Differentially Private Bootstrap: 
New Privacy Analysis and Inference Strategies

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Abstract

Differentially private (DP) mechanisms protect individual-level information by introducing randomness into the statistical analysis procedure. Despite the availability of numerous DP tools, there remains a lack of general techniques for conducting statistical inference under DP. We examine a DP bootstrap procedure that releases multiple private bootstrap estimates to infer the sampling distribution and construct confidence intervals (CIs). Our privacy analysis presents new results on the privacy cost of a single DP bootstrap estimate, applicable to any DP mechanisms, and identifies some misapplications of the bootstrap in the existing literature. Using the Gaussian-DP (GDP) framework (Dong et al., 2022), we show that the release of $B$ DP bootstrap estimates from mechanisms satisfying $(\mu/\sqrt{2-2/e})$-GDP asymptotically satisfies $\mu$-GDP as $B$ goes to infinity. Moreover, we use deconvolution with the DP bootstrap estimates to accurately infer the sampling distribution, which is novel in DP. We derive CIs from our density estimate for tasks such as population mean estimation, logistic regression, and quantile regression, and we compare them to existing methods using simulations and real-world experiments on 2016 Canada Census data. Our private CIs achieve the nominal coverage level and offer the first approach to private inference for quantile regression.

Keywords: Gaussian differential privacy, resampling, distribution-free inference, confidence interval, deconvolution.

1. Code is available at https://github.com/Zhanyu-Wang/Differentially_Private_Bootstrap
1 Introduction

In the big data era, individual privacy protection becomes more critical than ever because personal information is collected and used in many different ways; while the intention of the data collection is usually to improve the user experience or, more generally, to benefit society, there have also been rising concerns about malicious applications of these data. To protect individuals against arbitrary attacks on their data, Dwork et al. (2006) proposed differential privacy (DP) which has become the state-of-the-art framework in privacy protection.

DP is a probabilistic framework that measures the level of privacy protection of a mechanism; a mechanism satisfies DP if it is a randomized algorithm, i.e., its output is a realization from a random variable following a distribution determined by the mechanism and its input, and changing any individual in the input results in an output with its distribution similar to the original output distribution. Starting from the definition of $(\varepsilon, \delta)$-DP (Dwork et al., 2010), there have been many variants of DP definitions serving different needs, some of which are mentioned in the related work. For our results, we use $f$-DP (Dong et al., 2022), a hypothesis-testing perspective of DP formally defined in Definition 1 and 2, as it is the most informative DP notion satisfying the post-processing property (Dong et al., 2022, Theorem 2). For statistical analysis under DP guarantee, a great deal of prior work focused on producing private point estimates of a parameter, e.g., the sample mean, sample median, and the maximum of the data. In contrast, while some prior work aims to quantify the uncertainty of a DP procedure, their techniques are usually restricted to specific settings, and there is still a lack of general-purpose methods (see related work.)

One of the most widely used methods to approximate a sampling distribution is the bootstrap method (Efron, 1979), which can be used to quantify the uncertainty of an estimator in many statistical ways, such as by producing a non-parametric confidence interval (CI). Although the bootstrap has been studied very well in Statistics, it is still an open question of how to build and analyze a DP bootstrap for private statistical inference. Brawner and Honaker (2018) were the first to propose and analyze a DP bootstrap procedure and used it to produce a CI based on the private bootstrap outputs. However, a key step in their privacy proof is incorrect, and we show in Section 3 that their stated privacy guarantee does not hold. Furthermore, the performance of their CI is not satisfactory in terms of both coverage and width. Balle et al. (2018) developed the state-of-the-art analysis of resampling for $(\varepsilon, \delta)$-DP, both with and without replacement. As a particular case, their results can be used to analyze the privacy guarantees of mechanisms with their input being bootstrap samples. However, Balle et al. (2018) did not consider the cumulative privacy cost of multiple samples of the resampling methods, which restricts the usage of their results on the bootstrap. This is non-trivial, mainly because their results are presented in $(\varepsilon, \delta)$-DP, which does not have tight composition techniques, unlike $f$-DP. Moreover, we show that it is necessary to develop a new method for conducting statistical inference with the DP bootstrap samples while Balle et al. (2018) only focused on the privacy analysis.

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2. For easy reference, we include the definition of $(\varepsilon, \delta)$-DP in Definition 16 in Appendix A.

3. They use a subsampling result for zCDP (Bun and Steinke, 2016) in Section 6.2, while there is no such result for zCDP as discussed in (Bun et al., 2018).
Our contributions In this paper, we obtain a tight privacy analysis of a DP bootstrap method and develop inference strategies on the sampling distribution. Specifically, we derive the privacy guarantee of the DP bootstrap, which generalizes the result by Balle et al. (2018) from $(\varepsilon, \delta)$-DP to $f$-DP, and an aspect of our proof strategy applies to any mixture of DP mechanisms where the bootstrap is a special case. Our result also identifies misuses of resampling with replacement in the literature. We derive the asymptotic cumulative privacy cost of multiple DP bootstrap estimates via the central limit theorem (CLT) for $f$-DP (Dong et al., 2022). We demonstrate how to use deconvolution on the DP bootstrap results to obtain a private estimate of the non-private sampling distribution and develop CIs. We provide simulations to show the advantage of our method in terms of the coverage of the resulting CIs. We also conduct real-world experiments on the 2016 Canada Census Public Use Microdata, which reveals the dependence between individuals' income and shelter cost under DP guarantees by building CIs for the slope parameters of logistic regression and quantile regression. To the best of our knowledge, DP bootstrap is the first tool that can be used to perform valid private statistical inference for parameters in quantile regression.

Related work For DP CIs, D’Orazio et al. (2015) presented algorithms for releasing DP estimates of causal effects, particularly the difference between means estimators along with their standard errors and CIs. Sheffet (2017) presented a DP estimator for the $t$-values in ordinary least squares and derived the CI based on the $t$-values. Karwa and Vadhan (2018) gave DP CIs for the population mean of a normal distribution along with finite sample coverage and lower bounds on the size of the DP CI. Brawner and Honaker (2018) gave the first attempt to use the DP bootstrap to obtain a private CI. Wang et al. (2019b) developed algorithms for generating DP CIs for DP estimates from objective and output perturbation mechanisms in the empirical risk minimization framework. Awan and Slavković (2020) developed DP uniformly most powerful hypothesis tests and DP CIs for Bernoulli data. Du et al. (2020) proposed and compared different methods (including NoisyVar, which we discuss in our experimental sections) to build DP CIs for the mean of normally distributed data. Covington et al. (2021) used the bag of little bootstraps (BLB) and the CoinPress algorithm (Biswas et al., 2020) to privately estimate the parameters’ sampling distribution through its mean and covariance. Drechsler et al. (2021) proposed several strategies to compute non-parametric DP CIs for the median. Awan and Wang (2023) used a simulation-based method to produce finite-sample CIs and hypothesis tests from DP summary statistics.

Other literature involving the idea of bootstrap in DP includes (Ferrando et al., 2022) and (O’Keefe and Charest, 2019). Ferrando et al. (2022) used parametric bootstrap through resampling data from a distribution parametrized by estimated parameters, while we use nonparametric bootstrap through resampling data from the empirical data distribution. O’Keefe and Charest (2019) proposed a relaxed definition of differential privacy based on bootstrap, but they did not use bootstrap to perform statistical inference as we do.

Organization The remainder of this paper is organized as follows. In Section 2, we review the definition of $f$-DP and results used in our DP bootstrap analysis. In Section 3, we provide our privacy guarantee for the DP bootstrap along with a central limit theorem result for the cumulative privacy cost of many DP bootstrap outputs. In Section 4, we
propose to use deconvolution for performing statistical inference on the DP bootstrap; we show that the convergence rate of the estimates from DP bootstrap for building CI of the population mean is asymptotically satisfactory, and we improve the finite-sample coverage of the DP CI by using deconvolution to estimate the sampling distribution and construct percentile intervals. In Section 5, we demonstrate through a simulation study that the CIs derived from the DP bootstrap have better coverage than NoisyVar (Du et al., 2020). In Section 6, we analyze the dependence between market income and shelter cost in Ontario by building DP CIs for the slope parameters of logistic regression and quantile regression with the 2016 Canadian census dataset. We compare DP bootstrap with DP-CI-ERM (Wang et al., 2019b) only in logistic regression, as the DP-CI-ERM method is inapplicable to quantile regression. In Section 7, we discuss the implications of our work and highlight some directions for future research. Proofs and technical details are deferred to the appendix.

2 Background in differential privacy

In this section, we provide existing results that are closely related to our DP bootstrap method. First, we discuss the definition of $f$-DP. Then we introduce the Gaussian mechanism to guarantee DP, and prior results on subsampling, composition, and group privacy.

For a set $X$, a dataset $D \in X^n$ with cardinality $|D| = n$ is a finite collection of elements from $X$. We define $D_1 \simeq_k D_2$ if $|D_1| = |D_2|$ and they differ in $k$ entries. We call $D_1$ and $D_2$ as neighboring datasets if $D_1 \simeq_1 D_2$, and we also write it as $D_1 \simeq D_2$. A mechanism $M : X^n \rightarrow O$ is a randomized algorithm taking a dataset as input and outputting a value in $O$. Differential privacy measures how much the outputs of the mechanism differ when the inputs are two neighboring datasets.

In this paper, we use the $f$-DP framework, which measures the difference between two distributions by hypothesis testing. With an observation $M(D_{\text{in}}) = X$, we consider a hypothesis test between $H_0 : X \sim P$ versus $H_1 : X \sim Q$. For any rejection rule $\phi(X)$, we use $\alpha_\phi$ to denote the type I error and $\beta_\phi$ to denote the type II error. The tradeoff function $T_{P,Q}(\alpha) : [0, 1] \rightarrow [0, 1]$ is defined to be $T_{P,Q}(\alpha) := \inf_{\phi} \{\beta_\phi \mid \alpha_\phi \leq \alpha\}$. For a tradeoff function $f$, we denote its inverse by $f^{-1}(x) := \inf\{\alpha \in [0, 1] : f(\alpha) \leq x\}$. We define that $f$ is symmetric if $f = f^{-1}$.

Lower tradeoff functions indicate less privacy since an adversary can distinguish one distribution from the other with smaller type II error at a given type I error. The upper bound of the tradeoff function is $T(\alpha) = 1 - \alpha$, since it corresponds to the family of random rejection rules which reject $H_0$ with probability $\alpha$ for any observation $X$. 

**Definition 1 (tradeoff function (Dong et al., 2022))** Consider the hypothesis test $H_0 : X \sim P$ versus $H_1 : X \sim Q$. For any rejection rule $\phi(X)$, we use $\alpha_\phi$ to denote the type I error and $\beta_\phi$ to denote the type II error. The tradeoff function $T_{P,Q}(\alpha) : [0, 1] \rightarrow [0, 1]$ is defined to be $T_{P,Q}(\alpha) := \inf_{\phi} \{\beta_\phi \mid \alpha_\phi \leq \alpha\}$. For a tradeoff function $f$, we denote its inverse by $f^{-1}(x) := \inf\{\alpha \in [0, 1] : f(\alpha) \leq x\}$. We define that $f$ is symmetric if $f = f^{-1}$.
Definition 2 (f-DP (Dong et al., 2022)) Let f be a tradeoff function. We write $T_{M(D),M(D')} \geq f$ if $T_{M(D),M(D')}(\alpha) \geq f(\alpha) \forall \alpha \in [0,1]$. A mechanism $M$ is said to be $f$-differentially private (f-DP) if $T_{M(D),M(D')} \geq f$ for any datasets $D, D'$ with $D \simeq D'$.

Intuitively, for a mechanism $M$ satisfying f-DP where $f = T_{P,Q}$, testing $H_0 : X \sim M(D)$ versus $H_1 : X \sim M(D')$ is at least as hard as testing $H_0 : X \sim P$ versus $H_1 : X \sim Q$ when $D \simeq D'$. We visualize the definition of f-DP in Figure 1. Among all tradeoff functions, an important subclass is $G_{\mu}(\alpha) = T_{N(0,1),N(\mu,1)}(\alpha)$. $G_{\mu}$-DP is also called $\mu$-Gaussian DP (GDP), which is shown to be the limit of many DP procedures under composition by Dong et al. (2022). Another important subclass is $f_{\varepsilon,\delta}(\alpha) = \max\{0,1 - \delta - e^{\varepsilon}\alpha,e^{-\varepsilon}(1 - \delta - \alpha)\}$ since $f_{\varepsilon,\delta}$-DP is equivalent to $(\varepsilon,\delta)$-DP.

The Gaussian mechanism Let the $\ell_2$-sensitivity of a function $g : \mathcal{X}^n \to \mathbb{R}^d$ be $\Delta(g) = \sup_{D_1 \simeq D_2} \|g(D_1) - g(D_2)\|_2$. For any $g : \mathcal{X}^n \to \mathbb{R}^d$, the Gaussian mechanism on $g$ adds Gaussian noises to the output of $g$: $M_G(D,g,\sigma) = g(D) + \xi$ where $\xi \sim N(\mu = 0, \Sigma = \sigma^2 I_{d \times d})$. Dong et al. (2022) proved that $M_G(D,g,\sigma)$ satisfies $\mu$-GDP if $\sigma^2 = (\Delta(g)/\mu)^2$.

Proposition 3 (Subsampling (Dong et al., 2022)) Let $D \in \mathcal{X}^n$ be a dataset and $D' := \text{Sample}_{m,n}(D) \in \mathcal{X}^m$ be chosen uniformly at random among all the subsets of $D$ with size $m \leq n$ (sampling without replacement). For $M : \mathcal{X}^n \to \mathcal{O}$, $M \circ \text{Sample}_{m,n} : \mathcal{X}^n \to \mathcal{O}$ is the subsampled mechanism. For $0 \leq p \leq 1$, let $f_p := pf + (1 - p)\text{Id}$ and $C_p(f) := \min\{f_p, f_p^{-1}\}$ where $\text{Id}(x) = 1 - x$, $f^*(y) = \sup_{-\infty < x < \infty} xy - f(x)$, and $f^{**} = (f^*)^*$. If $M$ satisfies $f$-DP and $p = m/n$, then $M \circ \text{Sample}_{m,n}$ satisfies $C_p(f)$-DP.

Proposition 4 (Composition (Dong et al., 2022)) The composition property of DP quantifies the cumulative privacy cost of several DP outputs. If $f = T_{P,Q}$ and $g = T_{P',Q'}$, their tensor product is defined as $f \otimes g := T_{P \times P',Q \times Q'}$ where $P \times P'$ is the product measure of $P$ and $P'$, and $Q \times Q'$ is the product measure of $Q$ and $Q'$. If $M_i$ satisfies $f_i$-DP for $i = 1, \ldots, k$, then $M = (M_1, \ldots, M_k)$ satisfies $f_1 \otimes \ldots \otimes f_k$-DP.
Group privacy While $f$-DP guarantees protection of the privacy of each individual, it can be generalized to give a privacy guarantee for groups of size $k$. We say a mechanism $M$ satisfies $f_k$-DP for groups of size $k$ if $T(D_1, D_2) > f_k$ for all $D_1$ and $D_2$ with $D_1 \sim_k D_2$. If a mechanism is $\mu$-GDP, then it is $k\mu$-GDP for groups of size $k$ (Dong et al., 2022).

3 Privacy analysis of bootstrap resampling

In this section, we first use techniques from Dong et al. (2022) to convert the result from Balle et al. (2018) to $f$-DP. However, the resulting formula is computationally intractable. We then give a new $f$-DP bound with a direct proof, which agrees with the result from Balle et al. (2018) but is more transparent and computationally friendly. At the end of this section, we use the composition property to analyze the privacy cost of releasing several DP optimization problems for each $\alpha$. We then give a new $f$-DP result for $M \circ \text{boot}$ using the mixture of tradeoff functions. The fundamental idea of our proof is to view the bootstrap as a mixture distribution and decompose the overall tradeoff function into a mixture of tradeoff functions where each one is easy to obtain.

3.1 $f$-DP guarantee with one bootstrap sample as input

Bootstrap sampling is denoted by a randomized mapping $\text{boot}_n : \mathcal{X}^n \rightarrow \mathcal{X}^n$, where $D = (x_1, \ldots, x_n)$ is a database, $\text{boot}_n(D) = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ is a randomly generated dataset where $i_k \overset{i.i.d.}{\sim} \text{Uniform}\{1, 2, \ldots, n\}$, $k = 1, 2, \ldots, n$. Let $p_{i,n} = \binom{n}{i}(1/n)^i(1 - 1/n)^{n-i}$ which is the probability that a given entry of $D$ is included $i$ times in $\text{boot}_n(D)$. We also write $\text{boot}_n$ and $p_{i,n}$ as $\text{boot}$ and $p_i$ respectively when $n$ is known from the context.

We obtain an $f$-DP result using the primal-dual conversion (Dong et al., 2022) on the result in (Balle et al., 2018). The general conversion formula and the original result from Balle and Wang (2018) are included in the appendix Proposition 17 and Theorem 18.

**Proposition 5** For $i = 1, \ldots, n$, let $f_i$ be a symmetric tradeoff function. For $M$ satisfying group privacy $f_i$-DP with group size $i$, $M \circ \text{boot}$ satisfies $f_{M \circ \text{boot}}$-DP where $f_{M \circ \text{boot}} = C_{1-p_0}\left(\left(\sum_{i=1}^n \frac{p_n}{1-p_0} f_i^*\right)^{\alpha}\right)$, and $C_{1-p_0}(\cdot)$ and $f^*(\cdot)$ are introduced in Proposition 3.

Although this representation of $f_{M \circ \text{boot}}$ is seemingly simple, it is hard to compute or visualize this tradeoff function because evaluating $f_{M \circ \text{boot}}(\alpha)$ requires solving over $n$ optimization problems for each $\alpha$. It is also hard to derive composition results from this $f_{M \circ \text{boot}}(\alpha)$, which is crucial for using the bootstrap for statistical inference since multiple bootstrap samples will be used. Due to the intractability of Proposition 5, we prove a new $f$-DP result for $M \circ \text{boot}$ using the mixture of tradeoff functions. The fundamental idea of our proof is to view the bootstrap as a mixture distribution and decompose the overall tradeoff function into a mixture of tradeoff functions where each one is easy to obtain.

**Definition 6** (Mixture of tradeoff functions) For $i = 1, 2, \ldots, k$, let $f_i$ be a tradeoff function and $p_i \in (0, 1]$ satisfying $\sum_{i=1}^k p_i = 1$. We write $f = (f_1, \ldots, f_k)$ and $p = (p_1, \ldots, p_k)$. For a constant $C \in (-\infty, 0]$, define $A_i(C) := \{\alpha_i | \forall C \in \partial f_i(\alpha_i)\}$ where $\partial f_i(\alpha_i)$ is the sub-differential of $f_i$ at $\alpha_i$, and $A(C) := \{\sum_{i=1}^k p_i \alpha_i | \alpha_i \in A_i(C)\}$. The mixture of
Theorem 8 For $i = 1, 2, \ldots, k$, let $f_i$ be a tradeoff function and $\mathcal{M}_i : \mathcal{X}^n \to \mathcal{Y}_i$, $\mathcal{Y}_i \subset \mathcal{Y}$ be a mechanism satisfying $f_i$-DP. Let $\mathcal{M} : \mathcal{X}^n \to \mathcal{Y}$ be a mixture mechanism which randomly selects one mechanism from $k$ mechanisms, $\{\mathcal{M}_i\}_{i=1}^k$, with corresponding probabilities $\{p_i\}_{i=1}^k$ where $\sum_{i=1}^k p_i = 1$, and the output of $\mathcal{M}$ will be the output of $\mathcal{M}_i$ if $\mathcal{M}_i$ is selected. Then $\mathcal{M}$ satisfies $f$-DP with $f = \text{mix}(\underline{f}, \overline{f})$ where $\underline{f} = (f_1, \ldots, f_k)$ and $\overline{f} = (p_1, \ldots, p_k)$. 

Remark 7 If for $i = 1, \ldots, k$, $f_i$ has derivative $f_i'$ monotonically increasing for every $\alpha$ in $[0, 1]$, we can simplify Definition 6 as $\text{mix}(\underline{f}, \overline{f}) = (\sum_{i=1}^k (p_i f_i(0) \circ (f_i')^{-1})) \circ (\sum_{i=1}^k (p_i (f_i')^{-1})^{-1})$ since $(\sum_{i=1}^k p_i (f_i')^{-1})$ maps the slope $C$ to the type I error, and $(\sum_{i=1}^k (p_i f_i(0) \circ (f_i')^{-1}))$ maps the slope $C$ to the type II error.

Intuitively, as illustrated in Figure 2 by matching the slopes of each tradeoff function $f_i$, we minimize the overall type II error given a fixed type I error, and $\text{mix}(\underline{f}, \overline{f})$ is well-defined; see Lemma 21 in the appendix. In Theorem 8, we show that $\text{mix}(\underline{f}, \overline{f})$ always gives a lower bound on the privacy cost of an arbitrary mixture mechanism. Note that this general result applies to any mixture of DP mechanisms.

Figure 2: An illustration of the mixture of tradeoff functions. In the top row, the solid curves are the tradeoff functions $f_1, f_2, f_3$ corresponding to 1-GDP, 2-GDP, and 3-GDP, respectively, and the three dashed lines are the tangent lines with matched slopes. The mixture of $f = (f_1, f_2, f_3)$ with corresponding weights $p = (p_1, p_2, p_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is $\text{mix}(\underline{f}, \overline{f})$ shown as the curves in the figures on the bottom row. Each circle dot on the mixture curve is the average of the circle dots on $f_1, f_2, f_3$ weighted by $p_1, p_2, p_3$. Note that $\text{mix}(\underline{f}, \overline{f})$ is neither $\frac{1}{3}(f_1 + f_2 + f_3)$, nor is it a member of the GDP family.
The proof of Theorem 8 is based on using different rejection rules when different $M_i$ is selected to allocate the overall type I error to each rejection rule to minimize the overall type II error. We can combine such rejection rules as one if $Y_i$ are disjoint; therefore, Theorem 8 is not improvable. For the DP bootstrap setting, since each $M_i$ maps to the same $Y_i$, we can leverage this fact to strengthen the privacy guarantee in Theorem 9. We separately consider the case that a given entry of $D$ is not included in $\text{boot}(D)$ to improve our bound.

**Theorem 9** Let $f_i$ be a symmetric tradeoff function for $i = 1, 2, \ldots, n$. For a mechanism $M$ satisfying group privacy $f_i$-DP with group size $i$, $M \circ \text{boot}$ satisfies $f_{M_{\text{boot}}}$-DP where $f_{M_{\text{boot}}} = C_1 - p_0(\text{mix}(p, f))$ where $f = (f_1, \ldots, f_n)$ and $p = (\frac{p_1}{1-p_0}, \ldots, \frac{p_n}{1-p_0})$.

