On the well-posedness of relativistic viscous fluids

Marcelo M Disconzi

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA
E-mail: marcelo.disconzi@vanderbilt.edu

Received 21 February 2014, revised 6 April 2014
Accepted for publication 16 June 2014
Published 22 July 2014

Recommended by B Eckhardt

Abstract
Using a simple and well-motivated modification of the stress–energy tensor for a viscous fluid proposed by Lichnerowicz, we prove that Einstein’s equations coupled to a relativistic version of the Navier–Stokes equations are well-posed in a suitable Gevrey class if the fluid is incompressible and irrotational. These last two conditions are given an appropriate relativistic interpretation. The solutions enjoy the domain of dependence or a finite propagation speed property. We also derive a full set of equations describing a relativistic fluid that is not necessarily incompressible or irrotational, which is well-suited for comparisons with the system of an inviscid fluid.

Keywords: Einstein equations, Navier–Stokes equations, relativistic fluids
Mathematics Subject Classification: 83C05, 35L77, 35Q35

1. Introduction

It has been known for a long time that one cannot account for some important features of cosmological and astrophysical phenomena without incorporating dissipation into our physical models [15, 21, 28]. More recently, advances in the study of heavily dense objects, such as neutron stars [6, 8, 27], and in our understanding of the dynamics of the early universe [2, 19] point toward the necessity of a relativistic description of dissipative phenomena. Experience with general relativity suggests that the correct approach to this question should rely on the construction of a stress–energy tensor which subsumes characteristics due to the viscosity of the medium under consideration.

In spite of that, we still lack a satisfactory formulation of viscous phenomena within Einstein’s theory of general relativity. One of the main reasons for this is the lack of a
variational formulation of the classical (non-relativistic) Navier–Stokes equations. In the absence of a variational description of the equations of motion, one does not have a principle that uniquely defines the stress–energy tensor $T_{\alpha\beta}$ in the context of general relativity. As a consequence, there have been different proposals for what the correct $T_{\alpha\beta}$ should be. We refer the reader to [19] and references therein for a brief history of different attempts to formulate a viscous relativistic theory. A more complete and up-to-date discussion can be found in [24].

Still, in many respects, the natural choice for a viscous stress–energy tensor seems to be

$$T_{\alpha\beta}^N = (p + \varrho)u_\alpha u_\beta - pg_{\alpha\beta} + \kappa \pi_{\alpha\beta} \nabla_\mu u^\mu + \vartheta \pi^\rho_{\alpha\beta} (\nabla_\rho u_\mu + \nabla_\mu u_\rho),$$

(1.1)

where $p$ and $\varrho$ are, respectively, the pressure and density of the fluid, $u$ is its four-velocity, the bulk viscosity $\kappa$ and the shear viscosity $\vartheta$ are non-negative constants, $g$ is a Lorentzian metric$^1$ and $\pi_{\alpha\beta} = g_{\alpha\beta} - u_\alpha u_\beta$. $p$ and $\varrho$ are related by an equation known as an equation of state, the choice of which depends on the nature of the fluid and has to be given in order to close the system of the equations of motion (see below). We say that the choice $T_{\alpha\beta}^N$ is natural because it is a straightforward covariant generalization of the stress–energy tensor of a viscous non-relativistic fluid$^2$, and it reduces to the stress–energy tensor of an inviscid fluid$^3$ when $\kappa = \vartheta = 0$.

Unfortunately, as Pichon demonstrated [23], the equations of motion derived from Einstein’s equation coupled to (1.1) exhibit superluminal signals when $p + \varrho \gg 1$, a feature unacceptable for a relativistic theory. Other attempts to formulate a viscous relativistic theory based on a simple covariant generalization of the classical (i.e., non-relativistic) stress–energy tensor for the Navier–Stokes equations have also failed to produce a causal theory. See [24] for a detailed discussion.

One way of overcoming the lack of causality in such models consists of extending the variables of the theory. This leads to what is known as relativistic extended irreversible Thermodynamics. In these approaches, it is possible to show that, under certain circumstances, the equations of motion fall into known classes of hyperbolic equations, exhibiting, as a consequence, a finite propagation speed. It is not at all clear, however, that the equations remain hyperbolic under all physically realistic scenarios. Furthermore, the plethora of models that comes out of the extended thermodynamic approach suggests that it entails many ad hoc features, in sharp contrast to the usually unique way of coupling gravity to matter via the introduction of the stress–energy tensor of matter fields (when the latter is uniquely determined by a variational characterization). For the sake of this discussion, we shall refer to the procedure of coupling gravity to matter by solely introducing the stress–energy tensor of matter fields on the right-hand side of Einstein’s equations, without extending the space of variables as in relativistic extended irreversible thermodynamics, as the traditional approach to the coupling of gravity and matter. We refer the reader to [12, 22, 24] for a treatment of relativistic extended irreversible thermodynamics.

It is also important to mention the rather curious fact that some of the aforementioned recent advances in the modelling of heavy objects were accomplished by numerically solving Einstein’s equations coupled to essentially (1.1), and working in a regime where the superluminal problem could be avoided [8]. The ability to generate physically relevant results in a context where it is expected that viscosity will play a crucial role indicates that

$^1$ Our convention for the metric is $(+ − − −)$.

$^2$ We remind the reader that the stress–energy tensor for a non-relativistic viscous fluid is known, despite the absence of a variational formulation of the classical Navier–Stokes equations. It is constructed by exploring the conservation of mass, energy and momentum of the problem. A similar procedure becomes ambiguous in the setting of general relativity.

$^3$ Which is derived from a variational approach. We remark that since no other interactions will be added, we use the terms inviscid and ideal as synonymous.
the traditional approach to the problem should not be dismissed, despite all failed attempts in producing a causal theory when viscosity is present.

It is, therefore, worthwhile to take a fresh look at the question of whether there is a correct stress–energy tensor $T_{\alpha\beta}$ that describes relativistic viscous fluids, and that can be coupled to gravity in the traditional way. The above considerations suggest that if there is a correct choice for $T_{\alpha\beta}$, it should be close to $T^{(N)}_{\alpha\beta}$ for appropriate values of the quantities involved. ‘Appropriate’ here means choices of these quantities that lead to good agreement between models derived from (1.1) and observational data.

We propose the following guiding principles in the search for $T_{\alpha\beta}$—namely, that any candidate for a stress–energy tensor of a relativistic viscous fluid should satisfy the following:

(i) it reduces to $T_{\alpha\beta} = (p + \varrho)u_\alpha u_\beta - \rho g_{\alpha\beta}$, i.e., to the stress–energy tensor of an ideal fluid, when dissipation is absent; (ii) it reduces to the stress–energy tensor of the classical Navier–Stokes equations in the non-relativistic limit; (iii) it is close to $T^{(N)}_{\alpha\beta}$ in the sense mentioned above; (iv) the equations of motion derived from coupling Einstein’s equations to $T_{\alpha\beta}$ are well-posed and exhibit the domain of dependence property, with the speed of propagation of disturbances being at most the speed of light

Consider the following stress–energy tensor:

$$T_{\alpha\beta} = (p + \varrho)u_\alpha u_\beta - \rho g_{\alpha\beta} + \kappa \pi_{\alpha\beta} \nabla_\mu C^\mu + \vartheta \pi_\alpha^{\rho\mu} (\nabla_\rho C_\mu + \nabla_\mu C_\rho).$$

One immediately sees that (1.2) resembles (1.1), with the terms in derivatives of $u$ replaced by (derivatives of) what is known as the dynamic velocity $C$ ($C$ is also called the current of the fluid; $\pi_{\alpha\beta}$ is still given by $\pi_{\alpha\beta} = s_{\alpha\beta} - u_\alpha u_\beta$), defined by $C_\alpha = F u_\alpha$, where $F$ is the so-called index of the fluid. It is defined as follows. It is customary to introduce the rest mass density $\varrho$ by $\varrho = r(1 + \epsilon)$. Then $F = 1 + \epsilon + \frac{\varrho}{\varrho}$, so that $\dot{\varrho} + p = \varrho r (1 + \epsilon)$. The fluid is defined as follows. It is customary to introduce the rest mass density $\varrho$ by $\varrho = r(1 + \epsilon)$. Then $F = 1 + \epsilon + \frac{\varrho}{\varrho}$, so that $\dot{\varrho} + p = \varrho r (1 + \epsilon)$. The fluid is defined as follows. It is customary to introduce the rest mass density $\varrho$ by $\varrho = r(1 + \epsilon)$.

A fluid with the stress–energy tensor (1.2) will be called incompressible if $\nabla_\mu C^\mu = 0$, in which case $T_{\alpha\beta}$ becomes

$$T_{\alpha\beta} = (p + \varrho)u_\alpha u_\beta - \rho g_{\alpha\beta} + \vartheta \pi_\alpha^{\rho\mu} (\nabla_\rho C_\mu + \nabla_\mu C_\rho).$$

This definition of incompressibility is made so that it agrees with the notion of an incompressible fluid when $\kappa = \vartheta = 0$: a relativistic inviscid barotropic fluid is said to be incompressible if its acoustic waves propagate at the speed of light. It is possible to show that this is equivalent to the vanishing of the divergence of $C$.

We define the vorticity tensor by

$$\Omega_{\alpha\beta} = \nabla_\alpha C_\beta - \nabla_\beta C_\alpha \equiv \partial_\alpha C_\beta - \partial_\beta C_\alpha.$$
A fluid is called irrotational if $\Omega = 0$. For an incompressible irrotational fluid, \((1.2)\) simplifies to

$$T_{\alpha\beta} = (p + \varrho)u_\alpha u_\beta - pg_{\alpha\beta} + 2\theta \pi^\mu_\alpha \pi^\nu_\beta \nabla_\mu C_\nu.$$  \hfill (1.4)

Finally, we need to be more specific about the thermodynamic quantities $p$, $\varrho$, $r$, $\epsilon$ and $F$, and the relations among them. We suppose the validity of the first law of thermodynamics, in which case further thermodynamic variables, namely the specific entropy $s$ and the absolute temperature $\theta$, are introduced. In order to be consistent with guiding principle (i) above, we follow the standard approach of assuming that only two of the thermodynamic quantities are independent, with the other ones determined by thermodynamic relations among them coming from the first law of thermodynamics and an equation of state, which depends on the nature of the fluid. On physical grounds, all such relations should be invertible, which renders the question of which two quantities are independent a matter of choice. Here we shall assume that $r$ and $s$ are independent and postulate an equation of state of the form

$$\varrho = \mathcal{P}(r, s).$$  \hfill (1.5)

It follows that $p = p(r, s)$, $\theta = \theta(r, s)$, $\epsilon = \epsilon(r, s)$, and $F = F(r, s)$ are known if $r$ and $s$ are.

