NET MAP SLOPE FUNCTIONS

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Abstract. This paper studies NET map slope functions. It establishes Lipschitz-type conditions for them. It relates Lipschitz-type conditions to the half-space theorem. It gives bounds on the number of slope function fixed points. It provides examples of rational NET maps with many matings.

1. Introduction

This paper is part of a series of papers [2], [4], [5], [6], [7] on NET maps. A nearly Euclidean Thurston (NET) map is a Thurston map $f$ with exactly four postcritical points such that the local degree of $f$ at each of its critical points is 2. These are the simplest Thurston maps with nontrivial Teichmüller spaces. They are quite tractable and yet exhibit a wide range of behavior.

Although the emphasis is on NET maps, much of this paper applies to all Thurston maps with four postcritical points. Let $f: S^2 \to S^2$ be a Thurston map with postcritical set $P_f$ containing exactly four points.

Homotopy classes of simple closed curves in $S^2 \setminus P_f$ correspond to slopes, elements of $\mathbb{Q} = \mathbb{Q} \cup \{\infty\}$. We enlarge the set $\mathbb{Q}$ by adjoining one more element $\odot$, called the nonslope. Loosely speaking, the slope function $\mu_f: \mathbb{Q} \to \mathbb{Q} \cup \{\odot\}$ is defined (in Section 2) by starting with a slope, choosing a simple closed curve $\gamma$ with slope $s$ and defining $\mu_f(s)$ to be the slope of a simple closed curve in $f^{-1}(\gamma)$. The multiplier of $s$ is the sum of the reciprocals of the degrees of the restrictions of $f$ to the connected components of $f^{-1}(\gamma)$ which are neither null homotopic nor peripheral. Much of the interest in $\mu_f$ stems from the fact that W. Thurston’s characterization of rational maps implies in this situation that if the orbifold of $f$ is hyperbolic, then $f$ is equivalent to a rational map if and only if $\mu_f$ has no fixed point with multiplier at least 1. Slopes of mating equators are also fixed points of $\mu_f$.

There is an intersection pairing on simple closed curves and a corresponding intersection pairing on slopes. In Section 5 we establish a Lipschitz-type condition relative to the latter intersection pairing for slope functions of Thurston maps with exactly four postcritical points. In Section 7 we substantially improve this Lipschitz-type condition in the case of NET maps. This result is used in [7] as well as later in this paper.

We view the Teichmüller space associated to $(S^2, P_f)$ as the upper half-plane $\mathbb{H}$. Using $f$ to pull back complex structures on $(S^2, P_f)$ induces a pullback map $\sigma_f: \mathbb{H} \to \mathbb{H}$. Selinger proved in [13] that $\sigma_f$ extends continuously to the augmented Teichmüller space $\mathbb{H} \cup \overline{\mathbb{Q}}$. If $s$ is a slope such that $\mu_f(s)$ is a slope, not $\odot$, then $\sigma_f(-1/s) = -1/\mu_f(s)$. In this way a slope $s$ corresponds to a cusp $t = -1/s$. 

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Let $s$ be a slope such that $\mu_f(s) \notin \{s, \varnothing\}$. Then the half-space theorem of [2] provides an explicit open interval about $-1/s$ in $\partial \mathbb{H}$, called an excluded interval, which does not contain the negative reciprocal of the slope of a Thurston obstruction for $f$. The proof is based on an argument involving extremal lengths of families of simple closed curves. Theorem 8.4 gives another proof of this based on a slope function Lipschitz-type condition. Theorem 8.5, the NET map half-space theorem, improves on this for the special case in which $f$ is a NET map. Sometimes finitely many of these excluded intervals cover $\partial \mathbb{H}$, allowing one to conclude that $f$ is rational. This occurs in Example 6.8 of [2] as well as for Douady’s rabbit [4, Figure 4]. However, after the half-space theorem was first proved, computations suggested that there exist examples of rational NET maps for which every finite union of excluded intervals which arise from the half-space theorem omits an open interval of $\partial \mathbb{H}$. Theorem 8.10 allows for easy construction of many such examples, the first of which we are aware. See Remark 8.11. This possibility led to the extended half-space theorem, a qualitative version of which is discussed in [1, Section 10]. The extended half-space theorem provides an explicit open interval about an initial cusp $t = -1/s$ such that $\mu_f(s) \in \{s, \varnothing\}$. This open interval, except for $t$, is a union of infinitely many excluded intervals which arise from the half-space theorem. In a forthcoming paper [11] we prove for Thurston maps with four postcritical points having no obstructions with multipliers equal to 1 that finitely many excluded intervals arising from both the half-space and extended half-space theorems cover $\partial \mathbb{H}$. Thus in this case the union of all excluded intervals arising from the half-space theorem is a cofinite subset of $\partial \mathbb{H}$ (whose complement consists of elements of $\overline{\mathbb{Q}}$). However, Theorem 8.12 allows one to easily construct NET maps having obstructions with multiplier 1 for which the union of all excluded intervals arising from the half-space theorem is not a cofinite subset of $\partial \mathbb{H}$. See Remark 8.13. These are the first such examples of which we are aware.

If $f$ has an obstruction with multiplier 1, then $f$ commutes with a (nonzero power of a primitive) Dehn twist $\tau$ about the obstruction up to homotopy. So if $\mu_f$ fixes a slope $s$ other than the slope of the obstruction, then $\mu_f$ fixes every slope in the orbit of $s$ under the action of the infinite cyclic group generated by $\tau$. In this way it is possible for $\mu_f$ to have infinitely many fixed points. This occurs for the NET map presented in Figure 6 (See Remark 8.13 and the proof of Theorem 8.12.) On the other hand, if $f$ does not have an obstruction with multiplier 1, then Theorem 10.1 gives a bound on the number of slopes which $\mu_f$ can fix. In effect, this bounds the number of slope function fixed points in terms of the degree of $f$.

Finally, Section 11 deals with polynomial matings. See [1] for a thorough survey of this topic. Section 11 presents an infinite sequence of rational NET maps $f_n$ with degree $n$ such that $f_n$ has at least $\lceil (n - 1)/2 \rceil$ formal matings. Remark 11.2 shows how to modify these maps to obtain rational Thurston maps with many topological matings.

The paper [7] also deals somewhat with NET map slope functions. Although the present work deals primarily with NET maps, [7] deals almost exclusively with NET maps. Its goal, not quite achieved, is to find an effective algorithm which determines whether a given NET map is equivalent to a rational map. So its results are quantitative, whereas the results of the present paper are more qualitative.

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2. Slope functions

We define slope functions in this section.

We begin with a Thurston map \( f : S^2 \to S^2 \) whose postcritical set \( P_f \) has exactly four points. We wish to mark the pair \((S^2, P_f)\) in order to assign slopes to simple closed curves. For this, let \( \Gamma \) be the group of isometries of \( \mathbb{R}^2 \) of the form \( x \mapsto 2\lambda \pm x \) for \( \lambda \in \mathbb{Z}^2 \). By a marking of \((S^2, P_f)\) we mean an identification of \( S^2 \) with \( \mathbb{R}^2/\Gamma \) so that \( P_f \) is the image of \( \mathbb{Z}^2 \) in \( \mathbb{R}^2/\Gamma \). This choice of marking is arbitrary in this section. Later, when dealing with a NET map given by a presentation, the marking will be determined by the presentation. Every homotopy class of simple closed curves in \( S^2/P_f \) which are neither peripheral nor null homotopic contains the image of a line in \( \mathbb{R}^2 \). The slope of this line lies in the set \( Q = \mathbb{Q} \cup \{ \infty \} \) of extended rational numbers. This determines a bijection between the set of these homotopy classes and \( \overline{Q} \). (See \cite{3} Proposition 1.5 and Proposition 2.6, for example.) In addition to the slopes in \( \overline{Q} \), we introduce a quantity \( \circ \) not in \( \overline{Q} \), called the nonslope. This symbol is intended to suggest a loop homotopic to a point.

Now we define the slope function \( \mu_f : \overline{Q} \to \overline{Q} \cup \{ \circ \} \) of \( f \). Let \( s \in \overline{Q} \). Let \( \gamma \) be a simple closed curve in \( S^2 \setminus P_f \) with slope \( s \). If every connected component of \( f^{-1}(\gamma) \) is either peripheral or null homotopic, then we set \( \mu_f(s) = \circ \). Suppose that \( \delta \) and \( \delta' \) are connected components of \( f^{-1}(\gamma) \) which are neither peripheral nor null homotopic. Then \( \delta \) and \( \delta' \) are homotopic in \( S^2 \setminus P_f \). Hence they have the same slope \( t \). We set \( \mu_f(s) = t \). This defines the slope function \( \mu_f \).

3. Core arcs

Much can be learned about Thurston maps by studying their pullback actions on simple closed curves. More can be learned by also studying their pullback actions on core arcs. (See Pilgrim and Tan’s \cite{12} Theorem 3.2 for a result on arc pullbacks for general Thurston maps.) We introduce core arcs in this section. Let \( f : S^2 \to S^2 \) be a Thurston map whose postcritical set \( P_f \) contains exactly four points.

By a core arc for \((S^2, P_f)\) we mean a closed arc \( \alpha \) in \( S^2 \) whose endpoints are distinct elements of \( P_f \) and these endpoints are the only points of \( \alpha \) in \( P_f \). Let \( \gamma \) be a simple closed curve in \( S^2 \setminus P_f \) which is neither peripheral nor null homotopic. The complement of \( \gamma \) in \( S^2 \) consists of two open topological disks, each containing two elements of \( P_f \). So each of these disks contains a unique homotopy class of core arcs relative to \( P_f \). We say that such a core arc is a core arc for \( \gamma \). Conversely, starting with a core arc \( \alpha \), the boundary of a small regular neighborhood of \( \alpha \) is a simple closed curve \( \gamma \) in \( S^2 \setminus P_f \) which is neither peripheral nor null homotopic. So every core arc \( \alpha \) is the core arc of some simple closed curve, and there exists a core arc disjoint from \( \alpha \). Moreover, if \( \alpha \) is a core arc for two simple closed curves \( \gamma \) and \( \gamma' \), then \( \gamma \) and \( \gamma' \) are homotopic relative to \( P_f \). This allows us to define the slope of \( \alpha \) to be the slope of \( \gamma \). For every slope \( s \) there exist exactly two homotopy classes relative to \( P_f \) of core arcs with slope \( s \).

By a lift of a core arc \( \alpha \) we mean a closed arc \( \tilde{\alpha} \) in \( S^2 \) such that \( f \) maps \( \tilde{\alpha} \) homeomorphically onto \( \alpha \).

Let \( f \) be a NET map for the rest of this section. Let \( \alpha \) and \( \beta \) be disjoint core arcs for \((S^2, P_f)\). The rest of this section is devoted to an investigation of \( f^{-1}(\alpha \cup \beta) \).

For the present discussion, we view \( f^{-1}(\alpha \cup \beta) \) as a graph \( G \). The edges of \( G \) are the \( \deg(f) \) lifts of \( \alpha \) together with the \( \deg(f) \) lifts of \( \beta \). The vertices of \( G \) are the elements of
the set $f^{-1}(P_f)$. Since the local degree of $f$ at each of its critical points is 2, every vertex of $G$ has valence either 1 or 2. So every connected component of $G$ is either an arc or a simple closed curve.

