Zeros and Amoebas of Partition Functions

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Singular sectors $Z_{\text{sing}}$ (loci of zeros) for real-valued non-positively defined partition functions $Z$ of $n$ variables are studied. It is shown that $Z_{\text{sing}}$ have a stratified structure and each stratum is a set of certain hypersurfaces in $\mathbb{R}^n$. The concept of statistical amoebas is introduced and their properties are studied. Relation with algebraic amoebas is discussed. Tropical limit of statistical amoebas is considered too.

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1. Introduction

The partition function is a key object in various branches of physics. In statistical physics, due to the relation with the free energy \( F = -k_B T \ln Z \), all basic thermodynamic characteristics of a macroscopic system are encoded in the partition function

\[
Z = \sum_n g_n e^{-\frac{E_n}{k_B T}}
\]  

(1.1)

where \( \{E_n\} \) is the energy spectrum, \( g_n \) is the degeneracy of the \( n \)-th state, \( T \) is the temperature and \( k_B \) is the Boltzmann constant (see e.g.\(^1\,2\)). In equilibrium \( Z \) is finite and positive, otherwise it is an indication of instability of the system.

Zeros of the partition function as a function of physical parameters (temperature, magnetic field \textit{etc.}) are of particular interest since at such points the free energy becomes singular and, hence, the system changes of state, e.g. exhibits a phase transition\(^1,2\). Since the seminal papers of Lee and Yang\(^3,4\) it is known that for usual systems (with finite and positive \( g_n \) and real \( E_n \)) zeros of the partition functions lie in the complex plane (see e.g.\(^5\,14\)). These results have led to the intensive study of the partition function’s zero sets and associated phase transitions in the complex plane of physical parameters for a number of models in statistical physics, including those subject to quantum dynamics (see e.g.\(^15\,18\)).

Study of unstable or metastable states is another branch of statistical physics where complex-valued partition functions naturally arise\(^19,20\). Formally, for an unstable state the energy \( E_n \) is complex \( E_n = E_{n,0} + i \Delta E_n \) with width \( \Delta E_n \) and, hence, the partition function is complex-valued too. Wide classes of macroscopic systems like spin-glasses and other geometrically or dynamically frustrated systems have such peculiarity (see e.g.\(^21\,27\)).

In all these cases the situation when partition function’s zeros are real is of the greatest interest. The study of properties of macroscopic systems, in particular, structure of equilibrium and unstable domains, is simplified if the partition func-
tion is real-valued for all values of parameters. Such situation is realisable, for example, for spin-glasses and frustrated systems with different temperatures $T_n$ of microsystems (microbasins) if the widths $\Delta E_n$ of energy levels obey the condition 

$$\frac{\Delta E_n}{k_B T_n} = \ell_n \pi, \quad n = 1, 2, 3, \ldots$$

where $\ell_n$ are integers. Terms with odd $\ell_n$ acquire the factor $-1$ and, hence, the partition function $Z$ is of the form (1.1) and real-valued, but with $g_n$ assuming both positive and negative values. Negative degeneracies of energy levels can be interpreted as the contribution from sort of holes in spectrum. Formally, negativity of $g_n$ is closely connected with the concepts of negative probability and negative membership functions widely discussed in literature (see e.g. [28–30] and references therein).

This paper is devoted to the study of partition functions of such a type, more precisely, those of the form

$$Z(g; x) := \sum_{\alpha=1}^{N} g_{\alpha} \cdot e^{f_{\alpha}(x_1, \ldots, x_n)}$$

(1.2)

where factors $g_{\alpha}$ take values 1 or $-1$, $x_1, \ldots, x_n$ are real variables and $f_{\alpha}(x)$ are linear real-valued functions. Main attention is paid to an analysis of the singular sector $Z_{\text{sing}}$ (locus of zeros) of the partition function (1.2), its stratification and structure of stability ($Z(g; x) > 0$) and zero confinement domains in the space $\mathbb{R}^n$ of parameters $(x_1, \ldots, x_n)$.

Singular sector $Z_{\text{sing}}$ admits a natural stratification

$$Z_{\text{sing}} = \bigcup_{k=1}^{\lfloor N/2 \rfloor} Z_{\text{sing},k} :$$

(1.3)

the stratum $Z_{\text{sing},k}$ is composed by all hypersurfaces given by

$$Z_k(I; x) := -\sum_{\alpha \in I} e^{f_{\alpha}(x)} + \sum_{\beta \notin I} e^{f_{\beta}(x)} = 0, \quad k = 1, \ldots, \left[ \frac{N}{2} \right]$$

(1.4)

where $(I, [N]\setminus I)$ is any 2-partition of the set $[N] := \{1, 2, \ldots, N\}$ with cardinality $\#I = k$. These hypersurfaces for the $k$-stratum are contained in a certain domain
in $\mathbb{R}^n$ refered as the zero confinement domain $ZCD_k$. This domain is divided by hypersurfaces (1.4) into a number of subdomains $ZCD_{k,\delta}$ at which each of the \(\binom{N}{k}\) functions $Z_k$ in (1.4) is positive or negative. This allows to associate with each of these subdomains a set of \(\binom{N}{k}\) number 1 or $-1$ that can be viewed as the state of the system of \(\binom{N}{k}\) “spins” which take values 1 or $-1$.

The domain $D_{k,+}$, where all functions (1.4) are positive, is the stability (equilibrium) domain. The union $A_k := D_{k,+} \cup ZCD_k$ is called the statistical $k$-amoeba. A $k$-statistical amoeba is composed by the stable nucleus $D_{k,+}$ and intermittent shell $ZCD_k$ with varying degree of instability (number of signs $-1$). The complement $D_{k,-}$ of the $k$-amoeba in $\mathbb{R}^n$ is the domain with maximum number of signs $-1$, i.e. the domain of maximal instability.

The domains $D_{k,+}$, $ZCD_k$ and statistical $k$-amoebas exhibit a simple inclusion property, for instance, $D_{k,+} \supseteq D_{\hat{k},+}$ and $A_k \supseteq A_{\hat{k}}$ if $1 \leq k < \hat{k} < \frac{N}{2}$.

Statistical 1-amoebas $A_1$ coincide with the so-called non-lopsided amoebas $\mathcal{L}A$ introduced in algebraic geometry$^{31}$. Analogs of higher statistical $k$-amoebas ($k \geq 2$) seem to be not studied in algebraic geometry.

Tropical limits of statistical $k$-amoebas are considered. It is shown that all $k$-amoebas collapse into the same set of piecewise hyperplanes in $\mathbb{R}^n$ coinciding with that of tropical limit of $A_1$.

It should be noted that partition functions depending on several variables (multidimensional energy spectrum or several Hamiltonians) have been considered in$^{32,33}$ and$^{34}$. However these papers have addressed only the case (1.2) with all positive factors $g_{\alpha}$.

In a completely different setting zeros of superpositions of the form (1.2) ($\tau$-functions) with positive and negative factors $g_{\alpha}$ and very particular linear functions $f_{\alpha}$ arise within an analysis of singular solutions of integrable equations (see e.g.$^{35,36}$).

The paper is organized as follows. General definitions of strata of the singular
sector and some concrete examples for the first stratum $Z_{\text{sing},1}$ are given in section 2. Higher strata and properties of zero loci hypersurfaces are considered in section 3. Section 4 is devoted to the study of some general properties of equilibrium and zero confinement domains. Statistical amoebas and their relation to algebraic amoebas are discussed in section 5. Next section 6 is devoted to an analysis of the structure and properties of $ZCD_k$ domains and associated statistical systems of “spins”. Tropical limits of statistical amoebas are considered in section 7. In conclusion some peculiarities of partition function (1.2) with nonlinear functions $f_\alpha$ are noted.

2. Singular sector of partition function

So we will consider the family of partition functions of the form

$Z = \sum_{\alpha=1}^{N} g_\alpha \cdot e^{f_\alpha(x)}$  \hspace{1cm} (2.1)

where $f_\alpha(x) = b_\alpha + \sum_{i=1}^{n} a_{\alpha i} x_i$, all variables $g_\alpha$, $x_i$ and all functions $f_\alpha$ are real. Since $g_\alpha e^{b_\alpha} = \text{sign}(g_\alpha) \cdot e^{\log|g_\alpha|+b_\alpha}$ one can consider only the case $g_\alpha \in \{+1, -1\}$.

The space $V$ of parameters $g_1, \ldots, g_N, x_1, \ldots, x_n$ admits the stratification

$V = \bigcup_{\alpha=0}^{N} V_\alpha$  \hspace{1cm} (2.2)

where $V_\alpha$ is the union of subspaces of $V$ with $\alpha$ many negative $g_\beta$. For instance, $V_2 = \bigcup_{1 \leq \alpha < \beta \leq N} V_{2,\{\alpha\beta\}}$ where $V_{2,\{\alpha\beta\}} = \{(g_1, \ldots, g_N; x_1, \ldots, x_n) : g_\alpha = g_\beta = -1, \gamma \neq \alpha, \beta \Rightarrow g_\gamma > 0\}$, $\alpha \neq \beta$, $\alpha, \beta = 1, \ldots, N$. Inversion $P$ of all $g_\alpha$: $Pg_\alpha = -g_\alpha$, $\alpha = 1, \ldots N$, acts on strata $V_\alpha$ as $PV_\alpha = V_{N-\alpha}$.

Accordingly, singular sector $Z_{\text{sing}}$ of partition function also admits the stratification

$Z_{\text{sing}} = \bigcup_{k=1}^{N-1} Z_{\text{sing},k}$  \hspace{1cm} (2.3)
where $Z_{\text{sing},k}$ are subspaces of $V_k$ for which $Z|_{V_k} = 0$. Subspaces $V_0$ and $V_N$ are obviously regular and connected by inversion $P$. First singular stratum $Z_{\text{sing},1}$ is the union of the $N$ hypersurfaces defined by the equations

$$Z_1(\{\alpha\}; x) := \sum_{\beta=1}^{N} g_{\alpha(\beta)} e^{f_{\beta}(x)} = 0, \quad \alpha = 1, \ldots, N$$

(2.4)

with $g_{\alpha(\alpha)} = -1$ and $g_{\alpha(\beta)} = 1, \beta \neq \alpha$. Geometric characteristics of such hypersurfaces have been studied in the paper. Cases of linear functions $f_{\alpha}(x)$ were referred in as ideal statistical hypersurfaces. General statistical hypersurfaces considered in were associated with nonlinear functions $f_{\alpha}$ while super-ideal case corresponds to $N = n$ and $f_{\alpha}(x) \equiv x_{\alpha}$.

Generically the stratum $Z_{\text{sing},1}$ can be composed by $N$ hypersurfaces. In addition, it is easy to see that hypersurfaces given by $Z_1(\{\alpha\}; x) = 0$ with different $\alpha$ do not intersect at finite values of $x_1, \ldots, x_n$ and $N \geq 3$. Indeed, if there exists $\alpha \neq \beta$ such that $\{x : Z_1(\{\alpha\}; x) = 0\} \cap \{x : Z_1(\{\beta\}; x) = 0\}$ is not empty, then there exists $x_0$ in $\mathbb{R}^n$ such that $e^{f_{\beta}(x_0)} - e^{f_{\alpha}(x_0)} = \sum_{\gamma \notin \{\alpha, \beta\}} e^{f_{\gamma}(x_0)} = e^{f_{\alpha}(x_0)} - e^{f_{\beta}(x_0)}$, that is $f_{\beta}(x_0) = f_{\alpha}(x_0)$, so one has $\sum_{\gamma \neq \alpha, \beta} e^{f_{\gamma}(x_0)} = 0$ which is impossible if $N \geq 3$.

