega^1(E_{m_0}, S_{m_0})\]

and evaluating on a vector field \(X \in \mathfrak{X}(E_m)\) we obtain:

\[\iota_X D_{m_0}^{ad}(\xi) = T_\gamma \circ \left(\iota_{T_\gamma^{-1} X} D_{m_0}^{ad}(T_\gamma^{-1} \circ \xi \circ T_\gamma)\right) \circ T_\gamma^{-1} \in \Gamma(E_m, S_m).\]

Likewise, in the second step of equation (4.7) we use:

\[T_\gamma^{-1} \cdot \xi = T_\gamma^{-1} \circ \xi \circ T_\gamma \in \Gamma(E_{m_0}, S_{m_0}), \quad \xi \in \Gamma(E_m, S_m),\]

where \(T_\gamma^{-1}\) in the left hand side acts through “\(\cdot\)” as the isomorphism of modules:

\[T_\gamma^{-1} : \Gamma(E_m, S_m) \to \Gamma(E_{m_0}, S_{m_0}),\]

whereas \(T_\gamma^{-1}\) in the right-hand-side acts through “\(\circ\)” as composition with the unbased automorphism \(T_\gamma^{-1} : S_m \to S_{m_0}\) of vector bundles covering \(T_\gamma\).

The following proposition follows from the previous discussion and summarizes the isomorphism type of a fundamental bundle field \(\Psi\). Similar remarks apply for the fundamental bundle form \(\Theta\).

**Proposition 4.1.** Let \(\mathcal{D} = (\pi, \Phi, \Xi)\) be a scalar-electromagnetic structure of type:

\[\mathcal{D} = (\mathcal{M}, \mathcal{G}, \Phi, S, D, \omega, J)\]

Then the isomorphism type of the fundamental bundle form \(\Theta\) of \(\mathcal{D}\) is given by:

\[\Theta \overset{\text{def}}{=} \Theta_{m_0} = D^{ad} J \in \Omega^1(\mathcal{M}, \text{End}(\mathcal{S})),\]

while the isomorphism type of the fundamental bundle field \(\Psi\) of \(\mathcal{D}\) is given by:

\[\Psi \overset{\text{def}}{=} \Psi_{m_0} = (\mathcal{G} \otimes \text{Id}_{\text{End}(\mathcal{S})}) D^{ad} J \in \mathfrak{X}(\mathcal{M}, \text{End}(\mathcal{S})).\]

Now let \(s \in \Gamma(\pi)\) be a section of \(\pi: (E, h) \to (M, g)\). The pull-backs through \(s\) of \(\Phi\) and \(\Psi\) satisfy:

\[\Phi^s \in \Gamma(M, (V^s)^* \otimes \text{End}(\mathcal{S}^s)), \quad \Psi^s \in \Gamma(E, V^s \otimes \text{End}(\mathcal{S}^s)).\]

These objects will be used in section 5 in the formulation of GESM theories.

5. **Generalized Einstein-Section-Maxwell theories**

In this section, we define Generalized Einstein-Section-Maxwell theories, or GESM theories for short, in terms of a set of partial differential equations globally formulated on \((M, g)\). A GESM theory is a generalization of the generalized Einstein-Scalar-Maxwell theory introduced in reference [5], obtained by promoting the standard sigma model of the latter to a section sigma model in which the scalar map is replaced by a section of the corresponding Lorentzian submersion. When the horizontal distribution \(H \subset TE\) of the section sigma model is integrable, GESM theories can be used to systematically generalize in a globally nontrivial way the bosonic sector of four-dimensional ungauged supergravity theories.
5.1. The polarization condition and the electromagnetic equation. Before defining the complete GESM theory, we consider in this subsection a truncated version consisting of a section sigma model coupled to a generically non-trivial duality bundle. We call such theory a Generalized Section-Maxwell theory, or GSM theory for short.

Definition 5.1. Let $\mathcal{D} = (\pi, \Phi, \Xi)$ be a scalar-electromagnetic bundle with associated Kaluza-Klein space $\pi: (E, h) \to (M, g)$ and let $s \in \Gamma(\pi)$ be a section of $\pi$. An electromagnetic field strength is a two-form $\mathcal{V} \in \Omega^2(M, S^2)$ having the following properties:

1. $\mathcal{V}$ is positively-polarized with respect to $\mathcal{J}^s$, i.e. the following relation is satisfied:
   $$s^*\mathcal{V} = -\mathcal{J}^s \mathcal{V}.$$

2. $\mathcal{V}$ satisfies the electromagnetic equation with respect to $s$:
   $$d_{\mathcal{D}} \mathcal{V} = 0 \, . \quad (5.1)$$

We denote by $\Omega^{2+,s}_{\mathcal{J}}$ the sheaf of $S^2$-valued two-forms which are positively-polarized with respect to $\mathcal{J}^s$, thus $\Omega^{2+,s}_{\mathcal{J}}(M)$ denotes the space of $S^2$-valued two-forms on $M$ which are positively-polarized with respect to $\mathcal{J}^s$.

Definition 5.2. The sheaf of electromagnetic field strengths is the sheaf $\mathcal{E}^{s}_{\mathcal{J}}$ of vector spaces defined as follows:

$$\mathcal{E}^{s}_{\mathcal{J}}(U) \overset{\text{def}}{=} \{ \mathcal{V} \in \Omega^{2+,s}_{\mathcal{J}}(U), \ | \ d_{\mathcal{D}} \mathcal{V} = 0 \},$$

for every open set $U \subset M$. The vector space of global electromagnetic field strengths in $M$ is then given by $\mathcal{E}^{s}_{\mathcal{J}}(M)$.

Definition 5.3. Let $\mathcal{D} = (\pi, \Phi, \Xi)$ be a scalar-electromagnetic bundle with associated Kaluza-Klein space $\pi: (E, h) \to (M, g)$. The configuration sheaf $\text{Conf}_{\mathcal{D}}$ of a GSM theory associated to $\mathcal{D}$ is the sheaf of sets defined as follows:

$$\text{Conf}_{\mathcal{D}}(U) \overset{\text{def}}{=} \{ (s, \mathcal{V}), \ | \ s \in \Gamma(\pi|_U), \ \mathcal{V} \in \Omega^{2+,s}_{\mathcal{J}}(U) \},$$

for every open set $U \subset M$.

Definition 5.4. Let $\mathcal{D} = (\pi, \Phi, \Xi)$ be a scalar-electromagnetic bundle with associated Kaluza-Klein space $\pi: (E, h) \to (M, g)$. The GSM theory associated to $\mathcal{D}$ is defined by the following set of partial differential equations on $(M, g)^6$:

$$\mathcal{E}_S(s, \mathcal{V}) \overset{\text{def}}{=} \tau^v(h, s)+ (\text{grad}_h \Phi)^s - \frac{1}{2}(s \mathcal{V}, \Psi^s \mathcal{V}) = 0, \quad \mathcal{E}_K(s, \mathcal{V}) \overset{\text{def}}{=} d_{\mathcal{D}} \mathcal{V} = 0,$$

with unknowns given by pairs $(s, \mathcal{V}) \in \text{Conf}_{\mathcal{D}}(M)$.

Definition 5.5. The solution sheaf of a GSM theory associated to a scalar-electromagnetic bundle $\mathcal{D} = (\pi, \Phi, \Xi)$ is the sheaf of sets is given by:

$$\text{Sol}_{\mathcal{D}}(U) \overset{\text{def}}{=} \{ (s, \mathcal{V}) \in \text{Conf}_{\mathcal{D}}(U), \ | \ \mathcal{E}_S(s, \mathcal{V}) = 0, \ \mathcal{E}_K(s, \mathcal{V}) = 0 \},$$

for every open set $U \subset M$.

GSM theories generalize in a non-trivial way four-dimensional Maxwell-theory not only by coupling the theory to a section-sigma model instead of a standard sigma model, but also by coupling the field strengths appearing in the standard formulation of the theory to a generically non-trivial flat symplectic vector bundle. Many interesting aspects of these theories, such as their quantization and existence of solutions, remain to be explored even for the case when the theory is not coupled to a section sigma model.

\[6\] The meaning of the symbol $(-, -)$ will be explained in a moment, see definition 5.7.
5.2. The complete GESM theory. We are now ready to define GESM theories. In physics terms, a GESM-theory is a theory of gravity coupled to an arbitrary number of scalars and an arbitrary number of gauge fields with a potential which depends exclusively on the scalars. The formulation given below is appropriate for non-contractible space-times and should be sufficiently rich to accommodate the classical limit of string theory U-folds when the GESM theory is considered as a global formulation of the bosonic sector of four-dimensional supergravity on such space-times.

Definition 5.6. Let \( \mathcal{D} = (\pi, \Phi, \Xi) \) be a scalar-electromagnetic bundle with associated Lorentzian submersion \( \pi: (E, h) \to (M, g) \). The configuration sheaf of a GESM-theory defined by \( \mathcal{D} \) is the sheaf of sets given by:

\[
\text{Conf}_\mathcal{D}(U) \stackrel{\text{def}}{=} \left\{ (g, s, \mathbf{V}) \mid g \in \text{Met}_{3,1}(U), s \in \Gamma(\mathcal{D}^\pi|_U), \mathbf{V} \in \Omega_0^{2+} S_M(U) \right\},
\]

for every open set \( U \subset M \).

