Faddeev-Jackiw analysis for the charged compressible fluid in a higher-derivative electromagnetic field background

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received 8 June 2016; accepted in final form 11 November 2016
published online 1 December 2016

PACS 03.50.Kk – Other special classical field theories
PACS 11.10.Ef – Lagrangian and Hamiltonian approach
PACS 47.10.-g – General theory in fluid dynamics

Abstract – In the present paper we will discuss the Faddeev-Jackiw (FJ) symplectic approach in the analysis of a charged compressible fluid immersed in a higher-derivative electromagnetic field theory. In other words, we have analyzed, from the FJ point of view, the minimal coupling between the strength tensor of the fluid and the higher derivative of the electromagnetic field. We have obtained the full set of constraints directly from the zero-mode eigenvectors. Besides, we have computed the Dirac brackets for the dynamic variables of the compressible fluid. Finally, as a result of the coupling between the charged compressible fluid and the electromagnetic field we have calculated two Dirac brackets between the fluid and electromagnetic fields, which are both zero when there is no coupling between them.

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Introduction. – The search for a connection between fluid dynamics and electromagnetism is an old concept and it has played a crucial role in the development of the Maxwell equations \cite{1,2}. Thomson used analogous formulations to connect electrostatics, heat transfer and elasticity of solids, that later led Maxwell to formulate his theory of electricity and magnetism \cite{1}. This analogy was first applied to the set of Maxwell equations concerning fluid dynamics in the early 1962 \cite{3} only to the case of the one-dimensional Rayleigh problem. Recently, the generalization of the Maxwell set of fluid equations was introduced in terms of an incompressible flow, particularly with interest in turbulent flow \cite{4}. Even more recently, the fluid Maxwell equations were generalized to the compressible flow case \cite{5}. Besides, other generalizations of fluid dynamics have been constructed proposing noncommutative, non-Abelian and supersymmetric formulations, to mention a few \cite{6}.

In \cite{7}, some of the present authors have introduced a Lagrangian description for the compressible fluid together with the scenario where a charged fluid was immersed in an electromagnetic field. The interaction between them from the Lagrangian density was discussed. This analogy has been explored in the literature with applications in quark-gluons plasma (QGP) \cite{8–13} which is a dense liquid that flows with very little viscosity almost being an ideal fluid.

To motivate our work, for example, it is well known that we need to quantize QCD to see with in detail the difference between QED and QCD. Namely, the main difference between both is that the quantum effects become more important at low energy in QCD. Moreover, in the early Universe, where quantum effects are extremely relevant, it is believed to be formed by quark-gluon plasma (QGP). And since QGP is a state of matter in QCD, we believe that quantization methods applied to fluid models can be interesting theoretical laboratories in order to attack directly QGP.

Having said that, we can consider this work as part of a sequence of other ones from these authors upon the analysis of this mentioned analogy between the structure of the fluid dynamics and electrodynamics \cite{7,14,15}. The purpose of the present paper is to analyze the Lagrangian density which describes the charged compressible fluid...
imposed in an electromagnetic field, obtained in [7], from the point of view of the Faddeev-Jackiw (FJ) method [16] applied to this model.

The FJ [16] method is a symplectic description of constrained quantization, where the degrees of freedom are identified by means of the so-called symplectic variables. The essential point of the FJ method is to map the system into a first-order Lagrangian with some auxiliary fields, but the method does not depend on how the auxiliary fields are introduced to make the first-order Lagrangian. It was applied recently to non-Abelian theories [17].

The work is organized in the following way: in the next section we have briefly reviewed the FJ method. In the third section we have analyzed the theory via the Faddeev-Jackiw formalism. –

Faddeev-Jackiw formalism. – We will begin with a first-order time derivative Lagrangian, which arises from a standard second-order one with auxiliary fields. The first step is to construct the symplectic Lagrangian

\[ \mathcal{L} = a_i(\xi) \dot{\xi}^i - \mathcal{V}(\xi), \]

where \( a_i \) are the arbitrary one-form components and \( i = 1, \ldots, N \). Since the first-order system is constructed through a closed two-form, if it is non-degenerated, it defines a symplectic framework on the phase space, which is described by the coordinates \( \xi \). Besides, if this two-form is singular, with constant rank, it is defined as a pre-symplectic two-form. Hence, considering the components, the symplectic form can be defined by

