TOTALLY GEODESIC SPECTRA OF ARITHMETIC HYPERBOLIC SPACES

JEFFREY S. MEYER

Abstract. In this paper we analyze the extent to which totally geodesic submanifolds determine the geometry of standard arithmetic hyperbolic manifolds. We extend the results of McReynolds and Reid to show that totally geodesic subspaces determine the commensurability class of a standard arithmetic hyperbolic $n$-manifold, $n \geq 4$. In fact, most of the results are far more general and apply to irreducible arithmetic lattices in Lie groups of the form $\prod_{i=1}^{r} \text{SO}(p_i, m - p_i) \times (\text{SO}_m(\mathbb{C}))^s$. We produce a dictionary between the classical invariants of quadratic forms and the Tits index of its associated isometry group. We give an alternate proof of Maclachlan’s parametrization of commensurability classes of even dimensional arithmetic hyperbolic manifolds. We prove our Hyperbolic Subspace Dichotomy Theorem and extend our results to nonstandard hyperbolic spaces.

1. Introduction

The goal of this paper is to determine the extent to which the geometry of an arithmetic hyperbolic $n$-manifold, $n \geq 4$, is encoded in the collection of its totally geodesic submanifolds. To put this goal in a broader context, we step back a moment and ask a natural question, one going back over a century: What topological and geometric properties of a space $M$ are encoded in certain interesting collections of geometric data associated to $M$? One of the earliest examples of this line of inquiry was in 1911, when Weyl showed that the eigenvalues of the Laplace-Beltrami operator determine the dimension and volume of a closed Riemannian manifold [W]. In 1966, Kac popularized this question by asking “Can one hear the shape of a drum?” [Kac]. Since that time many different collections of data, called spectra, have been studied. Over the past few decades, one prominent spectrum has been the collection of lengths of closed geodesics. The weak length spectrum of a Riemannian manifold $M$, is the set

\[ L(M) := \{ \lambda \in \mathbb{R} \mid \lambda \text{ is the length of a closed geodesic in } M \}. \]

Observe that this collection can be equivalently formulated as

\[ L(M) = \{ \text{Isometry classes of closed geodesics in } M \}. \]

(1.1)

(1.2)

Two manifolds with the same weak length spectrum are said to be weakly iso-length-spectral.

Question 1: Does $L(M)$ determine the isometry class of $M$?

The answer is a resounding no, and since the 1960’s, there have been many constructions of weakly iso-length-spectral spaces which are not isometric, the most famous of which being:

- 16-dimensional flat tori (Milnor, 1964 [Mi]),
- 2- and 3-dimensional hyperbolic manifolds, and more generally spaces spaces coming from quaternion algebras (Vignéras, 1980 [Vig]),
- General method based on covering space theory (Sunada, 1985 [Su]).

However, the procedures used in these three papers always produce manifolds which are almost isometric in the sense that they are commensurable\(^1\).

\^1See Section 2 for a discussion of commensurability
When two Riemannian manifolds \( M_1 \) and \( M_2 \) are commensurable, then every length of a geodesic in \( M_1 \) is a rational multiple of a geodesic in \( M_2 \), and vice versa. Motivated by this, [CHLR] defined the \textit{rational length spectrum} to be the set
\[
\mathcal{Q}L(M) := \{ s\lambda \in \mathbb{R} \mid s \in \mathbb{Q} \text{ and } \lambda \text{ is the length of a closed geodesic in } M \}.
\] (1.3)

Again, we observe that this definition may be reformulated as follows:
\[
\mathcal{Q}L(M) = \{ \text{Commensurability classes of closed geodesics in } M \}.
\] (1.4)

Two manifolds with the same rational length spectrum are said to be \textit{length-commensurable}. In particular, commensurable manifolds are length-commensurable. One may then ask the following refined question.

**Question 2:** Does \( \mathcal{Q}L(M) \) determine the commensurability class of \( M \)?

When \( M_1 \) and \( M_2 \) are arithmetic hyperbolic \( n \)-manifolds, then \( \mathcal{Q}L(M_1) = \mathcal{Q}L(M_2) \) implies \( M_1 \) and \( M_2 \) are commensurable in each of the following cases:

- \( n = 2 \) (Reid, 1992 [Re92]),
- \( n = 3 \) (Chinburg, Hamilton, Long, and Reid, 2008 [CHLR]),
- \( n \neq 3, n \neq 7, n \not\equiv 1 \pmod{4} \) (Prasad and Rapinchuk, 2009 [PR09]),
- \( n = 7 \) (Garibaldi, 2013 [Ga]).

However, for each positive \( n \equiv 1 \pmod{4}, n > 1 \), [PR09] produced examples of noncommensurable length-commensurable arithmetic hyperbolic \( n \)-manifolds. More generally, there are many constructions of families of pairwise noncommensurable length-commensurable arithmetic locally symmetric spaces of the same Cartan-Killing type (see [LSV, Theorem 1], [PR09, Construction 9.15]).

Our motivation is to find a collection of data that is complementary to length spectra and that distinguishes commensurability classes. Recently there has been a push to look at certain higher dimensional analogues of geodesics: totally geodesic submanifolds. For us, it will be sufficient to only consider \textit{nonflat} totally geodesic subspaces. Furthermore, in analogy with looking at closed geodesics, we only want to look at \textit{finite volume} subspaces. With this in mind, we define the \textbf{weak totally geodesic spectrum} of a Riemannian manifold to be the set
\[
TG(M) := \left\{ \text{Isometry classes of nonflat finite volume totally geodesic submanifolds of } M \right\}.
\] (1.5)

McReynolds and Reid [McRe] prove that if \( M_1 \) and \( M_2 \) are arithmetic hyperbolic 3-manifolds such that \( TG(M_1) = TG(M_2) \), then either this set is empty or \( M_1 \) and \( M_2 \) are commensurable. As was the case for the weak length spectrum, \( TG(M) \) is not an invariant of commensurability class, and hence we define the \textit{totally geodesic commensurability spectrum} to be the set
\[
\mathcal{Q}TG(M) := \left\{ \text{Commensurability classes of nonflat finite volume totally geodesic submanifolds of } M \right\}.
\] (1.6)

Observe that \( TG(M) \) and \( \mathcal{Q}TG(M) \) are natural analogues of the second formulations of \( L(M) \) and \( \mathcal{Q}L(M) \) (see 1.2 and 1.4). If two Riemannian orbifolds \( M \) and \( M' \) have the same totally geodesic commensurability spectrum, we say they are \textit{totally-geodesic-commensurable}. The former is more rigid while the later is an invariant of the commensurability class of \( M \). The goal of this paper is to investigate the following question:

**Question 3:** Does \( \mathcal{Q}TG(M) \) determine the commensurability class of \( M \)?

In this paper we will address this question in the case of locally symmetric spaces of type \( B_n \) and \( D_n \). In particular, we will focus on standard arithmetic locally symmetric spaces associated to Lie groups of the form \( \prod_{i=1}^{r} \text{SO}(p_i, m - p_i) \times \text{(SO}_m(\mathbb{C})^s \), for \( m \geq 5 \). These spaces are constructed via the isometry groups quadratic forms over number fields (see Construction 4.8). We call such a locally symmetric space \textbf{simple} if its associates
Lie group is simple (i.e., if \(r + s = 1\)). Standard arithmetic hyperbolic \(n\)-manifolds are all simple in this sense. We begin by showing that when \(M\) is simple, \(\text{QTG}(M)\) determines the field of definition.

**Theorem A.** Let \(M_1\) and \(M_2\) be simple arithmetic locally symmetric spaces coming from quadratic forms of dimension \(m \geq 5\) over number fields \(k_1\) and \(k_2\) respectively. Then \(\text{QTG}(M_1) = \text{QTG}(M_2)\) implies \(k_1\) and \(k_2\) are \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-conjugate.

We go on to show when \(\text{QTG}(M)\) determines the commensurability class of \(M\).

**Theorem B.** Let \(M_1\) and \(M_2\) be arithmetic locally symmetric spaces coming from quadratic forms of dimension \(m \geq 5\) over number fields \(k_1\) and \(k_2\) respectively. Suppose that either \(M_1\) and \(M_2\) are simple or \(k_1\) and \(k_2\) are \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-conjugate. Then \(\text{QTG}(M_1) = \text{QTG}(M_2)\) implies \(M_1\) and \(M_2\) are commensurable.

It is worth noting that Theorem B holds for any \(\mathbb{R}\)-rank, and unlike \([PR09]\) and \([Ga]\), is not dependent in \(\mathbb{R}\)-rank \(\geq 2\) upon the truth of Schamel’s conjecture. Furthermore, the large class of nonsimple spaces had not been covered under the results of \([PR09]\). Specializing to the \(\mathbb{R}\)-rank \(1\) case, Theorem B gives the following theorem.

**Theorem C.** Let \(M_1\) and \(M_2\) be standard arithmetic hyperbolic manifolds of dimension \(n \geq 4\). Then \(\text{QTG}(M_1) = \text{QTG}(M_2)\) implies \(M_1\) and \(M_2\) are commensurable.

In fact, we show that the codimension-1 and codimension-2 totally geodesic subspaces determine the commensurability class of a standard arithmetic hyperbolic manifold. Furthermore, we show in Theorem 9.4 that the commensurability class of an even dimensional arithmetic hyperbolic orbifold is totally determined by its codimension-1 totally geodesic subspaces. To complement these results, in the next theorem we show that there are many commensurability classes of hyperbolic orbifolds with the exact same collection of totally geodesic subspaces in dimension greater than 2.

**Theorem D.** (Hyperbolic Subspace Dichotomy.) Let \(M_1\) and \(M_2\) be standard \(n\)-dimensional \((n \geq 4)\) finite volume arithmetic hyperbolic orbifolds. Then, up to commensurability, either

- for all \(j \in \mathbb{N}, 1 < j < n - 2\), \(M_1\) and \(M_2\) have the exact same collection of \(j\)-dimensional finite volume totally geodesic subspaces, or
- \(M_1\) and \(M_2\) do not share a single finite volume totally geodesic subspace of dimension \(\geq 2\).

Along the way, we construct several explicit examples of standard arithmetic hyperbolic manifolds. In particular, in Example [10.6] we construct a hyperbolic 3-manifold \(N\) and a hyperbolic 5-manifold \(M\) such that every totally geodesic surface in \(N\) is commensurable to a totally geodesic surface in \(M\), yet \(N\) is not commensurable to a totally geodesic subspace of \(M\).

While all even dimensional arithmetic hyperbolic manifolds come from quadratic forms, there are odd dimensional ones that do not. To this end we also address some results on spaces coming from skew hermitian forms over quaternion division algebras over number fields. Though there are considerable obstructions to finishing the analysis for groups coming from this construction, we do have the following partial results.

**Theorem E.** Let \(M_1\) and \(M_2\) be arithmetic locally symmetric spaces where \(M_1\) comes from a quadratic form of dimension \(m = 2n\) and \(M_2\) comes from a skew hermitian form of dimension \(n\) over a division algebra. Then \(\text{QTG}(M_1) \neq \text{QTG}(M_2)\).

**Theorem F.** Let \(M_1\) and \(M_2\) be simple arithmetic locally symmetric spaces coming from skew hermitian forms of dimension \(n \geq 4\) over over quaternion division algebras \(D_1\) and \(D_2\) over number fields \(k_1\) and \(k_2\) respectively. Then \(\text{QTG}(M_1) = \text{QTG}(M_2)\) implies \(k_1\) and \(k_2\) are \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-conjugate and this conjugacy induces an isomorphism between \(D_1\) and \(D_2\).
We now briefly go over the organization of this paper. In Section 2 we establish some basic notation and facts about commensurability, totally geodesic subspaces, and locally symmetric spaces. In Section 3 we introduce and analyze arithmetic subgroups of $\mathbb{Q}$-groups, arithmetic lattices in semisimple Lie groups, and arithmetic locally symmetric spaces. While many of the results on arithmetic locally symmetric spaces in this section are known, as of the time of writing this, we are unaware of references for them. As such, we state and prove many fundamental results on arithmetic locally symmetric spaces in the hope that this section will be a valuable reference for future research.

In Section 4 we give the results from the algebraic theory of quadratic forms over local and global fields that will be useful in our analysis. In particular, we introduce the classical invariants of forms over local fields and we then state the uniqueness and existence theorems of quadratic forms over local and global fields. In Section 5 we analyze the Tits index of groups coming from quadratic forms over a number field and create a dictionary between the local indices of such groups and the local invariants of the associated forms. This dictionary enables us in Section 6 to give an alternate proof of Maclachlan’s parametrization of commensurability classes of even dimensional arithmetic hyperbolic spaces (see [Mac]). Furthermore, this dictionary is essential in the proof of Theorem B.

In Section 7 we discuss fields of definition and prove Theorem A in Section 7. Section 8 is devoted to technical constructions of quadratic subforms with specific behavior. It is these subforms which enable us to distinguish between the totally geodesic spectra coming from noncommensurable arithmetic locally symmetric spaces coming from quadratic forms. In Section 9 we draw upon the results of the previous sections to prove Theorem B and Theorem C. In Section 10 we apply our techniques to prove some results about totally geodesic subspaces of standard arithmetic hyperbolic manifolds including the Hyperbolic Subspace Dichotomy Theorem (Theorem D). In Section 11 we address the issue totally geodesic subspaces of nonstandard arithmetic hyperbolic spaces, and their generalizations, and prove Theorem E and Theorem F.

2. Notation and Preliminary Results:

**Commensurability, Totally Geodesic Subspaces, and Locally Symmetric Spaces**

In this paper $F$ is a field which is not of characteristic 2, $\overline{F}$ is a fixed algebraic closure of $F$. We consider number fields as finite extensions of $\mathbb{Q}$ lying within $\overline{\mathbb{Q}}$, $k$ will denote a number field, $\mathcal{O}_k$ its ring of integers, and $\operatorname{Aut}(k/\mathbb{Q})$ is the group of field automorphisms of $k$.

Two subgroups $\Gamma_1$ and $\Gamma_2$ of a group $G$ are **commensurable** if $\Gamma_1 \cap \Gamma_2$ is finite index in both $\Gamma_1$ and $\Gamma_2$. This is an equivalence relation among subgroups of $G$. Following [PR09], we shall say two subgroups $\Gamma_1, \Gamma_2$ of a $G$ are **commensurable up to $G$-automorphism** if there exists a $G$-automorphism $\varphi$ such that $\Gamma_1$ and $\varphi(\Gamma_2)$ are commensurable. Note that some authors refer to this as **commensurable in the wide sense** [MaRe, Def. 1.3.4]. Again it is not hard to see that commensurability up to $G$-automorphism is an equivalence relation among subgroups of $G$. Two spaces topological spaces are **commensurable** if they share a finite sheeted cover. In what follows, when talking about commensurability of Riemannian manifolds, we shall always mean commensurable up to isometry.

Let $M$ be a Riemannian manifold and let $N \subset M$ be a connected immersed submanifold. Recall that $N$ is **geodesic at** $p \in N$ if every geodesic of $M$ starting at $p$ and tangent to $N$ at $p$ is a geodesic of $N$. If $N$ is geodesic at each of its points it is called **totally geodesic**. It is well known that totally geodesic subspaces of hyperbolic space are also hyperbolic [DC, Ch. 8, pg. 180 Exer. 2].

Following [Th, Chp 13], we call the quotient of a manifold by a properly discontinuous (not necessarily free) group action a **good orbifold**. Since discrete subgroups of semisimple Lie groups often have torsion, good orbifolds naturally appear in the commensurability classes of locally symmetric manifolds. When a good orbifold is a quotient of a Riemannian manifold, then we shall call it a good Riemannian orbifold. Every good Riemannian orbifold naturally has a Riemannian manifold universal cover. A subspace of a Riemannian orbifold is then defined to be totally geodesic if it is the image of a totally geodesic subspace in its universal
cover. It follows immediately that the sets $\overline{TG(M)}$ and $\mathbb{Q}TG(M)$ (Definitions 1.3 and 1.4) make sense for all good Riemannian orbifolds.

**Lemma 2.1.** Commensurable good Riemannian orbifolds are totally-geodesic-commensurable.

**Proof.** Let $M_1$ and $M_2$ be commensurable and $\tilde{M}$ a shared finite sheeted cover with projections $\pi_1$ and $\pi_2$. Pick a nonflat finite volume totally geodesic subspaces $N_1 \subset M_1$. Then $N_2 := \pi_2(N')$, where $N'$ is a connected component of $\pi_1^{-1}(N_1)$, is a totally geodesic submanifold of $M_2$. Since we are dealing with finite covers, $N_2$ is also nonflat and of finite volume. By symmetry of argument, the result holds.

In general, totally geodesic subspaces are rare, and we should only expect to find such subspaces when we are considering an ambient space with many symmetries. As such, in what follows, we shall only consider locally symmetric spaces. Recall that a Riemannian manifold $M$ is a globally symmetric space if each point $p \in M$ is an isolated fixed point of an involutive isometry of $M$. Totally geodesic subspaces of a globally symmetric space are also globally symmetric [He, Ch. IV Prop 7.1]. One of the advantages to working with globally symmetric spaces is that questions about the spaces can be translated into questions about its isometry group. A globally symmetric space is of noncompact type if $G := \text{Isom}^0(M)$ is a semisimple Lie group with no compact factors, in which case $M$ is isometric to $G/K$ where $K$ is a maximal compact subgroup of $G$.

**Lemma 2.2.** Let $M$ a connected globally symmetric space of noncompact type, $G = \text{Isom}^0(M)$ and $K$ a stabilizer of a point $p_0 \in M$.

1. Let $H \subset G$ be a semisimple Lie subgroup with no compact factors. Then $N_H := H/(H \cap K)$ is a totally geodesic submanifold of $M$.

2. Let $N \subset M$ be a totally geodesic submanifold of noncompact type such that $p_0 \in N$. Then there exists a semisimple Lie subgroup $H_N \subset G$ with no compact factors such that $H_N/(H_N \cap K) = N$.

**Proof.**

1. Note that $N_H$ is an immersed submanifold of $M$. Geodesics of $M$ arise from the exponential map of $G$. Given an element $X \in \text{Lie}(H)$ we know that $\exp_G(tX) \in H$ for all $t \in \mathbb{R}$, and hence $N$ must be totally geodesic.

2. Let $\text{Lie}(G) = \mathfrak{t} \oplus \mathfrak{p}$ be the Cartan decomposition. Let $\mathfrak{s} \subset \mathfrak{p}$ be the subspace associated with the tangent space of $N$. Then $\mathfrak{t}$ acts on $\mathfrak{p}$ by the adjoint representation and let $\mathfrak{t}' = N_\mathfrak{t}(\mathfrak{s}) = \{X \in \mathfrak{t} \mid \text{ad}(X)(\mathfrak{s}) \subset \mathfrak{s}\}$. Then $\mathfrak{h} := \mathfrak{t}' \oplus \mathfrak{s}$ is a Lie subalgebra of $\text{Lie}(G)$. Let $H_N$ be the unique connected Lie subgroup of $G$ with Lie algebra $\mathfrak{h}$. It follows that $H_N$ has the desired properties.

A good Riemannian orbifold $M$ is a locally symmetric space if $M$ has universal cover $\tilde{M}$ which is a globally symmetric space. In which case $M = \Gamma \backslash \tilde{M}$ where $\Gamma$ is a discrete subgroup of $\text{Isom}^0(\tilde{M})$. Locally symmetric spaces include complete hyperbolic manifolds. A locally symmetric space is of noncompact type if its universal cover is a globally symmetric space of noncompact type. The study of locally symmetric spaces of noncompact type translates to the study of discrete subgroups of semisimple Lie groups with no compact factors, as we shall now record with the following well known proposition. It quickly follows that a totally geodesic subspace of a locally symmetric space is also locally symmetric.

**Proposition 2.3.** Let $M_1 = \Gamma_1 \backslash G_1/K_1$ and $M_2 = \Gamma_2 \backslash G_2/K_2$ be locally symmetric spaces of noncompact type where $G_1$ and $G_2$ are connected, adjoint, semisimple Lie groups with no compact factors. Then $M_1$ and $M_2$ are isometric if and only if there is a Lie group isomorphism $\varphi : G_1 \to G_2$ such that $\varphi(K_1) = K_2$ and $\varphi(\Gamma_1) = \Gamma_2$.