**Remark 10** If $M$ satisfies $f_1$-DP, Dong et al. (2022) proved that $M$ also satisfies $[1 - (1 - f_1)^{\circ k}]$-DP for groups of size $k$ where $f^{\circ k}$ denote $f$ composed with itself for $k$ times, e.g., $f^{\circ 3}(x) = f(f(f(x)))$. We can use this result in Theorem 9 if no tighter result is known.

**Remark 11** The result by Balle et al. (2018) can be derived from our $f$-DP result in Theorem 9; see Proposition 25 in the appendix. Balle et al. (2018) showed that their results were based on a novel advanced joint convexity property used in $(\varepsilon, \delta)$-DP; similarly, our Theorem 8 and 9 reveal the advanced joint convexity property for $f$-DP. While Balle et al. (2018) provided specific settings attaining their bound, which also prove the optimality of our result, it is unknown for general settings how to construct rejection rules achieving each pair of the type I error and type II error given by our tradeoff function. Therefore, it may be possible to further improve the privacy analysis for the DP bootstrap. Nevertheless, our bound is fairly tight for the specific settings that we show in Figure 3a, and it should suffice for most purposes.

Our result in Theorem 9 is easier to compute when $\partial f_i$ are all known, since for any $C$, we can immediately obtain its corresponding $\alpha$ and $\beta$. For a given $\alpha$ or $\beta$, we can use the bisection method to search for $C$. From the following example and the composition result in Section 3.2, we see that our bound can be easily evaluated for the Gaussian mechanism.

**Example: DP bootstrap using the Gaussian mechanism** In Figure 3a, we assume that the Gaussian mechanism $M$ satisfies 1-GDP, then the DP bootstrap mechanism $M \circ \text{boot}$ satisfies $f_{M_{\text{boot}}}$-DP where $f_{M_{\text{boot}}}$ shown as the solid opaque curve is our lower bound of all tradeoff functions $T_{M_{\text{boot}}(D_1), M_{\text{boot}}(D_2)}$ when $D_1 \simeq D_2$. We also show the tradeoff functions from specific neighboring dataset pairs in Figure 3a (transparent curves) to illustrate misuses in the existing literature: 1) bootstrap cannot be used for free with the same privacy guarantee, i.e., $M \circ \text{boot}$ no longer satisfies 1-GDP, as opposed to (Brawner and Honaker, 2018); 2) DP bootstrap cannot be analyzed using the privacy loss distribution (PLD) method as in (Koskela et al., 2020). Details of this example are in Appendix C.

### 3.2 Composition of DP bootstrap with Gaussian mechanisms

In this section, we derive a composition result for the DP bootstrap because we need many bootstrap estimates to obtain the sampling distribution when we use the bootstrap method.
Figure 3: (a) An example showing the relationship between $f$, $f^{\text{Mboot}}$, and tradeoff functions from specific neighboring dataset pairs. We consider the Gaussian mechanism $M$ satisfying $1$-GDP shown as the dashed curve. The DP bootstrap mechanism $M \circ \text{boot}$ satisfies $f^{M,\text{boot}}$-DP by Theorem 9 shown as the solid opaque curve. The transparent curves are tradeoff functions $T^{M,\text{boot}}(D_1), M,\text{boot}(D_2)$ where $M(D) = \frac{1}{n} \sum_{i=1}^{n} x_i + \xi, D = (x_1, x_2, \ldots, x_n), \xi \sim \mathcal{N}(0, \frac{1}{n^2})$, and $D_1 = (a, 0, \ldots, 0), D_2 = (a-1, 0, \ldots, 0)$. The solid curve is tight as a lower bound of the transparent curves. The dashed and dotted dashed lines are misused as lower bounds in (Brawner and Honaker, 2018) and (Koskela et al., 2020). (b) The asymptotic price of using the bootstrap. Running a $1$-GDP mechanism on different bootstrap samples has a similar privacy guarantee to running a $(\sqrt{2} - \frac{2}{e})$-GDP mechanism on the original dataset for $B$ times if $B$ is large enough. ($\sqrt{2} - \frac{2}{e} \approx 1.125$).

Although numerical algorithms can be used to evaluate the privacy guarantee of the composition, the calculation is often burdensome (Zheng et al., 2020). We prove an asymptotic result to understand the behavior as the number of compositions goes to infinity. Note that for simplicity, we assume the initial mechanism satisfies GDP, but it may be possible to extend our result to mechanisms satisfying $f$-DP for other tradeoff functions $f$.

**Theorem 12** Let $\mu \in (0, \infty)$ be a given constant, and $\{\mu_B\}_{B=1}^{\infty}$ be a sequence such that $\mu_B \in (0, \infty)$ and $\lim_{B \to \infty} \mu_B \sqrt{(2 - \frac{2}{e})B} \to \mu$. Let $\{f_{Bi} : 1 \le i \le B\}_{B=1}^{\infty}$ be a triangular array of symmetric tradeoff functions where each $f_{Bi}$ corresponds to $\mu_B$-GDP, i.e., $f_{Bi} = G_{\mu_B}$. Call $f_{Bi,\text{boot}}$ as the lower bound for $f_{Bi}$ from Theorem 9. Then $\lim_{B \to \infty} f_{B1,\text{boot}} \otimes f_{B2,\text{boot}} \otimes \cdots \otimes f_{BB,\text{boot}} \ge G_\mu$. The operator $\otimes$ is introduced in Proposition 4.

Note that the asymptotic privacy guarantee of the composition result above is the same as running a $(\sqrt{2} - \frac{2}{e})\mu_B$-GDP mechanism on the original dataset (not on the bootstrap sample) for $B$ times (Dong et al., 2022). Therefore, the factor $(\sqrt{2} - \frac{2}{e}) = 1.12438 \ldots < 1.125$ is the price we pay for using the DP bootstrap (see Figure 3b). While there is a small increase in the privacy cost, the bootstrap samples now contain the randomness from sampling as well as from the privacy mechanism. In the next section, we will see how we can use this fact to perform statistical inference with the DP bootstrap estimates.
4 Private statistical inference with DP bootstrap

In this section, we first review two classic methods for statistical inference with bootstrap estimates; then, we analyze the convergence rates of the private unbiased estimates of the mean and variance of the sampling distribution from DP bootstrap, and the rates are shown to be comparable with other methods. Although the asymptotic rates demonstrate the theoretical potential of DP bootstrap, Brawner and Honaker (2018) showed that the empirical finite-sample coverage of the CI constructed from these estimates using normal approximation is unsatisfactory. Therefore, we propose another approach using deconvolution to recover the sampling distribution from the DP bootstrap estimates. The deconvolved distribution satisfies the same \( f \)-DP guarantee because of the post-processing property (Dong et al., 2022, Proposition 4). At the end of this section, we provide simulation results for deconvolution on the DP bootstrap with Gaussian mechanism.

Throughout the rest of this paper, to make our results comparable to other private inference methods, we use the Gaussian mechanism in DP bootstrap and its corresponding asymptotic privacy guarantee, \( \mu \)-GDP. We include remarks when other types of mechanisms can be used. Note that the asymptotic privacy guarantee can always be replaced by accurate composition results on our \( f \)-DP result in Theorem 9 if a strict privacy guarantee is needed (Zheng et al., 2020; Gopi et al., 2021; Zhu et al., 2022).

4.1 Statistical inference with Efron’s bootstrap

Before explaining how to conduct private statistical inference with the DP bootstrap, we briefly review two basic inference methods with the original Efron’s bootstrap estimates (Efron and Tibshirani, 1994), and we develop our private inference methods based on these classic methods. We denote the original dataset by \( D \in X^n \) and the estimator for a population parameter \( \theta \) by \( g(D) \). The sampling distribution of \( g(D) \) can be estimated from the bootstrap estimates, \( \{g(D_j)\}_{j=1}^B \) where \( D_j \) is the \( j \)th bootstrap sample of \( D \).

Using the normal approximation to the sampling distribution, we can obtain the standard interval for \( \theta \): we first estimate the standard error of \( g(D) \) by \( s_g,B^2 = \frac{1}{B-1} \sum_{j=1}^B (g(D_j) - \hat{m}_{g,B})^2 \) where \( \hat{m}_{g,B} = \frac{1}{B} \sum_{j=1}^B g(D_j) \), then approximate the sampling distribution of \( g(D) \) by \( N(\theta, s_g,B^2) \), and finally build CIs for \( \theta \) using \( g(D) \) and \( s_g,B^2 \). Alternatively, if we approximate the sampling distribution directly using the empirical distribution of \( \{g(D_j)\}_{j=1}^B \), we can build the percentile interval for \( \theta \) using the percentiles of \( \{g(D_j)\}_{j=1}^B \).

4.2 Convergence rate of DP bootstrap

Before developing techniques for building CI with DP bootstrap estimates, we derive the convergence rate of estimates used in the construction of standard intervals in DP bootstrap to demonstrate the potential of DP bootstrap.

The DP bootstrap estimates using the Gaussian mechanism are \( \{\tilde{g}(D_j)\}_{j=1}^B \) where \( \tilde{g}(D_j) = g(D_j) + \xi_j, \xi_j \overset{iid}{\sim} N(0, \frac{(\Delta(g))^2}{\mu^2_B}) \), and \( \Delta(g) \) is the \( \ell_2 \)-sensitivity of \( g \) on \( X^n \). If
we use \( \hat{m}_{g,B} = \frac{1}{B} \sum_{j=1}^{B} \tilde{g}(D_j) \) and \( \tilde{s}_{g,B}^2 = \frac{1}{B-1} \sum_{i=1}^{B} (\tilde{g}(D_j) - \hat{m}_{g,B})^2 \) to replace \( \hat{m}_{g,B}, s_{g,B}^2 \) and build the standard interval, we need to solve two problems: 1) \( \tilde{s}_{g,B} \) is a biased estimate for the standard error because \( \mathbb{E}[\tilde{s}_{g,B}^2] = \mathbb{E}[\tilde{s}_{g,B}^2] + \frac{(\Delta(g))^2}{\mu_B^2} \); 2) \( g(D) \) is no longer accessible due to the privacy constraint and cannot be used as the center of the CI. Therefore, we estimate \( g(D) \) using \( \hat{m}_{g,B} \) and replace \( s_{g,B}^2 \) with \((\tilde{s}_{g,B}^2)\)' = \( s_{g,B}^2 - \frac{(\Delta(g))^2}{\mu_B^2} \). Then we can build our private CI of \( \theta \) using \( \hat{m}_{g,B} \sim \mathcal{N}(\theta, (\tilde{s}_{g,B}^2)') + \frac{1}{B} s_{g,B}^2 \) approximately.

For better illustration, we look at a specific problem, the population mean estimation, and our results are in Table 1. In this table, the privacy guarantee is \( \mu \)-GDP; \( X, \tilde{X}, \tilde{X}' \), \( s_{g,B}^2 \), and \( \tilde{s}_{g,B}^2 \) correspond to \( D, g(D), \hat{m}_{g,B}, \tilde{s}_{g,B}^2, \) and \( \tilde{s}_{g,B}^2 \) respectively in our general analysis above, and \( n, \sigma_x^2, \) and \( s_X^2 \) are the sample size, population variance, and sample variance of \( X \) respectively. The derivation is in Appendix D where we assume the support of the population distribution is \([0,1]\), simplifying the sensitivity calculation. Note that our asymptotic privacy analysis requires a constant \( \mu \) with respect to \( n \) and \( B \). In Table 1, the variance of the mean estimate from DP bootstrap, \( \text{Var}(\hat{\theta}_4) \), is at the same rate as the variance of the other three estimates. When \( B \in O(n^2) \), the variance of the variance estimates from DP bootstrap, \( \text{Var}(\text{Var}(\hat{\theta}_4)) \), is at the same rate as the non-private bootstrap, \( \text{Var}(\text{Var}(\hat{\theta}_3)) \). If \( B \in O(n^2) \), then \( \text{Var}(\text{Var}(\hat{\theta}_i)), i = 1,2,3,4 \), are at the same rate of \( n \).

Although the estimates \( \hat{m}_{g,B} \) and \((\tilde{s}_{g,B}^2)\)' have nice convergence rates, they have severe problems: \( \frac{1}{B} \tilde{s}_{g,B}^2 + (\tilde{s}_{g,B}^2)' \) could be negative, and the coverage is lower than the nominal confidence level as shown in the Figure 7b in (Brawner and Honaker, 2018). In the following subsection, we develop a new inference method to solve this issue based on deconvolution.

### 4.3 Deconvolution for estimating sampling distribution

Instead of using the standard interval, which uses normal approximation and has potentially negative variance issues, we propose to use deconvolution on DP bootstrap estimates to recover the distribution of the non-private bootstrap estimates and build corresponding percentile intervals as CIs. Note that although we analyze this deconvolution method on DP bootstrap with Gaussian mechanism, it can be applied to DP bootstrap with any additive noise mechanisms, e.g., Laplace, Truncated-Uniform-Laplace (Tulap) (Awan and Slavković, 2020), canonical noise distributions (Awan and Vadhan, 2021).

Deconvolution is the major tool to solve the contaminated measurement problem, i.e., we want to estimate the distribution of \{\( X_i \)\}_{i=1}^{B} while our measurement is \{\( Y_i \)\}_{i=1}^{B} where \( Y_i = X_i + e_i \) and \( e_i \) is the measurement error. This is exactly the relationship between \( g(D_j) \) and \( \tilde{g}(D_j) \) since \( \tilde{g}(D_j) = g(D_j) + \xi_j \) and \( \xi_j \overset{\text{iid}}{\sim} \mathcal{N}(0, (\Delta(g))^2/\mu_B^2) \). Since DP allows for all details of the privacy mechanism to be publicly revealed (except the private dataset), the distribution of the added noises can be incorporated into our post-processing without raising any privacy concerns.

Deconvolution has been well-studied in Statistics, and it is known that the difficulty of deconvolution is determined by the smoothness of the noise distribution: the smoother, the harder (Fan, 1991). For the model \( Y = X + e \), if \( e \) follows a normal distribution \( \mathcal{N}(0, \sigma^2) \), which is super-smooth, the optimal rates for recovering \( X \) from \( Y \) is only \( O((\log N)^{-k/2}) \).
Table 1: Comparison of the convergence rates for the estimates of population mean $\theta$ and their corresponding uncertainty from four different methods. The upper table is about the estimates of $\theta$, and the lower table is the uncertainty quantification of the estimates; the first column is the estimate, and the second column is the variance of the estimate.

| Unbiased estimate of $\theta$ | Variance of the estimate |
|-------------------------------|--------------------------|
| Sample mean (non-private): $\hat{\theta}_1 = X$ | $\text{Var}(\hat{\theta}_1) = \frac{\sigma^2}{n}$ |
| Sample mean (Gaussian mechanism): $\theta_2 = X + \xi$ where $\xi \sim \mathcal{N}(0, \frac{1}{\mu^2 n^2})$ | $\text{Var}(\hat{\theta}_2) = \frac{\sigma^2}{n} + \frac{1}{\mu^2 n^2}$ |
| Bootstrap (non-private): $\hat{\theta}_3 = \bar{X}$ | $\text{Var}(\hat{\theta}_3) = \frac{\sigma^2}{n}$ |
| DP bootstrap (Gaussian mechanism): $\theta_4 = X'$ where $\xi_b \sim \mathcal{N}(0, \frac{(2-2/e)B}{\mu^2 n^2})$ | $\text{Var}(\hat{\theta}_4) = \frac{1+1/B-1/(nB)}{n} \sigma_x^2 + \frac{(2-2/e)}{\mu^2 n^2}$ |

| Unbiased estimate of $\text{Var}(\theta_i)$ | Variance of the estimate |
|-------------------------------------------|--------------------------|
| Sample variance (non-private): $\tilde{\text{Var}}(\hat{\theta}_1) = \frac{s_X^2}{n}$ | $\text{Var}(\tilde{\text{Var}}(\hat{\theta}_1)) \in O\left(\frac{1}{n^3}\right)$ |
| Sample variance (Gaussian mechanism): $\tilde{\text{Var}}(\hat{\theta}_2) = \frac{s_X^2 + \xi}{n} + \frac{1}{\mu^2 n^2}$ where $\xi \sim \mathcal{N}(0, \frac{1}{\mu^2 n^2})$ | $\text{Var}(\tilde{\text{Var}}(\hat{\theta}_2)) \in O\left(\frac{1}{n^3} + \frac{1}{\mu^2 n^2}\right)$ |
| Bootstrap (non-private): $\tilde{\text{Var}}(\hat{\theta}_3) = \frac{n}{n+1} s_B^2$ | $\text{Var}(\tilde{\text{Var}}(\hat{\theta}_3)) \in O\left(\frac{1}{n^2} + \frac{1}{n^3}\right)$ |
| DP bootstrap (Gaussian mechanism): $\tilde{\text{Var}}(\hat{\theta}_4) = \frac{nB + n - 1}{B(n+1)} s_B^2 - \frac{(2-2/e)B}{n(n+1)\mu^2}$ where $\xi_b \sim \mathcal{N}(0, \frac{(2-2/e)B}{\mu^2 n^2})$ | $\text{Var}(\tilde{\text{Var}}(\hat{\theta}_4)) \in O\left(\frac{1}{n^2} + \frac{1}{n^3} + \frac{B}{n^2 \mu^2} + \frac{1}{n^3}\right)$ |

where $k$ is the smoothness of $f_X$, the unknown density of $X$, and $N$ is the sample size of $Y$ (Fan, 1991). Although this slow rate indicates the difficulty of deconvolution with normal noises in general, if the scale of the normal noise is not too high, nonparametric deconvolution can still be practical (Fan, 1992): the optimal rate is $O(N^{-k/(2k+1)})$ when $\sigma = cN^{-1/(2k+1)}$ for some constant $c$, which is the same as the optimal rate for estimating $f_X$ directly from $X$. This optimal rate can be achieved by a kernel density estimator $\hat{f}_{N,h_N}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_K(th_N) \hat{\phi}_N(t) \ dt$ where $\hat{\phi}_N(t) = \frac{1}{N} \sum_{j=1}^{N} e^{itY_j}$ is the empirical characteristic function of $Y$, $\phi_K(\cdot)$ is the Fourier transform of a specified kernel function $K(\cdot)$ with $\phi_K(0) = 1$, $\phi_\epsilon$ is the characteristic function of $\epsilon$, $h_N = dN^{-1/(2k+1)}$, and $d > 0$ is a constant.

In our DP bootstrap setting, the sample size $N$ for deconvolution is the number of bootstrap estimates $B$, and $\hat{f}_{B,h_B}(x)$ generalizes the point estimates used in the standard interval: if we fix $B$ and define $\bar{\mu}_B = \lim_{h_B \to 0} \int_{-\infty}^{\infty} x^r \hat{f}_{B,h_B}(x) \ dx$, then $\bar{\mu}_1 = \frac{1}{B} \sum_{i=1}^{B} Y_i$, and $\bar{\mu}_2 = \frac{1}{B} \sum_{i=1}^{B} Y_i^2 - \mathbb{E}[\epsilon_i^2]$ are the estimates of $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$ (Hall and Lahiri, 2008); therefore, we can obtain the same $\bar{m}_{i,B}$ and $(\bar{s}_{i,B})^2$ for estimating $\mathbb{E}[g(D_i)]$ and $\text{Var}(g(D_i))$. 

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The deconvolution method is better since it does not require normal approximation to build CIs compared to the point estimates in Section 4.2. It also circumvents the problem caused by the possibly negative estimate of the variance of the sampling distribution. However, it is difficult to analyze the convergence rate of deconvolution in DP bootstrap with respect to $B$ and $n$ because the distribution of the added noises $e$ flattens when $B$ increases, and the distribution to be recovered is from bootstrap estimates. At the end of this section, we propose a signal-noise-ratio (SNR) measure as a rule-of-thumb for the choice of $B$.

We compared multiple numerical deconvolution methods through preliminary simulations with different settings on $n$ and $B$. Among different deconvolution methods, we choose to use deconvolveR (Efron, 2016) since it performs the best in our settings without tuning its hyper-parameters. We briefly summarize this method as follows. For the model $Y = X + e$, deconvolveR assumes that $Y$ and $X$ are distributed discretely with the sizes of their supports $|\mathcal{Y}| = k$ and $|\mathcal{X}| = m$. Then it models the distribution of $X$ by $f(\alpha) = e^{Q\alpha}/c(\alpha)$ where $Q$ is an $m \times p$ structure matrix with values from the natural spline basis with order $p$, $ns(X,p)$, and $\alpha$ is the unknown $p$-dimensional parameter vector; $c(\alpha)$ is the divisor necessary to make $f$ sum to 1. The estimation of the distribution of $X$ is obtained through the estimation of $\alpha$: We estimate $\alpha$ by maximizing a penalized log-likelihood $m(\alpha) = l(Y;\alpha) - s(\alpha)$ with respect to $\alpha$ where $s(\alpha)$ is the penalty term, and $l(Y;\alpha)$ is the log-likelihood function of $Y$ derived from $f(\alpha)$ and the known distribution of $e$.

After deconvolution, we construct the CI with confidence level $\alpha$ using the $\frac{\alpha}{2}$ and $1 - \frac{\alpha}{2}$ percentiles of the recovered sampling distribution. Figure 4 is an example of deconvolution on the DP bootstrap estimates with the following settings: $n = 10000$, $B = 1000$, $D = (x_1, \ldots, x_n)$, $x_i \in [0,1] = \mathcal{X}$, $g(D) = (\sum_{i=1}^n x_i)/n$; $\{D_j\}_{j=1}^B$ denotes $B$ bootstrap samples of $D$, and $\hat{g}(D_j) = g(D_j) + \xi_j$ is the DP bootstrap estimate where $\xi_j \sim \mathcal{N}(0, \sigma^2 = \frac{B}{n^2})$; $\{\hat{g}(D_j)\}_{j=1}^B$ on $\mathcal{X}^n$ asymptotically satisfies $(\sqrt{2-2/e})$-GDP; we generate $D$ by $x_i \sim \text{Unif}(0,1)$ for three different replicates. Since the ‘private bootstrap’ curve is much flatter than the ‘non-private bootstrap’ curve, conducting statistical inference directly with the distribution of the estimates from the DP bootstrap gives results inconsistent with the original bootstrap. We see that in Figure 4, deconvolution corrects this problem. Therefore, in the rest of this paper, we always use DP bootstrap with deconvolution for building CIs.

**Remark 13** Although the deconvolution method is specifically designed for additive noise mechanisms, we can potentially approximate more general mechanisms as additive ones asymptotically, e.g., exponential mechanism (Awan et al., 2019; Reimherr and Awan, 2019).
Remark 14 As it is difficult to analyze how to choose $B$ theoretically, we provide a rule of thumb where we define the signal-noise ratio (SNR) as the ratio between the variance of the sampling distribution and the variance of the noise added for DP: 1) If we have enough privacy budget, we choose the largest $B$ satisfying $\text{SNR} \geq 1$; 2) If we only have a very limited privacy budget, e.g., $\text{SNR} \leq 1$ for any $B \geq 10$, we choose the smallest $B$ such that $B \geq 2/\alpha$ for $1 - \alpha$ CIs. Note that for the case of the population mean inference in Section 4.2, the expectation of the variance of non-private bootstrap estimates is $(n-1)\sigma^2/n^2$, and for DP bootstrap, the added noise $\xi_j \sim N(0, (2-2/e)B\mu/n^2)$; therefore, our rule of thumb for building percentile intervals using deconvolution suggests $B \in \Theta(\mu^2n)$ which is consistent with our rate analysis in Section 4.2. The empirical results are in Appendix E.3.

