q-deformed Virasoro-Witt n-algebra

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Abstract

The q-deformation of the null Virasoro-Witt n-algebra is investigated. We construct a nontrivial q-deformed Virasoro-Witt n-algebra which satisfies the sh-Jacobi’s identity. This q-deformed infinite-dimensional n-algebra is nothing but a sh-n-Lie algebra. For the q-deformed Virasoro-Witt 3-algebra, we find that there exists a nontrivial finite-dimensional sub-3-algebra, i.e., $su_q(1,1)$ 3-algebra.
1 Introduction

Quantum algebras or more precisely quantized universal enveloping algebras first appeared in connection with the study of the inverse scattering problem. It is one parameter q deformation of Lie algebras which preserves the structure of a Hopf algebra and reduces to standard Lie algebra in the classical limit. The Virasoro algebra is an infinite dimensional Lie algebra and plays important roles in physics. Its q-deformation has been widely studied in the literature [1]-[9].

A q-deformation of the centerless Virasoro or Virasoro-Witt (V-W) algebra was first obtained by Curtright and Zachos [1]. Its central extension was later furnished by Aizawa and Sato [2]. Chaichian and Prešnajder [3] proposed a different version of the q-deformed Virasoro algebra by carrying out a Sugawara construction on a q-analogue of an infinite dimensional Heisenberg algebra. It is well-known that there is a remarkable connection between the Virasoro algebra and the Korteweg-de Vries (KdV) equation [10, 11]. For the q-deformed Virasoro algebra, Chaichian et al. [12] showed that it generates the sympletic structure which can be used for a description of the discretization of the KdV equation.

The Nambu 3-algebra was introduced in [13, 14] as a natural generalization of a Lie algebra for higher-order algebraic operations. Recently Bagger and Lambert [15], and Gustavsson [16] (BLG) found that 3-algebras play an important role in world-volume description of multiple M2-branes. Due to BLG theory, there has been considerable interest in the 3-algebra and its application. More recently there has been the progress in constructing the infinite-dimensional 3-algebras, such as V-W [17, 18], Kac-Moody [19] and $w_\infty$ 3-algebras [20, 21]. Moreover the relation between the infinite-dimensional 3-algebras and the integrable systems has also been paid attention [22, 23].

The structure and property of q-deformed algebra are now very well understood. But for the q-deformed 3-algebra, it has not been dealt with in such detail. Much less is known about its structure and property. Recently Curtright et al. [17], constructed a V-W algebra through the use of $su(1,1)$ enveloping algebra techniques. It is worthwhile to mention that this ternary algebra depends on a parameter $z$ and is only a Nambu-Lie algebra when $z = \pm 2i$. Ammar et al. [24] presented a q-deformation of this 3-algebra and noted it carrying the structure of ternary Hom-Nambu-Lie algebra.

In order to achieve a better understanding of the q-deformed n-algebra, in this Letter we focus on the q-deformation of the null V-W n-algebra. Based on the well-known structure of the q-deformed V-W algebra, we construct the nontrivial q-deformed V-W n-algebra and explore its intriguing features.
2 n-Lie algebra and sh-n-Lie algebra

To avoid too many technicalities, we will give here only the definitions of n-Lie algebra [25] and sh-n-Lie algebra [26].

The notion of n-Lie algebra or Filippov n-algebra was introduced by Filippov [25]. It is a natural generalization of Lie algebra. For a linear space $V$, an n-Lie algebra structure is defined by a multilinear map called Nambu bracket $[,]_{\ldots[,]_{\ldots}}: V^\otimes n \to V$ satisfying the following properties:

(1). Skew-symmetry

$$[A_{\sigma(1)}, \ldots, A_{\sigma(n)}] = (-1)^{\epsilon(\sigma)}[A_1, \ldots, A_n].$$

(2). Fundamental identity (FI) or Filippov condition

$$[A_1, \ldots, A_{n-1}, [B_1, \ldots, B_n]] = \sum_{k=1}^{n} [B_1, \ldots, B_{k-1}, [A_1, \ldots, A_{n-1}, B_k], B_{k+1}, \ldots, B_n].$$

For the case of 3-algebra, the corresponding fundamental identity is

$$[A, B, [C, D, E]] = [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]].$$

We have already seen that an n-Lie algebra $A$ is a vector space $A$ endowed with an n-ary skew-symmetric multiplication satisfying the FI condition. We now turn to the notion of sh-n-Lie algebra.

