SEVERAL COMMENTS ABOUT THE COMBINATORICS OF $\tau$-COVERS

BOAZ TSABAN

Abstract. In a previous work with Mildenberger and Shelah, we showed that the combinatorics of the selection hypotheses involving $\tau$-covers is sensitive to the selection operator used. We introduce a natural generalization of Scheepers’ selection operators, and show that:

(1) A slight change in the selection operator, which in classical cases makes no difference, leads to different properties when $\tau$-covers are involved.

(2) One of the newly introduced properties sheds some light on a problem of Scheepers concerning $\tau$-covers.

Improving an earlier result, we also show that no generalized Luzin set satisfies $U_{\text{fin}}(\Gamma, T)$.

1. Introduction

Topological properties defined by diagonalizations of open or Borel covers have a rich history in various areas of general topology and analysis, and they are closely related to infinite combinatorial notions, see [8, 12, 5, 13] for surveys on the topic and some of its applications and open problems.

Let $X$ be an infinite set. By a cover of $X$ we mean a family $\mathcal{U}$ with $X \notin \mathcal{U}$ and $X = \bigcup \mathcal{U}$. A cover $\mathcal{U}$ of $X$ is said to be

(1) a large cover of $X$ if: $(\forall x \in X) \{U \in \mathcal{U} : x \in U\}$ is infinite.
(2) an $\omega$-cover of $X$ if: $(\forall \text{ finite } F \subseteq X)(\exists U \in \mathcal{U})\ F \subseteq U$.
(3) a $\tau$-cover of $X$ if: $\mathcal{U}$ is a large cover of $X$, and $(\forall x, y \in X) \{U \in \mathcal{U} : x \in U \text{ and } y \notin U\}$ is finite, or $\{U \in \mathcal{U} : y \in U \text{ and } x \notin U\}$ is finite.
(4) a $\gamma$-cover of $X$ if: $\mathcal{U}$ is infinite and $(\forall x \in X) \{U \in \mathcal{U} : x \notin U\}$ is finite.

1991 Mathematics Subject Classification. 03E05, 54D20, 54D80.
Key words and phrases. combinatorial cardinal characteristics of the continuum, $\gamma$-cover, $\omega$-cover, $\tau$-cover, selection principles, Borel covers, open covers.

Partially supported by the Koshland Center for Basic Research.
Let $X$ be an infinite, zero-dimensional, separable metrizable topological space (in other words, a set of reals). Let $\Omega$, $\tau$, and $\Gamma$ denote the collections of all open $\omega$-covers, $\tau$-covers and $\gamma$-covers of $X$, respectively. Additionally, denote the collection of all open covers of $X$ by $\mathcal{O}$. Similarly, let $C_\Omega$, $C_\tau$, $C_\Gamma$, and $C$ denote the corresponding collections of clopen covers. Our restrictions on $X$ imply that each member of any of the above classes contains a countable member of the same class [11]. We therefore confine attention in the sequel to countable covers, and restrict the above four classes to contain only their countable members. Having this in mind, we let $B_\Omega$, $B_\tau$, $B_\Gamma$, and $B$ denote the corresponding collections of countable Borel covers.

Let $A$ and $B$ be any of the mentioned classes of covers (but of the same descriptive type, i.e., both open, or both clopen, or both Borel). Scheepers [7] introduced the following selection hypotheses that $X$ might satisfy:

- $S_1(A, B)$: For each sequence $\langle U_n : n \in \mathbb{N} \rangle$ of members of $A$, there exist members $U_n \in U_n$, $n \in \mathbb{N}$, such that $\{U_n : n \in \mathbb{N}\} \in B$.
- $S_{fin}(A, B)$: For each sequence $\langle U_n : n \in \mathbb{N} \rangle$ of members of $A$, there exist finite (possibly empty) subsets $F_n \subseteq U_n$, $n \in \mathbb{N}$, such that $\bigcup_{n \in \mathbb{N}} F_n \in B$.
- $U_{fin}(A, B)$: For each sequence $\langle U_n : n \in \mathbb{N} \rangle$ of members of $A$ which do not contain a finite subcover, there exist finite (possibly empty) subsets $F_n \subseteq U_n$, $n \in \mathbb{N}$, such that $\{\bigcup F_n : n \in \mathbb{N}\} \in B$.

