Pre-symplectic structures on the space of connections

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Abstract

Let $X$ be a four-manifold with boundary three-manifold $M$. We shall describe (i) a pre-symplectic structure on the space $\mathcal{A}(X)$ of connections on the bundle $X \times SU(n)$ that comes from the canonical symplectic structure on the cotangent space $T^*\mathcal{A}(X)$, and (ii) a pre-symplectic structure on the space $\mathcal{A}_0^0(M)$ of flat connections on $M \times SU(n)$ that have null charge. These two structures are related by the boundary restriction map. We discuss also the Hamiltonian feature of the space of connections $\mathcal{A}(X)$ with the action of the group of gauge transformations.

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0 Introduction

Let $X$ be an oriented Riemannian four-manifold with boundary $M = \partial X$ that may be empty. For the trivial principal bundle $P = X \times SU(n)$ we denote the space of irreducible $L^2_{s-\frac{1}{2}}$-connections by $\mathcal{A}(X)$. The tangent space of $\mathcal{A}(X)$ is

$$T_{\mathcal{A}}\mathcal{A}(X) = \Omega^{1}_{s-\frac{1}{2}}(X, \text{Lie } G).$$
Let $\mathcal{A}(M)$ be the space of irreducible $L^2_{s-1}$ connections on $M$. A connection is said to be flat if its curvature $F_A = dA + \frac{1}{2}[A \wedge A]$ vanishes. The space of flat connections over $X$ is denoted by $\mathcal{A}^\flat(X)$. Respectively that over $M$ is denoted by $\mathcal{A}^\flat(M)$. The tangent space of $\mathcal{A}^\flat(M)$ is given by

$$T_{A} \mathcal{A}^\flat(M) = \{a \in \Omega^1_{s-1}(M, \text{Lie} G); dAa = 0\}. \quad (0.1)$$

We consider the functional on $\mathcal{A}^\flat(M)$ defined by

$$CS(A) = \frac{1}{24\pi^3} \int_M Tr A^3, \quad A \in \mathcal{A}^\flat(M). \quad (0.2)$$

$\mathcal{A}^\flat(M)$ is decomposed into the disjoint union of connected components:

$$\mathcal{A}^\flat(M) = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_k^\flat(M)$$

with

$$\mathcal{A}_k^\flat(M) = \left\{ A \in \mathcal{A}^\flat(M); \int_M Tr A^3 = k \right\}. \quad (0.3)$$

The symplectic structure on the space of connections of a principal bundle over a Riemann surface was introduced in [1] and the geometric quantization theory of the moduli space of flat connections was investigated. In this paper we shall prove the following theorems.

**Theorem 0.1.** Let $P = X \times SU(n)$ be the trivial $SU(n)$–principal bundle on a four-manifold $X$. There exists a canonical pre-symplectic structure on the space of irreducible connections $\mathcal{A}(X)$ given by the 2-form

$$\sigma^{cs}_{A}(a, b) = \frac{1}{8\pi^3} \int_X Tr[(ab - ba)F_A] - \frac{1}{24\pi^3} \int_M Tr[(ab - ba)A], \quad (0.4)$$

for $a, b \in T_A \mathcal{A}(X)$

**Theorem 0.2.** Let $\omega$ be a 2-form on $\mathcal{A}(M)$ defined by

$$\omega_A(a, b) = -\frac{1}{24\pi^3} \int_M Tr[(ab - ba)A], \quad (0.5)$$

for $a, b \in T_A \mathcal{A}(M)$. Then $(\mathcal{A}_0^\flat(M), \omega|_{\mathcal{A}_0^\flat(M)})$ is a pre-symplectic manifold.

The author introduced in [7] the formula in the right hand side of (0.4), so the new feature of Theorem 0.1 is that this formula is obtained in a canonical manner. We explain why we call the 2-form $\sigma^{cs}$ canonical. On the
cotangent bundle $T^*\mathcal{A}(X)$ exists the canonical 1-form $\theta$ and the canonical 2-form $\sigma = \tilde{d}\theta$, where $\tilde{d}$ is the exterior differential on $\mathcal{A}(X)$. Let $s$ be a 1-form on $\mathcal{A}(X)$. Then $s$ gives a tautological section of the cotangent bundle $T^*\mathcal{A}(X)$ so that the pullback $\theta^s$ of $\theta$ is given by $\theta^s = s$. The pullback $\sigma^s$ of the canonical 2-form $\sigma$ is a closed 2-form on $\mathcal{A}(X)$. In particular if we take the 1-form given by

$$cs(A) = q(AF_A + F_A A - \frac{1}{2} A^3),$$

where $q = \frac{1}{24\pi^3}$, then $\sigma^{cs}$ is given by the equation (0.4). Thus, for a four-manifold $X$ there is a pre-symplectic form on $\mathcal{A}(X)$ that is induced from the canonical symplectic form on the cotangent bundle $T^*\mathcal{A}(X)$ by the generating function $cs : \mathcal{A}(X) \rightarrow T^*\mathcal{A}(X)$. This argument seems not to be adapted to the space of connections $\mathcal{A}(M)$ over a three-manifold $M$. Every principal $G$-bundle over a three-manifold $M$ is extended to a principal $G$-bundle over a four-manifold $X$ that cobord $M$, and for a connection $A \in \mathcal{A}(M)$ there is a connection $A \in \mathcal{A}(X)$ that extends $A$. So we detect a pre-symplectic structure on $\mathcal{A}(M)$ as the boundary restriction of the pre-symplectic structure $\sigma^{cs}$ on $\mathcal{A}(X)$. This does not work in general, that is, $\mathcal{A}(M)$ with the 2-form (0.5) gives only a presymplectic structure twisted by the Cartan 3-form $\kappa_A(a, b, c) = -\frac{1}{24\pi^3} \int_M Tr[(ab - ba)c]$. But restricted to the space of flat connections $\mathcal{M}^f(X)$ we can introduce the corresponding pre-symplectic structure.

Let $G_0(X)$ be the group of gauge transformations on $X$ that are identity on the boundary $M$. Then the closed 2-form $\sigma^{cs}$ in (0.4) is $G_0(X)$-invariant and the action of $G_0(X)$ on $\mathcal{A}(X)$ becomes a Hamiltonian action with the moment map given by the square of the curvature form $F_A^2$. [7] The generalization of the symplectic reduction to a pre-symplectic manifold is explained in [6]. Since 0 is not a regular value of the moment map $A \rightarrow F_A^2$ we can not apply this reduction theorem. But if the boundary $M$ is not empty the space of flat connections $\mathcal{A}^f(X)$ is a smooth manifold contained in the zero level set. Hence the canonical pre-symplectic structure on $\mathcal{A}(X)$ descends to the moduli space of flat connections $\mathcal{M}^f(X)$. The pre-symplectic structure on $\mathcal{M}^f(X)$ is given by the restriction of $\sigma^{cs}$ to the flat connections:

$$\sigma^{cs|_{\mathcal{M}^f(X)}}([a], [b]) = -\frac{1}{24\pi^3} \int_M Tr[(ab - ba)A]$$

(0.7) for $[A] \in \mathcal{M}^f(X)$ and $[a], [b] \in T_A\mathcal{M}^f(X)$, with $a, b \in T_A\mathcal{A}^f(X)$.

On the other hand we shall prove in section 4 the following
Theorem 0.3. The boundary restriction map

\[ \tau_X : \mathcal{M}^\flat(X) \rightarrow \mathcal{A}_0^\flat(M) \]  

(0.8)

is an isomorphism.

Hence we have the isomorphism of pre-symplectic manifolds

\[ (\mathcal{M}^\flat(X), \sigma_{cs}^\flat) \rightarrow (\mathcal{A}_0^\flat(M), \omega) . \]  

(0.9)

That proves Theorem 0.2.

1 Preliminaries on the space of connections

1.1 Calculation on the space of connections, [2, 3]

Let \( M \) be a compact, connected and oriented \( m \)-dimensional riemannian manifold possibly with boundary \( \partial M \). Let \( P^{\pi} \rightarrow M \) be a principal \( G \)-bundle, \( G = SU(N), N \geq 2 \).

We write \( \mathcal{A} = \mathcal{A}(M) \) the space of irreducible \( L^2\) connections over \( P \), which differ from a smooth connection by a \( L^2 \) section of \( T^*M \otimes \text{Lie}G \), hence the tangent space of \( \mathcal{A} \) at \( A \in \mathcal{A} \) is

\[ T_A\mathcal{A} = \Omega^{1}_{s-1}(M, \text{Lie}G) \].

