Some remarks on $\mathcal{U}_q(sl(2,\mathbb{R}))$ at root of unity

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Abstract
We discuss a modification of $\mathcal{U}_q(sl(2,\mathbb{R}))$ and a class of its irreducible representations when $q$ is a root of unity.

1 Introduction

Nowadays $q$-deformed universal enveloping algebras $\mathcal{U}_q(\mathfrak{g})$ are understood in depth in the case when $\mathfrak{g}$ is a complex simple Lie algebra belonging to one of the four principal series. The same is true for compact forms of these Lie algebras (see, e.g., monographs [1], [2], [3]). On the other hand, attempts to introduce $q$-deformed enveloping algebras for non-compact real Lie algebras frequently lead to serious difficulties though several particular cases have been already studied (see, e.g., [4], [5], [6]). In this note we discuss one of the simplest examples with $\mathfrak{g} = sl(2,\mathbb{R})$ as a real form of $sl(2,\mathbb{C})$. The deformation parameter $q$ is supposed to be a root of unity,

$$q = \exp(i\pi P/Q),$$

where $Q \in \mathbb{N}$ is odd, $P \in \{1,\ldots,Q-1\}$, and $P$ and $Q$ are relatively prime integers. So $q^{2j} \neq 1$, $j = 1,\ldots,Q-1$, and $q^{2Q} = 1$. 

1
We use the standard definition of the Hopf algebra $U_q(sl(2, \mathbb{C}))$ with the generators $K, K^{-1}, E, F$, the defining relations

$$
K K^{-1} = K^{-1} K = 1, \quad KE = q E K, \quad K F = q^{-1} F K,
$$

$$
[E, F] = \frac{1}{q-q^{-1}}(K^2 - K^{-2}),
$$

the comultiplication

$$
\Delta K = K \otimes K, \quad \Delta E = K \otimes E + E \otimes K^{-1}, \quad \Delta F = K \otimes F + F \otimes K^{-1},
$$

the antipode

$$
S(K) = K^{-1}, \quad S(E) = -q^{-1} E, \quad S(F) = -q F,
$$

and the counit

$$
\varepsilon(K) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0.
$$

A real form is determined by a $*$-involution; an element $X$ of a complex Hopf algebra belongs to a real form if and only if $X^* = S(X)$. Particularly, $U_q(sl(2, \mathbb{R}))$ is determined by the $*$-involution

$$
K^* = K, \quad E^* = -q^{-1} E, \quad F^* = -q F. \quad (1)
$$

Necessarily, $q$ is a complex unit, $\bar{q} = q^{-1}$.

Usually it is more convenient to deal with the complexification of a real form. In that case one regards the real form as the original complex Hopf algebra but endowed, in addition, with the $*$-involution in question. We shall adopt this point of view and treat $U_q(sl(2, \mathbb{R}))$ as the complex Hopf algebra $U_q(sl(2, \mathbb{C}))$ with the $*$-involution (1).

2 \quad \textbf{A modification of } U_q(sl(2, \mathbb{R}))

Let $U$ be a $*$-Hopf subalgebra of $U_q(sl(2, \mathbb{R}))$ generated by $X, Y, Z, Z^{-1}$, where

$$
X = -i q^{-1} E K^{-1}, \quad Y = -i q F K^{-1}, \quad Z = K^{-2}.
$$

Thus $U$ is defined by the relations

$$
ZX = q^{-2} X Z, \quad ZY = q^2 Y Z, \quad q^{-1} X Y - q Y X = -\frac{1}{q-q^{-1}}(1 - Z^2), \quad (2)
$$
with the comultiplication
\[ \Delta Z = Z \otimes Z, \quad \Delta X = 1 \otimes X + X \otimes Z, \quad \Delta Y = 1 \otimes Y + Y \otimes Z, \]
the antipode
\[ S(Z) = Z^{-1}, \quad S(X) = -X \; Z^{-1}, \quad S(Y) = -Y \; Z^{-1}, \]
and the counit
\[ \varepsilon(Z) = 1, \quad \varepsilon(X) = \varepsilon(Y) = 0. \]
Furthermore, all the generators are Hermitian,
\[ Z^* = Z, \quad X^* = X, \quad Y^* = Y. \]
It is also straightforward to check that
\[ C = X \; Y \; Z^{-1} - \frac{1}{(q - q^{-1})^2} (Z^{-1} + q^2 Z) \]
is an Casimir element in \( \mathcal{U} \).

Unfortunately, there exists no non-trivial irreducible representations \( \rho \) of \( \mathcal{U} \). Actually, \( Z^Q \) belongs to the center of \( \mathcal{U} \) and is Hermitian. Thus, by the Schur lemma, \( \rho(Z)^Q = c \; I \) for some real \( c \neq 0 \). Consequently, the self-adjoint operator \( \rho(Z) \) is a multiple of the identity as well. The commutation relations then imply that \( \rho(X) = \rho(Y) = 0, \rho(Z) = \pm I \).

