About the balancing principle for choice of the regularization parameter

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Abstract. We discuss the balancing principle (called also Lepskii principle) for choosing the regularization parameter in solving linear ill-posed problems. Our results show that the balancing principle is closely related to some known parameter choice rules. We prove the weak quasioptimality of balancing principle and give some recommendation to improve this rule. In addition, we give the general rule for choosing the regularization parameter and show that the balancing principle and some other known rules are special cases of general rule.

1. Introduction
We consider an operator equation

\[ Au = f, \quad f \in R(A), \]

where \( A \in L(H, F) \) is the linear continuous operator between real Hilbert spaces \( H \) and \( F \). In general our problem is ill-posed: the range \( R(A) \) may be non-closed, the kernel \( N(A) \) may be non-trivial. We suppose that instead of exact right-hand side \( f \) we have only an approximation \( f_\delta \in F, \quad \|f_\delta - f\| \leq \delta \). To get regularized solution \( u_r \) of equation \( Au = f \) we consider the regularization methods in the general form (see [1]), using in general case (in non-selfadjoint case) the approximation

\[ u_r = (I - A^*Ag_r(A^*A))u_0 + g_r(A^*A)A^*f_\delta, \]

and in case of selfadjoint operator \( (H = F, \quad A = A^* \geq 0) \) the approximation

\[ u_r = (I - Ag_r(A))u_0 + g_r(A)f_\delta. \]

Here \( u_0 \) is initial approximation, \( r \) is regularization parameter, \( I \) is the identity operator and the function \( g_r(\lambda) \) satisfies the conditions (4)-(6).

\[ \sup_{0 \leq \lambda \leq a} \sqrt{\lambda} g_r(\lambda) \leq \gamma r, \quad r \geq 0, \]

\[ \sup_{0 \leq \lambda \leq a} \lambda^p |1 - \lambda| g_r(\lambda) \leq \gamma_p r^{-p}, \quad r \geq 0, \quad 0 \leq p \leq p_0, \]

\[ \sup_{0 \leq \lambda \leq a} g_r(\lambda) \leq \gamma r, \quad r \geq 0. \]
Here $p_0, \gamma, \gamma_*$ and $\gamma_p$ are positive constants, $a \geq \|A^*A\|$ for the approximation (2) and $a \geq \|A\|$ for the approximation (3), $\gamma_0 \leq \gamma$ (see (5)) and the greatest value of $p_0$, for which the inequality (5) holds is called the qualification of method.

The following pairs of regularization methods are special cases of general methods (2),(3) for problems with non-selfadjoint and selfadjoint problems respectively.

**M1.** The Tikhonov method $u_\alpha = (\alpha I + A^*A)^{-1}A^*f_\delta$ and the Lavrentiev method $u_\alpha = (\alpha I + A)^{-1}f_\delta$. Here $u_0 = 0$, $r = \alpha^{-1}, g_\alpha(\lambda) = (\lambda + r^{-1})^{-1}$, $p_0 = 1, \gamma = 1, \gamma_* = 1/2, \gamma_p = p^\rho(1-p)^{1-p}$.

**M2.** The iterative variants of the Tikhonov method and of the Lavrentiev method. Let $u_n = u_{n-1} - \mu(Au_{n-1} - f_\delta)$, $\mu \in (0,1/\|A^*A\|), n = 1,2,...$ (method (2)),

$u_n = u_{n-1} - \mu(Au_{n-1} - f_\delta)$, $\mu \in (0,1/\|A\|), n = 1,2,...$ (method (3)).

Here $r = n, g_\alpha(\lambda) = \frac{1}{\lambda} \left(1 - \frac{1}{1 + r\lambda}\right)^m$, $p_0 = m, \gamma = m, \gamma_* = \sqrt{m}, \gamma_p = \left(\frac{p}{m}\right)^p \left(1 - \frac{p}{m}\right)^{m-p}$.

