Superconformal Quantum Mechanics
on a Quantum Computer

Yuan Feng\textsuperscript{(1)}, Michael McGuigan, Tristen White\textsuperscript{(2)}
(1) University of California, Berkeley
(2) Florida State University
email contact: michael.d.mcguigan@gmail.com

Abstract

We investigate superconformal quantum mechanics (SCQM) on a quantum computer. We study the ground state of the mass deformed SCQM on a quantum computer using the Variational Quantum Eigensolver (VQE) using a one boson and one boson - one fermion Hilbert space with and without noise and compare the accuracy of the results. We study the Feynman path integral for SCQM using the Evolution of Hamiltonian (EOH) algorithm on the quantum computer using the Trotter-Suzuki approximation and compare with the exact result. We consider an \(N\) boson and \(N\) fermion version of SCQM given be the Supersymmetric Calogero-Moser-Sutherland (SCMS) model. We compare the ground state of SCMS theory obtained using the VQE computation with the exact solution. Finally we discuss the implications of the the numerical simulation of SCQM on a quantum computer for the simulation of quantum gravity in light of the Anti-de Sitter/Superconformal field theory (AdS/CFT) correspondence.


1 Introduction

Superconformal Quantum Mechanics (SCQM) is part of a family of supersymmetric Hamiltonians such as Hamiltonians with quartic potentials in 3+1 dimensions and sixth order potentials 2+1 dimensions. Superconformal Quantum Mechanics has an inverse power of two potential in 0+1 dimensions $[1][2][3][4][5]$. We will consider the mass deformed version of the potential that has an addition quadratic term in the potential. Superconformal Quantum Mechanics has interesting applications including the study of Baryon Spectrum in the light front formalism $[6][7][8]$, black holes physics $[9][10][11]$, relations to $AdS_2$ and holography $[12][13]$ and also has application to quantum cosmology $[14]$. 

In this paper we study Superconformal Quantum mechanics using quantum computing. Quantum computing is a potentially disruptive form of computing that may excel at the simulation of quantum systems. As superconformal quantum mechanics is exactly soluble it can serve as a benchmark for scalability and accuracy of quantum computation in regimes beyond which can be verified using classical computers. In this paper we shall study quantum computations up to 9 qubits but the methods can in principle scaled much further and one should keep in mind that quantum computers already exist with capacities exceeding 100 qubits. We will use different quantum algorithms in our simulation of superconformal quantum mechanics such as the Variational Quantum Eigensover (VQE) and the Evolution of Hamiltonian (EOH) quantum algorithm. We study different bases to represent the Hamiltonian on the quantum computer and compare the accuracy with that of the classical computer. Finally we extend our analysis to the multi-particle supersymmetric Calogero-Moser-Sutherland model which may have applications to the study of Yang Mills Theories $[15][16]$, holographic quantum gravity $[17]$ and many body physics $[18]$.

2 Superconformal Quantum Mechanics

For superconformal quantum mechanics with mass deformation the superpotential is taken to be:

$$W(x) = m\omega x - \frac{g}{x}$$

(2.1)

Then the minus and plus partner potentials are defined as $[19][20][21]$:

$$V_-(x) = W^2(x) - W'(x)$$
$$V_+(x) = W^2(x) + W'(x)$$

(2.2)

and are given by:

$$V_-(x) = m^2\omega^2x^2 + (g^2 - g)x^{-2} - 2m\omega g - m\omega$$
$$V_+(x) = m^2\omega^2x^2 + (g^2 + g)x^{-2} - 2m\omega g + m\omega$$

(2.3)
Figure 1: Supersymmetric Conformal Quantum Mechanics (SCQM) potential and ground state wave function for $g = \sqrt{2}$.

The partner Hamiltonians are

$$H_- = p^2 + V_-(x)$$
$$H_+ = p^2 + V_+(x)$$

(2.4)

Rescaling the minus partner Hamiltonian by $\frac{1}{2m}$ we define the Hamiltonian $H$ as:

$$H = \frac{1}{2m} H_- = \frac{1}{2m} (p^2 + V_-(x))$$

(2.5)

For $m = \frac{1}{2}$, $\omega = 1$, $g = \sqrt{2}$ the ground state of the minus partner Hamiltonian is:

$$\psi_0^- = e^{-\int_0^x W(x) dx} = \frac{1}{2} (x) \sqrt{2} e^{-\frac{1}{4} x^2}$$