Since the image of a maximal compact (resp. discrete) subgroup under an automorphism is always a maximal compact (resp. discrete) subgroup, understanding isometry classes of locally symmetric spaces of noncompact type with universal cover $G/K$ reduces to understanding $\text{Aut}(G)$-orbits of discrete subgroups of $G$. In particular, understanding the commensurability classes of locally symmetric spaces (up to isometry) is equivalent to understanding the commensurability classes of discrete subgroups of $G$ up to $G$-automorphism.
Let $G$ be a semisimple Lie group and $\Gamma \subset G$ be a discrete subgroup. The Haar measure on $G$ naturally descends to a $G$-invariant measure on $\Gamma \backslash G$. When the Haar measure on $G$ descends to a measure of finite volume on $\Gamma \backslash G$, $\Gamma$ is called a lattice. When $\Gamma \backslash G$ is compact, $\Gamma$ is said to be cocompact or a uniform lattice. Cocompact discrete subgroups are always lattices. A lattice is irreducible if, up to commensurability, it is not a product of smaller lattices. Being cocompact, a lattice, or irreducible is an invariant of commensurability class.

Henceforth, all of our orbifolds will be good and all of our locally symmetric spaces shall be of noncompact type unless otherwise stated.

3. Arithmetic Groups and Arithmetic Locally Symmetric Spaces

We begin this section with a quick introduction to arithmetic groups and a few well known results which we will need later. We then discuss arithmetic locally symmetric spaces and state and prove results which are known, but as of writing this paper, we are unaware of references.

Arithmetic Subgroups of Algebraic $\mathbb{Q}$-Groups.

Let $G$ be an algebraic group defined over $\mathbb{Q}$. There exists a faithful $\mathbb{Q}$-rational embedding $\rho : G \to \text{GL}(V)$ for some $\mathbb{Q}$-vector space $V$ (B1 1.10). Let $L \subset V$ be a $\mathbb{Z}$-lattice of $V$, i.e., a free $\mathbb{Z}$-module such that $L \otimes_{\mathbb{Z}} \mathbb{Q} = V$. Define the group

$$G_{\rho,L} := \{ g \in G(\mathbb{Q}) \mid \rho(g)(L) = L \}.$$ 

Any subgroup $\Gamma \subset G(\mathbb{Q})$ commensurable with $G_{\rho,L}$ is an arithmetic subgroup of $G(\mathbb{Q})$. Were we to chose a different embedding, $\rho'$, and different $\mathbb{Z}$-lattice, $L'$, we would have obtained a different group $G_{\rho',L'}$, however, any such $G_{\rho',L'}$ is commensurable with $G_{\rho,L}$ (see [B2 7.12] and preceding discussion). It follows that the commensurability class of an arithmetic group is independent of the choices of $\rho$ and $L$. In other words, the $\mathbb{Q}$-isomorphism class of $G$ determines a commensurability class of arithmetic groups.

Often we will assume the existence of some embedding $\rho$ and lattice $L$, and we will denote $G(\mathbb{Z}) := G_{\rho,L}$. Note however that not all arithmetic groups arise as the stabilizer of a lattice. This can be seen from the fact that every lattice stabilizer contains a congruence subgroup [B2 7.12] but there are arithmetic groups which do not contain any congruence subgroups (for example, there are such groups in $\text{SL}_2(\mathbb{Z})$) [PR10 §2.1].

One way to construct algebraic $\mathbb{Q}$-groups is to start with a $k$-group, where $k$ is a number field, and then apply the Weil restriction of scalars functor $R_{k/\mathbb{Q}}$. ([PRa] §2.1.2, or [MaRe] §10.3)). This functor has the property that if $G$ is an algebraic $k$-group, then $R_{k/\mathbb{Q}}G$ is an algebraic $\mathbb{Q}$-group and there is an abstract group isomorphism between $G(k)$ and $(R_{k/\mathbb{Q}}G)(\mathbb{Q})$. With this identification, it makes sense to talk about arithmetic subgroups of $G(k)$. Furthermore, it is not hard to see that arithmetic subgroups of $G(k)$ are precisely the groups commensurable with the stabilizer of an $O_k$-lattice of a $k$-vector space $V$ where there is a $k$-rational embedding of $G$ into $\text{GL}(V)$.

An absolutely (resp. absolutely almost) simple algebraic $F$-group is an algebraic $F$-group that, upon extending scalars to $\overline{F}$, is (resp. isogenous to) a simple semisimple algebraic $\overline{F}$-group. For example the $\mathbb{C}$-group $\text{SL}_n$ is absolutely almost simple but not absolutely simple since it has nontrivial central equal to the group of $n^{th}$ roots of unity. The semisimple $\mathbb{R}$-group $R_{\mathbb{C}/\mathbb{R}}\text{SL}_2$, which is related to the study of hyperbolic 3-manifolds, is not absolutely almost simple, it is $\mathbb{C}$-isomorphic to $\text{SL}_2 \times \text{SL}_2$. If we start with an absolutely almost simple $k$-group, then $R_{k/\mathbb{Q}}(G)$ is always a semisimple $\mathbb{Q}$-group. An $F$-simple $F$-group is an algebraic $F$-group which, up to isogeny, does not contain a proper nontrivial normal $F$-subgroup. Absolutely almost simple $F$-groups are $F$-simple and $R_{\mathbb{C}/\mathbb{R}}\text{SL}_2$ is $\mathbb{R}$-simple. All semisimple $k$-groups are built from absolutely almost simple groups over number fields [BoTi 6.21(ii)] and [CGP Prop. A.5.14]. For the reader’s convenience, we record a corollary of [CGP Prop. A.5.14] which will be useful in what follows.

**Proposition 3.1.** Let $G$ be a semisimple $k$-simple $k$-group.
(1) (Existence) There exists a number field $k'$ containing $k$ and an absolutely almost simple $k'$-group $H'$ such that $G$ and $R_{k'/k}(H')$ are $k$-isogenous. Furthermore, if $G$ is adjoint, $G$ and $R_{k'/k}(H')$ are $k$-isomorphic.

(2) (Uniqueness) The pair $(H', k')$ is unique in the following sense: If $k''$ is a number field containing $k$, and $H''$ an absolutely almost simple $k''$-group such that $G$ and $R_{k''/k}(H'')$ are $k$-isogenous, then $k'$ and $k''$ are $\text{Gal}(\overline{k}/k)$-conjugate and there is an isogeny between $H$ and $H'$ defined over the Galois closure of $k'$.

We shall primarily be concerned with the case when $k = \mathbb{Q}$. Lastly, we recall a fundamental result on how arithmetic groups of a $\mathbb{Q}$-group $G$ behave inside the Lie group $G(\mathbb{R})$. Since $\mathbb{Z}$ is discrete in $\mathbb{R}$, it is not hard to show that an arithmetic subgroup $\Gamma \subset G(\mathbb{R})$ is discrete. The following result determines whether an arithmetic group is a lattice or cocompact.

**Theorem 3.2.** Let $G$ be a connected semisimple algebraic $\mathbb{Q}$-group and $\Gamma \subset G(\mathbb{Q})$ be an arithmetic group. Then

1. $\Gamma$ is a lattice in $G(\mathbb{R})$. \([\text{BoHC}] 7.8\)
2. $\Gamma$ is cocompact in $G(\mathbb{R})$ if and only if $G$ is $\mathbb{Q}$-anisotropic. \([\text{BoHC}] 11.8\) \([\text{MT}] 1.4\)

**Arithmetic Lattices in Semisimple Lie Groups.** Let $\mathcal{G}$ be a connected, adjoint, semisimple Lie group with no compact factors. Let $\Gamma \subset \mathcal{G}$ be a lattice.

Then $\Gamma$ is arithmetic if there exists a semisimple algebraic $\mathbb{Q}$-group $G$ and a surjective analytic homomorphism $\pi : G(\mathbb{R})^0 \to \mathcal{G}$ with compact kernel such that $\pi(G(\mathbb{Z}) \cap G(\mathbb{R})^0)$ and $\Gamma$ are commensurable up to $\mathcal{G}$-automorphism.

\[
\begin{array}{ccc}
G(\mathbb{Z}) \cap G(\mathbb{R})^0 & \xrightarrow{\pi} & G(\mathbb{R})^0 \xrightarrow{\pi} G(\mathbb{R}) \\
\xrightarrow{\pi} & G(\mathbb{Z}) \cap G(\mathbb{R})^0 & \xrightarrow{\sim} \varphi(\Gamma) \xrightarrow{\pi} \mathcal{G}
\end{array}
\]

In what follows, we shall say that $G$ gives rise to $\Gamma$. If $H \subset G$ is a $\mathbb{Q}$-simple factor, we may and will always assume that it is $\mathbb{R}$-isotropic, since otherwise $H(\mathbb{R})^0 \subset \ker(\pi)$, and we may just replace $G$ with $G/H$. It is not hard to see that if $\Gamma, \Gamma' \subset \mathcal{G}$ are two subgroups which are commensurable up to $\mathcal{G}$-automorphism and one in an arithmetic lattice, then so is the other.

It may appear as though arithmetic lattices are rather specific and potentially rare type of lattice. However, thanks to Margulis's arithmeticity theorem \([\text{Mar}]\) and the work of Gromov and Schoen \([\text{GS}]\), irreducible lattices in groups not locally isomorphic to $\text{SO}(n, 1)$ or $\text{SU}(n, 1)$ are always arithmetic. In particular, irreducible lattices in $\mathbb{R}$-rank 2 and higher are all automatically arithmetic.

**Arithmetic Locally Symmetric Spaces.** In what follows we shall adopt the following notation:

- $\mathcal{G}$ is a connected, adjoint, semisimple Lie group with no compact factors,
- $\mathcal{K} \subset \mathcal{G}$ is a maximal compact subgroup,
- $G$ is a semisimple algebraic $\mathbb{Q}$-group with no $\mathbb{R}$-anisotropic $\mathbb{Q}$-simple factors,
- $G(\mathbb{Z}) \subset G(\mathbb{Q})$ is the lattice stabilizer $G_{\rho, L}$ for some choice of $\rho$ and $L$,
- $\pi$ is projection $\pi : G(\mathbb{R})^0 \to \mathcal{G}$ with compact kernel,
- $\Gamma \subset \mathcal{G}$ is a subgroup commensurable up to $\mathcal{G}$-automorphism to $\pi(G(\mathbb{Z}) \cap G(\mathbb{R})^0)$,
- $\varphi \in \text{Aut}(\mathcal{G})$ is such that $\pi(G(\mathbb{Z}) \cap G(\mathbb{R})^0)$ and $\varphi(\Gamma)$ are commensurable,
- $K \subset G(\mathbb{R})$ is a maximal compact subgroup containing $\pi^{-1}(\varphi(\mathcal{K}))$. 

Given this data, we may define an arithmetic locally symmetric space (of noncompact type)\(^1\) to be a space \(M\) of the form \(\Gamma \backslash G/K\). When \(\Gamma\) is torsion-free, \(M\) is a Riemannian manifold, and since every \(\Gamma\) has a finite index torsion-free subgroup by Selberg’s Lemma [Sel], \(M\) is always a good Riemannian orbifold, in the sense of Thurston [Th Chp. 13].

The following two theorems are known, but as of the writing of this paper, we are unaware of references. As such, we provide proofs here.

**Theorem 3.3.** Let \(M\) be an arithmetic locally symmetric space and let \(N \subset M\) be a nonflat finite volume totally geodesic subspace. Then \(N\) is arithmetic.

**Proof.** By Lemma 2.2, there exists a connected, semisimple Lie subgroup \(G\) with no compact factors such that \(N = \overline{H}/(K \cap \overline{H})\) where \(\overline{H} := \Gamma \cap \overline{H}\) is a lattice in \(\overline{H}\). Let \(H\) denote the connected component of the intersection of \(\pi^{-1}(\varphi^{-1}(\overline{H}))\) with the noncompact factors of \(G(R)\). (This group can also be viewed as the unique connected Lie subgroup of \(G(R)\) with Lie algebra \(\text{Lie}(\varphi^{-1}(\overline{H}))\).) It follows that \(N' := \Lambda \backslash H/(K^0 \cap H)\), where \(\Lambda := G(Z) \cap H\), is commensurable with \(N\). Arithmeticality is an invariant of commensurability class so it suffices to show the arithmeticality of \(N'\). The result then follows by Proposition 3.4 below. \(\square\)

**Proposition 3.4.** Let

1. \(G\) be a semisimple \(Q\)-group,
2. \(H \subset G(R)\) be a connected semisimple Lie subgroup with no compact factors, and
3. \(\Delta \subset G(Z)\) be a subgroup which is also a lattice in \(H\).

Then \(H = H(R)^0\) where \(H \subset G\) is a semisimple \(Q\)-subgroup and \(\Delta \subset H(Q)\) is arithmetic.

**Proof.** Since \(H\) is a semisimple Lie group sitting inside the real points of a linear group, \(H\) is the connected component of the real points of some semisimple \(R\)-subgroup \(H \subset G\). By Borel’s Density Theorem [B60] \(\Lambda\) is Zariski dense in \(H\). The Zariski closure of an abstract subgroup sitting inside the \(Q\)-points of a group is also a \(Q\)-group [B1 Chp 1 Prop 1.3(b)]. Hence \(H\) is defined over \(Q\). Furthermore, let \(V := \text{Lie}(G)\) and \(W := \text{Lie}(H)\). The adjoint representation \(\text{Ad} : G \to \text{GL}(V)\) is defined over \(Q\). There is a lattice \(L \subset V\) which \(\Gamma\) stabilizes [B2 Prop 7.12]. Since \(\Lambda\) stabilizes \(W\), it stabilizes \(L \cap W\) and hence \(\Lambda\) is an arithmetic subgroup of \(H\). \(\square\)

**Theorem 3.5.** Let \(M_1\) and \(M_2\) be finite volume arithmetic locally symmetric spaces arising from the semisimple \(Q\)-groups \(G_1\) and \(G_2\) respectively. Then \(M_1\) and \(M_2\) are commensurable if and only if \(G_1\) and \(G_2\) are \(Q\)-isogenous.

**Proof.** First suppose \(G_1\) and \(G_2\) are \(Q\)-isogenous. Then \(\text{Ad}_{G_1}(G_1)\) and \(\text{Ad}_{G_2}(G_2)\) are \(Q\)-isomorphic via some \(Q\)-isomorphism \(\psi\). It quickly follows that \(M_i\) is commensurable with \(\text{Ad}_{G_i}(G_i(Z)) \backslash \text{Ad}_{G_i}(G_i(R)) \backslash \text{Ad}_{G_i}(K_i)\). The result then immediately follows from the fact that \(\psi(\text{Ad}_{G_1}(G_1(Z)))\) and \(\text{Ad}_{G_2}(G_2(Z))\) are commensurable [B2 Cor 7.13(2)].

Now suppose \(M_1\) and \(M_2\) are commensurable. By assumption, there exists a connected adjoint semisimple Lie group with no compact factors, \(\mathcal{G}\), and two arithmetic lattices \(\Gamma_1, \Gamma_2 \subset \mathcal{G}\) which are commensurable up to \(\mathcal{G}\)-automorphism, say \(\psi\), such that \(M_1 = \Gamma_1 \backslash \mathcal{G}/K\) and \(M_2 = \Gamma_2 \backslash \mathcal{G}/\psi(K)\) where \(K\) is a maximal compact subgroup. Replacing \(G_i\) with \(\text{Ad}_{G_i}(G_i)\), the result then follows from Proposition 3.4 below. \(\square\)

**Proposition 3.6.** Let \(\mathcal{G}\) be a connected adjoint semisimple Lie group with no compact factors. Let \(\Gamma_1, \Gamma_2 \subset \mathcal{G}\) be arithmetic lattices which are commensurable up to \(\mathcal{G}\)-automorphism. Let \(G_1\) and \(G_2\) be the connected adjoint semisimple \(Q\)-groups with no \(R\)-anisotropic \(Q\)-simple factors giving rise to \(\Gamma_1\) and \(\Gamma_2\) respectively. Then \(G_1\) and \(G_2\) are \(Q\)-isomorphic.

\(^1\)In the literature, these are sometimes also called arithmetically defined locally symmetric spaces. Furthermore, since all locally symmetric spaces in this paper are of noncompact type, we shall omit these words from here on out.
Proof. Let \( \psi \) be an analytic automorphism of \( \overline{G} \) for which \( \Gamma_1 \) and \( \psi(\Gamma_2) \) are commensurable. Let \( H_i \subset G_i \) be the product of the connected \( \mathbb{R} \)-simple \( \mathbb{R} \)-isotropic components of \( G_i \). Then \( \pi_i|_{H_i(\mathbb{R})^0} : H_i(\mathbb{R})^0 \to \overline{G} \) is an isomorphism. Picking sufficiently small finite index \( \Gamma'_i \subset \Gamma_i \) which are isomorphic via \( \psi \), we may identify \( \varphi_i^{-1}(\Gamma_i) \) with a finite index subgroup \( \Lambda_i := \pi_i^{-1}(\overline{H}_i(\mathbb{R})^0)(\varphi_i^{-1}(\Gamma_i)) \subset H_i(\mathbb{R})^0 \cap G_i(\mathbb{Q}) \).

Since \( \pi_i \) induces an \( \mathbb{R} \)-rational isomorphism between \( H_i \) and \( \text{Aut}(\text{Lie}(\overline{G}) \otimes \mathbb{R} \mathbb{C}) \), and \( \psi \) induces an \( \mathbb{R} \)-rational automorphism on \( \text{Aut}(\text{Lie}(\overline{G}) \otimes \mathbb{R} \mathbb{C}) \), it follows that there is an \( \mathbb{R} \)-rational isomorphism, which we also denote \( \psi_i \), from \( H_1_i \) to \( H_2 \) which sends \( \Lambda_1 \) to \( \Lambda_2 \).

For each \( i \), by \cite{BoLi} 6.21 (ii), \( G_i \cong \prod_{j=1}^{r_i} R_{k_{i,j}}/\mathbb{Q} S_{i,j} \) where \( S_j \) is an absolutely simple group over a number field \( k_{i,j} \). Furthermore, \( \Lambda_{i,j} := \Lambda_i \cap (R_{k_{i,j}}/\mathbb{Q}(S_{i,j}))((\mathbb{Q})) \) is an arithmetic group in \( (R_{k_{i,j}}/\mathbb{Q}(S_{i,j}))((\mathbb{Q})) = S_{i,j}(k_{i,j}) \) \cite{BoLi} 6.11. Borel’s Density Theorem \cite{B65} implies that \( \Lambda_{i,j} \) is Zariski dense in \( S_{i,j} \). Since each \( \Lambda_{i,j} \) is a normal subgroup of \( \Lambda_i \) and an irreducible lattice in \( (R_{k_{i,j}}/\mathbb{Q}(S_{i,j}))((\mathbb{R})) \), the isomorphism \( \psi \) must send each \( \Lambda_{1,j} \) to some \( \Lambda_{2,j'} \), from which we conclude \( r_1 = r_2 := r \) and \( \psi \) induces a permutation also denoted \( \psi \in S_r \).

Our assumption on \( \mathbb{Q} \)-simple factors implies that each \( R_{k_{i,j}}/\mathbb{Q}(S_{i,j}) \) contains an \( \mathbb{R} \)-simple \( \mathbb{R} \)-isotropic factor. Since \( \psi \) sends \( \mathbb{R} \)-isotropic \( \mathbb{R} \)-simple factors of \( R_{k_{i,j}}/\mathbb{Q}(S_{i,j}) \) to \( \mathbb{R} \)-isotropic \( \mathbb{R} \)-simple factors of \( R_{k_2,\psi(j)}/\mathbb{Q}(S_{2,\psi(j)}) \), we conclude \( S_{2,j} \) and \( S_{2,\psi(j)} \) have the same Cartan-Killing type. Let \( H_{i,j} \) be a fixed \( \mathbb{R} \)-simple \( \mathbb{R} \)-isotropic component of \( R_{k_{i,j}}/\mathbb{Q}(S_{i,j}) \). Then \( \psi \) induces an \( F \)-isomorphism between \( S_{1,j} \) and \( S_{2,\psi(j)} \), where \( F = \mathbb{R} \) when \( H_{1,j} \) is absolutely simple, and \( F = \mathbb{C} \) otherwise. Furthermore, this isomorphism sends \( \Lambda_{1,j} \) to \( \Lambda_{2,\psi(j)} \), hence by \cite{PR} Prop 2.5, \( k_{1,j} \) and \( k_{2,\psi(j)} \) are \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-conjugate and, letting \( k_j \) denote this isomorphism class, \( S_{1,j} \) and \( S_{2,\psi(j)} \) are \( k_j \)-isomorphic. The conclusion follows. \( \square \)

Fix a number field \( k \) and let \( G \) be an absolutely almost simple \( k \)-group. The following result, which will be essential in the rest of the paper, shows that the \( k \)-isomorphism class of \( G \), up to the action of \( \text{Aut}(k/\mathbb{Q}) \), determines a commensurability class of spaces.

