Surrogate models for efficient stability analysis of brake systems

Lyes Nechak, Frédéric Gillot, Sébastien Besset and Jean-Jacques Sinou
Laboratoire Tribologie et Dynamique de Systèmes, Ecole Centrale de Lyon-CNRS, 36 av. Collongue, 69134 Ecully cedex, France
E-mail: lyes.nechak, frédéric.gillot, sebastien.besset, jean-jacques.sinou@ec-lyon.fr

Abstract. This study assesses capacities of the global sensitivity analysis combined together with the kriging formalism to be useful in the robust stability analysis of brake systems, which is too costly when performed with the classical complex eigenvalues analysis (CEA) based on finite element models (FEMs). By considering a simplified brake system, the global sensitivity analysis is first shown very helpful for understanding the effects of design parameters on the brake system’s stability. This is allowed by the so-called Sobol indices which discriminate design parameters with respect to their influence on the stability. Consequently, only uncertainty of influent parameters is taken into account in the following step, namely, the surrogate modelling based on kriging. The latter is then demonstrated to be an interesting alternative to FEMs since it allowed, with a lower cost, an accurate estimation of the system’s proportions of instability corresponding to the influent parameters.

1. Introduction
The increasingly stringent requirements concerning tolerable noise levels in agglomerations and stations have made squeal prediction a very important issue for car and train manufacturers [1, 2, 3] since brake squeal is characterized by high and audible frequencies and high intensities that are discomfort for passengers while it does not affect the braking’s efficiency [4]. Two main groups of methods can be distinguished in brake squeal prediction’s framework. The nonlinear dynamic transient analysis is used to determine the time evolution of mechanical solutions (such as displacements, velocities, etc). However, it remains still prohibitive regarding to its time consuming even when associated with reduced and/or simplified FEMs [5]. So, the complex eigenvalue analysis remains the most used method [6]. Its main principle is to linearize the system’s model around its equilibrium state then to analyze the sign of the resulting linear system’s eigenvalues according to the Lyapunov theory. The main drawback of the CEA is that the associated stability analysis may lead to an underestimation or an over-estimation of the unstable modes observed in the nonlinear time simulation due to the fact that linear conditions (i.e. the linearized stability around an initial equilibrium point with a defined contact state for each node are not valid during transient oscillations [7]). However, the method remains less expensive than the nonlinear transient dynamic analysis but, because of the complexity of finite element models, it stays costly when robustness and/or optimal objectives are required. Therefore, it is necessary to search for alternative models that can be efficiently
used in stability analysis instead of FEMs. This is the main goal of this paper. Indeed, we propose the combination of the global sensitivity analysis together with kriging modeling to build simpler models helping for a more efficient stability analysis of brake systems. The global sensitivity analysis helps, by means of Sobol indices [8], for the quantification of individual and/or collective impacts of parameters on the brake system’s stability. This issue has been treated in other studies but sensitivities was calculated locally (in a relatively small neighborhood of nominal values of parameters, see [9] and references are therein). Based on the global sensitivity analysis, the kriging formalism [10] is then proposed to construct substituting models to FEMs, which can be used efficiently to analyze the stability of brake systems by taking into account the uncertainty of the influent parameters.

This paper is organized as follows. First, a simplified brake system is described in Section 2. The global sensitivity analysis of its stability is discussed in Section 3 while kriging is proposed to represent its instability proportions in Section 4.

2. Finite element model

This study concerns a simplified brake system represented by a finite element model (Fig.1) obtained from the assembling of finite element models of a pad and a disc with a contact surface modeled by nine uniformly spaced contact nodes. The inner surface of the disc is assumed clamped while the outline of the upper surface of the pad is just translating along the normal direction [11]. The simplified assembled model has 174 dofs issued from 80 dofs for the disc, 40 dofs for the pad and 54 dofs for the contact interface. As the main objective of this study is to assess the feasibility of the combination of global sensitivity analysis together with kriging, we consider in the following a non-damped dynamic behavior for the brake system. It is then described by the classic second order differential equation given by

\[ M \ddot{X} + (K + K_c)X = F_{\text{pressure}} \]  

(1)

where \( M \) is the mass matrix, \( K \) denotes the structural stiffness matrix while \( X \) is the displacement vector with the associated velocity \( \dot{X} \) and acceleration \( \ddot{X} \). Otherwise, \( F_{\text{pressure}} \) represents the force due to the brake pressure applied on the pad. \( K_c \) represents a non-symmetric matrix defined to model the contact interface between the disc and the pad at the selected interface nodes where normal and tangential contact forces are applied by the pad over the disc and conversely by the disc over the pad. The friction contact is assumed to follow a Coulomb friction law with a constant friction coefficient \( \mu \) and a permanent sliding. Normal and tangential contact forces are expressed for each node as follows [11].