Let $A = S^2 \setminus (\alpha \cup \beta)$, an annulus. Let $B$ be a connected component of $f^{-1}(A)$. Then $B$ is an annulus, and the restriction of $f$ to $B$ is a covering map of some degree $d$. The boundary components of $B$ are connected components of $G$. One of these boundary components is a connected component of $f^{-1}(\alpha)$, and the other is a connected component of $f^{-1}(\beta)$. If $B$ has a boundary component which is an arc, then this arc has $d$ edges in $G$. If $B$ has a boundary component which is a simple closed curve, then this simple closed curve has $2d$ edges in $G$.

Suppose that both boundary components of $B$ are arcs. Then the closure of $B$ is $S^2$. So $G$ has only two connected components and both of them are arcs with $d = \deg(f)$ edges.

Suppose that $B$ has a boundary component which is a simple closed curve $\delta$. Then $\delta$ is also a boundary component of a connected component $B'$ of $f^{-1}(A)$ other than $B$. Just as $\delta$ has $2d$ edges and the restriction of $f$ to $B$ has degree $d$, the restriction of $f$ to $B'$ must also have degree $d$. The union $B \cup \delta \cup B'$ is an annulus in $S^2$. If both of its boundary components are arcs, then the closure of this annulus is $S^2$. Otherwise there exists a connected component $B''$ of $f^{-1}(A)$ other than $B$ and $B'$ with a boundary component which is a simple closed curve equal to a boundary component of $B \cup \delta \cup B'$.

Continuing in this way, we obtain the following conclusions. Exactly two connected components of $f^{-1}(\alpha \cup \beta)$ are arcs, each containing $d$ lifts of either $\alpha$ or $\beta$ for some positive integer $d$. The remaining connected components of $f^{-1}(\alpha \cup \beta)$ are simple closed curves containing $2d$ lifts of either $\alpha$ or $\beta$. Let $\gamma$ be a simple closed curve in $S^2$ with core arcs $\alpha$ and $\beta$, so that $\gamma$ is a core curve for $A$. The connected components of $f^{-1}(\alpha)$ interlace the connected components of $f^{-1}(\beta)$, and every connected component of $f^{-1}(\alpha)$ is separated from every connected component of $f^{-1}(\beta)$ by a connected component of $f^{-1}(\gamma)$. We note that this discussion proves statement 2 of Lemma 1.3 of [2], namely, that $f^{-1}(P_f)$ contains exactly four points which are not critical points. This discussion also proves part of statement 1 of Theorem 4.1 of [2], namely, that every connected component of $f^{-1}(\gamma)$ maps to $\gamma$ with the same degree.

Figure 1 illustrates one possibility for $f^{-1}(\alpha \cup \beta \cup \gamma)$. The three dashed simple closed curves form $f^{-1}(\gamma)$. The solid black, respectively gray, curves form $f^{-1}(\alpha)$, respectively $f^{-1}(\beta)$. The dots are the elements of $f^{-1}(P_f)$.

4. INTERSECTION NUMBERS

In this section we fix notation and conventions concerning the intersection pairings that will be used later.
Corollary 6.2 of [2] shows that if

\begin{equation}
\psi \in \text{PSL}(2, \mathbb{Z})\end{equation}

the horoballs of two pairs of relatively prime integers. Suppose that

\begin{equation}
\iota(s, s') = 1/\sqrt{mm'}.
\end{equation}

Proof. Corollary 6.2 of [2] shows that if \( \varphi \in \text{PSL}(2, \mathbb{Z}) \) and \( t, t' \) are slopes such that \( \varphi(-1/t) = -1/t' \), then \( \varphi(B_m(t)) = B_m(t') \). This and the fact that \( \iota \) is invariant under the action of PSL(2,\text{Z}) imply that to prove the lemma, we may assume that \( s' = 0 \). This assumption and
the fact that $B_m(s)$ and $B_{m'}(s')$ are tangent to each other imply that $m'$ is the Euclidean diameter of $B_m(s)$. Section 6 of [2] determines this diameter, implying that $m' = \frac{1}{mp^2}$. Hence

$$\frac{1}{\sqrt{mm'}} = |p| = |p \cdot 1 - q \cdot 0| = \iota(s, s').$$

This proves Lemma 5.1.

Having proved the first lemma, we turn to the second. For this, let $P$ be a parabolic element of $SL(2, \mathbb{Z})$. Then $P$ acts on $\partial \mathbb{H}$ with exactly one fixed point. Every other point in $\partial \mathbb{H}$ moves in the same direction, either clockwise or counterclockwise. We say that $P$ is positive if every other point moves in the counterclockwise direction. For example, the element $[1 \ 0 \ 0 \ 1]$ of $SL(2, \mathbb{Z})$ generates the stabilizer of $\infty$ in $SL(2, \mathbb{Z})$ and it is positive. For every extended rational number $s$, there exists a positive generator of the stabilizer of $s$ in $SL(2, \mathbb{Z})$, and this generator is unique up to multiplication by $\pm 1$. This brings us to the second lemma, where we let $tr$ denote the trace map on $2 \times 2$ matrices.

**Lemma 5.2.** Let $P_1$ and $P_2$ be parabolic elements of $SL(2, \mathbb{Z})$. Suppose that $\pm P_i$ is the $n_i$th power of a positive generator of the stabilizer of the extended rational number $s_i$ which $P_i$ fixes for $i \in \{1, 2\}$. Then

$$|tr(P_1P_2^{-1})| = |2 + n_1n_2\iota(s_1, s_2)^2|.$$

**Proof.** The lemma is insensitive to conjugating $P_1$ and $P_2$ by the same element of $SL(2, \mathbb{Z})$. Hence we may assume that $s_1 = \infty$. The lemma is also insensitive to multiplying $P_1$ by $-1$. Hence we may assume that $P_1 = \begin{bmatrix} 1 & n_2 \\ 0 & 1 \end{bmatrix}$. We may also assume that $P_2 = Q \begin{bmatrix} 1 & n_2 \\ 0 & 1 \end{bmatrix}Q^{-1}$ for some matrix $Q = \begin{bmatrix} \frac{p}{q} & * \\ * & q \end{bmatrix} \in SL(2, \mathbb{Z})$. So $s_2 = \frac{p}{q}$. Then

$$P_2^{-1} = Q \begin{bmatrix} 1 & -n_2 \\ 0 & 1 \end{bmatrix} Q^{-1} = Q \begin{bmatrix} 0 & -n_2 \\ 1 & 0 \end{bmatrix} Q^{-1} = 1 + Q \begin{bmatrix} 0 & -n_2 \\ 0 & 0 \end{bmatrix} Q^{-1} = 1 + n_2 \begin{bmatrix} 0 & -p \\ 0 & -q \end{bmatrix} \begin{bmatrix} s & -r \\ -q & p \end{bmatrix} = 1 + n_2 \begin{bmatrix} pq & -p^2 \\ q^2 & -pq \end{bmatrix}.$$

So

$$tr(P_1P_2^{-1}) = tr\left(\left(1 + n_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \left(1 + n_2 \begin{bmatrix} pq & -p^2 \\ q^2 & -pq \end{bmatrix}\right)\right) = tr(1) + tr\left(n_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) + tr\left(n_2 \begin{bmatrix} pq & -p^2 \\ q^2 & -pq \end{bmatrix}\right) + tr\left(n_1n_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} pq & -p^2 \\ q^2 & -pq \end{bmatrix}\right) = 2 + n_1n_2q^2 = 2 + n_1n_2(1 \cdot q - 0 \cdot p)^2 = 2 + n_1n_2\iota(s_1, s_2)^2.$$

This proves Lemma 5.2.

With Lemmas 5.1 and 5.2 in hand, all that we need for Theorem 5.3 is the definition of a slope multiplier. For this, let $s \in \overline{Q}$. Let $\gamma$ be a simple closed curve in $S^2 \setminus P_f$ with slope $s$ with respect to a fixed marking (Section 2) of $S^2 \setminus P_f$. Let $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_k$ be the connected components of $f^{-1}(\gamma)$ which are neither peripheral nor null homotopic. Let $d_i$ be the degree with which $f$ maps $\tilde{\gamma}_i$ to $\gamma$ for $i \in \{1, \ldots, k\}$. Then the multiplier of $s$ is $\rho = \sum_{i=1}^k d_i^{-1}$.  

Because \( P_f \) contains only four points, the slopes of \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_k \) are equal. If this slope equals the slope of \( \gamma \), then \( \gamma \) forms an \( f \)-stable multicurve. It therefore has a Thurston matrix, which in this case is simply a \( 1 \times 1 \) matrix with entry \( \rho \). The spectral radius of this matrix is called a Thurston multiplier. W. Thurston’s characterization of rational maps in this situation implies that if the orbifold of \( f \) is hyperbolic, then \( f \) is equivalent to a rational map if and only if the multiplier of every \( f \)-stable multicurve is less than 1.

**Theorem 5.3.** Let \( f \) be a Thurston map whose postcritical set \( P_f \) has exactly four points. Let \( \mu_f \) be the slope function of \( f \) with respect to a fixed marking of \( S^2 \setminus P_f \). Let \( s_1 \) and \( s_2 \) be slopes with multipliers \( \rho_1 \) and \( \rho_2 \) such that \( \mu_f(s_1) \neq \circ \) and \( \mu_f(s_2) \neq \circ \). Then

\[
\underline{\nu}(\mu_f(s_1), \mu_f(s_2)) \leq \frac{1}{\sqrt[\rho_1\rho_2]} \nu(s_1, s_2).
\]

Furthermore, if \( s_1 \neq s_2 \), then the inequality is strict if and only if the orbifold of \( f \) is hyperbolic.

**Proof.** We first prove the inequality using Lemma 5.1. Then we give a second proof of the inequality using Lemma 5.2. The second argument also proves the statement concerning strictness of the inequality.

Line 6.5 of [2] deals with NET maps, but it holds whenever \( P_f \) contains exactly four points. It shows that the pullback map \( \sigma_f \) of \( f \) on the upper half-plane, naturally viewed as the Teichmüller space of \( f \), maps \( B_m(s_i) \) into \( B_{\rho_m(\mu_f(s_i))} \) for every positive real number \( m \) and \( i \in \{1, 2\} \). We choose positive real numbers \( m_1 \) and \( m_2 \) so that \( B_{m_1}(s_1) \) and \( B_{m_2}(s_2) \) are tangent. Then \( \sigma_f \) maps this point of tangency to the closures of both \( B_{\rho_1m_1}(\mu_f(s_1)) \) and \( B_{\rho_2m_2}(\mu_f(s_2)) \). Because the closures of \( B_{\rho_1m_2}(\mu_f(s_1)) \) and \( B_{\rho_2m_2}(\mu_f(s_2)) \) intersect nontrivially, Lemma 5.1 implies that

\[
\underline{\nu}(\mu_f(s_1), \mu_f(s_2)) \leq \frac{1}{\sqrt[\rho_1m_1\rho_2m_2]} = \frac{1}{\sqrt[\rho_1\rho_2]} \frac{1}{\sqrt{m_1m_2}} = \frac{1}{\sqrt[\rho_1\rho_2]} \nu(s_1, s_2).
\]

This completes our first proof of the inequality of Theorem 5.3.

We now give another proof of this inequality using Lemma 5.2 instead of Lemma 5.1. Let \( G \) be the pure modular group of \((S^2, P_f)\). Let \( G_f \) be the group of liftables for \( f \) in \( G \). So \( \psi \in G_f \) if and only if there exists \( \tilde{\psi} \in G \) such that \( \psi[f] = [f] \tilde{\psi} \), where \( [f] \) denotes the homotopy class of \( f \) relative to \( P_f \). As in Theorem 7.1 of [2], we have the following. Let \( i \in \{1, 2\} \). There exists \( \tau_i \in G_f \) whose pullback map \( \mu_{\tau_i} \) on slopes is the \( n_i \)th power of the positive element which generates the stabilizer of \( s_i \) in \( \text{PSL}(2, \mathbb{Z}) \) for some positive integer \( n_i \). Moreover, \( \tilde{\tau}_i = (\tau_i')^{n_i} \), where \( \tau_i' \) is the element of \( G \) such that \( \mu_{\tilde{\tau}_i} \) is the \( n_i \)th power of the positive element which generates the stabilizer of \( s_i \) in \( \text{PSL}(2, \mathbb{Z}) \).

