THRESHOLDS ON GROWTH OF NONLINEARITIES AND SINGULARITY OF INITIAL FUNCTIONS FOR SEMILINEAR HEAT EQUATIONS

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Abstract. Let $N \geq 1$ and let $f \in C([0, \infty))$ be a nonnegative nondecreasing function and $u_0$ be a possibly singular nonnegative initial function. We are concerned with existence and nonexistence of a local in time nonnegative solution in a uniformly local Lebesgue space of a semilinear heat equation

\[
\begin{cases}
\partial_t u = \Delta u + f(u) & \text{in } \mathbb{R}^N \times (0,T), \\
u(x,0) = u_0(x) & \text{in } \mathbb{R}^N,
\end{cases}
\]

under mild assumptions on $f$. A relationship between a growth of $f$ and an integrability of $u_0$ is studied in detail. Our existence theorem gives a sharp integrability condition on $u_0$ in a critical and subcritical cases, and it can be applied to a regularly or rapidly varying function $f$. In a doubly critical case existence and nonexistence of a nonnegative solution can be determined by special treatment. When $f(u) = u^{1+2/N}[\log(u+1)]^\beta$, a complete classification of existence and nonexistence of a nonnegative solution is obtained. We also show that the same characterization as in Laister et al. [11] is still valid in the closure of the space of bounded uniformly continuous functions in the space $L^r_0(\mathbb{R}^N)$. Main technical tools are a monotone iterative method, $L^p-L^q$ estimates, Jensen’s inequality and differential inequalities.

1. Introduction and main results

We are concerned with existence and nonexistence of a local in time solution for a semilinear heat equation

\[
\begin{cases}
\partial_t u = \Delta u + f(u) & \text{in } \mathbb{R}^N \times (0,T), \\
u(x,0) = u_0(x) & \text{in } \mathbb{R}^N,
\end{cases}
\]

where the domain is $\mathbb{R}^N$, $N \geq 1$, $f$ is a $C^1$ function and the initial function $u_0$ may be unbounded. When $u_0 \in L^\infty(\mathbb{R}^N)$, it is known that a solution can be constructed by contraction mapping theorem. On the other hand, when $u_0 \notin L^\infty(\mathbb{R}^N)$, the existence of a solution is not trivial, and it depends on the balance between a growth of $f$ and a strength of singularities of $u_0$, i.e., an integrability of $u_0$. Weissler [22] studied the power case $f(u) = |u|^{p-1}u$ and obtained the following:

Proposition 1.1. Let $f(u) = |u|^{p-1}u$, $p > 1$ and $r_c := N(p-1)/2$. Then the following hold:

(i) (Existence) The problem (1.1) admits a local in time solution $u(t) \in C([0,T), L^r(\mathbb{R}^N))$ if one of the following holds:

(a) (Subcritical case) $r > r_c$, $r \geq 1$ and $u_0 \in L^r(\mathbb{R}^N)$.

(b) (Critical case) $r = r_c > 1$ and $u_0 \in L^r(\mathbb{R}^N)$.

(ii) (Nonexistence) For each $1 \leq r < r_c$, there exists a nonnegative function $u_0 \in L^r(\mathbb{R}^N)$ such that (1.1) admits no nonnegative solution.

Let $u(x, t)$ be a solution of $\partial_t u = \Delta u + |u|^{p-1}u$. Let $\lambda > 0$ and $u_\lambda(x, t) = \lambda^{2/(p-1)}u(\lambda x, \lambda^2 t)$. Then, $u_\lambda$ also satisfies the same equation. We see that $\|u_\lambda(x, 0)\|_r = \|u(x, 0)\|_r$ if and only if $r = r_c$. Proposition 1.1 shows that $u_0 \in L^{r_c}(\mathbb{R}^N)$.

Date: May 3, 2021.

2010 Mathematics Subject Classification. primary 35K55, secondary 35A01, 46E30.

Key words and phrases. Existence and nonexistence; Doubly critical case; Uniformly local $L^p$ space; Regularly and rapidly varying functions.

The first author was supported by JSPS KAKENHI Grant Numbers 19H01797, 19H05599.

The second author was supported by Grant-in-Aid for JSPS Fellows No. 20J11985.
is an optimal integrability condition for the solvability. For the case $f(u) = |u|^{p-1}u$, much attention has been paid and a brief history can be found in [11]. See also [4][19][23] for various results.

As mentioned in [12], a tight correspondence between $f$ and the integrability of $u_0$ fails in the case where $f(u) \neq |u|^{p-1}u$. Then, two problems arise:

(A) given $f$, characterize the set $S$ of initial data for which ([11]) has a solution;

(B) given the set $S$ of initial data, characterize the nonlinearity $f$ for which ([11]) has a solution for every initial data in $S$.

With regards to (B), Laister et al. [11] gave a complete answer. In [11] the following was proved: Let $f$ be a nonnegative nondecreasing continuous function and $\Omega$ be a smooth bounded domain. Then, a Cauchy Dirichlet problem

$$
\begin{align*}
\partial_t u &= \Delta u + f(u) & \text{in } \Omega \times (0, T), \\
u &= 0 & \text{on } \partial \Omega \times (0, T), \\
u(x, 0) &= u_0(x) & \text{in } \Omega
\end{align*}
$$

admits a local in time nonnegative solution for every nonnegative initial data $u_0 \in L^q(\Omega)$ if and only if

$$
\limsup_{u \to \infty} \frac{|f(u)|}{u^{1-r}} < \infty \quad \text{if } 1 < r < \infty,
$$

$$
\inf_{u \to \infty} \frac{|f(u)|}{u^{1-r}} < \infty \quad \text{if } r = 1,
$$

where $\hat{f}(u) = \sup_{1 \leq r \leq u} \left( \frac{f(u)}{u^{1-r}} \right)$. In the case $r = 1$ various properties were studied in [12].

In this paper we mainly study Problem (A) and also study Problem (B) under a general integrability condition on $u_0$. We prepare some notation. Let $1 \leq r < \infty$. We define uniformly local $L^r$ spaces by

$$
L^r_u(\mathbb{R}^N) := \left\{ u \in L^1_{loc}(\mathbb{R}^N) \mid \| u \|_{L^r_u(\mathbb{R}^N)} < \infty \right\}.
$$

Here, for $\rho > 0$, $B(y, \rho) := \{ x \in \mathbb{R}^N \mid |x - y| < \rho \}$ and

$$
\| u \|_{L^r_u(\mathbb{R}^N)} := \left\{ \begin{array}{ll}
\sup_{y \in \mathbb{R}^N} \left( \frac{\int_{B(y, \rho)} |u(x)|^r \, dx}{\rho^r} \right)^{1/r} & \text{if } 1 \leq r < \infty, \\
\text{esssup}_{y \in \mathbb{R}^N} \| u \|_{L^\infty(B(y, \rho))} & \text{if } r = \infty.
\end{array} \right.
$$

We easily see that $L^\infty_u(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$ and that $L^1_u(\mathbb{R}^N) \subset L^\alpha_u(\mathbb{R}^N)$ if $1 \leq \alpha \leq \beta$. We define $L^r_u(\mathbb{R}^N)$ by

$$
L^r_u(\mathbb{R}^N) := BUC(\mathbb{R}^N)^\beta \cdot L^\alpha_u(\mathbb{R}^N),
$$

i.e., $L^r_u(\mathbb{R}^N)$ denotes the closure of the space of bounded uniformly continuous functions $BUC(\mathbb{R}^N)$ in the space $L^\alpha_u(\mathbb{R}^N)$. We assume

(f) $f \in C^1(0, \infty) \cap C[0, \infty)$, $f(u) > 0$ for $u > 0$, $f'(u) \geq 0$ for $u > 0$, $F(u) < \infty$ for $u > 0$,

where

$$
F(u) := \int_u^\infty \frac{d\tau}{f(\tau)}.
$$

We define $X_q$ by

$$
X_q := \left\{ f \in C[0, \infty) \mid f \text{ satisfies (f) and the limit } q := \lim_{u \to \infty} f'(u)F(u) \text{ exists.} \right\}.
$$

In [6][14] it was proved that if the limit $q$ exists, then $q \in [1, \infty]$. Let us explain the exponent $q$. If $f \in C^2$, then by L’Hospital’s rule we have

$$
q = \lim_{u \to \infty} \frac{F(u)}{1/f'(u)} = \lim_{u \to \infty} \frac{(F(u))'}{1/f''(u)} = \lim_{u \to \infty} \frac{f'(u)^2}{f(u)f''(u)}.
$$

The growth rate of $f$ can be defined by $p := \lim_{u \to \infty} u f'(u)/f(u)$. We apply L’Hospital’s rule. Then,

$$
\frac{1}{p} = \lim_{u \to \infty} \frac{f'(u)/f''(u)}{u} = \lim_{u \to \infty} \left( 1 - \frac{f'(u)^2}{f''(u)} \right) = 1 - \frac{1}{q}, \quad \text{and hence } \frac{1}{p} + \frac{1}{q} = 1.
$$

The $q$ exponent is the conjugate exponent of the growth rate $p$. For example, if $f(u) = u^p$ ($p > 1$), then $q = p/(p - 1)$. The leading term is not necessarily a pure power function $u^p$. If $f(u) = u^p[\log(u + e)]^q$
(\(p > 1, \beta \in \mathbb{R}\)), then \(q = p/(p-1)\). The case \(q = 1\) corresponds to the superpower case. For instance, the \(q\) exponent becomes 1 if
\[
f(u) = \exp(u^p) \quad (p > 0), \quad f(u) = \exp(\cdots \exp(u) \cdots) \quad \text{or} \quad f(u) = \exp(|\log u|^{p-1} \log u) \quad (p > 1).
\]

Fujishima-Ioku [6] studied Problem (A) for \(f \in X_q\) and obtained the following:

**Proposition 1.2.** The following hold:

(i) (Existence) Let \(u_0 \geq 0\). Suppose that \(f \in X_q\) and
\[
(1.4) \quad f'(u)F(u) \leq q \quad \text{for large } u > 0.
\]
Then \([1,2]\) has a local in time nonnegative solution if one of the following holds:

(a) (Subcritical case) \(r > N/2, q \leq 1 + r\) and \(F(u_0)^{-r} \in L^1_{\text{loc}}(\mathbb{R}^N)\).

(b) (Critical case) \(r = N/2, q < 1 + r\) and \(F(u_0)^{-r} \in L^1_{\text{loc}}(\mathbb{R}^N)\).

(ii) (Nonexistence) Suppose that \(f \in C^2[0,\infty) \cap X_q\) with \(q < 1 + N/2\) and that \(f''(u) \geq 0\) for \(u \geq 0\).

If \(0 < r < N/2\) and \(q < 1 + r\), then there exists a nonnegative initial function \(u_0\) such that \(F(u_0)^{-r} \in L^1_{\text{loc}}(\mathbb{R}^N)\) and \([1,2]\) admits no nonnegative solution.

**Remark 1.3.**

(i) In Proposition [1,2] (ii) we can take \(u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)\). See the proof of [6, Theorem 1.2] for details.

(ii) Proposition [1,2] shows that, for each \(f \in X_q\) with \([1,4]\), an optimal integrability condition is
\[
F(u_0)^{-N/2} \in L^1_{\text{loc}}(\mathbb{R}^N).
\]
When \(f(u) = u^p\), then \(F(u_0)^{-N/2} = (p-1)^{N/2}u^{N(p-1)/2} \geq 0\) for \(u > 0\). Therefore, the case \(r = N/2\) is a critical case which corresponds to Proposition [1,1] (i) (b).

(iii) Let \(qs := (N + 2)/4\) be the conjugate exponent of the critical Sobolev exponent \((N + 2)/(N - 2)\).
In \([1,4]\) a radial singular stationary solution \(u^*(x)\) of \([1,4]\) near the origin was constructed if \(f \in X_q\) with \(q > qs\).
Moreover, \(u^*\) is unique under a certain assumption on \(f\) (see [15,16]) and
\[
\left| u^*(x) \right| = F^{-1}\left( \frac{|x|^2}{2N - 4q}(1 + o(1)) \right) \quad \text{as } |x| \to 0.
\]
Since \(F(u^*)^{-r} \not\in L^1_{\text{loc}}(\mathbb{R}^N)\) for \(r > N/2\) and \(F(u^*)^{-r} \in L^1_{\text{loc}}(\mathbb{R}^N)\) for \(r < N/2\), \(u^*\) is on a border between Proposition [1,2] (i) and (ii).

Let
\[
S(t)[\phi](x) := \int_{\mathbb{R}^N} K(x, y, t)\phi(y)dy \quad \text{for } \phi \in L^1_{\text{loc}}(\mathbb{R}^N),
\]
where \(K(x, y, t) := (4\pi t)^{-N/2}\exp\left(-|x - y|^2/4t\right)\). Then, \(S(t)[\phi]\) gives a solution of the Cauchy problem of the linear heat equation \(\partial_t u = \Delta u\) with the initial function \(\phi(x)\).
We define a solution of \([1,4]\).

**Definition 1.4.** We call \(u(t)\) a solution of \([1,4]\) if there exists \(T > 0\) such that \(u(t) \in L^\infty((0, T), L^1_{\text{loc}}(\mathbb{R}^N)) \cap L^\infty_{\text{loc}}((0, T), L^\infty(\mathbb{R}^N))\) and \(u\) satisfies
\[
\infty > u(t) = F[u(t)] \quad \text{for a.e. } x \in \mathbb{R}^N, \quad 0 < t < T,
\]
where
\[
F[u(t)] := S(t)[u_0] + \int_0^t S(t-s)f(u(s))ds.
\]

We call a measurable finite almost everywhere function \(\bar{u} : \mathbb{R}^N \times (0, T) \to \mathbb{R}\) a supersolution for \([1,4]\) if there exists \(T > 0\) such that \(\bar{u}(t) \geq F[u(t)]\) for a.e. \(x \in \mathbb{R}^N, \quad 0 < t < T\).

The first result is a generalization of Proposition [1,2] (i). Specifically, the technical assumption \([1,4]\) can be removed as the following (i) and (iii) show.

**Theorem A.** Let \(u_0 \geq 0\) and \(f \in X_q\). Then \([1,1]\) has a local in time nonnegative solution \(u(t), 0 < t < T\), if one of the following holds:

(i) (Subcritical case 1) \(r > N/2, q < 1 + r\) and \(F(u_0)^{-r} \in L^1_{\text{loc}}(\mathbb{R}^N)\).

(ii) (Subcritical case 2) \(r > N/2, q = 1 + r\) and \(F(u_0)^{-r} \in L^1_{\text{loc}}(\mathbb{R}^N)\) and \([1,4]\) holds.

(iii) (Critical case) \(r = N/2, q < 1 + r\) and \(F(u_0)^{-r} \in L^1_{\text{loc}}(\mathbb{R}^N)\).
Moreover, in all cases, there exists $C > 0$ such that
\begin{equation}
\|F(u(t))^{-r}\|_{L^q_t(L^r_x(\mathbb{R}^N))} \leq C \text{ for } 0 < t < T.
\end{equation}

**Remark 1.5.**

(i) Theorem $A$ (i) with $q > 1$ was proved in [5] Theorem 1.4, and Theorem $A$ (ii) is included in Proposition 1.2 (i). However, in this paper we prove three cases in a unified way, using a different approach. See the proof of Theorem 1.2 for details.

(ii) If $q < 1 + r$ or if $q = 1 + r$ with \(1.4\), then $F(u_0)^{-r}$ is convex for large $u > 0$. Therefore, $F(u_0)^{-r} \in L^1(\mathbb{R}^N)$ always implies $u_0 \in L^1(\mathbb{R}^N)$ and $S(t)u_0$ is well defined.

(iii) If $q = 1 + r$ and \(1.4\) does not hold, then $F(u_0)^{-r}$ may not be nonconvex in $u$, and $F(u_0)^{-r} \in L^1(\mathbb{R}^N)$ does not necessarily imply $u_0 \in L^1(\mathbb{R}^N)$. Hence the case $q = 1 + r$ is critical in some sense.

(iv) Using the method used in the proof of [2] Theorem 1, we see that Proposition 1.2 (ii) also holds even if we adopt Definition 1.4. Theorem $A$ and Proposition 1.2 (ii) complete a classification of the existence and nonexistence problem for $f \in X_q$ in a reasonable region \{$(q, r) \mid 1 \leq q \leq 1 + r, r > 0$\} except $(q, r) = (1 + N/2, N/2)$.

Let us consider the case where $(q, r) = (1 + N/2, N/2)$. This case corresponds to the case $r = r_c = 1$ in Proposition 1.1 and it is not covered by Propositions 1.1, 1.2 or Theorem $A$. The simplest example is $f(u) = u^{1+2/N}$. Then, the integrability condition becomes $F(u_0)^{-r} = (2/N)^{N/2}u_0 \in L^1(\mathbb{R}^N)$. It is known that there exists a nonnegative initial data $u_0 \in L^1(\mathbb{R}^N)(\subset L^1_{\text{loc}}(\mathbb{R}^N))$ such that \(1.1\) admits no nonnegative solution. See [2, 5, 11, 12, 24] for nonexistence results. This case is quite delicate and referred as a doubly critical case in [2, Section 7.5], since $r = N/2$ and $q = 1 + r$. See Figure 1. A sufficient condition for existence is recently studied in [18]. Combining a nonexistence result [3] and an existence result [18], we have the following:

**Proposition 1.6.** Let $f(u) = |u|^{2/N}u$ and
\begin{equation}
Z_r := \left\{ \phi(x) \in L^1_{\text{loc}}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |\phi| \left[ \log(|\phi| + 1) \right]^r dx < \infty \right\}.
\end{equation}

Then the following hold:

(i) If $u_0 \in Z_r$ for some $r \geq N/2$, then \(1.7\) admits a local in time solution.

(ii) For each $0 \leq r < N/2$, there is a nonnegative initial function $u_0 \in Z_r$ such that \(1.7\) admits no nonnegative solution.

Since $Z_r \subset Z_{N/2}$ for $r \geq N/2$, by Proposition 1.6 we see that $u_0 \in Z_{N/2}$.
is an optimal integrability condition for the solvability when \( f(u) = |u|^{2\alpha}/u \). In particular, \( Z_{N/2} \) is a proper subset of \( L^1(\mathbb{R}^N) \).

We study existence of a solution in a doubly critical case when \( f \) is a general nonlinearity. The next main theorem is a generalization of Proposition \( \ref{prop:1.3} \) (i).

**Theorem B** (Existence, doubly critical case). Let \( u_0 \geq 0 \) and \( q = 1 + N/2 \). Suppose that \( f \in \mathcal{X}_q \) holds. Let

\[
g(u) := u[\log(u + e)]^\alpha, \quad h(u) := F(u)^{-N/2}
\]

and \( J_\alpha(u) := g(h(u)) = F(u)^{-N/2}\left(\log(F(u)^{-N/2} + e)\right)^\alpha \).

Then \( \ref{eq:1.1} \) admits a local in time nonnegative solution \( u(t), \ 0 < t < T \), if one of the following holds:

(i) There exists \( \alpha > N/2 \) such that \( J_\alpha(u_0) \in \mathcal{L}^1_{ul}(\mathbb{R}^N) \) and \( J_\alpha(u) \) is convex for large \( u > 0 \), i.e., \( J_\alpha''(u) \geq 0 \) for large \( u > 0 \).

(ii) \( J_\alpha(u_0) \in \mathcal{L}^1_{ul}(\mathbb{R}^N) \) for \( \alpha = N/2 \) and there exists \( \rho < 1 \) such that

\[
f'(u)F(u) - q \leq \frac{\rho}{2} \frac{\log(F(u)^{-N/2} + e)}{u} \quad \text{for large } u > 0.
\]

Moreover, in two cases, there exists \( C > 0 \) such that

\[
\|J_\alpha(u(t))\|_{\mathcal{L}^1_{ul}(\mathbb{R}^N)} \leq C \quad \text{for } 0 < t < T.
\]

In Corollary \( \ref{cor:5.1} \) it will be shown that we cannot take \( \rho = 1 \) in Theorem \( \ref{thm:B} \) (ii). Thus, the condition \( \rho < 1 \) is optimal.

