Globally Hyperbolic Regularization of Grad’s Moment System

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Abstract
In this paper, we propose a globally hyperbolic regularization to the general Grad’s moment system in multidimensional spaces. Systems with moments up to an arbitrary order are studied. The characteristic speeds of the regularized moment system can be analytically given and depend only on the macroscopic velocity and the temperature. The structure of the eigenvalues and eigenvectors of the coefficient matrix is fully clarified. The regularization together with the properties of the resulting moment systems is consistent with the simple one-dimensional case discussed in [1]. In addition, all characteristic waves are proven to be genuinely nonlinear or linearly degenerate, and the studies on the properties of rarefaction waves, contact discontinuities, and shock waves are included. © 2014 Wiley Periodicals, Inc.

1 Introduction
The kinetic gas theory, which is based on the Boltzmann equation, is one of the fundamental tools in modeling nonequilibrium processes. Nevertheless, in most cases, a direct numerical discretization of the Boltzmann equation leads to unacceptable computational costs. In 1940s, Grad [8] proposed the moment approximation of the distribution function, trying to establish a series of intermediate models between the fluid dynamics and the kinetic theory. However, due to a number of defects in Grad’s 13-moment equations, such as the appearance of unphysical subshocks, nonexistence of an entropy function, and lack of global hyperbolicity, not much attention was paid to the moment method in the last century.

During the past 20 years, as the investigation into the moment method has gone deeper, various “regularizations” have been proposed to challenge the traditional biases against the moment method. A list of relevant publications can be found in the references of [13]. Recently we became interested in the large moment system...
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together with its numerical methods $[2, 3, 4, 5]$, and it was found that the lack of the well-posedness due to the loss of global hyperbolicity is a major obstacle in our simulations, especially for large Mach number gas flows $[5]$. Torrilhon $[14]$ provided a 13-moment moment system based on the multivariate Pearson IV distributions, which is hyperbolic when reduced to one dimension, but it seems unlikely that the same technique can be extended to systems with a large number of moments.

As discussed in $[13]$, Levermore $[10]$ gave a partial answer to the question of hyperbolicity of the large moment system based on a maximal entropy distribution function. However, the analytical forms of Levermore’s equations cannot be obtained once the number of moments is greater than 10. While exploring the method ensuring the hyperbolicity of the moment system, we discovered $[11]$ that the structure of the characteristic polynomial of Grad’s moment equations with one-dimensional microscopic velocity is rather simple; thus a globally hyperbolic regularization can be achieved by simply adding two terms to the equation of the highest-order moment.

In this paper, the results in $[1]$ are extended to multidimensional space. For multidimensional moment systems, the regularization method is consistent with the one-dimensional case. Due to the complexity of the moment systems, this paper is mainly devoted to a rigorous proof of the hyperbolicity of the regularized moment system for any space dimension and an arbitrary order of moments. The result is obtained by first restricting the spatial variable in the one-dimensional space and then generalizing it to the multidimensional space using the rotation invariance of the regularized system.

For the case of one-dimensional spatial variable, we first reveal several useful properties of the coefficient matrix. Specifically, the sparsity pattern of the coefficient matrix shows that the structure of the coefficient matrix is similar to a Hessenberg matrix, which enables us to calculate the eigenvectors for a given eigenvalue. Then, by counting the number of linearly independent eigenvectors of the matrix, we prove that the matrix is diagonalizable with real eigenvalues; thus the regularized system is hyperbolic. By showing the Galilean invariance of the regularized moment system, the result is extended to the case of any space dimension and an arbitrary order of moments. After that, the expressions of all characteristic speeds are obtained, each of which is a sum of the macroscopic velocity and the square root of the temperature scaled by a root of the Hermite polynomial. In addition, we prove that each characteristic field of the hyperbolic moment systems is either genuinely nonlinear or linearly degenerate, and some properties of the rarefaction waves, the contact discontinuities, and the shock waves are investigated. Also, by studying the approximation of the convection term for both Grad’s moment system and the hyperbolic moment system, the regularized moment system is demonstrated to have the same convergence rate as Grad’s moment system in the truncation error point of view.
The rest of this paper is arranged as follows: in Section 2 a brief review of the moment methods of the Boltzmann equation and the results in [1] are presented. Section 3 gives the globally hyperbolic regularization for the moment system with one-dimensional spatial variable and multidimensional microscopic velocities. In Section 4, the result for a full multidimensional moment system is proved. The study of the characteristic waves is carried out in Section 5. Finally, Section 6 is devoted to a discussion of the convergence of Grad’s moment system and the hyperbolic moment system to the Boltzmann equation in the truncation error point of view.

2 Preliminaries

In this section, a concise introduction of the Boltzmann equation is presented, and some results of the work on the moment method in [1, 5] are briefly reviewed.

2.1 Moment Methods for the Boltzmann Equation

Let the motion of particles be depicted by the distribution function $f(t, x, \xi)$ governed by the Boltzmann transport equation

$$\frac{\partial f}{\partial t} + \sum_{j=1}^{D} \xi_j \frac{\partial f}{\partial x_j} = Q(f, f), \quad t \in \mathbb{R}^+, \quad x, \xi \in \mathbb{R}^D,$$

where $t$ denotes the time, and $x = (x_1, \ldots, x_D)$ and $\xi = (\xi_1, \ldots, \xi_D)$ stand for the spatial coordinates and the microscopic velocity, respectively. The right-hand side $Q(f, f)$ is the collision term describing the interaction between particles. In this paper, we are focusing on the transportation part; thus the collisionless Boltzmann equation with vanished $Q(f, f)$ is considered.

The moment method proposed by Grad [8] approximates the distribution function by a finite set of moments. To achieve this, we expand $f$ into the Hermite series as in [2]:

$$f(t, x, \xi) = \sum_{\alpha \in \mathbb{N}^D} f_{\alpha}(t, x) \mathcal{H}_{\theta(t,x),\alpha}\left(\frac{\xi - \mu(t, x)}{\sqrt{\theta(t, x)}}\right),$$

where $\alpha = (\alpha_1, \ldots, \alpha_D)$ is a $D$-dimensional multi-index, and the basis functions are defined as

$$\mathcal{H}_{\theta,\alpha}(z) = \prod_{d=1}^{D} \frac{1}{\sqrt{2\pi}} \theta^{-\frac{\alpha_d+1}{2}} He_{\alpha_d}(z_d) \exp\left(-\frac{z_d^2}{2}\right),$$

$z = (z_1, \ldots, z_D) \in \mathbb{R}^D,$

where $He_k$ is the $k$th-degree Hermite polynomial:

$$He_k(x) = (-1)^k \exp\left(\frac{x^2}{2}\right) \frac{d^k}{dx^k} \exp\left(-\frac{x^2}{2}\right), \quad k \in \mathbb{N}.$$
In (2.2), \( u(t, x) = (u_1(t, x), \ldots, u_D(t, x)) \) and \( \theta(t, x) \) denote the macroscopic velocity and temperature, respectively, and they are related to \( f \) by

\[
\rho(t, x) = \int_{\mathbb{R}^D} f(t, x, \xi) d\xi, \quad \rho(t, x)u(t, x) = \int_{\mathbb{R}^D} \xi f(t, x, \xi) d\xi.
\]

\[\Rightarrow \rho(t, x)|u(t, x)|^2 + D\rho(t, x)\theta(t, x) = \int_{\mathbb{R}^D} |\xi|^2 f(t, x, \xi) d\xi,
\]

where \( \rho \) stands for the density of the gas. The following relations can be deduced from the orthogonality of Hermite polynomials:

\[ f_0 = \rho, \quad f_{e_j} = 0, \quad \sum_{d=1}^{D} f_{2e_d} = 0, \quad j = 1, \ldots, D, \]

where \( e_j \) is the \( D \)-dimensional multi-index with its \( j \)th component the only nonzero component; it is equal to 1.

The moment system has been deduced in [5], and here we directly present the result therein: for all \( \alpha \in \mathbb{N}^D \),

\[
\left( \frac{\partial f_{\alpha}}{\partial t} + \sum_{d=1}^{D} \frac{\partial u_d}{\partial t} f_{\alpha-e_d} + \frac{1}{2} \frac{\partial \theta}{\partial t} \sum_{d=1}^{D} f_{\alpha-2e_d} \right) + \sum_{j=1}^{D} \left( \frac{\partial f_{\alpha-e_j}}{\partial x_j} + u_j \frac{\partial f_{\alpha}}{\partial x_j} + (\alpha_j + 1) \frac{\partial f_{\alpha+e_j}}{\partial x_j} \right) \\
+ \frac{D}{2} \sum_{j=1}^{D} \sum_{d=1}^{D} \frac{\partial \theta}{\partial x_j} \left( \frac{\partial f_{\alpha-2e_d-e_j}}{\partial x_j} + u_j f_{\alpha-2e_d} + (\alpha_j + 1) f_{\alpha-2e_d+e_j} \right) = 0.
\]

In this equation, \( f_\beta \) is taken as 0 if any component of \( \beta \) is negative. Some special choices of \( \alpha \) lead to the classic hydrodynamic equations:

\[
\frac{\partial \rho}{\partial t} + \sum_{j=1}^{D} \left( u_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial u_j}{\partial x_j} \right) = 0, \quad \frac{\partial u_i}{\partial t} + \sum_{j=1}^{D} \left( \rho u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p_{e_i+e_j}}{\partial x_j} \right) = 0, \quad \frac{D}{2} \rho \frac{\partial \theta}{\partial t} + \sum_{j=1}^{D} \left( \frac{D}{2} \rho u_j \frac{\partial \theta}{\partial x_j} + \frac{\partial q_{e_i+e_j}}{\partial x_j} \right) + \sum_{i=1}^{D} \sum_{j=1}^{D} p_{e_i+e_j} \frac{\partial u_j}{\partial x_j} = 0.
\]
where $p_{e_i+e_j}$ is the pressure tensor and $q_j$ is the heat flux. They are defined as

\begin{align}
(2.8a) \quad p_{e_i+e_j} &= \int_{\mathbb{R}^D} \left( \xi_i - u_i \right) \left( \xi_j - u_j \right) f \, d\xi = \delta_{ij} \rho \theta + (1 + \delta_{ij}) f_{e_i+e_j}, \\
(2.8b) \quad q_j &= \frac{1}{2} \int_{\mathbb{R}} |\xi - u|^2 \left( \xi_j - u_j \right) f \, d\xi = 2 f_{3e_j} + \sum_{d=1}^{D} f_{e_j+2e_d},
\end{align}  

where $\delta$ is Kronecker’s delta symbol. We refer the readers to [5] for the detailed derivation of (2.7).

Since (2.6) forms an infinite set of moment equations that are not suitable for practical use, moment closure is needed. The simplest way is to select an integer $M > 3$ and force $f_\alpha = 0$ if $|\alpha| > M$, and the result is the Grad-type system with $(M+D)$ moments.

### 2.2 Regularization with One-Dimensional Velocity Space

It is well-known that the lack of global hyperbolicity is one of the major defects of Grad’s moment equations. For the 13-moment case, the hyperbolicity region has been analytically obtained in [11]. The construction of globally hyperbolic moment systems is very meaningful to the robustness of fluid simulation using moment approximation. In this direction, a general method by Levermore [10] on the construction of symmetric hyperbolic moment systems was proposed. Later Torillhon [14] raised a clever idea to enlarge the hyperbolicity region of the 13-moment system by using Pearson IV distributions. In [1], we have studied the general one-dimensional moment systems and found a way to make globally hyperbolic regularization based on the characteristic speed correction. Here we are going to give a brief review on the results therein.

When $D = 1$, the multi-index $\alpha$ becomes a natural number. Substituting (2.7b) and (2.7c) into (2.6), we can eliminate the time derivative of the velocity and temperature. Thus the $M$th-order Grad’s moment system can be written in the form of a quasi-linear system

\begin{equation}
\frac{\partial w}{\partial t} + A(w) \frac{\partial w}{\partial x} = 0,
\end{equation}

where $A$ is a matrix dependent on $w$, and

$$w = (\rho, u, \theta, f_3, \ldots, f_M).$$

In [1], we have obtained the following results:

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1 In some of the literature, the pressure tensor is denoted as $p_{ij}, i, j = 1, \ldots, D$. Here the special subscript $e_i + e_j$ is used to match the form of general moments $f_{e_i+e_j}$ for convenience later.
(1) The characteristic polynomial of $A(w)$ is
\[
|\lambda I - A| = \theta^{M+1} \frac{\lambda^M}{\sqrt{\theta}} - \frac{(M+1)!}{2\rho} [((\lambda-u)^2 - \theta) f_{M-1} + 2(\lambda-u) f_M].
\]

(2) By adding the regularization term based on characteristic speed correction
\[
(2.10) \quad R_M = \frac{M+1}{2} \left( 2f_M \frac{\partial u}{\partial x} + f_{M-1} \frac{\partial \theta}{\partial x} \right)
\]
to the right-hand side of the last equation of (2.9), the system is regularized to be globally hyperbolic and the eigenvalues of the regularized moment system are
\[
u + C_{j,M+1}\sqrt{\theta}, \quad j = 1, \ldots, M + 1,
\]
where $C_{j,k}$ can be seen to be the $j$th root of the Hermite polynomial $He_k(x)$ by noticing that $He_k(x)$, $k \in \mathbb{N}$, has $k$ different zeros, which read $C_{1,k}, \ldots, C_{k,k}$ and satisfy $C_{1,k} < \cdots < C_{k,k}$.

The second result gives a practical implementation of a globally hyperbolic regularization.

2.3 Reformulation of the Moment System

In order to facilitate the study of the moment system when $D \geq 2$, we rewrite (2.6) in an alternative form. Let $p = \rho \theta$; then $p = \frac{1}{D} \sum_{d=1}^D p_{2e_d}$, and we have
\[
(2.11) \quad \frac{\partial \theta}{\partial x_j} = -\frac{\theta}{\rho} \frac{\partial \rho}{\partial x_j} + \frac{1}{D\rho} \sum_{d=1}^D \frac{\partial p_{2e_d}}{\partial x_j}, \quad j = 1, \ldots, D.
\]

By substituting (2.7) and (2.11) into (2.6), the following equation is obtained with some simplification:
\[
\begin{align*}
(2.12) \quad \frac{\partial f_a}{\partial t} &+ \sum_{j=1}^D \left( \frac{\partial f_{a-e_j}}{\partial x_j} + u_j \frac{\partial f_a}{\partial x_j} + (\alpha_j + 1) \frac{\partial f_{a+e_j}}{\partial x_j} \right) + \frac{\sum_{j=1}^D \frac{\theta}{2\rho} c_{\theta,a}^{(j)}}{\partial x_j} \frac{\partial \rho}{\partial x_j} \\
&+ \sum_{j=1}^D \sum_{d=1}^D \frac{\partial u_d}{\partial x_j} \left( \theta f_{a-e_d-e_j} + (\alpha_j + 1) f_{a-e_d+e_j} - \frac{C_a}{D\rho} p_{e_j+e_d} \right) \\
&+ \sum_{j=1}^D \sum_{d=1}^D \left( \frac{f_{a-e_d}}{\rho} \frac{\partial p_{e_j+e_d}}{\partial x_j} + \frac{C_{\theta,a}^{(j)} p_{2e_d}}{2D\rho} \frac{\partial \rho}{\partial x_j} \right) + \left( \frac{C_a}{D\rho} \right) \sum_{j=1}^D \frac{\partial q_j}{\partial x_j} = 0.
\end{align*}
\]
where \( C_\alpha \) and \( C_{\theta,\alpha}^{(j)} \) are defined as

\[
(2.13a) \quad C_\alpha = \sum_{k=1}^{D} f_{\alpha-2e_k},
\]

\[
(2.13b) \quad C_{\theta,\alpha}^{(j)} = \sum_{k=1}^{D} (\theta f_{\alpha-2e_k-e_j} + (\alpha_j + 1) f_{\alpha-2e_k+e_j}).
\]

Then collecting (2.7), (2.12), and (2.8a), we get for \( i = 1, \ldots, D \),

\[
\frac{\partial p_{2e_i}/2}{\partial t} + \sum_{j=1}^{D} u_j \frac{\partial p_{2e_i}/2}{\partial x_j} + \sum_{j=1}^{D} \left( \frac{1}{2} + \delta_{ij} \right) \rho \theta \frac{\partial u_j}{\partial x_j} + \sum_{j=1}^{D} \sum_{d=1}^{D} (2\delta_{ij} + 1) f_{2e_i-e_d+e_j} \frac{\partial u_d}{\partial x_j}
\]

\[
+ \sum_{j=1}^{D} (2\delta_{ij} + 1) \frac{\partial f_{2e_i+e_j}}{\partial x_j} = 0.
\]

Equations (2.7) together with (2.14) and (2.12) form a moment system with an infinite number of equations, which is equivalent to (2.6).

### 3 System in One-Dimensional Spatial Space

In order to derive the regularization term to achieve the hyperbolicity of the moment systems as in Section 2.3, we first consider the special case with homogeneous dependence of the distribution function on spatial coordinate \( x \) except for the \( x_1 \)-direction. Since the velocity space is multidimensional, the result in this section is essentially different from [1]. The general case in multidimensional spatial space is studied in the next section based on the results herein and the Galilean invariance of the regularization.

In the one-dimensional spatial space, the distribution function \( f(t, x_1, \xi) \) satisfies

\[
(3.1) \quad \frac{\partial f}{\partial t} + \xi_1 \frac{\partial f}{\partial x_1} = 0, \quad t \in \mathbb{R}^+, \ x_1 \in \mathbb{R}, \ \xi \in \mathbb{R}^D.
\]
The moment system in Section 2.3 degenerates to a simpler form. The conservation of mass, momentum, and energy (2.7) turn into

\[ \frac{\partial \rho}{\partial t} + u_1 \frac{\partial \rho}{\partial x_1} + \rho \frac{\partial u_1}{\partial x_1} = 0, \]

\[ \rho \frac{\partial u_1}{\partial t} + \rho u_1 \frac{\partial u_1}{\partial x_1} + \frac{\partial p_{e1+e1}}{\partial x_1} = 0, \quad i = 1, \ldots, D, \]

\[ \frac{D}{2} \rho \frac{\partial \theta}{\partial t} + \frac{D}{2} \rho u_1 \frac{\partial \theta}{\partial x_1} + \frac{\partial q_1}{\partial x_1} + \sum_{i=1}^{D} p_{e1+e1} \frac{\partial u_i}{\partial x_1} = 0. \]

The moment equations (2.12) become

\[ \frac{\partial f_\alpha}{\partial t} + \theta \frac{\partial f_\alpha - e_1}{\partial x_1} + u_1 \frac{\partial f_\alpha}{\partial x_1} + (\alpha_1 + 1) \frac{\partial f_{\alpha+e1}}{\partial x_1} - \frac{\theta}{2\rho} C^{(1)}_{\theta,\alpha} \frac{\partial \rho}{\partial x_1} \]

\[ + \sum_{d=1}^{D} \frac{\partial u_d}{\partial x_1} \left( \theta f_{\alpha-e_d-e_1} + (\alpha_1 + 1) f_{\alpha-e_d+e_1} - \frac{C_\alpha}{\rho} p_{e1+e_d} \right) \]

\[ + \sum_{d=1}^{D} \left( - \frac{f_{\alpha-e_d}}{\rho} \frac{\partial p_{e1+e_d}}{\partial x_1} + \frac{C^{(1)}_{\theta,\alpha}}{2\rho} \frac{\partial \rho_{e1+e_d}}{\partial x_1} \right) - \frac{C_\alpha}{\rho} \frac{\partial q_1}{\partial x_1} = 0, \]

where \( C_\alpha \) and \( C^{(1)}_{\theta,\alpha} \) are defined in (2.13). The governing equations of \( p_{e1} \) (2.14) turn into: for \( i = 1, \ldots, D, \)

\[ \frac{\partial p_{2e_i}}{\partial t} + u_1 \frac{\partial p_{2e_i}}{\partial x_1} + \left( \frac{1}{2} + \delta_{i1} \right) \rho \frac{\partial u_1}{\partial x_1} \]

\[ + \sum_{d=1}^{D} (2\delta_{i1} + 1) f_{2e_i-e_d+e_1} \frac{\partial u_d}{\partial x_1} + (2\delta_{i1} + 1) \frac{\partial f_{2e_i+e_1}}{\partial x_1} = 0. \]

Analogously to the moment system in Section 2.1, let \( f_\alpha = 0 \) and \( |\alpha| > M \) for \( M \geq 3 \); then (3.2) and (3.3) together with (3.4) form a closed moment system corresponding to (3.1).