Now we apply Lemma 5.2 to matrices \( P_1 \) and \( P_2 \) in \( \text{SL}(2, \mathbb{Z}) \) which represent \( \mu_{\tau_1} \) and \( \mu_{\tau_2} \), respectively, obtaining that

\[
|\text{tr}(P_1P_2^{-1})| = 2 + n_1n_2\nu(s_1, s_2)^2.
\]

We next apply Lemma 5.2 again, this time to matrices \( P'_1 \) and \( P'_2 \) in \( \text{SL}(2, \mathbb{Z}) \) which represent \( \mu_{\tilde{\tau}_1} \) and \( \mu_{\tilde{\tau}_2} \), respectively, obtaining that

\[
|\text{tr}(P'_1P'_2^{-1})| = 2 + \rho_1\rho_2n_1n_2\nu(\mu_f(s_1), \mu_f(s_2))^2.
\]

There is nothing to prove if \( s_1 = s_2 \), so suppose that \( s_1 \neq s_2 \). Then \( \nu(s_1, s_2) > 0 \) and \( |\text{tr}(P_1P_2^{-1})| > 2 \). So the matrix \( P_1P_2^{-1} \) in \( \text{SL}(2, \mathbb{Z}) \) which represents \( \mu_{\tau_1}\mu_{\tau_2}^{-1} \) is hyperbolic.
Just as \( f \) induces a pullback map \( \sigma_f : \mathbb{H} \to \mathbb{H} \), we have a pullback map \( \sigma_\psi : \mathbb{H} \to \mathbb{H} \) for every \( \psi \in G \). Section 6 of [4] shows that a fixed conjugation takes \( \mu_\psi \) to \( \sigma_\psi \) for every \( \psi \) in \( G \). Hence \( \sigma_{\tau_1} \sigma_{\tau_2}^{-1} \) is hyperbolic. Theorem 4.4 of [6] is proven in the context of NET maps, but it holds whenever \( P_f \) has exactly four points. So statement 3 of Theorem 4.4 of [6] implies that if the orbifold of \( f \) is hyperbolic, then

\[
\left| \text{tr}(P_1P_2^{-1}) \right| < \left| \text{tr}(P_1P_2^{-1}) \right|.
\]

If the orbifold of \( f \) is Euclidean, then this inequality is an equality. Thus

\[
\rho_1\rho_2\iota(\mu_f(s_1), \mu_f(s_2))^2 \leq \iota(s_1, s_2)^2.
\]

This easily gives our second proof of the inequality of Theorem 5.3 and also the statement about strictness of the inequality.

\( \square \)

6. Review of NET map slope function evaluation

When considering examples of NET maps later, we will want to evaluate NET map slope functions. So in this section we discuss how to evaluate the slope function \( \mu_f \) of a NET map \( f \) given by a presentation as in [5]. Specifically, we review Theorem 5.1 of [2] and its visual interpretation.

We begin with a slope \( s = \frac{p}{q} \), where \( p \) and \( q \) are relatively prime integers. Theorem 4.1 of [2] implies that if \( \gamma \) is a simple closed curve in \( S^2 \setminus P_f \) with slope \( s \), then every connected component of \( f^{-1}(\gamma) \) maps to \( \gamma \) with the same local degree \( d(s) \). We have a line segment \( S \) in \( \mathbb{R}^2 \) with slope \( s \) and endpoints \( v \) and \( w = v + 2d(s)(q, p) \). We choose \( \gamma \) and \( S \) so that \( S \) maps to one of the connected components of \( f^{-1}(\gamma) \) in the usual quotient space \( \mathbb{R}/\Gamma_1 \). The line segment \( S \) meets the spin mirrors of \( f \) transversely in finitely many points not equal to either \( v \) or \( w \). Let \( \lambda_1, \ldots, \lambda_n \) be the centers of these spin mirrors in order from \( v \) to \( w \). We have a basis \( B \) of the usual lattice \( \Lambda_1 \). The conclusion of Theorem 5.1 of [2] is that \( \mu_f(s) \) is the slope of the line segment joining \( v \) and \( w' = (−1)^n w + 2 \sum_{i=1}^n (−1)^{i+1} \lambda_i \) with respect to the basis \( B \). Although this conclusion is correct, it is peculiar because it fails to recognize that \( n \) is always even, and so \( (−1)^n = 1 \). In fact, this follows from the theorem. Indeed, it is implicit in the conclusion that \( v − w' \in \Lambda_1 \). If \( n \) is odd, then \( v − w \in \Lambda_1 \), \( v + w \in \Lambda_1 \) and hence \( 2v \in \Lambda_1 \). But we may perturb \( v \), and so \( \Lambda_1 \) contains a line segment in \( \mathbb{R}^2 \). This is absurd.

In this paragraph we give a perhaps clearer explanation of why the integer \( n \) in the previous paragraph is even. The line segment \( S \) maps to a simple closed curve \( \delta \) in \( \mathbb{R}^2/\Gamma_1 \), and \( f(\delta) = \gamma \). The curve \( \delta \) separates two elements of \( P_f \) from the other two elements of \( P_f \). We implicitly have that \( f = h \circ g \), where \( g \) is a Euclidean NET map and \( h \) is a push map. As in Figure 1, the curve \( \delta \) also separates two elements of \( P_g \) from the other two elements of \( P_g \). The image in \( \mathbb{R}^2/\Gamma_1 \) of a spin mirror is an arc \( \alpha \) with one endpoint in \( P_g \) and one endpoint in \( P_f \). The curve \( \delta \) separates the endpoints of \( \alpha \) if and only if the intersection number \( \iota(\alpha, \delta) \) is odd. Because these four arcs \( \alpha \) pair the elements of \( P_f \) and \( P_g \), the sum of the intersection numbers \( \iota(\alpha, \delta) \) is even. Thus \( S \) meets an even number of spin mirrors and \( n \) is even.

We next discuss by example what might be called the visual or geometric interpretation of NET map slope function evaluation, as discussed immediately after Remark 5.2 of [2]. Figure 2 shows part of \( \mathbb{R}^2 \). Elements of the lattice \( \mathbb{Z}^2 \) are marked with dots. The large dots are elements of the sublattice \( \Lambda_1 \) with basis consisting of \((4, 0)\) and \((4, 1)\). We assume that
(4, 0) and (4, 1) are part of a presentation for a NET map $f$ and that the dashed horizontal line segments in Figure 2 are spin mirrors determined by this presentation. Figure 2 shows a line segment $S$ with endpoints $v$, $w$ and slope $\frac{1}{3}$. The points where $S$ meets spin mirrors are labeled $A$, $B$, $C$, $D$. The image of $S$ in the usual quotient space $\mathbb{R}^2/\Gamma_1$ is a simple closed curve which $f$ maps to a simple closed curve with slope $\frac{1}{3}$ relative to our presentation. To compute $\mu_f(\frac{1}{3})$, we turn to Figure 3. We imagine a photon traveling along the initial segment of $S$ from $v$ to $A' = A$. At $A'$ the photon spins about the center of a spin mirror to $A''$. It then travels parallel to $S$ to $B'$, where it encounters another spin mirror. Then it spins to $B''$ and travels to $C'$. Then it spins to $C''$ and travels to $D'$. Then it spins to $D''$ and travels to $w'$, where it stops. The sum of the lengths of the line segments parallel to $S$ in Figure 3 is the length of $S$. The vector $v - w'$ is in $2\Lambda_1$. The slope of the line segment joining $v$ and $w'$ relative to the basis consisting of $(4, 0)$ and $(4, 1)$ is $\frac{2}{-2} = -1$. So $\mu_f(\frac{1}{3}) = -1$.

7. Lipschitz-type conditions for NET maps

Theorem 5.3 provides a Lipschitz-type condition for the slope function of a Thurston map with exactly four postcritical points. This section improves on this in the special case of NET maps. Most of the results of this section rest on the following elementary lemma, which we take for granted.

**Lemma 7.1.** Let $T^2$ be the standard torus $\mathbb{R}^2/\mathbb{Z}^2$ with its natural translation surface structure. Let $S^2$ be the 2-sphere half-translation surface gotten from $T^2$ by identifying every point $x \in T^2$ with $-x$. Then the following statements hold.

1. Two closed geodesics in $T^2$ with equal slopes have equal lengths.
2. Let $\gamma$ and $\delta$ be closed geodesics in $T^2$ with distinct slopes $s$ and $t$. Then the points in $\gamma \cap \delta$ partition $\gamma$ into $\nu(s, t)$ segments of equal length.
Let $\gamma$ and $\delta$ be closed geodesics in $S^2 \setminus P$ with distinct slopes $s$ and $t$, where $P$ is the set of four branch values of the quotient map from $T^2$ to $S^2$. Then the points in $\gamma \cap \delta$ partition $\gamma$ into $2\iota(s, t)$ segments whose lengths alternate between two (possibly equal) values.

We deal with a NET map $f$ with postcritical set $P_f$ for the rest of this section.

We continue by introducing some notation. Let $\gamma$ be a simple closed curve in $S^2 \setminus P_f$ which is neither peripheral nor null homotopic. It was noted at the end of the penultimate paragraph of Section 3 that $f$ maps every connected component of $f^{-1}(\gamma)$ to $\gamma$ with the same degree. Let $\tilde{d}(\gamma)$ denote this number. Let $c(\gamma)$ denote the number of these connected components which are neither peripheral nor null homotopic. We write $c(s) = c(\gamma)$ and $d(s) = d(\gamma)$, where $s$ is the slope of $\gamma$.

Here is the first theorem. Remark 7.4 shows that it is a strengthening of Theorem 5.3 in the case of NET maps.

**Theorem 7.2.** Let $\gamma$ and $\delta$ be simple closed curves in $S^2 \setminus P_f$ which are neither peripheral nor null homotopic. Let $\tilde{\gamma}$ and $\tilde{\delta}$ be connected components of $f^{-1}(\gamma)$ and $f^{-1}(\delta)$ which are neither peripheral nor null homotopic. Then

$$\iota(\tilde{\gamma}, \tilde{\delta}) \leq \frac{\tilde{d}(\gamma)\tilde{d}(\delta)}{\deg(f)} \iota(\gamma, \delta).$$

**Proof.** As in Theorem 2.1 of [2], we have that $f = h \circ g$, where $g$ is a Euclidean NET map and $h$ is a homeomorphism which maps the postcritical set $P_g$ of $g$ to $P_f$. The map $g$ preserves an affine structure on $S^2$ which is induced by a half-translation structure as in Lemma 7.1. The singular points of this half-translation structure are exactly the postcritical points of $g$. Let $\iota'$ denote the intersection pairing of $(S^2, P_g)$.