**M3.** Explicit iteration scheme (the Landweber’s method). Let $u_n = u_{n-1} - \mu(Au_{n-1} - f_\delta)$, $\mu \in (0,1/\|A^*A\|), n = 1,2,...$ (method (2)),

$u_n = u_{n-1} - \mu(Au_{n-1} - f_\delta)$, $\mu \in (0,1/\|A\|), n = 1,2,...$ (method (3)).

Here $r = n, g_\alpha(\lambda) = \frac{1}{\lambda} \left(1 - \frac{1}{1 + \mu\lambda}\right)^m$, $p_0 = \infty, \gamma = \mu, \gamma_* = \sqrt{\mu}, \gamma_p = \left(\frac{p}{\mu^e}\right)^p$.

**M4.** Implicit iteration scheme. Let $\alpha > 0$ be a constant and

$u_n = (\alpha I + A^*A)^{-1}(\alpha u_{n-1} + A^*f_\delta), n = 1,2,...$ (method (2)),

$u_n = (\alpha I + A)^{-1}(\alpha u_{n-1} + f_\delta), n = 1,2,...$ (method (3)).

Here $r = n, g_\alpha(\lambda) = \frac{1}{\lambda} \left(1 - \frac{\alpha}{\alpha + \lambda}\right)^m$, $p_0 = \infty, \gamma = \frac{1}{\alpha}, \gamma_* = \frac{b_0}{\sqrt{\alpha}}$, where

$b_0 = \sup_{0<\lambda<\infty} \lambda^{-1/2}(1 - e^{-\lambda}) \approx 0.6382$ and $\gamma_p = (\alpha\rho)^p$.

**M5.** The method of the Cauchy problem: approximation $u_\alpha$ solves the Cauchy problem

$u'(r) + A^*Au(r) = A^*f_\delta, \ u(0) = u_0$ (method (2)), $u'(r) + Au(r) = f_\delta, \ u(0) = u_0$ (method (3)).

Here $g_\alpha(\lambda) = \frac{1}{\lambda} \left(1 - e^{-\lambda}\right)$, $p_0 = \infty, \gamma = 1, \gamma_* = b_0, \gamma_p = (p/e)^p$.

Note that for methods M1-M5 the function $f_\alpha(r) = 1 - \lambda g_\alpha(\lambda)$ is monotonically decreasing and the function $f_2(r) = g_\alpha(\lambda)$ is monotonically increasing.

2. **Parameter choice rules**

In the regularization methods of the form (2),(3) the important problem is the choice of a proper regularization parameter $r$. If $r$ is too big, the numerical implementation will be unstable and the
approximation $u_r$ will be useless; if $r$ is too small, the approximation $u_r$ is dominated by the initial guess $u_0$. The most prominent a posteriori parameter choice rule is the discrepancy principle.

**Discrepancy principle.** Let $b_1, b_2$ be the constants such that $b_2 \geq b_1 > 1$. If $\|Au_0 - f_\delta\| \leq b_2 \delta$, then choose $r(\delta) = 0$. In the contrary case choose the parameter $r = r(\delta) > 0$ for which

$$b_1 \delta \leq \|Au_r - f_\delta\| \leq b_2 \delta.$$

To characterize the quality of the rule of the a posteriori regularization parameter choice we use in the following the property of the weak quasioptimality (see [2]). We say that rule R for the a posteriori choice of the regularization parameter $r = r(R)$ is weakly quasioptimal, if there exists a constant $C$ (which does not depend on $A$, $u_*$, $f_\delta$) such that for each $f_\delta, \|f_\delta - f\| \leq \delta$ it holds the error estimate