To facilitate the reading below in studying the moment system, some notation is introduced as follows:

- If \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \), then \( a(i : j) = (a_i, \ldots, a_j) \).
- If \( A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n} \), then \( A(i, j : k) = (a_{i,j}, \ldots, a_{i,k}) \) and

\[ A(i : l, j : k) = \begin{pmatrix} a_{i,j} & a_{i,j+1} & \cdots & a_{i,k} \\ a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{l,j} & a_{l,j+1} & \cdots & a_{l,k} \end{pmatrix}. \]

- \( D = \{1, 2, \ldots, D\} \),
\[ \tilde{\alpha} = (0, \alpha_2, \ldots, \alpha_D), \quad \alpha = (\alpha_2, \ldots, \alpha_D) \in \mathbb{N}^{D-1}, \quad \alpha! = \prod_{i=1}^{D} \alpha_i!, \quad |\alpha| = \sum_{i=1}^{D} \alpha_i, \]

- \( S_{D,M} = \{ \alpha \in \mathbb{N}^D \mid |\alpha| \leq M \}, \quad S_{D,M}(\tilde{\alpha}) = \{ \beta \in S_{D,M} \mid \beta = \tilde{\alpha} \}. \)

We permute the elements of \( S_{D,M} \) by lexicographic order. Then for any \( \alpha \in S_{D,M}, \)

\[ N_D(\alpha) = \sum_{i=1}^{D} \left( \sum_{k=D-i+1}^{D} \alpha_k + i - 1 \right) + 1 \tag{3.5} \]

holds, where \( N_D(\alpha) \) is the ordinal number of \( \alpha \) in \( S_{D,M} \), and the cardinal number of set \( S_{D,M} \) is \( N = N_D(M\mathbf{e}_D) = \binom{M+D}{D} \), which is the total number of moments if a truncation with \( |\alpha| \leq M \) is introduced. In addition, it is clear that for each \( \alpha, \beta \in S_{D,M} \) and \( \tilde{\alpha} \neq \tilde{\beta}, \)

\[ S_{D,M}(\tilde{\alpha}) \cap S_{D,M}(\tilde{\beta}) = \emptyset, \quad S_{D,M} = \bigcup_{\alpha \in S_{D,M}} S_{D,M}(\tilde{\alpha}). \]

simultaneously hold.

### 3.1 Structure of the Coefficient Matrix

Similar to the one-dimensional case, a truncation with \( |\alpha| \leq M, \quad M \geq 3 \), is applied. Let \( \mathbf{w} \in \mathbb{R}^N \) and for each \( i, j \in D \) and \( i \neq j, \)

\[ w_1 = \rho, \quad w_{N_D(e_i)} = u_i, \tag{3.6a} \]

\[ w_{N_D(2e_i)} = \frac{p_2 e_i}{2}, \quad w_{N_D(e_i + e_j)} = p e_i + e_j, \tag{3.6b} \]

\[ w_{N_D(\alpha)} = f_\alpha, \quad 3 \leq |\alpha| \leq M. \tag{3.6c} \]

Combining (3.2) with (3.4) and (3.3), we obtain

\[ \frac{\partial \mathbf{w}}{\partial t} + \mathbf{A}_M \frac{\partial \mathbf{w}}{\partial x_1} = 0, \tag{3.7} \]

where \( \mathbf{A}_M \) depends on (3.2), (3.4), and (3.3).

Clearly the entire matrix \( \mathbf{A}_M \) for any \( D \in \mathbb{N}^+, \quad 3 \leq M \in \mathbb{N}, \) is well-defined though quite complex. Here we first give some simple examples and conclude a few basic properties of the matrix \( \mathbf{A}_M \).

**Example 3.1.** If \( D = 2 \), the ordinal number of \( \alpha \) in \( S_{D,M} \) is

\[ N_D(\alpha) = \frac{(\alpha_1 + \alpha_2 + 1)(\alpha_1 + \alpha_2)}{2} + \alpha_2 + 1. \]
The permutation of $w$ is shown in Figure 3.4(a). As the simplest case, the matrix $A_3$ is

$$
(3.8) \quad A_3 = \begin{bmatrix}
  u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & u_1 & 0 & 2\rho^{-1} & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & u_1 & 0 & \rho^{-1} & 0 & 0 & 0 & 0 \\
  0 & \frac{3}{2}p_1 & 0 & u_1 & 0 & 0 & 3 & 0 & 0 \\
  0 & 2f_{11} & p_1 & 0 & u_1 & 0 & 0 & 2 & 0 \\
  0 & \frac{1}{2}p_2 & f_{11} & 0 & 0 & u_1 & 0 & 0 & 1 & 0 \end{bmatrix},
$$

where $p_1 = p_{2e_1}$ and $f_{mn} = f_{me_1+ne_2}$. If $M > 3$, for any $\alpha \in \mathbb{N}^2$ and $3 < |\alpha| \leq M$,

$$
(3.9a) \quad A_M(1:10, 1:10) = A_3, \\
(3.9b) \quad A_M(N_D(\alpha), N_D(\alpha)) = u_1, \\
(3.9c) \quad A_M(N_D(\alpha), N_D(\alpha - e_1)) = \theta \quad \text{if} \quad \alpha_1 > 0, \\
(3.9d) \quad A_M(N_D(\alpha), N_D(\alpha + e_1)) = \alpha_1 + 1 \quad \text{if} \quad |\alpha| < M,
$$

$$
A_M(N_D(\alpha), 1:9) = \left( -\frac{\theta}{2\rho} C_{\theta,\alpha}^{(1)} \right) , \quad \theta f_{a-2e_1} + (\alpha_1 + 1) f_a - \frac{C_{\alpha}}{2\rho} p_{2e_1}, \\
\theta f_{a-e_1-e_2} + (\alpha_1 + 1) f_{a+e_1-e_2} - \frac{C_{\alpha}}{2\rho} p_{e_1+e_2}, \\
-2 \frac{f_{a-e_1}}{\rho} + \frac{C_{\theta,\alpha}^{(1)}}{2\rho} - \frac{f_{a-e_2}}{\rho} - \frac{C_{\theta,\alpha}^{(1)}}{2\rho} - \frac{3C_{\alpha}}{2\rho}, \\
0, -\frac{C_{\alpha}}{2\rho} \right),
$$

where $C_{\alpha}$ and $C_{\theta,\alpha}^{(1)}$ are defined in equation (2.13). We remark that

- an entry $A_M(i, j)$ not defined above is taken to be 0;
- for $|\alpha| = 4$, some $A_M(i, j)$ may be doubly defined in (3.9c) and (3.9e), the value of which is the sum of both expressions.

For the reader’s convenience, we write the matrix $A_4$ as shown in Figure 3.1 to clarify the structure of $A_M$.

Figure 3.2 gives the sparsity pattern of $A_M$ with $M = 8$ and $D = 2$. It is observed that there is at most one nonzero component in $A_M(i, i + 1 : N)$ for each $i = 1, \ldots, N$. Specifically, $A_M(i, i + 1 : N)$ is a zero vector when $i >
\[ u_{n+1} = u_{n} f \quad \text{and} \quad z_{2d} = z_{d} \]

\[ \begin{vmatrix}
    1 & 0 & 0 & 0 & 0 & \frac{\sigma_f}{\alpha_f} & 0 & \frac{\sigma_f}{\alpha_f} & \frac{\sigma_f}{\alpha_f} & \frac{\sigma_f}{\alpha_f} & \frac{\sigma_f}{\alpha_f} + 1 & f & \frac{\sigma_f}{\alpha_f} + a f & \frac{\sigma_f}{\alpha_f} - \frac{\sigma_f}{\alpha_f} \\
    0 & 1 & 0 & 0 & 0 & \theta & \frac{\sigma_f}{\alpha_f} - 0 & \frac{\sigma_f}{\alpha_f} & \frac{\sigma_f}{\alpha_f} & \frac{\sigma_f}{\alpha_f} & \frac{\sigma_f}{\alpha_f} - \frac{\sigma_f}{\alpha_f} & \frac{\sigma_f}{\alpha_f} - \frac{\sigma_f}{\alpha_f} - 0 f/\theta - 2 z f \zeta & \frac{\sigma_f}{\alpha_f} - \frac{\sigma_f}{\alpha_f} - 1 f/\zeta & \frac{\sigma_f}{\alpha_f} - \frac{\sigma_f}{\alpha_f} - \frac{\sigma_f}{\alpha_f} - \frac{\sigma_f}{\alpha_f} - \frac{\sigma_f}{\alpha_f}
\end{vmatrix}
\]

Where

\[ \alpha_f = \alpha_f + \theta \]

\[ \beta_f = \beta_f + \frac{1}{\theta} \]

\[ \phi_f = \phi_f + \frac{1}{\theta} \]
$N_D((M - 1)e_D)$, and it contains a unique nonzero component when $1 \leq i \leq N_D((M - 1)e_D)$. Note that the column indices of the nonzero entries in the upper triangular part of $A_M$ on different rows are different from each other, which makes one recall the form of the lower Hessenberg matrix, of which the only nonzero entry in the upper triangular part of the $i^{th}$ row is located at position $(i, i + 1)$. This feature is very similar to the Hessenberg matrix, which makes it very convenient for one to calculate its eigenvectors once eigenvalues are given using a row-by-row sequential procedure. Here $A_M$ is essentially the same as the lower Hessenberg matrix on this point, and we notice that its lower triangular part is sparse; hence we are provided the approach to calculate the eigenvalues $A_M$ together with the corresponding eigenvectors using the same technique.

Furthermore, (3.9) shows that the diagonal entries of the matrix $A_M$ are all $u_1$, and the matrix $A_M - u_1 I$ is independent of $u_i, i \in D$, where $I$ is the $N \times N$ identity matrix. In fact, (3.7) can be written as

$$\frac{Dw}{Dt} + (A_M - u_1 I) \frac{\partial w}{\partial x_1} = 0,$$

FIGURE 3.2. The sparsity pattern of $A_M$ with $M = 8, D = 2$. Its nonzero entries are defined as in (3.9).
Figure 3.3. The permutation of moments while $D = 2, M = 8$. Each node stands for one moment. The marks in the lower right of the node shows the expression of the moment, while the number in the upper left represents the ordinal number in $w$ or $w'$. The dashed arrows depict the path of the corresponding permutation. The left one is the permutation of $w$, and the right one is a permutation of $w'$ defined in Example 3.4.

where $\frac{D}{Dt}$ is the material derivative defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x_1}.$$ 

Hence the fact that $A_M - u_1 I$ is independent of $u_i, i \in D$, indicates the moment system is translation invariant. On the other hand, the eigenvalues of $A_M$ can be written in the form $u_1 + a$, where $a$ is indeterminate and independent of $u_i, i \in D$, and its eigenvectors are independent of $u_i, i \in D$, too.

Example 3.2. Considering the case $D = 2$, we can write the matrix $A_M$ explicitly for any $3 \leq M \in \mathbb{N}$ according to Example 3.1. If we let $f_\alpha = 0$ for $\alpha$ satisfying

$$\alpha \in \mathbb{N}^2, \quad 1 \leq |\alpha| \leq M, \quad \text{and} \quad \alpha \neq Me_1.$$
direct calculation gives the characteristic polynomial of $A_M$ as
\[ |\lambda I - A_M| = \left( \prod_{i=1}^{M-1} H_{e_i} \left( \frac{\lambda - u_1}{\sqrt{\theta}} \right)^{\theta/2} \right) \times \left( H_{e_M} \left( \frac{\lambda - u_1}{\sqrt{\theta}} \right)^{\theta M/2} + (-1)^{M-1} M! f_{Me_1} \right) \times \left( H_{e_{M+1}} \left( \frac{\lambda - u_1}{\sqrt{\theta}} \right)^{\theta (M+1)/2} + (-1)^{M-1} (M + 1)! f_{Me_1} (\lambda - u_1) \right). \]

The matrix $A_M$ obviously has complex eigenvalues if the absolute value of $f_{Me_1}$ is sufficiently large.

Analogously, involved calculations with the help of a computer algebraic system show that the matrix $A_M$ has complex eigenvalues for some admissible $w$ for any $D \geq 3$. This reveals that $A_M$ is not diagonalizable with real eigenvalues for some $w$ like the case of $D = 1$ in [1].

Remark 3.3. Since moments $f_\alpha$ are related to $f(t, x, \xi)$ by (2.2), the positivity of the distribution function will impose some constraints on the moments $f_\alpha$. In particular, $\rho$ and $\theta$ given by (2.5) clearly satisfy
\[ (3.10) \quad \rho > 0, \quad \theta > 0. \]
Though (3.10) is not enough to provide a positive $f(t, x, \xi)$, the discussion in this paper requires no further constraints on other moments. Hence in this paper the admissible $w$ stands for the $w$ satisfying (3.10).

Example 3.4. Actually, it can be observed that the matrix $A_M$ is reducible if we rearrange $w$ as $w'$ using another permutation rule. In the case of $D = 2$, the rule reads:

1. The moments with $\alpha_2 \leq 2$ are arranged at first by the lexicographic order.
2. The rest of the moments are then arranged by lexicographic order based on the indices transformed as ($\alpha_2, \alpha_1$).

Clearly there exists a permutation matrix $P$ such that $w' = P w$. Figure 3.4(b) gives a schematic diagram of the permutation rule for $w'$ with $M = 8$. Let $A'_M = P A_M P^{-1}$; then
\[ \frac{\partial w'}{\partial t} + A'_M \frac{\partial w'}{\partial x_1} = 0, \]
holds. Figure 3.5 gives the sparsity pattern of $A'_M$. It is obvious that $A'_M$ is reducible (see, e.g., [17] for the definition) and can be reduced into $M - 1$ blocks. One of the blocks is $S_{D,M}(\hat{e}_1) \cup S_{D,M}(\hat{e}_2) \cup S_{D,M}(2\hat{e}_2)$, and $S_{D,M}^D(\hat{a})$, for each $\hat{a} \in \mathbb{N}$ and $\hat{\alpha} \neq \hat{e}_1, \hat{e}_2, 2\hat{e}_2$, is another block.
Figure 3.4. The permutation of moments for $D = 2, M = 8$. Each node stands for one moment. The marks in the lower right of the node shows the expression of the moment, while the number in the upper left represents the ordinal number in $w$ or $w'$. The dashed arrows depict the path of the corresponding permutation. The left one is the permutation of $w$, and the right one is a permutation of $w'$ defined in Example 3.4.

Figure 3.5. The sparsity pattern of $A_M'$ with $M = 8, D = 2$. $A_M'$ is reducible.
Properties. It can be directly verified that the matrix $A_M$ is equipped with the following properties:

1. For each $\alpha \in \mathbb{N}^D$, $|\alpha| \leq M$, let $i = N_D(\alpha)$; then no more than one entry of $A_M(i, i + 1 : N)$ is nonzero. In fact, there is a unique nonzero component as $|\alpha| < M$ and $A_M(i, i + 1 : N) = 0$ as $|\alpha| = M$.

2. The diagonal entries of the matrix $A_M$ are all $u_1$, and entries of the matrix $A_M - u_1 I$ are independent of $u_i$, $i \in D$.

3. $A_M(w)$ may be not diagonalizable with real eigenvalues for some admissible $w$.

4. $A_M$ is reducible, and can be reduced into $(D_{M+D-1}) - 2(D-1)$ blocks. One of the blocks is $S_{D,M}(\hat{\alpha}) \cup \bigcup_{k=2}^{D} S_M(2\hat{\alpha}_k)$, and $S_{D,M}(\hat{\alpha})$, for each $\hat{\alpha} \in \mathbb{N}^{D-1}$ and $\hat{\alpha} \neq \hat{\alpha}_1, 2\hat{\alpha}_k, k = 2, \ldots, D$, is one of the blocks.

3.2 Globally Hyperbolic Regularization

For the one-dimensional case, the regularization as described in (2.10) was proposed so that the moment system turns out to be globally hyperbolic (see [1] for details). Actually, the regularization therein can be extended to multiple dimensional systems. For $D \in \mathbb{N}^+$, let us start from the definition as below:

**Definition 3.5.** For any $|\alpha| = M$, let

$$R_{M,D}(\alpha) \triangleq \sum_{j=1}^{D} R_{j,M,D}(\alpha)$$

where

$$R_{j,M,D}(\alpha) = (\alpha_j + 1) \left[ \sum_{d=1}^{D} f_{\alpha-e_d+e_j} \frac{\partial u_d}{\partial x_j} + \frac{1}{2} \sum_{d=1}^{D} f_{\alpha-2e_d+e_j} \frac{\partial \theta}{\partial x_j} \right] .$$

As in the one-dimensional case [1], $R_{M,D}(\alpha)$ are the regularization terms based on the characteristic speed correction.

With (2.11), we have that

$$R_{j,M,D}(\alpha) = (\alpha_j + 1) \left[ \sum_{d=1}^{D} (\alpha_d - e_d + e_j) \frac{\partial u_d}{\partial x_j} \right]$$

$$+ \left( \sum_{d=1}^{D} \frac{\partial p_{2e_d+e_j}}{\partial x_j} \right) \left( \sum_{i=1}^{D} \frac{1}{D\rho} \frac{\partial \rho}{\partial x_j} - \frac{\theta}{2\rho} \frac{\partial \rho}{\partial x_j} \right) .$$

For the case where $f$ is independent of $x_2, \ldots, x_D$, we have that

$$R_{j,M,D}(\alpha) = 0 \quad \text{for} \quad j = 2, \ldots, D.$$

This leads to $R_{M,D}(\alpha) = \mathcal{R}_{M,D}(\alpha)$. The regularized system is obtained by subtracting $R_{M,D}(\alpha)$ from the governing equation of $f_\alpha$ in (3.7) for $|\alpha| = M$. 
DEFINITION 3.6. \( \hat{A}_M \) is called the \textit{regularized matrix} of the matrix \( A_M \) if it satisfies that for any admissible \( w \),

\[
\frac{\partial w}{\partial x_1} = A_M \frac{\partial w}{\partial x_1} - \sum_{|\alpha| = M} R_{M,D}^1 (\alpha) I_{N_D (\alpha)},
\]

where \( I_k \) is the \( k \)th column of the \( N \times N \) identity matrix.

With the definition of \( \hat{A}_M \), one can observe that \( \hat{A}_M \) is derived by changing some entries of \( A_M \); i.e., for \( |\alpha| = M \), let \( i = N_D (\alpha) \),

\[
\hat{A}_M(i, 1) = A_M(i, 1) + (\alpha_1 + 1) \frac{\theta}{2\rho} \sum_{d=1}^{D} f_{\alpha-e_d+e_1},
\]

\[
\hat{A}_M(i, j + 1) = A_M(i, j + 1) - (\alpha_1 + 1) f_{\alpha-e_j+e_1}, \quad j = 1, \ldots, D,
\]

\[
\hat{A}_M(i, N_D(2e_j)) = A_M(i, N_D(2e_j)) - \frac{(\alpha_1 + 1)}{D\rho} \sum_{d=1}^{D} f_{\alpha-e_d+e_1}, \quad j = 1, \ldots, D.
\]

Other entries of \( \hat{A}_M \) have the same values as those of \( A_M \).

