Abstract

It is known that there are four-manifolds which are not algorithmically recognizable. This implies that there exist triangulations of these manifolds which are separated by large barriers from the point of view of the computer algorithm. We have not observed these barriers for triangulations of $S^4$. 

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1 Introduction

A non-perturbative formulation of quantum gravity is one of the greatest challenges of theoretical physics today. One such suggestion which has attracted certain attention in the last couple of years is to use so-called “dynamical triangulations” [1, 2]. This approach provides us with a regularization of Euclidean quantum gravity, much in the spirit of the lattice regularization of Euclidean quantum field theories where the continuum limit is recovered at points in coupling constant space where the statistical systems have second order transitions. One difference is due to the dynamical nature of space-time in a theory of gravitation. This forces us to use “dynamical triangulations” of space-time rather than a fixed lattice: We have to sum over classes of different triangulations in the path integral. The partition function of Euclidean quantum gravity for a compact, closed manifold \( M \) can be written as:

\[
Z = \int \frac{Dg_{\mu\nu}}{\text{Vol}(\text{diff})} e^{-S[g]},
\]

where \( \text{Vol}(\text{diff}) \) is the “volume” of the diffeomorphism group of \( M \) and the integration is over equivalence classes of Riemannian structures on \( M \). It is no loss of generality to view \( M \) as a combinatorial or, equivalently, piecewise linear manifold, since there is a one-to-one correspondence between smooth and piecewise linear structures for manifolds of dimensions \( D \) less than seven. \( S[g] \) is the gravitational action, which we here will take to be the Einstein-Hilbert action:

\[
S[g] = \lambda \int d^D\tau \sqrt{g} - \frac{1}{16\pi G} \int d^D\tau \sqrt{g}R.
\]

The regularized version of this functional integral in the context of dynamical triangulations is replaced by

\[
Z = \sum_T \frac{1}{C_T} e^{-S_T}
\]

where the summation is over all triangulations of the manifold \( M \). To be more precise we consider two triangulations to be identical if there exists a mapping of the vertices of one triangulation on the vertices of the other triangulation, such that all simplexes (and sub-simplexes) of the first triangulation are mapped onto the corresponding simplexes (and sub-simplexes) of the second triangulation. The number \( C_T \) is the order of the automorphism group of the triangulation. This way of identifying triangulations is compatible with the introduction of a distance function on the triangulations by assigning a lattice length \( a \) to all links and considering the triangulations as piecewise linear manifolds. In the following we always take \( a=1 \), but the continuum limit should always refer to distances large compared to \( a \), which
in this way serves as a cut off. By this length assignment for each \( T \) there will be an associated metric assigned to \( M \), and by considering all triangulations of \( M \) we get a grid in space of metrics on \( M \). For a given triangulation \( T \) the discretized action \( S_T \) is evaluated by Regge calculus and after some trivial algebra the Regge version of (2) for a given triangulation \( T \) with the above distance function reads:

\[
S_T = k_4 N_4(T) - k_2 N_2(T),
\]

where \( N_4(T) \) denotes the number of 4-simplexes and \( N_2(T) \) the number of 2-simplexes in the triangulation \( T \). The coupling constant \( 1/k_2 \) is proportional to the bare Einstein coupling constant \( G \) in (2), while \( k_4 \) is related to the bare cosmological constant \( \lambda \) in (2).

If we consider two-dimensional Euclidean quantum gravity we can solve the continuum theory by conformal field theory methods. In addition the regularized version can be solved explicitly and at the critical point of the statistical model we recover the continuum results obtained by conformal field theory. When we move to higher dimensions we enter uncharted territory from the point of view of the continuum version of Euclidean quantum gravity. Nevertheless (3)-(4) provide us with a perfectly well defined statistical model and we can search for critical points where one can attempt to define a non-perturbative continuum limit.

Until now it has only been possible to analyze the model given by (3) and (4) by numerical methods, more specifically by Monte Carlo simulations\[^3,5,6\]. In order to get around in the class of all triangulations of a combinatorial manifold \( M \) one needs a set of moves which can be implemented on the computer and which are ergodic in the set of triangulations of \( M \). Two triangulations of \( M \) are combinatorially equivalent, i.e. they have a common subdivision (up to relabelling of vertices). It is by now well known that for manifolds of dimensions \( D \leq 4 \) there exists a finite set of “local” moves which are ergodic the in class of combinatorially equivalent triangulations of a given manifold \[^7\]. By “local” we mean that each of the moves will change the triangulations only in such a way that the number of integer algorithmic operations needed to implement a move is bounded by a fixed number independent of the triangulation.

