ON CLOSED GEODESICS IN NON-NEGATIVELY CURVED FINSLER STRUCTURES WITH LARGE VOLUME GROWTH

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ABSTRACT. We prove that a reversible Berwaldian Finsler structure \((M, F)\) with base-independent non-negative radial flag curvature and large volume growth does not have any closed geodesics.

1. INTRODUCTION

In the past few years, the study of metric-measure spaces (not necessarily smooth) that satisfy some curvature-dimension conditions (see [20][21][7]) has been an important theme in Mathematical research. If one is merely interested in understanding the limit of Riemannian manifolds with lower Ricci curvature bounds, then one can restrict this class to the metric-measure spaces satisfying Riemannian curvature-dimension conditions; namely, spaces that in addition to satisfying curvature-dimension conditions, also enjoy a linear Laplacian (see Ambrosio-Gigli-Savaré [1]).

Finsler manifolds which have their origins in Physics and have been back in the spotlight in the past couple of decades. For us, Finsler structures are important because they can be made into spaces satisfying the above mentioned curvature-dimension conditions (if we use the weighted Ricci curvature instead of the usual one [11]) though, It is well-known that the distribution Laplacian that is used in Finsler structures is a non-linear Laplacian therefore, Finsler manifolds are a source of examples for spaces that are close to Riemannian manifolds in some sense but do not satisfy Riemannian curvature-dimension bounds. Recently, there have been many new developments in Finsler geometry that provide many key tools that we need in order to do deeper geometric computations on these spaces. Among these new developments, are the Bochner-Weitzenböck formula which has been recently proven by Ohta-Sturm [12] and a version of Toponogov comparison theorem proven by Kondo-Ohta-Tanaka [5]. The author’s previous paper, [6], uses the Bochner-Weitzenböck formula to prove differential Harnack estimates for positive solutions of heat equation under Bao’s Finsler-Ricci flow (Introduced in Bao [3]). The current paper presents an application of the Kondo-Ohta-Tanaka’s Toponogov triangle comparison theorem [5].

There is a good deal of research done regarding Riemannian manifolds with large volume growth and some curvature or conjugate radius lower bounds. With a lower bound on the Ricci curvature, the key tools to use, usually are the Bishop-Gromov volume comparison, Abresch-Gromoll’s excess estimates and Perelman’s maximal volume result. For example, Perelman [15] proves that non-negative Ricci
curvature and large enough volume growth implies the contractibility of the underlying manifold and Munn [9] has found estimates for such thresholds both for Riemannian manifolds and Alexandrov spaces [10]. Shen [18] has proven that the manifold is of finite topological type provided that the sectional curvature is bounded away from $-\infty$ or with a lower bound on the conjugate radius. While, there is a well-developed comparison geometry and in particular, volume comparison theorem for Finsler structures (see Shen [19] and Wu-Xin [24]), the author is not aware of any clean cut excess estimate for Finsler metrics so generating similar results in the Finsler setting needs a lot more work.

With using a lower bound on the sectional curvature (or radial sectional curvature), one can benefit a lot from the various versions of Toponogov’s comparison theorems. One very clever use of Toponogov’s comparison result is the following:

**Theorem 1.1 (Wan [23]).** A complete non-compact Riemannian manifold, $M^n$ with non-negative sectional curvature and Euclidean volume growth, does not contain any closed geodesics.

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This paper is inspired by the above mentioned work of Wan [23]. In the Finsler setting, in general there does not yet exist a well-defined notion of angle between geodesics due to the fact that in the Finsler setting, lengths and “angles” could vary depending on the direction along which an observer is moving. A Toponogov type comparison theorem (that we are going to use in this paper) has been proven only recently in Kondo-Ohta-Tanaka [5].

We note that, in this paper, a closed geodesic is a geodesic loop which is smooth at all its points. Our main result is the following Finsler counterpart of Theorem 1.1.

**Theorem 1.2.** Let $(M, F)$ be a complete non-compact reversible Berwaldian Finsler structure that has base-independent non-negative radial flag curvature; then, $M$ does not contain any closed geodesics.

Since base-independent non-negative radial flag curvature condition is weaker than non-negative flag curvature, we readily obtain the following corollary:

**Corollary 1.3.** Any complete non-compact reversible Berwaldian Finsler structure with non-negative flag curvature and large volume growth can not contain any closed geodesics.

The structure of this paper is as follows: In section 2 we will briefly mention some basic facts regarding Finsler geometry; in section 3 we discuss the Finsler volume comparison theorem under lower Ricci curvature bounds and its straightforward generalization to annuli and we will prove a key lemma; section 4 discusses the Toponogov’s comparison theorem proven by Kondo-Ohta-Tanaka [5] and some of its consequences; In section 5 we will complete the proof of Theorem 1.2.

