Abstract

It is shown that the Poisson bracket with boundary terms recently proposed by Bering can be deduced from the Poisson bracket proposed by the present author if one omits terms free of Euler-Lagrange derivatives ("annihilation principle"). This corresponds to another definition of the formal product of distributions (or, saying it in other words, to another definition of the pairing between 1-forms and 1-vectors in the formal variational calculus). We extend the formula initially suggested by Bering only for the ultralocal case with constant coefficients onto the general non-ultralocal brackets with coefficients depending on fields and their spatial derivatives. The lack of invariance under changes of dependent variables (field redefinitions) seems a drawback of this proposal.
1 Introduction

Recently Bering [?] proposed a formula for the field theory Poisson bracket with boundary terms which are different from those proposed earlier by the present author [?]. The motivation for the new bracket arises from the fact that the well-known standard field theory Poisson bracket does not in general satisfy the Jacobi identity. The terms violating the Jacobi identity are, of course, of purely boundary (or divergence) nature and so can be killed by some boundary conditions. The problem addressed here as well as in publications [?, ?, ?, ?] consists in searching the Poisson bracket formula which exactly fulfils the Jacobi identity before putting any boundary conditions.

According to our knowledge the first observation of this problem and the first successful attempt to solve it was made by Lewis, Marsden, Montgomery and Ratiu (LMMR) in their treatment of the dynamics of ideal fluid with a free boundary [?]. Both formulae proposed in [?] and [?] are really two different extrapolations of the formula suggested in [?]. So, it would be better to remind first the pioneering approach.

The variations of functionals studied by LMMR are not free of boundary terms

\[ D_q F(q,p) \cdot \delta q = \int_\Omega \frac{\delta^\wedge F}{\delta q} \cdot \delta q \, dV + \oint_{\partial \Omega} \frac{\delta^\vee F}{\delta q} \cdot \delta q|_{\partial \Omega} dS, \tag{1} \]
\[ D_q F(q,p) \cdot \delta p = \int_\Omega \frac{\delta^\wedge F}{\delta p} \cdot \delta p \, dV + \oint_{\partial \Omega} \frac{\delta^\vee F}{\delta p} \cdot \delta p|_{\partial \Omega} dS. \tag{2} \]

The idea is to generalize the definition of the variational derivative by incorporating the boundary contribution

\[ \frac{\delta F}{\delta q} = \frac{\delta^\wedge F}{\delta q} + \delta(S) \cdot \frac{\delta^\vee F}{\delta q}. \tag{3} \]
\[ \frac{\delta F}{\delta p} = \frac{\delta^\wedge F}{\delta p} + \delta(S) \cdot \frac{\delta^\vee F}{\delta p}. \tag{4} \]

Then LMMR in fact proposed to use for the new Poisson bracket the old formula

\[ \{F,G\} = \int_\Omega \left[ \frac{\delta F}{\delta q(x)} \frac{\delta G}{\delta p(x)} - \frac{\delta G}{\delta q(x)} \frac{\delta F}{\delta p(x)} \right] dV \]

but with new variational derivatives (??), (??) in it

\[ \{F,G\} = \int_\Omega \left[ \frac{\delta^\wedge F}{\delta q(x)} \frac{\delta^\wedge G}{\delta p(x)} - \frac{\delta^\wedge G}{\delta q(x)} \frac{\delta^\wedge F}{\delta p(x)} \right] dV \]
\[ + \oint_{\partial \Omega} \left[ \frac{\delta^\wedge F}{\delta q(x)} \frac{\delta^\vee G}{\delta p(x)} + \frac{\delta^\vee F}{\delta q(x)} \frac{\delta^\wedge G}{\delta p(x)} \right] \bigg|_{\partial \Omega} dS \]
\[ - \oint_{\partial \Omega} \left[ \frac{\delta^\wedge G}{\delta q(x)} \frac{\delta^\vee F}{\delta p(x)} + \frac{\delta^\vee G}{\delta q(x)} \frac{\delta^\wedge F}{\delta p(x)} \right] \bigg|_{\partial \Omega} dS. \]
One immediately sees that the most dangerous term with the product of $\delta$-functions is absent above. In fact, to kill this term LMMR put a special boundary condition

$$\frac{\delta^N F}{\delta q} \frac{\delta^N G}{\delta p} - \frac{\delta^N G}{\delta q} \frac{\delta^N F}{\delta p} = 0,$$

which enforces zero value for the coefficient standing before this dangerous product. Unfortunately it is not quite clear whether the Poisson bracket $\{F, G\}$ preserves this property in general case even if the initial functionals $F$ and $G$ satisfy $(\ldots)$.

