Darboux Transformation and Classification of Solution for Mixed Coupled Nonlinear Schrödinger Equations

Liming Ling¹, Li-Chen Zhao², Boling Guo³

¹School of Mathematics, South China University of Technology, Guangzhou 510640, China; Email: linglm@scut.edu.cn
²Department of Physics, Northwest University, 710069, Xi’an, China;
³Institute of Applied Physics and Computational Mathematics, 100088, Beijing, China

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Abstract

We derive generalized nonlinear wave solution formula for mixed coupled nonlinear Schrödinger equations (mCNLSE) by performing the unified Darboux transformation. We give the classification of the general soliton formula on the nonzero background based on the dynamical behavior. Especially, the conditions for breather, dark soliton and rogue wave solution for mCNLSE are given in detail. Moreover, we analysis the interaction between dark-dark soliton solution and breather solution. These results would be helpful for nonlinear localized wave excitations and applications in vector nonlinear systems.

Key words: Darboux transformation, mCNLSE, Rogue wave, Breather, Dark-dark soliton.

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1 Introduction

Nonlinear Schrödinger equation (NLSE) is an important model in mathematical physics, which can be applied to hydrodynamics [1], plasma physics [2], molecular biology [3] and optics [4]. Recently, Peregrine soliton (rogue wave solution), Akhmediev breather, Kuznetzov-Ma breather and dark soliton were observed in experiments in succession. For instance, Kuznetzov-Ma soliton was confirmed in 2012 [3], the Akhmediev breather was verified in numerical experiment [5], the Peregrine soliton was experimentally observed in nonlinear fibre optics system [6], water tank [7, 8] and plasma [9]. Dark soliton was observed on the surface of water [10]. Indeed those exact solutions for the NLSE on the plane wave background were known well long time ago [3]. For the focusing NLSE, there exists Akhmediev breather, Kuznetzov-Ma soliton and Peregrine soliton. There exists dark soliton for the defocusing NLSE.

However, the nonlinear localized wave solutions for the coupled nonlinear Schrödinger equations (CNLSE) are more complexity than NLSE. Firstly, the spectral problem of NLSE is \(2 \times 2\), the coupled NLSE is \(3 \times 3\). The inverse scattering method of CNLSE on the nonzero background has not been solved completely [11]. Secondly, for the Darboux transformation (DT) method, the Darboux matrix for \(3 \times 3\) spectral problem is more complexity than \(2 \times 2\) spectral problem. What’s more, the Darboux matrix for the defocusing or mixed coupled NLSE is no longer positive or negative definite. In this work, we will use the matrix analysis method to deal with the condition of positive or negative definite of Darboux matrix.

Previous to introducing our work, we review a brief research history of CNLSE. The integrability of the CNLSE had been shown by Manakov who had also obtained the bright soliton in a focusing medium by applying the inverse scattering method [12]. An interesting fact for CNLSE is the collision of soliton on the vanishing background can be inelastic [13, 14] or even appear the soliton...
reflection [15]. For the nonzero background, there are lots of works for the focusing CNLSE [16, 17, 18, 19, 20, 21, 22]. But there are few results for the defocusing and mixed case [11]. The dark-dark soliton solution, bright-dark soliton solution and breather solution for defocusing CNLSE were given in reference [23] by inverse scattering method. The soliton solutions for the multi-component NLSE were given in reference [16, 24, 25] through DT. Recently, different types of soliton solutions on the nonzero background were obtained by algebraic geometry reduction method [26].

DT is a powerful method to construct the soliton solution for the integrable equations. There are different methods to derive the DT, for instance, operator decomposition method [27], gauge transformation [28, 29], loop group method [30] and Riemann-Hilbert method [31]. The DT for multi-component NLSE was given in reference [16, 24, 32, 33, 34] (and reference therein). Recently, combined Darboux dressing and tau function method for dark-dark soliton is given in reference [35]. Tsuchida gave a detailed analysis of different types of solutions by using the Darboux-Bäcklund transformation [25].

In this work, we focus on the nonlinear localized wave solutions of the mCNLSE

\[
iq_{1,t} + \frac{1}{2}q_{1,xx} + (|q_2|^2 - |q_1|^2)q_1 = 0, \\
iq_{2,t} + \frac{1}{2}q_{2,xx} + (|q_2|^2 - |q_1|^2)q_2 = 0,
\]

on the plane wave background. The classification of nonlinear localized wave solutions for mCNLSE (1) on the nonzero background is not an easy work but be important for nonlinear wave theory and physical applications. Indeed, there are few results about the classification of nonlinear localized wave solutions for NLSE, even for the focusing NLSE. Recently, there are lot of works about focusing CNLSE through DT. For instance, vector rogue wave (type-I and type-II) solution, bright-dark-rogue wave solution [17, 18, 19, 20] and high order solution [36] have been given. However, for the mCNLSE or the defocusing CNLSE, the Darboux matrix is no longer always positive definite or negative definite. We would confront with a new obstacle for the classification of solution through DT. Comparing with focusing CNLSE, a key step is giving the positive or negative definite condition for the Darboux matrix. To the best of our knowledge, this problem have never been solved with a proper method. In this work, we use the matrix analysis method to deal with this problem. Through this method, we can obtain the complete classification for the nonsingular solution for the mCNLSE (1). Besides the DT, the inverse scattering method [15] and the bilinear method [13, 37, 38] can be used to derive soliton solution of the CNLSE too.

This paper is organized as following. In section 2, we classify the soliton solution of mCNLSE (1) with two categories. Denote the wave vector of plane wave background for the i-th component is \(a_i\) respectively. The first case is \(a_1 = a_2\). In this case, we can obtain the degenerate rogue wave, degenerate breather (type-I and type-II), degenerate dark-dark soliton and bright-dark soliton. The second case is \(a_1 \neq a_2\). In this case, we can obtain the general rogue wave solution, general breather (type I and type II) solution and general dark-dark soliton solution. The coexistence of different types solution is given by two different categories. The rogue wave solutions for mCNLSE (1) are given with a compact form. The classification of rogue wave solutions is given based on the dynamics behavior. A method for looking for different types of rogue wave is given in detail. In section 3, we give the interaction between different types of soliton solution. Since the rogue wave solution is the limit of breather solution, we deem that rogue wave is the same kind with breather solution, although their dynamics behaviors are different. Thus we merely analysis the interaction between breather type solution and dark soliton. Finally, we give some discussions and conclusions.
2 Darboux transformation and classification of solution

As we well known that, the mCNLSE admits the following Lax pair:

\[
\Phi_x = U \Phi, \quad U(\lambda, Q) \equiv i(\lambda \sigma_3 + Q),
\]
\[
\Phi_t = V \Phi, \quad V(\lambda, Q) \equiv i\lambda^2 \sigma_3 + i\lambda Q - \frac{1}{2} \sigma_3(iQ^2 - Q_x),
\]

where

\[
\sigma_3 = \text{diag}(1, -1, -1), \quad Q = \begin{bmatrix} 0 & -\bar{q}_1 & \bar{q}_2 \\ q_1 & 0 & 0 \\ q_2 & 0 & 0 \end{bmatrix},
\]

the overbar represent the complex conjugation (similarly hereinafter). The compatibility condition of Lax pair (2) gives the mCNLSE (1). The unified DT is obtained in reference [39] with an integral form. Here we give another representation with a limit form:

**Theorem 1 ([39], Ling, Zhao and Guo)** The following unified DT

\[
\Phi[1] = T_1 \Phi, \quad T_1 = I - \frac{P_1}{\lambda - \lambda_1},
\]

where

\[
\begin{align*}
\text{if } \lambda_1 & \not\in \mathbb{R}, & P_1 &= \frac{(\lambda_1 - \bar{\lambda}_1)|y_1\rangle\langle y_1|J}{\langle y_1|J|y_1\rangle}, \\
\text{if } \lambda_1 & \in \mathbb{R}, & P_1 &= \lim_{\lambda_1 \to \bar{\lambda}_1} \frac{(\lambda_1 - \bar{\lambda}_1)|y_1\rangle\langle y_1|J}{\langle y_1|J|y_1\rangle},
\end{align*}
\]

\(J = \text{diag}(1, -1, -1), \quad |y_1\rangle \equiv v_1(x, t)\Phi_1, \quad \langle y_1| = |y_1|^\dagger, \) \(v_1(x, t)\) is an arbitrarily complex function, \(\Phi_1\) is a special vector solution for linear system (2) with \(\lambda = \lambda_1\), and \(^\dagger\) represents Hermite conjugation (similarly hereinafter), converts the above linear system (2) into a new linear system

\[
\begin{align*}
\Phi[1]_x &= U[1]\Phi[1], \quad U[1] = U(\lambda, Q[1]), \\
\Phi[1]_t &= V[1]\Phi[1], \quad V[1] = V(\lambda, Q[1]),
\end{align*}
\]

and transformation between potential functions is

\[
Q[1] = Q + [\sigma_3, P_1],
\]

where commutator \([A, B] \equiv AB - BA\).

To keep the results with the inverse scattering method, we restrict the parameter \(\lambda_1 \in \{z|\text{Im}(z) \geq 0\}\). The expression for \(P_1 (4b)\) is considered in the limited sense, since there exists special solutions which satisfy \(\langle y_1|J|y_1\rangle = 0\) when \(\lambda_1 \in \mathbb{R}\).

To obtain the nonlinear localized wave solutions on the plane wave background, we consider the following plane wave solutions as the seed solutions for DT

\[
q_1[0] = c_1 e^{i\theta_1}, \quad \theta_1 = \left[ a_1 x - \left( \frac{1}{2} a_1^2 + c_1^2 - c_2^2 \right) t \right],
\]
\[
q_2[0] = c_2 e^{i\theta_2}, \quad \theta_2 = \left[ a_2 x - \left( \frac{1}{2} a_2^2 + c_1^2 - c_2^2 \right) t \right],
\]

where \(a_1, a_2, c_1\) and \(c_2\) are real constants. Through above transformation (6) and the plane wave seed solution, we can obtain different types of nonlinear wave solutions. If \(\lambda_1 \not\in \mathbb{R}\), the DT can be
used to derive breather solution, rogue wave solution and bright-dark soliton solution. If $\lambda_1 \in \mathbb{R}$, the dark-dark soliton solution can be obtained through this transformation \(8\).

Next we give a method how to choose the special solution $\Phi_1$ to construct the nonsingular exact solution of mCNLSE \(1\). Through above DT \(3\) and \(6\), we can obtain many new nonlinear wave solutions for mCNLSE \(1\), which have not been reported before. To use the transformation \(3\), we firstly need to solve the linear system (2) with seed solution (7) by the gauge transformation method \[17\]:

$$\Phi = D \Psi, \quad D = \text{diag}(1, e^{i\theta_1}, e^{i\theta_2}).$$

Then matrix $\Psi$ satisfies the following linear system:

$$\Psi_x = iU_0 \Psi_t,$$

$$\Psi_t = i \left( \frac{1}{2} U_0^2 + \lambda U_0 - \frac{1}{2} \lambda^2 + c_1^2 - c_2^2 \right) \Psi,$$

where

$$U_0 = \begin{bmatrix} \lambda & -c_1 & c_2 \\ c_1 & -\lambda - a_1 & 0 \\ c_2 & 0 & -\lambda - a_2 \end{bmatrix}.$$

We can obtain the different kinds of solution by choosing different special solutions \(8\). The studies on coupled focusing or defocusing NLS have shown that the relative wave vector plays important role in determining dynamics of nonlinear waves \[19, 20, 40\]. Therefore, we classify them into two main cases according to the relative wave vector.

2.1 Case I: When $a_2 = a_1$

In this case, we have the following matrix decomposition

$$U_0 M = MD_0, \quad D_0 = \text{diag}(\chi - \lambda, \mu - \lambda, -(\lambda + a_1)), \quad \chi \neq \mu,$$

where

$$M = \begin{bmatrix} 1 & 1 & 0 \\ -c_1 & \chi + a_1 & c_2 \\ -c_2 & \mu + a_1 & \chi + a_1 \end{bmatrix},$$

and $\chi$ and $\mu$ are two different roots of the quadratic equation:

$$\xi^2 + (a_1 - 2\lambda)\xi - 2a_1\lambda + c_1^2 - c_2^2 = 0. \quad (10)$$

Then the fundamental solution \(8\) can be given as $\Phi = DMN$, where $N = \text{diag}(e^{iA_1}, e^{iB_1}, e^{iC_1})$,

$$A_1 = (\chi - \lambda)x + \left[ \frac{1}{2} \chi^2 + (c_1^2 - c_2^2 - \lambda^2) \right] t,$$

$$B_1 = (\mu - \lambda)x + \left[ \frac{1}{2} \mu^2 + (c_1^2 - c_2^2 - \lambda^2) \right] t,$$

$$C_1 = -(a_1 + \lambda)x + \left[ \frac{1}{2} a_1^2 + (c_1^2 - c_2^2 - \lambda^2) \right] t.$$

By the decomposition \(9\), we can simplify the solution form of mCNLSE \(1\) through using the following identity:

$$2\lambda - \chi + \frac{c_2^2 - c_1^2}{\chi + a_1} = 0,$$

$$2\lambda - \mu + \frac{c_2^2 - c_1^2}{\mu + a_1} = 0. \quad (11)$$
Interestingly, a further classification on nonlinear wave solutions can be made through relations between $c_1$ and $c_2$.

