Is there evidence for dimension-two corrections in QCD two-point functions?

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Abstract

The ALEPH data on the (non-strange) vector and axial-vector spectral functions, extracted from tau-lepton decays, is used in order to search for evidence for a dimension-two contribution, $C_{2V,A}$, to the Operator Product Expansion (other than $d = 2$ quark mass terms). This is done by means of a dimension-two Finite Energy Sum Rule, which relates QCD to the experimental hadronic information. The average $C_2 \equiv (C_{2V} + C_{2A})/2$ is remarkably stable against variations in the continuum threshold, but depends rather strongly on $\Lambda_{QCD}$. Given the current wide spread in the values of $\Lambda_{QCD}$, as extracted from different experiments, we conservatively conclude from our analysis that $C_2$ is consistent with zero.

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The Operator Product Expansion (OPE), extended beyond perturbation theory, is one of the pillars of the successful QCD sum rule method used extensively to link QCD and hadronic physics [1]. In analyzing e.g. two-point functions, one calculates QCD perturbative contributions up to a desired order in the running strong coupling, and then includes non-perturbative effects parametrized in terms of a series of vacuum to vacuum matrix elements of local, gauge invariant operators built from the quark and gluon fields entering the QCD Lagrangian. These so called vacuum condensates encompass the long distance dynamics. They are multiplied by Wilson coefficients, calculable in perturbation theory, which contain the short distance information. For a given dimension, these terms fall off as powers of the (Euclidean) momentum transfer $q^2$ ($q^2 < 0$). The lowest naive dimension in QCD is $d = 4$, corresponding to the gluon condensate and to the product of quark masses and the quark condensate. In the standard OPE approach there are no dimension-two corrections, other than the well known quark mass insertion contributions of the form $m_q^2/q^2$. Except possibly in the strange-quark sector, these terms can be safely neglected for light quark current correlators. On the other hand, some specific dynamical mechanisms have been suggested as potential sources of $d = 2$ corrections to the OPE, e.g. renormalons [2] or a tachyonic gluon mass [3]. If present, these dimension-two corrections are not expected to have much impact on most of the existing phenomenological predictions from QCD sum rules, as the level of precision is seldom better than 10-20%. However, they may have a non-negligible impact on the extraction of the QCD strong coupling at the tau-lepton mass scale, $\alpha_s(M^2_{\tau})$. In fact, current claims from this precision determination [4] have been questioned [5] on the grounds that they rely on the assumption of no $d = 2$ corrections to the OPE (other than quark mass insertions). Attempts have been made to determine phenomenologically the size of potential dimension-two terms using information on (i) the vector spectral function as obtained from $e^+e^-$ data [6], and (ii) the vector and the axial-vector spectral functions extracted from tau-lepton hadronic decays [7], as measured by the ARGUS collaboration [8]. The analysis based on tau decay data has the advantage of relying on two independent quantities ($d = 2$ corrections are expected to be chiral-symmetric), which can be constrained further by using theoretical information such
as the first Weinberg sum rule. The conclusion from this determination \cite{7} was that the
ARGUS data supported a non-zero dimension-two term in the OPE, which was consistent
with a dependence on the QCD scale of the form: $C_2 \propto \Lambda_{QCD}^2$. With the advent of the
ALEPH precision data \cite{9} on semileptonic tau decays it is now possible to analyze a vari-
ety of theoretical issues involving the vector and axial-vector spectral functions. One such
issue is, precisely, the possible presence of a dimension-two contribution in the OPE, which
is the subject of this note. These spectral functions are related to the discontinuities in
the complex energy plane of the two-point functions involving the vector and axial-vector
currents

$$
\Pi_{V}^{\mu\nu}(q^2) = i \int d^4 x e^{iqx} < 0|T(V_{\mu}(x) \ V_{\nu}^\dagger(0))|0 >
= (-g_{\mu\nu} q^2 + q_{\mu} q_{\nu}) \Pi_V(q^2),
$$

(1)

$$
\Pi_{AA}^{\mu\nu}(q^2) = i \int d^4 x e^{iqx} < 0|T(A_{\mu}(x) \ A_{\nu}^\dagger(0))|0 >
= (-g_{\mu\nu} q^2 + q_{\mu} q_{\nu}) \Pi_A(q^2) - q_{\mu} q_{\nu} \Pi_0(q^2),
$$