Here we see the bifurcation point for the following generalizations of LMMR result. The idea of [?] is that these dangerous terms with products of $\delta$-functions must be omitted independently of any boundary conditions ("annihilation principle"). Another idea advocated earlier in [?] is to find a reasonable formula for these terms.

To explain this in more detail we need first to introduce a relevant formalism for treating general variations of functionals depending on arbitrary (but finite!) number of spatial derivatives. In [?] the adequate mathematical machinery was found to be the so-called higher Eulerian operators $[\ldots, \ldots, \ldots]$. We shall follow notations of $[\ldots, \ldots, \ldots]$. Einstein rule is used, i.e. we omit the sign of summation over repeated indices and multi-indices.

The first variation of a general local ($\max|J| < \infty$) functional

$$F = \int_{\Omega} f(\phi_A(x), \phi^{(J)}_A(x)) d^n x$$

can be written in a form

$$\delta F = \int_{\Omega} \frac{\partial f}{\partial \phi^{(J)}_A} D_J \delta \phi_A d^n x \equiv \int_{\Omega} f_A'(\delta \phi_A) d^n x \equiv \int_{\Omega} D_J (E_A^J(f) \delta \phi_A) d^n x,$$

where in general $J$ denotes multi-index $J = (j_1, \ldots, j_n)$ and

$$\phi^{(J)}_A = \frac{\partial^{|J|} \phi_A}{\partial \phi^{(j_1)}_{A_1} \ldots \partial \phi^{(j_n)}_{A_n}} \equiv D_J \phi_A, \quad |J| = j_1 + \ldots + j_n,$$

but in the simplest case of one-dimensional space it is just the order of spatial derivative. We have introduced also the Fréchet derivative which is a differential operator

$$f'_A = \frac{\partial f}{\partial \phi^{(j)}_A} D_J.$$

The higher Eulerian operators $E_A^J$ are uniquely defined by the following formula

$$E_A^J(f) = \binom{K}{J}(-D)_K \frac{\partial f}{\partial \phi^{(K)}_A}.$$

Here binomial coefficients for multi-indices are

$$\binom{J}{K} = \frac{(j_1)}{k_1} \ldots \frac{(j_n)}{k_n}.$$
\[
\binom{j}{k} = \begin{cases} 
\frac{j!}{k!(j-k)!} & \text{if } 0 \leq k \leq j; \\
0 & \text{otherwise,}
\end{cases}
\]

and

\[
(-D)_K = (-1)^{|K|} D_K.
\]

Both [?] and [?] use the full variation (??) for the construction of the new Poisson brackets but in different ways. In [?] it was proposed to start from the formula

\[
\{F, G\} = \delta G F = \int_D (E^A_J(f) \delta G \phi_A) d^n x = \int_O f_A^J(\delta G \phi_A) d^n x,
\]

and to look for \(\delta G \phi_A\) of such a form which fulfils the equation

\[
\delta G F = -\delta F G.
\]

The following formula was derived in [?] after some calculations

\[
\{F, G\} = \int_D (E^A_J(f) \hat{I}_{AB} E^K_B(g)) d^n x = \int_O \left( f_A^J \hat{I}_{AB} g_B \right) d^n x.
\]

We can for the easier comparison with [?] first consider only the so-called ultralocal brackets then

\[
\{\phi_A(x), \phi_B(y)\} = I_{AB} \delta(x, y), \quad I_{AB} = -I_{BA}.
\]

In contrast, the proposal of [?] is to start with the already antisymmetric expression

\[
\{F, G\} = \Delta_G F - \Delta_F G - \{F, G\}_{\text{old}},
\]

where

\[
\{F, G\}_{\text{old}} = \int_O E_A^0(f) I_{AB} E_B^0(g) d^n x,
\]

\[
\Delta_G F = \int_D (E^A_J(f) \Delta_G \phi_A) d^n x = \int_O f_A^J(\Delta_G \phi_A) d^n x.
\]

Then it is possible to use the standard expression for the field variation

\[
\Delta_G \phi_A = I_{AB} E_B^0(g)
\]

and the resulting formula will be

\[
\{F, G\} = \int_O \left( E^A_J(f) I_{AB} E_B^0(g) - E_A^J(g) I_{AB} E_B^0(f) \right) d^n x - \{F, G\}_{\text{old}}.
\]

So, it is easy to see that the last formula contains only one summation over multi-index \(J\) whereas formula (??) contains a double sum over \(J\) and \(K\). If we omit all the terms without at least one of \(E^0\) operators in this double sum we immediately get (??).