**Corollary 3.7.** Let \( M_1 \) and \( M_2 \) be finite volume arithmetic locally symmetric spaces arising from the absolutely almost simple \( k \)-groups \( G_1 \) and \( G_2 \) respectively. Then \( M_1 \) and \( M_2 \) are commensurable if and only if the \( k \)-isogenous up to the action of \( \text{Aut}(k/\mathbb{Q}) \).

**Proof.** Suppose \( M_1 \) and \( M_2 \) are commensurable and replace \( G_i \) with their adjoint groups. Then by Theorem \cite{B3.5} there is a \( \mathbb{Q} \)-isomorphism \( \psi : R_{k/\mathbb{Q}}(G_1) \to R_{k/\mathbb{Q}}(G_2) \). By \cite{CGP} Prop A.5.14, \( \psi \) arises from a field isomorphism \( \alpha \in \text{Aut}(k/\mathbb{Q}) \) and a group isomorphism \( G_1 \to G_2 \) defined over \( \text{Spec}(\alpha) \). The result then follows. Now suppose \( G_1 \) and \( G_2 \) are \( k \)-isogenous up to the action of \( \text{Aut}(k/\mathbb{Q}) \). The group \( \text{Aut}(k/\mathbb{Q}) \) permutes the infinite places of \( k \). Since permuting the factors of a \( \mathbb{Q} \)-simple \( \mathbb{Q} \)-group gives a \( \mathbb{Q} \)-isomorphism, \( \text{Aut}(k/\mathbb{Q}) \) acts on \( R_{k/\mathbb{Q}}(G) \) by \( \mathbb{Q} \)-isomorphisms. The result then follows by Theorem \cite{B3.5} \( \square \)

4. Arithmetic Locally Symmetric Spaces Arising From Quadratic Forms

In this section we discuss the theory of quadratic forms and the results we need to construct and analyze arithmetic locally symmetric spaces coming from quadratic forms. For a complete treatment of the classical theory of quadratic forms over local and global fields, we refer the reader to \cite{OM, Sch, Lam}.

Recall \( F \) is a field which is not of characteristic 2. In what follows, \( (V,q) \) will denote a quadratic space over \( F \) where \( V \) is a finite dimensional vector space over \( F \) and \( q \) is a quadratic form on \( V \). When it will not cause confusion, we will omit \( V \) and simply refer to the quadratic form \( q \). We shall say \( q \) is a quadratic form over \( F \), or more succinctly, \( q \) is a quadratic \( F \)-form. If \( E/F \) is a field extension then \( (V,q) \) determines a quadratic space \( (V_E,q_{E}) \) over \( E \) by extending scalars (i.e., where \( V_E := V \otimes_F E \) and \( q_{E} \) is the extension of \( q \) to \( V_{E} \)). When it will not cause confusion, we will sometimes denote the extended form by the symbol \( q \) as well. Every quadratic space \( (V,q) \) determines an algebraic \( F \)-group, \( \text{SO}(V,q) \) whose \( E \) points are given by

\[
\text{SO}(V,q)(E) = \{ T \in \text{SL}(V_E) \mid q_{E}(Tv) = q_{E}(v) \text{ for all } v \in V_E \}.
\]
Definition 4.1. Let \((V_1, q_1)\) and \((V_2, q_2)\) be quadratic spaces over \(F\). Then \(q_1\) and \(q_2\) are

1. **isometric** if there some \(F\)-linear isomorphism \(T: V_1 \to V_2\) such that \(q_2(Tv) = q_1(v)\) for all \(v \in V_1\).
2. **similar** if there exists some \(a \in F^\times\) such that \(q_1\) and \(aq_2\) are isometric.
3. **isogroupic** if \(\text{SO}(q_1)\) and \(\text{SO}(q_2)\) are isomorphic as algebraic \(F\)-groups.

The first two definitions are standard, while the third we introduce in the paper. It is not hard to see that each of these determine an equivalence relation among quadratic \(F\)-forms. Furthermore, the following lemma begins to shows how they are related.

Lemma 4.2.

1. **Isometric forms are isogroupic.**
2. **Similar forms are isogroupic.**

Proof.

(1) Let \((V_1, q_1)\) and \((V_2, q_2)\) be isometric forms. By assumption there exists an \(F\)-linear isomorphism \(T: V_1 \to V_2\) preserving the forms. Then \(T\) induces an \(F\)-isomorphism \(T_*: \text{SL}(V_1) \to \text{SL}(V_2)\) via \(g \mapsto TgT^{-1}\). Upon restricting to \(\text{SO}(V_1, q_1)\), for any \(v \in V_2\), we have

\[
q_2(T_*(g)v) = q_2(TgT^{-1}v) = q_2(T(g(T^{-1}v))) = q_1(g(T^{-1}v)) = q_1(T^{-1}v) = q_2(v).
\]

Hence \(T_*(\text{SO}(V_1, q_1)) \subset \text{SO}(V_2, q_2)\) and by symmetry of argument it follows they are \(F\)-isomorphic.

(2) Let \((V_1, q_1)\) and \((V_2, q_2)\) be similar forms. By assumption there exists \(a \in F^\times\) such that \(aq_1\) and \(q_2\) are isometric. By part (1), it suffices to show that \(aq_1\) and \(q_1\) are isogroupic. Pick \(g \in \text{SO}(V_1, q_1)\) and \(v \in V_1\), then

\[
(aq_1)(gv) = a(q_1(gv)) = a(q_1(v)) = (aq_1)(v).
\]

Therefore \(g \in \text{SO}(V_1, aq_1)\), and by symmetry of argument, \(\text{SO}(V_1, q_1) = \text{SO}(V_1, aq_1)\). The result follows.  

In general there are many isometry classes in a given isogroupy class. If \(G := \text{SO}(q)\), then any \(q'\) in the isogroupy class of \(q\) shall be said to **represent** \(G\). A quadratic form \(r\) is a **subform** of a quadratic form \(q\) if there is some third form \(t\) such that \(r \oplus t\) is isometric to \(q\). We say a symmetric bilinear form \(b\) is **nondegenerate** when \(b(v, w) = 0\) for all \(w \in V\) implies that \(v = 0\). A quadratic form corresponding to a nondegenerate symmetric bilinear form is said to be **regular**. In this paper, all quadratic forms will be assumed to be regular unless explicitly stated otherwise. The **dimension** of \(q\), denoted \(\dim q\), is the dimension of its associated vector space. When possible, we shall reserve the symbol \(m\) to denote the dimension of \(q\). Upon choosing a basis, every quadratic form may be represented by an \(m \times m\) matrix. The **determinant** of \(q\), denoted \(\det q\), is the determinant of some \(Q \in \text{GL}_m(F)\) representing \(q\). Note however that since this should be be independent of the choice of basis and \(\det(TQT) = \det Q(\det T)^2\), the determinant is only well defined up to square class of \(F\), and hence we view \(\det q \in F^\times/(F^\times)^2\). Though the determinant is a square class, we will often omit the \((F^\times)^2\) and write \(\det q = a\) as opposed to \(\det q = a(F^\times)^2\), where \(a \in F^\times\). A common renormalization of the determinant is the **discriminant**, denoted \(\text{disc}(q)\), where \(\text{disc}(q) = (-1)^{\dim q/(\dim(q)-1)/2} \det(q)\). It contains the same information as the determinant if one knows the dimension, but often results in simpler expressions.

Let \(a, b \in F^\times\). Then the **Hilbert symbol** \((a, b)_F = (a, b)_F\) denotes the isomorphism class of the quaternion algebra defined by \(F[i, j]\) such that \(i^2 = a, j^2 = b,\) and \(ij = -ji\). When the field \(F\) is understood, we simply write \((a, b)\). The Hilbert symbol satisfies algebraic properties which may be found in [M, Chp III, Thm 4.4]. For the reader’s convenience, we list a few here:

1. **Defined up to square class:** \((a, bc^2) = (a, b)\)
2. **Symmetry:** \((a, b) = (b, a)\)
3. **Multiplicativity:** \((a_1a_2, b) = (a_1, b)(a_2, b)\)
4. **Nondegeneracy:** For \(a \in F^\times\) not a square, there exists \(b \in F^\times\) such that \((a, b) \neq 1\)
These properties can be more succinctly stated by saying that the Hilbert symbol is a symmetric nondegenerate bimultiplicative map from $F^\times/(F^\times)^2 \times F^\times/(F^\times)^2$ to $\text{Br}(F)$, the Brauer group of $F$.

Every isometry class of quadratic form can be represented by a diagonal matrix \[\text{Lam} \text{ I.2.4}\]. Choosing such a representation we write $q = \langle a_1, a_2, \ldots, a_m \rangle$ where the associated diagonal matrix is $\text{diag}(a_1, a_2, \ldots, a_m)$. Given a quadratic form $q = \langle a_1, a_2, \ldots, a_m \rangle$, we define the Hasse–Minkowski invariant\(^2\) $c(q)$ by

$$c(q) := \begin{cases} \prod_{i<j}(a_i, a_j) & \text{if } m \geq 2, \text{ and} \\ 1 & \text{if } m = 1. \end{cases}$$

A consequence of the definition and properties of the Hilbert symbol, the Hasse–Minkowski invariant satisfies the following product formula:

$$c(q_1 \oplus q_2) = c(q_1)c(q_2)(\det q_1, \det q_2).$$

The Hasse–Minkowski invariant is independent of the choice of isometry class representative and hence a well defined invariant of the isometry class of $q$ \[\text{Lam} \text{ V.3.8}\]. While $c(q)$ is not an invariant of the similarity class of $q$, we now show that there is a simple relationship between $c(q)$ and $c(\lambda q)$, for $\lambda \in F^\times$.

**Lemma 4.3.** Let $F$ be a any field not of characteristic 2, let $q$ be a quadratic form over $F$ of dimension $m$ and let $\lambda \in F^\times$. Then

$$c(\lambda q) = \left(\lambda, (-1)^{m(m-1)/2}(\det q)^{m-1}\right) c(q).$$

In particular this reduces to

$$c(\lambda q) = \begin{cases} (\lambda, \text{disc}(q)) c(q) & \text{when } m \text{ is even,} \\ (\lambda, (-1)^{m-1}) c(q) & \text{when } m \text{ is odd.} \end{cases}$$

**Proof.** A direct computation gives:

$$c(\lambda q) = \prod_{i<j}(\lambda a_i, \lambda a_j)$$

$$= \prod_{i<j}(\lambda, \lambda)(\lambda, a_i)(\lambda, a_j)(a_i, a_j)$$

$$= (\lambda, -1)^{m(m-1)/2}(\lambda, \det q)^{m-1} c(q)$$

$$= \left(\lambda, (-1)^{m(m-1)/2}(\det q)^{m-1}\right) c(q).$$

The reduction when $m$ is even and odd immediately follows. \(\square\)

The extent to which the Hasse–Minkowski invariant varies within an isogroupy class will be explored in Section 5. In general the Hasse–Minkowski invariant is difficult to compute, however, when $F$ is a nonarchimedean local field or $\mathbb{R}$, then $c(q)$ can only take values $\pm 1$, and over $\mathbb{C}$, $c(q)$ is identically 1.

When $F$ is an ordered field (e.g., $\mathbb{R}$), every isometry class of quadratic $F$-forms, can be diagonally represented with the first $m_+$ terms positive and the remaining $m_- := m - m_+$ terms negative. The signature\(^3\) of $q$ is the pair $\text{sgn}(q) := (m_+, m_-)$. Again, this value is independent of the choice of isometry class representative of $q$ and hence a well defined invariant of the isometry class of $q$. It is not hard to see that the unordered pair $\{m_+, m_-\}$ is an invariant of the similarity class of $q$.

\(^2\)Unfortunately there is a lack of uniformity in the literature when it comes to this invariant. In some texts and papers, this invariant is simply referred to as the Hasse invariant. Additionally some authors use the symbol $s(q)$ instead of $c(q)$. Lastly some references call $\prod_{i<j}(a_i, a_j) = c(q)(-1, \det q)$ the Hasse invariant.

\(^3\)Some authors define to the signature of $q$ to be the number $s = m_+ - m_-$. Observe that the two pairs $(m, s)$ and $(m_+, m_-)$ contain equivalent information.
These invariants totally determine the isometry classes of quadratic forms over local and global fields. Recall that local fields are \( \mathbb{C}, \mathbb{R}, \) or \( L \), an nonarchimedean local field and global fields are number fields or function fields in one variable over a finite field. For the reader’s convenience, we collect and record the uniqueness and existence theorems for quadratic forms over local and global fields. These will be absolutely essential in our analysis in later sections.

**Theorem 4.4** (Local Uniqueness). Let \( F \) be a local field and \( q \) and \( q' \) be quadratic \( F \)-forms. Then \( q \) and \( q' \) are isometric if and only if

1. When \( F = \mathbb{C} \), \( \dim q = \dim q' \).
2. When \( F = \mathbb{R} \), \( \dim q = \dim q' \) and \( \text{sgn}(q) = \text{sgn}(q') \).
3. When \( F = L \), \( \dim q = \dim q' \), \( \det(q) = \det(q') \), and \( c(q) = c(q') \).

**Theorem 4.5** (Local Existence).

1. For each \( m \in \mathbb{Z}_{\geq 1} \), there exists a quadratic \( \mathbb{C} \)-form \( q \) such that
   \[ \dim q = m. \]
2. For each pair \( (m_+, m_-) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \), there exists a quadratic \( \mathbb{R} \)-form \( q \) such that
   \[ \dim q = m := m_+ + m_- \quad \text{and} \quad \text{sgn}(q) = (m_+, m_-). \]
3. For each triple \( (m, d, c) \in \mathbb{Z}_{\geq 1} \times L^*(L^*)^2 \times \{\pm 1\} \), there exists a quadratic \( L \)-form \( q \) such that
   \[ \dim q = m, \quad \text{det} q = d \quad \text{and} \quad c(q) = c, \]
   (*) with the exception that \( c = 1 \) when either \( m = 1 \) or \( m = 2 \) and \( d = -1 \).

While the exceptional restrictions (*) on the Hasse–Minkowski invariant in dimensions \( m = 1 \) and \( m = 2 \) may seem inconsequential, they will play an integral role in the construction of subforms later in the paper. The proofs of the results over \( \mathbb{C} \) are straightforward. The proofs for the results over \( \mathbb{R} \) date back to Sylvester [Sy]. For proofs of the results over \( L \) we refer the reader to [OM VI.63:23]. Note that the statement in the text looks different because [OM] uses the alternate definition of the Hasse–Minkowski Invariant mentioned above and hence the values for this invariant differ by a factor of \((-1, \det q)\).

**Theorem 4.6** (Local-to-Global Uniqueness). Let \( k \) be a global field and \( q \) and \( q' \) be quadratic \( k \)-forms. Then \( q \cong q' \) if and only if \( q \otimes k_v \cong q' \otimes k_v \) for all \( v \in V_k \).

**Theorem 4.7** (Local-to-Global Existence). Let \( k \) be a number field and let

- \( m \in \mathbb{Z}_{\geq 1} \),
- \( d \in k^*/(k^*)^2 \), and
- \( S \subset V_k \) be a finite subset of even cardinality.

For each family \( \{q_v\}_{v \in V_k} \) where \( q_v \) is a quadratic form over \( k_v \) satisfying

- \( \dim q_v = m \),
- \( \text{det} q_v = d \), and
- \( c_v(q_v) = -1 \) if and only if \( v \in S \),
there exists a quadratic form \( q \) over \( k \) such that \( q \otimes k_v = q_v \) for all \( v \in V_k \).

For proofs of these two local-to-global theorems, we refer the reader to [OM VI.66:4] and [OM VII.72:1] respectively. We now move on to discuss how quadratic forms over number fields may be used to explicitly construct irreducible arithmetic lattices of semisimple Lie groups of the form

\[ G = \prod_{i=1}^r \text{SO}(p_i, m - p_i) \times (\text{SO}_m(\mathbb{C}))^s. \]
These lattices arise from the $\mathbb{Q}$-simple groups that are formed by applying the restriction of scalars functor to the isometry group of an $m$-dimensional quadratic form over a number field. Over the years, these lattices have been given many different names: “standard” [Lub97], “lattice of the simplest type” [Vin93], or just most descriptively, “coming from quadratic forms” [Me]. We shall use the terminology “standard” when convenient, or otherwise we shall say explicitly “coming from quadratic forms.”

**Construction 4.8.**

1. Let $k$ be a number field with infinite places $V_k^\infty$.
2. Let $(V,q)$ an $m$-dimensional quadratic $k$-space, and for $v \in V_k^\infty$, let $V_{k_v} := V \otimes_k k_v$ and $q_v := q \otimes k_v$.
3. Let $(m_+^{(v)}, m_-^{(v)})$ denote the signature of $(V_{k_v}, q_v)$ for each real $v \in V_k^\infty$.
4. Let $G := SO(V, q)$ be the absolutely almost simple $k$-group defined by $(V, q)$ and let $SO(q) := G(k)$.
5. Let $G_v$ denote the algebraic $k_v$-group $SO(V_{k_v}, q_v)$ for each $v \in V_k^\infty$.
   - If $v$ is real, then $G_v(\mathbb{R}) \cong SO(m_+^{(v)}, m_-^{(v)})$.
   - If $v$ is complex, then $G_v(\mathbb{C}) \cong SO_{m_v}(\mathbb{C})$.
6. Let $G' := R_k/\mathbb{Q}G$ be the semisimple $\mathbb{Q}$-group formed by restriction of scalars. Then $G'(\mathbb{R}) = \prod G_v(\mathbb{R})$ is a semisimple Lie group which has compact factors at precisely the real places where $q$ is anisotropic. By the construction of restriction of scalars, there is an isomorphism $G(k) \cong G'(\mathbb{Q})$. Hence there is a natural diagonal embedding $SO(q) \to G'(\mathbb{R})$.
7. Let $G$ be the projection of $G'(\mathbb{R})$ onto its noncompact factors and denote the projection map by $\pi : G'(\mathbb{R}) \to G$.
8. Fix an $\mathcal{O}_k$-lattice $L \subset V$ and let $G_L = \{ T \in G(k) \mid T(L) \subset L \}$. Then $G_L$ sits as a discrete arithmetic subgroup of the semisimple Lie group $G'(\mathbb{R})$.
9. Let $\Gamma \subset G$ be commensurable up to $G$-automorphism with $\pi(G_L)$. Then $\Gamma$ is a standard arithmetic lattice of $G$.
10. We have the following diagram illustrating our construction of irreducible arithmetic lattices in $G$ where we use the notation:
    - $r$ is the number of real places where $q$ is isotropic,
    - $s$ is the number of complex places,
    - $p_i = m_+^{(v_i)}$ where $\{v_1, \ldots, v_r\}$ is the set of real places where $q$ is isotropic

\[
\begin{array}{ccc}
G_L & \xrightarrow{\text{diagonal}} & \left( \prod_{v \text{ real} \text{ anisotropic}} SO(m_v) \times \prod_{q \text{ isotropic}} SO(m_+^{(v)}, m_-^{(v)}) \times \prod_{v \text{ complex}} SO_{m_v}(\mathbb{C}) \right) \\
& & \downarrow \pi \\
\Gamma \xleftarrow{\text{Commensurable (up to $G$-automorphism) with $\pi(G_L)$}} & \prod_{i=1}^r SO(p_i, m - p_i) \times (SO_{m_v}(\mathbb{C}))^s
\end{array}
\]

11. Let $K \subset G$ its maximal compact subgroup and let $M_\Gamma := \Gamma \backslash G/K$. This space $M_\Gamma$ is an arithmetic locally symmetric space coming from a quadratic form. We call $M_\Gamma$ simple if $G$ is simple as a Lie group (i.e., $r + s = 1$).
A choice of another $O_k$-lattice $L' \subset V$ and $\Gamma'$ commensurable up to $G$-automorphism with $\pi(G_{L'})$ will produce a space $M_{\Gamma'}$ which is commensurable with $M_{\Gamma}$. Hence choosing $q$ uniquely determines a commensurability (up to isometry) class which we will sometimes denote by $M_q$.

