АНАЛИТИЧЕСКИЕ И ЧИСЛЕННЫЕ МЕТОДЫ РАСЧЕТА КОНСТРУКЦИЙ

ANALYTICAL AND NUMERICAL METHODS OF ANALYSIS OF STRUCTURES

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RESEARCH ARTICLE / НАУЧНАЯ СТАТЬЯ

Stress state analysis of an equal slope shell under uniformly distributed tangential load by different methods

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Abstract. Nowadays there are various calculation methods for solving a wide range of problems in construction, hydrodynamics, thermal conductivity, aerospace research and many other areas of industry. Analytical methods that make up one class for solving problems, and numerical calculation methods that make up another class, including those implemented in computing complexes, are used for the design and construction of various thin-walled structures such as shells. Due to the fact that thin-walled spatial structures in the form of various shells are widely used in many areas of human activity it is useful to understand and know the capabilities of different calculation methods. Research works on the study of the stress-strain state of the torse shell of equal slope with an ellipse at the base are not widely available at the moment. For the first time the derivation of the differential equations of equilibrium of momentless theory of shells to determine the normal force $N_u$ from the action of uniformly distributed load tangentially directed along rectilinear generatrixes to the middle surface of the torse of equal slope with a directrix ellipse is presented in this article. The parameters of the stress state of the studied torse are also obtained by the finite element method and the variational-difference method. The SCAD software based on the finite element method and the program SHELLVRM written on the basis of the variational-difference method are used. The numerical results of the parameters of the stress state of the studied torse are analyzed, and the advantages and disadvantages of the analytical method and two numerical calculation methods are determined.

Keywords: thin shell theory, analytical method, momentless state, torse shell, surface of equal slope, finite element method, variational-difference method, SCAD Office computing system, Mathcad system

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Аналіз напруженно-деформованого стану оболочки однакового ската при дії рівномірно розподіленої касетальної навантаження

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Анотація. На сьогоднішній день є різні методи розрахунку для вирішення широкого спектра завдань в будівництві, гідродинаміці, тепловідведення, космічних досліджень та інших галузях. Для проєктування та зведення різноманітних тонких конструкцій типу оболочок використовують аналітичні методи, що складають один клас для розрахунку завдань, і числові методи розрахунку, що складають інший клас, в тому числі реалізовані в числових комплексах. В рамках цього виду, тонкі просторові конструкції в формі різноманітних оболочок широко використовуються в багатьох сферах діяльності людства, яким надається зрозуміння і знає можливості різних методів розрахунку. Роботи з існування напружено-деформованого стану торсу однакового ската з еліпсоїдальної основи представлені в мали обсягу. В статті вперше приводиться вивод диференціальних рівняння рівноваги безмоментної теорії оболочок для визначення нормального напруження 

\[ N_z \]

від дії рівномірно-рівномірної навантаження, направлена в напрямку осі конуса, який пряма, і напрямок нерухомої поверхні торсу однакового ската з направляючим еліпсоїдом. Також отримані параметри напружено-деформованого стану існуючого торсу методом конечних елементів та варіаційно-дискретним методом. Використовуються числові комплекси SCAD Office та система Mathcad.

Ключові слова: теорія тонких оболочок, аналітичне рішення, безмоментний стан, торсовая оболочка, поверхность одинакового ската, метод конечных элементов, вариационно-разностный метод, вычислительный комплекс SCAD Office, система Mathcad.

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Introduction

For the design of diverse engineering structures, various calculation methods are used, such as analytical, numerical and numerical-analytical. In the practice, to get the general parameters of the stress-strain state of spatial-structures, engineers use automated numerical calculation methods because analytical calculation methods are quite complex and time consuming.

The most common numerical calculation method is the finite element method (FEM). Originally, FEM was used for solving mathematical problems in a simpler form. The subsequent development of FEM and automated software systems based on this method such as SIMULIA (www.3ds.com), ANSYS (www.ansys.com), SAP2000 (www.csiamerica.com), SCAD (www.scadsoft.com), PROKON (www.prokon.com) and others, made it
possible to apply it to solve a wide range of problems in aerospace research, to model and take into account dynamic loads, to solve various problems in thermal conductivity, hydrodynamics, construction and many other areas.