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2. Preliminaries

In this section, we will briefly mention some basic definitions and notions in Finsler geometry. There are many good expository references for this subject, for example see [2] and [16].

2.1. Finslerian Structure. A Finslerian structure is a pair \((M^n, F)\) consisting of a connected \(C^\infty\) manifold \(M\) and a Finsler norm, \(F : TM \to \mathbb{R}\), that is required to satisfy the following conditions:

\(\text{(F1)}\) \(F\) is \(C^\infty\) on \(TM \setminus 0\),
\(\text{(F2)}\) \(F\) restricted to the fibres is positively \(1\)–homogeneous,
\(\text{(F3)}\) For any nonzero tangent vector \(y \in TM\), the approximated symmetric metric tensor, \(g_y\), defined by

\[
(1) \quad g_y(u, v) := \frac{1}{2} \left. \frac{\partial^2}{\partial s \partial t} \left( F^2 (y + su + tv) \right) \right|_{s=t=0},
\]

is positive definite.

2.2. Cartan Tensor. Cartan tensor, \(C_y : \otimes^3 TM \to \mathbb{R}\), measures the non-linearity of a Finsler structure and is defined by

\[
(2) \quad C_y(u, v, w) := \frac{1}{2} \frac{d}{dt} \left[ g_y + tw (u, v) \right].
\]

2.3. Legendre Transform. In order to define the gradient of a function, we need the Legendre transform, \(L^* : T^* M \to TM\); For \(\omega \in T^* M\), \(L^*(\omega)\) is the unique vector \(y \in TM\) such that,

\[
(3) \quad \omega(y) = F^*(\omega)^2 \quad \text{and} \quad F(y) = F^*(\omega),
\]

in which, \(F^*\) ia the dual norm to \(F\).

For a smooth function \(u : M \to \mathbb{R}\), the gradient of \(u\), \(\nabla u(x)\) is defined to be \(L^*(Du(x))\).

2.4. Geodesic Spray and Chern Connection and Curvature Tensor. In is straightforward to observe that the geodesic spray in the Finsler setting is of the form, \(G = y^i \frac{\partial}{\partial x_i} - 2 G^i(x, y) \frac{\partial}{\partial y^i}\) where,

\[
(4) \quad G^i(x, y) = \frac{1}{4} g^{ij} \left\{ 2 \frac{\partial g_{jk}}{\partial x_l} - \frac{\partial g_{jl}}{\partial x_k} \right\} y^j y^l.
\]

The non-linear connection that we will be using in this work is the Chern connection, the connection coefficients of which is given by

\[
(5) \quad \Gamma^i_{jk} = \Gamma^i_{kj} := \frac{1}{2} g^{il} \left\{ \frac{\partial g_{lj}}{\partial x_k} - \frac{\partial g_{lk}}{\partial x_j} + \frac{\partial g_{kl}}{\partial x_j} \frac{\partial g_{lj}}{\partial y^r} G^r_k + \frac{\partial g_{jk}}{\partial y^r} G^r_l - \frac{\partial g_{kl}}{\partial y^r} G^r_j \right\},
\]
where, $G_{ij} := \frac{\partial G^i}{\partial y^j}$.

2.5. **Berwaldian Structure.** For Berwald metrics, the geodesic coefficients, $G^i$ are quadratic in terms of $y$ (by definition) which immensely simplifies the formula for connection coefficients. In fact in for Berwald metrics we have $\Gamma_{ik}^j = \frac{\partial^2 G^i}{\partial y^i \partial y^j}$.

2.6. **Riemann Curvature.** Similar to the Riemannian setting, one uses the Chern connection or any other known Finsler connection (and the associated covariant differentiation) to define the curvature tensor (of course depending on a nonzero vector field $V$).

$$R^V(X,Y)Z := [\nabla^V_X, \nabla^V_Y]Z - \nabla^V_{[X,Y]}Z.$$  

2.7. **Flag and Ricci Curvatures.** Flag curvature is defined similar to the sectional curvature in the Riemannian setting. For a fixed flag pole, $v \in T_xM$ and for $w \in T_xM$, the flag curvature is defined as

$$K^v (v, w) := g_v(R^v(v,w)w, v) \frac{g_v(R^v(v,w)w, v)}{g_v(v,v)g_v(w,w) - g_v(v,w)^2}.$$  

The Ricci curvature is then the trace of the Flag curvature i.e.

$$\text{Ric}(v) := F(v)^2 \sum_{i=1}^{n-1} K^v (v, e_i),$$  

where, $\{e_1, \ldots, e_{n-1}, \frac{\nabla^V_x}{\tau_F(x)}\}$ constitutes a $g_v$-orthonormal basis of $T_xM$.