Let $[,]_{\ldots[,]_{\ldots}}$ be a n-ary skew-symmetric product on a vector space $A$. We say that $(A, [\ldots, \ldots])$ is a sh-n-Lie algebra if $[,]_{\ldots[,]_{\ldots}}$ satisfies the sh-Jacobi’s identity

$$\sum_{\sigma \in Sh(n, n-1)} (-1)^{\epsilon(\sigma)} [[x_{\sigma(1)}, \ldots, x_{\sigma(n)}], x_{\sigma(n+1)}, \ldots, x_{\sigma(2n-1)}] = 0,$$

for any $x_i \in A$, where $Sh(n, n-1)$ is the subset of $\Sigma_{2n-1}$ defined by

$$Sh(n, n-1) = \{ \sigma \in \Sigma_{2n-1}, \sigma(1) < \cdots < \sigma(n), \sigma(n+1) < \cdots < \sigma(2n-1) \}.$$

In terms of the Lévi-Cività symbol, i.e.,

$$\epsilon_{i_1 \cdots i_p}^{j_1 \cdots j_p} = \det \begin{pmatrix} \delta_{j_1}^{i_1} & \cdots & \delta_{j_p}^{i_1} \\ \vdots & & \vdots \\ \delta_{j_1}^{i_p} & \cdots & \delta_{j_p}^{i_p} \end{pmatrix},$$

the sh-Jacobi’s identity (4) can also be expressed as

$$\epsilon_{i_1 \cdots i_{2n-1}}^{m_1 \cdots m_{2n-1}} [[x_{i_1}, \ldots, x_{i_n}], x_{i_{n+1}}, \ldots, x_{i_{2n-1}}] = 0.$$

When $n = 3$, the sh-Jacobi’s identity becomes

$$[[A, B, C], D, E] - [[A, B, D], C, E] - [[A, B, E], C, D] + [[A, C, D], B, E]$$

$$-[[A, C, E], B, D] + [[A, D, E], B, C] - [[B, C, D], A, E] + [[B, C, E], A, D]$$

$$-[[B, D, E], A, C] + [[C, D, E], A, B] = 0.$$
We have briefly introduced the n-Lie algebra and sh-n-Lie algebra. It should be noted that any n-Lie algebra is a sh-n-Lie algebra, but a sh-n-Lie algebra is a n-Lie algebra if and only if any adjoint operator is a derivation.

3 q-deformed V-W 3-algebra

3.1 q-deformed V-W algebra

As a start before investigating the q-deformed 3-algebra, let us recall the case of q-deformed algebra. The deformation of the commutator is defined by

\[ [A, B]_{(p,q)} = pAB - qBA. \]  

(8)

It possesses the following properties [4, 9]:

\[ [A, B]_{(p,q)} = -[B, A]_{(q,p)}, \]
\[ [A + B, C]_{(p,q)} = [A, C]_{(p,q)} + [B, C]_{(p,q)}, \]
\[ [AB, C]_{(p,q)} = A[B, C]_{(p,r)} + [A, C]_{(r,q)}B, \]
\[ [A, BC]_{(p,q)} = B[A, C]_{(r,q)} + [A, B]_{(p,r)}C, \]  

(9)

and the q-Jacobi identity

\[ [A, [B, C]]_{(q_1,q_1^{-1})} + [B, [C, A]]_{(q_2,q_2^{-1})} + [C, [A, B]]_{(q_3,q_3^{-1})} = 0. \]  

(10)

The Virasoro algebra is an infinite dimensional Lie algebra and plays important roles in physics. The V-W algebra is indeed the centerless Virasoro algebra. It is given by

\[ [L_m, L_n] = (m - n)L_{m+n}. \]  

(11)

For the generators \( L_0, L_1 \) and \( L_{-1} \), it can be easily seen that they satisfy the \( SU(1,1) \) algebra:

\[ [L_0, L_1] = -L_1, \quad [L_0, L_{-1}] = L_{-1}, \quad [L_1, L_{-1}] = 2L_0. \]  

(12)

To construct the deformed V-W algebra, let us take the q-deformed generators

\[ L_m = -q^N (a^\dagger)^{m+1} a, \]  