We may, and do, choose $\gamma$ and $\delta$ so that $h^{-1}(\gamma)$ and $h^{-1}(\delta)$ are geodesics with respect to the half-translation structure on $(S^2, P_g)$. Then so are all of the connected components of $f^{-1}(\gamma) = g^{-1}(h^{-1}(\gamma))$ and $f^{-1}(\delta)$. Hence, letting $|X|$ denote the cardinality of a set $X$,

$$|\gamma \cap \delta| = |h^{-1}(\gamma) \cap h^{-1}(\delta)| = 2\iota'(h^{-1}(\gamma), h^{-1}(\delta)) = 2\iota(\gamma, \delta).$$

So

$$|f^{-1}(\gamma) \cap \tilde{\delta}| = \tilde{d}(\delta) |\gamma \cap \delta| = 2\tilde{d}(\delta) \iota(\gamma, \delta).$$

Viewing the connected components of $f^{-1}(\gamma)$ and $f^{-1}(\delta)$ as connected components of $g^{-1}(h^{-1}(\gamma))$ and $g^{-1}(h^{-1}(\delta))$, we see using Lemma 7.1 that every connected component of $f^{-1}(\gamma)$ meets $\tilde{\delta}$ in $|\gamma \cap \tilde{\delta}|$ points. Since the connected components of $f^{-1}(\gamma)$ all map to $\gamma$ with degree $d(\gamma)$, the number of them is $\deg(f)/d(\gamma)$. Hence

$$2\iota(\tilde{\gamma}, \tilde{\delta}) \leq |\gamma \cap \tilde{\delta}| = \frac{d(\gamma)}{\deg(f)} |f^{-1}(\gamma) \cap \tilde{\delta}| = \frac{2d(\gamma)d(\delta)}{\deg(f)} \iota(\gamma, \delta).$$

This proves Theorem 7.2.

**Corollary 7.3.** In the situation of Theorem 7.2,

$$\iota(\tilde{\gamma}, \tilde{\delta}) \leq \frac{d(\gamma)}{c(\delta)} \iota(\gamma, \delta).$$
Proof. This follows from Theorem 7.2 and the fact that \( c(\delta)d(\delta) \leq \deg(f) \).

\[
\text{□}
\]

**Remark 7.4.** It is possible to recover the inequality in Theorem 5.3 for NET maps from Corollary 7.3. Indeed, Corollary 7.3 implies that

\[
\iota(\tilde{\gamma}, \tilde{\delta}) \leq d(\gamma)c(\delta)\iota(\gamma, \delta)
\]

The inequality in Theorem 5.3 for NET maps results from this and the fact that if two positive real numbers are both greater than another, then so is their geometric mean. So Corollary 7.3 is stronger than Theorem 5.3 for NET maps.

We next give an extension of Corollary 7.3 to the case in which one of the simple closed curves is replaced by a core arc. This will be used in the proof of Theorem 8.7.

**Theorem 7.5.** Let \( \alpha \) be a core arc for \((S^2, P_f)\), and let \( \delta \) be a simple closed curve in \( S^2 \setminus P_f \) which is neither peripheral nor null homotopic. Let \( \tilde{\alpha} \) be a union of core arcs in \( f^{-1}(\alpha) \), and let \( \tilde{\delta} \) be a connected component of \( f^{-1}(\delta) \) which is neither peripheral nor null homotopic. Let \( d(\tilde{\alpha}) \) be the number of lifts of \( \alpha \) in \( \tilde{\alpha} \). Then

\[
\iota(\tilde{\alpha}, \tilde{\delta}) \leq d(\tilde{\alpha})c(\delta)\iota(\alpha, \delta).
\]

Proof. We may, and do, assume that \( \iota(\alpha, \delta) = |\alpha \cap \delta| \). So

\[
|\tilde{\alpha} \cap f^{-1}(\delta)| = d(\tilde{\alpha})\iota(\alpha, \delta).
\]

The set \( f^{-1}(\delta) \) contains \( c(\delta) \) connected components homotopic to \( \tilde{\delta} \). These curves each meet \( \tilde{\alpha} \) in no fewer than \( \iota(\tilde{\alpha}, \tilde{\delta}) \) points. Thus

\[
c(\delta)\iota(\tilde{\alpha}, \tilde{\delta}) \leq d(\tilde{\alpha})\iota(\alpha, \delta).
\]

This proves Theorem 7.3.

\[
\text{□}
\]

Just as Theorem 7.5 extends Corollary 7.3, the next theorem shows that if we replace one of the simple closed curves in Theorem 7.2 by a core arc, then we obtain almost the same result. The previous result is weakened mainly by the introduction of the ceiling function. Theorem 7.6 will be used in the proof of Theorem 11.1.

**Theorem 7.6.** Let \( \alpha \) be a core arc for \((S^2, P_f)\), and let \( \delta \) be a simple closed curve in \( S^2 \setminus P_f \) which is neither peripheral nor null homotopic. Let \( \tilde{\alpha} \) be a connected union of core arcs in \( f^{-1}(\alpha) \). Let \( d(\tilde{\alpha}) \) be the number of lifts of \( \alpha \) in \( \tilde{\alpha} \). Let \( \tilde{\delta} \) be a connected component of \( f^{-1}(\delta) \) which is neither peripheral nor null homotopic. Then

\[
\iota(\tilde{\alpha}, \tilde{\delta}) \leq 2 \left[ \frac{d(\tilde{\alpha})d(\delta)}{2 \deg(f)} \iota(\alpha, \delta) \right].
\]

Proof. Let \( \tilde{\alpha}' \) be the connected component of \( f^{-1}(\alpha) \) which contains \( \tilde{\alpha} \). Suppose that \( \tilde{\alpha}' \) is a simple closed curve. The case in which \( \tilde{\alpha}' \) is an arc will be left to the reader. Let \( \gamma \) be a simple closed curve in \( S^2 \setminus P_f \) with core arc \( \alpha \). We argue as in the proof of Theorem 7.2, reducing
to the case in which $\tilde{\alpha}'$ and $\tilde{\delta}$ are geodesics with respect to a half-translation structure. As in the proof of Theorem 7.2 we have that

$$\left|\tilde{\alpha}' \cap \tilde{\delta}\right| = \frac{2d(\gamma)d(\delta)}{\deg(f)} \iota(\gamma, \delta) = \frac{2d(\gamma)d(\delta)}{\deg(f)} \iota(\alpha, \delta).$$

Lemma 7.1 implies that the points in $\tilde{\alpha}' \cap \tilde{\delta}$ can be partitioned into two interlacing subsets $A$ and $B$ such that each partitions $\tilde{\alpha}'$ into segments of equal lengths. Since $\tilde{\alpha}$ contains $d(\tilde{\alpha})$ consecutive lifts of the $2d(\gamma)$ lifts of $\alpha$ in $\tilde{\alpha}'$ and these have equal lengths,

$$|\tilde{\alpha} \cap A| \leq \left\lceil \frac{d(\tilde{\alpha})}{2d(\gamma)} |A| \right\rceil$$

and

$$|\tilde{\alpha} \cap B| \leq \left\lceil \frac{d(\tilde{\alpha})}{2d(\gamma)} |B| \right\rceil.$$

Hence

$$\iota(\tilde{\alpha}, \tilde{\delta}) \leq |\tilde{\alpha} \cap \tilde{\delta}| = |\tilde{\alpha} \cap A| + |\tilde{\alpha} \cap B| \leq \left\lceil \frac{d(\tilde{\alpha})d(\delta)}{2 \deg(f)} \iota(\alpha, \delta) \right\rceil.$$

This yields Theorem 7.6 if $\tilde{\alpha}'$ is a simple closed curve. Essentially the same argument proves Theorem 7.6 when $\tilde{\alpha}'$ is an arc.

\[\square\]

8. The half-space theorem

In this section we reprove part of the half-space theorem [2, Theorem 6.7], we strengthen it in the case of NET maps and we discuss some of its limitations.

We return to the setting of a Thurston map $f$ whose postcritical set $P_f$ contains exactly four points. Let $\mu_f$ be the slope function of $f$ with respect to some marking (Section 2) of $S^2 \setminus P_f$. Let $s$ be a slope. If $\mu_f(s) \notin \{s, \circ\}$, then the half-space theorem provides an explicit open interval in $\partial \mathbb{H}$, called an excluded interval, containing $-1/s$ which does not contain the negative reciprocal of an obstruction for $f$. Theorem 8.4 shows that the following theorem in effect generalizes this result.

**Theorem 8.1.** Let $f$ be a Thurston map with exactly four postcritical points. Let $s = \frac{p}{q}$ be a slope in reduced form, and suppose that $\mu_f(s) = s' = \frac{p'}{q'}$, also an extended rational number in reduced form. Let $\rho$ be the multiplier for $s$. Let $\rho_0$ be a positive real number. Then the interval in $\partial \mathbb{H}$ defined by

$$|px + q|^2 < \rho \rho_0 |p'x + q'|^2$$

does not contain the negative reciprocal of a fixed point of $\mu_f$ with multiplier at least $\rho_0$.

**Proof.** Suppose that $s_2 = \frac{p_2}{q_2}$ is a fixed point of $\mu_f$ with multiplier $\rho_2$. Theorem 5.3 with $s_1 = s$ implies that

$$\sqrt{\rho \rho_2} \iota\left(\frac{p}{q}, \frac{p_2}{q_2}\right) \leq \iota\left(\frac{p}{q}, \frac{p_2}{q_2}\right).$$

Hence

$$\rho \rho_2 |p'q_2 - q'p_2|^2 \leq |pq_2 - qp_2|^2$$

and

$$\rho \rho_2 |p'x + q'|^2 \leq |px + q|^2,$$

where $x = -\frac{q_2}{p_2}$. This easily proves Theorem 8.1. \[\square\]
Corollary 8.2. Under the assumptions of Theorem 8.1, the interval in $\partial \mathbb{H}$ defined by
\[ |px + q|^2 < \rho |p'x + q'|^2 \]
does not contain the negative reciprocal of the slope of an obstruction for $f$.

Corollary 8.3. Under the assumptions of Theorem 8.1, the interval in $\partial \mathbb{H}$ defined by
\[ \deg(f) |px + q| < |p'x + q'| \]
does not contain the negative reciprocal of a fixed point of $\mu_f$.

We refer to the intervals which appear in Theorem 8.1, Corollary 8.2 and Corollary 8.3 as excluded intervals. The following theorem shows that the excluded interval in Corollary 8.2 equals the excluded interval in Theorem 6.7 of [2], thus providing a somewhat different interpretation of this interval.

Theorem 8.4. If $\frac{p'}{q'} \neq \frac{p}{q}$, then the excluded interval interval in Corollary 8.2 equals the excluded interval in Theorem 6.7 of [2].

Proof. We have distinct slopes $t = \frac{p}{q}$ and $t' = \frac{p'}{q'} = \mu_f(t)$, where $p$, $q$ and $p'$, $q'$ are relatively prime integers. The interval in Theorem 6.7 of [2] is described in terms of horoballs $B_m(t)$ and $B_{pm}(t')$, where $\rho$ is the multiplier for $t$. See Figure 4. These horoballs are tangent. There is a unique hyperbolic geodesic tangent to both of these horoballs at their point of tangency. This geodesic is the boundary of a hyperbolic half-space $H$ which contains $B_m(t)$. The ideal boundary of $H$ is the excluded interval in Theorem 6.7 of [2].

The statement of Corollary 8.2 leads us to consider the matrix $\begin{bmatrix} p & q \\ p' & q' \end{bmatrix}$. We multiply its second row by $-1$ if necessary so that its determinant is positive. Let
\[ \varphi(z) = \frac{pz + q}{p'z + q'} \]
We wish to identify $\varphi(H)$.

