Observability and null-controllability for parabolic equations in $L_p$-spaces

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Abstract

We study (cost-uniform approximate) null-controllability of parabolic equations in $L_p(R^d)$ and provide explicit bounds on the control cost. In particular, we consider systems of the form \[ \dot{x}(t) = -A_p x(t) + 1_E u(t), \quad t \in (0, T], \quad x(0) = x_0 \in L_p(R^d), \] where $-A_p$ is a strongly elliptic differential operator of order $m \in \mathbb{N}$ with constant coefficients, $1_E : L_p(E) \rightarrow L_p(R^d)$ is the embedding from a measurable set $E \subset R^d$ to $R^d$, $T > 0$, and where $u \in L_r((0, T); L_p(E))$ with some $r \in [1, \infty]$. Hence, the influence of the control function $u$ is restricted to the subset $E$. Note that we allow for lower order terms in the strongly elliptic differential operator. The focus of this paper is laid on null-controllability, that is, for any initial condition $x_0 \in L_p(R^d)$ there is a control function $u \in L_r((0, T); L_p(E))$ such that the mild solution of (1) at time $T$ equals zero. We will also be concerned with the notion of cost-uniform approximate null-controllability (or approximate null-controllability with uniformly bounded controls), which means that...

1. Introduction

We consider parabolic control systems on $L_p(R^d)$, $p \in [1, \infty)$, of the form

\[ \dot{x}(t) = -A_p x(t) + 1_E u(t), \quad t \in (0, T], \quad x(0) = x_0 \in L_p(R^d), \] (1)

where $-A_p$ is a strongly elliptic differential operator of order $m \in \mathbb{N}$ with constant coefficients, $1_E : L_p(E) \rightarrow L_p(R^d)$ is the embedding from a measurable set $E \subset R^d$ to $R^d$, $T > 0$, and where $u \in L_r((0, T); L_p(E))$ with some $r \in [1, \infty]$. Hence, the influence of the control function $u$ is restricted to the subset $E$. Note that we allow for lower order terms in the strongly elliptic differential operator. The focus of this paper is laid on null-controllability, that is, for any initial condition $x_0 \in L_p(R^d)$ there is a control function $u \in L_r((0, T); L_p(E))$ such that the mild solution of (1) at time $T$ equals zero. We will also be concerned with the notion of cost-uniform approximate null-controllability (or approximate null-controllability with uniformly bounded controls), which means that...
there exists $C > 0$ such that for all $\varepsilon > 0$ and all $x_0 \in L_p(\mathbb{R}^d)$ with $\|x_0\|_{L_p(\mathbb{R}^d)} \leq 1$ we can find a control function $u \in L_r((0, T); L_p(E))$ with $\|u\|_{L_r((0, T); L_p(E))} \leq C$ such that the mild solution of (1) at time $T$ has norm smaller than $\varepsilon$; in reflexive spaces, these two notions agree (see [Car88]). By linearity, (cost-uniform approximate) null-controllability implies that any target state in the range $\text{Ran}(e^{-TA_p})$ of the semigroup generated by $-A_p$ can be reached (up to an error $\varepsilon$) within time $T$.

We will show in Theorem 2.4 that if $E$ is a so-called thick set, then the system is cost-uniform approximately null-controllable for $p = 1$ and null-controllable if $p \in (1, \infty)$. Thus, we generalize the results of [GST20] in two ways. On the one hand, we are able to deal with the case $p = 1$ which is important for applications; just note that, e.g., the classical heat equation in $L_1$ yields interpretations of the solution as heat densities, while diffusing population models were also considered in $L_1$-spaces. On the other hand, we allow for lower order terms in the operator $-A_p$. Moreover, we provide explicit upper bounds on the control cost, i.e. on the norm of the control function $u$ which steers the system (approximately) to zero at time $T$, which are explicit in terms of geometric properties of the thick set $E$ and of the final time $T$.

Null-controllability for (linear) systems in $L_p(\Omega)$ with control function in $L_r$ has already been studied earlier in the literature, both in bounded as well as unbounded domains $\Omega \subseteq \mathbb{R}^d$. However, mostly the investigations are done in the Hilbert space context, i.e. $p = r = 2$. More precisely, for bounded domains $\Omega$, null-controllability has been considered in [FR71, LR95, FI96, EZ11, MRR14] with various methods. Note that in many applications $-A_2$ is self-adjoint and thus spectral theoretic methods are applicable. Null-controllability for unbounded domains $\Omega$ has been studied in [dT97, CdMZ01, MZ01a, MZ01b, CMV04, Mil05, KO20]. The fact that thickness of $E$ is necessary and sufficient to obtain null-controllability for the heat equation in $L_2(\mathbb{R}^d)$ has been realized in [EV18, WWZZ19]. Turning away from the Hilbert space case, we mention [GST20], where $\Omega = \mathbb{R}^d$, $p \in (1, \infty)$ and $r \in [1, \infty]$.

An equivalent formulation of cost-uniform approximate null-controllability is final-state observability of the dual system to (1). This means that there exists a constant $C_{\text{obs}} \geq 0$ such that for all $\varphi \in L_p(\mathbb{R}^d)'$ we have

$$\left\| S_{T}^t \varphi \right\|_{L_p(\mathbb{R}^d)'} \leq \begin{cases} C_{\text{obs}} \left( \int_{0}^{T} \left\| (S_t^r \varphi) |E| \right\|_{L_p(\mathbb{R}^d)'} \, dt \right)^{1/r'} & \text{if } r' \in [1, \infty), \\ C_{\text{obs}} \sup_{t \in [0, T]} \left\| (S_t^r \varphi) |E| \right\|_{L_p(\mathbb{R}^d)'} & \text{if } r' = \infty, \end{cases}$$

where $(S_t)_{t \geq 0}$ is the $C_0$-semigroup generated by $-A_p$ and $r' \in [1, \infty]$ is such that $1/r + 1/r' = 1$. This equivalence follows from Douglas’ lemma, see [Dou66] for Hilbert spaces, and [Emb73, DR77, Har78, CP78, Car85, Car88, For14] for Banach spaces.

In Section 2 we formulate our results on final-state observability in Theorem 2.2. Then the (cost-uniform approximate) null-controllability in Theorem 2.4 follows as a consequence of the above-mentioned duality. The proof of Theorem 2.2, which can be found in Section 3, rests on an abstract observability estimate stated in the appendix. There, Theorem A.1 provides a generalization of the abstract result in [GST20] for not necessarily strongly continuous semigroups. In view of [Lot85], this is particularly im-
portant for final-state observability in $L_\infty$ or, put differently, cost-uniform approximate null-controllability in $L_1$.

The main strategy we follow to prove observability has first been described for the Hilbert space case (i.e. $p=r=2$) in [Mil10] inspired by [LR95, LZ98, JL99], and further studied, e.g., in [TT11, WZ17, BPS18, NTTV20, BPZ21]. It combines an (abstract) uncertainty principle or unique continuation estimate with a dissipation estimate to obtain the final-state observability via an iterative argument; cf. Theorem A.1. However, far less is known about its generalization to Banach spaces; to the best of our knowledge, we are only aware of [GST20], which applied the strategy for the case of strongly continuous semigroups. To finally obtain Theorem 2.2 we thus need to show the uncertainty principle and the dissipation estimate. While the uncertainty principle is a consequence of the Logvinenko–Sereda theorem (see Theorem 3.1), the dissipation estimate is shown via explicit estimates on the kernel of the semigroup; cf. Proposition 3.2. This enables us to cover the case $p \in \{1, \infty\}$ as well, in contrast to the interpolation technique used in [GST20].

