ON THE PONTRYAGIN MAXIMUM PRINCIPLE
UNDER DIFFERENTIAL CONSTRAINTS OF HIGHER ORDER

FRANCO CARDIN     CRISTINA GIANNOTTI     ANDREA SPIRO

Abstract. Exploiting our previous results on higher order controlled Lagrangians in [Nonlinear Anal. 207 (2021), 112263], we derive here an analogue of the classical first order Pontryagin Maximum Principle (PMP) for cost minimising problems subjected to higher order differential constraints $\frac{d^k}{dt^k}x_j = f^j(t, x(t), \frac{d}{dt}x(t), \ldots, \frac{d^{k-1}}{dt^{k-1}}x(t), u(t)), t \in [0,T]$, where $u(t)$ is a control curve in a compact set $K \subset \mathbb{R}^m$. This result and its proof can be considered as a detailed illustration of one of the claims of that previous paper, namely that the results of that paper, originally established in a smooth differential geometric framework, yield directly properties holding under much weaker and more common assumptions. In addition, for further clarifying our motivations, in the last section we display a couple of quick indications on how the two-step approach of this paper (i.e., a preliminary easy-to-get differential geometric discussion followed by a refining analysis to weaken the regularity assumptions) might be fruitfully exploited also in the context of control problems governed by partial differential equations or in studies on the dynamics of controlled mechanical systems.

1. Introduction

In our previous paper [6], we considered the notion of controlled Lagrangians of higher order and, developing the differential geometric approach proposed in [5], we proved that – under certain strong regularity assumptions – a generalised version of the classical Pontryagin Maximum Principle (PMP) holds for control problems with higher order constraints of Euler-Lagrange type. Roughly speaking, on the one hand the results of [6] can be considered as generalisations to controlled Lagrangians of arbitrary order of certain facts on the first order Lagrangians, which were established by Ioffe and Tihomirov in their elementary proof of the PMP [9]. On the other hand, the results in [6] stem from a fresh differential geometric approach to variational problems – rooted in Stokes’ Theorem – which we think will reveal to be a fruitful addition to the traditional Hamiltonian tool box of control theory. In fact, our differential geometric approach, involving controlled Lagrangians rather than controlled Hamiltonians, admits straightforward generalisations in settings where several independent variables are involved [11] and/or lead to applications of the Noether Theorem on differential constraints with symmetries [12]. We therefore consider it much better suited than the traditional Hamiltonian approach for dealing with control problems in Continuum Mechanics (where systems are governed by

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partial differential equations) or for discussions on the dynamics of controlled systems, where information on symmetries and conservation laws can be exploited.

In this paper we use the above mentioned results of [6] to derive in a direct way an analogue of the classical PMP for cost minimising problems subjected to higher order differential constraints of the form \( \frac{d^k x^j}{dt^k} = f^j(t, x(t), \frac{dx}{dt}(t), \ldots, \frac{d^{k-1}x}{dt^{k-1}}(t), u(t)) \), \( t \in [0, T] \), with \( u(t) \) control curve in a compact set \( K \subset \mathbb{R}^m \). The contents of this paper can be taken as a detailed illustration of a claim we made in [6] (see also [5]), namely that, despite the fact that those results were established under strong regularity assumptions on solutions and control curves, they can be nonetheless used to directly derive results that hold under much weaker regularity conditions.

The class of controlled problems to which this paper is devoted is described in detail as follows. Consider the evolutions \( x = (x^i) : [0, T] \rightarrow Q = \mathbb{R}^n \) of a controlled dynamical system on a fixed time interval \([0, T]\) and the cost minimising problem, which is determined by the following three ingredients:

- a family of (possibly measurable) curves \( u : [0, T] \rightarrow K \subset \mathbb{R}^m \), \( u(t) = (u^a(t)) \) with values in a compact set \( K \subset \mathbb{R}^m \), playing the role of the control curves for the system;
- a system of \( k \)-th order differential constraints of the form

\[
\frac{d^k x^j}{dt^k} = f^j(t, x(t), \frac{dx}{dt}(t), \ldots, \frac{d^{k-1}x}{dt^{k-1}}(t), u(t)) \quad (1.1)
\]

and a set \( \mathcal{A}_{init} \) of standard \((k-1)\)-th order initial conditions \( s \) such that the following holds: for each pair \( U = (u(t), s) \) there exists a unique associated solution \( x^{(U)}(t) \) on \([0, T]\) satisfying (1.1) and the initial condition \( s \);
- a \( C^1 \) terminal cost function \( C = C\left(x(T), \frac{dx}{dt}(T), \ldots, \frac{d^{k-1}x}{dt^{k-1}}(T)\right) \), which depends on the \((k-1)\)-th order jets at the final time \( t = T \) of the solutions \( x^{(U)}(t) \).

For a fixed \( s_o \in \mathcal{A}_{init} \), a curve \( u_o(t) \) in \( K \), for which \( x_o(t) = x^{(u_o(t), s_o)}(t) \) has minimal cost among the controlled curves with identical initial condition \( s_o \), is called optimal control. If \( k = 1 \) and \( f = (f^i(t, x, u)) \) is a function on a set of the form \([0, T] \times \Omega \times K\), with \( \Omega \subset \mathbb{R}^n \) open, which is continuous and continuously differentiable with respect to \( x \), the described problem of determining optimal controls is one of the Mayer problems, to which the classical Pontryagin Maximum Principle (PMP) applies (see e.g. [13] [2] [4] [7] [8] [9] [17] and references therein). Let us briefly recall it. If \( k = 1 \), given a pair \( U_o = (u_o(t), s_o) \), let us denote by \( x_o(t) = x^{(U_o)}(t) \in Q \) the corresponding controlled curve and by \( p_o(t) = (p_{oi}(t)) \), \( t \in [0, T] \), the unique curve in \( Q^* \simeq \mathbb{R}^n \) satisfying the linear differential equations

\[
\dot{p}_j + \frac{\partial f^i}{\partial x^j}(t, x_o(t), u_o(t)) = 0 \quad \text{with the terminal condition} \quad p_j(T) = -\frac{\partial C}{\partial x^j}|_{x_o(T)} \quad (1.2)
\]

(here and throughout the paper we follow the Einstein convention on summations). Further, for any fixed \( \tau_o \in [0, T] \), let \( H = H^{(s_o, u_o, \tau_o)} : K \rightarrow \mathbb{R} \) be the Pontryagin function defined by

\[
H^{(s_o, u_o, \tau_o)}(\omega) := p_o(\tau_o)f^i(\tau_o, x_o(\tau_o), \omega). \quad (1.3)
\]

The classical PMP states that if \( u_o(t) \) is an optimal control for the considered cost problem, then \( H^{(s_o, u_o, \tau_o)}(u_o(\tau_o)) = \max_{\omega \in K} H^{(s_o, u_o, \tau_o)}(\omega) \) for almost all choices of \( \tau_o \in [0, T] \).
In many interesting cases this famous necessary condition is so restrictive that can be used to completely determine the optimal controls.

It is quite simple to check that, given a pair \( U_o = (u_o(t), s_o) \), at a fixed choice of time \( t_o \in [0, T] \), the equations which give the differential constraints on the \( x \) and the associated system on the \( p \) coincide with the Euler-Lagrange equations at \( t_o \) of the (degenerate) first order Lagrangian

\[
\mathcal{L}^{(u_o, t_o)}(t, x, \dot{x}, p) := p_j (\dot{x}^j - f^j(t, x, u_o(t_o))) .
\]

The function \( L(t, x, \dot{x}, p, u) = p_j (\dot{x}^j - f^j(t, x, u)) \), which gives the Lagrangians \( \mathcal{L}^{(u_o, t_o)} \), is called the controlled Lagrangian of order \( k \) of the considered control problem (\([5, 6]\)). We also recall that for any fixed \( \tau_o \in [0, T] \), the Pontryagin function \( H = \mathcal{H}^{(s_o, u_o, \tau_o)} \) is maximised exactly where the function

\[
\mathcal{P}^{(s_o, u_o, \tau_o)}(\omega) := -L(\tau_o, x_o(\tau_o), \dot{x}_o(\tau_o), p_o(\tau_o), \omega)
\]

is maximised. Indeed, the difference \( \mathcal{P}^{(s_o, u_o, \tau_o)}(\omega) - \mathcal{H}^{(s_o, u_o, \tau_o)}(\omega) \) is equal to

\[
-p_{o, j}(\tau_o) \dot{x}_o^j(\tau_o),
\]

a real number which is independent of \( \omega \). This remark is at the basis of the so-called Lagrangian version of the PMP (see e.g. \([9]\)).

We may now present the higher order analog of all this. Consider a control problem with differential constraints of the form (1.1) and a cost function \( C = C \left( x(T), \frac{dx}{dt}(T), \ldots, \frac{d^{k-1}x}{dt^{k-1}}(T) \right) \). Assume that \( f = (f^i) \) satisfies the following conditions:

(\( \alpha \)) it is \( C^{k-1} \); 

(\( \beta \)) its partial derivatives of order \( k - 1 \) are continuously differentiable with respect to each argument different from \( t \) and \( u = (u^a) \)

(note that (\( \alpha \)) and (\( \beta \)) reduce to the assumptions of the classical PMP when \( k = 1 \)).

Then, consider \( n \) auxiliary dual variables \( p = (p_i) \in Q^* \simeq \mathbb{R}^n \) and the controlled Lagrangian of order \( k \) on the jets of curves in \( Q \times Q^* \simeq \mathbb{R}^{2n} \) defined by

\[
L \left( t, x, \frac{dx}{dt}, \ldots, \frac{d^kx}{dt^k}, p, u \right) := p_j \left( \frac{d^kx^j}{dt^k} - f^j(t, x, \frac{dx}{dt}, \ldots, \frac{d^{k-1}x}{dt^{k-1}}, u) \right) .
\]

As before, for each fixed pair \( U_o = (u_o(t), s_o) \), given by a control curve \( u_o(t) \) and an initial condition \( s_o \), we denote by \( (x_o(t), p_o(t)) \in Q \times Q^* \) the unique curve which solves the Euler-Lagrange equations of the Lagrangian

\[
\mathcal{L}^{(u_o, t_o)} \left( t, x, \frac{dx}{dt}, \ldots, \frac{d^kx}{dt^k}, p \right) := L \left( t, x, \frac{dx}{dt}, \ldots, \frac{d^kx}{dt^k}, p, u_o(t_o) \right)
\]

at each fixed \( t_o \in [0, T] \) (they are explicitly given in (1.1) and (2.6)) and satisfying the end-point conditions defined as follows. The curve \( x_o(t) \) satisfies the initial condition at \( t = 0 \) given by \( s_o \), while \( p_o(t) \) satisfies the terminal conditions at \( t = T \) listed below (here, \( x^{(s)} \) stands for the \( s \)-th order derivative \( x^{(s)} := \frac{d^s x}{dt^s} \) and \( \frac{d}{dt} \) denotes the total differential
We may now state the result we are interested in:

\[
\begin{align*}
\frac{\partial C}{\partial x^{(k-1)}_{j^k-1}(x_o)} |_{p_i|t=T} & = (\partial C_{z(t)})_{j^k-1}(x_o), \\
\frac{d^np_i}{dt^n} |_{t=T} & = \left( (-1)^{\ell} \frac{\partial C}{\partial x^{(k-1-\ell)}_{j^k-1}} + \sum_{h=0}^{\ell-1} (-1)^{\ell+h} \frac{D^h}{Dt^h} \left( p_m \frac{\partial f^m}{\partial x^{(k-\ell-h)}_{j^k-1}} \right) \right)_{j^k-1}(x_o), \\
\frac{d^{k-1}p_i}{dt^{k-1}} |_{t=T} & = \left( (-1)^{k-1} \frac{\partial C}{\partial x^{(k-1)}} + \sum_{h=0}^{k-2} (-1)^{k-1+h} \frac{D^h}{Dt^h} \left( p_m \frac{\partial f^m}{\partial x^{(k-1)}_{j^k-1}} \right) \right)_{j^k-1}(x_o).
\end{align*}
\]

We remark that, for each curve \( x_o(t) \) solving the equations \((1.1)\), the corresponding differential problem on the \( p_j(t) \) is meaningful provided that all derivatives of \( x_o(t) \) up to order \( 2k-2 \) are at least almost everywhere defined. This is trivially true when \( k = 1 \) and \( u_o(t) \) is measurable, but when \( k > 1 \) other assumptions are needed. For this reason, we impose the following condition, which implies the desired property for any \( k \) (see Lemma 5.4):

\((\gamma)\) \( u_o(t) \) is measurable and, if \( k > 1 \), it satisfies the additional requirements:

- it is piecewise \( C^{k-1} \);
- each derivative \( u_{o(\ell)}(t) \), \( 1 \leq \ell \leq k-1 \), takes values in a fixed compact set \( K^{(\ell)} \subset \mathbb{R}^m \).

Finally, for any fixed \( \tau_o \in [0, T] \), we define

\[
H^{(s_0,u_o,\tau_o)} : K \to \mathbb{R}, \quad H^{(s_0,u_o,\tau_o)}(\omega) := p_{oj}(\tau_o)f^j(\tau,x_o(\tau_o),\ldots,x_{o(k-1)}(\tau_o),\omega).
\]

We may now state the result we are interested in:

**Theorem 1.1.** Assume that \( f \) satisfies \((\alpha)\) and \((\beta)\) and that \( u_o(t) \) satisfies \((\gamma)\). Then \( u_o(t) \) is an optimal control only if for almost all \( \tau_o \in [0, T] \)

\[
H^{(s_0,u_o,\tau_o)}(u_o(\tau_o)) = \max \{ H^{(s_0,u_o,\tau_o)}(\omega), \ 0 < \omega \in K \}.
\]

If \( f \) is \( C^\infty \) and the family of the control curves in which \( u(t) \) is allowed to vary is assumed to be the class of the \( C^\infty \) curves, the proof of \((1.10)\) is very short and elementary – it is essentially a direct consequence of [B] Cor.7.7.

An alternative way (but undoubtedly not direct) to prove Theorem 1.1 demands a translation of the whole setting into a corresponding problem with first order constraints, to which the classical PMP can be applied. More precisely, one first needs to introduce \( n \times (k-1) \) auxiliary variables, say \( y_1 = (y_1^i), \ldots, y_{k-1} = (y_{k-1}^i) \), and translate the constraints \((1.1)\) into the system of first order constraints

\[
\begin{align*}
\frac{dx^i}{dt} & = y_1^i, \quad \frac{dy_1^i}{dt} = y_2^i, \quad \frac{dy_2^i}{dt} = y_3^i, \quad \ldots \quad \frac{dy_{k-1}^i}{dt} = y_{k-1}^i, \\
\quad \frac{dy_{k-1}^i}{dt} & = f^i(t,x(t),y_1,\ldots,y_{k-1},u(t)).
\end{align*}
\]
Then, by introducing another set of additional variables

\[ \tilde{p}_0 = (\tilde{p}_0[i]) , \quad \tilde{p}_1 = (\tilde{p}_1[i]) , \ldots , \tilde{p}_{k-1} = (\tilde{p}_{k-1}[i]) , \]

(dual to the variables \( y_0 = (y_0^i = x^i) , y_1 = (y_1^i) , \ldots , y_{k-1} = (y_{k-1}^i) \)), one may considers an appropriate Pontryagin function \( H^{(\tilde{y}_0, \tilde{u}_0, \tau_0)} \) and a set of necessary conditions on optimal controls, which are directly implied by the classical PMP with first order constraints. Such conditions turn out to be different from those of Theorem 1.1. In fact, they involve a larger number auxiliary variables in the definition of the testing function (1.9). Nonetheless, with little additional effort, one can come back to the differential constraints on the auxiliary functions and replace each of them in an appropriate way. At the end one reaches a testing function involving a minimal set of auxiliary variables, namely the above defined function (1.10).

The main purpose of this paper is to show that, instead of adopting the above mentioned back and forth type argument (namely, introducing new auxiliary variables with the purpose of completely removing them in a second time) Theorem 1.1 can be proved in a direct way, by just extending to low regularity settings the simple differential geometric arguments – based on Stokes’ Theorem and of the Principle of Minimal Labour [5] – that hold when there are no restrictions on the level of regularity. In this paper the extensions to low regularity settings are reached by standard approximation techniques (similar to those considered by Gamkrelidze for first order control problems, see e.g. [8]) together with a couple of ad hoc lemmas for estimating variations of costs. All arguments are in principle quite simple but demand a number of tedious checks. Aiming to make the proof self-contained and ready to be used in future works, all proofs are presented in great detail. This is done also with the hope the proof can be taken as a convincing illustration that an approach to control problems, structured into

- a preliminary easy-to-get differential geometric proof, made under strong regularity assumptions, followed by
- a second step in which the previous claims are improved to new statements that are true under a minimal set of regularity assumptions,

can be fruitfully exploited in much more involved contexts of control theory. A couple of examples of settings where such two-step approach might provide interesting results are given in the last section.

**Remark 1.2.** The approach we followed for our proof of Theorem 1.1 made us aware that the minimal (or, more precisely, very close to the minimal) set of regularity conditions to be imposed is that the function \( f = (f^i) \) satisfies the conditions \((\alpha)\) and \((\beta)\) and the control curve \( u_0(t) \) satisfies \((\gamma)\). Such conditions become progressively weaker if the order of the system is reduced by introducing auxiliary variables. This phenomenon can be synthesised by saying that the stronger are the regularity properties of the differential constraints, the fewer auxiliary variables need to be considered. This property was unexpected to us and can be taken as an interesting by-product of our approach.

The paper is structured as follows. After a preliminary section, in §3 we review certain basic notions and results given in [6]. In §4 we present the two main lemmas that allow using approximations to extend the results of [6] to problems with weaker regularity.
assumptions. The proofs of Theorem 1.1 and of parts of the two main lemmas are given in §6 and §7, respectively. Suggestions for further investigations are given in §7.

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2. Preliminaries

2.1. Notational issues. The standard coordinates of the ambient space $\mathcal{Q} = \mathbb{R}^N$ are usually denoted by $q = (q^i)$. When $\mathcal{Q}$ is even dimensional and of the form $\mathcal{Q} = Q \times \mathbb{Q}^*$ for some $Q = \mathbb{R}^n$, the coordinates are indicated as pairs $q = (x, p)$, with $x = (x^i) \in Q$ and $p = (p_{ij}) \in \mathbb{Q}^*$. The elements of the controls set $K \subset \mathbb{R}^m$ are $m$-tuples $u = (u^a)$. Given a $k$-times differentiable curve $q(t) \in \mathcal{Q}$, $t \in I \subset \mathbb{R}$, its $r$-th order derivative is often indicated with the short-hand notation

$$q_{(r)}(t) := \left( q^i_{(r)}(t) \right) = \left( \frac{d^r q^i}{dt^r} \right)_{(r)} |_{t}. \quad \text{We also set} \quad q_{(0)}(t) := q(t) .$$

Accordingly, the $r$-th order jet of a curve $\gamma(t) := (t, q(t)) \in \mathbb{R} \times \mathcal{Q}$ is denoted by

$$j^r_t(\gamma) = (t, q(t), q(1)(t), \ldots, q(r)(t)) = (t, q(s)(t)) . \quad (2.1)$$

The manifold of all $r$-th order jets of curves, i.e. the jet bundle of order $r$ of the (trivial) bundle $\mathbb{R} \times \mathcal{Q}$ over $\mathbb{R}$, is denoted by $J^r(\mathcal{Q}|\mathbb{R})$.

If $g : J^r(\mathcal{Q}|\mathbb{R}) \times K \to \mathbb{R}$ is a smooth function of pairs $(j^r_t(\gamma), u)$, formed by a $r$-th order jet of a curve $\gamma(t) := (t, q(t))$ and a control $u \in K$, we denote by $\frac{Dg}{Dt} \big|_{(j^r_{t}, u)}$ the smooth real function on $J^{r+1}(\mathcal{Q}|\mathbb{R}) \times K$ defined by

$$\frac{Dg}{Dt} \big|_{(j^r_t(\gamma), u)} := \left. \frac{\partial g}{\partial t} \right|_{j^r_t(\gamma), u} + \sum_{\ell=0}^{r} \sum_{i=1}^{N} q_{(\ell+1)}^i \left. \frac{\partial g}{\partial q_{(\ell)}^i} \right|_{j^r_t(\gamma), u} . \quad (2.2)$$

Note that $\frac{Dg}{Dt}$ differs from the total derivative for the sections of $\pi : J^r(\mathcal{Q}|\mathbb{R}) \times K \to \mathbb{R}$ only because the term $u_0^a \frac{\partial g}{\partial u^a}$ is missing. We can also say that the operator $\frac{Dg}{Dt}$ is the pull-back on $J^r(\mathcal{Q}|\mathbb{R}) \times K$ of the classical total derivative operator $\frac{d}{dt}$ of $J^r(\mathcal{Q}|\mathbb{R})$.

2.2. Generalised Mayer problems with constraints of variational type. As we mentioned in the Introduction, any $k$-th order control system (1.1) on curves $x(t)$ can be considered as a sub-system of the Euler-Lagrange equations of an appropriate controlled $k$-th order Lagrangian for curves of the form $t \mapsto (x(t), p(t)) = (x^i(t), p_{ij}(t))$ with the curve $t \mapsto p(t)$ in an appropriate auxiliary space. Despite of the fact that (1.1) has order $k$, the full Euler-Lagrange system contains equations of order $2k$ (see §2.3). Moreover, a solution $t \mapsto (x(t), p(t))$, $t \in [0, T]$, of the controlled Euler-Lagrange equations is determined not only by the $(k-1)$-th order jet $\sigma_x(t)$ of $x(t)$ at $t = 0$ and by the control curve $u(t)$, but also by the $(2k-1)$-th order jet $\sigma_p(t)$ at $t = 0$ of the curve $p(t)$. Since in the curve $(x(t), p(t))$ only the part $x(t)$ is relevant for the cost problem, the part of the initial datum $\sigma$ for the curve $(x(t), p(t))$, which is not determined by $x(t)$, is freely specifiable and can be considered as an additional “controlling datum” for the problem on the curves $(x(t), p(t))$. In §3 we developed a theory of control problems with (smooth) differential constraints of variational type, which not only works for the classical first order Mayer
problems and the cost problems of this paper, but it is designed to be applicable to other contexts. With such second aim in mind and on the basis of the previous observation, in \cite{4} we were naturally led to consider the following definition (see also \cite{5}).

A generalised Mayer problem with smooth constraints of variational type of order \(k\) (for short, generalised Mayer problem) is a cost minimising problem determined by a triple \((\mathcal{X}, L, C)\) of the following kind.

- \(\mathcal{X}\) is a set of control pairs \(U = (u(t), \sigma)\), given by:
  
  a) a smooth curve \(u : [0, T] \to K \subset \mathbb{R}^m\) with values in a compact subset \(K \subset \mathbb{R}^m\);
  
  b) a \((2k - 1)\)-jet \(\sigma = j_{t=0}^{2k-1}(\gamma)\) at \(t = 0\) of a smooth curve \(t \mapsto \gamma(t) = (t, q_i(t))\). The jets \(\sigma\) of these pairs are constrained to be elements of a fixed set \(A_{\text{init}} \neq \emptyset\).