(2)

where $V_{\mu} =: (\bar{u}\gamma_{\mu} u - \bar{d}\gamma_{\mu} d) :/2$, and $A_{\mu} =: (\bar{u}\gamma_{\mu}\gamma_5 u - \bar{d}\gamma_{\mu}\gamma_5 d) :/2$. Considering these
(charge neutral) currents implies the normalization $\text{Im} \ \Pi_V = \text{Im} \ \Pi_A = 1/8\pi$, at leading
order in perturbative QCD. In order to determine the size of a potential dimension-two
contribution to the OPE we consider the following Finite Energy Sum Rule (FESR) \cite{10}

$$
I_{0 \ V,A} \equiv \frac{8\pi^2}{s_0} \int_{s_0}^{s_0} \rho_{V,A}(s) \ ds \ = \ \frac{C_2 \ V,A}{s_0} + F_2(s_0),
$$

(3)

where $C_2 \ V,A$ is the potential $d = 2$ term, $s_0$ is the continuum threshold, and $F_2(s_0)$ is the
radiative correction, identical in the vector and axial-vector channels, which is obtained
after a straightforward integration of the perturbative QCD results of \cite{11}; to four-loop
order it is given by

$$
F_2(s_0) = 1 + \frac{\alpha_s^{(1)}(s_0)}{\pi} + \left(\frac{\alpha_s^{(1)}(s_0)}{\pi}\right)^2 \left(F_3 - \frac{\beta_2}{\beta_1} \ln L - \frac{\beta_1}{2}\right)
+ \left(\frac{\alpha_s^{(1)}(s_0)}{\pi}\right)^3 \left[\frac{\beta_2}{\beta_1} (\ln^2 L - \ln L - 1) + \frac{\beta_3}{\beta_1} - 2 \left(F_3 - \frac{\beta_1}{2}\right) \frac{\beta_2}{\beta_1} \ln L + F_4 - F_3 \beta_1 - \frac{\beta_2}{2} + \frac{\beta_1^2}{2}\right],
$$

(4)
with
\[
\frac{\alpha_s^{(1)}(s_0)}{\pi} \equiv -\frac{2}{\beta_1 L},
\]
where \( L \equiv \ln(s_0/\Lambda_{QCD}^2) \), and for three flavours: \( \beta_1 = -9/2, \beta_2 = -8, \beta_3 = -3863/192, F_3 = 1.6398, F_4 = -10.2839 \). In writing Eq. (4) use has been made of the result [11]

\[
\frac{\alpha_s^{(3)}(s_0)}{\pi} = \left( \frac{\alpha_s^{(1)}(s_0)}{\pi} \right)^2 \left( -\frac{\beta_2}{\beta_1} \ln L \right) + \left( \frac{\alpha_s^{(1)}(s_0)}{\pi} \right)^3 \left( \frac{\beta_2^2}{\beta_1^2} (\ln^2 L - \ln L - 1) + \frac{\beta_3}{\beta_1} \right),
\]

and an expansion in powers of \( \alpha_s^{(1)} \) is to be understood. Alternatively, one may choose not to expand in such a way; in this case the radiative correction becomes

\[
F_2(s_0) = 1 + \frac{\alpha_s^{(3)}(s_0)}{\pi} + \left( \frac{\alpha_s^{(3)}(s_0)}{\pi} \right)^2 \left( F_3 - \frac{\beta_1}{2} \right) + \left( \frac{\alpha_s^{(3)}(s_0)}{\pi} \right)^3 \left( F_4 - F_3 \beta_1 - \frac{\beta_2}{2} + \frac{\beta_3}{2} \right),
\]

where \( \alpha_s^{(3)}(s_0) \) is given by Eq. (6). Numerically, these two alternatives have a non-negligible impact on the final result for \( C_2 \), as will be discussed later.

The quark mass insertion term, which contributes to this dimension-two FESR is of the form

\[
C_{2m} = -3 \frac{(\hat{m}_u^2 + \hat{m}_d^2)}{\left( \frac{1}{2} \ln s_0/\Lambda_{QCD}^2 \right)^{-4/\beta_1}}
\]

Using current values of the up- and down-quark masses, this term is negligible.

