Fixed Subgroups of Endomorphisms of Free Products

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Abstract

Let $G = \ast_{i=1}^n G_i$ and let $\phi$ be a symmetric endomorphism of $G$. If $\phi$ is a monomorphism or if $G$ is a finitely generated residually finite group, then the fixed subgroup $Fix(\phi) = \{g \in G : \phi(g) = g\}$ of $\phi$ has Kurosh rank at most $n$.

1 Introduction

In [1], Bestvina and Handel proved the Scott conjecture, which says that if $\phi$ is an automorphism of a free group of rank $n$, then the subgroup $Fix(\phi)$ of elements fixed by $\phi$ has rank at most $n$. Their result was generalized by several authors in various directions. See, for example, [6] [5] [7] [2] [9]. In particular, the result of Bestvina and Handel was generalized both to arbitrary endomorphisms of free groups by Imrich and Turner [6] and to automorphisms of free products by Collins and Turner [5].

In this note, following the main idea of [6], we show that in many interesting cases the study of fixed subgroups of endomorphisms of free products is reduced to that of automorphisms, thereby obtaining new generalizations of Bestvina-Hadel’s result.

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2 Preliminaries

Let $G = \ast_{i=1}^n G_i$ and let $H$ be a non-trivial subgroup of $G$. By the Kurosh subgroup theorem, $H$ is a free product $H = \ast_{i \in I} H_i \ast F$, where $F$ is a free group and every factor $H_i$ is the intersection of $H$ with a conjugate of a free factor $G_i$. In the case where the rank $r(F)$ of $F$ and the cardinality $|I|$ of $I$ (which may be empty) are finite, the Kurosh rank of $H$ with respect to the given splitting of $G$ is defined to be the sum $r(F) + |I|$. We will usually omit the phrase “with respect ... splitting of $G$”, when the splitting of $G$ is clear from the context.

Following [6], given a group $G$ and an endomorphism $\phi$ of $G$, we define the stable image $\phi^\infty(G)$ of $\phi$ to be the intersection $\cap_{n=1}^\infty \phi^n(G)$. Clearly $\phi^\infty(G)$ is invariant under $\phi$ and contains $\text{Fix}(\phi)$. Thus $\text{Fix}(\phi) = \text{Fix}(\phi_\infty)$, where $\phi_\infty : \phi^\infty(G) \to \phi^\infty(G)$ denotes the restriction of $\phi$ to $\phi^\infty(G)$. The key observation is that if $\phi$ is a monomorphism, then $\phi_\infty$ is an automorphism. To see this, let $g \in \phi^\infty(G)$ be any element. Then for every $n$ there exists an element $g_n$ of $\phi^n(G)$ such that $g = \phi(g_n)$. Since $\phi$ is injective, $g_1 = g_n$ for all $n$ and hence $g_1 \in \phi^\infty(G)$. This gives surjectivity of $\phi_\infty$.

Now, the basic idea can be described briefly as follows. Suppose that $G$ is a free product and that $\phi_\infty$ is an automorphism sending non-infinite-cyclic factors of the stable image onto conjugates of themselves. By [9, Theorem 6.12], the Kurosh-rank of $\text{Fix}(\phi_\infty)$ does not exceed the Kurosh rank of $\phi^\infty(G)$. Thus to find an upper bound for the Kurosh rank of $\text{Fix}(\phi)$, we need to know something about the kurosh rank of $\phi^\infty(G)$. By [10, Theorem 6.5], the Kurosh rank of $\phi^\infty(G)$ is bounded above by the maximum of the Kurosh ranks of the images $\phi^n(G)$. In the case where $G$ is a free group of rank $n$, it is immediate that the rank of every image $\phi^n(G)$ is less than or equal to $n$ while in the case of a free product is not. However, we will see that this happens in many cases, in which we obtain that the Kurosh rank of $\text{Fix}(\phi)$ does not exceed the Kurosh rank of $G$. 


3 Main Results

We start with the following result which has been obtained independently by Swarup [11].

Lemma 3.1. Let $G = \ast_{i=1}^n G_i \ast F$ and $H = \ast_{j=1}^m H_j \ast F'$, where each factor $G_i$ is not infinite cyclic and $F$, $F'$ are free groups, and let $\phi : G \to H$ be an epimorphism such that each factor $G_i$ is mapped by $\phi$ into a conjugate of some $H_j$. Then $n + r(F) \geq m + r(F')$ and $r(F) \geq r(F')$.