$$\|\psi(r) - u_*\| \leq C \inf_{r \geq 0} \psi(r) + O(\delta),$$

where the function $\psi(r) = \| (I - A^* A g_r(A^*) (u_0 - u_*)) + \gamma \sqrt{r \delta} \|$ (in selfadjoint case $\psi(r) = \| (I - A g_r(A)) (u_0 - u_*)) + \gamma r \delta \|$) is an upper bound of the error $\|u_r - u_*\|$. It was shown in [14] that if the rule R is weakly quasioptimal, then the regularization method with parameter choice by rule R is order optimal on the set $M_{p,p} = \{u \in H : u - u_0 = (A^r A)^{p/2} v, \|v\| \leq p, p > 0\}$ for the full range $p \in (0, 2p_0]$ (in selfadjoint case $p \in (0, p_0]$).

Discrepancy principle is weakly quasioptimal for methods M3-M5, but it can be shown that for methods M1 and M2 the discrepancy principle is not weakly quasioptimal. For these methods we consider the following modification of the discrepancy principle [3,4], which is weakly quasioptimal also for methods with finite qualification. Define the operator $B_r$:

$$B_r = \begin{cases} I, & \text{if } p_0 = \infty \\ (I - A A^* A g_r(A^*) (u_0 - u_*))^{1/p_0}, & \text{if } p_0 < \infty \text{ and } A^* \neq A \\ (I - A g_r(A))^{1/p_0}, & \text{if } p_0 < \infty \text{ and } A^* = A \geq 0. \end{cases}$$

**The modification of the discrepancy principle (MD rule).** Let $b_1, b_2$ be the constants such that $b_2 \geq b_1 > 1$. If $\|B_r (Au_0 - f_\delta)\| \leq b_2 \delta$, then choose $r(\delta) = 0$. In the contrary case choose the parameter $r = r(\delta) > 0$ for which

$$b_1 \delta \leq \|B_r (Au_r - f_\delta)\| \leq b_2 \delta.$$

For methods with infinite qualification this rule coincides with the discrepancy principle. For methods with finite qualification we apply to the discrepancy the operator $B_r$. Note, that for Tikhonov method $(m = 1)$ and its iterative variant $u_{r,m}$ it holds $\|B_r (Au_{r,m} - f_\delta)\| = (Au_{r,m} - f_\delta, Au_{r,m} - f_\delta)^{1/2}$.

If the equation (1) has only the quasisolution, then the previous rules can not be used. The following rule [5-8] is weakly quasioptimal for method M1-M5 in case $Qf \in R(A)$, where $Q$ is orthoprojector onto $R(A)$. Denote $\varphi(r) = \sqrt{r} \|A^r B^2_r (Au_r - f_\delta)\|$.  

**Rule R1.** Let $b_1, b_2$ be the constants such that $b_2 \geq b_1 > \tilde{y}_{1/2}$, where $\tilde{y}_{1/2} = y_{1/2}$ for methods $p_0 = \infty$ and $\tilde{y}_{1/2} = (y_{p_0/2(p_0+2)})^{1/p_0}$ for methods $p_0 < \infty$. For the regularization parameter we choose the parameter $r = r(\delta)$ for which
\[ \phi(r) \leq b_2 \delta \quad \text{for each } r \geq r(\delta), \quad (7) \]

\[ \phi(r(\delta)) \geq b_1 \delta. \quad (8) \]

Note that the function \( \phi(r) \) is in general non-monotone and there may exist different intervals

\[ [r_1^1, r_2^1], [r_1^2, r_2^2], \ldots, [r_1^m, r_2^m], \quad r_1^1 \leq r_2^1 < r_1^2 \leq r_2^2 < \ldots < r_1^m \leq r_2^m, \]

for which \( b_1 \delta \leq \phi(r) \leq b_2 \delta \). Therefore we must use the conditions (7),(8) instead of last inequalities and we choose the parameter from last interval \([r_1^m, r_2^m]\). In [5,6] similar rule to rule R1 was considered, choosing parameter from the first interval \([r_1^1, r_2^1]\). This rule guarantees convergence \( \|u_{r(\delta)} - u_0\| \to 0 \) for \( \tilde{\delta} \to 0 \) also in case of approximately given noise level \( \tilde{\delta} \) provided that the ratio \( \|f_\delta - f\|/\tilde{\delta} \) is bounded in process \( \tilde{\delta} \to 0 \).

