T-Stable-extending Modules and Strongly T-stable Extending Modules

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Abstract
In this paper we introduce the notions of t-stable extending and strongly t-stable extending modules. We investigate properties and characterizations of each of these concepts. It is shown that a direct sum of t-stable extending modules is t-stable extending while with certain conditions a direct sum of strongly t-stable extending is strongly t-stable extending. Also, it is proved that under certain condition, a stable submodule of t-stable extending (strongly t-stable extending) inherits the property.

Keywords: extending modules, S-extending module, t-stable extending modules, and strongly t-stable extending modules.

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In the previous paper, we defined the notions of t-stable extending and strongly t-stable extending modules. We explore properties and characterizations of each of these concepts. It is shown that a direct sum of t-stable extending modules is t-stable extending while with certain conditions a direct sum of strongly t-stable extending is strongly t-stable extending. Also, it is proved that under certain condition, a stable submodule of t-stable extending (strongly t-stable extending) inherits the property.

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Introduction
Let R be a ring with unity and M be a right R-module. A submodule \( N \) of \( M \) is called essential in \( M \) \( (N \leq_{ess} M) \) if \( N \cap K = \{0\} \), \( K \leq M \) implies \( K = \{0\} \). A submodule \( N \) of \( M \) is called closed in \( M \) if it has no proper essential extension in \( M \), that means if \( N \leq_{ess} W \), where \( W \leq M \), then \( N = W \) [1, 2]. It is known that for any submodule \( N \) of \( M \), there exists a submodule \( H \) of \( M \), such that \( N \leq_{ess} H \), hence \( H \) is a closed submodule of \( M \), \( H \) is called a closure of \( N \) [3]. Asgari [4] introduced the notion of t-essential submodule, where a submodule \( N \) of \( M \) is called t-essential (denoted by \( N \leq_{tes} M \)) if whenever \( W \leq M \), \( N \cap W \leq Z_2(M) \) implies \( W \leq Z_2(M) \), where \( Z_2(M) \) is the second

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singular submodule defined by $Z\left(\frac{M}{Z(M)}\right) = Z_2(M)Z(M)$ [1], where $Z(M) = \{x \in M: xl = 0\}$ for some essential ideal of $R$. Equivalently, $Z(M) = \{x \in M: ann(x) \leq_{ess} R\}$ and $ann(x) = \{r \in R: xr = 0\}$. $M$ is called singular (nonsingular) if $Z(M) = M(Z(M) = 0)$. Note that $Z_2(M) = \{x \in M: xl = 0\}$ for some t-essential ideal $I$ of $R$. $M$ is called $Z_2$-torsion if $Z_2(M) = M$. Asgari introduced the concept of t-closed submodule where a submodule $N$ is called t-closed ($\leq_{tc} M$) if $N$ has no proper t-essential extension in $M$ [4]. It is clear that every t-closed submodule is closed, but the converse is not true. However, under the class of nonsingular, the two concepts are equivalent. Asgari [5] stated that for any submodule $N$ of $M$, there exists a t-closed submodule $H$ of $M$ such that $N \subseteq_{tes} H$. $H$ is called a t-closure of $N$. A module $M$ is called extending if for every submodule $N$ of $M$ there exists a direct summand $W(W \leq M)$ such that $N \leq W$ [6]. Equivalently, $M$ is an extending module if every closed submodule is a direct summand. As a generalization of extending modules, Asgari [4] introduced the concept of t-extending module, where a module $M$ is t-extending if every t-closed submodule is a direct summand. Equivalently, $M$ is t-extending if every submodule of $M$ is t-essential in a direct summand. The notion of a strongly extending module is introduced in another study [7], which is a subclass of the class of extending module, where an $R$-module is called strongly extending if every stable submodule of $M$ is essential in a fully invariant direct summand of $M$, and a submodule $N$ of $M$ is called fully invariant if for each $f \in End(M)$, $f(N) \leq N$ [8]. A submodule $N$ of an $R$-module $M$ is called stable if for each $R$-homomorphism $f: N \rightarrow M, f(N) \leq N$ [9]. It is clear that every stable submodule is fully invariant but not conversely. An $R$-module $M$ is fully stable if every submodule of $M$ is stable [9]. An $R$-module $M$ is called strongly t-extending if every submodule of $M$ is t-essential in a stable direct summand. Equivalently, $M$ is strongly t-extending if every t-closed submodule is a fully invariant direct summand [10]. Saad [7] introduced the stable extending (S-extending) modules as a generalization of FI-extending modules. An $R$-module $M$ is called stable extending (S-extending) if every stable submodule of $M$ is essential in a direct summand of $M$. A ring $R$ is left (right) S-extending if $R$ is S-extending left (right) $R$-module and $M$ is called FI-extending if every fully invariant submodule of $M$ is essential in a direct summand of $M$ [11].

