Pointwise strong approximation of almost periodic functions in $S^1$

Włodzimierz Lenski and Bogdan Szal
University of Zielona Góra
Faculty of Mathematics, Computer Science and Econometrics
65-516 Zielona Góra, ul. Szafrana 4a, Poland
W.Lenski@wmie.uz.zgora.pl, B.Szal@wmie.uz.zgora.pl

Abstract
We consider the class $GM(2\beta)$ in pointwise estimate of the deviations in strong mean of $S^1$ almost periodic functions from matrix means of partial sums of their Fourier series.

Key words: Almost periodic functions; Rate of strong approximation; Summability of Fourier series

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1 Introduction
Let $S^p$ ($1 \leq p \leq \infty$) be the class of all almost periodic functions in the sense of Stepanov with the norm

$$
\|f\|_{S^p} := \begin{cases} 
\sup_{u} \left\{ \frac{1}{2} \int_{u}^{u+\pi} |f(t)|^p dt \right\}^{1/p} & \text{when } 1 \leq p < \infty \\
\sup_{u} |f(u)| & \text{when } p = \infty.
\end{cases}
$$

Suppose that the Fourier series of $f \in S^p$ has the form

$$
Sf(x) = \sum_{\nu=-\infty}^{\infty} A_{\nu}(f) e^{i\lambda_{\nu} x}, \quad \text{where } A_{\nu}(f) = \lim_{L \to \infty} \frac{1}{L} \int_{0}^{L} f(t)e^{-i\lambda_{\nu}t} dt,
$$

with the partial sums

$$
S_{\gamma_k} f(x) = \sum_{|\lambda_{\nu}| \leq \gamma_k} A_{\nu}(f) e^{i\lambda_{\nu} x}
$$

and that $0 = \lambda_0 < \lambda_{\nu} < \lambda_{\nu+1}$ if $\nu \in \mathbb{N} = \{1, 2, 3, \ldots\}$, $\lim_{\nu \to \infty} \lambda_{\nu} = \infty$, $\lambda_{-\nu} = -\lambda_{\nu}$, $|A_{\nu}| + |A_{-\nu}| > 0$. Let $\Omega_{\alpha,p}$, with some fixed positive $\alpha$, be the set of functions...
of class $S^p$ bounded on $U = (-\infty, \infty)$ whose Fourier exponents satisfy the condition

$$\lambda_{\nu+1} - \lambda_\nu \geq \alpha \ (\nu \in \mathbb{N}).$$

In case $f \in \Omega_{\alpha,p}$

$$S_{\lambda_k} f (x) = \int_0^\infty \{ f (x + t) + f (x - t) \} \Psi_{\lambda_k, \lambda_k+\alpha} (t) \, dt,$$

where

$$\Psi_{\lambda, \eta} (t) = \frac{2 \sin \left( \frac{(\eta - \lambda)t}{2} \right) \sin \left( \frac{(\eta + \lambda)t}{2} \right)}{\pi (\eta - \lambda) t^2} \quad (0 < \lambda < \eta, \ |t| > 0).$$

Let $A := (a_{n,k})$ be an infinite matrix of real nonnegative numbers such that

$$\sum_{k=0}^{\infty} a_{n,k} = 1, \text{ where } n = 0, 1, 2, \ldots.$$ \hspace{1cm} (1)

Let us consider the strong mean

$$H_{n,A,\gamma}^q f (x) = \left\{ \sum_{k=0}^{\infty} a_{n,k} |S_{\gamma_k} f (x) - f (x)|^q \right\}^{1/q} \quad (q > 0).$$ \hspace{1cm} (2)

As measures of approximation by the quantity (2), we use the best approximation of $f$ by entire functions $g_\sigma$ of exponential type $\sigma$ bounded on the real axis, shortly $g_\sigma \in B_\sigma$ and the moduli of continuity of $f$ defined by the formulas

$$E_\sigma (f)_{S^p} = \inf_{g_\sigma} \| f - g_\sigma \|_{S^p},$$

$$\omega f (\delta)_{S^p} = \sup_{|t| \leq \delta} \| f (\cdot + t) - f (\cdot) \|_{S^p},$$

and

$$w_x f (\delta)_{p} := \left\{ \frac{1}{\delta} \int_0^\delta |\varphi_x (t)|^p \, dt \right\}^{1/p},$$

$$G_x f (\delta)_{s,p} := \left\{ \sum_{k=0}^{[\pi/(\alpha \delta)]} \left( \frac{1}{(k+1) \delta} \int_{k\delta}^{(k+1)\delta} |\varphi_x (t)|^p \, dt \right)^{s/p} \right\}^{1/s}, \quad s > 1,$$

where $\varphi_x (t) := f (x + t) + f (x - t) - 2f (x)$, respectively.

Recently, L. Leindler [4] defined a new class of sequences named as sequences of rest bounded variation, briefly denoted by $RBV S$, i.e.

$$RBV S = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{\infty} |a_k - a_{k+1}| \leq K (a) |a_m| \text{ for all } m \in \mathbb{N} \right\},$$ \hspace{1cm} (3)

where here and throughout the paper $K (a)$ always indicates a constant depending only on $a$. 

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Denote by $MS$ the class of nonnegative and nonincreasing sequences. The class of general monotone coefficients, $GM$, will be defined as follows (see [11]):

$$GM = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leq K(a) |a_m| \text{ for all } m \in \mathbb{N} \right\}.$$  \hspace{1cm} (4)

It is obvious that $MS \subset RBVS \subset GM$.