To improve this situation we propose a modification of \( \mathcal{U} \) that we call here \( \mathcal{U}^\natural \). As a Hopf algebra, \( \mathcal{U} \) is extended to \( \mathcal{U}^\natural \) by adding another generator, \( T \), which satisfies
\[ T^2 = 1, \quad \Delta T = T \otimes T, \quad S(T) = T, \quad \varepsilon(T) = 1. \]
A \(*\)-involution on \( \mathcal{U}^\natural \) is defined as follows:
\[ X^* = T \; X \; T, \quad Y^* = T \; Y \; T, \quad Z^* = T \; Z \; T, \quad T^* = T. \]
So \( \mathcal{U} \) is a Hopf subalgebra of \( \mathcal{U}^\natural \) but not a \(*\)-Hopf subalgebra. On the other hand, \( \mathcal{U} \) may be obtained from \( \mathcal{U}^\natural \) by specializing \( T \) to 1.
3 A class of representations of $U^\natural$

Next we present a class of irreducible representations of the $\ast$-algebra $U^\natural$ while the question of a complete classification of irreducible representations of $U^\natural$ is proposed as an open problem. Though it is not excluded that the definition of $U^\natural$ should be further modified in order to get a reasonable theory. In this section most steps are only outlined with some details omitted.

The representation $\rho$ depends on an integer parameter $n \in \{1, 2, \ldots, Q\}$ and its dimension $d$ equals $Q + 1 - n$. The matrices $\rho(X)$, $\rho(Y)$, $\rho(Z)$ are tridiagonal with non-vanishing entries

$$\rho(Z)_{m-1,m} = -(q^2 - q^{-2})q^{2m+n-1}a_m,$$
$$\rho(Z)_{m+1,m} = (q^2 - q^{-2})q^{-2m-n-1}b_{m+1},$$
$$\rho(Z)_{mm} = (q + q^{-1})c_m,$$

$$\rho(X)_{m-1,m} = (q + q^{-1})a_m,$$
$$\rho(X)_{m+1,m} = (q + q^{-1})b_{m+1},$$
$$\rho(X)_{mm} = d_m,$$

$$\rho(Y)_{m-1,m} = (q + q^{-1})q^{2(2m+n-1)}a_m,$$
$$\rho(Y)_{m+1,m} = (q + q^{-1})q^{-2(2m+n+1)}b_{m+1},$$
$$\rho(Y)_{mm} = -d_m,$$

$m = 0, 1, \ldots d - 1$. Here

$$a_m = b_m = \frac{1}{q^{2m+n-1} + q^{-2m-n+1}} \sqrt{\frac{[m]_q [m + n - 1]_q}{[m]_q [m - n]_q}},$$
$$c_m = \frac{q^{n-1} + q^{-n+1}}{(q^{2m+n-1} + q^{-2m-n+1})(q^{2m+n+1} + q^{-2m-n-1})},$$
$$d_m = \frac{q^{n-1} + q^{-n+1}}{(q^{2m+n-1} + q^{-2m-n+1})(q^{2m+n+1} + q^{-2m-n-1})^{2m+n}[2m + n]_q}.$$

The quantum numbers are defined as usual,

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$
The matrix $\rho(T)$ is diagonal,

$$\rho(Z) = \text{diag}(\tau_0, \tau_1, \ldots, \tau_{d-1})$$

where $\tau_0 = 1$ and

$$\frac{\tau_m}{\tau_{m-1}} = \text{sgn} \left( \frac{[m]_q^2[m+n-1]_q^2}{(q^{2m+n-2} + q^{-2m-n+2})(q^{2m+n} + q^{-2m-n})} \right), \quad (6)$$

$m = 1, 2, \ldots, d - 1$.

Let us remark that a source of difficulties when working with real forms comes from the fact that the deformation parameter $q$ is forced to be a complex unit. In that case the sign $\tau_m/\tau_{m-1}$ in (6) may equal -1 for particular values of $m$. Concerning the representation $\rho$, it is worth mentioning that the matrix $\rho((qX - q^{-1}Y)Z^{-1})$ is diagonal and

$$\rho((qX - q^{-1}Y)Z^{-1})_{mm} = [2m + n]_q.$$

The verification of the commutation relations (2) is straightforward. This may be done even in the case when $q$ is generic and the tridiagonal matrices (3), (4), (5) are infinite with $m = 0, 1, 2, \ldots$. One then finds that relations (2) are satisfied if and only if the coefficients $c_m$ obey a recursive equation,

$$(q^{2m+n+3} + q^{-2m-n-3})c_{m+1} - (q + q^{-1})(q^{2m+n} + q^{-2m-n})c_m$$

$$+ (q^{2m+n-3} + q^{-2m-n+3})c_{m+1} = 0,$$

and $d_m$ and $a_m b_m$ are expressed in terms of $c_m$,

$$d_m = - \frac{(q^{2m+n-1} + q^{-2m-n+1}) c_m - (q^{2m+n-3} + q^{-2m-n+3}) c_{m-1}}{(q - q^{-1})^2},$$

