Testing Bell’s inequality using charmonium decays

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This paper discusses the feasibility of testing Bell’s inequality with the charmonium decays \( η_c \rightarrow \Lambda \bar{\Lambda}, \chi_{c0} \rightarrow \Lambda \bar{\Lambda} \) and \( J/\psi \rightarrow \Lambda \bar{\Lambda} \). We develop a new formulation of Bell’s inequality represented with particles’ orientations, which can be measured straightforwardly in experiment, and seek to ascertain if it is violated in these decay processes. It is shown that the \( η_c \) channel and the \( \chi_{c0} \) channel maximally violate Bell’s inequality, while the \( J/\psi \) channel gives no inconsistency with it. Simulations dealing with experimental aspects are also implemented. The expected statistical fluctuations and achievable significance are estimated as a function of sample size.

Subject Index A60, C50

1. Introduction

Without doubt, Bell’s inequality (BI) has played a significant role in modern physics. It establishes a clear discrimination between quantum mechanical and classical views of nature. Since it was first shown to be violated in a system of photons by optical experiments [1–3], people have been interested in whether this could be repeated with other types of particle or system. Cases involving massive particles are particularly interesting, since generally a massive particle displays more classical properties. At present, some experiments have been implemented dealing with proton pairs [4] or \( K^0 \bar{K}^0 \) and \( B^0 \bar{B}^0 \) oscillations [5,6], although the variety of the experiments is still limited.

In this context, charmonium decays \( c\bar{c} \rightarrow \Lambda \bar{\Lambda} \) have been anticipated to be one testing channel for BI [7,8] in a highly exotic system: one that involves high-energy massive, unstable fermions experiencing parity-violating weak interaction. In particular, \( η_c \rightarrow \Lambda \bar{\Lambda} \) and \( \chi_{c0} \rightarrow \Lambda \bar{\Lambda} \) are exact realizations of Bohm’s type of Einstein–Podolsky–Rosen experiment [9] in that these involve a spinless particle in the initial state decaying into two spin one half fermions with opposite spins. In quantum mechanics (QM) this is known as an entangled state, leading to a strong correlation between the two fermions that violates BI. \( J/\psi \rightarrow \Lambda \bar{\Lambda} \) is similar, and is thought to be the most promising channel because of the abundant statistics. However, this channel has the significant difference of having a spin 1 initial state. This allows the involvement of relative orbital angular momentum between the generated hyperons \( \Lambda \) and \( \bar{\Lambda} \), which disturbs the entanglement and weakens the particle correlation. It has not been clear if this weakened correlation is nonetheless sufficient to violate BI. (We discuss this and provide a conclusion in Sect. 4.)

Practically, this test seems to include a complication in that we have no means to directly measure the spin of decaying hyperons. For an indirect approach instead, it has been suggested by N.A. Tornqvist that the association between \( \Lambda (\bar{\Lambda}) \) spin and the decay distribution of its weak decay \( \Lambda \rightarrow p\pi^- \)
Fig. 1. An overview of the $c\bar{c} \rightarrow \Lambda \bar{\Lambda} \rightarrow p\pi^-\bar{p}\pi^+$ process. First, a charm meson decays into a $\Lambda\bar{\Lambda}$ pair. These travel back-to-back in the meson’s rest frame and decay into $p\pi^-$ and $\bar{p}\pi^+$ respectively. We measure the orientations of $p$ and $\bar{p}$ as the testing variables of BI. Experimentally, the charm mesons are supposed to be produced in $e^+e^-$ collisions. $J/\psi$ is obtained directly, while $\eta_c$ and $\chi_{c0}$ are generated via the decay of $J/\psi$ and $\psi'$ respectively.

$(\Lambda \rightarrow \bar{p}\pi^+)$ is useful for inferring the hyperon spins within the classical picture [7]. The distribution of this decay product proton (anti-proton) is known to be

$$\frac{d\Gamma_\Lambda}{d\Omega_p} \propto 1 + \alpha_\Lambda n \cdot s$$
$$\frac{d\Gamma_{\bar{\Lambda}}}{d\Omega_{\bar{p}}} \propto 1 - \alpha_{\bar{\Lambda}} n' \cdot s'$$

(1)

in the $\Lambda$ ($\bar{\Lambda}$) rest frame [10]. $n$ ($n'$) is the unit vector of the outgoing proton (anti-proton) direction and $s$ ($s'$) the polarization vector of the $\Lambda$ ($\bar{\Lambda}$), as shown in Fig. 1, all defined in the $\Lambda$ ($\bar{\Lambda}$) rest frame. $\alpha_\Lambda$ and $\alpha_{\bar{\Lambda}}$ are the decay parameters with experimental values of $\alpha_\Lambda = 0.642 \pm 0.013$ [11] and $\alpha_{\bar{\Lambda}} = \alpha_{\bar{\Lambda}}$, ignoring the small effect of CP violation. Due to the parity-violating nature of the weak interaction, the momentum of the outgoing proton (anti-proton) prefers the direction opposite to (along) the polarization of the $\Lambda$ ($\bar{\Lambda}$). In this sense, the hyperon’s decay is its own polarimeter.