### 3. Balancing principle

In last years the balancing principle (called also Lepskii principle) for a posteriori parameter choice are considered in many papers ([9-16]). In case of non-selfadjoint operator the approximations \( u_\alpha \) are computed for values \( r_0 = \delta^{-2} \) and \( r_i = r_0 q^i \) with \( q < 1 \) \((i = 1,2,\ldots,m+1)\) and we choose for the regularization parameter \( r(\delta) = r_m \), where \( m \) is the first index, for which a certain condition is fulfilled. This condition is in [11,13]

\[ \|u_{r_m} - u_{r_{m+1}}\| > c\gamma_* \sqrt{r_m} \delta \quad (9) \]

and in [14]

\[ \exists j \in 1,\ldots,m : \|u_{r_j} - u_{r_{j+1}}\| > c\gamma_* \sqrt{r_j} \delta, \quad (10) \]

where the constant \( c = 4 \) and the constant \( \gamma_* \) is determined by inequality (4).

In the following we show that the balancing principle of the form (9) in case of non-selfadjoint operator is closely related to the Rule R1. It is easy to show that for methods M1-M5 the following inequalities hold for functions \( g_r(\lambda) \)

\[ \gamma(1-q)r_0 \beta_r(\lambda) \leq g_r(\lambda) - g_{q,r}(\lambda) \leq \gamma(1-q)r_0 \beta_{q,r}(\lambda) (1-\lambda g_r(\lambda)), \quad 0 < q < 1, \quad (11) \]

where \( \beta_r(\lambda) = (I - \lambda g_r(\lambda))^{p_0}/p_0 \) for methods with \( p_0 < \infty \) and \( \beta_r(\lambda) = 1 \) for methods with \( p_0 = \infty \). Using last inequalities and the equalities

\[ (I - AA^* g_r(AA^*) (Au_0 - f_\delta), \quad g(A^* A) = A^* g(AA^*) \]

we can estimate the difference of the approximate solutions:

\[ \|u_{r_m} - u_{r_{m+1}}\| = \|g_{r_m}(A^* A) - g_{q_{r_m}}(A^* A) (f_\delta - Au_0)\| \leq \gamma(1-q)r_m \|A^* B_{r_{q_m}}^2 (Au_{q_m} - f_\delta)\| \]

(12)

Here \( B_r = (I - A^* A g_r(A^*)^{1/2})^{p_0} \) and

\[ \|u_{r_m} - u_{r_{m+1}}\| \geq \gamma(1-q)r_m \|A^* B_{r_m}^2 (Au_{r_m} - f_\delta)\| \quad (13) \]

If \( m \) is the first index for which (9) holds, then from (12),(13) follows

\[ \phi(q_{r_m}) = \sqrt{q_{r_m}} \|A^* B_{r_{q_m}}^2 (Au_{q_m} - f_\delta)\| \geq c\gamma_* \sqrt{q}/(\gamma(1-q)) \delta, \quad (14) \]

\[ \phi(r_i) = \sqrt{r_i} \|A^* B_{r_i}^2 (Au_{r_i} - f_\delta)\| \leq (c\gamma_*/(\gamma(1-q))) \delta, \quad i < m \quad (15) \]

