On Preemptive Scheduling of Unrelated Machines Using Linear Programming

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Abstract

We consider a basic problem of preemptive scheduling of \( n \) non-simultaneously released jobs on a group of \( m \) unrelated parallel machines so as to minimize maximum job completion time, the makespan. In the scheduling literature, the problem is commonly considered to be solvable in polynomial time by linear programming (LP) techniques proposed in Lawler and Labetoulle [4]. The authors in [4] give a LP formulation of the version with simultaneously released jobs and show how an optimal solution to this LP can be used to construct an optimal schedule to the latter problem. They also give a linear programming formulation of a related problem for non-simultaneously released jobs with the objective to minimize the maximum job lateness, and show how an optimal solution to this LP problem can be used to construct an optimal solution for non-simultaneously released jobs with the objective to minimize the maximum job lateness. As the current study shows, for non-simultaneously released jobs, unlikely, there exist a linear program such that a schedule with the minimum makespan can be constructed based on an optimal LP solution. We also prove that, in case no splitting of the same job on a machine is allowed (i.e., job part assigned to a machine is to be processed without an interruption on that machine), the problem is NP-hard. As a side result, we obtain that, whenever job splitting is not allowed, given an optimal LP solution, it is NP-hard to find an optimal schedule with the minimum makespan that agrees with that LP solution. As another side result, we obtain that it is NP-hard to find an optimal schedule with at most \( m - 1 \) preemptions if jobs are released non-simultaneously. We also present two positive results. First, we construct an optimal schedule in a low degree polynomial time in case an optimal solution to a modified LP formulation that we propose already possesses “enough” amount of integer components so that it yields no more than \( m \) preempted jobs parts. We also propose a stronger mixed integer linear program formulation. Finally, we extend the schedule construction procedure based on an optimal LP solution from Lawler and Labetoulle [4] for non-simultaneously released jobs.
The extended procedure, among all feasible schedules that agree with any feasible (not necessarily optimal) LP solution, generates one with the minimum makespan. Such procedure is helpful, in particular, because, as we show, there may exist no optimal schedule that agrees with an optimal LP solution if jobs are non-simultaneously released.

**Keywords:** Scheduling, unrelated machines, release time, linear programming, time complexity

## 1 Introduction

Scheduling unrelated machines to minimize the maximum job completion time is a well-known optimization problem. In a group of unrelated machines, a machine \( i \) has no universal speed characteristic (machine speed is job dependent), in contrast to a group of uniform machines, where each machine is characterized by a universal speed that extends to a whole set of jobs. A group of machines with the same speed is commonly referred to as a group of identical machines (the speed of every machine is the same for every job). In the scheduling problem that we consider here, commonly abbreviated as \( R|\text{r}_j;\text{pmtn}|\text{C}_{\text{max}} \), there are \( n \) jobs to be performed by \( m \) parallel unrelated machines. Job \( j \) becomes available at its (integer) release time \( r_j \) and it requires an (integer) processing time \( p_{ij} \) on machine \( i \), \( i = 1, \ldots, m \) (for the sake of simplicity, we will refer to jobs and machines by their corresponding indexes). A job, being processed by a machine, can be interrupted and resumed later on the same or on any other machine. In a feasible schedule a machine can process at most one job at a time and a job can be processed by at most one machine at a time, every job \( j \) is assigned to a machine no earlier than at time \( r_j \) and it is completely processed, i.e., \( \sum_{i=1}^{m} t_{ij} / p_{ij} = 1 \), where \( t_{ij} \) is the total amount of time that machine \( i \) spends on job \( j \) in that schedule. The objective is to find an optimal schedule, a feasible one in which the maximum job (machine) completion time \( C_{\text{max}} \) is the minimum possible.

In the scheduling literature, this classical scheduling problem is commonly considered to be solvable in polynomial time by linear programming technique from Lawler and Labetoulle [4] (we refer the reader to publicly accessible sources [9] and [10]). Lawler and Labetoulle [4] proposed a linear programming formulation for the version of the above described problem without job release times, and have adopted an open shop scheduling method of Gonzalez and Sahni [2] for the construction of an optimal schedule for that version of the problem based on an optimal LP solution. A feasible LP solution determines which part of each job is to be processed by each machine, i.e., it distributes job parts to machines without
specifying particular starting time of job part(s) on the corresponding machine(s). Hence, a scheduling stage identifying start time of every job part on the corresponding machine is required to transform such distribution to a feasible schedule.

It is commonly accepted that scheduling problems with non-simultaneously released jobs are considerably more difficult to solve than the corresponding versions with simultaneously released jobs (it is a typical scenario that a polynomially solvable scheduling problem with simultaneously released jobs becomes NP-hard if jobs are released non-simultaneously at arbitrary times). A distribution to linear program considered in Lawler and Labetoulle \[4\] “implicitly assumes” that job parts can be assigned to the machines at any time moment. This does not apply in case the jobs are non-simultaneously released. Such a “lack of information” essentially complicates the use an optimal fractional LP solution for the construction of an optimal preemptive schedule for non-simultaneously released jobs. We suggest two alternative linear programs for problem $R|r_j; pmtn|C_{\text{max}}$ that are more “flexible” since they take somehow into account job release times. The second LP, is however, a mixed integer linear program. Even the latter LP is not strong enough in the sense that an optimal schedule not necessarily agrees with an optimal distribution generated for that LP. In fact, there may exist no optimal schedule that agrees with any optimal distribution to any of the linear programs that we consider here. Moreover, as we argue, unlikely, there exists a “strong enough” linear program such that an optimal solution to that linear program can be used for the construction of an optimal schedule to problem $R|r_j; pmtn|C_{\text{max}}$ with non-simultaneously released jobs. If however, an optimal solution to a modified LP formulation that we propose, possesses “enough” amount of integer components (so that it yields no more than $m$ preempted jobs parts), then an optimal schedule can be constructed in a low degree polynomial time based on this optimal LP solution, as we show in Section 4. We also extend the schedule construction procedure based on an optimal LP solution of Lawler and Labetoulle \[4\] for non-simultaneously released jobs. The extended procedure, among all feasible schedules that agree with any feasible (not necessarily optimal) LP solution, generates one with the minimum makespan. The extended procedure is helpful, in particular, because any optimal schedule may only agree with a non-optimal LP solution if jobs are non-simultaneously released. As this study shows, for non-simultaneously released jobs, unlikely, there exist a linear program such that a schedule with the minimum makespan can be constructed based on an optimal solution to that linear program.

We also give NP-hardness results. In particular, if no splitting of the same job on a machine is allowed (i.e., job part assigned to a machine is to be processed without an interruption on that machine), then we show that the problem with non-simultaneously released jobs is NP-hard. As a side result, we obtain that, whenever job splitting is not allowed, given an optimal LP solution, it is NP-
hard to find an optimal schedule with the minimum makespan that agrees with that LP solution. As another side result, we obtain that it is NP-hard to find an optimal preemptive schedule with at most \( m - 1 \) preemptions. This result somehow extends an earlier known fact that scheduling identical machines with at most \( m - 2 \) preemptions, even for simultaneously released jobs, is NP-hard [8].

