Hydrodynamics of the Generalized N-urn Ehrenfest Model

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Abstract
In this paper we are concerned with a generalized N-urn Ehrenfest model, where balls perform independent random walks between N boxes uniformly laid on [0, 1]. After a proper scaling of the transition rates function of the aforesaid random walk, we derive the hydrodynamic limit of the model, i.e., the law of large numbers which the empirical measure of the model follows, under an assumption where the initial number of balls in each box independently follows a Poisson distribution. We show that the empirical measure of the model converges weakly to a deterministic measure with density driven by an integral equation. Furthermore, we derive non-equilibrium fluctuation of the model, i.e, the central limit theorem from the above hydrodynamic limit. We show that the non-equilibrium fluctuation of the model is driven by a time-inhomogeneous generalized O-U process on the dual of C[0, 1]. At last, we prove a large deviation principle from the hydrodynamic limit under an assumption where the transition rates function from [0, 1] × [0, 1] to [0, +∞) of the aforesaid random walk is a product of two marginal functions from [0, 1] to [0, +∞).

Keywords Hydrodynamic limit · N-urn Ehrenfest model · Non-equilibrium fluctuation · Large deviation

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1 Introduction
In this paper, we are concerned with the generalized N-urn Ehrenfest model. The model is a continuous-time Markov process defined as follows. Assume that there are N boxes symbolled as 1, 2, . . . , N. At t = 0, there are some balls in every box according to some probability distribution. Let \( \lambda \in C^{1,1}([0, 1] \times [0, 1]) \) be a positive function on [0, 1]×[0, 1] with continuous partial derivatives with respect to both coordinates. For each 1 \( \leq i \leq N \), let \( \lambda_N(i) = \sum_{j=1}^{N} \lambda(\frac{i}{N}, \frac{j}{N}) \). Each ball in box \( i \) is chosen independently at rate \( \frac{1}{N} \lambda_N(i) \). When
a ball \( A \) in box \( i \) is chosen, then for each box \( j \), \( A \) is put into \( j \) with probability \( \frac{\lambda(i, j)}{\lambda_N(i)} \). Note that \( A \) may be put into the box where it comes from.

The above process can be defined equivalently according to its generator. Let \( \mathbb{X} = \{0, 1, 2, \ldots\}^N \) and each \( x \in \mathbb{X} \) be written as \( x = (x(1), x(2), \ldots, x(N)) \), then there is a unique Markov process \( \{X_t^N\}_{t \geq 0} \) on \( \mathbb{X} \) with generator \( \mathcal{L}_N \) given by

\[
\mathcal{L}_N f(x) = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{x(i)\lambda(i, j)}{\lambda_N(i)} (f(x^{i,j}) - f(x))
\]

for any bounded \( f \) on \( \mathbb{X} \) and \( x \in \mathbb{X} \), where \( x^{i,j} \in \mathbb{X} \) satisfies \( x^{i,j} = x \) when \( i = j \) while

\[
x^{i,j}(l) = \begin{cases} 
  x(l), & \text{if } l \neq i, j, \\
  x(i) - 1, & \text{if } l = i, \\
  x(j) + 1, & \text{if } l = j
\end{cases}
\]

when \( i \neq j \). According to the definition of \( \mathcal{L}_N \), for the generalized \( N \)-urn Ehrenfest model, the number of balls in the \( i \)th box at moment \( t \) is given by \( X_t^N(i) \).

When \( \lambda \equiv 1 \), the model reduces to the standard \( N \)-Urn Ehrenfest model, where each ball is put into each box at rate \( \frac{1}{N} \).

There is a long history for the research of the classic Ehrenfest model (see [4, 21, 23, 25] and so on), where \( N = 2 \), since this 2-urn model describes the exchange of gaseous molecules between two containers. The case where \( N = 3 \) is investigated in [5], where an expected hitting time is given via electric networks. The main result in [5] is extended in [6] and [32], where analogous results of the expected hitting times are computed for the cases where \( N \geq 4 \). Furthermore, the distributions of hitting times for cases where \( N \geq 4 \) are given in [6].

In this paper, we are concerned with hydrodynamics of the generalized \( N \)-urn Ehrenfest model, i.e., limit theorems of \( \frac{1}{N} \sum_{i=1}^{N} X_t^N(i) \delta_a(dx) \) as \( N \to +\infty \), where \( \delta_a(dx) \) is Dirac measure concentrated on \( a \). The theory of hydrodynamics describes the phenomenon that the microscopic density field of the stochastic model is dominated macroscopically by a PDE after proper space-time scaling. There is a long history for the investigation of hydrodynamics of models such as exclusion processes (see Chapter 4 of [18]), zero-range processes (see [8]), voter models (see [26]) and so on. For a detailed survey of the theory of hydrodynamics, see [18].

Some macroscopic behaviors of the generalized \( N \)-urn Ehrenfest model can be derived from the results about hydrodynamics, which is our main motivation for the investigation of this topic. For instance, the hydrodynamic limit illustrates thermodynamic irreversibility of the model in some special cases. In detail, in some cases the hydrodynamic limit process has a convergence state \( \mu_\infty \) as \( t \to +\infty \), which implies that the density field of the process stays in a small neighborhood of \( \mu_\infty \) for a long time when \( N \) is large and hence \( \mu_\infty \) can be considered as an ‘absorbing state’ of our model. Of course the generalized \( N \)-urn Ehrenfest model has no absorbing state in strict sense. However, the hydrodynamic limit shows that in realistic observation it takes too long to wait for the process to escape from \( \mu_\infty \). This phenomenon is called as ‘thermodynamic irreversibility’ and \( \mu_\infty \) is called the metastable state of the process. A concrete example with a metastable state is given in Appendix A.3.

Our model can be considered as a special case of the zero-range process (see Chapter 5 of [18]), which belongs to a larger family of stochastic processes called misanthrope processes (see [7]). For previous investigations of hydrodynamics of misanthrope processes and zero-range processes, see [1–3, 10, 11, 29] as examples. The main difference between
the investigation in this paper with those in the above literature is that processes in the above literature are nearest-neighbour models while our model is a long-range model such that a particle can jump to any position directly. As a result, hydrodynamics of processes in above literature are driven by partial differential equations such as heat equations after a diffusive scaling, i.e., the process being speed up by $N^2$, while the hydrodynamic of our model given in this paper is driven by an integral equation without speeding up the process. For mathematical details, see Section 2.

For an interacting particle system with a non-stationary initial distribution, central limit theorem from the hydrodynamic of the model is called ‘non-equilibrium fluctuation’, which is a popular and important topic in recent years. Non-equilibrium fluctuations have already been widely discussed for models such as exclusion processes, voter models, reaction-diffusion models, binary contact path processes and so on. A common phenomenon of these models is that the non-equilibrium fluctuation is driven by some kind of generalized Ornstein-Uhlenbeck process on tempered distributions defined in [15]. For mathematical details, see [9, 17, 26, 35]. Inspired by above literature, we study non-equilibrium fluctuations of $N$-urn Ehrenfest models as a second step of our investigation. We will show that the non-equilibrium fluctuation of our model is driven by a time-inhomogeneous generalized O-U process on the dual of $C([0, 1])$. For a precise statement of our result, see Section 2.

The study of large deviations from hydrodynamics of interacting particle systems dates back to 1980s, when large deviations of simple exclusion processes are discussed in [19]. Proofs of main results in [19] given by Kipnis, Olla and Varadhan provide a later routine strategy for the research of large deviations from hydrodynamics and motivate many follow-up works such as [12, 13, 16] and so on. In this paper, inspired by [19], we prove a large deviation principle of our $N$-urn Ehrenfest model under an assumption that the transition rates function $\lambda(x, y)$ is a product of two marginal one-dimensional functions. For mathematical details, see Section 2.

2 Main Results

In this section we give our main results. For later use, we first introduce some notations and definitions. Throughout this paper, we let $T > 0$ be a given moment. We use $\mathcal{S}$ to denote the dual of $C[0, 1]$ with the topology such that $\mathcal{A}_n \to \mathcal{A}$ in $\mathcal{S}$ if and only if

$$\lim_{n \to +\infty} \mathcal{A}_n(f) = \mathcal{A}(f)$$

for any $f \in C[0, 1]$. For a bounded signed measure $m$, we use $\langle m, f \rangle$ to denote $\int_{[0,1]} f(x) m(dx)$ for any $f \in C[0, 1]$. Consequently, $m$ can be identified with an element in $\mathcal{S}$ such that $m(f) = \langle m, f \rangle$ for $f \in C[0, 1]$.

We use $D([0, T], \mathcal{S})$ to denote the Skorokhod space of càdlàg functions $F : [0, T] \to \mathcal{S}$. That is to say, for any $F \in D([0, T], \mathcal{S})$ and $0 < t < T$, $\lim_{s \downarrow t} F_s = F_t$ while $\lim_{s \uparrow t} F_s = F_{t-}$ exists in the sense that

$$\lim_{s \downarrow t} F_s(f) = F_t(f)$$

and

$$\lim_{s \uparrow t} F_s(f) = F_{t-}(f)$$

for any $f \in C[0, 1]$.

For $0 \leq t \leq T$ and $N \geq 1$, we define $\mu_t^N = \frac{1}{N} \sum_{i=1}^N X_t^N(i) \delta_{i/N}(dx)$, i.e.,

$$\mu_t^N(f) = \langle \mu_t^N, f \rangle = \frac{1}{N} \sum_{i=1}^N X_t^N(i) f\left(\frac{i}{n}\right)$$
for any \( f \in C[0, 1] \). Then, \( \mu^N := \{\mu^N_t\}_{0 \leq t \leq T} \) is a random element in \( D([0, T], \mathcal{S}) \). We define \( P_1, P_2 : C[0, 1] \to C[0, 1] \) as linear operators such that
\[
(P_1 f)(x) = \int_0^1 \lambda(x, y) f(y)dy \quad \text{and} \quad (P_2 f)(x) = f(x) \int_0^1 \lambda(x, y)dy
\]
for any \( f \in C[0, 1] \) and \( x \in [0, 1] \). We define \( P_1^*, P_2^* : S \to S \) as linear operators such that
\[
(P_1^* \mathcal{A})(f) = \mathcal{A}(P_1 f) \quad \text{and} \quad (P_2^* \mathcal{A})(f) = \mathcal{A}(P_2 f)
\]
for any \( \mathcal{A} \in S \) and \( f \in C[0, 1] \).

To give our main result of \( \mu^N \), we need the following assumption.

**Assumption (A):** For all \( N \geq 1 \), \( \{X^N_0 (i)\}_{i=1}^N \) are independent and \( X^N_0 (i) \) follows a Poisson distribution with mean \( \phi \left( \frac{i}{N} \right) \) for each \( i \), where \( \phi \in C[0, 1] \).

It is easy to check that, under Assumption (A), \( \frac{1}{N} \sum_{i=1}^N X^N_0 (i) f \left( \frac{i}{N} \right) \) converges weakly to \( \int_0^1 f(x)\phi(x)dx \) for any \( f \in C[0, 1] \) as \( N \) grows to infinity.

**Definition 2.1** \( \mu = \{\mu_t\}_{0 \leq t \leq T} \in D([0, T], \mathcal{S}) \) is called a weak solution of the following Equation
\[
\left\{ \begin{array}{l}
\frac{d}{dt} \mu_t = (P_1^* - P_2^*) \mu_t, \quad 0 \leq t \leq T, \\
\mu_0(dx) = \phi(x)dx,
\end{array} \right. \quad (2.1)
\]
if for any \( 0 \leq t \leq T \) and \( f \in C[0, 1] \),
\[
\mu_t (f) = \int_0^1 f(x)\phi(x)dx + \int_0^t \mu_s ((P_1 - P_2) f)ds. \quad (2.2)
\]

Equation (2.1) has the following property.

**Lemma 2.2** There exists a unique weak solution \( \mu \in D([0, T], \mathcal{S}) \) of Eq. 2.1. Furthermore, \( \mu_t(dx) = \rho(t, x)dx \) for any \( 0 \leq t \leq T \), where \( \rho \in C^{1,0}([0, T] \times [0, 1]) \) such that
\[
\left\{ \begin{array}{l}
\frac{d}{dt} \rho(t, x) = \int_0^1 \lambda(y, x) \rho(t, y)dy - \rho(t, x) \int_0^1 \lambda(x, y)dy, \quad 0 \leq t \leq T, \\
\rho(0, x) = \phi(x).
\end{array} \right. \quad (2.3)
\]

We put the proof of Lemma 2.2 in the appendix, which utilizes classic theory of ordinary differential equations on Banach spaces. Now we can state our first main result, which is the following limit theorem of \( \{\mu^N\}_{N \geq 1} \).

**Theorem 2.3** Under Assumption (A), \( \mu^N \) converges weakly to the weak solution \( \mu \) of Eq. 2.1 as \( N \) grows to infinity.

As we have introduced in Section 1, in some cases \( \lim_{t \to +\infty} \mu_t = \mu_\infty \) for some \( \mu_\infty \in \mathcal{S} \) and then \( \mu_\infty \) is called as the metastable state of the model. An example with a metastable state is given in Appendix A.3.

Our next result is regarding the non-equilibrium fluctuation, i.e., central limit theorem, corresponding to the hydrodynamic limit given in Theorem 2.3.

To give our result, we first introduce some notations. For any \( t \geq 0 \) and \( N \geq 1 \), we use \( V_i^N \) to denote \( \frac{1}{\sqrt{N}} \sum_{i=1}^N (X_i^N (i) - \mathbb{E} (X_i^N (i))) \delta_x (dx) \), where \( \mathbb{E} \) is the expectation. Then, \( V^N := \{V_i^N\}_{0 \leq i \leq T} \) is also a random element in \( D([0, T], \mathcal{S}) \). Let
(C ([0, 1] × [0, 1]))′ be the dual of C ([0, 1] × [0, 1]), then we use \( \mathcal{W} = \{ \mathcal{W}_t \}_{t \geq 0} \) to denote the \((C ([0, 1] × [0, 1]))′\)-valued standard Brownian motion, i.e., for any \( f \in C ([0, 1] \times [0, 1]) \), \( \{ \mathcal{W}_t (f) \}_{t \geq 0} \) is a real-valued Brownian motion such that \( \mathcal{W}_0 (f) = 0 \) and

\[
\text{Cov}(\mathcal{W}_t (f), \mathcal{W}_t (f)) = t \int_0^1 \int_0^1 f^2(x, y) \, dx \, dy
\]

for any \( t \geq 0 \) (see page 608 of [14]).

For any \( t \in [0, T] \), we define \( b_t : C[0, 1] \to C([0, 1] \times [0, 1]) \) as the linear operator such that

\[
(b_t f)(x, y) = \sqrt{\rho(t, x) \lambda(x, y)}(f(y) - f(x))
\]

for any \( f \in C[0, 1] \) and \( x, y \in [0, 1] \), where \( \rho \) is defined as in Eq. 2.3. Then, for any \( t \in [0, T] \), we define \( b_t^* : (C ([0, 1] \times [0, 1]))′ \to \mathcal{S} \) as the linear operator such that

\[
(b_t^* \mu)(f) = \mu(b_t f)
\]

for any \( f \in C[0, 1] \) and \( \mu \in (C ([0, 1] \times [0, 1]))′ \).

We also need to introduce the definition of a time-inhomogeneous generalized Ornstein-Uhlenbeck process. According to an analysis similar with that given in the proof of Theorem 1.4 of [15], the outline of which we put in Appendix A.2, there exists a unique stochastic element \( V = \{ V_t \}_{0 \leq t \leq T} \) in \( D ([0, T], \mathcal{S}) \) satisfying:

1. \( \{ V_t (H) \}_{0 \leq t \leq T} \) is a real-valued continuous function for any \( H \in C[0, 1] \).
2. For any \( H \in C[0, 1] \) and any \( G \in C_c^\infty (\mathbb{R}) \),

\[
\left\{ G(V_t (H)) - G(V_0 (H)) - \int_0^t G′(V_s (H)) \, V_s ((P_1 - P_2) H) \, ds - \frac{1}{2} \int_0^t G''(V_s (H)) \| b_s H \|_2^2 \, ds \right\}_{0 \leq t \leq T}
\]

is a martingale, where \( \| f \|_2 = \sqrt{\int_0^1 \int_0^1 f^2(x, y) \, dx \, dy } \) for any \( f \in L^2 ([0, 1] \times [0, 1]) \).

