Time Reversal of Broadband Signals in a Strongly Fluctuating MIMO Channel: Stability and Resolution

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We analyze the time reversal of a multiple-input-multiple-output (MIMO) system in a space-frequency-selective multi-path fading channel described by the stochastic Schrödinger equation with a random potential in the strong-fluctuation regime. We prove that in a broadband limit the conditions for stable super-resolution are the packing condition that the spacing among the transmitters and $M$ receivers be more than the coherence length $t_c$ and the consecutive symbols in the data-streams are separated by more than the inverse of the bandwidth $B^{-1}$ and the multiplexing condition that the number of the degrees of freedom per unit time at the transmitters ($\sim NB$) be much larger than the number of the degrees of freedom ($\sim MC$) per unit time in the ensemble of intended messages. Here $C$ is the number of symbols per unit time in the data-streams intended for each receiver. When the two conditions are met, all receivers receive simultaneously streams of statistically stable, sharply focused signals intended for them, free of fading and interference. This indicates the rough multiplexing gain of $NB$ in channel capacity, with the maximal gain per unit angular cross section given by $BL^d t_c^{-d}$ where $L$ is the distance from the transmitters to the receivers. We show that under the ideal packing condition time reversal can result in a high signal-to-interference ratio and low probability of intercept, and hence is an effective means for achieving the information capacity of multi-path channels in the presence of multiple users (receivers).

Introduction

Time reversal (TR) of waves is the process of recording the signal from a remote source and then retransmitting the signal in a time-reversed fashion to refocus on the source (see [13] and the references therein). The performance of TR depends on, among other factors, the reciprocity (or time symmetry) of the propagation channel. One of the most striking features of time reversal operation in a strongly scattering medium is super-resolution, the counterintuitive effect of scattering-enhancement of time reversal resolution [1, 11]. It highlights the great potential of time reversal in technological applications such as communications where the ability of steering and pinpointing signals is essential for realizing the information carrying capacity of a multi-path channel as well as achieving low probability of intercept [3, 20].

In order to take full advantage of the super-resolution effect in a random medium, one has to first achieve statistical stability which can be measured by the signal-to-interference ratio (SIR) and the signal-to-sidelobe ratio (SSR). Statistical stability and resolution are two closely related issues that should be analyzed side-by-side; together, they are the measure of performance of TR which depends on, but is not guaranteed by, the reciprocity (or time symmetry) of the propagation channel. It has been demonstrated experimentally that there are at least two routes to achieving statistical stability [2, 3]. One route is to use a time-reversal array (TRA) of sufficiently large aperture; the other is to use a broadband signal (even with one-element TRA of essentially zero aperture). There has been many advances in analytical understanding of the former situation (see [8] and references therein). In many interesting applications of time reversal, however, the aperture of TRA is typically small compared to the correlation length of the medium, therefore the technological potential of time reversal hinges more heavily on the second route to statistical stability. Compared to the case of large aperture the analytical understanding of the case of broadband signals in time reversal has been so far much less complete with the exception of a randomly layered medium [1].

In this paper we present the time reversal analysis for the MIMO broadband channel whose $k$-component is described by the stochastic Schrödinger equation

$$i \frac{\partial \Psi}{\partial t} + \frac{\gamma}{2} \Delta x \Psi + \frac{k}{\gamma} \chi_z \circ \Psi = 0, \quad x \in \mathbb{R}^d, \quad (1)$$

in the so called paraxial Markov approximation. Here the refractive index fluctuation $\chi_z(\cdot)$ is a $\delta$-correlated-in-$z$ stationary random field with a power spectral density $\Phi(p)$.
such that $E[\chi_z(x)\chi_z'(x')] = \delta(z-z') \int \Phi(p)e^{i p \cdot (x-x')} dp$ with $E$ standing for the ensemble average; $k$ is the (dimensionless) relative wavenumber to the center wavenumber $k_0$; the Fresnel number $\gamma = L_z/(k_0 L_z^2)$ is a dimensionless number constituting of the center wavenumber $k_0$ and the reference scales $L_z$ and $L_x$ in the longitudinal and transverse dimensions, respectively, see FIG. 1. The notation $\circ$ in eq. (1) means the Stratonovich product (v.s. Itô product). For simplicity of presentation we will assume isotropy, i.e. $\Phi(k) = \Phi(|k|), \forall k \in \mathbb{R}^d$ and smoothness of $\Phi$.

