Integrable supersymmetric correlated electron chain with open boundaries

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Abstract

We construct an extended Hubbard model with open boundaries from a $R$-matrix based on the $U_q[Osp(2|2)]$ superalgebra. We study the reflection equation and find two classes of diagonal solutions. The corresponding one-dimensional open Hamiltonians are diagonalized by means of the Bethe ansatz approach.

PACS numbers: 71.27.+a, 75.10.Jm

June 1999
1 Introduction

In recent years there has been substantial research devoted to the study of (1+1)-dimensional solvable lattice models with integrable boundary conditions. In one-dimensional theory with factorized scattering, the boundary effects can be described by scattering matrices satisfying the so-called reflection equation [1]. A systematic approach to construct such models has been developed by Sklyanin [2] who has investigated the six vertex model with boundary fields. Subsequently, this scheme has been generalized to handle a rather general class of models based on Lie algebras [3, 4] (see also ref. [5]). By now, variants of this method have been extensively used in the literature to study various integrable quantum field theories and lattice statistical mechanics models with open boundaries. See refs. [6, 7] just to mention a few examples.

Of particular interest are supersymmetric generalizations of the Hubbard model [8, 9, 10] due to their possible relevance in describing strongly correlated electron systems. These models are often derived from supersymmetric solutions of the Yang-Baxter equation invariant by the gl(2|1) and Osp(2|2) Lie superalgebras [11, 12]. A successful example is the supersymmetric free-parameter model with open boundaries constructed from the R-matrix associated to the four dimensional representation of gl(2|1) [13]. The purpose of this paper is to perform similar task for another Osp(2|2) R-matrix solution found some time ago by Deguchi et al [11]. This latter model, however, appears to be specially interesting, since its Hamiltonian provide a lattice regularization of a relevant integrable double sine-Gordon model [14]. This continuum field theory with open boundary is known to describe tunneling effects in quantum wires [15]. Therefore, the Bethe ansatz results for open boundaries we shall derive here could be a useful non-perturbative tool to investigate this condensed matter system as well.

This paper is organized as follows. The next section is concerned with diagonal solutions of the reflection equations [1, 2] associated with a particular Osp(2|2) R-matrix. We found two families of one parameter solutions leading to four possible choices of boundary conditions for integrable open chain Hamiltonians. In section 3 we formulate their Bethe ansatz solutions in terms of the coordinate Bethe ansatz approach. Section 4 is reserved for our conclusions.
and final remarks. For completeness, in appendix A we present another possible $R$-matrix embedding and discuss its boundary behaviour.

## 2 The $Osp(2|2)$ $R$-matrix and related $K$-matrices

The “bulk” part of (1+1)-dimensional integrable system with a boundary is governed by the two-particle scattering matrix $R(\lambda)$ satisfying the Yang-Baxter equation. The spectral parameter $\lambda$ plays the role of the difference of the particles pseudomomenta. The boundary effects are described in terms of two boundary $K_\pm(\lambda)$ matrices that characterize the scattering interactions at the boundary [1, 2]. The compatibility of the reflections and particle scatterings leads us to an algebraic condition, the so-called reflection equation [1, 2]

$$R_{12}(\lambda - \mu) \frac{1}{2} K^{(-)}(\lambda) R_{21}(\lambda + \mu) \frac{2}{1} K^{(+)}(\mu) = \frac{2}{1} K^{(+)}(\mu) R_{12}(\lambda + \mu) \frac{1}{2} K^{(-)}(\lambda) R_{21}(\lambda - \mu)$$

(1)

Here we will also require that $R$-matrix $R_{12}(\lambda)$ satisfies, besides the standard properties of regularity and unitarity, certain extra relations denominated $PT$ and crossing symmetries, namely

$$PT\text{-symmetry} : P_{12} R_{12}(\lambda) P_{12} = R_{12}^{t12}(\lambda)$$

(2)

$$crossing\text{-symmetry} : R_{12}(\lambda) = \frac{\rho(\lambda)}{\rho(-\lambda - \eta)} V R_{12}^{t12}(-\lambda - \eta) V^{-1}$$

(3)

where $P_{12}$ is the exchange operator, $t_k$ denotes the transpose in the space $k$, $\eta$ is the crossing parameter, $V$ is related to the crossing matrix by $M = V^t V$ and $\rho(\lambda)$ is a convenient normalization function.