2.8. **S-Curvature.** Associated to any Finsler structure, there is one canonical measure, called the Busemann-Hausdorff measure, given by

$$d\mu_F := \tau_F(x)dx_1 \wedge \cdots \wedge dx_n,$$

where $\tau_F(x)$ is the volume ratio

$$\tau_F(x) := \frac{\text{vol} (B_{R^n}(1))}{\text{vol} (y \in T_xM | F(y) \leq 1)};$$

The set whose volume comes in the denominator of (10) is called the indicatrix and there is often no known way to express its volume in terms of the equation of $F$.

The S-curvature, which is another measure of non-linearity, is then defined as

$$S(y) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x_i} (\ln \sigma_F(x)).$$

One important well-known fact that we will use later on is that if $(M, F)$ is Berwaldian, then the S-curvature vanishes identically (however the converse is not true). See Shen [19] for a proof. For more details regarding the S-curvature, please see [17] for example.
2.9. **Reversibility Constant.** One important factor in our estimates is the so called **reversibility constant**, $\rho$, of the Finsler structure $(M, F)$, which is defined by

$$ \rho = \sup_{TM \setminus 0} \frac{F(-y)}{F(y)}. $$

For $M$ compact, this constant will be finite. $(M, F)$ is called reversible if $\rho = 1$, i.e.,

$$ F(X) = F(-X) \ \forall X \in TM. $$

Using this constant, the triangle inequality gives us the inequalities

$$ F(y) - \rho F(z) \leq F(y + z) \leq F(y) + F(z), $$

for any $y, z \in T_xM$.

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3. **Volume Comparison and Volume Growth**

3.1. **Volume Comparison Theorem.** Volume comparison theorems for Finsler structures first appeared in Shen [19]. We will be using a particular form of the relative volume comparison for annuli; hence, we will need a little set up first.

Let $(M, F)$ be a Finsler structure. For any $p \in M$, the indicatrix, $I_p$ is

$$ I_p := \{ y \in T_pM \mid F(p, y) \leq 1 \}. $$

The indicatrix, is convex and hence its boundary can be thought of as the unit sphere in the tangent space, $T_pM$. The boundary of the indicatrix is of course given by

$$ \partial I_p = \{ y \in T_pM \mid F(p, y) = 1 \}. $$

Now, let $\Gamma \subset \partial I_p$ be a measurable subset and let

$$ A_{r, R}^\Gamma (p) := \left\{ x \in M \mid r \leq d(p, x) \leq R \text{ and there exists a unit speed geodesic, } \gamma(t), \text{ from } p \text{ to } x \text{ with } \dot{\gamma}(0) \in \Gamma. \right\} $$

On the other hand, for a different measurable subset of the unit sphere in the comparison space,

$$ \Sigma \subset S^{n-1}(1) \text{ and } \mathcal{H}^{n-1}(\Sigma) \neq 0, $$

let

$$ V_{r, R, \Sigma}^{\kappa, A} := \mathcal{H}^{n-1}(\Sigma) \int_r^R e^{\lambda} s_\kappa(t)^{n-1} \ dt. $$

where,

$$ s_\kappa(t) := \begin{cases} 
\sin (\sqrt{\kappa} t) & \kappa > 0 \\
\frac{1}{t} & \kappa \leq 0 \kappa = 0 \\
\sin (\sqrt{-\kappa} t) & \kappa < 0. 
\end{cases} $$
Theorem 3.1 (volume comparison for annuli). Let \((M^n, F, d_F)\) be a complete Finsler manifold. Suppose that
\[
\text{Ric} \geq (n - 1)\kappa \quad \text{and} \quad \|S\| \leq \lambda,
\]
then, for \(0 \leq r \leq s \leq S\) and \(0 \leq r \leq R \leq S\) we have
\[
\frac{\text{vol} \left( A^r_{s, S} \right)}{V^r_{s, s, \Sigma}} \leq \frac{\text{vol} \left( A^r_{r, R} \right)}{V^r_{r, r, \Sigma}}.
\]

Proof. Write the Busemann-Hausdorff volume form in polar coordinates as
\[
d\mu_F = \tau_F(r, \theta) dr \wedge d\theta \wedge \cdots \wedge d\theta^{n-1}.
\]
From [24, Theorem 6.3], we know that the radial function
\[
\frac{\tau_F(r, \theta)}{e^{\lambda r^2} S^{n-1}(r)}
\]
is non-increasing in \(r\). Now using a calculus lemma [25, Lemma 3.2], the result follows.

