Commutative Bezout domains of stable range 1.5

Dedicated to the 70-th birthday of Professor V.V. Sergeichuk

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Abstract

A ring $R$ is said to be of stable range 1.5 if for each $a, b \in R$ and $0 \neq c \in R$ satisfying $aR + bR + cR = R$ there exists $r \in R$ such that $(a + br)R + cR = R$. Let $R$ be a commutative domain in which all finitely generated ideals are principal, and let $R$ be of stable range 1.5. Then each matrix $A$ over $R$ is reduced to Smith’s canonical form by transformations $PAQ$ in which $P$ and $Q$ are invertible and at least one of them can be chosen to be a product of elementary matrices. We generalize Helmer’s theorem about the greatest common divisor of entries of $A$ over $R$.

Keywords: Commutative Bezout domain, Elementary divisor ring, Adequate ring, Stable range of a ring

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1. Introduction and main results

The problem of finding canonical forms of a matrix up to equivalency is classical. The rings over which matrices are equivalent to certain diagonal matrices have been studied extensively. In the present paper we investigate such question for some classes of commutative Bezout domains.

The matrix $\text{diag}(d_1, d_2, \ldots)$ means a (possibly rectangular) matrix having $d_1, d_2, \ldots$ on main diagonal and zeros elsewhere (by the main diagonal we mean the one beginning at the upper left corner). We use the following

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notations of commutative rings: \((a_1, \ldots, a_n)\) denotes the greatest common divisor of elements \(a_1, \ldots, a_n\) and \(a \mid b\) means that \(a\) is a divisor of \(b\). The set of all matrices of size \(n \times m\) over a ring \(R\) is denoted by \(R^{n \times m}\).

An associative (not necessary commutative) ring \(R\) is called an elementary divisor ring (introduced by I. Kaplansky in [12]) if every (not necessary square) matrix \(A\) over \(R\) admits a diagonal reduction, that is, there exist invertible matrices \(P\) and \(Q\) over the ring \(R\) such that

\[
P A Q = \text{diag}(\varphi_1, \ldots, \varphi_k, 0, \ldots, 0) = \Phi,
\]

in which each element \(\varphi_i\) is a total divisor of \(\varphi_{i+1}\) for \(i = 1, \ldots, k - 1\) (i.e. \(R\varphi_{i+1} \subseteq \varphi_i R \cap R\varphi_i\),

which is equivalent to \(\varphi_i \mid \varphi_{i+1}\) when \(R\) is commutative). The matrix \(\Phi\) is called the Smith normal form and \(\varphi_1, \ldots, \varphi_k\) are the invariant factors of the matrix \(A\). Examples of such rings are the ring of integers \(\mathbb{Z}\) (see [20]), Euclidean rings and principal ideal rings (see [11, 22]).

A ring \(R\) is a Bezout ring if each of its finitely generated ideals is principal. Each matrix from \(R^{1 \times n}\) and \(R^{n \times 1}\) over an elementary divisor ring \(R\) admits a diagonal reduction. This is equivalent to the condition that each finitely generated ideal in \(R\) is principal. Hence an elementary divisor ring is a Bezout ring. Gilman and Henriksen constructed an example of a commutative Bezout ring which is not an elementary divisor ring (see [6, Example 4.11, p. 382]). This raises the problem whether an arbitrary commutative Bezout domain is an elementary divisor ring. Euclidean rings and principal ideal rings satisfy the ascending chain condition on ideals. However Helmer [9] showed that this condition can be replaced by the less restrictive hypothesis that \(R\) is adequate.

A commutative Bezout domain \(R\) is adequate if for \(a, b \in R\) with \(a \neq 0\), there exist \(r, s \in R\) such that \(a = rs\), in which \((r, b) = 1\) and if \(s'\) is a non-unit divisor of \(s\), then \((s', b) \neq 1\). Commutative principal ideal domains and commutative regular rings with identity are adequate rings; see [7, Theorem 11, p. 365] (see also [5]).

The proof of the fact that an adequate ring is an elementary divisor ring (see [5, Theorem 3, p.234]) was based on [9, Theorem 1, p.228] which says that if \(A \in R^{n \times m}\) has maximal rank over an adequate ring \(R\), then there is a row \(u = [1, u_2, \ldots, u_n] \in R^{1 \times n}\) such that the g.c.d. of the entries of \(u A\) and the g.c.d. of entries of the matrix \(A\) coincide. It means that there is an
invertible matrix \( U = \begin{bmatrix} 1 & u_2 & \cdots & u_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \) such that g.c.d. of entries of the matrix \( A \) and g.c.d. of the elements of the first row of matrix \( UA \) coincide. This result was generalized for matrices with rank greater than one by Petrychkovych in [13, Lemma 3.1, p. 71].