It should be stressed at this point that the FESR are ideally suited, in principle, to extract the values of power corrections of a given dimensionality. Ignoring gluonic corrections to the condensates, the FESR involving \( \rho_{V,A} \) with kernel \( s^N \) (\( N=0,1,2,... \)) project out only condensates of dimension \( d = 2, 4, 6, ... \). In other words, in the FESR of lowest dimension all condensates of \( d = 4, 6, ... \) decouple. This should be contrasted with e.g. Laplace or Gaussian sum rules which receive contributions from all possible condensates. Since the numerical values of these power corrections are not well known, these other sum rules introduce an unnecessary additional uncertainty. On the other hand, the fact that FESR
tend to emphasize the high energy region, where the ALEPH data have larger errors, is of no importance here, as the uncertainty in $C_2$ turns out to be overwhelmingly dominated by the uncertainty in $\Lambda_{QCD}$, and to a lesser extent, by the way the perturbative expansion is organized. The experimental error in the hadronic integral in Eq.(3) can be safely neglected.

We show now that the dimension-two terms obtained from Eq.(3), i.e.

$$C_{2_{V,A}} = 8\pi^2 \int_0^{s_0} \rho_{V,A}(s) \, ds - s_0 F_2(s_0) ,$$

(9)

are actually identical in the vector and axial-vector channels, provided one takes the chiral limit. This result would follow trivially from e.g. the first Weinberg sum rule, provided this sum rule would be saturated by the data for $s_0 < \infty$ (actually, $s_0 < M_T^2$ in the case of tau decay data). However, this is not the case, as discussed in [12]. Instead, the data saturate much better the modified sum rule [12]

$$\int_0^{s_0} (1 - \frac{s}{s_0}) \left[ \rho_V(s) - \rho_A(s) \right] \, ds = 0 ,$$

(10)

Here, $\rho_A$ already contains the pion pole, i.e.

$$\rho_A(s) = f_\pi^2 \delta(s) + \rho_A(s)|_{RES}$$

(11)

where $\rho_A(s)|_{RES}$ is the resonance part of the spectral function. We make use of Eq.(9) in Eq.(10), and invoke the dimension-four FESR

$$I_{1_{V,A}} \equiv \frac{8\pi^2}{s_0} \int_0^{s_0} \rho_{V,A}(s) \, ds = \frac{F_4(s_0)}{2} - \frac{C_4 < O_4 >}{s_0^2}$$

(12)

where both the radiative correction $F_4(s_0)$, and the dimension-four condensate (equal to the gluon condensate in the chiral limit), are identical in the vector and axial-vector channel. One then obtains

$$\int_0^{s_0} (1 - \frac{s}{s_0}) \left[ \rho_V(s) - \rho_A(s) \right] ds = 0 = \frac{1}{8\pi^2} (C_{2V} - C_{2A}) ,$$

(13)

which completes the proof.
We proceed now to use the ALEPH data in Eq.(9) in order to check for any evidence for a dimension-two operator. The results of a numerical evaluation of the r.h.s. of Eq.(9), using the expanded form of the QCD integral, Eq. (4), and a fit to the ALEPH data as described in [12], are plotted as a function of $s_0$ in Fig.1, for $\Lambda_{QCD} = 300$ MeV. Curve (a) is the result in the vector channel, curve (b) in the axial-vector channel, and curve (c) is the average between the two, i.e. $C_2 = (C_{2V} + C_{2A})/2$. It can be seen that the results for $C_{2V}$ and $C_{2A}$, considered individually, are rather unstable against variations in $s_0$. This behaviour is simple to understand. When integrating the data, the vector integral at $s_0 \simeq 1$ GeV$^2$ has already picked up the contribution from the (narrow-width) rho-meson, while the axial-vector spectral function is still relatively small there. The contribution from the (broad-width) $a_1$-meson is important only for $s_0 > 1.5 - 1.6$ GeV$^2$. For this reason, the hadronic integral approaches the theoretical or QCD integral from above in the vector channel, and from below in the axial-vector channel at $s_0 < 1.5$ GeV$^2$. The expectation $C_{2V} = C_{2A}$ is only true asymptotically. However, making use of this constraint, it is natural to consider instead the average value. This turns out to be remarkably stable, and allows for a precise determination of $C_2$. Perhaps this should not be surprising, as the dimension-six four-quark condensate contributes with different signs to the vector and axial-vector correlators, and there is a tendency to an overall cancellation between the sum of the gluon and the four-quark condensates in $\Pi_V + \Pi_A$ (which, however, enter the sum rule under consideration only via radiative corrections). Nevertheless, there is a strong dependence of $C_2$ on $\Lambda_{QCD}$ as discussed next. The size of the current error bars in $\Lambda_{QCD}$ makes it the dominant source of uncertainty in the determination of $C_2$; in comparison, the small experimental uncertainty in the hadronic spectral functions plays a negligible role.