\[ F_{\text{normal},i}^{d} = \begin{cases} k_c(X_{d,i} - X_{p,i}) & \text{if } (X_{p,i} - X_{d,i}) > 0, \\ 0 & \text{otherwise.} \end{cases} \]  

(2)

\[ F_{\text{tangent},i}^{d} = \mu F_{\text{normal},i}^{d} \text{sign}(v_{r,i}) \]  

(3)
where \( X_{i,d} \) and \( X_{i,p} \) define the \( i^{th} \) node displacements of the disc and the pad denoted by \( d \) and \( p \) respectively, \( v_{i,r} \) is the corresponding velocity.

3. Stability and sensitivity analysis

The main question in this section is to quantify how the stability of the considered brake system is impacted by some design parameters \( p_i \in \{ e_d, e_p, \rho_d, \rho_p, \mu, k_c \} \) including Young modulus \( \{ e_d, e_p \} \) and densities \( \{ \rho_d, \rho_p \} \) of the pad and disc, the friction coefficient \( \{ \mu \} \) and the linear contact stiffness \( \{ k_c \} \), see Table 1 where intervals variations \( I_{p_i} \) of parameters \( p_i \) are given. These parameters are assumed to be driven by independent uniform laws within their respective intervals while the other parameters defining the geometry of the pad and disc are considered as determinist.

Stability of the considered brake system is characterized by its eigenvalues \( \lambda = a + jb \) which are obtained from the solution of the following CEA problem,

\[
(\lambda^2 M + (K + K_c)) \Psi = 0
\]

The Monte Carlo method is often used to analyze parameters effects on the stability properties. It consists of generating parameters samples following their probability density functions and calculating eigenvalues from the corresponding CEA problem. We have used this procedure so, real parts of the resulting eigenvalues are plotted against imaginary parts in Fig. 2. A stable eigenvalue \( \lambda = a + jb \) is said to be asymptotically stable if its real part \( a \) is strictly negative and unstable if \( a \) is strictly positive.

![Figure 2: Brake system’s eigenvalues in the complex plane obtained from Monte Carlo simulations](image)

From the obtained results, it is so difficult to objectively understand how the considered design parameters impact the distance of eigenvalues from the imaginary axis which defines the stability/instability border. Hence, we propose to carry out the global sensitivity analysis from the Sobol sense of the following distance function in design parameters:

\[
D_{\text{stability}} = \sum_{i=1}^{n} |\text{Re}(\lambda_i(e_d, e_p, \rho_d, \rho_p, \mu, k_c))|
\]

where \( \text{Re}(\cdot) \) denotes the real part of the \( i^{th} \) eigenvalue \( \lambda_i(\cdot) \).

The main idea is to determine Sobol indices which quantify the variability of the distance function induced by the variability of one or several parameters. From sensitivity analysis theory [8]), first order sensitivity indices (also named first order Sobol indices) quantifying the individual influence of parameter \( p_i \) on the stability function \( D_{\text{stability}} \) can be defined by
Table 1: Variability intervals of mechanical input parameters

| Parameters                        | Minimum value | Maximum value |
|-----------------------------------|---------------|---------------|
| Friction coefficient             | 0.2           | 0.7           |
| Disc Young modulus (pa)           | $14 \times 10^4$ | $22 \times 10^4$ |
| Pad Young modulus (pa)            | 0.5 $10^9$    | 2 $10^9$      |
| Disc density (kg.m$^{-3}$)        | 7100          | 7900          |
| Pad density (kg.m$^{-3}$)         | 1400          | 2600          |
| Linear contact stiffness (n.m$^{-1}$) | 1 $10^7$    | 4 $10^7$      |

\[
S_i = \frac{\text{Var}(E[D_{\text{stability}}|p_i])}{\text{Var}(D_{\text{stability}})} \tag{6}
\]

where \(\text{Var}(.)\) denotes the variance operator while \(E[\cdot|\cdot]\) is the conditional expectation operator.

Similarly, high order sensitivity indices measuring influences of couples or combinations of several parameters on the analyzed function can be defined. For example second order indices are given by:

\[
S_{ij} = \frac{\text{Var}(E[D_{\text{stability}}|p_i, p_j])}{\text{Var}(D_{\text{stability}})} \tag{7}
\]

Ultimately, a total index \(S_{T_i}\) can be obtained for a single parameter \(p_i\) by the summation of all indices including \(p_i\), that is:

\[
S_{T_i} = S_i + S_{ij} + S_{ijk} + ... \tag{8}
\]

A high value for \(S_i\) indicates that the corresponding parameter \(p_i\) influences strongly the variability of the stability function (5) while a small value for a total index \(S_{T_i}\) shows that the parameter \(p_i\) possesses a small impact on the stability function even if it is combined with other parameters.