The form of hypersurfaces which compose the sector $Z_{\text{sing},k}$ depends on $N$, $n$ and the choice of functions $f_{\alpha}(x)$. At the simplest case of $n = 1$ the stratum $Z_{\text{sing},1}$ is composed, in general, by at most $N$ points defined by the equations

$$Z_1(\{\alpha\}; x_1) = \sum_{\beta=1}^{N} g_{\alpha(\beta)} e^{a_{\beta}x_1 + b_{\beta}} = 0, \quad \alpha = 1, \ldots, N$$

(2.5)

with $g_{\alpha(\alpha)} = -1$ and $g_{\alpha(\beta)} = 1, \beta \neq \alpha$. In the case $N = 2$ the sector $Z_{\text{sing},1}$ contains only one hypersurface defined by the equation

$$Z_1(\{1\}; x) = -e^{f_1(x_1, \ldots, x_n)} + e^{f_2(x_1, \ldots, x_n)} = 0.$$ 

(2.6)

It is the hyperplane in $\mathbb{R}^n$ given by the equation

$$b_1 - b_2 + \sum_{i=1}^{n} (a_{1i} - a_{2i}) x_i = 0.$$ 

(2.7)
For $N \geq 3$ and $n = 2$ one has a family of curves on the plane $(x_1, x_2) =: (x, y)$. For example, at the choice $f_1 \equiv 0$, $f_2 \equiv x$, $f_3 \equiv y$ the sector $Z_{\text{sing},1}$ is composed by three curves shown in figure 1(a) (see also \textsuperscript{46}) where the curves 1, 2 and 3 are

\begin{align*}
(a) f_1 &\equiv 0, f_2 \equiv x, f_3 \equiv y. \\
(b) f_1 &\equiv 0, f_2 \equiv 3x, f_3 \equiv 3y, \quad f_4 \equiv x + y + \ln 6.
\end{align*}

Figure 1. Examples of 1-strata (red curves).

given by the equations

\begin{align*}
Z_1(\{1\}; x, y) &\equiv -1 + e^x + e^y = 0, \\
Z_1(\{2\}; x, y) &\equiv 1 - e^x + e^y = 0, \\
Z_1(\{3\}; x, y) &\equiv 1 + e^x - e^y = 0. \\
\end{align*}

(2.8)

Note that at $|x|, |y| \to \infty$ the curves 1 and 2 tend to the ray $x = 0$, $y < 0$, the curves 2 and 3 tend to the ray $x = y$, $x > 0$ while the curves 3 and 1 tend to the ray $y = 0$, $x < 0$. An example with a different homotopy and a bounded closed component (curve 4) is given by choice $f_1 \equiv 0$, $f_2 \equiv 3x$, $f_3 \equiv 3y$, $f_4 \equiv x + y + \ln 6$ and it is shown in figure 1(b).

The stratum $Z_{\text{sing},1}$ at $n = 2$ is composed by at most $N$ curves. For particular choice of functions $f_\alpha(x)$ this number can be smaller than $N$ when some equa-
tions in (2.4) define the empty set. To illustrate this let us consider the following examples with \( n = 2 \): if one chooses
\[
f_\alpha(x, y) \equiv \cos \left( \frac{2\pi (\alpha - 1)}{N} \right) \cdot x + \sin \left( \frac{2\pi (\alpha - 1)}{N} \right) \cdot y, \quad \alpha = 1, \ldots, N, \tag{2.9}
\]
then one gets \( N \) curves by symmetry. For instance, at \( N = 6 \) one gets 6 curves presented in figure 2(a). If, instead, one takes
\[
f_\alpha(x, y) \equiv (\alpha - 1) \cdot x + (N - \alpha) \cdot y, \quad \alpha = 1, \ldots, N, \tag{2.10}
\]
then for all \( 2 \leq \gamma \leq N - 1 \) the corresponding locus is empty, i.e. \( \{ Z_1(\{\gamma\}) \} = 0 \} = \emptyset \). Indeed, if \( x \leq y \) then \( e^{(\gamma-1)x} \leq e^{(\gamma-1)y} \) thus \( e^{f_\gamma(x,y)} < \sum_{\beta \neq \gamma} e^{f_\beta(x,y)} \). Similarly, if \( y \leq x \) then \( e^{f_\gamma(x,y)} \leq e^{f_N(x,y)} < \sum_{\beta \neq \gamma} e^{f_\beta(x,y)} \). The only two visible curves are given by \( Z_1(\{1\}; x, y) = 0 \) and \( Z_1(\{N\}; x, y) = 0 \): they are straight lines with slope 1 passing through \((\pm \ln \chi, 0)\) respectively, where \( \chi > 0 \) is uniquely defined by \(-1 + \sum_{\alpha=1}^5 \chi^\alpha = 0\). See figure 2(b) for the case at \( N = 6 \).

One has an intermediate case at \( f_1(x, y) \equiv 0 \), \( f_2(x, y) \equiv 3x \), \( f_3(x, y) \equiv 3y \), \( f_4(x, y) \equiv x + y + \ln 6 \), \( f_5(x, y) \equiv 2x + y + \ln 11 \), \( f_6(x, y) \equiv x + 3y + \ln 4 \). The stratum \( Z_{\text{sing},1} \) is composed by 5 curves given in figure 3. One can check directly that the set of solutions of the equation
\[
Z_1(\{4\}; x, y) \equiv 1 + e^{3x} + e^{3y} - 6 \cdot e^{x+y} + 11 \cdot e^{2x+y} + 4e^{x+3y} = 0 \tag{2.11}
\]
is empty. Indeed, if \( Z_1(\{4\}; x, y) = 0 \) then \( x + y + \ln 6 > 2x + y + \ln 11 \), that is \( x < \ln 6 - \ln 11 < 0 \). From the arithmetic-geometric means inequality one gets
\[
\frac{1}{2} + \frac{1}{2} + 11 \cdot e^{2x+y} + 4 \cdot e^{x+3y} \geq 4 \left[ \frac{1}{2} \cdot \frac{1}{2} \cdot (11 \cdot e^{2x+y}) \cdot (4 \cdot e^{x+3y}) \right]^{\frac{1}{4}} = 4 \cdot 11^{\frac{1}{4}} \cdot e^{\frac{3x+4y}{4}}.
\]
But \( 4 \cdot 11^{\frac{1}{4}} > 6 \) and \( x < 0 \), hence \( e^{\frac{3x+4y}{4}} > e^{x+y} \) and \( 1 + 11 \cdot e^{2x+y} + 4 \cdot e^{x+3y} \geq 4 \cdot 11^{\frac{1}{4}} \cdot e^{\frac{3x+4y}{4}} > 4 \cdot 11^{\frac{1}{4}} \cdot e^{x+y} > 6 \cdot e^{x+y} \). In particular, \( Z_1(\{4\}; x, y) \) is always positive.

At \( N = 3 \), \( n = 3 \) and \( f_1 \equiv x \), \( f_3 \equiv y \), \( f_4 \equiv z \) \((x_1 = x, x_2 = y, x_3 = z)\) the
(a) \( f_\alpha \equiv \cos \left( \frac{\pi (\alpha - 1)}{3} \right) x + \sin \left( \frac{\pi (\alpha - 1)}{3} \right) y, \)
\( \alpha = 1, \ldots, 6: \) all 6 components are visible.

(b) \( f_\alpha(x, y) := (\alpha - 1) \cdot x + (6 - \alpha) \cdot y, \)
\( \alpha = 1, \ldots, 6. \)

Figure 2. Extremal behaviors of number of visible curves.

Figure 3. 1-stratum for the choice \( f_1(x, y) \equiv 0, f_2(x, y) \equiv 3x, f_3(x, y) \equiv 3y, f_4(x, y) \equiv x + y + \ln 6, f_5(x, y) \equiv 2x + y + \ln 11, f_6(x, y) \equiv x + 3y + \ln 4. \)
stratum $Z_{\text{sing,1}}$ contains 3 super-ideal statistical surfaces defined by the equations

$$Z_1(\{1\}; x, y, z) \equiv -e^x + e^y + e^z = 0,$$
$$Z_1(\{2\}; x, y, z) \equiv e^x - e^y + e^z = 0,$$
$$Z_1(\{3\}; x, y, z) \equiv e^x + e^y - e^z = 0$$

(2.12)

and given in figure 4. Induced metric $g_{ik}$, Gauss curvature $K$ and mean curvature $\Omega$ of the surface given by the equation $Z_1(\{1\}) = 0$ are (with $y$ and $z$ as local coordinates)\(^{37}\)

$$g_{ik} = \delta_{ik} + w_iw_k, \quad i, k = 1, 2,$$

$$K = 0,$$

$$\Omega = \frac{1 - T_3}{(1 + T_2)^3}$$

(2.13)

where probability $w_1 = \frac{e^y}{e^y + e^z}$, $w_2 = \frac{e^z}{e^y + e^z}$ and $T_l = w_l^1 + w_l^2, \ l = 2, 3$. For surface given by the equation $Z_1(\{2\}) = 0$ and $Z_1(\{3\}) = 0$ one has similar results
with substitution $x \equiv y$ and $x \equiv z$, respectively. At large $x, y, z$ surfaces 1 and 2 tend to the half-plane $x = y, z < 0$, surfaces 1 and 3 tend to the half-plane $x = z, y < 0$ and surfaces 2 and 3 tend to the half-plane $y = z, x < 0$.

Last example is presented in figure 5 and corresponds to $N = 6$ and $n = 3$ with $f_1(x, y, z) = 0$, $f_2(x, y, z) = 3x$, $f_3(x, y, z) = 3y$, $f_4(x, y, z) = 2x + z + \log 6$, $f_5(x, y, z) = 2x + y + z + \log 11$, $f_6(x, y, z) = 3y + z + \log 4$.

![Figure 5. 1-stratum with $f_1 \equiv 0$, $f_2 \equiv 3x$, $f_3 \equiv 3y$, $f_4 \equiv 2x + z + \log 6$, $f_5 \equiv 2x + y + z + \log 11$, $f_6 \equiv 3y + z + \log 4$.](image)

3. Higher strata

Higher strata $Z_{\text{sing},k}$ have rather complicated structure. For instance, for $N = 6$, $n = 2$ and functions $f_a(x, y)$ given, as in figure 3, by $f_1(x, y) \equiv 0$, $f_2(x, y) \equiv 3x$, $f_3(x, y) \equiv 3y$, $f_4(x, y) \equiv x + y + \ln 6$, $f_5(x, y) \equiv 2x + y + \ln 11$, $f_6(x, y) \equiv x + 3y + \ln 4$, 

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the stratum $Z_{\text{sing},2}$ is the set of 15 curves defined by $\frac{N(N - 1)}{2} = 15$ equations

$$Z_2(\{\alpha, \beta\}; x, y) := \sum_{\gamma=1}^{6} g_{(\alpha\beta)(\gamma)} e^{f_\gamma(x,y)}, \quad \alpha \neq \beta, \; \alpha, \beta = 1, \ldots, 6 \quad (3.1)$$

with $g_{(\alpha\beta)(\gamma)} = -\delta_{\alpha\gamma} - \delta_{\beta\gamma}$ at $\gamma \in \{\alpha, \beta\}$, $g_{(\alpha\beta)(\gamma)} = 1$ at $\gamma \neq \alpha, \beta$. The stratum $Z_{\text{sing},3}$ is the set of 20 curves defined by $(\binom{N}{k}) = 20$ equations

$$Z_3(\{\alpha, \beta, \gamma\}; x, y) := \sum_{\eta=1}^{6} g_{(\alpha\beta\gamma)(\eta)} e^{f_\eta(x,y)}, \quad \alpha \neq \beta \neq \gamma \neq \alpha, \beta, \gamma = 1, \ldots, N \quad (3.2)$$

with $g_{(\alpha\beta\gamma)(\eta)} = -\delta_{\alpha\eta} - \delta_{\beta\eta} - \delta_{\gamma\eta}$ at $\eta \in \{\alpha, \beta, \gamma\}$, $g_{(\alpha\beta\gamma)(\eta)} = 1$ at $\eta \neq \alpha, \beta, \gamma$. These sets of curves are presented in figure 1(a) and 1(b) respectively.