3. \( V_0 (H) \) follows \( \mathbb{N} \left( 0, \int_0^1 H^2(x) \phi(x) \, dx \right) \) for any \( H \in C[0, 1] \), where \( \mathbb{N}(\mu, \sigma^2) \) is the normal distribution with mean \( \mu \) and variance \( \sigma^2 \).

Then, it is natural to define the above \( V \) as the solution to the \( \mathcal{S} \)-valued Equation:

\[
\begin{cases}
    dV_t = (P_1^* - P_2^*) V_t \, dt + b_t^* \, d\mathcal{W}_t, & 0 \leq t \leq T, \\
    V_0 (H) \text{ follows } \mathbb{N} \left( 0, \int_0^1 H^2(x) \phi(x) \, dx \right) & \text{for any } H \in C[0, 1], \\
    V_0 \text{ is independent of } \{ \mathcal{W}_t \}_{t \geq 0}.
\end{cases}
\]

(2.4)

where \( \{ \mathcal{W}_t \}_{t \geq 0} \) is the \((C ([0, 1] \times [0, 1]))′\)-valued Brownian motion as we have mentioned above.

Now we give our second main result, which gives non-equilibrium fluctuations of generalized \( N \)-urn Ehrenfest models.

**Theorem 2.4** Under Assumption (A), \( V^N \) converges weakly to \( V \) as \( N \to +\infty \), where \( V = \{ V_t \}_{0 \leq t \leq T} \) is the unique solution to Eq. 2.4.

**Remark 2.1** The solution of Eq. 2.4 can be defined equivalently through solving this Equation directly. As we have mentioned in the proof of Lemma 2.2, it is reasonable to
define
\[ e^{u(P_1 - P_2)} = \sum_{k=0}^{+\infty} \frac{u^k (P_1 - P_2)^k}{k!} \]
and hence define \( e^{u(P_1^* - P_2^*)} = \sum_{k=0}^{+\infty} \frac{u^k (P_1^* - P_2^*)^k}{k!} \) for any \( u \in \mathbb{R} \). Let \( V \) be the solution of Eq. 2.4, then
\[ d \left( e^{-t(P_1^* - P_2^*)} V_t \right) = e^{-t(P_1^* - P_2^*)} b_t^* dW_t \]
and hence
\[ V_t = e^{t(P_1^* - P_2^*)} V_0 + \int_0^t e^{(t-s)(P_1^* - P_2^*)} b_s^* dW_s, \]
i.e.,
\[ V_t(H) = V_0 \left( e^{t(P_1^* - P_2^*)} H \right) + \eta_t(H) \]
for any \( H \in C[0, 1] \), where \( \{\eta_t(H)\}_{0 \leq t \leq T} \) is a continuous martingale independent of \( V_0 \) and satisfies
\[ \langle \eta(H), \eta(H) \rangle_t = \int_0^t \left\| b_s \left( e^{(t-s)(P_1^* - P_2^*)} H \right) \right\|_2^2 ds, \]
where \( \|b_s(f)\|_2^2 = \int_0^1 \int_0^1 \rho(s, x) \lambda(x, y) (f(x) - f(y))^2 dxdy \) as we have defined above.

According to Remark 2.1, we have the following result as a direct corollary of Theorem 2.4.

**Corollary 2.5** Under Assumption (A), for any \( t > 0 \) and \( H \in C[0, 1] \),
\[ V_t^N(H) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( X^N_t(i) - \mathbb{E} \left( X^N_t(i) \right) \right) H \left( \frac{i}{N} \right) \]
converges weakly to \( \mathbb{N}(0, \theta^2(t, H)) \) as \( N \to +\infty \), where
\[ \theta(t, H) = \sqrt{\int_0^1 \left( e^{t(P_1 - P_2)} H \right)^2(x) \phi(x) dx + \int_0^t \| b_s \left( e^{(t-s)(P_1 - P_2)} H \right) \|_2^2 ds. \]

Our third main result is regarding the large deviation principle from the hydrodynamic limit given in Theorem 2.3. We first introduce some notations and definitions. For any \( v \in \mathcal{S} \), we define
\[ I_{ini}(v) = \sup \left\{ v(H) - \int_0^1 \phi(x) \left( e^{H(x)} - 1 \right) dx : H \in C[0, 1] \right\}. \]
We define nonlinear operator \( \mathcal{B} : C[0, 1] \to C[0, 1] \) as
\[ \mathcal{B} f(x) = \int_0^1 \lambda(x, y) \left( e^{f(y) - f(x)} - 1 \right) dy \]
for any \( f \in C[0, 1] \) and \( x \in [0, 1] \).
For any \( \mu = \{\mu_t\}_{0 \leq t \leq T} \in D([0, T], \mathcal{S}) \), we define
\[ I_{dyn}(\mu) = \sup \left\{ \mu_T(G_T) - \mu_0(G_0) 
- \int_0^T \mu_s (\partial_s + \mathcal{B}) G_s \ ds : G \in C^{1,0}([0, T] \times [0, 1]) \right\}. \]
Roughly speaking, $I_{ini}$ is the rate function of the large deviation with respect to the initial condition and $I_{dyn}$ is that with respect to the dynamic of the process. We use $D_0$ to denote the subset of $D([0, T], S)$ consisting of $\mu$ satisfying:

1. There exists nonnegative $\psi \in C^{1,0}([0, T] \times [0, 1])$ such that $\psi(t, \cdot)(x) = \frac{d\mu_t}{dx}$ for $0 \leq t \leq T$.
2. $\int_0^1 \partial_s \psi(s, x) dx = 0$ for $0 \leq s \leq T$.

For some technical reason, currently we need the following assumption to give our large deviation principle.

**Assumption (B):** There exists $\lambda_1, \lambda_2 \in C([0, 1])$ such that $\lambda(x, y) = \lambda_1(x)\lambda_2(y)$ for any $x, y \in [0, 1]$.

Now we give our main result.

**Theorem 2.6** Under Assumptions (A) and (B), for any closed set $C \subseteq D([0, T], S)$,

$$
\limsup_{N \to +\infty} \frac{1}{N} \log P \left( \mu_N \in C \right) \leq - \inf_{\mu \in C} \left( I_{ini}(\mu_0) + I_{dyn}(\mu) \right)
$$

(2.5)

while for any open set $O \subseteq D([0, T], S)$,

$$
\liminf_{N \to +\infty} \frac{1}{N} \log P \left( \mu_N \in O \right) \geq - \inf_{\mu \in D_0 \cap O} \left( I_{ini}(\mu_0) + I_{dyn}(\mu) \right).
$$

(2.6)

**Remark 2.2** (1) According to our current approach utilized in the proof of Theorem 2.6, Assumption (B) is required to prove the fact that $\mu \in D_0$ implies that there exists $G \in C([0, T] \times [0, 1])$ such that

$$
\partial_s \psi(s, x) = \int_0^1 \psi(s, y)\lambda(y, x)e^{G_s(x) - G_s(y)} - \psi(s, x)\lambda(x, y)e^{G_s(y) - G_s(x)} dy
$$

(2.7)

for any $(s, x) \in [0, T] \times [0, 1]$, where $\mu_t(dx) = \psi(t, x) dx$ for $0 \leq t \leq T$. We guess that $G$ defined in Eq. 2.7 still exists without Assumption (B) and consequently Theorem 2.6 holds for general $\lambda \in C([0, 1] \times [0, 1])$, We will work on this problem as a further investigation.

(2) We guess that $I_{dyn}(\mu) < +\infty$ if and only if $\mu \in D_0$. If this property holds, then Eq. 2.6 can be improved to

$$
\liminf_{N \to +\infty} \frac{1}{N} \log P \left( \mu_N \in O \right) \geq - \inf_{\mu \in D_0 \cap O} \left( I_{ini}(\mu_0) + I_{dyn}(\mu) \right).
$$

We will work on this problem as a further investigation.

**Remark 2.3** As an application of Theorem 2.6, a dynamical description of the metastability of the process under assumption (B) can be given. Let $\phi$ given in Assumption (A) be with the form that $\phi(x) = K \frac{\lambda_2(x)}{\lambda_1(x)}$ for any $x \in [0, 1]$, where $K > 0$, then Theorem 2.6 implies that for any $\epsilon > 0$ and $f \in C[0, 1]$, there exists $C = C(\epsilon, f) > 0$ such that

$$
\lim_{N \to +\infty} P \left( \sup_{0 \leq s \leq \exp\{CN\}} \left| \mu^N_t(f) - K \int_0^1 f(x) \frac{\lambda_2(x)}{\lambda_1(x)} dx \right| \leq \epsilon \right) = 1.
$$

(2.8)

Equation 2.8 shows that $\mu^N_t(f)$ stays in a small neighborhood of $\mu^\infty(f)$ for an amount of time with order $\exp\{O(N)\}$ with high probability under a specially given initial condition,
where \( \mu_\infty(dx) = K \frac{\lambda_2(x)}{\lambda_1(x)} dx \). Here we give an outline of the proof of Eq. 2.8. When \( \phi(\cdot) = K \frac{\lambda_2(\cdot)}{\lambda_1(\cdot)} \), it is easy to check that the unique weak solution to Eq. 2.1 is \( \mu_t = \mu_\infty \) and the initial distribution given in Assumption (A) is reversible and hence invariant. As a result,

\[
P \left( \sup_{k \leq t \leq k+1} \left| \mu_t^N(f) - K \int_0^1 f(x) \frac{\lambda_2(x)}{\lambda_1(x)} dx \right| > \epsilon \right) < \exp(-C_1N)
\]

is independent of the choice of positive integer \( k \). It is shown in [28] that \( I_{ini}(\cdot) + I_{dyn}(\cdot) \) is a good rate function and \( I_{ini}(\pi_0) + I_{dyn}(\pi) = 0 \) if and only if \( \pi \) is the weak solution to Eq. 2.1. As a result, by Theorem 2.6, there exists \( C_1 = C_1(f, \epsilon) > 0 \) such that

\[
P \left( \sup_{0 \leq t \leq 1} \left| \mu_t^N(f) - K \int_0^1 f(x) \frac{\lambda_2(x)}{\lambda_1(x)} dx \right| > \epsilon \right) < e^{-C_1N}
\]

when \( N \) is sufficiently large. Let \( C = \frac{C_1}{2} \), then

\[
P \left( \sup_{k \leq t \leq k+1} \left| \mu_t^N(f) - K \int_0^1 f(x) \frac{\lambda_2(x)}{\lambda_1(x)} dx \right| > \epsilon \right) < e^{-\frac{C_1}{2}N}
\]

and hence

\[
P \left( \sup_{0 \leq t \leq \exp(CN)} \left| \mu_t^N(f) - K \int_0^1 f(x) \frac{\lambda_2(x)}{\lambda_1(x)} dx \right| \leq \epsilon \right) \geq 1 - e^{-\frac{C_1}{2}N}.
\]

As a result, Eq. 2.8 holds.

The proofs of Theorems 2.3 and 2.4 are given in Sections 3 and 4 respectively. The strategies of the two proofs are similar. First we will show that \( \{\mu_t^N\}_{N \geq 1} \) and \( \{V_t^N\}_{N \geq 1} \) are tight. Then, by Dynkin’s martingale formula, we show that any limit point \( \bar{\mu} \) of subsequence of \( \{\mu_t^N\}_{N \geq 1} \) satisfies (2.2) for any \( f \in C[0, 1] \) while any limit point \( \bar{V} \) of subsequence of \( \{V_t^N\}_{N \geq 1} \) is a solution of Eq. 2.4. Consequently, \( \bar{V} = V \) (resp. \( \bar{\mu} = \mu \)) holds according to the uniqueness of the solution to Eq. 2.4 (resp. Equation 2.1).

The proof of Theorem 2.6 is divided into Sections 5 and 6. We follow a strategy similar with that introduced in [19], where an exponential martingale plays the crucial role. The proof of Eq. 2.5 is given in Section 5, where we first utilize the aforementioned exponential martingale, Chebyshev’s inequality and a minimax theorem given in [31] to show that Eq. 2.5 holds for all compact \( K \subseteq D([0, T], S) \). Then, we utilize the criteria introduced in [27] to show that \( \{\mu_t^N\}_{N \geq 1} \) are exponentially tight and consequently complete the proof. The proof of Eq. 2.6 is given in Section 6, where a critical step is the utilization of a generalized version of Girsanov’s theorem given in [30] to derive the law of large numbers \( \{\mu_t^N\}_{N \geq 1} \) obeys under a transformed measure with the aforementioned exponential martingale as the R-N derivative with respect to the original measure of \( \{X_t^N\}_{1 \leq t \leq N} \). The above strategy has also been utilized in the study of moderate deviations of stochastic models, see [13, 34, 36] as examples.

### 3 Proof of Theorem 2.3

In this section we prove Theorem 2.3. For simplicity, in proofs we use \( o(1) \) to denote a deterministic sequence which converges to 0 as \( N \to +\infty \) while use \( o_p(1) \) to denote a
stochastic sequence which converges weakly to 0 as $N \to +\infty$. We use $O(1)$ to denote a deterministic bounded sequence. First we show the tightness of $\{\mu_N^N\}_{N \geq 1}$.

**Lemma 3.1** Under Assumption (A), $\{\mu_N^N\}_{N \geq 1}$ is tight.

**Proof** According to Theorem 4.1 of [24] and Aldous’ criteria, to show the tightness of $\{\mu_N^N\}_{N \geq 1}$, we only need to check that the following two conditions hold.

1. $\lim_{M \to +\infty} \limsup_{N \to +\infty} P(\{|\mu_N^N(f)| \geq M\}) = 0$ for any $t \geq 0$ and $f \in C[0, 1]$.
2. For any $\epsilon > 0$ and $f \in C[0, 1]$,

$$\lim_{\delta \to 0} \limsup_{N \to +\infty} \sup_{t \in \mathcal{T}, s \leq \delta} P(|\mu_N^{t+s}(f) - \mu_N^t(f)| > \epsilon) = 0,$$

where $\mathcal{T}$ is the set of stopping times of $\{X_N^N\}_{t \geq 0}$ bounded by $T$.

To check Condition (1), we have

$$P(|\mu_N^N(f)| \geq M) \leq P\left(|\mu_N^N(f)| \geq M, \frac{1}{N} \sum_{i=1}^N X_0^N(i) \leq 2 \int_0^1 \phi(x)dx\right) + P\left(\frac{1}{N} \sum_{i=1}^N X_0^N(i) > 2 \int_0^1 \phi(x)dx\right).$$

Let $\|f\|_\infty = \sup_{0 \leq x \leq 1} |f(x)|$ be the $l_\infty$-norm of $f$ defined as in the appendix, then $|\mu_N^N(f)| \leq \|f\|_\infty \frac{1}{N} \sum_{i=1}^N X_i^N(i)$. According to the definition of $\{X_i^N\}_{t \geq 0}$, $\sum_{i=1}^N X_i^N(i) \equiv \sum_{i=1}^N X_0^N(i)$ for any $t \geq 0$. Hence

$$P(|\mu_N^N(f)| \geq M, \frac{1}{N} \sum_{i=1}^N X_0^N(i) \leq 2 \int_0^1 \phi(x)dx) = 0$$

when $M > 2\|f\|_\infty \int_0^1 \phi(x)dx$ and hence

$$\lim_{M \to +\infty} \limsup_{N \to +\infty} P(\{|\mu_N^N(f)| \geq M\}) \leq \limsup_{N \to +\infty} P\left(\frac{1}{N} \sum_{i=1}^N X_0^N(i) > 2 \int_0^1 \phi(x)dx\right).$$

Under Assumption (A), by Chebyshev’s inequality,

$$P\left(\frac{1}{N} \sum_{i=1}^N X_0^N(i) > 2 \int_0^1 \phi(x)dx\right) \leq \frac{1}{N\left(\int_0^1 \phi(x)dx + o(1)\right)^2} \to 0$$

as $N \to 0$ and hence Condition (1) holds.