and
\[
\varphi(r_m) = \sqrt{r_m}A^*B_{r_m}^2(Au_{r_m} - f_\delta) \leq (c'\gamma \sqrt{\gamma (1-q)})\delta,
\]
where \( c' = \left\| u_{r_m} - u_{r_{m+1}} \right\| / \sqrt{r_m} > c \). As the function \( t(r) = \left\| A^*B_r^2(Au_r - f_\delta) \right\| = \left\| A^*(I - AA^*)g_r(0, (AA^*)^{1/2}) \right\| \) is decreasing, then \( \varphi(r) \leq \varphi(qr) / \sqrt{q} \) and from last inequality and from (15), (16) follows that
\[
\varphi(r) \leq (c'\gamma \sqrt{\gamma (1-q)})\delta, r \geq r_m.
\]
Hence, the balancing principle (9) can be reduced to the Rule R1 with constants
\[
b_1 = c\gamma \sqrt{\gamma (1-q)}, \quad b_2 = c\gamma \sqrt{\gamma (1-q)}q,
\]
the only difference is that the inequality (14) holds for \( r = qr_m \), but in Rule R1 for \( r = r_m \).

Now using the results of [2] we can prove the weak quasi-optimality of the balancing principle (9).

**Theorem 1.** Let \( Qf \in R(\mathcal{A}) \), \( \left\| f_\delta - f \right\| \leq \delta \) and the parameter \( r(\delta) = r_m \) be chosen according to the balancing principle (9) with \( c > (1-q)\sqrt{\gamma_1/\gamma} \). Then for methods M1-M5 it holds the error estimate
\[
\left\| u_{r(\delta)} - u_* \right\| \leq C(b_1, b_2, q) \inf_{r \geq 20} \psi(r).
\]

**Proof.** Let \( r_* \) be the parameter for which the function \( \psi(r) = \left\| I - A^*Ag_r(\mathcal{A}^*)u_0 - u_* \right\| + \gamma \sqrt{r} \delta \) has a global minimum. In [2] the following results were proved:

(i) if \( r_* < r(\delta) \) and \( \varphi(r(\delta)) \geq b_1 \delta \), \( b_1 > \gamma_1/\gamma \) then
\[
\left\| I - A^*Ag_r(\mathcal{A}^*)u_0 - u_* \right\| + \gamma \sqrt{r(\delta)} \delta \leq C_1(b_1) \inf_{r \geq 20} \psi(r), \quad C_1(b_1) > 1
\]

(ii) if \( r_* \geq r(\delta) \) and \( \varphi(r) \leq b_2 \delta \) for \( r \geq r(\delta) \) then
\[
\left\| I - A^*Ag_{r(\delta)}(\mathcal{A}^*)u_0 - u_* \right\| + \gamma \sqrt{r(\delta)} \delta \leq C_2(b_2) \inf_{r \geq 20} \psi(r).
\]

To prove theorem 1 we consider separately three cases a) \( r_* < qr(\delta) \), b) \( qr(\delta) \leq r_* < r(\delta) \) and c) \( r_* \geq r(\delta) \). If \( r_* < qr(\delta) \), then using (i) we get
\[
\left\| u_{r(\delta)} - u_* \right\| \leq \left\| I - A^*Ag_{r(\delta)}(\mathcal{A}^*)u_0 - u_* \right\| + \gamma \sqrt{r(\delta)} \delta \leq
\]
\[
\frac{1}{\sqrt{q}} \left\| I - A^*Ag_{\gamma r(\delta)}(\mathcal{A}^*)u_0 - u_* \right\| + \gamma \sqrt{r(\delta)} \delta \leq C_1(b_1) \sqrt{q} \inf_{r \geq 20} \psi(r).
\]

If \( qr(\delta) \leq r_* < r(\delta) \), then
\[
\left\| u_{r(\delta)} - u_* \right\| \leq \left\| I - A^*Ag_{qr(\delta)}(\mathcal{A}^*)u_0 - u_* \right\| + \frac{1}{\sqrt{q}} + \gamma \sqrt{r(\delta)} \delta \leq \frac{1}{\sqrt{q}} \inf_{r \geq 20} \psi(r).
\]

If \( r_* \geq r(\delta) \) then the error estimate follows from (ii). As \( C_1(b_1) > 1 \) for all \( b_1 > \gamma_1/\gamma \) then theorem 1 is proved with constant
\[
C(b_1, b_2, q) = \max \{C_1(b_1) \sqrt{q}, C_2(b_2) \}.
\]

Note that the constant \( b_2 \) and the coefficient quasi-optimality \( C(b_1, b_2, q) \) depends on an a priori unknown constant \( c' \) and may be computed separately for each problem. But if we choose in balancing principle (9) for the regularization parameter \( r(\delta) = r_{m-1} \), then \( \varphi(r_{m-1}) \leq (c'\gamma \sqrt{\gamma (1-q)})\delta \) (see (15)) and we can show that this choice is weakly quasi-optimal with a priori constant
\[
C(b_1, b_2, q) = \max \{C_1(b_1) \sqrt{q}, C_2(b_2) \}, \quad b_1 = c\gamma \sqrt{\gamma (1-q)}, \quad b_2 = c\gamma \sqrt{\gamma (1-q)}q.
\]
We see that weak quasioptimality of balancing principle (9) holds if we take the constant \( c \) much less than 4. For example, if \( q = 1/2 \) then for Tikhonov method \((1-q)\sqrt[4]{1/q} = \sqrt{2} \sqrt[4]{3/4} = \sqrt{2} 3\sqrt{3}/16\) and the reasonable choice of \( c \) is for example \( \sqrt{2} \sqrt[4]{3}/16 \). On the other hand, to avoid that the lower bound of the constant \( c \) goes to zero, if \( q \to 1 \), then instead of the condition (9) we recommend the condition
\[
\left\| u_n - u_{n+1} \right\| > c\gamma\sqrt{2} \sqrt{m} \delta / \sqrt{q} = c\gamma \sqrt{m} \delta / \sqrt{q} = c\gamma (r_m - r_{m+1})\delta / \sqrt{r_{m+1}}
\]
with constant \( c > \gamma \sqrt[4]{3}/\gamma^* \). We also recommend to take for \( r_m = (M/\gamma^*)^2 \delta^{-2} \), where \( M \) is an upper bound of the initial error \( \| u_0 - u_* \| \).

For balancing principle (10) using inequality (holds for methods M1-M5)
\[
g_s(\lambda) - g_r(\lambda) \leq \gamma(s - r) \beta_s(\lambda) (1 - \lambda g_s(\lambda)), \quad 0 \leq r \leq s,
\]
we can show that from (10) follows that
\[
\left\| B_{\psi_n} (Au_{\psi_n} - f) \right\| \geq \frac{c\gamma^*}{(1-q)^{m+1-j}} \delta \geq \frac{c\gamma^*}{\gamma(\gamma^*)^{1/2}} \delta,
\]
\[
\phi(r) \leq \left( c\gamma^*/\gamma(1-q) \right) \delta, \quad r \geq r_m.
\]

**Theorem 2.** Let \( f \in R(A) \), \( \| f_0 - f \| \leq \delta \) and the parameter \( r(\delta) = r_m \) be chosen according to the balancing principle (10) with \( c > \gamma(\gamma^*)^{1/2}/\gamma^* \). Then for methods M1-M5 it holds the error estimate
\[
\left\| u_{r(\delta)} - u_* \right\| \leq C(b_1, b_2) \inf_{r \geq 0} \psi(r).
\]

In the following table the lower bounds of the constant \( c \) for quasioptimality of the balancing principle are presented.