Figure 6. Higher strata with $f_1 \equiv 0, f_2 \equiv 3x, f_3 \equiv 3y, f_4 \equiv x + y + \ln 6, f_5 \equiv 2x + y + \ln 11, f_6 \equiv x + 3y + \ln 4$. For each choice of 2- and 3- subsets, listed in lexicographical order, corresponding locus is shown. Components in $Z_{\text{sing},3}$ are listed twice: curve $\alpha$ coincides with curve $21 - \alpha, \alpha = 1, \ldots, 10$, since they correspond to the same partition.

Due to the involution $P : \; g_a \mapsto -g_a$ the strata $Z_{\text{sing},4}$ and $Z_{\text{sing},5}$ coincide with
strata $Z_{\text{sing},2}$ and $Z_{\text{sing},1}$, respectively. The stratum $Z_{\text{sing},3}$ is stable under involution, $P(Z_{\text{sing},3}) = Z_{\text{sing},3}$ and at most half of conditions in (3.2) are independent.

In the case $N = 6$ and $n = 3$ and with the same functions $f_\alpha$ as in figure 5, i.e. $f_1 \equiv 0$, $f_2 \equiv 3x$, $f_3 \equiv 3y$, $f_4 \equiv 2x + z + \log 6$, $f_5 \equiv 2x + y + z + \log 11$, $f_6 \equiv 3y + z + \log 4$, the strata $Z_{\text{sing},2}$ and $Z_{\text{sing},3}$ are given in figure 7.

In order to describe general properties of higher singular strata let us introduce some notation. We will denote by $(I_1, I_2)$ an ordered 2-partition of the set $[N] := \{1, \ldots, N\}$, i.e. the pair of two subsets $I_1, I_2 \subseteq [N]$ such that $I_1 \cup I_2 = [N]$ and $I_1 \cap I_2 = \emptyset$. Then for each ordered 2-partition $(I, [N]\setminus I)$ we define the function

$$Z_k(I; x) := -\sum_{\alpha \in I_1} e^{f_\alpha(x)} + \sum_{\beta \in I_2} e^{f_\beta(x)}$$

and the corresponding zero locus as

$$Z_{\text{sing},k}(I) := \{x : Z_k(I; x) = 0\}.$$  \hspace{1cm} (3.3)

In the following we will often write $Z_k(I)$ instead of $Z_k(I; x)$ for notational convenience.
Since $Z_{N-k}(I_2, I_1) = -Z_k(I_1, I_2)$ zero loci of $Z_k(I_1, I_2)$ and $Z_{N-k}(I_2, I_1)$ coincide. To avoid such redundancy we will focus on unordered partitions $\{I_1, I_2\}$ and we will assume in what follows for each partition $(I_1, I_2)$ the cardinality $\# I_1$ of the first subset $I_1$ is smaller than that of $I_2$. Since $I_1 \cup I_2 = [N]$ and $I_1 \cap I_2 = \emptyset$ one has $0 \leq \# I_1 \leq \left\lfloor \frac{N}{2} \right\rfloor$.

Further for each subset $I$ of $[N]$ of cardinality $k$ one has the equation

$$Z_k(I; x) = 0$$  \hspace{1cm} (3.5)

which defines the hypersurface (3.4) in $\mathbb{R}^n$. Union of all such hypersurfaces with $\# I = k$ is the stratum $Z_{\text{sing}, k}$. Denoting the set of all subsets $I$ of $[N]$ with $\# I = k$ as $P_k[N]$ one, hence, has

$$Z_{\text{sing}, k} = \bigcup_{I \in P_k[N]} Z_{\text{sing}, k}(I).$$  \hspace{1cm} (3.6)

Note also that

$$Z_k(I; x) = 0 \iff 2 \cdot \sum_{\alpha \in I} e_{f_\alpha}(x) = Z_0(x)$$  \hspace{1cm} (3.7)

and

$$Z_k(x) := \sum_{I \in P_k[N]} e_{f_I}(x) = \binom{N-1}{k-1} Z_0(x)$$  \hspace{1cm} (3.8)

where $Z_0(x) = \sum_{\alpha = 1}^{N} e_{f_\alpha}(x)$ and we denote

$$e_{f_I}(x) := \sum_{\alpha \in I} e_{f_\alpha}(x).$$  \hspace{1cm} (3.9)

Figure 8 indicates that curves and hypersurfaces which compose higher strata may intersect in contrast to the stratum $Z_{\text{sing}, 1}$. In general, one has

**Proposition 1.** Two hypersurfaces (3.5) belonging to the same stratum $Z_{\text{sing}, k}$ and different $I_1$ and $I_2$ intersect at finite $x$ only if $I_1 \cap I_2 \neq \emptyset$.

**Proof.** Consider two hypersurfaces $Z_{\text{sing}, k}(I_1)$ and $Z_{\text{sing}, k}(I_2)$ associated with two partitions of $[N]$ with $\# I_1 = \# I_2 = k$. Assume that they have a common point
Crossing of hypersurfaces can happen for a higher stratum $Z_{\text{sing},2}$. Not all hypersurfaces lie in the same halfspace defined by a certain hypersurface, as it is for the orange region defined by $Z_2(\{1, 2\}) > 0$. Red region is the set where $Z_2(\mathcal{I}) > 0$ for all $\mathcal{I}$ with $\#\mathcal{I} = 2$.

Figure 8. Comments on higher stratum $Z_{\text{sing},2}$ with $f_1 \equiv 0$, $f_2 \equiv 3x$, $f_3 \equiv 3y$, $f_4 \equiv x + y + \ln 6$, $f_5 \equiv 2x + y + \ln 11$, $f_6 \equiv x + 3y + \ln 4$.

$\tilde{x}$. Hence, one has

$$Z_k(\mathcal{I}_1; \tilde{x}) + Z_k(\mathcal{I}_2; \tilde{x}) = 2 \cdot \sum_{\alpha \in [N]\setminus(\mathcal{I}_1 \cup \mathcal{I}_2)} e^{f_\alpha(\tilde{x})} - 2 \cdot \sum_{\beta \in \mathcal{I}_1 \cap \mathcal{I}_2} e^{f_\beta(\tilde{x})} = 0.$$ 

If $\mathcal{I}_1 \cup \mathcal{I}_2 \neq [N]$ then this equation may have real solutions only if the second term is different from zero, i.e. $\mathcal{I}_1 \cap \mathcal{I}_2 \neq \emptyset$. If $\mathcal{I}_1 \cup \mathcal{I}_2 = [N]$ then $2k \geq N$, but $k \leq \frac{N}{2}$ so $k = \frac{N}{2}$; then $\mathcal{I}_1$ and $\mathcal{I}_2$ are complementary sets and they define the same equation $Z_k(\mathcal{I}_1, \mathcal{I}_2) = 0$. 

\[\square\]
Maximum number of intersections in stratum $\mathcal{Z}_{\text{sing},k}$ is bounded by one-half of the number of different pairs $\mathcal{I}_1$ and $\mathcal{I}_2$ with $\#\mathcal{I}_1 = \#\mathcal{I}_2 = k \leq \left\lfloor \frac{N}{2} \right\rfloor$ such that $\mathcal{I}_1 \cap \mathcal{I}_2 \neq \emptyset$. There are $\binom{N}{k}$ different partitions $(\mathcal{I}, [N] \setminus \mathcal{I})$ of $[N]$ with cardinality $\#\mathcal{I} = k$ and for each such $\mathcal{I}$ there are $\binom{N}{k} - 1 - \binom{N-k}{k}$ partitions $(\mathcal{J}, [N] \setminus \mathcal{J})$, $\mathcal{J} \neq \mathcal{I}$ and $\mathcal{I} \cap \mathcal{J} \neq \emptyset$. Then the number of intersections in the stratum $\mathcal{Z}_{\text{sing},k}$ is bounded from above by $\frac{1}{2} \cdot \binom{N}{k} \cdot \left[ \binom{N}{k} - 1 - \binom{N-k}{k} \right]$. It is not always a strict bound since some of these intersections might be unreachable. For example, at $k = 2$ one has $\#(\mathcal{I}_1 \cap \mathcal{I}_2) = 1$ and, hence, $\mathcal{I}_1 = \{\alpha, \gamma_1\}$ and $\mathcal{I}_2 = \{\alpha, \gamma_2\}$ for some $\alpha \neq \gamma_1 \neq \gamma_2 \neq \alpha$. Then, if $\bar{x}$ is in the intersection $\mathcal{Z}_{\text{sing},2}(\mathcal{I}_1) \cap \mathcal{Z}_{\text{sing},2}(\mathcal{I}_2)$ one has $e^{f_{\gamma_1}(\bar{x})} - e^{f_{\gamma_2}(\bar{x})} = -e^{f_{\alpha}(\bar{x})} + \sum_{\beta \neq \alpha, \gamma_1, \gamma_2} e^{f_{\beta}(\bar{x})} = e^{f_{\gamma_1}(\bar{x})} - e^{f_{\gamma_2}(\bar{x})}$, thus $f_{\gamma_1}(\bar{x}) = f_{\gamma_2}(\bar{x})$ and $e^{f_{\alpha}(\bar{x})} = \sum_{\beta \neq \alpha, \gamma_1, \gamma_2} e^{f_{\beta}(\bar{x})}$. This last equation not always has real solutions for general choice of functions.