We notice, however, that later on it will be more convenient to treat $s$ and $F$ as independent variables, in which case we shall assume that the equation of state takes the form $r = r(F, s)$. For physically relevant equations of state $F > 0$, which allows us to restrict to positive values when treating $F$ as an independent variable. In this situation, the following condition will be assumed to hold:

$$\frac{\partial r}{\partial F} \geq \frac{r}{F},$$  \hfill (1.6)

in particular $\frac{\partial r}{\partial F} > 0$ if $r > 0$. Condition (1.6) has to be satisfied if we want to recover the stress–energy tensor of an ideal fluid when $\kappa = \vartheta = 0$, in that it expresses the condition that sound waves in an ideal fluid travel at most at the speed of light. At last, we suppose that the equation of state is such that the temperature satisfies

$$\theta(r, s) > 0 \quad \text{if } r > 0, s \geq 0,$$

$$\theta(F, s) > 0 \quad \text{if } s \geq 0, F > 0,$$  \hfill (1.7)

expressing the positivity of the temperature regardless of the choice of independent variables.

The system of equations to be studied consists of Einstein’s equations coupled to \((1.2)\), or \((1.3)\), or \((1.4)\), and reads

\begin{align*}
R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} &= \mathcal{K}T_{\alpha\beta}, \quad \hfill (1.8a) \\
\nabla^a T_{\alpha\beta} &= 0, \quad \hfill (1.8b) \\
\nabla_\alpha (r u^\alpha) &= 0, \quad \hfill (1.8c) \\
u^\alpha u_\alpha &= 1. \quad \hfill (1.8d)
\end{align*}

$R_{\alpha\beta}$ and $R$ are, of course, the Ricci and scalar curvature of the metric $g$, and $\mathcal{K}$ is a constant.

Equation \((1.8d)\) is the standard normalization condition on the velocity of a relativistic fluid, whereas \((1.8c)\) expresses the condition that mass is locally conserved along the flow lines. We notice that without introducing \((1.8c)\), the motion of the fluid is underdetermined. The unknowns are the metric $g$, the fluid velocity $u$, which is a vector field, the specific entropy $s$, and the rest mass density $r$. These last two quantities are non-negative real-valued functions.

We suppose that we are also given a smooth function $\mathcal{P} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ that gives the equation of state \((1.5)\), with all the other thermodynamic quantities introduced above given as functions of $s$ and $r$ via relations derived from the first law of thermodynamics and the equation of state.
Definition 1.1. System (1.8a)–(1.8d) with \( T_{\alpha\beta} \) given by (1.2) will be called the Einstein–Navier–Stokes system; the incompressible Einstein–Navier–Stokes system when \( T_{\alpha\beta} \) is given by (1.3); and the incompressible irrotational Einstein–Navier–Stokes system when \( T_{\alpha\beta} \) assumes the form (1.4).

Assumption. We shall assume for the rest of the text that \( \theta > 0 \).

An initial data set for the Einstein–Navier–Stokes system consists of a three-dimensional manifold \( \Sigma \), a Riemannian metric \( g_0 \), a symmetric two-tensor \( \kappa \), two real-valued non-negative functions \( s_0 \) and \( r_0 \), and a vector field \( v \); these are all quantities defined on \( \Sigma \). As is well-known, these data cannot be arbitrary but must satisfy the constraint equations, which read in a coordinate system with \( \partial \alpha \) transversal and \( \partial i, i = 1, 2, 3 \), tangent to \( \Sigma \), as \( S_{\alpha\beta} = \mathcal{H} T_{\alpha\beta} \), where \( S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} \) is the Einstein tensor. By definition, an initial data set always satisfies these constraints.

We are now ready to state the main result. We refer the reader to the standard literature in general relativity for the terminology employed in theorem 1.2, and to the appendix or \([14,26]\) for the definition of the Gevrey spaces \( \gamma^{m,(\sigma)} \).

Theorem 1.2. Let \( \mathcal{I} = (\Sigma, g_0, \kappa, v, s_0, r_0) \) be an initial data set for the incompressible irrotational Einstein–Navier–Stokes system, with \( \Sigma \) compact, \( s_0 > 0, r_0 > 0 \), and an equation of state \( \mathcal{P} \) such that (1.6) and (1.7) are satisfied. Assume that the initial data is in \( \gamma^{0, (\sigma)}(\Sigma) \) for some \( 1 \leq \sigma < 2 \). Then there exists a space-time \( (M, g) \) that is a development of \( \mathcal{I} \), real-valued functions \( s > 0 \) and \( r > 0 \) defined on \( M \), and a vector field \( u \), such that \( g \in \gamma^{0, (\sigma)}(M), u \in \gamma^{1, (\sigma)}(M), s \in \gamma^{1, (\sigma)}(M), r \in \gamma^{1, (\sigma)}(M), \text{ and } (g, u, s, r) \) satisfy the incompressible irrotational Einstein–Navier–Stokes system in \( M \).

Furthermore, this solution satisfies the geometric uniqueness and domain of dependence properties, in the following sense. Let \( \mathcal{I}' = (\Sigma', g'_0, \kappa', v', s'_0, r'_0) \) be another initial data set, also with the equation of state \( \mathcal{P}' \), with corresponding development \( (M', g') \) and the solution \( (g', u', s', r') \) of the incompressible irrotational Einstein–Navier–Stokes equations in \( M' \). Assume that there exists a diffeomorphism between \( S \subset \Sigma \) and \( S' \subset \Sigma' \) that carries \( \mathcal{I}|_S \) onto \( \mathcal{I}'|_{S'} \), where \( S \) and \( S' \) are, respectively, domains in \( \Sigma \) and \( \Sigma' \). Then there exists a diffeomorphism between \( D_\mathcal{I}(S) \subset M \) and \( D_{\mathcal{I}'}(S') \subset M' \) carrying \( (g, u, s, r) \) onto \( (g', u', s', r') \), where \( D_\mathcal{I}(S) \) denotes the future domain of dependence of \( S \) in the metric \( g \); in particular \( D_{\mathcal{I}_0}(S) \) and \( D_{\mathcal{I}'_0}(S') \) are isometric.

We have chosen to work in the Gevrey class because the equations we shall derive form a Leray–Ohya system\(^8\) (see the appendix), and which, in general, are not well-posed in Sobolev spaces. The space–time \( M \) is diffeomorphic to \( \Sigma \times [0, T] \) for some \( T > 0 \), and to \( \Sigma \times [0, \bar{T}] \) for some \( \bar{T} > T \) if we require it to be a maximal Cauchy development. In light of the domain of dependence property, the compactness of \( \Sigma \) is not absolutely necessary, although in the case of a non-compact \( \Sigma \) without asymptotic conditions on the initial data, \( M \) may not contain any Cauchy surface other than \( \Sigma \) itself. The hypotheses \( s_0 > 0 \) and \( r_0 > 0 \) guarantee, by continuity, the positivity of \( s \) and \( r \) in the neighbourhood of \( \Sigma \), as stated in the theorem. The assumption \( s_0 > 0 \) could be weakened to \( s_0 \geq 0 \), but in this case the non-negativity of \( s \) in \( M \) would have to be derived from the equations of motion, a task we avoid for brevity. Allowing \( r_0 \) to vanish, however, causes severe difficulties, and the well-posedness of the corresponding problem is largely open, even in the case of an ideal fluid.

\(^7\) Except that with our conventions this metric is negative definite.

\(^8\) Called ‘hyperbolique non-stricts’ by Leray and Ohya. Sometimes these systems are called weakly or degenerate hyperbolic, although certainly these terms have been used to denote different types of systems in the literature.
The stress–energy tensor (1.2) was first introduced by Lichnerowicz [15], except that it contained an extra term of the form \( \vartheta \pi_{\alpha\beta} u^\mu \partial_\mu F \). As it was pointed out by Lichnerowicz himself and later by Pichon [23], this extra term leads to an indetermination in the computation of the pressure. Pichon proposed subtracting this term, leading in this way to (1.2). The reader is referred to their original works for the physical insights leading to the construction of (1.2). Choquet-Bruhat has also proposed a stress–energy tensor similar to (1.2) [5]. Her proposal, however, does not include the projection terms \( \pi_{\alpha\beta} \), and the viscous terms are, therefore, linear in the velocity.

We finish this introduction with some comments on the hypotheses and the thesis of theorem 1.2. Perhaps the first hypothesis one would like to remove is that of an irrotational fluid. While this assumption is certainly unsatisfactory from a physical point of view, we remind the reader that we are attempting, in a sense, to ‘reboot’ the traditional approach to the problem. In other words, we try to identify a suitable candidate for \( T_{\alpha\beta} \) that leads to a causal theory, without relying on the introduction of extra variables, as in the extended irreversible thermodynamic models. It is quite natural, therefore, to start by analysing a simpler situation\(^9\). Another hypotheses we would like to weaken is the use of initial data in Gevrey spaces\(^{10}\). These spaces have become an important tool in analysing the equations of fluid dynamics, specially when viscosity is present (see, e.g., [3, 4, 9, 26] and references therein). Hence, it is sensible that such spaces might play a role in the case of relativistic viscous fluids as well. Furthermore, Gevrey spaces are not completely foreign to the study of Einstein’s equations: in some relevant circumstances, the equations of ideal magneto-hydrodynamics appear to have been shown to be well-posed only in the Gevrey class [5, 10]\(^{11}\). On the other hand, the overwhelming success of Sobolev space techniques in the investigation of the Cauchy problem for Einstein’s equations\(^{12}\) almost demands that we employ Sobolev spaces in the study of the evolution problem. Moreover, in order to eventually settle the question of whether (1.2) can give a physically satisfactory description of relativistic viscous phenomena, we have to be able to explicitly compute several physical observables. For this, one has to solve the equations numerically, which, in turn, requires that the equations be well-posed in some function space characterized by a finite number of derivatives.

Such restrictions notwithstanding, one should not overlook the conclusion of theorem 1.2: it is possible, employing what we called a traditional approach (i.e., one the avoids the introduction of extra physical variables as in relativistic extended irreversible thermodynamics), to obtain a description of relativistic viscous fluids that satisfies the natural requirements (i)–(iv) discussed above. In particular, the equations of motion are well-posed, and they do not exhibit faster-than-light signals.