Let \( q = 1 + N/2 \) and \( \alpha > N/2 \). We can easily check that if \( f \in \mathcal{X}_q \) and \( \ref{eq:1.1} \) hold, then \( J_\alpha(u) \) is convex for large \( u > 0 \) and \( \ref{eq:1.8} \) holds for \( \rho = 0 \). Therefore, Theorem \( \ref{thm:B} \) immediately leads to the following simple sufficient condition:

**Corollary B**. Let \( u_0 \geq 0 \) and \( q = 1 + N/2 \). Suppose that \( f \in \mathcal{X}_q \) and \( \ref{eq:1.1} \) hold. Then \( \ref{eq:1.1} \) admits a local in time nonnegative solution if \( J_\alpha(u_0) \in \mathcal{L}^1_{ul}(\mathbb{R}^N) \) for some \( \alpha > N/2 \) or \( J_\alpha(u_0) \in \mathcal{L}^1_{ul}(\mathbb{R}^N) \) for \( \alpha = N/2 \).

We study nonexistence of a solution in a doubly critical case. For \( \beta \in \mathbb{R} \), we define

\[
f_\beta(u) := u^{1+2/N}[\log(u + e)]^\beta, \quad F_\beta(u) := \int_u^\infty \frac{dt}{f_\beta(t)} \quad \text{and} \quad h_\beta(u) := F_\beta(u)^{-N/2}.
\]

**Theorem C** (Nonexistence, doubly critical case). Let \( u_0 \geq 0 \). Suppose that \( f \) satisfies \( \ref{f:J} \) and there exist \( C_1 > 0, C_2 > 0, \beta > 0 \) and \( 0 < \delta < 1 \) such that the following hold:

(i) \( F^{-1} \circ F_\beta \) is convex on \( [C_1, \infty) \), i.e., \( f'(F^{-1}(v)) \geq f'_\beta(F^{-1}(v)) \) for \( 0 < v \leq F_\beta(C_1) \).

(ii) \( F(u) \leq C_2 u^{-2/N} [\log(u + e)]^\delta \) for \( u \geq C_1 \).

Let \( J_\alpha(u) := (F(u)^{-N/2} [\log(F(u)^{-N/2} + e)]^\alpha \). For each \( \alpha \in [0, N/2) \), there exists a nonnegative function \( u_0 \in \mathcal{L}^1_{ul}(\mathbb{R}^N) \) satisfying \( J_\alpha(u_0) \in \mathcal{L}^1_{ul}(\mathbb{R}^N) \) such that, for every \( T > 0 \), \( \ref{eq:1.1} \) admits no nonnegative solution.

**Remark 1.7.**

(i) If \( f \) satisfies \( \ref{f:J} \) and \( f'(u)F(u) \geq 1 + N/2 \) for large \( u > 0 \), then for each \( \beta > 0 \), the assumption of Theorem \( \ref{thm:C} \) (i) holds by taking \( C_1 > 0 \) sufficiently large.

(ii) A characterization of \( f \) for existence and nonexistence of a solution in \( \mathcal{L}^r_{ul}(\mathbb{R}^N) \), \( r \geq 1 \), is given in Corollary \( \ref{cor:7.2} \) and Theorem \( \ref{thm:7.3} \). This characterization is the same as \( \ref{eq:1.8} \) which was obtained in Corollary 4.5 and Theorem 3.4.

We study Problem (B) in Corollaries \( \ref{cor:5.2}, \ref{cor:5.3}, \ref{cor:5.3} \) and \( \ref{cor:5.5} \). These corollaries give existence and nonexistence conditions on \( f \) when integrability conditions on \( u_0 \) are given. These corollaries are not optimal, and could be improved. A threshold growth and a threshold integrability can be summarized as Table \( \ref{table:1} \).

We consider an example of a doubly critical case. Let \( f(u) = f_\beta(u) \). An elementary calculation shows that if \( \beta \geq -(1 + 2/N)\kappa \), then \( f_\beta(u) \) is nondecreasing for \( u > 0 \). Here, \( \kappa \) is the largest positive root of

\[
\log \kappa + 2 = \kappa, \quad \text{where} \quad \kappa \simeq 3.146.
\]

The following theorem is a complete classification of integrability conditions on \( u_0 \).
Table 1. Relationship between a threshold growth and a threshold integrability. Here, 
\[ g_{\beta}(u) = u(\log(u + e))^{N/2}, \] 
\[ q = \lim_{u \to \infty} f'(u)F(u) \] 
and 
\[ q_I = \lim_{u \to \infty} J'(u)^2/J(u)J''(u). \]

| Problem/Existence | Growth/Integrability | Existence/Nonexistence |
|-------------------|----------------------|------------------------|
| \(1 \leq q < 1 + \frac{N}{2} \) | \(f(u) \to F(u_0)^{-\frac{2}{N}} \in L_{ul}^1(\mathbb{R}^N)\) | Thm A | Prop 1.2(ii) |
| \(1 \leq q_I < \infty \) | \(J(u_0)^{1+\frac{2}{N}} \to J(u_0) \in L_{ul}^1(\mathbb{R}^N)\) | Cor 3.2 | Cor 1.8 |
| \(q = 1 + \frac{N}{2} \) | \(f(u) \to g_{\beta}(F(u_0)^{-\frac{2}{N}}) \in L_{ul}^1(\mathbb{R}^N)\) | Thm B | Thm C |
| \(q_I = \infty \) | \(J(u_0)^{1+\frac{2}{N}} \to J(u_0) \in L_{ul}^1(\mathbb{R}^N)\) | Cor 3.3 | Cor 1.8 |

**Theorem D** (Classification for \(f_\beta\)). Let \(u_0 \geq 0\),
\[ f(u) = f_\beta(u) \quad \text{and} \quad J_\alpha(u) = F_\beta(u)^{-N/2} \left[ \log \left( F_\beta(u)^{-N/2} + e \right) \right]^\alpha. \]

Then the following hold:

- **(i) Existence** \(\square\) with initial function \(u_0 \in L_{ul}^1(\mathbb{R}^N)\) admits a local in time nonnegative solution if one of the following holds:
  
  - (a) \(\alpha > N/2, \beta < -1\) and \(J_\alpha(u_0) \in L_{ul}^1(\mathbb{R}^N)\),
  - (b) \(\alpha = N/2, \beta > -1\) and \(J_\alpha(u_0) \in L_{ul}^1(\mathbb{R}^N)\),
  - (c) \(-1 + 2/N \leq \beta < -1\), where \(\kappa\) is given by \(1.10\).

- **(ii) Nonexistence**
  
  - (a) Let \(\beta > -1\). For each \(\alpha \in \{0, N/2\}\), there exists a nonnegative function \(u_0 \in L_{ul}^1(\mathbb{R}^N)\) such that \(J_\alpha(u_0) \in L_{ul}^1(\mathbb{R}^N)\) and that \(\square\) admits no nonnegative solution.
  - (b) Let \(\beta = -1\). For each \(\alpha \in \{0, N/2\}\), there exists a nonnegative function \(u_0 \in L_{ul}^1(\mathbb{R}^N)\) such that \(J_\alpha(u_0) \in L_{ul}^1(\mathbb{R}^N)\) and that \(\square\) admits no nonnegative solution.

**Remark 1.8.**

- **(i) In the case \(\beta > -1\) \(\square\) is solvable for all \(u_0 \geq 0\) satisfying \(J_{N/2}(u_0) \in L_{ul}^1(\mathbb{R}^N)\). However, in the case \(\beta = -1\) \(\square\) is not necessarily solvable even if \(J_{N/2}(u_0) \in L_{ul}^1(\mathbb{R}^N)\).
- **(ii) Theorem \(\square\) indicates that if \(\beta \geq -1\), then a threshold integrability condition is \(J_{N/2}(u_0) \in L_{ul}^1(\mathbb{R}^N)\).
- **(iii) If \(\beta = -1\), then there are \(C_2 > C_1 > 0\) such that \(C_1 < J_{N/2}(u) < C_2\) for \(u \geq 0\). Therefore, \(J_{N/2}(u_0) \in L_{ul}^1(\mathbb{R}^N)\) if and only if \(u_0 \in L_{ul}^1(\mathbb{R}^N)\).
- **(iv) In Section 4.4 it was proved that \(\square\) with \(f(u) = f_\beta(u)\) on a smooth bounded domain \(\Omega\) is always solvable (resp. is not always solvable) for a nonnegative function \(u_0 \in L^1(\Omega)\) if \(\beta < -1\) (resp. if \(-1 \leq \beta \leq 0\)).**

We characterize the class of nonlinearities \(X_q\), since Theorems \(\square\) and \(\square\) assume \(f \in X_q\).

**Definition 1.9.**

- **(i) Let \(RV_p, 0 \leq p < \infty\), denote the set of regularly varying functions, i.e., \(f \in RV_p\) if**
  \[ \lim_{u \to \infty} \frac{f(\lambda u)}{f(u)} = \lambda^p. \]
  In particular, if \(f \in RV_0\), then \(f\) is called a slowly varying function.

- **(ii) Let \(RV_\infty\) denote the set of rapidly varying functions, i.e., \(f \in RV_\infty\) if**
  \[ \lim_{u \to \infty} \frac{f(\lambda u)}{f(u)} = \begin{cases} \infty & \text{for } \lambda > 1, \\ 0 & \text{for } 0 < \lambda < 1. \end{cases} \]

The class \(RV_p\) is a generalization of a homogeneous function of degree \(p\) and \(RV_\infty\) is a generalization of a superpower function, e.g., \(e^{\alpha u}\). Readers can consult the book \(\square\) for details of \(RV_p\).

**Theorem E.** Assume that \(p\) and \(q\) satisfy the following: \(p := q/(q - 1)\) if \(q > 1\), and \(p := \infty\) if \(q = 1\). Then the following hold:

- **(i) If \(f \in X_q\) for some \(q \in [1, \infty)\), then \(f \in RV_p\).**
(ii) Suppose that $f$ satisfies $\text{(i)}$ and that $f'$ is nondecreasing. Then, $f \in X_q$ for $q \in (1, \infty)$ if and only if $f \in RV_p$ for $p \in (1, \infty)$.

(iii) Suppose that $f$ satisfies $\text{(i)}$ and that $f'(u)F(u)$ is nondecreasing. Then, $f \in X_1$ if and only if $f \in RV_\infty$.

Theorem 1 (ii) and (iii) say that $X_q$ ($1 \leq q < \infty$) and $RV_p$ ($1 < p \leq \infty$) are equivalent. Therefore, Theorems A and B can be applied to $f \in RV_p$ under additional assumptions. It follows from Karamata’s representation theorem, which is stated in Proposition 4.7, that for each function $f \in RV_p$, $1 < p < \infty$, $f(u)$ has a concrete form $\text{(7.1)}$ which explicitly describes a function of $X_q$. Moreover, it is known that $f \in RV_p$, $0 < p < \infty$, can be written as $f(u) = u^p L(u)$ for $u > 1$, where $L \in RV_0$, i.e., a slowly varying function.

Let us explain technical details. In the existence part a critical case (Theorem A (iii) without (1.4)) or a doubly critical case (Theorem B) were not covered by existing results. Since these cases are delicate, we introduce a new method. First, we separately treat the nonlinear term $f$ and a convex function $J$, which appears in an integrability condition $J(u_0) \in L^0_\alpha(\mathbb{R}^N)$. We introduce a simple but new supersolution $\text{(3.5)}$, using $J$. In $\text{[8, 10, 20]}$ similar functions were also used as supersolutions. However, these supersolutions were directly related to integrability conditions. In Theorem 3.1 we show that $\text{(3.5)}$ is actually a supersolution for $\text{(1.1)}$, and hence by monotone iterative method we can construct a nonnegative solution. In $\text{[6]}$ a change of variables was used to construct a supersolution, and $\text{(1.3)}$ was necessary. Essential conditions for $J$ are $\text{(3.1)}$ and $\text{(3.2)}$. Second, we relate $f$ and $J$. Specifically, we take $J(u) = F(u)^{-r}$ in Theorem A and $J(u) = J_\alpha(u)$ in Theorem B. This method can analyze in detail a relationship between the growth of $f$ and the integrability of $u_0$, and can treat superpower nonlinearities. Parabolic systems with superpower nonlinearities were studied in $\text{[10, 17, 21]}$. Theorem 3.1 is also useful in the study of Problem (B). Using Theorem 3.1 we give a necessary and sufficient condition on $f$ for a solvability in $L^r_1(\mathbb{R}^N)$, $r \geq 1$, in Section 9. Theorem 3.1 is used in the proof of the sufficient part. Main technical tools in the proof of Theorem 3.1 are a monotone iterative method (Proposition 2.4), $L^p-L^q$ estimates (Proposition 2.7), and Jensen’s inequality (Proposition 2.7).

It is not easy to obtain a nonexistence result in a doubly critical case. In $\text{[3, 9]}$ necessary conditions on $u_0$ were obtained for $f(u) = u^p$, and nonexistence results were established. In Theorem 3.2 we prove a nonexistence theorem for $f_\beta(u) = u^{1+2/N} [\log(u+e)]^\beta$, $\beta > 0$, which needs a more detailed analysis than previous studies. The function $f_\beta$ is not homogeneous, and the function $H(t)$ defined by $\text{[4, 12]}$, which is related to a local $L$-norm of a solution, is a key in the proof of Theorem 1.2. The behavior of $H(t)$ gives a necessary condition for the existence of a nonnegative solution. If we take $\text{(1.1)}$ as an initial condition, then we obtain a contradiction, and a nonexistence theorem for $f_\beta$ is proved. The proof of Theorem C is by contradiction. Suppose that $\text{(1.1)}$ has a nonnegative solution. Using a change of variables, we can construct a supersolution for $\text{(1.1)}$ with $f_\beta$ from a solution of $\text{(1.1)}$. Then, it follows from a monotone iterative method that $\text{(1.1)}$ with $f_\beta$ has a nonnegative solution. However, $\text{(1.1)}$ with $f_\beta$ does not have a nonnegative solution, because of Theorem 1.2. Therefore, the contradiction concludes the proof of Theorem C. Main technical tools in the proofs of Theorems 3.2 and C are the differential inequality $\text{(1.13)}$ and a comparison principle. Theorems B and C are used in the complete proof of the classification for $f_\beta(u)$ (Theorem 1).

In this paper Problem (B) is also studied. Specifically, we obtain growth conditions on $f$ for existence and nonexistence results when the integrability condition $J(u_0) \in L^1_0(\mathbb{R}^N)$ is given. Corollaries 3.2 and 3.3 are derived from Theorem 3.1. Corollaries 1.3 and 1.5 are counterparts of Proposition 1.2 (ii) and Theorem C respectively.

In $\text{[11]}$ a complete characterization for existence and nonexistence of a solution of $\text{(1.1)}$ in $L^r(\Omega)$ was obtained. Their definition of a solution is different from Definition 1.3 and requires that $u \in L^\infty(0, T)$, $L^r(\Omega)$). Only in Section 9 we adopt a similar definition of $\text{(11)}$ which is different from Definition 1.3 and obtain the same characterization in the $L^r_0(\mathbb{R}^N)$ framework.

This paper consists of ten sections. In Section 2 several examples to which Theorem A can be applied are given. We recall basic propositions and prove useful lemmas. They will be used in the proof of our Theorems A, B, C, and D. In Section 3 we prove an abstract existence theorem (Theorem 3.1) and prove Theorem A. Moreover, existence conditions on $f$ are obtained in Corollaries 3.2 and 3.3. In Section 4 we prove Theorem C. Nonexistence conditions on $f$ are obtained in Corollaries 1.3 and 1.5. In Section 5 we study a necessary and sufficient condition for a solvability of $\text{(1.1)}$ in $L^1_0(\mathbb{R}^N)$. Section 6 is devoted
to the proof of Theorem [D]. In Section 7 we prove Theorem [E]. Section 8 is a summary and problems. Sections 9 and 10 are appendices to [11] and [8], respectively.

## 2. Examples and preliminaries

We give four examples and recall known results which are useful in the proof of the main theorems.

### 2.1. Example 1

\( f(u) = \exp(u^p) \), \( p > 0 \). By direct calculation we have

\[
q := \lim_{u \to \infty} f'(u)F(u) = \lim_{u \to \infty} \frac{f'(u)^2}{f(u)f''(u)} = \lim_{u \to \infty} \frac{p}{p + (p - 1)u^p - 1} = 1.
\]

We have

\[
\frac{d}{du} \left( \frac{f'(u)}{f(u)^{1/q}} \right) = p(p-1)u^{p-2}.
\]

We consider the case \( p \geq 1 \). Since \( p \geq 1 \), \( f'(u)/f(u)^{1/q} \) is nondecreasing. Since

\[
\frac{f'(u)}{f(u)^{1/q}} \int_{u}^{\infty} \frac{ds}{f(s)} \leq f(u)^{1/q} \int_{u}^{\infty} \frac{f'(s)ds}{f(s)^{1/q+1}} = f(u)^{1/q} \left[ -qf(s)^{-1/q} \right]_{u}^{\infty} = q,
\]

we see that \( f'(u)F(u) \leq q \). Proposition [A] (i) and (ii) are applicable. Next, we consider the case \( 0 < p < 1 \). Since \( f'(u)/f(u)^{1/q} \) is decreasing, by calculation similar to \((2.1)\) we see that \( f'(u)F(u) > 1 \). Proposition [A] (i) is not applicable, while Theorem [A] (i) and (iii) are applicable. Using Theorem [A] (i) and (iii) and Proposition [A] (ii), we obtain the following:

**Theorem 2.1.** Let \( u_0 \geq 0 \) and \( f(u) = \exp(u^p) \) (\( p > 0 \)). Then the following hold:

(i) The problem \((1.1)\) admits a local in time nonnegative solution if \( F(u_0)^{-r} \in L^1_{ul}(\mathbb{R}^N) \) for some \( r > N/2 \) or \( F(u_0)^{-N/2} \in L^1_{ul}(\mathbb{R}^N) \).

(ii) For each \( r \in (0, N/2) \), there exists a nonnegative initial function \( u_0 \in L^1_{ul}(\mathbb{R}^N) \) such that \( F(u_0)^{-r} \in L^1_{ul}(\mathbb{R}^N) \) and \((1.1)\) admits no nonnegative solution.

By L’Hospital’s rule we see that

\[
\lim_{u \to \infty} \frac{F(u)}{u^{-1}(u + 1)^{-p+1}e^{-u^p}} = 1.
\]

Therefore, \( F(u_0)^{-r} \in L^1_{ul}(\mathbb{R}^N) \) if and only if \((u + 1)\log u \in L^1_{ul}(\mathbb{R}^N) \).

### 2.2. Example 2

\( f(u) = \exp([\log u]^{p-1} \log u) \), \( p > 1 \). By direct calculation we have

\[
q := \lim_{u \to \infty} f'(u)F(u) = \lim_{u \to \infty} \frac{f'(u)^2}{f(u)f''(u)} = \lim_{u \to \infty} \frac{1}{1 + \frac{p-1}{p\log u} - \frac{1}{p\log u^{p-1}}} = 1.
\]

We have

\[
\frac{d}{du} \left( \frac{f'(u)}{f(u)^{1/q}} \right) = p\left( \frac{\log u}{u} \right)^{p-2}\{ (p - 1) \log u < 0 \} \text{ for large } u > 0.
\]

Since \( f'(u)/f(u)^{1/q} \) is decreasing for large \( u > 0 \), by calculation similar to \((2.1)\) we see that \( f'(u)F(u) > 1 \). Proposition [A] (i) is not applicable, while Theorem [A] (i) and (iii) are applicable. Using Theorem [A] (i) and (iii) and Proposition [A] (ii), we obtain the following:

**Theorem 2.2.** Let \( u_0 \geq 0 \) and \( f(u) = \exp([\log u]^{p-1} \log u) \), \( p > 1 \). Then the same statements as Theorem [2.1] hold.