(a): When $c_1 > c_2$, the system (10) reflects the defocusing mechanism. With this case, there exists dark-dark soliton solution. Besides the dark-dark soliton, there exists breather solution. In what following, we give the explicit construction method for them. Firstly, solving equation (10), we have the following solution:

$$
\chi = \lambda - \frac{a_1}{2} + \sqrt{(\lambda + \frac{a_1}{2})^2 - (c_1 - c_2)}, \\
\mu = \lambda - \frac{a_1}{2} - \sqrt{(\lambda + \frac{a_1}{2})^2 - (c_1 - c_2)}.
$$

By above two roots, it is readily to obtain that $\mu + a_1 = \frac{(\lambda + a_1)(c_1^2 - c_2)}{\lambda + a_1}$, which implies that the roots $\chi$ and $\mu$ can not lay on the upper half plane or lower half plane simultaneously.

To give the nonsingular solution of (1), one must choose the special solution $\Phi_1$ such that $\Phi_1^\dagger J \Phi_1$ is negative or positive definitely. To find the special solution $\Phi_1$, it is meaningful to analysis the following matrix:

$$
M^\dagger J M = \begin{bmatrix}
\frac{2(\lambda - \lambda)}{\chi - \mu} & \frac{2(\lambda - \lambda)}{\chi - \mu} & 0 \\
\frac{2(\lambda - \lambda)}{\mu - \chi} & \frac{2(\lambda - \lambda)}{\mu - \mu} & 0 \\
0 & 0 & c_1^2 - c_2^2
\end{bmatrix},
$$

which can be obtained through equations (11). Indeed, the above matrix can not be positive definite or negative definite, since above matrix is nothing but the congruent matrix of $J$ and the congruent matrix can not change the sign of characteristic roots. However, we can look for some submatrix which could be positive or negative definite. By inspection, if $\text{Im}(\lambda)\text{Im}(\chi) > 0$, we can find that

$$
\begin{bmatrix}
1 & \frac{c_1}{\chi + a_1} & \frac{c_2}{\chi + a_1} \\
0 & \frac{1}{\chi + a_1} & c_1 \\
c_2 & c_1 & c_1
\end{bmatrix} J \begin{bmatrix}
1 & 0 \\
\frac{1}{\chi + a_1} & c_2 \\
\frac{c_1}{\chi + a_1} & c_1
\end{bmatrix} = \begin{bmatrix}
2\frac{\lambda - \lambda}{\chi - \mu} & 0 \\
0 & c_1^2 - c_2^2
\end{bmatrix}
$$

is positive definite. Thus, if we choose the special solution $\Phi_1$ and $v_1(x, t)$ such that

$$
|y_1\rangle = D \begin{bmatrix}
\frac{e^{i(\chi_1 x + \frac{1}{2} \chi_1^2 t)}}{\chi_1 + a_1} e^{i(\chi_1 x + \frac{1}{2} \chi_1^2 t)} + c_2 \alpha_1 e^{i(-a_1 x + \frac{1}{2} a_1^2 t)} \\
\frac{c_1}{\chi_1 + a_1} e^{i(\chi_1 x + \frac{1}{2} \chi_1^2 t)} + c_1 \alpha_1 e^{i(-a_1 x + \frac{1}{2} a_1^2 t)}
\end{bmatrix}, \quad (12)
$$

where $\alpha_1$ is a nonzero complex constant. Setting parameter

$$
\alpha_1 = \left(\frac{2\text{Im}(\lambda_1)}{(c_1^2 - c_2^2)\text{Im}(\chi_1)}\right)^{1/2} e^{\beta_1 t + i\gamma_1},
$$

where $\beta_1$ and $\gamma_1$ are real constants, $\text{Im}(\cdot)$ represents the image part of complex number $\cdot$ (similarly hereinafter), then we can obtain the following breather solution by theorem (1)}

$$
q_1[1] = c_1 \left[ \frac{C_1 + 1}{2} + \frac{C_1 - 1}{2} \tanh(\frac{A_1 - \beta_1}{2}) + \frac{c_2 D_1}{c_1} \text{sech}(\frac{A_1 - \beta_1}{2}) e^{iB_1} \right] e^{i\theta_1},
$$

$$
q_2[1] = c_2 \left[ \frac{C_1 + 1}{2} + \frac{C_1 - 1}{2} \tanh(\frac{A_1 - \beta_1}{2}) + \frac{c_1 D_1}{c_2} \text{sech}(\frac{A_1 - \beta_1}{2}) e^{iB_1} \right] e^{i\theta_1},
$$
where
\[
A_1 = -2 \text{Im}(\chi_1)[x + \text{Re}(\chi_1)t],
\]
\[
B_1 = -(2\text{Re}(\chi_1) + a_1)x + \left[\frac{1}{2}a_1^2 - \text{Re}(\chi_1^2)\right]t + \gamma_1 + \frac{3}{2}\pi,
\]
\[
C_1 = \frac{\bar{\chi}_1 + a_1}{\chi_1 + a_1}, \quad D_1 = \left(\frac{2\text{Im}(\lambda_1)\text{Im}(\chi_1)}{c_1^2 - c_2^2}\right)^{1/2},
\]
Re(\cdot) represents the real part of complex number \(\cdot\) (similarly hereinafter).

Through above expression of solution, we can see that this kind of breather solution is composed of a dark soliton solution and a bright soliton solution. Notably, this type breather solution can not be derived to rogue wave solution through the limit method. This breather solution never appear in the single component NLSE. Thus we call this type of breather as breather-II to distinguish the breather which can be reduced to rogue wave. By choosing special parameters, we can obtain the figure of breather-II (Fig. 1).

![Figure 1: (color online): Breather-II type solution: Parameters \(a_1 = a_2 = 0, c_1 = 2, c_2 = 1, \lambda_1 = i, \chi_1 = 3i\). It is seen that \(|q_2|^2\) possesses the feature of bright soliton and breather.](image)

If we take \(c_2 = 0\), we can obtain so-called bright-dark soliton solution
\[
q_1[1] = c_1 \left[\frac{C_1 + 1}{2} + \frac{C_1 - 1}{2} \tanh(\frac{A_1 - \beta_1}{2})\right] e^{i\theta_1},
\]
\[
q_2[1] = D_1 \text{sech}(\frac{A_1 - \beta_1}{2}) e^{i(B_1 + \theta_1)},
\]
where
\[
A_1 = -2 \text{Im}(\chi_1)[x + \text{Re}(\chi_1)t],
\]
\[
B_1 = -(2\text{Re}(\chi_1) + a_1)x + \left[\frac{1}{2}a_1^2 - \text{Re}(\chi_1^2)\right]t + \gamma_1 + \frac{3}{2}\pi,
\]
\[
C_1 = \frac{\bar{\chi}_1 + a_1}{\chi_1 + a_1}, \quad D_1 = \sqrt{2\text{Im}(\lambda_1)\text{Im}(\chi_1)}.
\]

(b): When \(c_1 = c_2\), there is no nontrivial solution of mCNLSE could be constructed through above theorem 1.

(c): When \(c_1 < c_2\), the system (11) reflects the focusing mechanism. By equation (11), we have the following matrix
\[
\begin{bmatrix}
1 & \frac{c_1}{\chi + a_1} & \frac{c_1}{\mu + a_1} \\
1 & \frac{\chi + a_1}{\mu + a_1} & \frac{\mu + a_1}{\mu + a_1}
\end{bmatrix} J
\begin{bmatrix}
1 & \frac{c_1}{\chi + a_1} & \frac{c_1}{\mu + a_1} \\
1 & \frac{\chi + a_1}{\mu + a_1} & \frac{\mu + a_1}{\mu + a_1}
\end{bmatrix} =
\begin{bmatrix}
\frac{2\bar{\lambda} - \lambda}{\bar{\chi} - \chi} & \frac{2\bar{\lambda} - \lambda}{\bar{\mu} - \mu} \\
\frac{2\bar{\lambda} - \lambda}{\bar{\chi} - \chi} & \frac{2\bar{\lambda} - \lambda}{\bar{\mu} - \mu}
\end{bmatrix}
\]
is positive definite, which could be used to construct the nonsingular solution. If we choose the special solution \( \Phi_1 \) and \( v_1(x, t) \) such that

\[
|y_1| = D \left[ \frac{e^{i(\lambda_1 x + \frac{1}{2} \lambda_1^2 t)}}{\lambda_1 + a_1} + \frac{\alpha_1 e^{i(\mu_1 x + \frac{1}{2} \mu_1^2 t)}}{\mu_1 + a_1} \right],
\]

(13)

where \( \alpha_1 \) is a complex parameter. Moreover, we introduce the following notation:

\[
\lambda_1 + \frac{a_1}{2} = \frac{\alpha}{2} (\zeta_1 - \zeta_1^{-1}), \quad \alpha = \sqrt{c_2 - c_1^2},
\]

it follows that

\[
\sqrt{(\lambda_1 + \frac{a_1}{2})^2 + \alpha^2} = \frac{\alpha}{2} (\zeta_1 + \zeta_1^{-1}),
\]

\[
\chi_1 = \alpha \zeta_1 - a_1,
\]

\[
\mu_1 = -\alpha \zeta_1^{-1} - a_1.
\]

Setting

\[
\alpha_1 = \exp[\alpha(1 - |\zeta_1|^{-2})] \zeta_1 i e_1 - i \alpha(1 + |\zeta_1|^2) \zeta_1 r f_1, \quad \zeta_1 = e^{\gamma_1 + i \beta_1},
\]

where \( \zeta_1 r = \text{Re}(\zeta_1), \zeta_1 i = \text{Im}(\zeta_1), e_1 \) and \( f_1 \) are real constants, then we can obtain

\[
q_1[1] = c_1 \left[ 1 + (e^{i \beta_1} - e^{-i \beta_1}) \frac{\sinh(C + \gamma_1 + i \beta_1) + i \sin(D + \beta_1 - i \gamma_1)}{\cosh(C + \gamma_1) + \frac{\sin(\beta_1)}{\cosh(\gamma_1)} \sin(D + \beta_1)} \right] e^{i \theta_1},
\]

\[
q_2[1] = \frac{c_2}{c_1} q_1[1],
\]

where

\[
C = \alpha(1 - |\zeta_1|^{-2}) \zeta_1 i \left[ x + \left( \frac{\alpha}{2} \zeta_1 r \left( (1 - |\zeta_1|^{-2}) + \frac{(1 + |\zeta_1^{-2}|^2)}{1 - |\zeta_1^{-2}|^2} - a_1 \right) t + e_1 \right],
\]

\[
D = -\alpha(1 + |\zeta_1|^{-2}) \zeta_1 r \left[ x + \left( \frac{\alpha}{2} (1 - |\zeta_1|^{-2}) \frac{(\zeta_1^2 r - \zeta_1^2 t)}{\zeta_1 r} - a_1 \right) t + f_1 \right].
\]

We can readily see that (suppose \( \zeta_1 i (1 - |\zeta_1|^{-2}) > 0 \))

\[
q_1[1] \to c_1 e^{2i \beta_1 + i \theta_1}, \quad \text{as} \; x \to +\infty,
\]

\[
q_2[1] \to c_2 e^{-2i \beta_1 + i \theta_1}, \quad \text{as} \; x \to -\infty.
\]

This kinds of solution can be used to obtain the rogue wave solution through limit technique. The above breather solution can be reduced to single component NLSE, thus we call it as breather-I to distinguish the above breather-II.

