RIESZ SPACES WITH GENERALIZED ORLICZ GROWTH

PETER HÄSTÖ, JONNE JUUSTI AND HUMBERTO RAFEIRO

ABSTRACT. We consider a Riesz $\varphi$-variation for functions $f$ defined on the real line when $\varphi : \Omega \times [0, \infty) \to [0, \infty)$ is a generalized $\Phi$-function. We show that it generates a quasi-Banach space and derive an explicit formula for the modular when the function $f$ has bounded variation. The resulting $BV$-type energy has previously appeared in image restoration models. We generalize and improve previous results in the variable exponent and Orlicz cases and answer a question regarding the Riesz–Medvedev variation by Appell, Banas and Merentes [Bounded Variation and Around, Studies in Nonlinear Analysis and Applications, Vol. 17, De Gruyter, Berlin/Boston, 2014].

1. INTRODUCTION

The classical total variation of $f : [a, b] \to \mathbb{R}$, defined as

$$\sup_{a=x_1<\ldots<x_n=b} \sum_{k=1}^{n-1} |f(x_{k+1}) - f(x_k)|,$$

is an intuitive way to measure the variation of a function in one dimension. Appell, Banas and Merentes provide many different versions of the variation, including what they call the Riesz–Medvedev variation ([2, Section 2.4], originally from [25]):

$$\sup_{a=x_1<\ldots<x_n=b} \sum_{k=1}^{n-1} \varphi\left(\frac{|f(x_{k+1}) - f(x_k)|}{|x_{k+1} - x_k|}\right) |x_{k+1} - x_k|.$$

When $\varphi(t) = t$, this reduces to the normal total variation, above. In 2016, Castillo, Guzmán and Rafeiro [7] generalized the Riesz–Medvedev variation to the variable exponent case. In this article, we further extend and improve their result to the generalized Orlicz case and answer a question regarding the Riesz–Medvedev variation by Appell, Banas and Merentes [2].

Generalized Orlicz spaces, also known as Musielak–Orlicz spaces, have been studied with renewed intensity recently [14, 23, 26, 28] as have related PDE [4, 5, 6, 9, 10, 17, 21, 29]. A contributing factor is that they cover both the variable exponent case $\varphi(x, t) := t^p(x)$ [11] and the double phase case $\varphi(x, t) := t^p + a(x)t^q$ [3], as well as their many variants: perturbed variable exponent, Orlicz variable exponent, degenerate double phase, Orlicz double phase, variable exponent double phase, triple phase and double variable exponent. For references see [19].

Chen, Levine and Rao [8] proposed a generalized Orlicz model for image restoration with the energy function

$$\varphi(x, t) := \begin{cases} \frac{1}{p(x)}t^p(x), & \text{when } t \leq 1, \\ t - 1 - \frac{1}{q(x)}, & \text{when } t > 1. \end{cases}$$

Based on input $u_0$, one seeks to minimize the sum of the regularization and the fidelity term:

$$\int_{\Omega} \varphi(x, |\nabla u|) + |u - u_0|^2 \, dx.$$
The variable exponent $p$ is chosen to be close to 1 in areas of potential edges in the image and close to 2 where no edges are expected. This allows us to avoid the so-called stair-casing effect whereby artificial edges are introduces in the image restoration process.

A feature of their functional is that $\varphi(x, t) \approx t$ for large $t$. Thus Chen, Levine and Rao could use the classical space $BV(\Omega)$ directly. Li, Li and Pi [24] suggested an image restoration model with variable exponent energy restricted away from 1, so that no $BV$-spaces are needed. In [15, 18], we considered pure variable exponent and double phase versions of this model with $p \to 1$. In this case, we cannot use the space $BV(\Omega)$, and are led to the regularization terms

$$\int_{\Omega} |\nabla f|^p(x) \, dx + |D^s f|(|\{p = 1\} \rangle \quad \text{and} \quad \int_{\Omega} |\nabla f| + a(x) |\nabla f|^q \, dx + |D^s f|(|\{a = 0\} \rangle,$$

where $\nabla f$ and $D^s f$ are the absolutely continuous and singular parts of the derivative, respectively. Here the singular part of the derivative (i.e. the edges in the image) is concentrated on the sets $\{p = 1\}$ or $\{a = 0\}$ and the exponent $p$ or coefficient $a$ should be chosen accordingly.

The papers [8, 15, 18] are all based on special structure of $\varphi$. The problem of defining a $BV$-type space based on generalized Orlicz growth has been recently attacked in [12, 13]. In this paper we show that the Riesz $\varphi$-variation gives the aforementioned energies in the variable exponent and double phase cases (Corollaries 1.2 and 1.3). This provides support for our formulation of the Riesz $\varphi$-variation as well as the generalized Orlicz growth models of image restoration.

To state some corollaries of the main result (Theorem 6.4) we define some variants of the Riesz $\varphi$-variation in generalized Orlicz spaces. For further definitions see the next section.

**Definition 1.1.** Let $I \subset \mathbb{R}$ be a closed interval, $\varphi \in \Phi_w(I)$ and $f : I \to \mathbb{R}$. We define the functional $V^\varphi_f : \mathbb{R}^I \to [0, \infty]$ by

$$V^\varphi_f := \sup_{(I_k)} V^\varphi(f, (I_k)) := \sup_{(I_k)} \sum_{k=1}^n \varphi^+_{I_k} \left( \frac{|\Delta_k f|}{|I_k|} \right) |I_k|,$$

where the supremum is taken over all partitions $(I_k)$ of $I$ by closed intervals (non-degenerate and with disjoint interiors). Here $\Delta_k f := \Delta f(I_k) := f(a_k) - f(b_k)$ for $I_k = [a_k, b_k]$. For a partition $(I_k)$ of $I$ we denote $|I_k| := \max\{|I_k|\}$ and using it we define

$$\nabla^\varphi f := \limsup_{|I_k| \to 0} \sum_{k=1}^n \varphi^+_{I_k} \left( \frac{|\Delta_k f|}{|I_k|} \right) |I_k| \quad \text{and} \quad \nabla^\varphi f := \limsup_{|I_k| \to 0} \sum_{k=1}^n \varphi^+_{I_k} \left( \frac{|\Delta_k f|}{|I_k|} \right) |I_k|.$$

When there is no dependence on $x$, all of these variants give the same end result (Lemma 5.1). However, with $x$-dependence, the variant with the limit superior gives a more precise result.

**Corollary 1.2** (Variable exponent). Let $f \in BV(I)$ be left-continuous. If $p : \Omega \to [1, \infty)$ is bounded and satisfies the strong log-Hölder condition

$$\lim_{x \to y} |p(x) - p(y)| \log(e + \frac{1}{|x-y|}) = 0,$$

uniformly in $y \in \Omega$, then, for $\varphi(x, t) := t^{p(x)}$, we have $|D^s f|(|\{p > 1\} \rangle = 0$ and

$$\nabla^\varphi f = \int_I |f'|^{p(x)} \, dx + |D^s f|(|\{p = 1\} \rangle.$$

**Corollary 1.3** (Double phase). Let $f \in BV(I)$ be left-continuous. If $a : \Omega \to [0, \infty)$ is bounded and $\alpha$-Hölder continuous with

$$q < 1 + \frac{\alpha}{n},$$

then, for $\varphi(x, t) := t^a$,
then, for \( \varphi(x, t) := t + a(x)t^p \), we have \( |D^s f|\{a > 0\} = 0 \) and

\[
\nabla^p_\varphi(f) = \int_a^t |f'| + a(x)|f'|^p \, dx + |D^s f|\{a = 0\}.
\]

We are also able to answer the question posed by Appell, Banaś and Merentes in [2, p. 168] with the help of the corollary for the Orlicz case.

**Corollary 1.4** (Orlicz). Let \( \varphi \in \Phi_c \) and \( K := \lim_{t \to \infty} \varphi(t)/t \). Suppose that \( V^\varphi_I(f) < \infty \).

1. If \( K < \infty \) and \( f \) is left-continuous, then \( f \in BV(I) \) and

\[
V^\varphi_I(f) = \int_I \varphi(|f'|) \, dx + K |D^s f|(I).
\]

2. If \( K = \infty \), then \( f \) is absolutely continuous and \( V^\varphi_I(f) = \varphi(|f'|) \).

To understand their question we recall Definition 2.11 from [2]:

\( \infty_p := \{ \varphi \in \Phi_c \mid \lim_{t \to \infty} \varphi(t)/t^p = \infty \} \).

In [2, Proposition 2.57] it is shown that \( V^\varphi_I \) and \( V^1_I \) generate the same space when \( \varphi \not\in \infty_1 \). We use the convention that when \( \varphi \) is replaced by a number \( p \), it indicates the case \( \varphi(x, t) = t^p \). Appell, Banaś and Merentes then ask whether \( V^\varphi_I \) and \( V^p_I \) generate the same space when \( \varphi \not\in \infty_p \) for some \( p > 1 \).

However, there is not a complete analogy between the cases \( p = 1 \) and \( p > 1 \). Since \( \varphi \) is convex and \( \varphi(0) = 0 \), the ratio \( \varphi(t)/t \) is increasing so its limit always exists. Thus \( \varphi \not\in \infty_1 \) is equivalent to \( K := \lim_{t \to \infty} \varphi(t)/t < \infty \). The same is not true when \( p > 1 \). In fact, \( \varphi \not\in \infty_p \) if and only if

\[
\lim \inf_{t \to \infty} \varphi(t)/t^p < \infty.
\]

However, this condition is satisfies for instance by \( \varphi(t) = t \) which generates the space \( BV(I) \) regardless of the value of \( p \).

In particular, this answers the question of Appell, Banaś and Merentes in the negative. From Corollary 1.4(2) we see by [11, Theorem 2.8.1] that \( V^\varphi_I \) and \( V^p_I \) generate the same space if and only if

\[
\frac{1}{c}t^p - c \leq \varphi(t) \leq ct^p + c
\]

for some constant \( c \geq 1 \).

