Steady growth of length function and Malcev algebras *

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Abstract

We introduce and investigate the algebras of steadily growing length, that is the class of algebras, where the length is bounded by a linear function of the dimension. In particular we show that Malcev algebras belong to this class and establish the exact upper bound for its length.

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1 Introduction

Let $\mathbb{F}$ be an arbitrary field. In this paper $A$ denotes a finite dimensional not necessarily unital nor necessarily associative $\mathbb{F}$-algebra with the multiplication $(\cdot)$ usually denoted by the concatenation. Let $S = \{a_1, \ldots, a_k\}$ be a finite generating set of $A$. Any product of a finite number of elements from $S$ is a word in $S$. The length of the word $w$, denoted $l(w)$, equals to the number of letters in the corresponding product. It is worth noting that different choices of brackets provide different words of the same length due to the non-associativity of $A$. If $A$ is unital, we consider 1 as a word in $S$ with the length 0.

The set of all words in $S$ with the lengths less than or equal to $i$ is denoted by $S^i$, here $i \geq 0$.

Note that similarly to the associative case, $m < n$ implies that $S^m \subseteq S^n$.

The set $L_i(S) = (S^i)$ is the linear span of the set $S^i$ (the set of all finite linear combinations with coefficients belonging to $\mathbb{F}$). We write $L_i$ instead of $L_i(S)$ if $S$ is clear from the context. It should be noted that for unital algebras $\mathcal{L}_0(\mathbb{F}) = \langle 1 \rangle = \mathbb{F}$ for any $\mathbb{F}$, and for non-unital algebras $\mathcal{L}_0 = \emptyset$. We denote $\mathcal{L}(S) = \bigcup_{i=0}^{\infty} L_i(S)$.

Note that if $S$ is a generating set of the algebra $A$, i.e., $A$ is the smallest subalgebra of itself containing $S$, then any element of $A$ can be expressed as a linear combination of words in $S$.

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associative algebras, Lie algebras, Leibniz algebras, Novikov algebras, and Zinbiel algebras, as class is considerably large. Among the other classes of algebras it contains finite dimensional algebras, by the length function. The first step in this direction was the notion of slowly growing length, introduced in [11]. The problem of the associative algebra length computation was first discussed in [26, 27] in the context of the mechanics of isotropic continua. Since then this important algebraic invariant became an active topic of investigations. From one side the investigations of length function is the area with a number of interesting open algebraic problems, see [12, 11, 10, 15, 23, 12]. For example, even the length of the matrix algebra is not known. From another side, the number of applications, where this invariant is used, is intensively growing nowadays, see [1, 2, 3, 8, 19, 25, 7, 23, 13, 18, 21, 22].

Recent results on the lengths of non-associative algebras were obtained in the works [9, 11, 11]. The key tool used across them was the notion of characteristic sequences which is defined as follows.

Definition 1.1 [11, Definition 1.5] The length \( l(A) \) of a finitely generated algebra \( A \) is the maximal length of its finite generating systems, 

\[ l(A) = \max\{l(S) : L(S) = A\}. \]

Notice that the unital algebra \( A \) has length 0 if and only if \( A = F \cdot 1_A \). Otherwise length is a positive integer or infinite.

The length of these algebras are obtained in [11]. Moreover, some polynomial conditions on the algebra elements that guarantee the slow growth of the length function are found.

Definition 1.2. [11, Definition 2.3] Given a set of generators \( S \) in an algebra \( A \), by the characteristic sequence of \( S \) in \( A \) we understand a monotonically non-decreasing sequence of non-negative integers \( (m_1, \ldots, m_N) \), constructed by the following rules:

1. Let \( s_0 = \dim L_0(S) \). If \( s_0 = 1 \), we set \( m_1 = 0 \).
2. Denoting \( s_1 = \dim L_1(S) - \dim L_0(S) \), we define \( m_{s_0+1} = \ldots = m_{s_0+s_1} = 1 \).
3. Let for some \( r > 0 \), \( t > 1 \) the elements \( m_1, \ldots, m_r \) be already defined and the sets \( L_0(S), \ldots, L_{t-1}(S) \) be considered. Then we inductively continue the process in the following way. Denote \( s_t = \dim L_t(S) - \dim L_{t-1}(S) \). Then we define \( m_{r+1} = \ldots = m_{r+s_t} = t \).

Lemma 1.3. [11, Lemma 2.6] The characteristic sequence of \( S \) contains exactly \( \dim A \) terms. Moreover, for the last term we have \( m_N = l(S) \).

In the associative case the characteristic sequences are not useful since for a given generating set \( S \) the dimension sequence \( (\dim L_i(S))_{i=0}^{\infty} \) is strictly monotone until the stabilization happens, in particular, there are no repetitions before stabilization in this sequence. By Definition 1.2, Item 3, it means that for all \( t = 2, 3, \ldots \), the value \( s_t \neq 0 \), so each integer \( t \) appear in the characteristic sequence, and thus neighboring elements of characteristic sequence cannot differ in their values by more than one. In the non-associative case this is not necessarily true, and due to this reason it was possible to produce a method of characteristic sequences to establish strict upper bounds on length for general non-associative algebras as well as for several known classes of non-associative algebras, see [9, 10].

Theorem 1.4. [9, Theorem 2.7]). Let \( A \) be a unital \( F \)-algebra, \( \dim A = n \geq 2 \). Then 

\[ l(A) \leq 2^{n-2}. \]

The above bound is strict, see [9, Example 2.8]. The next natural step is to consider different classes of algebras determined by certain special restrictions on the growth of the length function. The first step in this direction was the notion of slowly growing length, introduced in [11].

Definition 1.5. [11, Definition 1.5] We say that a class of algebras has slowly growing length, if for any representative \( A \) of this class it holds that \( l(A) \leq \dim(A) \).

In [11] certain general properties of this class are found. In particular, it is shown that this class is considerably large. Among the other classes of algebras it contains finite dimensional associative algebras, Lie algebras, Leibniz algebras, Novikov algebras, and Zinbiel algebras, as well as many other important classical finite dimensional algebras. Exact upper bounds for the length of these algebras are obtained in [11]. Moreover, some polynomial conditions on the algebra elements that guarantee the slow growth of the length function are found.

In this paper we introduce a more general class of algebras extending the notion above.
**Definition 1.6.** We say that a class of algebras has a steadily growing length, if for every representative $\mathcal{A}$ of this class it holds that $l(\mathcal{A}) \leq c \cdot \dim(\mathcal{A}) + b$, where $c$ and $b$ are constants which depend only on the class. We call $c$ an upper length velocity or u-velocity of this class.

By definition, classes of algebras with slowly growing length have steadily growing length and their u-velocity is 1. We show below that the opposite is not necessarily true, cf. Example 4.8.

We remark that there are classes of algebras that are not of steadily growing length. Certain particular examples can be found in [9, Examples 2.8 and 5.5] and [10, Example 4.4].

The main purpose of our paper is to investigate the algebras of steadily growing length. To do this we introduce the classes of $k$-mixing and $k$-sliding algebras that generalize mixing and sliding algebras introduced in [11] for arbitrary $k$ variables. These two properties guarantee that for a characteristic sequence $(m_1, \ldots, m_d)$ of a generating system of an algebra it holds that $m_j \leq m_{j-1} + k - 1$. This in turn easily provides an estimation for the length which is a linear function in the dimension with the coefficient $k$.

An interesting borderline case is the class of Malcev algebras introduced in 1955 by A.I. Malcev, [17].

**Definition 1.7.** An algebra $\mathcal{A}$ is called a Malcev algebra if

1. $xy = -yx$ for all $x, y \in \mathcal{A}$,
2. $(xy)(xz) = ((xy)x)x + ((yz)x)x + ((zx)x)y$ for all $x, y, z \in \mathcal{A}$.

This class is naturally connected to the class of Lie algebras via the following property.

**Proposition 1.8.** [24, Corollary 4.4] Any two elements $a, b$ of a Malcev algebra $\mathcal{A}$ are contained in a Lie subalgebra of $\mathcal{A}$.

The detailed and self-contained exposition of Malcev algebras can be found in [5, 6, 20].

This class does not fit in the framework provided by [11] and the classes of mixing and sliding algebras introduced there. However the generalization of its methods to $k$-sliding and $k$-mixing algebras allows to prove that Malcev algebras are 3-mixing, which guarantees that they are steadily growing. Then it is possible to tighten the length bound even further. In this paper we prove that Malcev algebras have slowly growing length using a detailed analysis of inner structure of its words by the new method proposed in the last section of the paper.

Our paper is organized as follows. In Section 2 we provide previously established results and notions, relevant for further proofs. In Section 3 polynomial properties which guarantee a steadily growth of length are introduced and studied. Namely, we introduce $k$-mixing and $k$-sliding algebras, relate them with mixing and sliding algebras, and investigate their length by means of the characteristic sequences. Section 4 contains several examples. Throughout Section 5 we examine inner structure of words in terms of lengths of their subwords and the bounds on these lengths for $k$-mixing and $k$-sliding algebras. In Section 6 we examine Malcev algebras in detail and compute the strict bound on their length.

**2 Basic results and notions**

**Definition 2.1.** A word $w$ from a generating set $S$ of an algebra $\mathcal{A}$ is irreducible, if for each integer $m$, $0 \leq m < l(w)$, it holds that $w \notin L_m(S)$.

**Remark 2.2.** For all algebras $\mathcal{A}$ and all generating sets $S$ we consider 0 to be an irreducible word of length $-\infty$.

If $\mathcal{A}$ is unital we consider 1 to be an irreducible word of length 0.

**Lemma 2.3.** [9, Lemma 2.14] Any irreducible word $w$, $l(w) > 1$, is a product of two irreducible words of non-zero lengths.

**Proposition 2.4.** [11, Corollary 2.5] 1. For any term $m_h$ of the characteristic sequence of $S$ there is an irreducible word in $\mathcal{L}(S)$ of length $m_h$. 2.
2. If there is an irreducible word in \( S \) of length \( k \), then \( k \) is included into the characteristic sequence of \( S \).

**Corollary 2.5.** Let \( A \) be an F-algebra, \( \dim A = n > 2 \). Assume \( S \) is a generating set for \( A \) and \( M = (m_0, m_1, \ldots, m_{n-1}) \) is the characteristic sequence of \( S \). Then for each \( h \) satisfying \( m_h \geq 2 \) it holds that there are indices \( 1 \leq t_1 \leq t_2 < h \) such that \( m_{t_1} = m_{t_2} \) and \( m_{t_1} + m_{t_2} > 0 \).

