Practical tracking control under actuator saturation for a class of flexible-joint robotic manipulators driven by DC motors

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Received: 12 December 2021 / Accepted: 3 June 2022 / Published online: 21 June 2022
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Abstract This paper is devoted to the practical tracking control for a class of flexible-joint robotic manipulators driven by DC motors. Different from the related literature where control constraint is neglected and the disturbances are excluded or only exist in one subsystem, actuator saturation is considered in this paper, while the disturbances are present in all the three subsystems. This leads to the incapability of the traditional schemes on this topic. For this, a novel control design scheme is proposed by skillfully incorporating adaptive dynamic compensation technique, constructive methods of command filters and an auxiliary system for the actuator saturation into the backstepping framework, and in turn to design a practical tracking controller which ensures that all the states of the resulting closed-loop system are bounded and the system output practically tracks the reference signal. It is worthwhile strengthening that a more wider class of reference signals can be tracked since they are only first-order continuously differentiable but twice or more in the related literature. Finally, a numerical example is provided to validate the effectiveness of the proposed theoretical results.

Keywords Flexible-joint robot · Tracking control · Actuator saturation · Adaptive control

Mathematics Subject Classification 93D21 · MSC 93C10

1 Introduction

The flexible-joint robots have been widely used in healthcare, aerospace, mechanics and other industry and social practice due to their flexibility, rapidity and accuracy ([1, 2]). Along with their widespread applications, certain control measurements are required to be designed so as to guarantee specific control objective and good performance. However, control design for such systems is usually rather challenging due to the presence of strong nonlinearity, serious uncertainties, multiple degrees of freedom and high coupling. Then, many control problems with notable practical and scientific significance remain unsolved which deserve further investigation.

In the past two decades, trajectory tracking control has been a hot topic in the field of the control for flexible-joint robots ([3–26]). Accordingly, many effective control methods have been proposed, such as neural network ([11, 19, 23, 26]), fuzzy control ([8, 15, 18, 20]), robust control ([3–5, 14, 21, 25]), and prescribed performance control ([6]). It is necessary to point out that the existing results are strongly restricted by the regardlessness of the control constraint and the
generality of reference signals as well as the system uncertainties. Specifically, (1) The control constraint is neglected. In fact, the constraint of control input is neglected in [6–10,12–18,20,22–24]. However, in practical engineering, for example in the control of a flexible-joint robot driven by DC motor, the control voltage cannot exceed the thresholds of some crucial devices (such as the switch). Then, certain constrains should be imposed on control input beforehand, which will lead to the ineffectiveness of the existing control methods in the literature. (2) High smoothness of the reference signal is required. The generality of the reference signals determines the applicability of the tracking controllers. Note that the applicability of the controllers in the existing literature is severely constrained since the reference signals must be high-order continuously differentiable (for example, twice in [3,8,11–14,16,19] or more in [3,7,9,10,17,18,21,25]), and hence exclude a large class of ones without such sufficient smoothness. Once the reference signals are not sufficiently smooth, such as only first-order continuous differentiable, the existing control methods will be inapplicable. (3) The disturbance uncertainties are neglected or constrained. In fact, the disturbance uncertainties coming from the external environment are completely excluded in [8,10,16–18,20,22,24] or only exist in one subsystem in [4,5,11,15,26]. However, disturbance inevitably exists in practical engineering which always bring essential obstacles in control design.

In view of the constrains of the existing results mentioned above, a meaningful and nontrivial control problem arises, i.e., for a class of flexible-joint robotic manipulators with certain control constraint and serious uncertainties, how to design a tracking controller to follow a more wider class of reference signals? This motivates the investigation of the paper, i.e., practical tracking control under actuator saturation for a class of flexible-joint robotic manipulators driven by DC motors. Different from the existing literature, remarkable characteristics of the system under investigation are existing which highlight the main novelties of the paper. First, the external disturbances exist in all the three subsystems (i.e., the dynamics of the link, joint and motor) rather than one or none as in the related literature. Second, a more wider class of reference signals are allowed since which are only once continuously differentiable rather than high-order (more than one) continuously differentiable ones. Third, actuator saturation is considered but neglected in most of the related literature. The presence of above three aspects of characteristics result into the ineffectiveness of the traditional schemes on this topic, and hence makes the control problem mentioned above unsolved.

To solve the control problem, a novel control framework is established which integrates the compensation of serious uncertainties and the guarantee of actuator saturation. Specifically, a group of command filters and an auxiliary dynamic system are skillfully designed to avoid the explosion of complexity in control design and cope with the actuator saturation, respectively. With these two systems in hands, a state transformation is constructed which gives a crucial error system. Then, for the error system, adaptive backstepping method is adopted to overcome the system uncertainties while derive an explicit controller. Finally, it is proved that the designed controller guarantees that all the states of the resulting closed-loop system are bounded and the system output practically tracks the reference signal, i.e., the tracking error enters and then remains in an arbitrary neighborhood of the origin after some time.

The rest of the paper is organized as follows: Section 2 presents the system description and control objective, Sect. 3 gives the procedure of control design, Sect. 4 analyzes the performance of the resulting closed-loop system, Sect. 5 provides an example to validate the effectiveness of the theoretical results, and Sect. 6 gives some concluding remarks.

## 2 Problem formulation

In this paper, tracking control is considered for the following $n$-link robotic manipulators with flexible-joint driven by DC motors (see Fig. 1):

\[
\begin{aligned}
D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) &= K(\theta - q) + d_1(t, q, \dot{q}), \\
\dot{\theta} + B\dot{\theta} - K(q - \theta) &= KT I + d_2(t, \theta, \dot{\theta}), \\
L\dot{I} + R I + K_B \dot{\theta} &= u + d_3(t, I),
\end{aligned}
\]

(1)

where $q, \theta, I \in \mathbb{R}^n$, respectively, describe the dynamics of the links, joints and motors, $d_i(\cdot), i = 1, 2, 3$ represent the lumped disturbance in respective paths including the model error and the effect suffered from the external environment, $u \in \mathbb{R}^n$ is the input voltage which is subject to saturation nonlinearities (see (2) below); $D(q), C(q, \dot{q}) \in \mathbb{R}^{n \times n}, G(q) \in \mathbb{R}^n$ are,
respectively, called inertia matrix, centripetal and Coriolis matrix, gravity vector; \( K = \text{diag}(K_i), K_T = \text{diag}(K_{Ti}), J = \text{diag}(J_i), B = \text{diag}(B_i), L = \text{diag}(L_i), R = \text{diag}(R_i), K_B = \text{diag}(K_{Bi}) \) are all diagonal and positive-definite matrices. Some variables and parameters mentioned above are given in Table 1.

In practice, due to the physical characteristics of actuator, the input \( u = (u_1, \ldots, u_n)^T \) satisfies saturation nonlinearities, and hence is given as follows:

\[
u_i = \text{sat}(\psi_i) = \begin{cases} 
  u_{\text{max}}, & \text{if } \psi_i \geq u_{\text{max}}, \\
  \psi_i, & \text{if } u_{\text{min}} < \psi_i < u_{\text{max}}, \\
  u_{\text{min}}, & \text{if } \psi_i \leq u_{\text{min}}, 
\end{cases}
\]

where \( u_{\text{max}} > 0 \) and \( u_{\text{min}} < 0 \) are the upper and lower bounds of saturation function, and \( \psi = (\psi_1, \ldots, \psi_n)^T \) is the input signal of nonlinear saturated function to be designed later.

Table 1 Description of parameters and variables

| Parameters or Variables | Description |
|-------------------------|-------------|
| \( q, \theta \)         | angular position of the link, joint |
| \( I \)                 | armature current |
| \( K_i \)               | stiffness coefficient of the \( i \)-th joint |
| \( K_{Ti} \)            | torque constant of the \( i \)-th motor |
| \( J_i \)               | inertia of the \( i \)-th motor |
| \( B_i \)               | damping of the \( i \)-th actuator |
| \( L_i \)               | inductance of the \( i \)-th motor |
| \( R_i \)               | resistance of the \( i \)-th motor |
| \( K_{Bi} \)            | back-emf constant of the \( i \)-th motor |

In the following, two properties are given which hold for most of the robotic manipulators and will be used in the later control design.