Some of the properties are never satisfied, and many equivalences hold among the meaningful ones. The surviving properties appear in Figure 1, where an arrow denotes implication [10]. It is not known whether any other implication can be added to this diagram – see [6] for a summary of the open problems concerning this diagram.

Below each property $P$ in Figure 1 appears its critical cardinality, $\text{non}(P)$, which is the minimal cardinality of a space $X$ not satisfying that property. The definitions of most of the cardinals appearing in this figure can be found in [2, 1], whereas $\mathfrak{od}$ is defined in [6], and the results were established in [4, 10, 9, 6].

A striking observation concerning Figure 1 is, that in the top plane of the figures, the critical cardinality of $\Pi(\Gamma, B)$ for $\Pi \in \{S_1, S_{fin}, U_{fin}\}$ is independent of $\Pi$ in all cases except for that where $B = \tau$. We demonstrate this anomaly further in Section 2, where we also give a partial answer to a problem of Scheepers. In Section 3 we show that no Luzin set satisfies $U_{fin}(\Gamma, T)$, improving a result from [10].
2. Generalized selection hypotheses

**Definition 2.1.** Let $\kappa < \lambda$ be any (finite or infinite) cardinal numbers. Denote

\[ S_{[\kappa, \lambda]}(\mathcal{A}, \mathcal{B}) : \text{For each sequence } \langle U_n : n \in \mathbb{N} \rangle \text{ of members of } \mathcal{A}, \text{ there exist subsets } F_n \subseteq U_n \text{ with } \kappa \leq |F_n| < \lambda \text{ for each } n \in \mathbb{N}, \text{ and } \bigcup_n F_n \in \mathcal{B}. \]

\[ U_{[\kappa, \lambda]}(\mathcal{A}, \mathcal{B}) : \text{For each sequence } \langle U_n : n \in \mathbb{N} \rangle \text{ of members of } \mathcal{A} \text{ which do not contain subcovers of size less than } \lambda, \text{ there exist subsets } F_n \subseteq U_n \text{ with } \kappa \leq |F_n| < \lambda \text{ for each } n \in \mathbb{N}, \text{ and } \bigcup_n F_n : n \in \mathbb{N} \in \mathcal{B}. \]

So that $S_{[1, 2]}(\mathcal{A}, \mathcal{B})$ is $S_1(\mathcal{A}, \mathcal{B})$, $S_{[0, \infty)}(\mathcal{A}, \mathcal{B})$ is $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$, and $U_{[0, \infty)}(\mathcal{A}, \mathcal{B})$ is $U_{\text{fin}}(\mathcal{A}, \mathcal{B})$.

**Definition 2.2.** Say that a family $\mathcal{A} \subseteq \{0, 1\}^{N \times N}$ is semi $\tau$-diagonalizable if there exists a partial function $g : \mathbb{N} \to \mathbb{N}$ such that:

1. For each $A \in \mathcal{A}$: $(\exists \in n \in \text{dom}(g)) A(n, g(n)) = 1$;
2. For each $A, B \in \mathcal{A}$:
   - Either $(\forall n \in \text{dom}(g)) A(n, g(n)) \leq B(n, g(n))$,
   - or $(\forall n \in \text{dom}(g)) B(n, g(n)) \leq A(n, g(n))$.

In the following theorem, note that $\min\{s, b, \text{add}(\mathcal{M})\} = \min\{s, \text{add}(\mathcal{M})\}$.

**Theorem 2.3.**
(1) $X$ satisfies $\mathcal{S}_{[0,2]}(\mathcal{B}_T, \mathcal{B}_T)$ if, and only if, for each Borel function $\Psi : X \to \{0, 1\}^{\mathbb{N} \times \mathbb{N}}$: If $\Psi[X]$ is a $\tau$-family, then it is semi $\tau$-diagonalizable (Definition 2.2). The corresponding clopen case also holds.

(2) The minimal cardinality of a $\tau$-family that is not semi $\tau$-diagonalizable is at least $\min\{s, b, \text{od}\}$.

(3) $\min\{s, b, \text{od}\} \leq \text{non}(\mathcal{S}_{[0,2]}(\mathcal{B}_T, \mathcal{B}_T)) = \text{non}(\mathcal{S}_{[0,2]}(T, T)) = \text{non}(\mathcal{S}_{[0,2]}(C_T, C_T))$.

