Noether Gauge Symmetry Approach in Quintom Cosmology

Adnan Aslam\textsuperscript{1} Mubaher Jamil\textsuperscript{1} Davood Momeni \textsuperscript{2}

and

Ratbay Myrzakulov \textsuperscript{2} Muneer Ahmad Rashid\textsuperscript{1} Muhammad Raza \textsuperscript{3}

Center for Advanced Mathematics and Physics (CAMP), National University of Sciences and Technology (NUST), H-12, Islamabad, Pakistan

Eurasian International Center for Theoretical Physics, Eurasian National University, Astana 010008, Kazakhstan

Department of Mathematics, COMSATS Institute of Information Technology (CIIT), Sahiwal Campus, Pakistan

Received \;\; accepted
ABSTRACT

In literature usual point like symmetries of the Lagrangian have been introduced to study the symmetries and the structure of the fields. This kind of Noether symmetry is a subclass of a more general family of symmetries, called Noether Gauge Symmetries (NGS). Motivated by this mathematical tool, in this article, we discuss the generalized Noether symmetry of Quintom model of dark energy, which is a two component fluid model of quintessence and phantom fields. Our model is a generalization of the Noether symmetries of a single and multiple components which have been investigated in detail before. We found the general form of the quintom potential in which the whole dynamical system has a point like symmetry. We investigated different possible solutions of the system for diverse family of gauge function. Specially, we discovered two family of potentials, one corresponds to a free quintessence (phantom) and the second is in the form of quadratic interaction between two components. These two families of potential functions are proposed from the symmetry point of view, but in the quintom models they are used as phenomenological models without clear mathematical justification. From integrability point of view, we found two forms of the scale factor: one is power law and second is de-Sitter. Some cosmological implications of the solutions have been investigated.

Subject headings: Cosmology; Noether symmetries; Quintom fields; Dynamical systems; Cosmography; Local stability
1. Introduction

Einstein gravity inspired from the equivalence principle, follows the Mach’s principle as the matter creates the geometry or space time concept. There is no reasonable and clear motive to believe that Einstein gravity must work beyond the solar system and compatible to describe the large scale structure of the whole universe, as well as the gravity in compact objects and solar system. Just on an adhoc basis and by assuming that the equivalence principle also works on large scale, the relativistic cosmology of Einstein gravity has been constructed. Because gravity as a theory for gravitation treats like a gauge theory, cosmology based on such a gauge theory of gravity is basically a highly non-linear system of differential equations. It means even there is no uniqueness principle or theorem for the solutions of the equations. This is a weak point or very bad freedom in the theory as a mathematical point of view. Symmetry is a key point for non linear differential equations. Modern approach is how to find the general point like symmetries of a given Lagrangian (holonomic or non-holonomic). This appropriate powerful method to find and investigate the solutions of linear (non linear) dynamical systems reduces the numbers of the unknown functions by construction of invariants, the quantities which remain invariance under gauge transformations. Cosmological models are dynamical systems with attractors and integrable families. It means, if we start from an initial value of fields then the cosmological time evolutionary scheme, in the infinite time domain, asymptotically tends to a state which is free from the initial conditions. The generic behavior of this type also appears in the other models of cosmology based on modifications of Einstein gravity.

Noether symmetry is a point like symmetry of Lagrangian and is defined on a tangent space of the configuration coordinates and their adjoint momenta. Recently, it has been applied widely in Cosmology from modified curvature gravities to Einstein-Cartan theories (see Capozziello et al. 2012 for a brief review).
The motivation to apply Noether symmetry for cosmological models is that in scalar fluid models of dark exotic fluid, quintessence, phantom or quintom, the potential function is not a unique and known function of fields. In fact, to choose a suitable and physically acceptable potential function, we have two major approaches: One is to select a potential function by reconstruction of a special cosmological behavior following a well known model like LCDM or phantom fluids. This is reconstruction method and can be used as a useful tool for fixing the potential function. Second method is that we have used some phenomenological facts invited from high energy physics. For example, Higgs bosonic models have been used in inflationary scenarios and the potential function is in the form that it has spontaneous symmetry breaking and also renormalizable in the quantum model. Beyond these two major links to the potential function in scalar field models, the better approach is to find the potential function by restriction of a general symmetry. Any symmetry in a Lagrangian defines a conserved charge and from these conservation laws, we can reduce the numbers of unknown functions, especially, we are able to find the explicit forms of potential function which posses a definite symmetry. Noether symmetry or a more generalized form of it, Noether gauge symmetry gives us such opportunity to reduce the system to a first order system of partial differential equations (PDEs) for generators. The solution of such first order PDEs give us the form of generators, conserved charges and especially the form of the interaction potential function.