**Property 1** [5,9,26] \( D(q) \) is a symmetric positive-definite matrix, and moreover, there exist positive constants \( D \) and \( \bar{D} \) (\( D \leq \bar{D} \)) such that \( 0 < D I_{n \times n} \leq D(q) \leq \bar{D} I_{n \times n} \), \( \forall q \in \mathbb{R}^n \).

**Property 2** [5,9,26] For all \( q, \dot{q} \in \mathbb{R}^n \), \( \dot{D}(q) - 2C(q, \dot{q}) \) is a skew-symmetric matrix, i.e., \( \eta^T (\dot{D}(q) - 2C(q, \dot{q})) \eta = 0, \forall \eta \in \mathbb{R}^n \).

The following gives three assumptions imposed on the plant, disturbances and the reference signal which will be used in the later control design and performance analysis.

**Assumption 1** The plant (1) is input-to-state stable (ISS).

**Assumption 2** The disturbances \( d_1(t, q, \dot{q}), d_2(t, q, \dot{q}), \dot{\theta} \) and \( d_3(t, I) \) are bounded, i.e., there exists an unknown positive constant \( \sigma \) such that \( \|d_1(t, q, \dot{q})\| \leq \sigma \), \( \|d_2(t, q, \dot{q})\| \leq \sigma \) and \( \|d_3(t, I)\| \leq \sigma \).

**Assumption 3** There exists an unknown positive constant \( \bar{\gamma}_r \) such that \( \|\gamma_r\| + \|\dot{\gamma}_r\| \leq \bar{\gamma}_r \).

**Remark 1** Assumption 2 shows that the three disturbances have the same upper bound. In practice, the three disturbances may have different bounds, such as \( \|d_i(t, q, \dot{q})\| \leq \sigma_i \) with \( \sigma_i \) being different unknown constants (for \( i = 1, 2, 3 \)). Then, there is a common unknown bound for all the three disturbances, such as \( \sigma = \max\{\sigma_1, \sigma_2, \sigma_3\} \). We use the same bound in the assumption for all the three disturbances so as to reduce the notation burden in the following control design.
Remark 2 Assumption 3 shows that the reference signal is only required to be first-order continuously differentiable, and hence includes those in the related literature (see [3, 5, 7–14, 16–19, 21, 25]) as special cases which are at least twice continuously differentiable. Thus, a more wider class of reference signals can be tracked in the paper.

The control objective of this paper is to design a controller $u$ such that all the states of the closed-loop system are bounded while the system output $q$ enter and then stay at a given neighborhood of reference signal $y_r$ after some time $T^*$, i.e., for any $\varepsilon > 0$, there exists a finite time $T^* > 0$ such that

$$\|q - y_r\| \leq \varepsilon, \quad t > T^*.$$  

\section{Control design}

In this section, controller is explicitly derived by backstepping control method. Remark that the applicability of the controller designed by the traditional backstepping method is restricted due to the explosion of complexity along with the increase in the system order. Then, command filters (i.e., (5) below) are designed and incorporated into the backstepping framework so as to avoid the explosion of complexity and hence enhance the applicability of the controller. Moreover, an auxiliary dynamic system (i.e., (6) below) is introduced to deal with input saturation, adaptive dynamic compensation technique with the smart choosing of the adaptive law is adopted to overcome the uncertainties of disturbance while guarantees the tracking objective.

To make control design convenient, we denote $x_1 = q$, $x_2 = \dot{q}$, $x_3 = \theta$, $x_4 = \dot{\theta}$, $x_5 = I$. Then, by system (1), it is easy to verify that the new states $x_i$’s satisfy the following equations:

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= D^{-1}(x_1) \left( K(x_3 - x_1) + d_1(t, x_1, x_2) - C(x_1, x_2)x_2 - G(x_1) \right), \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= J^{-1}(K_T x_5 + d_2(t, x_3, x_4) - Bx_4 + K(x_1 - x_3)), \\
\dot{x}_5 &= L^{-1}(u + d_3(t, x_5) - Rx_5 - K_B x_4).
\end{align*}$$  

For preparation of the backstepping control design, the following state transformations are introduced:

$$z_i = x_i - x_{i,c} - w_i, \quad i = 1, \ldots, 5,$$

where $x_{i,c} = y_r, x_i, c (i = 2, \ldots, 5)$ are generated by the following command filters:

$$\begin{align*}
\dot{e}_{i,1} &= \omega e_{i,2}, \\
\dot{e}_{i,2} &= -2\zeta \omega e_{i,2} - \omega (e_{i,1} - \alpha_i)
\end{align*}$$  

with $x_{i+1,c} = e_{i,1}$ and the filter initial conditions being $e_{i,1}(0) = \alpha_i(0), e_{i,2}(0) = 0$ ($i = 1, \ldots, 4$), $\omega > 0$ and $\zeta \in (0, 1]$ being the filter design parameters, $\alpha_i$’s being virtual control to be chosen later; $w_i$’s are introduced to cope with the input saturation which satisfy the following equations:

$$\begin{align*}
\dot{w}_1 &= w_2 - \rho_1 w_1, \\
\dot{w}_2 &= D^{-1}(x_1) K w_3 - \rho_2 w_2, \\
\dot{w}_3 &= w_4 - \rho_3 w_3, \\
\dot{w}_4 &= J^{-1} K_T w_5 - \rho_4 w_4, \\
\dot{w}_5 &= L^{-1} \Delta u + \rho_5 w_5.
\end{align*}$$  

By (3), (4) and (6), we obtain that $z_i$’s satisfy the following equations:

$$\begin{align*}
\dot{z}_1 &= z_2 + x_{2,c} - \dot{y}_r + \rho_1 w_1, \\
\dot{z}_2 &= D^{-1}(x_1) \left( K(x_3 - x_1) + d_1(t, x_1, x_2) - C(x_1, x_2)x_2 - G(x_1) \right) - \dot{x}_{2,c} - D^{-1}(x_1) K w_3 + \rho_2 w_2, \\
\dot{z}_3 &= z_4 + x_{4,c} - \dot{x}_{3,c} + \rho_3 w_3, \\
\dot{z}_4 &= J^{-1}(K_T x_5 + d_2(t, x_3, x_4) - Bx_4 + K(x_1 - x_3)) - \dot{x}_{4,c} - J^{-1} K_T w_5 + \rho_4 w_4, \\
\dot{z}_5 &= L^{-1}(u + d_3(t, x_5) - Rx_5 - K_B x_4) - \dot{x}_{5,c} - L^{-1} \Delta u + \rho_5 w_5.
\end{align*}$$  

Then, we define the compensated tracking error signals $v_i$ as follows:

$$v_i = z_i - \xi_i, \quad i = 1, \ldots, 5,$$
where $\xi_t$'s will be designed in the later.

Motivated by the related literature, the controller is derived by the following backstepping steps ([8, 9, 16, 30, 31]).

**Step 1:** Define $E_1 = \frac{1}{2}v_1^Tv_1$. Then, by the first equation of (7) and (8) with $i = 1$, we have

$$
\dot{E}_1 = v_1^T\dot{v}_1 = v_1^T(z_2 + x_{2,c} - \dot{y}_r + \rho_1 w_1 - \dot{\xi}_1).
$$

By Assumption 3 and using Young’s inequality, there holds that

$$
-v_1^T\dot{y}_r \leq \frac{v_1^2}{\kappa(t)} + \frac{\kappa(t)v_1^Tv_1}{4}.
$$

where $\kappa(t)$ is tuned online by the following adaptive law:

$$
\dot{\kappa}(t) = \begin{cases} 
\|z_1\| - \frac{\varepsilon}{4}, & \|z_1\| \geq \varepsilon, \\
0, & \|z_1\| < \varepsilon,
\end{cases}
$$

with $\kappa(0) = 1$.