In Section 4 we discuss further directions and developments, particularly focussing on control to trajectories, some nonlinear problems as well as the question of how good null controls can actually be found.

2. Observability and Null-controllability in $L_p$-Spaces

In order to formulate our main theorems, we review some basic facts from Fourier analysis. For details we refer, e.g., to the textbook [Gra14]. We denote by $S(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions, which is dense in $L_p(\mathbb{R}^d)$ for all $p \in [1, \infty)$. The space of tempered distributions, i.e. the topological dual space of $S(\mathbb{R}^d)$, is denoted by $S'(\mathbb{R}^d)$. For $f \in S(\mathbb{R}^d)$ let $Ff: \mathbb{R}^d \to \mathbb{C}$ be the Fourier transform of $f$ defined by

$$Ff(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} \, dx.$$ 

Then $F : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ is bijective, continuous and has a continuous inverse, given by

$$F^{-1}f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\xi) e^{i\xi \cdot x} \, d\xi$$

for all $f \in S(\mathbb{R}^d)$. For $u \in S'(\mathbb{R}^d)$ the Fourier transform is again denoted by $F$ and is given by $(F u)(\phi) = u(F\phi)$ for $\phi \in S(\mathbb{R}^d)$. By duality, the Fourier transform is bijective on $S'(\mathbb{R}^d)$ as well.

Let $m \in \mathbb{N}$ and

$$a(\xi) = \sum_{|\alpha|_1 \leq m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^d,$$

be a polynomial of degree $m$ with coefficients $a_\alpha \in \mathbb{C}$. We say that the polynomial $a$ is strongly elliptic if there exist constants $c > 0$ and $\omega \in \mathbb{R}$ such that $a$ satisfies for all $\xi \in \mathbb{R}^d$ the lower bound

$$\Re a(\xi) \geq c |\xi|^m - \omega. \quad (2)$$
Note that strong ellipticity implies that \( m \) is even.

Given a strongly elliptic polynomial \( a \) and \( p \in [1, \infty] \), we define the associated heat semigroup \( S : [0, \infty) \to \mathcal{L}(L_p(\mathbb{R}^d)) \) by

\[
S_tf = \mathcal{F}^{-1}e^{-t\alpha} \mathcal{F}f = \mathcal{F}^{-1}e^{-t\alpha} * f.
\]

(3)

Note that the second equality holds since \( e^{-t\alpha} \in \mathcal{S}(\mathbb{R}^d) \). It is well known that the operator semigroup \( (S_t)_{t \geq 0} \) is strongly continuous if \( p \in [1, \infty) \). For \( p = \infty \) the semigroup is the dual semigroup of a strongly continuous semigroup on \( L_1(\mathbb{R}^d) \) and hence it is only weak*-continuous in general. For details we refer, e.g., to [Are02]. By [TR96], the integral kernel \( k_t = \mathcal{F}^{-1}e^{-t\alpha} \) satisfies the following heat kernel estimate: There exist \( c_1, c_2 > 0 \) such that for all \( x \in \mathbb{R}^d \) and \( t > 0 \) we have

\[
|k_t(x)| \leq c_1 e^{c_2 t} t^{-d/m} e^{-c_2 \left( \frac{|x|^m}{t} \right)^{m-1}}.
\]

(4)

This implies that there is \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that for all \( p \in [1, \infty] \), \( f \in L_p(\mathbb{R}^d) \), and \( t \geq 0 \) we have

\[
\|S_tf\|_{L_p(\mathbb{R}^d)} \leq \|k_t\|_{L_1(\mathbb{R}^d)} \|f\|_{L_p(\mathbb{R}^d)} \leq M e^{c_2 t} \|f\|_{L_p(\mathbb{R}^d)}.
\]

(5)

In order to formulate our main result, we introduce the notion of a thick subset \( E \) of \( \mathbb{R}^d \).

**Definition 2.1.** Let \( \rho \in (0, 1] \) and \( L \in (0, \infty)^d \). A set \( E \subset \mathbb{R}^d \) is called \((\rho, L)\)-thick if \( E \) is measurable and for all \( x \in \mathbb{R}^d \) we have

\[
|E \cap \bigotimes_{i=1}^d (0, L_i) + x| \geq \rho \prod_{i=1}^d L_i.
\]

Here, \(|\cdot|\) denotes Lebesgue measure in \( \mathbb{R}^d \).

The following theorem yields a final-state observability estimate for \((S_t)_{t \geq 0}\) on thick sets.

**Theorem 2.2.** Let \( m \in \mathbb{N} \), \( a : \mathbb{R}^d \to \mathbb{C} \) a strongly elliptic polynomial of order \( m \), \( c > 0 \) and \( \omega \in \mathbb{R} \) as in (2), and \((S_t)_{t \geq 0}\) as in (3). Let \( \rho \in (0, 1] \), \( L \in (0, \infty)^d \), \( E \subset \mathbb{R}^d \) a \((\rho, L)\)-thick set, \( p, r \in [1, \infty] \), and \( T > 0 \). Then we have for all \( f \in L_p(\mathbb{R}^d) \)

\[
\|S_Tf\|_{L_p(\mathbb{R}^d)} \leq \begin{cases} C_{obs} \left( \int_0^T \|(S_tf)|_E\|_{L_p(E)}^r \, dt \right)^{1/r} & \text{if } r \in [1, \infty), \\
C_{obs} \limsup_{t \to 0} \|(S_tf)|_E\|_{L_p(E)} & \text{if } r = \infty,
\end{cases}
\]

where

\[
C_{obs} = \frac{K_a}{T^{1/r}} \left( \frac{K_d}{\rho} \right)^{K_d(1+|L|_1\lambda^*)} \exp \left( \frac{K_m(|L|_1 \ln(K_d/\rho))^{m/(m-1)}}{(cT)^{1/(m-1)}} + K \max \{\omega, 0\} T \right).
\]

Here, \( \lambda^* = (2^{n+3} \max \{\omega, 0\}/c)^{1/m} \), \( K > 0 \) is an absolute constant, and \( K_a, K_d, K_m > 0 \) are constants depending only on the polynomial \( a \), on \( d \), or on \( m \), respectively.
Remark 2.3 (Optimality of $C_{\text{obs}}$). In this remark we discuss the optimality of $C_{\text{obs}}$ with respect to the parameters $T > 0$ and $L \in (0, \infty)^d$.

We begin with optimality with respect to the time parameter $T$. Assume that $E \subset \mathbb{R}^d$ is a thick set, $r = 2$, and the generator $-A_2$ of the semigroup $(S_t)_{t \geq 0}$ is a self-adjoint operator in $L_2(\mathbb{R}^d)$ with $\min \sigma(A_2) = 0$. This is, e.g., the case if the strongly elliptic polynomial is given by $a(\xi) = |\xi|^2$. Then, the generator $-A_2$ of the semigroup $(S_t)_{t \geq 0}$ is the (self-adjoint) Laplacian on $L_2(\mathbb{R}^d)$. Let us define the optimal observability constant by

$$C_{\text{obs}}^*(T) := \sup_{f \in L_2(\mathbb{R}^d), f \neq 0} \frac{\|S_T f\|_{L_2(\mathbb{R}^d)}}{\left(\int_0^T \|S_t f\|_E^2 \|f\|_{L_2(E)}^2 dt\right)^{1/2}}.$$ 