Remark 3.2. The major difference between Riemannian volume comparison and Finsler volume comparison is the term \(e^{\lambda r^2}\) (\(\lambda\) is an upperbound for the S−curvature) in the denominator. This is due to the fact that \(S\) measures the change of the Hausdorff measures for nearby tangent spaces (which is 0 in the Riemannian setting.) Roughly, speaking it is like having some negative curvatures somewhere nearby and that would increase the volume exponentially and hence the exponential compensation term in Theorem 3.1.

3.2. Volume Growth (Asymptotic Volume). Let \((M, F)\) be a Finsler structure that satisfies
\[
\text{Ric} \geq (n - 1)\kappa \quad \text{and} \quad \|S\| \leq \lambda,
\]
and let \(p \in M\) be a point. Then Theorem 3.1 says that
\[
\nu_M := \lim_{r \to \infty} \frac{\text{vol} \left( B(p, r) \right)}{V^r_{0, r, S^{n-1}}},
\]
exists and is less than 1. Also it is a standard fact that \(\nu_M\) does not actually depend on the base point \(p\).

Definition 3.3 (volume growth). \(\nu_M\) is called the volume growth (or asymptotic volume) of \(M\).

Definition 3.4. We say \((M, F)\) has large volume growth whenever \(\nu_M > 0\).

It turns out that one can compute the volume growth by just measuring the volume of geodesic rays. Intuitively, this is not surprising since as we go off to infinity, the effective volume is the volume of rays. We will make this precise in the next two lemmas.

Definition 3.5. Let \((M, F)\) be a Finsler structure and \(p \in M\). Then the ray directions are defined as follows:
\[
\Gamma_{\text{ray}} := \{ y \in \partial I_p \mid \text{exp}_p(ty) \text{ is a geodesic ray} \}.
\]
Lemma 3.6. Let \((M, F)\) be a Finsler structure and \(p \in M\). Then, \(\Gamma_{ray} \subset \partial I_p\) is closed.

Proof. The proof follows from the simple fact that the limit of a sequence of geodesic rays is again a geodesic ray.

For any subset, \(\Gamma \subset \partial I_p\), let

\[
B \left( \Gamma_{ray}^c, r \right) := B(p, r) \cap C(\Gamma) \quad \text{where} \quad C(\Gamma) := \{ \exp_p ty \mid t \geq 0 \text{ and } y \in \Gamma \}.
\]

Lemma 3.7. Let \((M, F)\) be a Finsler structure that satisfies

\[
\text{(29) } \text{Ric} \geq 0 \text{ and } \|S\| \leq \lambda,
\]

\[
\text{(30) } \lim_{r \to \infty} \frac{\text{vol} \left( B \left( \Gamma_{ray}^c, r \right) \right)}{V^{0,\lambda}_{0,r,S^{n-1}}} = 0.
\]

Proof. For a small \(\delta\), take the \(\delta\)-neighbourhood, \(T_\delta \left( \Gamma_{ray} \right) \subset I_p\). Then, applying the volume comparison theorem (Theorem 3.1) to \(\Gamma := T_\delta \left( \Gamma_{ray} \right) \backslash \Gamma_{ray}\), we get

\[
\text{(31) } \frac{\text{vol} \left( B \left( T_\delta \left( \Gamma_{ray} \right) \backslash \Gamma_{ray}, r \right) \right)}{V^{0,\lambda}_{0,r,S^{n-1}}} \leq \lim_{r \to 0} \frac{\text{vol} \left( B \left( T_\delta \left( \Gamma_{ray} \right) \backslash \Gamma_{ray}, r \right) \right)}{V^{0,\lambda}_{0,r,S^{n-1}}} \leq \frac{\mathcal{H}^{n-1} \left( T_\delta \left( \Gamma_{ray} \right) \backslash \Gamma_{ray} \right)}{n}.
\]

Now, letting \(\delta \to 0\), we get the desired conclusion.

\[
\text{(32) } \text{Ric} \geq 0 \text{ and } \|S\| \leq \lambda,
\]

and let \(\Sigma \subset I_p\) be as above (determining rays). Then,

\[
\text{(33) } \lim_{r \to \infty} \frac{\text{vol} \left( B \left( \Gamma_{ray}, r \right) \right)}{V^{0,\lambda}_{0,r,S^{n-1}}} = v_M \text{ and } \text{vol} \left( B \left( \Gamma_{ray}, r \right) \right) \geq v_M V^{0,\lambda}_{0,r,S^{n-1}}.
\]

Proof. Proof is very similar to the Riemannian case (for example, see [14, Lema 4.1]). First we notice that, \(\Sigma\) is closed from Lemma 3.6. Then, we write

\[
\text{(34) } \text{vol} \left( B \left( p, r \right) \right) = \text{vol} \left( B \left( \Gamma_{ray}, r \right) \right) + \text{vol} \left( B \left( \Gamma_{ray}^c, r \right) \right).
\]

Now the result follows directly from Lemma 3.7.

