Learning sparse representations on the sphere

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Abstract

Many representation systems on the sphere have been proposed in the past, such as spherical harmonics, wavelets, or curvelets. Each of these data representations is designed to extract a specific set of features, and choosing the best fixed representation system for a given scientific application is challenging. In this paper, we show that we can learn directly a representation system from given data on the sphere. We propose two new adaptive approaches: the first is a (potentially multi-scale) patch-based dictionary learning approach, and the second consists in selecting a representation among a parametrized family of representations, the $\alpha$-shearlets. We investigate their relative performance to represent and denoise complex structures on different astrophysical data sets on the sphere.

Key words. Methods: statistical, Methods: data analysis, Methods: numerical

1. Introduction

Wavelets on the sphere (Starck et al. 2015) are now standard tools in astronomy and have been widely used for purposes such as FERMI-LAT data analysis (Schmitt et al. 2010, McDermott et al. 2016), the recovery of CMB and polarized CMB maps (Bobin et al. 2015, 2016), string detection (McEwen et al. 2017), point source removal in CMB data (Sureau et al. 2014), the detection of CMB anomalies (Naidoo et al. 2017, Rassat et al. 2014), or stellar turbulent convection studies (Bessolaz & Brun 2011). While wavelets are well suited for representing isotropic components in an image, they are far from optimal for analyzing anisotropic features such as filamentary structures. This has motivated in the past the construction of so called multiscale geometric decompositions such as ridgelets, curvelets (Candes & Donoho 2004, Starck et al. 2003), bandelets (Le Pennec & Mallat 2005), or shearlets (Labate et al. 2005a). Extensions to the sphere of ridgelets and curvelets were already presented in (Starck et al. 2006, Chan et al. 2017, McEwen 2015), and also for spherical vector field data sets in (Starck et al. 2009, Leistedt et al. 2017).

For a given data set, we therefore have the choice between many fixed representation spaces (pixel domain, harmonics, wavelets, ridgelets, curvelets, etc) which are also called dictionaries. A dictionary is a set of functions, named atoms, and the data can be represented as a linear combination of these atoms. The dictionary can be seen as a kind of prior (Beckouche et al. 1999a, Aharon et al. 2006a) have been proposed to design a dictionary directly from the data, in such a way that the data can be sparsely represented in that dictionary. DL has been used in astronomy for image denoising (Beckouche et al. 2013), stellar spectral classification (Diaz-Hernandez et al. 2014) and morphological galaxy classification (Diaz-Hernandez et al. 2016).

An alternative approach for adaptively choosing a dictionary is to start with a large parametrized family of dictionaries, and then to choose the parameter(s), either based on simulations or directly from the data. An example of such a parametrized family of dictionaries is the family of $\alpha$-shearlets (Labate et al. 2005a, Grohs et al. 2016, Voigtlaender & Pein 2017).

In this paper, we propose to extend to the sphere both adaptive representation methods, DL and $\alpha$-shearlets, and we compare the performance of the two approaches. More precisely, we are concerned with adaptive sparsifying representation systems for data defined on the sphere. In Section 2, we present our approach for performing DL on the sphere, while Section 3 is devoted to our extension of the $\alpha$-shearlet transform to data defined on the sphere. We present the scenarios for our comparison of the two approaches in Section 4, the results of this comparison are presented in Section 5. Finally, we conclude the paper in Section 6 and the necessary background related to $\alpha$-shearlets in the Euclidean setting is covered in Appendix A.

2. Dictionary learning on the sphere

Dictionary learning techniques have been proposed in the early 2000s (Olshausen & Field 1996, Engan et al. [1999b, Aharon et al. 2006b]) to build adapted linear representations that yield sparse decompositions of the signals of interest. Contrary to fixed dictionaries, in dictionary learning the atoms are estimated from the data (or a proxy, such as simulations or exemplars of the data), and can therefore model more complex geometrical content, which could ultimately result in sparser (and typically redundant) representations. The application of DL techniques
to many inverse problems in restoration, classification, and texture modeling has provided state-of-the-art results (see e.g. Elad & Aharon (2006); Mairal et al. (2008a,2009); Peyré (2009); Zhang & Li (2010)). A wide variety of dictionary learning techniques have been proposed to process multivariate data (Mairal et al. 2008a,b); to construct multiscale (Mairal et al. 2008b); translation-invariant (Jost et al. 2006; Aharon & Elad 2008), or hierarchical representations (Jenatton et al. 2011); to estimate coupled dictionaries (Rubinstein & Elad 2014); or to build analysis priors (Rubinstein et al. 2013). Also, online algorithms for dictionary learning have been considered (Mairal et al. 2010).

While fixed structured representations typically have fast direct and inverse transforms, dictionary learning techniques become computationally intractable even for signals of moderate size. Based on the observation that natural images exhibit nonlocal self-similarities, this computational problem is typically overcome by performing dictionary learning on patches extracted from the images that one wants to model. In this section we focus on this patch-based dictionary learning approach, and extend it for signals living on the sphere.

2.1. Sparse representation with patch-based dictionary learning

Given an \( n \times n = N \) image represented as a vector \( X \in \mathbb{R}^N \), we consider square overlapping patches \( x_{ij} \in \mathbb{R}^q \), with \( q = q \times q \), where \( q \) is typically small; in fact, in the present work we will always have \( q \leq 12 \). Formally,

\[
x_{ij} = R_{ij}X
\]

where the matrix \( R_{ij} \in \mathbb{R}^{q\times N} \) extracts a patch with upper left corner at position \((i,j)\).

From a training set \( T \) of such patches \( \{x_{ij}\}_{(i,j)\in T} \), a dictionary with \( M \) atoms \( D \in \mathbb{R}^{Q\times M} \) is then learned such that the codes \( \Lambda = \{\lambda_{ij}\}_{(i,j)\in T} \) satisfying \( x_{ij} = D\lambda_{ij} \) are sparse. To perform the training, one typically considers the following following inverse problem, or one of its variants:

\[
\arg\min_{\Lambda \in \mathcal{D} \cap \mathcal{C}} \sum_{(i,j)\in T} \|x_{ij} - D\lambda_{ij}\|_2^2 + \mu \cdot \|\lambda_{ij}\|_0
\]

where \( \mathcal{D} \) (resp. \( \mathcal{C} \)) is a non-empty convex set enforcing some constraints on the dictionary \( \mathcal{D} \) (resp. the codes \( \Lambda \)), and \( \mu \cdot \|\lambda_{ij}\|_0 \) is the weighted 0-pseudo-norm which enforces sparsity of the codes. To remove the scale indeterminacy in such a minimization problem—that is, if \( (D,\Lambda) \) is a solution, then so is \( (\alpha D, \alpha^{-1} \Lambda) \), at least if \( \alpha D \in \mathcal{D} \) and \( \alpha^{-1} \Lambda \in \mathcal{C} \)—the set \( \mathcal{D} \) is typically enforcing each atom (column) of the dictionary to belong to a unit \( L_2 \) ball, while \( \mathcal{C} \) can enforce constraints in the code (e.g. non-negativity in non-negative matrix factorization). More details can be found in Starck et al. (2013).

2.2. Extension of patch-based dictionary learning to the sphere

To extend patch-based dictionary learning to data defined on the sphere, we first need to specify how to construct patches on the sphere. We do so by introducing local charts on the sphere. Specifically, in this work we propose to consider the HEALPix framework (Górski et al. 1999,2005), widely used in astronomy, to construct these charts.

2.2.1. Defining patches on the sphere

HEALPix partitions the sphere into equal area pixels with curvilinear boundaries, defined hierarchically from a set of twelve base quadrilaterals (see Fig. 1). These twelve base elements (or faces) form an atlas of the sphere, and are further partitioned dyadically to obtain finer discretization levels. Consequently, each of the twelve faces is typically considered as a chart with HEALPix pixel positions mapped to a square grid in \([0,1] \times [0,1] \).

Using these charts to perform usual Euclidean patch-based dictionary learning is straightforward, and would have the main advantage of applying dictionary learning directly on the pixel values, without requiring any interpolation. This comes, however, with two drawbacks: first, this approach introduces boundary issues even when using overlapping patches on each face; second, it leads to distortions for band-limited functions defined on the sphere. While the second problem is inherent to the choice of HEALPix as a discretization of the sphere, we can however address the first problem in this framework: patches can be created based on local neighbors as defined in the HEALPix framework. Because of the regularity of the HEALPix sampling scheme, all pixels have eight neighbors, except for eight pixels on the sphere that are located at the vertices in between equatorial and polar faces, which only have seven neighbors.

