Brownian motors and stochastic resonance
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(Dated: 21 December 2013)

We study the transport properties for a walker on a ratchet potential. The walker consists of two particles coupled by a bistable potential that allow the interchange of the order of the particles while moving through a one-dimensional asymmetric periodic ratchet potential. We consider the stochastic dynamics of the walker on a ratchet with an external periodic forcing, in the overdamped case. The coupling of the two particles corresponds to a single effective particle, describing the internal degree of freedom, in a bistable potential. This double-well potential is subjected to both a periodic forcing and noise, and therefore is able to provide a realization of the phenomenon of stochastic resonance. The main result is that there is an optimal amount of noise where the amplitude of the periodic response of the system is maximum, a signal of stochastic resonance, and that precisely for this optimal noise the average velocity of the walker is maximal, implying a strong link between stochastic resonance and the ratchet effect.

PACS numbers: 05.40.-a; 05.40.Ca; 05.40.Jc; 05.60.Cd
Keywords: Brownian Motors; Ratchets; Stochastic Resonance; Classical Transport

Nowadays is well known that noise, instead of being an annoying feature that has to be removed, can play a constructive role for some nonlinear systems. Two important phenomena in this category are: Brownian motors and stochastic resonance. The former refers to the effect where an asymmetry in a nonlinear system can rectify an unbiased fluctuating force out of equilibrium, thereby generating a directional current; the latter refers to the effect where a weak coherent input subthreshold signal in a nonlinear system can be detected with the assistance of noise. It seems that these two effects must be related, however it is not clear under which circumstances this relation can be revealed. In this paper we present a model that establish such a connection. The model is inspired by the dynamics of motor proteins, like Kinesin or Myosin, and consists of a walker comprising two particles or Brownian motors, coupled through a bistable double-well potential, that walks through an asymmetric ratchet. For this model we show that for an optimal amount of noise the current generated by the ratchet effect is maximal and that this optimal noise intensity corresponds precisely to the maximal response characterizing stochastic resonance. Therefore, our model establishes a link between both phenomena.

I. INTRODUCTION

In this special issue we are paying homage to our dear friend Frank Moss, who throughout his long and productive career has contributed with a series of seminal papers in many different fields: biological physics, stochastic dynamics, condensed matter, stochastic resonance, to name but a few. In particular, his ideas and seminal experiments on different organisms were crucial to establish the presence of the phenomenon of stochastic resonance in the biological realm. His ideas and influence will endure and will be the inspiration for other generation of scientists. We will miss his cheerful and wit character that manage to shape this area of research and built a strong community around his leadership.

This paper is devoted to Brownian motors and stochastic resonance and is dedicated to Frank Moss as a humble tribute.

Brownian motors (or thermal ratchets) are nonlinear systems that can extract usable work from unbiased nonequilibrium fluctuations. The canonical example is a particle undergoing a random walk in a periodic asymmetric (ratchet) potential, and being acted upon by an external time-dependent force of zero average. The recent burst of work is motivated, for instance, by the challenge to model unidirectional transport of molecular motors within the biological realm and the potential for novel applications that enables an efficient scheme to shuttle, separate and pump particles on the micro- and even nanometer scale. In particular, it is worth mentioning the focus issue on “The constructive role of noise in fluctuation-driven transport and stochastic resonance”, in this very journal, edited by Dean Astumian and Frank Moss.

In this paper, we will study the transport properties of a walker (dimer) moving through a ratchet potential. This walker comprises two point particles coupled...
through a nonlinear force modeled by a bistable potential that opens the possibility of exchanging the order of the two particles while walking. This model was inspired by the physics of molecular motors, in particular Kinesin or Myosin, which are motor proteins that have two portions acting as “feet” that move through microtubules inside cells. Other authors have explored different models of Brownian walkers, like the model of Brownian steppers presented in [1], where each step is composed of two processes: an activation process describing the random attachment of a fuel molecule, followed by a conformational change of the stopper that leads to a forward unidirectional motion. The time-periodic modulation of the rate of the fuel concentration allows improving and regularizing the random motion of the stepper. For a recent review of the physics of molecular motors see [2]. Further aspects of the model discussed in [1] has been explored recently by other authors [3-10].

