A FAMILY OF SUMMATION FORMULAS INVOLVING GENERALIZED HARMONIC NUMBERS

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ABSTRACT. Combining the derivative operator with a binomial sum from the telescoping method, we establish a family of summation formulas involving generalized harmonic numbers.

1. INTRODUCTION

For \( x \in \mathbb{C} \) and \( l, n \in \mathbb{N}_0 \), define the functions \( H_{n}^{(l)}(x) \) by

\[
H_{0}^{(l)}(x) = 0 \quad \text{and} \quad H_{n}^{(l)}(x) = \sum_{k=1}^{n} \frac{1}{(x+k)^l} \quad \text{with} \quad n = 1, 2, \ldots .
\]

Fixing \( x = 0 \) in the functions just mentioned, we obtain the generalized harmonic numbers:

\[
H_{0}^{(l)}(0) = 0 \quad \text{and} \quad H_{n}^{(l)}(0) = \sum_{k=1}^{n} \frac{1}{k^l} \quad \text{with} \quad n = 1, 2, \ldots .
\]

When \( l = 1 \), they reduce to the classical harmonic numbers:

\[
H_{0}^{(1)} = 0 \quad \text{and} \quad H_{n}^{(1)} = \sum_{k=1}^{n} \frac{1}{k} \quad \text{with} \quad n = 1, 2, \ldots .
\]

There exist many elegant identities involving generalized harmonic numbers. They can be found in the papers [1]-[8].

For a differentiable function \( f(x) \), define the derivative operator \( D_x \) by

\[
D_x f(x) = \frac{d}{dx} f(x).
\]

Then it is not difficult to show the following two derivatives:

\[
D_x \left( \frac{x+n}{n} \right) = \left( \frac{x+n}{n} \right) H_n(x),
\]

\[
D_x H_n^{(l)}(x) = -l H_n^{(l+1)}(x).
\]
For a complex sequence \( \{ \tau_k \}_{k \in \mathbb{Z}} \), define the difference operator by
\[
\nabla \tau_k = \tau_k - \tau_{k-1}.
\]
Then we have the following relation:
\[
\nabla \left( \frac{(y+k+1)}{(x+k)} \right) = \frac{(y+k)}{(x+k)} \frac{y - x + 1}{y + 1}.
\]
Combining the last equation and the telescoping method:
\[
\sum_{k=1}^{n} \nabla \tau_k = \tau_n - \tau_0,
\]
we get the simple binomial sum:
\[
\sum_{k=1}^{n} \left( \frac{y+k}{n} \right) = \left( \frac{y+n+1}{n} \right) \frac{y + 1}{y - x + 1} - \frac{y + 1}{y - x + 1}.
\]

By means of the derivative operator \( D_x \) and the binomial sum \((\text{I})\), we shall explore systematically closed expressions for the family of sums:
\[
\sum_{k=1}^{n} k^i H_k^{(l)}(x) \quad \text{with} \quad i, l \in \mathbb{N}_0.
\]
When \( x = p \) with \( p \in \mathbb{N}_0 \), they give closed expressions for the following sums:
\[
\sum_{k=1}^{n} k^i H_p^{(l+k)}.
\]

### 2. Summation Formulas

**Theorem 1.** For \( x \in \mathbb{C} \) and \( l \in \mathbb{N}_0 \), there holds the summation formula:
\[
\sum_{k=1}^{n} H_k^{(l+1)}(x) = (x + n + 1)H_n^{(l+1)}(x) - H_n^{(l)}(x).
\]