Back to the cosmology, we know that numerous astrophysical observations indicate that the observable universe is undergoing accelerated expansion. This feature has not a unique full consistent description as a theoretical or phenomenological model. Another associated startling feature of this expansion is that the state parameter of dark energy is dynamical which progressively evolves from sub-negative values to super-negative values (i.e. cross of cosmological constant boundary $w = -1$), a phenomenon called the Phantom Crossing. This phantom crossing has been reported in many scenarios of
cosmology based on the dark energy. Recently modified theories of gravity proposed and one can find numerous theories of modified gravity in the literature, such as $f(R)$ ($R$ is the Ricci scalar) (Nojiri et al. 2011; Nojiri et al. 2006), $f(R,G)$ where $G$ is Gauss-Bonnet invariant, $f(R,T)$ gravity ($T$ is the trace of energy-momentum tensor) (Harko et al. 2011; Jamil et al. 2012b; Sharif et al. 2012), $f(R,L_m)$ gravity where $L_m$ is the matter Lagrangian (Lobo et al. 2012; Harko et al. 2013) and other ones.

Numerous exact solutions of black holes, wormholes and cosmological models have been found in these theories. Recently, the Noether symmetries of $f(R)$ and $f(\tau)$ gravity (where $\tau$ is the torsion of space) (Jamil et al. 2013a; Maluf 2013; Jamil et al. 2013b; Jamil et al. 2012c; Jamil et al. 2012d; Jamil et al. 2012e; Sadjadi 2012; Darabi 2012) have been investigated which help in restricting the astrophysically viable forms of these functions (Jamil et al. 2011; Hussain et al. 2012; Jamil et al. 2012f; Aslam et al. 2013; Jamil et al. 2012a; Shamir et al. 2012). Specifically, by adopting the Noether symmetry approach, power-law forms ($f(R) \sim R^n$ and $f(\tau) \sim \tau^m$, where $m$ and $n$ are finite constants of order unity) appear (see references in (Jamil et al. 2011; Hussain et al. 2012; Jamil et al. 2012f; Aslam et al. 2013; Jamil et al. 2012a; Shamir et al. 2012)).

One of the these alternative models and the most simple ones is scalar field models, in which we trust to the time evolution of the scalar field as a solver of the dark energy problem. In the literature, it has been introduced the idea of Quintom, in agreement with the current observations (Feng et al. 2005; Elizalde et al. 2004). Later, their model was constrained and fitted with the data of microwave background, supernova and galaxy clustering (Xia et al. 2005; Li et al. 2006). The quintom idea was then extended in other gravitational setups such as loop quantum cosmology, braneworlds, string theory and Gauss-Bonnet gravity (Wei et al. 2007; Saridakis et al. 2008a; Saridakis 2008b; Sadeghi et al. 2008; Zhang et al. 2010). Recently, wormhole solutions have been studied in Quintom scenario...
Quintom model has a generic unknown potential function of double fields which remains as a phenomenological function and time evolution of the model is not able to give any information on the form of it. In this paper, we calculate the Noether gauge symmetries of the quintom Lagrangian in the physical background of Friedmann-Robertson-Walker (FRW) space-time. We will propose two cases of potential with this kind of symmetry.

In this paper, our plan is as follows: In section-II, we write down the Lagrangian, the idea of Noether gauge symmetries and the corresponding system of equations. In section-III, we solve the system of differential equations under some assumptions and constraints on the model parameters and calculate the Noether symmetries and the first integrals. We discuss stability in section-IV. Finally we conclude in last section.