We complete this section by a very brief look at a few earlier known related results. Non-preemptive scheduling on identical machines of simultaneously released jobs \( P_{\text{||C_{max}}} \) is already NP-hard. This is in contrast to the preemptive case, even on a group of uniform machines \( Q_{|\text{pmtn}|\text{C_{max}}} \), for which a low-degree polynomial time algorithm is known Gonzalez and Sahni [1]. The setting with non-simultaneously released jobs \( Q_{|r_j,\text{pmtn}|\text{C_{max}}} \) remains polynomially solvable Labetoulle et al. [3], as well as the version with unrelated machines but simultaneously released jobs \( R_{|\text{pmtn}|\text{C_{max}}} \) Lawler and Labetoulle [4]. As already noted, the authors in [4] give a linear programming formulation of the problem and show how an optimal solution to this LP can be used to construct an optimal solution to the latter problem. Lawler and Labetoulle [4] also give a linear programming formulation of a related problem \( R_{|\text{pmtn}|L_{\text{max}}} \) where the objective is to minimize the maximum job lateness \( L_{\text{max}} \) (the lateness \( L_j \) of job \( j \) in schedule \( S \) is the difference between the completion time of that job in that schedule and its due-date \( d_j \), a “desirable” completion time for that job, and \( L_{\text{max}} \) is the maximum job lateness). This formulation allows to use their schedule construction procedure to generate an optimal schedule to that problem.

## 2 Alternative linear programs

In this section we present linear programs that have been used for scheduling unrelated machines. The following linear program \( \text{LP1}(\text{C_{max}}) \) was successfully used for an approximate solution of the non-preemptive version of the problem \( R_{||\text{C_{max}}} \) with simultaneously released jobs first by Potts [6] and later in [7]:

Minimize \( C_{\text{max}} \)

Subject to

\[
\sum_{j=1}^{n} x_{ij} p_{ij} \leq C_{\text{max}}, \quad i = 1, \ldots, m
\]

\[
\sum_{i=1}^{m} x_{ij} = 1, \quad j = 1, \ldots, n
\]

\[
x_{ij} \geq 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.
\]

In this linear program entry \( x_{ij} = t_{ij}/p_{ij} \) represents the part of job \( j \) to be processed by machine \( i \), for \( j = 1, \ldots, n \) and \( i = 1, \ldots, m \). These entries define the
corresponding distribution of job parts on machines. Unlike a schedule which is a mapping that assigns to each job specific time interval(s) on one or more machines, a distribution, instead of these time intervals, defines only their lengths on the corresponding machines. Because of real assignment variables, a distribution may split a job in different parts and assign these parts to different machines. Note that a distribution involves no start times and only assigns job parts to machines (hence, there is an infinite number of feasible schedules respecting a given distribution). In particular, a solution to a linear program is a distribution that explicitly indicates which fraction of each job is assigned to each machine. We refer the reader to [7] for related formal definitions, concepts and properties.

A distribution to linear program \( LP_1(C_{\text{max}}) \) has a nice property that it possesses a large amount of integer (zero) entries so that it yields at most \( m - 1 \) preempted jobs. Then such distribution can be rounded to (an integer) feasible approximate non-preemptive solution to problem \( R||C_{\text{max}} \), as suggested by Potts [6], where a complete enumeration of at most \( m - 1 \) preempted jobs on \( m \) machines is carried out. This results in a 2-approximation solution in time, polynomial in \( n \) and exponential in \( m \). Using a modified linear program combined with a binary search, a rounding that guarantees a 2-approximation solution in polynomial (in both \( n \) and \( m \)) time was achieved in Lenstra et al. [5]. A new method of rounding an optimal distribution to linear program \( LP_1(C_{\text{max}}) \) proposed in [7] yielded an improvement of the bound 2 to \( 2 - \frac{1}{m} \). This latter bound is the best possible that can be obtained by rounding a distribution to a feasible non-preemptive schedule.

In an optimal distribution to linear program \( LP_1(C_{\text{max}}) \), the total length of the parts of a job assigned to the machines can be longer than the minimized magnitude \( C_{\text{max}} \). Hence, such distribution is inappropriate for the preemptive case \( R|\text{pmtn}|C_{\text{max}} \). Linear program \( LP_2(C_{\text{max}}) \) which bounds the length of the assigned parts of each job was studied in Lawler and Labetoulle [4]. This linear program requires an additional set of \( n \) restrictions [2] (one additional restriction for each job):

Minimize \( C_{\text{max}} \)

Subject to

\[
\sum_{j=1}^{n} x_{ij} p_{ij} \leq C_{\text{max}}, \quad i = 1, \ldots, m
\]

\[
\sum_{i=1}^{m} x_{ij} p_{ij} \leq C_{\text{max}}, \quad j = 1, \ldots, n
\]

\[
\sum_{i=1}^{m} x_{ij} = 1, \quad j = 1, \ldots, n
\]

\[
x_{ij} \geq 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.
\]
Because of these additional $n$ restrictions \cite{2}, the total number of basic (non-zero) variables is no more bounded by $m - 1$ (in case an optimal distribution to linear program $LP_2(C_{\text{max}})$ still yields no more than $m - 1$ preemptions, it can easily be transformed to an optimal preemptive schedule with at most $m - 1$ preemptions, see Section 4). If an optimal distribution to linear program $LP_2(C_{\text{max}})$ yields more than $m$ preempted job parts, an optimal schedule to problem $R|\text{pmtn}|C_{\text{max}}$ can still be constructed in polynomial time. Lawler and Labetoulle \cite{4} adopted open shop scheduling technique from Gonzalez and Sahni \cite{2} for constructing an optimal feasible schedule from an optimal distribution to linear program $LP_2(C_{\text{max}})$ (note that an open shop instance can be already seen as a distribution). We describe this method in more detail in Section 4.

Suppose we have a feasible schedule for an instance of problem $R|\text{pmtn}|C_{\text{max}}$ respecting an optimal distribution $\{x_{ij}\}$ to linear program $LP_2(C_{\text{max}})$ with the makespan

$$C_{\text{max}} = \max \left\{ \max_{j=1}^n x_{ij}p_{ij}, \max_{i=1}^m x_{ij}p_{ij} \right\} \tag{3}$$

Then this schedule is clearly optimal. Such an optimal schedule is constructed in Lawler and Labetoulle \cite{4}.

During the construction of the schedule in \cite{4}, job parts can be assigned to machines at any time moment. Such an approach is not particularly useful if the jobs are not simultaneously released. Such “lack of required restrictions” questions the usefulness of an optimal fractional solution (an optimal distribution) to a linear program for the construction of an optimal preemptive schedule for problem $R|r_j;\text{pmtn}|C_{\text{max}}$. In particular, bound \cite{3} is no more attainable, i.e., for a given instance of problem $R|r_j;\text{pmtn}|C_{\text{max}}$, there may exist no feasible schedule with makespan $C_{\text{max}}$ respecting a given optimal distribution to linear program $LP_2(C_{\text{max}})$ (e.g., consider an optimal distribution in which the sum of the job parts assigned to some machine is $C_{\text{max}}$ but no job assigned to that machine is released at time 0). Furthermore, no feasible schedule respecting that optimal distribution might be optimal. This assertion is true for linear program $LP_2(C_{\text{max}})$, and is also true for more enhanced linear programs that we give later on.
3 NP-hardness results

Let \( R|r_j, pmtn − nosplit|C_{\text{max}} \) denote the version of problem \( R|r_j, pmtn|C_{\text{max}} \) in which no job part assigned to a machine can be split on that machine (it is to be processed continuously on that machine). No splitting is a reasonable assumption in a number of applications where it is undesirable to interrupt a currently running job on a machine in favor of another job (such split would cause unreasonable amount of machine reset and setup times).

**Theorem 1** \( R|r_j, pmtn − nosplit|C_{\text{max}} \) is NP-hard.

Proof. We show that the decision version of \( R|r_j, pmtn − nosplit|C_{\text{max}} \) is NP-complete using the reduction from an NP-complete PARTITION problem. Consider an arbitrary instance of this problem with \( k \) items \( \{z_1, \ldots, z_k\} \) and with \( M = \sum_{i=1}^{k} z_i \). Let \( P_1 \) and \( P_2 \) be a partition of the \( k \) items with \( \sum_{l \in P_1} z_l = \sum_{l \in P_2} z_l = M/2 \), a solution to PARTITION \( (P_1 \cup P_2 = \{z_1, \ldots, z_k\}, P_1 \cap P_2 = \emptyset) \).