The exterior pairing \((\ , \ , \ )_g\) is the pseudo-Euclidean metric induced by \( g \) on the exterior bundle \( \wedge_M \).

Definition 5.7. The twisted exterior pairing \((\ , \ , \ ) : (\ , \ , \ )_g, Q\) is the unique pseudo-Euclidean scalar product on the twisted exterior bundle \( \wedge_M (S^*) \) which satisfies:

\[
(p_1 \otimes \xi_1, p_2 \otimes \xi_2)_g, Q^s = (p_1, p_2)_g Q^s(\xi_1, \xi_2),
\]

for any \( p_1, p_2 \in \Omega^s(M) \) and any \( \xi_1, \xi_2 \in \Gamma(M, S^*) \). Here \( Q^s(\xi_1, \xi_2) = \omega^s(\Psi^s\xi_1, \xi_2) \) and the superscript denotes pull-back by \( s \).

For any vector bundle \( W \), we trivially extend the twisted exterior pairing to a \( W \)-valued pairing (which for simplicity we denote by the same symbol) between the bundles \( W \otimes (\wedge_M (S^*)) \) and \( \wedge_M (S^*) \).

Hence:

\[
(c \otimes \eta_1, c \otimes \eta_2)_g, Q = (c, c)_g, Q^s, \quad \forall c \in \Gamma(M, W), \quad \forall \eta_1, \eta_2 \in \wedge_M (S^*).
\]

The inner \( g \)-contraction of \( 2 \)-tensors is the bundle morphism \( \otimes_g : (\otimes^2 T^*M)^{\otimes 2} \to \otimes^2 T^*M \) uniquely determined by the condition:

\[
(\alpha_1 \otimes \alpha_2) \otimes_g (\alpha_3 \otimes \alpha_4) = (\alpha_2, \alpha_3) \alpha_1 \otimes \alpha_4, \quad \forall \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Omega^1(M).
\]

We define the inner \( g \)-contraction of \( 2 \)-forms to be the restriction of \( \otimes_g \) to \( \wedge^2 T^*M \otimes \wedge^2 T^*M \overset{\otimes}{\to} \wedge^2 T^*M \).

Definition 5.8. The twisted inner contraction of \( S^* \)-valued \( 2 \)-forms is the unique morphism of vector bundles \( \otimes : \wedge_M^2 (S^*) \times_M \wedge_M^2 (S^*) \to \otimes^2 (T^*M) \) which satisfies the condition:

\[
(p_1 \otimes \xi_1) \otimes (p_2 \otimes \xi_2) = Q^s(\xi_1, \xi_2)p_1 \otimes_g p_2,
\]

for all \( p_1, p_2 \in \Omega^2(M) \) and all \( \xi_1, \xi_2 \in \Gamma(M, S^*) \).

Definition 5.9. Let \( \mathcal{D} = (\pi, \Phi, \Xi) \) be a scalar-electromagnetic bundle with associated Kaluza-Klein space \( \pi: (E, h) \to (M, g) \). The GESM theory associated to \( \mathcal{D} \) is defined by the following set of partial differential equations on \( (M, g) \):

- The Einstein equations\(^{7}\):

\[
E_E(g, s, \mathbf{V}) \stackrel{\text{def}}{=} G(g) - \kappa T(g, s, \mathbf{V}) = 0,
\]

where \( T(g, s, \mathbf{V}) \in \Gamma(M, S^2 T^*M) \) is the energy-momentum tensor of the theory, which is given by:

\[
T(g, s, \mathbf{V}) = g e_0^\mu (g, h, s) - h_\mu^2 + \frac{g}{2} \Phi^s + 2 \mathbf{V} \otimes \mathbf{V}.
\]

- The scalar equations:

\[
E_\Phi(g, s, \mathbf{V}) \stackrel{\text{def}}{=} \tau^s(g, h, s) + (\text{grad}_h \Phi)^s - \frac{1}{2} (\ast \mathbf{V}, \Psi^s \mathbf{V}) = 0.
\]

\(^{7}\) We denote Einstein’s gravitational constant by \( \kappa \).
• The electromagnetic (or Maxwell) equations:

\[ E_K(g, s, \mathbf{V}) \overset{\text{def}}{=} \mathbf{d}_D \cdot \mathbf{V} = 0. \]  

(5.4)

with unknowns given by triples \((g, s, \mathbf{V}) \in \text{Conf}_D(M)\).

**Definition 5.10.** The solution sheaf \(\text{Sol}_D\) of a GESM theory associated to a scalar-electromagnetic bundle \(D = (\pi, \Phi, \Xi)\) is the sheaf of sets given by:

\[ \text{Sol}_D(U) \overset{\text{def}}{=} \{ (g, s, \mathbf{V}) \in \text{Conf}_D(U) \mid E_E(g, s, \mathbf{V}) = 0, \ E_S(g, s, \mathbf{V}) = 0, \ E_K(g, s, \mathbf{V}) = 0 \} \]

for every open set \(U \subset M\).

For any open set \(U \subset M\) and any scalar-electromagnetic structure \(D\), let \(\text{Conf}_D(U)\) and \(\text{Sol}_D(U)\) denote the configuration and solution sets of the generalized Einstein-Scalar-Maxwell theory defined in [5].

**Theorem 5.1.** Let \(D = (\pi, \Phi, \Xi)\) be an scalar-electromagnetic bundle of type \(D = (M, S, \omega, D, J)\), whose Kaluza-Klein space is integrable. Then any special trivializing atlas \((U_\alpha, q_\alpha)_{\alpha \in I}\) of \(D\) induces bijections:

\[ \mathbf{e}_\alpha : \text{Conf}_D(U_\alpha) \sim \to \text{Conf}_D(U_\alpha) \]

which restrict bijections:

\[ \mathbf{e}_\alpha : \text{Sol}_D(U_\alpha) \sim \to \text{Sol}_D(U_\alpha) \]

*Proof. Given \((g_\alpha, s_\alpha, \mathbf{V}_\alpha) \in \text{Conf}_D(U_\alpha)\), we set:

\[ \mathbf{e}_\alpha(g_\alpha, s_\alpha, \mathbf{V}_\alpha) \overset{\text{def}}{=} (g_\alpha, \phi^\alpha, \mathbf{V}^\alpha), \]

where \(\phi^\alpha\) and \(\mathbf{V}^\alpha\) are defined as follows:

• As explained in Subsection 4.3, the special trivializing atlas \((U_\alpha, q_\alpha)\) is underlined by a special trivializing atlas \((U_\alpha, g_\alpha)\) of the fiber bundle \(E \to M\), whose trivializing maps are diffeomorphisms \(q_\alpha : E_\alpha \overset{\text{def}}{=} \pi^{-1}(U_\alpha) \to E^0_\alpha = U_\alpha \times M\). This allows us to define smooth maps \(\phi^\alpha \in C^\infty(U_\alpha, M)\) such that \(s_\alpha = q_\alpha^{-1} \circ \text{ungraph} (\phi^\alpha)\) as in Subsection 3.4:

\[ \phi^\alpha \overset{\text{def}}{=} q_\alpha \circ s_\alpha = \text{ungraph}(q_\alpha \circ s_\alpha) = g_\alpha^{-1} \circ q_\alpha \circ s_\alpha. \]

• As explained in Subsection 4.3, the maps \(q_\alpha : (S_\alpha, \omega_\alpha, D_\alpha, J_\alpha) \sim \to (S_\alpha^0, \omega_\alpha^0, D_\alpha^0, J_\alpha^0)\) are unbased isomorphisms of electromagnetic structures which linearize the diffeomorphisms \(q_\alpha\). They induce isomorphisms:

\[ q_\alpha^{(s_\alpha)} : (S_\alpha^0, \omega_\alpha^0, D_\alpha^0, J_\alpha^0) \sim \to (S_\alpha^{s_\alpha}, \omega_\alpha^{s_\alpha}, D_\alpha^{s_\alpha}, J_\alpha^{s_\alpha}) \overset{\text{def}}{=} D^{s_\alpha}, \]

where we noticed that \((S_\alpha^{s_\alpha})^{s_\alpha} = (S_\alpha^0)^{s_\alpha} = S^{s_\alpha}\) and we used the fact that the horizontal part of \(D_\alpha^0\) is the trivial flat connection. This allows us to identify \(\mathbf{V}_\alpha\) with the \(S^{s_\alpha}\)-valued two-form defined on \(U_\alpha\) through:

\[ \mathbf{V}^{\alpha} \overset{\text{def}}{=} q_\alpha^{(s_\alpha)}(\mathbf{V}_\alpha) \in \mathcal{O}^2(U_\alpha, S^{s_\alpha}). \]

The relation \(J^{s_\alpha} \circ q_\alpha^{(s_\alpha)} = q_\alpha^{(s_\alpha)} \circ J_\alpha^{s_\alpha}\) implies:

\[ \mathbf{V}_\alpha \in \mathcal{O}^{2+s_\alpha, s_\alpha, J_\alpha^{s_\alpha}}(U_\alpha) \iff \mathbf{V}^{\alpha} \in \mathcal{O}^{2+s_\alpha, s_\alpha, J^{s_\alpha}}(U_\alpha) \]

(5.6)

The fact that \(\mathbf{e}_\alpha\) is bijective follows immediately from its definition. The fact that \(\mathbf{e}_\alpha\) induces a bijection between solution sets follows from the equivalences:

\[ \mathcal{E}_E^D(g_\alpha, s_\alpha, \mathbf{V}_\alpha) = 0 \iff \mathcal{E}_E^D(g_\alpha, \phi^\alpha, \mathbf{V}_\alpha) = 0, \quad \mathcal{E}_S^D(g_\alpha, s_\alpha, \mathbf{V}_\alpha) = 0 \iff \mathcal{E}_S^D(g_\alpha, \phi^\alpha, \mathbf{V}_\alpha) = 0 \]

(5.7)

This follows by a direct but somewhat lengthy computation using relations (5.6), which we will not reproduce here. For example, it is easy to see that the relation \(\mathbf{d}_D^{s_\alpha} \circ q_\alpha^{(s_\alpha)} = q_\alpha^{(s_\alpha)} \circ \mathbf{d}_D^{s_\alpha}\) implies that the electromagnetic equation \(\mathbf{d}_D^{s_\alpha} \mathbf{V}_\alpha = 0\) is equivalent with the equation \(\mathbf{d}_D^{s_\alpha} \mathbf{V}^{\alpha} = 0\).