- A (smooth) controlled Lagrangian \(L = L(t, q(u), u)\) of order \(k\), i.e. a \(C^\infty\) function

\[
L : J^k(Q|\mathbb{R}) \times K \to \mathbb{R}
\]

for which the following property holds: for each control pair \(U = (u(t), \sigma) \in \mathcal{X}\), there exists exactly one curve \(\gamma(t) = (t, q_i(t))\), \(t \in [0, T]\), with \(j_{t=0}^{2k-1}(\gamma) = \sigma\) and satisfying the Euler-Lagrange equations of the higher-order Lagrangian \(L^{(u, t_0)}(t, q(\beta)) := L(t, q(\beta), u(t_o))\) at each time \(t_o \in T\), i.e. the equations

\[
E_i(L)|_{(j^{2k}(\gamma(t)), u(t))} := 
\frac{\partial L}{\partial q^i}(j^{2k}(\gamma(t)), u(t)) + \sum_{\beta=1}^{k} (-1)^\beta \left( \frac{D}{Dt} \right)^{\beta} \left( \frac{\partial L}{\partial q_j(\beta)} \right)_{(j^{2k}(\gamma(t)), u(t))} = 0, \quad i = 1, \ldots, N, \quad (2.3)
\]

where \(\frac{D}{Dt}\) is the operator defined in (2.2). This curve is denoted \(\gamma^{(U)}(t)\).

- A (smooth) terminal cost function \(C : J_{t=T}^{2k-1}(Q|\mathbb{R}) \to \mathbb{R}\)

The equations (2.3) are called differential constraints of the triple \((\mathcal{X}, L, C)\) and the curves \(\gamma^{(U)}\), with \(U = (u(t), \sigma) \in \mathcal{X}\), are called \(\mathcal{X}\)-controlled curves.

A triple \((\mathcal{X}, L, C)\) as above is called a defining triple. The problem of determining the \(\mathcal{X}\)-controlled curves \(\gamma^{(U_o)}\), \(U_o \in \mathcal{X}\), for which the terminal cost \(C(j_{t=T}^{2k-1}(\gamma^{(U_o)}))\) is minimal, is called the generalised Mayer problem corresponding to the triple. The pairs \(U_o = (u_o(t), \sigma_o)\) giving the cost minimisation curves are called optimal controls.

In the next section (2.3) we explain how these notions are well fitted with the cost problems described in the Introduction.

Throughout the paper we restrict our discussion to the defining triples \((\mathcal{X}, L, C)\) satisfying the following additional technical hypothesis. We assume that there exists an open convex superset \(\hat{K} \supseteq K\) such that, denoting by \(\hat{\mathcal{X}} \supseteq \mathcal{X}\) the family of pairs \(U = (u(t), \sigma)\) with \(u(t)\) in \(\hat{K}\) but initial condition \(\sigma\) as in \(\mathcal{X}\), there still exists a unique solution to (2.3) for any \(U \in \hat{\mathcal{X}}\). The curves of this larger family are called \(\hat{\mathcal{X}}\)-controlled.

### 2.3. Defining triples for the cost problems of this paper.

Assume that all data of one of the cost minimisation problem of the Introduction are of class \(C^\infty\). We claim that in this case it is equivalent to the generalised Mayer problem given by the following defining triple \((\mathcal{X}, L, C)\). Consider the configuration space \(Q = Q \times Q^*\) with \(Q = \mathbb{R}^n\) and
coordinates \((x, p) = (x^i, p_j)\), as specified in \([2.1]\). The controlled Lagrangian \(L\) and the cost function \(C\) are the maps

\[
L(t, x, x(1), \ldots, x(k), p, u) = p_j \left( x^j_{(k)} - f^j(t, x, x(1), \ldots, x(k-1), u) \right) \quad \text{and} \quad C = C \left( x(T), x(1)(T), \ldots, x(k-1)(T) \right), \quad (2.4)
\]

where \(f = (f^j)\) is the function that gives the constraints \([1.1]\) and \(C\) is the cost of the Introduction, which depends only on the \((k - 1)\)-th order jet at \(t = T\) of the part \(x(t)\) of the \(\gamma(t) = (t, x(t), p(t)) \in [0, T] \times \mathbb{Q} \times \mathbb{Q}^n\). Note that, even if \(C\) depends only on the jets of order \(k - 1\), it can be trivially considered as a function on the space of \((2k - 1)\)-jets and hence is subsumed by the general notion of terminal cost function considered in \([6]\).

Finally, the set \(\mathcal{K}\) consists of the pairs \(U = (u(t), \sigma = (s, s))\) where: (a) \(s\) is an initial condition in a prescribed set \(A_{init} \subset J^{k-1}(\mathbb{Q}|\mathbb{R}|)\) for the curve \(x(t)\); (b) \(s\) is an initial condition (which can be freely chosen) for the curve \(p(t)\).

To show that the problem determined by \((\mathcal{K}, L, C)\) is equivalent to our original cost minimising problem, we first observe that the differential constraints determined by \((\mathcal{K}, L, C)\) are given by the controlled Euler-Lagrange equations (which are of normal type)

\[
E_{p_i}(L)|_{(t, j^2k(\gamma), u(t))} :=
\]

\[
= \left. \frac{\partial L}{\partial p_i} \right|_{(t, j^2k(\gamma), u(t))} + \sum_{\beta = 1}^k (-1)^\beta \left. \frac{\partial L}{\partial p_i} \right|_{(t, j^2k(\gamma), u(t))} = \left. x^i_{(k)} - f^i(t, x^i, x^i_{(1)}, \ldots, x^i_{(k-1)}, u^a) = 0 \right., \quad (2.5)
\]

\[
E_{x^i}(L)|_{(t, j^2k(\gamma), u(t))} :=
\]

\[
= \left. \frac{\partial L}{\partial x^i} \right|_{(t, j^2k(\gamma), u(t))} + \sum_{\beta = 1}^k (-1)^\beta \left. \frac{\partial L}{\partial x^i} \right|_{(t, j^2k(\gamma), u(t))} = (-1)^{-k} \left( p_j \frac{\partial f^j}{\partial x^i} - \frac{D}{Dt} \left( p_j \frac{\partial f^j}{\partial x^i_{(1)}} \right) + \frac{D^2}{Dt^2} \left( p_j \frac{\partial f^j}{\partial x^i_{(2)}} \right) \right)+
\]

\[
+ \ldots + (-1)^{k-1} \left. \frac{D^{k-1}}{Dt^{k-1}} \left( p_j \frac{\partial f^j}{\partial x^i_{(k-1)}} \right) \right|_{(t, j^2k-2(\gamma), u(t))} - p(k)i = 0. \quad (2.6)
\]

We immediately see that \((2.5)\) coincides with the differential constraints \([1.1]\). This fact together with the fact that \(C\) is independent of \(p(t) = (p_j(t))\) implies the claimed equivalence between our cost minimising problem and the problem determined by \((\mathcal{K}, L, C)\) – at least in the case of smooth data.

In the next sections, we establish some preliminary results on generalised Mayer problems. We will come back to this specific one in \([5]\).

2.4. Differential constraints of normal type. The smooth higher order constraints \([2.3]\) of a generalised Mayer problem are called of normal type if, using an appropriate
number of auxiliary variables, they can be reduced to a first order system of the type

\[ \frac{dy^A}{dt} = g^A(t, y^B, u^a(t)) , \quad 1 \leq A \leq \tilde{N} , \quad (2.7) \]

where the \( g^A \) are functions uniquely determined by the \( L \) of the defining triple. For such constraints, we denote by \( \mathcal{K}_{\text{meas}} \) (\( \cong \mathcal{K} \)) the set of the pairs \( U = (u(t), \sigma) \), in which the control curve \( u(t) \in K \) is merely measurable. The corresponding curves \( \gamma(U) \) (they exist by well known facts on first order equations) are called \( \mathcal{K}_{\text{meas}} \)-controlled.

Some well-known properties of controlled first order differential equations (see e.g. [4, Ch. 3]) have useful direct counterparts for the higher order constraints of this kind. We collect them in the next lemma, where we prove a Gronwall-type result for generalised Mayer problems of normal type, namely that two controlled curves are close whenever their initial conditions and controls are close. In the statement the following notation is used. For any pair of measurable control curves \( u, u' : [0, T] \to \tilde{K} \) we denote by \( \text{dist}(u, u') \) the distance

\[ \text{dist}(u, u') := \mu_{\text{Leb}}( \{ t \in [0, T] : u(t) \neq u'(t) \} ) , \quad (2.8) \]

where \( \mu_{\text{Leb}} \) is the Lebesgue measure on the subsets of \([0, T]\). We also use the jets coordinates \( (2.7) \) to identify \( J^{2k-1}(Q|R)|_{t=0} \approx \mathbb{R}^{2kN+1} \) and we use the classical Euclidean norm of \( \mathbb{R}^{2kN+1} \) to define distances \( |\sigma - \sigma'| \) between initial conditions \( \sigma, \sigma' \in A_{\text{init}} \).

**Lemma 2.1.** Let \( (\mathcal{K}, L, C) \) be a defining triple, giving differential constraints of order \( 2k \) of normal type, i.e. equivalent to first order equations of the form \( (2.7) \). Assume also that the initial conditions in \( A_{\text{init}} \) are in bijection with a set \( \tilde{A}_{\text{init}} \) of initial conditions for the problem \( (2.7) \) by means of a Lipschitz continuous map.

Given a \( \mathcal{K}_{\text{meas}} \)-controlled curve \( \gamma_o(t) = \gamma(U_o)(t) = (t, q_o(t)) \), with \( U_o = (u_o(t), \sigma_o) \in \mathcal{K}_{\text{meas}} \), there exist constants \( \rho, \kappa, \mathcal{C}, \mathcal{C}' > 0 \) such that for any \( U = (u(t), \sigma) \), \( U' = (u'(t), \sigma') \in \mathcal{K}_{\text{meas}} \) with

\[ |\sigma - \sigma_o|, |\sigma' - \sigma_o| < \rho , \quad \text{dist}(u, u_o), \text{dist}(u', u_o) < \rho , \]

the corresponding curves \( \gamma(U), \gamma(U') : [0, T] \to [0, T] \times Q \) are such that

\[ \| \gamma(U) - \gamma(U') \|_{C^{2k-1}} \leq c \text{ dist}(u, u') + \kappa |\sigma - \sigma'| \quad \text{with} \quad c := 4\mathcal{C}e^{2\mathcal{C}'T} . \quad (2.9) \]

The constants \( \rho, \kappa, \mathcal{C}, \mathcal{C}' \) depend only on

(a) the Lipschitz constant of the bijection between \( A_{\text{init}} \) and \( \tilde{A}_{\text{init}} \);
(b) the function \( g = (g^A) \) and thus the controlled Lagrangian \( L \);
(c) the choice of a cut-off function \( \varphi : \mathbb{R} \times \mathbb{R}^{\tilde{N}} \to \mathbb{R} \), which is identically equal to 1 on a relatively compact neighbourhood \( N \subset \mathbb{R}^{\tilde{N}+1} \) of the trace of the curve \( \tilde{\gamma}_o : [0, T] \to [0, T] \times \mathbb{R}^{\tilde{N}} \), which solves \( (2.7) \) and corresponds to \( \gamma_o(t) = (t, q_o(t)) \) in \([0, 1] \times Q \).

**Proof.** Let \( \tilde{\gamma}_o = (t, y^A_o(t)) \), \( N \) and \( \varphi \) as in (c) and denote by \( \tilde{\sigma}_o := (y^A_o(0)) \) the initial condition of \( \tilde{\gamma}_o \). The curve \( \tilde{\gamma}_o \) is a solution not only to \( (2.7) \), but also to the system

\[ \frac{dy^A}{dt} = h^A(t, y^B, u^a(t)) \quad \text{where} \quad h^A(t, y^B, u^a(t)) := \varphi(t, y^B, u^a(t)) \quad (2.10) \]
By construction, \( h|_N = g|_N \) and there are constants \( C, C' > 0 \) (depending on \( g \) and \( \varphi \)) such that

\[
\sup_{(t,y,\omega) \in \mathbb{R}^1 \times N \times \hat{K}} |h^A(t,y,\omega)| \leq C , \quad \sup_{(t,y,\omega) \in \mathbb{R}^1 \times N \times \hat{K}} \left| \frac{\partial h^A}{\partial y^B} (t,y,\omega) \right| \leq C'.
\]

By classical arguments based on Gronwall Lemma (see e.g. \cite[Prop. 3.2.2]{4}), for any two solutions \( \tilde{\gamma}^{(u,\tilde{\sigma})}(t) = (t,y^{(u,\tilde{\sigma})}(t)) \) and \( \tilde{\gamma}^{(u',\tilde{\sigma}')} (t) = (t,y^{(u',\tilde{\sigma}')}(t)) \) of \( (2.10) \), which are determined by pairs \( (u,\tilde{\sigma}), (u',\tilde{\sigma}') \), given by measurable curves \( u(t), u'(t) \in \hat{K} \) and initial conditions \( \tilde{\sigma}, \tilde{\sigma}' \in \hat{A}_{\text{init}} \), we have that

\[
\|y^{(u,\tilde{\sigma})} - y^{(u',\tilde{\sigma}')}\|_{C^0} \leq c \text{ dist}(u,u') + |\tilde{\sigma} - \tilde{\sigma}'| \quad \text{with} \quad c := 4Ce^{2C'T} .
\]

Therefore if \( \text{dist}(u,u_o), \text{dist}(u',u_o), |\tilde{\sigma} - \tilde{\sigma}_o| \) and \( |\tilde{\sigma}' - \tilde{\sigma}_o| \) are sufficiently small, then both curves \( \tilde{\gamma}^{(u,\tilde{\sigma})}, \tilde{\gamma}^{(u',\tilde{\sigma}')} \) have trace in \( N \) and are solutions to \( (2.7) \). Since the bijection from \( A_{\text{init}} \) to \( \hat{A}_{\text{init}} \) is Lipschitz, there is a \( \kappa > 0 \) such that \( |\tilde{\sigma} - \tilde{\sigma}'| \leq \kappa|\sigma - \sigma'| \) and the lemma follows.

\[\square\]

**Remark 2.2** (Stability under perturbation). The claim of Lemma \( 2.1 \) has the following extension, which is later used in the proof of our main result. Consider a one-parameter family of defining triples \( (K,L^{(\delta)},C^{(\delta)}) \), each of them with the same set of control pairs \( K \), but with (smooth) Lagrangians and cost functions, depending on a real parameter \( \delta \in (0,\delta_o] \). Assume that all of such triples give generalised Mayer problems of normal type and that the associated equivalent first order constraints

\[
\frac{dy^A}{dt} = g^{(\delta)}A(t,y^B,u^a(t)) , \quad (2.11)
\]

are such that the functions \( g^{(\delta)} \) tend uniformly on compacta, together with their first derivatives in the \( y^B \), to a limit map \( g(t,y^B,u) = \lim_{\delta \to 0} g^{(\delta)}(t,y^B,u) \). This limit map is clearly continuous and continuously differentiable with respect to the \( y^B \).

Consider now a pair \( U_o = (u_o(t),\sigma_o) \in \hat{K}_{\text{meas}} \) and the uniquely associated solution \( \tilde{\gamma}_o(t) = (t,y^A_{o}(t)) \) to

\[
\frac{dy^A}{dt} = g^A(t,y^B,u^a_o(t)) , \quad (2.12)
\]

with initial condition given by the point \( \tilde{\sigma}_o \) corresponding to \( \sigma_o \). Considering a neighbourhood \( N \) and a cut-off function \( \varphi \) as in (c) of Lemma \( 2.1 \) and setting

\[
h^{(\delta)}(t,y^B,u^a) := \varphi(t,y^B)g^{(\delta)}A(t,y^B,u^a(t)) , \quad h(t,y^B) := \varphi(t,y^B,u^a)g^A(t,y^B,u^a(t)) ,
\]

we have that also the \( h^{(\delta)} \) and their first derivatives with respect to the \( y^B \) tend uniformly on compacta to \( h \) and to its corresponding first derivatives. Due to this, for any sufficiently small interval \( (0,\delta_o] \), it is possible to select \( \delta \)-independent constants \( \rho, \kappa, C, C' > 0 \) such that the claim of Lemma \( 2.1 \) holds with such constants for any triple \( (K,L^{(\delta)},C^{(\delta)}) \), \( \delta \in (0,\delta_o] \).
3. The generalised PMP for Mayer problems with smooth differential constraints of variational type

Let $(\mathcal{X}, L, C)$ be a defining triple for a generalised Mayer problem with differential constraints of normal type. Given a control pair $U_o = (u_o(t), \sigma_o) \in \mathcal{X}$, with associated $\mathcal{X}$-controlled curve, and a triple $(\tau_o, \omega_o, \varepsilon) \in (0, T) \times K \times (0, +\infty)$ with $0 < \varepsilon < \min \{1, \frac{T}{2}, T - \tau_o\}$, we define

$$u^{(\tau_o, \omega_o, \varepsilon)} : [0, T] \to K, \quad u^{(\tau_o, \omega_o, \varepsilon)}(t) := \begin{cases} u_o(t) & \text{if } t \in [0, \tau_o - \varepsilon), \\ \omega_o & \text{if } t \in [\tau_o - \varepsilon, \tau_o), \\ u_o(t) & \text{if } t \in [\tau_o, T]. \end{cases} \quad (3.1)$$

We also select a constant $0 < \delta < \frac{1}{4}$ and with the (in general discontinuous) curve $u^{(\tau_o, \omega_o, \varepsilon)}$, we associate a smooth curve $\tilde{u}^{(\tau_o, \omega_o, \varepsilon)} : [0, T] \to \hat{K}$ satisfying the condition

$$\tilde{u}^{(\tau_o, \omega_o, \varepsilon)}(t) = u^{(\tau_o, \omega_o, \varepsilon)}(t) \quad \text{for any } t \notin [\tau_o - \varepsilon - \delta \varepsilon^2, \tau_o - \varepsilon] \cup [\tau_o, \tau_o + \delta \varepsilon^2]. \quad (3.2)$$

We assume that the smoothing algorithm which determines the smooth $\tilde{u}^{(\tau_o, \omega_o, \varepsilon)}(t)$ from the non-smooth $u^{(\tau_o, \omega_o, \varepsilon)}$ is fixed (the choice of the algorithm does not matter).

We call $u^{(\tau_o, \omega_o, \varepsilon)}$ the needle modification of $u_o(t)$ with peak time $\tau_o$, ceiling value $\omega_o$ and width $\varepsilon$. The associated smooth curve $\tilde{u}^{(\tau_o, \omega_o, \varepsilon)}$ is called the smooth needle modification of $u^{(\tau_o, \omega_o, \varepsilon)}$ (see Fig. 1 and Fig. 2).

![Fig. 1 Needle modification](image1)

![Fig. 2 Smoothed needle modification](image2)

The (non-smooth and smoothed) needle modifications are essential ingredients for the following definition, in which we combine the classical notion of needle variation, developed by Boltyanski for the original proof of the classical PMP, and the concept of homotopy variation.

**Definition 3.1.** Given a controlled curve $\gamma^{(U_o)}(t)$ and a triple $(\tau_o, \omega_o, \varepsilon_o)$ as above, consider a continuous map $\Sigma : [0, \varepsilon_o] \times [0, 1] \subset \mathbb{R}^2 \to A_{\text{init}} \subset J^{2k-1}(\mathbb{R})|_{t=0}$ such that $\Sigma(\varepsilon, 0) = \Sigma(0, s) = \sigma_o$ for any $\varepsilon$ and $s$. Moreover, for any $s \in [0, 1]$ and $\varepsilon \in [0, \varepsilon_o]$, let us denote by $u^{(\varepsilon, s)}$ (resp. $\tilde{u}^{(\varepsilon, s)}$) the control curve in the convex set $\hat{K}$ defined by

$$u^{(\varepsilon, s)}(t) = (1 - s)u_o(t) + su^{(\tau_o, \omega_o, \varepsilon)}(t), \quad s \in [0, 1] \quad (3.3)$$

for $\tilde{u}^{(\varepsilon, s)}(t) = (1 - s)u_o(t) + s\tilde{u}^{(\tau_o, \omega_o, \varepsilon)}(t), \quad s \in [0, 1] \quad (3.4)$

1From now till almost to the end, such an $\delta$ is a fixed number, say e.g. $\delta = \frac{1}{4}$. Only at the very end of §5.2 where a $\delta$-parameterised family of Mayer problems is taken into account, this constant $\delta$ will be taking depending on $\delta$ and tending to 0 for $\delta \to 0$. 


The needle variation (resp. smoothed needle variation) of $\gamma^{(U_0)}$ associated with $(\tau_0, \omega_0, \Sigma, \varepsilon_0)$ is the one-parameter family of maps

$$\text{Needle}^{(\tau_0, \omega_0, \Sigma, \varepsilon_0)}(\gamma^{(U_0)}) := \{ F^{(\tau_0, \omega_0, \Sigma)}(\varepsilon) : [0, T] \times [0, 1] \to [0, T] \times \Omega , \varepsilon \in (0, \varepsilon_0) \} \quad (3.5)$$

(resp. $\tilde{\text{Needle}}^{(\tau_0, \omega_0, \Sigma, \varepsilon_0)}(\gamma^{(U_0)}) := \{ \tilde{F}^{(\tau_0, \omega_0, \Sigma)}(\varepsilon) : [0, T] \times [0, 1] \to [0, T] \times \Omega , \varepsilon \in (0, \varepsilon_0) \} \)°

given by the homotopies of controlled curves

$$F^{(\tau_0, \omega_0, \Sigma)}(\varepsilon)(t, s) = \gamma^{(U_0)}(s)(t), \quad U^{\varepsilon}_0(s) := (u^{(\varepsilon, s)}(t), \Sigma(\varepsilon, s))$$

(3.6)

The class of needle variations of a fixed controlled curve contains the following important subclass, which plays a crucial role in the generalised PMP established in [6].

Consider a $K$-controlled curve $\gamma_0 = \gamma^{(U_0)}$ and a smoothed needle variation $\tilde{\text{Needle}}^{(\tau_0, \omega_0, \Sigma, \varepsilon_0)}(\gamma_0)$ as above. We introduce the following notation. For each homotopy $\tilde{F}^{(\tau_0, \omega_0, \Sigma)}(\varepsilon), \varepsilon \in [0, \varepsilon_0]$, we denote

- by $\tilde{F}^{(\tau_0, \omega_0, \Sigma)}(\varepsilon)(2k-1)$ the homotopy of the curves in $J^2 (\Omega ; \mathbb{R}) \times \tilde{K}$, given by the $s$-parameterised family of maps $t \to (\tilde{J}_t^{2k-1}(\gamma(\varepsilon, s)), \tilde{u}(\varepsilon, s)(t))$, made of the $(2k-1)$-jets $\tilde{J}_t^{2k-1}(\gamma(\varepsilon, s))$ of the curves $\gamma(\varepsilon, s)(t) = \tilde{F}^{(\tau_0, \omega_0, \Sigma)}(\varepsilon)(t, s)$ and the control curves $\tilde{u}(\varepsilon, s)(t)$;
- by $S(\varepsilon) = \tilde{F}^{(\tau_0, \omega_0, \Sigma)}(\varepsilon)(2k-1)([0, T] \times [0, 1])$ the 2-dimensional submanifold of $J^2 (\Omega ; \mathbb{R}) \times \tilde{K}$ spanned by the traces of the curves of the homotopy $\tilde{F}^{(\tau_0, \omega_0, \Sigma)}(\varepsilon)(2k-1)$;
- by $Y(\varepsilon) = Y^{(\varepsilon)}_t + Y^{(\varepsilon)}_s$ the field of tangent vectors of $S(\varepsilon)$ defined by

$$Y^{(\varepsilon)}_t := \frac{\partial \tilde{F}^{(\tau_0, \omega_0, \Sigma)}(\varepsilon)(2k-1)(t, s)}{\partial t} \bigg|_{(t, s)} . \quad (3.7)$$

We are now ready to define the particular class of needle variations, which are essential for our proof. In the subsequent Remark 3.3, a short explanation of the main ideas which motivates this definition is given (see [5, 6] for a discussion in greater detail).