Proof. By renumbering if necessary, we can assume that $H_1, \ldots, H_{m_0}, m_0 \leq m$ are the factors of $H$ whose conjugates contain the non-trivial images of $G_i$, $i = 1, \ldots, n$ under $\phi$. Note that $m_0 \leq n$. If $N$ and $K$ are the normal subgroups of $G$ and $H$ generated by $G_i, i = 1, \ldots, n$ and $H_j, j = 1, \ldots, m_0$ respectively, then $\phi(N) \subseteq K$, and so $\phi$ induces an epimorphism $\Phi : F \cong G/N \to H/K \cong H_{m_0+1} \ast \cdots \ast H_m \ast F'$. It follows that $r(F) \geq d(H_{m_0+1} \ast \cdots \ast H_m \ast F') = d(H_{m_0+1}) + \cdots + d(H_m) + r(F') \geq m - m_0 + r(F')$, and the lemma follows.

Let $G = \ast_{i=1}^n G_i$ and $H = \ast_{i=1}^m H_i$. A homomorphism $\phi : G \to H$ is said to be symmetric if each non-infinite-cyclic free factor of $G$ is mapped by $\phi$ into a conjugate of some non-infinite-cyclic free factor of $H$. For example, if each factor $G_i$ is freely indecomposable, then each injective homomorphism is symmetric.

The next lemma shows that symmetric automorphisms of free products map non-infinite-cyclic factors onto conjugates of themselves and therefore [9, Theorem 6.12] can be applied.

Lemma 3.2. Let $G = \ast_{i=1}^n G_i \ast F$ and let $\phi$ be an automorphism of $G$. If each factor $G_i$ is mapped by $\phi$ into a conjugate of some $G_j$, then $G_i$ is mapped by $\phi$ onto this conjugate.

Proof. Suppose on the contrary that there is a factor, say $G_1$, such that $\phi(G_1)$ is properly contained in $gG_1g^{-1}$, $g \in G$. By [3, Theorem 7] there is a free product decomposition $G = \ast_{i=1}^n G_i' \ast F'$ of $G$ such that $\phi(G_i') = G_i$, $i = 1, \ldots, n$ and $\phi(F') = F$. If $x = \phi^{-1}(g)$, then $\phi(x^{-1}G_1x) \subset G_{i_1} = \phi(G_{i_1}')$. 


and thus $x^{-1}G_1x \subset G'_{t_1}$. Since $G'_{t_1}$ properly contains $x^{-1}G_1x$, there is a free product decomposition $G'_{t_1} = x_1G_1x_1^{-1} * K$, obtained from the initial decomposition of $G$, where $K$ is a non-trivial subgroup of $G$. Thus $G = G_1 * \cdots * G_n * F = *_{i \neq i_1} G'_i * x_1G_1x_1^{-1} * K * F'$.

Now we consider the map $\psi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, defined as follows: $\psi(i) = j$ if and only if $\phi(G_i)$ is contained in a conjugate of $G_j$. The injectivity of $\phi$ implies that $\psi$ is well-defined, while the proof of Lemma 3.1 shows that $\psi$ is surjective, and hence bijective. We conclude that the normal subgroup $N$ of $G$ generated by $G_1, \ldots, G_n$ is contained in the normal subgroup $N'$ of $G$ generated by $G_1, G'_i$, $i \neq i_1$. Thus we have an epimorphism $\Phi : G/N \cong F \to G/N' \cong K * F'$. Since $K$ is non-trivial, it follows that $r(F) > r(F')$, which contradicts the fact that the groups $F$ and $F'$ are isomorphic. □

**Theorem 3.3.** Let $G = *_{i=1}^n G_i * F$, where each factor $G_i$ is not infinite cyclic and $F$ is a free group. If $\phi : G \to G$ is a symmetric monomorphism of $G$, then the fixed subgroup $\text{Fix}(\phi)$ of $\phi$ has Kurosh rank at most $n + r(F)$.

**Proof.** By the remarks preceding Lemma 3.1 it suffices to show that the stable image $\phi^\infty(G)$ of $\phi$ has Kurosh rank at most $n + r(F)$, and that the automorphism $\phi_\infty$ of $\phi^\infty(G)$ is symmetric, since the theorem is true for symmetric automorphisms of free products [9].