It is worth pointing out that this description is not valid in QM, in that it ignores the contribution from interference between the two spin states. In QM, the entire process $c\bar{c} \rightarrow \Lambda \bar{\Lambda} \rightarrow p\pi^-p\pi^+$ should be treated as a coherent process in which the spins of the intermediate state (hyperons in this case) cannot in principle be well defined.

While Tornqvist suggested the testing of QM using this tool [7,8], S.P. Baranov extended it for testing local realistic theory (LRT) by reformulating BI with respect to the orientation of the decay product $p$ and $\bar{p}$ in the final state [12] with the help of (1). We basically follow this line, attaching further, more comprehensive discussion. To be specific, the remainder of this paper consists of three sections:

- The development of BI into a representation with experimentally measurable observables, using LRT-based arguments, where the processes $c\bar{c} \rightarrow \Lambda \bar{\Lambda}$ and $\Lambda \bar{\Lambda} \rightarrow p\pi^-p\pi^+$ are considered separately.
- The evaluation of whether $c\bar{c} \rightarrow \Lambda \bar{\Lambda} \rightarrow p\pi^-p\pi^+$ has sufficient sensitivity to test this newly derived BI by QM-based calculations.
- An analysis of the achievable significance of such a test and its experimental feasibility.

2. Momentum representation of Bell’s inequality

In realistic theories, a particle’s spin is a definite physical quantity and is normally treated as a three-dimensional, continuously valued vector just like classical angular momentum. Consider a
two-particle system each with spins of one half. The polarization vectors \( \mathbf{s} \) and \( \mathbf{s}' \) follow the algebraic condition

\[
|\langle s_a s'_b \rangle - \langle s_a s'_c \rangle| \leq 1 + \langle s_b s'_c \rangle,
\]

where \( a, b, \) and \( c \) are arbitrary unit vector “guide axes”. \( \mathbf{s} \) and \( \mathbf{s}' \) have norms of 1, and \( s_a \) is its projection onto \( a \), i.e. \( s_a = \mathbf{s} \cdot \mathbf{a} \); \( s'_b \) is that of particle 2 onto \( b \); \( s_b \) and \( s'_c \) are defined similarly (Fig. 2). Here, \( \mathbf{s} \) and \( \mathbf{s}' \) are quoted as the realistic value of the polarizations, considering the case where they distribute probabilistically, and the ensemble average is weighted by their probability density function. These seem to be the most striking differences from the conventional BI [13], which has the exactly same form as (2):

\[
|\langle m_a m'_b \rangle - \langle m_a m'_c \rangle| \leq 1 + \langle m_b m'_c \rangle,
\]

with \( m_i \) and \( m'_i \) being the result of measuring \( s_i \) and \( s'_i \), \( i = a, b, c \). The inequality (3) is valid whether \( m_i \) (\( m'_i \)) is continuous or discrete, providing \(-1 \leq m_i (m'_i) \leq 1\), or if the realistic values \( s_i \) (\( s'_i \)) are different in each measurement, although Bell’s original discussion [13] takes discrete ones assuming that the same spin is measured by a Stern–Gerlach type of experiment. Thus, since both the inequalities (2) and (3) have the same essence, that they generally hold in a (local) realistic view of physics and the constraints are due to the realistic interpretation of spin, we call inequality (2) BI as well, and use it to test LRTs.

In extending to the relativistic case, in which particles 1 and 2 belong to different respective frames, the CHSH (Clauser–Horne–Shimony–Holt) version of BI gives a more appropriate description [14]:

\[
|\langle s_a s'_b \rangle + \langle s_a s'_d \rangle + \langle s_c s'_b \rangle - \langle s_c s'_d \rangle| \leq 2.
\]

Here, four guide axes are involved, two of which \( (a, c) \) are used for particle 1 and the other two \( (b, d) \) for particle 2. Each set of two guide axes is defined independently in their own frame. Recall that the original version of BI (2) has \( b \) common to both frames. This leads to confusion in the relativistic case, as pointed out by [12], while the CHSH inequality (4) remains well defined, with no such ambiguity.

At this point, it is clear that (4) is not capable of being tested by experiment because it is denoted by realistic values instead of measured values, which can yield different values due to disturbance by the involvement of hidden variables, generally in LRTs. However, we can translate it into a testable one, assuming the angular distribution (1) for the decay \( \Lambda \rightarrow p\pi^- (\bar{\Lambda} \rightarrow \bar{p}\pi^+) \). If the two decays are independent, the correlations of hyperon spin and the proton (anti-proton) orientation can be tagged
Table 1. Kinematics of $\Lambda$ and $\bar{\Lambda}$ from each charmonium decay. $\omega$ is the fraction that the two decays are in space-like configuration: $\omega = 2 \int_{0}^{\infty} dt \int_{1}^{\frac{1}{\beta}} \frac{1}{r} e^{-\frac{u}{r}} e^{-\frac{\gamma}{r}} = \beta$.

|       | $\eta_{c}$ | $\chi_{c}$ | $J/\psi$ |
|-------|------------|------------|----------|
| $\beta$ | 0.663      | 0.757      | 0.693    |
| $\gamma$ | 1.329      | 1.621      | 1.408    |
| $\omega$ | 0.663      | 0.757      | 0.693    |

as

$$\langle s_{\alpha} s'_{b} \rangle = -\frac{9}{\alpha_{\Lambda} \alpha_{\bar{\Lambda}}} \langle (n \cdot a)(n' \cdot b) \rangle = -\frac{9}{\alpha_{\Lambda} \alpha_{\bar{\Lambda}}} \langle n_{\alpha} n'_{b} \rangle.$$  \hspace{1cm} (5)