**Definition 3.2.** A good needle variation of $\gamma_0 = \gamma^{(U_0)}$ is a smoothed needle variation $\tilde{\text{Needle}}^{(\tau_0, \omega_0, \Sigma, \varepsilon_0)}(\gamma_0)$ which satisfies the following inequality for any $\varepsilon \in [0, \varepsilon_0]$:

$$\int_0^T \left( L_1^{(\tau_0, \omega_0, \Sigma)}(\varepsilon)(2k-1)(t, 1) - L_1^{(\tau_0, \omega_0, \Sigma)}(\varepsilon)(2k-1)(\gamma_0) \right) dt + \int_0^1 \left( - \frac{\partial C}{\partial q_{(\beta)}^{(\alpha)}} Y^{(\varepsilon)}_{(\alpha-\beta+1)} \bigg|_{(\tau_0, \omega_0, \Sigma)(\varepsilon)(2k-1)(T, s)} - \sum_{\alpha=1}^{k-1} \sum_{\beta=0}^{\alpha-1} (-1)^\beta \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial q_{(\alpha)}^{(\alpha)}} \right) Y^{(\varepsilon)}_{(\alpha-\beta+1)} \bigg|_{(\tau_0, \omega_0, \Sigma)(\varepsilon)(2k-1)(T, s)} \right) ds + \int_0^1 \sum_{\alpha=1}^{k-1} \sum_{\beta=0}^{\alpha-1} (-1)^\beta \frac{\partial}{\partial s} \left( \frac{\partial L}{\partial q_{(\alpha)}^{(\alpha)}} \right) Y^{(\varepsilon)}_{(\alpha-\beta+1)} \bigg|_{(\tau_0, \omega_0, \Sigma)(\varepsilon)(2k-1)(0, s)} ds \right) \geq 0 \quad (3.8)$$

With a small abuse of language, we sometimes call good also the (non-smooth) needle variations, for which the associated smoothed ones are good.
Remark 3.3. One of the key ideas of the approach developed in [6] (see in particular Sect. 2 of that paper) is the observation that the terminal cost of a $\mathcal{K}$-controlled curve $\gamma = \gamma^{(U)}(t)$, $U = (u(t), \sigma)$, is equal to the integral along the curve $t \mapsto (j_{t}^{2k-1}(\gamma), u(t))$ in $J^{2k-1}(\mathbb{R}) \times \mathcal{K}$ of the 1-form $\tilde{L}dt + d\tilde{C}$, where $\tilde{L}$ is an appropriate modification of $L$ and $\tilde{C}$ is a smooth extension of the original cost function $C : J^{2k-1}(\mathbb{R}) \times \mathcal{K} \to \mathbb{R}$ over the whole space $J^{2k-1}(\mathbb{R}) \times \mathcal{K}$, which vanishes at the submanifold $J^{2k-1}(\mathbb{R})_{t=0} \times \mathcal{K}$. The modification $\tilde{L}$ of $L$ is built in a way that it produces the same Euler-Lagrange equations of $L$ but has also the additional property of being identically equal to 0 along the solutions of the equations (for the cost problems of this paper, the Lagrangian $(2.4)$ has already this second property and one can just take $\tilde{L} = L$). The equality between the terminal cost of $\gamma$ and the integral of $\tilde{L}dt + d\tilde{C}$ along the curve $(j_{t}^{2k-1}(\gamma), u(t))$ is an immediate consequence of the fact that $\tilde{L}$ vanishes along the solutions.

This crucial observation implies that the difference between the terminal costs of two homotopic controlled curves is equal to the integral along of $-(\tilde{L}dt + d\tilde{C})$ along two arcs out of the four counter-clockwise oriented boundary of the surface $S \subset J^{2k-1}(\mathbb{R}) \times \mathcal{K}$ spanned by the jets and the controls of the curves of the homotopy. By Stokes’ Theorem, the integral along those two arcs (that is, the difference between the terminal costs) equals the sum of the integral of the 2-form $-d(\tilde{L}dt + d\tilde{C}) = -d\tilde{L} \wedge dt$ on $S$ and the integral of $\tilde{L}dt + d\tilde{C}$ along the other two oriented arcs of the boundary. Let us call such two arcs the “vertical part of the boundary” of the homotopy.

The condition $(3.8)$ is equivalent to requiring that the integral of the 1-form $\tilde{L}dt + d\tilde{C}$ along the (oriented) vertical part of the boundary is non-positive for any homotopy of a good needle variation.

Due to this, the difference between the terminal costs of two controlled curves related by an homotopy of a good needle variation is non-negative only if the integral of $-d\tilde{L} \wedge dt$ on the corresponding surface $S$ is non-negative. This implies that a necessary condition for a controlled curve to be a solution to the cost problem is that, for any good needle variation, the limit for $\varepsilon \to 0$ of the integral of the 2-form $-d\tilde{L} \wedge dt$ on the surfaces spanned by homotopies is non-negative. In [5] and [6] we made this necessary condition explicit and show that it reduces to the classical PMP in case of Mayer problems with smooth data and first order constraints. In the setting of this paper, the same necessary condition gives Theorem 3.4 below.

We finally remark that $(3.8)$ is basically a condition on the vector fields $Y^{(e)}$ we associated above with the homotopies $\tilde{F}(\sigma_{0}, \omega_{0}, \Sigma)(\varepsilon^{(2k-1)})$. The existence of good needle variations depends on the existence of a sufficient amount of freedom for constructing homotopies with such a property. In the next Lemma 5.1 it is shown that such a necessary freedom is granted for the cost problems of this paper. Roughly speaking this is essentially due to the fact that only the derivatives with respect to the $x_{i}^{(s)}$-variables are relevant for the terminal cost, while the auxiliary variables $p_{i}$ and their derivatives are freely specifiable. We expect that this is a general property, i.e. that the existence of good needle variations is always related with the existence of a sufficiently large number of (auxiliary and ineffective) variables.
Now, for any control pair $U_o = (u_o(t), \sigma_o)$, with corresponding curve $\gamma_o = \gamma(U_o)$, and for any $\tau_o \in (0, T)$, we define

$$
\mathcal{P}(\sigma_o, u_o, \tau_o) : K \rightarrow \mathbb{R}, \quad \mathcal{P}(\sigma_o, u_o, \tau_o)(\omega) := -L(\int_{\tau_o}^{\tau_o}(\gamma_o), \omega). \quad (3.9)
$$

By [6, Cor.7.7] the following holds:

**Theorem 3.4 (Generalised PMP).** If $U_o = (u_o(t), \sigma_o)$ is an optimal control, then

$$
\mathcal{P}(\sigma_o, u_o, \tau_o)(u_o(\tau_o)) \geq \mathcal{P}(\sigma_o, u_o, \tau_o)(\omega) \quad (3.10)
$$

for any pair $(\tau_o, \omega) \in (0, T) \times K$, for which there is at least one good needle variation of the form $\text{Needle}^{(\tau_o, \omega_o, \Sigma, \varepsilon_o)}(\gamma_o)$ for $\gamma_o$.

### 4. Two lemmas for the approximation technique

According to Theorem 3.4, if there is a pair $(\tau_o, \omega_o) \in (0, T) \times K$ for which there exists an associated good needle variation for the $K$-controlled curve $\gamma_o = \gamma(U_o)$ and such that the strict inequality $\mathcal{P}(\sigma_o, u_o, \tau_o)(u_o(\tau_o)) < \mathcal{P}(\sigma_o, u_o, \tau_o)(\omega_o)$ occurs, then $U_o$ cannot be an optimal control, i.e. there must exist an alternative $K$-controlled curve $\gamma \neq \gamma_o$ with a strictly smaller terminal cost. The proof in [6] of this is constructive and provides an explicit construction of curves with smaller costs.

In the next two lemmas, we present such a construction and we make explicit its dependence on the data of the Mayer problem and on the considered good needle variation. The first lemma holds for any generalised Mayer problem, with no particular assumptions on the controlled Lagrangian $L$. It consists of two claims: The first says that, for all sufficiently small widths $\varepsilon$, the jets of the controlled curves in the homotopies of a fixed good needle variations are in a prescribed neighbourhood of the jets of the undeformed curve; The second (and more important) claim gives an estimate for the terminal costs of the deformed curves of the needle variation, which depends on the value of the Pontryagin function at the parameter $\omega_o$ of the needle variation.

The second lemma holds only when the controlled Lagrangian and the good needle variations have very special forms and shows that certain constants, appearing in the statement of the first lemma, actually depend on much fewer data. Both lemmas apply to the cost minimising problems, on which we focus in this paper, and play a crucial role in the proof of our main result.
For the first lemma, we need to start introducing some useful notation. Given a $\mathcal{K}$-controlled curve $\gamma_o := \gamma^{(U_o)}$, $U_o = (u_o(t), \sigma_o) \in \mathcal{K}$, and a relatively compact neighbourhood $N \subset J^{2k-1}(\mathbb{Q}|\mathbb{R})$ of the $(2k - 1)$-jets of $\gamma_o$, for any $r \in \mathbb{N}$ we denote

$$
\left\| L \right\|_{r,N} := \sup_{j^{2k-1}(\gamma), u \in N \times K} \left\| \frac{\partial^r L}{(\partial t)^{\ell'} \partial q^1_{(m_1)} \cdots \partial q^i_{(m_{\ell'-i})}} \right\|
$$

$$
\left\| \frac{\partial L}{\partial u} \right\|_{r,N} := \sup_{j^{2k-1}(\gamma), u \in N \times K} \left\| \frac{\partial^{r+1} L}{(\partial t)^{\ell'} \partial q^1_{(m_2)} \cdots \partial q^i_{(m_{\ell'-i})} \partial y^a} \right\|
$$

$$
\left\| C \right\|_{C^1,N} := \sup_{j^{2k-1}(\gamma) \in N} \left( \left| C(j^{2k-1}(\gamma)) \right| + \sum_{i,s} \left| \frac{\partial C}{\partial q^i_{(s)}}(j^{2k-1}(\gamma)) \right| \right)
$$

Further, given a continuous two-parameters family of initial conditions $\Sigma = \Sigma(\varepsilon, s) \in \mathcal{A}_{\text{init}}$, $(\varepsilon, s) \in [0, \varepsilon_o] \times [0, 1]$, we denote by $\text{diam}(\Sigma)$ the diameter of the set of all such initial conditions. We finally recall that $\delta$ denotes a fixed constant, smaller than $\frac{1}{2}$, which appears in the definition of the smoothed needle modifications.

**Lemma 4.1.** Let $\gamma_o := \gamma^{(U_o)}$ and $N \subset J^{2k-1}(\mathbb{Q}|\mathbb{R})$ be a $\mathcal{K}$-controlled curve and a relatively compact neighbourhood of the $(2k - 1)$-jets of $\gamma_o$ as above and assume that there is at least one good needle variation $\tilde{\text{Needle}} = \tilde{\text{Needle}}^{(\tau_o, \omega_o, \tilde{\Sigma}, \tilde{\varepsilon}_o)}(\gamma_o)$ for a given choice of $(\tau_o, \omega_o) \in (0, T) \times K$. Let also $\rho, \kappa, \mathcal{E}, \mathcal{C}'$ be the constants that are determined as in Lemma 2.1 by $L$, the Lipschitz bijection between $\mathcal{A}_{\text{init}}$ and $\tilde{\mathcal{A}}_{\text{init}}$ and a cut-off function with support in the open set $\tilde{N} \subset \mathbb{R}^{N+1}$ corresponding to $N \subset J^{2k-1}(\mathbb{Q}|\mathbb{R})$ (so that (2.9) holds for the controlled curves, which are sufficiently close to $\gamma_o$). We finally assume that the map $\varepsilon \mapsto \text{diam}(\Sigma(\varepsilon, \cdot), s \in [0, 1])$ is continuous in the argument $\varepsilon$ and that there exists a constant $K(N,L)$, depending on $L$ and $N$, satisfying the following condition: the first order system (2.7) in normal form, which is equivalent to (2.3), is such that

$$
\sup_{(t,y,u) \in \tilde{N} \times K} \left\| \frac{\partial g^A}{\partial u^a} \right\| < K(N,L) \left\| L \right\|_{k+2,N}.
$$

Then:

1. There is a good needle variation $\tilde{\text{Needle}}^{(\tau_o, \omega_o, \Sigma, \varepsilon_o)}(\gamma_o)$ (obtained from $\tilde{\text{Needle}} = \tilde{\text{Needle}}^{(\tau_o, \omega_o, \tilde{\Sigma}, \tilde{\varepsilon}_o)}(\gamma_o)$ by appropriately reducing the width $\tilde{\varepsilon}_o$ to a smaller one $\varepsilon_o$) such that:
   a. $\text{diam}\Sigma$ and the distances between any two control curves $u(t)$, $u'(t)$, corresponding to two curves in a common homotopy $F(\tau_o, \omega_o, \Sigma)(\varepsilon)$, $\varepsilon \in (0, \varepsilon_o)$, of the family $\tilde{\text{Needle}}^{(\tau_o, \omega_o, \Sigma, \varepsilon_o)}(\gamma_o)$, are less than or equal to $\rho$;
   b. any $(2k-1)$-th order jet of a control curve in the homotopies of $\tilde{\text{Needle}}^{(\tau_o, \omega_o, \Sigma, \varepsilon_o)}(\gamma_o)$ is in $N$. 
(2) If \( \kappa_0 = \mathcal{P}(\sigma_o, u_o, \tau_o)(\omega_o) - \mathcal{P}(\sigma_o, u_o, \tau_o)(u_o) > 0 \), then there exists a constant \( M \), depending on \( N, \tau_o, \omega_o, \Sigma \), \( \|L\|_{k+2, N}, \|L\|_{k+1, N}, \|C\|_{C^1, N} \) and the infinitesimal \(^2\) \[
abla(V) := \frac{1}{\varepsilon} \int_{\tau_o - \varepsilon}^{\tau_o} \left( \mathcal{P}(\sigma_o, u_o, t)(u_o) - \mathcal{P}(\sigma_o, u_o, t)(u_o) \right) dt \tag{4.3}
\] such that the controlled curve \( \gamma = \gamma(L) \), determined by the pair \( \mathcal{U} = (\mathcal{U}(t), \sigma) \in \mathcal{K}_{\text{meas}} \) with \( \sigma := \Sigma \left( \frac{1}{2M}, 1 \right) \) and \[
(4.4)
\]
has a terminal cost that satisfies \[
C'(\gamma) \leq C'(\gamma_0) - \frac{\kappa_0}{2M} < C'(\gamma_0). \tag{4.5}
\]

There also exists a constant \( \tilde{M} > 0 \), for which \(^2\) holds for the smooth \( \mathcal{K} \)-controlled curve \( \tilde{\gamma} \), determined by the same initial condition and the smoothed version \( \tilde{\mathcal{U}}(t) \) of \( \mathcal{U}(t) \).

**Proof**. (1) Given a good needle variation \( \tilde{\text{Needle}} = \tilde{\text{Needle}}(\tau_o, \omega_o, \Sigma, \varepsilon_o)(\gamma_o) = \{ \tilde{F}(\tau_o, \omega_o, \Sigma)(\varepsilon), \varepsilon \in [0, \varepsilon_o] \} \), for any \( (\varepsilon, s) \in [0, \varepsilon_o] \times [0, 1] \) we denote by \( \tilde{\gamma}(\varepsilon, s)(t) \) and \( \tilde{\gamma}(\varepsilon, s)(t) \) the controlled curves \[
\tilde{\gamma}(\varepsilon, s)(t) := F(\tau_o, \omega_o, \Sigma)(\varepsilon)(t, \varepsilon), \quad \tilde{\gamma}(\varepsilon, s)(t) := F(\tau_o, \omega_o, \Sigma)(\varepsilon)(t, \varepsilon).
\]
We also use the short notation \( \gamma(\varepsilon)(t) := \gamma(\varepsilon, 1)(t) \) and \( \tilde{\gamma}(\varepsilon)(t) := \tilde{\gamma}(\varepsilon, 1)(t) \). By construction

- \( \gamma(\varepsilon, s)(t) \) is the \( \mathcal{K}_{\text{meas}} \)-controlled curve determined by \( \mathcal{U}(\varepsilon, s) := (\mathcal{U}(\varepsilon, s)(t), \Sigma(\varepsilon, s)) \) with \[
\mathcal{U}(\varepsilon, s)(t) = (1 - s)u_o(t) + su(\tau_o, \omega_o, \varepsilon)(t), \quad \mathcal{U}(\tau_o, \omega_o, \varepsilon)(t) := \begin{cases} u_o(t) & \text{if } t \in [\tau_o - \varepsilon, \tau_o], \\ \omega_o & \text{if } t \in (\tau_o - \varepsilon, \tau_o]. \end{cases}
\tag{4.6}
\]

- \( \tilde{\gamma}(\varepsilon, s)(t) \) is the \( \mathcal{K} \)-controlled curve determined by the same initial condition and the smooth version \( \tilde{\mathcal{U}}(\tau_o, \omega_o, \varepsilon)(t) \) of \( \mathcal{U}(\tau_o, \omega_o, \varepsilon)(t) \), defined in (4.3).

We now replace the good needle variation \( \tilde{\text{Needle}}, \) given in the statement, by a new one, in which the homotopies of curves are the same, but where the maximum value for the parameter \( \varepsilon \) is changed into a new value \( \varepsilon_o \leq \varepsilon_o \). This \( \varepsilon_o \) is chosen small enough to make the family of initial conditions \( \Sigma = \tilde{\Sigma}_{[0, \varepsilon_o] \times [0, 1]} \) such that \( \text{diam}(\Sigma) < \rho \) and, for any \( (\varepsilon, s) \in [0, \varepsilon_o] \times [0, 1] \),

\[
\text{dist}(\tilde{\mathcal{U}}(\varepsilon, s), u(\varepsilon, s)) \leq 2\varepsilon \rho^2 < \rho, \quad \text{dist}(\tilde{\mathcal{U}}(\varepsilon, s), u_o) \leq \varepsilon + 2\varepsilon \rho^2 < \rho.
\]

In this way (a) is satisfied. From Lemma \(^2\) by the fact that \( \Sigma(0, s) = \sigma_o \) for any \( s \) and from the Lipschitzian assumption on \( \Sigma \), it follows that for a sufficiently small \( \varepsilon_o \)

\[
\|\tilde{\gamma}(\varepsilon, s) - \gamma_o\|_{C^{2k - 1}} \leq C(\varepsilon + 2\varepsilon \rho^2) + \kappa|\Sigma(\varepsilon, s) - \Sigma(0, s)| < (2\varepsilon + \kappa C)\varepsilon,\tag{4.7}
\]

\[
\|\tilde{\gamma}(\varepsilon, s) - \gamma(\varepsilon, s)\|_{C^{2k - 1}} \leq 2\varepsilon \rho^2 < C\varepsilon^2, \tag{4.8}
\]

for a constant \( C \) determined by \( \Sigma \). By possibly taking a smaller \( \varepsilon_o \), also (b) is satisfied.

\(^2\)We use the short expression “infinitesimal” to mean that \( \lim_{\varepsilon_o \to 0} V(\varepsilon) = 0 \).
(2) Consider the good needle variation determined in (1) and its smoothed version. For any \( \varepsilon \in [0, \varepsilon_0] \), we denote \( \tilde{C}(\varepsilon) := C(J_{t=T}^{2k-1}(\tilde{\gamma}(\varepsilon))) \) and \( C(\varepsilon) := C(J_{t=T}^{2k-1}(\gamma(\varepsilon))) \). By [6, Cor. 6.5] we know that

\[
C(0) - \tilde{C}(\varepsilon) = \tilde{C}(0) - \tilde{C}(\varepsilon) = \int_0^T \left( \int_0^1 Y^{(\varepsilon)}(\tilde{\gamma}(\varepsilon), \tilde{\gamma}(\varepsilon), t) \frac{\partial p(\Sigma(\varepsilon, s), \tilde{\gamma}(\varepsilon), t)}{\partial a} \right) ds \, dt = \int_0^T \left( \int_0^1 \frac{\varepsilon^2 \tilde{\mu}}{\partial t \partial s} \right) ds \, dt \tag{4.9}
\]

where \( Y^{(\varepsilon)} \) are the \( \frac{\partial}{\partial a} \)-components of the vector field \( \tilde{\mu} \) and \( \tilde{\mu} : [0, T] \times [0, 1] \to \mathbb{R} \) is an appropriate function, which is completely determined by the needle variation considered. For our purposes, there is no need to recall the detailed definition of \( \tilde{\mu} \), but only to know that, since the needle variation is good, the value of \( \int_0^T \left( \int_0^1 \frac{\varepsilon^2 \tilde{\mu}}{\partial t \partial s} \right) ds \, dt \)

is non-positive ([6, Lemma 7.6]). In addition, the next sublemma, whose proof is quite technical and is postponed to §6.1 gives an estimate for the first term in (4.9).