When these properties are fulfilled one can follow the scheme devised by Mezincescu and Nepomechie [3]. One of the boundary matrices, say the $K_-(\lambda)$ matrix, is obtained by solving the reflection equation (1). The other $K_+(\lambda)$ matrix is automatically determined by the following isomorphism

$$K_+(\lambda) = K_-^{t}(-\lambda - \eta) M$$

(4)

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1 We are assuming the ordinary non-graded boundary framework, since we shall look only for diagonal solutions. For an application of the graded formalism to non-diagonal solutions, see for instance ref. [19].
Before proceeding, we remark that the original $Osp(2|2)$ $R$-matrix given in ref. [11] is not $PT$-symmetric. However, this difficulty is easily circumvent by applying a convenient unitary transformation. It turns out that the $PT$-symmetric $Osp(2|2)$ $R$-matrix can be presented in the following form

$$R(\lambda) = \sum_{\alpha=1}^{4} \frac{q}{q_\alpha} \frac{e^\lambda - q_\alpha^2}{1 - e^\lambda q_\alpha^2} E_{\alpha\alpha} \otimes E_{\alpha\alpha} + \frac{q(e^\lambda - 1)}{1 - e^\lambda q_\alpha^2} \sum_{\alpha, \beta = 1}^{4} \sum_{\alpha \neq \beta, \beta'} E_{\alpha\alpha} \otimes E_{\beta\beta}$$

$$+ \frac{1 - q^2}{1 - e^\lambda q_\alpha^2} \left[ e^\lambda \sum_{\alpha, \beta = 1}^{4} \sum_{\alpha \neq \beta, \beta'} \right] E_{\alpha\beta} \otimes E_{\beta\alpha} + \frac{e^\lambda - 1}{1 - e^\lambda q_\alpha^2} \sum_{\alpha=1}^{4} a_{\alpha\beta}(\lambda) E_{\alpha\beta} \otimes E_{\alpha'\beta'}$$

(5)

where $q$ is the deformation parameter, $E_{\alpha\beta}$ are the elementary $4 \times 4$ matrices and we set $\alpha' = 5 - \alpha$. The functions $a_{\alpha\beta}(\lambda)$ are

$$a_{\alpha\beta}(\lambda) = \begin{cases} \frac{q}{q_\alpha} \frac{e^\lambda + 1}{e^\lambda + 1} & \alpha = \beta, \\ e^\lambda \left[ \varepsilon_\alpha \varepsilon_\beta \frac{1 - q^2}{1 + e^\lambda} + \frac{1 - q^2}{e^\lambda - 1} \delta_{\alpha,\beta'} \right] & \alpha < \beta \\ -e^\lambda \varepsilon_\alpha \varepsilon_\beta \frac{1 - q^2}{1 + e^\lambda} + \frac{1 - q^2}{e^\lambda - 1} \delta_{\alpha,\beta'} & \alpha > \beta \end{cases}$$

(6)

and the parameters $q_\alpha$ and $\varepsilon_\alpha$ are defined by

$$q_\alpha = \begin{cases} q & \alpha = 1, 4 \\ -q^{-1} & \alpha = 2, 3 \end{cases}, \quad \varepsilon_\alpha = \begin{cases} 1 & \alpha = 1, 4 \\ -i & \alpha = 2, 3 \end{cases}$$

(7)

It is not difficult to verify that this $R$-matrix satisfies the properties (2,3), where

$$\eta = i\pi, \quad \rho(\lambda) = \frac{e^\lambda - 1}{1 - e^\lambda q^2}, \quad V = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

(8)