**Proof.** Applying the derivative operator \( D_x \) to \((\text{I})\), we achieve the identity:
\[
\sum_{k=1}^{n} \left( \frac{y+k}{x+k} \right) H_k(x) = \frac{y + 1}{y - x + 1} \left( \frac{y+n+1}{n} \right) \left( H_n(x) - \frac{1}{y - x + 1} \right) + \frac{y + 1}{(y - x + 1)^2}.
\]
Letting \( y = x \) in \((2)\), we attain the case \( l = 0 \) of Theorem \(\text{I}\)
\[
\sum_{k=1}^{n} H_k(x) = (x + n + 1)H_n(x) - n.
\]
Suppose that the following identity
\[
\sum_{k=1}^{n} H_k^{(l+1)}(x) = (x + n + 1)H_n^{(l+1)}(x) - H_n^{(l)}(x)
\]
is true. Applying the derivative operator \( D_x \) to the last equation, we have
\[
\sum_{k=1}^{n} H_k^{(l+2)}(x) = (x + n + 1)H_n^{(l+2)}(x) - H_n^{(l+1)}(x).
\]
This proves Theorem \(\text{I}\) inductively. \(\square\)

Making \( x = p \) in Theorem \(\text{I}\) we get the following equation.
Corollary 2. For \( l, p \in \mathbb{N}_0 \), there holds the summation formula:

\[
\sum_{k=1}^{n} H_{p+k}^{(l+1)} = (p + n + 1)H_{p+n}^{(l+1)} - (p + 1)H_p^{(l+1)} - H_{p+n}^{(l)} + H_p^{(l)}.
\]

3. Summation formulas with the factor \( k \)

Setting \( y = x + 1 \) in (2) and considering the relation:

\[
\sum_{k=1}^{n} \frac{(x+1)^k}{k} H_k(x) = \sum_{k=1}^{n} H_k(x) + \sum_{k=1}^{n} kH_k(x),
\]

we gain the following equation by using Theorem 1.

**Proposition 3.** For \( x \in \mathbb{C} \), there holds the summation formula:

\[
\sum_{k=1}^{n} kH_k(x) = \frac{(x + n + 1)(n - x)}{2} H_n(x) + \frac{(2x - n + 1)n}{4}.
\]

**Corollary 4** \((x = p \text{ with } p \in \mathbb{N}_0 \text{ in Proposition 3})\):

\[
\sum_{k=1}^{n} kH_{p+k} = \frac{(n - p)(p + n + 1)}{2} H_{p+n} + \frac{p(p + 1)}{2} H_p - \frac{n(n - 2p - 1)}{4}.
\]

**Theorem 5.** For \( x \in \mathbb{C} \) and \( l \in \mathbb{N}_0 \), there holds the summation formula:

\[
\sum_{k=1}^{n} kH_k^{(l+2)}(x) = \frac{(x + n + 1)(n - x)}{2} H_n^{(l+2)}(x) + \frac{2x + 1}{2} H_n^{(l+1)}(x) - \frac{H_n^{(l)}(x)}{2}.
\]

**Proof.** Applying the derivative operator \( D_x \) to Proposition 3, we achieve the case \( l = 0 \) of Theorem 5:

\[
\sum_{k=1}^{n} kH_k^{(2)}(x) = \frac{(x + n + 1)(n - x)}{2} H_n^{(2)}(x) + \frac{2x + 1}{2} H_n(x) - \frac{n}{2}.
\]

Suppose that the following identity

\[
\sum_{k=1}^{n} kH_k^{(l+2)}(x) = \frac{(x + n + 1)(n - x)}{2} H_n^{(l+2)}(x) + \frac{2x + 1}{2} H_n^{(l+1)}(x) - \frac{H_n^{(l)}(x)}{2},
\]

is true. Applying the derivative operator \( D_x \) to the last equation, we have

\[
\sum_{k=1}^{n} kH_k^{(l+3)}(x) = \frac{(x + n + 1)(n - x)}{2} H_n^{(l+3)}(x) + \frac{2x + 1}{2} H_n^{(l+2)}(x) - \frac{H_n^{(l+1)}(x)}{2}.
\]

This proves Theorem 5 inductively.

Taking \( x = p \) in Theorem 5, we attain the following equation.