Remark 15 To the best of our knowledge, we are the first to use deconvolution to recover the non-private sampling distribution from DP estimates and conduct statistical inference. Farokhi (2020) applied deconvolution to estimate the distribution of the sensitive data under local DP guarantees, which is different from the DP guarantee discussed in this paper.

5 Simulations

In this section, we use the confidence interval construction as a showcase for statistical inference with our DP bootstrap algorithm. First, we compare the non-private CI from the original bootstrap to the private CI from the recovered sampling distribution by using deconvolution on our DP bootstrap estimates. Then we compare our DP bootstrap with the NoisyVar (Du et al., 2020) and discuss the inherent difference between the two methods. The algorithms used in this section are in Appendix E.1.

5.1 Private CI compared to non-private CI for the population mean

Consider $D = (x_1, x_2, \ldots, x_n)$ where $x_i \in [0, 1]$ and $x_i \sim \text{iid } F_X$. We construct private CIs for $\mathbb{E}[x_i]$ with $D$. Let $g(D) = \frac{1}{n} \sum_{i=1}^{n} x_i$ be the non-private point estimate of $\mathbb{E}[x_i]$, and $\tilde{g}_B(D) = (\tilde{g}(D_1), \tilde{g}(D_2), \ldots, \tilde{g}(D_B))$ be DP bootstrap estimates where $\tilde{g}(D_j) = g(D_j) + \xi_j$, $\xi_j \sim N(0, (2-2/e)B\mu/n^2)$ is Gaussian mechanism, and $D_j \sim \text{boot}(D)$ is the $j$th bootstrap sample. From our privacy analysis, $\tilde{g}_B(D)$ asymptotically satisfies $\mu$-GDP. We construct private CIs for $\mathbb{E}[X]$ based on $\tilde{g}_B(D)$. For different privacy guarantees $\mu = 1, 0.5, 0.3, 0.1$, we choose $B = 2000\mu^2$ correspondingly to have a constant SNR (defined in Remark 14). The result is run with 2000 replicates where $x_i = \max(0, \min(1, z_i))$, $z_i \sim \text{iid } N(0.5, 1)$ and $n = 10000$.

The results for 90% CIs are in Table 2. The DP bootstrap’s coverage and width are similar to the non-private bootstrap, except that under the strongest privacy guarantee $\mu = 0.1$, DP bootstrap CI has over-coverage and larger width.

We examine the coverage of CIs with other confidence levels in Figure 5: Since the CI is built with the quantiles of the recovered distribution, and the coverage is determined by whether the true parameter value is in between the two quantiles, we evaluate the CDF of the recovered sampling distribution, $F^*$, at the true parameter value $\theta$, e.g., $0.05 \leq F^*(\theta) \leq 0.95$.
Table 2: Coverage and width of CIs with different privacy guarantees. The confidence level is 90%. The corresponding standard deviations among 2000 replicates are in the parenthesis.

| Privacy | Method               | Coverage | CI width   |
|---------|----------------------|----------|------------|
| N/A     | Bootstrap (B=2000)   | 0.905 (7e-3) | 0.014 (6e-6) |
| 1-GDP   | DP bootstrap (B=2000) | 0.896 (7e-3) | 0.014 (1e-5) |
|         | NoisyVar             | 0.803 (9e-3) | 0.011 (7e-6) |
| 0.5-GDP | DP bootstrap (B=500) | 0.898 (7e-3) | 0.014 (2e-5) |
|         | NoisyVar             | 0.806 (9e-3) | 0.011 (7e-6) |
| 0.3-GDP | DP bootstrap (B=180) | 0.901 (7e-3) | 0.015 (3e-5) |
|         | NoisyVar             | 0.804 (9e-3) | 0.011 (7e-6) |
| 0.1-GDP | DP bootstrap (B=20)  | 0.962 (4e-3) | 0.020 (1e-4) |
|         | NoisyVar             | 0.819 (9e-3) | 0.012 (7e-6) |

Figure 5: Coverage check of private CI with µ-GDP where µ = 1, 0.5, 0.3, 0.1.

is equivalent to the 90% CI covering θ. Therefore, the coverage at different confidence levels can be calculated by $E[\mathbb{1}(p_{\text{lower}} \leq F^*(\theta) \leq p_{\text{upper}})]$, and we want it to be close to the nominal confidence level, $p_{\text{upper}} - p_{\text{lower}}$. This is achieved if the CDF of $u := F^*(\theta)$ is close to the line $F(u) = u$, $\forall u \in [0, 1]$. In Figure 5, we can see that our DP bootstrap result aligns with the $F(u) = u$ when $B = 2000\mu^2$, similar to the non-private bootstrap.

In this simulation, our choice of $B$ varies from 20 to 2000. We use the DP bootstrap with a smaller $B$ under a stronger privacy guarantee because larger $B$ leads to smaller SNR, making deconvolution harder. If the coverage is satisfactory under many choices of $B$, e.g., $B = 20, 180, 500, 2000$ when $\mu = 1$, the largest $B$ gives the shortest CI since the deconvolution accuracy is determined by $B$. Detailed comparisons on the CI width, coverage, and corresponding SNR for different choices of $B$ are in Appendix E.3.

5.2 Comparison between DP bootstrap and NoisyVar

In the non-private setting, the parametric CI for the population mean $\theta$ can be built with the $t$-statistic $t = \frac{ar{X} - \theta}{\sqrt{s_X^2/n}}$ where $\bar{X}$ and $s_X^2$ are the sample mean and variance. Similarly, to construct a DP CI, one can obtain a DP $t$-statistic by replacing $\bar{X}$ and $s_X^2$ with their corresponding DP statistics, but this DP $t$-statistic may not follow the $t$-distribution because of the added noise for privacy. Du et al. (2020) adopted the idea of parametric bootstrap to construct the CI for the population mean: They plugged in the DP sample mean and the DP sample variance to generate normally distributed samples and compute the corresponding
DP sample means to estimate the sampling distribution of the DP sample mean. We compare our DP bootstrap with their algorithm, NoisyVar, under the settings we used in Section 5.1. We include NoisyVar in the appendix (Algorithm 3) for easier reference.

In Table 2, the performance of NoisyVar is not as satisfactory as in (Du et al., 2020) in terms of the coverage, which is because the distribution of our data, the clamped normal random variables, is not in the normal distribution family used by NoisyVar. Our comparison highlights the importance of non-parametric inference: The bootstrap CI does not assume the family of the sampling distribution or data distribution; therefore, our results do not suffer from the under-coverage issue. Note that the NoisyVar is also limited to building DP CIs for the population mean, while our DP bootstrap can be used on any DP statistic with additive noise mechanisms, which we demonstrate in the real-world experiments below.

6 Real-world experiments

In this section, we conduct experiments with the 2016 Census Public Use Microdata Files (PUMF), which provide data on the characteristics of the Canadian population (Canada, 2019). We analyze the dependence between market income and shelter cost in Ontario by the inference of logistic regression and quantile regression under DP guarantees. We use DP bootstrap with output perturbation mechanism (Chaudhuri et al., 2011) and compare our method with (Wang et al., 2019b, Algorithm 5), Differentially Private Confidence Intervals for Empirical Risk Minimization, which we abbreviate as DP-CI-ERM. Our results are in Figure 6, and more detailed comparisons are in Appendix E.3.

6.1 Experiment settings

The PUMF dataset contains 930,421 records of individuals, representing 2.7% of the Canadian population. Among the 123 variables in this dataset, we choose three variables: the province or territory of current residence (named PR), the market income (named MRKINC), and the shelter cost (named SHELCO). We extract the records of MRKINC and SHELCO belonging to the people in Ontario (according to the values in PR). After removing the records with unavailable values, the sample size is 217,360. We define the extracted data as the original dataset and analyze the relationship between MRKINC and SHELCO.

For the exploratory data analysis, we show the non-private empirical joint distribution between MRKINC and SHELCO in Figure 6a, and we assume that their maximum values are prior information since the dataset is top-coded, as shown in Figure 6a. We preprocess the data by scaling MRKINC and SHELCO so their ranges are $[0, 1]$.

To evaluate the performance of different statistical inference methods, we calculate the coverage and width of the CIs from 2000 simulations for each setting where the input datasets are sampled from the original dataset with replacement with size $n = 1000, 3000,$

\footnote{For the Line 5 in (Wang et al., 2019b Algorithm 5), we replace $c$ with 0 since in their analysis, the eigenvalues of the Hessian matrix is no less than $2c > 0$ while the eigenvalues of the covariance matrix only need to be non-negative. This modification greatly improves the performance of DP-CI-ERM when $n$ is large, as overestimating the covariance matrix leads to over-coverage and wider CIs.}
Figure 6: (a) Joint distribution between MRKINC and SHELCO in Ontario, Canada. The polygon region is the convex hull of all data points. (b) Results of 90% CIs for the slope parameters in logistic regression and quantile regression between MRKINC and SHELCO. Note that DP-CI-ERM cannot be used in the inference of quantile regression.

10000, 30000, 100000. We also calculate the probability that the CI of the slope parameter covers 0, which indicates that there is not enough evidence to reject the independence between MRKINC and SHELCO. The privacy guarantee is set to be 1-GDP, the confidence level is 90%, and we use $B = 100$ for bootstrap and DP bootstrap.

6.2 Logistic regression

We set the response $y_i = 1$ if SHELCO $\geq 0.5$, otherwise $y_i = -1$. In logistic regression, the true model is $P(Y|X) = \frac{1}{1+\exp(-\theta^T X \cdot Y)}$, and the empirical risk minimizer (ERM), also the maximum likelihood estimate of $\theta$, is $\hat{\theta} = \arg\min_{\theta} R(\theta)$ where $R(\theta) := \frac{1}{n} \sum_{i=1}^{n} -\log(P(y_i|x_i))$.

To obtain DP estimates, we implement the output perturbation mechanism following (Wang et al., 2019b, Algorithm 5), which replaces $R(\theta)$ by a regularized empirical risk $R(\theta) + c\|\theta\|^2_2$ and adds noise to the output: $\tilde{\theta} = \hat{\theta} + \xi$ where $\hat{\theta} = \arg\min_{\theta} (R(\theta) + c\|\theta\|^2_2)$. As the sensitivity of the regularized ERM is $\Delta(\hat{\theta}) = \frac{1}{nc}$ (Wang et al., 2019b), we use Gaussian mechanism, $\xi \sim \mathcal{N}(0, (\frac{1}{\mu nc})^2)$, then $\tilde{\theta}$ satisfies $\mu$-GDP; to satisfy the constraint, $\|x_i\|^2_2 \leq 1$, in the sensitivity analysis, we let the covariate be $x_i = (1/\sqrt{2}, MRKINC/\sqrt{2})$. Following (Wang et al., 2019b), we define the true parameter $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ as the regularized ERM estimated with the original dataset under the same $c$. We build CI for the slope parameter $\theta_2$: If the 90% CI does not cover 0, we are confident that MRKINC is related to SHELCO.

The results are shown in the upper figures of Figure 6b where $c = 1$. The CIs by DP bootstrap are wider than the ones by DP-CI-ERM when the sample size $n$ is small: As a non-parametric method, DP bootstrap does not fully utilize the structure of the private ERM as opposed to DP-CI-ERM, so its CIs are often not as tight as the ones by DP-CI-ERM. Both types of private CIs are as wide as non-private bootstrap CI when $n = 100000$, do not suffer from the under-coverage issue, and have $P(\text{CI covers 0}) \approx 0$ when $n \geq 30000$.  

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6.3 Quantile regression

We use quantile regression as an example to demonstrate the advantage of DP bootstrap as a non-parametric method. We set the response $y_i = \text{SHELCO}$ and the covariate $x_i = (1, \text{MRKINC})$. Following (Reimherr and Awan, 2019), we assume that the conditional quantile function of $Y$ given $X$ is $Q_{Y|X}(\tau) = X^\top \theta_\tau$, we estimate $\theta_\tau$ also by ERM with the objective function $R(\theta) = \frac{1}{n} \sum_{i=1}^n l(z_i)$ where $l(z_i) = (\tau - 1(z_i \leq 0))z_i$ and $z_i = y_i - x_i^\top \theta$. Similar to our experiment with logistic regression, we define the true parameter $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ as the regularized ERM estimated with the original dataset under the same regularization parameter $c$. By (Chaudhuri et al., 2011, Lemma 7), the sensitivity of the regularized ERM, $\hat{\theta} = \text{argmin}_\theta \{R(\theta) + c\|\theta\|^2\}$, is bounded by $\Delta(\hat{\theta}) = \max\{2\tau, 2(1-\tau), \sqrt{2}\}$; the derivation is in Appendix E.2. Then $\tilde{\theta} = \hat{\theta} + \xi$ satisfies $\mu$-GDP when $\xi \sim \mathcal{N}(0, \frac{\Delta(\hat{\theta})^2}{\mu^2})$ (Gaussian mechanism.)

To the best of our knowledge, this is the first result of building private CI for the coefficients of quantile regression. As DP-CI-ERM uses Taylor expansion of the gradient of $R(\theta)$ to characterize the difference between the estimate $\hat{\theta}$ and the true parameter $\theta^*$, i.e., $\theta^* - \hat{\theta} \approx H[R(\hat{\theta})]^{-1}(\nabla R(\theta^*) - \nabla R(\hat{\theta}))$, it is not usable in quantile regression as the Hessian of $R(\theta)$ is always 0 when it exists. The results for DP bootstrap and non-private bootstrap are shown in the lower figures of Figure 6b where we set $c = 1$ and $\tau = 0.5$. Similar to our experiment in logistic regression, DP bootstrap never suffers from the under-coverage issue. It performs similarly to non-private bootstrap when $n \geq 30000$. We can see that $P(\text{CI covers 0}) \approx 0$ when $n \geq 1000$; therefore, we are confident that MRCINC and the median of corresponding SHELCO are not independent with 90% confidence.

7 Conclusion

Our analysis of the DP bootstrap provides a new perspective on resampling in DP by considering the output distribution as a mixture distribution which gives a tractable lower bound for the DP bootstrap in $f$-DP. Furthermore, our composition result for DP bootstrap using the Gaussian mechanism gives a simple asymptotic privacy guarantee and highlights the minimal cost of DP bootstrap. We are the first to use deconvolution to recover the non-private sampling distribution from DP bootstrap estimates for statistical inference. Our simulations and experiments show that the CIs generated by the deconvolved distribution achieve the nominal coverage, and our results are not only comparable to existing methods like NoisyVar and DP-CI-ERM but also applicable to the inference problems such as quantile regression where existing methods cannot be used.

One direction of future work is on improving the privacy analysis of the DP bootstrap: The lower bound in Theorem 9 could be tightened by considering all $\alpha_i$ jointly rather than individually. Furthermore, our composition result is user-friendly but asymptotic, and it remains to develop a computable tight finite-sample composition result.

For the statistical inference using the DP bootstrap, the choice of the number of bootstrap samples $B$ can be further optimized for a given sample size $n$ and privacy parameter $\mu$. One may pursue finite sample utility results for the deconvolved distribution or the CIs ob-
tained by the DP bootstrap deconvolution procedure. Furthermore, to handle non-additive noise mechanisms, new inference techniques may be needed to replace deconvolution. We also notice in Table 2 and Figure 6b that with a strong privacy guarantee or small \( n \), our CI may still be wider and have higher coverage than the non-private CI or the private CI from other methods, indicating that the deconvolution procedure can be further optimized to get a tighter estimate and improve the width of the CIs while maintaining the nominal coverage. Apart from the standard interval and the percentile interval, there are also other inference methods based on the bootstrap estimates such as \( \text{BC}_a \) (bias-corrected and accelerated) (Efron, 1987) or \( \text{ABC} \) (approximate bootstrap confidence) (Diciccio and Efron, 1992), we leave it as future work to investigate the private versions of these methods.

We can also use the DP bootstrap framework for high-dimensional inference. Our privacy analysis applies to any mechanism, including those multivariate mechanisms, e.g., the Gaussian mechanism on each dimension of the estimate. Future work in this direction is to apply existing methods of multivariate deconvolution (Youndjé and Wells, 2008; Hazelton and Turlach, 2009; Sarkar et al., 2018) and bootstrap (Hall, 1987; Aelst and Willems, 2005) to our DP bootstrap framework.

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Appendix A. Further discussion on differential privacy

In this section, we first restate the definition of $(\varepsilon, \delta)$-DP and the primal-dual result between $(\varepsilon, \delta)$-DP and $f$-DP. Then we discuss the existing results for subsampling and composition.

**Definition 16** ($((\varepsilon, \delta)$-DP; Dwork et al., 2006) A mechanism $\mathcal{M} : \mathcal{X}^n \rightarrow \mathcal{O}$ is $(\varepsilon, \delta)$-differentially private ($(\varepsilon, \delta)$-DP) if for any neighboring datasets $D, D' \in \mathcal{X}^n$, and every measurable set $S \subseteq \mathcal{O}$, the following inequality holds: $\Pr[\mathcal{M}(D) \in S] \leq e^{\varepsilon} \Pr[\mathcal{M}(D') \in S] + \delta$.

For any $\mathcal{M}$ and $\varepsilon \geq 0$, there exists $0 \leq \delta \leq 1$ such that $\mathcal{M}$ is $(\varepsilon, \delta)$-DP. The privacy profile (Balle et al., 2018), $\delta_{\mathcal{M}}(\varepsilon)$, maps $\varepsilon$ to the smallest $\delta$ such that $\mathcal{M}$ is $(\varepsilon, \delta)$-DP.

**Proposition 17** (primal-dual view between $(\varepsilon, \delta)$-DP and $f$-DP; Dong et al., 2022) A mechanism is $(\varepsilon_i, \delta_i)$-DP for all $i \in I$ if and only if it is $f$-DP with $f = \sup_{i \in I} f_{\varepsilon_i, \delta_i}$ where $f_{\varepsilon_i, \delta_i} = \max\{0, 1 - \delta - \varepsilon_i \alpha, e^{-\varepsilon_i}(1 - \delta - \alpha)\}$. For a symmetric tradeoff function $f$, a mechanism is $f$-DP if and only if it is $(\varepsilon, \delta)$-DP for all $\varepsilon \geq 0$ with $\delta(\varepsilon) = 1 + f^*( -e^\varepsilon)$ where $f^*(y) = \sup_{-\infty < x < \infty} xy - f(x)$, also known as the convex conjugate of $f$.

It is common to design DP mechanisms using either 1) subsampling: replacing the input with a dataset resampled from the original dataset where each entry is resampled for at most once, and/or 2) composition: running multiple mechanisms and releasing all their outputs. For $(\varepsilon, \delta)$-DP, the classical composition results are given by Dwork et al. (2010); Ullman (2017) derived a tight bound for subsampling, and Balle et al. (2018) gave the state-of-the-art results for resampling with and without replacement. For privacy loss distributions (PLD), Koskela et al. (2020) derived the PLDs for resampling with and without replacement for the Gaussian mechanism. For Rényi-DP, the state-of-the-art in composition results are given by Mironov (2017), and Wang et al. (2019a) derived a tight bound for subsampling. For concentrated DP (CDP) and zero concentrated DP (zCDP), there are tight composition results (Dwork and Rothblum, 2016; Bun and Steinke, 2016), but no result for resampling. Bun et al. (2018) proposed truncated CDP (tCDP) as an extension of zCDP, which has results for both composition and resampling without replacement. For $f$-DP, Dong et al. (2022) gave results for both composition and subsampling, along with a central limit theorem under composition. Zhu et al. (2022) derived ‘dominating pairs’ for composition and subsampling (subset subsampling and Poisson subsampling), which could be used to calculate any of the above DP guarantees.

Appendix B. Proofs for Section 3

In this section, we provide the proofs for the theorems and propositions in Section 3.

B.1 Proofs for Section 3.1

We first restate the $(\varepsilon, \delta)$-DP results in (Balle et al., 2018) and provide some useful results for the proof of Proposition 5.
Theorem 18 (Theorem 10 in (Balle et al., 2018)) Given $\varepsilon \geq 0$, assume $\mathcal{M}$ satisfies $(\varepsilon, \delta_{\mathcal{M},i}(\varepsilon))$-DP with group size $i$. Let $p_i = \left(\frac{n}{n_i}\right)^i(1-\frac{1}{n})^{n-i}$ and $\varepsilon' = \log(1+1-(1-p_0)(\varepsilon'-1))$, then $\mathcal{M} \circ \text{boot}$ satisfies $(\varepsilon', \delta_{\mathcal{M} \circ \text{boot}}(\varepsilon'))$-DP where $\delta_{\mathcal{M} \circ \text{boot}}(\varepsilon') = \sum_{i=1}^{n} p_i \delta_{\mathcal{M},i}(\varepsilon)$.

Definition 19 Let $f$ be a tradeoff function, $\bar{x} = \inf\{x \in [0,1] : -1 \in \partial f(x)\}$. The symmetrization operator which maps a possibly asymmetric tradeoff function to a symmetric tradeoff function is defined as

$$\text{Symm}(f) := \begin{cases} \min\{f, f^{-1}\}, & \text{if } \bar{x} \leq f(\bar{x}), \\ \max\{f, f^{-1}\}, & \text{if } \bar{x} > f(\bar{x}). \end{cases}$$

Proposition 20 (Proposition E.1 in Dong et al., 2022) Let $f$ be a tradeoff function. Suppose a mechanism is $(\varepsilon, 1 + f^*(-e^\varepsilon))$-DP for all $\varepsilon \geq 0$, then it is $\text{Symm}(f)$-DP.

Proof [Proof of Proposition 5] We use $f$-DP to restate the $\delta_{\mathcal{M} \circ \text{boot}}(\varepsilon')$ in Theorem 18: Let $p = 1 - p_0$, $\delta_{\mathcal{M} \circ \text{boot}}(\varepsilon') \leq \sum_{i=1}^{n} p_i \delta_{\mathcal{M},i}(\varepsilon) = \sum_{i=1}^{n} p_i (1 + f_{\mathcal{M},i}^*(-e^\varepsilon)) = p \left(1 + \sum_{i=1}^{n} \frac{p_i}{1-p_0} f_{\mathcal{M},i}^*(-e^\varepsilon)\right)$ where $\mathcal{M}$ satisfies $f_{\mathcal{M},i}$-DP with group size $i$ and $f_{\mathcal{M},i}$ is symmetric.