This paper is organized as follows. In section 2 we derive from ((1.8a)–(1.8d)) new systems of equations for both the incompressible and incompressible irrotational systems. Initial data for these systems are calculated from the original initial data set in section 3. The characteristics of both systems are also studied in this section. In section 4, we prove theorem 1.2. For this we shall use the results of Leray and Ohyaa [14] that are reviewed in the appendix for the reader’s convenience. Finally, in section 5, we derive yet another system of equations that is suited for comparisons with the case of an inviscid fluid. We remark that

---

\(^9\) Although, as the reader can check below, already under the present assumptions, the system of equations is rather involved.

\(^{10}\) It will be shown in a future work that, upon restricting to a smaller Gevrey class than as in theorem 1.2, the irrotational hypothesis can be removed [7].

\(^{11}\) Although it is very likely that the formulation of [1] would carry over, with almost no modifications, to the coupling with Einstein’s equations. A proof of this statement, however, does not seem to be available in the literature.

\(^{12}\) The literature on this topic is too vast; see, e.g., the monographs [5,25].
although theorem 1.2 applies only to the incompressible irrotational Einstein–Navier–Stokes system, many of the arguments below (e.g., the derivation of the initial data) are carried out in the more general case of a fluid that is only incompressible, with the condition of zero vorticity introduced at a later point. Along the way, we shall obtain that at least for analytic Cauchy data, the incompressible (not necessarily irrotational) Einstein–Navier–Stokes system can be solved.

In the following, we adopt:

Convention. Greek indices run from 0 to 3 and Latin indices run from 1 to 3.

2. A new system of equations

In this section, we assume that we are given a solution to the Einstein–Navier–Stokes system. We suppose that \( s > 0 \) and \( F > 0 \) are the two independent thermodynamic variables, and that the equation of state reads \( r = r(F, s) \) and satisfies the hypotheses stated in theorem 1.2.

Using thermodynamic relations to express \( \rho \) and \( p \) in terms of \( s \) and \( F \) and equation (1.8c), it is seen that (1.8b) decomposes as

\[
\frac{rF}{F}C^\alpha \partial_\alpha s + \kappa L^{(s)} + \theta V^{(s)} = 0,
\]

(2.1)

and

\[
\frac{r}{F}C^\alpha \Omega_{\alpha\beta} + \theta r \partial_\beta s - \frac{rF}{F}C_\beta C^\alpha \partial_\alpha s + \kappa L_\beta + \theta V_\beta = 0,
\]

(2.2)

where \( \theta = \theta(F, s) \) is the temperature,

\[ L^{(s)} = F^{-2} \left( \frac{1}{F} C^\mu \nabla_\tau C_\mu - F \nabla_\mu C^\mu \right) \nabla_\rho C^\rho, \]

\[ V^{(s)} = -\frac{1}{F} \left( g_\mu^\nu - F^{-2} C_\mu C^\nu \right) \left( \nabla_\nu C_\mu + \nabla_\mu C_\nu \right) C^{\beta \gamma} \left( F^{-2} C_{\rho} C^{\gamma} \right), \]

\[ L_\beta = (g_\beta^\mu - F^{-2} C_\beta C^\mu) \nabla_\alpha \nabla_\beta C^\mu - F^{-2} \left( g_\rho^\gamma - F^{-2} C_\rho C^\gamma \right) C^\alpha \nabla_\alpha C_\gamma \nabla_\mu C^\mu, \]

(2.3)

\[ V_\beta = -(g_\mu^\nu - F^{-2} C_\mu C^\nu) \left( \nabla_\rho C_\mu + \nabla_\mu C_\rho \right) \nabla_\alpha \left( F^{-2} C_\beta C^\alpha \right) \]

\[ -\left( g_\rho^\gamma - F^{-2} C_\rho C^\gamma \right) \left( \nabla_\nu C_\mu + \nabla_\mu C_\nu \right) \left( g_\beta^\gamma - F^{-2} C_\beta C^\gamma \right) \nabla_\alpha \left( F^{-2} C_\mu C_\gamma \right) \]

\[ + \left( g_\beta^\mu - F^{-2} C_\beta C^\mu \right) \left( g_\rho^\alpha - F^{-2} C_\rho C^\alpha \right) \left( \nabla_\alpha \nabla_\beta C_\mu + \nabla_\beta \nabla_\mu C_\rho \right). \]

(2.4)

We use the superscript \(^{(s)}\) to emphasize that \( L^{(s)} \) and \( V^{(s)} \) are scalars that come from the same part of the stress–energy tensor as the terms \( L_\alpha \) and \( V_\alpha \). Equation (2.2) can be written in invariant form as

\[ \iota_C \Omega + \theta F \text{d}s = \frac{\theta}{F} C_\tau \text{d}s + \kappa \frac{r}{F} L + \theta \frac{r}{F} V = 0, \]

where \( \iota_C \) is the interior product with \( C \). Taking the exterior derivative and recalling

\[ L_\tau \Omega = (d\iota_C + \iota_C d) \Omega = d\iota_C \Omega, \]

(2.5)

where \( L_X \) is the Lie derivative in the direction of the vector field \( X \) and we have used \( d\Omega = 0 \) (\( \Omega \) is a closed form by the way it was defined), we obtain, after expressing the Lie derivative
in terms of covariant derivatives,
\[ C^\mu \nabla_\mu \Omega_{\alpha \beta} + \nabla_\alpha C^\mu \Omega_{\mu \beta} + \nabla_\beta C^\mu \Omega_{\alpha \mu} + \partial_\alpha (\theta F) \nabla_\beta \delta s - \partial_\beta (\theta F) \partial_\alpha s \]
\[ - C^\mu \partial_\mu \left( \frac{\partial}{\partial F} C_\beta - \frac{\partial_\beta}{\partial F} C_\alpha \right) - \frac{\theta}{F} C^\mu \partial_\mu s \Omega_{\alpha \beta} \]
\[ - \frac{\theta}{F} \left( (C^\mu \nabla_\mu \partial_\alpha s + \partial_\mu s \nabla_\alpha C^\mu) C_\beta - (C^\mu \nabla_\mu \partial_\beta s + \partial_\mu s \nabla_\beta C^\mu) C_\alpha \right) \]
\[ + \left[ d \left( \kappa \frac{\nabla L}{r} + \vartheta \frac{\nabla V}{r} \right) \right]_{\alpha \beta} = 0. \] (2.6)

Equation (2.6) will be called the relativistic viscous Helmholtz equation, as it reduces to the standard Helmholtz equation for inviscid relativistic fluids when \( \kappa = \vartheta = 0 \).

To complete the system, we shall consider the Laplacian of the current \( C \), computing
\[ \Delta C = (\delta + \delta d)C = d\delta C + \delta \Omega, \] (7)
where \( \delta \) is co-differentiation. But
\[ \Delta C_\alpha = -g^{\mu \rho} \nabla_\mu \nabla_\rho C_\alpha + R^\alpha_\mu C_\mu. \] (8)

Hence, using (8),
\[ g^{\mu \rho} \nabla_\mu \nabla_\rho C_\alpha = \nabla^\alpha \Omega_{\mu \rho} + \nabla_\alpha \nabla_\mu C^\mu + R^\alpha_\mu C_\mu. \] (9)

Since \( C^\alpha C_\mu = F^2 \), we can consider \( F \) as a function of \( g_{\alpha \beta} \) and \( C_\alpha \). In this case, the system of equations to be studied consists of the usual Einstein equations (1.8a), the entropy equation (2.1), the viscous relativistic Helmholtz equation (2.6) and the current equation (2.9). These are 21 equations for the 21 variables: ten \( g_{\alpha \beta} \), one \( s \), six \( \Omega_{\alpha \beta} \) and four \( C_\alpha \).

In order to proceed further, we need to compute the term \[ d \left( \kappa \frac{\nabla L}{r} + \vartheta \frac{\nabla V}{r} \right) \] of order \( \Omega_{\alpha \beta} \). Keeping all of these terms renders the system unpleasantly complex and difficult to analyse. We shall make simplifying assumptions according to the statement of theorem 1.2

### 2.1. The system of an incompressible fluid.

We make our first simplifying assumption, namely, that the fluid is incompressible:
\[ \nabla_\mu C^\mu = 0. \] (10)

It follows that \( L^{(\cdot)} = 0 = L_{\alpha \cdot} \). Also, \( \nabla_\mu \nabla_\nu C^\mu = 0 \) and, therefore,
\[ \nabla_\mu \nabla_\nu C^\mu = R^\mu_\nu C_\mu. \] (11)

Equation (9) then reads
\[ g^{\mu \rho} \partial_\mu \partial_\rho C_\alpha = B_\alpha (g, \partial g, \partial^2 g, \Omega, \partial \Omega, C, \partial C), \] (12)
where from now on we adopt the following:

**Convention.** The letters \( B \) and \( B' \), with indices attached when necessary, will be used to denote expressions where the number of derivatives of the variables \( g, s, \Omega \) and \( C \) are indicated in their arguments. The expression represented by the same letter \( B \) or \( B' \) can vary from equation to equation.

From (2.4) we can then write
\[ V_\alpha = (g^\mu_\mu - F^{-2} C^\mu C_\mu)(g^{\rho \rho} - F^{-2} C^\rho C^\rho)(\nabla_\sigma \nabla_\rho C_\mu + \nabla_\sigma \nabla_\mu C_\rho) + B_\alpha (g, \partial g, C, \partial C). \]

Using (11) produces, after some algebra,
\[ V_\alpha = g^{\rho \sigma} \nabla_\rho C_\sigma - F^{-2} C^\rho C^\sigma \nabla_\rho C_\sigma + B_\alpha (g, \partial g, \partial^2 g, \partial^3 g, C, \partial C, \partial^2 C). \] (13)
where the terms $\partial^2 g$ and $\partial^3 g$ are due to the presence of $R_{ab}$ and its derivative, respectively. 

