JACOBIAN CONJECTURE IN $\mathbb{R}^2$

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Abstract. Jacobian conjecture states that if $F: \mathbb{C}^n(\mathbb{R}^n) \to \mathbb{C}^n(\mathbb{R}^n)$ is a polynomial map such that the Jacobian of $F$ is a nonzero constant, then $F$ is injective.

This conjecture is still open for all $n \geq 2$, and for both $\mathbb{C}^n$ and $\mathbb{R}^n$. Here we provide a positive answer to the Jacobian conjecture in $\mathbb{R}^2$ via the tools from the theory of dynamical systems.

Let $(f, g): \mathbb{R}^2(\mathbb{C}^2) \to \mathbb{R}^2(\mathbb{C}^2)$ be a polynomial map. We denote by $J(f, g)$ the Jacobian matrix of the map $(f, g)$, and by $D(f, g)$ the Jacobian of $(f, g)$, i.e. $D(f, g) = \det J(f, g)$. In what follows $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

The classical Jacobian conjecture states that if the Jacobian $D(f, g) = 1$, then $F$ is injective. This conjecture was first posed as a question by Keller [33] in 1939. For more information on the history of this conjecture, see e.g. the survey paper [6, 23, 53]. Nowadays, the Jacobian conjecture is formulated in the next form (see e.g. Bass et al [6] and Essen [23] pages XV and 82)).

Jacobian conjecture. If $F: \mathbb{R}^n(\mathbb{C}^n) \to \mathbb{R}^n(\mathbb{C}^n)$ is a polynomial map such that the Jacobian $DF \in \mathbb{R}^*(\mathbb{C}^*)$, then $F$ is injective.

Associated to the Jacobian conjecture, Randall [41] in 1983 posed the so called real Jacobian conjecture.

Real Jacobian conjecture. If $F: \mathbb{R}^2 \to \mathbb{R}^2$ is a polynomial map such that the Jacobian $DF$ of $F$ does not vanish, then $F$ is injective.

This conjecture is not correct in general, as illustrated by Pinchuck [39] in 1994, which we will recall it again later on.

A general formulation in $\mathbb{C}^n$ of the real Jacobian conjecture was posed by Smale [46] in 1998 as his 16th problem on a list of 18 open mathematical problems. For distinguishing it from the Jacobian conjecture mentioned above and according to Pinchuck [39], we call it Strong Jacobian conjecture.

Strong Jacobian conjecture. If $F: \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map such that the Jacobian $DF$ of $F$ does not vanish, then $F$ is injective.

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For the polynomial maps, there is a natural connection between injective and surjective. Bia\l{}ynicki-Birula and Rosenlicht [8] in 1962 proved the next result (see also Rudin [42] in 1995).

**Bia\l{}ynicki-Birula and Rosenlicht Theorem.** If \(F: \mathbb{K}^n \rightarrow \mathbb{K}^n\) is an injective polynomial map, with \(\mathbb{K}\) an algebraically closed field of characteristic zero, then \(F\) is a polynomial automorphism from \(\mathbb{K}^n\) to itself.

Pinchuck [39] in 1994 constructed a counterexample to the real Jacobian conjecture, which is a polynomial map in \(\mathbb{R}^2\) with nonvanishing Jacobian, and it is not injective. As mentioned by the author, one of the components of the constructed map has its level curves containing more than one branches.

According to the Pinchuck’s example, in order that the real Jacobian conjecture holds, there needs some additional conditions. Along this direction, there appeared some additional sufficient conditions ensuring the real Jacobian conjecture holds. See for instance [9, 10, 11, 12, 17, 23, 31, 44], parts of which were proceeded via tools from the theory of dynamical systems on the characterization of global centers for real planar polynomial vector fields. But, at the moment there is no a necessary and sufficient condition for which the real Jacobian conjecture holds. In addition, there are also many other partial results on both Jacobian and strong Jacobian conjectures, and on their relations with other subjects, see e.g. the survey papers by Bass et al [6] in 1982 and Essen [23] in 2000, and the recent publications [5, 7, 13, 14, 19, 21, 24, 26, 27, 29, 32, 35, 38, 40, 47, 48, 49, 50, 55, 56]. On the history of the Jacobian and strong Jacobian conjectures we should also mention the papers by Abhyankar and Moh [3] in 1975 for two dimensional case, by Wang [51] in 1980 for the map \(F\) with degrees no more than 2, by Bass et al [6] in 1982, Yagzhev [54] in 1980 and Drużkowski [20] in 1983 via reduction of degrees of the polynomial maps, and by Hubbers [30] on cubic maps in dimension 4.

In any case, as our knowledge, the Jacobian and strong Jacobian conjectures are in general open for all \(n \geq 2\), see Essen [23] in 2000, the recent paper [10] in 2016 on the first paragraph of page 5251, and also the item Jacobian conjecture in Wikipedia.

Here we prove the Jacobian conjecture in \(\mathbb{R}^2\), using the tools from the theory of dynamical systems related with the commuting Hamiltonian vector fields and the Poincaré compactification.