The structure of the paper is as follows. In the next section we present necessary background information. In Section 3 we define functions of bounded Riesz \( \varphi \)-variation when \( \varphi : \Omega \times [0, \infty) \to [0, \infty) \) is a generalized \( \Phi \)-function (Definition 1.1), and show that the variation defines a quasi-seminorm. In Section 4 we show that the space of bounded Riesz \( \varphi \)-variation is complete. Then we consider variants of the definition of \( \varphi \)-variation in Section 5, in particular the effect of replacing \( \sup \) by \( \lim \sup \). Finally, in Section 6 we prove a Riesz representation lemma, which connects the Riesz \( \varphi \)-variation seminorm with the \( L^p \)-norm of the derivative of the function and prove the aforementioned corollaries. We do this by connecting the Riesz variation in these cases to modern \( BV \)-spaces as presented by Ambrosio, Fusco and Pallara in Section 3.2 of [1].
2. Preliminaries

We briefly introduce our assumptions. More information about \( L^\varphi \)-spaces can be found in [14]. Our previous works were based on conditions defined for almost every point \( x \in \Omega \). In this article we consider not equivalence classes of functions but the functions themselves, and so the following assumptions have been correspondingly adjusted.

We always use \( I = [a, b] \) to denote a closed interval with end-points \( a \) and \( b \). \textit{Almost increasing} means that a function satisfies \( f(s) \leq Lf(t) \) for all \( s < t \) and some constant \( L \geq 1 \). If there exists a constant \( c > 0 \) such that \( \frac{1}{c}g(x) \leq f(x) \leq cg(x) \) for every \( x \), then we write \( f \approx g \). Two functions \( \varphi \) and \( \psi \) are \textit{equivalent}, \( \varphi \simeq \psi \), if there exists \( L \geq 1 \) such that \( \psi(x, \frac{1}{L}t) \leq \varphi(x, t) \leq \psi(x, Lt) \) for every \( x \in \Omega \) and every \( t > 0 \). Equivalent \( \Phi \)-functions give rise to the same space with comparable norms. By \( \beta \) we indicate a generic positive constant whose value may change between appearances. By \( \beta \) we indicate a parameter from \((0,1)\) which may appear in several assumptions; since the assumptions are all monotone in \( \beta \), there is no loss of generality in assuming the same \( \beta \).

2.1. \( \Phi \)-functions.

\textbf{Definition 2.1.} We say that \( \varphi : \Omega \times [0, \infty) \to [0, \infty) \) is a \textit{weak \( \Phi \)-function}, and write \( \varphi \in \Phi_w(\Omega) \), if the following conditions hold:

- For every measurable function \( f : \Omega \to \mathbb{R} \), the function \( x \mapsto \varphi(x, f(x)) \) is measurable.
- For every \( x \in \Omega \), the function \( t \mapsto \varphi(x, t) \) is non-decreasing.
- For every \( x \in \Omega \), \( x(0,0) = \lim_{t \downarrow 0^+} \varphi(x, t) = 0 \) and \( \lim_{t \to \infty} \varphi(x, t) = \infty \).
- For every \( x \in \Omega \), the function \( t \mapsto \frac{\varphi(x,t)}{t} \) is \( L \)-almost increasing on \((0, \infty)\) with \( L \) independent of \( x \).

If \( \varphi \in \Phi_w(\Omega) \) is additionally convex and left-continuous, then \( \varphi \) is a \textit{convex \( \Phi \)-function}, and we write \( \varphi \in \Phi_c(\Omega) \). If \( \varphi \) does not depend on \( x \), then we omit the set and write \( \varphi \in \Phi_w \) or \( \varphi \in \Phi_c \).

We denote \( \varphi_A^+(t) := \sup_{x \in A \cap \Omega} \varphi(x, t) \) and \( \varphi_A^-(t) := \inf_{x \in A \cap \Omega} \varphi(x, t) \). We say that \( \varphi \) (or \( \varphi_A^\pm \)) is non-degenerate if \( \varphi_A^+ \), \( \varphi_A^- \in \Phi_w \); if \( \varphi \) is degenerate, then \( \varphi_A^+|_{(0,\infty)} \equiv 0 \) or \( \varphi_A^-|_{(0,\infty)} \equiv \infty \). We define several conditions. Let \( p, q > 0 \). We say that \( \varphi : \Omega \times [0, \infty) \to [0, \infty) \) satisfies

(A0) if there exists \( \beta \in (0, 1) \) such that \( \varphi(x, \beta) \leq 1 \leq \varphi(x, \frac{1}{\beta}) \) for all \( x \in \Omega \);

(A1) if for every \( K > 0 \) there exists \( \beta \in (0, 1) \) such that, for every ball \( B \) and \( x, y \in B \cap \Omega \),

\[ \varphi(x, \beta t) \leq \varphi(y, t) + 1 \quad \text{when} \quad \varphi(y, t) \in \left[ 0, \frac{K}{|B|} \right] ; \]

(VA1) if for every \( K > 0 \) there exists a modulus of continuity \( \omega \) such that, for every ball \( B = B_r \) and \( x, y \in B \cap \Omega \),

\[ \varphi(x, \frac{t}{1+\omega(r)}) \leq \varphi(y, t) + \omega(r) \quad \text{when} \quad \varphi(y, t) \in \left[ 0, \frac{K}{|B|} \right] ; \]

(\text{aInc}) \( p \) if \( t \mapsto \frac{\varphi(x,t)}{t^p} \) is \( L_p \)-almost increasing in \((0, \infty)\) for some \( L_p \geq 1 \) and all \( x \in \Omega \);

(\text{aDec}) \( q \) if \( t \mapsto \frac{\varphi(x,t)}{t^q} \) is \( L_q \)-almost decreasing in \((0, \infty)\) for some \( L_q \geq 1 \) and all \( x \in \Omega \).

We say that (\text{aInc}) holds if (\text{aInc}) \( p \) holds for some \( p > 1 \), and similarly for (\text{aDec}). If in the definition of (\text{aInc}) \( p \), we have \( L_p = 1 \), then we say that \( \varphi \) satisfies (\text{Inc}) \( p \), similarly for (\text{Dec}) \( q \).

\textbf{Example 2.2} (Variable exponent growth). Let \( p : \Omega \to [1, \infty) \) and let \( \varphi(x, t) := t^{p(x)} \) be the variable exponent functional with \( p^\infty := \infty \chi_{(1,\infty)}(t) \). It was shown in [14, Proposition 7.1.2] that \( \varphi \) satisfies (A1) if and only if

\[ \left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \leq \frac{c}{\log(e + \frac{1}{|x-y|})} , \]
Note that this result does not require $p$ to be bounded. One can show that $\varphi$ satisfies (VA1) if
\[
\left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \leq \omega(|x - y|) \log(e + \frac{1}{|x - y|})
\]
where $\omega$ is a function with $\lim_{r \to 0^+} \omega(r) = 0$.

It is easy to see that the variable exponent $\Phi$-function $\varphi$ satisfies (Inc)$_{p^-}$ and (Dec)$_{p^+}$ provided $p^- \leq p(x) \leq p^+$ for every $x \in \Omega$ and that (A0) always holds.

The stronger continuity condition (VA1) (“vanishing (A1)”) was introduced in [19, 20]. If $\varphi$ satisfies (Dec), then (VA1) implies that $\varphi$ is continuous. The condition also allows us to get sharper estimates. An example is the following Jensen’s inequality with constant close to $1$ that is needed later on. Without (VA1), the inequality only holds for some $\beta > 0$ which need not be small since $\varphi^-$ is not convex [14, Corollary 2.2.2].

**Theorem 2.3** (Jensen’s inequality). If $\varphi \in \Phi_c(\Omega)$ satisfies (VA1) and $B = B_r \subset \Omega$, then there exists $L > 0$ such that
\[
\varphi_B \left( \frac{1}{1 + \omega(r)} \int_B |f| \, dx \right) \leq \int_B \varphi(x, f) \, dx + \omega(r),
\]
for $\varrho_\varphi(Lf) \leq 1$.

**Proof.** Let $t_0 := \|f\|_{L^1(B)} / |B|$. Since $\varphi_B$ is equivalent to a convex function [14, Lemma 2.2.1], we find that
\[
\varphi_B^{-}(t_0) = \varphi_B \left( \int_B |f| \, dx \right) \leq \int_B \varphi_B^{-}(L|f|) \, dx \leq \frac{1}{|B|},
\]
where the constant $L$ is determined by the equivalence. This is a Jensen inequality, but the constant does not approach one for small balls. We obtain $\omega$ from (VA1) and define $\beta_r := \frac{1}{1 + \omega(r)}$. We denote by $\varphi'$ a function, non-decreasing in $s$, such that
\[
\varphi(x, t) = \int_0^t \varphi'(x, s) \, ds.
\]
Such function exists since $\varphi$ is convex in the second variable. Fix $x_0 \in B$ with
\[
\varphi'(x_0, \beta_r t_0) \leq (\varphi')_B^{-}(t_0)
\]
and define
\[
\psi(t) := \int_0^t \varphi'(x_0, \min\{s, \beta_r t_0\}) \, ds.
\]
Since $\psi'$ is increasing, $\psi$ is convex. Furthermore, $\psi(t) = \varphi(x_0, t)$ when $t \leq \beta_r t_0$. It follows from Jensen’s inequality that
\[
\varphi_B^- \left( \beta_r \int_B |f| \, dx \right) \leq \psi \left( \beta_r \int_B |f| \, dx \right) \leq \int_B \psi(\beta_r |f|) \, dx.
\]
When $t \leq t_0$ we use (VA1) to conclude that $\psi(\beta_r t) = \varphi(x_0, \beta_r t) \leq \varphi(x, t) + \omega(r)$. When $t > t_0$ we estimate
\[
\psi(\beta_r t) = \psi(\beta_r t_0) + \varphi'(x_0, \beta_r t_0) \beta_r (t - t_0) \leq \varphi(x, t_0) + \omega(r) + (\varphi')_B^{-}(t_0) \beta_r (t - t_0)
\]
\[
\leq \varphi(x, t_0) + \omega(r) + \varphi'(x, t_0) (t - t_0) \leq \varphi(x, t) + \omega(r),
\]
where we also used the convexity of $\varphi$ in the last step. Thus
\[
\varphi_B^- \left( \beta_r \int_B |f| \, dx \right) \leq \int_B \psi(\beta_r |f|) \, dx \leq \int_B \varphi(x, |f|) \, dx + \omega(r). \]
\]
2.2. Modular spaces. The following results are from [16]; the proofs follow [11, 22, 27].