4. TOPONOGOV TYPE COMPARISON

4.1. "Von Mongoldt" Model Surfaces. In this section, we introduce the comparison spaces that will be used in the finslerian Toponogov comparison theorem. For us, a von Mongoldt surface is a triple \((\mathbb{R}^2, \tilde{g}, G)\) where

\[
\tilde{g} = dt^2 + f^2(t)d\theta^2 \text{ for } (t, \theta) \in (0, \infty) \times S^1.
\]
is a smooth metric (hence, satisfying \( f(0) = 0 \) and \( f'(0) = 1 \)), and the non-increasing function, \( G(t) \), is the sectional curvature of \( \tilde{g} \) which is given by

\[
G(t) = -\frac{f''(t)}{f(t)}.
\]

**Definition 4.1** (radial flag curvature bound). Let \( (M, F, p) \) be a pointed Finsler manifold. We say that \( M \) has radial flag curvature bounded below by that of a von Mongoldt surface \( (\mathbb{R}^2, \tilde{g}, G) \) whenever

\[
K_x \geq G(d_F(x, p)),
\]

in which, \( K_x \) denotes the sectional curvature(s) at \( x \).

Let \( (M, F) \) be a forward complete Finsler structure and let \( p \in M \) be a point. We can define notions of forward and backward angles based on the first variation of length.

**Definition 4.2** (forward and backward angles, Kondo-Ohta-Tanaka [5]). Let \( \gamma : [0, l] \to M \) be a unit speed minimal geodesic joining \( a \) to \( c \) and let \( b \) be a point on \( \gamma \);

Then, Forward angle \( \xrightarrow{\gamma} pbc \) is defined via

\[
\cos \left( \xrightarrow{\gamma} pbc \right) := -\lim_{h \to 0^+} \frac{d(p, b + h) - d(p, b)}{d_{max}(b, b + h)},
\]

and similarly, backward angle via

\[
\cos \left( \xleftarrow{\gamma} pba \right) := \lim_{h \to 0^+} \frac{d(p, b) - d(p, b - h)}{d_{max}(b - h, b)},
\]

where,

\[
d_{max}(x, y) := \max\{d(x, y), d(y, x)\}
\]

\[
= \frac{d(x, y) + d(y, x) + |d(x, y) - d(y, x)|}{2}
\]

\[
= \int_0^{d(x, y)} \max\{F(\dot{c}(t)), F(-\dot{c}(t))\} \, ds.
\]

Now we can state the Toponogov comparison of Kondo-Ohta-Tanaka [5]

**Theorem 4.3** (Toponogov Type Comparison, Kondo-Ohta-Tanaka [5]). Suppose \( (M, F) \) is a Berwaldian forward complete connected Finsler structure and \( p \in M \). Assume that the radial flag curvature is bounded below by that of \( (\mathbb{R}^2, \tilde{g}, G) \) in the sense of Definition 4.1. Furthermore, assume \( f'(0) = 0 \) for a unique value of \( \rho \in (0, \infty) \). Suppose \( \triangle \left( \overrightarrow{pa}, \overrightarrow{pb}, \overrightarrow{ab} \right) \) is a forward triangle in \( M \) such that for a small neighborhood, \( N(c) \) of \( c \), we have

(i) \( c \left( [0, d_F(x, y)] \right) \subset M \setminus \overline{B_{B_{c}^{p}}(p)} \) (where, \( B_{c}^{p}(p) \) is the closure of the forward ball of radius \( \rho \)),

(ii) \( g_v(\omega, \omega) \geq F(\omega)^2 \), for all \( z \in N(c) \), \( v \in TV_p(z) \) and \( \omega \in T_z M \) (where, \( TV_p(z) \) is as in Definition 4.9).
If \( \Delta \left( \overrightarrow{pa}, \overrightarrow{pb}, \overrightarrow{ab} \right) \) admits a comparison triangle \( \Delta \left( \overrightarrow{\tilde{a}a}, \overrightarrow{\tilde{b}b}, \overrightarrow{\tilde{a}b} \right) \), then,

\[
\tilde{a} \geq a \quad \text{and} \quad \tilde{b} \geq b.
\]

**Remark 4.4.** Assumption (ii) in Theorem 4.3 (which expresses some uniform convexity) always holds due to Ohta \[13\].

A very crucial lemma that we will need in our approach is the following double triangle lemma which allows us to use the Toponogov’s triangle comparison in situations where one of geodesics is not minimal (which will be the case in our setting).