In [5, 11, 12, 13] was defined the class of $\beta$–general monotone sequences as follows:

**Definition 1** Let $\beta := (\beta_n)$ be a nonnegative sequence. The sequence of complex numbers $a := (a_n)$ is said to be $\beta$–general monotone, or $a \in GM(\beta)$, if the relation

$$\sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leq K(a) \beta_m$$

holds for all $m$.

In the paper [13] Tikhonov considered, among others, the following examples of the sequences $\beta_n$:

1. $\beta_n = |a_n|,$

2. $\beta_n = \sum_{k=\lfloor n/c \rfloor}^{\lfloor n \rfloor} |a_k|/c$ for some $c > 1$.

It is clear that $GM(1\beta) = GM$ and (see [13, Remark 2.1])

$$GM(1\beta + 2\beta) \equiv GM(2\beta).$$

Moreover, we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant $K$ such that

$$0 \leq K(\alpha_n) \leq K$$

holds for all $n$, where $K(\alpha_n)$ denote the sequence of constants appearing in the inequalities (3)-(5) for the sequences $\alpha_n := (\alpha_{n,k})_{k=0}^{\infty}$.

Now we can give the conditions to be used later on. We assume that for all $n$

$$\sum_{k=m}^{2m-1} |a_{n,k} - a_{n,k+1}| \leq K \sum_{k=\lfloor m/c \rfloor}^{\lfloor cm \rfloor} \frac{g_{n,k}}{k}$$

holds if $\alpha_n = (\alpha_{n,k})_{k=0}^{\infty}$ belongs to $GM(2\beta)$, for $n = 1, 2, ...$

We have shown in [7] the following theorem:

**Theorem 2** If $f \in \Omega_{\alpha,p}$, $p > 1$, $p \geq q$, $\alpha > 0$, $(\alpha_{n,k})_{k=0}^{\infty} \in GM(2\beta)$ for all $n$, \cite{11} and $\lim_{n \to \infty} a_{n,0} = 0$ hold, then

$$\left\| H^{q}_{n,A,\gamma} f \right\|_{S^p} \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \omega^{q} f \left( \frac{\pi}{k+1} \right) S^p \right\}^{1/q},$$

for $n = 0, 1, 2, ...$, where $\gamma = (\gamma_k)$ is a sequence with $\gamma_k = \frac{\alpha_k}{2}$.
In this paper we consider the class $GM (2\beta)$ in pointwise estimate of the quantity $H^q_{n,A,\gamma} f$ for $f \in S^1$. Thus we present some analog of the following result of P. Pych-Taberska (see [10, Theorem 5]):

**Theorem 3** If $f \in \Omega_{\alpha,\infty}$ and $q \geq 2$, then

$$\| H^q_{n,A,\gamma} f \|_{S^\infty} \ll \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \bigg[ \omega f \left( \frac{\pi}{k+1} \right) \bigg]_{S^\infty}^q \right\}^{1/q} + \| f \|_{S^\infty} \left( n+1 \right)^{1/q},$$

for $n = 0, 1, 2, \ldots$, where $\gamma = (\gamma_k)$ is a sequence with $\gamma_k = \frac{\alpha k}{2}$, $a_{n,k} = \frac{1}{n+1}$ when $k \leq n$ and $a_{n,k} = 0$ otherwise.

We shall write $I_1 \ll I_2$ if there exists a positive constant $K$, sometimes depended on some parameters, such that $I_1 \leq KI_2$.

**2 Statement of the results**

Let us consider a function $w_x$ of modulus of continuity type on the interval $[0, +\infty)$, i.e. a nondecreasing continuous function having the following properties:

$$w_x(0) = 0, w_x(\delta_1 + \delta_2) \leq w_x(\delta_1) + w_x(\delta_2)$$

for any $\delta_1, \delta_2 \geq 0$ with $x$ such that the set

$$\Omega_{\alpha,p,s} (w_x) = \left\{ f \in \Omega_{\alpha,p} : \left[ \frac{1}{\delta} \int_0^\delta |\varphi_x(t) - \varphi_x(t \pm \gamma)|^p dt \right]^{1/p} \ll w_x(\gamma) \right\}$$

and $G_x f(\delta)_{s,p} \ll w_x(\delta)$, where $\gamma, \delta > 0$

is nonempty.

We start with proposition

**Proposition 4** If $f \in \Omega_{\alpha,1,2} (w_x)$, $\alpha > 0$ and $0 < q \leq 2$, then

$$\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| S_{\frac{\pi k}{n+1}} f(x) - f(x) \right|^q \right\}^{1/q} \ll w_x \left( \frac{\pi}{n+1} \right) + E_{an/2} (f)_{S^1},$$

for $n = 0, 1, 2, \ldots$

Our main results are following

**Theorem 5** If $f \in \Omega_{\alpha,1,2} (w_x)$, $\alpha > 0$, $0 < q \leq 2$, $(a_{n,k})_{k=0}^{\infty} \in GM (2\beta)$ for all $n$, $[1]$ and $\lim_{n \to \infty} a_{n,0} = 0$ hold, then

$$H^q_{n,A,\gamma} f(x) \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[ w_x \left( \frac{\pi}{k+1} \right) + E_{\frac{n k}{2n+1}} (f)_{S^1} \right] \right\}^{1/q},$$

for some $c > 1$ and $n = 0, 1, 2, \ldots$, where $\gamma = (\gamma_k)$ is a sequence with $\gamma_k = \frac{\alpha k}{2}$. 

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Theorem 6 If \( f \in \Omega_{\alpha,1,2}(w_x) \), \( \alpha > 0 \), \( 0 < q \leq 2 \), \((a_{n,k})_{k=0}^{\infty} \in MS \) for all \( n \), and \( \lim_{n \to \infty} a_{n,0} = 0 \) hold, then

\[
H_{n,A,\gamma}^q f(x) \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[ w_x \left( \frac{\pi}{k+1} \right) + E_{\alpha k}^+ (f) S_1 \right] \right\}^{1/q}
\]

for \( n = 0, 1, 2, \ldots \), where \( \gamma = (\gamma_k) \) is a sequence with \( \gamma_k = \frac{\alpha k}{2} \).