Example 3.7. In the case of \( D = 2 \), the matrix \( A_4 \) is provided in Figure 3.1 and its regularized matrix \( \hat{A}_4 \) is as in Figure 3.6.

Clearly only the entries in the boxes in Figure 3.6 are different from the entries in Figure 3.1. For general cases, the regularization terms only change a few entries in the lower triangular part of \( A_M \), and the effect is that only the equations for \( f_\alpha \) with \( |\alpha| = M \) are altered. Therefore, properties 1, 2, and 4 of \( A_M \) are also valid for \( \hat{A}_M \), while property 3 becomes the diagonalizability of \( A_M \) over \( \mathbb{R} \). We have the following theorem:

THEOREM 3.8. The regularized moment system

\[
\frac{\partial w}{\partial t} + \hat{A}_M \frac{\partial w}{\partial x_1} = 0
\]

is hyperbolic for any admissible \( w \).

The definition of the hyperbolicity shows that this theorem is equivalent to the diagonalizability of \( \hat{A}_M \) with real eigenvalues for any admissible \( w \). Before proving this result, we first write the matrix \( \hat{A}_M \) in a simplified form with a translation and a similarity transformation and give several useful lemmas.
\[ \hat{A}_4 = \begin{pmatrix}
  u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & u_1 & 0 & 2\rho^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & u_1 & 0 & \rho^{-1} & 0 & 0 & 0 & 0 & 0 \\
  0 & \frac{3}{2} p_1 & 0 & u_1 & 0 & 0 & 3 & 0 & 0 & 0 \\
  0 & 2 f_{11} & p_1 & 0 & u_1 & 0 & 0 & 2 & 0 & 0 \\
  0 & \frac{1}{2} p_2 & f_{11} & 0 & 0 & u_1 & 0 & 0 & 1 & 0 \\
  \frac{\rho \theta^2 - 2\theta p_1}{2\rho} & 4 f_{30} & 0 & \theta & 0 & \frac{2 f_{20}}{\rho} & u_1 & 0 & 0 & 4 \\
  -\frac{3\theta f_{11}}{2\rho} & 3 f_{21} & 3 f_{30} & -\frac{f_{11}}{2\rho} & \frac{\rho \theta - f_{30}}{\rho} & \frac{3 f_{11}}{2\rho} & 0 & u_1 & 0 & 0 \\
  -\frac{\theta^2}{2} & 2 f_{12} & 2 f_{21} & \frac{2 f_{20}}{\rho} & -\frac{f_{11}}{\rho} & \theta & 0 & 0 & u_1 & 0 \\
  -\theta f_{11} & f_{03} & f_{12} & \frac{f_{11}}{2\rho} & \frac{f_{20}}{\rho} & \frac{f_{11}}{2\rho} & 0 & 0 & 0 & u_1 \\
  0 & \frac{p_1 f_{20}}{2p} & -\frac{f_{11} f_{20}}{2\rho} & -\frac{f_{30}}{\rho} & 0 & 0 & -\frac{3p_1 \theta + \theta p_1}{4\rho} & 0 & -\frac{f_{20}}{2p} & u_1 & 0 \\
  0 & \frac{p_1 f_{11}}{2p} & \theta f_{20} & -\frac{f_{11}^2}{2\rho} & -\frac{f_{21}}{\rho} & -\frac{f_{30}}{\rho} & 0 & -\frac{3 f_{11}}{2\rho} & \theta & -\frac{f_{11}}{2\rho} & 0 \\
  0 & -\theta f_{20} & \theta f_{11} & -\frac{f_{11}}{\rho} & -\frac{f_{21}}{\rho} & 0 & 0 & 0 & \theta & 0 & 0 \\
  0 & -\frac{f_{11} p_1}{2\rho} & -\theta f_{20} & -\frac{f_{11}^2}{2\rho} & -\frac{f_{03}}{\rho} & -\frac{f_{12}}{\rho} & 0 & -\frac{3 f_{11}}{2\rho} & 0 & -\frac{f_{11}}{2\rho} & \theta & 0 \\
  0 & p_1 f_{20}/2p & f_{20} f_{11} & 0 & -\frac{f_{03}}{\rho} & 0 & 0 & \frac{3 f_{20}}{2p} & 0 & \frac{f_{20}}{2p} & 0 & 0 & 0 & 0 & u_1
\end{pmatrix} \]

Figure 3.6.
Let us denote that
\[ \Lambda = \text{diag}\{d_1, d_2, \ldots, d_N\}, \quad d_{N_D}(\alpha) = \rho^{-1} \theta^{-|\alpha|/2}, \quad 1 \neq |\alpha| \leq M, \]
\[ d_{N_D}(\alpha) = \theta^{-1/2}, \quad |\alpha| = 1, \quad \hat{p}_{e_i+e_k} = \frac{p_{e_i+e_k}}{\rho \theta}, \quad i, k = D, \]
\[ g_\alpha = \frac{f_\alpha}{\rho \theta |\alpha|}, \quad 2 \leq |\alpha| \leq M. \]

By virtue of property \[\text{2}\] we let
\[
(3.13) \quad \hat{\Lambda}_M = u_1 I + \sqrt{\theta} \Lambda^{-1} \tilde{\Lambda}_M \Lambda.
\]

Since \( \Lambda \) is diagonal, property \[\text{1}\] also holds for \( \tilde{\Lambda}_M \). Hence it is convenient to calculate the eigenvectors of matrix \( \tilde{\Lambda}_M \). In order to clearly demonstrate the process of the calculation of the eigenvectors, we first consider an relatively simple example.

Example 3.9. When \( D = 2 \) and \( M = 4 \), the matrices \( \Lambda_4 \) and \( \tilde{\Lambda}_4 \) are given by Figures 3.1 and 3.6, respectively. Then matrix \( \tilde{\Lambda}_4 \) is as in Figure 3.7.

Since property \[\text{1}\] is valid for \( \tilde{\Lambda}_4 \), we can calculate its right eigenvectors once the eigenvalues are given. Suppose \( \lambda \) is an eigenvalue of \( \tilde{\Lambda}_4 \) and \( R \) is the corresponding eigenvector. Then
\[
(3.14) \quad \tilde{\Lambda}_4(i, 1 : 15) \cdot R = \lambda R_i, \quad i = 1, \ldots, 15.
\]

Let \( R_1 = v_1 \), where \( v_1 \) is an undetermined parameter. Considering the cases \( i = 1, 2, 4 \) in (3.14), one has
- \( \tilde{\Lambda}_4(1, 1 : 15) \cdot R = \lambda R_1 \implies R_2 = \lambda v_1; \)
- \( \tilde{\Lambda}_4(2, 1 : 15) \cdot R = \lambda R_2 \implies R_4 = \frac{1}{2} \lambda^2 v_1; \)
- \( \tilde{\Lambda}_4(4, 1 : 15) \cdot R = \lambda R_4 \implies R_7 = (\frac{\lambda^3 - 3\lambda}{6} - g_{20}) v_1. \)

Analogously, let \( R_3 = v_2 \) be a parameter to be determined; then (3.14) with \( i = 3, 5 \) gives
\[ R_5 = \lambda v_2, \quad R_8 = -g_{11} \lambda v_1 + \frac{\lambda^2 - \hat{p}_1}{2} v_2. \]

respectively. Let \( R_6 = v_3, R_{10} = v_4, \) and \( R_{15} = v_5 \) be the parameters; then (3.14) with \( i = 6, \ldots, 10 \) gives
\[ R_9 = -\frac{1}{2} \hat{p}_2 \lambda v_1 - g_{11} v_2 + \lambda v_3, \]
\[ R_{11} = \left( \frac{He_4(\lambda)}{4!} - \frac{(\lambda^2 - 2) g_{20}}{4} - g_{30} \lambda \right) v_1 - \frac{g_{20}}{2} \lambda v_2, \]
\[ R_{12} = -\left( \frac{\lambda^2 - 2}{4} g_{11} + g_{21} \lambda \right) v_1 + \left( \frac{He_3(\lambda)}{3!} - g_{30} \right) v_2 - \frac{g_{11}}{2} v_3. \]
\[ \tilde{A}_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} \tilde{p}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \tilde{p}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]
\[ R_{13} = \frac{He_2(\lambda)}{2!} \left( v_3 - \frac{v_1}{2} \right) - g_{12} \lambda v_1 - g_{21} v_2. \]
\[ R_{14} = \left( -\frac{g_{11}}{4} \lambda^2 - g_{03} \lambda + \frac{g_{11}}{2} \right) v_1 + \left( g_{02} \lambda - g_{12} \right) v_2 - \frac{g_{11}}{2} v_3 + \lambda v_4. \]

Now, once \( v_j, \ j = 1, \ldots, 5, \) are determined, the complete eigenvector \( R \) is obtained. Of course, \( v_1, \ldots, v_5 \) have to satisfy the last five equations in (3.14). For \( i = 11, \ldots, 15, \) a general expression of (3.14) can be given as follows: Let
\[ G(\alpha) = \sum_{d=1}^{2} g_{\alpha - e_d} R_{N_D(e_d)} + \left( \sum_{d=1}^{2} \frac{R_{N_D(2e_d)} - R_1}{2} \right) \sum_{k=1}^{2} g_{\alpha - 2e_k}. \]

Then, if we substitute the expression of \( \tilde{A}_4(i, 1:15), \ i = 11, \ldots, 15, \) into (3.14), we have that for \( j = 4 \) and \( i = N_D(\alpha), \) the following equality holds:
\[ \lambda R_i - \tilde{A}_4(i, 1:15) \cdot R = \frac{He_{\alpha_1+1}(\lambda)}{(\alpha_1 + 1)!} \left( R_{N_D(\alpha)} + G(\alpha) \right). \]

Note that when \( |\alpha| = 4, \) the index \( i \) satisfies \( 11 \leq i \leq 15. \) Thus, the parameters \( \lambda, \ v_1, \ldots, v_5 \) satisfy
\[ (3.15) \quad \frac{He_{\alpha_1+1}(\lambda)}{(\alpha_1 + 1)!} \left( R_{N_D(\alpha)} + G(\alpha) \right) = 0 \quad \text{for all } |\alpha| = 4. \]

In conclusion, the vector \( R \) is an eigenvector of \( \tilde{A}_4 \) for the eigenvalue \( \lambda \) if and only if \( \lambda, \ v_1, \ldots, v_5, \) satisfy (3.13).

For the general case, we first define some symbols:
\[ (3.16a) \quad r_{w_{N_D(\alpha)}} = v_{N_D-1(\alpha)}, \quad \text{if } \alpha_1 = 0, \]
\[ (3.16b) \quad r_{u_1} = \lambda r_{\rho}, \quad r_{p_{2e_1}/2} = \frac{\lambda^2}{2} r_{\rho}, \]
\[ (3.16c) \quad r_{f_\alpha} = 0, \quad \text{if at least one } \alpha_j < 0, \ j \in D, \]
\[ (3.16d) \quad r_{p_{e_1+e_k}} = \lambda r_{u_k}, \quad k \in D \setminus \{1\}, \]
\[ r_{f_{2e_i}} = r_{p_{2e_i}/2} - \sum_{d=1}^{D} \frac{r_{p_{2e_d}/2}}{D}, \quad r_{f_{e_i}} = 0, \quad i \in D, \]
\[ r_{w_{N_D(\alpha)}} = \frac{He_{\alpha_1}(\lambda)}{\alpha_1!} (r_{f_\alpha} + G(\alpha)) - G(\alpha) \quad \text{if } \alpha_1 \neq 0, \ |\alpha| \geq 3. \]

where \( v_{N_D-1(\alpha)} \) and \( \lambda \) are undetermined parameters, and
\[ (3.17) \quad G(\alpha) = \sum_{d=1}^{D} g_{\alpha - e_d} r_{u_d} + \left( \sum_{d=1}^{D} \frac{r_{p_{2e_d}/2}}{D} - \frac{r_{\rho}}{2} \right) \sum_{k=1}^{D} g_{\alpha - 2e_k}. \]
For better readability, here we adopt notation such as $w_k$, $f_{e_i}$, and $f_{2e_i}$ as subscripts of $r$. However, two symbols $r_A$ and $r_B$ denote the same quantity if and only if $A = B$. For example, $r_{w_1}$ and $r_{\rho}$ are the same quantity since $w_1$ always stands for the density $\rho$. However, $r_{\rho}$ and $r_{f_{2e_1}}$ are two different quantities, even if, for some $t$ and $x$, $\rho(t, x) = f_{2e_1}(t, x)$. Similarly, $r_{p_{2e_1}/2}$ and $r_{w_{D+2}}$ are the same, while $r_{p_{2e_1}}$ is not defined above.

We collect these $r_{w_k}$, with $w_k$ as a component of $w$, $k = 1, \ldots, N$, to produce a vector $\mathbf{r} \in \mathbb{R}^N$ as

$$
(3.18) \quad \mathbf{r} = (r_{w_1}, r_{w_2}, \ldots, r_{w_N}),
$$

where $N$ is the total number of moments. Here it is clear that $\mathbf{r}$ is prescribed after $v_{N,D-1}(\bar{a})$ and $\lambda$ are all given. With the particular setup of these parameters, $\lambda$ and $\mathbf{r}$ turn out to be an eigenvalue/eigenvector pair of $\tilde{A}_M$.

Specifically, we have the following lemma:

**Lemma 3.10.** $\mathbf{r} \neq 0$ is the right eigenvector of the matrix $\tilde{A}_M$ for the eigenvalue $\lambda$ if

$$
(3.19) \quad \frac{He_{\alpha_1+1}(\lambda)}{(\alpha_1 + 1)!} (r_{w_{N,D}(\bar{a})} + G(\bar{a})) = 0
$$

holds, for all $|\alpha| = M$.

**Proof.** Let $i = \mathcal{N}_D(\alpha)$, with $|\alpha| \leq M$. Then we need only to verify that

$$
(3.20) \quad \tilde{A}_M(i, 1:N) \cdot \mathbf{r} = \lambda r_{w_i}
$$

is always valid. Since $A_M$ is determined by (3.2), (3.3), and (3.4), and $\tilde{A}_M$ and $\tilde{A}_M$ are defined as in (3.11) and (3.13), respectively, we can write all entries of $\tilde{A}_M$. Now let us verify equation (3.20) case by case:

1. For $\alpha = 0$, $\tilde{A}_M(1, \mathcal{N}_D(e_1)) = 1$ is the only nonzero entry of $\tilde{A}_M(1, 1:N)$; hence,

$$
\tilde{A}_M(i, 1:N) \cdot \mathbf{r} = 1 \cdot r_{u_1} = \lambda r_{\rho} = \lambda r_{w_i}.
$$

2. For $\alpha = e_1$,

$$
\tilde{A}_M(i, 1:N) \cdot \mathbf{r} = 2 \cdot r_{p_{2e_1}/2} = \lambda^2 r_{\rho} = \lambda r_{u_1} = \lambda r_{w_i}.
$$

3. For $\alpha = e_k$, $k = 2, \ldots, D$,

$$
\tilde{A}_M(i, 1:N) \cdot \mathbf{r} = 1 \cdot r_{p_{2e_1+e_k}/2} = \lambda r_{u_k} = \lambda r_{w_i}.
$$
(4) For $\alpha = 2e_1$,

$$\tilde{A}_M(i, 1:N) \cdot r = \frac{3}{2} r_{u_1} + \sum_{d=1}^{D} 3g_{\alpha + e_1 - e_d} r_{u_d} + 3r_{f_{2e_1}}$$

$$= \frac{3}{2} r_{u_1} + 3g_{2e_1} r_{u_1} + 3 \left( \frac{He_3(\lambda)}{6} r_\rho - g_{2e_1} r_{u_1} \right)$$

$$= \frac{\lambda^3}{2} r_\rho = \lambda r_{p_{2e_1}} = \lambda r_{w_i}.$$ 

(5) For $\alpha = 2e_k$, $k = 2, \ldots, D$,

$$\tilde{A}_M(i, 1:N) \cdot r = \frac{1}{2} r_{u_1} + \sum_{d=1}^{D} g_{\alpha + e_1 - e_d} r_{u_d} + r_{f_{e_1 + 2e_k}}$$

$$= \frac{1}{2} r_{u_1} + g_{\alpha} r_{u_1} + g_{e_1 + e_k} r_{u_k}$$

$$+ \left( \lambda \left( r_{p_{2e_k}/2 - 1/2} r_\rho \right) - g_{\alpha} r_{u_1} - g_{e_1 + e_k} r_{u_k} \right)$$

$$= \lambda r_{p_{2e_k}/2} = \lambda r_{w_i}.$$ 

(6) For $\alpha = e_1 + e_k$, $k = 2, \ldots, D$,

$$\tilde{A}_M(i, 1:N) \cdot r = 2r_{f_{2e_1 + e_k}} + 2g_{e_1 + e_d} r_{u_1} + (1 + 2g_{2e_1}) r_{u_k}$$

$$= (\lambda^2 - 1) r_{u_k} - 2g_{e_1 + e_k} r_{u_1} - 2g_{2e_1} r_{u_k}$$

$$+ 2g_{e_1 + e_k} r_{u_1} + (1 + 2g_{2e_1}) r_{u_k}$$

$$= \lambda r_{p_{e_1 + e_k}} = \lambda r_{w_i}.$$ 

(7) For $3 \leq |\alpha| < M$, $\alpha_1 > 0$,

$$\tilde{A}_M(i, 1:N) \cdot r = 1 \cdot r_{f_{a - e_1}} + (\alpha_1 + 1) r_{f_{a + e_1}} - \frac{1}{2} \tilde{C}^{(1)}_{\theta, a} r_\rho$$

$$+ \sum_{d=1}^{D} \left( g_{a - e_d - e_1} + (\alpha_1 + 1) g_{a - e_d + e_1} - \frac{\tilde{C}_{\theta, a}}{D} \tilde{\beta}_{e_1 + e_d} \right) r_{u_d}$$

$$+ \sum_{d=1}^{D} \left( -g_{a - e_d} r_{p_{e_1 + e_d}} + \frac{\tilde{C}^{(1)}_{\theta, a}}{D} r_{p_{2e_1}/2} \right) - \frac{\tilde{C}_{\theta, a}}{D} r_{q_1}$$

$$\triangleq X_1 + X_2.$$
where

\[ \tilde{C}_\alpha = \sum_{k=1}^{D} g_{\alpha-2e_k}, \]

\[ \tilde{C}^{(1)}_\theta,\alpha = \sum_{k=1}^{D} (g_{\alpha-2e_k-e_1} + (\alpha_1 + 1)g_{\alpha-2e_k+e_1}), \]

\[ r_{q1} = 3r_{f_{3e_1}} + \sum_{k=2}^{D} r_{f_{e_1+2e_k}} = -\frac{D}{2} \lambda r_\rho + \sum_{k=1}^{D} (\lambda r_{p_{2e_k}/2} - \tilde{\rho}_{e_1+e_k} r_{uk}), \]

\[ X_1 = r_{f_{\alpha-e_1}} + (\alpha_1 + 1)r_{f_{\alpha+e_1}}, \quad X_2 \text{ are the remaining terms.} \]

