A note on simplified SINR expressions for OFDM with insufficient CP

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Abstract

This note provides derivation details of simplified OFDM transmission equation and resulting signal-to-interference plus noise ratio (SINR) for the case of an insufficient CP. Each channel component after demodulation is expressed as a single sum which can be interpreted a weighted Fourier transform of the channel impulse response. Specifically, for CP length $N_{cp}$, FFT size $N_{fft}$, $N_{sc}$ consecutively allocated subcarriers, and channel impulse response $h[n] = \sum_{l=-L_d}^{L_u} h_l \delta_{n-l}$ with $0 \leq L_d, L_u \leq N_{fft} - 1$; the SINR at the $i$th subcarrier is

$$\text{SINR}_i = \frac{|\mathcal{H}_{0,i,i}|^2}{\sum_{l=0}^{N_{sc}-1} |\mathcal{H}_{0,0,l,i}|^2 + \sum_{l=0}^{N_{sc}-1} |\mathcal{H}_{-1,1,l,i}|^2 + |\mathcal{H}_{1,1,l,i}|^2 + 1/\text{SNR}}$$

where

$$\mathcal{H}_{0,i,i} = \sum_{m=-L_d}^{L_u} c[m] h[m] e^{-j2\pi \frac{m}{N_{fft}}}$$

$$\mathcal{H}_{-1,1,i} = \sum_{m=N_{sc}}^{L_u} (1 - c[m]) h[m] e^{-j2\pi \frac{m}{N_{fft}}}$$

$$\mathcal{H}_{1,1,i} = \sum_{m=-L_d}^{0} (1 - c[m]) h[m] e^{-j2\pi \frac{m}{N_{fft}}}$$

with weight functions

$$c[m] = \begin{cases} \frac{N_{sc}+m}{N_{fft}}, & -N_{fft} \leq m \leq 0 \\ 1, & 0 \leq m \leq N_{cp} \\ \frac{N_{sc}-(m-N_{cp})}{N_{fft}}, & N_{cp} \leq m \leq N_{cp} + N_{fft} \end{cases}$$

$$\tilde{c}_{l,i}[m] = \begin{cases} \frac{1-j2\pi \frac{m-l}{N_{fft}}}{N_{fft}(1-e^{-j2\pi \frac{l}{N_{fft}}})}, & -N_{fft} \leq m \leq 0 \\ \frac{e^{-j2\pi \frac{(m-N_{cp})l}{N_{fft}}}}{N_{fft}(1-e^{-j2\pi \frac{l}{N_{fft}}})}, & 0 \leq m \leq N_{cp} \end{cases}$$

Moreover, if the channel is causal, i.e., $L_d = 0$, the SINR depends only on ICI terms as

$$\text{SINR}_i = \frac{|\mathcal{H}_{0,i,i}|^2}{|\mathcal{H}_i - \mathcal{H}_{0,i,i}|^2 + \sum_{l=0}^{N_{sc}-1} 2|\mathcal{H}_{0,0,l,i}|^2 + 1/\text{SNR}}$$

where $\mathcal{H}_i = \sum_{m=0}^{L_u} h[m] e^{-j2\pi \frac{m}{N_{fft}}}$ is the Fourier transform of the channel at the $i$th subcarrier.

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Simplifications of OFDM channels can be found, but scattered, in classical 90’s OFDM literature [1–3]. Still, these derivation steps are often reproduced in recent works for specific scenarios, see e.g. [4, 5]. In [6], detailed derivations of OFDM channels as single sum expressions were summarized for the case of a causal channel. Writing the channel coefficients as such directly greatly simplifies the resulting SINR expression.

In this note, we review these derivations with some additional simplifications and extensions.

I. TRANSMISSION EQUATIONS

An OFDM modulation is defined by a subcarrier spacing $\Delta_f$, an IFFT size $N_{\text{fft}}$, and a CP length $N_{\text{cp}}$. An OFDM symbol (without CP) has then a time duration of $T_s = 1/\Delta_f$ with sampling period $T_{\text{sp}} = T_s/N_{\text{fft}}$. We consider that a total of $N_{\text{sc}}$ consecutive subcarriers are allocated by i.i.d. data symbols with average power $P$.

