Interpolatory pointwise estimates for convex polynomial approximation\textsuperscript{*}

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Abstract

This paper deals with approximation of smooth convex functions \( f \) on an interval by convex algebraic polynomials which interpolate \( f \) and its derivatives at the endpoints of this interval. We call such estimates “interpolatory”. One important corollary of our main theorem is the following result on approximation of \( f \in \Delta^2 \), the set of convex functions, from \( W^r \), the space of functions on \([-1,1]\) for which \( f^{(r-1)} \) is absolutely continuous and \( \| f^{(r)} \|_\infty := \text{ess sup}_{x \in [-1,1]} | f^{(r)}(x) | < \infty \):

For any \( f \in W^r \cap \Delta^2, r \in \mathbb{N} \), there exists a number \( N = N(f,r) \), such that for every \( n \geq N \), there is an algebraic polynomial of degree \( \leq n \) which is in \( \Delta^2 \) and such that

\[
\left\| f - P_n \right\|_{\infty} \leq \frac{c(r)}{n^r} \left\| f^{(r)} \right\|_{\infty},
\]

where \( \varphi(x) := \sqrt{1-x^2} \).

For \( r = 1 \) and \( r = 2 \), the above result holds with \( N = 1 \) and is well known. For \( r \geq 3 \), it is not true, in general, with \( N \) independent of \( f \).

1 Introduction and main results

We start by recalling some standard notation. As usual, \( C^r(I) \) denotes the space of \( r \) times continuously differentiable functions on a closed interval \( I \), \( C^0(I) := C(I) \) is the space of continuous functions on \( I \), equipped with the uniform norm which will be denoted by \( \| \cdot \|_I \). For \( k \in \mathbb{N} \) and an interval \( I \), \( \Delta_k^q(f,x;I) := \sum_{i=0}^k (-1)^i \binom{k}{i} f(x+(k/2-i)u) \) if \( x \pm ku/2 \in I \) and := 0, otherwise, and \( \omega_k(f,t;I) := \sup_{0 < \xi \leq t} \| \Delta_k^q(f,\xi;I) \|_I \) is the \( k \)th modulus of smoothness of \( f \) on \( I \). When dealing with \( I = [-1,1] \), we suppress referring to the interval and use the notation \( \| \cdot \| := \| \cdot \|_{[-1,1]} \), \( \omega_k(f,t) := \omega_k(f,t;[-1,1]) \), \( C^r := C^r([-1,1]) \), etc. We denote by \( \Delta^q \) the class of all \( q \)-monotone functions on \([-1,1] \), i.e., continuous functions such that \( \Delta_k^q(f,x) \geq 0 \) for all \( x \in [-1,1] \) and \( u > 0 \). In particular, \( \Delta^1 \) and \( \Delta^2 \) are the classes of all monotone and convex functions on \([-1,1] \), respectively. Also,

\[
\varphi(x) := \sqrt{1-x^2} \quad \text{and} \quad \rho_n(x) := \varphi(x)n^{-1} + n^{-2}, \quad n \in \mathbb{N},
\]

\( \rho_0(x) \equiv 1 \), and \( \Pi_n \) denotes the space of algebraic polynomials of degree \( \leq n \).

The following classical Timan-Dzyadyk-Freud-Brudnyi direct theorem for the approximation by algebraic polynomials (see e.g. [8] Theorem 8.5.3) shows that the order of approximation becomes significantly better near the endpoints of \([-1,1] \): if \( k \in \mathbb{N} \), \( r \in \mathbb{N}_0 \) and \( f \in C^r \), then for each \( n \geq k+r \) there is a polynomial \( P_n \in \Pi_n \) satisfying

\[
|f(x) - P_n(x)| \leq \frac{c(k,r)}{n^r} \rho_n(x) \omega_k(f^{(r)},\rho_n(x)), \quad x \in [-1,1].
\]

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Clearly, if we require that the approximating polynomials interpolate \( f \) as well as its derivatives at the endpoints, and we are successful, then the estimates should become even better.

Indeed, the following Telyakovskii-Gopengauz-type (i.e., “interpolatory”-type) theorem is an immediate consequence of \( \mathbf{3} \) Corollary 2-3.4 (see e.g. \( \mathbf{3} \) for the history of this problem).

**Theorem 1.1** (see \( \mathbf{3} \) Corollary 2-3.4). Let \( r \in \mathbb{N}_0, k \in \mathbb{N} \) and \( f \in C^r \). Then for any \( n \geq \max \{ k + r - 1, 2r + 1 \} \), there is a polynomial \( P_n \in \Pi_n \) such that (1.2) is valid and, moreover,

\[
(1.3) \quad |f(x) - P_n(x)| \leq c(r,k)\varphi^{2r}(x)\omega_k(f^{(r)}, \varphi^{2/k}(x)n^{-2(k-1)/k}), \quad \text{if} \quad 1 - n^{-2} \leq |x| \leq 1.
\]

It follows from \( \mathbf{3} \) Theorem 3] that, for any \( \gamma \in \mathbb{R} \), the quantity \( \varphi^{2/k}(x)n^{-2(k-1)/k} \) in (1.3) cannot be replaced by \( \varphi^{2\beta}(x)n^\gamma \) with \( \beta > 1/k \). Hence, the estimate (1.3) provides the optimal rate of approximation near the endpoints of \([-1,1]\).

It is a natural question if these estimates are valid if we approximate \( q \)-monotone functions by \( q \)-monotone polynomials. Of course, as is rather well known, (1.2) may not be valid in the \( n \) case even if \( n \) is allowed to depend on the function \( f \) that is approximated. For example, this is the case if \( i. 1 \leq q \leq 3, 0 \leq r \leq q - 1 \) and \( r + k \geq q + 2 \) if \( q = 1 \), [21] if \( q = 2 \) or \( q = 3 \), and (ii) \( q \geq 4 \) and \( r + k \geq 3 \).

Moreover, for any \( q, r, k, n \in \mathbb{N} \), there exists a function \( f_n \in C^r \cap \Delta(q) \) such that (1.3) is not valid for any polynomial \( P_n \in \Pi_n \cap \Delta(q) \) (the construction of such an \( f_n \) is the same as in \( \mathbf{14} \), see also \( \mathbf{13} \) [19]). This means that, in the case \( r \geq 1 \), (1.3) cannot be true for all functions \( f \in C^r \cap \Delta(q) \) and all \( n \geq N(k, r, q) \). We emphasize that this does NOT mean that, for each fixed \( f \in C^r \cap \Delta(q) \), (1.3) is invalid for sufficiently large \( n \), i.e., (1.3) may still be valid if \( n \geq N(f) \) (in fact, the proof of this fact in the case \( q = 2 \) is the main result of this paper).

If \( r = 0 \) and \( k \) is “small”, then the situation is different: for any \( q, n \in \mathbb{N} \), if \( r = 0 \) and \( 1 \leq k \leq 2 \), then (1.2) and (1.3) with \( k = 3 \) are both valid for \( q \)-monotone approximation (it is possible to show that the case for \( k = 1 \) follows from that for \( k = 2 \)). Indeed, the following interpolatory estimate follows from \( \mathbf{1} \) (\( q = 1 \)), \( \mathbf{15} \) [20] (\( q = 2 \)) and \( \mathbf{2} \) (\( q \geq 3 \)): for any \( q, n \in \mathbb{N} \) and \( f \in C \cap \Delta(q) \), there exists a polynomial \( P_n \in \Pi_n \cap \Delta(q) \) such that

\[
(1.4) \quad |f(x) - P_n(x)| \leq c(q)\omega_2(f, \varphi(x)/n), \quad x \in [-1,1],
\]

where \( c \) is an absolute constant. Additionally, (1.2) and (1.3) with \( n \geq 2 \) are valid for convex approximation (i.e., \( q = 2 \)) if \( r = 0 \) and \( k = 3 \) (8), and the case \( q = 3, r = 0 \) and \( k = 3 \) or \( k = 4 \) is still unresolved (in fact, it is not even known if (1.2) holds if \( k = 3 \)).

Recently, we were able to show (see \( \mathbf{13} \)) that (1.2) and (1.3) hold for monotone approximation \( (q = 1) \) if \( r \in \mathbb{N}, k = 2 \) and \( n \geq N(f, r) \), and the main purpose of this paper is to prove an analogous result for convex approximation \( (q = 2) \). In fact, we follow similar ideas and apply some of the construction in \( \mathbf{13} \), but there are some additional rather significant technical difficulties that we have to overcome in this case (for example, proofs in the cases for \( r = 1 \) and \( r \geq 2 \) turn out to be completely different). Also, one of the important tools that we are using is our recent result \( \mathbf{14} \) on convex approximation of \( f \in C^r \cap \Delta(2) \), by convex piecewise polynomials (see Theorem \( \mathbf{5.1} \) below).

The following theorem is the main result in this manuscript.

**Theorem 1.2.** Given \( r \in \mathbb{N} \), there is a constant \( c = c(r) \) with the property that if \( f \in C^r \cap \Delta(2) \), then there exists a number \( N = N(f, r) \), depending on \( f \) and \( r \), such that for every \( n \geq N \), there is \( P_n \in \Pi_n \cap \Delta(2) \) satisfying

\[
(1.5) \quad |f(x) - P_n(x)| \leq c(r) \varphi^{2r}(x)/(\varphi(x)/n)^r \omega_2(f^{(r)}, \varphi(x)/n), \quad x \in [-1,1].
\]

Moreover, for \( x \in [-1, -1 + n^{-2}] \cup [1 - n^{-2}, 1] \) the following stronger estimates are valid:

\[
(1.6) \quad |f(x) - P_n(x)| \leq c(r)\varphi^{2r}(x)\omega_2(f^{(r)}, \varphi(x)/n)
\]

and

\[
(1.7) \quad |f(x) - P_n(x)| \leq c(r)\varphi^{2r}(x)\omega_1(f^{(r)}, \varphi^2(x)).
\]

**Remark 1.3.** [14] Theorem 2.3] implies that Theorem 1.2 is NOT valid with \( N \) independent of \( f \).
We now discuss some corollaries and applications of Theorem 1.2.

Recall that, given a number \( \alpha > 0 \), \( \text{Lip}^* \alpha \) denotes the class of all functions \( f \) on \([-1, 1]\) such that \( \omega_2(f^{[\alpha-1]}, t) = O(t^{\alpha-|\alpha|+1}) \). Together with the classical inverse theorems (see e.g. [12] Theorem 5 and Corollary 6), (1.2) implies that, if \( \alpha > 0 \), then a function \( f \) is in \( \text{Lip}^* \alpha \) if and only if

\[
\inf_{P_n \in \Pi_n} \| \rho_n \alpha (f - P_n) \| = O(1).
\]

**Corollary 1.4.** If \( \alpha > 0 \) and \( f \in \text{Lip}^* \alpha \cap \Delta(2) \), then there exists a constant \( C = C(\alpha) \) such that, for all sufficiently large \( n \), there are polynomials \( P_n \in \Pi_n \cap \Delta(2) \) satisfying

\[
|f(x) - P_n(x)| \leq C \left( \varphi(x)/n \right)^\alpha, \quad x \in [-1, 1].
\]

For \( 0 < \alpha < 2 \), (1.9) follows from (1.4) (and was stated in [13]).

In order to state another consequence of Theorem 1.2, we recall that \( W^r \) denotes the space of functions on \([-1, 1]\) for which \( f^{(r-1)} \) is absolutely continuous and \( \| f^{(r)} \|_\infty := \text{ess sup}_{x \in [-1,1]} |f^{(r)}(x)| < \infty \).

**Corollary 1.5.** For any \( f \in W^r \cap \Delta(2) \), \( r \in \mathbb{N} \), there exists a number \( N = N(f, r) \), such that

\[
\sup_{n \geq N} \inf_{P_n \in \Pi_n \cap \Delta(2)} \left\| \frac{f - P_n}{\varphi^n(\min\{1/n, \varphi\})^r} \right\|_\infty \leq c(r) \left\| f^{(r)} \right\|_\infty.
\]

In particular,

\[
\sup_{n \geq N} \inf_{P_n \in \Pi_n \cap \Delta(2)} \left\| \frac{f - P_n}{\varphi^n} \right\|_\infty \leq \frac{c(r)}{n^r} \left\| f^{(r)} \right\|_\infty.
\]

It follows from [14] Theorem 2.3 that, if \( r \geq 2 \) and \( r \geq 3 \), then, respectively, inequalities (1.10) and (1.11) are not true, in general, with \( N \) independent of \( f \). For all other \( r \in \mathbb{N} \), these inequalities hold with \( N = 1 \) which is a corollary of (1.4) with \( q = 2 \).