We will study the dynamics, in the overdamped regime, of the above-mentioned walker, in a ratchet potential. Each of the two particles is characterized by a coordinate that obeys a Langevin equation that includes thermal noise, the force due to the ratchet potential, an external periodic force, and a coupling force between the two particles. This coupling is given by a bistable (double-well) potential in terms of an internal degree of freedom given by the difference of the coordinates of the two particles. Therefore, this internal degree of freedom experience a double well potential subjected to a periodic forcing and noise, and thus is able to manifest the phenomenon of stochastic resonance.

Stochastic resonance involves the interplay of nonlinearity and noise, in which a signal detection can be amplified and optimized by the assistance of noise. It involves the following essential features: an energetic barrier or threshold, a coherent (periodic) signal, and noise [11-13]. The field of stochastic resonance began in the early eighties and start to flourish in the nineties, specially due to the breakthrough experiments of Frank Moss and collaborators, that show the effect of stochastic resonance in living organisms [14-16].

The paradigmatic model is a particle in a bistable double well potential that is rocked periodically and subjected to stochastic noise. In the absence of noise and for a weak periodic signal, the particle is in a subthreshold regime and cannot visit the two wells in the double well potential, being confined to only one of them. But, with the aid of a noisy signal, the particle is now capable of surmounting the potential barrier and start to visit both wells. If the noise is very weak, this particle remains confined in one of the minima and for a very strong noise the resulting dynamics becomes fully stochastic, masking the periodic signal. However, between these two extremes, there is an optimal amount of noise where the dynamics fully reveals the subthreshold periodic signal. That is, noise helps to detect the otherwise hidden periodicity. This is, in a nutshell, the phenomenon of stochastic resonance.

II. A WALKER WITH TWO BROWNIAN MOTORS AND STOCHASTIC RESONANCE

The model considers a walker moving on an asymmetric ratchet that has two feet that are represented by two particles coupled nonlinearly through a bistable potential [17-20]. Additionally, there is an external periodic driving and thermal noise. The equations of motion for the two particles, represented by \( x \) and \( y \), in the overdamped regime, are

\[
m\gamma \dot{x} = -\frac{dV(x)}{dx} - m\gamma \frac{\partial V_b(x-y)}{\partial x} + m\gamma \sqrt{2D} \xi_1(t) - F_D \cos(\omega_D t),
\]

\[
m\gamma \dot{y} = -\frac{dV(y)}{dy} - m\gamma \frac{\partial V_b(x-y)}{\partial y} + m\gamma \sqrt{2D} \xi_2(t) + F_D \cos(\omega_D t),
\]

where \( m \) is the mass of the particle, \( \gamma \) is the friction coefficient, \( V(x) \) is the asymmetric periodic ratchet potential, \( V_b(x-y) \) is the bistable potential, and \( F_D \) and \( \omega_D \) represent the amplitude and the frequency of the external driving force, respectively.

These equations represent two coupled particles on a ratchet potential, which is given by

\[
V(x) = V_1 - V_r \left[ \sin \left( \frac{2\pi(x-x_0)}{L} \right) + \frac{1}{4} \sin \left( \frac{4\pi(x-x_0)}{L} \right) \right],
\]

where \( L \) is the periodicity of the potential, \( V_r \) is the amplitude, and \( V_1 \) is an arbitrary constant. The potential is shifted by an amount \( x_0 \) in order that the minimum of the potential is located at \( 0 \) [20-21].

The bistable potential is given by

\[
V_b(x-y) = V_b' + V_b \left[ \frac{(x-y)^4}{l^4} - \frac{2(x-y)^2}{l^2} \right],
\]

Here, \( V_b \) is the amplitude of the bistable potential and represents the coupling strength between the particles, and \( 2l \) is the distance between the two minima.