Proof. (1) is proved as usual, (2) is shown in the proof of Theorem 4.15 of [6], and (3) follows from (1) and (2).

Definition 2.4 ([9]). For functions $f, g, h \in \mathbb{N}^{\mathbb{N}}$, and binary relations $R, S$ on $\mathbb{N}$, define subsets $[f R g]$ and $[h R g S f]$ of $\mathbb{N}^{\mathbb{N}}$ by:

$[f R g] = \{n : f(n) R g(n)\}$, $[f R g S h] = [f R g] \cap [g S h]$.

For a subset $Y$ of $\mathbb{N}^{\mathbb{N}}$ and $g \in \mathbb{N}^{\mathbb{N}}$, we say that $g$ avoids middles in $Y$ with respect to $\langle R, S \rangle$ if:

(1) for each $f \in Y$, the set $[f R g]$ is infinite;

(2) for all $f, h \in Y$ at least one of the sets $[f R g S h]$ and $[h R g S f]$ is finite.

$Y$ satisfies the $\langle R, S \rangle$-excluded middle property if there exists $g \in \mathbb{N}^{\mathbb{N}}$ which avoids middles in $Y$ with respect to $\langle R, S \rangle$.

In [10] it is proved that $U_{\text{fin}}(\mathcal{B}_T, \mathcal{B}_T)$ is equivalent to having all Borel images in $\mathbb{N}^{\mathbb{N}}$ satisfying the $\langle <, \leq \rangle$-excluded middle property (the statement in [10] is different but equivalent).

Theorem 2.5. For a set of reals $X$, the following are equivalent:

(1) $X$ satisfies $U_{[1,\aleph_0]}(\mathcal{B}_T, \mathcal{B}_T)$.

(2) Each Borel image of $X$ in $\mathbb{N}^{\mathbb{N}}$ satisfies the $\langle <, \leq \rangle$-excluded middle property.

The corresponding assertion for $U_{[1,\aleph_0]}(\mathcal{C}_T, \mathcal{C}_T)$ holds when “Borel” is replaced by “continuous”.

Proof. The proof is similar to the one given in [10] for $U_{\text{fin}}(\mathcal{B}_T, \mathcal{B}_T)$, but is somewhat simpler.

$1 \Rightarrow 2$: Assume that $Y \subseteq \mathbb{N}^{\mathbb{N}}$ is a Borel image of $X$. Then $Y$ satisfies $U_{[1,\aleph_0]}(\mathcal{B}_T, \mathcal{B}_T)$. For each $n$, the collection $\mathcal{U}_n = \{U_m^n : m \in \mathbb{N}\}$, where $U_m^n = \{f \in \mathbb{N}^{\mathbb{N}} : f(n) \leq m\}$, is a clopen $\gamma$-cover of $\mathbb{N}^{\mathbb{N}}$. By standard arguments (see (1 $\Rightarrow$ 2) in the proof of Theorem 2.3 of [6]) we may assume that no $\mathcal{U}_n$ contains a finite cover. For all $n$, the sequence $\{U_m^n : m \in \mathbb{N}\}$ is monotonically increasing with respect to $\subseteq$, therefore—as large subcovers of $\tau$-covers are also $\tau$-covers—we may use $\mathcal{S}_1(\mathcal{B}_T, \mathcal{B}_T)$
instead of $U_{[1,\aleph_0]}(B_T, B_T)$ to get a $\tau$-cover $U = \{\Psi^{-1}[U_n] : n \in \mathbb{N}\}$ for $X$. Let $g \in \mathbb{N}^\mathbb{N}$ be such that $g(n) = m_n$ for all $n$. Then $g$ avoids middles in $Y$ with respect to $\langle \leq, < \rangle$.

2 $\Rightarrow$ 1: Assume that $U_n = \{U^n_m : m \in \mathbb{N}\}$, $n \in \mathbb{N}$, are Borel covers of $X$ which do not contain a finite subcover. Replacing each $U^n_m$ with the Borel set $\bigcup_{k \leq m} U^n_k$ we may assume that the sets $U^n_m$ are monotonically increasing with $m$. Define $\Psi : X \to \mathbb{N}^\mathbb{N}$ by: $\Psi(x)(n) = \min\{m : x \in U^n_m\}$. Then $\Psi$ is a Borel map, and so $\Psi[X]$ satisfies the $\langle \leq, < \rangle$-excluded middle property. Let $g \in \mathbb{N}^\mathbb{N}$ be a witness for that. Then $U = \{U^n_m : n \in \mathbb{N}\}$ is a $\tau$-cover of $X$.