## 2 Notations and some inequalities for the Chebyshev partition

Most symbols used in this paper were introduced and discussed in [13]. For convenience, we list them in the following table which also includes symbols introduced in the previous section. Note that, in the proofs below (but not in definitions and statements), we often omit writing index “\( n \)” if it does not create any confusion (thus, we write “\( \rho \)” instead of “\( \rho_n \)” “\( x_j \)” instead of “\( x_{j,n} \)” etc.).

| Chebyshev knots and Chebyshev partition | k-majorants |
|----------------------------------------|-------------|
| \( x_j := x_{j,n} \) := \cos(j\pi/n) \), \( 0 \leq j \leq n \); 1 for \( j < 0 \) and \(-1 \) for \( j > n \) (Chebyshev knots) | \( \Phi^k := \{ \psi \in C[0, \infty) \mid \psi \uparrow, \psi(0) = 0 \), and \( t_2^{-k} \psi(t_2) \leq t_1^{-k} \psi(t_1) \) for \( 0 \leq t_1 \leq t_2 \} \). Note: if \( f \in C^r \), then \( \phi(t) := t^r \omega_k(f^{(r)}, t) \) is equivalent to a function from \( \Phi^{k+r} \) |
| \( T_n := (x_j)_{j=0}^n \) (Chebyshev partition) | |
| \( I_j := I_{j,n} := [x_j, x_{j-1}] \) | |
| \( h_j := h_{j,n} := |I_{j,n}| = x_{j-1} - x_j \) | |
| \( I_{i,j} := \bigcup_{k=\min\{i,j\}}^{\max\{i,j\}} I_k = [x_{\min\{i,j\}}, x_{\min\{i,j\}}-1], \) \( 1 \leq i,j \leq n \) (the smallest interval containing both \( I_i \) and \( I_j \)) | |
| \( h_{i,j} := |I_{i,j}| = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} h_k = x_{\min\{i,j\}} - x_{\max\{i,j\}} \) | |
| \( \psi_j := \psi_j(x) := \frac{|I_j|}{|x - x_j| + |I_j|} \) | |
| \( \varphi(x) := \sqrt{1 - x^2} \) | |
| \( \rho_0(x) := \varphi(x)n^{-1} + n^{-2} \), \( n \in \mathbb{N} \), and \( \rho_0(x) \equiv 1 \) | |
| \( \delta_n(x) := \min\{1, \rho_0(x)\} \) | |
### Piecewise polynomials on Chebyshev partition

| Expression | Description |
|------------|-------------|
| \( \Sigma_k := \Sigma_{k,n} \) | the set of continuous piecewise polynomials of degree \( \leq k-1 \) with knots at \( x_j \), \( 1 \leq j \leq n-1 \) |
| \( \Sigma_k^{(1)} := \Sigma_{k,n}^{(1)} \) | the set of continuously differentiable piecewise polynomials of degree \( \leq k-1 \) with knots at \( x_j \), \( 1 \leq j \leq n-1 \) |
| \( p_j := p_j(S) \) | := \( S|_{I_j} \), \( 1 \leq j \leq n \) (polynomial piece of \( S \) on the interval \( I_j \)) |
| \( b_{i,j}(S, \phi) \) | := \( \|p_i - p|_{I_j}\|_{I_j} \left( \frac{h_{j}}{h_{i,j}} \right)^k \), where \( \phi \in \Phi^k \), \( \phi \neq 0 \) and \( S \in \Sigma_k \) |
| \( b_k(S, \phi, A) \) | := \( \max_{1 \leq i,j \leq n} \{b_{i,j}(S, \phi) \mid I_i \subset A \text{ and } I_j \subset A \} \), where an interval \( A \subseteq [-1, 1] \) contains at least one interval \( I_\nu \) |
| \( b_k(S, \phi) \) | := \( b_k(S, \phi, [-1, 1]) = \max_{1 \leq i,j \leq n} b_{i,j}(S, \phi) \) |

### Constants

| \( C(\gamma_1, \ldots, \gamma_\mu) \) | positive constants depending only on parameters \( \gamma_1, \ldots, \gamma_\mu \) that may be different on different occurrences |
| \( c \) | positive constants that are either absolute or may only depend on the parameters \( k \) and \( r \) (if present) |
| \( C_1 \) | positive constants that are fixed throughout this paper |
| \( \gamma_1 \sim \gamma_\mu \) | \( A \sim \gamma_1 \sim \gamma_\mu \ B \) if \( C^{-1}B \leq A \leq CB \), for some positive constant \( C = C(\gamma_1, \ldots, \gamma_\mu) \) |

### Indicator functions and truncated powers

| \( \chi_j(x) \) | := \( \chi_{[x_j, 1]}(x) := 1 \), if \( x_j \leq x \leq 1 \), and := 0, otherwise |
| \( \Phi_j(x) \) | := \( (x - x_j)_+ := (x - x_j)\chi_j(x) = \int_{-1}^x \chi_j(t)dt \) |

We now collect all facts and inequalities for the Chebyshev partition that we need throughout this paper. Many of them are checked by straightforward calculations (also, see e.g. \([5, 13, 14] \) for references). Unless specified otherwise, it is assumed that \( 1 \leq j \leq n \), \( x, y \in [-1, 1] \).

\[(2.1) \quad n^{-1}\varphi(x) < \rho_n(x) < h_j < 5\rho_n(x), \quad x \in I_j \]
\[(2.2) \quad h_{j+1} < 3h_j \]
\[(2.3) \quad \rho_n^2(y) < 4\rho_n(|x-y| + \rho_n(x)) \]
\[(2.4) \quad (|x-y| + \rho_n(x))/2 < (|x-y| + \rho_n(y)) < 2(|x-y| + \rho_n(x)) \]
\[(2.5) \quad \rho_n(x) \leq |x - x_j|, \text{ for any } 0 \leq j \leq n \text{ and } x \notin (x_{j+1}, x_{j-1}) \]
\[(2.6) \quad \delta_n(x) \leq n\varphi(x) < \pi\delta_n(x), \text{ if } x \in [-1, x_{n-1}] \cup [x_1, 1], \text{ and } \delta_n(x) = 1, \text{ if } x \in [x_{n-1}, x_1] \]
\[(2.7) \quad \rho_n^2(x) < 8h_j(|x - x_j| + \rho_n(x)) \]
\[(2.8) \quad \left( \frac{\rho_n(x)}{\rho_n(x) + |x - x_j|} \right)^2 < c\psi_j(x) \]
\[(2.9) \quad \rho_n(x) + |x - x_j| \sim \rho_n(x) + \text{dist}(x, I_j) \]
\[(2.10) \quad \sum_{j=1}^n \psi_j(x) \leq c \]
\[(2.11) \quad \sum_{j=1}^n \left( \frac{\rho_n(x)}{\rho_n(x) + \text{dist}(x, I_j)} \right)^4 \leq c \]
\[(2.12) \quad c\psi_j^2(x)\delta_n^2(x) \leq \frac{1-x^2}{(1+x_{j-1})(1-x_j)} \leq c\delta_n^2(x)\psi_j^{-2}(x) \]
\[(2.13) \quad c\psi_j^2(x)\rho_n(x) \leq h_j \leq c\psi_j^{-1}(x)\rho_n(x) \]
\[(2.14) \quad c\psi_j^{2k}(x)\phi(\rho_n(x)) \leq \phi(h_j) \leq c\psi_j^{-k}(x)\phi(\rho_n(x)) \]
3 Auxiliary results on polynomial approximation of indicator functions and truncated powers

Recall the notation

\[(3.1) \quad t_j(x) := \left(\frac{\cos 2n \arccos x}{x - x_j} \right)^2 + \left(\frac{\sin 2n \arccos x}{x - x_j} \right)^2, \]

where \(x_j := \cos((j - 1/2)\pi/n)\) for \(1 \leq j \leq n\), \(x_j^0 := \cos((j - 1/4)\pi/n)\) for \(1 \leq j < n/2\), \(x_j^0 := \cos((j - 3/4)\pi/n)\) for \(n/2 \leq j \leq n\), and note that \(t_j \in \Pi_{4n-2}\) and, for all \(1 \leq j \leq n\),

\[(3.2) \quad t_j(x) \sim (|x - x_j| + 1)^{-2}, \quad x \in [-1, 1], \]

(see e.g. [19] or [10], (22), Proposition 5).

For \(\gamma_1, \gamma_2 \in \mathbb{N}_0\), \(\xi, \mu \in \mathbb{N}\), and \(1 \leq j \leq n\), we let

\[\mathcal{T}_j(x) := \mathcal{T}_{j,n}(x; \gamma_1, \gamma_2, \xi, \mu) := d_j^{-1} \int_{-1}^{x} (y - x_j)^{\gamma_1} (x_{j-1} - y)^{\gamma_2} (1 - y^2)^{\xi} t_j^\mu(y) \, dy,\]

where \(d_j := d_j(\gamma_1, \gamma_2, \xi, \mu)\) is the normalizing constant such that \(\mathcal{T}_j(1) = 1\). Then, it is possible to show (see e.g. [9], Proposition 4) that, for sufficiently large \(\mu\), function \(\mathcal{T}_j\) is well defined and is a polynomial of degree \(\leq \mu n\) (with some absolute constant \(c\)), and

\[d_j \sim (1 + x_{j-1})^{\xi} (1 - x_j)^{\xi} h_j^{-2\mu + 1 + \gamma_1 + \gamma_2}.\]

Also,

\[(3.3) \quad 1 - x_{j-1} < \int_{-1}^{1} \mathcal{T}_j(t) \, dt < 1 - x_j, \quad 1 \leq j \leq n.\]

Indeed, denoting for convenience \(\vartheta(y) := (y - x_j)^{\gamma_1} (x_{j-1} - y)^{\gamma_2} (1 - y^2)^{\xi} t_j^\mu(y)\), we have

\[\int_{-1}^{1} \mathcal{T}_j(t) \, dt < 1 - x_j \iff \int_{-1}^{1} \vartheta(t) \, dt < (1 - x_j) \int_{-1}^{1} \vartheta(t) \, dt \iff \int_{-1}^{1} (t - x_j) \vartheta(t) \, dt = d_j(\gamma_1 + 1, \gamma_2, \xi, \mu) > 0\]

(the other inequality is proved similarly).

Now, for the polynomials

\[\tau_j(x) := \mathcal{T}_{j,n}(x; 0, 0, \xi, \mu) \quad \text{and} \quad \bar{\tau}_j(x) := \mathcal{T}_{j,n}(x; 1, 1, \xi, \mu),\]

the following lemma was proved in [13].

**Lemma 3.1** ([13] Lemmas 4.1 and 4.2). If \(\alpha, \beta \geq 1\), then for sufficiently large \(\xi\) and \(\mu\) depending only on \(\alpha\) and \(\beta\) and for each \(1 \leq j \leq n - 1\), polynomials \(\tau_j\) and \(\bar{\tau}_j\) of degree \(\leq C(\alpha, \beta)n\) satisfy

\[(3.4) \quad \tau_j^\alpha(x) \geq C(\alpha, \beta)|I_j|^{-1} \delta_n^{\alpha\mu}(x) \psi_j^{30(\alpha + \beta)}(x), \quad x \in [-1, 1],\]

\[(3.5) \quad \bar{\tau}_j^\alpha(x) \leq 0, \quad \text{for} \quad x \in [-1, x_j] \cup [x_{j-1}, 1],\]

and, for all \(x \in [-1, 1]\),

\[(3.6) \quad \max \{|\tau_j^\alpha(x)|, |\bar{\tau}_j^\alpha(x)|\} \leq C(\alpha, \beta)|I_j|^{-1} \delta_n^{\alpha}(x) \psi_j^{\beta}(x)\]

and

\[(3.7) \quad \max \{|\chi_j(x) - \tau_j(x)|, |\chi_j(x) - \bar{\tau}_j(x)|\} \leq C(\alpha, \beta) \delta_n^{\alpha}(x) \psi_j^{\beta}(x).\]
Remark 3.2. The statement of this lemma is not valid if \( j = n \) since \( \chi_n \equiv 1, \tau_n(-1) = 0 \) and \( \delta_n(-1) = 0 \).