To check Condition (2), note that $|\mu_{u+s}^N(f) - \mu_u^N(f)| \leq \frac{2\|f\|_\infty}{N}$ when $\{X_i^N\}_{t \geq 0}$ jumps at moment $u$, hence, by strong Markov property, $|\mu_{t+s}^N(f) - \mu_t^N(f)|$ is stochastically dominated from above by $\frac{2\|f\|_\infty}{N} Y(2N\delta \|\lambda\|_\infty \int_0^1 \phi(x)dx)$ conditioned on $\frac{1}{N} \sum_{i=1}^N X_0^N(i)$, where

$\delta$ Springer
\[ \int_0^1 \phi(x) dx, \quad \text{where } \{Y(t)\}_{t \geq 0} \text{ is a Poisson process with rate one and } \|\lambda\|_\infty = \sup_{0 \leq x, y \leq 1} \lambda(x, y). \] As a result, by Chebyshev’s inequality,

\[
\lim_{\delta \to 0} \limsup_{N \to +\infty} \sup_{\tau \in T, \delta \leq s \leq t} P \left( \left| \mu^N_{\tau+s}(f) - \mu^N_{\tau}(f) \right| > \epsilon \right) 
\leq \lim_{\delta \to 0} \limsup_{N \to +\infty} \frac{2\|f\|_\infty}{N} Y \left( 2N\delta \|\lambda\|_\infty \int_0^1 \phi(x) dx \right) > \epsilon
\]

\[ + \lim_{N \to +\infty} P \left( \frac{1}{N} \sum_{i=1}^N X^N_0(i) > 2 \int_0^1 \phi(x) dx \right) \]

\[
\leq \lim_{\delta \to 0} \limsup_{N \to +\infty} e^{N[2\delta \|\lambda\|_\infty(e^{2\|f\|_\infty} - 1) \int_0^1 \phi(x) dx - \epsilon]} + \lim_{N \to +\infty} \frac{1}{N(e^{\int_0^1 \phi(x) dx + o(1)})^2} \]

\[= 0 \]

and hence Condition (2) holds.

Now we give the proof of Theorem 2.3.

**Proof of Theorem 2.3** As we have shown in Lemma 2.2, Eq. 2.1 has a unique weak solution. Hence, by Lemma 3.1, we only need to show that any limit point \( \bar{\mu} \) of subsequence of \( \{\mu^N\}_{N \geq 1} \) satisfies (2.2) for any \( f \in C[0, 1] \).

For any \( f \in C[0, 1] \) and \( N \geq 1 \), we define

\[ M^N_t(f) = \mu^N_t(f) - \mu^N_0(f) - \int_0^t \mathcal{L}_N(\mu^N_s(f)) ds, \]

then, by Dynkin’s martingale formula, \( \{M^N_t(f)\}_{t \geq 0} \) is a martingale with \( \langle M^N(f) \rangle_t \) given by

\[ \langle M^N(f) \rangle_t = \int_0^t \mathcal{L}_N \left( \left( \mu^N_s(f) \right)^2 \right) - 2\mu^N_s(f) \mathcal{L}_N \left( \mu^N_s(f) \right) ds. \]

According to the definition of \( \mathcal{L}_N \),

\[
\mathcal{L}_N \left( \mu^N_s(f) \right) = \sum_{i=1}^N \sum_{j=1}^N X^N_s(i) \lambda(i, j) N \left( \mu^N_s(f) + \frac{f(j)}{N} - f(i) \right) \mu^N_s(f) - f(i) \mu^N_s(f) \)
\]

\[= -\frac{1}{N} \sum_{i=1}^N X^N_s(i) f \left( i \frac{j}{N} \right) \left( \frac{1}{N} \sum_{j=1}^N \lambda(i, j) \right) + \frac{1}{N} \sum_{i=1}^N X^N_s(i) \left( \frac{1}{N} \sum_{j=1}^N \lambda(i, j) f \left( \frac{j}{N} \right) \right) \]

\[= -\frac{1}{N} \sum_{i=1}^N X^N_s(i) f \left( i \frac{j}{N} \right) \left( P_2 f \left( \frac{i}{N} \right) + o(1) \right) + \frac{1}{N} \sum_{i=1}^N X^N_s(i) \left( P_1 f \left( \frac{i}{N} \right) + o(1) \right). \]

Note that the \( o(1) \)s in the above equation can be chosen uniformly according to the absolute continuity of \( \lambda \). According to the fact that \( \sum_{i=1}^N X^N_s(i) \) are conserved and Chebyshev’s inequality, under Assumption (A),

\[
\frac{1}{N} \sum_{i=1}^N X^N_s(i) = \frac{1}{N} \sum_{i=1}^N X^N_0(i) = \int_0^1 \phi(y) dy + o_p(1).
\]
Hence, \( \mathcal{L}_N(\mu_s^N(f)) = \mu_s^N((P_1 - P_2)f) + o_p(1) \). Similarly,

\[
\mu_0^N(f) = \int_0^1 f(y)\phi(y)dy + o_p(1)
\]

and

\[
\mathcal{L}_N \left( \left( \mu_s^N(f) \right)^2 \right) - 2\mu_s^N(f)\mathcal{L}_N(\mu_s^N(f)) = \sum_{i=1}^N \sum_{j=1}^N X_s^N(i)\lambda(i,j)N^{-\frac{1}{3}}(f(i/N) - f(j/N))^2
\]

\[
\leq 4\|f\|_\infty^2\|\lambda\|_\infty \sum_{i=1}^N X_s^N(i) = 4\|f\|_\infty^2\|\lambda\|_\infty \left( \int_0^1 \phi(y)dy + o_p(1) \right) = o_p(1).
\]

Then, \( M^N_t(f) = o_p(1) \) by Doob’s inequality. Consequently,

\[
\mu_t^N(f) = \mu_0^N(f) + \int_0^t \mathcal{L}_N(\mu_s^N(f))ds + M^N_t(f)
\]

\[
= \int_0^1 f(y)\phi(y)dy + \int_0^t \mu_s^N((P_1 - P_2)f)ds + o_p(1).
\]

Therefore, for any limit point \( \bar{\mu} \) of subsequence of \( \{\mu^N\}_{N\geq 1} \) and \( f \in C[0, 1] \),

\[
\bar{\mu}_t(f) = \int_0^1 f(y)\phi(y)dy + \int_0^t \bar{\mu}_s((P_1 - P_2)f)ds,
\]

i.e., \( \bar{\mu} \) satisfies (2.2) and hence the proof is complete. \( \square \)

4 Proof of Theorem 2.4

In this section, we prove Theorem 2.4. First we show the tightness of \( \{V^N\}_{N\geq 1} \).

Lemma 4.1 Under Assumption (A), \( \{V^N\}_{N\geq 1} \) is tight.

To prove Lemma 4.1, we need the following lemma.

Lemma 4.2 There exists a constant \( C_1 < +\infty \) such that

\[
\mathbb{E} \left( V^N_t(f) \right)^2 = \text{Cov} \left( V^N_t(f), V^N_t(f) \right) \leq C_1 \|f\|_\infty^2
\]

for all \( N \geq 1 \) and any \( f \in C[0, 1] \).

We first utilize Lemma 4.2 to prove Lemma 4.1.

Proof of Lemma 4.1 Similarly with the proof of Lemma 3.1, we only need to check that the following two conditions holds.

1. \( \lim_{M \to +\infty} \limsup_{N \to +\infty} P \left( |V_t^N(f)| \geq M \right) = 0 \) for any \( t \geq 0 \) and \( f \in C[0, 1] \).

2. For any \( \epsilon > 0 \) and \( f \in C[0, 1] \),

\[
\lim_{\delta \to 0} \limsup_{N \to +\infty} \sup_{\tau \in T, \tau \leq \delta} P \left( |V_{\tau+s}^N(f) - V_{\tau}^N(f)| > \epsilon \right) = 0.
\]
By Dynkin’s martingale formula, \( V_N^t(f) = V_N^0(f) + \alpha_N^t + \beta_N^t \), where \( \alpha_N^t = \int_0^t (L_N + \partial_s) V_s^N(f) ds \) while \( \{\beta_N^t\}_{t \geq 0} \) is a martingale with

\[
\langle \beta_N^t \rangle = \int_0^t L_N ((V_s^N(f))^2) - 2V_s^N(f) L_N V_s^N(f) ds.
\] (4.1)

Then, to check conditions (1) and (2), we only need to check that the following four conditions hold.

1. \( \lim_{M \to +\infty} \limsup_{N \to +\infty} P(|\alpha_N^t| \geq M) = 0 \) for any \( t \geq 0 \) and \( f \in C[0, 1] \).
2. For any \( \epsilon > 0 \) and \( f \in C[0, 1] \),
   \[
   \lim \limsup_{\delta \to 0} \sup_{N \to +\infty} P(|\alpha_N^t| > \epsilon) = 0.
   \]
3. \( \lim_{M \to +\infty} \limsup_{N \to +\infty} P(|\beta_N^t| \geq M) = 0 \) for any \( t \geq 0 \) and \( f \in C[0, 1] \).
4. For any \( \epsilon > 0 \) and \( f \in C[0, 1] \),
   \[
   \lim \limsup_{\delta \to 0} \sup_{N \to +\infty} P(|\beta_N^t| > \epsilon) = 0.
   \]

We first check Condition (3). According to the expression of the generator \( L_N \) of \( \{X_N^t\}_{t \geq 0} \) given in Eq. 1.1,

\[
\partial_t \mathbb{E}(X_N^{t}(i)) = -\frac{1}{N} \sum_{j=1}^{N} \mathbb{E}(X_N^{t}(i)) \lambda(i, j) + \frac{1}{N} \sum_{j=1}^{N} \mathbb{E}(X_N^{t}(j)) \lambda(j, i),
\]

for all \( 1 \leq i \leq N \). As a result,

\[
(L_N + \partial_s)V_s^N(f) = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{X_s^N(i) \lambda(i, j)}{N} \frac{f(i) - f(j)}{\sqrt{N}} - \frac{1}{\sqrt{N}} \sum_{j=1}^{N} f(i) \partial_s \mathbb{E}(X_s^N(i)) = V_s^N((P_1^N - P_2^N) f),
\] (4.2)

where

\[
(P_1^N f)(x) = \frac{1}{N} \sum_{j=1}^{N} \lambda(x, j) f(j) \quad \text{and} \quad (P_2^N f)(x) = \frac{f(x)}{N} \sum_{j=1}^{N} \lambda(x, j)
\]

for \( x \in [0, 1] \). Then, by Markov’s inequality, Hölder inequality and Lemma 4.2,

\[
P(|\alpha_N^t| \geq M) \leq \frac{1}{M^2} \mathbb{E} \left( \left( \int_0^t V_s^N((P_1^N - P_2^N) f) ds \right)^2 \right)
\]

\[
\leq \frac{1}{M^2} \int_0^t \int_0^t \text{Cov}(V_s^N((P_1^N - P_2^N) f), V_u^N((P_1^N - P_2^N) f)) du dv
\]

\[
\times \sqrt{\text{Cov}(V_s^N((P_1^N - P_2^N) f), V_u^N((P_1^N - P_2^N) f))}
\]

\[
\leq \frac{1}{M^2} \int_0^t \int_0^t \sqrt{C_1 \| (P_1^N - P_2^N) f \|_\infty^2} \sqrt{C_1 \| (P_1^N - P_2^N) f \|_\infty^2} du dv
\]

\[
= \frac{C_1 f^2}{M^2} \| (P_1^N - P_2^N) f \|_\infty^2 \leq \frac{4C_1 f^2}{M^2} \| \lambda \|_\infty^2 \| f \|_\infty^2
\]
and hence Condition (3) holds.

Then we check Condition (4). According to Eq. 4.2,

$$\alpha_{\tau+s}^N - \alpha_{\tau}^N = \int_\tau^{\tau+s} V_u^N ((P_1^N - P_2^N) f) du.$$ 

Therefore, by Hölder inequality, Markov’s inequality and Lemma 4.2, for $s \leq \delta$ and $\tau \in T$,

$$P \left( |\alpha_{\tau+s}^N - \alpha_{\tau}^N| > \epsilon \right) \leq \frac{1}{\epsilon} \mathbb{E} \left( \int_\tau^{\tau+s} |V_u^N ((P_1^N - P_2^N) f)| du \right)$$

$$= \frac{1}{\epsilon} \mathbb{E} \left( \int_0^{T+\delta} |V_u^N ((P_1^N - P_2^N) f)| 1_{\tau \leq u \leq \tau+s} du \right)$$

$$= \frac{1}{\epsilon} \int_0^{T+\delta} \mathbb{E} \left( |V_u^N ((P_1^N - P_2^N) f)| 1_{\tau \leq u \leq \tau+s} \right) du$$

$$\leq \frac{\sqrt{C_1 \| (P_1^N - P_2^N) f \|_\infty}}{\epsilon} \int_0^{T+\delta} \sqrt{P(\tau \leq u \leq \tau+s)} du$$

$$\leq \frac{2 \sqrt{C_1 (T+\delta) \| \lambda \| \| f \|_\infty}}{\epsilon} \sqrt{\int_0^{T+\delta} P(\tau \leq u \leq \tau+s) du}$$

$$= \frac{2 \sqrt{C_1 (T+\delta) \| \lambda \| \| f \|_\infty}}{\epsilon} \sqrt{\mathbb{E} \left( \int_0^{T+\delta} 1_{\tau \leq u \leq \tau+s} du \right)}$$

$$= \frac{2 \sqrt{C_1 (T+\delta) \| \lambda \| \| f \|_\infty}}{\epsilon} \leq 2 \sqrt{C_1 (T+\delta) \| \lambda \| \| f \|_\infty}$$

and hence Condition (4) holds.

Then we check Condition (5). According to the definition of $\mathcal{L}_N$,

$$\mathcal{L}_N ((V_s^N (f))^2) - 2 V_s^N (f) \mathcal{L}_N V_s^N (f) = \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N X_s^N (i) \lambda (\frac{j}{N}, \frac{i}{N}) (f (\frac{j}{N}) - f (\frac{i}{N}))^2$$

$$\leq \left( \frac{4}{N} \sum_{i=1}^N X_s^N (i) \right) \| \lambda \|_\infty \| f \|_\infty^2 = \left( \frac{4}{N} \sum_{i=1}^N X_0^N (i) \right) \| \lambda \|_\infty \| f \|_\infty^2.$$ 

Hence,

$$\mathbb{E} \left( (\beta_t^N)^2 \right) = \mathbb{E} \left( (\beta_t^N) \right) \leq \frac{4t}{N} \left( \sum_{i=1}^N \phi (\frac{i}{N}) \right) \| \lambda \|_\infty \| f \|_\infty^2$$

$$= \left( 4t \int_0^1 \phi (x) dx + o(1) \right) \| \lambda \|_\infty \| f \|_\infty^2$$

and Condition (5) follows from Chebyshev’s inequality.
At last we check Condition (6). Since \( \mathbb{E}[(\beta_{t+s}^N - \beta_t^N)^2] = \mathbb{E}( (\beta_t^N)_{t+s} - (\beta_t^N)_{t} ) \) while
\[
(\beta_{t+s}^N - \beta_t^N) \leq \int_{t}^{t+s} \left( \frac{4}{N} \sum_{i=1}^{N} X_0^N(i) \right) \|\lambda\|_\infty \|f\|_\infty^2 \, du
\]
for any \( s \leq \delta \),
\[
\mathbb{E}[(\beta_{t+s}^N - \beta_t^N)^2] \leq 4\delta \left( \int_{0}^{1} \phi(x)dx + o(1) \right) \|\lambda\|_\infty \|f\|_\infty^2.
\]
As a result, Condition (6) follows from Chebyshev’s inequality.