**Proof.** By Proposition 2.4 Item 1 each term \( m_k \) of the characteristic sequence corresponds to an irreducible word of the length \( m_h \). Denote it by \( w_{m_h} \). By Lemma 2.3 each irreducible word of the length \( m_h \geq 2 \) can be represented as a product of two irreducible words of positive lengths. Thus, \( w_{m_h} = w_{k_1}w_{k_2} \) for some irreducible words \( w_{k_1}, w_{k_2} \) of lengths \( k_1, k_2 > 0 \), correspondingly. Assume \( k_1 < k_2 \). Then by Proposition 2.4 Item 2 there are indices \( t_1, t_2 \geq 1 \) such that \( m_{t_1} = k_1 \) and \( m_{t_2} = k_2 \). Since \( M \) is non-decreasing, \( 1 \leq t_1 \leq t_2 < h \). Assume \( k_1 > k_2 \). Then by Proposition 2.4 Item 2 there are indices \( t_1, t_2 \geq 1 \) such that \( m_{t_1} = k_2 \) and \( m_{t_2} = k_1 \). Since \( M \) is non-decreasing, \( 1 \leq t_1 \leq t_2 < h \). Assume \( k_1 = k_2 \). Then by Proposition 2.4 Item 2 there are indices \( t_1 = t_2 \geq 1 \) such that \( m_{t_1} = m_{t_2} = k_1 = k_2 \). Since \( M \) is non-decreasing, \( 1 \leq t_1 \leq t_2 < h \). In all cases, the additivity of word length gives the proof.

\[ \square \]

Below we introduce the notions of mixing and sliding algebras from our paper [11] which provide a large class of algebras with the slowly growing length. We present here complete definitions to prepare the reader for the corresponding sets in \( k \) variables. These sets are provided in the next section in order to determine classes of algebras having steadily growing length.

Let \( x, y, z \) be variables. To introduce the following definition we need the special sets \( Q_l \) and \( Q_r \) of monomials:

\[ Q_l(x, y, z) = \{ x(zy), x(yz), y(xz), g(xz), xy, yx, zy, z, xz, zy, x, y, z \} \]

contains all monomials of degrees 1 and 2 and only those monomials of degree 3 in which \( z \) is an argument of the first multiplication and the multiplier with \( z \) is the second factor of the second multiplication.

\[ Q_r(x, y, z) = \{ (xz)y, (zx)y, (yz)x, (zy)x, xy, yx, zx, xz, xy, yz, zy, x, y, z \} \]

contains all monomials of degrees 1 and 2 and only those monomials of degree 3, in which \( z \) is an argument of the first multiplication and the multiplier with \( z \) is the first factor of the second multiplication.

**Definition 2.6.** [11] Definition 3.1] Let \( A \) be an F-algebra such that at least one of the following statements holds:

1. \( z(xy) \in (Q_r(x, y, z)) \) for all \( x, y, z \in A \), if \( A \) is non-unital; \( z(xy) \in (Q_r(x, y, z), 1) \) for all \( x, y, z \in A \), if \( A \) is unital.

2. \( (xy)z \in (Q_l(x, y, z)) \) for all \( x, y, z \in A \), if \( A \) is non-unital; \( (xy)z \in (Q_l(x, y, z), 1) \) for all \( x, y, z \in A \), if \( A \) is unital.

Then we call \( A \) a sliding algebra.

To introduce the next class of algebras we need the monomial set: \( P(x, y, z) = Q_l(x, y, z) \cup Q_r(x, y, z) = \)

\[ \{ (xz)y, (zx)y, (yz)x, (zy)x, x(zy), x(yz), y(xz), y(zx), xy, yx, zx, xz, yz, zy, x, y, z \} \]

i.e. we consider those monomials of degree 3 that have \( z \) inside the brackets.

**Definition 2.7.** [11] Definition 3.2] Let \( A \) be an F-algebra such that for all \( x, y, z \in A \) it holds that \( (xy)z, z(xy) \in (P(x, y, z)) \) if \( A \) is unital, and \( (xy)z, z(xy) \in (P(x, y, z)) \) if \( A \) is non-unital. Then we call \( A \) a mixing algebra.

**Theorem 2.8.** [11] Theorem 3.6] The length of a mixing or a sliding algebra \( A \) of dimension \( d \geq 2 \) is less than or equal to \( d \).
3 \( k \)-sliding and \( k \)-mixing algebras

In this section we plan to generalize Definitions 2.6 and 2.14 for \( k \) variables. To do this we need analogs of the sets \( Q_l \) and \( Q_\ell \) and to introduce them we fix the following notations.

Let \( Z = \{ z_0, z_1, \ldots, z_k \} \) be a certain set of variables, and \( \Sigma = (z_0, z_1, \ldots, z_k) \) be the ordered sequence of the variables from \( Z \). We denote \( Z_0 = \{ z_1, \ldots, z_k \} \subset Z \). Let \( T \subset Z \) be a certain subset. Recall that \(|T|\) as usual denotes the cardinality of the set \( T \). Then we define the following monomial sets.

**Definition 3.1.**

- \( W(T) \) is the set of all multilinear words in variables from \( T \) of length \(|T|\). We set \( W(\emptyset) = \{ 1 \} \) in the unital case and \( W(\emptyset) = \emptyset \) in the non-unital case.
- \( D(T) = \bigcup_{T' \subseteq T} W(T') \) is the set of all multilinear words in variables from \( T \).
- \( D'(T) = D(T) \setminus W(T) \) is the set of all multilinear words in \( T \) of length strictly less than \(|T|\). Note that \( D'(T) = \bigcup_{T' \subseteq T} W(T') \).
- \( D_0(\Sigma) = D(Z) \setminus (z_0 W(Z_0) \cup W(Z_0) z_0) \) is the set of all multilinear words in \( Z \) except the words of the length \( k + 1 \) with \( z_0 \) being a factor of the last multiplication.
- \( D_1(\Sigma) = \bigcup_{T' \subset Z_0} \{ w_1 w_2 | w_1 \in W(T'), w_2 \in W(Z \setminus T') \} \cup D'(Z) \) is the set of all multilinear words in \( Z \) except the words of the length \( k + 1 \) having \( z_0 \) in the first factor of the last multiplication.
- \( D_r(\Sigma) = \bigcup_{T' \subset Z_0} \{ w_1 w_2 | w_1 \in W(Z \setminus T'), w_2 \in W(T') \} \cup D'(Z) \) is the set of all multilinear words in \( Z \) except the words of the length \( k + 1 \) having \( z_0 \) in the second factor of the last multiplication.

Now we are ready to define the classes of algebras which are crucial for our study.

**Definition 3.2.** Let \( A \) be an \( F \)-algebra such that at least one of the following statements holds:

1. \( W(\{ y_1, \ldots, y_k \}) x \subset \langle D_l(x, y_1, \ldots, y_k) \rangle \),
2. \( x W(\{ y_1, \ldots, y_k \}) \subset \langle D_l(x, y_1, \ldots, y_k) \rangle \).

Then we call \( A \) a \( k \)-sliding algebra.

**Definition 3.3.** Let \( A \) be an \( F \)-algebra such that for all \( x, y_1, \ldots, y_k \in A \) it holds that \( x W(\{ y_1, \ldots, y_k \}) \cup W(\{ y_1, \ldots, y_k \}) x \subset \langle D_0(x, y_1, \ldots, y_k) \rangle \). Then we call \( A \) a \( k \)-mixing algebra.

Let us show that these notions generalize the notions introduced in [14].

**Proposition 3.4.** Let \( A \) be a finite dimensional \( F \)-algebra.

1. \( A \) is mixing if and only if \( A \) is 2-mixing.
2. \( A \) is sliding if and only if \( A \) is 2-sliding.

**Proof.** Below we consider only non-unital case. The unital case can be considered in a similar way.

1. Assume the algebra \( A \) is mixing. Consider arbitrary \( x, y_1, y_2 \in A \). By Definition 2.14 it holds that \( (y_1 y_2) x, x (y_1 y_2) \in \langle P(y_1, y_2, x) \rangle \) and \( (y_2 y_1) x, x (y_2 y_1) \in \langle P(y_2, y_1, x) \rangle \). Since \( x W(\{ y_1, y_2 \}) \cup W(\{ y_1, y_2 \}) x = \{ (y_1 y_2) x, x (y_1 y_2), (y_2 y_1) x, x (y_2 y_1) \} \), \( P(y_2, y_1, x) = P(y_1, y_2, x) \) and \( P(y_1, y_2, x) = D_0(x, y_1, y_2) \), we have \( x W(\{ y_1, y_2 \}) \cup W(\{ y_1, y_2 \}) x \subset \langle D_0(x, y_1, y_2) \rangle \), i.e. \( A \) is 2-mixing.
Now let \( \mathcal{A} \) be 2-mixing. Consider arbitrary \( x, y, z \in \mathcal{A} \). By Definition 3.3, it holds that \( zW(W(x, y))z \subseteq D_0(\{x, y\}) \). Since \( \{xy\}z \leq zW(W(x, y))z \) and \( D_0(z, x) = P(x, y, z) \), we have \( \{xy\}z \leq P(x, y, z) \), i.e. \( \mathcal{A} \) is mixing.

2. Assume \( \mathcal{A} \) is sliding. If Item 1 of Definition 2.6 holds for \( \mathcal{A} \), then for arbitrary \( x, y_1, y_2 \in \mathcal{A} \) we have \( x(y_1y_2) \in Q_r(y_1, y_2, x) \). Since \( zW(\{y_1, y_2\}) = \{(x(y_1y_2), x(y_2y_1))\} \),

\[ Q_r(y_2, y_1, x) = Q_r(y_1, y_2, x), \text{ and } Q_r(y_1, y_2, x) = D_r(x, y_1, y_2) \]

this means \( zW(\{y_1, y_2\}) \subseteq (D_r(x, y_1, y_2)) \), i.e. \( \mathcal{A} \) is 2-sliding. The case Item 1 of Definition 2.6 holds for the algebra \( \mathcal{A} \) is similar.

If \( \mathcal{A} \) is 2-sliding and Item 2 of Definition 2.6 holds for \( \mathcal{A} \), then we consider arbitrary \( x, y, z \in \mathcal{A} \). We have \( zW(W(x, y)) \subseteq (D_r(x, y)) \). Since \( z(xy) \in zW(\{x, y\}) \) and \( D_r(x, y) = Q_r(x, y, z) \), this means \( z(xy) \in \{Q_r(x, y, z)\} \), i.e. \( \mathcal{A} \) is sliding. The other case is similar.

For a given element of a \( k \)-mixing algebra

\[ w \in W(\{y_1, \ldots, y_k\}) \cap \bigcup \{W(\{y_1, \ldots, y_k\}) \mid x \}

by \( R(w) \) we denote the set of all words from \( W(\{x, y_1, \ldots, y_k\}) \) (i.e. monomials of degree \( k+1 \)) of the representations of \( w \) as a linear combination of elements of \( D_0(x, y_1, \ldots, y_k) \).

The following lemma is a key tool to prove the linearity of the growth length.

**Lemma 3.5.** Let \( \mathcal{A} \) be a \( k \)-mixing \( \mathbb{F} \)-algebra (\( k \geq 2 \)), \( \mathcal{S} \) be its generating set and \( M = \{m_1, \ldots, m_d\} \) be a characteristic sequence of \( \mathcal{S} \). Then \( m_{j+1} - m_j \leq k-1 \) for all \( j = 1, \ldots, d-1 \).

**Proof.** Assume the contrary. Let \( \mathcal{A} \) be a \( k \)-mixing algebra, \( \mathcal{S} \) be its generating set, and assume that there exists \( j, k \leq j \leq d-1 \) such that the inequality \( m_{j+1} - m_j \leq k-1 \) does not hold. Let \( p \) be the smallest index such that \( m_{p+1} - m_p \geq k \).