First, we have found a strong dependence of $C_2$ on the order at which the radiative correction $F_2$, Eq.(4), is computed. As the right hand side of Eq.(9) is the (small) difference of two similar numbers, it is not surprising that the transition from one-loop to two-loops of perturbative QCD in Eq.(4) leads to relatively large differences. In other words, the result for $C_2$ obtained using Eq.(4) truncated to one loop, i.e. using only the first term in
that equation, differs substantially from the result for $C_2$ if the truncation is to two-loops, i.e. including the first two terms in Eq.(4). Higher order corrections (three-, four-, and five-loops) show, however, the expected convergence, i.e. there is little difference between the values of $C_2$ obtained after truncation at the three-loop, four-loop, and five-loop level. In the latter case, Eq.(7) becomes

$$F_2(s_0) = 1 + \frac{\alpha_s^{(4)}(s_0)}{\pi} + \left(\frac{\alpha_s^{(4)}(s_0)}{\pi}\right)^2 \left(F_3 - \frac{\beta_1}{2}\right)$$

$$+ \left(\frac{\alpha_s^{(4)}(s_0)}{\pi}\right)^3 \left(F_4 - F_3\beta_1 - \frac{\beta_2}{2} + \frac{\beta_1^2}{2}\right)$$

$$+ \left(\frac{\alpha_s^{(4)}(s_0)}{\pi}\right)^4 \left[k_3 - \frac{3}{2} \beta_1 F_4 + \frac{\beta_1^2}{2} F_3 (3 - \frac{\pi^2}{2}) - \frac{3}{4} \beta_1^3 - \beta_2 F_3 + \frac{5}{4} \beta_1 \beta_2 (1 - \frac{\pi^2}{6}) - \frac{\beta_3}{2}\right],$$

(14)

where $\alpha_s^{(4)}(s_0)$ is given by [13]

$$\frac{\alpha_s^{(4)}(s_0)}{\pi} = \frac{\alpha_s^{(3)}(s_0)}{\pi} - \left(\frac{\alpha_s^{(1)}(s_0)}{\pi}\right)^4 \left[\frac{\beta_1^3}{\beta_1^2} (\ln^2 L - \frac{5}{2} \ln^2 L - 2\ln L + \frac{1}{2}) + 3 \frac{\beta_1^2 \beta_3}{\beta_1^2} \ln L + \frac{b_3}{\beta_1}\right],$$

(15)

and

$$b_3 = \frac{1}{4^4} \left[\frac{149753}{6} + 3564 \zeta_3 - \frac{1678361}{162} + \frac{6508}{27} \zeta_3\right] n_F + \left[\frac{50065}{162} + \frac{6472}{81} \zeta_3\right] n_F^2 + \frac{1093}{729} n_F^3,$$

(16)