Since Conditions (3)-(6) all hold, the proof is complete.

Now we give the proof of Lemma 4.2.

Proof of Lemma 4.2 By direct calculation,
\[
\text{Cov} \left(V_t^N(f), V_t^N(f)\right) \leq I + II,
\]
where
\[
I = \frac{\|f\|_\infty^2}{N} \sum_{i=1}^{N} \mathbb{E} \left( \left( X_t^N(i) \right)^2 \right)
\]
and
\[
II = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq i} f \left( \frac{i}{N} \right) f \left( \frac{j}{N} \right) \text{Cov} \left( X_t^N(i), X_t^N(j) \right).
\]
Conditioned on \( X_0^N \), since each ball jumps to the \( i \)th box at rate at most \( \|\lambda\|_\infty \), \( X_t^N(i) - X_0^N(i) \) is stochastically dominated from above by a Poisson process \( \{K_t\}_{t \geq 0} \) with rate
\[
\|\lambda\|_\infty \frac{1}{N} \sum_{j=1}^{N} X_0^N(j).
\]
Hence,
\[
\mathbb{E} \left( \left( X_t^N(i) \right)^2 \bigg| X_0^N \right) \leq \left( X_0^N(i) \right)^2 + 2X_0^N(i) \mathbb{E}K_t + \mathbb{E} \left( K_t^2 \right)
\]
\[
= \left( X_0^N(i) \right)^2 + 2X_0^N(i)\|\lambda\|_\infty \frac{1}{N} \sum_{j=1}^{N} X_0^N(j) + \|\lambda\|_\infty \frac{1}{N} \sum_{j=1}^{N} X_0^N(j)
\]
\[
+ \frac{t^2\|\lambda\|_\infty^2}{N^2} \left( \sum_{j=1}^{N} X_0^N(j) \right)^2,
\]
and hence
\[
\mathbb{E} \left( \left( X_t^N(i) \right)^2 \right) \leq \|\phi\|_\infty^2 \left( 1 + 2\|\lambda\|_\infty T + T^2\|\lambda\|_\infty^2 \right) + \|\phi\|_\infty \left( 1 + 3\|\lambda\|_\infty T + T^2\|\lambda\|_\infty^2 \right).
\]
for all $0 \leq t \leq T$ according to Assumption (A).

For $i \neq j$, $\text{Cov} \left( X^N_t (i), X^N_t (j) \right) = \text{III} + \text{IV}$, where

$$\text{III} = \mathbb{E} \left( \mathbb{E} \left( X^N_t (i) X^N_t (j) \mid X^N_0 \right) - \mathbb{E} \left( X^N_t (i) \mid X^N_0 \right) \mathbb{E} \left( X^N_t (j) \mid X^N_0 \right) \right)$$

and

$$\text{IV} = \text{Cov} \left( \mathbb{E} \left( X^N_t (i) \mid X^N_0 \right), \mathbb{E} \left( X^N_t (j) \mid X^N_0 \right) \right).$$

For a given ball initially in box $i$, we use $p_{ij}(t)$ to denote the probability that this ball is in box $j$ at moment $t$. Then,

$$\mathbb{E} \left( X^N_t (i) \mid X^N_0 \right) = \sum_{k=1}^N X^N_0 (k) p_{ki}(t).$$

Hence, by Assumption (A),

$$\text{IV} = \sum_{k=1}^N \sum_{l=1}^N \text{Cov} \left( X_0^N (k), X_0^N (l) \right) p_{ki}(t) p_{lj}(t)$$

$$= \sum_{k=1}^N \text{Cov} \left( X_0^N (k), X_0^N (k) \right) p_{ki}(t) p_{kj}(t) = \sum_{k=1}^N \phi \left( \frac{k}{N} \right) p_{ki}(t) p_{kj}(t).$$

For $1 \leq i, j \leq N$ and $1 \leq l \leq X_0^N (i)$, we use $A^i_t (l, j)$ to denote the event that the $l$-th ball initially in box $i$ is in box $j$ at moment $t$. Then, $X^N_t (i) = \sum_{k=1}^N \sum_{l=1}^{X_0^N (k)} 1_{A^i_t (l, i)}$ and

$$\mathbb{E} \left( X^N_t (i) X^N_t (j) \mid X^N_0 \right) - \mathbb{E} \left( X^N_t (i) \mid X^N_0 \right) \mathbb{E} \left( X^N_t (j) \mid X^N_0 \right)$$

$$= \sum_{k=1}^N X_0^N (k) \sum_{l=1}^{N} \sum_{(u,s) \neq (k,l)} P \left( A^k_t (l, i) \right) P \left( A^u_t (s, j) \right)$$

$$- \sum_{k=1}^N \sum_{l=1}^{X_0^N (k)} \sum_{u=1}^{N} \sum_{s=1}^{N} P \left( A^k_t (l, i) \right) P \left( A^u_t (s, j) \right)$$

$$= - \sum_{k=1}^N \sum_{l=1}^{X_0^N (k)} P \left( A^k_t (l, i) \right) P \left( A^k_t (l, j) \right) = - \sum_{k=1}^N X_0^N (k) p_{ki}(t) p_{kj}(t).$$

Therefore, by Assumption (A),

$$\text{III} = - \sum_{k=1}^N \phi \left( \frac{k}{N} \right) p_{ki}(t) p_{kj}(t) = -\text{IV}$$

and hence $\text{II} = 0$. As a result, Lemma 4.2 follows from Eq. 4.3. \qed

At last, we give the proof of Theorem 2.4.

**Proof of Theorem 2.4** By Lemma 4.1, we only need to show that any limit point $\tilde{V}$ of subsequence of $\{V^N_N\}_{N \geq 1}$ is a solution of Eq. 2.4. Equivalently, as we have introduced in Section 2, we need to check that $\tilde{V}$ has those three properties corresponding to the
definition of the unique solution of Eq. 2.4. The first property that \( \tilde{V}(H) \) is continuous for any \( H \in C[0, 1] \) follows from the fact that

\[
\sup_{0 \leq t \leq T} |V^N_t(H) - V^N_t(X)| \leq \frac{2\|H\|_{\infty}}{\sqrt{N}}.
\]

The third property that \( \tilde{V}_0(H) \) follows \( N \left( 0, \int_0^1 H^2(x) \phi(x) dx \right) \) is a consequence of Assumption (A). Hence, we only need to check that \( \tilde{V}_0(H) \) has the second property, i.e.,

\[
\left\{ G \left( \tilde{V}_t(H) \right) - G \left( \tilde{V}_0(H) \right) - \int_0^t G' \left( \tilde{V}_s(H) \right) \tilde{V}_s ((P_1 - P_2)H) ds \right. \\
- \frac{1}{2} \int_0^t G'' \left( \tilde{V}_s(H) \right) \| b_s H \|_2^2 ds \right\}_{0 \leq t \leq T}
\]

is a martingale for \( H \in C[0, 1] \) and \( G \in C^\infty(R) \).

For any \( G \in C^\infty(R) \) and \( H \in C[0, 1] \), we define

\[
\mathcal{M}_t^N(G, H) = G \left( V^N_t(H) \right) - G \left( V^N_0(H) \right) - \int_0^t (\mathcal{L}_N + \partial_s) G \left( V^N_s(H) \right) ds,
\]

then, by Dynkin’s martingale formula, \( \{\mathcal{M}_t^N(G, H)\}_{t \geq 0} \) is a martingale. According to the definition of \( \mathcal{L}_N \) and Taylor’s expansion up to the third order,

\[
(\mathcal{L}_N + \partial_s) G \left( V^N_s(H) \right)
= \sum_{i=1}^N \sum_{j=1}^N X^N_s(i) \lambda \left( \frac{i}{N} , \frac{j}{N} \right) \frac{H (\frac{i}{N}) - H (\frac{j}{N})}{\sqrt{N}}
- \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N X^N_s(i) \lambda \left( \frac{i}{N} , \frac{j}{N} \right) G'' \left( V^N_s(H) \right) \frac{H (\frac{i}{N}) - H (\frac{j}{N})}{N}^2
+ \xi^N_s
\]

\[
= G' \left( V^N_s(H) \right) V^N_s \left( \left( P^N_1 - P^N_2 \right) H \right) + \xi^N_s
+ \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N X^N_s(i) \lambda \left( \frac{i}{N} , \frac{j}{N} \right) G'' \left( V^N_s(H) \right) \frac{H (\frac{i}{N}) - H (\frac{j}{N})}{N}^2.
\]
where \( \xi_s^N \) is bounded from above by

\[
\frac{\|G''\|_\infty}{6} \sum_{i=1}^N \sum_{j=1}^N \frac{X_s^N(i)\lambda \left( \frac{i}{N}, \frac{j}{N} \right)}{N} \left| H \left( \frac{j}{N} \right) - H \left( \frac{i}{N} \right) \right|^3 
\]

\[
\leq \frac{4\|G''\|_\infty \|\lambda\|_\infty \|H\|_\infty}{3N^{\frac{1}{2}}} \left( \frac{1}{N} \sum_{i=1}^N X_0^N(i) \right) 
\]

\[
= \frac{4\|G''\|_\infty \|\lambda\|_\infty \|H\|_\infty}{3N^{\frac{1}{2}}} \left( \int_0^1 \phi(x)dx + o_p(1) \right) = o_p(1). 
\]

According to the definition of \( P_1^N, P_2^N \) and the fact that \( \lambda \in C^{1,1} ([0, 1] \times [0, 1]) \), it is easy to check that \( \| (P_1^N - P_2^N) H - (P_1 - P_2) H \|_\infty = O(1/N) \) while

\[
V_s^N \left( \left( P_1^N - P_2^N \right) H \right) - V_s^N \left( (P_1 - P_2) H \right) = o_p(1) 
\]

and this \( o_p(1) \) can be chosen uniformly for \( 0 \leq s \leq T \). Since \( \frac{1}{N} \sum_{i=1}^N X_s^N(i) = \frac{1}{N} \sum_{j=1}^N X_0^N(i) = \int_0^1 \phi(x)dx + o_p(1) \) and

\[
\sup_{1 \leq i \leq N} \left| \sum_{j=1}^N \frac{X_s^N(i)\lambda \left( \frac{i}{N}, \frac{j}{N} \right)}{N} \left( H \left( \frac{j}{N} \right) - H \left( \frac{i}{N} \right) \right)^2 \right| = o(1), 
\]

by Theorem 2.3,

\[
\sum_{i=1}^N \sum_{j=1}^N \frac{X_s^N(i)\lambda \left( \frac{i}{N}, \frac{j}{N} \right)}{N} \left( H \left( \frac{i}{N} \right) - H \left( \frac{j}{N} \right) \right)^2 
\]

\[
= \int_0^1 \rho(s, x) \left[ \int_0^1 \lambda(x, y) (H(y) - H(x))^2 dy \right] dx + o_p(1) = \|b_s H\|_2 + o_p(1) 
\]

and this \( o_p(1) \) can be chosen uniformly for \( 0 \leq s \leq T \).

Consequently,

\[
(L_N + \partial_s)G(V_s^N(H)) = G'(V_s^N(H))V_s^N((P_1 - P_2)H) + \frac{1}{2} G''(V_s^N(H))\|b_s H\|_2^2 + o_p(1) 
\]

and

\[
M_t^N(G, H) = G(V_t^N(H)) - G(V_0^N(H)) - \int_0^t G'(V_s^N(H))V_s^N((P_1 - P_2)H)ds 
\]

\[
- \int_0^t \frac{1}{2} G''(V_s^N(H))\|b_s H\|_2^2 ds + o_p(1). 
\]

By Eq. 4.5, a subsequence of \( \{M_t^N(G, H)\}_{0 \leq t \leq T} \) converges weakly to

\[
\left\{ G(\bar{V}_t(H)) - G(\bar{V}_0(H)) - \int_0^t G'(\bar{V}_s(H))\bar{V}_s((P_1 - P_2)H)ds 
\right. 
\]

\[
- \frac{1}{2} \int_0^t G''(\bar{V}_s(H))\|b_s H\|_2^2 ds \right\}_{0 \leq t \leq T}. 
\]
Since $\{M^N_t(G, H)\}_{t \geq 0}$ is a martingale for each $N \geq 1$, according to Theorem 5.3 of [33], we only need to show that $\{M^N_t(G, H)\}_{N \geq 1}$ are uniformly integrable for each $t \geq 0$ to complete this proof. Since $G \in C^\infty_c(\mathbb{R})$, we only need to show that

$$\sup_{0 \leq s \leq T, N \geq 1} \mathbb{E}\left(\left(\left(\mathcal{L}_N + \partial_s\right)G\left(V^N_s(H)\right)\right)^2\right) < +\infty \tag{4.6}$$

to derive the uniform integrability of $\{M^N_t(G, H)\}_{N \geq 1}$. According to Eq. 4.4, Lemma 4.2 and the upper bound of $\varepsilon^N_s$ given above, Eq. 4.6 follows from the fact that

$$\sup_{N \geq 1} \mathbb{E}\left(\left(\frac{1}{N} \sum_{i=1}^N X^N_0(i)\right)^2\right) \leq \|\phi\|_\infty^2 + \|\phi\|_\infty$$
given by Assumption (A) and the proof is complete. \qed

5 Proof of Eq. 2.5

In this section, we give the proof of Eq. 2.5. As we have mentioned in Section 2, an exponential martingale is crucial for our proof, so we first introduce this martingale. For any $H \in C^{1,0}([0, T] \times \mathbb{R})$ and $0 \leq t \leq T$, we use $\mathcal{U}^N_t(H)$ to denote

$$H(t, X^N_t) - H(0, X^N_0) - \int_0^t (\partial_s + \mathcal{L}_N) H(s, X^N_s) ds.$$ 

Then, by Dynkin’s martingale formula, $\{\mathcal{U}^N_t(H)\}_{0 \leq t \leq T}$ is a martingale. For any $G \in C^{1,0}([0, T] \times [0, 1])$ and $x \in \mathbb{R}$, we define

$$f^N_G(t, x) = e^{\sum_{i=1}^N x(i)Gt(i)}$$

and hence

$$f^N_G(t, X^N_t) = e^{N\mu^N_t(G_t)}.$$ 

For any $0 \leq t \leq T$ and $G \in C^{1,0}([0, T] \times [0, 1])$, we define

$$\Lambda^N_t(G) = \frac{f^N_G(t, X^N_t)}{f^N_G(0, X^N_0)} e^{-\int_0^t (\partial_s + \mathcal{L}_N) f^N_G(s, X^N_s) ds f^N_G(t, X^N_t)}.$$ 

By the definition of $\mathcal{L}_N$ and direct calculation,

$$\Lambda^N_t(G) = \exp \left\{ N \mu^N_t(G_t) - N \mu^N_0(G_0) - N \int_0^t \mu^N_s \left( (\partial_s + \mathcal{B}^N)G_s \right) ds \right\}, \tag{5.1}$$

where

$$(\mathcal{B}^N g)(x) = \frac{1}{N} \sum_{j=1}^N \lambda(x, j) \left( e^{g(\frac{j}{N})} - e^{g(x)} - 1 \right)$$

for any $g \in C[0, 1]$ and $x \in [0, 1]$. We have the following lemma.