Substituting (10) into (9) arrives at

$$
\dot{E}_1 \leq v_1^T(z_2 + x_{2,c} - \alpha_1 + \alpha_1 + \frac{\kappa v_1}{4} + \rho_1 w_1 - \dot{\xi}_1) + \frac{\dot{v}_1^2}{\kappa}.
$$

Then, by choosing virtual controller $\alpha_1$ and compensation signal $\xi_1$ as follows:

$$
\begin{cases}
\alpha_1 = -c_1 z_1 - \frac{\kappa v_1}{4} - \rho_1 w_1, \\
\dot{\xi}_1 = -c_1 \dot{\xi}_1 + x_{2,c} - \alpha_1 + \dot{\xi}_2,
\end{cases}
$$

with $c_1$ will be determined later, we obtain that

$$
\dot{E}_1 \leq -c_1 v_1^Tv_1 + v_1^Tv_2 + \frac{\dot{v}_1^2}{\kappa}.
$$

**Step 2:** Define $E_2 = E_1 + \frac{1}{2}v_2^TD(x_1)v_2$. Then, by using Property 2, the second equation of (7) and (8) with $i = 2$, we have

$$
\dot{E}_2 = \dot{E}_1 + \frac{1}{2}v_2^TD(x_1)v_2 + v_2^TD(x_1)\dot{v}_2
$$

$$
= \dot{E}_1 + \frac{1}{2}v_2^TD(x_1)v_2 + v_2^T(K(x_3 - x_1) + d_1(t, x_1, x_2) - C(x_1, x_2)x_2 - G(x_1) - D(x_1)\dot{x}_{2,c} - Kw_3 + \rho_2 D(x_1)w_2 - D(x_1)\dot{\xi}_2)
$$

$$
\leq -c_1 v_1^Tv_1 + \frac{1}{2}v_2^TD(x_1)v_2 + v_2^T(v_1 + K(z_3 + x_{3,c} + w_3 - z_1) - y_r - w_1) + d_1(t, x_1, x_2) - C(x_1, x_2)(v_2 + \dot{\xi}_2 + x_{2,c} + w_2) - G(x_1) - D(x_1)\dot{x}_{2,c} - Kw_3 + \rho_2 D(x_1)w_2
$$

$$
= -c_1 v_1^Tv_1 + v_2^T(v_1 + K(z_3 + x_{3,c} - z_1 - y_r - w_1) + d_1(t, x_1, x_2) - C(x_1, x_2)(\dot{\xi}_2 + x_{2,c} + w_2) - G(x_1) - D(x_1)\dot{x}_{2,c} + \rho_2 D(x_1)w_2
$$

$$
= -c_1 v_1^Tv_1 + v_2^T(v_1 + K(z_3 + x_{3,c} - z_1 - y_r - w_1) + d_1(t, x_1, x_2) - C(x_1, x_2)(\dot{\xi}_2 + x_{2,c} + w_2) - G(x_1) - D(x_1)\dot{x}_{2,c} + \rho_2 D(x_1)w_2
$$

$$
- D(x_1)\dot{\xi}_2) + \frac{\dot{v}_2^2}{\kappa},
$$

(14)

The estimations of some terms on the right-hand side of above equality are given by Assumptions 2, 3 and the well-known Young’s inequality, i.e.,

$$
\begin{cases}
-v_2^T K y_r \leq \frac{\dot{y}_r^2}{\kappa} + \frac{\kappa}{4} v_2^Tv_2\|K\|^2, \\
v_2^T d_1(t, x_1, x_2) \leq \frac{\sigma^2}{\kappa} + \frac{1}{4} v_2^Tv_2.
\end{cases}
$$

Substituting the above inequalities into (14) arrives at

$$
\dot{E}_2 \leq -c_1 v_1^Tv_1 + v_2^T(v_1 + K(z_3 + x_{3,c} - \alpha_2 + \alpha_2 - z_1 - w_1) + \frac{\kappa}{4} v_2^Tv_2\|K\|^2 + \frac{\kappa}{4} v_2^Tv_2 - C(x_1, x_2)(\dot{\xi}_2 + x_{2,c} + w_2) + Kw_3) - G(x_1) - D(x_1)\dot{x}_{2,c} + \rho_2 D(x_1)w_2 - D(x_1)\dot{\xi}_2)
$$

$$
+ \frac{1}{\kappa}(2\dot{\xi}_r^2 + \sigma^2).
$$

(15)

By choosing virtual controller $\alpha_2$ and compensation signal $\xi_2$ as follows:

$$
\begin{cases}
\alpha_2 = K^{-1}(-c_2 z_2 - v_1 + K(z_1 + w_1) - \frac{\kappa}{4} v_2^Tv_2\|K\|^2 - \frac{\kappa}{4} v_2^Tv_2 - G(x_1) + C(x_1, x_2)(\dot{\xi}_2 + x_{2,c} + w_2) + D(x_1)\dot{x}_{2,c} - \rho_2 D(x_1)w_2)
\end{cases}
$$

with $c_2$ will be determined later, we get

$$
\dot{E}_2 \leq -c_1 v_1^Tv_1 - c_2 v_2^Tv_2 + v_2^T(K v_3 + \frac{1}{\kappa}(2\dot{\xi}_r^2 + \sigma^2)).
$$
Step 3: Define $E_3 = E_2 + \frac{1}{2}v_T^2 v_3$. Then, by the third equation of (7) and (8) with $i = 3$, we have
\[ \dot{E}_3 = \dot{E}_2 + v_T^T \dot{v}_3 \]
\[ \leq -c_1 v_1^T v_1 - c_2 v_2^T v_2 + v_3^T (K v_2 + z_4 + x_{4,c} - \dot{x}_{3,c} + \rho_3 w_3 - \xi_3) \]
\[ + \frac{1}{\kappa} (2y_r^2 + \sigma^2) \]
\[ = -c_1 v_1^T v_1 - c_2 v_2^T v_2 + v_3^T (K v_2 + z_4 + x_{4,c} - \alpha_3 + \dot{x}_{3,c} + \rho_3 w_3 - \xi_3) \]
\[ + \frac{1}{\kappa} (2y_r^2 + \sigma^2). \] (17)

By choosing the following virtual controller $\alpha_3$ and compensation signal $\xi_3$:
\[ \begin{align*}
\alpha_3 &= -c_2 z_3 - K v_2 - \dot{x}_{3,c} - \rho_3 w_3, \\
\xi_3 &= -c_3 \xi_3 + x_{4,c} - \alpha_3 + \xi_4.
\end{align*} \] (18)

with $c_3$ will be determined later, we have
\[ \dot{E}_3 \leq -c_1 v_1^T v_1 - c_2 v_2^T v_2 - c_3 v_3^T v_3 + v_3^T v_4 + \frac{1}{\kappa} (2y_r^2 + \sigma^2). \]

Step 4: Define $E_4 = E_3 + \frac{1}{2}v_T^2 J v_4$. Then, by the fourth equation of (7) and (8) with $i = 4$, we have
\[ \dot{E}_4 = \dot{E}_3 + v_T^T J \dot{v}_4 \]
\[ \leq -c_1 v_1^T v_1 - c_2 v_2^T v_2 - c_3 v_3^T v_3 + v_3^T v_4 + v_4^T v_3 + v_4^T v_4^T v_4 + \frac{1}{\kappa} (2y_r^2 + \sigma^2) \]
\[ = -c_1 v_1^T v_1 - c_2 v_2^T v_2 - c_3 v_3^T v_3 + v_4^T \left( v_3 + K_T x_5 + d_2(t, x_3, x_4) - B x_4 \right) \]
\[ + K(x_1 - x_3) - J \dot{x}_{4,c} - K_T w_5 + \rho_4 J w_4 \]
\[ + J \dot{\xi}_4 + \frac{1}{\kappa} (2y_r^2 + \sigma^2) \]
\[ = -c_1 v_1^T v_1 - c_2 v_2^T v_2 - c_3 v_3^T v_3 + v_4^T \left( v_3 + K_T (x_5 + x_{5, c} + w_5) + d_2(t, x_3, x_4) - K_T w_5 + \rho_4 J w_4 \right) \]
\[ + J \dot{\xi}_4 + \frac{1}{\kappa} (2y_r^2 + \sigma^2). \] (19)