For this, note that because $\gcd(p', q') = 1$, there exist integers $r$ and $s$ such that $\begin{bmatrix} r & s \\ p' & q' \end{bmatrix} \in \text{SL}(2, \mathbb{Z})$. So there exist integers $a$, $b$, $c$, $d$ such that
\[ \begin{bmatrix} p & q \\ p' & q' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} r & s \\ p' & q' \end{bmatrix} \]
Let
\[ \psi(z) = \frac{rz + s}{p'z + q'} \]
Because \( \varphi(-\frac{q'}{p'}) = \psi(-\frac{q'}{p'}) = \infty \), we have that \( c = 0 \). Now by computing the second row of the above matrix product, we find that \( d = 1 \). By taking determinants, we find that \( a = \iota(t, t') \). Corollary 6.2 of [2] implies that \( \psi(B_{pm}(\frac{p'}{q'})) = B_{pm}(0) \). It follows that \( \varphi(B_{pm}(\frac{p'}{q'})) = B_{apm}(0) \).

It is clear that \( \varphi(B_m(\frac{p}{q})) \) is a horoball tangent to \( \partial \mathbb{H} \) at \( 0 \). So \( \varphi(H) \) is the half-space tangent to \( \{ z \in \mathbb{H} : \text{Im}(z) > apm \} \) at the point \( apm \sqrt{-1} \) and whose ideal boundary contains \( 0 \). Lemma 5.1 implies that \( m \sqrt{\rho} = \iota(t, t')^{-1} = a^{-1} \). So \( apm = \sqrt{\rho} \). Therefore \( H \) is the half-space defined by the inequality in Corollary 8.2.

This proves Theorem 8.4.

So for Thurston maps with four postcritical points, the excluded interval in Corollary 8.2 is the same as the excluded interval in the half-space theorem. Since the proof of Corollary 8.2 is based on the Lipschitz-type inequality in Theorem 5.3 and Theorem 7.2 strengthens Theorem 5.3 for NET maps, one might expect to also obtain a strengthening of Corollary 8.2 for NET maps. This is what the following theorem does, as can easily be verified.

**Theorem 8.5** (The NET map half-space theorem). Let \( f \) be a NET map. Let \( s = \frac{p}{q} \) be a slope in reduced form, and suppose that \( \mu_f(s) = s' = \frac{p'}{q'} \), also an extended rational number in reduced form. Let \( d = d(s) \), defined early in Section 7. Let \( e \) be the smallest positive divisor of \( \deg(f) \) such that \( e^2 \geq \deg(f) \). Then the interval in \( \partial \mathbb{H} \) defined by

\[
d|px + q| < e|p'x + q'|
\]

does not contain the negative reciprocal of the slope of an obstruction for \( f \).

**Proof.** Let \( \gamma \) be a simple closed curve in \( S^2 \setminus P_f \) with slope \( s \), so that \( d = d(\gamma) \). Let \( \delta \) be a simple closed curve in \( S^2 \setminus P_f \) which is an obstruction for \( f \). The multiplier of \( \delta \) is \( \frac{c(\delta)}{d(\delta)} \). Because \( \delta \) is an obstruction, \( \frac{c(\delta)}{d(\delta)} \geq 1 \). Because \( f \) is a NET map, Theorem 4.1 of [2] implies that \( d(\delta) \) divides \( \deg(f) \). Of course, \( c(\delta)d(\delta) \leq \deg(f) \).

Thus

\[
\left( \frac{\deg(f)}{d(\delta)} \right)^2 \geq \frac{c(\delta)d(\delta)}{d(\delta)^2} \deg(f) \geq \frac{c(\delta)}{d(\delta)} \cdot \deg(f) \geq \deg(f).
\]

So the minimality of \( e \) implies that \( \frac{\deg(f)}{d(\delta)} \geq e \).

Now we apply Theorem 7.2 to \( \gamma \) and \( \delta \). The curve \( \tilde{\gamma} \) in Theorem 7.2 has slope \( \frac{p'}{q'} \), and \( \tilde{\delta} \) is homotopic to \( \delta \) relative to \( P_f \). Theorem 7.2 and the previous paragraph imply that

\[
\iota(\gamma, \tilde{\delta}) \leq \frac{d(\gamma)d(\delta)}{\deg(f)} \iota(\gamma, \delta) \leq \frac{d}{e} \iota(\gamma, \delta).
\]

It is a straightforward matter to complete the proof of Theorem 8.5.

In the same way, we have the following NET map analog of Corollary 8.3.

**Theorem 8.6.** Under the assumptions of Theorem 8.5, the interval in \( \partial \mathbb{H} \) defined by

\[
d|px + q| < |p'x + q'|
\]
does not contain the negative reciprocal of a fixed point of \( \mu_f \).
The rest of this section is devoted to investigating limitations of the half-space theorem. The main results are in Theorems 8.10 and 8.12. Theorems 8.10 and 8.12 rest on Theorem 8.7.

This uses the notation \( c(s) \) and \( d(s) \) from early in Section 7.

**Theorem 8.7.** Let \( f \) be a NET map with postcritical set \( P_f \). Let \( \alpha \) be a core arc for \((S^2, P_f)\) having slope \( s \) with respect to a fixed marking of \((S^2, P_f)\). Suppose that there is a core arc \( \tilde{\alpha} \) in \( f^{-1}(\alpha) \) with slope \( s \) which \( f \) maps to \( \alpha \) with degree 1. Then we have the following.

1. If \( t \) is a slope with \( t \neq s \) which is fixed by \( \mu_f \), then \( c(t) = 1 \).
2. Every excluded open interval for \( f \) arising from the half-space theorem [2, Theorem 6.7] omits \(-1/s\).
3. If \( c(t)d(t) > 1 \) for every slope \( t \) such that \( c(t) \neq 0 \), then the closure in \( \partial \mathbb{H} \) of every excluded interval for \( f \) which arises from the half-space theorem [2, Theorem 6.7] omits \(-1/s\).

**Remark 8.8.** The assumptions imply that \( \alpha \) has a lift which is a core arc either homotopic relative to \( P_f \) to \( \alpha \) or homotopic relative to \( P_f \) to a core arc disjoint from \( \alpha \). In both cases, statement 1 follows from the proof of Theorem 3.2 of [12].

**Proof.** To prove statement 1, we apply Theorem 7.5. We choose the simple closed curve \( \delta \) there to have slope \( t \). The curve \( \tilde{\delta} \) is homotopic to \( \delta \) relative to \( P_f \). We have by assumption that \( d(\tilde{\alpha}) = 1 \). So Theorem 7.5 implies that \( c(t)\nu(s,t) \leq \nu(s,t) \). But \( \nu(s,t) \neq 0 \) because \( t \neq s \). Hence \( c(t) = 1 \). This proves statement 1.

To prove statement 2, we apply Theorem 7.5 again. We take the core arc \( \alpha \) there to be the present \( \alpha \). Suppose that the simple closed curve \( \delta \) in Theorem 7.5 has slope \( \frac{p}{q} = t \) and that \( \tilde{\delta} \) has slope \( \frac{p'}{q'} \neq t \), where \( p, q \) and \( p', q' \) are two pairs of relatively prime integers. Theorem 7.5 implies that \( c(t)\nu(s,t') \leq \nu(s,t) \). Hence

\[
c(t)|p'(-1/s) + q'| \leq |p(-1/s) + q|,
\]

and so

\[
|p(-1/s) + q|^2 \geq c(t)^2|p'(-1/s) + q'|^2 \geq \frac{c(t)}{d(t)}|p'(-1/s) + q'|^2.
\]

This and Theorem 8.4 imply that \( -\frac{1}{s} \) is not in the excluded interval arising from the half-space theorem applied to slope \( t \). This proves statement 2.

To prove statement 3, we focus on the second inequality in the last display. If it is an equality, then either \( s = t' \) or \( c(t) = d(t) = 1 \). Since \( c(t)d(t) > 1 \),

\[
|p(-1/s) + q|^2 > \frac{c(t)}{d(t)}|p'(-1/s) + q'|^2.
\]

This proves statement 3.

**Remark 8.9.** The condition \( c(t)d(t) > 1 \) in statement 3 of Theorem 8.7 is satisfied if both elementary divisors [3, Section 8] of \( f \) are greater than 1. Indeed, if both elementary divisors of \( f \) are greater than 1, then \( d(t) > 1 \) for every slope \( t \). To prove this, let \( t = \frac{p}{q} \), expressed in reduced form. Statement 1 of Theorem 4.1 in [2] implies that \( d(t) \) equals the order of the image of \((q,p) \in \mathbb{Z}^2 \) in the group \( \mathbb{Z}^2/\Lambda_1 \), with \( \Lambda_1 \) [3, Section 3] as usual. We have that \( \Lambda_1 \subseteq n\mathbb{Z}^2 \), where \( n \) is the smaller elementary divisor of \( f \). By assumption, \( n > 1 \). Because \( p \) and \( q \) are relatively prime, \( n|d(t) \). In particular, \( d(t) > 1 \). Thus \( c(t)d(t) > 1 \) if \( c(t) \neq 0 \).
Now we strengthen the assumptions in Theorem 8.7 and obtain a stronger conclusion.

**Theorem 8.10.** Let $f$ be a NET map with postcritical set $P_f$. Suppose that neither elementary divisor of $f$ equals 1. Let $\alpha$ be a core arc for $(S^2, P_f)$ having slope $s$ with respect to a fixed marking of $(S^2, P_f)$. Suppose that there is a core arc in $f^{-1}(\alpha)$ with slope $s$ which $f$ maps to $\alpha$ with degree 1. Suppose that $s$ is not the slope of an obstruction. Then $f$ is Thurston equivalent to a rational map. Moreover, the closure of every excluded interval for $f$ which arises from the half-space theorem [2, Theorem 6.7] omits $-1/s$. Hence every finite union of these excluded intervals omits an interval of $\partial \mathbb{H}$.

**Proof.** We begin by proving that the orbifold of $f$ is hyperbolic. We proceed by contradiction. Suppose that the orbifold of $f$ is Euclidean. In this case, the assumption that $\bar{\alpha}$ maps to $\alpha$ with degree 1 implies that $d(s) = 1$. This contradicts Remark 8.9. Thus the orbifold of $f$ is hyperbolic.

So to prove that $f$ is rational, it suffices to prove that it has no obstruction. We have assumed that $s$ is not the slope of an obstruction. Statement 1 of Theorem 8.7 implies that if $t$ is a slope with $t \neq s$ which is fixed by $\mu_f$, then $c(t) = 1$. This contradicts Remark 8.9. Thus the orbifold of $f$ is hyperbolic.

The rest of Theorem 8.10 follows from statement 3 of Theorem 8.7. 

**Remark 8.11.** It is easy to satisfy the conditions of Theorem 8.10. For example, the NET map with the presentation diagram (for which see [5]) in Figure 5 satisfies the conditions of Theorem 8.10. We may take $s$ to be either 0 or $\infty$. Moreover, one can verify using Lemma 4.2 of [2] that every multiplier is less than 1 for this example, and so the modular group Hurwitz class (for which see [6]) of this map is completely unobstructed.

The next theorem provides conditions under which the union of the excluded intervals arising from the half-space theorem omits infinitely many points.

**Theorem 8.12.** Let $f$ be a NET map with postcritical set $P_f$. Let $\alpha$ be a core arc for $(S^2, P_f)$ having slope $s$ with respect to a fixed marking of $(S^2, P_f)$. Suppose that there is a core arc in $f^{-1}(\alpha)$ with slope $s$ which $f$ maps to $\alpha$ with degree 1. Suppose that $t \neq s$ is the slope of an obstruction. Then there exists an infinite sequence of points in $\partial \mathbb{H}$ converging to $-1/t$ such that no element in this sequence is contained in an excluded interval for $f$ which arises from the half-space theorem [2, Theorem 6.7].