Substituting (3.17) into \( X_1 \) yields

\[
(3.22) \quad X_1 = \frac{He_{\alpha_1-1}(\lambda)}{(\alpha_1 - 1)!} (r_{f_{\bar{\alpha}}} + G(\bar{\alpha})) - G(\alpha - e_1) + (\alpha_1 + 1) \left( \frac{He_{\alpha_1+1}(\lambda)}{(\alpha_1 + 1)!} (r_{f_{\bar{\alpha}}} + G(\bar{\alpha})) - G(\alpha + e_1) \right) \\
= \lambda \frac{He_{\alpha_1}(\lambda)}{\alpha_1!} (r_{f_{\bar{\alpha}}} + G(\bar{\alpha})) - G(\alpha - e_1) - (\alpha_1 + 1)G(\alpha + e_1). 
\]

For \( X_2 \), using (3.16d), we get

\[
(3.23) \quad X_2 = -\frac{r_\rho}{2} \sum_{k=1}^{D} (g_{\alpha-e_1-2e_k} + (\alpha_1 + 1)g_{\alpha+e_1-2e_k} - \lambda g_{\alpha-2e_k}) \\
+ \sum_{d=1}^{D} r_{u_d} (g_{\alpha-e_1-e_d} + (\alpha_1 + 1)g_{\alpha+e_1-e_d} - \lambda g_{\alpha-e_d}) + \left( \sum_{d=1}^{D} r_{p_{2e_d}/2} \frac{D}{D} \right) \\
\times \sum_{k=1}^{D} (g_{\alpha-e_1-2e_k} + (\alpha_1 + 1)g_{\alpha+e_1-2e_k} - \lambda g_{\alpha-2e_k}).
\]
Now we calculate $G(\alpha - e_1) + (\alpha_1 + 1)G(\alpha + e_1) - \lambda G(\alpha)$. Some simplification gives

$$G(\alpha - e_1) + (\alpha_1 + 1)G(\alpha + e_1) - \lambda G(\alpha)$$

$$= \sum_{d=1}^{D} r_{ud} (g_{\alpha-e_1-e_d} + (\alpha_1 + 1)g_{\alpha+e_1-e_d} - \lambda g_{\alpha-e_d})$$

$$+ \left( \sum_{d=1}^{D} \frac{r_{p2e_d}/2}{D} - \frac{r_p}{2} \right)$$

$$\times \sum_{k=1}^{D} (g_{\alpha-e_1-2e_k} + (\alpha_1 + 1)g_{\alpha+e_1-2e_k} - \lambda g_{\alpha-2e_k})$$

$$= X_2. \tag{3.24}$$

(3.22), (3.23), and (3.24) show

$$X_1 + X_2 = \frac{\lambda}{\alpha_1!} \left( \frac{He_{\alpha_1}(\lambda)}{\alpha_1} (r_{f\alpha} + G(\alpha)) - \lambda G(\alpha) = \lambda r_{u_1}. \right)$$

(8) For $3 \leq |\alpha| < M$, $\alpha_1 = 0$ or $\alpha = e_k + e_j$, $j > k > 1$, one lets $r_{f\alpha-e_1} = 0$ in (3.21), which is actually part of (3.16c). Hence (3.20) is valid in this case.

(9) For $|\alpha| = M$, if $\alpha_1 > 0$, then (3.11) and (3.13) show this case is equivalent to letting

$$r_{f\alpha+e_1} + \sum_{d=1}^{D} g_{\alpha-e_d+e_1}r_{u_d}$$

$$+ \left( \sum_{d=1}^{D} g_{\alpha-2e_d+e_1} \right) \left( \sum_{i=1}^{D} \frac{1}{D} r_{p2e_i}/2 - \frac{1}{2} r_p \right) = 0 \tag{3.25}$$

in (3.21). Since

$$G(\alpha + e_1) = \sum_{d=1}^{D} g_{\alpha-e_d+e_1}r_{u_d} + \left( \sum_{d=1}^{D} g_{\alpha-2e_d+e_1} \right) \left( \sum_{i=1}^{D} \frac{1}{D} r_{p2e_i}/2 - \frac{1}{2} r_p \right).$$

we need only to prove

$$r_{f\alpha+e_1} + G(\alpha + e_1) = 0.$$

Actually, it is what (3.19) tells.

If $\alpha_1 = 0$, (3.11) and (3.13) show this case is equivalent to letting $r_{f\alpha-e_1} = 0$ and (3.25) is valid in (3.21). The former is part of (3.16c), while the latter is proved above. Hence (3.20) is valid in this case.

Collecting all the cases above, we conclude that (3.20) is valid for arbitrary $\alpha$. The lemma is proved. \qed
For any $\alpha$, let $\beta = \alpha + ke_1$, $k \in \mathbb{N}$; then $\bar{\alpha} = \alpha - \alpha_1 e_1 = \beta - \beta_1 e_1 = \bar{\beta}$ holds. Therefore, $r_{\omega_{N_D}(\bar{\alpha})}, |\alpha| \leq M$ is equivalent to $r_{\omega_{N_D}(\bar{\alpha})}, |\alpha| = M$. Hence, in Lemma 3.10, parameters $r_{\bar{\alpha}}, |\alpha| = M$, and $\lambda$ are all undetermined. Let $N_v = N_{\bar{D}}^{-1}(M \bar{\epsilon}_D)$ and

$$v = (v_1, v_2, \ldots, v_{N_v}) \in \mathbb{R}^{N_v}.$$ 

Since $r$ is determined by $v$ and $\lambda$, by studying the space of the parameters $v$ and $\lambda$, we can fully clarify the structure of the eigenvectors of $\bar{A}_M$.

Now let us consider the case in Example 3.9. In Example 3.9, the relation between the eigenvalues and the eigenvectors of $\bar{A}_4$ with $D = 2$ is given in (3.15). Now these eigenvalues and eigenvectors will be described in more detail. In order to get a closer look at the constraints (3.15), we expand it as below:

\begin{align*}
(3.26a) \quad & \frac{He_5(\lambda)}{5!} (R_{N_D}(0) + G(0)) = \frac{He_5(\lambda)}{5!} v_1 = 0, \\
(3.26b) \quad & \frac{He_4(\lambda)}{4!} (R_{N_D}(e_2) + G(e_2)) = \frac{He_4(\lambda)}{4!} 2v_2 = 0, \\
(3.26c) \quad & \frac{He_3(\lambda)}{3!} (R_{N_D}(e_2) + G(2e_2)) = \frac{He_3(\lambda)}{3!} \left( \frac{3}{2} v_3 + \frac{\lambda^2 - 2}{4} v_1 \right) = 0, \\
(3.26d) \quad & \frac{He_2(\lambda)}{2!} (R_{N_D}(3e_2) + G(2e_2)) = \frac{He_2(\lambda)}{2!} (v_4 + g_{2e_2} v_2) = 0, \\
(3.26e) \quad & \frac{He_1(\lambda)}{1!} (R_{N_D}(4e_2) + G(4e_2)) = \frac{He_1(\lambda)}{1!} (v_5 + g_{3e_2} v_2 + \frac{(\lambda^2 - 2)v_1 + 2v_3}{4} g_{2e_2}) = 0.
\end{align*}

For conciseness, the following symbols are introduced:

$$\hat{v}_1 = v_1, \quad \hat{v}_2 = 2v_2, \quad \hat{v}_3 = \frac{3}{2} v_3 + \frac{\lambda^2 - 2}{4} v_1,$$

$$\hat{v}_4 = v_4 + g_{2e_2} v_2, \quad \hat{v}_5 = v_5 + g_{3e_2} v_2 + \frac{(\lambda^2 - 2)v_1 + 2v_3}{4} g_{2e_2}.$$ 

Since all the equations in (3.26) must be satisfied, at least one of the following conditions should be met:

\begin{align*}
(3.27a) \quad & He_5(\lambda) = 0, \quad \hat{v}_2 = 0, \quad \hat{v}_3 = 0, \quad \hat{v}_4 = 0, \quad \hat{v}_5 = 0; \\
(3.27b) \quad & He_4(\lambda) = 0, \quad \hat{v}_1 = 0, \quad \hat{v}_3 = 0, \quad \hat{v}_4 = 0, \quad \hat{v}_5 = 0; \\
(3.27c) \quad & He_3(\lambda) = 0, \quad \hat{v}_1 = 0, \quad \hat{v}_2 = 0, \quad \hat{v}_4 = 0, \quad \hat{v}_5 = 0; \\
(3.27d) \quad & He_2(\lambda) = 0, \quad \hat{v}_1 = 0, \quad \hat{v}_2 = 0, \quad \hat{v}_3 = 0, \quad \hat{v}_5 = 0; \\
(3.27e) \quad & He_1(\lambda) = 0, \quad \hat{v}_1 = 0, \quad \hat{v}_2 = 0, \quad \hat{v}_3 = 0, \quad \hat{v}_4 = 0.
\end{align*}
For the first case, (3.27a), the condition $He_5(\lambda) = 0$ provides five choices of $\lambda$, and for each $\lambda$, the equations $\hat{v}_2 = 0$, $\hat{v}_3 = 0$, $\hat{v}_4 = 0$, and $\hat{v}_5 = 0$ form a non-degenerate linear system of $v_i, i = 2, \ldots, 5$, where $v_1$ is considered as a constant here. Thus, $v_2, \ldots, v_5$ can all be represented as a function of $v_1$.

In order for $R$ to be a nonzero vector, $v_1$ must be nonzero. This means that five different eigenvalues and five linearly independent eigenvectors are implied by (3.27a), and for each eigenvector, we have $v_1 \neq 0$. Analogously,

- (3.27b) determines four linearly independent eigenvectors, and for each eigenvector $v_1 = 0, v_2 \neq 0$ holds;
- (3.27c) determines three linearly independent eigenvectors, and for each eigenvector $v_1 = v_2 = 0, v_3 \neq 0$ holds;
- (3.27d) determines two linearly independent eigenvectors, and for each eigenvector $v_1 = v_2 = v_3 = 0, v_4 \neq 0$ holds;
- (3.27e) determines one eigenvector, and for this eigenvector $v_1 = v_2 = v_3 = v_4 = 0, v_5 \neq 0$ holds.

In total, $\tilde{A}_4$ has $5 + 4 + 3 + 2 + 1 = 15 = \binom{4 + 2}{2}$ linearly independent eigenvectors.

Actually, the procedure above can be extended to the general case, and the result is the following lemma:

**Lemma 3.12.** $\tilde{A}_M$ has $N$ linearly independent eigenvectors.

**Proof.** For $|\alpha| = M$, (3.19) can be written as

- if $\alpha = 0$, then $N_{D-1}(\alpha) = 1$ and
  \begin{equation}
  v_1He_{M+1}(\lambda) = 0;
  \end{equation}

- if $\alpha = e_k, k \in \mathcal{D}\setminus\{1\}$, then $N_{D-1}(\alpha) = k$ and
  \begin{equation}
  v_kHe_M(\lambda) = 0;
  \end{equation}

- if $\alpha = 2e_k, k \in \mathcal{D}\setminus\{1\}$, then
  \begin{equation}
  0 = \left( v_{N_{D-1}(2\hat{e}_k)} + \sum_{d=1}^{D} \frac{ru_{N_{D-1}(2\hat{e}_d)}}{D} - \frac{v_1}{2} \right)He_{M-1}(\lambda) \\
  = \left( v_{N_{D-1}(2\hat{e}_k)} + \sum_{d=2}^{D} \frac{v_{N_{D-1}(2\hat{e}_d)}}{D} + \frac{\lambda^2}{2D} v_1 - \frac{v_1}{2} \right)He_{M-1}(\lambda);
  \end{equation}

- if $\alpha = e_k + e_l, k \neq l$, and $k, l \in \mathcal{D}\setminus\{1\}$, then
  \begin{equation}
  v_{N_{D-1}(\alpha)}He_{M-1}(\lambda) = 0;
  \end{equation}
otherwise (3 ≤ |α| ≤ M),

\[
\begin{align*}
&v_N D^{-1}(\alpha) + \sum_{d=2}^{D} g_{\alpha-e_d} v_d \\
&+ \sum_{i=1}^{D} g_{\alpha-2e_i} \left( \sum_{d=2}^{D} \frac{v_N D^{-1}(2e_d)}{D} + \frac{\lambda^2}{2D} v_1 - \frac{v_1}{2} \right) \right) He_{\alpha_1+1}(\lambda) = 0.
\end{align*}
\]

Let

\[
z_{\lambda} = (He_{M+1}(\lambda), \ldots, He_{k}(\lambda), \ldots, He_{k}(\lambda), \ldots, He_{1}(\lambda)),
\]

where the \(N D^{-1}(\alpha)^{th}\) component of \(z_{\lambda}\) is \(He_{\alpha_1+1}(\lambda)\), and the cardinal number of the set is

\[
\#\{\alpha \mid |\alpha| = M, \alpha_1 = k - 1\} = \binom{D-1 + M - k}{M + 1 - k}.
\]

Equations (3.28), (3.29), (3.30), (3.31), and (3.32) can be collected as

\[
z_{\lambda} \circ B v = 0,
\]

where \(c = a \circ b\) stands for \(c_i = a_i b_i, i = 1, \ldots, n\), and \(B\) is an \((N_v + 1) \times (N_v + 1)\) real matrix. Specifically, the formation of \(B\) is

\[
B = \begin{bmatrix}
I & 0 & 0 \\
B_{21} & B_{22} & 0 \\
B_{31} & B_{32} & I
\end{bmatrix}
\]

where \(I\) is an identity matrix whose dimension depends on the context. The first \(D\) rows of \(B\) arise from (3.28) and (3.29); the following \(D(D-1)/2\) rows arise from (3.30) and (3.31), and the remaining \(N_v - D(D+1)/2\) rows arise from (3.32).

The properties of \(B\) are here further clarified. We denote the entry of \(B\) located at the \((i, j)\) position as \(b_{ij}\), where \(i, j = 0, \ldots, N_v\). Noticing that entries of \(B_{21}\) are from (3.30) and (3.31), we have

\[
b_{ij} = \begin{cases} 
\frac{\lambda^2}{2D} - \frac{1}{2} & \text{if } j = 1 \text{ and } i = N D^{-1}(2e_l) \text{ for some } l = 2, \ldots, D, \\
0 & \text{otherwise.}
\end{cases}
\]

By equations (3.30) and (3.31), it is clear that \(B_{22}\) can be written as

\[
B_{22} = I + \frac{1}{D} \Omega.
\]
where $\Omega \in \mathbb{R}^{\frac{D(D-1)}{2} \times \frac{D(D-1)}{2}}$ is a matrix with its entries $\omega_{ij}$ being

$$
\omega_{ij} = \begin{cases} 
1 & \text{if } i = N_{D-1}(2\hat{e}_k) - D, \; j = N_{D-1}(2\hat{e}_l) - D \text{ for some } k, l \in D \setminus \{1\}, \\
0 & \text{otherwise}.
\end{cases}
$$

Since there are at most $D - 1$ nonzero entries in each row of $\Omega$, it is clear that $B_{22}$ is strictly diagonally dominant and thus nonsingular, and so $B$ is nonsingular. With the entry value of $\Omega$ as given above, one can check that $\Omega^2 = (D - 1)\Omega$ holds. Hence we can get the inverse of $B_{22}$, namely,

$$
(3.36) \quad B_{22}^{-1} = I - \frac{1}{2D - 1} \Omega.
$$

Meanwhile, since $B$ is a nonsingular block lower triangular matrix, we can get its inverse as

$$
B^{-1} = \begin{bmatrix}
I & 0 & 0 \\
-\bar{B}_{22}^{-1}B_{21} & \bar{B}_{22}^{-1} & 0 \\
* & * & I
\end{bmatrix}
$$

Let

$$
(3.37) \quad \hat{\mathbf{B}} = \text{diag}\{I, \ B_{22}, \ I\}
$$

be the diagonal blocks of $B$. The inverse of $\hat{\mathbf{B}}$ is $\hat{\mathbf{B}}^{-1} = \text{diag}\{I, \ B_{22}^{-1}, \ I\}$, and we have

$$
(3.38) \quad \hat{\mathbf{B}}B^{-1} = \begin{bmatrix}
I & 0 & 0 \\
-\bar{B}_{21} & I & 0 \\
* & * & I
\end{bmatrix}.
$$

Since $B$ is nonsingular, for an arbitrary $j \in \{1, \ldots, N_v\}$, we let

$$
(3.39) \quad \mathbf{v}^{(j)} = \hat{\mathbf{B}}B^{-1}I_j,
$$

where $I_j$ is the $j$th column of the $N_v \times N_v$ identity matrix. Actually, $\mathbf{v}^{(j)}$ is the $j$th column of $\hat{\mathbf{B}}B^{-1}$. Notice in (3.33) for any $\alpha$ where $|\alpha| = M$, the $N_{D-1}(\hat{\alpha})$th component of $\mathbf{z}_{\hat{\alpha}}$ is $He_{\alpha_1 + 1}(\lambda)$. For the $\alpha$ satisfying

$$
|\alpha| = M \quad \text{and} \quad j = N_{D-1}(\hat{\alpha}), \quad k = \alpha_1 + 1,
$$

we choose $\lambda$ such that

$$
He_k(\lambda) = 0.
$$

Then we have that the $j$th component of $\mathbf{z}_{\hat{\alpha}}$ vanishes,

$$
z_{\hat{\alpha},j} = He_k(\lambda) = 0.
$$

Therefore (3.34) holds if

$$
(3.40) \quad \hat{\mathbf{B}}B^{-1}\mathbf{v}^{(j)} = \rho I_j, \quad z_{\hat{\alpha},j} = 0, \quad j = 1, \ldots, N_v.
$$
Since \( r \) depends only on \( v \) and \( \lambda \), we denote by \( r_{\tilde{\alpha},i} \) the vector prescribed by the given \( v^{(j)}, j = \mathcal{N}_{D-1}(\tilde{\alpha}) \), and \( \lambda = C_{i,k} \), when \( k = \alpha_1 + 1 \), for arbitrary \( |\alpha| = M \), \( i = 1, \ldots, k \). It is clear that \( C_{i,k} \) and \( r_{\tilde{\alpha},i} \) are an eigenvalue/eigenvector pair of \( \tilde{\Lambda}_M \) such that

\[
\tilde{\Lambda}_M r_{\tilde{\alpha},i} = C_{i,k} r_{\tilde{\alpha},i}.
\]

The eigenvectors of \( \tilde{\Lambda}_M \) can be divided into a cluster of classes, each of which is, for arbitrary \( j = \mathcal{M}, i = 1, \ldots, k \),

\[
\{ r_{\tilde{\alpha},i} \mid i = 1, \ldots, k, k = \alpha_1 + 1 \}.
\]

This fact essentially stems from the reducibility of the matrix \( \tilde{\Lambda}_M \).

Notice the following:

1. The components of \( v \) are a subset of \( r \)'s components, and linearly independent \( v^{(j)} \)'s determine linearly independent \( r \)'s.
2. Eigenvectors belonging to different eigenvalues are orthogonal and the \( k \) zeros of the Hermite polynomial \( H_k(\lambda) \) are different.

We have that \( r_{\tilde{\alpha},i}, i = 1, \ldots, k \), when \( |\alpha| = M \) and \( k = \alpha_1 + 1 \) are linearly independent and the matrix \( \tilde{\Lambda}_M \) has

\[
\sum_{k=1}^{M+1} k \binom{D - 1 + M - k}{M + 1 - k} = \binom{M + D}{D} = N
\]

differently independent eigenvectors.

