Schwarzschild-Like Wormholes as Accelerators

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In a stationary spacetime $S$ consider a pair of free falling particles that collide with the energy $E_{\text{c.m.}}$ (as measured in the center-of-mass system). Let the metric of $S$ or/and the trajectories of the particles depend on a parameter $k$. Then $S$ is said to be a “(super) accelerator” if $E_{\text{c.m.}}$ grows unboundedly with $k$, even though the energies of the particles at infinity remain bounded. The existence of naturally occurring super accelerators would make it possible to observe otherwise inaccessible phenomena. This is why in recent years a lot of spacetimes were tested on being super accelerators.

In this paper a wormhole $W$ of an especially simple—and hence, hopefully, realistic—geometry is considered: it is static, spherically symmetric, its matter source is confined to a compact neighbourhood of the throat, and the $tt$-component (in the Schwarzschild coordinates) of its metric has a single minimum. It is shown that such a wormhole is a super accelerator with $k \equiv \frac{1}{3} \ln |g_{tt}\min|$. In contrast to the rotating Teo wormhole, considered by Tsukamoto and Bambi, $W$ cannot accelerate the collision products on their way to a distant observer. On the other hand, in contrast to the black hole colliders, $W$ does not need such acceleration to make those products detectable.

I. INTRODUCTION

Suppose, in a stationary gravitational field a pair of particles with mass $m$ move on the geodesics $\gamma_{1,2}$ and then collide. The energy of the collision—let us measure it in the center-of-mass system and denote $E_{\text{c.m.}}$—is not bounded by their initial Killing energies at infinity $E_1$ and $E_2$ (to see this consider the case when both particles are initially at rest at infinity, $\varepsilon_i \equiv E_i/m = 1$, $i = 1, 2$: imagine two comets moving on parabolic orbits and colliding head-on). This makes one wonder if the gain in energy can be so large (in the vicinity of a

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black hole, say) that $E_{\text{c.m.}}$ gives one a chance to observe trans-Planckian effects.

The answer proved to be non-trivial. On the one hand, it is negative in the Schwarzschild space, see \[1\] for the case of $\varepsilon_{1,2} = 1$. On the other hand, the energies in question do diverge in the case of the non-extreme Kerr black holes as was demonstrated by Piran, Shaham, and Katz (PSK), who considered collisions of a special type in those spaces and discovered \[1\] that $E_{\text{c.m.}}$ of such collisions increases beyond bounds (even though $\varepsilon_{1,2} = 1$) as the spin parameter $a$ tends to the mass $M$ of the black hole (i.e., as the black hole “tends to the extreme one”):

$$\lim_{n \to \infty} a_n = M \Rightarrow \lim_{n \to \infty} E_{\text{c.m.}}(\gamma_{1n}, \gamma_{2n}) = \infty$$

(it is understood of course that the mass $m$ does not change with $n$). Note that we speak of different $a$’s, so it is meant that each pair $\gamma_{1n,2n}$ belongs to its own spacetime $S_n$:

$$a_n \neq a_{n'} \Rightarrow \gamma_{1n,2n} \subset S_n \neq S_{n'}.$$  

As proven by Bañados, Silk, and West \[2\], the divergence is “continuous” in the sense that in the limiting (extreme) spacetime there also are geodesics $\gamma_{1n,2n}$ satisfying \(\Pi\) with $S_n = S$, $a_n = M$ for all $n$ (the phenomenon known as the BSW effect).

The PSK and BSW effects have the disadvantages that in “reasonable astrophysical situations”

1. Kerr black holes are believed to obey “Thorne’s bound” $a \lesssim 0.998$ \[3\];

2. it takes too long (by the Killing time) for a particle to reach the collision point. For example \[4\], if the particle is initially at rest at a few gravitational radii from the horizon, then the corresponding time $T$ is

$$\left( \frac{T}{10 \text{ Gyr}} \right) \approx \left( \frac{E_{\text{c.m.}}}{2.5 \times 10^{20} \text{ eV}} \right)^2 \left( \frac{M}{M_\odot} \right) \left( \frac{1 \text{ GeV}}{m} \right)^2;$$

3. “…the photon formed from a collision just outside the horizon can suffer a diverging redshift with decreasing radius at a rate that exceeds the divergence of the CM energy, and thereby results in a vanishing energy reaching infinity” \[5\].

