Supernova Constraints on a holographic dark energy model

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In this paper, we use the type Ia supernova data to constrain the model of holographic dark energy. For $d = 1$, the best fit result is $\Omega^0_m = 0.25$, the equation of the state of the holographic dark energy $w^0_\Lambda = -0.91$ and the transition between the decelerating expansion and the accelerating expansion happened when the cosmological red-shift was $z_T = 0.72$. If we set $d$ as a free parameter, the best fit results are $d = 0.21$, $\Omega^0_m = 0.46$, $w^0_\Lambda = -2.67$, which sounds like a phantom today, and the transition redshift is $z_T = 0.28$. 

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The type Ia supernova (SN Ia) observations \cite{1,2,3} provide evidence that the expansion of our universe at the present time appears to be accelerating which is due to dark energy with negative pressure. The kinematic interpretation of the relationship between SN Ia luminosity distance and red-shift is most consistent with two distinct epochs of expansion: a recent accelerated expansion and a previous decelerated expansion with a transition around $z_T \sim 0.4$ between them \cite{3}. This is a generic requirement of a mixed dark matter and dark energy universe. The cosmic background microwave (CMB) observations \cite{4} hint a spatially flat universe. However, the unusual small value of the cosmological constant, which is extremely smaller than the estimate from the effective local quantum field theory, is perhaps one of the biggest puzzles and deepest mysteries in modern physics. Alternatively, many dynamical dark energy models with cosmological constant like behavior were proposed in the literature \cite{5} and \cite{6}. For a review, see, for example \cite{6} and references therein.

't Hooft \cite{7} and Susskind \cite{8} showed that the effective local quantum field theories greatly over-count degrees of freedom because the entropy scales extensively for an effective quantum field theory in a box of size $L$ with UV cutoff $\Lambda$. In order to solve the problem, A. Cohen et al. \cite{9} proposed a relationship between UV and IR cut-offs corresponding to the assumption that the effective field theory describes all states of the system excluding those for which have already collapsed to a black hole. If the sum of the zero-point energies of all normal modes of the fields is $\rho_\Lambda$, we must have $L^3 \rho_\Lambda \leq L M_p^2$ or $\rho_\Lambda \leq M_p^2 L^{-2}$, this means that the maximum entropy is in the order of $S_{BH}^{3/4}$. The magnitude of the holographic energy proposed by Cohen et al. may be the same as that from cosmological observations. But Hsu recently pointed out that the equation of state is not correct for describing the accelerating expansion of our Universe in \cite{10}. In other words, the original holographic energy couldn’t give an accelerating universe. The idea was later generalized to make the gravitational constant varying with time in \cite{11}.

The origin of the Bekenstein-Hawking constraint on the entropy of a black hole is the existence of the event horizon, which serves as a natural boundary for all processes inside a black hole. However, there is no event horizon in a non-inflationary universe and we should replace it with the particle horizon, which has been discussed in \cite{12}. But there is an event horizon in the Universe with accelerating expansion and it is a natural choice that the event horizon acts as the boundary of the Universe. Very recently, Li suggested that we should use the proper future event horizon of our Universe to cut-off the large scale and bring about an accelerating expansion of our Universe in \cite{13}. On the other
hand, Banks and Fischler have pointed out that the the number of the e-foldings during inflation is bounded, which is due to the bound on the entropy, if we take the event horizon as the boundary of our Universe and the present acceleration of the Universe is due to an asymptotically de Sitter universe with small cosmological constant in [14].

In this paper, we use the new SN Ia data compiled by Riess et al. to constrain the holographic dark energy model proposed by Li. Firstly we take a short trip on the holographic energy model proposed in [13]. According to [13], the energy density of the holographic dark energy is

\[ \rho = \frac{3}{d^2} M_p^2 R_h^{-2}, \]

(1)

where we keep \( d \) as a free parameter (the author of [13] favored \( d = 1 \)) and \( R_h \) is the proper size of the future event horizon,

\[ R_h(t) = a(t) \int_0^t \frac{dt'}{a(t')} = a \int_a^\infty \frac{d a'}{H' a'^2}. \]

(2)