Now we have the basic ingredients to built up integrable boundary models associated with the $R$-matrix (5). In this paper we are interested to look for diagonal solutions of the reflection
equation, which will play the role of boundary fields in the context of spin chains. Substituting the ansatz

\[
K_{-}^{(l)}(\lambda, \xi_{-}) = \begin{pmatrix}
A^{(l)}(\lambda, \xi_{-}) & 0 & 0 & 0 \\
0 & B^{(l)}(\lambda, \xi_{-}) & 0 & 0 \\
0 & 0 & C^{(l)}(\lambda, \xi_{-}) & 0 \\
0 & 0 & 0 & D^{(l)}(\lambda, \xi_{-})
\end{pmatrix}
\] (9)

in equation (1), and after some algebra we find two classes of diagonal solutions given by

\[
A^{(1)}(\lambda, \xi_{-}) = (e^{\lambda} + e^{\xi_{-}})(e^{\lambda} - e^{\xi_{-}}q^{-2})
\]
\[
B^{(1)}(\lambda, \xi_{-}) = (e^{-\lambda} + e^{\xi_{-}})(e^{\lambda} - e^{\xi_{-}}q^{-2})
\]
\[
C^{(1)}(\lambda, \xi_{-}) = (e^{-\lambda} + e^{\xi_{-}})(e^{\lambda} - e^{\xi_{-}}q^{-2})
\]
\[
D^{(1)}(\lambda, \xi_{-}) = (e^{-\lambda} + e^{\xi_{-}})(e^{-\lambda} - e^{\xi_{-}}q^{-2})
\] (10)

and

\[
A^{(2)}(\lambda, \xi_{-}) = (e^{\lambda} + e^{\xi_{-}})(e^{\lambda} - e^{\xi_{-}}q^{-2})(e^{\lambda} - e^{-\xi_{-}}q^{2})
\]
\[
B^{(2)}(\lambda, \xi_{-}) = (e^{-\lambda} + e^{\xi_{-}})(e^{\lambda} - e^{\xi_{-}}q^{-2})(e^{-\lambda} - e^{-\xi_{-}}q^{2})
\]
\[
C^{(2)}(\lambda, \xi_{-}) = (e^{\lambda} + e^{\xi_{-}})(e^{\lambda} - e^{\xi_{-}}q^{-2})(e^{-\lambda} - e^{-\xi_{-}}q^{2})
\]
\[
D^{(2)}(\lambda, \xi_{-}) = (e^{\lambda} + e^{\xi_{-}})(e^{-\lambda} - e^{\xi_{-}}q^{-2})(e^{-\lambda} - e^{-\xi_{-}}q^{2})
\] (11)

The corresponding \(K_{+}^{(l)}(\lambda, \xi_{+})\) matrices are easily derived from the isomorphism (4) with the help of crossing property (8). However, it is convenient to introduce suitable \(K_{+}^{(l)}(\lambda, \xi_{+})\) matrices to make the interactions at the boundaries be as much as possible symmetric. With this in mind, we choose the following expression for these matrices

\[
K_{+}^{(l)}(\lambda, \xi_{+}) = K_{-}^{(l)}(-\lambda - i\pi, i\pi + 2 \log[q] - \xi_{+})^{t}M
\] (12)

where \(\xi_{\pm}\) are two independent parameters that characterize the interactions at the right and left ends of the open chain, respectively.

Equipped with the refection matrices \(K_{\pm}^{(l)}(\lambda, \xi_{\pm})\), an integrable model with open boundary condition can be obtained through the double-row transfer matrix \(t(\lambda)\) formulated by Sklyanin.
where \( T(\lambda) = R_{aL}(\lambda) \cdots R_{a1} \) is the standard monodromy matrix of the corresponding closed chain with \( L \) sites. We note that the fact \( K^{(l)}(\lambda, \xi) \) can be taken from either \( K^{(1)}(\lambda, \xi) \) or \( K^{(2)}(\lambda, \xi) \) gives us four possible choices of boundary transfer matrices. In next section we will present the Bethe ansatz solution of the corresponding four one-dimensional Hamiltonians with open boundaries.

### 3 Integrable open boundary Hamiltonians

In order to obtain solvable Hamiltonians with open boundaries we have to expand the double-row transfer matrix \(t^{(l,m)}(\lambda)\) up to the second order in the spectral parameter \(\lambda\). This is because the relevant term in the first order expansion, which is proportional to the trace of \(K^{(l)}(\lambda)\), turns out to be zero for both classes of \(K\)-matrices of section 2. Following ref. [13], it can be shown that the corresponding Hamiltonian commuting with \(t^{(l,m)}(\lambda)\) is given by