By Assumptions 2,3 and Young’s inequality, we obtain
\[ \begin{align*}
v_T^T d_2(t, x_3, x_4) &\leq \frac{\sigma^2}{\kappa} + \frac{\kappa}{4} v_4^T v_4, \\
v_T^T K y_r &\leq \frac{\sigma^2}{\kappa} + \frac{\kappa}{4} v_4^T v_4 K ||K||^2.
\end{align*} \]

Substituting the above inequalities into (19) gives that
\[ \dot{E}_4 = -c_1 v_1^T v_1 - c_2 v_2^T v_2 - c_3 v_3^T v_3 \]
\[ + v_4^T \left( v_3 + K_T (x_5 + x_{5, c} - \alpha_4 + \alpha_4) \right) \]
\[ + \frac{\kappa}{4} v_4 - B x_4 + K(z_1 + w_1 - x_3) \]
\[ + \frac{\kappa}{4} v_4 \cdot ||K||^2 - J \dot{x}_{4,c} + \rho_4 J w_4 \]
\[ + J \dot{\xi}_4 + \frac{1}{\kappa} (3y_r^2 + 2\sigma^2). \] (20)

By choosing the following virtual controller $\alpha_4$ and compensation signal $\xi_4$:
\[ \begin{align*}
\alpha_4 &= K_T^{-1} \left( -c_4 z_4 - v_3 - \frac{\kappa}{4} v_4 + B x_4 \right) \\
&- K(z_1 + w_1 - x_3) \\
&- \frac{\kappa}{4} v_4 \cdot ||K||^2 + J \dot{x}_{4,c} - \rho_4 J w_4, \\
\dot{\xi}_4 &= J^{-1} \left( -c_4 \xi_4 + K_T (x_{5, c} - \alpha_4 + K_T \xi_3) \right),
\end{align*} \] (21)

with $c_4$ will be determined later, we obtain that
\[ \dot{E}_4 \leq -c_1 v_1^T v_1 - c_2 v_2^T v_2 - c_3 v_3^T v_3 - c_4 v_4^T v_4 \]
\[ + v_4^T K_T v_5 + \frac{1}{\kappa} (3y_r^2 + 2\sigma^2). \]

Step 5: Define $E_5 = E_4 + \frac{1}{2}v_T^2 L v_5$. Then, by the fifth equation of (7) and (8) with $i = 5$, we have
\[ \dot{E}_5 = \dot{E}_4 + v_T^T L v_5 \]
\[ \leq -c_1 v_1^T v_1 - c_2 v_2^T v_2 - c_3 v_3^T v_3 - c_4 v_4^T v_4 + v_5^T v_3 + v_5^T v_4 + v_5^T v_5 \]
\[ + v_5^T \left( K_T v_4 + u \right) \]
\[ + d_3(t, x_5) - R x_5 - K_B x_4 - L \dot{x}_{5, c} - \Delta u \]
\[ + \rho_5 L w_5 - L \dot{\xi}_5 \]
\[ + \frac{1}{\kappa} (3y_r^2 + 2\sigma^2). \] (22)

By Assumption 2 and Young’s inequality, we obtain
\[ v_T^T d_3(t, x_5) \leq \frac{\sigma^2}{\kappa} + \frac{\kappa}{4} v_5^T v_5. \]

Substituting the above inequality into (22) while noting that $\Delta u = u - \psi$ gives that
\[ \dot{E}_5 \leq -c_1 v_1^T v_1 - c_2 v_2^T v_2 - c_3 v_3^T v_3 - c_4 v_4^T v_4 \]
\[ + v_5^T \left( K_T v_4 + \psi + \frac{\kappa}{4} v_5 \right) \]
\[ - R x_5 - K_B x_4 - L \dot{x}_{5, c} + \rho_5 L w_5 \]
\[ - L \dot{\xi}_5 + \frac{1}{\kappa} (3y_r^2 + 3\sigma^2). \] (23)
By choosing the following control law $\psi$ and compensation signal $\xi_5$:

$$
\begin{cases}
\psi = -c_5 z_5 - K_T v_4 - \frac{\rho}{\tau} v_5 + R x_5 + K_B x_4 + L \dot{x}_{5,c} \\
\dot{\xi}_5 = -c_5 L^{-1} \xi_5,
\end{cases}
$$

(24)

with $c_5$ will be determined later, we get

$$
\dot{E}_5 \leq -c_1 \xi_1^T \xi_1 - c_2 \xi_2^T \xi_2 - c_3 \xi_3^T \xi_3 - c_4 \xi_4^T \xi_4
$$

$$
- c_5 \xi_5^T \xi_5 + \frac{\tau}{\kappa} \xi_5
\leq -\varrho E_5(t) + \frac{\tau}{\kappa} \xi_5.
$$

(25)

with $\varrho = \min \left\{ 2 c_1, \frac{2}{D} c_2, 2 c_3, \frac{2}{\max(D)} c_4, \frac{2}{\max(L)} c_5 \right\}$, $\tau = 3 \gamma r^2 + 3 \sigma^2$.

4 Performance analysis

In order to prove the desirable performance of the resulting closed-loop system, three propositions are first given which collects the properties of command filters signals, compensation signals and time-varying gain, respectively.

Proposition 1. For any given $\mu > 0$, $\alpha_i$’s defined in the first equations of (13), (16), (18), (21) and the command filters defined in (5) satisfy that $\|x_{i+1,c} - \alpha_i\| \leq \mu$ on $[0, +\infty)$.

Proof. Define $\hat{e} = (e_{i,1}, e_{i,2}, e_{i,3}, e_{i,4}, e_{i,5}, e_{i,6}) \in \mathbb{R}^8$. For the following command filter (5) with $\omega = \frac{1}{\epsilon}$, we have

$$
\begin{cases}
\epsilon \dot{\xi}_{i,1} = e_{i,2} \\
\epsilon \dot{\xi}_{i,2} = -2 \zeta \epsilon \xi_{i,2} - (e_{i,1} - \alpha_i),
\end{cases}
$$

(26)

whose equivalent point can be easily obtained, i.e., $e_{i,1} = \alpha_i$, $e_{i,2} = 0$ which are lumped as $\hat{h} = h$ with $h_{i,1} = \alpha_i$, $h_{i,2} = 0$, $i = 1, \ldots, 4$.

Define the following transformation (for $i = 1, \ldots, 4$)

$$
\begin{cases}
y_{i,1} = e_{i,1} - h_{i,1}, \\
y_{i,2} = e_{i,2},
\end{cases}
$$

By the similar derivation of (11.4) of [27] or the proof of Theorem 2 in [28], we obtain the following two equations:

$$
\begin{cases}
\epsilon \dot{y}_{i,1} = y_{i,2}, \\
\epsilon \dot{y}_{i,2} = -2 \zeta y_{i,2} - (e_{i,1} - h_{i,1} + h_{i,1} - \alpha_i)
= -2 \zeta y_{i,2} - (e_{i,1} - h_{i,1})
\end{cases}
$$

(27)

which is denoted as $\dot{y}_i = A_i y_i$ with $A_i$ being

$$
A_i = \frac{1}{\epsilon} \begin{pmatrix} 0 & 1 \\ -1 & -2 \zeta \end{pmatrix}, \quad i = 1, \ldots, 4.
$$

Clearly, $A_i$ is Hurwitz, then we obtain that $\lim_{t \to \infty} y_i = 0$. Noting that $y_{i,1} = e_{i,1} - h_{i,1}$, $e_{i,1} = x_{i+1,c}$ and $h_{i,1} = \alpha_i$, then we can obtain that for any $\mu > 0$, there exists $\mu > 0$ such that $\|x_{i+1,c} - \alpha_i\| \leq \mu$, $\tau > T_\mu$. Moreover, by Lemma 2 of [29], we obtain that $\|x_{i+1,c} - \alpha_i\| \leq \mu, \forall t \in [0, T_\mu)$. Therefore, there holds that $\|x_{i+1,c} - \alpha_i\| \leq \mu, \forall t \in [0, +\infty)$. \hfill \square

Proposition 2. The compensation signals $\xi_i$ ($i = 1, \ldots, 5$) are bounded on $[0, +\infty)$.