**Definition 2.4.** Let $X$ be a real vector space. A function $\varrho : X \to [0, +\infty]$ is called a quasi-semimodular on $X$ if:

1. the function $\lambda \mapsto \varrho(\lambda x)$ is increasing on $[0, \infty)$ for every $x \in X$;
2. $\varrho(0_x) = 0$;
3. $\varrho(-x) = \varrho(x)$ for every $x \in X$;
4. there exists $\beta \in (0, 1]$ such that $\varrho(\beta(\theta x + (1-\theta)y)) \leq \theta \varrho(x) + (1-\theta)\varrho(y)$ for every $x, y \in X$ and every $\theta \in [0, 1]$.

If (4) holds with $\beta = 1$, then $\varrho$ is a semimodular.

If $\varrho$ is a quasi-semimodular in $X$, then the set defined by

$$X_\varrho := \{ x \in X \mid \lim_{\lambda \to 0} \varrho(\lambda x) = 0 \}$$

is called a modular space. We define the Luxemburg quasi-seminorm on $X_\varrho$ by

$$\|x\|_\varrho := \inf \left\{ \lambda > 0 \mid \varrho\left(\frac{x}{\lambda}\right) \leq 1 \right\}.$$  

Note that our terminology differs from Musielak [27]. Our justification comes from the correspondence with standard terminology for norms, as demonstrated in the following proposition.

**Proposition 2.5.** Let $X$ be a real vector space.

1. If $\varrho$ is a quasi-semimodular in $X$, then $\| \cdot \|_\varrho$ is a quasi-seminorm.
2. If $\varrho$ is a semimodular in $X$, then $\| \cdot \|_\varrho$ is a seminorm.
3. If $\varrho$ is a quasi-modular in $X$, then $\| \cdot \|_\varrho$ is a quasi-norm.
4. If $\varrho$ is a modular in $X$, then $\| \cdot \|_\varrho$ is a norm.

The next proposition contains the main properties that we need regarding modular spaces.

**Lemma 2.6** ([16]). Let $X$ be a real vector space, $\varrho$ be a quasi-semimodular on $X$ and $x \in X$. Denote by $\beta$ the constant in property (4) of Definition 2.4. Then

1. $\|x\|_\varrho < 1 \implies \varrho(x) \leq 1 \implies \|x\|_\varrho \leq 1$;
2. $\|x\|_\varrho < 1 \implies \beta \varrho(x) \leq \|x\|_\varrho$;
3. $\|x\|_\varrho > 1 \implies \varrho(x) \geq \beta \|x\|_\varrho$;
4. $\|x\|_\varrho \leq \beta^{-1} \varrho(x) + 1$.

As special cases we have generalized Orlicz and Orlicz–Sobolev spaces.

**Definition 2.7.** Let $\varphi \in \Phi_w(\Omega)$ and define the quasi-semimodular $\varrho_\varphi$ for $f \in L^0(\Omega)$, the set of measurable functions in $\Omega$, by

$$\varrho_\varphi(f) := \int_\Omega \varphi(x, |f(x)|) \, dx.$$  

The generalized Orlicz space, also called Musielak–Orlicz space, is defined as the set

$$L^\varphi(\Omega) := (L^0(\Omega))_{\varrho_\varphi} = \{ f \in L^0(\Omega) \mid \lim_{\lambda \to 0^+} \varrho_\varphi(\lambda f) = 0 \}$$

equipped with the (Luxemburg) quasi-seminorm

$$\|f\|_{L^\varphi(\Omega)} := \|f\|_{\varrho_\varphi} = \inf \left\{ \lambda > 0 \mid \varrho_\varphi\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$  

If the set is clear from context we abbreviate $\|f\|_{L^\varphi(\Omega)}$ by $\|f\|_\varphi$.

**Definition 2.8.** A function $f \in L^\varphi(\Omega)$ belongs to the Orlicz–Sobolev space $W^{1,\varphi}(\Omega)$ if its weak derivative $f'$ exists and belongs to $L^\varphi(\Omega)$. For $f \in W^{1,\varphi}(\Omega)$, we define the norm

$$\|f\|_{W^{1,\varphi}(\Omega)} := \|f\|_{L^\varphi(\Omega)} + \|f'\|_{L^\varphi(\Omega)}.$$
3. BOUNDED VARIATION IN THE RIESZ SENSE

We introduce the space of bounded Riesz \( \varphi \)-variation based on \( V^\varphi_I \) from the introduction. Note that we do not assume that \( f \) is measurable.

**Definition 3.1.** Let \( I \subset \mathbb{R} \) be a closed interval and \( \varphi \in \Phi_w(I) \). The space of *bounded \( \varphi \)-variation in Riesz’ sense* is defined by the quasi-semimodular \( V^\varphi_I \):

\[
\text{RBV}^\varphi(I) := \{ f : I \to \mathbb{R} \mid \lim_{\lambda \to 0} V^\varphi_I(\lambda f) = 0 \}.
\]

Often two \( \Phi \)-functions \( \varphi, \psi \in \Phi_w(I) \) are considered to be the same, if \( \varphi(x, t) = \psi(x, t) \) for almost everywhere \( x \) and every \( t \). In our setting we cannot use this convention, as the following example demonstrates.

**Example 3.2.** Let \( (A_j)_{j=1}^\infty \) be a sequence of pairwise disjoint, countable and dense subsets of \( I \). Denote \( A := \bigcup_{j=1}^\infty A_j \). Then \( |A| = 0 \) since \( A \) is countable. Define \( \varphi, \psi : I \times [0, \infty) \to [0, \infty] \) by \( \varphi(x, t) := t^2 \) and \( \psi(x, t) := \begin{cases} t^2, & \text{if } x \in I \setminus A, \\ j t^2, & \text{if } x \in A_j \text{ for some } j \in \mathbb{N}. \end{cases} \)

Then \( \psi(x, t) = \varphi(x, t) \) for every \( x \in I \setminus A \) and every \( t \geq 0 \), so that \( \varphi = \psi \) a.e. Let \( f : I \to \mathbb{R} \) be the identity function and let \( (I_k) \) be any partition of \( I \). Then \( |\Delta_k f|/|I_k| = 1 \) for every \( k \). Since the sets \( A_j \) are dense, the intersection \( A_j \cap I_k \) is non-empty for every \( j \) and every \( k \). Thus \( \psi^\varphi_{I_k}(1) \geq j \) for every \( j \), and therefore

\[
V^\psi_I(f) \geq \sum_{k=1}^n \psi^\varphi_{I_k} \left( \frac{|\Delta_k f|}{|I_k|} \right) |I_k| \geq \sum_{k=1}^n j |I_k| = j |I|.
\]

Letting \( j \to \infty \), we see that \( V^\psi_I(f) = \infty \). On the other hand,

\[
\sum_{k=1}^n \varphi^\varphi_{I_k} \left( \frac{|\Delta_k f|}{|I_k|} \right) |I_k| = \sum_{k=1}^n \varphi^\varphi_{I_k}(1) |I_k| = \sum_{k=1}^n |I_k| = |I|.
\]

This implies that \( V^\varphi_I(f) = |I| \).

Next, we show that equivalent \( \Phi \)-functions give rise to the same space of bounded variation.

**Lemma 3.3.** Suppose that \( \varphi, \psi \in \Phi_w(I) \) and \( \varphi \simeq \psi \) with constant \( L \geq 1 \). Then

\[
V^\varphi_I(L^{-1} f) \leq V^\psi_I(f) \leq V^\psi_I(L f)
\]

for every \( f : I \to \mathbb{R} \) and \( \text{RBV}^\varphi(I) = \text{RBV}^\psi(I) \).

**Proof.** By the definition of equivalence,

\[
\psi(x, L^{-1} t) \leq \varphi(x, t) \leq \psi(x, L t)
\]

for every \( x \in I \) and every \( t \in [0, \infty) \). Thus

\[
V^\varphi_I(f) = \sup_{(I_k)} \sum_{k=1}^n \varphi^\varphi_{I_k} \left( \frac{|\Delta_k f|}{|I_k|} \right) |I_k| \leq \sup_{(I_k)} \sum_{k=1}^n \psi^\varphi_{I_k} \left( L \frac{|\Delta_k f|}{|I_k|} \right) |I_k| = V^\psi_I(L f).
\]

Similarly \( V^\psi_I(L^{-1} f) \leq V^\varphi_I(f) \). To see that \( \text{RBV}^\varphi(I) = \text{RBV}^\psi(I) \), we note that if \( V^\varphi_I(\lambda f) \to 0 \), then \( V^\psi_I(L^{-1} \lambda f) \to 0 \) and if \( V^\psi_I(\lambda f) \to 0 \), then \( V^\varphi_I(L^{-1} \lambda f) \to 0 \).

We now show that \( V^\varphi_I \) is a quasi-semimodular.
Lemma 3.4. Let \( \varphi \in \Phi_w(I) \). Then \( V^\varphi_I : \mathbb{R}^I \to \mathbb{R}_+ \) is a quasi-semimodular. If \( \varphi \in \Phi_c(I) \), then \( V^\varphi_I \) is a semimodular.

**Proof.** The properties \( V^\varphi_I(0) = 0 \) and \( V^\varphi_I(-f) = V^\varphi_I(f) \) are clear. Since \( t \mapsto \varphi(x, t) \) is increasing for every \( x \in I \) it follows that \( t \mapsto \varphi_A^+(t) \) is increasing whenever \( A \subset I \). Since

\[
V^\varphi_I(\lambda f) = \sup_{(I_k)} \sum_{k=1}^n \varphi^+_k \left( \frac{\lambda |\Delta_k f|}{|I_k|} \right) |I_k|,
\]

the function \( \lambda \mapsto V^\varphi_I(\lambda f) \) is increasing on \([0, \infty)\) for every \( f : I \to \mathbb{R} \).

Note that \( \varphi_A^+ \) satisfies (aInc) with the same constant \( L \geq 1 \) as \( \varphi \). Let \( (I_k) \) be a partition of \( I \). If \( \theta \in [0, 1] \), then

\[
\varphi^+_k \left( \frac{1}{2L} |\Delta_k (\theta f + (1 - \theta)g)| \right) \leq \varphi^+_k \left( \frac{\theta |\Delta_k f|}{2L |I_k|} + \frac{1 - \theta |\Delta_k g|}{2L |I_k|} \right)
\]

\[
\leq \varphi^+_k \left( \frac{\theta |\Delta_k f|}{L |I_k|} \right) + \varphi^+_k \left( \frac{1 - \theta |\Delta_k g|}{L |I_k|} \right)
\]

\[
\leq \theta \varphi^+_k \left( \frac{|\Delta_k f|}{|I_k|} \right) + (1 - \theta) \varphi^+_k \left( \frac{|\Delta_k g|}{|I_k|} \right),
\]

where the last inequality follows from (aInc). This implies that \( V^\varphi_I(\beta(\theta f + (1 - \theta)g)) \leq \theta V^\varphi_I(f) + (1 - \theta) V^\varphi_I(g) \) with \( \beta := \frac{1}{2L} \).

