Spin Transmutation in (2 + 1) Dimensions

Wei Chen∗
Department of Physics
University of North Carolina
Chapel Hill, NC 27599-3255

Chigak Itoi†
Department of Physics and Atomic Energy Research Institute
College of Science and Technology, Nihon University
Kanda Surugadai, Chiyoda-ku, Tokyo 101, Japan

Abstract

We study a relativistic anyon model with a spin-$j$ matter field minimally coupled to a statistical gauge potential governed by the Chern-Simons dynamics with a statistical parameter $\alpha$. A spin and statistics transmutation is shown in terms of a continuous random walk method. An integer or odd-half-integer part of $\alpha$ can be reabsorbed by change of $j$. We discuss the equivalence of a large class of (infinite number) Chern-Simons matter models for given $j$ and $\alpha$.

∗ chen@physics.unc.edu
† itoi@phys.cst.niho-u.ac.jp
1 Introduction

It is well-known now that in (2+1) spacetime dimensions there are anyons which have arbitrary spin and statistics [1]. The statistics of particles is changed if an interaction with a statistical gauge potential governed by the Chern-Simons term is introduced [2]. While this issue is settled for non-relativistic matters, there has been debate as far as the correct formalism of relativistic anyons is concerned. In this context, the spin degree of freedom plays an essential role. In the framework of quantum field theory, a Fermi-Bose transmutation has been observed in a (2+1) dimensional system [3]. Namely that a charged scalar field coupled to a Chern-Simons gauge potential at a specific value of Chern-Simons coefficient (or statistical parameter) turns out to be a free fermion field theory. In this example, one has to consider spin transmutation as well as statistics transmutation. These refer to apparently different objects: a statistical transformation affects the Aharonov-Bohm phase of the wave-function of two identical particles moving around one to the other on a plane, while any spin transmutation should cause corresponding change in the Lorentz group representation of the matter fields. Though there was investigation on anomalous magnetic momentum in a perturbation theory in the literature [4], a clear understanding for spin transmutation seems still lacking.

The problem with spin transmutation is discussed in a recent letter [5]. The present paper is to continue the discussion. We see that in a general Chern-Simons matter field theory model, any integer or odd-half-integer part of the Chern-Simons coefficient can be reabsorbed by changing the character of the Poincaré representation describing the matter. This is an observation through a continuous random walk method with certain assumptions with respect to a regularization. In this formalism, the partition function of an anyon system can be represented in particle path integrals with some phase factor with terms including a spin factor from the path integral of the matter field and a self-energy and a relative energy from that of the Chern-Simons gauge field. And, a topological relation between the spin factor and self-energy enables us to show the spin and statistics transmutation. Moreover, we see that, due to the quantization of the Chern-Simons relative energy, only integer or odd-half-integer part of the coupling constant endows the anyons with a change of spin and a “small” part of it ($< 1/2$) remains representing a residual Chern-Simons interaction.

There is a nice application with the spin and statistics transmutation. As we
know, the Chern-Simons coefficient is used quite often as a parameter over which a perturbation expansion is conducted. However, when the Chern-Simons coefficient is required large in some systems, the perturbation expansion is not reliable. Now, with the spin and statistics transmutation, one can trade a large part of the Chern-Simons coefficient for a change of the spin of the matter and keep the remaining Chern-Simons coupling weak. In this way, a perturbation expansion becomes controllable.

As in any relativistic field theory, the fundamental fields are required to carry irreducible unitary representations of the universal covering group of Poincaré group. One way to realize this is to construct the fundamental fields transforming as linear representations of the Lorentz group, and to subject them to constraints that eliminate unphysical degrees of freedom. A striking feature of the Poincaré representations in $(2+1)$ dimensions is that each of them has only one independent component. This can be understood with the following argument. Define the value of spin $j$, which can be arbitrary in $(2+1)$ dimensions \cite{[6]}, via $j^\mu j_\mu = j(j+1)$ with $j^\mu$ the spin part of the Lorentz generators. To construct an irreducible linear representation, one needs $2j + 1$ components for an integer or odd-half-integer spin-$j$. However, a constraint, the Pauli-Lubanski condition, kills all but one component.

Like statistics transmutation, spin transmutation is obviously a phenomenon at the quantum level. A key step to understand the spin transmutation in a Chern-Simons matter model is to establish a relation between the spin $j$ and the Chern-Simons coefficient $\alpha$, while a place we feel convenient to do so is the partition function of an anyon system, where the matter fields and gauge potential are all integrated out. From proposing an anyon model with an arbitrary spin-$j$ spinning matter field minimally coupled to a Chern-Simons gauge potential (a local four-body matter self-interaction is also introduced), we offer a step-by-step analysis and obtain a $j-\alpha$ relation that displays the spin transmutation. We then consider some consequences with emphasis on the equivalence of a large class of (infinite number) Chern-Simons matter models. We also shed lights on the debate about the role a Chern-Simons gauge field plays, and on the possible role the Chern-Simons spinning field theory model plays in describing the critical phenomena.

In section 2, we briefly review the Poincaré group in $(2+1)$ dimensions and its representations, with details for $j = 1/2, 1, \text{and } 3/2$. In section 3, we define a spin factor for matter fields, and introduce $SU(2)$ coherent states to formulate the free-energy of spin-$j$ matter fields, treating the Chern-Simons gauge potential as an
external field. We then carry out, in section 4, the functional integral over the Chern-Simons field; we analyze the relative and self-energies from the Chern-Simons gauge potential and verify their relations to a topological quantity, the Gaussian linking number, and to the spin factor associated with the spin of matter fields. From these the spin transmutation is explicitly observed. We discuss the equivalence of a sort of Chern-Simons matter models with a local four-body interaction in section 5, while concluding remarks are given in section 6.