2. Model

We begin from the simple double field quintom action in four dimensions (Elizalde et al. 2004), (Feng et al. 2005):

$$S = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ R + \phi,\mu \phi^{;\mu} - \sigma,\mu \sigma^{;\mu} - 2V(\phi, \sigma) \right].$$  \hspace{1cm} (1)  

Here $g$ denotes the determinant of the Riemannian metric, $\{\phi, \sigma\}$ are the pair of fields and $V(\phi, \sigma)$ is potential function of fields which is an unknown function of the model. By observational data the spatial curvature of the space is negligible. So we adopt the spatially flat FRW metric as the following

$$g_{\mu\nu} = diag(1, -a^2(t)\Sigma_3).$$

Here $\Sigma_3$ is metric of three dimensional Euclidean space. We can rewrite the following field equations

$$2\frac{\dddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -p,$$  \hspace{1cm} (2)
\[
\dot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \\
\dot{\sigma} + 3H\dot{\sigma} - \frac{dV}{d\sigma} = 0.
\]

Here the effective pressure and dark energy read

\[
\rho = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\sigma}^2 + V(\phi, \sigma), \\
p = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\sigma}^2 - V(\phi, \sigma).
\]

The point like Lagrangian of (1) is

\[
L(a, \dot{a}, \phi, \dot{\phi}, \sigma, \dot{\sigma}) = -3a\ddot{a} + a^3(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\sigma}^2 - V(\phi, \sigma)).
\]

Lagrangian is holonomic and it posses time translation as trivial symmetry.

We define a vector field

\[
X = T \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \phi} + \gamma \frac{\partial}{\partial \sigma}
\]

is a Noether gauge symmetry (NGS) of the Lagrangian, if

\[
X\mathcal{L} + \mathcal{L}D_t T = D_t G,
\]

where the coefficients \( T, \alpha, \beta, \gamma \) and gauge function \( G \) (all functions of \( (a, \phi, \sigma) \)) are determined from the Noether symmetry conditions.

Symmetries of the Lagrangian (Noether symmetries), also called the symmetries of the action integral, are very important as these give double reduction in the order of the corresponding Euler-Lagrange equation and also provide conserved quantities (Stephani 1989; Bluman 2010; Ibragimov 1999; Aslam et al. 2012).

Applying (7) in (9) we obtain the following system of the PDEs

\[
T_a = 0, T_\phi = 0, T_\sigma = 0,
\]
\[ \beta_\sigma - \gamma_\phi = 0, \]
\[ 6\alpha_\sigma - a^2\gamma_a = 0, \]
\[ -6\alpha_\phi + a^2\beta_a = 0, \]
\[ a^3\gamma_t + G_\sigma = 0, \]
\[ a^3\beta_t - G_\phi = 0, \]
\[ 6a\alpha_t + G_a = 0, \]
\[ \alpha + 2a\alpha_a - a\mathcal{T}_t = 0, \]
\[ 3\alpha + 2a\beta_\phi - a\mathcal{T}_t = 0, \]
\[ 3\alpha + 2a\gamma_\sigma - a\mathcal{T}_t = 0, \]
\[ 3a^2V\alpha + a^3\beta_\phi + a^3\gamma_\sigma + a^3V\mathcal{T}_t + G_t = 0. \] (10)

This is a linear system of partial differential equations. The interaction potential \( V(\phi, \sigma) \) will be determined by the above partial differential equations. This potential is in fact a phenomenological form with parameters which can be adjusted using the data and in favor of fitting to the astrophysical values. As we know the form of \( V(\phi, \sigma) \) is just a typical function and in a classical level it has any arbitrary form. Restriction of the form of \( V(\phi, \sigma) \) is one of the most important results of this paper. We will show how NGS helps us to fix the form of \( V(\phi, \sigma) \) without any reference to the phenomenological facts.

3. Solutions

Finding all solutions of (10) is a hard job. Although, we have constraints on the forms of the functions but the number of functions and freedom to fix them is wide. In this section, we will study some particular solutions of (10) in different cases. In each case, we will find the corresponding generators and show the closed algebra of the generators.
Case 1: For the gauge function to be arbitrary constant, in which the (10) is integrable for set of functions to give the following particular solutions

\[ G = \text{constant} \]
\[ V = V(\phi, \sigma) = \text{arbitrary} \]

This case gives us no information about the generic form of interaction potential between quintom components. Moreover, in this case explicitly, the generators can not be obtained directly from (10). In this special case with a constant gauge, the only possible symmetry

\[ X = \frac{\partial}{\partial t}. \]

It defines the time translation symmetry corresponds to the conservation of first integral (energy) of system. This case is the minimal symmetry of system and here we do not gain much more information than before. This corresponds to the families of potential which people put as adhoc phenomenological functions. Due to arbitrary of the gauge such models are gauge invariant in the language of symmetry.