$n$ ($n'$) is the traveling direction of the proton (anti-proton). The equation, originally provided by S.P. Baranov [12], takes a different picture of classical spin reality from the one in this paper, however. The derivation for our spin interpretation is given in Appendix, yielding the same result as [12]. Equation (4) can therefore be written as

$$|\langle n_{\alpha} n'_{b} \rangle + \langle n_{\alpha} n'_{d} \rangle + \langle n_{c} n'_{b} \rangle - \langle n_{c} n'_{d} \rangle| \leq 2 \frac{\alpha_{\Lambda} \alpha_{\bar{\Lambda}}}{9}.$$  \hspace{1cm} (6)

We call this “the momentum representation” of BI. Here, $n$ and $n'$ are measured values, and have corresponding observables in QM. Therefore, (6) is eventually an inequality which can be evaluated by experiment and by QM.

Several comments should be added to this reformulation:

- On deriving (5), we assumed that the decay of the first hyperon only depends on the polarization of that particular hyperon. This seems reasonable according to the kinematic properties of $\Lambda$ and $\bar{\Lambda}$ from each meson decay, as shown in Table 1. The two decays have a space-like separation for $66\% \sim 76\%$ of events, across which no interaction can occur. Experimentally, these space-like events can be selectively extracted, which realizes a complete isolation of the decays.

- In conventional tests with direct spin measurement, guide axes play an important physical role in that only one direction of spin can be chosen to be measured. In our test, however, the guide axes are just arbitrary unit vectors, as seen in (2). This causes no problem because momentum components can be simultaneously determined within one measurement, and it even has the advantage that we will not suffer from the “free will” problem in choosing guide axes.

- On the other hand, it has no scheme comparable to “delayed-choice measurement” in optical experiments which prevent the hyperons from deciding how to decay in advance, or exchanging information with each other just after their production so that they can have a larger correlation in their orientations. These are expected to be the loopholes of this test.

- The development from (4) to (6) proceeds based wholly on the classical picture. Although (1) can be derived by a standard QM calculation, we accept (1) as just an experimental fact without assuming any background theories underlying it.

- There has been a claim that BI with respect to commuting variables cannot violate even in QM due to their incoherency [15]. However, this is not the case because the inequality (6) is essentially a BI with respect to spins. We just transform and represent it with $n$ and $n'$, and as a result the upper limit of the BI is lower by a factor of $\frac{\alpha_{\Lambda} \alpha_{\bar{\Lambda}}}{9} (\sim \frac{1}{20})$ than that of a BI which naively takes variables as $n$ ($n'$): $|\langle n_{\alpha} n'_{b} \rangle + \langle n_{\alpha} n'_{d} \rangle + \langle n_{c} n'_{b} \rangle - \langle n_{c} n'_{d} \rangle| \leq 2$. Thus the BI (6) does violate, as we show in Sect. 4.
with the constraint conditions
\( a \) and \( b \). Theoretically this is true since the upper limit—the right-hand side of (6)—drops accordingly. The conceptual essence of the test is not how much \( n' \) acts as the polarimeter of hyperons but that the correlations \( \langle s_i s_j' \rangle \) and \( \langle n_i n_j' \rangle \) have an exact relation (5), and thus we can set the upper limit on \( \langle n_i n_j' \rangle \) as (6). Practically, however, the small correlation amplitude of \( \langle n_i n_j' \rangle \) is difficult to measure precisely, and therefore we finally prefer the decay channel with as large \( \alpha_\Lambda \) as possible.

3. Bilinear expression for the momentum representation of BI

For later analysis, it is convenient to transform the BI (6) into one written with a “correlation matrix” \( \hat{C} \). \( \hat{C} \) is a \( 3 \times 3 \) real-valued matrix given by the correlation amplitude,

\[
\hat{C}_{ij} := \langle n_i n_j' \rangle \quad (i, j = 1, 2, 3),
\]

where the indices \( i, j \) label the \( x, y, z \) components of a vector in Cartesian coordinates. \( \langle n_i n_j' \rangle \) can be written in a simple bilinear form using matrix \( \hat{C} \) as

\[
\langle n_i n_j' \rangle = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \langle n_i n_j' \rangle = a^T \hat{C} b.
\]

We now define \( \hat{Q} \) as the left-hand side of (6), which can be written as a sum of bilinears:

\[
\hat{Q} = |a^T \hat{C} (b + d) + c^T \hat{C} (b - d)|.
\]

The advantage of this representation can be seen in that the physical part (\( \hat{C} \)) and physics-independent part (guide axes) are well separated.

Next we try to specify the maximum value of \( \hat{Q} \) and to remove the guide axes from the inequality since they have no physical importance in our formulation. It is easy to show that the maximum value of \( \hat{Q} \) is the same as that of \( a^T \hat{C} (b + d) + c^T \hat{C} (b - d) \), and thus considering the case of \( \hat{Q} = a^T \hat{C} (b + d) + c^T \hat{C} (b - d) \) is sufficient. The method of Lagrange multipliers (MLM) can be utilized.