An upper bound on $C_{\text{obs}}^*(T)$ is given in Theorem 2.2. As $\min \sigma(A_2) = 0$, for $\varepsilon > 0$ we can choose a function $0 \neq f_\varepsilon \in L_2(\mathbb{R}^d)$ from the spectral subspace of the interval $[0, \varepsilon]$, and calculate, using spectral calculus,

$$C_{\text{obs}}^*(T) \geq \frac{e^{-\varepsilon T} \|f_\varepsilon\|_{L_2(\mathbb{R}^d)}}{\left(\int_0^T \|f_\varepsilon\|_E^2 dt\right)^{1/2}} = \frac{e^{-\varepsilon T}}{T^{1/2}}.$$ 

As $\varepsilon > 0$ was arbitrary, our $C_{\text{obs}}$ in Theorem 2.2 is optimal with respect to $T$ for large $T$. We refer to [NTTV20, Theorem 2.13] for a more general statement. In order to see that our bound is optimal for small $T$ as well, we might argue as follows. If one considers a linear control problem in $L_2(\Omega)$ with bounded open $\Omega \subset \mathbb{R}^d$ instead of $\mathbb{R}^d$ and with $(S_t)_{t \geq 0}$ being the heat-semigroup, it has been shown that

$$\frac{\sup_{B(\rho) \subset \Omega \cap E} \rho^2 / 4}{\ln T} \leq \liminf_{T \to 0} T \ln C_{\text{obs}}^*(T),$$

see [FCZ00, Zua01, Mil04]. This shows that the exponential blowup for $T \to 0$ has to occur for the controlled heat equation on bounded open subsets $\Omega \subset \mathbb{R}^d$. In order to extend (6) to the case $\Omega = \mathbb{R}^d$ it seems feasible to apply the method obtained in [SV20]. In that paper, the authors show that if the controlled heat equation on $\Omega = \mathbb{R}^d = (-L/2, L/2)^d$ satisfies an observability estimate with a constant independent of $L > 0$, then, using a continuity argument, the corresponding system on $\Omega = \mathbb{R}^d$ satisfies an observability estimate as well with the same upper bound. By an analogous argument, the lower bound (6) should hold in the case $\Omega = \mathbb{R}^d$ as well. This suggests that $C_{\text{obs}}$ in Theorem 2.2 is optimal also in the regime $T \to 0$.

Let us now turn to a discussion of the optimality with respect to $L \in (0, \infty)^d$. We consider the case $r = p = 2$ and $(S_t)_{t \geq 0}$ being the heat-semigroup. As above, we assume that the lower bound (6) holds in the case $\Omega = \mathbb{R}^d$. Then Theorem 2.2 implies (using $c = 1$ and $\omega = 0$)

$$\liminf_{T \to 0} T \ln C_{\text{obs}} = K_m\left(|L|_1 \ln(K_d / \rho)^2\right)$$

Now we consider an example from [NTTV20, Remark 4.14]. Let $d = 2$, $l > 0$ and

$$E = \bigcup_{k \in \mathbb{Z}} ((k - l/2, k) \times \mathbb{R}).$$
The set $E$ is $(1/2, L)$-thick with $L = (l, l)$ and the left hand side of (6) is given by
\[
\sup_{B(\rho) \subset \mathbb{R}^d \setminus E} \rho^2 / 4 = \frac{l^2}{64}.
\] (8)

From (7) and (8) we conclude that $c_{\text{obs}}$ in Theorem 2.2 is optimal with respect to $L$.

For $p \in [1, \infty)$ let $-A_p$ be the generator of the $C_0$-semigroup $(S_t)_{t \geq 0}$ on $L_p(\mathbb{R}^d)$. Note that for all $f \in S(\mathbb{R}^d)$ we have
\[
A_p f = \sum_{|\alpha| \leq m} a_\alpha (-i)^{|\alpha|} \partial^\alpha f.
\]
Moreover, by (2), the differential operator $A_p$ is strongly elliptic, i.e., there is $c > 0$ such that
\[
\text{Re} \left( \sum_{|\alpha| = m} a_\alpha \xi^\alpha \right) \geq c |\xi|^m.
\]

Then, the statement of Theorem 2.2 corresponds to a final-state observability estimate for the system
\[
\dot{x}(t) = -A_p x(t), \quad t \in (0, T], \quad x(0) = x_0 \in L_p(\mathbb{R}^d),
\]
\[
y(t) = x(t)|_E, \quad t \in [0, T].
\]

Let us now turn to the discussion on null-controllability. For a measurable set $E \subset \mathbb{R}^d$ and $T > 0$ we consider the linear control problem
\[
\dot{x}(t) = -A_p x(t) + 1_E u(t), \quad t \in (0, T], \quad x(0) = x_0 \in L_p(\mathbb{R}^d)
\]
where $u \in L_r((0, T); L_p(E))$ with $r \in [1, \infty]$. The unique mild solution is given by Duhamel’s formula
\[
x(t) = S_t x_0 + B^t u, \quad \text{where} \quad B^t u = \int_0^t S_{t-\tau} 1_E u(\tau) d\tau.
\] (9)

By Theorem 2.2 and duality, we obtain (cost-uniform approximate) null-controllability for (1).

**Theorem 2.4.** Let $m \in \mathbb{N}$, $a: \mathbb{R}^d \to \mathbb{C}$ a strongly elliptic polynomial of order $m$, $c > 0$ and $\omega \in \mathbb{R}$ as in (2), and $-A_p$ the generator of the $C_0$-semigroup $(S_t)_{t \geq 0}$ as in (3). Let $\rho \in (0, 1]$, $L \in (0, \infty)^d$, $E \subset \mathbb{R}^d$ a $(\rho, L)$-thick set, $r \in [1, \infty]$, and $T > 0$.

(a) For any $x_0 \in L_1(\mathbb{R}^d)$ and any $\varepsilon > 0$ there exists $u \in L_r((0, T); L_1(E))$ with
\[
\|u\|_{L_r((0, T); L_1(E))} \leq c_{\text{obs}} \|x_0\|_{L_1(\mathbb{R}^d)} \quad \text{and} \quad \|x(T)\|_{L_1(\mathbb{R}^d)} < \varepsilon,
\]
where $x$ is the solution of (1) given by (9).
(b) Let \( p \in (1, \infty) \). Then for any \( x_0 \in L_p(\mathbb{R}^d) \) there exists \( u \in L_r((0,T);L_p(E)) \) with
\[
\|u\|_{L_r((0,T);L_p(E))} \leq C_{\text{obs}} \|x_0\|_{L_p(\mathbb{R}^d)} \quad \text{and} \quad x(T) = 0,
\]
where \( x \) is the solution of (1) given by (9).

Here, \( C_{\text{obs}} \) is as in Theorem 2.2 with \( r \) replaced by \( r' \) where \( r' \in [1, \infty] \) such that \( 1/r + 1/r' = 1 \).

The statement (a) of Theorem 2.4 corresponds to cost-uniform approximate null-controllability in time \( T \), whereas the statement (b) corresponds to null-controllability in time \( T \). Note that in case \( p \in (1, \infty) \) null-controllability and cost-uniform approximate null-controllability are equivalent, see, e.g., [Car88].

It is a standard duality argument that Theorem 2.4 follows from Theorem 2.2 by means of Douglas’ lemma. For the sake of completeness, we give a short proof.