**Corollary 6.** For \( l, p \in \mathbb{N}_0 \), there holds the summation formula:

\[
\sum_{k=1}^{n} kH_{p+k}^{(l+2)} = \frac{(p + n + 1)(n - p)}{2} H_{p+n}^{(l+2)} + \frac{p(p + 1)}{2} H_p^{(l+2)} + \frac{2p + 1}{2} \left( H_{p+n}^{(l+1)} - H_p^{(l+1)} \right) - \frac{H_{p+n}^{(l)} - H_p^{(l)}}{2}.
\]
4. Summation formulas with the factor $k^2$

Letting $y = x + 2$ in (2) and considering the relation:

$$
\sum_{k=1}^{n} \frac{(x+2)^k}{(x+k)} H_k(x) = \sum_{k=1}^{n} H_k(x) + \frac{2x + 3}{(x+1)(x+2)} \sum_{k=1}^{n} k H_k(x)
$$

$$
+ \frac{1}{(x+1)(x+2)} \sum_{k=1}^{n} k^2 H_k(x),
$$

we get the following equation by using Theorem 1 and Proposition 3.

**Proposition 7.** For $x \in \mathbb{C}$, there holds the summation formula:

$$
\sum_{k=1}^{n} k^2 H_k(x) = \frac{x(x+1)(2x+1) + n(n+1)(2n+1)}{6} H_n(x)
$$

$$
- \frac{(12x^2 + 12x - 6xn + 4n^2 - 3n - 1)n}{36}.
$$

**Corollary 8** ($x = p$ with $p \in \mathbb{N}_0$ in Proposition 7).

$$
\sum_{k=1}^{n} k^2 H_{p+k} = \frac{(p+1)(2p+1)(2p^2 + n - 2pn + p + 2p^2)}{6} H_{p+n}
$$

$$
- \frac{p(p+1)(2p+1)}{6} H_p - \frac{n(4n^2 - 3n + 6pn + 12p + 12p^2 - 1)}{36}.
$$

Applying the derivative operator $\mathcal{D}_x$ to Proposition 7, we gain the following equation.

**Proposition 9.** For $x \in \mathbb{C}$, there holds the summation formula:

$$
\sum_{k=1}^{n} k^2 H_k^{(2)}(x) = \frac{x(x+1)(2x+1) + n(n+1)(2n+1)}{6} H_n^{(2)}(x)
$$

$$
- \frac{6x^2 + 6x + 1}{6} H_n(x) + \frac{(4x + 2 - n)n}{6}.
$$

**Corollary 10** ($x = p$ with $p \in \mathbb{N}_0$ in Proposition 9).

$$
\sum_{k=1}^{n} k^2 H_{p+k}^{(2)} = \frac{p(p+1)(2p+1) + n(n+1)(2n+1)}{6} H_{p+n}^{(2)} - \frac{p(p+1)(2p+1)}{6} H_p^{(2)}
$$

$$
- \frac{6p^2 + 6p + 1}{6} (H_{p+n} - H_p) + \frac{(4p + 2 - n)n}{6}.
$$

**Theorem 11.** For $x \in \mathbb{C}$ and $l \in \mathbb{N}_0$, there holds the summation formula:

$$
\sum_{k=1}^{n} k^2 H_k^{(l+3)}(x) = \frac{x(x+1)(2x+1) + n(n+1)(2n+1)}{6} H_n^{(l+3)}(x)
$$

$$
- \frac{6x^2 + 6x + 1}{6} H_n^{(l+2)}(x) + \frac{2x + 1}{2} H_n^{(l+1)}(x) - \frac{H_n^{(l)}(x)}{3}.
$$