Thus in search for particle super accelerators one had to look at more exotic processes, such as collisions of charged particles \[6\] and “multiple scattering” \[7, 8\] or more exotic spacetimes such as five-dimensional black holes \[9\], naked singularities \[10\], etc., see \[4\] for a review.
The latter list contains, in particular, the rotating wormhole [11] considered first by Teo [12]:

$$\text{d}s^2 = -N^2 \text{d}t^2 + \frac{r}{r-b} \text{d}r^2 + r^2 N^2 \left[ \text{d}\theta^2 + \sin^2 \theta (\text{d}\varphi - 2ar^{-3}\text{d}t)^2 \right],$$

where $N \equiv 1 + (4a \cos \theta)^2 d/r$, $a,b,d$—constants

(from now on we use the Planck units: $G = c = \hbar = 1$). There is good reason to study the Teo wormholes: a wormhole rotating sufficiently fast, i.e., having a sufficiently large $a$, possesses an ergoregion. The Penrose effect makes such a wormhole a possible source of high energy cosmic rays. At the same time the non-rotating wormholes do not offer such a possibility and perhaps for this reason have not been considered so far\(^1\) as potential super colliders. The present paper aims to fill this gap and to demonstrate that the PSK effect takes place in some non-rotating wormholes too. This may be important for the following reasons:

a. We do not know today whether wormholes even exist, so it is a little premature to compare their advantages. Still, being static, spherically symmetric, and empty outside some compact region, the wormholes discussed in this paper are much simpler than Teo’s, they also need much ($10^{21}$ times) less matter as source. Arguably this makes them more realistic;

b. In contrast to the situation with black holes, see item\(^3\) above, high energy collisions generated by wormholes do not become invisible to a distant observer even in the absence of the Penrose acceleration. Suppose, for example, that two identical, but opposite moving, particles with energy $\varepsilon_1 = \varepsilon_2$ collide at the throat and transform into another pair of identical particles, this time with $\varepsilon_3 = \varepsilon_4$. To escape the wormhole these newborn particles will have to get redshifted, indeed, but the energy conservation guarantees that they will lose exactly the same energy that was gained by the incoming ones. So, $\varepsilon_{3,4} = \varepsilon_{1,2}$: initially relativistic particles generate relativistic ejecta which, in principle, can be detected by a terrestrial observer.

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\(^1\) The two exceptions are the Teo wormhole with $a = 0$ and the Ellis wormhole. As shown in [11] neither of them can accelerate particles unboundedly.
FIG. 1: The gray curve is the graph of $\frac{1}{2} \ln(1 - 1/x)$. The dashed curve is $\psi(x)$.

II. SCHWARZSCHILD COLLIDER

A. The metric

Consider a spacetime $W$ which is a Morris-Thorne wormhole with the metric

$$ds^2 = -e^{2h(l)} dl^2 + dl^2 + r^2(l)(d\theta^2 + \sin^2 \theta \, d\phi^2), \quad l \in \mathbb{R},$$

where $r(l)$ and $h(l)$ are smooth even functions monotone increasing at positive $l$ and obeying the conditions

$$dr = \sqrt{1 - 1/x} \, dl, \quad x \equiv r/r(0)$$

and

$$h(x) \big|_{x > 3} = \frac{1}{2} \ln(1 - 1/x)$$

(the positive constant $r_g \equiv r(0)$ is called, for obvious reason, the radius of the wormhole).

Thus, up to the additional condition (3) (which is added in order to make the wormhole indistinguishable by its lensing properties from the Schwarzschild black holes) the spacetime (2) is what was called the Schwarzschild wormhole in [13]. It consists of the spherical layer $|l| \leq |l(r = 3r_g)|$ filled with matter and two empty asymptotically flat regions $r > 3r_g$ with Schwarzschild metric.