Turning to the Helmholtz equation, we compute $(dV)_{ab} = \partial_a V_b - \partial_b V_a$ using (2.13) and the definition of $\Omega_{ab}$, obtaining 

$$g^{\alpha\sigma} \partial_{\alpha\sigma} \Omega_{ab} = B_{ab}(g, \partial g, \partial^2 g, \partial^3 g, s, \partial s, \partial^2 s, \Omega, \partial \Omega, C, \partial C, \partial^3 C, \partial^4 C).$$  

(2.14)

Assume, as usual, that we wish to solve the reduced Einstein system, which corresponds to Einstein’s equations in harmonic coordinates, in which case (1.8a) reads 

$$g^\alpha\beta \partial_{\alpha\beta} \Omega_{ab} = B_{ab}(g, \partial g, s, C, \partial C).$$  

(2.15)

Finally, the entropy equation (2.1) is of the form 

$$C^\alpha \partial_\alpha s = B(g, \partial g, s, C, \partial C).$$  

(2.16)

Understanding, as before, that $F$ is a function of $C^\alpha$ and $g_{ab}$, our system of equations for an incompressible fluid is given by (2.12), (2.14), (2.15) and (2.16); the unknowns are $g_{ab}$, $s$, $\Omega_{ab}$ and $C_\alpha$. The system, however, is not yet in a suitable form for an application of the Leray–Ohya theorem stated in the appendix. We shall take further derivatives of the equations and make one more simplifying hypothesis.

2.2. The system of an incompressible irrotational fluid

We consider now our second simplifying assumption, namely, that the fluid is irrotational: 

$$\Omega = 0.$$  

(2.17)

Applying $g^{\lambda\sigma} \partial_\lambda \partial_\sigma$ to (2.15) gives 

$$g^{\lambda\sigma} g^{\alpha\beta} \partial_{\lambda\sigma} \Omega_{ab} = B_{ab}(g, \partial g, \partial^2 g, \partial^3 g, s, \partial s, \partial^2 s, C, \partial C, \partial^3 C).$$  

(2.18)

Notice that the right-hand side does not contain third derivatives of $C$ because the fluid is incompressible and irrotational. In fact, these terms would come from 

$$\pi^\alpha_\sigma \pi^\mu_\beta g^{\lambda\sigma} \partial_{\lambda\sigma} \left( \nabla_\rho C_\mu + \nabla_\mu C_\rho \right) = 2\pi^\alpha_\sigma \pi^\mu_\beta g^{\lambda\sigma} \partial_{\lambda\sigma} \left( 2\nabla_\rho C_\mu + \Omega_{\mu\rho} \right) = 2\pi^\alpha_\sigma \pi^\mu_\beta g^{\lambda\sigma} \partial_{\lambda\sigma} \nabla_\rho C_\mu = 2\pi^\alpha_\sigma \pi^\mu_\beta \pi^\lambda_\nu \partial_{\lambda\nu} C_\mu + B_{ab}(g, \partial g, \partial^2 g, \partial^3 g, C, \partial C, \partial^3 C).$$

But 

$$\nabla_\lambda \nabla_\sigma \nabla_\rho C_\mu = \partial_{\lambda\sigma\rho} C_\mu + B_{\lambda\sigma\rho}(g, \partial g, \partial^2 g, \partial^3 g, C, \partial C, \partial^3 C),$$

and commuting the covariant derivatives gives 

$$\nabla_\lambda \nabla_\sigma \nabla_\rho C_\mu = \nabla_\rho \nabla_\lambda \nabla_\sigma C_\mu + B_{\lambda\sigma\rho}(g, \partial g, \partial^2 g, \partial^3 g, C, \partial C),$$

so that 

$$\partial_{\lambda\sigma\rho} C_\mu = \nabla_\rho \nabla_\lambda \nabla_\sigma C_\mu + B_{\lambda\sigma\rho}(g, \partial g, \partial^2 g, \partial^3 g, C, \partial C, \partial^3 C).$$

Contracting with $g^{\sigma\alpha}$ and invoking (2.8) produces 

$$g^{\lambda\sigma} \partial_{\lambda\sigma} C_\mu = \nabla_\rho \left( -\Delta C_\mu + R^\alpha_\mu C_\alpha \right) + B_{\rho\mu}(g, \partial g, \partial^2 g, \partial^3 g, C, \partial C, \partial^3 C) = B_{\rho\mu}(g, \partial g, \partial^2 g, \partial^3 g, C, \partial C, \partial^3 C),$$

since $\Delta C = 0$ by (2.7), (2.10) and (2.17).

Apply $g^{\rho\alpha} \partial_{\rho\alpha}$ to (2.1) and use $\frac{\partial^2 C}{\partial t^2} > 0$ to find 

$$g^{\rho\alpha} C^\alpha \partial_{\rho\alpha} s = B(g, \partial g, \partial^2 g, \partial^3 g, s, \partial s, \partial^2 s, C, \partial C, \partial^3 C),$$  

(2.19)

where the third derivatives of $C$ are not present by an argument similar to the one used above involving $\Delta C$ in the derivation of (2.18).
Finally, apply \( g^{\mu\nu} \partial_{\mu\nu} \) to (2.12) and use (2.17) to get
\[
g^{\mu\nu} g^{\alpha\beta} \partial_{\mu\nu} C_\alpha = B_\alpha(g, \partial g, \partial^2 g, \partial^3 g, C, \partial C, \partial^2 C, \partial^3 C). \tag{2.20}
\]
As we shall see, the system of 15 equations (2.18), (2.19), and (2.20), for the ten \( g_{\alpha\beta} \), one \( s \) and four \( C_\alpha \) (again, \( F \) is considered as a function of \( C_\alpha \) and \( g_{\alpha\beta} \)) forms a Leray–Ohya system with an index structure (see the appendix for definitions)
\[
\begin{aligned}
m(\text{equation (2.18)}) &= 4, & m(\text{equation (2.19)}) &= 3, & m(\text{equation (2.20)}) &= 4, \\
s(g_{\alpha\beta}) &= 5, & s(s) &= 4, & s(C_\alpha) &= 4, \\
t(\text{equation (2.18)}) &= 2, & t(\text{equation (2.19)}) &= 2, & t(\text{equation (2.20)}) &= 1.
\end{aligned}
\tag{2.21}
\]
In order to confirm this and apply theorem A.1 from the appendix, we need to first turn our attention to the Cauchy data and the system’s characteristics.

3. Initial data and characteristics

3.1. Characteristics

Consider a regular hypersurface \( S \) that is locally given as the zero set of a sufficiently differentiable function, i.e., \( S = \{ f(x) = 0 \} \). Looking at the left-hand side of the system (2.12), (2.14), (2.15), (2.16), we see at once that \( S \) will be characteristic if any of the two following conditions hold:
\[
C_\alpha \partial_\alpha f = 0, \tag{3.1}
\]
or
\[
g^{\alpha\beta} \partial_\alpha f \partial_\beta f = 0. \tag{3.2}
\]
A quick inspection shows that these are also the characteristics of the system (2.18), (2.19) and (2.20), although the characteristics of (2.18) and (2.20) have multiplicity two.

Recalling that \( C^\alpha = F u^\alpha \), we see that (3.1) expresses that \( S \) is spanned by the flow-lines of the fluid, whereas (3.2) means that \( S \) is tangent to the light-cone of the metric \( g \) at each point. The reader should contrast the present characteristic surfaces with those of a relativistic inviscid fluid. In the latter case, besides the flow-lines and surfaces tangent to the light-cone, a third family of physically meaningful characteristic surfaces is present, namely, those corresponding to the sound waves of the fluid. As is discussed in section 5, if no simplifying assumption is made, and we consider the full set of equations (2.1), (2.6), (2.9), and (2.15), a third family of characteristics also appears in the viscous case, but these are non-physical and do not correspond to the propagation of any physical quantity. In fact, if one insists on computing the speed of propagation of would-be acoustic waves along such surfaces, it is found to be infinite. Hence, here, as in the non-relativistic case, there is not a well-defined notion of sound speed for a viscous fluid.

3.2. Cauchy data for an incompressible fluid

We are interested in obtaining initial conditions for the system (2.18), (2.19), and (2.20). Initially, we do not necessarily assume (2.17), and along the way, initial data for equations (2.12), (2.14), (2.15) and (2.16) will be obtained. Hence, we suppose we are given a solution to the incompressible Einstein–Navier–Stokes equations. As in section 2, we continue to assume that \( s > 0 \) and \( F > 0 \) are the two independent thermodynamic variables, with an equation of state \( r = r(F, s) \) satisfying the hypotheses stated in theorem 1.2.
The Cauchy data\(^{13}\) that is given for system (1.8a)–(1.8d) consist of the values of 
\[ g_{\alpha\beta}, \partial_0 g_{\alpha\beta}, u_\alpha, s, \text{ and } F \text{ on } \Sigma. \]  
(3.3)

We naturally suppose that the constraints are satisfied and that \( \Sigma \) is non-characteristic. As a consequence, in the principal part of system (2.12), (2.14), (2.15), (2.16), and of system (2.18), (2.19) and (2.20), one can always solve for the highest time derivatives appearing on the left-hand side in terms of quantities on the right-hand side, and this fact will be extensively used below; derivatives along \( \Sigma \) can always be computed and present no problem in determining the Cauchy data.

We shall say that some expression is equal to \( C.D. \) (Cauchy data) when it can be expressed solely in terms of quantities that are written in terms of the Cauchy data on \( \Sigma \). \( \partial_i \) will symbolically denote spatial derivatives \( \partial_i \). We shall denote by \( Z \) a general smooth function of its arguments, which may vary from expression to expression.

Recall that
\[ C^\alpha = F u^\alpha, \]  
(3.4)
so in particular \( C^\alpha \) is known on \( \Sigma \).

We derive the values of \( \partial_0 u_\alpha \) on \( \Sigma \). Differentiating (1.8d) yields \( u^\alpha \nabla_0 u_\alpha = 0 \), so that
\[ u^\alpha \partial_0 u_\alpha = Z(g, \partial g, u, \partial u). \]
Pichon [23] has shown that if \( u \) is time-like, then the above relation used in conjunction with the constraints \( S_0^\alpha = \mathcal{N} T_0^\alpha \) and (1.8d) allow us to solve for the values of \( \partial_0 u_\alpha \) by first changing to a coordinate system where \( u' = 0 \). We have therefore determined \( \partial_0 u_\alpha \) on \( \Sigma \), i.e.,
\[ \partial_0 u_\alpha|_\Sigma = Z(g, \partial g, u, \partial u)|_\Sigma = C.D. \]  
(3.5)

Using (3.4) in (2.1) yields
\[ r \theta u^\alpha \partial_\theta s - F \partial (\pi^{\alpha\rho} \nabla_\rho u_\beta \nabla_\alpha u^\beta + \nabla_\alpha u^\mu \nabla_\mu u^\alpha) = 0. \]
Using (3.5), \( r, \theta > 0 \),
\[ \partial_0 s|_\Sigma = C.D. \]  
(3.6)

Notice that (3.6) could not have been determined directly from (2.16) in that this last equation involves \( \partial_\mu C_\alpha \), which, as we next show, requires \( \partial_\mu s|_\Sigma \) to be known on \( \Sigma \).