In this subsection, the results can be concluded as follows:

- If \( a_1 = a_2 \) and \( c_1 > c_2 \), there exists dark-dark soliton and breather-II soliton. Moreover, if \( c_2 = 0 \), the breather solution degenerate as bright-dark soliton.
- If \( a_1 = a_2 \) and \( c_1 = c_2 \), there is no nontrivial solution.
- If \( a_1 = a_2 \) and \( c_1 < c_2 \), there exists breather-I solution and rogue wave solution. But there is no dark-dark soliton.
2.2 Case II: When $a_1 \neq a_2$

In this case, performing the similar way with above subsection, we have the matrix decomposition:

$$U_0M = MD_0, \quad D_0 = \text{diag}(\chi - \lambda, \mu - \lambda, \nu - \lambda),$$

(14)

where $\text{Im}(\chi) \geq \text{Im}(\mu) \geq \text{Im}(\nu)$, $\text{Im}(\chi) \neq \text{Im}(\mu) \neq \text{Im}(\nu)$,

$$M = \begin{bmatrix} 1 & 1 & 1 \\ \frac{c_1}{\chi + a_1} & \frac{c_1}{\mu + a_1} & \frac{c_1}{\nu + a_1} \\ \frac{c_2}{\chi + a_2} & \frac{c_2}{\mu + a_2} & \frac{c_2}{\nu + a_2} \end{bmatrix},$$

and $\chi$, $\mu$ and $\nu$ are three different roots of the following cubic equation

$$\xi^3 + (a_1 + a_2 - 2\lambda)\xi^2 + [a_1a_2 + c_1^2 - c_2^2 - 2(a_1 + a_2)\lambda]\xi + a_2c_1^2 - a_1c_2^2 - 2a_1a_2\lambda = 0.$$ (15)

The fundamental solution is $\Phi = DMN$, where $N = \text{diag}(e^{iA_1}, e^{iB_1}, e^{iC_1})$,

$$A_1 = (\chi - \lambda)x + \left[\frac{1}{2}\chi^2 + (c_1^2 - c_2^2 - \lambda^2)\right]t,$$

$$B_1 = (\mu - \lambda)x + \left[\frac{1}{2}\mu^2 + (c_1^2 - c_2^2 - \lambda^2)\right]t,$$

$$C_1 = (\nu - \lambda)x + \left[\frac{1}{2}\nu^2 + (c_1^2 - c_2^2 - \lambda^2)\right]t.$$

Through the matrix decomposition (14), we have the following equations:

$$2\lambda - \chi - \frac{c_1^2}{\chi + a_1} + \frac{c_2^2}{\chi + a_2} = 0,$$

$$2\lambda - \mu - \frac{c_1^2}{\mu + a_1} + \frac{c_2^2}{\mu + a_2} = 0,$$

$$2\lambda - \nu - \frac{c_1^2}{\nu + a_1} + \frac{c_2^2}{\nu + a_2} = 0.$$ (16)

**Theorem 2** For any $\lambda \in \mathbb{C}_+ = \{z|\text{Im}(z) > 0\}$, there exists two roots $\chi, \mu \in \mathbb{C}_+$, one root $\nu \in \mathbb{C}_- = \{z|\text{Im}(z) < 0\}$ for the cubic equation (15).

**Proof:** Since matrix $M^\dagger JM$ is congruent with matrix $J$. By above equation (16), we can obtain

$$M^\dagger JM = \begin{bmatrix} 2(\lambda - \lambda) & 2(\lambda - \lambda) & 2(\lambda - \lambda) \\ 2(\lambda - \lambda) & 2(\lambda - \lambda) & 2(\lambda - \lambda) \\ 2(\lambda - \lambda) & 2(\lambda - \lambda) & 2(\lambda - \lambda) \end{bmatrix},$$

which possesses two positive roots and one negative root. We can arrange roots with the order $\text{Im}(\chi) \geq \text{Im}(\mu) \geq \text{Im}(\nu)$. We merely need to prove that $\text{Im}(\mu) > 0$. It is evident that if $\lambda \neq \bar{\lambda}$, we can not obtain the real root. So $\text{Im}(\mu) \neq 0$. We prove it by contradiction. Assuming $\text{Im}(\mu) < 0$, we can know that the matrix

$$M_1 = \begin{bmatrix} 2(\lambda - \lambda) & 2(\lambda - \lambda) \\ \frac{\mu - \mu}{2(\lambda - \lambda)} & \frac{\mu - \nu}{2(\lambda - \lambda)} \\ \frac{\nu - \mu}{2(\lambda - \lambda)} & \frac{\nu - \nu}{2(\lambda - \lambda)} \end{bmatrix}$$
is negative definite. Thus this matrix possesses two negative roots. Together with some simple linear algebra, which implies that $M^1JM$ possesses two negative roots and one positive root. A contradiction emerges. Thus we complete the proof. □

Based on above theorem, one can know that the matrix

$$
\begin{bmatrix}
1 & \frac{c_1}{\lambda+a_1} & \frac{c_2}{\lambda+a_2} \\
\frac{1}{\lambda+a_1} & \frac{1}{\lambda+a_2} & \frac{1}{\lambda+\mu+a_1} \\
\frac{c_1}{\lambda+a_1} & \frac{c_2}{\lambda+a_2} & \frac{1}{\lambda+\mu+a_2}
\end{bmatrix}
J
\begin{bmatrix}
1 & \frac{c_1}{\lambda+a_1} & \frac{c_2}{\lambda+a_2} \\
\frac{1}{\lambda+a_1} & \frac{1}{\lambda+a_2} & \frac{1}{\lambda+\mu+a_1} \\
\frac{c_1}{\lambda+a_1} & \frac{c_2}{\lambda+a_2} & \frac{1}{\lambda+\mu+a_2}
\end{bmatrix}
= \begin{bmatrix}
\frac{2(\lambda-\lambda_1)}{\lambda-\lambda_1} & \frac{2(\lambda-\lambda_1)}{\lambda-\lambda_1} \\
\frac{2(\lambda-\lambda_1)}{\lambda-\lambda_1} & \frac{2(\lambda-\lambda_1)}{\lambda-\lambda_1}
\end{bmatrix}
$$

is positive definite. This could be used to construct the nonsingular solution of mCNLSE (11).

We take special solution $\Phi_1$ and $v_1(x, t)$ such that

$$
|\varphi_1| = D \left[ \begin{array}{c} \varphi_1 \\ c_1 \psi_1 \\ c_2 \phi_1 \end{array} \right], \quad |\psi_1| = \frac{1}{\lambda_1 + a_1} \left[ \begin{array}{c} \varphi_1 \\ \psi_1 \\ \phi_1 \end{array} \right] = \frac{1}{\lambda_1 + a_1} \left[ \begin{array}{c} \varphi_1 \\ \psi_1 \\ \phi_1 \end{array} \right], \quad |\phi_1| = \frac{1}{\lambda_1 + a_2} \left[ \begin{array}{c} \varphi_1 \\ \psi_1 \\ \phi_1 \end{array} \right],
$$

where $A_1 = \chi_1 [(x + x_1) + \frac{\chi_1}{2}(t + t_1)]$, $B_1 = \mu_1 [(x + x_1) + \frac{\chi_1}{2}(t + t_1)]$. For simplicity, we take $x_1 = t_1 = 0$. Then, we give the following calculations

$$
|\varphi_1|^2 - c_1^2 |\psi_1|^2 + c_2^2 |\phi_1|^2 = \frac{2(\lambda_1 - \lambda_1)}{\lambda_1 - \lambda_1} e^{i(\lambda - \lambda_1)} Y^\dagger \begin{bmatrix}
1 + \frac{\chi_1 + a_1}{\mu_1 + a_1} & \frac{\chi_1 + a_2}{\mu_1 + a_2} \\
\frac{\chi_1 + a_1}{\mu_1 + a_1} & \frac{\chi_1 + a_2}{\mu_1 + a_2}
\end{bmatrix} Y,
$$

where $Y = [1, e^{C_1}^T]$, $C_1 = i(B_1 - A_1)$. By the formula (13), we have

$$
q_1[1] = c_1 \left( 1 + \frac{2(\lambda_1 - \lambda_1)}{|\varphi_1|^2 - c_1^2 |\psi_1|^2 + c_2^2 |\phi_1|^2} \right) e^{i\theta_1},
$$
$$
q_2[1] = c_2 \left( 1 + \frac{2(\lambda_1 - \lambda_1)}{|\varphi_1|^2 - c_1^2 |\psi_1|^2 + c_2^2 |\phi_1|^2} \right) e^{i\theta_2}.
$$

It follows that the breather solutions of mCNLSE (11) are

$$
q_1[1] = c_1 \left( 1 + \frac{\chi_1}{\lambda_1 + a_1} \frac{1}{\mu_1 + a_1} e^{2Re(C_1)} + \frac{\chi_1}{\mu_1 + a_1} e^{C_1} + e^{C_1} \right) e^{i\theta_1},
$$
$$
q_2[1] = c_2 \left( 1 + \frac{\chi_1}{\lambda_1 + a_2} \frac{1}{\mu_1 + a_2} e^{2Re(C_1)} + \frac{\chi_1}{\mu_1 + a_2} e^{C_1} + e^{C_1} \right) e^{i\theta_2}.
$$

Rogue wave solution It is well known that rogue wave solution can be reduced from certain type breather solution. In what following, we give the limit process(when $\chi_1 \to \mu_1$) to obtain the rogue wave solution. It follows that

$$
|\varphi_1|^2 - c_1^2 |\psi_1|^2 + c_2^2 |\phi_1|^2 = \frac{e^{i(\chi_1 - \mu_1) |x + \frac{\chi_1 + \chi_1}{2}| t}}{\chi_1 - \chi_1} Y^\dagger \begin{bmatrix}
1 & -1 \\
-1 & \frac{1}{\chi_1 - \chi_1}
\end{bmatrix} Y,
$$

$$
\varphi_1 \psi_1 = \frac{e^{i(\chi_1 - \mu_1) |x + \frac{\chi_1 + \chi_1}{2}| t}}{\chi_1 + a_1} Y^\dagger \begin{bmatrix}
1 & -1 \\
-1 & \frac{1}{\chi_1 + a_1}
\end{bmatrix} Y,
$$
$$
\varphi_1 \phi_1 = \frac{e^{i(\chi_1 - \mu_1) |x + \frac{\chi_1 + \chi_1}{2}| t}}{\chi_1 + a_2} Y^\dagger \begin{bmatrix}
1 & -1 \\
-1 & \frac{1}{\chi_1 + a_2}
\end{bmatrix} Y.
where $\epsilon = \mu_1 - \chi_1$, $Y_1 = \left[ e^{i(1 + \frac{1}{2}\epsilon)t} \right]$.

Taking limit for above equation, we can obtain the rogue wave solution

\[
q_1[1] = \lim_{\epsilon \to 0} c_1 \left( 1 + \frac{2(\bar{\lambda}_1 - \lambda_1)\bar{\varphi}_1\varphi_1}{|\varphi_1|^2 - c_1^2|\psi_1|^2 + c_2^2|\chi_1|^2} \right) e^{i\theta_1},
\]

\[
= c_1 \left( 1 + \frac{-2i r_1}{p_1 + i r_1 + a_1} \frac{(x + p_1 t)^2 + r_1^2 t^2 + \frac{i}{p_1 + a_1 + i r_1} (x + p_1 t - i r_1 t)}{(x + p_1 t + \frac{1}{2r_1})^2 + r_1^2 t^2 + \frac{1}{4r_1^2}} \right) e^{i\theta_1},
\]

and

\[
q_2[1] = \lim_{\epsilon \to 0} c_2 \left( 1 + \frac{2(\bar{\lambda}_1 - \lambda_1)\bar{\varphi}_1\varphi_1}{|\varphi_1|^2 - c_1^2|\psi_1|^2 + c_2^2|\chi_1|^2} \right) e^{i\theta_2},
\]

\[
= c_2 \left( 1 + \frac{-2i r_1}{p_1 + i r_1 + a_2} \frac{(x + p_1 t)^2 + r_1^2 t^2 + \frac{i}{p_1 + a_2 + i r_1} (x + p_1 t - i r_1 t)}{(x + p_1 t + \frac{1}{2r_1})^2 + r_1^2 t^2 + \frac{1}{4r_1^2}} \right) e^{i\theta_2},
\]

where $p_1 = \text{Re}(\chi_1)$, $r_1 = \text{Im}(\chi_1)$ and $\chi_1$ is two multiple root.