(13)

where the q-deformed oscillator is deformed by the following relations [27-29]:

\[ aa^\dagger - qa^\dagger a = q^{-N}, \quad aa^\dagger = [N], \]
\[ [N, a] = -a, \quad [N, a^\dagger] = a^\dagger. \]  

(14)
Substituting the q-generators (13) into the commutator (8) and using the q-deformed oscillator (14), it leads to the so-called q-deformed V-W algebra [11]

\[ [L_m, L_n]_{(q^{m-n}, q^{n-m})} = q^{m-n}L_m L_n - q^{n-m} L_n L_m = [m-n] L_{m+n}, \quad (15) \]

where \( [k] = \frac{q^k - q^{-k}}{q - q^{-1}} \). In the limit \( q \to 1 \), (15) reduces to the V-W algebra [11]

From q-deformed V-W algebra (15), we note that the generators \( L_0, L_1 \) and \( L_{-1} \) of (13) comprise the \( SU_q(1,1) \) algebra, 

\[ [L_0, L_1]_{(q^{-1}, q)} = -L_1, \quad [L_0, L_{-1}]_{(q, q^{-1})} = L_{-1}, \quad [L_1, L_{-1}]_{(q^2, q^{-2})} = [2] L_0. \quad (16) \]

This q-deformed \( su(1,1) \) algebra has been well investigated in the literature [1, 30].

Let us define the star product by

\[ L_n \ast [L_m, L_k]_{(q^{m-k}, q^{k-m})} = q^{2n+m-k}L_n[L_m, L_k]_{(q^{m-k}, q^{k-m})}, \]

\[ [L_m, L_k]_{(q^{m-k}, q^{k-m})} \ast L_n = q^{m+k-2n}[L_m, L_k]_{(q^{m-k}, q^{k-m})}L_n. \quad (17) \]

Then we have

\[ L_n \ast [L_m, L_k]_{(q^{m-k}, q^{k-m})} = [L_m, L_k]_{(q^{m-k}, q^{k-m})} \ast L_n \]

\[ = [L_n, [L_m, L_k]_{(q^{m-k}, q^{k-m})}]_{(q^{2n-m-k}, q^{m+k-2n})}. \quad (18) \]

By means of (18), one can confirm the following q-Jacobi identity [4] satisfied by the q-deformed V-W algebra (15):

\[ [L_n, [L_m, L_k]_{(q^{m-k}, q^{k-m})}]_{(q^{2n-m-k}, q^{m+k-2n})} + cycl. \text{perms.} = 0. \quad (19) \]

### 3.2 q-deformed V-W 3-algebra and \( su_q(1,1) \) 3-algebra

We have introduced the q-deformed algebra in the previous subsection. Let us now turn our attention to the case of 3-algebra. The operator Nambu 3-bracket is defined to be a sum of single operators multiplying commutators of the remaining two [13], i.e.,

\[ [A, B, C] = A[B, C] + B[C, A] + C[A, B], \quad (20) \]

where \([A, B] = AB - BA\).

For the q-deformed V-W algebra (15), we have already seen that the q-Jacobi identity (19) is guaranteed to hold. It is worth to emphasize that the star product (17) plays a pivotal role in the q-Jacobi identity. In terms of the star product (17), let us define the q-3-bracket as follows:

\[ [L_m, L_n, L_k] = L_m \ast [L_n, L_k]_{(q^{m-k}, q^{k-n})} + L_n \ast [L_k, L_m]_{(q^{k-m}, q^{m-k})} + L_k \ast [L_m, L_n]_{(q^{n-m}, q^{m-n})}. \quad (21) \]
By means of (15) and (17), we may derive the following q-deformed 3-algebra from (21):

\[
\llbracket L_m, L_n, L_k \rrbracket = \frac{1}{q - q^{-1}} \left( [2m - 2k] + [2k - 2n] + [2n - 2m] \right) L_{m+n+k}
\]

\[
= \left( q - q^{-1} \right) \left( [m - n][m - k][n - k] \right) L_{m+n+k}
\]

\[
= -\frac{1}{(q - q^{-1})^2} \det \begin{pmatrix} q^{-2m} & q^{-2n} & q^{-2k} \\ 1 & 1 & 1 \\ q^{2m} & q^{2n} & q^{2k} \end{pmatrix} L_{m+n+k}.
\]

(22)

Performing lengthy but straightforward calculations, we find that (22) satisfies the sh-Jacobi’s identity (7), but the FI condition (3) does not hold. It is easy to verify that the skew-symmetry holds for this ternary algebra

\[
\llbracket L_m, L_n, L_k \rrbracket = -\llbracket L_n, L_k, L_m \rrbracket = -\llbracket L_k, L_m, L_m \rrbracket.
\]

(23)

Therefore the q-deformed V-W 3-algebra (22) is indeed a sh-3-Lie algebra. In the limit \( q \to 1 \), (22) reduces to the null 3-algebra derived in [18],

\[
[L_m, L_n, L_k] = 0.
\]

(24)

The FI condition (3) is trivially satisfied for this null 3-algebra.