Consider a word \( w \leq w' \) with length at least \( 2k \). It is equal to a product \( w' \cdot w'' \), where \( w' \) and \( w'' \) have non-zero lengths. We denote \( s(w) = \min(l(w'), l(w'')) \).

1. Consider an irreducible word \( w \in \mathcal{A} \) of length \( m_{p+1} \). Then \( s(w) \geq k \). Indeed, if \( s(w) \leq k-1 \) then by Lemma 2.6 the word \( w \) is a product of irreducible words of lengths \( s(w) \) and \( m_{p+1} - s(w) \). Hence by Proposition 2.1 Item 2, there is an element equal to \( m_{p+1} - s(w) \) in the characteristic sequence \( M \). This is impossible, since \( M \) is non-decreasing and \( m_p < m_{p+1} - k + 1 \leq m_{p+1} - s(w) \) by the assumptions.

2. Let us choose such an irreducible word \( w_0 \) of length \( m_{p+1} \) that \( s(w_0) \) is the smallest. If there are several such words, we take any one of them. The chosen word is a product of two irreducible words, \( w_0' \) and \( w_0'' \), such that \( l(w_0') = s(w_0) \). Thus we have the following 2 cases:

   Case 1. \( w_0 = w_0' \cdot w_0'' \). By Item 1 of the proof \( s(w_0) \geq k \) holds. This means that \( w_0' \) is a product of exactly \( k \) words of positive length. Let us denote them as \( w_1', \ldots, w_k' \). Consider the set \( R(w_0', w_0'') \) chosen with respect to \( D_0(w_0', w_0'', \ldots, w_k') \). At least one element \( r(w) \) of this set must be an irreducible word, as words from \( D_0(w_0', w_0'', \ldots, w_k') \) have length strictly less than \( l(w_0) \). However, by the definition of \( D_0 \), \( r(w) \) is a product of two words, one in \( W(\{w', i \in I\}) \) and the other in \( W(\{w', i \in I \} \cup \{w_i', i \in \{1, \ldots, k\} \setminus I\}) \) for some non-empty \( I \subseteq \{1, \ldots, k\} \). The length of \( r(w) \) is equal to \( l(w_0) \), while \( s(r(w)) \) is equal to the length of the element in \( W(\{w_i', i \in I\}) \), which is strictly less than \( l(w_0) = s(w_0) \). This contradicts the choice of \( w_0 \). Thus the initial assumption is false, i.e. \( k \)-mixing algebra cannot have a generating set with such a characteristic sequence that the difference between neighboring element is greater than \( k-1 \).

   Case 2. \( w_0 = w_0' \cdot w_0'' \). We obtain the same contradiction similarly, considering an irreducible element of \( R(w_0', w_0'') \) chosen with respect to the set \( D_0(w_0', w_0'', \ldots, w_k') \).

For a given element of a \( k \)-sliding algebra \( w \in W(\{y_1, \ldots, y_k\}) \) (or \( w \in W(\{y_1, \ldots, y_k\})x \)) by \( U_i(w) \) (or \( U_i(w) \)) we denote the set of all words from \( W(\{x, y_1, \ldots, y_k\}) \) which are included with non-zero coefficients in at least one representation of \( w \) as a linear combination of elements of \( D_1(x, y_1, \ldots, y_k) \) (or \( D_1(x, y_1, \ldots, y_k) \)).
Lemma 3.6. Let $\mathcal{A}$ be a $k$-sliding $F$-algebra ($k \geq 2$), $\mathcal{S}$ be its generating set and $M = (m_1, \ldots, m_d)$ be a characteristic sequence of $\mathcal{S}$. Then $m_{j+1} - m_j \leq k - 1$ for all $j = 1, \ldots, d - 1$.

Proof. Assume the contrary: let $\mathcal{A}$ be a $k$-sliding algebra satisfying Item 2 of Definition 2.4 (the case of Item 1 can be proven similarly) and let $\mathcal{S}$ be a generating set of $\mathcal{A}$ such that for its characteristic sequence $M$ the inequality $m_{j+1} - m_j \leq k - 1$ does not hold for all $j = 1, \ldots, d - 1$. Let $k$ be the smallest index such that $m_{p+1} - m_p \geq k$.

Consider a word $w$ of length at least two. It is equal to a product $w' \cdot w''$, where $w'$ and $w''$ have non-zero lengths. We denote $l_r(w) = l(w'')$.

Let us choose such an irreducible word $w_0$ of length $m_{p+1}$ in $\mathcal{S}$ that $l_r(w)$ is minimal (if there are multiple possible candidates, we can choose one at random). By Lemma 2.9, $w_0$ is equal to $w'_0 \cdot w''_0$, where both factors have non-zero length.

1. $l_r(w_0) = l(w''_0) \leq k - 1$, cannot hold. This would mean that $l(w''_0) = m_{p+1} - l(w'_0)$ which by Proposition 2.4 Item 2 would mean that there is an element of characteristic sequence equal to $m_{p+1} - l(w'_0)$, and that is impossible: $M$ is non-decreasing and $m_p < m_{p+1} - k + 1 \leq m_{p+1} - l(w'_0)$.

2. If $l(w''_0) > k - 1$, $w''_0$ is equal to a product of exactly $k$ words $w''_0^1, \ldots, w''_0^k$ of positive length. Consider the set $D_r(w''_0^1, w''_0^2, \ldots, w''_0^k)$. As all of the elements of $D_r(w''_0^1, w''_0^2, \ldots, w''_0^k)$ have lengths strictly less than $l(w_0)$, at least one element of $U_r(w_0)$ must be irreducible, since otherwise $w_0$ is a linear combination of shorter words or reducible words of the same lengths, i.e. reducible as well. Let us denote this element by $w(w_0)$. Note that $l(w(w_0)) = l(w_0)$, while the second factor in the last multiplication of $w(w_0)$ has length strictly less than $l(w''_0)$, which contradicts the choice of $w_0$. Thus the initial assumption is false. \qed

Theorem 3.7. Let $\mathcal{A}$, $\dim \mathcal{A} = d \geq 2$, be a $k$-mixing or a $k$-sliding algebra, $k \geq 2$. Then the length of $\mathcal{A}$ has steady growth and $(k - 1)$ is its $u$-velocity.

Proof. Follows directly from Lemma 3.6 or for $k$-mixing algebras or from Lemma 3.8 for $k$-sliding algebras. Namely, for a generating set $\mathcal{S}$ of $\mathcal{A}$ with $l(\mathcal{S}) = l(\mathcal{A})$ and characteristic sequence $(m_1, \ldots, m_d)$ of $\mathcal{S}$ we have $m_1 \leq 1$ and $l(\mathcal{S}) = m_d \leq m_{d-1} + k - 1 \leq \ldots \leq m_1 + (d-1)(k-1) \leq 1 + (d-1)(k-1) \leq d(k-1)$. \qed

4 Examples

Below we present two classes of algebras satisfying, respectively, $k$-sliding and $k$-mixing properties such that $k - 1$ is their minimal $u$-velocity. We call them $k$-round and $k$-based algebras, respectively. Some examples of $k$-round algebras appeared in [11] as examples of algebras that have linear in dimension growth of the length but are not of slowly growing length, cf. [11] Proposition 4.25.

Definition 4.1. We say that an algebra $\mathcal{A}$ is $k$-round ($k \geq 2$) if for all $x, y_1, \ldots, y_k \in \mathcal{A}$ and for any product $v = y_1 \cdots y_k$ with any placement of parentheses it holds that $xv = 0$.

Proposition 4.2. Any $k$-round algebra is $k$-sliding.

Proof. Follows immediately from Definition 4.1 Item 2. \qed

An example of a $k$-round algebra is constructed below. In order to check polynomial identities for algebras we use the following lemma.

Lemma 4.3. [11] Lemma 4.1] Consider a finite-dimensional algebra $\mathcal{A}$ over the field $F$, its basis $\{e_1, \ldots, e_d\}$ and multilinear function $G$ of $k$ arguments such that $G(e_{i_1}, \ldots, e_{i_k}) = 0$ for all $i_u \in \{1, \ldots, d\}$. Then for all $a_1, \ldots, a_k \in \mathcal{A}$ it holds that $G(a_1, \ldots, a_k) = 0$.

Example 4.4. [11] Example 4.26] Consider an algebra $\mathcal{E}_d$ with the basis $x_1, \ldots, x_d$, $d \geq k \geq 2$ and the following multiplication law:

$$x_j x_1 = x_{j+1}, \ j = 1, \ldots, k - 2,$$
Let $E$ be the basis of $E_d$. We have
\[ l(E_d) \geq l(\{x_1\}) = (k-1)d - (k-2)(k-1). \]

Let us prove that $E_d$ satisfies Definition 4.4. At first, we consider $x, y_1, \ldots, y_k \in \{x_1, \ldots, x_d\}$. A word $xy$, where $v$ is a product of $y_1, \ldots, y_k$, is indeed zero as $v$ cannot be neither $x_1$ nor $x_{k-1}$. So, the required condition holds for the basis of $E_d$. Then by Lemma 4.3, it is satisfied for other elements as well.

**Proposition 4.4.** Let $k \geq 2$ be integer. The minimal $u$-velocity for the class of $k$-round algebras is $k-1$.

**Proof.** Note that $k$-round algebras are $k$-sliding and by Theorem 3.7, their $u$-velocity is $k-1$. Consider a number $c < k - 1$ and an arbitrary $b \in \mathbb{R}$. There exists an integer $d \geq k$ such that $cd + b < (k-1)d - (k-2)(k-1)$. Since $E_d$ belongs to the class of $k$-round algebras and $(k-1)d - (k-2)(k-1) \leq l(E_d)$, this means that the inequality $l(A) \leq c \cdot \text{dim} A + b$ does not hold universally for the members of this class. Since we can pick $b$ arbitrarily, this means that $c$ is not a $u$-velocity of the class of $k$-round algebras. Thus, $k-1$ is indeed a minimal $u$-velocity of the class of $k$-round algebras. 

\[ \square \]

**Definition 4.6.** We say that an algebra $A$ is $k$-based, $k \geq 2$, if for all $x, y_1, \ldots, y_k \in A$ and any fixed placement of parentheses $uv = y_1 \cdots y_k$, it holds that $x(uv) = u(xv), (uv)x = (ux)v$.

**Proposition 4.7.** A $k$-based algebra is $k$-mixing.

**Proof.** Follows immediately from Definition 3.6. 

The following is an example of a $k$-based algebra.