\[
H^{(l,m)} = \sum_{j=1}^{L-1} H_{j,j+1} + \frac{\zeta}{2} \frac{d}{d\lambda} K_+^{(l)}(0) + \frac{1}{\varrho^{(m)}} \left\{ \operatorname{Tr}_a \left[ \frac{d}{d\lambda} K_+^{(m)}(0) H_{La} \right] \right\} + \frac{1}{2} \operatorname{Tr}_a \left[ K_+^{(m)}(0) \frac{d^2}{d\lambda^2} R_{aL}(0) P_{aL} \right] + \frac{1}{2} \operatorname{Tr}_a \left[ K_+^{(m)}(0) H_{La}^2 \right] \]

where

\[
H_{j,j+1} = P_{j,j+1} \frac{d}{d\lambda} R_{j,j+1}(0), \quad R_{12}(0) = \zeta P_{12}, \quad \varrho^{(m)} = \operatorname{Tr}_a \left[ \frac{d}{d\lambda} K_+^{(m)}(0) \right] + \frac{2}{\zeta} \operatorname{Tr}_a \left[ K_+^{(m)}(0) H_{La} \right] \]

We recall that in the derivation of expression (14) it is implicitly required that \(K^{(l)}(\lambda, \xi)\) has been normalized by a scalar function such that \(K^{(l)}(0, \xi) = Id\). For both cases (10,11), the only non-trivial contributions to the boundaries come from the first two terms of equation (14). The remaining ones are proportional to the identity, playing the role of irrelevant additive constants. Considering the physical motivations of the introduction, one would like to express these open Hamiltonians in terms of fermionic creation and annihilation fields \(c_{j,\sigma}^\dagger\) and \(c_{j,\sigma}\).
acting on the site $j$ and carrying spin index $\sigma = \pm$. After some algebra, the final expression for the Hamiltonians can be written as follows

$$H^{(l,m)} = \sum_{j=1}^{L-1} H_{j,j+1} + U(\xi_-) n_{j+1} n_1 - U(\xi_+) n_{L+} n_{L-} + \sum_{\sigma = \pm} \mu^{(l)}_{1\sigma} n_{1\sigma} + \mu^{(m)}_{L\sigma} n_{L\sigma}$$

(16)

where $n_{j,\sigma} = c_{j,\sigma}^\dagger c_{j,\sigma}$ is the number operator for electrons with spin $\sigma$ on site $j$. The expression for the bulk part is

$$H_{j,j+1} = \sum_{\sigma = \pm} \begin{bmatrix} c_{j,\sigma}^\dagger c_{j+1,\sigma} + h.c. [1 - n_{j,-\sigma} (1 + \sigma V_1) - n_{j+1,-\sigma} (1 - \sigma V_1)] \\
+ V_2 \left[ c_{j,+}^\dagger c_{j,-} - c_{j,+}^\dagger c_{j+1,-} + c_{j,-}^\dagger c_{j+1,+} - h.c. \right] + i V_1 \sum_{\sigma = \pm} [n_{i\sigma} - n_{i+1\sigma}] \\
+ V_2 \left[ n_{j,+} n_{j,-} + n_{j+1,+} n_{j,-} + n_{j,+} n_{j+1,-} + n_{j,-} n_{j+1,+} - \sum_{\sigma = \pm} (n_{j,\sigma} + n_{j+1,\sigma}) \right] \right)$$

(17)

where the couplings $V_1$ and $V_2$ are determined in terms of the parameter $q = \exp(i\gamma)$ by

$$V_1 = \sin(\gamma), \quad V_2 = \cos(\gamma).$$

(18)

Turning now to the boundary interactions we found that the on-site Coulomb coupling $U(\xi_\pm)$ is given by

$$U(\xi_\pm) = -i \frac{\sin(2\gamma)}{\sinh(\xi_\pm/2 - i\gamma) \cosh(\xi_\pm/2)}$$

(19)

while the boundary chemical potentials are

$$\mu^{(l)}_{1\sigma} = \begin{cases} \mu^{(1)}_{1+} = \mu^{(1)}_{1-} = i V_1 \frac{e^{-\xi_-^+/2 + i\gamma}}{\sinh(\xi_-^+/2 - i\gamma)} \\
\mu^{(2)}_{1+} = i V_1 \frac{e^{\xi_-^+/2 - i\gamma}}{\sinh(\xi_-^+/2 - i\gamma)}, \quad \mu^{(2)}_{1-} = -i V_1 \frac{e^{-\xi_-^+/2}}{\cosh(\xi_-^+/2)} \end{cases}$$