By L’Hospital’s rule we see that

\[
\lim_{u \to \infty} \frac{F(u)}{u^{1+e^{-1}}e^{-u^{p-1}\log u}} = 1.
\]

Therefore, \( F(u_0)^{-r} \in L^1_{ul}(\mathbb{R}^N) \) if and only if

\[
\frac{[\log(u + e)]^{(p-1)r}}{(u + e)^r} e^{r\log u \log u^{p-1}} \in L^1_{ul}(\mathbb{R}^N).
\]
2.3. Example 3. \( f(u) = (u + a)^p / ((p - 1) \log(u + a) - 1) \), \( p > 1 + 2/N \). We define \( a := e^{2/(p - 1)} \) so that \((p - 1) \log(u + a) - 1 \geq 1\) for \( u \geq 0\). Let \( q := p/(p - 1) \). By direct calculation we have

\[
F(u) = \frac{\log(u + a)}{(u + a)^{p - 1}},
\]
hence,

\[
f'(u)F(u) = \frac{p}{p - 1} + \frac{(p - 1)(2p - 1) \log(u + a) - p}{(p - 1)((p - 1) \log(u + a) - 1)^2} \rightarrow q \quad \text{as} \quad u \rightarrow \infty.
\]

Since \( f'(u)F(u) > q \), Proposition \ref{prop:1} (i) is not applicable. The statements of Theorem \ref{thm:A} (i) and (iii) and Proposition \ref{prop:2} (ii) hold. Here,

\[
F(u)^{-r} = \left( \frac{u + a}{(p - 1)r} \right)^{\log(u + a)^{r - 1}}.
\]

2.4. Example 4. The \( n \)-th iterated exponential function. Let \( f(u) := \exp(\cdots \exp(u)) \), \( n \geq 1 \).

It is easy to show that \( q = 1 \) and \( f'(u)^2 / f(u) f''(u) \leq 1 \). See \cite{13} for details. Integrating \( 1 / f(u) \leq f''(u) / f(u)^2 \) over \([u, \infty)\), we have \( f'(u)F(u) \leq 1 \). Using Proposition \ref{prop:2} we obtain the following:

**Theorem 2.3.** Let \( N \geq 1 \), \( u_0 \geq 0 \) and \( f(u) := \exp(\cdots \exp(u)) \), \( n \geq 1 \). Then the same statements as Theorem \ref{thm:2} hold.

2.5. Preliminaries. For any set \( X \) and the mappings \( a = a(x) \) and \( b = b(x) \) from \( X \) to \([0, \infty)\), we say

\[
a(x) \lesssim b(x) \quad \text{for all} \quad x \in X.
\]

We recall a monotone iterative method.

**Proposition 2.4.** Let \( 0 < T \leq \infty \) and let \( f \) be a continuous nondecreasing function such that \( f(0) \geq 0 \). The problem \( \ref{prob:1} \) has a nonnegative solution for \( 0 < t < T \) if and only if \( \ref{prob:1} \) has a nonnegative supersolution \( \bar{u}(t) \in L^\infty([0, T), L^1_{ul}(\mathbb{R}^N)) \cap L^\infty_{loc}((0, T), L^\infty(\mathbb{R}^N)) \). Moreover, if a nonnegative supersolution \( \bar{u}(t) \) exists, then the solution \( u(t) \) obtained satisfies \( 0 \leq u(t) \leq \bar{u}(t) \).

We show the proof for readers’ convenience. See e.g. \cite{20} Theorem 2.1 for details.

**Proof.** If \( \ref{prob:1} \) has a nonnegative solution, then the solution is also a supersolution. Thus, it is enough to show that \( \ref{prob:1} \) has a nonnegative solution if \( \ref{prob:1} \) has a supersolution. Let \( \bar{u} \) be a supersolution for \( 0 < t < T \). Let \( u_1 = S(t)u_0 \). We define \( u_n, n = 1, 2, 3, \ldots \), by

\[
u_n = F[u_{n-1}],
\]

Then we can show by induction that

\[
0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq \bar{u} < \infty \quad \text{for a.e.} \quad x \in \mathbb{R}^N, \quad 0 < t < T.
\]

This indicates that the limit \( \lim_{n \to \infty} u_n(x, t) \) which is denoted by \( u(x, t) \) exists for almost all \( x \in \mathbb{R}^N \) and \( 0 < t < T \). By the monotone convergence theorem we see that

\[
\lim_{n \to \infty} F[u_{n-1}] = F(u),
\]

and hence \( u = F(u) \). It is clear that \( 0 \leq u(t) \leq \bar{u}(t) \). Since \( \bar{u}(t) \in L^\infty((0, T), L^1_{ul}(\mathbb{R}^N)) \cap L^\infty_{loc}((0, T), L^\infty(\mathbb{R}^N)) \), we see that \( u(t) \in L^\infty((0, T), L^1_{ul}(\mathbb{R}^N)) \cap L^\infty_{loc}((0, T), L^\infty(\mathbb{R}^N)) \). Thus, \( u \) is a solution of \( \ref{prob:1} \). \( \square \)

**Proposition 2.5.** The following hold:

(i) Let \( N \geq 1 \) and \( 1 \leq \alpha \leq \beta \leq \infty \). There is \( C > 0 \) and \( t_0 > 0 \) such that, for \( \phi \in L^\alpha_{ul}(\mathbb{R}^N) \),

\[
\| S(t) \phi \|_{L^\beta_{ul}(\mathbb{R}^N)} \leq Ct^-\frac{\alpha}{\beta} \| \phi \|_{L^\alpha_{ul}(\mathbb{R}^N)} \quad \text{for} \quad 0 < t < t_0.
\]

(ii) Let \( N \geq 1 \) and \( 1 \leq \alpha < \beta \leq \infty \). Then, for each \( \phi \in L^\alpha_{ul}(\mathbb{R}^N) \) and \( C_* > 0 \), there is \( t_0 = t_0(C_*, \phi) \) such that

\[
\| S(t) \phi \|_{L^\beta_{ul}(\mathbb{R}^N)} \leq C_* t^-\frac{\alpha}{\beta} \quad \text{for} \quad 0 < t < t_0.
\]

A proof of Proposition \ref{prop:2} which is based on \cite{13} Corollary 3.1 and \cite{2} Lemma 8, can be found in \cite{8} Propositions 2.4 and 2.5. Note that \( C_* > 0 \) in (ii) can be chosen arbitrary small.
Proposition 2.6. Let \(1 \leq \alpha < \infty\). The following are equivalent:

(i) \(\phi \in L_0^\alpha(R^N)\).

(ii) \(\lim_{|y| \to 0} ||\phi(\cdot + y) - \phi(\cdot)||_{L_0^\alpha(R^N)} = 0\).

(iii) \(\lim_{t \to 0} ||S(t)\phi - \phi||_{L_0^\alpha(R^N)} = 0\).

Fundamental properties of \(S(t)\) in \(L_0^\alpha(R^N)\) were studied in [13]. For details of Proposition 2.6 see [13, Proposition 2.14].

Proposition 2.7. (cf. [6] Lemma 2.4) Let \(C \geq 0\). The following (i) and (ii) hold:

(i) Suppose that \(J : [C, \infty) \to [0, \infty)\) is a convex function. If \(\phi \in L_0^1(R^N)\), \(J(\phi) \in L_0^1(R^N)\) and \(\phi \geq C\) in \(R^N\), then
\[
J(S(t)[\phi](x)) \leq S(t)[J(\phi)](x) \quad \text{in } R^N \times (0, \infty).
\]

(ii) Suppose that \(K : [C, \infty) \to [0, \infty)\) is a concave function. If \(\phi \in L_0^1(R^N)\) and \(\phi \geq C\) in \(R^N\), then
\[
K(S(t)[\phi](x)) \geq S(t)[K(\phi)](x) \quad \text{in } R^N \times (0, \infty).
\]

Proposition 2.7 follows from Jensen’s inequality. See [3, Proposition 2.9] for a proof of Proposition 2.7.

Hereafter in this section we collect useful lemmas.

Lemma 2.8. Let \(C > 0\). If \(u \in L_0^1(R^N)\), then \(\max\{u, C\} \in L_0^1(R^N)\).

Proof. Since \(u \in L_0^1(R^N)\), there exists a sequence \(\{u_n\} \subset BUC(R^N)\) such that \(u_n \to u\) in \(L_0^1(R^N)\) as \(n \to \infty\). Let \(\{v_n(x)\}_{n=1}^\infty\) be defined by \(v_n(x) := \max\{u_n(x), C\}\). We see that \(\{v_n\} \subset BUC(R^N)\) and obtain
\[
|\max\{u, C\} - v_n| \leq |u - u_n| \to 0 \quad \text{in } L_0^1(R^N) \quad \text{as } n \to \infty.
\]
Thus \(\max\{u, C\} \in L_0^1(R^N)\).

Lemma 2.9. Let \(q \geq 1\) and \(\varepsilon > 0\). If \(f \in X_q\), then \(F(u) \lesssim u^{-1/(q-1)+\varepsilon}\) for large \(u > 0\).

Proof. Since \(f \in X_q\), we see that \(f'(u)F(u) \leq q + \varepsilon\) for large \(u > 0\). By this together with \(f'(u) = F''(u)/F'(u)^2\) and \(F'(u) = -1/f(u) < 0\) we have
\[
\frac{F''(u)}{F'(u)} \geq \frac{q + \varepsilon}{F'(u)},
\]
which implies that \(-F''(u) \gtrsim F(u)^{q+\varepsilon}\) for large \(u > 0\). Then we obtain \(F(u) \lesssim u^{-1/(q-1)+\varepsilon}\) for large \(u > 0\).

Lemma 2.10. Let \(N \geq 1, \beta > 0\) and \(h_\beta(u) := F_\beta(u)^{-N/2}\). Put \(h_\beta(u) := (4N)^{N/2}u[\log(u + e)]^{-N\beta/2}\). Then \(\hat{h}_\beta(u) \leq h_\beta^{-1}(u)\) for large \(u > 0\).

Proof. Let \(u > 0\) be sufficiently large. Then we have
\[
F_\beta(u) = \int_u^\infty \frac{d\tau}{\tau^{1+2/N}[\log(\tau + e)]^\beta} \geq \frac{N}{4} \int_u^\infty \frac{\frac{2}{N} \log(\tau + e)^{N/2} + \frac{1}{N+1}}{\tau^{1+2/N}[\log(\tau + e)]^{N\beta/2}} d\tau = \frac{N}{4} u^{-2/N}[\log(u + e)]^{-\beta}.
\]
Hence, we obtain \(h_\beta(u) \leq (4N)^{N/2}u[\log(u + e)]^{N\beta/2}\). We see that
\[
h_\beta(\hat{h}_\beta(u)) \leq \left(\frac{4}{N}\right)^{N/2} \hat{h}_\beta(u)[\log(\hat{h}(u) + e)]^{N\beta} u[\log(u + e)]^{-N\beta/2} \leq u.
\]
Since \(h_\beta\) is increasing, \(\hat{h}_\beta(u) \leq h_\beta^{-1}(u)\) for large \(u > 0\).

Lemma 2.11. Let \(N \geq 1\). Suppose that \(f\) satisfies all the assumptions of Theorem 3. Let \(h(u) := F(u)^{-N/2}\). Then there exists \(0 < \delta' < N/2\) such that \(h^{-1}(u) \lesssim u[\log(u + e)]^{\delta'}\) for large \(u > 0\).

Proof. Let \(\delta\) be given by the assumption of Theorem 3. Since \(0 < \delta < 1\), we choose \(\varepsilon > 0\) such that \(N\delta/2 + \varepsilon < N/2\). Put \(\delta' := N\delta/2 + \varepsilon > 0\) and \(\hat{h}(u) := C_2^{N/2}u[\log(u + e)]^{\delta'}\). Since \(h(u) \geq C_2^{N/2}u[\log(u + e)]^{-N\delta/2}\) for \(u \geq C_1\), we obtain
\[
(2.2) \quad h(\hat{h}(u)) \geq C_2^{-N/2}h(u)[\log(\hat{h}(u) + e)]^{-N\delta} = u \left(\frac{\log(u + e)}{\log(h(u) + e)}\right)^{\frac{N\delta}{2}} [\log(u + e)]^{\varepsilon} \quad \text{for large } u > 0.
\]
We see that
\[ \log u = \log h(u) \quad \text{for large } u \to \infty, \]
which yields
\[ \left( \frac{\log(u + e)}{\log(h(u) + e)} \right)^{\frac{2}{\delta}} = \left( \frac{\log(u + e) - \log u}{\log h(u) - \log h(u) + e} \right)^{\frac{2}{\delta}} \to 1 \quad \text{as } u \to \infty. \]
By this together with (2.2) we have \( h(\hat{h}(u)) \geq u \) for large \( u > 0 \). Since \( h \) is increasing, \( h^{-1}(u) \leq \hat{h}(u) \) for large \( u > 0 \).

### 3. Existence

In this section a function \( J \) satisfies the following:

\( J \in C^2[0, \infty), \lim_{u \to \infty} J(u) = \infty, \ J(u) > 0 \) for \( u > 0 \), \( J'(u) > 0 \) for \( u > 0 \)

and \( J'(u) \) is nondecreasing for large \( u > 0 \).

The main theorem in this section is the following:

**Theorem 3.1.** Let \( N \geq 1 \) and \( u_0 \geq 0 \). Suppose that \( f \in C[0, \infty), \ f \) is nonnegative and \( f \) is nondecreasing for \( u > 0 \) and that \( J \) satisfies [4]. Suppose that there exist \( \theta \in (0, 1] \) and \( \xi \geq 0 \) such that one of the following holds:

(i) \( J(u_0) \in L^1_u({\mathbb{R}}^N) \) and

\[
\lim_{\eta \to \infty} \tilde{J}(\eta) \int_{\eta}^{\infty} \frac{\tilde{f}(\tau)J'(\tau)d\tau}{J(\tau)^{1+2/N}} = 0,
\]

(ii) \( J(u_0) \in L^1_u({\mathbb{R}}^N) \) and

\[
\limsup_{\eta \to \infty} \tilde{J}(\eta) \int_{\eta}^{\infty} \frac{\tilde{f}(\tau)J'(\tau)d\tau}{J(\tau)^{1+2/N}} < \infty,
\]

where

\[
\tilde{f}(u) := \sup_{\xi \leq \tau \leq u} \frac{f(\tau)}{\tau^\theta} \quad \text{and} \quad \tilde{J}(u) := \sup_{\xi \leq \tau \leq u} \frac{J'(\tau)}{\tau^{1-\theta}}.
\]

Then \( J(u) \) admits a local in time nonnegative solution \( u(t) \) for \( 0 < t < T \). Moreover, there exists \( C_0 > 0 \) such that

\[
\| J(u(t)) \|_{L^1_u({\mathbb{R}}^N)} \leq C_0 \quad \text{for } 0 < t < T.
\]

**Proof.** Let \( C_1 > 0 \) be large such that \( J(u) \) is convex for \( u \geq C_1 \). Let \( \sigma > 0 \) be a constant and \( u_1(\cdot) := \max \{ u_0(\cdot), C_1, 1, \xi \} \). We see that \( J(u_1) \in L^1_u({\mathbb{R}}^N) \) in the case (i). By Lemma 2.8 we see that \( J(u_1) \in L^1_u({\mathbb{R}}^N) \) in the case (ii). Since \( J \) is convex for \( u > C_1 \), in two cases (i) and (ii) \( J(u_1) \in L^1_u({\mathbb{R}}^N) \) implies \( u_1 \in L^1_u({\mathbb{R}}^N) \), and hence it follows from Proposition 2.5 that \( \| S(t)u_1 \|_{L^\infty} < \infty \) for \( t > 0 \). The first term of \( F \) is well defined.

We define

\[
\tilde{u}(t) := J^{-1}((1 + \sigma)S(t)J(u_1)).
\]

We show that

\[
\tilde{u}(t) \in L^\infty((0, T), L^1_u({\mathbb{R}}^N)) \cap L^\infty((0, T), L^\infty({\mathbb{R}}^N))
\]

for sufficiently small \( T > 0 \). Since \( J(u_1) \in L^1_u({\mathbb{R}}^N) \), it follows from Proposition 2.5 that

\[
\| \tilde{u}(t) \|_{L^\infty({\mathbb{R}}^N)} \leq J^{-1}((1 + \sigma) \| S(t)J(u_1) \|_{L^1_u({\mathbb{R}}^N)}) < \infty \quad \text{for small } t > 0.
\]

Hence, \( \tilde{u}(t) \in L^\infty((0, T), L^\infty({\mathbb{R}}^N)) \) for small \( T > 0 \). Since \( J^{-1}(u) \) is concave for \( u \geq J(C_1) \), by [45] and Proposition 2.5 (i) we have

\[
\| \tilde{u}(t) \|_{L^1_u({\mathbb{R}}^N)} \leq \| S(t)J(u_1) \|_{L^1_u({\mathbb{R}}^N)} + \| S(t)J(u_1) \|_{L^1_u({\mathbb{R}}^N)} + \| 1 \|_{L^1_u({\mathbb{R}}^N)} \leq \| u_1 \|_{L^1_u({\mathbb{R}}^N)} + 1,
\]

and hence \( \tilde{u} \in L^\infty((0, T), L^1_u({\mathbb{R}}^N)) \). We have proved 3.6.
By Proposition 2.7 (i) we have
\[ \bar{u}(t) - S(t)u_0 \geq \bar{u}(t) - S(t)\bar{u}(t) - J^{-1}(S(t)J(u_1)) \]
\[ = J^{-1}((1 + \sigma)S(t)J(u_1)) - J^{-1}(S(t)J(u_1)) = (J^{-1}')((1 + \rho\alpha)S(t)J(u_1)) \sigma S(t)J(u_1) \]
for some \( \rho = \rho(x, t) \in [0, 1]. \) Here, we used the mean value theorem. Since \( J(u) \) is convex for \( u \geq C_1, \) \( J^{-1}(u) \) is concave for \( u \geq J(C_1). \) We have
\[ \text{By (3.7) and (3.8) we have} \]
\[ \bar{u}(t) - S(t)u_0 \geq \frac{\sigma}{1 + \sigma} J(\bar{u}(t)) \]
On the other hand, let \( s \in (0, t). \) Since \( 0 < \theta \leq 1, \) it follows from Proposition 2.7 (ii) that
\[ S(t - s)J(\bar{u}(s)) = S(t - s) \left( (1 + \sigma)S(s)J(u_1) \right)^\theta \]
\[ \leq \left\{ S(t - s) \left( (1 + \sigma)S(s)J(u_1) \right) \right\}^\theta \leq \left\{ (1 + \sigma)S(t)J(u_1) \right\}^\theta = J(\bar{u}(t)) \theta. \]
Using (3.10), we have
\[ \int_0^t S(t - s)f(\bar{u}(s))ds \leq \int_0^t S(t - s) \left[ \frac{f(\bar{u}(s))}{J(\bar{u}(s))} \right] \theta ds \]
\[ \leq \int_0^t \left\| \frac{f(\bar{u}(s))}{J(\bar{u}(s))} \right\|^{\theta} \left\{ S(t - s)J(\bar{u}(s)) \right\}^\theta \leq J(\bar{u}(t)) \theta \int_0^t \left\| \frac{f(\bar{u}(s))}{J(\bar{u}(s))} \right\| ds. \]
We prove the case (i). We have
\[ J(\bar{u}(t)) \theta \int_0^t \left\| \frac{f(\bar{u}(s))}{J(\bar{u}(s))} \right\| ds \leq J(\bar{u}(t)) \theta \int_0^t \left\| \frac{f(\bar{u}(s))}{J(\bar{u}(s))} \right\| ds \]
We define \( \eta \) by
\[ \eta := J^{-1}(C(1 + \sigma)t^{-\frac{N}{2}}\|J(u_1)\|_{L^1_{loc}(\mathbb{R}^N)}). \]
Since \( \|\bar{u}(t)\|_{\infty} \leq \eta, \) we have
\[ \bar{J}(\|\bar{u}(t)\|_{\infty}) \int_0^t \bar{f}(\|\bar{u}(s)\|_{\infty})ds \leq \frac{2}{N}(C(1 + \sigma)\|J(u_1)\|_{L^1_{loc}(\mathbb{R}^N)})^{2/N} \bar{J}(\eta) \int_\eta^\infty \frac{\bar{f}(\tau)J'(\tau)d\tau}{J(\tau)^{1+2/N}}, \]
where we used a change of variables \( \tau := J^{-1}(C(1 + \sigma)s^{-\frac{N}{2}}\|J(u_1)\|_{L^1_{loc}(\mathbb{R}^N)}). \) Because of (3.11), there exists a large \( \eta > 0 \) such that
\[ \frac{2}{N} \{ C(1 + \sigma) \|J(u_1)\|_{L^1_{loc}(\mathbb{R}^N)} \}^{2/N} \bar{J}(\eta) \int_\eta^\infty \frac{\bar{f}(\tau)J'(\tau)d\tau}{J(\tau)^{1+2/N}} \leq \frac{\sigma}{1 + \sigma}. \]
Therefore, if \( \eta > 0 \) is large, then by (3.14), (3.15), (3.12) and (3.11) we have
\[ \int_0^t S(t - s)f(\bar{u}(s))ds \leq \frac{\sigma}{1 + \sigma} J(\bar{u}(t)) \]
By (3.16) and (3.9) we have
\[ \bar{u}(t) - S(t)u_0 \geq \int_0^t S(t - s)f(\bar{u}(s))ds. \]
Since \( \eta \) and \( t \) are related by (3.13), we have shown that there is a small \( t > 0 \) such that \( \bar{u}(t) \) is a supersolution which satisfies (3.6). By Proposition 2.4 we see that (1.1) has a nonnegative solution \( u(t) \) and that \( u(t) \leq \bar{u}(t) \) for \( 0 < t < T. \) Since \( \sigma > 0 \) is a constant, (3.11) follows from (3.9).
calculation as (3.14) we have (3.14) with $C$ replaced by $C_*$. Because of (3.2) and Proposition 2.5(ii) we can take $C_*>0$ such that if $0 < t < t_0(C_*)$, then

$$
J \left( f'(u) \right) \geq 0 \text{ for large } u.
$$

In the case (ii) we see that $r + 1 = q \geq f'(u)F(u)$ for large $u$, and hence $J''(u) \geq 0$ for large $u$.

Next, we show that

$$
\frac{d}{du} \left( \frac{f(u)}{J(u)^{1-\theta}} \right) \geq 0 \text{ for large } u \text{ and } \frac{d}{du} \left( \frac{J'(u)}{J(u)^{1-\theta}} \right) \geq 0 \text{ for large } u.
$$

We consider the cases (i) and (iii). Then, $1 \leq q < 1 + r$ and $r \geq N/2$, and there exists $\theta > 0$ such that $\frac{1}{\theta} < \theta < \min \{1, 2\}$. Then,

$$
\frac{d}{du} \left( \frac{J'(u)}{J(u)^{1-\theta}} \right) = (1 - \theta)J(u)\theta^{-1}J''(u) \left( \frac{1}{1 - \theta} - \frac{J'(u)^2}{J(u)^{1-\theta}} \right).
$$

Then,

$$
\frac{J'(u)^2}{J(u)^{1-\theta}} = \frac{r}{r + 1 - q} F(u) \to \frac{r}{r + 1 - q} \text{ as } u \to \infty.
$$

We see that $\frac{1}{\theta} = \frac{r}{r - 1 - q} > 0$, and hence $J'(u)/J(u)^{1-\theta}$ is nondecreasing. Thus, (3.19) holds in the cases (i) and (iii). We consider the case (ii). Then $q = 1 + r$ and $r > N/2$. We take $\theta = 1$. Since $q - r = 1$, (3.20) holds. Since $\theta = 1$, by (1.3) we see that

$$
\frac{d}{du} \left( \frac{J'(u)}{J(u)^{1-\theta}} \right) = J''(u) = \frac{r(q - f'(u)F(u))}{f(u)^2F(u)^{r+2}} \geq 0 \text{ for large } u.
$$

Thus, (3.19) holds in the case (ii).

We prove (i) and (ii). Then, $r > N/2$. We check (3.1). Because of (3.19), we can take $\xi > 0$ such that $f(u)/J(u)^\theta$ and $J'(u)/J(u)^{1-\theta}$ are nondecreasing for $u > \xi$. If $\eta > \xi$ is large, then

$$
J(\eta) \int_\eta^\infty \frac{\tilde{f}(\tau)J'(\tau)}{J(\tau)^{1-2/N}} d\tau \leq \int_\eta^\infty \frac{f(\tau)J'(\tau)J(\tau)^{2/N}}{J(\tau)^{2+2/N}} r^2 \int_\eta^\infty \frac{F(\tau)^{2r/N-2}}{f(\tau)} d\tau = r^2 F(\eta)^{2r/N-1} 2r/N - 1 \to 0 \text{ as } \eta \to \infty.
$$

Since $J$ satisfies all the assumptions of Theorem 3.1 (i), by Theorem 3.1 (i) we see that (1.1) has a nonnegative solution and (1.3) holds.

We prove (iii). Then, $r = N/2$. We check (3.2). Since $q - 1 < r\theta$, we can choose $\varepsilon > 0$ such that $q - 1 + \varepsilon < r\theta$. By Lemma 2.5 we see that

$$
\int_\eta^\infty F(\tau)^{r\theta} d\tau < C \int_\eta^\infty \tau^{-r\theta/(q - 1 + \varepsilon)} d\tau < \infty.
$$

Integrating $f'(u)/f(u) \leq (q + \varepsilon)/f(u)F(u)$, we have $f(u) \leq CF(u)^{-q - \varepsilon}$ for large $u$, and hence

$$
f(u)F(u)^{1+r\theta} \leq CF(u)^{-q - \varepsilon + 1+r\theta} \to 0 \text{ as } u \to \infty.
$$

Proof of Theorem A. In three cases (i) (ii) and (iii) we easily see that $f \in C[0, \infty), f$ is nonnegative and $f$ is nondecreasing for $u \geq 0$, since $f \in X_\eta$. Let $J(u) := F(u)^{-r}$. We show that $J$ satisfies (3.6). It is enough to show that $J''(u) \geq 0$ for large $u$, since other properties follow from the definition of $J(u)$. In the cases (i) and (iii) we see that $q < 1 + r$, and hence

$$
J''(u) = \frac{r(r + 1 - f'(u)F(u))}{f(u)^2F(u)^{r+2}} \geq 0 \text{ for large } u.
$$

Since $\xi > \varepsilon > 0$ such that $f(u)/J(u)^\theta$ and $J'(u)/J(u)^{1-\theta}$ are nondecreasing for $u > \xi$. If $\eta > \xi$ is large, then

$$
J(\eta) \int_\eta^\infty \frac{\tilde{f}(\tau)J'(\tau)}{J(\tau)^{1-2/N}} d\tau \leq \int_\eta^\infty \frac{f(\tau)J'(\tau)J(\tau)^{2/N}}{J(\tau)^{2+2/N}} r^2 \int_\eta^\infty \frac{F(\tau)^{2r/N-2}}{f(\tau)} d\tau = r^2 F(\eta)^{2r/N-1} 2r/N - 1 \to 0 \text{ as } \eta \to \infty.
$$

Since $J$ satisfies all the assumptions of Theorem 3.1 (i), by Theorem 3.1 (i) we see that (1.1) has a nonnegative solution and (1.3) holds.

We prove (iii). Then, $r = N/2$. We check (3.2). Since $q - 1 < r\theta$, we can choose $\varepsilon > 0$ such that $q - 1 + \varepsilon < r\theta$. By Lemma 2.5 we see that

$$
\int_\eta^\infty F(\tau)^{r\theta} d\tau < C \int_\eta^\infty \tau^{-r\theta/(q - 1 + \varepsilon)} d\tau < \infty.
$$

Integrating $f'(u)/f(u) \leq (q + \varepsilon)/f(u)F(u)$, we have $f(u) \leq CF(u)^{-q - \varepsilon}$ for large $u$, and hence

$$
f(u)F(u)^{1+r\theta} \leq CF(u)^{-q - \varepsilon + 1+r\theta} \to 0 \text{ as } u \to \infty.
Because of (3.19), we can take \( \xi > 0 \) such that \( f(u)/J(u)^\theta \) and \( J'(u)/J(u)^{1-\theta} \) are nondecreasing for \( u \geq \xi \). If \( \eta > \xi \) is large, then

\[
\frac{J'(\eta)}{J(\eta)^{1-\theta}} \int_\eta^\infty \frac{f(\tau)J'(\tau)d\tau}{J(\tau)^{1-\theta+2/N}} = \frac{J'(\eta)}{J(\eta)^{1-\theta}} \int_\eta^\infty \frac{f(\tau)J'(\tau)d\tau}{J(\tau)^{1-\theta+2/N}} = \frac{r^2}{f(\eta)F(\eta)^{1+\theta}} \int_\eta^\infty F(\tau)^{\theta}d\tau.
\]

Because of (3.21) and (3.22), L'Hospital's rule is applicable in the following limit:

\[
\lim_{\eta \to \infty} \frac{F(\tau)^{\theta}d\tau}{f(\eta)F(\eta)^{1+\theta}} = \lim_{\eta \to \infty} \frac{\frac{d}{d\eta} \left( \frac{F(\tau)^{\theta}d\tau}{f(\eta)F(\eta)^{1+\theta}} \right)}{1} = \lim_{\eta \to \infty} \frac{-1}{r\theta + 1 - q} = \frac{1}{r\theta + 1 - q} > 0.
\]

By Theorem 3.4 (ii) we see that (1.1) has a nonnegative solution and that (1.9) holds.

\[\square\]

**Proof of Theorem 3.5**. In two cases (i) and (ii) we easily see that \( f \in C[0, \infty) \), \( f \) is nonnegative and \( f \) is nondecreasing for \( u > 0 \), since \( f \in X_q \). Let \( J(u) := J_s(u) \) and \( \theta = 1 \). In both cases (i) and (ii) we have

\[
\frac{d}{du} \left( \frac{f(u)}{J(u)^\theta} \right) = \frac{h(u)^{-1+2/N} \left( f'(u)F(u) - N/2 \left( 1 + \frac{\alpha h(u)}{(h(u) + e) \log(h(u) + e)} \right) \right)}{[\log(h(u) + e)]^2} \geq 0 \quad \text{for large } u,
\]

since \( f'(u)F(u) \to 1 + N/2 \) as \( u \to \infty \). In the case (i) we see that \( \frac{d}{du} \left( \frac{J'(u)}{J(u)^{1+\theta}} \right) = J''(u) \geq 0 \), since \( J \) is convex. In the case (ii) we let \( \tilde{\rho} > \rho \). By direct calculation we have

\[
\frac{d}{du} \left( \frac{f(u)}{J(u)^\theta} \right) = \frac{N}{2f(u)} \frac{[\log(h(u) + e)]^{\alpha - 1}}{h(u) + e} \cdot j(u),
\]

where

\[
j(u) := \frac{N}{2} \tilde{\alpha} \tilde{\rho} \left( 1 + \frac{e}{h(u) + e} + \frac{(\alpha - 1)h(u)}{(h(u) + e) \log(h(u) + e)} \right) + (q - f'(u)F(u)) \left( 1 + \frac{e}{h(u)} \log(h(u) + e) + \alpha \right) \geq \frac{N}{2} \tilde{\alpha} \tilde{\rho} \left( 1 + \frac{e}{h(u) + e} + \frac{(\alpha - 1)h(u)}{(h(u) + e) \log(h(u) + e)} \right) - \frac{N}{2} \tilde{\alpha} \tilde{\rho} \left( 1 + \frac{e}{h(u)} + \frac{\alpha}{\log(h(u) + e)} \right) > 0 \quad \text{as } u \to \infty.
\]

Note that we use (1.5). Considering the case where \( \tilde{\rho} = 1 \), we see that \( J''(u) \geq 0 \) for large \( u > 0 \). In both cases (i) and (ii) we have checked (J) and (3.19).

We prove (i). Because of (3.19), we can take a large \( \xi > 0 \) such that \( \tilde{f}(u) = f(u)/J(u)^\theta \) for \( u \geq \xi \) and \( \tilde{J}(u) = J'(u)/J(u)^{1-\theta} \) for \( u \geq \xi \). Then,

\[
\frac{J'(\eta)}{J(\eta)^{1-\theta}} \int_\eta^\infty \frac{f(\tau)J'(\tau)d\tau}{J(\tau)^{1-\theta+2/N}} \leq \frac{N}{2} \frac{2c}{\sigma + e}[\log(\sigma + e)]^{2\alpha/N} \leq \frac{NC}{2\alpha/N - 1}[\log(h(\eta) + e)]^{1-2\alpha/N} \to 0 \quad \text{as } \eta \to \infty,
\]

and hence (5.1) holds. By Theorem 5.4 (i) we see that (1.1) has a nonnegative solution and that (1.9) holds.

We prove (ii). Because of (3.19), we can take a large \( \xi > 0 \) such that \( \tilde{f}(u) = f(u)/J(u)^\theta \) for \( u \geq \xi \) and \( \tilde{J}(u) = J'(u)/J(u)^{1-\theta} \) for \( u \geq \xi \). We choose \( \tilde{\rho} \in (\rho, 1) \). It follows from (3.23) and (3.24) that \( g'(h(u))h'(u) \) is increasing for large \( u > 0 \). When \( \eta > 0 \) is large, we have

\[
\frac{J'(\eta)}{J(\eta)^{1-\theta}} \int_\eta^\infty \frac{f(\tau)J'(\tau)d\tau}{J(\tau)^{1+\theta+2/N}} = \frac{g'(h(\eta))}{g(h(\eta))} \int_\eta^\infty \frac{f(\tau)g'(h(\eta))h'(\tau)d\tau}{g(h(\eta))} \leq \frac{N}{2} \frac{g'(h(\eta))}{g(h(\eta))} \int_\eta^\infty \frac{\sigma^{1+2/N}}{g(\sigma)^{1+2/N}} \frac{d\sigma}{g(h(\eta))} \leq \frac{N}{2} \frac{C[\log(h(\eta) + e)]^{\alpha-1}}{1 - \tilde{\rho}} \int_{h(\eta)}^{\infty} \frac{\sigma^{1+2/N}}{g(\sigma)^{1+2/N}} \frac{d\sigma}{g(h(\eta))} \leq \frac{C}{1 - \tilde{\rho}}.
\]

By Theorem 3.4 (ii) we see that (1.1) has a nonnegative solution and that (1.9) holds.
**Corollary 3.2.** Let \( u_0 \geq 0 \). Suppose that \( f \in C[0, \infty) \), \( f \) is nonnegative and \( f \) is nondecreasing for \( u > 0 \). Suppose that \( J \in C^2[0, \infty) \) satisfies (J) and

\[
(3.25) \quad \text{the limit } \gamma := \lim_{u \to \infty} \frac{J(u)J''(u)}{J'(u)^2} \text{ exists.}
\]

If there exists a small \( \varepsilon > 0 \) such that

\[
(3.26) \quad \limsup_{u \to \infty} \frac{f(u)J'(u)}{J(u)^{1+2/N-\varepsilon}} < \infty,
\]

then \( J \) admits a local in time nonnegative solution for \( u_0 \) satisfying \( J(u_0) \in L^1_{ul}(\mathbb{R}^N) \).

**Proof.** We define \( q_J \) by

\[
q_J := \lim_{u \to \infty} \frac{J'(u)^2}{J(u)J''(u)},
\]

which is a conjugate exponent of the growth rate of \( J \). In [6,14] it was shown that \( 1 \leq q_J \leq \infty \) if the limit \( q_J \) exists. Therefore, \( 0 \leq \gamma \leq 1 \).

Let \( \tilde{f}(u) := J(u)^{1+2/N-\gamma}/J'(u) \). First, we consider the case \( 0 < \gamma \leq 1 \). Since \( 0 < \gamma \leq 1 \), we can take \( \theta \in (0, 1] \) such that \( 1 - \gamma < \theta < 1 - \gamma + 2/N - \varepsilon \). Then, for large \( u > 0 \),

\[
(3.27) \quad \frac{d}{du} \left( \frac{f(u)}{J(u)^\theta} \right) = J(u)^{1+2/N-\theta-\varepsilon} \left( 1 + \frac{2}{N} - \theta - \varepsilon - \frac{J(u)''(u)}{J(u)'} \right) > 0,
\]

\[
\frac{d}{du} \left( \frac{J'(u)}{J(u)^{1-\theta}} \right) = J(u)^2 \left( \frac{J(u)''(u)}{J(u)'^2} - \frac{1}{\theta} \right) > 0.
\]

Let \( \xi \) be large, and let \( \tilde{f} \) and \( \tilde{J} \) be defined by (3.26). Then, there exists \( \xi > 0 \) such that

\[
(3.28) \quad \tilde{f}(u) = \frac{\tilde{f}(u)}{J(u)^\theta} \quad \text{for } u \geq \xi \quad \text{and} \quad \tilde{J}(u) = \frac{J'(u)}{J(u)^{1-\theta}} \quad \text{for } u \geq \xi.
\]

Second, we consider the case \( \gamma = 0 \). Let \( \theta = 1 \). Then we see that \( (3.27) \) holds for large \( \tau > 0 \). Since \( \theta = 1 \),

\[
\frac{d}{d\tau} \left( \frac{J'(u)}{J(u)^{1-\theta}} \right) = J''(u) \geq 0 \quad \text{for large } u > 0.
\]

Then there exists \( \xi > 0 \) such that \( (3.28) \) holds.

We prove the corollary in the case \( 0 \leq \gamma \leq 1 \). It follows from (3.26) that

\[
(3.29) \quad f(u) \leq C_0 \tilde{f}(u) \quad \text{for large } u > 0.
\]

Since

\[
\tilde{J}(\eta) \int_{\eta}^{\infty} \tilde{f}(\tau) J'(\tau) d\tau = \int_{\eta}^{\infty} \hat{J}(\tau)^{1+2/N-\gamma} J'(\tau)^2 d\tau = \int_{\eta}^{\infty} \hat{J}'(\tau) d\tau \quad \text{as } \eta \to \infty,
\]

by (3.29) we see that \( (3.3) \) holds for \( f \). Hence, it follows from Theorem 3.3.1 (i) that \( J \) with \( f \) admits a nonnegative solution if \( J(u_0) \in L^1_{ul}(\mathbb{R}^N) \). The proof is complete.

**Corollary 3.3.** Let \( u_0 \geq 0 \). Suppose that \( f \in C[0, \infty) \), \( f \) is nonnegative and \( f \) is nondecreasing for \( u > 0 \). Suppose that \( J \in C^2[0, \infty) \) satisfies (J) and

\[
(3.30) \quad \lim_{u \to \infty} \frac{J(u)J''(u)}{J'(u)^2} = 0.
\]

If there exists \( \gamma > N/2 \) such that

\[
(3.31) \quad \limsup_{u \to \infty} \frac{f(u)J'(u)[\log(J(u) + e)]^{2\gamma/N}}{J(u)^{1+2/N}} < \infty,
\]

then \( J \) admits a local in time nonnegative solution for \( u_0 \) satisfying \( J(u_0) \in L^1_{ul}(\mathbb{R}^N) \).

**Proof.** Let \( \tilde{f}(u) = J(u)^{1+2/N}/J'(u)[\log(J(u) + e)]^{2\gamma/N} \) and let \( \theta = 1 \). By the same argument as in the proof of Corollary 3.2 we see that \( \tilde{J}(u) = J'(u)/J(u)^{1-\theta} \) for \( u \geq \xi \) if \( \xi > 0 \) is large. Since

\[
\frac{d}{d\tau} \left( \frac{J(u)^{2/N}}{J(u)[\log(J(u) + e)]^{2\gamma/N}} \right) = \left( \frac{J(u)^{2/N-1} \log(J(u) + e)}{J'(u)} \right) \left( \frac{J(u)J''(u)}{J(u)^2} - 2\gamma \frac{J'(u)}{J(u) + e}\log(J(u) + e) \right) > 0
\]
for large $u > 0$, we see that $\tilde{f}(u) = \tilde{f}(u)/J(u)^\beta$ for $u \geq \xi$ if $\xi > 0$ is large. Thus, there exists $\xi > 0$ such that (3.28) holds.

It follows from (3.31) that

\[(3.32) \quad f(u) \leq C_0 \tilde{f}(u) \quad \text{for large } u > 0.\]

Since

\[
\hat{J}(\eta) \int_0^\infty \frac{\tilde{f}(\tau) J'(\tau) d\tau}{J(\tau)^{1+2/N}} \leq \int_0^\infty \frac{J(\tau)^{1+2/N-\theta} J'(\tau)^2 d\tau}{J(u)^{2+2/N-\theta} J(u) |\log(J(u) + e)|^{2+2/N}} \leq \frac{C}{2\gamma/N - 1} |\log(J(\eta) + e)|^{1-2\gamma/N} \to 0 \text{ as } \eta \to \infty.
\]

by (3.32), we see that (3.1) holds for $f$. Hence, it follows from Theorem 3.1 (i) that (1.11) with $f$ admits a nonnegative solution if $J(u_0) \in L^1_{ul}(\mathbb{R}^N)$. The proof is complete. □

4. Nonexistence

In this section let $N \geq 1$ and $0 \leq \alpha < N/2$. We begin to consider the case where $f(u) = f_\beta(u), \beta > 0$.

We recall that

\[
F_\beta(u) := \int_u^\infty \frac{dr}{f_\beta(r)} \quad \text{and} \quad h_\beta(u) := F_\beta(u)^{-N/2}.
\]

Let $\varepsilon \in (0, N/2 - \alpha)$. There exists $C_0 > 0$ such that $f_\beta(u)$ is convex on $[C_0, \infty)$. Then there exists $m \in (0, 1/e)$ such that $|x|^{-N}(\log(1/|x|))^{-N/2} \geq h_\beta(C_0)$ for $|x| \leq m$. We define

\[(4.1) \quad u_0(x) := \begin{cases} \hbar^{-1}(|x|^{-N} \log(1/|x|))^{-N/2-1+\varepsilon} & \text{if } |x| \leq m, \\ \hbar^{-1}(m^{-N} \log(1/m)^{-N/2-1+\varepsilon}) & \text{if } |x| > m. \end{cases}
\]

Lemma 4.1. Let $J_\alpha(u)$ and $g(u)$ be defined by (1.4). Let $f(u) = f_\beta(u), \beta > 0$ and let $u_0(x)$ be defined by (4.1). Then, the following hold:

(i) $J_\alpha(u_0) = g(h_\beta(u_0)) \in L^1_{ul}(\mathbb{R}^N)$.

(ii) $u_0 \in L^1_{ul}(\mathbb{R}^N)$.

Proof. (i) Let $\rho \in (0, m]$ be fixed. It suffices to prove $\int_{B(0, \rho)} J_\alpha(u_0(x)) dx < \infty$. If $|x| \leq m$, then

\[(4.2) \quad \log |x|^{-N} \left( \log \frac{1}{|x|} \right)^{-N/2-1+\varepsilon} \leq \log |x|^{-N} \left( \log \frac{1}{|x|} \right)^{-N/2-1+\varepsilon} \leq 2N \log \frac{1}{|x|},
\]

which yields

\[
J_\alpha(u_0(x)) = g \left( |x|^{-N} \left( \log \frac{1}{|x|} \right)^{-N/2-1+\varepsilon} \right) \leq |x|^{-N} \left( \log \frac{1}{|x|} \right)^{-N/2-1+\varepsilon} \left( 2N \log \frac{1}{|x|} \right)^\alpha.
\]

We deduce that

\[
\int_{B(0, \rho)} J_\alpha(u_0(x)) dx \leq \int_0^\rho \frac{1}{r} \left( \log \frac{1}{r} \right)^{-N/2-1+\varepsilon + \alpha} dr = \left( \log \frac{1}{\rho} \right)^{-N/2-1+\varepsilon + \alpha} < \infty.
\]

Thus, $J_\alpha(u_0) \in L^1_{ul}(\mathbb{R}^N)$.

(ii) Let $\{\phi_n(x)\}_{n=1}^\infty$ be defined by

\[
\phi_n(x) := \begin{cases} u_0(2^{-n}m) & \text{if } |x| \leq 2^{-n}m, \\ u_0(x) & \text{if } |x| > 2^{-n}m. \end{cases}
\]

We see that $\{\phi_n\} \subset \text{BUC}(\mathbb{R}^N)$. Since $f'(u)F(u) \leq q$ for large $u > 0$, $h_\beta^{-1}(u)$ is concave for large $u > 0$, which implies that $h_\beta^{-1}(u) \leq u$ for large $u > 0$. Therefore, if $n$ is large, then

\[
\int_{\mathbb{R}^N} |u_0(x) - \phi_n(x)| dx \leq \int_{B(0, 2^{-n}m)} h_\beta^{-1} \left( |x|^{-N} \left( \log \frac{1}{|x|} \right)^{-N/2-1+\varepsilon} \right) dx \
\leq \int_0^{2^{-n}m} \frac{1}{r} \left( \log \frac{1}{r} \right)^{-N/2-1+\varepsilon} dr = \frac{1}{N/2 - \varepsilon} \left( \log \frac{2n}{m} \right)^{-N/2+\varepsilon} \to 0 \text{ as } n \to \infty.
\]
Thus, \( u_0 \in L^1_{ul}(\mathbb{R}^N) \).

**Theorem 4.2.** Let \( N \geq 1, 0 \leq \alpha < N/2 \) and \( \beta > 0 \). Let \( u_0 \) be defined by (4.1). Then (1.7) with \( f(u) = f_{\beta}(u) \) admits no local in time nonnegative solution.

We postpone the proof of Theorem 4.2.

**Proof of Theorem 4.2.** We construct an initial function \( u_0 \). Choose \( \varepsilon \) such that \( 0 < \varepsilon < N/2 - \max\{\alpha, \delta'\} \), where \( 0 < \delta' < N/2 \) is chosen later. We also choose \( C_0 \geq C_1 \) such that \( f_{\beta}(u) \) is convex on \([C_0, \infty)\). Then we define \( v_0(x) \) by the right hand side of (4.1). Put \( J := F^{-1} \circ f_{\beta} \) and \( u_0(x) := J(v_0(x)) \).

We see that \( u_0(x) = h^{-1} \left( |x|^{-N} \left( \log \frac{1}{|x|} \right)^{-N/2-1+\varepsilon} \right) \) for \( |x| \leq m \). Then we can obtain \( J_\alpha(u_0) \in L^1_{ul}(\mathbb{R}^N) \) in the same way as Lemma 4.1 (i).

We show that \( u_0 \in L^1_{ul}(\mathbb{R}^N) \). It suffices to prove \( \int_{B(0,\rho)} u_0(x)dx < \infty \) for small \( \rho > 0 \). By the assumption (ii) we can apply Lemma 2.11. Using Lemma 2.11 and 4.2 we see that there exists \( \delta' \in (0, N/2) \) independent of \( \varepsilon \) such that

\[
 h^{-1} \left( |x|^{-N} \left( \log \frac{1}{|x|} \right)^{-N/2-1+\varepsilon} \right) \leq |x|^{-N} \left( \log \frac{1}{|x|} \right)^{-N/2-1+\varepsilon} \left( 2N \log \frac{1}{|x|} \right)^{\delta'}
\]

for small \( |x| > 0 \). Then we deduce that

\[
 \int_{B(0,\rho)} u_0(x)dx \leq \int_0^\rho \frac{1}{r} \left( \log \frac{1}{r} \right)^{-N/2-1+\varepsilon+\delta'} dr = \frac{1}{N/2 - \varepsilon - \delta'} \left( \log \frac{1}{\rho} \right)^{-N/2+\varepsilon+\delta'}
\]

for small \( \rho > 0 \). Thus, \( u_0 \in L^1_{ul}(\mathbb{R}^N) \).

The proof is by contradiction. Suppose that there exists \( T > 0 \) such that (1.1) has a nonnegative solution. Since \( u(t) \geq S(t)u_0 \geq J(C_0) \) in \( \mathbb{R}^N \times (0, T) \), we can define \( v(x, t) := J^{-1}(u(x, t)) \). Since \( J''(v) \geq 0 \) for \( v \geq C_0 \) the assumption (i), we have \( u(t) \geq C_0 \) and

\[
 \partial_t v = \frac{J''(v)}{J'(v)} \nabla v^2 + \Delta v + \frac{f'(J(v))}{J'(v)} \geq \Delta v + f_{\beta}(v) \text{ in } \mathbb{R}^N \times (0, T).
\]

Here we use \( f(J(v)) = f_{\beta}(v)J'(v) \). Then it follows (see [19 p.77] for details) that

\[
 (4.3) \quad v(t) \geq S(t-\tau)v(\tau) + \int_{\tau}^t S(t-s)f_{\beta}(v(s))ds \text{ in } \mathbb{R}^N \times (\tau, T).
\]

We also obtain

\[
 J(v(\tau)) = u(\tau) \geq S(\tau)u_0 = S(\tau)J(v_0) \geq J(S(\tau)v_0) \text{ in } \mathbb{R}^N \times (0, T).
\]

Since \( J \) is increasing on \([C_0, \infty)\), we see that \( v(\tau) \geq S(\tau)v_0 \) in \( \mathbb{R}^N \times (0, T) \). Letting \( \tau \to 0 \) in (4.3), we have

\[
 v(t) \geq S(t)v_0 + \int_0^t S(t-s)f_{\beta}(v(s))ds \text{ in } \mathbb{R}^N \times (0, T),
\]

and hence Proposition 2.3 says that (1.1) has a nonnegative solution. It contradicts the nonexistence result in Theorem 4.2. We complete the proof. \( \square \)

**Proof of Theorem 4.2.** The proof is by contradiction. Suppose that there exists \( T > 0 \) such that (1.1) with \( f(u) = f_{\beta}(u) \) possesses a local in time nonnegative solution on \((0, T)\). Let \( 0 < \xi < t < T \). It follows from the Fubini theorem and (4.5) that

\[
 (4.4) \quad u(t) = S(t-\xi)u(\xi) + \int_\xi^t S(t-s)f_{\beta}(u(s))ds.
\]

By (4.3) and \( u_0 \geq C_0 \) we have \( u(t) \geq S(t)u_0 \geq C_0 \) in \( \mathbb{R}^N \times (0, T) \). Let \( G(x,t) := K(x,0,t) \). Then, by (4.4) we can obtain in a similar way to [19 Eq. (3.22)] that if \( \rho > 0 \) is sufficiently small, then

\[
 (4.5) \quad w(t) \geq c_sM_3^{-N/2}G(0,1)t^{-N/2} + 2^{-N/2} \int_0^t s^{N/2}f_{\beta}(w(s))ds
\]
holds for almost all $0 < \tau < \rho^2$ and $\rho^2 < t < (T - 4\rho^2)/3$, where $c_0 > 0$ is a constant depending only on $N$,  
\[ w(t) := \int_{\mathbb{R}^N} u(x, t + 4\rho^2)G(x, t)dx \quad \text{and} \quad M_\tau := \int_{B(\tau, \rho)} u(y, \tau)dy. \]

We show that if $0 < \rho < 1$ is sufficiently small, then there exist $C_1 > 0$ and $\delta > 0$ such that

\begin{equation}
(4.6) \quad w(s) \geq C_1 s^{-N/2} \left( \log \frac{2}{3\delta^2 \rho^2} \right)^{-N(\beta+1)/2-1+\varepsilon} \quad \text{for} \quad \rho^2 < s < \rho.
\end{equation}

Let $0 < \rho < 1$ and $\rho^2 < s < \rho$. We see that

\begin{equation}
(4.7) \quad w(s) \geq \int_{\mathbb{R}^N} S(s + 4\rho^2)[u_0](x)G(x, s)dx
\end{equation}

\begin{align*}
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G(x - y, s + 4\rho^2)u_0(y)dy G(x, s)dx \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G(x - y, s + 4\rho^2)G(x, s)dx u_0(y)dy \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G(y, s + 4\rho^2)u_0(y)dy \\
&= \int_{\mathbb{R}^N} G(y, 2s + 4\rho^2)u_0(y)dy.
\end{align*}

Here we use $G(y, t = \int_{\mathbb{R}^N} G(y - x, t - z)G(x, s)dx$ for $y \in \mathbb{R}^N$ and $0 < s < t$. By Lemma 2.10 there exists a small $m' > 0$ such that for $|y| \leq m'$,

\begin{equation}
(4.8) \quad h_{\beta}^{-1} \left( |y|^{-N} \left( \frac{1}{|y|} \right)^{-N/2-1+\varepsilon} \right) \geq \tilde{h}_{\beta} \left( |y|^{-N} \left( \frac{1}{|y|} \right)^{-N/2-1+\varepsilon} \right) \geq |y|^{-N} \left( \frac{1}{|y|} \right)^{-N(\beta+1)/2-1+\varepsilon}.
\end{equation}

Put $s_* := 2s + 4\rho^2$ and choose $\delta > 0$ such that $\sqrt{3}\delta \leq \min\{m, m'\}$. Then we obtain

\begin{equation}
(4.9) \quad \int_{\mathbb{R}^N} G(y, 2s + 4\rho^2)u_0(y)dy
\end{equation}

\begin{align*}
&\geq (4\pi s_*)^{-N/2} \int_{|z| \leq \sqrt{3}\delta} e^{-\frac{|z|^2}{4s_*}} \cdot |y|^{-N} \left( \frac{1}{|y|} \right)^{-N(\beta+1)/2-1+\varepsilon} \ dy \\
&\geq (4\pi)^{-N/2} \int_{|z| \leq \delta} e^{-\frac{|z|^2}{4s_*}} \cdot s_*^{-N/2} |z|^{-N} \left( \frac{1}{s_*^{1/2}} \right)^{-N(\beta+1)/2-1+\varepsilon} \ dz \\
&\geq (4\pi)^{-N/2} \int_{|z| \leq \delta} e^{-\frac{|z|^2}{4s_*}} \cdot s_*^{-N/2} |z|^{-N} \left( \frac{2}{s_*^{1/2}} \right)^{-N(\beta+1)/2-1+\varepsilon} \ dz \\
&\geq s_*^{-N/2} \left( \frac{4}{s_*^{1/2}} \right)^{-N(\beta+1)/2-1+\varepsilon}.
\end{align*}

Here we put $y = \sqrt{s_*^2}$. By (4.7) and (4.9) we have

\begin{equation}
(4.10) \quad s^{N/2}w(s) \geq \left( \frac{s}{s_*} \right)^{N/2} \left( \frac{4}{s_*^{1/2}} \right)^{-N(\beta+1)/2-1+\varepsilon}.
\end{equation}

Since the right hand side of (4.10) is nondecreasing with respect to $s$, we have

\begin{equation*}
\begin{aligned}
\int_{s_*}^{\rho^2} &s^{N/2}w(s) \geq \left( \frac{1}{6} \right)^{N/2} \left( \frac{2}{3\delta^2 \rho^2} \right)^{-N(\beta+1)/2-1+\varepsilon}.
\end{aligned}
\end{equation*}

Thus there exists $C_1 > 0$ such that (4.6) holds. We observe from $s < \rho$ that

\begin{equation*}
\begin{aligned}
s^{-N/2} \left( \frac{2}{3\delta^2 \rho^2} \right)^{-N(\beta+1)/2-1+\varepsilon} &> \rho^{-N/2} \left( \frac{2}{3\delta^2 \rho^2} \right)^{-N(\beta+1)/2-1+\varepsilon} \to \infty \quad \text{as} \quad \rho \to 0.
\end{aligned}
\end{equation*}
Hence, we choose $\rho$ sufficiently small and the right hand side of (4.4) is greater than 1. Then we have

$$\log(w(s) + \varepsilon) \geq \log \left\{ C_1 s^{-N/2} \left( \log \frac{2}{\delta^\beta \tau} \right)^{N(\beta + 1)/2 - 1 + \varepsilon} \right\} = \frac{N}{2} \left( \log \frac{1}{s} + C_2(\rho) \right) \tag{4.11}$$

for $\rho^2 < s < \rho$, where $C_2(\rho) := \frac{2}{\delta^\beta \tau} \log C_1 + \frac{2}{\delta^\beta \tau} (-N(\beta + 1)/2 - 1 + \varepsilon) \log \left( \log \frac{2}{\delta^\beta \tau} \right)$. By (4.5) and (4.11) we have

$$t^{N/2} w(t) \geq c_* M_* 3^{-N/2} G(0, 1) + 2^{-N/2} \left( \frac{N}{2} \right)^{\beta} \int_{\mathbb{R}^d} s^{N/2} \left( \log \frac{1}{s} + C_2(\rho) \right)^\beta w(s)^{1+2/N} ds =: H(t) \tag{4.12}$$

for almost all $0 < \tau < \rho^2$ and $\rho^2 < t < \rho$. Note that $\rho < (T - 4\rho^2)/3$ holds since $\rho$ is sufficiently small. Put $C_3 := 2^{-N/2}(N/2)^\beta$. By (4.12) we have

$$\frac{dH}{dt}(t) = C_3 t^{N/2} \left( \log \frac{1}{\tau} + C_2(\rho) \right)^\beta w(t)^{1+2/N} \geq C_3 t^{N/2} \left( \log \frac{1}{\tau} + C_2(\rho) \right)^\beta (t^{-N/2} H(t))^{1+2/N} = \frac{C_3}{t} \left( \log \frac{1}{\tau} + C_2(\rho) \right)^\beta H(t)^{1+2/N}, \tag{4.13}$$

which yields

$$-\frac{N}{2} \cdot \frac{d}{dt} \left( H(t)^{-2/N} \right) \geq -\frac{C_3}{\beta + 1} \cdot \frac{d}{dt} \left\{ \left( \log \frac{1}{\tau} + C_2(\rho) \right)^{\beta + 1} \right\}. \tag{4.14}$$

This implies that

$$\frac{N}{2} H(\rho^2)^{-2/N} - \frac{N}{2} H(\rho)^{-2/N} \geq \frac{C_3}{\beta + 1} \left\{ \left( 2 \log \frac{1}{\rho} + C_2(\rho) \right)^{\beta + 1} - \left( \log \frac{1}{\rho} + C_2(\rho) \right)^{\beta + 1} \right\} \tag{4.15}$$

for almost all $0 < \tau < \rho^2$. We see that

$$M_\tau \lesssim \left\{ \left( 2 \log \frac{1}{\rho} + C_2(\rho) \right)^{\beta + 1} - \left( \log \frac{1}{\rho} + C_2(\rho) \right)^{\beta + 1} \right\}^{-N/2} \tag{4.16}$$

by Lemma 4.1(ii) and Proposition 2.6. Thus it follows from Fatou’s lemma, (4.14) and (4.15) that

$$\int_{B(0, \rho)} u(y, \tau) dy \geq \int_{B(0, \rho)} S(\tau)[u_0](y) dy. \tag{4.17}$$

By Lemma 4.1(ii) and Proposition 2.6 we see that $\|S(\tau)u_0 - u_0\|_{L^1_1} \to 0$ as $\tau \to 0$. Then there exists a subsequence $\{S(\tau)u_0\}_{\tau \geq 0}$ such that $(S(\bar{\tau})u_0)|_{B(0, \rho)} \to u_0|_{B(0, \rho)}$ as $\bar{\tau} \to 0$ a.e. in $B(0, \rho)$. Thus (4.16) holds since

$$\left\{ \left( 2 \log \frac{1}{\rho} + C_2(\rho) \right)^{\beta + 1} - \left( \log \frac{1}{\rho} + C_2(\rho) \right)^{\beta + 1} \right\}^{-N/2} \geq \liminf_{\tau \to 0} \int_{B(0, \rho)} S(\bar{\tau})[u_0](y) dy \geq \int_{B(0, \rho)} u_0(y) dy. \tag{4.18}$$

On the other hand, by (4.8) we have

$$\int_{B(0, \rho)} u_0(y) dy \geq \int_{B(0, \rho)} |y|^{-N} \left( \log \frac{1}{|y|} \right)^{-N(\beta + 1)/2 - 1 + \varepsilon} dy \tag{4.19}$$

for $0 < \rho \leq 2 \min\{m, m'\}$. By (4.16) and (4.17) we have

$$\left\{ \left( 2 \log \frac{1}{\rho} + C_2(\rho) \right)^{\beta + 1} - \left( \log \frac{1}{\rho} + C_2(\rho) \right)^{\beta + 1} \right\}^{-N/2} \geq \left( \log \frac{1}{\rho} \right)^\varepsilon \to \infty \text{ as } \rho \to 0. \tag{4.18}$$

Here we use $C_2(\rho)/\log(1/\rho) \to 0$ as $\rho \to 0$. Then the left hand side of (4.18) converges to $(2^{\beta + 1} - 1)^{-N/2}$ as $\rho \to 0$. This is a contradiction. The proof is complete.
Using Proposition \[1.2\] (ii), we can obtain a nonexistence result corresponding to Theorem \[3.1\].