**Sublemma 4.2.** In the hypotheses of the lemma, there exists a constant \( n = n_{(\tau_0, N, L, \frac{\varepsilon_0}{\varepsilon})} \), depending on \( \tau_0 \), \( N \), \( \|L\|_{k+2,N} \) and \( \|\frac{\partial p}{\partial u}\|_{k+1,N} \) such that for any \( \varepsilon \in [0, \varepsilon_0] \)

\[
\int_0^T \left( \int_0^1 Y^{(\varepsilon)}(\tilde{\gamma}(\varepsilon), \tilde{\gamma}(\varepsilon), t) \frac{\partial p(\Sigma(\varepsilon, s), \tilde{\gamma}(\varepsilon), t)}{\partial a} \right) ds \, dt > \int_{\tau_0 - \varepsilon}^{\tau_0 + \varepsilon} \left( p(\sigma(\varepsilon), \tilde{\gamma}(\varepsilon)) - \mu(\sigma, u_0)(t) \right) dt - n\varepsilon^2 \tag{4.10}
\]

where \( \sigma(\varepsilon) := \Sigma(\varepsilon, 1) \) and \( \tilde{\gamma}(\varepsilon) = \tilde{\gamma}(\varepsilon, 1) \)

Let us now focus on the right hand side of (4.10). The first integral decomposes into

\[
\int_{\tau_0 - \varepsilon}^{\tau_0 + \varepsilon} \left( p(\sigma(\varepsilon), \tilde{\gamma}(\varepsilon)) - \mu(\sigma, u_0)(t) \right) dt = \int_{\tau_0 - \varepsilon}^{\tau_0 - \varepsilon} \left( p(\sigma(\varepsilon), \tilde{\gamma}(\varepsilon)) - \mu(\sigma, u_0)(t) \right) dt + \int_{\tau_0 - \varepsilon}^{\tau_0 + \varepsilon} \left( p(\sigma(\varepsilon), \tilde{\gamma}(\varepsilon)) - \mu(\sigma, u_0)(t) \right) dt + \int_{\tau_0 + \varepsilon}^{\tau_0 + \varepsilon} \left( p(\sigma(\varepsilon), \tilde{\gamma}(\varepsilon)) - \mu(\sigma, u_0)(t) \right) dt \tag{4.11}
\]

Since \( L \) is continuous, for any \( (t, \omega) \in [0, T] \times \tilde{K} \), we have that \( \left| p(\Sigma(\varepsilon, s), \tilde{\gamma}(\varepsilon), t)(\omega) \right| < \varepsilon'' \), with \( \varepsilon''_{(N, L)} := \|L\|_{\infty, N \times K} \), and the sum of the first and the third terms in (4.11) is bounded.
by $4\epsilon_{(N,L)}^m h r^2$. On the other hand, the second decomposes into
\[
\int_{\tau_0-\varepsilon}^{\tau_0} \left( p(\sigma_{(e)} u_{(t)})(\omega_0) - p(\sigma_{(u_0,t)})(u_0(t)) \right) dt = \\
= \int_{\tau_0-\varepsilon}^{\tau_0} \left( p(\sigma_{(e)} u_{(t)})(\omega_0) - p(\sigma_{(u_0,t)})(\omega_0) + p(\sigma_{(u_0,t)})(\omega_0) - p(\sigma_{(u_0,t)})(\omega_0) \right) dt + \\
+ \int_{\tau_0-\varepsilon}^{\tau_0} \left( p(\sigma_{(u_0,t)})(u_0(\tau_0)) - p(\sigma_{(u_0,t)})(u_0(\tau_0)) \right) dt + \\
+ \int_{\tau_0-\varepsilon}^{\tau_0} \left( p(\sigma_{(u_0,t)})(u_0(\tau_0)) - p(\sigma_{(u_0,t)})(u_0(\tau_0)) \right) dt \\
\tag{4.12}
\]
By (4.7) the integrands of the first and the third terms are bounded by
\[
\left| p(\sigma_{(e)} u_{(t)})(\omega_0) - p(\sigma_{(u_0,t)})(\omega_0) \right|, \quad \left| p(\sigma_{(u_0,t)})(u_0(\tau_0)) - p(\sigma_{(u_0,t)})(u_0(\tau_0)) \right| < \epsilon_{(N,L)}^{m} \epsilon^{(e)}
\tag{4.13}
\]
for some $\epsilon_{(N,L,S)}^m$ depending on $\|L\|_1N$ and the constants $\epsilon, \kappa, C$. On the other hand, by continuity, the fourth term is an infinitesimal of higher order than $\varepsilon$, that is (see (4.3))
\[
\int_{\tau_0-\varepsilon}^{\tau_0} \left( p(\sigma_{(u_0,t)})(u_0(\tau_0)) - p(\sigma_{(u_0,t)})(u_0(\tau_0)) \right) dt = V(\varepsilon) \varepsilon .
\tag{4.14}
\]
Combining (4.9) with Sublemma 4.2 and the above discussion, we obtain that
\[
C^{(0)} - C^{(e)} \geq \varepsilon(\kappa_0 - 4\epsilon_{(N,L)}^m h r^2 - 2\epsilon_{(N,L,S)}^m \epsilon - |V(\varepsilon)| - n\epsilon) .
\tag{4.15}
\]
On the other hand, by (4.6) and the smoothness of the cost function, there is a constant $m = m(S,C)$, depending on $\|C\|_{C^{1,N}}$, such that for all sufficiently small $\varepsilon$
\[
| \tilde{C}(\varepsilon) - C(\varepsilon) | < m \| \gamma^{(e)} - \gamma^{(e)} \|_{C^{2k-1}} \leq cm\varepsilon^2 .
\tag{4.16}
\]
Hence, from (4.15) and (4.16) we get
\[
C^{(e)} \leq C^{(0)} - \varepsilon(\kappa_0 - \tilde{d} \varepsilon - |V(\varepsilon)|)
\tag{4.17}
\]
where $\tilde{d} = \tilde{d}_{(\tau_0,\omega_0),L,S,C} := (4\epsilon_{(N,L)}^m h + 2\epsilon_{(N,L,S)}^m + cm + n) .
\]
Since the map $\varepsilon \mapsto \tilde{d} \varepsilon + |V(\varepsilon)|$ is infinitesimal for $\varepsilon \to 0^+$, there exists $M > 0$ such that
\[
\kappa_0 - (\tilde{d} \varepsilon + |V(\varepsilon)|) > \kappa_0 \frac{1}{2} \quad \text{for all } \varepsilon \in \left[0, \frac{1}{M} \right] .
\tag{4.18}
\]
Thus, setting $\underline{\varepsilon} := \frac{1}{2M}$, we have $C^{(e)} \leq C^{(0)} - \frac{\kappa_0}{2} = C^{(0)} - \frac{\kappa_0}{4M}$, that is (4.5). The last claim is proved similarly, using just (4.15) in place of (4.17).

We now present the second advertised lemma, which gives a radical improvement of Lemma 4.1 under additional assumptions on $L$ and on the considered good needle variation. More precisely, we assume that the configuration space has the form $Q = Q \times \mathbb{Q}^*$ for an $n$-dimensional affine space $Q = \mathbb{R}^n$ (the coordinates are thus pairs $q = (x, p)$ with $x = (x^i) \in Q$ and $p = (p_j) \in \mathbb{Q}^*$) and $L$ has the form
\[
L = p_i(x_{(k)}^i - f^i(t, x^i, x^j_{(1)}, \ldots, x^j_{(k-1)}, u^a(i))) .
\tag{4.18}
\]
Lemma 4.3. Assume the hypotheses of Lemma 4.1 with the exception of the condition (4.2) and let \( Q = Q \times Q^* \) and \( L \) are as above. Moreover, for any \( \epsilon \in [0, \epsilon_0] \), let \( \mathcal{G}(t, s) := \left( \tilde{t}^{2k-1}(\tilde{\tau}(s)), u(\tilde{\tau}(s))(t) \right) \) and \( Y \) the field of tangent vectors to the surface \( S := L([0, T] \times [0, 1]) \) defined by \( Y|_{\mathcal{G}(t, s)} = \frac{\partial \mathcal{G}}{\partial \tilde{t}}(t, s) \).

If the family of initial conditions \( \Sigma \), occurring in the definition of \( \text{Needle}^{(\tau_o, \omega_o, \Sigma, \epsilon_0)}(\gamma_0) \), is such that

\[
\sum_{\delta=1}^{k} \sum_{\eta=0}^{\delta-1} (-1)^{\eta} \int_0^1 \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial q^i(\delta)} \right) Y_i(\delta-(\eta+1)) \bigg|_{\mathcal{G}(t, s)} \quad ds = \\
= \sum_{\delta=1}^{k} \sum_{\eta=0}^{\delta-1} (-1)^{\eta} \int_0^1 \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial q^i(\delta)} \right) Y_i(\delta-(\eta+1)) \bigg|_{\mathcal{G}(t, s)} \quad ds = 0 , \quad (4.19)
\]

then claim (2) of Lemma 4.1 holds with constants \( M, \bar{M} \) that depend just on \( \tau_o, \omega_o, N, \|L\|_{1,N} \) and \( |C|_{1,N} \) and not on \( \|L\|_{k+2,N} \) and \( \|\frac{\partial L}{\partial q^i}\|_{k+1,N} \).

Due to the technicalities in the arguments, the proof is given later in \( \S \) 6.2.

Remark 4.4. Let \((K, L^{(\delta)}, C^{(\delta)}) \), \( \delta \in (0, \delta_0] \) be a one-parameter family of defining triples as in Remark 2.2. Assume that all associated generalised Mayer problems are with smooth data and of normal type and that, for \( \delta \) tending to 0, the Lagrangians \( L^{(\delta)} \) and the cost functions \( C^{(\delta)} \) tend uniformly on compacta to continuous functions \( L^{(0)} \) and \( C^{(0)} \). Assume also that:

(A) The partial derivatives

\[
\frac{\partial^\ell L^{(\delta)}}{\partial q^i(\alpha_1) \ldots \partial q^i(\alpha_\ell)} , \quad \frac{\partial^\ell L^{(\delta)}}{\partial q^i(\alpha_1) \ldots \partial q^i(\alpha_{\ell-\ell'})} \quad \text{with} \quad 1 \leq \ell \leq k + 2
\]

and

\[
\frac{\partial^{\ell+1} L^{(\delta)}}{\partial u^o(\tilde{\tau})(\alpha_1) \ldots \partial q^i(\alpha_\ell)} , \quad \frac{\partial^{\ell+1} L^{(\delta)}}{\partial q^i(\alpha_1) \ldots \partial q^i(\alpha_{\ell-\ell'})} \quad \text{with} \quad 1 \leq \ell \leq k + 1
\]

tend uniformly on compacta to the corresponding partial derivatives of \( L^{(0)} \);

(B) There is a one-parameter family \( U^{(\delta)} = (u_0^{(\delta)}(t), \sigma_0^{(\delta)}) \in K \), whose associated controlled curves \( \gamma^{(\delta)}(t) := \gamma^{(U^{(\delta)})} \) converge in the norm of \( C^{k-1}([0, T]) \) to a curve \( \gamma^{(0)}(t) \) such that: (a) it is a solution to the differential constraints determined by \( L^{(0)} \), (b) it has \( \sigma_0^{(\delta)} = \lim_{\delta \to 0} \sigma_0^{(\delta)} \) as initial condition; (c) it is determined by a measurable control curve \( u_0^{(0)}(t) \in K \) with \( u_0^{(0)}(t) = \lim_{\delta \to 0} u^{(\delta)}_0(t) \) a. e.;

(C) There exists a \( \delta \)-parameterised family of initial data maps \( \Sigma^{(\delta)}(\epsilon, s) \), converging uniformly on \([0, \epsilon_0] \times [0, 1]\) to a limit map \( \Sigma^{(0)}(\epsilon, s) \), and a corresponding \( \delta \)-parameterised family of good needle variations, determined by the maps \( \Sigma^{(\delta)} \) and a pair \((\tau_o, \omega_o)\), in which \( \tau_o \) is one of the points where \( u_0^{(\delta)}(\tau_o) = \lim_{\delta \to 0} u_0^{(\delta)}(\tau_o) \);

(D) The real value

\[
\kappa_0^{(\delta)} := \left( p(\sigma_0^{(\delta)}, u_0^{(\delta)}(\tau_o))(\omega_0) - p(\sigma_0^{(\delta)}, u_0^{(\delta)}(\tau_o))(\tau_o) \right) = \lim_{\delta \to 0} \kappa_0^{(\delta)}
\]

is strictly positive.
(E) The functions \( V^{(\delta)}(\varepsilon), \varepsilon \in (0, \varepsilon_0] \), which are defined by (4.3) for each \( \delta \), tend uniformly on compacta to a function \( \tilde{V}(\varepsilon) : (0, \varepsilon_0] \rightarrow \mathbb{R} \), which is an infinitesimal for \( \varepsilon \rightarrow 0 \).

Note that the \( \delta \)-parameterised family of control pairs \( U^{(\delta)}(\varepsilon, s) := (u^{(\delta)(\varepsilon,s)}(t), \Sigma^{(\delta)}(\varepsilon, s)) \), which determine the curves \( \gamma^{(\delta)(\varepsilon,s)}(t) \) of the needle variation in (C), have the following property: for any \( (\varepsilon, s) \in [0, \varepsilon_0] \times [0, 1] \), the a.e. limit curve

\[
\lim_{\delta \rightarrow 0} u^{(0)(\varepsilon,s)}(t) := \lim_{\delta \rightarrow 0} u^{(\delta)(\varepsilon,s)}(t)
\]

is a needle modification of \( u_0(t) \). We may therefore consider also the following additional assumptions:

(F) For any \( \varepsilon \in [0, \varepsilon_0] \) the controlled curves \( \gamma^{(\delta;\varepsilon)}(t) := \gamma^{(\delta;\varepsilon,1)}(t) \) converge uniformly to the solution of the differential constraints for \( \delta = 0 \), which is determined by the control pair \( (u^{(0)(\varepsilon,1)}(t), \Sigma^{(0)}(\varepsilon, 1)) \):

(G) There is a relatively compact neighbourhood of the set

\[
\mathcal{Ends} := \{ j_T^{2k-1}((\delta;\varepsilon,s)), \ (\delta, \varepsilon, s) \in [0, \delta_0] \times [0, \varepsilon_0] \times [0, 1] \},
\]

given by the \((2k-1)\)-jets at \( t = T \) of the controlled curves of the needle variations, on which \( C^{(\delta)} \) tend to \( C^{(0)} \) in the \( C^1 \)-norm.

For each of the above Lagrangian \( L^{(\delta)} \), we may follow the proof of Lemma 4.3 and derive the inequality (4.17) for any sufficiently small \( \varepsilon \), i.e. the inequality

\[
C^{(\delta;\varepsilon)}(\varepsilon) \leq C^{(\delta;0)}(\varepsilon) - \varepsilon (\kappa^{(\delta;0)}_0 - \delta^{(\delta;\varepsilon)} - |V^{(\delta;\varepsilon)}(\varepsilon)|)
\]

(4.20)

relating the cost \( C^{(\delta;0)}(\varepsilon) \) of the controlled curve \( \gamma^{(\delta;\varepsilon)}(t) \) with the cost \( C^{(\delta;\varepsilon)}(\varepsilon) \) of the controlled curves \( \gamma^{(\delta;\varepsilon)}(t) \). Notice that, under the assumptions (A) – (G), for \( \delta \) sufficiently small, we may assume that the constants \( \delta^{(\delta;\varepsilon)} \) are independent of \( \delta \), say \( \delta^{(\delta;\varepsilon)} = \delta^{(\delta)} \), so that, letting \( \delta \rightarrow 0 \),

\[
C^{(\delta=0;\varepsilon)}(\varepsilon) \leq C^{(\delta=0;0)}(\varepsilon) - \varepsilon (\kappa^{(\delta=0;0)}_0 - \delta^{(\delta;\varepsilon)} - |\tilde{V}(\varepsilon)|)
\]

(4.21)

where \( C^{(\delta=0;\varepsilon)}(\varepsilon) \) is the terminal cost of the controlled curve \( \gamma^{(\delta=0;\varepsilon)}(t) \). From this, using the same concluding argument of the proof of Lemma 4.1 we obtain the existence of a constant \( M > 0 \) and an associated needle modification for the (merely measurable) limit curve \( u_0^{(0)}(t) \), such that the corresponding cost satisfies

\[
C^{(\delta=0;\varepsilon)}(\varepsilon) \leq C^{(\delta=0;0)}(\varepsilon) - \frac{\kappa^{(\delta=0;0)}_0}{4M}.
\]

(4.22)

This fact is crucially exploited in the proof of our main result, given in the last section.

We finally observe that, by Lemma 4.3 if the Lagrangians \( L^{(\delta)} \) have the special form (4.18) and all maps \( \Sigma^{(\delta)} \) satisfy the condition (4.19), the above conclusion on the costs of the limit controlled curve \( \gamma^{(0;0)}_0 \) and of its needle modification holds also if (A) is replaced by the following weaker assumption:

(A') the partial derivatives \( \frac{\partial L^{(\delta)}}{\partial q^{(0;0)}_i} \), \( \frac{\partial L^{(\delta)}}{\partial t} \), tend uniformly on compacta to the corresponding partial derivatives of \( L^{(0)} \).
5. The proof of Theorem 1.1

5.1. A preliminary “smooth” version of the main result. As we pointed in §2.3 when all of its data are of class \( C^\infty \), the cost problem presented in the Introduction is equivalent to the generalised Mayer problem determined by a defining triple \((\mathcal{K}, L, C)\) given in that section. We recall that the configuration space has the form \( \mathcal{Q} = Q \times Q^* \), \( Q = \mathbb{R}^n \), the controlled Lagrangian \( L \) and the cost function \( C \) in \( 2.4 \), and the set \( \mathcal{K} \) consists of the pairs \( U = (u(t), \sigma = (s, \tilde{s})) \) where (a) \( s \) is an initial condition in a prescribed set \( A_{\text{init}} \subset J^{k-1}(\mathcal{Q}^{\mathbb{R}})_{t=0} \) for the curve \( x(t) \) and (b) \( \tilde{s} \) is an initial condition, which can be arbitrary, for the curve \( p(t) \). We finally recall that the controlled Euler-Lagrange equations are of normal type and are given in \( 2.5 \) and \( 2.6 \).

Let us now consider the following subset of \( \mathcal{K} \). Given \( s \in A_{\text{init}} \) and a control curve \( u(t) \in K \), we denote by \( x^{(u,s)}(t) \) the unique solution to \( 2.5 \) with initial condition \( s \). We then denote by \( p^{(u,s)}(t) \) the unique solution to \( 2.6 \) with \( x(t) = x^{(u,s)}(t) \), that satisfies the terminal conditions \( 1.6 \) – \( 1.8 \). Finally, we set \( \tilde{s}^{(u,s)} \) to be the initial jet \( \tilde{s}^{(u,s)} = j_{t=0}^{d-1}(p^{(u,s)}) \) of \( p^{(u,s)}(t) \). By construction, the pair

\[
U^{(u,s)} := \left( u(t), \sigma = (s, \tilde{s}^{(u,s)}) \right),
\]

has \( \gamma^{(U)}(t) := (t, x^{(u,s)}(t), p^{(u,s)}(t)) \) as associated \( \mathcal{K} \)-controlled curve. The pairs \( 5.1 \) and the corresponding controlled curves are called good. The subset of the good pairs in \( \mathcal{K} \), \( \mathcal{K} \), \( \mathcal{K} \) are denoted by \( \mathcal{K}_{\text{good}}, \mathcal{K}_{\text{good}}, \mathcal{K}_{\text{good meas}} \), respectively.

Our interest in the good \( \mathcal{K} \)-controlled curves comes from the following lemma.

**Lemma 5.1.** Let \( \gamma_o = \gamma^{(U_o)} \) be a \( \mathcal{K}_{\text{good}} \)-controlled curve and \( \text{Needle}^{(\tau_0, \omega_0, \Sigma, \varepsilon_o)}(\gamma_o) \) a needle variation, whose associated smoothed needle variation \( \tilde{\text{Needle}}^{(\tau_0, \omega_0, \Sigma, \varepsilon_o)}(\gamma_o) := \{ \tilde{F}^{(\tau_0, \omega_0, \Sigma)}(\varepsilon) \}, \varepsilon \in [0, \varepsilon_o] \} \) satisfies the following two conditions:

(a) all control pairs \( U(s, \varepsilon) \) that determine the \( \tilde{\mathcal{K}} \)-controlled curves \( \gamma^{(s, \varepsilon)} = \tilde{F}^{(\tau_0, \omega_0, \Sigma)}(\varepsilon) \) are good;

(b) the initial conditions for the \( x \)-components \( x^{(s, \varepsilon)}(t) \) of the curves \( \gamma^{(s, \varepsilon)} \) are constant and independent of \( (s, \varepsilon) \in [0, 1] \times [0, \varepsilon_o] \).

Then \( \tilde{\text{Needle}}^{(\tau_0, \omega_0, \Sigma, \varepsilon_o)}(\gamma_o) \) is a good needle variation in the sense of Definition 3.2.

**Proof.** First of all, we claim that if a controlled curve \( \gamma(t) := (t, x^{(u,s)}, p^{(u,s)}(t)) \) is good, then

\[
\left( \frac{\partial C}{\partial x^{(\beta)}_{i}} + \sum_{\ell=0}^{k-\beta-1} (-1)^{\ell} \frac{d^{\ell}}{dt^{\ell}} \left( \frac{\partial (p_m(x^{(m)}_{(k)} - f^{m})}{\partial x^{(\beta+\ell+1)}_{i}} \right) \right)_{j=2^{k-\beta}}^{2^{k-2}(x^{(u,s)})} = 0
\]

for \( 0 \leq \beta \leq k-1 \) and \( 0 \leq i \leq n \).
This is a consequence of the fact that, setting $\rho := k - \beta - 1$, the conditions (5.2) become
\[
\left. \left( \frac{\partial C}{\partial x_i^{(k-\rho-1)}} + \sum_{\ell=0}^{\rho} (-1)^{\ell} \frac{d^\ell}{dt^\ell} \left( \frac{\partial (p_m(x_i^{(k)} - f_i^m))}{\partial x_i^{(k-\rho+\ell)}} \right) \right) \right|_{t=T}^{x(u,s)} = 0 ,
\]
and these are precisely the conditions (1.0) - (1.8). Consider now an arbitrary needle variation $\text{Needle}^{(\tau_o, \omega_o, \Sigma, \epsilon_o)}(\gamma_o)$ of $\gamma_o := \gamma(U_o)$ with $U_o = (u_o(t), s_o)$. By the particular form of the differential constraints, the value of the controlled Lagrangian $L$ is 0 at all $(k-1)$-th order jets of the $\mathcal{K}$-controlled curves. Hence, using the short-hand notation $\gamma^{(s,\epsilon)}(t) := j^{k-1}(\gamma(s,\epsilon))$, we may write
\[
\int_0^T (L|_{\gamma^{(s,\epsilon)}(k-1)(t)} - L|_{\gamma_o^{(k-1)}(t)}) \, dt = 0 \quad \text{for any } s \in [0, 1] .
\] (5.4)
From this and the fact that the only non-trivial derivatives $\frac{\partial L}{\partial q^{(s,\epsilon)}}$ with respect to the jet coordinates $q^{(s,\epsilon)} = (x_i^{(s,\epsilon)}, p^{(s,\epsilon)}_j)$, $\alpha \geq 1$, are those with $q_i^{(s,\epsilon)} = x_i^{(s)}$, it follows that (5.3) is satisfied if and only if
\[
\int_0^1 \left( - \sum_{\beta=1}^{k-1} \frac{\partial C}{\partial x_i^{(\beta)}} \left( \frac{\partial \gamma^{(x_i)}}{\partial \gamma^{(k-1)}} \right) \right|_{\gamma^{(s,\epsilon)}(k-1)(T)} - \sum_{\delta=1}^{k} \sum_{\eta=0}^{\delta-1} (-1)^{\eta} \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial x_i^{(\delta)}} \right) Y^{(x_i)}_{\gamma^{(s,\epsilon)}(k-1)(T)} ds + \int_0^1 \sum_{\delta=1}^{k} \sum_{\eta=0}^{\delta-1} (-1)^{\eta} \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial x_i^{(\delta)}} \right) Y^{(x_i)}_{\gamma^{(s,\epsilon)}(k-1)(0)} ds = 0 .
\] (5.5)
From (5.2) and the definition of $Y$, if the needle variation satisfies (a) and (b), both integrals in (5.5) are zero and the inequality is satisfied. \hfill \Box

Remark 5.2. By definition, for any control curve $u_o(t)$ in $\mathcal{K}$ and any $s_o \in \mathcal{A}_{null}$, there exists a uniquely associated good pair $U^{(u_o, s_o)} := (u_o(t), \sigma_o = (s_o, s_o^{(u_o, s_o)}))$. Then for any good $\mathcal{K}$-controlled curve $\gamma_o$ and any $(\tau_o, \omega_o) \in (0, T] \times K$, it is possible to construct a smoothed needle variation $\text{Needle}^{(\tau_o, \omega_o, \Sigma, \epsilon_o)}(\gamma_o)$ satisfying both conditions of Lemma
This means that for any good controlled curve $\gamma_o$ and any $(\tau_o, \omega_o) \in (0, T] \times K$, there is a good needle variation $\tilde{\mathcal{N}}_o(\tau_o, \omega_o; \Sigma, \varepsilon_o)(\gamma_o)$ associated with $(\tau_o, \omega_o)$.