Inequalities (3.1) imply that, for each \( 1 \leq j \leq n - 1 \), there exists a constant \( 0 < \lambda_j < 1 \) such that the polynomial

\[
Q_j(x) := Q_{j,n}(x) := Q_{j,n}(x; \gamma_1, \gamma_2, \xi, \mu) := \int_{-1}^{x} (\lambda_j T_j(t) + (1 - \lambda_j) T_{j+1}(t)) \, dt
\]

satisfies \( Q_j(1) = 1 - x_j \). This implies that, if \( T_j \) is such that (3.1) is satisfied, then \( Q_j \) provides a “good” approximation of \( \Phi_j \), \( 1 \leq j \leq n - 2 \). The proof of this fact is rather standard. Indeed, first note that, for \( 1 \leq j \leq n - 2 \) and \( x \in [-1, 1] \),

\[
|\chi_j(x) - \chi_{j+1}(x)| \leq |\chi_{j+1}(x)| \leq C \delta_n(x) \psi_j^\beta(x) \quad \text{and} \quad \psi_j(x) \sim \psi_{j+1}(x).
\]

Now, if \( x \leq x_j \), then (assume that \( \beta > 1 \))

\[
|\Phi_j(x) - Q_j(x)| \leq \left| \int_{-1}^{x} (\lambda_j |T_j(t) - \chi_j(t)| + (1 - \lambda_j) |T_{j+1}(t) - \chi_j(t)|) \, dt \right|
\]

\[
\leq C \int_{-1}^{x} \delta_n(t) \psi_j^\beta(t) \, dt \leq C \delta_n(x) \int_{-\infty}^{x} |I_j|^\beta (x_j - t + |I_j|)^{-\beta} \, dt
\]

\[
\leq C |I_j| \delta_n(x) \psi_j^{\beta - 1}(x)
\]

and, if \( x > x_j \), then, similarly,

\[
|\Phi_j(x) - Q_j(x)| = \left| \int_{x}^{1} (\chi_j(t) - Q'_j(t)) \, dt \right|
\]

\[
\leq \left| \int_{x}^{1} (\lambda_j |T_j(t) - \chi_j(t)| + (1 - \lambda_j) |T_{j+1}(t) - \chi_j(t)|) \, dt \right|
\]

\[
\leq C |I_j| \delta_n(x) \psi_j^{\beta - 1}(x).
\]

Now, for \( 1 \leq j \leq n - 1 \), defining

\[
T_j(x) := T_{j,n}(x) := \Omega_{2,j,2n}(x; 0, 0, \xi, \mu) \quad \text{and} \quad \tilde{T}_j(x) := \tilde{T}_{j,n}(x) := \Omega_{2,j-1,2n}(x; 1, 1, \xi, \mu),
\]

and noting that \( x_j,n = x_{2j,2n}, h_{j,n} \sim h_{2j,2n} \sim h_{2j-1,2n}, \psi_{j,n} \sim \psi_{2j,2n} \sim \psi_{2j-1,2n}, \delta_n \sim \delta_{2n} \), we have the following result which follows from Lemma 3.1.

Lemma 3.3. If \( \alpha, \beta \geq 1 \), then for sufficiently large \( \xi \) and \( \mu \) depending only on \( \alpha \) and \( \beta \) and for each \( 1 \leq j \leq n - 1 \), polynomials \( T_j \) and \( \tilde{T}_j \) of degree \( \leq C(\alpha, \beta)n \) defined in (3.1) satisfy

\[
T_j(x) \geq C(\alpha, \beta)|I_j|^{-1} \delta_n(x) \psi_j^{30(\alpha+\beta)}(x), \quad x \in [-1, 1],
\]

(3.10)

\[
\tilde{T}'_j(x) \leq 0, \quad \text{for} \quad x \in [-1, x_j] \cup [x_{j-1}, 1],
\]

(3.11)

\[
\max \left\{ |T''_j(x)|, |\tilde{T}''_j(x)| \right\} \leq C(\alpha, \beta)|I_j|^{-1} \delta_n(x) \psi_j^\beta(x)
\]

(3.12)

\[
\max \left\{ |\chi_j(x) - T_j(x)|, |\chi_j(x) - \tilde{T}_j(x)| \right\} \leq C(\alpha, \beta) \delta_n(x) \psi_j^\beta(x)
\]

(3.13)

\[
\max \left\{ |\Phi_j(x) - T_j(x)|, |\Phi_j(x) - \tilde{T}_j(x)| \right\} \leq C(\alpha, \beta)|I_j| \delta_n(x) \psi_j^{\beta - 1}(x)
\]

(3.14)
4 Auxiliary results on properties of piecewise polynomials

Lemma 4.1 ([13] Lemma 5.1). Let \(k \in \mathbb{N}, \phi \in \Phi^k, f \in C[-1,1] \) and \(S \in \Sigma_{k,n} \). If \(\omega_k(f,t) \leq \phi(t)\) and \(|f(x) - S(x)| \leq \phi(\rho_n(x)), x \in [-1,1]\), then \(b_k(S,\phi) \leq c(k)\).

Lemma 4.2 ([6] Lemma 2.1). Let \(k \geq 3, \phi \in \Phi^k \) and \(S \in \Sigma_{k,n}^{(1)} \). Then \(b_k(S,\phi) \leq c(k)\|\rho_n^2 \phi^{-1}(\rho_n)S''\|_\infty\).

The following lemma on simultaneous polynomial approximation of piecewise polynomials and their derivatives is an immediate corollary of [13] Lemma 8.1 ([with \(q = r = 2\) and \(k \geq 2\)].

Lemma 4.3 ([13] Lemma 8.1). Let \(\gamma > 0, k \in \mathbb{N}, \phi \in \Phi^k, \) and let \(n, n_1 \in \mathbb{N}\) be such that \(n_1\) is divisible by \(n\). If \(S \in \Sigma_{k,n}\), then there exists a polynomial \(D_{n_1}(\cdot, S)\) of degree \(\leq Cn\) such that

\[
(4.1) \quad |S(x) - D_{n_1}(x,S)| \leq C\delta_n(x)\phi(\rho_n(x))b_k(S,\phi).
\]

Moreover, if \(S \in C^1 \) and \(A := [x_{\mu^*}, x_{\mu^*}^*], 0 \leq \mu^* < \mu^* \leq n, \) then for all \(x \in A \setminus \{x_j\}_{j=1}^{n-1}\), we have

\[
(4.2) \quad |S''(x) - D''_{n_1}(x,S)| \leq C\delta_n^2(x)\phi(\rho_n(x))\left(b_k(S,\phi,A) + b_k(S,\phi)\times \frac{n}{n_1} \left(\frac{\rho_n(x)}{\delta_n(x)}\right)^{\gamma+1}\right).
\]

All constants \(C\) may depend only on \(k\) and \(\gamma\) and are independent of the ratio \(n_1/n\).

5 Convex polynomial approximation of piecewise polynomials with “small” derivatives

Lemma 5.1. Let \(\alpha > 0, k \in \mathbb{N}\) and \(\phi \in \Phi^k\), be given. If \(S \in \Sigma_{k,n} \cap \Delta^{(2)}\) is such that

\[
(5.1) \quad |S''(x)| \leq \frac{\phi(\rho_n(x))}{\rho_n^2(x)}, \quad x \in [x_{n-1}, x_1] \setminus \{x_j\}_{j=1}^{n-1},
\]

\[
(5.2) \quad 0 \leq S''(x_j) - S''(x_{j-1}) \leq \frac{\phi(\rho_n(x_j))}{\rho_n(x_j)}, \quad 1 \leq j \leq n - 1,
\]

and

\[
(5.3) \quad S''(x) = 0, \quad x \in [-1, x_{n-1}) \cup (x_1, 1],
\]

then there is a polynomial \(P \in \Delta^{(2)} \cap \Pi_{Cn}, C = C(k, \alpha), \) such that

\[
(5.4) \quad |S(x) - P(x)| \leq C(k, \alpha)\delta_n^\alpha(x) \phi(\rho_n(x)), \quad x \in [-1,1].
\]

Proof. Denote by \(S_1\) the piecewise linear continuous function interpolating \(S\) at the points \(x_j, 0 \leq j \leq n, \) and let \(I_j := S_1|_{I_j}\). Then \(S_1 \in \Delta^{(2)}\),

\[
(5.5) \quad S_1(x) = S(x), \quad x \in I_1 \cup I_n,
\]

and, for \(x \in I_j, 1 \leq j \leq n, \) we have by Whitney’s inequality and (2.1)

\[
|S(x) - S_1(x)| \leq c\omega_2(S, h_j; I_j) \leq ch_j^2\|S''\|_{L_\infty(I_j)} \leq c\phi(h_j),
\]

which can be rewritten as

\[
(5.6) \quad |S(x) - S_1(x)| \leq c\phi(\rho_n(x)), \quad x \in [-1,1].
\]

We now write \(S_1\) as

\[
S_1(x) = S_1(-1) + S_1'(-1)(x+1) + \sum_{j=1}^{n-1} \alpha_j \Phi_j(x), \quad \alpha_j := S_1'(x_j) - S_1'(x_{j-1}),
\]
note that, by Markov and Whitney inequalities,
\[ 0 \leq \alpha_j = \frac{\ell_j(x_j) - \ell_{j+1}(x_j)}{l_j - l_{j+1}} \leq ch_j^{-1} \|l_j - l_{j+1}\|_{L^1(I_j \cup I_{j+1})} \leq ch_j^{-1} \omega_2(S, h_j; I_j \cup I_{j+1}) \]
\[ \leq ch_j \left( \|S''\|_{L^\infty(I_j)} + \|S''\|_{L^\infty(I_{j+1})} \right) + c(S'(x_j) - S'(x_j^-)) \]
\[ \leq ch_j^{-1} \phi(h_j), \quad 1 \leq j \leq n - 1. \]

Now, if
\[ P(x) := S_1(-1) + S'_1(-1)(x + 1) + \sum_{j=1}^{n-1} \alpha_j \mathcal{F}_j(x), \]
then \( P \) is a convex polynomial of degree \( \leq Cn \) and, in view of \eqref{2.10} and \eqref{2.13}, we only need to estimate \( |S_1(x) - P(x)| \). Note that \eqref{2.13} implies, for all \( 1 \leq j \leq n \) and \( x \in [-1, 1] \),
\[ \phi(h_j) \leq \phi(\psi_j^{-1}(x)\rho_n(x)) \leq C \psi_j^{-k}(x)\phi(\rho_n(x)). \]

Hence, by Lemma 3.3 and \eqref{2.10}, we have
\[ |S_1(x) - P(x)| \leq \sum_{j=1}^{n-1} \alpha_j |\mathcal{F}_j(x) - \mathcal{F}_j(x)| \leq C \sum_{j=1}^{n-1} \phi(h_j) \delta_n^\alpha(x) \psi_j^\beta(x) \]
\[ \leq C \delta_n^\alpha(x) \phi(\rho_n(x)) \sum_{j=1}^{n-1} \psi_j^{-k}(x) \leq C \delta_n^\alpha(x) \phi(\rho_n(x)), \]
provided \( \beta \geq k + 2. \)

\[ \square \]

6 One particular polynomial with controlled second derivative

All constants \( C \) in this section may depend on \( k, \alpha \) and \( \beta \).

We start with the following auxiliary lemma.

**Lemma 6.1** (Lemma 9). Let \( A := \{j_0, \ldots, l_0 + l_0\} \) and let \( A_1, A_2 \subset A \) be such that \# \( A_1 = 2l_1 \) and \# \( A_2 = l_2 \). Then, there exist 2l constants \( a_i, i \in A_1 \), such that \( |a_i| \leq (l_0/l_1)^2 \) and
\[ \frac{1}{l_2} \sum_{j \in A_2} (x - x_j) + \frac{1}{l_1} \sum_{j \in A_1} a_j(x - x_j) = 0. \]

**Lemma 6.2.** Let \( \alpha > 0, k \in \mathbb{N}, k \geq 2, \beta > 0 \) be sufficiently large (\( \beta \geq k + 7 \) will do) and let \( \phi \in \Phi^k \) be of the form \( \phi(t) := t\psi(t), \psi \in \Phi^{k-1} \). Also, let \( E \subset [-1, 1] \) be a closed interval which is the union of \( m_E \geq 100 \) of the intervals \( I_j \), and let a set \( J \subset E \) consist of \( m_J \) intervals \( I_j \), where \( 1 \leq m_J \leq m_E/4 \). Then there exists a polynomial \( Q_n(x) = Q_n(x, E, J) \) of degree \( \leq Cn \), satisfying
\[ Q_n''(x) \geq \frac{\delta_n^\alpha(x) \phi(\rho_n(x))}{\rho_n^2(x)} \left( \frac{\rho_n(x)}{\max\{\rho_n(x), \text{dist}(x, E)\}} \right)^{\beta_1}, \quad x \in J \cup([-1, 1] \setminus E), \]
\[ Q_n''(x) \geq -\delta_n^\alpha(x) \frac{\phi(\rho_n(x))}{\rho_n^2(x)}, \quad x \in E \setminus J, \]
and
\[ |Q_n(x)| \leq C m_E^k \delta_n^\alpha(x) \rho_n(x) \phi(\rho_n(x)) \sum_{j: I_j \subset E} \frac{h_j}{(x - x_j + \rho_n(x))^2}, \quad x \in [-1, 1], \]
where \( \alpha_1 = 8\alpha, \beta_1 = 60(\alpha + \beta) + k + 1 \) and \( k_1 = k + 6 \).
Proof. As in the proof of [13] Lemma 9.1, we may assume that \( I_n \not\subseteq E \) provided that the condition \( m_J < m_E/4 \) is replaced by \( m_J \leq m_E/4 \). Also, we use the same notation that was used in [13]:

\[
\rho := \rho_n(x), \quad \delta := \delta_n(x), \quad \psi_j := \psi_j(x),
\]

\[
\mathcal{E} := \{1 \leq j \leq n \mid I_j \subset E\}, \quad \mathcal{J} := \{1 \leq j \leq n \mid I_j \subset J\},
\]

\[
J_* := \min \{j \mid j \in \mathcal{E}\}, \quad J^* := \max \{j \mid j \in \mathcal{E}\},
\]

\[
\mathcal{A} := \mathcal{J} \cup \{J_*, J^*\} \quad \text{and} \quad \mathcal{B} := \mathcal{E} \setminus \mathcal{A}.
\]

Now, let \( \tilde{E} \subset E \) be the subinterval of \( E \) such that

(i) \( \tilde{E} \) is a union of \( \lfloor m_E/3 \rfloor \) intervals \( I_j \), and

(ii) \( \tilde{E} \) is centered at 0 as much as \( E \) allows it, i.e., among all subintervals of \( E \) consisting of \( \lfloor m_E/3 \rfloor \) intervals \( I_j \), the center of \( \tilde{E} \) is closest to 0.

Then (see [13]),

\[
\text{if } I_j \subset \tilde{E} \text{ and } I_i \subset E \setminus \tilde{E}, \text{ then } |I_j| \geq |I_i|,
\]

\[
|I_j| \sim \frac{|\tilde{E}|}{m_E} \quad \text{for all } I_j \subset \tilde{E},
\]

and, with \( \tilde{\mathcal{E}} := \left\{1 \leq j \leq n \mid I_j \subset \tilde{E}\right\} \) and \( \tilde{\mathcal{B}} := \mathcal{B} \cap \tilde{\mathcal{E}} = \tilde{\mathcal{E}} \setminus \mathcal{A} \),

\[
\# \tilde{\mathcal{B}} \geq m_E/20.
\]

Note that index \( j = n \) is in none of the sets \( \mathcal{A}, \mathcal{B} \) and \( \tilde{\mathcal{B}} \).

It follows from Lemma [6.1] \((l_0 \sim m_E, l_1 \sim \lfloor \# \tilde{\mathcal{B}}/2 \rfloor \sim m_E, l_2 \sim m_J)\) that there exist constants \( \lambda_i, i \in \tilde{\mathcal{B}}, \) such that \( |\lambda_i| \leq c, i \in \tilde{\mathcal{B}}, \) and

\[
\frac{m_E}{m_J} \sum_{j \in \mathcal{A}} (x - x_j) + \sum_{j \in \tilde{\mathcal{B}}} \lambda_i (x - x_j) \equiv 0.
\]

We now let \( i_* \) be such that \( l_{i_*} \) is the largest interval in \( \tilde{E} \) and \( h_* := h_{i_*} = |l_{i_*}| \), and

\[
Q_n(x) := \kappa \frac{\phi(h_*)}{h_*} \left( \frac{m_E}{m_J} \sum_{j \in \mathcal{A}} \mathcal{J}_j(x) + \sum_{j \in \tilde{\mathcal{B}}} \lambda_j \mathcal{J}_j(x) \right),
\]

where \( \kappa \) is a sufficiently small absolute constant to be prescribed and

\[
\mathcal{J}_j := \begin{cases} \mathcal{J}_j, & \text{if } \lambda_j < 0, \\ \mathcal{J}_j, & \text{if } \lambda_j \geq 0. \end{cases}
\]

It follows from (6.4) that

\[
h_j \leq h_*, \quad j \in \mathcal{E},
\]

and so \( \rho \leq h_*, \phi(\rho)/\rho = \psi(\rho) \leq \psi(h_*) = \phi(h_*)/h_* \), for all \( x \in E \) as well as all \( x \not\in E \) such that \( h_* \geq \rho \). If \( x \not\in E \) and \( h_* < \rho \), then by (2.1), (2.3) and (2.4)

\[
\frac{\phi(h_*)}{h_*} \geq \frac{\phi(\rho)/h_*^{k-1}}{\rho^k} \geq \phi(\rho)/\rho^k \max\{h_j^{k-1}, h_j^{k-1}\} \geq \phi(\rho)/\rho^k \frac{\rho^{2k-2}}{\min\{|x - x_j|, |x - x_j'| + \rho\}^k} \geq c \phi(\rho) \frac{\rho^{2k-2}}{\max\{\rho, \text{dist}(x, E)\}^{k-1}}.
\]
Hence,

\[
\frac{\phi(h_*)}{h_*} \geq c \frac{\phi(\rho)}{\rho} \left( \frac{1}{\max\{\rho, \text{dist}(x, E)\}} \right)^{k-1}, \quad \text{for all } x \in [-1, 1].
\]

We now note that \( \lambda_j \mathcal{I}'(x) \geq 0 \) if \( j \in \mathcal{B} \) and \( x \in J \cup \{[-1, 1] \setminus E\} \) (as well as for any \( x \in I_j \cup I_{j'} \)). Hence, for these \( x \), using Lemma 3.3, 2.13, 2.8 and 6.8 we have

\[
Q''_n(x) \geq \kappa \frac{\phi(h_*)}{h_*} \frac{m_E}{m_J} \sum_{j \in A} \mathcal{I}'(x) \\
\geq C_k \delta_8 \frac{\phi(h_*)}{h_*} \frac{m_E}{m_J} \sum_{j \in A} h_j^{-1} \psi_j^{30(\alpha+\beta)} \\
\geq C_k \delta_8 \frac{\phi(h_*)}{\rho h_*} \frac{m_E}{m_J} \sum_{j \in A} \psi_j^{30(\alpha+\beta)+1} \\
\geq C_k \delta_8 \frac{\phi(h_*)}{\rho h_*} \frac{m_E}{m_J} \sum_{j \in A} \left( \frac{1}{\rho + |x - x_j|} \right)^{60(\alpha+\beta)+2} \\
\geq C_k \delta_8 \frac{\phi(h_*)}{\rho^2} \frac{m_E}{m_J} \frac{\phi(\rho)}{\rho^2} \left( \frac{1}{\max\{\rho, \text{dist}(x, E)\}} \right)^{60(\alpha+\beta)+k+1},
\]

since, for \( x \notin E, \max\{\rho, \text{dist}(x, E)\} \sim \min \{|x - x_j^*, |x - x_{j^*}|\} + \rho \), and, if \( x \in J \), then \( x \in I_j \) for some \( j \in A \), so that \( \rho/(|x - x_j| + \rho) \sim 1 \) for that \( j \).

If \( x \in E \setminus J \) and \( x \notin I_j \cup I_{j'} \), then there exists \( j_0 \in \mathcal{B} \) such that \( x \in I_{j_0} \). If \( j_0 \notin \mathcal{B} \), or if \( j_0 \notin \mathcal{B} \) and \( \lambda_{j_0} \geq 0 \), then, clearly, \( Q''_n(x) \geq 0 \). Otherwise, since \( h_* \sim h_{j_0} \) by (6.5), we have using (6.12)

\[
Q''_n(x) \geq \kappa \lambda_{j_0} \phi(h_*) h_{j_0}^{-1} \mathcal{I}'(x) \geq -C_k \phi(\rho) h_{j_0}^{-2} \delta^\alpha \psi_j^{\beta},
\]

for sufficiently small \( \kappa \).

We now estimate \(|Q_n(x)|\). Let

\[
L(x) := \kappa \frac{\phi(h_*)}{h_*} \left( \frac{m_E}{m_J} \sum_{j \in A} \Phi_j(x) + \sum_{j \in \mathcal{B}} \lambda_j \Phi_j(x) \right),
\]

It follows from (9.8) that, for any \( j \in \mathcal{E}, cm_E \leq |E|/h_j \leq m^2_E \). This implies that \( h_* \leq cm_E, j \in \mathcal{E}, \) \( \phi(h_*) \leq cm^2_E \phi(h_j), j \in \mathcal{E} \). Hence, using (3.44) as well as the estimate (see 13 pp. 1282-1283)

\[
\sum_{j \in \mathcal{E}} \phi(h_j) \psi_j^{\beta - 1} \leq C \phi(\rho) \sum_{j \in \mathcal{E}} \frac{h_{j} \rho}{(|x - x_j| + \rho)^2}
\]

which is true if \( \beta \geq k + 7 \), we have

\[
|Q_n(x) - L(x)| \\
= \kappa \frac{\phi(h_*)}{h_*} \left| \frac{m_E}{m_J} \sum_{j \in A} \Phi_j(x) - \Phi_j(x) \right| + \sum_{j \in \mathcal{B}} \lambda_j \left| \Phi_j(x) - \Phi_j(x) \right| \\
\leq C m_E \delta \frac{\phi(h_*)}{h_*} \sum_{j \in \mathcal{E}} \frac{h_j \rho}{(|x - x_j| + \rho)^2} \psi_j^{\beta - 1} \\
\leq C m^k E \delta^\alpha \phi(\rho) \sum_{j \in \mathcal{E}} \frac{h_j \rho}{(|x - x_j| + \rho)^2}.
\]

It remains to estimate \(|L(x)|\). First assume that \( x \notin E \). If \( x \leq x_{j^*} \), then \( \Phi_j(x) = 0, j \in A \cup \mathcal{B} \), and \( L(x) = 0 \). If, on the other hand, \( x > x_{j^*} \), then \( \Phi_j(x) = x - x_j, j \in A \cup \mathcal{B} \), so that (6.7) implies that \( L(x) = 0 \). Hence, in particular, \( L(x) = 0 \) for \( x \in I_1 \cup I_n \).
Suppose now that \( x \in E \setminus I_1 \) (recall that we already assumed that \( E \) does not contain \( I_n \)). Then, as above, \( h_\ast \leq c|E|/m_E \leq cpm_E \) and so \( \phi(h_\ast) \leq cm_{k,\ast}^2 \phi(\rho) \). Also, \( h_\ast \geq |E|/m_E^2 \). Hence, since \( \delta = 1 \) on \([-x_{n-1}, x_1]\),

\[
|L(x)| \leq \frac{c\phi(h_\ast)}{h_\ast} \left( \frac{m_E}{m_j} \sum_{j \in A} |x - x_j| + c \sum_{j \in B} |x - x_j| \right) \leq Cm_{k,\ast}^{k+3} \phi(\rho) |E| \sum_{j \in \mathcal{E}} |x - x_j| \leq Cm_{k,\ast}^{k+3} \phi(\rho) |E| \sum_{j \in \mathcal{E}} |x - x_j| \leq Cm_{k,\ast}^{k+4} \delta^\alpha \phi(\rho).
\]

It remains to note that

\[
1 = |E| \sum_{j \in \mathcal{E}} \frac{h_j}{|E|^2} \leq c|E| \sum_{j \in \mathcal{E}} \frac{h_j}{(|x - x_j| + \rho)^2} \leq cm_{k,\ast}^2 \sum_{j \in \mathcal{E}} \frac{\rho h_j}{(|x - x_j| + \rho)^2},
\]

and the proof is complete. \( \square \)

7 Convex polynomial approximation of piecewise polynomials

**Lemma 7.1** ([6 Lemma 4.3]). Let \( k \geq 3, \phi \in \Phi^k \) and \( S \in \Sigma_{k,n}^{(1)} \) be such that \( b_k(S, \phi) \leq 1 \). If \( 1 \leq \mu, \nu \leq n \) are such that the interval \( I_{\mu,\nu} \) contains at least \( 2k - 5 \) intervals \( I_i \) and points \( x_i^* \in (x_i, x_{i+1}) \) so that

\[
\rho_n^2(x_i^*) \phi^{-1}(\rho_n(x_i^*)) |S''(x_i^*)| \leq 1,
\]

then, for every \( 1 \leq j \leq n \), we have

\[
\|\rho_n^2 \phi^{-1}(\rho_n) S''\|_{L_\infty(I_j)} \leq c(k) \left( (j - \mu)^{4k} + (j - \nu)^{4k} \right).
\]

**Theorem 7.2.** Let \( k, r \in \mathbb{N}, r \geq 2, k \geq r + 1 \), and let \( \phi \in \Phi^k \) be of the form \( \phi(t) := t^r \psi(t), \psi \in \Phi^{k-r} \). Also, let \( d_+ \geq 0, d_- \geq 0 \) and \( \alpha \geq 0 \) be given. Then there is a number \( N = N(k, r, \phi, d_+, d_-, \alpha) \) satisfying the following assertion. If \( n \geq N \) and \( S \in \Sigma_{k,n}^{(1)} \cap \Delta^{(2)} \) is such that

\[
b_k(S, \phi) \leq 1,
\]

and, additionally,

\[
\text{if } d_+ > 0, \text{ then } d_+ |I_2|^{r-2} \leq \min_{x \in I_2} S''(x),
\]

\[
\text{if } d_+ = 0, \text{ then } S^{(i)}(1) = 0, \text{ for all } 2 \leq i \leq k - 2,
\]

\[
\text{if } d_- > 0, \text{ then } d_- |I_{n-1}|^{r-2} \leq \min_{x \in I_{n-1}} S''(x),
\]

\[
\text{if } d_- = 0, \text{ then } S^{(i)}(-1) = 0, \text{ for all } 2 \leq i \leq k - 2,
\]

then there exists a polynomial \( P \in \Delta^{(2)} \cap \Pi_{C,n}, C = C(k, \alpha) \), satisfying, for all \( x \in [-1, 1] \),

\[
|S(x) - P(x)| \leq C(k, \alpha) \delta_n^\alpha \phi(\rho_n(x)), \quad \text{if } d_+ > 0 \text{ and } d_- > 0,
\]

\[
|S(x) - P(x)| \leq C(k, \alpha) \delta_n^{\min(\alpha, 2k-2)} \phi(\rho_n(x)), \quad \text{if } \min\{d_+, d_-\} = 0.
\]

The proof of Theorem 7.2 is quite long and technical and is similar (with some rather significant changes) to that of [13 Theorem 10.2]. It is given in the last section of this paper.

8 Convex approximation by smooth piecewise polynomials

The following theorem was proved in [14].

**Theorem 8.1** ([14 Theorem 2.1]). Given \( r \in \mathbb{N} \), there is a constant \( c = c(r) \) such that if \( f \in C^r[-1, 1] \) is convex, then there is a number \( N = N(f, r) \), depending on \( f \) and \( r \), such that for \( n \geq N \), there
are convex piecewise polynomials $S$ of degree $r + 1$ with knots at the Chebyshev partition $T_n$ (i.e., $S \in \Sigma_{r+2,n} \cap \Delta^{(2)}$), satisfying

\[
|f(x) - S(x)| \leq c(r)(\varphi(x)/n)^r \omega_2 \left( f^{(r)}, \varphi(x)/n \right), \quad x \in [-1, 1],
\]

and, moreover, for $x \in [-1, -1 + n^{-2}] \cup [1 - n^{-2}, 1]$,

\[
|f(x) - S(x)| \leq c(r) \varphi^{2r}(x) \omega_2 \left( f^{(r)}, \varphi(x)/n \right)
\]

and

\[
|f(x) - S(x)| \leq c(r) \varphi^{2r}(x) \omega_1 \left( f^{(r)}, \varphi^2(x) \right).
\]

As was shown in [14], $N$ in the statement of Theorem 8.1 in general, cannot be independent of $f$.

We will now show that the following “smooth analog” of this result also holds.

**Theorem 8.2.** Given $r \in \mathbb{N}$, there is a constant $c = c(r)$ such that if $f \in C^r[-1, 1]$ is convex, then there is a number $N = N(f, r)$, depending on $f$ and $r$, such that for $n \geq N$, there are continuously differentiable convex piecewise polynomials $S$ of degree $r + 1$ with knots at the Chebyshev partition $T_n$ (i.e., $S \in \Sigma_{r+2,n} \cap \Delta^{(2)}$), satisfying (8.1), (8.2) and (8.3).

Let $\mathcal{S}_r(z_m)$ denote the space of all piecewise polynomial functions (ppf) of degree $r - 1$ (order $r$) with the knots $z_m := (z_i)_{i=0}^m$, $a := z_0 < z_1 < \cdots < z_m-1 < z_m := b$. Also, the scale of the partition $z_m$ is denoted by

\[
\vartheta(z_m) := \max_{0 \leq j \leq m-1} \frac{|J_{j+1}|}{|J_j|},
\]

where $J_j := [z_j, z_{j+1}]$.

In order to prove Theorem 8.2 we need the following lemma which is an immediate corollary of a more general result in [11].

**Lemma 8.3** (see [11] Lemma 3.8)). Let $r \in \mathbb{N}$, $z_m := (z_i)_{i=0}^m$, $a := z_0 < z_1 < \cdots < z_m-1 < z_m := b$ be a partition of $[a, b]$, and let $s \in \Delta^{(2)} \cap \mathcal{S}_{r+2}(z_m)$. Then, there exists $\bar{s} \in \Delta^{(2)} \cap \mathcal{S}_{r+2}(z_m) \cap C^1[a, b]$ such that, for any $1 \leq j \leq m - 1$,

\[
\|s - \bar{s}\|_{[z_{j-1}, z_{j+1}]} \leq c(r, \vartheta(z_m)) \omega_{r+2}(s, z_{j+2} - z_{j-2}; [z_{j-2}, z_{j+2}]),
\]

where $z_j := z_0, j < 0$ and $z_j := z_m, j > m$. Moreover,

\[
\bar{s}^{(ν)}(a) = s^{(ν)}(a) \quad \text{and} \quad \bar{s}^{(ν)}(b) = s^{(ν)}(b), \quad ν = 0, 1.
\]

**Proof of Theorem 8.2.** Let $n$ be a sufficiently large fixed number, and let $S_0 \in \Sigma_{r+2,n} \cap \Delta^{(2)}$ be a piecewise polynomial from the statement of Theorem 8.1 for which estimates (8.1), (8.2), (8.3) hold. Let $a := x_{2n-1,2n}, b := x_{1,2n}$ and let $z_m := (z_i)_{i=0}^m$ be such that $z_0 := a, z_n := b$ and $z_i := x_{n-i}, 1 \leq i \leq n - 1$ (note that $z_m \subset T_{2n}$). Clearly, $S_0 \in \mathcal{S}_{r+2}(z_m), \vartheta(z_m) \sim 1$, and Lemma 8.3 implies that there exists $\bar{S}_0 \in \Delta^{(2)} \cap \mathcal{S}_{r+2}(z_m) \cap C^1[a, b]$ such that, for any $1 \leq j \leq n$,

\[
\|S_0 - \bar{S}_0\|_{\bar{I}_j} \leq c(r) \omega_{r+2}(S_0, h_j; \bar{J}_j),
\]

where $\bar{I}_j := I_j \cap [a, b]$ and $\bar{J}_j := [x_{j+2}, x_{j-2}] \cap [a, b]$, and

\[
\bar{S}_0^{(ν)}(a) = S_0^{(ν)}(a) \quad \text{and} \quad \bar{S}_0^{(ν)}(b) = S_0^{(ν)}(b), \quad ν = 0, 1.
\]

We now define

\[
S(x) :=
\begin{cases}
S_0(x), & \text{if } x \in [-1, 1] \setminus [a, b], \\
\tilde{S}_0(x), & \text{if } x \in [a, b].
\end{cases}
\]
Clearly, $S \in \Sigma_{r+2,2n}^{(1)} \cap \Delta^{(2)}$, estimates (8.3) and (8.4) hold (with $n$ replaced by $2n$), and (8.1) also holds (clearly, it does not matter if we use $n$ or $2n$ there) since $\varphi(x)/n \sim h_j$, for any $x \in \delta_j, \ 1 \leq j \leq n$. Thus, for $x \in I_j, \ 1 \leq j \leq n$,

$$|f(x) - S(x)| \leq |f(x) - S_0(x)| + \left\| S_0 - \tilde{S}_0 \right\|_{I_j} \leq c \left\| f - S_0 \right\|_{I_j} + \omega_{r+2}(f, h_j; \delta_j)$$

$$\leq ch_j \omega_2(f^{(r)}, h_j) \leq c (\varphi(x)/n)^r \omega_2 \left( f^{(r)}, \varphi(x)/n \right).$$

\[\square\]

**Remark 8.4.** It follows from [11, Corollary 3.7] that Theorem 8.2 is, in fact, valid for splines of minimal defect, i.e., for $S \in \Sigma_{r+2,2n}^{(1)} \cap \Delta^{(2)}$. 

### 9 Proof of Theorem 1.2

**9.1 The case for $r \geq 2$**

Let $S$ be the piecewise polynomial from the statement of Theorem 8.2. Without loss of generality, we can assume that $S$ does not have knots at $x_1$ and $x_{n-1}$ (it is sufficient to treat $S$ as a piecewise polynomial with knots at the Chebyshev partition $T_{2n}$). Then,

$$l_1(x) := S(x)|_{I_1 \cup I_2} = f(1) + \frac{f'(1)}{1!} (x-1) + \cdots + \frac{f^{(r)}(1)}{r!} (x-1)^r + a_+(n; f)(x-1)^{r+1}$$

and

$$l_n(x) := S(x)|_{I_n \cup I_{n-1}} = f(-1) + \frac{f'(-1)}{1!} (x+1) + \cdots + \frac{f^{(r)}(-1)}{r!} (x+1)^r + a_-(n; f)(x+1)^{r+1},$$

where $a_+(n, f)$ and $a_-(n, f)$ are some constants that depend only on $n$ and $f$.

We will now show that

$$n^{-2} \max\{|a_+(n, f)|, |a_-(n, f)|\} \to 0 \quad \text{as} \quad n \to \infty. \quad (9.1)$$

Indeed, it follows from (8.3) that, for all $x \in I_1 \cup I_2$,

$$|a_+(n, f)(1-x)| \leq \frac{|l_1(x) - f(x)|}{(1-x)^r} + \left\| f(x) - f(1) - \frac{f'(1)}{1!} (x-1) - \cdots - \frac{f^{(r)}(1)}{r!} (x-1)^r \right\|$$

$$\leq \omega_1(f^{(r)}, 1-x) + \frac{1}{(r-1)! (1-x)^r} \left| \int_x^1 f^{(r)}(t) - f^{(r)}(1) \right| (t-x)^{r-1} dt$$

$$\leq \omega_1(f^{(r)}, 1-x),$$

and, in particular, $n^{-2} |a_+(n, f)| \leq \omega_1(f^{(r)}, n^{-2}) \to 0$ as $n \to \infty$. Analogously, one draws a similar conclusion for $|a_-(n, f)|$.

