Existence and concentration of solutions for Kirchhoff-type fractional Dirichlet problem with $p$-Laplacian

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Abstract: In this paper, we investigate the existence and concentration of solutions for a class of $p$-Laplacian fractional order Kirchhoff-type system with Riemann-Liouville fractional derivatives and a parameter $\lambda$. By mountain pass theorem, we prove that system has at least one non-trivial weak solution $u_\lambda$ under some local superquadratic conditions for each given large parameter $\lambda$. We get a concrete lower bound of the parameter $\lambda$, and then obtain two estimates of weak solutions $u_\lambda$. We also obtain that $u_\lambda \to 0$ if $\lambda$ varies to $\infty$. Finally, we present an example as an application of our results.

Keywords: Kirchhoff-type system; Fractional $p$-Laplacian; Local superquadratic nonlinearity; Mountain pass theorem; Existence and concentration

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1. Introduction and main results

In this paper, we are concerned with the following system

$$
\begin{aligned}
\begin{cases}
A(u(t))[tD_t^\alpha \phi_p(0D_t^\alpha u(t)) + V(t)\phi_p(u(t))] = \lambda \nabla F(t,u(t)), & \text{a.e. } t \in [0,T], \\
u(0) = u(T) = 0,
\end{cases}
\end{aligned}
$$

(1.1)

where

$$
A(u(t)) = \left[a + b \int_0^T (|0D_t^\alpha u(t)|^p + V(t)|u(t)|^p)dt\right]^{p-1},
$$

$a, b, \lambda > 0, p > 1$ and $1/p < \alpha \leq 1$ are constants, $u(t) = (u_1(t), \cdots, u_N(t))^T \in \mathbb{R}^N$ for a.e. $t \in [0,T]$ and $N$ is a given positive integer, $(\cdot)^T$ denote the transpose of a vector, $V(t) \in C([0,T], \mathbb{R})$ with $\min_{t \in [0,T]} V(t) > 0$, $0D_t^\alpha$ and $tD_t^\alpha$ are the left and right Riemann-Liouville fractional derivatives, respectively, and the order $\alpha \in \left(\frac{1}{p}, 1\right]$.

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\[ \phi_p(s) := |s|^{p-2}s, \nabla F(t, x) \] is the gradient of \( F \) with respect to \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \), that is, \( \nabla F(t, x) = \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_N} \right) \), and \( F : [0, T] \times \mathbb{R}^N \to \mathbb{R} \) satisfies the following condition:

\((H0)\) there exists a constant \( \delta > 0 \) such that \( F(t, x) \) is continuously differentiable in \( x \in \mathbb{R}^N \) with \( |x| \leq \delta \) for a.e. \( t \in [0, T] \), measurable in \( t \) for every \( x \in \mathbb{R}^N \) with \( |x| \leq \delta \), and there exist \( a \in C(\mathbb{R}^+, \mathbb{R}^+) \) and \( b \in L^\infty([0, T]; \mathbb{R}^+) \) such that

\[ |F(t, x)| + |\nabla F(t, x)| \leq a(|x|)b(t) \]

for all \( x \in \mathbb{R}^N \) with \( |x| \leq \delta \) and a.e. \( t \in [0, T] \).

When \( \alpha = 1 \), the operator \( \mathcal{D}_T^\alpha (\mathcal{D}_t^\lambda u(t)) \) reduces to the usual second order differential operator \(-d^2/dt^2\). Hence, if \( \alpha = 1, p = 2, N = 1, \lambda = 1 \) and \( V(t) = 0 \) for all \( t \in [0, T] \), system \((1.1)\) becomes the equation with Dirichlet boundary condition

\[ \begin{cases} - \left( a + b \int_0^T |u'(t)|^2 dt \right) u''(t) = f(t, u(t)), & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \]

where \( f(t, x) = \frac{\partial F(t, x)}{\partial x} \) and \( F : [0, T] \times \mathbb{R} \to \mathbb{R} \). It is well known that equation \((1.2)\) is related to the stationary problem of a classical model introduced by Kirchhoff [1]. To be precise, in [1], Kirchhoff introduced the model

\[ \rho \frac{\partial^2 u}{\partial t^2} = \left( P_0 + \frac{Eh}{2L} \int_0^L \left( \frac{\partial u}{\partial y} \right)^2 dy \right) \frac{\partial^2 u}{\partial y^2}, \]

where \( 0 \leq y \leq L, \ t \geq 0, \ u \) is the lateral deflection, \( \rho \) is the mass density, \( h \) is the cross-sectional area, \( L \) is the length, \( E \) is the Young’s modulus and \( P_0 \) is the initial axial tension. \( \text{(Notations: in model (1.3), (1.7) and (1.8) below,} \ t \text{ is time variable and } y \) is spatial variable, which are conventional notations in partial differential equations. One need to distinguish them to \( t \) in \((1.4), (1.5), (1.6)\) and \((1.7)\) below, where \( t \) corresponds to the spatial variable \( x \). \) The model \((1.3)\) is used to describe small vibrations of an elastic stretched string. Equation \((1.3)\) has been studied extensively, for instance, \([2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]\) and reference therein. For \( p > 1 \), the reader can consult \([15], [16], [17], [18], [19]\) and references therein.

When \( \alpha < 1, \mathcal{D}_t^\alpha \) and \( \mathcal{D}_t^\lambda \) are the left and right Riemann-Liouville fractional derivatives, respectively, which has been given some physical interpretations in [20]. Moreover, it is also applied to describe the anomalous diffusion, Lévy flights and traps in [21] and [22]. Fractional differential equations have been proved to provide a natural framework in the modeling of many real phenomena such as viscoelasticity, neurons, electrochemistry, control, porous media, electromagnetic (the reader can consult [23] in which a collection of references is given).

In [23], Jiao and Zhou considered the system

\[ \begin{cases} \mathcal{D}_T^\alpha (\mathcal{D}_t^\lambda u(t)) = \nabla F(t, u(t)), & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0. \end{cases} \]
They successfully applied critical point theory to investigate the existence of weak solutions for system (1.4). To be precise, they obtained that system has at least one weak solution when $F$ has a quadratic growth or a superquadratic growth by using the least action principle and mountain pass theorem. Subsequently, this topic related to system (1.4) attracted lots of attention, for instance, [24], [25], [26], [27], [28], [29] and references therein. It is obvious that system (1.1) is much more complicated than system (1.4) since the appearance of nonlocal term $A(u(t))$ and $p$-Laplacian term $\phi_p(s)$. Recently, in [30], the following fractional Kirchhoff equation with Dirichlet boundary condition was investigated:

\[
\begin{align*}
\left\{ \begin{array}{l}
(a + b \int_0^T |D^\alpha_t u(t)|^2 dt)^{p-1} D^\alpha_t \phi_p(D^\alpha_t u(t)) = f(t, u(t)), \quad t \in [0, T], \\
u(0) = u(T) = 0,
\end{array} \right.
\end{align*}
\]

where $a, b, \lambda > 0$, $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$. By using the mountain pass theorem in [42] and the linking theorem in [43], the authors established some existence results of nontrivial solutions for system (1.5) if $f$ satisfies

\( (f1) \) there exist constants $\mu > 4$, $0 < \tau < 2$ and a nonnegative function $g \in L^{\tau \over 2-\tau}$ such that

\[ F(t, x) - {1 \over \mu} f(t, x)x \leq g(t)|x|^{\tau}, \quad \text{for a.e. } t \in [0, T], x \in \mathbb{R}; \]

\( (f2) \) there exists $\theta > 2$ such that $\lim_{|x| \to \infty} \inf_{t \in [0, T]} \frac{F(t, x)}{|x|^{\theta}} > 0$;

\( \text{(or } (f2)' \text{ There exists } \theta > 4 \text{ such that } \lim_{|x| \to \infty} \inf_{t \in [0, T]} \frac{F(t, x)}{|x|^{\theta}} > 0); \)

\( (f3) \) there exists $\sigma > 2$ such that $\lim_{|x| \to 0} \sup_{t \in [0, T]} \frac{F(t, x)}{|x|^\sigma} < \infty$, and some other reasonable conditions.