Maybe it will be of some interest to add that in ultralocal case for the local functionals depending on the spatial derivatives of the fields of order up to \(N\), Bering’s bracket involves
spatial derivatives of order $3N$, whereas the bracket proposed in [?] involves $2N$, as also the standard bracket does.

The point of difficulty with Bering’s formula seems to be the lack of invariance under the changes of dependent variables (differential substitutions of fields).

2 Differential substitutions

Let us consider the invariance properties of the field theory Poisson brackets under field redefinitions of the type

$$
\phi_A \rightarrow \tilde{\phi}_B = \xi_B(\phi_A, D_J \phi_A),
$$

(differential substitutions).

If we initially have some local Poisson brackets for fields $\phi_A(x)$, i.e.

$$
\{\phi_A(x), \phi_B(y)\} = \hat{I}_{AB}(x)\delta(x,y),
$$

where $\hat{I}_{AB} = \hat{I}_{AB}^K D_K$ is a differential operator of a finite order with field-dependent coefficients

$$
\hat{I}_{AB}^K = \hat{I}_{AB}^K(\phi_C, D_J \phi_C),
$$

then as a result of the differential substitution (??) we get a result

$$
\{\tilde{\phi}_C(x), \tilde{\phi}_D(y)\} = (\xi_C)'_A (x) (\xi_D)'_B (y) \hat{I}_{AB}(x)\delta(x,y).
$$

To transform this expression to the form similar to (??) we need a definition of the “adjoint” operator

$$
\tilde{J}_{AB}(x)\delta(x,y) = \tilde{J}_{AB}^{\text{adjoint}}(y)\delta(x,y),
$$

then we will have

$$
(\xi_C)'_A (x) (\xi_D)'_B (y) \hat{I}_{AB}(x)\delta(x,y) = (\xi_C)'_A (x) \hat{I}_{AB}(x)\left[(\xi_D)'_B \right]^{\text{adjoint}} (x)\delta(x,y),
$$

and

$$
\hat{J}_{CD}(x) = (\xi_C)'_A (x) \hat{I}_{AB}(x)\left[(\xi_D)'_B \right]^{\text{adjoint}} (x).
$$

The approach which we consider here is different from the standard one by the treatment of boundary (or divergence) terms. All of them should be preserved in the calculations. This means that we require the exact equality

$$
\int_{\Omega} \xi_A \hat{J}_{AB} \eta_B d^m x = \int_{\Omega} \eta_A \hat{J}_{AB} \xi_B d^m x,
$$

without discarding any boundary (divergence) terms. In contrast the standard approach requires only equality up to boundary terms

$$
\xi_A \hat{J}_{AB} \eta_B \approx \eta_A \hat{J}_{AB} \xi_B \quad (\text{mod divergences}),
$$
or, in notations of Appendix C,
\[ \langle \xi | \hat{J} | \eta \rangle = \langle \eta | \hat{J}^* | \xi \rangle. \]

From the last definition we get a usual relation
\[ \hat{J}^*_{AB} = (-D) K \circ J^K_{BA}. \]  
(12)

But the former one gives a different result:
\[ \hat{J}'_{AB} = (-D) K \circ \theta_K J^K_{BA}. \]  
(13)

Here we use the characteristic function of the domain of integration (physical domain)
\[ \theta_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega; \\ 0 & \text{otherwise}; \end{cases} \]
to codify the divergences. There is an apparent relation
\[ \int_{\mathbb{R}^n} (-D)_K \theta_\Omega f(x) d^n x = \int_{\Omega} \theta_\Omega D_K f(x) d^n x \equiv \int_{\Omega} D_K f(x) d^n x. \]

So, with \( \theta_\Omega \) we are able to write all the spatial integrals not as integrals over the physical domain but as integrals over the whole infinite coordinate space \( \mathbb{R}^n \).