**Theorem 1.** If \(F: \mathbb{R}^2 \rightarrow \mathbb{R}^2\) is a polynomial map such that the Jacobian \(DF \in \mathbb{R}^2\), then \(F\) is injective.

**Proof.** Set \(F(x, y) = (f(x, y), g(x, y)) \in (\mathbb{R}[x, y])^2\). Then \(DF = D(f, g) := f_x g_y - f_y g_x\). Let \(H_f = (-f_y, f_x)^T\) and \(H_g = (-g_y, g_x)^T\) be respectively the Hamiltonian vector fields associated to \(f\) and \(g\), where \(T\) represents the transpose of a matrix. Denote by \([H_f, H_g]\) the Lie bracket of the two Hamiltonian vector fields \(H_f\) and \(H_g\). Recall that the Lie bracket of two smooth vector fields \(X\) and \(Y\) is by definition \([X, Y] = (JY)X - (JX)Y\), where \(JX\) and \(JY\) are the Jacobian matrices of the vector fields \(X\) and \(Y\), respectively. For more information on Lie brackets, see e.g. [11, 37].
The next seven claims will complete the proof of Theorem 1.

Claim 1. \(D(f,g) \in \mathbb{R}^2\) if and only if \([H_f, H_g] \equiv 0\).

This statement can be found for example in Remark 3.2 of [16]. For completeness, we present its proof here.

Direct calculations show that
\[
D_x = -f_y g_{x^2} + f_x g_y + f_{xxy} g_x - f_{xy} g_x,
\]
\[
D_y = -f_y g_{xy} + f_x g_y^2 + f_{yy} g_y - f_{y^2} g_x,
\]
where \(D_x\) represents the partial derivative of \(D(f,g)\) with respect to \(x\). In addition, some computations verify that
\[
[H_f, H_g] = (JH_g) H_f - (JH_F) H_g = (-D_y, D_x)^T = H_D.
\]
If follows that \([H_f, H_g] \equiv 0\) if and only if \(D_x \equiv 0\) and \(D_y \equiv 0\), and if and only if \(D(f,g)\) is a constant. The claim follows. \(\Box\)

We remark that for a given polynomial or an analytic vector field \(X\), the set of vector fields \(Y\) satisfying \([X, Y] = 0\) is called centralizer of \(X\). Recently Cerveau and Lins Neto [16] and Walcher and Zhang [34] characterized the structure and dimension of the centralizer.

Claim 2. If \(D(f,g) \in \mathbb{R}^*,\) then the Hamiltonian vector fields \(H_f\) and \(H_g\) both have no singularities and have no periodic orbits. Consequently, all orbits of \(H_f\) and \(H_g\) have their positive and negative limits going to infinity.

Indeed, if the Hamiltonian vector field \(H_f\) has a singularity at \(P_0\), then \((-f_y(P_0), f_x(P_0)) = 0\). So the gradient vector field \(\nabla f := (f_x, f_y)\) vanishes at \(P_0\). It forces that \(D(f,g)(P_0) = 0\), a contradiction with the assumption of the claim.

If the polynomial Hamiltonian vector field \(H_f\) has a periodic orbit, saying \(\Gamma_0\), then it will have a period annulus containing \(\Gamma_0\) in its interior. By Ye [57] Theorem 1.6] the outer and inner boundaries of a maximal period annulus of an analytic vector fields both contain singularities (those on the outer boundary may be at infinity). Or using the well known fact that in the region limited by a periodic orbit there is at least one singularity of the planar vector field, we will be in a contradiction with the last argument that \(H_f\) cannot have singularities in the finite plane.

As a consequence of these last two facts, all orbits of \(H_f\) are regular and have their positive and negative limits both at infinity. The claim holds. \(\Box\)

Claim 3. If \(D(f,g) \in \mathbb{R}^*,\) then the Hamiltonian vector fields \(H_f\) and \(H_g\) intersect transversally everywhere. Moreover, the flow of \(H_g\) maps any orbit of \(H_f\) to a different one and preserves the time.

Indeed, the first argument follows from the fact that
\[
\det(H_f, H_g) = \det J(f, g) = D(f, g) \neq 0.
\]
Claim 1 has shown that \([H_f, H_g] \equiv 0\). Then the second argument is a well known fact in the world of dynamical systems, and is from the commutation of the flows of two commuting vector fields \(H_f\) and \(H_g\) and their
transversality everywhere. For more information on dynamical properties of commuting vector fields, see e.g. [37] Theorem 1.34, or [43], or [4]. The claim follows.

\textbf{Claim 4.} For the polynomial \( f \in \mathbb{R}[x,y] \), its associated Hamiltonian vector field \( H_f \) has finitely many singularities at the infinity, i.e. on the equator of the Poincaré disc.

Poincaré compactification is useful in the study of global dynamics, including the infinity, of polynomial vector fields in \( \mathbb{R}^n \). For more information on Poincaré compactification, see e.g. [18], [22, Chapter 5].