If \( \varphi \in \Phi_c(I) \), then \( \varphi_A^+ \) is convex as the supremum of convex functions. Thus

\[
\varphi^+_k \left( \frac{|\Delta_k (\theta f + (1 - \theta)g)|}{|I_k|} \right) \leq \varphi^+_k \left( \frac{\theta |\Delta_k f|}{|I_k|} + (1 - \theta) \frac{|\Delta_k g|}{|I_k|} \right)
\]

\[
\leq \theta \varphi^+_k \left( \frac{|\Delta_k f|}{|I_k|} \right) + (1 - \theta) \varphi^+_k \left( \frac{|\Delta_k g|}{|I_k|} \right),
\]

and it follows that \( V^\varphi_I \) is convex. \( \square \)

By Proposition 2.5, we can define the Luxemburg quasi-seminorm in \( \text{RBV}^\varphi(I) \) by

\[
\|f\|_{\text{RBV}^\varphi(I)} = \inf \left\{ \lambda > 0 \mid V^\varphi_I \left( \frac{f}{\lambda} \right) \leq 1 \right\}.
\]

This is not a quasi-norm, as is easily seen by considering constant functions: if \( f \) is constant, then \( V^f_I(f/\lambda) = 0 \) for every \( \lambda > 0 \) (since \( \Delta_k f = 0 \) for every partition and every \( k \)), and therefore \( \|f\|_{\text{RBV}^\varphi(I)} = 0 \). It is also easy to see that \( V^\varphi_I(f) = V^\varphi_I(g) \) whenever \( f - g \) is constant. Thus, in addition to \( \text{RBV}^\varphi(I) \), we also consider the sub-space

\[
\text{RBV}^\varphi_0([a,b]) := \{ f \in \text{RBV}^\varphi([a,b]) \mid f(a) = 0 \}.
\]

Then \( \text{RBV}^\varphi_0(I) \) is a quasi-normed space with the quasi-norm \( \| \cdot \|_{\text{RBV}^\varphi(I)} \), as we will see in the next theorem.

**Theorem 3.5.** Let \( \varphi \in \Phi_w(I) \). Then \( \text{RBV}^\varphi_0(I) \) is a quasi-normed space which is non-trivial if and only if \( \varphi_A^+ \) is non-degenerate.

**Proof.** By Proposition 2.5 and Lemma 3.4, \( \text{RBV}^\varphi_0(I) \) is a quasi-seminormed space. For \( \text{RBV}^\varphi_0(I) \) to be a quasi-normed space, we check that \( \|f\|_{\text{RBV}^\varphi_0(I)} = 0 \) only if \( f = 0 \). This is equivalent to the condition that \( \|f\|_{\text{RBV}^\varphi(I)} \) is non-zero for non-constant \( f \) since \( f(a) = 0 \).
Let $f \in \text{RBV}^{\varphi}(I)$ be an arbitrary non-constant function. Then we can choose a partition $(I_k)$ of $I$ such that $\Delta_{k_0} f \neq 0$ for some $k_0$. Note that $\lim_{t \to \infty} \varphi_{k_0}^+(t) = \infty$ by the definition of $\Phi_w(I)$. Hence, by the definition of $V^\varphi_I$, we get that

$$\lim_{\lambda \to 0^+} V^\varphi_I \left( \frac{f}{\lambda} \right) \geq \lim_{\lambda \to 0^+} \sum_{k=1}^{n} \varphi^+_k \left( \frac{\left| \Delta_k f \right|}{\lambda |I_k|} \right) |I_k| \geq \lim_{\lambda \to 0^+} \varphi^+_k \left( \frac{|\Delta_{k_0} f|}{\lambda |I_{k_0}|} \right) |I_{k_0}| = \infty.$$ 

Thus $V^\varphi_I (f/\lambda) > 1$ when $\lambda$ is small enough, and therefore $\|f\|_{\text{RBV}^\varphi(I)} > 0$.

We now prove the claim concerning the non-triviality of $\text{RBV}^\varphi_0(I)$. Suppose first that $\varphi^+_I$ is degenerate. Let $f$ be a non-constant function. Let us show that $V^\varphi_I (f) = \infty$. If $f(a) \neq f(b)$, then $|\Delta f(I)|/|I| > 0$, and

$$V^\varphi_I (f) \geq \varphi^+_I \left( \frac{|\Delta f(I)|}{|I|} \right) |I| = \infty.$$ 

If $f(a) = f(b)$, then there exists $c \in I$ with $f(c) \neq f(a)$, since $f$ is not constant. Let $I_1 := [a, c]$ and $I_2 := [c, b]$. Since $\varphi^+_I$ is degenerate, it follows that at least one of $\varphi^+_{I_1}$ or $\varphi^+_{I_2}$ must also be degenerate. Since $|\Delta_k f|/|I_k| > 0$, $k = 1, 2$, we get that

$$V^\varphi_I (f) \geq \sum_{k=1}^{2} \varphi^+_k \left( \frac{|\Delta_k f|}{|I_k|} \right) |I_k| = \infty.$$ 

Thus $V^\varphi_I (\lambda f) = \infty$ for every non-constant function $f$ and every $\lambda \in (0, \infty)$. Hence $\text{RBV}^\varphi(I)$ is just the space of constant functions, which implies that $\text{RBV}^\varphi_0(I)$ is trivial.

Suppose then that $\varphi^+_I$ is non-degenerate. Thus there exists $t_0 \in (0, \infty)$ with $\varphi^+_I (t_0) < \infty$. Let $g(x) := t_0 (x - a)$. If $(I_k)$ is any partition of $I$, then

$$\sum_{k=1}^{n} \varphi^+_k \left( \frac{|\Delta_k g|}{|I_k|} \right) |I_k| = \sum_{k=1}^{n} \varphi^+_k (t_0) |I_k| \leq \sum_{k=1}^{n} \varphi^+_k (t_0) |I_k| = \varphi^+_I (t_0) |I| < \infty,$$

which, together with \text{(aInc)}$_1$, implies that $g \in \text{RBV}^\varphi(I)$. Since $g(a) = 0$, it follows that $g \in \text{RBV}^\varphi_0(I)$. Thus $\text{RBV}^\varphi_0(I)$ is non-trivial. \hfill \Box

### 4. Completeness

In this section, we show the completeness of the space of bounded Riesz $\varphi$-variation.

**Lemma 4.1.** Let $\varphi \in \Phi_w(I)$ and $L$ be the constant from \text{(aInc)}$_1$. Then, for $\alpha \geq 0$ and $\beta > 0$,

$$V^\varphi_I \left( \frac{f}{\beta} \right) \leq \alpha \iff \|f\|_{\text{RBV}^\varphi(I)} \leq \begin{cases} L \alpha \beta, & \alpha > 1, \\ \beta, & \alpha \leq 1. \end{cases}$$

**Proof:** The case $\alpha \leq 1$ follows immediately from the definition of norm.

Let now $\alpha > 1$. Since $\varphi^+_I$ satisfies \text{(aInc)}$_1$ and $\varphi^+_I (0) = 0$, we have

$$\varphi^+_I \left( \frac{|\Delta_k f|}{L \alpha \beta |I_k|} \right) \leq \frac{1}{\alpha} \varphi^+_I \left( \frac{|\Delta_k f|}{\beta |I_k|} \right)$$

for each sub-interval of $I$, from which we obtain

$$V^\varphi_I \left( \frac{f}{L \alpha \beta} \right) \leq \frac{1}{\alpha} V^\varphi_I \left( \frac{f}{\beta} \right) \leq 1.$$ 

The result now follows from the definition of the norm. \hfill \Box
Lemma 4.2. Let \( \varphi, \psi \in \Phi_\circ(I) \). Suppose that there exists \( K > 0 \) such that
\[
\varphi(x, \frac{t}{K}) \leq \varphi(x, t) + 1.
\]
Then
\[
\text{RBV}^{\varphi}(I) \hookrightarrow \text{RBV}^{\psi}(I).
\]

Proof. Let \( \lambda_\varepsilon := \|f\|_{\text{RBV}^\varphi(I)} + \varepsilon \). Then \( V^\varphi_f(f/\lambda_\varepsilon) \leq 1 \). From the assumption and the definition of the modular, we have \( V^\varphi_f(f/\lambda_\varepsilon K) \leq V^\varphi_f(f/\lambda_\varepsilon) + |I| \leq 1 + |I| \). By Lemma 4.1,
\[
\|f\|_{\text{RBV}^\varphi(I)} \leq LK(1 + |I|)\lambda_\varepsilon = LK(1 + |I|)(\|f\|_{\text{RBV}^\varphi(I)} + \varepsilon),
\]
which concludes the proof.

We now show that \( \text{RBV}^{\varphi}(I) \) contains all Lipschitz functions and is contained in the set of absolutely continuous functions when \( \varphi \) satisfies suitable conditions. We denote by \( \text{Lip}(I) \) the space of Lipschitz functions on \( I \) and by \( \text{AC}(I) \) the space of absolutely continuous functions in \( I \).

Lemma 4.3. Let \( \varphi \in \Phi_\circ(I) \) satisfy (A0), (aInc) and (aDec). Then
\[
\text{Lip}(I) \subseteq \text{RBV}^{\varphi}(I) \subseteq \text{AC}(I).
\]

Proof. From [2, Proposition 2.52], we know that
\[
\text{Lip}(I) \subseteq \text{RBV}^s(I) \subseteq \text{AC}(I),
\]
with \( s \in (1, \infty) \), where \( \text{RBV}^s(I) \) corresponds to the choice \( \varphi(x, t) = t^s \). When \( t \geq \frac{1}{\beta} \), it follows from (aInc) that \((\beta t)^p \varphi(x, \frac{1}{\beta}) \leq L \varphi(x, t)\) and so it follows from (A0) that
\[
(\beta t)^p \leq L \varphi(x, t) + 1.
\]
From (aDec), we conclude that \( \varphi(x, \beta t) \leq L^{nt} \varphi(x, \beta) \) when \( t \geq 1 \). Hence (A0) implies that
\[
\varphi(x, \beta t) \leq L^{nt} + 1.
\]
By Lemma 4.2 and these inequalities, we conclude that
\[
\text{Lip}(I) \subseteq \text{RBV}^\varphi(I) \subseteq \text{RBV}^p(I) \subseteq \text{RBV}^p(I) \subseteq \text{AC}(I).
\]

Theorem 4.4. If \( \varphi \in \Phi_\circ(I) \), then \( \text{RBV}^{\varphi}_0(I) \) is a quasi-Banach space.