Furthermore, we can classify the rogue wave solution with four different types by the dynamics behavior. Since $|q_2[1]|^2$ possesses the similar characteristic with $|q_1[1]|^2$, we merely consider $|q_1[1]|^2$. We first solve the following equation

\[
(|q_1[1]|^2)_x = 0, \quad (|q_1[1]|^2)_t = 0.
\]

Then we have the stationary point

\[
(x, t) = \left( -\frac{1}{2r_1}, 0 \right),
\]

\[
(x, t) = \left( -\frac{A + (2p_1 + a_1)B_1}{2Ar_1}, \frac{B_1}{2Ar_1} \right),
\]

\[
(x, t) = \left( -\frac{Ar_1 + (a_1 p_1 + p_1^2 - r_1^2)B_2}{2Ar_1^2}, \frac{(p_1 + a_1)B_2}{2Ar_1} \right),
\]

where

\[
A = (p_1 + a_1)^2 + r_1^2,
\]

\[
B_1 = \pm \sqrt{3(p_1 + a_1)^2 - r_1^2},
\]

\[
B_2 = \pm \sqrt{3r_1^2 - (p_1 + a_1)^2}.
\]

So there are five extreme points when $\frac{1}{3}r_1^2 < (p_1 + a_1)^2 < 3r_1^2$, or there are three extreme points. Another standard for classification of rogue wave solution is the value of $K = |q_1[1]|^2|_{x=-\frac{1}{2r_1}, t=0} = \left[ 1 - \frac{4r_1^2}{(p_1 + a_1)^2 + r_1^2} \right]^2$.

When $K > 1$, the point is higher than the background; or the point is lower than the background. Thus we can classify the rogue wave to the following four different types:

- If $\frac{(p_1 + a_1)^2}{r_1^2} \geq 3$, then the rogue wave is called dark rogue wave.
- If $1 \leq \frac{(p_1 + a_1)^2}{r_1^2} < 3$, then the rogue wave is four petals type [19].
• If $\frac{1}{3} < \frac{(p_1 + a_1)^2}{r_1^2} < 1$, then the rogue wave is called two peaks type.

• If $\frac{1}{3} \leq \frac{(p_1 + a_1)^2}{r_1^2} < \frac{1}{3}$, then the rogue wave is called bright rogue wave.

It is pointed that similar properties hold for the $|q_2[1]|^2$.

In order to looking for the different types of rogue wave, we give the following methods. For convenience, we set $a_2 = -a_1$, since the system possesses the Galieo transformation. Suppose $\chi_1 = \gamma_1 r_1 - a_1 + i r_1$, where $\gamma_1$ is a real parameter (we can obtain the different type rogue wave through parameter $\gamma_1$), substituting $\chi_1$ into characteristic equation (15), we have

$$r_1^3 + br_1^2 + cr_1 + d = 0,$$

where

$$b = -\frac{2\lambda + 3a_1}{\gamma_1 + i},$$
$$c = \frac{4\lambda a_1 + 2a_1^2 + c_1^2 - c_2^2}{(\gamma_1 + i)^2},$$
$$d = \frac{-2a_1 c_1^2}{(\gamma_1 + i)^3}.$$

The discriminant for above cubic equation is

$$\Delta = 18bcd - 4b^3d + b^2c^2 - 4c^3 - 27d^2 = 0.$$

Solving above two equations about $c_1^2$ and $c_2^2$, we have

$$c_1^2 = -\frac{(2ir_1 - 2a_1 + 2r_1 \gamma_1 - 3a_1)(\gamma_1 + i)^2}{2a_1} r_1^2,$$
$$c_2^2 = -\frac{(2ir_1 - 2a_1 + 2r_1 \gamma_1 - a_1)(r_1 \gamma_1 + ir_1 - 2a_1)^2}{2a_1}.$$

Assuming that $\lambda_1 = \lambda_{1r} + i\lambda_{1i}$, and solving the equations $\text{Im}(c_1^2) = 0$ and $\text{Im}(c_2^2) = 0$, we have

$$\lambda_{1r} = \frac{(r_1 \gamma_1 - a_1)(3r_1^2 \gamma_1^2 - 6a_1 \gamma_1 + r_1)}{2(r_1 \gamma_1^2 + r_1 - 2a_1 \gamma_1)},$$
$$\lambda_{1i} = \frac{r_1^2}{r_1 \gamma_1^2 + r_1 - 2a_1 \gamma_1}.$$

Finally, we have

$$c_1^2 = \frac{r_1^2 (1 + \gamma_1^2)^2 (r_1 \gamma_1 - 2a_1)}{2a_1 [r_1 (1 + \gamma_1^2) - 2a_1 \gamma_1]} > 0,$$
$$c_2^2 = \frac{\gamma_1 [r_1^2 + (r_1 \gamma_1 - 2a_1)^2]^2}{2a_1 [r_1 (1 + \gamma_1^2) - 2a_1 \gamma_1]} > 0. \quad (19)$$

Thus the above parameters must satisfy the following condition if $\frac{2a_1}{r_1} > 0$, then

$$\frac{2a_1}{r_1^2} < \frac{\gamma_1}{r_1} < \frac{1 + \gamma_1^2}{2a_1}, \quad (20)$$

or if $\frac{2a_1}{r_1} < 0$, then

$$\frac{1 + \gamma_1^2}{2a_1} < \frac{\gamma_1}{r_1} < \frac{2a_1}{r_1^2}. \quad (21)$$
For instance, we give some exact rogue wave solutions without solving the cubic equation \((15)\). Choosing parameters 
\[
\gamma_1 = 1, \
 r_1 = 8 \quad \text{and} \quad a_1 = 1 \quad \text{such that condition (20) or condition (21), it follows that}
\]
\[
p_1 = \gamma_1 r_1 + a_1 = 7 \quad \text{and} \quad |q_1[1]|^2 \quad \text{is four petal type, then we can obtain}
\]
\[
c_1 = \frac{32}{7} \sqrt{21},
\]
\[
c_2 = \frac{50}{7} \sqrt{7}.
\]
On the other hand, we have \(\frac{1}{3} < \left(\frac{p_1 - a_1}{r_1}\right)^2 = \frac{9}{19} < 1\). It follows that \(|q_2[1]|^2\) is two-peaks type.

We can plot the above rogue wave solution by soft Maple (Fig. 2).

![Rogue wave solutions](image1.png)

**Figure 2:** (color online): Rogue wave solution: Parameters \(a_1 = -a_2 = 1, c_1 = \frac{32}{7} \sqrt{21}, c_2 = \frac{50}{7} \sqrt{7}, \)
\(\lambda_1 = \frac{13}{2} + \frac{32}{7}, \chi_1 = 7 + 8i\). It is seen that the solution \(|q_1|^2\) possesses four-petals structure and the solution \(|q_2|^2\) possesses two-peaks structure.

Finally, we need to consider the method to determine the multiple roots. We set

\[
a_1 = \alpha + \beta, \\
a_2 = \alpha - \beta,
\]
where

\[
\alpha = \frac{a_1 + a_2}{2}, \\
\beta = \frac{a_1 - a_2}{2}. \quad (22)
\]

It follows that the matrix \(U_0 - \xi\) can be represented as

\[
\begin{bmatrix}
2\kappa - \zeta & -c_1 & c_2 \\
c_1 & -\zeta - \beta & 0 \\
c_2 & 0 & -\zeta + \beta
\end{bmatrix}
\]
where \(\zeta = \xi + \alpha, \kappa = \lambda_1 + \frac{1}{2} \alpha\). Then the determinant of above matrix is

\[
F = \zeta^3 - 2\kappa \zeta^2 + [c_1^2 - c_2^2 - \beta^2] \zeta + 2\beta^2 \kappa - \beta(c_1^2 + c_2^2) = 0. \quad (23)
\]

The discriminant of above matrix is

\[
D(F) = 64\beta^2 \left( \kappa^4 - \frac{c_1^2 + c_2^2}{2\beta} \kappa^3 + A_2 \kappa^2 + A_1 \kappa + A_0 \right) \quad (24)
\]
To obtain the rogue wave solution, another condition is the spectral parameters must be nonreal. Indeed, the quartic equation
\[
\beta^4 + 3(\beta^2 + c_2^2 - c_1^2)\beta^2 + \frac{27}{4}c_2^4\beta^2 + (c_2^2 - c_1^2)^3 = 0.
\]

Proof: We need to analyze the solution of \(D(F) = 0\). The discriminant of \(D(F) = 0\) is
\[
E = -\frac{c_1c_2}{2}\frac{1}{\beta^6} \left[(4\beta^2 + c_2^2 - c_1^2)^3 + 27c_2^6(4\beta^2)\right] - 4
\]
If \(E < 0\), then \(D(F) = 0\) possesses two real roots and a pair of complex conjugate root. If \(E > 0\), then then \(D(F) = 0\) possesses four real roots or two pairs of complex conjugate root. Since \(E > 0\), then we can deduce that \(c_1 > c_2\) and the solution of \(E = 0\) is \(\beta^2 = \frac{1}{4}(c_1^{2/3} - c_2^{2/3})\).

In what following, we illustrate the equation \(D(F) = 0\) never possesses two pairs of complex conjugate roots. If quartic equation has two pairs of complex conjugate roots, then we have
\[
G(\beta^2) = 16A_0\beta^2 = (\beta^2 + c_2^2 - c_1^2)^3 - \frac{27}{4}\beta^2(c_1^2 + c_2^2)^2
\]

On the other hand, we have the discriminant of \(G(\beta^2) = 0\) is
\[
\Delta = \frac{3^9}{4}c_1^2c_2^2(c_2^2 + c_1^2)^4 > 0,
\]
then the equation \(G(\beta^2) = 0\) possesses three different real roots \(\beta_1^2, \beta_2^2\) and \(\beta_3^2\). By the Vieta formula, we have
\[
\beta_1^2 + \beta_2^2 + \beta_3^2 = 3(c_1^2 - c_2^2) > 0,
\]
\[
\beta_1^2\beta_2^2 + \beta_2^2\beta_3^2 + \beta_3^2\beta_1^2 = -3\left(\frac{5}{2}c_2^2 + \frac{1}{2}c_1^2\right)^3\left(\frac{5}{2}c_1^2 + \frac{1}{2}c_2^2\right) < 0,
\]
\[
\beta_1^2\beta_2^2\beta_3^2 = (c_2^2 - c_1^2)^3 > 0,
\]
it follows that the above equation possesses one positive root and two negative roots. Since \(\beta_1^2 > 0\), then there is merely one positive roots. The following inequality
\[
G\left(\frac{1}{4}(c_1^{2/3} - c_2^{2/3})^3\right) = -\frac{27}{16} (c_1^{2/3} - c_2^{2/3})^3 \left(\frac{1}{4}(c_1^{2/3} + c_2^{2/3})^6 + (c_1^2 + c_2^2)^2\right) < 0,
\]
illustrate that when \(E > 0\), then \(G(\beta^2) < 0\). It follows that when \(E > 0\), there is four real roots for the equation (24). □

Indeed, the quartic equation \(D(F) = 0\) is condition of multiple roots for the cubic equation (23). To obtain the rogue wave solution, another condition is the spectral parameters must be nonreal. So we can obtain the existence condition of rogue wave is \(E < 0\) i.e. \(a_1 \neq a_2\) and \(\beta^2 > \frac{1}{4}(c_1^{2/3} - c_2^{2/3})^3\), through above theorem.

For the focusing CNLSE, there is three multiple seed for the characteristic equation (15). However, for mixed case or the defocusing case, there is no three multiple seed. So there is no type-II rogue wave (17). This fact can be verified by the following elementary fact.
Theorem 4  The characteristic equation \((\ref{15})\) never possesses three multiple root.

Proof: If characteristic equation \((\ref{15})\) possesses three multiple roots, then we have
\[
\zeta^3 - 2\kappa \zeta^2 + (c_1^2 - c_2^2 - \beta^2)\zeta + 2\beta^2\kappa - \beta(c_1^2 + c_2^2) = (\zeta - \frac{2\kappa}{3})^3.
\]

It follows that
\[
c_1^2 - c_2^2 - \beta^2 - \frac{4}{3}\kappa^2 = 0,
\]
\[
-c_2^2\beta + 2\kappa\beta^2 + \frac{8}{27}\kappa^3 = 0.
\]

Moreover, we have
\[
c_1^2 = \frac{\left(\frac{2\kappa + \beta}{2\beta}\right)^3}{0}, \quad \kappa \neq \bar{\kappa},
\]
\[
c_2^2 = \frac{\left(\frac{2\kappa - \beta}{2\beta}\right)^3}{0}.
\]

If \(\beta > 0\), we have \(\frac{2}{3}\kappa \pm \beta \in \omega \mathbb{R}^+ \) or \(\omega^2 \mathbb{R}^+\). If \(\beta < 0\), we have \(\frac{2}{3}\kappa \pm \beta \in \omega \mathbb{R}^- \) or \(\omega^2 \mathbb{R}^-\). Indeed, this is no possible. Thus there is no three multiple root. \(\Box\)

Since the type-II rogue wave solution is obtained by three multiple roots, there exists not type-II rogue wave solution for the mCNLSE \((\ref{11})\) by above theorem.