(20)

and

$$\mu^{(m)}_{L\sigma} = \begin{cases} \mu^{(1)}_{L+} = \mu^{(1)}_{L-} = i V_1 \frac{e^{\xi_+^+/2 - i\gamma}}{\sinh(\xi_+^+/2 - i\gamma)} \\
\mu^{(2)}_{L+} = i V_1 \frac{e^{-\xi_+^+/2 - i\gamma}}{\sinh(\xi_+^+/2 - i\gamma)}, \quad \mu^{(2)}_{L-} = i V_1 \frac{e^{-\xi_+^+/2}}{\cosh(\xi_+^+/2)} \end{cases}$$

(21)

Our next task is to diagonalize the Hamiltonian (16) by the coordinate Bethe ansatz formalism. The number of electrons $N_e^\sigma$ with spin $\sigma$ are conserved quantities and they label the possible disjoint sectors of the Hilbert space. For a sector of a given number of particles $N_e = N_e^+ + N_e^-$ the Bethe wave function assumes the following form

$$|\Psi\rangle = \sum_{x_{Q_1}, \ldots, x_{Q_N}, \sigma} \text{sgn}(P) \prod_{j=1}^{N_e} e^{(k_{P_{Q_1}}, x_{Q_1}, \ldots, k_{P_{Q_N}}, x_{Q_N}, \sigma)} c_{x_{Q_1}, \ldots, x_{Q_N}, c}^\dagger |0\rangle$$

(22)
where \(|0\rangle\) denotes a reference state containing none particles, \(1 \leq x_{Q_1} \leq x_{Q_2} \leq \cdots \leq x_{Q_{N_e}} \leq L\) indicate the positions of the electrons, \(P\) is the sum over all the permutations of the momenta \((P_1 \cdots P_{N_e})\) and the symbol \(sgn\) denotes the sign of the permutation. For configurations such that \(|x_{Q_i} - x_{Q_j}| \geq 2\) the Hamiltonian (16) behaves as a free-theory and the solution of the eigenvalue problem \(H^{(l,m)}|\Psi\rangle = E^{(l,m)}(L)|\Psi\rangle\) is

\[
E^{(l,m)}(L) = \sum_{j=1}^{N_e} 2 \cos(k_j)
\]

up to some additive constants proportional to the number of electrons \(N_e^\pm\). It is standard in Bethe ansatz approach that configurations in which the electrons are nearest neighbors, at the same site or even at the boundaries impose constraints on the amplitudes of the wave function. For previous similar computations to other models with boundary, see refs. [17, 18]. We found that such consistency condition on the “bulk” provide us to the following relation

\[
A_{\cdots\sigma_j,\sigma_i\cdots}(\cdots, k_j, k_i, \cdots) = S_{i,j}(k_i, k_j)A_{\cdots\sigma_i,\sigma_j\cdots}(\cdots, k_i, k_j, \cdots)
\]

while the reflection at the left and right ends of the chain gives us

\[
A_{\sigma_i\cdots(-k_j, \cdots)} = S_l(k_j, P_i^{(l)})A_{\sigma_i\cdots}(k_j, \cdots)
\]
\[
A_{\cdots,\sigma_i(\cdots, -k_j)} = S_r(k_j, P_i^{(m)})A_{\cdots,\sigma_i(\cdots, k_j)}
\]

The two-body \(S\)-matrix \(S_{i,j}(k_i, k_j)\) connects the scattering amplitudes between the states \(((k_i, \sigma_i); (k_j, \sigma_j))\) and \(((k_j, \sigma'_j); (k_i, \sigma'_i))\) and its non-null elements are [11]

\[
S_{++}(\lambda) = S_{--}(\lambda) = 1
\]
\[
S_{+-}(\lambda) = S_{-+}(\lambda) = \frac{\sinh(\lambda)}{\sinh(\lambda + 2i\gamma)}
\]
\[
S_{++}(\lambda) = S_{-+}(\lambda) = \frac{\sinh(2i\gamma)}{\sinh(\lambda + 2i\gamma)}
\]

where the rapidities \(\lambda_j(\lambda = \lambda_1 - \lambda_2)\) are related to the momenta \(k_j\) by