**Lemma 5.1** For any $G \in C^{1,0}([0, T] \times [0, 1])$, $\{\Lambda^N_t(G)\}_{0 \leq t \leq T}$ is a martingale.
\textbf{Proof} Let \( V^N_t (G) = \frac{1}{f^N_t (0,X^N_0)} e^{-\int_0^t \frac{(\lambda + L_N)(x,X^N_s)}{f^N_t (x,X^N_s)} ds} \), then, by Itô's formula,
\[ d\Lambda^N_t (G) = V^N_t (G) dU^N_t (f^N_t) . \tag{5.2} \]
As a result, \( \{ \Lambda^N_t (G) \}_{0 \leq t \leq T} \) is a local martingale. By Eq. 5.1,
\[ \sup_{0 \leq t \leq T} \Lambda^N_t (G) \leq \exp \sum_{i=1}^N X^N_0 (i) \left( 2 + \| \lambda \|_\infty T \left( e^{2\|g\|_\infty} - 1 \right) + \| \partial_t G_t \|_\infty \right) . \tag{5.3} \]
As a result, \( \{ \Lambda^N_t (G) \}_{0 \leq t \leq T} \) being a martingale follows from the fact that
\[ \mathbb{E} \left( e^{\theta \sum_{i=1}^N X^N_0 (i)} \right) < +\infty \]
for any \( \theta > 0 \) given by Assumption (A).

We will first show that Eq. 2.5 holds for any compact \( C \). A crucial step is to bound the error from above when we replace \( \mu^N_t (B^N G_t) \) by \( \mu^N_t (BG_t) \). So we need the following lemma.

\textbf{Lemma 5.2} For any \( G \in C^{1,0} ([0, T] \times [0, 1]) \) and \( \varepsilon > 0 \),
\[ \limsup_{N \to +\infty} \frac{1}{N} \log P \left( \sup_{0 \leq t \leq T} \left| \mu^N_t (BG_t) - \mu^N_t (B^N G_t) \right| > \varepsilon \right) = -\infty . \]

\textbf{Proof} According to the uniform continuity of \( \lambda \) and \( G \), for any \( \delta > 0 \), there exists \( N(\delta) < +\infty \) such that
\[ \sup_{0 \leq t \leq T, 0 \leq x \leq 1} \left| BG_t (x) - B^N G_t (x) \right| < \delta \]
when \( N \geq N(\delta) \). Hence, when \( N \geq N(\delta) \),
\[ \sup_{0 \leq t \leq T} \left| \mu^N_t (B^N G_t) - \mu^N_t (BG_t) \right| \leq \frac{\delta}{N} \sum_{i=1}^N X^N_0 (i) . \]
According to Assumption (A) and Chebyshev’s inequality,
\[ P \left( \frac{\delta}{N} \sum_{i=1}^N X^N_0 (i) \geq \varepsilon \right) \leq \exp \left\{ - \frac{N \varepsilon}{\delta} + (e - 1) \sum_{i=1}^N \phi \left( \frac{i}{N} \right) \right\} \]
\[ = \exp \left\{ N \left( - \frac{\varepsilon}{\delta} + (e - 1) \int_0^1 \phi (x) dx + o(1) \right) \right\} . \tag{5.4} \]
Since \( \delta \) is arbitrary, Lemma 5.2 follows from Eq. 5.4.

According to Lemmas 5.1 and 5.2, we have the following conclusion.

\textbf{Lemma 5.3} Under Assumptions (A), for any compact set \( C \subseteq D([0, T], S) \),
\[ \limsup_{N \to +\infty} \frac{1}{N} \log P \left( \mu^N \in C \right) \leq - \inf_{\mu \in C} \left( I_{ini}(\mu_0) + I_{dyn}(\mu) \right) . \]
Proof For any \( \epsilon > 0 \), by Lemma 5.2,
\[
\limsup_{N \to +\infty} \frac{1}{N} \log P \left( \mu^N \in C \right) = \limsup_{N \to +\infty} \frac{1}{N} \log P \left( \mu^N \in C, \sup_{0 \leq t \leq T} \left| \mu^N_t (BG_t) - \mu^N_t (B^N G_t) \right| \leq \epsilon \right).
\] (5.5)

By Lemma 5.1 and Assumption (A), for any \( H \in C[0, 1] \) and \( G \in C^{1,0} ([0, T] \times [0, 1]) \),
\[
\mathbb{E} \left( e^{N \mu^N_0 (H) \Lambda^N_T (G)} \right) = \mathbb{E} \left( e^{N \phi(x)} \right) = e^N \left( \int_0^1 \phi(x) (e^{H(x)} - 1) dx + o(1) \right). \tag{5.6}
\]

For simplicity, we use \( r^N_{C, \epsilon} \) to denote the event
\[
\left\{ \mu^N \in C, \sup_{0 \leq t \leq T} \left| \mu^N_t (BG_t) - \mu^N_t (B^N G_t) \right| \leq \epsilon \right\}.
\]

On \( r^N_{C, \epsilon} \),
\[
e^{N \mu^N_0 (H) \Lambda^N_T (G)} \geq \exp \left\{ N \mu^N_0 (H) + N \mu^N_T (G_T) - N \mu^N_0 (G_0) - N \int_0^T \mu^N_s ((\partial_s + B)G_s) ds - NT \epsilon \right\}
\]
\[
\geq \exp \left\{ N \inf_{\mu \in C} \left\{ \mu_0 (H) + \mu_T (G_T) - \mu_0 (G_0) - \int_0^T \mu_s ((\partial_s + B)G_s) ds \right\} - NT \epsilon \right\}.
\]

Therefore, by Eq. 5.6,
\[
\mathbb{E} \left( e^{N \mu^N_0 (H)} \right) = e^N \left( \int_0^1 \phi(x) (e^{H(x)} - 1) dx + o(1) \right)
\]
\[
= \mathbb{E} \left( e^{N \mu^N_0 (H) \Lambda^N_T (G)} \right) \geq \mathbb{E} \left( e^{N \mu^N_0 (H) \Lambda^N_T (G)} 1_{r^N_{C, \epsilon}} \right)
\]
\[
\geq \mathbb{P} (r^N_{C, \epsilon}) \times \exp \left\{ N \inf_{\mu \in C} \left\{ \mu_0 (H) + \mu_T (G_T) - \mu_0 (G_0) - \int_0^T \mu_s ((\partial_s + B)G_s) ds \right\} - NT \epsilon \right\}
\]
and hence
\[
\limsup_{N \to +\infty} \frac{1}{N} \log \mathbb{P} (r^N_{C, \epsilon}) \leq \epsilon T - \inf_{\mu \in C} \left\{ \mu_0 (H) - \int_0^1 \phi(x) (e^{H(x)} - 1) dx \right\} 
\]
\[
+ \mu_T (G_T) - \mu_0 (G_0) - \int_0^T \mu_s ((\partial_s + B)G_s) ds \right\}.
\]

By Eq. 5.5,
\[
\limsup_{N \to +\infty} \frac{1}{N} \log \mathbb{P} (\mu^N \in C) \leq \epsilon T - \inf_{\mu \in C} \left\{ \mu_0 (H) - \int_0^1 \phi(x) (e^{H(x)} - 1) dx \right\} 
\]
\[
+ \mu_T (G_T) - \mu_0 (G_0) - \int_0^T \mu_s ((\partial_s + B)G_s) ds \right\}.
\]
Since $\epsilon, G, H$ are arbitrary,
\[
\limsup_{N \to +\infty} \frac{1}{N} \log P \left( \mu^N \subseteq C \right) \leq - \sup_{H \in C[0,1], \ G \in C^{1,0}(0,T) \times [0,1]} \inf_{\mu \in C} \left\{ \mu_0(H) - \int_0^1 \phi(x) \left( e^{H(x)} - 1 \right) dx + \mu_T(G_T) - \mu_0(G_0) - \int_0^T \mu_s ((\partial_s + B) G_s) ds \right\}.
\]

Since $C$ is compact, according to the minimax theorem given in [31],
\[
\begin{aligned}
\sup_{H \in C[0,1], \ G \in C^{1,0}(0,T) \times [0,1]} \inf_{\mu \in C} & \left\{ \mu_0(H) - \int_0^1 \phi(x) \left( e^{H(x)} - 1 \right) dx + \mu_T(G_T) - \mu_0(G_0) - \int_0^T \mu_s ((\partial_s + B) G_s) ds \right\} \\
= & \inf_{\mu \in C} \sup_{H \in C[0,1], \ G \in C^{1,0}(0,T) \times [0,1]} \left\{ \mu_0(H) - \int_0^1 \phi(x) \left( e^{H(x)} - 1 \right) dx + \mu_T(G_T) - \mu_0(G_0) - \int_0^T \mu_s ((\partial_s + B) G_s) ds \right\} \\
= & \inf_{\mu \in C} (I_{ini}(\mu_0) + I_{dyn}(\mu))
\end{aligned}
\]

and the proof is complete. \(\square\)

At last we give the proof of Eq. 2.5 for all closed $C$.

**Proof of Eq. 2.5** By Lemma 5.3, we only need to show that $\{\mu^N\}_{N \geq 1}$ are exponentially tight, i.e., for any $\epsilon > 0$, there exists compact $K = K(\epsilon) \subseteq D([0, T], S)$ such that
\[
\sup_{N \geq 1} \left\{ \left( P(\mu^N \notin K) \right)^{\frac{1}{N}} \right\} < \epsilon.
\]

According to the criteria given in [27], we only need to show that
\[
\limsup_{M \to +\infty} \limsup_{N \to +\infty} \frac{1}{N} \log P \left( \sup_{0 \leq t \leq T} \mu^N_t(H) > M \right) = -\infty \quad (5.7)
\]
for any $H \in C[0, 1]$ and
\[
\limsup_{\delta \to 0} \limsup_{N \to +\infty} \frac{1}{N} \log \sup_{\tau \in \mathcal{T}} P \left( \sup_{0 \leq t \leq \delta} \left| \mu^N_{t+\tau}(H) - \mu^N_{t}(H) \right| > \epsilon \right) = -\infty \quad (5.8)
\]
for any $\epsilon > 0$ and $H \in C[0, 1]$, where $\mathcal{T}$ is the set of stopping times of $\{X^N_t\}_{t \geq 0}$ bounded by $T$.  

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We first check (5.7). For any \( H \in C[0, 1], \)

\[
\sup_{0 \leq t \leq T} |\mu_t^N (H)| \leq \frac{\|H\|_\infty}{N} \sum_{i=1}^N X_0^N (i).
\]

Then, by Assumption (A) and Chebyshev’s inequality,

\[
P \left( \sup_{0 \leq t \leq T} |\mu_t^N (H)| > M \right) \leq e^{-N \left( M - \int_0^1 \phi(x)dx (e^{\|H\|_\infty} - 1) + o(1) \right)}
\]

and hence (5.7) holds.

Now we check (5.8). For any \( M \geq 1, \) we use \( B_M^N \) to denote the event \( \big\{ \frac{1}{N} \sum_{N=1}^N X_0^N (i) \leq M \big\}. \) For \( c > 0, \) by Eq. 5.1,

\[
\frac{\Lambda_{t+\tau}^N (cH)}{\Lambda_t^N (cH)} = \exp \left\{ cN \left( \mu_{t+\tau}^N (H) - \mu_t^N (H) - \int_{\tau}^{t+\tau} \mu_s^N (B_s^N H) ds \right) \right\}
\]

and hence

\[
\left\{ \sup_{0 \leq t \leq \delta} \left( \mu_{t+\tau}^N (H) - \mu_t^N (H) \right) > \epsilon \right\} \cap B_M^N
\]

\[
\subseteq \left\{ \sup_{0 \leq t \leq \delta} \frac{\Lambda_{t+\tau}^N (cH)}{\Lambda_t^N (cH)} \geq \exp \left\{ cN\epsilon - cNM\delta \|\lambda\|_\infty (e^{2\|H\|_\infty} - 1) \right\} \right\}.
\]

Then, since \( \left\{ \frac{\Lambda_{t+\tau}^N (cH)}{\Lambda_t^N (cH)} \right\}_{t \geq 0} \) is a martingale according to Lemma 5.1, by Doob’s inequality,

\[
\frac{1}{N} \log P \left( \sup_{0 \leq t \leq \delta} \left( \mu_{t+\tau}^N (H) - \mu_t^N (H) \right) > \epsilon, B_M^N \right) \leq -c\epsilon + cM\delta \|\lambda\|_\infty (e^{2\|H\|_\infty} - 1)
\]

and hence

\[
\lim_{\delta \to 0} \limsup_{N \to +\infty} \frac{1}{N} \log P \left( \sup_{0 \leq t \leq \delta} \left( \mu_{t+\tau}^N (H) - \mu_t^N (H) \right) > \epsilon, B_M^N \right) \leq -c\epsilon.
\]

Since \( c \) is arbitrary,

\[
\lim_{\delta \to 0} \limsup_{N \to +\infty} \frac{1}{N} \log P \left( \sup_{0 \leq t \leq \delta} \left( \mu_{t+\tau}^N (H) - \mu_t^N (H) \right) > \epsilon, B_M^N \right) = -\infty
\]

and hence

\[
\lim_{\delta \to 0} \limsup_{N \to +\infty} \frac{1}{N} \log P \left( \sup_{0 \leq t \leq \delta} \left( \mu_{t+\tau}^N (H) - \mu_t^N (H) \right) > \epsilon \right)
\]

\[
\leq \limsup_{N \to +\infty} \frac{1}{N} \log P \left( \left( B_M^N \right)^c \right).
\]

By Assumption (A) and Chebyshev’s inequality,

\[
\limsup_{N \to +\infty} \frac{1}{N} \log P \left( \left( B_M^N \right)^c \right) \leq -M + (e - 1) \int_0^1 \phi(x)dx.
\]
Since $M$ is arbitrary, we have
\[
\limsup_{\delta \to 0} \limsup_{N \to +\infty} \frac{1}{N} \log \sup_{\tau \in T} P \left( \sup_{0 \leq t \leq \delta} \left( \mu^N_{t+\tau}(H) - \mu^N_{\tau}(H) \right) > \epsilon \right) = -\infty. \tag{5.9}
\]

For $c > 0$, since \( \left\{ \frac{\Delta^N_{t+\tau}(-cH)}{\Delta^N_{t}(-cH)} \right\}_{t \geq 0} \) is also a martingale, an analysis similar with that leading to Eq. 5.9 shows that
\[
\limsup_{\delta \to 0} \limsup_{N \to +\infty} \frac{1}{N} \log \inf_{0 \leq t \leq \delta} \left( \mu^N_{t+\tau}(H) - \mu^N_{\tau}(H) \right) < -\infty \tag{5.10}
\]
Equation 5.8 follows from Eqs. 5.9 and 5.10 while the proof is complete.

6 Proof of Eq. 2.6

In this section, we give the proof of Eq. 2.6. For simplicity, in proofs we write a random variable $\mathcal{Q}_N$ as $o_{\exp(\frac{1}{N})}$ when $\{\mathcal{Q}_N\}_{N \geq 1}$ satisfies
\[
\limsup_{N \to +\infty} \frac{1}{N} \log P (|\mathcal{Q}_N| > \epsilon) = -\infty
\]
for any $\epsilon > 0$. We first give a more clear expression of $I_{ini}(\mu_0)$ and $I_{dyn}(\mu)$ for $\mu \in D_0$, where $D_0$ is defined as in Section 2.

**Lemma 6.1** Under Assumption (B), for any $\mu \in D_0$,
\[
I_{ini}(\mu_0) = \int_0^1 \psi(0, x) \log \psi(0, x) - \psi(0, x) \log \phi(x) + \phi(x) - \psi(0, x) dx 
\]
where $\psi(0, \cdot)(x) = \frac{d\mu_0}{dx}$ and there exists $G \in C([0, T] \times [0, 1])$ such that
\[
\partial_s \psi(s, x) = \int_0^1 \psi(s, y) \lambda(y, x) e^{G_s(x) - G_s(y)} - \psi(s, x) \lambda(x, y) e^{G_s(y) - G_s(x)} dy \tag{6.1}
\]
and
\[
I_{dyn}(\mu) = \int_0^T \int_0^1 \partial_s \psi(s, x) G_s(x) ds dx 
\]
\[
- \int_0^T \int_0^1 \psi(s, x) \lambda(x, y)(e^{G_s(y) - G_s(x)} - 1) ds dx dy,
\]
where $\psi(s, \cdot)(x) = \frac{d\mu_s}{dx}$.