From the Supplement to (Dong et al., 2022) (Page 42), we know that $\varepsilon' = \log(1-p+pe^\varepsilon)$ and $\delta' = p(1 + f^*(-e^\varepsilon))$ can be re-parameterized into $\delta' = 1 + f_p^*(-e^\varepsilon)$ where $f_p = pf + (1-p)\text{Id}$. For symmetric $f$, we have $\text{Symm}(f_p) = C_p(f)$ since $\bar{x} \leq f_p(\bar{x})$ where $\bar{x} = \inf\{x \in [0,1] : -1 \in \partial f(x)\}$. Using Proposition 20, we have $f_{\mathcal{M} \circ \text{boot}} = C_p \left((\sum_{i=1}^{n} \frac{p_i}{1-p_0} f_{\mathcal{M},i}^*)\right)$.

Lemma 21 For $i = 1, 2, \ldots, k$, let $f_i$ be a tradeoff function and $p_i \in (0,1]$ satisfying $\sum_{i=1}^{k} p_i = 1$. We write $f = (f_1, \ldots, f_k)$ and $p = (p_1, \ldots, p_k)$.

1. $\text{mix}(p, f) : [0,1] \to [0,1]$ is a well-defined tradeoff function.
2. If the tradeoff functions $f_i$ are all symmetric, then $\text{mix}(p, f)$ is symmetric.

We first state some useful properties of tradeoff functions for proving Lemma 21.

Proposition 22 (Proposition 1 in Dong et al., 2022) A function $f : [0,1] \to [0,1]$ is a tradeoff function if and only if $f$ is convex, continuous, non-increasing, and $f(x) \leq 1 - x$.

Proposition 23 For any tradeoff function $f$,

1. $f(\alpha)$ is strictly decreasing for $\alpha \in \{\alpha : f(\alpha) > 0\}$.
2. $0 \notin \partial f(\alpha)$ if and only if $\alpha \in \{\alpha : f(\alpha) > 0\}$.
3. If $f^{-1}(f(y)) \neq y$, then $f(y) = 0$. 

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4. If \( f = f^{-1} \) and there exists \( C < 0 \) such that \( C \in \partial f(\alpha)|_{\alpha=f(\alpha_0)} \), then we have \( f(f(\alpha_0)) = \alpha_0 \) and \( \frac{1}{C} \in \partial f(\alpha)|_{\alpha=f(\alpha_0)} \).

5. There is exactly one \( \bar{\alpha} \) such that \( f(\bar{\alpha}) = \alpha \).

6. If \( f = f^{-1} \) and \( f(\bar{\alpha}) = \bar{\alpha} \), then \(-1 \in \partial f(\alpha)|_{\alpha=\bar{\alpha}} \).

7. If \( f = f^{-1} \) and \(-1 \in \partial f(\alpha)|_{\alpha=\alpha_0} \), then \(-1 \in \partial f(\alpha)|_{\alpha=f(\alpha_0)} \).

**Proof** [Proof of Proposition 23] We provide the proofs for each property of \( f \) below.

1. If there are \( 0 \leq \alpha_1 < \alpha_2 \leq 1 \) such that \( f(\alpha_1) = f(\alpha_2) = \beta > 0 \), since \( g \) is convex, we will have \( f(\alpha) \geq \beta \) \( \forall \alpha \geq \alpha_1 \). Therefore, \( f(1) \geq \beta > 0 \) which contradicts with \( f(1) \leq 1 - 1 = 0 \). Since \( f(\alpha) \) is not increasing, it is strictly decreasing for \( \alpha \in \{ \alpha : f(\alpha) > 0 \} \).

2. If \( 0 \in \partial f(\alpha) \) and \( f(\alpha) > 0 \), we have \( f(y) \geq f(\alpha) > 0 \) \( \forall y \in [0, 1] \). This contradicts with the fact that \( f(1) = 0 \). If \( 0 \notin \partial f(\alpha) \) and \( f(\alpha) = 0 \), there exists \( z \in [0, 1] \) such that \( f(y) < 0 \cdot (y - \alpha) + f(\alpha) = 0 \) which contradicts with the fact that \( f : [0, 1] \to [0, 1] \).

3. If \( f^{-1}(f(y)) = \inf \{ t \in [0, 1] : f(t) \leq f(y) \} < y \), since \( f \) is not increasing, \( f(f^{-1}(f(y))) = f(y) \), which holds only when \( f(y) = 0 \) (otherwise \( f(y) \) is strictly decreasing).

4. From the fact that \( C \) is a sub-differential value of \( f \) at \( \alpha_0 \), we have \( f(z) \geq C(z - \alpha_0) + f(\alpha_0) \) \( \forall z \in [0, 1] \). If there exists \( z \) such that \( z < \alpha_0 \) and \( f(z) \leq f(\alpha_0) \), it contradicts with \( f(z) \geq C(z - \alpha_0) + f(\alpha_0) \). Therefore,

\[
  f(f(\alpha_0)) = f^{-1}(f(\alpha_0)) = \inf \{ t \in [0, 1] : f(t) \leq f(\alpha_0) \} = \alpha_0.
\]

We prove \( \frac{1}{C} \in \partial f(\alpha)|_{\alpha=f(\alpha_0)} \) by showing \( f(y) \geq \frac{1}{C}(y - f(\alpha_0)) + f(f(\alpha_0)) \forall y \in [0, 1] \). Since \( C \in (-\infty, 0) \), if \( f(y) \geq \frac{1}{C}(y - f(\alpha_0)) + f(f(\alpha_0)) \) holds when \( y = f(0) \), then it also holds for \( y > f(0) \) as \( f(y) = 0 = f(f(0)) \geq \frac{1}{C}(f(0) - f(\alpha_0)) + \alpha_0 \). If \( y \in [0, f(0)] \), we define \( z := f(y) \). Since \( f = f^{-1} \) and \( f \) is strictly decreasing when \( f > 0 \), we know that \( f(y) > 0 \) and \( f(z) = f^{-1}(f(y)) = y \) for \( y \in [0, f(0)] \). We also know that \( f(z) = y \) when \( y = f(0) \) since \( z = f(y) = f(f(0)) = 0 \). From the fact that \( C \) is a sub-differential value of \( f \) at \( \alpha \), we have \( f(z) \geq C(z - \alpha) + f(\alpha) \). Now we show \( f(y) \geq \frac{1}{C}(y - f(\alpha_0)) + f(f(\alpha_0)) \forall y \in [0, f(0)] \) through the analysis below.

- If \( f(0) \geq y > f(\alpha_0) \), we have \( z = f(y) < \alpha_0, f(z) = y, \) and \( C \geq \frac{f(z) - f(\alpha_0)}{z - \alpha_0} \). Therefore, \( \frac{1}{C}(y - f(\alpha_0)) + f(f(\alpha_0)) \leq \frac{z - \alpha_0}{f(z) - f(\alpha_0)}(y - f(\alpha_0)) + \alpha_0 = z = f(y) \).
- If \( 0 \leq y < f(\alpha_0) \), we have \( z = f(y) > \alpha_0, f(z) = y, \) and \( C \leq \frac{f(z) - f(\alpha_0)}{z - \alpha_0} \). Therefore, \( \frac{1}{C}(y - f(\alpha_0)) + f(f(\alpha_0)) \leq \frac{z - \alpha_0}{f(z) - f(\alpha_0)}(y - f(\alpha_0) + \alpha_0 = z = f(y) \).
- If \( y = f(\alpha_0) \), we have \( \frac{1}{C}(y - f(\alpha_0)) + f(f(\alpha_0)) = f(y) \).

5. If there are \( \bar{\alpha}_1 < \bar{\alpha}_2 \) that \( \bar{\alpha}_1 = f(\bar{\alpha}_1) \) and \( \bar{\alpha}_2 = f(\bar{\alpha}_2) \), then since \( f \) is non-increasing, we have \( f(\bar{\alpha}_1) \geq f(\bar{\alpha}_2) \) which contradicts with \( \bar{\alpha}_1 < \bar{\alpha}_2 \). Since \( f(\alpha) \) is continuous and \( f(0) - 0 \geq 0 - 0 = 0, f(1) - 1 = -1 < 0 \), there exists \( \bar{\alpha} \in [0, 1] \) such that \( f(\bar{\alpha}) - \bar{\alpha} = 0 \).
6. If \(-1 \notin \partial f(\bar{\alpha})\), there exists \(y \in [0, 1]\) such that \(f(y) < -(y-\bar{\alpha}) + f(\bar{\alpha})\) and \(f(f(y)) = y\) because \(f = f^{-1}\) (if \(f(f(y)) \neq y\), we have \(f(y) = 0\), then we can replace \(y\) with \(f(0) \leq y\) and we still have \(f(y) < -(y-\bar{\alpha}) + f(\bar{\alpha})\)). Therefore, \((y, f(y))\) and \((f(y), y)\) are both on the curve of \(f\). Since \(y \neq \bar{\alpha}\), we know \(y \neq f(y)\). Without the loss of generality, we assume that \(y > f(y)\). Then we know that \(y > \bar{\alpha} > f(y)\) since otherwise we will have contradictions: \(\bar{\alpha} \geq y > f(y) \geq f(\bar{\alpha}) = \bar{\alpha}\) or \(\bar{\alpha} \leq f(y) < y = f(f(y)) \leq f(\bar{\alpha}) = \bar{\alpha}\).

We denote \(q = \frac{\bar{\alpha} - f(y)}{y - f(y)} > 0\) and \(1 - q = \frac{y - \bar{\alpha}}{y - f(y)} > 0\). Then \(\bar{\alpha} = qy + (1 - q)f(y)\), and by the convexity of \(f\), we have \(\bar{\alpha} = f(\bar{\alpha}) \leq qf(y) + (1 - q)f(y) = qf(y) + (1 - q)y\). Therefore, \(f(\bar{\alpha}) + \bar{\alpha} \leq (qy + (1 - q)f(y)) + (qf(y) + (1 - q)y) = y + f(y)\). But from \(f(y) < -(y-\bar{\alpha}) + f(\bar{\alpha})\), we know \(f(\bar{\alpha}) + \bar{\alpha} > y + f(y)\), which leads to a contradiction. Therefore, we have \(-1 \in \partial f(\bar{\alpha})\).

7. As \(-1 \notin \partial f(\alpha)\), we have \(f(y) \geq -(y-\alpha_0) + f(\alpha_0)\) \(\forall y \in [0, 1]\) and \(f(f(\alpha_0)) = \alpha_0\). Therefore, \(f(y) \geq -(y-f(\alpha_0)) + f(f(\alpha_0))\) \(\forall y \in [0, 1]\) and we have \(-1 \notin \partial f(\alpha)\).

**Proof** [Proof of Lemma 21] For part 1, first we show that for every \(\alpha \in (0, 1)\), there exists \(C \in (0, C] = A(C)\). Since each \(f_i\) is a convex and non-increasing, its sub-differential \(\partial f_i(\alpha_i)\) is in \((0, C]\) and non-decreasing with respect to \(\alpha_i\). Therefore, for any \(-\infty < C_1 < C_2 \leq 0\), we have \(a_1 \leq a_2 \forall a_1 \in A_i(C_1), a_2 \in A_i(C_2)\), and for any \(0 < a_1 < a_2 < 1\), we have \(C_1 \leq C_2 \forall C_1 \in \partial f_i(a_1), C_2 \in \partial f_i(a_2)\). We name these two properties as the monotonicity of the sub-differential mapping.

Since each \(f_i\) is convex and continuous, its sub-differential \(\partial f_i(\alpha_i)\) is also continuous in the sense that for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that \(\partial f_i(\alpha_i') \subset \partial f_i(\alpha_i) + (-\varepsilon, \varepsilon)\) whenever \(\|\alpha_i' - \alpha_i\| < \delta\) (see Exercise 2.2.22(a) in (Borwein and Vanderwerff, 2010)).

From the continuity and monotonicity of the sub-differential mapping, we know that for any \(C \in (0, C] = A_i(C)\) is a closed interval, e.g., \([a, b]\). Note that \(A_i(C)\) is always nonempty: let \(C_{i, \text{range}} := \bigcup_{\alpha \in (0, 1)} \partial f_i(\alpha)\); if \(C < C_i \forall C_i \in C_{i, \text{range}}\), we have \(A_i(C) = \{0\}\); if \(C > C_i \forall C_i \in C_{i, \text{range}}\), we have \(A_i(C) = \{1\}\); if \(C \in C_{i, \text{range}}\), there must be an \(\alpha \in (0, 1)\) such that \(C \in \partial f_i(\alpha)\). By the same reasoning, we also have \((0, 1) \subset \bigcup_{C \in (0, C]} A_i(C)\) for any \(i\). As \(A(C) = \{\sum_{i=1}^{k} p_i \alpha_i | \alpha_i \in A_i(C)\}\), we have \((0, 1) \subset \bigcup_{C \in (0, C]} A(C)\), and we also have the monotonicity of \(A(C)\) with respect to \(C\).

Next, we show that \(\text{mix}(p, f)\) is a well-defined function, i.e., for a given \(\alpha\), although there could be multiple choices of \(\{\alpha_i\}_{i=1}^{k}\) such that \(\sum_{i=1}^{k} p_i \alpha_i = \alpha\), we will obtain the same value of \(\sum_{i=1}^{k} p_i f_i(\alpha_i)\) for all choices. Consider two choices, \(\{\alpha_i\}_{i=1}^{k}\) and \(\{\alpha'_i\}_{i=1}^{k}\), that correspond to the same \(\alpha\). Let \(C\) and \(C'\) correspond to \(\{\alpha_i\}_{i=1}^{k}\) and \(\{\alpha'_i\}_{i=1}^{k}\) respectively. As there exist \(i \neq j\) such that \(\alpha_i < \alpha_i'\) and \(\alpha_j > \alpha_j'\) (since \(\sum_{i=1}^{k} p_i \alpha_i = \sum_{i=1}^{k} p_i \alpha_i'\) and \(p_i > 0\) for \(i = 1, 2, \ldots, k\), from the monotonicity of the sub-differential mapping, we know that \(C = C'\) since \(C \leq C'\) and \(C' \leq C\). For \(\{\alpha_i\}_{i=1}^{k}\) and \(\{\alpha_i'\}_{i=1}^{k}\), since \(\partial f_i(\alpha_i) = \partial f_i(\alpha_i') = C\), we have \(f_i(\alpha_i) \geq C(\alpha_i - \alpha_i') + f_i(\alpha_i') \geq f_i(\alpha_i)\) for \(i = 1, 2, \ldots, n\). Therefore, \(f_i(\alpha_i) - f_i(\alpha_i') = C(\alpha_i - \alpha_i')\). As we know \(\sum_{i=1}^{k} p_i \alpha_i = \sum_{i=1}^{k} p_i \alpha_i'\), we have \(\sum_{i=1}^{k} p_i f_i(\alpha_i) - \sum_{i=1}^{k} p_i f_i(\alpha_i') = C \sum_{i=1}^{k} p_i (\alpha_i - \alpha_i') = 0\), which means that \(\text{mix}(p, f)\) is well-defined.
Finally, we show that mix($p, f$) is a tradeoff function.

Let $f = \text{mix}(p, f)$. We can see that $f(x) \in [0, 1]$ for all $x \in [0, 1]$ and $f(x) \leq 1 - x$. We also know $f$ is non-increasing due to the monotonicity of $A(C)$, the monotonicity of the sub-differential mapping, and $f_i$ being non-increasing.

Now we prove that $f$ is continuous. For a fixed $\alpha$ and $\delta > 0$, we can find the $\{\alpha_i\}_{i=1}^k$ corresponding to this $\alpha$, and find $\varepsilon_i$ such that $|f_i(\alpha'_i) - f_i(\alpha_i)| < \delta$ whenever $|\alpha'_i - \alpha_i| < \varepsilon_i$; then we let $\varepsilon = \min_{i \in \{1, 2, \ldots, k\}} \varepsilon_i p_i$, and we have $|f(\alpha') - f(\alpha)| < \delta$ whenever $|\alpha' - \alpha| < \varepsilon$.

To prove this, without loss of generality, we assume $\alpha < \alpha'$, and we can find $\{\alpha_i\}_{i=1}^k$ and $\{\alpha'_i\}_{i=1}^k$ corresponding to $\alpha$ and $\alpha'$ respectively, where $\alpha_i \leq \alpha'_i$ for $i = 1, 2, \ldots, k$. Then if $\alpha' - \alpha < \varepsilon$, we must have $\alpha'_i - \alpha_i < \varepsilon_i$ for $i = 1, 2, \ldots, k$, therefore $|f(\alpha') - f(\alpha)| < \delta$.

Now we prove that $f$ is convex. By the definition of convexity, we only need to show that for any $\alpha, \alpha', \tau \in [0, 1]$, we have $tf(\alpha) + (1-t)f(\alpha') \geq f(t\alpha + (1-t)\alpha')$. From the construction of mix($p, f$), we can find $\{\alpha_i\}_{i=1}^k, \{\alpha'_i\}_{i=1}^k$, and $\{\tilde{\alpha}_i\}_{i=1}^k$ with their matched sub-differential being $C, C'$, and $\tilde{C}$ corresponding to $\alpha, \alpha'$, and $\tilde{\alpha}$ respectively. Then

$$
tf(\alpha) + (1-t)f(\alpha') = \sum_{i=1}^k p_i (tf_i(\alpha_i) + (1-t)f_i(\alpha'_i)) \geq \sum_{i=1}^k p_i (t\alpha_i + (1-t)\alpha'_i) \quad \text{(convexity)}$$

$$\geq \sum_{i=1}^k p_i (\tilde{C}(t\alpha_i + (1-t)\alpha'_i - \tilde{\alpha}_i) + f_i(\tilde{\alpha}_i)) \quad \text{(since $\tilde{C} \in \partial f_i(\tilde{\alpha}_i)$)}$$

$$= \tilde{C}(t\alpha + (1-t)\alpha' - \tilde{\alpha}) + f(t\alpha + (1-t)\alpha') = f(t\alpha + (1-t)\alpha').$$

Therefore, $f$ is convex, and we have proved that $f$ is a tradeoff function.

For part 2, let $g = \text{mix}(p, f)$. We prove that $g$ is symmetric by showing $g^{-1} = \text{mix}(p, f)$.

By definition, $g^{-1}(\beta) = \inf\{\alpha \in [0, 1] : g(\alpha) \leq \beta\}$. By the construction of mix($p, f$), for each $\alpha \in [0, 1]$, there exist a constant $C$ and $\{\alpha_i\}_{i=1}^k$ such that $C \in \partial f_i(\alpha_i), \alpha = \sum_{i=1}^k p_i \alpha_i$, and $g(\alpha) = \sum_{i=1}^k p_i f_i(\alpha_i)$.

If $\beta = 0$, then $g(\alpha) = 0$, and $f_i(\alpha_i) = 0$. Therefore, $g^{-1}(0) = \inf\{\alpha \in [0, 1] : \alpha = \sum_{i=1}^k p_i \alpha_i, f_i(\alpha_i) = 0\}$. Since $f_i$ is symmetric, we have that $g(\alpha_i) = 0$ for $i \in \{1, 2, \ldots, k\}$.

If $\beta \geq g(0)$, then from the definition of $g^{-1}$, we have $g^{-1}(\beta) = 0$, and we need to prove $g(\alpha) = 0$ for $\alpha \geq g(0)$. From the construction of mix($p, f$), we have $g(\alpha) = \sum_{i=1}^k p_i f_i(\alpha_i)$ where $\alpha = \sum_{i=1}^k p_i \alpha_i$. Let $\alpha \geq g(0) = \sum_{i=1}^k p_i f_i(0)$. Then, if there exists $i$ that $\alpha_i > f_i(0)$ which means that $f_i(\alpha_i) = 0$, then $f_i(\alpha_i) = 0$ and $f_i(\alpha_i) = \{0\}$; therefore, $\partial f_i(\alpha_j) = \{0\}$ and $f_j(\alpha_j) = 0$ for $j = 1, 2, \ldots, k$. We have $g(\alpha) = 0$. If $\alpha_i \leq f_i(0)$ for $i = 1, 2, \ldots, k$, since $\sum_{i=1}^k p_i \alpha_i = \alpha \geq \sum_{i=1}^k p_i f_i(0)$, we have $\alpha_i = f_i(0) \forall i$, which means $f_i(\alpha_i) = 0 \forall i$; therefore, $g(\alpha) = 0$.

If $g(0) > \beta > 0$, since $g$ is a tradeoff function, there exists only one $\alpha \in [0, 1]$ such that $g(\alpha) = \beta$. From the construction of $g$, i.e., $g(\alpha) = \sum_{i=1}^k p_i f_i(\alpha_i) = \beta > 0$, for the $\{\alpha_i\}_{i=1}^k$ corresponding to $\alpha$, there exists $i_0$ such that $f_{i_0}(\alpha_{i_0}) > 0$. Since $f_{i_0}(\alpha_{i_0}) > 0$, we have $0 \notin \partial f_{i_0}(\alpha_{i_0})$. Therefore, the corresponding constant $C$ that $C \in \partial f_i(\alpha_i)$ for $i = 1, 2, \ldots, k$, and $C$ is not 0. Therefore, $f_i(\alpha_i) \neq 0$ for $i = 1, 2, \ldots, k$. 

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From Proposition 23, we know that $\frac{1}{c} \in \partial f_i(\alpha)|_{\alpha=f_i(\alpha)}$. Let $\alpha = \sum_{i=1}^{k} p_i \alpha_i$, and $g(\alpha) = \sum_{i=1}^{k} p_i f_i(\alpha_i)$. As $g$ is a tradeoff function, $g(\alpha)$ is strictly decreasing in $\{\alpha|g(\alpha) > 0\}$. Therefore, for any $g(\alpha) > 0$, there is a one-to-one mapping between $\alpha$ and $g(\alpha)$, and $g^{-1}(g(\alpha)) = \alpha$. Now we can view $g^{-1}$ as a mixture of $f_i$ at the values $f_i(\alpha_i)$: we let $\beta_i := f_i(\alpha_i)$; since $f_i(\alpha_i) = \alpha_i$, we have $g^{-1}(\sum_{i=1}^{k} p_i \beta_i) = \sum_{i=1}^{k} p_i f_i(\beta_i)$, and there exists a constant $\frac{1}{c}$ such that $\frac{1}{c} \in \partial f_i(\alpha)|_{\alpha=\beta_i}$ for $i = 1, \ldots, k$. Therefore, $g^{-1} = \text{mix}(p, f)$. Since the mixture operation is well-defined, we know that $g = g^{-1}$.

**Proof** [Proof of Theorem 8] Consider the neighboring datasets $D_1, D_2$, and a rejection rule $\psi$ giving the type I error $\alpha = E_{M(D_1)}\psi$, and type II error $\beta = E_{M(D_2)}(1 - \psi)$. We write $\alpha_i = E_{M_i(D_1)}\psi$, $\beta_i = E_{M_i(D_2)}(1 - \psi)$. Then $\alpha = \sum_{i=1}^{k} p_i \alpha_i$, $\beta = \sum_{i=1}^{k} p_i \beta_i$.