**Remark 5.1.** Together with the results of [5], Theorem 5.1 shows that a GESM theory represents an admissible global extension of the locally-formulated bosonic theories considered by physicists in the context of supergravity.
5.3. U-fold interpretation of globally-defined solutions in the integrable case. Let $\mathcal{D} = (\pi, \Phi, \Xi)$ be an integrable scalar-electromagnetic bundle and let $(U_{\alpha}, q_{\alpha})_{\alpha \in I}$ be a special trivializing atlas for $\pi$. Let $s \in \Gamma(\pi)$ be a global section of $\pi$ and $\mathcal{V} \in \Omega^2(M, S^\ast)$ be a two-form defined on $M$ and valued in the pulled-back bundle $S^\ast$. Let:

$$g_\alpha \overset{\text{def}}{=} g|_{U_\alpha}, \quad s_\alpha \overset{\text{def}}{=} s|_{U_\alpha}, \quad \mathcal{V}_\alpha \overset{\text{def}}{=} \mathcal{V}|_{U_\alpha}$$

and $\mathfrak{c}_\alpha(g_\alpha, s_\alpha, \mathcal{V}_\alpha) = (g_\alpha, \varphi^\alpha, \mathcal{V}_\alpha)$. Relations (5.5) imply the gluing conditions:

$$\mathcal{V}_\beta|_{U_{\alpha\beta}} = f_{\alpha\beta}^\beta \mathcal{V}_\alpha|_{U_{\alpha\beta}}, \quad (5.8)$$

which accompany the gluing conditions (3.5) for $\varphi^\alpha$ and the trivial gluing conditions:

$$g_\alpha|_{U_{\alpha\beta}} = g_\beta|_{U_{\alpha\beta}}, \quad (5.9)$$

for $g_\alpha$. When $(g, s, \mathcal{V}) \in \text{Sol}_\mathcal{D}(M)$, we have $(g_\alpha, s_\alpha, \mathcal{V}_\alpha) \in \text{Sol}_\mathcal{D}(U_\alpha)$ and Theorem 5.1 implies $(g_\alpha, \varphi^\alpha, \mathcal{V}^\alpha) \in \text{Sol}_\mathcal{D}(U_\alpha)$.

Conversely, any family $(g_\alpha, \varphi^\alpha, \mathcal{V}^\alpha)_{\alpha \in I}$ with $(g_\alpha, \varphi^\alpha, \mathcal{V}^\alpha) \in \text{Sol}_\mathcal{D}(U_\alpha)$ which satisfies conditions (5.9), (3.5) and (5.8) is obtained by restriction to $U_\alpha$ of a uniquely-determined global solution $(g, s, \mathcal{V})$ of the equations of motion of the GESM theory defined by the scalar-electromagnetic bundle $\mathcal{D}$. Once again, this can be formulated in sheaf language, a formulation whose details we leave to the reader. These observations justify the following:

**Definition 5.11.** Let $\mathcal{D}$ be a scalar-electromagnetic structure. A classical locally-geometric U-fold of type $\mathcal{D}$ is a global solution $(g, s, \mathcal{V}) \in \text{Sol}_\mathcal{D}(M)$ of the equations of motion of a GESM theory defined by a scalar-electromagnetic bundle $\mathcal{D}$ of type $\mathcal{D}$ with integrable Kaluza-Klein space. We say that a locally-geometric U-fold is trivial if the corresponding extended holonomy group $G$ is the trivial group.

6. A simple example

In this section, we show that the celebrated Scherk-Schwarz construction [30] can be understood as an instance of the integrable case of the general models considered in this paper. Notice from the outset that, when referring to the Scherk-Schwarz construction, we do not assume the presence of any continuous isometry group of the scalar manifold and hence that we do not gauge any such putative group.

Let $M = \mathbb{R}^3 \times S^1$, where $S^1 = \{ \sigma = e^{i\theta} | \theta \in \mathbb{R} \} = U(1)$ denotes the unit circle and let $\pi_1 : M \to \mathbb{R}^3$ and $\pi_2 : M \to S^1$ be the canonical projections. Let $1 \in S^1$ be the point which corresponds to $\theta = 0$ and let $g \in \text{Met}_{3,1}(M)$ be a Lorentzian metric of the form:

$$ds_g^2 = ds_{g_3}^2 + (2\pi)^2 R^2 d\theta^2,$$

where $g_3$ is a Lorentzian metric on $\mathbb{R}^3$ and $R$ is a positive parameter. In this example, we have $\pi_1(M) \simeq \pi_1(S^1) \simeq \mathbb{Z}$. Setting $x = (x^0, x^1, x^2) \in \mathbb{R}^3$, let $C \overset{\text{def}}{=} \{0\} \times S^1 \subset M$ be the circle defined inside $M$ by the equation $x = 0$. Orienting $C$ in the direction of decreasing $\theta$ (the “clockwise” orientation), its free homotopy class gives a generator of $\pi_1(M)$.

Given an $n$-dimensional Riemannian manifold $(M, \mathcal{G})$, an integrable Kaluza-Klein space $(E, h)$ over $(M, \mathcal{G})$ with fibers isometric to $(M, \mathcal{G})$ is determined by a morphism of groups $\rho : \mathbb{Z} \to \text{Iso}(M, \mathcal{G})$, i.e. by an element $U_{\mathcal{G}} \overset{\text{def}}{=} \rho(1) \in \text{Iso}(M, \mathcal{G})$ which describes the Ehresmann holonomy of $E$ along $C$. Let $\pi_E : E \to M$ be the projection of $E$ and $U$ be its Ehresmann transport.

Consider the restriction of $E$ to $C$, which has total space $F \overset{\text{def}}{=} \pi^{-1}_E(C)$ and projection $\pi_F \overset{\text{def}}{=} \pi_E|_F$. For simplicity, we denote the fiber of $F$ above a point $(0, \sigma) \in C$ by $F_\sigma$. For each point $m = (x, \sigma) \in M$, the Ehresmann transport along the curve $\gamma_m : [0, 1] \to M$ defined through $\gamma_m(t) = ((1-t)x, \sigma)$ gives an isometry $U_m : E_m \to F_\sigma$. These isometries allow us to identify $E$ with the pull-back bundle $\pi_2^*F$ by sending any point $e \in E_m$ to the point $(m, U_m(e)) \in \pi_2^*F | m = \{ m \} \times F_\sigma$. Moreover, we can identify the fiber $F_1$ with the Riemannian manifold $(M, \mathcal{G})$.

For any $\sigma \in S^1 = U(1)$, let $\ell_\sigma : [0, 1] \to C$ be the curve in $C$ defined by $\ell_\sigma(t) = 0, e^{i(1-t)\arg(\sigma)}$, where $\arg(\sigma) \in [0, 2\pi)$. Notice that this “clockwise”-oriented curve satisfies $\ell_\sigma(0) = (0, \sigma)$ and $\ell_\sigma(1) = (1, 0, 0)$. 

The concatenation $\ell_\sigma \cdot \gamma_m$ is a curve which connects $m$ to the point $(0,1) \in C$. The Ehresmann transport along this curve gives an isometry $V_\sigma \circ U_m : E_m \to F_1$, where $V_\sigma : F_\sigma \to F_1$ is the Ehresmann transport along $\ell_\sigma$. Thus sections $s \in \Gamma(\pi_E)$ of $E$ correspond to maps $\varphi : M \to F_1 = M$ defined through the relation:

$$\varphi(x,\sigma) \overset{\text{def}}{=} (V_\sigma \circ U_m)(s(x,\sigma)) \quad (6.1)$$

Recall that $U_C = \rho(1)$ denotes the Ehresmann holonomy of $(E,h)$ around $C$, where $C$ is oriented “clockwise”. Identifying the free fundamental group $\pi_1(M)$ with the based first homotopy group $\pi_1(M,(0,1))$, we have $U_C = \lim_{\epsilon \to 0^+} V_\sigma$, where for $\epsilon \in (0,1)$ we defined $\sigma_\epsilon \overset{\text{def}}{=} e^{2\pi i(1-\epsilon)}$. Using this in (6.1) gives:

$$\lim_{\epsilon \to 0^+} \varphi(x,\sigma_\epsilon) = (U_C \circ U_m)(s(x,1)) = U_C(\varphi(x,1)) \quad ,$$

where noticed that $\varphi(x,1) = U_m(s(x,1))$ since $V_1 = \text{id}_M$. Thus $\varphi(x,\sigma)$ has a branch cut in $\sigma$ at $\sigma = 1$, with monodromy $U_C = \rho(1)$ along $S^1$. Accordingly, $\varphi(x,\sigma)$ can be extended to a multivalued function of $\sigma$ having this monodromy.