Case 2: As we can check easily, system given by (10) has another very important solution for potential even when we set gauge as constant. The second non trivial family is

\[ G = \text{constant}, \]
\[ V = V(\phi, \sigma) = F\left(\frac{1}{2}c_1(\sigma^2 - \phi^2) + c_2 \sigma - c_3 \phi\right). \]

The following form of interaction reported before as a viable model of acceleration expansion in the frame of quintom models (Cai et al. 2010). It corresponds to two family of quadratic potentials. In spite of the previous case, here the dynamical behavior of whole system is determined by a closed set of equations of motion. The corresponding Noether symmetries are

\[ X_1 = \frac{\partial}{\partial t}. \]
\[
X_2 = (c_1 \sigma + c_2) \frac{\partial}{\partial \phi} + (c_1 \phi + c_3) \frac{\partial}{\partial \sigma}.
\]

These symmetry generators form the simple commutative algebra

\[
[X_1, X_2] = 0. \tag{12}
\]

This family has the crossing phantom line, stable attractors and also a gauge invariance description. We will study it more later.

**Case 3:** With constant gauge there also exists another family, in which we treat one scalar field to be free and another to move in a specific potential field. The family of solutions here refers to a noninterative quintom models, and also, the generic form of the interaction as a single value function remains undetermined. Just to report the result we write here

\[
G = \text{constant},
\]

\[
V = V(\phi, \sigma) = F(\phi).
\]

One possibility is to take \(F(\phi) \sim \phi^2\) and investigate the dynamical behavior of fields.

In this case, the corresponding Noether symmetries are

\[
X_1 = \frac{\partial}{\partial t},
\]

\[
X_2 = \frac{\partial}{\partial \sigma},
\]

that form the commutative algebra

\[
[X_1, X_2] = 0. \tag{13}
\]

The case if on a free phantom, a constraint quintessence field is subjected to an unknown potential function \(F(\phi)\). Due to the free phantom \(\sigma\) it is not very interesting as a double scalar field model.
Case 4: One special non-trivial gauge is that when we consider a time translational (boost) symmetry through the gauge function. The simple linear time translation admits the following form of gauge and a non trivial interaction function

\[ G = c_1 t + c_2, \]
\[ V = V(\phi, \sigma) = -c_1 \phi + F(\sigma). \]

The first term in interaction potential acts as a linear term and has no non-trivial contribution to the dynamical behavior of the system. Although it exerts a constant force field but we can absorb it inside the fields equation by redefining the scalar field as \( \phi \rightarrow \phi + c_1 t^2 / 2 \). In this case, the Lagrangian becomes time dependent and the Hamiltonian is not conserved. It implies a friction in the system due to the existence of a time dependence Hamiltonian. Here the corresponding Noether symmetry is

\[ X = \frac{\partial}{\partial t} + \frac{1}{a^2} \frac{\partial}{\partial \phi}. \]  

(14)

Case 5: By a weak rotation scheme if we change \( t \rightarrow \phi \), another non zero gauge has been obtained as the following

\[ G = c_1 \phi + c_2, \]
\[ V = V(\phi, \sigma) = F(\sigma). \]

It defines a single mode dynamical quintom model and also, here we cannot fix the interaction function. We have a gauge freedom and one possibility is to obtain it by taking \( F(\sigma) \sim \sigma^2 \). The corresponding Noether symmetries are

\[ X_1 = \frac{\partial}{\partial t}, \]
\[ X_2 = (c_1 \frac{t}{a^3} + F(a)) \frac{\partial}{\partial \phi}. \]
and the corresponding Lie algebra is

\[ [X_1, X_2] = c_1 \frac{1}{a^3} \frac{\partial}{\partial \phi} \]  

(15)

This case is the \( \sigma \to \phi \) version of the case 4. By the same reason we are not interesting to it.