In order to obtain the lower bound of $\beta$ given $\alpha$, which we denote as $f_{\text{mix}}(\alpha)$, we not only need the tradeoff between $\alpha_i$ and $\beta_i$, which is $f_i$, but also consider the tradeoff between $\alpha_i$ and $\alpha_j$ because of the constraint $\sum_{i=1}^{k} p_i \alpha_i = \alpha$. Therefore, we consider $f_{\text{mix}}(\alpha) = \min\{\sum_{i=1}^{k} p_i \beta_i | \beta_i \geq f_i(\alpha_i), \alpha = \sum_{i=1}^{k} p_i \alpha_i, \alpha_i \in [0,1]\}$, which is a convex optimization problem since the objective function is linear, and the constraints $f_i$ are all convex (Proposition 22). Therefore, by Karush–Kuhn–Tucker theorem (Boyd and Vandenberghe, 2004), let the Lagrangian function be $L(\{\alpha_i, \beta_i, \nu_i, \kappa_i\}_{i=1}^{k}, \lambda) = \sum_{i=1}^{k} p_i \beta_i + \sum_{i=1}^{k} (\nu_i f_i(\alpha_i) - \beta_i) + \nu_i(-\alpha_i) + \kappa_i(\alpha_i - 1)) + \lambda (\alpha - \sum_{i=1}^{k} p_i \alpha_i)$, and the minimum of $\sum_{i=1}^{k} p_i \beta_i$ is achieved at $\{\alpha_i, \beta_i\}_{i=1}^{k}$ if and only if the following conditions are satisfied

Stationarity: $p_i - \mu_i = 0$, $i = 1, 2, \ldots, k$;

$$0 \in \mu_i \partial f_i(\alpha_i) - \nu_i + \kappa_i - \lambda p_i, \ i = 1, 2, \ldots, k.$$ Primal feasibility: $\sum_{i=1}^{k} \alpha_i = \alpha$, $\beta_i \geq f_i(\alpha_i)$, $0 \leq \alpha_i \leq 1$, $i = 1, 2, \ldots, k$;

Dual feasibility: $\nu_i \geq 0$, $\nu_i \geq 0$, $\kappa_i \geq 0$, $i = 1, 2, \ldots, k$.

Complementary slackness: $\sum_{i=1}^{k} (\nu_i f_i(\alpha_i) - \beta_i) + \nu_i(-\alpha_i) + \kappa_i(\alpha_i - 1)) = 0$.

Therefore, we have $\mu_i = p_i \neq 0$ from the stationarity condition, and $f_i(\alpha_i) = \beta_i$ from the complementary slackness condition.

- If $0 < \alpha_i < 1$, we have $\nu_i = \kappa_i = 0$, therefore, $\lambda \in \partial f_i(\alpha_i)$;

- If $\alpha_i = 0$, we have $\kappa_i = 0$ and $\partial f_i(\alpha_i) \ni \frac{\alpha_i}{\mu_i} + \lambda \geq \lambda$, therefore, for any $\alpha \in [0,1]$, we have $f_i(\alpha) \geq (\alpha - \alpha_i)(\frac{\alpha_i}{\mu_i} + \lambda) + f_i(\alpha_i) \geq (\alpha - \alpha_i)\lambda + f_i(\alpha_i)$, i.e., $\lambda \in \partial f_i(\alpha_i)$;

- If $\alpha_i = 1$, we have $\nu_i = 0$ and $\partial f_i(\alpha_i) \ni -\frac{\alpha_i}{\mu_i} + \lambda \leq \lambda$; similarly, we have $\lambda \in \partial f_i(\alpha_i)$.

As $f_i$ is a tradeoff function, $\partial f_i(\alpha_i)$ is non-decreasing when $\alpha_i$ increases. We write $\partial f_i(\alpha_i) > \lambda$ if $a > \lambda \forall a \in \partial f_i(\alpha_i)$. For a given $\lambda$,
• if there exist two constants $0 < \alpha_{i,1} < \alpha_{i,2} < 1$ such that $\partial f_i(\alpha_{i,1}) \leq \lambda \leq \partial f_i(\alpha_{i,2})$, we define $\alpha_i^{\lambda,\text{lower}}$ and $\alpha_i^{\lambda,\text{upper}}$ such that $\alpha_i \in [\alpha_i^{\lambda,\text{lower}}, \alpha_i^{\lambda,\text{upper}}]$ if and only if $\lambda \in \partial f_i(\alpha_i)$;

• if $\partial f_i(\alpha_i) > \lambda$ for all $\alpha_i \in (0, 1)$, we define $\alpha_i^{\lambda,\text{lower}} = \alpha_i^{\lambda,\text{upper}} = 0$; intuitively, we do not want to have any of the type I error of $\alpha$ on the $f_i$ part since the corresponding type II error would be larger otherwise;

• if $\partial f_i(\alpha_i) < \lambda$ for all $\alpha_i \in (0, 1)$, we define $\alpha_i^{\lambda,\text{lower}} = \alpha_i^{\lambda,\text{upper}} = 1$; intuitively, we want to have as much of the type I error of $\alpha$ on the $f_i$ part as possible since the corresponding type II error would be larger otherwise.

Define $\alpha^{\lambda,\text{lower}} = \sum_{i=1}^k p_i \alpha_i^{\lambda,\text{lower}}$, and $\alpha^{\lambda,\text{upper}} = \sum_{i=1}^k p_i \alpha_i^{\lambda,\text{upper}}$. By definition, if $\lambda = -\infty$, we have $\alpha^{\lambda,\text{upper}} = \alpha^{\lambda,\text{lower}} = 0$, and if $\lambda = +\infty$, we have $\alpha^{\lambda,\text{upper}} = \alpha^{\lambda,\text{lower}} = 1$. For $\lambda \in (-\infty, +\infty)$, we have $[0, 1] \subseteq \cup_{\lambda \in (-\infty, +\infty)} [\alpha^{\lambda,\text{lower}}, \alpha^{\lambda,\text{upper}}]$. Therefore, for any $\alpha \in [0, 1]$, we can find $\lambda$ such that $\alpha \in [\alpha^{\lambda,\text{lower}}, \alpha^{\lambda,\text{upper}}]$, and we can determine $\alpha_i$ by the $\lambda$.

From the procedure above, we know $f_{\min} = \text{mix}(p, f)$, and $\mathcal{M}$ satisfies $f_{\min}$-DP. ■

We will use Lemma 24 in the proof of Theorem 9.

Lemma 24 (Equation (13) in (Dong et al., 2022)) For a symmetric tradeoff function $f$, define $f_p := pf + (1 - p)Id$ for $0 \leq p \leq 1$, where $Id(x) = 1 - x$. Let $x^*$ be the unique fixed point of $f$, that is $f(x^*) = x^*$, we have

$$C_p(f)(x) = \begin{cases} f_p(x), & x \in [0, x^*] \\ x^* + f_p(x^*) - x, & x \in [x^*, f_p(x^*)] \\ f_p^{-1}(x), & x \in [f_p(x^*), 1]. \end{cases}$$

Proof [Proof of Theorem 9] We are going to find a lower bound of $T_{\mathcal{M}(\text{boot}(D_1)), \mathcal{M}(\text{boot}(D_2))}$ uniformly for any neighboring datasets $D_1 = (x_1, x_2, \cdots, x_n)$ and $D_2 = (x_1', x_2', \cdots, x_n')$ (without loss of generality, we let $x_1$ be the different data point in $D_1$ and $D_2$). We use $\text{boot}^i$ to denote the conditional bootstrap subsampling where $x_1$ or $x_1'$ is drawn for exactly $i$ times from $D_1$ or $D_2$. Note that both $\text{boot}^i(D_1)$ and $\text{boot}^i(D_2)$ are random variables for any $i = 0, 1, 2, \cdots, n$. Furthermore, we define $\text{boot}^>$ to denote the conditional bootstrap subsampling where $x_1$ or $x_1'$ is drawn for at least once from $D_1$ or $D_2$.

From Theorem 8, we know $\text{mix}((\{ q_i, g_i \})_{i \in I}) = g$ for any $g$, $I$ and $q_i$. Therefore, we have $T_{\mathcal{M}(\text{boot}^0(D_1)), \mathcal{M}(\text{boot}^0(D_2))}(\alpha) = f_0$ where $f_0(\alpha) = 1 - \alpha$ since $\text{boot}^0(D_1) = \text{boot}^0(D_2)$, and we also have $T_{\mathcal{M}(\text{boot}^k(D_1)), \mathcal{M}(\text{boot}^k(D_2))} \geq f_k$ since $\text{boot}^k(D_1)$ and $\text{boot}^k(D_2)$ are neighboring datasets with respect to group size $k$. Now we consider $\mathcal{M} \circ \text{boot}^>$ as a mixture of $\mathcal{M} \circ \text{boot}^1$. Using Theorem 8 again, we have $T_{\mathcal{M}(\text{boot}^>(D_1)), \mathcal{M}(\text{boot}^>(D_2))} \geq f_>$ where $f_> := \text{mix}((\{ \frac{p_i}{1 - p_0}, f_i \})_{i = 1}^n)$ (here we use $\frac{p_i}{1 - p_0}$ instead of $p_i$ because $\sum_{i=1}^n \frac{p_i}{1 - p_0} = 1$).

In order to obtain a better lower bound of $T_{\mathcal{M}(\text{boot}(D_1)), \mathcal{M}(\text{boot}(D_2))}$, we find the mixture of $f_>$ and $f_0$ with considering the bootstrap resampling context because $\mathcal{M} \circ \text{boot}^0$ is a mixture of $\mathcal{M} \circ \text{boot}^>$ and $\mathcal{M} \circ \text{boot}^0$.

5. If there are multiple choices of $\{ \alpha_i \}_{i=1}^n$, all of them correspond to the same $\beta$.  

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Consider a rejection rule $\psi$. Let

$$
\begin{align*}
\alpha & := \mathbb{E}_{\mathcal{M}(\text{boot}(D_1))} \psi, \\
\alpha_0 & := \mathbb{E}_{\mathcal{M}(\text{boot}^0(D_1))} \psi = \mathbb{E}_{\mathcal{M}(\text{boot}^0(D_2))} \psi, \\
\beta & := \mathbb{E}_{\mathcal{M}(\text{boot}(D_2))} (1 - \psi), \\
\beta_0 & := \mathbb{E}_{\mathcal{M}(\text{boot}^0(D_2))} (1 - \psi) = 1 - \alpha_0.
\end{align*}
$$

We prove that when calculating the mixture of $f_>$ and $f_0$, there are two additional constraints, $\beta_0 \geq f_>(\alpha_>)$ and $\beta_0 \geq f_>(\alpha_0)$, as the hypothesis testing between $\mathcal{M}(\text{boot}^0(D_1))$ and $\mathcal{M}(\text{boot}^>(D_1))$ is similar to the one between $\mathcal{M}(\text{boot}^> (D_2))$ and $\mathcal{M}(\text{boot}^>(D_1))$.

We consider the hypothesis testing between $\mathcal{M}(\text{boot}^0(D_1))$ and $\mathcal{M}(\text{boot}^> (D_1))$. Similar to the idea in (Balle et al., 2018), we consider replacing each $x_1$ in $\mathcal{M}(\text{boot}^> (D_1))$ with a data point independently and uniformly drawn from $(x_2, x_3, \ldots, x_n)$. We denote the distribution of $\mathcal{M}(\text{boot}^> (D_1))$ as $\omega_1$, the distribution of $\mathcal{M}(\text{boot}^0(D_1))$ as $\omega_0$, the distribution of $\mathcal{M}(\text{boot}(D_1))$ as $\omega_{0\&1}$, and the replacement procedure as replace. Since Balle et al. (2018) did not provide proof for why this replacement procedure can transform $\omega_1$ to $\omega_0$, i.e., replace$(\omega_1) = \omega_0$, we prove it below.

For one element in $\omega_1$, let its histogram be $h^> = (h_1, h_2, \ldots, h_n)$ where $h_i$ is the number of occurrences of $x_i$ in this element, $h_1 \geq 1$, $h_2 \geq 0$ for $i = 2, 3, \ldots, n$, $\sum_{i=1}^n h_i = n$. For one element in $\omega_0$, we let its histogram be $h^0 = (h_1 = 0, h_2, \ldots, h_n)$. Then

$$
\begin{align*}
P_{\omega_{0\&1}}(H = h) &= \frac{1}{n^n} \binom{n}{h_1} h_2 \cdots h_n, \\
P_{\omega_0}(H = h^0) &= \frac{1}{(n-1)^n} \binom{n}{h_2} h_3 \cdots h_n, \\
P_{\omega_1}(H = h^>) &= \frac{1}{1 - \left(1 - \frac{1}{n}\right)^n} \frac{1}{n^n} \binom{n}{h_1} h_2 \cdots h_n = \frac{1}{n^n - (n-1)^n} \binom{n}{h_1} h_2 \cdots h_n.
\end{align*}
$$

For the replacement procedure, we replace the $h_1$ replicates of $x_1$ in $h^>$ with elements independently and uniformly drawn from $(x_2, x_3, \ldots, x_n)$, where the histogram of the replacement is $h' = (h'_2, h'_3, \ldots, h'_n)$.

Since for any element in $\omega_0$, the replacement does not change it, i.e., replace$(\omega_0) = \omega_0$, we can perform the replacement on $\omega_{0\&1}$ and show replace$(\omega_{0\&1}) = \omega_0$, then we also have

$$
\text{replace}(\omega_1) = \text{replace} \left( \frac{\omega_{0\&1} - p_0 \omega_0}{1 - p_0} \right) = \frac{\text{replace}(\omega_{0\&1}) - p_0 \text{replace}(\omega_0)}{1 - p_0} = \omega_0.
$$

We prove replace$(\omega_{0\&1}) = \omega_0$ below.

---

6. The replacement procedure is defined deliberately: if it is defined to be replacing all $x_1$ with the same data point uniformly randomly drawn from $(x_2, x_3, \ldots, x_n)$, e.g., $x_1$ are all replaced by $x_2$, one can verify that this procedure will not transform $\omega_1$ to $\omega_0$ even for $n = 3$. 

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\[ P_{\text{replace}(\omega_0 \& 1)}(H^0 = h^0) = \sum_{h'} P_{\text{replace}(h_1)}(H' = h') \cdot P_{\omega_0 \& 1}(H = (h_1, h^0 - h')) \]

\[ \begin{align*}
&= \sum_{h_1 = 0}^{n} \frac{1}{(n-1)!} \frac{1}{h_1! n^n} \sum_{h'_i \sum_i=2}^{n} (h'_2! h_2 - h'_2)! \cdots (h'_n! (h_n - h'_n))! \\
&= \frac{n!}{h_2! h_3! \cdots h_n!} \sum_{h'_i = 0}^{h_2} (n-1)! h'_2! h_2 - h'_2)! \cdots (n-1)! h'_n! (h_n - h'_n))! \\
&= \frac{n!}{h_2! h_3! \cdots h_n!} \left( \frac{h_2}{n} \left( \frac{n}{n-1} \right) \cdots \left( \frac{n}{n-1} \right) \right) = \frac{n!}{h_2! h_3! \cdots h_n! (n-1)^n} \\
\Rightarrow & \quad \text{replace}(\omega_0 \& 1) = \omega_0.
\end{align*} \]

Now we are ready to prove \( \beta_0 \geq f_\geq(\alpha_\geq) \) and \( \beta_\geq \geq f_\geq(\alpha_0) \). From the result \( \text{replace}(\omega_1) = \omega_0 \), if we consider the tradeoff function between the two distributions \( \mathcal{M}(\text{boot}^0(D_1)) \) and \( \mathcal{M}(\text{boot}^r(D_1)) \), we can break each of the two mixture distributions into parts following the replacement procedure so that there is a one-to-one mapping between the parts. When the number of occurrence of \( x_1 \) is \( h_1 \) in the outcome of \( \text{boot}^r(D_1) \), two parts in such a mapping pair have distance \( h_1 \) (between \( \text{boot}^h(D_1) \) and \( \text{replace}(\text{boot}^h(D_1)) \)), so the tradeoff function between \( \mathcal{M}(\text{boot}^h(D_1)) \) and \( \mathcal{M}(\text{replace}(\text{boot}^h(D_1))) \) is \( f_{h_1} \). We use Theorem 8 to obtain the mixture of those \( f_{h_1} \). Since \( f_{h_1} \) only depends on \( h_1 \), its corresponding probability in \( \text{boot}^r(D_1) \) is \( \frac{h_1}{n} \). Therefore, the tradeoff function between \( \mathcal{M}(\text{boot}^r(D_1)) \) and \( \mathcal{M}(\text{boot}^0(D_1)) \) is \( \text{mix}((\omega_i, f_i))_{i=1}^n = f_\geq \). Recall that

\[ \alpha_0 := \mathbb{E}_{\mathcal{M}(\text{boot}^0(D_1))} \psi = \mathbb{E}_{\mathcal{M}(\text{boot}^r(D_1))} \psi, \quad \beta_0 := \mathbb{E}_{\mathcal{M}(\text{boot}^0(D_1))}(1 - \psi) = 1 - \alpha_0, \]

\[ \alpha_\geq := \mathbb{E}_{\mathcal{M}(\text{boot}^r(D_1))} \psi, \quad \beta_\geq := \mathbb{E}_{\mathcal{M}(\text{boot}^r(D_1))}(1 - \psi), \]

we have \( \beta_0 \geq f_\geq(\alpha_\geq) \). Similarly, \( \beta_\geq \geq f_\geq(\alpha_0) \).

Now we have established the additional constraints for the mixture of \( f_0 \) and \( f_\geq \). We are ready to derive the final mixture tradeoff function. Notice that \( \alpha = p_0 \alpha_0 + (1 - p_0) \alpha_\geq \) and \( \alpha_0, \alpha_\geq \in [0, 1] \): For \( \alpha = 0 \) and \( \alpha = 1 \), we have \( \alpha_0 = \alpha_\geq = 0 \) and \( \alpha_0 = \alpha_\geq = 1 \) respectively.

Now we consider the constrained optimization problem for \( \alpha \in (0, 1) \) where we replace \( \alpha_\geq \) and \( \beta_0 \) with \( \frac{\alpha - \alpha_0}{1 - p_0} \) and \( 1 - \alpha_0 \) respectively: \( f_{\text{min}}(\alpha) = \min(p_0(1 - \alpha_0) + (1 - p_0) \beta_0 | \beta_0 \leq 1, \alpha_0 \geq 0, \beta_\geq \geq f_\geq(\alpha_0), \beta_\geq \geq f_\geq(\frac{\alpha - \alpha_0}{1 - p_0}), 1 - \alpha_0 \geq f_\geq(\frac{\alpha - \alpha_0}{1 - p_0}) \). We ignore the constraint \( \alpha_0 \leq 1 \) and \( \beta_\geq \geq 0 \) because they can be derived from \( 1 - \alpha_0 \geq f_\geq(\frac{\alpha - \alpha_0}{1 - p_0}) \) and \( \beta_\geq \geq f_\geq(\alpha_0) \) respectively. Since \( f_\geq : \mathbb{R} \mapsto \mathbb{R} \) is a convex function, we use the Karush–Kuhn–Tucker theorem to solve to the convex optimization problem: let the Lagrangian function be \( L(\alpha_0, \beta_\geq, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5) = p_0(1 - \alpha_0) + (1 - p_0) \beta_\geq + \mu_1(\beta_\geq - 1) + \mu_2(\alpha_0 - \alpha_\geq) + \mu_3(\beta_\geq - 1) + \mu_4(\alpha_\geq - 1) + \mu_5(1 - \alpha_0) \).
Complementary slackness: µ ≤ C, (µ Dual feasibility: if α of the objective function: Notice that 1 − which means that the new choice (α, β) is achieved by the previous analysis, and the second inequality holds due to f when β Stationarity: 0 ≤ p ∈ [0, 1], we verify that (α > 0, β ≤ 0). By Proposition 23, there is exactly one ¯ α that satisfies the conditions: p = 0, 1, and the objective function value p = 0, 1. Similarly, we show that we only need to consider β ≥ 1. If β = 1, since α = α = α, β = f(α) where p = (0, 1). Therefore, if α < α, we have α < α, and β ≥ f(α) ≥ f(α) where the first inequality holds from previous analysis, and the second inequality holds due to f β being decreasing. Therefore, p(1 - α) + (1 - p)β = p(1 - α) + (1 - p)f(α) + p(α - α0) + (1 - p)(β - f(α)) > p0(1 - α) + (1 - p0)f(α),

which means that the choice (α0 = α, β = f(α)) satisfies all the constraints and also achieves a value of the objective function lower than the choice (α0, β). This contradicts the optimality of (α0, β). Therefore, α0 ≥ α > 0.

With α0 ≥ α > 0, β < 1, from the complementary slackness, we know µ1 = µ2 = 0.

By Proposition 23, there is exactly one α satisfying α = f(α) and −1 ∈ ∂f(α). We denote C ∈ ∂f(α), and C′ ∈ ∂f(α)|α = α = (1 - p0)(1 - p0/C).

- If 0 < α ≤ ¯ α, we verify that (α0 = α1, β = f(α), µ3 = (1 - p0)(p0 + p0/C), µ4 = (1 - p0)(1 - p0 - p0/C)), µ5 = 0) satisfies the KKT conditions which means (α0 = α, β = f(α)) is a minimizer. First, we see that α = α = α.
  - The primal feasibility is satisfied because f(α) ≤ 1 − α.
  - The complementary slackness is satisfied as µ1 = µ2 = µ5 = 0 and f(α) = β.
  - Since 0 < α ≤ ¯ α, we know that C ≤ −1. The dual feasibility conditions, µ3 ≥ 0 and µ4 > 0, hold because p0 ∈ (0, 1).
  - The stationarity conditions also hold when we plug in the value C for ∂f(α).
If \( \bar{\alpha} < \alpha \leq (1 - p_0) \bar{\alpha} + p_0 (1 - \bar{\alpha}) \), we verify that

\[
\alpha_0 = \frac{\alpha - (1 - p_0) \bar{\alpha}}{p_0}, \beta_0 = \bar{\alpha}, \mu_3 = 0, \mu_4 = 1 - p_0, \mu_5 = 0
\]

satisfies the KKT conditions. First, we see that

- The primal feasibility is satisfied: since \( \alpha > \bar{\alpha} \), we have \( \alpha_0 > \bar{\alpha} \) and \( f_>(\alpha_0) \leq f_>(\bar{\alpha}) = \beta_0 \). Since \( \alpha \leq (1 - p_0) \bar{\alpha} + p_0 (1 - \bar{\alpha}) \), we have \( \alpha_0 \leq 1 - \bar{\alpha} \) and \( 1 - f_>(\bar{\alpha}) = 1 - \bar{\alpha} \geq \alpha_0 \).
- The complementary slackness is satisfied as \( \mu_1 = \mu_2 = \mu_3 = \mu_5 = 0, f_>(\bar{\alpha}) = \beta_0 \).
- The dual feasibility condition, \( \mu_4 > 0 \), holds because \( p_0 \in (0, 1) \).
- The stationarity conditions also hold: We plug in the value \(-1\) for \( \partial f_>(\bar{\alpha}) \), then the right-hand sides of both conditions are 0.