**Example 4.8.** Consider the algebra $X_d$ with the basis $x_1, \ldots, x_d$, $d \geq k \geq 2$ and the following multiplication law:
\[
x_1x_j = x_{j+1}, \quad j = 1, \ldots, k-2, \\
x_{k-1}x_i = x_{i+1}, \quad i = k-1, \ldots, d-1,
\]
with other products being zero. We have
\[ l(X_d) \geq l(\{x_1\}) = (k-1)d - (k-2)(k-1). \]

Let us prove that $X_d$ is indeed a $k$-based algebra, namely, $X_d$ satisfies Definition 4.0. We consider $x, y_1, \ldots, y_k \in \{x_1, \ldots, x_d\}$. If the required condition holds for the basis of $X_d$, then by Lemma 4.3, it is satisfied for other elements as well. For the word $x(uv)$, where $uv = y_1 \cdots y_k$ with a certain fixed placement of parentheses, there are three possibilities:

- Assume $u = x_j, j < k - 1$. This means $uv = 0$ and $u(xv) = 0$ by the multiplication laws, from which it follows that $x(uv) = 0 = u(xv)$.
- Assume $u = x_{k-1}$. In this case the following three possibilities appear:
  - If $x = x_j, j < k - 1$, then similarly to Case 1 both $x(uv)$ and $u(xv)$ are equal to zero,
  - If $x = x_{k-1}$, then $u = x$ and the equality of $x(uv)$ and $u(xv)$ is trivial,
  - If $x = x_i, i > k - 1$, then $x(uv) = 0$ and $xv = 0$ by the multiplication laws, from which follows $x(uv) = 0 = u(xv)$.
- Assume $u = x_i, i > k - 1$. This means that $uv = 0$ and $u(xv) = 0$ by the multiplication laws, from which follows $x(uv) = 0 = u(xv)$.

Also for the word $(uv)x$, where $uv = y_1 \cdots y_k$ with a certain fixed placement of parentheses, there are three possibilities:
- Assume \( u = x_j, j < k - 1 \). This means \( uv = 0 \) and \( ux = 0 \) by the multiplication laws, from which follows \( (uv)x = 0 = (ux)v \).
- Assume \( u = x_{k-1} \). This means \( uv \) and \( ux \) are either zero or some \( x_i, i > k - 1 \) which leads by multiplication laws to \( (uv)x = 0 = (ux)v \).
- Assume \( u = x_i, i > k - 1 \). This means \( uv = 0 \) and \( ux = 0 \) by the multiplication laws, from which follows \( (uv)x = 0 = (ux)v \).

This concludes the proof that \( \mathcal{X}_d \) is indeed a \( k \)-based algebra.

**Proposition 4.9.** The minimal \( u \)-velocity for the class of \( k \)-based algebras is \( (k - 1) \).

*Proof.* Note that \( k \)-based algebras are \( k \)-mixing and by Theorem 3.8, \( k - 1 \) is a \( u \)-velocity for this class.

Consider a number \( c \) such that \( cd + b < (k - 1)d - (k - 2)(k - 1) \). Since \( \mathcal{X}_d \) belongs to the class of \( k \)-based algebras and \( (k - 1)d - (k - 2)(k - 1) \), this means that the inequality \( l(A) \leq c \cdot \dim A + b \) does not hold universally for the members of this class. Since we can pick \( b \) arbitrarily, this means that \( c \) is not a \( u \)-velocity of the class of \( k \)-based algebras.

Thus, \( k - 1 \) is indeed a minimal \( u \)-velocity of the class of \( k \)-based algebras.

The following class of algebras shows that there are algebras of steadily growing lengths that are neither mixing nor sliding, so these classes of algebras provide a useful tool to investigate algebra length. The general question of characterization for these algebras remains open.

**Definition 4.10.** An algebra \( \mathcal{A} \) is called a Vinberg algebra if

\[
(xyz - x(yz)) = (xz)y - x(zy)
\]

for all \( x, y, z \in A \).

These algebras are sometimes called right-symmetric algebras.

**Example 4.11.** Consider an algebra \( \mathcal{V}_d \) with the basis \( x_1, \ldots, x_d, d \geq 4 \) and the following multiplication law:

\[
x_i x_j = x_{i+j}, i + j \leq d - 1,
\]

\[
x_d - 1 x_d - 2 = x_d.
\]

with other products being zero.

It is easy to see that \( \mathcal{V}_d \) is neither \( k \)-mixing nor \( k \)-sliding for any \( k \in \{2, \ldots, d - 2\} \). The characteristic sequence of the set \( \{x_1\} \) is \( (1, 2, \ldots, d - 2, d - 1, 2d - 3) \) and \( (2d - 3) - (d - 1) = d - 2 \).

However, \( \{x_1\} \) is a class of steadily growing algebras with \( u \)-velocity 2 since it is straightforward to check that \( l(\mathcal{V}_1) = 2d - 3 \).

### 5 Sprout sequences

The aim of this section is to establish new methods to investigate the inner structure of words in algebras which allow to compute the length. The definitions below are the first steps in this direction.

**Definition 5.1.** Consider an algebra \( \mathcal{A} \), its generating set \( \mathcal{S} \) and a word \( w \) in \( \mathcal{S} \). We say that a word \( w' \) in \( \mathcal{S} \) is a subword of \( w \) if it is a factor of one of the multiplication inside of \( w \) or it is \( w \) itself. We also say that \( w' \) is a proper subword of \( w \) if it is a subword which is not \( w \) itself.

**Example 5.2.** Let \( \mathcal{A} \) be an algebra and \( \mathcal{S} \) be its generating set such that there exist \( x, y \in \mathcal{S} \) with \( x \neq y \). For a word \( w = (xy)(y(xx)) \) in \( \mathcal{S} \), words \( x, y, xx, xy, (y(xx)), (xy)(y(xx)) \) are its subwords, while \( yy \) or \( yx \) are not. Note that there is no subword of length 4 in \( w \).
Lemma 5.3. Let \( A \) be an algebra, \( S \) be its generating set and \( w \) be a word in \( S \) such that \( w = uv \) where \( u, v \) are also words in \( S \) of positive length. Then the set of proper subwords of \( w \) is the union of the sets of subwords of \( u \) and \( v \).

Proof. It is straightforward to see that a subword of \( u \) or \( v \) is also a subword of \( w \), moreover, a proper subword, as it is shorter than \( w \).

On the other hand, a proper subword of \( w \) is a factor in one of the multiplications inside of \( w \). If it is a factor of the last multiplication, then it is either \( u \) or \( v \). Otherwise, it is a factor in one of the multiplications inside of \( u \) or \( v \). In both cases it is a subword of \( u \) or \( v \). □

Definition 5.4. Let \( A \) be an \( F \)-algebra, \( S \) be its generating set and \( w \) be a word of length at least \( 2 \) in \( S \). We construct inductively two sequences \((w_1, \ldots, w_r)\) and \((w'_1, \ldots, w'_r)\) of subwords of \( w \) as follows:

1. Set \( w_1 \) and \( w'_1 \) to be the two factors of the last multiplication in \( w \) (i.e. \( w = w_1w'_1 \) or \( w = w'_1w_1 \)) such that \( l(w_1) \leq l(w'_1) \).
2. Assume \( w_1, \ldots, w_{j-1} \) and \( w'_1, \ldots, w'_{j-1} \) are constructed and \( l(w'_{j-1}) \geq 2 \). Set \( w_j \) and \( w'_j \) to be the two factors of the last multiplication in \( w'_{j-1} \) such that \( l(w_j) \leq l(w'_j) \).
3. The process stops if \( l(w'_j) = l(w_j) = 1 \) for a certain \( r \).

We call the sequence \((w_1, \ldots, w_r)\) a sprout sequence, or just a sprout, of \( w \) and the sequence \((w'_1, \ldots, w'_r)\) a supporting sequence of \( w \).

If \((w_1, \ldots, w_r)\) is a sprout sequence, we say that \((l(w_1), \ldots, l(w_r))\) is an \( l \)-sprout sequence.

Let us start to analyze sprouts and their properties with an example.

Example 5.5. Let \( A \) be an \( F \)-algebra, \( S \) be its generating set and \( x, y, z \in S \). Then both sequences \(((xy)(zy), y, z, x)\) and \(((xz)zy, xy, z)\) are sprout sequences of the word \( w = ((xy)(zy))((xz)zy) \) and \((4, 1, 1, 1)\) and \((4, 2, 1)\) are \( l \)-sprout sequences of \( w \).

The supporting sequence has the following useful property.

Lemma 5.6. Let \( A \) be an \( F \)-algebra, \( S \) be its generating set, \( w \) be a word of length at least \( 2 \) in \( S \), \((w_1, \ldots, w_r)\) and \((w'_1, \ldots, w'_r)\) be sprout and supporting sequences of \( w \), and \( j \) be an integer, \( 2 \leq j \leq r \). Then the sequence \((w_j, \ldots, w_r)\) is a sprout sequence of \( w'_{j-1} \).

Proof. Follows directly from Definition 5.4. □

The next statement is converse to Lemma 5.6.

Lemma 5.7. Let \( A \) be an \( F \)-algebra, \( S \) be its generating set, \( w, w' \) be words in \( S \), \( l(w) \geq l(w') > 0 \). We denote by \((w_1, \ldots, w_r)\) a sprout sequence of \( w \). Then \((w, w', w, \ldots, w_r)\) is a sprout sequence of both \( w w' \) and \( w' w \). Additionally, \((l(w'), l(w_1), \ldots, l(w_r))\) is an \( l \)-sprout sequence for both words \( w w' \) and \( w' w \).

Proof. Follows directly from Definition 5.4. □

The following concept further expands the applications of the idea of the sprout sequence. Namely, sprout sequence consists of 'shorter' subwords of a given word. In the next definition we directly restrict their lengths.

Definition 5.8. We say that a word \( w \) is \( k \)-bounded, \( k \geq 2 \), if there exists an \( l \)-sprout sequence for \( w \) that does not contain elements of the length greater than \( k \) or the word itself has length \( 1 \). We call the respective sequence \( k \)-bounded as well.

Lemma 5.9. Let \( A \) be an \( F \)-algebra, \( S \) be its generating set, and \( k \geq 2 \) be an integer. Assume that \( w \) is a word in \( S \) such that \( l(w) \leq 2k - 1 \). Then \( w \) is \( k \)-bounded.

Proof. If \( w \) has length \( 1 \), then the statement is evident. Otherwise it is enough to note that the elements of its sprout sequence are the shorter factors of last multiplication in the respective subwords of \( w \), which means that their lengths are less than or equal to \( \frac{l(w)}{2} \leq \frac{2k-1}{2} < k \). □
Lemma 5.10. Let $A$ be an $F$-algebra, $S$ be its generating set, and $k \geq 2$ be an integer. Assume, $w$ be a $k$-bounded word in $S$. Then every subword of $w$ is also $k$-bounded.

Proof. We prove this statement using induction on the length $l$ of the word $w$.

The base. For $l = 1, \ldots, 2k - 1$ the statement is evident by Lemma 5.9.

The step. Assume that the statement holds for $l = 1, \ldots, t$ and consider $l = t + 1$, $t \geq 2k - 1$. There are two possibilities for a subword $w'$: either it is a factor of $w$ or it is a subword of one of factors of $w$. Let us consider each of these cases separately:

a. Let $w = w'w''$ or $w = w''w'$. If $l(w') \leq l(w'')$, then $l(w') < k$ since it is a first element in every $l$-sprout sequence of a $k$-bounded word. If $l(w') > l(w'')$, we consider a sprout sequence of $w$ corresponding to the $k$-bounded $l$-sprout sequence. By Lemma 5.10 it has the form $(w'', T)$, where $T$ is a sprout sequence of $w'$. Thus, the sequence of lengths of elements of $T$ is $k$-bounded itself as a subsequence of a $k$-bounded sequence.

b. Let $w'$ be a factor of $w$, i.e. a proper subword of one of factors of $w$. Note that factors of the last multiplication are $k$-bounded by Item a. of this lemma and shorter than $w$. Thus $w'$ is $k$-bounded by the induction hypothesis.