For $f \in C^r, \ r \geq 2$, let $i_+ \geq 2$, be the smallest integer $2 \leq i \leq r$, if it exists, such that $f^{(i)}(1) \neq 0$, and denote

$$D_+(r, f) := \begin{cases} (2r!)^{-1} |f^{(i)}(1)|, & \text{if } i_+ \text{ exists}, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, let $i_- \geq 2$, be the smallest integer $2 \leq i \leq r$, if it exists, such that $f^{(i)}(-1) \neq 0$, and denote

$$D_-(r, f) := \begin{cases} (2r!)^{-1} |f^{(i)}(-1)|, & \text{if } i_- \text{ exists}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, if $n$ is sufficiently large, then

$$S''(x) \geq D_+(r, f)(1-x)^{-r-2}, \quad x \in (x_2, 1], \quad (9.2)$$

and

$$S''(x) \geq D_-(r, f)(x+1)^{-r-2}, \quad x \in [-1, x_{n-2}). \quad (9.3)$$
**Proof of Theorem 1.2 in the case** \( r \geq 2 \). Given \( r \in \mathbb{N}, r \geq 2 \), and a convex \( f \in C^r \), let \( \psi \in \Phi^2 \) be such that \( \omega_2(f^{(r)}, t) \sim \psi(t) \), denote \( \phi(t) := t^r \psi(t) \), and note that \( \phi \in \Phi^{r+2} \).

For a sufficiently large \( N \in \mathbb{N} \) and each \( n \geq N \), we take the piecewise polynomial \( S \in \Sigma_{r+2,n} \) of Theorem 8.2 satisfying (9.2) and (9.3), and observe that

\[
\omega_{r+2}(f, t) \leq t^r \omega_2(f^{(r)}, t) \sim \phi(t),
\]

so that by Lemma 4.1 with \( k = r + 2 \), we conclude that

\[
b_{r+2}(S, \phi) \leq c.
\]

Now, it follows from (9.2) and (2.2) that

\[
\min_{x \in I_2} S''(x) \geq D_+(r, f)|I_2|^{-r-2} \geq 3^{-r+2}D_+(r, f)|I_2|^{-r-2}
\]

and, similarly, (9.3) yields

\[
\min_{x \in I_{n-1}} S''(x) \geq 3^{-r+2}D_-(r, f)|I_{n-1}|^{-r-2}.
\]

Hence, using Theorem 7.2 with \( k = r + 2, d_+ := 3^{-r+2}D_+(r, f), d_- := 3^{-r+2}D_-(r, f) \) and \( \alpha = 2k - 2 = 2r + 2 \), we conclude that there exists a polynomial \( P \in \Pi_{c,n} \cap \Delta^{(2)} \) such that

\[
|S(x) - P(x)| \leq c_0^2(x)\rho_n(x)\psi(\rho_n(x)), \quad x \in [-1, 1].
\]

In particular, for \( x \in I_1 \cup I_n, x \neq -1, 1 \), using the fact that \( \rho_n(x) \sim n^{-2} \) for these \( x \), and \( t^{-2}\psi(t) \) is nonincreasing we have

\[
|S(x) - P(x)| \leq c(n\varphi(x))^{2r+2}\rho_n^2(x)\psi(\rho_n(x))
\]

\[
\leq cn^2\varphi^{2r+2}(x)\left(\frac{n\rho_n(x)}{\varphi(x)}\right)^2 \psi\left(\frac{\varphi(x)}{n}\right)
\]

\[
\leq c\varphi^{2r}(x)\omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right).
\]

In turn, this implies for \( x \in I_1 \cup I_n \), that

\[
|S(x) - P(x)| \leq c\left(\frac{\varphi(x)}{n}\right)^r \omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right),
\]

which combined with (9.3) implies

\[
|S(x) - P(x)| \leq c\left(\frac{\varphi(x)}{n}\right)^r \omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right), \quad x \in [-1, 1].
\]

Now, (9.6) together with (8.1) yield (1.5), and (9.5) together with (8.2) yield (1.6). In order to prove (1.7), using the fact that \( t^{-1}\omega_1(f^{(r)}, t) \) is nonincreasing we have, for \( x \in I_1 \cup I_n, x \neq -1, 1 \),

\[
|S(x) - P(x)| \leq c(n\varphi(x))^{2r+2}\rho_n(x)\omega_1(f^{(r)}, \rho_n(x))
\]

\[
\leq cn^2\varphi^{2r+2}(x)\frac{\rho_n(x)}{\varphi^2(x)}\omega_1(f^{(r)}, \varphi^2(x))
\]

\[
\leq c\varphi^{2r}(x)\omega_1\left(f^{(r)}, \varphi^2(x)\right),
\]

which together with (8.3) completes the proof of Theorem 1.2.

**9.2 The case for** \( r = 1 \)

It is possible to show that, in order to prove Theorem 1.2 in the case \( r = 1 \), it is sufficient to construct a convex polynomial \( P_n \) that approximates the quadratic spline \( S \) from Theorem 8.1 (with \( r = 1 \)) so that

\[
|S(x) - P_n(x)| \leq c\varphi(f, \rho_n(x)),
\]
and

\[ P_n(\pm 1) = S(\pm 1) \quad \text{and} \quad P_n'(\pm 1) = S'(\pm 1). \]

In order to construct such a polynomial, one can use exactly the same method as in [8] (using \( S \) instead of \( f \) and \( 2n \) instead of \( n \)) with the only difference that extra factors \((1 - y^2)\) should appear inside integrals in the definitions of \( Q_j, \overline{Q}_j \) and \( T_j \) (see [8 p. 158]). All estimates are then still valid and either follow from the statements in Section 3 or are proved similarly. This change is needed in order to guarantee that (9.8) holds because addition of these factors implies that the first derivatives of these modified polynomials \( Q_j, \overline{Q}_j \) and \( T_j \) are 0 at \( \pm 1 \) which, in turn, implies that the first derivatives of other auxiliary polynomials \( \sigma_j, R_j \) and \( \overline{R}_j \) at \( \pm 1 \) are “correct”. We omit details.

10 Appendix: proof of Theorem 7.2

Throughout the proof, we fix \( \beta := k + 7 \) and \( \gamma := \beta_1 - 1 = 60(\alpha + \beta) + k \). Hence, the constants \( C_1, \ldots, C_6 \) (defined below) as well as the constants \( C \), may depend only on \( k \) and \( \alpha \). We also note that \( S \) does not have to be twice differentiable at the Chebyshev knots \( x_j \). Hence, when we write \( S''(x) \) (or \( S''(x) \), \( 1 \leq i \leq 4 \)) everywhere in this proof, we implicitly assume that \( x \neq x_j, 1 \leq j \leq n - 1 \).

Let \( C_1 := C \), where the constant \( C \) is taken from (6.1) (without loss of generality we assume that \( C_1 \leq 1 \)), and let \( C_2 := C \) with \( C \) taken from (4.2). We also fix an integer \( C_3 \) such that

\[ C_3 \geq 8k/C_1. \]

Without loss of generality, we may assume that \( n \) is divisible by \( C_3 \), and put \( n_0 := n/C_3 \).

We divide \([-1, 1]\) into \( n_0 \) intervals

\[ E_q := [x_qC_6, x_{(q-1)C_6}] = I_qC_6 \cup \cdots \cup I_{(q-1)C_6+1}, \quad 1 \leq q \leq n_0, \]

consisting of \( C_3 \) intervals \( I_q \) each (i.e., \( m_{E_q} = C_3 \), for all \( 1 \leq q \leq n_0 \)).

We write “\( j \in UC \)” (where “UC” stands for “Under Control”) if there is \( x_j^* \in (x_j, x_{j-1}) \) such that

\[ S''(x_j^*) \leq \frac{5C_2\phi(\rho(x_j^*))}{\rho^2(x_j^*)}. \]

(10.10)

We say that \( q \in G \) (for “Good”), if the interval \( E_q \) contains at least \( 2k - 5 \) intervals \( I_j \) with \( j \in UC \). Then, (10.10) and Lemma 7.1 imply that,

\[ S''(x) \leq \frac{C\phi(\rho)}{\rho^2}, \quad x \in E_q, \quad q \in G. \]

Set

\[ E := \cup_{q \notin G} E_q, \]

and decompose \( S \) into a “small” part and a “big” one, by setting

\[ s_1(x) := \begin{cases} 
S''(x), & \text{if } x \notin E, \\
0, & \text{otherwise},
\end{cases} \]

and

\[ s_2(x) := S''(x) - s_1(x) = \begin{cases} 
0, & \text{if } x \notin E, \\
S''(x), & \text{otherwise},
\end{cases} \]

and putting

\[ S_1(x) := S(-1) + (x+1)S'(-1) + \int_{-1}^x (x-u)s_1(u)du \quad \text{and} \quad S_2(x) := \int_{-1}^x (x-u)s_2(u)du. \]

(Note that \( s_1 \) and \( s_2 \) are well defined for \( x \neq x_j, 1 \leq j \leq n - 1 \), so that \( S_1 \) and \( S_2 \) are well defined everywhere and possess second derivatives for \( x \neq x_j, 1 \leq j \leq n - 1 \).)
Evidently, 

$$S_1, S_2 \in \Sigma_{k,n},$$

and 

$$S_1''(x) \geq 0 \quad \text{and} \quad S_2''(x) \geq 0, \quad x \in [-1,1].$$

Now, (10.11) implies that 

$$S_1''(x) \leq \frac{C\phi(\rho)}{\rho^2}, \quad x \in [-1,1],$$

which, in turn, yields by Lemma 4.2 

$$b_k(S_1, \phi) \leq C.$$ 

Together with (7.1), we obtain 

(10.12) 

$$b_k(S_2, \phi) \leq b_k(S_1, \phi) + b_k(S, \phi) \leq C + 1 \leq [C + 1] =: C_4.$$ 

The set $E$ is a union of disjoint intervals $F_p = [a_p, b_p]$, between any two of which, all intervals $E_q$ are with $q \in G$. We may assume that $n > C_3C_4$, and write $p \in AG$ (for “Almost Good”), if $F_p$ consists of no more than $C_4$ intervals $E_q$, that is, it consists of no more than $C_4C_4$ intervals $I_j$. Hence, by Lemma 7.1 

(10.13) 

$$S_2''(x) \leq \frac{C\phi(\rho)}{\rho^2}, \quad x \in F_p, \ p \in AG.$$

One may think of intervals $F_p, p \notin AG$, as “long” intervals where $S''$ is “large” on many subintervals $I_i$ and rarely dips down to 0. Intervals $F_p, p \in AG$, as well as all intervals $E_q$ which are not contained in any $F_p$’s (i.e., all “good” intervals $E_q$) are where $S''$ is “small” in the sense that the inequality $S''(x) \leq C\phi(\rho)/\rho^2$ is valid there.

Set 

$$F := \cup_{p \notin AG} F_p,$$

note that $E = \cup_{p \in AG} F_p \cup F$, and decompose $S$ again by setting 

$$s_4 := \begin{cases} 
S''(x), & \text{if } x \in F, \\
0, & \text{otherwise},
\end{cases}$$

and 

$$s_3(x) := S''(x) - s_4(x) = \begin{cases} 
0, & \text{if } x \in F, \\
S''(x), & \text{otherwise},
\end{cases}$$

and putting 

(10.14) 

$$S_3(x) := S(-1) + (x + 1)S'(-1) + \int_{-1}^x (x - u)s_3(u)du \quad \text{and} \quad S_4(x) := \int_{-1}^x (x - u)s_4(u)du.$$ 

Then, evidently, 

(10.15) 

$$S_3, S_4 \in \Sigma_{k,n}^{(1)}, \quad S_3 + S_4 = S,$$

and 

(10.16) 

$$S_3''(x) \geq 0 \quad \text{and} \quad S_4''(x) \geq 0, \quad x \in [-1,1].$$ 

We remark that, if $x \notin \cup_{p \notin AG} F_p$, then $s_1(x) = s_3(x)$ and $s_2(x) = s_4(x)$. If $x \in \cup_{p \in AG} F_p$, then $s_1(x) = s_4(x) = 0$ and $s_2(x) = s_3(x) = S''(x)$.

For $x \in \cup_{p \in AG} F_p$, (10.13) implies that 

$$S_3''(x) = S_4''(x) \leq \frac{C\phi(\rho)}{\rho^2}.$$
For all other $x$'s, 
\[ S''_3(x) = S''_4(x) \leq \frac{C \phi(\rho)}{\rho^2}. \]

We conclude that
\[ S''_3(x) \leq \frac{C_5 \phi(\rho)}{\rho^2}, \quad x \in [-1, 1], \tag{10.17} \]
which by virtue of Lemma 4.2 yields that $b_k(S_3, \phi) \leq C$. As above, we obtain
\[ b_k(S_4, \phi) \leq b_k(S_3, \phi) + b_k(S, \phi) \leq C + 1 \leq [C + 1] =: C_6. \tag{10.18} \]

We will approximate $S_3$ and $S_4$ by convex polynomials that achieve the required degree of pointwise approximation.

**Approximation of $S_3$:**

If $d_+ > 0$, then there exists $N^* \in \mathbb{N}$, $N^* = N^*(d_+, \psi)$, such that, for $n > N^*$,
\[ \frac{\phi(\rho)}{\rho^2} = \rho^{-2} \psi(\rho) < \frac{d_+ |I_2|^{-2}}{C_5} \leq C_5^{-1} S''(x), \quad x \in I_2, \]
where the first inequality follows since $\psi(\rho) \leq \psi(2/n) \to 0$ as $n \to \infty$, and the second inequality follows by (7.2). Hence, by (10.17), if $n > N^*$, then $s_3(x) \neq S''(x)$ for $x \in I_2$. Therefore, since $s_3(x) = S''(x)$, for all $x \notin F$, we conclude that $I_2 \subset F$, and so $E_1 \subset F$, and $s_3(x) = 0$, $x \in E_1$. In particular, $s_3(x) \equiv 0$, $x \in I_1$.

Similarly, if $d_- > 0$, then using (7.4) we conclude that there exists $N^{**} \in \mathbb{N}$, $N^{**} = N^{**}(d_-, \psi)$, such that, if $n > N^{**}$, then $s_3(x) \equiv 0$ for all $x \in I_n$.

Thus, we conclude that for $n \geq \max\{N^*, N^{**}\}$, we have
\[ s_3(x) = 0, \quad \text{for all} \ x \in I_1 \cup I_n. \tag{10.19} \]

Therefore, in view of (10.16) and (10.18), it follows by Lemma 5.1 combined with (10.17) that, in the case $d_+ > 0$ and $d_- > 0$, there exists a convex polynomial $r_n \in \Pi_{C_5}$, such that
\[ |S_3(x) - r_n(x)| \leq C \delta^\alpha \phi(\rho), \quad x \in [-1, 1]. \tag{10.20} \]

Suppose now that $d_+ = 0$ and $d_- > 0$. First, proceeding as above, we conclude that $s_3 \equiv 0$ on $I_n$. Additionally, if $E_1 \subset F$, then, as above, $s_3 \equiv 0$ on $I_1$ as well. Hence, (10.19) holds which, in turn, implies (10.20).

If $E_1 \not\subset F$, then $s_3(x) = S''(x)$, $x \in I_1$, and so it follows from (7.3) that, for some constant $A_1 \geq 0$,
\[ s_3(x) = S''(x) = A_1(1 - x)^{k-3}, \quad x \in I_1. \]

Note that $A_1$ may depend on $n$, but by (7.16) we conclude that,
\[ A_1 \leq C_5 \frac{\phi(\rho_n(x_1))}{(1 - x_1)^{k-3} \rho^2(x_1)} \sim n^{2k-2} \phi(n^{-2}). \]

Hence, for $x \in I_1$,
\[ S''_3(x) = s_3(x) = A_2 n^{2k-2} \phi(n^{-2})(1 - x)^{k-3}, \tag{10.21} \]
where $A_2$ is a nonnegative constant that may depend on $n$ but $A_2 \leq C$.

We now construct $\tilde{S}_3 \in \Sigma_{k,2n}$ which satisfies all conditions of Lemma 5.1 (with $2n$ instead of $n$). Note that $x_j := x_{j,n} = x_{2j,2n}$, denote $\xi := x_{1,2n}$ and define
\[ \tilde{S}_3(x) := \begin{cases} 
S_3(x), & \text{if } x < x_1, \\
S_3(1) + (x - 1)S_3'(1) =: L(x), & \text{if } \xi < x \leq 1, \\
\ell(x), & \text{if } x \in [x_1, \xi]. 
\end{cases} \]
where \( \ell(x) \) is the linear polynomial chosen so that \( \bar{S}_3 \) is continuous on \([-1, 1] \), i.e., \( \ell(x_1) = S_3(x_1) \) and \( \ell(\xi) = S_3(1) + (\xi - 1)S_3'(1) = L(\xi) \). Clearly, \( \bar{S}_3 \in C[-1, 1] \) (and, in fact, is in \( C^1[-1, 1] \)), \( \bar{S}_3''(x) \leq C\rho^{-2}\phi(\rho), x \notin \{x_j\}_{j=1}^{n-1} \cup \{\xi\} \), and \( \bar{S}_3'' \equiv 0 \) on \( I_{1,2n} \cup I_n \). Note that \( \bar{S}_3 '' \) may be discontinuous at \( x_1 \) and \( \xi \), but, evidently, the slope of \( L \) is no less than the slope of \( \ell \), so that \( \bar{S}_3 \) is convex in \([x_1, 1]\).

Denote
\[
S_L := S_3 - L, \quad \bar{S}_L := \bar{S}_3 - L \quad \text{and} \quad \ell_L := \ell - L,
\]
and note that,
\[\bar{S}_L(x) \equiv 0, \quad x \in [\xi, 1], \quad \text{and} \quad \bar{S}_L(x) = \ell_L(x), \quad x \in [x_1, \xi].\]
Also, \( \ell_L(x_1) = S_L(x_1) \) and \( \ell_L(\xi) = \bar{S}_L(\xi) = 0 \), and in view of (10.21),
\[
0 \leq S_L(x) = S_3(x) - S_3(1) - (x - 1)S_3'(1) = \int_x^1 (u - x)S_3(u)du = A_3n^{2k-2}\phi(n^{-2}) (1 - x)^{k-1}, \quad x \in I_1,
\]
where \( A_3 \leq C \). Now, the tangent line to \( S_L \) at \( x = x_1 \) is
\[
y(x) = S_L(x_1) + (x - x_1)S_L'(x_1) = A_3n^{2k-2}\phi(n^{-2})(1 - x_1)^{k-2} (1 - x_1 - (k - 1)(x - x_1)),
\]
which intersects the \( x \) axis at
\[x_1 + \frac{1 - x_1}{k - 1} \leq \frac{1 + x_1}{2} < x_{1,2n} = \xi.
\]
Hence, the slope of \( \ell_L \) is no less than the slope of that tangent and, in turn, we conclude that the slope of \( \ell \) is no less than \( S_3'(x_1) \), so that \( \bar{S}_3 \) is convex in \([-1, 1]\).

Further, we have,
\[
|S_3(x) - \bar{S}_3(x)| = |S_L(x) - \bar{S}_L(x)| \leq S_L(x) + \bar{S}_L(x) \leq S_L(x_1) + \bar{S}_L(x_1) = 2S_L(x_1) \leq C\phi(n^{-2}), \quad x \in [x_1, \xi],
\]
and
\[
|\bar{S}_3(x) - S_3(x)| = |L(x) - S_3(x)| = S_L(x) = A_3\phi(n^{-2})n^{2k-2}(1 - x)^{k-1} \leq C\delta^{2k-2}\phi(n^{-2}), \quad x \in [\xi, 1].
\]
Note that \( \bar{S}_3' \) may have (nonnegative) jumps at \( x_1 \) and \( \xi \). However,
\[
\bar{S}_3'(\xi+) - \bar{S}_3'(\xi-) + \bar{S}_3'(x_1+) - \bar{S}_3'(x_1-) = S_3'(1) - S_3'(x_1) = -S_3'(x_1) \leq Cn^2\phi(n^{-2}),
\]
so that Lemma 5.1 implies that there exists a convex polynomial \( r_n \in \Pi_{CN} \) such that,
\[
|\bar{S}_3(x) - r_n(x)| \leq C\delta^n\phi(\rho), \quad x \in [-1, 1].
\]
Observing that \( \bar{S}_3 \equiv S_3 \) on \([-1, x_1] \), and combining with (10.23) and (10.24) (recalling that \( n^{-2} \leq \rho \), we conclude that
\[
|\bar{S}_3(x) - S_3(x)| \leq C\delta^{2k-2}\phi(\rho), \quad x \in [-1, 1],
\]
so that
\[
|S_3(x) - r_n(x)| \leq C\delta^{\min\{n,2k-2\}}\phi(\rho), \quad x \in [-1, 1].
\]
Finally, if \( d_- = 0 \) and \( d_+ > 0 \), then the considerations are completely analogous and, if \( d_- = 0 \) and \( d_+ = 0 \), then \( \bar{S}_3 \) can be modified further on \( I_n \) using (10.24) and the above argument.

Hence, we’ve constructed a convex polynomial \( r_n \in \Pi_{CN} \) such that, in the case when both \( d_+ \) and \( d_- \) are strictly positive, (10.20) holds, and (10.26) is valid if at least one of these numbers is 0.

**Approximation of \( S_3 \):**

Given a set \( A \subset [-1, 1] \), denote
\[
A^c := \bigcup_{I_j \cap A \neq \emptyset} I_j \quad \text{and} \quad A^{2c} := (A^c)^c,
\]

where $I_0 = \emptyset$ and $I_{n+1} = \emptyset$. For example, $[x_7, x_3]^c = [x_8, x_2]$, $I_1^c = I_1 \cup I_2$, etc.

Also, given subinterval $I \subset [-1, 1]$ with its endpoints at the Chebyshev knots, we refer to the rightmost and the left-most intervals $I_i$ contained in $I$ as $EP_+(I)$ and $EP_-(I)$, respectively (for the “End Point” intervals). More precisely, if $1 \leq \mu < \nu \leq n$ and

$$I = \bigcup_{i=\mu}^{\nu} I_i,$$

then $EP_+(I) := I_{\mu}$, $EP_-(I) := I_{\nu}$ and $EP(I) := EP_+(I) \cup EP_-(I) = I_{\mu} \cup I_{\nu}$. For example, $EP_+[-1, 1] := I_1$, $EP_-[-1, 1] := I_n$, $EP_+[x_7, x_3] = [x_4, x_3] = I_4$, $EP_-[x_7, x_3] = [x_7, x_6] = I_7$, $EP[x_7, x_3] = I_4 \cup I_7$, etc. Here, we simplified the notation by using $EP_+([a, b]) := EP_+([a, b])$ and $EP([a, b]) := EP([a, b])$.

In order to approximate $S_4$, we observe that for $p \notin AG$,

$$S_4''(x) = S_4''(x), \quad x \in F_{p}^{2e},$$

so that by virtue of (10.12), we conclude that

$$(10.27) \quad b_k(S_4, \phi, F_{p}^{2e}) = b_k(S_2, \phi, F_{p}^{2e}) \leq b_k(S_2, \phi) \leq C_4.$$  

(Note that, for $p \in AG$, $S_4$ is linear in $F_{p}^{2e}$ and so $b_k(S_4, \phi, F_{p}^{2e}) = 0$.)

We will approximate $S_4$ using the polynomial $D_{n_1}(\cdot, S_4) \in \Pi_{Cn_1}$ defined in Lemma 4.3 (with $n_1 := C_6n$), and then we construct two “correcting” polynomials $Q_n, M_n \in \Pi_{Cn}$ (using Lemma 6.2) in order to make sure that the resulting approximating polynomial is convex.

We begin with $Q_n$. For each $q$ for which $E_q \subset F$, let $J_q$ be the union of all intervals $I_j \subset E_q$ with $j \in UC$; then we construct two “correcting” polynomials $\overline{Q}_n, \overline{M}_n \in \Pi_{Cn}$ (using Lemma 6.2) in order to make sure that the resulting approximating polynomial is convex.