Until now the setup seems perfect from a computational point of view: We have a set of local moves which can connect any two triangulations of \( M \) in a finite number of steps. We have an action, and by Monte Carlo simulations we should now be able to make an importance sampling of the triangulations with the weight provided by \( e^{-S_T} \), \( S_T \) given by (4). However, the fact that there exist four-manifolds which are algorithmically unrecognizable casts some doubts on this program. Denote such a manifold by \( M_0 \) and let it be finitely presented by a combinatorial
triangulation $T(M_0)$. The algorithmic unrecognizability of $M_0$ means that there exists no algorithm which allows us to decide whether another manifold $M$, again finitely presented by a triangulation $T(M)$ is combinatorially equivalent to $M_0$. When this is combined with the existence of the finite set of local moves which are able to connect any two triangulations of $M_0$ in a finite number of steps, but where this number is a function of the chosen triangulations, one can prove the following theorem [8]:

The number of moves needed to connect two triangulations of $M_0$, $T$ and $T'$ with $N_4(T) = N_4(T')$, cannot be bounded by any recursive function $r(N_4)$.

Recall that $\exp(N)$ or $\exp(\exp(\ldots \exp(N)))$ (the exponentiation $N$ times) are recursive functions. Effectively this implies that there will be very large barriers between some classes of triangulations of $M_0$ and there would be triangulations which could never be reached in any reasonable number of steps even for quite moderate values of $N_4$. Of course the number of configurations which are separated from some standard triangulation of $M_0$ by such barriers could vanish relative to the total number of configurations as a function of $N_4$. In ref. [8] it was conjectured that it will not be the case and some plausibility arguments in favor of the conjecture were given. If the conjecture is correct, a Monte Carlo method based on the finite set of local moves will never get around effectively in the class of triangulations of $M_0$. We can say that the moves, although ergodic in the class of triangulations, will not be computationally ergodic [8].

How is the situation for $S^4$ which is the manifold which until now has been used in the Monte Carlo simulations? It is unknown whether $S^4$ is algorithmically recognizable in the class of four-manifolds. If $S^4$ is algorithmically unrecognizable the arguments given above for $M_0$ apply. Since the number of different triangulations with a fixed $N_4$ is bounded by some number $N_4^n$, and for triangulations of a fixed topology, like $S^4$, probably even exponentially bounded, the only way the number of steps needed to connect any two triangulations with $N_4$ simplexes can end up not being bounded by a recursive function is the following: In the process of connecting two triangulations by a sequence of moves we will be forced to very high values of $N_4$. In fact this number itself cannot be bounded by a recursive function.

It would be unnatural if this phenomenon was not present at essentially all scales and one would therefore expect to be able to observe it in the following way: Let us by Monte Carlo simulations generate a number of independent configurations for some large values of $N_4$. Now “shrink” (again by Monte Carlo simulations) these configurations to the minimal triangulation of $S^4$, consisting of 6 4-simplexes. In case we never get seriously stuck in this shrinking procedure there can be no barrier
separating two triangulations since we can first move to the minimal configuration and then out to another triangulation by the reverse set of moves.

We have never observed that the triangulations get stuck in the process of a “reasonable shrinking procedure”. We take this as some evidence in favor of $S^4$ being computationally recognizable.

In the rest of this paper we explain in what sense we are scanning the space of triangulations of $S^4$ by Monte Carlo simulations, what we mean by “shrinking” and we give a tentative “experimental” upper limit of the number of moves needed to connect any two triangulations of $S^4$. Finally we discuss some implications for quantum gravity.

2 Scanning the configuration space

A four-dimensional triangulation is characterized by the number of vertices, links, triangles, tetrahedra and 4-simplexes which constitute the triangulation, and most importantly, the information about the way they are glued together to form a combinatorial four-manifold. This last requirement means that the numbers $N_n$ of $n$-simplexes, $n \leq 4$, can not be chosen arbitrary: They have to satisfy the Dehn-Sommerville relations:

$$N_n = \sum_{i=n}^{4} \frac{(-1)^{i+1}(i+1)!}{(n+1)!(i-n)!} N_i. \quad (5)$$

These relations are valid in dimensions $D$ other than four if we replace 4 with $D$. They express the manifold requirement that the set of $D$-simplexes having a $n$-simplex in common should constitute a combinatorial $D-n$-ball. If we define $N_{-1} \equiv \chi$, the Euler characteristics of the manifold, (5) gives an additional constraint when $D$ is even. For $D = 4$ this implies that at most two of the $N_n$'s can be chosen independently. Let it be $N_4$ and $N_2$. It is now truly remarkable that these are precisely the quantities which enter in the Einstein-Hilbert action (4). The interpretation of the Monte Carlo simulations where we include the gravitational action is now that the choice of coupling constants $k_4$ and $k_2$ will determine the average value of the $N_n$'s but that all triangulations with the same values of $N_n$ will be chosen with the same probability. By monitoring $k_2$ and $k_4$ we can explore the neighborhood of the class of triangulations characterized by a given allowed choice of $N_n$'s. If we increase the average volume $\langle N_4 \rangle$ the other $\langle N_n \rangle$, $n = 0, 1, 2, 3$ will increase too. While from (5) it follows that there exist positive constants $a_n, b_n$ such that $a_n N_4 \leq N_n \leq b_n N_4$ for $n > 0$, the number of vertices, $N_0$ behaves differently, and it is easy to construct explicitly triangulations of $S^4$ such that $N_0 \sim N_4^{1/2}$ as well as triangulations where $N_0 \sim N_4$ for $N_4 \to \infty$. The first kind of triangulations seems
a little pathological from the point of view of smooth manifolds since the number of vertices per unit volume goes to zero. Nevertheless they might be numerous and cannot be dismissed \textit{a priori} in a theory of quantum gravity.

It is easy to understand that triangulations with as few vertices as possible are favored in the limit \( k_2 \to -\infty \) while triangulations with a maximal number of vertices are favored for \( k_2 \to \infty \). If we use \((\ref{eq:5})\) we can write the Einstein-Hilbert action \((\ref{eq:4})\) as:

\[
S_T = k_4' N_4 - 2k_2(N_0 - 2), \quad k_4' = k_4 - 2k_2, \quad (6)
\]

and it is seen that \( k_2 \) acts as a “chemical” potential not only for the number of triangles but also for the number of vertices \( N_0 \). In addition it is easy to understand the two limits in a qualitative way. If there are very few vertices relative to the number of \( N_n \)’s, \( n > 0 \), the order of some of the vertices must be quite high and most likely it will be possible to move between any two vertices along the links in few steps. This indicates that the Hausdorff dimension of the triangulation might be high, maybe even infinite. On the other hand we get a maximum number of vertices relative to the number of four-simplexes if we glue four-simplexes together in an almost one-dimensional structure. These qualitative aspects of the triangulations as functions of the bare gravitational coupling constant \( 1/k_2 \) are clearly seen in the Monte Carlo simulations, and from this point of view the simulations certainly pick up “typical” configurations.

Intuitively it seems as if large barriers may appear more easily if the manifold is highly crumpled and of large Hausdorff dimension. \textit{A posteriori} these were indeed the manifolds which it took the longest time to “cool” to the minimal volume configuration of \( S^4 \). We therefore concentrated on simulations with small values of \( k_2 \). For \( k_2 = 0 \) we are well into the region of small \( k_2 \) from a practical point of view and simulations performed there will generate “typical” crumpled configurations. The choice \( k_2 = 0 \) has the additional nice feature that it weights all triangulations equally. We performed the main series of numerical experiments on thermalized configurations with \( k_2 = 0 \) and \( N_4 = 16000, 32000 \) and \( 64000 \). These configurations were obtained in numerical experiments, where the system was forced to stay in the neighborhood of a chosen volume \( N_4^0 \) by modifying the action \((\ref{eq:6})\) to:

\[
S_T = k_4 N_4(T) + \Delta|N_4 - N_4^0| - k_2 N_2(T), \quad (7)
\]

with small \( \Delta \) and \( k_4 \) close to the pseudo–critical value \( k_4^c(k_2, N_4) \). The dependence on \( N_4 \) is a finite–size effect. For \( k_2 = 0 \) the \( k_4^c \) values for \( N_4 = 16000, 32000 \) and \( 64000 \) were found to be respectively 1.134(2), 1.152(2) and 1.168(2). The numerical simulation means performing “moves”. A number of successfully performed moves
can be used as a measure of time or numerical distance covered by the simulation. This number for thermalized configurations is typically of the order $10^9$.