We now prove the claim. It is well known [22, Chapter 5] that the infinity of a polynomial vector field of degree \( m \) either contains at most \( m + 1 \) singularities on the equator of the Poincaré disc or fulfils singularities on the equator of the Poincaré disc, where each pair of diametral points on the equator is numerated as one point. Let a polynomial vector field of degree \( m \) be of the form \( \mathcal{X}_P := \left( \sum_{j=0}^{m} p_j(x,y), \sum_{j=0}^{m} q_j(x,y) \right) \) with \( p_j, q_j \) being homogeneous polynomials of degree \( j \) or naught, and with \( p_m \) and \( q_m \) not both identically zero. By [15] [45] [58] [59] in order that the infinity of \( \mathcal{X}_P \) on the Poincaré disc fulfils singularities, it is necessary that \( (p_m(x,y), q_m(x,y)) \) is of the form \( (x h_{m-1}(x,y), y h_{m-1}(x,y)) \), where \( h_{m-1} \) is a homogeneous polynomial of degree \( m - 1 \). But it is not possible for the polynomial Hamiltonian vector field \( H_f \). In fact, suppose that \( f \) is of degree \( \ell \), and its highest order homogenous part \( f_\ell(x,y) = \sum_{j=0}^{\ell} \alpha_j x^j y^{\ell-j}, \alpha_j \in \mathbb{R} \). Then the highest order term of the Hamiltonian vector field \( H_f \) is \( H_{f_\ell} := \left( -\frac{\partial_y f_\ell, \partial_x f_\ell} \right) = \left( - \sum_{j=0}^{\ell} \alpha_j(\ell-j)x^j y^{\ell-j-1}, \sum_{j=0}^{\ell} \alpha_j j x^{j-1} y^{\ell-j} \right) \), where \( \partial_y f_\ell \) is the partial derivative of \( f_\ell \) with respect to \( y \). In order for its first (resp. second) component of \( H_{f_\ell} \) to have a factor \( x \) (resp. \( y \)), one has \( \alpha_0 = \alpha_{\ell} = 0 \). So \( H_{f_\ell} = \left( x(- \sum_{j=1}^{\ell-1} (\ell-j)\alpha_j x^{j-1} y^{\ell-j-1}), y(\sum_{j=1}^{\ell-1} j\alpha_j x^{j-1} y^{\ell-j-1}) \right) \). Clearly, \( H_{f_\ell} \) is not of the form \( (x h_{\ell-2}, y h_{\ell-2}) \) with \( h_{\ell-2} \) being homogeneous of degree \( \ell - 2 \). This proves that the highest order term of the Hamiltonian vector field \( H_f \) does not have the form such that its infinity fulfils singularities. Consequently, the claim holds.

In order to prove the injectivity of the polynomial map \( F = (f,g) \) in \( \mathbb{R}^2 \), it is equivalent to prove \( F(p_0) \neq F(q_0) \) for all \( p_0, q_0 \in \mathbb{R}^2 \) and \( p_0 \neq q_0 \), i.e. \( f(p_0) \neq f(q_0) \) or \( g(p_0) \neq g(q_0) \).

For \( p, q \in \mathbb{R}^2 \) satisfying \( p \neq q \), if \( f(p) \neq f(q) \), we are down. Assume that there exist some \( p_0, q_0 \in \mathbb{R}^2 \) with \( p_0 \neq q_0 \), such that \( f(p_0) = f(q_0) \). Set \( u_0 := f(p_0) = f(q_0) \) and \( S := \{ r \in \mathbb{R}^2 | f(r) = u_0 \} \).

Let \( \phi_t \) and \( \psi_s \) be the flows of the Hamiltonian vector fields \( H_f \) and \( H_g \), respectively.
Claim 5. If $D(f,g) \in \mathbb{R}^s$, then any connected branch of $S$ sweeps the full plane $\mathbb{R}^2$ under the action of the flow $\psi_s$ of $H_g$.

We now prove this claim. Let $S_1$ be one of the connected branches of $S$ and assume without loss of generality that $p_0$ is located on $S_1$. Set $v_0 = g(p_0)$ and $\mathcal{R} = \{ r \in \mathbb{R}^2 | g(r) = v_0 \}$. Let $\mathcal{R}_1$ be the connected branch of $\mathcal{R}$ with $p_0$ located on it.

Now we work on the Poincaré compactification for planar polynomial vector fields, see e.g. [18], [22, Chapter 5]. Recall from Claim 2 that $\mathcal{R}$ and $S$ are nonsingular curves and are not closed in the finite plane. Then $\mathcal{R}_1$ and $S_1$ all have their endpoints at the infinity. Thus, $S_1$ separates the Poincaré disc in two disjoint regions, saying $D^+_{S_1}$ and $D^-_{S_1}$, and its two endpoints at the infinity are denoted by $I_{11}$ and $I_{12}$ (they maybe coincide, i.e. $I_{11} = I_{12}$), which are singularities of the Hamiltonian vector field $H_f$ at the infinity.