Homoclinical orbits solution The homoclinic orbit solution is a kind of space periodical and time exponential decay solution, or called the Akhmediev breather. When time tends to \(\pm \infty\), it tends to a plane wave solution with different phase. From the solution expression of \((\ref{18})\), the homoclinic orbit solution can be obtained through choosing parameters \(\text{Im}(\chi_1) = \text{Im}(\tau_1)\).

In the following, we present a way to looking for the homoclinic orbit solution for mCNLSE \((\ref{11})\). Then, it is necessary to analysis the following characteristic equation \((\ref{23})\)
\[
\zeta^3 - 2\kappa \zeta^2 + (c_1^2 - c_2^2 - \beta^2)\zeta + 2\beta^2\kappa - \beta(c_1^2 + c_2^2) = 0,
\]
where \(\kappa = \lambda_1 + \frac{1}{2}\alpha, \zeta = \tau_1 + \alpha, \alpha \) and \(\beta \) are given in equations \((\ref{22})\). Suppose another root is \(\chi_1 = \tau_1 + \delta, \delta \in \mathbb{R}, \) then we have
\[
3\delta\zeta^2 + (3\delta^2 - 4\delta\kappa)\zeta + \delta^3 - 2\kappa\delta + \delta(c_1^2 - c_2^2 - \beta^2) = 0.
\]

It follows that
\[
\zeta = -\frac{\delta}{2} + \frac{2\kappa}{3} \pm \frac{1}{6}\sqrt{16\kappa^2 - 3\delta^2 + 12\beta^2 + 12(c_2^2 - c_1^2)},
\]
and it also satisfies the following equation
\[
(16\kappa^2 - 3\delta^2 - 12c_1^2 + 12c_2^2 + 12\beta^2) [3(c_1^2 - c_2^2) + 3(\delta^2 - \beta^2) - 4\kappa^2]^2
= [16\kappa^3 + 18(c_2^2 - c_1^2 - 2\beta^2)\kappa + 27\beta(c_1^2 + c_2^2)]^2.
\]

If we obtain a pair of conjugate complex roots, exact values of \(\delta, \beta, c_1 \) and \(c_2 \) are substituted into above equation. Then we can obtain the homoclinic orbit solution through substituting the above mentioned parameters into solutions \((\ref{18})\).

Then, we give an explicit example to illustrate the method. Suppose \(\alpha = 0, \beta = 1, \delta = 1, c_1 = 1 \) and \(c_2 = 2, \) substitute these parameters into \((\ref{25})\) and solve it about \(\kappa, \) we can obtain
that $\kappa = \lambda_1 \approx 0.633263953 + 1.812212393i$. And then substituting above parameters into characteristic equation (23), we have

$$\chi_1 = 1.625455953 + 1.933383832i,$$

$$\tau_1 = 0.625455953 + 1.933383832i.$$  

We give the explicit figure for homoclinic orbit solution by choosing special parameters (Fig. 3).

![Figure 3: (color online): Homoclinic orbit solution: Parameters $a_1 = -a_2 = 1$, $c_1 = 1$, $c_2 = 2$, $\lambda_1 = 0.633263953 + 1.812212393i$, $\chi_1 = 1.625455953 + 1.933383832i$, $\tau_1 = 0.625455953 + 1.933383832i$. It is seen that the solution $|q_1|^2$ and $|q_2|^2$ possess the periodical behavior in space.](image)

If $\lambda_1 \in \mathbb{R}$, we can obtain the dark-dark soliton solution. Through theorem 3, the quartic equation (24) admits two real roots at least. So there always exists $\lambda_1$ such that $D(F) < 0$, it follows that there is a pair of complex roots for cubic equation (23). Thus the dark-dark soliton always exist. The details for the dark-dark soliton through DT refer to reference [39]. Similar as above, we choose the special solution $\Phi_1$ and $v_1(x,t)$ such that

$$|y_1| = D \left[ \frac{1}{\chi_1 + a_1} \left( 1 + \frac{1}{\chi_1 + a_1} \right) \right] e^{i\chi_1(x + \frac{1}{2} \chi_1 t)} \
\alpha_1 (\bar{\lambda}_1 - \lambda_1) e^{i\chi_1(x + \frac{1}{2} \chi_1 t)} \right],$$  

where $\text{Im}(\chi_1) > 0$. Substituting above special solution into (6), we have the dark-dark soliton solution

$$q_1[1] = \frac{c_1}{2} \left[ 1 + \frac{\bar{x}_1 + a_1}{\chi_1 + a_1} + \left( 1 - \frac{\bar{x}_1 + a_1}{\chi_1 + a_1} \right) \tanh(A_1) \right] e^{i\theta_1},$$

$$q_2[1] = \frac{c_2}{2} \left[ 1 + \frac{\bar{x}_1 + a_2}{\chi_1 + a_2} + \left( 1 - \frac{\bar{x}_1 + a_2}{\chi_1 + a_2} \right) \tanh(A_1) \right] e^{i\theta_2},$$

where $A_1 = \text{Im}(\chi_1)[x + \text{Re}(\chi_1)t] + \frac{1}{2} \ln \beta_1$ and $\beta_1 = -\text{Im}(\chi_1) \text{Im} \left[ \alpha_1 \left( 1 - \frac{c^2_1}{(\chi_1 + a_1)^2} + \frac{c^2_2}{(\chi_1 + a_2)^2} \right) \right] > 0$.

In summary, the results in this subsection can be concluded as follows:

- If $a_1 \neq a_2$ and $\beta^2 > \frac{1}{4}(c_1^{2/3} - c_2^{2/3})^3$, there exists rogue wave, breather-I solution and dark-dark soliton solution.
- If $a_1 \neq a_2$ and $\beta^2 \leq \frac{1}{4}(c_1^{2/3} - c_2^{2/3})^3$, there exists breather-II solution and dark-dark soliton solution.
3 General localized wave solution formula and interaction between two types of localized wave solution

The general Darboux matrix for above linear system (2) is given in reference [39]. We can summarize as the following theorem:

**Theorem 5** Suppose we have \( N \) different vector solutions \( \Phi_i \) for linear system (2) with \( \lambda = \lambda_i \) \((i = 1, 2, \cdots, N)\), denote \(|y_i\rangle = v_i \Phi_i\), where \( v_i \) is an appropriate function of \( x \) and \( t \), then the \( N \)-fold DT is

\[
T_N = I - YM^{-1}(\lambda - S)^{-1}Y^\dagger J,
\]

where

\[
Y = [|y_1\rangle, |y_2\rangle, \cdots, |y_N\rangle],
\]

\[
S = \text{diag} (\lambda_1, \lambda_2, \cdots, \lambda_N),
\]

\[
M = \left( \begin{array}{c|c|c|c|} 
(\langle y_j| J | y_i \rangle) & \lambda_i - \lambda_j 
\end{array} \right)_{N \times N},
\]

and \(|y_i\rangle^\dagger\), if \( \lambda_i = \bar{\lambda}_i \) the element \( \frac{\langle y_i| J | y_i \rangle}{\lambda_i - \bar{\lambda}_i} \) is considered as the mean of limit, the transformation between potential functions is

\[
Q[N] = Q + [\sigma_3, P], \quad P = YM^{-1}Y^\dagger J.
\]

In what following, we consider an explicit application of the above theorem. We choose the seed solution as the plane wave solution (7), then there are three kinds of special solutions for \(|y_i\rangle\) (equation (12), (17) and (26)), which can be used to construct different types of localized wave solution. Denote

\[
|y_i\rangle = \begin{bmatrix} \varphi_i \\ c_1e^{i\theta_1}\psi_i \\ c_2e^{i\theta_2}\phi_i \end{bmatrix}.
\]

Through above theorem, we can obtain that the general localized wave solution formula for mCNLS (1) on the nonzero background:

\[
q_1[N] = c_1e^{i\theta_1} \left( 1 - 2\psi M^{-1}\varphi^\dagger \right),
\]

\[
q_2[N] = c_2e^{i\theta_2} \left( 1 - 2\phi M^{-1}\varphi^\dagger \right),
\]

where

\[
\varphi = [\varphi_1, \varphi_2, \cdots, \varphi_N],
\]

\[
\psi = [\psi_1, \psi_2, \cdots, \psi_N],
\]

\[
\phi = [\phi_1, \phi_2, \cdots, \phi_N].
\]

Furthermore, by simple linear algebra formula, setting \( \hat{M} = -\frac{1}{2}M \), we have determinant representation for above general localized wave solution formula:

\[
q_1[N] = c_1 \left( \frac{\det(\hat{M} + X_1)}{\det(\hat{M})} \right) e^{i\theta_1}, \quad X_1 = \varphi^\dagger \psi,
\]

\[
q_2[N] = c_2 \left( \frac{\det(\hat{M} + X_2)}{\det(\hat{M})} \right) e^{i\theta_2}, \quad X_2 = \varphi^\dagger \phi.
\]

In what following, we prove that the above general localized wave solutions are nonsingular. We can establish the following theorem:
Theorem 6 The matrix $\hat{M} = i\hat{M}$ is negative definite.

Proof: We merely give the case when $a_1 \neq a_2$, since the case $a_1 = a_2$ is similar. We prove this fact by analysis the elements of matrix $\hat{M}$. The elements $|y_i\rangle$ possesses two different choices (here we do not consider the limit case),

$$\|y_i\rangle = D \begin{bmatrix}
\frac{1}{\chi_i + a_1} & \frac{1}{\chi_i + a_2} \\
\frac{1}{\mu_i + a_1} & \frac{1}{\mu_i + a_2}
\end{bmatrix} \begin{bmatrix}
e^{-i\chi_i[x+x_i + \frac{1}{2}\mu_i(t+t_i)]} \\
e^{-i\mu_i[x+x_i + \frac{1}{2}\mu_i(t+t_i)]} 
\end{bmatrix}, i = 1, 2, \cdots, N_1,$$

$\text{Im}(\chi_i) > 0, \text{Im}(\mu_i) > 0, x_i, t_i \in \mathbb{R},$

$$\|y_j\rangle = D \begin{bmatrix}
\frac{1}{\chi_j + a_1} & \frac{1}{\chi_j + a_2} \\
\frac{1}{\mu_j + a_1} & \frac{1}{\mu_j + a_2}
\end{bmatrix} \begin{bmatrix}e^{i\chi_j(x+\frac{1}{2}\chi_j t)} \\
\alpha_j(\bar{\chi}_j - \chi_j) e^{i\chi_j(x+\frac{1}{2}\chi_j t)} 
\end{bmatrix}, j = N_1 + 1, N_1 + 2, \cdots, N_1 + N_2,$

$N_1 + N_2 = N, \text{Im}(\chi_j) > 0, \lambda_j \in \mathbb{R}, \beta_j = -\text{Im}(\chi_j) \text{Im} \left[ \alpha_j \left( 1 - \frac{\alpha_j^2}{(\chi_j + a_1)^2} + \frac{c_j^2}{(\chi_j + a_2)^2} \right) \right] > 0.$

Then we have the following results: If $i \neq j$, we have (here we merely consider the case $1 \leq i \leq N_1, N_1 + 1 \leq j \leq N$, the other case can be obtained with a parallel way)

$$\frac{\langle y_i|J|y_j \rangle}{2i(\lambda_i - \lambda_j)} = \left[ e^{-i\chi_i[x+x_i + \frac{1}{2}\chi_i(t+t_i)]} - e^{-i\mu_i[x+x_i + \frac{1}{2}\mu_i(t+t_i)]} \right] \left[ \begin{bmatrix}1 \\
\frac{\alpha_j(\bar{\chi}_j - \chi_j)}{(\mu_j - \mu_i)} 
\end{bmatrix} \right] e^{i\chi_j(x+\frac{1}{2}\chi_j t)} = -\int_x^{+\infty} \left( e^{-i\chi_i[s+x_i + \frac{1}{2}\chi_i(t+t_i)]} + e^{i\mu_i[s+x_i + \frac{1}{2}\mu_i(t+t_i)]} + e^{-i\mu_i[s+x_i + \frac{1}{2}\mu_i(t+t_i)]} + e^{i\chi_j(s+\frac{1}{2}\chi_j t)} \right) ds.$$

Similarly, we have

$$\frac{\langle y_i|J|y_i \rangle}{2i(\lambda_i - \lambda_i)} = -\int_x^{+\infty} \left| e^{i\chi_i[s+x_i + \frac{1}{2}\chi_i(t+t_i)]} + e^{i\mu_i[s+x_i + \frac{1}{2}\mu_i(t+t_i)]} \right|^2 ds,$$

$$\frac{\langle y_j|J|y_j \rangle}{2i(\lambda_j - \lambda_j)} = -\int_x^{+\infty} \left| e^{i\chi_j(s+\frac{1}{2}\chi_j t)} \right|^2 ds - \frac{\beta_j}{2\text{Im}(\chi_j)}.$$

It follows that, for any nonzero vector $v = (v_1, v_2, \cdots, c_n)^T$, we have

$$v^\dagger M v = \int_x^{+\infty} \sum_{i=1}^{N_1} v_i \left( e^{i\chi_i[s+x_i + \frac{1}{2}\chi_i(t+t_i)]} + e^{i\mu_i[s+x_i + \frac{1}{2}\mu_i(t+t_i)]} \right) + \sum_{j=1}^{N_2} v_j \left( e^{i\chi_j(s+\frac{1}{2}\chi_j t)} \right) ds$$

$$- \sum_{j=1}^{N_2} \frac{v_j \beta_j}{2\text{Im}(\chi_j)} < 0.$$ 

Thus we complete the proof. □

3.1 Interaction between two types of localized wave solution

In this subsection, we consider the cases that different types of localized waves coexist and interplay with each other. Firstly, we consider the case $a_1 \neq a_2$ and $c_1c_2 \neq 0$. 