(2.6)

and the energy spectrum of the partner Hamiltonians are:

$$\varepsilon_n^- = 0, 2, 4, 6, 8, \ldots$$
$$\varepsilon_n^+ = 2, 4, 6, 8, \ldots$$

(2.7)

The potential and ground state wave function is plotted in figure 1. One can form $A$ and $A^\dagger$ operators through

$$A = \frac{1}{\sqrt{2m}} (ip + m\omega x - gx^{-1})$$
$$A^\dagger = \frac{1}{\sqrt{2m}} (-ip + m\omega x - gx^{-1})$$

(2.8)

Then the Hamiltonian is simply:

$$H = A^\dagger A$$

(2.9)

This form will be used in our quantum computations and turns out to yield highly accurate results.
Quantum computing basis representations

Quantum computing for supersymmetric models has recently attracted interest because of their many mathematically and physically interesting features [22][23][24][25][26]. In quantum computing one represents the quantum operators as finite matrices in a given basis. In this paper we study the oscillator basis, position basis and finite difference basis described below:

Once we have operators represented as matrices one performs an expansion of these matrices in terms of tensor products of the three Pauli matrices and the identity matrix called Pauli terms in order to represent them on a quantum computer. The quantum algorithms we will use in this paper are the Variational Quantum Eigensolver (VQE) to estimate the ground state energy and wave function and the Evolution of Hamiltonian (EOH) algorithm to estimate the Kernel or path integral.

Gaussian or Simple Harmonic Oscillator basis

This is a very useful basis based on the matrix treatment of the simple harmonic oscillator which is sparse in representing the position and momentum operator. For the position operator we have:

\[
X_{osc} = \frac{1}{\sqrt{2}} \begin{bmatrix}
0 & \sqrt{1} & 0 & \cdots & 0 \\
\sqrt{1} & 0 & \sqrt{2} & \cdots & 0 \\
0 & \sqrt{2} & \ddots & \ddots & 0 \\
0 & 0 & \ddots & 0 & \sqrt{N-1} \\
0 & 0 & \cdots & \sqrt{N-1} & 0 \\
\end{bmatrix}
\]  

while for the momentum operator we have:

\[
P_{osc} = \frac{i}{\sqrt{2}} \begin{bmatrix}
0 & -\sqrt{1} & 0 & \cdots & 0 \\
\sqrt{1} & 0 & -\sqrt{2} & \cdots & 0 \\
0 & \sqrt{2} & \ddots & \ddots & 0 \\
0 & 0 & \ddots & 0 & -\sqrt{N-1} \\
0 & 0 & \cdots & \sqrt{N-1} & 0 \\
\end{bmatrix}
\]  

The SCQM Hamiltonian \( H_- \) is then

\[
H_- = P_{osc}^2 + m^2 \omega^2 X_{osc}^2 + (g^2 - g)X_{osc}^{-2} - (m\omega + g2m\omega)I
\]  

where \( I \) is the \( N \times N \) identity matrix.
Position basis

In the position basis the position matrix is diagonal but the momentum matrix is dense and constructed from the position operator using a Sylvester matrix $F$. In the position basis the position matrix is:

$$(X_{\text{pos}})_{j,k} = \sqrt{\frac{2\pi}{4N}} (2j - (N+1))\delta_{j,k} \quad (3.4)$$

and the momentum matrix is:

$$P_{\text{pos}} = F^{\dagger}X_{\text{pos}}F \quad (3.5)$$

where

$$F_{j,k} = \frac{1}{\sqrt{N}} e^{\frac{2\pi i}{4N} (2j-(N+1))(2k-(N+1))} \quad (3.6)$$

The SCQM Hamiltonian is then

$$H_{-} = P_{\text{pos}}^2 + m^2 \omega^2 X_{\text{pos}}^2 + (g^2 - g) X_{\text{pos}}^{-2} - (m\omega + g^2 m\omega)I \quad (3.7)$$

and in this case the matrix potential is very simple as it is a function of a diagonal matrix.