**Lemma 4.5** (double triangle lemma, Kondo-Ohta-Tanaka \[5\]). Suppose \( \Delta \tilde{p}\tilde{x}\tilde{y} \) and \( \Delta \tilde{p}\tilde{y}\tilde{z} \) be two geodesic triangles in \( \tilde{M} \) such that the arguments of their vertices satisfies

\[
0 = \arg(\tilde{x}) < \arg(\tilde{y}) < \arg(\tilde{z}),
\]

and such that,

\[
\angle \tilde{p}\tilde{x}\tilde{y} + \angle \tilde{p}\tilde{y}\tilde{z} \leq \pi.
\]

If there is a geodesic triangle \( \Delta \tilde{p}\tilde{a}\tilde{b} \) in \( \tilde{M} \) for which,

\[
\tilde{d} (\tilde{p}, \tilde{a}) = \tilde{d} (\tilde{p}, \tilde{x}) \quad \text{and} \quad \tilde{d} (\tilde{p}, \tilde{b}) = \tilde{d} (\tilde{p}, \tilde{z}) \quad \text{and} \quad \tilde{d} (\tilde{a}, \tilde{b}) = \tilde{d} (\tilde{x}, \tilde{y}) + \tilde{d} (\tilde{y}, \tilde{z}),
\]

then we have

\[
\angle \tilde{x} \geq \angle \tilde{a} \quad \text{and} \quad \angle \tilde{y} \geq \angle \tilde{b}.
\]

We know that for Riemannian manifolds with non-negative sectional curvature (or for metric spaces with non-negative curvature in the sense of Alexandrov), the comparison angles are monotonically non-increasing as the length of geodesic increases. In the following lemma, we will prove a similar behaviour in the Finsler setting.

**Lemma 4.6** (monotonicity of the comparison angle). Suppose \((M, F)\) is a Finsler structure with base-independent non-negative radial flag curvature. Let \( \sigma(t) \) be a geodesic emanating from a (i.e. \( \sigma(0) = a \)). Consider the forward geodesic triangle \( \Delta \left( \overrightarrow{p\sigma(t)}, \overrightarrow{pa\sigma(t)}, \overrightarrow{a\sigma(t)} \right) \) and let \( b_t := \sigma(t) \). Then the comparison angle \( \angle \tilde{p}\tilde{a}b_t \) is non-increasing in \( t \).

**Proof.** Consider the forward geodesic triangle \( \Delta \left( \overrightarrow{pa}, \overrightarrow{pb_t}, \overrightarrow{ab_t} \right) \).

**Step I: Small \( t \).** For small \( t, \sigma(t) \) is minimizing and a comparison triangle in the model space \((\mathbb{R}^2, g_{Euc})\) exists and monotonicity of the comparison angle is standard.

**Step II: Cut and paste argument.** Suppose \( 0 < s < t \) and \( \sigma|_{[0,s]} \) and \( \sigma|_{[s,t]} \) are minimizing. Also assume that comparison triangles, \( \Delta \left( \tilde{p}\tilde{a}\tilde{b}_s \right) \) and \( \Delta \left( \tilde{p}\tilde{b}_s\tilde{b}_t \right) \) exists. Since, these two triangles are in \( \mathbb{R}^2 \) we can paste them together along the sides of common length. Also by the definition of angles we have

\[
\angle \tilde{p}\tilde{a}\tilde{b}_s + \angle \tilde{p}\tilde{b}_s\tilde{b}_t \leq \pi,
\]

and from Theorem 4.3 we get

\[
\angle \tilde{p}\tilde{a}\tilde{b}_t + \angle \tilde{p}\tilde{b}_t\tilde{b}_t \leq \pi.
\]
Notice that, we are in the setting of Lemma 4.5. So, if \( \tilde{p}\tilde{q}\tilde{r} \) is a comparison triangle with
\[
\tilde{p}\tilde{q} = \tilde{p}\tilde{a} \quad \text{and} \quad \tilde{p}\tilde{r} = \tilde{p}\tilde{b}_t \quad \text{and} \quad \tilde{q}\tilde{r} = \tilde{a}\tilde{b}_s + \tilde{b}_s\tilde{b}_t,
\]
then,
\[
\angle \tilde{p}\tilde{q}\tilde{r} \leq \angle \tilde{p}\tilde{a}\tilde{b}_s,
\]
and of course, by Theorem 4.3, one has
\[
\angle \tilde{p}\tilde{a}\tilde{b}_s \leq \angle \tilde{pab}_s.
\]
Combining 49 and 50, we see that
\[
\angle \tilde{p}\tilde{q}\tilde{r} \leq \angle \tilde{pab}_s.
\]