(1.1)

The cotangent space of \( \mathcal{A} \) at \( A \) is

\[ T^*_A\mathcal{A} = \Omega^{m-1}_{s-1}(M, \text{Lie}G) \],

(1.2)

where the pairing \( \langle \alpha, a \rangle_A \) of \( \alpha \in T^*_A\mathcal{A} \) and \( a \in T_A\mathcal{A} \) is given by the symmetric bilinear form \( (X, Y) \rightarrow \text{tr}(XY) \) of \( \text{Lie}G \) and the Sobolev norm \( (\cdot, \cdot)_{s-1} \) on the Hilbert space \( L^2_{s-1}(M) \):

\[ \langle \psi \otimes X, \phi \otimes Y \rangle = (\psi, \phi)_{s-1} \text{tr}(XY) \],

for \( \psi \in \Omega^{m-1}(M), \phi \in \Omega^{1}(M) \), and \( X, Y \in \text{Lie}G \). We shall write it by \( \langle \alpha, a \rangle_A = \int_M \text{tr}(\alpha \land a) \), or simply by \( \int_M \text{tr}(\alpha a) \).

A vector field \( a \) on \( \mathcal{A} \) is a section of the tangent bundle; \( a(\mathcal{A}) \in T_A\mathcal{A} \), and a 1-form \( \varphi \) on \( \mathcal{A} \) is a section of the cotangent bundle; \( \varphi(\mathcal{A}) \in T^*_A\mathcal{A} \).

For a smooth map \( F = F(\mathcal{A}) \) on \( \mathcal{A} \) valued in a vector space \( V \) the derivation \( \partial_A F \) is defined by the functional variation of \( A \in \mathcal{A} \):

\[ \partial_A F : T_A\mathcal{A} \rightarrow V , \]

(1.3)

\[ (\partial_A F) a = \lim_{t \rightarrow 0} \frac{1}{t} ( F(A + ta) - F(A) ) , \]  

for \( a \in T_A\mathcal{A} \).  

(1.4)
For example, 

$$(\partial_A A) a = a,$$

since the derivation of an affine function is defined by its linear part. The curvature of $A \in \mathcal{A}$ is given by 

$$F_A = dA + \frac{1}{2} [A \wedge A] \in \Omega^2_{s-2}(M, \text{Lie } G).$$

So it holds that 

$$F_{A+a} = F_A + d_A a + a \wedge a,$$

and we have 

$$(\partial_A F_A) a = d_A a.$$

The derivation of a vector field $v$ on $\mathcal{A}$ and that of a 1-form $\varphi$ are defined similarly:

$$(\partial_A v)_a \in T_A \mathcal{A}, \quad (\partial_A \varphi)_a \in T^*_A \mathcal{A}, \quad \forall a \in T_A \mathcal{A}.$$

It follows that the derivation of a function $F = F(A)$ by a vector field $v$ is given by 

$$(vF)_A = (\partial_A F)(v_A).$$

We have the following formulas, [2].

\[ [v, w]_A = (\partial_A v)_A w - (\partial_A w)_A v, \tag{1.5} \]

\[ (v \langle \varphi, u \rangle)_A = \langle \varphi_A, (\partial_A u)_A \rangle + \langle (\partial_A \varphi)_A v, u_A \rangle. \tag{1.6} \]

Let $\tilde{d}$ be the exterior derivative on $\mathcal{A}(M)$. For a function $F$ on $\mathcal{A}(M)$, 

$$(\tilde{d}F)_A a = (\partial_A F)_a.$$ 

For a 1-form $\Phi$ on $\mathcal{A}(M)$, 

\[ (\tilde{d}\Phi)_A(a, b) \]

\[ = (\partial_A < \Phi, b >)a - (\partial_A < \Phi, a >)b - < \Phi, [a, b] > 
\]

\[ = < (\partial_A \Phi)a, b > - < (\partial_A \Phi)b, a >. \tag{1.7} \]

This follows from [1.5] and [1.6]. Likewise, if $\varphi$ is a 2-form on $\mathcal{A}(M)$ then it holds that 

\[ (\tilde{d}\varphi)_A(a, b, c) = (\partial_A \varphi(b, c))a + (\partial_A \varphi(c, a))b + (\partial_A \varphi(a, b))c. \tag{1.8} \]

We write the group of $L^2_s$-gauge transformations by $\mathcal{G}'(M)$: 

\[ \mathcal{G}'(M) = \Omega^0_s(M, \text{Ad } P). \tag{1.9} \]
Where $AdP = P \times_G G$ is the adjoint bundle associated to the principal bundle $P$. In this paper we shall mainly deal with the trivial principal bundle. In this case $\mathcal{G}'(M) = \Omega^0_s(M, G)$. $\mathcal{G}'(M)$ acts on $\mathcal{A}(M)$ by

$$g \cdot A = g^{-1}dg + g^{-1}Ag = A + g^{-1}dAg. \quad (1.10)$$

By Sobolev lemma one sees that $\mathcal{G}'(M)$ is a Banach Lie Group and its action is a smooth map of Banach manifolds.

In the following we choose a fixed point $p_0 \in M$ and deal with the group of gauge transformations that are identity at $p_0$:

$$\mathcal{G} = \mathcal{G}(M) = \{g \in \mathcal{G}'(M); g(p_0) = 1\}.$$  

$\mathcal{G}$ act freely on $\mathcal{A}$. Let $\mathcal{C}(M) = \mathcal{A}(M)/\mathcal{G}(M)$ be the quotient space of this action. It is a smooth infinite dimensional manifold.

We have

$$\text{Lie} (\mathcal{G}) = \Omega^0_s(M, adP).$$

Where $adP = P \times_G \text{Lie} G$ is the derived bundle of $AdP$. When $P$ is trivial $Lie (\mathcal{G}) = \Omega^0_s(M, Lie G)$. The infinitesimal action of $\mathcal{G}$ on $\mathcal{A}$ is described by

$$dA = d + [A \wedge ] : \Omega^0_s(M, adP) \longrightarrow \Omega^1_{s-1}(M, Lie G). \quad (1.11)$$

The fundamental vector field on $\mathcal{A}$ corresponding to $\xi \in \text{Lie}(\mathcal{G})$ is given by

$$d_A\xi = \frac{d}{dt}|_{t=0}(\exp t\xi) \cdot A,$$

and the tangent space to the orbit at $A \in \mathcal{A}$ is

$$T_A(\mathcal{G} \cdot A) = \{d_A\xi; \xi \in \Omega^0_s(M, adP)\}. \quad (1.12)$$

We have the following orthogonal decomposition of the tangent space:

$$T_A\mathcal{A}(M) = \{d_A\xi; \xi \in \Omega^0_s(M, adP)\} \oplus \{a \in \Omega^1_{s-1}(M, Lie G); d^*a = 0\}. \quad (1.13)$$

### 1.2 Moduli spaces $\mathcal{A}/\mathcal{G}$ and $\mathcal{A}/\mathcal{G}_0$

When $M$ has the boundary there are two types of gauge groups. Let $\mathcal{G}'(\partial M)$ be the group of $L^2_s-M$ gauge transformations on the boundary $\partial M$. We have the restriction map to the boundary:

$$r : \mathcal{G}'(M) \longrightarrow \mathcal{G}'(\partial M).$$
Let $G_0 = G_0(M)$ be the kernel of the restriction map. It is the group of gauge transformations that are identity on the boundary. $G_0$ acts freely on $A$ and $A/G_0$ is therefore a smooth infinite dimensional manifold, while the action of $G'$ is not free.

In the following we choose a fixed point $p_0 \in M$ on the boundary $\partial M$ and deal with the group of gauge transformations based at $p_0$:

$$G = G(M) = \{ g \in G'(M); g(p_0) = 1 \}.$$ 

If $\partial M = \phi$, $p_0$ may be any point of $M$. $G$ acts freely on $A$ and the orbit space $A/G$ is a smooth infinite dimensional manifold.