**Proof.** Applying the derivative operator $\mathcal{D}_x$ to Proposition 9, we achieve the case $l = 0$ of Theorem 11.

$$
\sum_{k=1}^{n} k^2 H_k^{(3)}(x) = \frac{x(x+1)(2x+1) + n(n+1)(2n+1)}{6} H_n^{(3)}(x)
$$

$$
- \frac{6x^2 + 6x + 1}{6} H_n^{(2)}(x) + \frac{2x + 1}{2} H_n(x) - \frac{n}{3}.
Suppose that the following identity
\[
\sum_{k=1}^{n} k^2 H_k^{(l+3)}(x) = \frac{x(x+1)(2x+1) + n(n+1)(2n+1)}{6} H_n^{(l+3)}(x)
\]
\[
- \frac{6x^2 + 6x + 1}{6} H_n^{(l+2)}(x) + \frac{2x + 1}{2} H_n^{(l+1)}(x) - \frac{H_n^{(l)}}{3}
\]
is true. Applying the derivative operator \(D_x\) to the last equation, we have
\[
\sum_{k=1}^{n} k^2 H_k^{(l+4)}(x) = \frac{x(x+1)(2x+1) + n(n+1)(2n+1)}{6} H_n^{(l+4)}(x)
\]
\[
- \frac{6x^2 + 6x + 1}{6} H_n^{(l+3)}(x) + \frac{2x + 1}{2} H_n^{(l+2)}(x) - \frac{H_n^{(l+1)}}{3}.
\]
This proves Theorem 13 inductively.

Making \(x = p\) in Theorem 13 we attain the following equation.

**Corollary 14.** For \(l, p \in \mathbb{N}_0\), there holds the summation formula:
\[
\sum_{k=1}^{n} k^2 H_k^{(l+3)} = \frac{p(p+1)(2p+1) + n(n+1)(2n+1)}{6} H_p^{(l+3)}
\]
\[
- \frac{p(p+1)(2p+1)}{6} H_p^{(l+3)} - \frac{6p^2 + 6p + 1}{6} \left( H_{p+n}^{(l+2)} - H_p^{(l+2)} \right)
\]
\[
+ \frac{2p + 1}{2} \left( H_{p+n}^{(l+1)} - H_p^{(l+1)} \right) - \frac{H_{p+n}^{(l)} - H_p^{(l)}}{3}.
\]

5. **Summation Formulas with the Factor \(k^3\)**

Setting \(y = x + 3\) in (2) and considering the relation:
\[
\sum_{k=1}^{n} \left( \frac{x+k}{k} \right) H_k(x) = \sum_{k=1}^{n} H_k(x) + \frac{3x^2 + 12x + 11}{(x+1)(x+2)(x+3)} \sum_{k=1}^{n} kH_k(x)
\]
\[
+ \frac{3}{(x+1)(x+3)} \sum_{k=1}^{n} k^2 H_k(x)
\]
\[
+ \frac{1}{(x+1)(x+2)(x+3)} \sum_{k=1}^{n} k^3 H_k(x),
\]
we get the following equation by using Theorem 13, Proposition 3 and Proposition 7.

**Proposition 13.** For \(x \in \mathbb{C}\), there holds the summation formula:
\[
\sum_{k=1}^{n} k^3 H_k(x) = \frac{(n-x)(x+n+1)(x^2 + x + n + n^2)}{4} H_n(x)
\]
\[
- \frac{(12x^3 + 18x^2 - 6nx^2 + 2x - 6xn + 4n^2x - 2 + 3n + 2n^2 - 3n^3)n}{48}.
\]

**Corollary 14 (\(x = p\) with \(p \in \mathbb{N}_0\) in Proposition 13).**
\[
\sum_{k=1}^{n} k^3 H_{p+k} = \frac{(n-p)(p+n+1)(p^2 + p + n + n^2)}{4} H_{p+n} + \frac{p^2(p+1)^2}{4} H_p
\]
\[
- \frac{(12p^3 + 18p^2 + 2p - 6pn + 4n^2p - 2 + 3n + 2n^2 - 3n^3)n}{48}.
\]
Corollary 18. For $x = p$ with $p \in \mathbb{N}_0$ in Proposition 15,

$$
\sum_{k=1}^{n} k^3 H_p^{(2)}(x) = \frac{(n-p)(p+n+1)(p^2 + p + n + n^2)}{4} H_p^{(2)} + \frac{p(p+1)(2p+1)}{2} (H_{p+n} - H_p) - \frac{6p^2 + 6p + 1}{4} (H_{p+n} - H_p) + \frac{6p + 3 - n}{8}.
$$

Applying the derivative operator $D_x$ to Proposition 15, we achieve the following equation.