\[
\exp[ik_j] = \frac{\sinh(\lambda_j/2 - i\gamma/2)}{\sinh(\lambda_j/2 + i\gamma/2)}
\]
Finally, the boundary scattering matrices are

\[ S_l(k_j, P^{(l)}_{lr}) = \frac{1 + [P^{(l)}_{lr} - 2 \cos(\gamma)] e^{ik_j}}{1 + [P^{(l)}_{lr} - 2 \cos(\gamma)] e^{-ik_j}} \]

\[ S_r(k_j, P^{(m)}_{rs}) = \frac{1 + [P^{(m)}_{rs} - 2 \cos(\gamma)] e^{-ik_j}}{1 + [P^{(m)}_{rs} - 2 \cos(\gamma)] e^{ik_j}} \ e^{2(L+1)k_j} \]

(31)

(32)

where

\[ P^{(l)}_{lr} = \begin{cases} P^{(1)}_{1+} = \frac{\cosh(\xi_+ / 2 + i\gamma)}{\cosh(\xi_- / 2)} \\ P^{(1)}_{1-} = \frac{\sinh(\xi_+ / 2 - 2i\gamma)}{\sinh(\xi_- / 2 - 2i\gamma)} \end{cases} \]

(33)

and

\[ P^{(m)}_{rs} = \begin{cases} P^{(1)}_{r+} = \frac{\cosh(\xi_+ / 2 + i\gamma)}{\cosh(\xi_- / 2)} \\ P^{(1)}_{r-} = \frac{\sinh(\xi_+ / 2 - 2i\gamma)}{\sinh(\xi_- / 2 - 2i\gamma)} \end{cases} \]

(34)

So far we managed to solve the charge degrees of freedom of the system but we still have to diagonalize the spin sector associated with the scattering matrices \( S_{ij}(k_i, k_j), S_l(k_i, P^{(l)}_{lr}) \) and \( S_r(k_j, P^{(m)}_{rs}) \). These amplitudes, however, are easily related to those of the six-vertex model and the spin part of the problem is reduced to the diagonalization of an inhomogeneous 6-vertex model with open boundaries \([8]\). In the course of solution one has to introduce a second Bethe ansatz for the spin rapidities \( \mu_j, j = 1, \cdots, N_e^+ \). Since this problem has been discussed in many different contexts in the literature \([7, 4, 8]\) we restrict ourselves to present only the final Bethe ansatz results. For the four possible boundary case, we find that the pseudomomenta \( \lambda_j \) and the spin variables \( \mu_j \) satisfy the following nested Bethe ansatz equations

\[ \left[ \frac{\sinh(\lambda_j / 2 - i\gamma / 2)}{\sinh(\lambda_j / 2 + i\gamma / 2)} \right]^{2L} F(\lambda_j, \xi_{\pm}) = \prod_{k=1}^{N_e^+} \frac{\sinh(\lambda_j - \mu_k - i\gamma) \sinh(\lambda_j + \mu_k - i\gamma)}{\sinh(\lambda_j - \mu_k + i\gamma) \sinh(\lambda_j + \mu_k + i\gamma)}, \ j = 1, \cdots, N_e^+ \]

(35)

\[ \prod_{k=1}^{N_e} \frac{\sinh(\mu_j - \lambda_k - i\gamma) \sinh(\mu_j + \lambda_k - i\gamma)}{\sinh(\mu_j - \lambda_k + i\gamma) \sinh(\mu_j + \lambda_k + i\gamma)} = G(\mu_j, \xi_{\pm}) \prod_{k=1}^{N_e^+} \frac{\sinh(\mu_j - \mu_k - 2i\gamma) \sinh(\mu_j + \mu_k - 2i\gamma)}{\sinh(\mu_j - \mu_k + 2i\gamma) \sinh(\mu_j + \mu_k + 2i\gamma)}, \ j = 1, \cdots, N_e^+ \]