Proof. Fix \( I = [a, b] \). We first prove that
\[
(4.5) \quad \sup_{I} |f| \leq C \|f\|_{\text{RBV}^\varphi(I)}
\]
when \( f \in \text{RBV}^\varphi_0(I) \). Denote \( I_1 = [a, x] \). Since \( f(a) = 0 \), it follows from \( \Delta f(I_1) = f(x) \) that
\[
\varphi\left(a, \frac{|f(x)|}{|I_1|}\right) |I_1| \leq \varphi_{i_1}^+( \frac{\Delta I_1 f}{|I_1|}) |I_1| \leq V^\varphi_f(f).
\]
We apply this inequality to the function \( \frac{f}{\|f\|_{\text{RBV}^\varphi(I)} + \varepsilon} \) where \( \varepsilon > 0 \). Thus
\[
\varphi\left(a, \frac{|f(x)|}{|I_1|(|f|_{\text{RBV}^\varphi(I)} + \varepsilon)}\right) \leq V^\varphi_f\left(\frac{f}{\|f\|_{\text{RBV}^\varphi(I)} + \varepsilon}\right) \leq 1.
\]
Since \( \varphi(a, t) \to \infty \) as \( t \to \infty \), we conclude that the argument of \( \varphi \) on the left-hand side is bounded. Thus
\[
|f(x)| \leq C |I| (\|f\|_{\text{RBV}^\varphi(I)} + \varepsilon).
\]
The estimate (4.5) now follows as \( \varepsilon \to 0^+ \).

Let \( (f_i) \) be a Cauchy sequence in \( \text{RBV}^\varphi_0(I) \). By (4.5), it is also a Cauchy sequence in \( L^\infty(I) \), which ensures that \( f := \lim_{i \to \infty} f_i \) exists. For \( \varepsilon > 0 \), there exists \( N_\varepsilon \in \mathbb{N} \) such that \( i, j > N_\varepsilon \).
implies $V^\psi_f\left(\frac{f_i-f_j}{L^\psi_f}\right) \leq 1$, since $\|f_i-f_j\|_{RBV^\psi_f(I)} < \varepsilon$. By [14, Lemma 2.2.1] there exists $\psi \in \Phi_c(I)$ with $\psi \simeq \varphi$. Since $\psi$ is left-continuous, $\psi_{k+1}^-$ is left-continuous (see comment after the proof of [14, Lemma 2.1.8]) and lower semicontinuous by [14, Lemma 2.1.5]. Taking this into account, we get

$$V^\psi\left(\frac{f_i-f_j}{L^\varepsilon}, (I_k)\right) = \sup \left\{ V^\psi\left(\frac{f_i-f_j}{L^\varepsilon}, (I_k)\right) \right\} \leq \lim_{j \to \infty} V^\psi\left(\frac{f_i-f_j}{L^\varepsilon}, (I_k)\right).$$

When $i > N_\varepsilon$, it follows from Lemma 3.3 that

$$V^\psi\left(\frac{f_i-f_j}{L^\varepsilon}, (I_k)\right) \leq V^\psi\left(\frac{f_i-f_j}{L^\varepsilon}, (I_k)\right) \leq \lim_{j \to \infty} V^\psi\left(\frac{f_i-f_j}{L^\varepsilon}, (I_k)\right) \leq \lim_{j \to \infty} V^\psi\left(\frac{f_i-f_j}{L^\varepsilon}, (I_k)\right) \leq 1.$$

Taking the supremum over partitions $(I_k)$, we find that

$$V^\psi_f\left(\frac{f_i-f_j}{L^\varepsilon}, (I_k)\right) \leq 1$$

and so $\|f_i-f_j\|_{RBV^\psi_f(I)} \leq L^\varepsilon$. Thus $f \in RBV^\phi_f(I)$ and $(f_i)$ converges to $f$ in $RBV^\phi_f(I)$.

5. Variants of the Variation

In Definition 1.1 we defined $V^\psi_f$, $\nabla^\psi f$ and $\nabla^\phi f$ by taking supremum or limit superior of partitions and using either $\varphi^+$ or $\varphi^-$. In this section, we consider the impact of these choices. Since all the partitions in $\nabla^\psi f$ are allowed in the supremum in $V^\psi_f$ and $\varphi^- f_k \leq \varphi^+ f_k$, we see that

$$\nabla^\psi f \leq \nabla^\psi f \leq \nabla^\psi f.$$

When $\varphi$ is convex and independent of $x$, the opposite inequalities hold, as well. Indeed, $\nabla^\psi f = \nabla^\psi f(f)$ is trivial in this case.

**Lemma 5.1.** If $\varphi \in \Phi_c$, then

$$V^\psi_f(f) = \nabla^\psi f(f).$$

**Proof.** Let $(I_k)$ be a partition of $I$ and $(I_k')$ be its subpartition. Suppose $I_k = \bigcup_{j=j_k}^{j_{k+1}-1} I'_j$. Then

$$\frac{|\Delta f(I_k)|}{|I_k|} \leq \sum_{j=j_k}^{j_{k+1}-1} \frac{|\Delta f(I'_j)|}{|I'_j|} = \sum_{j=j_k}^{j_{k+1}-1} \frac{|I'_j|}{|I'_j|} \frac{|\Delta f(I'_j)|}{|I'_j|}.\frac{\varphi}{|I'_j|}$$

Since the coefficients’ sum equals 1, it follows from convexity that

$$\varphi\left(\frac{|\Delta f(I_k)|}{|I_k|}\right) |I_k| \leq \sum_{j=j_k}^{j_{k+1}-1} |I'_j| \varphi\left(\frac{|\Delta f(I'_j)|}{|I'_j|}\right) |I'_j| = \sum_{j=j_k}^{j_{k+1}-1} \varphi\left(\frac{|\Delta f(I'_j)|}{|I'_j|}\right) |I'_j|.$$

Thus $V^\psi_f(f, (I_k')) \leq V^\psi_f(f, (I'_j))$. Since this holds for any subpartition, we see that we can always move to a partition with half as large $|\langle I_k \rangle|$ but no smaller $V^\psi_f(f, (I_k)).$ Hence $V^\psi_f \leq V^\psi_f.$

We just showed that $\nabla^\psi f = V^\psi f$ when $\varphi \in \Phi_c$. However, if $\varphi$ depends on $x$, then it is possible that $\nabla^\psi f < V^\psi f$.

**Example 5.2.** Let $I := [0, 1]$ and define $\varphi \in \Phi_c(I)$ by $\varphi(x, t) := (x + 1)t^2$. Note that $\varphi$ satisfies (A0), (A1), (alnc)$_2$ and (aDec)$_2$. Let $f$ be the identity function. Then

$$V^\psi_f(f) \geq \varphi^+\left(\frac{|\Delta f(I)|}{|I|}\right) |I| = \varphi^+(1) = 2.$$
Let next \( \epsilon \in (0, 1) \) and let \((I_k)\) be a partition with \(|(I_k)| < \epsilon\). If \( I_k = [a_k, b_k] \), then
\[
\varphi(a_k, \frac{|\Delta_k f|}{|I_k|}) \leq \varphi^+_k \left( \frac{|\Delta_k f|}{|I_k|} \right) \leq \varphi(b_k, \frac{|\Delta_k f|}{|I_k|}) \leq \varphi(a_k + \epsilon, \frac{|\Delta_k f|}{|I_k|})
\]
since \( \varphi \) is increasing in \( x \). If \( f \) is the identity function, then \( \frac{|\Delta f|}{|I_k|} = 1 \) and
\[
\int_0^1 \varphi(x, 1) \, dx \leq \sum_{k=1}^n \varphi^+_k \left( \frac{|\Delta_k f|}{|I_k|} \right) |I_k| \leq \int_{\epsilon}^{1+\epsilon} \varphi(x, 1) \, dx.
\]
Since \( \varphi(x, 1) = x + 1 \), we obtain as \( \epsilon \to 0 \) that
\[
\nabla^\varphi(f) = \int_0^1 \varphi(x, 1) \, dx = \frac{3}{2}.
\]

We next show that \( \varphi^+ \) and \( \varphi^- \) give the same result at the limit under the stronger continuity condition (VA1). Example 5.4 shows that the result does not hold under (A1).

**Lemma 5.3.** If \( \varphi \in \Phi_{c}(\Omega) \) satisfies (VA1) and (Dec), then
\[
\nabla^\varphi(f) = \nabla^\varphi(f).
\]

**Proof.** Clearly, \( \nabla^\varphi \leq \nabla^\varphi \) so we show that \( \nabla^\varphi \leq \nabla^\varphi \). We assume that \( \nabla^\varphi(f) < \infty \), since otherwise there is nothing to prove. For \( \epsilon \in (0, 1) \) we choose \( \delta \in (0, \epsilon) \) such that
\[
\sup_{|(I_k)| < \delta} \sum_{k=1}^n \varphi^-_k \left( \frac{|\Delta_k f|}{|I_k|} \right) |I_k| \leq (1 + \epsilon) \nabla^\varphi(f).
\]
Let \((I_k)\) be a partition with \(|(I_k)| < \delta\). By the previous line,
\[
\varphi^-_k \left( \frac{|\Delta_k f|}{|I_k|} \right) < \frac{2 \nabla^\varphi(f)}{|I_k|}.
\]
Therefore it follows by (VA1) and (Dec) that
\[
\varphi^+_k \left( \frac{|\Delta_k f|}{|I_k|} \right) \leq (1 + \omega(\epsilon))q \left[ \varphi^-_k \left( \frac{|\Delta_k f|}{|I_k|} \right) + \omega(\epsilon) \right],
\]
where \( \omega \) is a modulus of continuity from (VA1) with \( K := 2 \nabla^\varphi(f) \). Hence
\[
\sum_{k=1}^n \varphi^+_k \left( \frac{|\Delta_k f|}{|I_k|} \right) |I_k| \leq (1 + \omega(\epsilon))q \left[ \sum_{k=1}^n \varphi^-_k \left( \frac{|\Delta_k f|}{|I_k|} \right) |I_k| + \omega(\epsilon) |I| \right]
\]
\[
\leq (1 + \omega(\epsilon))q \left[ (1 + \epsilon) \nabla^\varphi(f) + \omega(\epsilon) |I| \right].
\]
Since this holds for all \((I_k)\) with \(|(I_k)| < \delta\), we conclude that
\[
\nabla^\varphi(f) \leq (1 + \omega(\epsilon))q [(1 + \epsilon) \nabla^\varphi(f) + \omega(\epsilon) |I|] \to \nabla^\varphi(f),
\]
as \( \epsilon \to 0 \). Combined with \( \nabla^\varphi \leq \nabla^\varphi \), this gives \( \nabla^\varphi = \nabla^\varphi \).