Provided some care is taken on defining the respective position of each neighbour to a central pixel across the sphere, overlapping patches can be created—even in between the twelve HEALPix faces—without any interpolation, except at the patches crossing the specific points on the HEALPix grid which only have seven neighbors. Interpolation strategies to compensate for these “missing” neighbors can be envisioned; but in this work we choose not to interpolate, which implies that for a few pixels around these points, we do not construct all overlapping patches. The final covering of the map is illustrated in Fig. 2. Once these patches are extracted, classical dictionary learning techniques can be used to learn a sparse adapted representation.

![Figure 1. The HEALPix grid (visualizing Nside = 16) in orthographic projection on the left and Mollweide projection on the right. Faint right lines indicate the circles of latitude \( \theta = \cos^{-1}(\pm \frac{1}{2}) \). The right image also introduces the numbering of the faces, used in the following illustrations.](image)

2.2.2. Learning a multi-scale representation on the sphere

Our proposed approach for dictionary learning on the sphere can be extended to capture multiscale information as proposed in Ophir et al. (2011): a dictionary is learned from patches extracted from a multiscale decomposition of the data.

At lower scales, capturing information would require to increase the patch size, and would ultimately lead to a computational burden impossible to handle. To capture this information without increasing the patch size, the decomposition is subsampled.
Figure 2. Example of our coverage of the sphere with overlapping patches based on HEALPix neighborhoods. The plotted value indicates the number of overlapping patches including each pixel, for $N_{side} = 128$ and patch width $q = 5$. Because the patch width is usually small with respect to the number of pixels per face, only a small fraction of patches is not considered (in this work, $N_{side} = 2048$ and the patch width is $q = 8$ or $q = 12$).

Table 1. Parameters used for learning the multiscale dictionary for thermal dust data. For each Starlet scale, the maximal multipole $ℓ_{max}$, the $N_{side}$ parameter, the number of patches, their width $q$, the number of atoms $M^{(s)}$, the maximal sparsity $K^{(s)}$, and the number iterations $N_t$ are displayed.

| Scale | $ℓ_{max}$ | $N_{side}$ | NPatch | $q$ | $M^{(s)}$ | $K^{(s)}$ | $N_t$ |
|-------|------------|------------|--------|-----|----------|--------|------|
| 0     | n.a.       | 2048       | 200k   | 12  | 256      | 10     | 100  |
| 1     | 1024       | 512        | 50k    | 12  | 256      | 20     | 100  |
| 2     | 512        | 256        | 25k    | 12  | 256      | 30     | 100  |

In this work, we use the Starlet decomposition for data on the sphere (Starck et al. 2006), with one dictionary learned per wavelet scale. Since all scales except the first one are band-limited, subsampling can be performed without loosing information by adapting the $N_{side}$ parameter to the maximal multipole at the level considered (typically dyadically decreasing, as illustrated in Table 1).

The resulting minimization problem for the multiscale dictionary learning problem reads:

$$
\arg\min_{\{D^{(s)}\}_{s=0}^{S}, \{\Lambda^{(s)}\}_{s=0}^{S}, \{\ell\}} \sum_{s=0}^{S} \sum_{(i,j) \in T^{(s)}} \|R_{ij}/W^{(s)}X - D^{(s)}\Lambda_{ij}^{(s)}\|_2^2 + \mu^{(s)}\|\Lambda_{ij}^{(s)}\|_0 \quad (3)
$$

where $X$ is the signal on the sphere, $W^{(s)}$ extracts the scale $s$ of the wavelet transform on the sphere according to the $N_{side}$ chosen for that scale, $R_{ij}$ is now extracting patches according to neighbors on the sphere for the patch indexed by $(i,j)$ at scale $s$ in training set $T^{(s)}$, and $S$ is the total number of wavelet scales. For each scale $s = 0, \ldots, S$, a dictionary $D^{(s)}$ is therefore learned, giving coefficients $\Lambda_{ij}^{(s)}$ collected in $\Lambda^{(s)}$; the hyperparameter $\mu^{(s)}$ is also allowed to change with the scale.

Because the cost function is separable per scale, the minimization problem (3) is equivalent to solving $S + 1$ dictionary learning sub-problems associated to each wavelet scale.

2.3. Our algorithm for patch-based dictionary learning on the sphere

In the training phase, the joint nonconvex problems described in Eqs. (2)-(3) are typically handled by alternating sparse coding steps and dictionary update steps.

Here, a sparse coding step means that one minimizes Eq. (2) (resp. Eq. (3)) with respect to $\Lambda$ (resp. $\Lambda^{(s)}$), with a fixed previously estimated dictionary. Similarly, a dictionary update step means that one minimizes Eq. (2) (resp. Eq. (3)) with respect to $D$ (resp. $D^{(s)}$), with the fixed previously estimated codes. Note that both sub-problems can be minimized with standard algorithms. In this work, we will use the classical dictionary learning technique K-SVD (Aharon et al. 2006b) with Orthogonal Matching Pursuit (OMP) (Mallat & Zhang 1993), Pati & Krishnaprasad 1993 as a sparse coder. For denoising applications, the sparse coding step will encompass both a maximal sparsity level, and an approximation threshold based on the $\ell_2$ norm of the residual, similar to the approach in Elad & Aharon (2006). This approach resulted in adapted sparse representations, while not being sensitive to small fluctuations below the targeted level of approximation, and in practice led to faster algorithms.

The resulting multiscale dictionary learning algorithm is described in Algorithm 1, from which its variant without the multiscale transform can be obtained for $S = 0$ and $W^{(0)} = I_d$.

**Algorithm 1 Multiscale Dictionary Learning on the Sphere**

1: **Initialization**: For each scale $s = 0, \ldots, S$, choose the number of atoms $M^{(s)}$, a maximal sparsity degree $K^{(s)}$, a maximal approximation error $\epsilon^{(s)}$. Initialize the dictionary. Choose the number of iterations $N_t$.

2: **Patch Extraction**: For each scale $s$, extract randomly patches $[R_{ij}/W^{(s)}X]_{j \in T^{(s)}}$ on the sphere. Subtract from each patch its mean value.

3: for $s = 0$ to $S$ do (Subproblem for scale $s$)

5: for $n = 0$ to $N_t$ do (Main Learning Loop)

6: Compute the sparse code $\Lambda_{ij}^{(s)}$ using OMP with stopping criterion $\|R_{ij}/W^{(s)}X - D^{(s)}\Lambda_{ij}^{(s)}\|_2 < \epsilon^{(s)}$ or $\|\Lambda_{ij}^{(s)}\|_0 > K^{(s)}$

7: end for

8: Update $D^{(s)}$ using K-SVD (Aharon et al. 2006b) (Dictionary Update)

9: end for

10: end for

11: return $[D^{(s)}]_{s=0}^{S}$

The first critical choice for this dictionary learning technique is to adapt the patch size $q$ to capture information at the scale of the patch without impacting too much the computational burden of the algorithm ($q$ is at most 12 in this work). The maximal sparsity degree $K^{(s)}$ and the number of atoms $M^{(s)}$ should be selected so that the dictionary leads to small approximation errors, while being able to capture the important features with only a few atoms, in particular for denoising applications. The parameter $\epsilon^{(s)}$ is the level of the noise expected in the denoising application at the considered wavelet scale, and the number of iterations is in practice chosen sufficiently large so that the average approximation error does not change with iterations. Because this problem is non-convex, it is crucial to initialize the algorithm with a meaningful dictionary; in our case, the initial dictionary is chosen to be an overcomplete discrete cosine transform (DCT) dictionary as in Elad & Aharon (2006).

3. $α$-shearlets on the sphere

3.1. Euclidean $α$-shearlets

$α$-shearlets are a family of representations that generalizes wavelets and shearlets; the family is parameterized by the
anisotropy parameter $\alpha \in [0,1]$). To each parameter $\alpha$ corresponds a dictionary characterized by:

- atoms with a “shape” governed by height $= \text{width}^\alpha$ (see Fig. A.2);
- a directional selectivity: on scale $j$, an $\alpha$-shearlet system can distinguish about $2^{(1-\alpha)j}$ different directions (see Fig. A.3);
- a specific frequency support for the atoms (see Fig. A.3).

A key result (Voigtländer & Pein 2017) is that $\alpha$-shearlets are almost optimal for the approximation of so-called $C^\beta$-cartoon-like functions, a model class for natural images. More precisely, the $N$-term $\alpha$-shearlet approximation error (that is, the smallest approximation error that can be obtained using a linear combination of $N$ $\alpha$-shearlets) for a $C^\beta$-cartoon-like function is decreasing at (almost) the best rate that any dictionary can reach for the class of such functions. For this to hold, the anisotropy parameter $\alpha$ needs to be adapted to the regularity $\beta$, that is, one needs to choose $\alpha = 1/\beta$. For more details on this, we refer to Appendix A.

In general, given a certain data set, or a certain data model, different types of $\alpha$-shearlet systems will be better adapted to the given data than other $\alpha'$-shearlet systems. Thus, having such a versatile, parametrized family of representation systems is valuable to adapt to a variety of signals to recover.

### 3.2. Extending $\alpha$-shearlets to the sphere

In order to define the $\alpha$-shearlet transform on the sphere, similarly to what was discussed for the dictionary learning approach, we need to define the charts on which the Euclidean $\alpha$-shearlet transform will be applied. HEALPix faces are again an obvious candidate since these bases are composed of $N_{\text{side}}^2$ equally spaced pixels, although their shape is contorted in different ways on the sphere (see Fig. 1). We could map the sphere to these twelve square faces and then take the $\alpha$-shearlet transform on every one of them individually. However, as for dictionary learning, this approach to the processing of HEALPix data (e.g. for the task of denoising) is deemed to introduce boundary artefacts due to the disjoint nature of the partition. An example of such artefacts can be seen in the upper-left part of Fig. 1, shown in Section 5. Note also that contrary to the patch-based dictionary learning where the patch size remains typically small compared to a face size, the increasing size of the $\alpha$-shearlet atoms when going to lower scales can introduce large border effects.

In the following two subsections, we discuss two approaches for handling this problem.

#### 3.2.1. The rotation-based approach

The first strategy to alleviate the block artefacts was proposed for curvelets in Starck et al. (2006). This approach relies on considering overlapping charts that are obtained by considering HEALPix faces after resampling the sphere through a small number of rotations.

More precisely, for a given Euclidean $\alpha$-shearlet system, a HEALPix face $f$, and a rotation $r$, the redundant coefficients are obtained by:

$$\lambda_{\alpha,x,f} = S_\alpha \left( H_f R_r (X) \right), \quad (4)$$

where $R_r$ is computing the resampled map by a rotation $r$ of the sphere, $H_f$ is a matrix extracting the pixels that belong to the HEALPix face $f$, and $S_\alpha$ is computing the Euclidean $\alpha$-shearlet transform on this face. In practice, a bilinear interpolation is performed by the HEALPix rotation routines that are used for the resampling.

The reconstruction is performed using a partition of unity on the sphere (see Fig. 3), which is obtained from weights that are smoothly decaying from 1 in a central region of the faces to 0 at their borders and therefore mitigating border effects. Formally, the reconstruction reads:

$$\tilde{X} = N \sum_{r \in \mathbb{R}} \sum_{j=1}^{12} R_{r}(H_j^\alpha M^T \alpha_j \lambda_{\alpha,x,f}), \quad (5)$$

where $R_{r}$ resamples the sphere with the inverse rotation matrix, $T_{\alpha}$ is computing the inverse $\alpha$-shearlet transform, $M$ applies weights, and the normalization matrix $N$ is chosen such that $N \sum_{r,f} R_{r}(H_j^\alpha M^T) = I$ where $I$ is a vector with all entries equal to 1. An example of the weights and normalization maps used to construct this partition of unity are illustrated in Fig. 3.

#### Figure 3. Partition of unity for the rotation-based reconstruction. The weights smoothly decaying toward the border are presented in the top left and are copied to each HEALPix face in the top right. In the bottom left figure, resampling was first performed using a rotation and bilinear interpolation, and the image shows the weights that would be applied in the original reference coordinates. The resulting covering of the sphere using 5 rotations is illustrated in the last part of the figure.

Note that since the rotations $R_r$ and $R_{r'}$ are implemented using interpolation, it is not true exactly that $R_{r'} R_r X = X$. Therefore, even if the coefficients $\lambda_{\alpha,r,f}$ are obtained through Eq. (4), the reconstruction in Eq. (5) will only satisfy $\tilde{X} \approx X$, not $\tilde{X} = X$. However, the error introduced by the inexact inverse rotation is often negligible, at least for sufficiently smooth signals; see Section 5.2 for a further comment on this.

#### 3.2.2. The “patchwork” approach

The “patchwork” approach is another strategy to eliminate artefacts that arise if one naively uses the disjoint HEALPix faces. In contrast to the rotation-based technique, where an interpolation is performed during the resampling, the patchwork approach is based on extending the HEALPix faces using parts of the surrounding faces so as to avoid interpolation. Similar to the rotation-based approach, the six resulting extended faces (see Fig. 4) form a redundant covering of the sphere, which is beneficial for avoiding boundary artefacts. Once these six extended
faces are computed, the $\alpha$-shearlet transform and all further processing are performed on these faces. Of course, for the reconstruction, the last step consists in combining the redundant faces to get back a proper HEALPix map.

Formally, the decomposition can be described as follows:

$$\lambda_{s,f} = \mathcal{S}_s \left( \mathcal{P}_f (X) \right),$$  \hspace{1cm} (6)

where $\mathcal{P}_f$ is now the operator that extracts the extended face $f$ from the HEALPix map $X$. Similarly, the reconstruction reads:

$$\bar{X} = \mathcal{M} \left[ \left( T_{\alpha} \left( \lambda_{s,f} \right) \right)_{f=1,...,6} \right],$$  \hspace{1cm} (7)

where $\mathcal{M}$ is the operator that reconstructs a HEALPix map from data on the six extended faces.

The rest of this section explains how precisely the extended faces are obtained from the original HEALPix faces, and conversely how a HEALPix map can be obtained from data on these six extended faces. For an accompanying visual explanation of the procedure, the reader should consult Figures 1, 4, and 5.

Each of the six extended faces consists of an inner square with HEALPix pixels that are unique to this extended face, and a border zone with HEALPix pixels that appear in several of the extended faces. The border itself is again subdivided in an outer margin that is disregarded after the reconstruction step so that the artefacts at the boundary are cut off (not mapped to the sphere), and an inner part that forms a transition zone, where the values of neighboring faces are blended together, to prevent visible discontinuities between them.

Instead of extending all twelve original faces, we combine them to six bigger composite faces and extend those. This reduces the number of additional pixels that have to be processed (when using a border of the same size), at the cost of increased memory requirements. The first two composite faces cover the bulk of the north and south polar regions, and particularly the poles itself. Since the four faces of each polar region meet at the poles, we can arrange those four faces to form a square around the pole. It only remains to clip this area to the requested size. Although there is much freedom to set the extent of the individual composite faces, we prefer all squares to be of equal size, so that they can be processed without distinction. The remaining four composite faces are obtained by expanding the equatorional faces. An expansion of the equatorial faces by $\frac{N_p}{16}$ in each direction results in areas of width $\frac{N_p}{16}$, that each contain a fourth of every surrounding polar face. By removing those parts from the polar areas, constructed earlier, those are truncated to the same width (see Fig. 5). Thus, we get six areas of equal size that cover the sphere. Chosen this way, there is still no overlap between the polar and equatorial composite faces; therefore we extend each face further by half the requested width of the transition zone. We chose an extension of width $\frac{N_p}{16}$ (that is $c_1$ in Fig. 5). Since each face enters their neighbors territory by that amount, this results in a transition zone of width $\frac{N_p}{16}$ between each face. Additionally each face is extended by a margin (that is $c_2$ in Fig. 5) to avoid border artefacts. Here, a margin of width $\frac{N_p}{16}$ was chosen.

However, to extend the equatorial faces, we have to address the problem that there are eight vertexes where two faces of a polar region meet a face of the equatorial region (located on the circles of latitude $\theta = \cos^{-1}(\pm 2/3)$, depicted in Fig. 1). By arranging the twelve faces as a net in the plane—as illustrated in Fig. 4—it becomes clear that there are gaps between the polar faces, where no values exist; these areas are marked in red in Fig. 4. We need to fill those gaps in order to obtain rectangular extended faces, to which we can apply the $\alpha$-shearlet transform. In the end, these parts will be cut away and disregarded like the outer margin of the extension, so the filled in values will not actually be used for the reconstruction. Nevertheless, we need to be careful, since otherwise we might introduce additional artefacts like the ones at the boundary.

For the sake of simplicity, we will describe the situation at the edge between faces 1 and 2 (see Figures 1, 4, and 5), which is exemplary for all gaps: From the perspective of face 2, the missing square is expected to feature a rotated copy of face 1, while conversely face 1 expects a rotated copy of face 2. To fabricate a weighted blending of those anticipated values, we divide the empty square, interpreted as $[0,1]^2$, along the lines $2x = y$, $x = y$, and $y = 2x$, into quarters, as demonstrated in Fig. 6. On both outer quarters the full weight is assigned to the face which the adjoining face expects, while the two middle quarters serve to produce a smooth transition. All weights are normalized in such a way that every pixel is a convex combination of the pixels of the two faces; that is, the weights are non-negative and their sum is one at each pixel.

![Figure 4](image4.png)

**Figure 4.** Left: The twelve squares corresponding to the faces of the HEALPix framework (see Fig. 1) arranged as a net in the plane. The areas that are covered by multiple of the extended faces—the transition zones—are displayed in gray. The areas where pixels are “missing” are displayed in red. Right: The six extended faces produced by the patchwork procedure. The two polar faces form the top row, followed by the four equatorial faces below. The shaded area around the transition zone of each composite face indicates the margin, which is later discarded.

![Figure 5](image5.png)

**Figure 5.** Detailed view of two of the six extended faces. The dark outer boundary with width $c_2$ is the margin that is discarded after the reconstruction step, and the two dark squares in the corners of the equatorial face on the right are treated likewise. The remaining part of the extended faces has a gray outer boundary of width $2c_1$. In conjunction with the gray squares in the corners of the equatorial face, this boundary forms the transition zone that contains the values shared with the neighboring extended faces.
To evaluate $\alpha$-shearlets and dictionary learning, we have selected two different simulated data sets on the sphere:

- Thermal dust map: a full sky thermal dust map from the Planck Sky Model (100 GHz map) (Planck Collaboration 2016), obtained through the Planck Legacy Archive (http://pla.esac.esa.int/pla/#maps).
- Horizon full sky maps: a series of full sky maps from the Horizon N-body simulations describing the dark matter halo distribution between redshift 0 and 1 (Teyssier, R. et al. 2009) (see http://www.projet-horizon.fr).

While in the former scenario, the signal is smooth and expected to be best represented by multi-scale transforms, in the latter the signal is more discontinuous and geometrically composed of filamentary structures joining clusters, with density changing with redshift. These two simulations are therefore illustrative of different scenarios where such adaptive transforms would be useful.

To evaluate the respective performance of DL and $\alpha$-shearlets for denoising, we have added to the thermal dust map an additive white Gaussian noise with standard deviation $45\mu K$, which corresponds to the expected level of CMB at such frequency. The resulting map can be seen in Fig. 7.

The galactic mask used for quantitative comparisons to separate regions of high dust amplitude from regions with lower values at higher galactic latitude is displayed in Fig. 8 along with the location of a region close to the galactic plane where the differences in between the methods could be better visualized.

For the dark matter halo distribution, we select the first slice of the data cube, and adjust the white noise level at 5, so that filamentary structures are of a similar amplitude as the noise, as can be observed in Fig. 9. This noise does not correspond to something realistic in our actual experiments, but our goal here is only to evaluate how different adaptive representations behave when extracting features embedded in Gaussian noise.
In the following two subsections, we outline the precise choice of the hyperparameters that we used, respectively, for the $\alpha$-shearlets and for the dictionary learning based denoising.

### 4.1. $\alpha$-shearlet parameters

For the two $\alpha$-shearlet approaches, we used 11 values of $\alpha$, sampled uniformly with a density of 0.1 ranging from 0 to 1. We used 4 scales of decomposition, using either the rotation-based approach (Eq. (4)), or the patchwork approach (Eq. (6)). For the actual denoising, we performed a hard thresholding of the $\alpha$-shearlet coefficients. For this, we used different detection thresholds on different scales. Precisely, we used a $4\sigma$ detection threshold for scale 0 with a lower signal to noise ratio, and a detection threshold of $3\sigma$ for the other scales; for the coarse scale, however, we did not do any thresholding. The reconstruction was then performed using either Eq. (5) or (7).

For the rotation-based approach, 5 rotations were selected as a balance between having "more uniform" weights and the computational burden of this approach. The weight maps were built using a margin and transition (smooth trigonometric variation in between 0 and 1) of size $N_{\text{side}}$. For the patchwork approach, we set the size of both the utilized extension and the margin to $N_{\text{side}}$, which results in increasing the number of pixels that have to be processed by about half (53.1%). A little less than half of the added pixels are used for the sake of redundancy, and the rest is disregarded.

### 4.2. Dictionary learning parameters

For the thermal dust data where the information is present at several scales, we chose the multiscale dictionary learning technique. 3 scales of the Starlet transform on the sphere (Starck et al. 2006) were first computed from the input simulated dust map without noise. Note that the finest wavelet scale has not been directly computed through its spherical harmonic decomposition to avoid artefacts for a non band-limited signal. We followed Algorithm 1 for the learning procedure, with the parameters listed in Table 1. The patch size, the number of atoms, and the maximal sparsity were selected experimentally by choosing values that lead to the lowest average approximation error during the training phase.

An example of a dictionary learned for this adaptive multiscale representation of thermal dust is shown in Fig. 10. The dictionaries have captured at various scales both directional and more isotropic structures.

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**Figure 9.** Dark matter halo distribution for the first slice, without (top) and with the additive white Gaussian noise added (bottom), for evaluation of the methods. The colorscale has been stretched to visualize filamentary structures.

**Figure 10.** Atoms learned in the multiscale dictionary learning approach. On the left: scale 0, on the right: scale 1. The dictionaries have departed from the original redundant DCT dictionary and have learned specific features related to the scale. Note that due to the change of the $N_{\text{side}}$ parameter with the scale, the actual distance between two adjacent pixels has increased, and the atoms for scale 1 are indeed smoother than those for scale 0.

In the second scenario, because information is localized in space, the dictionary was learned directly on patches extracted from the first slice describing the dark matter halo distribution, from a training set of 200,000 patches of size $8 \times 8$. As in the previous experiment, a stopping criterion was set for the approximation error (which should be less than the targeted level of noise), and a maximal sparsity of 7 was set for OMP. K-SVD was then run for 100 iterations. The learned dictionary is presented in Fig. 11. Note that the atoms are essentially containing high frequency information in this case, in contrast to the previously learned distribution on thermal dust.

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**Figure 11.** Atoms learned in the dictionary learning approach, applied to the dark matter halo distribution data. The dictionary elements are composed of point-like structures and edges.

Once these dictionary are learned, the sparse decomposition step with this representation is used for denoising. The same parameters as above were used for the sparse coding, except for the targeted approximation error which was set to a value that would
not be exceeded by a patch of pure noise with a probability of 0.9545.

5. Results

5.1. Denoising Experiments

We tested our adaptive approaches to denoise the data in the two denoising scenario presented in the previous section, using the parameters described in sections 4.1 and 4.2.

For the thermal dust simulation, the full sky denoised maps using the three approaches are displayed in Fig. 12, with a zoom to a region close to the galactic plane in Fig. 13 to visually inspect the differences between methods. Residuals on the full sphere are also shown in Fig. 14, and the performance of each approach are quantitatively evaluated in Table 2 in the full sky as well as in regions defined by the galactic mask.

Similarly, for the dark matter halo distribution, the full sky denoised maps are displayed in Fig. 15 and the residuals are presented in Fig. 16. To better inspect the recovery of the filamentary structures as well as the core regions, a zoom in was also performed for this dataset in Fig. 17. Finally, the results were quantitatively evaluated in Table 3.

To inspect the impact of the anisotropy parameter on the recovery of geometrical structures in the different redshift slices,
Table 2. Statistics on the recovery of spherical thermal dust maps with the proposed approaches. Bias, root mean square error (RMSE) and mean absolute deviation (MAD) are presented, for the overall map (All), the region not in the mask (Out) and the galactic region (Gal.) defined by the mask of Fig. 8. The best results are in bold, the best results among ω-shearlet and dictionary learning atoms are underlined. Units in μK.

| Method   | Bias   | RMSE   | MAD    |
|----------|--------|--------|--------|
|          |        | All    | Out    | Gal.   | All    | Out    | Gal.   |
| α = 0    | 0.008  | 0.005  | 0.016  | 4.266  | 3.028  | 6.270  | 3.020  | 2.392  | 4.490  |
| α = 0.1  | 0.008  | 0.005  | 0.016  | 4.264  | 3.025  | 6.268  | 3.018  | 2.389  | 4.488  |
| α = 0.2  | 0.008  | 0.005  | 0.016  | 4.261  | 3.022  | 6.264  | 3.016  | 2.387  | 4.485  |
| α = 0.3  | 0.008  | 0.005  | 0.016  | 4.256  | 3.019  | 6.257  | 3.012  | 2.384  | 4.480  |
| α = 0.4  | 0.008  | 0.005  | 0.016  | 4.256  | 3.021  | 6.255  | 3.012  | 2.384  | 4.480  |
| α = 0.5  | 0.008  | 0.005  | 0.016  | 4.258  | 3.024  | 6.257  | 3.012  | 2.384  | 4.481  |
| α = 0.6  | 0.008  | 0.005  | 0.016  | 4.252  | 3.017  | 6.252  | 3.008  | 2.380  | 4.477  |
| α = 0.7  | 0.008  | 0.005  | 0.016  | 4.256  | 3.020  | 6.256  | 3.010  | 2.381  | 4.480  |
| α = 0.8  | 0.008  | 0.005  | 0.016  | 4.257  | 3.019  | 6.261  | 3.010  | 2.380  | 4.483  |
| α = 0.9  | 0.008  | 0.005  | 0.016  | 4.260  | 3.019  | 6.266  | 3.011  | 2.380  | 4.486  |
| α = 1    | 0.008  | 0.005  | 0.016  | 4.267  | 3.027  | 6.273  | 3.012  | 2.380  | 4.489  |
| Dict. Learn. | 0.008  | 0.006  | 0.014  | 4.034  | 2.343  | 6.440  | 2.570  | 1.750  | 4.487  |

we also computed for the patchwork approach the non-linear approximation curves which display the evolution of the RMSE as a function of given thresholds. This allows to give a more comprehensive view of the best α for different density levels thresholds. These non-linear approximation curves are illustrated in linear and log scale in Figs 19 and 20, respectively.

5.2. Discussion

In the following, we discuss several questions concerning the results; in particular, we analyze the relative performance of our different approaches to sparsifying representations on the sphere.

Block artefacts

The first challenge in extending the representation from the Euclidean framework to data defined on the sphere was to avoid the border effects due to considering disjoint charts processed independently. Fig. 18 illustrates that all our proposed redundant representations, based on different overlapping charts, are free of these block artefacts when denoising the thermal dust map. A similar result is obtained for denoising the dark matter maps.

Visual inspection

Qualitatively, Figs 13 and 17 illustrate the different shapes captured by ω-shearlet and dictionary learning atoms. In particular, for the thermal dust maps, the noise appears as curvelet-like structures for the former and more isotropic structures for the dictionary learning approach.

For the first slice of the dark matter halo distribution simulations, the dictionary learning approach visually seems to best recover the structures in the data, in particular the filamentary structures and the compact cores.

Which approach is best?

This is confirmed quantitatively in Tables 2 and 3 where the dictionary learning approach outperforms overall both ω-shearlet techniques in the denoising of thermal dust (with a multiscale approach) and dark matter halo distribution. For thermal dust, when looking at specific regions (region inside or outside the galactic mask), the rotation-based approach gives however the lowest residuals in the galactic region, while using the learned representation gave the best results outside this region. This could be explained by the wide diversity of amplitudes in the galactic plane, not captured in our training set of 200,000
patches for the first wavelet scale, which corresponds only to 0.4% of the total number of patches over the full sky. Improving performance for dictionary learning in the galactic region would require either to train the dictionary with a larger training set so that it encompasses more patches from the galactic center, or to sample more densely the galactic region than higher galactic latitudes in this training set.

Is the rotation-based or the patchwork approach preferable?

The rotation-based approach outperforms the patchwork approach in the thermal dust denoising scenario, but conversely the patchwork approach outperforms the rotation-based technique in the dark matter halo distribution scenario. The last result is due to the bilinear interpolation performed when resampling the sphere with rotations, which leads to severe approximation errors when the signal varies greatly at the scale of a few pixels.

What is the best $\alpha$-value?

Tables 2 and 3 show that for $\alpha$-shearlets in the denoising of thermal dust, $\alpha = 0.6$ (system close to the curvelets) gives the best performance, while for the dark matter halo distribution scenario, $\alpha = 1.0$ (system close to the wavelets) gave the best performance.

However, the second scenario displays a diversity of structures with both high density cores and numerous less dense filaments, with distribution changing in different slices of data corresponding to different redshifts. It would therefore be reductive to investigate a single noise level scenario to set a best $\alpha$ for one of this slice.

We therefore computed for the patchwork approach the non-linear approximation curves for the different slices in redshift. These non-linear approximation curves are illustrated in linear and log scale in Figs 19 and 20, respectively. These curves illustrate that for large threshold values, corresponding to selecting dense core regions, the $\alpha = 0.9$-shearlet system is most suitable. For slice 600 and 605 (higher redshift), when decreasing the threshold, there is a transition from $\alpha = 0.9$ to $\alpha = 0$ (very elongated shearlets) for the best $\alpha$ value. This can be understood as including more and more filamentary structures when the threshold decreases.

For lower redshift slices on the other hand, the best values are obtained more consistently across thresholds for $\alpha = 0.9$ or $\alpha = 1$ because more core structures and less filaments are visible.

Figure 14. Residuals for the maps displayed in Fig. 12. Units in $\mu K$.

Figure 15. Denoised dark matter maps for all three approaches. Top and middle: $\alpha$-shearlet denoising with rotation-based (top) or patchwork (middle) approach, both with $\alpha = 1$; bottom: representation learned with dictionary learning.
5.3. Computing Requirements

All codes were run on the same cluster so that we can assess the relative computing time requirements for the three approaches.

For the rotation-based approach, on the current python implementation using pyFFTW (https://pypi.org/project/pyFFTW/) and also based on a parallelized transform using 6 cores, denoising a $N_{\text{side}} = 2048$ map using 5 rotations and 4 scales of decomposition takes about 35 minutes for $\alpha = 1$ and 1 hour for $\alpha = 0$ (the most redundant transform). Note that time to perform the rotation-based approach scales linearly with the number of rotations. In comparison, denoising with the patchwork approach a $N_{\text{side}} = 2048$ map using 4 scales of decomposition (with the same parallelization of the transform as for the rotation-based approach) takes about 9 minutes for $\alpha = 1$ and 20 minutes for $\alpha = 0$.

In the data, overall, this illustrates how adaptive to diverse structures in the data the $\alpha$-shearlets can be. Furthermore, it shows that the anisotropy parameter $\alpha$ can be used to characterize different types of structure present in the data.

Figure 16. Amplitude of the residuals for all three approaches, for the dark matter map scenario. Top and middle: $\alpha$-shearlet denoising with rotation-based (top) or patchwork (middle) approach, both with $\alpha = 1$; bottom: representation learned with dictionary learning.

Figure 17. Dark Matter Map amplitudes for all three approaches in a zoomed region. From top to bottom and left to right: original map, noisy map, rotation-based approach with $\alpha = 1$, patchwork approach with $\alpha = 0$, patchwork approach with $\alpha = 1$, representation learned from data.

Figure 18. Cartesian projection of the denoised thermal dust maps centered at the intersection of 4 faces. From left to right and top to bottom: denoising each face independently using $\alpha$-shearlets with $\alpha = 1$, restoration via the rotation-based approach, patchwork approach with $\alpha = 1$, dictionary learning with patch width of 12. The colorscale has been stretched to visualize the artefacts seen as a cross-shape discontinuity at the boundaries of the 4 HEALPix faces in the upper left figure. All of our proposed approaches are free from these artefacts. Units in $\mu$K.

For the multiscale dictionary learning algorithm, computing time for the learning phase ranged from about 2.5 hours for scale 0 to about 3.5 hours for scale 2, when using our C++ code with 4 cores for the sparse coding. This increase is due to the low value for $\epsilon^{(2)}$ and large value for the maximal sparsity $K^{(2)}$, even if the training set is smaller than for scale 0. Note that learning these dictionaries can be performed in parallel, which was done in practice. For the dark matter scenario, the learning took about 65 minutes.

Once the dictionary was learned, sparse coding all patches took typically from 15 minutes (scale 2) to about 22 minutes (scale 0) for the thermal dust map, and 9 minutes for the dark matter halo distribution, using 24 cores.
Table 3. Statistics on the recovery of dark matter halo distribution with the proposed approaches. Bias, root mean square error (RMSE) and mean absolute deviation (MAD) are presented. The best results for RMSE and MAD are in bold, the best results among $\alpha$-shearlets are underlined.

| Method      | Bias  | RMSE | MAD  |
|-------------|-------|------|------|
| Rotation    |       |      |      |
| $\alpha = 0$| 0.0002| 3.09 | 0.83 |
| $\alpha = 0.1$| 0.0002| 3.05 | 0.81 |
| $\alpha = 0.2$| 0.0002| 3.02 | 0.80 |
| $\alpha = 0.3$| 0.0002| 3.00 | 0.80 |
| $\alpha = 0.4$| 0.0002| 2.97 | 0.79 |
| $\alpha = 0.5$| 0.0002| 2.95 | 0.78 |
| $\alpha = 0.6$| 0.0002| 2.94 | 0.78 |
| $\alpha = 0.7$| 0.0002| 2.92 | 0.77 |
| $\alpha = 0.8$| 0.0002| 2.92 | 0.77 |
| $\alpha = 0.9$| 0.0002| 2.91 | 0.77 |
| $\alpha = 1$ | 0.0002| 2.90 | 0.77 |

| Patchwork  |       |      |      |
|------------|-------|------|------|
| $\alpha = 0$| 0.0002| 1.64 | 0.86 |
| $\alpha = 0.1$| 0.0002| 1.58 | 0.84 |
| $\alpha = 0.2$| 0.0002| 1.53 | 0.82 |
| $\alpha = 0.3$| 0.0002| 1.49 | 0.81 |
| $\alpha = 0.4$| 0.0002| 1.45 | 0.80 |
| $\alpha = 0.5$| 0.0002| 1.43 | 0.79 |
| $\alpha = 0.6$| 0.0002| 1.39 | 0.78 |
| $\alpha = 0.7$| 0.0002| 1.37 | 0.78 |
| $\alpha = 0.8$| 0.0002| 1.35 | 0.77 |
| $\alpha = 0.9$| 0.0002| 1.34 | 0.77 |
| $\alpha = 1$ | 0.0002| 1.35 | 0.77 |

| Dict. Learn.| 0.0002| 1.32 | 0.72 |

Overall, the two $\alpha$-shearlet approaches are therefore easier to set up, with less parameters to optimize that depend directly on the data, and result in faster denoising than the dictionary learning based approach.

6. Conclusions

We have proposed two new types of adaptive representations on the sphere: a patch-based dictionary learning approach and choosing among a parametrized family of representations, the $\alpha$-shearlets. To extend these constructs from the Euclidean setting to data defined on the sphere, we proposed to use overlapping charts based on the HEALPix framework. For the dictionary learning technique, a possible multi-scale extension was presented by learning dictionaries on each scale after performing a subsampled wavelet decomposition on the sphere. For the $\alpha$-shearlets, we proposed two approaches to construct the charts: resampling the sphere according to various rotations associated with a partition of unity not sensitive to border effects, or constructing 6 overlapping charts based on composite extended HEALPix faces.

We evaluated all three approaches by conducting denoising experiments on thermal dust maps, and dark matter maps.

Our main findings are as follows:

- thanks to the use of overlapping charts, all of our proposed approaches are free of the block artefacts that typically appear if one naively uses the disjoint HEALPix faces for doing denoising;
- in both scenarios investigated, the dictionary learning approach gave the best performance by providing atoms
adapted to the structure present in the images, for a given noise level;
- the performance of the dictionary learning approach depends on setting several hyper-parameters that depend on the signal observed (multiscale or not), and on the training set. This approach therefore requires more computing and tuning time than the other approaches;
- which of the two $\alpha$-shearlet approaches performed better depended on the chosen scenario; the rotation-based approach involves interpolation which is detrimental to capturing signals that vary significantly at the scale of just a few pixels, but it achieved better results for the thermal dust simulations;
- for different values of the anisotropy parameter $\alpha$, the $\alpha$-shearlet system is adapted to different structures (filaments, dense cores) present in the dark matter halo distribution simulation.

The respective performance of these approaches depends on the criteria used: the dictionary learning approach provided the best denoising results in both scenarios, but has a higher number of parameters to set and requires more computing time; among the $\alpha$-shearlets, the rotation-based approach is best for smooth signals, but the converse is true for signals with significant variation at the scale of a few pixels. The three proposed approaches can therefore be used to process data living on the sphere, and choosing the "best" approach will depend on the scenario considered as well as the computing resources available.

Reproducible Research

In the spirit of reproducible research, we make public our codes on the sphere on the common repository [github.com/[florentsureau/ARES] The dictionary learning and alpha-shearlet codes on the sphere are associated with tutorial jupyter notebooks illustrating how to use them for denoising.

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Appendix A: Review of Euclidean $\alpha$-shearlets

$\alpha$-shearlets are a family or representations that generalizes wavelets and shearlets. Like shearlets—originally introduced in Labate et al. (2005a); Guo et al. (2006)—they are a directionally sensitive multiscale system in $\mathbb{R}^2$, improving upon wavelets when it comes to handling data that is governed by directional features like edges.

$\alpha$-shearlets are characterized by an anisotropy parameter $\alpha \in [0, 1]$, and were designed to yield optimally sparse representations for the class of $C^\alpha$-cartoon-like functions (Kutyniok & Labate 2012; Kutyniok & Lim 2011; Guo & Labate 2007; Voigtlaender & Pein 2017), a model class for natural images (Candès & Donoho 2004) as illustrated in Fig. A.1.

Figure A.1. An example of a cartoon-like function. Such a function $f$ is smooth, apart from a jump continuity along a curve $\gamma$. Even though $f$ might be discontinuous along $\gamma$, the boundary curve $\gamma$ itself is required to be smooth.

In the remainder of this section, we briefly motivate the choice of $\alpha$-shearlet systems, discuss the most important mathematical properties of $\alpha$-shearlet systems, and then comment on the implementation that we used.

A.1. Motivation

Before giving a formal definition of $(\alpha)$-shearlet systems, it is instructive to roughly compare the operations used for their construction to the ones used for defining wavelet systems (Daubechies 1992). Recall (see e.g. Daubechies 1992) that for a scaling function $\phi \in L^2(\mathbb{R}^d)$ and a mother wavelet $\psi \in L^2(\mathbb{R}^d)$, the associated (discrete) wavelet system with sampling density $\delta > 0$ is given by

$$W(\phi, \psi; \delta) := \left\{ (\phi \ast \delta \cdot k)_{k \in \mathbb{Z}^d} \cup \left( 2^{j/2} \cdot \psi(2^j \cdot \delta \cdot k) \right)_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} \right\}.$$ 

In other words, the wavelet system consists of all translates of the scaling function $\phi$ along the lattice $\delta \mathbb{Z}^d$, together with certain translates of the isotropically dilated scaling functions $\psi_j := 2^{j/2} \cdot \psi(2^j \cdot \delta)$, Here, the wavelet $\psi_j$ on the $j$-th scale is translated along the lattice $\delta \cdot 2^j \mathbb{Z}^d$, which is adapted to the “size” of $\psi_j$.

It is crucial to note that even in dimension $d > 1$, wavelets use the isotropic dilations $x \mapsto 2x$ which treat all directions in the same way. Therefore, wavelet systems are not optimally suited for representing functions governed by features with different directions. Admittedly, instead of using one mother wavelet $\psi$, it is common to employ wavelet systems that use finitely many mother wavelets $\psi^{(1)}, \ldots, \psi^{(N)}$; usually these are obtained by choosing each $\psi^{(j)}$ as a certain tensor product of one-dimensional scaling functions and mother wavelets. But such a modified wavelet system is again only able to distinguish a fixed number of directions, independent of the scale $j$, and therefore does not admit a satisfactory directional sensitivity.

To overcome this problem, shearlets (like curvelets) use the parabolic dilation matrices $D_j^{(\alpha)} := \begin{pmatrix} 2^j & 0 \\ 0 & 2^\alpha \end{pmatrix}$ for $j \in \mathbb{N}_0$.

As shown in Fig. A.2 dilating a function $\psi$ with these matrices $D_j^{(\alpha)}$ produces functions $\psi^{(j)} := \psi(D_j^{(\alpha)} \cdot)$ which are more elongated along the $x_2$-axis than along the $x_1$-axis, where the anisotropy is more pronounced for larger values of $\alpha$ or $j$. The support of the dilated function satisfies $2^{-j\alpha} \approx \text{height} \approx \text{width}^\alpha$.

Figure A.2. The effect of dilating a “prototype function” $\psi$ (shown at the top of each row) with the matrices $D_j^{(\alpha)}$ to obtain $\psi(D_j^{(\alpha)} \cdot)$, for different values of the scale $j$ (going from $j = 0$ (top) to $j = 2$ (bottom)) and of the “anisotropy parameter” $\alpha \in [0, 1]$.

It is apparent from Fig. A.2 that for $\alpha < 1$ and large $j \in \mathbb{N}_0$, the functions $\psi^{(j)}$ have a distinguished direction. More precisely, if (as in the figure) $\psi$ oscillates along the $x_1$-axis, then $\psi^{(j)}$ is similar to a sharp jump along the $x_2$-axis. Since we want our dictionary to be able to represent jumps along arbitrary directions, we have to allow some way of changing the direction of the elements $\psi^{(j)}$. The most intuitive way for achieving this is to use rotations, as was done in the construction of (second generation) curvelets (Candès & Donoho 2004). But later on, it was noted in Labate et al. (2005a); Guo et al. (2006) that from an implementation point of view, rotations have the disadvantage that they do not leave the digital grid $\mathbb{Z}^d$ invariant. Therefore, instead of rotations, $(\alpha)$-shearlets use the shear matrices

$$S_x := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

to adjust the direction of the functions $\psi^{(j)}$. Note though that the shear matrices $S_x$, $x \in (-\infty, \infty)$ can never cause an effect similar to a rotation with angle $\theta$ for $|\theta| > 90^\circ$. Therefore, for the definition of a cone-adapted shearlet system, one only uses
shearings corresponding to rotations with angle $|\theta| \leq 45^\circ$, and then uses a modified mother shearlet $\psi_\delta$ to cover the remaining directions.

Collecting all previously described constructs, the cone-adapted $\alpha$-shearlet system with sampling density $\delta > 0$, associated to a low-pass filter $\varphi \in L^2(\mathbb{R}^2)$, and mother shearlet $\psi \in L^2(\mathbb{R}^2)$ is defined as:

$$
\text{SH}_\alpha(\varphi, \psi; \delta) := \left\{ (\varphi(\cdot - \delta k))_{k \in \mathbb{Z}} \cup \left\{ 2^{(1+\alpha)/2} \psi(R^\alpha D_j^{(\alpha)} S_\ell \cdot - \delta k) \right\}_{(j,\ell,i) \in \mathbb{E}} \right\}.
$$

(A.1)

with $R := \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$, and

$$
I := I^{(\alpha)} := \left\{ (j, \ell, i) \in \mathbb{N}_0 \times \mathbb{Z} \times [0, 1] : |\ell| \leq \left\lfloor 2^{(1-\alpha)} \right\rfloor \right\}.
$$

For brevity, let us set $\psi^{(\alpha)} := 2^{(1+\alpha)/2} \psi(R^\alpha D_j^{(\alpha)} S_\ell \cdot)$, and observe with this notation that

$$
2^{(1+\alpha)/2} \psi(R^\alpha D_j^{(\alpha)} S_\ell \cdot - \delta k) = \psi^{(\alpha)} \left( R^\alpha A_j^{-1} \ell, \delta \right), \quad (A.2)
$$

with $A_{j,\ell,i} := R^\alpha D_j^{(\alpha)} S_\ell$.

A.2. Mathematical properties

The most basic property of $\alpha$-shearlets that we will be interested in is that they indeed form a (redundant) representation system for $L^2(\mathbb{R}^2)$. In mathematical terms, this means that the $\alpha$-shearlet system forms a frame (Christensen 2016), for a suitable choice of the generators $\varphi, \psi$. In particular, if $\varphi, \psi \in L^2(\mathbb{R}^2)$ have compact support and satisfy certain decay and smoothness conditions (see Voigtlaender & Pein (2017) Theorem 5.10 for details), then there is a “minimal sampling density” $\delta_0 > 0$, such that the $\alpha$-shearlet system is indeed a frame for $L^2(\mathbb{R}^2)$, for all $0 < \delta \leq \delta_0$.

The main motivation for introducing ($\alpha$)-shearlets was the wish for a representation system which is better adapted to data governed by directional features, which are often present in natural images, and also in astronomical images. One key result relates ($\alpha$)-shearlets to $C^\alpha$-cartoon-like functions.

Roughly speaking, a function $f \in L^2(\mathbb{R}^2)$ is called a $C^\alpha$-cartoon-like function, written $f \in E^\alpha(\mathbb{R}^2)$ (with $\beta \in \mathbb{R}$), if $f = f_1 + f_2 \cdot 1_B$ for certain $f_1, f_2 \in C^\alpha([0,1]^2)$ and such that the set $B \subset [0,1]^2$ has a boundary curve of regularity $C^\alpha$. For a more formal definition, we refer to Voigtlaender & Pein (2017) (Definition 6.1).

Using this notion, we have the result that the best $N$-term approximation error with such a frame of $\alpha$-shearlets (that is, the smallest approximation error obtained by a linear combination of $N$ $\alpha$-shearlets) is decaying at (almost) the best rate that any dictionary $\Psi$ can reach for $C^\alpha$-cartoon-like functions; see Voigtlaender & Pein (2017) Theorem 6.3) for a more precise formulation of this result. To obtain this optimal approximation rate, the anisotropy parameter $\alpha$ needs to be adapted to the regularity $\beta$ of the $C^\alpha$-cartoon-like functions, that is, $\alpha = \beta/2$. In general, given a certain data set, or a certain data model, different types of $\alpha$-shearlet systems will be better adapted to the given data than other $\alpha'$-shearlet systems. In Section 4 we will verify this for specific sets of data living on the sphere.

We close our discussion of the mathematical properties of $\alpha$-shearlet systems with a brief discussion of the frequency concentration of such systems. To this end, assume for the moment that the “mother shearlet” $\psi$ is concentrated in frequency to the set

$$
Q := \{ \xi \in \mathbb{R}^2 : 3^{-3} \leq |\xi_1| \leq 3 \text{ and } |\xi_2| \leq |\xi_1| \},
$$

which is a union of two opposing “wedges” (highlighted in green in Fig. A.3). From elementary properties of the Fourier transform, one then sees that each $\alpha$-shearlet $\psi_{j,\ell}^{(\alpha)}$ has frequency support in $S_j^\alpha D_j^{(\alpha)} R Q$, where we denote by $A^T$ the transpose of a matrix $A$. The resulting coverings of the frequency plane for different values of the anisotropy parameter $\alpha$ are shown in Fig. A.3.

![Figure A.3. The frequency concentration of $\alpha$-shearlets for different values of $\alpha$. One sees that each “dyadic annulus” $\{ \xi : |\xi| = |2^j\xi| \}$ is split into a number $N_j^{(\alpha)}$ of “wedges” representing the different directions. In fact, $N_j^{(\alpha)} \approx 2^{2(1-\alpha)}$.](image)

Together, Figs A.2 and A.3 show that the parameter $\alpha$ has three different, but related effects:

- It affects the “shape” of the elements of the $\alpha$-shearlet system. Indeed, Fig. A.2 shows that height $\approx$ width$^\alpha$.
- It affects the directional selectivity: As seen in Fig. A.3 on scale $j$, an $\alpha$-shearlet system can distinguish about $2^{2(1-\alpha)}$ different directions.
- It affects the frequency support of the elements of the $\alpha$-shearlet system, see Fig. A.3.