When $m \geq 3$ is odd, all irreducible arithmetic lattices of $G$ arise from this construction \cite{Tits02} \cite{Lubotzky97} §3. When $m > 3$, $m \neq 8$, is even, all other are irreducible arithmetic lattices of $G$ come from skew hermitian forms over quaternion division algebras over number fields. When $m = 8$, in addition to lattices coming from skew hermitian forms, there are also lattices which come from triality.

**Construction 4.9.** Let $(W, r)$ be a quadratic $k$-subspace of $(V, q)$. Then $H = \text{SO}(W, r)$ is an absolutely almost simple $k$-subgroup of $G$. Let $H' := R_{k/Q} H$. Then $H'$ is a semisimple $\mathbb{Q}$-subgroup of $G'$. It follows that $L \cap W$ is an $O_k$-lattice of $W$, hence $G_L \cap H'(\mathbb{R})$ is an arithmetic subgroup of $H'(\mathbb{R})$. Let $H$ be the image of $H'(\mathbb{R})$ under the projection map $\pi$ onto the noncompact factors of $G'(\mathbb{R})$. Then $\pi(G_L \cap H'(\mathbb{R}))$ is an arithmetic subgroup of $H$. Note that $H$ may be trivial. It follows that $N_{\Gamma \cap H} := (\Gamma \cap H)/H(\Gamma \cap K)$ is a totally geodesic submanifold of $M_{\Gamma}$. We denote this commensurability class $N_r$. In what follows, we shall call such totally geodesic subspaces **subform subspaces**.

We have just proven the following result:

**Proposition 4.10.** Let $k$ be a number field and $q$ a quadratic form over $k$. Every quadratic subform $r$ of $q$ produces a commensurability class of totally geodesic submanifolds $N_r \subset M_q$. Furthermore, if $\dim r > 2$ and $r$ is isotropic at a real place of $k$, then $N_r$ is a commensurability class of nontrivial, nonflat, finite volume, locally symmetric spaces of noncompact type.

5. **The Index of Isometry Groups of Quadratic Forms**

Let $G$ be an absolutely almost simple algebraic $k$-group. In this paper, we will use the conventions of \cite{Tits02} and denote the **Tits index** of $G$ by $gX^{(d)}_{n,r}$ where:

- $X_n$ is the Cartan–Killing type of $G \otimes k^{\text{sep}}$,
- $n$ is the $k^{\text{sep}}$-rank of $G$,
- $g$ is the order of the image of the $*$-action map $[\cdot]$, 
- $r$ is the $k$-rank of $G$, and
- $d$ is an additional invariant.

An immediate consequence of the Tits Classification Theorem \cite{Tits02} Theorem 2.7.1] is the following corollary which will be useful in our analysis.

**Corollary 5.1.** If two semisimple $k$-groups have nonisomorphic indices, they cannot be $k$-isomorphic.

We refer the reader to \cite{Tits02} for an in depth discussion of the index and the above results. For the reader’s convenience, we now recall two basic results relating a form’s invariants and whether or not it is isotropic which will be used in establishing our dictionary.

**Proposition 5.2.** \cite{Ca} Chp. 4 Lem 2.5 & Lem 2.6] Let $L$ be a nonarchimedean local field.

1. Let $q'$ be a 3-dimensional quadratic form over $L$. Then $q'$ is isotropic if and only if $c(q') = (-1, -\det q')$.
2. Let $q'$ be a 4-dimensional quadratic form over $L$. Then $q'$ is anisotropic if and only if $\text{disc}(q') = 1$ and $c(q') = (-1, -1)$.

For proofs, see \cite{Ca}. Though the proofs are explicitly written with $k = \mathbb{Q}$, they are generalizable to an arbitrary number field. We now use these results to relate a form’s invariants to its index. See the Table \cite{Table} below to see the summary of this section’s results.

---

\footnote{In the cases we are analyzing in this paper, $g = 1$ or $g = 2$ depending on whether $G$ in an inner or outer form respectively.}

\footnote{In the cases we are analyzing in this paper, $d$ is the degree of the division algebra associated with the group. In particular, for quadratic forms this is always 1. When $d$ is 1, we often leave the spot blank.}
**Proposition 5.3.** Let $k$ be a number field. Let $q$ be a quadratic form of dimension $2n + 1$. Then the Tits index of $\text{SO}(q_v)$ at a finite place $v \in V_k$ is $B_{n,n}$ if and only if

\begin{equation}
(5.1) \quad c_v(q_v) = (-1, -1)\frac{n(n-3)}{2} (-1, \det(q_v))^n.
\end{equation}

**Proof.** We will show that the following statements are equivalent:

1. $\text{SO}(q_v)$ is of type $B_{n,n}$.
2. $q_v \cong (1, -1)^{n-1} \oplus q'_v$ where $q'_v$ is an isotropic 3-dimensional form.
3. $q_v \cong (1, -1)^{n-1} \oplus q'_v$ where $c_v(q'_v) = (-1, -\det(q'_v))$.
4. $c_v(q_v) = (-1, -1)\frac{n(n-3)}{2} (-1, \det(q_v))^n$.

First (1) is equivalent to (2) by the classification of algebraic $k$-groups in [13]. Next, (2) is equivalent to (3) by Proposition 5.2 (1). Lastly (3) is equivalent to (4) by the following computation:

\begin{align*}
c_v(q_v) &= c_v((1, -1)^{n-1} \oplus q'_v) \\
&= c_v((1, -1)^{n-1}) c_v(q'_v) ((-1)^{n-1}, \det(q'_v)) \\
&= (-1, -1)\frac{(n-1)(n-2)}{2} (-1, -\det(q'_v)) (-1, \det(q'_v))^{n-1} \\
&= (-1, -1)\frac{(n-1)(n-2)+1}{2} (-1, \det(q'_v))^n \\
&= (-1, -1)\frac{n^2-3n+2+2}{2} (-1, (-1)^{n-1} \det(q_v))^n \\
&= (-1, -1)\frac{n^2-3n+2}{2} (-1, \det(q_v))^n \\
&= (-1, -1)\frac{n(n-3)}{2} (-1, \det(q_v))^n.
\end{align*}

\[\square\]

**Proposition 5.4.** Let $k$ be a number field. Let $q$ be a quadratic form of dimension $2n$. Then the Tits index at a finite place $v \in V_k$ of $\text{SO}(q_v)$ is $1D_{n,n-2}$ if and only if

\begin{equation}
(5.2) \quad \text{disc}(q_v) = 1 \quad \text{and} \quad c_v(q_v) = -(-1, -1)\frac{n(n-1)}{2}.
\end{equation}

**Proof.** We will show that the following statements are equivalent:

1. $\text{SO}(q_v)$ is of type $1D_{n,n-2}$.
2. $q_v \cong (1, -1)^{n-2} \oplus q'_v$ where $q'_v$ is an anisotropic 4-dimensional form.
3. $q_v \cong (1, -1)^{n-2} \oplus q'_v$ where $\text{disc}(q'_v) = 1$ and $c_v(q'_v) = (-1, -1)$.
4. $\text{disc}(q_v) = 1$ and $c_v(q_v) = -(-1, -1)\frac{n(n-1)}{2}$.
First (1) is equivalent to (2) by the classification of algebraic $k$-groups in [Ti]. Next, (2) is equivalent to (3) by Proposition 5.2 (2). Lastly (3) is equivalent to (4) by the following computations:

$$\text{disc}(q_v) = \text{disc}((1, -1)^{-n-2} \oplus q_v')$$

$$= \text{disc}((1, -1)^{-n-2}) \text{disc}(q_v')$$

$$= 1.$$

$$c_v(q_v) = c_v((1, -1)^{-n-2} \oplus q_v')$$

$$= c_v((1, -1)^{-n-2}) c_v(q_v') ((-1)^{n-2}, \det(q_v'))$$

$$= (-1, -1)^{(n-2)(n-3)} - (-1, -1)$$

$$= -(-1, -1)^{(n-2)(n-3)} + 1$$

$$= -(-1, -1)^{(n^2-5n-6+2)}$$

$$= -(-1, -1)^{(n^2-n)} - 2(n-1)$$

$$= -(-1, -1)^{n(n-1)}.$$

\[ \square \]

**Proposition 5.5.** Let $q$ and $q'$ be $m$-dimensional quadratic forms where $m$ is odd. Then $q$ and $q'$ are isogroupic if and only if they are similar.

**Proof.** In Lemma 4.2 we showed similar forms are isogroupic. Now suppose $q'$ represents $G := \text{SO}(q)$. Let $a \in k^*/(k^*)^2$ such that $\det(q') = a \det(q)$. We shall show $aq$ and $q'$ are isometric. Note that $aq$ also represents $G$, and since $m$ is odd, $\det(aq) = a \det q = \det q'$. We now look at the forms locally.

- At each complex place $v \in V_k$, $aq$ and $q'$ have the same dimension, and hence are isometric by Theorem 4.4 (a).
- At each real place $v \in V_k$, since $m$ is odd, the index of $G$ together with the determinant $\det(q')$ uniquely determines the signature of $q' \otimes k_v$. Hence at each finite place, $\text{sgn}(q') = \text{sgn}(aq)$. Hence they are isometric by Theorem 4.4 (b).
- At each finite place $v \in V_k$, since $m$ is odd, equation (5.1) shows that the index $G$ together with $\det(q')$ uniquely determines $c(q')$. Hence at each finite place, $c(q') = c(aq)$. Hence they are isometric by Theorem 4.4 (c).

Hence by Theorem 4.6 $aq$ and $q'$ are isometric and the result follows. \[ \square \]

In [GPS 2.6], there is an analogous result for forms of any dimension as long as $q$ and $q'$ have very specific behavior at the real places. Their proof heavily uses the hyperbolic geometry associated their assumptions on the real places. In contrast, our proof is algebraic in nature and applies to all even dimensional forms. Using similar techniques, we prove the following theorem.

**Theorem 5.6.** Let $k$ be a number field, $q$ and $q'$ be $m = 2n + 1$-dimensional quadratic forms over $k$, and $G_i = \text{SO}(q_i)$. Then $G_1$ and $G_2$ are $k$-isomorphic if and only if the groups $G_1 \otimes k_v$ and $G_2 \otimes k_v$ have the same index for all $v \in V_k$.

In particular, if $q$ is an $m = 2n + 1$-dimensional quadratic forms over $k$, then the $k$-isomorphism class of $G := \text{SO}(q)$ is determined by its index at all places.
| Type | Classical Invariants | Tits Index |
|------|---------------------|------------|
| $B_{n,n}$ | $\dim(q) = 2n + 1$  
$\det(q) = \text{anything}$  
$c(q) = (-1, -1)^{\frac{2n(n-1)}{2}}(-1, \det(q))^n$ | ![Diagram] |
| $B_{n,n-1}$ | $\dim(q) = 2n + 1$  
$\det(q) = \text{anything}$  
$c(q) = (-1, -1)^{\frac{n(n-3)}{2}}(-1, \det(q))^n$ | ![Diagram] |
| $1D_{n,n}^{(1)}$ | $\dim(q) = 2n$  
$\det(q) = (-1)^n$ (i.e. $\text{disc}(q) = 1$)  
$c(q) = (-1, -1)^{\frac{n(n-1)}{2}}$ | ![Diagram] |
| $1D_{n,n-2}^{(1)}$ | $\dim(q) = 2n$  
$\det(q) = (-1)^n$ (i.e. $\text{disc}(q) = 1$)  
$c(q) = (-1, -1)^{\frac{n(n-1)}{2}}$ | ![Diagram] |
| $2D_{n,n-1}^{(1)}$ | $\dim(q) = 2n$  
$\det(q) \neq (-1)^n$ (i.e. $\text{disc}(q) \neq 1$)  
$c(q) = \text{anything}$ | ![Diagram] |

Table 1: Dictionary between the classical invariants of $q$ and index of $\text{SO}(q)$.

**Proof.** If $G_1$ and $G_2$ are $k$-isomorphic, then $G_1 \otimes k_v$ and $G_2 \otimes k_v$ are $k_v$-isomorphic for all $v \in V_k$, and hence by the Tits Classification Theorem [Ti, Theorem 2.7.1], they have the same index at every place.

We now prove the other direction and suppose that $G_1 \otimes k_v$ and $G_2 \otimes k_v$ have the same index for all $v \in V_k$. We may replace $q_2$ with the similar form $\frac{\det q_1}{\det q_2} q_2$, and since $m$ is odd, we may now assume $\det q_1 = \det q_2$. As we observed in the proof of the previous proposition, at local places the index and the determinant determine the isometry class of a representing form. Therefore $q_1 \otimes k_v$ and $q_2 \otimes k_v$ are isometric for all $v \in V_k$, and hence by Theorem 4.6, $q_1$ and $q_2$ are isometric. The result follows from Lemma 4.2.

**Remark 5.7.** Theorem 5.6 says that the local index determines groups over number fields of Cartan-Killing type $B_n$. Unfortunately, a similar result cannot hold for groups of type $D_n$. In particular, there exists a number field $k$ and $k$-groups $G_1$ and $G_2$ of type $D_n$, $n \equiv 1 \pmod 4$, which have the same index at every place $v \in V_k$, yet are not $k$-isomorphic. The existence of such examples is related to the existence of noncommensurable length-commensurable arithmetic locally symmetric spaces of type $D_n$ for $n$ odd. See [PR09, 9.15] for details.
In this section, we show how the computations in Section 5 may be used to parametrize even dimensional arithmetic hyperbolic manifolds. In so doing, we shall provide an alternate proof of the results of Maclachlan in [Mac]. For the reader’s convenience we recall Maclachlan’s parametrization here.

**Theorem 6.1** (Maclachlan [Mac] Theorem 1.1). The commensurability classes of discrete arithmetic subgroups of \( \text{Isom}(\mathbb{H}^{2n}) \), \( n \geq 1 \), are parametrized for each totally real number field \( k \) by sets \( \{p_1, p_2, \ldots, p_r\} \) of prime ideals in the ring of integers \( \mathcal{O}_k \) where

\[
 r \equiv \begin{cases} 
 0 \pmod{2} & \text{if } n \equiv 0 \pmod{4}, \\
 [k : \mathbb{Q}] - 1 \pmod{2} & \text{if } n \equiv 1 \pmod{4}, \\
 [k : \mathbb{Q}] \pmod{2} & \text{if } n \equiv 2 \pmod{4}, \\
 1 \pmod{2} & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

(6.1)

Maclachlan’s method uses the theory of quaternion algebras and Clifford algebras. We will now use equation (5.1) of the previous section to rederive his results. We will need the following lemma. Note that a place \( v \in V_k \) is called **dyadic** if \( k_v \) is nonarchimedean with residue field of characteristic 2. For example, the place associated with the prime 2 is dyadic over \( \mathbb{Q} \) since \( \mathbb{Q}_2 \) is nonarchimedean with residue field \( \mathbb{Z}/2\mathbb{Z} \).

**Lemma 6.2.** Let \( k/\mathbb{Q} \) be a totally real number field. Let

\[
 \delta(k) := \left\{ \text{number of dyadic places where } \left( \frac{-1,-1}{q} \right) \text{ ramifies} \right\}.
\]

Then \( \delta(k) \equiv [k : \mathbb{Q}] \pmod{2} \).

**Proof.** Over \( \mathbb{Q} \), Hamilton’s quaternions ramify at precisely 2 and \( \infty \). Hence over \( k \), Hamilton’s quaternions ramify at precisely \( \delta(k) \) places over 2, \( [k : \mathbb{Q}] \) places over \( \infty \), and nowhere else. Since a quaternion algebra ramifies at an even number of places, the result follows. \( \square \)

**Proof of Theorem 6.1.** We shall first show that \( k \)-isomorphism classes of groups giving rise to standard arithmetic hyperbolic manifolds are parametrized by sets of the form

\[
 (v, \{p_1, p_2, \ldots, p_r\})
\]

where \( v \in V_k \) is a real place and \( \{p_1, p_2, \ldots, p_r\} \) is a set of prime ideals satisfying (6.1). The theorem then follows from Corollary 3.7. In Proposition 5.5, we showed that similarity classes of quadratic forms of dimension \( 2n + 1 \) parametrize groups of Cartan–Killing type \( B_n \) over \( k \). Picking the determinant 1 representative of each similarity class, the set

\[
 \mathcal{F} := \{ q \mid \text{dim } q = 2n + 1, \det q = 1, \text{ and } q \text{ gives rise to a hyperbolic manifold} \}
\]

parametrizes \( k \)-isomorphism classes of groups giving rise to standard arithmetic hyperbolic manifolds.

For \( q \in \mathcal{F} \), there is a unique real place \( v_q \) where \( q \) is isotropic, and at all other real places, is anisotropic. Let \( v_1, \ldots, v_l \) denote the real embeddings of \( k \). We now fix \( v_i \), for \( 1 \leq i \leq l \), and analyze all forms in

\[
 \mathcal{F}_i := \{ q \in \mathcal{F} \mid q \text{ is isotropic at } v_i \}.
\]

For \( q \in \mathcal{F}_i \), the fact that \( \det q = 1 \) now implies that \( q \) has signature \( (1, 2n) \) at \( v_i \) and signature \( (2n + 1, 0) \) at all other real places. A basic computation shows that the Hasse–Minkowski invariants at the real places are then

\[
 c_{v_j}(q) = \begin{cases} 
 (-1)^n & i = j \\
 1 & i \neq j.
\end{cases}
\]
Let $V^s_k = \{ v \in V_k \mid (-1, -1)_v = +1 \}$ and $V^r_k = \{ v \in V_k \mid (-1, -1)_v = -1 \}$. These sets correspond to the finite places where Hamilton’s quaternions split and ramify, respectively. For $q \in \mathcal{F}$, let $e_s(q)$ (resp. $e_r(q)$) denote the number of finite places in $V^s_k$ (resp. $V^r_k$) where $SO(q)$ is not split. Clearly $r(q) := e_s(q) + e_r(q)$ is the total number of finite places where $SO(q)$ is not split. (Note that this is always finite because any $k$-group is quasi-split at all but finitely many places and quasi-split groups of type $B_n$ are split.)

We now use (5.1) to relate $r(q)$ to the local Hasse-Minkowski invariants of $q$. Since $q$ has determinant 1, (5.1) may be simplified to state that $SO(q)$ splits over $v$ if and only if
\[
c_v(q) = (-1, -1)^{\frac{n(n-3)}{2}}.
\]
Let $f_s(q)$ (resp. $f_r(q)$) denote the number of finite places $v$ in $V^s_k$ (resp. $V^r_k$) where $e_v(q) = -1$. If as in Lemma 6.2 $\delta(k)$ is the number of dyadic places where $\left(\frac{-1}{q}\right)$ ramifies, then it follows that:

- $f_s(q) = e_s(q)$, and
- $f_r(q) = \left\{ \begin{array}{ll} e_r(q) & \text{if } n \equiv 0, 3 \pmod{4}, \\ \delta(k) - e_r(q) & \text{if } n \equiv 1, 2 \pmod{4}. \end{array} \right.$

By Theorem 4.7, the local Hasse-Minkowski invariants of $q$ must satisfy the compatibility condition that $\prod_{v \in V_k} c_v(q) = 1$. It follows that
\[
(-1)^n (-1)^{f_s(q)} (-1)^{f_r(q)} = 1
\]
and hence
\[
n + f_s(q) + f_r(q) \equiv 0 \pmod{2}.
\]

Putting the pieces together, we now have the following four cases:

- **Case 1:** $n \equiv 0 \pmod{4}$
  
  Equation (6.2) immediately gives $r(q) \equiv 0 \pmod{2}$.

- **Case 2:** $n \equiv 1 \pmod{4}$
  
  Equation (6.2) gives
  \[
  n + e_s(q) + \delta(k) - e_r(q) \equiv 0 \pmod{2}.
  \]
  By Lemma 6.2 and simplifying,
  \[
  1 + e_s(q) + [k : \mathbb{Q}] - e_r(q) \equiv 0 \pmod{2},
  \]
  and hence
  \[
r(q) \equiv [k : \mathbb{Q}] - 1 \pmod{2}.
  \]

- **Case 3:** $n \equiv 2 \pmod{4}$
  
  Again using Lemma 6.2 equation (5.2) gives
  \[
  0 + e_s(q) + [k : \mathbb{Q}] - e_r(q) \equiv 0 \pmod{2},
  \]
  and hence
  \[
r(q) \equiv [k : \mathbb{Q}] \pmod{2}.
  \]

- **Case 4:** $n \equiv 3 \pmod{4}$
  
  Equation (6.2) immediately gives $r(q) \equiv 1 \pmod{2}$.

We have shown that every form $q \in \mathcal{F}$ uniquely determines a set $(v_q, \{v_1, v_2, \ldots, v_{r(q)}\})$ where $v_q$ is the unique real place where $q$ is isotropic, and $\{v_1, v_2, \ldots, v_{r(q)}\}$ is precisely the set of finite places where $SO(q)$ is not split over $k_v$, where $r(q)$ satisfies equation (6.1).
We now show that any collection \((v_0, \{v_1, v_2, \ldots, v_r\})\) where \(v_0 \in V_k\) is a real place, \(\{v_1, v_2, \ldots, v_r\}\) is a set of finite places, and \(r\) satisfies equation (6.1), determines a form in \(F\). Let \(\{q_v\}_{v \in V_k}\) be a family of \((2n + 1)\)-dimensional forms of determinant 1 satisfying the following:

- \(q_{v_0}\) has signature \((1, 2n)\),
- \(q_v\) has signature \((2n + 1, 0)\) at all other real places,
- for \(v \in V_k\) finite, \(\text{SO}(q_v)\) is not split if and only if \(v \in \{v_1, v_2, \ldots, v_r\}\), and hence \(c_v(q_v)\) is determined by equation (6.1).

The above computations show that this family satisfies the compatibility condition of Theorem 4.7 and hence there exists a global form \(q \in F\) with localizations \(q_v\).

It follows that sets of the form \((v_0, \{v_1, v_2, \ldots, v_r\})\) where \(v_0 \in V_k\) is a real place, \(\{v_1, v_2, \ldots, v_r\}\) is a set of finite places, and \(r\) satisfies equation (6.1), parametrize \(F\) and hence the theorem follows. \(\square\)

With proper modification, these techniques may be used to rederive Maclachlan’s parametrization of commensurability classes of odd dimensional arithmetic hyperbolic spaces coming from quadratic forms [Mac Cor. 7.5]. Again with additional modification, these techniques are generalizable to give parametrizations of commensurability classes of certain higher rank locally symmetric spaces.

7. Fields of Definition and the Proof of Theorem A

Let \(G\) be a semisimple algebraic group over \(\mathbb{C}\) and let \(\Gamma \subset G(\mathbb{C})\) be a Zariski-dense subgroup. A field of definition\(^7\) for \(\Gamma\) is a field \(F \subset \mathbb{C}\) for which there exists an \(F\)-form \(G'\) of \(G\) and an isomorphism \(\varphi : G \to G'\) defined over a finite extension of \(F\) such that \(\varphi(\Gamma) \subset G'(F)\) [MaRe 10.3.10]. Vinberg showed [Vin71] that for Zariski-dense groups, there is a unique minimal field of definition

\[
k_G(\Gamma) := \mathbb{Q}(\text{Tr}((\text{Ad}_G(\gamma))) \mid \gamma \in \Gamma),
\]

where \(\text{Ad}_G\) is the adjoint representation of \(G\). Furthermore this is an invariant of the commensurability class. In general, the minimal field of definition of a Zariski-dense \(\Gamma\) need not coincide with the field that \(G\) is defined over. Furthermore, the same abstract group can have different fields of definition depending on the ambient group. However, [PR09, Prop. 2.6] showed that for an absolutely almost simple group \(G\) over a number field \(k\), and \(\Gamma \subset G(k)\) arithmetic and Zariski-dense, the minimal field of definition of \(\Gamma\) coincides with the field of definition of the group (i.e. \(k_G(\Gamma) = k\)).

With this in mind, we define the field of definition of a finite volume arithmetic locally symmetric space arising from an absolutely almost simple \(k\)-group \(G\) to be \(k(M) := k\). In particular, the field of definition of an arithmetic locally symmetric space \(M\) coming from a quadratic form is the field over which its associated quadratic form is defined.

Remark 7.1. A standard arithmetic hyperbolic \(n\)-orbifold \(M\), \(n \geq 2\), is compact if and only if its field of definition \(k := k(M)\) is strictly larger than \(\mathbb{Q}\). If \(|k : \mathbb{Q}| > 1\), then there is at least one real place where its associated form is anisotropic, and hence the form must be anisotropic over \(k\). By Theorem 3.2 \(M\) is then compact. Conversely, if \(k = \mathbb{Q}\), then the form must be \(\mathbb{Q}\)-isotropic, and again by Theorem 3.2 \(M\) is not compact.

For an arbitrary totally geodesic subspace \(N \subset M\), we do not expect to see a relationship between \(k(N)\) and \(k(M)\) as is demonstrated by the following example.

Example 7.2. We show three basic algebraic methods of constructing of totally geodesic subspaces \(N \subset M\) where both \(N\) and \(M\) come from quadratic forms and where each realizes a different relationship between \(k(N)\) and \(k(M)\).

---

\(^7\)There are many equivalent definitions for “field of definition” or more generally a “ring of definition.” For one such definition, we refer the reader to [Vin71].
(1) Subforms produce $N \subset M$ such that $k(N) = k(M)$.

Let $k$ be an arbitrary number field and let $q$ be a quadratic form over $k$ of dimension $\geq 4$. Let $r \subset q$ be a subform of dimension $\geq 3$. Then $\text{SO}(r)$ naturally sits inside $\text{SO}(q)$ as a $k$-subgroup. Then $k(N) = k = k(M)$.

(2) Extension of scalars produce $N \subset M$ such that $k(N) \subset k(M)$.

Let $k'/Q$ be a nontrivial finite extension and let $q$ be a quadratic form over $Q$ of dimension $\geq 3$. Then $\text{SO}(q)$ naturally sits as a $Q$-subgroup in the diagonal of $R_{k'/Q}(\text{SO}(q_{\otimes Q} k'))$. Then $k(N) = Q \subset k = k(M)$.

(3) Killing form produces $N \subset M$ such that $k(N) \supset k(M)$.

Let $k/Q$ be a nontrivial finite extension, let $q$ be a quadratic form over $k$ of dimension $\geq 3$, let $H = \text{SO}(q)$, and $G = \text{SO}(\text{Lie}(R_{k/Q}(H)), \kappa)$ where $\kappa$ is the Killing form on $\text{Lie}(R_{k/Q}(H))$. Then, via the adjoint representation, $H(k) = (R_{k/Q}(H))(Q) \subset (\text{Aut}(\text{Lie}(R_{k/Q}(H))))^\kappa(Q) \subset G(Q)$. Then $k(N) = k \supset Q = k(M)$.

Observe that in the above examples, when $k(N) \neq k(M)$, the difference between $\dim N$ and $\dim M$ was quite large. As the next results show, if the dimensions of $N$ and $M$ are sufficiently close, there is a relationship between their fields of definition.

**Lemma 7.3.** For $i = 1, 2$, let $H_i$ be semisimple $k_i$-groups such that $H_1$ is absolutely almost simple and $R_{k_1/Q}(H_1)$ is $Q$-isogenous to $R_{k_2/Q}(H_2)$.

1. Then $k_2$ is a subfield of a $\text{Gal}(\overline{Q}/Q)$-conjugate if $k_1$.
2. If $\dim H_2 < 2 \dim H_1$, then $k_1$ and $k_2$ are $\text{Gal}(\overline{Q}/Q)$-conjugate and $H_1$ and $H_2$ are isogenous over the Galois closure of $k_1$.

**Proof.**

(1) Replacing $H_i$ by their adjoint groups, we have $R_{k_1/Q}(H_1)$ and $R_{k_2/Q}(H_2)$ are $Q$-isomorphic. Since $H_1$ is absolutely simple, $R_{k_1/Q}(H_1)$ is $Q$-simple and hence $R_{k_2/Q}(H_2)$ is $Q$-simple. It follows that $H_2$ must be $k_2$-simple, and by Proposition 5.1 (1), there exists a field extension $k_1'/k_2$ and absolutely simple $k_1'$-group $H_1'$ such that $R_{k_1'/k_2}(H_1')$ and $H_2$ are $k_2$-isomorphic. It follows that $R_{k_1/Q}(H_1)$ and $R_{k_1'/k_2}(R_{k_2/Q}(H_1')) \cong R_{k_1/Q}(H_1')$ are $\text{Gal}(\overline{Q}/Q)$-conjugate. By Proposition 5.1 (2), $k_1$ and $k_1'$ are $\text{Gal}(\overline{Q}/Q)$-conjugate and $H_1$ and $H_1'$ are isomorphic over the Galois closure of $k_1$.

(2) Our initial assumptions imply that $H_2$ is $\overline{Q}$-isomorphic to $\text{deg}_{k_2}(k_1')$ copies of $H_1$. The restriction on dimension implies that $H_2$ has precisely one such simple factor. Hence $k_1' = k_2$ and the result follows. 

**Proposition 7.4.** Let $H_1$ be an absolutely almost simple $k_1$-group and $G$ be absolutely almost simple $k_2$-group, both of which are isotropic at precisely one infinite place, such that $\dim G < 2 \dim H_1$. Suppose $R_{k_1/Q}(H_1)$ is $Q$-isogenous to a $Q$-subgroup of $R_{k_2/Q}(G)$. Then $k_1$ and $k_2$ are $\text{Gal}(\overline{Q}/Q)$-conjugate.

**Proof.** Replace $H_1$ and $G$ by their adjoint groups and let $v_1$ (resp. $v_2$) denote the unique infinite place of $k_1$ (resp. $k_2$) where $H_1$ (resp. $G$) is isotropic. Then there is an injective $Q$-rational map,

$$\varphi : R_{k_1/Q}(H_1) \rightarrow R_{k_2/Q}(G),$$

which induces an injective map of absolutely simple Lie groups

$$\varphi : H_1(k_{1,v_1}) \rightarrow G(k_{2,v_2}).$$

Let $H_2$ denote the Zariski-closure of $\varphi(H_1(k))$ in $G(k_{2,v_2})$. Since $\varphi(H_1(k)) \subset G(k_2)$, $H_2$ is defined over $k_2$. Observe that $R_{k_1/Q}(H_1)$ is $Q$-isogenous to $R_{k_2/Q}(H_2)$ and by our assumption on dimension, $\dim H_2 \leq \dim G < 2 \dim H_1$. Therefore by Lemma 7.3 (2), the result follows. 

\[\Box\]
Observe that in the proof of Proposition 7.4, we use the fact that our groups are isotropic at precisely one infinite place to ensure that \( H_2 \) is defined over \( k_2 \), instead of a proper subfield, and that the dimension of \( H_2 \) satisfies the bounds of Lemma 7.3 (2). We now show that Proposition 7.4 applies to our situation of locally symmetric spaces coming from quadratic forms.

**Proposition 7.5.** Let \( M_1 \) and \( M_2 \) be arithmetic locally symmetric spaces coming from quadratic forms \( q_1 \) and \( q_2 \) of dimension \( \geq 4 \) over number fields \( k_1 \) and \( k_2 \) respectively. Then \( \mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2) \) implies \( \dim q_1 = \dim q_2 \).

**Proof.** We shall prove the contrapositive. If \( \dim q_1 \neq \dim q_2 \), then (potentially after relabeling), \( \dim q_1 > \dim q_2 \). Let \( v_0 \in V_{k_1} \) be a real place where \( q_1 \otimes k_{1,v_0} \) is isotropic. Then by deleting one entry in a diagonal representation of \( q_1 \) we have a \( (\dim q_1 - 1) \)-dimensional form \( r \) which is isotropic at \( v_0 \), and by dimensional considerations, there is no which no proper subform of \( q_2 \) which can represent \( H := SO(r) \). Since \( \dim r \geq 3 \), \( r \) gives rise to a nonflat finite volume totally geodesic submanifold \( N \) of \( M_1 \) which \( N \) cannot be a proper totally geodesic submanifold of \( M_2 \). The result then follows. \( \square \)

**Proof of Theorem A.** Since \( \mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2) \), Proposition 7.5 implies \( \dim q_1 = \dim q_2 =: m \). Let \( r \) be an \( (m - 1) \)-dimensional quadratic \( k_1 \)-subform of \( q_1 \) which is isotropic at the real place \( q_1 \) is isotropic. Let \( H_1 := SO(r) \). Observe that

\[
\dim SO(q_2) = \frac{m(m - 1)}{2} = \frac{(m - 1)(m - 2)}{2} + m < 2 \left( \frac{(m - 1)(m - 2)}{2} \right) = 2 \dim SO(r)
\]

Since \( M_1 \) and \( M_2 \) are simple, we may apply Proposition 7.4 and the result follows. \( \square \)

Notice that it is sufficient for \( M_1 \) and \( M_2 \) to both contain the same (up to commensurability) totally geodesic space coming from a codimension-1 form. In particular, for arithmetic hyperbolic spaces, we immediately have the following corollary.

**Corollary 7.6.** Let \( M_1 \) and \( M_2 \) be standard arithmetic hyperbolic \( n \)-manifolds of dimension \( \geq 4 \) over number fields \( k_1 \) and \( k_2 \) respectively. Suppose there is a totally geodesic \((n - 1)\)-manifold \( N_1 \subset M_1 \) commensurable to a totally geodesic \((n - 1)\)-manifold \( N_2 \subset M_2 \). Then \( k_1 \) and \( k_2 \) are \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-conjugate.

In fact, we shall prove a stronger version of this result for arithmetic hyperbolic \( n \)-orbifolds in Section 10. It is interesting to note that the larger \( m \) is, the higher in codimension we can go to still be able to apply Proposition 7.4. Geometrically, this translates to smaller subspaces still containing the information of the field of definition. Let \( d_q \) denote the maximum codimension of a subform of \( q \) for which we can apply the proposition. Here is a table of some smaller dimensions with exact values of \( d_q \).

| \( \dim q \) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \dim SO(q) \) | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 | 55 | 66 | 78 | 91 | 105 |
| \( d_q \) | - | - | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 4 | |

Table 2: Small values of \( \dim q \) vs. \( d_q \).

In general,

\[
(m - d_q - 1)(m - d_q - 2) \leq \frac{m(m - 1)}{2} < (m - d_q)(m - d_q - 1)
\]

hence

\[
(m - d_q - 2)^2 < \frac{m^2}{2} \quad \text{and} \quad \frac{(m - 1)^2}{2} < (m - d_q)^2
\]
and we get bounds
\[
\left(1 - \frac{1}{\sqrt{2}}\right)m - 2 < d_q < \left(1 - \frac{1}{\sqrt{2}}\right)m + \frac{1}{\sqrt{2}}.
\]
In particular, \(d_q\) grows linearly with \(m\).

8. Technical Results: Construction of Subforms of Quadratic Forms

This section is dedicated to showing that over number fields, nonisogroupic forms cannot have the same isogroupy classes of subforms. Toward these ends, we construct proper quadratic subforms with very specific local properties which will exploit the exceptional restrictions on the Hasse–Minkowski invariant in dimensions 1 and 2. Many of the results of this section heavily rely upon the following fundamental lemma.

Lemma 8.1 (Square Existence Lemma). Let

1. \(k\) be a number field,
2. \(S\) be a finite set of places of \(k\), and
3. for each \(v \in S\), let \(\alpha_v\) be a square class in \(k_v^\times\).

Then there exists an \(s \in k^\times\) for which \(s \in \alpha_v\) for all \(v \in S\).

Proof. Each nontrivial (resp. trivial) square class \(\alpha_v\) corresponds to a unique quadratic (resp. trivial) extension \(L_v/k_v\). By local class field theory, this corresponds to a character \(\chi_v\) of \(k_v^\times\) of order 2 (resp. order 1). Then by the famous Grunwald–Wang Theorem [M Chp VIII Thm 2.4], there exists a character \(\chi\) of \(GL_1(k_v)/GL_1(k)\) whose restriction to \(k_v^\times\) is \(\chi_v\), for all \(v \in S\). Since \(n = 2\) and \(k[\sqrt{2}] = k\) is trivially cyclic, we may choose \(\chi\) to have order 2. By global class field theory, this gives a quadratic extension \(L/k\) where \(L = k(s)\). Then \(s \in \alpha_v\) for all \(v \in S\). \(\Box\)

Constructing Nonrepresentable Subforms 1: Nonisogroupic At a Real Place. Let \(k\) be a number field and let \(q\) be an \(m\)-dimensional quadratic form over \(k\). If \(v \in V_k\) is a real place, then we shall say \(q\) is ordered at \(v\) if the signature \((m_+^{(v)}, m_-^{(v)})\) of \(q \otimes k_v\) satisfies \(m \geq m_+^{(v)} \geq m_-^{(v)} \geq 0\). We call \(q\) ordered if it is ordered at all real places. We now show that every similarity class contains an ordered representative.

Lemma 8.2. Let \(k\) be a number field and \(q\) a quadratic form over \(k\). Then there exists an \(a \in k^\times\) so that \(aq\) is ordered.

Proof. Let \(S \subset V_k\) denote the set of all real places and let \(S_0 \subset S\) denote the set of all real places where \(q\) is not ordered. For each \(v \in S\), let
\[
\alpha_v = \begin{cases} 
-(k_v^\times)^2 & \text{if } v \in S_0, \\
(k_v^\times)^2 & \text{if } v \notin S_0.
\end{cases}
\]
By Lemma 8.1 there exists \(a \in k^\times\) such that \(a(k_v^\times)^2 = \alpha_v\) for all \(v \in S\) and hence \(aq\) is ordered. \(\Box\)

Note that two quadratic forms over \(\mathbb{R}\) are isogroupic if an only if they are similar. Using this fact we begin constructing subform of one form which are not similar to a subform of the other.

Lemma 8.3. Let \(q_1\) and \(q_2\) be nonisometric \(m\)-dimensional quadratic forms over \(\mathbb{R}\) with signatures \((m_1, n_1)\) and \((m_2, n_2)\) respectively such that \(m_1 > m_2 \geq n_2 > n_1\). Then for all \(j \in \mathbb{Z}_{\geq 1}\) such that
\[
n_1 + n_2 < j < m
\]
there exists an isotropic \(j\)-dimensional form dividing \(q_2\) that is not similar to a form dividing \(q_1\). Furthermore, this form can be realized by deleting \(m - j\) entries in a diagonal representation of \(q_2\).
Proof. The idea of the proof is that we pick a subform $r$ of $q_2$ such that neither $r$ nor $-r$ divides $q_1$. We may represent

$$q_1 = \langle a_1, \ldots, a_{m_1}, a_{m_1+1}, \ldots, a_m \rangle \quad \text{and} \quad q_2 = \langle b_1, \ldots, b_{m_2}, b_{m_2+1}, \ldots, b_m \rangle,$$

with $a_i, b_j \in \mathbb{R}$. The desired subform may be obtained by deleting the first $m - j$ entries of $q_2$, namely let

$$r := \langle b_{m-j+1}, b_{m-j+2}, \ldots, b_{m-1}, b_m \rangle.$$

By construction, $r$ has signature $(j - n_2, n_2)$ from which we can see that $r$ is always isotropic and both

- $j - n_2 > n_1 + n_2 - n_2 = n_1$, and
- $n_2 > n_1$.

Hence neither $r$ nor $-r$ is a subform of $q_1$. \qed

Remark 8.4. The more isotropic both forms are, the fewer subforms arise from this construction. In particular, there are no subforms precisely when $m$ is even and the two forms have signatures

$$\left(\frac{m}{2} - 1, \frac{m}{2} + 1\right) \quad \text{and} \quad \left(\frac{m}{2}, \frac{m}{2}\right).$$

In the end, our goal is to construct locally symmetric spaces of noncompact type, and hence want isotropic subforms. Hence Lemma 8.3 largely succeeds, but we need to address the case in the above remark.

Lemma 8.5. Let $q_1$ and $q_2$ be nonisometric $m$-dimensional quadratic forms over $\mathbb{R}$ with signatures $(m_1, n_1)$ and $(m_2, n_2)$ respectively such that $m_1 > m_2 \geq n_2 > n_1 > 0$. Then for all $j \in \mathbb{Z}_{\geq 1}$ such that

$$m_1 < j < m$$

there exists an isotropic $j$-dimensional form dividing $q_1$ that is not similar to a form dividing $q_2$. Furthermore, this form can be realized by deleting $m - j$ entries in a diagonal representation of $q_1$.

Proof. Again we may represent

$$q_1 = \langle a_1, \ldots, a_{m_1}, a_{m_1+1}, \ldots, a_m \rangle \quad \text{and} \quad q_2 = \langle b_1, \ldots, b_{m_2}, b_{m_2+1}, \ldots, b_m \rangle,$$

with $a_i, b_j \in \mathbb{R}$. This time the desired subform may be obtained by deleting the last $m - j$ entries of $q_1$, namely let

$$r := \langle a_1, a_2, \ldots, a_j \rangle.$$

By construction, $r$ has signature $(m_1, n_1 - m + j)$ from which we can see that $r$ is always isotropic and by our initial assumptions, both

- $m_1 > m_2$.
- $m_1 > n_2$.

Hence neither $r$ nor $-r$ is a subform of $q_2$. \qed

Remark 8.6. The more anisotropic $q_1$ is, the fewer subforms arise from this construction. In particular, there are no subforms arising from this construction precisely when $m_1 = m - 1$.

Combining Lemma 8.3 and Lemma 8.5 we obtain the following corollary.

Corollary 8.7. Let $q_1$ and $q_2$ be nonisomorphism quadratic forms over $\mathbb{R}$ of dimension $m \geq 5$. Then there exists an isotropic $(m - 1)$-dimensional subform of one which is not similar to a subform of the other. Furthermore, this form can be realized by deleting one entry in a diagonal representation of either $q_1$ or $q_2$. 
We should note that the bound \( m \geq 5 \) is strict because neither Lemma 8.3 nor Lemma 8.5 may be applied to the nonisometric 4-dimensional real forms \( q_1 \) and \( q_2 \) with signatures \((3,1)\) and \((2,2)\) respectively. It is not hard to see that every isotropic subform of one is a subform of the other. This failure is related to subtleties related to comparing the groups \( \text{SO}(3,1) \) and \( \text{SO}(2,2) \).

We now apply these results on quadratic forms over \( \mathbb{R} \) to obtain the following result over number fields.

**Theorem 8.8.** Let

1. \( k \) be a number field,
2. \( m \geq 5 \), and
3. \( q_1 \) and \( q_2 \) be ordered \( m \)-dimensional quadratic \( k \)-forms such that there is a real place \( v_0 \in V_k \) over which \( q_1 \) and \( q_2 \) are not isogroupic.

Then there exists an \((m-1)\)-dimensional quadratic \( k \)-form \( r \), isotropic at \( v_0 \), which is a subform of one and not isogroupic to a subform of the other.

**Proof.** Begin by representing \( q_1 = \langle a_1, \ldots, a_m \rangle \) and \( q_2 = \langle b_1, \ldots, b_m \rangle \), \( a_i, b_j \in k \). Then by Corollary 8.7, we may delete one entry to get an \((m-1)\)-dimensional subform which over \( k_{v_0} \) is not similar to a subform of the other, and the result then follows. \( \Box \)

**Proposition 8.9.** Let

1. \( k \) be a number field,
2. \( m \geq 4 \), and
3. \( q_1 \) and \( q_2 \) be ordered \( m \)-dimensional quadratic \( k \)-forms such that the set of real places where \( q_1 \) is isotropic is disjoint from the set of real places where \( q_2 \) is isotropic.

If \( r \) is a \( j \)-dimensional subform of \( q_1 \), where \( 0 < j < m \) which is isotropic at a real place \( v_0 \in V_k \), then no subform of \( q_2 \) is isogroupic to \( r \).

**Proof.** By assumption, \( \text{SO}(r \otimes_k k_{v_0}) \) is an \( \mathbb{R} \)-isotropic \( \mathbb{R} \)-group, but \( \text{SO}(q_2 \otimes_k k_{v_0}) \) is \( \mathbb{R} \)-anisotropic. \( \Box \)

**Constructing Nonrepresentable Subforms 2: Isogroupic At All Real Places.** We continue our analysis of forms by assuming they are isogroupic at all infinite places but differ at a finite place.

**Theorem 8.10.** Let

1. \( k \) be a number field,
2. \( m = 2n + 1 \) for \( n \geq 2 \),
3. \( q_1 \) and \( q_2 \) be nonisometric ordered \( m \)-dimensional quadratic forms over \( k \) such that \( q_1,v \cong q_2,v \) for each infinite place \( v \in V_k \), and
4. there be some finite place \( v_0 \in V_k \) where:
   - \( \det_{v_0} q_1 = 1 = \det_{v_0} q_2 \),
   - \( c_{v_0}(q_1) \neq c_{v_0}(q_2) \).

Then there exist \((m-1)\)-dimensional quadratic forms \( r_i \) dividing \( q_i \) such that no subform of \( q_j \) represents \( \text{SO}(r_i) \) for \( i \neq j \). Furthermore if the \( q_i \) are isotropic at a real place, then the \( r_i \) can be chosen to be isotropic at that real place as well.

**Proof.** We are going to construct the desired forms locally and then use the existence, uniqueness, and local-to-global results of Section 8 to create the desired global forms. Let

\[
S = \{v_0\} \cup \{\text{infinite real places of } k\}
\]

For each \( v \in S \), we pick square classes \( \alpha_v \in k_v^\times/(k_v^\times)^2 \) as follows:

- For \( v_0 \), let \( \alpha_{v_0} \) be such that \( \alpha_{v_0} = (-1)^n \).
- For each infinite \( v \in S \) let \( \alpha_v = \det q_1(k_v^\times)^2 = \det q_2(k_v^\times)^2 \).

\[
S = \{v_0\} \cup \{\text{infinite real places of } k\}
\]
By Lemma 8.1 above, we may choose an $s \in k^\times$ such that $s \in \alpha_v$ for all $v \in S$.

For each finite place $v \in V_k$, define $t_{1,v}, t_{2,v}, r_{1,v}, r_{2,v}$ to be the quadratic $k_v$-forms with invariants given by:

\[
\begin{align*}
\dim t_{i,v} &= 1 \\
\det t_{i,v} &= \frac{\det q_i}{s} \\
c_v(t_{i,v}) &= 1
\end{align*}
\]

\[
\begin{align*}
\dim r_{i,v} &= m - 1 \\
\det r_{i,v} &= s \\
c_v(r_{i,v}) &= c_v(q_i) \left(\frac{\det q_i}{s}\right)_v
\end{align*}
\]

We know such forms exist by Theorem 4.5 (3).

For each infinite place $v \in V_k$, define forms $t_{1,v}, t_{2,v}$ by:

\[
t_{i,v} = \left\langle \frac{\det q_i}{s} \right\rangle.
\]

At each complex place $t_{i,v}$ divides $q_i \otimes k_v$. By assumption, $q_1$ and $q_2$ are ordered at each real place $v \in V_k$, and hence $t_{i,v}(=1)$ is a subform of $q_i \otimes k_v$. Therefore at each infinite place it makes sense to take the complement of $t_{i,v}$ in $q_i \otimes k_v$ and we may define forms $r_{1,v}, r_{2,v}$ by

\[
r_{i,v} = t_{i,v}^\perp.
\]

At each complex place $v \in V_k$, we trivially have $c_v(r_{i,v}) = 1 = c_v(q_i)$. For each real $v \in V_k$, let $(m_+^{(v)}, m_-^{(v)})$ denote the signature of $q_i \otimes k_v$. Observe that $r_{i,v}$ has signature $(m_+^{(v)} - 1, m_-^{(v)})$, and hence is isotropic whenever $q_i \otimes k_v$ is isotropic. Also note that

\[
c_v(r_{i,v}) = (-1)^{m_+^{(v)}(m_-^{(v)} - 1)} = c_v(q_i).
\]

We shall now show that for each place $v \in V_k$, $t_{i,v} \oplus r_{i,v} \cong q_i \otimes k_v$. This is true by construction at the infinite places. Now suppose $v$ is finite. Clearly

\[
\dim(t_{i,v} \oplus r_{i,v}) = 1 + (n - 1) = n = \dim(q_i \otimes k_v)
\]

\[
\det(t_{i,v} \oplus r_{i,v}) = (\det q_i/s)s = \det(q_i \otimes k_v)
\]

and by the product formula for the Hasse–Minkowski invariant

\[
c(t_{i,v} \oplus r_{i,v}) = c(t_{i,v})c(r_{i,v}) \left(\frac{\det q_i}{s}, s\right) = c_v(q_i) \left(\frac{\det q_i}{s}, s\right) = c(q_i \otimes k_v),
\]

and hence by Theorem 4.4 (3), they are isomorphic.

We now wish to build a global form, and hence must check that our forms satisfy the compatibility criteria of Theorem 4.7. Observe that $c_v(t_{i,v}) = 1$ for all $v \in V_k$ and hence $\prod_{v \in V_k} c_v(t_{i,v}) = 1$. Next observe that by
our choice of \( s, \left( s, \frac{\det q_i}{s} \right) \) = 1 at each infinite place, and hence

\[
\prod_{v \in V_k} c_v(r_{i,v}) = \left( \prod_{v \in V_k \text{ finite}} c_v(r_{i,v}) \right) \times \left( \prod_{v \in V_k \text{ real}} c_v(r_{i,v}) \right) \times \left( \prod_{v \in V_k \text{ complex}} c_v(r_{i,v}) \right)
\]

\[
= \left( \prod_{v \in V_k \text{ finite}} c_v(q_i) \left( s, \frac{\det q_i}{s} \right)_v \right) \times \left( \prod_{v \in V_k \text{ real}} c_v(q_i) \right) \times \left( \prod_{v \in V_k \text{ complex}} c_v(q_i) \right)
\]

\[
= \prod_{v \in V_k} c_v(q_i) \times \prod_{v \in V_k} \left( s, \frac{\det q_i}{s} \right)_v .
\]

(8.1)

Where we know the final product is trivial because both the Hasse–Minkowski invariant and the Hilbert symbol of global objects satisfy the product formula.

By Theorem 4.7, there exist quadratic forms \( t_i \) and \( r_i \) over \( k \) such that for all \( v \in V_k \), \( t_i \otimes k_v \cong t_{i,v} \) and \( r_i \otimes k_v \cong r_{i,v} \). Furthermore, for each \( v \in V_k \), we have shown that \( t_{i,v} \oplus r_{i,v} \cong q_i \otimes k_v \), so by Theorem 4.6 we conclude \( t_i \oplus r_i \cong q_i \), and hence \( r_i \) is a subform of \( q_i \).

Let \( H_i = \text{SO}(r_i) \). We must show that \( H_i \subset G_j \) if and only if \( i = j \), and hence this reduces to showing that there are no representatives \( r'_i \) of \( H_i \) such that \( r'_i \subset q_j \) for \( j \neq i \). Now \( H_i \) is a group of type \( D_n \) over \( k \). Let \( r'_i \) be any representative of \( H_i \). Then \( H_i \) determines the following invariants of \( r' \):

1. \( \dim(r'_i) = 2n = \dim(r_i) \).
2. \( \text{disc}_v(r'_i) = 1 \) at precisely the places \( v \in V_k \) where \( H_i \otimes k_v \) is a group of inner type (i.e., the *-action is trivial). This means that \( \text{disc}_v(r'_i) = 1 \) if and only if \( \text{disc}_v(r_i) = 1 \), or in other words, at the places where \( \text{disc}_v(r_i) = 1 \), then \( \det r' = \det r \).
3. \( c_v(r'_i) = c_v(r_i) \) at each place \( v \) where \( \text{disc}_v(r_i) = 1 \) (see equation (5.2)).

Now let \( r'_i \) be any quadratic form satisfying these three. Suppose there exists some form \( t'_i \) such that \( r'_i \oplus t'_i \cong q_j \) for \( i \neq j \). It immediately follows that \( \dim t'_i = 1 \), \( \det t'_i = \det q_j / \det r'_i \) and by the exceptional restriction, \( c(t'_i) = 1 \).

Observe that our choice of \( s \) implies that

\[
\text{disc}_v(r_i) = (-1)^n \det r_i = (-1)^{2n} = 1
\]

Hence at \( v_0 \), we have \( \det r_i = \det r'_i \) and \( c_{v_0}(r_i) = c_{v_0}(r'_i) \), which we use in the following computation of \( c_{v_0}(q_j) \):

\[
c_{v_0}(q_j) = c_{v_0}(r'_i \oplus t'_i)
\]

\[
= c_{v_0}(r'_i)c_{v_0}(t'_i) \left( \det r'_i, \frac{\det q_j}{\det r'_i} \right)_{v_0}
\]

\[
= c_{v_0}(r_i) \left( \det r_i, \frac{\det q_j}{\det r_i} \right)_{v_0}
\]

\[
= \left( c_{v_0}(q_i) \left( \det r_i, \frac{\det q_i}{\det r_i} \right)_{v_0} \right) \left( \det r_i, \frac{\det q_j}{\det r_i} \right)_{v_0}
\]

\[
= c_{v_0}(q_i) \left( \det r_i, \frac{\det q_j}{(\det r_i)^2} \right)_{v_0}
\]

\[
= c_{v_0}(q_i) \det r_i, 1)_{v_0}
\]

\[
= c_{v_0}(q_i).
\]
However this contradicts our initial assumption that \( c_{v_0}(q_1) \neq c_{v_0}(q_2) \) and the conclusion follows. \( \square \)

**Example 8.11.** Consider the following 5-dimensional quadratic forms over \( \mathbb{Q} \):

\[
q_1 = \langle 1, 1, 1, 1, -5 \rangle \quad \text{and} \quad q_2 = \langle 1, 1, 3, 3, -5 \rangle.
\]

Observe that \( \det q_1 = -5 = \det q_2 \), which in \( \mathbb{Q}_3 \) is a square. Furthermore, a quick computation shows \( c_3(q_1) = 1 \) and \( c_3(q_2) = -1 \). Hence by Theorem 8.10 there exists a 4-dimensional quadratic form \( r \subset q_1 \) so that \( H := \text{SO}(r) \subset \text{SO}(q_1) \) but \( H \) is not \( \mathbb{Q} \)-isomorphic to a subgroup of \( \text{SO}(q_2) \). It is not hard to check that \( r = \langle 1, 1, 1, -5 \rangle \) is such a form.

**Theorem 8.12.** Let

1. \( k \) be a number field,
2. \( m = 2n \) for \( n \geq 2 \),
3. \( q_1 \) and \( q_2 \) be nonisometric ordered \( m \)-dimensional quadratic forms over \( k \) such that
   a. \( \det q_1 = \det q_2 \) (and hence \( \text{disc}(q_1) = \text{disc}(q_2) \)),
   b. \( q_{1,v} \cong q_{2,v} \) at each infinite place \( v \),
4. there be some finite place \( v_0 \in V_k \) where:
   a. \( \text{disc}_{v_0}(q_1) = 1 = \text{disc}_{v_0}(q_2) \),
   b. \( c_{v_0}(q_1) = (-1, -1)_{v_0} \neq (-1, -1)_{v_0} = c_{v_0}(q_2) \).

Then there exists an \( (m - 1) \)-dimensional quadratic form \( r \) dividing \( q_1 \) such that no subform of \( q_2 \) represents \( \text{SO}(r) \). Furthermore if the \( q_1 \) is isotropic at a real place, then the \( r \) can be chosen to be isotropic at that real place as well.

**Proof.** Again we are going to construct the desired forms locally and then use the existence, uniqueness, and local-to-global results of Section 4 to create the desired global forms.

Let \( S = \{ \text{infinite real places of } k \} \). For each \( v \in S \), we pick the trivial square class \( \alpha_v \in k_v^\times/(k_v^\times)^2 \). By Lemma 8.1 above, we may choose an \( s \in k_v^\times \) for which \( s \in \alpha_v \) for all \( v \in S \).

For each finite place \( v \in V_k \), define \( t_v, r_v \) to be the quadratic \( k_v \)-forms with invariants given by:

\[
\begin{align*}
\dim t_v &= 1 \\
\det t_v &= \frac{\det q_1}{s} \\
\dim r_v &= m - 1 \\
\det r_v &= s \\
c_v(t_v) &= 1 \\
c_v(r_v) &= c_v(q_1) \left( s, \frac{\det q_1}{s} \right)_v.
\end{align*}
\]

We know such forms exist by Theorem 14.5 (3).

For each finite place \( v \in V_k \), define form \( t_v \) by:

\[
t_v = \left( \frac{\det q_1}{s} \right)_v.
\]

At each complex place \( v \) divides \( q_1 \otimes k_v \). By assumption, \( q_1 \) is ordered at each real place \( v \in V_k \), and hence \( t_v (= \langle 1 \rangle) \) is a subform of \( q_1 \otimes k_v \). Therefore at each infinite place it makes sense to take the complement of \( t_v \) in \( q_1 \otimes k_v \) and we may define forms \( r_v \) by:

\[
r_v = t_v^\perp.
\]

At each complex place \( v \in V_k \), we trivially have \( c_v(r_v) = 1 = c_v(q_1) \). For each real \( v \in V_k \), let \( (m_+^{(v)}, m_-^{(v)}) \) denote the signature of \( q_1 \otimes k_v \). Observe that \( r_v \) has signature \( (m_+^{(v)} - 1, m_-^{(v)}) \), and hence is isotropic whenever \( q_1 \otimes k_v \) is isotropic. Also note that:

\[
c_v(r_v) = (-1)^{m_-^{(v)}(m_+^{(v)} - 1)} = c_v(q_1).
\]
Just as in the proof of Theorem 8.10, we have:

- The families \( \{ t_v \}_{v \in V_k} \) and \( \{ r_v \}_{v \in V_k} \) satisfy the global compatibility conditions (see 8.1), and hence by Theorem 4.7 there exist quadratic forms \( t \) and \( r \) over \( k \) such that for all \( v \in V_k \), \( t \otimes k_v \cong t_v \) and \( r \otimes k_v \cong r_v \).
- By Theorem 4.4 (3), \( t_v \oplus r_v \) and \( q_1 \otimes k_v \) are isometric at each place \( v \in V_k \).
- By Theorem 4.6 we conclude \( t \oplus r \cong q_1 \) and hence \( r \) is a subform of \( q_1 \).

We claim that \( H := \text{SO}(r) \) is the split group \( B_{n-1,n-1} \) at \( v_0 \). By equation (5.1), this means we must show

\[
 c_{v_0}(r) = (-1, -1)_{v_0}^{(n-1)/(n-3)} (-1, s)_{v_0}^{n-1} = (-1, -1)_{v_0}^{(n-1)/2} (-1, s)_{v_0}^{n-1}
\]

A direct computation yields

\[
 c_{v_0}(r) = c_{v_0}(q_1) \left( \frac{s}{s} \right)_{v_0}^{\frac{(n-1)n}{2}} (-1, s)_{v_0}^{n-1} = (-1, -1)_{v_0}^{(n-1)/2} (-1, s)_{v_0}^{n-1}
\]

As we have just seen, \( H \) is split at \( k_{v_0} \), hence

\[
 \text{rank}_{k_{v_0}}(H) = \frac{(m-1) - 1}{2} = \frac{(n-1) - 1}{2} = n - 1 > n - 2 = \text{rank}_{k_{v_0}}(G_2).
\]

We have just shown that \( H \otimes k_{v_0} \) cannot be a subgroup of \( G_2 \otimes k_{v_0} \), and hence \( H \) cannot be a subgroup of \( G_2 \).

An interesting consequence of the proof is the following result.

**Corollary 8.13.** Over a local field, the split group of type \( ^1D_n^{(1)} \) cannot contain a subgroup of type \( B_{n-1,n-2} \).

**Example 8.14.** Consider the following 4-dimensional quadratic forms over \( \mathbb{Q} \):

\[
 q_1 = (1, 1, 5, -1) \quad \text{and} \quad q_2 = (3, 3, 5, -1).
\]

Observe that \( \det q_1 = -5 = \det q_2 \), which in \( \mathbb{Q}_3 \) is a square. Hence these have discriminant 1 in \( \mathbb{Q}_3 \). Furthermore, a quick computation shows \( c_3(q_1) = 1 \) and \( c_3(q_2) = -1 \). Hence by Theorem 8.12 there exists a 3-dimensional quadratic form \( r \subset q_1 \) so that \( H := \text{SO}(r) \subset \text{SO}(q_1) \) but \( H \) is not \( \mathbb{Q} \)-isomorphic to a subgroup of \( \text{SO}(q_2) \). It is not hard to check that \( r = (1, 1, -1) \) is such a form.