**Proof of Theorem 2.4 (assuming Theorem 2.2).** Let \( p \in [1, \infty) \) and \( r \in [1, \infty] \). Moreover, let \( B^T: L_r((0,T);L_p(E)) \rightarrow L_p(\mathbb{R}^d) \) be given by
\[
B^Tu = \int_0^T S_{T-t}1_Eu(t) \, dt.
\]
Then, by [Vie05, Theorem 2.1] we have for all \( g \in L_{p'}(\mathbb{R}^d) \)
\[
\|(B^T)'g\|_{L_r((0,T);L_{p'}(E))'} = \sup_{\tau \in [0,T]} \|(S_{T-t}^\tau g) \|_{L_{p'}(E)} = \sup_{\tau \in [0,T]} \|(S_t^\tau g) \|_{L_{p'}(E)}
\]
if \( r = 1 \), and
\[
\|(B^T)'g\|_{L_r((0,T);L_{p'}(E))'} = \left( \int_0^T \|(S_{T-t}^\tau g) \|_{L_{p'}(E)}^r \, dt \right)^{1/r'} = \left( \int_0^T \|(S_t^\tau g) \|_{L_{p'}(E)}^{r'} \, dt \right)^{1/r'}
\]
if \( r \in (1, \infty] \), where \( r' \in [1, \infty] \) is such that \( 1/r + 1/r' = 1 \) and \( p' \in (1, \infty] \) is such that \( 1/p + 1/p' = 1 \). Since \( \mathcal{F}S_t^\tau = e^{-t\omega(a)\tau} \mathcal{F} \), we have that \( (S_t^\tau)_{\tau \geq 0} \) is associated to the symbol \( a(\cdot) \) which is strongly elliptic with the same constant \( c > 0 \). Moreover, since the associated heat kernel is given by \( (\mathcal{F}^{-1}e^{-t\omega(a)\cdot}) \mathcal{F} \), we have \( \|S_t^\tau\| \leq Me^{ct} \) with the same \( M \) and \( \omega \) as in (5). Thus, Theorem 2.2 and the above equalities imply for all \( g \in L_{p'}(\mathbb{R}^d) \)
\[
\|S_T^\tau g\|_{L_{p'}} \leq C_{\text{obs}} \|(B^T)'g\|_{L_r((0,T);L_{p'}(E))'} = C_{\text{obs}} \|(B^T)'g\|_{L_r((0,T);L_{p'}(E))'}
\]
where \( C_{\text{obs}} \) is as in Theorem 2.2 with \( r \) replaced by \( r' \). By Douglas’ lemma, see e.g. [Har78, Car85, Car88], we conclude
\[
\{S_Tx_0: \|x_0\|_{L_p(\mathbb{R}^d)} \leq 1 \} \subset \{B^Tu: \|u\|_{L_r((0,T);L_p(E))} \leq C_{\text{obs}} \} \quad \text{if} \quad p = 1
\]
and
\[
\{S_Tx_0: \|x_0\|_{L_p(\mathbb{R}^d)} \leq 1 \} \subset \{B^Tu: \|u\|_{L_r((0,T);L_p(E))} \leq C_{\text{obs}} \} \quad \text{if} \quad p \in (1, \infty).
\]
By scaling and linearity, this implies the statement of the theorem. 

\( \square \)
3. Proof of Theorem 2.2

For the proof of Theorem 2.2 we apply the abstract observability estimate in Theorem A.1. For this purpose, we define a family of operators $P_\lambda$, and verify the uncertainty principle (21) and the dissipation estimate (22).

We start with defining the operators $P_\lambda$ as in [GST20]. Let $\eta \in C_0^\infty([0, \infty))$ with $0 \leq \eta \leq 1$ such that $\eta(r) = 1$ for $r \in [0, 1/2]$ and $\eta(r) = 0$ for $r \geq 1$. For $\lambda > 0$ we define $\chi_\lambda : \mathbb{R}^d \rightarrow \mathbb{R}$ by $\chi_\lambda(\xi) = \eta(|\xi|/\lambda)$. Since $\chi_\lambda \in \mathcal{S}(\mathbb{R}^d)$, we have $\mathcal{F}^{-1} \chi_\lambda \in \mathcal{S}(\mathbb{R}^d) \subset L_1(\mathbb{R}^d)$ and for all $p \in [1, \infty]$ we define $P_\lambda : L_p(\mathbb{R}^d) \rightarrow L_p(\mathbb{R}^d)$ by $P_\lambda f = (\mathcal{F}^{-1} \chi_\lambda) * f$. By Young’s inequality we have for all $f \in L_p(\mathbb{R}^d)$

$$
\|P_\lambda f\|_{L_p(\mathbb{R}^d)} = \|\mathcal{F}(\mathcal{F}^{-1} \chi_\lambda) \ast f\|_{L_p(\mathbb{R}^d)} \leq \|\mathcal{F}^{-1} \chi_\lambda\|_{L_1(\mathbb{R}^d)} \|f\|_{L_p(\mathbb{R}^d)}.
$$

Moreover, the norm $\|\mathcal{F}^{-1} \chi_\lambda\|_{L_1(\mathbb{R}^d)}$ is independent of $\lambda > 0$. Indeed, by the scaling property of the Fourier transform and by change of variables we have for all $\lambda > 0$

$$
\|\mathcal{F}^{-1} \chi_\lambda\|_{L_1(\mathbb{R}^d)} = |\lambda|^d \|\mathcal{F}^{-1} \chi_\lambda(\lambda \cdot)\|_{L_1(\mathbb{R}^d)} = \|\mathcal{F}^{-1} \chi_1\|_{L_1(\mathbb{R}^d)}.
$$

Next, we observe that the uncertainty principle (21) is a consequence of the following Logvinenko–Sereda theorem from [Kov01], see also [LS74, Kov00] for predecessors.

**Theorem 3.1** (Logvinenko–Sereda theorem). There exists $K \geq 1$ such that for all $p \in [1, \infty]$, $\lambda > 0$, $\rho \in (0, 1]$, $L \in (0, \infty)^d$, $(\rho, L)$-thick sets $E \subset \mathbb{R}^d$, and $f \in L_p(\mathbb{R}^d)$ satisfying $\text{supp} \mathcal{F} f \subset [-\lambda, \lambda]^d$ we have

$$
\|f\|_{L_p(\mathbb{R}^d)} \leq d_0 e^{d_1 \lambda} \|f\|_{L_p(E)},
$$

where

$$
d_0 = e^{Kd \ln(Kd/\rho)} \quad \text{and} \quad d_1 = 2|L|_1 \ln(Kd/\rho).
$$

Concerning the dissipation estimate (22), we prove the following Proposition.

**Proposition 3.2.** Let $m \in \mathbb{N}$, $a : \mathbb{R}^d \rightarrow \mathbb{C}$ a strongly elliptic polynomial of order $m$, $c > 0$ and $\omega \in \mathbb{R}$ as in (2), $(S_t)_{t \geq 0}$ as in (3), and $(P_\lambda)_{\lambda > 0}$ as above. Then for all $p \in [1, \infty]$, $f \in L_p(\mathbb{R}^d)$, $\lambda > 2^{(m+3)/m}(\max \{\omega, 0\}/c)^{1/m}$, and $t \geq 0$ we have

$$
\|(\text{Id} - P_\lambda)S_t f\|_{L_p(\mathbb{R}^d)} \leq K_a e^{-2^{-m-3}c t \lambda^m} \|f\|_{L_p(\mathbb{R}^d)},
$$

where $K_a \geq 0$ is a constant depending only on $a$ (and therefore also on $m$ and $d$).

**Proof.** For all $f \in L_p(\mathbb{R}^d)$, $\lambda > 0$ and $t \geq 0$ we have

$$
(\text{Id} - P_\lambda)S_t f = \mathcal{F}^{-1}((1 - \chi_\lambda)e^{-ta}) * f,
$$

and by Young’s inequality, we thus obtain

$$
\|(\text{Id} - P_\lambda)S_t f\|_{L_p(\mathbb{R}^d)} \leq \|\mathcal{F}^{-1}((1 - \chi_\lambda)e^{-ta})\|_{L_1(\mathbb{R}^d)} \|f\|_{L_p(\mathbb{R}^d)}.
$$
We write
\[ k_{t,\lambda} = \mathcal{F}^{-1}((1 - \chi_\lambda)e^{-ta}). \]

By Young’s inequality, (10) and (4), there exists \( K_a \geq 0 \) such that
\[ \|\mathcal{F}^{-1}(\chi_\lambda e^{-ta})\|_{L^1(\mathbb{R}^d)} = \|\mathcal{F}^{-1}\chi_\lambda \ast \mathcal{F}^{-1}e^{-ta}\|_{L^1(\mathbb{R}^d)} \leq \|\mathcal{F}^{-1}\chi_1\|_{L^1(\mathbb{R}^d)}\|\mathcal{F}^{-1}e^{-ta}\|_{L^1(\mathbb{R}^d)} \leq K_a e^{cta}. \]

Hence we find for all \( \lambda > 0 \) and \( t \geq 0 \) the bound
\[ \|k_{t,\lambda}\|_{L^1(\mathbb{R}^d)} \leq \|\mathcal{F}^{-1}e^{-ta}\|_{L^1(\mathbb{R}^d)} + \|\mathcal{F}^{-1}(\chi_\lambda e^{-ta})\|_{L^1(\mathbb{R}^d)} \leq K_a e^{cta}. \] (12)

Setting
\[ \sigma_{t,\lambda} = ((1 - \chi_\lambda)e^{-ta})(t^{-1/m}), \]
for \( \lambda > 0 \) and \( t > 0 \), by linear substitution \( \eta = t^{1/m}\xi \) it follows that
\[ k_{t,\lambda}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{it^{-1/m}x\eta}((1 - \chi_\lambda)e^{-ta})(t^{-1/m}\eta) t^{-d/m}d\eta = t^{-d/m}(\mathcal{F}^{-1}\sigma_{t,\lambda})(t^{-1/m}x). \]

Therefore,
\[ \|k_{t,\lambda}\|_{L^1(\mathbb{R}^d)} = \|\mathcal{F}^{-1}\sigma_{t,\lambda}\|_{L^1(\mathbb{R}^d)}. \]

Let now \( \lambda > \lambda_0 = 2^{(m+3)/m}(\max\{\omega, 0\}/c)^{1/m} \) and \( \alpha \in \mathbb{N}_0^d \) with \( |\alpha| \leq d + 1 \). By differentiation properties of the Fourier transform we have
\[ (-1)^{|\alpha|}x^\alpha \mathcal{F}^{-1}\sigma_{t,\lambda}(x) = \mathcal{F}^{-1}(\partial^\alpha\sigma_{t,\lambda})(x), \ x \in \mathbb{R}^d. \] (13)

We apply the product rule and the triangle inequality to obtain
\[ |\partial^\alpha\sigma_{t,\lambda}| \leq 1_{\{\xi \geq t^{1/m}\lambda / 2\}} |\partial^\alpha\chi_\lambda e^{-ta(t^{-1/m} \cdot)}| + \sum_{\beta \in \mathbb{N}_0^d, \beta < \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) |\partial^{\alpha-\beta}(1 - \chi_\lambda t^{1/m})| |\partial^\beta e^{-ta(t^{-1/m} \cdot)}|. \] (14)

By (2), for \( t \geq 0 \) and \( \xi \in \mathbb{R}^d \) with \( |\xi| \geq \lambda/2 \) we observe
\[ |e^{-ta(\xi)}| = e^{-t \text{Re } a(\xi)} \leq e^{ct\xi} e^{-c t|\xi|^m} \leq e^{ct\xi} e^{-ct|\xi|^m/2} e^{-ct\lambda^m/2m+1}. \]

Hence, by possibly increasing \( K_a \), for all \( \beta \in \mathbb{N}_0^d \) with \( \beta \leq \alpha \) and all \( \xi \in \mathbb{R}^d \) with \( |\xi| \geq t^{1/m}\lambda / 2 \) we obtain
\[ |(\partial^\beta e^{-ta(t^{-1/m} \cdot)})(\xi)| \leq K_a \left( 1 + t^{(m-1)/m} |\xi|^{m-1} \right)^{|\beta|_1} |e^{-ta(t^{-1/m} \cdot)}| \leq K_a e^{ct\xi} \left( 1 + t^{(m-1)/m} |\xi|^{m-1} \right)^{|\beta|_1} e^{-ct|\xi|^m/2} e^{-ct\lambda^m/2m+1} \]
\[ \leq K_a e^{ct\xi} \left( 1 + t^{(m-1)/m} |\xi|^{m-1} \right)^{|\beta|_1} e^{-ct|\xi|^m/2} e^{-ct\lambda^m/2m+2}. \]
Since, for all $\beta \in \mathbb{N}_0^d$ with $\beta \leq \alpha$ we have
\[
\sup_{t \geq 0, \xi \in \mathbb{R}^d} (1 + t^{(m-1)/m} |\xi|^{m-1})^{\beta} e^{-c|\xi|^{m}/4} e^{-ct\lambda_0^{m}/2^{m+2}} < \infty,
\]
we may again increase $K_a$ such that for all $\lambda > \lambda_0$, $t > 0$, $\beta \in \mathbb{N}_0^d$ with $\beta \leq \alpha$ and $|\xi| \geq t^{1/m} \lambda/2$ we have
\[
|\partial^\beta e^{-ta(t^{-1/m})}(\xi)| \leq K_a e^{\omega t} e^{-c|\xi|^{m}/4} e^{-ct\lambda^{m}/2^{m+2}}.
\] (15)
For all $\beta \in \mathbb{N}_0^d$ with $\beta < \alpha$ we have for all $\lambda > 0$, $t \geq 0$ and $\xi \in \mathbb{R}^d$ that
\[
|\partial^\beta (1 - \chi_{t^{1/m}}(\lambda))(\xi)| \leq (t^{1/m} \lambda)^{-|\alpha| - \beta} (\partial^\alpha \chi_1(\xi/t^{1/m} \lambda)) 1_{\{t^{1/m} \lambda/2 \leq |\xi| \leq t^{1/m} \lambda\}}(\xi)
\]
\[
\leq C(t^{1/m} \lambda)^{-|\alpha| - \beta} 1_{\{t^{1/m} \lambda/2 \leq |\xi| \leq t^{1/m} \lambda\}}(\xi),
\] (16)
where $C = \max_{\beta < \alpha} \|\partial^\beta \chi_1\|_\infty$. Thus, from (14), (15) and (16) and the definition of $\lambda_0$, we may increase $K_a$ such that for all $\lambda > \lambda_0$ and $t > 0$ such that $t^{1/m} \lambda \geq 1$ and all $\xi \in \mathbb{R}^d$ we have
\[
|\partial^\alpha \sigma_{t, \lambda}(\xi)| \leq K_a e^{\omega t} e^{-c|\xi|^{m}/4} e^{-ct\lambda^{m}/2^{m+2}} \leq K_a e^{-c|\xi|^{m}/4} e^{-ct\lambda^{m}/2^{m+3}},
\]
and hence, from (13), we may increase $K_a$ such that for all $x \in \mathbb{R}^d$ we obtain
\[
|x^\alpha \mathcal{F}^{-1} \sigma_{t, \lambda}(x)| \leq K_a e^{-ct\lambda^{m}/2^{m+3}}.
\] (17)
In particular, for $j \in \{1, 2, \ldots, d\}$ and $\alpha_j = (d + 1) e_j$, where $e_j$ denotes the $j$-th canonical unit vector in $\mathbb{R}^d$, we obtain $|x_j|^{d+1} |\mathcal{F}^{-1} \sigma_{t, \lambda}(x)| \leq K_a e^{-ct\lambda^{m}/2^{m+3}}$, hence $||x||^{d+1} |\mathcal{F}^{-1} \sigma_{t, \lambda}(x)| \leq K_a e^{-ct\lambda^{m}/2^{m+3}}$, and consequently for all $\lambda > \lambda_0$, $t > 0$ such that $t^{1/m} \lambda \geq 1$ we find for all $x \in \mathbb{R}^d \setminus \{0\}$
\[
|\mathcal{F}^{-1} \sigma_{t, \lambda}(x)| \leq K_a e^{-ct\lambda^{m}/2^{m+3}} |x|^{d-1}.
\] (18)
From (17) with $\alpha = 0$ and (18) we obtain for all $\lambda > \lambda_0$, $t > 0$ such that $t^{1/m} \lambda \geq 1$ that
\[
\|k_{t, \lambda}\|_{L_1(\mathbb{R}^d)} = \|\mathcal{F}^{-1} \sigma_{t, \lambda}\|_{L_1(\mathbb{R}^d)} \leq K_a e^{-ct\lambda^{m}/2^{m+3}} \left( \int_{|x| \leq 1} dx + \int_{|x| > 1} |x|^{-d-1} dx \right)
\]
\[
\leq K_a e^{-ct\lambda^{m}/2^{m+3}},
\]
where we again increased $K_a$ in the last estimate. In view of (12), for $\lambda > \lambda_0$, $t > 0$ such that $t^{1/m} \lambda \leq 1$ we have
\[
\|k_{t, \lambda}\|_{L_1(\mathbb{R}^d)} \leq K_a e^{\omega t} \leq K_a e^{\omega / \lambda_0^{m}} \leq K_a e^{\omega / \lambda_0^{m}} e^{c/2^{m+3}} e^{-ct\lambda^{m}/2^{m+3}}.
\]
Hence, increasing $K_a$ again, we finally obtain for all $\lambda > \lambda_0$ and $t > 0$ that
\[
\|k_{t, \lambda}\|_{L_1(\mathbb{R}^d)} \leq K_a e^{-ct\lambda^{m}/2^{m+3}}. \]
We can finally prove Theorem 2.2.

**Proof of Theorem 2.2.** Let \((P_\lambda)_{\lambda>0}\) be the family of operators defined at the beginning of this section. Then we have \(\text{supp}\mathcal{F}(P_\lambda f) \subset [-\lambda, \lambda]^d\) for all \(\lambda > 0\) and all \(f \in L_p(\mathbb{R}^d)\). Thus, Theorem 3.1 implies that for all \(f \in L_p(\mathbb{R}^d)\) and all \(\lambda > 0\) we have

\[
\|P_\lambda f\|_{L_p(\mathbb{R}^d)} \leq d_0 e^{d_1 \lambda} \|P_\lambda f\|_{L_p(E)},
\]

where \(d_0\) and \(d_1\) are as in (11). Moreover, according to Proposition 3.2, for all \(\lambda > \lambda^*\) and all \(f \in L_p(\mathbb{R}^d)\) we have

\[
\|(I - P_\lambda)S_t f\|_{L_p(\mathbb{R}^d)} \leq d_2 e^{-d_3 \lambda^* t} \|f\|_{L_p(E)},
\]

where \(\lambda^* := \left(2^{m+3} \max\{\omega, 0\}/c\right)^{1/m}\), \(d_2 \geq 1\) depends only on the polynomial \(a\), and where \(d_3 := 2^{-m-3}c\). Moreover, the function \(t \mapsto \|(S_t f)|_E\|_{L_p(E)}\) is Borel-measurable for all \(f \in L_p(\mathbb{R}^d)\). Indeed, if \(p \in [1, \infty)\) the semigroup \((S_t)_{t \geq 0}\) is strongly continuous and the measurability follows. If \(p = \infty\) the semigroup \((S_t)_{t \geq 0}\) is the dual of a strongly continuous semigroup on \(L_1(\mathbb{R}^d)\). By means of the Hahn–Banach theorem the function \(t \mapsto \|(S_t f)|_E\|_{L_\infty(E)}\) is, as the supremum of continuous functions, lower semicontinuous and hence measurable. Thus, we can apply Theorem A.1 with \(X = L_p(\mathbb{R}^d)\), \(Y = L_p(E)\), \(C: X \to Y\) given by the restriction map on \(E\), and obtain that the statement of the theorem holds with \(C_{\text{obs}}\) replaced by

\[
\tilde{C}_{\text{obs}} := \frac{C_1}{T^{1/r}} \exp \left(\frac{C_2}{T^{m-1}} + C_3 T\right),
\]

where \(T^{1/r} = 1\) if \(r = \infty\), and

\[
C_1 := (4M d_0 \max\left\{\left(4d_2 M^2 (d_0 + 1)\right)^{8/(e \ln 2)}, e^{4d_3 2\lambda^*}\right\}),
\]

\[
C_2 := 4\left(2 \cdot 8^{m-1} d_3 \frac{d_1}{d_2}\right)^{m-1},
\]

\[
C_3 := \max\{\omega, 0\}(1 + 10/(e \ln 2)),
\]

with \(M\) as in (5). We denote by \(K_d, K_m,\) and \(K_a\) positive constants which depend only on the dimension \(d\), on \(m\), or on the polynomial \(a\), respectively. A straightforward calculation shows that

\[
C_1 \leq K_a \left(\frac{K_d}{\rho}\right)^{K_d (1 + |L| \lambda^*)} \quad \text{and} \quad C_2 \leq K_m (|L| \ln(K_d/\rho))^{m/(m-1)} \left(\frac{1}{e^{1/(m-1)}}\right).\]

Thus we obtain

\[
\tilde{C}_{\text{obs}} \leq \frac{K_a}{T^{1/r}} \left(\frac{K_d}{\rho}\right)^{K_d (1 + |L| \lambda^*)} \exp \left(\frac{K_m (|L| \ln(K_d/\rho))^{m/(m-1)}}{(e T)^{1/(m-1)}} + C_3 T\right) := C_{\text{obs}}. \quad \square
\]
4. Discussion on related questions and further research

The focus of the whole paper is put more or less on observability and cost-uniform (approximate) null-controllability for linear control problems in Banach spaces. In particular, we study controllability to one single state at a given fixed time. In this section, we discuss related questions and possible generalizations, in particular, we address

(i) control to a given trajectory defined on a time interval;
(ii) possible generalization to non-linear problems;
(iii) how to determine a control function.

4.1. Control to trajectories

We consider parabolic control systems on \( L_p(\mathbb{R}^d) \), \( p \in [1, \infty) \), of the form

\[
\dot{x}(t) = -A_p x(t) + 1_E u(t), \quad t \in (0, T], \quad x(0) = x_0 \in L_p(\mathbb{R}^d),
\]

where \(-A_p\) is a strongly elliptic differential operator of order \( m \in \mathbb{N} \) with constant coefficients, \( 1_E : L_p(E) \to L_p(\mathbb{R}^d) \) is the embedding from a measurable set \( E \subset \mathbb{R}^d \) to \( \mathbb{R}^d \), \( T > 0 \), and where \( u \in L_r((0, T); L_p(E)) \) with some \( r \in (1, \infty) \). (For the sake of simplicity we restrict the discussion in this section to \( r \in (1, \infty) \) only and exclude the cases \( r = 1 \) and \( r = \infty \).) Moreover, we introduce the cost functional

\[
J(u) = \int_0^T \alpha(t) \|u(t)\|_{L_p(E)}^r \, dt + \int_0^T \beta(t) \|x(t) - x^d(t)\|_{L_p(\mathbb{R}^d)}^r \, dt,
\]

with some time-dependent weights \( 0 < \alpha_1 \leq \alpha(t) \leq \alpha_2 < \infty \) and \( 0 < \beta_1 \leq \beta(t) \leq \beta_2 < \infty \) for all \( t \in [0, T] \), and some desired trajectory \( x^d : (0, T) \to L_p(\mathbb{R}^d) \). Thus, the first term describes the (weighted) cost of the control function, while the second penalizes deviations from a given trajectory. Let \( \varepsilon > 0 \), \( x_0 \in L_p(\mathbb{R}^d) \) be our initial state, and \( x^* \in L_p(\mathbb{R}^d) \) a given target state. We are interested in the optimal control problem

\[
\min_{u \in L_r((0, T); L_p(E))} \left\{ J(u) : \|x(T) - x^*\|_{L_p(\mathbb{R}^d)} \leq \varepsilon \right\}. \tag{19}
\]

That is, we are interested in finding a cost-optimal control function that steers our system up to an error \( \varepsilon \) to a given target state. If \( r, p \in (1, \infty) \), by standard convex optimization wisdom it follows that the problem (19) has a unique solution which we denote by \( u^{\text{opt}} \). Indeed, since the domain of \( J \) is a reflexive space, and the functional \( J \) is proper, convex, coercive, lower-semicontinuous and strictly convex in the sense of [Pey15, Theorem 2.19], it follows that \( \text{argmin} J \) is a singleton. We denote the global minimizer of \( J \) by \( u^{\text{min}} \). In order to see that the constrained minimization problem (19) has a unique solution as well, one introduces the indicator function

\[
I(x) = \begin{cases} 
0 & \text{if } \|x - x^*\|_{L_p(\mathbb{R}^d)} \leq \varepsilon, \\
\infty & \text{else}.
\end{cases}
\]
Then, our problem (19) is equivalent to the problem
\[
\min_{u \in L_r((0,T);L_p(E))} \hat{J}(u), \quad \text{where} \quad \hat{J}(u) = J(u) + I(x(T)).
\]

Since \( \hat{J} \) is again proper, convex, coercive, lower-semicontinuous and strictly convex, [Pey15, Theorem 2.19] implies that \( \text{argmin} \ \hat{J} \) is a singleton. Note that the unique solution \( u^{\text{opt}} \) of (19) depends on the initial state \( x_0 \) and the target state \( x^* \). Some remarks are in order:

(i) If \( \alpha = 1, \beta = 0, \) and \( x^* = 0 \), then the latter minimization problem corresponds to the control problem studied in the paper under consideration. In particular, \( u^{\text{opt}} \) corresponds to our control function, and \( J(u^{\text{opt}})^{1/r} \) to the \( x_0 \)-dependent control cost. In particular, in this paper we prove an upper bound on the control cost \( J(u^{\text{opt}})^{1/r} \) which is uniform in the choice of the initial state \( x_0 \in L_p(\mathbb{R}^d) \). Moreover, we give explicit dependence of the upper bound on the geometric properties of the control set \( E \).

(ii) If \( \beta \neq 0 \), then the second term in \( J \) models controls to trajectories, as it punishes large deviations from the desired trajectory. Such problems have been already studied in the case of Hilbert spaces, i.e. where \( r = p = 2 \), see [LM21] and the references therein. The paper [LM21] provides an abstract formula for \( u^{\text{opt}} \), which is used to derive numerical algorithms for the approximation of \( u^{\text{opt}} \) in certain situations.

Let us finish this section by addressing some possible future projects. One possible goal might be to prove an abstract formula for \( u^{\text{opt}} \) in the Banach space setting, just in the manner of, e.g., the results in [LM21]. Such an abstract formula can then be applied to numerical studies. Another goal might be to generalize our upper bound on the control cost to the setting of control to trajectories, i.e. providing an upper bound on \( J(u^{\text{opt}}) \). The first question appears to be well worth studying: However, it seems that the methods to answer this question are entirely different from those used for our results. The second question is to us at least as interesting, but we believe it to be more challenging. Recall that in the case \( \beta = 0 \) we derive our result by giving an observability estimate. By duality such an observability constant gives an upper bound on \( J(u^{\text{opt}}) \). To the best of our knowledge, if \( \beta \neq 0 \) this duality argument, and hence our whole approach, fails. This means it is not clear how to easily employ our main result on an observability estimate to control to trajectories.

4.2. Controllability of non-linear problems

A general strategy to treat non-linear problems stems from linearisations (i.e. first considering linear problems) and fixed point arguments under suitable assumptions on the non-linearity (such as small Lipschitz constants). Let us restrict to semilinear problems of the form
\[
\dot{x}(t) = -A_p x(t) + f(x(t), \nabla x(t)) + 1_E u(t), \quad t \in (0,T], \quad x(0) = x_0 \in L_p(\mathbb{R}^d),
\]
where \( f: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) is smooth and locally Lipschitz with \( f(0,0) = 0 \) and a suitable growth condition at infinity. For the case of the semilinear heat equation (sometimes with no gradient dependence of \( f \)), where \( -A_p = \Delta \) is the Laplacian, cost-uniform null-controllability was studied both in bounded [FPZ95, Fer97, FCZ00] as well as unbounded domains [Zua01, CdMZ01, Zua07, GBdT07]. Since we are interested in unbounded domains, we will focus on this case here. There, a typical condition needed is that \( \mathbb{R}^d \setminus E \) is bounded. Note that in the linear case we only require that \( E \) is thick, which is much weaker.

Let us sketch the method in the case of the semilinear heat equation. Controllability of (20) is derived in two steps. First, one proves an observability estimate for the dual system of the linearised system, i.e. of a system of the form

\[
\dot{x}(t) = -A'_p x(t) + a x(t) - \text{div}(b x(t)), \quad t \in (0,T], \quad x(0) = x_0 \in L_{p'}(\mathbb{R}^d) \\
y(t) = x(t)|_E, \quad t \in (0,T].
\]

Here, \( a \) and \( b \) come from linearising \( f \). This yields cost-uniform null-controllability of the linearised system

\[
\dot{x}(t) = -A_p x(t) + a x(t) + b \cdot \nabla x(t) + 1_E u(t), \quad t \in (0,T], \quad x(0) = x_0 \in L_p(\mathbb{R}^d).
\]

For the second step, we note that for controllability of (20) it suffices to obtain controllability of

\[
\dot{x}(t) = -A_p x(t) + 1_{\mathbb{R}^d \setminus E} f(x(t), \nabla x(t)) + 1_E u(t), \quad t \in (0,T], \quad x(0) = x_0 \in L_p(\mathbb{R}^d);
\]

cf. e.g. [CdMZ01, Section 3]. Writing \( f(x, \nabla x) = F(x) + G(x) \cdot \nabla x \), for fixed \( \tilde{x} \) we get the linearised control problem with \( a = F(\tilde{x}) \) and \( b = G(\tilde{x}) \). One then establishes a map \( N: \tilde{x} \mapsto x \), which maps \( \tilde{x} \) to the solution state \( x \) of the linearised control problem. The goal is then to show that \( N \) has a fixed point, which may be done by applying Schauder’s fixed point theorem.