In this paper, we introduce the concepts of t-stable extending and strongly t-stable extending modules. The class of t-stable extending modules contains the class of stable extending, and the class of strongly t-stable contains the class of t-stable extending and it is contained in the class of strongly t-extending.

In section two we study t-stable extending modules and their relationships with other related modules. Among other results in this section, we prove that an $R$-module $M$ is a t-stable-extending $R$-module if and only if for each stable submodule $A$ of $M$, there is a decomposition $M = M_1 \oplus M_2$ such that $A \leq M_1$ and $A + M_2 \leq_{tes} M$. An $R$-module $M$ is t-stable extending if and only if for each stable submodule $K$ of $M$, there exist $e = e^2 \in End(E(M))$ such that $K \leq_{tes} e^2(E(M))$ and $e(M) \leq M$ where $E(M)$ is the injective hull of $M$. Let $M$ be a stable injective relative to a stable submodule $X$. If $M$ is t-stable extending, then so is $X$.

In section three, we study strongly t-stable extending modules. Many properties are given.

2. T-Stable-extending Modules

In this section we introduce the concept of t-stable extending modules which is a generalization of $S$-extending modules.

First we give the following definitions.

**Definition 2.1:** An $R$-module $M$ is called t-stable extending if every stable submodule of $M$ is t-essential in a direct summand. A ring $R$ is called right t-stable extending if $R$ is a right t-stable extending $R$-module.

Recall that an $R$-module is t-uniform if every submodule of $M$ is t-essential in $M$ [12]. As a generalization of t-uniform module, we present the following concept.

**Definition 2.2:** An $R$-module is called stable-t uniform if every stable submodule of $M$ is t-essential in $M$.

**Remarks and Examples 2.3:**

1. It is clear that every $S$-extending module (or t-extending module) is t-stable extending, for example:
(i) For arbitrary $Z$-module $M$, $E(M)\oplus Z_2\oplus Z_8$ is t-extending [4], so it is t-stable extending. Also $Z_2\oplus Q$ as $Z$-module is S-extending, so it is t-stable extending.

Recall that an $R$-module $M$ is called t-continuous if $M$ satisfies the following: $M$ is t-extending, and every submodule of $M$ which contains $Z_2(M)$ and isomorphic to direct summand of $M$ is itself a direct summand [3]. Hence, every t-continuous module is t-stable extending. Hence, we can give the following examples:

(I) By [6, Example 2.6(2)], Let $R$ be a $Z_2$-torsion ring (e.g $R = \frac{Z}{p^2Z}$, for a prime number $P$) and set $T = (R_R, R_R, T, T_2)$. $T_2$ t-continuous T-module. It follows that $T_2$ is a t-stable extending module. However, $T_2$ is not stable extending. Hence $T_2$ is not stable extending.

(II) Let $R$ be a ring and $M$ be an $R$-module and $I \leq \text{ess} R$. The $R$-module $E(M)\oplus Z_2$ is t-continuous [6, Example 2.6(1)], so it is t-stable extending. In particular if $M = Z_p$ as $Z$-module. Then $Z_p\oplus \frac{Z}{<4>} \cong Z_p\oplus Z_4$ is t-stable.

(2) Let $M$ be a nonsingular $R$-module. Then $M$ is S-extending if and only if $M$ is t-stable extending.

**Proof:** since $M$ is non-singular, then the two concepts essential and t-essential coincide [5]. Hence the two concepts, S-extending and t-stable extending, are equivalent.

(3) If $M$ is a singular module then $M$ is t-stable extending.

**Proof:** since $M$ is a singular module then $Z_2(M)=M$ and for every submodule $N$ of $M, N+ Z_2 \cong N+M=MS_{\text{ess}} M$, hence $N \leq \text{tes} M$ by [5,Prop1.1]. But $M$ is a direct summand of $M$, so every stable submodule of $M$ is t-essential in a direct summand. Thus $M$ is t-stable extending.

(4) Every FI-t-extending is t-stable-extending where $M$ is FI-t-extending if every fully invariant is t-essential in a direct summand.

**Proof:** Let $N$ be a stable submodule of $M$. Then $N$ is fully invariant, hence $N$ is t-essential in a direct summand.

(5) The converse of (4) holds if $M$ is FI-quasi-injective, where an $R$-module $M$ is called FI-quasi-injective if for each fully invariant submodule $N$ of $M$, each $R$-homomorphism $f: N \rightarrow M$ can be extended to an $R$-endomorphism $g: M \rightarrow M$ [7].

**Proof:** Let $N$ be a fully invariant submodule of $M$. By [7, Proposition 3.1.19] $N$ is stable. Hence by t-stable extending property of $M$, $N$ is t-essential in direct summand. Thus $M$ is a FI-t-extending.

(6) t-stable extending module need not be extending, for example the $Z$-module $Z_6 \oplus Z_2$ is not extending but it is $S$-extending by [7, Remarks and Examples 3.1.3(3)] hence it is t-stable extending.

(7) Every stable t-uniform (hence every t-uniform) is t-stable extending.

**Proof:** Let $N$ be a stable submodule of $M$. Hence $N \leq \text{tes} M$. But $M \leq \oplus M$, so $N$ is t-essential in a direct summand.

Recall that an $R$-module $M$ is called an S-indecomposable if $(0)$, $M$ are the only stable direct summand. $M$ is S-extending and S-indecomposable if $M$ is S-uniform. "An $R$-module $M$ is called stable uniform (shortly, S-uniform) if every stable submodule of $M$ is essential in $M"$ [7]. However we have:

**Proposition 2.4:** If $M$ is t-stable extending and indecomposable, then $M$ is stable t-uniform.

**Proof:** Let $N$ be a stable submodule in $M$. Then $N \leq \text{tes} W$ for some $W \leq \oplus M$. Since $M$ is indecomposable, $W = M$. Thus $N \leq \text{tes} M$ and so $M$ is a t-stable uniform.

Note that a stable t-uniform module does not imply indecomposable, for example $Z_6$ as $Z$-module is stable t-uniform, but $Z_6$ is not indecomposable. Also, $Z_6$ is not S-indecomposable.

**Proposition 2.5:** Let $M$ be an R-module. If $M$ is t-stable extending, then every stable t-closed submodule is a direct summand and the converse holds if every t-closure of stable submodule is stable.

**Proof:** Let $N$ be a stable t-closed submodule. Since $M$ is t-stable extending, $N \leq \text{tes} W$ for some $W \leq \oplus M$. Hence $N = W \leq \oplus M$, since $N$ is a t-closed. Now if $N$ is a stable submodule of $M$, then $N \leq \text{tes} W$, where $W$ is a t-closure of $N$ [5, Lemma 2.3]. By hypothesis, $W$ is stable, and so $W$ is stable t-closed, which implies $W \leq \oplus M$. Thus $N$ is t-essential in a direct summand and $M$ is t-stable extending.
**Proposition 2.6:** Let $M$ be an $R$-module which satisfies that the t-closure of any submodule is stable. Then $M$ is $t$-stable extending if and only if $M$ is $t$-extending.

**Proof:** Assume $M$ is $t$-stable extending. Let $K$ be a stable submodule of $M$. Then there exists $D \leq \operatorname{tes}M$ of $M$ such that $K \leq \operatorname{tes}D$ and so there is $H \leq M$ such that $D = D \oplus H$. Hence $E(M) = E(D) \oplus E(H)$. Let $e : E(M) \to E(D)$ be the projection endomorphism from $E(M)$ onto $E(D)$. Clearly $e^2 = e$ (it is idempotent). Thus we have $e(M) \leq (D \oplus H)$. Also, $K \leq \operatorname{tes}D \leq \operatorname{ess}E(D)$ implies $K \leq \operatorname{tes}E(D) = e(E(M))$.

Let $K$ be a stable submodule of $M$. By hypothesis, there exists $e \in \operatorname{End}(E(M))$, $e^2 = e$ such that $K \leq \operatorname{tes}e(E(M))$ and $e(M) \leq M$. Since $M \leq \operatorname{tes}M$, then $K \cap M \leq \operatorname{tes}e(E(M)) \cap M = e(M)$. It is easy to see that $e(E(M)) \cap M = e(M)$. Also, since $K \cap M = K$, hence $K \leq \operatorname{tes}e(M)$. But $e(M) \leq \operatorname{tes}M$ by Lemma 1.1.22, so $K$ is $t$-essential in stable direct summand. Thus $M$ is $t$-stable extending.

**Theorem 2.9:** An $R$-module $M$ is $t$-stable extending if and only if for each stable submodule $A$ of $M$, there is a decomposition $M = M_1 \oplus M_2$ such that $A \leq M_1$ and $A + M_2 \leq M$.

**Proof:** Suppose $M$ is $t$-stable extending. Let $A$ be a stable submodule of $M$. Then $A \leq \operatorname{tes}M_1 \leq \operatorname{tes}M$, hence $M_1 \oplus M_2 = M$ for some $M_2 \leq M$. It follows that $A \oplus M_2 \leq \operatorname{tes}M_1 \oplus M_2 = M$ (since $A \leq \operatorname{tes}M_1$ and $M_2 \leq \operatorname{tes}M_2$ [5, Corollary 1.3]).

Let $A$ be a stable submodule of $M$. By hypothesis, there is a decomposition $M = M_1 \oplus M_2$ with $A \leq M_1$ and $A + M_2 \leq M = M_1 \oplus M_2$. It follows that $A \leq \operatorname{tes}M_1$ by [5, Corollary 1.3]. Thus $A \leq \operatorname{tes}M_1 \leq \operatorname{tes}M$. Therefore $M$ is $t$-stable extending.

The following are another characterization of $t$-stable extending modules.

**Theorem 2.10:** An $R$-module $M$ is $t$-stable extending if and only if each stable submodule $K$ of $M$, there exists $e = e^2 \in \operatorname{End}(E(M))$ such that $K \leq \operatorname{tes}e(E(M))$ and $e(M) \leq M$ where $E(M)$ is the injective hull of $M$.

**Proof:** Assume $M$ is $t$-stable extending. Let $K$ be a stable submodule of $M$. Then there exists $D \leq \operatorname{tes}M$ of $M$ such that $K \leq \operatorname{tes}D$ and so there is $H \leq M$ such that $D = D \oplus H$. Hence $E(M) = E(D) \oplus E(H)$. Let $e : E(M) \to E(D)$ be the projection endomorphism from $E(M)$ onto $E(D)$. Clearly $e^2 = e$ (it is idempotent). Thus we have $e(M) \leq (D \oplus H)$. Also, $K \leq \operatorname{tes}D \leq \operatorname{ess}E(D)$ implies $K \leq \operatorname{tes}E(D) = e(E(M))$.

Let $K$ be a stable submodule of $M$. By hypothesis, there exists $e \in \operatorname{End}(E(M))$, $e^2 = e$ such that $K \leq \operatorname{tes}e(E(M))$ and $e(M) \leq M$. Since $M \leq \operatorname{tes}M$, then $K \cap M \leq \operatorname{tes}e(E(M)) \cap M = e(M)$. It is easy to see that $e(E(M)) \cap M = e(M)$. Also, since $K \cap M = K$, hence $K \leq \operatorname{tes}e(M)$. But $e(M) \leq \operatorname{tes}M$ by Lemma 1.1.22, so $K$ is $t$-essential in stable direct summand. Thus $M$ is $t$-stable extending.

**Lemma 2.11:** Let $M = \bigoplus_{i \in I} M_i$. Let $N$ be a stable submodule of $M$. Then $N = \bigoplus_{i \in I} (N \cap M_i)$ where $N \cap M_i$ is stable in $M_i$, $\forall i \in I$. 

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**Corollary 2.7:** Let $M$ be a fully stable $R$-module. Then the following statements are equivalent:
1. $M$ is a $t$-stable extending module;
2. $M$ is a $t$-extending module;
3. $M$ is a strongly $t$-extending module.

**Proof:** Since $M$ is a fully stable $R$-module, and the t-closure of any submodule of $M$ is stable. Then (1) $\implies$ (2) follows by Proposition 2.6.

(1)$\implies$ (3) Let $N \leq M$. Since $M$ is fully stable, then $N$ is stable. Hence $N$ is $t$-essential in a direct summand $W$. But $W$ is stable in $M$. Then $N$ is $t$-essential in a stable direct summand and so $M$ is strongly $t$-extending.

(3)$\implies$ (2) obvious.

**Proposition 2.8:** Let $M$ be an $R$-module that satisfies that the t-closure of any submodule is stable. Then the following statements are equivalent:
1. $M$ is a $t$-stable extending module;
2. Every stable t-closed submodule of $M$ is a direct summand;
3. Every stable submodule is $t$-essential in stable direct summand.

**Proof:** (1)$\implies$ (2) Let $N$ be a stable t-closed submodule. Condition (1) implies $N$ is $t$-essential in a direct summand $W$. Hence $N = W \leq M$ since $N$ is a t-closed.

(2)$\implies$ (3) Let $N$ be a stable submodule in $M$. Then $N$ has a t-closure $W$; such that $N \leq \operatorname{tes}W$ and $W$ is a t-closed. But $W$ is stable by hypothesis, so that $W$ is t-closed stable. Then by condition (2) $W \leq M$ and hence $N$ is $t$-essential in a stable direct summand.

(3)$\implies$ (1) clear.

The following are characterizations of the $t$-stable extending modules.
Proof: Let \( W \) be a stable submodule. Then \( W = \bigoplus_{i \in I} (W \cap M_i) \) by [9, Proposition 4.5] we claim that \( N \cap M_i \) is stable in \( M_i \) for each \( i \in I \). To prove this, let \( g: W \cap M_i \rightarrow M_i \) be any \( R \)-homomorphism. Then \( g(W \cap M_i) \subseteq M_i \). Consider the following \( W = \bigoplus_{i \in I} (W \cap M_i) \subseteq W \cap M_i \), where \( \rho \) is the natural projection and \( i \) is the inclusion mapping. Then \( (i \circ g \circ \rho)(W) \subseteq W \) (since \( W \) is stable in \( M \)). But \( (i \circ g \circ \rho)(W) = i \circ g(W \cap M_i) = (g(W \cap M_i) = g(W \cap M_i) \subseteq M_i \). Thus \( (W \cap M_i)(W) \subseteq W \). From above \( g(W \cap M_i) \subseteq M_i \), so we get \( g(W \cap M_i) \subseteq W \cap M_i \) and \( W \cap M_i \) is a stable submodule of \( M_i \), for each \( i \in I \).

Theorem 2.12: A direct sum of \( t \)-stable extending modules is \( t \)-stable extending.

Proof: Suppose that \( M = \bigoplus_{i \in I} M_i \), \( M_i \) is \( t \)-stable extending for each \( i \in I \). Let \( W \) be a stable submodule of \( M \). Then \( W = \bigoplus_{i \in I} (W \cap M_i) \) and \( W \cap M_i \) is stable in \( M_i \) for each \( i \in I \) by Lemma 2.11 and so by the \( t \)-stable extending property of \( M_i \), \( W \cap M_i \) is \( t \)-essential in a direct summand \( N_i \) of \( M_i \) for each \( i \in I \). Then \( \bigoplus_{i \in I} (W \cap M_i) \leq_{\text{tes}} \bigoplus_{i \in I} N_i \) by [5, Corollary 1.3]. Put \( N = \bigoplus_{i \in I} N_i \), so \( N \leq_{\Theta} M \). Thus \( N \leq_{\text{tes}} N \leq_{\Theta} M \) and \( \square \) is \( t \)-stable extending.

Note that any direct sum of extending is \( S \)-extending [7, Corollary 3.2.2], hence by Remarks and Examples 2.4(2), it is \( t \)-stable extending.

By applying Theorem 2.12, each of \( Z_p \otimes Z, Z_p \otimes Z \) (for each prime number \( P \)) \( Z \otimes Z, Z \otimes Z, Z \otimes Z \) ... as \( Z \)-module is \( t \)-stable extending. Not that \( Z_p \otimes Z \) and \( Z \otimes Z \) are not extending. Note that by [7, Corollary 3.2.4] every finitely generated \( Z \)-module is \( S \)-extending, hence it is \( t \)-stable extending.

Proposition 2.13: Let \( M \) be an \( R \)-module which satisfies that the \( t \)-closure of any submodule is stable. If \( M \) is \( t \)-stable extending, then every \( t \)-essential summand is \( t \)-stable extending.

Proof: Let \( N \leq_{\Theta} M \). Since \( M \) is \( t \)-stable extending, then \( M \) is \( t \)-extending by Proposition 2.6. Hence \( N \) is \( t \)-extending by [4, Proposition 2.14(1)]. It follows that \( N \) is \( F \)-extending and hence by Remarks and Examples 2.3(3), \( N \) is \( t \)-stable extending.

Corollary 2.14: Let \( M \) be a fully stable \( R \)-module. If \( M \) is \( t \)-stable extending, then every \( t \)-essential summand is \( t \)-stable extending.

Recall that an \( R \)-module \( M \) has the summand intersection property (SIP) if the intersection of two direct summands of \( M \) is a direct summand [13]. Since \( S \)-extending and \( t \)-stable extending are equivalent in the class of nonsingular modules, thus we have every direct summand of \( t \)-stable extending module \( M \) (where \( M \) is nonsingular with SIP) is \( t \)-stable extending module. Also, we have by [2, Corollary 3.2.7, Corollary 3.2.8 and Corollary 3.2.9] the following:

1. Let \( M \) be a nonsingular \( SS \)-module (that is every direct summand is stable). If \( M \) is \( t \)-stable extending, then every \( t \)-essential summand is \( t \)-stable extending.

2. Every direct summand right ideal of a nonsingular \( t \)-stable extending commutative ring is \( t \)-stable extending.

3. Every direct summand of nonsingular cyclic \( Z \)-module is \( t \)-stable extending.

An \( R \)-module \( M \) is called stable-injective relative to \( X \) (simply, \( S \)-X-injective) if for each stable submodule \( A \) of \( X \), each \( R \)-homomorphism \( f: A \rightarrow M \) can be extended to an \( R \)-homomorphism \( g: X \rightarrow M \).” [7, Definition 3.2.10].

By using the procedure of the proof of Theorem 2.14 [7], we have the following Lemma.

Lemma 2.15: Let \( M \) be a stable injective module relative to a stable submodule \( X \) of \( M \). If \( A \subseteq X \) such that \( A \) is a stable in \( X \), then \( A \) is stable in \( M \).

Proof: Let \( f \in Hom(A, M) \). Since \( M \) is stable injective relative to \( X \), there exists an \( R \)-homomorphism \( g: X \rightarrow M \) such that \( g \circ i = f \) where \( i \) is the inclusion mapping from \( A \) into \( X \). It follows that \( g(X) \subseteq X \), since \( X \) is stable in \( M \). So \( g \circ i(A) = g(A) \subseteq X \); that is \( g|_A : A \rightarrow X \). But \( A \) is stable in \( X \), so that \( g|_A (A) \subseteq A \). Thus \( f(A) \subseteq A \) and \( A \) is stable in \( M \).

Proposition 2.16: Let \( M \) be a stable injective relative to a stable submodule \( X \). If \( M \) is \( t \)-stable extending, then so is \( X \).

Proof: To prove \( X \) is \( t \)-stable. Let \( A \) be a stable submodule of \( X \). By Lemma 2.15, \( A \) is stable in \( M \). Since \( M \) is \( t \)-stable extending, there exists \( D \leq_{\Theta} M \) such that \( A \leq_{\text{tes}} D \) it follows that \( M = D \oplus D' \) for some \( D' \leq \Theta M \) and \( A = X \cap \leq_{\text{tes}} D \cap D' \leq_{\Theta} M \) by (5, Corollary 1.3).

3. Strongly \( t \)-stable extending modules
In this section, we extend the notion of t-stable extending modules into strongly t-stable extending modules. We study these classes of modules and their relations with some related concepts.

**Definition 3.1:** An $R$-module $M$ is called strongly t-stable extending if each stable submodule $N$ of $M$. $N$ is t-essential in a stable direct summand.

**Remarks and Examples 3.2:**

(1) It is clear that every strongly t-stable extending is t-stable extending.

(2) Every strongly t-extending (hence every $\mathbb{Z}_2$- torsion) module is strongly t-stable extending. In particular, each of $\mathbb{Z}$-module $M = \mathbb{Z}_n \oplus \mathbb{Z}$ where $n$ is a positive integer is strongly t-extending (see [10, Example 3.3]. Thus $M$ is strongly t-stable extending.

(3) The converse of (2) is not true as the following example shows: Let $M$ be the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$. Let $N$ be a stable submodule of $M$. Then $N = (\mathbb{Z} \cap \mathbb{Z}) \oplus (\mathbb{Z} \cap \mathbb{Z})$, where $\mathbb{Z} \cap \mathbb{Z}$ is stable in $\mathbb{Z}$ by Lemma 2.11. Since the only stable submodules of $\mathbb{Z}$ are $\mathbb{Z}$, $(0)$, then $N = \mathbb{Z} \oplus \mathbb{Z}$ or $N = (0) \oplus (0)$ and hence $N \leq_{tes} N \leq^{\oplus} M$. Thus $M$ is a strongly t-stable extending module. On the other hand, $N = \mathbb{Z} \oplus (0)$ is t-closed(closed) and $N$ is not a fully invariant direct summand, since there exists $f: M \mapsto M$, such that $f(x, y) = (y, x)$ for each $(x, y) \in M$ and so $f(N) = f(\mathbb{Z} \oplus (0)) = (0) \oplus \mathbb{Z} \not\leq N$.

(4) Recall that an $R$-module $M$ is called weak duo if every direct summand is fully invariant [14]. Let $M$ be a week duo. Then $M$ is strongly t-stable extending if and only if $M$ is a t-stable extending module.

**Proof:** $\Rightarrow$ It follows by (1)

$\Leftarrow$ Let $N$ be a stable submodule of $M$. Then $N \leq_{tes} W \leq^{\oplus} M$. Since $M$ is weak duo, $W$ is a fully invariant in $M$ and then by [7, Lemma 2.1.6] $W$ is stable. Thus $M$ is strongly t-stable extending.

(5) Let $M$ be a fully stable module. Then the following are equivalent:

(1) $M$ is t-stable extending;

(2) $M$ is t-extending;

(3) $M$ is strongly t-stable extending;

(4) $M$ is strongly t-extending;

(6) Every stable t-uniform module is strongly t-stable extending.

(7) If $M$ is S-indecomposable and $M$ is strongly t-stable extending, then $M$ is a stable t-uniform.

**Proof:** Let $N$ be a stable submodule of $M$. Since $M$ is strongly t-stable extending, $N \leq_{tes} W \leq^{\oplus} M$. $W$ is a fully invariant in $M$. Then by [7, Lemma 2.1.6], $W$ is stable in $M$, but $W$ is S-indecomposable, so $W = M$. Thus $N \leq_{tes} M$ and $M$ is a stable t-uniform.

(8) If $M$ is S-uniform, then $M$ is strongly t-stable extending and $M$ is S-indecomposable.

(9) Let $M$ be an indecomposable module. Then $M$ is strongly t-stable extending if and only if $M$ is t-stable extending.

(10) If $M$ is a FI-t-extending, then $M$ is strongly t-stable extending. The converse holds if $M$ is FI-quasi injective.

**Proof:** Let $N$ be a stable submodule of $M$. Then $N$ is fully invariant, hence by [11, Theorem 2.2 (1) $\Rightarrow$ (7)] $N$ is t-essential in a fully invariant direct summand, say $W$. By [7, Lemma 2.1.6] $W$ is stable. Thus $M$ is strongly t-stable extending.

**Proposition 3.3:** Let $M$ be an $R$-module which satisfies that the t-closure of any submodule is equivalent. Then the following statements are equivalent:

(1) $M$ is strongly t-stable extending;

(2) $M$ is t-stable extending;

(3) $M$ is t-extending;

(4) Every stable t-closed is a direct summand;

(5) $M$ is strongly t-extending.

**Proof:** (1) $\Rightarrow$ (2) Let $N$ be a stable submodule of $N$. Then by definition of strongly t-stable extending, $N$ is a t-essential in a fully invariant direct summand. Thus $M$ is t-stable extending.

(3) $\Rightarrow$ (4) Since $M$ is t-extending, every t-closed is a direct summand, so it is clear that every stable t-closed is a direct summand.

(2) $\Leftrightarrow$ (4) It follows by Proposition 2.8.

(2) $\Leftrightarrow$ (3) It follows by Proposition 2.6.
(4) ⇒ (1) Let \( N \) be a stable submodule of \( M \). Then there exists a t-closure of \( N \) say \( W \) such that \( N \subseteq_{t\text{-}\text{ess}} W \). By hypothesis, \( W \) is stable t-closed of \( M \), hence \( W \subseteq \oplus M \). Thus \( M \) is strongly t-stable extending.

(5) ⇒ (1) It follows by Remarks and Examples 3.2(2).

(1) ⇒ (5) Let \( N \) be a t-closure of \( M \). Hence \( N \) is a t-closure of \( N \) and so by hypothesis \( N \) is stable. Since \( M \) is strongly t-stable extending, \( N \subseteq_{t\text{-}\text{ess}} W \) for some stable direct summand \( W \). It follows that \( N = W \), since \( N \) is t-closed. Thus \( N \) is a stable direct summand and \( M \) is strongly t-extending.

Recall that an \( R \)-module \( M \) is a multiplication module if for each \( N \leq M \), there exists an ideal \( I \) of \( R \) such that \( N = MI \) [15].

**Proposition 3.4:** Let \( M \) be a multiplication t-extending. Then \( M \) is strongly t-stable extending.

**Proof:** Let \( N \) be a stable submodule of \( M \). Since \( M \) is t-stable extending, then there exists \( H \leq \oplus M \) such that \( N \subseteq_{t\text{-}\text{ess}} H \leq \oplus M \). But \( M \) is a multiplication module implies \( H \) is a fully invariant submodule of \( M \) and so by [7, Lemma 2.1.6], \( H \) is stable. Thus \( M \) is t-essential in stable direct summand of \( M \). Therefore, \( M \) is strongly t-stable extending.

**Corollary 3.5:** Every cyclic t-stable extending module over a commutative ring is strongly t-stable extending.

**Corollary 3.6:** Every commutative t-stable extending ring is strongly t-stable extending.

The following is a characterization of strongly t-stable extending modules.

**Theorem 3.7:** Let \( M \) be an \( R \)-module. \( M \) is strongly t-stable extending if for each stable submodule \( A \) of \( M \), there is a decomposition \( M = M_1 \oplus M_2 \) such that \( A \leq M_1 \) and \( M_1 \) is a stable submodule of \( M \) and \( A + M_2 \leq_{t\text{-}\text{ess}} M \).

**Proof:** Let \( A \) be a stable submodule of \( M \). Since \( M \) is strongly t-stable extending, \( A \leq_{t\text{-}\text{ess}} M_1 \leq \oplus M \) and \( M_1 \) is stable in \( M \). Hence \( M = M_1 \oplus M_2 \) for some \( M_2 \leq M \). Since \( A \leq_{t\text{-}\text{ess}} M_1 \leq M_2 \leq_{t\text{-}\text{ess}} M_2 \), then \( A + M_2 \leq_{t\text{-}\text{ess}} M \), by [5, Corollary 1.3].

\( \Leftarrow \) Let \( A \) be a stable submodule of \( M \). By hypothesis, there is a decomposition \( M = M_1 \oplus M_2 \) such that \( A \leq M_1 \), \( M_1 \) is stable in \( M \) and \( A + M_2 \leq_{t\text{-}\text{ess}} M \). Since \( A + M_2 = A \oplus M_2 \leq_{t\text{-}\text{ess}} M = M_1 \oplus M_2 \), then \( A \leq_{t\text{-}\text{ess}} M_1 \). But \( M_1 \) is a stable direct summand of \( M \). Thus \( M \) is strongly t-stable extending.

**Theorem 3.8:** Let \( M = M_1 \oplus M_2 \), where \( M_1 \) and \( M_2 \) are \( R \)-module, such that \( M \) is an abelian module \((\text{ann}M_1 \oplus \text{ann}M_2 = R)\). If \( M_1 \) and \( M_2 \) are strongly t-stable extending, then \( \square = \square_1 \oplus \square_2 \) is strongly t-stable extending.