\[ f_{ij} = \frac{\partial}{\partial \xi^i} a_j(\xi) - \frac{\partial}{\partial \xi^j} a_i(\xi), \]

and the equations of motion are

\[ f_{ij} \dot{\xi}^i = \frac{\partial}{\partial \xi^j} \mathcal{V}(\xi), \]

where the two-form \( f_{ij} \) can be either singular or non-singular. In this last case it has an inverse \( f^{ij} \)

\[ \dot{\xi}^i = f^{ij} \frac{\partial}{\partial \xi^j} \mathcal{V}(\xi), \]

where we have that \( \{ \xi^i, \xi^j \} = f^{ij} \). To consider a constrained system described by (1), it means that the symplectic matrix is singular. And the constraints of the system have to be determined, of course. Consider that the rank of \( f_{ij} \) is \( 2n \). In this case we have \( N - 2n = M \) zero-mode vectors \( \nu^\alpha \), \( \alpha = 1, \ldots, M \). The system is then constrained through \( M \) equations with no time derivatives. We will have constraints that reduce the degrees of freedom number. Hence, multiplying (3) by the (left) zero-modes \( \nu^\alpha \) of \( f_{ij} \) we have the (symplectic) constraints with the structure of algebraic relations

\[ \Omega^\alpha \equiv \nu^\alpha \frac{\partial}{\partial \xi^i} \mathcal{V}(\xi) = 0. \]

So, we can construct the first-iterated Lagrangian by including the corresponding Lagrange multipliers relative to the obtained constraints

\[ \mathcal{L} = a_i(\xi) \dot{\xi}^i + \Omega^\alpha \lambda_\alpha - \mathcal{V}(\xi). \]

The Lagrange multipliers \( \lambda \) can be considered as the symplectic variables which can increase the symplectic variables set. This move reduces the number of \( \xi \)'s. After that, the procedure can be entirely repeated until all the constraints can be eliminated and the completely reduced, unconstrained and canonical system remains. But notice that in the case of gauge theories, we have no new constraint through the zero-mode. And the symplectic matrix remains singular. Hence, we can consider mandatory to introduce gauge condition(s) to highlight the singularity. In this way the procedure can be finished in terms of the original variables and the basic brackets can be determined.

Faddeev-Jackiw analysis for the charged compressible fluid immersed in an electromagnetic field. – The effective Lagrangian density which describes the charged compressible fluid immersed in an electromagnetic field is defined, valid for each species (\( \epsilon \)), by

\[ \mathcal{L} = -\frac{1}{4} T_{\mu\nu} T^{\mu\nu} - \frac{1}{4} g^2 F_{\mu\nu} F^{\mu\nu} - \frac{g}{2} T_{\mu\nu} F^{\mu\nu}, \]

where \( T_{\mu\nu} = \partial_\mu U^{(\epsilon)} - \partial_\nu U^{(\epsilon)} \) is the strength tensor of the fluid, the four-vector potential \( U^{(\epsilon)} \equiv (U_0^\epsilon, \vec{U}^\epsilon) - U_0^\epsilon \) is the energy function and \( \vec{U}^\epsilon \) is the average velocity field [7]—

and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the strength tensor for the electromagnetic field. The spacetime metric elements are \( \eta_{\mu\nu} = (-+\ldots+) \). The coupling constant is \( g = e_\epsilon / m_\epsilon \), where \( e_\epsilon \) is the charge and \( m_\epsilon \) is the mass of the charge. Note that, when \( g = 0 \), we have two uncoupled theories. The Euler-Lagrange equations of motion are

\[ (1 + g^2) \partial_\mu F^{\mu\nu} + g \partial_\nu T^{\mu\nu} = 0, \]

and it is easy to see that (7) is invariant under the gauge transformations, \( A_\mu \rightarrow A_\mu + \partial_\mu \Lambda \), for the electromagnetic fields, and \( U^{(\epsilon)} \rightarrow U^{(\epsilon)} + \partial_\mu \Lambda \), for the compressible fluid field. In terms of the potentials, \( U^{(\epsilon)} \) and \( A_\alpha \), the above equation reads

\[ (1 + g^2) [\Box A_\mu - \partial_\nu \partial_\mu A_\nu] + g [\Box U^{(\epsilon)} - \partial_\nu \partial_\mu U^{(\epsilon)}] = 0. \]