**Step III: Iterating step II.** Consider a partition
\[
0 = s_0 < s_1 < \cdots < s_k = t,
\]
such that \( \sigma|_{[s_j,s_{j+1}]} \) is minimizing and let \( b_j := \sigma(s_j) \). Iterating step II, we get
\[
\angle \tilde{pab}_1 \geq \angle \tilde{p\tilde{x}\tilde{y}},
\]
where, \( \triangle (\tilde{p}\tilde{x}\tilde{y}) \) is a comparison triangle with
\[
\tilde{p}\tilde{x} = \tilde{p}\tilde{a} \quad \text{and} \quad \tilde{p}\tilde{y} = \tilde{p}\tilde{b}_t \quad \text{and} \quad \tilde{x}\tilde{y} = \sum_{i=0}^{k-1} \tilde{b}_i\tilde{b}_{i+1} = \text{Length}(\sigma|_{[0,t]}).
\]

**Lemma 4.7 (Riemannian rays \( \perp \) closed geodesics [14]).** Let \( \sigma(t) : [0,l] \to M^n \) be a unit speed closed geodesic on a complete open manifold \( M^n \) with non-negative sectional curvature. Suppose \( \gamma \) is a ray emanating from \( a = \sigma(0) \), then,
\[
\dot{\gamma}(0) \perp \sigma(0) = \sigma(l).
\]

**Proof.** For the sake of clarity, we repeat the proof that has appeared in Ordway-Stephens-Yang [14]. Using the Riemannian Toponogov’s comparison theorem, for a point \( b = \sigma(s) \), one has
\[
\cos \theta \leq \frac{t^2 + s^2 - \left(\gamma(t)b\right)^2}{2ts} \quad \text{where}, \quad \theta := \angle \gamma(t)\sigma(0)b.
\]
Notice that despite the notation, \( \theta \) does not depend on \( b \).

Letting \( s \to l \) and noting that \( \sigma(l) = a = \gamma(0) \), we get
\[
\cos \theta \leq \frac{1}{2t}.
\]
Now letting \( t \to \infty \), we get \( \cos \theta \leq 0 \) therefore, \( \theta \geq \frac{\pi}{2} \). Since \( \sigma \) is a closed geodesic and smooth at all its points, repeating the above argument for the closed geodesic \( \bar{\sigma} \) defined as \( \bar{\sigma}(t) := \sigma(l-t) \) will tell us that \( \pi - \theta \leq \frac{\pi}{2} \) which means \( \theta \geq \frac{\pi}{2} \) and we are done. □
The following lemma is the Finsler counterpart of Lemma 4.7. As you will see, in contrast to the Riemannian setting, we need to apply the Finsler Toponogov comparison to a diverging sequence of points on the ray. Though neither of the results so far needed reversibility, for the following Lemma, we have to assume reversibility for the reasons that will become clear in the proof.

**Lemma 4.8.** Let $(M, F)$ be a Berwaldian and reversible forward complete Finsler structure with base-independent non-negative radial flag curvature. Suppose $\sigma : [0, l] \to M$ is a unit speed closed geodesic in $M$ and suppose $\gamma(t)$ is a ray emanating from $a := \sigma(0) = \gamma(0)$. Let $b := \sigma(s)$ then,

\[
\lim_{t \to \infty} \angle \gamma(t)ab = \frac{\pi}{2}.
\]

**Proof.** Suppose not. Suppose for $t_j \nearrow +\infty$ and a fixed $\epsilon > 0$ we have

\[
\angle \gamma(t_j)ab \leq \frac{\pi}{2} - \epsilon.
\]

Notice that by the definition of a forward angle, $\angle \gamma(t)ab(s)$ does not depend on $s$ so from Lemma 4.6, we get

\[
\angle \gamma(t_j)ab(s) \geq \angle \tilde{o}\tilde{a}\tilde{b},
\]

where, $\triangle (\tilde{o}\tilde{a}\tilde{b})$ is a comparison triangle that satisfies

\[
\tilde{o}\tilde{x} = \tilde{o}\tilde{a} \text{ and } \tilde{o}\tilde{y} = \tilde{o}\tilde{b} \text{ and } \tilde{x}\tilde{y} = s.
\]

Now, first letting $s \to l$, and then $j \to \infty$, by Lemma 4.7, we get

\[
\lim_{j \to \infty} \angle \gamma(t_j)ab(s) \geq \frac{\pi}{2},
\]

which is a contradiction. Since, $F$ is assumed to be reversible, $\sigma(s) := \sigma(l - s)$ is again a closed geodesic and applying the same argument to $\sigma(s)$ completes the proof. \(\square\)

4.2. Finsler Angles and First Variation. In this section, we will mention the relation between the forward and backward angles and the first variation formula in the Finsler setting.