In [31], Chen-Liu investigated the Kirchhoff-type fractional Dirichlet problem with $p$-Laplacian:

\[
\begin{align*}
\left\{ \begin{array}{l}
(a + b \int_0^T |D^\alpha_t u(t)|^p dt)^{p-1} D^\alpha_t \phi_p(D^\alpha_t u(t)) = f(t, u(t)), \quad t \in (0, T), \\
u(0) = u(T) = 0,
\end{array} \right.
\end{align*}
\]

where $a, b, \lambda > 0$, $f \in C^1([0, T] \times \mathbb{R}, \mathbb{R})$. By using the Nehari method, they established the existence result of ground state solution for system (1.6) if $f$ satisfies

\( (f4) \) $f(t, x) = o(|x|^{p-1})$ as $|x| \to 0$ uniformly for all $t \in [0, T]$,

and the following well-known Ambrosetti-Rabinowitz (AR for short) condition

\( (AR) \) there exist two constants $\mu > p^2$, $R > 0$ such that

\[ 0 < \mu F(t, x) \leq xf(t, x), \quad \text{for } \forall t \in [0, T], x \in \mathbb{R} \text{ with } |x| \geq R, \]

where $F(t, x) = \int_0^x f(t, s)ds$, and some additional conditions. It is easy to see that all of these conditions (f1), (f2), (f2)' and (AR) imply that $F(t, x)$ need to have a growth near the infinity for $x$, and (f3) and (f4) imply that $F(t, x)$ need to have a growth near 0 for $x$. 

3
In this paper, we investigate the existence and concentration of solutions for system (1.1) under local assumptions only near 0 for the nonlinear term \(F\). Our work is mainly motivated by [32] and [12]. In [32], Costa and Wang investigated the multiplicity of both signed and sign-changing solutions for the one-parameter family of elliptic problems

\[
\begin{aligned}
\Delta u &= \lambda f(u) \quad \text{in } \Omega \\
u(y) &= 0 \quad \text{in } \partial \Omega,
\end{aligned}
\]

where \(\lambda > 0\) is a parameter, \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^N (N \geq 3)\) and \(f \in C^1(\mathbb{R}, \mathbb{R})\). They assumed that the nonlinearity \(f(u)\) has superlinear growth in a neighborhood of \(u = 0\) and then obtained the number of signed and sign-changing solutions which are dependent on the parameter \(\lambda\). The idea in [32] has been applied to some different problems, for example, [33] and [34] for quasilinear elliptic problems with \(p\)-Laplacian operator, [34] for an elliptic problem with fractional Laplacian operator, [35] for Schrödinger equations, [11] for Neumann problem with nonhomogeneous differential operator and critical growth, and [38] for quasilinear Schrödinger equations. Especially, in [12], Li and Su investigated the Kirchhoff-type equations

\[
\begin{aligned}
-\left[1 + \int_{\mathbb{R}^3} (|\nabla u|^2 + V(y)|u|^2)dy\right] [\Delta u + V(y)u] &= \lambda Q(y)f(u), \quad y \in \mathbb{R}^3, \\
u(y) &\to 0, \quad \text{as } |y| \to \infty,
\end{aligned}
\]

where \(\lambda > 0\), \(V, Q\) are radial functions and \(f \in C((-\delta_0, \delta_0), \mathbb{R})\) for some \(\delta_0 > 0\). Via the idea in [32], they also established the existence result of solutions when \(f(u)\) has superlinear growth in a neighborhood of \(u = 0\). It is worthy to note that \(\lambda\) usually needs to be sufficiently large, that is, \(\lambda\) has a lower bound \(\lambda^*\). However, the concrete values of \(\lambda^*\) are not given in these references. Similar to system (1.8), comparing with equation (1.5) and equation (1.6), we add a nonlocal term \(\int_0^T V(t)u(t)dt\) in system (1.1) where \(\min_{t \in [0,T]} V(t) > 0\), and multiply \(V(t)\phi_p(u(t))\) by the nonlocal part \(A(u(t))\). Moreover, we consider the high-dimensional case, that is, \(N \geq 1\). Since \(\min_{t \in [0,T]} V(t) > 0\), system (1.1) is different from equation (1.2), (1.5), (1.6) and system (1.4). More importantly, we present a concrete value of the lower bound \(\lambda^*\) for system (1.1) and then obtain two estimates of the solutions family \(\{u_\lambda\}\) for all \(\lambda > \lambda^*\). Next, we make some assumptions for \(F\).

(H1) there exist constants \(q_1 > p^2\), \(q_2 \in (p^2, q_1)\), \(M_1 > 0\) and \(M_2 > 0\) such that

\[M_1|x|^{q_1} \leq F(t, x) \leq M_2|x|^{q_2}\]

for all \(x \in \mathbb{R}^N\) with \(|x| \leq \delta\) and a.e. \(t \in [0, T]\);

(H2) there exists a constant \(\beta > p^2\) such that

\[0 \leq \beta F(t, x) \leq (\nabla F(t, x), x)\]

for all \(x \in \mathbb{R}^N\) with \(|x| \leq \delta\) and a.e. \(t \in [0, T]\).
Theorem 1.1. Suppose that (H0)-(H2) hold. Then system (1.7) has at least a nontrivial solution \( u_\lambda \) for all \( \lambda > \lambda^* := \max\{\Lambda_1, \Lambda_2, \Lambda_3\} \) and

\[
\|u_\lambda\|_V^p \leq \frac{p^2 \theta}{a p - (\theta - p^2)} C_* \lambda^{-\frac{p-1}{p-\theta}} \leq \frac{p^2 \theta}{a p - (\theta - p^2)} C_* \max\{\Lambda_1, \Lambda_2, \Lambda_3\}^{-\frac{p-1}{q_1 - p}},
\]

\[
\|u_\lambda\|_\infty \leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha q - q + 1)^{\frac{1}{p}}} \cdot \frac{p^2 \theta}{a p - (\theta - p^2)} C_* \lambda^{-\frac{p-1}{q_1 - p}}
\]

\[
\leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha q - q + 1)^{\frac{1}{p}}} \cdot \frac{p^2 \theta}{a p - (\theta - p^2)} C_* \max\{\Lambda_1, \Lambda_2, \Lambda_3\}^{-\frac{p-1}{q_1 - p}},
\]

\[
\lim_{\lambda \to \infty} \frac{\|u_\lambda\|_V}{\|u_\lambda\|_\infty} = 0 = \lim_{\lambda \to \infty} \|u_\lambda\|_\infty,
\]

where \( \theta = \min\{\beta, q_2\} \),

\[
\|u_\lambda\|_V = \left( \int_0^T |a p u_\lambda(t)|^p dt + \int_0^T V(t)|u_\lambda(t)|^p dt \right)^{1/p}, \quad \|u_\lambda\|_\infty = \max_{t \in [0, T]} u_\lambda(t),
\]

\[
A_1 = \max \left\{ \frac{V\infty_{p-1}}{\Gamma(\alpha q - q + 1)^{\frac{1}{p}}} (\delta \min\{1, V\infty\}(D^p + G^p) \delta)^{\frac{1}{p}} \right\}, \quad (1.9)
\]

\[
A_2 = a + b \max\{1, V\infty\} \delta^p (D^p + G^p) \delta^{q_1 - p}, \quad (1.10)
\]

\[
A_3 = \left( \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha q - q + 1)^{\frac{1}{p}}} \right)^{\frac{p-1}{p}}, \quad (1.11)
\]

\[
\delta = \begin{cases} \left( \frac{T^{p+1}}{p+1} \frac{2(p-1)!}{p!} \right)^{\frac{1}{p}}, & \text{if } \alpha \text{ is odd}, \\ \left( \frac{T^{p+1}}{p!} \frac{(p-1)!}{p!} \right)^{\frac{1}{p}}, & \text{if } \alpha \text{ is even}, \end{cases}
\]

\[
G = \left( \frac{T^{p+1-p^1p}}{\Gamma(2-\alpha)^p(p+1-p\alpha)} \right)^{\frac{1}{p}},
\]

\[
C_* = \frac{1}{p} \frac{M_1}{(M_1 q_1)^{\frac{1}{p} - 1}} \left( \frac{\max\{1, V\infty\}}{T^{\alpha - \frac{1}{p}} \delta D} \right)^{\frac{p-1}{q_1 - p}}, \quad (1.12)
\]

\[
D = \left\{ \begin{array}{ll} \left( \frac{T^{p+1}}{p+1} \frac{2(p-1)!}{p!} \right)^{\frac{1}{p}}, & \text{if } p \text{ is odd}, \\ \left( \frac{T^{p+1}}{p!} \frac{(p-1)!}{p!} \right)^{\frac{1}{p}}, & \text{if } p \text{ is even}, \end{array} \right.
\]

\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad (\text{for all } z > 0).
\]

We organize this paper as follows. In section 2, we recall some preliminary results including the definitions of Riemann-Liouville fractional derivatives and working spaces, some conclusions for the working space and mountain pass theorem. In section 3, we give the proof of Theorem 1.1. In section 4, we apply Theorem 1.1 to an example and compute the value of lower bound \( \lambda^* \).
2. Preliminaries

In this section, we mainly recall some basic definitions and results.

**Definition 2.1.** (Left and Right Riemann-Liouville Fractional Integrals) Let $f$ be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional integrals of order $\gamma > 0$ for function $f$ denoted by $aD_t^{-\gamma}f(t)$ and $bD_t^{\gamma}f(t)$, respectively, are defined by

$$
aD_t^{-\gamma}f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds, \quad t \in [a, b], \gamma > 0,
$$

$$\quad bD_t^{\gamma}f(t) = \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} f(s) ds, \quad t \in [a, b], \gamma > 0,$$

provided the right-hand sides are pointwise defined on $[a, b]$, where $\Gamma > 0$ is the Gamma function.

**Definition 2.2.** (Left and Right Riemann-Liouville Fractional Derivatives) Let $f$ be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional derivatives of order $\gamma > 0$ for function $f$ denoted by $aD_t^n f(t)$ and $bD_t^n f(t)$, respectively, are defined by

$$
aD_t^n f(t) = \frac{d^n}{dt^n} aD_t^{-n} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \left( \int_a^t (t-s)^{n-\gamma-1} f(s) ds \right),
$$

$$
\quad bD_t^n f(t) = \frac{d^n}{dt^n} bD_t^{n-\gamma} f(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \left( \int_t^b (s-t)^{n-\gamma-1} f(s) ds \right),
$$

where $t \in [a, b], n - 1 \leq \gamma < n$ and $n \in \mathbb{N}$.

**Definition 2.3.** ([23]) Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0, T], \mathbb{R}^N)$ with the norm

$$
\|u\| = \left( \int_0^T \|aD_t^\alpha u(t)\|^p dt + \int_0^T \|bD_t^{n-\alpha} u(t)\|^p dt \right)^{1/p}, \quad \forall u \in E_0^{\alpha,p}.
$$

From the definition of $E_0^{\alpha,p}$, it is apparent that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u : [0, T] \to \mathbb{R}^N$ which is absolutely continuous and has an $\alpha$-order left and right Riemann-Liouville fractional derivative $aD_t^\alpha u \in L^p([0, T], \mathbb{R}^N)$ and $u(0) = u(T) = 0$ and one can define the norm on $L^p([0, T], \mathbb{R}^N)$ as

$$
\|u\|_{L^p} = \left( \int_0^T \|u(t)\|^p dt \right)^{1/p}.
$$

$E_0^{\alpha,p}$ is uniformly convex by the uniform convexity of $L^p$.

**Remark 2.1.** It is easy to see that $\|u\|_V$ is also a norm on $E_0^{\alpha,p}$ and $\|u\|_V$ and $\|u\|$ are equivalent and

$$
\min\{1, V_\infty\} \|u\|^p \leq \|u\|_V^p \leq \max\{1, V_\infty\} \|u\|^p.
$$

**Lemma 2.1.** ([23]) Let $0 < \alpha \leq 1$ and $1 < p < \infty$. $E_0^{\alpha,p}$ is a reflexive and separable Banach space.
Lemma 2.2. (23) Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, there has
\[
\|u\|_{L^p} \leq C_p \|\partial_t^\alpha u\|_{L^p},
\]
where
\[
C_p = \frac{T^\alpha}{\Gamma(\alpha + 1)} > 0.
\]
Moreover, if $\alpha > \frac{1}{p}$, then
\[
\|u\|_{\infty} \leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)(aq - q + 1)\theta} \|\partial_t^\alpha u\|_{L^p}, \quad \frac{1}{p} + \frac{1}{q} = 1.
\]
(2.14)

Lemma 2.3. (23) Let $1/p < \alpha \leq 1$ and $1 < p < \infty$. The imbedding of $E_0^{\alpha,p}$ in $C([0, T], \mathbb{R}^N)$ is compact.

Let $X$ be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. A sequence $\{u_n\} \subset X$ is called (PS)$_c$ sequence (named after R. Palais and S. Smale) if the sequence $\{u_n\}$ satisfies
\[
\varphi(u_n) \to c, \quad \varphi'(u_n) \to 0.
\]

Lemma 2.4. (Mountain Pass Theorem [44, 40]) Let $X$ be a Banach space, $\varphi \in C^1(X, \mathbb{R})$, $w \in X$ and $r > 0$ be such that $\|w\| > r$ and
\[
b := \inf_{\|u\| = r} \varphi(u) > \varphi(0) \geq \varphi(w).
\]
Then there exists a (PS)$_c$ sequence with
\[
c := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \varphi(\gamma(t)),
\]
\[
\Gamma := \{\gamma \in ([0, 1], X) : \gamma(0) = 0, \gamma(1) = w\}.
\]
As usual, for each $\lambda > 0$, if we define the functional $I_\lambda : E_0^{\alpha,p} \to \mathbb{R}$ as
\[
I_\lambda(u) = \frac{1}{bp^2} \left( a + b \int_0^T \left( \|\partial_t^\alpha u(t)\|^p + V(t)|u(t)|^p \right) dt \right)^p - \lambda \int_0^T F(t, u(t)) dt - \frac{a^p}{bp^2},
\]
it is easy to see that the assumption (H0)-(H2) can not ensure that $I_\lambda$ is well defined on $E_0^{\alpha,p}$. So we follow the idea in [32] and sketch the outline of proof simply here. We use Lemma 2.4 to complete the proof. Since $F$ satisfies the growth condition only near 0 by (H0)-(H2), in order to use the conditions globally, we modify and extend $F$ to $\tilde{F}$, and the corresponding functional is defined as $\tilde{I}_\lambda$. Next we prove that $\tilde{I}_\lambda$ has mountain pass geometry on $E_0^{\alpha,p}$. Then Lemma 2.4 implies that $\tilde{I}_\lambda$ has a (PS)$_{c_\lambda}$ sequence. Then by a standard analysis, a convergent subsequence of the (PS)$_{c_\lambda}$ sequence is obtained to ensure that $c_\lambda$ is the critical value of $\tilde{I}_\lambda$. Finally, by an estimate about $\|u_\lambda\|_{\infty}$, we obtain that the critical point $u_\lambda$ of $\tilde{I}_\lambda$ with $\|u_\lambda\|_{\infty} \leq \delta/2$ is just right the solution of system (1.1).
3. Proofs