From now on, for simplicity, we will not use the species index, and much of what follows is true for each species. In our model, the symmetric energy-momentum tensor is given by

\[ \Theta^{\alpha\beta} = (1 + g^2) \left[ \eta^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} \eta^{\alpha\beta} F_{\mu\lambda} F^{\mu\lambda} \right] + \left[ \eta^{\alpha\mu} T_{\mu\lambda} T^{\lambda\beta} + \frac{1}{4} \eta^{\alpha\beta} T_{\mu\lambda} T^{\mu\lambda} \right] + g \left[ \eta^{\alpha\mu} T_{\mu\gamma} F^{\gamma\beta} + \eta^{\alpha\mu} F_{\mu\gamma} T^{\gamma\beta} + \frac{1}{2} \eta^{\alpha\beta} T^{\mu\lambda} F_{\mu\lambda} \right]. \]
and it follows directly that

$$\Theta^{00} = \frac{1}{2} \left( \dot{r}^2 + \dot{\omega}^2 \right) + \frac{1}{2} \left( E^2 - B^2 \right) + g \dot{r} \cdot \dot{E} + g \dot{\omega} \cdot \dot{B}$$

which is the energy of the model, where the first term in (11) is the energy of the fluid and the second one is the energy of the electromagnetic field. The last two terms are the contributions of the interaction between the two fields.

As we said before, in this paper we want to discuss the FJ methodology [16] applied to a solution of a higher-derivative theory which, in this case, has the higher derivative in the Maxwell sector. So, rewriting (7) in the form

$$L = -\frac{1}{4} T^{\mu \nu} T_{\mu \nu} - \frac{(1 + g^2)}{4} F^{\mu \nu} F_{\mu \nu} - g U_\mu \partial_\nu F^{\mu \nu},$$

we can introduce another set of canonical pair $(\Sigma_\mu, \phi_\mu)$ to have a correct extended phase space in order to proceed with the canonical analysis. Therefore, we have that

$$L = -\frac{1}{2} \left( \dot{U} - \nabla U_0 \right)^2 + \frac{1}{2} \left( \nabla \times \dot{U} \right)^2 + \frac{1}{2} (1 + g^2) \left( \dot{\Sigma} - \nabla A_0 \right)^2 + \frac{1}{2} (1 + g^2) \left( \nabla \times \dot{\Sigma} \right)^2 - g \dot{U} \cdot (\nabla \Sigma_0 - \dot{\Sigma}) - g U_0 \left( \nabla \cdot \nabla \Sigma - \nabla^2 A_0 \right) - g \dot{U} \cdot (\nabla \times \nabla \times \dot{S}),$$

and, to write a first-order Lagrangian, we will use an auxiliary field, which is chosen to be the canonical momentum due to an algebraic simplification. In this case, we have a set of canonical pairs $(U_\mu, p_\mu), (A_\mu, \pi_\mu)$ and $(\Sigma_\mu, \phi_\mu)$ and we have directly that

$$p_\mu = \frac{\partial L}{\partial (\partial_\mu U_\mu)}, \quad \phi_\mu = \frac{\partial L}{\partial (\partial_\mu \Sigma_\mu)},$$

$$\pi_\mu = \frac{\partial L}{\partial (\partial_\mu \Sigma_\mu)} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \Sigma_\mu)} \right) - 2 \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \Sigma_\mu)} \right),$$

which results in the following expressions

$$p_\mu = T_{\mu 0}, \quad \phi_\mu = g \eta_{\mu k} U_k,$$

$$\pi_\mu = (1 + g^2) F_{\mu 0} - g \eta_{\mu k} T_{0 k} + g \eta_{\mu 0} \partial_\mu U_k.$$ (15)

Therefore, making use of the equation of motion for the canonical momenta associated with the fields $U_\mu, A_\mu$ and $\Sigma_\mu$, we have

$$L^{(0)} = -\vec{p} \cdot \dot{\vec{U}} + \vec{\phi} \cdot \dot{\vec{\Sigma}} + \pi_\mu \dot{A}_\mu - V^{(0)},$$

where the potential density is

$$V^{(0)} = \pi_\mu \Sigma_\mu - \frac{1}{2} \vec{p}^2 - \vec{p} \cdot \nabla U_0 - \frac{1}{2} (\nabla \times \dot{U})^2 - \frac{1}{2} (1 + g^2) (\dot{\Sigma} - \nabla A_0)^2 + \frac{1}{2} (1 + g^2) (\nabla \times \dot{\Sigma}) - g U_0 (\nabla \cdot \dot{\Sigma} - \nabla^2 A_0) + \vec{\phi} \cdot (\nabla \times \dot{\Sigma} - \nabla \times \dot{\Sigma}).$$