Our scheduling instance consists of \( 4 + k \) jobs on three machines, where \( Z_1, \ldots, Z_k \) are partition jobs. All partition jobs are released at time 3. The processing times of these jobs are such that \( p_{2Z_j} = 2z_j/M \) and \( p_{iZ_j} = \infty \), for \( j = 1, \ldots, k \) and \( i = 1, 3 \) (note that the total length of the partition jobs is 2). The processing times of the remaining four jobs are defined as follows. \( p_{11} = p_{21} = p_{31} = 6, p_{13} = 2, p_{24} = 3 \) and \( p_{32} = 5 \). All the unspecified processing times are infinity (large enough numbers). Job 3 is released at time 4 and the remaining jobs are released at time 0 (except the partition ones which are released at time 3).

As it is easy to see, the processing time 6 of job 1 is a lower bound on the optimal schedule makespan. In a schedule with this makespan (see Fig. 1):

On machine 1, job \( J^3 \) cannot be started earlier than at its release time 4 and can be completed at time 6, hence job 1 starts at its release time 0 and competes at time 4. On machine 3, job \( J^2 \) is to occupy 5 time units. Hence, only one time unit is left where job \( J^1 \) can be processed on that machine. Therefore, one unit of time of job \( J^1 \) is to be processed on machine 2. All partition jobs are to be processed also on machine 2. None of the partition jobs can start earlier than at time 3 whereas job \( J^4 \) is to occupy 3 time units on machine 2. Since machine 1 is running job \( J^1 \) in interval \([0, 4]\), job \( J^4 \) is needs to occupy the first 3 time units on machine 2 and is to be followed by the partition jobs. Since the assigned to machine 3 part of job \( J^2 \) cannot be interrupted on that machine, the remaining unprocessed unit time of job \( J^1 \) can only be executed within the interval \([4, 5]\) on machine 2. Hence, exactly the intervals \([3, 4]\) and \([5, 6]\) are left for the partition jobs.
It should now be apparent that there exist a schedule of length 6 if and only if the partition instance has a “yes” answer, i.e., there exists a partition of set \( \{z_1, \ldots, z_k\} \) into sets \( P_1 \) and \( P_2 \) with equal length 1. In other words, an optimal schedule with makespan 6 provides a solution to PARTITION, and vice-versa, if PARTITION has a solution then the above optimal schedule can be constructed in polynomial time.

**Corollary 1** Given an instance of problem \( R|r_j,pmtn|C_{max} \), it is NP-hard to find an optimal solution with at most \( m-1 \) preemptions.

Proof. By Theorem 1, \( R|r_j,pmtn - nosplit|C_{max} \) is NP-hard. No split on any machine yields at most \( m \) preempted job parts (one job part on every machine), hence at most \( m-1 \) preemptions.

**Corollary 2** Given an optimal distribution, it is NP-hard to find a schedule for problem \( R|r_j,pmtn - nosplit|C_{max} \) with the minimum makespan respecting that distribution.

Proof. From the proof of Theorem 1 the PARTITION instance is to be solved if job \( J^2 \) is not allowed to be split on machine 3. The corollary follows as the distribution respected by the schedule of Fig. 1 is optimal.

4 Scheduling little-preemptive distributions

In this section we show how an optimal schedule can be constructed in case an optimal distribution to linear program LP2(\( C_{max} \)) yields at most \( m \) preempted job
parts. We first consider the case of simultaneously released jobs. Then we intro-
duce a new liner program and consider the case when jobs are not simultaneously
released.

4.1 Simultaneously released jobs

Suppose an optimal distribution to linear program LP2($C_{\text{max}}$) yields no more than
$m - 1$ preemptions, i.e., there are at most $m$ (preempted) job parts, hence at most
one such part on any machine. We can convert such distribution to an optimal
preemptive schedule for problem $R|\text{pmtn}|C_{\text{max}}$ with at most $2m - 4$ preemptions as
follows. Sort the preempted jobs in non-increasing order of their total processing
times (the total processing time of job $j$ in distribution $\{x_{ij}\}$ is $\sum_{i=1}^{m} x_{ij} p_{ij}$). Take the
first job $j$ in the list and consider its parts in the ascending order of the machine
indexes to which these parts are assigned. Let $i_1 < \ldots < i_k$ be these indexes
(ones of the corresponding machines). Schedule the first part of job $j$ at time 0
on machine $i_1$, schedule the second part at the completion time of the first part
on machine $i_2$, and so on, schedule the last $k$th part at the completion time of the
preceding $(k - 1)$th part on machine $i_k$. Due to inequalities (2), the completion
time of the last scheduled part of job $j$ cannot be larger than $C_{\text{max}}$. Now take the
second job in the list and schedule its parts similarly, and so on. Since there is
at most one preempted job part on any machine, no conflicts between preempted
parts of different jobs will occur. The remaining entire job parts are scheduled
in the remained idle time slots from time 0 in any order without creating an idle
time in between the jobs. In case an overlapping of an entire part with an earlier
included preempted job part occurs, the entire job part is preempted and resumed
at the completion time of the latter preempted job part. Due to inequalities (1),
the makespan of the resultant feasible schedule will not surpass $C_{\text{max}}$. Note that no
additional preemption will occur on any machine on which a preempted job part
is scheduled the first or the last on that machine. In the worst case, there is only
one preempted job distributed among all the machines. Then $m - 3$ additional
preemptions will occur, hence the $2m - 4$ is an upper bound on the number of
preemptions.

4.2 Non-simultaneously released jobs

First, let us modify linear program LP2($C_{\text{max}}$) by replacing inequalities (2) with
the following set of inequalities that take into account the release time of each job:
\[ r_j + \sum_{i=1}^{m} x_{ij} p_{ij} \leq C_{\text{max}}, \quad j = 1, \ldots, n. \] (4)

The resultant new linear program LP3(\(C_{\text{max}}\)) properly reflects the desired restriction for each job.

Suppose now an optimal distribution to linear program LP3(\(C_{\text{max}}\)) yields no more than \(m - 1\) preemptions. The above described schedule construction procedure for simultaneously released jobs can easily be extended for non-simultaneously released jobs for problem \(R|r_j; \text{pmtn}|C_{\text{max}}\). We again consider job parts in ascending order of the corresponding machine indexes. Now, instead of starting each first (according to this order) part of the next job from the list at time 0, it is scheduled at the release time of that job. The entire (non-preempted) jobs are scheduled in non-decreasing order of their release times on each machine. Due to the modified inequalities (4) (and the fact that there is at most one preempted job part on any machine), again, any of the preempted jobs will complete no later than at time \(C_{\text{max}}\).

5 “Weaknesses” of linear programs

For a given feasible schedule to an instance of the scheduling problem \(R|r_j; \text{pmtn}|C_{\text{max}}\) with makespan \(C_{\text{max}}\), let \(\{x_{ij}\}\) be the distribution that respects this schedule. The values \(C_{\text{max}}\) and \(\{x_{ij}\}\) form a feasible solution to linear programs LP3(\(C_{\text{max}}\)) and LP2(\(C_{\text{max}}\)). A less obvious question is, given a distribution to linear program LP3(\(C_{\text{max}}\)), whether there is an optimal schedule respecting that distribution with the makespan \(C_{\text{max}}\). If this assertion is true, then a feasible schedule respecting an optimal distribution will be optimal.