From (1.8c), we obtain
\[ u^\alpha \frac{\partial r}{\partial s} \partial_\alpha s + \frac{\partial r}{\partial F} u^\alpha \partial_\alpha F + r \nabla_\alpha u^\alpha = 0. \]
Since \( \frac{\partial r}{\partial F} > 0 \) by (1.6) and \( \partial_0 s \) is known on \( \Sigma \) by (3.6), we also have
\[ \partial_0 F|_\Sigma = C.D. \]  
(3.7)

It follows from (3.5) and (3.7) that
\[ \partial_\alpha C_\alpha|_\Sigma = C.D. \]  
(3.8)

From (2.15) and the above, we get
\[ \partial_{00} g_{\alpha\beta}|_\Sigma = C.D. \]
From \( \pi^{\rho\mu} \nabla^\alpha T_{\alpha\mu} = 0 \), we obtain
\[ \partial \pi^{\rho\mu} \pi^{\alpha\beta} (\partial_\rho g_{\alpha\beta} + \partial_\beta g_{\alpha\rho}) = Z(g, \partial g, \partial^2 g, F, \partial F, s, \partial s, \partial C, \partial C, \partial \partial C). \]  
(3.9)

\(^{13}\) We remark that we shall eventually consider harmonic coordinates, in which case only the spatial components \( g_{ij}, \partial_0 g_{ij} \) and \( u' \) on \( \Sigma \) are given in the initial Cauchy data. The remaining components are obtained by the use of harmonic coordinates and (1.8d). See section 4.
We shall use a similar procedure to that employed by Pichon [23]. We can choose coordinates near $\Sigma$ such that
\[ u^0 \neq 0, \ u^i = 0, \ \pi^0_\mu = 0, \ \pi^{00} \neq 0, \]  
so that (3.9) becomes, after lowering the index $\gamma$,
\[ \vartheta(\pi^{00}_\gamma \pi^0_\mu + \pi^0_\gamma \pi^{00}_\mu) \partial_{00} C_\mu = Z(g, \partial g, \partial^2 g, F, \partial F, s, \partial s, C, \partial C, \bar{\partial} \partial C). \]  
Setting $\gamma = 0$ in (3.11) and using (1.8d) yields
\[ \vartheta(u^0)^2 \partial_{00} C_0 = Z(g, \partial g, \partial^2 g, F, \partial F, s, \partial s, C, \partial C, \bar{\partial} \partial C). \]  
Putting $\gamma = i$ in (3.11):
\[ \vartheta(\delta^j_i \pi^{00} + \pi^{00}_i \pi^0_j) \partial_{00} C_j = Z(g, \partial g, \partial^2 g, F, \partial F, s, \partial s, C, \partial C, \bar{\partial} \partial C). \]  
The determinant of the matrix on the left-hand side is
\[ 2\vartheta^3(\pi^{00})^3 \neq 0, \]  
where we used that $\pi^0_\gamma \pi^{00}_\gamma = \pi^{00}_i \pi^0_i = \pi^{00}$ implying that $\partial_{00} C_\mu|_\Sigma$ is known, since, in light of (3.8), $\partial_{00} C_\mu|_\Sigma$ and the other terms on the right-hand side are known on $\Sigma$ as shown above. For a general coordinate system, one changes coordinates to satisfy (3.10), where all second derivatives of $C$ on $\Sigma$ can be found. Changing coordinates back gives that
\[ \partial_{00} C_\mu|_\Sigma = C.D. \]  
Notice that we could not have determined $\partial_{00} C_\mu|_\Sigma$ from (2.12), since this involves first derivatives of $\Omega$. In addition to (3.3), we have therefore determined
\[ \vartheta^2 g, \partial s, C, \partial C, \]  and $\vartheta^3 C$ on $\Sigma$.

Also, from (2.2), (2.3), (2.4), (3.3) and (3.13), we see that
\[ \Omega_{\alpha\beta}|_\Sigma = C.D. \]  
Differentiating (2.16), using (3.3) and (3.13), gives
\[ \partial_{00} s|_\Sigma = C.D., \]  
and differentiating (2.15), using (3.3), (3.13) and (3.15), gives
\[ \partial^3 s|_\Sigma = C.D. \]  
Differentiating (3.9), using a similar argument to the one leading to (3.12) and invoking (3.3), (3.13), (3.15) and (3.16), yields
\[ \partial^5 C_\mu|_\Sigma = C.D. \]  
Differentiating (2.2), using (2.3), (2.4), (3.3), (3.13), (3.14), (3.15), (3.16) and (3.17), gives
\[ \partial_0 \Omega_{\alpha\beta}|_\Sigma = C.D. \]  
Finally, taking two derivatives of (2.15), use (3.3), (3.13), (3.15), (3.16) and (3.17) to get
\[ \partial^4 s|_\Sigma = C.D. \]  
Together, (3.3), and (3.13), (3.14), (3.15), (3.16), (3.17), (3.18) and (3.19) are the desired Cauchy data for the system (2.12), (2.14), (2.15) and (2.16). It is clear that we can continue the above process, and determine all the derivatives of the unknowns on $\Sigma$. If the data is analytic, we obtain an analytic solution in a neighbourhood of $\Sigma$. 

1926
3.3. Cauchy data for an incompressible irrotational fluid

We now turn our attention to system (2.18), (2.19) and (2.20), so we suppose from now on that $\Omega = 0$. In view of (2.21) and theorem A.1, we need to determine the derivatives of $g_{\alpha\beta}$ up to order 4; of $s$ up to order 3; and of $C_{\alpha}$ up to order 3.

Noticing that the initial data derived in section 3.2 is compatible with equations (2.18), (2.19) and (2.20), it remains for us to find the third derivatives of $s$ on $\Sigma$. Using (2.19), (3.3), (3.13), (3.15), (3.16) and (3.17) gives

$$\partial_0^3 s \big|_{\Sigma} = C.D.$$  \hspace{1cm} (3.20)

Here (3.3), (3.13), (3.15), (3.16), (3.17), (3.19) and (3.20) are the initial data for equations (2.18), (2.19) and (2.20). These have to satisfy further compatibility conditions, as explained in appendix Appendix A. These are determined by plugging the initial data into equations (2.18), (2.19) and (2.20), and taking the $t$ derivative of equation (2.18) and the $t$ derivative of equation (2.19); no further derivative of equation (2.20) is necessary since $t$ derivative of equation (2.20) $= 1 - 1 = 0$.

4. Proof of theorem 1.2

We shall show that under the hypotheses stated in theorem 1.2, the system (2.18), (2.19) and (2.20) is a Leray system, and theorem A.1 can be applied. The reader is referred to appendix Appendix A for the notation and terminology employed in this section in connection with theorem 1.2.

Let $I$ be given as in the statement of theorem 1.2. Following a standard procedure, we embed $\Sigma$ into the product $\mathbb{R} \times \Sigma$, and consider a coordinate chart $U \subset \Sigma$, where harmonic coordinates have been chosen such that on $U$, $g_{ij} = (g_0)_{ij}$, $\partial_0 g_{ij} = K_{ij}$, $g_{00} = 1$, $g_{0i} = 0$, with $\partial_0 g_{ij}$ determined by the conditions of harmonic coordinates on $U$. On $U$ we also have $u^i = v^i$ and $u^0$ determined by the condition (1.8). The values of $s$ and $r$ are known on $U$ from the initial data, which determines, via the equation of state, the values of $F$ on $U$. We have, therefore, the Cauchy data (3.3). As shown in section 3.3, from this the Cauchy data for the system (2.18), (2.19) and (2.20), is known.

Let $A$ be the principal part of our system, i.e., the matrix formed by the left-hand side of equations (2.18), (2.19) and (2.20). Symbolically:

$$A(x, g, s, C, \partial) = \begin{pmatrix} g_{\lambda\sigma} g^{\mu\rho} \partial_{\lambda\sigma\mu\rho} & 0 & 0 \\ 0 & g^{\mu\alpha} C_\alpha \partial_{\mu\alpha} & 0 \\ 0 & 0 & g^{\lambda\sigma} g^{\mu\rho} \partial_{\lambda\sigma\mu\rho} \end{pmatrix}.$$  

Let $a_1 = a_3 = g_{\lambda\sigma} g^{\mu\rho} \partial_{\lambda\sigma\mu\rho}$, $a_2 = g^{\mu\alpha} C_\alpha \partial_{\mu\alpha}$, and let $h_1(x, \xi) = h_3(x, \xi)$ and $h_2(x, \xi)$ be their characteristic polynomials when $g_{\alpha\beta}$, $s$ and $C_{\alpha}$ are given. To identify the characteristic cones, set

$$h_2(x, \xi) = g^{\rho\mu}(x) C_{\alpha}(x) \xi_{\rho} \xi_{\mu} \xi_{\alpha} = 0,$$

for $x \in U$, with $g^{\rho\mu}$ and $C_{\alpha}$ replaced by the corresponding Cauchy data on $U$. It follows that $h_2(x, \xi)$ is hyperbolic at $x$ if $g^{\rho\mu}(x) C_{\alpha}(x) C^\beta(x) > 0$, i.e., if $C$ is time-like with respect to the metric $g$. Since $C_{\alpha} = F u^\alpha$, this is the case by the hypotheses of theorem 1.2. A consequence is that the half-cone defined by $C_{\alpha} \xi_{\alpha} > 0$ is exterior to the one given by $g^{\rho\beta} \xi_{\rho} \xi_{\beta} > 0$, hence $\Gamma_+^0(a_2)$ coincides with the half-cone $g^{\rho\beta} \xi_{\rho} \xi_{\beta} > 0$ of the metric $g$. For $a_1$ and $a_3$, we have

$$h_1(x, \xi) = g^{\rho\mu}(x) g^{\lambda\sigma}(x) \xi_{\rho} \xi_{\mu} \xi_{\lambda} \xi_{\sigma} = (g^{\rho\mu}(x) \xi_{\rho} \xi_{\mu})^2 = 0.$$
The orders of the operators are the hypotheses of theorem \( A.1 \) with \( \sigma \) in \( f \) of these variables and their derivatives. The denominator of the rational expressions appearing \( \leq 1 \) for any \( \sigma \) in view of our choice of indices (2.21), it follows that the coefficients of system (4.1) satisfy them. By our hypotheses, all such terms are uniformly bounded away from zero on \( \Sigma \). Thus \( w = 2 \), and the initial data belongs to Gevrey spaces satisfying the condition \( 1 \leq \sigma < 2 = \frac{w}{w-1} \).