The initial set of symplectic variables defining the extended space is given by the set $\xi^{(0)} = (U_k, p_k, U_0; A_k, \pi_k, A_0, \pi_0; \Sigma_\mu, \phi_\mu, \Sigma_\mu),$ and the corresponding canonical non-zero one-form is

$$U^{(0)} = -p_k; \quad \Sigma^{(0)} = \phi_k; \quad A^{(0)} = -\pi_k; \quad A^{(0)} = \pi_0.$$

Using this result in the symplectic two-form matrix $f^{(0)}$ we have that

$$f^{(0)}(x, y) = \begin{pmatrix} F_{ij} & 0_{6 \times 3} & 0_{6 \times 3} & M_{ij} & 0_{3 \times 3} & 0_{3 \times 3} \end{pmatrix} \delta(x - y)$$

with

$$F_{ij} = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_{ij} = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where we can note that the matrix $f^{(0)}$ is singular, which means that there are constraint and it has two zero-mode $\nu_\mu \equiv (0, 0, \nu^{0 \mu}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \nu^{\Sigma_\mu})$ where $\nu^{0 \mu}$ and $\nu^{\Sigma_\mu}$ are arbitrary functions. From this two-zero-mode, we have the following constraints:

$$1 \Omega^{(0)} = \int d^3 x dU_0 (x) \left[ \delta \frac{\partial L}{\partial (\partial_\mu U_\mu)} \left( \frac{\partial L}{\partial (\partial_\mu U_\mu)} \right) \right]$$

$$= \int d^3 x dU_0 (x) \left[ \nabla \cdot \vec{p}(x) + g (\nabla \cdot \vec{\Sigma}(x) - \nabla^2 A_0(x)) \right]$$

$$= 0,$$ (21)

and

$$2 \Omega = \int d^3 x \nu^{\Sigma_\mu} (x) \left[ \delta \frac{\partial L}{\partial (\partial_\mu \Sigma_\mu)} \left( \frac{\partial L}{\partial (\partial_\mu \Sigma_\mu)} \right) \right]$$

$$= \int d^3 x \nu^{\Sigma_\mu} (x) \left[ \delta (\nabla \cdot \phi(x)) \right] = 0.$$ (22)

Since $\nu^{U_0}$ and $\nu^{\Sigma_\mu}$ are arbitrary functions, we obtain the constraints

$$1 \Omega = \nabla \cdot \vec{p} + g (\nabla \cdot \vec{\Sigma} - \nabla^2 A_0) = 0,$$ (23)

and

$$2 \Omega = \pi_0 - \nabla \cdot \phi = 0.$$ (24)

According to the symplectic algorithm, the constraints (23) and (24) are introduced in the Lagrangian density by using the Lagrangian multipliers. Thus, the first-iterated Lagrangian density is written as

$$L^{(1)} = -\vec{p} \cdot \dot{\vec{U}} + \vec{\phi} \cdot \dot{\vec{\Sigma}} + \pi_\mu \dot{A}_\mu + \dot{\lambda}_1 \Omega + \dot{\lambda}_2 \nabla \Omega - V^{(1)},$$ (25)
where $\lambda_1$ and $\lambda_2$ are the Lagrangian multipliers, and the first-iterated symplectic potential density is
\[ V^{(1)} = V^{(0)} \big|_{\Omega = 0, \Sigma = 0} = -\vec{F} \cdot \hat{\Sigma} = \frac{1}{2} \lambda^2 - \frac{1}{2} (\nabla \times \vec{U})^2, \]
\[ -\frac{1}{2} (1 + g^2) (\hat{\Sigma} - \nabla A_0)^2 - \frac{1}{2} (1 + g^2) (\nabla \times \vec{A})^2 + \vec{\phi} \cdot (\nabla \times \nabla \times \vec{A}). \] (26)

It should be noted that when the constraints $\Omega$ and $\Sigma$ are imposed, the dependence in $U_0$ and $\Sigma_0$ disappears, once the terms in $U_0$ and $\Sigma_0$ were incorporated in the term introduced into the kinetic part, which was done by redefining the Lagrange multipliers.

From the above Lagrangian we have the following set of symplectic variables defined by $g^{(1)} = (U_k, p_k; A_k, \pi_k; A_0, \pi_0; \Sigma_k, \phi_k; \lambda_1, \lambda_2)$, with the new canonical one-form defined by
\[ U^{(0)} = -p_k; \quad \Sigma^{(0)} = \phi_k; \quad A^{(0)} = -\pi_k; \quad A_0^{(0)} = \pi_0; \quad \lambda_1^{(0)} = 1\Omega; \quad \lambda_2^{(0)} = 2\Omega. \] (27)

Hence, the first-iterated symplectic matrix is written as
\[ f^{(1)}_{ij}(\vec{x}, \vec{y}) = \left( A_{ij} - B_{i,y} \vec{G}_{ij} \right) \delta(\vec{x} - \vec{y}) \] (28)
where
\[ A_{ij} = \begin{pmatrix} 0 & \delta_{ij} & 0 & 0 & 0 \\ 0 & -\delta_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_{ij} & 0 \\ 0 & 0 & -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ B_{j,y} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -g \delta_{j,y}^2 & 0 \end{pmatrix}, \] (29)
\[ \vec{G}_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -\delta_{ij} & 0 & 0 & -g \delta_{j,y} & 0 \\ 0 & 0 & -\delta_{ij} & 0 & 0 \\ -g \delta_{i,x} & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \]
and we can see that $f^{(1)}_{ij}$ is a singular matrix. From this result, we can determine its zero-mode as being
\[ \vec{\nu}_a = (\nu^{U}_i, \vec{0}, \vec{0}, \nu^{A_0}, \vec{0}, \nu^{\Sigma}, \nu^{\phi}, \nu^{\lambda_1}, \nu^{\lambda_2}), \] (30)
where $\nu^{U}_i = \partial_i \nu^{\lambda_1}$, $\nu^{\Sigma} = \partial_i \nu^{\lambda_2}$, $\nu^{\phi} = -g \partial_i \nu^{\lambda_1}$, $\nu^{A_0} = -\nu^{\lambda_2}$ and $\nu^{\lambda_1}, \nu^{\lambda_2}$ are arbitrary functions. Thus, from this zero-mode in eq. (30) we have that
\[ 3\Omega = \int d^3\vec{x} \left( \frac{\delta}{\delta U^i_1(\vec{x})} - \frac{\delta}{\delta U^i_0(\vec{x})} \right) \frac{\delta}{\delta \nu^{\lambda_1}(\vec{x})} + \frac{\delta}{\delta A_0(\vec{x})} \frac{\delta}{\delta \nu^{\lambda_2}(\vec{x})} \int d^3\vec{y} V^{(1)}(\vec{y}) \]
\[ = \int d^3\vec{x} \nu^{\lambda_1}(\vec{x}) [\nabla \cdot (\vec{F} - \vec{\phi})] = 0. \] (31)

Once again, as $\nu^{\lambda_2}$ is an arbitrary function, we obtain a new set of constraint relations given by
\[ \nabla \cdot \vec{F} = 0. \] (32)

Now, following the FJ method, the second-iterated Lagrangian can be written as
\[ L^{(2)} = -\vec{p} \cdot \vec{F} + \phi \cdot \hat{\Sigma} + \pi_\mu A^\mu + \lambda_1 \Omega + \lambda_2 \Omega + 3\Omega - V^{(2)}, \] (33)
where
\[ V^{(2)} = V^{(1)} \big|_{\Omega = 0} = V^{(1)}. \] (34)

From the above Lagrangian we can find the following canonical non-zero one-form:
\[ U^{(0)} = -p_k; \quad \Sigma^{(0)} = \phi_k; \quad A^{(0)} = -\pi_k; \quad A_0^{(0)} = \pi_0; \quad \lambda_1^{(0)} = 1\Omega; \quad \lambda_2^{(0)} = 2\Omega; \quad \lambda_3^{(0)} = 3\Omega, \] (35)
which leads to the corresponding third-iterated symplectic matrix,
\[ f^{(2)}_{ij}(\vec{x}, \vec{y}) = \left( A_{ij} - B_{i,y} \vec{G}_{ij} \right) \delta(\vec{x} - \vec{y}), \] (36)
where $A_{ij}$ has the same expression given in (29), and
\[ B_{j,y} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -g \delta_{j,y} & 0 \end{pmatrix}, \] (37)
\[ \vec{G}_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -\delta_{ij} & 0 & 0 & -g \delta_{j,y} & 0 \\ 0 & 0 & -\delta_{ij} & 0 & 0 \\ -g \delta_{i,x} & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \]
and once again, we can see that $f^{(2)}$ is singular and the zero-mode associated with this matrix is
\[ \tilde{\nu}_a = (\tilde{\nu}_a, \tilde{\nu}^{\lambda_1}), \] (38)
where $\tilde{\nu}_a$ has the same expression given by eq. (30). However, the zero-mode $\tilde{\nu}_a$ generates the constraint $3\Omega$ again, the zero-mode does not generate any new constraints and, consequently, the symplectic matrix remains singular. It characterizes the theory as a gauge theory.
In order to obtain a regular symplectic matrix a gauge fixing term must be added to the theory. The choice of this condition can be suggested by many reasons, the most important being the simplification that it may introduce in the theory. In the Maxwell theory, the condition usually employed in the gauge fixing procedure is the Coulomb gauge

\[ A_0 = 0, \quad \nabla \cdot \vec{A} = 0. \quad (39) \]

However, concerning the theory described by the Lagrangian in (7), where a charged compressible fluid is immersed in an electromagnetic field, the condition (39) is not sufficient to promote a gauge fixing, to do that we need an extra condition. In this case, an appropriate choice is the “Lorentz gauge” to a compressible fluid [7], where

\[ \partial_\alpha U^\alpha = 0 \quad \text{or} \quad \nabla \cdot \vec{U} + \Gamma_0 = 0, \quad (40) \]

where \( \Gamma_0 \) is part of the gauge condition. For example, in [7], \( \Gamma_0 \) is the time derivative of the scalar potential where \( \vec{U} \), also in [7], is the vector potential. The expressions in (40) are directly related to the condition relative to the compressibility of the fluid, where \( \Gamma_0 \) will be defined [7]. Thus, considering eqs. (39) and (40) with gauge fixing conditions \( ^4 \Omega = \nabla \cdot \vec{A} \), and \( ^4 \Omega = \nabla \cdot \vec{U} + \Gamma_0 \), we then obtain a new Lagrangian density

\[ L^{(3)} = -\vec{p} \cdot \vec{U} + g \cdot \vec{\Sigma} - \vec{\pi} \cdot \vec{A} + \lambda_1 (\nabla \cdot \vec{p}) + g (\nabla \cdot \vec{\Sigma}) + \lambda_2 (\nabla \cdot \vec{\pi}) + \lambda_4 (\nabla \cdot \vec{A}) + \lambda_5 (\nabla \cdot \vec{U} + \Gamma_0) - V^{(3)}, \quad (41) \]

where

\[ V^{(3)} = -\vec{\pi} \cdot \vec{\Sigma} - \frac{1}{2} \vec{p}^2 - \frac{1}{2} (\nabla \times \vec{U})^2 \]

\[ - \frac{1}{2} (1 + g^2) \Sigma^2 + \frac{1}{2} (1 + g^2) \vec{A} \cdot (\nabla^2 \vec{A}) \]

\[ - \vec{\pi} \cdot (\nabla^2 \vec{A}) \quad (42) \]

is associated with the symplectic variables \( \xi^{(3)} = (U_k, p_k; A_k, \pi_k; \Sigma_k, \phi_k; \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \). From the expression for the potential \( V^{(3)} \) we can see what theory’s dynamical variables are. They are part of the canonical set \((p_k, U_k), (\pi_k, A_k)\) and \((\phi_k, \Sigma_k)\). The new canonical one-form is defined by

\[ U^{(0)} = -p_k; \quad \Sigma^{(0)} = \phi_k; \quad A^{(0)} = -\pi_k; \]

\[ \lambda_1^{(0)} = \partial_\pi p_i + g \partial_\Sigma \Sigma_i; \quad \lambda_2^{(0)} = \pi_0 - \partial_\pi \phi_i; \]

\[ \lambda_3^{(0)} = \partial_\pi \pi_i; \quad \lambda_4^{(0)} = \partial_\pi A_i; \quad \lambda_5^{(0)} = \partial_\pi U_i + \Gamma_0. \quad (43) \]