Proof. Define $E_\xi = \frac{1}{2} \sum_{i=1}^5 \xi_i^T \xi_i$. By using the second equations of (13), (16), (18), (21) and (24), we have

$$
\dot{E}_\xi = \xi_1^T \dot{\xi}_1 + \xi_2^T \dot{\xi}_2 + \xi_3^T \dot{\xi}_3 + \xi_4^T \dot{\xi}_4 + \xi_5^T \dot{\xi}_5
$$

$$
= \xi_1^T (e_{1,1} + x_{2,c} - \alpha_1 - \xi_1)
+ \xi_2^T D^{-1}(x_1)(-c_2 \dot{\xi}_2 + K(x_3,c)
- \alpha_2) + \xi_3^T (c_3 \dot{\xi}_3 + x_{4,c} - \alpha_3)
+ \xi_4^T J^{-1}(x_4) + K_T(x_5,c - \alpha_4) + K_T \xi_5
+ \xi_5^T (-c_5 L^{-1} \xi_5).
$$

(28)

Some terms of the above inequality will be estimated. By Proposition 1, we obtain $\|x_{i+1,c} - \alpha_i\| \leq \mu, \forall t \in [0, +\infty)$. Then, by using Young’s inequality and Property 1, we have

$$
\begin{align*}
\xi_1^T \xi_2 &\leq \frac{\xi_1^T \xi_1}{2} + \frac{\xi_2^T \xi_2}{2}, \\
\xi_1^T (x_{2,c} - \alpha_1) &\leq \frac{\xi_1^T \xi_1}{2} + \mu^2, \\
\xi_2^T D^{-1}(x_1) K \xi_3 &\leq \|K\| \xi_2^T \xi_2 + \frac{\xi_3^T \xi_3}{2}, \\
\xi_2^T D^{-1}(x_1) K (x_{3,c} - \alpha_2) &\leq \|K\| \xi_2^T \xi_2 + \frac{\xi_3^T \xi_3}{2}, \\
\xi_3^T \xi_4 &\leq \frac{\xi_3^T \xi_3}{2} + \frac{\xi_4^T \xi_4}{2}, \\
\xi_3^T (x_{4,c} - \alpha_3) &\leq \frac{\xi_3^T \xi_3}{2} + \mu^2, \\
\xi_4^T J^{-1} K_T \xi_5 &\leq \|J^{-1}\| K_T \xi_2^T \xi_2 + \frac{\xi_5^T \xi_5}{2}, \\
\xi_4^T J^{-1} K (x_{5,c} - \alpha_4) &\leq \|J^{-1}\| K_T \xi_2^T \xi_2 + \frac{\xi_5^T \xi_5}{2} + \mu^2. 
\end{align*}
$$

\hfill \square

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Substituting above inequalities into (28) leads to that
\[ \dot{E}_\xi \leq -(c_1 - 1)\xi_1^T \xi_1 - \left( \frac{c_2}{D} - \frac{\|K\|^2}{D^2} - \frac{1}{2} \right) \xi_2^T \xi_2 \]
\[ - \left( c_3 - \frac{3}{2} \right) \xi_3^T \xi_3 \]
\[ - (c_4 \lambda_{\min}(J^{-1}) - \frac{1}{2} - \|J^{-1}\|\|K_T\|) \xi_4^T \xi_4 \]
\[ - (c_5 \lambda_{\min}(L^{-1}) - \frac{1}{2}) \xi_5^T \xi_5 + 2\mu^2 \]
\[ \leq -\sigma E_\xi(t) + 2\mu^2, \]
with \( c_i \)'s satisfying \( c_1 > 1, c_2 > \frac{\|K\|^2}{D^2}, c_3 > \frac{3}{2}, c_4 > \frac{1}{2\lambda_{\min}(J^{-1})} + \frac{\|J^{-1}\|^2\|K_T\|^2}{\lambda_{\min}(J^{-1})}, c_5 > \frac{1}{2\lambda_{\min}(L^{-1})} \).
\( \sigma = \min\{2c_1 - 2, 2D^{-1}c_2 - 2D^{-2}\|K\|^2 - 1, 2c_3 - 3, 2c_4\lambda_{\min}(J^{-1}) - 1 - 2\|J^{-1}\|^2\|K_T\|^2, 2c_5\lambda_{\min}(J^{-1}) - 1\}. \)
Integrating both sides of the above inequality on \([0, t]\) leads to
\[ E_\xi(t) \leq E_\xi(0)e^{-\sigma t} + \frac{2\mu^2}{\sigma}, \quad (29) \]
which implies that \( E_\xi \) and hence \( \xi_i (i = 1, \ldots, 5) \) are bounded on \([0, +\infty)\). \( \square \)

**Proposition 3** The time-varying gain \( \kappa(t) \) updated by (11) is bounded on \([0, +\infty)\).

**Proof** Such proposition is proved by contradiction. Suppose that \( \kappa(t) \) is unbounded on \([0, +\infty)\). Then, for some positive constant \( \frac{16\varepsilon^2}{\sigma^2} \), there exists \( t_1 > 0 \) such that \( \kappa(t_1) > \frac{16\varepsilon^2}{\sigma^2} \). Since \( \kappa(t) \) is monotone increasing by noting that \( \ddot{k}(t) \geq 0 \) from (11), there holds that \( \kappa(t) \geq \kappa(t_1) > \frac{16\varepsilon^2}{\sigma^2}, \forall t > t_1 \). Then, integrating both sides of (25) over \([t_1, t]\) leads to
\[ E_5(t) < E_5(t_1)e^{-\omega(t-t_1)} + \frac{\varepsilon^2}{16}, \quad t > t_1. \]
Since \( \lim_{t \to +\infty} e^{-\omega(t-t_1)} = 0 \), there exists \( t_2 > 0 \) such that \( E_5(t_1)e^{-\omega(t-t_1)} < \frac{\varepsilon^2}{16}, t > t_2 \). Choosing \( T_1 = \max\{t_1, t_2\} \), there holds that
\[ \frac{1}{2} \dot{v}_1^T v_1 < E_5(t) < \frac{\varepsilon^2}{8}, \quad t > T_1. \quad (30) \]
which implies that \( \|v_1\| < \frac{\varepsilon}{2}, t > T_1 \). Similarly, \( \lim_{t \to +\infty} e^{-\omega t} = 0 \) implies that there exists \( t_3 > 0 \)
such that \( E_5(0)e^{-\omega t} < \frac{\varepsilon^2}{16}, t > t_3 \). Then, for a certain \( \mu = \frac{\varepsilon}{4\sqrt{\frac{\sigma}{2}}} \), (29) gives that
\[ \frac{1}{2} \xi_1^T \xi_1 < E_\xi(t) < \frac{\varepsilon^2}{8}, \quad t > t_3. \quad (31) \]
which implies that \( \|\xi_1\| < \frac{\varepsilon}{2} \). Then, choosing \( T_2 = \max\{t_1, t_2, t_3\} \), (8) with \( i = 1 \) gives that
\[ \|z_1\| < \|v_1\| + \|\xi_1\| < \varepsilon, \quad t > T_2. \]
which leads to that \( \dot{k}(t) \equiv 0, t > T_2 \) by using (11). Then, \( \kappa(t) \) remains a constant after \( T_2 \). This contradicts with the induce assumption, and hence \( \kappa(t) \) is bounded on \([0, +\infty)\). \( \square \)

It is a position to give the main results of the closed-loop stability, which are summarized as follows:

**Theorem 1** For system (1) with Assumptions 1–3, under the command filters and auxiliary dynamic system given, respectively, by (5) and (6), the designed (virtual) control with the error compensation signals given by (13), (16), (18), (21), (24) as well as the adaptive law given by (11) guarantee the following two properties of the resulting closed-loop system:

(1) all the states of the closed-loop system are bounded on \([0, +\infty)\).