Next, write the system as
\[
A(x, g, s, C, \partial)(g, s, C) = B(x, g, s, C),
\]
where \( B \) is given by the expression on the right-hand side of equations (2.18), (2.19) and (2.20).

The matrix \( A \) depends polynomially on the functions \( g, s \) and \( C \), while \( B \) is a rational function of these variables and their derivatives. The denominator of the rational expressions appearing in \( B \) are of the form \( F = \sqrt{C^\alpha C_\alpha}, F^2 = C^\alpha C_\alpha, r = r(F, s), \theta = \theta(F, s) \) or products of them. By our hypotheses, all such terms are uniformly bounded away from zero on \( \Sigma \). And in view of our choice of indices (2.21), it follows that the coefficients of system (4.1) satisfy the hypotheses of theorem \( A.1 \) with \( m = 3 \).

It remains to verify that (4.1) is indeed a Leray system with the index structure (2.21). The orders of the operators are \( m(\text{equation }2.18) = 4 = s(g_{\alpha\beta}) - t(\text{equation }2.18) + 1 = \text{order of } a_1; m(\text{equation }2.19) = 3 = s(s) - t(\text{equation }2.19) + 1 = \text{order of } a_2; m(\text{equation }2.20) = 4 = s(C_\alpha) - t(\text{equation }2.20) + 1 = \text{order of } a_3 \). The coefficients of \( a_j, j = 1, 2, 3 \), do not depend on derivatives of \( g, s \) or \( C \), so it suffices to verify that each \( b_i(x, g, s, C) \) depends on at most \( s(k) - t(l) \) derivatives of the corresponding \( i \)th unknown.

Below, we list, for each equation, the difference \( s(k) - t(l) \) and the corresponding highest-order derivative appearing on the right-hand side of the equation.

**equation (2.18)**
\[
\begin{align*}
&\varnothing, & a_1^0 g, \\
&\varnothing, & a_1^1 s, \\
&\varnothing, & a_1^2 C, \\
\end{align*}
\]

**equation (2.19)**
\[
\begin{align*}
&\varnothing, & a_2^0 g, \\
&\varnothing, & a_2^1 s, \\
&\varnothing, & a_2^2 C, \\
\end{align*}
\]

**equation (2.20)**
\[
\begin{align*}
&\varnothing, & a_3^0 g, \\
&\varnothing, & a_3^1 s, \\
&\varnothing, & a_3^2 C, \\
\end{align*}
\]

where \( \varnothing \) indicates that the corresponding variable does not appear. We have therefore verified all of the hypotheses of theorem \( A.1 \), obtaining in this way a solution to equations (2.18), (2.19) and (2.20) in a neighbourhood of \( \mathcal{U} \). This solution is the unique solution in the Gevrey class indicated in theorem 1.2.

We need to show that this solution yields a solution to the original incompressible irrotational Einstein–Navier–Stokes system. Suppose first that the initial data given in the hypotheses of theorem 1.2 is analytic, and consider the Einstein–Navier–Stokes system with the reduced Einstein equations in place of (1.8a), i.e., suppose that the system is written in harmonic coordinates. Pichon [23] has shown how the analytic Cauchy problem for the Einstein–Navier–Stokes system, with the reduced Einstein equations in (1.8a), can be solved by successively computing higher-order time derivatives in terms of the Cauchy data on \( \Sigma \). His work only treated the case of an equation of state \( p = p(\rho) \), i.e., without including entropy, but it is not difficult to see that his procedure can be generalized to allow for the more general system we are treating here. By the way the Cauchy data was derived in section 3.2, and upon setting \( C^\alpha = Fu^\alpha \) and \( \Omega_{\alpha\beta} = \nabla_\alpha C_\beta - \nabla_\beta C_\alpha \), this solution satisfies equations (2.12),
(2.14), (2.15) and (2.16) when (1.2) reduces to (1.3), i.e., when the fluid is incompressible. If, furthermore, the fluid is also irrotational, i.e., (1.3) reduces to (1.4), then this solution also satisfies (2.18), (2.19) and (2.20) in light of the way its initial data was obtained in section 3.3, and it necessarily agrees with the solution given by the Leray–Ohya theorem.

Summing up, if we are given analytic initial data for the incompressible irrotational Einstein–Navier–Stokes equations, then we obtain an analytic solution to the system ((1.8a)–(1.8d)) written in harmonic coordinates, and this solution satisfies the system (2.18), (2.19) and (2.20), and coincides with the solution given by the Leray–Ohya theorem.

Returning now to the general, i.e., non-analytic case, we approximate, in the Gevrey topology, the given initial data \( I \) from theorem 1.2 by a sequence of analytic initial data \( \{I_\nu\} \), obtaining a family of analytic solutions \( \{Z_\nu = (g_\nu, u_\nu, s_\nu, r_\nu)\} \) to the incompressible irrotational Einstein–Navier–Stokes system written in harmonic coordinates. This yields a family of solutions \( \{U_\nu = (g_\nu, s_\nu, C_\nu)\} \) to system (2.18), (2.19) and (2.20). When \( Z_\nu \rightarrow Z \), one gets \( U_\nu \rightarrow U \), where \( U \) is the solution obtained above by application of the Leray–Ohya theorem. The energy-type of estimates derived by Leray and Ohya [14] guarantee that the sequence \( Z_\nu \) also has a limit \( Z \) that lies in the stated Gevrey spaces and satisfies the system ((1.8a)–(1.8d)) written in harmonic coordinates, with the stress–energy tensor given by (1.4)\(^{14}\). It is well-known that a solution to the reduced Einstein equations satisfies the full system if and only if the constraints are satisfied, which is the case by our hypotheses on the initial data.

We also notice that from \( \pi^{\alpha\beta}\nabla_\alpha T_{\beta\gamma} = 0 \), it follows that

\[
0 = u^\alpha u^\gamma \nabla_\alpha u_\gamma = \frac{1}{2} u^\alpha \partial_\alpha (u^\gamma u_\gamma),
\]

and therefore \( u \), being unitary at time zero, remains unitary.

The existence of a domain of dependence, as stated in theorem 1.2, follows at once from the domain of dependence property of theorem A.1, using the fact (shown above) that the half-cones \( \Gamma^\pm_1(A) \) agree with those of the metric \( g \). With the domain of dependence property at hand, a standard gluing argument now gives a global, in space, solution that is geometrically unique. This finishes the proof of theorem 1.2.

**Remark 4.1.** A natural question is whether the condition \( \Omega = 0 \) is preserved under the time-evolution, i.e., whether \( \Omega \) remains zero if it vanishes on \( \Sigma \). This is known to be the case for the non-relativistic Navier–Stokes equations [20], and also for relativistic inviscid fluids [16]. In order to establish this here, we reason as follows. As mentioned in section 3.2, the incompressible, not necessarily irrotational, system can be solved for analytic data; notice that such a solution is not necessarily derived from theorem 1.2. In this case, the vorticity satisfies (2.14), and this equation remains valid when \( \Omega = 0 \), and in particular when we consider (2.14) with the (analytic) solution given by theorem 1.2. On the other hand, the analytic solution to (2.14), with initial data given as in theorem 1.2 and \( \Omega = 0 \) at \( t = 0 \), agrees, by uniqueness, with that given by the theorem. Thus \( \Omega = 0 \), if it is zero initially.

**Remark 4.2.** The argument of the previous remark is rather indirect. It is of importance to obtain a simple evolution equation for \( \Omega \) only, as in the case of non-relativistic [20] and relativistic inviscid fluids [16]. Not only would this give a direct argument for the propagation of the zero vorticity condition, but perhaps more importantly, it would give us a direct way of studying the evolution of the vorticity, a problem of clear physical importance.

\(^{14}\) We notice that this approximation argument is essentially the same one used by Lichnerowicz to produce solutions in the Gevrey class for the equations of magneto-hydrodynamics [18].
5. Comparison with the inviscid case.

In this section, we derive a different system of equations suited for comparisons with the case of an ideal fluid. This follows the spirit of our guiding principle (i) (see introduction), and it is also a tentative first step in addressing the (notably hard) problem of the convergence of solutions of the viscous equations to solutions of the inviscid ones when $\kappa, \theta \to 0$.

We once again consider the Laplacian of the current $C$, i.e., equation (5.2), but now a different route will be taken. Using $\delta C = -\nabla_a C^a = -\nabla_a (F u^a)$ and $\nabla_a (r u^a) = 0$, we get

$$\delta C = C^a \left( \frac{\partial_a r}{r} - \frac{\partial_a F}{F} \right).$$

Set $\tilde{F} = F^2$ and $\mathcal{F}(\tilde{F}, s) = \log \tilde{F}$ (recall that $r = r(F, s)$). So

$$\delta C = \mu d\mathcal{F},$$

from which follows, upon computing $d\delta C$, that

$$\Delta C = \mathcal{L}_c d\mathcal{F} + \delta \Omega.$$  \hspace{1cm} (5.1)

We recall that $\mathcal{L}$ is the Lie derivative. Next, we compute $\mathcal{L}_c d\mathcal{F}$. Computing $\partial_a \mathcal{F}$ directly, using the definition of $\Omega$, and equation (2.2) yields

$$\partial_a \mathcal{F} = \frac{\partial F}{\partial F} C^\mu C^\nu \nabla_\mu C_a + \left( \frac{\partial F}{\partial s} + 2 \frac{\partial F}{\partial F} \frac{\partial \mathcal{F}}{\partial F} \right) \partial_a s - \frac{2}{\omega_1} \frac{\partial F}{\partial F} C_a C^\sigma \partial_\sigma s + 2 \frac{\partial F}{\partial F} \left( \frac{\partial w_\mu}{\partial F} \right) \left( \kappa \mu_a + \theta \nu_a \right).$$