**Example 5.4.** Let \( p(x) := 1 + \frac{1}{\log(1/2)} \) for \( x \in (0, \frac{1}{2}) \) and \( p(0) := 1 \). Set \( \varphi(x, t) := t^p(x) \). We consider the function \( f := \chi_{(0, \frac{1}{2})} \) and a partition \((I_k)\) of \( I := [0, \frac{1}{2}] \). Then \( \Delta f(I_k) = 0 \) unless \( k = 1 \). Suppose that \( I_1 := [0, x], \quad x \in (0, \frac{1}{2}) \). Then
\[
\varphi^+_1 \left( \frac{|\Delta f(I_1)|}{|I_1|} \right) |I_1| = x^{1-p^+_1} = x^{-\frac{1}{\log(1/2)}} = \epsilon
\]
and
\[
\varphi_{I_k}^-(\frac{|\Delta f(I_k)|}{|I_k|}) |I_k| = x^{1-p_{I_k}} = x^0 = 1.
\]
Since all other terms vanish, we see that \( \nabla \varphi_{I_k}^-(f) = e \) and \( \nabla \varphi_{I_k}^+(f) = 1 \). Note that \( p \) is log-Hölder continuous so that \( \varphi \) satisfies (A1) [14, Proposition 7.1.2].

The next example shows that \( \varphi^+ \) and \( \varphi^- \) do not give the same result without the limiting process, i.e. for \( \nabla \varphi_I \) and a version of it with \( \varphi_{I_k}^- \).

**Example 5.5.** Let \( I := [0, 3] \) and define \( \varphi \in \Phi_c(I) \) by
\[
\varphi(x, t) := t^{\max\{2, x\}}.
\]
Note that \( \varphi \) satisfies (A0), (A1), (VA1), (alnc)\(_2\) and (aDec)\(_3\). For \( \alpha > \frac{3}{\beta} \) we define \( f_\alpha : I \to \mathbb{R} \) by \( f_\alpha(x) := \alpha \min\{x, 1\} \). Then
\[
\nabla \varphi_I^+(\beta f_\alpha) \geq \frac{\beta |\Delta f_\alpha(I)|}{|I|} |I| = 3 \varphi_I^+ \left( \frac{3\alpha}{3} \right) = \frac{(\beta \alpha)^3}{9}.
\]
Let \( (I_k) \) be a partition of \( I \). If \( I_k \cap [0, 1] \) is non-empty, then
\[
\varphi_{I_k}^- \left( \frac{|\Delta f_\alpha|}{|I_k|} \right) \leq \varphi_{I_k}^-(\alpha) \leq \alpha^2,
\]
since \( |\Delta f_\alpha| \leq \alpha |I_k| \). If \( I_k \cap [0, 1] \) is empty, then \( \Delta f_\alpha = 0 \). Thus
\[
\sum_{k=1}^{n} \varphi_{I_k}^- \left( \frac{|\Delta f_\alpha|}{|I_k|} \right) |I_k| \leq \sum_{k=1}^{n} \alpha^2 |I_k| = 3\alpha^2,
\]
and hence
\[
\sup_{(I_k)} \sum_{k=1}^{n} \varphi_{I_k}^- \left( \frac{|\Delta f_\alpha|}{|I_k|} \right) |I_k| \leq 3\alpha^2 \leq \frac{27}{\beta^3 \alpha} \nabla \varphi_I^+(\beta f_\alpha).
\]
As \( \alpha \to \infty \), this shows that the inequality \( \nabla \varphi_I^+(\beta f_\alpha) \leq \sup_{(I_k)} \sum_{k=1}^{n} \varphi_{I_k}^- \left( \frac{|\Delta f_\alpha|}{|I_k|} \right) |I_k| \) does not hold for any fixed \( \beta > 0 \).

The next lemma shows that \( \nabla \varphi_I^+ \) and \( \nabla \varphi_I^- \) do define equivalent norms, even though they are not themselves equivalent.

**Lemma 5.6.** If \( \varphi \in \Phi_c(\Omega) \) satisfies (A1), then there exist a constant \( \beta \in (0, 1] \) such that
\[
\nabla \varphi_I^+(f) \leq 1 \implies \nabla \varphi_I^- (f) \leq 1.
\]
Furthermore, \( \| \cdot \| \nabla \varphi_I^- \approx \| \cdot \| \nabla \varphi_I^- \).

**Proof.** Let \( \varepsilon > 0 \) and choose \( \delta > 0 \) such that
\[
\sup_{|I_k| < \delta} \sum_{k=1}^{n} \varphi_{I_k}^+ \left( \frac{|\Delta f|}{|I_k|} \right) |I_k| \leq \nabla \varphi_I^+(f) + \varepsilon \leq 1 + \varepsilon.
\]
Let \( (I_k) \) be a partition of \( I \) and \( (I'_k) \) be a subpartition of \( (I_k) \) with \(|\{I'_k\}|| \leq \varepsilon \). As in Lemma 5.1, we find that
\[
\varphi_{I_k}^- \left( \frac{|\Delta f(I_k)|}{|I_k|} \right) |I_k| \leq \sum_{j=k}^{j+1-1} |I'_j| \frac{|\Delta f(I'_j)|}{|I'_j|} \varphi_{I_k}^- \left( \frac{|\Delta f(I'_j)|}{|I'_j|} \right) |I_k| \leq \sum_{j=k}^{j+1-1} \varphi_{I'_j}^- \left( \frac{|\Delta f(I'_j)|}{|I'_j|} \right) |I'_j|,
\]
that

\( V(A1) \)

and

\( V(A0) \)

the homogeneity of the norm.

We start with approximate equality. Note that we do not assume (aInc) and (aDec); thus we generalize also the previous results from the variable exponent case \([7]\). Note that the assumption \( f \in AC(I) \) can be replaced by (aInc), since together with (A0) it implies that the function is absolutely continuous.

**Theorem 6.1.** Let \( \varphi \in \Phi_w(I) \) satisfy (A0) and (A1). If \( f \in AC(I) \) and \( f' \in L^\varphi(I) \), then \( f \in \overline{RBV}^\varphi(I) \) and

\[
\|f\|_{\overline{RBV}^\varphi(I)} \leq c \|f'\|_{L^\varphi(I)}.
\]

**Proof.** Let \( \|f'\|_{L^\varphi(I)} < 1 \). Since \( f \) is absolutely continuous,

\[
\left| \frac{\Delta f(I_k)}{|I_k|} \right| = \left| \int_{I_k} f' \, dx \right| \leq \int_{I_k} |f'| \, dx.
\]

By [14, Theorem 4.3.2] there exists \( \beta > 0 \) such that

\[
\varphi^+_I \left( \beta \int_{I_k} |f'| \, dx \right) \leq \int_{I_k} \varphi(x, |f'|) \, dx + 1.
\]

It follows that

\[
V_I^\varphi(\beta f, (I_k)) \leq \sum_k \varphi^+_I \left( \beta \int_{I_k} |f'| \, dx \right) |I_k| \leq \int_I \varphi(x, |f'|) \, dx + |I| \leq 1 + |I|.
\]

Since this holds for any partition \( (I_k) \), we find that \( V_I^\varphi(\beta f) \leq 1 + |I| \). By (aInc), \( V_I^\varphi(\frac{\beta}{1+|I|} f) \leq \frac{1}{1+|I|} V_I^\varphi(\beta f) \leq 1 \). Hence \( \|f\|_{\overline{RBV}^\varphi(I)} \leq \frac{1}{\beta(1+|I|)} \); the claim for general \( \|f'\|_\varphi \) follows from this by the homogeneity of the norm.

\( \square \)
We next derive the corresponding upper bound. Note that here we use only the limit $|(I_k)| \to 0$, so the result holds also with $V_I^\varphi$ in place of $V_I^\varphi$.

**Theorem 6.2.** Let $\varphi \in \Phi_w(I)$. If $f \in \text{RBV}^\varphi(I) \cap AC(I)$, then $f' \in L^\varphi(I)$ and
\[
\|f'\|_{L^\varphi(I)} \leq \|f\|_{\text{RBV}^\varphi(I)}.
\]

**Proof.** We assume first that $V_I^\varphi(f) \leq 1$. Since $f \in AC(I)$, the derivative $f'$ exists almost everywhere in $I$. Let $((I_k^n)_k)_n$ be a sequence of partitions of $I$ with $|(I_k^n)| \to 0$ as $n \to \infty$. Define a step-function
\[
F_n := \sum_k \frac{\Delta f(I_k^n)}{|I_k^n|} \chi_{I_k^n}.
\]
Since $\lim_n F_n = f'$ a.e. and $\varphi$ is increasing, we see that $\varphi(x, \beta |f'(x)|) \leq \liminf_n \varphi(x, |F_n(x)|)$ for a.e. $x \in I$ and fixed $\beta \in (0, 1)$. Hence Fatou’s lemma implies that
\[
\int_I \varphi(x, \beta |f'|) \, dx \leq \liminf_n \int_I \varphi(x, |F_n|) \, dx \leq \liminf_n \sum_k \int_{I_k^n} \varphi_{I_k^n}^{\varphi}(|F_n|) \, dx.
\]
By the definition of $F_n$,
\[
\int_{I_k^n} \varphi_{I_k^n}(|F_n|) \, dx = \varphi_{I_k^n}^{\varphi} \left( \frac{|\Delta f(I_k^n)|}{|I_k^n|} \right) |I_k^n|.
\]
Thus
\[
\int_I \varphi(x, \beta |f'|) \, dx \leq \liminf_n V(f, (I_k^n)) \leq V_I^\varphi(f) \leq 1.
\]
This implies that $\|f'\|_{\varphi} \leq \frac{1}{\beta}$ and the general case follows by homogeneity as $\beta \to 1^-$. \qed

Combining the previous two results, we obtain the following:

**Corollary 6.3.** Let $\varphi \in \Phi_w(I)$ satisfy (A0) and (A1). Then
\[
\text{RBV}^\varphi(I) \cap L^\varphi(I) \cap AC(I) = W^{1, \varphi}(I).
\]

We next derive an exact formula for the Riesz semi-norm. In this case, we have to restrict our attention to the lim sup-version $\nabla^\varphi$. This result has no analogue in [7], so it is new even in the variable exponent case.