Remark 1 Since, by the Jackson type theorem

\[
E_\sigma (f) S_p \ll \omega f \left( \left\{ \frac{1}{\sigma} \right\} S_p \right)
\]

and

\[
\left\| \frac{1}{\delta} \int_0^\delta |\varphi_\sigma (t) - \varphi_\sigma (t \mp \gamma)| \, dt \right\|_{S_p} \leq \omega f (\gamma)_{S_p},
\]

\[
\left\| G_\sigma (\delta)_{2,p} \right\|_{S_p} \leq \omega f (\delta)_{S_p},
\]

the analysis of the proof of Proposition 4 shows that, the estimate from Theorem 5 implies the estimate from Theorem 2 with \( p \geq 2 \) and \( 0 < q \leq 2 \). Thus, taking \( a_{n,k} = \frac{1}{n+1} \) when \( k \leq n \) and \( a_{n,k} = 0 \) otherwise, in the case \( p \in [2, \infty] \) we obtain the better estimate than this one from Theorem 3 with \( q = 2 \).

3 Proofs of the results

3.1 Proof of Proposition 4

In the proof we will use the following function \( \Phi_x f (\delta, \nu) = \frac{1}{\delta} \int_\nu^{\nu+\delta} \varphi_x (u) \, du \), with \( \delta = \delta_n = \frac{\alpha n}{n+1} \) and its estimate from [6, Lemma 1, p. 218]

\[
|\Phi_x f (\zeta_1, \zeta_2)| \leq w_x (\zeta_1) + w_x (\zeta_2)
\]

for \( f \in \Omega_{\alpha,1,2}(w_x) \) and any \( \zeta_1, \zeta_2 > 0 \).

We can also note that by monotonicity in \( q \in (0,2] \)

\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |S_{\frac{n}{2}} f(x) - f(x)|^q \right\}^{1/q} \leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| S_{\frac{n}{2}} f(x) - f(x) \right|^2 \right\}^{1/2}.
\]

Moreover, for \( n = 0 \) our estimate is evident, therefore we give the estimate of the quantity \( H_{n, A, \gamma}^q f(x) \) with \( q = 2 \) and \( n > 0 \), only.

Denote by \( S_k^f \) the sums of the form

\[
S_k^f = \sum_{|\lambda| \leq \frac{\alpha k}{2}} A_{\nu} (f) e^{i\lambda x}
\]
such that the interval \((\frac{\alpha k}{2}, \frac{\alpha (k+1)}{2})\) does not contain any \(\lambda_\nu\). Applying Lemma 1.10.2 of \[8\] we easily verify that

\[
S_k^* f(x) - f(x) = \int_0^\infty \varphi_x(t) \Psi_k(t) \, dt,
\]

where \(\Psi_k(t) = \Psi_{\frac{\alpha k}{2}, \frac{\alpha (k+1)}{2}}(t)\), i.e.

\[
\Psi_k(t) = \frac{4 \sin \frac{\alpha t}{4} \sin \frac{\alpha (2k+1)t}{4}}{\alpha \pi t^2},
\]

(see also \[2\], p.41). Evidently, if the interval \((\frac{\alpha k}{2}, \frac{\alpha (k+1)}{2})\) contains a Fourier exponent \(\lambda_\nu\), then

\[
S_k^* f(x) = S_{k+1}^* f(x) - (A_\nu(f) e^{i\lambda_\nu x} + A_{-\nu}(f) e^{-i\lambda_\nu x}).
\]

Analyzing the proof of Proposition 1.2.2 from \[1\] p. 8] we can write

\[
|A_{\pm \nu}(f)| = |A_{\pm \nu}(f - g_{\alpha \mu/2})| = \lim_{L \to \infty} \frac{1}{L} \int_0^L \left| \left( f(t) - g_{\alpha \mu/2}(t) \right) e^{-i\lambda_\nu t} \right| dt \leq \lim_{L \to \infty} \sup_{T \geq L} \frac{1}{T} \int_0^T \left| \left( f(t) - g_{\alpha \mu/2}(t) \right) e^{-i\lambda_\nu t} \right| dt \leq \lim_{L \to \infty} \sup_{T \geq L} \frac{1}{T} \int_U^{U+T} \left| f(t) - g_{\alpha \mu/2}(t) \right| dt = \left\| f - g_{\alpha \mu/2} \right\|_W \leq \left\| f - g_{\alpha \mu/2} \right\|_{S_1} = E_{\alpha \mu/2}(f)_{S_1},
\]

for some \(g_{\alpha \mu/2} \in B_{\alpha \mu/2}\), with \(\alpha k/2 < \alpha \mu/2 < \lambda_\nu\), where \(\| \cdot \|_W\) is the Weyl norm. Therefore, the deviation

\[
\left\{ \frac{1}{n + 1} \sum_{k=n}^{2n} \left| S_{\alpha \nu/2}^* f(x) - f(x) \right|^2 \right\}^{1/2}
\]

can be estimated from above by

\[
\left\{ \frac{1}{n + 1} \sum_{k=n}^{2n} \int_0^\infty \varphi_x(t) \Psi_{k+n}(t) \, dt \right\}^{1/2} + \left\{ \frac{1}{n + 1} \sum_{k=n}^{2n} \left( E_{\alpha k/2}(f)_{S_1} \right)^2 \right\}^{1/2} \leq \left\{ \frac{1}{n + 1} \sum_{k=n}^{2n} \int_0^\infty \varphi_x(t) \Psi_{k+n}(t) \, dt \right\}^{1/2} + E_{\alpha \mu/2}(f)_{S_1},
\]