With the constraint conditions \( a^T a = 1, b^T b = 1, c^T c = 1, \) and \( d^T d = 1 \), a scalar function \( L \) can be constructed using four multipliers \( \xi_a, \xi_b, \xi_c, \) and \( \xi_d \):

\[
L = a^T \hat{C} (b + d) + c^T \hat{C} (b - d)
- \frac{1}{2} \xi_a (a^T a - 1) - \frac{1}{2} \xi_b (b^T b - 1) - \frac{1}{2} \xi_c (c^T c - 1) - \frac{1}{2} \xi_d (d^T d - 1).
\]

Setting all the derivatives to zero,

\[
\frac{\partial L}{\partial a} = 0 \quad \iff \quad \hat{C} (b + d) - \xi_a a = 0 \quad (8a)
\]
\[
\frac{\partial L}{\partial b} = 0 \quad \iff \quad (a + c)^T \hat{C} - \xi_b b^T = 0 \quad (8b)
\]
\[
\frac{\partial L}{\partial c} = 0 \quad \iff \quad \hat{C} (b - d) - \xi_c c = 0 \quad (8c)
\]
\[
\frac{\partial L}{\partial d} = 0 \quad \iff \quad (a - c)^T \hat{C} - \xi_d d^T = 0 \quad (8d)
\]

\[
\frac{\partial L}{\partial \xi_a} = 0 \quad \iff \quad \text{(constraint conditions)} \quad (\alpha = a, b, c, d). \quad (9)
\]
Multiplying $a^T$, $c^T(b, d)$ from the left (right)-hand side of (8a)–(8d) respectively, and using the constraint conditions (9), the multipliers $\xi_\alpha$ can be written as:

$$\xi_a = a^T \hat{C}(b + d) \quad \xi_c = c^T \hat{C}(b - d)$$

$$\xi_b = (a + c)^T \hat{C}b \quad \xi_d = (a - c)^T \hat{C}d.$$ \hspace{1cm} (10)

Substituting these back into (8b), (8d) gives:

$$b = \frac{\hat{C}^T(a + c)}{(a + c)^T \hat{C}b} = \frac{\hat{C}^T(a + c)}{\rho}$$

$$d = \frac{\hat{C}^T(a - c)}{(a - c)^T \hat{C}d} = \frac{\hat{C}^T(a - c)}{\sigma}.$$ \hspace{1cm} (11)

Note that $b$ and $d$ are parallel to $\hat{C}^T(a + c)$ and $\hat{C}^T(a - c)$. The normalization factors $\rho$ and $\sigma$ are chosen to satisfy $b^Tb = 1$ and $d^Td = 1$, i.e.

$$\rho = \sqrt{(a + c)^T \hat{C}(a + c)} \quad \sigma = \sqrt{(a - c)^T \hat{C}(a - c)}.$$ \hspace{1cm} (12)

Now we define a symmetric matrix $S$ as $S := \hat{C}^T \hat{C} = \hat{C}^T \hat{C}$. With (10) and (11), we eliminate $b$ and $d$ in (8a), (8c), giving

$$(\sigma + \rho)Sa + (\sigma - \rho)Sc = \mu a$$

$$(\sigma - \rho)Sa + (\sigma + \rho)Sc = \nu c,$$

where

$$\mu := (\sigma + \rho)a^T Sa + (\sigma - \rho)a^T Sc$$

$$\nu := (\sigma + \rho)c^T Sa + (\sigma - \rho)c^T Sc.$$

These can be expressed as eigenequations for the vectors $a$ and $c$:

$$\begin{bmatrix} -4\rho\sigma S^2 + (\sigma + \rho)(\mu + \nu)S \end{bmatrix} a = \mu v a$$

$$\begin{bmatrix} 4\rho\sigma S^2 + (\sigma + \rho)(\mu + \nu)S \end{bmatrix} c = \nu v c.$$ \hspace{1cm} (13)

It can easily be verified that the matrices appearing in the left-hand sides of (13), $\pm 4\rho\sigma S^2 + (\sigma + \rho)(\mu + \nu)S$, have identical eigenvectors to those of $S$, as

$$\begin{bmatrix} \pm 4\rho\sigma S^2 + (\sigma + \rho)(\mu + \nu)S \end{bmatrix} v_i \bigg| = \bigg[ \pm 4\rho\sigma \lambda_i^2 + (\sigma + \rho)(\mu + \nu)\lambda_i \bigg] v_i,$$

with $\lambda_i$, $v_i$ being the eigenvalues and eigenvectors satisfying $Sv_i = \lambda_i v_i$. Since any three-dimensional matrix can only have three independent eigenvectors at most, $v_i(i = 1, 2, 3)$ give a full description of all solutions of (13). Therefore,

$$a = \pm v_i$$

$$c = \pm v_j$$

$$|v_i| = 1 \quad (i, j = 1, 2, 3)$$ \hspace{1cm} (14)

are required, and we see that all of these satisfy (13). Note that the norms of $a$ and $c$ are confirmed to be 1. Substituting these into (11), (12), $b$, $d$ and all other coefficients are determined. Setting
\((a, c) = (v_i, v_j)\) for simplicity,

\[
\rho = \sqrt{(\lambda_i + \lambda_j)(1 + v_i^T \cdot v_j)}
\]
\[
\sigma = \sqrt{(\lambda_i + \lambda_j)(1 - v_i^T \cdot v_j)}
\]

(i) \(a = c \ (i = j)\)

\[
\rho = 2\sqrt{\lambda_i} \quad \sigma = 0
\]
\[
b = \frac{\hat{C}^T v_i}{\sqrt{\lambda_i}} \quad d = \text{(arbitrary unit vector)}
\]
\[
\hat{Q} = 2a^T \hat{C} b = 2\sqrt{\lambda_i}
\]

(ii) \(a \neq c \ (i \neq j)\)

\[
\rho = \sigma = \sqrt{\lambda_i + \lambda_j}
\]
\[
b = \frac{\hat{C}^T (v_i + v_j)}{\sqrt{\lambda_i + \lambda_j}} \quad d = \frac{\hat{C}^T (v_i - v_j)}{\sqrt{\lambda_i + \lambda_j}}
\]
\[
\hat{Q} = 2\sqrt{\lambda_i + \lambda_j}
\]

In deriving (16) we used the fact that \(S = \hat{C}^T \hat{C}\) is a symmetric matrix with orthogonal eigenvectors \((v_i^T \cdot v_j = \delta_{ij})\). The same analysis can be applied to the cases of \((a, c) = (v_i, -v_j), (-v_i, v_j), (-v_i, -v_j)\). The maximum \(\hat{Q}\) value comes from (16) when we set \(\lambda_i\) and \(\lambda_j\) to the largest two eigenvalues of \(S\). One point to be noted is that solutions of MLM generally include local maxima where the real maximum value is given not at extrema but at the boundary of the parameter space. However, this is not the case here, since the parameter space of \(a, b, c, d\) are independent "spheres" that have no boundary.

With all these considerations, we finally reach the form

\[
\hat{Q}_{\text{max}} = 2\sqrt{\lambda_1 + \lambda_2} \leq 2\alpha / \Lambda_1, \quad (\lambda_1, \lambda_2: \text{the largest two eigenvalues of } \hat{C}^T \hat{C})
\]

where \(\lambda_1\) and \(\lambda_2\) are the largest two eigenvalues of \(\hat{C}^T \hat{C}\). For later convenience, we define \(C\) and \(Q\) by dividing by the right-hand side of (17):

\[
C_{ij} = \langle n_i n'_j \rangle \frac{9}{2\alpha / \Lambda_1} \quad (i, j = 1, 2, 3)
\]

\[
Q_{\text{max}} = 2\sqrt{\lambda_1 + \lambda_2} \quad (\lambda_1, \lambda_2: \text{the largest two eigenvalues of } C^T C)
\]

\[
Q_{\text{max}} \leq 1.
\]

This (18) is our target expression. The classical limit of \(Q_{\text{max}}\) is \(Q_{\text{CL}} = 1\), while the quantum limit reaches \(\sqrt{2}\).

4. Quantum mechanical calculation of \(C\) and \(Q_{\text{max}}\)

In this section, we perform a QM-based computation of the correlation matrix \(C\) and \(Q_{\text{max}}\) for each channel \(\eta_c \rightarrow \Lambda \bar{\Lambda} \rightarrow p\pi^- \bar{p}\pi^+, \chi_c \rightarrow \Lambda \bar{\Lambda} \rightarrow p\pi^- \bar{p}\pi^+, \text{and } J / \psi \rightarrow \Lambda \bar{\Lambda} \rightarrow p\pi^- \bar{p}\pi^+,\text{ to determine if the BI (18) is held or violated in QM.}
The matrix elements for each channel are as below, according Feynman’s rules and the effective Lagrangian prescription:

\[ \mathcal{M}_{\Lambda c} = \mathcal{M}_A(p_A, s_A, p_p, s_p) \bar{u}(p_A, s_A) \gamma^5 v(p_A, s_A) \mathcal{M}\bar{\Lambda}(p_A, s_A, p_p, s_p) \]

\[ \mathcal{M}_{\Lambda 0} = \mathcal{M}_A(p_A, s_A, p_p, s_p) \bar{u}(p_A, s_A) v(p_A, s_A) \mathcal{M}\bar{\Lambda}(p_A, s_A, p_p, s_p) \]

\[ \mathcal{M}_{J/\psi} \propto \mathcal{M}_A(p_A, s_A, p_p, s_p) \bar{u}(p_A, s_A) \epsilon_\mu \left[ \gamma^\mu + \frac{a_\psi}{m_\psi}(p_\Lambda^\mu - p_\bar{\Lambda}^\mu) \right] v(p_A, s_A) \mathcal{M}\bar{\Lambda}(p_A, s_A, p_p, s_p) \]

\[ \mathcal{M}_A(p_A, s_A, p_p, s_p) = \bar{u}(p_p, s_p)(1 + c_\Lambda \gamma^5)u(p_A, s_A) \]

\[ \mathcal{M}_{\bar{\Lambda}}(p_A, s_A, p_p, s_p) = \bar{u}(p_A, s_A) (1 - c_\Lambda \gamma^5)v(p_p, s_p). \]