For a spatially flat, isotropic and homogeneous universe with an ordinary matter and dark energy, the Friedmann equation is

\[ \Omega_\Lambda + \Omega_m = 1, \quad \Omega_m = \frac{\rho_m}{\rho_{cr}} \quad \text{and} \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{cr}}, \]

(3)

where \( \rho_m \) (\( \rho_\Lambda \)) is the energy density of matter (dark energy) and the critical density \( \rho_{cr} = 3M_p^2 H^2 \). Using Friedmann equation (3) and \( \rho_m = \rho_m^0 a^{-3} = 3M_p^2 H_0^2 \Omega_m^0 a^{-3} \), where we set \( a_0 = 1 \) and \( a = (1 + z)^{-1} \), we have \( \Omega_m = 1 - \Omega_\Lambda = (H_0/H)^2 \Omega_m^0 a^{-3} \),

\[ \frac{1}{aH} = a^{1/2} \frac{1}{\sqrt{\Omega_m^0 H_0}} (1 - \Omega_\Lambda)^{1/2}, \]

(4)

and

\[ \rho_\Lambda = \Omega_\Lambda \rho_{cr} = \frac{\Omega_\Lambda}{1 - \Omega_\Lambda} \Omega_m \rho_{cr} = \frac{\Omega_\Lambda}{1 - \Omega_\Lambda} \rho_m = \frac{\Omega_\Lambda}{1 - \Omega_\Lambda} \rho_m^0 a^{-3}. \]

(5)

Combining equations (4) and (5), we find

\[ R_h(t) = a^{3/2} \frac{d}{\sqrt{\Omega_m^0 H_0}} \left( \frac{1 - \Omega_\Lambda}{\Omega_\Lambda} \right)^{1/2}. \]

(6)

Substituting equations (4) and (5) into (2),

\[ \int_x^\infty dx' e^{x'/2} (1 - \Omega_\Lambda)^{1/2} = d e^{x/2} \left( \frac{1 - \Omega_\Lambda}{\Omega_\Lambda} \right)^{1/2}, \]

(7)
where \( x = \ln a \). Taking derivative with respect to \( x \) in both sides of equation (7), we get

\[
\Omega'_\Lambda = \Omega_\Lambda(1 - \Omega_\Lambda)(1 + \frac{2}{d}\sqrt{\Omega_\Lambda}),
\]

(8)

where the prime denotes the derivative with respect to \( x \). We can get the analytic solution of equation (8) as,

\[
\ln \Omega_\Lambda - \frac{d}{2 + d} \ln(1 - \sqrt{\Omega_\Lambda}) + \frac{d}{2 - d} \ln(1 + \sqrt{\Omega_\Lambda}) - \frac{8}{4 - d^2} \ln(d + 2\sqrt{\Omega_\Lambda}) = -\ln(1 + z) + y_0,
\]

(9)

where \( y_0 \) can be determined by the value of \( \Omega^0_\Lambda \) through equation (9).

Because of the conservation of the energy-momentum tensor, the evolution of the energy density of dark energy is governed by

\[
\frac{d}{da}(a^3\rho_\Lambda) = -3a^2p_\Lambda.
\]

(10)

Thus we obtain

\[
p_\Lambda = -\frac{1}{3}\frac{d\rho_\Lambda}{d\ln a} - \rho_\Lambda.
\]

(11)

Using equations (5) and (8), after a lengthy but straightforward calculation, we find the pressure of dark energy can be expressed as

\[
p_\Lambda = -\frac{1}{3}(1 + \frac{2}{d}\sqrt{\Omega_\Lambda})\rho_\Lambda,
\]

(12)

and the equation of state of dark energy is

\[
w_\Lambda = \frac{p_\Lambda}{\rho_\Lambda} = -\frac{1}{3}(1 + \frac{2}{d}\sqrt{\Omega_\Lambda}),
\]

(13)

and

\[
\frac{dw_\Lambda}{dz} = \frac{1}{3d}\sqrt{\Omega_\Lambda}(1 - \Omega_\Lambda)(1 + \frac{2}{d}\sqrt{\Omega_\Lambda})\frac{1}{1 + z}.
\]

(14)

Since \( 0 \leq \Omega_\Lambda \leq 1 \), we find the equation of state of dark energy \(- (1 + 2/d)/3 \leq w_\Lambda \leq -1/3\) and the evolution of \( w_\Lambda \) is slow. If we use \( \Omega^0_\Lambda = 0.73 \) and \( d = 1 \), we obtain \( w^0_\Lambda = -0.90 \) and \( dw^0_\Lambda/dz = 0.21 \). In the future, our Universe will be dominated by dark energy with \( w_\Lambda = -(1 + 2/d)/3 \), this result is the same as Eq. (9) in [13]. The expansion of our Universe will be accelerating forever.