In general, intersections of hypersurfaces (3.5) for the stratum $\mathcal{Z}_{\text{sing},k}$ are not necessarily transversal. Moreover, the assumption of real-valued functions opens up the way to reductions. For example, take two functions $g(\bar{x})$ and $h(\bar{x})$ and consider $f_1(\bar{x}) \equiv 4 \cdot g(\bar{x})$, $f_2(\bar{x}) \equiv 4 \cdot h(\bar{x})$, $f_3(\bar{x}) \equiv \ln 6 + 2 \cdot g(\bar{x}) + 2 \cdot h(\bar{x})$, $f_4(\bar{x}) = \ln 4 + 3 \cdot g(\bar{x}) + h(\bar{x})$, $f_5(\bar{x}) = \ln 4 + g(\bar{x}) + 3 \cdot h(\bar{x})$. Then $\mathcal{Z}_2(\{4, 5\}; \bar{x}) = \left( e^{2g(\bar{x})} + e^{2h(\bar{x})} - e^{\ln 2 + g(\bar{x}) + h(\bar{x})} \right)^2 \geq 0$ and it vanishes if and only if $e^{2g(\bar{x})} + e^{2h(\bar{x})} = e^{\ln 2 + g(\bar{x}) + h(\bar{x})}$. From Arithmetic-Geometric inequality, this is equivalent to the algebraic constraint $g(\bar{x}) = h(\bar{x})$. Occurrence of such non-transversal crossings or reductions is a particular case and influences the investigation on equilibrium and non-equilibrium regions. We assume hereafter that pairwise intersections between hypersurfaces defined by $\mathcal{Z}_k(\mathcal{I}) = 0$ or $f_\alpha(\bar{x}) - f_\beta(\bar{x}) = 0$, $1 \leq \alpha < \beta \leq N$ are transversal.
4. Higher strata. General properties

All hypersurfaces (3.4) which compose the stratum $Z_{\text{sing},k}$ divide $\mathbb{R}^n$ in a number of regions which we will call domains. Then, let us denote the subdomain in $\mathbb{R}^n$ where all functions $Z_k(I) > 0$, $I \in \mathcal{P}_k[N]$, as $\mathcal{D}_{k+}$. For example, blue regions in figures 1, 2 and 3 represent $\mathcal{D}_{1+}$ and the red region in 8(a) represents $\mathcal{D}_{2+}$. The domain $\mathbb{R}^n \setminus \mathcal{D}_{k+}$ is divided by hypersurfaces $\{Z_k(I) = 0\}$ into subdomains where some of functions $Z_k(I)$ are positive and others are negative. Let us denote $\mathcal{D}_{k-}$ as the domain where the number of negative functions $Z_k(J)$ with $J \in \mathcal{P}_k[N]$ is maximal. The hypersurfaces of the $k$-stratum are confined in certain domain which we will refer as zeros confinement domain $Z_{CD}^k := \mathbb{R}^n \setminus (\mathcal{D}_{k+} \cup \mathcal{D}_{k-})$. The domain $Z_{CD}^k$ is a sort of intermittent shell which separate the domains $\mathcal{D}_{k+}$ and $\mathcal{D}_{k-}$ and the boundary of $\mathcal{D}_{k+} \cup Z_{CD}^k$ will be referred as the extremal points of hypersurfaces composing $Z_{\text{sing},k}$. For the first stratum $V_1$ defined in (2.2) the domain $Z_{CD}^1$ generically has dimension $n - 1$ and consists of hypersurfaces $\{x : Z_1(\{\alpha\}; x) = 0\}$, $\alpha = 1, \ldots, N$ themselves. For higher $V_k$, $k \geq 2$, the domain $Z_{CD}^k$ has generically dimension $n$ and its boundary is tipically formed by pieces of different hypersurfaces belonging to $Z_{\text{sing},k}$.

Since at $\mathcal{D}_{k+}$ the partition function $Z_k \equiv Z|_{V_k} > 0$ it is natural to refer to the domain $\mathcal{D}_{k+}$ as stability (equilibrium) domain. It is surrounded by the domain $Z_{CD}^k$ with rather complicated singularity structure. The domain $\mathcal{D}_{k-}$ is an ambient instability domain.

These domains for different strata exhibit a simple inclusion chain.

**Proposition 2.** Let $1 \leq k < \hat{k} \leq \left\lfloor \frac{N}{2} \right\rfloor$ then $\mathcal{D}_{\hat{k}+} \subseteq \mathcal{D}_{k+}$.

**Proof.** Through the proof is an immediate consequence of the definition of $\mathcal{D}_{k+}$, we present it here for completeness. Let $\hat{k} > k$ and $\mathcal{I}_k \subset \mathcal{I}_{\hat{k}}$ be two subsets of $[N]$
such that $\#I_k = k$ and $\#I_k = \hat{k}$. Then for any $x \in \mathbb{R}^n$ one has the identity
\[
Z_k(I_k; x) - Z_k(I_k; x) = 2 \cdot \sum_{\alpha \in \mathbb{Z}_{k+1} \setminus I_k} e^{f_{\alpha}(x)} > 0. \tag{4.1}
\]
In the domain $D_{k+1}$ one has $Z_k(I_k; x) > 0$ for all subsets $I_{k+1}$. Inequality (4.1) implies that all $Z_k(I_k; x) > 0$ in $D_{k+1}$ too. So $D_{k+} \subseteq D_{k+}$.

Thus, one has the inclusion chain
\[
D_0 += \mathbb{R}^n \supseteq D_{1+} \supseteq D_{2+} \supseteq \cdots \supseteq D_{\lfloor \frac{N}{2} \rfloor +}. \tag{4.2}
\]
An example with $N = 7$ and $f_{\alpha}(x, y)$ as in (2.9) is given in figure 9, where the domain $D_{1+}$ is shown in blue color, $D_{2+}$ in red and $D_{3+}$ in green.

![Figure 9](image_url)

(a) Singular sectors $Z_{\text{sing,1}}$ (blue), $Z_{\text{sing,2}}$ (red) and $Z_{\text{sing,3}}$ (green).

(b) Domains $D_{3+} \subseteq D_{2+} \subseteq D_{1+}$.

Figure 9. Inclusion chain for equilibrium domains in the case $f_{\alpha} \equiv \cos \left( \frac{2\pi(\alpha-1)}{7} \right) x + \sin \left( \frac{2\pi(\alpha-1)}{7} \right) y$, $\alpha = 1, \ldots, 7$.

Furthermore, one also has
Proposition 3. Let us take \(1 \leq k < \hat{k} \leq \left\lfloor \frac{N}{2} \right\rfloor\). Then, there exists a dense subset of \(Z_{\text{sing},k}\) such that each ray \((\vec{r})_i = x_{0,i} + t \cdot e_i\) originated from this set, \(e \in \mathbb{R}^n\), intersects \(Z_{\text{sing},k}\).

In other words, if \(\{f_\alpha(x) : \alpha \in [N]\}\) are pairwise different linear functions, then \(Z_{\text{sing},k}\) lies inside a region of \(\mathbb{R}^n\) delimited by some components of \(Z_{\text{sing},k}\), \(1 \leq k < \hat{k} \leq \left\lfloor \frac{N}{2} \right\rfloor\).

Proof. The proof is based on the following

Lemma 1. Consider any ray \(\vec{r}\) with base point \(x\), slopes \(e := (e_1, \ldots, e_n)\) and parametrization \(\vec{r}(t) := x + t \cdot e\), \(t \geq 0\). If none of functions \(f_\alpha(\vec{r}(t)) - f_\beta(\vec{r}(t))\) vanishes identically, \(\alpha \neq \beta\), then there exists \(t_0 \in \mathbb{R}_+\) such that \(\vec{r}(t)\) belongs to a region in \(\mathbb{R}^n\) where there is only one dominant function \(f_\alpha, \alpha \in [N]\), at \(t \geq t_0\), i.e. 
\[
\# \{\alpha \in [N] : \forall \beta \in [N], f_\alpha(\vec{r}(t)) \geq f_\beta(\vec{r}(t))\} = 1.
\]

Proof. Let us consider the \(\frac{N(N - 1)}{2}\) functions
\[
d_{\alpha\beta}(t) := f_\alpha(\vec{r}(t)) - f_\beta(\vec{r}(t)) \quad 1 \leq \alpha < \beta \leq N. \tag{4.3}
\]
Since none of these linear functions vanishes identically, each of them has a finite number of roots, so the set of points
\[
\Omega := \bigcup_{1 \leq \alpha < \beta \leq N} \{t : d_{\alpha\beta}(t) = 0\} \tag{4.4}
\]
is finite. Thus, for \(t_0 > \max(\Omega)\), all \(d_{\alpha\beta}(t_0)\) will be definitely different from zero, hence all \(\{f_\alpha(\vec{r}(t_0)) : \alpha \in [N]\}\) are pairwise different. In particular, there will be one and only one \(\alpha_0 \in [N]\) such that \(f_{\alpha_0}(\vec{r}(t_0)) > f_\alpha(\vec{r}(t_0))\) for all \(\alpha \neq \alpha_0\). So \(f_{\alpha_0}(\vec{r}(t)) - f_\alpha(\vec{r}(t)) > 0\) at \(t > t_0\) too, since a change of sign would imply an additional zero of \(d_{\alpha_0\alpha}\) by continuity. \(\square\)

Now, let \((\mathcal{J}, [N] \setminus \mathcal{J})\) be any partition of \([N]\) such that \#\(\mathcal{J} = \hat{k} \leq \left\lfloor \frac{N}{2} \right\rfloor\) and \(\vec{r}(t) := (x_i + t \cdot e_i), t \geq 0\) be a ray with base point \(x \in Z_{\text{sing},k}\). If \(d_{\alpha\beta}(t)\) in (4.3)
does not vanish identically for all \( \alpha < \beta \), then lemma 1 implies that there exist \( t_0 > 0 \) and \( \alpha_0 \in [N] \) such that one has \( d_{\beta \alpha_0}(t) < 0 \) at \( t \geq t_0 \) and \( \beta \neq \alpha_0 \). In particular, \( d_{\beta \alpha_0} \) are linear functions of \( t \) and they are negative at \( t \geq t_0 \). Hence one has

\[
\lim_{t \to +\infty} f_{\beta}(\vec{r}(t)) - f_{\alpha_0}(\vec{r}(t)) = -\infty, \quad \beta \neq \alpha_0.
\]

So there exists \( t_1 > t_0 \) such that \( \sum_{\beta \neq \alpha_0} e^{f_{\beta}(\vec{r}(t)) - f_{\alpha_0}(\vec{r}(t))} < 1 \) at \( t > t_1 \). The index \( \alpha_0 \) belongs to only one subset \( \mathcal{J} \) or \([N] \setminus \mathcal{J} \), call it \( \mathcal{J}(\alpha_0) \): from \( k < \hat{k} \leq N - \hat{k} \) it follows that \( k < \# \mathcal{J}(\alpha_0) \) and one can always choose a subset \( \mathcal{I} \subset \mathcal{J}(\alpha_0) \) with \( k \) elements such that \( \alpha_0 \in \mathcal{I} \). One has \( \vec{r}(t)|_{t=0} = x \) so \( \sum_{\alpha \in \mathcal{I}} e^{f_{\alpha}(x)} < \sum_{\alpha \in \mathcal{J}(\alpha_0)} e^{f_{\beta}(x)} < \sum_{\beta \neq \alpha_0} e^{f_{\beta}(x)} \). On the other hand, at \( t = t_1 \) one has \( \sum_{\alpha \in \mathcal{I}} e^{f_{\alpha}(\vec{r}(t_1))} > e^{f_{\alpha_0}(\vec{r}(t_1))} > e^{f_{\beta}(\vec{r}(t_1))} > \sum_{\beta \neq \alpha_0} e^{f_{\beta}(\vec{r}(t_1))} \). Then, there exists a point \( 0 < \bar{t} < t_1 \) such that \( \sum_{\alpha \in \mathcal{I}} e^{f_{\alpha}(\vec{r}(\bar{t}))} = \sum_{\beta \neq \alpha_0} e^{f_{\beta}(\vec{r}(\bar{t}))} \) by continuity. Thus \( \vec{r}(\bar{t}) \) is in the set \( \mathcal{Z}_{\text{sing},k}(\mathcal{I}) \subseteq \mathcal{Z}_{\text{sing},k} \). If instead \( d_{\alpha_0 \beta_0}(t) \equiv 0 \) for some \( \alpha_0 \neq \beta_0 \) at the point \((x_0; e)\), then \( f_{\alpha_0}(x_0) - f_{\beta_0}(x_0) = d_{\alpha_0 \beta_0}(0) = 0 \) and \( x_0 \in \mathcal{Z}_{\text{sing},k} \cap \{ x \in \mathbb{R}^n : f_{\alpha_0}(x) = f_{\beta_0}(x) \} \). From the generic hypothesis of transversal crossing, the complement of set of such points \( x_0 \) in \( \mathcal{Z}_{\text{sing},k} \) is dense. □

Some concrete examples are presented in figure 10.