We begin with $\overline{Q}_n$. For each $q$ for which $E_q \subset F$, let $J_q$ be the union of all intervals $I_j \subset E_q$ at the endpoints of $E_q$. In other words,

$$J_q := \bigcup_j \{I_j \mid j \in UC \quad and \quad I_j \subset E_q \} \cup EP(E_q).$$

Since $E_q \subset F$, then $q \notin G$ and so the number of intervals $I_j \subset E_q$ with $j \in UC$ is at most $2k - 6$. Hence, by (10.9),

$$m_{J_q} \leq 2k - 4 < 2k \leq \frac{C_1C_3}{4} \leq \frac{C_3}{4},$$

Recalling that the total number $m_{E_q}$ of intervals $I_j$ in $E_q$ is $C_3$ we conclude that Lemma 6.2 can be used with $E := E_q$ and $J := J_q$. Thus, set

$$\overline{Q}_n := \sum_{q : E_q \subset F} \overline{Q}_n(\cdot, E_q, J_q),$$

where $Q_n$ are polynomials from Lemma 6.2 and denote

$$J := \bigcup_{q : E_q \subset F} J_q.$$

Then (6.1) through (6.3) imply that $\overline{Q}_n$ satisfies

$$(10.28) \quad \begin{align*}
(a) & \quad \overline{Q}_n''(x) \geq 0, \quad x \in [-1, 1] \setminus F, \\
(b) & \quad \overline{Q}_n''(x) \geq -\frac{\phi(\rho)}{\rho^2} \quad x \in F \setminus J, \\
(c) & \quad \overline{Q}_n''(x) \geq \frac{4\phi(\rho)}{\rho^2} g_{8\alpha}, \quad x \in J.
\end{align*}$$

Note that the inequalities in (10.28) are valid since, for any given $x$, all relevant $Q_n''(x; E_q, J_q)$, except perhaps one, are nonnegative, and

$$C_1 \frac{m_{E_q}}{m_{J_q}} \geq \frac{C_1C_3}{2k} \geq 4.$$
Also, it follows from (6.3) that, for any $x \in [-1, 1],$

\begin{equation}
|Q_n(x)| \leq C \delta \alpha \phi(\rho) \sum_{q \in F \cap J_i \subset E_q} \frac{h_j}{(|x - x_j| + \rho)^2}
\end{equation}

\begin{align*}
&\leq C \delta \alpha \phi(\rho) \sum_{j=1}^{n} \frac{h_j}{(|x - x_j| + \rho)^2} \\
&\leq C \delta \alpha \phi(\rho) \int_{0}^{\infty} \frac{du}{(u + \rho)^2} \\
&= C \delta \alpha \phi(\rho).
\end{align*}

Next, we define the polynomial $M_n$. For each $F \notin AG$, let $J_p^-$ denote the union of the two intervals on the left side of $F_p^e$ (or just the interval $I_1$ if $-1 \in F_p$), and let $J_p^+$ denote the union of the two intervals on the right side of $F_p^e$ (or just one interval $I_1$ if $1 \in F_p$), i.e.,

\begin{equation}
J_p^- = EP_-(F_p^e) \cup EP_-(F_p) \quad \text{and} \quad J_p^+ = EP_+(F_p^e) \cup EP_+(F_p).
\end{equation}

Also, let $F_p^-$ and $F_p^+$ be the closed intervals each consisting of $m_{F_p^e} := C_3 C_4$ intervals $I_j$ and such that $J_p^- \subset F_p^- \subset F_p^e$ and $J_p^+ \subset F_p^+ \subset F_p^e$, and put

\begin{equation}
J_p^* := J_p^- \cup J_p^+ \quad \text{and} \quad J^* := \bigcup_{p \notin AG} J_p^*.
\end{equation}

Now, we set

\begin{equation}
M_n := \sum_{p \notin AG} (Q_n(\cdot, F_p^+, J_p^+) + Q_n(\cdot, F_p^-, J_p^-)).
\end{equation}

Since $m_{F_p^+} = m_{F_p^-} = C_3 C_4$ and $m_{J_p^+}, m_{J_p^-} \leq 2$, it follows from (10.30) that

\begin{equation}
C_1 \min \left\{ \frac{m_{F_p^+}}{m_{J_p^+}}, \frac{m_{F_p^-}}{m_{J_p^-}} \right\} \geq C_1 C_3 C_4 \geq 2C_4.
\end{equation}

Then Lemma 6.2 implies

\begin{equation}
|M_n(x)| \leq C \delta \alpha \phi(\rho)
\end{equation}

(this follows from (6.3) using the same sequence of inequalities that was used to prove (10.29) above), and

\begin{align*}
(a) & \quad M_n''(x) \geq -2 \frac{\phi(\rho)}{\rho^2}, \quad x \in F \setminus J^*, \\
(b) & \quad M_n''(x) \geq 2C_4 \delta \alpha \phi(\rho) \frac{\phi(\rho)}{\rho^2}, \quad x \in J^*, \\
(c) & \quad M_n''(x) \geq 2C_4 \delta \alpha \phi(\rho) \frac{\phi(\rho)}{\rho^2} \left( \frac{\rho}{\text{dist}(x, F)} \right)^{\gamma + 1}, \quad x \in [-1, 1] \setminus F^e,
\end{align*}

where in the last inequality we used the fact that

\[ \max \{\rho, \text{dist}(x, F^e)\} \leq \text{dist}(x, F), \quad x \in [-1, 1] \setminus F^e, \]

which follows from (2.5).

The third auxiliary polynomial is $D_{n1} := D_{n1}(\cdot, S_4)$ with $n_1 = C_6 n$ from Lemma 4.3. By (10.48), (4.1) yields

\begin{equation}
|S_4(x) - D_{n1}(x)| \leq C \delta \gamma \phi(\rho) \leq C \delta \alpha \phi(\rho), \quad x \in [-1, 1],
\end{equation}

since $\gamma > \alpha$, and (4.2) implies that, for any interval $A \subset [-1, 1]$ having Chebyshev knots as endpoints,

\begin{equation}
|S_4''(x) - D_{n1}''(x)| \leq C_2 \delta \gamma \phi(\rho) \frac{\phi(\rho)}{\rho^2} b_h(S_4, \phi, A)
\end{equation}

\begin{equation}
+ C_2 C_6 \delta \gamma \phi(\rho) \frac{n}{\rho} \left( \frac{\rho}{\text{dist}(x, [-1, 1] \setminus A)} \right)^{\gamma + 1}, \quad x \in A.
\end{equation}
We now define

\[(10.34) \quad R_n := D_{n_1} + C_2 \Omega_n + C_2 M_n.\]

By virtue of (10.29), (10.30), and (10.32) we obtain

\[|S_4(x) - R_n(x)| \leq C \delta^n \phi(\rho), \quad x \in [-1, 1],\]

which combined with (10.28)(b), (10.31)(a) and (10.35), proves (7.6) and (7.7) for \(P := R_n + r_n\).

Hence, by virtue of (10.28)(b), (10.31)(a) and (10.35), we get

\[|S_4(x) - R_n(x)| \leq C \delta^n \phi(\rho), \quad x \in [-1, 1],\]

Thus, in order to conclude the proof of Theorem 7.2 we should prove that \(P\) is convex. We recall that \(r_n\) is convex, so it is sufficient to show that \(R_n\) is convex as well.

Note that (10.34) implies

\[R_n''(x) \geq C_2 \Omega_n''(x) + C_2 M_n''(x) - |S_4''(x) - D_{n_1}''(x)| + S_4''(x), \quad x \in [-1, 1],\]

(this inequality is extensively used in the three cases below), and that (10.33) holds for any interval \(A\) with Chebyshev knots as the endpoints, and so we can use different intervals \(A\) for different points \(x \in [-1, 1]\). We consider three cases depending on whether (i) \(x \in F \setminus J^*\), or (ii) \(x \in J^*\), or (iii) \(x \in [-1, 1] \setminus F^*\).

**Case (i):** If \(x \in F \setminus J^*\), then, for some \(p \notin AG\), \(x \in F_p \setminus J_p^*\), and so we take \(A := F_p\). Then, the quotient inside the parentheses in (10.33) is bounded above by 1 (this follows from (2.5)). Also, since \(s_4(x) = S''(x), \quad x \in F\), it follows that \(b_k(S_4, \phi, F_p) = b_k(S, \phi, F_p) \leq 1\). Hence,

\[(10.35) \quad |S_4''(x) - D_{n_1}''(x)| \leq C_2 \frac{\phi(\rho)}{\rho^2} b_k(S_4, \phi, F_p) + C_2 C_6 \frac{\phi(\rho)}{\rho^2} \frac{n}{n_1},\]

\[\leq 2C_2 \frac{\phi(\rho)}{\rho^2}, \quad x \in F \setminus J^*\).

Note that \(x \notin I_1 \cup I_n\) (since \(F \setminus J^*\) does not contain any intervals in \(EP(F_p), \quad p \notin AG\)), and so \(\delta = 1\).

It now follows by (10.28)(c), (10.31)(a), (10.35) and (10.16), that

\[R_n''(x) \geq C_2 \frac{\phi(\rho)}{\rho^2} (4 - 2 - 2) = 0, \quad x \in J \setminus J^*\).

If \(x \in F \setminus (J \cup J^*)\), then (10.10) is violated and so

\[S_4''(x) = S''(x) > \frac{5C_2 \phi(\rho)}{\rho^2}\]

Hence, by virtue of (10.28)(b), (10.31)(a) and (10.35), we get

\[R_n''(x) \geq C_2 \frac{\phi(\rho)}{\rho^2} (-1 - 2 - 2 + 5) = 0, \quad x \in F \setminus (J \cup J^*).\]

**Case (ii):** If \(x \in J^*\), then, \(x \in J_p^*\), for some \(p \notin AG\), and we take \(A := F_p^{2^*}\). Then, (10.27) and (10.33) imply (again, (2.5) is used to estimate the quotient inside the parentheses in (10.33)),

\[(10.36) \quad |S_4''(x) - D_{n_1}''(x)| \leq C_2 \delta^n \frac{\phi(\rho)}{\rho^2} b_k(S_4, \phi, F_p^{2^*}) + C_2 C_6 \delta^n \frac{\phi(\rho)}{\rho^2} \frac{n}{n_1},\]

\[\leq 2C_2 C_4 \delta^n \frac{\phi(\rho)}{\rho^2}, \quad x \in J^*\).

Now, we note that \(EP(F_p) \subset J\), for all \(p \notin AG\), and so \(F \cap J^* \subset J\). Hence, using (10.28)(a,c), (10.31)(b), (10.36) and (10.10), we obtain

\[R_n''(x) \geq 2C_2 C_4 \delta^n \frac{\phi(\rho)}{\rho^2} - 2C_2 C_4 \delta^n \frac{\phi(\rho)}{\rho^2} \geq 0,\]

since \(\gamma > 8\alpha\), and so \(\delta^n \leq \delta^{8\alpha}\).
Theorem 7.2: If \( x \in [-1, 1] \setminus F^e \), then we take \( A \) to be the connected component of \([-1, 1] \setminus F\) that contains \( x \). Then by (10.33),

\[
|S_n'(x) - D_{n_1}'(x)|
\]

(10.37)

\[
\leq C_2 \delta^n \frac{\phi(n)}{\rho^2} b_k(S_4, \phi, A) + C_2 C_6 \delta^n \frac{\phi(n)}{\rho^2} \frac{n}{n_1} \left( \frac{\rho}{\text{dist}(x, [-1, 1] \setminus A)} \right)^{\gamma + 1}
\]

where we used the fact that \( S_4 \) is linear in \( A \), and so \( b_k(S_4, \phi, A) = 0 \).

Now, (10.28)(a), (10.31)(c), (10.37) and (10.16) imply,

\[
R_n''(x) \geq \frac{\phi(n)}{\rho^2} \left( \frac{\rho}{\text{dist}(x, F)} \right)^{\gamma + 1} (2C_2 C_4 \delta^n - C_2 \delta^n) \geq 0,
\]

since \( C_4 \geq 1 \) and \( \gamma > 8\alpha \).

Thus, \( R_n''(x) \geq 0 \) for all \( x \in [-1, 1] \), and so we have constructed a convex polynomial \( P \), satisfying (7.7) and (7.8), for each \( n \geq N \). This completes the proof of Theorem 7.2.

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