In the past, the expansion of our Universe experienced deceleration due to domination by radiation or matter. With the evolution, our Universe will be dominated by the
dark energy and the expansion of our Universe starts to be accelerating. This transition happened when
\[
\frac{\ddot{a}}{a} = -\frac{1}{6M_p^2}(\rho_\Lambda + 3p_\Lambda + \rho_m) = 0.
\] (15)

Using equations (3) and (12), we obtain
\[
\Omega_\Lambda^T + \frac{2}{d} \Omega_\Lambda^T \sqrt{\Omega_\Lambda^T} = 1.
\] (16)

If \(d = 1\), solving this equation, we find that the turning point is corresponding to \(\Omega_\Lambda^T = 0.4320\). For \(d \to \infty\), \(\Omega_\Lambda^T \approx 1\). If \(\Omega_\Lambda^0\) is finite, \(y_0\) in equation (3) will be finite. Using equation (3) again, we obtain the red-shift of turning point must be \(z_T \approx -1\). This result can be understood easily. Since \(d \to \infty\) and \(\Omega_\Lambda\) must be finite, \(w_\Lambda \to -1/3\). But the energy density of matter will be red-sifted faster than the holographic dark energy. So in the far future our Universe will be dominated by the dark energy and its expansion will be accelerating because of \(w_\Lambda < -1/3\). On the other hand, for \(d \to 0\), equation (16) tells us
\[
\Omega_\Lambda^T \approx (d/2)^2/3.
\] (17)

Also assuming \(\Omega_\Lambda^0\) is finite, applying equation (3), we have \(y_0 = -2 \ln 2\). Substituting equation (17) into (3), we obtain the red-shift of turning point is \(z_T \approx 0\). In this case, \(w_\Lambda \to -\infty\) for finite \(\Omega_\Lambda\) and the energy density of the holographic dark energy will increase very fast. Therefore our Universe was dominated by this dark energy and its expansion started to be accelerating very recently, if \(\Omega_\Lambda^0\) is finite. We show the relation between the red-shift of the turning point and the value of the parameter \(d\) for some finite \(\Omega_\Lambda^0\) in Fig. 1. This figure is consistent with our previous analysis.

![Figure 1](image)

**Figure 1.** \(z = z_T\) is the cosmological red-shift corresponding to the turning point between decelerating expansion and accelerating expansion, here the blue line corresponds to \(\Omega_\Lambda^0 = 0.75\) and the red one corresponds to \(\Omega_\Lambda^0 = 0.54\).
The luminosity distance $d_L$ expected in a spatially flat Friedmann-Robertson-Walker (FRW) cosmology with mass density $\Omega_m$ and the holographic dark energy density $\Omega_\Lambda$ is

$$d_L(z) = c(1 + z) \int_t^{t_0} \frac{dt'}{a(t')} = c(1 + z)[(1 + z)R_h(t) - R_h(t_0)]$$

$$= c d H_0^{-1} \left[ \sqrt{\frac{1 - \Omega_\Lambda}{\Omega_m \Omega_0^m}} (1 + z)^{1/2} - (1 - \Omega_0^m)^{-1/2}(1 + z) \right]. \quad (18)$$