Note that the rays that do not come out \( ZCD_k \) are connected with locus of coincident dominant functions \( f_\alpha(x) = f_\beta(x) = \max \{ f_\gamma(x) \} \), \( \alpha \neq \beta \), see figure 11(b).

### 5. Statistical amoebas vs. algebraic amoebas

We will refer to the domains \( \mathcal{A}_k := D_{k+} \cup ZCD_k \) described in previous section as the statistical \( k \)-amoebas. They, generically, are composed by the internal stable nuclei (domains \( D_{k+} \)) and enveloping shells \( ZCD_k \) which contain singular
hypersurfaces $Z_{\text{sing},k}(I)$ and subdomains with some number of positive and negative partition functions. The statistical amoeba $A_k$ is surrounded by the domain $D_{k-}$ of maximal instability (as we will demonstrate in next section).

For $k > \left\lfloor \frac{N}{2} \right\rfloor$ the domain $ZCD_k$ coincides with that of $ZCD_{N-k}$ while the domains $D_{k-}$ and $D_{k+}$ exchange their roles, namely $D_{k-} := D_{(N-k)+}$ and $D_{k+} := D_{(N-k)-}$. With increasing $k$ the stability domain shrinks (proposition 2) while instable domain $D_{k-}$ expands. At $k = N$ the whole space $\mathbb{R}^n$ is the domain of instability. It would be natural to refer to the domain $D_{k-} \cup ZCD_k$ at $k > \left\lfloor \frac{N}{2} \right\rfloor$ as the statistical $k$-antiamoeba.

The name amoeba is borrowed from algebraic geometry. The amoeba of the
algebraic variety $\mathcal{V}_n$ given by the algebraic equation

$$
\sum_{m_1,\ldots,m_n} c_{m_1,\ldots,m_n} z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} = 0
$$

with complex $z_1, \ldots, z_n$ and $c_{m_1,\ldots,m_n}$ is defined\textsuperscript{38} as the image of $\mathcal{V}_n$ under logarithmic map $(z_1, \ldots, z_n) \mapsto (\log |z_1|, \ldots, \log |z_n|)$. Amoebas of algebraic varieties and their properties have been intensively studied since their introduction by Gelfand, Kapranov and Zelevinsky (see e.g.\textsuperscript{31,39–44}).

In the simplest example of the complex plane given by the equation $1+z_1+z_2 = 0$
the amoeba is defined by the triangle inequalities
\[ e^x + e^y > 1, \]
\[ 1 + e^y > e^x, \]
\[ 1 + e^x > e^y \] (5.2)
and is presented in figure 1(a) (colored region). So in this case statistical and algebraic amoebas coincide. However, in general, it is not so. Indeed let us, firstly, rewrite equation (5.1) in the form
\[ \sum_{\alpha=1}^{N} \exp \left( b_\alpha + \sum_{i=1}^{n} a_{\alpha i} x_i + i \left( \arg a_\alpha + \sum_{i=1}^{n} a_{\alpha i} \varphi_i \right) \right) = 0 \] (5.3)
where \( x_i = \log |z_i|, \varphi_i = \arg z_i, a_{\alpha i} = e^{b_{\alpha i} + i \arg a_{\alpha i}} \) and rows of \( a_{\alpha i} \) are given by integers \( m_i \) (\( a_{\alpha i} = m_i \) for given monomial indexed by \( \alpha \)). Projection of the \( 2n - 2 \) dimensional real hypersurface given by (5.3) onto the space \( \mathbb{R}^n \) with coordinates \((x_1, \ldots, x_n)\) is the amoeba \( A \) of this hypersurface.

On the other hand applying the triangle inequality to (5.3), one gets the set of inequalities
\[ -e^{f_\alpha(x)} + \sum_{\beta \neq \alpha} e^{f_\beta(x)} > 0 \] (5.4)
where \( f_\alpha(x) = b_\alpha + \sum_{i=1}^{n} a_{\alpha i} x_i \). The domain in \( \mathbb{R}^n \) defined by \( N \) inequalities (5.4) is called approximated amoeba (non-lopsided set) \( LA \). In general \( LA \) does not coincide with the amoeba \( A \), namely \( LA \supseteq A \), but “\( LA \) is a very good approximation for \( A \)”.

Comparing the set of inequalities (5.4) with our definition of the domain \( D_{1+} \), we can conclude that the statistical 1-amoeba with integer-valued elements \( a_{\alpha i} \) and \( f_\alpha(x) = b_\alpha + \sum_{i=1}^{n} a_{\alpha i} x_i \) in (2.4) coincide with \( LA \) amoeba for the hypersurface (5.3). The triangle inequalities reasoning becomes rather involved for partitions different from \( I_1 = \{ \alpha \} \) and \( I_2 = [N] \setminus \{ \alpha \} \), \( \alpha = 1, \ldots, N \). Anyway, equilibrium domains \( D_{k+} \) have a simple geometrical interpretation. First, one has the following well-known
Lemma 2. If $x$ is in the 1-equilibrium region $D_{1+}$, then one can construct a polygonal closed path with $N$ sides of lengths $(e^{f_\alpha(x)} : \alpha \in [N])$ in some order.

Proof. The base case $N = 3$ is equivalent to triangle inequality. Then we assume that the assertion holds for all integers $k$ such that $N - 1 \geq k \geq 3$ and will proceed by induction on $N$. One can fix $f_1(x) \geq f_2(x) \geq \cdots \geq f_N(x)$ without loss of generality and choose a number $\ell$ such that

$$\max \left\{ e^{f_2(x)}, e^{f_1(x)} - e^{f_N(x)} \right\} \leq \ell \leq \min \left\{ e^{f_1(x)}, \sum_{\beta=2}^{N-1} e^{f_\beta(x)} \right\}.$$  (5.5)

Note that this definition is well-posed: $e^{f_1(x)} - e^{f_N(x)} < e^{f_1(x)}$ since $f_N(x)$ is real, $e^{f_1(x)} - e^{f_N(x)} < \sum_{\beta=2}^{N-1} e^{f_\beta(x)}$ since $x$ belongs to the equilibrium region, $e^{f_2(x)} \leq e^{f_1(x)}$ by hypothesis and $e^{f_2(x)} < \sum_{\beta=2}^{N-1} e^{f_\beta(x)}$ since $N > 3$. So there is at least one positive term. Let us consider

$$\Lambda_1 := \left( e^{f_1(x)}, \ell, e^{f_N(x)} \right), \quad \Lambda_2 := \left( \ell, e^{f_2(x)}, \ldots, e^{f_{N-1}(x)} \right).$$  (5.6)

One has $e^{f_1(x)} = e^{f_1(x)} - e^{f_N(x)} + e^{f_N(x)} < \ell + e^{f_N(x)}$ and $e^{f_1(x)} = \max \left\{ e^{f_1(x)}, e^{f_N(x)}, \ell \right\}$ so $\Lambda_1$ is non-lopsided. One can see that $\ell = \max \Lambda_2$ and $\ell < \sum_{\beta=2}^{N-1} e^{f_\beta(x)}$ by construction. Then both $\Lambda_1$ and $\Lambda_2$ are non-lopsided and their cardinalities are 3 and $N - 1$. Thus, by induction hypothesis there exists two polygonal closed paths $C_1$ and $C_2$ with sides $\left( e^{f_1(x)}, \ell, e^{f_N(x)} \right)$ and $\left( \ell, e^{f_2(x)}, \ldots, e^{f_{N-1}(x)} \right)$ in some order. Finally, one can join $C_1$ and $C_2$ along the side of length $\ell$ and get a closed polygonal path with $N$ sides of lengths $\left( e^{f_\alpha(x)} : \alpha \in [N] \right)$. \hfill \Box

Stratification (4.2) can now be seen as a refinement of the triangle inequality property.

Proposition 4. The equilibrium domain $D_{k+}$ relative to the $k$-amoeba is the set of all points $x$ in $\mathbb{R}^n$ that satisfy the following condition: there exists a planar polygon
with \( g \) sides, for all \( N - k + 1 \leq g \leq N \), and with lengths of sides \((\ell_1, \ldots, \ell_g)\) equal to \( \left( \sum_{\alpha \in I_1} e^{f_\alpha(x)}, \ldots, \sum_{\alpha \in I_g} e^{f_\alpha(x)} \right) \), where \( \{I_1, \ldots, I_g\} \) is any partition of \([N]\) in \( g \) disjoint non-empty subsets.

**Proof.** Let us suppose that \( x \in \mathcal{D}_{k+} \) and consider any \( g \)-partition \( \{I_1, \ldots, I_g\} \) of \([N]\). Since all subsets in \( I_1, \ldots, I_g \) are not empty, then \( \#I_u \geq 1 \) for all \( 1 \leq u \leq g \).

Thus

\[
\#I_u = N - \sum_{w \neq u} \#I_w \leq N - (g - 1) \leq N - (N - k + 1 - 1) = k, \quad 1 \leq u \leq g. \tag{5.7}
\]

Assuming that \( x \in \mathcal{D}_{k+} \) one gets

\[
\sum_{\alpha \in I_u} e^{f_\alpha(x)} < \sum_{\beta \notin I_u} \sum_{\alpha \in I_w} e^{f_\beta(x)}, \quad 1 \leq u \leq g.
\]

By lemma 2, this is equivalent to the existence of a closed planar polygon with \( g \) sides whose lengths are (in some order) \( \sum_{\alpha \in I_1} e^{f_\alpha(x)}, \ldots, \sum_{\alpha \in I_g} e^{f_\alpha(x)} \).

Now assume that the existence hypothesis holds. In particular, it holds at \( g = N - k + 1 \). For any \( I \in \mathcal{P}_k[N] \) one can consider the \( g \)-partition \( \{\{\beta_1\}, \ldots, \{\beta_{g-1}\}, I\} \) where \([N]\setminus I =: \{\beta_1, \ldots, \beta_{g-1}\} \). The hypothesis in such a case implies that

\[
\sum_{\alpha \in I} e^{f_\alpha(x)} < \sum_{u=1}^{g-1} e^{f_{\beta_u}(x)} = \sum_{\beta \notin I} e^{f_\beta(x)}. \tag{5.4}
\]

This means that \( Z_k(I; x) > 0 \) for all \( I \in \mathcal{P}_k[N] \), that is \( x \in \mathcal{D}_{k+} \).

It seems that these higher amoebas, i.e. those defined by sets of inequalities of the type (5.4) with more than one minus sign were not discussed before in this context.

Some important features of algebraic amoebas are not preserved in the general \( k \)-amoeba case. For example, the extremal boundary of standard 1-amoebas is the set of points \( x \in \mathbb{R}^n \) such that \( x \in Z_{\text{sing},1}(\{\alpha\}) \) for certain \( \alpha \in [N] \). All non-extremal points are partitioned in two sets defined by the sign of \( Z_i(\{\alpha\}) \). As already noted, such partitions at different values of \( \alpha \) are compatible, in the sense
that if $Z_1(\{\alpha\}) < 0$ then one knows that $Z_1(\{\beta\}) > 0$ for all $\beta \neq \alpha$. Conversely, crossing points in $Z_{\text{sing},k}(I_1) \cap Z_{\text{sing},k}(I_2)$ between distinct components of the $k$-singular locus at $k \geq 2$ open the way for more sign combinations, see figure 8(a).

Furthermore, each connected component of the boundary of a standard algebraic amoeba bounds a certain convex (finite or infinite) region of the space (see e.g.\textsuperscript{36}). This property does not hold in general for $k$-singular loci $Z_{\text{sing},k}(I)$, $I \in \mathcal{P}_k[N]$. Such a case is pointed out in figure 8(b).

Anyway, a generalization of these properties to $k$-amoebas can be done taking in account all $k$-singular loci $Z_{\text{sing},k}(I)$ simultaneously. Thus, in our real-valued approach higher statistical amoebas arise in a natural way.

We note also that the relations between 1-statistical amoebas and statistical physics have been discussed in\textsuperscript{47} and\textsuperscript{32,34}.

6. Structure of ZCD domains

Zeros confinement domain $ZCD_k$ for $k$-stratum separating the domains $D_{k+}$ and $D_{k-}$ has rather complicated structure in general. Here we will consider some of their simplest properties.

For the first stratum $k = 1$ the $ZCD_1$ collapses into the set of hypersurfaces of zeros $Z_{\text{sing},1}(\{\alpha\})$. Let $k \geq 2$. Each zero hypersurface $Z_{\text{sing},k}(I_k)$ for given subset $I_k$ divides the $ZCD_k$ in two subdomains where the function $Z_k(I_k; x)$ have definite, positive or negative, sign. The set of all $\binom{N}{k}$ hypersurfaces $Z_{\text{sing},k}(I_1)$ with all possible partitions of $[N]$ $I_1 \cup I_2 = [N]$ and cardinality $\#I_1 = k$ divides the $ZCD_k$ into a finite, say $M$, number of subdomains $ZCD_{k; \delta}$ and inside each of them each of functions $Z_k(I_k; x)$ has definite sign.

So, one can associate with each such subdomain $ZCD_{k; \delta}$ a set of $\binom{N}{k}$ numbers
1 and −1 coinciding with values of sign function defined as
\[ s(I; x) := \text{sign} \left[ \sum_{\alpha \in I} e^{f_\alpha(x)} - \sum_{\beta \in [N] \setminus I_1} e^{f_\beta(x)} \right] \in \{-1, 0, +1\} \] (6.1)
evaluated for each partition \( I_1 \cup I_2 = [N] \) with \( \#I_1 = k \) and \( x \in \mathbb{R}^n \). If one chooses an order for the subdomains \( \{ZCD_{k,\delta}\} \to [M] \) and for elements of \( \mathcal{P}_k [N] \), e.g. lexicographical order, then one has the set of mappings
\[ (S_k(x))_\tau := s(I_\tau; x), \quad \tau = 1, \ldots, \binom{N}{k} \] (6.2)
and
\[ S_{k,\delta} := S_k(x) \] (6.3)
which assigns to a subdomain \( ZCD_{k,\delta} \) a vector of \( \binom{N}{k} \) components, whose \( \tau \)-th component is the sign of \( Z_k(I_\tau; x) \) evaluated at an interior point \( x \in ZCD_{k,\delta} \).

With a slight abuse of notation, we will also use \( \delta \in [M] \) to denote the corresponding \( ZCD_{k,\delta} \) with the chosen order. For example, \( S_{2,\delta} = (-1, 1, 1, 1, -1, -1, -1, 1, 1, 1, 1, 1, 1) \) where \( \{f_\alpha\} \) are as in figure 3 and \( \delta \) is the subdomain containing the point \((x, y) \equiv (2, -2)\). In the domain \( \mathcal{D}_{k+} \) one has \( S_{k;\mathcal{D}_{k+}} = (1, 1, 1, \ldots, 1) \).

Number of signs −1 in \( S_{k,\delta} \) varies in \( ZCD_k \cup \mathcal{D}_{k-} \). One has

**Proposition 5.** The maximum number of −1 in \( S_{k,\delta} \) at fixed \( k < \frac{N}{2} \) and varying \( \delta \in [M] \) is equal to \( \binom{N-1}{k} \). If \( 2k = N \) then the number of −1 signs in \( S_{\frac{N}{2}} \) is identically equal to \( \binom{2k-1}{k} \) on \( \mathbb{R}^n \setminus \mathcal{Z}_{\text{sing}, \frac{N}{2}} \).

**Proof.** At fixed \( k < \frac{N}{2} \) and for any subdomain \( ZCD_{k,\delta} \) one can consider the family
\[ \mathcal{F}_{k,\delta-} := \{ I \in \mathcal{P}_k [N] : s_{k,\delta}(I) = -1 \} \] (6.4)
The intersection between two elements of \( \mathcal{F}_{k,\delta-} \) is non-empty. Indeed, let us assume that \( I \in \mathcal{F}_{k,\delta-} \) and \( I \cap J = \emptyset \), with \( \#J = k \). In particular, one has \( J \subseteq [N] \setminus I \) and \( I \subseteq [N] \setminus J \). This implies that
\[ \sum_{\alpha \in [N] \setminus J} e^{f_\alpha(x)} > \sum_{\alpha \in I} e^{f_\alpha(x)} > \sum_{\beta \in [N] \setminus I} e^{f_\beta(x)} \geq \sum_{\beta \in J} e^{f_\beta(x)} \] (6.5)
Thus, $Z_k(\mathcal{J}) > 0$ and $\mathcal{J} \notin \mathcal{F}_{k,\delta-}$. Then, the family $\mathcal{F}_{k,\delta-}$ of all $k$-subsets, $k < \frac{N}{2}$, corresponding to a $-1$ sign is an intersecting family, that is a family of subsets with same cardinality $k$ and pairwise non-empty intersections. Hence, Erdős–Ko–Rado theorem for intersecting family (see e.g. 48) holds and so $\mathcal{F}_{k,\delta-}$ has at most $\binom{N-1}{k-1}$ elements. Moreover, this maximum is reached exactly if all elements of the family contain a certain $\alpha_0 \in [N]$. This maximum is indeed attained. Let us consider 1-domains $D_{1-}(\alpha), \alpha \in [N]$, defined as

$$D_{1-}(\alpha) := \left\{ x \in \mathbb{R}^n : Z_1(\{\alpha\}; x) = -e^{f_\alpha(x)} + \sum_{\beta \neq \alpha} e^{f_\beta(x)} < 0 \right\}, \quad \alpha = 1, \ldots, N.$$  

(6.6)

The linearity of functions $f_\alpha$ assures that not all $D_{1-}(\alpha)$ are empty since the assumptions of proposition 3 are satisfied and $\vec{r}(t_1) \in D_{1-}(\alpha_0)$. If $x \in D_{1-}(\alpha)$ then one has $|Z_k(\mathcal{I}; x)| = -\sum_{\alpha \in \mathcal{I}} e^{f_\alpha(x)} + \sum_{\beta \in [N] \setminus \mathcal{I}} e^{f_\beta(x)} > e^{f_\alpha(x)} - \sum_{\beta \neq \alpha} e^{f_\beta(x)} > 0$ for all $\mathcal{I} \in \mathcal{P}_k[N], \text{so } D_{1-}(\alpha) \notin Z_{\text{sing},k}$ and all components of $S_k(x)$ are not vanishing. Let us denote a subdomain $ZCD_{k,\delta}$ such that $D_{1-}(\alpha) \cap \delta(\alpha) \neq \emptyset$ as $\delta(\alpha)$. Then, $S_{k;\delta(\alpha)}$ coincides with $S_k(x)$, $x \in D_{1-} \cap \delta(\alpha)$. The definition of $D_{1-}(\alpha)$ implies that $Z_k(\mathcal{I}; x) < 0$ if and only if $\alpha \in \mathcal{I}$ and the number of $-1$ signs in $S_{k;\delta(\alpha)}$ is equal to the number of $k$-subsets $\mathcal{I} \subset [N]$ containing $\alpha$, that is the maximum $\binom{N-1}{k-1}$.

At $2k = N$ and for any point $x \in \mathbb{R}^n \setminus Z_{\text{sing},\frac{N}{2}}$ one has $Z_k(\mathcal{I}) < 0$ if and only if $Z_k([N] \setminus \mathcal{I}) > 0$. Since both $\mathcal{I}$ and $[N] \setminus \mathcal{I}$ have cardinality $\frac{N}{2}$, there is the same number of negative and positive terms in $S_k(x)$ for all $x \in \mathbb{R}^n \setminus Z_{\text{sing},\frac{N}{2}}$, that is $\binom{2k-1}{k} = \binom{2k-1}{k-1} = \frac{1}{2} \binom{2k}{k}$.  

Thus, the ambient domain $D_{k-}$ for each statistical $k$-amoeba is the domain of maximal instability. Previous proposition implies following

**Corollary 1.** One has

$$D_{k-} \subseteq D_{k-}$$  

(6.7)
for all \(1 \leq k < \hat{k} < \frac{N}{2}\).

**Proof.** Let us take \(x \in D_{k-}\) at \(1 \leq k < \hat{k} < \frac{N}{2}\). From Erdős–Ko–Rado theorem and proposition 5 it follows that there exists \(\alpha_0 \in [N]\) such that \(Z_k(I; x) < 0\) if and only if \(\alpha_0 \in I\). Then, let us consider \(J \subset [N]\) with \(\#J = \hat{k}\) and \(\alpha_0 \in J\). One can choose a subset \(I_J \subset J\) such that \(\#I_J = k\) and \(\alpha_0 \in I_J\), thus \(Z_{\hat{k}}(J; x) < Z_k(I_J; x) < 0\) where last inequality holds since \(\alpha_0 \in I_J\). Then, \(Z_{\hat{k}}(J) < 0\) if \(\alpha_0 \in J\) and \(Z_{\hat{k}}(J) > 0\) otherwise, since additional \(-1\) signs would contradict the bound \(\left\lfloor \frac{N-1}{\hat{k}-1} \right\rfloor\) in proposition 5. This means that \(x \in D_{\hat{k}-}\).

Consequently one also has the chain

\[
D_1- \subseteq D_2- \subseteq \cdots \subseteq D_{\left\lfloor \frac{N}{2} \right\rfloor-}.
\]

which is dual with respect to (4.2). It is equivalent to

\[
D_0+ \cup ZCD_1 \supseteq D_2+ \cup ZCD_2 \supseteq \cdots \supseteq D_{\left\lfloor \frac{N}{2} \right\rfloor+} \cup ZCD_{\left\lfloor \frac{N}{2} \right\rfloor-}.
\]

So, the domain of complete stability \(D_{k+}\) (possible) equilibrium shrinks in transition to higher strata while the domain of instability expands. Note that there is no domain in \(\mathbb{R}^n\) where all \(Z_k(I)\) are negative if \(k \leq \left\lfloor \frac{N}{2} \right\rfloor\).