The identification (6.1) shows that the section sigma model defined by $(M,g)$ and $(E,h)$ can be interpreted as a variant of the ordinary sigma model with source $(M,g)$ having this monodromy. Mathematically, this is equivalent with an ordinary smooth map $\tilde{\varphi} : \mathbb{R}^4 \to M$ defined on the universal covering space $\mathbb{R}^4$ of $M$ and which satisfies the quasi-periodicity condition:

$$\tilde{\varphi}(x,\theta + 1) = U_C(\tilde{\varphi}(x,\theta)) \quad . \quad (6.2)$$

Hence the section sigma model of this simple example gives a geometric encoding of the Scherk-Schwarz construction applied to the ordinary sigma model with source $(M,g)$ and target $(M,G)$. One can similarly check that the Abelian vector fields of the full GESM model can be described in this example through Scherk-Schwarz type monodromy conditions — except that, as in [5,6,7], we allow for a twist by a continuous group of isometries (in particular, the group $\text{Iso}(M,G)$ can be discrete). Due to this fact, the geometric U-folds constructed in this paper need not be solutions of gauged supergravity and the Abelian gauge fields of a GESM theory need not originate from “gauging” of any continuous isometry group.

7. Conclusions and further directions

The present paper extends ordinary Einstein-Scalar-Maxwell theory in a few directions so it may be useful to summarize our main results:

1. We generalized the ordinary scalar sigma model from a theory of maps to a theory of sections of a Lorentzian Kaluza-Klein space, whose solutions have an interpretation as “classical U-folds” in the integrable case. This generalized model is what we call the section sigma model.

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8 Notice that the the fundamental group $\pi_1(M)$ may not even be finitely-generated, since typically the space-time $M$ is not compact.
2. We gave a global mathematical formulation of the coupling of the section sigma model to Abelian gauge fields, consistent with electromagnetic-duality and without assuming topological triviality of space-time, of the fibers of the Kaluza-Klein space or of the duality bundle. This produces what we call a *Generalized Einstein-Section-Maxwell (GESM) theory*, a theory that provides a non-trivial global realization of the generic bosonic sector of four-dimensional supergravity in a manner which is consistent with electromagnetic duality.

3. When the Kaluza-Klein space is integrable, we showed that the GESM theory is *locally indistinguishable* from the generalized ESM theory of [5] and that its global solutions correspond to *classical locally-geometric U-folds*. This can be viewed as a rigorous mathematical definition of classical locally-geometric U-folds with scalar and Abelian gauge fields in the language of global differential geometry.

4. When the Kaluza-Klein space is not integrable, we showed that GESM theories are locally equivalent with modified sigma models of maps into the fiber coupled to Abelian gauge fields, which differ *locally* from ordinary ESM theories.

The locally-geometric classical U-folds described in this paper further generalize those constructed in in [5]. From the perspective of the present paper, the special situation discussed in op. cit. corresponds to the particular case when \( \pi \) is a topologically trivial fiber bundle whose Ehresmann connection is the trivial integrable connection but whose typical fiber \( (M, G) \) is not simply-connected.

The present work opens up various directions for further research. First, it is natural to consider the supersymmetrization of our theories. When the Kaluza-Klein space is integrable, this can be achieved by building fibered and twisted versions of four-dimensional supergravities coupled to matter, which are locally-indistinguishable (but globally different) from the latter. Imposing the appropriate supersymmetry constraints on a GESM theory turns out to be quite subtle. When the Kaluza-Klein space is not integrable, our bosonic theories differ *locally* from the bosonic sector of ordinary four-dimensional supergravities coupled to matter and hence may lead to new locally-defined supergravity models provided that one can find appropriate local supersymmetrizations. In this regard, we mention that all known “no-go theorems” regarding uniqueness of (ungauged) supergravity theories in four dimensions assume that the bosonic sector is described locally by an *ordinary* sigma model coupled to Abelian gauge fields. This is *not* the case if one takes the bosonic sector to be described by a section sigma model whose Kaluza-Klein space is not integrable.

Our formulation allows classical locally-geometric U-folds to be approached using the well-developed tools of global differential geometry and global analysis. This could be used to shed light on the classification of U-fold solutions, on moduli spaces of such and to construct new classes of classical locally-geometric U-folds. It would be interesting to extend our constructions to dimensions different from four as well as to the case of Euclidean signature and to study how they may relate to “dimensional reduction” on generalized Kaluza-Klein spaces. It would also be interesting to study various deformation problems for the structures arising in this construction. As a long-term project, it would be very interesting to gauge our models and to relate them to the appropriate extensions of gauged supergravity theories. The reader should also notice that GESM theories contain a “twisted” generalization of the standard Maxwell theory of Abelian gauge fields on Lorentzian four-manifolds which are not simply-connected, a theory which may itself be of some interest.

A. Pseudo-Riemannian submersions and Kaluza-Klein metrics

A.1. **Horizontal distributions and related notions.** For any surjective submersion \( \pi : E \to M \) with \( \dim E > \dim M \) and connected fibers, let \( V \stackrel{\text{def}}{=} \ker(d\pi) \subset TE \) denote the vertical distribution of \( \pi \). The differential of \( \pi \) induces an unbased linear epimorphism of vector bundles \( TE \to TM \) which covers \( \pi \). This can be identified with a based epimorphism \( \beta_\pi : TE \to (TM)^\pi \). We have a short exact sequence of vector bundles over \( E \) and based morphisms of vector bundles:

\[
0 \to V \to TE \xrightarrow{\beta_\pi} (TM)^\pi \to 0.
\] (A.1)

A *horizontal distribution* for \( \pi \) is a distribution \( H \subset TE \) such that \( TE = H \oplus V \); giving such a distribution amounts to giving a splitting of the sequence (A.1). Let \( \mathcal{H}(\pi) \) denote the set of horizontal distributions...
for π. Given a horizontal distribution H, let P_V and P_H denote the complementary projectors of TE on V and H and define X_H \text{def}= P_H X and X_V \text{def}= P_V X for any vector field X \in \mathcal{X}(E). The differential d\pi induces a based isomorphism of vector bundles (d\pi)_H \text{def}= \beta_\pi|_H : H \rightsquigarrow (TM)^{\pi}. The horizontal lift of a vector field X \in \mathcal{X}(M) is the vector field \overline{X} \in \mathcal{X}(E) defined through:

\[ \overline{X} \text{def}= (d\pi)_H^{-1}(X^\pi) \in \Gamma(M,H). \]

Given a smooth section s \in \Gamma(\pi), let ds : TM \to TE be the differential of s, viewed as an unbased morphism of vector bundles from TM to TE covering s. The vertical differential of s is the unbased morphism of vector bundles d^v s : TM \to V defined through:

\[ d^v s \text{def}= P_V \circ ds, \]

which covers s. Notice that d^v s can also be viewed as a V^* -valued one-form \widehat{d^v s} \in \text{Hom}(TM,V^*) \simeq \Gamma(M,T^a(M \otimes V^*)). When no confusion is possible, we will denote \widehat{d^v s} simply by d^v s.

A vertical tensor field defined on E is a section t \in \Gamma(E,\otimes^k V^*), which can also be viewed as an element t \in \text{Hom}(\otimes^k V^*,\mathbb{R}_E), where \mathbb{R}_E denotes the trivial real line bundle defined on the total space of E. The vertical differential pullback of t through a section s \in \Gamma(\pi) of π is the tensor field defined on M through:

\[ s^*_h(t) \text{def}= t^s \circ (\otimes^k \overline{d^v s}) \in \text{Hom}(\otimes^k TM,\mathbb{R}_M) \simeq \Gamma(M,\otimes^k T^a M), \]

where we noticed that \mathbb{R}_E^1 = \mathbb{R}.

Any connection ∇ on TE induces connections ∇^V = P_V \circ ∇ on V and ∇^H = P_H \circ ∇ on H, called the vertical and horizontal connections induced by ∇ relative to H.

A.2. Vertically non-degenerate metrics. A pseudo-Riemannian metric h on E is called vertically non-degenerate with respect to π if its restriction h_V to V is non-degenerate. Then the bundle H(h) of tangent vectors to E which are h-orthogonal to V is a complement of V inside TE:

\[ TE = V \oplus H(h) \]

called the horizontal distribution determined by π and h. Moreover, h restricts to a non-degenerate metric h_H on H(h).

Let h be a vertically non-degenerate metric with respect to π and let H \text{def}= H(h). Let ∇^V and ∇^H denote the vertical and horizontal covariant derivatives induced by the Levi-Civita connection of (E,h). Notice that h_V \text{def}= h|_V \in \Gamma(E,\text{Sym}^2(V^*)) is a symmetric vertical 2-tensor field defined on E.