Deep studies of the theory of elementary divisor rings increasingly suggest that methods of pure ring theory are insufficient. Promising studies were based on the concept of stable range of rings, introduced by Bass [2] as an important \( K \)-theory invariant.

According to [3, Property (7.2), p. 106] (see also the definition after Lemma 1 in [21]), the stable range of a ring \( R \) is the smallest positive integer \( n \) such that the following condition holds:

\[(a_1 + a_{n+1}b_1)R + \cdots + (a_n + a_{n+1}b_n)R = R.\]

If such \( n \) does not exist, then the stable range of \( R \) is infinity.

The concept of the stable range of a ring turned out to be useful in the study of elementary divisor rings. In particular, Zabavsky [23, Theorem 1, p. 666] proves that each elementary divisor ring has stable range \( \leq 2 \). His survey [24] contains results on the problem when a commutative Bezout domain is an elementary divisor ring.

We say that associative ring \( R \) has stable range 1.5 if for each \( a, b \in R \) and \( 0 \neq c \in R \) satisfying \( aR + bR + cR = R \) there exists \( r \in R \) with

\[(a + br)R + cR = R.\]

This notion was introduced by the second author [18] and studied in [16, 19]. Commutative principal ideal domains, adequate rings (see [14, Propositions 3.15 and 3.14] and [1, Proposition 4]), rings of \( 2 \times 2 \) matrices over rings listed before (see [18, Theorem 5, p. 856]) has stable range 1.5.

Evidently each ring with stable range 1.5 has Bass stable range 2. The converse is not always true. For instance, the subring \( \mathbb{Z} + x\mathbb{Q}[[x]] \) of the ring of formal power series \( \mathbb{Q}[[x]] \) over the field of rational numbers \( \mathbb{Q} \) (see [10, Example 1, p. 160]) has stable range 2 but not 1.5 (see [17, Example 1.1, p. 22]). This shows that the rings of stable range 1.5 are between the rings of stable range 1 and 2, respectively.
Note that the notion of stable range 1.5 is closely related to the concept of almost stable range 1, introduced by McGovern [14]. A ring \( R \) has almost stable range 1 if each proper homomorphic image of \( R \) has stable range 1. In such rings if \( aR + bR + cR = R \), where \( c \) does not belong to the Jacobson radical \( \mathfrak{J}(R) \) of \( R \), then there exists \( r \in R \) (see [14, Theorem 3.6]) such that 
\[(a + br)R + cR = R.\]

In commutative rings of almost stable range 1 the condition \( c \notin \mathfrak{J}(R) \) can be replaced to \( c \neq 0 \) (see [1, Proposition 4]). Using this result and [14, Theorem 3.7] we conclude that each commutative Bezout domain of stable range 1.5 is an elementary divisor ring.

Several authors define and study rings of idempotent stable range \([4, 14]\), rings of unit stable range \([8]\), rings of neat stable range \([25]\), and rings of square stable range \([13]\).

Our first result is a generalization of Helmer’s result \([9, \text{Theorem } 1]\).

**Theorem 1.** If \( R \) is a commutative Bezout domain, then the following conditions are equivalent:

(i) \( R \) has stable range 1.5;

(ii) for each \( A \in R^{n \times m} \) with \( \text{rank}(A) > 1 \), there exists \( u = [1, u_2, \ldots, u_n] \in R^{1 \times n} \) such that \( uA = [b_1, b_2, \ldots, b_m] \), in which \((b_1, b_2, \ldots, b_m)\) coincides with the g.c.d. of entries of the matrix \( A \).

The following example shows that the condition \( \text{rank}(A) > 1 \) in Theorem 1(ii) is essential. Indeed, let \( A := [\frac{5}{7} 0] \in \mathbb{Z}^{2 \times 2} \) and \( u = [1, u_2] \in \mathbb{Z}^{1 \times 2} \). Then \( uA = [5 + 7u_2, 0] \). The g.c.d. of entries of matrix \( A \) is equal to 1. But
\[(5 + 7u_2, 0) = 5 + 7u_2 \neq \pm 1\]
for any \( u_2 \in \mathbb{Z} \). However the ring \( \mathbb{Z} \) has stable range 1.5.

Matrices \( P \) and \( Q \) satisfying equality (1) are called transforming matrices of the matrix \( A \). The set of all transforming matrices \( P \) and \( Q \) are denoted by \( T_l(A) \) and \( T_r(A) \), respectively. An elementary matrix is a matrix which is obtained from the identity matrix by elementary transformations.

**Theorem 2.** Let \( R \) be a commutative Bezout domain of stable range 1.5. If \( A = [a_{ij}] \in R^{n \times m} \) with \( \text{rank}(A) > 1 \), then both of the sets \( T_l(A), T_r(A) \) contain an elementary matrices.
2. Proofs

If $A, B$ are matrices such that $A = UBV$ for some invertible matrices $U$ and $V$, then we say $A \sim B$. We use the following result proved in [18, Property 6, p. 50].