**Corollary 4.3.** Suppose that \( f \in C[0, \infty) \), \( f \) is nonnegative and \( f \) is nondecreasing for \( u > 0 \). Suppose that \( J \in C^3[0, \infty) \) satisfies \( \[4.1\] \) and

\[
\text{(4.19)} \quad \text{the limit } \delta := \lim_{u \to \infty} \frac{J(u)J''(u)}{J'(u)} \text{ exists.}
\]

If there exists \( \varepsilon > 0 \) such that

\[
\text{(4.20)} \quad \liminf_{u \to \infty} \frac{\int f(u)J'(u)}{(1+\varepsilon)^{2/N+\varepsilon}} > 0,
\]

then there exists \( u_0 \geq 0 \) such that \( J(u_0) \in L_{u_0}^1(\mathbb{R}^N) \) and \( \text{(4.1)} \) admits no nonnegative solution.

**Proof.** Because of \( \text{(4.20)} \), there exist \( C_0 > 0 \) and \( C_1 > 0 \) that

\[
\text{(4.21)} \quad f(u) > C_0 \frac{J(u)^{1+2/N+\varepsilon}}{J'(u)} \quad \text{for } u \geq C_1.
\]

Here, \( C_0 > 0 \) can be arbitrary large, since \( C_1 > 0 \) can be arbitrary large and \( \varepsilon > 0 \) can be arbitrary small. Let \( \hat{f}(u) := \rho J(u)^{1+1/\rho} / J'(u) \), \( 0 < \rho < N/2 \). Here, \( \rho \) is determined later. Then \( \hat{F}(u) := \int_0^u dr \hat{f}(r) = C J(u)^{-1/\rho} \). First, we consider the Cauchy problem

\[
\text{(4.22)} \quad \begin{cases} \partial_t u = \Delta u + \hat{f}(u) & \text{in } \mathbb{R}^N \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}
\]

By direct calculation we have

\[
\text{(4.23)} \quad \hat{f}'(u)\hat{F}(u) = \rho + 1 - \rho \frac{J(u)J''(u)}{J'(u)^2} \to \rho + 1 - \rho \gamma \quad \text{as } u \to \infty.
\]

By \( \text{(4.23)} \) we see that \( \hat{f}'(u) > 0 \) for large \( u > 0 \). We can take \( C_1 > 0 \) such that \( \hat{f}'(u) > 0 \) for \( u > C_1 \). Then, we can modify \( \hat{f}(u) \), \( 0 \leq u \leq C_1 \), such that \( \hat{f} \) satisfies \( \text{(4.1)} \). Hereafter, we do not use \( \hat{f}(u) \) in \( 0 \leq u \leq C_1 \).

By L'Hospital's rule we see the following limit \( \gamma \) exists:

\[
\gamma = \lim_{u \to \infty} \frac{J(u)J''(u)}{J'(u)^2} = \lim_{u \to \infty} \left( \frac{1}{2} + \frac{J(u)J''(u)}{2J'(u)J''(u)} \right) = \frac{1}{2} + \frac{\delta}{2}.
\]

Moreover,

\[
\eta = \lim_{u \to \infty} \frac{J(u)^2J'''(u)}{J'(u)^3} = \lim_{u \to \infty} \frac{J(u)J''(u)J(u)^2}{J'(u)^2} = \delta \gamma.
\]

Therefore, \( \eta = \gamma - 2\gamma^2 \). By direct calculation we have

\[
\lim_{u \to \infty} \frac{\hat{f}''(u)}{\rho J(u) J'(u)} = \lim_{u \to \infty} \left( \frac{1}{\rho} \left(1 + \frac{1}{\rho}\right) - \left(1 + \frac{1}{\rho}\right) \frac{J(u)J''(u)}{J'(u)^2} + 2 \left( \frac{J(u)J''(u)}{J'(u)^2} \right)^2 - \frac{J(u)^2J'''(u)}{J'(u)^3} \right) = \frac{1}{\rho} \frac{(1-\gamma)}{\rho^2} + \frac{1}{\rho^2} > 0,
\]

and hence \( \hat{f}''(u) \geq 0 \) for large \( u > 0 \). Here, we see that \( 0 \leq \gamma \leq 1 \) as mentioned in the proof of Corollary \[3.2\]. We can take \( C_1 > 0 \) such that \( \hat{f}''(u) > 0 \) for \( u > C_1 \). Then, we again modify \( \hat{f}(u) \), \( 0 \leq u \leq C_1 \), such that \( \hat{f}''(u) \) for \( u \geq 0 \). Hereafter, we do not use \( \hat{f}(u) \) in \( 0 \leq u \leq C_1 \).

Since \( \hat{f} \in C^2 \), by \( \text{(1.23)} \) we see that \( \hat{f} \in C^2[0, \infty) \cap X_q \) with \( q = 1 + \rho(1-\gamma) \). We see that \( 1 < q < 1+N/2 \).

We have checked all the assumptions of Proposition \[1.2\] (ii). It follows from Proposition \[1.2\] (ii) that, for each \( r \in (q-1, N/2) \), there exists a nonnegative function \( u_0 \in L_{u_0}^1(\mathbb{R}^N) \) such that \( \hat{F}(u_0)^{-r} \in L_{u_0}^1(\mathbb{R}^N) \) and \( \text{(4.1)} \) admits no nonnegative solution. Without loss of generality, we can assume that \( u_0 \geq C_1 \). Since \( q-1 \leq r < N/2 \), we can take \( r = \rho \). We have

\[
\text{(4.24)} \quad \hat{F}(u_0)^{-r} = J(u_0) \in L_{u_0}^1(\mathbb{R}^N).
\]
Second, we consider (1.1). We take \( \rho = N/(N\varepsilon + 2) \). Then, \( q - 1 = \rho(1 - \gamma) \leq \rho < N/2 \), and hence all the conditions before are satisfied. Because of (4.21), we have
\[
f(u) > C_0 \frac{J(u)^{1+2/N + \varepsilon}}{J'(u)} > \rho \frac{J(u)^{1+1/\rho}}{J'(u)} = \tilde{f}(u) \quad \text{for } u \geq C_1.
\]
Suppose that (1.1) with the initial function \( u_0 \) has a solution \( u(t) \). Then,
\[
u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))ds \geq S(t)u_0 + \int_0^t S(t-s)\tilde{f}(u(s))ds,
\]
and hence \( u(t) \) is a supersolution for (4.22). By Proposition 2.4 we see that (4.22) has a nonnegative solution. However, it contradicts the nonexistence of a nonnegative solution of (4.22) with \( u_0 \). Thus, (1.1) with \( u_0 \) admits no nonnegative solution. By (4.24) we obtain the conclusion of the corollary.

Remark 4.4.

(i) When \( f(u) = J(u)^{1+2/N} / J'(u) \), it follows from Theorem (1.7) that (1.7) admits a local in time nonnegative solution for every \( u_0 \geq 0 \) satisfying \( J(u_0) \in \mathcal{L}^{1\ast}_u(\mathbb{R}^N) \). Therefore, we cannot take \( \varepsilon = 0 \) in Corollary 4.3.

(ii) Remark (1.7) (ii) indicates that a threshold growth is \( f(u) = J(u)^{1+2/N} / J'(u) \) when the integrability condition is \( J(u_0) \in \mathcal{L}^{1\ast}_u(\mathbb{R}^N) \). On the other hand, Theorem (1.7) and Proposition (1.2) (ii) indicate that a threshold integrability condition is \( F(u_0)^{1-N/2} \in \mathcal{L}^{1\ast}_u(\mathbb{R}^N) \) when the growth is \( f(u) \).

(iii) Since \( q = 1 + \rho (1 - \gamma) < 1 + N/2 \), the extremal case is \( \gamma = 0 \) and \( \rho = N/2 \). In this case the \( q \) exponent of \( \tilde{f}(u) \), which is given by (4.27), is \( 1 + N/2 \). Since \( (q, r) = (1 + N/2, N/2) \), (4.22) is a doubly critical case.

Using Theorem (1.7) we can obtain a nonexistence result corresponding to Theorem (5.1)

Corollary 4.5. Let \( g_\alpha(u) = u[\log(u + e)]^\alpha \). Suppose that \( f \in C[0, \infty), f \) is nondecreasing and \( f \) is nonnegative for \( u > 0 \). Suppose that \( J \in C^2[0, \infty) \) satisfies (1.7) and
\[
\lim_{u \to \infty} \frac{J(u)J''(u)}{J'(u)^2} = 0.
\]
If there exists \( \gamma \in (0, N/2) \) such that
\[
\liminf_{u \to \infty} \frac{f(u)J'(u) [\log(J(u) + e)]^{2\gamma/N}}{J(u)^{1+2/N}} > 0,
\]
and
\[
\frac{d^2}{du^2} \left( g_\gamma^{-1}(J(u)) \right) \leq 0 \quad \text{for large } u > 0,
\]
then there exists a nonnegative function \( u_0 \) such that \( J(u_0) \in \mathcal{L}^{1\ast}_u(\mathbb{R}^N) \) and (1.7) admits no nonnegative solution.

In particular, if \( J(u) = g_\alpha(u) \) for some \( \alpha \in (0, N/2) \) and \( \liminf_{u \to \infty} f(u)/u^{1+2/N} > 0 \), then there exists a nonnegative function \( u_0 \) such that \( J(u_0) \in \mathcal{L}^{1\ast}_u(\mathbb{R}^N) \) and (1.7) admits no nonnegative solution.

Proof. Because of (4.25), there exist \( C_0 > 0 \) and \( C_1 > 1 \) such that
\[
f(u) > C_0 \frac{J(u)^{1+2/N}}{J'(u)\log(J(u) + e)^{2\gamma/N}} \quad \text{for } u \geq C_1.
\]
Note that, for each large \( C_0 > 0 \), we can retake \( \gamma \in (0, N/2) \) and \( C_1 > 0 \) such that the above inequality holds. Let \( g_\gamma^{-1}(u) \) denote the inverse function of \( g_\gamma \). We define
\[
\tilde{f}(u) := \frac{N g_\gamma^{-1}(J(u))^{1+2/N} g_\gamma^{-1}(J(u))}{J(u)}.
\]
Then, \( \tilde{F}(u) := \int_u^\infty d\tau / \tilde{f}(\tau) = g_\gamma^{-1}(J(u))^{-2/N} \). First, we consider the Cauchy problem
\[
\begin{cases}
\partial_t u = \Delta u + \tilde{f}(u) & \text{in } \mathbb{R}^N \times (0, T), \\
u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N.
\end{cases}
\]
By direct calculation we have

\[(4.29) \quad \hat{f}'(u)\hat{F}(u) = 1 + \frac{N}{2} + \frac{N}{2} \frac{v\gamma g_\gamma'(v)}{g_\gamma(v)} \left( \frac{g_\gamma(v)g_\gamma''(v)}{g_\gamma'(v)^2} - \frac{J(u)J''(u)}{J'(u)^2} \right),\]

where \(v := g_\gamma^{-1}(J(u))\). As \(v \to \infty\),

\[(4.30) \quad \frac{v\gamma g_\gamma'(v)}{g_\gamma(v)} = \frac{v[\log(v + e)]\gamma(1 + o(1))}{v[\log(v + e)]\gamma} \to 1 \quad \text{and} \quad \frac{g_\gamma(v)g_\gamma''(v)}{g_\gamma'(v)^2} = \frac{v[\log(v + e)]\gamma^{-\gamma} \log(v + e)]^{-1}(1 + o(1))}{[\log(v + e)]^{2\gamma}(1 + o(1))} \to 0.\]

By (4.30), (4.29) and (4.25) we have \(\lim_{u \to \infty} \hat{f}'(u)\hat{F}(u) = 1 + N/2\). Because of (4.27), we have

\[(4.31) \quad \frac{g_\gamma(v)g_\gamma''(v)}{g_\gamma'(v)^2} - \frac{J(u)J''(u)}{J'(u)^2} = \frac{g_\gamma(v)}{g_\gamma'(v)} \left( \frac{g_\gamma'(v)}{\gamma} \right)^2 \geq 0.\]

By (4.31) and (4.29) we have

\[(4.32) \quad \hat{f}'(u)\hat{F}(u) = 1 + \frac{N}{2} + \frac{N}{2} \left( \frac{\frac{\partial^2 g_\gamma}{\partial u^2}}{\frac{\partial g_\gamma}{\partial u}} \right)^2 \geq 1 + \frac{N}{2}.\]

By (4.32) we see that \(\hat{f}'(u) > 0\) for large \(u > 0\). Hence, we can take \(C_1 > 0\) such that \(\hat{f}'(u) > 0\) for \(u > C_1\). Moreover, we can modify \(f(u), 0 \leq u \leq C_1\), such that \(f\) satisfies (1). Hereafter, we do not use \(\hat{f}(u)\) in \(0 \leq u \leq C_1\).

Now we prove Remark 14.7 (i). Assume that there exists \(c > 0\) such that \(f'(F^{-1}(v))F'(F^{-1}(v)) \geq 1 + N/2\) for \(v < c \leq 0\). Since \(f'' / f'(u)^{1/(1 + N/2)}\) is nondecreasing for \(u > 0\), we obtain in the same way as (2.1) that \(f''(u)/f'(u) \leq 1 + N/2\) for \(u > 0\). This implies that \(f''(F^{-1}(v))/F'(F^{-1}(v)) \leq 1 + N/2\) for \(0 < v < F_\gamma(0) = \infty\). Thus we obtain \(f'(F^{-1}(v)) \geq (1 + N/2)/v \geq f''(F^{-1}(v)) \geq 0\), for \(0 < v \leq c\).

By Remark 14.7 (i) and (4.32) we see that the assumption Theorem 3(i) is satisfied. Because of (1), we see that \(J(u) \geq C_2u\) for large \(u > 0\). Since \(0 < \gamma < N/2\), there exist \(\delta \in (0, 1)\) and \(C_3 > 0\) such that

\[(4.33) \quad J(u) \geq C_2u \geq C_3u[\log(u + e)]^{-\gamma - N\delta/2} \quad \text{for large } u > 0.\]

Then,

\[\hat{F}(u) = g_\gamma^{-1}(J(u))^{-\frac{\gamma}{\delta}} \leq g_\gamma^{-1}(C_3u[\log(u + e)]^{-\gamma - N\delta/2})^{-\frac{\gamma}{\delta}} \leq C_4u^{-\frac{\gamma}{\delta}}[\log(u + e)]^{\delta},\]

and hence the assumption Theorem 3(ii) is satisfied. Using Theorem 3, we see that for each \(\alpha \in (0, N/2)\), there exists \(u_0 \geq 0\) such that \(J_{\alpha}(u_0) = g_\alpha(\hat{F}(u_0)^{-N/2}) \in L^1_{ul}(\mathbb{R}^N)\) and (4.33) admits no nonnegative solution. We take \(\alpha = \gamma\). Then,

\[(4.34) \quad J(u) = \gamma u \in L^1_{ul}(\mathbb{R}^N).\]

We consider (4.28). Since

\[\frac{N}{2} g_\gamma^{-1}(J(u))^{1 + 2/N} g_\gamma'(g_\gamma^{-1}(J(u))) \leq C_5(J(u)[\log(J(u) + e)]^{-\gamma} + 2/N)[\log(J(u) + e)]^{-\gamma} = C_5 J(u)^{1 + 2/N} [\log(J(u) + e)]^{-2\gamma/N}\]

for large \(u > 0\),

\[\text{there exists } C_6 > C_1 \text{ such that } C_6 > C_5 \text{ and}

\[f(u) > C_6 \frac{J(u)^{1 + 2/N}}{J'(u)[\log(J(u) + e)]^{2\gamma/2N}} \geq C_5 \frac{J(u)^{1 + 2/N}}{J'(u)[\log(J(u) + e)]^{2\gamma/2N}} \geq \frac{g_\gamma^{-1}(J(u))^{1 + 2/N} g_\gamma''(g_\gamma^{-1}(J(u)))}{2 J'(u)} = \hat{f}(u) \quad \text{for } u \geq C_6.\]

We can assume that \(u_0 \geq C_6\). Suppose that (1.1) with the initial function \(u_0\) has a solution \(u(t)\). Then

\[u(t) = S(t)u_0 + \int_0^t S(t - s)f(u(s))ds \geq S(t)u_0 + \int_0^t S(t - s)\hat{f}(u(s))ds,\]

and hence \(u(t)\) is a supersolution for (1.2). By Proposition 2.1 we see that (1.2) has a nonnegative solution. However, it contradicts the nonexistence of a nonnegative solution of (1.2) with \(u_0\). Thus, (1.1) with \(u_0\) admits no nonnegative solution. By (4.33) we obtain the first statement of the corollary.

Next, we prove the second statement. We take \(\gamma = \alpha\). Then, (1.2) holds, and \(\liminf_{u \to \infty} f(u)/u^{1+2/N} > 0\) implies (1.28). Thus, the second statement follows from the first statement. □
5. Solvability in $L_{ul}^1(\mathbb{R}^N)$

Let $J(u) = u$, $\theta = 1$ and $\xi = 1$. We use Theorem 5.3(i) to obtain the following:

**Corollary 5.1.** Let $u_0 \geq 0$. Suppose that $f \in C([0, \infty))$, and $f$ is nonnegative and nondecreasing. If

$$\int_1^\infty \tilde{f}(u)du < \infty, \quad \text{where } \tilde{f}(u) := \sup_{1 \leq \tau \leq u} \frac{f(\tau)}{\tau},$$

then (5.1) admits a local in time nonnegative solution $u(t)$, $0 < t < T$, for each $u_0 \in L_{ul}^1(\mathbb{R}^N)$. Moreover, $\|u(t)\|_{L_{ul}^1(\mathbb{R}^N)} \leq C$ for $0 < t < T$.

As mentioned in Section 1, (11) obtained a necessary and sufficient condition on $f$ for a solvability of (1.2) in $L^1(\Omega)$. Here, we use the following nonexistence result:

**Proposition 5.2** (11 Theorem 4.1 and Lemma 4.2). Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. Let $f \in C([0, \infty))$ be nonnegative and nondecreasing. If

$$\int_1^\infty \tilde{f}(u)du = \infty, \quad \text{where } \tilde{f}(u) := \sup_{1 \leq \tau \leq u} \frac{f(\tau)}{\tau},$$

then there is a nonnegative function $u_0 \in L^1(\Omega)$ such that (5.3) admits no local in time nonnegative solution in $L^1(\Omega)$. Specifically, for each small $t > 0$,

$$\int_0^t \int_0^t S_\Omega(t-s)f(S_\Omega(s)u_0)dsdx = \infty.$$

Here, $S_\Omega(t)[\phi](x) := \int_\Omega K_\Omega(x, y, t)\phi(y)dy$ and $K_\Omega(x, y, t)$ denotes the Dirichlet heat kernel on $\Omega$.