2 Poincaré Group and Representations

The three-dimensional Poincaré group \( \pi \) is defined as the set of real transformations

\[
(a, \Lambda) : \ x'^\mu = \Lambda^\mu_\nu x'^\nu + a^\mu
\]

in the three-dimensional Minkowski space which leave \( g_{\mu\nu}(x - y)^\mu(x - y)^\nu \) invariant, where the metric \( g_{\mu\nu} = \text{diag}(1, -1, -1) \). The group \( \pi \) is actually a semidirect product of the translation and Lorentz groups, \( N \) and \( L \), as applying two successive transformations on \( x \), one has \((a', \Lambda')(a, \Lambda) = (a' + \Lambda'a, \Lambda'\Lambda)\).

The Hermitean generators of (infinitesimal) Lorentz transformations \( L_{\mu\nu} \), realized as \( i(x_\mu \partial_\nu - x_\nu \partial_\mu) \) with \( \partial_\mu = \partial/\partial x_\mu = (\partial_0, \nabla) \), obey the Lie algebra of \( SO(2,1) \)

\[
[L_{\mu\nu}, L_{\rho\tau}] = i(g_{\mu\tau}L_{\nu\rho} - g_{\nu\tau}L_{\mu\rho} + g_{\mu\rho}L_{\nu\tau} - g_{\nu\rho}L_{\mu\tau}).
\]

(2.2)

The most general representation of the generators of \( SO(2,1) \) satisfying (2.2) is

\[
M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu},
\]

(2.3)

where the Hermitean operators \( S_{\mu\nu} \), introduced for describing spin of particles and thus called spin matrix, satisfy the same Lie algebra as \( L_{\mu\nu} \) and commute with them. Formally, the discussion so far looks similar to that in the \( (d + 1) \) dimensions for \( d > 2 \). But there is an essential difference, i.e. in \( (2+1) \) dimensions, the little group \( M_{ij} \) \((i, j = 1, 2)\) of the Lorentz rotation is Abelian (\( SO(2) \)) and moreover the universal covering group of \( SO(2) \) can be imbedded in the universal covering group of \( SO(2,1) \), and thus its eigenvalue can be any real number \( \mathbb{R} \). Therefore spin of particles in \((2+1)\) dimensions can be arbitrary, while spin of particles in higher dimensions must be quantized.
Introducing a new operator

\[ J^\mu = \frac{1}{2} \epsilon^{\mu\sigma\lambda} M_{\sigma\lambda}, \]  

(2.4)

the Lorentz algebra takes a form

\[ [J^\mu, J^\nu] = i\epsilon^{\mu\nu\lambda} J^\lambda. \]  

(2.5)

The Hermitean generators of translations \( P_\mu \), realized as \( i \partial_\mu \), satisfy

\[ [P_\mu, P_\nu] = 0, \]  

(2.6)

\[ [J^\mu, P_\nu] = i\epsilon^{\mu\nu\lambda} P_\lambda. \]  

(2.7)

(2.5)-(2.7) form the three-dimensional Poincaré algebra.

As in any field theory, the matter fields that describe the elementary particles are normally required to carry irreducible unitary representations (UIR’s) of Poincaré group, so that the particles are indivisible and the probability amplitudes calculated from the theories are invariant. A complete set of UIR’s of the three-dimensional Poincaré group is given in [8]. Alternatively, one may begin with covariant fields, fields transforming as

\[(a, \Lambda): \Phi(p) \rightarrow e^{ip \cdot a} D(\Lambda) \Phi(\Lambda^{-1} p), \]  

(2.8)

in the momentum space where \( P_\mu \) is replaced by the eigenvalue \( p_\mu \). The exponential \( e^{ip \cdot a} \) is the UIR of the translation group \( N \), and \( D(\Lambda) \) is an appropriate representation of the Lorentz group. Since a covariant field is in general not irreducible, one subjects the covariant field to constraint(s) to remove unphysical degrees of freedom. This latter approach that we shall follow is convenient for constructing interactions in local terms.

In the (2+1) dimensional Poincaré algebra, the invariants, or the Casimir operators that commute with all the generators, are \( P^2 \) and the Pauli-Lubanski scalar \( P \cdot J \).

The spin part of \( J_\mu \), denoted as \( j_\mu \), defines the value of spin \( j \) via \( j_\mu j_\mu = j(j + 1) \). \( P^2 \) gives the mass-shell condition,

\[ [P^2 - (j/k)^2 M^2] \Phi = 0, \]  

(2.9)

where \( k (0 < k \leq j) \) labels the non-zero eigenvalues of \( j_\mu \), for any given spin-\( j \). Apparently the mass spectrum now is \( \pm jM/k \) for all \( k \). And the Pauli-Lubanski scalar provides an additional constraint

\[ (P \cdot J + jM) \Phi = 0, \]  

(2.10)
which specifies the helicity, as the Pauli-Lubanski scalar \( P \cdot J \) concerns no orbit angular momentum \( (P_\mu \epsilon^{\mu\nu\lambda} L_{\nu\lambda} = 0) \).

By these constraints, one can construct linear representations of the Poincaré group. For an integer or odd-half-integer \( j \), \( 2j + 1 \) components are usually required for the wave-function \( \Phi \). However, as the Pauli-Lubanski condition consists of \( 2j + 1 \) homogeneous equations for the \( 2j + 1 \) components, and as a \( (2j + 1) \times (2j + 1) \) Pauli-Lubanski matrix \( P \cdot J + jM \) is of rank \( 2j \) under a mass-shell condition, one and only one of these components is independent. For an arbitrary \( j \), infinite number components are needed for a linear representations, while constraints kill all but one component \([10]\). Moreover, for a given \( j \) (> 1 if \( j \) is an integer or odd-half-integer), there are more than one such wave-function characterized by \( k \). All these span a complete basis of the spin-\( j \) irreducible representation.