If \( (1 - p_0) \bar{\alpha} + p_0 (1 - \bar{\alpha}) < \alpha < p_0 + (1 - p_0) f_>(0) \), we let \( \alpha^*_0 \) be the solution of

\[
p_0 \alpha_0 + (1 - p_0) f_>(1 - \alpha_0) = \alpha.
\]

Since \( f_> \) is continuous and not increasing, we know \( g(\alpha_0) = p_0 \alpha_0 + (1 - p_0) f_>(1 - \alpha_0) \) is continuous and strictly increasing with respect to \( \alpha_0 \). We know \( g(1) = p_0 + (1 - p_0) f_>(0) \) and \( g(1 - \bar{\alpha}) = (1 - p_0) \bar{\alpha} + p_0 (1 - \bar{\alpha}) \). Therefore, \( p_0 \alpha_0 + (1 - p_0) f_>(1 - \alpha_0) = \alpha \) has only one solution which is \( \alpha^*_0 \) with \( 1 > \alpha^*_0 > 1 - \bar{\alpha} \).

Now we verify that \( \alpha_0 = \alpha^*_0, \beta_0 = 1 - \alpha^*_0, \mu_3 = 0, \mu_4 = 1 - p_0, \mu_5 = \frac{p_0 (1 + C')}{1 - p_0} \) satisfies the KKT conditions. First, we see that

\[
\frac{\alpha - p_0 \alpha_0}{1 - p_0} = f_>(1 - \alpha_0) < f_>(\bar{\alpha}) = \bar{\alpha} \text{ and }
\frac{\alpha - p_0 \alpha_0}{1 - p_0} = f_>(1 - \alpha_0) < f_>(0).
\]

Therefore, \( 0 > C' \geq -1 \).

- The primal feasibility is satisfied: since \( \alpha_0 \geq \alpha \geq \bar{\alpha} \), we have \( 1 - \alpha_0 < \bar{\alpha} \).

Therefore, by Proposition 23, since \( f_> \) is convex and symmetric, there exists \( C \leq -1 \) such that \( C \in \partial f_>(1 - \alpha_0) \) and

\[
\beta_0 = 1 - \alpha_0 = f_>(1 - \alpha_0) = f_>(\frac{\alpha - p_0 \alpha_0}{1 - p_0}).
\]

We also have \( \beta_0 = 1 - \alpha_0 = f_>(\frac{\alpha - p_0 \alpha_0}{1 - p_0}) \).

- The dual feasibility conditions hold as \( p_0 \in (0, 1), C' \in [-1, 0] \).

- The stationarity conditions also hold: We plug in the value \( C' \) for \( \partial f_>(\alpha) \bigg|_{\alpha = \frac{\alpha - p_0 \alpha_0}{1 - p_0}} \), then the right hand side is 0.

If \( \alpha \geq p_0 + (1 - p_0) f_>(0) \), we let \( \alpha_0 = 1, \beta_0 = 0, \mu_5 = p_0, \mu_3 = (1 - p_0)/2, \mu_4 = (1 - p_0)/2 \). Notice that \( \frac{\alpha - p_0 \alpha_0}{1 - p_0} \geq f_>(0) \). Therefore, \( f_>(\frac{\alpha - p_0 \alpha_0}{1 - p_0}) = 0 \).

- The primal feasibility is satisfied.

- The complementary slackness is also satisfied because \( f_>(\alpha_0) - \beta_0 = 0, f_>(\frac{\alpha - p_0 \alpha_0}{1 - p_0}) - \beta_0 = 0, f_>(\frac{\alpha - p_0 \alpha_0}{1 - p_0}) - 1 + \alpha_0 = 0 \).

- The dual feasibility conditions hold because \( p_0 \in (0, 1) \).

- The stationarity conditions also hold: We use \( 0 \in \partial f_>(\alpha_0) \) and \( 0 \in \partial f_>(\alpha) \bigg|_{\alpha = \frac{\alpha - p_0 \alpha_0}{1 - p_0}} \) to plug in the first condition, and it holds because \(-p_0 + \mu_5 = 0\). The second condition holds because \((1 - p_0) - \mu_3 - \mu_4 = 0\).
Therefore, we have the tradeoff function \( f_{\min} \) from \( f > \) as follows:

\[
\begin{align*}
f_{\min}(\alpha) =
\begin{cases}
  p_0 + (1 - p_0)f_>(0), & \text{if } \alpha = 0 \\
  p_0(1 - \alpha) + (1 - p_0)f_>(\alpha), & \text{if } 0 < \alpha \leq \bar{\alpha} \\
  p_0 - \alpha + 2(1 - p_0)\bar{\alpha}, & \text{if } \bar{\alpha} < \alpha \leq (1 - p_0)\bar{\alpha} + p_0(1 - \bar{\alpha}) \\
  1 - \alpha^*_0, & \text{if } (1 - p_0)\bar{\alpha} + p_0(1 - \bar{\alpha}) < \alpha < p_0 + (1 - p_0)f_>(0) \\
  0, & \text{if } \alpha \geq p_0 + (1 - p_0)f_>(0)
\end{cases}
\]

where \( \bar{\alpha} \) satisfies \( f_>(\bar{\alpha}) = \bar{\alpha} \) and \( \alpha^*_0 \) is the only solution of \( p_0\alpha_0 + (1 - p_0)f_>(1 - \alpha_0) = \alpha \) with respect to \( \alpha_0 \).

Now we verify the \( f_{\min} \) above is the same as \( C_{1-p_0}(f_>) \). In Lemma 24, we replace \( p \) and \( f \) with \( 1 - p_0 \) and \( f_\approx \) respectively:

- For \( \alpha \in [0, \bar{\alpha}] \), we have \( f_{\min}(\alpha) = (f_\approx)_{1-p_0}(\alpha) = C_{1-p_0}(f_\approx)(\alpha) \).
- For \( \alpha \in [\bar{\alpha}, (f_\approx)_{1-p_0}(\alpha)] \), we have \( f_{\min}(\alpha) = p_0 - \alpha + 2(1 - p_0)\bar{\alpha} = \bar{\alpha} + ((1 - p_0)\bar{\alpha} + p_0(1 - \bar{\alpha})) - \alpha = \bar{\alpha} + (f_\approx)_{1-p_0}(\bar{\alpha}) - \alpha = C_{1-p_0}(f_\approx)(\alpha) \).
- For \( \alpha \in [(f_\approx)_{1-p_0}(\bar{\alpha}), p_0 + (1 - p_0)f_>(0)] \), since \( \alpha^*_0 \) satisfies \( p_0\alpha_0 + (1 - p_0)f_>(1 - \alpha_0) = \alpha \), we let \( t = 1 - \alpha^*_0 \), and we have \( (f_\approx)_{1-p_0}(t) = (1 - p_0)f_>(t) + p_0(1 - t) = \alpha \). Since \( f_\approx \) is non-increasing, we have that \( (f_\approx)_{1-p_0}(t) \) is strictly decreasing because \( p_0 \in (0, 1) \). Therefore, \( (f_\approx)_{1-p_0}^{-1}(\alpha) = 1 - \alpha^*_0 = f_{\min}(\alpha) \).
- For \( \alpha \in [p_0 + (1 - p_0)f_>(0), 1] \), we know that \( \alpha \geq (f_\approx)_{1-p_0}(0) \). Since \( (f_\approx)_{1-p_0} \) is strictly decreasing, we have \( (f_\approx)_{1-p_0}(t) \leq \alpha \forall t \in [0, 1] \). Therefore, \( (f_\approx)_{1-p_0}^{-1}(\alpha) = \inf\{t \in [0, 1] : (f_\approx)_{1-p_0}(t) \leq \alpha\} = \inf\{t \in [0, 1] \} = 0 \). We have \( (f_\approx)_{1-p_0}^{-1}(\alpha) = 0 = f_{\min}(\alpha) \).

\[\blacksquare\]

B.2 Derive \((\varepsilon, \delta)\)-DP from \( f\)-DP for DP bootstrap

In this section, we transform our Theorem 9 to (Balle et al., 2018, Theorem 10).

**Proposition 25** The \((\varepsilon, \delta)\)-DP result in Theorem 18 can be derived from Theorem 9.

**Proof** We use the primal-dual transformation (see Proposition 17) to obtain the \((\varepsilon', \delta')\)-DP result by our \( f_{\text{Mo boot}}\)-DP result: \( \delta'(\varepsilon') = 1 + f_{\text{Mo boot}}(-e^{\varepsilon'}) \).

Since \( f \) is convex function when \( f \) is a tradeoff function, we have \( \alpha y - f(\alpha) \leq \alpha_0 y - f(\alpha_0) \) for any \( \alpha \) and \( (\alpha_0, y) \) satisfying \( y \in \partial f(\alpha_0) \). Therefore, we have \( f^*(y) = \sup_{\alpha} \alpha y - f(\alpha) = \alpha_0 y - f(\alpha_0) \) where \( y \in \partial f(\alpha_0) \), and \( 1 - \delta(\varepsilon) = -f^*(-e^{\varepsilon'}) = e^{\varepsilon'}\alpha + f(\alpha_\varepsilon) \) where \( -e^{\varepsilon'} \in \partial f(\alpha_\varepsilon) \).

We let \( f_\approx = \text{mix}\{((1/p_{\text{boot}}, f_i), \alpha_i)\}_{i=1}^n \} \), and our bootstrap privacy guarantee is \( f_{\text{Mo boot}} = C_{1-p_0}(f_\approx) \). We let \( \bar{\alpha} \) be the solution of \( \alpha = f_\approx(\alpha) \). By Proposition 23 and Lemma 24, we have \(-1 \in \partial f_\approx(\bar{\alpha}) \) and \(-1 \in \partial f_{\text{Mo boot}}(\bar{\alpha}) \). Since \(-e^{\varepsilon'} \leq -1 \) for any \( \varepsilon' \geq 0 \), we can find
Let \( \alpha_{e'} \leq \bar{\alpha} \) that \( -e^{e'} \in \partial f_{\text{Mboot}}(\alpha_{e'}) \). By Lemma 24, we also have \(-e^{e'} \in \partial((f_{\gtrsim})_{1-p_0})(\alpha_{e'}) \) and \( f_{\text{Mboot}}(\alpha_{e'}) = C_{1-p_0}(f_{\gtrsim})(\alpha_{e'}) = (f_{\gtrsim})_{1-p_0}(\alpha_{e'}) = (1-p_0)f_{\gtrsim}(\alpha_{e'}) + p_0(1-\alpha_{e'}) \), and

\[
\delta'(e') = 1 + f_{\text{Mboot}}^*(\alpha_{e'}) = \left( -e^{e'} \alpha_{e'} - f_{\text{Mboot}}(\alpha_{e'}) \right) = -e^{e'} \alpha_{e'} - \left( (f_{\gtrsim})_{1-p_0}(\alpha_{e'}) = 1 + ((f_{\gtrsim})_{1-p_0})^*(e^-e') \right)
\]

From the Supplement to (Dong et al., 2022) (Page 42), we know the following two equations, \( e' = \log(1-p + pe^e) \), \( \delta' = p(1 + f^*(-e')) \) can be re-parameterized into \( \delta' = 1+f_p^*(-e') \) where \( f_p = pf + (1-p)\Id \). We can re-parameterize \( \delta'(e') = 1+((f_{\gtrsim})_{1-p_0})^*(-e'^e') \) and \( e' = \log(p_0 + (1-p_0)e^e) \) into \( \delta'(e') = (1-p_0)(1 + f_p^*(-e')) \). Since \( e' = \log(p_0 + (1-p_0)e^e) \) is also the relationship between \( e' \) and \( \varepsilon \) in Theorem 18, in order to prove Theorem 18, we only need to show that \( (1-p_0)(1 + f_p^*(-e')) = \sum_{i=1}^np_i\delta_M,i(\varepsilon) \).

As we have \( f_{\gtrsim} = \text{mix}((\frac{p_i}{1-p_0}, f_i)_1^n) \), from the construction of the mixture tradeoff function, we let \( C = -e^e \) and find \( \{\alpha_i\}_{i=1}^n \) such that \( C \in \partial f_i(\alpha_i) \), then we know that \( \alpha = \sum_{i=1}^n\frac{p_i}{1-p_0}\alpha_i \), we also have \( C \in \partial f_{\gtrsim}(\alpha) \). This is because for any \( \alpha' = \sum_{i=1}^n\frac{p_i}{1-p_0}\alpha'_i \) and \( f_{\gtrsim}(\alpha') = \sum_{i=1}^n\frac{p_i}{1-p_0}f_i(\alpha'_i) \), we have \( f_{\gtrsim}(\alpha') \geq f_{\gtrsim}(\alpha) + C(\alpha' - \alpha) \) by the fact that \( C \in \partial f_i(\alpha) \). Therefore, we can find \( \alpha = \sum_{i=1}^n\frac{p_i}{1-p_0}\alpha_i \) such that \( -e^e \in \partial f_{\gtrsim}(\alpha_i) \) and \( -e^e \in \partial f_i(\alpha,e,i) \) for \( i = 1, 2, \ldots, n \). Then we can prove \((1-p_0)(1 + f_p^*(-e')) = \sum_{i=1}^np_i\delta_M,i(\varepsilon) \) using the primal-dual transformation of \( \delta_M,i(\varepsilon) \) and the fact that \( (1-p_0)(-e^e \alpha - f_{\gtrsim}(\alpha)) = \sum_{i=1}^np_i\left(-e^e \alpha,e,i - f_i(\alpha,e,i)\right) \) due to \( \alpha = \sum_{i=1}^n\frac{p_i}{1-p_0}\alpha_i \) and \( f_{\gtrsim}(\alpha) = \sum_{i=1}^n\frac{p_i}{1-p_0}f_i(\alpha,e,i) \).

\[ \square \]

### B.3 Proofs for Section 3.2

We first restate the results in (Dong et al., 2022) for comparing the results of \( f_{\gtrsim} \) and \( C_{1-p_0}(f_{\gtrsim}) \) which are used in our proof.

**Theorem 26 (Berry-Esseen CLT for the composition of f-DP: Dong et al., 2022)**

Define \( k_3(f) := f_0 \log|f'(x)| dx \), \( \kappa_2(f) := f_0 \log^2|f'(x)| dx \), \( \kappa_3(f) := f_0 \log|f'(x)|^3 dx \), \( \kappa_3(f) := f_0 \log|f'(x)|^{1+\kappa_2(f)} dx \), Let \( f_1, \ldots, f_n \) be symmetric tradeoff functions such that \( \kappa_3(f_i) < \infty \) for all \( 1 \leq i \leq n \). Denote \( \kappa_1 := (\kappa_1(f_1), \ldots, \kappa_1(f_n)) \), \( \kappa_2 := (\kappa_2(f_1), \ldots, \kappa_2(f_n)) \), \( \kappa_3 := (\kappa_3(f_1), \ldots, \kappa_3(f_n)) \), \( \mu := \frac{2\|\kappa_1\|_1}{\sqrt{\kappa_2\|1-\kappa_1\|_2}} \), \( \gamma := \frac{\sqrt{0.56\|\kappa_3\|_1}}{\sqrt{\kappa_2\|1-\kappa_1\|_2}} \), and assume \( \gamma < \frac{1}{2} \).

Then, for all \( \alpha \in [\gamma, 1-\gamma] \), we have

\[
G_\mu(\alpha + \gamma) - \gamma \leq f_1 \otimes f_2 \otimes \cdots \otimes f_n(\alpha) \leq G_\mu(\alpha - \gamma) + \gamma.
\]

Let \( \{f_{ni} : 1 \leq i \leq n\}_{n=1}^\infty \) be a triangular array of symmetric tradeoff functions and assume for some constants \( K \geq 0 \) and \( s > 0 \) that \( \lim_{n \to \infty} \sum_{i=1}^n \kappa_1(f_{ni}) = K \), \( \lim_{n \to \infty} \max_{1 \leq i \leq n} \kappa_2(f_{ni}) = 0 \), \( \lim_{n \to \infty} \sum_{i=1}^n \kappa_3(f_{ni}) = s^2 \), \( \lim_{n \to \infty} \sum_{i=1}^n \kappa_3(f_{ni}) = 0 \). Then, uniformly for all \( \alpha \in [0, 1] \), we have

\[
\lim_{n \to \infty} f_{n1} \otimes f_{n2} \otimes \cdots \otimes f_{nn}(\alpha) = G_{2K/s}(\alpha).
\]
Lemma 27 (Lemma F.2 and F.3 in the Supplement to Dong et al., 2022) Suppose \( f \) is a symmetric tradeoff function with \( f(0) = 1 \) and \( x^* \) is its unique fixed point. Then

\[
\begin{align*}
\kappa_2(f) &= \int_0^{x^*} (|f'(x)| + 1) (\log |f'(x)|)^2 \, dx \\
\kappa_3(f) &= \int_0^{x^*} \left( \log |f'(x)| + \kappa(f) \right)^3 + |f'(x)| \cdot \log |f'(x)| - \kappa(f) |^3 \, dx \\
\kappa_3(f) &= \int_0^{x^*} (|f'(x)| + 1) (\log |f'(x)|)^3 \, dx.
\end{align*}
\]

Let \( g(x) = -f'(x) - 1 = |f'(x)| - 1 \). Then

\[
\begin{align*}
\kappa(C_p(f)) &= p \int_0^{x^*} g(x) \log (1 + pg(x)) \, dx \\
\kappa_2(C_p(f)) &= \int_0^{x^*} (2 + pg(x)) [\log (1 + pg(x))]^2 \, dx \\
\kappa_3(C_p(f)) &= \int_0^{x^*} (2 + pg(x)) [\log (1 + pg(x))]^3 \, dx.
\end{align*}
\]

For DP bootstrap with Gaussian mechanism, we can apply our Theorem 9 as follows.

Corollary 28 Let \( M : \mathcal{X}^n \rightarrow \mathcal{O} \) satisfy \( \mu \cdot \text{GDP} \). Then \( M \circ \text{boot} \) satisfies \( f_{\text{boot}} \cdot \text{DP} \) where \( f_{\text{boot}} = C_{1-p_0}(f_{\text{p}}) \), \( f_{\text{p}} = \text{mix}(\{ \frac{p_i}{1-p_0}, f_{i_i} \}_{i=1}^n) \), \( p_i = \binom{n}{i} (1/n)^i (1-1/n)^{n-i} \), \( f_i = G_{\mu} \).

Now we provide the exact representation of \( f_{\text{p}} \) in Corollary 28.

Lemma 29 Let \( f_{\text{p}} = \text{mix}(\{ \frac{p_i}{1-p_0}, f_i \}_{i=1}^n) \) where \( p_i = \binom{n}{i} (1/n)^i (1-1/n)^{n-i} \), \( f_i = G_{\mu} \). Then \( f_{\text{p}} \) is the tradeoff function between \( \sum_{i=1}^n \frac{p_i}{1-p_0} N \left( -\frac{i^2 \mu^2}{2}, i^2 \mu^2 \right) \) and \( \sum_{i=1}^n \frac{p_i}{1-p_0} N \left( \frac{1}{i^2 \mu^2}, i^2 \mu^2 \right) \).

Proof [Proof of Lemma 29] First, we know that \( f_i(\alpha) = G_{\mu} \) where \( \Phi \) is the cumulative distribution function of the standard normal distribution. We first obtain the subdifferential of \( f_i \): \( \frac{df_i(\alpha)}{\alpha_i} = -\exp(-\alpha_i \mu^2/2) + i\mu \Phi^{-1}(1-\alpha_i) \). We let the type I error and type II error in \( f_i \) be \( \alpha_i \) and \( \beta_i \), and we have \( f_i(\alpha_i) = \beta_i \). We let \( \frac{df_i(\alpha)}{\alpha_i} = C \). Then we have \( \alpha_i = 1 - \Phi(\log(-C)/(i\mu) + i\mu/2) \), \( \beta_i = \Phi(\log(-C)/(i\mu) - i\mu/2) \). This setting of \( (\alpha_i, \beta_i) \) can also be achieved by using the rejection rule \( \phi(x) = I_{x \geq \log(-C)} \) for the hypothesis testing between \( H_0 : x \sim N \left( -\frac{\mu^2}{2}, \mu^2 \right) \) and \( H_1 : x \sim N \left( \frac{\mu^2}{2}, \mu^2 \right) \).

For \( f_{\text{p}} = \text{mix}(\{ \frac{p_i}{1-p_0}, f_i \}_{i=1}^n) \), we let \( f_{\text{p}}(\alpha) = \beta \) where \( \alpha = \sum_{i=1}^n \frac{p_i}{1-p_0} \alpha_i, \beta = \sum_{i=1}^n \frac{p_i}{1-p_0} \beta_i, \beta_i = f_i(\alpha_i), \) and \( \frac{df_{\text{p}}(\alpha)}{\alpha_i} = C \). Then \( \alpha \) and \( \beta \) as type I error and type II error can be achieved by using the rejection rule \( \phi(x) = I_{x \geq \log(-C)} \) for the hypothesis testing between \( H_0 : x \sim \sum_{i=1}^n \frac{p_i}{1-p_0} N \left( -\frac{\mu^2}{2}, \mu^2 \right) \) and \( H_1 : x \sim \sum_{i=1}^n \frac{p_i}{1-p_0} N \left( \frac{\mu^2}{2}, 2 \mu^2 \right) \).
We let $h_1(x) = \sum_{i=1}^{n} \frac{p_i}{1-p_0} \phi\left(\frac{x - i\mu_B}{\nu_B} \right)$ and $h_2(x) = \sum_{i=1}^{n} \frac{p_i}{1-p_0} \phi\left(\frac{x - i\mu_B}{\nu_B} \right)$ be the density functions of $\sum_{i=1}^{n} \frac{p_i}{1-p_0} N\left(-\frac{i^2 \mu^2}{2}, i^2 \mu^2 \right)$ and $\sum_{i=1}^{n} \frac{p_i}{1-p_0} N\left(-\frac{\nu^2 \mu^2}{2}, \nu^2 \mu^2 \right)$ respectively. We have $\log \left( \frac{h_1(x)}{h_2(x)} \right) = \log(e^{x}) = x$. Therefore, by the Neyman-Pearson Lemma, the most powerful rejection rule for the test between $H_0$ and $H_1$ is $\phi(x) = I(x \geq \lambda)$ where $\lambda$ is a constant. This aligns with our previous rejection rule $\phi(x) = I(x \geq \log(-C))$. Therefore, $f_>$ is the tradeoff function between $\sum_{i=1}^{n} \frac{p_i}{1-p_0} N\left(-\frac{\nu^2 \mu^2}{2}, \nu^2 \mu^2 \right)$ and $\sum_{i=1}^{n} \frac{p_i}{1-p_0} N\left(-\frac{i^2 \mu^2}{2}, i^2 \mu^2 \right)$.