**Definition A.1.** The vertical first fundamental form of a section s \in \Gamma(\pi) is the symmetric 2-tensor field \text{Sym}^2(T^a M):

\[ s^*_h(h_V)(X,Y) \text{def}= h_V((d^v s)(X), (d^v s)(Y)), \quad \forall X,Y \in \mathcal{X}(M). \]

Given a metric g on M, the vertical second fundamental form of s is the tensor field \text{Sym}^2(s) \text{def}= \nabla d^v s, where ∇ is the connection induced on the bundle T^a M \otimes V^* by the Levi-Civita connection of (M,g) and by the s-pullback of ∇^V, while d^v s is the vertical differential of s relative to the horizontal distribution H(h).

**Remark A.1.** Notice that \text{Sym}^2(s) need not be symmetric (even when π is a pseudo-Riemannian submersion with totally geodesic fibers, see below).

A.3. The Kaluza-Klein correspondence. Recall the notion of pseudo-Riemannian submersion [17, Chapter 7, Definition 44]:

**Definition A.2.** Let π : E \to M be a surjective submersion and h, g be pseudo-Riemannian metrics on E and M respectively. We say that π is a pseudo-Riemannian submersion from (E,h) to (M,g) if the following two conditions are satisfied:
1. $h$ is vertically non-degenerate with respect to $\pi$, i.e. the restriction of $h$ to $V$ is non-degenerate.

2. The based isomorphism $(d\pi)_H : H \cong (TM)^\pi$ is an isometry from $(H, h_H)$ to $((TM)^\pi, g^\pi)$. Let $\pi : E \to M$ be a surjective submersion. Given a pseudo-Riemannian metric $g$ on $M$, the pseudo-Riemannian metrics $h$ on $E$ which make $\pi : E \to M$ into a pseudo-Riemannian submersion from $(E, h)$ to $(M, g)$ are parameterized by horizontal distributions $H \in \mathcal{H}(\pi)$ and by non-degenerate metrics on $V$, as we explain below.

**Definition A.3.** Let $g$ be a pseudo-Riemannian metric on $M$. A pseudo-Riemannian metric $h$ on $E$ is called compatible with $\pi$ and $g$ if $\pi$ is a pseudo-Riemannian submersion from $(E, h)$ to $(M, g)$. It is called compatible with $\pi$ if there exists a pseudo-Riemannian metric $g$ on $M$ such that $h$ is compatible with $\pi$ and $g$.

Let $\text{Met}_{\pi,g}(E)$ denote the set of all pseudo-Riemannian metrics on $E$ which are compatible with $\pi$ and $g$ and $\text{Met}_{\pi}(E)$ denote the set of all pseudo-Riemannian metrics on $E$ which are compatible with $\pi$. For any $h \in \text{Met}(E)$ which is vertically non-degenerate, there exists at most one $g \in \text{Met}(M)$ such that $h$ is compatible with $\pi$ and $g$. Indeed, $g$ is determined by $\pi$ and $h$ through the relation:

$$g(X, Y) = h(\overline{X}, \overline{Y}) \quad \forall X, Y \in \mathcal{X}(M),$$

where $\overline{X}$ and $\overline{Y}$ denote the horizontal lifts of $X$ and $Y$ relative to $H(h)$. Thus $\text{Met}_{\pi}(E) = \cup_{g \in \text{Met}(M)} \text{Met}_{\pi,g}(E)$. Let $\text{Met}(V)$ denote the set of pseudo-Riemannian metrics on $V$.

**Definition A.4.** Let $g \in \text{Met}(M)$ be a pseudo-Riemannian metric on $M$, $\mathfrak{h} \in \text{Met}(V)$ be a pseudo-Riemannian metric on $V$ and $H \in \mathcal{H}(\pi)$ be a horizontal distribution for $\pi$. The Kaluza-Klein metric determined by $g$, $\mathfrak{h}$ and $H$ is the pseudo-Riemannian metric on $E$ defined through:

$$h_{g,\mathfrak{h},H}(X, Y) \overset{\text{def}}{=} \mathfrak{h}(X_V, Y_V) + g^\pi((d\pi)_H(X_H), (d\pi)_H(Y_H)), \quad \forall X, Y \in \mathcal{X}(E). \quad (A.2)$$

Notice that $h_{g,\mathfrak{h},H}|_V = \mathfrak{h}$, so the Kaluza-Klein metric is vertically non-degenerate with respect to $\pi$. Moreover, we have $h_{g,\mathfrak{h},H}|_H = g^\pi \circ ((d\pi)_H \otimes (d\pi)_H)$, hence $(d\pi)_H$ is an isometry between the pseudo-Euclidean distributions $(H, h_{g,\mathfrak{h},H}|_H)$ and $((TM)^\pi, g^\pi)$. Thus the Kaluza-Klein metric is compatible with $\pi$ and $g$, i.e. we have $h_{g,\mathfrak{h},H} \in \text{Met}_{\pi,g}(E) \subset \text{Met}_{\pi}(E).

**Definition A.5.** The Kaluza-Klein correspondence of $\pi$ is the map $\mathcal{K}_\pi : \text{Met}(M) \times \text{Met}(V) \times \mathcal{H}(\pi) \to \text{Met}_{\pi}(E)$ defined through:

$$\mathcal{K}_\pi(g, \mathfrak{h}, H) \overset{\text{def}}{=} h_{g,\mathfrak{h},H} \in \text{Met}_{\pi,g}(E) \subset \text{Met}_{\pi}(E).$$

The proof of the following statement is obvious:

**Proposition A.1.** Let $h$ be a pseudo-Riemannian metric on $E$ and $g$ be a pseudo-Riemannian metric on $M$. Then the following statements are equivalent:

1. $h$ is compatible with $\pi$ and $g$, i.e. we have $h \in \text{Met}_{\pi,g}(E)$.

2. $h$ is vertically non-degenerate with respect to $\pi$ and coincides with the Kaluza-Klein metric determined by $H \overset{\text{def}}{=} H(h)$, $g$ and by the metric $h_V \overset{\text{def}}{=} h_V |_V \in \text{Met}(V)$.

In particular, $h$ is uniquely determined by $H(h)$, $g$ and $h_V$.

**Corollary A.1.** For every $g \in \text{Met}(M)$, the map $\mathcal{K}_\pi(g, \cdot, \cdot) : \text{Met}(V) \times \mathcal{H}(\pi) \to \text{Met}_{\pi,g}(E)$ is bijective. In particular, the Kaluza-Klein correspondence $\mathcal{K}_\pi$ is a bijection from $\text{Met}(M) \times \text{Met}(V) \times \mathcal{H}(\pi)$ to $\text{Met}_{\pi}(E)$. 
Hence any $\pi$-compatible metric $h$ on $E$ determines a unique metric $g$ on $M$ such that $h$ is compatible with $\pi$ and $g$. Moreover, $h$ is the Kaluza-Klein metric determined by $\pi$, $g$, a unique vertical metric $h \in \text{Met}(V)$ and a unique horizontal distribution $H$ for $\pi$, namely $h = h_V$ and $H = H(\pi)$.

If $g \in \text{Met}(M)$ has signature $(p,q)$ and $h \in \text{Met}(V)$ has signature $(p_V,q_V)$, then the Kaluza-Klein metric $h = \mathcal{K}_\pi(g,h)$ determined by $g$, $h$ and any horizontal distribution $H$ has signature $(p+p_V,q+q_V)$. When $g$ is Lorentzian, the Lorentzian metric $h$ is Lorentzian iff $h$ is positive-definite, i.e. iff $q_V = 0$. Let $\dim M = d$ and $\dim E = d + n$ where $n > 0$. Then $\text{rk} V = n$. The restriction of the Kaluza-Klein correspondence gives a bijection:

$$\mathcal{K}_\pi : \text{Met}_{d-1,1}(M) \times \text{Met}_{n,0}(V) \times \mathcal{H}(\pi) \sim \text{Met}^{d+n-1,1}_g(E),$$

where $\text{Met}^{d+n-1,1}_g(E)$ is the set of all Lorentzian metrics on $E$ which are compatible with $\pi$.

**Definition A.6.** A Lorentzian submersion from $(E,h)$ to $(M,g)$ is a surjective pseudo-Riemannian submersion from $(E,h)$ to $(M,g)$, where $(E,h)$ and $(M,g)$ are Lorentzian manifolds.

The remarks above show that the set of Lorentzian metrics $h$ on $E$ for which $\pi : (E,h) \rightarrow (M,g)$ is a Lorentzian submersion is in bijection with the set $\text{Met}_{d,0}(V) \times \mathcal{H}(\pi)$ through the Kaluza-Klein correspondence.

**B. Local description in adapted coordinates**

Let $\pi : (E,h) \rightarrow (M,g)$ be a Lorentzian Kaluza-Klein space. Let $(W,u^0,\ldots,u^3,y^1,\ldots,y^n)$ be a $\pi$-adapted coordinate chart on $E$ with corresponding coordinate chart $(U,x^0,\ldots,x^3)$ on the base, where $U = \pi(W)$ and $u^\mu = x^\mu \circ \pi$. We use lowercase Greek letters for indices corresponding to the base, uppercase Latin letters for indices related to the fiber and lowercase Latin letters for indices related to the total space $E$.