Remark [5.2] and Theorem [5.3] easily imply the following $C^\infty$ version of Theorem 1.1.

**Theorem 5.3.** Let $f = (f^i)$ and $C$ of class $C^\infty$ and $U_o = (u_o(t), (s_o, \tilde{\omega}(u_o, s_o))) \in \mathcal{K}_{\text{good}}$ with $u_o(t)$ smooth. If $U_o$ is an optimal control, then (1.10) holds for any $(\tau_o, \omega_o) \in (0, T) \times K$.

More precisely, for any such $(\tau_o, \omega_o)$, there exist constants $M, \tilde{M}$, which depend on $f$, $C$, $\tau_o$ and $\omega_o$, such that if

$$\kappa_o := H^{(u_o, \omega_o, \tau_o)}(\omega_o) - H^{(u_o, \omega_o, \tau_o)}(u_o(\tau_o)) > 0,$$

then there is a needle modification $\tilde{u}_o(t)$ of $u_o(t)$, with associated smoothed needle modification $\tilde{\omega}_o(t)$, such that

$$C(j^{k-1}_{t=T}(x(u_o, s_o))) \leq C(j^{k-1}_{t=T}(x(u_o, s_o))) - \frac{\kappa_o}{4M} < C(j^{k-1}_{t=T}(x(u_o, s_o))) \leq C(j^{k-1}_{t=T}(x(u_o, s_o))) - \frac{\kappa_o}{4M} < C(j^{k-1}_{t=T}(x(u_o, s_o))).$$

**Proof.** If $U_o = (u_o(t), (s_o, \tilde{\omega}(u_o, s_o)))$ is good, the function (3.9) is equal to

$$p((s_o, \tilde{\omega}(u_o, s_o)), u_o, \tau_o) = -p_i^{(u_o, s_o)}(\tau_o) x^{(u_o, s_o)i}(\tau_o) + p_i^{(u_o, s_o)}(\tau_o) f^i(\tau, x(u_o, s_o)(\tau_o), x^{(u_o, s_o)}(\tau_o), \ldots, x^{(u_o, s_o)}(\tau_o), \omega) = -p_i^{(u_o, s_o)}(\tau_o) x^{(u_o, s_o)i}(\tau_o) + H^{(u_o, s_o, \tau_o)}(\omega).$$

Thus $\omega$ is a maximum point for $p((s_o, \tilde{\omega}(u_o, s_o)), u_o, \tau_o)$ if and only if it is a maximum point for $H^{(u_o, s_o, \tau_o)}$. The claim then follows from Theorem 3.4 Lemma 1.1 and Remark 5.2.

5.2. **The proof of Theorem 1.1**

First of all, we assume the following condition, which causes no loss of generality. Let $\overline{B}_{\tilde{R}} \subset \mathbb{R}^m$ and $\overline{B}_R \subset \mathbb{R}^{mk}$ be two closed balls centred at the origin and of radius $\tilde{R}$, which contain the compact sets $K \subset \mathbb{R}^m$ and $K^{(k-1)} := K \times K^{(1)} \times \ldots K^{(k-1)} \subset \mathbb{R}^{mk}$, respectively. Then, we set $\tilde{K} := \overline{B}_{2\tilde{R}}, \tilde{K}' := \overline{B}_{3\tilde{R}}$ and $\tilde{K}'' := \overline{B}_{4\tilde{R}}$. We also assume that $f$ is extended to a map on $\Omega \times \tilde{K}''$, which still satisfies (a) and (b).

As a preliminary step, we need the following lemma.

**Lemma 5.4.** For each pair $(u_o(t), s_0)$, with $s_0 \in \mathcal{A}_{\text{init}}$ and $u_o : [0, T] \to \tilde{K} \subset \mathbb{R}^m$ satisfying the condition (c) of the Introduction, there exist:

- A unique solution $x^{(u_0, s_0)} : [0, T] \to \mathbb{R}^n$ with initial condition $j^{k-1}_{t=0}(x^{(u_0, s_0)}) = s_0$. If $k = 1$, this solution is $C^0$ with bounded measurable first derivative. If $k \geq 2$, the solution is piecewise $C^{2k-2}$.
- A unique solution $p^{(u_0, s_0)} : [0, T] \to \mathbb{R}^n$ with terminal conditions (1.6) - (1.8). This solution is of class $C^{k-1}$ and with bounded measurable $k$-th derivative.
Proof. Consider the auxiliary variables $x^i_\ell$, $1 \leq \ell \leq k-2$, $1 \leq i \leq n$, and the first order differential problem
\begin{equation}
\begin{aligned}
\frac{dx^i_1}{dt} &= x^i_1, & \frac{dx^i_2}{dt} &= x^i_2, & \ldots, & \frac{dx^i_{k-2}}{dt} &= x^i_{k-1}, \\
\frac{dx^i_{k-1}}{dt} &= f^i(t, x^j_\ell, \ldots, x^j_{k-1}, u^o_\ell(t)),
\end{aligned}
\end{equation}
with initial conditions $(x^i_\ell, x^j_\ell)|_{t=0}$ determined by the jet $s_\ell = (x^i_\ell, x^j_\ell)|_{t=0}$. This problem is equivalent to the system (2.5) with initial condition \[ J^{k-1}_{t=0}(x^{(u_\ell, s_\ell)}) = s_\ell. \] Hence the existence and uniqueness of a $C^{k-1}$ solution $x^{(u_\ell, s_\ell)}$ is a consequence of a well-known result on first order differential systems with control parameters in normal form (see e.g. \cite{[4] Th. 3.2.1}). The $(k-1)$-th derivative of this solution is absolutely continuous with bounded derivative. Moreover, for $k \geq 2$, on each subinterval on which $u_\ell(t)$ is $C^{k-1}$, the curve $x_{k-1}(t)$ is $C^{k-1}$. It follows that $x^{(u_\ell, s_\ell)}$ is piecewise $C^{2k-2}$.

The existence and uniqueness of $p^{(u_\ell, s_\ell)}$ is checked by considering (2.6) as a system of equations on the functions $p_j(t)$, depending on the control curve $u(t) := (u_\ell(t), j^{2k-2}(x^{(u_\ell, s_\ell)}))$ taking values in $K \times j^{2k-2}(Q, R)$. Since the curve $u_\ell(t)$ is bounded and measurable and $x^{(u_\ell, s_\ell)}$ is of class $C^{2k-2}$, the result follows from the above mentioned facts on systems with control parameters.

We are now ready to prove the following crucial result, which implies Theorem 1.1

**Theorem 5.5.** Let $u_\ell : [0, T] \rightarrow K$ be a measurable control curve and $s_\ell \in A_{\text{init}}$, as in Theorem 1.1, and assume that $(\tau_0, \omega_0) \in (0, T) \times K$ is a pair, in which $\tau_0$ satisfies the following condition:

- if $k = 1$, the time $\tau_0$ is one of the points for which
  \begin{equation}
  \lim_{\varepsilon \to 0^+} \frac{1}{\tau_0 - \tau_0 - \varepsilon} \int_{\tau_0 - \varepsilon}^{\tau_0} \left| f(t, x^{(s_\ell, u_\ell)}(t), u_\ell(t)) - f(t, x^{(s_\ell, u_\ell)}(\tau_0), u_\ell(\tau_0)) \right| dt = 0,
  \end{equation}
  i.e. $\tau_0$ is a density (Lebesgue) point of the map $f(t, x(t), u(t))$;
- if $k \geq 2$, $\tau_0$ is an inner point of a subinterval $I \subset [0, T]$ on which $u_\ell(t)$ is $C^{k-1}$.

Then there is a constant $M = M_{f, C, \tau_0, \omega_0} > 0$, depending on $f, C, \tau_0, \omega_0$, such that if
\begin{equation}
\kappa_0 := H^{(u_\ell, s_\ell, \tau_0)}(\omega_0) - H^{(u_\ell, s_\ell, \tau_0)}(u_\ell(\tau_0)) > 0,
\end{equation}
then there is a needle modification $u'(t)$ of $u_\ell(t)$ with peak time $\tau_0$ and ceiling $\omega_0$, satisfying
\begin{equation}
C(j_T^{k-1}(x^{(u', s_\ell)})) \leq C(j_T^{k-1}(x^{(u_\ell, s_\ell)})) - \frac{\kappa_0}{4M} < C(j_T^{k-1}(x^{(u_\ell, s_\ell)}))
\end{equation}
and $U_\ell = (u_\ell(t), s_\ell)$ cannot be an optimal control.

**Proof.** The proof is based on a three-step approximation procedure, which allows inferring the theorem from its previous “smooth” version, Theorem 5.3. For reader’s convenience, here is an outline of the arguments which we are going to use in case $k \geq 2$.

(I) First we introduce a one-parameter family of globally $C^{k-1}$ curves $v^{(\eta)} : [0, T] \rightarrow \hat{K}$, which tends to the piecewise $C^{k-1}$ curve $u_\ell(t)$ for $\eta \to 0$ with respect to the distance (2.8). This family is constructed in such a way that: (1) $v^{(\eta)}(t)$ coincides with $u_\ell(t)$
on a neighbourhood of $\tau_o$, for any $\eta$; (2) the associated controlled curves with initial datum $s_o$ tend to the curve $x^{(u_o, s_o)}$ in $C^{2k-2}$ norm.

(II) Second, for any fixed value $\delta_1$ for the parameter $\eta$ and the corresponding curve $v^{(\eta=\delta_1)}(t)$ of step (I), we consider a one-parameter family of polynomials (in the $t$-variable) $v^{(\delta_1, \eta)} : [0, T] \to \tilde{K}'$, which converge to $v^{(\delta_1)}(t)$ in the $C^{k-1}$-norm for $\eta \to 0$ and such that, the associated controlled curves $x^{(v^{(\delta_1, \eta)}, s_o)}(t)$ and $p^{(v^{(\delta_1, \eta)}, s_o)}(t)$, defined in Lemma 5.4, converge to the curves $x^{(v^{(\delta_1)}, s_o)}(t)$ and $p^{(v^{(\delta_1)}, s_o)}(t)$ in the $C^{2k-2}$ and $C^{k-1}$ norm, respectively.

(III) Third, for any fixed choice of $\delta_1$, $\delta_2 > 0$, we consider a one parameter family of smooth functions $f^{(\delta_1, \delta_2)} : \Omega \times \tilde{K}' \to \mathbb{R}^n$, which converges in the $C^{k-1}$ norm to the function $f(t, \tilde{r}^{k-1}(x), u)$ for $\eta \to 0$. The family is constructed in such a way that the smooth solutions $x^{(\delta_1, \delta_2, \eta)}(t)$ to the constraint given by $f^{(\delta_1, \delta_2, \eta)}$, the polynomial control curve $v^{(\delta_1, \eta=\delta_2)}(t)$ and the initial datum $s_o$, tend in the $C^{2k-2}$ norm to the solution of the constraint determined by $f$, $v^{(\delta_1, \delta_2)}(t)$ and $s_o$.

After these preliminary constructions, we show that:

(a) For any triple $(\delta_1, \delta_2, \delta_3)$, the controlled curve $x^{(\delta_1, \delta_2, \delta_3)}(t)$, determined by the smooth constraint given by $f^{(\delta_1, \delta_2, \delta_3)}$, the smooth control curve $v^{(\delta_1, \delta_2)}(t)$ and the initial datum $s_o$ satisfies the hypotheses of Theorem 5.3. This implies the existence of an appropriate needle modification $v^{(\delta_1, \delta_2, \eta)}(t)$ of the control curve $v^{(\delta_1, \delta_2)}(t)$ (with $\eta = e^{\delta_3}$ depending on $\delta_3$), which determines a controlled curve with a smaller terminal cost.

(b) We then show that the $\delta$-parameterised family of needle modifications $v^{(\delta, \delta, \eta)}(t)$ converges in the $L^1$ norm to a needle modification $u'(t)$ of $u_o(t)$, whose associated controlled curve $x^{(u', s_o)}(t)$ gives a terminal cost satisfying (5.12).

The scheme of the proof for the case $k = 1$ is similar, but requires a preparatory additional step. Before starting with the whole construction, we replace $u_o(t)$ by the curve $u^{(\delta_0)} : [0, T] \to \tilde{K}$, which is constant and equal to the value $u_o(\tau_o)$ on the interval $[\tau_o - \delta_0, \tau_o + \delta_0]$ and is equal to $u_o(t)$ at all other points. Then, working with the modified curve $u^{(\delta_0)}$, we perform the analogs of the three steps (I), (II) and (III). This leads to the construction of a $\delta$-parameterised family of needle modifications of the smooth control curves, which converge in $L^1$-norm to a needle modification $u'(t)$ of $u_o(t)$ with associated controlled curve $x^{(u', s_o)}$ with terminal cost satisfying (5.12), as desired.

Let us now proceed with the proof for the case $k \geq 2$. Let us consider the following derived system of order $2k - 1$ associated with (2.5). It is the system of equations which can be obtained from (2.5) by differentiating $k-1$ times with respect to $t$ and by replacing
any $k$-th order derivative $x^i_{(k)}$ by $f^i(t, j^{k-1}_t(x), u_o(t))$ at all places:

$$
x^i_{(k)} = f^i,
$$

$$
x^i_{(k+1)} = \frac{\partial f^i}{\partial t} + \sum_{r=0}^{k-2} x^i_{(r+1)} \frac{\partial f^i}{\partial x^i_{(r)}} + f^i \frac{\partial f^i}{\partial x^i_{(k-1)}} + \frac{\partial f^i}{\partial u_o} u_o^{(1)} ,
$$

(5.13)

Let us synthetically denote these equations by

$$
x^i_{(\ell)} = F^i_{(\ell)}(t, j^{k-1}_t(x), j^{k-1}_t(u_o)) , \quad k \leq \ell \leq 2k - 1 .
$$

(5.14)

Then, consider the analogue of the first order system (5.9) that gives the reduction to the first order of the last line of (5.13). Using the shorter notation $y := (x^i_j)$, this system can be written as

$$
\dot{y} = g(t, y(t), u_o(t), u_o^{(1)}(t), \ldots u_o^{(k-1)}(t))
$$

(5.15)

where $g$ is a map $g : \bar{\Omega} \times \bar{K}^{(k-1)} \to \mathbb{R}^{n(2k-1)}$, for an appropriate open set $\bar{\Omega} \subset \mathbb{R}^{n(2k-1)+1}$, which is uniquely determined by $f : \Omega \times K \to \mathbb{R}^n$. By the above described technical assumptions on $f$, we may assume that $g$ is defined on a larger domain $\bar{\Omega} \times \bar{\bar{K}}^{(k-1)}$ with $\bar{\bar{K}}^{(k-1)} \supset K^{(k-1)}$. Such extension is continuously differentiable in each variable $y^A$.

Finally, for any pair $(u(t), s)$, let us denote by $y^{(u,s)} : [0, T] \to \mathbb{R}^{n(2k-1)}$ the unique solution to (5.15), which is controlled by $u(t)$ and with the initial condition that correspond to the initial condition $s$ for $x(t)$.

We may now construct the one-parameter family described in (I), using the following

**Lemma 5.6.** Let $(t_0 = 0, t_1), (t_1, t_2), \ldots , (t_{P-1}, t_P = T)$ be the intervals on which $u_o(t)$ is $C^{k-1}$. Given $\eta > 0$ there exists a $C^k$ curve $v^{(\eta)} : [0, T] \to \bar{K}$, which coincides with $u_o(t)$ on the subintervals $\left(t_{i-1} + \frac{\eta}{P}, t_{i} - \frac{\eta}{P}\right)$ (and, in particular, on some neighbourhood of $\tau_o$) and such that

$$
\|x^{(u_o,s_o)}(t) - x^{(v^{(\eta)},s_o)}(t)\|_{C^{2k-2}} , \quad \|p^{(u_o,s_o)}(t) - p^{(v^{(\eta)},s_o)}(t)\|_{C^{k-1}} < \eta .
$$

(5.16)

**Proof.** It is almost immediate to realise that, for any choice of $\delta$, there exists a curve $\tilde{v}(t)$ which is $C^{k-1}$ over the whole domain $[0, T]$ and coincides with $u_o(t)$ on the subintervals $\left(t_{i-1} + \frac{\delta}{P}, t_{i} - \frac{\delta}{P}\right)$. This implies that all distances dist$(u_o, \tilde{v})$, dist$(u_o^{(1)}, \tilde{v}^{(1)})$, \ldots , dist$(u_o^{(k-1)}, \tilde{v}^{(k-1)})$ are less than $\delta$. Now, by the assumptions on $f$ and on its derivatives (which completely determine the function $g$ in (5.13)), there exists a unique solution to the reduced-to-the-first-order system (5.15) for the pair $(\tilde{v}, s_o)$. By Lemma 2.7 there are constants $\rho > 0$ and $\epsilon$ (depending on $f = (f^i)$ and on a cut-off function $\varphi$ as described in the statement of that lemma) such that, if $\delta \leq \rho$, then

$$
\sup_{t \in [0,T]} |y^{(u_o,s_o)}(t) - y^{(\tilde{v},s_o)}(t)| < \epsilon \text{dist}(u_o, \tilde{v}) < \epsilon \delta
$$

with dist$(\cdot , \cdot )$ given by (2.8). Selecting a $\delta_{\eta}$ such that $\delta_{\eta} < \min \left\{ \frac{\eta}{P}, \rho, \eta \right\}$, we get

$$
\sup_{t \in [0,T]} |y^{(u_o,s_o)}(t) - y^{(\tilde{v},s_o)}(t)| < \eta ,
$$

meaning that $v^{(\eta)}(t) := \tilde{v}$ satisfies the first upper bound in (5.16). By considering a possibly smaller $\delta_{\eta}$ also the second bound is satisfied.
This is because \( p^{(\tilde{v}, s_0)} \) is a solution of a system of controlled differential equations, where
the controls are given by the curve \( \tilde{v}(t) \) and the curve of the \((2k - 2)\)-jets of \( x^{(\tilde{v}, s_0)}(t) \). □

Let us now fix a control curve \( v^{(\delta_1)} \) as in the previous lemma. The family of polynomials
control curves in (II) is constructed using the next

**Lemma 5.7.** Given \( \eta > 0 \), there exists a polynomial curve \( v^{(\delta_1, \eta)}(t) \) in \( \tilde{K}' \supset \tilde{K} \) with

\[
|v^{(\delta_1, \eta)}(\tau_0) - v^{(\delta_1)}(\tau_0)| = |v^{(\delta_1, \eta)}(\tau_0) - u_0(\tau_0)| < \eta
\]

and such that the solutions \( x^{(v^{(\delta_1, \eta)}), s_0} \) and \( x^{(v^{(\delta_1)}, s_0)} \) to the differential problem (1.1)
(which is the same of \( 2.5 \)) and the associated curves \( p^{(v^{(\delta_1, \eta)}, s_0)}(t) \) and \( p^{(v^{(\delta_1)}, s_0)}(t) \),
defined in Lemma 5.4, satisfy

\[
\|x^{(v^{(\delta_1, \eta)}, s_0)} - x^{(v^{(\delta_1)}, s_0)}\|_{C^{2k-2}} , \quad \|p^{(v^{(\delta_1, \eta)}, s_0)} - p^{(v^{(\delta_1)}, s_0)}\|_{C^{k-1}} < \eta.
\]

**Proof.** By a well known result on interpolation of continuous functions (see e.g. [14, Thm. 7.1.6]),
we may consider a family of Bernstein polynomials converging to \( v^{(\delta_1)}(t) \) in \( C^{k-1} \).
Thus, for any choice of a sufficiently small \( \delta > 0 \), we may select a polynomial \( \tilde{v}^{(\delta_1)}(t) \) which satisfies (5.17),
takes values in \( \tilde{K}' \) and such that

\[
\int_{[0,T]} |v^{(\delta_1)}(t) - \tilde{v}^{(\delta_1)}(t)| dt < \delta , \quad 0 \leq \ell \leq k - 1.
\]

Now, by [4] Prop. 3.2.5 (i), there exists a \( \tilde{\delta} > 0 \) (depending on \( \eta \)) such that (5.19) implies
that the corresponding solutions \( y^{(v^{(\delta_1)}, s_0)}(t) \) and \( y^{(\tilde{v}^{(\delta_1)}, s_0)}(t) \) of the reduced-to-the-first-order system [5.15]
satisfy the inequality \( \sup_{t \in [0,T]} |y^{(v^{(\delta_1)}, s_0)}(t) - y^{(\tilde{v}^{(\delta_1)}, s_0)}(t)| < \eta \). Hence, the
polynomial \( v^{(\delta_1, \eta)}(t) := \tilde{v}^{(\delta_1)}(t) \) is such that the first upper bound in (5.18) holds.
Using the same arguments for the equations on the \( p(t) \), the second bound of (5.18) can be satisfied as well. □

It is now the turn to present the family of functions described in (III).

**Lemma 5.8.** Let \( v^{(\delta_1, \delta_2)} : [0, T] \to \tilde{K}' \) be one of the polynomials described in Lemma 5.7
converging to the \( C^{k-1} \) control curve \( v^{(\delta_1)} \). Let also \( x^{(v^{(\delta_1, \delta_2)}, s_0)}(t) \) be the unique solution
to (1.1) determined by the pair \( (v^{(\delta_1, \delta_2)}, s_0) \), as discussed in Lemma 5.4.