Finite difference basis

This is the type of basis that comes up when realized differential equations in terms of finite difference equations. In this case the position operator is again diagonal but the momentum operator although not diagonal is still sparse. In the finite difference basis the position matrix is:

$$(X_{\text{fd}})_{j,k} = \sqrt{\frac{1}{2N}} (2j - (N+1))\delta_{j,k}$$

and the momentum-squared matrix is:

$$P_{\text{fd}}^2 = \frac{N}{2} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \quad (3.8)$$

The SCQM Hamiltonian is then:

$$H_{-} = P_{\text{fd}}^2 + m^2 \omega^2 X_{\text{fd}}^2 + (g^2 - g) X_{\text{fd}}^{-2} - (m\omega + g^2 m\omega)I \quad (3.9)$$
Annihilation and Creation operator basis

The $A$ operator basis is similar to the oscillator basis. One forms

$$A = \frac{1}{\sqrt{2m}}(iP_{osc} + m\omega X_{osc} - gX_{osc}^{-1})$$

$$A^\dagger = \frac{1}{\sqrt{2m}}(-iP_{osc} + m\omega X_{osc} - gX_{osc}^{-1})$$  \hspace{1cm} (3.10)

Then the Hamiltonian is simply:

$$H = A^\dagger A$$  \hspace{1cm} (3.11)

Whatever basis one uses one needs to map the Hamiltonian to an expression in terms of a sum of tensor products of Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$ plus the identity matrix which are the Pauli terms. As there are four such matrices the maximum number of terms in this expansion is $4^n$ where $n$ is the number of qubits. In most of our simulations the number of qubits was fixed at 4 so that the maximum number of Pauli terms was 256.

4 Quantum computing results - One boson formulation

The one boson formulation is the traditional formulation used in supersymmetric quantum mechanics. In this section we use the aop basis and write the Hamiltonian as:

$$H = \frac{1}{2m} (-ip + m\omega x - gx^{-1}) (ip + m\omega x - gx^{-1})$$  \hspace{1cm} (4.1)

for $g = \sqrt{2}$. The size of the matrix we used was $16 \times 16$ so the Hamiltonian in the one boson formulation used 4 qubits. We use the Variational Quantum Eigensolver (VQE) quantum algorithm which is part of IBM Qiskit, a toolkit developed by IBM to run quantum algorithms on quantum computers and simulators. The VQE quantum algorithm works by representing a variational quantum state in terms of rotation gates parametrized by angles. Then one uses an optimizer to vary the angles from one iteration to the next to try to find the lowest energy which is determined by the algorithm to lie above the exact ground state energy. We used two types of simulations, one using the state-vector simulator and no noise, and another that used the QASM simulator with realistic noise. The results are shown in tables 1 and 2 with convergence graphs in figures 2 and 3. For the no noise simulation we found highly accurate results using the Limited-memory BFGS Bound (L-BFGS-B) optimizer. For the noisy simulation not surprisingly the simulation was less accurate but still produced a ground state energy estimate close to zero. The accuracy of the no noise simulation was somewhat surprising given the relatively large value of the coupling. This means the parametrization of the ground state wave function in terms of parametrized rotation gates had a strong overlap with the exact ground state wave function. Previous studies of supersymmetric quantum mechanics showed a strong overlap with this type of parametrization only for weak coupling.
Figure 2: Convergence graph for the VQE computation of the ground state energy of SCQM using the one boson formulation. The VQE result was obtained using the State-vector simulator with no noise and the Limited-memory BFGS Bound (L-BFGS-B) optimizer.

Figure 3: Convergence graph for the VQE computation of the ground state energy of SCQM using the one boson formulation. The VQE result was obtained using the QASM simulator including realistic noise and the Simultaneous Perturbation Stochastic Approximation (SPSA) optimizer.
SCQM one boson formulation | Ground State Energy
---|---
Exact | 0.0
Exact Discrete | 0.001904019686
VQE-State-vector (L-BFGS-B optimizer) | 0.001904019785

Table 1: Exact, Exact discrete, and VQE computation of the ground state energy of SCQM using the one boson one fermion formulation. The VQE result was obtained using the State-vector simulator with no noise and the Limited-memory BFGS Bound (L-BFGS-B) optimizer.

SCQM one boson formulation | Ground State Energy
---|---
Exact | 0.0
Exact Discrete | 0.001904019686
VQE-QASM-Simulator (SPSA optimizer) | 0.178733653271

Table 2: Exact, Exact discrete, and VQE computation of the ground state energy of SCQM using the one boson one fermion formulation. The VQE result was obtained using the QASM simulator including realistic noise and the Simultaneous Perturbation Stochastic Approximation (SPSA) optimizer.

5 Quantum computing results - One boson and one fermion formulation

The one boson one fermion formulation of supersymmetric conformal quantum mechanics is more in line with higher dimensional formulations of supersymmetric quantum field theory. In this case we the oscillator basis and use tensor products to define the boson-fermion Hilbert space from:

\[
X = X_{osc} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
P = P_{osc} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
C = I_{16} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]  

(5.1)

The supercharge is then defined by:

\[
Q_{susy} = C \left( iP + X - \sqrt{2}X^{-1} \right)
\]  

(5.2)

and the Hamiltonian is then:

\[
H = \frac{1}{2} \left( Q_{susy}Q_{susy}^\dagger + Q_{susy}^\dagger Q_{susy} \right)
\]  

(5.3)
We found that the L-BFGS-B optimizer again yielded the best results for the ground state energy. Using the VQE and 5 qubits (4 for the boson and 1 for the fermion) we obtain for the no noise simulation the convergence graph in figure 4 and the results in table 3 and for the noisy simulation the convergence graph in figure 5 and the results in table 4. Again we found that the L-BFGS-B optimizer again yielded the best results for the ground state energy. The results for the no noise simulation were again very accurate but were slightly less accurate than the one boson formulation. The noisy simulation of the one boson one fermion formulation was the least accurate of the quantum computations of superconformal quantum mechanics.

| SCQM one boson one fermion formulation | Ground State Energy |
|----------------------------------------|---------------------|
| Exact                                  | 0.0                 |
| Exact Discrete                         | 0.00190358          |
| VQE-State-vector (L-BFGS-B optimizer)  | 0.00190402          |

Table 3: Exact, Exact discrete, and VQE computation of the ground state energy of SCQM using the one boson one fermion formulation. The VQE result was obtained using the State-vector simulator with no noise and the Limited-memory BFGS Bound (L-BFGS-B) optimizer.
Figure 5: Convergence graph for the VQE computation of the ground state energy of SCQM using the one boson one fermion formulation. The VQE result was obtained using the QASM simulator including realistic noise and the Simultaneous Perturbation Stochastic Approximation (SPSA) optimizer.

6 Evolution of Hamiltonian and Kernel for SCQM on a quantum computer

The Kernel or Greens function for a quantum system tells how a system evolves from one state to another as a function of time. Like the energy spectrum it is a fundamental quantity to compute for any quantum system. It is given for position states by [27]:

\[
K(x, x'; T) = \langle x' | x, T \rangle = \langle x' | e^{-iHT} | x \rangle = \int_{x(0)=x}^{x(T)=x'} Dxe^{i \int_0^T L(x, \dot{x}) dt} \tag{6.1}
\]

Where \(H\) is the Hamiltonian and \(L\) is the Lagrangian. For Superconformal quantum mechanics (ignoring constant terms) this is given by:

\[
H = \frac{1}{2m} p^2 + \frac{m^2 \omega^2}{2m} x^2 + \frac{g^2 - g}{2m x^2}
\]

\[
L(x, \dot{x}) = \frac{m}{2} \dot{x}^2 - \frac{m^2 \omega^2}{2m} x^2 - \frac{g^2 - g}{2m x^2} \tag{6.2}
\]

Like the energy spectrum for Superconformal quantum mechanics the kernel or Greens function for superconformal quantum mechanics can be computed exactly. The result is
| SCQM one boson one fermion formulation | Ground State Energy |
|--------------------------------------|---------------------|
| Exact                                | 0.0                 |
| Exact Discrete                       | 0.00190358          |
| VQE-QASM-Simulator (SPSA optimizer)  | 0.29056190          |

Table 4: Exact, Exact discrete, and VQE computation of the ground state energy of SCQM using the one boson one fermion formulation. The VQE result was obtained using the QASM simulator including realistic noise and the Simultaneous Perturbation Stochastic Approximation (SPSA) optimizer.

given by \[28\]:

\[
K(x,x';T) = \frac{m\omega \sqrt{xx'}}{i\hbar \sin \omega T} \exp \left\{ \frac{i m \omega}{2\hbar} (x^2 + x'^2) \cot \omega T \right\} I_\gamma \left( \frac{m\omega xx'}{i \hbar \sin \omega T} \right)
\]  
(6.3)

with

\[
\gamma = \frac{1}{2} \sqrt{1 + \frac{8m(g^2 - g)}{2m\hbar^2}}
\]  
(6.4)

and where \( I \) is a modified Bessel function. The Evolution of Hamiltonian (EOH) quantum algorithm is based on the quantum walks quantum algorithm. The quantities computed are the transition probability from one position to the same or another position as a function of time which defines the Kernel or propagator \[29\]. We use the Qiskit EOH quantum algorithm to calculate the evolution of Hamiltonian. To construct the Hamiltonian we use the finite difference basis discussed in section 2 where the position and momentum matrices are represented by sparse matrices.