We have $(dx)(\frac{\partial}{\partial u^\mu}) = \frac{\partial}{\partial x^\mu}$ and $(dx)(\frac{\partial}{\partial y}) = 0$. Thus $(\frac{\partial}{\partial x^0},\ldots,\frac{\partial}{\partial x^3})$ is a local frame of the vertical distribution $V$ defined on $W$. Any local frame $e_0,\ldots,e_3$ of the horizontal distribution $H$ which is defined on $W$ has the form:

$$e_\mu = e_\nu^\rho \frac{\partial}{\partial u^\rho} + v_\mu^A \frac{\partial}{\partial y^A}, \quad (B.1)$$

for some coefficient functions $e_\nu^\rho, v_\mu^A \in C^\infty(W,\mathbb{R})$, where the determinant of the matrix-valued function $e \equiv (e_\nu^\rho)_{\mu,\nu = 0,\ldots,3} \neq 0$ does not vanish on $W$. Setting $f := (f_\mu^\nu)_{\mu,\nu = 0,\ldots,3} \equiv e^{-1} \in C^\infty(W,\text{Mat}(4,\mathbb{R}))$, we have $f_\nu^\rho e_\nu^\mu = e_\rho^\mu f_\mu^\nu = \delta_\rho^\mu$ and:

$$\frac{\partial}{\partial u^\mu} = f_\mu^\nu e_\nu = f_\mu^A \frac{\partial}{\partial y^A}, \quad (B.2)$$

where we defined:

$$f_\mu^A \equiv f_\mu^\nu v_\nu^A.$$

We also have $(dx)(e_\mu) = e_\nu^\rho \frac{\partial}{\partial x^\rho}$. Given a vector field on $M$ locally expanded as $X = X^\mu \frac{\partial}{\partial x^\mu}$, its horizontal lift defined by $H$ is given by $\tilde{X} = X^\mu f_\mu^\nu e_\nu$. Let $f_\mu^\nu$ are the coefficients of the vertical lift morphism with respect to the local frames $\frac{\partial}{\partial x^\mu}$ of $TX$ and $e_\mu$ of $H$:

$$\frac{\partial}{\partial x^\mu} = f_\mu^\nu e_\nu.$$ 

Let:

$$h_{\mu\nu} \overset{\text{def}}{=} W h(e_\mu, e_\nu), \quad h_{AB} \overset{\text{def}}{=} W h\left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B}\right), \quad g_{\mu\nu} \overset{\text{def}}{=} U g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right),$$

where $h_{\mu\nu}$ and $h_{AB}$ are smooth functions defined on $W$ while $g_{\mu\nu}$ are smooth functions defined on $U$. Since $H$ and $V$ are $h$-orthogonal, we have $h(e_\mu, \frac{\partial}{\partial y}) = 0$. Since $d\pi$ restricts to an isometry between $H$ and $(TM)^*$, we have:

$$h_{\mu\nu} = g^{\rho\sigma} e_\mu^\rho e_\nu^\sigma.$$
where \( g_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu} \circ \pi \). In the local coordinates chosen, this means:

\[
h_{\mu\nu}(u, y) = h_{\mu\nu}(u) = h_{\mu\nu}(x \circ \pi) = h_{\mu\nu}(x).
\]

Setting:

\[
e^\mu \overset{\text{def}}{=} f^\mu_\rho du^\rho, \quad e^A \overset{\text{def}}{=} dy^A - f^A_\mu du^\mu,
\]

we have:

\[
e^\mu(e_\nu) = \delta^\mu_\nu, \quad e^\mu \left( \frac{\partial}{\partial y^B} \right) = 0
\]

\[
e^A(e_\nu) = 0, \quad e^A \left( \frac{\partial}{\partial y^B} \right) = \delta^A_B,
\]

which shows that \((e^\mu, e^A)\) is the coframe dual to the frame \((e_\mu, \frac{\partial}{\partial y^\pi})\). The metric \(h\) has the well-known Kaluza-Klein form:

\[
h = W h_{\mu\nu} e^\mu \otimes e^\nu + h_{AB} e^A \otimes e^B = g_{\mu\nu}^\pi du^\mu \otimes du^\nu + h_{AB} (dy^A - f^A_\mu du^\mu) \otimes (dy^B - f^B_\mu du^\mu), \tag{B.3}
\]

which depends only on \(f^A_\mu\). Thus:

\[
h_V = W h_{AB} e^A \otimes e^B = h_{AB} (dy^A - f^A_\mu du^\mu) \otimes (dy^B - f^B_\mu du^\mu)
\]

\[
\equiv h_{AB}(x, y)(dy^A - f^A_\mu(x, y)dx^\mu) \otimes (dy^B - f^B_\mu(x, y)dx^\mu),
\]

\[
h_H = W h_{\mu\nu} e^\mu \otimes e^\nu = g_{\mu\nu}^\pi du^\mu \otimes du^\nu \equiv g_{\mu\nu}(x)dx^\mu \otimes dx^\nu,
\]

where after the sign \(\equiv\) we identified \(u\) with \(x\) by abuse of notation. Here \(h_{AB}\) and \(f^A_\mu\) depend on both \(x\) and \(y\).

**B.1. Local expression of the vertical differential and of the vertical first fundamental form of a section.**

Any section \(s \in \Gamma(\pi)\) satisfies \(\pi \circ s = \text{id}_M\), which implies \(u^\mu \circ s = u^\mu x^\mu\). Defining:

\[
\varphi^A \overset{\text{def}}{=} y^A \circ s|_U \in C^\infty(U, \mathbb{R}),
\]

the section has the local expression:

\[
u^\mu(s(x)) = x^\mu, \quad y^A = \varphi^A(x)
\]

in the coordinate charts \((U, x)\) of \(M\) and \((W, x, y)\) of \(E\). Setting \(\partial_\mu \varphi^A \overset{\text{def}}{=} \frac{\partial \varphi^A}{\partial x^\mu}\), we have:

\[
(ds) \left( \frac{\partial}{\partial x^\mu} \right) = \frac{\partial}{\partial u^\mu} + \partial_\nu \varphi^A \frac{\partial}{\partial y^A} = f^\nu_\mu e_\nu + (\partial_\mu \varphi^A - f^A_\mu) \frac{\partial}{\partial y^A},
\]

which gives the local expression of the vertical differential of \(s\):

\[
(du^\nu)(\frac{\partial}{\partial x^\mu}) = (\partial_\mu \varphi^A - f^A_\mu) \frac{\partial}{\partial y^A}.
\]

In turn, this gives the local expression of the vertical first fundamental form:

\[
s^*_V(h_V)(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}) = (h_{AB} \circ s)(\partial_\nu \varphi^A - f^A_\mu \circ s)(\partial_\nu \varphi^B - f^B_\nu \circ s).
\]

**B.2. Local form of the Lagrange density of the section sigma model.** Using the formulas obtained above, we find that the Lagrange density of the section sigma model has the coordinate expression:

\[
e^\mu_{\Phi}(s) = U g^\mu_\nu(h_{AB} \circ s)(\partial_\mu \varphi^A - f^A_\mu \circ s)(\partial_\nu \varphi^B - f^B_\nu \circ s) + \Phi(s), \tag{B.4}
\]
i.e.:
\[ e^\nu(s)(x) = U \ g^{\mu\nu}(x) h_{AB}(x, \varphi(x)) [\partial_\mu \varphi^A(x) - f^A_\mu(x, \varphi(x))] [\partial_\nu \varphi^B - f^B_\nu(x, \varphi(x))] + \Phi(x, \varphi(x)) \]  
\[ \Phi(x, \varphi(x)) = \Phi^0(x) + \Phi^1(x) \]

Hence the action for a relatively compact subset \( U_0 \subset U \) takes the form:
\[ S_{\omega, U_0}[s] = S_{\omega, U_0}[\varphi] = \int_{U_0} d^4x \sqrt{|g|} \left\{ g^{\mu\nu}(x) h_{AB}(x, \varphi(x)) [\partial_\mu \varphi^A(x) - f^A_\mu(x, \varphi(x))] [\partial_\nu \varphi^B - f^B_\nu(x, \varphi(x))] + \Phi(x, \varphi(x)) \right\}, \]
where \( |g| = \det |g_{\mu\nu}|. \)