The idea of discretization on which the FEA is based is very old. Before 1922, Courant used the finite element ideas in Dirichlet’s principle. The period of 1962–1972 is known as the golden age of FEM [1]. There are five groups of papers (Courant, Argyris, Turner et al., Clough and Zienkiewicz) which may be considered in the development of the FEM and in one of these the name originated [2]. Clough coined the term “finite elements”, Turner perfected the direct stiffness method and the works of Huges, Bathe and Zienkiewicz [3] laid the foundation for further progress of the FEM [1]. In [4], a method for calculating bending plates by the finite element method in stresses is proposed, and a comparison with the results of the finite element method in displacements is made. The solution of plane problems of the theory of elasticity based on the approximation of stresses is considered, the calculations of a cantilever beam and a plate with a hole are performed for various finite element meshes, and comparison is made with solutions by the method of finite elements in displacements and with exact solutions in the work [5]. A special issue including 35 papers is devoted to research in the field of development and application of FEM [6].

The finite-difference energy method (FDEM) [7–10] or so called variational-difference method (VDM) [9–14] is also referred to numerical calculation methods [15; 16]. This method takes into account the geometric characteristics of the middle surface of the shell, which allows a more accurate representation of the stress-strain state of thin-walled structures of complex geometry. The VDM (FDEM) is based on the idea put forward by Courant in 1943 [9; 17; 18], which was continued by Houbolt in 1958 [8], who performed static analysis of beams and plates combining finite difference analog of derivatives with a variational formulation [19]. Further developed by Griffin and Varga in 1963 [20] who introduced finite difference into the variational formulation of strain compatibility and boundary conditions for the analysis of plane elasticity problems [19]. Further Bushnell in 1973, and Brush and Almroth in 1975 [21] who extended the approach to the analysis of other type of structures [22].

The successful application of VDM largely depends on how well the system of basic functions allows the qualitative characteristics of the solution. Consequently, it can be expected that the efficient solution of these variational problems will require numerical schemes that differ from traditional techniques based on continuous approximations [23].

In the Department of Construction of the Academy of Engineering of the RUDN University of Russia, the Doctor of Technical Sciences, Professor V.N. Ivanov together with his postgraduate students (currently PhD) Nasr Younis Ahmed Abboushi (Palestine), Muhammad Rizwan (Pakistan), Bock Hyeng Christian Alain (Cameroon), Govind Prasad Lamichhane (Nepal) led the development of SHELLVRM, a new Variational-Difference Method based program. This program allows to determine the stress-strain state of plates and various types of shells with an orthogonal coordinate system, which middle surfaces are described by analytical equations. The program includes such classes of shells as: flat shells on rectangular and curved planes, shells of revolution, shells in the form of Joachimsthal’s channel surfaces, shells in the form of Monge surfaces and normal cyclic surfaces. The program includes a system of plane curves, on the basis of which sections of surface classes are formed and coefficients of quadratic forms are calculated. The basics of the VDM and the text of the program for plate calculations are given in [15].

Analytical calculation methods are used for spatial structures in the form of various surfaces [24]. Analytical methods are quite complex and time-consuming. More than 600 analytical surfaces are described in the Encyclopedia of Analytical Surfaces [25]. The geometry of surfaces and automated possibilities of their construction are considered in the monograph [26].

Among an extensive variety of analytical surfaces, the torse shells of equal slope possesses the ability to unfold onto a plane without folds and breaks [27], and this type of shells are widely used in many areas of industry and manufacturing [28–31].

This article is part of a series of research papers devoted to the study of the geometry and stress state of torse shell of equal slope with an ellipse at the base under the action of different loads. In previous works, the authors have performed calculations this shell under the action of a linear load on the upper edge and under the action of self-weight [32; 33] and with a different restraint of the base ellipse [34]. Also, a design of an awning in the shape of a torse of equal slope was proposed and new results were obtained in the field of geometric studies [35; 36]. In this article, we consider the uniformly distributed load directed along rectilinear generatrixes of the torse. The choice of the load type is determined by the possibilities of the momentless shell theory. The main task of this article is to find an analytical solution and determine the parameters of the stress state of the torse by
the momentless theory (MLT), followed by comparison with the results of two numerical methods (the finite element method and the variational-difference method).

The surface of equal slope is a ruled surface generated by a straight line moving in the normal plane of a flat directrix-curve with a constant angle of inclination to the plane of the directrix. If we take an ellipse as a flat directrix-curve, then straight lines of equal inclination to the plane of the ellipse will generate the torse surface of equal slope (Figure 1). The surfaces of equal slope are surfaces of zero Gaussian curvature \(K = 0\). The papers [37; 38] describe the basic properties of these surfaces. The equal slope surface also belongs to the class of Monge surfaces [24; 27].

Figure 1. Torse shell of equal slope with an ellipse at the base

As its shown in [27], the directrix ellipse can be defined by parametric equations (1):

\[
x = x(v) = a \cos v, \quad y = y(v) = b \sin v,
\]

where \(a\) and \(b\) are the dimensions of the semi-axes of the directrix ellipse at the base of the torse, and the parameter \(v\) must be in the limits \(0 \leq v \leq 2\pi\).

According to [27], the parametric form of setting the torse surface with a directrix ellipse is:

\[
x = x(u, v) = a \cos v - \frac{ub \cos \alpha \cos v}{\sqrt{a^2 \sin^2 v + b^2 \cos^2 v}};
\]

\[
y = y(u, v) = b \sin v - \frac{u \alpha \cos \alpha \sin v}{\sqrt{a^2 \sin^2 v + b^2 \cos^2 v}};
\]

\[
z = z(u) = u \alpha \sin \alpha.
\]

The family of \(u\) lines is the rectilinear generatrices of the torse surface of equal slope, while the coordinate line \(u = 0\) coincides with the ellipse at the base, \(\alpha\) is the angle between the principal normal \(\mathbf{n} = -\mathbf{e} \times \mathbf{k}\) directed inwards of the directrix ellipse and the straight generatrix \(u\) (Figure 1).

The coefficients of the basic quadratic forms of a given surface and its main curvatures are [27]:

\[
A = 1; \quad B = \mu^{1/2} - u \frac{\beta}{\mu}; \quad F = 0; \quad L = M = 0; \quad N = B \frac{ab \sin \alpha}{\mu};
\]

\[
k_1 = k_u = 0; \quad k_2 = k_v = \frac{ab \sin \alpha}{B \mu},\]

where \(\mu = \mu(v) = a^2 \sin^2 v + b^2 \cos^2 v, \quad \beta = ab \cos \alpha.\)
Let us consider the application of the momentless theory of shell calculation, the finite element method and the variational-difference method on the example of a thin torse shell of equal slope with an ellipse at the base under the action of a uniformly distributed load \( q = 1 \text{kN/m}^2 \) tangentially directed along rectilinear generatrixes to the middle surface of the torse (Figure 2). Thus, the external surface load is \( X = X = Y = Z = 0 \).

The geometric parameters of the torse are: \( a = 3 \text{ m}, b = 2 \text{ m}, \alpha = 60^\circ \), the length of the straight generatrixes is \( u = 2 \text{ m} \). The boundary conditions at the level of the directrix ellipse \((u = 0 \text{ m})\) are simple (movable) supports, and at the top \((u = 2 \text{ m})\) the edge is free.

![Figure 2. Torse under the action of external distributed surface load](image)

To determine the parameters of the stress state of the considered torse, the momentless theory of shell calculation [24; 27], the SCAD integrated system for finite element structural analysis (FEA), and the SHELLVRM program based on the variational-difference method [15; 16] are used.

The differential equilibrium equations of the momentless theory are obtained from the general equilibrium equations of the moment shell theory [24; 27].

### Differential equations of equilibrium of the momentless torse shell

The momentless theory is a simplified version of the general theory of thin elastic shells, which neglects the influence of transverse forces and moments. At the same time, the possibility of existence of the momentless stress state of the shell depends on a number of conditions [24; 27]. The shell should have the form of a smoothly changing continuous surface, the load on the shell should be continuous and smooth, and the supports of the edges should allow the shell to move freely in the direction normal to the middle surface, normal movements and rotation angles at the edges of the shell should not be restrained.

We obtain differential equations of equilibrium for determining the normal force under the action of a uniformly distributed load acting in the direction of a tangent along rectilinear generatrixes to the middle surface of the considered torse.

General differential equations of equilibrium of the momentless theory [24; 27] have the form:

\[
\frac{\partial}{\partial u} \left( BN_u \right) - \frac{\partial B}{\partial u} N_v + \frac{1}{A} \frac{\partial}{\partial v} \left( A^2 S \right) + ABX = 0;
\]

\[
\frac{\partial}{\partial v} \left( AN_v \right) - \frac{\partial A}{\partial v} N_u + \frac{1}{B} \frac{\partial}{\partial u} \left( B^2 S \right) + ABY = 0;
\]

\[
\frac{N_v}{R_v} + \frac{N_u}{R_u} - Z = 0.
\]

\[(4)\]
For the considered case of load application (Figure 2), we obtain \( X = -q \) and \( Y = Z = 0 \). The differential equations of equilibrium (4), taking into account expressions (3), are transformed as following:

\[
\frac{\partial}{\partial u} (BN_u) - \frac{\partial B}{\partial u} N_u + \frac{\partial S}{\partial v} + XB = 0;
\]

\[
\frac{\partial N_v}{\partial v} + \frac{1}{B} \frac{\partial}{\partial u} (B^2 S) = 0;
\]

\[
\frac{N_v}{R_v} = 0.
\]

The resulting system of differential equations (5) is of second order. To solve it, it is sufficient to have one boundary condition at each point of the torse shell contour. Thus, at the top of the shell at \( u = 2 \) m the force \( N_u = 0 \). Moreover, from the second and third equations of system (5) the forces \( N_v = 0 \) and \( S = 0 \).

By integrating the first equation of system (5), we obtain the expression for the values of normal force \( N_u \) along the rectilinear generatrixes \( u \):

\[
N_u = \frac{1}{B(u, v)} \int qB(u, v) \, du + X_1(v).
\]

(6)

Here \( X_1(v) \) is an arbitrary function of integration.

Then, by integrating of (6):

\[
\int B(u, v) \, du = u^{1/2} - \frac{u^2}{2\mu} = \frac{u}{2} \left( B(u, v) + \mu^{1/2} \right) = \frac{\mu}{2\beta} (u - B^2(u, v)).
\]

(7)

To satisfy the boundary condition \( N_u = 0 \) on the upper free edge under \( u = \eta = 2 \) m, the arbitrary function of integration \( X_1(v) \) in (7) must be equal to:

\[
X_1(v) = -q \left( \eta^{1/2} \frac{\eta^2}{2\mu} \right).
\]

(8)

The equation (6) for the calculation of numerical values of the normal forces \( N_u \) along the rectilinear generatrixes taking into account the value \( X_1(v) \) of the arbitrary integration function (8) takes the following form:

\[
N_u = \frac{q}{B(u, v)} \left[ \mu^{1/2} (u - \eta) - \frac{\beta}{2\mu} (u^2 - \eta^2) \right].
\]

(9)

To find numerical results of normal force \( N_u \) (9) we use the engineering math software Mathcad.

**Numerical methods for investigation of the stress state of the shell**

The investigation of the stress state of the torse of equal slope was performed by the finite element method and the variational-difference method. The first calculation is performed by using SCAD software. The view of the 3D computational model when approximating the middle surface by a set of quadrangular planar shell elements is shown in Figure 2. The maximum distance between the nodes of the finite elements of the computational model is 0.228 m. The number of finite elements is 1680 and of nodes is 1760.

For the implementation of simple (movable) supports, which is a necessary condition for the momentless work of the torse, the SCAD program has added short bar elements with hinges (Figure 3). The introduction of hinges in these support rod elements releases linear movements along the normal to the torse middle surface (Figure 3, direction \( z_1 \)), angular movements tangent to the surface (Figure 3, direction \( y_1 \)) and normal to the surface of the shells (Figure 3, direction \( z_1 \)), as well as angular movements in the direction of rectilinear generatrixes \( u \) (Figure 3, direction \( x_1 \)).
The second calculation is performed in the program SHELLVRM, based on the variational-difference method. The calculated grid is similar to the grid in FEM. This calculation also takes into account and implements all the necessary conditions for the momentless state of the shell. The calculation is performed for a 1/4 segment of the torse shell, taking into account two planes of symmetry.

**Results and discussion**

The obtained results of the analytical calculation are compared with the results of numerical methods (by the finite element method and the variational-difference method) for 11 cross-sections (Figure 4).

![Figure 4. Cross-sections of the torse to compare the results](image)

![Figure 5. Normal stress σ(Nu) by FEM, kN/m²](image)

The maximum deviations of the analytical results of normal force $N_u$ along the rectilinear generatrixes from the results of two numerical methods are: 7.4% in section 1–1 (Table 1), 5.0% in section 2–2, 1.9% in section 3–3, 3.7% in section 4–4, 4.1% in section 5–5 (Table 2), 3.6% in section 6–6, 2.8% in section 7–7, 2.2% in section 8–8, 2.0% in section 9–9, 1.9% in section 10–10, and 1.9% in section 11–11 (Table 3). At nodes of coordinates $u = 2.00$ m in FEM and VRM the values are different from zero when compared with MLT.

For an overall picture of the stress state of torse shell under the action of uniformly distributed load $q$ tangentially applied along rectilinear generatrixes to the torse middle surface, the contour graph distribution of normal stress $\sigma(N_u)$ obtained in the SCAD software is shown in Figure 5.
### Table 1

| U-axis coordinate, m | $N_u$, MLT, section 1–1, kN/m | $N_u$, FEM, section 1–1, kN/m | Deviation, $N_u$, MLT and FEM, section 1–1, % | $N_u$, VDM, section 1–1, kN/m | Deviation, $N_u$, MLT and VDM, section 1–1, % |
|---------------------|-------------------------------|-------------------------------|---------------------------------------------|-------------------------------|---------------------------------------------|
| 0.000               | −1.2500                       | −1.2438                       | 0.50                                        | −1.2980                       | 3.70                                        |
| 0.200               | −1.1432                       | −1.1906                       | 3.98                                        | −1.1920                       | 4.09                                        |
| 0.400               | −1.0353                       | −1.0841                       | 4.50                                        | −1.0860                       | 4.67                                        |
| 0.600               | −0.9258                       | −0.9757                       | 5.11                                        | −0.9767                       | 5.21                                        |
| 0.800               | −0.8143                       | −0.8640                       | 5.75                                        | −0.8645                       | 5.81                                        |
| 1.000               | −0.7000                       | −0.7477                       | 6.39                                        | −0.7480                       | 6.42                                        |
| 1.200               | −0.5818                       | −0.6253                       | 6.96                                        | −0.6255                       | 6.99                                        |
| 1.400               | −0.4579                       | −0.4942                       | 7.35                                        | −0.4944                       | 7.38                                        |
| 1.600               | −0.3250                       | −0.3503                       | 7.22                                        | −0.3504                       | 7.25                                        |
| 1.800               | −0.1769                       | −0.1871                       | 5.45                                        | −0.1870                       | 5.40                                        |
| 2.000               | 0.0000                        | −0.0563                       | –                                           | −0.0002                       | –                                           |

### Table 2

| U-axis coordinate, m | $N_u$, MLT, section 5–5, kN/m | $N_u$, FEM, section 5–5, kN/m | Deviation, $N_u$, MLT and FEM, section 5–5, % | $N_u$, VDM, section 5–5, kN/m | Deviation, $N_u$, MLT and VDM, section 5–5, % |
|---------------------|-------------------------------|-------------------------------|---------------------------------------------|-------------------------------|---------------------------------------------|
| 0.000               | −1.5623                       | −1.5004                       | 4.12                                        | −1.5340                       | 1.84                                        |
| 0.200               | −1.4292                       | −1.4004                       | 2.06                                        | −1.4010                       | 2.01                                        |
| 0.400               | −1.2930                       | −1.2647                       | 2.23                                        | −1.2650                       | 2.21                                        |
| 0.600               | −1.1531                       | −1.1262                       | 2.39                                        | −1.1270                       | 2.32                                        |
| 0.800               | −1.0090                       | −0.9840                       | 2.54                                        | −0.9842                       | 2.52                                        |
| 1.000               | −0.8599                       | −0.8375                       | 2.68                                        | −0.8376                       | 2.66                                        |
| 1.200               | −0.7050                       | −0.6859                       | 2.78                                        | −0.6860                       | 2.77                                        |
| 1.400               | −0.5432                       | −0.5284                       | 2.80                                        | −0.5285                       | 2.78                                        |
| 1.600               | −0.3731                       | −0.3637                       | 2.58                                        | −0.3638                       | 2.56                                        |
| 1.800               | −0.1928                       | −0.1892                       | 1.88                                        | −0.1899                       | 1.53                                        |
| 2.000               | 0.0000                        | −0.0572                       | –                                           | −0.0002                       | –                                           |