The Hamiltonian evolution is approximated using the Trotter-Suzuki decomposition. The simplest Trotter-Suzuki decomposition is given by

\[
e^{x(A+B)} = e^{xA} e^{xB} + O(x^2)
\]  
(6.5)

where \( x \) is a parameter and \( A \) and \( B \) are arbitrary operators with some commutation relation \([A, B] \neq 0\). To deal with higher order case, we need more parameters to do the corrections, with the generalized form of Trotter formula as

\[
e^{x(A+B)} = e^{p_1 x A} e^{p_2 x B} e^{p_3 x A} e^{p_4 x B} \cdots e^{p_M x B} + O(x^{m+1})
\]  
(6.6)

The set of parameters \( \{p_1, p_2, \cdots, p_M\} \) corresponding to order of \( x^{m+1} \). To calculate time evolution over the time slice as \( t/n \) of a Hamiltonian system with the kinetic energy \( K(p) \), we could use the Trotter decomposition approximates the time evolution with the operator

\[
e^{itH} \simeq \left( e^{\frac{i}{n}K} e^{\frac{i}{n}\nu} \right)^n
\]  
(6.7)
Implementations on IBM quantum computers

We use Qiskit to build the circuits for the EOH algorithm and simulate the time evolution. First the Hamiltonian matrix operator is converted to an expansion of Pauli terms that can read into the quantum computer. A larger number of time slices in each calculation would give a more accurate result, especially for a longer time interval. We used the Trotter-Suzuki expansion for all the calculations, with the expansion order equal to 3. We used the state vector simulator to run the EOH circuit. Each run of the circuit returns a complex vector as a result, which can be used to define the probability distribution. To get a series results, we used a for-loop in the code to return the results corresponding to different time intervals. In this way, we see how system is evolving step by step in a long time interval.

The results from the EOH quantum algorithm for Superconformal Quantum Mechanics are shown in figure 6 and for a time evolution from one position back to the same position or other positions. We were not able to take the evolution out far enough to exhibit the periodicity in $T$ that is present in the exact solution (6.3). Nevertheless the time evolution compares well with the exact time evolution for smaller time intervals that we plot in figure 7. The use of large matrices and more qubits together with smaller time slices should improve the representation of time evolution using the quantum computer.

![Figure 6](image.png)

Figure 6: (Left) Result of Evolution of Hamiltonian quantum algorithm for Supersymmetric Conformal Quantum Mechanics for $g = \sqrt{2}$ from one position back to the same position. (Right) Result of Evolution of Hamiltonian quantum algorithm for Supersymmetric Conformal Quantum Mechanics for $g = \sqrt{2}$ from one position back to the same position or another position.
Figure 7: Exact solution for the time evolution of the Hamiltonian of Supersymmetric Conformal Quantum Mechanics potential for $g = \sqrt{2}$ form one position back to the same position.

7 Bosonic Calogero-Moser-Sutherland Model on a Quantum Computer

The bosonic Calogero-Moser-Sutherland model is a generalization of conformal quantum mechanics to a system with $N$ bosons. It is defined by the Hamiltonian [4]:

$$H_B = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + V_B(x_1, x_2, \ldots, x_N)$$ (7.1)

with

$$V_B(x_1, x_2, \ldots, x_N) = \frac{\omega^2}{2N} \sum_{i<j} (x_i - x_j)^2 + g^2 \sum_{i<j} (x_i - x_j)^{-2}$$ (7.2)

The ground state energy can be determined exactly and is given by:

$$E_0 = \frac{1}{2} (N - 1) \left(1 + N\alpha\right) \omega$$ (7.3)

with

$$\alpha = \frac{1}{2} + \sqrt{\frac{1}{4} + g^2}$$ (7.4)