In the above derivation, we used Eqs. (2) and (6). The parameter $\Omega_0^m$ of the model and the nuisance parameter $H_0$ are determined by minimizing

$$\chi^2 = \sum_i \frac{[\mu_{\text{obs}}(z_i) - \mu(z_i)]^2}{\sigma_i^2}, \quad (19)$$

where the extinction-corrected distance moduli $\mu(z) = 5 \log_{10}(d_L(z)/\text{Mpc}) + 25$ and $\sigma_i$ is the total uncertainty in the observation. The nuisance parameter $H_0$ is marginalized. If $d = 1$, the best fit to the 157 gold SN sample in [3] is $\Omega_0^m = 0.25^{+0.04}_{-0.03}$ with $\chi^2 = 176.7$ or $\chi^2/\text{dof} = 1.133$, and the best fit to the whole 186 gold and silver SN sample is $\Omega_0^m = 0.25 \pm 0.03$ with $\chi^2 = 232.8$ or $\chi^2/\text{dof} = 1.258$. By using the best fit $\Omega_0^m$, we find that $w_0^\Lambda = -0.91 \pm 0.01$ and the red-shift corresponding to transition is $z_T = 0.72^{+0.11}_{-0.13}$. For comparison, the best fit to the gold SN sample for the $\Lambda$-model is $\Omega_0^m = 0.31 \pm 0.04$ with $\chi^2 = 177.1$ or $\chi^2/\text{dof} = 1.135$. So the transition redshift for $\Lambda$-model is $z_T = 0.65^{+0.11}_{-0.10}$.

If we set $d$ as a free parameter, we find the best fit to the gold SN sample is $\Omega_0^m = 0.46^{+0.08}_{-0.13}$ and $d = 0.21^{+0.45}_{-0.14}$ with $\chi^2 = 173.45$ or $\chi^2/\text{dof} = 1.119$. In this case the red-shift corresponding to transition is $z_T = 0.28^{+0.23}_{-0.13}$. The best fit contour for $\Omega_0^m$ and $d$ is plotted in Fig. 2. The best fit to the gold and silver SN sample is $\Omega_0^m = 0.46^{+0.14}_{-0.11}$ and $d = 0.20^{+0.28}_{-0.10}$ with $\chi^2 = 226.4$ or $\chi^2/\text{dof} = 1.230$. And the red-shift corresponding to transition is $z_T = 0.27^{+0.19}_{-0.14}$. 5
Figure 2. The best fit contour for $\Omega_m^0$ and $d$ to the gold sample SNe.

Using the fit parameters $\Omega_m^0 = 0.75$ and $d = 1$, we get $y_0 = -1.67$. With the best fit parameters $\Omega_m^0 = 0.54$ and $d = 0.21$, we get $y_0 = -1.47$. Combining equations (9), (13) and (18), we show the equation of state of the dark energy $w_\Lambda$ in Fig. 3 and the extinction-corrected distance moduli in Fig. 4.

Figure 3. The evolution of $w = w_\Lambda(z)$, here the blue line corresponds to $\Omega_m^0 = 0.75$, $d = 1$ and the red one corresponds to $\Omega_m^0 = 0.54$, $d = 0.21$. 

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Figure 4. The relation between the extinction-corrected distance moduli and redshift for the best fit model, here \( d = 1 \).

In [3], Riess et al. found that \( w_0^\Lambda < -0.76 \) at the 95% confidence level by using SN Ia data, our result \( w_0^\Lambda = -0.91 \) with \( d = 1 \) and \( w_0^\Lambda = -2.67 \) with \( d = 0.21 \) are consistent with that. Recently, Tegmark et al. found that \( \Omega_m^0 \approx 0.30 \pm 0.04 \) by using the Wilkinson Microwave Anisotropy Probe (WMAP) data in combination with the Sloan Digital Sky Survey (SDSS) data [15]. If we use this prior, then we find the best fit parameters are \( \Omega_m^0 = 0.32 \pm 0.06 \) and \( d = 0.64 \pm 0.36 \) with \( \chi^2 = 175.93 \). Then the transition redshift is \( z_T = 0.53 \). The plot in Fig. 3 is consistent with the model independent analysis over the evolution of \( \Omega_\Lambda \) by using the WMAP and SN Ia data in [16]. More recent model independent analysis favor a phantom like dark energy model and lower transition redshift \( z_T \sim 0.3 \) or \( z_T \sim 0.4 \) [17]. The two parameter representation of dark energy models also favor a higher value for \( \Omega_m^0 \). Our best fit result is consistent with those analysis.

In conclusion, the holographic dark energy model is consistent with current observations and the more precise cosmological observations will be taken to be the decided constraints on this model. The model is also a better fit to observations than the ΛCDM model.

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