Now we connect the notion of the sprout sequences to the previously introduced classes of $k$-mixing and $k$-sliding algebras, namely, we are going to prove that in these classes of algebras any irreducible word $w$ can be represented as a linear combination of $k$-bounded words of the length not greater than $l(w)$.

Proposition 5.11. Let $k \geq 2$ be an integer, $A$ be a $k$-mixing $F$-algebra, and $S$ be its generating set. Assume that $w$ is an irreducible word in $S$, $l(w) \geq 2$. Then there exist irreducible $k$-bounded words $w_1, \ldots, w_n$ such that $l(w_i) \leq l(w)$ and $w \in \langle w_1, \ldots, w_n \rangle$.

Proof. We will prove this statement by induction on the length $l$ of the word $w$.

The base for $l$. For $l = 2, \ldots, 2k - 1$ the statement is trivial, as $w$ itself is $k$-bounded by Lemma 5.9.

The step for $l$. Assume that the statement holds for $l = 2, \ldots, r$ where $r \geq 2k - 1$ and consider an irreducible word $w$ of length $r + 1$. By Lemma 2.3 we can write $w = v \cdot u$, where $v$ and $u$ are irreducible words of non-zero lengths.

Now we need to start another induction based on the parameter $j = s(w) = \min\{l(u), l(v)\}$ in order to use the $k$-mixing property of $A$.

1. The base for $j$. Assume $j = 1, \ldots, k - 1$. There are two possibilities.

1.1. Let $l(v) = j$, i.e. $l(u) \geq l(v)$ and $l(v) < k$. Note that $l(u) \geq \frac{j + 1}{2} \geq k \geq 2$, which means we can apply the induction hypothesis for $l$ to $u$. Let $u = f_1 w_1 + \ldots + f_n w_n$ be the resulting decomposition, where $f_i$ are scalar coefficients from $F$, and $w_i$ are $k$-bounded irreducible words. This allows us to write $w = vu = f_1 (vw_1) + \ldots + f_n (vw_n)$.

If $l(u_i) \geq k$, then by Lemma 5.9 there is a sprout sequence of $w_i$ of the form $(v, T)$, where $T$ is a $k$-bounded sprout sequence of $u_i$. This means that $vu_i$ is $k$-bounded as $l(v) < k$ and length of every element of $T$ is less than $k$ as well.

If $l(u_i) \leq k - 1$, then $vu_i$ is $k$-bounded by Lemma 5.9.

Thus, $w = f_1 (vw_1) + \ldots + f_n (vw_n)$ is a representation of the initial word as a linear combination of $k$-bounded words. We will prove that each element $w' = (vw_1)$ in this combination can be reduced to the linear combination of $k$-bounded irreducible words.

1.1.a. If $w'$ is reducible, we can decompose it as a linear combination of irreducible words of lengths less than $r$ and apply the induction hypothesis for $l$ to them.

1.1.b. If $w'$ is irreducible, it is enough to note that it is already $k$-bounded.

1.2. The case $l(u) = j$ is similar, which concludes the proof of the base for $j$.

2. The step for $j$. Assume that the statement holds for values of $j = 1, \ldots, p$ with $p \geq k - 1$. For $j = p + 1$ there are again two possibilities.

2.1. Let $l(v) = p + 1$. Then it holds that $p + 1 \geq k$, which means that $u$ is equal to a product of exactly $k$ words of non-zero length. Let us denote them as $v_1, \ldots, v_k$. Consider the representation of $w = v \cdot u$ as a linear combination of elements of $D_0(u, v_1, \ldots, v_k)$. We will prove that each element $w'$ in this combination can be reduced to a linear combination of $k$-bounded irreducible words.

2.1.a. If $w'$ is reducible, we can decompose it as a linear combination of irreducible words of lengths less than $r$ and apply the induction hypothesis for $l$ to them.
2.1.b. If $w'$ is irreducible and $l(w) \leq r$, then we can apply the induction hypothesis for $l$ to $w'$ directly.

2.1.c. If $w'$ is irreducible and $l(w) = r + 1$, i.e. an element of $R(v \cdot u)$, then $s(w') < s(w)$ by the definition of this set and the induction hypothesis for $j$ can be applied.

2.2. $l(w) = p + 1$ is similar.

This concludes the induction by $j$ and thus by induction by $l$ concludes the proof. □

Proposition 5.12. Let $k \geq 2$ be an integer, $A$ be a $k$-sliding $F$-algebra, $S$ be its generating set. Assume that $w$ is an irreducible word in $S$, $l(w) \geq 2$. Then there exist irreducible $k$-bounded words $w_1, \ldots, w_n$ such that $l(w_i) \leq l(w)$ for each $i = 1, \ldots, n$, and $w \in \langle w_1, \ldots, w_n \rangle$.

Proof. This statement can be demonstrated similarly to the proposition above. □

Corollary 5.13. Let $k \geq 2$ be an integer, $A$ be a $k$-sliding or a $k$-mixing $F$-algebra, $S$ be its generating set. Assume that $w$ is an irreducible word in $S$, $l(w) \geq 2$. Then there exists a $k$-bounded irreducible word in $A$ of the length $l(w)$.

Proof. By successive application of Propositions 5.11 and 5.12, the word $w$ belongs to $(w_1, \ldots, w_n)$, where every $w_i$ is a $k$-bounded irreducible word. Since $w$ is irreducible as well, there exists an index $i$ such that $l(w_i) = l(w)$, as otherwise $w$ would be equal to a linear combination of shorter words. □

6 Malcev algebras

The following definition of Malcev algebras in the multilinear form is equivalent to Definition 1.7 and are already given, see [24].

Proposition 6.1. [24] Proposition 2.21. An algebra $A$ of characteristic not 2 is a Malcev algebra if and only if $A$ satisfies the identities:

1. $xy = -yx$ and
2. $(xy)(zw) = x((wy)z) + w((yz)x) + y((zx)w) + z((xw)y)$

for all $x, y, z, w \in A$.

Now we provide two consequences of this representation that we need in our further considerations.

Lemma 6.2. Let $A$ be a Malcev algebra over a field $F$ with char $F \neq 2$ and $a, b, c, w'$ be its arbitrary elements. It holds that

$$(ab)(cw') = -(ac)(bw') + a((w'b)c) + b((ca)w') + c((aw')b) + a((w'c)b) + c((ba)w') + b((aw')c).$$

(6.1)

Proof. By Proposition 6.1 we can write $(ab)(cw') = a((w'b)c) + w'((bc)a) + b((ca)w') + c((aw')b)$. Note that $w'((bc)a) = -w'((cb)a)$ and by the same proposition $w'((cb)a) = (ac)(bw') - a((w'c)b) - c((ba)w') - b((aw')c)$. By substituting this result in the first equality we obtain the desired identity. □

Lemma 6.3. Let $A$ be a Malcev algebra over a field $F$ with char $F \neq 2$ and $a, b, c, d, w'$ be its arbitrary elements. Then we have

$$(ab)((cd)w') = -(cd)((ab)w') + ((cw')b)(da) - a((bd)(cw')) - b((dc)w')a - d((w'(ab)c) - (ab)((dw')c) - c((db)(aw')) + c(d((w'a)b) + c(b((ad)w')) + c(a((dw')b) - c((dw')(ab)).$$

(6.2)
Proof. By Proposition 6.1 we can write
\[(ab)((cd)w') = (d(ab))(cw') - d((w'(ab))c) - w'((ab)c) - c((dw')(ab)).\] (6.3)

Now we consider the words on the right hand side of this equality.

Note that the first word \((d(ab))(cw') = (cw')(d(ab)).\) By the same proposition, \((cw')(d(ab)) = ((cw')(d(a)) - a((bd)(cw')) - b(d(cw'))a) - d((cw')(ab)).\)

The third word \(w'((ab)c) = w'((ab)c) - d((w'(ab))c)\) and by the same proposition \(w'((ab)c) = (dc)((ab)x') - d((w'(ab))c) - ((ab)w') - (ab)((dw')(c)).\)

Now, \((ab)d = w'((ba)w).\) By the same proposition, \(w'((ba)w) = (db)(aw') - b((ad)w') - a((dw')(b)).\)

By substituting these results in the expression (6.3) and noting that \((cd)((ab)w') = -(dc)((ab)w')\) we obtain the desired identity.

\[\square\]

Corollary 6.4. A Malcev algebra over a field \(F\) with \(\text{char } F \neq 2\) is 3-mixing and has steadily growing length with u-velocity 2.

Proof. By Proposition 6.1 a Malcev algebra \(A\) over a field \(F\) with \(\text{char } F \neq 2\) satisfies
\[(xy)(zw) = x((uw)y)z + w((y)z)x + y((zx)w) + z((xw)y)\]
for all \(x, y, z, w \in A.\) By rearranging words we achieve
\[x((uw)y)z = (xy)zw - w((y)z)x - y((zx)w) - z((xw)y) \in D_0(x, y, z, w).\] (6.4)

Since \(A\) is anticommutative the inclusion \([xy] = (xy)zw - w((y)z)x - y((zx)w) - z((xw)y)\) implies
\[x(z(wy)), ((wy)z)x, (z(wy))x \in D_0(x, y, z, w)\]
for all \(x, y, z, w \in A.\) Thus, we can conclude that \(A\) is 3-mixing by Definition 6.8 which in turns leads to the class having steadily growing length with u-velocity 2 by Theorem 6.7. \(\square\)

The above result can be substantially improved using the notions introduced in Section 5. Namely, our next goal is to demonstrate that Malcev algebras have a slowly growing length.

In the paper \(11\) we proved that 2-mixing and 2-sliding algebras have slowly growing length. This was done by the direct application of a particular case of Theorem 6.4 for \(k = 2,\) see \(11\) Theorem 3.6. However, since an arbitrary Malcev algebra \(A\) is 3-mixing, we cannot guarantee that \(m_j - m_{j-1} \leq 1\) for the characteristic sequence \((m_1, \ldots, m_d)\) of a generating set \(S\) of \(A.\) Indeed, Theorem 6.7 implies just that the inequality \(m_j - m_{j-1} \leq 2\) for each \(j = 1, \ldots, d.\) We are going to demonstrate that in a Malcev algebra if for some \(l, 1 \leq l \leq d\) the equality \(m_l - m_{l-1} = 2\) holds, then \(m_{l-1} = m_{l-2}.\) This implication provides the bound \(l(S) = m_d \leq m_1 + (d - 1) \leq d = \dim A\) since each addition of 2 is preceded by the addition of 0.

To realize this plan we need the following definition.

Definition 6.5. Consider an algebra \(A\) over a field \(F\) and its generating set \(S.\) We say that the words \(u\) and \(v\) in \(S\) are equivalent if \(u, v\) are linearly dependent modulo \(L_{\max(l(u), l(v))} - 1(S)\), i.e., a certain linear combination of \(u\) and \(v\) belongs to the subspace \(L_{\max(l(u), l(v))} - 1(S)\). We denote this as \(u \sim v.\)

For the relation \(\sim\) we can prove several useful properties.