A. OFDM transmission

The OFDM signal is a consecutive transmission of OFDM blocks as

$$s[k] = \sum_b s_b[k - b(N_{\text{cp}} + N_{\text{fft}})]$$

where the data symbol $x_{b,l}$ for the $l$th subcarrier of the $b$th OFDM block is modulated as

$$s_b[k] = \frac{1}{\sqrt{N_{\text{sc}}}} \sum_{l=0}^{N_{\text{sc}}-1} x_{b,l} e^{j2\pi\frac{lk}{N_{\text{fft}}}} \text{rect}(k)$$

with OFDM rectangular block window $\text{rect}(n) = \begin{cases} 1, & -N_{\text{cp}} \leq n \leq N_{\text{fft}} - 1 \\ 0, & \text{otherwise} \end{cases}$.

B. Channel

The transmitted signal is convolved with the channel $h[n]$ where its channel impulse response (CIR) is $h[n] = \sum_{l=-L_d}^{L_u} h_l \delta_{n-l}$ with $-L_d \leq 0 \leq L_u$. The channel taps are assumed to be i.i.d. with average energy of the $p$th channel tap being $E_p = E[|h[p]|^2]$. 

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The time \( n = 0 \) corresponds to the time of reference (TOR) for demodulation, resulting from synchronization. Therefore, the channel as observed by the receiver may not be causal. This can be, for example, the consequence of using non-causal pulse shaping filtering at the transmitter, such as a sinc shaped filter, and synchronizing the receiver with the filter’s maximum peak.

C. Received and demodulated signal

The received signal is

\[ r[k] = \sum_{m=-L_d}^{L_u} h[m]s[k-m] + z[k] \]  

(3)

where \( z[k] \sim \mathcal{CN}(0, \sigma_z^2) \) is a zero-mean additive white Gaussian noise (AWGN) with variance \( \sigma_z^2 \). After CP removal, for \( 0 \leq k \leq (N_{\text{fft}} - 1) \) we have

\[ r[k] = \sum_{b} \sum_{m=-L_d}^{L_u} h[m]s_b[k-m-b(N_{\text{fft}} + N_{\text{cp}})] \]  

(4)

and then by substitution of (2)

\[ r[k] = \frac{1}{\sqrt{N_{\text{sc}}}} \sum_{b} \sum_{l=0}^{N_{\text{sc}}-1} x_{b,l} \left( \sum_{m=-L_d}^{L_u} h[m]e^{-j2\pi \frac{m}{N_{\text{fft}}}} \text{rect}[k-m-b(N_{\text{fft}} + N_{\text{cp}})] \right) e^{j2\pi \frac{[k-bN_{\text{cp}}]}{N_{\text{sc}}}} + z[k]. \]  

(5)

The received signal is then demodulated by FFT which gives the demodulated symbol for the \( i \)th subcarrier

\[ y[i] = \frac{\sqrt{N_{\text{sc}}}}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} r[k]e^{-j2\pi \frac{k}{N_{\text{fft}}}} \]

\[ = \mathcal{H}_{0,i,i}x_{0,i} + \sum_{l=0}^{N_{\text{sc}}-1} \mathcal{H}_{0,l,i}x_{0,l} + \sum_{l=0}^{N_{\text{sc}}-1} \sum_{b \neq 0} \mathcal{H}_{b,l,i}x_{b,l} + n[i] \]

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ISI

where for any \( b \)

\[ \mathcal{H}_{b,l,i} = \frac{1}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} \sum_{m=-L_d}^{L_u} h[m]e^{-j2\pi \frac{(m-k(l-i)+blN_{\text{cp}})}{N_{\text{sc}}}} \text{rect}[k-m-b(N_{\text{fft}} + N_{\text{cp}})], \]  

(6)

and \( n[i] = \frac{\sqrt{N_{\text{sc}}}}{N_{\text{sc}}} \sum_{k=0}^{N_{\text{sc}}-1} z[k]e^{-j2\pi \frac{k}{N_{\text{fft}}}} \) is the post-processed AWGN with variance \( \sigma_n^2 = \frac{N_{\text{sc}}}{N_{\text{sc}}} \sigma_z^2 \). 

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II. SINR

Accordingly, the signal to interference plus noise ratio (SINR) on the \( i \)th subcarrier is

\[
\text{SINR}_i = \frac{|H_{0,i,i}|^2}{\sum_{l=0}^{N_{sc}-1} |H_{0,l,i}|^2 + \sum_{l=0}^{N_{sc}-1} \sum_{b\neq 0} |H_{b,l,i}|^2 + 1/\text{SNR}},
\]

(7)

with \( H_{b,l,i} \) given in (6) and \( \text{SNR} = P/\sigma_n^2 \).