### Table 3

| U-axis coordinate, m | $N_u$, MLT, section 11–11, kN/m | $N_u$, FEM, section 11–11, kN/m | Deviation, $N_u$, MLT and FEM, section 11–11, % | $N_u$, VDM, section 11–11, kN/m | Deviation, $N_u$, MLT and VDM, section 11–11, % |
|---------------------|-------------------------------|-------------------------------|---------------------------------------------|-------------------------------|---------------------------------------------|
| 0.000               | −1.7778                       | −1.7439                       | 1.95                                        | −1.7870                       | 0.51                                        |
| 0.200               | −1.6159                       | −1.6240                       | 0.50                                        | −1.6240                       | 0.50                                        |
| 0.400               | −1.4512                       | −1.4585                       | 0.50                                        | −1.4590                       | 0.53                                        |
| 0.600               | −1.2833                       | −1.2899                       | 0.51                                        | −1.2900                       | 0.52                                        |
| 0.800               | −1.1122                       | −1.1178                       | 0.50                                        | −1.1180                       | 0.52                                        |
| 1.000               | −0.9375                       | −0.9420                       | 0.47                                        | −0.9420                       | 0.48                                        |
| 1.200               | −0.7590                       | −0.7622                       | 0.42                                        | −0.7622                       | 0.42                                        |
| 1.400               | −0.5763                       | −0.5783                       | 0.35                                        | −0.5784                       | 0.36                                        |
| 1.600               | −0.3892                       | −0.3901                       | 0.23                                        | −0.3902                       | 0.26                                        |
| 1.800               | −0.1972                       | −0.1973                       | 0.07                                        | −0.1974                       | 0.10                                        |
| 2.000               | 0.0000                        | −0.0553                       | –                                           | 0.0000                        | –                                           |
Comparison of the obtained results of normal force $N_u$ by three calculation methods shows good convergence. The concentration of the largest deviations of the numerical values of the normal force $N_u$ by the momentless theory from the VDM and FEM is in the region of the shell with the largest change in the radius of curvature along the curvilinear directrices, i.e., in the upper nodes of sections 1–1 and 2–2 (Figure 4).

According to the Theory of Strength of Materials, the numerical values of the normal force $N_u$ under the action of uniformly distributed load tangentially along rectilinear generatrixes to the torse middle surface at the nodes of all sections at coordinate $u = 2.00$ m must be $N_u = 0$. However, the values of the normal force $N_u$ in the FEM and VDM are different from zero, and the results of the VDM are more accurate compared to the FEM. It is well known that the accuracy of the results of FEM and VDM calculations depends on the correct choice of the size of the finite elements (mesh). Moreover, it is noted in [15] that a comparison of the results of VDM and FEM calculations at the same mesh shows close accuracy, and in some cases, VDM gives even higher accuracy results.

FEM and VDM allow obtaining numerical values also for normal forces $N_y$ along curved directrices, bending moments $M_u, M_v$, tangential forces $S$ and shear forces $Q_u, Q_v$. The normal forces $N_v$ by VDM range from $-0.0246$ to $0.0148$ kN/m, and by FEM ranges from $0.0490$ to $0.0216$ kN/m. The shear forces $Q_u, Q_v$ range from $-0.01$ to $0.01$ kN/m. The tangential forces $S$ ranges from $-0.0354$ to $0.0354$ kN/m by VDM, and from $-0.0162$ to $0.0067$ kN/m by VDM. The values of bending moment $M_u$ range from $-0.0261$ to $0.4244$ N·m/m by VDM, and by FEM from $-0.1143$ to $0.4733$ N·m/m. The values of bending moment $M_v$ by VDM range from $-0.4140$ to $1.4030$ N·m/m, and by FEM range from $-0.4398$ to $1.5562$ N·m/m.

The bending moments $M_u$ and $M_v$ are of particular interest, since the values of bending stresses when compared with normal stresses can be used to infer the bending state of the torse shell under the action of the considered load.

The normal stress $\sigma_M$ and $\sigma_N$ from normal forces $N_{u,v}$ and moments $M_{u,v}$ are determined as follows:

$$\sigma_N = \frac{N_{u,v}}{h}; \quad \sigma_M = \frac{6M_{u,v}}{h^2}. \quad (10)$$