First we need the to define the terminal velocity set of minimizing geodesics.

**Definition 4.9** (terminal velocity set). For two points $p, x \in M$, the terminal velocity set from $p$ to $x$ is defined as follows

\[
TV_p(x) := \{ \dot{\mu}(l) \in T_xM \mid \mu : [0, l = d(p, x)] \to M \text{ is a minimal geodesic from } p \text{ to } x. \}.
\]

The following first variation theorem of Tanaka-Sabau \[22\] will tell us how to compute the Finsler angles.

**Theorem 4.10** (First variation Formula, Tanaka-Sabau \[22\]). Let $(M^n, F)$ be a Finsler structure and fix a point $p \in M$. Let $\{\gamma_i\}$ with $\text{Length}(\gamma_i) = l_i$ are unit speed minimizing
geodesics from $p$ to their end points $x_i := \gamma(l_i)$ that are converging to the minimizing geodesic, $\gamma$. Suppose the limit,

$$y_{\text{var}} := \lim_{i \to \infty} \frac{1}{F(x_i)} \exp^{-1}_x(x_i),$$

exists; then, we have the following generalized first variation formula,

$$\lim_{i \to \infty} \frac{d_F(p, x_i) - d_F(p, x)}{d_F(x, x_i)} = \min_{\omega \in TV_p(x)} g_\omega(y_{\text{var}}, \omega).$$

5. Main result

In this section, we complete the proof of Theorem 1.2.

Lemma 5.1. Suppose $\gamma$ is a ray emanating from $p$, then

$$TV_{\gamma(t)}(p) \to \{-\dot{\gamma}(0)\},$$

with respect to the Hausdorff distance.

Proof. Suppose not, then we will have $v \neq w \in \partial I_p$ for which

$$\lim_{t \to \infty} (\exp(tv), \exp(tw)) = 0.$$

Applying Theorem 4.3 based at $q := \exp(tv)$ (or $r := \exp(tw)$) and to the forward triangle $\triangle (\overrightarrow{qp}, \overrightarrow{qr}, \overrightarrow{qr})$ and then letting $t \to \infty$ will imply that $v = w$ which is a contradiction. □

5.1. Proof of Theorem 1.2

Lemma 5.2. Let $p := \sigma(0)$ and consider the map $h: \partial I_p \to \mathbb{R}$ given by

$$h(v) = g_v(\dot{\sigma}(0), v).$$

Then, we have

$$\Gamma_{\text{ray}} \subset h^{-1}(0).$$

Proof. Take $x_i := \sigma(1/i)$, then in the virtue of Theorem 4.10 we have $y_{\text{var}} = \dot{\sigma}(0)$ and by Lemma 5.2 and (65), we get the desired result. □

This means that $H^{n-1}(\Gamma_{\text{ray}}) = 0$ (in fact, the Hausdorff dimension of $\Gamma_{\text{ray}}$ is $n - 2$ since $h$ is not locally constant). Since, $\exp$ is smooth, Sard’s theorem will ensure that

$$\operatorname{vol}(B(\Gamma_{\text{ray}}, r)) = 0 \quad \forall r > 0.$$

Using Lemma 3.8, we get $v_M = 0$, which is a contradiction. Q.E.D.

Remark 5.3. One consequence of this result is the topological observation that, if $(M, F)$ is Berwaldian, reversible and non-negatively curved with large volume growth and if $\operatorname{injrad}_M > 0$, then $M$ is simply connected.
The author is unaware of any theorems generalizing Cheeger-Gromoll’s soul theorem to the Finsler setting. Though, if the Riemannian soul theorem was to hold in the Berwaldian and reversible non-negatively curved Finsler manifolds, then we would have the following consequence (the Finsler counterpart of Marenich-Toponogov’s theorem ([8])) due to the fact that existence of a compact soul would guarantee the existence of a closed geodesic (with a little work as in Klingenberg [4]).

**Conjecture 5.4.** Let $(M, F)$ be Berwaldian, reversible and non-negatively curved with large volume growth. Then, $M$ is diffeomorphic to $\mathbb{R}^n$.

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