From this and the standard formula $\mathcal{L}_X w_a = X^\mu \nabla_a w_a + w_\mu \nabla_a X^\mu$, we find

$$[\mathcal{L}_c d\mathcal{F}]_a = 2 \frac{\partial F}{\partial F} C^\mu C^\nu \nabla_\mu \nabla_\nu C_a + \left( \frac{\partial F}{\partial s} + 2 \frac{\partial F}{\partial F} \frac{\partial \mathcal{F}}{\partial F} \right) \nabla_a \left( C^\mu \nabla_\mu C_a \right) - 2 \frac{\partial F}{\partial F} \left( \frac{\partial \mathcal{F}}{\partial F} \right) \nabla_a \left( \kappa \mu_a + \theta \nu_a \right) \nabla_\nu C_a.$$ 

Combining this with (5.1), using (2.8), and applying $C^\sigma \nabla_\sigma$ to the resulting expression, leads to

\begin{align*}
\left( \tilde{g}^{\mu\nu} - \left( 1 - \frac{F}{r} \frac{\partial r}{\partial F} \right) \left( \frac{C^\mu C^\nu}{F^2} \right) C^\sigma C^\rho \nabla_\sigma C_\rho \nabla_\mu C_a + C^\sigma C^\rho \nabla_\rho \nabla_\sigma \left[ \frac{2}{\omega_1} \frac{\partial \mathcal{F}}{\partial F} \left( \kappa \mu_a + \theta \nu_a \right) \right] \right) \\
- \nabla_\mu \left( C^\sigma \nabla_\rho \Omega^\sigma \right) + \left( \frac{\partial F}{\partial s} + 2 \frac{\partial F}{\partial F} \frac{\partial \mathcal{F}}{\partial F} \right) C^\sigma \nabla_\sigma \left( C^\mu \nabla_\mu C_a \right) \\
+ 2 C^\sigma \nabla_\sigma \left( \frac{\partial F}{\partial F} C^\mu \right) \nabla_\mu \nabla_\sigma C_a + \nabla_\mu C^\sigma \nabla_\rho \Omega^\sigma \nabla_\rho C_a + C^\sigma R_{\rho\alpha} C^\rho \Omega^\sigma \nabla_\alpha + R_{\mu\rho} \partial_\sigma \Omega^\rho \nabla_\sigma C_a \\
+ C^\sigma \left( \frac{\partial F}{\partial F} + 2 \frac{\partial F}{\partial F} \frac{\partial \mathcal{F}}{\partial F} \right) \nabla_a \left( C^\sigma \nabla_\mu C_a \right) - C^\sigma \nabla_a \left( R_{\mu\rho} C_{\rho a} \right) \\
+ C^\sigma \left( \frac{\partial F}{\partial F} + 2 \frac{\partial F}{\partial F} \frac{\partial \mathcal{F}}{\partial F} \right) \nabla_a \left( C^\sigma \nabla_\mu C_a \right) + C^\sigma \nabla_a \left( \kappa \mu_a + \theta \nu_a \right) \nabla_\sigma C_a - 2 C^\sigma \nabla_\sigma \left( \theta \frac{\partial \mathcal{F}}{\partial F} \nabla_\rho C_{\rho a} \right) + C^\sigma \nabla_\sigma \left( d \mathcal{F} \nabla_\sigma C^\mu \right) \\
+ C^\sigma \nabla_\sigma C_{\rho a} \left[ \frac{2}{\omega_1} \frac{\partial \mathcal{F}}{\partial F} \left( \kappa \mu_a + \theta \nu_a \right) \right] = 0.
\end{align*}
where the Riemann tensor appears due to the commutation of covariant derivatives. Taking equations (2.1), (2.6), (2.15) and (5.2) as the set of equations for a relativistic viscous fluid on the unknowns $s$, $\Omega$, $g$ and $C$ (again, $F$ is treated as a function of $C$ and $g$), we obtain a system that has the convenient property of directly reducing, upon setting $\kappa = \vartheta = 0$, to the system of an inviscid relativistic fluid studied by Lichnerowicz [16, 17].

Unfortunately, system (2.1), (2.6), (2.15) and (5.2) does not seem amenable to a treatment via the techniques here employed: there are just too many derivatives present in order to accommodate the necessary index structure of a Leray system. That is why the system (2.1), (2.6), (2.15) and (5.2), which would seem natural from the point of view of the equations obtained when $\kappa = \vartheta = 0$, has not been used, and we had to derive a different set of equations in order to prove theorem 1.2.

One could try to circumvent the above problem by incorporating some of the higher-order derivatives into the principal part of the system, i.e., into $A(x, U, \partial)U$. But if one attempts to do this, the operator $A(x, U, \partial)$ will contain terms of the form

$$(g^{\lambda\rho} - C^\lambda C^\rho) \partial_{\lambda\rho},$$ (5.3)

whose characteristic polynomial fails to be hyperbolic. In fact, the characteristic surfaces defined by (5.3) are space-like and correspond to the would-be acoustic waves mentioned in section 3.1. We believe that more specific techniques, tailored to the special structure of the above equations, will have to be developed in order to prove, for a general relativistic viscous fluid, the existence of solutions that enjoy the domain of dependence property.

Acknowledgment

The author is supported by NSF grant 1305705.

Appendix A. Leray–Ohya systems.

For the reader’s convenience, in this section we briefly recall the definition of a Leray–Ohya system and state the Leray–Ohya theorem used in our proof.

Let $M$ be a smooth manifold, $a = a(x, \partial)$, $x \in M$, a differential operator of order $m$ acting on sufficiently regular real-valued functions defined on $M$, and $h(x, \partial)$ the principal part of $a(x, \partial)$ at $x$. For a given $x \in M$, consider the characteristic polynomial of $a$ at $x$, denoted by $h(x, \xi)$, where $\xi \in T^*_x M$; it is a homogeneous polynomial of degree $m$. The cone $V_x(h)$ of $h$ in $T^*_x M$ is defined by the equation

$$h(x, \xi) = 0.$$ 

Here $h(x, \xi)$ is called a hyperbolic polynomial if there exists $\zeta \in T^*_x M$ such that every straight line through $\xi$ that does not contain the origin intersects the cone $V_x(h)$ at $m$ real distinct points. Under these conditions, the set of points $\xi$ with this property forms the interior of two opposite convex half-cones $\Gamma^*_x(a)$, $\Gamma^*_x(-a)$, with $\Gamma^*_x(\pm a)$ non-empty, with boundaries that belong to $V_x(h)$.

Consider the $\ell \times \ell$ diagonal matrix

$$A(x, \partial) = \begin{pmatrix}
a_1(x, \partial) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & a_\ell(x, \partial)
\end{pmatrix}.$$ 

Each $a_t(x, \partial)$, $t = 1, \ldots, \ell$ is a differential operator of order $m_t$. The operator $A(x, \partial)$ is called Leray–Ohya hyperbolic at $x$ if:
(i) The characteristic polynomial \( h_t(x, \xi) \) of each \( a_t(x, \partial) \) is a product of hyperbolic polynomials, i.e.
\[
h_t(x, \xi) = h_t^1(x, \xi) \cdots h_t^p(x, \xi), \ t = 1, \ldots, \ell,
\]
where each \( h_t^q(x, \xi), q = 1, \ldots, p_t, t = 1, \ldots, \ell, \) is a hyperbolic polynomial.

(ii) The two opposite convex half-cones,
\[
\Gamma_+^c(A) = \bigcap_{t=1}^{\ell} \bigcap_{q=1}^{p_t} \Gamma_+^c(a_t^q), \quad \text{and} \quad \Gamma_-^c(A) = \bigcap_{t=1}^{\ell} \bigcap_{q=1}^{p_t} \Gamma_-^c(a_t^q),
\]
have a non-empty interior. Here, \( \Gamma_+^c(a_t^q) \) are the half-cones associated with the hyperbolic polynomials \( h_t^q(x, \xi), q = 1, \ldots, p_t, t = 1, \ldots, \ell. \)

We define the convex half-cone \( C_t(x) \) at \( T_x M \) as the set of \( v \in T_x M \) such that \( \xi(v) \geq 0 \) for every \( \xi \in \Gamma_+^c(A); C_t^-(x) \) is analogously defined, and we set \( C_t(x) = C_t^+(x) \cup C_t^-(x) \). If the convex cones \( C_t^+(x) \) and \( C_t^-(x) \) can be continuously distinguished with respect to \( x \in M \), then \( M \) is called time-oriented (with respect to the hyperbolic form provided by the operator \( A \)). A path in \( M \) is called future (past) time-like with respect to \( A \) if its tangent at each point \( x \in M \) belongs to \( C_t^+(x) (C_t^-(x)) \); future (past) causal if its tangent at each point \( x \in M \) belongs or is tangent to \( C_t^+(x) (C_t^-(x)) \). A regular surface \( \Sigma \) is called space-like with respect to \( A \) if \( T_x \Sigma (\subseteq T_x M) \) is exterior to \( C_t(x) \) for each \( x \in \Sigma \). It follows that for a time-oriented \( M \), the concepts of causal past, future, domains of dependence and influence of a set can be defined in the same way one does when the manifold is endowed with a Lorentzian metric. We refer the reader to [13] for details; here we shall need only the following: the causal past \( J^-(x) \) of a point \( x \in M \) is the set of points that can be joined to \( x \) by a past causal curve.

Next, we consider the following quasi-linear system of differential equations
\[
A(x, U, \partial)U = B(x, U),
\]
where \( A(x, U, \partial) \) is the \( \ell \times \ell \) diagonal matrix
\[
A(x, U, \partial) = \begin{pmatrix}
a_1(x, U, \partial) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & a_{\ell}(x, U, \partial)
\end{pmatrix},
\]
with \( a_t(x, U, \partial), t = 1, \ldots, \ell \) differential operators of order \( m(t) \), \( B(x, U) \) the vector
\[
B(x, U) = (b_t(x, U)), \ t = 1, \ldots, \ell,
\]
and the vector
\[
U(x) = (u_k(x)), \ k = 1, \ldots, \ell
\]
is the unknown.

The system \( A(x, U, \partial)U = B(x, U) \) is called a Leray system if it is possible to attach to each unknown \( u_k \) an integer \( s(k) \geq 1 \), and to each equation \( i \) of the system an integer \( t(i) \geq 1 \), such that\textsuperscript{15}:

(i) \( m(t) = s(t) - t(i) + 1, \ t = 1, \ldots, \ell; \)

(ii) the functions \( b_t \) and the coefficients of the differential operators \( a_t \) are sufficiently regular functions of \( x \), of \( u_k \), and of the derivatives of \( u_k \) of order less than or equal to \( s(k) - t(i) \), \( k, t = 1, \ldots, \ell \). If, for some \( k \) and some \( t, s(k) - t(i) < 0, \) then the corresponding \( a_t \) and \( b_t \) do not depend on \( u_k \).