(36)

where the boundary factors \( F(\lambda_j, \xi_{\pm}) \) and \( G(\mu_j, \xi_{\pm}) \) are given by

\[ F(\lambda_j, \xi_{\pm}) = \frac{\cosh(\lambda_j / 2 + i\gamma / 2 - \xi_- / 2)}{\cosh(\lambda_j / 2 - i\gamma / 2 + \xi_- / 2)} \]

\[ \frac{\cosh(\lambda_j / 2 + i\gamma / 2 - \xi_+ / 2)}{\cosh(\lambda_j / 2 - i\gamma / 2 + \xi_+ / 2)} \]

(37)
and, in terms of the rapidities $\lambda_j$, the eigenvalues $E^{(l,m)}(L)$ are given (modulo additive constants) by

$$E^{(l,m)}(L) = \sum_{i=1}^{N_s} \frac{2 \sin^2(\gamma)}{\cos(\gamma) - \cosh(\lambda_i)}$$

We close this section commenting on two special open boundary conditions. The quantum group is obtained from the results for $l = m = 2$ in the limits $\xi_+ \to -\infty$ and $\xi_- \to +\infty$. We remark that there is a more transparent way to derive such boundary condition, however we have to use a different $R$-matrix embedding. For further details see Appendix A. Other interesting boundary, concerning critical behaviour, is the free-boundary condition. We note that this case is achieved by setting $\xi_+ = \xi_- = i\pi + 2i\gamma$ in the model $l = m = 1$.

### 4 Conclusions

We have completed the analysis of the integrability of an interesting supersymmetric Hubbard-like model in the presence of boundary fields. This was accomplished by first deriving diagonal solutions of the reflection equation associated with a particular $U_{q}[Osp(2|2)]$ invariant $R$-matrix. This leads us to four boundaries conditions for the corresponding one-dimensional Hamiltonian, which have been diagonalized by the Bethe ansatz approach. Quantum-group invariant solutions have been discussed either as a special limit of the free-parameter $\xi_\pm$ or by the analysis of other possible $R$-matrix embedding.

The Bethe ansatz equations of section 3 provide us a tool to compute the thermodynamic behaviour and the finite-size corrections to the spectrum of the system. In principle, this allows us to determine the scattering of the physical excitations and the bulk and the boundary critical properties of the underlying field theory. These computations could be of interest as an
alternative way to rederive the results of ref. [15] for the integrable double sine-Gordon model. This also opens the possibility to obtain extra information concerning the operator content of this system which should provide further insight to the problem of tunneling in quantum wires.

Finally, we mention that one possible generalization of this work is to investigate operator valued solutions of the reflection equation associated with the $Osp(2|2)$ $R$-matrix (5) [20, 19, 21]. This will leads us to an electronic system with Kondo impurities [20, 19] which hopefully could be the lattice analog of an interesting double sine-Gordon model with Kondo impurity. We plan to investigate these problems in future publications.

**Acknowledgements**

This research has been supported by *Fapesp* (Fundação de Amparo a Pesquisa do Estado de S.Paulo) and partially by *CNPq* (Brazilian research program).

**Appendix A : Other $R$-matrix embedding**

The purpose of this appendix is to discuss an extra $R$-matrix embedding for the $Osp(2|2)$ vertex model. The $R$-matrix has the same structure of equation (5) but with new weights $a_{\alpha\beta}(\lambda)$ for $\alpha \neq \beta$, namely

$$a_{\alpha\beta}(\lambda) = \begin{cases} 
\frac{q_{\alpha\bar{\alpha}} e^\lambda + 1}{q_{\alpha} e^\lambda + 1} & \alpha = \beta \\
q e^\lambda \left[ \varepsilon_{\alpha\beta} q^\alpha - \frac{\beta}{1 + e^\lambda} + \frac{1 - q^2}{e^\lambda - 1} \delta_{\alpha,\beta'} \right] & \alpha < \beta, \\
-\varepsilon_{\tilde{\alpha}} q^\tilde{\alpha} - \frac{\beta}{1 + e^\lambda} + \frac{1 - q^2}{e^\lambda - 1} \delta_{\alpha,\beta'} & \alpha > \beta
\end{cases}$$

(A.1)

where $\varepsilon_1 = -\varepsilon_4 = q$, $\varepsilon_2 = -\varepsilon_3 = i$ and $\tilde{\alpha}$ is defined by

$$\tilde{\alpha} = \begin{cases} 
\alpha - \frac{1}{2} & 1 \leq \alpha \leq 2 \\
\alpha + \frac{1}{2} & 3 \leq \alpha \leq 4
\end{cases}$$