Define \( m(s) \in C^1(\mathbb{R}, [0, 1]) \) as an even cut-off function satisfying \( sm'(s) \leq 0 \) and

\[
m(s) = \begin{cases} 
1, & \text{if } |s| \leq \delta/2, \\
0, & \text{if } |s| \geq \delta.
\end{cases}
\] (3.1)

Define \( \bar{F} : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \) as

\[
\bar{F}(t, x) = m(|x|)F(t, x) + (1 - m(|x|))M_2|x|^{q_2}.
\]

We define the variational functional corresponding to \( \bar{F} \) as

\[
\bar{I}_\lambda(u) = \frac{1}{b p^2} \left( a + b \int_0^T (|\alpha D^p_t u(t)|^p + V(t)|u(t)|^p) dt \right)^p - \lambda \int_0^T \bar{F}(t, u(t)) dt - \frac{a p}{bp^2}
\]

for all \( u \in E_0^{\alpha,p} \). By (H0) and the definition of \( \bar{F} \), it is easy to obtain that \( \bar{F} \) satisfies

(H0)' \( \bar{F}(t, x) \) is continuously differentiable in \( x \in \mathbb{R}^N \) for a.e. \( t \in [0, T] \), measurable in \( t \) for every \( x \in \mathbb{R}^N \), and there exist \( a \in C(\mathbb{R}^+, \mathbb{R}^+) \) and \( b \in L^\infty([0, T]; \mathbb{R}^+) \) such that

\[
|\bar{F}(t, x)| + |\nabla \bar{F}(t, x)| \leq a(|x|)b(t) + M_2|x|^{q_2}
\]

for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \).

Hence, a standard argument shows that \( \bar{I}_\lambda \in C^1(E_0^{\alpha,p}, \mathbb{R}) \) and

\[
\langle \bar{I}'_\lambda(u), v \rangle = (a + b\|u\|_V^p)^{p-1} \left( \int_0^T \left[ |\alpha D^p_t u(t)|^{p-2} \alpha D^p_t u(t), \alpha D^p_t v(t) \right] dt + V(t)|u(t)|^{p-2}(u(t), v(t)) dt \right) - \lambda \int_0^T (\nabla \bar{F}(t, u(t)), v(t)) dt
\]

for all \( u, v \in E_0^{\alpha,p} \). Hence

\[
\langle \bar{I}'_\lambda(u), u \rangle = (a + b\|u\|_V^p)^{p-1} \|u\|_V^p - \lambda \int_0^T (\nabla \bar{F}(t, u(t)), u(t)) dt
\]

for all \( u \in E_0^{\alpha,p} \).

**Lemma 3.1.** Assume that (H1)-(H2) hold. Then

(H1)

\[
0 \leq \bar{F}(t, x) \leq M_2|x|^{q_2}, \text{ for all } x \in \mathbb{R}^N;
\]
\[(H2)'

\[
0 < \theta \bar{F}(t,x) \leq (\nabla \bar{F}(t,x), x), \quad \text{for all } x \in \mathbb{R}^N/\{0\},
\]

where \(\theta = \min\{q_2, \beta\}\).

**Proof.**

- If \(|x| \leq \frac{\delta}{2}\), then by (H1), the conclusion \((H1)'\) holds;
  - If \(\frac{\delta}{2} < |x| \leq \delta\), by (H1), we have

  \[
  0 \leq \bar{F}(t,x) = m(|x|)F(t,x) + (1 - m(|x|))M_2|x|^{q_2} \leq m(|x|)M_2|x|^{q_2} + (1 - m(|x|))M_2|x|^{q_2} = M_2|x|^{q_2};
  \]

  If \(|x| \geq \delta\), then by the definition of \(m\), we have \(\bar{F}(t,x) = M_2|x|^{q_2}\).

- For all \(x \in \mathbb{R}^N/\{0\}\), we have

  \[
  \nabla \bar{F}(t,x) = m'(|x|)\frac{x}{|x|} F(t,x) + m(|x|)\nabla F(t,x) + (1 - m(|x|))q_2 M_2 |x|^{q_2 - 2} x - m'(|x|) \frac{x}{|x|} M_2|x|^{q_2}.
  \]

Then

\[
(\nabla \bar{F}(t,x), x) = |x|m'(|x|)(F(t,x) - M_2|x|^{q_2}) + m(|x|)(\nabla F(t,x), x) + (1 - m(|x|))q_2 M_2 |x|^{q_2}.
\]

and

\[
\theta \bar{F}(t,x) - (\nabla \bar{F}(t,x), x) = m(|x|)(\theta F(t,x) - (\nabla F(t,x), x)) + (\theta - q_2)(1 - m(|x|))M_2 |x|^{q_2} - |x|m'(|x|)(F(t,x) - M_2|x|^{q_2}).
\]

Apparently, the conclusion holds for \(0 \leq |x| \leq \delta/2\) and \(|x| \geq \delta\). If \(\delta/2 < |x| \leq \delta\), by using \(\theta \leq q_2\), the conclusion \((H1), (H2)\) and the fact \(sm'(s) \leq 0\) for all \(s \in \mathbb{R}\), we can get the conclusion \((H2)'\). \(\square\)

**Lemma 3.2.** \(I_\lambda\) satisfies the mountain pass geometry for all \(\lambda > \Lambda_1\), where \(\Lambda_1\) is defined in \((1.7)\).

**Proof.** Note that \(q_2 > p^2 > p\). By Lemma 3.1 and \((2.14)\), we have

\[
I_\lambda(u) = \frac{1}{bp^2} (a + b\|u\|^p_{V_1})^p - \frac{\lambda}{bp^2} \int_0^T \bar{F}(t,u(t))dt - \frac{a^p}{bp^2}
\]

\[
\geq \frac{a^p}{bp^2} + \frac{a^{p-1}}{p^2} \|u\|^p_{V_1} - \lambda M_2 \int_0^T |u(t)|^{q_2} dt - \frac{a^p}{bp^2}
\]

\[
\geq \frac{a^{p-1}}{p^2} \|u\|^p_{V_1} - \lambda M_2 \|u\|^{q_2-p}_{\infty} \int_0^T |u(t)|^p dt - \frac{a^p}{bp^2}
\]

\[
\geq \frac{a^{p-1}}{p^2} \|u\|^p_{V_1} - \lambda M_2 \Gamma(\alpha)(\alpha q + 1)^{\frac{1}{q}} \|u\|^{q_2-p}_V \|u\|^p_{L^p}
\]

\[
\geq \frac{a^{p-1}}{p^2} \|u\|^p_{V_1} - \lambda M_2 \frac{T^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)(\alpha q + 1)^{\frac{1}{q}}} \|u\|^{q_2-p}_V \|u\|^p_{L^p}.
\]
We choose \( \nu_\lambda = \left( \frac{a^{p-1} V_\infty}{2 p^2 M_2 \left( \frac{x^{-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q + 1)} \right)^{\frac{q_2 - p}{p}}} \right)^{\frac{1}{q_2 - p}} \) for any given \( \lambda > 0 \). Then we have

\[
\bar{I}_\lambda(u) > d_\lambda := \frac{a^{p-1}}{p^2} \nu_\lambda^p - \lambda \frac{M_2}{V_\infty} \left( \frac{T^{\alpha-rac{1}{p}}}{\Gamma(\alpha)(\alpha q + 1)} \right)^{\frac{q_2 - p}{p}} \nu_\lambda^{q_2 - p} > 0, \quad \text{for all } \|u\|_V = \nu_\lambda.
\] (3.3)