The proof in the clopen case is similar. □

**Corollary 2.6.** The critical cardinalities of $U_{[1,\aleph_0]}(B_T, B_T)$, $U_{[1,\aleph_0]}(\Gamma, T)$, and $U_{[1,\aleph_0]}(C_T, C_T)$, are all equal to $b$.

**Proof.** This follows from Theorem 2.5 and the corresponding combinatorial assertion, which was proved in [9]. □

Recall from Figure 1 that the critical cardinality of $U_{\text{fin}}(\Gamma, T) = U_{[0,\aleph_0]}(\Gamma, T)$ is $\max\{s, b\}$. Contrast this with Corollary 2.6.

According to Scheepers [12, Problem 9.5], one of the more interesting problems concerning Figure 1 is whether $S_1(\Omega, T)$ implies $U_{\text{fin}}(\Gamma, \Gamma)$. If $U_{[1,\aleph_0]}(\Gamma, T)$ is preserved under taking finite unions, then we get a positive solution to Scheepers’ Problem. (Note that $S_1(\Omega, T)$ implies $S_1(\Gamma, T)$.)

**Corollary 2.7.** If $U_{[1,\aleph_0]}(\Gamma, T)$ is preserved under taking finite unions, then it is equivalent to $U_{\text{fin}}(\Gamma, \Gamma)$ and $S_1(\Gamma, T)$ implies $U_{\text{fin}}(\Gamma, \Gamma)$.

**Proof.** The last assertion of the theorem follows from the first since $S_1(\Gamma, T)$ implies $U_{[1,\aleph_0]}(\Gamma, T)$.

Assume that $X$ does not satisfy $U_{\text{fin}}(\Gamma, \Gamma)$. Then, by Hurewicz’ Theorem [3], there exists an unbounded continuous image $Y$ of $X$ in $\mathbb{N}^\mathbb{N}$. For each $f \in Y$, define $f_0, f_1 \in \mathbb{N}^\mathbb{N}$ by $f_0(2n+i) = f(n)$ and $f_1(2n+(1-i)) = 0$. For each $i \in \{0, 1\}$, $Y_i = \{f_i : f \in Y\}$ is a continuous image of $Y$. It is not difficult to see that $Y_0 \cup Y_1$ does not satisfy the $\langle \leq, < \rangle$-excluded middle property [9]. By Theorem 2.5, $Y_0 \cup Y_1$ does not satisfy $U_{[1,\aleph_0]}(\Gamma, T)$, thus, by the theorem’s hypothesis, one of the sets $Y_i$ does not satisfy that property. Therefore $Y$ (and therefore $X$) does not satisfy $U_{[1,\aleph_0]}(\Gamma, T)$ either. □

We do not know whether $U_{[1,\aleph_0]}(\Gamma, T)$ is preserved under taking finite unions. We also do not know the situation for $U_{\text{fin}}(\Gamma, T)$. The following theorem is only interesting when $s < b$. 

SEVERAL COMMENTS ABOUT THE COMBINATORICS OF $\tau$-COVERS

5
Theorem 2.8. If there exists a set of reals $X$ satisfying $U_{\text{fin}}(\Gamma, T)$ but not $U_{\text{fin}}(\Gamma, \Gamma)$, then $U_{\text{fin}}(\Gamma, T)$ is not preserved under taking unions of $s$ many elements.

Proof. The proof is similar to the last one, except that here we define $s$ many continuous images of $Y$ as we did in [9] to prove that the critical cardinality of $U_{\text{fin}}(\Gamma, T)$ is $\max\{s, b\}$. □

3. Luzin sets

A set of reals $L$ is a generalized Luzin set if for each meager set $M$, $|L \cap M| < |L|$. In [10] we constructed (assuming a portion of the Continuum Hypothesis) a generalized Luzin set which satisfies $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ but not $U_{\text{fin}}(\Gamma, T)$. We now show that the last assertion always holds.

Theorem 3.1. Assume that $L \subseteq \mathbb{N}^\mathbb{N}$ is a generalized Luzin set. Then $L$ does not satisfy the $\langle <, \leq \rangle$-excluded middle property. In particular, $L$ does not satisfy $U_{\text{fin}}(C_\Gamma, C_T)$.