Using Corollary 5.1 and Proposition 5.2 we obtain the following characterization:

**Theorem 5.3.** Let $u_0 \geq 0$. Suppose that $f \in C([0, \infty))$, $f$ is nonnegative and nondecreasing. Then, (11) admits a local in time nonnegative solution for all $u_0 \in L_{ul}^1(\mathbb{R}^N)$ if and only if (5.3) holds.

**Proof.** The sufficient part follows from Corollary 5.1. Hereafter, we prove the necessary part. Since $C_0^\infty(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$ is dense and $C_0^\infty(\mathbb{R}^N) \subset BUC(\mathbb{R}^N)$, we see that $L^1(\mathbb{R}^N) \subset L_{ul}^1(\mathbb{R}^N)$. We assume (5.3). Suppose the contrary, i.e., (11) always has a nonnegative solution. Let $u_0 \in L^1(\Omega)$ be the initial function given in Proposition 5.2. Here, we define $u_0 = 0$ in $\mathbb{R}^N \setminus \Omega$. Then, $u_0 \in L_{ul}^1(\mathbb{R}^N)$. Since

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))ds,$$

we have

$$u(t) \geq S(t)u_0.$$

By (5.5) and (5.4) we have

$$u(t) \geq \int_0^t S(t-s)f(u(s))ds \geq \int_0^t S(t-s)f(S_\Omega(s)u_0)ds.$$

Since two Green functions satisfy $K(x, y, t) \geq K_\Omega(x, y, t)$ (see, e.g., (11 Corollary 2.2)), we see that $S(t)u_0 \geq S_\Omega(t)u_0$, and hence

$$u(t) \geq \int_0^t S_\Omega(t-s)f(S_\Omega(s)u_0)ds.$$

By (5.3) we see that $u(t) \notin L_{ul}^1(\mathbb{R}^N)$ for small $t > 0$, and hence (11) admits no nonnegative solution. This is a contradiction. Thus, the proof of the necessary part is complete, and the whole proof is also complete. \(\square\)

Using Theorem 5.3 we show that the condition $\rho < 1$ in Theorem 5.1 is optimal. Specifically, we cannot take $\rho = 1$ in Theorem 5.1.

**Corollary 5.4.** Let $g(u) := u[\log(u + e)]^\alpha$, $\alpha \geq 0$, and let $f(u) := \frac{N}{2}g'(g^{-1}(u))g^{-1}(u)^{1+2/N}$. Then the following hold:
(i) \( q = 1 + N/2, \ f \in X_q \) and

\[ f'(u)F(u) - q = \frac{N \alpha}{2} f(\frac{u}{(h+e) \log(u + e)}) - 1 + \frac{1 + \frac{1}{N \alpha}}{h(\alpha + \frac{h+e}{(h+e) \log(h+e)})} \]

where \( h := F(u)^{-N/2} \). In particular, \((\ref{eq:5.6})\) with \( \rho = 1 \) holds.

(ii) Let \( J_\alpha(u) := g(F(u)^{-N/2}), \ \alpha := N/2 \). Then, \( J_\alpha(u) = u \) and there exists a nonnegative function \( u_0 \in L_1^1(\mathbb{R}^N) \) such that \((\ref{eq:1.1})\) admits no nonnegative solution.

**Proof.** First, we prove (i). By direct calculation we have \( F(u) = (g^{-1}(u))^{-2/N} \) and \( g(F(u)^{-N/2}) = u \). Differentiating \( g(F(u)^{-N/2}) = u \) with respect to \( u \) twice, we obtain \((\ref{eq:5.6})\). Since \( h(u) := g^{-1}(u) \to \infty \) \((u \to \infty)\), by \((\ref{eq:5.6})\) we see that \( \lim_{u \to \infty} f'(u)F(u) = q \). Since all other conditions on \( f \) clearly hold, \( f \) satisfies \((\ref{eq:4.1})\), and hence \( f \in X_q \) with \( q = 1 + N/2 \). Using \((\ref{eq:5.6})\), a direct calculation reveals that \((\ref{eq:4.1})\) with \( \rho = 1 \) holds. The proof of (i) is complete.

Second, we prove (ii). Let \( u_1 > 1 \) be large. Since \( J_\alpha(u) = u \), we have

\[
\int_{1}^{\infty} \frac{f(u)du}{u^{1+2/N}} \geq \int_{u_1}^{\infty} \frac{f(u)du}{u^{1+2/N}} = \frac{N}{2} \int_{F(u_1)^{-N/2}}^{\infty} \frac{g'(\tau)2^{1+2/N}d\tau}{g(\tau)2^{1+2/N}} = \frac{\log(\log(\tau + e))}{\log(\log(u + e))} = \infty,
\]

where we used the change of variables \( \tau = F(u)^{-N/2} \). By Theorem \((\ref{thm:5.3})\) we see that there exists a nonnegative function \( u_0 \in L_1^1(\mathbb{R}^N) \), which obviously implies \( J_\alpha(u_0) \in L_1^1(\mathbb{R}^N) \), such that \((\ref{eq:1.1})\) admits no nonnegative solution. The proof of (ii) is complete. \( \square \)

It follows from a direct calculation that \( f \) in Corollary \((\ref{cor:4.4})\) behaves as follows:

\[ f(u) = \frac{N}{2} u^{1+2/N}[\log(u + e)]^{-2N/1 + o(1)} \quad \text{as} \quad u \to \infty. \]

6. **The Case \( f'(u)F(u) - q \log(h(u) + e) \)**

**Proof of Theorem \((\ref{thm:4.3})\).** We prove (i). By \((\ref{eq:3.23})\) with \( \hat{\rho} = 1 \) and \((\ref{eq:3.24})\) with \( \hat{\rho} = 1 \) we see that if there exists \( \rho < 1 \) such that \((\ref{eq:4.3})\) holds, then \( J_\rho(u) \) is convex for large \( u > 0 \).

It suffices to show that \((\ref{eq:4.3})\) holds in both cases (a) and (b). By direct calculation we have

\[
(f'(u)F(u) - q) \log(h(u) + e) = (f'(u)F(u) - q) \log(h(u) + e) \cdot \log(h(u) + e)
\]

\[
= \left\{ \begin{array}{ll}
\beta u & \text{if } u + e < u^{-2/N}[\log(u + e)]^{-\beta} \\
u^{-2/N}[\log(u + e)]^{-\beta} & \text{otherwise}
\end{array} \right.
\]

\[
= \left(1 + \frac{2}{N}\right) \frac{F(u)}{u^{-2/N}[\log(u + e)]^{-\beta}} \log(u + e) - \frac{F(u)}{u^{-2/N}[\log(u + e)]^{-\beta}} \log(u + e) \cdot \log(h(u) + e).
\]

By L'Hospital's rule we have

\[
\lim_{u \to \infty} \frac{F(u)}{u^{-2/N}[\log(u + e)]^{-\beta}} = \lim_{u \to \infty} \frac{\frac{d}{du} F(u)}{\frac{d}{du} (u^{-2/N}[\log(u + e)]^{-\beta})}
\]

\[
= \lim_{u \to \infty} \left( \frac{2}{N} + \frac{\beta u}{u + e} \cdot [\log(u + e)]^{-1} \right)^{-1} = \frac{N}{2}.
\]

We see that

\[
\left(\frac{N}{2} - \frac{F(u)}{u^{-2/N}[\log(u + e)]^{-\beta}} \right) \log(u + e) = \frac{N}{2} u^{-2/N}[\log(u + e)]^{-\beta} - F(u) \frac{u^{-2/N}[\log(u + e)]^{-\beta - 1}}{u^{-2/N}[\log(u + e)]^{-\beta - 1}},
\]

which implies that

\[
\lim_{u \to \infty} \left( \frac{N}{2} - \frac{F(u)}{u^{-2/N}[\log(u + e)]^{-\beta}} \right) \log(u + e) = \lim_{u \to \infty} \frac{\frac{d}{du} \left( \frac{N}{2} u^{-2/N}[\log(u + e)]^{-\beta} - F(u) \right)}{\frac{d}{du} (u^{-2/N}[\log(u + e)]^{-\beta - 1})}
\]

\[
= \lim_{u \to \infty} \frac{-\frac{N}{2} \beta \cdot u^{-2/N}[\log(u + e)]^{-\beta - 1} \cdot \frac{1}{u + e} \cdot [\log(u + e)]^{-1}}{\frac{1}{u + e} + \beta \cdot u^{-2/N}[\log(u + e)]^{-\beta - 1} \cdot \frac{1}{u + e} \cdot [\log(u + e)]^{-1}} = \left(\frac{N}{2}\right)^2 \beta.
\]
Moreover, since \( \frac{d}{du}(u + e)/\{ \frac{d}{du}(f(u)F(u)) \} = 1/(f'(u)F(u) - 1) \to 2/N \) as \( u \to \infty \), we obtain
\[
\frac{d}{du} \left\{ \log(h(u) + e) \right\} = \frac{N}{1 + F(u)N/2} \to 1 \text{ as } u \to \infty,
\]
and hence \( \log(h(u) + e)/\log(u + e) \to 1 \) as \( u \to \infty \). Therefore, we have
\[
(6.1) \quad \lim_{u \to \infty} (f'(u)F(u) - q) \log(h(u) + e) = \left( \frac{N}{2} \right)^2 \beta.
\]
Since \( \beta \geq -1 \) and \( \alpha > N/2 \), or \( \beta > -1 \) and \( \alpha = N/2 \), we have \(-N\beta/(2\alpha) < 1\). Then we can choose \( \rho \) such that \(-N\beta/(2\alpha) < \rho < 1\), which leads to \(- (\frac{N}{2})^2 \beta < \frac{N}{2} \rho \). By this together with (6.1) we obtain (1.8) in both cases (a) and (b).

We prove (c). Let \(-(1 + 2/N)\kappa < \beta < -1\), \( \theta = 1 \) and \( J(u) = u \). In order to apply Theorem 3.1 (i) we check all the assumptions. Since \( \frac{d}{du} \left( \frac{J'(u)}{J(u)^\theta} \right) = 0 \) and
\[
\frac{d}{du} \left( \frac{f(u)}{J(u)^\theta} \right) = \frac{2}{N} u^{-\frac{1}{N}} \{ \log(u + e) \}^{\beta} \left( 1 + \frac{\beta u}{(u + e) \log(u + e)} \right) > 0 \text{ for large } u > 0,
\]
\( f(u)/J(u)^\theta \) and \( J'(u)/J(u)^{1-\theta} \) are nondecreasing for large \( u > 0 \). If \( \eta > 0 \) is large, then
\[
\int_{\eta}^\infty \frac{\tilde{J}(\eta)}{\tilde{J}(\tau)^{1+2/N}} d\tau \geq \frac{J'(\eta)}{J(\eta)^{1-\theta}} \int_{\eta}^\infty \frac{f(\tau)J'(\tau) d\tau}{J(\tau)^{1+2/N}} = \int_{\eta}^\infty \frac{[\log(\tau + e)]^\beta d\tau}{\tau^\theta} \leq \int_{\eta}^\infty \frac{2[\log(\tau + e)]^\beta d\tau}{\tau + e} = \frac{2}{-\beta - 1}[\log(\eta + e)]^{\beta+1} \to 0 \text{ as } \eta \to \infty.
\]

Then, (6.1) holds. It follows from Theorem 3.1 (i) that (1.1) has a nonnegative solution. A proof of (c) is complete.

We prove (a) of (ii). It suffices to show that \( f = f_\beta \) satisfies all the assumptions of Theorem C. We easily see that \( f \) satisfies (6.1). We check the assumption Theorem C (i). When \( \beta > 0 \), since \( F_{\beta}^{-1} \circ F_\beta(u) = u \), Theorem C (i) holds. When \( -1 < \beta \leq 0 \), we have \( f'(u)F(u) \geq 1 + N/2 \) for large \( u > 0 \). Thus, Theorem C (i) follows from Remark 1.7. We check the assumption Theorem C (ii). We choose \( \delta \) such that \( \max\{-\beta, 0\} < \delta < 1 \). Since \( \beta + \delta < 0 \), we have
\[
\lim_{u \to \infty} \frac{dF(u)}{du} \left\{ u^{-2/N} [\log(u + e)]^\delta \right\} = \lim_{u \to \infty} \frac{[\log(u + e)]^{\beta+\delta} (1 + 2u/(u + e) \log(u + e))}{u^{1+2/N}} = 0.
\]
By L'Hospital's rule we have \( F(u)/\{ u^{-2/N} [\log(u + e)]^\delta \} \to 0 \) as \( u \to \infty \). Thus, Theorem C (ii) also holds. Applying Theorem C we obtain (a) of (ii).

We prove (b) of (ii). Since \( \beta \geq -(1 + 2/N)\kappa \), \( f_\beta(u) \) is nondecreasing, and Theorem 5.3 is applicable. Since there is \( \sigma > 1 \) such that \( f(\tau) = f(\tau)/\tau \) for \( \tau \geq \sigma \), we have
\[
\int_{1}^{\infty} \frac{\tilde{f}(\tau) d\tau}{\tau^{1+2/N}} \geq \int_{\sigma}^{\infty} \frac{\tilde{f}(\tau) d\tau}{\tau^{1+2/N}} \geq \int_{\sigma}^{\infty} \frac{d\tau}{(\tau + e) \log(\tau + e)} = [\log \log(\tau + e)]_{\sigma} \to \infty,
\]
and hence it follows from Theorem 5.3 that there exists a nonnegative function \( u_0 \in L^1_{\text{ul}}(\mathbb{R}^N) \) such that (1.1) admits no nonnegative solution. Next, we show that \( J_\alpha(u_0) \in L^1_{\text{ul}}(\mathbb{R}^N) \). We have
\[
F(u) \geq \log(u + e) \int_{u}^{\infty} \frac{d\tau}{\tau^{1+2/N}} = \frac{N \log(u + e)}{2 u^{2/N}}.
\]
Hence,
\[
(6.2) \quad h(u) := F(u)^{-N/2} \leq C \frac{u}{[\log(u + e)]^{N/2}}.
\]
Since \( 0 \leq \alpha \leq N/2 \), by (6.2) we see that
\[
(6.3) \quad \frac{\log(h(u) + e)}{[\log(u + e)]^{N/2}} \leq C \quad \text{for } u \geq 0.
\]
By (6.2) and (6.3) we have

\[
(6.4) \quad 0 \leq J'_\alpha(u) = \frac{N}{2} \left( 1 + \frac{\alpha h(u)}{(h(u) + e) \log(h(u) + e)} \right) \frac{[\log(h(u) + e)]^\alpha}{f(u)F(u)^{1+N/2}} \leq C \frac{[\log(h(u) + e)]^\alpha}{u^{1+2/N}[\log(u + e)]^{N/2}} \leq C \quad \text{for } u \geq 0.
\]

Since \( u_0 \in \mathcal{L}^1_{\infty}(\mathbb{R}^N) \), by (6.4) we see that \( J_\alpha(u_0) \in \mathcal{L}^1_{\infty}(\mathbb{R}^N) \).

Proofs of all the cases are complete. \( \square \)

7. Regularly varying functions and rapidly varying functions

In this section we always assume the two exponents \( p \) and \( q \) always satisfy

\[
\begin{cases}
\frac{1}{p} + \frac{1}{q} = 1 & \text{if } 1 < q < \infty, \\
p = \infty & \text{if } q = 1.
\end{cases}
\]

The following theorem is a fundamental property of \( \text{RV}_p \):

**Proposition 7.1** (Karamata’s representation theorem). There exist functions \( a(s) \) and \( b(u) \) with

\[
\lim_{u \to \infty} b(u) = b_0 \quad (0 < b_0 < \infty) \quad \text{and} \quad \lim_{s \to \infty} a(s) = p \quad (0 \leq p < \infty)
\]

and \( u_0 \geq 0 \) such that for \( u > u_0 \),

\[
\tag{7.1} f(u) = b(u) \exp \left( \int_{u_0}^u \frac{a(s)}{s} ds \right)
\]

if and only if \( f \in \text{RV}_p \) \((0 \leq p < \infty)\).

See [7, Theorem 1.5] for details. Note that in this section \( u_0 \) does not stand for an initial function. Hereafter, we assume that \( f \) satisfies \([\text{I}]\).

**Lemma 7.2.** Suppose that \( f \in X_q \) for some \( q \in (1, \infty) \). Then there exist \( a(s) \) and \( b(u) \) with

\[
\lim_{u \to \infty} b(u) = b_0 \quad (0 < b_0 < \infty) \quad \text{and} \quad \lim_{s \to \infty} a(s) = p
\]

and \( u_0 > 0 \) such that (7.1) holds for \( u > u_0 \).

**Proof.** Let \( \eta(u) \in C[0, \infty) \) such that \( f'(u)F(u) = q + \eta(u) \). Then \( \eta(u) \to 0 \) as \( u \to \infty \). We have \( (f(u)F(u))' = q - 1 + \eta(u) \). Integrating it over \([u_0, u]\), we have

\[
\tag{7.2} f(u)F(u) = (q - 1)u + h(u),
\]

where \( h(u) := \int_{u_0}^u \eta(s) ds + f(u_0)F(u_0) - (q - 1)u_0 \). Integrating \( \frac{1}{f(u)F(u)} = \frac{1}{(q-1)u + h(u)} \) over \([u_0, u]\), we have \(- \log F(u) = \int_{u_0}^u \frac{ds}{(q - 1)s + h(s)} \). Hence,

\[
\tag{7.3} \frac{1}{F(u)} = \frac{1}{F(u_0)} \exp \left( \int_{u_0}^u \frac{ds}{(q - 1)s + h(s)} \right).
\]

By (7.2) and (7.3) we have

\[
f(u) = \frac{(q - 1)u + h(u)}{F(u_0)} \exp \left( \int_{u_0}^u \frac{ds}{(q - 1)s + h(s)} \right).
\]

Thus, we obtain (7.1), where

\[
a(u) := \frac{p + \rho(u)}{1 + \rho(u)}, \quad b(u) := \frac{(q - 1)u_0}{F(u_0)}(1 + \rho(u)), \quad p := \frac{q}{q - 1} \quad \text{and} \quad \rho(u) := \frac{h(u)}{(q - 1)u_0}.
\]

Since \( \rho(u) \to 0 \) \((u \to \infty)\), we see that \( b(u) \to b_0 > 0 \) \((u \to \infty)\) and \( a(u) \to p \) \((u \to \infty)\). The proof is complete. \( \square \)
Proof of Theorem 7.3 (i). We consider the case $1 < q < \infty$. Let $f \in X_q$. It follows from Lemma 7.2 that there exist $a(s)$ and $b(u)$ such that $f(u) = b(u) \exp \left( \int_{u_0}^{u} a(s)ds/s \right)$, where $b(u) \to b_0 > 0$ ($u \to \infty$) and $a(u) \to p := q/(q-1)$ ($u \to \infty$). By Proposition 7.1 we see that $f \in RV_p$.