Choose

\[
e = \left( \frac{T}{\pi} \sin \frac{\pi t}{T}, 0, \ldots, 0 \right) \in E_0^\alpha.
\] (3.4)

Then

\[
\|e\|_{L^p} = D := \begin{cases}
\left( \frac{T^{p+1} (2p-1)!}{\pi^{p+1} p!} \right)^{\frac{1}{p}}, & \text{if } p \text{ is odd,} \\
\left( \frac{T^{p+1} (p-1)!}{\pi^p p!} \right)^{\frac{1}{p}}, & \text{if } p \text{ is even}
\end{cases}
\] (3.5)

and

\[
\|D_\lambda^\alpha e\|_{L^p} \leq G := \frac{T^{p+1-a\gamma}}{\Gamma(2-a)(p+1-\gamma)}.
\] (3.6)

By (3.4),

\[
\|e\|_\infty \leq \frac{T^{\alpha-rac{1}{p}}}{\Gamma(\alpha)(\alpha q + 1)^\frac{1}{p}} \|D_\lambda^\alpha e\|_{L^p} \leq G_0 := \frac{T^{\alpha-rac{1}{p}}}{\Gamma(\alpha)(\alpha q + 1)^\frac{1}{p}} G.
\] (3.7)

Note that

\[
A_1 = \max \left\{ \frac{V_\infty a^{p-1} (\Gamma(\alpha)(\alpha q + 1)^\frac{1}{p}) G_0^{q_2 - p}}{2 p^2 M_2 T^{\alpha - \frac{1}{p}} (q_2 - p) (\delta \min\{1, V_\infty\} D) a^{q_2 - p}}, \frac{1}{b p^2} \left( a + \frac{b p}{c_0^\alpha} \max\{1, V_\infty\} (D^p + G^p) \right)^p \right\}.
\]

Then

\[
\|\delta \| G_0 e\|_V \geq \frac{\delta \min\{1, V_\infty\}}{G_0} \|e\|_{L^p} \geq \nu_\lambda
\] (3.8)

for all \( \lambda > A_1 \). By (3.7), we have \( \|\delta \| G_0 e\|_\infty \leq \delta \). By the definition of \( \bar{F} \) and (H1), we have \( \bar{F}(t, x) = F(t, x) \geq M_1 |x|^{q_1} \) for all \( |x| \leq \delta/2 \), and

\[
\bar{F}(t, x) = m(|x|) F(t, x) + (1 - m(x)) M_2 |x|^{q_2} \geq m(|x|) M_1 |x|^{q_1} + (1 - m(x)) M_1 |x|^{q_1} = M_1 |x|^{q_1}
\]

for all \( \frac{\delta}{2} < |x| \leq \delta \). Hence, by Hölder inequality, we have

\[
\bar{I}_\lambda \left( \frac{\delta}{G_0} e \right) = \frac{1}{b p^2} \left( a + b \|G_0 e\|_V \right)^p - \lambda \int_0^T \bar{F}(t, \frac{\delta}{G_0} e(t)) dt - \frac{a^p}{b p^2} \leq \frac{1}{b p^2} \left( a + b \|G_0 e\|_V \right)^p - \lambda M_1 \int_0^T \|\frac{\delta}{G_0} e(t)\|^{q_1} dt - \frac{a^p}{b p^2}.
\]

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\[ \leq \frac{1}{bp^2}(a + \frac{b\delta_p}{G_0^2})\max\{1, V^\infty\} \|e\|^p - \lambda M_i \frac{\delta_{q_1}}{G_0^2} T^{1 - \frac{q_1}{p}} \|e\|^{q_1}_p \]
\[ \leq \frac{1}{bp^2}(a + \frac{b\delta_p}{G_0^2})\max\{1, V^\infty\}(D^p + G^p)^p - \lambda M_i \frac{\delta_{q_1}}{G_0^2} T^{1 - \frac{q_1}{p}} D^{q_1} \]
\[ < 0 \]

for all \( \lambda > \Lambda_1 \).

Let \( w = \frac{\delta}{G_0^2} \) and \( \varphi = \tilde{I}_\lambda \). Then for any given \( \lambda > \Lambda_1 \), Lemma 3.2 and Lemma 2.4 imply that \( \tilde{I}_\lambda \) has a 
(PS)\(_c_\lambda\) sequence \( \{u_n\} := \{u_{n,\lambda}\} \), that is, there exists a sequence \( \{u_n\} \) satisfying
\[ \tilde{I}_\lambda(u_n) \to c_\lambda, \quad \tilde{I}_\lambda'(u_n) \to 0, \quad \text{as} \ n \to \infty, \]

where
\[ c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \tilde{I}_\lambda(\gamma(t)), \]
\[ \Gamma := \{ \gamma \in ([0,1], X) : \gamma(0) = 0, \gamma(1) = w \}. \]

**Lemma 3.3.** The (PS)\(_c_\lambda\) sequence \( \{u_n\} \) has a convergent subsequence.

**Proof.** By virtue of Lemma 3.1, \([5,9]\) and \( \theta = \min\{q_2, \beta\} > p^2 \), there exists a positive constant \( M > 0 \) such that
\[
M + \|u_n\|_V \geq \tilde{I}_\lambda(u_n) - \frac{1}{\theta} \langle \tilde{I}_\lambda'(u_n), u_n \rangle \\
= (a + b\|u_n\|_V)^{p-1} \left[ \frac{1}{bp^2}(a + b\|u_n\|_V^p) - \frac{1}{\theta} \|u_n\|_V^p \right] \\
- \lambda \int_0^T \left[ \tilde{F}(t, u_n) - \frac{1}{\theta} (\nabla \tilde{F}(t, u_n), u_n) \right] dt - \frac{a^p}{bp^2} \\
\geq (a + b\|u_n\|_V)^{p-1} \left[ \frac{1}{bp^2}(a + b\|u_n\|_V^p) - \frac{1}{\theta} \|u_n\|_V^p \right] - \frac{a^p}{bp^2} \\
\geq a^{p-1} \left[ \frac{a}{bp^2} + \left( \frac{1}{p^2} - \frac{1}{\theta} \right) \|u_n\|_V^p \right] - \frac{a^p}{bp^2} \\
= a^{p-1} \left( \frac{1}{p^2} - \frac{1}{\theta} \right) \|u_n\|_V^p \]

for \( n \) large enough, which shows that \( \{u_n\} \) is bounded in \( E_0^{\alpha,p} \) by \( p > 1 \). By Lemma 2.1, we can assume that, up to a subsequence, for some \( u_\lambda \in E_0^{\alpha,p} \),
\[ u_n \to u_\lambda \quad \text{in} \quad E_0^{\alpha,p}, \]
\[ u_n \to u_\lambda \quad \text{in} \quad C([0,T], \mathbb{R}^N). \]

Since
\[ \langle \tilde{I}_\lambda'(u_n), u_n - u_\lambda \rangle \]
we have

\begin{align*}
\langle \mathcal{T}_\lambda (u_n) - \mathcal{T}_\lambda (u_\lambda), u_n - u_\lambda \rangle &= (a + b)\|u_n\|_V^p \left( \int_0^T (|_0^T \|D_t^\alpha u_n\|^p + V(t)|u_n|^p)dt \right)^{p-1}
+ \int_0^T V(t)(|u_n|^{p-2}u_n, u_n - u_\lambda)dt - \lambda \int_0^T (\nabla \tilde{F}(t, u_n), u_n - u_\lambda)dt,
\end{align*}