**Proof.** Statement 1 of Theorem 8.7 implies that $c(t) = 1$. Since $t$ is an obstruction, $c(t)/d(t) \geq 1$. So $d(t) = 1$ and the multiplier of $t$ is 1.
Now let $\tau$ be a (nonzero power of a primitive) Dehn twist about a simple closed curve with slope $t$ whose homotopy class in the modular group of $f$ is liftable. The fact that $t$ is an obstruction for $f$ with multiplier 1 implies that $f$ and $\tau$ commute up to homotopy relative to $P_f$. It follows that some core arc in $f^{-1}(\tau^n(\alpha))$ with the same slope as $\tau^n(\alpha)$ maps to $\tau^n(\alpha)$ with degree 1 for every integer $n$. Let $\sigma_\tau$ denote the pullback map induced by $\tau$ on $H$, much as $f$ induces a pullback map $\sigma_f$ on $H$. Now statement 2 of Theorem 8.7 implies that every excluded interval for $f$ which arises from the half-space theorem omits $\sigma_\tau^{-1}(-1/s)$ for every integer $n$. These points converge to the fixed point of $\sigma_\tau$, namely $-1/t$, as $n$ tends to $\infty$.

This proves Theorem 8.12.

Remark 8.13. It is easy to satisfy the conditions of Theorem 8.12. For example, the NET map with the presentation diagram (for which see [5]) in Figure 6 satisfies the conditions of Theorem 8.12 with $s = 0$ and $t = \infty$. This is $21H\text{Class3}$ in the census of modular group Hurwitz class representatives in [10]. The proof of Theorem 8.12 implies that no excluded interval for this NET map contains the reciprocal of an integer. In fact, the output for the computer program NETmap in [10] suggests that no excluded interval for this NET map contains 2 times the reciprocal of an integer.

9. A property of the modular group virtual multi-endomorphism

The goal of this section is to prove a somewhat technical result in Lemma 9.1. This will be used in Section 10 to bound the number of slope function fixed points. In this section and the next, we work with a general Thurston map with four postcritical points, not necessarily a NET map. We need some preparations to state Lemma 9.1.

Let $f$ be a Thurston map with exactly four postcritical points. Let $G$ be the modular group of $f$. It is well known [3, Proposition 2.7] that $G \cong \text{PSL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}/2\mathbb{Z})^2$. Following Section 3 of [6], we say that an element of $G$ is elliptic, parabolic or hyperbolic according to whether its PSL(2,\mathbb{Z})-factor is elliptic, parabolic or hyperbolic. The remaining elements of $G$ are translations in $T = (\mathbb{Z}/2\mathbb{Z})^2$. Suppose that we have an element $\psi \in G$ whose first factor is the image of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2,\mathbb{Z})$. Then Proposition 4.1 of [6] shows that the pullback map $\sigma_\psi: \mathbb{H} \to \mathbb{H}$ induced by $\psi$ is given by $\sigma_\psi(z) = \frac{dz+b}{cz+a}$. So, as in Corollary 4.2 of [6], the linear fractional transformation $\sigma_\psi$ is elliptic, parabolic or hyperbolic if and only if $\psi$ is elliptic, parabolic or hyperbolic.

An element $\psi \in G$ is liftable for $f$ if and only if there exists $\tilde{\psi} \in G$ such that $\psi[f] = [f] \tilde{\psi}$, where $[f]$ is the homotopy class of $f$ relative to its postcritical set. The element $\tilde{\psi}$ is a lift of $\psi$. The set $G_f$ of all liftables for $f$ in $G$ is a subgroup of $G$. The element $\tilde{\psi}$ need not be unique, and so the assignment $\psi \mapsto \tilde{\psi}$ is a multifunction, the modular group virtual multi-endomorphism for $f$. This multi-endomorphism maps the identity element of $G_f$ to a
subgroup $\text{DeckMod}(f)$ of $G$. For every element of $G_f$, the set of all its lifts is a right coset of $\text{DeckMod}(f)$.

Now we assume that the Thurston pullback map $\sigma_f$ is not constant. If $\psi \in \text{DeckMod}(f)$, then $[f]\psi = [f]$. Hence $\sigma_\psi \circ \sigma_f = \sigma_f$. Thus $\sigma_\psi$ is the identity element because $\sigma_f$ is not constant. It follows that the map $\psi \mapsto \sigma_\psi$ for $\psi \in G_f$ is a group antihomomorphism: $\sigma_\varphi \varphi^{-1} = \sigma_\psi \sigma_\varphi^{-1}$ for $\varphi, \psi \in G_f$. In particular, it is a function even if the modular group virtual multi-endomorphism is not. The same argument shows that this multi-endomorphism maps translations in $G_f$ to translations. This brings us to Lemma 9.1.

**Lemma 9.1.** Let $f$ be a Thurston map with exactly four postcritical points. We assume that the orbifold of $f$ is hyperbolic and that the Thurston pullback map $\sigma_f$ of $f$ is not constant. Let $\varphi$ be an element of $\text{PSL}(2, \mathbb{Z})$. Then the set $K$ of all elements $\psi \in G_f$ such that $\sigma_\psi = \varphi^{-1} \sigma_\varphi \varphi$ is a subgroup of $G_f$. Furthermore, $T \cap G_f$ is a normal subgroup of $K$, $K/(T \cap G_f)$ is cyclic and $K$ contains no hyperbolic elements.

**Proof.** Using the fact that the map $\psi \mapsto \sigma_\psi$ for $\psi \in G_f$ is a group antihomomorphism, one verifies that $K$ is a subgroup of $G_f$. Because the modular group virtual multi-endomorphism of $f$ maps translations in $G_f$ to translations, $T \cap G_f \subseteq K$. This is a normal subgroup of $G_f$ as well as $K$ because $T$ is a normal subgroup of $G$. Although stated for NET maps, Theorem 4.4 of [6] holds in the present more general situation. Statement 3 of Theorem 4.4 of [6] implies that if $\sigma_\psi$ is hyperbolic, then the absolute value of the trace of $\sigma_\psi$ is less than the absolute value of the trace of $\sigma_\varphi$. Because conjugation preserves trace, it follows that $K$ contains no hyperbolic elements. Thus to prove Lemma 9.1 it only remains to prove that $K/(T \cap G_f)$ is cyclic.

To prove that $K/(T \cap G_f)$ is cyclic, it suffices to prove that every subgroup of $\text{PSL}(2, \mathbb{Z})$ which is not cyclic contains a hyperbolic element. So let $H$ be a subgroup of $\text{PSL}(2, \mathbb{Z})$ which is not cyclic. We use the fact that $\text{PSL}(2, \mathbb{Z})$ is the free product of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$. (See [14 I.4.2].) The corollary in [14 I.4.3] implies that if $H$ is finite, then it is conjugate to a subgroup of one of these free factors. Since $H$ is not cyclic, it must be infinite. Hence it has a nontrivial intersection with the kernel of a surjective group homomorphism from $\text{PSL}(2, \mathbb{Z})$ to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ arising from the free product decomposition of $\text{PSL}(2, \mathbb{Z})$. Proposition 18 of [14 I.4.3] implies that this kernel is a free group. Hence $H$ contains an element of infinite order. Since $H$ contains an element of infinite order, it contains either a hyperbolic element or a parabolic element. In the former case we are done. So suppose that $H$ contains a parabolic element. Then $H$ contains a positive parabolic element $\eta_1$ in the sense of Lemma 5.2. Suppose that $\eta_1$ fixes the cusp $s_1$. Since $H$ is not cyclic, it contains an element $\gamma$ such that $\gamma(s_1) = s_2 \neq s_1$. Then $\eta_2 = \gamma \eta_1 \gamma^{-1}$ is a positive parabolic element of $H$ which fixes $s_2$. Lemma 5.2 now implies that the trace of a matrix in $\text{SL}(2, \mathbb{Z})$ which represents $\eta_1 \eta_2^{-1}$ is greater than 2. So $\eta_1 \eta_2^{-1}$ is a hyperbolic element of $H$. This proves that every subgroup of $\text{PSL}(2, \mathbb{Z})$ which is not cyclic contains a hyperbolic element.

$\square$

## 10. Bounding the number of fixed points

This section is devoted to bounding the number of slope function fixed points of a Thurston map with four postcritical points. The statement of the result in Theorem 10.1 requires some definitions.
We prepare for these definitions by discussing two modular group actions on \( \mathbb{Q} = \mathbb{Q} \cup \{ \infty \} \). Let \( f \) be a Thurston map with postcritical set \( P_f \) containing exactly four points. Let \( \mu_f : \mathbb{Q} \to \mathbb{Q} \cup \{ \infty \} \) be its slope function and let \( \sigma_f : \mathbb{H} \to \mathbb{H} \) be its Thurston pullback map with respect to some marking (Section 2) of \((S^2, P_f)\). Selinger proved in [13] that \( \sigma_f \) extends continuously to augmented Teichmüller space \( \mathbb{H} \cup \mathbb{Q} \). If \( s \in \mathbb{Q} \) and \( \mu_f(s) \in \mathbb{Q} \), then \( \mu_f(s) = -1/\sigma_f(-1/s) \).

Let \( G \) be the modular group of \((S^2, P_f)\). Every element \( \psi \in G \) induces a pullback map \( \mu_\psi : \mathbb{Q} \to \mathbb{Q} \) on slopes and a pullback map \( \sigma_\psi : \mathbb{H} \cup \mathbb{Q} \to \mathbb{H} \cup \mathbb{Q} \) on augmented Teichmüller space. The map \( z \mapsto -1/z \) conjugates these linear fractional transformations to each other. So, we obtain a slope (right) action of \( G \) on \( \mathbb{Q} \) by means of the maps \( \mu_\psi \) and we obtain a cusp (right) action of \( G \) on \( \mathbb{Q} \) by means of the maps \( \sigma_\psi \). The map \( z \mapsto -1/z \) conjugates one action to the other.

Now we make the aforementioned definitions. As in the previous section, let \( G_f \) be the group of liftables for \( f \) in the modular group \( G \). We define three positive integers \( C_f, D_f \) and \( E_f \). Proposition 3.1 of [8] implies that the index of \( G_f \) in \( G \) is finite. Because \( G \) acts transitively on \( \mathbb{Q} \) and \( [G : G_f] < \infty \), the group \( G_f \) acts on \( \mathbb{Q} \) via either its action on slopes or its action on cusps with the same finite number of orbits. Let \( C_f \) be the number of these orbits. Let \( D_f \) be 2 times the least common multiple of the positive integers less than or equal to \( \deg(f) \). If \( G_f \) contains an elliptic element of order 3, then \( E_f = 3 \). Otherwise, if \( G_f \) contains an elliptic element of order 2, then \( E_f = 2 \). Otherwise, \( E_f = 1 \). This notation is chosen because \( C_f \) deals with \( G_f \)-orbits of cusps, \( D_f \) deals with the degree of \( f \) and \( E_f \) deals with the elliptic elements of \( G_f \).

When referring to slope function fixed points, we are referring to elements of \( \mathbb{Q} \), not the nonslope. This brings us to Theorem 10.1.

**Theorem 10.1.** Let \( f \) be a Thurston map with exactly four postcritical points whose orbifold is hyperbolic. Let \( \mu_f \) be its slope function, and let \( G_f \) be the subgroup of liftables in its modular group \( G \).