### B.3. Local form of the vertical tension field

Let \( \Gamma^\nu_{\mu\rho} = \Gamma^\nu_{\mu\rho} \) denote the Levi-Civita coefficients of \( h \) in the coordinate chart \((W, x, y)\). Thus:

\[ \nabla_{\sigma\mu\nu} \left( \frac{\partial}{\partial \sigma} \right) = \Gamma^\rho_{\mu\nu} \frac{\partial}{\partial \rho} + \Gamma^\lambda_{\mu\rho} \frac{\partial}{\partial \lambda} \]

\[ = \Gamma^\rho_{\mu\nu} \frac{\partial}{\partial \rho} + \Gamma^\lambda_{\mu\rho} \frac{\partial}{\partial \lambda} \]

\[ = \Gamma^\rho_{\mu\nu} \frac{\partial}{\partial \rho} + \Gamma^\lambda_{\mu\rho} \frac{\partial}{\partial \lambda} \]

\[ = \Gamma^\rho_{\mu\nu} \frac{\partial}{\partial \rho} + \Gamma^\lambda_{\mu\rho} \frac{\partial}{\partial \lambda} \]

with the symmetry properties \( \Gamma^\nu_{\mu\rho} = \Gamma^\rho_{\nu\mu} \), \( \Gamma^C_{AB} = \Gamma^C_{BA} \), \( \Gamma^A_{\mu\nu} = \Gamma^A_{\nu\mu} \) and \( \Gamma^B_{\mu\nu} = \Gamma^B_{\nu\mu} \). This gives:

\[ \nabla_{\sigma\mu\nu} \left( \frac{\partial}{\partial \sigma} \right) = \left( \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} f^A_{\rho} \right) \frac{\partial}{\partial \rho} \]

\[ = \left( \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} f^A_{\rho} \right) \frac{\partial}{\partial \rho} \]

\[ = \left( \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} f^A_{\rho} \right) \frac{\partial}{\partial \rho} \]

On the other hand, we have:

\[ \nabla_{\sigma\mu\nu} \left( \frac{\partial}{\partial \sigma} \right) = K^\rho_{\mu\nu} \frac{\partial}{\partial \sigma} \]

Let \( \partial_A f^C_{\nu} \overset{\text{def}}{=} \frac{\partial f^C_{\nu}}{\partial y^A} \). Using the relations above, we compute:

\[ \nabla_{\sigma\mu\nu} \left( \frac{\partial}{\partial \sigma} \right) = \nabla_{\sigma\mu\nu} \left( \frac{\partial}{\partial \sigma} \right) + \partial_\nu \varphi^A \frac{\partial}{\partial y^A} \left( \partial_\nu \varphi^B - f^B_\nu \right) \frac{\partial}{\partial y^B} \]

\[ = \nabla_{\sigma\mu\nu} \left( \frac{\partial}{\partial \sigma} \right) + \partial_\nu \varphi^A \nabla_{\sigma\mu\nu} \left( \frac{\partial}{\partial y^A} \right) \]

\[ = \nabla_{\sigma\mu\nu} \left( \frac{\partial}{\partial \sigma} \right) + \partial_\nu \varphi^A \nabla_{\sigma\mu\nu} \left( \frac{\partial}{\partial y^A} \right) \]

and:

\[ d^4x s(\nabla_{\sigma\mu\nu} \left( \frac{\partial}{\partial \sigma} \right)) = K^\rho_{\mu\nu} \nabla_{\sigma\mu\nu} \left( \frac{\partial}{\partial \sigma} \right) \frac{\partial}{\partial y^C} \]

Hence the vertical tension field of \( s \) is given by:

\[ \tau^\nu(s) = g^{\mu\nu} \left[ \nabla_{\sigma\mu\nu} \left( \frac{\partial}{\partial \sigma} \right) \right] - d^4x s \left( \nabla_{\sigma\mu\nu} \left( \frac{\partial}{\partial \sigma} \right) \right) \]

where:

\[ \tau^\nu(s)^C = g^{\mu\nu} \left[ \partial_\nu (\partial_\mu \varphi^C - f^C_\mu) - \partial_\nu \varphi^A \partial_A f^C_\nu + (\Gamma^C_{\mu A} - \Gamma^C_{\nu A} f^C_\rho)(\partial_\nu \varphi^A - f^A_\nu) + (\Gamma^C_{A B} - \Gamma^C_{A B} f^C_\rho)(\partial_\mu \varphi^A - f^A_\mu) \right] \]
This can also be written as:

$$
\tau^v(s)^C = \varphi^B \partial_\mu \varphi^C + (\Gamma^B_{AB} - \Gamma^B_{AB\mu} f^B_\mu) \varphi^A \partial_\mu \varphi^B + [\Gamma^C_{\mu A} - \Gamma^C_{\mu A B} - \partial_\mu f^C_\nu + (\Gamma^B_{\mu A} f^C_\nu - \Gamma^C_{\mu A B}) f^B_\nu] \partial_\mu \varphi^A.
$$

(B.6)

In particular, the harmonic section equation for $s$ amounts to a system of second order PDEs for the functions $\varphi^C$, which contain terms of order zero and can be viewed as deformations of the d’Alembert equation.

### B.4. Local form of the curvature of $H$.

The Lie bracket of the vector fields given in (B.1) has the form:

$$
[e_\mu, e_\nu] = \begin{pmatrix}
e_\nu^\rho & e_\lambda^\rho & e_\lambda^\nu & e_\lambda^\mu 
\end{pmatrix} = \varphi^B \partial_\mu \varphi^C + \varphi^A \partial_A \varphi^B,
$$

where:

$$
\Gamma^C = \begin{pmatrix}
\mu & \nu & \rho
\end{pmatrix}.
$$

Thus $H$ is integrable iff $\mu = \nu$ and the Kaluza-Klein metric reduces to:

$$
\text{Hence: }
\mathcal{F}(e_\mu, e_\nu) = \mathcal{F}^C_{\mu \nu} \frac{\partial}{\partial y^C}.
$$

where:

$$
\mathcal{F}^C_{\mu \nu} = \varphi^B \partial_\mu \varphi^C + \varphi^A \partial_A \varphi^B - \varphi^A \partial_A \varphi^C = - f^C_{\sigma} (e_\nu^\rho \partial_\sigma \varphi^\rho - e_\nu^\rho \partial_\sigma \varphi^\rho + \varphi^A \partial_A \varphi^\sigma - \varphi^A \partial_A \varphi^\sigma).
$$

Thus $H$ is integrable if $\mathcal{F}^C_{\mu \nu} = 0$. The previous calculation uses a local frame $\{e_\mu\}_{\mu=0, \ldots, 3}$ of $H$. Any other local frame of $H$ defined above $W$ has the form:

$$
e_\mu' = a_\mu^\nu e_\nu,
$$

where $a = \begin{pmatrix} a_{\mu \nu} \end{pmatrix}$ and $a_{\mu \nu} = \text{inv}$ is an invertible matrix. We have $e' = ae$ and $(e')_{\mu}^A = a_{\mu \nu} e^A_\nu$, thus $f' = f a^{-1}$ and $(f')_{\mu}^A = f^A_{\nu}$. The metric (B.3) is invariant under such changes of the local frame of $H$. Since $e$ is non-degenerate, by taking $a = e^{-1}$ we can always insure that $e' = I_4$. Hence $H$ always admits a local frame with $e_\mu' = \delta_{\mu}^\nu$, which we call a special frame. For such a frame of $H$, we have $f_{\nu}^A = \delta_{\nu}^A$, $f^A_{\mu} = v_{\mu}^A$, and:

$$
e_\mu = \frac{\partial}{\partial u^\mu} + v_{\mu}^A \frac{\partial}{\partial y^A} = + \frac{\partial}{\partial u^\mu} + f^A_{\mu} \frac{\partial}{\partial y^A}.
$$

When expressed using a special frame of $H$, the curvature coefficients become:

$$
\mathcal{F}^C_{\mu \nu} = \partial_\mu f^C_{\nu} - \partial_\nu f^C_{\mu} + f^A_{\mu} \partial_A f^C_{\nu} - f^C_{\nu} (e_\nu^\rho \partial_\rho \varphi^\rho - e_\nu^\rho \partial_\rho \varphi^\rho + \varphi^A \partial_A \varphi^\sigma - \varphi^A \partial_A \varphi^\sigma).
$$

(B.7)

### B.5. The case when $H$ is integrable.

A simple particular case arises when $H|_W$ is Frobenius integrable. In this situation, one can find adapted coordinates $(W, u, y)$ such that $H$ identifies locally with the integrable distribution $H = (TM)^A$ (which is spanned by $\partial_1 \partial_2 \cdots \partial_{3A}$ and such that $\partial_{3A}$ are invariant under the Ehresmann transport of $H$). We can then take $e_\mu = \delta_{\mu}^\nu$, which corresponds to $e_\mu' = \delta_{\mu}^\nu$ and $e_\mu = 0$. Thus $f_{\mu}^A = 0$ and the Kaluza-Klein metric reduces to:

$$
h = g^{w}_{\mu \nu} du^\mu \odot du^\nu + h_{AB} dy^A \odot dy^B.
$$

(B.8)

Since $h_{\nu}$ is invariant under Ehresmann transport (which proceeds through isometries of the fibers) the coefficients $h_{AB}$ are independent of $x$ and coincide with the metric coefficients of the fiber:

$$
h_{AB} = g_{AB}(y).
$$

(B.9)
Hence the metric (B.8) reduces to the product metric $g \times \mathcal{G}$. We have $\Gamma_{\mu A}^C = 0$, hence the coefficients (B.6) of the vertical tension field reduce to the local form of an ordinary sigma model:

$$\tau^v(s)^C = \partial^\mu \partial_\mu \varphi^C + \Gamma_{AB}^C \partial_\mu \varphi^A \partial^\mu \varphi^B - \varphi^{\nu \rho} \partial^\rho \partial_\mu \varphi^C$$

On the other hand, the restriction of $\Phi$ to $W$ is a function:

$$\Phi|_W : W \to \mathbb{R},$$

which in principle depends both on $u^\mu$ and $y^A$. Equation (2.2) implies:

$$d\Phi|_W = \frac{\partial}{\partial u^\mu} \Phi|_W du^\mu + \frac{\partial}{\partial y^A} \Phi|_W dy^A = \left( \frac{\partial}{\partial y^A} \Phi|_W - f^A_{\mu} \frac{\partial}{\partial y^A} \Phi|_W \right) dy^A.$$  \hspace{1cm} (B.11)

Since integrability of $H|_W$ implies $f^A_{\mu} = 0$ (see above), we find that $\Phi|_W$ only depends on $y^A$:

$$\frac{\partial}{\partial u^\mu} \Phi|_W = 0, \quad d\Phi|_W = \frac{\partial}{\partial y^A} \Phi|_W dy^A.$$ \hspace{1cm} (B.12)

Thus $\Phi|_W(x, y) = \Phi_W(y)$ for some function $\Phi_W$. This shows that the section scalar sigma model is locally equivalent with the ordinary scalar sigma model when $H$ is integrable. A more geometric explanation of this fact is given in Section 3.