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where \( \kappa \) equals 0 or 1. Applying the Minkowski inequality we obtain

\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left| \int_{0}^{\pi/\alpha} |\varphi_x(t)\Psi_{k+\kappa}(t)|^2 dt \right| \right\}^{1/2}
\]

\[
= \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} \left( \int_{0}^{\pi/\alpha} + \int_{\pi/\alpha}^{\infty} \right) |\varphi_x(t)\Psi_{k+\kappa}(t)|^2 dt \right\}^{1/2}
\]

\[
\leq \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_1(k)|^2 \right\}^{1/2} + \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_2(k)|^2 \right\}^{1/2}
\]

So, for the first term we have

\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_1(k)|^2 \right\}^{1/2} \leq \left\{ \frac{2e^2}{n+1} \sum_{k=n}^{2n} \left( 1 - \frac{1}{n+1} \right)^{2n+\kappa} |I_1(k)|^2 \right\}^{1/2}
\]

\[
\leq \left\{ \frac{2e^2}{n+1} \sum_{k=n}^{2n} \left( 1 - \frac{1}{n+1} \right)^{2n+\kappa} |I_1(k)|^2 \right\}^{1/2}
\]

\[
\leq \left\{ \frac{2e^2}{n+1} \sum_{k=0}^{\infty} \left( 1 - \frac{1}{n+1} \right)^{k+\kappa} |I_1(k)|^2 \right\}^{1/2}
\]

\[
= \left\{ \frac{2e^2}{n+1} \sum_{k=0}^{\infty} \left( 1 - \frac{1}{n+1} \right)^{k+\kappa} \left| \int_{0}^{\pi/\alpha} \varphi_x(t)\Psi_{k+\kappa}(t) dt \right|^2 \right\}^{1/2}
\]

\[
\ll \left\{ \frac{2e^2}{n+1} \int_{0}^{\pi/\alpha} \int_{0}^{\pi/\alpha} \varphi_x(u)\varphi_x(v) \sum_{k=0}^{\infty} \left( 1 - \frac{1}{n+1} \right)^{k+\kappa} \Psi_{k+\kappa}(u)\Psi_{k+\kappa}(v) du dv \right\}^{1/2}
\]

\[
= \left\{ \frac{2e^2}{n+1} \int_{0}^{\pi/\alpha} \int_{0}^{\pi/\alpha} \varphi_x(u)\varphi_x(v) \sum_{k=0}^{\infty} \left( 1 - \frac{1}{n+1} \right)^{k+\kappa} \frac{\sin \alpha u (2 (k+\kappa) + 1)}{4} \frac{\sin \alpha v (2 (k+\kappa) + 1)}{4} du dv \right\}^{1/2}
\]

\[
\leq \left\{ \frac{2e^2}{n+1} \int_{0}^{\pi/\alpha} \int_{0}^{\pi/\alpha} \varphi_x(u)\varphi_x(v) \sum_{k=0}^{\infty} \left( 1 - \frac{1}{n+1} \right)^{k+\kappa} \frac{\sin \alpha u (2k+1)}{4} \frac{\sin \alpha v (2k+1)}{4} du dv \right\}^{1/2}
\]
Hence, taking $y = \frac{\pi}{2}$, $z = \frac{\pi}{2}$ and $r = 1 - \frac{1}{n+1}$ in the relation (see [3] and [9])

\[
\sum_{k=0}^{\infty} r^k \sin \frac{y(2k+1)}{2} \sin \frac{z(2k+1)}{2} = \sin \frac{y}{2} \sin \frac{z}{2} (1 - r) \left[ (1 + r)^2 + 2r (\cos y + \cos z) \right]
\]

and using the inequality $\sin \frac{y+z}{2} \geq \frac{y+z}{2}$ (if $y, z \leq \pi$), we obtain

\[
\left| \sum_{k=0}^{\infty} \left( 1 - \frac{1}{n+1} \right)^k \sin \frac{\alpha(2k+1)}{4} \sin \frac{\alpha(2k+1)}{4} \right| \leq \frac{1}{n+1} \left[ (1-r)^2 + (u+v)^2 \right] \left[ (1-r)^2 + (u-v)^2 \right].
\]

Hence, taking $u - v = t$, by the Gabisoniya idea [3]

\[
\frac{1}{n+1} \sum_{k=0}^{2n} |I_1(k)|^2 \leq \frac{1}{(n+1)^2} \int_0^{\pi/n} \int_0^u \frac{\left| \varphi_x (u) \varphi_x (v) \right| du dv}{\left[ (n+1)^{-2} + (u+v)^2 \right] \left[ (n+1)^{-2} + (u-v)^2 \right]}
\]

\[
\leq \frac{1}{(n+1)^2} \int_0^{\pi/n} \int_0^u \frac{\left| \varphi_x (u) \varphi_x (v) \right| du dv}{\left[ (n+1)^{-2} + u^2 \right] \left[ (n+1)^{-2} + (u-v)^2 \right]}.
\]

\[
\leq \frac{1}{(n+1)^2} \int_0^{\pi/n} \int_0^u \frac{\left| \varphi_x (u) \varphi_x (u-t) \right| du dt}{\left[ (n+1)^{-2} + u^2 \right] \left[ (n+1)^{-2} + t^2 \right]}.
\]