For the later use, in the rest of this section, we consider \( j = 1/2, 1, \) and \( 3/2 \) as examples. For spin-1/2 particles, it is convenient to choose

\[
j^\mu = \frac{1}{2} \gamma^\mu ,
\]

where \( 2 \times 2 \) matrices \( \gamma^\mu = (\sigma^3, -i \sigma^1, i \sigma^2) \), with \( \sigma^1, \sigma^2 \) and \( \sigma^3 \) the Pauli matrices. \( j^\mu \) obey the Lorentz algebra \((2.3)\). The two-component Dirac field \( \psi \) transforms as

\[
\Lambda : \psi(p) \rightarrow e^{i\omega \cdot j} \psi(\Lambda^{-1} p) .
\]

The Pauli-Lubanski condition \((2.10)\) then is precisely the Dirac equation:

\[
i \gamma \cdot \partial \psi + M \psi = 0 .
\]

Above the Dirac indefinite scalar product is used so that \((2.11)\) is self-conjugated and the representation of the Poincaré group is unitary. The (positive energy) solution of \((2.13)\) in the momentum space is

\[
\psi(p) = \frac{1}{\sqrt{2M(E + M)}} \left( \frac{p^\mu - ip^x}{E + M} \right) \phi(p) ,
\]

where \( E = \sqrt{p^2 + M^2} \) and the normalization is so chosen that, in the rest-frame \( \hat{p}^\mu = (M, 0, 0) \), \( \psi^\dagger \psi = \phi^* \phi \), with \( \phi(p) \) a scalar function. \((2.14)\) provides the spin-1/2 representation. Obviously, there exist \([10]\) a Lorentzian transformation that boosts \( \hat{p}^\mu \) from its rest-frame \( \hat{p}^\mu \) and an associated transformation that boosts the solution \((2.14)\) from \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \phi(\hat{p}) \).
For spin-1 particles, the covariant vector field $B_\mu$ transforms as

$$\Lambda : B_\mu(p) \to \Lambda_\nu B_\nu (\Lambda^{-1} p) .$$ (2.15)

Accordingly, the generators $j^\mu$ can be chosen as

$$(j^\mu)_{\sigma\lambda} = -i\epsilon^\mu_{\sigma\lambda} ,$$ (2.16)

which satisfy the Lorentz algebra (2.5). Acting on the three-vector $B^\mu$, the Pauli-Lubanski condition (2.10) with $j = 1$ is

$$- \epsilon^{\mu\nu\lambda} \partial_\nu B_\lambda + MB^\mu = 0 .$$ (2.17)

It is not difficult to check that the on-shell solution of (2.17) is

$$B^\mu(p) = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix} + \frac{p^x + ip^y}{M(E + M)} \begin{pmatrix} E + M \\ p^x \\ p^y \end{pmatrix} \right] \phi(p) .$$ (2.18)

(2.17) is actually the equation of motion of a field theory described by the Lagrangian

$$\frac{1}{2} B_\mu [-\epsilon^{\mu\nu\lambda} \partial_\nu + Mg^{\mu\lambda}] B_\lambda .$$ (2.19)

On the other hand, in the present work we treat the vector $B_\mu$ as a fundamental charged spin-1 matter field, and couple it minimally to a Chern-Simons gauge potential $a_\mu$ and an external gauge fields $C_\mu$. Namely, we consider the local gauge invariant Lagrangian

$$\mathcal{L}_1 = \frac{1}{2} B_\mu [-\epsilon^{\mu\nu\lambda} (\partial_\nu + ia_\nu + iC_\nu) + Mg^{\mu\lambda}] B_\lambda + \text{CS term of } a_\mu .$$ (2.20)

The corresponding $U(1)$ gauge transformations are

$$a_\mu \to a_\mu - \partial_\mu \epsilon_1 , \quad C_\mu \to C_\mu - \partial_\mu \epsilon_2 , \quad B_\mu \to e^{i(\epsilon_1 + \epsilon_2)} B_\mu .$$ (2.21)

The dynamics of the gauge field $a_\mu$ is governed by the Chern-Simons term. Due to the topological nature of Chern-Simons action, $a_\mu$ field carries no independent degree of freedom. As we shall explicitly see in the following sections, the Chern-Simons interaction changes the spin of matter field [11].
For $j = 3/2$, we construct the spin matrix satisfying the Lorentz algebra (2.5) as the $4 \times 4$ matrices
\[
\frac{1}{2} \begin{pmatrix} 3 & 1 \\ -1 & -3 \end{pmatrix}, \quad -\frac{i}{2} \begin{pmatrix} \sqrt{3} & 2 \\ 2 & \sqrt{3} \end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix} -\sqrt{3} & 2 \\ 2 & -\sqrt{3} \end{pmatrix},
\]
for $j_0$, $j_1$ and $j_2$, respectively. It is readily to check that $j^2 = (3/2)(3/2+1)$. The four-component spin-$3/2$ field $\Psi$ is subject to the mass-shell and Pauli-Lubanski conditions. The latter condition is actually the equation of motion for the field $\Psi$:
\[
ij \cdot \partial \Psi + \frac{3}{2} M \Psi = 0.
\]
(2.23)

The on-shell positive energy solutions of (2.23) in the momentum space take a form
\[
\Psi_k(p) = \frac{\sqrt{3}}{2M(E_k + jM/k)} \begin{pmatrix} (E_k - jM/k)/\sqrt{3} \\ p^\mu + ip^x \\ p^\mu - ip^x \\ (E_k + jM/k)/\sqrt{3} \end{pmatrix} \phi(p),
\]
(2.24)
where $E_k = \sqrt{p^2 + (jM/k)^2}$, $j = 3/2$, $k = 1/2$, or $3/2$. These two states span a complete basis of spin-$3/2$ irreducible representation.

In the above, we have assumed the mass parameter $M$ positive and worked out the representations for spin $j = 1/2, 1, \text{ and } 3/2$, respectively. One may set $M$ as $-|M|$ and keep all the others same, then the representations constructed will be for $j = -1/2, -1, \text{ and } -3/2$. In other words, with the sign of $j$ fixed, the sign of mass differs the spin “up” and “down” [12], corresponding to the two helicity directions. For concreteness, we assume $j$ positive thereafter.