The stochastic parabolic wave equation (1) is a fundamental equation for wave propagation in a randomly inhomogeneous continuum such as underwater acoustic and electromagnetic waves in atmospheric turbulence as an approximation to the wave equation with random coefficients when backscattering and depolarization are weak. It also models the cross-phase-modulation in nonlinear optical fibers when backscattering and depolarization are weak. The main assumption is the 4-th order sub-Gaussianity property (12). The Gaussian-like behavior for 4-th order correlations is widely believed to occur in the strong-fluctuation regime, defined by $\alpha_\ast = D_2 L_s \gg 1, \sigma_2^* = D_2 L_z^3 \gg 1$. We will point out some independent evidences for this in our calculation. Here $L$ is the (longitudinal) distance between the TRA and the receivers and $D_2 = d^{-1} \int |p|^2 \Phi(p) dp$ is the angular diffusion coefficient (hence $\alpha_\ast = \sqrt{D_2 L}$ is the angular spread). In the strong-fluctuation regime (11), $\gamma^{-1} \alpha_\ast$ is the spread in the so called spatial frequency, $\sigma_\ast = \sqrt{D_2 L_z^3}$ the spatial spread and their product $\gamma^{-1} D_2 L_z^2$ the spatial-spread-bandwidth product (SSB) which, as we will show, is exactly $\gamma^{-1} \beta_c^{-1}$. By the duality principle for the strong-fluctuation regime, proved in (11), the effective aperture is $2\pi$ times the spatial spread $\sigma_\ast$ (independent of the numerical aperture of TRA and hence super-resolution), and its dual quantity $\gamma L/(k \sigma_\ast) \approx \gamma/\alpha_\ast$ (the inverse of spatial-frequency spread) is the coherence length $\ell_c$ of the forward propagation (as well as the time reversal resolution). Hence the ratio $\sigma_\ast/\ell_c$ equals the spatial-spread-bandwidth product and is roughly the number of uncorrelated sub-channels (paths) per transverse dimension in the cross section of diameter $\sigma_\ast$ at the receiver plane, which will place upper bound on

Our main theorem says that in the strong-fluctuation regime and the broadband limit (8) the MIMO-TRA system achieves stable super-resolution in the sense that both the SIR and SSR tend to infinity and that the signal received by each receiver is focused to within a circle of the coherence length $\ell_c$ when the additional multiplexing condition is also met, namely $NB \gg MT \beta_c$ where $\beta_c$ is the coherence bandwidth.
the capacity gain per unit angle of the channel (see more on this in the Conclusion).

In what follows, we first formulate the problem and develop the essential tool for analyzing TR, the one- and two-frequency mutual coherence functions, and then carry out the stability and resolution analysis for the single-input-single-output (SISO), multiple-input-single-output (MISO), single-input-multiple-output (SIMO) and the multiple-input-multiple-output (MIMO) cases. Both MISO- and SIMO-TRA systems have been demonstrated to be feasible for ocean acoustic communication [22, 19] and the MIMO-TRA system with $N > M$ has been shown to work well for ultrasound [6]. We will discuss the implications of our results on the channel capacity in the Conclusion. We have by and large neglected the effect of noise in our analysis, assuming that the TRA operates in a high signal-to-noise ratio (SNR) situation as is the case for the experiments reported in [1], [8]. The robustness of TR in the presence of noises has been well documented, see e.g. [24].

**MIMO-Time reversal**

We extend the time-reversal communication scheme [1] to the setting with multiple users. Let the $M$ receivers located at $(L, r_j), j = 1, ..., M$ first send a pilot signal $\int e^{i B g(k)} \delta(r_j - a_i) \mathrm{d}k$ to the $N$-element TRA located at $(0, a_i), i = 1, ..., N$ which then use the time-reversed version of the received signals $\int e^{i B g(k)} G_L(r_j, a_i; k) \mathrm{d}k$ to modulate streams of symbols and send them back to the receivers. Here $G_L$ is the Green function of eq. (1) and $g^2(k)$ is the power density at $k$. As shown in [1], [8], when the TRA has an infinite time-window (see the Conclusion for the case of finite time-window), the signal arriving at the receiver plane with delay $L + t$ is given by

$$S(r, t) = \sum_{l=1}^{T} \sum_{i=1}^{N} \sum_{j=1}^{M} m_j(\tau_l) \int e^{-i B g(k)} \frac{\mathrm{d}k}{B \gamma} g(k) G_L(r, a_i; k) G_L^*(r, a_i; k) \mathrm{d}k$$

(2)

where $m_j(\tau_l), l = 1, ..., T \leq \infty$ are a stream of $T$ symbols intended for the $j$-th receiver transmitted at times $\tau_1 < \tau_2 < ... < \tau_T$. We assume for simplicity that $|m_j(\tau_l)| = 1, \forall j, l$. We have chosen the time scale such that the speed of propagation is unity (thus wavenumber=frequency).

