The fate of conformal symmetry 
in the non-linear Schrödinger theory

M.O. de Kok
Inst. Lorentz, University of Leiden
P.O. Box 9506
2300 RA Leiden NL
e-mail: mdekok@lorentz.leidenuniv.nl

J.W. van Holten
Nikhef
P.O. Box 41882
1009 DB Amsterdam NL
e-mail: v.holten@nikhef.nl

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Abstract
The free Schrödinger theory in $d$ space dimensions is a non-relativistic conformal field theory. The interacting non-linear theory preserves this symmetry in specific numbers of dimensions at the classical (tree) level. This holds in particular for the $|\Phi|^4$-theory in $d = 2$. We compute the full quantum corrections to the 1PI 4-point function in $d = 2 - \epsilon$ dimensions and find a non-trivial $\beta$-function completely given by the 1-loop result. We exhibit an explicit Ward-identity showing that scale-invariance is broken in the limit $d = 2$ by an anomalous contribution proportional to the $\beta$-function.
1. Conformal symmetry of the free Schrödinger theory

Conformal symmetry, in exact or broken form, plays an important role in field theory. As it governs the scale dependence of a theory, it implicitly determines the regime of applicability of the theory. String theory, as an important candidate for a unified theory of quantum gravity, has an exact conformal invariance when formulated as a 2-D field theory on the world sheet. On the other hand, effective field theories in four space-time dimensions, like QED or the standard model, possess an approximate scale invariance, broken explicitly by mass terms and/or by quantum effects. Indeed, very few theories are known to be exactly scale invariant, except in 2 dimensions \[1\]. In 4 dimensions \( N = 4 \) Yang-Mills theory is known to have vanishing \( \beta \)-function to all orders \[2\], but in general gauge theories which are classically scale invariant, such as QCD with massless quarks, have this symmetry broken at the quantum level.

In this paper we consider conformal symmetry in the context of non-relativistic field theory. Such symmetries were first identified in \[3, 4\]. The symmetries and their realization in classical and quantum field theory have been studied by several authors \[5\]-\[10\]. We consider in particular theories describing Bose gases in \( d \) space dimensions. Conformal symmetry is an exact symmetry of the free theory, but it can be implemented in certain interacting models as well. The quantum field theoretical aspects of Bose gases in general have been studied widely in the literature; reviews and references can be found e.g. in \[11, 12\].

In \( d \) space dimensions the free theory is defined by the action

\[
S_0 = \int dt \int d^d x \left( \frac{i}{2} \bar{\Psi} \partial_t \Psi - \frac{1}{2m} \nabla \bar{\Psi} \cdot \nabla \Psi \right).
\]

Without loss of generality the theory can be simplified by rescaling the time variable

\[
\tau = \frac{t}{m},
\]

which is equivalent to choosing units in which \( m = 1 \). The action then reads

\[
S_0 = \int d\tau \int d^d x \left( \frac{i}{2} \bar{\Psi} \partial_\tau \Psi - \frac{1}{2} \nabla \bar{\Psi} \cdot \nabla \Psi \right),
\]

(1)

which is stationary when \( \Psi \) satisfies the linear Schrödinger equation for free particles with unit mass:

\[
i \partial_\tau \Psi = -\frac{1}{2} \Delta \Psi.
\]

(2)
This equation and the action $[1]$ are invariant under the Schrödinger group of space-time transformations $[4, 9]$, which includes time and space translations, spatial rotations, Galilei boosts, dilatations and special conformal transformations. The explicit form of these transformations is presented in the appendix. The associated constants of motion are:

a. the hamiltonian

$$H_0 = \frac{1}{2} \int d^d x \nabla \Psi^* \cdot \nabla \Psi; \quad (3)$$

b. the momentum

$$P = \frac{i}{2} \int d^d x \Psi^* \nabla \Psi; \quad (4)$$

c. the angular momentum

$$M_{ij} = \frac{i}{2} \int d^d x \left( x_i (\Psi^* \nabla_j \Psi) - x_j (\Psi^* \nabla_i \Psi) \right); \quad (5)$$

d. the galilean boost operator

$$G = \tau P + \int d^d x x \Psi^* \Psi; \quad (6)$$

e. the scaling operator

$$D_0 = 2\tau H_0 - \frac{i}{2} \int d^d x x \cdot (\Psi^* \nabla \Psi); \quad (7)$$

f. the special conformal charge

$$K_0 = \tau^2 H_0 - \tau D_0 - \frac{1}{2} \int d^d x x^2 \Psi^* \Psi; \quad (8)$$

g. the particle number

$$N = \int d^d x \Psi^* \Psi. \quad (9)$$

By the fundamental Poisson bracket

$$\{\Psi(x, \tau), \Psi^*(y, \tau)\} = -i\delta^d(x - y), \quad (10)$$

the constants of motion (a-g) generate the infinitesimal transformations of the fields $\Psi$ and $\Psi^*$ which leave the action $[\square]$ invariant. As a result they
realize the Schrödinger algebra extended with a central charge $N$:
\[
\{M_{ij}, M_{kl}\} = \delta_{il} M_{jk} - \delta_{ik} M_{jl} - \delta_{jl} M_{ik} + \delta_{jk} M_{il},
\]
\[
\{H_0, D_0\} = 2H_0, \quad \{H_0, G\} = P,
\]
\[
\{P, D_0\} = P, \quad \{P_i, M_{jk}\} = \delta_{ij} P_k - \delta_{ik} P_j,
\]
\[
\{H_0, K_0\} = -D_0, \quad \{P_i, G_j\} = \delta_{ij} N,
\]
\[
\{G, D_0\} = -G, \quad \{G_i, M_{jk}\} = \delta_{ij} G_k - \delta_{ik} G_j,
\]
\[
\{K_0, D_0\} = -2K_0, \quad \{K_0, P\} = G,
\]
all other brackets vanishing. Notice, that the central charge is the particle number $N$, which generates a phase transformation of the complex field $\Psi$.

Equivalently, the theory can be defined by the hamiltonian generating the time evolution of phase-space functions $F[\Psi, \Psi^*; \tau]$:
\[
\frac{dF}{d\tau} = \frac{\partial F}{\partial \tau}_{\Psi, \Psi^*} + \{F, H_0\},
\]
(12)
where the partial time derivative refers to time dependence other than through $(\Psi, \Psi^*)$. For the constants of motion this implies directly that
\[
\{P, H_0\} = \{M_{ij}, H_0\} = \{N, H_0\} = 0,
\]
(13)
whilst the explicit time dependence in the definition of $G$, $D_0$ and $K_0$ implies
\[
\{G, H_0\} = -P, \quad \{D_0, H_0\} = -2H_0, \quad \{K_0, H_0\} = D_0,
\]
(14)
in agreement with eqs. (11).

The conformal symmetry has direct relevance for the physical content of the theory. In particular, consider the quantities
\[
I \equiv \int d^d x \Psi^* \Psi = G - \tau P,
\]
\[
I_1 \equiv \frac{1}{2} \int d^d x x^2 \Psi^* \Psi = \tau^2 H_0 - \tau D_0 - K_0,
\]
\[
I_2 \equiv \frac{i}{2} \int d^d x \cdot \Psi^* \overrightarrow{\nabla} \Psi = D_0 - 2\tau H_0.
\]
(15)
As \((H_0, P, G, D_0, K_0)\) are constants of motion, it follows that
\[
\frac{dI}{d\tau} = -P, \quad \frac{dI_1}{d\tau} = -I_2, \quad \frac{dI_2}{d\tau} = -2H_0.
\] (16)

Therefore, if there would be any static (i.e. time-independent) solutions of the theory, such that \((I, I_1, I_2)\) are constant themselves, they would necessarily have zero energy and momentum. Of course, such solutions don’t exist in the present case: all the solutions of the free Schrödinger theory represent scattering states, superpositions of \(\delta\)-function normalizable plane waves with a continuous energy spectrum.

2. Conformal symmetry in the non-linear Schrödinger model

In general the conformal transformations which are part of the Schrödinger group and a symmetry of the free theory, are broken by interactions. In particular, \(U(1)\)-invariant polynomial interactions of the form
\[
S_{\text{int}} = -\frac{g^2}{n} \int d\tau \int d^d x (\Psi^* \Psi)^n,
\] (17)

where \(g\) is a coupling constant, break the scale invariance unless \(n\) is such that the dimension of \(|\Psi|^n\) matches that of the space-time integration measure:
\[
nd = d + 2,
\] (18)

implying that \(g\) is dimensionless. In the case \(d = 1\) this is satisfied for \(n = 3\), in the case \(d = 2\) for \(n = 2\); also, for \(d \to \infty\) we get \(n \to 1\). In all other cases \(n\) must take on non-integer values. With the interactions included, the hamiltonian becomes
\[
H_n = \int d^d x \left( \frac{1}{2} \nabla \Psi^* \cdot \nabla \Psi + \frac{g^2}{n} (\Psi^* \Psi)^n \right).
\] (19)

Systems described by such a hamiltonian appear frequently in the context of condensed matter systems \[13\], where the corresponding non-linear Schrödinger equation
\[
i\partial_\tau \Psi = -\frac{1}{2} \Delta \Psi + g^2 |\Psi|^{n-1} \Psi,
\] (20)

is also known as the Gross-Pitaevskii equation \[14\] \[15\].

\(^1\) The Weyl weight of the field \(\Psi\) being \(d/2\); see eq. \[99\].
The non-linear theory defined by eqs. (17), (19) is still invariant under
time- and space-translations, spatial rotations and \( U(1) \) phase transforma-
tions. As a result the generators \( \{ P, M_{ij}, G, N \} \) as in (4)-(9) also represent
constants of motion in the interacting theory. The generators of scale and
special conformal transformations are replaced by

\[
D_n = 2\tau H_n + I_2, \tag{21}
\]

\[
K_n = \tau^2 H_n - \tau D_n - I_1,
\]

with \( (I_1, I_2) \) defined by the explicit first expressions in eqs. (15). They indeed
generate the infinitesimal transformations

\[
\{ \Psi, D_n \} = \left( 2\tau \partial_\tau + \mathbf{x} \cdot \nabla + \frac{d^2}{2} \right) \Psi,
\]

\[
\{ \Psi, K_n \} = \left( -\tau^2 \partial_\tau - \tau \left( \frac{d^2}{2} + \mathbf{x} \cdot \nabla \right) + \frac{i}{2} \mathbf{x}^2 \right) \Psi.
\]

However, \( D_n \) and \( K_n \) are constants of motion only if \( n \) and \( d \) are related by (18). To show this, we note the results

\[
\{ I, H_n \} = -P, \quad \{ I_1, H_n \} = -I_2,
\]

\[
\{ I_2, H_n \} = -2H_n + \left( \frac{2 + d - nd}{n} \right) \int d^dx g^2 (\Psi^* \Psi)^n. \tag{23}
\]

The first of these equations implies the conservation of \( G \) for all \( n \) and \( d \); the
other two equations imply

\[
\frac{dK_n}{d\tau} = -\tau \frac{dD_n}{d\tau}, \quad \frac{dD_n}{d\tau} = \left( \frac{2 + d - nd}{n} \right) \int d^dx g^2 (\Psi^* \Psi)^n. \tag{24}
\]

Eq. (18) is indeed the necessary and sufficient condition for the right-hand
side of these equations to vanish. In these cases we obtain an interacting
classical field theory invariant under the full Schrödinger group of tran-
fonnations. For these theories one can derive equations similar to (16):

\[
\frac{dI}{d\tau} = -P, \quad \frac{dI_1}{d\tau} = -I_2, \quad \frac{dI_2}{d\tau} = -2H_n, \tag{25}
\]
and conclude that again any static solutions of the interacting theories have zero energy and momentum. Now it is obvious, that both the free hamiltonian $H_0$ as well as $H_n$ in the interacting theory with $g^2 > 0$ are non-negative. Therefore the only zero-energy solution is $\Psi = 0$; it follows immediately, that the conformal models have no non-trivial static solutions, neither the $|\Psi|^6$-model in $d = 1$, nor the $|\Psi|^4$-model in $d = 2$. As in the free theory, the physical solutions represent scattering states with a continuous energy spectrum. A more complete discussion is presented in appendix B.

3. Quantum non-linear Schrödinger model

In the remainder of this paper we consider the non-linear Schrödinger model with $|\Psi|^4$ interactions. As we have seen, the classical field theory is conformally invariant in $d = 2$ space dimensions. The problem we wish to address is whether the conformal symmetry is preserved in the corresponding quantum field theory.

The quantum field theory is obtained by taking $\Psi$ and $\Psi^*$ to be conjugate operators with the equal-time commutation relation

$$[\Psi(x, \tau), \Psi^*(y, \tau)] = \delta^d(x - y).$$

Furthermore we take the hamiltonian and other generators of the Schrödinger group to be the normal-ordered operator version of the corresponding classical ones:

$$H_\mu[\Psi, \Psi^*] = \int d^d x \left( \frac{1}{2} \nabla \Psi^* \cdot \nabla \Psi + \mu \Psi^* \Psi + \frac{g^2}{2} \Psi^* \Psi^2 \Psi \Psi \right);$$

The bilinear term $\mu \Psi^* \Psi$ arises from the operator-ordering ambiguity in going from the classical to the quantum theory. As in the classical theory there exist conserved operators $(P, M_{ij}, G, N)$, defined by the expressions (4)-(9) with $(\Psi, \Psi^*)$ interpreted as field operators. It follows immediately, that we can subtract a constant term $\mu N$ from the energy and define another $\mu$-independent conserved energy operator

$$H_s = H - \mu N = \int d^d x \left( \frac{1}{2} \nabla \Psi^* \cdot \nabla \Psi + \frac{g^2}{2} \Psi^* \Psi^2 \Psi \Psi \right).$$

Equivalently, we can redefine the fields:

$$\Phi(x, t) = e^{-i\mu t} \Psi(x, t), \quad \Phi^*(x, t) = e^{i\mu t} \Psi^*(x, t),$$

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with the same equal-time commutation relations
\[ [\Phi(x, t), \Phi^*(y, t)] = \delta^d(x - y). \] (30)

In terms of these fields the Hamiltonian (27) becomes
\[ H[\Phi, \Phi^*] = \int d^dx \left( \frac{1}{2} \nabla \Phi^* \cdot \nabla \Phi + \frac{g^2}{2} \Phi^* \Phi \right); \] (31)

More generally, a shift in the value of \( \mu \) in the Hamiltonian (27) can be compensated by a multiplicative field renormalization of the type (29).

Assuming such a (finite) renormalization of the fields and the \( \mu \) term to have been performed, we consider the theory as defined by (30) and (31). In addition to the conserved operators of the Galilean transformations \( (P, M_{ij}, G, N) \), which take the same form in terms of the new fields \( (\Phi, \Phi^*) \), we can then also construct the conformal operators
\[ D = 2\tau H + I_2, \] \[ K = \tau^2 H - \tau D - I_1, \] (32)

where \( I_{1,2} \) are the normal-ordered expressions (15).

The transformations of the field operators under the full Schrödinger algebra are obtained by computing their commutator with the generators:
\[ -i [\Phi, H] = \partial_\tau \Phi, \quad -i [\Phi, P] = \nabla \Phi, \]
\[ -i [\Phi, M_{ij}] = (x_i \nabla_j - x_j \nabla_i) \Phi, \quad -i [\Phi, G] = (\tau \nabla - i x) \Phi, \]
\[ -i [\Phi, D] = \left( 2\tau \partial_\tau + x \cdot \nabla + \frac{d}{2} \right) \Phi, \quad -i [\Phi, N] = -i \Phi, \] \[ -i [\Phi, K] = \left( \tau^2 \partial_\tau - \tau \left( x \cdot \nabla + \frac{d}{2} \right) + \frac{i}{2} x^2 \right) \Phi. \] (33)

For any operator \( F[\Phi, \Phi^*; \tau] \) the Heisenberg equation of motion now reads
\[ \frac{dF}{d\tau} = \frac{\partial F}{\partial \tau} \bigg|_{\Phi, \Phi^*} - i [F, H]. \] (34)
As in the classical theory, for the full set of Schrödinger operators to be conserved: \(dF/d\tau = 0\), imposes additional constraints. Indeed, we reobtain the results (24) as normal-ordered operator equations:

\[
\frac{dK}{d\tau} = -\tau \frac{dD}{d\tau}, \quad \frac{dD}{d\tau} = \frac{(2 - d)}{2} g^2 \int d^d x \Phi^* \Phi^2. \tag{35}
\]

The relation between symmetries and constants of motion in the non-linear Schrödinger theory is a specific case of Noether’s general theorem. It can be formulated locally in terms of a charge and current density \((\rho, j)\), which for scale transformations take the form

\[
\rho = \tau \left( \nabla \Phi^* \cdot \nabla \Phi + g^2 \Phi^* \Phi^2 \right) + \frac{i}{2} \mathbf{x} \cdot \left( \Phi^* \nabla \Phi \right), \tag{36}
\]

and

\[
j = -\frac{d}{4} \nabla \left( \Phi^* \Phi \right) - \frac{1}{2} \left( \mathbf{x} \cdot \nabla \Phi^* \nabla \Phi + \nabla \Phi^* \mathbf{x} \cdot \nabla \Phi \right)
+ \frac{1}{2} \mathbf{x} \left( \frac{1}{2} \Delta (\Phi^* \Phi) - g^2 \Phi^* \Phi^2 \right)
- \frac{i\tau}{2} \left( \nabla \Phi^* \Delta \Phi - \Delta \Phi^* \nabla \Phi + g^2 \left( \Phi^* \nabla \Phi^2 \right) \right). \tag{37}
\]

The operator equations of motion

\[
i\partial_\tau \Phi = [\Phi, H] = -\frac{1}{2} \Delta \Phi + g^2 \Phi^* \Phi^2, \tag{38}
\]

and its hermitean conjugate, then imply the equation of continuity

\[
\partial_\tau \rho + \nabla \cdot j = \frac{(2 - d)}{2} g^2 \Phi^* \Phi^2. \tag{39}
\]

By taking the integral over 2-dimensional space and in the absence of boundary terms one then directly reproduces the conservation law (35):

\[
D = \int d^d x \rho \quad \Rightarrow \quad \frac{dD}{d\tau} = \frac{(2 - d)}{2} g^2 \int d^d x \Phi^* \Phi^2 = (2 - d) H_{int}, \tag{40}
\]

where \(H_{int}\) is the interaction part of the hamiltonian:

\[
H_{int}(\tau) = \frac{g^2}{2} \int d^d x \Phi^* (\tau, \mathbf{x}) \Phi^2 (\tau, \mathbf{x}), \tag{41}
\]
Using the expression \(32\) for \(D\), this can be rewritten in the form
\[
\tau \frac{dD}{d\tau} = \frac{(2 - d)}{2} g^2 \frac{\partial D}{\partial g^2}.
\] (42)

4. Quantum effects

The results of quantum field theory are expressed in terms of the time-ordered correlation functions, which can be calculated in various ways. In the interaction representation of the canonical formulation the connected parts of the correlation functions are given by
\[
G_n(z_1, ..., z_{2n}) = \mathcal{N}_c \langle 0 | T \left( \Phi(z_1) ... \Phi(z_n) \Phi^*(z_{n+1}) ... \Phi^*(z_{2n}) e^{-i \int_{-\infty}^{\infty} d\tau H_{\text{int}}} \right) | 0 \rangle,
\] (43)

with \(z = (x, \tau)\). In the expression on the right-hand side \(T\) is the time-ordering operator, whilst \(\mathcal{N}_c\) is a normalization factor dividing out the vacuum-to-vacuum contributions
\[
\mathcal{N}_c^{-1} = \langle 0 | e^{-i \int_{-\infty}^{\infty} d\tau H_{\text{int}}} | 0 \rangle.
\] (44)

Equivalently, the generating functional for the correlation functions
\[
W[\eta, \eta^*] = -i \log Z[\eta, \eta^*],
\]
\[
= \sum_n \int d^{d+1}z_1 ... d^{d+1}z_{2n} G_n(z_1, ..., z_{2n}) \, \eta^*(z_1) ... \eta^*(z_{n+1}) \eta(z_{n+1}) ... \eta(z_{2n}),
\] (45)
can also be computed from the path integral
\[
Z[\eta, \eta^*] = \int D\Phi \Phi^* e^{iS[\Phi, \Phi^*] + i \int_{-\infty}^{\infty} (\eta^* \Phi + \Phi^* \eta)},
\] (46)
with \(S[\Phi, \Phi^*]\) the classical action
\[
S = \int d\tau \int d^d x \left( \frac{i}{2} \Phi^* \frac{\partial}{\partial \tau} \Phi - \frac{1}{2} \nabla \Phi^* \cdot \nabla \Phi - \frac{g^2}{2} |\Phi|^4 \right).
\] (47)

For consistency of presentation, in the following we take the operator expression \(43\) as our starting point for the perturbative calculation of correlation functions as a series in powers of the coupling constant \(g^2\). As the direct naive perturbative calculation of the correlation functions \(G_n\) in general leads to
divergences due to integrals over the infinite volume of momentum space, a
renormalization procedure is necessary to obtain finite results which can be
related to measurable quantities. It will be argued below that in the present
case the only modification needed in $d = 2$ is a multiplicative renormalization
of the coupling constant

$$g^2 = Z_g g_R^2, \quad Z_g = 1 + \sum_{n \geq 1} z_n g_R^{2n}, \quad (48)$$

where the coefficients $z_n$ depend on the regularization scheme and become
infinite in the limit where the regularization is removed. If one then calculates
correlation functions order by order in the renormalized coupling constant
$g_R^2$, the resulting expressions remain finite upon removing the regulator.

In principle we could regularize the theory by introducing a cut-off $\Lambda$
in momentum space, restricting momentum integrals to the sphere $k^2 < \Lambda^2$. However, such a cut-off would introduce an explicit violation of the
conformal symmetry (scaling and conformal boosts), directly spoiling the
conservation of the scaling operator $D$. Instead in this paper we follow a
different route, doing computations in $d \neq 2$ dimensions, treating $d$ as a
continuous parameter and taking the limit $d \to 2$ only at the end. This
procedure is closely related to the standard dimensional regularization used
in relativistic QFT [16, 17], except that we continue only the number of
spatial dimensions $d$, treating the time dimension separately.

According to eq. (35) the scaling charge is again not conserved during
the calculation in $d = 2 - \epsilon$ dimensions, be it in a very controled way. The
question then is, whether the conservation of $D$ is restored in the limit $\epsilon \to 0$.
Below we show that this is not the case: quantum effects spoil the scaling
symmetry in $d = 2$. Indeed, following standard renormalization group argu-
ments let us introduce a scale $\Lambda$ at which the renormalized coupling $g_R^2(\Lambda)$
is defined, such that the bare coupling remains fixed:

$$g^2 = f(g_R^2(\Lambda), \Lambda, \epsilon), \quad \frac{dg^2}{d\Lambda} = \frac{dg_R^2}{d\Lambda} \frac{\partial f}{\partial g_R^2} |_{\Lambda} + \frac{\partial f}{\partial \Lambda} |_{g_R^2} = 0. \quad (49)$$

This result defines the $\beta$-function:

$$\beta(g_R^2, \Lambda, \epsilon) = \Lambda \frac{dg_R^2}{d\Lambda} = -\Lambda \frac{\partial f/\partial \Lambda}{\partial f/\partial g_R^2}. \quad (50)$$
Then in the renormalized theory the dimensionless scaling charge can be a function only of the renormalized coupling \( g_R^2 \) and the dimensionless time-parameter \( y = \Lambda^2 \tau \):

\[
D = D_R(g_R^2, y). 
\]  

(51)

Now as \( D \) cannot depend on the renormalization scale \( \Lambda \), we must have

\[
\Lambda \frac{dD}{d\Lambda} = \Lambda \frac{d g_R^2}{d\Lambda} \frac{\partial D_R}{\partial g_R^2} + 2y \frac{\partial D_R}{\partial y} = 0.
\]  

(52)

It then follows that

\[
\tau \frac{dD}{d\tau} = y \frac{\partial D_R}{\partial y} = -\frac{\beta}{2} \frac{\partial D_R}{\partial g_R^2}.
\]  

(53)

Comparing this with eq. (42) we find that consistency requires

\[
\frac{\epsilon g_R^2}{2} \frac{\partial D}{\partial g^2} = -\frac{\beta}{2} \frac{\partial D_R}{\partial g_R^2}.
\]  

(54)

Eqs. (53) and (54) imply, that if the \( \beta \)-function does not vanish in the limit \( \epsilon \to 0 \), then there is a scale anomaly proportional to the \( \beta \)-function; this is precisely what one would expect in a non-finite renormalizable quantum field theory.

5. Computation of the 2- and 4-point function

In this section we compute the quantum contributions to the propagator and the 2-particle scattering amplitude to all orders in perturbation theory in \( d = 2 - \epsilon \) dimensions. We can then compute the \( \beta \)-function, and show that equation (54) becomes an identity. From the explicit expression for the \( \beta \)-function it follows that it does not vanish in \( d = 2 \), and there is a scale anomaly. We start by defining the tree-level propagator

\[
D_+(\tau, x) = \theta(\tau) \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x - \frac{1}{2}k^2\tau}
\]  

(55)

\[
= \frac{i}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dk_0 \int d^d k \frac{e^{ik\cdot x - ik_0 \tau}}{k_0 - \frac{1}{2} k^2 + i\varepsilon},
\]

which is a solution of the inhomogeneous free Schrödinger equation for positive time intervals \( \tau > 0 \):

\[
(i\partial_\tau + \frac{1}{2} \Delta) D_+(\tau, x) = i\delta(\tau) \delta^d(x).
\]  

(56)
As such it represents the time-ordered two-point correlation function of the free theory:

\[ \langle 0| T(\Phi(\tau_2, x_2)\Phi^*(\tau_1, x_1)) |0 \rangle_{g^2=0} = D_+(\tau_2 - \tau_1, x_2 - x_1), \quad (57) \]

which naturally vanishes for \( \tau_2 < \tau_1 \). In the interacting \( |\Phi|^4 \)-theory the connected two-point function is defined by

\[ G_2(z_1, z_2) = \mathcal{N}_c \langle 0| T(\Phi(z_1)\Phi^*(z_2)e^{-i\int H_{\text{int}}}|0) \rangle, \quad (58) \]

where \( z_i = (\tau_i, x_i) \). It reduces to the tree-level result (57) for \( g^2 = 0 \). The quantum corrections to the tree-level propagator are represented by the 1-loop diagram of fig. 1.

![Fig. 1: 1-loop propagator correction](image)

It is straightforward to see, that this contribution vanishes. Indeed, the loop represents the contraction of two field operators at the same point in spacetime. In such an equal-time contraction, as in \( H_{\text{int}} \) itself, the operators are normal ordered, hence their contribution to the vacuum-to-vacuum amplitude vanishes [11]. Alternatively, we may evaluate the loop in the diagram

\[
D_+(0, 0) = \frac{i}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} dk_0 \int d^dk \frac{1}{k_0 - \frac{1}{2}k^2 + i\varepsilon}
\]

\[
= \frac{1}{2(2\pi)^d} \int d^dk = \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(d/2)} \int_0^\infty dk \ (k^2)^{\frac{d-1}{2}} = 0, \quad (59)
\]

where the last result is obtained by employing a rotation-invariant regularization scheme such as dimensional regularization [16, 17]. Higher-order quantum corrections consist either of insertions of these seagull-type propagator corrections, which all vanish by the argument above, or of bubbles connecting two propagators (\( t \)-channel bubbles), as in fig. 2:
Again, the time-ordering of the fields in these diagrams always forces these contributions to vanish; this has been made explicit in the figure by drawing the vertices always in a time-ordered way, with two incoming lines from the left, end two outgoing lines to the right. As a result of this analysis of propagator corrections, neither the fields nor chemical potential $\mu$ are renormalized by quantum loop effects. This shows that we can without loss of generality take $\mu = 0$ from the start, as we have done in (29) by switching to the fields $(\Phi, \Phi^*)$.

Next we consider the connected 4-point correlation function

$$G_4(z_1, z_2, z_3, z_4) = N_c \langle 0| T \left( \Phi(z_1) \Phi(z_2) \Phi^*(z_3) \Phi^*(z_4) e^{-i \int_t^T H_{int}} \right) | 0 \rangle. \quad (60)$$

In view of the vanishing of the seagull-diagrams and $t$-channel bubbles, the full first and second order contributions in perturbation theory are represented by the diagrams of fig. 3:

$$\times \qquad \times$$

Fig. 3: 4-point diagrams

The expression corresponding to the combined set of diagrams is

$$\frac{-2ig^2}{(2\pi)^{3(d+1)}} \int_{k_1} \cdots \int_{k_4} \delta^{d+1}(k_1 + k_2 - k_3 - k_4) A^{(2)}(k_1, k_2) \times \prod_{i=1}^4 \left( \frac{e^{ik_{i,0} - \frac{1}{2} k_i^2 + i\epsilon}}{k_{i,0} - \frac{1}{2} k_i^2 + i\epsilon} \right),$$

$$\quad (61)$$
where $k \cdot z = k \cdot x - k_0 \tau$, and

\[
A^{(2)}(k_1, k_2) = 1 - \frac{ig^2}{(2\pi)^{d+1}} \int dk_0 \int d^d k \ B(k; k_1, k_2)
\]

\[
B(k; k_1, k_2) = \frac{1}{(k_0 - \frac{1}{2} k^2 + i\varepsilon)(k_0 - k_{1,0} - k_{2,0} + \frac{1}{2}(k - k_1 - k_2)^2 - i\varepsilon)}.
\]

Now we can also compute all higher-order corrections, as the only new contribution at each order $g^{2n}$ is another loop attached to the diagram of order $g^{2(n-1)}$, as in fig. 4.

---

**Fig. 4:** $n^{th}$-order contribution to $G_4$.

Thus to $n^{th}$-order in the perturbative expansion of $G_4$ we get

\[
G^{(n)}_4(z_1, z_2, z_3, z_4) = \frac{-2ig^2}{(2\pi)^{3(d+1)}} \int_{k_1} \ldots \int_{k_4} \delta^{d+1}(k_1 + k_2 - k_3 - k_4) \ A^{(n)}(k_1, k_2)
\]

\[
\times \prod_{i=1}^4 \left( e^{ik_i z_i} \left( k_{i,0} - \frac{1}{2} k_{i,0}^2 + i\varepsilon \right) \right),
\]

with

\[
A^{(n)}(k_1, k_2) = \sum_{p=0}^{n-1} \left( -\frac{ig^2}{(2\pi)^{d+1}} \int dk_0 \int d^d k \ B(k; k_1, k_2) \right)^p.
\]

This implies that the all-order expression for the sum of bubble diagrams representing $G_4$ is given by a momentum amplitude $A(k_1, k_2)$ which is the sum of a geometric series

\[
g^2 A(k_1, k_2) \equiv g^2 A^{(\infty)}(k_1, k_2) = \frac{g^2}{1 + \frac{ig^2}{(2\pi)^{d+1}} \int dk_0 \int d^d k \ B(k; k_1, k_2)}.
\]

To evaluate this expression, we first perform the $k_0$-integral by closing the contour in the upper-half plane; as only the pole at in the second factor is
inside the contour, we then get after taking the external energies on-shell\footnote{I.e., $k_0 = \frac{1}{2}k^2$.}

\[ A^{-1}(k_1, k_2) - 1 = \frac{g^2}{(2\pi)^d} \int d^d k \frac{1}{k^2 - \frac{1}{4}(k_1 - k_2)^2 - i\varepsilon}. \] (66)

To properly perform the integral, we make the integrand and the integration measure dimensionless by introducing a reference momentum scale $\Lambda$, and writing

\[ k = \kappa \Lambda, \] (67)

where $\kappa$ is dimensionless. Then

\[ A^{-1}(k_1, k_2) - 1 = \frac{g^2 \Lambda^{d-2}}{(2\pi)^d} \int d^d \kappa \frac{1}{\kappa^2 - \frac{1}{4}(\kappa_1 - \kappa_2)^2 - i\varepsilon}. \] (68)

By performing the integrals over the angles, and switching to a single remaining integration variable

\[ s = \frac{4\kappa^2}{(\kappa_1 - \kappa_2)^2}, \] (69)

whilst continuing to $d = 2 - \epsilon$ dimensions, we then find that

\[ A^{-1}(k_1, k_2) - 1 = \frac{g^2 \Lambda^{-\epsilon}}{4\pi} \frac{\Lambda^{d-2}}{\Gamma(d/2)} \left( \frac{(\kappa_1 - \kappa_2)^2}{16\pi} \right)^{\frac{d-2}{2}} \int_0^\infty ds \frac{s^{\frac{d-2}{2}}}{s - 1 - i\varepsilon}. \] (70)

A result equivalent at one loop has been obtained in the context of the Jackiw-Pi model in [7]; a similar one-loop calculation for the fermion gas has been performed in [10].

We can rewrite the result (70) as

\[ g^2 \Lambda^{-\epsilon} A(k_1, k_2) = \frac{4\pi}{g^2 \Lambda^{-\epsilon}} \left( \frac{4\pi}{g^2 \Lambda^{-\epsilon}} + \frac{2}{\epsilon} - \gamma_E + i\pi - \ln \frac{(k_1 - k_2)^2}{16\pi \Lambda^2} + \mathcal{O}[\epsilon] \right). \] (71)

Clearly, the denominator diverges in the limit $\epsilon \to 0$. This divergence can be absorbed in the coupling constant. Indeed, noting that $g^2 \Lambda^{-\epsilon}$ is dimensionless
for any $d$, we can introduce a dimensionless renormalized coupling constant $g_R^2(\Lambda)$ by
\[ \frac{4\pi}{g^2\Lambda^{-\epsilon}} + \frac{2}{\epsilon} - \gamma_E = \frac{4\pi}{g_R^2}. \] (72)
This renormalization procedure is similar to the MS-scheme in relativistic field theory. In the limit $\epsilon \to 0$ we then finally get
\[ g^2 A(k_1, k_2) = \frac{4\pi}{g_R^2} - \ln \left[ \frac{(k_1 - k_2)^2}{16\pi\Lambda^2} \right] + i\pi, \] (73)
which is finite for finite $g_R^2(\Lambda)$ at all momenta. Observe, that the real part vanishes for
\[ \frac{1}{4} (k_1 - k_2)^2 = 4\pi \Lambda^2 \, e^{4\pi/g_R^2}, \] (74)
whilst the imaginary part satisfies
\[ |g^2 A(k_1, k_2)|^2 = -4 \text{Im} \left( g^2 A(k_1, k_2) \right), \] (75)
as required by unitarity in $d = 2$. The natural scale at which to define the renormalized coupling constant is the particle mass: $\Lambda = m$, which in the present conventions is normalized to unity. Defining $g_R^* \equiv g_R(1)$, the final result for the 4-point function then is
\[ G_4(z_1, z_2, z_3, z_4) = -\frac{2i}{(2\pi)^9} \int_{k_1} \cdots \int_{k_4} \prod_{i=1}^4 \left( \frac{e^{ik_i z_i}}{k_{i,0} - \frac{1}{2}k_i^2 + i\epsilon} \right) \times \frac{g_R^{*2} \delta^{d+1}(k_1 + k_2 - k_3 - k_4)}{1 - g_R^2/4\pi \ln \left[ \frac{(k_1 - k_2)^2}{16\pi} \right] + ig_R^2/4}. \] (76)

6. Scale anomaly

Using the definition of the renormalized coupling \[72\] we can compute the exact $\beta$-function:
\[ \beta(g_R^2, \epsilon) = \Lambda \frac{dg_R^2}{d\Lambda} \bigg|_{g_R^2, \epsilon} = g_R^4 \frac{d}{2\pi} - \epsilon g_R^2 \left( 1 + \frac{\gamma_E}{4\pi} g_R^2 \right) \xrightarrow{\epsilon \to 0} \frac{g_R^4}{2\pi}. \] (77)
It is easy to check, that the same result is obtained if one only renormalizes the theory in the one-loop approximation. We also note that our result is
equivalent to the scaling behaviour of the 4-point coupling found in ref. [6]. In particular it implies that in dimensions $d < 2$ there is a non-trivial fixed point

$$g_R^2 = \frac{2\pi \epsilon}{1 - \frac{\gamma_E \epsilon}{2}}.$$ (78)

In the following we study the limit $d = 2$ by taking $\epsilon \to 0$. Rewriting eq. (72) in the form

$$g^2 = \frac{g_R^2 \Lambda^4}{1 - \frac{g_R^2}{2\pi \epsilon} + \frac{\gamma_E}{4\pi} g_R^2},$$ (79)

it is straightforward to show that

$$\epsilon g^2 \frac{\partial}{\partial g^2} = \left( \epsilon g_R^2 \left( 1 + \frac{\gamma_E}{4\pi} g_R^2 \right) - \frac{g_R^4}{2\pi} \right) \frac{\partial}{\partial g_R^2} = -\beta \frac{\partial}{\partial g_R^2},$$ (80)

which proves the identity (54). To make the existence of a scale anomaly explicit, we consider the correlation function

$$\langle \Phi(z_1)\Phi(z_2)\Phi^*(z_3)\Phi^*(z_4) \frac{dD}{d\tau} \rangle$$

$$\equiv \mathcal{N}_c \langle 0 | T \left( \Phi(z_1)\Phi(z_2)\Phi^*(z_3)\Phi^*(z_4) \frac{dD}{d\tau} e^{-i\int_{H_{int}}} \right) | 0 \rangle,$$ (81)

and prove that it is non-zero. The easiest way to do this, is to use the result (40) and integrate the equation over time:

$$\int d\tau \langle \Phi(z_1)\Phi(z_2)\Phi^*(z_3)\Phi^*(z_4) \frac{dD}{d\tau} \rangle = \epsilon \int d\tau \langle \Phi(z_1)\Phi(z_2)\Phi^*(z_3)\Phi^*(z_4) H_{int} \rangle$$

$$= i\epsilon g^2 \frac{\partial}{\partial g^2} \langle \Phi(z_1)\Phi(z_2)\Phi^*(z_3)\Phi^*(z_4) \rangle = -i\beta \frac{\partial}{\partial g_R^2} G_4(z_1, z_2, z_3, z_4).$$ (82)

The step from the first to the second line is not quite trivial; more precisely,
one finds
\[
\frac{i\epsilon g^2}{2}\frac{\partial}{\partial g^2} \langle \Phi(z_1)\Phi(z_2)\Phi^*(z_3)\Phi^*(z_4) \rangle
\]
\[
= i\epsilon N_c g^2 \frac{\partial}{\partial g^2} \langle 0 \left| T \left( \Phi(z_1)\Phi(z_2)\Phi^*(z_3)\Phi^*(z_4)e^{-i\int d\tau H_{int}} \right) \right| 0 \rangle
\]
\[
- \epsilon \langle \Phi(z_1)\Phi(z_2)\Phi^*(z_3)\Phi^*(z_4) \rangle \int d\tau \langle H_{int} \rangle
\]
\[
= \epsilon \int d\tau \langle \Phi(z_1)\Phi(z_2)\Phi^*(z_3)\Phi^*(z_4) H_{int} \rangle,
\]
where we use \( \langle H_{int} \rangle = 0 \). Eq. (82) shows explicitly that the scaling charge \( D \) is not conserved in the full quantum theory, the anomaly being proportional to the \( \beta \)-function.

7. Discussion

The free (linear) Schrödinger theory in \( d \) space dimensions is invariant under the full Schrödinger group, and this holds for the non-linear classical \( |\Phi|^4 \) theory with \( d = 2 \) as well. As we have shown however, in the non-linear quantum field theory the conformal part of the symmetry, comprising the dilatations and special conformal transformations, is anomalous; the anomaly is proportional to the \( \beta \)-function:

\[
\frac{dK}{d\tau} = -\tau \frac{dD}{d\tau} = \frac{\beta}{2} \frac{\partial D_R}{\partial g^2_R}.
\]

Of course, the subgroup of galilean symmetries: time- and space-translations, rotations and galilean boosts, is still manifest in the full theory. Other conformal field theories of the same type are \( |\Phi|^6 \) in \( d = 1 \), which we have not analyzed, and the \( |\Phi|^2 \)-model for \( d \to \infty \), which is a free theory and therefore has no anomaly.

Taking the \( \beta \)-function (77) and solving for the running coupling, again taking the mass \( m \) as reference scale, we get:

\[
\alpha_R(\Lambda) \equiv \frac{g^2_R(\Lambda)}{2\pi} = \frac{\alpha^*_R}{1 - \alpha^*_R \ln \Lambda},
\]
showing that the theory is infrared free and has a Landau-type singularity (in units of $m$) for
\[ \Lambda_s = e^{1/\alpha_R} = \Lambda e^{1/\alpha_R(\Lambda)}. \] (86)

The behaviour of $\alpha_R(\Lambda)$ has been plotted in fig. 5. For $0 < \Lambda < e^{-1} \Lambda_s$ the theory is in the perturbative regime $0 < \alpha_R < 1$. In this regime it represents an effective theory for a weakly interacting Bose-gas with repulsive hardcore (i.e., $\delta$-function) interactions at $T = 0$ in an infinite volume. It is important that the zero in the real part of the 4-point function (80), or equivalently in the momentum amplitude $g^2 A(k_1, k_2)$ of eq. (73), is independent of the renormalization scale; indeed it is reached for
\[ \frac{1}{4} (k_1 - k_2)^2 = 4\pi (\Lambda e^{1/\alpha_R})^2 = 4\pi e^{2/\alpha_R}. \] (87)

![Fig. 5: Running coupling of the 2-d $|\Phi|^4$-model](image)

It is to be noted that there is also a perturbative regime $|\alpha_R| < 1$ for large momentum scales $\Lambda > e\Lambda_s$. In this regime the coupling constant is negative: the point-like interactions are attractive. Such an attractive interaction is relevant in the Jackiw-Pi model, an extension of the non-linear Schrödinger model with Chern-Simons gauge interactions [5, 18]. Indeed, the Jackiw-Pi model preserves the conformal symmetries of the Schrödinger group at tree level, and possesses static classical solutions with zero energy in the regime of negative coupling.
Observe, that replacing the bare coupling \( g^2 \rightarrow -\bar{g}^2 \) after renormalization in the minimal subtraction scheme leads to the relation

\[
\bar{g}^2 \Lambda^{-\epsilon} = \frac{\bar{g}_R^2}{1 + \frac{\bar{g}_R^2}{2\pi \epsilon} + \frac{\gamma_E}{4\pi} \bar{g}_R^2}
\]

and a negative \( \beta \)-function:

\[
\beta(\bar{g}_R^2) = -\frac{\bar{g}_R^4}{2\pi}.
\]

The upshot is just a change of sign \( \alpha_R \rightarrow -\bar{\alpha}_R \), such that

\[
\bar{\alpha}_R(\Lambda) = \frac{\bar{\alpha}^*_R}{1 + \bar{\alpha}^*_R \ln \Lambda}.
\]

The behaviour of this function is as in fig. 5, but reflected across the \( \Lambda \)-axis by interchanging the positive and negative \( \alpha_R \)-values. This leads to exactly the same physics: repulsive interactions, now corresponding to \( \bar{\alpha}_R < 0 \), in the low momentum regime below \( \Lambda_s = e^{-1/\bar{\alpha}_R} \), and attractive interactions with \( \bar{\alpha}_R > 0 \) in the high-momentum domain above \( \Lambda_s \).

Of course, as a non-relativistic theory we do not expect the model to be valid on scales of the order \( \Lambda = 1 \) (the mass scale), where pair creation becomes relevant. However, if \( \Lambda_s \ll 1 \), there still could exist a perturbative regime for negative coupling \( e\Lambda_s < \Lambda < 1 \). Finally we observe, that in the limit \( \Lambda \rightarrow \infty \) the running coupling vanishes, as does the \( \beta \)-function. In this limit the scale invariance is restored.

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Appendix A: the Schrödinger group

The Schrödinger group is a set of space and time transformations including space and time translations, spatial rotations, Galilei boosts, dilatations and special conformal transformations, acting on the co-ordinates \((x, t)\) as

\[
\begin{align*}
\tau' &= \frac{\alpha \tau + \beta}{\gamma \tau + \delta}, \\
x' &= \frac{R \cdot x + u \tau + a}{\gamma \tau + \delta},
\end{align*}
\]  
(91)

where \((\alpha, \beta, \gamma, \delta)\) are scalar parameters restricted by \(\alpha \delta - \beta \gamma = 1\), whilst \(R\) is a \(d\)-dimensional orthogonal matrix, and \((u, a)\) are \(d\)-dimensional vectors parametrizing boosts and translations.

The transformations (91) can be realized on the complex scalar fields \((\Psi, \Psi^*)\) such that the free action (1) remains invariant. This realization includes an additional \(U(1)\) phase transformation, acting as a central charge \([4]\). First, the subgroup of transformations with \(\alpha = \delta = 1\) and \(\gamma = 0\) consist of time translations parametrized by \(\beta\), space translations parametrized by \(a\), spatial rotations parametrized by \(d(d-1)/2\) parameters \(\omega_{ab} = -\omega_{ba}\) such that

\[
R_{ij} = \left( e^{-\frac{1}{2} \omega_{ab} L_{ab}} \right)_{ij}, \quad (L_{ab})_{ij} = \delta_{ia} \delta_{jb} - \delta_{ib} \delta_{ja},
\]  
(92)

and galilean boosts parametrized by \(u\). For rotations and space and time translations, the transformation rule for the scalar field is by definition

\[
\Psi(x', \tau') = \Psi(x, \tau).
\]  
(93)

However, under galilean boosts parametrized by \(u\), the field transforms with an additional space-time dependent phase factor

\[
\Psi(x', \tau') = e^{i(u \cdot x + \frac{1}{2} u^2 \tau')} \Psi(x, \tau).
\]  
(94)

The remaining transformations rescale time in a non-trivial way. Under dilatations the field has a dimension-dependent non-zero Weyl weight, which also turns up in the special conformal transformations. Taking \(\alpha = 1/\delta = e^{\eta}\), \(\beta = \gamma = 0\) the dilatations are realized on complex \(\Psi\) as

\[
\Psi(x', \tau') = \alpha^{-d/2} \Psi'(x, \tau) = e^{-d\eta/2} \Psi'(x, \tau),
\]  
(95)

whilst the special conformal transformations with \(\alpha = \delta = 1\) and \(\beta = 0\) take the form

\[
\Psi(x', \tau') = (1 + \gamma \tau)^{d/2} e^{-\frac{i}{2} \frac{\gamma x^2}{(1 + \gamma \tau)}} \Psi'(x, \tau).
\]  
(96)
Finally, the $U(1)$ phase transformations leave the space-time co-ordinates invariant, and take the standard form on the complex scalar field:

$$\Psi(x', \tau') = \Psi(x, \tau) = e^{i\theta} \Psi'(x, \tau).$$

(97)

The infinitesimal forms of these transformations

$$\delta \Psi(x, \tau) = \Psi'(x, \tau) - \Psi(x, \tau)$$

$$= \beta \partial_\tau \Psi + a \cdot \nabla \Psi + \omega_{ij} x_i \nabla_j \Psi + u \cdot (\tau \nabla - i x) \Psi$$

$$+ \eta \left(2\tau \partial_\tau + x \cdot \nabla + \frac{d}{2}\right) \Psi - i \theta \Psi$$

$$+ \gamma \left(-\tau^2 \partial_\tau - \tau \left(x \cdot \nabla + \frac{d}{2}\right) + \frac{i}{2} x^2 \right) \Psi,$$

(98)

are precisely those generated by the constants of motion (3)-(9) of the free Schrödinger theory:

$$\{\Psi, H\} = \frac{i}{2} \Delta \Psi = \partial_\tau \Psi,$$

$$\{\Psi, M_{ij}\} = (x_i \nabla_j - x_j \nabla_i) \Psi,$$

$$\{\Psi, P\} = \nabla \Psi,$$

$$\{\Psi, D\} = \left(2\tau \partial_\tau + x \cdot \nabla + \frac{d}{2}\right) \Psi,$$

$$\{\Psi, G\} = (\tau \nabla - i x) \Psi,$$

$$\{\Psi, K\} = \left(-\tau^2 \partial_\tau - \tau \left(\frac{d}{2} + x \cdot \nabla\right) + \frac{i}{2} x^2 \right) \Psi,$$

$$\{\Psi, N\} = -i \Psi.$$

(99)

**Appendix B: stationary solutions**

We have proved in section 2, that in conformally invariant non-linear Schrödinger models there are no non-trivial time-independent classical solutions. However, one can still look for stationary solutions, i.e. solutions for which the modulus of $\Psi$ is time-independent:

$$\partial_\tau (\Psi^* \Psi) = 0.$$  (100)

Such fields are solutions of the time-independent non-linear Schrödinger equation:

$$i \partial_\tau \Psi = E \Psi,$$

$$-\frac{1}{2} \Delta \Psi + \left(g^2 |\Psi|^{2(n-1)} - E\right) \Psi = 0.$$  (101)
In the free theory a complete set of solutions consists of the plane waves:

$$\Psi_k(x, \tau) = \frac{1}{(2\pi)^{d/2}} e^{i k \cdot x - i E_k \tau}, \quad E_k = \frac{1}{2} k^2. \quad (102)$$

These solutions are $\delta$-function normalized:

$$\int d^d x \Psi^*_q \Psi_k = \delta^d(k - q). \quad (103)$$

In general, also in the interacting theory there exist solutions

$$\Psi(x, \tau) = e^{-i E \tau} \Psi(x, 0). \quad (104)$$

We decompose the time-independent field in modulus and phase

$$\Psi(x, 0) = a(x)e^{i \theta(x)}. \quad (105)$$

The second equation (101) then implies two real equations

$$2 \nabla a \cdot \nabla \theta + a \Delta \theta = 0,$$

$$-\frac{1}{2} \Delta a + \frac{1}{2} a (\nabla \theta)^2 + g^2 a^{2(n-1)} = Ea. \quad (106)$$

For $a \neq 0$ the first equation is equivalent to

$$\nabla \cdot (a^2 \nabla \theta) = 0. \quad (107)$$

In particular, for constant $a$ there is a set of plane-wave solutions with

$$\theta(x) = i k \cdot x, \quad E_k = \frac{1}{2} k^2 + g^2 a^{2(n-1)}. \quad (108)$$

To normalize these solutions we take

$$a = \frac{1}{(2\pi)^{d/2}}, \quad E_k = \frac{1}{2} k^2 + \frac{g^2}{(2\pi)^d} + \frac{g^2}{(2\pi)^2}, \quad (109)$$

where we have used eq. (18). Thus the plane-wave solutions are

$$\Psi_k = \frac{1}{(2\pi)^{d/2}} e^{i k \cdot x - i E_k \tau}, \quad (110)$$

normalized according to the $\delta$-function norm

$$\int d^d x \Psi^*_q \Psi_k = \delta^d(k - q). \quad (111)$$

This result amounts to a completeness theorem for the plane-wave solutions (110) in the sense that any solution of the non-linear Schrödinger equation can be expanded as a linear combination of these plane waves, even in the absence of a superposition principle as holds in the linear theory.
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