The objective of the experiment was to reduce the volume $N_4$ in the “cooling” experiment. One can imagine many possible setups for such experiment. As is clear from the discussion above small values of $N_4$ will be favored for large $k_4$, provided $k_4 > k_4^c(k_2, N_4)$. If we choose this value to be very large, the system can be frozen into one of the metastable configurations and we can try to measure the height of the barrier separating it from the path leading to the bottom of the configuration space – a configuration with a minimal four–volume. In all experiments presented here we chose $k_2 = 0$, but the same qualitative behaviour is seen for other values of $k_2$. The outcome of a series of typical experiments is presented on figure 1. In the first step of the experiment the configurations were cooled with $k_4 = 8$. The unit of “time” on the horizontal axis is 5000 accepted moves. In few time steps (typically below 10) the system reaches a stable volume, where only “canonical” moves can be performed. Further reduction of the volume becomes impossible, because the number of points with coordination 5 and links with coordination 4, necessary to perform the volume–reducing moves becomes zero and the inverse moves are exponentially suppressed. The situation can be viewed as reaching the boundary of the configuration space. In the second step of the experiment the value of $k_4$ is raised to 6.0. This is still far above the critical value, which is of the order 1. It is nevertheless sufficient for the system to find it’s way down, eventually reaching the minimal configuration. The height of the barrier is finite, the system has to increase the volume only by few (typically 2 – 8) simplexes before it can be reduced again.

A picture which emerges from this experiment is that of a very smooth configuration space boundary rather than that with many very deep valleys. The volume falls down almost linearly with “time”. For the number of vertices this dependence is more complicated: the decrease gets faster for smaller volumes. The corresponding plots are shown on figure 2.

We tried to modify in various ways the first step of this experiment to get different starting points. In all cases the qualitative behaviour was the same, although we observed some dependence of the time necessary for a complete cooling of the initially frozen configuration.

On figure 3 we show results of a series of experiments for the system with 32000 simplices. In all experiments the same thermalized $k_2 = 0$ configuration was cooled with different values of $k_4 > k_4^c$. We started with $k_4 = 2.0$, where we observe a smooth volume dependence; the system never reaches the boundary of the configuration space and reaches the minimal configuration after 135 steps. For $k_4 \geq 3.0$
the cooling process has two phases. In the first one (approximately 10 time steps, independent of \(k_4\)) the system reaches the boundary with no vertices with coordination 5 and links with coordination 4. In the second phase the system slides down, eventually to reach the minimal configuration. The structures, necessary to perform the volume-reducing moves are dynamically created (typically one link with coordination 4 and more rarely a vertex with coordination 5) which is enough to find a path down. The process gets more difficult the bigger is the value of \(k_4\) when the volume-increasing moves become more suppressed. Figure 4 shows the corresponding dependence of the number of vertices.

We repeated the experiment for other thermalized configurations for systems with 32000 simplexes. In the cooling process we set \(k_2 = 0.0\). As expected, for configurations typical for larger values of \(k_2\) the complete cooling of the configuration is achieved faster. Already for the \(k_2 = 1.0\) configuration the cooling time with \(k_4 = 6.0\) takes less than 20 steps. In the other extreme, we studied configurations for negative \(k_2\). The cooling of the thermalized configuration for \(k_2 = -1.0\) looks almost identical to that of the \(k_2 = 0.0\) configuration. It should be noticed that for the \(k_2 = -1.0\) the number of points \(N_0 \approx 500\), so we are dealing with an extremely crumpled manifold with many points of very high order.

3 Discussion

We have not seen any trace of the very large distances, measured in the number of local moves, which separate certain configurations in algorithmically unrecognizable manifolds. This indicates that \(S^4\) is either algorithmically recognizable in the class of four-manifolds, or that the class of configurations separated from the trivial minimum configuration is small, maybe of measure zero, in the class of all configurations of \(S^4\).

Since we here talk about numerical “experiments” the above results can not constitute a proof in any way. However we find it remarkable that we have not seen any sign at all of even small barriers separating parts of the configuration space from the trivial minimum configuration. Rather, it seems as if the number of moves needed to connect any two configurations of volume \(N_4\) is simply proportional to \(N_4\).

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Fig. 1

$N_4$ vs. time dependence in the cooling experiments for systems with 16000, 32000 and 64000 simplices. In the first step systems were cooled with $(\kappa_2 = 0.0, \kappa_4 = 8.0)$ to reach a stable pseudo–minimum. The rest of the cooling was done with $(\kappa_2 = 0.0, \kappa_4 = 6.0)$ Unit of time is 5000 moves.
Fig. 2
The same as figure 1, dependence $N_0$ vs. time.
Fig. 3

$N_4$ vs. time dependence in the cooling experiments for a system with 32000 simplices. In all cases $\kappa_2 = 0.0$. Plots correspond to $\kappa_4 = 2.0, 3.0, 4.0, 5.0$ and 6.0. Above $\kappa_4 = 3.0$ system reaches the "boundary" in the first few steps. In all cases the minimal configuration was reached.
Fig. 4
The same as figure 3, dependence $N_0$ vs. time.