It seems an interesting open problem whether our result for the linear case can be extended to the semilinear situation as sketched above.

4.3. Determining control functions

Our result states (approximate) null-controllability for (1), i.e. for all \( x_0 \) there exists \( u \) such that \( x(T) = 0 \) (has arbitrarily small norm). Thus, we obtain an existence statement.

Of course, in applied contexts, one would not only want to know that such a control function \( u \) exists but how one can construct it. For the Hilbert space case \( p = r = 2 \), one can either employ that control functions can be considered as orthogonal projections on an affine subspace, see [NTTV20, Remark 2.6], or consult the practical algorithm obtained in [LM21, Sections 4,5] to construct the optimal minimizer \( u^{opt} \). This changes the perspective from (functional analytic) mathematical control theory to PDE-constrained optimization. Note that, to the best of our knowledge, some arguments used there require to work in Hilbert spaces, so they are not available in our general Banach space situation. Since this question is of interest nonetheless, it may be dealt with in a forthcoming paper.
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A. Sufficient criteria for observability in Banach spaces

We provide an abstract sufficient criteria for final-state observability in Banach space, which is a slight generalization of Theorem 2.1 in [GST20], see also [Mil10, TT11, WZ17, BPS18, NTTV20, BPZ21] for earlier results. In particular, it does not assume strong continuity of the semigroup. For the proof, we comment only on the necessary modifications compared to [GST20].

Theorem A.1. Let $X$ and $Y$ be Banach spaces, $C : X \to Y$ a bounded linear operator, $(S_t)_{t \geq 0}$ a semigroup on $X$, $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S_t\| \leq Me^{\omega t}$ for all $t \geq 0$, and assume that for all $x \in X$ the mapping $t \mapsto \|CS_t x\|_Y$ is measurable. Further, let $\lambda^* \geq 0$, $(P_\lambda)_{\lambda > \lambda^*}$ a family of bounded linear operators in $X$, $r \in [1, \infty]$, $d_0, d_1, d_3, \gamma_1, \gamma_2, \gamma_3, T > 0$ with $\gamma_1 < \gamma_2$, and $d_2 \geq 1$, and assume that

\[
\forall x \in X \ \forall \lambda > \lambda^* : \quad \|P_\lambda x\|_X \leq d_0 e^{d_1 \lambda^* \gamma_1} \|CP_\lambda x\|_Y, \quad (21)
\]

and

\[
\forall x \in X \ \forall \lambda > \lambda^* \ \forall t \in (0, T/2] : \quad \|(\text{Id} - P_\lambda)S_t x\|_X \leq d_2 e^{-d_3 \lambda^* \gamma_2 t^3} \|x\|_X. \quad (22)
\]

Then we have for all $x \in X$

\[
\|S_T x\|_X \leq \begin{cases} 
  C_{\text{obs}} \left( \int_0^T \|CS_t x\|_Y^r \, dt \right)^{1/r} & \text{if } r \in [1, \infty), \\
  C_{\text{obs}} \text{ess sup}_{t \in [0, T]} \|CS_t x\|_Y & \text{if } r = \infty,
\end{cases}
\]

with

\[
C_{\text{obs}} = \frac{C_1}{T^{1/r}} \exp \left( \frac{C_2}{T^{1/r}} + C_3 T \right),
\]

where $T^{1/r} = 1$ if $r = \infty$, and

\[
C_1 = (4Md_0) \max \left\{ (4d_2 M^2 (d_0 \|C\| + 1))^{8/(e \ln 2)}, e^{4d_1 (2\lambda^*) \gamma_1} \right\},
\]

\[
C_2 = 4 \left( 2^{\gamma_1} (2 \cdot 4^{\gamma_1}) \frac{\gamma_2}{\gamma_2 - \gamma_1} \frac{d_1^2}{d_3} \frac{1}{\gamma_2 - \gamma_1} \right),
\]

\[
C_3 = \max \{\omega, 0\} \left( 1 + 10/(e \ln 2) \right).
\]

The assumption in (21) is an abstract uncertainty principle (sometimes also called spectral inequality), while (22) is a dissipation estimate. Thus, Theorem A.1 can be rephrased that an uncertainty principle together with a dissipation estimate implies a final-state observability estimate.
Remark A.2. In the situation of Theorem A.1, if we assume that \( t \mapsto CS_tx \) is Bochner measurable, we can rewrite the statement of the theorem as

\[
\| S_Tx \|_X \leq C_{\text{obs}} \| CS_t(\cdot)x \|_{L_t((0,T);Y)}.
\]

Proof of Theorem A.1. Since we do not assume the semigroup \((S_t)_{t \geq 0}\) to be strongly continuous, we cannot apply \([GST20, \text{Theorem 2.1}]\) directly. The strong continuity of \((S_t)_{t \geq 0}\) was assumed in \([GST20]\) in order to ensure that for all \( x \in X \) and \( \lambda > \lambda^* \) the functions

\[
F(t) = \| S_tx \|_X, \quad F_\lambda(t) = \| P_\lambda S_tx \|_X, \quad F_\lambda^t(t) = \| (\text{Id} - P_\lambda)S_tx \|_X,
\]

\[
G(t) = \| CS_tx \|_Y, \quad G_\lambda(t) = \| C\lambda S_tx \|_Y, \quad G_\lambda^t(t) = \| C(\text{Id} - P_\lambda)S_tx \|_Y,
\]

are measurable. The measurability of these six functions was used to obtain the estimate

\[
F(t) \leq 2Me^{\omega_+ T} d_0 e^{d_1 \lambda^*} \int_{t/2}^t G(\tau) d\tau + d_2 M^2 e^\omega T/4 e^{d_1 \lambda^*} \| C \|_Y (d_0 \| C \| + 1) F(t/4),
\]

where \( \omega_+ = \max\{0, \omega\} \). Such an inequality implies the statement of the theorem by iteration, see \([GST20]\). Thus it suffices to show the last displayed inequality by assuming merely measurability of the mapping \( t \mapsto G(t) \). Let \( t > 0, \tau \in [t/2, t] \) and \( x \in X \). Since \( F(\tau) \leq F_\lambda(\tau) + F_\lambda^t(\tau) \), by our assumptions and by the semigroup property we obtain

\[
F(\tau) \leq d_0 e^{d_1 \lambda^*} G_\lambda(\tau) + d_2 e^{-d_3 \lambda/2(\tau/2)^3} F(\tau/2).
\]

Using \( G_\lambda(\tau) \leq G(\tau) + G_\lambda^t(\tau) \leq G(\tau) + \| F_\lambda^t(\tau) \|_X \), our assumption, \( e^{d_1 \lambda^*} \geq 1 \), and \( F(\tau/2) \leq M e^{\omega_+ T/4} F(t/4) \) we obtain

\[
F(\tau) \leq d_0 e^{d_1 \lambda^*} G(\tau) + (d_0 \| C \| + 1)d_2 e^{-d_3 \lambda/2(\tau/2)^3} e^{d_1 \lambda^*} M e^{\omega_+ T/4} F(t/4).
\]

Since \( F(t) \leq M e^{\omega_+ T} F(\tau) \), we obtain

\[
F(t) \leq M e^{\omega_+ T} d_0 e^{d_1 \lambda^*} G(\tau) + (d_0 \| C \| + 1)d_2 e^{-d_3 \lambda/2(\tau/2)^3} e^{d_1 \lambda^*} M^2 e^{\omega_+ T/4} F(t/4).
\]

Since the mapping \( \tau \mapsto G(\tau) \) is measurable by assumption, we can integrate this inequality with respect to \( \tau \), and obtain the desired estimate.

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