Lemma 6.6. 1. \(\sim\) is an equivalence relation.
2. The word \(u\) is reducible if and only if \(u \sim 0.\)
3. If \(u\) is irreducible and \(u \sim v,\) then \(v\) is also irreducible.
4. If \(u \sim v\) and \(w\) is a word in \(S\) then \(uw \sim vw\) and \(uw \sim vw.\)

Proof. 1. Reflexivity and symmetry of \(\sim\) are evident, and it can be easily checked that it satisfies transitivity as \(u, v\) linearly dependent modulo \(L_h\) implies that they are linearly dependent modulo \(L_{h'},\) where \(h \leq h'.\)
2. Follows directly from Definition 6.7.
3. Follows directly from the first two items.
4. If \( u \sim v \), i.e. \( f_u + f_v + x = 0 \) where \( f_u \) and \( f_v \) belong to \( F \) and are simultaneously zero, while \( x \in \mathcal{L}_{\max(l(u),l(v))−1}(\mathcal{S}) \), then \( f_u(uw) + f_v(wv) + uw = 0 \) and \( f_u(wu) + f_v(wv) + vw = 0 \). Since \( uw \) and \( wv \) belong to \( \mathcal{L}_{\max(l(u),l(v))−1+l(w)} = \mathcal{L}_{\max(l(u)+l(v),l(v)+l(u))−1} \), this implies \( uw \sim vw \) and \( wu \sim vw \).

Thus, the equivalent words behave similarly in terms of being irreducible and generating new irreducible words.

**Lemma 6.7.** Let \( \mathcal{A} \) be a finite dimensional \( F \)-algebra, \( \mathcal{S} \) be its generating set, and \( M = (m_1, \ldots, m_d) \) be the characteristic sequence of \( \mathcal{S} \). The equality \( m_{j−1} = m_{j−2} \) holds if and only if there exist at least two non-equivalent irreducible words of length \( m_{j−1} \) in \( \mathcal{S} \).

**Proof.** By the definition of \( M \), there are exactly \( \dim \mathcal{L}_k(\mathcal{S}) − \dim \mathcal{L}_{k−1}(\mathcal{S}) \) elements in \( M \) that are equal to a given \( k \). The condition \( m_{j−1} = m_{j−2} \) means that there are at least two equal values in the characteristic sequence, so it is equivalent to \( \dim \mathcal{L}_{m_{j−1}}(\mathcal{S}) − \dim \mathcal{L}_{m_{j−1}−1}(\mathcal{S}) \geq 2 \). The last condition means the existence of at least two non-equivalent irreducible words of length \( m_{j−1} \) in \( \mathcal{S} \). \( \Box \)

Now we introduce two concepts specific to the 3-mixing and 3-sliding cases.

**Definition 6.8.** Consider a 3-mixing or a 3-sliding algebra \( \mathcal{A} \) and its generating set \( \mathcal{S} \). For a 3-bounded word \( w \) in \( \mathcal{S} \), \( l(w) \geq 2 \), we define the step function

\[
\sigma(w) = \min \{ t : w \text{ has a subword of length } l−2t−1 \}.
\]

**Lemma 6.9.** The notion of step function is defined correctly.

**Proof.** For a 3-bounded word \( w \) by Definition 6.8 there exists an \( l \)-sprout sequence consisting only of 1 and 2, meaning that \( w \) can be factored into a product of words of lengths 1 and 2. If the last multiplication in \( w \) has a factor of length 1, then \( w \) has a subword of length \( l(w)−1 = l(w)−2·0−1 \). Otherwise \( w \) is a product of words of length 2 and \( l(w)−2 \). If the latter has a factor of length 1, then \( w \) has a subword of length \( l(w)−2 = l(w)−2·1−1 \). Otherwise, \( w \) has a subword of length \( l(w)−2·2 \), to which the same argument can be applied. Since \( l(w) \) is finite, by continuing this process we get a word of length \( l(w)−2t \) for some \( t \geq 0 \) such that it has a subword of length 1. So, \( w \) has a subword of length \( l(w)−2t−1 \). \( \Box \)

**Remark 6.10.** The definition above is formulated only in 3-mixing and 3-sliding cases since by Propositions 5.11 and 5.12 each word in these algebras is a sum of 3-bounded words, so Lemma 6.9 is applicable.

**Definition 6.11.** Let \( \mathcal{A} \) be a 3-mixing or a 3-sliding algebra and \( \mathcal{S} \) be its generating set. If there exists an irreducible 3-bounded word \( w \) of length \( l \) in \( \mathcal{S} \) such that

1. \( \sigma(w) = p \) and
2. \( \sigma(v) \geq p \) for each irreducible 3-bounded words \( v \) of length \( l \),

then \( w \) is called a \( p \)-step word.

The key idea behind these notions stems from the previously discussed concept of the gap \( m_j−m_{j−1}=2 \) and can be formulated as follows.

**Lemma 6.12.** Let \( \mathcal{A} \) be a finite dimensional 3-mixing \( F \)-algebra, \( \mathcal{S} \) be its generating set, and \( M = (m_1, \ldots, m_d) \) be the characteristic sequence of \( \mathcal{S} \). If there exists \( j \) such that \( m_j−m_{j−1}=2 \), then there exists an irreducible 3-bounded \( p \)-step word of length \( m_j \) with \( p \geq 1 \).

**Proof.** Consider an arbitrary 3-bounded irreducible word \( w \) of length \( m_j \). Our aim is to demonstrate that \( \sigma(w) \geq 1 \).

Firstly, as \( m_{j−1} \geq 0 \), we have \( l(w) = m_j \geq 2 \) and \( \sigma(w) \) is well-defined, see Definition 6.8. Additionally this implies \( w = w_1w_2 \), where \( w_1 \) and \( w_2 \) are irreducible words of positive lengths. There are three possibilities:
1. \( l(w_1) < l(w_2) \). In this case \( l(w_1) < 3 \) as all of \( l \)-sprout sequences of the word \( w \) start with \( l(w_1) \) and \( w \) is 3-bounded. However, if \( l(w_1) = 1 \), \( l(w_2) \) would be equal to \( m_j - 1 \). By Corollary 2.4 this would mean that \( m_j - 1 \) belongs to \( M \), which is impossible as \( M \) is non-decreasing and \( m_j - 1 < m_j - 1 \). Thus, \( l(w_1) = 2 \), \( l(w_2) = l(w) - 2 \) and \( w \) does not contain subwords of length \( m_j - 1 = l(w) - 1 \), i.e. \( \sigma(w) > 0 \).

2. \( l(w_2) > l(w_1) \) is similar to the first case.

3. \( l(w_1) = l(w_2) \). In this case \( l(w_1) = l(w_2) < 3 \) as all of \( l \)-sprout sequences of the word \( w \) start with \( l(w_1) = l(w_2) \) and \( w \) is 3-bounded. However, if \( l(w_1) = l(w_2) = 1 \), then \( m_j = l(w) = 2 \), which means that \( m_{j-1} = 0 \). It is impossible for \( M \) to not have elements equal to 1, thus \( l(w_1) = l(w_2) = 2\), \( m_j = 4 \) and \( w \) does not contain subwords of length \( 3 \), \( m_j - 1 \), i.e. \( \sigma(w) > 0 \).

So, in each of the cases \( \sigma(w) > 0 \). Then \( p \geq \sigma(w) \geq 1 \), since \( p \) is the minimal value of \( \sigma(w) \).

Since \( m_j \) is in the characteristic sequence, there exists an irreducible word of length \( m_j \) in \( S \) by Corollary 2.4. By Corollary 5.13 there exists at least one 3-bounded irreducible word of length \( m_j \). From the observations above it follows that the step function of all such words is at least 1. Thus for the word \( w_0 \) with minimal possible value of the step function we have \( \sigma(w_0) \geq 1 \). Finally, it follows from Definition 6.8 that \( w_0 \) is a \( \sigma(w_0) \)-step word.

The following lemmas establish important properties of the step function \( \sigma \).

**Lemma 6.13.** Let \( A \) be a finite dimensional 3-mixing \( \mathcal{F} \)-algebra, \( S \) be its generating set and \( w \) be a word of length \( l \) in \( S \). If \( \sigma(w) = p \geq 0 \) and \( l > 2p + 3 \) then:

1. There exists an \( l \)-sprout sequence \( R(w) \) of \( w \) such that the elements of \( R(w) \) belong to the set \( \{1, 2\} \) and the first \( (p+1) \) elements of this sequence are \( \{2, 2, 2\} \).

2. For every \( j \in \{1, \ldots, p\} \) there exists a subword \( w' \) of \( w \) such that \( l(w') = l - 2j + 2 \) and \( \sigma(w') = p - j + 1 \).

**Proof.** We will prove this statement using induction on \( p \).

The base. For \( p = 1 \) the only possible \( j \) is 1 and Item 2 is trivial. For Item 1 consider an \( l \)-sprout sequence \( R \) of \( w \) which satisfies Definition 6.8. By our choice it contains only the entries 1 and 2. Since \( w \) does not contain subwords of length \( l - 1 \), the first element of \( R \) is 2, meaning that \( w = w \) where \( l(u_j) = 2 \leq l(v) \) or \( l(v) = 2 \leq u \). Assume that the first case holds. As \( w \) has a subword of length \( l - 3 \), by Lemma 5.13 this subword is a subword of \( v \) (as \( u \) is too short), meaning that \( v \) is a product of two words of length \( l - 3 \) and \( (l-2)-(l-3) = 1 \).

In particular, the second element of \( R \) is 1. The second case is similar.

The step. Assume that the statement holds for \( p = 1, \ldots, q \) and consider \( p = q + 1 \), \( q \geq 1 \). Since \( w \) does not contain subwords of length \( l - 1 \) but it is 3-bounded, one of the factors in the last multiplication of \( w \) is a subword of \( w \) with length \( l - 2 \). Denote this subword by \( w_0 \). As a subword of \( w \), \( w_0 \) is also 3-bounded. We will demonstrate that \( \sigma(w_0) = p - 1 \).

Firstly, observe that \( w_0 \) must contain a subword of \( w \) with length \( l - 2p - 1 \) or \( (l - 2) - 2(p - j - 1) - 1 \). This follows from the fact that \( l - 2p - 1 > 2 \), hence the other factor of the last multiplication in \( w \), which has length 2, cannot contain it. Secondly, \( w_0 \) cannot contain subwords of length \( l - 3 \), \( l - 2p + 1 \) as they would be subwords of \( w \) as well. Thus, we can apply the induction hypothesis to \( w_0 \).

For Item 1, since \( l(w_0) = l(w) - 2 = l - 2 \geq 2p + 1 \), by Lemma 5.13 the sequence \( (2, R(w_0)) \) would be an \( l \)-sprout sequence of \( w \) with the first \( p+1 \) elements equal to \( \{2, 2, 2, \ldots, 2\} \), which allows to set \( R(w) = (2, R(w_0)) \).