We have also the group $G(\partial M) = \{ g \in G'(\partial M); g(p_0) = 1 \}$, and the restriction map $r: G(M) \rightarrow G(\partial M)$ with the kernel $G_0$. We have

$$\text{Lie}(G_0) = \{ \xi \in \text{Lie}(G); \xi|_{\partial M} = 0 \}. \quad (1.14)$$

We have two moduli spaces of irreducible connections:

$$\mathcal{B}(M) = A/G_0, \quad \mathcal{C}(M) = A/G.$$ 

(1.15)

$\mathcal{B}(M)$ is a $G/G_0$-principal bundle over $\mathcal{C}(M)$. $\mathcal{C}(M)$ coincides with $\mathcal{B}(M)$ if $M$ has no boundary. $\mathcal{C}(M)$ is finite dimensional but in general $\mathcal{B}(M)$ is infinite dimensional, in fact it contains the orbit of $G(\partial M)$.

$\mathcal{B}(M)$ is a smooth manifold modelled locally on the ball in the subspace $\ker d^*_A$ of the Hilbert space $\Omega^2_{s-1}(M, \text{ad} P)$. $\mathcal{C}(M)$ is a smooth manifold modelled locally on the ball in the Hilbert subspace $\ker d^*_A \cap \ker(\ast|_{\partial M}) \subset \Omega^2_{s-1}(M, \text{ad} P)$. The reader can find the precise and technical description of these facts in [4, 5]. We shall supply a few aspect related to the Dirichlet and Neumann boundary value problems, and its relation to the horizontal subspaces of the fibrations $\mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{A} \rightarrow \mathcal{C}$. These facts have not direct necessity for our following argument but will make precise the role of two moduli spaces $\mathcal{B}$ and $\mathcal{C}$.

The Stokes formula is stated as follows:

$$\int_{\partial M} < f, \ast u > = \int_M < d_A f, \ast u > - \int_M < f, \ast d_A^* u >,$$

for $f \in \Omega^0_s(M, \text{Lie} G)$, $u \in \Omega^1_{s-1}(M, \text{Lie} G)$. If $M$ is a compact manifold without boundary we have the following decomposition:

$$T_A A = \{ d_A \xi; \xi \in \text{Lie}(G) \} \oplus H^0_A,$$ 

(1.16)

where

$$H^0_A = \{ a \in \Omega^1_{s-1}(M, \text{Lie} G); d_A^* a = 0 \}.$$
In this case we have

$$T_a \mathcal{B} = T_a \mathcal{C} \simeq H^0_A. \quad (1.17)$$

We shall look at the case when $M$ has the boundary. Let $\Delta_A$ be the covariant Laplacian defined as the closed extension of $d_A^* d_A$ with the domain of definition $\mathcal{D}_{\Delta_A} = \{ u \in \Omega^0_s(M, \text{Lie} G); u|\partial M = 0 \}$. Since $A \in \mathcal{A}$ is irreducible $\Delta_A : \mathcal{D}_{\Delta_A} \rightarrow \Omega^0_s(M, \text{Lie} G)$ is an isomorphism. Let $G_A = (\Delta_A)^{-1}$ be the Green operator of the Dirichlet problem:

$$\begin{align*}
\Delta_A u &= f \\
|u|\partial M &= 0
\end{align*}$$

Let $A \in \mathcal{A}$. We have the following orthogonal decomposition:

$$T_A \mathcal{A} = \{d_A \xi; \xi \in \text{Lie}(\mathcal{G}_0) \} \oplus H^0_A, \quad (1.18)$$

with

$$H^0_A = \{a \in \Omega^1_{s-1}(M, \text{Lie} G); d_A^* a = 0\}.$$

$a \in \Omega^1_{s-1}(M, \text{Lie} G)$ is decomposed to

$$a = d_A \xi + b, \quad \text{with} \quad \xi = G_A d_A^* a \in \text{Lie}(\mathcal{G}_0), \quad b \in H^0_A.$$

From this we see that the $\mathcal{G}_0$-principal bundle $\pi : \mathcal{A} \rightarrow \mathcal{B}$ has a natural connection defined by the horizontal subspace $H^0_A$, which is given by the connection 1-form $\gamma^0_A = G_A d_A^*$. The curvature form of $\gamma^0$ is given by

$$F^0_A(a, b) = G_A(*[a, *b]) \quad \text{for} \ a, b \in H^0_A.$$

Now we proceed to the fibration $\mathcal{A} \rightarrow \mathcal{C} = \mathcal{A}/\mathcal{G}$.

For a 1-form $v$ on $M$, let $g = G_A^{(v)} v$ denote the solution of the following Neuman boundary value problem:

$$\begin{align*}
\Delta_A g &= 0 \\
*d_A g|\partial M &= *v|\partial M.
\end{align*}$$

Let $A \in \mathcal{A}$. We have the orthogonal decomposition:

$$T_A \mathcal{A} = \{d_A \xi; \xi \in \text{Lie}(\mathcal{G})\} \oplus H^{(n)}_A, \quad (1.19)$$

where

$$H^{(n)}_A = \{a \in \Omega^1_{s-1}(M, \text{Lie} G); d_A^* a = 0, \text{ and } *a|\partial M = 0\}.$$
In fact let \( a \in \Omega^1(M, \text{Lie} G) \) and \( a = d_A \xi + b \) be the decomposition of \((1.18)\), then \( \xi = \gamma_A^0 a \in \text{Lie}(G_0) \) and \( b \in H^0_{\Delta_R} \). Put \( \eta = G_A^{(n)} b \). Then we have the orthogonal decomposition

\[
a = d_A (\xi + \eta) + c,
\]

with \( c \in H^{(n)}_{\Delta} \) and \( \xi + \eta \in \text{Lie}(G) \).

If we write

\[
\gamma_A = \gamma_A^0 + G_A^{(n)} (I - d_A \gamma_A^0),
\]

where \( I \) is the identity transformation on \( T_A \mathcal{A} \), then \( \gamma_A \) is a \( \text{Lie}(G) \)-valued 1-form which vanishes on \( H^{(n)}_{\Delta} \) and \( \gamma_A d_A \xi = \xi \), that is, \( \gamma_A \) is the connection 1-form of the fibration \( \mathcal{A} \rightarrow \mathcal{C} \). The curvature form of \( \gamma_A \) is given by

\[
F_A(a, b) = N_A(*[a, *b]) \quad \text{for } a, b \in H_A.
\]

Where \( N_A = (\Delta_A^{(n)})^{-1} \) is the Green operator of Neuman problem:

\[
\begin{cases}
\Delta_A^{(n)} \, g = f \\
* d_A g|_{\partial M} = 0 \quad \text{on } \partial M,
\end{cases}
\]

\( \Delta_A^{(n)} \) being the closed extension of \( d_A^* d_A \) with the domain of definition \( \mathcal{D}_{\Delta_A^{(n)}} = \{ u \in \Omega^0(M, \text{ad} \, P) ; \, * d_A u|_{\partial M} = 0 \} \).

1.3 Moduli space of flat connections

In the sequel we shall suppress the Sobolev indices. So \( \mathcal{A} \) is always the space of irreducible \( L^2_{s-1} \) connections and \( \mathcal{G} \) is the group of based \( L^2_s \) gauge transformations.

The space of flat connections is

\[
\mathcal{A}^\flat(M) = \{ A \in \mathcal{A}(M) ; F_A = 0 \},
\]

which we shall often abbreviate to \( \mathcal{A}^\flat \). The tangent space of \( \mathcal{A}^\flat \) is given by

\[
T_A \mathcal{A}^\flat = \{ a \in \Omega^1_{s-1}(M, \text{Lie} G) ; d_A a = 0 \}.
\]

The moduli space of flat connections is by definition

\[
\mathcal{M}^\flat = \mathcal{A}^\flat / \mathcal{G}_0.
\]

When there is a doubt about which manifold is involved, we shall write \( \mathcal{M}^\flat(M) \) for the orbit space \( \mathcal{M}^\flat = \mathcal{A}^\flat(M) / \mathcal{G}_0(M) \).
We know that $\mathcal{M}^\flat$ is a smooth manifold. In fact, the coordinate mappings are described by the implicit function theorem [5]. For $A \in \mathcal{A}^\flat$ there is a slice for the $G_0$-action on $\mathcal{A}^\flat$ given by the Coulomb gauge condition:

$$V_A = \{ a \in \Omega^1(M, \text{Lie} G); |a| < \epsilon, d_A a + a \wedge a = 0, d_A^* a = 0 \}. \tag{1.23}$$

Let

$$H_A^1 = \{ \Omega^1(M, \text{Lie} G); d_A a = 0, d_A^* a = 0 \}.$$

The Kuranishi map is defined by

$$K_A : \Omega^1(M, \text{Lie} G) \ni \alpha \mapsto K_A(\alpha) = \alpha + d_A^* G_A(\alpha \wedge \alpha) \in \Omega^1(M, \text{Lie} G).$$