We consider the case $q = 1$. Let $f \in X_1$. Since $f'(u)F(u) \to 1$ ($u \to \infty$), for any $\varepsilon > 0$, there is $u_{\varepsilon} > 0$ such that $|f'(u)F(u) - 1| < \varepsilon$ for $u > u_{\varepsilon}$. By the mean value theorem we see that $0 \leq f'(u)F(u) \leq f(u_{\varepsilon})F(u_{\varepsilon}) + \varepsilon(u - u_{\varepsilon})$ for $u > u_{\varepsilon}$. We have

$$
\frac{u}{f(u)F(u)} \geq \frac{u}{f(u_{\varepsilon})F(u_{\varepsilon}) + \varepsilon(u - u_{\varepsilon})} \to \frac{1}{\varepsilon} \text{ as } u \to \infty.
$$

Since $\varepsilon > 0$ can be chosen arbitrary small, we see that $\lim_{u \to \infty} u/f(u)F(u) = \infty$. Then,

$$
(7.4) \quad \lim_{u \to \infty} \frac{uf'(u)}{f(u)} = \lim_{u \to \infty} f'(u)F(u) \frac{u}{f(u)F(u)} = \infty.
$$

Let $a(u) := uf'(u)/f(u)$. Then, we easily see that $f(u) = f(u_0) \exp \left( \int_{u_0}^{u} a(s)ds/s \right)$. It follows from (7.4) that for any $M > 0$, there is $u_M > 0$ such that $a(u) > M$ for $u > u_M$. Let $\lambda > 1$. For $u > u_M$,

$$
\frac{f(\lambda u)}{f(u)} = \exp \left( \int_{u}^{\lambda u} \frac{a(s)}{s}ds \right) \geq \exp \left( \int_{u}^{\lambda u} \frac{M}{s}ds \right) = \lambda^M.
$$

Since $M$ is arbitrary large, we see that $f(\lambda u)/f(u) \to \infty$ as $u \to \infty$. When $0 < \lambda < 1$, by similar way we can show that $f(\lambda u)/f(u) \to 0$ as $u \to \infty$. Thus, $f \in RV_{\infty}$.

\begin{lemma}
Lemma 7.3. The following hold:
(i) Suppose that $f'$ is nondecreasing. If $f \in RV_p$ for some $p \in (1, \infty)$, then $f \in X_q$.
(ii) Suppose that $f'(u)F(u)$ is nondecreasing. If $f \in RV_{\infty}$, then $f \in X_1$.
\end{lemma}

\textbf{Proof.} (i) We follow the strategy used in [11 Proposition 1.7.11]. Let $1 < q < \infty$ and $f \in RV_p$. Let $\lambda > 1$. Since $f'$ is nondecreasing,

$$
\frac{u(\lambda - 1)f'(u)}{f(u)} \leq \int_{1}^{\lambda} f'(\mu u)F(\mu u) d\mu = \frac{f(\lambda u) - f(u)}{f(u)}.
$$

Since $f \in RV_p$, we see that $\limsup_{u \to \infty} \frac{uf'(u)}{f(u)} \leq \frac{\lambda^p - 1}{\lambda - 1}$ for all $\lambda > 1$. Letting $\lambda \downarrow 1$, we have $\limsup_{u \to \infty} \frac{uf'(u)}{f(u)} = p$. Let $0 < \lambda < 1$. Since

$$
(7.5) \quad \frac{u(1 - \lambda)f'(u)}{f(u)} \geq \int_{1}^{\lambda} f'(\mu u)F(\mu u) d\mu = \frac{f(u) - f(\lambda u)}{f(u)},
$$

we have $\liminf_{u \to \infty} \frac{uf'(u)}{f(u)} \geq \frac{\lambda^p - 1}{\lambda - 1}$. Letting $\lambda \uparrow 1$, we have $\liminf_{u \to \infty} \frac{uf'(u)}{f(u)} \geq p$. Thus, $\liminf_{u \to \infty} \frac{uf'(u)}{f(u)} = p$. Since $p > 1$, we can show that $\lim_{u \to \infty} \frac{F(u)}{f(u)} = 0$. By L'Hospital's rule we have

$$
\lim_{u \to \infty} \frac{f'(u)}{f(u)} = \lim_{u \to \infty} \frac{-\frac{1}{f(u)}}{-\frac{uf'(u)}{f(u)} - \frac{uf'(u)}{f(u)^2}} = \frac{1}{p - 1}.
$$

Then,

$$
\lim_{u \to \infty} f'(u)F(u) = \lim_{u \to \infty} \frac{uf'(u)}{f(u)} \frac{F(u)}{f(u)} = \frac{p}{p - 1},
$$

and hence $f \in X_q$.

(ii) Let $f \in RV_{\infty}$. Since $F(u)$ is decreasing and $f'(u)F(u)$ is nondecreasing, $f'(u)$ is nondecreasing. Let $0 < \lambda < 1$. Then, (7.3) holds. Since $f \in RV_{\infty}$ and $0 < \lambda < 1$, we see that $\lim_{u \to \infty} f(\lambda u)/f(u) = 0$. Then,

$$
\frac{uf'(u)}{f(u)} \geq \frac{1}{1 - \lambda} \left( 1 - \frac{f(\lambda u)}{f(u)} \right) \to \frac{1}{1 - \lambda} \text{ as } u \to \infty.
$$

Letting $\lambda \uparrow 1$, we have $\lim_{u \to \infty} uf'(u)/f(u) = \infty$. By L'Hospital's rule we have

$$
(7.6) \quad \lim_{u \to \infty} \frac{f(u)F(u)}{u} = \lim_{u \to \infty} \frac{F(u)}{f(u)} = \lim_{u \to \infty} \frac{1}{f(u)} \frac{uf'(u)}{f(u)} = 0.
$$
Let $\lambda > 1$. Since $f'(u)F(u)$ is nondecreasing, by (4.10) we have
\[(\lambda - 1)(f'(u)F(u) - 1) \leq \int_1^\lambda (f'(\mu u)F(\mu u) - 1) d\mu = \int_1^\lambda \frac{1}{\mu} \frac{d}{d\mu} (f(\mu u)F(\mu u)) d\mu = \frac{f(\lambda u)F(\lambda u)}{\lambda u} \lambda - \frac{f(u)F(u)}{u} \rightarrow 0 \quad \text{as} \quad u \rightarrow \infty,
\]
and hence $\limsup_{u \rightarrow \infty}(f'(u)F(u) - 1) \leq 0$. Let $0 < \lambda < 1$. Then
\[(1 - \lambda)(f'(u)F(u) - 1) \geq \int_\lambda^1 (f'(\mu u)F(\mu u) - 1) d\mu = \frac{f(u)F(u)}{u} - \frac{f(\lambda u)F(\lambda u)}{\lambda u} \lambda \rightarrow 0 \quad \text{as} \quad u \rightarrow \infty,
\]
and hence $0 \leq \liminf_{u \rightarrow \infty}(f'(u)F(u) - 1)$. Thus,
\[0 \leq \liminf_{u \rightarrow \infty}(f'(u)F(u) - 1) \leq \limsup_{u \rightarrow \infty}(f'(u)F(u) - 1) \leq 0,
\]
and hence $\lim_{u \rightarrow \infty} f'(u)F(u) = 1$. We see that $f \in X_1$. □

**Proof of Theorem E** (ii) and (iii). Theorem E (ii) and (iii) follow from Theorem E (i) and Lemma 7.3. □

8. **Summary and problems**

In this paper we study integrability conditions on $u_0$ which determines existence and nonexistence of a local in time nonnegative solution of (1.1). In a critical and subcritical cases existence and nonexistence integrability conditions on $u_0$ are given by Theorem A and Proposition 1.2 (ii). In the doubly critical case these conditions are given by Theorems B and C. See Figure 1. When $f(u) = u^{1+2/N}[\log(u + \epsilon)]^\beta$, $\beta \geq -(1+2/N)\kappa$, where $\kappa$ is given by (1.10), the problem becomes a doubly critical case and a complete classification is given by Theorem E. Theorems A and B can be applied to a nonlinearity in $X_q$. A characterization of $X_q$ is given in Theorem E.

We also study Problem (B) stated in Section 1. Corollaries 3.2 and 3.3 are sufficient conditions on $f$ for existence when $J$ is given. Corollaries 4.3 and 4.5 are sufficient conditions on $f$ for nonexistence when $J$ is given. In Sections 5 and 9 we give a necessary and sufficient condition on $f$ for an existence of a nonnegative solution of (1.1) for every nonnegative function $u_0 \in L^1_{x_0}(\mathbb{R}^N)$. Section 5 (resp. 9) is for the case $r = 1$ (resp. $r > 1$). This necessary and sufficient condition corresponds to [11] Corollary 4.5 and Theorem 3.4 which studied in the $L^r(\Omega)$ framework.

An objective of this study is to prove Table 1 under mild assumptions on $f$ and $J$. This problem derives several concrete problems.

In the proof of Theorems A and E we use Theorem 5.1 which relates the nonlinearity $f$ and the integrability $J(u_0) \in L^1_{x_0}(\mathbb{R}^N)$. The condition (3.2) is a sufficient condition for an existence. Since there is a gap between (3.2) and (4.20) (or between (3.2) and (4.26)), it is natural to ask the following:

**Problem 8.1.** Suppose that (3.2) does not hold. Does there exist $u_0 \geq 0$ such that $J(u_0) \in L^1_{x_0}(\mathbb{R}^N)$ and (1.1) admits no nonnegative solution?

Corollaries 3.2, 3.3, 4.3 and 4.5 are partial answers to Problem (B) and they are not optimal. We do not know whether (3.2), (3.30), (4.19) and (4.20) are technical conditions or not.

**Problem 8.2.** Can one prove a theorem similar to Corollary 3.3 (resp. Corollary 4.3) without assuming (1.19) (resp. (1.27))?

There is a gap between (3.2) and (4.20) (or between (3.31) and (4.20)).

**Problem 8.3.** Can one obtain a growth condition on $f$ which is shaper than (3.20), (3.31), (4.20) or (4.26)\(^2\)?

Theorem C is a sufficient condition for nonexistence in a doubly critical case. The assumption Theorem C (i) and (ii) seem technical.

**Problem 8.4.** Can one obtain a nonexistence result for a wide class of nonlinearities in a doubly critical case?
9. Appendix to [11]: Solvability in $\mathcal{L}_{ul}^r(\mathbb{R}^N)$

We recover [11] Theorem 3.4 and Corollary 4.5 in a framework of uniformly local Lebesgue spaces. Only in this section we adopt the following definition of a solution:

**Definition 9.1.** Let $r \geq 1$. We call $u(t)$ a solution of (9.1) if $u(t)$ is a solution in the sense of Definition 1.4 and $u(t) \in L^\infty((0,T), L_{ul}^1(\mathbb{R}^N))$.

In Theorem 5.3 we already obtained a necessary and sufficient condition for an existence of a nonnegative solution of (9.1) in $L_{ul}^1(\mathbb{R}^N)$ in the sense of Definition 1.4. Since the solution $u(t)$ satisfies $u(t) \in L^\infty((0,T), L_{ul}^1(\mathbb{R}^N))$, $u(t)$ is also a solution in the sense of Definition 9.1 with $r = 1$.

**Corollary 9.2.** Suppose that $f \in C[0, \infty)$, $f$ is nonnegative and nondecreasing. Then (9.1) admits a local in time nonnegative solution in the sense of Definition 9.1 for every nonnegative function $u_0 \in L_{ul}^1(\mathbb{R}^N)$ if and only if (9.1) holds.

Hereafter, we consider the case $r > 1$.

**Theorem 9.3.** Let $r > 1$. Suppose that $f \in C[0, \infty)$, $f$ is nonnegative and nondecreasing. Then, (9.1) admits a local in time nonnegative solution in the sense of Definition 9.1 for every nonnegative function $u_0 \in L_{ul}^1(\mathbb{R}^N)$ if and only if

$$\limsup_{u \to \infty} \int_0^t \frac{|f(u)|}{u^{1+2r/N}} < \infty. \quad (9.1)$$

When the function space is $L^r(\Omega)$, Theorem 9.3 was obtained in [11]. Theorem 9.3 corresponds to [11] Theorem 3.4. Since we work in $L_{ul}^r(\mathbb{R}^N)$, we do not have to care about a behavior of $f(u)$ near $u = 0$.

**Proof.** First, we prove the sufficient part. Specifically, we prove the existence of a solution provided that (9.1) holds. Let $J(u) := u^r$. We consider the case $r \geq N/(N-2)$ and $N \geq 3$. Let $\theta := 1/r + 2/N$. Then, $\theta r = 1 + 2r/N$, and $0 < \theta \leq 1$. Since there is $C > 0$ such that

$$\tilde{f}(u) = \sup_{1 \leq \tau \leq u} \frac{f(\tau)}{\tau^{r\theta}} < C,$$

we have

$$\tilde{J}(\eta) \int_0^\infty \tilde{f}(\tau)J'(\tau) d\tau \leq r^2 \eta^{2/N} \int_0^\infty \tilde{f}(\tau) d\tau \leq r^2 \eta^{2/N} \frac{C N^2}{2} = \frac{C N^2}{2}.$$

Since $u_0 \in L_{ul}^1(\mathbb{R}^N)$ implies $u_0^r \in L_{ul}^1(\mathbb{R}^N)$, it follows from Theorem 5.1 (ii) that (9.1) has a nonnegative solution $u(t)$ in the sense of Definition 1.4 and $\|u(t)\|_{L_{ul}^1(\mathbb{R}^N)} < C_1$ for small $t > 0$. Since $\|u(t)\|_{L_{ul}^r(\mathbb{R}^N)} < C_1$ for small $t > 0$, $u(t)$ is a nonnegative solution in the sense of Definition 9.1. We consider the case $N = 1, 2$ or $1 < r < N/(N-2)$ and $N \geq 3$. Let $\theta = 1$. Then, $1/r + 2/N > 1$. Since

$$\tilde{f}(\tau) \leq \sup_{1 \leq \tau \leq u} \left( \frac{f(\sigma) J(\tau)}{J(\sigma)^{1/r+2/N-1}} \right) = \sup_{1 \leq \sigma \leq \tau} \left( \frac{f(\sigma)}{J(\sigma)^{1/r+2/N-1}} \right) \tilde{J}(\eta) \int_0^\infty \frac{\tilde{f}(\tau)J'(\tau)d\tau}{J(\tau)^{1+2/N}} \leq r\eta^{r-1} \int_0^\infty \frac{C \tau^{r-1}}{\tau^{2r-1}} \leq r^2 \eta^{r-1} \leq C \frac{r^2}{r-1},$$

we have

$$\tilde{J}(\eta) \int_0^\infty \frac{\tilde{f}(\tau)J'(\tau)d\tau}{J(\tau)^{1+2/N}} \leq r\eta^{r-1} \int_0^\infty \frac{C \tau^{r-1}}{\tau^{2r-1}} \leq r^2 \eta^{r-1} \frac{C}{r-1} = \frac{C r^2}{r-1}.$$

Since $u_0^r \in L_{ul}^1(\mathbb{R}^N)$, it follows from Theorem 5.1 (ii) and the same argument above that (9.1) has a nonnegative solution in the sense of Definition 9.1. The proof of the sufficient part is complete.

Second, we prove the necessary part. Specifically, we prove that for a certain nonnegative function $u_0 \in L_{ul}^1(\mathbb{R}^N)$, (9.1) admits no nonnegative solution provided that (9.1) does not hold. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. In [11] Theorem 3.3] it was shown that if (9.1) does not hold, i.e.,

$$\limsup_{u \to \infty} \int_0^t \int_0^r S_\Omega(t-s)f(S_\Omega(s)u_0)ds \, dx = \infty \text{ for small } t > 0,$$

then there exists a nonnegative function $u_0 \in L^r(\Omega)$ such that

$$\int_0^t \int_0^r S_\Omega(t-s)f(S_\Omega(s)u_0)ds \, dx = \infty \text{ for small } t > 0.$$
We can easily see that $u_0 \in L^r(\mathbb{R}^N)$, because $u_0 \in L^r(\mathbb{R}^N)$ and $C^0_0(\mathbb{R}^N)$ is dense in $L^r(\mathbb{R}^N)$. If a solution $u(t)$ of (1.1) exists, then by the same argument as in the proof of Theorem 5.3 we see that

$$\|u(t)\|_{L^r(\Omega)} \geq \int_0^t \left| \int_\Omega S_\Omega(t-s)f(S_\Omega(s)u_0)ds \right|^r dx = \infty \text{ for small } t > 0,$$

which indicates that (1.1) with $u_0$ admits no nonnegative solution in the sense of Definition 9.1. The proof is complete.

□

10. APPENDIX TO [8]

In the proof of [8] Theorems 1.4 (ii) and 1.6 (ii)] the following was claimed: Let $0 < r < N/2$ and $2 < \alpha < N/r$. The function

$$u_0(x) := \begin{cases} F^{-1}(|x|\alpha) & \text{if } F(0) = \infty, \\ F^{-1}(\min\{|x|\alpha, F(0)|) & \text{if } F(0) < \infty. \end{cases}$$

satisfies $F(u_0)^{-r} \in L^1_{ul}(\mathbb{R}^N)$ and (1.1) admits no nonnegative solution.

As mentioned in Remark 1.5 (vi), $F(u_0)^{-r} \in L^1_{ul}(\mathbb{R}^N)$ does not necessarily imply $u_0 \in L^1_{ul}(\mathbb{R}^N)$ if $q > 1 + r$. In that case it may occur that $S(t)u_0 \notin L^\infty((0, T), L^1_{ul}(\mathbb{R}^N)) \cap L^\infty_{loc}((0, T), L^\infty(\mathbb{R}^N))$, and hence (1.5) does not hold. Thus, the assumption $q \leq 1 + r$ should be added in [8] Theorems 1.4 (ii) and 1.6 (ii)] as follows:

**Proposition 10.1.** Let $f \in X_q$. If $q \leq 1 + r$, then for each $r \in (0, N/2)$, there is a nonnegative function $u_0 \in L^1_{ul}(\mathbb{R}^N)$ such that $F(u_0)^{-r} \in L^1_{ul}(\mathbb{R}^N)$ and (1.1) admits no nonnegative solution.

**Proof.** If we prove

$$(1.1) \quad u_0 \in L^1_{ul}(\mathbb{R}^N),$$

then by Proposition 2.3 we see that $S(t)u_0 \in L^\infty((0, T), L^1_{ul}(\mathbb{R}^N)) \cap L^\infty_{loc}((0, T), L^\infty(\mathbb{R}^N))$.

Hereafter, we prove (1.1). Since $f'(u)F(u) \to q$ for large $u$, there is $\epsilon_1 > 0$ such that

$$f'(u)F(u) \leq q + \epsilon_1$$

for large $u$. Integrating $f'(u)/f(u) \leq (q + \epsilon_1)/f(u)F(u)$ over $[u_0, u]$, we have $f(u)F(u)^{q+\epsilon_1} \leq C_1$. Integrating $1/C_1 \leq 1/f(u)F(u)^{q+\epsilon_1}$ over $[u_0, u]$, we have $F(u) \leq C_2(u - C_3u^{1/(q-1+\epsilon_1)})$. Therefore, there is a large $C_4 > 0$ such that $F(u) \leq C_4u^{1/(q-1+\epsilon_1)}$ for large $u$. Since $F^{-1}$ is decreasing, $u \geq F^{-1}(C_4u^{1/(q-1+\epsilon_1)}),$ and hence $F^{-1}(u) \leq C_5u^{1/(q-1+\epsilon_1)}.$ Let $\rho > 0$ be small. Then,

$$(1.2) \quad \int_{|x| \leq \rho} F^{-1}(|x|^\alpha)dx \leq C_6 \int_0^\rho r^{-\alpha(q-1+\epsilon_1)+N-1}dr.$$

Since $q - 1 \leq r$ and $\alpha = N(1 - \epsilon_2)/r$ for some $\epsilon_2 > 0$, we have

$$-\alpha(q - 1 + \epsilon_1) + N - 1 \geq -\frac{N}{r}(1 - \epsilon_2)(1 + \epsilon_1) + N - 1 = -1 + \epsilon_2 N - \epsilon_1(1 - \epsilon_2)\frac{N}{r}.$$ 

Since $\epsilon_1 > 0$ can be taken arbitrary small, we can take $\epsilon_1 > 0$ such that $-\alpha(q - 1 + \epsilon_1) + N - 1 < -1$. By (1.2) we see that $\int_{|x| \leq \rho} F^{-1}(|x|^\alpha)dx \leq \infty$ and it indicates (1.1). By the proof of [8] Theorems 1.4 (ii) and 1.6 (ii)] we see that the conclusion of the proposition holds. □

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