$$a_m b_m = \frac{(q - q^{-1})^{-4}(q + q^{-1})^{-2}}{(q^{2m+n} + q^{-2m-n})^{-1}(q^{2m+n-2} + q^{-2m-n+2})^{-1}}$$

$$\times ((q^{2m+n+1} + q^{-2m-n-1})^2 c_m + (q^{2m+n-3} + q^{-2m-n+3})^2 c_{m-1})$$

$$- (q^2 + q^{-2})(q^{2m+n-3} + q^{-2m-n+3})(q^{2m+n+1} + q^{-2m-n-1})c_m c_{m-1}$$

$$+ (q - q^{-1})^2).$$

Equivalently,

$$d_m = - \frac{(q^{2m+n+3} + q^{-2m-n-3}) c_{m+1} - (q^{2m+n+1} + q^{-2m-n}) c_m}{(q - q^{-1})^2}.$$
To verify the irreducibility we shall show that even the restriction of \( \rho \) to the subalgebra \( \mathcal{U} \) is irreducible. This will become obvious as soon as we prove that \( \rho \) is equivalent to \( \tilde{\rho} \) with

\[
\tilde{\rho}(X) = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
x_1 & 0 & \ldots & 0 & 0 \\
0 & x_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & x_{d-1} & 0
\end{pmatrix}, \quad \tilde{\rho}(Y) = \begin{pmatrix}
0 & y_1 & 0 & \ldots & 0 \\
0 & 0 & y_2 & \ldots & 0 \\
0 & 0 & 0 & \ldots & y_{d-1}
\end{pmatrix},
\]

and

\[
\tilde{\rho}(Z) = \begin{pmatrix}
z_0 & 0 & \ldots & 0 \\
0 & z_1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & z_{d-1}
\end{pmatrix},
\]

where

\[
x_j = q^{-n-2j+1}[n+j-1]_q, \quad y_j = [j]_q, \quad z_j = q^{-n-2j}.
\]

Note that \( x_j \neq 0, \ y_j \neq 0 \), for \( j = 1, \ldots, d-1 \).

The equivalence in turn follows from a more geometrical realization of the representation \( \rho \) which is closely related to the twisted adjoint action [7], [8].

The vector space \( \mathcal{M} \) of meromorphic functions in variable \( w \) on the complex plane becomes a left \( \mathcal{U} \) module with respect to the action

\[
X \cdot f(w) = -i \frac{q^{-1} w}{q - q^{-1}} \left( q^n f(w) - q^{-n} f(q^{-2} w) \right),
\]

\[
Y \cdot f(w) = i \frac{q^{n+1}}{(q - q^{-1}) w} \left( f(w) - f(q^{-2} w) \right),
\]

\[
Z \cdot f(w) = q^{-n} f(q^{-2} w).
\]

Set

\[
\psi_m(w) = \frac{\prod_{j=0}^{n-1} (q^{2j+n} w - i)}{\prod_{j=0}^{n+m-1} (q^{2j-2m-n} w + i)}, \quad m = 0, 1, 2, \ldots.
\]

Then the vector space \( \text{span}\{\psi_0, \psi_1, \ldots, \psi_{d-1}\} \), \( d = Q+1-n \), may be checked to be \( \mathcal{U} \) invariant. After renormalization of the basis vectors, \( \tilde{\psi}_m = \lambda_m \psi_m \), with the factors \( \lambda_m \) being determined by \( \lambda_0 = 1 \) and

\[
\lambda_m = \sqrt{q^{2m+n} + q^{-2m-n}} [m+n-1]_q^2 \lambda_{m-1}.
\]
we get the representation $\rho$.

Consider now a point set,
\[ M = \{1, q^2, \ldots, q^{2Q-2}\} \subset \mathbb{C}. \]

Note that for any function $f$, the values of the function $A \cdot f$ on the set $M$ depend only on the restriction $f|_M$ where $A$ is any of the generators $X, Y$ or $Z$. Thus the vector space $\mathcal{F} \cong \mathbb{C}^Q$ of functions on $M$ becomes a $\mathcal{U}$ module and the restriction map $\mathcal{M} \to \mathcal{F}: f \mapsto f|_M$ is a surjective morphism of $\mathcal{U}$ modules. The representation $\rho$ corresponds to the submodule $\mathcal{R} = \text{span}\{\tilde{\psi}_0|_M, \tilde{\psi}_1|_M, \ldots, \tilde{\psi}_{d-1}|_M\}$ with the distinguished basis. Omitting the details we claim that another basis in $\mathcal{R}$ may be chosen as $\{\phi_0|_M, \phi_1|_M, \ldots, \phi_{d-1}|_M\}$ where
\[ \phi_j(w) = \left(\frac{1}{i}\right)^j q^{j(j-1)+nj}w^{-Q+j}. \]

Expressing operators in the latter basis we get the representation $\tilde{\rho}$. This proves the equivalence of $\rho$ and $\tilde{\rho}$ and consequently that the representation $\rho$ is irreducible.

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