On the other hand, for arbitrary \( r_{\tilde{\alpha},i} \), there exists a unique \( \beta \) satisfying \( \beta = (i - 1)e_1 + \tilde{\alpha}; \) hence there is a one-to-one mapping between \( r_{\tilde{\alpha},i} \) and \( \alpha \) with \( |\alpha| \leq M \). So we can also get that \( \tilde{\Lambda}_M \) has \( N \) linearly independent eigenvectors. This completes the proof. \( \square \)

With the help of Lemma 3.12 it is not difficult to get the following result:

**Lemma 3.13.** Let

\[
\tilde{\Phi}_{1,m} = H_{m+1}(\lambda), \quad m \in \mathbb{N},
\]

\[
\tilde{\Phi}_{D,m} = \prod_{k=0}^{m} \tilde{\Phi}_{D-1,k}, \quad 1 < D \in \mathbb{N}^+.
\]

\( \tilde{\Phi}_{D,M} \) is the characteristic polynomial of \( \tilde{\Lambda}_M \).

**Proof.** In the case of \( D = 1 \), the result has been proved in [1]. Here we give the proof for \( D \geq 2 \). In the proof of Lemma 3.12 (3.40) shows that the characteristic polynomial of \( \tilde{\Lambda}_M \) is

\[
\tilde{\Phi}_{D,M} = \prod_{k=1}^{M+1} H_k(\lambda)^{D-1+M-k} = \prod_{k=1}^{M+1} H_k(\lambda)^{D-1+M-k}.
\]
Now we need only to prove that $\overline{P}_{D,M}$ satisfies (3.43). Here we use an induction argument on $D$. As $D = 2$, (3.44) can be written as

$$\overline{P}_{2,M} = \prod_{k=1}^{M+1} \text{He}_k(\lambda) = \prod_{m=0}^{M} \overline{P}_{1,m}.$$  

We assume that (3.43) holds for $D - 1$, $D \geq 3$. With the induction hypothesis, we have

$$\prod_{m=0}^{M} \overline{P}_{D-1,m} = \prod_{m=0}^{M} \left( \prod_{k=1}^{m+1} \text{He}_k(\lambda) \left( \frac{D-1-1+m-k}{D-1-2} \right) \right)$$

$$= \prod_{k=1}^{M+1} \text{He}_k(\lambda) \sum_{m=k-1}^{M} \left( \frac{D-2+m-k}{D-3} \right)$$

$$= \prod_{k=1}^{M+1} \text{He}_k(\lambda) \left( \frac{D-1+M-k}{D-2} \right) = \overline{P}_{D,M}.$$  

This completes the proof. \hfill \Box

With the relation of $\tilde{A}_M$ and $\tilde{A}_M$ (3.13), we have the following theorem:

**Theorem 3.14.** Let

$$P_{1,m} = \text{He}_{m+1} \left( \frac{\lambda - u_1}{\sqrt{\theta}} \right)^{(m+1)/2}, \quad m \in \mathbb{N},$$

$$P_{D,m} = \prod_{k=0}^{m} P_{D-1,k}, \quad 1 < D \in \mathbb{N}^+.$$  

$P_{D,M}$ is the characteristic polynomial of $\tilde{A}_M$, which has $N$ linearly independent eigenvectors that can be written as

$$(3.45) \quad \tilde{r}_{i,i} = A^{-1} r_{i,i} \quad \text{for eigenvalue} \lambda_{i,k} = u_1 + C_{i,k} \sqrt{\theta}$$

for all $|\alpha| = M$, $i = 1, \ldots, k$, where $k = \alpha_1 + 1$.

**Proof.** Since $\tilde{A}_M = u_1 I + \sqrt{\theta} A^{-1} A^T A$ and $A$ is nonsingular, so any $r_{i,i} \in \mathbb{R}^N$ is the eigenvector of $\tilde{A}_M$ for the eigenvalue $C_{i,k}$, then $\tilde{r}_{i,i} = A^{-1} r_{i,i}$ is the eigenvector of $\tilde{A}_M$ for the eigenvalue $u_1 + C_{i,k} \sqrt{\theta}$. Using Lemma 3.10 and the discussion in Lemma 3.12, we obtain (3.45). Lemma 3.12 shows $\tilde{A}_M$ has $N$ linearly independent eigenvectors, so $\tilde{A}_M$ also has $N$ linearly independent eigenvectors and (3.45) gives a basis set.

Lemma 3.12 and (3.45) show that the characteristic polynomial of $\tilde{A}_M$ is

$$\overline{P}_{D,M} = \prod_{k=1}^{M+1} \left( \text{He}_k \left( \frac{\lambda - u_1}{\sqrt{\theta}} \right) \theta^{k/2} \right) \left( \frac{D-1+M-k}{D-2} \right).$$
Using the same idea as in the proof of Lemma \ref{lem:3.13}, we obtain $P_{D,M} = \overline{P}_{D,M}$ is thus the characteristic polynomial of $\hat{A}_M$.

Theorem \ref{thm:3.8} is now straightforward:

**Proof of Theorem \ref{thm:3.8}** With Theorem \ref{thm:3.14}, we declare that $\hat{A}_M$ is diagonalizable with real eigenvalues directly; that is, the moment system (3.12) is hyperbolic.

4 System in Multidimensional Spatial Space

As the main result of this paper, here we give the general hyperbolic moment system containing all moments with orders lower than $M$. Without the assumption that the dependence of $f$ on $x_2, \ldots, x_D$ is homogeneous, according to the discussions in Section \ref{sec:2.3} Grad’s moment system can be written in the following form:

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{D} M_j(w) \frac{\partial w}{\partial x_j} = 0,$$

where $w$ has the same definition as in the one-dimensional case, (3.6), and $M_j, j = 1, \ldots, D$, are square matrices depending on $w$. Comparing with (3.7), one immediately has $M_1 = A_M$. We give the following definition, which is similar to Definition \ref{def:3.6}:

**Definition 4.1.** For $j = 1, \ldots, D$, $\hat{M}_j$ is called the regularized matrix of the matrix $M_j$ if it satisfies that for any admissible $w$,

$$\hat{M}_j \frac{\partial w}{\partial x_j} = M_j \frac{\partial w}{\partial x_j} - \sum_{|\alpha|=M} R_{M,D}(\alpha) I_{N,D}(\alpha),$$

where $I_k$ is the $k$th column of the $N \times N$ identity matrix.

Now the multidimensional regularized moment equations can be written as

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{D} \hat{M}_j(w) \frac{\partial w}{\partial x_j} = 0.$$  

Recalling that

$$R_{M,D}(\alpha) = \sum_{j=1}^{D} R_{M,D}^j(\alpha).$$

one finds that the multidimensional regularized moment system is obtained by subtracting $R_{M,D}(\alpha)$ from (4.1) for all $|\alpha| = M$. Applying such an operation to (2.6),
we can reformulate the regularized moment system as

\[
\left( \frac{\partial f_\alpha}{\partial t} + \sum_{d=1}^{D} \frac{\partial u_d}{\partial t} f_{\alpha-e_d} + \frac{1}{2} \frac{\partial \theta}{\partial t} \sum_{d=1}^{D} f_{\alpha-2e_d} \right) \\
+ \sum_{j=1}^{D} \left( \theta \frac{\partial f_{\alpha-e_j}}{\partial x_j} + u_j \frac{\partial f_\alpha}{\partial x_j} + (1 - \delta_{|\alpha|,M})(\alpha_j + 1) \frac{\partial f_{\alpha+e_j}}{\partial x_j} \right) \\
+ \sum_{j=1}^{D} \sum_{d=1}^{D} \frac{\partial u_d}{\partial x_j} \left( \theta f_{\alpha-e_d-e_j} + u_j f_{\alpha-e_d} \right) \\
+ (1 - \delta_{|\alpha|,M})(\alpha_j + 1) f_{\alpha-e_d+e_j} \\
+ \frac{1}{2} \sum_{j=1}^{D} \sum_{d=1}^{D} \frac{\partial \theta}{\partial x_j} \left( \theta f_{\alpha-2e_d-e_j} + u_j f_{\alpha-2e_d} \right) \\
+ (1 - \delta_{|\alpha|,M})(\alpha_j + 1) f_{\alpha-2e_d+e_j} = 0, \quad |\alpha| \leq M.
\]

Actually, (4.2) can be obtained by a linear transformation of (4.4). More precisely, if we use (2.7) to eliminate the time derivatives of \(u_d\) and \(\theta\) in (4.4), then (4.2) is immediately obtained. Such an operation can be inverted, which means there exists an invertible matrix \(T(w)\) depending on \(w\) such that (4.4) is identical to the following system:

\[
T(w) \frac{\partial w}{\partial t} + \sum_{j=1}^{D} T(w) \hat{M}_j(w) \frac{\partial w}{\partial x_j} = 0.
\]

If we let all partial derivatives with respect to \(x_j\) with \(j > 1\) be 0, (4.4) reduces to the one-dimensional hyperbolic moment system (3.12) in Section 3. Comparison of (4.2) and (3.12) clearly shows that \(\hat{M}_1 = \hat{A}_M\).

The following theorem declares the hyperbolicity\(^2\) of the multidimensional regularized moment system (4.2):

**Theorem 4.2.** The regularized moment system (4.2) is hyperbolic for any admissible \(w\). Specifically, for a given unit vector \(n = (n_1, \ldots, n_D)\), there exists a constant matrix \(R\) partially dependent on \(n\) such that

\[
\sum_{j=1}^{D} n_j \hat{M}_j(w) = R^{-1} \hat{A}_M(Rw)R.
\]

and this matrix is diagonalizable and its eigenvalues are

\[
u \cdot n + C_{n,m} \sqrt{\theta}, \quad 1 \leq n \leq m \leq M + 1.
\]

\(^2\)For multidimensional quasi-linear systems, we refer the readers to [9] for the definition of hyperbolicity.
Actually, this theorem gives the rotation invariance of the regularized moment system and its global hyperbolicity. Since the translation invariance of the system is apparent, we conclude that the regularized system is Galilean invariant. Specifically, if another set of coordinates \((\tilde{x}_1, \ldots, \tilde{x}_D)\) is chosen and the vector \(\mathbf{n}\) is along the \(\tilde{x}_1\)-axis, then the rotated moment system is equivalent to the original one. This result is easy to understand: on one hand, Grad’s moment system is rotationally invariant, since the full \(M\)-degree polynomials are used in the truncated Hermite expansion; on the other hand, our regularization is symmetric in every direction, which can be considered as “isotropic” in some sense. However, a rigorous proof of this theorem is rather tedious.

In the literature, two types of indices have been used in the moment methods. In Grad’s paper [8], indices such as

\[
\vartheta = (\vartheta_1, \ldots, \vartheta_m) \in \mathbb{D}^m
\]

are used to denote the \(m\)th-order moments, while in [2], the symbols

\[
\alpha = (\alpha_1, \ldots, \alpha_D) \in \mathbb{N}^D
\]

are used as the subscripts of \(|\alpha|^\text{th}\)-order moments. The former is convenient for mathematical proofs, while the latter is easier to use in the numerical implementation, since for (4.8), the map from the index set to the moment set is a bijection, while this is not true for (4.7). If (4.8) and (4.7) represent the same moment, then one has

\[
\alpha = e_{\vartheta_1} + \cdots + e_{\vartheta_m}, \quad m = |\alpha|.
\]

Below, both types of indices are needed in the proof of rotation invariance, and we will always use the variant forms of Greek letters such as \(\vartheta\) and \(\varphi\) to denote the Grad-type indices, and normal Greek letters such as \(\alpha\) and \(\beta\) will be used to denote indices like (4.8). The Greek letter “sigma” denotes the conversion between them. Supposing (4.9) holds, we write

\[
\alpha = \sigma(\vartheta), \quad \vartheta = \varsigma(\alpha).
\]

That is, the normal form of sigma \(\sigma(\cdot)\) converts indices like (4.7) to indices like (4.8), and the variant form of sigma \(\varsigma(\cdot)\) does the inverse conversion. Note that for a given \(\alpha\), the Grad-type index \(\vartheta\) satisfying (4.9) is not uniquely determined. Define

\[
\Sigma(\alpha) = \{\vartheta \in \mathbb{D}^{|\alpha|} \mid \sigma(\vartheta) = \alpha\};
\]

then in most cases \(\Sigma(\alpha)\) has more than one element. For example, if \(D = 2\) and \(\alpha = (2, 2)\), then

\[
\Sigma(\alpha) = \{(1, 1, 2, 2), (1, 2, 1, 2), (1, 2, 2, 1), (2, 2, 1, 1), (2, 1, 2, 1), (2, 1, 1, 2)\}.
\]

Thus \(\varsigma(\alpha)\) has multiple values. However, there is always one special element \(\vartheta \in \Sigma(\alpha)\) satisfying

\[
\vartheta_1 \leq \cdots \leq \vartheta_{|\alpha|},
\]
and we use this element as the value of $\zeta(\alpha)$. It is easy to find
\[ \sigma(\zeta(\alpha)) = \alpha. \]
Additionally, we use $\sigma_i(\vartheta)$ to denote the $i^{th}$ component of $\sigma(\vartheta)$.

Based on these symbols, we have the following lemma:

**Lemma 4.3.** Suppose $\alpha \in \mathbb{N}^D$ and $F(\cdot)$ is a function on $\mathcal{D}_m$. If $F$ satisfies that $F(\varphi)$ is 0 when $\sigma_i(\varphi) < \alpha_i$ for some $i \in D$, then the following equality holds:
\[
\sum_{\varphi \in \mathcal{D}_m} F(\varphi) = \sum_{\beta \in \mathbb{N}^D \mid |\beta| = m - |\alpha|} \sum_{\varphi \in \Sigma(\beta + \alpha)} F(\varphi).
\]

**Proof.** It is obvious that
\[ I \triangleq \bigcup_{\beta \in \mathbb{N}^D \mid |\beta| = m - |\alpha|} \Sigma(\beta + \alpha) \subset \mathcal{D}_m, \]
and there are no duplicate elements in the union since $\Sigma(\beta + \alpha) \cap \Sigma(\tilde{\beta} + \alpha) = \emptyset$ if $\beta \neq \tilde{\beta}$. Thus it only remains to prove that $\varphi \in I$ if
\[ \varphi \in \mathcal{D}_m \quad \text{and} \quad \sigma_i(\varphi) \geq \alpha_i \quad \forall i \in D. \]
This is true since $\varphi \in \Sigma(\beta + \alpha)$ for $\beta = \sigma(\varphi) - \alpha$. \[ \square \]

As a special case of Lemma 4.3, we set $\alpha = 0$ and have
\[
(4.11) \sum_{\varphi \in \mathcal{D}_m} F(\varphi) = \sum_{\beta \in \mathbb{N}^D \mid |\beta| = m} \sum_{\varphi \in \Sigma(\beta)} F(\varphi).
\]

Here $F(\cdot)$ is an arbitrary function on $\mathcal{D}_m$.

Some more symbols are introduced as follows. All $m$-permutations of the set $\{1, \ldots, n\}$ form the following set:
\[
A^m_n = \{ \varpi = (\varpi_1, \ldots, \varpi_m) \in \{1, \ldots, n\}^m \mid \varpi_i \neq \varpi_j \text{ if } i \neq j \}
\]
\[
\forall m, n \in \mathbb{N}, \quad n \geq m,
\]
which contains $n!/m!$ elements. Thus when we want to construct a short vector using the components of a long vector, we will use the following notation:
\[ \vartheta_{\varpi} = (\vartheta_{\varpi_1}, \ldots, \vartheta_{\varpi_m}) \in \mathcal{D}_m \quad \forall \vartheta \in \mathcal{D}_n, \quad \varpi \in A^m_n. \]
The remaining part is denoted as $\vartheta \setminus \vartheta_{\varpi}$. For example, if $\vartheta = (1, 3, 2, 3, 1, 2, 1)$ and $\varpi = (5, 2, 4)$, then
\[ \vartheta_{\varpi} = (1, 3, 3), \quad \vartheta \setminus \vartheta_{\varpi} = (1, 2, 2, 1). \]

Below, $G = (g_{ij})_{D \times D}$ stands for the rotation matrix, and we suppose $G$ is orthogonal and its determinant is 1. Define
\[
(4.13) \prod_{g}(\vartheta, \varphi) = \prod_{i=1}^{n} g_{\vartheta_i \varphi_i} \quad \forall \vartheta, \varphi \in \mathcal{D}_n.
\]
Then we have the following lemma:

**Lemma 4.4.** For a given matrix $G$ and multi-indices $\alpha, \beta \in \mathbb{N}^D$, the following equality holds for arbitrary $\vartheta \in \mathcal{D}^{\alpha + |\beta|}$:

\[ \sum_{\varphi \in \Sigma(\alpha + \beta)} \frac{\sigma(\varphi)}{\sigma(\vartheta)}! \prod_g (\vartheta, \varphi) = \]

\[ \frac{\alpha!}{\sigma(\vartheta)!} \prod_{\varpi \in \mathcal{A}^{\alpha + |\beta|}} \prod_g (\vartheta \setminus \vartheta \varpi \varphi) \]

\[ \sum_{\varphi \in \Sigma(\alpha + \beta)} \frac{\sigma(\varphi)}{\sigma(\vartheta)}! \prod_g (\vartheta, \varphi) = \frac{\alpha!}{\sigma(\vartheta)!} \sum_{\varphi \in \Sigma(\alpha + \beta)} \prod_g (\vartheta \setminus \vartheta \varpi \varphi). \]

**Proof.** We first consider the case $|\beta| = 1$. Suppose $\beta = e_d$; then (4.14) becomes

\[ \sum_{\varphi \in \Sigma(\alpha + e_d)} \frac{\sigma(\varphi)}{\sigma(\vartheta)}! \prod_g (\vartheta, \varphi) = \frac{\alpha!}{\sigma(\vartheta)!} \sum_{i=1}^{\alpha + 1} g_{\vartheta i} d \prod_{\varphi \in \Sigma(\alpha)} \prod_g (\vartheta \setminus \vartheta i \varphi). \]

For $\varphi \in \Sigma(\alpha + e_d)$, one has $\sigma(\varphi) = (\alpha_d + 1)\alpha!$. Thus (4.15) is equivalent to

\[ (\alpha_d + 1) \sum_{\varphi \in \Sigma(\alpha + e_d)} \prod_g (\vartheta, \varphi) = \sum_{i=1}^{\alpha + 1} g_{\vartheta i} d \prod_{\varphi \in \Sigma(\alpha)} \prod_g (\vartheta \setminus \vartheta i \varphi). \]

For an arbitrary $\varphi \in \Sigma(\alpha + e_d)$, if $\varphi_i = d$, then $\prod_g (\vartheta, \varphi) = g_{\vartheta i} d \prod_g (\vartheta \setminus \vartheta i \varphi \setminus \varphi i)$ and $\varphi \setminus \varphi_i \in \Sigma(\alpha)$. Since there are $(\alpha_d + 1)$ choices of $i$ such that $\varphi_i = d$, the product $\prod_g (\vartheta, \varphi)$ appears $(\alpha_d + 1)$ times in the right-hand side of (4.16). This proves (4.15).