(3.13)
\[-(a+b\|u_n\|_V^p)^{p-1}\|u_n\|_V^p - (a+b\|u_\lambda\|_V^p)^{p-1}\|u_\lambda\|_V^p - \lambda \int_0^T (\nabla \tilde{F}(t,u_n) - \nabla \tilde{F}(t,u_\lambda), u_n - u_\lambda) dt\]

\[= (a+b\|u_n\|_V^p)^{p-1}\|u_n\|_V^p - (a+b\|u_\lambda\|_V^p)^{p-1}\|u_\lambda\|_V^p - \lambda \int_0^T (\nabla \tilde{F}(t,u_n) - \nabla \tilde{F}(t,u_\lambda), u_n - u_\lambda) dt\]

\[= \left( (a+b\|u_n\|_V^p)^{p-1}\|u_n\|_V^p - (a+b\|u_\lambda\|_V^p)^{p-1}\|u_\lambda\|_V^p \right) (\|u_n\|_V - \|u_\lambda\|_V) \]

\[-\lambda \int_0^T (\nabla \tilde{F}(t,u_n) - \nabla \tilde{F}(t,u_\lambda), u_n - u_\lambda) dt. \quad (3.14)\]

Note that

\[\lambda \int_0^T (\nabla \tilde{F}(t,u_n) - \nabla \tilde{F}(t,u_\lambda), u_n - u_\lambda) dt \leq \lambda \int_0^T |\nabla \tilde{F}(t,u_n) - \nabla \tilde{F}(t,u_\lambda)||u_n - u_\lambda| dt \to 0, \quad (3.15)\]

by \(u_n \to u_\lambda\) in \(C([0,T], \mathbb{R}^N)\) and \(|\nabla \tilde{F}(t,u_n) - \nabla \tilde{F}(t,u_\lambda)|\) is bounded in \([0,T]\) because of \((H0)'\) and the boundedness of \(\{u_n\}\) in \(E_{0}^{\alpha,p}\), and (3.9) and (3.12) imply that

\[\left\langle \tilde{T}_\lambda (u_n) - \tilde{T}_\lambda (u_\lambda), u_n - u_\lambda \right\rangle \to 0, \text{ as } n \to \infty. \quad (3.16)\]

So by (3.14), (3.15) and (3.16), we have

\[\|u_n\|_V \to \|u_\lambda\|_V, \text{ as } n \to \infty.\]

By the uniform convexity of \(E_{0}^{\alpha,p}\) and \(u_n \to u_\lambda\), it follows from the Kadec-Klee property (see \(\text{[H]}\) and \(\text{[2.13]}\), \(u_n \to u_\lambda\) in \(E_{0}^{\alpha,p}\).

By the continuity of \(\tilde{I}_\lambda\), we obtain that \(\tilde{I}_\lambda (u) = c_\lambda\), where \(c_\lambda\) is defined by (3.10). Then (3.3) implies that \(c_\lambda \geq d_\lambda > 0\). Hence \(u_\lambda\) is a nontrivial critical point of \(\tilde{T}_\lambda\) in \(E_{0}^{\alpha,p}\) for any given \(\lambda > \Lambda_1\).

Next, we show that \(u_\lambda\) precisely is the nontrivial weak solution of system (1.1) for any given \(\lambda > \lambda^*\). In order to get this, we need to make an estimate for the critical level \(c_\lambda\). We introduce the functional \(\tilde{J}_\lambda : E_{0}^{\alpha,p} \to \mathbb{R}\) as follows

\[\tilde{J}_\lambda (u) = \frac{1}{bp^2} (a+b\|u\|_V^p)^{p-1}\|u\|_V^p - \lambda M_1 \int_0^T |u(t)|^p dt - \frac{a^p}{bp^2}.\]

**Lemma 3.3.** For all \(\lambda \geq \Lambda_2\),

\[c_\lambda \leq C_* \lambda^{-\frac{p-1}{\alpha-\gamma}},\]

where \(C_*\) is defined by (1.12) which is obviously independent of \(\lambda\).

**Proof.** Define \(f_i : [0, \infty) \to \mathbb{R}, i = 1, 2,\) by

\[f_1(s) = \frac{1}{bp^2} (a+bs^p\|e_1\|_V^p)^{p-1}\|e_1\|_V^p s^p - \frac{a^p}{bp^2}, \]

\[f_2(s) = \frac{1}{bp^2} (a+b|s|^p u_\lambda^p)^{p-1}|s|^p u_\lambda^p - \frac{a^p}{bp^2}, \]
\[ f_2(s) = -\lambda M_1 s \int_0^T |e_1|^q(t) \, dt + \lambda \frac{\lambda}{\lambda_1} \|e_1\|_{V^*}^{p} \|s^p\],

where \( e_1 = \frac{\lambda}{\lambda_0} e \) and \( e \) is defined in (3.3). Then \( f_1(s) + f_2(s) = \bar{J}_\lambda(se_1) \). Let

\[ f_2'(s) = -\lambda M_1 q_1 \|e_1\|_{L^1} \|s^p\|^{q_1-1} + \lambda \frac{\lambda}{\lambda_1} \|e_1\|_{V^*}^{p} \|s^p\|^{q_1-1} = 0. \]

Thus for each given \( \lambda > 0 \), we have

\[
\max_{s \geq 0} f_2(s) = \left( \frac{1}{p(M_1 q_1)^{\frac{1}{n-1}}} - \frac{M_1}{(M_1 q_1)^{\frac{1}{n-1}}} \right) \left( \frac{\|e_1\|_{V^*}^{p} \|s^p\|^{q_1-1}}{\|e_1\|_{L^1}} \right) \lambda^{-\frac{n-1}{n-1}}. 
\]

Obviously, \( f_1(0) = 0 \) and

\[ f_1'(s) = (a + bs^p \|e_1\|_{V^*}^{p})^{q_1-1} \|e_1\|_{V^*}^{p} s^{q_1-1} - \lambda \frac{\lambda}{\lambda_1} \|e_1\|_{V^*}^{p} s^{q_1-1}. \]

So if

\[ \lambda > \Lambda_2 := \left[ a + b \max\{1, V^{\infty}\} \frac{\delta}{G_0} (D^p + G^p) \right] q_1 \]

\[ = \left( a + b \max\{1, V^{\infty}\} \frac{\delta}{G_0} \|e\|^p \right) q_1 \]

\[ \geq (a + bs^p \|e_1\|_{V^*}^{p})^{q_1-1}, \]

\( f_1(s) \) is decreasing on \( s \in [0, 1] \) and then \( f_1(s) < 0 \) for all \( s \in [0, 1] \). By (3.7), we have

\[
\|se_1\|_{\infty} \leq \|\frac{\delta}{G_0} e\|_{\infty} \leq \delta
\]

for all \( s \in [0, 1] \). Then for all \( \lambda > \Lambda_2 \), by (H1)’, (3.5), (3.6) and Hölder inequality, we have

\[ c_{\lambda} \leq \max_{s \in [0, 1]} \bar{I}_\lambda(se_1) \leq \max_{s \in [0, 1]} \bar{J}_\lambda(se_1) \leq \max_{s \geq 0} f_1(s) + \max_{s \geq 0} f_2(s)
\]

\[ \leq \max_{s \geq 0} f_2(s) = \left( \frac{1}{p(M_1 q_1)^{\frac{1}{n-1}}} - \frac{M_1}{(M_1 q_1)^{\frac{1}{n-1}}} \right) \left( \frac{\|e_1\|_{V^*}^{p} \|s^p\|^{q_1-1}}{\|e_1\|_{L^1}} \right) \lambda^{-\frac{n-1}{n-1}}
\]

\[ \leq \left( \frac{1}{p(M_1 q_1)^{\frac{1}{n-1}}} - \frac{M_1}{(M_1 q_1)^{\frac{1}{n-1}}} \right) \left( \frac{\max\{1, V^{\infty}\}^{1/p} (D^p + G^p)^{1/p}}{T_{\frac{1}{n-1} - \frac{1}{p}} \|e\|_{L^p}} \right) \lambda^{-\frac{n-1}{n-1}}
\]

\[ = C_{\lambda} \lambda^{-\frac{n-1}{n-1}}. \]