Finally, the parameter \( D \) is the intensity of the zero-mean statistically independent Gaussian white noises \( \xi_1(t) \) and \( \xi_2(t) \) acting on particles \( x \) and \( y \), respectively. Being statistically independent, they satisfy:

\[
\langle \xi_i(t) \xi_j(s) \rangle = \delta_{ij} \delta(t-s).
\]

Let us define the following dimensionless units: \( x' = x/L, \ x_0' = x_0/L, \ y' = y/L, \ y_0' = y_0/L, \ t' = \gamma t, \ l' = l/L, \ \omega_D' = \omega_D/\gamma, \ F' = F/mL^2\gamma^2, \ F_D' = F_D/mL^2\gamma^2, \ V_b' = V_b/mL^2\gamma^2, \ V_1' = V_1/mL^2\gamma^2 \), and \( V_b'' = V_b/mL^2\gamma^2 \).

Thus, we are using the periodicity of the potential \( L \) as the natural length scale and the inverse of the friction coefficient \( \gamma \) defines the natural time scale. With these
two quantities, the natural force is given by \( mL\gamma^2 \) and the associated energy by \( mL^2\gamma^2 \).

The dimensionless equation of motion, after renaming the variables again without the primes, becomes

\[
\dot{x} = \frac{dV(x)}{dx} - \frac{\partial V_b(x-y)}{\partial x} + \sqrt{2D}\xi_1(t) - F_D \cos(\omega_D t),
\]

(6)

\[
\dot{y} = \frac{dV(y)}{dy} - \frac{\partial V_b(x-y)}{\partial y} + \sqrt{2D}\xi_2(t) + F_D \cos(\omega_D t),
\]

(7)

where the dimensionless ratchet potential can be written as

\[
V(x) = V_1 - V_r \left(\sin(2\pi(x-x_0)) + \frac{1}{4}\sin(4\pi(x-x_0))\right),
\]

(8)

and the dimensionless bistable potential is

\[
V_b(x-y) = V_b + V_b \left[\frac{(x-y)^4}{l^4} - 2\frac{(x-y)^2}{l^2}\right].
\]

(9)

It is convenient to rewrite the equations of motion in terms of an effective time-dependent bistable potential that incorporates the periodic forcing as

\[
\dot{x} = -\frac{dV(x)}{dx} - \frac{\partial U_b(x-y,t)}{\partial x} + \sqrt{2D}\xi_1(t),
\]

(10)

\[
\dot{y} = -\frac{dV(y)}{dy} - \frac{\partial U_b(x-y,t)}{\partial y} + \sqrt{2D}\xi_2(t),
\]

(11)

where the potential \( U_b \) is given by

\[
U_b(x-y,t) = V_b(x-y) + (x-y)F_D \cos(\omega_D t).
\]

(12)

In this way, it is clear that the dynamics corresponds to a rocking bistable potential for the relative coordinate \( x-y \).

In Fig. 1, we show the walker on the ratchet potential. The “feet” of the walker are indicated by the particles at the coordinates \( x \) and \( y \). In the inset we depict the bistable potential that couples both particles. This model, at variance with many others that consider only a linear coupling, incorporates a nonlinear coupling between the two particles, as has been discussed before. The bistable potential depends on the variable \( x-y \), that can be positive, negative or zero. When \( x-y > 0 \), the \( x \) particle is leading, and when \( x-y < 0 \), the \( y \) particle is the leading one. Thus, the transitions between the two wells in the bistable potential correspond to an exchange of the order between the particles. The minima are located at \( x-y = \pm 1 \), and correspond to the two stable equilibrium configurations for the walker; on the other hand, the maximum at the origin \( x-y = 0 \) is unstable. So, we can think of a state oscillating in the bistable potential back and forth between the two minima, as the walker alternating theirs two feet. In Fig. 1a, we show the case when \( y \) (red or black) is larger that \( x \) (green or gray) and in (b) we depict the opposite situation after a single step. In the inset we show the rocking bistable potential as a function of the distance \( x-y \); (a) for \( x-y < 0 \), the foot \( x \) (green or gray) is behind the foot \( y \) (red or black) and, (b) for \( x-y > 0 \), the foot \( y \) is behind the foot \( x \). When the walker makes a step by alternating the positions of the feet, that corresponds to a transition between the wells of the bistable potential.