Proof. We use the following easy observation.

Lemma 3.2 ([10]). Assume that $A$ is an infinite set of natural numbers, and $f \in \mathbb{N}^\mathbb{N}$. Then the sets

$$M_{f, A} = \{g \in \mathbb{N}^\mathbb{N} : [g \leq f] \cap A \text{ is finite}\}$$

$$\tilde{M}_{f, A} = \{g \in \mathbb{N}^\mathbb{N} : [f < g] \cap A \text{ is finite}\}$$

are meager subsets of $\mathbb{N}^\mathbb{N}$. □

Fix any $f \in \mathbb{N}^\mathbb{N}$. We will show that $f$ does not avoid middles in $Y$ with respect to $\langle <, \leq \rangle$. The sets $M_{f,N} = \{g \in \mathbb{N}^\mathbb{N} : [g \leq f] \text{ is finite}\}$ and $\tilde{M}_{f,N} = \{g \in \mathbb{N}^\mathbb{N} : [f < g] \text{ is finite}\}$ are meager, thus there exists $g_0 \in L \setminus (M_{f,N} \cup \tilde{M}_{f,N})$. Now consider the meager sets $M_{f,[f<g_0]} = \{g \in \mathbb{N}^\mathbb{N} : [g \leq f < g_0] \text{ is finite}\}$ and $\tilde{M}_{f,[g_0>f]} = \{g \in \mathbb{N}^\mathbb{N} : [g_0 \leq f < g] \text{ is finite}\}$, and choose $g_1 \in L \setminus (M_{f,[f<g_0]} \cup \tilde{M}_{f,[g_0>f]})$. Then both sets $[g_0 < f \leq g_1]$ and $[g_1 < f \leq g_0]$ are infinite. □

References

[1] A. R. Blass, Combinatorial cardinal characteristics of the continuum, in: Handbook of Set Theory (M. Foreman, A. Kanamori, and M. Magidor, eds.), Kluwer Academic Publishers, Dordrecht, to appear.

[2] E. K. van Douwen, The integers and topology, in: Handbook of Set Theoretic Topology (eds. K. Kunen and J. Vaughan), North-Holland, Amsterdam: 1984, 111–167.

[3] W. Hurewicz, Über Folgen stetiger Funktionen, Fundamenta Mathematicae 9 (1927), 193–204.
SEVERAL COMMENTS ABOUT THE COMBINATORICS OF $\tau$-COVERS

[4] W. Just, A. W. Miller, M. Scheepers, and P. J. Szeptycki, *The combinatorics of open covers II*, Topology and its Applications 73 (1996), 241–266.

[5] Lj. D. Kočinac, *Selected results on selection principles*, in: *Proceedings of the 3rd Seminar on Geometry and Topology* (Sh. Rezapour, ed.), July 15-17, Tabriz, Iran, 2004, 71–104.

[6] H. Mildenberger, S. Shelah, and B. Tsaban, *The combinatorics of $\tau$-covers*, Topology and its Applications 154 (2007), 263–276.

[7] M. Scheepers, *Combinatorics of open covers I: Ramsey theory*, Topology and its Applications 69 (1996), 31–62.

[8] M. Scheepers, *Selection principles and covering properties in topology*, Note di Matematica 22 (2003), 3–41.

[9] S. Shelah and B. Tsaban, *Critical cardinalities and additivity properties of combinatorial notions of smallness*, Journal of Applied Analysis 9 (2003), 149–162.

[10] B. Tsaban, *Selection principles and the minimal tower problem*, Note di Matematica 22 (2003), 53–81.

[11] B. Tsaban, *The combinatorics of splittability*, Annals of Pure and Applied Logic 129 (2004), 107–130.

[12] B. Tsaban, *Selection principles in Mathematics: A milestone of open problems*, Note di Matematica 22 (2003), 179–208.

[13] B. Tsaban, *Some new directions in infinite-combinatorial topology*, in: *Set Theory* (J. Bagaria and S. Todorcevic, eds.), Trends in Mathematics, Birkhäuser, 2006, 225–255.

Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel
E-mail address: boaz.tsaban@weizmann.ac.il
URL: http://www.cs.biu.ac.il/~tsaban