**Proof of Theorem 1.1.** Note that \( u_\lambda \) is a critical point of \( \bar{I}_\lambda \) with critical value \( c_{\lambda} \). Since \( \langle \bar{P}(u_\lambda), u_\lambda \rangle = 0 \), similar to the argument in (3.11) and by Lemma 3.3, we have

\[
\|u_\lambda\|_{V^*}^{p} \leq \frac{p^2 \theta}{a^{p-1} (\theta - p^2)} \bar{I}_\lambda(u_\lambda)
\]
we have

\[
\frac{p^2\theta}{a^{p-1}(\theta - p^2)} C_\lambda
\leq \frac{p^2\theta}{a^{p-1}(\theta - p^2)} C_* \lambda^{-\frac{p-1}{n-p}}.
\] (3.18)

If

\[
\lambda > \Lambda_3 = \left( \frac{T^\alpha_{\alpha^2}}{\Gamma(\alpha)(\alpha q - 1)^\frac{1}{q}} \right)^{\frac{1}{\beta}} \left( \frac{p^2 \theta C_*}{a^{p-1}(\theta - p^2)} \right)^{\frac{2p}{\beta}}
\]

we have

\[
\|u_\lambda\|_\infty \leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)(\alpha q - 1)^{\frac{1}{q}}} \|u_\lambda\|_V \leq \delta/2.
\] (3.19)

So for all \( \lambda > \Lambda_3 \), \( |u_\lambda(t)| \leq \|u_\lambda\|_\infty \leq \delta/2 \) for a.e. \( t \in [0, T] \) and then \( \tilde{F}(t, u(t)) = \tilde{F}(t, u(t)) \) for a.e. \( t \in [0, T] \). Furthermore, \( \tilde{I}_\lambda(u_\lambda) = I_\lambda(u_\lambda) = c_\lambda > 0 \) and \( \langle \tilde{I}'(u_\lambda), v \rangle = \langle I'(u_\lambda), v \rangle = 0 \) for all \( v \in E_0^{\alpha, p} \). Thus \( u_\lambda \) is precisely the nontrivial weak solution of system (3.1) when \( \lambda > \lambda^* := \max\{\Lambda_1, \Lambda_2, \Lambda_3\} \). Note that \( p > 1 \) and \( q_1 > p \). By (3.18) and (3.19), it is obvious that

\[
\lim_{\lambda \to \infty} \|u_\lambda\|_V = 0 = \lim_{\lambda \to \infty} \|u_\lambda\|_\infty.
\]

\[\square\]

4. Example

Assume that \( N = 2, a = b = T = 1 \) and \( p = 3 \). Then \( q_1 = 12, q_2 = 10, F(t, u) = (t + 1)|u|^{11} \) for a.e. \( t \in [0, 1] \), and therefore \( \nabla F(t, u) = 11(t + 1)|u|^{9}u \), where \( u = (u_1, u_2) \). Let \( V(t) = 7t^2 + 1 \) for all \( t \in [0, T] \). Then \( V^\infty = 8 \) and \( V_\infty = 1 \). Choose \( \alpha = \frac{1}{4} \). Consider the system

\[
\begin{aligned}
\begin{cases}
A(u(t))[\alpha D_1^{1/2} \phi_3(u) D_1^{1/2} u(t)] + (7t^2 + 1)\phi_3(u(t))] = 11\lambda(t + 1)|u|^{9}u, & \text{a.e. } t \in [0, 1], \\
u(0) = u(1) = 0,
\end{cases}
\end{aligned}
\] (4.1)

where

\[
A(u(t)) = \left[ 1 + \int_0^1 (|D_1^{1/2} u(t)|^3 + (7t^2 + 1)|u(t)|^3) dt \right]^2.
\]

By Theorem 1.1, we can obtain that system (4.1) has at least a nontrivial solution \( u_\lambda \) in \( E_0^{3,3} \) for each \( \lambda > 183.46^{24} \) and \( \lim_{\lambda \to \infty} \|u_\lambda\|_V = 0 = \lim_{\lambda \to \infty} \|u_\lambda\|_\infty \).

In fact, we can verify that \( F(t, u) \) satisfies the assumption (H0)-(H2) as follows.

i) Let \( \delta = 1 \), and then we have

\[
|F(t, x)| + |\nabla F(t, x)| = (t + 1)|x|^{11} + 11(t + 1)|x|^{10} \leq 12(t + 1)|x|^{10}
\]
for all $|x| \leq \delta$. Set $a(|x|) = |x|^{10}, b(t) = 12(t + 1)$. Then assumption (H0) is satisfied.

ii) Note that $q_1 = 12 > q_2 = 10 > p^2 = 9$, and

$$|x|^{12} \leq F(t, x) = (t + 1)|x|^{11} \leq 2|x|^{10},$$

for all $|x| \leq \delta$ and a.e. $t \in [0, 1]$. Set $M_1 = 1$ and $M_2 = 2$. Then assumption (H1) is also satisfied.

iii) Let $\beta = 10 > p^2 = 9$. Then

$$0 \leq 10(t + 1)|x|^{11} = \beta F(t, x) \leq 11(t + 1)|x|^{11} = (\nabla F(t, x), x)$$

holds for all $x \in \mathbb{R}^2$ and a.e. $t \in [0, 1]$, and so assumption (H3) is satisfied. Next, we compute the value of $\lambda^*$ by the formulas in Theorem 1.1. Note that $\Gamma(\frac{1}{2}) = \sqrt{\pi}, \Gamma(2 - \frac{1}{2}) = \frac{\sqrt{\pi}}{2}$. We obtain

$$D = \left( \frac{T^{p+1} 2(p-1)!!}{\pi^{p+1} p!} \right)^{\frac{1}{p}} = \left( \frac{4}{3} \right)^{\frac{1}{p}} \pi^{-\frac{1}{2}},$$

$$G = \left( \frac{T^{p+1-p\alpha}}{\Gamma(p(2-\alpha)(p+1-p\alpha))} \right)^{\frac{1}{p}} = \left( \frac{16}{5} \right)^{\frac{1}{p}} \pi^{-\frac{1}{2}},$$

$$G_0 = \frac{T^{\gamma-\frac{1}{p}}}{\Gamma(\alpha)(aq - q + 1)} G = 5^{-\frac{1}{2}} \cdot 16^\frac{1}{2} \pi^{-1}.$$

Then by $\theta = \min\{\beta, q_2\} = 10$, (1.20), (1.10) and (1.11), we have

$$\Lambda_1 = \max \left\{ \sqrt{\frac{768}{78125}} \pi^{35/6}, \frac{1611/2 \cdot 3\pi^4}{625} \left(1 + \frac{5}{24\pi} + \frac{\pi^{3/2}}{2}\right)^3 \right\},$$

$$\Lambda_2 = \left( 1 + \frac{40}{3} \pi^{-1} + 32\pi^{3/2} \right)^{24},$$

$$\Lambda_3 = \left( \frac{720 \cdot 16 C_*}{\pi^{3/2}} \right)^{\frac{2}{3}},$$

and by (1.12),

$$C_* = 16 \left( \frac{1}{3 \cdot \sqrt[3]{12}} - \frac{1}{\sqrt[6]{12^4}} \right) \left( 1 + \frac{12}{5} \pi^{5/2} \right)^{4/3}.$$

Compared $\Lambda_1, \Lambda_2$ and $\Lambda_3$, it is easy to see $\lambda^* = \Lambda_2 \approx 183.46^{24}$.  

\[ \Box \]

**Authors’ contributions**

All authors have equal contribution and they read and approve the final manuscript.