**Proof** For the first part of Lemma 6.1,
\[
I_{ini}(\mu_0) = \int_0^1 \psi(0, x) \log \psi(0, x) - \psi(0, x) \log \phi(x) + \phi(x) - \psi(0, x) dx
\]
follows from the fact that
\[
\Xi(\theta) = \psi(0, x) \theta - \phi(x)(e^\theta - 1)
\]
gets maximum $\psi(0, x) \log \psi(0, x) - \psi(0, x) \log \phi(x) - \psi(0, x) + \phi(x)$ at $\theta = \log \frac{\psi(0, x)}{\phi(x)}$. 
For the second part, if $\psi \equiv 0$, then it is easy to check that this part holds with $G \equiv 1$. So we only need to deal with the case where $\psi \not\equiv 0$. Since $\int_0^1 \partial_s \psi(s, x) dx = 0$, $\psi \not\equiv 0$ implies that $\psi(s, \cdot) \not\equiv 0$ for any $0 \leq s \leq T$. Then, for any $0 \leq s \leq T$, it is easy to check that there exists a unique $C_s > 0$ such that

$$2C_s = \int_0^1 \sqrt{\left( \frac{\partial}{\partial s} \psi(s, y) \right)^2 + 4\psi(s, y)\lambda_1(y)\lambda_2(y)C_s dy}$$

according to the fact that

$$\varphi(x) = 2x - \int_0^1 \sqrt{\left( \frac{\partial}{\partial s} \psi(s, y) \right)^2 + 4\psi(s, y)\lambda_1(y)\lambda_2(y)xdy}$$

is a convex function with respect to $x$ while $\varphi(0) \leq 0$, $\varphi(+\infty) = +\infty$. Let

$$G_s(x) = \log \left( \frac{\partial_s \psi(s, x) + \sqrt{\left( \frac{\partial}{\partial s} \psi(s, x) \right)^2 + 4\psi(s, x)\lambda_1(x)\lambda_2(x)C_s}}{2C_s\lambda_2(x)} \right),$$

then, by direct calculation, it is easy to check that $G$ satisfies (6.1). For $\mu \in D_0$, according to the formula of integration by parts and the fact that $C^{1,0}([0, T] \times [0, 1])$ is dense in $C([0, T] \times [0, 1])$, $I_{dyn}(\mu) = \sup \left\{ \int_0^T \int_0^1 \partial_s \psi_s(x)H_s(x)dxds \right. - \int_0^T \int_0^1 \psi(s, x)\lambda(x, y)\left( e^{H_s(y) - H_s(x)} - 1 \right)dsdxdy : H \in C([0, T] \times [0, 1]) \right\}.$ For simplicity, we use $\vartheta(H)$ to denote

$$\int_0^1 \int_0^T \partial_s \psi_s(x)H_s(x)dxds - \int_0^T \int_0^1 \psi(s, x)\lambda(x, y)\left( e^{H_s(y) - H_s(x)} - 1 \right)dsdxdy.$$

For any $\epsilon > 0$ and any $H \in C([0, T] \times [0, 1])$, let $k_\epsilon(G, H) = \vartheta(G + \epsilon(H - G))$, then it is easy to check that $k_\epsilon$ is a concave function with respect to $\epsilon$ and $\frac{dk_\epsilon}{d\epsilon} = 0$ when $\epsilon = 0$ since $G$ satisfies (6.1). Hence, $\vartheta(G) = k_0 \geq k_1 = \vartheta(H)$ and

$$I_{dyn}(\mu) = \sup_{H \in C([0, T] \times [0, 1])} \vartheta(H) = \vartheta(G).$$

As a second step, we need to give the hydrodynamic limit of $\{X^N_t\}_{t \geq 0}$ under a transformed measure with the exponential martingale introduced in Section 5 as R-N derivative with respect to the original probability measure $P$. To give the precise statement of our result, we introduce some notations. For any positive $\gamma \in C[0, 1]$, we use $P_\gamma$ to denote the probability measure of our process $\{X^N_t\}_{t \geq 0}$ with initial condition where $\{X^N_0(i)\}_{1 \leq i \leq N}$ are independent and $X^N_0(i)$ follows Poisson distribution with mean $\gamma(i/N)$ for each $1 \leq i \leq N$. For any $G \in C^{1,0}([0, T] \times [0, 1])$, we use $\widehat{P}_\gamma^G$ to denote the probability measure such that

$$\frac{d\widehat{P}_\gamma^G}{dP_\gamma} = \Lambda_T^N(G).$$
For given $G \in C^{1,0}([0,T] \times [0,1])$ and $0 \leq t \leq T$, let $Q^G_t$ be the linear operator from $C[0,1]$ to $C[0,1]$ such that
\[ Q^G_t(f)(x) = \int_0^1 f(y)\lambda(y,x)e^{G_s(x) - G_s(y)} - f(x)\lambda(x,y)e^{G_s(y) - G_s(x)} \, dy \]
for any $x \in [0,1]$ and $f \in C[0,1]$, then it is easy to check that there exists $K < +\infty$ independent of $0 \leq t \leq T$ such that
\[ \|Q^G_t(f_1) - Q^G_t(f_2)\|_\infty \leq K \|f_1 - f_2\|_\infty \]
for any $f_1, f_2 \in C[0,1]$ and $0 \leq t \leq T$. As a result, according to the theory of ordinary differential equations on Banach spaces (see Theorem 1.2 of Chapter 19 of [20]), for positive $\gamma \in C[0,1]$ and $G \in C^{1,0}([0,T] \times [0,1])$, there exists a unique $\gamma^G \in C^{1,0}([0,T] \times [0,1])$ such that
\[
\begin{cases}
\frac{\partial}{\partial s}\gamma^G(s, x) = \int_0^1 \gamma^G(s, y)\lambda(y, x)e^{G_s(x) - G_s(y)} - G_s(y) - \gamma^G(s, x)\lambda(x, y)e^{G_s(y) - G_s(x)} \, dy \\
\gamma^G(0, x) = \gamma(x) \text{ for all } 0 \leq x \leq 1.
\end{cases}
\]
(6.2)

Then, we have the following lemma.

**Lemma 6.2** For positive $\gamma \in C[0,1]$, $\{\mu^N\}_{N \geq 1}$ converges in $\tilde{P}_\gamma^G$-probability to $\mu^\gamma,G$, where
\[ \mu^\gamma,G(t, dx) = \gamma^G(t, x) \, dx \]
for $0 \leq t \leq T$.

To prove Lemma 6.2, we need the following lemma.

**Lemma 6.3** For given $G \in C^{1,0}([0,T] \times [0,1])$ and positive $\gamma \in C[0,1]$, under $\tilde{P}_\gamma^G$,
\[ \sup_{H \in C[0,1], \|H\|_\infty = 1} \sum_{0 \leq t \leq T} \left( \mu^N_t(H) - \mu^{-N}_t(H) \right)^2 = o(\exp(1/N)). \]
(6.4)

*Proof of Lemma 6.3* For any $H \in C[0,1]$ with $\|H\|_\infty = 1$, on each jumping moment $s$ of $\{X^N_t\}_{t \geq 0}$,
\[ |\mu^N_s(H) - \mu^{-N}_s(H)|^2 \leq \left( \frac{2\|H\|_\infty}{N} \right)^2 = \frac{4}{N^2}. \]
For simplicity, we use $\zeta^N$ to denote $\sup_{H \in C[0,1], \|H\|_\infty = 1} \sum_{0 \leq t \leq T} \left( \mu^N_t(H) - \mu^{-N}_t(H) \right)^2$. For $M \geq 1$, since each ball jumps at rate at most $\|\lambda\|_\infty$ under $P_\gamma$, condition on
\[ \frac{1}{N} \sum_{i=1}^N X^N_0(i) \leq M, \]
$\zeta^N$ is stochastically dominated from above by $\frac{4}{N^2} Y(NM\|\lambda\|_\infty T)$ under $P_\gamma$, where $\{Y(t)\}_{t \geq 0}$ is the Poisson process with rate 1 defined as in Section 3. Therefore, by Chebyshev’s inequality,
\[ P_\gamma \left( \zeta^N \geq \epsilon, \frac{1}{N} \sum_{i=1}^N X^N_0(i) \leq M \right) \leq e^{-\frac{4N^2}{\epsilon^2} \epsilon^2 \|\lambda\|_\infty^2 MNT} \]
and
\[
\lim_{N \to +\infty} \frac{1}{N} \log P_\gamma \left( \zeta^N \geq \epsilon, \frac{1}{N} \sum_{i=1}^N X_0^N(i) \leq M \right) = -\infty.
\]
Therefore,
\[
\lim_{N \to +\infty} \frac{1}{N} \log P_\gamma \left( \zeta^N \geq \epsilon \right) \leq \lim_{N \to +\infty} \frac{1}{N} \log P_\gamma \left( \frac{1}{N} \sum_{i=1}^N X_0^N(i) \geq M \right).
\]
Under \(P_\gamma\),
\[
P_\gamma \left( \frac{1}{N} \sum_{i=1}^N X_0^N(i) \geq M \right) \leq e^{-MN} e^{(e-1) \sum_{i=1}^N \gamma(x)}
\]
and hence
\[
\lim_{N \to +\infty} \frac{1}{N} \log P_\gamma \left( \zeta^N \geq \epsilon \right) \leq -M + (e-1) \int_0^1 \gamma(x) dx.
\]
Since \(M\) is arbitrary, Eq. 6.4 holds under \(P_\gamma\). By Eq. 5.3, it is easy to check that
\[
\mathbb{E}_\gamma \left( \left( \Lambda_T^N(G) \right)^2 \right) = e^{NO(1)}.
\]
Then, Eq. 6.4 holds under \(\hat{P}_\gamma^G\) according to Hölder’s inequality. \(\square\)

Now we give the proof of Lemma 6.2. The proof follows a strategy similar with those introduced in [34, 36], where a generalized version of Girsanov’s theorem introduced in [30] will be utilized.

**Proof of Lemma 6.2** According to the definition of \(\mu_{\gamma, G}^t\) and the formula of integration by parts, for any \(0 \leq t \leq T\) and \(H \in C[0, 1]\) with \(\|H\|_{\infty} = 1\),
\[
\mu_{\gamma, G}^t(H) = \int_0^1 \gamma(x) H(x) dx + \int_0^t \mu_{\gamma, G}^s \left( \nu_s^G H \right) ds,
\]
where
\[
(\nu_s^G H)(x) = \int_0^1 \lambda(x, y) e^{G_s(y) - G_s(x)} (H(y) - H(x)) dy
\]
for any \(x \in C[0, 1]\).
For any \(H \in C[0, 1]\) with \(\|H\|_{\infty} = 1\), we define \(\Theta^N_H \in C(X)\) as
\[
\Theta^N_H(x) = \frac{1}{N} \sum_{i=1}^N x(i) H \left( \frac{i}{N} \right),
\]
then \(\Theta^N_H(X_i^N) = \mu_i^N(H)\) and, as we have defined in Section 5,
\[
\mathcal{U}_i^N (\Theta^N_H) = \mu_i^N(H) - \mu_0^N(H) - \int_0^t \mathcal{L}_N \nu_i^N(H) ds
\]
is a martingale under \(P_\gamma\). Let \(f_G^N\) be defined as in Section 5, then we define
\[
\tilde{\mathcal{U}}_i^N (f_G^N) = \int_0^t \frac{1}{f_G^N(s, X_s^N)} d\mathcal{U}_i^N (f_G^N)
\]
for \(0 \leq t \leq T\). Then, by Eq. 5.2,
\[
d\Lambda_t^N(G) = \Lambda_{i_0}^N(G) d\tilde{\mathcal{U}}_i^N (f_G^N). \quad (6.6)
\]
By Eq. 6.6, Lemma 5.1 and Theorem 3.2 of [30],
\[
\hat{U}^N_t(\Theta^N_H) := U^N_t(\Theta^N_H) - \langle U^N(\Theta^N_H), \hat{U}^N(f_G^N) \rangle_t,
\]
is a martingale under \( \hat{P}^G_y \) and
\[
\left[ \hat{U}^N(\Theta^N_H), \hat{U}^N(\Theta^N_H) \right]_t = \left[ U^N(\Theta^N_H), U^N(\Theta^N_H) \right]_t = \sum_{0 \leq s \leq t} \left( \mu^N_s(H) - \mu^N_s(H) \right)^2
\]
under both \( P_y \) and \( \hat{P}^G_y \). By Lemma 6.3 and Doob’s inequality,
\[
\sup_{0 \leq t \leq T, H \in \mathcal{C}[0,1]} \left| \hat{U}^N_t(\Theta^N_H) \right| = o_p(1) \quad (6.7)
\]
under \( \hat{P}^G_y \). According to the definition of \( \hat{U}^N_t(f_G^N) \),
\[
d(\langle U^N(\Theta^N_H), \hat{U}^N(f_G^N) \rangle_t) = \frac{1}{f_G^N(t - X^N_t)} d(\langle U^N(\Theta^N_H), U^N(f_G^N) \rangle_t).
\]
By Dynkin’s martingale formula,
\[
d(\langle U^N(\Theta^N_H), U^N(f_G^N) \rangle_t) = \left( \mathcal{L}_N(f_G^N \Theta^N_H) - \Theta^N_H \mathcal{L}_N f_G^N - f_G^N \mathcal{L}_N \Theta^N_H \right) (t, X^N_t) dt.
\]
By the definition of \( \mathcal{L}_N \) and direct calculations, it is not difficult to check that
\[
\left( \mathcal{L}_N(f_G^N \Theta^N_H) - \Theta^N_H \mathcal{L}_N f_G^N - f_G^N \mathcal{L}_N \Theta^N_H \right) (t, X^N_t)
\]
\[
= f_G^N(t, X^N_t) \left( \mu_t^{N, u G,N}(H) - \mathcal{L}_N \mu_t^{N, H} \right),
\]
where
\[
(v_t^{G,N}(H))(x) = \sum_{j=1}^N \lambda \left( x, \frac{j}{N} \right) e^{G_t \left( \frac{j}{N} \right) - G_t(x)} \left( H \left( \frac{j}{N} \right) - H(x) \right)
\]
for any \( x \in [0,1] \). As a result, under \( \hat{P}^G_y \),
\[
\mu_t^{N}(H) = \mu_0^{N}(H) + \int_0^t \mu_s^{N, u G,N}(H) ds + o_p(1)
\]
\[
= \int_0^1 \gamma(x) H(x) dx + \int_0^t \mu_s^{N, u G,N}(H) ds + o_p(1).
\]
Note that in the above equation we utilize the fact that \( \mu_0^{N}(H) = \int_0^1 \gamma(x) H(x) dx + o_p(1) \) under \( \hat{P}^G_y \) and this \( o_p(1) \) can be chosen uniformly for all \( H \in \mathcal{C}[0,1] \) with \( \|H\|_\infty = 1 \).