We assume that $g$ is a smooth and rapidly decaying function with effective support of size $B \gamma$. For simplicity we take $g^2(k) = \exp(-\frac{|k-1|^2}{2B^2\gamma^2})$. The broadband limit may be formulated as the double limit

$$\gamma \to 0, \quad B \to \infty, \quad \lim B \gamma = 0$$

(3)

so that in the limit $g^2(k)$ becomes narrowly focused around $k = 1$. The idea underlying the definition is to view the broadband limit as a sequence of narrow-bands with indefinitely growing center frequency and bandwidth. This is particularly well suited to the framework of parabolic approximation described by [4]. The apparent narrow-banding of [4] is deceptive: the delay-spread-bandwidth product (DSB) turns out to be $B \beta_c^{-1}$ and is doubly divergent as $B \to \infty$ (the broadband limit) and $\beta_c \to 0$ (the strong fluctuation regime). Note that since $k$ is the relative wavenumber, the product $B \gamma$ should always be uniformly bounded between zero and unity, independent of $\gamma > 0$.

In the case $d = 1$ this has the intuitive implication that the number $B \beta_c^{-1}$ of degrees of freedom at each TRA-element is less than or equal to the number $\gamma^{-1} \beta_c^{-1}$ (SSB) of uncorrelated propagation paths in the medium.

**Packing condition.** We assume that the spacing within the $N$ TRA-elements and the $M$ receivers be much larger than the coherence length $\ell_c$ and that the separation of the successive symbols be much larger than $(2B)^{-1}$. Though there is no technical limitation on $M, N, T$, it suffices to consider the case where all the $N$ TRA-elements and all the $M$ receivers are located within one circle of diameter $\sigma_*$ (implying $M, N \ll \gamma^{-d} \beta_c^{-d}$), and all the $T$-datum streams are within one interval of the delay spread $\sim \beta_c^{-1}$ (implying $T \ll B \beta_c^{-1}$) since the signals separated by much more than one spatial spread $\sigma_*$ or one delay spread $\delta_*$ are essentially uncorrelated.

For simplicity, we have assumed that all the receivers lie on the plane parallel to the TRA. When this is not the case, then the above spacing of antennas refers to the transverse separation parallel to the TRA.

**SIR/SSR.** Anticipating a singular limit we employ the coupling with smooth, compactly supported test functions. Denote the mean by $E(r, t) = \gamma^{-d} \int \theta^\alpha((x - r) / \ell_c) \mathcal{E} \delta(x, t) \mathrm{d}x$ where the coupling with the test function $\theta$ can be viewed as the averaging induced by measurement. Denote the variance by $V(r, t) = \gamma^{-2d} \int \theta^\alpha((x - r) / \ell_c) \mathcal{E}^2 \delta(x, t) \mathrm{d}x$ - $E^2(r, t)$. We have made the test function $\theta$ act on the scale of the coherence length $\ell_c$, the smallest spatial scale of interest (the speckle size) in the present context. Different choices of scale would not affect the conclusion of our analysis.

The primary object of our analysis is

$$\rho(r, t) = \frac{E^2(r_j, \tau_l)}{V(r, t)}, \quad j = 1, ..., M, l = 1, ..., T$$

(4)

which is the SIR if $r = r_j, t = \tau_l$ and the SSR if $|r - r_j| \gg \ell_c, \forall j$ (spatial sidelobes) or $|t - \tau_l| \gg B^{-1}, \forall l$ (temporal sidelobes) (as $V(r, \tau) \approx E^2(r, \tau)$ as we will see below).

In the special case of $r = r_j$ and $|t - \tau_l| \gg B^{-1}, \forall l,
\( \rho^{-1} \) is a measure of intersymbol interference. To show stability and resolution, we shall find the precise conditions under which \( \rho \to \infty \) and \( \mathbb{E}S(r, t) \) is asymptotically

\[
S_{\beta}(r, t) \approx \sum_{i=1}^{N} \int e^{-\frac{k(t-\tau_i)}{2}} g(k) \mathbb{E}\left[ G_L(r, a; k)G_*^L(r_j, a; k) \right] dk
\]

is a sum of \( \delta \)-like functions around \( r_j \) and \( \tau_i = 0, \forall l \). In other words, we employ the TRA as a multiplexer to transmit the \( M \) scrambled data-streams to the receivers and we hope to turn the medium into a demultiplexer by employing the broadband time reversal technique.

**Mutual coherence functions**

A quantity repeatedly appearing in the subsequent analysis is the mutual coherence function \( \Gamma \) between the Green functions at two different wavenumbers \( k_1 = k - \gamma \beta/2, k_2 = k + \gamma \beta/2 \)

\[
\Gamma_\gamma(x + r, y - r; k, \beta) = \mathbb{E}\left[ G_z(x, a; k - \gamma \beta/2) \times G_z^*(r, a'; k + \gamma \beta/2) \right].
\]

We shall omit writing \( k, \beta, a, a' \) when no confusion arises. Here we have chosen \( x, r \) to be the pair of variables of concern and left out \( a, a' \) as parameters. By the reciprocity of the Green function, we can choose one variable from \( \{x, a\} \) and the other from \( \{r, a'\} \) as the variables of \( \Gamma \), and leave the others as parameters.