**Proof:** Let \( N \) be a stable submodule of \( M \). By Lemma 2.11, \( N = (N \cap M_1) \oplus (N \cap M_2) \) where \( N \cap M_1 \) is stable in \( M_1 \), \( N \cap M_2 \) is stable in \( M_2 \). Put \( N_1 = (N \cap M_1), N_2 = (N \cap M_2) \). Since \( M_1 \) and \( M_2 \) are strongly t-stable extending, there exist \( W_1 \leq \oplus M_1, W_2 \leq \oplus M_2 \) and \( W_i \) is stable in \( M_i \) for \( i = 1, 2 \) and \( N_i \leq_{t\text{-}\text{ess}} W_i \). It follows that \( N_1 \oplus N_2 \leq_{t\text{-}\text{ess}} W_1 \oplus W_2 \) by [5, Corollary 1.3]. Since \( W_1 \leq \oplus M_1, W_2 \leq \oplus M_2 \), then \( W_1 \oplus W_2 \leq \oplus M \). On other hand, \( M \) is abelian (or \( \text{ann}M_1 \oplus \text{ann}M_2 = R \)) implies \( \text{Hom}(M_1, M_2) = 0 \), \( \text{Hom}(M_2, M_1) = 0 \), by [14, Theorem 4.6]. Hence \( \text{End}(M) \cong \left( \begin{array}{cc} \text{End}(M_1) & 0 \\ \text{Hom}(M_1, M_2) & \text{End}(M_2) \end{array} \right) \bigoplus \left( \begin{array}{cc} 0 & \text{End}(M_2) \\ \text{Hom}(M_2, M_1) & 0 \end{array} \right) \). Hence for each \( f \in \text{End}(M) \), \( f = \left( \begin{array}{c} f_1 \\ 0 \end{array} \right), f_1 \in \text{End}(M_1) \), \( f_2 \in \text{End}(M_2) \) and \( f(W_1 \oplus W_2) = f(W_1) \oplus f(W_2) \). But \( W_1 \) and \( W_2 \) are stable in \( M_1 \), \( M_2 \) respectively and so that \( f(W_1) \subseteq W_1, f(W_2) \subseteq W_2 \). Thus \( f(W_1 \oplus W_2) \subseteq W_1 \oplus W_2 \), hence \( W_1 \oplus W_2 \) is a fully invariant in \( M \), \( W_1 \oplus W_2 \leq \oplus M \), then [2, Lemma 2.1.6] \( W_1 \oplus W_2 \) is stable in \( M \).

Now we ask the following: Is the property of being strongly t-stable extending inherit to a submodule?

**Definition 3.9:** An \( R \)-module \( M \) is said to be stable-injective if \( M \) is stable-injective to \( N(M) = S-N \)-injective, where \( N \) is any \( R \)-module.

**Theorem 3.10:** Let \( M \) be a stable-injective \( R \)-module. If \( M \) is strongly t-stable extending, then every stable submodule of \( M \) is strongly t-stable extending.

**Proof:** Let \( X \) be a stable submodule of \( M \). To prove \( X \) is strongly t-stable extending, let \( A \) be a stable submodule of \( X \). Since \( M \) is stable-injective, then \( M \) stable-injective relative to \( X \) and hence by Lemma 2.15, \( A \) is strongly t-stable extending and \( A \) is stable in \( M \) imply there
exists a stable direct summand D such that $A \leq_{\text{tes}} D \leq^{\oplus} M$. Thus $M = D \oplus D'$ for some $D' \leq M$.

Since X is stable in X, $X = (X \cap D) \oplus (X \cap D')$ where $X \cap D$ is stable of D, $X \cap D'$ is stable of $D'$ by Lemma 2.11. Now $A \leq_{\text{tes}} D$ implies $A = X \cap A \leq_{\text{tes}} X \cap D$ by [3, Corollary 1.3]. But $(X \cap D) \leq^{\oplus} X$, so that $A \leq_{\text{tes}} X \cap A \leq^{\oplus} X$. We claim that $X \cap D$ is stable in X. Since $X \cap D$ is stable of $X \cap D$ and $X \cap D'$ is stable in M, then $X \cap D$ is stable of M by Lemma 2.15. But $X \cap D$ is stable in M and $X \cap D \subseteq X$ imply $X \cap D$ is stable in X.

**Proposition 3.11:** Let M be an R-module which satisfies that the t-closure of any submodule is stable. If $M$ is strongly t-stable extending, then every direct summand is strongly t-stable extending.

**Proof:** Let $W \leq^{\oplus} M$. Since $M$ satisfies that the t-closure of any submodule is stable, then by (Proposition 3.3) $M$ is strongly t-extending and so by [8, Theorem 3.5] $W$ is strongly t-extending. Thus by Remarks and Examples 3.2(2), $W$ is strongly t-stable extending.

**Corollary 3.12:** Let M be a fully stable R-module. If $M$ is strongly t-stable extending, then every direct summand is strongly t-stable extending.

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