Suppose the lemma holds for $|\beta| = m - 1$, and we are going to prove the case $|\beta| = m$. In order to use the technique of induction, we choose $d \in \{1, \ldots, D\}$ such that $\beta_d > 0$, and let $\beta' = \beta - e_d$. Thus $|\beta'| = m - 1$. Applying (4.15), one has

\[ \sum_{\varphi \in \Sigma(\alpha + \beta)} \frac{\sigma(\varphi)}{\sigma(\vartheta)}! \prod_g (\vartheta, \varphi) \]

\[ = \sum_{\varphi \in \Sigma(\alpha + \beta' + e_d)} \frac{\sigma(\varphi)}{\sigma(\vartheta)}! \prod_g (\vartheta, \varphi) \]

\[ = \frac{\sigma(\vartheta')!}{\sigma(\vartheta)!} \sum_{i=1}^{\alpha + |\beta|} g_{\vartheta i} d \prod_{\varphi \in \Sigma(\alpha + \beta')} \prod_g (\vartheta \setminus \vartheta i \varphi) \]

\[ = \sum_{i=1}^{\alpha + |\beta|} \frac{\sigma(\vartheta \setminus \vartheta i \varphi)!}{\sigma(\vartheta)!} \sum_{\varphi \in \Sigma(\alpha + \beta')} \frac{\sigma(\varphi)!}{\sigma(\vartheta \setminus \vartheta i \varphi)!} \prod_g (\vartheta \setminus \vartheta i \varphi). \]
Defining $\hat{\vartheta}^i = \vartheta \setminus \vartheta_i$ and using the inductive assumption, one obtains

$$
(4.17) \quad \sum_{\varphi \in \Sigma(\alpha + \beta)} \frac{\sigma(\varphi)!}{\sigma(\vartheta)!} \Pi_g(\vartheta, \varphi) = \\
\frac{\alpha!}{\sigma(\vartheta)!} \sum_{i=1}^{|\alpha| + |\beta|} g_{\delta_i} d \sum_{\varphi \in \Sigma(\alpha)} \Pi_g(\hat{\vartheta}^i \setminus \vartheta, \sigma(\beta')) \sum_{\varphi \in \Sigma(\alpha)} \Pi_g(\hat{\vartheta}^i \setminus \vartheta, \varphi).
$$

It is evident that the right-hand sides of (4.14) and (4.17) are the same. Thus the lemma is proved. □

Now let us start the rotation. We first define

$$
(4.18) \quad \bar{x}_i = \sum_{j=1}^{D} g_{ij} x_j, \quad i = 1, \ldots, D,
$$

and denote by $\bar{\rho}$, $\bar{u}$, and $\bar{\vartheta}$ the density, macroscopic velocity, and temperature in the new coordinates $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_D)$. If we define $\bar{\xi} = G\xi$, then the orthogonality of $G$ shows

$$
\bar{\rho} = \int_{\mathbb{R}^D} f(\xi) d\bar{\xi} = \int_{\mathbb{R}^D} f(\xi) d\xi = \rho,
$$

$$
\bar{\rho} \bar{u} = \int_{\mathbb{R}^D} \bar{\xi} f(\xi) d\bar{\xi} = \int_{\mathbb{R}^D} G\xi f(\xi) d\xi = \rho Gu,
$$

$$
\bar{\rho} \bar{\vartheta} = \frac{1}{D} \int_{\mathbb{R}^D} |\bar{\xi} - \bar{u}|^2 f(\xi) d\bar{\xi} = \frac{1}{D} \int_{\mathbb{R}^D} |\xi - u|^2 f(\xi) d\xi = \rho \vartheta,
$$

and it follows immediately that

$$
(4.19) \quad \bar{\vartheta} = \vartheta, \quad \bar{u}_i = \sum_{j=1}^{D} g_{ij} u_j, \quad i = 1, \ldots, D.
$$

Now we consider the general moments $\bar{f}_\alpha$ in the coordinates $\bar{x}$. Define $z = (\xi - u)/\sqrt{\vartheta}$ and $\bar{z} = (\bar{\xi} - \bar{u})/\sqrt{\vartheta}$. Then $\bar{z} = Gz$. The orthogonality of Hermite polynomials gives

$$
(4.20) \quad \bar{f}_\alpha = \frac{(2\pi)^D |\alpha| + D}{\alpha!} \int_{\mathbb{R}^D} f(u + \sqrt{\vartheta}z) \mathcal{H}_{\vartheta, \alpha}(z) \exp\left(-\frac{|z|^2}{2}\right) dz.
$$
From the definition of Hermite polynomials (2.4), it is easy to see that (2.3) can be rewritten as

\[ \mathcal{H}_{\theta,\alpha}(z) = (-1)^{\vert \alpha \vert} (2\pi)^{-\frac{D}{2}} \theta^{-\frac{D+\vert \alpha \vert}{2}} \frac{\partial^{\vert \alpha \vert}}{\partial^{\alpha} z} \exp\left( -\frac{|z|^2}{2} \right). \]

Applying the chain rule of differentiation, we obtain

\[ \mathcal{H}_{\theta,\alpha}(z) = (-1)^{\vert \alpha \vert} (2\pi)^{-\frac{D}{2}} \theta^{-\frac{D+\vert \alpha \vert}{2}} \]

\[ \times \sum_{\varphi \in \mathcal{D}^{\vert \alpha \vert}} \Pi_g(\zeta(\alpha), \varphi) \frac{\partial^{\vert \alpha \vert}}{\partial \varphi(z)} \exp\left( -\frac{|z|^2}{2} \right) \]

\[ = \sum_{\varphi \in \mathcal{D}^{\vert \alpha \vert}} \Pi_g(\zeta(\alpha), \varphi) \mathcal{H}_{\theta,\varphi}(z). \]

Collecting (4.20) and (4.21), one obtains

\[ \tilde{f}_\alpha = \frac{(2\pi)^D \theta^{\vert \alpha \vert+D}}{\alpha!} \]

\[ \times \sum_{\varphi \in \mathcal{D}^{\vert \alpha \vert}} \Pi_g(\zeta(\alpha), \varphi) \int_{\mathbb{R}^D} f(u + \sqrt{\theta}z) \mathcal{H}_{\theta,\varphi}(z) \exp\left( -\frac{|z|^2}{2} \right) \mathrm{d}z \]

\[ = \sum_{\varphi \in \mathcal{D}^{\vert \alpha \vert}} \frac{\sigma(\varphi)!}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) f_\varphi. \]

As in (3.6), all the rotated moments can also be collected into a vector denoted as \( \tilde{\mathbf{w}} \). The equations (4.22) and (4.19) directly give the following result:

**Lemma 4.5.** Based on the expressions of the rotated moments (4.22) and (4.19), the following equalities hold for arbitrary \( \alpha \in \mathbb{N}^D \):

\[ \sum_{d=1}^{D} \tilde{u}_d \frac{\partial \tilde{f}_\alpha}{\partial x_d} = \sum_{d=1}^{D} \sum_{\varphi \in \mathcal{D}^{\vert \alpha \vert}} \frac{\sigma(\varphi)!}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) \cdot u_d \frac{\partial f_\varphi}{\partial x_d}. \]  

\[ \sum_{d=1}^{D} (\alpha_d + 1) \frac{\partial \tilde{f}_\alpha + e_d}{\partial x_d} = \]

\[ \sum_{d=1}^{D} \sum_{\varphi \in \mathcal{D}^{\vert \alpha \vert}} \frac{\sigma(\varphi)!}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) \cdot (\sigma_d(\varphi) + 1) \frac{\partial f_\varphi + e_d}{\partial x_d}, \]

where \( \sigma_d(\varphi) \) is the \( d^{th} \) component of \( \sigma(\varphi) \).
PROOF. Using (4.22) and (4.19) directly, we get

\[(4.25)\]
\[
\sum_{d=1}^{D} \tilde{u}_d \frac{\partial \tilde{f}_{\alpha}^{d}}{\partial \tilde{x}_d} = \sum_{d=1}^{D} \sum_{\varphi \in \mathfrak{D}^{[\alpha]}} \frac{\sigma(\varphi)!}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) \cdot \sum_{j=1}^{D} g_{d j} u_j \frac{\partial f_{\sigma(\varphi)}}{\partial \tilde{x}_d}.
\]

Equation (4.18) shows that

\[(4.26)\]
\[
\frac{\partial}{\partial x_j} = \sum_{d=1}^{D} g_{d j} \frac{\partial}{\partial \tilde{x}_d}.
\]

Thus (4.23) is the direct result of (4.25) and (4.26).

The proof of (4.24) is also straightforward:

\[
\sum_{d=1}^{D} (\alpha_d + 1) \frac{\partial \tilde{f}_{\alpha+e_d}}{\partial \tilde{x}_d}
\]
\[
= \sum_{d=1}^{D} (\alpha_d + 1) \sum_{\varphi \in \mathfrak{D}^{[\alpha]+1}} \frac{\sigma(\varphi)!}{(\alpha+\epsilon_d)!} \Pi_g(\zeta(\alpha+\epsilon_d), \varphi) \frac{\partial f_{\sigma(\varphi)}}{\partial \tilde{x}_d}.
\]
\[
= \sum_{d=1}^{D} \sum_{j=1}^{D} g_{d j} \sum_{\varphi \in \mathfrak{D}^{[\alpha]}} \frac{\sigma(\varphi) + \epsilon_j}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) \frac{\partial f_{\sigma(\varphi)+\epsilon_j}}{\partial \tilde{x}_d}
\]
\[
= \sum_{j=1}^{D} \sum_{\varphi \in \mathfrak{D}^{[\alpha]}} \frac{\sigma(\varphi)!}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) \cdot (\sigma_j(\varphi) + 1) \frac{\partial f_{\sigma(\varphi)+\epsilon_j}}{\partial x_j}.
\]

This equality is identical to (4.24). \(\square\)

By using Lemma 4.4, it is not difficult to prove the following lemma:

L EMMA 4.6. The following equalities hold for arbitrary \(\alpha \in \mathbb{N}^D\):

\[(4.27)\]
\[
\sum_{d=1}^{D} \frac{\partial \tilde{u}_d}{\partial t} \tilde{f}_{\alpha-e_d} = \sum_{d=1}^{D} \sum_{\varphi \in \mathfrak{D}^{[\alpha]}} \frac{\sigma(\varphi)!}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) \frac{\partial u_d}{\partial t} f_{\sigma(\varphi)-e_d},
\]

\[(4.28)\]
\[
\sum_{d=1}^{D} \frac{\partial \tilde{\theta}}{\partial t} \tilde{f}_{\alpha-2e_d} = \sum_{d=1}^{D} \sum_{\varphi \in \mathfrak{D}^{[\alpha]}} \frac{\sigma(\varphi)!}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) \frac{\partial \theta}{\partial t} f_{\sigma(\varphi)-2e_d},
\]

\[(4.29)\]
\[
\sum_{d=1}^{D} \frac{\partial \tilde{f}_{\alpha-e_d}}{\partial \tilde{x}_d} = \sum_{d=1}^{D} \sum_{\varphi \in \mathfrak{D}^{[\alpha]}} \frac{\sigma(\varphi)!}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) \frac{\partial f_{\sigma(\varphi)-e_d}}{\partial x_d}.
\]
\begin{align}
(4.30) \quad & \sum_{j=1}^{D} \sum_{d=1}^{D} \frac{\partial \tilde{u}_d}{\partial x_j} f_{\alpha - e_d - e_j} = \\
& \sum_{j=1}^{D} \sum_{d=1}^{D} \sum_{\varphi \in \mathcal{D}[\alpha]} \frac{\sigma(\varphi)!}{\alpha!} \pi_g(\zeta(\alpha), \varphi) \frac{\partial u_d}{\partial x_j} f_{\sigma(\varphi) - e_d - e_j},
\end{align}

\begin{align}
(4.31) \quad & \sum_{j=1}^{D} \sum_{d=1}^{D} \frac{\partial \tilde{\theta}}{\partial x_j} f_{\alpha - 2e_d - e_j} = \\
& \sum_{j=1}^{D} \sum_{d=1}^{D} \sum_{\varphi \in \mathcal{D}[\alpha]} \frac{\sigma(\varphi)!}{\alpha!} \pi_g(\zeta(\alpha), \varphi) \frac{\partial \theta}{\partial x_j} f_{\sigma(\varphi) - 2e_d - e_j}.
\end{align}

**Proof.** Recalling that \( f_\beta = 0 \) if \( \beta \) has a negative component and using Lemma 4.3, we have the following equality:

\[ I \triangleq \sum_{d=1}^{D} \sum_{\varphi \in \mathcal{D}[\alpha]} \frac{\sigma(\varphi)!}{\alpha!} \pi_g(\zeta(\alpha), \varphi) \frac{\partial u_d}{\partial t} f_{\sigma(\varphi) - e_d} \]
\[ = \sum_{d=1}^{D} \sum_{\beta \in \mathcal{N}^D} \sum_{\varphi \in \mathcal{N}(\beta + e_d)} \frac{\sigma(\varphi)!}{\alpha!} \pi_g(\zeta(\alpha), \varphi) \frac{\partial u_d}{\partial t} f_\beta. \]

Let \( \partial = \zeta(\alpha) \) and use (4.15); we then have

\[ I = \sum_{d=1}^{D} \sum_{\beta \in \mathcal{N}^D} \frac{\beta!}{\alpha!} \sum_{i=1}^{\alpha} g_{\partial_i d} \sum_{\varphi \in \mathcal{N}(\beta)} \pi_g(\zeta \setminus \partial_i, \varphi) \frac{\partial u_d}{\partial t} f_\beta. \]

Now we employ (4.11) to join two of the summation symbols in the above equation:

\[ I = \sum_{d=1}^{D} \frac{\partial u_d}{\partial t} \sum_{i=1}^{\alpha} g_{\partial_i d} \sum_{\varphi \in \mathcal{D}[\alpha]} \frac{\sigma(\varphi)!}{\alpha!} \pi_g(\zeta \setminus \partial_i, \varphi) f_{\sigma(\varphi)}. \]

Using (4.22) and (4.19) again, we get

\[ I = \sum_{d=1}^{D} \frac{\partial u_d}{\partial t} f_{\alpha - e_d} = \sum_{d=1}^{D} \frac{\partial u_d}{\partial t} f_{\alpha - e_d}. \]
Thus (4.27) is proved. Equation (4.28) can be proved in a similar way. Setting \( \vartheta = \zeta(\alpha) \) and using Lemma 4.3, Lemma 4.4, and (4.22), we obtain

\[
\Pi \equiv \sum_{d=1}^{D} \sum_{\varphi \in D(\alpha)} \frac{\sigma(\varphi)}{\alpha!} \Pi_{G}(\zeta(\alpha), \varphi) \frac{\partial \vartheta}{\partial \varphi} f_{\varphi(e^{-2e_{d}})}
\]

\[
= \sum_{d=1}^{D} \sum_{\varphi \in D(\alpha)} \frac{\sigma(\varphi)}{\alpha!} \Pi_{G}(\zeta(\alpha), \varphi) \frac{\partial \vartheta}{\partial \varphi} f_{\varphi(e^{-2e_{d}})}
\]

(4.32)

\[
= \sum_{d=1}^{D} \sum_{\varphi \in D(\alpha)} \frac{\sigma(\varphi)}{\alpha!} \Pi_{G}(\zeta(\alpha), \varphi) \frac{\partial \vartheta}{\partial \varphi} f_{\varphi(e^{-2e_{d}})}
\]

\[
= \sum_{d=1}^{D} \sum_{\varphi \in D(\alpha)} \frac{\sigma(\varphi)}{\alpha!} \Pi_{G}(\zeta(\alpha), \varphi) \frac{\partial \vartheta}{\partial \varphi} f_{\varphi(e^{-2e_{d}})}
\]

Since \( G \) is an orthogonal matrix, we have

\[
\sum_{d=1}^{D} g_{\varphi_{i}, d} g_{\varphi_{j}, d} = \delta_{\varphi_{i}, \varphi_{j}}.
\]

Thus (4.32) can be further simplified as

\[
\Pi = \sum_{i, j=1}^{D} \frac{1}{\alpha \varphi_{i} (\alpha \varphi_{i} - 1)} \frac{\partial \vartheta}{\partial \varphi_{i}} f_{\varphi(e^{-2e_{d}})} = \sum_{d=1}^{D} \frac{\partial \vartheta}{\partial \varphi_{i}} f_{\varphi(e^{-2e_{d}})}
\]

which completes the proof of (4.28). Equations (4.29)–(4.31) can be proved using exactly the same technique. The detailed proofs are omitted here to avoid redundancy.

It is not difficult to find that (4.27) and (4.28) still hold if we replace \( t \) with \( x_{j} \) or \( \bar{x}_{j} \) for any \( j = 1, \ldots, D \). Such observation leads to the following two lemmas:

**Lemma 4.7.** The following equalities hold for arbitrary \( \alpha \in \mathbb{N}^{D} \):

(4.33) \[
\sum_{j=1}^{D} \sum_{d=1}^{D} \frac{\partial \bar{u}_{d}}{\partial \bar{x}_{j}} \bar{u}_{j} f_{\varphi(e^{-2e_{d}})} = \sum_{j=1}^{D} \sum_{d=1}^{D} \frac{\sigma(\varphi)}{\alpha!} \Pi_{G}(\zeta(\alpha), \varphi) \frac{\partial \vartheta}{\partial \varphi} f_{\varphi(e^{-2e_{d}})}
\]

(4.34) \[
\sum_{j=1}^{D} \sum_{d=1}^{D} \frac{\partial \bar{u}_{d}}{\partial \bar{x}_{j}} \bar{u}_{j} f_{\varphi(e^{-2e_{d}})} = \sum_{j=1}^{D} \sum_{d=1}^{D} \frac{\sigma(\varphi)}{\alpha!} \Pi_{G}(\zeta(\alpha), \varphi) \frac{\partial \vartheta}{\partial \varphi} f_{\varphi(e^{-2e_{d}})}
\]
Proof. Replacing \( t \) with \( x_j \) in (4.27), we obtain
\[
\sum_{j=1}^{D} \sum_{d=1}^{D} \frac{\sigma(\psi)!}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) \frac{\partial u_d}{\partial x_j} f_{\sigma(\psi)-e_d} =
\]
\[
= \sum_{d=1}^{D} \sum_{j=1}^{D} \sum_{i=1}^{D} g_{ij} \frac{\partial u_d}{\partial x_i} f_{\sigma(\psi)-e_d} = \sum_{d=1}^{D} \sum_{i=1}^{D} \frac{\partial u_d}{\partial x_i} \tilde{u}_i f_{\sigma(\psi)-e_d}.
\]
This equation is the same as (4.33). The proof of (4.34) is almost the same. □

Lemma 4.8. The following equalities hold for arbitrary \( \alpha \in \mathbb{N}^D \):
\[
\sum_{j=1}^{D} \sum_{d=1}^{D} (\alpha_j + 1) \frac{\partial \tilde{u}_d}{\partial \tilde{x}_j} f_{\alpha-e_d+e_j} =
\]
\[
\sum_{j=1}^{D} \sum_{d=1}^{D} \sum_{\varphi \in \mathbb{N}^{D(\alpha)}} \frac{\sigma(\psi)!}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) \cdot (\sigma_j(\psi) + 1) \frac{\partial u_d}{\partial x_j} f_{\sigma(\psi)-e_d+e_j},
\]
\[
\sum_{j=1}^{D} \sum_{d=1}^{D} (\alpha_j + 1) \frac{\partial \tilde{\theta}}{\partial \tilde{x}_j} f_{\alpha-2e_d+e_j} =
\]
\[
\sum_{j=1}^{D} \sum_{d=1}^{D} \sum_{\varphi \in \mathbb{N}^{D(\alpha)}} \frac{\sigma(\psi)!}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) \cdot (\sigma_j(\psi) + 1) \frac{\partial \theta}{\partial x_j} f_{\sigma(\psi)-2e_d+e_j}.
\]