**Proof** [Proof of Theorem 12] Let $n$ be the sample size for the bootstrap resampling. From the result of Corollary 28 and Lemma 29, the tradeoff function is $f_{Bi, boot} = C_{1-p_0}(f_>)$ where $p_0 = (1 - 1/n)^n$ and $f_>$ is the tradeoff function between two Gaussian mixtures, $\sum_{i=1}^{n} \frac{p_i}{1-p_0} N\left(\mu = \frac{i^2 \mu^2}{2}, i^2 \mu^2 \right)$ vs $\sum_{i=1}^{n} \frac{p_i}{1-p_0} N\left(\mu = \frac{\nu^2 \mu^2}{2}, \nu^2 \mu^2 \right)$ where $p_i = (\frac{n}{i}) (1/n)^i (1 - 1/n)^{n-1}$. We let $h_1(x) = \sum_{i=1}^{n} \frac{p_i}{1-p_0} \phi\left(\frac{x - i\mu_B}{\nu_B} \right)$ and $h_2(x) = \sum_{i=1}^{n} \frac{p_i}{1-p_0} \phi\left(\frac{x - i\mu_B}{\nu_B} \right)$. From the proof of Lemma 29, we can parameterize the tradeoff function $f_{\alpha}(\alpha C) = \beta_C$ using $C \in (-\infty, \infty)$ with $\alpha_C = f_C^{\infty} h_1(x) \, dx$ and $\beta_C = f_\infty^{C} h_2(x) \, dx$. We have $\frac{d\alpha_C}{dC} = -h_1(C)$, $\frac{d\beta_C}{dC} = h_2(C)$, $h_1(0) = e^{-C}$, $h_1(0)/h_2(0) = 1$, $\alpha_\infty = \beta_\infty = 1$, $\alpha_0 = \beta_0 = 0$, and $f'(\alpha C) = \frac{d\beta_C}{dC} = -e^{-C}$. We can transform the result in Lemma 27 to

\[
\begin{align*}
\text{kl}(f_>) &= \int_{\alpha_\infty}^{\alpha_0} (e^C - 1) C \, d\alpha_C = \int_{0}^{\infty} (e^C - 1) C h_1(C) \, dC, \\
\kappa_2(f_>) &= \int_{\alpha_\infty}^{\alpha_0} (e^C - 1) C \, d\alpha_C = \int_{0}^{\infty} (e^C - 1) C^2 h_1(C) \, dC, \\
\kappa_3(f_>) &= \int_{\alpha_\infty}^{\alpha_0} (e^C - 1) C \, d\alpha_C = \int_{0}^{\infty} (e^C - 1) C^3 h_1(C) \, dC.
\end{align*}
\]

Now we consider $\mu_B \to 0$. Since $h_1(x) = \sum_{i=1}^{n} \frac{p_i}{1-p_0} \phi\left(\frac{x - i\mu_B}{\nu_B} \right)$,

\[
\begin{align*}
\text{kl}(f_>) &= \sum_{i=1}^{n} \frac{p_i}{1-p_0} \int_{0}^{\infty} (e^x - 1) \frac{x}{\nu_B} \phi\left(\frac{x - i\mu_B}{\nu_B} \right) \, dx \\
&= \sum_{i=1}^{n} \frac{p_i}{1-p_0} \int_{0}^{\infty} (e^x - 1) \frac{x}{\nu_B} \sqrt{2\pi} e^{-\left(\frac{x - i\mu_B}{\nu_B}\right)^2/2} \, dx \\
&= \sum_{i=1}^{n} \frac{p_i}{1-p_0} \int_{0}^{\infty} (e^{i\mu_B y} - 1)(i\mu_B y) \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{y + i\mu_B}{\nu_B}\right)^2/2} \, dy \quad \text{(let } y = \frac{x}{i\mu_B}).
\end{align*}
\]
Similarly, we have
\[
\text{kl}(C_{1-p_0}(f_x)) = \sum_{i=1}^{n} p_i \int_{0}^{\infty} (e^{i\mu_B y} - 1) \log (1 + (1 - p_0)(e^{i\mu_B y} - 1)) \frac{1}{\sqrt{2\pi}} e^{-\frac{(y + \frac{i\mu_B}{2})^2}{2}} dy.
\]
Now we have \( \lim_{\mu_B \to 0} \frac{\text{kl}(C_{1-p_0}(f_x))}{\text{kl}(f_x)} = (1 - p_0)^2 \) from the two following facts
\[
\lim_{\mu_B \to 0} \int_{0}^{\infty} e^{i\mu_B y} \log (1 + (1 - p_0)(e^{i\mu_B y} - 1)) \frac{1}{\sqrt{2\pi}} e^{-\frac{(y + \frac{i\mu_B}{2})^2}{2}} dy = 1 - p_0,
\]
\[
\lim_{\mu_B \to 0} \int_{0}^{\infty} (i\mu_B y)^2 e^{-\frac{(y + \frac{i\mu_B}{2})^2}{2}} dy = 1 - p_0
\]
which are obtained using L'Hospital's rule and the Leibniz integral rule (i.e., the interchange of the integral and partial differential operators).

Similarly, \( \lim_{\mu_B \to 0} \frac{\kappa_2(C_{1-p_0}(f_x))}{\kappa_2(f_x)} = (1 - p_0)^2 \) and \( \lim_{\mu_B \to 0} \frac{\kappa_3(C_{1-p_0}(f_x))}{\kappa_3(f_x)} = (1 - p_0)^3 \).

We re-parameterize the definition of \( \text{kl}(f_x) \) in Theorem 26 with \( C \) and calculate it using \( h_1(x) = \sum_{i=1}^{n} p_i \frac{1}{1 - p_0} \frac{1}{i\mu_B} \phi(\frac{x}{i\mu_B} + \frac{i\mu_B}{2}) \):
\[
\text{kl}(f_x) = -\int_{-\infty}^{\alpha_0} C \text{d}C = \int_{-\infty}^{\infty} C \text{d}C = \sum_{i=1}^{n} p_i \int_{-\infty}^{\infty} \frac{x}{i\mu_B} \phi(\frac{x}{i\mu_B} + \frac{i\mu_B}{2}) dx
\]
\[
= \sum_{i=1}^{n} p_i \int_{-\infty}^{\infty} (-i\mu_B y) \phi(y + \frac{i\mu_B}{2}) dy = \sum_{i=1}^{n} p_i \frac{i^2 \mu_B^2}{2} = \frac{(2 - 1/n)\mu_B^2}{2(1 - (1 - 1/n)^n)}.
\]
Similarly, for any \( n = 1, 2, \ldots, \) and \( \mu_B \to 0 \), we have
\[
\kappa_2(f_x) = \sum_{i=1}^{n} p_i \int_{-\infty}^{\infty} \frac{x^2}{i\mu_B} \phi(\frac{x}{i\mu_B} + \frac{i\mu_B}{2}) dx = \sum_{i=1}^{n} p_i \frac{1}{1 - p_0} \left( \frac{i^4 \mu_B^4}{4} + i^2 \mu_B^2 \right)
\]
\[
= \frac{(2 - 1/n)\mu_B^2}{1 - (1 - 1/n)^n} + \Theta(\mu_B^4),
\]
\[
\kappa_3(f_x) = \sum_{i=1}^{n} p_i \int_{-\infty}^{\infty} |x|^3 \phi(\frac{x}{i\mu_B} + \frac{i\mu_B}{2}) dx = \sum_{i=1}^{n} p_i \frac{1}{1 - p_0} \left( \frac{i^6 \mu_B^6}{4} + i^4 \mu_B^4 \right) \int_{-\infty}^{\infty} \frac{1}{2} \left( \phi(z) \right) dz
\]
\[
\leq \sum_{i=1}^{n} p_i \frac{1}{1 - p_0} (i\mu_B)^3 \int_{-\infty}^{\infty} \left( \frac{1}{2} |z|^3 + 3 \frac{i\mu_B}{2} |z|^2 + 3 \left( \frac{i\mu_B}{2} \right)^3 \right) \phi(z) dz \in \Theta(\mu_B^3).
\]
By Theorem 26, we have \( \lim_{B \to \infty} f_{\text{boot}}^{B} = G_{2K/s} \) if
\[
\lim_{B \to \infty} \sum_{i=1}^{B} \text{kl}(f_{B_{i,\text{boot}}}) = K, \quad \lim_{B \to \infty} \max_{1 \leq i \leq B} \text{kl}(f_{B_{i,\text{boot}}}) = 0,
\]
\[
\lim_{B \to \infty} \sum_{i=1}^{B} \kappa_2(f_{B_{i,\text{boot}}}) = s^2, \quad \lim_{B \to \infty} \sum_{i=1}^{B} \kappa_3(f_{B_{i,\text{boot}}}) = 0.
\]
Since we assume that \( \sqrt{B} \mu_B \to \mu' \), we have \( \mu_B \in \Theta(B^{-\frac{1}{2}}) \) and

\[
\lim_{B \to \infty} \frac{1}{B} \sum_{i=1}^{B} \text{kl}(f_{B;i}, \text{boot}) = \lim_{B \to \infty} B(1-p_0)^2 \frac{(2-1/n)\mu_B^2}{2(1-(1-1/n)^n)} = (1-p_0)^2 \frac{(2-1/n)(\mu')^2}{2(1-(1-1/n)^n)},
\]

\[
\lim_{B \to \infty} \max_{1 \leq i \leq B} \frac{\text{kl}(f_{B;i}, \text{boot})}{B} = \lim_{B \to \infty} (1-p_0)^2 \frac{(2-1/n)\mu_B^2}{2(1-(1-1/n)^n)} = 0,
\]

\[
\lim_{B \to \infty} \frac{1}{B} \sum_{i=1}^{B} \kappa_2(f_{B;i}, \text{boot}) = \lim_{B \to \infty} B(1-p_0)^2 \left( \frac{(2-1/n)\mu_B^2}{1-(1-1/n)^n} + \Theta(\mu_B^2) \right) = (1-p_0)^2 \frac{(2-1/n)(\mu')^2}{1-(1-1/n)^n},
\]

\[
\lim_{B \to \infty} \frac{1}{B} \sum_{i=1}^{B} \kappa_3(f_{B;i}, \text{boot}) = \lim_{B \to \infty} B(1-p_0)^3 \Theta(\mu_B^3) = 0.
\]

Therefore, \( \frac{2K}{\kappa} = \frac{2(1-p_0)^2 (2-1/n)\mu'/(1-p_0)}{\sqrt{(1-p_0)^2 (2-1/n)\mu'^2/(1-(1-1/n)^n)}} = \sqrt{(2-1/n) (1-(1-1/n)^n)} \mu' < \sqrt{2-2/e} \mu' \).

This bound holds for \( n \in \{1, 2, 3, \ldots\} \) since the function is monotonically increasing with respect to \( n \) and \( \lim_{n \to \infty} \sqrt{(2-1/n) (1-(1-1/n)^n)} = \sqrt{2-2/e} = 1.12438 \ldots \)

Appendix C. Privacy analysis of DP bootstrap with Gaussian mechanism

In this section, we first derive the curve for our lower bound on DP bootstrap with Gaussian mechanism in Figure 3a; then we show why the privacy analysis by Brawner and Honaker (2018) is incorrect; Finally, we show why the PLDs by Koskela et al. (2020) is incorrect.

C.1 Our lower bound

In this section, we explain how the curves in Figure 3a were derived. We evaluate our lower bound \( C_1-p_0(f_{\triangledown}) \) based on Corollary 28 and Lemma 29 and visualize it in Figure 3a. From the proof of Lemma 29, the tradeoff function \( \beta = f_{\triangledown}(\alpha) \) is parameterized by \( C \in (-\infty, 0) \) where \( \alpha = \sum_{i=1}^{n} \frac{p_i}{1-p_0} \alpha_i, \alpha_i = 1 - \Phi(\log(-C)/(i\mu) + i\mu/2) \), \( \beta = \sum_{i=1}^{n} \frac{p_i}{1-p_0} \beta_i, \beta_i = \Phi(\log(-C)/(i\mu) - i\mu/2) \); Then we use Lemma 24 to obtain \( C_1-p_0(f_{\triangledown}) \).

In Figure 3a, we also visualize the tradeoff functions for specific neighboring datasets. We let \( D = (x_1, x_2, \ldots, x_n) \), and \( \mathcal{M}_G(D) = \frac{1}{n} \sum_{i=1}^{n} x_i + \xi \) where \( \xi \sim N(0,1/(n\mu)^2) \). We study the tradeoff function between \( \mathcal{M}_G \circ \text{boot}(D_1) \) and \( \mathcal{M}_G \circ \text{boot}(D_2) \) where \( D_1 = (a,0,0,\ldots,0), D_2 = (a-1,0,0,\ldots,0), |D_1| = |D_2| = n \). The tradeoff function between \( \mathcal{M}_G(D_1) \) and \( \mathcal{M}_G(D_2) \) is \( G_\mu \). By the number of occurrences of \( a \) and \( 1-a \), we have

\[
\mathcal{M}_G \circ \text{boot}(D_1) \sim \sum_{i=0}^{n} p_i N \left( \frac{ia}{n}, \frac{1}{(n\mu)^2} \right), \quad \mathcal{M}_G \circ \text{boot}(D_2) \sim \sum_{i=0}^{n} p_i N \left( \frac{i(a-1)}{n}, \frac{1}{(n\mu)^2} \right),
\]

where we are referring to the distribution of the output of \( \mathcal{M}_G \) applied to one bootstrap sample which includes the randomness of \( \mathcal{M}_G \) as well as the randomness of \( \text{boot} \). Since the
Definition 30 (Zero-Concentrated DP (zCDP): Bun and Steinke, 2016) and Honaker (2018). Their result is under zCDP, a variant of DP. In this section, we show an example disproving the statement ‘bootstrap for free’ by Brawner C.2 Counterexample of the privacy analysis by Brawner and Honaker (2018)

ρα if for all neighboring datasets D

In Figure 3a, we show the curves of (α, β) for μ = 1, n = 1000, a ∈ 0, 0.2, 0.4, 0.6, 0.8, 1. We can see that the curve for a = 0 is not lower bounded by 1-GDP which shows that the bootstrap cannot be used for free with the same f-DP guarantee.

C.2 Counterexample of the privacy analysis by Brawner and Honaker (2018)

In this section, we show an example disproving the statement ‘bootstrap for free’ by Brawner and Honaker (2018). Their result is under zCDP, a variant of DP.

Definition 30 (Zero-Concentrated DP (zCDP): Bun and Steinke, 2016) M is ρ-zCDP if for all neighboring datasets D₁ and D₂ and all α ∈ (1, ∞), Dα(M(D₁)||M(D₂)) ≤ ρα where Dα(P||Q) is the α-Rényi divergence, Dα(P||Q) = \( \frac{1}{α-1} \log \left[ \mathbb{E}_{x \sim Q} \left( \frac{dp}{dQ}(x)^α \right) \right] \), and \( \frac{dp}{dQ} \) is the Radon-Nikodym derivative of P with respect to Q.

We consider \( X = [0, 1] \). For any dataset D containing two individuals from X, i.e., D ∈ \( X^2 \), D = (x₁, x₂), we define the Gaussian mechanism on the sample sum: M(D) := x₁ + x₂ + ξ where ξ ∼ \( N(0, 1) \). From the results in (Bun and Steinke, 2016), we know that M satisfies \( \frac{1}{2} \)-zCDP, i.e., ρ = \( \frac{1}{2} \). Now we show that M ◦ boot does not satisfy \( \frac{1}{2} \)-zCDP.

To find a counterexample, we consider the neighboring datasets D₁ = (1, 0), D₂ = (0, 0). Under this case, we have M ◦ boot(D₁) ∼ \( \frac{1}{4} N(0, 1) + \frac{1}{2} N(1, 1) + \frac{1}{4} N(2, 1) \) and M ◦ boot(D₂) ∼ \( N(0, 1) \). We check the α-Rényi divergence when α = 2:

\[
D_2 \left( \frac{1}{4} N(0, 1) + \frac{1}{2} N(1, 1) + \frac{1}{4} N(2, 1) \mid\mid N(0, 1) \right) = \log \left( \frac{1 + 4e + e^4 + 4 + 2 + 4e^2}{16} \right) \approx 1.85265 \ldots > ρα = 1.
\]

Therefore, there exists a mechanism M satisfying \( \frac{1}{2} \)-zCDP such that M ◦ boot does not satisfy \( \frac{1}{2} \)-zCDP which disproves the result in (Brawner and Honaker, 2018).

C.3 Counterexample of the privacy analysis by Koskela et al. (2020)

In this section, we show that the privacy loss distribution (PLD) of the DP bootstrap with Gaussian mechanism cannot be the one shown in Koskela et al. (2020).

The privacy loss function and privacy loss distribution are defined as below.

Definition 31 (Definition 3 and 4 (Koskela et al., 2020)) Let M : \( X^N \to \mathbb{R} \) be a randomised mechanism and let X ∼ Y. Let \( f_X(t) \) denote the density function of M(X) and \( f_Y(t) \) the density function of M(Y). Assume \( f_X(t) > 0 \) and \( f_Y(t) > 0 \) for all t ∈ \( \mathbb{R} \). We define the privacy loss function of \( f_X \) over \( f_Y \) as \( L_{X/Y}(t) = \log \frac{f_X(t)}{f_Y(t)} \).
Using the notation of Def. 31, suppose that \( \mathcal{L}_{X/Y} : \mathbb{R} \rightarrow D, D \subset \mathbb{R} \) is a continuously differentiable bijective function. The privacy loss distribution (PLD) of \( M \) is defined to be a random variable which has the density function

\[
\omega_{X/Y}(s) = \begin{cases} 
  f_X \left( \mathcal{L}_{X/Y}^{-1}(s) \right) \frac{d\mathcal{L}_{X/Y}^{-1}(s)}{ds}, & s \in \mathcal{L}_{X/Y}(\mathbb{R}), \\
  0, & \text{else.}
\end{cases}
\]

Although Koskela et al. (2020) did not mention in this definition of the privacy loss distribution, their density function requires \( \mathcal{L}_{X/Y}(s) \) to be a monotonically increasing function with respect to \( s \). When \( \mathcal{L}_{X/Y}(s) \) monotonically decreases, we need to replace \( \frac{d\mathcal{L}_{X/Y}^{-1}(s)}{ds} \) by \( \left| \frac{d\mathcal{L}_{X/Y}^{-1}(s)}{ds} \right| \). Then they state the following result on privacy profile.

**Lemma 32 (Lemma 5 (Koskela et al., 2020))** Assume \( (\varepsilon, \infty) \subset \mathcal{L}_{X/Y}(\mathbb{R}) \). \( \mathcal{M} \) is tightly \((\varepsilon, \delta)\)-DP for \( \delta(\varepsilon) = \max_{X \succsim Y} \{ \delta_{X/Y}(\varepsilon) \} \), where \( \delta_{X/Y}(\varepsilon) = f_\varepsilon^{\infty}(1 - e^{-\varepsilon}) \omega_{X/Y}(s) \) ds.

Their result on subsampling with replacement is as below.

**Proposition 33 (Section 6.3 in (Koskela et al., 2020))** Consider the sampling with replacement and the \( \succsim \)-neighbouring relation. The number of contributions of each member of the dataset is not limited. The size of the new sample is \( m \). Then \( \ell \), the number of times the differing sample \( x' \) is in the batch, is binomially distributed, i.e., \( \ell \sim \text{Binomial}(1/n, m) \), where \( n \) is the size of the original sample. Then the subsampled Gaussian mechanism, \( \mathcal{M}(D) = \sum_{x \in D} f(x) + \mathcal{N}(0, \sigma^2 I_d) \) where \( B \) is a uniformly randomly drawn subset (with replacement) of \( D = \{ x_1, \ldots, x_n \} \) and \( \| f(x) \|_2 \leq 1 \) for all \( x \in X \), satisfies \((\varepsilon, \delta(\varepsilon))\)-DP where \( \delta(\varepsilon) = \delta_{Y/X}(\varepsilon) = \delta_{X/Y}(\varepsilon) \) which is derived from

\[
f_X(t) = \frac{1}{\sqrt{2\pi \sigma^2}} \sum_{\ell=0}^{m} \left( \frac{1}{n} \right)^\ell (1 - \frac{1}{n})^{m-\ell} \left( \frac{m}{\ell} \right) e^{-\frac{(t-\sigma^2)^2}{2\sigma^2}}, \quad f_Y(t) = \frac{1}{\sqrt{2\pi \sigma^2}} \sum_{\ell=0}^{m} \left( \frac{1}{n} \right)^\ell (1 - \frac{1}{n})^{m-\ell} \left( \frac{m}{\ell} \right) e^{-\frac{(t+\sigma^2)^2}{2\sigma^2}}.
\]

After restating the results by Koskela et al. (2020), we prove that the privacy profile \( \delta(\varepsilon) \) from the above theorem is not a valid privacy guarantee for some neighboring datasets.

Consider the Gaussian mechanism \( \mathcal{M} \) on \( f(D) = x_1 + x_2 \) as \( \mathcal{M}(D) = f(D) + \xi \) where \( \xi \sim \mathcal{N}(0, 1) \), \( D = (x_1, x_2) \), \( x_1, x_2 \in X := [-1, 1] \). We consider two neighboring datasets, \( D_1 = (1, 1) \) and \( D_2 = (-1, 1) \), and bootstrap, i.e., sampling with replacement when sample size remaining the same. We have the two output distributions as \( \mathcal{M}(\text{boot}(D_1)) \sim \mathcal{N}(2, 1) \) and \( \mathcal{M}(\text{boot}(D_2)) \sim \frac{1}{4} \mathcal{N}(-2, 1) + \frac{1}{2} \mathcal{N}(0, 1) + \frac{1}{4} \mathcal{N}(2, 1) \) with their PDFs being

\[
f_{D_1,1}(t) = \frac{1}{\sqrt{2\pi}} \left( e^{-\frac{(t-2)^2}{2}} \right), \quad f_{D_2,1}(t) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{4} e^{-\frac{(t-2)^2}{2}} + \frac{1}{2} e^{-\frac{t^2}{2}} + \frac{1}{4} e^{-\frac{(t+2)^2}{2}} \right).
\]

However, Proposition 33 gives the privacy loss distribution from the following (which can also be derived from \( D'_1 = (1, 0) \) and \( D'_2 = (-1, 0) \)):

\[
f_{D_1,2}(t) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{4} e^{-\frac{t^2}{2}} + \frac{1}{2} e^{-\frac{(t-1)^2}{2}} + \frac{1}{4} e^{-\frac{(t+1)^2}{2}} \right), \quad f_{D_2,2}(t) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{4} e^{-\frac{t^2}{2}} + \frac{1}{2} e^{-\frac{(t+1)^2}{2}} + \frac{1}{4} e^{-\frac{(t+2)^2}{2}} \right).
\]
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Now, we can calculate the \( \delta \) when \( \varepsilon = 1 \).

\[
\delta_{X/Y}(\varepsilon) = \int_{\varepsilon}^{\infty} (1 - e^{-s}) \omega_{X/Y}(s) \, ds = \int_{\varepsilon}^{\infty} (1 - e^{-s}) f_X(L_{X/Y}^{-1}(s)) \frac{dL_{X/Y}^{-1}(s)}{ds} \, ds \\
= \int_{\varepsilon}^{\infty} (1 - e^{-s}) f_X(L_{X/Y}^{-1}(s)) \, dL_{X/Y}^{-1}(s) = \int_{L_{X/Y}^{-1}(\varepsilon)}^{\infty} (f_X(z) - e^{-\varepsilon} f_Y(z)) \, dz.
\]

For \( f_{D_{1,1}} \) vs \( f_{D_{2,1}} \), we know that \( \frac{f_{D_{2,1}}}{f_{D_{1,1}}} \) is monotonically decreasing since the ratio between each of the three parts in \( f_{D_{2,1}} \) and \( f_{D_{1,1}} \) is decreasing. Therefore, \( \log(f_{D_{2,1}}/f_{D_{1,1}}) \) is decreasing, and we let it be \( L_{X/Y} \) with \( f_X = f_{D_{2,1}} \) and \( f_Y = f_{D_{1,1}}. \) To calculate the \( \delta \), we find \( t_1 \) such that \( t_1 = L_{X/Y}^{-1}(1) \) and \( t_2 = L_{X/Y}(\infty) \), i.e., \( \frac{f_{D_{2,1}}(t_1)}{f_{D_{1,1}}(t_1)} = e \) and \( \frac{f_{D_{2,1}}(t_2)}{f_{D_{1,1}}(t_2)} = \infty \). Using \( R \), we have \( t_1 = 0.2228743 \), \( t_2 = -\infty \). Then \( \delta_1 = F_{D_{2,1}}(t_1) - e \ast F_{D_{1,1}}(t_1) \) where \( F \) is the cumulative distribution function corresponding to \( f \). Using \( R \), we have \( \delta_1 = 0.4475773 \).