According to an analysis similar with that given in the proof of Lemma 5.2, it is not difficult to check that
\[
\sup_{0 \leq t \leq T, H \in \mathcal{C}[0,1], \|H\|_\infty = 1} \left| \mu_t^{N, u G,N}(H) - \mu_t^{N, u G,H} \right| = o_{\exp}(\frac{1}{N})
\]
under \( P_y \). Then, according to Hölder’s inequality,
\[
\sup_{0 \leq t \leq T, H \in \mathcal{C}[0,1], \|H\|_\infty = 1} \left| \mu_t^{N, u G,N}(H) - \mu_t^{N, u G,H} \right| = o_{\exp}(\frac{1}{N}) = o_p(1)
\]
As a result, for $0 \leq t \leq T$ and $H \in C[0, 1]$ with $\|H\|_\infty = 1$,

$$
\mu_i^N(H) = \int_0^1 \gamma(x) H(x) dx + \int_0^t \mu_i^N(\nu_i^G H) ds + o_p(1),
$$

(6.8)

where $o_p(1)$ can be chosen uniformly for all $H \in C[0, 1]$ with $\|H\|_\infty$. According to Eqs. 6.5, 6.8 and the fact that $\|\nu_i^G H\|_\infty \leq 2 \|H\|_\infty \|\lambda\|_\infty e^{2\|G\|_\infty}$,

$$
\sup_{H \in C[0, 1], \|H\|_1} \left| \mu_i^{\nu_i^G H} - \mu_i^N(H) \right| 
\leq o_p(1) + 2 \|\lambda\|_\infty e^{2\|G\|_\infty} \int_0^t \sup_{H \in C[0, 1], \|H\|_1} \left| \mu_i^{\nu_i^G H} - \mu_i^N(H) \right| ds
$$

for any $0 \leq t \leq T$ under $\mathcal{F}_\gamma$. Then, by Gronwall’s inequality,

$$
\sup_{H \in C[0, 1], \|H\|_1} \left| \mu_i^{\nu_i^G H} - \mu_i^N(H) \right| \leq o_p(1) \exp \left\{ 2t \|\lambda\|_\infty e^{2\|G\|_\infty} \right\}
$$

(6.9)

under $\mathcal{F}_\gamma$. Lemma 6.2 follows from Eq. 6.9 directly.

At last we give the proof of Eq. 2.6.

**Proof of Eq. 2.6** For any $\epsilon > 0$, there exists $\mu^\epsilon \in D_0 \cap O$ such that

$$
I_{ini}(\mu_0^\epsilon) + I_{dyn}(\mu^\epsilon) \leq \inf_{\mu \in D_0 \cap O} \left( I_{ini}(\mu_0) + I_{dyn}(\mu) \right) + \epsilon.
$$

By Lemma 6.1, there exists $G^\epsilon \in C([0, T] \times [0, 1])$ such that

$$
\partial_s \psi^\epsilon(s, x) = \int_0^1 \psi^\epsilon(s, y) \lambda(y, x) e^{G_s^\epsilon(y) - G_t^\epsilon(x)} - \psi^\epsilon(s, x) \lambda(x, y) e^{G_s^\epsilon(y) - G_t^\epsilon(x)} dy
$$

and

$$
I_{dyn}(\mu^\epsilon) = \int_0^T \int_0^1 \partial_s \psi^\epsilon(s, x) G_s^\epsilon(x) ds dx
$$

$$
- \int_0^T \int_0^1 \psi^\epsilon(s, x) \lambda(x, y) \left( e^{G_s^\epsilon(y) - G_t^\epsilon(x)} - 1 \right) ds dy
$$

$$
= \mu_0^\epsilon(G_0^\epsilon) - \mu_0^\epsilon(G_0^\epsilon) - \int_0^T \mu_s^\epsilon ((\partial_s + B) G_s^\epsilon) ds
$$

while

$$
I_{ini}(\mu_0^\epsilon) = \int_0^1 \psi^\epsilon(0, x) \log \psi^\epsilon(0, x) - \psi^\epsilon(0, x) \log \phi(x) + \phi(x) - \psi^\epsilon(0, x) dx,
$$

where $\psi_s^\epsilon = \frac{d\psi^\epsilon_s}{dx}$. Since $C_{1,0}^1([0, T] \times [0, 1])$ is dense in $C([0, T] \times [0, 1])$ under the $l_\infty$ norm, there exists $C_{1,0}^1([0, T] \times [0, 1])$-valued sequence $\{G^n\}_{n \geq 1}$ such that $\|G^n - G^\epsilon\|_\infty \to 0$ as $n \to +\infty$. For each $n \geq 1$ and $x \in [0, 1]$, we define $\gamma^n(x) = \psi^n(0, x) + \frac{\epsilon}{n}$ and then $\gamma^n$ is
strictly positive. We use $\zeta^n$ to denote $\mu^{\gamma^n,G^n}$ defined as in Eq. 6.3. Then, by Lemma 6.1,

$$I_{dyn}(\zeta^n) = \int_0^T \int_0^1 \partial_s \psi^n_s(x)G^n_s(x)dsdx$$

$$- \int_0^T \int_0^1 \psi^n_s(x)\lambda(x,y)\left(e^{G^n_s(y)}-G^n_s(x)\right)dsdx$$

$$= \zeta^n_T(G^n_T) - \zeta^n_0(G^n_0) - \int_0^T \zeta^n_s((\partial_s + B)G^n_s)ds$$

(6.10)

and

$$I_{ini}(\zeta^n_0) = \int_0^1 \gamma^n(x)\log\gamma^n(x) - \gamma^n(x)\log\phi(x) - \gamma^n(0,x)dx,$$

where $\psi^n_s = \frac{d\zeta^n_s}{dx}, \psi^n_0 = \frac{d\zeta^n_0}{dx} = \gamma^n$ while

$$\partial_s \psi^n(s,x) = \int_0^1 \psi^n(s,y)\lambda(y,x)e^{G^n_s(y)} - \psi^n(s,x)\lambda(x,y)e^{G^n_s(y)}dy.$$ 

Furthermore, according to Gronwall’s inequality, it is not difficult to check that $\zeta^n \rightarrow \mu^\epsilon$ in $D([0,T],S)$ and $I_{dyn}(\zeta^n) \rightarrow I_{dyn}(\mu^\epsilon), I_{ini}(\zeta^n_0) \rightarrow I_{ini}(\mu^\epsilon_0)$. Since $O$ is open while $\mu^\epsilon \in O$, there exists integer $m = m(\epsilon) \geq 1$ such that

$$I_{ini}(\zeta^n_0) + I_{dyn}(\zeta^n) \leq I_{ini}(\mu^n_0) + I_{dyn}(\mu^n) + \epsilon$$

and $\mu^n \in O$.

For $N \geq 1$, we use $W_N$ to denote the subset

$$\left\{ \mu \in D([0,T],S) : \sup_{0 \leq s \leq T} \mu_s(BG^n_s) - \mu_s(B^NG^n_s) < \epsilon \right\}$$

of $D([0,T],S)$ while use $Q$ to denote the subset

$$\left\{ \mu \in D([0,T],S) : \left| (\zeta^m - \mu)_T(G^n_T) \right. \right.$$<br>$$- (\zeta^m - \mu)_0(G^n_0) - \int_0^T (\zeta^m - \mu)_s((\partial_s + B)G^n_s)ds \left| < \epsilon \right. \}$$

of $D([0,T],S)$. According to Lemma 5.2 and Hölder’s inequality, it is easy to check that

$$\lim_{N \rightarrow +\infty} P_{\gamma^m}^{G^n_m}(\mu^N \in W_N) = \lim_{N \rightarrow +\infty} P_{\gamma^m}^{G^n_m}(\mu^N \in W_N) = 1.$$ 

(6.11)

By Lemma 6.2, $\mu^N \rightarrow \mu^{\gamma^m,G^m} = \tau^m \in O$ in $P_{\gamma^m}^{G^n_m}$-probability and hence

$$\lim_{N \rightarrow +\infty} P_{\gamma^m}^{G^n_m}(\mu^N \in Q \cap O) = 1.$$ 

Then, by Eq. 6.11,

$$\lim_{N \rightarrow +\infty} P_{\gamma^m}^{G^n_m}(\mu^N \in W_N \cap Q \cap O) = 1.$$ 

(6.12)

On the event $\{\mu^N \in W_N \cap Q\}$, by Eqs. 5.1 and 6.10,

$$\Lambda_T^N(G^m) \leq \exp\left\{ N (I_{dyn}(\zeta^m) + (T + 1)\epsilon) \right\}$$

and hence

$$\frac{dP_{\gamma^m}^{G^n_m}}{dP_{\gamma^m}^{G^n_m}} \geq \exp\left\{ -N (I_{dyn}(\zeta^m) + (T + 1)\epsilon) \right\}.$$
According to the definition of $P_{\gamma m}$ and Assumption (A),
\[ \frac{dP}{dP_{\gamma m}} = \prod_{i=1}^{n} \left[ e^{-\phi(i) + \gamma_m(i) \left( \frac{\phi(i)}{\gamma_m(i)} \right)^N(i)} \right] \]
\[ = \exp \left\{ -N \left[ \mu_0^N (\log \gamma_m - \log \phi) + \int_0^1 \phi(x)dx - \int_0^1 \gamma_m(x)dx + o(1) \right] \right\}. \]

Under $\hat{G}_\gamma^m$, since $\xi_0^m(dx) = \gamma_m(x)dx$, it is easy to check that $\mu_0^N \to \xi_0^m$. As a result, let $Q_1$ be the subset
\[ \{ \mu \in D([0, T], S) : |\mu_0 (\log \phi - \log \gamma_m) - \xi_0^m (\log \phi - \log \gamma_m)| < \epsilon \} \]
of $D([0, T], S)$, then
\[ \lim_{N \to +\infty} \hat{G}_\gamma^m \left( \mu^N \in Q_1 \right) = 1 \]
and hence
\[ \lim_{N \to +\infty} \hat{G}_\gamma^m \left( \mu^N \in O \cap W_N \cap Q \cap Q_1 \right) = 1 \] (6.13)
by Eq. 6.12. On $Q_1$, by Lemma 6.1,
\[ \frac{dP}{dP_{\gamma m}} \geq \exp \left\{ -N \left[ I_{ini}(\xi_0^m) + \epsilon + o(1) \right] \right\}. \]

Hence, on $W_N \cap Q \cap Q_1$,
\[ \frac{dP}{dP_{\gamma m}} \geq \exp \left\{ -N \left[ I_{dyn}(\xi_0^m) + I_{ini}(\xi_0^m) + (T + 2)\epsilon + o(1) \right] \right\}. \]
As a result, by Eq. 6.13,
\[ P \left( \mu^N \in O \right) \geq P \left( \mu^N \in O \cap W_N \cap Q \cap Q_1 \right) \]
\[ = \mathbb{E}_\gamma^m \left( \frac{dP}{dP_{\gamma m}} \right)_{O \cap W_N \cap Q \cap Q_1} \]
\[ \geq \exp \left\{ -N \left[ I_{dyn}(\xi_0^m) + I_{ini}(\xi_0^m) + (T + 2)\epsilon + o(1) \right] \right\} (1 + o(1)). \]

Therefore,
\[ \liminf_{N \to +\infty} \frac{1}{N} \log P \left( \mu^N \in O \right) \geq - \left( I_{dyn}(\xi_0^m) + I_{ini}(\xi_0^m) \right) - (T + 2)\epsilon \]
\[ \geq - \left( I_{dyn}(\mu^\epsilon) + I_{ini}(\mu_0^\epsilon) \right) - (T + 3)\epsilon \]
\[ \geq - \inf_{\mu \in D_0 \cap O} \left( I_{ini}(\mu_0) + I_{dyn}(\mu) \right) - (T + 4)\epsilon. \]

Since $\epsilon$ is arbitrary, Eq. 2.6 holds.

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A Appendix

A.1 Proof of Lemma 2.2

**Proof of Lemma 2.2** We first construct a weak solution \( \mu \) of Eq. 2.1 to show the existence. We use \( \|f\|_\infty \) to denote the \( l_\infty \)-norm of \( f \) for any \( f \in C[0,1] \) and \( \|\lambda\|_\infty \) to denote \( \sup_{0 \leq x,y \leq 1} \lambda(x,y) \). For each \( f \in C[0,1] \), we define

\[
(P_3 f)(x) = \int_0^1 \lambda(y,x) f(y) dy
\]

for any \( x \in [0,1] \). Since \( \|P_2 f\|_\infty, \|P_3 f\|_\infty \leq \|\lambda\|_\infty \|f\|_\infty \) for any \( f \in C[0,1] \), the linear \( C[0,1] \)-valued ODE

\[
\frac{d}{dt} \rho_t = (P_3 - P_2) \rho_t, \quad 0 \leq t \leq T,
\]

\[
\rho_0(x) = \phi(x), \quad 0 \leq x \leq 1,
\]

satisfies Lipschitz’ condition and hence has the unique solution

\[
\rho_t = e^{t(P_3 - P_2)} \phi = \sum_{k=0}^{+\infty} \frac{t^k (P_3 - P_2)^k}{k!} \phi.
\]

Let \( \mu_t(dx) = \rho_t(x) dx \) for any \( 0 \leq t \leq T \), then \( \mu = \{\mu_t\}_{0 \leq t \leq T} \) is a weak solution of Eq. 2.1 since

\[
\int_0^1 f(x) (P_3 \rho_t)(x) dx = \int_0^1 (P_1 f)(x) \rho_t(x) dx.
\]

Now we only need to show the uniqueness. Let \( \mu^1, \mu^2 \) be two weak solutions of Eq. 2.1 and \( \nu = \mu^1 - \mu^2 \), then

\[
\nu_t(f) = \int_0^t \nu_s ((P_1 - P_2) f) ds
\]

for any \( 0 \leq t \leq T \). For any \( \mathcal{A} \in S \), we define \( \|\mathcal{A}\|_\infty = \sup \{\mathcal{A}(f) : \|f\|_\infty = 1\} \). Then, for any \( f \) satisfying \( \|f\|_\infty = 1 \),

\[
|\nu_t ((P_1 - P_2) f) | \leq \|v_t\|_\infty \|P_1 - P_2\|_1 \leq 2 \|v_t\|_\infty \|\lambda\|_\infty.
\]

Therefore, for any \( 0 \leq t \leq T \),

\[
\|\nu_t\|_\infty \leq 2 \|\lambda\|_\infty \int_0^t \|v_s\|_\infty ds.
\]

Then, by Gronwall’s inequality, \( \|\nu_t\|_\infty \leq 0 e^{2t\|\lambda\|_\infty} = 0 \) for any \( 0 \leq t \leq T \) and hence \( \mu^1 = \mu^2 \).

A.2 The uniqueness of solution to Eq. 2.4

In this subsection we give an outline of the proof of the uniqueness of solution to Eq. 2.4. Our proof follows an analysis similar with that given in the proof of Theorem 1.4 of [15].