**Theorem 8.15.** Let

1. \( k \) be a number field,
2. \( m = 2n \) for \( n \geq 3 \),
3. \( q_1 \) and \( q_2 \) be nonisometric ordered \( m \)-dimensional quadratic forms over \( k \) such that
   a. \( \det q_1 \neq \det q_2 \) (and hence \( \text{disc}(q_1) \neq \text{disc}(q_2) \)),
   b. \( q_1, v \cong q_2, v \) at each infinite place \( v \),
4. there be some finite place \( v_0 \in V_k \) where:
   a. \( \text{disc}_{v_0} q_1 = 1 \),
(b) $\text{disc}_{v_0} q_2 \neq 1$,

(c) $c_{v_0}(q_1) \neq c_{v_0}(q_2)(-1, \text{disc}(q_2))^{\frac{m-2}{2}}$

Then there exists an $(m-2)$-dimensional quadratic form $r$ dividing $q_2$ such that no subform of $q_2$ represents $\text{SO}(r)$. Furthermore if the $q_2$ is isotropic at a real place, then the $r$ can be chosen to be isotropic at that real place as well.

Proof. As we did in Theorems 8.10 and 8.15 we construct the desired forms locally and use the results of Section 4 to create global forms. Let

$$S = \{v_0\} \cup \{\text{infinite real places of } k\}$$

For each $v \in S$, we pick square classes $\alpha_v \in k_v^\times/(k_v^\times)^2$ as follows:

- For $v_0$, let $\alpha_{v_0}$ be such that $\alpha_{v_0} = (-1)^{\frac{m-2}{2}}(k_{v_0}^\times)^2$.
- For infinite $v \in S$ let $\alpha_v = \det q_2(k_v^\times)^2$.

By Lemma 8.1 above, we may choose an $s \in k_v^\times$ for which $s \in \alpha_v$ for all $v \in S$.

For each finite place $v \in V_k$, define $t_v, r_v$ to be the quadratic $k_v$-forms with invariants given by:

$$\dim t_v = 2 \quad \dim r_v = m - 2$$

$$\det t_v = \frac{\det q_2}{s} \quad \det r_v = s$$

$$c_v(t_v) = 1 \quad c_v(r_v) = c_v(q_2) \left( s, \frac{\det q_2}{s} \right)_v$$

We know such forms exist by Theorem 4.5 (3).

For each infinite place $v \in V_k$, define form $t_v$ by:

$$t_v = \left( 1, \frac{\det q_2}{s} \right)_v$$

At each complex place $t_v$ divides $q_2 \otimes k_v$. By assumption, $q_2$ is ordered at each real place $v \in V_k$, and hence $t_v(= \langle 1, 1 \rangle)$ is a subform of $q_2 \otimes k_v$. Therefore at each infinite place it makes sense to take the complement of $t_v$ in $q_2 \otimes k_v$ and we may define forms $r_v$ by

$$r_v = t_v^\perp$$

At each complex place $v \in V_k$, we trivially have $c_v(r_v) = 1 = c_v(q_2)$. For each real $v \in V_k$, let $(m_+^{(v)}, m_-^{(v)})$ denote the signature of $q_2 \otimes k_v$. Observe that $r_v$ has signature $(m_+^{(v)} - 2, m_-^{(v)})$, and hence is isotropic whenever $q_2 \otimes k_v$ is isotropic. Also note that

$$c_v(r_v) = (-1)^{\frac{m_-^{(v)}(m_+^{(v)} - 1)}{2}} c_v(q_2)$$

We shall now show that for each place $v \in V_k$, $t_v \oplus r_v \cong q_2 \otimes k_v$. This is true by construction at the infinite places. Now suppose $v$ is finite. Clearly

$$\dim(t_v \oplus r_v) = 1 + (n - 1) = n = \dim(q_2 \otimes k_v),$$

$$\det(t_v \oplus r_v) = (\det q_2/s)s = \det(q_2 \otimes k_v),$$

and by the product formula for the Hasse–Minkowski invariant

$$c(t_v \oplus r_v) = c(t_v)c(r_v) \left( \frac{\det q_2}{s}, s \right) = c_v(q_2) \left( \frac{\det q_2}{s}, s \right)^2 = c(q_2 \otimes k_v),$$

and hence by Theorem 4.3 (3), they are isomorphic.
We now wish to build a global form, and hence must check that our forms satisfy the compatibility criteria of Theorem 4.7. Observe that $c_v(t_v) = 1$ for all $v \in V_k$ and hence $\prod_{v \in V_k} c_v(t_v) = 1$. Next observe that by our choice of $s$, $(s, \frac{\det q_2}{s})_v = 1$ at each infinite place, and hence

$$
\prod_{v \in V_k} c_v(r_v) = \left( \prod_{v \in V_k \text{ finite}} c_v(r_v) \right) \times \left( \prod_{v \in V_k \text{ real}} c_v(r_v) \right) \times \left( \prod_{v \in V_k \text{ complex}} c_v(r_v) \right)
$$

$$
= \left( \prod_{v \in V_k \text{ finite}} c_v(q_2) \left( s, \frac{\det q_2}{s} \right)_v \right) \times \left( \prod_{v \in V_k \text{ real}} c_v(q_2) \right) \times \left( \prod_{v \in V_k \text{ complex}} c_v(q_2) \right)
$$

$$
= \prod_{v \in V_k} c_v(q_2) \prod_{v \in V_k} \left( s, \frac{\det q_2}{s} \right)_v
$$

(8.2)

$$
= 1.
$$

Where we know the final product is trivial because both the Hasse–Minkowski invariant and the Hilbert symbol of global objects satisfy the product formula.

By Theorem 4.7, there exist quadratic forms $t$ and $r$ over $k$ such that for all $v \in V_k$, $t \otimes k_v \cong t_v$ and $r \otimes k_v \cong r_v$. Furthermore, for each $v \in V_k$, we have shown that $t_v \oplus r_v \cong q_2 \otimes k_v$, so by Theorem 4.6 we conclude $t \oplus r \cong q_2$, and hence $r$ is a subform of $q_2$.

Let $H = \text{SO}(r)$. We will show that $H \not\subseteq G_1 = \text{SO}(q_1)$, and hence that there are no representatives $r'$ of $H$ such that $r' \subset q_1$. Again $H$ is a group of type $D_n$ over $k$. Let $r'$ be any representative of $H$. As in the proof of Theorem 8.10 the group $H$ determines the following invariants of $r'$:

1. $\dim(r') = 2n - 2 = \dim(r)$.
2. $\text{disc}_v(r') = 1$ at precisely the places $v \in V_k$ where $H \otimes k_v$ is a group of inner type (i.e., the $*$-action is trivial). This means that $\text{disc}_v(r') = 1$ if and only if $\text{disc}_v(r) = 1$, or in other words, at the places where $\text{disc}_v(r) = 1$, then $\det r' = \det r$.
3. $c_v(r') = c_v(r)$ at each place $v$ where $\text{disc}_v(r) = 1$ (see equation (5.2)).

Let $r'$ be any quadratic form satisfying these three properties. Suppose there exists some form $t'$ such that $r' \oplus t' \cong q_1$. It follows that $\dim t' = 2$, $\det t' = \det q_1/\det r'$.

Observe that our choice of $s$ implies that

$$
\text{disc}_v(r) = (-1)^{(n-1)} \det r = (-1)^{2n-2} = 1
$$

Hence at $v_0$, we have $\det r = \det r'$ and $c_{v_0}(r) = c_{v_0}(r')$. Furthermore we have

$$
\det_{v_0} t' = \frac{\det_{v_0} q_1}{\det_{v_0} r'}
$$

$$
= \frac{(-1)^{\frac{n}{2}} \text{disc}(q_1)}{(-1)^{\frac{n-1}{2}} \text{disc}(r')}
$$

$$
= \frac{(-1)^{\frac{n}{2}}}{(-1)^{\frac{n-1}{2}}}
$$

$$
= -1,
$$
and thus by the exceptional restriction, \( c_{v_0}(t') = 1 \). Now the product formula at \( v_0 \) yields the following contradiction:

\[
c_{v_0}(q_1) = c_{v_0}(r' \oplus t')
= c_{v_0}(r') \left( \frac{\det q_1}{\det r'} \right)_{v_0}
= c_{v_0}(q_2) \left( \frac{\det q_2}{\det r'} \right)_{v_0}
\]

\[
= c_{v_0}(q_2) \left( \frac{\det q_1 \det q_2}{(\det r')^2} \right)_{v_0}
\]

\[
= c_{v_0}(q_2) \left( \frac{m - 2}{2}, (1) \frac{m - 2}{2} \right)_{v_0}
\]

Hence no representative of \( H \) can be a subform of \( q_1 \), concluding the proof. \( \square \)

**Example 8.16.** Consider the following 6-dimensional quadratic forms over \( \mathbb{Q} \):

\[
q_1 = \langle 1, 1, 1, 3, 3, -1 \rangle \quad \text{and} \quad q_2 = \langle 1, 1, 1, 1, 1, -5 \rangle.
\]

Observe that \( \det q_1 = -1 \neq -5 = \det q_2 \). Furthermore, \( \text{disc}_3(q_1) = 1 \), but \( \text{disc}_3(q_2) = 5 \) which is not a square in \( \mathbb{Q}_3 \). Furthermore, a quick computation shows \( \text{disc}_3(q_1) = -1 \) and \( \text{disc}_3(q_2) = 1 \). Hence by Theorem 8.15 there exists a 4-dimensional quadratic form \( r \subset q_2 \) so that \( H := \text{SO}(r) \subset \text{SO}(q_2) \) but \( H \) is not \( \mathbb{Q} \)-isomorphic to a subgroup of \( \text{SO}(q_1) \). It is not hard to check that \( r = \langle 1, 1, 1, -5 \rangle \) is such a form.

**Constructing Subforms In Codimension \( > 2 \).** We have shown that given certain non-isometric forms, we may find codimension 1 or 2 subforms of one that are not represented in the other. In this section we show that this is the best we can hope for.

**Proposition 8.17.** Let \( k \) be a number field and let \( q_1 \) and \( q_2 \) be \( m \)-dimensional quadratic forms over \( k \), \( m \geq 4 \), which are isometric at each infinite place. If \( r \) is a \( j \)-dimensional subform of \( q_1 \), where \( 0 < j < m - 2 \), then \( r \) is also a subform of \( q_2 \).

**Proof.** As usual, we construct forms locally from which we will obtain a global form. For each finite \( v \in V_k \), let \( t_v \) be the \( k_v \) form uniquely determined by

- \( \dim t_v = n - m \),
- \( \det t_v = \frac{\det q_2}{\det r} \), and
- \( c_v(t_v) = c_v(q_2) c_v(r) \left( \frac{\det(r)}{\det(q_2)} \right)_{v} \).

We know such forms exist by Theorem 4.15 (3). Since \( q_1 \otimes k_v \) and \( q_2 \otimes k_v \) are isometric at each infinite \( v \in V_k \), then \( r \otimes k_v \) is a subform of \( q_2 \otimes k_v \), and hence it makes sense to take its complement. We therefore define

- \( t_v := (r \otimes k_v)^\perp \).

From this definition, it immediately follows that at each infinite place

\[
c_v(t_v) = c_v(q_2) c_v(r) \left( \frac{\det(r)}{\det(q_2)} \right)_{v}.
\]

We now wish to build a global form, and hence must check that our forms satisfy the compatibility criteria of Theorem 4.7. This can be seen with the following computation:
\[
\prod_{v \in V_k} c_v(t_v) = \left( \prod_{v \in V_k} c_v(q_2) c_v(r) \left( \frac{\det(r)}{\det(q_2)} \right)_v \right) = \left( \prod_{v \in V_k} c_v(q_2) \right) \times \left( \prod_{v \in V_k} c_v(r) \right) \times \left( \prod_{v \in V_k} \left( \frac{\det(r)}{\det(q_2)} \right)_v \right) = 1.
\]

Where we know the final product is trivial because both the Hasse–Minkowski invariant and the Hilbert symbol of global objects satisfy the product formula. Hence we may now use Theorem 8.7 to obtain a quadratic form \( t \) over \( k \) such that for all \( v \in V_k, t \otimes k_v \cong t_v \). Furthermore, for each \( v \in V_k \), \( t_v \oplus r_v \) and \( q_2 \otimes k_v \) have the same local invariants so by Theorem 4.4 they are isometric, and by Theorem 4.6 we conclude \( t \oplus r \cong q_2 \), and hence \( r \) is a subform of \( q_2 \).

\[\Box\]

9. The Semisimple Subgroup Spectrum and the Proofs of Theorems B and C

In this section, we bring together the results of the earlier sections to prove Theorems B and C. In preparation for these proofs, we introduce the notion of the semisimple subgroup spectrum of an algebraic group. If \( G \) is an algebraic group defined over a number field \( k \), let its semisimple subgroup spectrum be the set

\[SS_k(G) = \left\{ \text{Isomorphism classes of proper semisimple } k \text{-subgroups of } G \right\} \]

and its reduced semisimple subgroup spectrum be the related set

\[SSS_k(G) = \left\{ \text{Aut}(k/\mathbb{Q})\text{-orbits of isomorphism classes of proper semisimple } k \text{-subgroups of } G \right\} \]

Since we are interested in commensurability classes of spaces, in light of Corollary 3.7 we state and prove several theorems for \( SS_k(G) \). However, with the obvious modifications, most of the result also hold for \( SSS_k(G) \). We are now in position to state and prove the following theorem.

Theorem 9.1. Let \( k \) be a number field and \( G_1 \) and \( G_2 \) be absolutely almost simple \( k \)-groups coming from quadratic forms of dimension \( \geq 5 \). Then \( SS_k(G_1) = SS_k(G_2) \) implies \( G_1 \) and \( G_2 \) are \( k \)-isomorphic.

To prove Theorem 9.1 we will in fact prove the contrapositive. We assume we have two nonisomorphic groups which give nonisometric forms. We then chose certain subforms and, using the classical invariants, check the Tits index at local places to guarantee these forms give rise to the desired subgroups.

Theorem 9.2. Let \( k \) be a number field and \( G_1 \) and \( G_2 \) be semisimple \( k \)-groups coming from quadratic forms of dimension \( \geq 5 \). If \( G_1 \) and \( G_2 \) are not \( k \)-isomorphic up to the action of \( \text{Aut}(k/\mathbb{Q}) \), then there exists a semisimple \( k \)-subgroup \( H \) which, up to the action of \( \text{Aut}(k/\mathbb{Q}) \), is a \( k \)-subgroup of one but not the other. Furthermore, if either \( G_1 \) or \( G_2 \) is isotropic at a real place, the \( H \) can be chosen to be isotropic at a real place.

Proof. Let \( q_1 \) and \( q_2 \) represent \( G_1 \) and \( G_2 \) respectively such that at each real infinite place \( v \) the signature of \( q_i \) is \((m_{+;i}^{(v)}, m_{-;i}^{(v)})\). By Lemma 8.2 we may assume that \( m \geq m_{+;i}^{(v)} \geq m_{-;i}^{(v)} \geq 0 \) for all real places \( v \). If no form in the \( \text{Aut}(k/\mathbb{Q}) \)-orbit of \( q_1 \) is isometric to \( q_2 \) at every real place, then by Theorem 8.8 the result follows.

Now suppose we have fixed representatives \( q_1 \) and \( q_2 \) that are isometric at all infinite places. Since the groups \( G_1 \) and \( G_2 \) are not \( k \)-isomorphic, the Hasse principle for special orthogonal groups [PlRa, pg. 348] implies that there exists some finite place \( v_0 \) where \( G_1 \otimes k_{v_0} \) and \( G_2 \otimes k_{v_0} \) are not \( k_{v_0} \)-isomorphic. Since the groups are not isometric at \( v_0 \), the forms are not isometric over \( k \).
If $m$ is odd, then by Lemma 8.1 we may replace $q_1$ and $q_2$ with similar forms as necessary to guarantee that $\det v_0 q_1 = \det v_0 q_2 = 1$ while not altering the signatures at the infinite places. Hence $c_{v_0}(q_1) \neq c_{v_0}(q_2)$ and then by Theorem 8.10 the result follows.

Now suppose $m = 2n$ is even. If $\det q_1 = \det q_2$ but $\disc_{v_0}(q_i) \neq 1$, then by Lemma 4.3 and Lemma 8.1 we may replace $q_2$ with a similar form while not altering the signatures at the infinite place and for which $c_{v_0}(q_1) = c_{v_0}(q_2)$. This would imply $q_1$ and $q_2$ are isomorphic over $k_{v_0}$, contradicting our choice of $v_0$. Hence if $\det q_1 = \det q_2$, then after possibly relabeling, their invariants must satisfy both of the following:

1. $\disc_{v_0}(q_1) = 1 = \disc_{v_0}(q_2)$, and
2. $c_{v_0}(q_1) = \lambda q_1 = (1, -1)^{\frac{n(n-1)}{2}}_{v_0} \neq (-1, -1)^{\frac{n(n-1)}{2}}_{v_0} = c_{v_0}(q_2)$.

By Theorem 8.12 the result follows. Otherwise, if $\det q_1 \neq \det q_2$ then in terms of their forms this means, after possible relabeling:

1. $\disc_{v_0} q_1 = 1$,
2. $\disc_{v_0} q_2 \neq 1$.

Furthermore, if $c_{v_0}(q_1) = c_{v_0}(q_2)(-1, \disc(q_2))_{v_0}^{\frac{n-2}{2}}$, then we will replace $q_2$ with a similar form in the following way. Let $S = \{v_0\} \cup \{\text{infinite real places of } k\}$ and for each $v \in S$, we pick a square class $\alpha_v \in k_v^\times/(k_v^\times)^2$ as follows:

- at $v_0$, $(\alpha_{v_0}, \disc(q_2))_{v_0} = -1$ (note that such a class exists by the nondegeneracy of the Hilbert symbol and the fact that $\disc(q_2) \neq 1$), and
- for all $v \in S$ real, $\alpha_v$ is trivial.

Then by Lemma 4.3 it follows that $c_{v_0}(\lambda q_2) = -c_{v_0}(q_2)$ and replacing $q_2$ by $\lambda q_2$, it follows that $c_{v_0}(q_1) \neq c_{v_0}(q_2)(-1, \disc(q_2))_{v_0}^{\frac{n-2}{2}}$. Then by Theorem 8.15 the result follows. \hfill $\Box$

**Theorem 9.3.** Let $M_1$ and $M_2$ be arithmetic locally symmetric spaces coming from quadratic forms of dimension $m \geq 5$ such that $k(M_1)$ and $k(M_2)$ are $\Gal(\overline{Q}/Q)$-conjugate. Then $\Q\TG(M_1) = \Q\TG(M_2)$ implies $M_1$ and $M_2$ are commensurable.

**Proof.** Let $k$ be a fixed representative if the isomorphism class of $k(M_1)$ and $k(M_2)$. By assumption there exists quadratic forms $q_1$ and $q_2$ over $k$ such that $M_i$ arises from the absolutely almost simple $k$-groups $G_i := \SO(q_i)$. By Proposition 7.5 $\dim q_1 = \dim q_2$. We shall prove the contrapositive. Suppose $M_1$ and $M_2$ are not commensurable. Then $G_1$ and $G_2$ are not $k$-isomorphic up to the action of $\Aut(k/Q)$. By Theorem 9.2 there exists an $i \in \{1, 2\}$ where $G_i$ contains a semisimple $k$-subgroup $H$ which, up to the action of $\Aut(k/Q)$, is isotropic at a real place and which is not contained in $G_j$, $j \neq i$. Hence $M_i$ contains a totally geodesic submanifold not commensurable to a totally geodesic submanifold of $M_j$ and the resulting contradiction shows $M_1$ and $M_2$ are commensurable. \hfill $\Box$

**Proof of Theorem B.** By Theorem A and Theorem 9.3. \hfill $\Box$

**Proof of Theorem C.** This is an immediate corollary to Theorem B upon specializing to the $\R$-rank 1 case. \hfill $\Box$

Unravelling the proof of Theorem B and Theorem 9.2 we see that we can tell apart noncommensurable even dimensional arithmetic hyperbolic spaces using only totally geodesic hypersurfaces, as we record in the following theorem.

**Theorem 9.4.** Let $M_1$ and $M_2$ be even dimensional arithmetic hyperbolic manifolds of dimension $n \geq 4$. Suppose every totally geodesic hypersurface in one is commensurable to a totally geodesic hypersurface in the other. Then $M_1$ and $M_2$ are commensurable.
Note that our constructions show noncommensurable arithmetic locally symmetric spaces coming from quadratic forms have different small codimension totally geodesic subspaces, while (due to Proposition 5.17) these spaces have the same commensurability classes of totally geodesic subspaces in high codimensions.