It is known that the \( su(1,1) \) algebra is a subalgebra of V-W algebra. From (24), we have the null \( su(1,1) \) 3-algebra,

\[
[L_{-1}, L_0, L_1] = 0.
\]

(25)

Let us turn to discuss the q-deformation of (25). Taking the generators to be \( L_0, L_1 \) and \( L_{-1} \) in (22), it leads to the \( su_q(1,1) \) 3-algebra

\[
[L_{-1}, L_0, L_1] = -2(q - q^{-1})L_0 = \frac{1 - q^2}{q^2} L_0.
\]

(26)

An intriguing feature is that for the null \( su(1,1) \) 3-algebra, its q-deformed 3-algebra is nontrivial. Moreover it is worth to emphasize that this \( su_q(1,1) \) 3-algebra satisfies the FI condition (3).

4 q-deformed V-W n-algebra

Now encouraged by the possibility of constructing the nontrivial sh-3-Lie algebra (22), it would be interesting to study further and see whether one could construct the q-deformed V-W n-algebra with a genuine sh-n-Lie algebra structure. In this section we give affirmative answer to this question.

The \( n \)-bracket with \( n \geq 3 \) is defined by

\[
[L_{i_1}, L_{i_2}, \cdots, L_{i_n}] = \sum_{s=1}^{n} (-1)^{s+1} L_{i_s} [L_{i_1}, L_{i_2}, \cdots, \hat{L}_{i_s}, \cdots, L_{i_n}].
\]

(27)
Here we denote a notational convention used frequently in the rest of this paper. Namely for any arbitrary symbol \( Z \), the hat symbol \( \hat{Z} \) stands for the term that is omitted.

Let us define a q-n-bracket as follows:

\[
\{ L_{i_1}, L_{i_2}, \ldots, L_{i_n} \} = \frac{\text{sign} (n)}{(q - q^{-1})^{n-1}} \left( q^{-2 \left[ \frac{n+1}{4} \right]}_{i_1} q^{-2 \left[ \frac{n+1}{4} \right]}_{i_2} \ldots q^{-2 \left[ \frac{n+1}{4} \right]}_{i_n} \right) \left( q^2(-\left[ \frac{n+1}{2} \right] +1)_{i_1} q^2(-\left[ \frac{n+1}{2} \right] +1)_{i_2} \ldots q^2(-\left[ \frac{n+1}{2} \right] +1)_{i_n} \right) \left( \begin{array}{c} \vdots \\ \vdots \\ q^2(\left[ \frac{n}{2} \right] -1)_{i_1} q^2(\left[ \frac{n}{2} \right] -1)_{i_2} \ldots q^2(\left[ \frac{n}{2} \right] -1)_{i_n} \\ q^2(\frac{n}{2})_{i_1} q^2(\frac{n}{2})_{i_2} \ldots q^2(\frac{n}{2})_{i_n} \end{array} \right) \right] L_{\Sigma_{i=1}^n i_i}, \tag{30}
\]

where \([n] = \text{Max} \{ m \in \mathbb{Z} | m \leq n \}\) is the floor function, \(\text{sign} (n)\) is the signature function, i.e.,

\[
\text{sign} (n) = \begin{cases} 1, & \text{for } n \text{ mod } 4 = 0, 1 \\ -1, & \text{for } n \text{ mod } 4 = 2, 3 \end{cases}.
\]