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Taking
\[
\begin{bmatrix}
\varphi_1 \\
\psi_1 \\
\phi_1
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\chi_1 + a_1} & \frac{1}{\chi_1 + a_2} & e^{i\chi_1[(x+x_1)+\frac{1}{2}\chi_1(t+t_1)]} \\
\frac{1}{\tau_1 + a_1} & \frac{1}{\tau_1 + a_2} & e^{i\tau_1[(x+x_1)+\frac{1}{2}\tau_1(t+t_1)]}
\end{bmatrix},
\]
and
\[
\begin{bmatrix}
\varphi_2 \\
\psi_2 \\
\phi_2
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\chi_2 + a_1} & \frac{1}{\chi_2 + a_2} & e^{i\chi_2[x+\frac{1}{2}\chi_2 t]} \\
\frac{1}{\tau_2 + a_1} & \frac{1}{\tau_2 + a_2} & e^{i\tau_2[x+\frac{1}{2}\tau_2 t]}
\end{bmatrix},
\]
(27)
where
\[
\beta = -\text{Im}(\chi_2)\text{Im}\left[\alpha \left( 1 - \frac{c_1^2}{(\chi_2 + a_1)^2} + \frac{c_2^2}{(\chi_2 + a_2)^2} \right) \right] > 0,
\]
x_1 and t_1 are real constants (for simplicity, we set x_1 = t_1 = 0), we can obtain the general formulas between breather solution and dark-dark soliton solution:
\[
q_1[2] = c_1 \left( \frac{\det(M + X_1)}{\det(M)} \right) e^{i\theta_1},
\]
\[
q_2[2] = c_2 \left( \frac{\det(M + X_2)}{\det(M)} \right) e^{i\theta_2},
\]
(28)
where
\[
M = \begin{bmatrix}
M_1 & M_2 \\
M_3 & M_4
\end{bmatrix}, \quad X_1 = \begin{bmatrix}
X_{11} & X_{12} \\
X_{13} & X_{14}
\end{bmatrix}, \quad X_2 = \begin{bmatrix}
X_{21} & X_{22} \\
X_{23} & X_{24}
\end{bmatrix},
\]
and
\[
M_1 = Y_1^\dagger \begin{bmatrix}
\frac{1}{\chi_1 - \chi_1} & \frac{1}{\chi_1 - \tau_1} \\
\frac{1}{\chi_1} & \frac{1}{\tau_1 - \tau_1}
\end{bmatrix} Y_1, \quad Y_1 = \begin{bmatrix}
1 \\
e^{i(\tau_1 - \chi_1)[x+\frac{1}{2}(\tau_1 + \chi_1)t]}
\end{bmatrix},
\]
\[
M_2 = Y_1^\dagger \begin{bmatrix}
\frac{1}{\chi_2 - \chi_2} \\
\frac{1}{\chi_2 - \tau_2}
\end{bmatrix} e^{i\chi_2(x+\frac{1}{2}\chi_2 t)}, \quad M_3 = e^{-i\chi_2(x+\frac{1}{2}\chi_2 t)} \begin{bmatrix}
\frac{1}{\chi_2 - \chi_1} & \frac{1}{\chi_2 - \tau_1} \\
\frac{1}{\chi_2} & \frac{1}{\tau_2 - \tau_1}
\end{bmatrix} Y_1,
\]
\[
M_4 = \beta + \frac{1}{\chi_2 - \chi_2} e^{-2\text{Im}(\chi_2)(x+\text{Re}(\chi_2)t)},
\]
and
\[
X_{11} = Y_1^\dagger \begin{bmatrix}
\frac{1}{\chi_1 + a_1} & \frac{1}{\tau_1 + a_1} \\
\frac{1}{\chi_1 + a_1} & \frac{1}{\tau_1 + a_1}
\end{bmatrix} Y_1, \quad X_{12} = Y_1^\dagger \begin{bmatrix}
\frac{1}{\chi_2 + a_1} \\
\frac{1}{\chi_2 + a_1}
\end{bmatrix} e^{i\chi_2(x+\frac{1}{2}\chi_2 t)},
\]
\[
X_{13} = e^{-i\chi_2(x+\frac{1}{2}\chi_2 t)} \begin{bmatrix}
\frac{1}{\chi_1 + a_1} & \frac{1}{\tau_1 + a_1} \\
\frac{1}{\chi_2} & \frac{1}{\tau_2 - \tau_1}
\end{bmatrix} Y_1, \quad X_{14} = \frac{1}{\chi_2 + a_1} e^{-2\text{Im}(\chi_2)(x+\text{Re}(\chi_2)t)},
\]
and \(X_{2i} = X_{1i}(a_1 \rightarrow a_2), \ i = 1, 2, 3, 4.\)

### 3.1.1 Interaction between dark-dark soliton and breather

Since \(q_2[2]\) is similar with \(q_1[2]\), we merely consider the asymptotical behavior for \(q_1[2]\). From above section, we know that \(\text{Im}(\chi_1) \geq \text{Im}(\tau_1) > 0\) and \(\text{Im}(\chi_2) > 0\). At the same time, we can know that the velocity of dark-dark soliton is \(v_d = -\text{Re}(\chi_2)\), and the velocity of breather solution is
\[
v_b = \frac{\text{Im}(\tau_1)\text{Re}(\tau_1) - \text{Im}(\chi_1)\text{Re}(\chi_1)}{\text{Im}(\chi_1) - \text{Im}(\tau_1)}.
\]
We do not consider they possess the same velocity $v_d = v_b$. So we can assume $v_d < v_b$. To analysis the interaction between dark-dark soliton and breather, we use the standard method of asymptotical analysis.

Firstly, we fixed $x - v_d t = c_1$, where $c_1$ is a real constant.

**Case 1** If $t \to +\infty$, then $\Re(i(\tau_1 - \chi_1)[x + \frac{1}{2}(\tau_1 + \chi_1)t]) \to -\infty$, it follows that we have

$$q_1[2] \to c_1 \left(\frac{\tilde{x}_1 + a_1}{\chi_1 + a_1}\right) \left[\frac{\beta}{\chi_1 - \chi_1} + \frac{\tilde{x}_2 + a_1}{\chi_2 + a_1} A\right] e^{i\theta_1},$$

where

$$A = \left(\frac{1}{|\chi_1 - \chi_2|^2} + \frac{1}{(\chi_1 - \chi_1)(\tilde{\chi}_2 - \chi_2)}\right) e^{-2\Re(\chi_2)(x + \Re(\chi_2)t)}.$$  

By the explicit asymptotical expression, we analyze the phase difference between two sides of dark-dark soliton. By simple algebra, we can obtain that

- If $\Re(i\chi_2(x + \frac{1}{2}\chi_2 t)) \to -\infty$, then
  $$q_1[2] \to c_1 e^{i(\theta_1 + \varphi_1)}, \quad \ln \left(\frac{\tilde{x}_1 + a_1}{\chi_1 + a_1}\right) = i\varphi_1.$$  

- If $\Re(i\chi_2(x + \frac{1}{2}\chi_2 t)) \to +\infty$, then
  $$q_1[2] \to c_1 e^{i(\theta_1 + \varphi_2)}, \quad \ln \left(\frac{\tilde{x}_1 + a_1 \tilde{\chi}_2 + a_1}{\chi_1 + a_1 \chi_2 + a_1}\right) = i\varphi_2.$$  

We can see that the phase between two sides are different.

**Case 2** If $t \to -\infty$, then $\Re(i(\tau_1 - \chi_1)[x + \frac{1}{2}(\tau_1 + \chi_1)t]) \to +\infty$, then we have

$$q_1[2] \to c_1 \left(\frac{\tilde{x}_1 + a_1}{\tau_1 + a_1}\right) \left[\frac{\beta}{\tau_1 - \tau_1} + \frac{\tilde{x}_2 + a_1}{\chi_2 + a_1} A\right] e^{i\theta_1},$$

where

$$A = \left(\frac{1}{|\tau_1 - \chi_2|^2} + \frac{1}{(\tau_1 - \tau_1)(\tilde{\chi}_2 - \chi_2)}\right) e^{-2\Re(\chi_2)(x + \Re(\chi_2)t)}.$$  

Similarly, we can obtain that phase difference between two sides. It follows that

- If $\Re(i\chi_2(x + \frac{1}{2}\chi_2 t)) \to -\infty$, then we have
  $$q_1[2] \to c_1 e^{i(\theta_1 + \varphi_3)}, \quad \ln \left(\frac{\tilde{x}_1 + a_1}{\tau_1 + a_1}\right) = i\varphi_3.$$  

- If $\Re(i\chi_2(x + \frac{1}{2}\chi_2 t)) \to +\infty$, we have
  $$q_1[2] \to c_1 e^{i(\theta_1 + \varphi_4)}, \quad \ln \left(\frac{\tilde{\chi}_2 + a_1 \tilde{x}_1 + a_1}{\chi_2 + a_1 \tau_1 + a_1}\right) = i\varphi_4.$$  

Secondly, we fixed $x - v_b t = c_2$, where $c_2$ is a real constant.
Case 3 If $t \to -\infty$, then $\text{Re}(i\chi_2(x + \frac{1}{2}\chi_2 t)) \to +\infty$, it follows that

$$q_1[2] \to c_1 \left( \frac{\bar{\chi}_2 + a_1}{\chi_2 + a_1} \right) \left[ \frac{\frac{\bar{\chi}_1 + a_1}{\chi_1 + a_1} B_1 + \frac{\bar{\chi}_1 + a_1}{\chi_1 + a_1} B_2 + \frac{\bar{\chi}_1 + a_1}{\chi_1 + a_1} B_3 + \frac{\bar{\chi}_1 + a_1}{\chi_1 + a_1} B_4}{B_1 + B_2 + B_3 + B_4} \right] e^{i\theta_1},$$

where

$$B_1 = \frac{1}{(\bar{\chi}_2 - \chi_2)(\bar{\chi}_1 - \chi_1)} + \frac{1}{|\chi_1 - \chi_2|^2},$$

$$B_2 = \left[ \frac{1}{(\bar{\chi}_2 - \chi_2)(\bar{\chi}_1 - \chi_1)} - \frac{1}{(\bar{\chi}_1 - \chi_2)(\bar{\chi}_2 - \chi_1)} \right] e^{-i(\bar{\tau}_1 - \chi_1)[x + \frac{1}{2}(\bar{\tau}_1 + \chi_1)t]},$$

$$B_3 = \left[ \frac{1}{(\bar{\chi}_2 - \chi_2)(\bar{\tau}_1 - \chi_1)} - \frac{1}{(\bar{\tau}_1 - \chi_2)(\bar{\chi}_2 - \chi_1)} \right] e^{i(\tau_1 - \chi_1)[x + \frac{1}{2}(\tau_1 + \chi_1)t]},$$

$$B_4 = \left[ \frac{1}{(\bar{\chi}_2 - \chi_2)(\bar{\tau}_1 - \chi_1)} + \frac{1}{|\bar{\chi}_2 - \tau_1|^2} \right] e^{2\text{Re}(i(\tau_1 - \chi_1)[x + \frac{1}{2}(\tau_1 + \chi_1)t])}.$$

Similar as case 1, we can obtain that phase difference between two sides. It follows that

- If $\text{Re}(i(\tau_1 - \chi_1)[x + \frac{1}{2}(\tau_1 + \chi_1)t]) \to -\infty$, then
  $$q_1[2] \to c_1 e^{i(\theta_1 + \varphi_2)}.$$