Thus a feasible schedule respecting an optimal distribution is trivially optimal if its makespan is \(C_{\text{max}}\). Lawler and Labetoulle [4] constructed schedules respecting optimal distributions to linear program LP2(\(C_{\text{max}}\)) with makespan is \(C_{\text{max}}\) for simultaneously released jobs. In contrary to the case with simultaneously released jobs, an optimal schedule respecting an optimal distribution to linear program LP3(\(C_{\text{max}}\)) (and linear program LP2(\(C_{\text{max}}\)) not necessarily attains makespan \(C_{\text{max}}\) if jobs are non-simultaneously released. In particular, inequalities (1) no more reflect actual restrictions imposed by job release times on the completion time of each machine, since it may not be possible to schedule a job assigned to a machine at the earliest idle-time moment on that machine. In particular, restrictions (1) are no more necessarily satisfied in an optimal schedule where the completion time of a machine may be larger than \(C_{\text{max}}\) (recall an earlier mentioned simple scenario...
where no job assigned to a machine is released at time 0). Even restrictions (4) may not be satisfied in an optimal schedule.

We illustrate these points on small problem instances in the following examples. For none of these instances the procedure from Section 4.2 can guarantee the construction of an optimal schedule with makespan $C_{\text{max}}$. Our examples illustrate that in an optimal schedule constructed from an optimal distribution to linear program LP3($C_{\text{max}}$), neither restrictions (1) nor restrictions (4) might be satisfied (the completion time of at least one job in that schedule can be larger than $C_{\text{max}}$). More importantly, a bit later we will see that an optimal schedule not necessarily respects an optimal distribution.

**Example 1.** Let us consider a problem instance where an optimal distribution assigns just two different job parts to the same machine (hence the schedule construction procedure from Section 4.2 already cannot be applied). We have five jobs $2, \ldots, 6$ on four machines $1, \ldots, 4$ such that:

$r_3 = 3, r_2 = 2, r_4 = 8, r_5 = 0, r_6 = 5,$ $p_{13} = p_{33} = 15, p_{22} = p_{32} = p_{42} = 16, \text{ and } p_{14} = p_{26} = p_{45} = 13.$

The processing times of these jobs on the remaining machines are infinities (large enough numbers). Note that in any feasible schedule, jobs 4, 6 and 5 can only be processed by machines 1, 2 and 4, respectively. Job 2 is to be distributed among machines 2, 3 and 4, and job 3 is to be distributed on machines 1 and 3.

There are a few optimal distributions to linear program LP2($C_{\text{max}}$) with $C_{\text{max}} = 18$. We consider two of them which assign (preempted) parts of jobs 3 and 2 to machine 3. In distribution 1, the processing times are as follows:

$t_{13} = 5, t_{14} = 13, t_{22} = 4, t_{26} = 13, t_{32} = 8, t_{33} = 10 \text{ and } t_{45} = 13, t_{42} = 4.$

Note that this is not an optimal distribution to linear program LP3($C_{\text{max}}$) due to inequalities (4), where $C_{\text{max}} = r_4 + p_{14} = 8 + 13 = 21$ is attained for job 4. A non-feasible schedule respecting distribution 1 is depicted in Fig. 2a, and an optimal schedule respecting the same distribution with makespan 24 is depicted in Figure 2b.

An optimal distribution 2 is identical to distribution 1 except that $t_{22} = 5, t_{32} = 6 \text{ and } t_{42} = 5$ (see Fig. 2c). An optimal schedule respecting this distribution has the makespan 23, see Fig. 2d.

As can see, distribution 2 possesses better properties, so that an optimal schedule respecting that distribution has a smaller makespan than an optimal schedule respecting distribution 1. At the same time, none of these schedules is (globally) optimal (see below). Moreover, one can easily verify that there exists no optimal schedule respecting any optimal distribution to linear program LP2($C_{\text{max}}$). (From here on we refer to a schedule with minimum makespan respecting a given opti-
(a) A non-feasible schedule respecting distribution 1

(b) A feasible schedule respecting distribution 1

(c) A non-feasible schedule respecting distribution 2

(d) A feasible schedule respecting distribution 2

(e) A globally optimal schedule that respects a non-optimal distribution 3

(f) A globally optimal schedule that respects a non-optimal distribution 4

Figure 2

mal distribution as an optimal schedule respecting that distribution; such schedule, may not be (globally) optimal, i.e., there may exist no optimal schedule respecting this distribution. Moreover, there may exist no optimal schedule respecting any optimal distribution.)
Now we consider a modification of the above considered distributions, a non-optimal distribution 3 with $C_{\text{max}} = 20$, in which $t_{22} = 5$, $t_{32} = 4$, $t_{42} = 7$. A schedule with the makespan 21 respecting this distribution is depicted in Fig. 2e. Distribution 3 is not optimal for linear program LP2($C_{\text{max}}$) and it is not feasible to linear program LP3($C_{\text{max}}$) (e.g., $r_4 + t_{14} = 21 > 20$, see inequalities [4]). It is easy to see that the schedule of Fig. 2e is (globally) optimal.

In Fig. 2f another optimal schedule with the makespan 21 is depicted. This schedule respects distribution 4 with $C_{\text{max}} = 21$, which is not optimal for linear program LP2($C_{\text{max}}$) but it is (feasible and) optimal for linear program LP3($C_{\text{max}}$) (due to inequalities [4], $C_{\text{max}} = 8 + 13 = 21$ is attained for job 4, and 21 is also the load of machine 4). However, this schedule is not feasible for an instance of $R|r_j, pmtn - nosplit|L_{\text{max}}$. Distribution 4 differs from the above optimal distributions on machines 3 and 4. Job processing times are distributed as follows: $t_{13} = 5$, $t_{14} = 13$, $t_{22} = 5$, $t_{26} = 13$, $t_{32} = 3$, $t_{33} = 10$ and $t_{45} = 13$, $t_{42} = 8$. 

As we saw from the above example, optimal schedules respecting different optimal distributions may have different makespan. “Guessing” an optimal distribution and also a “suitable” linear program is an important and also difficult task. Furthermore, as we argue in the next section, unlikely, there exists a universal “suitable” linear program for the studied scheduling problem, such that an optimal schedule respecting an optimal distribution to that linear program can be guaranteed to be (globally) optimal. Below we give a smaller problem instance illustrating similar points.

As another example, consider a smaller problem instance with four jobs on two machines. Jobs 1 and 4 are released at times 3 and 1, respectively, and jobs 2 and 3 are released behind these jobs at time, say 5, i.e., $r_1 = 3$, $r_2 = 1$ and $r_3 = r_4 = 5$.  

![Figure 3: A non-feasible schedule respecting an optimal distribution (a) and an optimal schedule (b)](image)

**Example 2**. As another example, consider a smaller problem instance with four jobs on two machines. Jobs 1 and 4 are released at times 3 and 1, respectively, and jobs 2 and 3 are released behind these jobs at time, say 5, i.e., $r_1 = 3$, $r_2 = 1$ and $r_3 = r_4 = 5$.  

13
Job processing times are as follows:
\[ p_{11} = 9, \quad p_{12} = \infty, \quad p_{14} = \infty, \quad p_{24} = 10, \quad p_{13} = p_{23} = 3 \quad \text{and} \quad p_{12} = p_{22} = 2. \]
An optimal distribution with \( C_{\text{max}} = 12 \) defines the processing times \( t_{11} = 9, \ t_{13} = 3 \) and \( t_{24} = 10, \ t_{22} = 2. \) This distribution is optimal for both linear programs \( \text{LP2}(C_{\text{max}}) \) and \( \text{LP3}(C_{\text{max}}) \). A non-feasible schedule with makespan 12 respecting this distribution is depicted in Figure 3a, whereas an optimal feasible schedule with makespan 14 is depicted in Figure 3b. The latter schedule respects another distribution with processing times \( t_{11} = 9, \ t_{12} = 2 \) and \( t_{24} = 10, \ t_{23} = 3 \) (in which the roles of jobs 2 and 3 are interchanged) and it is not optimal for linear programs \( \text{LP2}(C_{\text{max}}) \) and \( \text{LP3}(C_{\text{max}}) \). The latter distribution is however optimal for linear program \( \text{MILP}(C_{\text{max}}) \) that we introduce in the next section.