1. If \( \mu_f \) has more than \( C_f D_f E_f \) fixed points, then \( f \) has an obstruction with multiplier 1.
2. If \( f \) has an obstruction with multiplier 1 and \( \mu_f \) has more than one fixed point, then \( \mu_f \) has infinitely many fixed points.
3. If \( f \) has an obstruction with multiplier 1, then the stabilizer in \( G_f \) of this obstruction acts on the set of fixed points of \( \mu_f \) with at most \( C_f D_f \) orbits.

**Remark 10.2.** We note that \( f \) has an obstruction with multiplier 1 if and only if there exists a Dehn twist in the modular group of \( f \) which commutes with \( f \) up to homotopy relative to the postcritical set of \( f \).

**Proof.** Rather than working directly with slopes, we work with cusps, that is, negative reciprocals of slopes in the boundary of the upper half-plane \( \mathbb{H} \). Let \( \sigma_f \) be the Thurston pullback map of \( f \) on \( \mathbb{H} \). Similarly, we let \( \sigma_\psi \) denote the pullback map of \( \psi \in G \). We use the action of \( G \) on \( \mathbb{Q} \) which arises from the group antihomomorphism \( \psi \mapsto \sigma_\psi \). As discussed before Lemma 9.1 in the previous section, the assignment \( \psi \mapsto \sigma_\psi \) is a function, a group antihomomorphism, even when the modular group virtual multi-endomorphism \( \psi \mapsto \tilde{\psi} \) is not a function.

To prove statement 1, suppose that \( \mu_f \) has more than \( C_f D_f E_f \) fixed points. Then \( \sigma_f \) fixes more than \( C_f D_f E_f \) cusps in \( \mathbb{Q} \). Since the cusp action of \( G_f \) on \( \mathbb{Q} \) has \( C_f \) orbits, one of these
orbits contains more than \(D_f E_f\) cusps \(t_1, t_2, t_3, \ldots, t_N\) fixed by \(\sigma_f\). So \(N > D_f E_f\). Since all obstructions are homotopic, we may, and do, choose \(t_1\) so that its negative reciprocal is not the slope of an obstruction.

Since \(t_1, \ldots, t_N\) are in the same orbit for the action of \(G_f\) on \(\mathbb{Q}\), there exist elements \(\psi_1, \ldots, \psi_N\) in \(G_f\) such that \(\sigma_{\psi_n}(t_1) = t_n\) for every \(n \in \{1, \ldots, N\}\). Let \(n \in \{1, \ldots, N\}\). Then

\[
 t_n = \sigma_f(t_n) = \sigma_f(\psi_n(t_1)) = \sigma_{\psi_n}(\sigma_f(t_1)) = \sigma_{\psi_n}(t_1).
\]

So \(\sigma_{\psi_n}^{-1} \sigma_{\psi_n}\) fixes \(t_1\). The image in \(\text{PSL}(2, \mathbb{Z})\) of the stabilizer of \(t_1\) in \(G_f\) is a cyclic (maximal parabolic) subgroup of \(\text{PSL}(2, \mathbb{Z})\). Let \(\tau\) be the square root of a Dehn twist in \(G_f\) such that \(\sigma_\tau\) generates this cyclic subgroup. Then \(\sigma_{\psi_n}^{-1} \sigma_{\psi_n} = \sigma_\tau^k\) for some integer \(k\).

In this paragraph we normalize \(\psi_n\) to bound the absolute value of \(k\) independently of \(n\). Let \(P_f\) be the postcritical set of \(f\). Let \(\gamma\) be a simple closed curve in \(S^2 \setminus P_f\) with slope \(t_1\). Consider the connected components of \(f^{-1}(\gamma)\). Let \(d\) be the least common multiple of the degrees with which \(f\) maps them to \(\gamma\). The multiplier \(\rho\) of \(t_1\) is the sum of the reciprocals of these degrees for which the associated connected component of \(f^{-1}(\gamma)\) is neither peripheral nor null homotopic. So \(\rho \cdot d = c \in \mathbb{Z}\). Then, as in Theorem 7.1 of [2], \(\tau^{2d} \in G_f\) and \(\tau^{2d} = \tau^{2c}\).

We maintain \(n \in \{1, \ldots, N\}\) and the integer \(k\) in the previous paragraph. Let \(\varphi_n = \tau^{2dr} \psi_n\) be the postcritical set of \(f\). Then \(\varphi_n \in G_f\),

\[
 \sigma_{\varphi_n}(t_1) = \sigma_{\psi_n}(\sigma_{\tau^{2dr}}(t_1)) = \sigma_{\psi_n}(t_1) = t_n
\]

and

\[
 \sigma_{\psi_n}^{-1} \sigma_{\psi_n} = \sigma_{\tau^{2d}}^{-1} \sigma_{\psi_n}^{-1} \sigma_{\tau^{2cr}} = \sigma_{\tau^{2d}}^{-1} \sigma_{\tau^{2cr}} = \sigma_{\tau^{2d}}^{-1} \sigma_{\tau^{2cr}} = \sigma_{\tau^{2d}}^{-1} \sigma_{\tau^{2cr}} = \sigma_{\tau^{2d}} \sigma_{\tau^{2cr}} = \sigma_{\tau^{k+2r(c-d)}}.
\]

Because \(t_1\) is not the negative reciprocal of an obstruction, \(d > c\). In particular, \(c < 0\). This shows that by replacing \(\psi_n\) by \(\tau^{2dr} \psi_n\) for the appropriate choice of \(r\), we have that \(0 \leq k < 2(d - c)\). We do this so that this inequality holds for every \(n \in \{1, \ldots, N\}\).

Clearly \(2(d - c) < 2d \leq D_f\). We also have that \(N > D_f E_f\). So there exists an integer \(k\) and a subset \(\mathcal{N}\) of \(\{1, \ldots, N\}\) with \(|\mathcal{N}| > E_f\) such that \(\sigma_{\psi_n}^{-1} \sigma_{\psi_n} = \sigma_\tau^k\) for every \(n \in \mathcal{N}\).

Now let \(m, n \in \mathcal{N}\). Let \(\psi = \psi_m^{-1} \psi_n\). Then

\[
 \sigma_\psi = \sigma_{\psi_m} \sigma_{\psi_n}^{-1} = \sigma_{\psi_m} \sigma_\tau^{-k} \sigma_{\psi_n}^{-1} = \sigma_{\psi_m} \sigma_{\psi_n}^{-1} = \sigma_\psi.
\]

It follows that \(\psi_m^{-1} \psi_n\) belongs to the set \(K\) of all elements \(\psi \in G_f\) such that \(\sigma_\psi = \sigma_\psi\) for every \(m, n \in \mathcal{N}\). Lemma [9.1] with \(\varphi = 1\) implies that \(K\) is a subgroup of \(G\) and that the image of \(K\) in \(\text{PSL}(2, \mathbb{Z})\) is a cyclic group which contains no hyperbolic elements. Since \(|\mathcal{N}| > E_f\), this image is infinite, generated by a parabolic element. Let \(\phi\) be an element of \(K\) which maps this parabolic element in the image of \(K\). Then \(\phi\) fixes the homotopy class of a simple closed curve \(\gamma\) in \(S^2 \setminus P_f\). Because \(\sigma_\phi = \sigma_\phi\), the curve \(\gamma\) is \(f\)-stable and its multiplier is 1. Thus \(\gamma\) is an obstruction with multiplier 1. This proves statement 1 of Theorem [10.1].

Minor modifications of the above argument prove statement 3.

We turn our attention to statement 2. Suppose that \(f\) has an obstruction with multiplier 1 and more than one slope function fixed point. Let \(t_0\) be the negative reciprocal of the obstruction’s slope. Then \(\sigma_f\) fixes \(t_0\) and one other extended rational number \(t\). The group \(G_f\) contains the homotopy class \(\tau\) of a (nonzero power of a primitive) Dehn twist about the obstruction. Because the obstruction’s multiplier is 1, this homotopy class is fixed by the modular group virtual multi-endomorphism \(\psi \mapsto \psi\) of \(f\). So

\[
 \sigma_f \sigma_\tau = \sigma_\tau \sigma_f = \sigma_\tau \sigma_f.
\]
Hence for every integer \( n \) we have that
\[
\sigma_f(\sigma^n_x(t)) = \sigma^n_f(\sigma_f(t)) = \sigma^n_x(t).
\]
Thus \( \sigma_f \) fixes every element in the orbit of \( t \) under the action of \( \langle \sigma_x \rangle \) on the set of extended rational numbers. This implies that \( \mu_f \) has infinitely many fixed points.

This completes the proof of Theorem 10.1. \( \square \)

11. NET maps with many formal matings

Using [6], Bill Floyd enumerated all possible NET map dynamic portraits [6, Section 9] through degree 40 and ran the computer program NETmap on NET maps having these dynamic portraits. The results are in the NET map website [10]. These computations revealed that some NET maps have many formal matings. The examples presented in this section are based on these computations.

To describe these NET maps, let \( n \) be an integer with \( n \geq 4 \). Let \( f_n \) be the NET map with the following presentation (for which see [5]). The associated affine map \( \Phi(x) = Ax + b \) has matrix \( A = \begin{bmatrix} n & -1 \\ 0 & 1 \end{bmatrix} \) and translation term \( b = (n, 0) \). There is a green line segment joining \((0, 0)\) and \((1, 0)\). There is a green line segment joining \((n, 0)\) and \((2, 0)\). The green line segments containing \((-1, 1)\), \((n - 1, 1)\) and \((2n - 1, 1)\) are trivial. This defines a NET map \( f_n \) with degree \( n \) up to Thurston equivalence. The presentation diagram for \( f_5 \) is shown in Figure 7.

**Theorem 11.1.** Let \( n \) be an integer with \( n \geq 4 \). Then the map \( f_n \) is Thurston equivalent to a rational map and it can be expressed as a formal mating in at least \( \left\lceil \frac{n}{2} \right\rceil \) ways. The slopes of \( \left\lceil \frac{n}{2} \right\rceil \) mating equators have the form \( \frac{2m}{n - 2m - 1} \), where \( m \) is an integer with \( 0 \leq m \leq \left\lfloor \frac{n-2}{2} \right\rfloor \).

**Proof.** To prove that \( f_n \) is Thurston equivalent to a rational map, we apply W. Thurston’s characterization theorem. The map \( f_n \) has a hyperbolic orbifold. So it suffices to prove that \( f_n \) has no obstruction.

We proceed by contradiction. Suppose that \( f_n \) has an obstruction \( \delta \) with slope \( t \). We maintain the notation \( c(t) = c(\delta) \) and \( d(t) = d(\delta) \) from early in Section 7. By assumption, \( \mu_{f_n}(t) = t \) and \( c(t)/d(t) \geq 1 \), where \( \mu_{f_n} \) is the slope function of \( f_n \). Although \( \mu_{f_n}(0) = 0 \), it is clear that \( c(0) = 1 \) and \( d(0) = n \). Hence \( t \neq 0 \).

Now we apply Theorem 8.7. We take \( \alpha \) there to be the image in \( S^2 \) of the line segment joining \((1, 0)\) and \((2, 0)\), so that \( \alpha \) has slope \( s = 0 \). We may, and do, take \( \tilde{\alpha} = \alpha \). Then \( d(\tilde{\alpha}) = 1 \). Theorem 8.7 implies that \( c(\delta) = 1 \). Moreover, \( d(\delta) = 1 \) because \( c(\delta)/d(\delta) \geq 1 \).