For Item 2, the case \( j = 1 \) is trivial, the case \( j = 2 \) is already established by \( w_0 \) and for greater \( j \) we can find a subword \( w' \) of \( w_0 \) (which in turn means that \( w' \) is a subword of \( w \) with the length \( l - 2(j - 1) + 2 = l - 2j + 2 \) and \( \sigma(w') = (p - 1) - (j - 1) + 1 = p - j + 1 \).

**Lemma 6.14.** Let \( A \) be a finite dimensional 3-mixing \( \mathcal{F} \)-algebra, \( S \) be its generating set and \( w \) be a word of length \( l \) in \( S \), such that \( w \) is equal to a product of words \( v \) and \( w_0 \) satisfying the following properties:

1. \( v \) has length 2.
2. \( w_0 \) is 3-bounded.
3. \( \sigma(w_0) = p - 1 \)
4. \( l > 2p + 3 \).

Then \( w \) is 3-bounded and \( \sigma(w) = p \).

**Proof.** Consider the \( l \)-sprout sequence \( R(w_0) \) obtained in Lemma \ref{lem:6.18}. By Lemma \ref{lem:6.14} the sequence \( (2, R(w_0)) \) would be an \( l \)-sprout sequence of \( w \), which means that \( w \) is also 3-bounded.

Note that \( w \) contains a subword of length \( l - 2p - 1 \) as its subword \( w_0 \) contains a subword of length \( l - 2p - 2(p - 1) - 1 \). Also it does not contain any subwords of length \( l - 1 \), and subwords of lengths \( l - 3, \ldots, l - 2p + 1 \) would have been subwords of \( w_0 \) by Lemma \ref{lem:6.20} since \( l - 2p > 2 = l(v) \). However, \( w_0 \) does not have subwords of such lengths, which means \( \sigma(w) = p \). \( \Box \)

The final set of lemmas covers properties of 3-bounded words specific to Malcev algebras.

**Lemma 6.15.** Let \( A \) be a Malcev algebra, \( S \) be its generating set and \( w \) be an irreducible 3-bounded word in \( S \). If \( \hat{w} \) is a word obtained from \( w \) using transpositions of factors in the products, then \( \hat{w} \) is also 3-bounded and irreducible, while \( \sigma(w) = \sigma(\hat{w}) \).

**Proof.** As \( \hat{w} = \pm w \) by anticommutativity and has the same length, the word \( \hat{w} \) is also irreducible. Also note that the application of a transposition of factors does not affect the set of possible lengths of subwords or possible \( l \)-sprout sequences. Thus, \( \hat{w} \) is also 3-bounded and \( \sigma(w) = \sigma(\hat{w}) \). \( \Box \)

**Lemma 6.16.** Let \( A \) be a Malcev algebra over a field \( \mathbb{F} \) with \( \text{char } \mathbb{F} \neq 2 \) and \( S \) be its generating set. If \( a, b, c \in S \), \( w' \) is a 3-bounded word of length at least 2 in \( S \), then the following items are true:

- \( \sigma((ab)(cw')) = 1 \).
- All of the summands on the right-hand side of Identity \ref{eq:6.1} are 3-bounded.
- \( \sigma((ac)(bw')) = 1 \).
- \( \sigma(z) = 0 \) where \( z \) is any summand other than \( (ac)(bw') \) on the right-hand side of Identity \ref{eq:6.1}.

**Proof.** It is straightforward to see that \( \sigma((ab)(cw')) = 1 \). Denote by \( R \) a 3-bounded \( l \)-sprout sequence of \( w' \).

We check all the elements on the right-hand side of Identity \ref{eq:6.1} one by one.

1. \( (ac)(bw') \) has the same set of lengths of subwords as \( (ab)(cw') \) and the same possible \( l \)-sprout sequences, which means that it is 3-bounded and the values of their step function are equal. Hence, they are equal to 1.
2. \( a((w'b)c) = 3 \)-bounded as it has an \( l \)-sprout sequence \( (1, 1, 1, R) \). Additionally, \( \sigma(a((w'b)c)) = 0 \).
3. \( b((ca)w') \) is 3-bounded as it has an \( l \)-sprout sequence \( (1, 2, R) \). Additionally, \( \sigma(b((ca)w')) = 0 \).
4. \( c((aw')b) \) is 3-bounded as it has an \( l \)-sprout sequence \( (1, 1, 1, R) \). Additionally, \( \sigma(c((aw')b)) = 0 \).
5. \( a((w'c)b) \) is 3-bounded as it has an \( l \)-sprout sequence \( (1, 1, 1, R) \). Additionally, \( \sigma(a((w'c)b)) = 0 \).
6. \( c((ba)w') \) is 3-bounded as it has an \( l \)-sprout sequence \( (1, 2, R) \). Additionally, \( \sigma(c((ba)w')) = 0 \).
7. \( b((aw')c) \) is 3-bounded as it has an \( l \)-sprout sequence \( (1, 1, 1, R) \). Additionally, \( \sigma(b((aw')c)) = 0 \).

**Lemma 6.17.** Let \( A \) be a Malcev algebra over a field \( \mathbb{F} \) with \( \text{char } \mathbb{F} \neq 2 \) and \( S \) be its generating set. If \( a, b, c, d \in S \), and \( w' \) is a 3-bounded word of length at least 2 in \( S \) and \( \sigma((ab)((cd)w')) = p \) then the following items are true:

- All of the summands on the right-hand side of the identity \ref{eq:6.2} are 3-bounded.
Lemma 6.18. \(s\) is any summand other than \((cd)((ab)w')\) in the right-hand side of the identity [6.2].

Proof. It is easy to see that \(p \geq 2\). Denote by \(R\) a 3-bounded \(l\)-sprout sequence of \(w'\). We consider all the elements on the right-hand side of [6.2] one by one and check that each statement holds true for each word.

1. \((cd)((ab)w')\) has the same set of lengths of subwords as \((ab)((cd)w')\) and the same possible \(l\)-sprout sequences, which means that it is 3-bounded and the values of step function are equal.

2. \((cw')b)(da)\) is 3-bounded as it has an \(l\)-sprout sequence \((2, 1, 1, R)\). Additionally, \(\sigma(((cw')b)(da)) = 1\).

3. \(a((bd)(cw'))\) is 3-bounded as it has an \(l\)-sprout sequence \((1, 2, 1, R)\). Additionally, \(\sigma(a((bd)(cw'))) = 0\).

4. \(b((cd)(cw'))a)\) is 3-bounded as it has an \(l\)-sprout sequence \((1, 1, 1, R)\). Additionally, \(\sigma(b((cd)(cw'))a) = 0\).

5. \(d((w')(ab))c\) is 3-bounded as it has an \(l\)-sprout sequence \((1, 1, 2, R)\). Additionally, \(\sigma(d((w')(ab))c) = 0\).

6. \((ab)((dw')c)\) is 3-bounded as it has an \(l\)-sprout sequence \((2, 1, 1, R)\). Additionally, \(\sigma((ab)((dw')c)) = 1\).

7. \(c((db)(aw'))\) is 3-bounded as it has an \(l\)-sprout sequence \((2, 1, 1, R)\). Additionally, \(\sigma(c((db)(aw'))) = 0\).

8. \(c(d((w'a)b))\) is 3-bounded as it has an \(l\)-sprout sequence \((1, 1, 1, R)\). Additionally, \(\sigma(c(d((w'a)b)) = 0\).

9. \(c(b((ad)(w')))\) is 3-bounded as it has an \(l\)-sprout sequence \((1, 1, 2, R)\). Additionally, \(\sigma(c(b((ad)(w'))) = 0\).

10. \(c(a((ad)(w')))\) is 3-bounded as it has an \(l\)-sprout sequence \((1, 1, 1, R)\). Additionally, \(\sigma(c(a((ad)(w'))) = 0\).

11. \(c((dw')(ab))\) 3-bounded as it has an \(l\)-sprout sequence \((2, 1, 1, R)\). Additionally, \(\sigma(c((dw')(ab))) = 0\).

So, each condition is justified, and the lemma is proved. \(\square\)

Lemma 6.18. Let \(A\) be a Malcev algebra over a field \(F\) with \(\text{char } F \neq 2\), \(S\) be its generating set and \(w\) be a \(p\)-step word of length \(l\) in \(S\) with \(p \geq 2\) which has the form 
\[
w = (s_1s_2)((s_3s_4)\ldots((s_{2p-1}s_{2p})(s_{2p+1}w'))\ldots),
\]
where \(s_i \in \mathcal{S}\) and \(w'\) has the length \((l - 2p - 1)\). Then the word 
\[
(s_3s_4)\ldots((s_{2p-1}s_{2p})(s_{2p+1}w'))\ldots
\]
is also \(p\)-step and equivalent to \(w\).

Proof. We proceed by the induction on the index \(t\) such that 
\[
w = (s_3s_4)\ldots((s_{2t+1}s_{2t+2})(s_1s_2)(s_{2t+3}s_{2t+4})\ldots((s_{2p-1}s_{2p})(s_{2p+1}w'))\ldots)
\]
is a \(p\)-word and equivalent to \(w\) for all \(t = 2, \ldots, p - 1\).

The base. For \(t = 1\) we have \(w_1 = (s_3s_4)((s_1s_2)\ldots((s_{2p-1}s_{2p})(s_{2p+1}w'))\ldots)\). By Lemma 6.3 \(w = w_1 + x\), where \(x\) is a linear combination of words with the value of step function being 0 or 1. As \(w\) is a \(p\)-step this means that summands of \(x\) are reducible and \(x \in L_{l-1}\). Thus, \(w_1 \sim w\), and \(w_1\) is irreducible by Lemma 6.4. Additionally, for the word \(v_1 = (s_3s_4)\ldots((s_{2p-1}s_{2p})(s_{2p+1}w'))\ldots\) we have \(\sigma(v_1) = p - 2\) by Lemma 6.2, Item 2, as it is the only subword of \(w\) of the length \(l - 4\). Applying Lemma 6.14 to \((s_1s_2)v_1\) and \(w_1 = (s_3s_4)((s_1s_2)v_1)\) we get that the word \(w_1\) is 3-bounded and \(\sigma(w_1) = p\), which means \(w_1\) is also a \(p\)-step word.

The step. Assume that the statement holds for \(t = T\), with \(T < p - 1\). For \(t = T + 1\) consider the word \(w_{T+1} = (s_3s_4)\ldots((s_{2T+3}s_{2T+4})(s_1s_2)(s_{2T+5}s_{2T+6})\ldots((s_{2p-1}s_{2p})(s_{2p+1}w'))\ldots)\).
The word \( w_{T+1} \) has a subword \( w_{T+1} = (s_{2T+3}s_{2T+4})((s_1s_2)v_{T+1}) \), where
\[
v_{T+1} = (s_{2T+5}s_{2T+6})((s_{2p-1}s_{2p})(s_{2p+1}w')) \ldots .
\]
Note that by the induction hypothesis the word
\[
w_T = (s_3s_4)(\ldots ((s_{2T+3}s_{2T+4})((s_1s_2)((s_{2T+5}s_{2T+6})((s_{2p-1}s_{2p})(s_{2p+1}w')) \ldots ))
\]
is a \( T \)-step and is equivalent to \( w \). The word \( w_T \) has its own subword of length equal to \( l(w_{T+1}) \), that is \( (s_1s_2)(\ldots ((s_{2T+3}s_{2T+4})\ldots )) =: w'_T \). By Lemma 6.13 Item 2, \( \sigma(w'_T) = p - T \geq 2 \) as it is the only subword of \( w_T \) of length \( l - 2T \).