(2) the system output \( q \) practically track the reference signal \( y_r \), that is, for any \( \varepsilon > 0 \), there exists a finite time \( T^* > 0 \) such that
\[ \|q - y_r\| \leq \varepsilon, \quad t > T^*. \]

**Proof** (1) Boundedness of the closed-loop system signals

Integrating both sides of (25) over \([0, t]\) gives that
\[ E_5(t) \leq E_5(0)e^{-\omega t} + \frac{t}{\sigma}, \]
which shows that \( E_5(t) \) is bounded on \([0, +\infty)\). So are \( v_1 \)'s by noting the definition of \( E_5(t) \). Then, (8) implies that \( z_i \)'s are bounded by noting the boundedness of \( \xi_i \)'s (see Proposition 2). Moreover, Assumption 1 implies that \( x_i \)'s are bounded since \( u = sat(\psi) \) is bounded.

In the following, the boundedness of \( w_i \)'s and \( x_i e \)'s will be derived by recursion. First, noting that \( x_{1,e} = y_r \) which is bounded by Assumption 3, (4) with \( i = 1 \)

\[ \square \]
implies that \( w_1 \) is bounded. So is \( \alpha_1 \) by the first equation
of (13). Then, the inequality \( ||x_{2,c} - \alpha_1|| < \mu \) (see it in
Proposition 1) that \( x_{2,c} \) is bounded, and hence \( w_2 \) is bounded
by (4) with \( i = 2 \). Noting that \( ||e_{1,1} - \alpha_1|| < \mu \), the second equation of (5) gives that \( e_{1,2} \) is bounded,
so is \( \dot{e}_{1,1} \) (i.e., \( \dot{x}_{2,c} \)) by the first equation of (5). Then,
\( \alpha_2 \) is bounded by the first equation of (16), and hence
\( x_{3,c} \) is bounded by noting the fact that \( ||x_{3,c} - \alpha_2|| < \mu \),
and the boundedness of \( w_3 \) follows by (4) with \( i = 3 \).
The derivation of the boundedness of \( w_4, w_5 \) and \( x_{4,c} \),
\( x_{5,c} \) is omitted since which are similar to those of \( w_3 \)
and \( x_{3,c} \). Finally, the first equation of (24) implies that
\( \psi \) is bounded.

(2) Practical tracking of system output to reference signal
As preparations, the estimation of \( ||w_i|| \) is first given.
In fact, by defining \( E_w = \frac{1}{2} \sum_{i=1}^{5} \gamma_i w_i^T w_i \), the auxiliary
system (6) gives that
\[
\dot{E}_w = \sum_{i=1}^{5} \gamma_i w_i^T \dot{w}_i = \gamma w_1^T (w_2 - \rho_1 w_1) + \gamma w_2^T (D^{-1}(x_1)K w_3 - \rho_2 w_2) + \gamma w_3^T (w_4 - \rho_3 w_3) + \gamma w_4^T (J^{-1} K T w_5 - \rho_4 w_4) + \gamma w_5^T (L^{-1} \Delta u - \rho_5 w_5).
\] (32)
By Young’s inequality and Property 1, some terms on
the right-hand side of above equality are estimated
which are given as follows:
\[
\{ w_1^T w_2 \leq \frac{w_1^T w_1}{2} + \frac{w_2^T w_2}{2}, \ w_2^T D^{-1}(x_1) K w_3 \leq \frac{D^{-2} ||K||^2 w_2^T w_2}{2} + \frac{w_4^T w_3}{2}, \ w_3^T w_4 \leq \frac{w_3^T w_3}{2} + \frac{w_4^T w_4}{2}, \ w_4^T J^{-1} K T w_5 \leq \frac{J^{-1} ||K T||^2 w_4^T w_4}{2} \ + \frac{w_5^T w_5}{2}, \ w_5^T L^{-1} \Delta u \leq \frac{w_5^T w_5}{2} + \frac{L^{-1} ||\Delta u||^2}{2}, \}
\] where the fact that \( ||\Delta u|| = ||sat(\psi) - \psi|| \leq \zeta \) with
\( \zeta \) being some positive constant by noting the boundedness
of \( \psi \) has been used. Substituting the above inequalities into (32) leads to that
\[
\dot{E}_w \leq -\gamma \left( \rho_1 - \frac{1}{2} \right) w_1^T w_1 - \gamma \left( \rho_2 - \frac{D^{-2} ||K||^2}{2} - \frac{1}{2} \right) w_2^T w_2 - \gamma (\rho_3 - 1) w_3^T w_3 - \gamma \left( \rho_4 - \frac{1}{2} \right) w_4^T w_4 - \gamma (\rho_5 - 1) w_5^T w_5 + \frac{\gamma L^{-1} ||\psi||^2}{2} \ .
\] (33)

with \( \rho = \min \{2 \rho_1 - 1, 2 \rho_2 - ||K||^2 D^{-2} - 1, 2 \rho_3 - 2, 2 \rho_4 - 1 - ||J^{-1}||^2 ||K T||^2, 2 \rho_5 - 2 \} \).

Integrating both sides of (33) over \([0, t]\) arrives at
\[
E_w(t) \leq E_w(0) e^{-\rho t} + \frac{\gamma L^{-1} ||\psi||^2}{2 \rho}.
\] (34)

Then, it is ready to show that the system output prac-
tically tracks the given reference signal. It suffices to show
that \( \lim_{t \to +\infty} \dot{k}(t) = 0 \). First, \( \dot{z}_1 \) is bounded since
all the terms on the right-hand side of the first equation
of (7) are bounded, and hence \( z_1 \) is uniformly continuous.
So is \( \dot{k}(t) \) by using (11). Then, the boundedness of \( \kappa(t) \) (see Proposition 3) implies that
\[
\int_0^{+\infty} \dot{k}(t) dt = \kappa(+\infty) - \kappa(0) < +\infty,
\] which show that \( \dot{k}(t) \) is integrable. Therefore, by the
well-known Barbalat’s Lemma, we obtain that \( \dot{k}(t) \rightarrow 0(t \to +\infty) \). This implies that there exists a finite time \( t_5 > 0 \) such that
\[
|\dot{k}(t)| \leq \frac{\epsilon}{4}, \ t > t_5,
\]
which, together with (11), gives that \( \|z_1\| \leq \frac{\xi}{2}, t > t_5 \). Then, by choosing \( T^* = \max\{t_4, t_5\} \), the first equation of (4) and (34) with \( i = 1 \) give that
\[
\|q - y_r\| \leq \|z_1\| + \|w_1\| \leq \varepsilon, \quad t > T^*.
\]

\[ \square \]

5 Simulation results

In this section, simulation results by the proposed method are first given to validate the effectiveness of the proposed theoretical results. Then, comparison with the existing methods is given to show the advantages of the proposed method.

We choose a two-link manipulator with flexible-joint actuated by DC motors for the simulation.