Following [1, Section 3.2] we consider functions $f$ of bounded variation on the real line whose derivative can be described as a signed measure $Df$ with finite total variation, $|Df|(I) < \infty$. The measure $Df$ can be split into an absolutely continuous part represented by $f' \, dx$ and a singular part $D^s f$ (with respect to the Lebesgue measure). In [1, (3.24)], it is shown that
\[
\inf_{g = f \text{ a.e.}} V_I^1(g) =: EV_I^1(f) = |Df|(I);
\]
the left-hand side is called the essential variation. Without the almost everywhere equivalence, the equality does not hold since we may take a function $f$ equal to zero except at one point so that $V_I^1(f) > 0 = |Df|(I)$. Functions for which the essential variation equals the variation (i.e. $V_I^1 = EV_I^1$) are called good representatives in [1].

A left-continuous function of bounded variation is an example of a good representative and can be expressed as $f(x) = Df([a, x]) + f(a)$. Furthermore, by [1, Theorem 3.28], the left-continuous representative of a function of bounded variation is a good representative. For simplicity, we restrict our attention to left-continuous functions. Note that [1] defined variations on open intervals. Where necessary, we can treat the first interval $[a, x]$ separately by a direct calculation.
Following [13], we define, for $\varphi \in \Phi_c$,

$$
\varphi'_\infty(x) := \lim_{t \to \infty} \frac{\varphi(x, t)}{t}.
$$

Note that the limit exists since $\frac{\varphi(x, t)}{t}$ is increasing. If $\varphi$ is differentiable and convex, then $\varphi'_\infty(x) = \lim_{t \to \infty} \varphi'(x, t)$, hence the notation.

**Theorem 6.4** (Riesz representation theorem). *Let $f \in BV(I)$ be left-continuous. If $\varphi \in \Phi_c(I)$ satisfies $(VA1)$ and $(\text{Dec})$, then*

$$
\nabla^r (x) = \int_I \varphi(x, |f'|) \, dx + \int_I \varphi'_\infty \, d|D^s f|.
$$

**Proof.** We prove the inequality “$\leq$” and assume that the right-hand side is finite. By Lemma 5.3, we may replace $\nabla^r (x)$ by $\nabla^r$ for the lower bound. Let $\varepsilon > 0$ and choose $\delta \in (0, \varepsilon)$ such that

$$
\int_A \varphi(x, |f'|) \, dx < \varepsilon
$$

for any set $A \subset I$ with $|A| < \delta$. Since the support of the singular part $D^s f$ has measure zero we can choose by the definition of the Lebesgue measure a finite union $A := \bigcup_{i=1}^m [a_i, a'_i]$ with $|A| < \delta$ and

$$
\int_I \varphi'_\infty \, d|D^s f| < \varepsilon.
$$

By left-continuity, $f(x) - f(y) = Df([a, x]) - Df([a, y]) = Df([y, x])$ for $x > y$. Thus

$$
\frac{\Delta f(I_k)}{|I_k|} = \left| \int_{I_k} f' \, dx + \frac{D^s f(I_k)}{|I_k|} \right| \leq \int_{I_k} |f'| \, dx + \frac{|D^s f(I_k)|}{|I_k|}.
$$

Since $t + s \leq \max\{(1 + \theta)t, (1 + \theta^{-1})s\}$ we obtain by $(\text{Dec})$ that

$$
\varphi'_\infty(t + s) \leq (1 + \theta)^q \varphi'_\infty(t) + (1 + \theta^{-1})^q \varphi'_\infty(s).
$$

By the previous estimates, Theorem 2.3 (with $\beta := \frac{1}{1 + \omega(|A|)}$) and $\varphi(x, t) \leq \varphi'_\infty(x) \, t$ we obtain

$$
\varphi'^{-}_k \left( \frac{\Delta f(I_k)}{|I_k|} \right) \leq (1 + \theta)^q \varphi'^{-}_k \left( \int_{I_k} |f'| \, dx \right) + (1 + \theta^{-1})^q \varphi'^{-}_k \left( \frac{|D^s f(I_k)|}{|I_k|} \right)
$$

$$
\leq (1 + \theta)^q \int_{I_k} \varphi'^{-}_k (|f'|) \, dx + \omega(|I_k|) \, dx + (1 + \theta^{-1})^q \int_{I_k} \varphi'_\infty \, d|D^s f|.
$$

We first apply the previous inequality to the set $A_i := [a_i, a'_i]$ from $A$ defined above, assumed to be so small that $\omega(|A_i|) \leq \varepsilon$, and choose $\theta := e^{-1/(2q)}$:

$$
\sum_i \varphi^A_i \left( \frac{\Delta f(A_i)}{|A_i|} \right) |A_i| \leq (1 + \theta)^q \int_A \varphi(x, |f'|) \, dx + \varepsilon + (1 + \theta^{-1})^q \int_A (\varphi'_\infty \, \lambda_k \, d|D^s f|)
$$

$$
\leq (1 + e^{-1/(2q)})^q \beta^{-q} \varepsilon (1 + |I|) + (1 + e^{1/(2q)})^q \int_I \varphi'_\infty \, d|D^s f|.
$$

Choose sufficiently small complementary closed intervals $B_i = [b_i, b'_i]$, i.e. $\cup_i A_i = \cup_i B_i = I \setminus \{b\}$ and $A_i \cap B_j = \emptyset$, such that $\omega(B_i) \leq \varepsilon$. We use the same estimate but now choose $\theta := e^{1/(2q)}$ and use (6.5) to obtain

$$
\sum_i \varphi^B_i \left( \frac{\Delta f(B_i)}{|B_i|} \right) |B_i| \leq (1 + e^{1/(2q)})^q \beta^{-q} \int_I \varphi(x, |f'|) + \varepsilon \, dx + (1 + e^{-1/(2q)})^q \varepsilon.
$$
Adding these two estimates and letting $\varepsilon \to 0$ and $\beta \to 1^-$, we obtain that

$$\nabla_i^\varphi(f) = \nabla_i^\psi(f) \leq \int_A \varphi(x, |f'|) \, dx + \int_I \varphi_\infty \, d|D^s f|.$$  

For the opposite inequality we use $\nabla_i^\varphi$ and start by observing that

$$\frac{|\Delta f(B_i)|}{|B_i|} \geq \left| \int_{B_i} f' \, dx \right| - \frac{|D^s f|(B_i)}{|B_i|}.$$  

This time we set $t = u - s$ in (6.6) and use the resulting inequality

$$\varphi_B^+(u - s) \geq (1 + \theta)^{-q} \varphi_B^+(u) - \theta^{-q} \varphi_B^+(s).$$  

With $B_i$ as before, we now obtain that

$$\sum_i \varphi_{B_i}^+ \left( \frac{|\Delta f(B_i)|}{|B_i|} \right) |B_i| \geq (1 + \theta)^{-q} \sum_i \varphi_{B_i}^+ \left( \left| \int_{B_i} f' \, dx \right| \right) |B_i| - \theta^{-q} \varepsilon$$

$$\geq (1 + \theta)^{-q} \left[ \int_I \varphi(x, |f'|) \, dx - \varepsilon \right] - \theta^{-q} \varepsilon,$$

where the second inequality follows as in Theorem 6.2 (possibly after restricting $\delta$ to a smaller value). In the case $\int_I \varphi(x, |f'|) \, dx = \infty$, we replace the square bracket with $\frac{1}{\gamma}$ and obtain a lower bound tending to $\infty$. Otherwise, we choose $\theta := \varepsilon^{1/(2q)}$ and continue with the estimate of the singular part of the derivative.

Fix $\lambda > 1$. Assume that $\delta \leq \lambda^{-2q}$ so that $|A| \leq \lambda^{-2q}$. By absolute continuity of the non-singular part, we may assume $\delta$ is so small that

(6.7) \[|Df|(A) \leq |D^s f|(A) + \frac{1}{\varphi_I^+(\lambda)}.\]

Since $u$ is a good representative, $V^+_A(f) = |Df|(A)$, which implies that

$$\sum_k |\Delta f(A_{i,k})| > |Df|(A) - \frac{1}{i_{\max} \varphi_I^+(\lambda)},$$

for any subpartition $(A_{i,k})$ of $A_i$ with sufficiently small $\max\{|A_{i,k}|\}$, where $i_{\max}$ is the number of intervals $A_i$. Now if $\frac{|\Delta f(A_{i,k})|}{|A_{i,k}|} \geq \lambda$, then (Inc) from the convexity of $\varphi$ implies that

$$\varphi_{A_{i,k}}^+ \left( \frac{|\Delta f(A_{i,k})|}{|A_{i,k}|} \right) \geq \varphi_{A_{i,k}}^+(\lambda) \frac{|\Delta f(A_{i,k})|}{|A_{i,k}|}.$$  

If, on the other hand, $\frac{|\Delta f(A_{i,k})|}{|A_{i,k}|} < \lambda$, then

$$\varphi_{A_{i,k}}^+ \left( \frac{|\Delta f(A_{i,k})|}{|A_{i,k}|} \right) \geq 0 \geq \frac{\varphi_{A_{i,k}}^+(\lambda) |\Delta f(A_{i,k})|}{\lambda |A_{i,k}|} - \varphi_{A_{i,k}}^+(\lambda).$$  

Therefore in either case we have the inequality

$$\varphi_{A_{i,k}}^+ \left( \frac{|\Delta f(A_{i,k})|}{|A_{i,k}|} \right) \geq \frac{\varphi_{A_{i,k}}^+(\lambda) |\Delta f(A_{i,k})|}{\lambda |A_{i,k}|} - \varphi_I^+(\lambda).$$
By (Dec)$_q$, (A0) and $|A_i| \leq \delta \leq \lambda^{-2q}$, we obtain $\varphi^+_A(\lambda) \leq (\lambda/\beta)^q \varphi^+_A(\beta) \leq \frac{K}{|A_i|}$. Hence by (A0), (VA1) and $\lambda > 1$, we see that $\varphi^+_A(\lambda) \leq (1+\varepsilon)\varphi^-_A(\lambda) \leq (1+\varepsilon)\varphi^+_A(\lambda)$. We conclude that

$$
\sum_{i,k} \varphi^+_A(\lambda) \left( |\Delta f(A_{i,k})| \right)_{|A_{i,k}|} \geq \sum_{i,k} \left( \frac{\varphi^+_A(\lambda)}{\lambda} |\Delta f(A_{i,k})| - \varphi^+_f(\lambda) |A_{i,k}| \right) \\
\geq \sum_{i} \varphi^+_A(\lambda) \left( |Df|(A_i) - \frac{1}{\max \varphi^-_A(\lambda) |A_i|} \right) - \varphi^+_f(\lambda) |A| \\
\geq \sum_{i} \varphi^+_A(\lambda) \left( |Df|(A_i) - \frac{1}{\lambda} \right) - \lambda^{-2q} \varphi^+_f(\lambda).
$$

Again, by (Dec)$_q$ and (A0), $\lambda^{-2q} \varphi^+_f(\lambda) \leq \lambda^{-q} \varphi^+_f(1) \leq \frac{c}{\lambda}$.

Next we observe by (6.7) that

$$
\sum_{i} \varphi^+_A(\lambda) \frac{|Df|(A_i)}{\lambda} \geq \sum_{i} \varphi^+_A(\lambda) |D^s f|(A_i) - \varphi^+_A(\lambda) \sum_{i} \left[ |Df|(A_i) - |D^s f|(A_i) \right] \\
\geq \sum_{i} \varphi^+_A(\lambda) \frac{|D^s f|(A_i)}{\lambda} - \frac{1}{\lambda} = \int \sum_{i} \varphi^+_A(\lambda) \frac{|D^s f|}{\lambda} d|D^s f| - \frac{1}{\lambda}.
$$

Since $\varphi^+_A(\lambda) \geq \varphi(x,\lambda)$ for every $x \in A_i$, we obtain that

$$
\sum_{i,k} \varphi^+_A(\lambda) \left( |\Delta f(A_{i,k})| \right)_{|A_{i,k}|} \geq \int \varphi(x,\lambda) \frac{d|D^s f|}{\lambda} - \frac{c}{\lambda}.
$$

Finally, we combine the estimate over $A$ and $B$ and have thus show that

$$
\nabla^\varphi(f) \geq (1+\varepsilon^{1/(2q)})^{-q} \int \varphi(x,|f'|) \, dx + \int \varphi(x,\lambda) \frac{d|D^s f|}{\lambda} - \frac{c}{\lambda} - c\sqrt{\varepsilon}.
$$

We obtain the desired lower bound as $\varepsilon \to 0$ and $\lambda \to \infty$ by monotone convergence, since $\frac{\varphi(x,\lambda)}{\lambda} \nearrow \varphi'(x)$, also using (6.5).

In the situation of the previous theorem, we precisely regain the $\varphi$-norm of the derivative of the function. Note that this result is new also in the variable exponent case. Furthermore, Example 5.2 shows that the result does not hold for $V^\varphi_f$.

**Corollary 6.8.** Let $f \in AC(I)$. If $\varphi \in \Phi_c(I)$ satisfies (VA1) and (Dec), then

$$
\nabla^\varphi(f) = \int_I \varphi(x,|f'|) \, dx.
$$

We conclude by commenting on the corollaries from the introduction. If $p : \Omega \to [1,\infty)$ is a bounded variable exponent, then (VA1) is equivalent to the strong log-Hölder condition. Further, $\varphi' = 1 + \infty \chi_{\{p>1\}}$ so that

$$
\int_I \varphi' \, d|D^s f| = \int_I 1 + \infty \chi_{\{p>1\}} \, d|D^s f| = |D^s f|(\{p = 1\})
$$

when $\chi_{\{p>1\}} = 0$ $|D^s f|$-a.e. In the double phase case $\varphi(x,t) = t + a(x)t^q$, we similarly obtain $\varphi' = 1 + \infty \chi_{\{a>0\}}$, and the corollary follows. Finally, in the Orlicz case $\varphi'_\infty$ is a constant. If the constant is infinity, then the singular part must vanish in order that $\infty |D^s f|(I)$ be finite, and so the function is absolutely continuous.
REFERENCES

[1] L. Ambrosio, N. Fusco and D. Pallara: *Functions of Bounded Variation and Free Discontinuity Problems*, Clarendon Press, Oxford, 2000.
[2] J. Appell, J. Banaś and N. Merentes: *Bounded Variation and Around*, Studies in Nonlinear Analysis and Applications, Vol. 17, De Gruyter, Berlin/Boston, 2014.
[3] P. Baroni, M. Colombo and G. Mingione: Regularity for general functionals with double phase, *Calc. Var. Partial Differential Equations* 57 (2018), no. 2, article 62, 48 pp.
[4] A. Benyaiche, P. Harjulehto, P. Hästö, and A. Karpipinen: The weak Harnack inequality for unbounded supersolutions of equations with generalized Orlicz growth, *J. Differential Equations* 275 (2019), 790–814.
[5] A. Benyaiche and I. Khlift: Harnack inequality for quasilinear elliptic equations in generalized Orlicz-Sobolev spaces, *Potential Anal.* 53 (2020), 631–643.
[6] T.A. Bui: Regularity estimates for nondivergence parabolic equations on generalized Orlicz spaces, *Int. Math. Res. Notices IMRN* 2021 (2021), no. 14, 11103–11139.
[7] R.E. Castillo, O.M. Guzmán and H. Rafeiro: Variable exponent bounded variation spaces in the Riesz sense, *Nonlinear Anal.* 132 (2016), 173–182.
[8] Y. Chen, S. Levine and M. Rao: Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.* 66 (2006), no. 4, 1383–1406.
[9] I. Chlebicka and A. Zatorska-Goldstein: Generalized superharmonic functions with strongly nonlinear operator, *Potential Anal.*, to appear. https://doi.org/10.1007/s11118-021-09920-5
[10] I. Chlebicka, P. Gwiazda and A. Zatorska-Goldstein: Renormalized solutions to parabolic equation in time and space dependent anisotropic Musielak–Orlicz spaces in absence of Lavrentiev’s phenomenon, *J. Differential Equations* 267 (2019), no. 2, 1129–1166.
[11] L. Diening, P. Harjulehto, P. Hästö and M. Růžička: *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, vol. 2017. Springer, Heidelberg, 2011.
[12] M. Eleuteri, P. Harjulehto and P. Hästö: Minimizers of abstract generalized Orlicz–bounded variation energy, submitted. arXiv:2112.06622
[13] M. Eleuteri, P. Harjulehto and P. Hästö: Generalized Orlicz-bounded variation spaces, in preparation.
[14] P. Harjulehto and P. Hästö: *Orlicz spaces and Generalized Orlicz spaces*, Lecture Notes in Mathematics, vol. 2236, Springer, Cham, 2019.
[15] P. Harjulehto and P. Hästö: Double phase image restoration, *J. Math. Anal. Appl.* 501 (2021), no. 1, article 123832
[16] P. Harjulehto, P. Hästö and J. Juuti: Revisiting basic assumptions of generalized Orlicz spaces, in preparation.
[17] P. Harjulehto, P. Hästö and M. Lee: Hölder continuity of quasiminimizers and ω-minimizers of functionals with generalized Orlicz growth, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) XXII* (2021), no. 2, 549–582.
[18] P. Harjulehto, P. Hästö, V. Latvala and O. Toivanen: Critical variable exponent functionals in image restoration, *Appl. Math. Letters* 26 (2013), 56–60.
[19] P. Hästö and J. Ok: Maximal regularity for local minimizers of non-autonomous functionals, *J. Eur. Math. Soc.* 24 (2022), no. 4, 1285–1334.
[20] P. Hästö and J. Ok: Regularity theory for non-autonomous partial differential equations without Uhlenbeck structure, arXiv:2110.14351
[21] A. Karpipinen and M. Lee: Hölder continuity of the minimizer of an obstacle problem with generalized Orlicz growth, *Int. Math. Res. Not. IMRN*, to appear.
[22] W. Kołodski: *Modular Function Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 122, Marcel Dekker, Inc., New York, 1988.
[23] J. Lang and O. Mendez: *Analysis on Function Spaces of Musielak-Orlicz Type*, Monographs and Research Notes in Mathematics, Chapman & Hall/CRC, 2019.
[24] F. Li, Z. Li, and L. Pi: Variable exponent functionals in image restoration, *Appl. Math. Comput.* 216 (2010), no. 3, 870–882.
[25] Yu.T. Medvedev: A generalization of a certain theorem of Riesz (Russian), *Uspekhi Mat. Nauk.* 6 (1953), 115–118.
[26] Y. Mizuta, T. Ohno and T. Shimomura: Sobolev’s theorem for double phase functionals, *Math. Inequal. Appl.* 23 (2020), no. 1, 17–33.
[27] J. Musielak: *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math., 1034, Springer-Verlag, 1983.
[28] T. Ohno and T. Shimomura: Sobolev’s inequality for Musielak-Orlicz-Morrey spaces over metric measure spaces, *J. Austral. Math. Soc.* 110 (2021), no. 3, 371–385.
[29] B. Wang, D. Liu and P. Zhao: Hölder continuity for nonlinear elliptic problem in Musielak-Orlicz-Sobolev space, *J. Differential Equations* 266 (2019), no. 8, 4835–4863.

P. HÄSTÖ  
Department of Mathematics and Statistics, FI-20014 University of Turku, Finland  
peter.hasto@utu.fi

J. JUUSTI  
Department of Mathematics and Statistics, FI-20014 University of Turku, Finland  
jthjuu@utu.fi

H. RAFEIRO  
Department of Mathematical Sciences, United Arab Emirates University, College of Science, UAE  
rafeiro@uaeu.ac.ae