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References

[1] G. Kirchhoff, Vorlesungen über Mechanik, Lectures on Mechanics, Teubner, Stuttgart, 1883.

[2] S. Bernstein, Sur une classe d’équations fonctionelles aux dérivées partielles (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 4(1940), 17-26.

[3] P. D’Ancona and S. Spagnolo, A class of nonlinear hyperbolic problems with global solutions, Archive for Rational Mechanics & Analysis 124(1993), 201-219.

[4] J. M. Greenberg and S. C. Hu, The initial value problem for a stretched string, Quarterly of Applied Mathematics 38(1980), 289-311.

[5] S. I. Pokhozhaev, On a class of quasilinear hyperbolic equations, Math USSR Sbornik 25(1975), 145-158.

[6] A. Arosio and S. Panizzi, On the well-posedness of the Kirchhoff string, Transactions of the American Mathematical Society 348(1)(1996), 305-330.

[7] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation, Adv. Differential Equations 6(2001), 701-730.

[8] P. D’Ancona and S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, Invent. Math. 108(1992), 247-262.

[9] C. O. Alves, F. Corrêa and T. F. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, Computers and Mathematics with Applications 49(1)(2005), 85-93.

[10] K. Perera and Z. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, Journal of Differential Equations 221(1)(2006), 246-255.

[11] Y. He, G. B. Li and S. J. Peng, Concentrating bound states for Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents, Adv. Nonlinear Stud. 14(2014), 483-510.

[12] Anran Li and Jiabao Su, Existence and multiplicity of solutions for Kirchhoff-type equation with radial potentials in $\mathbb{R}^3$, Z. Angew. Math. Phys. 66(2015), 3147-3158.

[13] X. H. Tang and Bitao Cheng, Ground state sign-changing solutions for Kirchhoff type problems in bounded domains, Journal of Differential Equations 261(2016), 2384-2402.

[14] Zhijian Yang and Fang Da, Stability of attractors for the Kirchhoff wave equation with strong damping and critical nonlinearities, J. Math. Anal. Appl. 469(2019), 298-320.
[15] M. Dreher, The Kirchhoff equation for the $p$-Laplacian, Rendiconti Del Seminario Matematico 64(2)(2006), 217-238.

[16] Júlio Francisco, S. A. Corrêa and G. M. Figueiredo, On an elliptic equation of $p$-Kirchhoff type via variational methods, Bulletin of the Australian Mathematical Society 74(2)(2006), 263-277.

[17] Duchao Liu, On a $p$-Kirchhoff equation via Fountain Theorem and Dual Fountain Theorem, Nonlinear Analysis 72(2010), 302-308.

[18] Giuseppina Autuori and Patrizia Pucci, Kirchhoff systems with dynamic boundary conditions, Nonlinear Analysis 73(7)(2010), 1952-1965.

[19] Chunhan Liu, Jianguo Wang and Qingling Gao, Existence of nontrivial solutions for $p$-Kirchhoff type equations, Boundary Value Problems 2013(1)(2013), 279.

[20] Nicole Heymans and Igor Podlubny, Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives, Rheol Acta 45(2006), 765-771.

[21] V. J. Ervin, N. Heuer and J. P. Roop, Numerical approximation of a time dependent, nonlinear, spacefractional diffusion equation, SIAM Journal on Numerical Analysis 45(2)(2007), 572-591.

[22] P. Zhuang, F. Liu, V. Anh and I. Turner, New solution and analytical techniques of the implicit numerical method for the anomalous subdiffusion equation, SIAM Journal on Numerical Analysis 46(2)(2008), 1079-1095.

[23] F. Jiao and Y. Zhou, Existence results for fractional boundary value problem via critical point theory, International Journal of Bifurcation and Chaos 22(04)(2012), 1250086.

[24] Y. Zhou, Basic theory of fractional differential equations, World Scientific Publishing Company, 2014.

[25] Y. Zhao, H. Chen and B. Qin, Multiple solutions for a coupled system of nonlinear fractional differential equations via variational methods, Applied Mathematics & Computation 257(2015), 417-427.

[26] G. Bonanno, R. Rodriguez-López and S. Tersian, Existence of solutions to boundary value problem for impulsive fractional differential equations, Fractional Calculus & Applied Analysis 17(3)(2014), 717-744.

[27] Y. Zhao and L. Tang, Multiplicity results for impulsive fractional differential equations with $p$-Laplacian via variational methods, Boundary Value Problems 2017(1)(2017), 123.

[28] J. Xie and X. Zhang, Infinitely many solutions for a class of fractional impulsive coupled systems with $(p,q)$-Laplacian, Discrete Dynamics in Nature and Society, 2018, 1-14.
[29] N. Nyamoradi and Y. Zhou, Existence results to some damped-like fractional differential equations, International Journal of Nonlinear Sciences and Numerical Simulation 18(3-4)(2017), 88-103.

[30] Guoqing Chai and Weiming Liu, Existence of solutions for the fractional Kirchhoff equations with sign-changing potential, Boundary Value Problems 2018(1)(2018), 125.

[31] Taiyong Chen and Wenbin Liu, Ground state solutions of Kirchhoff-type fractional Dirichlet problem with p-Laplacian, Advances in Difference Equations 2018(1)(2018), 436.

[32] D. Costa and Z. Q. Wang, Multiplicity results for a class of superlinear elliptic problems, Proc. Am. Math. Soc. 133(2005), 787-794.

[33] E. S. Medeiros and U. B. Severo, On the existence of signed solutions for a quasilinear elliptic problem in $\mathbb{R}^N$, Mat. Contemp. 32(2007), 193-205.

[34] Yongqiang Xu, Zhong Tan and Daoheng Sun, Multiplicity results for a nonlinear elliptic problem involving the fractional laplacian, Acta Mathematica Scientia 36(6)(2016), 1793-1803.

[35] Nikolaos S. Papageorgiou, Vicentiu D. Radulescu and Dusan D. Repov, Double-phase problems with reaction of arbitrary growth, Z. Angew. Math. Phys. 69(2018), 108.

[36] João Marcos do ó, Everaldo Medeiros and Uberlandio Severo, On the existence of signed and sign-changing solutions for a class of superlinear Schrödinger equations, J. Math. Anal. Appl. 342(2008), 432-445.

[37] Tieshan He, Zheng-an Yao and Zhaohong Sun, Multiple and nodal solutions for parametric Neumann problems with nonhomogeneous differential operator and critical growth, J. Math. Anal. Appl. 449(2017), 1133-1151.

[38] Chen Huang and Jia Gao, Existence of positive solutions for supercritical quasilinear Schrödinger elliptic equations, J. Math. Anal. Appl. 472(2019), 705-727.

[39] A. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, Elsevier Science Limited, 2006.

[40] M. Willem, Minimax theorems, Springer Science and Business Media, 1997.

[41] E. Hewitt and K. Stromberg, Real and abstract analysis, American Mathematical Monthly volume 60(3)(1965), 317-318.

[42] I. Ekeland, Convexity Methods in Hamiltonian Mechanics, Springer, 1990.
[43] G. Li and C. Wang, The existence of a nontrivial solution to a nonlinear elliptic problem of linking type without the Ambrosetti-Rabinowitz condition, Annales Academiae Scientiarum Fennicae Mathematica 36(2)(2011),461-480.

[44] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, American Mathematical Society, Providence, RI, 1986.