The matrices of system (1) are defined as follows:
\[
D(q) = \begin{pmatrix}
(m_1 + m_2)l_1^2 & m_2l_1l_2\cos(q_2 - q_1) \\
m_2l_1l_2\cos(q_2 - q_1) & m_2l_2^2
\end{pmatrix},
\]
\[
C(q, \dot{q}) = \begin{pmatrix}
0 & -m_2l_1l_2\dot{q}_2\sin(q_2 - q_1) \\
m_2l_1l_2\dot{q}_1\sin(q_2 - q_1) & 0
\end{pmatrix},
\]
\[
G(q) = \begin{pmatrix}
(m_1 + m_2)g_1\sin q_1 \\
m_2g_2\sin q_2
\end{pmatrix},
\]

where \( q = (q_1, q_2)^T \), \( \dot{q} = (\dot{q}_1, \dot{q}_2)^T \), \( m_i \) \((i = 1, 2)\) indicates the mass of link \( i \), \( l_i \) \((i = 1, 2)\) indicates the length of link \( i \), the actual values of these parameters as well as other parameters in the simulation are given in Table 2. The initial conditions of the system are chosen as \( q(0) = (0.6, -0.4)^T \), \( \theta(0) = (1.2, -0.8)^T \), \( I(0) = (0.3, 0.2)^T \), \( \dot{q}(0) = (2.1)^T \), \( \dot{\theta}(0) = (1.1, -0.5)^T \). Suppose that the lumped uncertainties in system (1) are defined as follows:
\[
\begin{align*}
d_1 &= (3\cos x_1 + \sin t, 2.5\cos 2t + \sin x_2)^T, \\
d_2 &= (2\sin t + \cos x_3, 3\cos x_4 + 2\sin t)^T, \\
d_3 &= (4\sin x_5 + 2\cos t, 3.5\cos 2t + \cos x_3)^T.
\end{align*}
\]

The reference signal is chosen as \( y_r = (y_{r1}, y_{r2})^T \) with
\[
\begin{align*}
y_{r1} &= \begin{cases} 
0.1, & 0 \leq t < \frac{\pi}{2}, \\
0.1\sin t, & t \geq \frac{\pi}{2}.
\end{cases} \\
y_{r2} &= 0.2\cos 2t.
\end{align*}
\]

Remark that the above reference signal is just first-order continuously differentiable. For the tracking accuracy parameter \( \varepsilon = 0.05 \), simulation results are given as follows:

Table 2 Description of parameters and variables

| Parameter (unit) | Value | Parameter (unit) | Value |
|------------------|-------|------------------|-------|
| \( m_1(\text{kg}) \) | 0.8   | \( m_2(\text{kg}) \) | 1     |
| \( l_1(\text{m}) \) | 1.5   | \( g(\text{m/s}^2) \) | 9.8   |
| \( K_1(\text{Nm/\text{rad}}) \) | 2     | \( K_2(\text{Nm/\text{rad}}) \) | 4     |
| \( J_1(\text{kg \cdot m}^2) \) | 0.2   | \( J_2(\text{kg \cdot m}^2) \) | 0.1   |
| \( B_1(\text{Nm \cdot s/\text{rad}}) \) | 0.5   | \( B_2(\text{Nm \cdot s/\text{rad}}) \) | 0.9   |
| \( K_t_1(\text{Nm/A}) \) | 6     | \( K_t_2(\text{Nm/A}) \) | 9     |
| \( L_1(\text{mH}) \) | 1     | \( L_2(\text{mH}) \) | 2.5   |
| \( R_{11}(\Omega) \) | 2     | \( R_{22}(\Omega) \) | 5     |
| \( K_{B_1}(\text{Nm/A}) \) | 2     | \( K_{B_2}(\text{Nm/A}) \) | 3     |

| Table 2 \( Description \) of parameters and variables |
|------------------------------------------------------|
| Parameter (unit) | Value | Parameter (unit) | Value |
|------------------|-------|------------------|-------|
| \( m_1(\text{kg}) \) | 0.8   | \( m_2(\text{kg}) \) | 1     |
| \( l_1(\text{m}) \) | 1.5   | \( g(\text{m/s}^2) \) | 9.8   |
| \( K_1(\text{Nm/\text{rad}}) \) | 2     | \( K_2(\text{Nm/\text{rad}}) \) | 4     |
| \( J_1(\text{kg \cdot m}^2) \) | 0.2   | \( J_2(\text{kg \cdot m}^2) \) | 0.1   |
| \( B_1(\text{Nm \cdot s/\text{rad}}) \) | 0.5   | \( B_2(\text{Nm \cdot s/\text{rad}}) \) | 0.9   |
| \( K_t_1(\text{Nm/A}) \) | 6     | \( K_t_2(\text{Nm/A}) \) | 9     |
| \( L_1(\text{mH}) \) | 1     | \( L_2(\text{mH}) \) | 2.5   |
| \( R_{11}(\Omega) \) | 2     | \( R_{22}(\Omega) \) | 5     |
| \( K_{B_1}(\text{Nm/A}) \) | 2     | \( K_{B_2}(\text{Nm/A}) \) | 3     |

1) Simulation by the proposed control method

By using the adaptive controller (11), (24) with \( \xi_i(0) = 0(i = 1, \cdots, 4) \), \( \xi_5(0) = (10, 10)^T \), the amplitudes of input saturation being \( u_{\text{max}} = 50 \), \( u_{\text{min}} = -50 \). Implementing the above controller with \( c_1 = 2 \), \( c_2 = 850 \), \( c_3 = 3 \), \( c_4 = 1624 \), \( c_5 = 2 \), \( \omega = 60 \), \( \zeta = 0.6 \), \( \rho_1 = 1 \), \( \rho_2 = 50 \), \( \rho_3 = 2 \), \( \rho_4 = 8102 \), \( \rho_5 = 5 \) in MATLAB, eight simulation figures are obtained and given below. Figure 2 shows that the tracking error \( q - y_r \) enters and then stays at the prescribed \( \varepsilon \) neighborhood (i.e., \([-0.05, 0.05]\)) of the origin after 10s. Figure 3 demonstrates that the control gain function \( \kappa(t) \) is bounded. Figure 4 shows that the control law \( \psi \) is bounded but somewhat large. Figure 5 shows that the saturation input \( u \) is bounded and always stays at the prescribed range of the saturation function. Also, Fig. 5 shows that the control input \( u \) reaches the saturation boundary values many times before 1s. This may be related to the excessive lumped uncertainties selected or the large derivation of the tracking error to the desirable neighborhood (as shown in Fig. 2). Figure 6 indicates that the closed-loop states \( q, \dot{q}, \theta, \dot{\theta}, I \) are bounded. Figures 7 and 8, respectively, show that \( w_i \)’s and \( \xi_i \)’s are bounded. Figure 9 demonstrates that the command filters signals \( x_{i,c} \)’s are bounded.

2) Comparison with the existing methods

Since the reference signal given by (35) is only first-order continuously differentiable but those in [3,5,7–14,16–19,21,25] must be at least twice continuously differentiable, the proposed controllers in the literature are incapable. Noting the capability of the traditional PID control scheme for general reference signals and serious system uncertainties, the comparison of the
simulation results between the proposed method and the PID control method is given in this section.

In fact, for the given system and the reference signal, one can choose the following PID controller:

\[ u = k_p e_1 + k_i \int_0^t e_1(t) \, dt + k_d \dot{e}_1, \tag{36} \]

where \( e_1 = q - y_r \), \( k_p, k_i, k_d \) are controller parameters. It is worth pointing out that different from this paper which is smart in the choosing of controller parameters since any controller parameters satisfying certain constraints depending on system parameters (i.e., \( \omega > 0 \), \( \zeta \in (0, 1) \), \( c_i > 0 \), \( \rho_i \) satisfying some inequalities below (6), \( i = 1 \ldots 5 \) can guarantee the desirable control performance, the control performance of PID controller severely depends on the choice of the controller parameters \( k_p, k_i, k_d \) which should be carefully chosen, and random choice of a group of controller parameters cannot guarantee the desired control performance. For example, by selecting \( k_p = 2.2 \), \( k_i = -4.4 \) and \( k_d = 5 \) which are not carefully chosen, Fig. 10 shows that the tracking error cannot enter and then stay at the prescribed \( \varepsilon \) neighborhood (i.e., \([-0.05, 0.05]\)) of the origin after some time.
6 Concluding remarks

In this paper, practical tracking control under actuator saturation has been solved for a class of flexible-joint robotic manipulators driven by motors. A state feedback controller is explicitly designed by skillfully combing backstepping scheme, adaptive technique and the constructive methods of command filters and an auxiliary system for the actuator saturation, which guarantees that all the states of the resulting closed-loop system are bounded, while the system output practically tracks a large class of the reference signals.