**Case 6:** The case of constant gauge has the following possibility:

\[
G = \text{constant},
\]
\[
V = V(\phi, \sigma) = \text{constant}.
\]

It corresponds to two free scalar degrees and cannot explain the cosmological behavior of the model in accelerated expansion era. We have the corresponding Noether symmetries:

\[
X_1 = \frac{\partial}{\partial t},
\]
\[
X_2 = \sigma \frac{\partial}{\partial \phi} + \phi \frac{\partial}{\partial \sigma},
\]
\[
X_3 = \frac{\partial}{\partial \sigma},
\]
\[
X_4 = F(a) \frac{\partial}{\partial \phi}.
\]

The closed complete commutative algebra corresponding to these Noether symmetries is

\[
[X_1, X_2] = [X_1, X_3] = [X_1, X_4] = [X_2, X_4] = [X_3, X_4] = 0,
\]
\[ (16) \]

\[
[X_2, X_3] = -\frac{\partial}{\partial \phi}.
\]
\[ (17) \]

4. **Exact solutions**

In a dynamical system, when the symmetries are fixed by a tool like NGS, which we applied here, the next reasonable question is how to find the exact solutions for different
fields using the symmetry generators. Especially Noether symmetries are useful when we can find the interaction potential functions of the system. If the generators form a complete non commutative and fully associative algebra, it is possible to find exact solutions due to the wide class of symmetries of the original Lagrangian. In our case, because the system has six different families of symmetries, so we only examine two simple cases to find exact solutions for scale factor and fields.

For this purpose, we find the invariants and scale parameter for the case (4) for $c_1 = 0$. As, there is only one Noether symmetry, so the corresponding conserved quantity or invariant is

$$I = 3a\ddot{a} - \frac{1}{2}a^3\dot{\phi}^2 + \frac{1}{2}a^3\dot{\sigma}^2 - a^3F(\sigma) + \dot{\phi}. \quad (18)$$

$$I$$ is a conserved quantity, so it must be equal to some constant $c$ say. Field equations (2)-(4) must satisfy this constraint. We have the following solutions of the field equations (2)-(4), for the constant $c = 0$

$$\phi = c_3$$
$$a = c_2 \exp(\pm \sqrt{6}/3 \cdot t)$$
$$F = 2$$
$$\sigma = c_1$$
and for the constant \( c \neq 0 \)

\[
\begin{align*}
\phi &= c_2 \\
a &= \frac{1}{2} (6c)^{\frac{1}{4}} t^{\frac{3}{4}} \\
F &= 0 \\
\sigma &= \int \frac{\sqrt{2a(c - 3a\dot{a}^2)} dt}{a^2} + c_1,
\end{align*}
\]

where \( c_1, c_2 \) and \( c_3 \) are constants. The solutions, which we have found here, have interesting cosmological implications. For example, the exponential scale factor denotes a de-Sitter epoch and the power law family corresponds to a fluid with equation of state of stiff fluid \( w = -2 \). In the later case, the explicit form of field \( \sigma \) reads as the following:

\[
\sigma(t) = c_1 + \frac{2\sqrt{2}}{3} \log t.
\] (20)

The corresponding Hubble parameter is:

\[
H(t) = \frac{2}{3t}.
\] (21)

The model mimics LCDM model.

5. Cosmography

The form of the quintom interaction given in Case2, is very interesting. In this section, we want to find the full numerical time evolutionary scheme of the model, with this potential. Before the full cosmography analysis based on different scenarios for evolution of the universe in an accelerating universe has been investigated\(^{[\text{Bamba et al. 2012}]}.\) Attractors and cosmological predictions have been investigated in details. As a very special case, due to the specific form of the potential function which we obtained by NGS, we will study cosmological predictions of our restricted model as a special two fluids scenario.
for dark energy. So, in our case we have the explicit form of \( V(\phi, \sigma) \), so the numerical analysis is done more easily. In our case, specifically we want to know, how the scale factor, quintessence and phantom fields evolve in time when we fix the form of the potential as Case2, by symmetry. Further, as a cosmological result, we want to study the behaviors of deceleration parameter and EoS parameter, given by \( w_{eff} = \frac{p_{eff}}{\rho_{eff}} \) as a function of time. In (12) we make choice \( F = X \) as an arbitrary function, also for numerical reasons we choose \( c_1 = 1, c_2 = c_3 = \frac{1}{2} \), so that the potential becomes

\[
V(\sigma, \phi) = \frac{1}{2}(\sigma - \phi)(\sigma + \phi + 1).
\]