**Proposition 16.** For $x = p$ with $p \in \mathbb{N}_0$ in Proposition 15,

$$
\sum_{k=1}^{n} k^3 H_p^{(2)}(x) = \frac{(n-p)(p+n+1)(p^2 + p + n + n^2)}{4} H_p^{(2)} + \frac{p(p+1)(2p+1)}{2} (H_{p+n} - H_p) - \frac{6p^2 + 6p + 1}{4} (H_{p+n} - H_p) + \frac{6p + 3 - n}{8}.
$$

Applying the derivative operator $D_x$ to Proposition 17, we gain the following equation.

**Proposition 17.** For $x = p$ with $p \in \mathbb{N}_0$ in Proposition 17.

$$
\sum_{k=1}^{n} k^3 H_p^{(3)}(x) = \frac{(n-p)(p+n+1)(p^2 + p + n + n^2)}{4} H_p^{(3)} + \frac{p(p+1)(2p+1)}{2} (H_{p+n} - H_p) - \frac{6p^2 + 6p + 1}{4} (H_{p+n} - H_p) + \frac{6p + 3 - n}{8}.
$$

Applying the derivative operator $D_x$ to Proposition 17, we attain the case $l = 0$ of Theorem 19.

**Theorem 19.** For $x \in \mathbb{C}$ and $l \in \mathbb{N}_0$, there holds the summation formula:

$$
\sum_{k=1}^{n} k^3 H_k^{(l+4)}(x) = \frac{(n-x)(x+n+1)(x^2 + x + n + n^2)}{4} H_n^{(l+4)}(x) + \frac{x(x+1)(2x+1)}{2} H_n^{(l+3)}(x) - \frac{6x^2 + 6x + 1}{4} H_n^{(l+2)}(x) + \frac{2x + 1}{4} H_n^{(l+1)}(x) - \frac{n}{4}.
$$

**Proof.** Applying the derivative operator $D_x$ to Proposition 17, we attain the case $l = 0$ of Theorem 19.
Suppose that the following identity
\[
\sum_{k=1}^{n} k^3 H_k^{(l+4)}(x) = \frac{(n-x)(x+n+1)(x^2 + x + n + n^2)}{4} H_n^{(l+4)}(x) + \frac{x(x+1)(2x+1)}{2} H_n^{(l+3)}(x) - \frac{6x^2 + 6x + 1}{4} H_n^{(l+2)}(x) + \frac{2x+1}{2} H_n^{(l+1)}(x) - \frac{H_n^{(l)}}{4}
\]
is true. Applying the derivative operator \(D_x\) to the last equation, we have
\[
\sum_{k=1}^{n} k^3 H_k^{(l+5)}(x) = \frac{(n-x)(x+n+1)(x^2 + x + n + n^2)}{4} H_n^{(l+5)}(x) + \frac{x(x+1)(2x+1)}{2} H_n^{(l+4)}(x) - \frac{6x^2 + 6x + 1}{4} H_n^{(l+3)}(x) + \frac{2x+1}{2} H_n^{(l+2)}(x) - \frac{H_n^{(l+1)}}{4}.
\]
This proves Theorem 19 inductively.

Taking \(x = p\) in Theorem 19 we get the following equation.