**Proof of uniqueness to Eq. 2.4** For each \( t \geq 0 \) and \( H \in C([0,1]) \), we define

\[
A_t(H) = V_t(H) - V_0(H) - \int_0^t V_s ((P_1 - P_2) H) ds
\]
\[ \Xi_t(H) = V_t^2(H) - V_0^2(H) - \int_0^t 2V_s(H)V_s((P_1 - P_2)H)\,ds - \int_0^t \|b_sH\|^2_2\,ds. \]

According to the second statement in the definition of \( \{V_t\}_{t \geq 0} \), it is easy to check that \( \{A_t(H)\}_{t \geq 0} \) and \( \{\Xi_t(H)\}_{t \geq 0} \) are both local martingales. Since
\[ dV_t(H) = V_t((P_1 - P_2)H)\,dt + dA_t(H), \]
by Itô’s formula,
\[ dV_t^2(H) = 2V_t(H)dV_t(H) + d\langle A(H)\rangle_t \]
and hence
\[ \left\{ V_t^2(H) - V_0^2(H) - \int_0^t 2V_s(H)V_s((P_1 - P_2)H)\,ds - \langle A(H)\rangle_t \right\}_{t \geq 0} \]
is a local martingale. Then, since \( \{\Xi_t\}_{t \geq 0} \) is also a local martingale,
\[ \langle A(H)\rangle_t = \int_0^t \|b_sH\|^2_2\,ds. \]

Let \( i = \sqrt{-1} \) and
\[ J_t(H) = \exp \left\{ iA_t(H) + \frac{1}{2} \langle A(H)\rangle_t \right\} = \exp \left\{ iA_t(H) + \frac{1}{2} \int_0^t \|b_sH\|^2_2\,ds \right\} \]
for any \( t \geq 0 \), then \( \{J_t(H)\}_{0 \leq t \leq T} \) is a local martingale according to Itô’s formula and hence a martingale since \( \{J_t(H)\}_{0 \leq t \leq T} \) are uniformly bounded. As a result, let \( \Upsilon^t_s(H) = \frac{J_s(H)}{J_t(H)} \) for any \( 0 \leq s < t \), then
\[ \mathbb{E} \left( \Upsilon^t_s(H)\big|V_u, u \leq s \right) = 1 \quad \text{(A.1)} \]
for any \( 0 \leq s < t \) and \( H \in C([0, 1]) \). For given \( 0 \leq t_1 < t_2 \leq T \) and integers \( 0 \leq k \leq n \), let \( \sigma_{n,k} = t_1 + \frac{k(t_2 - t_1)}{n} \). For any \( t \geq 0 \), let \( H_t = e^{i(P_1 - P_2)H} \), i.e., \( \{H_t\}_{t \geq 0} \) is the solution to
\[ \left\{ \begin{array}{l}
\frac{d}{dt} H_t = (P_1 - P_2)H_t, \\
H_0 = H.
\end{array} \right. \]

It is easy to check that \( \{H_t\}_{0 \leq t \leq T} \) is continuous in \( C([0, 1]) \) and hence
\[ \lim_{n \to +\infty} \left( \sum_{k=0}^{n-1} \int_{\sigma_{n,k}}^{\sigma_{n,k+1}} V_u \left( (P_1 - P_2)H_{t_2-u} \right) du \right. \]
\[ \left. - \sum_{k=0}^{n-1} \int_{\sigma_{n,k}}^{\sigma_{n,k+1}} V_{\sigma_{n,k}} \left( (P_1 - P_2)H_{t_2-u} \right) du \right) = 0 \]
almost surely. Furthermore, since
\[ \sum_{k=0}^{n-1} \int_{\sigma_{n,k}}^{\sigma_{n,k+1}} V_{\sigma_{n,k}} \left( (P_1 - P_2)H_{t_2-u} \right) du = \sum_{k=0}^{n-1} V_{\sigma_{n,k}} \left( \int_{\sigma_{n,k}}^{\sigma_{n,k+1}} (P_1 - P_2)H_{t_2-u} \,du \right) \]
\[ = - \sum_{k=0}^{n-1} V_{\sigma_{n,k}} \left( H_{t_2-\sigma_{n,k+1}} - H_{t_2-\sigma_{n,k}} \right), \]

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we have
\[
\lim_{n \to +\infty} \prod_{k=0}^{n-1} \gamma_{\sigma_{n,k}}^{\sigma_{n,k+1}} (H_{t_2} - \sigma_{n,k+1}) = \frac{Y_{t_2}}{Y_{t_1}}
\]
almost surely, where
\[
Y_t = \exp \left\{ iV_t (H_{t_2} - t) + \frac{1}{2} \int_0^t \| b_u H_{t_2} - u \|_2^2 du \right\}
\]
for \(0 \leq t \leq t_2\). Since \(\prod_{k=0}^{n-1} \gamma_{\sigma_{n,k}}^{\sigma_{n,k+1}} (H_{t_2} - \sigma_{n,k+1})\) are uniformly bounded, the convergence above is also in \(L^1\). As a result,
\[
E(Y_t, u \leq t_1) = \lim_{n \to +\infty} \mathbb{E}(Y_t, \prod_{k=0}^{n-1} \gamma_{\sigma_{n,k}}^{\sigma_{n,k+1}} (H_{t_2} - \sigma_{n,k+1}) | V_u, u \leq t_1)
\]
in \(L^1\). Repeatedly utilizing (A.1) by \(n\) times, we have
\[
E(Y_t, \prod_{k=0}^{n-1} \gamma_{\sigma_{n,k}}^{\sigma_{n,k+1}} (H_{t_2} - \sigma_{n,k+1}) | V_u, u \leq t_1) = Y_{t_1}
\]
and hence
\[
E(\exp(iV_t (H)) | V_u, u \leq t_1) = e^{iV_t (H_{t_2} - t_1) - \frac{1}{2} \int_0^t \| b_u H_{t_2} - u \|_2^2 du}
\] (A.2)
for any \(0 \leq t_1 < t_2 \leq T\) and \(H \in C([0, 1])\). The uniqueness of \(\{V_t\}_{t \geq 0}\) follows from Eq. A.2 and the given distribution of \(V_0\).

**A.3 A generalized N-urn Ehrenfest model with a metastable state**

Throughout this subsection we assume that our model is under Assumptions (A) and (B) given in Section 2. We will show that in this case the model has a metastable state, i.e, \(\lim_{t \to +\infty} \mu_t = \mu_\infty\) for some \(\mu_\infty \in S\), where \(\mu_{t \geq 0}\) satisfies that \(\{\mu_t\}_{0 \leq t \leq T}\) is the solution to Eq. 2.1 for any \(T > 0\).

**The proof of the existence of \(\mu_\infty\)** By Lemma 2.2, \(\mu_t(dx) = \rho(t, x) dx\), where
\[
\frac{d}{dt} \rho(t, x) = -m\lambda_1(x) \rho(t, x) + \lambda_2(x) r(t)
\]
under Assumption (B), where \(m = \int_0^1 \lambda_2(y) dy\) and \(r(t) = \int_0^1 \lambda_1(y) \rho(t, y) dy\). Let \(K(t, x) = \frac{\lambda_1(x)}{\lambda_2(x)} \rho(t, x), \hat{\lambda}_1(x) = m\lambda_1(x), \hat{\lambda}_2(x) = \frac{\lambda_2(x)}{m}\), then
\[
\frac{d}{dt} K(t, x) = \hat{\lambda}_1(x) \int_0^1 (K(t, y) - K(t, x)) \hat{\lambda}_2(y) dy.
\] (A.3)

Let \(\{\mathcal{R}_t\}_{t \geq 0}\) be the Markov process with state space \([0, 1]\) and generator \(\mathcal{A}\) given by
\[
\mathcal{A} f(x) = \hat{\lambda}_1(x) \int_0^1 (f(y) - f(x)) \hat{\lambda}_2(y) dy
\]
for any \(f \in C[0, 1]\) and \(x \in [0, 1]\), i.e, conditioned on \(\mathcal{R}_t = x\), the process waits for an exponential time with rate \(\hat{\lambda}_1(x)\) to escape from \(x\) and choose the next state \(y\) according to the probability measure \(\hat{\lambda}_2(y) dy\). Then, by Eq. A.3,
\[
K(t, x) = E_x K(0, \mathcal{R}_t).
\] (A.4)
It is easy to check that $\pi(x)dx$ is a reversible distribution of $\{R_t\}_{t \geq 0}$, where

$$\pi(x) = \frac{\hat{\lambda}_2(x)}{\hat{\lambda}_1(x)} \int_0^1 \frac{\hat{\lambda}_2(y)}{\hat{\lambda}_1(y)} dy$$

for any $x \in [0, 1]$. For any probability measure $\nu$ on $[0, 1]$ absolutely continuous with respect to Lebesgue measure, we claim that $R_t$ converges weakly to $\pi(x)dx$ as $t \to +\infty$ conditioned on $R_0$ is with distribution $\nu$. We prove this claim later. As a result, for any positive $f \in C[0, 1]$ with $\int_0^1 f(x)dx = 1$,

$$\lim_{t \to +\infty} \int_0^1 f(x)K(t, x)dx = \lim_{t \to +\infty} \int_0^1 f(x)E_x K(0, R_t)dx = \int_0^1 \pi(x)K(0, x) dx \quad (A.5)$$

by Eq. A.4. Since $K(0, x) = \frac{\hat{\lambda}_1(x)}{\hat{\lambda}_2(x)} \rho(0, x) = \frac{\hat{\lambda}_1(x)}{\hat{\lambda}_2(x)} \phi(x)$ and $\frac{h(x)}{\int_0^1 h(x)dx}$ is a probability density for any positive $h \in C[0, 1]$, Eq. A.5 implies that $\lim_{t \to +\infty} \mu_t = \mu_{\infty}$, where

$$\mu_{\infty}(dx) = \frac{\int_0^1 \phi(u)du \hat{\lambda}_2(x)}{\int_0^1 \frac{\hat{\lambda}_2(u)}{\hat{\lambda}_1(u)} du \hat{\lambda}_1(x)} dx.$$  

At last we prove our claim. The proof follows the entropy method given in Section 2.4 of [22] and hence we only give an outline. For any $\nu$ absolutely continuous with respect to Lebesgue measure, let $E(\nu)$ be the entropy of $\nu$ relative to $\pi$, i.e,

$$E(\nu) = \int_0^1 q \left( \frac{f_\nu(x)}{\pi(x)} \right) \pi(x)dx,$$

where $q(x) = x \log x$ and $f_\nu$ is the probability density of $\nu$. For any $t \geq 0$, let $\nu_t$ be the probability measure of $R_t$ conditioned on $R_0$ with distribution $\nu$. By Chapman-Kolmogorov equation, $\nu_t$ is also absolutely continuous with respect to Lebesgue measure and its probability density $\{f_\nu(t, x) : x \in [0, 1]\}$ is the solution to the equation

$$\begin{cases} \frac{d}{dt} f_\nu(t, x) = -\hat{\lambda}_1(x) f_\nu(t, x) + \hat{\lambda}_2(x) \int_0^1 f_\nu(t, y) \hat{\lambda}_1(y) dy, \\ f_\nu(0, x) = f_\nu(x). \end{cases} \quad (A.6)$$

For any $y \in [0, 1]$, let $p_t(y, du)$ be the probability measure of $R_t$ conditioned on $R_0 = y$, then for any Borel-measurable set $B \subset [0, 1]$, we define

$$m(t, B) = \int_0^1 p_t(y, B)dy,$$

i.e, $m(t, du)$ is the probability measure of $R_t$ conditioned on $R_0$ is uniformly distributed on $[0, 1]$. Then, $m(t, du)$ is absolutely continuous with respect to Lebesgue measure and its probability density $f_m(t, x)$ is the solution to Eq. A.6 with initial condition $f_m(0, x) = 1$ for all $x \in [0, 1]$. By classic theory of measures on product spaces, for each $u \in [0, 1]$, there exists probability measure $m_2(t, u, dy)$ such that

$$\int_A p_t(y, B)dy = \int_B m_2(t, u, A)m(t, du) = \int_B m_2(t, u, A)f_m(t, u)du$$

for any Borel-measurable $A, B \subset [0, 1]$. Since

$$E_\nu h(R_t) = \int_0^1 f_\nu(y) \left( \int_{[0,1]} h(u)p_t(y, du) \right) dy.$$
for any $h \in C[0, 1]$, we have
\[ f_v(t, x) = f_m(t, x) \int_{[0,1]} f_v(y)m_2(t, x, dy) \] (A.7)
and especially
\[ \pi(x) = f_m(t, x) \int_{[0,1]} \pi(y)m_2(t, x, dy) \] (A.8)
since $\pi(x)dx$ is an invariant measure of $\{R_t\}_{t \geq 0}$. By Eqs. A.7, A.8 and Jensen’s inequality,
\[ E(\nu_t) = \int_0^1 q \left( \frac{f_v(t, x)}{\pi(x)} \right) \pi(x) dx \]
\[ = \int_0^1 q \left( \int_{[0,1]} \frac{f_m(t, x)f_v(y)}{\pi(x)} m_2(t, x, dy) \frac{m_2(t, x, dy)}{\pi(x)} \right) \pi(x) dx \]
\[ \leq \int_0^1 \left( \int_{[0,1]} \frac{\pi(y)q \left( \frac{f_v(y)}{\pi(y)} \right) m_2(t, x, dy)}{\pi(x)} \right) m(t, dx) \]
\[ = \int_0^1 \pi(y)q \left( \frac{f_v(y)}{\pi(y)} \right) \left( \int_{[0,1]} 1 p_t(y, dx) \right) dy \]
\[ = \int_0^1 \pi(y)q \left( \frac{f_v(y)}{\pi(y)} \right) dy = E(v). \]
As a result, $E(v_t)$ is decreasing with $t$ and hence
\[ \lim_{t \to +\infty} E(v_t) \]
exists. Furthermore, by Chapman-Kolmogorov equation,
\[ p_t(y, du) = l_1(t, y)\delta_y(du) + l_2(t, y, u)du, \]
where $\delta_y(du)$ is the Dirac measure concentrated on $y$ and $l_1(t, y), l_2(t, y, \cdot)$ is the solution to the equation
\[
\begin{aligned}
\frac{d}{dt} l_1(t, y) &= -\hat{\lambda}_1(y)l_1(t, y), \\
\frac{d}{dt} l_2(t, y, u) &= \hat{\lambda}_2(u)l_1(t, y)\hat{\lambda}_1(y) + \hat{\lambda}_2(u)\int_0^1 l_2(t, y, x)\hat{\lambda}_1(x) dx - l_2(t, y, u)\hat{\lambda}_1(u), \\
l_1(0, y) &= 1, \\
l_2(0, y, u) &= 0.
\end{aligned}
\]
Hence, $l_1(t, y) = e^{-\hat{\lambda}_1(y)t}$ and $l_2(t, y, u) > 0$ for any $t > 0, u \in [0, 1]$. According to the definition of $m_2(t, x, dy)$,
\[ m_2(t, x, dy) f_m(t, x) = l_1(x)\delta_x(dy) + l_2(t, y, x)dy. \]
As a result, $m_2(t, x, B) = 0$ implies that the Lebesgue measure of $B$ is 0 for any $t > 0$ and $x \in [0, 1]$. Consequently, $E(v_t) = E(v)$ when and only when $v(dx) = \pi(x)dx$. 
Since \( \{\nu_t\}_{t \geq 0} \) are probability measures on \([0, 1]\), they are tight. Let \( \tilde{\nu} \) be the weak limit of a subsequence \( \{\nu_k\}_{k \geq 1} \) of \( \{\nu_t\}_{t \geq 0} \). By Eq. A.6,
\[
f_\nu(t, x) = f_\nu(x) e^{-\hat{\lambda}_1(x)t} + \int_0^t e^{\hat{\lambda}_1(x)(s-t)} \hat{\lambda}_2(x) F_\nu(s) ds
\] (A.9)
where \( F_\nu(s) = \int_0^s \hat{\lambda}_1(y) f_\nu(s, y) dy \). By Eq. A.9, it is easy to check that \( \{f_\nu(t, \cdot)\}_{t \geq 0} \) are uniformly bounded and equicontinuous. As a result, \( \tilde{\nu} \) is absolutely continuous with respect to Lebesgue measure and its probability density \( f_{\tilde{\nu}} \) is a C\([0, 1]\)-limit of a subsequence of \( \{f_\nu(t, \cdot)\}_{t \geq 0} \). Since \( q \) is continuous on \([0, 1]\),
\[
E(\tilde{\nu}) = \lim_{t \to +\infty} E(\nu_t).
\]
For any given \( s > 0 \), \( \tilde{\nu}_s \) is the weak-limit of the subsequence \( \{\nu_{k+s}\}_{k \geq 1} \) of \( \{\nu_t\}_{t \geq 0} \). As a result,
\[
E(\tilde{\nu}_s) = \lim_{t \to +\infty} E(\nu_t) = E(\tilde{\nu})
\]
and hence \( \tilde{\nu}(dx) = \pi(x)dx \). As a result, \( \nu_t \) converges weakly to \( \pi(x)dx \) as \( t \to +\infty \) and the proof is complete.

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