Then there is a \( \eta_0 > 0 \) such that for \( \eta \in (0, \eta_0] \) there are \( C^\infty \) maps \( f^{(\delta_1, \delta_2, \eta)} : \Omega \times \tilde{K}^n \to \mathbb{R}^n \)
satisfying the following conditions: they converge uniformly on compacta to \( f \) together
with all partial derivatives up to order \( k-1 \) for \( \eta \to 0 \) and, for each \( \eta \), the unique solution
\( x^{(v^{(\delta_1, \delta_2)}, s_0, \eta)}(t) \) to the differential problem

\[
x^{(k)} = f^{(\delta_1, \delta_2, \eta)} \left( t, j_{t}^{k-1}(x, v^{(\delta_1, \delta_2)}(t)) \right) , \quad j_{t=0}^{k-1}(x) = s_0
\]

and the associated curves \( p^{(v^{(\delta_1, \delta_2, \eta)}, s_0)}(t) \) defined in Lemma 5.4 satisfy

\[
\|x^{(v^{(\delta_1, \delta_2)}, s_0, \eta)} - x^{(v^{(\delta_1, \delta_2), s_0})}\|_{C^{2k-2}} < \eta , \quad \|p^{(v^{(\delta_1, \delta_2, \eta)}, s_0)} - p^{(v^{(\delta_1, \delta_2), s_0})}\|_{C^{k-1}} < \eta ,
\]

\[
\left| f^{(\delta_1, \delta_2, \eta)}(t, j_{t}^{k-1}(x^{(v^{(\delta_1, \delta_2)}, s_0, \eta)}(t), v^{(\delta_1, \delta_2)}(t)) - f(t, j_{t}^{k-1}(x^{(v^{(\delta_1, \delta_2), s_0)}(t), v^{(\delta_1, \delta_2)}(t))) \right| < \eta
\]

for any \( t \in [0, T] \).
Proof. Since \( v^{(\delta_1, \delta_2)}(t) \) is a polynomial, it satisfies the condition \((\gamma)\) of the Introduction and the corresponding solution \( x^{(v^{(\delta_1, \delta_2)}, s_0)}(t) \) to (1.1) determines a curve of jets and controls

\[
\gamma^{(v^{(\delta_1, \delta_2)}, s_0)}(k-1)(t) := \left( t, x^{(v^{(\delta_1, \delta_2)}, s_0)}(t), x_1^{(v^{(\delta_1, \delta_2)}, s_0)}(t), \ldots, x_{(k-1)}^{(v^{(\delta_1, \delta_2)}, s_0)}(t), v^{(\delta_1, \delta_2)}(t) \right) \in J^{k-1}(\mathbb{R}[\mathbb{R}])[0, T] \times \hat{K}',
\]

which is of course continuous and with compact image. Hence, there is a \( \eta_0 > 0 \) such that the set

\[
\Pi_{\eta_0} := \left\{ (t, s, u) \in J^{k-1}(\mathbb{R}^n[\mathbb{R}])[0, T] \times \mathbb{R}^n : |s_t - j^{k-1}_t(x^{(v^{(\delta_1, \delta_2)}, s_0)})| \leq \eta_0 \text{ and } |u - v^{(\delta_1, \delta_2)}(t)| \leq \eta_0 \right\}
\]

is compact and with \( u \in \bigcup_{v \in \overline{B}_{\eta_0}} B_{\eta_0}(v) \subset \hat{K}'' = B_{\eta_0}. \) By the assumptions \((\alpha)\) and \((\beta)\) on \( f \), there is a constant \( \mathcal{C} \) such that

\[
\|f\|_{C^{k-1}(\Pi_{\eta_0})}, \max_{0 \leq w_0 + |w| + |\ell| = k, 0 \leq w_0 + |w| \leq k-1, (t, s, u) \in \Pi_{\eta_0}} \left\{ \left| \frac{\partial^{w_0+|w|+|\ell|} f}{(\partial t)^{w_0} (\partial u)^w (\partial x_r)^\ell (t, s, u)} \right| \right\} < \mathcal{C}, \quad (5.22)
\]

where we denote

\[
\frac{\partial^{|w|}}{(\partial u)^w} = \frac{\partial^{|w|}}{(\partial u_1)^{w_1} \ldots (\partial u_m)^{w_m}}, \quad \frac{\partial^{|\ell|}}{(\partial x_r)^\ell} = \frac{\partial^{|\ell|}}{(\partial x_1)_{\ell_1} \ldots (\partial x_n)_{\ell_n}},
\]

with \( w \) and \( \ell \) multiindices \( w = (w_1, \ldots, w_m) \) and \( \ell = (\ell_1, \ldots, \ell_1, \ldots, \ell_{k-1}, \ell_1, \ell_2, \ldots, \ell_{|\ell|-1}, \ldots, \ell_n, \ldots, \ell_{|\ell|-1}) \). Moreover, for any \( \delta > 0 \), there is \( \hat{f}^{(\delta_1, \delta_2, \delta)} \in C^\infty(J^{k-1}(\mathbb{R}^n[\mathbb{R}])[0, T] \times \hat{K}'') \) such that

\[
\|\hat{f}^{(\delta_1, \delta_2, \delta)} - f\|_{C^{k-1}(\Pi_{\eta_0})} < \delta, \quad (5.23)
\]

(it is a consequence of a classical approximation procedure; see e.g. [19 Ch. 15]).

We now want to prove that there exists a constant \( \mathcal{C} \), depending on \( k, n, \mathcal{L} \) and

\[
\sup \left\{ |j^{k-1}_t(x) : j^{k-1}_t(x) \in \Pi_{\eta_0}(j^{k-1}_t(x^{(v^{(\delta_1, \delta_2)}, s_0)}), t \in [0, T]) \right\}, \quad (5.24)
\]

with the following property: for any \( \eta \in (0, \eta_0] \) with \( \eta_0 < \frac{\delta}{\max_{|w|+|\ell|+1} \left\{ \frac{\partial^{|w|}}{(\partial u)^w} \left( \frac{\partial^{|\ell|}}{(\partial x_r)^\ell} \right) \right\}} \), if \( \delta_\eta \) is sufficiently small, then the corresponding function \( f^{(\delta_1, \delta_2, \delta_\eta)} := \hat{f}^{(\delta_1, \delta_2, \delta_\eta)} \) satisfies (5.23) and the associated solution \( x^{(0)}(t) := x^{(v^{(\delta_1, \delta_2)}, s_0)}(t) \) to the system (5.20) satisfies (5.21).

To see this, consider the two systems of first order of the form (5.9) (obtained by introducing the auxiliary variables \( y'_i \)), which correspond to the derived differential systems of order \( 2k - 1 \) associated with the system (1.1) and the system \( x^{(k)} = f^{(\delta_1, \delta_2)}(t, j^{k-1}_t(x), v^{(\delta_1, \delta_2)}(t)) \). Then, for each choice of \( \delta \), let us denote by

\[
y^{(v^{(\delta_1, \delta_2)}, s_0)}(t) = (y^1(t), y^1_1(t), \ldots, y^1_{2k-2}(t)),
\]

\[
y^{(v^{(\delta_1, \delta_2)}, s_0)}(t) = (y^1(t), y^1_1(t), \ldots, y^1_{2k-2}(t))
\]
the solutions to such two systems, corresponding to the solutions $x^{(v(\delta_1, \delta_2), s_0)}(t)$ and $x^{(v(\delta_1, \delta_2), s_0; \delta)}(t)$, respectively. We synthetically denote the two systems of equations, of which they are solutions, by

$$
y^i_{t-1} = g^i_\ell(t, y^i_m, v^{(\delta_1, \delta_2)}(t)) \quad \text{and} \quad \hat{y}^i_{t-1} = g^i_\ell(t, y^i_m, v^{(\delta_1, \delta_2)}(t))$$

with $1 \leq \ell \leq 2k - 1$.

By construction $g^i_m(t, y^i_j, v^{(\delta_1, \delta_2)}(t)) = g^i_m(t, y^i_j, v^{(\delta_1, \delta_2)}(t)) = y^i_m$ for any $0 \leq m \leq 2k - 2$. On the other hand, the functions $g^{(\delta)i}_{2k-1}(t, y^i_j, v^{(\delta_1, \delta_2)}(t))$ and $g^{(\delta)i}_{2k-1}(t, y^i_j, v^{(\delta_1, \delta_2)}(t))$ are in general different, because they are given by the $(k - 1)$-th order total derivatives of the functions $f$ and $f^{(\delta_1, \delta_2, \delta)}$, respectively, evaluated at the points $(t, y^i_m, v^{(\delta_1, \delta_2)}(t))$.

The initial values of the curves $y(t) := y^{(v(\delta_1, \delta_2), s_0)}(t)$ and $y^{(\delta)}(t) := y^{(v(\delta_1, \delta_2), s_0; \delta)}(t)$ are denoted by $\overline{y} = (\overline{y}_1, \ldots, \overline{y}_{2k-2})$ and $\overline{y}^\delta = (\overline{y}_{1i}, \overline{y}_{2i}, \ldots, \overline{y}_{2k-2i})$, respectively. Note that the components of $\overline{y}$ and $\overline{y}^\delta$, determined by the derivatives up to order $k - 1$ at $t = 0$ of the two curves, are the same and uniquely determined by $s_0$. By construction of the derived system of order $2k - 1$, the remaining components of $\overline{y}$ and $\overline{y}^\delta$ might be different, but also such that $\overline{y}^\delta \to \overline{y}$ for $\delta \to 0$.

Let us denote $z^i_r(t) = y^i_\ell(t) - y^i_r(t)$ for any $0 \leq \ell \leq 2k - 2$. We observe that, for any $t \in [0, T]$

$$z^i_r(t) = z^i_{r+1}(t) \quad \text{if} \quad 0 \leq r \leq 2k - 3 ,$$

$$z^i_{2k-2}(t) = g^{(\delta)i}_{2k-1}(t, y^i_\ell(t), v^{(\delta_1, \delta_2)}(t)) - g^{\delta i}_{2k-1}(t, y(t), v^{(\delta_1, \delta_2)}(t))$$

and therefore

$$|z^i_r(t)| \leq \sum_{t, i} |z^i_r(t)| , \quad 0 \leq r \leq 2k - 3 ,$$

$$|z^i_{2k-2}(t)| \leq |g^{(\delta)i}_{2k-1}(t, y^i_\ell(t), v^{(\delta_1, \delta_2)}(t)) - g^{\delta i}_{2k-1}(t, y(t), v^{(\delta_1, \delta_2)}(t))| +$$

$$+ |g^{\delta i}_{2k-1}(t, y^i_\ell(t), v^{(\delta_1, \delta_2)}(t)) - g^{\delta i}_{2k-1}(t, y(t), v^{(\delta_1, \delta_2)}(t))| \leq$$

$$\leq \text{Const} \cdot \mathcal{C} \sum_{t, i} |z^i_r(t)| ,$$

$$\frac{d}{dt} \sum_{t, i} |z^i_r(t)| \leq \sum_{t, i} |z^i_r(t)| \leq \mathcal{C}(\delta + \max\{\mathcal{L}, 1\} \sum_{t, i} |z^i_r(t)|) ,$$

where Const is a constant, which depends only on $\mathcal{L}$ and $\delta$ and $\mathcal{C} := n(2k - 1)\text{Const}$.

Hence if we take $\delta = \delta_0$ so that $\sum_{t, i} |z^i_r(0)| \leq \eta^2 \leq \eta_0$ and $\delta_0 \leq \frac{\eta_0 e^{\epsilon \max\{\mathcal{L}, 1\}T}}{4\epsilon T}$, then by Gronwall’s inequality we obtain that

$$|y^{2k-2}_{t}(x^{(v(\delta_1, \delta_2), s_0; \delta_0)}) - y^{2k-2}_{t}(x^{(v(\delta_1, \delta_2), s_0))})| \leq \sum_{t, i} |z^i_r(0)| \leq$$

$$\leq (\delta_0 e T + \sum_{t, i} |z^i_r(0)|) e^{\epsilon \max\{\mathcal{L}, 1\}T} \leq \delta_0 e T e^{\epsilon \max\{\mathcal{L}, 1\}T} + \eta_0 e^{\epsilon \max\{\mathcal{L}, 1\}T} \leq \frac{\eta}{2} < \eta .$$

From this and $\eqref{5.23}$, all three estimates in $\eqref{5.21}$ follow.
We are now ready to conclude the proof following the arguments described in (a) and (b) above. Using Lemmas \[5.6, 5.7\] and \[5.8\] we may consider the families of control curves and functions, parameterised by a positive \(\delta\) tending to 0,

\[v^\delta(t) := v^{(\delta, \delta)}(t), \quad f^\delta(t, x^\ell(t), u) := f^{(\delta, \delta, \delta)}(t, x^\ell(t), u).\]

They have the following properties:

- each map \(v^\delta : [0, T] \to K^\prime\) is polynomial, it satisfies \(|v^\delta(\tau_0) - u_0(\tau_0)| < \delta\) and the corresponding solution \(x^{(v^\delta, s_0)}(t)\) to the equations (1.1) satisfies

\[\|x^{(u_0, s_0)} - x^{(v^\delta, s_0)}\|_{C^{2k-2}} < \delta;\] \hspace{1cm} (5.25)

- \(f^\delta : \Omega \times K^\prime \to \mathbb{R}^n\) is a smooth function and the solution \(x^{(v^\delta, s_0, \delta)}(t)\) to the differential problem

\[x^{i(\delta)}(t) = f^{\delta i}(t, j_t^{-1}(x^\delta(t)), v^\delta(t))\]

satisfies

\[\|x^{(v^\delta, s_0, \delta)}(t) - x^{(u_0, s_0)}(t)\|_{C^{2k-2}} < \delta\]

and

\[|f^\delta(t, j_t^{-1}(x^{(v^\delta, s_0, \delta)}), v^\delta(\tau_0)) - f(t, j_t^{-1}(x^{(u_0, s_0)}), u_0(\tau_0))| < \delta.\] \hspace{1cm} (5.27)

Therefore, for any sufficiently small \(\delta, \varepsilon > 0\) we may also consider:

- the real function \(H^{\delta} : K \to \mathbb{R}\) which is defined by

\[H^{\delta}(\omega) := p_i^{(v^\delta, s_0, \delta)}(\tau_0) f^{\delta i}(\tau_0, j_t^{-1}(x^{(v^\delta, s_0, \delta)}), \omega),\]

where we denote by \(p^{(v^\delta, s_0, \delta)}(t)\) the solution to (2.6) with \(f\) replaced by the smooth \(\delta\) and determined by the control curve \(v^\delta\) and the initial value \(s_0\);

- the needle modification \(v^\delta\varepsilon := v^{\delta(\tau_0, \omega_0, \varepsilon)}\) of the polynomial curve \(v^\delta(t)\), with peak time \(\tau_0\), ceiling value \(\omega_0\) and width \(\varepsilon\);

- the needle modification \(u_0^\varepsilon := u_0(\tau_0, \omega_0, \varepsilon)\) of the (merely piecewise \(C^{k-1}\)) \(u_0(t)\), also with peak time \(\tau_0\), ceiling value \(\omega_0\) and width \(\varepsilon\).

By the Lemmas \[5.6, 5.7\] and \[5.8\] for \(\delta \to 0\) the functions \(f^\delta\), the curves \(p^{(v^\delta, s_0, \delta)}(t)\) and the curves of jets \(j_t^{-1}(x^{(v^\delta, s_0, \delta)})\) tend uniformly on compacta to the map \(f\), to the curve \(p^{(u_0, s_0)}(t)\) and to the curve of jets \(j_t^{-1}(x^{(u_0, s_0)})\), respectively. Therefore, if we set

\[\kappa^\delta_0 := H^{\delta}(\omega_0) - H^{\delta}(\tau_0)\] \hspace{1cm} (5.29)

we have that \(\lim_{\delta \to 0} \kappa^\delta_0 = \kappa_0 > 0\) and hence there is \(\delta_0 > 0\) such that \(\kappa^\delta_0 > 0\) for any \(\delta \in (0, \delta_0]\).

We now observe that the (restrictions to an appropriate relatively compact neighbourhood of the \(k\)-jets of the curve \(\gamma(t) = (t, x^{(u_0, s_0)}(t), p^{(u_0, s_0)}(t))\) of the) Lagrangians

\[L^{(\delta)}(t, x^\ell, \ldots, x^\ell(k), p_j, u) := p_j(x^j(k) - f^\delta j(t, x^\ell, \ldots, x^\ell(k-1), u))\]

and the one-parameter families of control pairs \(U^{(\delta)} = (v^\delta, (s_0, s_0^{(v^\delta, s_0)}))\) are such that the conditions (A'), (B) and (D) of Remark \[1.4\] are satisfied with \(\sigma_0^{(0)} := (s_0, s_0^{(u_0, s_0)})\), \(u_0^{(0)} := u_0\) and that the limit Lagrangian for \(\delta \to 0\) is

\[L^{(0)}(t, x^\ell, \ldots, x^\ell(k), p_j, u) := p_j(x^j(k) - f^j(t, x^\ell, \ldots, x^\ell(k-1), u)).\]
Moreover, since each Lagrangian (5.30) is smooth, for each \( \delta \) we may consider a one-parameter family of good needle variations \( \text{Needle}^{(\tau_0, \omega_0, \Sigma^{(\delta)}, \varepsilon_0)}_{\gamma_0} \) for the pair \((\tau_0, \omega_0)\) as defined in Lemma 5.1. By definition of the good needle variations, the family of initial data maps \( \Sigma^{(\delta)} \) converges uniformly to a limit initial data map \( \Sigma^{(0)} \) and satisfies conditions (C) and (F) of Remark 4.4. We may also consider an appropriate family of smooth cost functions \( C^{\delta} \), which depend on the \((k - 1)\)-jets at \( t = T \) of the controlled curves and converge in the \( C^1 \) norm to the cost function \( C = C^{(0)} \) on an appropriate relatively compact neighborhood of the end \( k - 1 \)-jets of the curves of the needle variations determined by \( \text{Needle}^{(\tau_0, \omega_0, \Sigma^{(\delta)}, \varepsilon_0)}_{\gamma_0} \). This would imply that also condition (G) is satisfied.

We finally observe that the functions \( V^{(\delta)} \), associated with the above described Lagrangians, control pairs and initial valued maps, satisfy also condition (E) of Remark 4.4. This is in fact a direct consequence of the property that, by construction of the polynomial curves \( v^{\delta}(t) = v^{(\delta, \delta)}(t) \), the \( v^{\delta}(t) \) converge in \( C^{k-1} \) norm to the function \( u_o(t) \) on a fixed interval containing \( \tau_o \). Since all conditions of Remark 4.4 are satisfied, we infer the existence of a constant \( M > 0 \) and a needle modification \( u'(t) \) of \( u_o(t) \), with peak time \( \tau_o \) and ceiling \( \omega_o \), such that \( (5.12) \) holds. This concludes the proof with \( k \geq 2 \).

Let us now focus on the case \( k = 1 \), i.e. on the situation in which Theorem 1.1 reduces to the classical PMP. As the reader will shortly see, the approximation procedure which we used for \( k > 1 \) is valid also for \( k = 1 \), but requires some nontrivial adjustments. We give here such adjustments in full detail mainly with the following purposes: (a) showing that, in the classical setting, our approximation technique has the same power of the standard approach; (b) paving the way for future developments in different contexts – see §7.

For the time being, we assume that \( f(t, x, u) \) is continuously differentiable not only with respect to \( x \) but also with respect to \( t \) (we show how to remove this assumption later). As announced above, for each sufficiently small \( \delta_0 \) let us denote by \( u^{(\delta_0)} : [0, T] \to \hat{K} \) the control curve defined by

\[
    u^{(\delta_0)}(t) = \begin{cases} 
    u_o(\tau_o) & \text{if } t \in [\tau_o - \delta_0, \tau_o], \\
    u_o(t) & \text{otherwise}.
    \end{cases}
\]

Note that \( \text{dist}(u_o, u^{(\delta_0)}) \leq \delta_0 \). Hence, by the usual circle of ideas, the corresponding control curve \( x^{(u^{(\delta_0)}, s_0)}(t) \) and the associated curve \( p^{(u^{(\delta_0)}, s_0)}(t) \) uniformly converge to the curves \( x^{(u_o, s_0)}(t) \) and \( p^{(u_o, s_0)}(t) \), respectively, for \( \delta_0 \to 0 \). So, by continuity of \( f \) and \( C \), if we set

\[
    \kappa^{(\delta_0)} :=
    p_i^{(u^{(\delta_0)}, s_0)}(\tau_o) f^i(\tau_o, x^{(u^{(\delta_0)}, s_0)}(\tau_o), \omega_o) - p_i^{(u^{(\delta_0)}, s_0)}(\tau_o) f^i(\tau_o, x^{(u^{(\delta_0)}, s_0)}(\tau_o), u^{(\delta_0)}(\tau_o)) =
    p_i^{(u^{(\delta_0)}, s_o)}(\tau_o) f^i(\tau_o, x^{(u^{(\delta_0)}, s_0)}(\tau_o), \omega_o) - p_i^{(u^{(\delta_0)}, s_0)}(\tau_o) f^i(\tau_o, x^{(u^{(\delta_0)}, s_0)}(\tau_o), u_o(\tau_o)) ,
\]

we directly obtain that \( \lim_{\delta_0 \to 0} \kappa^{(\delta_0)} = \kappa_o \) and \( \lim_{\delta_0 \to 0} C(x^{(u^{(\delta_0)}, s_0)}(T)) = C(x^{(u_o, s_0)}(T)) \).

Now, let us consider the following analog of Lemma 5.6.
Lemma 5.9. Given the curve \(u^{(δ_0)}\), for any \(η > 0\), there exists a continuous curve \(v^{(δ_0,η)} : [0, T] \rightarrow \hat{K}\), which coincides with \(u^{(δ_0)}(t)\) on the interval \([τ_o - δ, τ_o]\) and such that
\[
dist(u^{(δ_0)}, u^{(δ_0,η)}) < η, \quad \|x(u^{(δ_0)}(s_o)) - x(u^{(δ_0,η)}(s_o))\|_{C^0} < η, \quad \|P(u^{(δ_0)}(s_o)) - P(u^{(δ_0,η)}(s_o))\|_{C^0} < η. \tag{5.33}
\]

Proof. We recall that, by the Lusin Theorem (see e.g. [1] p.14]), for any choice of \(\tilde{δ} > 0\) there exists a \(v^{(\tilde{δ})} \in C^0([0, T], \mathbb{R}^m)\) such that
\[
\sup_{t \in [0, T]} |v^{(\tilde{δ})}(t)| \leq \sup_{t \in [0, T]} |u_o(t)| \quad \text{and} \quad dist(u^{(δ_0)}, v^{(\tilde{δ})}) < \tilde{δ}. \tag{5.34}
\]
Note that such a curve can be taken equal to \(u^{(δ_0)}(t)\) on the interval \([τ_o - δ, τ_o]\). Indeed, one can start with a continuous curve \(v^{(\tilde{δ})}(t)\) satisfying the first inequality in (5.33) and with \(dist(u^{(δ_0)}(s_o), v^{(\tilde{δ})}(s_o)) < \frac{1}{2}\). Then one can modify \(v^{(\tilde{δ})}(t)\) just on the closed interval \([τ_o - δ - \frac{1}{4}, τ_o + \frac{1}{4}]\), determining a continuous curve \(v^{(δ)}\) that satisfies also the desired additional requirement. Now, given \(η\), by taking \(\tilde{δ}\) sufficiently small, the curve \(v^{(δ_0,η)}(t) := v^{(δ)}(t)\) satisfies all required inequalities. \(\blacksquare\)

The same arguments of Lemma 5.7 imply that, for any continuous control curve \(v^{(δ_0,δ_1)}(t)\) and any \(η > 0\) there exists a polynomial curve \(v^{(δ_0,δ_1,η)}(t)\) in \(\hat{K'} \supset \hat{K}\) with
\[
|v^{(δ_0,δ_1,η)}(τ_o) - v^{(δ_0,δ_1)}(τ_o)| = |v^{(δ_0,δ_1,η)}(τ_o) - u_o(τ_o)| < η \tag{5.35}
\]
and such that the solutions \(x^{(v^{(δ_0,δ_1)}(s_o))}\) and \(x^{(v^{(δ_0,δ_1,η)}(s_o))}\) and the associated curves \(p^{(v^{(δ_0,δ_1)}(s_o))}(t)\) and \(p^{(v^{(δ_0,δ_1,η)}(s_o))}(t)\) satisfy
\[
\|x^{(v^{(δ_0,δ_1)}(s_o))} - x^{(v^{(δ_0,δ_1,η)}(s_o))}\|_{C^0}, \quad \|p^{(v^{(δ_0,δ_1)}(s_o))}(t) - p^{(v^{(δ_0,δ_1,η)}(s_o))}(t)\|_{C^0} < η. \tag{5.36}
\]
This can be considered as the analog of (II) for \(k = 1\) and with control curve given by \(u^{(δ_0)}(t)\) in place of \(u_o(t)\). It is also quite immediate to check that, for any given polynomial control curve \(v^{(δ_0,δ_1,δ_2)}(t)\) and for any \(η > 0\), there exists a smooth function \(f^{(δ_0,δ_1,δ_2)} : Ω \times \hat{K}' \rightarrow \mathbb{R}^n\), with the properties given in Lemma 5.8 for \(k = 1\). In other words, the analog of (III) is also true.