Now we apply Theorem 7.6 with the same \( \alpha \) and \( \delta \). We easily conclude that \( \iota(\alpha, \delta) \leq 2 \). So if we write \( t = \frac{p}{q} \), where \( p \) and \( q \) are relatively prime integers, then \( |p| \leq 2 \). We next apply Theorem 7.6 with \( \alpha \) there equal to the image of the line segment with slope 2 joining
We finally prove that \( \mu_{f_n}(s) = s \). To do this, we use the visual interpretation of slope function evaluation in Section \( \ref{sec:slope-evaluation} \). So we have a line segment \( S \) in \( \mathbb{R}^2 \) with slope \( s \) and endpoints \( v \) and \( w \) such that \( S \) maps to a simple closed curve in \( \mathbb{R}^2/\Gamma_1 \). We assume that \( S \) contains no elements of \( \mathbb{Z}^2 \). We are led to the action of \( 2\Lambda_1 \) on \( \mathbb{R}^2 \). The closed parallelogram with corners \((0,0), (2n,0), (-1, 1) \) and \((2n-1,1)\) is a fundamental domain for \( \Gamma_1 \). Doubling this, we obtain the closed parallelogram with corners \((0,0), (2n,0), (-2,2) \) and \((2n-2,2)\), which is a fundamental domain for \( 2\Lambda_1 \). The translates of the latter parallelogram by elements of \( 2\Lambda_1 \) form a tessellation \( \mathcal{T} \) of \( \mathbb{R}^2 \).

In the visual interpretation of the computation of \( \mu_{f_n}(s) \), a photon starts at \( v \) and traverses an initial segment of \( S \) until it hits a spin mirror. Our nontrivial spin mirrors have two sizes, small and large. Whether small or large, the photon spins about the center of this spin mirror. Then it travels parallel to \( S \) but in the direction opposite to the direction that it was
traveling just before it hits the spin mirror. It continues in this way. Let \( P \) be the photon’s path, the set of points traversed by the photon. There exists a canonical map \( \phi: P \to S \).

To analyze \( P \), we use the basis \( B = \mathbb{R}^2 \) consisting of \((1,0)\) and \((-1,1)\). We use the notation \((x,y)_B\) to denote the point \(x(1,0) + y(-1,1) \in \mathbb{R}^2\). So for every \((x,y) \in \mathbb{R}^2\), we have that \((x,y) = (x+y,y)_B\). Since the vector \((q,p)\) gives the direction of \( S \), the slope of \( S \) with respect to \( B \) is \( \frac{p}{p+q} = \frac{p}{n-1} \).

We put conditions on \( v \) and \( w \) in this paragraph. It is easy to see that \( \mu_f(0) = 0 \). So we assume that \( s \neq 0 \). The line segment joining \((1,0)\) and \((2,0)\) in \( \mathbb{R}^2 \) maps to a core arc for \((S^2,P_{f_n})\) with slope \( 0 \). Since \( s \neq 0 \), every simple closed curve in \( S^2 \setminus P_f \) with slope \( s \) meets this core arc in its interior. It follows that we may choose \( v \) so that \( v = (x,0)_B \) with \( 1 < x < 2 \). We choose \( w \) so that its second \( B \)-coordinate is positive.

We say that real numbers \( x \) and \( y \) are congruent modulo an integer \( k \) if and only if \( x-y \in k\mathbb{Z} \). If \( u \in P \) and \( \phi(u) \) are moving in the same direction, then their first \( B \)-coordinates are congruent modulo \( 2n \) and their second \( B \)-coordinates are congruent modulo \( 4 \). If \( u \) and \( \phi(u) \) are moving in opposite directions, then their first \( B \)-coordinates are congruent to minus each other modulo \( 2n \) and their second \( B \)-coordinates are congruent to minus each other modulo \( 4 \). In particular, a point \( u \in P \) is in a spin mirror if and only if \( \phi(u) \) is in a spin mirror.

We are now ready to analyze \( P \). The analysis separates into two cases.

**Case 1.** The photon enters a parallelogram \( T \in \mathcal{T} \) at a point \( u_0 \in P \) which is in the gap between spin mirrors in the lower left corner of \( T \). This is the case when \( u_0 = v \). This case is illustrated in the top diagram of Figure 8. For both of the diagrams in Figure 8, the photon enters the parallelogram at \( A \), travels to \( B \), spins to \( B' \) and continues in this way until it exits the parallelogram at either \( G \) or \( H \). Let \( u \in P \) be the photon’s position, and suppose that \( \phi(u) = (x,y)_B \). As \( \phi(u) \) moves toward \( w \) from \( \phi(u_0) = (x_0,y_0)_B \), the photon remains in \( T \) until one of three conditions is met. One condition is that \( y \in 2\mathbb{Z} \) and \( \pm x \) is congruent modulo \( 2n \) to a real number between 0 and 1. Another condition is that \( y \in 2\mathbb{Z} \) and \( \pm x \) is congruent modulo \( 2n \) to a real number between 1 and 2. The remaining condition is that \( x \) is congruent to 0 modulo \( 2n \). If the first condition is met, then the photon leaves \( T \) by spinning about a small spin mirror. If the second condition is met, then the photon leaves \( T \) through either its top or bottom between two spin mirrors. If the third condition is met, then the photon leaves \( T \) through a side of \( T \).

Consider what happens when the second \( B \)-coordinate of \( \phi(u) \) increases from \( y_0 \) by a nonnegative real number \( r \). Since the slope of \( S \) with respect to \( B \) is \( \frac{p}{n-1} \), as the second \( B \)-coordinate of \( \phi(u) \) increases by \( r \), its first \( B \)-coordinate increases by \( \frac{n-1}{p}r \). We have that \( 1 \leq p \leq 2 \lfloor \frac{n-2}{2} \rfloor \leq n-1 \), so \( \frac{n-1}{p} \geq 1 \). In particular, when \( r = 2 \) the first \( B \)-coordinate of \( \phi(u) \) is at least 3. So the photon does not leave \( T \) through the gap between the spin mirrors in the upper left corner of \( T \) when \( r = 2 \). Furthermore, if \( r \leq 2p-2 \), then

\[
x = x_0 + \frac{n-1}{p}r \leq x_0 + 2(n-1) - 2\frac{n-1}{p} \leq x_0 + 2n - 4.
\]

So for such values of \( r \), the photon remains in \( T \). Because \( p = 2m \), we have that \( 2p-2 \equiv 2 \mod 4 \). Hence the photon is in the top of \( T \) when \( r = 2p-2 \). It is in the large spin mirror in the top of \( T \). When \( r = 2p \), we have that \( x = x_0 + 2n - 2 \). So the photon is in a small spin mirror in the bottom of \( T \). This spin mirror must be the one in the lower left corner of \( T \), as in the top diagram in Figure 8. The photon then leaves \( T \) by spinning about this
spin mirror. It quickly re-enters $T$ through the left side of $T$. Now there are essentially two possibilities. It might proceed directly to the gap between the spin mirrors in the upper left corner of $T$. We are back in Case 1. Otherwise, we note that $p = 2m$, $q = n - 2m - 1$ and $\frac{p}{q} = s$. We have dismissed the case in which $s = 0$. So $p \geq 2$ and $q \leq n - 3$, which implies that $\frac{p}{q} > \frac{2}{n-2}$. Hence if we do not return to Case 1, then the photon proceeds directly to the left half of the large spin mirror in the top of $T$. This puts us in Case 2.

Case 2. The photon enters a parallelogram $T \in \mathcal{T}$ through its left side and proceeds directly to the left half of the large spin mirror in the top of $T$. This case is illustrated in the second diagram in Figure 8. Suppose that the photon enters $T$ at the point $u_1$. Suppose that $\phi(u_1) = (x_1, y_1)_B$. Then $x_1$ is congruent to 0 modulo $2n$. From $u_1$ the photon proceeds directly to a point $u_2$ in the top of $T$. Suppose that $\phi(u_2) = (x_2, y_2)_B$. Then $x_2$ is congruent modulo $2n$ to a real number between 2 and $n$. We argue as in Case 1. We allow the second $B$-coordinate of $\phi(u)$ to increase by a nonnegative real number $r$ beyond $y_1$. If $r \leq 2p$, then $x \leq x_1 + 2n - 2$. So the photon remains in $T$ as $r$ increases from 0 to $2p$. Because $2p \equiv 0 \mod 4$, the photon is moving up and right as $r$ approaches $2p$. It is near the right side of $T$. As $r$ increases from $2p$ to $2p + 2$, the number $x$ assumes the value $x_2 + 2n - 2 > 2n$. So the photon passes out of the right side of $T$ into a parallelogram $T'$, as in the second diagram in Figure 8.

Now there are essentially three possibilities. The photon might proceed directly to the large spin mirror in the top of $T'$. This puts us back in Case 2. It might proceed directly to the gap between the spin mirrors in the upper left corner of $T'$. This puts us back in Case 1. Finally, it might proceed directly to the small spin mirror in the upper left corner of $T'$, as in the second diagram in Figure 8. It then spins and re-enters $T$. As it continues from here, its behavior is opposite to the behavior in Case 1. Hence it leaves $T$ through the gap between the spin mirrors in the upper right corner of $T$. After the photon passes through this gap, we quickly return to the situation at the beginning of this paragraph. This concludes Case 2.

The above analysis of $P$ has the following consequence. The photon never leaves a parallelogram $T \in \mathcal{T}$ by passing through the left side or a gap in the bottom of $T$. However, these parallelograms have the defect that the photon might spin from a parallelogram $T$ to the parallelogram immediately left of $T$. This defect can be repaired by replacing the parallelograms with rectangles, as in Figure 9. The midpoints of the sides of the rectangles equal the midpoints of the sides of the parallelograms. For these rectangles it can be said that once the photon enters a rectangle $R$, it is never later in the rectangle immediately left of $R$ or a rectangle immediately below $R$.
We are finally prepared to evaluate $\mu_{f_n}(s)$. The photon begins at $v$. Suppose that it ends at $w'$. Then $\mu_{f_n}(s) = \frac{a}{b}$, where $a$ and $b$ are the integers such that $w' - v = b(n, 0) + a(-1, 1)$. We have that $s = \frac{p'}{q'}$, where $p'$ and $q'$ are relatively prime nonnegative integers. The photon moves in line segments with slope $\frac{p'}{q'}$. The integer $a$ is 2 times the number of times that the photon passes through the top of a rectangle. The number of times that the photon passes through the top of a rectangle is the number of times that the photon passes through a gap between spin mirrors. This is the number of times that the line segment $S$ passes through a gap between spin mirrors. The image of $S$ in $\mathbb{R}^2/\Gamma_1$ is a simple closed curve which $f$ maps to a simple closed curve with slope $s$. The gap between two spin mirrors is spanned by a line segment whose image in $\mathbb{R}^2/\Gamma_1$ is a core arc with slope 0 which $f$ maps with degree 1 to a core arc with slope 0. It follows that $a = 2\iota(s, 0) = 2p'$. Similarly, $b = 2\iota(s, \infty) = 2q'$. Therefore $\mu_{f_n}(s) = \frac{2p'}{2q'} = s$.

This completes the proof of Theorem 11.1.

**Remark 11.2.** Theorem 11.1 provides examples of NET maps with many formal matings. These NET maps have translation term $\lambda_1 = (n, 0)$. This choice was made to satisfy the orientation condition for equators. The other equator conditions are satisfied by the NET maps with the other translation terms. So if we compose one of these other NET maps with itself, we obtain another Thurston map with many formal matings. Two simple computations show that the NET maps with translation terms $\lambda_2 = (-1, 1)$ and $\lambda_1 + \lambda_2 = (n-1, 1)$ are hyperbolic. So Meyer’s Theorem 4.2 in [9] implies that these NET map iterates even have many topological matings.

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