These relations can lead us to the corresponding third-iterated symplectic matrix

\[ f^{(3)} \equiv \left( \begin{array}{cccccc}
0 & \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} & 0 & 0 & 0 & 0 \\
-\delta_{ij} + \frac{\partial_i \partial_j}{\nabla^2} & 0 & 0 & 0 & 0 & \frac{\partial_i \partial_j}{\nabla^2} \\
0 & 0 & 0 & 0 & \frac{\partial_i \partial_j}{\nabla^2} & 0 \\
0 & 0 & \frac{\partial_i \partial_j}{\nabla^2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial_i \partial_j}{\nabla^2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial_i \partial_j}{\nabla^2} & 0 \\
\end{array} \right) \delta^{(3)}(\vec{x} - \vec{y}). \quad (44) \]

We can observe that \( f^{(3)} \) is not singular, therefore we can construct its inverse. The inverse of \( f^{(3)} \) is called the symplectic tensor see eq. (46) above

\[ 20004-5 \]
Moreover, we can relate $\lambda_1 = U_0$, $\lambda_2 = \Sigma_0$ and $\lambda_3 = A_0$. In this way, from (46) it is possible to identify the following FJ generalized brackets given by

$$
\begin{align*}
\{A_i(\vec{x}), \pi_j(\vec{y})\} &= \left(-\delta_{ij} + \frac{\partial_i}{\partial \vec{x} \cdot \vec{y}}\right) \delta(\vec{x} - \vec{y}), \\
\{\Sigma_i(\vec{x}), \phi_j(\vec{y})\} &= \delta_{ij} \delta(\vec{x} - \vec{y}), \\
\{U_i(\vec{x}), p_j(\vec{y})\} &= \left(\delta_{ij} - \frac{\partial_i}{\partial \vec{x} \cdot \vec{y}}\right) \delta(\vec{x} - \vec{y})
\end{align*}
$$

and

$$
\begin{align*}
\{p_i(\vec{x}), \phi_j(\vec{y})\} &= g \delta_{ij} \delta(\vec{x} - \vec{y}), \\
\{p_i(\vec{x}), \pi_0(\vec{y})\} &= g \partial_i \delta(\vec{x} - \vec{y}),
\end{align*}
$$

where the Dirac brackets for the electromagnetic fields correspond to the one we have found in previous works [7,14], as well as the Dirac brackets for the higher-derivative terms in the electromagnetic fields, $\Sigma$ and $\phi$. Besides, we have found the Dirac brackets for the dynamic variables of the compressible fluid $p$ and $U$, the last of eqs. (47).

As a result of the coupling between the charged compressible fluid and the electromagnetic field we found two Dirac brackets between the fluid and electromagnetic fields, eqs. (48), which are both zero when there is no coupling between them.

Finally, one can say that the analysis of the final reduction of constraints by means of a gauge condition is not finished yet. It can be shown that the final dynamics is independent of the choice of the gauge condition. A complete analysis was performed concerning the non-Abelian case [18], where it is shown that the last step involves a reduction to the quotient space of the gauge orbits which, in the Abelian case, can be obtained by a suitable choice of a global gauge condition. This demonstration is out of the scope of this work, which describes the FJ technique applied to the fluid model depicted here.

**Conclusions.** – In this paper we have analyzed the gauge invariance of the theory which describes the charged compressible fluid interacting with an electromagnetic field (7) by using the FJ method. We have found the constraints, the gauge transformations and we have obtained the generalized FJ brackets.

We can clearly see from eqs. (47) and (48) that the Dirac brackets involve quantities such as the strength tensor. This last one is formed by the four-vector $U^\mu$, which has the velocity vector as one of its components together with the conjugated momenta. Having said that, a future work can be to construct the dynamics of the compressible fluid model analyzed here. This task would be based on these dynamical elements such as $U^\mu$ and the other ones described just above. This is an ongoing research.

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The authors thank CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico), Brazilian scientific support federal agency, for partial financial support, Grants Nos. 302155/2015-5, 302156/2015-1 and 442369/2014-0 and EMCA acknowledges the hospitality of the Theoretical Physics Department at Federal University of Rio de Janeiro (UFRJ), where part of this work was carried out.

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