\textsuperscript{15} Notice that the indices \( s(k) \) and \( t(i) \) of a Leray system are defined up to an additive constant.
A Leray–Ohya system is a Leray system where the matrix \( A \) is Leray–Ohya hyperbolic. In the quasi-linear case, we need to specify the function \( U \) that is plugged into \( A(x, U, \partial) \) and then talk about a Leray–Ohya system for a function \( U \). Naturally, the case of interest is when \( U \) assumes the values of the given Cauchy data.

In order to simplify the statements of the Cauchy problem and of the Leray–Ohya theorem, we shall assume from now on that \( M \) is diffeomorphic to \( \mathbb{R}^{n+1} \). This will not affect the applications of these ideas to the Einstein–Navier–Stokes system because, in light of the domain of dependence or finite propagation speed property stated in the theorem below, it suffices to address the well-posedness of the system from a local point of view.

Let \( \Sigma \) be a regular hypersurface in \( M \), which we assume for simplicity to be given by \( x^0 = 0 \). The Cauchy data on \( \Sigma \) for a Leray system in \( M \) consists of the values of \( U = (u_k) \) and their derivatives of order less than or equal to \( s(k) - 1 \) on \( \Sigma \), i.e., \( \partial^a u_k \big|_{\Sigma} \mid \alpha \mid \leq s(k) - 1 \), \( k = 1, \ldots, \ell \). The Cauchy data is required to satisfy the following compatibility conditions. If \( V = (v_k) \) is an extension of the Cauchy data defined in a neighbourhood of \( \Sigma \), i.e., \( \partial^a v_k \big|_{\Sigma} = \partial^a u_k \big|_{\Sigma} \mid \alpha \mid \leq s(k) - 1 \), \( k = 1, \ldots, \ell \), then the difference \( a_i(x, V, \partial)V - b_i(x, V) \) and its derivatives of order less than \( t(t) \) vanish on \( \Sigma \), for \( t = 1, \ldots, \ell \). When to a Leray system \( A(x, U, \partial)U = B(x, U) \) we prescribe initial data satisfying these conditions, we say that we have a Cauchy problem for \( A(x, U, \partial)U = B(x, U) \). Notice that by definition, the Cauchy data for a Leray system satisfies the aforementioned compatibility conditions.

For a number \( |X| > 0 \), let \( X \) be the strip \( 0 \leq x^0 \leq |X| \). We denote by \( \gamma^{m, (\sigma)}_2 (X) \) and \( \gamma^{m, (\sigma)}_{2, u.l.} (X) \) the Gevrey and uniformly local\(^{16} \) Gevrey spaces of functions defined on \( X \), respectively. More precisely, let \( S_i = \{x^0 = i\} \),

\[
|D^k u|_i = c(n, k) \sup_{|\alpha| \leq k} \| D^\alpha u \|_{L^2(S_i)},
\]

\[
|D^k u|_{u.l.}^{m, i} = c(n, k) \sup_{|\alpha| \leq k, x \in S_i} \| D^\alpha u \|_{L^2(B^i_x)}),
\]

where \( B^i_x \) is the ball of radius one in \( S_i \) centred at \( x \) and \( c(n, k) \) is a normalization constant. Then, for \( \sigma \geq 1 \), and \( m \) a non-negative integer, \( u \in \gamma^{m, (\sigma)}_2 (X) \) means that \( u \in C^\infty (X) \), and

\[
\sup_{|\alpha| \leq m, u \in X} \frac{1}{(1 + |\alpha|)\sigma} (|D^\alpha u|_1)^{\frac{1}{m}} < \infty,
\]

where the sup over \( \alpha \) is taken over multi-indices such that \( \alpha_0 = 0 \). Replacing \( |\cdot| \) by \( |\cdot|_1 \) gives \( \gamma^{m, (\sigma)}_{2, u.l.} (X) \). Analogously one defines such spaces when \( X \) is an open set of some \( \mathbb{R}^N \), and also \( \gamma^{m, (\sigma)}_2 (X \times Y) \) and \( \gamma^{m, (\sigma)}_{2, u.l.} (X \times Y) \). See [14] for details.

We can now state Leray–Ohya’s result.

**Theorem A.1.** (Leray–Ohya) On the strip \( X \subset \mathbb{R}^{n+1} \), consider a Leray system \( A(x, U, \partial)U = B(x, U) \), for \( U = (u_k) \), \( k = 1, \ldots, \ell \), and its Cauchy problem

\[
\begin{align*}
A(x, U, \partial)U &= B(x, U), & \text{in } X, \\
D^\alpha u_k &= \varphi^\alpha, & |\alpha| \leq s(k) - 1, & \text{on } \Sigma = \{x^0 = 0\}.
\end{align*}
\]

Suppose that \( \varphi^\alpha \in \gamma^{m, (\sigma)}(\Sigma) \) for some \( \sigma \). Assume that

\[
\begin{align*}
a_i(x, y, D) &\in \gamma^{t(i) + m, (\sigma)}_{2, u.l.} (X \times Y), \\
b_i(x, y) &\in \gamma^{t(i) + m, (\sigma)}_{2, u.l.} (X \times Y),
\end{align*}
\]

where \( Y \) is an open set of \( \mathbb{R}^N \), containing the closure of the values of the Cauchy data \( \{\varphi^\alpha\} \), with \( N \) being the number of derivatives of \( u_k \) of orders less than or equal to max\(_i\)(s(k) - t(t)),

\(^{16}\) The terminology uniformly local for Gevrey spaces is not standard and has not been employed by Leray and Ohya.
When $p_t = 1$ for all $i$, then the operator $A(x, \partial)$ is called strictly hyperbolic. In this case, the system is well-posed in Sobolev spaces.

We point out that we have not stated the theorem in its most general fashion—see [14]. When $p_t = 1$ for all $i$, then the operator $A(x, \partial)$ is called strictly hyperbolic. In this case, the system is well-posed in Sobolev spaces.

References

[1] Anile A M and Pennisi S 1987 On the mathematical structure of test relativistic magnetofluid dynamics Ann. Inst. H. Poincaré Phys. Théor. 46 27–44
[2] Banerjee N and Beesham A 1996 Causal dissipative cosmology Pramana J. Phys. 46 213–7
[3] Bae H, Biswas A and Tadmor E 2012 Analyticity and decay estimates of the Navier–Stokes equations in critical Besov spaces Arch. Ration. Mech. Anal. 205 963–91
[4] Cao C, Rammaha M A and Titi E S 1999 The Navier–Stokes equations on the rotating 2D sphere: Gevrey regularity and asymptotic degrees of freedom Z. Angew. Math. Phys. 50 341–60
[5] Choquet-Bruhat Y 2009 General Relativity, the Einstein Equations. (New York: Oxford University Press)
[6] Cook J N, Shapiro S L and Stephens B C 2003 Magnetic braking and viscous damping of differential rotation in cylindrical Stars Astrophys. J. 599 1272–92
[7] Czubak M and Disconzi M M 2014 On the well-posedness of relativistic viscous fluids with non-zero vorticity in preparation
[8] Duez M D, Liu Y T, Shapiro S L and Stephens B C 2004 General relativistic hydrodynamics with viscosity: contraction, catastrophic collapse and disk formation in hypermassive neutron stars Phys. Res. D 69 104030
[9] Ferrari A B and Titi E S 1998 Gevrey regularity for nonlinear analytic parabolic equations Commun. Partial Diff. Eqs 23 1–16
[10] Friedrich H and Rendall A D 2000 The Cauchy problem for the Einstein equations. Einstein’s field equations and their physical implications Lect. Notes Phys. 540 127–223
[11] Geroch R 1991 Causal theories of dissipative relativistic fluids Ann. Phys. 207 394–416
[12] Jou D, Casas-Vazquez J and Lebon G 2009 Extended Irreversible Thermodynamics (Berlin: Springer)
[13] Leray J 1953 Hyperbolic differential equations (Princeton, NJ: The Institute for Advanced Study) 240pp
[14] Leray J and Ohya Y 1970 Équations et systèmes non linéaires, hyperboliques non-stricts. (Rencontres, Battelle Res. Inst., Seattle, Wash., 1968) (Berlin: Springer) pp 331–69
[15] Lichnerowicz A 1955 Théories Relativistes de la Gravitation et de l’Électromagnétisme. (Paris: Masson et Cie)
[16] Lichnerowicz A 1965 Théorèmes d’existence et d’unicité pour un fluide thermodynamique relativiste C. R. Acad. Sci. Paris 260 3291–9
[17] Lichnerowicz A 1966 Étude mathématique des fluides thermodynamiques relativistes Commun. Math. Phys. 1 328–73
[18] Lichnerowicz A 1965 Étude mathématique des équations de la magnétohydrodynamique relativistes C. R. Acad. Sci. Paris 260 4449–53
[19] Mavriplis R 1995 Dissipative cosmology Class. Quantum Grav. 12 1455
[20] Majda A J and Bertozzi A L 2001 Vorticity, Incompressible Flow. (Cambridge Texts in Applied Mathematics) (Cambridge: Cambridge University Press)
[21] Misner C W, Thorne K S and Wheeler J A 1973 Gravitation 1st edn (San Francisco, CA: Freeman)
[22] Mühlner I and Ruggeri T 1998 Rational Extended Thermodynamics (Berlin: Springer)
[23] Pichon G 1965 Étude relativiste de fluides visqueux et chargés Ann. Inst. Henri Poincaré A 2 21–85
[24] Rezzolla L and Zanotti O 2013 Relativistic Hydrodynamics (New York: Oxford University Press)

17 Terms like hyperbolic and strictly hyperbolic have been used in the literature to denote different concepts. Here we adopt the conventions of Leray and Ohya.
[25] Ringstrom H 2009 *The Cauchy Problem in General Relativity*. (ESI Lectures in Mathematics and Physics) (Zurich: European Mathematical Society)

[26] Rodino L 1993 *Linear Partial Differential Operators in Gevrey Spaces* (Singapore: World Scientific)

[27] Saijo M and Gourgoulhon E 2006 Viscosity driven instability in rotating relativistic stars *Phys. Rev.* D **74** 084006

[28] Weinberg S 1972 *Gravitation, Cosmology: principles, applications of the General Theory of Relativity* 1st edn (New York: Wiley)