Let us confirm this by the mathematical induction for \( n \). Equation (22) indicates that (30) is satisfied for \( n = 3 \). We suppose (30) is satisfied for \( n \). By means of (27), we have

\[
\{ L_{i_1}, L_{i_2}, \ldots, L_{i_{n+1}} \} = \frac{\text{sign} (n)}{(q - q^{-1})^{n-1}} A \left( q^{-2 \left[ \frac{n+1}{4} \right]}_{i_1} q^{-2 \left[ \frac{n+1}{4} \right]}_{i_2} q^{-2 \left[ \frac{n+1}{4} \right]}_{i_3} \ldots q^{-2 \left[ \frac{n+1}{4} \right]}_{i_{n+1}} \right) \left( \begin{array}{c} \vdots \\ \vdots \\ q^2(-\left[ \frac{n+1}{2} \right] +1)_{i_1} q^2(-\left[ \frac{n+1}{2} \right] +1)_{i_2} \ldots q^2(-\left[ \frac{n+1}{2} \right] +1)_{i_{n+1}} \\ q^2(\left[ \frac{n}{2} \right] -1)_{i_1} q^2(\left[ \frac{n}{2} \right] -1)_{i_2} \ldots q^2(\left[ \frac{n}{2} \right] -1)_{i_{n+1}} \\ q^2(\frac{n}{2})_{i_1} q^2(\frac{n}{2})_{i_2} \ldots q^2(\frac{n}{2})_{i_{n+1}} \end{array} \right) \right] L_{\Sigma_{i=1}^{n+1} i_i}, \tag{31}
\]
A but the FI condition (3) does not hold. Let us consider the case of the q-n-bracket (30). Taking the second determinate in (31), we obtain the explicit form of (30).

Then let us use the expression (32) to calculate

\[
[L_{i_1}, L_{i_2}, \ldots, L_{i_{n+1}}] = \frac{\text{sign}(n+1)}{(q - q^{-1})^n}
\]

\[
\begin{pmatrix}
q^{x_{i_1}+(2+y)\Sigma_{j=1}^n i_j+2} & \cdots & q^{x_{i_1}+(2+y)\Sigma_{j=1}^n i_j+2} & \cdots & q^{x_{i_1}+(2+y)\Sigma_{j=1}^n i_j+2} \\
q^{-2}[\frac{n-1}{2}] i_1 & \cdots & q^{-2}[\frac{n-1}{2}] i_s & \cdots & q^{-2}[\frac{n-1}{2}] i_{n+1} \\
q^2(-[\frac{n-1}{2}]+1) i_1 & \cdots & q^2(-[\frac{n-1}{2}]+1) i_s & \cdots & q^2(-[\frac{n-1}{2}]+1) i_{n+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
q^2[\frac{n}{2}] i_1 & \cdots & q^2[\frac{n}{2}] i_s & \cdots & q^2[\frac{n}{2}] i_{n+1}
\end{pmatrix}
\]

Substituting \((x = n - 1, y = -2)\) for odd \(n\) and \((x = n, y = 0)\) for even \(n\) into (31), respectively, we find that the determinate \(A\) is zero. After a straightforward calculation for the second determinate in (31), we obtain the explicit form of \((n + 1)\)-bracket (31)

\[
[L_{i_1}, L_{i_2}, \cdots, L_{i_{n+1}}] = \frac{\text{sign}(n+1)}{(q - q^{-1})^n}
\]

\[
\begin{pmatrix}
q^{-2}[\frac{n}{2}] i_1 & q^{-2}[\frac{n}{2}] i_2 & \cdots & q^{-2}[\frac{n}{2}] i_{n+1} \\
q^2(-[\frac{n}{2}] + 1) i_1 & q^2(-[\frac{n}{2}] + 1) i_2 & \cdots & q^2(-[\frac{n}{2}] + 1) i_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
q^2[\frac{n}{2}] i_1 & q^2[\frac{n}{2}] i_2 & \cdots & q^2[\frac{n}{2}] i_{n+1}
\end{pmatrix}
\]

which shows that (30) is satisfied for \(n + 1\). Now the proof is completed.

For the q-3-bracket (22), we already recognize that it satisfies the sh-Jacobi’s identity (7), but the FI condition (3) does not hold. Let us consider the case of the q-n-bracket (30). Taking \(A_i = L_{n+i}, i = 1, 2, \cdots, n-3, A_{n-2} = L_n, A_{n-1} = L_{n+2} \cdots \Sigma_{j=1}^n\) and \(B_j = L_j, j = 1, 2, \cdots, n\) in (2), straightforward calculation shows that the right-hand side of (2) equals zero, but its left-hand side does not. Therefore the FI condition (2) does not hold for (30).