As it is well known, the flow of an analytic vector field on a compact space is complete, i.e. any of its solutions is defined on all $\mathbb{R}$ (see e.g. [22]). So the Poincaré compactifications of the Hamiltonian vector fields $H_f$ and $H_g$ have their flows complete. That is, the flows $\phi_t$ of $H_f$ and $\psi_s$ of $H_g$ are complete on the Poincaré disc. Since we are working in Poincaré compactification, in what follows all the proofs related to the flows will be processed in the full $\mathbb{R}$. In fact, we will show in Remark 3 that in our setting the flows of the Hamiltonian vector fields $H_f$ and $H_g$ are also complete in $\mathbb{R}^2$.

By Claim 3 the vector fields $H_f$ and $H_g$ are transversal everywhere in the finite plane, and by Claim 1 one has $[H_f, H_g] \equiv 0$ and so the flows of $H_f$ and $H_g$ commute, it follows that under the action of the flow $\psi_s$ of $H_g$, the invariant curve $S_1$ of $H_f$ moves towards either into $D^+_{S_1}$ or into $D^-_{S_1}$. For fixing and without loss of generality, we assume that $\psi_s(S_1) \subset D^+_{S_1}$ when $s > 0$ and $\psi_s(S_1) \subset D^-_{S_1}$ when $s < 0$. By commutation of the flows $\phi_t$ and $\psi_s$, it follows that for any fixed $s \in \mathbb{R}$, $\psi_s(S_1)$ is an invariant curve of $H_f$. Recall that invariant curves of a planar Hamiltonian vector field are all given by the level curves of its associated Hamiltonian function. It forces that $\psi_s(S_1)$ is a level curve of $f$, and so it is regular and has its two endpoints at the infinity.

Note that when the time $s$ continuously increases or decreases, the invariant curve $\psi_s(S_1)$ of $H_f$ continuously runs forward from $S_1$ inside $D^+_{S_1}$ or moves backward from $S_1$ inside $D^-_{S_1}$. By Claim 4, the singularities at the infinity of $H_f$ on the Poincaré disc are finite. So it follows from commutation of the Hamiltonian vector fields $H_f$ and $H_g$ that when $s$ continuously varies the two endpoints (may coincide) of $\psi_s(S_1)$ will keep the same as those of $S_1$. In addition, the two endpoints (may coincide) of $\mathcal{R}_1$ are also at the infinity of $H_g$ on the Poincaré disc, we denote them by $J_{11}$ and $J_{12}$ (may $J_{11} = J_{12}$), which are singularities of the Hamiltonian vector field $H_g$ at the infinity. Then it follows from the transversality of the vector fields $H_f$ and $H_g$ that the locations of $I_{11}$, $I_{12}$, $J_{11}$ and $J_{12}$ have four possibilities:
(a₁) $I_{11} \neq I_{12}$, $J_{11} \neq J_{12}$, and $\{J_{11}, J_{12}\} \cap \{I_{11}, I_{12}\} = \emptyset$. So $J_{11}$ and $J_{12}$ are located on the different sides of $S_1$, saying $J_{11} \in D_{S_1}^+$ and $J_{12} \in D_{S_1}^-$. The simple map $(f, g) = (x, y)$ is in this situation.

(a₂) $I_{11} \neq I_{12}$, $J_{11} \neq J_{12}$, and $\{J_{11}, J_{12}\} \cap \{I_{11}, I_{12}\} = \emptyset$. So $J_{11}$ and $J_{12}$ coincide with $I_{11}$ and $I_{12}$, saying for example $J_{11} = I_{11}$. The map $(f, g) = (x, y - x^3)$ illustrates this case.

(a₃) $I_{11} \neq I_{12}$ and $J_{11} = J_{12}$. Then one of $I_{11}$ and $I_{12}$ coincides with $J_{11} = J_{12}$. The map $(f, g) = (x, y - x^2)$ is in this case.

(a₄) $I_{11} = I_{12}$ and $J_{11} \neq J_{12}$. This is also in the situation (a₃).

(a₅) $I_{11} = I_{12}$ and $J_{11} = J_{12}$. Then $I_{11} = I_{12} = J_{11} = J_{12}$. The map $(f, g) = (y - x^2, y + x - x^2)$ illustrates this case.

Now we consider the actions of the flow $\phi_t$ on $S_1$, and of the flow $\psi_s$ on $R_1$. Fig. 1 illustrates these actions.

![Figure 1](image)

**Figure 1.** Here we illustrate for $s_2 > s_1 > 0$ and $t_2 > t_1 > 0$

When $s$ increases from 0 to $\infty$, the flow $\psi_s$ acts on $p_0$ and pushes it along $R_1$ to the infinity $J_{11}$, because $R_1$ is invariant under the action of the flow $\psi_s$ of $H_g$. Denote by $R_1^+$ the part of $R_1$ from $p_0$ to $J_{11}$. Applying the similar arguments as presented in the previous two paragraphs, one gets that for
Remark 1. The proofs given in (a1) and (a2) verify the next facts.