By Lemma 6.3
\[
w'_T = w_{T+1} + f_1x_1 + \ldots + f_{r}x_{r}
\] (6.5)

where \( f_i = \pm 1 \) and \( x_i \) are the other summands from Identity 6.2. By Lemma 6.17 each of \( x_i \) is \( 3 \)-bounded and \( \sigma(x'_i) = p - T > 1 \geq \sigma(x_i) \), while \( \sigma(w_{T+1}) = p - T \). By sequentially multiplying the equality \( \sigma(w'_{T+1}) \) by \((s_{2T+1}s_{2T+2}), \ldots, (s_{3}s_4)\) on the left we would get \( w_T = w_{T+1} + f_1x_1 + \ldots + f_{r}x_{r} \), where all words are \( 3 \)-bounded and \( \sigma(w_{T+1}) = p \). For each \( i \in \{1, \ldots, r\} \) we can consider sequence \( x^{(j)}_i \), defined inductively: \( x^{(0)}_i = x_i \) and \( x^{(j+1)}_i = (s_{2T-1-2j}s_{2T+2-2j})x^{(j)}_i \), \( j \leq T - 1 \). It is straightforward to see that \( X_i = x^{(T)}_i \).

By Lemma 6.14 \( \sigma(x^{(j+1)}_i) = \sigma(x^{(j)}_i) + 1 \), meaning that \( \sigma(X_i) = \sigma(x^{(T)}_i) = \sigma(x^{(T-1)}_i) + 1 = \ldots = \sigma(x^{(0)}_i) + T = \sigma(x_i) + T < p - T + T = p \).

Since \( \sigma(X_i) < p \), these words cannot be irreducible and \( w_{T+1} \sim w_T \). Thus, \( w_{T+1} \) is a \( T \)-step word and it is equivalent to \( w \).

This induction allows us to conclude that the word
\[
(s_3s_4)(\ldots ((s_{2p-1}s_{2p})(s_1s_2)(s_{2p+1}w'))) \ldots
\]
corresponding to \( t = p - 1 \) is a \( T \)-step word and is equivalent to \( w \).

After all the preparations above we are ready to prove the key result of this section.

**Proposition 6.19.** Let \( A \) be a Malcev algebra over a field \( \mathbb{F} \) with char \( \mathbb{F} \neq 2 \), \( S \) be its generating set and \( w \) be a \( T \)-step word in \( S \) of length \( l \) with \( p \geq 2 \). Then there are at least two irreducible words of length \( l - 2 \) linearly independent modulo \( L_{1-3} \).

**Proof.** Firstly since \( w \) is a \( p \)-word, by Lemma 6.13 it has an \( l \)-sprout sequence starting with \( p + 1 \) elements equal to \((2, \ldots, 2, 1)\). Thus, using the anticommutativity and Lemma 6.15 we can construct the word
\[
\bar{w} = (s_1s_2)((s_3s_4)((s_{2p-1}s_{2p})(s_{2p+1}w'))) \ldots,
\] (6.6)

where \( s_i \in S \) and \( w' \) has length \( l - 2p - 1 \), such that \( \bar{w} = w \) in \( A \) and \( \bar{w} \) has the same length and value of the step function as \( w \). This means that now we can assume that \( w \) is already in the form \( (6.6) \).

Consider firstly the case \( p = 1 \). This means \( w = (s_1s_2)(s_3w') \). By Lemma 6.2 we get \( w = (s_1s_3)(s_2w') + f_1x_1 + \ldots + f_{r}x_{r} \), where \( f_i = \pm 1 \) and \( x_i \) are the other summands from the right-hand side of Identity 6.11. Note that by Lemma 6.16 we have \( \sigma(x_i) = 0 < 1 = \sigma(w) \), which means that \( x_i \) is reducible for each \( i = 1, \ldots, r \). Thus, the word \( (s_1s_3)(s_2w') \) is irreducible.

Note that the words \( s_2w' \) and \( s_3w' \) are irreducible as subwords of irreducible words. Assume that \( s_2w' \) and \( s_3w' \) are linearly independent modulo \( L_{1-3} \). Hence by Lemma 6.18 the word \( (s_1s_2)(s_2w') \) is irreducible. However \( (s_1s_2)(s_3w') = s_1(s_2w' + (s_{2p+1}w') + s_2(s_1w' - s_2)) \), where the second word on the right-hand side is equal to zero as \( s_{2p+1} = 0 \) and other words are reducible as the value of the step function on them is again 0. Thus the assumption is incorrect and \( s_2w' \) and \( s_3w' \) are linearly independent modulo \( L_{1-3} \).

Now we consider the case \( p \geq 2 \). Consider the given word in the form \( (6.6) \):
\[
w = (s_1s_2)((s_3s_4)((s_{2p-1}s_{2p})(s_{2p+1}w'))) \ldots
\]

By Lemma 6.18 the word \( w_T = (s_3s_4)((s_{2p-1}s_{2p})(s_{2p+1}w'))) \ldots \) is a \( T \)-step word equivalent to \( w \). By Lemma 6.2
\[
(s_1s_2)(s_{2p+1}w') = (s_1s_{2p+1}) + f_1y_1 + \ldots + f_{r}y_{r}
\] (6.7)
where \( f_i = \pm 1 \) and \( y_i \) are the other summands from the right-hand side of Identity \ref{eq:identity}

Note that by Lemma \ref{lem:property}
a0 all of the words on the right hand side are 3-bounded, and \( \sigma((s_1 s_2)(s_{2p+1}) s_2 w') = 1 \), while \( \sigma(y_i) = 0 \). By sequentially multiplying the equality \ref{eq:identity} by \((s_{2p-1} s_{2p}) \ldots , (s_3 s_4) \) on the left we would get \( w_p = w'_p + f_i Y_i + \ldots + f_r Y_r \), where \( w'_p = (s_3 s_4) \ldots ((s_1 s_{2p}) (s_2 w')) \ldots \), all words are 3-bounded and \( \sigma(w'_p) = p \).

Moreover, by Lemma \ref{lem:bound} applied \( p-1 \) times \( \sigma(Y_i) = \sigma(y_i) + p - 1 = p - 1 < p \). Thus all \( Y_i, i = 1, \ldots , r \), are reducible, and \( w'_p \) is irreducible.

Assume that \((s_1 s_4) \ldots ((s_1 s_2)(s_{2p+1}) s_2 w')) \ldots \) (the irreducible subword of \( w \) of length \( l-2 \)) and \((s_5 s_6) \ldots ((s_1 s_{2p+1}) s_2 w')) \ldots \) (the irreducible subword of \( w'_p \) of length \( l-2 \)) are linearly dependent modulo \( L_{l-3} \). By Lemma \ref{lem:linear} it follows that \((s_1 s_2)((s_5 s_6) \ldots ((s_1 s_{2p+1}) s_2 w')) \ldots \) is an irreducible word. However, as this word has the same set of lengths of subwords as \( w \), it also has the same value of the step function. By applying Lemma \ref{lem:irreducible} to this word we would get \((s_5 s_6) \ldots ((s_1 s_2)(s_2 w')) \ldots \) is an irreducible word. Thus its subword \((s_1 s_2)(s_2 w') \) would be irreducible, which is incorrect due to the same reasoning as in the case \( p = 1 \). Thus the assumption is false and the two words are not linearly dependent modulo \( L_{l-3} \).

\( \blacksquare \)

**Theorem 6.20.** Let \( \mathcal{A} \) be a Malcev algebra over a field \( \mathbb{F} \) with char \( \mathbb{F} \neq 2 \). Then it holds that \( l(A) \leq d - 1 \) where \( d = \text{dim} \ A \). If \( A \) is not Lie algebra, then it holds that \( l(A) \leq d - 2 \).

**Proof.** Consider a generating set \( S \) of algebra \( \mathcal{A} \) such that \( l(S) = l(A) \). If \( \text{dim} \ L_1(S) \leq 2 \), then by Proposition \ref{prop:dim} \( \mathcal{A} \) is a Lie algebra. In this case the result follows from \cite{11} Proposition 4.7.

Thus below we assume \( \text{dim} \ L_1(S) \geq 3 \).

Consider the characteristic sequence \( M = (m_1, \ldots , m_d) \) of \( S \).

Note that
\[ m_d = m_3 + (m_4 - m_3) + \ldots + (m_d - m_{d-1}). \]

(6.8)

Since a Malcev algebra is 3-mixing, by Lemma \ref{lem:mixing} we have \( m_j - m_{j-1} \leq 2 \). As the characteristic sequence is non-decreasing by Definition \ref{def:characteristic} \( m_j - m_{j-1} \geq 0 \). Thus, for \( j \in \{4, \ldots , d\} \) we have \( m_j - m_{j-1} \in \{0, 1, 2\} \).

Define the sets \( I_0 = \{j|m_j - m_{j-1} = 0\}, I_1 = \{j|m_j - m_{j-1} = 1\}, I_2 = \{j|m_j - m_{j-1} = 2\} \subseteq \{4, \ldots , d\} \) and denote their cardinalities by \( J_0 = |I_0|, J_1 = |I_1|, J_2 = |I_2| \). By the observation above, \( I_0 \cup I_1 \cup I_2 = \{4, \ldots , d\} \), thus \( J_0 + J_1 + J_2 = d - 3 \). Additionally, by grouping respective terms of the equality \ref{eq:identity} we get
\[ m_d = m_3 + J_0 + J_1 + 2J_2, \]

and, since \( m_3 = 1 \), this means
\[ m_d = 1 + 0J_0 + 1J_1 + 2J_2. \]

Assume \( j \in I_2 \), i.e. \( m_j - m_{j-1} = 2 \). In this case we have \( j \geq 5 \) as \( m_3 = m_4 = m_1 = 1 \), and either \( m_4 = 1 \) or \( m_4 \geq 2 \). By Corollary \ref{cor:mixed} \( m_4 \) is equal to the sum of two previous elements, i.e. \( m_4 = 2 \). By Lemma \ref{lem:mixed} \( m_j - m_{j-1} = 2 \) implies that there exists an irreducible \( p \)-step word of length \( m_1 \) in \( S \) with \( p \geq 1 \). Thus by Proposition \ref{prop:irreducible} there are at least two non-equivalent words of length \( m_i - 1 = m_i - 2 \), which implies \( m_{j-1} = m_{j-2} \) by Lemma \ref{lem:irreducible}.

Thus, for each \( j \in I_2 \) we have \( j - 1 \in I_0 \) and the inequality \( J_2 \leq J_0 \) is true, which allows us to conclude by Lemma \ref{lem:characteristic} that
\[ l(A) = l(S) = m_d = 1 + J_1 + 2J_2 \leq 1 + J_0 + J_1 + J_2 = d - 2. \]

\( \blacksquare \)

**Data availability statement**

All data generated or analyzed during this study are included in this published article.

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