Proof. Replacing \( t \) with \( \tilde{x}_j \) and substituting \( \alpha + e_j \) for \( \alpha \) in (4.27), we get
\[
\sum_{j=1}^{D} \sum_{d=1}^{D} (\alpha_j + 1) \frac{\partial \tilde{u}_d}{\partial \tilde{x}_j} \tilde{u}_j f_{\alpha-e_d+e_j} =
\]
\[
= \sum_{j=1}^{D} \sum_{d=1}^{D} (\alpha_j + 1) \sum_{\varphi \in \mathbb{N}^{D(\alpha)+1}} \frac{\sigma(\psi)!}{(\alpha + e_j)!} \Pi_g(\zeta(\alpha + e_j), \varphi) \frac{\partial u_d}{\partial x_j} f_{\sigma(\psi)-e_d} =
\]
\[
= \sum_{j=1}^{D} \sum_{d=1}^{D} \sum_{i=1}^{D} g_{ji} \sum_{\varphi \in \mathbb{N}^{D(\alpha)}} \frac{\sigma(\psi) + e_i}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) \frac{\partial u_d}{\partial x_i} f_{\sigma(\psi)+e_i-e_d} =
\]
\[
= \sum_{d=1}^{D} \sum_{i=1}^{D} \sum_{\varphi \in \mathbb{N}^{D(\alpha)}} \frac{\sigma(\psi)!}{\alpha!} \Pi_g(\zeta(\alpha), \varphi) \cdot (\sigma_i(\psi) + 1) \frac{\partial u_d}{\partial x_i} f_{\sigma(\psi)+e_i-e_d}.
\]
This proves the first equality. The second equality can be similarly proved; we omit the details. □

Now the proof of Theorem 4.2 is given as follows:
Proof of Theorem 4.2. Since \((n_1, \ldots, n_D)\) is a unit vector, we let \(G = (g_{ij})_{D \times D}\) be an orthogonal matrix with its first row being \((n_1, \ldots, n_D)\). Now we use this matrix as the rotation matrix and define \(\tilde{w}\) as (4.22) and (4.19). It is obvious that the relation between \(\tilde{w}\) and \(w\) is linear. Therefore, there exists a constant matrix \(R\) (see (4.22)) depending on \(G\) such that

\[\tilde{w} = Rw,\]

and \(R\) is invertible since \(w\) can be obtained from \(\tilde{w}\) by applying the rotation matrix \(G^{-1}\). Lemmas 4.6–4.8 have clearly shown that the “rotated equations”

\[(4.35)\]

\[T(\tilde{w}) \frac{\partial \tilde{w}}{\partial t} + \sum_{j=1}^{D} T(\tilde{w}) \tilde{M}_j(\tilde{w}) \frac{\partial \tilde{w}}{\partial x_j} = 0\]

can be deduced from (4.5) by linear operations. Thus there exists a square matrix \(H(w)\) such that

\[(4.36)\]

\[H(w)T(w) \frac{\partial w}{\partial t} + \sum_{j=1}^{D} H(w)T(w) \tilde{M}_j(w) \frac{\partial w}{\partial x_j} = 0\]

is identical to (4.35). Matching the terms with time derivatives, one finds \(H(w) = T(\tilde{w})RT(\tilde{w})^{-1}\). Thus (4.36) becomes

\[T(\tilde{w}) \frac{\partial \tilde{w}}{\partial t} + \sum_{j=1}^{D} T(\tilde{w})R \tilde{M}_j(w) \frac{\partial w}{\partial x_j} = 0.\]

Using (4.26), the above equation can be rewritten as

\[T(\tilde{w})R \frac{\partial w}{\partial t} + \sum_{j=1}^{D} \sum_{d=1}^{D} g_{dj} T(\tilde{w})R \tilde{M}_j(w) \frac{\partial w}{\partial x_d} = 0.\]

Compared with (4.35), one concludes

\[\sum_{j=1}^{D} g_{1j} T(\tilde{w})R \tilde{M}_j(w) = T(\tilde{w}) \tilde{M}_1(Rw)R.\]

Multiplying both sides by \(R^{-1}T(\tilde{w})^{-1}\), (4.6) is attained. Recalling \(\tilde{M}_1 = \tilde{A}_M\) and that the first component of the macroscopic velocity after the rotation is \(u \cdot n\) (see (4.19)), the diagonalizability and the eigenvalues of the matrix (4.6) are naturally obtained using Theorem 3.14. □

5 Riemann Problem

Though the regularized moment system (4.2) is given by the moment expansion up to an arbitrary order \(M\) thus is extremely complex, we can clarify appreciably the structures of the elementary waves of this system with Riemann initial value, including the rarefaction wave, contact discontinuity, and the shock wave. The
structure of the elementary wave is fundamental to further investigation into the behavior of the solution of the system. Furthermore, the solution structure of the Riemann problem is instructive for studying the approximate Riemann solver, which is the basis of the numerical methods using Godunov-type schemes. The analysis below shows that the structure of the elementary wave of the Riemann problem is a natural extension of that of Euler equations, which indicates that the regularized moment system \((4.2)\) is actually a very reasonable high-order moment approximation of the Boltzmann equation. Following \([12]\) where the multidimensional Euler equations are studied, we consider the \(x_1\)-split, \(D\)-dimensional Riemann problem as below:

\[
\begin{aligned}
\frac{\partial \mathbf{w}}{\partial t} + \hat{\Lambda}_M \frac{\partial \mathbf{w}}{\partial x_1} &= 0, \\
\mathbf{w}(x_1, t = 0) &= \begin{cases} 
\mathbf{w}_L & \text{if } x_1 < 0, \\
\mathbf{w}_R & \text{if } x_1 > 0.
\end{cases}
\end{aligned}
\]

The Riemann problem with one-dimensional velocity space has been studied in \([1]\) in detail. Here we focus on the case of \(D \geq 2\).

Let us first recall the definitions of the notations \(\hat{\Lambda}_M, \hat{\Lambda}_N, \hat{B}, \hat{\lambda}_i, \hat{\rho}_i, \hat{\lambda}_i, \mathbf{v}(\hat{j})\), and \(\lambda_{i,k}\) in Section \(3\). In particular, we need the expressions of \(\hat{\mathbf{B}}\hat{\mathbf{B}}^{-1}\) and \(\mathbf{v}(\hat{j})\):

\[
\hat{\mathbf{B}}\hat{\mathbf{B}}^{-1} = \begin{bmatrix}
I & 0 & 0 \\
-B_{21} & I & 0 \\
* & * & I
\end{bmatrix}
\]

and for any \(|\alpha| = M\), \(k = \alpha_1 + 1\), \(j = N_{D-1}(\hat{\alpha})\):

\[
\mathbf{v}(\hat{j}) = \hat{\mathbf{B}}\hat{\mathbf{B}}^{-1} I_j, \quad \text{He}_k(C) = 0,
\]

where \(I_j\) is the \(j\)th column of the \(N_v \times N_v\) identity matrix, \(\hat{\mathbf{r}}_\hat{i,\hat{j}}\), which depends on \(\mathbf{v}(\hat{j})\) and \(\lambda_{i,k} = u_1 + C_{i,k} \sqrt{\hat{\theta}}\), is the eigenvector of \(\hat{\Lambda}_M\) for the eigenvalue \(\lambda_{i,k} = u_1 + C_{i,k} \sqrt{\hat{\theta}}\), where \(j = N_{D-1}(\hat{\alpha})\), \(k = M + 1 - |\hat{\alpha}|\). As the first result on the Riemann problem \((5.1)\), we have the following theorem:

**Theorem 5.1.** Each characteristic field of \((5.1)\) is either genuinely nonlinear or linearly degenerate. One characteristic field is genuinely nonlinear if and only if \(\mathbf{v}\) (determined by the right eigenvector through \((3.16)\)) and the eigenvalue \(\lambda = u_1 + C \sqrt{\hat{\theta}}\) satisfy one of the following two conditions:

1. \(\mathbf{v} = \mathbf{v}(1)\) and \(C\) is subject to \(\text{He}_{M+1}(C) = 0\) and \(C \neq 0\);
2. \(\mathbf{v} = \mathbf{v}(\hat{j}), \ j = N_{D-1}(2\hat{e}_k), \ k \in \mathcal{D} \setminus \{1\}\), and \(C\) is subject to \(\text{He}_{M-1}(C) = 0\) and \(C \neq 0\).
PROOF. Let $\hat{\bf r}$ denote an eigenvector of $\hat{\lambda}_M$ with the eigenvalue $\lambda = u_1 + C\sqrt{\theta}$ and $v$ be the corresponding vector determined by (3.16). Since

$$
\lambda = u_1 + C\sqrt{\theta} = u_1 + C\sqrt{\sum_{d=1}^{D} \frac{p_{2e_d}}{D\rho}}
$$

depends only on $\rho$, $u_1$, and $p_{2e_d}/2$, $d \in \mathcal{D}$, we have

$$
\nabla_{\bf w} \cdot \hat{\bf r} = -\frac{C\sqrt{\theta}}{2\rho} \cdot \rho r_\rho + 1 \cdot C\sqrt{\theta}r_\rho + \frac{C}{D\rho \sqrt{\theta}} \cdot \frac{C^2\theta}{2} \rho r_\rho
$$

$$
+ \sum_{d=2}^{D} \frac{C}{D\rho \sqrt{\theta}} \cdot \rho r_{p_{2e_d}/2}
$$

$$
= \frac{\sqrt{\theta}C}{2} \left[ \left( 1 + \frac{C^2}{D} \right) v_1 + \sum_{d=2}^{D} \frac{2}{D} v_{N_{D-1}(2\hat{e}_d)} \right].
$$

(5.2)

(1) If $v = \bf v^{(1)}$, then (3.39) shows that $\bf v^{(1)} = \hat{\bf B}\hat{\bf B}^{-1} I_1$ and $HeM_{+1}(C) = 0$. From (3.38), we get

$$
v_1 = 1 \quad \text{and} \quad v_{N_{D-1}(2\hat{e}_d)} = \cdots = v_{N_{D-1}(2\hat{e}_D)} = \frac{C^2}{2D} - \frac{1}{2}.
$$

Thus (5.2) can be written as

$$
\nabla_{\bf w} \lambda \cdot \hat{\bf r} = \frac{(D + 1)\sqrt{\theta}C}{4D^2} (C^2 + D).
$$

Hence,

$$
\begin{cases}
\nabla_{\bf w} \lambda \cdot \hat{\bf r} \equiv 0 & \text{if } C = 0, \\
\nabla_{\bf w} \lambda \cdot \hat{\bf r} \not\equiv 0 & \text{otherwise}.
\end{cases}
$$

(2) If $v = \bf v^{(j)}$, $j = N_{D-1}(2\hat{e}_k)$ for any $k \in \mathcal{D} \setminus \{1\}$, (3.39) shows $v = \hat{\bf B}\hat{\bf B}^{-1} I_j$ and $HeM_{-1}(C) = 0$. From (3.38), we can get

$$
v_j = 1 \quad \text{and} \quad v_1 = v_l = 0, \quad l = N_{D-1}(2\hat{e}_d) \quad \text{for any } d \in \mathcal{D} \setminus \{1, k\}.
$$

Then (5.2) can be simplified as

$$
\nabla_{\bf w} \lambda \cdot \hat{\bf r}_{\hat{e}_i} = \frac{\sqrt{\theta}}{D} C.
$$

Again, we have

$$
\begin{cases}
\nabla_{\bf w} \lambda \cdot \hat{\bf r} \equiv 0 & \text{if } C = 0, \\
\nabla_{\bf w} \lambda \cdot \hat{\bf r} \not\equiv 0 & \text{otherwise}.
\end{cases}
$$

(3) Otherwise, (3.38) indicates $v_1 = v_{N_{D-1}(2\hat{e}_k)} = 0$ for each $k \in \mathcal{D} \setminus \{1\}$. Hence $\nabla_{\bf w} \lambda \cdot \hat{\bf r} \equiv 0$ always holds.

This completes the proof. \(\Box\)
This theorem reveals that for each characteristic field, the eigenvalue is constant or varies monotonically along the integral curve, resulting in simple wave structures. Below some elementary waves, including the rarefaction waves, contact discontinuities, and shock waves, are studied in detail, and the basic relations across these waves are established.

The analysis below is based on the fact that an eigenvector \( y^r \) of \( \hat{A}^M \) for the eigenvalue \( \lambda = u_1 + C\sqrt{\theta} \) depends only on \( v \) and \( C \). With Theorem 5.1 and the forms of \( B \) and \( BB \) in Lemma 3.12, we can divide characteristic fields into three cases:

CASE 1: \( v = v^{(1)} \), and \( C \) subject to \( He_M + 1(C) = 0 \) and \( C \neq 0 \).

CASE 2: \( v = v^{(j)} \), \( j = N_{D-1}(2\hat{c}_k) \) for any \( k \in \mathcal{D} \setminus \{1\} \) and \( C \) subject to \( He_{M-1}(C) = 0 \), and \( C \neq 0 \).

CASE 3: otherwise.

For convenience, let characteristic field \( \alpha \) denote the characteristic field corresponding to the eigenvector \( \hat{r}_{\alpha,i} \) for the eigenvalue \( \lambda_{i,k} = u_1 + C_{i,k}\sqrt{\theta} \) with \( i = \alpha_1 \), \( k = M + 1 - |\alpha| \). Below, the rarefaction waves, contact discontinuities, and shock waves will be studied, respectively.

### 5.1 Rarefaction Waves

For the regularized moment system, if two states \( \mathbf{w}^L \) and \( \mathbf{w}^R \) are connected by a rarefaction wave in a genuinely nonlinear field \( \alpha \), then the following two conditions must be met:

1. constancy of the *generalized Riemann invariants* across the wave, which means the integral curve \( \tilde{w}(\xi) = (\tilde{w}_1(\xi), \tilde{w}_2(\xi), \ldots, \tilde{w}_N(\xi)) \) in the \( N \)-dimensional phase space satisfies

\[
\tilde{w}'(\xi) = \hat{r}_{\alpha,i}(\tilde{w})
\]

with \( i = \alpha_1 \);

2. divergence of characteristics

\[
\lambda_{i,k}(\mathbf{w}^L) < \lambda_{i,k}(\mathbf{w}^R).
\]

Fortunately, for a given point \( \mathbf{w}^0 = (\rho^0, u_1^0, \ldots, w_j^0, \ldots, w_N^0) \) in the phase space, the integral curve across \( \mathbf{w}^0 \) can be given. Since \( p = \frac{1}{\mathcal{D}} \sum_{d=1}^{\mathcal{D}} p_{2\epsilon_d} \), we let \( p^0 = \frac{1}{\mathcal{D}} \sum_{d=1}^{\mathcal{D}} p_{2\epsilon_d}^0 \). The results are rather tedious, and here the integral curves are only partially given in three cases as below:

1. If \( v = v^{(1)} \), we have

\[
\begin{align*}
 r_p &= \rho, \quad \rho u_1 = C_{i,k} \sqrt{\theta}, \quad r_{u_d} = 0, \\
r_{p_{2\epsilon_1}/2} &= \frac{C_{i,k}^2}{2} \rho \theta, \quad r_{p_{2\epsilon_d}/2} = \frac{D - C_{i,k}^2}{2(2D - 1)} \rho \theta, \quad d \in \mathcal{D} \setminus \{1\}.
\end{align*}
\]
Let $\Gamma = \frac{c_{i,k} + D^{-1}}{2D^{-1}}$; then we have

\begin{align}
(5.5a) \quad \bar{\rho}(\xi) &= \rho^0 \exp(\xi), \\
(5.5b) \quad \bar{u}_1(\xi) &= u_1^0 + \frac{2c_{i,k}\sqrt{\rho^0}}{\Gamma - 1} \left[ \exp\left( \frac{\Gamma - 1}{2} \xi \right) - 1 \right], \\
(5.5c) \quad \bar{u}_d(\xi) &= u_d^0, \quad d = 2, \ldots, D, \\
(5.5d) \quad \bar{p}_{2e_1}(\xi) &= p_{2e_1}^0 + \frac{c_{i,k}^2}{\Gamma} p^0 [\exp(\Gamma \xi) - 1], \\
(5.5e) \quad \bar{p}(\xi) &= p^0 \exp(\Gamma \xi).
\end{align}

(2) If $v = v^{(j)}$, $j = N_{D-1}(2\sigma_k)$, $k \in D \setminus \{1\}$, we have

$\begin{align}
\rho = 0, \quad u_1 = 0, \quad d \in D, \quad r_{p_{2e_1}} = \rho \theta, \quad r_{p_{2e_2}/2} = 0, \quad d \in D \setminus \{k\}.
\end{align}$

Hence, the integral curve satisfies

\begin{align}
(5.6a) \quad \bar{\rho}(\xi) &= \rho^0, \\
(5.6b) \quad \bar{u}_d(\xi) &= u_d^0, \quad d = 1, \ldots, D, \\
(5.6c) \quad \bar{p}_{2e_1}(\xi) &= p_{2e_1}^0, \\
(5.6d) \quad \bar{p}(\xi) &= p^0 \exp\left( \frac{2\xi}{D} \right).
\end{align}

(3) Otherwise ($v = v^{(j)}$, $j \neq N_{D-1}(2\sigma_k)$ for any $k \in D$),

$r_\rho = r_{u_1} = r_{p_{2e_d}/2} = 0, \quad d \in D.$

Hence we have

\begin{align}
(5.7a) \quad \bar{\rho}(\xi) &= \rho^0, \quad \bar{u}_1(\xi) = u_1^0, \\
(5.7b) \quad \bar{p}_{2e_1}(\xi) &= p_{2e_1}^0, \quad \bar{p}(\xi) = p^0.
\end{align}

One can check that (5.3), (5.7), and (5.6) satisfy (5.3). An eigenvalue that satisfies (3.45) of $\hat{A}_M(\bar{\omega}(\xi))$ is

$s_{i,k}(\bar{\omega}(\xi)) = \bar{u}_1(\xi) + C_{i,k} \sqrt{\bar{\rho}(\xi)} \bar{\rho}(\xi)$

\begin{align}
&= \begin{cases}
  u_1^0 + C_{i,k} \sqrt{\rho^0} + \frac{c_{i,k}}{\Gamma - 1} C_{i,k} \sqrt{\rho^0} \left[ \exp\left( \frac{\Gamma - 1}{2} \xi \right) - 1 \right] & \text{for } v^{(1)}, \\
  u_1^0 + C_{i,k} \sqrt{\rho^0} \exp\left( \frac{\xi}{\Gamma} \right) & \text{for } v^{(j)}, j = N_{D-1}(2\sigma_k), k \in D \setminus \{1\}, \\
  u_1^0 + C_{i,k} \sqrt{\rho^0} & \text{otherwise.}
\end{cases}
\end{align}

It is convenient to verify that $s_{i,k}(\bar{\omega}(\xi)) \geq s_{i,k}(\omega^0)$ if and only if $C_{i,k} \xi \geq 0$, and $v$ and $C_{i,k}$ satisfy case 1 or case 2. Therefore, if the left state $\omega^L$ and the right state $\omega^R$ are connected by a rarefaction wave and $\omega^0 = \omega^L$, (5.4) indicates $s_{i,k}(\omega^L) < s_{i,k}(\omega^R)$; hence $C_{i,k} \xi > 0$ and $v, C_{i,k}$ satisfy case 1 or case 2. Therefore, we have the following:
For case 1,
\[ u_d^L = u_d^R, \quad d = 2, \ldots, D, \]
and
\[
\begin{align*}
&\text{if } C_{i,k} > 0, \quad u_1^L < u_1^R, \quad p_1^L < p_1^R, \\
&\text{if } C_{i,k} < 0, \quad u_1^L < u_1^R, \quad p_1^L > p_1^R.
\end{align*}
\]

For case 2,
\[ u_d^L = u_d^R, \quad d = 2, \ldots, D, \]
and
\[
\begin{align*}
&\text{if } C_{i,k} > 0, \quad u_1^L = u_1^R, \quad p_1^L < p_1^R, \\
&\text{if } C_{i,k} < 0, \quad u_1^L = u_1^R, \quad p_1^L > p_1^R.
\end{align*}
\]

### 5.2 Contact Discontinuities

For contact discontinuities, (5.3) is still valid, and the divergence of characteristics is replaced by
\[
(5.10) \quad \lambda_{i,k}(w^L) = \lambda_{i,k}(w^R).
\]
According to Theorem 5.1 and the analysis in Section 5.1, contact discontinuities can be found if and only if \( v \) and \( C_{i,k} \) satisfy case 3.