Since the differential of $K_A$ at $\alpha = 0$ becomes the identity transformation on $\Omega^1(M, \text{Lie} G)$, the implicit function theorem in Banach space yields that $K_A$ gives an isomorphism on a small neighborhood of 0. Thus we see that the slice $A + V_A$ is a neighborhood of $A$ that is homeomorphic to the following subset of $H_A^1$:

$$\{ \beta \in H_A^1; |\beta| < \epsilon, \lambda_A(\beta) = 0 \},$$

where

$$\lambda_A(\beta) = (I - G_A \Delta_A)(\alpha \wedge \alpha), \quad \alpha = K_A^{-1} \beta.$$

We can also consider the moduli space of flat connections modulo the total gauge transformation group $\mathcal{G}$,

$$\mathcal{N}^\flat = \mathcal{A}^\flat / \mathcal{G}. \tag{1.24}$$

A slice in a neighborhood of $A \in \mathcal{A}^\flat$ in this case is

$$W_A = \{ a \in \Omega^1(M, \text{Lie} G); |a| < \epsilon, d_A a + a \wedge a = 0, d_A^* = 0, \text{and } *a|\partial M = 0 \}. \tag{1.25}$$

The Kuranishi map is defined by

$$L_A : \Omega^1(M, \text{Lie} G) \ni \alpha \mapsto L_A(\alpha) = \alpha + d_A^* N_A(\alpha \wedge \alpha) \in \Omega^1(M, \text{Lie} G).$$

The same argument as above yields that there is a slice through $A$ in $\mathcal{N}^\flat$ that is homeomorphic to

$$\{ \beta \in H_A^1; |\beta| < \epsilon, \mu_A(\beta) = 0 \},$$

where

$$\mu_A(\beta) = (I - N_A \Delta_A)(\alpha \wedge \alpha), \quad \alpha = L_A^{-1} \beta.$$

The dimension of $\mathcal{N}^\flat$ is finite but the dimension of $\mathcal{M}^\flat$ is in general not finite.
2 Canonical structure on $T^*A$

On the cotangent bundle of any manifold we have the notion of canonical symplectic form, and the standard theory of Hamiltonian mechanics and its symmetry follows from it [10]. Here we apply these standard notions to our infinite dimensional manifold $A(M)$ and write up their explicit formulas.

2.1 Canonical 1-form and 2-form on $T^*A$

Let $M$ be a manifold of dim $M = m$ possibly with the non-empty boundary. Let $T^*A \xrightarrow{\pi} A$ be the cotangent bundle. The tangent space to the cotangent space $T^*A$ at the point $(A, \lambda) \in T^*A$ becomes

$$T_{(A,\lambda)}T^*A = TA \oplus T^*_AA = \Omega^1(M, \text{Lie}G) \oplus \Omega^{m-1}(M, \text{Lie}G).$$

The canonical 1-form on the cotangent space is defined as follows. For a tangent vector $\left( \begin{array}{c} a \\ \alpha \end{array} \right) \in T_{(A,\lambda)}T^*A$,

$$\theta_{(A,\lambda)}\left( \begin{array}{c} a \\ \alpha \end{array} \right) = \langle \lambda, \pi^*\left( \begin{array}{c} a \\ \alpha \end{array} \right) \rangle_A = \int_M tr a \wedge \lambda. \quad (2.1)$$

Let $\phi$ be a 1-form on $A$. By definition, $\phi$ is a section of the cotangent bundle $T^*A$, so the pullback by $\phi$ of $\theta$ is a 1-form on $A$. We have the following tautological relation:

$$\phi^*\theta = \phi. \quad (2.2)$$

**Lemma 2.1.** The derivation of the 1-form $\theta$; is given by

$$\partial_{(A,\lambda)}\theta\left( \begin{array}{c} a \\ \alpha \end{array} \right) = \alpha, \quad \forall \left( \begin{array}{c} a \\ \alpha \end{array} \right) \in T_{(A,\lambda)}T^*A. \quad (2.3)$$

In fact,

$$(\partial_{(A,\lambda)}\theta)\left( \begin{array}{c} a \\ \alpha \end{array} \right) = \lim_{t \to 0} \frac{1}{t} \int_M (tr a \wedge (\lambda + t\alpha) - tr a \wedge \lambda) = \int_M tr a \wedge \alpha.$$

The canonical 2-form is defined by

$$\sigma = \tilde{d}\theta. \quad (2.4)$$

**Lemma 2.1** and (1.7) yields the following
Proposition 2.2.

\[
\sigma_{(A,\lambda)}\left(\begin{pmatrix} a \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ \beta \end{pmatrix}\right) = \int_M \text{tr}\left[b \wedge \alpha - a \wedge \beta\right] \quad (2.5)
\]

\(\sigma\) is a non-degenerate closed 2-form on the cotangent space \(T^*A\). We see the non-degeneracy as follows. Let \(\begin{pmatrix} a \\ \alpha \end{pmatrix}\) \(\in T_{(A,\lambda)} T^*A\), then \(a \in \Omega^1(M,\text{Lie}G)\) and \(\alpha \in \Omega^{m-1}(M,\text{Lie}G)\). Hence \(*\alpha \in \Omega^1(M,\text{Lie}G)\) and \(*a \in \Omega^{m-1}(M,\text{Lie}G)\) and we have

\[
\sigma_{(A,\lambda)}\left(\begin{pmatrix} a \\ \alpha \end{pmatrix}, \begin{pmatrix} *\alpha \\ *a \end{pmatrix}\right) = ||\alpha||^2 - ||a||^2.
\]

The formula implies the non-degeneracy of \(\sigma\).

For a function \(\Phi = \Phi(A,\lambda)\) on \(T^*A\) corresponds the Hamitonian vector field \(X_{\Phi}\)

\[
(\tilde{d}\Phi)_{(A,\lambda)} = \sigma(X_{\Phi}(A,\lambda), \cdot). \quad (2.6)
\]

Let \(\Phi = \Phi(A,\lambda)\) be a function on the cotangent space \(T^*A\). The directional derivative of \(\Phi\) at the point \((A,\lambda)\) to the direction \(a \in T_AA\) is given by \(\delta_A\Phi \in T_AA\) that is defined by the formula

\[
\langle \delta_A\Phi, a \rangle_A = \lim_{t \to 0} \frac{1}{t}\left(\Phi(A + ta, \lambda) - \Phi(A, \lambda)\right).
\]

Similarly the directional derivative of \(\Phi\) at the point \((A, \lambda)\) to the direction \(\alpha \in T^*_A A\) is \(\delta_\lambda\Phi \in T\bar{\lambda}_\alpha\) given by

\[
\langle \alpha, \delta_\lambda\Phi \rangle_A = \lim_{t \to 0} \frac{1}{t}\left(\Phi(A, \lambda + t\alpha) - \Phi(A, \lambda)\right)
\]

For any \(\begin{pmatrix} a \\ \alpha \end{pmatrix} \in T_{(A,\lambda)} T^*A\), it holds that

\[
(\tilde{d}\Phi)_{(A,\lambda)} \left(\begin{pmatrix} a \\ \alpha \end{pmatrix}\right) = \langle \delta_A\Phi, a \rangle_A + \langle \alpha, \delta_\lambda\Phi \rangle_A, \quad (2.7)
\]

So the Hamiltonian vector field of \(\Phi\) is given by

\[
X_{\Phi} = \begin{pmatrix} -\delta_\lambda\Phi \\ \delta_A\Phi \end{pmatrix}. \quad (2.8)
\]

In particular if we take the Hamitonian function

\[
H(A, \lambda) = \frac{1}{2} \int_M \text{tr}[F_A \wedge *F_A] + \frac{1}{2} \int_M \text{tr}[\lambda \wedge *\lambda], \quad (2.9)
\]
then, since $\delta_A H = d_A^* F_A$ and $\delta_A H = \ast \lambda$,

$$X_H = \left( - \ast \lambda \atop d_A(\ast F_A) \right).$$

(2.10)

It follows that the critical points of the Hamiltonian function $H(A, \lambda) = \frac{1}{2} \int_M tr[F_A \wedge \ast F_A]$ are given by the Yang-Mills equation: $d_A F_A = d_A^* F_A = 0$.

The group of (pointed) gauge transformations $G(M) = \Omega^0(M, \text{Lie} G)$ acts on $T_A A$ by the adjoint representation; $a \rightarrow \text{Ad}_{a} = a^{-1} g^{-1}$, and on $T_A^* A$ by its dual $\alpha \rightarrow g \alpha g^{-1}$. Hence the canonical 1-form and 2-form are $G$-invariant. The infinitesimal action of $\xi \in \text{Lie} G$ on the cotangent space $T^* A$ gives a vector field $\xi T A$ (called fundamental vector field) on $T^* A$ that is defined at the point $(A, \lambda)$ by the equation:

$$\xi_{T^* A}(A, \lambda) = \frac{d}{dt} \exp t \xi \cdot \left( \begin{array}{c} A \\ \lambda \end{array} \right) = \left( \begin{array}{c} d_A \xi \\ [\xi, \lambda] \end{array} \right).$$

(2.11)

The moment map of the action of $G$ on the symplectic space $(T^* A, \sigma)$ is described as follows. For each $\xi \in \text{Lie} G$ we define the function

$$J^\xi(A, \lambda) = \theta_{(A, \lambda)}(\xi_{T^* A}) = \int_M tr (d_A \xi \wedge \lambda).$$

(2.12)

Then the correspondence $\xi \rightarrow J^\xi(A, \lambda)$ is linear and defines a element of $J(A, \lambda) \in \text{Lie} G^*$ and we have a map

$$\Phi : T^* A \ni (A, \lambda) \longrightarrow J^\xi(A, \lambda) \in (\text{Lie} G)^*.$$  

(2.13)

Hence we have the following

**Theorem 2.3.**

1. The action of the group of gauge transformations $G(M)$ on the symplectic space $(T^* A(M), \sigma)$ is an hamiltonian action and the moment map is given by

$$J^\xi(A, \lambda) = \int_M tr (d_A \xi \wedge \lambda), \quad \forall \xi \in G(M).$$

(2.14)

2. In the case when $M$ has a boundary, the action of the group of gauge transformations $G_0(M)$ on the symplectic space $(T^* A(M), \sigma)$ is an hamiltonian action and the moment map is given by

$$J_0(A, \lambda) = d_A \lambda.$$

(2.15)

The second assertion follows from Stokes’ theorem since any $\xi \in \text{Lie} G_0$ has the boundary value 0.
2.2 Generating functions

Let
\[ \tilde{s} : A \rightarrow T^*A \]
be a local section of \( T^*A \). We write it by \( \tilde{s}(A) = (A, s(A)) \) with \( s(A) \in T_A^*A \).

Where "local" means that we consider the space of connections restricted to coordinate neighborhoods: \( P|U \rightarrow U \subset M \), and we abbreviate to notify the set \( U \subset M \).

The pullback of the canonical 1-form \( \theta \) by \( \tilde{s} \) defines a 1-form \( \theta^*s \) on \( A \):
\[
\theta^*_A(a) = (\tilde{s}^*\theta)_Aa, \quad a \in T_A^*A. \tag{2.16}
\]
From the definition we have the following tautological fact.

**Lemma 2.4.**
\[
\theta^*s = s. \tag{2.17}
\]
That is,
\[
(\theta^*)_Aa = \langle s(A), a \rangle. \tag{2.18}
\]
for \( a \in T_A^*A \).

Let \( \sigma^* = \tilde{s}^*\sigma \) be the pullback by \( \tilde{s} \) of the canonical 2-form \( \sigma \).
\[
\sigma^*_A(a, b) = \sigma_\tilde{s}(\tilde{s}^*a, \tilde{s}^*b) = \sigma_{(A,s(A))}(a, (\tilde{s}^*a), (\tilde{s}^*b)). \tag{2.19}
\]
\( \sigma^* \) is a closed 2-form on \( A \). From Lemma 2.4 we see
\[
\sigma^* = \tilde{d}s. \tag{2.20}
\]
s is a so-called (local) generating function. It is not necessarily non-degenerate.

**Remark 2.1.** If the 1-form \( \theta^* \) is invariant by the action of \( G(M) \) (respectively \( G_0(M) \)) then the pull-back of the moment map \( J \) on \( T^*A \), \( \tilde{s}^*J \), gives also the moment map \( (\tilde{s})^*J \) under the action of \( G(M) \) (respectively \( G_0(M) \)).

But this seems to be seldom. While it seems that the 2-form \( \sigma^* \) is always invariant under the action of \( G_0(M) \). See the next example.

**Example** ([Atiyah-Bott, 1982])
Let \( \Sigma \) be a surface (2-dimensional manifold).
\[
T_A^*A(\Sigma) \simeq T^*_A\Sigma(\Sigma) \simeq \Omega^1(\Sigma, LieG)
\]
Define the generating function
\[
s : A \ni A \rightarrow s(A) = A \in \Omega^1(\Sigma, LieG) = T^*_A\Sigma
\]
Then
\[ (\theta^s)_A a = \int_{\Sigma} tr(Aa), \]
and
\[ \omega_A(a, b) \equiv \sigma^s_A(a, b) = (\tilde{d}\theta^s)_A(a, b) = (\langle \partial_A \theta^s \rangle a, b) - (\langle \partial_A \theta^s \rangle b, a) \]
(2.21)
\[ = \int_{\Sigma} tr(ba) - \int_{\Sigma} tr(ab) = 2 \int_{\Sigma} tr(ba). \]
(2.22)

Then \((\mathcal{A}(\Sigma), \omega)\) is a symplectic manifold, in fact \(\omega\) is non-degenerate.

\section{Pre-symplectic structure on the space of connections on a four-manifold}

Let \(X\) be an oriented Riemannian four-manifold with boundary \(M = \partial X\) that may be empty. For the trivial principal bundle \(P = X \times SU(n)\) we denote as before the space of irreducible \(L^2_{s-\frac{1}{2}}\)-connections by \(\mathcal{A}(X)\) which is abbreviated to \(\mathcal{A}\) when there is no confusion. The tangent space is

\[ T_{A}(X) = \Omega^1_{s-\frac{1}{2}}(X, \text{Lie}\ G). \]

We define a section \(\tilde{c}s\) of the cotangent bundle of \(\mathcal{A}\) by

\[ \tilde{c}s(A) = (A, cs(A)) = \left( A, q(AF_A + F_A A - \frac{1}{2} A^3) \right). \]
(3.1)

so \(cs(A) = q(AF_A + F_A A - \frac{1}{2} A^3)\) is a 3-form on \(X\) valued in \(su(n)\), where \(q_3 = \frac{1}{24\pi^3}\).

The differential of \(\tilde{c}s\) becomes

\[ (\tilde{c}s)_A a = \left( q\left( aF_A + F_A a + A d_A a + d_A a A - \frac{1}{2}(aA^2 + AaA + A^2 a) \right) \right), \]

for any \(a \in T_A\).

\textbf{Lemma 3.1.} Let \(\theta^s = \tilde{c}s^* \theta\) and \(\sigma^s = \tilde{c}s^* \sigma\) be the pullback of the canonical forms by \(\tilde{c}s\). Then we have

\[ \theta^s_A(a) = \frac{1}{24\pi^3} \int_X tr\left[ (AF + F_A a - \frac{1}{2} A^3) a \right], \quad a \in T_A, \]
(3.2)

and

\[ \sigma^s_A(a, b) = \frac{1}{8\pi^3} \int_X tr[(ab - ba) F] - \frac{1}{24\pi^3} \int_{\partial M} tr[(ab - ba) A]. \]
(3.3)
The first equation follows from the definition: \((\tilde{c}s^* \theta)_{\mathcal{A}} = \langle c_s(A), a \rangle\). For \(a, b \in T_{\mathcal{A}} \mathcal{A}\),

\[
(\tilde{d} \theta^\alpha)_{\mathcal{A}}(a, b) = \langle (\partial_{\mathcal{A}} \theta^\alpha)a, b \rangle - \langle (\partial_{\mathcal{A}} \theta^\alpha)b, a \rangle
= \frac{1}{24\pi^3} \int_X \text{tr} [2(ab - ba)F - (ab - ba)A^2]
- (bd_{\mathcal{A}}a + d_{\mathcal{A}}ab - d_{\mathcal{A}}ba - d_{\mathcal{A}}b)a].
\]

But since

\[
d\text{tr}[(ab - ba)A] = \text{tr}[(bd_{\mathcal{A}}a + d_{\mathcal{A}}ab - d_{\mathcal{A}}ba - d_{\mathcal{A}}b)a] + \text{tr}[(ab - ba)(F + A^2)],\]

we have

\[
\sigma^c_{\mathcal{A}}(a, b) = \frac{1}{8\pi^3} \int_X \text{tr}[(ab - ba)F] - \frac{1}{24\pi^3} \int_M \text{tr}[(ab - ba)A], \quad (3.4)
\]

for \(a, b \in T_{\mathcal{A}} \mathcal{A}\).

**Theorem 3.2.** [7] Let \(P = X \times SU(n)\) be the trivial \(SU(n)\)-principal bundle on a four-manifold \(X\). There exists a pre-symplectic structure on the space of irreducible connections \(\mathcal{A}(X)\) given by the 2-form

\[
\sigma^c_{\mathcal{A}}(a, b) = \frac{1}{8\pi^3} \int_X \text{tr}[(ab - ba)F] - \frac{1}{24\pi^3} \int_M \text{tr}[(ab - ba)A], \quad (3.5)
\]

The 2-form \(\sigma^c\) is \(G_0(X)\)-invariant, so we have the following

**Corollary 3.3.** There exists a pre-symplectic structure on the moduli space of connections \(\mathcal{B}(X)\).

**Remark 3.1.** \(\tilde{c}s\) is not \(G_0(X)\)-equivariant. In fact we gave in [7] the formula of variation of \(\theta^\alpha = \tilde{c}s^* \theta\) by the action of \(G_0(X)\) that was important to construct an hermitian line bundle with connection over \(\mathcal{B}(X)\).

The pullback \(\tilde{c}s^* J\) of the moment map \(J : T^* \mathcal{A} \longrightarrow (\text{Lie } G_0)^*\), \((2.15)\), does not give a moment map under the action of \(G_0(X)\). But we have the following theorem.

**Theorem 3.4.** [7] The action of \(G_0(X)\) on \(\mathcal{A}(X)\) is a Hamiltonian action with the corresponding moment map given by

\[
\Phi : \mathcal{A}(X) \longrightarrow (\text{Lie } G_0)^* = \Omega^4(X, \text{Lie } G); A \longrightarrow F_A^2, \quad (3.6)
\]

\[
\langle \Phi(A), \xi \rangle = \Phi^\xi(A) = \frac{1}{8\pi^3} \int_X \text{tr}(F_A^2 \xi), \quad \forall \xi \in \text{Lie } G_0. \quad (3.7)
\]
If $X$ has no boundary and $A$ is a flat connection then $\sigma_A^s = 0$, so we have the following

**Proposition 3.5.** Let $X$ be a compact 4-manifold without boundary then

$$L^c = \{ \tilde{c}_s(A); \ A \in A^p(X) \}$$

is a Lagrangian submanifold of $T^*\mathcal{A}(X)$.

In fact the derivative $\partial_A \tilde{c}_s$ of the map $\tilde{c}_s$ is an isomorphism, so $L^c$ becomes a submanifold of $T^*\mathcal{A}$.

4 The space of flat connections on a three-manifold

In this section we study the space of connections on a 3-manifold $M$ by looking at the space of connections on a 4-manifold $X$ that cobord $M; \partial X = M$.

4.1 Chern-Simons function

It is a well known fact that given a principal $G$–bundle $P$ over a 3-manifold $M$ there exist an oriented 4-manifold $X$ with the boundary $\partial X = M$ and a $G$–bundle $\mathcal{P}$ over $X$ that extends $P$. And any connection $A$ on $P$ has an extension to a connection $A$ on $\mathcal{P}$. These are essentially the consequence of Tietz’s extension theorem of continuous function on a closed subset of a space $[9]$.

We denote by

$$r_X : \mathcal{A}(X) \longrightarrow \mathcal{A}(M),$$

the restriction map to the boundary of connections on $X$:

$$r_X(A) = A|M, \ A \in \mathcal{A}(X).$$

The tangent map of $r_X$ at $A \in \mathcal{A}(X)$ is

$$\rho_{X,A} : T_A \mathcal{A}(X) = \Omega^{1}_{s-\frac{1}{2}}(X, \text{Lie } G) \longrightarrow T_A \mathcal{A}(M) = \Omega^{1}_{s-1}(M, \text{Lie } G),$$

where $A = r_X(A)$.

The group of $L^2_s$-gauge transformations on $X$ is denoted by $\mathcal{G}(X)$. Similarly the group of $\widetilde{L}^2_s$-gauge transformations on $M$ is denoted by $\mathcal{G}(M)$. 17
\( \mathcal{G} = \mathcal{G}(M) \) is not connected and is divided into denumerable sectors labeled by the mapping degree

\[
\deg f = \frac{1}{24\pi^2} \int_M Tr(df^{-1})^3.
\]

(4.2)

We have the following relation:

\[
\deg(gf) = \deg(f) + \deg(g).
\]

(4.3)

The group of \( L^2_{s+1/2} \)-gauge transformations on \( X \) that are identity on the boundary \( M \) is denoted by \( \mathcal{G}_0(X) \). It is the kernel of the restriction map \( r_X : \mathcal{G}(X) \rightarrow \mathcal{G}(M) \).

If \( X \) is simply connected then \( f \in \mathcal{G}(M) \) is the restriction to \( M \) of a \( f \in \mathcal{G}(X) \) if and only if \( \deg f = 0 \). Thus we have the following exact sequence:

\[
1 \rightarrow \mathcal{G}_0(X) \rightarrow \mathcal{G}(X) \xrightarrow{r_X} \Omega_0^M G \rightarrow 1,
\]

(4.4)

here we denote

\[
\Omega_0^M G = \{ g \in \mathcal{G}(M) ; \deg g = 0 \}.
\]

On a 3-manifold any principal bundle has a trivialization. We choose a trivialization so that a connection becomes identified with a Lie algebra-valued 1-form. We define the 3-dimensional Chern-Simons function:

\[
CS_{(3)}(A) = \frac{1}{8\pi^2} \int_M Tr(AF - \frac{1}{3} A^3), \quad A \in \mathcal{A}(M).
\]

(4.5)

It depends on the trivialization only up to an integer. From the Stokes’ theorem, we have the well known relation:

\[
\int_X Tr[F^2_A] = \int_M Tr[AF_A - \frac{1}{3} A^3].
\]

(4.6)

The Chern-Simons function descends to define a map from \( \mathcal{B}(M) \) into \( \mathbb{R}/\mathbb{Z} \), and the critical points of the Chern-Simons function are the gauge equivalence classes of flat connections on \( P \).

**Proposition 4.1.** For \( A \in \mathcal{A}(M) \) and \( g \in \mathcal{G}(M) \), we have

\[
CS_{(3)}(g \cdot A) = CS_{(3)}(A) + \deg g.
\]

(4.7)
4.2 A twisted pre-symplectic structure on flat connections

It seems impossible to have a pre-symplectic structure on the space of connections $\mathcal{A}(M)$ that is induced from the canonical structure of the cotangent space $T^*\mathcal{A}(M)$. For example, if we take the generating function $\tilde{f} : \mathcal{A} \to T^*_A\mathcal{A} \simeq \Omega^2(M, \text{Lie} G)$ defined by the curvature;

$$\tilde{f}(A) = (A, F_A),$$

then

$$\sigma^f(a, b) = \sigma_{(A,F_A)}(\begin{pmatrix} a \\ d_Aa \\ \hline b \\ d_Ab \end{pmatrix}) = \int_M tr(b \wedge d_Aa - a \wedge d_Ab) = \int_M d(tr(ab)) = 0.$$

so $\tilde{d}F = \sigma^f = 0$ and every connection is a critical point of the generating function $F = F_A$. There would not be a good choice of a generating function.

On the other hand Theorem 3.2 presents us a 2-form on $\mathcal{A}(M)$ that is related to the boundary restriction of the canonical pre-symplectic form $\sigma^{cs}$ on $\mathcal{A}(X)$ of the four-manifold $X$ that cobord $M$. Things being so we shall investigate the following differential 2-form and 3-form on $\mathcal{A}(M)$:

$$\omega_A(a, b) = -q \int_M Tr[(ab - ba)A], \quad \kappa_A(a, b, c) = -3q \int_M Tr[(ab - ba)c],$$

for $a, b \in T_A\mathcal{A}$. Then

$$\tilde{d}\omega_A = \kappa_A.$$  \hspace{1cm} (4.10)

In fact, for $a, b, c \in T_A\mathcal{A}$, we have

$$\tilde{d}\omega_A(a, b, c) = 3\partial_A(\omega_A(a, b))(c) = -3q \int_M Tr[(ab - ba)c] = \kappa_A(a, b, c).$$

$(\mathcal{A}(M), \omega, \kappa)$ is a pre-symplectic manifold twisted by the 3-form $\kappa$.

Remark 4.1. For $G = SU(2)$, $\kappa$ and $\omega$ vanishes identically, \hspace{1cm} \Box Lemma 1.3. So in the following we consider mainly for the case $G = SU(n)$ with $n \geq 3$. 

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Let $\mathcal{A}^\flat = \mathcal{A}^\flat(M)$ be the space of flat connections;

$$\mathcal{A}^\flat(M) = \{ A \in \mathcal{A}(M); \; F_A = 0 \}.$$ 

The tangent space of $\mathcal{A}^\flat$ at $A \in \mathcal{A}^\flat$ is given by

$$T_A \mathcal{A}^\flat = \{ a \in \Omega^1(M, \text{Lie} G); \; d_A a = 0 \}. \quad (4.11)$$

From (1.13) we have the orthogonal decomposition

$$T_A \mathcal{A}^\flat = \{ d_A \xi; \; \xi \in \text{Lie} G(M) \} \oplus H_A^\flat, \quad (4.12)$$

where $H_A^\flat = \{ a \in \Omega^1(M, \text{ad} P); \; d^* a = d_A a = 0 \}$. $\mathcal{A}^\flat(M)$ is $G(M)$-invariant and $d_A \xi$ for $\xi \in \text{Lie} G(M)$ is a vector field along $\mathcal{A}^\flat(M)$ since $d_A d_A \xi = [F_A, \xi] = 0$. Moreover the action of $G(M)$ on $\mathcal{A}^\flat(M)$ is infinitesimally pre-symplectic. In fact, we have the following lemma:

**Lemma 4.2.** Let $i_{d_A \xi}$ and $L_{d_A \xi}$ denote respectively the inner derivative and the Lie derivative by the fundamental vector field $d_A \xi$. We have

$$i_{d_A \xi} \kappa = 0, \quad L_{d_A \xi} \omega = 0. \quad (4.13)$$

on $\mathcal{A}^\flat(M)$.

*Proof*

We have, for $a, b \in T_A \mathcal{A}^\flat$,

$$i_{d_A \xi} \kappa_A (a, b) = -3q \int_M Tr[(ab - ba)d_A \xi] = -3q \int_M dTr[(ab - ba)\xi] = 0,$$

because $d_A a = d_A b = 0$. Then $i_{d_A \xi} \tilde{d} \omega = i_{d_A \xi} \kappa = 0$ and

$$(L_{d_A \xi} \omega)_A (a, b) = (\tilde{d} i_{d_A \xi} \omega)_A (a, b) = \partial_A (i_{d_A \xi} \omega_A(b))(a) - \partial_A (i_{d_A \xi} \omega_A(a))(b)$$

$$= -\frac{1}{24\pi^3} \int_M Tr[(b \tilde{d} A \xi - d_A \xi b)a] + \frac{1}{24\pi^3} \int_M Tr[(a \tilde{d} A \xi - d_A \xi a)b]$$

$$= -\frac{1}{12\pi^3} \int_M Tr[(ab - ba)d_A \xi] = -\frac{1}{12\pi^3} \int_M dTr[(ab - ba)\xi]$$

$$= 0,$$

for $A \in \mathcal{A}^\flat$ and for $a, b \in T_A \mathcal{A}^\flat$. \hfill \Box

Note that the 1-form $i_{d_A \xi} \omega$ is explicitly given by

$$(i_{d_A \xi} \omega)_A (a) = q_3 \int_M Tr[(A^2 \xi + \xi A^2)a], \quad \text{for } a \in T_A \mathcal{A}^\flat(M).$$
4.3 pre-symplectic sectors of flat connections

Let $X$ be a 4-manifold that cobord $M$; $\partial X = M$. Let $\mathcal{A}(X)$ be the space of connections over the trivial bundle $X \times G$. Let $\mathcal{G}_0(X)$ is the group of gauge transformations that are trivial on the boundary. The tangent space of $\mathcal{A}(X)$ has the following orthogonal decomposition:

$$T_{\mathcal{A}}\mathcal{A}(X) = \{d_A\xi; \xi \in \text{Lie} \mathcal{G}_0(X)\} \oplus \{a \in \Omega^1(X, \text{Lie} G); d_A^* a = 0\}. \quad (4.14)$$

Let $\mathcal{A}^b(X)$ be the space of flat connections. We call $M^b(X) = \mathcal{A}^b(X)/\mathcal{G}_0(X)$ the moduli space of flat connections over $X$. The tangent space of $M^b(X)$ is identified from (4.14) with

$$T_{[A]}M^b(X) \simeq \{a \in \Omega^1(X, \text{Lie} G); d_A a = d_A^* a = 0\}, \quad (4.15)$$

where we suppressed the Sobolev index.

We denote by

$$r_X : \mathcal{A}^b(X) \rightarrow \mathcal{A}^b(M)$$

the restriction to the boundary of a connection $A \in \mathcal{A}^b(X)$:

$$r_X(A) = A|_M.$$ 

The tangent map of $r_X$ at $A \in \mathcal{A}^b(X)$ becomes

$$\rho_{X,A} \colon \{a \in \Omega^1(X, \text{Lie} G); d_A a = 0\} \rightarrow \{a \in \Omega^1(M, \text{Lie} G); d_A a = 0\}.$$ 

We often use bold face for connections on a manifold that extend connections on its boundary. But this is not definitive and we use plain symbols when no confusion occurs.

Next we shall investigate the range of $r_X : \mathcal{A}^b(X) \rightarrow \mathcal{A}^b(M)$ that is independent of the cobording 4-manifold $X$.

Let $A \in \mathcal{A}^b(M)$. Let $\tilde{X}$ be the universal covering of $X$ and $\tilde{M}$ be the subset of $\tilde{X}$ that lies over $M$. Let $f_A$ be the parallel transformation by $A$ along the paths starting from $m_0 \in M$. It defines a smooth map on the covering space $\tilde{M}$: $f = f_A \in \text{Map}(\tilde{M}, G)$, such that $f^{-1} df = A$. Then the degree of $f$ is equal to

$$\deg f = \frac{1}{24\pi^2} \int_M Tr A^3 = \text{CS}_3(A). \quad (4.16)$$

If the integral vanishes:

$$\int_M Tr A^3 = 0,$$

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then there is a $f \in \mathcal{G}(\tilde{X})$ that extends $f$. Therefore $A = f^{-1}df \in \mathcal{A}^\flat(X)$ gives a flat extension of $A$ over $X$ such that $r_X(A) = A$.

For $A \in \mathcal{A}^\flat(M)$ and $a \in T_A \mathcal{A}^\flat(M)$, we have

$$\left(\tilde{d}\text{CS}_{(3)}\right)_A a = \frac{1}{8\pi^2} \int_M Tr(A^3a) = \frac{1}{8\pi^2} \int_M dTr(Aa) = 0.$$  \hspace{1cm} (4.17)

Hence $\text{CS}_{(3)}$ is constant on every connected component of $\mathcal{A}^\flat(M)$.

We introduce the following subspace of connections on $M$.

**Definition 4.1.** For each $k \in \mathbb{Z}$ we define

$$\mathcal{A}^\flat_k(M) = \left\{ A \in \mathcal{A}^\flat(M) ; \int_M TrA^3 = k \right\}.$$ \hspace{1cm} (4.18)

We call $\mathcal{A}^\flat_k(M)$ the $k$-sector of the flat connections.

By virtue of Proposition 4.1 we see that $\mathcal{A}^\flat_0(M)$ is invariant under the action of $G^M$.

**Proposition 4.3.** For any 4-manifold $X$ with the boundary $M$ we have the following properties:

1. The image of $r_X$ is precisely $\mathcal{A}^\flat_0(M)$.

2. $d_A(LieG(M)) \in T_A\mathcal{A}^\flat_0(M)$.

3. The action of the group of gauge transformations $G(M)$ on $\mathcal{A}^\flat_0(M)$ is infinitesimally symplectic.

Proof

It follows from the above discussion that any $A \in \mathcal{A}^\flat_0(M)$ is the boundary restriction of a $A \in \mathcal{A}^\flat(X)$. Conversely let $A = r_X(A)$ for a $A \in \mathcal{A}^\flat(X)$. Then

$$\int_M TrA^3 = \int_X TrA^4 = 0,$$

and $A \in \mathcal{A}^\flat_0(M)$. Thus, for any 4-manifold $X$ that cobord $M$ the image of $r_X$ is precisely $\mathcal{A}^\flat_0(M)$. The properties 2 and 3 are restatement of the facts

$$d_A \xi \in T_A\mathcal{A}^\flat_0(M), \quad L_{d_A \xi} \omega = 0.$$

The orthogonal complement $H^\flat_A(M)$ of $d_A(LieG(M))$ in $T_A\mathcal{A}^\flat_0(M)$, (4.12), is identified with $H^\flat_A(M, LieG)$. This is non-zero if and only if the connection can be deformed infinitesimally within $\mathcal{M}^\flat(M)$.

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Lemma 4.4. Let $X$ be a 4-manifold with $\partial X = M$ then $r_X$ is a submersion.

Take $A \in \mathcal{A}^0(M)$ and $\mathbf{A} \in \mathcal{A}^b(X)$ such that $r_X(\mathbf{A}) = A$. Let $a \in T_A \mathcal{A}^b(M)$. From (4.12), $a$ is decomposed into

$$a = d_A \xi + b,$$

by $\xi \in \Omega^0(M, \text{Lie}_G)$ and $b \in H^b_A(M)$. Let $\eta \in \Omega^0(X, \text{Lie}_G)$ be an extension to $X$ of $\xi$, then

$$\rho_{X, A}(d_A \eta) = d_A \xi.$$ 

On the other hand the spaces of $\Delta_A$-harmonic 1-forms $H^b_A(M)$ and $H^b_A(X)$ are isomorphic to the cohomology group $H^1_A(M, \text{Lie}_G)$ and $H^1_A(X, \text{Lie}_G)$ respectively. Since the cohomology groups with compact support of $X$; $H^k_{\Delta_A,c}(X, \text{Lie}_G)$ vanishes for $k = 1, 2$, we have

$$H^1_A(X, \text{Lie}_G) \simeq H^1_A(M, \text{Lie}_G).$$

Hence there is $b \in H^1_A(X, \text{Lie}_G) = H^b_A(X)$ such that

$$b = (\rho_X)_A b + d_A \alpha = (\rho_X)_A (b + d_A \beta),$$

with $\beta \in \Omega^0(X, \text{Lie}_G)$ and $\alpha = r_X(\beta) \in \Omega^0(M, \text{Lie}_G)$. ($b + d_A \beta$) being in $T_A \mathcal{A}^b(X)$ the lemma is proved.

$$(\rho_X)_A (d_A \eta + b + d_A \beta) = d_A \xi + b = a.$$

\[\blacksquare\]

Theorem 4.5. $(\mathcal{A}^0_0(M), \omega)$ is a pre-symplectic manifold.

Proof

We must show

$$\overline{d} \omega_A = \kappa_A = 0,$$

for any $A \in \mathcal{A}^0_0(M)$. Let $X$ be a 4-manifold with boundary $\partial X = M$ and let $\mathbf{P}$ be a $G$–bundle over $X$ with a connection $\mathbf{A}$ such that $A = r_X \mathbf{A}$.

Let $a, b, c \in T_A \mathcal{A}^b(M)$. $\rho_{X, A}$ being surjective, there are $a, b, c \in T_A \mathcal{A}^b(X)$ that extend $a, b, c$ respectively. Then we have

$$\kappa_A(a, b, c) = -q \int_M Tr[ (ab - ba)c]$$

$$= -q \int_X Tr[ (d_Aa b - ad_Ab - d_A b a + b d_A a)c + (ab - ba)d_A c]$$

$$= 0,$$

(4.19)
because of $d_{A}a = 0$, etc..

Let $\mathcal{M}^{0}(X)$ be as was introduced in 1.3 the moduli space of flat con-
nexions over $X$. Because of Theorem 3.2 $\mathcal{M}^{0}(X)$ is endowed with the pre-
symplectic structure

$$\sigma^{*}_{[A]}(a, b) = -q \int_{M} Tr[(ab - ba)A],$$ \hspace{1cm} (4.20)

for $A \in \mathcal{A}^{0}(X)$ and $a, b \in T_{A}\mathcal{A}^{0}(X)$, where $A = r_{X}(A)$ and $a = \rho_{X}(a), b = \rho_{X}(b)$. The right hand side is the pre-symplectic form on $\mathcal{A}^{0}(M)$ that coincides with $\omega_{A}(a, b)$.

We have evidently $r_{X}(g \cdot A) = r_{X}(A)$ for $g \in G_{0}$. Hence it induces the map

$$\tau_{X} : \mathcal{M}^{0}(X) \longrightarrow \mathcal{A}^{0}(M).$$ \hspace{1cm} (4.21)

**Proposition 4.6.** $\tau_{X}$ gives a diffeomorphism of $\mathcal{M}^{0}(X)$ to $\mathcal{A}^{0}_{0}(M)$.

**Proof**

We have already seen that $r_{X} : \mathcal{A}^{0}(X) \longrightarrow \mathcal{A}^{0}(M)$ is a surjective submersion. Hence it is enough to prove that $\tau_{X}$ is injective immersion. In fact, let $r_{X}(A_{1}) = r_{X}(A_{2})$ for $A_{1}, A_{2} \in \mathcal{A}^{0}(X)$, and let $f_{A_{i}}, i = 1, 2$, be the parallel transformations by $A_{i}, i = 1, 2$, respectively, along the paths starting from $m_{0} \in M$. It defines a smooth map on the universal covering space $\tilde{X} \xrightarrow{\pi} X$; $f_{i} = f_{A_{i}} \in Map(\tilde{X}, G)$, such that $f_{i}^{-1}df_{i} = A_{i}$. Since $r_{X}(A_{1}) = r_{X}(A_{2})$ these parallel transformations coincide along the paths contained in $M$, that is, $f_{1}$ and $f_{2}$ coincide on the covering space $M = \pi^{-1}(M)$ of $M$. Let $\tilde{g} \in Map(\tilde{X}, G)$ be such that $f_{2} = \tilde{g} \cdot f_{1}$. Then $\tilde{g}$ descends to a $g \in G_{0}(X)$ such that $A_{2} = g \cdot A_{1}$. Therefore $\tau_{X}$ is injective.

The restriction of $d_{A}LieG_{0}(X)$ on the boundary $M$ is obviously 0. From (4.14) the orthogonal complement of $d_{A}LieG_{0}(X)$ in $T_{A}\mathcal{A}^{0}(X)$ consists of those $a \in \Omega^{1}(X, LieG)$ that satisfies $d_{A}a = d_{A}^{*}a = 0$. Therefore $a = 0$ if $a|M = 0$, hence

$$\ker \rho_{X,A} = d_{A}LieG_{0}(X).$$

Thus $\tau_{X}$ is an injective immersion.

**Theorem 4.7.**

$$\tau_{X} : \mathcal{M}^{0}(X) \longrightarrow \mathcal{A}^{0}_{0}(M)$$

gives an isomorphism of pre-symplectic manifolds;

$$(\mathcal{M}^{0}(X), \sigma^{cs}) \simeq (\mathcal{A}^{0}_{0}(M), \omega).$$ \hspace{1cm} (4.22)
The group of gauge transformations $G(X)$ acts on $\mathcal{A}(X)$ and restricted to the space $\mathcal{A}^\mathfrak{b}(X)$ of flat connections the action is infinitesimally symplectic. This is seen by exactly the same calculation as in Lemmas 4.2 where it is proved that the action of $\Omega_0^0 G$ on $\mathcal{A}_0^0(M)$ is infinitesimally symplectic. Since

$$\mathcal{N}^\mathfrak{b}(X) = \mathcal{A}^\mathfrak{b}(X)/G \simeq \mathcal{M}(X)/\Omega_0^0 G,$$

we have the presymplectic reduction $(\mathcal{N}^\mathfrak{b}(X), \sigma^{cs} = \omega)$ and the following equivalence of the moduli spaces of flat connections on $X$ and $M$ holds.

**Proposition 4.8.**

$$\mathcal{N}^\mathfrak{b}(X) \simeq \mathcal{A}_0^0(M)/\Omega_0^0 G. \quad (4.23)$$

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