- The finite plane $\mathbb{R}^2$ is foliated by $\{\psi_s(S_1)\}$, $s \in \mathbb{R}$, each of which is a regular orbit of $H_f$. 

The next is a proof to this last argument. By contrary, if not, set $\Omega_1 := \bigcup_{s=0}^{\infty} \psi_s(S_1)$ and $\Omega_0 := D_{S_1}^+ \setminus \Omega_1$, then $\Omega_0$ will be a nonempty closed domain and $\Omega_1 \setminus \{S_1\}$ is open. Denote by $\partial \Omega_0$ the boundary of $\Omega_0$ which is located in $\mathbb{R}^2$ (excluding the part at the infinity). It is well known that the boundary of $\Omega_0$ is formed by orbits of $H_f$. Let $B_1^*$ be one of the regular orbits of $\partial \Omega_0$, and its endpoints be $I_1^*$ and $I_2^*$ (they may coincide), which are at the infinity. Hence $B_1^*$ must be a heteroclinic or a homoclinic orbit of $H_f$. Since the Hamiltonian vector fields $H_f$ and $H_g$ are transversal everywhere in the finite plane, it follows that under the action of the flow $\psi_s$, $B_1^*$ will positively get into the interior of $\Omega_0$, and negatively moves into $\Omega_1$. Now the flows $\psi_s$ and $\phi_t$ commute, it forces that the negative action of $B_1^*$ under the flow $\psi_s$ will be an orbit of $H_f$, choose one of them, and denote it by $O^*$. Then, as shown in the previous proof, $O^*$ will have its endpoints same as those of $B_1^*$. Since $B_1^*$ belongs to the boundary of $\Omega_1 = \bigcup_{s=0}^{\infty} \psi_s(S_1)$, the orbit $O^*$ of $H_f$ must be in the region $\Omega_1$. By the uniqueness theorem of solutions of a smooth differential system with respect to the initial point, it follows that $O^* = \psi_{s_0}(S_1)$ for some $s_0 \in (0, \infty)$. Hence the endpoints $I_1^*$ and $I_2^*$ of $B_1^*$ must coincide with $I_{11}$ and $I_{12}$. This means that $B_1^*$ is a heteroclinic orbit connecting $I_{11}$ and $I_{12}$ when $I_{11} \neq I_{12}$, or a homoclinic one when $I_{11} = I_{12}$. Again by the transversality of $H_f$ and $H_g$ it follows that $B_1^*$ under the action of the flow $\psi_s$ will positively get into the interior of $\Omega_0$. It is in contradiction with the definition of $\Omega_0$. Hence, the heteroclinic (or homoclinic) orbit $B_1^*$ for $H_f$ does not exist, and so $\Omega_0$ does not have a boundary in $\mathbb{R}^2$. Consequently, $\Omega_0$ is empty. Hence $\bigcup_{s=0}^{\infty} \psi_s(S_1) = D_{S_1}^+$, which is exactly what we want to prove.

Now we turn to $S_1$ under the action of $\psi_s$ with $s < 0$. Applying the same arguments as above to $S_1$ with $s \leq 0$ yields that $\bigcup_{s=0}^{-\infty} \psi_s(S_1) = D_{S_1}^-$. Hence, $\bigcup_{s=--\infty}^{-\infty} \psi_s(S_1)$ is the full finite plane $\mathbb{R}^2$. This proves the claim. 

Remark 1. The proofs given in (a1) and (a2) verify the next facts.

- The finite plane $\mathbb{R}^2$ is foliated by $\{\psi_s(S_1)\}$, $s \in \mathbb{R}$, each of which is a regular orbit of $H_f$. 

• All orbits of $H_f$ except the line at infinity are heteroclinic to the same singularities at the infinity, or homoclinic to the same singularity at the infinity. The same argument applies to $H_g$.

**Claim 6.** If $D(f, g) \in \mathbb{R}^*$, then $S$ has a unique connected branch.

In fact, by contrary we suppose that $S$ has more than one connected branches. Let $S_1$ and $S_2$ be two different connected branches of $S$, with $p_0 \in S_1$. By Claim 5 there exists an $s_0 \in \mathbb{R}$ such that $S_2 = \psi_{s_0}(S_1)$. Let $r_0 \in S_2$ be the image of $p_0$ under the diffeomorphism $\psi_{s_0}$. Then $f(p_0) = f(r_0)$ by the assumption of the contrary. In addition, $g(\psi_s(p_0)) \equiv g(p_0)$ for all $s \in \mathbb{R}$ because $g$ and $\psi_s$ are respectively the first integral and the flow of $H_g$.

Since $f$ is a first integral of the Hamiltonian vector field $H_f$ and $\psi_s(S_1)$ is an orbit of $H_f$ for any fixed $s$, it follows that $f(\psi_s(S_1))$ is a constant for any fixed $s \in \mathbb{R}$. Set

$$\omega(s) = f(\psi_s(S_1)), \quad s \in [0, s_0].$$

Then $\omega(s)$ is well defined and is an analytic function, where we have used the facts that

- $f$ is a polynomial,
- $\psi_s$ is an analytic flow, because the Hamiltonian vector field $H_g$ is polynomial, and so is analytic.
- $S_1$ is an analytic curve, because it is a regular orbit of the polynomial vector field $H_f$.

By definition one has $\omega(0) = \omega(s_0)$. It forces that there exists an $s^* \in (0, s_0)$ at which $\omega$ takes a maximum or a minimum value on $[0, s_0]$. Then $\omega'(s^*) = 0$. On the other hand,

$$\omega'(s^*) = \frac{df(\psi_s(p_0))}{ds}
\bigg|_{s=s^*} = \nabla f(\psi_s(p_0)) \frac{d\psi_s(p_0)}{ds}
\bigg|_{s=s^*}
\begin{align*}
&= \nabla f(\psi_s(p_0)) H_g(\psi_s(p_0))
\bigg|_{s=s^*}
= -(f_yg_y - f_xg_x)(\psi_s^*(p_0))
&= -D(f, g)(\psi_s^*(p_0)) \neq 0,
\end{align*}$$

a contradiction. This verifies that $S$ cannot have two different connected branches. Consequently, the claim follows.

Since $S$ has a unique connected branch, the points $p_0, q_0 \in \mathbb{R}^2$ with $p_0 \neq q_0$ satisfying $f(p_0) = f(q_0)$ must be located on the same orbit, i.e. $S$, of $H_f$.

**Claim 7.** If $D(f, g) \in \mathbb{R}^*$ and $f(p_0) = f(q_0)$, then $g(p_0) \neq g(q_0)$.

Indeed, by contrary we assume that $g(p_0) = g(q_0)$, and let this common value to be $\nu_0$. Define $\mathcal{R} := \{r \in \mathbb{R}^2 | g(r) = \nu_0\}$. The similar proof to Claim 6 yields that $\mathcal{R}$ contains a unique connected branch, which is an orbit of $H_g$. This implies that the orbits $S$ of $H_f$ and $\mathcal{R}$ of $H_g$ intersect at the two different points $p_0$ and $q_0$. But it is impossible, because the flows $\phi_t$ and $\psi_{s_0}$ of the vector fields $H_f$ and $H_g$ commute because $[H_f, H_g] \equiv 0$, and so one of the flows, saying $\psi_{s_0}$, maps any orbit of $\phi_t$ to a different one, and vice versa. Another explanation on this impossibility is that there will exist
at least one point in between \( p_0 \) and \( q_0 \) along \( S \) at which the Hamiltonian vector fields \( H_f \) and \( H_g \) are tangent, a contradiction to Claim 3. This proves the claim. \( \square \)

Summarizing Claims 1 to 7 achieves that for any two points \( p, q \in \mathbb{R}^2 \) with \( p \neq q \), one has \( f(p) \neq f(q) \) or \( g(p) \neq g(q) \). Consequently, the map \( F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2 \) is injective.

It completes the proof of Theorem 1. \( \square \)

As a consequence of the results obtained in the proof of Theorem 1, one gets the geometry of the level curves of \( f \) and \( g \) for the planar real polynomial map \( F = (f, g) \).

**Corollary 1.** Assume that \( F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2 \) is a polynomial map such that the Jacobian \( DF \in \mathbb{R}^* \), the following statements hold.

(a) Each of the level curves of \( f \) (and also of \( g \)) has a unique connected branch with its two endpoints at the infinity, and the level curves (which are one-dimensional smooth manifolds) foliate the real plane \( \mathbb{R}^2 \).

(b) Each of the level curves of \( g \) intersects all the level curves of \( f \) transversally, and vice versa.

(c) Each of the level curves of \( f \) and \( g \) is closed in \( \mathbb{R}P^2 \) with possibly a singularity at the infinity.

**Proof.** Statements (a) and (b) follow directly from the results given in the proof of Theorem 1. Statement (c) is a consequence of Claim 2 and the facts that a nonsingular variety of a real homogeneous polynomial \( h(x, y, z) \) in \( \mathbb{R}P^2 \) is a compact one dimensional manifold (see e.g. Milnor [36] and Wilson [52]), and that a polynomial in \( \mathbb{R}^2 \) can be projectivized in \( \mathbb{R}P^2 \). Statement (c) can also be obtained partially by identifying the diametrally opposite points on the boundary of the Poincaré disc, which produce \( \mathbb{R}P^2 \).

We mention that as shown in the following Remark 3 a nonsingular real planar curve in \( \mathbb{R}^2 \) could have a singularity at the infinity of \( \mathbb{R}P^2 \). \( \square \)

**Example 1.** Here we provide some other examples illustrating the geometry of the level curves of \( f \) and \( g \), which satisfy the Jacobian conjecture in \( \mathbb{R}^2 \).