To specify the metric of $W$ (that is to define $h$ in the non-empty region) split the ray $x \in [1, \infty)$ into three intervals by the points $x = 2$ and $x = 3$, pick a positive constant $k$, 

and
and define an auxiliary function \( \psi \) by its restrictions to those intervals:

\[
\psi(x) = \frac{1}{2}[\ln(x - 1) - \ln x] \quad \text{at } x \in [3, \infty); \tag{5a}
\]

\[
\psi(x) = -\frac{1}{2}(k - \frac{1}{12})(x - 3)^2 + \frac{1}{12}(x - 3) + \frac{1}{2} \ln \frac{2}{3} \quad \text{at } x \in [2, 3]; \tag{5b}
\]

\[
\psi(x) = kx + c, \quad c \equiv -2k + \psi(2 + 0) \quad \text{at } x \in [1, 2] \tag{5c}
\]

[the graph of \( \psi(x) \) is a straight segment connected by a piece of a parabola to the Schwarzschild “tail” \( \psi \mid_{x>3} \), see (4) and figure 1]. As easily seen, \( \psi(x) \) is \( C^1 \) and piecewise smooth, but its second derivative jumps in the points \( x = 2, 3 \). So, the spacetime (2) with \( h = \psi \) can be interpreted as describing a throat enclosed by a pair of spherical layers with sharp boundaries\(^2\). \( \psi \), however, can be arbitrarily well approximated by smooth functions, i.e. for an arbitrary \( \epsilon \) a smooth function \( h_\epsilon \) can be found such that

\[
h_\epsilon(x) \mid_{[3, \infty)} = \psi(x), \quad |h_\epsilon^{(0)}(x) - \psi^{(0)}(x)| < \epsilon \quad |h_\epsilon''(x)| \leq |\psi''(x)|, \quad \forall x. \tag{6}
\]

We define \( W \) to be the spacetime (2) with \( h \equiv h_\epsilon \), \( \epsilon \) sufficiently small, and \( k \gg 1 \). The last two conditions imply, in particular, that \( c \approx -\frac{5}{2}k \) [we have substituted (5b) into (5c)] and hence

\[
h_{\min} \approx \psi(1) \approx -\frac{3}{2}k. \tag{7}
\]

B. Head-on collisions at the throat

Consider a particle of mass \( m \) falling from the infinity \( l = -\infty \) on a radial geodesic \( \gamma = \left(t(\tau), l(\tau)\right) \subset W \), where \( \tau \) is the proper time. The spacetime is static and therefore

\[
\varepsilon \equiv -g(\partial t, \partial \tau) = e^{2h} \dot{t} \quad \text{is constant along } \gamma. \tag{8}
\]

\( \varepsilon \) is the initial specific energy of the particle as measured by a Schwarzschild observer and, correspondingly, \( \varepsilon \geq 1 \).

Further, by (2),

\[
\dot{t}^2 = e^{2h_\epsilon} - 1 = \varepsilon^2 e^{-2h} - 1 \tag{9}
\]

\(^2\) The stress-energy tensor here has no delta-function singularities, so such a space is not to be confused with the “thin-shall wormhole” [14].
must hold on $\gamma$. In the case at hand $h < 0$, so on any finite interval $\dot{l}$ is bounded away from zero, which means that the particle never turns back\footnote{Unlike what happens sometimes in the Damour–Solodukhin wormhole\cite{15}, which otherwise is very similar to ours.} and sooner or later reaches the throat $l = 0$.

Now suppose that another, identical, particle falls in the same manner from $l = +\infty$. Then these two particles will collide at the throat. The energy of this collision is

$$E_{\text{c.m.}} = m\sqrt{2(1 - v_{(1)}^\mu v_{(2)}^\mu)},$$

where $v_{(i)}^\mu, i = 1, 2$ is the velocity of the $i$-th particle at the throat, see\cite{2}. Using consequently (2), (8), (9), and (7) we obtain

$$-v_{(1)}^\mu v_{(2)}^\mu|_{l=0} = e^{2h_{\text{min}}} \hat{l}_{(1)} \hat{l}_{(2)} - \dot{l}_{(1)} \dot{l}_{(2)} = \varepsilon^2 e^{-2h_{\text{min}}} + \sqrt{\varepsilon^2 e^{-2h_{\text{min}}} - 1} \sqrt{\varepsilon^2 e^{-2h_{\text{min}}} - 1},$$

$$\approx 2\varepsilon^2 e^{-2h_{\text{min}}} \approx 2\varepsilon^2 e^{3k} \quad (10)$$

(note that the whole effect discussed in this paper originates from the sign $+$ before the radical, which in its turn is due to the fact that $\dot{l}_{(1)} = -\dot{l}_{(2)}$ in a head-on collision).