**Lemma 1.** Let $R$ be a commutative Bezout domain of stable range 1.5. Let $a_1, \ldots, a_n$ be a collection of relatively prime elements in $R$ and $0 \neq \psi \in R$. Then there exist $u_1, \ldots, u_n \in R$ such that

(i) $u_1a_1 + \cdots + u_na_n = 1$;

(ii) $(u_1, \ldots, u_k) = (\psi, u_k) = 1$ for each fixed $2 \leq k \leq n$.

**Proof of Theorem**. (i) $\implies$ (ii). Let $A \in R^{n \times m}$ has rank greater than 1. Without loss of generality, assume that $m \geq n$. A ring $R$ is an elementary divisor ring. Therefore the equation (1) holds for some invertible $P$ and $Q$. Since rank$(A) > 1$, we have $k \geq 2$ in (1). Consider

$$U := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & I_{k-2} & 0 & 0 & 0 \\ 1 & 0 & 0 & I_{n-k} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & I_{m-k} & 0 \end{bmatrix} \quad \text{and} \quad V := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & I_{k-2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

in which $I_s$ $(s \geq 1)$ is the identity $s \times s$ matrix and $I_0$ is an empty matrix. It is easy to check that

$$\Phi' := (UP)A(QV) = \text{diag}(\varphi_k, \varphi_2, \varphi_3, \ldots, \varphi_{k-1}, \varphi_1, 0, \ldots, 0).$$

If $P_1 := (UP)\det(UP)^{-1}$ and $Q_1 := (QV)\det(UP)$, then $P_1AQ_1 = \Phi'$ and from $\det(P_1^{-1}) = 1$ we obtain that $\det(P_1^{-1}) = \sum_{i=1}^{n}(-1)^{i+1}p_{ij}\Delta_i = 1$, where $P_1^{-1} := [p_{ij}]$ and $\Delta_1, \ldots, \Delta_n$ are the corresponding minors. There exist $s_{11}, \ldots, s_{1n}$ by Lemma 1 such that $\sum_{i=1}^{n}s_{1i}\Delta_i = 1$ and

$$(s_{11}, s_{12}, \ldots, s_{1k}) = (\varphi_k, s_{1k}) = 1. \quad (2)$$

Moreover (see [18, Property 3, p. 48])

$$[s_{11}, s_{12}, \ldots, s_{1n}] = [1, t_2, \ldots, t_n]P_1^{-1}, \quad (3)$$

for some $t_2, \ldots, t_n \in R$. The equality $P_1AQ_1 = \Phi'$ implies $AQ_1 = P_1^{-1}\Phi'$, and so by (3) we have

$$\begin{bmatrix} 1 & t_2 & \cdots & t_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} AQ_1 = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \Phi'$$

$$= \begin{bmatrix} s_{11}\varphi_k & s_{12}\varphi_2 & \cdots & s_{1k-1}\varphi_{k-1} & s_{1k}\varphi_1 & 0 & \cdots & 0 \\ p_{21}\varphi_k & p_{22}\varphi_2 & \cdots & p_{2k-1}\varphi_{k-1} & p_{2k}\varphi_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ p_{n1}\varphi_k & p_{n2}\varphi_2 & \cdots & p_{n,k-1}\varphi_{k-1} & p_{nk}\varphi_1 & 0 & \cdots & 0 \end{bmatrix}. \quad (4)$$
According to (2), the greatest common divisor \( \tau \) of elements of the first row of the last matrix in (4) is equal to

\[
\tau = (s_{11}\varphi_k, s_{12}\varphi_2, \ldots, s_{1,k-1}\varphi_{k-1}, s_{1k}\varphi_1)
\]

\[
= \varphi_1 \left( s_{11} \frac{\varphi_k}{\varphi_1}, s_{12} \frac{\varphi_2}{\varphi_1}, \ldots, s_{1,k-1} \frac{\varphi_{k-1}}{\varphi_1}, s_{1k} \frac{\varphi_1}{\varphi_1} \right)
\]

\[
= \varphi_1 \left( s_{1k}, s_{11} \frac{\varphi_k}{\varphi_1}, s_{12} \frac{\varphi_2}{\varphi_1}, \ldots, s_{1,k-1} \frac{\varphi_{k-1}}{\varphi_1} \right)
\]

\[
= \varphi_1(s_{11}, s_{12}, \ldots, s_{1k}) = \varphi_1.
\]

Using (4) and reasoning as above, we obtain that

\[
(1, t_2, \ldots, t_n)AQ_1 = (s_{11}, s_{12}, \ldots, s_{1n}) \Phi'
\]

\[
= (s_{11}\varphi_k, s_{12}\varphi_2, \ldots, s_{1,k-1}\varphi_{k-1}, s_{1k}\varphi_1, 0, \ldots, 0)
\]

\[
\sim (\varphi_1, 0, \ldots, 0),
\]

so \((1, t_2, \ldots, t_n)A \sim (\varphi_1, 0, \ldots, 0)\), in which \(\varphi_1\) is the g.c.d. of entries of \(A\).