Let us turn to the case of the sh-Jacobi’s identity with respect to the q-n-bracket (30). We first focus on (30) with odd \(n\). In terms of the Lévi-Civitá symbol (5), we can rewrite \((2n + 1)\)-bracket (30) as

\[
\llbracket L_{i_1}, \cdots, L_{i_{2n+1}} \rrbracket = \epsilon_{i_1 \cdots i_{2n+1}} q^{-2n j_1 + 2(-n+1)j_2 + \cdots + 2(n-1)j_{2n} + 2n j_{2n+1}} \epsilon_{i_1 \cdots i_{2n+1}} L_{\Sigma_{j=1}^{2n+1} i_j}.
\]

(32)

Then let us use the expression (32) to calculate \(\llbracket L_{i_1}, \cdots, L_{i_{2n+1}} \rrbracket, L_{i_{2n+2}}, \cdots, L_{i_{4n+1}} \rrbracket\). It leads to

\[
\llbracket L_{i_1}, \cdots, L_{i_{2n+1}} \rrbracket, L_{i_{2n+2}}, \cdots, L_{i_{4n+1}} \rrbracket = \sum_{k=2}^{2n+2} (-1)^k \epsilon_{i_1 \cdots i_{2n+1}} \epsilon_{i_2 \cdots i_{2n+2}} \cdots \epsilon_{i_{4n+1} i_{4n+2}} q^{-2n_k + 2(-n+k-1)j_2 + \cdots + (k-2)j_{2n+1}} q^{-n j_{2n+2} + \cdots + (-n+k-3)j_{2n+k-1} + (-n+k-2)j_{2n+k} + (-n+k-1)j_{2n+k+1} + \cdots + n j_{4n+2}} \epsilon_{i_1 \cdots i_{2n+1}} L_{\Sigma_{j=1}^{4n+1} i_j}.
\]

(33)
Substituting (33) into the left-hand side of (3), we obtain
\[ \epsilon_{m_1 \cdots m_{4n+1}}^{1 \cdots 4n+1} \left[ [L_{i_1}, \cdots, L_{i_{2n+1}}], L_{i_{2n+2}}, \cdots, L_{i_{4n+1}} \right] \]
\[ = (2n+1)! \langle 2n \rangle \sum_{k=2}^{2n+2} (-1)^k j_1^{i_1} j_2^{i_2} \cdots j_{4n}^{i_{4n+1}} q^{\alpha L_{\sum_{i=1}^{4n+1} i_i}}, \]  
(34)
where the power of q is given by
\[ \alpha = (-2n + k - 2) j_1 + (-2n + k - 1) j_2 + \cdots + (k - 2) j_{2n+1} - nj_{2n+2} + \cdots + (-n + k - 2) j_{2n+k} + \cdots + nj_{4n+2}, \]  
(35)
and the following formula is useful in simplifying expression:
\[ \epsilon_{m_1 \cdots m_n}^{i_1 \cdots i_k} \epsilon_{n_1 \cdots n_k}^{j_1 \cdots j_k} = k! \epsilon_{m_1 \cdots m_n}^{i_1 \cdots i_k} \epsilon_{n_1 \cdots n_k}^{j_1 \cdots j_k}. \]  
(36)
From the expression of \( \alpha \) (35), we observe that the coefficients of two different \( j_\mu \) should be equal. Since \( \epsilon_{1 \cdots 4n+1}^{j_1 \cdots j_{4n+2}} \) is completely antisymmetric, it is easy to see that (34) equals zero. It indicates that the sh-Jacobi’s identity is satisfied by (30) with odd \( n \). For the case of (30) with even \( n \), by the similar way, we can confirm the corresponding sh-Jacobi’s identity. Taking the above results, we may conclude that the sh-Jacobi’s identity (11) does hold for (30). Since the structure constants are determined by the the determinate, n-bracket (30) is anticommutative. Based on the above analysis, it is clear that the q-deformed V-W n-algebra is indeed a sh-n-Lie algebra.