For each subdomain \(ZCD_{k;\delta}\) and the corresponding set \(S_{k;\delta}\) one can introduce also its integral characteristic

\[
\bar{S}_{k;\delta} = \frac{1}{\binom{N}{k}} \sum_{\tau=1}^{\binom{N}{k}} (S_{k;\delta})_\tau.
\]

This quantity take values in the interval \([1 - 2\frac{k}{N}; 1], k \leq \left\lfloor \frac{N}{2} \right\rfloor\). The maximum is reached in the stability domain \(D_{k+}\) while the minimum \(-1 + 2\frac{k}{N}\) is achieved at the ambient domain \(D_{k-}\) of maximal instability. An absolute minimum of \(\bar{S}_{k;\delta}\)
equal to 0 is reached in $\mathbb{R}^n$ when $N$ is even and $N = 2k$. An example of the function $\bar{S}_{k,\delta}$ is presented in the figure 12.

\[ (a) \bar{S}_{1,\delta} = \frac{1}{6} \cdot \sum_{\tau=1}^{6} (S_{1,\delta})_{\tau}. \]

\[ (b) \bar{S}_{2,\delta} = \frac{1}{15} \cdot \sum_{\tau=1}^{15} (S_{2,\delta})_{\tau}. \]

**Figure 12.** $\bar{S}_{1,\delta}$ and $\bar{S}_{2,\delta}$ in the case $f_1 \equiv 0, f_2 \equiv 3x, f_3 \equiv 3y, f_4 \equiv x + y + \ln 6, f_5 \equiv 2x + y + \ln 11, f_6 \equiv x + 3y + \ln 4.$

The formula (6.10) suggests also a natural interpretation of $\bar{S}_{k,\delta}$. Indeed, let us view values of sign function (6.1) as two projections +1 and −1 of a “spin” associated with the subdomain $ZCD_{k,\delta}$ and certain functions $Z_{k,\tau}(x)$. So at the subdomain $ZCD_{k,\delta}$ one has a set of $\binom{N}{k}$ “spins” with different projections. Assuming that projections associated with functions $Z_{k,\tau}$ at different $\tau$ are realised with the same probability $w_{N,k} = \frac{1}{\binom{N}{k}}$ then $\bar{S}_{k,\delta}$ defined by (6.10) is just the mean value of spin at the subdomain $ZCD_{k,\delta}$.

Further, one can view the collection of $S_{k,\delta}$ for all subdomain $ZCD_{k,\delta}$ and domains $D_{k,+}, D_{k,-}$ as the set of states of the statistical system of $\binom{N}{k}$ spins. Considering the interaction of spins with external (magnetic) field $H$ as for the standard spin systems (see e.g.\(^1,\)\(^2\)), one defines the energy

$$E_{k,\delta} = -H \cdot \sum_{\tau=1}^{\binom{N}{k}} (S_{k,\delta})_{\tau}, \quad \delta = 1, \ldots, M. \quad (6.11)$$
Finally, for the partition function of the spin system one has

\[ Z_{k, \text{spin}} = \sum_{\delta=1}^{M} \exp \left( -\beta H \cdot \sum_{\tau=1}^{N_k} (S_{k, \delta})_{\tau} \right) , \quad k = 1, \ldots, \left\lceil \frac{N}{2} \right\rceil \]  

(6.12)

where \( \beta \) is a parameter (say inverse of “temperature” \( T \)).

Energy \( E_{k} \) has minimum at the domain \( D_{k+} \) and maximum in the domain \( D_{k-} \).

Excited transition states are associated with subdomains of \( ZCD_k \).

Introducing interaction between spins of the form

\[ E_{k; \delta, \text{int}} = \gamma \cdot \sum_{\tau, \nu=1}^{N_k} (S_{k, \delta})_{\tau} \cdot (S_{k, \delta})_{\nu} , \]

one gets a partition function of the Ising type model.

7. Tropical limit and tropical zeros

The amoebas viewed at large distance are essentially the sets of thinning tentacles which become certain piecewise linear objects in the tropical limit for algebraic amoebas, see e.g. \(^{42,49}\). Such images of statistical \( k \)-amoebas are associated with the limiting behaviour at large functions \( f_{\alpha}(x) \) in the partition function (1.2). In the case of linear functions \( f_{\alpha}(x) \) as in (2.1) there are different ways to realise such a limit. The first one is to consider large values of the variables \( x_i \) introducing slow variables \( \tilde{x}_i := \varepsilon \cdot x_i \), \( i = 1, \ldots, n \) with \( \varepsilon \to 0 \). For the \( k \)-th stratum the functions \( Z_k(I_{\tau}; x) \) at \( \varepsilon \to 0 \) are the superpositions of highly singular terms and the corresponding hypersurfaces are defined as

\[ \sum_{\alpha=1}^{N} g_{I_{\tau}}(\alpha) \exp \left( \frac{1}{\varepsilon} \cdot \sum_{i=1}^{n} a_{\alpha i} \tilde{x}_i \right) = 0 , \quad \tau = 1, \ldots, \left( \frac{N_k}{k} \right) \]  

(7.1)

as \( \varepsilon \to 0 \). For the first stratum \( (k = 1) \) equations (7.1) are of the form

\[ \exp \left( \frac{1}{\varepsilon} \cdot \sum_{i=1}^{n} a_{\alpha i} \tilde{x}_i \right) = \sum_{\beta \neq \alpha} \exp \left( \frac{1}{\varepsilon} \cdot \sum_{i=1}^{n} a_{\beta i} \tilde{x}_i \right) \]  

(7.2)

and in the limit \( \varepsilon \to 0 \) one gets the set of hyperplanes in \( \mathbb{R}^n \) given by

\[ \sum_{i=1}^{n} a_{\alpha i} \tilde{x}_i = \max_{\beta \neq \alpha} \left\{ \sum_{i=1}^{n} a_{\beta i} \tilde{x}_i \right\} . \]  

(7.3)
All these hyperplanes pass through the origin $\tilde{x}_i = 0$, $i = 1, \ldots, n$. They are the tropical limit of the ideal statistical hypersurfaces considered in $^{37}$.

For higher strata and each partition $(I_1, I_2)$ the limit $\varepsilon \to 0$ of equations (3.4), (3.5) is given by the set of hyperplanes

$$\max_{a \in I_1} \left\{ \sum_{i=1}^{n} a_{ai} \tilde{x}_i \right\} = \max_{b \in I_2} \left\{ \sum_{i=1}^{n} b_{bi} \tilde{x}_i \right\}. \quad (7.4)$$

The second way to realise the limit $f_\alpha \to \infty$, more close to the standard tropical limit in algebraic geometry $^{42,49}$, is to make the parameter $b_\alpha$ in $f_\alpha$ large too, i.e. to consider the limit $x_i = \frac{\tilde{x}_i}{\varepsilon}$, $b_\alpha = \frac{\tilde{b}_\alpha}{\varepsilon}$, with finite $\tilde{x}_i$, $\tilde{b}_\alpha$, and $\varepsilon \to 0$. In this case the tropical limit of the hypersurfaces (3.4), (3.5) is given by

$$\max_{a \in I_1} \left\{ \tilde{b}_\alpha + \sum_{i=1}^{n} a_{ai} \tilde{x}_i \right\} = \max_{b \in I_2} \left\{ \tilde{b}_\beta + \sum_{i=1}^{n} a_{bi} \tilde{x}_i \right\}. \quad (7.5)$$

Now the hyperplanes (7.5) do not pass, in general, through the origin $\tilde{x}_i = 0$, $i = 1, \ldots, n$.

The third way is to keep variables $x_i$ finite, but to send to infinity the parameters $a_{ai}$ and $b_\alpha$ as $a_{ai} = \frac{\tilde{a}_{ai}}{\varepsilon}$, $b_\alpha = \frac{\tilde{b}_\alpha}{\varepsilon}$, with $\varepsilon \to 0$ and finite $\tilde{a}_{ai}$, $\tilde{b}_\alpha$. Such a limit of hypersurfaces (3.4), (3.5) is given by the set of hyperplanes defined by equations

$$\max_{a \in I_1} \left\{ \tilde{b}_\alpha + \sum_{i=1}^{n} \tilde{a}_{ai} x_i \right\} = \max_{b \in I_2} \left\{ \tilde{b}_\beta + \sum_{i=1}^{n} \tilde{a}_{bi} x_i \right\}. \quad (7.6)$$

Equations (7.5) and (7.6) are related via exchange $a_{ai} \leftrightarrow \tilde{a}_{ai}$, $\tilde{x}_i \leftrightarrow x_i$ keeping in both cases the product $a_{ai} x_i \sim \frac{1}{\varepsilon}$.

For different strata the sets of equations (7.5) or (7.6), defining the tropical limit of hypersurfaces (3.4), (3.5) are quite different. However, one has

**Proposition 6.** In the tropical limits considered above, zeros loci of $Z_{k, \text{trop}}(\mathcal{I})$ given by equations (7.5) or (7.6) are the same for all strata. All domains $Z_{\text{CD}} D_k$ collapse into a single set of piecewise hyperplanes given e.g. by equations (7.5) or (7.6) for the first stratum $Z_{\text{sing,1}}$. 

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Proof. Let us denote \( I_1 := I \) and \( I_2 := [N] \setminus I \). The equation \( Z_k(I_1; x) = 0 \), \( \# I_1 = k \), is equivalent to \( \sum_{\alpha \in I_1} e^{f_{\alpha}(x)} = \sum_{\beta \in I_2} e^{f_{\beta}(x)} \). In term of slow variables it becomes \( \sum_{\alpha \in I_1} \exp \left( \frac{f_{\alpha}(x)}{\varepsilon} \right) = \sum_{\beta \in I_2} \exp \left( \frac{f_{\beta}(x)}{\varepsilon} \right) \). Let us take \( \bar{\alpha}_i \in I_i \) such that \( f_{\bar{\alpha}_i}(x) = \max_{\alpha \in I_i} \{ f_{\alpha}(x) \} \), \( i \in \{1, 2\} \). Then, previous equation is equivalent to

\[
\exp \left( \frac{f_{\bar{\alpha}_1}(x)}{\varepsilon} \right) \cdot \left[ \sum_{\beta \in I_1} \exp \left( \frac{f_{\beta}(x) - f_{\bar{\alpha}_1}(x)}{\varepsilon} \right) \right] = \exp \left( \frac{f_{\bar{\alpha}_2}(x)}{\varepsilon} \right) \cdot \left[ \sum_{\gamma \in I_2} \exp \left( \frac{f_{\gamma}(x) - f_{\bar{\alpha}_2}(x)}{\varepsilon} \right) \right].
\]  

(7.7)

Both the factors in square bracket in (7.7) lie in the interval \([1, N-k]\) independently on \( \varepsilon \in \mathbb{R}_+ \). Hence they are finite and non-vanishing. Thus, \( \exp \left( \frac{f_{\bar{\alpha}_1}(x)}{\varepsilon} \right) \cdot \left[ \sum_{\beta \in I_1} \exp \left( \frac{f_{\beta}(x) - f_{\bar{\alpha}_1}(x)}{\varepsilon} \right) \right] \) lies in \( \left[ \frac{1}{N-k}; N-k \right] \) for all \( \varepsilon \in \mathbb{R}_+ \). Considering the limit \( \varepsilon \to 0 \) one gets

\[
\max_{\alpha \in I_1} \{ f_{\alpha}(x) \} = \max_{\beta \in I_2} \{ f_{\beta}(x) \}.
\]  

(7.8)

Note that \( \bar{\alpha}_1 \neq \bar{\alpha}_2 \) since they belong to different parts of the partition. For any \( \gamma \in [N], \gamma \in I_i \) for exactly one \( i \in \{1, 2\} \), so \( f_{\gamma}(x) \leq \max_{\alpha \in I_1} \{ f_{\alpha}(x) \} = f_{\bar{\alpha}_1}(x) \). Thus, \( f_{\bar{\alpha}_1}(x) = f_{\bar{\alpha}_2}(x) = \max_{\gamma \in [N]} \{ f_{\gamma}(x) \} \) so the maximum \( \max_{\gamma \in [N]} \{ f_{\gamma}(x) \} \) is attained at least twice, once for each index \( i \in \{1, 2\} \) of \( I_i \). Considering all such partitions with \( \# I = k \), one gets the union of all these tropical loci. This is the set of all points \( x \in \mathbb{R}^n \) such that maximum of \( \{ f_1(x), \ldots, f_N(x) \} \) is attained at least twice and it is independent of the stratum \( k \) considered.