\[
\leq \left( \frac{\pi(n+1)/\alpha}{n+1} \right) \sum_{i=0}^{[\pi(n+1)/\alpha]} \sum_{j=0}^{i} \int_{\frac{i+j}{n+1}}^{\frac{i+j+1}{n+1}} \int_{\frac{i+j+1}{n+1}}^{\frac{i+j+2}{n+1}} \frac{\left| \varphi_x (u) \varphi_x (u-t) \right| du dt}{\left[ (n+1)^{-2} + (1+t^2) \right] \left[ (n+1)^{-2} + t^2 \right]}.
\]

\[
\leq \sum_{i=0}^{[\pi(n+1)/\alpha]} \sum_{j=0}^{i} \left( \frac{n+1}{1+i^2} \right) \int_{\frac{i+j}{n+1}}^{\frac{i+j+1}{n+1}} \int_{\frac{i+j+1}{n+1}}^{\frac{i+j+2}{n+1}} \left| \varphi_x (u) \right| du \int_{\frac{i+j+1}{n+1}}^{\frac{i+j+2}{n+1}} \left| \varphi_x (u-t) \right| dt.
\]

\[
\leq \sum_{i=0}^{[\pi(n+1)/\alpha]} \sum_{j=0}^{i} \left( \frac{n+1}{1+i^2} \right) \int_{\frac{i+j}{n+1}}^{\frac{i+j+1}{n+1}} \int_{\frac{i+j+1}{n+1}}^{\frac{i+j+2}{n+1}} \left| \varphi_x (u) \right| du \int_{\frac{i+j+1}{n+1}}^{\frac{i+j+2}{n+1}} \left| \varphi_x (v) \right| dv.
\]
For the second term, using the /suppress Lenski method [6], we obtain

\[
\sum_{i=0}^{\lfloor \pi(n+1)/\alpha \rfloor} \sum_{j=0}^{i} \frac{(n+1)^2}{(1 + i^2)(1 + j^2)} \left[ \left( \int_{\frac{i+1}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(u)| \, du \right)^2 + \left( \int_{\frac{i+1}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(v)| \, dv \right)^2 \right]
\]

\[
\sum_{i=0}^{\lfloor \pi(n+1)/\alpha \rfloor} \left( \frac{n+1}{1 + i} \int_{\frac{i+1}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(u)| \, du \right)^2 
+ \sum_{j=0}^{\lfloor \pi(n+1)/\alpha \rfloor} \frac{1}{(1 + j)^2} \sum_{i=j}^{\lfloor \pi(n+1)/\alpha \rfloor} \left( \frac{n+1}{1 + i} \int_{\frac{i+1}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(v)| \, dv \right)^2
\]

\[
\sum_{i=0}^{\lfloor \pi(n+1)/\alpha \rfloor} \left( \frac{n+1}{1 + i} \int_{\frac{i+1}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(u)| \, du \right)^2 
+ \sum_{\nu=0}^{\lfloor \pi(n+1)/\alpha \rfloor} \left( \frac{n+1}{1 + \nu} \int_{\frac{i+1}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(v)| \, dv \right)^2
\]

\[
\sum_{i=0}^{\lfloor \pi(n+1)/\alpha \rfloor} \left( \frac{n+1}{1 + i} \int_{\frac{i+1}{n+1}}^{\frac{i+1}{n+1}} |\varphi_x(u)| \, du \right)^2 = \left[ G_x f \left( \frac{1}{n + 1} \right) \right]^2 \ll \left[ w_x \left( \frac{\pi}{n+1} \right) \right]^2.
\]

For the second term, using the Lenski method [6], we obtain

\[
\left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_2(k)|^2 \right\}^{1/2} \leq \left( \frac{1}{n+1} \sum_{k=n}^{2n} \left| \sum_{\mu=1}^{\infty} \int_{\mu \pi/\alpha}^{(\mu+1)\pi/\alpha} [\varphi_x(t) - \Phi_x f(\delta_k, t)] \Psi_{k+\kappa} (t) \, dt \right|^2 \right)^{1/2}
\]

\[
+ \left( \frac{1}{n+1} \sum_{k=n}^{2n} \left| \sum_{\mu=1}^{\infty} \Phi_x f(\delta_k, t) \Psi_{k+\kappa} (t) \, dt \right|^2 \right)^{1/2}
\]

\[
= \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{21}(k)|^2 \right\}^{1/2} + \left( \frac{1}{n+1} \sum_{k=n}^{2n} |I_{22}(k)|^2 \right)^{1/2}
\]

\[
= \left\{ \frac{1}{n+1} \sum_{k=n}^{2n} |I_{21}(k)|^2 \right\}^{1/2} + \left( \frac{1}{n+1} \sum_{k=n}^{2n} |I_{22}(k)|^2 \right)^{1/2}
\]
\[
|I_{21}(k)| \leq \frac{4}{\alpha \pi} \sum_{\mu=1}^{\infty} \int_{\mu \pi/\alpha}^{(\mu+1)\pi/\alpha} |\varphi_x(t) - \Phi_x f(\delta_k, t)| t^{-2} dt
\]

\[
\leq \frac{4}{\alpha \pi} \sum_{\mu=1}^{\infty} \int_{\mu \pi/\alpha}^{(\mu+1)\pi/\alpha} \left[ \frac{1}{\delta_k^2} \int_{0}^{\delta_k} |\varphi_x(t) - \varphi_x(t+u)| du \right] dt
\]

\[
= \frac{4}{\alpha \pi} \frac{1}{\delta_k} \int_{0}^{\delta_k} \sum_{\mu=1}^{\infty} \left\{ \int_{\mu \pi/\alpha}^{(\mu+1)\pi/\alpha} \left[ \frac{1}{t^2} \int_{0}^{t} |\varphi_x(s) - \varphi_x(s+u)| ds \right]_{s=\mu \pi/\alpha}^{s=(\mu+1)\pi/\alpha} + 2 \int_{0}^{\delta_k} \left[ \frac{1}{t^3} \int_{0}^{t} |\varphi_x(s) - \varphi_x(s+u)| ds \right] dt \right\} du
\]

\[
\leq \frac{1}{\delta_k} \int_{0}^{\delta_k} \sum_{\mu=1}^{\infty} \left\{ \frac{1}{(\mu+1)\pi/\alpha} \int_{0}^{(\mu+1)\pi/\alpha} |\varphi_x(s) - \varphi_x(s+u)| ds \right\} du
\]