Below, the terms \( H_{b,l,i} \) are simplified.

A. Simplified channel coefficients

Each channel component after demodulation can be expressed as a single sum, which can be interpreted as a Fourier transform of a weighted channel impulse response. Precisely, the desired signal channel on the \( i \)th subcarrier is

\[
H_{0,i,i} = \sum_{m=-L_d}^{L_u} c[m] h[m] e^{-j2\pi \frac{im}{N_{\text{fft}}}},
\]

(8)

the ICI channel coefficient from the \( l \neq i \) subcarrier is

\[
H_{0,l,i} = \sum_{m=-L_d}^{L_u} \tilde{c}_{l,i}[m] h[m] e^{-j2\pi \frac{im}{N_{\text{fft}}}}
\]

(9)

the ISI channel from the \( l \)th subcarrier of the \( b \)-block is

\[
H_{b,l,i} = \sum_{m=-L_d}^{L_u} a_{b,l,i}[m] h[m] e^{-j2\pi \frac{im}{N_{\text{fft}}}}
\]

(10)

with weight functions

\[
c[m] = \begin{cases} 
\frac{N_{\text{fft}}+m}{N_{\text{fft}}} & -N_{\text{fft}} \leq m \leq 0 \\
1 & 0 \leq m \leq N_{cp} \\
\frac{N_{\text{fft}}-(m-N_{cp})}{N_{\text{fft}}} & N_{cp} \leq m \leq N_{cp} + N_{\text{fft}} \\
0 & \text{otherwise}
\end{cases}
\]

(11)
\[
\tilde{c}_{l,i}[m] = \begin{cases} 
\frac{1 - e^{j2\pi \frac{m(l-i)}{N_{\text{fft}}}}}{N_{\text{fft}}(1 - e^{j2\pi \frac{N_{\text{fft}}}{N_{\text{fft}}}})} & -N_{\text{fft}} \leq m \leq 0 \\
0 & 0 \leq m \leq N_{\text{cp}} \\
e^{j2\pi \frac{(m-N_{\text{cp}})(l-i)}{N_{\text{fft}}}} \frac{1}{N_{\text{fft}}(1 - e^{j2\pi \frac{N_{\text{fft}}}{N_{\text{fft}}}})} & N_{\text{cp}} \leq m \leq N_{\text{fft}} + N_{\text{cp}} \\
0 & \text{otherwise}
\end{cases},
\]

(12)

and

\[
a_{b,i,i}[m] = \begin{cases} 
eg c_l[i][m] + b(N_{\text{fft}} + N_{\text{cp}}) & l = i \\
e^{-j2\pi \frac{b N_{\text{cp}}}{N_{\text{fft}}}} \tilde{c}_{l,i}[m + b(N_{\text{fft}} + N_{\text{cp}})] & l \neq i
\end{cases}.
\]

(13)

B. Further simplification with \(L_d, L_u \leq N_{\text{fft}} - 1\)

If one can assume that \(L_d\) and \(L_u\) are both less than a symbol length, i.e., \(L_d, L_u \leq N_{\text{fft}} - 1\), the interference inside one block then only depends of the previous and consecutive blocks. Thus, we can limit the analysis to \(b = -1, 0\) and 1.

The resulting ISI channel coefficients from the previous block are

\[
\mathcal{H}_{-1,i,i} = e^{j2\pi \frac{L_{\text{cp}}}{N_{\text{fft}}}} \times \begin{cases} 
\sum_{m=N_{\text{cp}}}^{L_u} h[m] e^{-j2\pi \frac{m}{N_{\text{fft}}} (1 - c[m])} & l = i \\
- \sum_{m=N_{\text{cp}}}^{L_u} h[m] e^{-j2\pi \frac{m}{N_{\text{fft}}} \tilde{c}_{l,i}[m]} & l \neq i
\end{cases},
\]