- If $\text{Re}(i(\tau_1 - \chi_1)[x + \frac{1}{2}(\tau_1 + \chi_1)t]) \to +\infty$, then
  $$q_1[2] \to c_1 e^{i(\theta_1 + \varphi_1)}.$$

Case 4 If $t \to +\infty$, then $\text{Re}(i\chi_2(x + \frac{1}{2}\chi_2 t)) \to -\infty$, it follows that

$$q_1[2] \to c_1 \left[ \frac{\frac{\bar{\chi}_1 + a_1}{\chi_1 + a_1} B_1 + \frac{\bar{\chi}_1 + a_1}{\chi_1 + a_1} B_2 + \frac{\bar{\chi}_1 + a_1}{\chi_1 + a_1} B_3 + \frac{\bar{\chi}_1 + a_1}{\chi_1 + a_1} B_4}{B_1 + B_2 + B_3 + B_4} \right] e^{i\theta_1},$$

where

$$B_1 = \frac{1}{\bar{\chi}_1 - \chi_1}, \quad B_2 = \frac{1}{\bar{\chi}_1 - \bar{\tau}_1} e^{-i(\bar{\tau}_1 - \chi_1)[x + \frac{1}{2}(\bar{\tau}_1 + \chi_1)t]},$$

$$B_3 = \frac{1}{\bar{\tau}_1 - \chi_1} e^{i(\tau_1 - \chi_1)[x + \frac{1}{2}(\tau_1 + \chi_1)t]},$$

$$B_4 = \frac{1}{\bar{\tau}_1 - \bar{\tau}_1} e^{2\text{Re}(i(\tau_1 - \chi_1)[x + \frac{1}{2}(\tau_1 + \chi_1)t])}.$$

Similar as case 1, we can obtain that phase difference between two sides. It follows that

- If $\text{Re}(i(\tau_1 - \chi_1)[x + \frac{1}{2}(\tau_1 + \chi_1)t]) \to -\infty$, then
  $$q_1[2] \to c_1 e^{i(\theta_1 + \varphi_1)}.$$

- If $\text{Re}(i(\tau_1 - \chi_1)[x + \frac{1}{2}(\tau_1 + \chi_1)t]) \to +\infty$, then
  $$q_1[2] \to c_1 e^{i(\theta_1 + \varphi_3)}.$$

Finally, we can see that the asymptotical analysis between case 1, case 2 and case 3, case 4 is consistence. To demonstrate the dynamics of this type solution explicitly, we plot some figures with setting related parameters by Maple. In Fig 4 we show the interaction between one homoclinic orbit solution and dark-dark solution. In Fig 5 we show the interaction between one breather-II type solution and dark-dark solution.
we can have the following expression

\[ \tau \]

To derive the dark-dark-rogue solution, we use the limit technique. In this paragraph, we analysis the interaction between dark-dark soliton and rogue wave solution.

### 3.1.2 Dark-dark-rogue solution

In this paragraph, we analysis the interaction between dark-dark soliton and rogue wave solution. To derive the dark-dark-rogue solution, we use the limit technique \( \tau_1 \rightarrow \chi_1 \). Denote \( \epsilon = \tau_1 - \chi_1 \), we can have the following expression

\[
\begin{align*}
\frac{1}{\chi_1 - \chi - \epsilon} &= \frac{1}{\chi_1 - \chi} + \frac{\epsilon}{(\chi_1 - \chi)^2} + o(\epsilon^2), \\
\frac{1}{\epsilon + \chi_1 - \chi} &= \frac{1}{\chi_1 - \chi} - \frac{\epsilon}{(\chi_1 - \chi)^2} + o(\epsilon^2), \\
\frac{1}{\chi_1 - \chi - \epsilon + \epsilon} &= \frac{1}{\chi_1 - \chi} + \frac{\epsilon}{(\chi_1 - \chi)^2} - \frac{2\epsilon\epsilon}{(\chi_1 - \chi)^3} + o(\epsilon^2, \epsilon^2), \\
\frac{1}{\epsilon + \chi_1 - \chi_2} &= \frac{1}{\chi_1 - \chi_2} - \frac{\epsilon}{(\chi_1 - \chi_2)^2} + o(\epsilon^2), \\
\frac{1}{\chi_2 - \chi - \epsilon} &= \frac{1}{\chi_2 - \tau_1} + \frac{\epsilon}{(\chi_2 - \tau_1)^2} + o(\epsilon^2), \\
\frac{1}{\epsilon + \chi_1 + a_1} &= \frac{1}{\chi_1 + a_1} - \frac{\epsilon}{(\chi_1 + a_1)^2} + o(\epsilon^2),
\end{align*}
\]

and

\[
\begin{align*}
e^{i\epsilon(x + (\chi_1 + \frac{1}{2}t))} &= 1 + i\epsilon(x + \chi_1 t) + o(\epsilon^2), \\
e^{-i\epsilon(x + (\chi_1 + \frac{1}{2}t))} &= 1 - i\epsilon(x + \chi_1 t) + o(\epsilon^2), \\
e^{i\epsilon(x + (\chi_1 + \frac{1}{2}t))} - i^{2}(x + (\chi_1 + \frac{1}{2}t)) &= 1 + i\epsilon(x + \chi_1 t) - i\epsilon(x + \chi_1 t) + \frac{\epsilon\epsilon}{2} |x + \chi_1 t|^2 + o(\epsilon^2, \epsilon^2).
\end{align*}
\]

By the following expansion and (28), we can obtain the dark-dark-rogue solution

\[
q_1[2] = c_1 \left( \frac{F_1 + F_2 F_3}{D_1 + D_2 + D_3} \right) e^{i\theta_1}, \quad q_2[2] = c_2 \left( \frac{G_1 + G_2 G_3}{D_1 + D_2 + D_3} \right) e^{i\theta_2}, \tag{29}
\]
In this subsection, we consider the special case \( a_1 = a_2 = \frac{1}{27}, \ c_1 = 2, \ c_2 = 1, \ \lambda_1 = 2 + 5i, \ \lambda_2 = 0, \ \chi_1 = 3.901220880 + 10.25418804i, \ \tau_1 = 0.07993030051 + 0.01536696563i, \ \chi_2 = -0.0416053126 + 1.7328280i, \ \beta = 0.2885456543i. \) It is seen that there is breather-II type solution collision with dark-dark soliton, and their interaction is elasticity too.

where

\[
D_1 = \frac{\beta}{\chi_1 - \chi_1} \left( |x + \chi_1 t|^2 - \frac{1}{(\chi_1 - \chi_1)^2} \right),
\]

\[
D_2 = \frac{1}{(\chi_1 - \chi_1)(\chi_2 - \chi_2)} \left( |x + \chi_1 t|^2 - \frac{1}{(\chi_1 - \chi_1)^2} \right) e^{-2i\text{Im}(\chi_2)[x + \text{Re}(\chi_2)t]},
\]

\[
D_3 = \frac{1}{(\chi_1 - \chi_2)(\chi_2 - \chi_1)} \left( i(x + \chi_1 t) + \frac{1}{\chi_1 - \chi_2} \right) \left( i(x + \chi_1 t) + \frac{1}{\chi_2 - \chi_1} \right),
\]

and

\[
F_1 = \frac{(\chi_1 + a_1)|x + \chi_1 t|^2}{(\chi_1 + a_1)(\chi_1 - \chi_1)} \left( \beta + \frac{(\chi_1 + a_1)e^{-2i\text{Im}(\chi_2)[x + \text{Re}(\chi_2)t]}}{(\chi_1 + a_1)(\chi_1 - \chi_1)} \right),
\]

\[
F_2 = \frac{i(x + \chi_1 t)}{\chi_2 + a_1} + \frac{1}{(\chi_1 - \chi_2)^2} + \frac{i(x + \chi_1 t)}{\chi_1 - \chi_1} \ e^{i\chi_2(x + \chi_2 t)},
\]

\[
F_3 = \frac{i(x + \chi_1 t)}{\chi_1 + a_1} - \frac{1}{(\chi_1 + a_1)^2} + \left( \frac{1}{(\chi_2 - \chi_1)^2} + \frac{i(x + \chi_1 t)}{\chi_2 - \chi_1} \right) e^{-i\chi_2(x + \chi_2 t)},
\]

and \( G_i = F_i(a_1 \rightarrow a_2), \ i = 1, 2, 3. \) The asymptotical analysis of the above subsubsection is still valid for the dark-dark-rogue solution, since the dark-dark-rogue solution is nothing but the limit for solution (28). Finally we show the explicit dynamics by plotting figure (Fig. 6).

### 3.2 Two dark-one bright soliton solution

In this subsection, we consider the special case \( a_1 = a_2. \) Firstly, we consider \( c_1 > c_2 > 0, \) choosing the following special solution

\[
|y_1\rangle = \begin{bmatrix}
\frac{e^{i(\chi_1 x + \chi_2^2 t)}}{\chi_1 + \alpha_1} e^{i(\chi_1 x + \chi_2^2 t)} + c_2 \alpha_1 e^{i(-\alpha_1 x + \chi_2^2 t)} \\
\frac{e^{i(\chi_1 x + \chi_2^2 t)}}{\chi_1 + \alpha_1} e^{i(\chi_1 x + \chi_2^2 t)} + c_1 \alpha_1 e^{i(-\alpha_1 x + \chi_2^2 t)} \\
\end{bmatrix}
\]
Figure 6: (color online): Dark-dark-rogue solution: Parameters $a_1 = -a_2 = 1$, $c_1 = 2$, $c_2 = 1$, $\lambda_1 = 1.24185466772002 + .636002000756738i$, $\lambda_2 = 0.8333333333$, $\chi_1 = 1.356709486 + 1.087820879i$, $\chi_2 = 1.414213562i$, $\beta = .3535533907i$. It is seen that there is dark-dark rogue wave solution collision with rogue wave solution.

and $|y_2\rangle$ as above subsection, for simplicity we take $\alpha = 1$, we can obtain the solution in above subsection with

$$M_1 = \frac{1}{\chi_1 - \chi_1} + \frac{(c_1^2 - c_2^2)}{2(\lambda_1 - \lambda_1)} e^{2\text{Re}[i(\chi_1 + a_1)[x + \frac{1}{2}(\chi_1 - a_1)t]]},$$

$$M_2 = \frac{1}{\chi_1 - \chi_2} e^{i\chi_2(x + \frac{1}{2}\chi_2t)},$$

$$M_3 = \frac{1}{\chi_2 - \chi_1} e^{-i\chi_2(x + \frac{1}{2}\chi_2t)},$$

$$M_4 = \beta + \frac{1}{\chi_2 - \chi_2} e^{-2\text{Im}(\chi_2)[x + \text{Re}(\chi_2)t]},$$

and

$$X_{11} = \frac{1}{\chi_1 + a_1} + \frac{c_2}{c_1} e^{-i(\chi_1 + a_1)[x + \frac{1}{2}(\chi_1 - a_1)t]},$$

$$X_{12} = \frac{1}{\chi_2 + a_1} e^{i\chi_2(x + \frac{1}{2}\chi_2t)},$$

$$X_{13} = e^{-i\chi_2(x + \frac{1}{2}\chi_2t)} \left( \frac{1}{\chi_1 + a_1} + \frac{c_2}{c_1} e^{-i(\chi_1 + a_1)[x + \frac{1}{2}(\chi_1 - a_1)t]} \right),$$

$$X_{14} = \frac{1}{\chi_2 + a_1} e^{-2\text{Im}(\chi_2)[x + \text{Re}(\chi_2)t]},$$

and $X_{21} = X_{11}(c_1 \leftrightarrow c_2)$, $X_{23} = X_{13}(c_1 \leftrightarrow c_2)$, $X_{22} = X_{12}$, $X_{24} = X_{14}$.

We can perform similar asymptotical analysis on them. Choosing special parameters, we can obtain the interaction figure between dark-dark soliton and breather-II solution in the degenerate case. Here we give an example for the dark-dark soliton and breather-II soliton possesses the same velocity (Fig. 7).