3 Path Integral of Spinning Particles

We turn to consider how the Chern-Simons interaction affects the spin of matter fields. The sort of Chern-Simons matter models of interest can be uniformly described by the Lagrangian
\[
\mathcal{L}(j, \alpha) = \bar{\Phi} [j^{-1} j_\mu (\partial_\mu + ia_\mu + iC_\mu) + M] \Phi - \frac{i}{8\pi \alpha} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda.
\]
(3.1)
Besides the mass parameter $M$, the system is characterized by two real parameters, $j$, which we call the spin of the field $\Phi$, the character of irreducible representations of the Poincaré group and $\alpha$ the so-called statistical parameter. The spin $j$ is defined by $j_\mu j^\mu = j(j+1)$ with the spin matrix $j_\mu$ obeying the Lorentz algebra \((2.5)\). The statistical parameter $\alpha$ represents a change of the statistics of the matter field brought by the Chern-Simons interaction. It represents also a change of the spin of the matter field, as we come to see now. Viewing \((3.1)\) as an interacting field theory, on the other hand, $\alpha$ reflects the strength of the interaction among the matter fields and Chern-Simons gauge potential. To manifest this, one rescales the Chern-Simons field $a_\mu \rightarrow \sqrt{4\pi \alpha}a_\mu$ so that the Chern-Simons term is normalized to $1/2$, and then the gauge coupling constant is $\sqrt{4\pi \alpha}$. To be more general, we have introduced an external gauge field $C_\mu$ also minimally coupled to the matter fields.

Let’s calculate the partition function of the Chern-Simons matter model \((3.1)\). For this purpose, we make a Wick rotation and come to the Euclidean spacetime. We have the partition function

$$Z(j, \alpha) = \int DA_\mu D\bar{\Phi} D\Phi e^{-\int d^3x \mathcal{L}(j, \alpha)}, \quad (3.2)$$

keeping in mind that the partition function is a functional of the external gauge field $C_\mu$, and pending a gauge fixing concerning the gauge freedom with the Chern-Simons field until the integral over $a_\mu$ field is to be carried out. With the minimal coupling, the path integrals over the matter fields and over the Chern-Simons gauge potential both are of Gaussian type. We choose to perform the former first. By a standard method, we obtain

$$Z(j, \alpha) = \int DA_\mu \sum_{n=0}^{\infty} \frac{1}{n!} (-W_j)^n \exp\left[\frac{i}{8\pi \alpha} \int d^3x \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda\right], \quad (3.3)$$

where the free energy of the spin-$j$ matter field $W_j$ is

$$W_j = \text{Tr} \log[-i(\partial_\mu + ia_\mu + iC_\mu)j_\mu j^{-1} + M]. \quad (3.4)$$

To render the remaining integral over $a_\mu$ in \((3.3)\) an explicit Gaussian, we invoke the path integral representation of spinning particles \([13]\) to formulate $W_j$. To start, we introduce the SU(2) spin coherent states in the spin-$j$ representation \([14]\), which are parametrized by a unit vector $e$ in three dimensions

$$|e > \equiv \exp[-i\theta(e_3 \times e) \cdot j|e_3 \times e |^{-1}]|0 >, \quad (3.5)$$
where \( e_3 = (0, 0, 1) \), \( \theta \) is the angle between \( e_3 \) and \( e \), and \( |0> \) is the highest weight state in the spin-\( j \) representation. Note, this definition involves the spin matrix \( j_\mu \) with the magnitude \( j \). The spin coherent state has the following three properties:

1. **Unity partition**
   \[
   \int d\mathbf{e} |\mathbf{e}><\mathbf{e}| = 1 , \tag{3.6}
   \]
   where \( d\mathbf{e} \) is the rotationally invariant measure on the two dimensional unit sphere.

2. **Area law**
   \[
   <|\mathbf{e} + \delta\mathbf{e}|\mathbf{e}> = \exp\{ijA[\mathbf{e}, \mathbf{e} + \delta\mathbf{e}, e_3]\} + O(\delta\mathbf{e}^2) , \tag{3.7}
   \]
   where \( A[\mathbf{e}, \mathbf{e} + \delta\mathbf{e}, e_3] \) is the area of a spherical triangle with the vertices \( \mathbf{e} \), \( \mathbf{e} + \delta\mathbf{e} \) and \( e_3 \).

3. **Expectation value**
   \[
   <\mathbf{e}|j|\mathbf{e}> = j |\mathbf{e}| . \tag{3.8}
   \]
   With these properties, the trace \( \text{Tr} \) can be represented in terms of path integral over random paths that are closed on a two dimensional sphere, such as

\[
\text{Tr}[T \exp\left( \int_0^L dt \mathbf{j} \cdot \mathbf{S}(t) \right)] = \lim_{N \to \infty} \text{Tr} \prod_{i=1}^N (1 + \Delta t \mathbf{j} \cdot \mathbf{S}(t_i))
\]
\[
= \int \mathcal{D}\mathbf{e} \exp \left( ij \Xi[\mathbf{e}] + j \int_0^L d\mathbf{e} \cdot \mathbf{S} \right) , \tag{3.9}
\]
where \( \mathbf{S}(t) \) is a c-number source, and
\[
\Xi[\mathbf{e}] = \int_D dt ds \mathbf{e} \cdot [\partial_s \mathbf{e} \times \partial_t \mathbf{e}] \tag{3.10}
\]
is defined as spin factor. \( \mathbf{e}(t, s) \) is an extention of \( \mathbf{e}(t) \) satisfying \( \mathbf{e}(t, 0) = \text{const} \) and \( \mathbf{e}(t, 1) = \mathbf{e}(t) \). Geometrically, the spin factor is the area enclosed by the closed path \( \mathbf{e}(t) (0 \leq t \leq L) \) on the two dimensional unit sphere.