\((E_1)\) The map \( F_1 = (f_1, g_1) = (y - (2x - y)^4, 2x - y) : \mathbb{R}^2 \to \mathbb{R}^2 \) has its determinant equal to \(-2\). The level curves of \( f_1 \) all have their endpoints at the infinity of the Poincaré disc in the direction \( y = 2x \) with \( x > 0 \), and are closed in the Poincaré compactification with a singularity at the infinity. Whereas the level curves of \( g_1 \) all have their endpoints at the infinities of the Poincaré disc in the two directions of the line \( y = 2x \), and are all nonsingular and closed in \( \mathbb{R}P^2 \). For a definition on singular point of a curve, see for example, [28, page 31].

The argument that the nonsingular level curve \( f_1 = c_1 \) in \( \mathbb{R}^2 \) is singular at the infinity of \( \mathbb{R}P^2 \) follows from the facts that the projectivization of \( f_1 = c_1 \) in the projective coordinate \([X : Y : Z]\) is
$f_1^*(X,Y,Z) = YZ^3 - (2X - Y)^4 - c_1Z^4 = 0$ with the gradient of $f_1^*$ vanishing at $[1 : 2 : 0]$, and its local expression at the infinity $[1 : 2 : 0]$ is $2Z^3 - 16X^4 - c_1Z^4 + O(|X,Z|^5) = 0$, which has two different branches at $[1 : 2 : 0]$ for $Z \geq 0$.

The argument that the level curve $g_1 = d_1$ is nonsingular and closed in $\mathbb{R}^2P$ follows from the facts that the projectivization of $g_1 = d_1$ in the projective coordinate $[X : Y : Z]$ is $g_1^*(X,Y,Z) = 2X - Y - d_1Z = 0$, which is nonsingular in $\mathbb{R}P^2$.

$\langle E_2 \rangle$ The map $F_2 = (f_2, g_2) = (y - x^3, y - x - x^3) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has its determinant equal to 1. The level curves of $f_2$ and $g_2$ all have their endpoints at the infinity of the Poincaré disc with one in the positive $y$ direction and another in the negative $y$ direction, and they are all closed and singular in $\mathbb{R}P^2$.

$\langle E_3 \rangle$ The map $F_3 = (f_3, g_3) = (y - x^2, y - x - x^2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has its determinant equal to 1. The level curves of $f_3$ and $g_3$ all have their endpoints at the infinity of the Poincaré disc in the positive $y$ direction, and they are all closed and nonsingular in $\mathbb{R}P^2$. Here we have used the fact that any irreducible conic is nonsingular in $\mathbb{R}P^2$ [28, page 55], or that an irreducible conic is isomorphic to $\mathbb{R}P^1$ [28, page 78].

These examples show that the level curves of the two polynomials in the polynomial maps $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying the Jacobian conjecture can be either all nonsingular closed ones, or all singular closed ones, or nonsingular closed ones for one polynomial and singular closed one for another polynomial.

**Remark 2.** As an application of the results from the proof of Theorem 1, one gets that if $f = 1 + x - x^2y$ (see [39]), then for any polynomial $g(x, y)$, the map $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is not injective, because $f = c$ with $c \in \mathbb{R}$ has more than one distinct connected branches. In addition, one could compute via Mathematica that a polynomial $g(x, y)$ of higher degree such that $|H_f, H_g| \equiv 0$ is generated by $f$, i.e. $g$ is a polynomial of $f$.

**Remark 3.** After completing the proof of Theorem 1 one wants to apply our tools to prove the Jacobian conjecture in $\mathbb{C}^2$. But the complicated topology of planar algebraic curves in $\mathbb{C}^2$ maybe prevent its direct application, because any polynomial $f(x, y)$ in $\mathbb{C}^2$ can be projectivized in $\mathbb{C}P^2$, and as it is well known that for any real homogeneous polynomial $F(x, y, z)$ of degree $m$, if $\mathcal{V}(F) := \{ [x : y : z] \in \mathbb{C}P^2 \mid F(x, y, z) = 0 \}$ is nonsingular in $\mathbb{C}P^2$, then it is a Riemann surface of genus $g = (m - 1)(m - 2)/2$. See e.g. Wilson [52].

In addition, as shown in Example 1 a nonsingular curve in $\mathbb{C}^2$ could have singularities at the infinity. This implies that in general, a nonsingular planar curve in $\mathbb{C}^2$ could have more complicated topology in $\mathbb{C}P^2$.

**Remark 4.** Our proofs, especially those on Claim 5, provide also some information of the Hamiltonian vector field $H_f$ (and of $H_g$) at the infinity. In this direction we should mention the next results, which were originally proved for $\mathbb{C}[x, y]$, but one can check that they also work for real polynomials.

By definition, a polynomial $f \in \mathbb{R}[x, y]$ has a *Jacobian mate* if there exists a $g \in \mathbb{R}[x, y]$ such that $D(f, g) = 1$. 
Theorem A. (Abhyankar [1, 2]). The following two statements are equivalent.