**Corollary 20.** For \(l, p \in \mathbb{N}_0\), there holds the summation formula:
\[
\sum_{k=1}^{n} k^3 H_{p+k}^{(l+4)} = \frac{(n-p)(p+n+1)(p^2 + p + n + n^2)}{4} H_{p+n}^{(l+4)} + \frac{p^2(p+1)^2}{4} H_p^{(l+4)} + \frac{p(p+1)(2p+1)}{2} (H_{p+n}^{(l+3)} - H_p^{(l+3)}) - \frac{6p^2 + 6p + 1}{4} (H_{p+n}^{(l+2)} - H_p^{(l+2)}) + \frac{2p+1}{2} (H_{p+n}^{(l+1)} - H_p^{(l+1)}) - \frac{H_{p+n}^{(l)}}{4} + \frac{H_p^{(l)}}{4}.
\]

6. **Summation formulas with the factor \(k^4\)**

Letting \(y = x + 4\) in (2) and considering the relation:
\[
\sum_{k=1}^{n} \frac{(x+4+k)}{(x+k)} H_k(x) = \sum_{k=1}^{n} H_k(x) + \frac{2(2x + 5)(x^2 + 5x + 5)}{(x+1)(x+2)(x+3)(x+4)} \sum_{k=1}^{n} k H_k(x)
\]
\[
+ \frac{6x^2 + 30x + 35}{(x+1)(x+2)(x+3)(x+4)} \sum_{k=1}^{n} k^2 H_k(x)
\]
\[
+ \frac{4x + 10}{(x+1)(x+2)(x+3)(x+4)} \sum_{k=1}^{n} k^3 H_k(x)
\]
\[
+ \frac{1}{(x+1)(x+2)(x+3)(x+4)} \sum_{k=1}^{n} k^4 H_k(x),
\]
we gain the following equation by using Theorem 1, Proposition 3, Proposition 7 and Proposition 13.
Proposition 21. For $x \in \mathbb{C}$, there holds the summation formula:

$$
\sum_{k=1}^{n} k^4 H_k(x) = \frac{6x^5 + 15x^4 + 10x^3 - x - n + 10n^3 + 15n^4 + 6n^5}{30} H_n(x) \\
- \frac{(72n^4 - 45n^3 - 130n^2 + 75n + 28)n}{1800} \\
- \frac{(12x^3 + 24x^2 + 7x - 6nx^2 - 9nx + 4n^2x - 5 + 2n + 4n^2 - 3n^3)nx}{60}.
$$

Corollary 22 ($x = p$ with $p \in \mathbb{N}_0$ in Proposition 21).

$$
\sum_{k=1}^{n} k^4 H_{p+k}(x) = \frac{6p^5 + 15p^4 + 10p^3 - p - n + 10n^3 + 15n^4 + 6n^5}{30} H_{p+n}^{(2)}(x) \\
- \frac{6p^5 + 15p^4 + 10p^3 - p}{30} H_p - \frac{(72n^4 - 45n^3 - 130n^2 + 75n + 28)n}{1800} \\
- \frac{(12p^3 + 24p^2 + 7p - 6np^2 - 9np + 4n^2p - 5 + 2n + 4n^2 - 3n^3)np}{60}.
$$

Applying the derivative operator $D_x$ to Proposition 21 we achieve the following equation.

Proposition 23. For $x \in \mathbb{C}$, there holds the summation formula:

$$
\sum_{k=1}^{n} k^4 H_k^{(2)}(x) = \frac{6x^5 + 15x^4 + 10x^3 - x - n + 10n^3 + 15n^4 + 6n^5}{30} H_n^{(2)}(x) \\
- \frac{30x^2(x + 1)^2 - 1}{30} H_n(x) \\
+ \frac{(48x^3 + 72x^2 - 18nx^2 + 14x - 18nx + 8n^2x - 5 + 2n + 4n^2 - 3n^3)n}{60}.
$$

Corollary 24 ($x = p$ with $p \in \mathbb{N}_0$ in Proposition 23).

$$
\sum_{k=1}^{n} k^4 H_{p+k}^{(2)}(x) = \frac{6p^5 + 15p^4 + 10p^3 - p - n + 10n^3 + 15n^4 + 6n^5}{30} H_{p+n}^{(2)}(x) \\
- \frac{6p^5 + 15p^4 + 10p^3 - p}{30} H_p - \frac{30p^2(p + 1)^2 - 1}{30} (H_{p+n} - H_p) \\
+ \frac{(48p^3 + 72p^2 - 18np^2 + 14p - 18np + 8n^2p - 5 + 2n + 4n^2 - 3n^3)n}{60}.
$$

Applying the derivative operator $D_x$ to Proposition 23 we attain the following equation.