### 3 The standard bracket

Let us first consider transformations of the standard Poisson bracket
\[ \{F, G\}_\text{old} = \frac{1}{2} \int_{\Omega} \left( E_A^0(f) \hat{I}_{AB} E_B^0(g) - E_A^0(g) \hat{I}_{AB} E_B^0(f) \right) d^n x, \]  
(14)

under field redefinitions of the type (??). We will use a formula
\[ E_A^0(f) = \left[ (\xi_C)'_A \right]^* E_C^0(f) \equiv \left( E_A^0(\xi_C) - E_A^1(\xi_C) D + E_A^2(\xi_C) D^2 - \ldots \right) E_C^0(f) \equiv (-1)^{[K]} E_A^K(\xi_C) D_K E_C^0(f), \]  
(15)

derived in Appendix C. Then in the integrand of (??) we will have the following expressions
\[ E_A^0(f) \hat{I}_{AB} E_B^0(g) = \left[ (\xi_C)'_A \right]^* E_C^0(f) \hat{I}_{AB} \left[ (\xi_D)'_B \right]^* E_D^0(g) = \]  
(16)

But, as we discussed above, the two definitions of the “adjoint” differential operator are not equivalent
\[ \left[ (\xi_C)'_A \right]^* \hat{I}_{AB} \left[ (\xi_D)'_B \right]^* \neq \hat{I}_{CD}, \]
and so
\[ \left[ (\xi_C)'_A \right]^* \hat{I}_{AB} \left[ (\xi_D)'_B \right]^* \neq \hat{I}_{CD}. \]
Of course, the difference has a form of divergences. Therefore we see that the standard field theory Poisson bracket is invariant under field redefinitions of the form (??) (i.e. differential substitutions) only up to boundary terms. Surely, this is enough because all the other requirements (Jacobi identity, antisymmetry, Leibnitz rule) are also fulfilled only up to boundary terms and this bracket does not pretend to be adequate for nontrivial boundary problems.

4 The general approach to both proposals [1] and [2]

Dealing with local functionals and local Poisson brackets in field theory we always get a general bracket of two functionals in the following form

\[ \{F,G\} = \int d^n x \int d^n y f'_A(x)g'_B(y) \{\phi_A(x),\phi_B(y)\}, \]

where Frechet derivatives \( f'_A(x), g'_B(y) \) are differential operators (??). If we represent each integral not as the integral over finite domain but as an integral over all the infinite \( \mathbb{R}^n \) with a characteristic function of the domain \( \Omega \)

\[ \{F,G\} = \int_{\mathbb{R}^n} \theta(\Omega(x)) d^n x \int_{\mathbb{R}^n} \theta(\Omega(y)) d^n y f'_A(x)g'_B(y) \{\phi_A(x),\phi_B(y)\}, \]

then it easy to integrate by parts formally and get

\[ \{F,G\} = \int_{\mathbb{R}^n} d^n x \int_{\mathbb{R}^n} d^n y \frac{\delta F}{\delta \phi_A(x)} \frac{\delta G}{\delta \phi_B(y)} \{\phi_A(x),\phi_B(y)\}, \]

where

\[ \frac{\delta F}{\delta \phi_A(x)} = (f'_A(x))^\dagger \theta = (-D)_K \left( \theta \frac{\partial f}{\partial \phi_A^{(K)}} \right) \equiv E_A^f((-D)_J \theta_\Omega), \]

with the the analogous formula valid for the variational derivative of \( G \). Let us restrict the consideration to ultralocal case for simplicity, then

\[ \{F,G\} = \int_{\mathbb{R}^n} d^n x \int_{\mathbb{R}^n} d^n y \frac{\delta F}{\delta \phi_A(x)} I_{AB} \frac{\delta G}{\delta \phi_B(y)} \delta(x,y), \]

and in both approaches it is believed that

\[ \{F,G\} = \int_{\mathbb{R}^n} d^n x \frac{\delta F}{\delta \phi_A(x)} I_{AB} \frac{\delta G}{\delta \phi_B(x)}. \]

But, of course, this expression includes products of distributions of the form \( D_J \theta_\Omega \times D_K \theta_\Omega \) and here these two proposals are different:

- in Bering’s approach [?]:

\[ D_J \theta_\Omega \times D_K \theta_\Omega = \delta_J \delta_K \theta_\Omega + \delta_K \delta_J \theta_\Omega - \delta_J \delta_K \theta_\Omega, \]
• in the approach of [2]:

\[ D_J \theta_\Omega \times D_K \theta_\Omega = D_{J+K} \theta_\Omega. \]

Apparently it is possible to avoid these distributions completely as they serve only to codify divergences. This is demonstrated in publications [3]. Then the key transformation from the double spatial integral to the single spatial integral with the help of \( \delta \)-function can be interpreted simply as a pairing between 1-forms and 1-vectors of the formal variational calculus [4, 5, 6]. The pairing defined in [4, 5] is compatible with the grading related to divergences.

Now if we use the above formula for Bering’s pairing it is possible to derive the Poisson brackets in the most general (not treated in [3]) non-ultralocal case where \( \hat{I}_{AB} \) is a differential operator of a finite order with field dependent coefficients. Really, this is the same formula with hats added\(^2\)

\[
\{ f, g \} = f'_A \left( \hat{I}_{AB} E_B^0(g) \right) - g'_A \left( \hat{I}_{AB} E_B^0(f) \right) - \frac{1}{2} \left( E_A^0(f) \hat{I}_{AB} E_B^0(g) - E_A^0(g) \hat{I}_{AB} E_B^0(f) \right).
\]

Moreover, it is possible to demonstrate that the Jacobi identity is fulfilled for this bracket for any local operator \( \hat{I}_{AB} \) with constant coefficients. In the case of ultralocal Poisson brackets with the field dependent coefficients (but the dependence on field derivatives is excluded) in Appendix B we derive the following condition of the fulfilment of the Jacobi identity for the bracket constructed according to Bering proposal

\[ I_{AB,C} I_{CD} + I_{DA,C} I_{CB} + I_{BD,C} I_{CA} = 0. \]

In Appendix B we also give the condition for the most general case of non-ultralocal brackets with coefficients depending on the spatial derivatives of fields also.

### 5 Differential substitutions and Bering’s proposal [1]

Now let us consider the formula derived in the previous Section as a further development of the initial proposal by Bering [3]

\[
\{ F, G \}_B = \int_\Omega f'_A \hat{I}_{AB} E_B^0(g) d^n x - \int_\Omega g'_A \hat{I}_{AB} E_B^0(f) d^n x - \{ F, G \}_\text{old},
\]

where \( \{ F, G \}_\text{old} \) is the standard Poisson bracket treated before.

The Fréchet derivative transforms under differential substitutions (??) as follows (see Appendix C)

\[
f'_A = f'_C (\xi_C)^*_A.
\]

\(^2\)It is possible to derive a more general formula if we do not suggest that the operator \( I_{AB} \) should be antisymmetric with respect to the standard definition of the adjoint. But this formula does not fulfill the Jacobi identity even for non-ultralocal brackets with constant coefficients.
so we obtain

\[ f'_A \hat{I}_{AB} E_B^0 = f'_C (\xi_C)'_A \hat{I}_{AB} (\xi_D)'_B + E_B^0 (g). \]

It means that the first and second terms of the bracket will be invariant if we suppose

\[ \hat{I}_{CD} = (\xi_C)'_A \hat{I}_{AB} (\xi_D)'_B \]

so, the old definition of the adjoint operator \((?)\) should be used here in accordance with the treatment given by Bering (see Subsection (5.5) of [?]).

Unfortunately, this bracket contains also a term \(\{F, G\}_\text{old}\) (the standard Poisson bracket) with another transformation properties. As we have demonstrated in Section 3 this term is invariant under field redefinitions only up to divergences. So, taken as a whole, Bering’s formula is not invariant.

6 Differential substitutions and the bracket proposed in [2]

Let us show that in contrast the formula

\[ \{F, G\} = \int_\Omega \text{Tr} \left( f'_A \hat{I}_{AB} g'_B \right) dx \]

is precisely invariant under field redefinitions of the form \((?)\). We remind that the trace is used here to denote the rules of composition of the differential operators \(f'_A\), \(\hat{I}_{AB}\) and \(g'_B\):

\[
\begin{align*}
  f'_A &= \frac{\partial f}{\partial \phi_A^{(j)}} D_J, \\
  \hat{I}_{AB} &= I_{K}^{BA} D_K, \\
  g'_B &= \frac{\partial g}{\partial \phi_B^{(j)}} D_J.
\end{align*}
\]

Operator \(f'_A\) acts on everything to the right of it, so does operator \(\hat{I}_{AB}\), and operator \(g'_B\) acts on everything to the left of it, i.e. acts on everything besides its own coefficients,

\[ \text{Tr} \left( f'_A \hat{I}_{AB} g'_B \right) = \left( \begin{array}{c} J \\ L \end{array} \right) \left( \begin{array}{c} K \\ M \end{array} \right) D_M \frac{\partial f}{\partial \phi_A^{(j)}} D_J + K - L - M \hat{I}_{AB} D_L \frac{\partial g}{\partial \phi_B^{(j)}}. \]

After the field redefinition we get

\[ \{F, G\} = \int_\Omega \text{Tr} \left( f'_C (\xi_C)'_A \hat{I}_{AB} (\xi_D)'_B \right) d^nx. \]

So, if we use here the adjoint operator to \((\xi_D)'_B\) defined by \((?)\) then it will act only onto \(g'_D\)

\[ \{F, G\} = \int_\Omega \text{Tr} \left( f'_C (\xi_C)'_A \hat{I}_{AB} (\xi_D)'_B \right) d^nx. \]

But according to our definitions given in Section 2

\[ \hat{I}_{CD} = (\xi_C)'_A \hat{I}_{AB} (\xi_D)'_B. \]
As a result we see that this definition of the field theory Poisson bracket with boundary terms is exactly invariant under differential substitutions.

In publication [?] this invariance was demonstrated for the concrete example — the Ashtekar transformation [?] of the gravitational variables.

7 Conclusion

We considered above an interesting proposal made by Bering on the boundary terms in the field theory Poisson bracket. We generalize this proposal to the most general local Poisson brackets and find the conditions necessary to fulfil the Jacobi identity. According to our treatment given in more detail in previous publications [?, ?] there are three separate ingredients of the Poisson bracket construction which should be revised: the differential of the local functional, the Poisson bivector and the pairing operation. Bering uses the same definition for the differential, but changes the pairing and the bivector. It occurs so that to change the pairing alone means to get into trouble with the Jacobi identity in the non-ultralocal case.

Really, the paper [?] suggest a lot of new ideas which deserve more discussion. Here we only concentrated on the drawback that it seemingly had. Probably, the further investigation will show whether these drawback could be overcome in Bering’s approach. But anyhow it is absent if we use another formula suggested in [?].

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Appendix A. Useful relations

Let us suppose that \( \hat{I}_{AB} = I_{AB}^{M}D_{M} \), where \( I_{AB}^{M} = I_{AB}^{M}(\phi^{(J)}) \), and

\[
\hat{I}_{AB} = (-D)_{M} \circ I_{BA}^{M}.
\]

Then the following useful relations can be proved by using the technique of higher Eulerian operators [?, ?] compiled in Lemmas 2.5 – 2.12 of [?]:

\[
\begin{align*}
\langle E^{0}_{B}(g) \rangle_{C} & \langle \hat{I}_{CD}E^{0}_{D}(h) \rangle = (-D)_{L} \left[ \frac{\partial^{2}g}{\partial\phi_{C}^{(L)} \partial\phi_{C}^{(J)}} D_{J}(\hat{I}_{CD}E^{0}_{D}(h)) \right], \\
E^{0}_{B} \left( g_{C}(\hat{I}_{CD}E^{0}_{D}(h)) \right) & = (-D)_{L} \left[ \frac{\partial^{2}g}{\partial\phi_{B}^{(L)} \partial\phi_{C}^{(J)}} D_{J}(\hat{I}_{CD}E^{0}_{D}(h)) \right] +
\end{align*}
\]
+ \frac{\partial^2 h}{\partial \phi_{B}^{(L)} \partial \phi_{C}^{(J)}} D_{J} (I_{CD}^{0} E_{D}^{0}(g)) + \\
+ E_{C}^{0}(g) \frac{\partial \chi^{(M)}_{CD B}}{\partial \phi_{B}^{(L)}} D_{M} E_{B}^{0}(h) \right],

E_{B}^{0} \left( E_{C}^{0}(g) \hat{I}_{CD} E_{D}^{0}(h) \right) = (-D)_{L} \left[ \frac{\partial^2 g}{\partial \phi_{B}^{(L)} \partial \phi_{C}^{(J)}} D_{J} (I_{CD}^{0} E_{D}^{0}(g)) + \\
+ \frac{\partial^2 h}{\partial \phi_{B}^{(L)} \partial \phi_{C}^{(J)}} D_{J} (I_{CD}^{0} E_{D}^{0}(g)) + \\
+ E_{C}^{0}(g) \frac{\partial \chi^{(M)}_{CD B}}{\partial \phi_{B}^{(L)}} D_{M} E_{B}^{0}(h) \right],

\text{Let also } \hat{I}_{AB} = -\hat{I}_{AB}, \text{ then from the above it follows}

E_{B}^{0} \left( g_{C}^{'} (I_{CD} E_{D}^{0}(h)) \right) = -E_{B}^{0} \left( h_{C}^{'} (I_{CD} E_{D}^{0}(g)) \right) = \\
= E_{B}^{0} \left( E_{C}^{0}(g) \hat{I}_{CD} E_{D}^{0}(h) \right) = (-D)_{L} \left[ E_{C}^{0}(g) \frac{\partial \chi^{(M)}_{CD B}}{\partial \phi_{B}^{(L)}} D_{M} E_{B}^{0}(h) \right] + \\
+ (E_{B}^{0}(g))_{C}^{'} (I_{CD} E_{D}^{0}(h)) - (E_{B}^{0}(h))_{C}^{'} (I_{CD} E_{D}^{0}(g)).

\text{We will use these relations in checking the Jacobi identity in Appendix B.}

\text{Let us illustrate our results by the less general case of ultralocal Poisson brackets}

\hat{I}_{AB} = I_{AB} = -I_{BA},

\text{and suppose for the simplicity that functions } I_{AB} \text{ depend only on fields and do not depend on their derivatives}

I_{AB} = I_{AB}(\phi_{C}).

\text{Then}

\frac{\partial I_{CD}^{0}}{\partial \phi_{B}^{(L)}} = \delta_{M0} \delta_{L0} I_{CD,B},

\text{and so,}

(-D)_{L} \left[ E_{C}^{0}(g) \frac{\partial I_{CD}^{0}}{\partial \phi_{B}^{(L)}} D_{M} E_{B}^{0}(h) \right] = I_{CD,B} E_{C}^{0}(g) E_{D}^{0}(h).

\textbf{Appendix B. The Jacobi identity}

\text{By using Bering’s formula for the Poisson bracket we get}

\{F,G\} = \int_{\mathbb{R}^{n}} \{f,g\} d^{n}x,

\{f,g\} = f_{A}^{'}(\hat{I}_{AB} E_{B}^{0}(g)) - g_{A}^{'}(\hat{I}_{AB} E_{B}^{0}(f)) - \\
- \frac{1}{2} E_{A}^{0}(f) \hat{I}_{AB} E_{B}^{0}(g) + \frac{1}{2} E_{A}^{0}(g) \hat{I}_{AB} E_{B}^{0}(f),
\[
\{\{f, g\}, h\} = \{f, g\}_C(\hat{i}_{CD}E_D^0(h)) - h_C(\hat{i}_{CD}E_D^0(\{f, g\})) - \\
- \frac{1}{2}E_C^0(\{f, g\})\hat{i}_{CD}E_D^0(h) + \frac{1}{2}E_C^0(h)\hat{i}_{CD}E_D^0(\{f, g\}) = \\
\quad f''_{AC} \left( \hat{i}_{AB}E_B^0(g), \hat{i}_{CD}E_D^0(h) \right) - g''_{AC} \left( \hat{i}_{AB}E_B^0(f), \hat{i}_{CD}E_D^0(h) \right) + \\
\quad f'_A \left( (i_{AB})_C(\hat{i}_{CD}E_D^0(h))E_B^0(g) + \hat{i}_{AB}(E_B^0(g))_C(\hat{i}_{CD}E_D^0(h)) \right) - \\
\quad g'_A \left( (i_{AB})_C(\hat{i}_{CD}E_D^0(h))E_B^0(f) + \hat{i}_{AB}(E_B^0(f))_C(\hat{i}_{CD}E_D^0(h)) \right) - \\
- \frac{1}{2}(E_A^0(f))_C(\hat{i}_{CD}E_D^0(h))\hat{i}_{AB}E_B^0(g) - \\
- \frac{1}{2}E_A^0(f)(i_{AB})_C(\hat{i}_{CD}E_D^0(h))E_B^0(g) - \\
- \frac{1}{2}E_A^0(g)(i_{AB})_C(\hat{i}_{CD}E_D^0(h))E_B^0(f) + \\
+ \frac{1}{2}(E_A^0(g))_C(\hat{i}_{CD}E_D^0(h))\hat{i}_{AB}E_B^0(f) + \\
+ \frac{1}{2}E_A^0(g)(i_{AB})_C(\hat{i}_{CD}E_D^0(h))E_B^0(f) + \\
+ \frac{1}{2}E_A^0(g)\hat{i}_{AB}(E_B^0(f))_C(\hat{i}_{CD}E_D^0(h)) + \\
+ E_A^0(f)\hat{i}_{AB}(E_B^0(f))(g)'(i_{AB}E_B^0(f)) + \\
- f'_A (\hat{i}_{AB}E_B^0(g)) + \\
- \frac{1}{2}E_A^0(f)(i_{AB}E_B^0(g)) - g'_A(\hat{i}_{AB}E_B^0(f)) - \\
- \frac{1}{2}E_A^0(f)\hat{i}_{AB}E_B^0(g) + \frac{1}{2}E_A^0(g)\hat{i}_{AB}E_B^0(f) \right) \hat{i}_{CD}E_D^0(h) + \\
+ \frac{1}{2}E_C^0(h)\hat{i}_{CD}E_D^0 \left( f'_A (\hat{i}_{AB}E_B^0(g)) - g'_A(\hat{i}_{AB}E_B^0(f)) \right) - \\
- \frac{1}{2}E_A^0(f)\hat{i}_{AB}E_B^0(g) + \frac{1}{2}E_A^0(g)\hat{i}_{AB}E_B^0(f) \right)
\]

Here we use notation
\[
f''_{AB}(\xi_A, \eta_B) = \frac{\partial^2 f}{\partial \phi_A^{(1)} \partial \phi_B^{(1)}} D_s \xi_A D_K \eta_B.
\]

Then by making cyclic permutations and applying formulae from Appendix A we get a result
\[
\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = \\
= f'_A \left( \frac{\partial I_{AB}}{\partial \phi_C^{(1)}} D_L (\hat{i}_{CD}E_D^0(h)D_M E_B^0(g)) \right) \\
- g'_A \left( \frac{\partial I_{AB}}{\partial \phi_C^{(1)}} D_L (\hat{i}_{CD}E_D^0(h)D_M E_B^0(f)) \right) \\
- h_C \left( \hat{i}_{CD}(-D)_L \left[ \frac{1}{2}E_A^0(f) \frac{\partial I_{AB}}{\partial \phi_D^{(1)}} D_M E_B^0(g) - \frac{1}{2}E_A^0(g) \frac{\partial I_{AB}}{\partial \phi_D^{(1)}} D_M E_B^0(f) \right] \right) \\
- \frac{1}{2}E_A^0(f) \frac{\partial I_{AB}}{\partial \phi_C^{(1)}} D_L (\hat{i}_{CD}E_D^0(h))D_M E_B^0(g)
\]
From the above expression it is apparent that in the case of constant coefficients \( I_{AB} \) the Jacobi identity is satisfied. It is straightforward to check that in the case of ultralocal Poisson brackets with the coefficients depending on the fields (but not on their spatial derivatives) we get a well-known condition for the fulfillment of the Jacobi identity

\[ I_{AB,C} I_{CD} + \text{cyclic permutation of } (A, B, D) = 0. \]

**Appendix C. The transformation rules**

Here we derive the transformation rules for Euler-Lagrange and Fréchet derivatives under differential substitutions of fields (??).

First, let us consider the variation of an arbitrary function of the fields

\[ \delta f = f'_A \delta \phi_A \equiv \frac{\partial f}{\partial \phi_A} D_J \delta \phi_A. \]

If we use the transformed fields

\[ \tilde{\phi}_B = \xi_B(\phi_A, D_J \phi_A), \]

then we get

\[ \delta f = f'_B \delta \tilde{\phi}_B \equiv \frac{\partial f}{\partial \tilde{\phi}_B} D_K \delta \tilde{\phi}_B, \]

where

\[ \delta \tilde{\phi}_B = (\xi_B)'_A \delta \phi_A. \]

Therefore,

\[ f'_A = f'_B \circ (\xi_B)'_A. \]

Second, let us consider an expression

\[ \langle 1 | f'_A | \delta \phi_A \rangle = \langle \delta \phi_A | (f_A')^\dagger | 1 \rangle \equiv E^0_A(f) \delta \phi_A, \]

where the angle brackets denote the standard integrand, defined up to divergences, and make a change of variables

\[ \phi_A \to \tilde{\phi}_B = \xi_B(\phi_A, D_J \phi_A), \]
then
\[ E_\lambda^0(f)\delta\phi_A = \langle 1|f_B' \circ (\xi_B)'_A|\delta\phi_A|\rangle = \langle 1|f_B'(\xi_B)'_A\delta\phi_A|\rangle = \]
\[ = \langle (\xi_B)'_A\delta\phi_A|(f_B')^*|1\rangle = E_B^0(f)(\xi_B)'_A\delta\phi_A = \]
\[ = (E_B^0(f))(\xi_B)'_A|\delta\phi_A| = \langle (\xi_B)'_A|E_B^0(f)\rangle, \]
or,
\[ E_\lambda^0(f) = (\xi_B)'_A^* E_B^0(f). \]

This result can be checked by more tedious but straightforward calculation by using formulae for Eulerian operators given in article [?].

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