At this point, if for any \(δ = δ_0 > 0\) we define
\[
u_o^{(δ)}(t) := u^{(δ_0)}(t), \quad \tilde{u}_o^{(δ)}(t) := v^{(δ,δ)}(t), \quad v^{(δ)}(t) := v^{(δ,δ,δ)}(t), \quad f^{(δ,δ,δ)}(t, x, u) := f^{(δ,δ,δ,δ)}(t, x, u). \tag{5.37}
\]
then, for all \(δ\) sufficiently small, we have:

- each \(u^{(δ)}_o : [0, T] \rightarrow K\) is a measurable curve which coincides with \(u_o(t)\) outside of \([τ_o - δ, τ_o]\) and it is constant and equal to \(u_o(τ_o)\) on such interval;
- each \(\tilde{u}_o^{(δ)} : [0, T] \rightarrow \hat{K}\) is a continuous curve which is constant equal to \(u_o(τ_o)\) on \([τ_o - δ, τ_o]\) and with \(dist(\tilde{u}_o^{(δ)}, u^{(δ)}_o) < δ\);
- each \(v^{(δ)} : [0, T] \rightarrow \hat{K}'\) is a polynomial with \(|v^{(δ)}(τ_o) - \tilde{u}_o^{(δ)}(τ_o)| = |v^{(δ)}(τ_o) - u_o(τ_o)| < δ\);
- \(f^{(δ)} : Ω \times \hat{K}' \rightarrow \mathbb{R}^n\) is a smooth function and the solution \(x^{(v^{(δ)}, s_o)}(t)\) to
\[
x^{(1)}(t) = f^{(δ)} \left(t, x(t), v^{(δ)}(t)\right), \quad x(0) = s_o \tag{5.38}
\]
satisfies
\[
\|x^{(v_0, s_0; 0)}(t) - x^{(\tilde{v}_0, s_0)}(t)\|_{\mathcal{C}^0}, \|x^{(v_0, s_0; 0)}(t) - x^{(u_0, s_0)}(t)\|_{\mathcal{C}^0} < \delta,
\]
\[
\left| f^\delta(t, x^{(v_0, s_0; 0)}(t), v^\delta(t)) - f(t, x^{(\tilde{v}_0, s_0)}(t), \tilde{v}_0(t)) \right| < \delta, \quad \text{for any } t \in [0, T],
\]
\[
\left| f^\delta(t, x^{(v_0, s_0; 0)}(t), \omega) - f(t, x^{(u_0, s_0)}(t), \omega) \right| < \delta, \quad \text{for any } \omega \in K.
\]

Therefore, for any sufficiently small \( \delta, \varepsilon > 0 \) we may consider:

- the real function \( H^\delta : K \to \mathbb{R} \) defined by
  \[
  H^\delta(\omega) := p_i^{(v_0, s_0; 0)}(\tau_0) f^\delta i(\tau_0, x^{(v_0, s_0; 0)}(\tau_0), \omega),
  \]
  where, as usual, we denote by \( p_i^{(v_0, s_0; 0)}(t) \) the solution to (2.6) with \( f \) replaced by the smooth \( f^\delta \) and determined by the control curve \( v^\delta \) and the initial value \( s_0; 
- the needle modification \( v^\delta,\varepsilon(t) := v^\delta(\tau_0, \omega_0, \varepsilon) \) of the polynomial curve \( v^\delta(t) \), with peak time \( \tau_0 \), ceiling value \( \omega_0 \) and width \( \varepsilon; 
- the needle modification \( u_0^\delta,\varepsilon(t) := u_0^\delta(\tau_0, \omega_0, \varepsilon) \) of \( u_0^\delta(t) \) with peak time \( \tau_0 \), ceiling value \( \omega_0 \) and width \( \varepsilon; 
- the needle modification \( u_0^\varepsilon(t) := u_0^\varepsilon(\tau_0, \omega_0, \varepsilon) \) of \( u_0(t) \) with peak time \( \tau_0 \), ceiling value \( \omega_0 \) and width \( \varepsilon. 

Note that for \( \delta < \varepsilon \), the needle modifications \( u_0^\delta,\varepsilon(t) \) and \( u_0^\varepsilon(t) \) coincide.

By the Lemmas 2.1, 5.9 and the above remarks, for \( \delta \to 0 \) the functions \( f^\delta \), the curves \( x^{(v_0, s_0; 0)}(t) \) and \( p_i^{(v_0, s_0; 0)}(t) \) tend uniformly on compacta to the map \( f \) and to the curves \( x^{(u_0, s_0)}(t) \) and \( p_i^{(u_0, s_0)}(t) \), respectively. Therefore, setting \( \kappa_0^\delta \) as in (5.29), we have that \( \lim_{\delta \to 0} \kappa_0^\delta = \kappa_0 > 0 \) and hence there is \( \delta_0 > 0 \) such that \( \kappa_0^\delta > 0 \) for any \( \delta \in (0, \delta_0]. \)

As for the previous case, it is straightforward to check that the conditions (A'), (B), (C), (D) and (F) of Remark 4.4 are satisfied also for the new \( \delta \)-parameterised family of control curves and Lagrangians \( L^\delta \), determined by the functions \( f^\delta \). As before, we can also consider an appropriate family of smooth cost functions \( C^\delta \) which converge in the \( C^1 \) norm to \( C = C^{(0)} \) on an appropriate relatively compact neighbourhood of the endpoints of the curves of the needle variations. So, in order to conclude, it remains to prove that also condition (E) is satisfied, i.e. that the functions \( V^\delta(\varepsilon) \) tend to some function, which is an infinitesimal for \( \varepsilon \to 0 \). In fact, we claim that the \( V^\delta(\varepsilon) \) tend uniformly on compacta of \( (0, \varepsilon_0] \) to the function \( V_0(\varepsilon) \) defined by
\[
V_0(\varepsilon) = \frac{1}{\varepsilon} \int_{\tau_0 - \varepsilon}^{\tau_0} \left( p_i^{(u_0, s_0)}(t) f^i(t, x^{(u_0, s_0)}(t), u_0(\tau_0)) - p_i^{(u_0, s_0)}(t) f^i(t, x^{(u_0, s_0)}(t), u_0(t)) \right) dt,
\]
and that such a function is an infinitesimal for \( \varepsilon \to 0 \), as required.

To prove this, on a fixed interval \( [\rho, \varepsilon_0], \rho > 0 \), we need to show that
\[
|V^\delta(\varepsilon) - V_0(\varepsilon)| =
\]
\[
= \frac{1}{\varepsilon} \int_{\tau_0 - \varepsilon}^{\tau_0} \left( p_i^{(v_0, s_0; 0)}(t) f^\delta i(t, x^{(v_0, s_0; 0)}(t), v^\delta(\tau_0)) - p_i^{(v_0, s_0; 0)}(t) f^\delta i(t, x^{(v_0, s_0; 0)}(t), v^\delta(t)) -
\]
\[
- p_i^{(u_0, s_0)}(t) f^i(t, x^{(u_0, s_0)}(t), u_0(\tau_0)) + p_i^{(u_0, s_0)}(t) f^i(t, x^{(u_0, s_0)}(t), u_0(t)) \right) dt \]
\[
(5.42)
\]
is uniformly bounded by some constant $C_\delta$ tending to 0 for $\delta \to 0$. In this regard, we recall that:

- $\hat{u}_o^\delta(\tau_o) = u_o^\delta(\tau_o) = u_o(\tau_o)$ for any $\delta$ and $v^\delta(\tau_o)$ tends to $u_o(\tau_o)$ for $\delta \to 0$;
- the maps $x^{(v^\delta,s_o,\delta)}(t)$, $p^{(v^\delta,s_o,\delta)}(t)$ and $f^\delta(t,x,u)$ converge to $x^{(u_o,s_o)}(t)$, $p^{(u_o,s_o)}(t)$ and $f(t,x,u)$, respectively, in the $C^0$ norm.

Due to this, it suffices to check that the function $\Delta V^\delta(\varepsilon) : [\rho, \varepsilon_o] \to \mathbb{R}$ defined by

$$
\Delta V^\delta(\varepsilon) := \frac{1}{\varepsilon} \int_{\tau_o - \varepsilon}^{\tau_o} \left( p_i^{(u_o,s_o)}(t)f^i(t, x^{(u_o,s_o)}(t), u_o(\tau_o)) - p_i^{(u_o,s_o)}(t)f^i(t, x^{(u_o,s_o)}(t), \hat{u}_o^\delta(t)) - p_i^{(u_o,s_o)}(t)f^i(t, x^{(u_o,s_o)}(t), u_o(\tau_o)) + p_i^{(u_o,s_o)}(t)f^i(t, x^{(u_o,s_o)}(t), u_o(t)) \right) dt = $$

$$
= \frac{1}{\varepsilon} \int_{\tau_o - \varepsilon}^{\tau_o} \left( p_i^{(u_o,s_o)}(t)f^i(t, x^{(u_o,s_o)}(t), \hat{u}_o^\delta(t)) - p_i^{(u_o,s_o)}(t)f^i(t, x^{(u_o,s_o)}(t), u_o(t)) \right) dt + \text{Const } \delta
$$

(5.43)

is such that $\sup_{\varepsilon \in [\rho, \varepsilon_o]} \Delta V^\delta(\varepsilon)$ tends to 0 for $\delta \to 0$. In order to prove this, we first observe that, since $\hat{u}_o^\delta(t)$ differs from $u_o^\delta(t)$ just on a set of measure less than $\delta$, we may replace $\hat{u}_o^\delta(t)$ by $u_o^\delta(t)$ and get the inequality

$$
\Delta V^\delta(\varepsilon) \leq \frac{1}{\varepsilon} \int_{\tau_o - \varepsilon}^{\tau_o} \left( p_i^{(u_o,s_o)}(t)f^i(t, x^{(u_o,s_o)}(t), u_o^\delta(t)) - p_i^{(u_o,s_o)}(t)f^i(t, x^{(u_o,s_o)}(t), u_o(t)) \right) dt + \text{Const } \delta
$$

(5.44)

for an appropriate constant Const. Secondly, we recall that $\{ u_o^\delta(t) \neq u_o(t) \} = [\tau_o - \delta, \tau_o]$ and that $u_o^\delta_{\varepsilon} = u_o(\tau_o)$. Hence, if we set $\text{Const} := \max_{t \in [0,T]} |p^{(u_o,s_o)}(t)|$ and take $\delta < \rho \leq \varepsilon$, we get

$$
\Delta V^\delta(\varepsilon) \leq \frac{1}{\varepsilon} \int_{\tau_o - \varepsilon}^{\tau_o} \left( p_i^{(u_o,s_o)}(t)f^i(t, x^{(u_o,s_o)}(t), u_o^\delta(t)) - f^i(t, x^{(u_o,s_o)}(t), u_o(t)) \right) dt + \text{Const } \delta
$$

$$
\leq \text{Const} \varepsilon \int_{\tau_o - \varepsilon}^{\tau_o} \left| f(t, x^{(u_o,s_o)}(t), u_o^\delta(t)) - f(t, x^{(u_o,s_o)}(t), u_o(t)) \right| dt + \text{Const } \delta
$$

$$
= \text{Const} \varepsilon \int_{\tau_o - \delta}^{\tau_o} \left| f(t, x^{(u_o,s_o)}(t), u_o^\delta(t)) - f(t, x^{(u_o,s_o)}(t), u_o(t)) \right| dt + \text{Const } \delta
$$

$$
\leq \text{Const} \varepsilon \int_{\tau_o - \delta}^{\tau_o} \left| f(t, x^{(u_o,s_o)}(t), u_o^\delta(t)) - f(t, x^{(u_o,s_o)}(t), u_o(t)) \right| dt + \text{Const } \delta
$$

$$
+ \text{Const } \delta
$$

(5.45)
It follows that
\[
\Delta V^{(\delta)}(\varepsilon) \leq \frac{\text{Const}}{\delta} \int_{\tau_0 - \delta}^{\tau_0} \left| f(t, x^{(u_\delta, s_\delta)}(t), u_\delta(t)) - f(t, x^{(u_\delta, s_\delta)}(\tau_0), u_\delta(\tau_0)) \right| dt + \\
+ \frac{\text{Const}}{\delta} \int_{\tau_0 - \delta}^{\tau_0} \left| f(t, x^{(u_\delta, s_\delta)}(\tau_0), u_\delta(\tau_0)) - f(t, x^{(u_\delta, s_\delta)}(t), u_\delta(t)) \right| dt + \\
+ \text{Const} \delta
\]
and the right hand side can be assumed to be smaller than any desired quantity because of the continuity of \( f \) and \( x^{(u_\delta, s_\delta)}(t) \) and the condition (5.10).

This proves that for any choice of \( \rho \in (0, \varepsilon_0) \), the restriction \( \Delta V^{(\delta)}|_{[\rho, \varepsilon_0]} \) converges uniformly to 0. Note also that, by a similar argument, the assumption (5.10) and the continuity of \( p^{(u_\delta, s_\delta)}(t), x^{(u_\delta, s_\delta)}(t) \) and \( f(t, x, u) \) imply that the function
\[
V_\varepsilon(\varepsilon) := \frac{1}{\varepsilon} \int_{\tau_0 - \varepsilon}^{\tau_0} \left\{ p_i^{(u_\delta, s_\delta)}(t)f^i(t, x^{(u_\delta, s_\delta)}(t), u_\delta(t)) - p_i^{(u_\delta, s_\delta)}(t)f^i(t, x^{(u_\delta, s_\delta)}(t), u_\delta(t)) \right\} dt
\]
is an infinitesimal for \( \varepsilon \to 0 \). This concludes the proof that also condition (E) is satisfied. By Remark 4.4 we conclude that the theorem holds also in case \( k = 1 \) and under the additional assumption that \( f \) is continuously differentiable with respect to \( t \).

In order to conclude, it is now necessary to remove this assumption. This can be done by noting that the term (4.12) considered in the proof of Lemma 4.1 can be written as
\[
\int_{\tau_0 - \epsilon}^{\tau_0} \left( \mathcal{P}(\sigma(\varepsilon), \tilde{u}(\varepsilon), t) - \mathcal{P}(\sigma(\varepsilon), t) \right) dt = \varepsilon K_0 + \varepsilon W(\varepsilon)
\]
with
\[
W(\varepsilon) := \frac{1}{\varepsilon} \int_{\tau_0 - \epsilon}^{\tau_0} \left( \mathcal{P}(\sigma(\varepsilon), \tilde{u}(\varepsilon), t) - \mathcal{P}(\sigma(\varepsilon), t) \right) - \\
- \left( \mathcal{P}(\sigma(\varepsilon), t) - \mathcal{P}(\sigma(\varepsilon), \tau_0) \right) dt .
\]

Considering this new function instead of the function \( V(\varepsilon) \), the family of control curves and differential constraints defined in (5.37) satisfies the following analogs of the (4.20), which involve the \( \delta \)-parametrised family of new functions \( W^{(\delta)}(\varepsilon) \) instead of the \( V^{(\delta)}(\varepsilon) \):
\[
C^0(\delta) \leq C^0(\delta) - \varepsilon (K_0(\delta) - \tilde{C}(\delta) \varepsilon - |W^{(\delta)}(\varepsilon)|)
\]
where \( \tilde{C}(\delta) \) is now a constant which does not depend on the derivative \( \frac{\partial f^{\delta}}{\partial x} \). The explicit expression of the function \( W^{(\delta)}(\varepsilon) \) can be directly derived from (5.47). One finds
\[
W^{(\delta)}(\varepsilon) := \\
\int_{\tau_0 - \epsilon}^{\tau_0} \left\{ p_i^{(u_\delta, s_\delta)(\varepsilon)}(t)f^i(t, x^{(u_\delta, s_\delta)(\varepsilon)}(t), \omega_\delta(t)) - \\
- p_i^{(u_\delta, s_\delta)(\varepsilon)}(\tau_0)f^i(t, x^{(u_\delta, s_\delta)(\varepsilon)}(\tau_0), \omega_\delta(t)) \right\} dt .
\]
where, according to the notational conventions of Lemma 4.1, \( \tilde{v}^\delta_{(\varepsilon)} \) is the smoothed needle variation of the polynomial curve \( v^\delta(t) \) of width \( \varepsilon \) and peak time \( \tau_\varepsilon \), while \( s_{(\varepsilon)} \) is the corresponding initial datum for the curve \( \gamma(t) = (x(t), p(t)) \). This initial datum \( s_{(\varepsilon)} \) is determined so that the initial value for \( x(t) \) is \( s_0 \), while the initial value for \( p(t) \) is prescribed in order to have the usual terminal conditions at \( t = T \). We also assume that the constant \( h \), used in the definition of the smoothed needle modifications (see (5.2)) is chosen differently for each value of \( \delta \) and in a way that \( h = h(\delta) \) tends to 0 for \( \delta \to 0 \). In order to conclude, it is now sufficient to show (in analogy with what we did above for the functions \( V^\delta(\varepsilon) \)) that the function \( W^\delta(\varepsilon) \) converge uniformly on compacta on \( (0, \varepsilon_0] \) to a function \( W(0)(\varepsilon) \) and that the limit function \( W(0)(\varepsilon) \) is an infinitesimal for \( \varepsilon \to 0 \). This can be checked directly. More precisely, following the same circle of ideas as above, one can see that on any interval \( [\rho, \varepsilon_0], \rho > 0 \) the function \( W^\delta \) converges in \( C^0 \) norm to the function

\[
W^\delta(\varepsilon) := \frac{1}{\varepsilon} \int_{\tau_\varepsilon - \varepsilon}^{\tau_\varepsilon} \left\{ \left[ \frac{s(\varepsilon)}{\varepsilon} \right] \cdot \left( x(t, s, p, \delta) - f(t, x(s, p, \delta)) \right) \right\} dt
\]

and that such a function is an infinitesimal. For brevity, we omit the details. \( \Box \)

6. The proofs of Sublemma 4.2 and Lemma 4.3

6.1. The proof of Sublemma 4.2 Let us denote by \( \omega^\delta_{(\delta)} \) the 1-forms on the manifold \( J^{2k+1}(\Omega|\mathbb{R}) \times \mathbb{R}^M \) defined by

\[
\omega^\delta_{(\delta)} := dq_{(\delta)} - q_{(\delta+1)} dt , \quad \delta = 0, \ldots, 2k .
\]

Using these 1-forms, we can introduce the controlled Poincaré-Cartan form \( \beta^{PC} \) associated with the controlled Lagrangian \( L \) (\cite{6} Sect. 5)

\[
\beta^{PC} := L dt + \sum_{\delta=1}^{k} \sum_{\eta=0}^{\delta-1} (-1)^\eta \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial \dot{q}^\delta_{(\delta)}} \right) \omega^\delta_{(\delta-(\eta+1))} .
\]

By basic facts on variationally equivalent 1-forms (see e.g. \cite{16} Prop. A2 and \cite{6} Proof of Lemma 5.2), the exterior differential \( d\beta^{PC} \) has the form

\[
d\beta^{PC} = E(L) \omega^0_{(0)} \wedge dt + \frac{\partial L}{\partial u^a} du^a \wedge dt + \text{linear combinations of wedges of pairs of 1-forms of the kind (6.1)},
\]

where the functions \( E(L) \) are the controlled Euler-Lagrange expressions defined in \( (2.3) \). Consider the (smooth) map

\[
\mathcal{G} = \mathcal{G}_{(\varepsilon)} : [0, T] \times [0, 1] \to J^{2k+1}(\Omega|\mathbb{R}) \times \mathbb{R}^M , \quad \mathcal{G}(t, s) := \left( \frac{\partial^{2k+1}}{dt^{2k+1}}(\gamma(s)), u^\varepsilon(s)(t) \right) .
\]
and the fields of tangent vectors of the surface $S := \mathcal{G}([0, T] \times [0, 1])$ defined by

$$ X|_{\mathcal{G}(t,s)} := \left. \frac{\partial \mathcal{G}}{\partial t} \right|_{(t,s)}, \quad Y|_{\mathcal{G}(t,s)} := \left. \frac{\partial \mathcal{G}}{\partial s} \right|_{(t,s)} $$

(6.5)

By construction, each vector $X|_{\mathcal{G}(t,s)}$ has the first components that are tangent to the curve of jets, determined by a $\mathcal{K}$-controlled curve $\tilde{\gamma}^{(e,s)}(t) = (t, q^{(e,s)}(t))$. In particular, the $\frac{\partial}{\partial t}$-component of $X|_{\mathcal{G}(t,s)}$ is 1 for any $(t, s)$. By the same reason the $\frac{\partial}{\partial s}$-component of $Y|_{\mathcal{G}(t,s)}$ is identically 0. Due to this and the vanishing of the 1-forms on the tangent vectors of curves of jets given by curves in $\mathcal{Q} \times \mathbb{R}$, we have that for any $(t, s) \in [0, T] \times [0, 1]$

$$ \beta^{PC}(X|_{\mathcal{G}(t,s)}) = L_{[\mathcal{G}(t,s)]}^k, $$

$$ d\beta^{PC}(X|_{\mathcal{G}(t,s)}, Y|_{\mathcal{G}(t,s)}) = -\left. \frac{\partial L}{\partial u^a} \right|_{\mathcal{G}(t,s)} Y^a|_{\mathcal{G}(t,s)}, \quad \text{with } Y^a|_{\mathcal{G}(t,s)} := du^a(Y|_{\mathcal{G}(t,s)}) $$

(6.6)