**One-frequency version.** When \( \beta = 0, \) \( \Gamma \) satisfies

\[
\frac{\partial}{\partial z} \Gamma_\gamma + \frac{i}{k} \nabla_x \cdot \nabla_y \Gamma_\gamma + \frac{k^2}{2\gamma^2} D(\gamma y) \Gamma_\gamma = 0
\]

where the structure function of the medium fluctuation \( D(x) \) is given by \( D(x) = \int \mathcal{F}(k) \left[ 1 - e^{ik\cdot x} \right] dk \geq 0, \forall x \in \mathbb{R}^d \). Eq. \( \Box \) is exactly solvable by the Fourier transform in \( x \). For \( \alpha_s \ll \gamma^{-1} \) and \( \alpha_s \ll \gamma^{-1} \) we can use the approximation

\[
k^2 \gamma^{-2} \int_0^L D(\gamma y - p z \gamma/k) dz \approx \int_0^L D_2 ky - p z|^2 dz
\]

to obtain

\[
\Gamma_L(x, y; k, 0) \approx \int e^{ip \cdot x} \Gamma_0(p, y - \frac{Lp}{k}; k, 0) e^{-\int_0^L D_2 ky - p z|^2 dz} dp
\]

where \( \tilde{y} = y \alpha_s k \) and \( \tilde{p} = p \sigma_s \). It is clear from \( \Box \) that \( \Gamma_L \) has a Gaussian-tail in \( y \) (the difference coordinates) and, by rescaling, an effective support \( \sim \alpha_s^{-1} \) and hence \( \ell_c = \gamma/\alpha_s \) (recall that \( y \) is the coordinate on the scale \( \gamma^{-1} \)).

**Two-frequency version.** The two-frequency mutual coherence function is not exactly solvable except for some special cases. Fortunately the asymptotic for \( \gamma \ll 1 \) has a universal form and can be calculated exactly. Without loss of generality we assume \( \beta > 0 \) in what follows.

Using the so called two-frequency Wigner distributions we have proved in \( \Box \) that in the limit \( \gamma \to 0, \) \( \Gamma \) satisfies the equation

\[
\frac{\partial \Gamma_z}{\partial z} - \frac{i}{k} \nabla_y \cdot \nabla_x \Gamma_z = -D_2 \left| k y + \frac{\beta}{2} \right|^2 \Gamma_z - \frac{\beta^2}{2} D_0 \Gamma_8(8)
\]

where \( D_0 = \int \mathcal{F}(k) dk \). The key to understanding eq. \( \Box \) is the rescaling:

\[
\tilde{x} = \frac{x}{\alpha_s}, \quad \tilde{y} = y \alpha_s, \quad \tilde{z} = z/L, \quad \tilde{\beta} = \beta/\beta_c
\]

with \( \beta_c = D_2^{-1} L^{-2} \) which transforms eq. \( \Box \) into the form

\[
\frac{\partial \Gamma_z}{\partial z} - i \nabla_y \cdot \nabla_x \Gamma_z = -\left| \tilde{y} + \frac{\tilde{\beta}}{2} \right|^2 \Gamma_z - \frac{\beta^2}{2} D_0 \Gamma_z(10)
\]

By another change of variables \( z_1 = \tilde{y} + \beta \tilde{x}/2, z_2 = \tilde{y} - \beta \tilde{x}/2 \) eq. \( \Box \) is then transformed into that of the quantum harmonic oscillator and solved exactly. The solution is given by

\[
\Gamma_L(x, y; k, \beta) = \frac{(2\pi)^{d/2} (1 + i)^{d/2} \beta^{d/4}}{\sin d/2 (\beta^{1/2} (1 + i))} e^{-\frac{\beta^2 D_0}{2\sigma_s^2}} \int d\xi d\nu e^{i(\xi^2 - \beta \nu^2)/2} |\tilde{\beta} \nu^2 \tan(\sqrt{\beta}(1+i))|^{1/2} \Gamma_0(\sigma_s \xi, \nu^2/\beta)\)

Several remarks are in order: (i) The Green function for \( \Gamma_L \) is of the Gaussian form in \( x, y, \) consistent with the (sub-)Gaussianity assumption; (ii) In the vanishing fluctuation limit \( D_0, \beta \to 0 \) the free-space two-frequency mutual coherence function is recovered; (iii) The apparent singular nature of the limit \( \beta \to 0 \) in \( \Box \) is deceptive. Indeed, the small \( \beta \) limit is regular and yields the result obtained from eq. \( \Box \) with \( \beta = 0 \); (iv) In the strong-fluctuation regime, \( D_0 \) is typically much smaller than \( D_2^2 L^3 \gg 1 \) so the factor \( \exp(-\beta^2 D_0/(2\sigma_s^2)) \) is negligible in the strong-fluctuation regime. On the other hand, the rapidly decaying factor \( \sin d/2 (\beta^{1/2} (1 + i)) \) is crucial for the stability argument below; (v) \( \Gamma_L(x, y; k, \beta) \) is slowly varying in \( x \) on the scale \( \alpha_s \) for \( \beta \sim \beta_c \) and more rapidly varying in \( x \) for \( \beta \gg \beta_c \).
Fourth order sub-Gaussianity. The strong-fluctuation regime $\alpha_s \gg 1, \sigma_s \gg 1$ can result from either long distance propagation and/or large medium fluctuation. It is widely accepted that, in this regime, the statistics of the wave fields (for at least lower moments) become Gaussian-like resulting in, for $d = 2$, an exponential PDF for the intensity $16, 29, 28, 15, 26$. The Gaussian statistics follows heuristically from Central-Limit-Theorem as the number of uncorrelated sub-channels (paths) per transverse dimension in the cross section of diameter $\sigma_s$ increases linearly with the spatial-spread-bandwidth product, as explained in the Introduction. This is consistent with the experimental finding of the saturation of intensity fluctuation with the scintillation index approaching unity $17$.

In what follows we shall make the 4-th order sub-Gaussianity hypothesis, namely that the fourth moments of the Green function at different frequencies $\{G_L(k)\}$ can be estimated by those of the Gaussian process of the same covariance. More specifically, we assume that

$$\begin{align*}
|\mathbb{E}[G_L(k_1) \otimes G^*_L(k_1) \otimes G_L(k_2) \otimes G^*_L(k_2)]| \\
- \mathbb{E}[G_L(k_1) \otimes G^*_L(k_1)] \otimes \mathbb{E}[G_L(k_2) \otimes G^*_L(k_2)] | \\
\leq K|\mathbb{E}[G_L(k_1) \otimes G^*_L(k_2)] \otimes \mathbb{E}[G^*_L(k_1) \otimes G_L(k_2)] | \\
+ K|\mathbb{E}[G^*_L(k_1) \otimes G^*_L(k_2)] \otimes \mathbb{E}[G^*_L(k_1) \otimes G_L(k_2)] |
\end{align*}$$

(12)

for some constant $K$ independent of $\gamma \to 0, |k_1 - 1| = O(B\gamma), |k_2 - 1| = O(B\gamma)$ and all the variables. For a jointly Gaussian process, the constant $K = 1$. Note that, in view of the scaling in the two-frequency mutual coherence the first term on the RHS of (12) is much smaller than the second term due to difference in wavenumber for $G_L(k) = G^*_L(-k)$.

The sub-Gaussianity assumption will be used to estimate the 4-th order correlations of Green functions appearing in the calculation for $V$ by the two-frequency mutual coherence function in the strong-fluctuation regime.

From SISO to MIMO

Our first application of the mutual coherence functions is the estimate for the delay spread. Consider the band-limited impulse response $u(x, t) = \int g(k)e^{-i(kx-Lt)} G_L(x, 0; k)dk$. It follows easily using the preceding results that the mean delay is $L$ and the asymptotic for the delay spread $\delta_s$, when $B \gg \beta_c$, is given by

$$\begin{align*}
\delta_s &= \sqrt{\int (t-L)^2 \mathbb{E}[|u(x,t)|^2] dt / \int \mathbb{E}[|u(x,t)|^2] dt} \\
&\approx \sqrt{-\frac{d^2}{d\beta^2}}\bigg|_{\beta=0} \Gamma_L(x,0;1,\beta)/\Gamma_L(x,0;1,0) \sim \beta_c^{-1}
\end{align*}$$

which is slowly varying in $x$ on the scale $\sigma_s$. As commented before it suffices to consider the case with a finite $T$ such that $|\gamma_1 - \tau_T| \sim \beta_c^{-1}$, implying the number of symbols in each data-stream $T \ll B\beta_c^{-1}$, the DSBI. In what follows, due to $\beta_c \ll 1$ the temporal component of the signals is essentially decoupled from the spatial component and determined by the power distribution $g^2$.

SISO. This case corresponds to $N = 1, M = 1$. Let $a_0 = 0$. In the calculation of $E(x,t)$, the expression

$$\langle \theta, \Gamma_L \rangle (r) = \int \theta^* \left( \frac{r_1 - r}{\epsilon_c} + \frac{y}{\epsilon_c} \right) \Gamma_L(r_1 + \frac{y}{2}, y; k, 0) dy$$

arises and involves only the one-frequency mutual coherence. Using $7$ with $\Gamma_0(x,y) = \delta(x + \frac{2y}{\gamma})\delta(x - \frac{2y}{\gamma})$ and making the necessary rescaling of variables we obtain the following asymptotic

$$\langle \theta, \Gamma_L \rangle (r) \approx C_0(r, r_1) \beta_c^d$$

(13)

$$C_0 = \int dp \theta^* (p + \frac{r_1 - r}{\epsilon_c}) e^{-|p|^2/\gamma} e^{-|p|^2/3}. \quad (14)$$

To derive (14) we have used the defining conditions of the strong-fluctuation regime. Note that the transfer function in (14) is Gaussian in $p$ and that $C_0(r, r_1)$ has a Gaussian-tail in $|r - r_1|/\epsilon_c$ and $C_0(r_1, r_1)$ is bounded away from zero and slowly varying in $r_1$ on the scale $\sigma_s$. That is, after proper normalization $C_0(r, r_1)$ behaves like a $\delta$-function centered at $r_1$. By (13) $C_0$ we obtain the mean field asymptotic $E(x,t) \approx 0$ for $|r - r_1| \gg \epsilon_c$ (spatial sidelobes) or $|t - \tau_1| \gg B^{-1}$, $\forall l$ (temporal sidelobes) and $E(r_1, \tau_1) \approx \sqrt{4\pi} C_0(r_1, r_1) \beta_c^d B^{-1}$.

The calculation for $V$ involves the four-point correlation of the Green functions at different frequencies. Under the sub-Gaussianity condition $12$ the calculation reduces to that of two-frequency mutual coherence functions.

Using $11$ with $\Gamma_0(x,y) = \delta(x + \frac{2y}{\gamma})\delta(x - \frac{2y}{\gamma})$ we obtain the asymptotic for the dominant term in the calculation for $V(x, \tau)$ prior to the $k$-integration

$$\Gamma_L(r_1, 0; k, \beta) \int \Gamma_L(\frac{x_1 + x_2}{2}, \frac{x_1 - x_2}{\gamma}, k, \beta) \times \theta^* \left( \frac{x_1 - r}{\epsilon_c} \right) \theta \left( \frac{x_2 - r}{\epsilon_c} \right) e^{-|x_1|^2/\gamma} e^{-|x_2|^2/\gamma} \approx C \beta_c^d$$

(15)
with the constant $C_\beta$ given by

$$C_\beta = (2\pi)^{2d}(1+i)^d\beta^{d/2}\sin^{-d}(\sqrt{\beta}(1+i))e^{-\beta^2c_0^2/\pi^2} \times e^{2\beta\gamma} \frac{1}{\sin^2(\beta\gamma/2)} \int \theta^2(\tilde{y} + \frac{\gamma}{2})\theta(\tilde{y} - \frac{\gamma}{2}) \times e^{\frac{\gamma^2}{2}} \sin^2(\beta\gamma) |\beta|^2 d\tilde{y}. $$

Due to the rapidly decaying factor $\sin^{-d}(\sqrt{\beta}(1+i))$ the $\beta$-integration of $C_\beta$ is convergent as $B \to \infty$. Because $\beta \ll 1$, in the $(k_1,k_2)$-integration the power distribution $g(k_1)g(k_2)$ and $C_\beta$ are decoupled after the change of variables: $(k_1,k_2) = (k - \beta\gamma/2,k + \beta\gamma/2)$, so we conclude that $V(x,t) \leq 2\sqrt{2\pi}K^{-2}\beta^{2d+1}BT \int C_\beta d\beta$. Note that the variance increases linearly with the number $T$ of symbols in each data-stream.

The asymptotic SIR/SSR for the SISO-TRA is given by $\rho = O(B\beta^{-1}T^{-1})$. Note that the SIR/SSR is slowly varying in the test point $r$ and the receiver location $r_1$ on the scale of $\sigma_s$.

SIMO. Let us turn to the SIMO case with $N = 1$ element TRA located at $a_1 = 0$.

The mean field calculation is analogous to the SISO case. Namely, $E(r_j,\tau_j) \approx \sqrt{4\pi}C_0(r_j,\tau_j)\beta \gamma B$ and zero in the temporal or spatial sidelobe.

In view of the the remark following (12) the variance of $S$ is dominated by the contribution from the diagonal terms in the summation over receivers given by

$$\sum_{l,l'=1} \int e^{-k(\gamma - \gamma')}g^2(k)dk \sum_{j=1}^M \int \theta(\frac{x_1 - r}{\ell_c})\theta(\frac{x_2 - r}{\ell_c}) \times d\beta L(x_1 + x_2,0;\beta,\gamma) \gamma \int x_1 x_2 d\beta \approx \sqrt{2\pi}B\gamma^2T \int C_\beta d\beta,$$

because $|r_i - r_j| \gg \ell_c$ regardless whether the test point is near or away from any receiver. Therefore we have the estimate: $\rho = O(B\beta^{-1}M^{-1}T^{-1})$.

MISO. The case corresponds to $M = 1$. Each term in the summation over the $N$ TRA-elements has the same asymptotic as that of the SISO case. Hence $E(r_j,\tau_j) \approx \sqrt{4\pi}NC_0(r_j,\tau_j)\beta \gamma B$ and zero in the spatial or temporal sidelobes.

For the variance calculation, let us first note that the correlations of two Green functions starting with two TRA-elements located at $a_i, a_j$ satisfy eq. (8) in the variables $(a_i, a_j)$ by the reciprocity of the time-invariant channel, and hence vanish as $|a_i - a_j| \gg \ell_c$. The variance of the signal at $r$ (whether at $r_1$ or away from it) before performing the $k$-integration is then dominated by the following diagonal terms in the summation over receivers

$$\sum_{j=1}^M \int \theta(\frac{x_1 - r}{\ell_c}) \times d\beta L(x_1 + x_2,0;\beta,\gamma) \gamma \int x_1 x_2 d\beta \approx NMC_\beta \beta^{2d}.$$

The $k$-integration induces the additional factor of $\sqrt{2\pi}B\gamma^2T$. Hence $V(r,t) \leq 2\sqrt{2\pi}K^{-2}\beta^{2d+1}BT \int C_\beta d\beta$ since $|r - r_j| \ll \sigma_s, \forall j$. We conclude that $\rho = O(NB\beta^{-1}T^{-1})$.

MIMO. The analysis for the MIMO case combines all the previous cases. The mean signal has the same asymptotic as that of the MISO case, i.e., linearly proportional to $BN$. The variance of the signal prior to performing the $k$-integration is dominated by

$$\sum_{i,j=1}^{M,N} \int \theta(\frac{x_2 - r}{\ell_c}) \times d\beta L(x_1 + x_2,0;\beta,\gamma) \gamma \int x_1 x_2 d\beta \approx NMC_\beta \beta^{2d}. $$

and therefore $V \leq TMN2K^{-2}B\beta^{2d+1} \int C_\beta d\beta$.

We collect the above analysis in the following statement.

Summary. Let the $N$-element TRA, $M$ receivers and the number of symbols $T$ satisfy the packing condition. Assume the $4$-th order sub-Gaussianity condition (12) in the strong-fluctuation regime and let $1 \ll \alpha_s \ll \gamma^{-1}, 1 \ll \sigma_s \ll \gamma^{-1}$. Then in the broadband limit the asymptotic SIR/SSR $\sim N^{-1}M^{-1}T^{-1}B\beta^{-1}$ is valid uniformly for all $r_j, j = 1, \ldots, M$, with the constant of proportionality $2^{-1}(2\pi)^{-1/2}K^{-1}(\int C_\beta d\beta)^{-1}|C_0|^2$ where $C_0$ as given by (14) is not zero for $\theta \neq 0$.

The asymptotic signal at the receiver plane within the distance $\ll \sigma_s$ from the receivers is $\sum_{i=1}^T \sum_j^M m_j(\gamma)S_{ji}(x,t)$ where $S_{ji}(x,t)$ given by (8).

Conclusion and discussion

The strong-fluctuation regime constitutes the so called space-frequency-selective multi-path fading channels in wireless communications [22]. In such a channel, TR has the super-resolution given by $\ell_c = \gamma/\sqrt{D_2T}$. We have established firmly the packing and multiplexing conditions for stable super-resolution for the MIMO-TRA communication system under the $4$-th order sub-Gaussianity assumption. The experimental evidence for our result in the case of $M = 1$ has been demonstrated in [3].
We have argued that statistical stability is crucial for multi-receiver TR communications, especially when the multiple receivers do not have channel state information, as the multiuser interference is essentially indistinguishable from the intended signal, the only difference being their statistical properties. The latter is in the mean field while the former is primarily in the fluctuating field. Our result implies that the time-reversal communication can be realized stably in principle with up to $M \sim NB\beta_c^{-1}T^{-1}$ receivers simultaneously at the rate $T \beta_c$ with low probability of intercept due to super-resolution. Concerning the channel capacity, our result is analogous to the finding in [14, 30, 21, 22] based on the random matrix modeling and theory that the ergodic capacity with complete channel state information at the receiver with $M$ receive antennas (but not at the $N$ transmit antennas) scales like $\min (M, N) \log_2 \text{SNR}$ (per unit frequency) at high SNR. After taking into account the frequency multiplexing gain [2, 30], the multi-frequency channel capacity then scales like $B \min (M, N) \log_2 \text{SNR}$. Note, however, this result does not include the interference due to noncooperating multiuser receivers as we do here. Also, these works consider only narrow-band signals for which statistical stability is rarely valid in practice and consequently the ergodic capacity is an average, not almost sure, quantity.

In the present set-up with the $B$-band-limited channel state information at the transmitters but not the receivers, the multiplexing gain is, up to a logarithmic factor, roughly $MT \beta_c \sim BN$, the number of degrees of freedom per unit time at TRA (see [10] for more analysis on TR capacity in multi-path Rayleigh fading channels). The packing condition also points to the maximal capacity per unit angular cross section $B \gamma T \delta_\gamma^{-1} \log_2 \gamma$ when $N$ reaches the saturation point $\sigma_\gamma^2/\ell_\gamma^2$ in the angular spread $\alpha_\gamma$. Here $L \delta_\gamma^{-d} = \sigma_\gamma^d \gamma^{-d}$ has the physical meaning of the angular density of uncorrelated propagation paths in the medium.

Let us point out several possible extensions of our results. First, the case of even broader bandwidth of $0 < \lim B \gamma \leq 1$ can easily be treated by partitioning the full passband into many sub-bands with their own $B$ and $\gamma$ satisfying [3]. Since the self-averaging takes place in each sub-band and the whole process is linear, stable super-resolution is valid in the full passband. Second, in the case of a finite time-window, the output signals, unlike [22], involve a coupling of nearby wavenumbers [12]. If the time window is sufficiently large, $\gg \beta_c^{-1}$, then the coupling takes place only between wavenumbers of separation much smaller than $\beta_c$ and our result carries over without major adjustment. Finally, our results may also be extended to time-varying channels, prevalent in mobile wireless communications, with a low spread factor $T^{-1} \delta_\gamma < 1$ where $T_c$ is the coherence time $\sigma_\gamma^2/\ell_\gamma^2$.

[1] P. Blomgren, G. Papanicolaou and H. Zhao, J. Acoust. Soc. Am. 111(2002), 230-248.
[2] T.M. Cover and J.A. Thomas, Elements of Information Theory Wiley, New York, 1991.
[3] A. Derode, E. Larose, M. Tanter, J. de Rosny, A. Tourin, M. Campillo and M. Fink, J. Acoust. Soc. Am.113 (2003), 2973.
[4] A. Derode, A. Tourin and M. Fink, Ultrasoundics 40(2002), 275-280.
[5] A. Derode, A. Tourin, J. de Rosny, M. Tanter, S. Yon, and M. Fink, Phys. Rev. Lett.90(2003), 014301.
[6] A. Derode, A. Tourin and M. Fink, Phys. Rev. E 64 (2001), 036606.
[7] G. Edelmann, T. Akal, W. S. Hodgkiss, S. Kim, W. A. Kuperman, H. C. Song, IEEE J. of Oceanic Eng. 27 (2002), 602-609.
[8] A. Fannjiang, Arch. Rat. Mech. Anal. 175:3(2005), 343 - 387.
[9] A. Fannjiang, J. Stat. Phys. (2005), in press.
[10] A. Fannjiang, Preprint (2005).
[11] A. Fannjiang and K. Solna, Physics Letters A352:1-2 (2005), 22-29.
[12] A. Fannjiang and K. Solna, preprint, 2005.
[13] M. Fink, D. Cassereau, A. Derode, C. Prada, P. Roux, M. Tanter, J.L. Thomas and F. Wu, Rep. Progr. Phys. 63(2000), 1933-1995.
[14] G.J. Foschini and M.J. Gans, Wireless Personal Communication 6 (1998), 311-335.
[15] J. P. Fouque, G. Papanicolaou and Y. Samuelides, Waves Rand. Media 8 (1998) 303-314.
[16] J.W. Goodman, Statistical Optics, John Wiley & Sons, 1985.
[17] A. Ishimaru: Wave Propagation and Scattering in Random Media, Vol. II. Academic, New York, 1978.
[18] L. Kazovsky, S. Benedetto, A. Willner, Optical Fiber Communication Systems, Artech House, 1996.
[19] S. Kim, W. A. Kuperman, W. S. Hodgkiss, H. C. Song, G. Edelmann and T. Akal, J. Acoust. Soc. Am. 114 (2003), 145-157.
[20] A. D. Kim, P. Kyritsi, P. Blomgren and G. Papanicolaou, preprint, 2004.
[21] A.L. Moustakas, H.U. Baranger, L. Balents, A.M. Sengupta and S.H. Simon, Science 287 (2000), 287-290.
[22] A. Paulraj, R. Nabar and D. Gore, Introduction to Space-Time Wireless Communications, Cambridge University Press, 2003.
[23] D. Rouseff, D. R. Jackson, W. L. J. Fox, C. D. Jones, J. A. Ritcey and D. R. Dowling, IEEE J. Oceanic Eng. 26 (2001), 821-831.
[24] K.G. Sabra, S.R. Khosla and D.R. Dowling, J. Acoust. Soc. Am. 111(2) (2002), 823-830.
[25] P. Sebbah, B. Hu, A.Z. Genack, R. Pnini and B. Shapiro, Phys. Rev. Lett. 88 (2002), 123901.
[26] B. Shapiro, Phys. Rev. Lett. 57 (1986), 21682-171.
[27] S.H. Simon, A.L. Moustakas, M. Stoychev and H. Safar, Phys. Today 54:9 (2001), 38.
[28] Tatarskii V I, Ishimaru A and Zavorotny V U (eds), Wave Propagation in Random Media (Scintillation), (Bellingham, WA: SPIE and Bristol: Institute of Physics Publishing), 1993.
[29] Tatarskii V I and Zavorotny V U, Progress in Optics 18 (1980), 20756.
[30] I.E. Telatar, European Trans. Tel. 10 (1999), 585-595.