The system of equations of motion becomes

\[
\ddot{\phi} + 3H\dot{\phi} - \frac{1}{2}(1 + \phi) = 0,
\]

\[
\ddot{\sigma} + 3H\dot{\sigma} - \frac{1}{2}(1 + \sigma) = 0,
\]

\[
2\dot{H} + 3H^2 + \frac{1}{2}(\dot{\phi}^2 - \dot{\sigma}^2) - \frac{1}{2}(\sigma - \phi)(\sigma + \phi + 1) = 0,
\]

which can be integrated numerically for a set of functions \( [H, \phi, \sigma] \). We put the following initial condition: \( t = 0, H(t) = H_0, \phi(0) = 1, \dot{\phi}(0) = 0.2, \sigma(0) = -0.2, \dot{\sigma}(0) = -1 \).

Here we give the illustration of graphs:

From Fig. 1, we deduce that the behavior of Hubble parameter closely mimics the standard cold dark matter model with vacuum energy. The Hubble parameter has the highest value in the early Universe while it suddenly decreases.

In Fig. 2, we plot \( \phi \) and \( \sigma \) over cosmic time. Asymptotically their evolution becomes the same, while for small values of time, \( \phi \) approaches zero (initial state) while \( \sigma \) approaches -5.

We plotted the equation of state parameter against cosmic time in Fig. 3. Now this \( w \) parameter has contributions from the pressure and energy density of both quintessence and
Fig. 1.— (Top Left) Plot of $H$ vs $t$. It resembles LCDM model. (Top Right) Plot of $[\phi, \sigma]$ vs $t$. (Middle Left) Plot of effective EoS $w$ vs $t$. It denotes phantom cross line clearly. (Middle Right) Plot of energy density $\rho$ vs $t$ for fields.
phantom fields. For the suitable choice of parameters, it appears that $w$ begins to evolve from a highly negative value and gradually rising to positive values. During this stage, it obviously crosses the $w = -1$ boundary. It turns out that $w$ follows an oscillatory behavior which gradually diminishes until $t = 7$. We remark that this value of time does not coincide with the present time since the choice of parameters is kept arbitrary. It is noticeable that $w$ remains less than $-1$ at large time scales.

From Fig. 4, we observe that the total conserved energy density increases with time globally. However locally, there are fluctuations in the energy density over time. It turns out that energy density increases when the phantom scalar field dominates, while it decreases when the quintessence field decays. For large cosmic time, the energy density is dominated by the phantom energy. we also observe that the total energy density increases with time globally. However locally, there are fluctuations in the energy density over time. It turns out that energy density increases when the phantom scalar field dominates while it decreases when the quintessence field decays. For large cosmic time, the energy density dominates.

6. Stability

In double fields model stability problem has been studied before (Ito et al. 2012). We are interesting in studying the local stability of the system. Formally, we want to know that the de-Sitter space is stable or not. We write the system of the dynamical equations in the following autonomous system of equations:

\[
\begin{align*}
\dot{x}_1 &= -3Hx_1 + \frac{1 + \phi}{2}, \\
\dot{x}_2 &= -3Hx_2 + \frac{1 + \sigma}{2}, \\
\dot{\sigma} &= x_1,
\end{align*}
\]
\[ \dot{\phi} = x_2, \quad (28) \]
\[ \dot{H} = -\frac{1}{2} \left( 3H^2 + \frac{1}{2}(x_1^2 - x_2^2) - \frac{1}{2}(\sigma - \phi)(\sigma + \phi + 1) \right). \quad (29) \]

Stationary (critical point) of the system is:

\[ X_c = (x_1 = 0, x_2 = 0, \phi = -1, \sigma = -1, H = 0). \]

The model with \( H = 0 \) is a sub case of de-Sitter so called as static model or Einstein universe. We perturb the system around this critical point up to first order to find:

\[ \delta \ddot{\chi} - \frac{1}{2} \delta \chi = 0, \quad \chi = \sigma - \phi, \quad (30) \]
\[ \delta \dot{H} + \frac{1}{4} \delta \chi = 0. \quad (31) \]

General solution of the model is written as the following:

\[ \delta \chi = \delta \chi_0 \sin(t/\sqrt{2} + \theta_0), \quad \delta H = \frac{\sqrt{2}}{4} \delta \chi_0 \cos(t/\sqrt{2} + \theta_0). \]

Oscillatory solutions indicate that the model is stable under small perturbation in first order. So our quintom scenario predicts the stability scenario of de-Sitter case.