For the special case $N = 3$ the Hamiltonian becomes:

$$H = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + \frac{\omega^2}{6} \left((x_{12})^2 + (x_{23})^2 + (x_{31})^2\right) + g^2 \left((x_{12})^{-2} + (x_{23})^{-2} + (x_{31})^{-2}\right)$$ (7.5)

where $x_{ij} = x_i - x_j$. Here we take $N = 3$, $\omega = 1$ and $g = \sqrt{2}$ so that $\alpha = 2$ and the exact value $E_0 = 7$. To perform the quantum computation of the bosonic CMS model
we construct the position and momentum operator matrices. Here we choose the position basis and represent each boson by a $4 \times 4$ matrix. Defining the position

$$X_p = \begin{pmatrix}
-\frac{3}{2} \sqrt{\frac{\pi}{2}} & 0 & 0 & 0 \\
0 & -\frac{\sqrt{\pi}}{2} & 0 & 0 \\
0 & 0 & \frac{\sqrt{\pi}}{2} & 0 \\
0 & 0 & 0 & \frac{3}{2} \sqrt{\frac{\pi}{2}}
\end{pmatrix}$$

(7.6)

and the momentum using the Sylvester matrix $F$ discussed in section 2 as

$$P_p = F^\dagger X_p F$$

(7.7)

We can construct the position and momentum matrices using the Kronecker product through:

$$x_1 = X_p \otimes I_4 \otimes I_4$$

$$x_2 = I_4 \otimes X_p \otimes I_4$$

$$x_3 = I_4 \otimes I_4 \otimes X_p$$

$$p_1 = P_p \otimes I_4 \otimes I_4$$

$$p_2 = I_4 \otimes P_p \otimes I_4$$

$$p_3 = I_4 \otimes I_4 \otimes P_p$$

(7.8)

The total Hilbert space for the three bosons is then $4 \times 4 \times 4 = 64$ dimensional. We used the VQE algorithm to compute the ground state energy on the quantum computer using six qubits. For the bosonic CMS model we found that the L-BFGS-B optimizer yields the best results. The convergence graph for this optimizer is shown in figure 8. In table 5 we record the results of the quantum computation. The results were not as accurate as the single boson SCQM or one boson and one fermion SCQM computations. This may be because the variational wave functions in the three boson case do not overlap with the true ground state as strongly for the bosonic CMS as in the SCQM case. It would be interesting to examine the true ground state in more detail and explore different variational ansatz for the trial wave functions to see if one can improve the accuracy of the result for the bosonic CMS model.
Table 5: Exact, Exact discrete, and VQE computation of the ground state energy of bosonic CMS model for $g = \sqrt{2}$ using the position basis. The VQE result was obtained using the State-vector simulator with no noise and the Limited-memory BFGS Bound (L-BFGS-B) optimizer.

| Bosonic CMS Model | Ground State Energy |
|-------------------|---------------------|
| Exact             | 7                   |
| Exact Discrete    | 6.59198266          |
| VQE (L-BFGS-B)    | 7.12370445          |

Figure 8: Convergence graph for the VQE computation of the ground state energy of bosonic CMS model for $g = \sqrt{2}$ using the position basis. The VQE result was obtained using the State-vector simulator with no noise and the Limited-memory BFGS Bound (L-BFGS-B) optimizer.

8 Quantum computing for Supersymmetric Calogero-Moser model

The Supersymmetric Calogero-Moser-Sutherland model is a generalization of SCQM with $N$ bosons and $N$ fermions. It is defined by the Hamiltonian [4][5]:

$$H = \frac{1}{2} \sum_{i=1}^{N} \left[ p_i^2 + \omega^2 x_i^2 \right] + \sum_{1 \leq i < j \leq N} \frac{g(g - 1 + \theta_{ij} \theta_{ij}^\dagger)}{x_{ij}^2} - \frac{N}{2} [g(N-1) - \omega]$$ (8.1)

where

$$x_{ij} = x_i - x_j$$

$$\theta_{ij} = \theta_i - \theta_j$$ (8.2)
Defining the supercharge $Q$ so that:

$$H = \frac{1}{2} \left\{ Q, Q^\dagger \right\}$$  \hspace{1cm} (8.3)

where

$$Q = \sum_{j=1}^{N} \theta_j \left( ip_j + \omega x_j - g \sum_{1 \leq k \leq N, k \neq j} \frac{1}{x_{jk}} \right)$$  \hspace{1cm} (8.4)

and

$$Q = \sum_{j=1}^{N} \theta_j \left( -ip_j + \omega x_j - g \sum_{1 \leq k \leq N, k \neq j} \frac{1}{x_{jk}} \right)$$  \hspace{1cm} (8.5)