(a) Every polynomial $f \in \mathbb{R}[x, y]$ which has a Jacobian mate has a point at infinity.

(b) The Jacobian conjecture for polynomials in $\mathbb{R}[x, y]$ holds.

Theorem A was also proved in [23, Theorem 10.2.23] for complex polynomials in $\mathbb{C}[x, y]$ with its proof adapting for real polynomials of $\mathbb{R}[x, y]$.

By Theorem A it follows that the Jacobian conjecture only holds if the Hamiltonian vector fields $H_f$ and $H_g$ have only one pair of equilibria at the infinity.

Our proofs do not provide exact information on the number of pairs of singularities at the infinity. But combining Remark 1 and Claim 5 and its proofs, it follows that all orbits of $H_f$ (resp. $H_g$) in $\mathbb{R}^2$ are heteroclinic to the same two singularities of $H_f$ (resp. $H_g$) at the infinity, or homoclinic to the same singularity of $H_f$ (resp. $H_g$) at the infinity. Then applying Theorem A one has that the global topological phase portrait of $H_f$, and also of $H_g$ is one of the two pictures illustrated in Fig. 2.

![Figure 2. Global phase portraits of $H_f$, and of $H_g$](image)

Remark 5. In our setting, i.e. $D(f, g) = c$ a nonzero real number, the flows of the Hamiltonian vector fields $H_f$ and $H_g$ are both complete in $\mathbb{R}^2$.

Indeed, as before let $\phi_t$ and $\psi_s$ be the flows of the Hamiltonian vector fields $H_f$ and $H_g$, respectively. For any $p \in \mathbb{R}^2$, and let $I_p = (\alpha_p, \beta_p)$ be the maximal open interval of $\mathbb{R}$ on which the solution $\phi_t(p)$ is defined. Then

$$f_x(\phi_t(p))g_y(\phi_t(p)) - f_y(\phi_t(p))g_x(\phi_t(p)) = c,$$

for all $t \in I_p$.

This last equality can be written as

$$c = \langle H_f(\phi_t(p)), \nabla g(\phi_t(p)) \rangle$$

$$= \left\langle \frac{d\phi_t(p)}{dt}, \nabla g(\phi_t(p)) \right\rangle = \frac{dg(\phi_t(p))}{dt}, \quad t \in I_p,$$

where $\langle \cdot, \cdot \rangle$ represents the inner product of two vectors. This shows that along any orbit of $H_f$, the value of the polynomial $g$ monotonically increases if $c > 0$, and decreases if $c < 0$.  


By contrary, if there exists a $p_\ast \in \mathbb{R}^2$ and the orbit $\phi_t(p_\ast)$ positively or negatively approaches infinity in a finite time, without loss of generality we assume that $\beta_{p_\ast}$ is finite. Let $\Gamma_{p_\ast}^+$ be the positive orbit of $H_f$ passing $p_\ast$, and let $\Upsilon_\ast$ be the orbit of $H_g$ passing $p_\ast$. Then applying the similar techniques as those in the proof of Claim 5 but independent of that proof we get via commutation of the flows $\phi_t$ and $\psi_s$ that $\bigcup_{t=0}^{\beta_{p_\ast}} \phi_t(\Upsilon_\ast)$ covers one of the half spaces limited by $\Upsilon_\ast$. Denote this half space by $\Omega_{\ast}$. Integrating (1) for $p$ replaced by $\Upsilon_{\ast}$ in $t$ from 0 to $\beta_{p_\ast}$ gives

$$g(\phi_{\beta_{p_\ast}}(\Upsilon_{\ast})) = g(\phi_0(\Upsilon_{\ast})) + c_{\beta_{p_\ast}} = g(\Upsilon_{\ast}) + c_{\beta_{p_\ast}} = g(p_\ast) + c_{\beta_{p_\ast}}.$$  

This implies that $g$ is bounded in the half space $\Omega_{\ast}$. But it is impossible because $g$ is a polynomial. This contradiction verifies that $\beta_{p_\ast} = \infty$.

Consequently, the flow of $H_f$ is complete in $\mathbb{R}^2$. It completes the proof of the argument of Remark 5. □

Finally we illustrate that for Strong Jacobian Conjecture, the flow of the Hamiltonian vector field $H_f$ or $H_g$ could be not complete in $\mathbb{R}^2$. Take $(f, g) = (-(1 + x^2)y, x)$. Then $D(f, g) = 1 + x^2 \geq 1$, and the Hamiltonian vector field associated to $f$ is

$$H_f = \begin{pmatrix} 1 + x^2 \\ -2xy \end{pmatrix}.$$  

It has the flow

$$\phi_t(c_1, c_2) = \left( \tan(t + \arctan(c_1)), (1 + c_1^2)c_2 \cos(t + \arctan(c_1)) \right)$$

satisfying $\phi_0(c_1, c_2) = (c_1, c_2)$, defined on $(\arctan(c_1 - \pi/2), \arctan(c_1 + \pi/2))$ for arbitrary $(c_1, c_2) \in \mathbb{R}^2$.

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