**Table 1.** Minimal coefficients for weak quasioptimality of the balancing principle

| Method                  | M1            | M2            | M3            | M4            | M5            |
|-------------------------|---------------|---------------|---------------|---------------|---------------|
| Balancing principle (9) | \( \frac{3\sqrt{3}}{8} \) \( 1 - q \) \sqrt{q} \( 1 - q \) \( 1 - q \) \( 1 - q \) | \( \frac{\sqrt{m}}{\sqrt{q}} \) \( 1 - q \) \( 1 - q \) \( 1 - q \) \( 1 - q \) \( 1 - q \) | \( \frac{1}{\sqrt{2e}} \) \( \sqrt{q} \) \( \sqrt{q} \) \( \sqrt{q} \) \( \sqrt{q} \) \( \sqrt{q} \) | \( \frac{1}{\sqrt{2e}} \) \( b_0 \) \( b_0 \) \( b_0 \) \( b_0 \) \( b_0 \) |
| Balancing principle (10)| 2             | \( (m-1)/m \) \( (m-1)/m \) \( (m-1)/m \) \( (m-1)/m \) \( (m-1)/m \) | \( e^{-1} \) \( b_0^{-1} \) \( b_0^{-1} \) \( b_0^{-1} \) \( b_0^{-1} \) \( b_0^{-1} \) |

One can show, that if in Tikhonov approximation parameter \( r \) is chosen by balancing principle and we consider error of Tikhonov approximation as the function of constant \( c \), then this error is the increasing function of \( c \), if \( r \) is chosen by balancing principle (9) with \( c > 2 \sqrt{1 - q}/q \) or by balancing principle (10) with \( c > \sqrt{1 - q}/q \). Instead of the condition \( \left\| u_{r_j} - u_{n+1} \right\| > c\gamma \sqrt{r_j} \delta \) in (10) we recommend the condition \( \left\| u_{r_j} - u_{n+1} \right\| > c\gamma \sqrt{r_j} \delta \), \( c > \gamma(\gamma^*)^{1/2} \).

In case of approximately given error bound \( \tilde{\delta} \) the following variant of the balancing principle is useful.
**Rule R2.** The approximations \( u_{\kappa} \) are computed for values \( r_0 = 1 \) and \( r_i = r_0 q^{-i} \) with \( q < 1 \) \((i = 1, 2, \ldots, m)\) and we choose for the regularization parameter \( r(\delta) = r_m \) where \( m \) is the first index, for which \( \| u_{\kappa} - u_{\kappa-1} \| \leq c \sqrt{r_m \delta} \).

It can be shown that rule R2 is stable parameter choice in this sense that \( u_{\kappa}(\delta) \to u_* \) if only the ratio \( \| f_\delta - f \| / \delta \) of the actual error of the right hand-side \( \| f_\delta - f \| \) and the supposed error level \( \delta \) is bounded in the process \( \delta \to 0 \).

Note that if the sequence of Tikhonov approximations \( u_{\kappa} \) is computed, for approximate solution one can take instead of some \( u_{\kappa} \) the linear combination of \( n \) Tikhonov approximations. In case of proper coefficients the resulting approximation has qualification \( p_0 = n \) (see [17]).

In case of selfadjoint operator the balancing principle (9) is the following: the approximations \( u_{\kappa} \) are computed for values \( r_0 = \delta^{-1} \) and \( r_i = r_0 q^i \) with \( q < 1 \) \((i = 1, 2, \ldots, m + 1)\) and we choose for the regularization parameter \( r(\delta) = r_m \) where \( m \) is the first index, for which \( \| u_{\kappa} - u_{\kappa-1} \| > c \gamma r_m \delta \).

This rule is closely related to the modification of the discrepancy principle. Using (11), we get
\[
B_{m_\kappa} (A u_{\kappa} - f_\delta) \geq c \delta / (1 - q),
\]
\[
B_{m} (A u_{\kappa} - f_\delta) \leq c' \delta / (1 - q), \quad r \leq r_m,
\]
where \( c' := \| u_{\kappa} - u_{\kappa-1} \| / (r_\kappa \delta) > c \). Hence, the balancing principle (9) in case of selfadjoint operator can be reduced to the modified discrepancy principle with constants \( b_1 = c/(1 - q), \quad b_2 = c'/(1 - q) \); the only difference is that here the inequality (19) holds for \( r = q r_m \), but in MD Rule for \( r = r_m \).