and $\zeta_3 = 1.202$. While the QCD beta-function is known to four loops [13], the imaginary part of the vector (axial-vector) correlator to five-loop order involves the unknown constant $k_3$ in Eq. (14). We have estimated this constant assuming a geometric series behaviour for those constants not determined by the renormalization group, i.e. $k_3 \simeq k_2^2/k_1 \simeq 25$, with $k_1 \equiv F_3$, and $k_2 \simeq F_4 + \pi^2 \beta_1^2/12$. This is in good qualitative agreement with other estimates [14]. In this case we find that the difference between the four-loop and the five-loop results for $C_2$ is again quite small. Nonetheless, the four- and five-loop determinations (represented in Fig.2) depend strongly on the QCD scale parameter $\Lambda_{QCD}$ (here $\Lambda_{QCD} \equiv \Lambda_{MS}(n_F = 3)$). As $\Lambda_{QCD}$ is varied in the range [13] 300 MeV $\leq \Lambda_{QCD} \leq$ 400 MeV, the extracted value of $C_2$ remains small and changes sign after crossing zero for $\Lambda_{QCD} \simeq 330 - 360$ MeV, for typical values of $s_0$ in the stability region ($s_0 \simeq 1.5 - 2.7$ GeV$^2$). In Fig. 3 we show the dependence of $C_2$ on $\Lambda_{QCD}$ for a typical value $s_0 = 2$ GeV$^2$ at the center of the stability
region. Such a strong dependence of $C_2$ on $\Lambda_{QCD}$ could be a welcome feature, if for some theoretical reason one could rule out completely the presence of a dimension-two operator in QCD. One could then determine $\Lambda_{QCD}$ with great accuracy. Finally, the difference in the QCD integral after expanding in powers of $\alpha_s^{(1)}$, as in Eq.(4), or not expanding them, as in Eq.(7), though small, does affect the value of $C_2$, as this is the result of the subtraction of two very similar numbers. For $\Lambda_{QCD}$ as determined by the PDG \cite{15}, i.e. $\Lambda_{QCD} = 389 \pm 35$ MeV, we find $C_2 = -(0.08 \pm 0.06)$ GeV$^2$ if the expansion in $\alpha_s^{(1)}$ is done as in Eq. (4), and $C_2 = -0.05 \pm 0.05$ GeV$^2$ if one does not expand, as in Eq.(7). Alternatively, we consider the value of $\alpha_s$ extracted in \cite{9}, which implies $\Lambda_{QCD} = 367 \pm 40$ MeV, for three flavours. This analysis was performed assuming $C_2 = 0$. In view of our results, this assumption is not justified a priori. However, such a value of $\Lambda_{QCD}$ leads to $C_2 = -(0.05 \pm 0.06)$ GeV$^2$ using Eq.(4), and $C_2 = -0.03 \pm 0.06$ GeV$^2$ using Eq.(7), which are consistent with zero, and thus justifies the assumption a posteriori. All the above numerical calculations were performed using an analytical chi-squared fit to the ALEPH data, and also by direct integration of these data, as they are available in bin-interval format \cite{9}. The difference between the hadronic integrals in Eq.(3) from these two procedures is less than 1 %, which provides a reasonable estimate of the error in integrating the data. While this small difference translates into a larger difference in the values of $C_2$, due to the cancellation between the hadronic and the QCD integrals, the conclusion remains the same, i.e. that given the present uncertainty in $\Lambda_{QCD}$ the ALEPH data imply a value of $C_2$ consistent with zero. This conclusion is rather different from that reached in a previous analysis \cite{7}, along the same lines as here but using instead the ARGUS tau-decay data \cite{8} to determine the hadronic spectral functions. This discrepancy is due in part to the much larger error bars of the ARGUS data. However, this does not fully explain the differences, which are mostly due to different values of the vector and axial-vector spectral functions at, and in the vicinity of the respective resonances (rho- and $a_1$-mesons). This translates into different areas under the hadronic spectral functions. For instance, the ARGUS data saturated the first and second Weinberg sum rules reasonably well \cite{16}, while this is no longer the case for the ALEPH data\cite{9}, \cite{12}. 
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Figure Captions

Figure 1. The right hand side of Eq.(9) as a function of the continuum threshold $s_0$ in the vector channel (curve (a)), the axial-vector channel (curve(b)), and the average $C_2 \equiv (C_{2V} + C_{2A})/2$ (curve (c)), all for $\Lambda_{QCD} = 300$ MeV. The expanded expression, Eq. (4), has been used.

Figure 2. The average $C_2$ as a function of the continuum threshold $s_0$ for $\Lambda_{QCD} = 300$ MeV (curve (a)), and $\Lambda_{QCD} = 400$ MeV (curve (b)). The expanded expression, Eq.(4), has been used.

Figure 3. The dependence of the average $C_2$ on $\Lambda_{QCD}$ for a typical value $s_0 = 2$ GeV$^2$ at the center of the stability region. The expanded expression, Eq.(4) has been used.
Figure 1:
Figure 2:
Figure 3: