CLASSIFICATION OF LOG ENRIQUES SURFACES
WITH $\delta = 2$

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Abstract. Log Enriques surfaces with $\delta = 2$ are classified.

Introduction

Let $S$ be a projective surface with klt singularities and with numerically trivial canonical class $K_S \equiv 0$. Then $S$ is called a log Enriques surface.

Let us consider the following invariant

$$\delta(S) = \# \{ E \mid E \text{ is exceptional divisor with discrepancy } a(E, 0) \leq \frac{6}{7}\}.$$

By theorem [7, 5.1] we have $0 \leq \delta(S) \leq 2$. In this paper the classification of such surfaces with $\delta = 2$ is given (see theorems 2.5 and 2.6).

The log Enriques surfaces often appear in many problems. Such of them are the following ones: the study of surface degeneration, the study of $K3$ surfaces (in particular, see the last section of §2), the study of Calabi–Yao varieties and the problem of inductive classification of strictly log canonical singularities.

In the latter problem the log Enriques surfaces can be realized as the exceptional divisors of purely log terminal blow-ups of three-dimensional strictly log canonical singularities. For example, in the case of hypersurface singularities we have $K_S \sim 0$ and $S$ is a $K3$ surface with Du Val singularities or $S$ is a surface obtained by the contraction of section of elliptic surface [3].

The method applied in this paper was developed by V.V. Shokurov in [7]. It works in any dimension for the classification of the varieties with Kodaira dimension $-\infty$ or 0, extremal contractions and singularities.

The number of different log Enriques surfaces with $\delta = 2$ is more then 1000 and hence (for example) to enumerate their minimal resolutions is difficult. This approach allows to obtain the short and complete classification.
The other methods of log Enriques surface study were given in [1], [8] and [9]. In many papers the automorphisms of $K3$ surfaces were investigated and hence some results about the structure of log Enriques surfaces were obtained (for example, see [4], [5]).

In this paper we can also see that the classification of the "models" of exceptional log Del Pezzo surfaces (see definitions of exceptionality and model in [7], see also theorem 1.3) implies the classification of log Enriques surface.

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1. Preliminary facts and results

All varieties are algebraic and are assumed to be defined over $\mathbb{C}$, the complex number field. The main definitions, terminology and notations used in the paper are given in [2], [6].

**Definition 1.1.** Let $X$ be a normal variety and let $D = S + B$ be a subboundary on $X$ such that $B$ and $S$ have no common components, $S$ is an effective integral divisor and $\langle B \rangle \leq 0$. Then we say that $K_X + D$ is $n$-complementary if there is a $\mathbb{Q}$-divisor $D^+$ such that

1. $n(K_X + D^+) \sim 0$ (in particular, $nD^+$ is an integral divisor);
2. $K_X + D^+$ is lc;
3. $nD^+ \geq nS + \langle n + 1 \rangle B$.

In this situation the $n$-complement of $K_X + D$ is $K_X + D^+$. The divisor $D^+$ is called $n$-complement too.

**Theorem 1.2.** [7, 3.1] Let $(X/Z \ni P, D)$ be a log surface of local type (i.e. $Z$ is not point), where $f : X \to Z \ni P$ is a contraction. Assume that $-(K_X + D)$ is $f$-nef and $K_X + D$ is lc. Then there exists an 1, 2, 3, 4 or 6-complement of $K_X + D$ near $f^{-1}(P)$.

**Theorem 1.3.** [7, §5] Let $S$ be a log Enriques surface with $\delta = 2$. Denote the exceptional curves with discrepancy $a(\tilde{C}_i, 0) \leq -\frac{6}{7}$ by $\tilde{C}_1$ and $\tilde{C}_2$. Let $f : \tilde{S} \to S$ be an extraction of $\tilde{C}_i$ (i.e. $f$ is a birational contraction such that $\text{Exc} f = \tilde{C}_1 \cup \tilde{C}_2$).

Then there exists a birational contraction $g : \tilde{S} \to \overline{S}$ with the following properties: $\rho(\overline{S}) = 1$, $g$ doesn't contract the curves $\tilde{C}_i$. Put
The main steps of proof. The divisor $K_{\tilde{S}} - a(\tilde{C}_1,0)\tilde{C}_1 - a(\tilde{C}_2,0)\tilde{C}_2$ is klt. Therefore by theorem 1.2 there exists an 1,2,3,4 or 6-complement near any point of $\tilde{S}$. Hence $K_{\tilde{S}} + \tilde{C}_1 + \tilde{C}_2$ is lc by the definition of complement. It is clear that $-(K_{\tilde{S}} + \tilde{C}_1 + \tilde{C}_2)$ is not nef.

Then there exists an extremal ray $R$ such that $(K_{\tilde{S}} + \tilde{C}_1 + \tilde{C}_2) \cdot R > 0$. The contraction of this ray is birational and the curves $\tilde{C}_i$ are not contracted. To prove these statements the theorem 1.2 is used. We get a required model $\overline{S}$ by repeating such procedure.

Then such models are classified. Taking into account the condition $-K_{\overline{S}} + a(\tilde{C}_1,0)C_1 + a(\tilde{C}_2,0)C_2 \equiv 0$ we obtain only two cases for $\overline{S}$.

Since klt singularities are rational then the curves $C_i$ must be rational. Hence in the case $(\text{A}_2^6)$ the curve $C_2$ must have an ordinary double point.

\begin{definition}
Let $S$ be a log Enriques surface. The minimal index of complementary $I$ is called a canonical index of $S$, i.e. $I = \min \{ n \in \mathbb{Z}_{>0} \mid nK_S \sim 0 \}$. It is known that $I \leq 21$ [1], [8].
\end{definition}

\begin{corollary}
Let $S$ be a log Enriques surface with $\delta = 2$. Then $I = 7$.
\end{corollary}

So, the problem of classification of log Enriques surface with $\delta = 2$ is to describe the following procedures. At first we consider the extraction $\tilde{S} \to \overline{S}$ such that every exceptional divisor $E$ has the discrepancy $a(E, \frac{6}{7}C_1 + \frac{6}{7}C_2) = 0$. Then we contract the proper transforms of $C_1$ and $C_2$. The number of such procedures is finite by the following easy fact.

Let $(X,D)$ be a klt pair. Then the number of divisors $E$ of the function field $\mathcal{K}(X)$ with $a(E,D) \leq 0$ is finite [6, lemma 3.1.9].

2. Classification of log Enriques surfaces with $\delta = 2$

\begin{proposition}
1. Let us consider the pair $(X, \frac{6}{7}C) \simeq (\mathbb{C}^2, \frac{6}{7} \{ x = 0 \})/\mathbb{Z}_2(1,1)$. Then the extraction of all exceptional curves with discrepancies 0
The proper transform of \( C \) is denoted by \( \overline{C} \). The numbers over vertexes denote the self-intersection indexes of corresponding curves. The numbers under vertexes denote the discrepancies with opposite sign of corresponding curves. The empty circle denotes the required exceptional curve with discrepancy 0. Its self-intersection index is always equal to \(-1\) (on a minimal resolution).

Note that \( \overline{C^2} = C^2 - \frac{7}{2} \).

2. Let us consider the pair \((X, \frac{6}{7}C) \simeq (\mathbb{C}^2, \frac{6}{7}\{x = 0\})/\mathbb{Z}_3(1,2)\). Then the extraction of all exceptional curves with discrepancies 0 is shown in the following figure.

The notations are as in the point (1). Note that \( \overline{C^2} = C^2 - \frac{14}{3} \).

3. Consider the pair \((X, \frac{6}{7}C_1 + \frac{6}{7}C_2) \simeq (\mathbb{C}^2, \frac{6}{7}\{xy = 0\})\). Then the extraction of all exceptional curves with discrepancies 0 is shown in the following figure.

The notations are as in the point (1). Notice that \( \overline{C_i^2} = C_i^2 - 6 \) if \( C_1 + C_2 \) is a (globally) reducible curve and \( \overline{C^2} = C^2 - 14 \) if \( C = C_1 + C_2 \) is an (globally) irreducible curve with an ordinary double point.

Proof. In the cases (1) and (2) at first we consider the minimal resolution of singularity \( f: Y \to X \). Then \( K_Y + \sum d_i D_i = f^*(K_X + \frac{6}{7}C) \), where \( D_i \) are the irreducible divisors. If \( D_i \cdot D_j \neq 0 \) for some \( i \neq j \) and \( d_i + d_j \geq 1 \) then let us blow-up their intersection point. After the finite number of such blow-ups we will extract all exceptional divisors with discrepancies 0. Note also that the exceptional divisor with discrepancy 0 appears if and only if \( d_i + d_j = 1 \). \( \square \)
This proposition trivially implies the next two corollaries.

**Corollary 2.2.** Let us consider the case \((A^{6}_{2})\) of theorem 1.3. Then the extraction of all exceptional curves with discrepancies 0 for the pair \((\mathcal{S}, \frac{6}{7}C_{1} + \frac{6}{7}C_{2})\) is shown in the following figure.

![Diagram](image)

The notations are as in proposition 2.1. Every curve with discrepancy 0 has a number between 1 and 15. Their self-intersection indexes (on a minimal resolution) are equal to \(-1\).

**Corollary 2.3.** Let us consider the case \((V^{2}_{2})\) of theorem 1.3. Then the extraction of all exceptional curves with discrepancies 0 for the pair \((\mathcal{S}, \frac{6}{7}C_{1} + \frac{6}{7}C_{2})\) is shown in the following figure.
The notations are as in proposition 2.1. Every curve with discrepancy 0 has a number between 1 and 15. Their self-intersection indexes (on a minimal resolution) are equal to $-1$.

**Classification of log Enriques surfaces in the case $(A_6 \mathbb{2})$.** Let $S$ be a log Enriques surface with $\delta = 2$. Assume that its model has the type $(A_6 \mathbb{2})$ (see theorem 1.3). Then $S$ can be constructed by the following way: at first we extract some set $\mathcal{T}$ of exceptional curves with discrepancy 0. After it we contract the proper transforms of $C_1$ and $C_2$. So, the classification of log Enriques surfaces with $\delta = 2$ is reduced to the description of sets $\mathcal{T}$.

**Definition 2.4.** Let us extract some set $\mathcal{T}$ of exceptional curves with discrepancy 0. Then $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, where $\mathcal{T}_1 \subset \{1, 2, 3\}$, $\mathcal{T}_2 \subset \{4, 5, 6, 7, 8, 9\}$ and $\mathcal{T}_3 \subset \{10, 11, 12, 13, 14, 15\}$. Every number denotes the corresponding exceptional curve with discrepancy 0 (see the figure in corollary 2.2). Let $T_i = |\mathcal{T}_i|$ be a number of elements of set $\mathcal{T}_i$ and $\overline{T}_i = \max\{t \mid t \in \mathcal{T}_i\}$, $\underline{T}_i = \min\{t \mid t \in \mathcal{T}_i\}$.
In the following first classification theorem the set $T_3$ is considered up to symmetry. For example, if $10 \notin T_3$ then $15 \notin T_3$.

**Theorem 2.5.** In the case $(A^6_3)$ the set $T$ must satisfy the condition (1) and be one of the following sets:

1. Always $T_1 + T_2 \geq 1$ and $T_3 \geq 1$. If $T_1 = 0$ then $T_2 \neq \{9\}$.
2. Let $T_2 = 0$ and $T_3 = 1$. Then $10 \in T_3$.
3. Let $T_2 \geq 1$ and $T_3 = 1$. If $\overline{T}_2 \leq 6$ then $10 \in T_3$. If $\overline{T}_2 = 7$ then $\{10, 11\} \cap T_3 \neq \emptyset$. If $\overline{T}_2 = 8$ or $9$ then $T_3 \neq \emptyset$ is arbitrary.
4. Let $T_2 = 0$ and $T_3 = 2$. Then either $10 \in T_3$, or $T_3 = \{11, 14\}$ and $\overline{T}_1 \neq \{3\}$.
5. Let $T_2 \geq 1$ and $T_3 = 2$. Then either $10 \in T_3$, or $T_3 = \{11, 14\}$, or $T_3 = \{11, 13\}$ and $\overline{T}_2 \geq 5$, or $T_3 = \{11, 12\}$ and $\{7, 8, 9\} \cap T_2 \neq \emptyset$, or $T_3 = \{12, 13\}$ and $\{8, 9\} \cap T_2 \neq \emptyset$.
6. Let $T_2 = 0$ and $T_3 = 3$. Then either $10 \in T_3$, or $T_3 = \{11, 12, 14\}$ and $\overline{T}_1 \neq \{3\}$.
7. Let $T_2 \geq 1$ and $T_3 = 3$. Then $10 \in T_3$, or $T_3 = \{11, 12, 14\}$, or $T_3 = \{11, 12, 13\}$ and $\overline{T}_2 \geq 5$.
8. Let $T_2 = 0$ and $T_3 = 4$. Then either $10 \in T_3$, or $T_3 = \{11, 12, 13, 14\}$ and $\overline{T}_1 \neq \{3\}$.
9. If either $T_2 \geq 1$ and $T_3 \geq 4$, or $T_3 \geq 5$ then $T$ is arbitrary (of course, taking into account the condition (1)).

**Proof.** Let $f : Y \to X$ be an extraction of some set $T$. The proper transforms of $C_i$ are denoted by $\overline{C}_i$. We have $C^1_2 = \frac{1}{6}$ and $C^2_2 = 6$. The obvious requirement for the set $T$ is $\overline{C}^2_1 < 0$. Therefore $T_1 + T_2 \geq 1$ and $T_3 \geq 1$. Hence, always $\overline{C}^2_1 < 0$, except one case $T_1 = 0$ and $T_2 = \{9\}$. For this case $\overline{C}^2_1 = 0$.

The remaining cases are written taking into account the requirements $\overline{C}^2_2 < 0$. We can contract $\overline{C}_1$ and $\overline{C}_2$ except one case, which appears in the conditions 4, 6, 8.

For this case $T_2 = 0$, $2 \leq T_3 \leq 4$, $T_1 = \{3\}$, $10 \notin T_3 \supset \{11, 14\}$. Consider the minimal resolution. Then our configuration is illustrated in the following figure.

Indeed, the determinant of intersection matrix is equal to 0, although $\overline{C}^2_1 = \overline{C}^2_2 = -1$. □
Classification of log Enriques surfaces in the case $I_2^2$. We have the same notations as in definition 2.4. The corresponding figure is given in corollary 2.3. The case $T_2 = 0, T_3 ≥ 1$ is symmetric to the case $T_2 ≥ 1, T_3 = 0$ and therefore it isn’t considered.

**Theorem 2.6.** In the case $(I_2^2)$ the set $T$ must satisfy the condition (1) and be one of the following sets:

1. $T_2 + T_3 ≥ 1$.
2. Let $T_3 = 0$ and $T_1 = 0$. Then $T_2 ≤ 5$ and $9 ∈ T_2$.
3. Let $T_3 = 0$ and $T_1 = 1$. Then either $T_1 = \{1\}$ and $9 ∈ T_2$, or $T_1 = \{2\}$ and $T_2 ≤ 5$, or $T_1 = \{3\}$ and $4 ∈ T_2 \cap \{8, 9\} ≠ \emptyset$, or $T_1 = \{3\}$ and $\{5, 9\} ⊂ T_2$.
4. Let $T_3 = 0$ and $T_1 = 2$. Then either $T_1 = \{1, 2\}$, or $T_1 = \{1, 3\}$ and $T_2 \cap \{8, 9\}$, or $T_1 = \{2, 3\}$ and $T_2 ≤ 5$.
5. Let $T_3 = 0$ and $T_1 = 3$. Then $T$ is arbitrary.
6. Let $T_2, T_3 ≥ 1$ and $T_1 = 0$. Then the set $T$ must satisfy the following two conditions $Y_1$ and $Y_2$.

   - **Condition $Y_1$.** $(T_2 ≤ 5, T_3$ is arbitrary), or $(T_2 = 6, T_3 ≤ 14)$, or $(7 ≤ T_2 ≤ 8, T_3 ≤ 12)$, or $(T_2 = 9, T_3 ≤ 11)$.
   - **Condition $Y_2$.** $(T_2 = 4, T_3 = 15)$, or $(5 ≤ T_2 ≤ 7, T_3 ≥ 14)$, or $(T_2 = 8, T_3 ≥ 11)$, or $(T_2 = 9, T_3$ is arbitrary).

7. Let $T_2, T_3 ≥ 1$ and $T_1 = 1$. Then the set $T$ must satisfy the condition $Y_2$.
8. Let $T_2, T_3 ≥ 1$ and $2 ∈ T_1$. Then the set $T$ must satisfy the condition $Y_1$.
9. Let $T_2, T_3 ≥ 1$ and $T_1 = \{3\}$. Then the set $T$ must satisfy the condition $Y_1$ and the next condition $Y_3$.

   - **Condition $Y_3$.** $(T_2 = 4, T_3 ≥ 14)$, or $(5 ≤ T_2 ≤ 6, T_3 ≥ 13)$, or $(T_2 = 7, T_3 ≥ 11)$, or $(T_2 = 8, T_3$ is arbitrary).
10. Let $T_2, T_3 ≥ 1$ and $\{1, 2\} ⊂ T_1$. Then $T$ is arbitrary.
11. Let $T_2, T_3 ≥ 1$ and $T_1 = \{1, 3\}$. Then the set $T$ must satisfy the condition $Y_3$.

**Proof.** Let $f : Y → X$ be an extraction of some set $T$. The proper transforms of $C_i$ are denoted by $\tilde{C}_i$. We have $C_1^2 = \frac{3}{2}$ and $C_2^2 = \frac{8}{3}$. If $T_2 = T_3 = 0$ then $\tilde{C}_1^2 < 0$ and $\tilde{C}_2^2 < 0$ in the case $\{1, 2\} ⊂ T_1$ only. Consider the minimal resolution. Then our configuration is illustrated in the following figure.

```
\begin{center}
\begin{tikzpicture}
  \node at (0,0) [circle,fill,inner sep=2pt] {};
  \node at (0,-1) [circle,fill,inner sep=2pt] {};
  \draw (-0.3,-0.3) -- (0.3,0.3);
  \draw (0.3,-0.3) -- (-0.3,0.3);
\end{tikzpicture}
\end{center}
```
The determinant of intersection matrix is equal to 0 and therefore always \( T_2 + T_3 \geq 1 \). The rest cases are written taking into account the requirements \( \tilde{C}_1^2 < 0 \) and \( \tilde{C}_2^2 < 0 \). We can contract \( \tilde{C}_1 \) and \( \tilde{C}_2 \) except one case: \( T_3 = 0, T_1 = \{3\}, T_2 = 5, \tilde{T}_2 = 8 \). Consider the minimal resolution. Then our configuration is illustrated in the figure of theorem 2.5 proof. It appears in the case (3).

By theorems 2.5 and 2.6 we get the following corollary.

**Corollary 2.7.** Let \( S \) be a log Enriques surface with \( \delta = 2 \). Then \( 1 \leq \rho(S) \leq 14 \).

**Log Enriques surfaces and K3 surfaces.** Let \( S \) be a log Enriques surface with \( \delta = 2 \). Consider its canonical cover \( \varphi: \hat{S} = \text{Spec} \mathcal{O}_S(\oplus_{i=1}^I\mathcal{O}_S(-iK_S)) \to S \), where \( I \) is an index of \( S \). Since \( I = 7 \) (see corollary 1.5) then \( \hat{S} \) is a K3 surface with at worst Du Val singularities [1], [8] and

1. \( \varphi \) is cyclic Galous cover of degree 7, which is etale over \( S \setminus \text{Sing} S \).
2. There exists a generator \( g \) of \( \text{Gal}(\hat{S}/S) \cong \mathbb{Z}_7 \) such that \( g^*\omega_{\hat{S}} = \varepsilon_7\omega_{\hat{S}} \), where \( \varepsilon_7 = \exp(2\pi\sqrt{-1}/7) \) is a primitive root and \( \omega_{\hat{S}} \) is a nowhere vanishing regular 2-form on \( \hat{S} \).

Let \( \Delta(S) \) be an exceptional set of \( \chi \), where \( \chi \) is a minimal resolution of \( \hat{S} \). Then \( \Delta(S) \) is a disconnected sum of divisors of Dynkin’s type \( A_i, D_j, E_k \). So \( \Delta(S) = (\oplus A_\alpha) \oplus (\oplus D_\beta) \oplus (\oplus E_\gamma) \). Let us define \( \text{rank} \Delta(S) = \sum \alpha + \sum \beta + \sum \gamma \).

**Theorem 2.8.** Let \( S \) be a log Enriques surface with \( \delta = 2 \). Then \( \text{rank} \Delta(S) + \rho(S) = 16 \).

**Proof.** Let \( T = \{1, 2, \ldots, 15\} \). The surface corresponding to this set is denoted by \( S' \). In the cases \( (A_6^3) \) and \( (I_2^2) \) the set \( \Delta(S') \) has the same type \( A_1 \oplus A_1 \), i.e. \( \text{rank} \Delta(S') + \rho(S') = 2 + 14 = 16 \). Every surface \( S \) can be obtained from \( S' \) by the contractions of some curves in \( T \). If we contract any curve in \( T \) then \( \text{rank} \Delta \) is increased by 1. Therefore the required equality is reserved.

**Example 2.9.** Let \( T = \{8, 12\} \) then \( \rho(S) = 1 \) and \( \Delta(S) = E_7 \oplus E_8 \) in both cases \( (A_6^3) \) and \( (I_2^2) \).

**Remark 2.10.** Other approach to the classification was given in [9]. In particular, see [9, §5] in the case of surfaces with index 7.

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