FIG. 1. The walker on the ratchet potential. In (a) we show the case when \( y \) (red or black) is larger that \( x \) (green or gray) and in (b) we depict the opposite situation after a single step.
III. NUMERICAL RESULTS

We used the Fox algorithm given by Eqs. (10, 11) for the walker on a ratchet. We used the Fox algorithm, that integrates the deterministic part of the equations using a fourth-order Runge-Kutta algorithm, whereas the stochastic part is integrated with a Taylor scheme; this method is called the exact propagator.

Once we solve the system, we calculate the position of each of the particles of the walker. We will fix throughout the paper the following parameters: the amplitude of the ratchet potential as $V_r = 1/2 \pi$, the amplitude of the bistable potential as $V_0 = V_r$, that is, both amplitudes are equal; we fix $l = 0.5$, $F_D = 1$ and $\omega_D = 0.1$. The initial conditions are $x(t = 0) = 0$ and $y(t = 0) = 1$.

The dynamics of the walker can be describe as two coupled particles on a one-dimensional ratchet potential, or as a single particle in an effective time-dependent two-dimensional potential $U(x, y, t)$, given by

$$U(x, y, t) = V(x) + V(y) + U_0(x - y, t). \tag{13}$$

In Fig. 2, we show a three-dimensional plot of this effective potential $U(x, y, t)$ as a function of $x$ and $y$, fixing the time as $t = T$, where $T = 2\pi/\omega_D$, and the contour lines at the bottom. Due to the coupling between the particles, the effective potential resemble a channel with maxima and minima. Since this potential varies in time, the entire profile changes accordingly in a periodic fashion. In Fig. 3, we show the 2D contour map and a typical trajectory when the noise intensity is $D = 0.02$. When this trajectory crosses the diagonal line $x = y$, the walker performs a step that exchange the order of the two feet. It is clear from this trajectory that the walker is stepping through the potential in a systematic and synchronized way. For clarity, in Fig. 4, we show the same trajectory depicted in Fig. 3, using the same parameters, but this time showing the dynamics of the two particles $x$ and $y$ as a function of time. Notice here that there is a systematic positive current where the order of the feet exchange periodically on average.

In Fig. 5, we show the internal degree of freedom $(x - y)/l$ as a function of time. We scale this internal degree freedom with the characteristic length $l$ of the bistable potential. The green (dashed) line corresponds to the deterministic case, without noise ($D = 0$), where the dynamics is confined and oscillates around the minimum at $x - y = -l$ with the period $T = 2\pi/\omega_D$. In this case, the velocity of the walker is zero and there is no net transport despite the presence of the asymmetric ratchet potential (see Fig. 6). However, in the presence of a small amount of noise with $D = 0.02$, we have a completely different behavior, as depicted in the red (full) line. Now, assisted by the noise, the internal degree of freedom fluctuates around each of the two minima of the bistable potential and jumps periodically from one to the other.
minimum to the other, surmounting the potential barrier at \( x - y = 0 \). Notice that the average periodicity of this stochastic dynamics is precisely the same periodicity of the external driving force \( T = 2\pi/\omega_D \). In this case, thanks to the presence of noise, the average velocity of the walker is different from zero and we obtain a net transport through the ratchet (see Fig. 6). Thus, the dynamics depicted in this figure illustrates very clearly the phenomenon of stochastic resonance for the internal dynamics and its connection with the ratchet effect.

The central result of this paper is depicted in Fig. 6, where we show two quantities that characterized two different phenomena: stochastic resonance and the ratchet effect. The latter is characterized by the rectification, due to symmetry breaking, of external unbiased forces, giving as a result a net transport (current) or a finite average velocity. On the other hand, stochastic resonance is characterized by a maximum in the periodic response of the system to a periodic input signal, as a function of noise.