Thus, a Schwarzschild-like wormhole does exhibit the PSK effect:

$$\lim_{h_{\text{min}} \to -\infty} \left( \frac{E_{\text{c.m.}}}{\varepsilon m} \right) \to \infty \quad (11)$$

In particular, a wormhole with $h_{\text{min}} = \ln(2m_{\text{proton}}/E_{\text{c.m.}}) \approx -44$ can collide two initially non-relativistic ($\varepsilon_{(i)} \approx 1$) protons with the Planck-scale energy $E_{\text{c.m.}} \approx 1$.

### C. The technical characteristics of the collider

In this section we roughly estimate two parameters of our wormhole $W$. That has to be done because too large values of these parameters are obvious arguments against this type of super colliders. Define $\mathcal{A}$ and $\mathcal{T}$ as

$$\mathcal{T} \equiv \mathcal{T}(1, 3), \quad \text{where } \mathcal{T}(x_1, x_2) \equiv 2r_g \int_{x_1}^{x_2} e^{-h} \sqrt{x} \frac{\sqrt{x}}{\sqrt{x} - 1} \, dx \quad (12)$$

and

$$\mathcal{A} \equiv \mathcal{A}(1, \infty), \quad \text{where } \mathcal{A}(x_1, x_2) \equiv r_g \int_{x_1}^{x_2} x^2 \max(h_{xx}^2(x), |h_{x,x}(x)|) \sqrt{x} \frac{1}{\sqrt{x} - 1} \, dx. \quad (13)$$
The chain
\[ r_g \int_{x_1}^{x_2} \frac{e^{-h} \sqrt{x}}{\sqrt{x-1}} \, dx \approx \int_{t_1}^{t_2} e^{-\sqrt{h}} \, dt \approx \int_{t_1}^{t_2} t/\sqrt{h} \, dt = t_2 - t_1 \]
shows that \( T \) is the delay (measured by a Schwarzschild observer located at \( r = 3r_g \)) between dropping a particle with \( \varepsilon = 1 \) into the wormhole and receiving the same particle after it bounced off something at the throat.

The physical meaning of \( A \) is less clear. The scalar curvature in \( W \) is
\[ R = \frac{2}{r^2} \left( -2 h_{,r} r - h_{,rr} r^2 + 2 h_{,r} r_g - h_{,rr} r^2 + h_{,rr} r r_g + 3/2 h_{,r} r_g \right). \tag{14} \]
So, it is a sum of a few terms each is of the order of
\[ r_g^{-2} h_{,x}, \quad r_g^{-2} h_{,xx}, \quad \text{or} \quad r_g^{-2} h_{,xx} \tag{15} \]
or less. It follows from the Einstein equations and eqs. (2), (15) that
\[ A \gtrsim \int_{t=0} \left| T^\alpha_{\alpha}(x) \right| \, d^3V, \tag{16} \]
where \( T^{\alpha\beta} \) is the stress-energy tensor. Thus, \( A \) can be interpreted as the “total amount of matter” filling the wormhole. It is a generalization of the concept of mass to the situation where the energy density in some non-empty regions is zero (as in our case, see eq. (14,43) in [16]) or even negative. A lot can be said against \( A \) as a characteristic of a wormhole, cf. [17], however, for lack of a better choice we adopt in this paper the point of view that the larger \( A \) is, the less feasible the corresponding wormhole is.

Example. The Teo wormhole. In this spacetime (in the case \( d = 0 \)) the scalar curvature is
\[ R = 18a^2 \sin^2 \theta (r - b) r^{-7} \]
and the energy of a head-on collision at the throat of two particles radially moving in the equatorial plane is
\[ E_{c.m.} \lesssim m \varepsilon (\sqrt{a/b})^3. \]
So,
\[ A \approx \int_{t=\text{const}} R \, d^3V \approx 10a^2 \int_b^\infty (r - b) r^{-5} \left( 1 - \frac{b}{r} \right)^{-1/2} \, dr \gtrsim 10a^2 \int_b^\infty (r - b)^{1/2} r^{-9/2} \, dr = 10a^2 b^{-3} \int_1^{\infty} (y - 1)^{1/2} y^{-9/2} \, dy \approx a^2 b^{-3} \gtrsim \left( \frac{E_{c.m.}}{m \varepsilon} \right)^{4/3} \tag{17} \]
To get a feeling for what these estimates mean, suppose we wish to study the Planckian physics and contemplate colliding initially non-relativistic protons (which are the most available particles) accelerated by a Teo wormhole to Planck scale energies. Then we have to take $b$ greater than the Compton wavelength of proton (that is than $1/m_{\text{proton}}$) to make legitimate our classical analysis of the protons’ trip through the throat. Thus, taking into account (17) we need a wormhole with