6 Another linear program

Linear program \( \text{LP3}(C_{\text{max}}) \), although it properly reflects the desired restriction for each job, it does not reflect actual restrictions imposed by job release times on machine start times due to the nature of inequalities \([1]\) dealing with a mere sum of processing times of jobs assigned to each machine. This causes potential conflicts while scheduling machines. In particular, there may exist no optimal schedule to problem \( R|r_j;\text{pmtn}|C_{\text{max}} \) respecting an optimal distribution to linear program \( \text{LP3}(C_{\text{max}}) \). An appropriate modification of restrictions \([1]\) would require a kind of “prediction” of an actual completion time of each machine given the job parts assigned to this and other machines possessing parts of the same jobs.

Such a prediction however seems to be unrealistic since it would actually require the outcome of scheduling process on each machine. Nevertheless, we may stall make some assumptions on the scheduling strategy on each machine. In particular, we easily observe that no avoidable gap is created on any machine in an optimal schedule. It is easy to see that, among the job parts assigned to a machine, the first included one corresponds to an earliest released job in an optimal schedule.

In general, whenever an idle time is unavoidable on a machine, an earliest released job is scheduled immediately after that idle time on that machine. However, in a linear program with these kind of restrictions the use of Boolean variables seems to be unavoidable.

Define \( nm \) 0-1 variables, \( z_{ij} \) being 1 if a (non-empty) part of job \( j \) is assigned to machine \( i \), and 0 otherwise. Let, further, \( J(> r) \) denote the set of jobs having the release time no smaller than \( r \). Then we can rewrite \( m \) constraints \([1]\) into \( nm \) constraints as follows.
\[ z_{ij} r_j + \sum_{l \in J(> r_j)} x_{il} p_{il} \leq C_{\text{max}}, \quad j = 1, \ldots, n, \quad i = 1, \ldots, m \]  

(5)

By incorporating these restrictions instead of constraints (1) and introducing additional \(nm\) 0-1 constraints

\[ z_{ij} \in \{0, 1\} \]

into linear program \(LP3(C_{\text{max}})\), we obtain a mixed 0-1 integer linear program \(\text{MILP}(C_{\text{max}})\).

Example 1 (continuation). Returning to the problem instance of Example 1, we can easily observe that, for an optimal distribution to the new linear program \(\text{MILP}(C_{\text{max}})\).
MILP\((C_{\text{max}})\), \(C_{\text{max}} = 21\). In particular, distribution 4 from the previous section (Fig. 2f) is optimal also for linear program MILP\((C_{\text{max}})\). There exist other optimal distributions to linear program MILP\((C_{\text{max}})\). In one of them, 
\[ t_{13} = 3, \ t_{14} = 13, \ t_{22} = 4, \ t_{26} = 13, \ t_{32} = 5, \ t_{33} = 12 \text{ and } t_{45} = 13, \ t_{42} = 7, \]
see Fig. 4a representing a non-feasible schedule that respects this distribution. A feasible schedule with makespan 23 respecting the same distribution is depicted in Fig. 4b. In a slight modification of the latter optimal distribution, job 3 is redistributed on machines 1 and 3 so that its processing time on machine 1 is increased by 2 and its processing time on machine 3 is decreased by the same amount. This results in a globally optimal schedule with makespan 21 depicted in Fig. 4c. In another optimal distribution \(t_{22}\) is reduced to 3 and \(t_{42}\) is increased to 8. Another globally optimal schedule with makespan 21 respecting the latter optimal distribution is depicted in Fig 4d.

As we can see, even for a very small sized problem instance, a number of different optimal distributions to the new linear program MILP\((C_{\text{max}})\) exist, some of them leading to an optimal schedule and some not. We again need to “guess” a “correct” optimal distribution among all possible optimal ones. Moreover, as we show below, not necessarily there exists a globally optimal schedule that respects an optimal distribution to the new linear program MILP\((C_{\text{max}})\), as it was the case for linear programs LP2\((C_{\text{max}})\) and LP3\((C_{\text{max}})\).

**Example 1a.** Consider a slight modification of the problem instance of Example 1 in which all job parameters remain the same except that \(r_4 = 0\). This reduces the makespan of an optimal distribution to linear program MILP\((C_{\text{max}})\) from \(C_{\text{max}} = 21\) to \(C_{\text{max}} = 20\). Now job 4 can be partitioned on machine 1 in two parts, hence \(t_{13}\) can be increased to 7. An optimal schedule with makespan 20 respecting this optimal distribution is depicted in Fig. 5a.

**Example 1b.** For the second modification of Example 1, let \(r_4 = 6\). For an optimal distribution to this modified instance \(C_{\text{max}} = 20\), which is the completion time of machine 4 (note that the completion time on machine 1, compared to that in the schedule of Fig. 4b, is reduced by the length of the gap [6, 8]). A non-feasible schedule respecting an optimal distribution is depicted in Fig. 5b. An optimal feasible schedule with makespan 23 respecting the same distribution is depicted in Fig. 5c. The latter schedule is not globally optimal. An optimal schedule with makespan 21 is illustrated in Fig 5d; this schedule respects a distribution with the same makespan \(C_{\text{max}} = 21\). Observe that the latter distribution is not optimal for linear program MILP\((C_{\text{max}})\).
Figure 5: (a): An optimal schedule for the first modified instance; (b): a non-feasible schedule respecting an optimal distribution for the second modified instance; (c): an optimal schedule respecting the latter distribution; (d): an optimal schedule, that respects a non-optimal distribution

7 The scheduling method of Lawler and Labetoulle

In the following section we describe the method of Lawler and Labetoulle [4] for the construction of an optimal schedule from an optimal distribution to linear program LP2($C_{\text{max}}$) for problem $R|\text{pmtn}|C_{\text{max}}$, and then we show how it can be extended to construct an optimal schedule respecting any distribution to problem $R|r_j;\text{pmtn}|C_{\text{max}}$.

Lawler and Labetoulle [4] adopted open shop scheduling technique of Gonzalez and Sahni [2] for the construction of an optimal feasible schedule with makespan $C_{\text{max}}$ respecting an optimal distribution to program LP2($C_{\text{max}}$) for simultaneously released jobs (note that an open shop instance can already be seen as a distribution). Recall that an optimal distribution $\{x_{ij}\}$ defines an $m \times n$ non-negative processing time matrix $T = \{t_{ij}\} = x_{ij}p_{ij}$. The so-called decrementing sets are iteratively formed from iteration 1 by selecting one entry in each tight row and in each tight column: a row/column is tight iff the sum of the entries in that row/column is exactly $C_{\text{max}}$. The initial processing time matrix $T = T^1$, defined by an optimal distribution, is iteratively transformed into the $m \times n$ 0-matrix. At each iteration $h > 0$, the matrix $T^{h-1}$ of the previous iteration is updated according to the formed decrementing set $D^h$ at iteration $h$; in particular, each entry in matrix $T^{h-1}$ corresponding to an element of set $D^h$ is decreased by a suitably chosen (small enough) number $\delta^h$ resulting in the updated matrix $T^h$ of iteration $h$. $\delta^h$ is chosen in such a way that the new matrix $T^h$ possesses similar properties
as its predecessor matrix $T^{h-1}$: The sum of the elements in each row and column of the updated matrix is no larger than $C_{\text{max}} - \sum_{i=1}^{h} \delta_i$. The elements in decrementing set $D^h$ define a partial schedule of iteration $h$ of length $\delta^h$ according to the selected portions of processing times. $\sigma^h$ is the partial schedule generated by iteration $h$, which is obtained by a mere merging of the partial schedules of each of the iterations $1, \ldots, h$, so that the makespan of partial schedule $\sigma^h$ is $\tau_{h+1} = \sum_{i=1}^{h} \delta_i$. We will refer to $\tau_{h+1}$ as the scheduling time of iteration $h + 1$, which is the time moment at which the partial schedule of iteration $h + 1$ starts. The whole procedure halts at iteration $h$ such that matrix $T^h$ is a 0-matrix. At that iteration, $\sigma^h$ is a complete feasible schedule. The optimality of this complete schedule immediately follows from the fact that its makespan is $C_{\text{max}}$ (a lower bound on the optimum schedule makespan).