First, we note that for each $k \geq 0$, the epimorphism $\phi_k : \phi^k(G) \to \phi^{k+1}(G)$ obtained by restricting $\phi$ to $\phi^k(G)$ is symmetric (where $\phi^0(G) = G$), which implies that $\phi^k(G)$ has Kurosh rank at most $n + r(F)$ for all $k$ by Lemma 3.1. To see this, let $\phi^k(G) \cap xG_ix^{-1}$ be a non-infinite-cyclic free factor of $\phi^k(G)$ (with respect to the free product decomposition of $\phi^k(G)$ inherited from this one of $G$). The assumption that $\phi$ is symmetric implies that there is an index $j(i) \in \{1, \ldots, n\}$ and an element $g_i \in G$ such that $\phi(G_i) \subseteq g_iG_{j(i)}g_i^{-1}$. Thus $\phi(\phi^k(G) \cap xG_ix^{-1}) \subseteq \phi^{k+1}(G) \cap \phi(x)\phi(G_i)\phi(x)^{-1} \subseteq \phi^{k+1}(G) \cap \phi(x)\phi(G_i)\phi(x)^{-1}$. The latter group is a subgroup of $\phi^{k+1}(G)$ which stabilizes a vertex in any $G$-tree constructed from the given free product decomposition of $G$. It follows that $\phi(\phi^k(G) \cap xG_ix^{-1})$ is contained in a $\phi^{k+1}(G)$-conjugate of a free factor of $\phi^{k+1}(G)$ and hence $\phi_k$ is symmetric. The same argument shows that the automorphism $\phi_\infty$ is symmetric as well.
Since each term of the decreasing sequence of subgroups
\[ G \supseteq \phi(G) \supseteq \phi^2(G) \supseteq \cdots \supseteq \phi^k(G) \supseteq \cdots \]
has Kurosh rank at most \( n + r(F) \), [10, Theorem 6.5] implies that the stable image \( \phi^\infty(G) \) of \( \phi \) also has Kurosh rank at most \( n + r(F) \).

Corollary 3.4. Let \( \phi \) be a monomorphism of a free product \( \ast_{i=1}^n G_i \) of freely indecomposable groups. Then \( \text{Fix}(\phi) \) has Kurosh rank at most \( n \).

In view of the preceding theorem, it is natural to seek conditions under which a free product endomorphism becomes “finally” a monomorphism. The second proof of the Hopficity of finitely generated residually finite groups sketched in [3], actually shows that the restriction of an endomorphism of a residually finite group to its stable image is a monomorphism (see also [4, Lemma 1]). For completeness, we include the argument here.

Lemma 3.5. Let \( \phi \) be an endomorphism of a finitely generated residually finite group \( G \). Then the restriction \( \phi^\infty : \phi^\infty(G) \to \phi^\infty(G) \) of \( \phi \) to \( \phi^\infty(G) \) is a monomorphism.

Proof. Let \( 1 \neq g \in \ker(\phi^\infty) \). Then \( \phi(g) = 1 \) and for each positive integer \( n \) there is \( g_n \in G \) such that \( g = \phi^n(g_n) \). Since \( G \) is residually finite there is a finite group \( \Gamma \) and a homomorphism \( \pi : G \to \Gamma \) with \( \pi(g) \neq 1 \). We consider the sequence of homomorphisms \( \pi_n = \pi \circ \phi^n : G \to \Gamma \). Then \( 1 \neq \pi(g) = \pi(\phi^n(g_n)) = \pi_n(g_n) \). On the other hand, \( \pi_m(g_n) = \pi(\phi^m(g_n)) = \pi(\phi^{m-n}(g)) = 1 \) whenever \( m > n \). It follows that there are infinitely many distinct homomorphisms from the finitely generated group \( G \) to the finite group \( \Gamma \), a contradiction.

Theorem 3.6. Let \( G = \ast_{i=1}^n G_i \ast F \) be a finitely generated residually finite group, where each factor \( G_i \) is not infinite cyclic and \( F \) is a free group. If \( \phi \) is a symmetric endomorphism of \( G \), then the fixed subgroup \( \text{Fix}(\phi) \) of \( \phi \) has Kurosh rank at most \( n + r(F) \).

Proof. The arguments of Theorem 3.3 show that the stable image \( \phi^\infty(G) \) of \( \phi \) has Kurosh rank at most \( n + r(F) \) and that the restriction \( \phi^\infty \) of \( \phi \) to
\(\phi^\infty(G)\) is a symmetric endomorphism. By Lemma 3.5, \(\phi^\infty\) is a monomorphism, so Theorem 3.3 applies.

\[\text{Theorem 3.7. Let } G = \ast_{i=1}^n G_i \text{ be a free product of finitely generated nilpotent and finite groups. If } \phi \text{ is an endomorphism of } G, \text{ then the fixed subgroup } \text{Fix}(\phi) \text{ of } \phi \text{ has Kurosh rank at most } n.\]

\[\text{Proof. Since each quotient of a nilpotent group is freely indecomposable, each of the epimorphisms } \phi_k : \phi^k(G) \to \phi^{k+1}(G) \text{ satisfies the hypothesis of Lemma 3.1. This implies that } \phi^\infty(G) \text{ has Kurosh rank at most } n. \text{ By Lemma 3.5, } \phi^\infty \text{ is a monomorphism. Also, it is easy to see that } \phi^\infty \text{ is symmetric. The theorem now follows by Theorem 3.3.}\]

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