However, the balancing principle is computationally less demanding then rule R1 or MD rule.

4. General rule for weakly quasioptimal regularization parameter choice

In the following we formulate a new general weakly quasioptimal parameter choice rule for methods M1-M5, including as special cases many known rules.

In the error \( u_{\kappa} - u_* = (I - A^* A g_r (A^* A) (u_0 - u_*) + g_r (A^* A) (f_\delta - f) = w_\kappa + v_\kappa \) term \( w_\kappa \) is monotonically decreasing function of \( r \) and \( \| v_\kappa \| \) is monotonically increasing function of \( r \) for methods M1-M5. Therefore proper choice of parameter \( r \) is such, for which the inequality
\[
\frac{d(u_{\kappa} - u_*)}{dr} = \frac{du_{\kappa}}{dr} \geq \frac{dv_\kappa}{dr}, \quad \text{still holds.}
\]

**Gradient principle.** Let \( b > 1 \) be the constant. For the regularization parameter we choose the largest parameter \( r \), for which
\[
\frac{du_{\kappa}}{dr} \geq b \sup_{A, \alpha \in \mathbb{R} \setminus (1, F)} \left\| \frac{dv_\kappa}{dr} \right\|.
\]

It is easy to show that for methods M1-M5 in case of selfadjoint operator \( A = A^* \geq 0 \)
\[
\sup_{A, \alpha \in \mathbb{R} \setminus (1, F)} \left\| \frac{dv_\kappa}{dr} \right\| = \sup_{A, \alpha \in \mathbb{R} \setminus (1, F)} \left\| \frac{dg_r (A)}{dr} (f_\delta - f) \right\| = \gamma \delta.
\]
and in the general case $A \in L(H, F)$

$$
\sup_{A,\sigma \in (H, F)} \frac{d\gamma}{dr} = \sup_{A,\sigma \in (H, F)} \frac{d\gamma(A)A^*}{dr} \frac{(f_\sigma - f)}{\gamma} = \frac{\gamma}{\sqrt{r}}.
$$

Therefore, in case of selfadjoint operator the gradient principle for methods M1-M5 has form $\|u'_r\| \geq b \gamma \delta$ and in the general case $\sqrt{r} \|u'_r\| \geq b \gamma \sqrt{1/2} \delta$, where $u'_r = du_r / dr$.

On the other hand, for methods M1 and M2 it holds $\|u'_r\| = \gamma \|DB^*(Au_r - f_\sigma)\|$, for method M3 $\gamma \|D(Au_r - f_\sigma)\| \leq \|u'_r\| \leq \gamma \|D(Au_{r-1} - f_\sigma)\|$, for method M4 $\gamma \|B(Au_{r+1} - f_\sigma)\| \leq \|u'_r\| \leq \gamma \|D(Au_r - f_\sigma)\|$ and for method M5 $\|u'_r\| = \gamma \|D(Au_r - f_\sigma)\|$, where $D = I$ in case of selfadjoint operator and $D = A^*$ in the general case.

Thus, the gradient principle in case of selfadjoint operator for methods M1-M2 coincides with modified discrepancy principle and for method M5 with discrepancy principle. In case of non-selfadjoint operator for methods M1, M2 and M5 gradient principle coincides with Rule R1. For methods M3 and M4 the gradient principle is approximately same (the difference of chosen parameters is less or equal one) as discrepancy principle in case of selfadjoint operator and as the Rule R1 in the general case. Now from results about the modified discrepancy principle and the Rule R1 easily follows weak quasioptimality of the gradient principle.

The balancing principle of the form (9) follows from gradient principle, if we use instead of $\|u'_r\|$ the approximation $\|u'_{r_m} - u'_{r_{m+1}}\|/ (r_m - r_{m+1})$.

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