7. Conclusions

To explain the acceleration expansion era of the current universe quintom model proposed as a double field model, contained of two fields, quintessence and phantom field. The typical form of the potential function \( V(\sigma, \phi) \) remains as a phenomenological unknown function. In this article, we studied Noether gauge symmetry of quintom cosmology with arbitrary potentials via Noether Gauge symmetry, as a generalization of the popular Noether symmetry. Such an approach has been previously investigated in the quintom cosmology for dynamical system analysis. We presented the Noether symmetry generators for a typical interactive quintom model. In general, we obtained six families of quintom
models based on different generators. In case 1, we took the gauge field constant and with the time translation symmetry. Family 2, corresponded to two families of quadratic potentials where the dynamical behavior of whole system determined by a closed set of equations of motion. In this family $V(\phi, \sigma) = F(\frac{1}{2}c_1(\sigma^2 - \phi^2) + c_2\sigma - c_3\phi)$. For this case, we performed cosmography analysis in details. It is interesting to note that the behavior of Hubble parameter closely mimics the standard cold dark matter model with vacuum energy. Furthermore, the total energy density increases with time monotonically, globally. Also, numerically, we deduced that EoS parameter $w$ began to evolve from a highly negative value and gradually rising to positive values. Also we showed that Einstein space is stable as a special case. During this stage, it obviously crosses the $w = -1$ boundary. So, we extend the cosmology of quintom models beyond the phenomenological potential functions by a new approach of Noether Gauge Symmetry. Further, as we know the equivalent presentation of quintom theory gives specific fluid with a pair of effective energy density and pressure. Consequently, the exact solutions for the generators and the potential function found, remain to be solutions for field equations in the other equivalent description via fluids with effective quantities. In this new representation, conservation of energy in terms of the Liouville’s form is $\frac{D\rho_{\text{eff}}}{Dt} = 0$ where it is equivalent to the usual conservation equation $\rho_{\text{eff}}' + 3H(\rho_{\text{eff}} + p_{\text{eff}}) = 0$. In this equivalent form, NGS of point like scalar field Lagrangian now represent a kind of the symmetry in fluid description. Existence of a closed algebra between the generators defines a sub class of Lie symmetries. This Lie symmetry now transfers to the fluid’s equivalent description and so the corresponding Noether charges (fluid’s invariants) remain conservative for effective fluid. So, the quintom model inspired from the Noether gauge symmetry is able to give a set of reasonable predictions. Finally, we mention here that this method may be applied to generalized quintom model introduced in (Nojiri et al. 2005). In such generalized quintom models by using NGS, we will be able to find explicitly the form of the potential function.
The work of M. Jamil is supported from the financial grant No. 20-2166/NRPU/RD/HEC/12-5699 of the Higher Education Commission, Pakistan.
REFERENCES

Aslam, A., Jamil, M., Momeni, D., Myrzakulov, R.: Canadian J. Phys. 91, 93 (2013) 
[arXiv:1212.6022]

Aslam, A., Qadir, A.: Journal of applied Mathematics, 532690, 14 (2012)

Bamba, K., Capozziello, S., Nojiri, S., Odintsov, S.D.: Astrophys.Space Sci. 342 (2012) 155-228 [arXiv:1205.3421]

Bluman, G. W., Cheviakov, A. F. Anco, S. C.: Applications of Symmetry Methods to Partial Differential Equations, Springer Science, New York, 2010

Cai, Y. F., Saridakis, E. N., Setare, M. R., Xia, J. Q.: Phys. Rept. 493, 1 (2010) [arXiv:0909.2776 [hep-th]]