Introducing Schwinger parameter \( L \), we write the free energy of spin-\( j \) particles \( W_j \) as

\[
W_j = \text{Tr} \log[(\partial_\mu + ia_\mu + iC_\mu)j_\mu j^{-1} + M]
\]
\[
= \int_\epsilon^{\infty} dL \frac{dL}{L} \text{Tr} e^{-L(i(p+a+C)j^{-1}+M)} , \tag{3.11}
\]
where \( \epsilon \) is an ultraviolet cutoff. Using the \( SU(2) \) coherent states, we have

\[
W_j = \int_\epsilon^{\infty} dL \frac{dL}{L} \int \prod_{i=1}^N dx_i d\mathbf{e}_i <\mathbf{x}_i| <\mathbf{e}_i|e^{-\Delta L(i(p+a+C)j^{-1}+M)}|\mathbf{e}_{i-1}> |\mathbf{x}_{i-1}> . \tag{3.12}
\]
The path integrals over \( x_i \) and \( e_i \) are subject to periodic boundary conditions \( x_N = x_0 \) and \( e_N = e_0 \). The infinitesimal kernel in terms of the coherent states and the momentum eigenstates is

\[
\langle x_i | e_i | e_{i-1} \rangle = \int \frac{d^3p_i}{(2\pi)^3} e^{-iL(p + a + C) \cdot e_{i-1}} e_{i-1} \exp[-iL (i(p_i + a_i) \cdot e_i + M + i\mathbf{p} \cdot (x_i - x_{i-1})) + O(\Delta L^2)].
\]

(3.13)

Therefore, the free energy takes a form of path integrals

\[
W_j = \int_\tau^\infty \frac{dL}{L} \int D\mathbf{x} D\mathbf{p} D\mathbf{e} \exp \left[- \int_0^1 dt \left\{ i(p + a + C) \cdot e + Mh - i\mathbf{p} \cdot \dot{x} + i\mathbf{e} \cdot \dot{x} \right\} + ij\Xi[\mathbf{e}] \right].
\]

(3.14)

where the parameter \( t \) is rescaled as \( t \to Lt \), and \( \dot{x} = dx(t)/dt \). Next, we try to integrate out \( \mathbf{p} \), \( \mathbf{e} \) and \( L \). Let’s map the integral over \( L \) to the integral over an einbein \( h \), which by definition is an one-form in one dimensional space. Doing so, we take the advantage of the integral thus being explicitly diffeomorphism invariant in the one dimensional space. In the Fadeev-Popov procedure, the integral over the Schwinger parameter \( L \) is regarded as a gauge fixed path integral with respect to the diffeomorphism transformation, with \( L \) the zero mode of the einbein. Namely, we have the mapping

\[
\lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{dL}{L} \to \int \frac{Dh}{V_{Diff}}.
\]

(3.15)

With this replacement, the path integral reads

\[
W_j = \int \frac{Dh}{V_{Diff}} D\mathbf{x} D\mathbf{p} D\mathbf{e} \exp \left[- \int_0^1 \left\{ i(p + a + C) \cdot e + Mh - i\mathbf{p} \cdot \dot{x} + i\mathbf{e} \cdot \dot{x} \right\} + ij\Xi[\mathbf{e}] \right].
\]

(3.16)

Here, \( p \) is readily integrated out

\[
W_j = \int \frac{Dh}{V_{Diff}} D\mathbf{e} D\mathbf{x} \delta(\dot{x} - e) \exp \left[- \int_0^1 dt \left\{ i(a + C) \cdot e + Mh \right\} + ij\Xi[\mathbf{e}] \right].
\]

(3.17)

In this expression we see \( h = \sqrt{\dot{x}^2} \) and

\[
e(t) = \dot{x}/\sqrt{\dot{x}^2},
\]

(3.18)

which is the unit tangent vector of the path parametrized by the real variable \( t \). Carrying out the integrals over \( e \) and \( h \), finally we obtain

\[
W_j = \int D\mathbf{x} \exp \left[- \int_0^1 dt \left\{ M\sqrt{\dot{x}^2} + i\mathbf{a} \cdot \dot{x} + i\mathbf{C} \cdot \dot{x} \right\} + ij\Xi[\dot{x}/\sqrt{\dot{x}^2}] \right].
\]

(3.19)
It is worth of notice that the free energies of different spin-$j$ matter fields differ from each other just by a coefficient, the value of spin $j$, to the spin factor $\Xi$.

4 Spin Transmutation via Chern-Simons Term

With $W_j$ in the form expressed in (3.19), the integral over $a_\mu$ in the partition function (3.3) is readily to carry out, it gives

$$Z(j, \alpha) = \sum_{n=1}^{\infty} \prod_{i=1}^{n} Dx_i \exp \left[ - \sum_{i=1}^{n} \left\{ \int dt (M\sqrt{|\dot{x}_i^2|} + iC \cdot \dot{x}_i) - ij\Xi[\dot{x}_i] - i\frac{\alpha}{2} \Theta_{ii} + i\alpha \sum_{i<k} \Theta_{ik} \right\} \right],$$

where $x_i(t)$ is the position vector of the $i$th path, and

$$\Theta_{ik} = \frac{1}{\alpha} \int_0^1 dt \int_0^1 ds \frac{dx_i^\mu}{dt} \frac{dx_k^\nu}{ds} <a_\mu(x_i) a_\nu(x_k)>.$$

A convenient gauge choice is the Landau gauge, under which,

$$<a_\mu(x) a_\nu(y)> = \int Da_\lambda a_\mu(x) a_\nu(y) \exp \left[ \frac{i}{8\pi\alpha} \int dx e^{\sigma\tau} a_\sigma \partial_\tau a_\eta \right]$$

$$= 8\pi\alpha \epsilon_{\mu\nu\lambda} \frac{x^\lambda - y^\lambda}{|x - y|^3}.$$

Using notation of the unit vector

$$e(s, t) = \frac{x_i(s) - x_k(t)}{|x_i(s) - x_k(t)|} \in S^2,$$

$\Theta_{ik}$ takes a compact form

$$\Theta_{ik} = \int_0^1 ds \int_0^1 dt e \cdot [\partial_s e \times \partial_t e].$$

$\Theta_{ik}$ has been extensively discussed in ref. [13] and [15], here we present its main features with some comments. First, if $i \neq k$, i.e. the closed paths denoted by $i$ and $k$ do not coincide, $\Theta_{ik}$, called relative energy, is the Gauss linking number and thus is quantized

$$\Theta_{ik} \in 4\pi Z.$$

From (4.1) and (4.6), we see that to any system that has an integer or odd-half-integer statistical parameter, $\alpha \in Z/2$, the relative energy is irrelevant. In other words, the
contribution to the partition function from the relative energy is associated with the ‘small’ portion of statistics $\alpha$ modular integer and odd-half-integer.

Secondly, when the paths $i$ and $k$ do coincide, $\Theta_{ii}$, called self-energy, is not quantized. Instead it is related to the spin factor $\Xi$ (see (3.10)) via

$$\Theta_{ii} - 2\Xi = 4\pi \pmod{8\pi} .$$

(4.7)

It seems when $i = j$ the definition (4.4) of $e(t, s)$ at the point $s = t$ is ambiguous. To fix this problem, one could use the “ribbon-splitting technique” [15], in which one splits the two coinciding paths a little bit, and takes a wisely chosen limit to merge them again later. Here we prefer a resolution to well define the self-energy by introducing a limitation procedure without this splitting [13]. Let’s change the variables of $e(s, t)$ from $(s, t)$ to $(u, t)$ with $u = s - t \in [0, 1]$, and consider the connection conditions at $u = 0$ and $u = 1$. At $u = 0$, one naturally defines a limit

$$e(u = 0, t) = \lim_{\epsilon \to +0} e(\epsilon, t) = e(t) \equiv \frac{\dot{x}(t)}{|x|} ;$$

(4.8)

then at $u = 1$ it must be

$$e(u = 1, t) = \lim_{\epsilon \to +0} e(1 - \epsilon, t) = -e(t) .$$

(4.9)

Therefore, $e(u, t)$ satisfies the anti-periodic boundary condition for the variable $u$

$$e(0, t) = -e(1, t) = e(t) ,$$

(4.10)

comparing the periodic boundary condition for the variable $t$

$$e(u, t + 1) = e(u, t) .$$

(4.11)

In this way, $e$ and thus the self-energy $\Theta_{ii}$ are well-defined on the whole (smooth) path. Moreover, the factor 2 on the left hand side of (4.7) reflects the fact that the self-energy $\Theta_{ii}$ has two boundaries at $u = 0$ and $u = 1$ while the spin factor $\Xi$ has only one. Due to the diffeomorphism invariance of $\Theta_{ii} - 2\Xi$ defined on a closed path, one is free to deform the path continuously onto a plane, where (4.7) is readily to obtain. Finally, we should mention that in this approach, one assumes that, when the random paths intersect, a suitable point splitting can always be used to regularize the intersecting point(s).
The quantization of the relative energy $\Theta_{ik}$, (4.6), and the simple relation of the self-energy $\Theta_{ii}$ with the spin factor, (4.7), have very interesting implication that the Chern-Simon coupling therefore can be related to the spin of the matter field coupled. This can be seen clearly from (4.1). Decreasing $\alpha$ by an integer or odd-half-integer and increasing $j$ at the same time by the same amount (or vise versa), one doesn’t change the partition function at all. Namely, we have for any $n \in \mathbb{Z}$

$$Z(j, \alpha) = Z(j + n/2, \alpha - n/2). \quad (4.12)$$

(4.12) is our main result here, which explicitly exhibits the spin transmutation. As a special case, when $\alpha$ is an integer or odd-half-integer (so that the relative energy is irrelevant), one may transmute all the Chern-Simons coupling into the spin of matter field by using (4.7), and obtain a free theory of anyon with a spin $j + \alpha$.

5 Equivalence of Chern-Simons Matter Models

Treating anyons as fundamental fields faces great difficulties [17]. More practical approach in dealing with anyons is to use integer or odd-half-integer spin-$j$ fields as the fundamental fields and couple them to Chern-Simons fields. In this way, traditional and powerful methods to solve interacting field theories can be used directly. In particular, to calculate quantum fluctuations, perturbation expansion over the Chern-Simons coupling $\sqrt{4\pi\alpha}$ is normally used. In many realistic systems, unfortunately, the Chern-Simons interactions are very strong, i.e. $\alpha$ has to be so large [18] that an expansion in it is not well convergent. With the spin transmutation mechanism seen here, one can now trade a major part of Chern-Simons coupling for higher spin. Then, one may have an effective theory with a remaining small Chern-Simons coupling.

For example, it is straightforward now to verify the following three models being equivalent one to the others, for any given $\alpha$ [5]. The first model has a spin-1/2 two-component Dirac field $\psi$ as the fundamental field,

$$\mathcal{L}(\frac{1}{2}, \alpha) = \bar{\psi} [\gamma_\mu (\partial_\mu + ia_\mu + iC_\mu) + M] \psi - \frac{i}{8\pi\alpha} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda, \quad (5.1)$$

where the Dirac matrices in three (Euclidean) dimensions $\gamma_\mu = \sigma_\mu$ with $\sigma_\mu$ ($\mu = 1, 2, 3$) Pauli matrices, and $C_\mu$ again an external gauge field. And the second involves the
spin-1 complex vector field $B_{\mu}$,

$$\mathcal{L}(1, \alpha - \frac{1}{2}) = \frac{1}{2} B_{\mu}^* [ - i e^{\mu \lambda} (\partial_{\nu} + i a_{\nu} + i C_{\nu}) + M \delta^{\mu \lambda} ] B_{\lambda} - \frac{i}{8\pi (\alpha - 1/2)} \epsilon_{\mu \nu \lambda} a_{\mu} \partial_{\nu} a_{\lambda} .$$

(5.2)

Finally, the third contains a spin-3/2 four-component fermion field $\Psi$

$$\mathcal{L}\left(\frac{3}{2}, \alpha - 1\right) = \bar{\Psi} \left[ \frac{2}{3} L_{\mu} (\partial_{\mu} + i a_{\mu} + i C_{\mu}) + M \right] \Psi - \frac{i}{8\pi (\alpha - 1)} \epsilon_{\mu \nu \lambda} a_{\mu} \partial_{\nu} a_{\lambda} ,$$

(5.3)