- For \( v^{(1)} \), (5.10) means \( C_{i,k} = 0 \). Substituting it into (5.3), we can get \( u_d, \quad d \in D, \quad p_{2e_1} \) are invariant, while \( p \) is not (otherwise, (5.5e) gives us \( \zeta = 0 \), thus \( w^L = w^R \)).

- For \( v^{(j)}, j = N_{D-1}(2\widehat{e}_k), k \in D \setminus \{1\}, \) (5.10) means \( C_{i,k} = 0 \) again. (5.6) shows \( \rho, u_d \) (\( d \in D \)), and \( p_{2e_1} \) are invariant, while \( p \) is not (otherwise, (5.6d) gives us \( \zeta = 0 \), which results in \( w^L = w^R \)).

- Otherwise, (5.7) shows that \( u_1, p \), and \( p_{2e_1} \) are all invariant.

Summarizing the discussion above, we conclude that if \( C_{i,k} \neq 0 \), then \( u_1, p \), and \( p_{2e_1} \) are invariant across the contact discontinuities, while if \( C_{i,k} = 0, u_1 \) and \( p_{2e_1} \) are invariant and \( p \) is not. However, \( u_d, d = 2, \ldots, D \), may change discontinuously across a contact discontinuity. In fact, the case \( v = B^{-1}I_d, d = 2, \ldots, D \), corresponds to a contact discontinuity where \( u_d \) is discontinuous. This is similar to the Euler equations.

### 5.3 Shock Waves

One should be more careful while studying the shock waves. As is well known, the jump condition on the shock wave is sensitive to the form of the hyperbolic equations. Thus, before we give the Rankine-Hugoniot condition, it is necessary to rewrite (5.1) in an appropriate form. However, (5.1) cannot be written as conservation laws due to the presence of \( R_{M,D}^1(\alpha) \). Nevertheless, (5.1) can still keep the conservation of the conservative moments with orders from 0 to \( M - 1 \). Therefore, (5.1) can be reformulated by \( N_D((M - 1)e_D) \) conservation laws and \( N - N_D((M - 1)e_D) \) nonconservative equations.
Let 
\[ F = (F_0, F_{e_1}, F_{e_2}, \ldots, F_{M e_D}), \quad F_\alpha = \frac{1}{\alpha!} \int_{\mathbb{R}^D} \xi^\alpha f \, d\xi, \quad |\alpha| \leq M, \]
where \( \xi^\alpha = \prod_{d=1}^D \xi_d^{\alpha_d} \) and \( F_0 \) stands for \( F_\alpha|_{\alpha=0} \). Then (5.1) can be written as
\[
\begin{aligned}
\frac{\partial F_\alpha}{\partial t} + (\alpha_1 + 1) \frac{\partial F_{\alpha+e_1}}{\partial x_1} &= 0, \quad |\alpha| < M, \\
\frac{\partial F_\alpha}{\partial t} + (\alpha_1 + 1) \frac{\partial \hat{F}_\alpha}{\partial x_1} - \hat{R}_M^1 (\alpha) &= 0, \quad |\alpha| = M.
\end{aligned}
\]
(5.11)

The relation between \( F \) and \( w \) is
\[
f_\alpha = \sum_{|\beta| \leq |\alpha|} (-1)^{|\alpha-\beta|} F_\beta \frac{He_{\alpha-\beta}(u/\sqrt{\theta})}{(\alpha-\beta)!} |\alpha-\beta|/2,
\]
\[
u_i = \frac{F_{e_i}}{F_0}, \quad p_{2e_i} = 2F_{2e_i} - \frac{F_{e_i}^2}{F_0}, \quad i \in D,
\]
where
\[
He_\alpha \left( \frac{u}{\sqrt{\theta}} \right) = \prod_{d=1}^D He_{\alpha_d} \left( \frac{u_d}{\sqrt{\theta}} \right)
\]
and \( He_\alpha (x) = 0 \) if at least one \( \alpha_j \) is negative. In addition,
\[
\hat{F}_\alpha = \sum_{|\beta| \leq |\alpha|} (-1)^{|\alpha-\beta|} F_\beta \frac{He_{\alpha+e_1-\beta}(u/\sqrt{\theta})}{(\alpha + e_1 - \beta)!} |\alpha+e_1-\beta|/2.
\]

For convenience, the quasi-linear form of (5.11) is written as
\[
(5.12) \quad \frac{\partial F}{\partial t} + \Gamma (F) \frac{\partial F}{\partial x_1} = 0,
\]
where \( \Gamma (F) \) is an \( N \times N \) matrix and depends on (5.11).

Since (5.12) is not a conservative system, we have to adopt the DLM theory [6] to study the shock wave. For a shock wave the two constant states \( F_L \) and \( F_R \) are connected through a single jump discontinuity in a genuinely nonlinear field \( \alpha \) traveling at the speed \( S_\alpha \), and the following two conditions apply:

- Generalized Rankine-Hugoniot condition:
\[
(5.13) \quad \int_0^1 [S_\alpha I - \Gamma (\Phi(v; F_L, F_R)) \frac{\partial \Phi}{\partial v} (v; F_L, F_R)] \, dv = 0,
\]
where \( I \) is the \( N \times N \) identity matrix, and \( \Phi(v; F_L, F_R) \) is a locally Lipschitz mapping satisfying
\[
\Phi(0; F_L, F_R) = F_L, \quad \Phi(1; F_L, F_R) = F_R.
\]
We refer the readers to [6] for details.
Entropy condition:
\[
\lambda_{i,k}(F^L) > S_\alpha > \lambda_{i,k}(F^R),
\]
where \( i = \alpha_1 \) and \( k = M + 1 - |\hat{\alpha}|. \)

For conservation laws, (5.13) is the same as the classical Rankine-Hugoniot condition. Thus the first \( N_D ((M - 1)e_D) \) rows of (5.13) are independent of \( \Phi \).
This allows us to analyze the properties of the shock waves without regard for the form of \( \Phi \).

The first equation and the \((D + 1)\)th equation of (5.13) are
\[
\rho^L u_1^L - \rho^R u_1^R = S_\alpha (\rho^L - \rho^R),
\]
\[
\rho^L (u_1^L)^2 + p_{2e_1}^L - \rho^R (u_1^R)^2 - p_{2e_1}^R = S_\alpha (\rho^L u_1^L - \rho^R u_1^R).
\]

If \( \rho^L \neq \rho^R \), (5.15) and (5.16) give
\[
S_\alpha = \frac{\rho^L u_1^L - \rho^R u_1^R}{\rho^L - \rho^R}.
\]

Substituting (5.17a) into (5.14) and multiplying both sides by \((\rho^L - \rho^R)^2\), we get
\[
\rho^L (u_1^L - u_1^R)(\rho^L - \rho^R) > C_{i,k} (\rho^L - \rho^R)^2 \sqrt{\theta^R}. \tag{5.18a}
\]
\[
\rho^R (u_1^L - u_1^R)(\rho^L - \rho^R) < C_{i,k} (\rho^L - \rho^R)^2 \sqrt{\theta^L}. \tag{5.18b}
\]

If \( C_{i,k} > 0 \), (5.18a) gives
\[
(u_1^L - u_1^R)(\rho^L - \rho^R) > 0.
\]
Therefore, we can divide (5.18) by \((u_1^L - u_1^R)(\rho^L - \rho^R)\) and obtain
\[
\frac{\rho^L}{\sqrt{\theta^L}} > \frac{C_{i,k} (\rho^L - \rho^R)}{u_1^L - u_1^R} > \frac{\rho^R}{\sqrt{\theta^R}}.
\]
Thus we have
\[
(\rho^L)^2 \theta^L > (\rho^R)^2 \theta^R. \tag{5.20}
\]

Furthermore, (5.17) gives us the relation
\[
(\rho^L - \rho^R)(p_{2e_1}^L - p_{2e_1}^R) = \rho^L \rho^R (u_1^L - u_1^R)^2. \tag{5.21}
\]

If \( \rho^L < \rho^R \), (5.19) indicates \( u_1^L < u_1^R \). (5.20) can be written as \( \rho^L p^L > \rho^R p^R \), so we have \( p^L > p^R \). If \( \rho^L > \rho^R \), (5.19) implies \( u_1^L > u_1^R \).

Summarizing these results, we get
- if \( \rho^L < \rho^R \), then \( u_1^L < u_1^R \) and \( p^L > p^R \).
- if \( \rho^L > \rho^R \), then \( u_1^L > u_1^R \).
Wave type | Eigenvalue | Velocity and Pressure
--- | --- | ---
**Rarefaction wave** | \( C_{i,k} > 0 \) | \( u_1^L \leq u_1^R, \ p_L < p_R \)
| \( C_{i,k} < 0 \) | \( u_1^L \leq u_1^R, \ p_L > p_R \)

**Shock wave** | \( C_{i,k} > 0 \) | \( u_1^L \leq u_1^R, \ p_L > p_R \)
| \( C_{i,k} < 0 \) | \( u_1^L \leq u_1^R, \ p_L < p_R \)
| \( C_{i,k} \neq 0 \) | \( u_1^L > u_1^R \)

**Contact discontinuity** | \( C_{i,k} = 0 \) | \( u_1^L = u_1^R \)
| \( C_{i,k} \neq 0 \) | \( u_1^L = u_1^R, \ p_L = p_R \)

| Table 5.1. The relation between the type classification of elementary wave and the eigenvalue, macroscopic velocity, and pressure. |

Analogously, if \( C_{i,k} < 0 \), we have

\[(u_1^L - u_1^R)(\rho^L - \rho^R) < 0, \ (\rho^L)^2\vartheta^L < (\rho^R)^2\vartheta^R,\]

and (5.21) still holds. Hence we get that

- if \( \rho^L > \rho^R \), then \( u_1^L < u_1^R \) and \( p_L < p_R \),
- if \( \rho^L < \rho^R \), then \( u_1^L > u_1^R \).

- If \( \rho^L = \rho^R \), (5.15) and (5.16) imply that \( u_1^L = u_1^R \) and \( p_L^e = p_R^e \), respectively. Therefore, (5.14) indicates

\[(5.22) \quad C_{i,k} \sqrt{\vartheta^L} > C_{i,k} \sqrt{\vartheta^R}.\]

We then obtain the following result:

- if \( C_{i,k} > 0 \), then \( u_1^L = u_1^R \), \( p_L > p_R \),
- if \( C_{i,k} < 0 \), then \( u_1^L = u_1^R \), \( p_L < p_R \).

Now we summarize our discussion of the entropy conditions of the three types of elementary waves in the following theorem:

**Theorem 5.2.** For the Riemann problem (5.1), for the wave of the \( \alpha \)th family, \( C_{i,k} \), the macroscopic velocities and pressures on both sides of the wave have the relation with the type of the wave as in Table 5.1 where \( C_{i,k} \) corresponds to the eigenvalue \( \lambda_{i,k} = u_1 + C_{i,k} \sqrt{\vartheta} \), \( i = \alpha_1 \), and \( k = M + 1 - |\alpha| \).

## 6 Discussion of Convergence

As a necessary requirement, the solution of the moment system must converge to the solution of the Boltzmann equation if the expansion (2.2) is valid. Currently we cannot prove this yet. As an alternative, we analyze below the approximations to the convection term for both the hyperbolic moment system and Grad’s
moment system, since the only difference between these two models is found in the approximations to the convection term. In the current stage, we assume that the distribution function is smooth enough in both the \(x\) - and \(\xi\) -directions, and all infinite series encountered are convergent.

Let us consider the Grad moment system first. We define

\[
G(t, x, \xi) = f(t, x, \xi), \quad F(t, x, \xi) = \nabla_x \cdot G(t, x, \xi).
\]

Then the expansion of \(G\) and \(F\) can be written as

\[
G(t, x, \xi) = \sum_{\alpha \in \mathbb{N}^D} G_{\alpha}(t, x) \mathcal{H}_{\theta(t, x), \alpha} \left( \frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right),
\]

\[
F(t, x, \xi) = \sum_{\alpha \in \mathbb{N}^D} F_{\alpha}(t, x) \mathcal{H}_{\theta(t, x), \alpha} \left( \frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right),
\]

where

\[
G_{\alpha} = (G_{1, \alpha}, \ldots, G_{D, \alpha})^T, \quad G_{j, \alpha} = \theta f_{\alpha-e_j} + u_j f_{\alpha} + (\alpha_j + 1) f_{\alpha+e_j},
\]

\[
F_{\alpha} = \nabla_x \cdot G_{\alpha} + \sum_{j=1}^D \left( \nabla_x u_j \cdot G_{\alpha-e_j} + \frac{1}{2} \nabla_x \theta \cdot G_{\alpha-2e_j} \right).
\]

The detailed calculation can be found in [5]. For Grad’s moment system, the approximation to \(F\) can be interpreted as follows:

1. Approximate \(G\) with

\[
\hat{G}(t, x, \xi) = \sum_{|\alpha| < M} G_{\alpha}(t, x) \mathcal{H}_{\theta(t, x), \alpha} \left( \frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right)
+ \sum_{|\alpha| = M} \hat{G}_{\alpha}(t, x) \mathcal{H}_{\theta(t, x), \alpha} \left( \frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right),
\]

where

\[
\hat{G}_{\alpha} = (\hat{G}_{1, \alpha}, \ldots, \hat{G}_{D, \alpha})^T, \quad \hat{G}_{j, \alpha} = G_{j, \alpha} - (\alpha_j + 1) f_{\alpha+e_j}.
\]

2. Approximate \(F\) with

\[
\hat{F}(t, x, \xi) = \sum_{|\alpha| < M} F_{\alpha}(t, x) \mathcal{H}_{\theta(t, x), \alpha} \left( \frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right)
+ \sum_{|\alpha| = M} \hat{F}_{\alpha}(t, x) \mathcal{H}_{\theta(t, x), \alpha} \left( \frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right),
\]

where

\[
\hat{F}_{\alpha} = \nabla_x \cdot \hat{G}_{\alpha} + \sum_{j=1}^D \left( \nabla_x u_j \cdot \hat{G}_{\alpha-e_j} + \frac{1}{2} \nabla_x \theta \cdot \hat{G}_{\alpha-2e_j} \right).
\]
Now we give a rough estimation of $G - \hat{G}$. Let $\omega(t, x, \xi)$ be the weight function

$$\omega(t, x, \xi) = \exp\left(\frac{|\xi - u(t, x)|^2}{2\theta(t, x)}\right).$$

Then one has

$$\|G - \hat{G}\|_{\omega}^2 \triangleq \int_{\mathbb{R}^D} \|G(t, x, \xi) - \hat{G}(t, x, \xi)\|^2 \omega(t, x, \xi) d\xi \leq \sum_{|\alpha| > M} C_{\theta, \alpha} \|G_\alpha\|^2 + \sum_{|\alpha| = M} D \sum_{j=1}^D [\alpha_j + 1] f_{\alpha + e_j}^2,$$

where $C_{\theta, \alpha} = \alpha!(2\pi)^{-D/2}\theta^{-|\alpha|}$. Now the above equation can be estimated by

$$\|G - \hat{G}\|_{\omega}^2 \leq \sum_{|\alpha| > M} C_{\theta, \alpha} \|G_\alpha\|^2 + D(M + 1) \theta \sum_{|\alpha| = M + 1} C_{\theta, \alpha} |f_\alpha|^2 \leq \sum_{|\alpha| > M} C_{\theta, \alpha} \|G_\alpha\|^2 + D(M + 1) \theta |f_\alpha|^2 \|

\begin{align*}
\|G - G^M\|^2_{\omega} + D(M + 1) \theta \|f - f^M\|^2_{\omega},
\end{align*}

where

(6.3) $G^M(t, x, \xi) = \sum_{|\alpha| \leq M} G_\alpha(t, x) H_{\theta(t, x), \alpha} \left(\frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}}\right),$

(6.4) $f^M(t, x, \xi) = \sum_{|\alpha| \leq M} f_\alpha(t, x) H_{\theta(t, x), \alpha} \left(\frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}}\right).$

According to the theory of the spectral method and the assumption that the distribution function has enough smoothness, there exists a positive number $k$ such that

$$\|G - G^M\|^2_{\omega} \lesssim M^{-k}, \quad \|f - f^M\|^2_{\omega} \lesssim M^{-k},$$

where the constant is dependent on $G$ or $f$. Thus

(6.5) $\|G - \hat{G}\|_{\omega} \lesssim M^{-(k-1/2)}.$

Equations (6.1) and (6.2) show that $\hat{F}$ is deduced from $G$ by direct differentiation, which indicates that $\hat{F}$ cannot converge faster than $\hat{G}$ as $M$ goes to infinity.

Now we consider the regularized moment system (4.4). Define $E_j(t, x, \xi) = \partial f(t, x, \xi)/\partial x_j$; then $E_j$ can be expanded as

$$E_j(t, x, \xi) = \sum_{\alpha \in \mathbb{N}^D} E_{j, \alpha}(t, x) H_{\theta(t, x), \alpha} \left(\frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}}\right).$$
where
\[ E_j^{\alpha} = \frac{\partial f_\alpha}{\partial x_j} + \sum_{d=1}^D \frac{\partial u_d}{\partial x_j} f_\alpha - e_d + \frac{1}{2} \sum_{d=1}^D \frac{\partial \theta}{\partial x_j} f_\alpha - 2e_d. \]

It is not difficult to find from (4.4) that we approximate \( F \) with \( z^F \).

Now using a similar method to that used in the case of Grad’s moment system, we obtain
\[
\| F - \tilde{F} \|_\omega^2 = \sum_{|\alpha| > M} C_{\theta,\alpha} |F_\alpha|^2 + \sum_{|\alpha| = M} C_{\theta,\alpha} \left( \sum_{j=1}^D (\alpha_j + 1) E_j^{\alpha+e_j} \right)^2
\]
\[
\leq \sum_{|\alpha| > M} C_{\theta,\alpha} |F_\alpha|^2 + 2 \sum_{|\alpha| = M} C_{\theta,\alpha} \sum_{j=1}^D (\alpha_j + 1) E_j^{\alpha+e_j} \left( \sum_{j=1}^D \sum_{|\alpha|=M+1} C_{\theta,\alpha} |E_j^{\alpha}| \right)^2
\]
\[
\leq \| F - F^M \|_\omega^2 + 2D(M + 1)\theta \sum_{|\alpha| = M+1} \sum_{j=1}^D C_{\theta,\alpha} |E_j^{\alpha}|^2
\]
\[
\leq \| F - F^M \|_\omega^2 + 2D(M + 1)\theta \sum_{j=1}^D \| E_j - E_j^M \|_\omega^2.
\]

Here \( F^M \) and \( E_j^M \) are defined similarly as in (6.3) and (6.4). Again, it is reasonable to suppose
\[
\| F - F^M \|_\omega \lesssim M^{-k}, \quad \| E_j - E_j^M \|_\omega \lesssim M^{-k}.
\]

Therefore, we have
\[
(6.6) \quad \| F - \tilde{F} \|_\omega \lesssim M^{-(k-1/2)}.
\]

Comparing (6.6) with (6.5), it is clear that the hyperbolic moment system has the same convergence rate as Grad’s moment system to the Boltzmann equation in the truncation error point of view.

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