Capozziello, S., De Laurentis, M., Odintsov, S. D.: Eur. Phys. J. C 72, 2068 (2012) [arXiv:1206.4842 [gr-qc]]

Darabi, F.: [arXiv:1207.0212] [gr-qc] (2012)

Elizalde, E., Nojiri, S., Odintsov, S.D.: Phys. Rev. D 70 (2004) 043539 [hep-th/0405034]

Feng, B., Wang, X., Zhang, X.: Phys. Lett. B 607, 35 (2005)

Harko, T., Lobo, F. S.N., Nojiri, S., Odintsov, S. D.: Phys. Rev. D 84, 024020 (2011)

Harko, T., Lobo, F. S. N., Minazzoli, O.: Phys. Rev. D 87, 047501 (2013)

Hussain, I., Jamil, M., Mahomed, F. M.: Astrophys. Space Sci. 337, 373 (2012)

Ibragimov, N.H.: Elementary Lie Group Analysis and Ordinary Differential Equations, John Wiely, 1999

Ito, Y., Nojiri, S., Odintsov, S. D: [arXiv:1111.5389]
Jamil, M., Ali, S., Momeni, D., Myrzakulov, R.: Eur. Phys. J. C 72, 1998 (2012a)
Jamil, M., Mahomed, F. M., Momeni, D.: Phys. Lett. B 702, 315 (2011)
Jamil, M., Momeni, D., Myrzakulov, R.: Chin. Phys. Lett. 29, 109801 (2012b)
Jamil, M., Momeni, D., Myrzakulov, R.: Eur. Phys. J. C 73, 2267 (2013a)
Jamil, M., Momeni, D., Myrzakulov, R.: Gen. Relat. Grav. 45, 263 (2013b)
Jamil, M., Momeni, D., Myrzakulov, R., Rudra, P.: J. Phys. Soc. Jpn. 81, 114004 (2012c)
Jamil, M., Momeni, D., Myrzakulov, R.: Eur. Phys. J. C 72, 2122 (2012d)
Jamil, M., Momeni, D., Myrzakulov, R.: Eur. Phys. J. C 72, 1959 (2012e)
Jamil, M., Momeni, D., Myrzakulov, R.: Eur. Phys. J. C 72, 2137 (2012f)
Kuhfittig, P. K.F., Rahaman, F., Ghosh, A.: Int. J. Theor. Phys. 49, 1222 (2010)
Lazkoz, R., LeûZn, G., Quiros, I.: Phys. Lett. B 649, 103 (2007)
Li, H., Feng, B., Xia, J-Q., Zhang, X.: Phys. Rev. D 73, 103503 (2006)
Lobo, F. S. N., Harko, T.: arXiv:1211.0426 [gr-qc] (2012)
Maluf, J.W.: arXiv:1303.3897 [gr-qc] (2013)
Nojiri, S., Odintsov, S.D.: Gen. Rel. Grav. 38 (2006) 1285 [hep-th/0506212]
Nojiri, S., Odintsov, S.D.: eConf C 0602061 (2006) 06 [Int. J. Geom. Meth. Mod. Phys. 4 (2007) 115] [hep-th/0601213]
Nojiri, S., Odintsov, S.D.: Phys. Rept. 505, 59 (2011)
Sadeghi, J., Setare, M. R., Banijamali, A., Milani,F.: Phys. Lett. B 662, 92 (2008)
Sadjadi, H. M.: Phys. Lett. B 718, 270 (2012)

Saridakis, E.N.: JCAP 0804, 020 (2008a)

Saridakis, E.N.: Phys. Lett. B 661, 335 (2008b)

Shamir, M. F., Jhangeer, A., Bhatti, A. A.: Chin. Phys. Lett. 29, 080402 (2012)

Sharif, M., Zubair, M.: JCAP 03, 028 (2012)

Stephani, H.: Differential Equations: Their Solutions Using Symmetry, Cambridge University Press, New York, 1989

Wei, H., Zhang, S. N.: Phys. Rev. D 76, 063005 (2007)

Xia, J-Q., Feng, B., Zhang, X.: Mod. Phys. Lett. A 20, 2409 (2005)

Zhang, J., Gui, Y-X.: Commun. Theor. Phys. 54, 380 (2010)