$L_{\mu}$ are $4 \times 4$ matrices satisfying the Lorentz algebra and $L_{\mu} L_{\mu} = (3/2)(3/2 + 1)$. Obviously (5.1), (5.2), and (5.3) can be good perturbation theories only around $\alpha = 0, 1/2$, and 1, respectively. This claim of equivalence is based on (4.12) which takes a form in the present case

$$Z\left(\frac{1}{2}, \alpha\right) = Z(1, \alpha - \frac{1}{2}) = Z\left(\frac{3}{2}, \alpha - 1\right) = \cdots .$$

(5.4)

“…” here means obvious extension of the equivalence to ($\infty$ number) models with “higher” matter spins and “weaker” Chern-Simons interactions. Moreover, the equivalence can be generalized to some matter field self-interactions. Let’s take the four-body interaction as an example. Assume a local four-fermion interaction term $(\bar{\psi} \psi)^2$ added to the Chern-Simons Dirac fermion model (5.1), the Lagrangian now reads

$$\mathcal{L}\left(\frac{1}{2}, \alpha, g\right) = \bar{\psi} \left[ a_{\mu} (\partial_{\mu} + i a_{\mu} + i C_{\mu}) + M \right] \psi + \frac{g}{4} (\bar{\psi} \psi)^2 - \frac{i}{8\pi \alpha} \epsilon_{\mu \nu \lambda} a_{\mu} \partial_{\nu} a_{\lambda} ,$$

(5.5)

where $g$ is the coupling constant for the four-body interactions.

We introduce an auxiliary field $b(x) = i(g/2) \bar{\psi} \psi$ and turn the four-body interaction into a Yukawa interaction:

$$\mathcal{L}_{b}\left(\frac{1}{2}, \alpha, g\right) = \bar{\psi} \left[ a_{\mu} (\partial_{\mu} + i a_{\mu} + i C_{\mu}) + M + i b \right] \psi - \frac{i}{8\pi \alpha} \epsilon_{\mu \nu \lambda} a_{\mu} \partial_{\nu} a_{\lambda} + \frac{1}{g} b^2 .$$

(5.6)

The four-body interaction in (5.5) is recovered by performing the Gaussian functional integral over $b$ in the partition function

$$Z\left(\frac{1}{2}, \alpha, g\right) = \int \mathcal{D} b \mathcal{D} a_{\mu} \mathcal{D} \bar{\psi} \mathcal{D} \psi e^{-\int \mathcal{L}_{b}(1/2, \alpha, g)} .$$

(5.7)

Integrating out the Dirac fields, we have the free energy for spin-1/2 field

$$W_{1/2} = \int \mathcal{D} x \exp \left[ - \int_{0}^{1} dt \left\{ (M + i b(x)) \sqrt{\dot{x}^2} + i a \cdot \dot{x} + i C \cdot \dot{x} \right\} + \frac{i}{2} \left[ \frac{\dot{x}}{|\dot{x}|} \right]^2 \right] .$$

(5.8)
Compared to (3.19) (taken \(j = 1/2\)), which involves no self-interaction, in (5.8), the change is only to replace the mass term \(M\) with \(M + b(x)\). Then, carrying out the integral over \(a_\mu\) as done in the previous section, we obtain

\[
Z(\frac{1}{2}, \alpha, g) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathcal{D}b \exp\left[-\int d^3xb^2/g\right] \int \prod_{i=1}^{n} \mathcal{D}x_i \exp \left[-\sum_{i=1}^{n} \left\{ \int dt \{ (M + b(x))\sqrt{\dot{x}_i^2 + iC \cdot \dot{x}_i} \} - \frac{i}{2} \Xi[\dot{x}_i/|\dot{x}_i|] - \frac{i}{2}\Theta_{ii} + i\alpha \sum_{i<j} \Theta_{ik} \right\} \right].
\]

(5.9)

Now, we use (4.6) and (4.7) to shift \(\alpha\) by an integer or odd-half-integer. For any \(n \in \mathbb{Z}\), we obtain

\[
Z(\frac{1}{2}, \alpha, g) = Z(\frac{1+n/2}{2}, \alpha - n/2, g).
\]

(5.10)

Taking \(n = 1\) as an example, the right hand side above, \(Z(1, \alpha - 1/2, g)\), is the partition function of the vector Chern-Simons model with four-body interaction

\[
\mathcal{L}(1, \alpha - 1/2, g) = -i\epsilon_{\mu\nu\lambda} B^*_\mu (\partial_\nu + ia_\nu + iC_\nu + M) B_\lambda + \frac{g}{4}(B^*_\mu B_\mu)^2 - \frac{i}{8\pi(\alpha - 1/2)} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda.
\]

(5.11)

Now, we have seen the equivalence between the models (5.5) and (5.11).

It is straightforward to play this game over again for more versions of the Chern-Simons matter model for any given \(\alpha\).

6 Concluding Remarks

In this paper, we have shown explicitly how the spin of particles transmutes when they interact to the Chern-Simons gauge field. Doing so, we have uncovered the intrinsic relation, the equivalence, of a large class Chern-Simons matter models. An immediate consequence of the equivalence is that one is free to select one of the equivalent anyon models (though they may look like very different) that is most suitable for exploring the features of the anyon system one is interested in.

As the higher-spin matter fields in general carry more degrees of freedom, there seems a problem with a counting of degrees of freedom in equivalent models. The answer is that the Chern-Simons interaction affects the number of degrees of freedom of the whole anyon system. As is seen in section 4, an integer or odd-half-integer part
of the Chern-Simons coupling $\alpha$ can be reabsorbed by changing the spin of matter field. This change of spin is accompanied by a change of numbers of degrees of freedom carried by the matter field. What happens here is a transfer but not creation of degrees of freedom. Since equivalent models can be derived from each other, all these have the same amount degrees of freedom and describe the same many-body system with arbitrary $\alpha$.