Since \(f \in \Omega_{\alpha,1,2}(w_x)\), for any \(x\)

\[
\lim_{\zeta \to \infty} \frac{1}{\zeta^2} \int_{0}^{\zeta} |\varphi_x(s) - \varphi_x(s+u)| ds \leq \lim_{\zeta \to \infty} \frac{1}{\zeta} w_x(u) \leq \lim_{\zeta \to \infty} \frac{1}{\zeta} w_x(\delta_k) \leq \lim_{\zeta \to \infty} \frac{1}{\zeta} w_x(\pi) = 0,
\]

and therefore

\[
|I_{21}(k)| \leq \frac{1}{\delta_k} \int_{0}^{\delta_k} \frac{\alpha}{\pi} \int_{0}^{\pi/\alpha} |\varphi_x(s) - \varphi_x(s+u)| du \]

\[
+ \frac{1}{\delta_k} \int_{0}^{\delta_k} w_x(u) du \sum_{\mu=1}^{\infty} \left\{ \int_{\mu \pi/\alpha}^{(\mu+1)\pi/\alpha} \frac{1}{t^2} dt \right\}
\]

\[
\ll \frac{1}{\delta_k} \int_{0}^{\delta_k} w_x(u) du + w_x(\delta_k) \sum_{\mu=1}^{\infty} \frac{\alpha}{\pi \mu^2}
\]

\[
\ll w_x(\delta_k).
\]
Next, we will estimate the term $|I_{22}(k)|$. So,

\[
I_{22}(k) = 2 \sum_{\mu = 1}^{\infty} \int_{\mu \pi / \alpha}^{(\mu + 1) \pi / \alpha} \frac{\Phi_x f (\delta_k, t)}{t^2} \left( - \cos \frac{\alpha t (k + \kappa)}{2} + \cos \frac{\alpha t (k + \kappa + 1)}{2} \right) dt
\]

\[
= 2 \sum_{\mu = 1}^{\infty} \int_{\mu \pi / \alpha}^{(\mu + 1) \pi / \alpha} \Phi_x f (\delta_k, t) \left( - \cos \frac{\alpha t (k + \kappa)}{2} + \cos \frac{\alpha t (k + \kappa + 1)}{2} \right) dt
\]

\[
+ 2 \sum_{\mu = 1}^{\infty} \int_{\mu \pi / \alpha}^{(\mu + 1) \pi / \alpha} \frac{d}{dt} \left( \Phi_x f (\delta_k, t) t^2 \right) \left( \cos \frac{\alpha t (k + \kappa)}{2} - \cos \frac{\alpha t (k + \kappa + 1)}{2} \right) dt
\]

\[
= I_{221}(k) + I_{222}(k)
\]

Since $f \in \Omega_{n,1,2} (w_x)$, for any $x$ (using (7))

\[
\lim_{\zeta \to \infty} \left| \Phi_x f (\delta_k, \alpha \zeta) \right| \left( - \cos \left[ \frac{\pi \zeta}{2} (k + \kappa) \right] + \cos \left[ \frac{\pi \zeta}{2} (k + \kappa + 1) \right] \right)
\]

\[
\ll \lim_{\zeta \to \infty} \frac{w_x (\delta_k) + w_x (\alpha \zeta)}{\zeta^2 k} \ll \lim_{\zeta \to \infty} \frac{w_x (\delta_k) + \zeta w_x (\alpha \zeta)}{\zeta^2 k} \ll w_x (\pi) \lim_{\zeta \to \infty} \frac{1 + \zeta}{\zeta^2} = 0,
\]

and therefore

\[
I_{221}(k) = 2 \sum_{\mu = 1}^{\infty} \int_{\mu \pi / \alpha}^{(\mu + 1) \pi / \alpha} \Phi_x f (\delta_k, \alpha \mu (\mu + 1)) \left( - \cos \left[ \frac{\pi \mu}{2} (k + \kappa) \right] \right)
\]

\[
+ \cos \left[ \frac{\pi \mu}{2} (k + \kappa + 1) \right] \frac{\alpha (k + \kappa + 1)}{2}
\]

\[
- \frac{\Phi_x f (\delta_k, \alpha \mu)}{\left[ \frac{\pi \mu}{2} \right]^2} \left( - \cos \left[ \frac{\pi \mu (k + \kappa)}{2} \right] + \cos \left[ \frac{\pi \mu (k + \kappa + 1)}{2} \right] \right)
\]

\[
= - \frac{2}{\alpha \pi} \sum_{\mu = 1}^{\infty} \int_{\mu \pi / \alpha}^{(\mu + 1) \pi / \alpha} \Phi_x f (\delta_k, \pi / \alpha) \left( - \cos \left[ \frac{\pi \mu (k + \kappa)}{2} \right] + \cos \left[ \frac{\pi \mu (k + \kappa + 1)}{2} \right] \right)
\]

\[
= - \frac{4}{\pi^3} \Phi_x f (\delta_k, \pi / \alpha) \left( \cos \frac{\pi \mu (k + \kappa + 1)}{2} - \cos \frac{\pi \mu (k + \kappa)}{2} \right).
\]