As an example, let us list first few q-deformed V-W n-algebras as follows:

- \([L_{i_1}, L_{i_2}, L_{i_3}, L_{i_4}]\)
\[ = (q - q^{-1})^{-3} \det \begin{pmatrix} q^{-2i_1} & q^{-2i_2} & q^{-2i_3} & q^{-2i_4} \\ 1 & 1 & 1 & 1 \\ q^{2i_1} & q^{2i_2} & q^{2i_3} & q^{2i_4} \\ q^{4i_1} & q^{4i_2} & q^{4i_3} & q^{4i_4} \end{pmatrix} L_{\sum_{k=1}^{4} i_k} \]
\[ = (q - q^{-1})^{-3} q^{\sum_{k=1}^{4} i_k} \prod_{1 \leq m < n \leq 4} [i_m - i_n] L_{\sum_{k=1}^{4} i_k}. \]  
(37)

- \([L_{i_1}, L_{i_2}, L_{i_3}, L_{i_4}, L_{i_5}]\)
\[ = (q - q^{-1})^{-4} \det \begin{pmatrix} q^{-4i_1} & q^{-4i_2} & q^{-4i_3} & q^{-4i_4} & q^{-4i_5} \\ q^{-2i_1} & q^{-2i_2} & q^{-2i_3} & q^{-2i_4} & q^{-2i_5} \\ 1 & 1 & 1 & 1 & 1 \\ q^{2i_1} & q^{2i_2} & q^{2i_3} & q^{2i_4} & q^{2i_5} \\ q^{4i_1} & q^{4i_2} & q^{4i_3} & q^{4i_4} & q^{4i_5} \end{pmatrix} L_{\sum_{k=1}^{5} i_k} \]
\[ = (q - q^{-1})^{-6} q^{\sum_{k=1}^{5} i_k} \prod_{1 \leq m < n \leq 5} [i_m - i_n] L_{\sum_{k=1}^{5} i_k}. \]  
(38)
• \[ [L_{i_1}, L_{i_2}, L_{i_3}, L_{i_4}, L_{i_5}, L_{i_6}] \]

\[
= - (q - q^{-1})^{-5} \det \begin{pmatrix}
q^{-4i_1} & q^{-4i_2} & q^{-4i_3} & q^{-4i_4} & q^{-4i_5} & q^{-4i_6} \\
q^{-2i_1} & q^{-2i_2} & q^{-2i_3} & q^{-2i_4} & q^{-2i_5} & q^{-2i_6} \\
1 & 1 & 1 & 1 & 1 & 1 \\
q^{2i_1} & q^{2i_2} & q^{2i_3} & q^{2i_4} & q^{2i_5} & q^{2i_6} \\
q^{4i_1} & q^{4i_2} & q^{4i_3} & q^{4i_4} & q^{4i_5} & q^{4i_6} \\
q^{6i_1} & q^{6i_2} & q^{6i_3} & q^{6i_4} & q^{6i_5} & q^{6i_6}
\end{pmatrix}
L_{\sum_{k=1}^{6} i_k}
\]

\[
= - (q - q^{-1})^{10} q^{\sum_{k=1}^{6} i_k} \prod_{1 \leq m < n \leq 6} [i_m - i_n] L_{\sum_{k=1}^{6} i_k}. \tag{39}
\]

For the q-deformed V-W 3-algebra, we note that there exists a nontrivial sub-3-algebra, i.e., \( su_q(1, 1) \)-3-algebra. As to the case of the q-deformed V-W n-algebra \( \text{[30]} \), we can easily see that this sh-n-Lie algebra only admits the null \( su_q(1, 1) \)-n-algebra for \( n \geq 4 \).

5 Summary

The V-W algebra is the centerless Virasoro algebra. Its q-deformation has been well investigated in the literature. One has already known that in the usual way, the V-W n-algebra is null. In this paper, we investigated the q-deformation of the null V-W n-algebra and constructed the nontrivial q-deformed V-W n-algebra. It is of interest to note that it satisfies the sh-Jacobi’s identity, but the FI condition fails. Thus this q-deformed V-W n-algebra is indeed a sh-n-Lie algebra. Furthermore we pointed out that a special case is that of \( n = 3 \). For the q-deformed Virasoro-Witt 3-algebra, we found that there exists a nontrivial finite-dimensional sub-3-algebra, i.e., \( su_q(1, 1) \)-3-algebra.

Our investigation revealed a deep connection between the q-deformed infinite-dimensional n-algebra and the sh-n-Lie algebra. It sheds new light on the sh-n-Lie algebra. It would be interesting to study further and see whether there exist the central extension terms for the sh-n-Lie algebra derived in this paper. Furthermore the application of this sh-n-Lie algebra in physics might be of interest.

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