sA denotes the helicity of particle A, u(pA, sA) and v(pA, sA) the 4-spinor, and eμ the polarization vector of the meson J/ψ. The momenta appearing here, pA, p̅A, are all defined in the decaying meson rest frame. MA and M̅A are the matrix elements responsible for the A → pπ− and A̅ → ̅pπ+ sector, which gives cA = c̅A providing the CP conservation argument. The distribution of A decay and the decay parameter αA in (1) is associated as

\[ \frac{d\Gamma_A}{d\Omega_p} \propto \sum_{s_p} |M_A|^2 \]

\[ \alpha_A = \frac{-2|p|c_A}{E(1 + c_\Lambda^2) + m_p(1 - c_\Lambda^2)}, \]

with which the parameter cA is derived to cA = c̅A = −6.79 ± 0.18. E, p, and m_p are the energy and the momentum of the outgoing proton in the Λ rest frame and its rest mass respectively. M_{J/ψ} includes an additional parameter a_ψ associated with the form factor of J/ψ. This has been experimentally determined from the unpolarized A distribution of e⁺e⁻ → J/ψ → ΛA̅, where e⁺e⁻ act as chiral fermions leading J/ψ polarization parallel or anti-parallel to the beam axis according to the helicity conservation in a high-energy system:

\[ \frac{d\Gamma_{J/\psi}}{d\Omega_A} \propto \sum_{s_A, s_{J/\psi}} |M_{J/\psi}|^2 = 1 + F \cos^2 \theta \]

\[ F = \frac{(M^2 - 4m^2)(1 - r^2)}{(1 + r^2)(M^2 + 4m^2) - 8Mr} \quad r := \frac{a_\psi}{2m^2_\psi a_\psi + 1}. \]

M and m are the masses of J/ψ and Λ respectively. θ is the angle between the e⁺e⁻ beam axis and the direction of Λ. The observed value of F is 0.65 ± 0.11 [16], which gives two possible solutions: a_ψ = 2.46 ± 0.18 and a_ψ = 0.54 ± 0.18. In our calculation here, both lead to identical results. (It probably depends only on F.)

The angular distribution for the coherent process c̅c → Λ̅Λ → pπpπ is therefore obtained, assuming unpolarized initial and final states:

\[ \left( \frac{d\sigma}{d\Omega_A d\Omega_p d\Omega_\bar{p}} \right) \propto \text{ave} \left( \sum_{s_A, s_{\bar{\Lambda}}} |\mathcal{M}|^2 \right) = |\mathcal{M}_0|^2. \]

\[ d\Omega_p (d\Omega_\bar{p}) \] is the solid angle element in the proton (anti-proton) momentum space in the Λ (̅Λ) rest frame. The average is taken over the spins of the particles in their initial and final states. Note that the sums over s_A and s_̅Λ are taken before squaring, which leads to the terms representing quantum
interference between the two intermediating spin states. The correlation matrix \( C \) is calculated as:

\[
C_{ij} = \frac{n_i n'_j}{2a_i^2} = \frac{p_i p'_j}{2a_i^2} \int d\Omega_\Lambda d\Omega_{\tilde{\Lambda}} p_i p'_j \text{Prob}(p, p', \Lambda) = \frac{9}{2a_i^2 |p|^2} \int d\Omega_\Lambda d\Omega_{\tilde{\Lambda}} p_i p'_j \left( \frac{d\sigma}{d\Omega_\Lambda d\Omega_{\tilde{\Lambda}}} / \sigma_{\text{tot}} \right) = \frac{9}{2a_i^2 |p|^2} \int d\Omega_\Lambda d\Omega_{\tilde{\Lambda}} p_i p'_j |M_0|^2 / \int d\Omega_\Lambda d\Omega_{\tilde{\Lambda}} d\Omega p |M_0|^2.
\]  

(23)

\( p (p') \) is defined as the proton (anti-proton) momentum in the \( \Lambda (\tilde{\Lambda}) \) rest frame. (Recall that the momenta \( p_p (p_{\tilde{p}}) \) used in computing \( |M_0|^2 \) above are in the decaying meson rest frame.) The results for \( C_{ij} \) and \( Q_{\text{max}} \) for each channel are shown in Table 2. The off-diagonal components are always 0, reflecting the symmetry in the processes, thus \( Q_{\text{max}} = 2\sqrt{C_{11}^2 + C_{33}^2} \) following (18).

The systematic uncertainties are estimated by shifting the parameters used in the calculation (e.g. the particle masses, \( c_\Lambda, a_\psi, \) etc.) within their 1\( \sigma \) experimental uncertainties. The main contribution is from the uncertainty of \( a_\psi \), while those from the other parameters have almost no effect on \( C \) or \( Q_{\text{max}} \). One sees that \( Q_{\text{max}} \) in the \( \eta_c \) and the \( \chi_{c0} \) channel well surpass the classical limit \( Q_{\text{CL}} = 1 \) and even reach the quantum limit \( \sqrt{2} \). In contrast, the \( J/\psi \) channel gives no significant excess in \( Q_{\text{max}} \), and therefore is insensitive to test BI. The \( \eta_c \) and the \( \chi_{c0} \) channels conserve their entanglement throughout the process whereas the \( J/\psi \) channel loses the spin correlation in \( J/\psi \rightarrow \Lambda \tilde{\Lambda} \) where the relative orbital angular momentum between the two hyperons dilutes the spin correlation by some fraction.