Theorem B shows that for spaces coming from quadratic forms, the totally-geodesic-commensurability-spectrum determines the commensurability class, which in turn determines the rational length spectrum $\mathbb{Q}L(M)$. We have just shown the following theorem.

**Theorem 9.5.** Let $M_1$ and $M_2$ be simple arithmetic locally symmetric spaces coming from quadratic forms of dimension $\geq 5$. Then $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$ implies $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$.

Hence the set of totally geodesic subspaces determines the rational multiples of the lengths of all closed geodesics, even though there exist closed geodesics which do not lie in any proper nonflat totally geodesic subspace. (The existence of such geodesics follows from the existence of $\mathbb{R}$-regular elements in these arithmetic lattices. See [Pr94] for an elementary proof of this fact.)

10. Hyperbolic Subspace Dichotomy and Other Applications

We have seen that quadratic subforms give totally geodesic subspaces, which we call subform subspaces. We shall now see that in the case of standard arithmetic hyperbolic spaces coming from quadratic forms, these are all that may arise.

**Proposition 10.1.** If $M$ is a standard arithmetic hyperbolic $n$-orbifold, $n \geq 4$, and $N \in \mathbb{Q}TG(M)$ then

1. $k(N) = k(M)$ and
2. $N$ is a subform subspace.

**Proof.** By assumption, $M = M_q$ where $(V, q)$ is a quadratic $m$-space, $m = n + 1 \geq 5$, over a totally real number field $k$ with a unique real place $v$ where $q$ is isotropic. Let $G = \text{SO}(V, q)$ and we will denote $k_v$ by $\mathbb{R}$. Let $H \subset G := G(\mathbb{R})$ be the connected semisimple Lie subgroup giving rise to $N$. Since $M_q$ is hyperbolic, it follows that $H = H(\mathbb{R})^s$ where $H = \text{SO}(W', r')$ for some $\mathbb{R}$-subspace $W' \subset V_\mathbb{R}$ and $r'$ the restriction of $q_\mathbb{R}$ to $W'$. Let $L \subset V$ be an $O_k$-lattice and let $G_L$ be its stabilizer in $G$. Since $L := G_L \cap H$ is a lattice in $H$, it is Zariski-dense in $H$. It follows that the $\mathbb{R}$-span of $L \cap W'$ must be all of $W'$. Let $W$ denote the $k$-span of $L \cap W'$ and let $r$ be the restriction of $q$ to $W$. Then $N = N_r$ and hence the result follows.

We will now give some geometric consequences of these results. The first if the proof of Theorem D our Hyperbolic Dichotomy Theorem.

**Proof of Theorem D.** If $M_1$ and $M_2$ share a single finite volume totally geodesic subspace, Proposition 10.1 implies $k(M_1)$ and $k(M_2)$ are $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugate. Let $k$ be a fixed representative of the isomorphism class of $k(M_1)$ and $k(M_2)$. By Proposition 8.9, there are quadratic $k$-forms $q_1$ and $q_2$ which represent the commensurability classes of $M_1$ and $M_2$ respectively and are isotropic at precisely the same place $v \in V_k$. The result follows by Proposition 8.17.

In particular, since all noncompact arithmetic hyperbolic spaces come from $k = \mathbb{Q}$, which only has real place, it follows that all standard noncompact, finite volume, arithmetic, hyperbolic spaces have the exact same collection of finite volume totally geodesic subspaces of noncompact type of codimension $> 2$, which we record in the following corollary.

**Corollary 10.2.** Let $M_1$ and $M_2$ be $n$-dimensional $(n \geq 4)$ noncompact finite volume arithmetic hyperbolic spaces coming from quadratic forms. Then, up to commensurability, $M_1$ and $M_2$ have the exact same collection of finite volume totally geodesic subspaces of noncompact type of codimension $> 2$.

Recent work by McReynolds [Mc] shows that certain noncommensurable arithmetic manifolds arising from the semisimple Lie groups of the form $(\text{SL}_d(\mathbb{R}))^r \times (\text{SL}_d(\mathbb{C}))^s$ have the same commensurability classes of totally geodesic surfaces coming from a fixed field. An immediate consequence of our work above proves the following...
Proposition 10.3. For each $n \geq 4$, there exist noncommensurable standard arithmetic hyperbolic $n$-manifolds $M_1$ and $M_2$ that have the same commensurability classes of totally geodesic surfaces.

We conclude this section by addressing the following question was posed to us by Jean-François Lafont:

Question 10.4. Let $M_1$ and $M_2$ be Riemannian manifolds. When is it the case that $\mathbb{Q}T \Gamma(M_1) \subset \mathbb{Q}T \Gamma(M_2)$ implies $M_1 \subset M_2$?

It turns out that for arithmetic hyperbolic spaces, we can largely answer this question. When the difference $\dim M_2 - \dim M_1$ is large, we have a positive result as we shall now see.

Proposition 10.5. Let $M_1$ and $M_2$ be standard arithmetic hyperbolic spaces. Suppose that $3 \leq \dim M_1 \leq \dim M_2 - 3$ and $\mathbb{Q}T \Gamma(M_1) \subset \mathbb{Q}T \Gamma(M_2)$. Then up to commensurability $M_1 \subset M_2$.

Proof. By assumption, every totally geodesic surface of $M_1$ is totally geodesic in $M_2$. Corollary 7.6 implies that $k(M_1) = k(M_2) =: k$. Let $q_i$ be quadratic forms over $k$ which give rise to $M_i$. Our assumption on $\mathbb{Q}T \Gamma$ shows that $q_1$ and $q_2$ are isotropic at the same real place of $k$. Then by Proposition 8.17, it follows that $q_1$ is a subform of $q_2$ and the result follows. \hfill \Box

However when $\dim M_2 - \dim M_1$ is small we can have negative results. Hence there do exist counterexamples to the above question.

Example 10.6. Consider following quadratic forms over $\mathbb{Q}$ described in Example 8.16

$$q_1 = \langle 1, 1, 1, -5 \rangle \quad \text{and} \quad q_2 = \langle 1, 1, 1, 3, 3, -1 \rangle.$$

By Theorem 8.15 the 3-dimensional hyperbolic space $M_{q_1}$ is not commensurable to a totally geodesic subspace of the five dimensional space $M_{q_2}$, yet by Proposition 8.17 they contain precisely the same totally geodesic surfaces.

Hence we have proven the following.

Proposition 10.7. There exist arithmetic hyperbolic manifolds $M_1$ and $M_2$ for which $\mathbb{Q}T \Gamma(M_1) \subset \mathbb{Q}T \Gamma(M_2)$ but $M_1$ is not commensurable to a totally geodesic submanifold of $M_2$.

11. Nonstandard Arithmetic Hyperbolic Manifolds and Proof of Theorems E and F

We now discuss the construction of arithmetic lattices in groups of Cartan–Killing type $D_n$ arising from skew hermitian forms over division algebras over number fields. It is similar to Construction 4.8 for quadratic forms.

Construction 11.1.

1. Let $k$ be a number field.
2. Let $D$ be a quaternion division algebra with center $k$.
3. Let $(V, h)$ an $n$-dimensional skew Hermitian space over $D$.
4. Let $G = SU(V, h)$, $G' := (R_{k/\mathbb{Q}}G)(\mathbb{R})$, where $R$ denotes restriction of scalars.
5. Let $G$ be the projection of $G'$ and onto its noncompact factors and $K \subset G$ its maximal compact subgroup.
6. Fix an order $\mathcal{O}_D$ in $D$ and an $\mathcal{O}_D$-lattice $L \subset V$, and let $G_L = \{ T \in G(k) \mid T(L) \subset L \}$.
7. Define $\Gamma_L$ to be the projection of $G_L$ to $G$ and let $M_L = \Gamma_L \backslash G/K$.

A choice of another order in $D$ and another lattice $L' \subset V$ will produce a space $M_{L'}$, which is commensurable with $M_L$. Hence choosing $h$ uniquely determines a commensurability class which we denote by $M_h$. More on this construction can be found in [LM].
Theorem 11.2. Let $k$ be a number field and $G_1$ and $G_2$ be semisimple $k$-groups of type $D_n$ such that one comes from a quadratic form of dimension $m \geq 3$ and the other from a skew hermitian form. Then $SS_k(G_1) \neq SS_k(G_2)$.

We will show the contrapositive, which is a consequence of the following proposition.

Proposition 11.3. Let $G_i$ be algebraic $k_i$-groups, $i \in \{1, 2\}$, such that $G_1$ comes from a $2n$-dimensional quadratic form over $k_1$, $n \geq 2$, and $G_2$ comes from an $n$-dimensional skew hermitian forms over a division algebra $D$ over $k_2$. Then there exists a semisimple $k_1$-group $H$ which is a subgroup of $G_1$ but no group in its $\text{Aut}(k/\mathbb{Q})$-orbit is a subgroup of $G_2$. Furthermore, if $G_1$ is isotropic at a real place then $H$ can be chosen to be isotropic at a real place.

Proof. Choose a form $q$ to represent $G_1$ such that $\langle a_1, a_2, \ldots, a_{2n} \rangle$ is a diagonal representation of $q$. Let $q' = \langle a_1, a_2, \ldots, a_j \rangle$ for $n/2 + 2 < j < 2n$ and let $H = \text{SO}(q) \subset G_1$. We shall show that $H$ cannot be a subgroup of $G_2$. Let $v \in V_k$ be a finite place where $D$ ramifies.

$$\text{rank}_{k_v}(G_2) \leq \frac{n}{2} \leq j - 2 \leq \text{rank}_{k_v}(H).$$

Hence by rank considerations $H$ cannot be a subgroup of $G_2$. Furthermore, if $q$ is isotropic at a real place, then we may pick a $q'$ which is also isotropic and the result follows. □

Proof of Theorem E. The result follows from Corollary 3.7 and Theorem 11.2 □

The case when both groups come from skew hermitian forms over division algebras is more difficult. What we can say is the following.

Proposition 11.4. Let $G_1$ and $G_2$ be algebraic $k$-groups coming from $n$-dimensional skew hermitian forms over division algebras $D_1$ and $D_2$ respectively. If $D_1$ and $D_2$ are not isomorphic up to the action of $\text{Aut}(k/\mathbb{Q})$, then $SS_k(G_1) \neq SS_k(G_2)$.

Proof. Let $\langle a_1, a_2, \ldots, a_n \rangle$ be any diagonal representation of $h_1$. Let $h'_i = \langle a_1, a_2, \ldots, a_j \rangle$ for $n/2 + 2 < j < n$. Let $H = \text{SU}(h) \subset G_1$. We shall show that $H$ cannot be a subgroup of $G_2$. Since $D_1$ and $D_2$ are not isomorphic up to the action of $\text{Aut}(k/\mathbb{Q})$, there is a finite place $v \in V_k$ where one splits and the other ramifies. After relabeling if necessary, we may assume $D_1$ splits and $D_2$ ramifies.

$$\text{rank}_{k_v}(G_2) \leq \frac{n}{2} \leq j - 2 \leq \text{rank}_{k_v}(H).$$

Hence by rank considerations $H$ cannot be a subgroup of $G_2$. □

Proof of Theorem F. Let $G_i = \text{SU}(h_i)$, $i = 1, 2$, be groups giving rise to $M_i$, where $h_i$ is an $n$-dimensional skew hermitian form over $D_i$. Let $r$ be an $(n - 1)$-dimensional hermitian subform of $h_1$ which is isotropic at the real place $h_1$ is isotropic. Let $H_1 := \text{SU}(r)$. Observe that, when $n \geq 4$,

$$\dim \text{SU}(h_2) = n(2n - 1) < 2(n - 1)(2n - 3) = 2\dim \text{SU}(r).$$

By Proposition 7.4, $k_1$ and $k_2$ are $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugate. We now prove the contrapositive of the remaining statement. Let $k$ be a fixed representative of this isomorphism class and suppose that $D_1$ and $D_2$ are not isomorphic up to the action of $\text{Aut}(k/\mathbb{Q})$. We may now apply Proposition 11.4 and the result follows. □

However the following question remains open.

Question 11.5. Let $G_1$ and $G_2$ be algebraic $k$-groups coming from $n$-dimensional skew hermitian forms over the same division algebras $D$. If $G_1$ and $G_2$ are not $k$-isomorphic, then must $SS_k(G_1) \neq SS_k(G_2)$?

The primary difficulty to addressing this question is the lack of local and global existence theorems for skew hermitian forms over division algebras. We may immediately rephrase the above question to one about locally symmetric spaces.
Question 11.6. Let $M_1$ and $M_2$ be arithmetic locally symmetric spaces coming from skew hermitian forms over a division algebra $D$ over a number field $k$. Does $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$ imply $M_1$ and $M_2$ are commensurable?

Answering this would complete the analysis of $\mathbb{Q}TG(M)$ for simple arithmetic spaces of Cartan–Killing type $D_n$ for all $n \geq 5$, and all those of type $D_n$, $2 \leq n \leq 4$, not arising from exceptional isomorphisms (e.g., $D_4$ triality).

12. Acknowledgments

We thank Matthew Stover for suggesting this problem and for countless helpful conversations. The author was supported by the NSF RTG grant 1045119. We would like to also thank Jean-François Lafont, Lucy Lifschitz, Ben Linowitz, and Ben McReynolds for many valuable and interesting discussions.

REFERENCES

[B1] A. Borel Linear Algebraic Groups. Graduate Texts in Mathematics, 126, Springer-Verlag, (1991).
[B2] A. Borel Introduction aux groupes arithmétiques. Publications de l’Institut de Mathématique de l’Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, 1341, Paris: Hermann, (1969).
[B60] A. Borel Density properties for certain subgroups of semi-simple groups without compact components. Ann. of Math. (2) 72 (1960) 179-188.
[B65] A. Borel Density and maximality of arithmetic subgroups. J. Reine Angew. Math. 224 (1966) 78-89.
[BoHC] A. Borel and Harish-Chandra Arithmetic Subgroups of Algebraic Groups. Annals of Mathematics, second Series, 75, No. 3, (1962).
[BoPr] A. Borel and G. Prasad Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups. Publ. Math. I.H.E.S., 69 (1989), 119-171.
[BoTi] A. Borel and J. Tits Groupes réductifs. Publ. Math. I.H.E.S., 27 (1965), 55-150.
[Ca] J. W. S. Cassels Rational Quadratic Forms. Dover Books on Mathematics, (2008).
[CHLR] T. Chinburg, E. Hamilton, D. D. Long, and A. W. Reid Geodesics and commensurability classes of arithmetic hyperbolic 3-manifolds. Duke Math. J. 145 No. 1 (2008), 25-44.
[CGP] B. Conrad, O. Gabber, and G. Prasad Pseudo-reductive groups. New Mathematical Monographs, 17. Cambridge University Press, Cambridge, (2010).
[DC] M.P. Do Carmo. Riemannian Geometry. Birkhäuser, Boston, MA, (1992).
[Ga] S. Garibaldi Outer automorphisms of algebraic groups and determining groups by their maximal tori. Michigan Mathematical Journal, 61, #2 (2012), 227-237.
[GPS] M. Gromov and I. Piatetski-Shapiro Nonarithmetic groups in Lobachevsky spaces. Inst. Hautes Études Sci. Publ. Math. No. 66 (1988), 93-103.
[GS] M. Gromov and R. Schonen Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. Inst. Hautes Études Sci. Publ. Math. No. 76 (1992), 165-246.
[He] S. Helgason Differential Geometry, Lie Groups, and Symmetric Spaces. AMS, Providence, (2001).
[Kac] M. Kac Can one hear the shape of a drum? Amer. Math. Monthly, 73 (1966) no. 4.
[Lam] T.Y. Lam Introduction To Quadratic Forms Over Fields. Graduate Studies in Mathematics, 67, AMS, Providence, RI (2005).
[LM] J.-S. Li and J. Millson On the first Betti number of a hyperbolic manifold with an arithmetic fundamental group. Duke Math. J. 71 (1993) no. 2, 365-401.
[Lub97] A. Lubotzky Eigenvalues of the Laplacian, the First Betti Number and the Congruence Subgroup Problem. Ann. of Math. (2) 145 (1997) 441-452.
[LSV] A. Lubotzky, B. Samuels, and U. Vishne Division algebras and noncommensurable isospectral manifolds. Duke Math. J. 135 (2006), no. 2, 361-379.
[Mac] C. Maclachlan Commensurability classes of discrete arithmetic hyperbolic groups. Groups Geom. Dyn., 5, No. 4, (2011).
[MaRe] C. Maclachlan and A. W. Reid The Arithmetic of Hyperbolic 3-Manifolds. Graduate Texts in Mathematics, 219, Springer-Verlag, (2003).
[Mar] G.A. Margulis Arithmeticity of the irreducible lattices in the semi-simple groups of rank greater than 1. Invent. Math, 76, (1984) 93-120.
[Mc] D. B. McReynolds Geometric spectra and commensurability, to appear in Cand. Jour. Math, (2014).
[McRe] D. B. McReynolds and A. W. Reid The genus spectrum of hyperbolic 3-manifolds. to appear in Math. Res. Lett., (2014).
[Me] J. Meyer Ph. D. thesis. Univ. of Michigan (2013).
[M] J.S. Milne [Class Field Theory]. Course Notes. Version 4.02, (2013).

[Mi] J. Milnor Eigenvalues of the Laplace operator on certain manifolds. Proc. Nat. Acad. Sci. U.S.A., 51, (1964) 542.

[MT] G. Mostow and T. Tamagawa On the compactness of arithmetically defined homogeneous spaces. Ann. of Math. (2) 76 (1962) 446-463.

[OM] O.T. O’Meara Introduction to Quadratic Forms. Grundlehren der mathematischen Wissenschaften, 117, Springer-Verlag, Berlin (1973).

[PlRa] V. Platonov and A. Rapinchuk. Algebraic groups and number theory. Translated from Russian by Rachel Rowen. Pure and Applied Mathematics, 139. Academic Press, Inc., Boston, MA, (1994).

[Pr89] G. Prasad Volumes of $S$-arithmetic quotients of semi-simple groups. Inst. Hautes Études Sci. Publ. Math. 69 (1989), no. 91-117.

[Pr94] G. Prasad $R$-regular elements in Zariski-dense subgroups. Quart. J. Math. Oxford Ser. (2) 45 (1994), no. 180, 541-545.

[PR09] G. Prasad and A. S. Rapinchuk Weakly commensurable arithmetic groups and isospectral locally symmetric spaces. Inst. Hautes Études Sci. Publ. Math. 109 (2009), 113-184.

[PR10] G. Prasad and A. S. Rapinchuk Developments on the congruence subgroup problem after the work of Bass, Milnor and Serre. Collected Papers of John Milnor: V. Algebra. AMS, (2010), 307-326.

[Re87] A. W. Reid Ph. D. thesis. Univ. of Aberdeen (1987).

[Re92] A. W. Reid Isospectrality and commensurability of arithmetic hyperbolic 2- and 3- manifolds. Duke Math. J, 65 (1992), 215-228.

[Sch] W. Scharlau Quadratic and Hermitian Forms. Grundlehren der mathematischen Wissenschaften, 270, Springer-Verlag, Berlin (1985).

[Sel] A. Selberg On discontinuous groups in higher-dimensional symmetric spaces. “Contributions to Function Theory,” Tata Institute of Fundamental Research, Bombay (1960), 147-164.

[Su] T. Sunada Riemann coverings and isospectral manifolds. Ann. Math. (2), 121 (1985), 169-186.

[Sy] J.J. Sylvester A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares. Philosophical Magazine (Ser. 4) 4 (23) (1852), 138-142.

[Th] W. Thurston The Geometry and Topology of 3-Manifolds. Lecture Notes, Princeton University Math. Dept. (1978).

[Ti] J. Tits Classification of algebraic semisimple groups. Proc. Summer Inst on Algebraic Groups and Discontinuous Groups (Boulder, 1965), Proc. Sympos. Pure Math., 9, Amer. Math Soc., Providence, R.I., (1966), 33-62.

[Vig] M.-F. Vignéras Variétés Riemanniennes isospectrales et non isométriques. Ann. Math. (2), 112 (1980), 21-32.

[Vin71] E. B. Vinberg Rings of definition of dense subgroups of semisimple linear groups. Math. USSR Izv., 5 (1971), 45-55.

[Vin93] E. B. Vinberg Geometry II, Encyclopedia of Mathematical Sciences. Vol. 29 Springer-Verlag, New York (1993).

[W] H. Weyl Über die asymptotische Verteilung der Eigenwerte. Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen (1911), 110-117.