$$A \gtrsim \left( \frac{m_{\text{Planck}}}{m_{\text{proton}}} \right)^{4/3} b \gtrsim \left( \frac{m_{\text{Planck}}}{m_{\text{proton}}} \right)^{7/3} \frac{m_{\text{Planck}}}{m_{\text{proton}}} \gtrsim 10^6 M_\odot.$$  (18)

This, as we shall see, is $10^{21}$ times more than in the case of the Schwarzschild-like wormhole.

To estimate $T$ and $A$ of the wormhole $W$ start with the segment $[2,3]$ (since $R = 0$ at $x > 3$). It’s contribution to the total amount of matter is

$$A(2,3) \lesssim r_g \int_2^3 x^2 \max(\psi_x^2(x), |\psi_{xx}(x)|) \frac{\sqrt{x}}{\sqrt{x-1}} \, dx \lesssim \frac{1}{4} r_g \int_2^3 x^2 k^2 \frac{\sqrt{x}}{\sqrt{x-1}} \, dx \lesssim 10 r_g k^2$$  (19)

[we have used (5b) in deriving the first inequality] and to the particle’s trip time—

$$T(2,3) = 2 r_g \int_2^3 \frac{e^{-h(x)} \sqrt{x}}{\sqrt{x-1}} \, dx \lesssim r_g \int_2^3 \frac{e^{-h(x)} \sqrt{x}}{\sqrt{x-1}} \, dx \lesssim 2 r_g e^{-h(2)}$$  (20)

Now note that $\psi_x = k$ in the interval $[1,2]$. So, again

$$A(1,2) \lesssim r_g \int_1^2 x^2 k^2 \frac{\sqrt{x}}{\sqrt{x-1}} \, dx \lesssim 10 r_g k^2$$  (21)

and

$$T(1,2) = r_g \int_1^2 \frac{e^{-h(x)} \sqrt{x}}{\sqrt{x-1}} \, dx \lesssim r_g \int_1^2 \frac{e^{-h_{\text{min}}(x)} \sqrt{x}}{\sqrt{x-1}} \, dx < 10 r_g e^{-h_{\text{min}}}$$  (22)

Gathering the estimates (19)–(22) we obtain

$$T \lesssim 10 r_g e^{-h_{\text{min}}} \approx 5 r_g E_{\text{Planck}} / m_{\text{proton}} \approx 2 \times 10^7 \cdot \left( \frac{r_g}{3 \text{ km}} \right) \text{ yr};$$  (23)

$$A \lesssim 10 r_g k^2 \approx 10^4 r_g \approx 10^4 \cdot M_\odot \left( \frac{r_g}{3 \text{ km}} \right).$$  (24)

**III. CONCLUSIONS**

We have considered a special class of exotic compact objects—the Schwarzschild-like wormhole with a particular profile parametrized by the radius of the throat $r_g$ and the
“depth” \( h_{\text{min}} \). Such wormholes have the properties of a super accelerator—two identical particles of mass \( m \) falling radially into the wormhole towards each other, collide in the throat and the energy of the collision (in the center-of-mass system) is

\[
E_{\text{c.m.}} \approx 2\varepsilon me^{-h_{\text{min}}}.
\]

So, dropping at some moment \( t_0 \) a relativistic proton—emitted from an accretion disk, say—in either of the mouths one makes the protons collide at the throat with the Planck-scale energy if the wormhole has \( h_{\text{min}} \approx -44 \).

Collisions in kilometer-size wormholes of this type can in principle be observable. As the collision products escape the wormhole they experience the red shift and come out—in ten millions of years after \( t_0 \)—with moderate (but still relativistic if the products are few in number) energies. Such cosmic rays may well be registered with modern equipment.

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