In the initial matrix $T$ and in each following matrix $T^h$ there exists a decrementing set. This follows from a known Birkhoff and von-Neumann theorem stating that every doubly stochastic matrix is a linear combination of permutation matrices. Indeed, by completing matrix $T$ with additional slack rows and columns, an $m + n \times m + n$ matrix with the entries of each its row and column summing up to $(C_{\text{max}})$ can be obtained. Dividing all the entries of this matrix by $(C_{\text{max}})$, a doubly stochastic matrix is obtained. Then a permutation matrix from the theorem defines a decrementing set.

![Figure 6: A non-feasible schedule respecting an optimal distribution (a), a feasible non-optimal schedule (b) and two optimal schedules (c) and (d) respecting the same distribution.](image)

**Example 4.** Let us illustrate the scheduling method of Lawler and Labetoulle on a small instance of problem $R|\text{pmtn}|C_{\text{max}}$. It is basically the instance of Example 1 adopted for the case when all jobs are simultaneously released. To maintain $C_{\text{max}} = 18$, we increase the processing time of jobs 2 and 3 to 18 and decrease the
Table 1: Flowchart of the procedure with processing time matrices and the corresponding decrementing sets (marked with circles)

|           | 2 | 3 | 4 | 5 | 6 |
|-----------|---|---|---|---|---|
| **1**     | 0 | **13** | 0 | 0 |   |
| **2**     | 5 | 0 | 0 | 0 | **43** |
| **3**     | **5** | 13 | 0 | 0 | 0 |
| **4**     | 8 | 0 | 0 | **10** | 0 |

|           | 2 | 3 | 4 | 5 | 6 |
|-----------|---|---|---|---|---|
| **1**     | 0 | 0 | **43** | 0 | 0 |
| **2**     | 5 | 0 | 0 | 0 | **8** |
| **3**     | 0 | **43** | 0 | 0 | 0 |
| **4**     | 0 | 0 | 0 | 5 | **0** |

|           | 2 | 3 | 4 | 5 | 6 |
|-----------|---|---|---|---|---|
| **1**     | 0 | 0 | **5** | 0 | 0 |
| **2**     | 5 | 0 | 0 | 0 | 0 |
| **3**     | 0 | 0 | 0 | **5** | 0 |
| **4**     | 0 | 0 | 0 | 0 | 0 |

processing time of job 5 to 10. It can be easily verified that in an optimal distribution to linear program LP2($C_{\text{max}}$) with $C_{\text{max}} = 18$, we have $t_{13} = 5$, $t_{14} = 13$, $t_{22} = 5$, $t_{26} = 13$, $t_{32} = 5$, $t_{33} = 13$, and $t_{42} = 8$, $t_{45} = 10$. A non-feasible schedule respecting this distribution is depicted in Fig. 6a. This schedule can straightforwardly be converted to a feasible schedule of Fig. 6b with makespan 23 by imposing a gap [0, 5) on machine 3. This schedule is not optimal. An optimal one respecting the above optimal distribution is depicted in Fig. 6c.

The scheduling method of Lawler and Labetoulle [4] will generate an optimal schedule as follows. The initial matrix $T = T^1$ of job processing times defined by the optimal distribution is represented in the first quarter of Table 1, where the entries corresponding to the elements of the decrementing set $D^1$ and the following decrementing sets are circled. $\delta^1$ can be chosen to be equal to 5. The updated matrix $T^2$ and the entries corresponding to the elements of the decrementing set $D^2$ are shown in the second quarter of Table 1. The first partial schedule with length 5 can be seen as the initial (first) part of the schedule of Fig. 6d corresponding to interval [0, 5). The computations in the following iterations $h = 2, 3$ with $\delta^2 = 8$, $\delta^3 = 5$ are reflected in the next quarters of Table 1 and in the upper parts in the schedule of Fig. 6d. Although schedule $\delta^3$ of Fig. 6d with makespan $C_{\text{max}} = 5 + 8 + 5 = 18$ is somewhat similar to that of Fig. 6c, it splits job 5 on machine 4 (hence it is not a feasible solution to problem $R|r_j, pmtu - nosplit|L_{\text{max}}$).

7.1 Scheduling non-simultaneously released jobs

One naturally wishes to extend the schedule construction technique of Lawler and Labetoulle [4] for non-simultaneously released jobs. As we saw, an optimal schedule respecting an optimal distribution to any of linear programs that we considered is not necessarily optimal. Without giving a formal proof that there may exist no other “proper” linear program for the problem, we argued in Section 6.1 that it is unlikely that such linear program exists. As we also observed, there may exist no globally optimal schedule respecting an optimal distribution, i.e., any
such schedule may respect a non-optimal distribution.

Due to these observations, now we would like to find a schedule with the minimum makespan among all schedules respecting a given (not necessarily optimal) distribution, i.e., find an optimal schedule respecting that distribution. The schedule construction technique of Lawler and Labetoulle [4] can be generalized by maintaining an extra information on which job parts can be scheduled at every scheduling time \( \tau^h \). Note that such care is to be taken only on the first scheduled part of each job. For that, we introduce an additional row 0 in processing time matrices.

Initially, in the extended matrix \( T^1 \), the entry \( t^0_{0j} \) in column \( j \) of row 0 is \( r_j \); iteratively, \( t^h_{0j} := \max\{0, r_j - \tau^h\} \). During the scheduling process, we impose an additional restriction that forbids scheduling of job \( j \) at time \( \tau^h \) if \( t^h_{0j} > 0 \) (independently of whether the corresponding entries are from a tight row or tight column).

As a result, not necessarily a decrementing set will contain one entry from a tight row or a tight column. In particular, set \( D^h \) will contain no entry from a (tight) row \( i \) if among yet unscheduled job parts assigned to machine \( i \) no job is yet released by time \( \tau^{h+1} \) (i.e., for any positive entry in row \( i \), the corresponding entry in row 0 is positive); likewise, the decrementing set of iteration \( h \) will contain no entry from a (tight) column \( j \) if job \( j \) is not yet released by time \( \tau^{h+1} \).

We will refer to a row \( i \) from matrix \( T^h \) as ready at iteration \( h \) if it contains a positive entry \( t^h_{ij} > 0 \) such that \( t^h_{0j} = 0 \); we will refer to such an entry as valid for row \( i \) at iteration \( h \).

Let \( t^0_{0j} = 0 \), and let \( M_h(j) \) be the set of machines such that the entry \( t^h_{ij} \) is valid for \( i \in M_h(j) \). Then the ready rows from set \( M_h(j) \) are said to be conflicting by job \( j \) at iteration \( h \). Note that two or more rows may be conflicting by two or more different jobs.

A decrementing set \( D^h \) of iteration \( h \) contains one entry from a ready row such that no two entries from the same column are included into that set (since the same job cannot be scheduled on different machines at a time); whenever a ready row \( i \) contains an element from a tight column \( j \), the corresponding part of job \( j \) can be selected if entry \( t_{ij} \) is valid at iteration \( h \).