\((i) \iff (ii)\). Let \(A := \begin{bmatrix} a & c \\ b & 0 \end{bmatrix} \in \mathbb{R}^{2 \times 2}\), where \(a, b \in R\), \(0 \neq c \in R\), and \((a, b, c) = 1\). There is \([1, r] \in \mathbb{R}^{1 \times 2}\) such that \([1, r]A = [b_1, b_2]\), where

\[
(b_1, b_2) = (a, b, c) = 1.
\]

Hence \((a + br, c) = 1\) and so \(R\) has stable range 1.5.

**Proof of Theorem 2.** By Theorem 1 there exists \(u = [1, u_2, \ldots, u_n] \in \mathbb{R}^{1 \times n}\) such that \(uA = [b_1, \ldots, b_n]\), in which \(\varphi_1 := (b_1, \ldots, b_m)\) is equal to the g.c.d. of entries of \(A\). Thus \(\varphi_1\) is the first invariant factor of \(A\) (see (1)). Clearly

\[
U_1A = \begin{bmatrix} b_1 & b_2 & \ldots & b_m \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nm} & a_{n2} & \ldots & a_{nm} \end{bmatrix}, \quad \text{where } U_1 := \begin{bmatrix} 1 & u_2 & \ldots & u_n \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}.
\]

There exists an invertible \(V_1\) such that \(U_1AV_1 = \begin{bmatrix} \varphi_1 & 0 & \ldots & 0 \\ a_{21}' & a_{22}' & \ldots & a_{2m}' \\ \vdots & \vdots & \ddots & \vdots \\ a_{nm}' & a_{n2}' & \ldots & a_{nm}' \end{bmatrix} \begin{bmatrix} a_{11}' \\ \varphi_1' \\ 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ \varphi_1' & 0 & \ldots & 1 \end{bmatrix} \). Then \(\varphi_1\) is also the g.c.d. of entries of \(A_1 := U_1AV_1\). Thus, \(\varphi_1\) for \(i = 2, \ldots, n\) and

\[
U_1' = \begin{bmatrix} \varphi_1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ c_{21} & c_{22} & \ldots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{n2} & \ldots & c_{nm} \end{bmatrix}, \quad \text{where } U_1' := \begin{bmatrix} 1 & 0 & \ldots & 0 \\ a_{11}' & \varphi_1' & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ a_{n1}' & \varphi_1' & 0 & \ldots & 1 \end{bmatrix}.
\]
Note that $U_1$ and $U'_1$ are elementary matrices.

Consider the submatrix $B := \begin{bmatrix} c_{22} & \cdots & c_{2m} \\ \vdots & \ddots & \vdots \\ c_{n2} & \cdots & c_{nm} \end{bmatrix} \in \mathbb{R}^{(n-1) \times (m-1)}$ of $U'_1A_1$. Using the same technique as above, we can find an elementary matrix $Z$ and an invertible matrix $W$ such that $ZB_2W = \begin{bmatrix} \varphi_2 & 0 & \cdots & 0 \\ 0 & d_{33} & \cdots & d_{3m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & d_{n3} & \cdots & d_{nm} \end{bmatrix}$, in which $\varphi_2$ is the second invariant factor of $A$.

Evidently $U_2 := 1 \oplus Z$ is elementary, $V_2 := 1 \oplus W \in \text{GL}_m(\mathbb{R})$ and

$$(U_2U'_1U_1)A(V_1V_2) = \text{diag}(\varphi_1, \varphi_2) \oplus F, \quad (F \in \mathbb{R}^{(n-2)\times(m-2)}).$$

Continuing this process we obtain that there exist $P \in \text{GL}_n(\mathbb{R})$ and $Q \in \text{GL}_m(\mathbb{R})$ such that $PAQ = \text{diag}(\varphi_1, \ldots, \varphi_k)$ in which $P$ is a product of elementary matrices.

Taking $A^T$ instead of $A$ and applying the same reduction, we construct "new" matrices $P$ and $Q$ such that $\Phi := PA^TQ$ is the Smith canonical matrix and $P$ is a product of elementary matrices. Since $\Phi$ is symmetric, $\Phi = \Phi^T = Q^TAP^T$, where $P^T$ is a product of elementary matrices.

Note that from Theorem 2 does not imply that in (1) both of $P$ and $Q$ are elementary.

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