So, in the tropical limit the statistical amoebas collapse into the \((n - 1)\)-dimensional objects \( A_{\text{trop}} \) formed by pieces of hyperplanes and the maximal instability domains \( D_{k-} \) expand to the almost whole space \( \mathbb{R}^n \), namely to \( \mathbb{R}^n \setminus A_{\text{trop}} \). Points of the piecewise hyperplanes \( A_{\text{trop}} \) are tropical zeros of partition function.

These different kinds of tropical limit provide different structures for the same underlying model. For example, the tropical limit of the first kind (7.4) highlights the degree 1 homogeneous part of linear functions \( f_{\alpha} \). More in general, it gives the dominant homogeneous parts of functions \( f_{\alpha} \) and can be applied in the study
of emergence of degenerate metrics from tropical limit, see e.g.\(^{37}\). An advantage of tropical limit of the first kind is that it has a rather simple geometry.

**Lemma 3.** If \(f_\alpha\) are \(N\) real functions then
\[
\sum_{\alpha=1}^{N} e^{\lambda f_\alpha(x)} \leq \left(\sum_{\alpha=1}^{N} e^{f_\alpha(x)}\right)^{\lambda}
\]
for all \(\lambda \geq 1\).

**Proof.** For all \(\lambda \geq 1\) one has
\[
0 < \left(\sum_{\beta} e^{f_\beta(x)}\right)^{\lambda} \leq \sum_{\alpha} e^{f_\alpha(x)} < 1, \quad \alpha \in [N],
\]
which implies
\[
0 < \sum_{\alpha=1}^{N} \left(\frac{e^{f_\alpha(x)}}{\sum_{\beta} e^{f_\beta(x)}}\right)^{\lambda} \leq \sum_{\alpha=1}^{N} e^{f_\alpha(x)} = 1 \Rightarrow \sum_{\alpha=1}^{N} e^{\lambda f_\alpha(x)} \leq \left(\sum_{\alpha=1}^{N} e^{f_\alpha(x)}\right)^{\lambda}. \quad (7.10)
\]

**Proposition 7.** Connected components of the complement of the tropical graph of the first kind are unbounded. For homogeneous functions, connected components of \(D_{1-}\) are unbounded too.

**Proof.** Given \(N\) linear functions \(f_1, \ldots, f_N\), let \(\varphi_\alpha(x) = f_\alpha(x) - f_\alpha(0)\) be the 1-homogeneous part of \(f_\alpha\), \(\Delta_{1-}(\alpha) := \left\{ y \in \mathbb{R}^n : e^{\varphi_\alpha(y)} > \sum_{\beta \neq \alpha} e^{\varphi_\beta(y)} \right\}\) be the instability domain where \(\varphi_\alpha\) dominates and \(\Delta_{1-}^{\text{trop}}(\alpha) := \left\{ y \in \mathbb{R}^n : \varphi_\alpha(y) > \max_{\beta \neq \alpha} \left\{ \varphi_\beta(y) \right\} \right\}\) be the tropical limit of \(\Delta_{1-}(\alpha)\). In particular, it easily follows from the definitions that \(\Delta_{1-}(\alpha) \subseteq \Delta_{1-}^{\text{trop}}(\alpha)\). From (7.4), \(\{f_\alpha\}\) and \(\{\varphi_\alpha\}\) have the same tropical limit of the first kind, so we focus on the latter set of functions. If \(x \in \Delta_{1-}(\alpha)\) and \(\lambda \geq 1\) then
\[
e^{\varphi_\alpha(\lambda x)} = \left(e^{\varphi_\alpha(x)}\right)^{\lambda} > \left(\sum_{\beta \neq \alpha} e^{\varphi_\beta(x)}\right)^{\lambda} \geq \sum_{\beta \neq \alpha} e^{\lambda \varphi_\beta(x)} = \sum_{\beta \neq \alpha} e^{\varphi_\beta(\lambda x)}
\]
where the second inequality follow from lemma 3 applied to homogeneous functions \(\varphi_\beta, \beta \neq \alpha\). Hence \(\lambda \cdot x \in \Delta_{1-}(\alpha)\) for all \(x \in \Delta_{1-}(\alpha)\) and \(\lambda \geq 1\). In the same way
one can show that $\lambda \cdot x \in \Delta_{1-\text{trop}}(\alpha)$ for all $x \in \Delta_{1-\text{trop}}(\alpha)$ and $\lambda \geq 1$. So, let $C$ (respectively, $C^*$) be a connected component of $\Delta_{1-}(\alpha)$ (respectively, of $\Delta_{1-\text{trop}}(\alpha)$) and choose $x \in C$ (respectively, $x^* \in C^*$). One has $\{\lambda \cdot x : \lambda \geq 1\} \subseteq C$ since the ray $\{\lambda \cdot x : \lambda \geq 1\}$ is a connected subset of $\Delta_{1-}(\alpha)$ intersecting $C$ and $C$ is maximal among connected subsets of $\Delta_{1-}(\alpha)$. Similarly, $\{\lambda \cdot x^* : \lambda \geq 1\} \subseteq C^*$. Thus both $C$ and $C^*$ are unbounded since they contain an unbounded subset.

Thus, tropical limit of the first kind has simple topological properties. For example, in two-dimensional case, the result of proposition 7 means a trivial homotopy for the resulting tropical graph.

It is worth mentioning that terms in (7.4) coincide with $f_\alpha(x)$ if $f_\alpha(0) = 0$ for all $\alpha \in [N]$. Homogeneous linear functions $f_\alpha(x) \equiv \sum_{i=1}^{n} \kappa_i^\alpha x_i$ with real distinct parameters $\kappa_1^\alpha < \cdots < \kappa_N^\alpha$ represent a particular example. These functions arise in the study of Wronskian soliton solutions of KP II equation where $e^{f_\alpha(x)}$ are special solutions of the heat hierarchy. If one considers the tropical limit of the second kind (7.5) instead of (7.4), then the resulting object has a more refined structure and many combinatorial properties (see e.g.50).

Tropical limits discussed above are quite meaningful in the statistical physics of macrosystems. Tropical limit of free energy considered in51 corresponds to $n = 1$, $\tilde{x}_1 = \frac{1}{k_B T}$, $a_{\alpha 1} = -E_\alpha$, $b_\alpha = \frac{S_\alpha}{k_B}$, $\varepsilon = k_B$ where $T$ is the temperature, $\{E_\alpha\}$ is the energy spectrum, $\exp \left( \frac{S_\alpha}{k_B} \right)$ are degenerations of energy levels and $k_B$ is the Boltzmann constant.

One can consider also more complicated situations when some of the products $a_{\alpha i} \cdot x_i$ remain finite, for instance, when $x_{i_0} = \frac{\tilde{x}_{i_0}}{\varepsilon}$ and $a_{\alpha i_0} = \varepsilon \cdot \tilde{a}_{\alpha i_0}$. In such a case the product $a_{\alpha i_0} \cdot x_{i_0}$ does not contribute in the limit $\varepsilon \to 0$ and the corresponding equation (7.4), or (7.5), will not contain the variable $\tilde{x}_{i_0}$. So in the tropical limit the zero locus is a piecewise hyperplane of cylindrical type.

Such non-uniform scaling behaviour of the variables $x_i$ or parameters $a_{\alpha i}$ and
its connections with the multiscale tropical limit will be discussed elsewhere.

8. Conclusion

In this paper partition functions (1.2) with linear $f_\alpha(x)$ have been studied. The case of nonlinear functions $f_\alpha(x)$ is of great interest too. Many general properties of singular sectors described above, e.g. stratification of statistical $k$-amoebas, remain unchanged for more general polynomial functions $f_\alpha(x)$. Specifically, for polynomials $f_\alpha(x)$ the set of roots (4.4) is finite and at least one $D_1_-(\alpha)$ in (6.6) is not empty. These hypotheses are crucial for propositions as 3, 5 and corollary 1 to be valid. Proposition 7 can be generalized to polynomials by considering their degree $d$ homogeneous parts, where $d := \max_{\alpha \in \mathbb{N}} \{\deg f_\alpha\} < \infty$. Figure 13 presents an example of such a type.

![Figure 13. Stratification of $Z_{\text{sing},2}$ (blue) and $Z_{\text{sing},1}$ (orange) in a nonlinear polynomial case: $f_1 \equiv 0$, $f_2 \equiv 3x^2$, $f_3 \equiv 3y^3$, $f_4 \equiv x + xy + \ln 6$, $f_5 \equiv 2x + y^2 + \ln 11$, $f_6 \equiv x + 3y + xy + \ln 4$.](image)

However for general nonlinear functions situation is quite different. An example
with non-polynomial functions

\[
f_\alpha(x) \equiv \begin{cases} 
  c_\alpha \cdot \eta(||x|| - \frac{\alpha+1999}{1000}) & ||x|| > \frac{\alpha+1999}{1000} \\
  d_\alpha \cdot \eta(||x|| - \frac{\alpha+1999}{1000}) & ||x|| \geq \frac{\alpha+1999}{1000}
\end{cases}
\]  

(8.1)

where \( \eta(z) = \begin{cases} 
  1 - \exp\left(-\frac{z^2}{1-z^2}\right), & 1 > |z| \\
  1, & |z| \geq 1
\end{cases} \), \((c_1, d_1) = (\ln 20, \ln 8), (c_2, d_2) = (\ln 20, \ln 2)\) and \((c_\alpha, d_\alpha) = (\ln 2, \ln 2), \alpha = 3, \ldots, 10,\) is shown in figure 14. Let us consider \( S_k(x) := C_k^{10} \cdot \bar{S}_k(x) \) at \( k = 3, 4. \) At \( ||x|| \leq 1 \) one has \( Z_k(I; x) < 0 \) iff \( \{1, 2\} \subset I. \) Thus \( S_k(x) = \left(\begin{smallmatrix} 10 \\ k \end{smallmatrix}\right) - 2 \cdot \left(\begin{smallmatrix} 8 \\ k-2 \end{smallmatrix}\right) \). At \( 1 \leq ||x|| < 2 \) \( S_k(x) \) is not decreasing. At \( ||x|| \geq 2 \) one has \( Z_3(I; x) > 0 \) for all \( I \in \mathcal{P}_3[10], \) then \( S_3(x) = C_3^{10}. \) On the other hand, at \( ||x|| \geq 4 \) one has \( Z_4(I; x) < 0 \) iff \( 1 \in I, \) hence \( S_4(I; x) \) has a minimum according to Erdos-Ko-Rado theorem\(^{48}. \) In conclusion, 

\[
\min_x S_3(x) = \left(\begin{smallmatrix} 10 \\ 3 \end{smallmatrix}\right) - 2 \cdot \left(\begin{smallmatrix} 8 \\ 1 \end{smallmatrix}\right) = 104 \text{ is attained only if } ||x|| < 2. \text{ Vice versa, } \\
\min_x S_4(x) = \left(\begin{smallmatrix} 10 \\ 4 \end{smallmatrix}\right) - 2 \cdot \left(\begin{smallmatrix} 9 \\ 3 \end{smallmatrix}\right) = 42 \text{ is attained only if } ||x|| > 2. \text{ In particular, } \\
\mathcal{D}_{3-} \not\subseteq \mathcal{D}_{4-}.
\]

Figure 14. A non-polynomial case when chain stratification of instability domains fails.

Singular sectors of partition functions (1.2) with nonlinear \( f_\alpha(x) \) will be considered in a separate publication.
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