Similarly, we know \( \frac{f_{D_{2,2}}}{f_{D_{1,2}}} \) is monotonically decreasing from the explanation in the Section 6.3 in (Koskela et al., 2020). We find \( t_3 \) such that \( \frac{f_{D_{2,2}}(t_3)}{f_{D_{1,2}}(t_3)} = e \), then \( \delta_2 = F_{D_{2,2}}(t_3) - e \ast F_{D_{1,2}}(t_3) \). Using \( R \), we have \( t_3 = -0.7830073 \), and \( \delta_2 = 0.369344 \).

Since we have \( \delta_1 > \delta_2 \), the claim by Koskela et al. (2020) that \( f_{D_{1,2}} \) and \( f_{D_{2,2}} \) gives an upper bound of \( \delta(\varepsilon) \) is not true.

**Appendix D. Inference of the population mean with unknown variance**

In this section, we analyze the inference of population mean to obtain results in Table 1.

**D.1 Rate analysis of the unbiased estimates**

We denote the original sample by \( D = X = (x_1, x_2, \ldots, x_n) \). Assume that \( x_i \in [0, 1] \), \( \mathbb{E}(x_i) = \theta \) and \( \text{Var}(x_i) = \sigma^2 \) for \( i = 1, 2, \ldots, n \). We let \( g(D) = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i \), and its \( \ell_2 \)-sensitivity is \( \frac{1}{n} \). Note that for the analysis in this subsection, we do not require any normal assumption or approximation.

We estimate \( \theta \) by \( \hat{\theta}_1 = g(D) \) when there is no privacy constraint. If \( \mu \)-GDP is required, we use Gaussian mechanism on \( g(D) \) and release \( \hat{\theta}_2 = g(D) + \xi, \xi \sim \mathcal{N}(0, \frac{1}{\mu^2}) \) as an estimate of \( \theta \). For the original bootstrap method which is non-private, we still estimate \( \theta \) by \( \hat{\theta}_3 = g(D) \), and we let \( X_b = (x_{b,1}, x_{b,2}, \ldots, x_{b,n}) \) be the \( b \)-th bootstrap sample with the \( b \)-th bootstrap estimate being \( \bar{X}_b = \frac{1}{n} \sum_{i=1}^{n} x_{b,i} \). By our privacy analysis of DP bootstrap, we can release \( \{\bar{X}_b\}_{b=1}^{B} \) with an approximate \( \mu \)-GDP privacy guarantee where \( \bar{X}_b = \bar{X}_b + \xi_b, \xi_b \sim \mathcal{N}(0, \frac{(2-2/e)B}{\mu^2}) \) for \( b = 1, 2, \cdots, B \), and we estimate \( \theta \) by \( \hat{\theta}_4 = \bar{X}' = \frac{1}{B} \sum_{b=1}^{B} \bar{X}_b \).

In our analysis below, the mean of \( B \) bootstrap estimates is \( \bar{X}' = \frac{1}{B} \sum_{b=1}^{B} \bar{X}_b \). The sample variance of \( X \) is \( s_X^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_i - \bar{X})^2 \). The sample variance of \( B \) bootstrap estimates is \( s_B^2 = \frac{1}{(B-1)} \sum_{b=1}^{B} (\bar{X}_b - \bar{X}')^2 \). The sample variance of \( B \) DP bootstrap estimates
For $i = 1, 2, 3, 4$, we need $\hat{\theta}_i$ to construct confidence intervals of $\theta$ based on $\theta_i$. Since we do not have the population variance $\sigma^2_x$, we replace it with its unbiased estimates. Since $\mathbb{E}(s^2_X) = \sigma^2_x$, when there is no privacy constraint, we use $\tilde{\text{Var}}(\hat{\theta}_1) = \frac{s^2_X}{n}$ to estimate $\text{Var}(\hat{\theta}_1)$. If $\mu$-GDP is required, we use Gaussian mechanism on $\sigma^2_x$. Since we assume $x_i \in [0, 1]$ for $i = 1, 2, \ldots, n$, the $\ell_2$-sensitivity of $\sigma^2_x$ is $\frac{1}{n}$. Therefore, we estimate $\text{Var}(\hat{\theta}_2)$ by $\tilde{\text{Var}}(\hat{\theta}_2) = \frac{s^2_X + \xi}{\mu.n^2} + \frac{1}{\mu.n^2}$ where $\xi \sim \mathcal{N}(0, \frac{1}{\mu.n^2})$. Note that if we release both $\hat{\theta}_2$ and $\tilde{\text{Var}}(\hat{\theta}_2)$, the overall privacy guarantee becomes $2\mu$-GDP. For the bootstrap method, we estimate $\text{Var}(\hat{\theta}_3)$ by $\tilde{\text{Var}}(\hat{\theta}_3) = \frac{n}{n - 1} \bar{s}_B^2$. For the DP bootstrap method, we estimate $\text{Var}(\hat{\theta}_4)$ by $\tilde{\text{Var}}(\hat{\theta}_4) = \frac{n.B + n - 1}{B(n - 1)} \bar{s}_B^2 - \frac{(2 - 2/e)B}{n(n - 1)n^2} \sigma^2_x$.

Now we prove that the estimates $\tilde{\text{Var}}(\hat{\theta}_i)$ are unbiased for $i = 1, 2, 3, 4$.

We have $\mathbb{E}(\tilde{\text{Var}}(\hat{\theta}_1)) = \mathbb{E}(\frac{s^2_X}{n}) = \frac{\sigma^2_x}{n}$ and $\mathbb{E}(\tilde{\text{Var}}(\hat{\theta}_2)) = \mathbb{E}(\frac{s^2_X + \xi}{\mu.n^2} + \frac{1}{\mu.n^2}) = \frac{\sigma^2_x}{n} + \frac{1}{\mu.n^2}$.

The expectation of the sample variance of $B$ bootstrap estimates is

$$
\mathbb{E}(\bar{s}_B^2) = \mathbb{E}(\frac{1}{B - 1} \sum_{b=1}^{B} (\bar{X}_b - \bar{X}')^2) = \mathbb{E}(\frac{1}{B - 1} \sum_{b=1}^{B} (\bar{X}_b - \bar{X}')^2|X))
$$

$$
= \mathbb{E}(\text{Var}(\bar{X}_b|X)) = \mathbb{E}(\frac{1}{n} \text{Var}(x_{bi}|X)) = \mathbb{E}(\frac{1}{n} (\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{X})^2) = \frac{n - 1}{n^2} \sigma^2_x.
$$

Therefore, $\mathbb{E}(\tilde{\text{Var}}(\hat{\theta}_3)) = \mathbb{E}(\frac{n}{n - 1} \bar{s}_B^2) = \frac{\sigma^2_x}{n}$.

Similarly, the expectation of the sample variance of $B$ DP bootstrap estimates is

$$
\mathbb{E}(\bar{s}_B^2) = \mathbb{E}(\frac{1}{B - 1} \sum_{b=1}^{B} (\bar{X}_b + \xi_b - \bar{X}')^2) = \mathbb{E}(\frac{1}{B - 1} \sum_{b=1}^{B} (\bar{X}_b + \xi_b - \bar{X}')^2|X))
$$

$$
= \mathbb{E}(\text{Var}(\bar{X}_b + \xi_b|X)) = \mathbb{E}(\text{Var}(\bar{X}_b|X) + \text{Var}(\xi_b))
$$

$$
= \mathbb{E}(\frac{1}{n} (\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{X})^2) + \frac{(2 - 2/e)B}{\mu^2.n^2}) = \frac{n - 1}{n^2} \sigma^2_x + \frac{(2 - 2/e)B}{\mu^2.n^2}.
$$

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Therefore, we use \( \hat{\sigma}_x^2 = \frac{n^2}{n-1}s_B^2 - \frac{(2-2/e)B}{\mu^2(n-1)} \) to estimate \( \sigma_x^2 \), and we have \( \mathbb{E}(\hat{\sigma}_x^2) = \sigma_x^2 \). Now we use \( \tilde{s}_B^2 \) to obtain the variance of the mean of \( B \) DP bootstrap estimates: 
\[
\hat{\text{Var}}(\hat{\theta}_1) = \frac{(B+1)n-1}{n^2B} \tilde{s}_B^2 + \frac{2-2/e}{n^2\mu^2} = \frac{(B+1)n-1}{n^2B} \left( \frac{n^2}{n-1}s_B^2 - \frac{(2-2/e)B}{\mu^2(n-1)} \right) + \frac{2-2/e}{n^2\mu^2} = \left( \frac{n}{n-1} + \frac{1}{B} \right) \tilde{s}_B^2 - \frac{(2-2/e)B}{n(n-1)\mu^2}.
\]
Therefore, \( \mathbb{E} \left( \text{Var} (\hat{\theta}_1) \right) = \mathbb{E} \left( \frac{n+1}{B(n-1)} \tilde{s}_B^2 - \frac{(2-2/e)B}{n(n-1)\mu^2} \right) = \left( \frac{B+1}{n^2B} \right) \sigma_x^2 + \frac{2-2/e}{n^2\mu^2} \).

Now we analyze the variance of \( \text{Var}(\hat{\theta}_i) \) for \( i = 1, 2, 3, 4 \). Let \( \mu_{4,X} = \mathbb{E}((X - \mathbb{E}(X))^4) \).

\[
\mathbb{E}((\tilde{X}_b - \mathbb{E}(\tilde{X}_b))^4) = \mathbb{E}((\tilde{X}_b - \mathbb{E}(\tilde{X}_b))^4|X) + \mathbb{E}((\tilde{X}_b - \mathbb{E}(\tilde{X}_b))^4)X) = \mu_{4,X} + 2(n-1)\mathbb{E}(X) + 6\mathbb{E}((\tilde{X}_b - \mathbb{E}(\tilde{X}_b))^2|X).
\]

Since \( \xi_b \sim N(0, (2-2/e)B/(n^2\mu^2)) \), we denote \( \tilde{X}_b = \bar{X}_b + \xi_b \), and we have \( \mathbb{E}(\tilde{X}_b|X) = \bar{X} \).

\[
\mathbb{E}((\tilde{X}_b - \mathbb{E}(\tilde{X}_b))^4|X) = \mathbb{E}((\tilde{X}_b - \mathbb{E}(\tilde{X}_b))^4|X) + \mathbb{E}(\xi_b^4) + 6\mathbb{E}((\tilde{X}_b - \mathbb{E}(\tilde{X}_b))^2\xi_b^2|X)
= \mu_{4,X} + 2(n-1)\mathbb{E}(X) + \left( \frac{2-2/e}{n^2\mu^2} \right) \mathbb{E}(X)^2 + \left( \frac{2-2/e}{n^2\mu^2} \right)^2 \mathbb{E}(X)^4
\in O \left( \frac{1}{n^2} + \frac{B^2}{n^4\mu^4} + \frac{B}{n^3\mu^2} \right).
\]

\[
\text{Var}(\tilde{s}_B^2) = \mathbb{E}(\text{Var}(\tilde{s}_B^2)) = \mathbb{E}(\mathbb{E}((\tilde{X}_b - \mathbb{E}(\tilde{X}_b))^4|X)) = \mathbb{E} \left( \frac{(\tilde{X}_b - \mathbb{E}(\tilde{X}_b))^4}{B} \right) + \mathbb{E}((\tilde{X}_b - \mathbb{E}(\tilde{X}_b))^4|X) = \mathbb{E} \left( \frac{(\tilde{X}_b - \mathbb{E}(\tilde{X}_b))^4}{B} \right) + \mathbb{E}((\tilde{X}_b - \mathbb{E}(\tilde{X}_b))^4|X)
\in O \left( \frac{1}{n^2} + \frac{B}{n^4\mu^2} + \frac{B}{n^3\mu^4} + \frac{1}{n^3} \right).
\]

Therefore,

\[
\text{Var}(\text{Var}(\hat{\theta}_3)) = \mathbb{E} \left( \frac{n}{n-1} \tilde{s}_B^2 \right) = O \left( \frac{1}{n^2} + \frac{1}{n^3} \right),
\]

\[
\text{Var}(\text{Var}(\hat{\theta}_4)) = \mathbb{E} \left( \frac{nB + n - 1}{B(n-1)} \tilde{s}_B^2 - \frac{(2-2/e)B}{n(n-1)\mu^2} \right) = O \left( \frac{1}{n^2} + \frac{1}{n^3} + \frac{B}{n^4\mu^2} + \frac{1}{n^3} \right).
\]
If we set $B \in O(n)$, the rate of the width of confidence intervals from the DP bootstrap $\sqrt{\max(0, \text{Var}(\hat{\theta}_4))}$ will be the same as those from the non-private bootstrap $\sqrt{\text{Var}(\hat{\theta}_3)}$. If we have $B \in \Theta(n)$, the rate of the width is $O(\frac{1}{n^3})$ for all four methods.

Appendix E. Details of the simulation and real-world experiment

In this section, we show our DP bootstrap method in Algorithm 1 for the Gaussian Mechanism, and based on it, we can obtain the DP sampling distribution estimate (Algorithm 2). We also include the NoisyVar (Algorithm 3) used in our simulation. Then, we derive the sensitivity of the regularized ERM of quantile regression for using DP bootstrap with output perturbation in quantile regression. In the end, we explain how we obtain the rule-of-thumb in Remark 14 by the results in the inference of the population mean and the slope parameters in logistic regression and quantile regression.

E.1 DP bootstrap algorithm for the Gaussian Mechanism

Note that our privacy analysis in this paper (on Algorithm 1) is valid for $g : \mathcal{X}^n \rightarrow \mathbb{R}^d$ for any $d \in \{1, 2, 3, \ldots \}$, but the deconvolution (Algorithm 2) can only be applied when $d = 1$ because of deconvolveR. For $d \in \{2, 3, \ldots \}$, one may try other deconvolution methods. This will not affect the DP guarantee because of the post-processing property.

Algorithm 1 DP_bootstrap (DP bootstrap with Gaussian mechanism)

1: **Input** Dataset $D \in \mathcal{X}^n$, Statistic $g : \mathcal{X}^n \rightarrow \mathbb{R}$ with $\ell_2$ sensitivity $\Delta(g)$, $\mu > 0$ for $\mu$-GDP, number of bootstrap samples $B$.
2: Let $\mathcal{M}(D) = s(D) + \varepsilon$, $\varepsilon \sim \mathcal{N}(0, \sigma^2)$, $\sigma = (\sqrt{2 - 2/e})\Delta(g)\sqrt{B}/\mu$.
3: **for** $b = 1, \ldots, B$ **do**
4: Obtain Bootstrap sample $D^{*b}$ and compute the DP statistic for it: $\tilde{g}^{*b} = \mathcal{M}(D^{*b})$
5: **end for**
6: **Return** $(\tilde{g}^{*1}, \tilde{g}^{*2}, \ldots, \tilde{g}^{*B}, \sigma^2)$ which satisfies approximately $\mu$-GDP.

Algorithm 2 DP_Sampling_Distribution (DP estimation of the sampling distribution)

1: **Input** Dataset $D \in \mathcal{X}^n$, Statistic $g : \mathcal{X}^n \rightarrow \mathbb{R}$ with $\ell_2$ sensitivity $\Delta(g)$, $\mu > 0$ for $\mu$-GDP, number of bootstrap samples $B$.
2: $(\tilde{g}^{*1}, \tilde{g}^{*2}, \ldots, \tilde{g}^{*B}, \sigma^2) = \text{DP_bootstrap}(D, g, \mu, B)$.
3: The DP sampling distribution of $g(D)$ is $f_g = \text{deconvolveR}((\tilde{g}^{*1}, \tilde{g}^{*2}, \ldots, \tilde{g}^{*B}), \sigma^2)$.
4: **Return** $f_g$ which satisfies approximately $\mu$-GDP.
Algorithm 3 NoisyVar (DP CI for population mean (Du et al., 2020))

1: **Input** Dataset $D = \{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$, $l(z_i) = (\tau - 1(z_i \leq 0))z_i$ and $z_i = y_i - x_i^\top \theta$. The regularized empirical risk function $R^c_D(\theta) = \frac{1}{n} \sum_{i=1}^{n} l(z_i) + c \|\theta\|^2$ is 2c-strongly convex.
2: Consider a neighboring dataset to $D$: $D' = \{(x'_1, y'_1), (x_2, y_2), \ldots, (x_n, y_n)\}$, where $x_i = (1, w_i)$, $x'_i = (1, w'_i)$, and $w_i \in [0, 1]$, $w'_i \in [0, 1]$. Let $\theta^c_D = \text{argmin}_{\theta} R^c_D(\theta)$. Denote $a = 1(y_1 \leq \theta^\top x_1), b = 1(y'_1 \leq \theta^\top x'_1)$. By (Chaudhuri et al., 2011, Lemma 7), $\|\theta^c_D - \theta^c_{D'}\| \leq \frac{1}{2c} \max \|\nabla(l(z) - l(z'))/n\| = \frac{1}{2nc} \|\tau(x_1 - x'_1) + ax_1 - bx'_1\| = \frac{1}{2nc} \|\tau w - ax - bx'_1\|.$
3: If $a = b = 1$, $\|\tau(w_1 - w'_1) + aw_1 - bw'_1\| = \|\tau(1 - w_1)w'_1\| \leq 2(1 - \tau)$;
4: If $a = b = 0$, $\|\tau(w_1 - w'_1) + aw_1 - bw'_1\| = \|\tau(1 - w_1)w'_1\| \leq 2\tau$;
5: If $a = 0, b = 1$, $\|\tau(w_1 - w'_1) + aw_1 - bw'_1\| = \|\tau(1 - w_1)w'_1\| \leq \sqrt{2}$;
6: If $a = 1, b = 0$, $\|\tau(w_1 - w'_1) + aw_1 - bw'_1\| = \|\tau w_1 + w'_1\| \leq \sqrt{2}$.
Therefore, $\|\theta^c_D - \theta^c_{D'}\| \leq \frac{1}{2nc} \max \{2\tau, 2(1 - \tau), \sqrt{2}\}$ which is an upper bound of the sensitivity of the regularized ERM.

E.2 Sensitivity of the regularized ERM of quantile regression

Let $D = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$, $l(z_i) = (\tau - 1(z_i \leq 0))z_i$ and $z_i = y_i - x_i^\top \theta$. The regularized empirical risk function $R^c_D(\theta) = \frac{1}{n} \sum_{i=1}^{n} l(z_i) + c \|\theta\|^2$ is 2c-strongly convex.

E.3 More comparison results between DP bootstrap and other methods

Figure 7 shows more simulation results of the population mean inference following the setting in Section 5.1 and Table 2. In the legend, “Bootstrap (B=1000)” is the non-private baseline, and “B=…” is DP bootstrap with B bootstrap estimates. The first row of subfigures shows that the CI width decreases when the sample size n increases. The second row shows that DP bootstrap has over-coverage when n is small, while NoisyVar often has under-coverage. In the third row, we define Extra width := \frac{\text{private CI width}}{\text{non-private CI width}} - 1 and use a log scale for the y-axis in the subfigures (replacing Extra width with max\{10^{-4}, \text{Extra width}\}) to better show the relationship between the extra width and the sample size. In the fourth row, we define $\sqrt{\text{SNR}} := \frac{\text{non-private CI width}(2\Phi(0.95))}{\sigma_{\text{DP noise}}}$ as the confidence level is 90% and we assume the sampling distribution is approximately normal. We can see that if $\sqrt{\text{SNR}} \geq 1$,
Figure 7: Results for various choices of $B$ and $\mu$-GDP for the population mean inference.

DP bootstrap does not have over-coverage, and the largest $B$ satisfying $\sqrt{\text{SNR}} \geq 1$ gives the smallest extra width.

Figure 8 and 9 show more results of the slope parameter inference in logistic regression following the setting in Section 6.2 and Figure 6. The regularization parameter is $c = 1$ and $c = 0.01$ in Figure 8 and 9 respectively. In the figures, the third row is $P(\text{CI covers } 0)$, indicating the power of the private CI for rejecting the independence between MRKINC and SHELCO, and the other rows are the same as in Figure 7. We observe the same conclusion as from Figure 7. Additionally, with $n$ being large enough, DP bootstrap has better performance (less extra width) than DP-CI-ERM if $B$ is chosen to satisfy $\sqrt{\text{SNR}} \approx 1$. By comparing the second and third rows in Figure 8 and 9, we find that DP bootstrap has less over-coverage but also less power when the regularization parameter $c$ is larger.

Similarly, Figure 10 and 11 show more results of quantile regression. We observe that DP bootstrap has better SNR and more power for rejecting the independence between MRKINC and SHELCO in quantile regression than in logistic regression under the same choice of $c$. This resonates with the comparison between private linear regression and quantile regression in (Reimherr and Awan, 2019, Figure 1 and 2), which shows that robust methods like quantile regression are less affected by the DP constraints to some extent.
Figure 8: Results for various choices of $B$ and $\mu$-GDP for logistic regression with $c = 1$.

Figure 9: Results for various choices of $B$ and $\mu$-GDP for logistic regression with $c = 0.01$. 

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Figure 10: Results for various choices of $B$ and $\mu$-GDP for quantile regression with $c = 1$.

Figure 11: Results for various choices of $B$ and $\mu$-GDP for quantile regression with $c = 0.01$. 
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