The main step is then to compute the Sobol indices. The referential Monte carlo method is used in the present study \([8, 12]\). The sample size ensuring a convergent calculus of Sobol indices is obtained equal to \(N = 10000\). Table 2 gives both the computed first order Sobol indices and the total indices of input parameters \((p_i \in \{e_d, e_p, \rho_d, \rho_p, \mu, k_c\})\). Two sets of parameters can be

Table 2: First order Sobol indices estimated by Monte Carlo method

| Parameters                        | \(S_i\) with \(N = 10000\) | \(S_{T_i}\) with \(N = 10000\) |
|-----------------------------------|-----------------------------|-------------------------------|
| Friction coefficient             | 0.756                       | 0.865                         |
| Disc Young modulus               | -0.007                      | 0.021                         |
| Pad Young modulus                | -0.007                      | 0.002                         |
| Disc density                     | -0.007                      | 0.004                         |
| Pad density                      | 0.001                       | 0.055                         |
| Linear contact stiffness         | 0.204                       | 0.226                         |
distinguished. The first one includes the friction coefficient and linear contact stiffness while the second contains the remaining parameters. The friction coefficient and the linear contact stiffness present the most significant influences on the stability function (5) since the corresponding first order Sobol indices are significantly high. However, Young modulus and density parameters possess very small first order Sobol indices (negative sign denotes that the corresponding indices are closed to zero,[12]) which denote the non-significant impact of parameters on the considered stability function. Otherwise, relatively small differences between total and first orders Sobol indices especially for friction and linear contact stiffness coefficients show that these parameters influence stability function individually while the collective impacts can be considered as negligible.

4. Kriging based meta-modeling

Based on conclusions of the performed sensitivity analysis, the relevance of using of kriging based models to predict instability proportions of the used brake system is assessed. More accurately, we are interested by the prediction, under the uncertainty of the friction coefficient μ and the linear contact stiffness k_l previously shown as influent parameters, of instability proportions denoted by prop(μ, k_l) which depend implicitly on Young modulus (e_d, e_p) of the disc and the pad, while the corresponding densities of the pad and disc are omitted for the sake of simplicity.

The instability proportion corresponds to the percentage of instability given by the number of times the brake system is unstable within the sample data set \{μ_j, k_j\} for each couple(e_d, e_p). Fig. 3 represents prop(μ, k_l) computed from the solution of 10^7 CEA problems. The final solution is obtained after 21 days. Kriging is then proposed to estimate instability proportions with a lower cost.

4.1. Mathematical formulation of kriging

From kriging theory, it is possible to determine a function at unknown points from some calculus of correlations between couples of sample points generated from the function domain [10]. The correlation is built such as two infinitely distant points be not correlated while two identical points have a unitary correlation. We present in the following main principles of kriging based modeling through its exploiting to represent the instability proportion based function prop(μ, k_l). So, let consider m sample points \(e_{prop} = [e_1, e_2, ..., e_m]\) with \(e = [e_d, e_p]\) and the m outputs \(Y = [y_1, y_2, ..., y_m]^T\) defining instability proportions for each input sample such that \(y_i = prop(e_i)\). Thus, from kriging theory [10], the instability proportion prop(\(e\)) can be written as follows:

\[
prop(e) = f(e)^T \beta + z(e) \quad (9)
\]

The first part in the summation is a constant, linear or polynomial regression model with q basis polynomial functions \(f_i, i = 1, ..., q\) weighted by the regression parameters \(\beta_i\). It defines the mean structure of the function while the second part \(z\) is the realization of a normally distributed random process with a zero mean value and a covariance function given by:

\[
E[z(w)z(e)] = \sigma^2 R(\theta, w, e) \quad (10)
\]
where $\sigma^2$ is the process variance and $\mathcal{R}(\theta, w, e) \in [0, 1]$ is a spatial correlation model with the scaling vector $\theta = [\theta_1, \theta_2]$ which has to be estimated later, $E[.]$ is the expectation operator. This model is a monotone function depending on the distance positive or null between two points $w$ and $e$. It is built such as two infinitely distant points be not correlated while two identical points have a unitary correlation. Correlation models (see Table 3) used are of the following form:

$$\mathcal{R}(\theta, w, e) = \prod_{j=1}^{2} \mathcal{R}_j[-\theta (w_j - e_j)^2]$$