(14)

the ISI channel coefficients from the consecutive block are

\[
\mathcal{H}_{-1,i,i} = e^{j2\pi \frac{L_{\text{cp}}}{N_{\text{fft}}}} \times \begin{cases} 
\sum_{m=-L_d}^{0} h[m] e^{-j2\pi \frac{m}{N_{\text{fft}}} (1 - c[m])} & l = i \\
- \sum_{m=-L_d}^{0} h[m] e^{-j2\pi \frac{m}{N_{\text{fft}}} \tilde{c}_{l,i}[m]} & l \neq i
\end{cases},
\]

(15)

and

\[
\mathcal{H}_{b,i,i} = 0 \text{ for } |b| > 1.
\]

(16)
C. Further simplification with \( L_d = 0 \) and \( L_u \leq N_{ft} - 1 \)

If the channel is causal, i.e., the ISI comes only from the previous block with \( b = -1 \). The ISI channel coefficients become

\[
\mathcal{H}_{b,i,i} = 0 \quad \text{for } b \neq \{-1, 0\}
\]

and

\[
\mathcal{H}_{-1,i,i} = e^{j2\pi \frac{N_{cp} l}{N_{ft}}} \times \begin{cases} \mathcal{H}_i - \mathcal{H}_{0,i,i} & l = i \\ -\mathcal{H}_{0,l,i} & l \neq i \end{cases}
\]

where

\[
\mathcal{H}_i = \sum_{m=0}^{L_u} h[m] e^{-j2\pi \frac{ml}{N_{ft}}}
\]

is the Fourier transform of the channel at the \( i \)th subcarrier.

This leads to an interference term in the SINR expression that only depends on the ICI power as

\[
\text{SINR}_i = \frac{|\mathcal{H}_{0,i,i}|^2}{|\mathcal{H}_i - \mathcal{H}_{0,i,i}|^2 + \sum_{l=0}^{N_{sc}-1} 2|\mathcal{H}_{0,l,i}|^2 + 1/\text{SNR}}
\]

III. AVERAGE-SIGNAL TO AVERAGE-INTERFERENCE PLUS NOISE RATIO (aSaINR)

Sometimes in literature, see e.g. [3, 4, 7, 8], the SINR is instead defined as the average-signal-power to average-interference-power-plus-noise ratio (aSaINR). The information-theoretic justification of this quantity is less clear but it provides often a practical and convenient OFDM design parameter.

Here, we consider again \( L_u, L_d \leq N_{ft} - 1 \). The aSaINR is then

\[
\Gamma_i = \frac{E[|\mathcal{H}_{0,i,i}|^2]}{\sum_{l=0}^{N_{sc}-1} E[|\mathcal{H}_{0,l,i}|^2] + \sum_{l=0}^{N_{sc}-1} E[|\tilde{\mathcal{H}}_{-1,l,i}|^2] + E[|\tilde{\mathcal{H}}_{1,l,i}|^2] + 1/\text{SNR}}
\]

Given the average channel tap power as \( E_m = E[|h_m|^2] \), this simplifies as

\[
\Gamma_i = \frac{\sum_{m=1}^{L_u} c[m]^2 E_m}{\sum_{m=1}^{L_u} (1 - c[m])^2 + \sum_{l=0}^{N_{sc}-1} 2|\tilde{e}_{l,i}[m]|^2} E_m + 1/\text{SNR}
\]
which, if $N_{sc} = N_{\text{fft}}$, further simplifies as

$$\Gamma_i = \frac{\sum_{m=-L_d}^{L_u} c[m]^2 E_m}{\sum_{m=-L_d}^{L_u} (1 - c[m]^2) E_m + 1/\text{SNR}}.$$ \tag{23}

IV. DERIVATIONS

A. Simplification of $H_{0,l,i}$

1. Case $l = i$

We start with the simplest case which is desired signal channel

$$H_{0,i,i} = \frac{1}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} \sum_{m=-L_d}^{L_u} h[m] e^{-j2\pi \frac{im}{N_{\text{fft}}} \text{rect}(k-m)}$$ \tag{24}

$$= \sum_{m=-L_d}^{L_u} h[m] e^{-j2\pi \frac{im}{N_{\text{fft}}}} \times \frac{1}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} \text{rect}(k-m)$$ \tag{25}

$$= \sum_{m=-L_d}^{L_u} c[m] h[m] e^{-j2\pi \frac{im}{N_{\text{fft}}}}$$ \tag{26}

where

$$c[m] = \frac{1}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} \text{rect}(k-m)$$ \tag{27}

which can be computed as

$$c[m] = \begin{cases} 
\frac{N_{\text{fft}}+m}{N_{\text{fft}}} & -N_{\text{fft}} \leq m \leq 0 \\
1 & 0 \leq m \leq N_{cp} \\
\frac{N_{\text{fft}}-(m-N_{cp})}{N_{\text{fft}}} & N_{cp} \leq m \leq N_{cp} + N_{\text{fft}} \\
0 & \text{otherwise} 
\end{cases}.$$ \tag{28}

2. Case $l \neq i$

Now for the ICI channels, we have
For the ISI channels we have

\[ H_{b,l,i} = \frac{1}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} \sum_{m=-L_d}^{L_u} h[m] e^{-j2\pi \frac{(m-k(l-i))}{N_{\text{fft}}}} \text{rect}[k-m] \] \quad (29)\]

\[ = \sum_{m=-L_d}^{L_u} h[m] e^{-j2\pi \frac{m}{N_{\text{fft}}}} \times \frac{1}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} e^{j2\pi \frac{k(l-i)}{N_{\text{fft}}}} \text{rect}[k-m] \] \quad (30)\]

\[ = \sum_{m=-L_d}^{L_u} \tilde{c}_{l,i}[m] h[m] e^{-j2\pi \frac{m}{N_{\text{fft}}}} \] \quad (31)\]

where

\[ \tilde{c}_{l,i}[m] = \frac{1}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} e^{j2\pi \frac{k(l-i)}{N_{\text{fft}}}} \text{rect}[k-m] \] \quad (32)\]

\[ = \begin{cases} 
\frac{1}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1+m} e^{j2\pi \frac{k(l-i)}{N_{\text{fft}}}} & -N_{\text{fft}} \leq m \leq 0 \\
\frac{1}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} e^{j2\pi \frac{k(l-i)}{N_{\text{fft}}}} & 0 \leq m \leq N_{\text{cp}} \\
\frac{1}{N_{\text{fft}}} \sum_{k=-(N_{\text{cp}}-m)}^{N_{\text{fft}}-1} e^{j2\pi \frac{k(l-i)}{N_{\text{fft}}}} & N_{\text{cp}} \leq m \leq N_{\text{fft}} + N_{\text{cp}} 
\end{cases} \] \quad (33)\]

Using the formula

\[ \sum_{k=b}^{a} r^k = \frac{r^{a+1} - r^b}{1-r} \] \quad (34)\]

we get

\[ \tilde{c}_{l,i}[m] = \begin{cases} 
\frac{1-e^{j2\pi \frac{m(l-i)}{N_{\text{fft}}}}}{N_{\text{fft}}(1-e^{j2\pi \frac{(l-i)}{N_{\text{fft}}}})} & -N_{\text{fft}} \leq m \leq 0 \\
0 & 0 \leq m \leq N_{\text{cp}} \\
\frac{e^{j2\pi \frac{(m-N_{\text{cp}})(l-i)}{N_{\text{fft}}}}}{N_{\text{fft}}(1-e^{j2\pi \frac{(l-i)}{N_{\text{fft}}}})} & N_{\text{cp}} \leq m \leq N_{\text{fft}} + N_{\text{cp}} \\
0 & \text{otherwise} 
\end{cases} \] \quad (35)\]

B. Simplification of \( H_{b,l,i}, b \neq 0 \)

For the ISI channels we have

\[ H_{b,l,i} = \sum_{m=-L_d}^{L_u} h[m] e^{-j2\pi \frac{m}{N_{\text{fft}}}} \times \frac{1}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} e^{j2\pi \frac{k(l-i)}{N_{\text{fft}}}} \text{rect}[k-m - b(N_{\text{fft}} + N_{\text{cp}})] \] \quad (36)\]

\[ = \sum_{m=-L_d}^{L_u} a_{b,l,i}[m] h[m] e^{-j2\pi \frac{m}{N_{\text{fft}}}} \] \quad (37)\]
where

$$a_{b,l,i}[m] = \frac{e^{-j2\pi \frac{blN_{\text{cp}}}{N_{\text{fft}}}}}{N_{\text{fft}}} e^{j2\pi \frac{k(l-i)}{N_{\text{fft}}}} \text{rect}[k - m - b(N_{\text{fft}} + N_{\text{cp}})]. \quad (38)$$

Two cases can be distinguished.