We also have that

$$ \beta^{PC}(Y|_{\mathcal{G}(t,s)}) = \sum_{\delta=1}^k \sum_{\eta=0}^{\delta-1} (-1)^{\eta} d^\eta \left( \frac{\partial L}{\partial q^i(\delta)} \right) Y^i|_{\mathcal{G}(t,s)} $$

(6.7)

where $Y^i|_{(\alpha)}$, $0 \leq \alpha \leq k-1$, are the $\frac{\partial}{\partial q^i(\alpha)}$-components of $Y$. From (6.6), (6.7), the definition of $\mathcal{P}(\Sigma^{(e,s)}, \tilde{\gamma}^{(e,s), t})$ and Stokes’ Theorem, we have

$$ \int_0^T \int_{\tau_0 - \varepsilon - h^2 \tau_0 + h^2 \varepsilon} \left( \int_0^{1} Y^a \left. \frac{\partial \mathcal{P}(\Sigma^{(e,s)}, \tilde{\gamma}^{(e,s), t})}{\partial u^a} \right|_{\tilde{\gamma}^{(e,s)}} \right) ds \right) dt = $$

$$ = - \int_{[\tau_0 - \varepsilon - h^2 \tau_0 + h^2 \varepsilon]}^{\mathcal{G}(t,s)} \left( \int_{\tau_0 - \varepsilon - h^2 \tau_0 + h^2 \varepsilon}^{1} \left. \frac{\partial L}{\partial u^a} \right|_{\mathcal{G}(t,s)} Y^a \right) dt ds = $$

$$ = \int_{\tau_0 - \varepsilon - h^2 \tau_0 + h^2 \varepsilon}^{\mathcal{G}(t,s)} \left( L|_{\mathcal{G}(t,s)} - L|_{\mathcal{G}(t,s)} \right) dt + $$

$$ + \sum_{\delta=1}^k \sum_{\eta=0}^{\delta-1} (-1)^{\eta} \left( \frac{d^\eta}{dt} \left( \frac{\partial L}{\partial q^i(\delta)} \right) Y^i|_{\mathcal{G}(t,s)} \right) ds = $$

$$ = \int_{\tau_0 - \varepsilon - h^2 \tau_0 + h^2 \varepsilon}^{\mathcal{G}(t,s)} \left( \mathcal{P}(\sigma^{(e)}, \tilde{u}(\tau_0 - \varepsilon, t)) - \mathcal{P}(\sigma^{(e)}, \tilde{u}(\tau_0 + \varepsilon, t)) \right) dt + $$

$$ + \sum_{\delta=1}^k \sum_{\eta=0}^{\delta-1} (-1)^{\eta} \left( \frac{d^\eta}{dt} \left( \frac{\partial L}{\partial q^i(\delta)} \right) Y^i|_{\mathcal{G}(t,s)} \right) ds. $$
The claim is therefore proven if we can show that for any \( s \in [0, 1] \) the absolute value
\[
\left| \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial q_i^{(\delta)}} \right) \right|_{\mathcal{G}(\tau_0 + \varepsilon \delta^2, \varepsilon^2)} - \left| \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial q_i^{(\delta)}} \right) \right|_{\mathcal{G}(\tau_0 - \varepsilon \delta^2, \varepsilon^2)}
\]
is bounded above by \( \varepsilon^2 \) times a constant depending on \( \tau_0, N, \| L \|_{k+2, N} \) and \( \| \frac{\partial L}{\partial u} \|_{k+1, N} \).
To check this, we first observe that
\[
\left| \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial q_i^{(\delta)}} \right) \right|_{\mathcal{G}(\tau_0 + \varepsilon \delta^2, \varepsilon^2)} - \left| \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial q_i^{(\delta)}} \right) \right|_{\mathcal{G}(\tau_0 - \varepsilon \delta^2, \varepsilon^2)} \leq \left( \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial q_i^{(\delta)}} \right) \right)_{\mathcal{G}(\tau_0 + \varepsilon \delta^2, \varepsilon^2)} + \left( \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial q_i^{(\delta)}} \right) \right)_{\mathcal{G}(\tau_0 - \varepsilon \delta^2, \varepsilon^2)} \right)
\]
We now recall that for any \( (t_0, s_o) \in [0, T] \times [0, 1] \)
\[
Y_i^{(\alpha)} \|_{\mathcal{G}(t_0, s_0)} = \frac{\partial}{\partial s} \left( q^{(\varepsilon, s)i}(t) \right),
\]
where \( q^{(\varepsilon, s)i}(t) \) stands for the \( i \)-th component of the jet \( J_{t=t_0}^{2k-1}(\gamma^{(\varepsilon, s)}) \) of the \( \hat{K} \)-controlled curve \( \gamma^{(\varepsilon, s)} \). Combining (6.9), the differentiability of \( L \) with respect to \( u \), the explicit expressions of the Euler-Lagrange equations (which are obtained by taking at most \( k+1 \) derivatives of \( L \) with respect to the jets coordinates) and a straightforward generalisation of a classical fact on solutions to controlled differential equations (see e.g. [4] Thm. 3.2.6) and the proof of Lemma 2.1, one can check that for any \( 0 \leq \beta \leq k \) and any \( (t_0, s_o) \in [\tau_0 - \varepsilon - \delta^2, \tau_0 + \delta^2] \times [0, 1] \)
\[
\left| Y_i^{(\beta)} \|_{\mathcal{G}(t_0, s_0)} \right| \leq \sup_{[\tau_0 - \varepsilon - \delta^2, \tau_0 + \delta^2]} \left| \gamma^{(\tau_0, \omega, \varepsilon)}(t) - u_o(t) \right| \leq (\varepsilon + 2\delta^2) \text{diam}(\hat{K}) e^{(\varepsilon + 2\delta^2) K(N,L)} \| L \|_{k+2, N} K(N,L) \left| \frac{\partial L}{\partial u} \right|_{k+1, N},
\]
where \( \text{diam}(\hat{K}) \) is the diameter of the relatively compact set \( \hat{K} \subset \mathbb{R}^M \). Consequently, for any \( 0 \leq \alpha \leq k - 1 \)
\[
\left| Y_i^{(\alpha)} \|_{\mathcal{G}(t_0, s_0)} - Y_i^{(\alpha)} \|_{\mathcal{G}(\tau_0 - \varepsilon - \delta^2, \varepsilon^2)} \right| = \int_{t \in [\tau_0 - \varepsilon - \delta^2, t_0]} \frac{\partial}{\partial t} Y_i^{(\alpha)} \|_{\mathcal{G}(t, s_0)} dt = \int_{t \in [\tau_0 - \varepsilon - \delta^2, t_0]} Y_i^{(\alpha + 1)} \|_{\mathcal{G}(t, s_0)} dt \leq (\varepsilon + 2\delta^2)^2 \text{diam}(\hat{K}) e^{(\varepsilon + 2\delta^2) K(N,L)} \| L \|_{k+2, N} K(N,L) \left| \frac{\partial L}{\partial u} \right|_{k+1, N}.
\]
Hence, for any $0 \leq \alpha \leq k - 1$, $0 \leq \beta \leq k$,
\[ \left| \frac{d^\alpha}{dt^\alpha} \left( \frac{\partial L}{\partial q^\beta_i(\delta)} \right) \right|_{\mathbb{G}(\tau_0 - \varepsilon - \delta^2, s)} \cdot \left| \frac{Y^\alpha_i(\tau_0 - \varepsilon - \delta^2, s)}{\mathbb{G}(\tau_0 - \varepsilon - \delta^2, s)} \right| \leq \varepsilon \left( 2 + 2h\varepsilon^2 \right) \text{diam}(\bar{K}) \|L\|_{k+1, N} \|L\|_{k+2, N} |K_N| \left\| \frac{\partial L}{\partial \bar{u}} \right\|_{k+1, N}. \quad (6.11) \]

A similar line of arguments yields to the estimate
\[ \left| \frac{d^\alpha}{dt^\alpha} \left( \frac{\partial L}{\partial q^\beta_i(\delta)} \right) \right|_{\mathbb{G}(\tau_0 + \delta^2, s)} \cdot \left| \frac{Y^\alpha_i(\tau_0, s)}{\mathbb{G}(\tau_0 - \varepsilon - \delta^2, s)} \right| \leq 2 \left( 2 + 2h\varepsilon^2 \right) \text{diam}(\bar{K}) \|L\|_{k+1, N} \|L\|_{k+2, N} |K_N| \left\| \frac{\partial L}{\partial \bar{u}} \right\|_{k+1, N}. \quad (6.12) \]

From (6.8), (6.11), (6.12), the conclusion follows. \[\square\]

6.2. The proof of Lemma 4.3 It suffices to prove that the constant $n_{(\tau_0, N, L, \frac{\partial L}{\partial \bar{u}})}$ of Sublemma 4.2 is independent on $\|L\|_{k+2, N}$ and $\|\frac{\partial L}{\partial \bar{u}}\|_{k+1, N}$. By (6.8), this is proven if we can show that, under the assumption (4.19), then
\[
\sum_{\delta=1}^{k} \sum_{\eta=0}^{\delta-1} (-1)^{\eta} \int_0^1 \left( \frac{d^0}{dt^0} \left( \frac{\partial L}{\partial q^\delta_i(\delta)} \right) \right) Y^\alpha_i(\delta - (\eta+1)) \left|_{\mathbb{G}(\tau_0 + \delta^2, s)} \right| - \left( \frac{d^0}{dt^0} \left( \frac{\partial L}{\partial q^\delta_i(\delta)} \right) \right) Y^\alpha_i(\delta - (\eta+1)) \left|_{\mathbb{G}(\tau_0 - \varepsilon - \delta^2, s)} \right| ds = 0. \quad (6.13) \]

For this, we observe that, by Stokes’ Theorem, (6.6) and (6.7),
\[
\sum_{\delta=1}^{k} \sum_{\eta=0}^{\delta-1} (-1)^{\eta} \int_0^1 \left( \frac{d^0}{dt^0} \left( \frac{\partial L}{\partial q^\delta_i(\delta)} \right) \right) Y^\alpha_i(\delta - (\eta+1)) \left|_{\mathbb{G}(\tau_0 + \delta^2, s)} \right| - \left( \frac{d^0}{dt^0} \left( \frac{\partial L}{\partial q^\delta_i(\delta)} \right) \right) Y^\alpha_i(\delta - (\eta+1)) \left|_{\mathbb{G}(0, s)} \right| = \\
= \int_{[0, \tau_0 - \varepsilon - \delta^2] \times [0, 1]} \beta^{PC}(X, Y) (t, s) dtds - \int_0^{\tau_0 - \varepsilon - \delta^2} \left( L|_{(j_{\delta}^\alpha(\tau_0), u_0(t))} - L|_{(j_{\delta}^\alpha(\tau_0 - \varepsilon - \delta^2), u(\tau_0, \omega_0, c)(t))} \right) dt = - \int_0^{\tau_0 - \varepsilon - \delta^2} \left( L|_{(j_{\delta}^\alpha(\tau_0), u_0(t))} - L|_{(j_{\delta}^\alpha(\tau_0 - \varepsilon - \delta^2), u(\tau_0, \omega_0, c)(t))} \right) dt, \quad (6.14) \]

where $\beta^{PC}$ is the 1-form (6.3). We now recall that in the region $[0, \tau_0 - \varepsilon - \delta^2] \times [0, 1]$ the components $Y^\alpha$ of the vector field $Y$ are identically 0. Moreover, if $L$ has the form (4.18), the controlled Euler-Lagrange equations imply that the value of $L$ is 0 at all jets of each $\mathcal{X}$-controlled curve $\gamma$, so that
\[
\int_0^{\tau_0 - \varepsilon - \delta^2} \left( L|_{(j_{\delta}^\alpha(\tau_0), u_0(t))} - L|_{(j_{\delta}^\alpha(\tau_0 - \varepsilon - \delta^2), u(\tau_0, \omega_0, c)(t))} \right) dt = 0. \]
Hence (6.14) and (4.19) imply
\[
\sum_{\delta=1}^{k} \sum_{\eta=0}^{\delta-1} (-1)^{\eta} \int_{0}^{1} \left( \frac{d^{\eta}}{dt^{\eta}} \left( \frac{\partial L}{\partial q_{(\delta)}} \right) \right) Y_{(\delta-(\eta+1))}^{i} \bigg|_{(\tau_{0} - \varepsilon, \tau_{2}, s)} = \sum_{\delta=1}^{k} \sum_{\eta=0}^{\delta-1} (-1)^{\eta} \int_{0}^{1} \left( \frac{d^{\eta}}{dt^{\eta}} \left( \frac{\partial L}{\partial q_{(\delta)}} \right) \right) Y_{(\delta-(\eta+1))}^{i} \bigg|_{(0,s)} = 0. \tag{6.15}
\]

A similar argument yields
\[
\sum_{\delta=1}^{k} \sum_{\eta=0}^{\delta-1} (-1)^{\eta} \int_{0}^{1} \left( \frac{d^{\eta}}{dt^{\eta}} \left( \frac{\partial L}{\partial q_{(\delta)}} \right) \right) Y_{(\delta-(\eta+1))}^{i} \bigg|_{(\tau_{0} + \varepsilon, \tau_{2}, s)} = \sum_{\delta=1}^{k} \sum_{\eta=0}^{\delta-1} (-1)^{\eta} \int_{0}^{1} \left( \frac{d^{\eta}}{dt^{\eta}} \left( \frac{\partial L}{\partial q_{(\delta)}} \right) \right) Y_{(\delta-(\eta+1))}^{i} \bigg|_{(T,s)} = 0. \tag{6.16}
\]

From this (6.13) follows.

7. Suggested investigations

As we mentioned in the Introduction, here we want to point out some problems of Control Theory where our two-steps approach (= a preliminary analysis based on classical results of Differential Geometry, followed by arguments devoted to reduce the regularity assumptions) has good chances to produce new results or to enlighten some particular aspects of the dynamics of controlled systems. The discussion is intentionally very sketchy, because its purpose is merely to provide suggestions and motivations for future studies.

7.1. Maximum Principles in Continuum Dynamics. Consider the following toy problem. Let $\mathcal{E}$ be an (unbounded) elastic continuum, whose elements are described by just one space-variable, denoted by $s \in \mathbb{R}$, that evolves in the time $t$. The deformations of such continuum are represented by functions $x(t, s)$ of the time and space variables and are assumed to satisfy a hyperbolic equations of the form
\[
\frac{\partial^{2} x}{\partial t^{2}} - \frac{\partial^{2} x}{\partial s^{2}} = f(t, s, u(t, s)), \tag{7.1}
\]
where $f : \mathbb{R}^{3} \to \mathbb{R}$ is a fixed smooth function and $u(t, s)$ is a control map with values in a compact set $K \subset \mathbb{R}$. The composed map $f(t, s, u(t, s))$ might be physically interpreted as a (density of a) dead load attached at the elements of the continuum and varying in time. Note also that, when $f(s, t, u)$ is linear in $u$, the equation (7.1) is in the class of controlled hyperbolic equations, which is intensively studied in the theory of control problems governed by partial differential equations (see [11][18]).

Following our usual two-step approach, let us at first restrict the discussion of this toy problem to deformations $x(s, t)$ and control maps $u(s, t)$ of class $C^\infty$ and satisfying all needed assumptions (as, for instance, rapidly decreasing properties for $s \to 0$) that may guarantee that all subsequent arguments are meaningful.
Consider the following problem: given an initial condition for \( x(t, s) \) at \( t = 0 \)
\[
x(0, s) = \varphi(s), \quad \frac{\partial x}{\partial t}(0, s) = \psi(s),
\]
(7.2)
look for a load \( u_o(t, s) \) such that the corresponding solution to (7.1) satisfying (7.2) minimises the integral at \( t = T \) (= the terminal cost)
\[
C(x_o(T, s)) = \int_{\mathbb{R}} \ell(x(T, s))ds
\]
(7.3)
where \( \ell(x) \) is a prescribed smooth real function. This can be classified as a control problem, whose optimal controls are the loads \( u_o(t, s) \) satisfying the above minimising requirement. Inspired by the discussions in [5, 6] about the control problems involving just one independent variable, it is natural to start studying this new type of control problem by considering the controlled Lagrangian density on the 2-jets of maps \((t, s) \mapsto (x(t, s), p(t, s))\), defined by
\[
\mathcal{L}^{(u(t,s))}(t, s, x, x_t, x_{tt}, x_s, x_{ss}, p) := p(x_t - x_{ss} - f(t, s, u(t, s))) + \left( \ell(x) + t \frac{\partial \ell(x)}{\partial x} x_t \right) .
\]
(7.4)

One can check that, for any fixed choice of the control function \( u(t, s) \),

1. The Euler-Lagrange equations determined by \( \mathcal{L}^{(u(t,s))} \) give a system of two partial differential equations, the first equal to (7.1), the second equal to the hyperbolic equation on \( p(t, s) \)
\[
\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial s^2} = 0 ;
\]
(7.5)
2. If the pair \((x(t, s), p(t, s))\) is a solution to the Euler-Lagrange equations in (1), then
\[
\int \int_{S=\{0 \leq t \leq T, s \in \mathbb{R}\}} \mathcal{L}^{(u(t,s))} \left( t, s, x(t, s), \frac{\partial x}{\partial t}, \frac{\partial^2 x(t, s)}{\partial t^2}, \frac{\partial x}{\partial s}, \frac{\partial^2 x(t, s)}{\partial s^2}, p(t, s) \right) dt ds = C(x(T, s)) .
\]

All this shows that the new setting is extremely close to what is considered in [5, 6] for control problems with differential constraints involving just one independent variable. We are confident that the same line of arguments considered there (and in particular the “road map” presented in [5, Sect. 2.2]) can be followed for this toy problem and many other control problems with constraints given by partial differential equations. This would lead to analogs of the PMP (compare, for instance. [11, 3]) under strong regularity assumptions, results which can be considered as the first step of differential-geometric type of the approach we are promoting. The direct proof of Theorem 1.1 given in this paper can be then considered as guiding line for extending the results of the “first step” to reach results under low regularity assumptions.

### 7.2. Dynamics of controlled systems with higher order constraints.

Consider a dynamical system which is subjected to a second order differential constraint in normal form and independent on time, that is of the form
\[
\frac{d^2 x_j}{dt^2} = f^j(x(t), u(t)) , \quad 1 \leq j \leq n .
\]
with control curve \( u(t) = (u^a(t)) \) taking values in a relatively compact set \( K \subset \mathbb{R}^m \).

Following the first step of our two-step approach, let us at first assume that all data satisfy strong regularity assumptions (i.e. assume that \( f \) is smooth, \( u(t) \) varies in the class of smooth curves, \( K \) has smooth boundary, etc.), so that the most common differential geometric tools might be used. Let us also denote by \( \mathcal{L}(x, x(1), x(2), p, u) \) the second order controlled Lagrangian \( \mathcal{L}(x) \) associated with this control problem:

\[
\mathcal{L}(x, x(1), x(2), p, u) := p_j \left( x^j_{(2)} - f^j(x, u(t)) \right).
\]

We remark that, for any fixed choice of a control curve \( u_o(t) \), the Euler-Lagrange equations of \( \mathcal{L}(x, x(1), x(2), p, u_o(t)) \) for \( x \) and \( p \) coincide with the Euler-Lagrange equations of the equivalent Lagrangian (their difference is a null Lagrangian)

\[
\tilde{\mathcal{L}}(x, x(1), x(2), p, p(1), u_o(t)) := p(1) j x^j_{(1)} - p_j f^j(x, u_o(t)).
\]

If we consider the coordinates \((\tilde{x}^i, \tilde{p}_j)\) related with the \((x^i, p_j)\) by

\[
x^i = \frac{1}{\sqrt{2}} \tilde{x}^i + \frac{1}{\sqrt{2}} \tilde{p}_i, \quad p_j = \frac{1}{\sqrt{2}} \tilde{x}_j - \frac{1}{\sqrt{2}} \tilde{p}_j,
\]

the new Lagrangian \( \tilde{\mathcal{L}}(x) \) takes a very familiar form, namely

\[
\tilde{\mathcal{L}}(x, x(1), x(2), p, p(1), u_o(t)) := \frac{1}{2} \sum_{i=1}^n \left( (\tilde{x}_{(1)i})^2 - \frac{1}{2} (\tilde{p}(1)i)^2 \right) + V(\tilde{x}, \tilde{p}, u_o(t))
\]

where \( V(\tilde{x}, \tilde{p}, u) := -\left( \frac{1}{\sqrt{2}} \tilde{x}_j - \frac{1}{\sqrt{2}} \tilde{p}_j \right) f^j \left( \frac{1}{\sqrt{2}} \tilde{x} + \frac{1}{\sqrt{2}} \tilde{p}^t, u \right) \).

This is a Lagrangian that describes the dynamics on a Lorentzian 2-manifold of a system subjected to force with time-dependent potential \( V(\tilde{x}, \tilde{p}, u_o(t)) \). It is therefore possible to use a variety of well known mathematical physics tools to study the dynamics of such controlled systems. For instance, for any given choice of a smooth \( u_o(t) \), studying symmetries of \( \mathcal{V} \) and using Noether Theorem (\( \mathcal{L} \)), all conservation laws that are satisfied (or, more interesting, violated) can be explicitly determined. In particular, in the time intervals on which \( u_o(t) \) is constant (recall that, in several classical settings, the optimal control \( u_o(t) \) is constant a.e.) an appropriate non-positively defined energy is conserved by the corresponding controlled evolution.

Furthermore, if we denote by \((Q^I) := \left( \begin{array}{c} x^i \\ p_j \end{array} \right)\), we may observe that the Hessian

\[
\frac{\partial^2 \tilde{\mathcal{L}}}{\partial Q_{(1)}^i \partial Q_{(1)}^j} \]

is non-degenerate, a property that allows a formulation of the differential constraints into a Hamiltonian formulation in the phase space spanned by the coordinates \( Q = (Q^I) = (x^i, p_j) \) and their duals \( P = (P_K) \). In a sense, this would be a “true Hamiltonian presentation” of the constraints of the controlled system, very much different from the traditional Pontryagin’s Hamiltonian type presentation. We think that it would be quite important to get a clear view of the relations between these two distinct Hamiltonian type presentations of the differential constraints and of their dependences on the needle variations. As usual, answers to any question in this topic can at first be obtained via differential geometric tools under strong regularity assumptions. Secondly one can extend the results to the lowest possible regularity assumptions following the ideas of this paper. Similar investigations might – and, in our opinion, should – be made.
for controlled systems with differential constraints of order higher than two and/or by means of the alternative presentations of Lagrangian type, which are discussed in [6].

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FRANCO CARDIN
DIPARTIMENTO DI MATEMATICA “TULLIO LEVI-CIVITA”
 UNIVERSITÀ DEGLI STUDI DI PADOVA
 VIA TRIESTE, 63
 I-35121 PADUA
 ITALY

E-mail: cardin@math.unipd.it

Cristina Giannotti & Andrea Spiro
SCUOLA DI SCIENZE E TECNOLOGIE
UNIVERSITÀ DI CAMERINO
VIA MADONNA DELLE CARCERI
I-62032 CAMERINO (MACERATA)
ITALY

E-mail: cristina.giannotti@unicam.it
E-mail: andrea.spiro@unicam.it