Using (7), we get

\[
|I_{221}(k)| \ll \frac{1}{k + 1} |\Phi_x f (\delta_k, \pi / \alpha)| \leq \frac{1}{(k + 1)} \left( w_x (\delta_k) + w_x (\pi / \alpha) \right).
\]

Similarly

\[
I_{222}(k) = 2 \sum_{\mu = 1}^{\infty} \int_{\mu \pi / \alpha}^{(\mu + 1) \pi / \alpha} \frac{d}{dt} \Phi_x f (\delta_k, t) \left( - \frac{2}{t^3} \Phi_x f (\delta_k, t) \right)
\]

\[
\cdot \left( \cos \frac{\alpha t (k + \kappa)}{2} - \cos \frac{\alpha t (k + \kappa + 1)}{2} \right) dt
\]

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and

\[ |I_{222}(k)| \ll \frac{8}{\alpha^2 (k + 1) \pi} \sum_{\mu=1}^{\infty} \left[ \int_{\mu \pi / \alpha}^{(\mu+1) \pi / \alpha} \frac{|\varphi_x (t + \delta_k) - \varphi_x (t)|}{\delta_k t^2} dt \right.
+ 2 \int_{\mu \pi / \alpha}^{(\mu+1) \pi / \alpha} \frac{|\Phi_x f (\delta_k, t)|}{t^3} \left. dt \right] \]

\[ \leq \frac{8}{\alpha^2 (k + 1) \pi \delta_k} \sum_{\mu=1}^{\infty} \left[ \int_{\mu \pi / \alpha}^{(\mu+1) \pi / \alpha} \frac{|\varphi_x (t + \delta_k) - \varphi_x (t)|}{t^2} dt \right.
+ \frac{16}{\alpha^2 (k + 1) \pi} \sum_{\mu=1}^{\infty} \left[ \int_{\mu \pi / \alpha}^{(\mu+1) \pi / \alpha} \frac{|\varphi_x (t + \delta_k) - \varphi_x (t)|}{t^3} dt \right. \\
\left. \right] \]

\[ \ll \frac{1}{(k + 1) \delta_k} w_x (\delta_k) + \frac{1}{k + 1} \sum_{\mu=1}^{\infty} \left[ \left( w_x (\delta_k) + w_x \left( \frac{\pi (\mu + 1)}{\alpha} \right) \right) \frac{\alpha^2}{\pi^2 \mu^3} \right] \]

\[ \ll w_x (\delta_k) + \frac{1}{k + 1} \left[ w_x (\delta_k) \sum_{\mu=1}^{\infty} \frac{1}{\mu^3} + \sum_{\mu=1}^{\infty} \frac{w_x \left( \frac{\pi (\mu + 1)}{\alpha} \right)}{\mu^3} \right] \]

\[ \ll w_x (\delta_k) + \frac{1}{k + 1} \left( w_x (\delta_k) + w_x \left( \frac{2 \pi}{\alpha} \sum_{\mu=1}^{\infty} \frac{\mu + 1}{\mu^3} \right) \right) \]

\[ \ll w_x (\delta_k) + \frac{1}{k + 1} \left( w_x (\delta_k) + w_x \left( \frac{2 \pi}{\alpha} \right) \right) . \]

Summing up

\[ |I_2 (k)| \ll w_x (\delta_k) + \frac{1}{k + 1} \left( w_x (\delta_k) + w_x \left( \frac{\pi}{\alpha} \right) + w_x \left( \frac{2 \pi}{\alpha} \right) \right) , \]

whence

\[ \left\{ \frac{1}{n + 1} \sum_{k=n}^{2n} |I_2 (k)| \right\}^{1/2} \ll \left\{ \frac{1}{n + 1} \sum_{k=n}^{2n} \left( w_x \left( \frac{\pi}{k + 1} \right) + \frac{1}{k + 1} w_x \left( \frac{\pi}{\alpha} \right) \right) \right\}^{1/2} \]

\[ \ll \left\{ \frac{1}{n + 1} \sum_{k=n}^{2n} \left( w_x \left( \frac{\pi}{k + 1} \right) \right)^2 \right\}^{1/2} \leq w_x \left( \frac{\pi}{n + 1} \right) \]

and thus the desired result follows. \( \square \)

### 3.2 Proof of Theorem 5

For some \( c > 1 \)

\[ H^q_{n, A, \gamma} f (x) = \left\{ \sum_{k=0}^{2^{|c|} - 1} a_{n,k} \left| S_{2^k} f (x) - f (x) \right|^q \right\}^{1/q} \]
Using Proposition 4 and denoting the left hand side of the inequality from its
by \( F_n \), i.e. \( F_n = w \left( \frac{2}{m+1} \right) + E_{an/2} (f)_{S1} \), we get

\[
I_1 (x) \leq \left\{ \sum_{k=0}^{2^{[c]-1}} a_{n,k} \left[ S_{\frac{k}{k+1}} f (x) - f (x) \right]^q \right\}^{1/q} \leq \left\{ \sum_{m=\lfloor c \rfloor}^{2m+1-1} \sum_{k=2^m}^{2^m+1-1} a_{n,k} \left[ S_{\frac{k}{k+1}} f (x) - f (x) \right]^q \right\}^{1/q} \leq \left\{ \sum_{k=0}^{2^{[c]-1}} a_{n,k} F^q \right\}^{1/q}.
\]

By partial summation, our Proposition 4 gives

\[
I_2^q (x) = \sum_{m=\lfloor c \rfloor}^{2m+1-1} \sum_{k=2^m}^{2^m+1-1} (a_{n,k} - a_{n,k+1}) \sum_{l=2^m}^{2^{m+1}-1} \left| S_{\frac{k}{k+1}} f (x) - f (x) \right|^q.
\]