1. **Case $l = i$**

For the ISI resulting from the same subcarrier index, we have

$$a_{b,i,i}[m] = \frac{e^{-j2\pi \frac{biN_{\text{cp}}}{N_{\text{fft}}}}}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} \text{rect}[k - m - b(N_{\text{fft}} + N_{\text{cp}})] \quad (39)$$

$$= e^{-j2\pi \frac{biN_{\text{cp}}}{N_{\text{fft}}}} c[m + b(N_{\text{fft}} + N_{\text{cp}})] \quad (40)$$

where the last equality follows from $c[m] = \frac{1}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} \text{rect}(k - m)$.

Precisely,

$$a_{b,i,i} = e^{-j2\pi \frac{biN_{\text{cp}}}{N_{\text{fft}}}} \begin{cases} \frac{m+(b+1)N_{\text{fft}}+bN_{\text{cp}}}{N_{\text{fft}}} & - (1 + b)N_{\text{fft}} - bN_{\text{cp}} \leq m \leq -b(N_{\text{fft}} + N_{\text{cp}}) \\ 1 & -b(N_{\text{fft}} + N_{\text{cp}}) \leq m \leq -bN_{\text{fft}} + (1 - b)N_{\text{cp}} \\ \frac{(1-b)(N_{\text{fft}}+N_{\text{cp})}-m}{N_{\text{fft}}} & -bN_{\text{fft}} + (1 - b)N_{\text{cp}} \leq m \leq (1 - b)(N_{\text{fft}} + N_{\text{cp}}) \\ 0 & \text{otherwise} \end{cases}. \quad (41)$$

2. **Case $l \neq i$**

For the ISI resulting from the other subcarrier indices, we get

$$a_{b,l,i}[m] = \frac{e^{-j2\pi \frac{blN_{\text{cp}}}{N_{\text{fft}}}}}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} e^{j2\pi \frac{k(l-i)}{N_{\text{fft}}}} \text{rect}[k - m - b(N_{\text{fft}} + N_{\text{cp}})] \quad (42)$$

$$= e^{-j2\pi \frac{blN_{\text{cp}}}{N_{\text{fft}}}} \tilde{c}_{l,i}[m + b(N_{\text{fft}} + N_{\text{cp}})] \quad (43)$$
which is precisely given as

\[
a_{b,l,i} = e^{-j2\pi \frac{bN_{cp}}{N_{ff}}}
\begin{cases}
1-e^{j2\pi \frac{(m+bN_{cp})(l-i)}{N_{ff}}} & \text{if } (1+b)N_{ff} - bN_{cp} \leq m \leq -b(N_{ff} + N_{cp}) \\
0 & \text{otherwise}
\end{cases}
\]

\[
e^{-j2\pi \frac{(m+(b-1)N_{cp})(l-i)}{N_{ff}}}
\begin{cases}
1-e^{j2\pi \frac{l-i}{N_{ff}}} & \text{if } -b(N_{ff} + N_{cp}) \leq m \leq -bN_{ff} + (1-b)N_{cp} \\
0 & \text{otherwise}
\end{cases}
\]

\[
e^{-j2\pi \frac{mN_{cp}}{N_{ff}}} & \text{if } -bN_{ff} + (1-b)N_{cp} \leq m \leq (1-b)(N_{ff} + N_{cp})
\]

\(, \quad (44)\)

C. Special cases \(L_u, L_d \leq N_{ff} - 1\)

Remark that

\[
a_{-1,i,i} = e^{j2\pi \frac{iN_{cp}}{N_{ff}} (1 - c[m])} \quad \text{for } 0 \leq m \leq N_{ff} + 2N_{cp}, \quad (45)
\]

\[
a_{1,i,i} = e^{j2\pi \frac{iN_{cp}}{N_{ff}} (1 - c[m])} \quad \text{for } -N_{ff} \leq m \leq 0, \quad (46)
\]

\[
a_{-1,i}[m] = -e^{j2\pi \frac{iN_{cp}}{N_{ff}} \tilde{c}_{l,i}[m]} \quad \text{for } 0 \leq m \leq N_{ff} + 2N_{cp}, \quad (47)
\]

\[
a_{1,i}[m] = -e^{-j2\pi \frac{iN_{cp}}{N_{ff}} \tilde{c}_{l,i}[m]} \quad \text{for } -N_{ff} \leq m \leq 0. \quad (48)
\]