Secondly, we consider $c_1 > 0$ and $c_2 = 0$. Indeed, the explicit expression of this kind solution can be obtained by choosing parameter $c_2 = 0$ on the above. Thus we can have two dark-one bright soliton solution:

$$q_1[2] = c_1 \frac{\det(M + X_1)}{\det(M)} e^{i\theta_1},$$

$$q_2[2] = \frac{\det(X_2)}{\det(M)} e^{i\theta_2},$$

where

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}, \quad X_1 = \begin{bmatrix} X_{11} & X_{12} \\ X_{13} & X_{14} \end{bmatrix}, \quad X_2 = \begin{bmatrix} X_{21} & X_{22} \\ X_{23} & X_{24} \end{bmatrix}.$$
Figure 7: (color online): Dark-dark-breather-II solution: Parameters $a_1 = a_2 = 0$, $c_1 = 2$, $c_2 = 1$, $\lambda_1 = i$, $\lambda_2 = 0$, $\chi_1 = 3i$, $\chi_2 = 1.732050808i$, $\beta = 0.2886751345 \exp(10)i$. It is seen that there exists dark-dark solution and breather-II solution with the stationary. and

\[ M_1 = \frac{1}{\chi_1 - \chi_1} + \frac{c_1^2}{2(\lambda_1 - \lambda_1)} e^{2\Re[i(\bar{\chi}_1 + a_1)(x + \frac{1}{2}(\bar{\chi}_1 - a_1)t)]}, \]
\[ M_2 = \frac{1}{\chi_1 - \chi_2} e^{i\chi_2(x + \frac{1}{2}\chi_2 t)}, \quad M_3 = \frac{1}{\chi_2 - \chi_1} e^{-i\chi_2(x + \frac{1}{2}\chi_2 t)}, \]
\[ M_4 = \beta + \frac{1}{\chi_2 - \chi_2} e^{-2\Im(\chi_2)[x + \Re(\chi_2)t]}, \]

and

\[ X_{11} = \frac{1}{\chi_1 + a_1}, \quad X_{12} = \frac{1}{\chi_2 + a_1} e^{i\chi_2(x + \frac{1}{2}\chi_2 t)}, \]
\[ X_{13} = \frac{1}{\chi_1 + a_1} e^{-i\chi_2(x + \frac{1}{2}\chi_2 t)}, \quad X_{14} = \frac{1}{\chi_2 + a_1} e^{-2\Im(\chi_2)[x + \Re(\chi_2)t]}, \]

and

\[ X_{21} = e_1 e^{-i(\chi_1 + a_1)(x + \frac{1}{2}(\chi_1 - a_1)t)}, \quad X_{22} = \left( \frac{1}{\chi_2 + a_1} + \frac{1}{\chi_1 - \chi_2} \right) e^{i\chi_2(x + \frac{1}{2}\chi_2 t)}, \]
\[ X_{23} = e_1 e^{-i(\chi_1 + a_1)(x + \frac{1}{2}(\chi_1 - a_1)t) - i\chi_2(x + \frac{1}{2}\chi_2 t)}, \quad X_{24} = \left( \frac{1}{\chi_2 + a_1} + \frac{1}{\chi_2 - \chi_2} \right) e^{-2\Im(\chi_2)[x + \Re(\chi_2)t]} + \beta. \]

To show its dynamics behavior, we choose two different groups of parameters. The figure 8 and figure 9 show that the component $|q_1[2]|^2$ possesses two dark soliton and $|q_2[2]|^2$ possesses one bright soliton. In figure 8 the two dark solitons possess different velocities. In figure 9 the two dark solitons possess same velocity with stationary.

4 Discussions and Conclusions

In this paper, we provide a method to derive nonsingular nonlinear wave solutions of mixed coupled nonlinear Schrödinger equations for which it is essential to deal with indefinite Darboux matrix. Furthermore, we present one possible classification for nonlinear localized wave solutions of the model through combining Darboux transformation and matrix analysis methods. The explicit conditions and ideal excitation forms for these nonlinear waves are presented in detail, which are
meaningful for further physical studies on them. The high order solution can be obtained by the
generalized DT \cite{41,42,43,44}. Indeed, high-order solution can be obtained through limit technique
from the solution formula in \textbf{theorem 5} too. We would like to consider the general high order
solution with a proper form in the future. The methods here can be extended directly to the
defocusing CNLSE or even general multi-component NLSE, the mixed or defocusing Sasa-Satuma
system, three wave system, long wave-short wave model and other AKNS reduction system with
indefinite Darboux matrix cases.

Recently, a classification on soliton solution for multi-component NLSE was presented in ref.
\cite{25}, which mainly involving the dark-dark soliton, the bright-dark soliton and the breather solution
(called by bright soliton there). The nonlinear wave solutions and the derivation method presented
here are distinctive from the results.

It should be pointed out that the non-singularity condition has been given through the general
algebro-geometric scheme in reference \cite{45}. The differences between our work and reference \cite{45}
have been given in \cite{46}.
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References

[1] V. E. Zakharov, and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. Zh. Eksp. Teor. Fiz. 61, 118-134 (1971).
[2] H. Bailung and Y. Nakamura, Observation of modulational instability in a multi-component plasma with negative ions. 50, 231-242 (1993)
[3] B. Kibler, J. Fatome, C. Finot, et al. Observation of Kuznetsov-Ma soliton dynamics in optical fibre. Scientific reports, 2, (2012).
[4] G.P. Agrawal, Nonlinear Fiber Optics. (4th Edition, Academic Press, Boston, 2007).
[5] J.M. Dudley, G. Genty, F. Dias, et al. Modulation instability, Akhmediev Breathers and continuous wave supercontinuum generation. Optics express, 17: 21497-21508 (2009)
[6] B. Kibler, J. Fatome, C. Finot, et al. The Peregrine soliton in nonlinear fibre optics. Nature Physics, 6, 790-795 (2010).
[7] A. Chabchoub, N.P. Hoffmann and N. Akhmediev, Rogue wave observation in a water wave tank. Phys. Rev. Lett. 106: 204502 (2011)
[8] A Chabchoub, N Hoffmann, M Onorato, et al. Super rogue waves: observation of a higher-order breather in water waves. Phys. Rev. X, 2 011015 (2012)
[9] H. Bailung, S.K. Sharma and Y. Nakamura, Observation of Peregrine solitons in a multicomponent plasma with negative ions. Phys. Rev. Lett. 107 255005 (2011)
[10] A. Chabchoub, O. Kimmoun, H. Branger, et al. Experimental observation of dark solitons on the surface of water. Phys. Rev. Lett. 110 124101 (2013)
[11] B. Primari, M.J. Ablowitz and G. Biondini, Inverse scattering transform for the vector nonlinear Schrödinger equation with nonvanishing boundary conditions. J. Math. Phys. 47 063508 (2006)
[12] S.V. Manakov, Zh. Éksp. Teor. Fiz. 65, 505 1973[Sov. Phys. JETP 38, 248 1974.]
[13] T. Kanna, M. Lakshmanan, Exact soliton solutions, shape changing collisions, and partially coherent solitons in coupled nonlinear Schrödinger equations. Phys. Rev. Lett. 86 5043 (2001)
[14] T. Kanna, M. Lakshmanan, P. Tchofo Dinda, et al. Soliton collisions with shape change by intensity redistribution in mixed coupled nonlinear Schrödinger equations. Phys. Rev. E, 73 026604: 1-15. (2006)
[15] D.S. Wang, D.J. Zhang and J. Yang, Integrable properties of the general coupled nonlinear Schrödinger equations. J. Math. Phys. 51 023510. (2010)
[16] M.G. Forest, O C. Wright, An integrable model for stable: unstable wave coupling phenomena. Physica D: Nonlinear Phenomena, 178 173-189 (2003)
[17] B. Guo and L. Ling, Rogue wave, breathers and bright-dark-rogue solutions for the coupled Schrödinger equations. Chin. Phys. Lett., 28 110202 (2011)
[18] F. Baronio, A. Degasperis, M. Conforti, et al. Solutions of the vector nonlinear Schrödinger equations: evidence for deterministic rogue waves. Phys. Rev. Lett. 109 044102 (2012)
[19] L. Zhao, J. Liu, Localized nonlinear waves in a two-mode nonlinear fiber. JOSA B, 29 3119-3127 (2012)
[20] L. Zhao, J. Liu, Rogue-wave solutions of a three-component coupled nonlinear Schrödinger equation. Phys. Rev. E, 87 013201 (2013)
[21] X. Wang, Y. Li , Y. Chen, Generalized Darboux transformation and localized waves in coupled Hirota equations. Wave Motion, (2014)
[22] J. He, L. Guo, Y. Zhang and A. Chabchoub, Theoretical and experimental evidence of nonsymmetric doubly localized rogue waves. Accepted by Proceedings of the Royal Society A, (2014)
[23] G. Dean, T. Klotz, B. Prinari, et al. *Dark-dark and dark-bright soliton interactions in the two-component defocusing nonlinear Schrödinger equation*. Applicable Analysis, 92 379-397 (2013)

[24] Q.H. Park and H.J. Shin, *Systematic construction of multicomponent optical solitons*. Phys. Rev. E, 61 3093 (2000)

[25] T. Tsuchida, *Exact solutions of multicomponent nonlinear Schrödinger equations under general plane-wave boundary conditions*. arXiv preprint arXiv:2013.1308.6623, 2013.

[26] C. Kalla, *Breathers and solitons of generalized nonlinear Schrödinger equations as degenerations of algebro-geometric solutions*. J. Phys. A: Math. Theor. 44 335210 (2011)

[27] P. Deift and E. Trubowitz, *Inverse scattering on the line*. Comm. Pure and Appl. Math. 32 121-251 (1979)

[28] V. B. Matveev and M A. Salle, *Darboux transformations and solitons*. (Berlin: Springer-Verlag, 1991)

[29] C.H. Gu, H.S. Hu, Z. Zhou, *Darboux transformations in integrable systems: theory and their applications to geometry*. (Springer, 2006)

[30] C.-L. Terng and K. Uhlenbeck, *Bäcklund transformations and loop group actions*. Comm. Pure Appl. Math. 53, 1-75 (2000)

[31] S. P. Novikov, S. V. Manakov, V. E. Zakharov, and L. P. Pitaevskii, *Theory of solitons: the inverse scattering method*. (Springer, 1984)

[32] O.C. Wright and Gregory M. Forest, *On the Bäcklund-gauge transformation and homoclinic orbits of a coupled nonlinear Schrödinger system*. Physica D: Nonlinear Phenomena, 141 104-116 (2000)

[33] A. Degasperis, S. Lombardo, *Multicomponent integrable wave equations: I. Darboux-dressing transformation*. J. Phys. A: Math. Theor. 40 961 (2007)

[34] A. Degasperis, S. Lombardo, *Multicomponent integrable wave equations: II. Soliton solutions*. J. Phys. A: Math. Theor. 42 385206 (2009)

[35] A. de O Assuncao, H. Blas and M. da Silva, *New derivation of soliton solutions to the AKNS2 system via dressing transformation methods*. J. Phys. A: Math. Theor. 45 085205 (2012)

[36] L. Ling, B. Guo, and L. Zhao, *High-order rogue waves in vector nonlinear Schrödinger equations*. Phys. Rev. E 89, 041201(R) (2014)

[37] B. Feng, *General N-soliton solution to a vector nonlinear Schrödinger equation* (private communication and preprint).

[38] D. Zhang, S. Zhao, Y. Sun, and J. Zhou, *Solutions to the modified Korteweg-de Vries equation*. Rev. Math. Phys. 26 1430006 (2014)

[39] L. Ling, L. Zhao, and B. Guo, *Darboux transformation and multi-dark soliton for N-component coupled nonlinear Schrödinger equations*. arXiv preprint arXiv:2013.09103 (2013).

[40] F. Baronio, M. Conforti, A. Degasperis, et al., *Rogue wave solutions for coupled defocusing nonlinear Schrödinger equations*. Phys. Rev. Lett. 113, 034101 (2014).

[41] B. Guo, L. Ling, and Q.P. Liu, *Nonlinear Schrödinger equation: Generalized Darboux transformation and rogue wave solutions*. Physical Review E, 85, 026607 (2012)

[42] J. He, H. Zhang, L. Wang, and A. Fokas, *Generating mechanism for higher-order rogue waves*. Phys. Rev. E, 87: 052914 (2013)

[43] B. Guo, L. Ling, and Q.P. Liu, *High-Order Solutions and Generalized Darboux Transformations of Derivative Nonlinear Schrödinger Equations*. Stud. Appl. Math. 130 317-344 (2013)

[44] D. Bian, B. Guo, and L. Ling, *High-Order Soliton Solution of Landau-Lifshitz Equation*. Stud. Appl. Math. To appear (2014)

[45] B. A. Dubrovin, T. M. Malanyuk, I. M. Krichever, and V. G. Makhankov, *Exact solutions of the time-dependent Schrödinger equation with self-consistent potentials*. Sov. J. Part. Nucl. 19 252-269 (1988)

[46] L. Ling, L. Zhao, and B. Guo, *Reply to “Comment on “Darboux transformation and classification of solution for mixed coupled nonlinear Schrödinger equations””.* arXiv:2014.08223 (2014)