Since (iii) holds, we have

\[
a_{n,s+1} - a_{n,r} \leq \sum_{k=r}^{s} |a_{n,k} - a_{n,k+1}| \leq 2^{m+1-2} \sum_{k=2^m}^{2^m+1-2} |a_{n,k} - a_{n,k+1}| + 2^m a_{n,2m+1-1} \leq \sum_{m=\lfloor c \rfloor}^{2m+1-1} \sum_{k=2^m}^{2^m+1-1} \left| a_{n,k} - a_{n,k+1} \right| + a_{n,2^{m+1}-1},
\]

\[
\sum_{k=2^m}^{2^m+1-2} |a_{n,k} - a_{n,k+1}| \ll \sum_{k=\lfloor 2^m/c \rfloor}^{c/2^m} \frac{a_{n,k}}{k} \quad (2 \leq 2^m \leq r \leq s \leq 2^{m+1} - 2),
\]

\[
\ll \left\{ \sum_{k=0}^{2^{[c]-1}} a_{n,k} \left[ S_{\frac{k}{k+1}} f (x) - f (x) \right]^q \right\}^{1/q}.
\]
whence
\[ a_{n,s+1} \ll a_{n,r} + \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} \quad (2 \leq 2^m \leq r \leq 2^{m+1} - 2) \]
and
\[ 2^m a_{n,2^{m+1}-1} = \frac{2^m}{2^m - 1} \sum_{r=2^m}^{2^{m+1}-2} a_{n,2^{m+1}-1} \]
\[ \ll \sum_{r=2^m}^{2^{m+1}-2} \left( a_{n,r} + \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} \right) \]
\[ \ll \sum_{r=2^m}^{2^{m+1}-1} a_{n,r} + 2^m \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k}. \]
Thus
\[ L^q(2) \ll \sum_{m=[c]}^{\infty} \left\{ 2^m \frac{F^q_{\alpha2^m}}{\alpha2^m/2} \sum_{k=[2^m/c]}^{[c2^m]} \frac{a_{n,k}}{k} + F^q_{\alpha2^m/2} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \right\}. \]
Finally, by elementary calculations we get
\[ L^q(2) = \sum_{m=[c]}^{\infty} \left\{ \sum_{k=[2^m/c]}^{2^{m+1}-c} a_{n,k} \right\} \]
\[ = \sum_{m=[c]}^{\infty} \left\{ \sum_{k=2^m-c}^{2^{m+1}-c} a_{n,k} + \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \right\} \]
\[ \ll \sum_{m=[c]}^{\infty} \sum_{k=2^m-c}^{2^{m+1}-c} a_{n,k} F^q_{\alpha k/2} + \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} F^q_{\alpha k/2^{2^m}} \]
\[ = \sum_{m=[c]}^{\infty} \sum_{k=2^m-c}^{2^{m+1}-c} a_{n,k} F^q_{\alpha k/2} + \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} F^q_{\alpha k/2^{2^m}} \]
\[ = \sum_{m=[c]}^{\infty} \sum_{r=1}^{2^{m-r+1}-1} a_{n,k} F^q_{\alpha k/2} + \sum_{m=[c]}^{\infty} \sum_{r=0}^{2^{m+r+1}-1} \sum_{k=2^m-r}^{2^{m+r}} a_{n,k} F^q_{\alpha k/2^{2^m+r}} \]
\[ + \sum_{m=[c]}^{\infty} F^q_{\alpha 2^m} a_{n,2^{m+|c|}} \]
\[
\begin{align*}
\leq & \sum_{r=1}^{[c]} \sum_{k=2^{2^{r}-1}}^{\infty} a_{n,k} F_{\alpha k/2}^q + \sum_{r=0}^{[c]-1} \sum_{k=2^{2^{r}}}^{\infty} a_{n,k} F_{\alpha k/2}^q + \sum_{k=2^{2^{[c]}}}^{\infty} a_{n,k} F_{\alpha k/2}^q \\
\ll & \sum_{k=0}^{\infty} a_{n,k} F_{\alpha k}^q \frac{2^{k}}{2^{k+1}}.
\end{align*}
\]

Thus we obtain the desired result. □

3.3 Proof of Theorem 6

If \((a_{n,k})_{k=0}^{\infty} \in MS\) then \((a_{n,k})_{k=0}^{\infty} \in GM(\alpha \beta)\) and using Theorem 5 we obtain

\[
H_{n,A,\gamma}^q f(x) \leq \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[ w_x \left( \frac{\pi}{k+1} \right) \right]^q \right\}^{1/q} + \left\{ \sum_{k=0}^{\infty} \sum_{m=k2^{[c]}}^{(k+1)2^{[c]}-1} a_{n,m} \left[ E_{\frac{\alpha m}{2^{k+1}}}(f)_{S_p} \right]^q \right\}^{1/q}
\]

\[
\leq \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[ w_x \left( \frac{\pi}{k+1} \right) \right]^q \right\}^{1/q} + \left\{ \sum_{k=0}^{\infty} \sum_{m=k2^{[c]}}^{2^{k+1}} a_{n,m} \left[ E_{\frac{\alpha m}{2^{k+1}}}(f)_{S_p} \right]^q \right\}^{1/q}
\]

\[
\ll \left\{ \sum_{k=0}^{\infty} a_{n,k} \left[ w_x \left( \frac{\pi}{k+1} \right) + E_{\frac{\alpha k}{2^{k+1}}}(f)_{S_p} \right]^q \right\}^{1/q}
\]

This ends our proof. □

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