Thus with \(L_u, L_d \leq N_{ff} - 1\), we have as special cases

\[
a_{-1,i,i}[m] = \begin{cases}
0 & \text{if } -N_{ff} \leq m \leq 0 \\
e^{j2\pi \frac{iN_{cp}}{N_{ff}} (1 - c[m])} & 0 \leq m \leq N_{ff}
\end{cases}, \quad (49)
\]

\[
a_{-1,i}[m] = \begin{cases}
0 & \text{if } -N_{ff} \leq m \leq N_{cp} \\
-e^{j2\pi \frac{iN_{cp}}{N_{ff}} \tilde{c}_{l,i}[m]} & N_{cp} \leq m \leq N_{ff}
\end{cases}, \quad (50)
\]

and similarly

\[
a_{1,i,i}[m] = \begin{cases}
e^{-j2\pi \frac{iN_{cp}}{N_{ff}} (1 - c[m])} & N_{ff} \leq m \leq 0 \\
0 & 0 \leq m \leq N_{ff}
\end{cases}, \quad (51)
\]

\[
a_{1,i}[m] = \begin{cases}
e^{-j2\pi \frac{iN_{cp}}{N_{ff}} \tilde{c}_{l,i}[m]} & -N_{ff} \leq m \leq 0 \\
0 & 0 \leq m \leq N_{ff}
\end{cases}. \quad (52)
\]

The resulting simplified channel coefficients directly follows from these.
D. aSaINR derivations

By direct averaging we have the average desired signal power

\[
E[|H_{0,i,i}|^2] = \sum_{m=L_d}^{L_u} c[m]^2 E_m. \tag{53}
\]

Similarly we can average the ICI terms as

\[
E[|H_{0,i,i}|^2] = \sum_{m=L_d}^{L_u} E_m |\tilde{c}_{l,i}[m]|^2, \tag{54}
\]

and the ISI terms as

\[
E[|H_{-1,i,i}|^2] = \sum_{m=N_{cp}}^{L_u} (1 - c[m])^2 E_m, \quad E[|H_{-1,i,i}|^2] = \sum_{m=N_{cp}}^{L_u} |\tilde{c}_{l,i}[m]|^2 E_m, \tag{55}
\]

\[
E[|H_{1,i,i}|^2] = \sum_{m=L_d}^{0} (1 - c[m])^2 E_m, \quad E[|H_{1,i,i}|^2] = \sum_{m=L_d}^{0} |\tilde{c}_{l,i}[m]|^2 E_m. \tag{56}
\]

Therefore, the total interference power is

\[
E[I] = \sum_{l=0}^{N_{sc} - 1} E[|H_{0,i,i}|^2] + \sum_{l=0}^{N_{sc} - 1} E[|H_{-1,i,i}|^2] + E[|H_{1,i,i}|^2] \tag{57}
\]

\[
= \sum_{m=L_d}^{L_u} (1 - c[m])^2 E_m + \sum_{m=L_d}^{L_u} \sum_{l=0}^{N_{sc} - 1} 2|\tilde{c}_{l,i}[m]|^2 E_m \tag{58}
\]

\[
= \sum_{m=L_d}^{L_u} \left( (1 - c[m])^2 + \sum_{l=0}^{N_{sc} - 1} 2|\tilde{c}_{l,i}[m]|^2 \right) E_m. \tag{59}
\]

In the case, \(N_{sc} = N_{ff}\), we can simplify the total interference term as

\[
E[I] = \sum_{m=L_d}^{L_u} (1 - c[m])^2 E_m \tag{60}
\]

which follows from the lemma below.
Lemma 1

\[
\sum_{l=0 \atop l \neq i}^{N_{\text{fft}}-1} |\tilde{e}_{l,i}[m]|^2 = c[m] - c[m]^2 \tag{61}
\]