It might not be possible to select an entry from every ready row at a given iteration \( h \) since two or more rows may be conflicting by the same jobs in such a way that all entries cannot be selected. For example, if we have two ready rows with valid entries only in, say, column \( j \), then only one of these entries can be selected; likewise, if there are three ready rows with valid entries only in, say, columns \( j \) and \( j' \), then only two of these entries, one from column \( j \) and the other
from column \( j' \), can be selected in decrementing set \( D^h \). We break ties by selecting the corresponding entry from row \( i \) with the maximum current load, i.e., with the maximum \( \sum_{l=1}^{n} t^h_{il} \). If such a row contains several valid entries, then the entry from a column \( j \) with the maximum remaining processing time, i.e., with the maximum \( \sum_{l=1}^{m} t^h_{lj} \) is selected. Further ties are broken by selecting an entry from column \( l \) such that \( t^h_{il} \in D^{h-1} \) (i.e., part of job \( l \) was included in the decrementing set of the previous iteration). The latter tie breaking rule avoids unnecessary preemption of an already running job. (At every iteration \( h \), the rows can be considered in non-increasing order of their loads and the corresponding valid entries can be selected from a column \( j \) with the maximum remaining processing time.)

Let \( r \) be the minimum positive element in row 0 at iteration \( h \) in matrix \( T^h \), i.e., \( r = \min_{j} t_{0j}^h \). The jump \( \delta^h \) at iteration \( h \) is now defined as the minimum between \( \delta^h \) as defined in Lawler and Labetoulle \([4]\) and \( r \).

Since there may exist no more than \( n \) district job release times, the extended method yields an additional term \( n \) bounding the number of iterations and hence has the same polynomial time complexity as the method of Lawler and Labetoulle \([4]\). It is not difficult to see that \( \tau^{h*} \) is an optimal schedule makespan, where \( h^* \) is the last iteration in the procedure. First note that the extended procedure will work as the basic one from the earliest scheduling time \( \tau^h \) such that \( t^h_{0j} = 0 \), for all \( j = 1, \ldots, n \), since the corresponding decrementing sets will contains \( m \) elements with exactly one element from each tight column (by the construction, all entries in column \( j \) will be valid at any iteration \( h \) with \( \tau^h \geq r_j \)). In general, the extended procedure will work as the basic one if at every iteration there are \( m \) non-conflicting ready rows with an entry in a tight column so that the entries from each tight column are included into the corresponding decrementing sets. Otherwise, in case there is an entry in a tight column that was not selected at an iteration \( h \), the corresponding entry in row 0 should have been positive, i.e., the corresponding job is not released by the current scheduling time \( \tau^{h-1} \). In other words, if an entry from a tight column \( j \) was not included in decrementing set \( D^h \) then there is no valid entry from that column at iteration \( h \), equivalently, \( r_j > \tau^{h-1} \). No feasible schedule may include such a job at time \( \tau^{h-1} \). Likewise, if there are only \( l < m \) non-conflicting ready rows at iteration \( h \), then there are only \( l \) jobs that can feasibly be scheduled at time \( \tau^{h-1} \). Our tie breaking rule will include an entry corresponding to each of these \( l \) jobs in decrementing set \( D^h \), in total \( l \) entries from \( l \) rows corresponding to \( l \) most loaded machines will be included. Finally, note that among conflicting rows, ties can easily be broken by selecting valid entries from the rows corresponding to most loaded machines.

**Example 3.** Let us first illustrate the extended scheduling method for the scheduling instance from Theorem \([\ref{thm:main}]\). Recall an optimal distribution respected
by the schedule of Fig. 1. Let us first assume that we have a solution to the corresponding partition instance, i.e., we have sets $P_1$ and $P_2$. Then, for the sake of simplicity, we can represent all partition jobs in one column marked as $P$, see Table 2 below. We update the entries in this column according to the made selections. All partition jobs from sets $P_1$ and $P_2$ are scheduled consequently without creating any machine idle time in two different (aggregated) iterations 2 and 4. In the first quarter of Table 2 an initial extended processing time matrix $T^0$ is presented. For this instance, all three rows are ready from the beginning of iteration 1. The valid entries corresponding to the elements in the decrementing set $D_1$ are circled (note that the corresponding entries in row 0 are 0). Row 1 and column 1 are tight, hence element $t_{11}$ is circled. From rows 2 and 3 elements $t_{24}$ and $t_{32}$ are similarly selected. We have $\delta^1 = 3$ and $\tau^1 = 3$. The part of the schedule of Fig. 1 corresponding to the interval $[0,3)$ is the partial schedule $\sigma^1$ of iteration 1. Matrix $T^1$ is similarly represented in the second quarter of Table 2 (note that the entries in row 0 are updated correspondingly). Now, $\delta^2 = 1$ and hence $\tau^2 = 4$. The part of the schedule of Fig. 1 corresponding to the interval $[0,4)$ is partial schedule $\sigma^2$. Similarly, in the following iterations 3 and 4, $\delta^3 = 1$, $\tau^3 = 5$, and $\delta^4 = 1$ and $\tau^4 = 6$.

Schedule $\sigma^4$ is the resultant complete schedule of Fig. 1 which extends through the interval $[0,6)$. Since column 1 is tight, the corresponding parts of job $J^1$ are included in the decrementing sets in iterations 1, 2, 3 and 4 (until this job is completely scheduled). Note that $C_{\text{max}} = 6$ is attained by both, job $J^1$ and machine 2 in the optimal distribution of Fig. 1 (i.e., the total length of job $J^1$ and the load of machine 2, see inequalities 4 and 5). Hence, schedule $\sigma^4$ is optimal.

Given an optimal distribution of Fig. 1, the extended scheduling procedure will create an optimal schedule without the knowledge of a solution to the PARTITION instance. Then the jumps $\delta^i$ will be determined by the lengths of the selected partition jobs (without our aggregated presentation, the number of iterations would depend on $k$, in our case, it would be $k + 2$). Note that the procedure may split a partition job or/and job $J^2$, which is not allowed for setting $R|r_j,pmtn − nsplit|L_{\text{max}}$. For instance, in this example, job $J^2$ was not split on machine 3 the corresponding parts of that job being selected in consecutive iterations 1, 2 and 3 (in fact, we solved an instance of problem $R|r_j,pmtn − nsplit|L_{\text{max}}$ given a solution to the PARTITION instance, see Corollary 2). Finally, note that, depending on the made selections of the decrementing sets, the procedure may create different optimal schedules respecting the same distribution.

Example 1 (continuation 2). Next, we illustrate the extended schedule construction procedure on the problem instance of our basic Example 1. We first construct an optimal schedule of Fig. 2d respecting optimal distribution 2 (recall
that this schedule is not globally optimal). Table 3 represents the flowchart of the procedure in its 10 iterations (we omit the table with all 0 entries of the last iteration 11). Initially in matrix $T^1$ all entries in row 0 are positive except that of column (job) 5. In particular, only row 4 is ready. Since 2 is the minimum entry in row 0, $\delta^1 = 2$, hence $\tau^1 = 2$. The partial schedule of iteration 1 corresponds to the segment $[0,2)$ of the schedule in Fig. 7. In the next matrix $T^2$ there arises one additional ready row 2 with the entry 5 in column 2 (recall that the minimum entry in column 0 of matrix $T^2$ was precisely in column 2). Now the minimum entry in row 0 is 3, hence $\delta^2 = \min\{5,1\} = 1$ and $\tau^1 = 2+1 = 3$, see again Fig 2d.

The entries in the decrementing set $D^2$ correspond to columns 2 and 5 (both, jobs 2 and 5 are released by time 2). Again, in the processing time matrix $T^3$ there arises one additional ready row 1 corresponding to job 3. The decrementing set $D^3$ contains now 3 jobs so that already 3 machines, 1, 2 and 4 become busy. At the next iteration 4 job 6 becomes available on machine 2, but the last unscheduled part of job 2 is scheduled on that machine by our tie breaking rule. The decrementing set of iteration 5 already contains 4 elements, hence all four machines become busy; $\delta^5 = 1$ and $\tau^5 = 2 + 1 + 2 + 2 + 1 = 8$, see Fig. 7. At iteration 6 all entries in row 0 in matrix $T^6$ are already 0, hence all job parts become released. The procedure continues in the same fashion until it constructs a complete feasible schedule of Fig. 2d respecting an optimal distribution 2 to linear program MILP($C_{\text{max}}$).