Proposition 25. For $x \in \mathbb{C}$, there holds the summation formula:

$$
\sum_{k=1}^{n} k^4 H_k^{(3)}(x) = \frac{6x^5 + 15x^4 + 10x^3 - x - n + 10n^3 + 15n^4 + 6n^5}{30} H_n^{(3)}(x) \\
- \frac{30x^2(x + 1)^2 - 1}{30} H_n^{(2)}(x) + x(x + 1)(2x + 1)H_n(x) \\
- \frac{(72x^2 + 72x - 18nx + 7 - 9n + 4n^2)n}{60}.
$$
Corollary 26 \((x = p \text{ with } p \in \mathbb{N}_0 \text{ in Proposition } [25])\).

\[
\sum_{k=1}^{n} k^4 H_{p+k}^{(3)} = \frac{6p^5 + 15p^4 + 10p^3 - p - n + 10n^3 + 15n^4 + 6n^5}{30} H_{p+n}^{(3)} \\
- \frac{6p^5 + 15p^4 + 10p^3 - p}{30} H_p^{(3)} - \frac{30p^2(p + 1)^2 - 1}{30} (H_{p+n}^{(2)} - H_p^{(2)}) \\
+ p(p + 1)(2p + 1)(H_{p+n} - H_p) - \frac{(72p^2 + 72p - 18np + 7 - 9n + 4n^2)n}{60}.
\]

Applying the derivative operator \(\mathcal{D}_x\) to Proposition [25], we get the following equation.

Proposition 27. For \(x \in \mathbb{C}\), there holds the summation formula:

\[
\sum_{k=1}^{n} k^4 H_k^{(4)}(x) = \frac{6x^5 + 15x^4 + 10x^3 - x - n + 10n^3 + 15n^4 + 6n^5}{30} H_n^{(4)}(x) \\
- \frac{30x^2(x + 1)^2 - 1}{30} H_n^{(3)}(x) + x(x + 1)(2x + 1)H_n^{(2)}(x) \\
- \frac{6x^2 + 6x + 1}{3} H_n(x) + \frac{(8x + 4 - n)n}{10}.
\]

Corollary 28 \((x = p \text{ with } p \in \mathbb{N}_0 \text{ in Proposition } [27])\).

\[
\sum_{k=1}^{n} k^4 H_{p+k}^{(4)} = \frac{6p^5 + 15p^4 + 10p^3 - p - n + 10n^3 + 15n^4 + 6n^5}{30} H_{p+n}^{(4)} \\
- \frac{6p^5 + 15p^4 + 10p^3 - p}{30} H_p^{(4)} - \frac{30p^2(p + 1)^2 - 1}{30} (H_{p+n}^{(3)} - H_p^{(3)}) \\
+ p(p + 1)(2p + 1)(H_{p+n}^{(2)} - H_p^{(2)}) - \frac{6p^2 + 6p + 1}{3} (H_{p+n} - H_p) \\
+ \frac{(8p + 4 - n)n}{10}.
\]

Theorem 29. For \(x \in \mathbb{C}\) and \(l \in \mathbb{N}_0\), there holds the summation formula:

\[
\sum_{k=1}^{n} k^4 H_k^{(l+5)}(x) = \frac{6x^5 + 15x^4 + 10x^3 - x - n + 10n^3 + 15n^4 + 6n^5}{30} H_n^{(l+5)}(x) \\
- \frac{30x^2(x + 1)^2 - 1}{30} H_n^{(l+4)}(x) + x(x + 1)(2x + 1)H_n^{(l+3)}(x) \\
- \frac{6x^2 + 6x + 1}{3} H_n^{(l+2)}(x) + \frac{2x + 1}{2} H_n^{(l+1)}(x) - \frac{H_n^{(l)}(x)}{5}.
\]