### 5. Estimation of necessary event number and the corresponding significance

For the \( \eta_c \) and \( \chi_{c0} \) channels, the BI is violated with the large \( Q_{\text{max}} \) values which, however, experience statistical fluctuations with a limited number of events. In this section, we show how the components of \( C \) and \( Q_{\text{max}} \) fluctuate statistically. Using Monte-Carlo (MC) simulations based on matrix (19), the distributions of \( C_{11}, C_{33}, \) and \( Q_{\text{max}} \) are calculated as in Fig. 3, where we set the guide axes to the configuration that gives the maximum \( Q \) value as we discussed in Sect. 3, i.e. \( Q_{\text{max}} = 2\sqrt{C_{11}^2 + C_{33}^2} \).
Fig. 3. Distributions of $C_{11}, C_{33}$, and $Q_{\text{max}}$ in the $\eta_c$ and $\chi_{c0}$ channels.

$C_{11}$ and $C_{33}$ fluctuate according to Gaussian distributions for all $n$. On the other hand, the $Q_{\text{max}}$ distribution has a slight positive bias and can be approximated well by a Gaussian distribution providing $n \gtrsim 1000$. The corresponding mean, RMS, and significance are given in Table 3. The significance is calculated as the corresponding deviation in a Gaussian distribution of the p-value that $Q_{\text{max}}$ fluctuates under the classical limit $Q_{\text{CL}} = 1$, as plotted in Fig. 4. The two channels have the same $n$ dependency and we see about 2000 events are sufficient to announce evidence of the BI violations.

The number of experimentally available events and the measurement feasibility were also studied. The branching fractions of each relevant decay are listed in Table 4, assuming that $\eta_c$ and $\chi_{c0}$ are all produced via $J/\psi \to \eta_c + \gamma$ and $\psi' \to \chi_{c0} + \gamma$ respectively. In the BES3 experiment, $1 \times 10^9$ of $J/\psi$ and $4 \times 10^8$ of $\psi'$ are planned to be produced by the end of 2012 [17]. Using only the space-like separated event, $s$ which account for around 70% of all events (Table 1), the event yields are about $6500 \times \epsilon$ and $4000 \times \epsilon$ for the $\eta_c$ and the $\chi_{c0}$ channel respectively, with event acquisition efficiency $\epsilon$. The $\eta_c$ channel has enough yield even with conservative selection of events (efficiency $\epsilon \sim 0.3$), while the $\chi_{c0}$ channel is available with a looser selection of $\epsilon \gtrsim 0.6$. 
Table 3. Some characteristics of the $Q_{\text{max}}$ distribution.

| $\eta_c$ channel | $\chi_{c0}$ channel |
|------------------|---------------------|
| $n$              | mean  | RMS  | significance | $n$              | mean  | RMS  | significance |
| 100              | 1.619 | 0.653| 1.33         | 100              | 1.617 | 0.652| 1.33         |
| 300              | 1.477 | 0.403| 1.55         | 300              | 1.479 | 0.405| 1.55         |
| 500              | 1.454 | 0.316| 1.77         | 500              | 1.456 | 0.314| 1.77         |
| 1000             | 1.432 | 0.223| 2.20         | 1000             | 1.433 | 0.225| 2.20         |
| 2000             | 1.423 | 0.159| 2.86         | 2000             | 1.423 | 0.159| 2.86         |
| 3000             | 1.420 | 0.131| 3.43         | 3000             | 1.421 | 0.131| 3.38         |
| 5000             | 1.417 | 0.100| 4.24         | 5000             | 1.418 | 0.101| 4.28         |

Fig. 4. Achievable significance with respect to the number of events is estimated by MC simulation. We use a large number of samples where MC fluctuation is negligibly small.

Table 4. Branching fractions of each decay in $J/\psi \rightarrow \gamma \eta_c$; $\eta_c \rightarrow \Lambda \Lambda \rightarrow p\pi \bar{p}\pi \pi$ and $\psi' \rightarrow \gamma \chi_{c0}$; $\chi_{c0} \rightarrow \Lambda \Lambda \rightarrow p\pi \bar{p}\pi$.

| channel                      | branching fraction |
|------------------------------|--------------------|
| $J/\psi \rightarrow \eta_c + \gamma$ | $(1.7 \pm 0.4) \times 10^{-2}$ |
| $\psi' \rightarrow \chi_{c0} + \gamma$ | $(9.7 \pm 0.3) \times 10^{-2}$ |
| $\eta_c \rightarrow \Lambda + \bar{\Lambda}$ | $(1.41 \pm 0.17) \times 10^{-3}$ |
| $\chi_{c0} \rightarrow \Lambda + \bar{\Lambda}$ | $(3.3 \pm 0.4) \times 10^{-4}$ |
| $\Lambda \rightarrow p + \pi^-$ | $(6.39 \pm 0.05) \times 10^{-1}$ |
| $\bar{\Lambda} \rightarrow \bar{p} + \pi^+$ | $(6.39 \pm 0.05) \times 10^{-1}$ |
| $J/\psi \rightarrow \gamma \eta_c$; $\eta_c \rightarrow \Lambda \bar{\Lambda}$ | $(9.8 \pm 2.6) \times 10^{-6}$ |
| $\psi' \rightarrow \gamma \chi_{c0}$; $\chi_{c0} \rightarrow \Lambda \bar{\Lambda}$ | $(1.31 \pm 0.16) \times 10^{-5}$ |