The future research directions are the following twofold: (1) Tack control under more serious uncertainties. Although this paper considers the lumped
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Fig. 10 Trajectory of tracking error $q - y_r$ by PID

uncertainties (i.e., $d_i, i = 1, 2, 3$ in system (1)), certain nominal parts require to be known (such as the known matrices $D(q), C(q, \dot{q}), G, K, L$, etc.) and hence restrict the uncertainties. Then, it is interesting to consider the tracking control under more serious uncertainties (both external disturbances and the system matrices/parameters without known nominal parts are allowed). For this, a powerful dynamic compensation for the serious uncertainties requires to be developed. Track control via output feedback. Note that the controller designed in the paper require all the states to be available for feedback, and hence leads to high burden of measurement in implementation. Thus, how to design an output feedback controller by using less measurements (such as only $q$) will be rather interesting. For this, a state observer should be designed to reconstruct the unobservable states and hence challenges the control design.

Availability of data and material Not applicable.

Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

Code availability Not applicable.

References

1. Albu-Schaffer, A., Ott, C., Hirzinger, G.: A unified passivity-based control framework for position, torque and impedance control of flexible joint robots. Int. J. Robot. Res. 26(1), 23–39 (2007)
2. Palleschi, A., Mengacci, R., Angelini, F., Caporale, D., Pallottino, L., De Luca, A., Garabini, M.: Time-optimal trajectory planning for flexible joint robots. IEEE Robot. Autom. Lett. 5(2), 938–945 (2020)
3. Chang, Y.C., Yen, H.L.: Robust tracking control for a class of electrically driven flexible-joint robots without velocity measurements. Int. J. Control 85(2), 194–212 (2012)
4. Fateh, L.L.: Robust control of flexible-joint robots using voltage control strategy. Nonlinear Dyn. 67(2), 1525–1537 (2012)
5. Izadbakhsh, A.: Robust control design for rigid-link flexible-joint electrically driven robot subjected to constraint: theory and experimental verification. Nonlinear Dyn. 85(2), 751–765 (2016)
6. Li, J., Ma, K., Wu, Z.: Prescribed performance control for uncertain flexible-joint robotic manipulators driven by DC motors. Int. J. Control Autom. Syst. 19(4), 1640–1650 (2021)
7. Kim, J., Croft, E.A.: Full-state tracking control for flexible joint robots with singular perturbation techniques. IEEE Tran. Control Syst Technol. 27(1), 63–73 (2017)
8. Li, Y., Tong, S., Li, T.: Adaptive fuzzy output feedback control for a single-link flexible robot manipulator driven DC motor via backstepping. Nonlinear Anal. Real World Appl. 14(1), 483–494 (2013)
9. Cui, L., Wu, Z.: Trajectory tracking of flexible joint manipulators actuated by DC-motors under random disturbances. J. Franklin Inst. 356(16), 9330–9343 (2019)
10. Mbede, J.B., Mvogo Ahanda, J.J.B.: Exponential tracking control using backstepping approach for voltage-based control of a flexible joint electrically driven robot. J. Robot. (2014)
11. Izadbakhsh, A., Khorashadizadeh, S.: Single-loop PID controller design for electrical flexible-joint robots. J. Braz. Soc. Mech. Sci. Eng. 42(2), 1–12 (2020)
12. Chien, L.C., Huang, A.C.: Adaptive impedance controller design for flexible-joint electrically-driven robots without computation of the regressor matrix. Robotica 30(1), 133–144 (2012)
13. Liu, Z.G., Wu, Y.Q.: Modelling and adaptive tracking control for flexible joint robots with random noises. Int. J Control 87(12), 2499–2510 (2014)
14. Oya, L., Su, C.Y., Kobayashi, T.: State observer-based robust control scheme for electrically driven robot manipulators. IEEE Trans. Robot. 20(4), 796–804 (2004)
15. Hwang, J.P., Kim, E.: Robust tracking control of an electrically driven robot: adaptive fuzzy logic approach. IEEE Trans. Fuzzy Syst. 14(2), 232–247 (2006)
16. Liu, Z.G., Huang, J.L.: A new adaptive tracking control approach for uncertain flexible joint robot system. Int. J. Autom. Comput. 12(5), 559–566 (2015)
17. Burg, T., Dawson, D., Hu, J., De Queiroz, L.: An adaptive partial state-feedback controller for RLED robot manipulators. IEEE Trans. Autom. Control 41(7), 1024–1030 (1996)
18. Sun, W., Su, S.F., Xia, J., Nguyen, V.T.: Adaptive fuzzy tracking control of flexible-joint robots with full-state constraints. IEEE Trans. Syst. Man Cybern. Syst. 49(11), 2201–2209 (2018)
19. Yoo, S.J., Park, J.B., Choi, Y.H.: Adaptive output feedback control of flexible-joint robots using neural networks: dynamic surface design approach. IEEE Trans. Neural Netw. 19(10), 1712–1726 (2008)
20. Diao, S., Sun, W., Yuan, W.: Adaptive fuzzy practical tracking control for flexible-joint robots via command filter design. Measur. Control 53(5–6), 814–823 (2020)
21. Yan, Z., Lai, X., Meng, Q., Zhang, P., Wu, L.: Tracking control of single-link flexible-joint manipulator with unmodeled dynamics and dead zone. Int. J. Robust Nonlinear Control 31(4), 1270–1287 (2021)
22. Chien, L.C., Huang, A.C.: Adaptive control for flexible-joint electrically driven robot with time-varying uncertainties. IEEE Trans. Ind. Electron. 54(2), 1032–1038 (2007)
23. Yoo, S.J., Park, J.B., Choi, Y.H.: Adaptive dynamic surface control of flexible-joint robots using self-recurrent wavelet neural networks. IEEE Trans. Syst. Man Cybern. Part B Cybern. 36(6), 1342–1355 (2006)
24. Kim, L.S., Lee, J.S.: Adaptive tracking control of flexible-joint manipulators without overparametrization. J. Robot. Syst. 21(7), 369–379 (2004)
25. Wang, H., Zhang, Y., Zhao, Z., Tang, X., Yang, J., Chen, I.: Finite-time disturbance observer-based trajectory tracking control for flexible-joint robots. Nonlinear Dyn. 106(1), 459–471 (2021)
26. Peng, J.Z., Ding, S., Yang, Z.Q., Xin, J.B.: Adaptive neural impedance control for electrically driven robotic systems based on a neuro-adaptive observer. Nonlinear Dyn. 100(2), 1359–1378 (2020)
27. Khalil, H.: Nonlinear Systems, 3rd edn. Prentice-Hall, Englewood Cliffs, NJ (2002)
28. Farrell, J.A., Polycarpou, L., Sharma, L., Dong, W.: Command filtered backstepping. IEEE Trans. Autom. Control 54(6), 1391–1395 (2009)
29. Dong, W.J., Farrell, J.A., Polycarpou, L.L., Djapic, V., Sharma, L.: Command filtered adaptive backstepping. IEEE Trans. Control Syst. Technol. 20(3), 566–580 (2011)
30. Liu, Y.J., Zhao, W., Liu, L., Li, D., Tong, S., Chen, C.P.: Adaptive neural network control for a class of nonlinear systems with function constraints on states. IEEE Trans. Neural Networks Learn. Syst. (2021)
31. Liu, L., Liu, Y.J., Chen, A., Tong, S., Chen, C.L.: Integral barrier Lyapunov function-based adaptive control for switched nonlinear systems. Sci.China Inf. Sci. 63(3), 1–14 (2020)

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