The ground state satisfies $Q \psi_0 = 0$ and is given by:

$$\psi_0(x) = \left( \prod_{1 \leq j < k \leq N} (x_{jk})^g \right)^{\frac{N}{2}} \prod_{i=1}^{N} e^{-\omega x_i^2}$$  \hspace{1cm} (8.6)

To perform the quantum computation we again choose the position basis where each position and momentum matrix is represented using 2 qubits so as $4 \times 4$ matrices. In addition we add three fermion matrices which use 1 qubit each so are represented by $2 \times 2$ matrices. The entire Hilbert space is $4 \times 4 \times 4 \times 2 \times 2 = 512$ dimensional so uses 9 qubits total. This is the largest quantum computation studied in this paper. The position and fermion operators are represented as:

$$x_1 = X_p \otimes I_4 \otimes I_4 \otimes I_8$$
$$x_2 = I_4 \otimes X_p \otimes I_4 \otimes I_8$$
$$x_3 = I_4 \otimes I_4 \otimes X_p \otimes I_8$$

$$\theta_1 = I_{64} \otimes \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \otimes I_2 \otimes I_2$$

$$\theta_2 = I_{64} \otimes \sigma_3 \otimes \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \otimes I_2$$

$$\theta_3 = I_{64} \otimes \sigma_3 \otimes \sigma_3 \otimes \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)$$  \hspace{1cm} (8.7)

From these one can construct the Hamiltonian from (8.1). Using this Hamiltonian for the VQE we find that the Constrained Optimization By Linear Approximation (COBYLA) optimizer yielded the best results which we list table 6. The reason the exact discrete vacuum energy is not zero is due to to the approximation of infinite dimensional position and momentum matrices by finite matrices. This discretization error can be reduced by going to higher number of qubits.
Table 6: Exact, Exact discrete, and VQE computation of the ground state energy of
supersymmetric CMS model for $g = \sqrt{2}$ using the position basis. The VQE result was
obtained using the State-vector simulator with no noise and the Constrained Optimization
By Linear Approximation (COBYLA) optimizer.

| Susy CMS Model | Ground State Energy |
|----------------|---------------------|
| Exact          | 0.0                 |
| Exact Discrete | -0.04716555         |
| VQE (COBYLA)   | 0.75686596          |

9 Conclusions

In this paper we have investigated the simplest superconformal field theory in 0+1 di-
men-sion on a quantum computer which is superconformal quantum mechanics (SCQM). We
studied the ground state of the mass deformed SCQM on a quantum computer using the
Variational Quantum Eigensolver (VQE) quantum algorithm using a one boson and a one
boson - one fermion Hilbert space and compared the results. We studied the Feynman
path integral for SCQM using the EOH algorithm on the quantum computer using the
Trotter-Suzuki approximation and compared with the exact result. Finally we considered
an N boson and N fermion version of SCQM given be the Supersymmetric Calogero-Moser
model. We compared the ground state of theory obtained using the VQE computation with
the exact solution known for the theory. The accuracy of the one boson formulation of
SCQM was the best, followed by the one boson - one fermion formulation, bosonic Calogero-
Moser-Sutherland model and the Supersymmetric Calogero-Moser Sutherland model was
the least accurate. Improved variational ansatz should improve the accuracy of the VQE
computations for the multiparticle simulations as it does for quantum chemistry simula-
tions of molecules. For the time evolution finer divisions of the time interval and higher
order expansions used in the Trotter-Suzuki approximation should improve the accuracy,
but at the cost of longer and more complex quantum circuits used in the computations.

We emphasize that superconformal field theories, including SCQM which can be con-
sidered a superconformal field theory in 0+1 dimensions, are thought to be dual to a
Universe with negative cosmological constant or Anti-de Sitter space (AdS) due to the
AdS/CFT correspondence [30]. Thus if we are simulating a superconformal field theory we
can, also in the same computation, simulating a Universe [31] [32], albeit one with negative
cosmological constant. Furthermore the reduced computational complexity of simulating
the superconformal field theory instead of the full quantum gravity may cause one to re-
think the computational complexity of gravitational simulations [33] [34] and how one can
simulate the Universe on a quantum computer [35] [36] [37].
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