Besides the statistics and spin, one more interesting character of anyons is the scaling dimension, which governs the asymptotical behavior of anyon systems. Due to the Fermi-Bose transmutation, one is led to expect the scaling dimensions of anyons be dependent of the statistics parameter, i.e. $d = d(\alpha)$. Take the Chern-Simons Dirac field model (5.1) as an example. When $\alpha$ is an integer or odd-half-integer the model describes free fermions or bosons, respectively. There must be for instance $d = d(\alpha = 0) = 1$ at the Dirac fermion point and $d = d(\alpha = -1/2) = 1/2$ at the scalar point, since for free fields the scaling dimensions are just the engineering dimensions. However, when $\alpha$ varies away from integer and odd-half-integer, the scaling dimension of anyon should be a sum of the engineering dimension of fundamental field and the $\alpha$ dependent correction from Chern-Simons interaction. This is again a quantum-level phenomenon – the asymptotical behavior of a system is usually changed by quantum fluctuations. In renormalizable field theories, a correction to the scaling dimension of field, called an anomalous dimension, is computed normally by the renormalization group method.

The calculations to two-loop \cite{20} show that the anomalous dimensions of the matter fields are a continuous decreasing function of $\alpha^2$ \cite{21}. The perturbation results are reliable near $\alpha = 0$, but less and less reliable as $|\alpha|$ goes larger and larger. As an example in applying the equivalence, the authors of \cite{3} calculated the anomalous dimension of the matter field in a system where each fermion particle carries about one flux tube, i.e. the model (5.1) with $\alpha \sim 1/2$, which reflects a strong coupling. The calculation then is done by mapping (5.1) to the Chern-Simons field coupled to the vector matter model (5.2). The anomalous dimension of the vector matter field is calculated as a decreasing function of $(\alpha - 1/2)^2$, which is well convergent near $\alpha = 1/2$ \cite{22}.

There has been debate on whether the only effect of Chern-Simons gauge field is to endow the particle with arbitrary spin or whether residual interactions are present \cite{23}. We are now able to shed some lights on this issue. Looking at the partition
function (4.1) and the conditions (4.6) and (4.7), it is clear that only the integer or odd-half-integer part of the Chern-Simons coefficient can be absorbed into the spin of the fundamental field. In other words, our analysis here suggests that: only when $\alpha$ is an integer or odd-half-integer, the sole role the Chern-Simons field plays is to change the spin of particle by an amount $\alpha$; otherwise, a residual Chern-Simons interaction, minimally the part of $\alpha$ that is less than $1/2$, must be present. The complication is apparently due to the relative energy $\Theta_{ik}$, which is quantized in a way shown in (4.6).

We conclude this paper with remarks on two more issues. The relativistic anyon theory we have considered, (3.1), is rather general as it represents infinite number models characterized by two parameters $j$ and $\alpha$. However, this model doesn’t admit the scalar as a fundamental field, as $j \neq 0$. A local linear scalar theory is somehow special in the sense that its kinetic term must involve a second derivative. On the other hand, based on the equivalence discussed here, it is possible to construct a kind of spin-0 field theory by coupling spinning fields to Chern-Simons field with $j + \alpha = 0$ for exactly free boson particles, or $= \lambda \sim 0$ for boson particles near free. A sample model is that the (spin-1/2) Dirac field couples to Chern-Simons field with the coupling $\alpha \sim -1/2$. It is worth of notice that, since the fundamental Dirac field obeys the Pauli exclusion principle, so do the constructed spin-0 fields. The particles described by such a theory are known as hard core bosons.

One might be tempted to understand whether the class of models we discussed here can serve as a critical model that describes the critical behavior or phase transition of anyon systems. For $j = 1/2$, i.e. for the Chern-Simons Dirac fermion model, it is indeed the case. In ref. [24], the authors analyzed a lattice model of anyons in a periodic potential and an external magnetic field which exhibits a second order transition from a Mott insulator to a quantum Hall fluid. The continuum limit of this lattice model is shown exactly the Chern-Simons Dirac fermion model (5.1) (with $C_\mu$ missed), and the transition is characterized by the anyon statistics, $\alpha$. We would guess that all odd-half-integer $j$ models are in this nature.

On the other hand, however, for $j = 1$, i.e. for the Chern-Simons vector boson model (5.2), and likely for all the integer $j$ models, there appears a kind of discontinuity when the mass parameter $M$ is taken to be zero. In the model (5.2), the only degree of freedom is carried by the mass term of the vector field (so this degree of freedom is of longitudinal). When the mass parameter $M = 0$ is taken, the only degree of freedom is missing and the U(1) Chern-Simons (vector) matter theory becomes a
topological field theory. To see this, we set

\[ a_\mu = e A^1_\mu, \quad B_\mu = A^2_\mu + i A^3_\mu, \]

(6.1)

where \( e = \sqrt{4\pi(\alpha - 1/2)}, \) and substitute these into (5.2) with \( C_\mu = 0. \) Then we obtain

\[ \mathcal{L} = -\frac{i}{2} \epsilon_{\mu\nu\lambda} (A^a_\mu \partial_\nu A^a_\lambda + \frac{e}{3} \epsilon^{abc} A^a_\mu A^b_\nu A^c_\lambda), \]

(6.2)

upto a total derivative term. (6.2) is the well-known \( SU(2) \) Chern-Simons field theory, it is independent of a choice of the space-time metric.

The seeming singular behavior of the spin-1 model at \( M = 0 \) raises a question whether the spin-1 field theory exists as a continuum limit of a lattice model? This question can be answered only by studying the corresponding lattice model. A research of the quantum diffusion process with the spin factor as a free field theory on three dimensional lattice suggests there exists a continuum limit for the spin-1 model, as it does for the spin-1/2 model \[25\]. Meanwhile, the research of diffusion process also shows that \( M = 0 \) is indeed a singular point for spin-1 model. This explains why, as continuum field theories, the two models describe phase transitions of different nature. As \( M \to 0 \), in the spin-1 model it restores topological and non-Abelian gauge invariance \[24\], but in the spin-1/2 model it concerns only a global conformal symmetry. Obviously, the equivalence between the spin-1 and spin-1/2 models can not be simply extrapolated to the \( M = 0 \) case.

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Notice: after this work was finished, we noticed ref. \[27\] where a similar issue was discussed in a different way.

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