(11)

| Correlation model | $\mathcal{R}(\theta, w_j, e_j)$ |
|-------------------|-------------------------------|
| Exponential       | $\exp(-\theta_j |w_j - e_j|)$ |
| Gaussian          | $\exp(-\theta_j (w_j - e_j)^2)$ |
| Linear            | $\max\{0, 1 - \theta_j |w_j - e_j|\}$ |

Table 3: Correlation functions

Denoting by $G$ a matrix whose entries are given by $g_{i,:} = [f(e^i)]^T$, the regression parameter vector $\beta$ given by

$$\beta^* = (G^T \Phi^{-1} G)^{-1} G^T \Phi^{-1} Y$$

(12)

is the least square solution of the following classical regression problem

$$G \beta \simeq Y$$

(13)

Otherwise, the associated optimal variance $\sigma^2$ is given by

$$\sigma^2 = \frac{1}{m} (Y - G \beta^*)^T \Phi^{-1} (Y - G \beta^*)$$

(14)

where $\Phi$ is the correlation matrix with entries defined by $\phi_{i,j} = \mathcal{R}(\theta, e^i, e^j)$ and $r(x) = [\mathcal{R}(\theta, e^1, x), ..., \mathcal{R}(\theta, e^m, x)]^T$ is a vector containing the values of the correlation between each of the $m$ points of the input sample data set and a variable point $x$ in the design space. The previous optimal parameters $\beta^*$ and $\theta^*$ depend on the inverse matrix $\Phi^{-1}$ which depends in turn on the process correlation parameter vector $\theta$. The optimal choice $\theta^*$ is defined as the solution with respect to $\theta$ of the following optimization problem [13].

$$\min f(\theta) = |\Phi|^{-\frac{1}{m}} \sigma^2$$

(15)

where $|\Phi|$ is the determinant of $\Phi$. Based on the decomposition (9) and the obtained $\theta^*$ from the solution of (15), we are interested by the prediction of the unbiased instability proportion at a given sample point $e$ in the design space. The kriging based predictor [10] of instability proportions is defined by

$$\hat{y}(e) = f^T(e) \beta^* + r(e)^T \gamma^*$$

(16)
where $\beta^*$ is the solution (12) of the regression problem (13) while $\gamma^* = \Phi^{-1}(Y - G\beta^*)$.

4.2. Prediction of instability proportions

In this sub-section kriging models are used to estimate the referential map of instability proportions shown in Fig. 3.

![Figure 3: Instability proportions from FEM based computing](image)

The sample data size $m$ is varied to observe its effect on the accuracy of kriging based estimations. Otherwise, the gaussian spatial correlation model is chosen. In fact, no criterion helping for determining the most suitable correlation model is found. A more complete study would consider other correlation models to select the best one. Kriging estimations of instability proportions obtained by using Dace toolbox [13] for different sample data sizes $m = 100 : 100 : 1000$ are shown in Fig. 4.

![Figure 4: Instability proportions estimated via kriging based models using different sample data sizes $m \in \{100, 300, 600, 900\}$ with exponential correlation model](image)

It can be observed that zones of estimated instability proportions against $(e_d, e_p)$ presented in Fig.4 globally show the same shape as the referential map plotted in Fig.3. The accuracy of the estimated instability proportions depends on the sample data size $m$. Indeed, it increases with $m$. A relatively high sample data size is required for a suitable estimation of stability/instability borders. We could expect this result since zones defining borders between stability and instability
proportions are submitted to fast variations that can also be nonlinear. Otherwise, kriging based estimations have required much smaller time that the referential calculus. For example with \( m = 300 \) the corresponding kriging model has cost \( 3 \times 10^5 \) CEA problems solved after 54432 seconds (almost 15 hours). The cost difference is logically paid by less accurate predictions than the reference. A suitable compromise between the accuracy of kriging based estimated instability proportions and the time consuming has to be defined in practice.

5. Conclusion
This study focused on capacities of the global sensitivity analysis and kriging modeling to be combined together in order to generate simpler models that can be used instead of finite element models for a more efficient robust stability analysis of brake systems. The feasibility of the proposed strategy is tested on a simplified brake system. Firstly, the global sensitivity analysis is shown to be very important in the understanding of the stability behaviour against parameters variabilities. Secondly, kriging is demonstrated to be an interesting alternative to finite element models in brake systems’s robust stability analysis. Indeed, with a fewer number of CEA solutions than the one required by the referential method, a kriging based model can be built and used to suitably estimated instability proportions corresponding to uncertain parameters. Finally, it can be noted that the proposed strategy can be considered for other classes of dynamic systems as rotors.

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