Proof: First, by direct expansion we get

\[
|\tilde{e}_{l,i}[m]|^2 = \frac{1}{N_{\text{fft}}^2} \left| \sum_{k=0}^{N_{\text{fft}}-1} e^{j2\pi \frac{k(l-i)}{N_{\text{fft}}}} \text{rect}[k-m] \right|^2 \tag{62}
\]

\[
= \frac{1}{N_{\text{fft}}^2} \left( \sum_{k=0}^{N_{\text{fft}}-1} e^{j2\pi \frac{k(l-i)}{N_{\text{fft}}}} \text{rect}[k-m] \right) \left( \sum_{h=0}^{N_{\text{fft}}-1} e^{j2\pi \frac{-h(l-i)}{N_{\text{fft}}}} \text{rect}[h-m] \right) \tag{63}
\]

\[
= \frac{1}{N_{\text{fft}}^2} \sum_{k=0}^{N_{\text{fft}}-1} \sum_{h=0}^{N_{\text{fft}}-1} e^{j2\pi \frac{(k-h)(l-i)}{N_{\text{fft}}}} \text{rect}[k-m]\text{rect}[h-m] \tag{64}
\]

From this, it follows

\[
\sum_{l=0 \atop l \neq i}^{N_{\text{fft}}-1} |\tilde{e}_{l,i}[m]|^2 = \frac{1}{N_{\text{fft}}^2} \sum_{k=0}^{N_{\text{fft}}-1} \sum_{h=0}^{N_{\text{fft}}-1} \sum_{l=0 \atop l \neq i}^{N_{\text{fft}}-1} e^{j2\pi \frac{(k-h)(l-i)}{N_{\text{fft}}}} \text{rect}[k-m]\text{rect}[h-m] \tag{65}
\]

\[
= \frac{1}{N_{\text{fft}}^2} \sum_{k=0}^{N_{\text{fft}}-1} \sum_{h=0}^{N_{\text{fft}}-1} \sum_{\eta=1}^{N_{\text{sc}}-1} e^{j2\pi \frac{(k-h)\eta}{N_{\text{fft}}}} \text{rect}[k-m]\text{rect}[h-m] \tag{66}
\]

where in the last equality we used the change of variable \(\eta = l - i\).

Now assuming \(N_{\text{sc}} = N_{\text{fft}}\), recalling that \(c[m] = \frac{1}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} \text{rect}(k-m)\), and remarking that

\[
\sum_{\eta=1}^{N_{\text{sc}}-1} e^{j2\pi \frac{(k-h)\eta}{N_{\text{fft}}}} = \begin{cases} 
-1 & (k-h) \neq 0 \\
N_{\text{fft}} - 1 & (k-h) = 0 
\end{cases} \tag{67}
\]

, we get
\[
\sum_{l=0}^{N_{\text{fft}}-1} |\tilde{c}_{l,i}[m]|^2 = \frac{1}{N_{\text{fft}}} \sum_{k=0}^{N_{\text{fft}}-1} \left( (N_{\text{fft}} - 1) \times \text{rect}[k-m]^2 + \sum_{h=0}^{N_{\text{fft}}-1} (-1) \times \text{rect}[k-m] \text{rect}[h-m] \right)
\]

\[
= \left( \frac{N_{\text{fft}} - 1}{N_{\text{fft}}} \right) c[m] - \frac{1}{N_{\text{fft}}^2} \sum_{k=0}^{N_{\text{fft}}-1} \text{rect}[k-m] \sum_{h=0}^{N_{\text{fft}}-1} \text{rect}[h-m]
\]

(68)

\[
= \left( \frac{N_{\text{fft}} - 1}{N_{\text{fft}}} \right) c[m] - \frac{1}{N_{\text{fft}}^2} \sum_{k=0}^{N_{\text{fft}}-1} \text{rect}[k-m] \left( \sum_{h=0}^{N_{\text{fft}}-1} \text{rect}[h-m] - \text{rect}[k-m] \right)
\]

(69)

\[
= \left( \frac{N_{\text{fft}} - 1}{N_{\text{fft}}} \right) c[m] - \frac{1}{N_{\text{fft}}^2} \sum_{k=0}^{N_{\text{fft}}-1} \sum_{h=0}^{N_{\text{fft}}-1} \text{rect}[k-m] \text{rect}[h-m]
\]

(70)

\[+ \frac{1}{N_{\text{fft}}^2} \sum_{k=0}^{N_{\text{fft}}-1} \text{rect}[k-m] \text{rect}[k-m]
\]

(71)

\[
= \left( \frac{N_{\text{fft}} - 1}{N_{\text{fft}}} \right) c[m] - c[m]^2 + \frac{c[m]}{N_{\text{fft}}}
\]

(72)

\[
= c[m] - c[m]^2
\]

(73)

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