### B.6. Local expression of the fundamental bundle form $\Theta$ and of the fundamental bundle field $\Psi$.

Let $\mathcal{D}$ be a scalar-electromagnetic bundle of type $\mathcal{D} = (\pi, \Phi, \Xi)$ with associated Kaluza-Klein space $\pi : (E, h) \to (M, g)$. Let $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ be the corresponding duality bundle, namely $\mathcal{S}$ is a real vector bundle of rank $2r$ over $E$ endowed with a symplectic structure $\omega$ and a compatible flat connection $\mathcal{D}$. As explained in Section 4, a taming $J$ of $(\mathcal{S}, \omega)$ is said to be vertical if it satisfies $D_X \circ J = J \circ D_X$ for all $X \in \Gamma(E, H)$. Let $(W, u^0, \ldots, u^3, y^1, \ldots, y^r)$ be a $\pi$-adapted coordinate chart on $E$, with corresponding coordinate chart $(U, x^0, \ldots, x^3)$ on $M$, where $U = \pi(W) \subset M$ and $u^\mu = x^\mu \circ \pi$. Let $\mathcal{E} : = (e_M)_{M=1,\ldots,2r}$ be a local flat symplectic frame of $\mathcal{S}$ over $W \subset E$ and let $\mathcal{E}^* : = (e^*_M)_{M=1,\ldots,2r}$ denote the dual frame of $\mathcal{S}^*$. In terms of $\mathcal{E}$ and $\mathcal{E}^*$, the restriction of $J$ to $W$ can be written as follows:

$$J|_W = J^N_M e_N \otimes e^*_M,$$ \hspace{1cm} (B.13)

where $J^N_M \in C^\infty(W, \mathbb{R})$. Let $H = h(h) \subset TE$ be the horizontal distribution of the Kaluza-Klein space $\mathcal{D}$. As already explained, $H$ is locally spanned on $W$ by the tangent vector fields $e_\mu$ of equation (B.1). Since $\mathcal{E}$ is a local flat frame, we have:

$$D(e_M) = 0.$$  \hspace{1cm}

Using this relation, the verticality condition for $J$ amounts to:

$$D_{e_\mu} J|_W = 0,$$  \hspace{1cm}

which in turn is equivalent with:

$$e^*_\nu \frac{\partial}{\partial u^\nu} J^N_M + v^A_{\mu} \frac{\partial}{\partial y^A} J^N_M = 0,$$ \hspace{1cm} (B.14)

where we used (B.13) and (B.1).

Assume now that $H$ is integrable. Without loss of generality, we can than take $e^*_\mu = \delta^*_\mu$ and $v^A_{\mu} = 0$, thus $e_\mu = \frac{\partial}{\partial u^\mu}$ (see Subsection B.5). In this case, the verticality condition (B.14) for $J$ reduces to:

$$\frac{\partial}{\partial u^\mu} J^N_M = 0, \quad \mu = 0, \ldots, 3.$$ \hspace{1cm} (B.15)

This shows that $J^N_M$ depend only on $y^A$. Using the definition of the fundamental bundle form $\Theta$, we obtain:

$$\Theta|_W(e_M) = (D^{\text{ad}}J)|_W(e_M) = D(J|_W(e_M)) = \left( \frac{\partial}{\partial y^A} J^N_M \right) dy^A \otimes e_N.$$
Similarly, the fundamental bundle field $\Psi$ is given by:

$$\Psi|_W(e_M) = h^{AB} \left( \frac{\partial}{\partial y^A} J^N_M \right) \frac{\partial}{\partial y^B} \otimes e_N.$$  

Let $s$ be a section of $\pi$. The pull-backs by $s$ of $\Theta$ and $\Psi$ are given by:

$$\Theta^o|_U(e_M^o) = \left( \frac{\partial J^N_M}{\partial y^A} \right) \circ \varphi \, d\varphi^A \otimes e_N^o, \quad \Psi^o|_{u_o}(e_M^o) = \mathcal{G}^{AB} \circ \varphi \left( \frac{\partial J^N_M}{\partial y^A} \right) \circ \varphi \left( \frac{\partial}{\partial y^B} \right)^s \otimes e_N^o,$$

where we used equation (B.9) and we defined:

$$\varphi^A = y^A \circ s|_U \in C^\infty(U, \mathbb{R}).$$

Notice that the pullback of $J$ by $s$ has the local form:

$$J^o|_U = J^N_M \circ \varphi e^N_M \otimes e^M.$$

Let $D = (S, \omega, D, J)$ be the type of $\mathbf{D}$ and consider a special trivializing atlas $(U_o, q_o)_{o \in I}$ for the scalar-electromagnetic bundle $\mathbf{D}$. This induces local isometric trivializations:

$$q_o : E_o \rightarrow U_o \times \mathcal{M}$$

of $\pi : (E, h) \rightarrow (M, g)$ and unbiased isomorphisms of electromagnetic structures:

$$q_o : (S_o, \omega_o, D_o, J_o) \rightarrow (S^0_o, \omega^0_o, D^0_o, J^0_o),$$

where $S_o = \mathbf{S}|_{U_o}$ etc. and:

$$S^0_o = S_o^\alpha, \quad \omega^0_o = \omega_o^\alpha, \quad D^0_o = D_o^\alpha, \quad J^0_o = J_o^\alpha.$$

In turn, $q_o$ induce isomorphisms:

$$q_o(s) : (S^0_o, \omega^0_o, D^0_o, J^0_o) \rightarrow (S^\alpha, \omega^\alpha, D^\alpha, J^\alpha).$$

Setting $U = U_o$ in equation (B.16) gives:

$$\Theta^o(e_M^o) = \left( \frac{\partial}{\partial \varphi^A} J^N_M(\varphi^o) \right) \circ d\varphi^A \otimes e_N^o,$$

$$\Psi^o(e_M^o) = \mathcal{G}^{AB}(\varphi^o) \left( \frac{\partial}{\partial \varphi^A} J^N_M(\varphi^o) \right) \circ \frac{\partial}{\partial y^B} \otimes e_N^o.$$

(B.17)

Here $\Theta^o \equiv (\varphi^o)^\ast \Theta$ and $\Psi^o \equiv (\varphi^o)^\ast \Psi$ (where $\Phi$ and $\Psi$ are the fundamental form and fundamental vector field of $\mathbf{D}$ defined in [5]), while $e_M^o = q_o(e_M)$ form a flat symplectic frame $E^o$ of $S^\alpha$. We conclude that the local form of $\Theta^o$ and $\Psi^o$ is consistent with that of generalized Einstein-Scalar-Maxwell theories.

We can identify $\mathbf{V}|_{U_o}$ with the bundle-valued form:

$$V^o = q_o^s(\mathbf{V}_o) = F^A e_A^o + G_A \varphi^o \in \Omega^2(\mathcal{U}_o, S^\alpha),$$

where $F_A, G_A \in \Omega^2(\mathcal{U}_o)$ for $A = 1, \ldots, r$. In the symplectic frame $E^o$, the form $\varphi^o(\omega) \in \Omega^2(\mathcal{U}_o)$ has matrix components $\omega^o_{MN}$ ($M, N = 1, \ldots, 2r$) corresponding to the standard symplectic matrix. The previous formulas imply:

$$(\ast \mathbf{V}, \Psi^o \mathbf{V}) = u_o \ast (\mathbf{V}^N, \mathbf{V}^P) \omega^o_{NM} \left( \frac{\partial}{\partial \varphi^A} J^M_P(\varphi) \right) = -2(\partial A G_A, \ast F^A)_g,$$

$$\mathbf{V} \otimes \mathbf{V} = u_o \ast (\omega^o_{MN} J^M_P) \mathbf{V}^N_P \mathbf{V}^P \mathbf{V} \otimes \mathbf{V},$$

as required by local consistency with standard ESM theories. It is now easy to see that the positive polarization condition:

$$\ast_g \mathbf{V} = -J^o \mathbf{V},$$

takes the following form when restricted to $U_o$:

$$\ast_g \mathbf{V} = -J^o \mathbf{V}, \quad \text{(B.18)}$$

thus reducing to the local twisted self-duality condition by of an ordinary ESM theory defined on $U_o$. 

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