Let us define these two quantities in more detail. The current is defined as the average velocity of the center of mass of the walker. The center of mass is simply \((x+y)/2\) and knowing \( x(t) \) and \( y(t) \) we calculate the asymptotic velocity of the center of mass. Then we perform an ensemble average over 1500 realizations of the noise. In this way we obtain the average velocity of the center of mass \( \langle v \rangle \) and rescale this quantity as \( 2\pi \langle v \rangle / \omega_D \). This is the main quantifier for the ratchet effect of the walker with two Brownian motors moving on the ratchet potential. In Fig. 6, we plot this current as a function of the noise intensity \( D \) and notice that in the limit of zero noise (deterministic case) the current is zero; on the other hand, for a large intensity of noise the current tends to zero. However, for a certain optimal amount of noise the current is different from zero and we obtain a net transport through the ratchet (see Fig. 6). Thus, the phenomenon of stochastic resonance for the internal degree of freedom is jumping periodically between the two minima in the bistable potential, that is, \[ \langle x - y \rangle(t) = \langle A \rangle \cos(\omega_D t - \phi), \] with an amplitude \( \langle A \rangle \) and a phase lag \( \phi \). In Fig. 6, we show as a green (dashed) line the amplitude \( \langle A \rangle/l \) as a function of the intensity of noise \( D \). As expected, we obtain a maximum that is a clear signature of stochastic resonance. Not only that, the optimal noise is precisely around \( D = 0.02 \). This means that, at this value, the internal degree of freedom is given by a rocking bistable potential in which the average periodicity \( T = 2\pi/\omega_D \) as a function of noise \( D \). The question is: Why at this particular noise? In order to answer this, we have to consider the internal dynamics given by the internal degree of freedom of the walker. As we have discussed before, the dynamics of this internal degree of freedom is given by a rocking bistable potential in the presence of noise, that is, the equation of motion corresponds to an overdamped particle in a double well potential tilted with a periodic function \( F_D \cos(\omega_D t) \). The mean value of the response is obtained by averaging over an ensemble of noise realizations (1500 in our case) and for small amplitudes this response is also periodic, with the same period, and can be written as\(^{31}\):

\[ \langle A \rangle \cos(\omega_D t - \phi). \]
FIG. 7. The same two quantities as in Fig. 6, but instead of an asymmetric ratchet potential, we used a symmetric potential of the form \( \cos(2\pi x) \).

still have the effect of stochastic resonance for the internal degree of freedom, but now the current is zero (with some fluctuations due to the finiteness of the ensemble of realizations of noise). Thus, even though the walker is alternating their feet, it is not moving systematically in any direction since there is no symmetry breaking in the potential that rectifies the motion. A similar case of a neutral dipole in a symmetric 1D substrate has been studied by \( ^{34} \).

IV. CONCLUDING REMARKS

We have analyzed the stochastic dynamics of a walker comprising two particles (Brownian motors) coupled through a bistable potential. This double-well potential is rocked by a periodic force in the presence of thermal noise, and thus is able to manifest the phenomenon of stochastic resonance. The walker is moving on a one-dimensional asymmetric ratchet potential by interchanging the order of the two particles and an ensemble of these walkers show an average directed current in the presence of noise. In this paper we show first that the internal degree of freedom of the walker exhibits stochastic resonance and then we show that the average current has a maximum for an optimal amount of noise. On the other hand, stochastic resonance is characterized by the amplitude of the periodic response as a function of noise; this amplitude shows a maximum for a particular value of the noise intensity. As we demonstrate in this paper, the maximum of the current and the maximum of the stochastic resonance occurs at the same amount of noise. Therefore, the main conclusion is that the synchronized walking due to stochastic resonance corresponds to an optimal transport of the walker. In this way, we show the intimate relation between stochastic resonance and the ratchet effect.

Acknowledgements

FRA gratefully acknowledges financial support from C3-UNAM posdoctoral fellowship.

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