The results of the VDM for the maximum ratio of stresses $\sigma_{M_{\text{fem}}}$ to $\sigma_{N_u}$ are: in the cross section 1–1 is 156.5% in the node of coordinate $u = 2.00$ m, 27.2% in the node with the coordinate $u = 1.80$ m, 13.1% in the node with the coordinate $u = 1.60$ m, 7.2% in the node with the coordinate $u = 1.40$ m, in other nodes does not exceed 4.4%. In cross section 2–2 is 61.9% in the node of coordinate $u = 2.00$ m, 25.0% in the node with coordinate $u = 1.80$ m, 12.3% in the node with the coordinate $u = 1.60$ m, 6.8% in the node with the coordinate $u = 1.40$ m, other nodes do not exceed 4.2%. In section 3–3 is 25.9% in the node with the coordinate $u = 2.00$ m, 18.5% in the node with the coordinate $u = 1.80$ m, 9.9% in a node with coordinate $u = 1.60$ m, and in the other

![Figure 6. Bending moment $M_u$ by FEM, N·m/m](image_url)
nodes does not exceed 5.8%. In section 4–4 is 7.0% in node with coordinate $u = 2.00$ m, 9.3% in the node with coordinate $u = 1.80$ m, 6.7% in the node with coordinate $u = 1.60$ m, and in all other nodes does not exceed 4.5%. In section 5–5 is 45.5% in the node with the coordinate $u = 2.00$ m; in section 6–6 is 13.3% in the node with the coordinate $u = 2.00$ m; in section 7–7 is 17.4% in the node with the coordinate $u = 2.00$ m; in section 8–8 is 10.9% in the node with the coordinate $u = 2.00$ m; in section 10–10 is 8.9% in the node with the coordinate $u = 2.00$ m; in section 11–11 is 34.8% in the node with the coordinate $u = 2.00$ m, 5.8% in the node with the coordinate $u = 1.80$ m. In other nodes of sections 5–5 to 11–11, the stress ratio does not exceed 5.2%.

The bending stresses $\sigma_{Mv}$ arising from a uniformly distributed load directed tangentially along rectilinear generatrixes to the middle surface, in the VDM have an even greater influence on the bending state of the considered torse shell with a directrix ellipse at the base.

The results of studying the influence of bending stresses $\sigma_{Mu}$ and $\sigma_{Mv}$ in FEM show a similar character. Figures 6 and 7 show the contour graph distribution of bending moments $M_u$ and $M_v$ obtained in the SCAD software.

**Conclusion**

The research is carried out at the Academy of Engineering of the Peoples' Friendship University of Russia (RUDN University). In the field of geometry and stress-strain state of various shells, in particular torse shells class, works at RUDN University have been carried out since 1960's. An undeniable contribution to modern theory of shells was made by Prof. V.G. Rekach, Prof. S.N. Krivoshapko and Prof. V.N. Ivanov and their postgraduate students (today PhD in Technical Sciences). Currently, S.N. Krivoshapko and V.N. Ivanov continue their research in the field of shell theory [39–41].

This paper for the first time presents the differential equations of equilibrium for a torse shell of equal slope with a directrix ellipse and the expression for the normal force $N_u$ determination under the action of uniformly distributed load tangentially directed along rectilinear generatrixes to the torse middle surface.

Determination of the internal force $N_u$ of the investigated torse shell by the analytical method is a complex and time-consuming task that requires a lot of time and increased concentration of attention on its implementation, since a slight inaccuracy can lead to incorrect results. The comparison of the results of the momentless theory with the results of the finite element method and the variational-difference method shows good convergency, which indicates the correctness of the obtained differential equilibrium equations and the expression for determining the values of the normal force $N_u$. The use of SHELLVRM and SCAD programs simplifies the solution of this task. However, the calculation in the SHELLVRM program is possible if there is the program text for its implementation, and in the SCAD program it becomes difficult to implement a momentless state (introduction of simple-movable supports). When choosing a method of solving the problem, SCAD program, based on the finite element method, is the simplest and most versatile way for solving the research problem.
The values of the normal force $N_u$ along the rectilinear generatrixes of the shell indicate that the considered shell is working in compression. Thus, it is a big plus when selecting materials for the design and manufacture of torse shells. Considering the property of this class of shells to be flattened on the plane without folds and breaks, this is also an advantage when selecting torse shells among similar shaped.

Due to the results of the FEM and VDM, it was found that the bending stresses $\sigma_{M_u}$ and $\sigma_{M_s}$ have a significant influence on the torse shell stress state. Therefore, it is necessary to consider the bending moments $M_u$ and $M_s$ when designing different structures in the form of this class of shells. The momentless theory does not allow us to obtain these parameters of the stress state of the torse. Thus, it may be concluded that the momentless theory is not applicable for the considered torse shell of equal slope with ellipse directrix.

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