Proof. Applying the derivative operator \(\mathcal{D}_x\) to Proposition [27], we gain the case \(l = 0\) of Theorem [29].

\[
\sum_{k=1}^{n} k^4 H_k^{(5)}(x) = \frac{6x^5 + 15x^4 + 10x^3 - x - n + 10n^3 + 15n^4 + 6n^5}{30} H_n^{(5)}(x) \\
- \frac{30x^2(x + 1)^2 - 1}{30} H_n^{(4)}(x) + x(x + 1)(2x + 1)H_n^{(3)}(x) \\
- \frac{6x^2 + 6x + 1}{3} H_n^{(2)}(x) + \frac{2x + 1}{2} H_n(x) - \frac{n}{5}.
\]
Suppose that the identity
\[ \sum_{k=1}^{n} k^4 H_{k}^{(l+5)}(x) = \frac{6x^5 + 15x^4 + 10x^3 - x - n + 10n^3 + 15n^4 + 6n^5}{30} H_{n}^{(l+5)}(x) \]
\[ - \frac{30x^2(x+1)^2 - 1}{30} H_{n}^{(l+4)}(x) + x(x+1)(2x+1) H_{n}^{(l+3)}(x) \]
\[ - \frac{6x^2 + 6x + 1}{3} H_{n}^{(l+2)}(x) + \frac{2x + 1}{2} H_{n}^{(l+1)}(x) - \frac{H_{n}^{(l)}(x)}{5} \]
is true. Applying the derivative operator \( \mathcal{D}_x \) to the last equation, we have
\[ \sum_{k=1}^{n} k^4 H_{k}^{(l+6)}(x) = \frac{6x^5 + 15x^4 + 10x^3 - x - n + 10n^3 + 15n^4 + 6n^5}{30} H_{n}^{(l+6)}(x) \]
\[ - \frac{30x^2(x+1)^2 - 1}{30} H_{n}^{(l+5)}(x) + x(x+1)(2x+1) H_{n}^{(l+4)}(x) \]
\[ - \frac{6x^2 + 6x + 1}{3} H_{n}^{(l+3)}(x) + \frac{2x + 1}{2} H_{n}^{(l+2)}(x) - \frac{H_{n}^{(l+1)}(x)}{5} \]
This proves Theorem 29 inductively. \( \square \)

Making \( x = p \) in Theorem 29, we obtain the following equation.

**Corollary 30.** For \( l, p \in \mathbb{N}_0 \), there holds the summation formula:
\[ \sum_{k=1}^{n} k^4 H_{p+k}^{(l+5)} = \frac{6p^5 + 15p^4 + 10p^3 - p - n + 10n^3 + 15n^4 + 6n^5}{30} H_{p+n}^{(l+5)} \]
\[ - \frac{6p^5 + 15p^4 + 10p^3 - p}{30} H_{p+n}^{(l+4)} - \frac{30p^2(p+1)^2 - 1}{30} (H_{p+n}^{(l+4)} - H_{p}^{(l+4)}) \]
\[ + p(p+1)(2p+1)(H_{p+n}^{(l+3)} - H_{p}^{(l+3)}) - \frac{6p^2 + 6p + 1}{3} (H_{p+n}^{(l+2)} - H_{p}^{(l+2)}) \]
\[ + \frac{2p + 1}{2} (H_{p+n}^{(l+1)} - H_{p}^{(l+1)}) - \frac{H_{p+n}^{(l)} - H_{p}^{(l)}}{5}. \]

**Remark:** Further summation formulas with the factor \( k^i \), where \( i \) is a positive integer greater than 4, can also be derived in the same way. Considering that the resulting identities will become more complicated, we shall not lay out them here.

**References**

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