Now we apply our schedule construction procedure to a non-optimal distri-
Table 4: caption of table 4.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2 | 3 | 8 | 0 | 5 | 0 | 1 | 6 | 0 | 3 | 0 | 1 | 6 | 0 | 3 | 0 | 5 | 0 | 2 | 0 | 1 | 0 | 0 |
| 1 | 0 | 5 | 13 | 0 | 0 | 0 | 5 | 13 | 0 | 0 | 0 | 5 | 13 | 0 | 0 | 0 | 5 | 13 | 0 | 0 | 0 | 5 |
| 2 | 5 | 0 | 0 | 0 | 13 | 2 | 0 | 0 | 0 | 13 | 2 | 0 | 0 | 0 | 13 | 2 | 0 | 0 | 0 | 0 | 13 |
| 3 | 4 | 10 | 0 | 0 | 0 | 4 | 10 | 0 | 0 | 0 | 4 | 10 | 0 | 0 | 0 | 4 | 10 | 0 | 0 | 0 | 0 |
| 4 | 7 | 0 | 0 | 5 | 0 | 7 | 0 | 0 | 10 | 0 | 7 | 0 | 0 | 10 | 0 | 7 | 0 | 0 | 0 | 13 |

Table 4: caption of table 4.

Table 4: caption of table 4.

bution to obtain a globally optimal schedule respecting that distribution. We illustrate this using a non-optimal distribution 3 from Example 1 (see Fig. 2e). We represent the flowchart of the procedure in Table 4. Initially at iteration 1 we have one ready row 4 with a single valid entry 13 (corresponding to job 2) which is included into the decrementing set $D_1; \delta_1 = t_{02}^4 = 2$ and $\tau_1 = 2$. At iteration 2 row 2 also gets ready with a single valid entry 5 corresponding to job 2; $D_2 = \{t_{22}^2, t_{45}^2\} = \{5, 11\}, \delta_2 = t_{03}^2 = 1$ and $\tau_2 = 2 + 1 = 3$. At iteration 3 one additional row 1 with a single valid entry 5 corresponding to job 2 gets ready; $D_3 = \{t_{13}^3, t_{22}^3, t_{45}^3\} = \{5, 4, 10\}$; now $\delta_3 = t_{06}^3 = 2$, but since the remaining processing time of job 2 is 13, the same as that of job 6, by our tie breaking rule we include $t_{22}^3$ (and not $t_{36}^3$) in set $D_3$ (without creating unnecessary preemption of job 2). We respectively skip one iteration by setting $\delta_3 = 4$ and hence letting $\tau_3 = 2 + 1 + 4 = 7$. At iteration 4 all 4 rows are ready and the corresponding four entries marked in the fourth quarter of Table 4 are included in set $D_4$. The next four iterations are similarly reflected in Table 4 (we omitted gain the 0-matrix of iteration 9). The resultant complete optimal schedule with makespan $\tau_8 = 2 + 1 + 4 + 1 + 3 + 2 + 7 + 1 = 21$ coincides with that of Fig. 2e.

8 Concluding remarks

Optimal distributions to linear programs that we considered here, including linear program MILP($C_{\text{max}}$), do not “completely capture” all required features for the construction of an optimal schedule to problem $R|r_j; \text{pmtn}|C_{\text{max}}$. Some optimal distributions may posses better features than others for the creation of an optimal schedule, but it is difficult to predict what kind of an optimal distribution an
LP solver will deliver and what kind of an optimal distribution is preferable, in general. Moreover, a non-optimal distribution may suit better a given problem instance than any optimal one so that an optimal schedule respecting that non-optimal distribution may be globally optimal, whereas no globally optimal schedule respecting an optimal distribution may exist (Section 6).

The extended schedule construction procedure from Section 7.1 may be applied to any (not necessarily optimal) distribution to obtain an optimal schedule respecting that distribution. Such a schedule may also be globally optimal (e.g., the schedule of Fig. 4d). The method of Lawler and Labetoulle [4] for the construction of an optimal schedule relies on the fact that the makespan of a (globally) optimal schedule to linear program LP2($C_{\text{max}}$) equals to the corresponding $C_{\text{max}}$ for problem $R|\text{pmtn}|C_{\text{max}}$. As we saw, similar property does not hold for problem $R|r_j; \text{pmtn}|C_{\text{max}}$, as the makespan of a globally optimal schedule may be larger than the corresponding $C_{\text{max}}$.

A linear program, “more intelligent” than the linear programs studied here, would “correctly” estimate the starting time of the first scheduled job on each machine. However, this kind of estimation looks unrealistic without actually carrying out the scheduling of assigned job parts. Hence unlikely, there exists a linear programming such that an optimal schedule respecting an optimal distribution to that linear program can be guaranteed to be optimal. Because of this, we suggested an alternative schedule construction procedure that delivers an optimal schedule to problem $R|r_j, \text{pmtn}|L_{\text{max}}$ respecting any distribution. Such distribution can of course be obtained by a linear program, but it might be possible to create it using some alternative way. In contrast with the studied here scheduling problem $R|r_j, \text{pmtn}|L_{\text{max}}$, for the setting $R|\text{pmtn}|L_{\text{max}}$ without job release times, an optimal schedule respecting an optimal distribution to linear program LP2($C_{\text{max}}$) is guaranteed to be optimal. In this sense, the latter problem possesses more accessible structural properties than the former one.

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References

[1] Gonzalez T. and S. Sahni, “Preemptive scheduling of uniform processor systems”, Journal of the ACM 25, 92-101, 1978.

[2] Gonzalez T. and S. Sahni, “Open Shop Scheduling to Minimize Finish time”, Journal of the ACM 23, 665-679 (1976)

[3] Labetoulle, J.; Lawler, E.L.; Lenstra, J.K.; Rinnooy Kan, A.H.G. “Preemptive scheduling of uniform machines subject to release dates”, in Pulleyblank, H.R. (ed.) Progress in Combinatorial Optimization, New York, Academic Press, 245–261, 1984.

[4] Lawler, E.L.; Labetoulle, J.: On preemptive scheduling of unrelated parallel processors by linear programming, Journal of the Association of Computing Machinery 1978, 25 (4), 612–619.

[5] J.K. Lenstra, D.B. Shmoys and E. Tardos. ”Approximation algorithms for scheduling unrelated parallel machines” Math. Programming, 46, 259-271, 1990

[6] C.N. Potts. “Analysis of a linear programming heuristic for scheduling unrelated parallel machines” Discrete Appl. Math., 10, 155-164, 1985

[7] Shchepin E. and N. Vakhania. “An optimal rounding gives a better approximation for scheduling unrelated parallel machines”, Operations Research Letters 33, p.127-133 (2005)

[8] E. Shchepin and N. Vakhania. “On the geometry, preemptions and complexity of multiprocessor and open shop scheduling”. Annals of Operations Research 159, p.183-213, 2008.

[9] http://www2.informatik.uni-osnabrueck.de/knust/class/ (Complexity results for scheduling problems by Peter Brucker and Sigrid Knust: a classification of optimal scheduling problems by what is known on their runtime complexity.)

[10] http://schedulingzoo.lip6.fr/ (Scheduling Zoo by Christoph Durr, Sigrid Knust, Damien Prot, Oscar C. Vasquez: an online tool for searching an optimal scheduling problem using the notation.)