6. Conclusion

The charmonium decays $\eta_c \rightarrow \Lambda \bar{\Lambda}$, $\chi_{c0} \rightarrow \Lambda \bar{\Lambda} \pi \pi$, and $J/\psi \rightarrow \Lambda \bar{\Lambda}$ are possible probes of Bell’s inequality, due to the strongly correlated spins of the hyperon pair. As the hyperons undergo decays in which the angular distribution is associated with the hyperon spins, Bell’s inequality can be developed into a new expression in terms of a momentum correlation of the decay products p and $\bar{p}$. By QM calculation we find that the correlation in the $J/\psi$ channel is not strong enough to test Bell’s inequality,
while it is sufficient in the other two channels. The violation can be experimentally confirmed with around 2000 events for each channel; this number of events would already have been produced at BES, and this experiment is therefore now feasible.

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Appendix. The derivation of Eq. (5)

We start with the angle distribution of the protons (anti-protons) from $\Lambda$ ($\bar{\Lambda}$) decays—the same equation as (1) in Sect. 1:

$$ P(n|s) = 1 + \alpha_n n \cdot s \quad P(n'|s') = 1 - \alpha_n n' \cdot s'. $$

(A1)

$n(n')$ is the unit vector of the proton (anti-proton) orientation. $P(n|s) (P(n'|s'))$ indicates the conditional probability density of $n (n')$ with the given polarization of the hyperon $s (s')$. Note that these distributions are normalized to 1 with $n (n')$ integration over the solid angle $\int \Omega_n/4\pi (\int \Omega_{n'}/4\pi)$. The correlation amplitude $\langle (n \cdot a)(n' \cdot b) \rangle$ can be calculated in terms of these distributions:

$$ \langle (n \cdot a)(n' \cdot b) \rangle = \int \frac{d\Omega_n d\Omega_{n'}}{4\pi} \frac{d\Omega_\alpha}{4\pi} \frac{d\Omega'_\alpha}{4\pi} (n \cdot a)(n' \cdot b) P(n, n') $$

$$ = \int \frac{d\Omega_n d\Omega_{n'}}{4\pi} \frac{d\Omega_\alpha}{4\pi} \frac{d\Omega'_\alpha}{4\pi} (n \cdot a)(n' \cdot b) P(n, n'|s, s') P(s, s'). $$

$P(x)$ represents the probability density function which a set of variables $x$ follow. When the two hyperon decays are isolated in space-time, as confirmed in Sect. 2, the joint probability density $P(n|s) P(n'|s')$ should be identical to the combined probability density $P(n, n'|s, s')$ since there is no correlation between them, according to the locality principle. Using (A1), therefore,

$$ \langle (n \cdot a)(n' \cdot b) \rangle $$

$$ = \int \frac{d\Omega_n d\Omega_{n'}}{4\pi} \frac{d\Omega_\alpha}{4\pi} \frac{d\Omega'_\alpha}{4\pi} (n \cdot a)(n' \cdot b) P(n|s) P(n'|s') P(s, s') $$

$$ = \int \frac{d\Omega_n d\Omega_{n'}}{4\pi} \frac{d\Omega_\alpha}{4\pi} \frac{d\Omega'_\alpha}{4\pi} (n \cdot a)(n' \cdot b)(1 + \alpha n \cdot s)(1 - \alpha n' \cdot s') P(s, s'). $$

The $d\Omega_n d\Omega_{n'}$ integration can be performed using spherical polar coordinates $(\theta, \phi)$, $(\theta', \phi')$ with $s, s'$ being the zeniths:

$$ \langle (n \cdot a)(n' \cdot b) \rangle $$

$$ = \int \frac{d\cos \theta d\phi d\cos \theta' d\phi'}{4\pi} \frac{d\Omega_\alpha}{4\pi} \frac{d\Omega'_\alpha}{4\pi} (a_1 \sin \theta \cos \phi + a_2 \sin \theta' \sin \phi + a_3 \cos \theta) $$

$$ \times (b_1 \sin \theta' \cos \phi' + b_2 \sin \theta' \sin \phi' + b_3 \cos \theta')(1 + \alpha \cos \theta)(1 - \alpha \cos \theta') P(s, s'). $$
Here, the $\cos \phi$ and $\sin \phi$ terms vanish in the $\phi$ integral, and thus only $\cos \theta$ terms remain. As $a_3 = a \cdot s = s_a$ and $b_3 = b \cdot s' = s'_b$, we obtain the desired form:

$$
\langle (n \cdot a)(n' \cdot b) \rangle = \left( \int \frac{d \cos \theta \ d \cos \theta'}{2}(1 + \alpha \Lambda \cos \theta)(1 - \alpha \bar{\Lambda} \cos \theta') \right) \cos \theta \cos \theta' \left( \int \frac{d \Omega_s \ d \Omega_{s'}}{4\pi} s_a s'_b P(s, s') \right) = -\frac{\alpha \Lambda \alpha \bar{\Lambda}}{9} \langle s_a s'_b \rangle.
$$

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