(A.2)
This $R$-matrix satisfies the properties (2,3), but now the crossing matrix $V$ is

$$V = \begin{pmatrix} 0 & 0 & 0 & q^{-1} \\ 0 & 0 & iq^{-1} & 0 \\ 0 & -iq & 0 & 0 \\ q & 0 & 0 & 0 \end{pmatrix} \quad (A.3)$$

We note that some of the new Boltzmann weights $a_{\alpha\beta}(\lambda)$ have indeed a different functional form as compared to those of equation (6). For periodic boundary conditions, however, we have checked that such differences are not important as long as Bethe ansatz analysis is concerned. The corresponding Bethe ansatz equations of this “new” vertex model (or associated quantum spin chain) are precisely the same as that found in ref. [10]. The situation for open boundary conditions is, however, not so rich as in section 2. Although we managed to find two classes of diagonal $K$-matrices solutions, none of them possess a free-parameter. The first solution is the standard quantum-group invariant one

$$K^{(1)}_-(\lambda) = Id \quad (A.4)$$

while the second class is given by

$$K^{(2)}_-(\lambda) = \begin{pmatrix} A_1(\lambda) & 0 & 0 & 0 \\ 0 & A_2(\lambda) & 0 & 0 \\ 0 & 0 & A_2(\lambda) & 0 \\ 0 & 0 & 0 & A_3(\lambda) \end{pmatrix} \quad (A.5)$$

where

$$A_1(\lambda) = (e^\lambda + \epsilon iq^3)(e^\lambda - \epsilon iq^{-3})$$

$$A_2(\lambda) = (e^{-\lambda} + \epsilon iq^3)(e^{-\lambda} - \epsilon iq^{-3}) \quad (A.6)$$

$$A_3(\lambda) = (e^{-\lambda} + \epsilon iq^3)(e^{-\lambda} - \epsilon iq^{-3})$$

where $\epsilon = \pm 1$. The Bethe ansatz solution for such open boundaries follows closely the steps of section 3. The only difference is concerned with the bulk scattering matrix of the spins degree
of freedom. Now, this matrix possesses the quantum-group invariant form:

\begin{align}
S^{++}_{++}(\lambda) &= S^{--}_{--}(\lambda) = 1, \\
S^{--}_{+-}(\lambda) &= S^{++}_{-+}(\lambda) = \frac{\sinh(\lambda)}{\sinh(\lambda + 2i\gamma)}, \\
S^{--}_{++}(\lambda) &= e^\lambda \frac{\sinh(2i\gamma)}{\sinh(\lambda + 2i\gamma)}, \\
S^{+-}_{--}(\lambda) &= e^{-\lambda} \frac{\sinh(2i\gamma)}{\sinh(\lambda + 2i\gamma)}. 
\end{align}

Having this information, it is easy to derive that the Bethe ansatz equations associated with the first boundary (A.4) are of quantum-group type, i.e. $F(\lambda_j, \xi_\pm) = G(\mu_j, \xi_\pm) = 1$. Similarly, the Bethe ansatz equations for the second boundary (A.5) gives us $F(\lambda_j, \xi_\pm) = f_\epsilon(\lambda_j) f_{\epsilon+}(\lambda_j)$ and $G(\mu_j, \xi_\pm) = 1$ where

\begin{equation}
 f_\epsilon(\lambda) = \begin{cases} 
\frac{\cosh[\lambda/2-i(\gamma+\pi/4)]}{\cosh[\lambda/2+i(\gamma+\pi/4)]} & \epsilon = +1 \\
\frac{\sinh[\lambda/2-i(\gamma+\pi/4)]}{\sinh[\lambda/2+i(\gamma+\pi/4)]} & \epsilon = -1
\end{cases} 
\end{equation}

We remark that (A.11) can be recovered from our previous results for the “mixed” boundary conditions $l = 1, m = 2$ and $l = 2, m = 1$ via fine tuning of the parameters $\xi_\pm$. The conclusions of this appendix suggest that such different embedding may be formulated as a twisting of Deguchi et al [11] original solution [3]. It would be interesting to explore this possibility further since this may lead us to new integrable multiparametric spin chains [22].

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We thank J. Links for pointing out this possibility.
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