A.3. Implementation

The git repository of our implementation of the Euclidean $\alpha$-shearlet transform can be found at [github.com/dedale-fet/alpha-transform](https://github.com/dedale-fet/alpha-transform) with extensive documentation available at [rawgit.com/dedale-fet/alpha-transform/master/build/html/AlphaTransform.html](https://rawgit.com/dedale-fet/alpha-transform/master/build/html/AlphaTransform.html). Our software package is implemented in Python3 (Van Rossum & Drake Jr 1995), using NumPy (van der Walt et al. 2011).

In this section, we give a rough overview over what the transform computes, and how it can be used. Our software package implements two different versions of the $\alpha$-shearlet transform: A
fully-sampled (non-decimated) version, and a subsampled (decimated) version. For the fully-sampled version, the computed coefficients are the (discrete) convolutions \( f * \varphi \) and \( \psi_{\ell, \jmath, \iota} \) (for a certain range of scales \( j = 0, \ldots, j_{\text{max}} \)), where the filters \( \varphi \) and \( \psi_{\ell, \jmath, \iota} \) are chosen as in Equations (A.1) and (A.2). Thus, for a given input image \( f \in \mathbb{C}^{N \times N} \), the resulting coefficients form a three-dimensional tensor of dimension \( N_{\ell, \jmath, \iota} \times N \times N \), where the integer \( N_{\ell, \jmath, \iota} \) is the total number of \( \alpha \)-shearlet filters that is used, and where each \( N \times N \) component of the tensor is the discrete convolution of \( f \) with one of the \( \alpha \)-shearlet filters. When considering \( j_{\text{max}} \) many scales (i.e., \( j = 0, \ldots, j_{\text{max}} - 1 \)) and if \( \alpha < 1 \), then

\[
N_{\ell, \jmath, \iota} = 1 + 2 \sum_{j=0}^{j_{\text{max}}-1} |\{-2^{(1-\alpha)/2}, \ldots, 2^{(1-\alpha)/2}\}| \times 2^{(1-\alpha)j_{\text{max}}}.
\]

(A.3)

In particular, for \( \alpha = 0 \), note \( N_{\ell, \jmath, \iota} \approx 2^{j_{\text{max}}} \), so that the redundancy of the fully sampled \( \alpha \)-shearlet frame grows very quickly when increasing the number of scales.

To motivate the subsampled transform, note that according to Eq. (A.1), the \( \alpha \)-shearlet system does not contain all translations of the functions \( \varphi \) and \( \psi_{\ell, \jmath, \iota} \). Rather, \( \varphi \) is shifted along the lattice \( \Delta \mathbb{Z}^2 \), and—as seen in Eq. (A.2)—\( \psi_{\ell, \jmath, \iota} \) is shifted along the lattice \( \Delta \mathbb{Z}^2 \), with \( \Delta \mathbb{Z}^2 = R D^{(\alpha)} \mathbb{Z}^2 \). Effectively, this means that the full convolution \( f * \psi_{\ell, \jmath, \iota} \) is only sampled at certain points, where the sampling density gets more dense as the scale \( j \) increases. The subsampled version of the \( \alpha \)-shearlet transform computes these coefficients. Internally, this is achieved by using the “frequency wrapping” approach outlined in Candès et al. (2006 Sections 3.3 and 6), [2010 Chapter 4], and [2011] for the case of the curvelet transform. Since each convolution is sampled along a different lattice, the subsampled transform of a given image \( f \) is a list of rectangular matrices of varying dimension. This will become more clear in the example below. One can show for the subsampled transform that the total number \( M = M(\alpha, j_{\text{max}}, N) \) of \( \alpha \)-shearlet coefficients for an \( N \times N \) image is bounded, i.e., \( M(\alpha, j_{\text{max}}, N) \leq M_0 \cdot N^2 \), with \( M_0 \) independent of \( \alpha, j_{\text{max}}, N \). This is in stark contrast to the fully sampled transform (at least for \( \alpha < 1 \)), where the total number of coefficients is \( \approx 2(1-\alpha)j_{\text{max}} \cdot N^2 \), see Equation (A.3).

The main effect of choosing the fully sampled transform is that one gets a translation-invariant transform (i.e., taking the transform of a shifted image is the same as shifting each component of the coefficient tensor), and the increased redundancy. This increased redundancy can actually be beneficial for certain tasks like denoising, but it can greatly impact the memory footprint and the runtime: Computations using the subsampled transform are usually much faster and require much less memory, but yield slightly worse results.

We close this section with a short IPython session showing how our implementation of the \( \alpha \)-shearlet transform can be used.

```python
>>> from AlphaTransform import AlphaShearletTransform as AST
>>> import numpy as np; from scipy import misc
>>> im = misc.face(gray=True); im.shape (768, 1024)
>>> # Setting up the transform.
```

In the line marked with #1, we set up the \( \alpha \)-shearlet transform object \( \alpha \)-traj. Roughly speaking, this will precompute all necessary \( \alpha \)-shearlet filters, which are stored in the \( \alpha \)-traj object. The first two parameters of the constructor simply determine the shape of the images for which the \( \alpha \)-traj object can be used, while the third parameter determines the number of scales \( j_{\text{max}} \) to be used, as well as the value of the anisotropy parameter \( \alpha \). Passing \( [\alpha_0] = 0.5 \) will construct an \( \alpha \)-shearlet transform with \( N \) scales (plus the low-pass) and with \( \alpha \) given by \( \alpha_0 \). The verbose parameter simply determines how much additional output (like a progress bar) is displayed. The subsampled parameter determines whether the non-decimated, or the decimated transform is used. Finally, the real parameter determines whether real-valued or complex-valued \( \alpha \)-shearlet filters are used. Essentially, real-valued filters have frequency support in the union of two opposing wedges (as shown in Fig. A.3), while for complex-valued filters, one gets two filters for each real-valued one: one complex-valued filter has frequency support in the “left” wedge, while the other one is supported in the “right” wedge.
In line #2, we use the `transform()` method of the constructed `trafo` object to compute the $\alpha$-shearlet transform of $\mathbf{im}$. As seen, the result is an ordinary NumPy array of dimension $N_{\alpha,j_{\text{max}}} \times N_1 \times N_2$, where the input image has dimension $N_1 \times N_2$, and where $N_{\alpha,j_{\text{max}}}$ is the total number of $\alpha$-shearlet filters used by the transform.

The `indices` property of the `trafo` object (see line #3) can be used to determine to which $\alpha$-shearlet filter the individual components of the `coeff` array are associated. The value -1 represents the low-pass filter, while a tuple of the form $(j, 1, c)$ represents the shearlet filter $\psi_{j,\ell,\iota}$ as in Equation (A.1), where $\ell = 0$ if $c$ is ‘h’ (which stands for the *horizontal* frequency cone), and where $\iota = 1$ if $c$ is ‘v’ (*vertical* frequency cone).

To explain the differences between the fully sampled and the subsampled transform, in line #5, we set up a subsampled transform object `trafo2`. The only difference to the construction of the `trafo` object is that we pass `subsampled=True`, and `real=False`. The reason for this second change is that—at least with the current implementation—the subsampled transform can only be used with complex-valued shearlet filters. We then compute the coefficients (see line #6) just as for the fully sampled transform. Note, however, that the coefficients for the fully sampled transform were a single 3-dimensional NumPy array. For the subsampled transform, however, the coefficients are a list of 2-dimensional NumPy arrays. The reason for this is that the number of coefficients varies from scale to scale for the subsampled transform.

The `indices` property (see line #7) for the subsampled transform also differs from that of the fully sampled transform. The reason for this is that we use *complex* shearlets; therefore, the frequency plane is divided into four cones (top, or ‘t’; right, or ‘r’; bottom, or ‘b’; and left, or ‘l’), instead of the two cones that are used for real-valued shearlet filters.

The main advantage of the subsampled transform is revealed in line #8: The *redundancy* (that is, the number of $\alpha$-shearlet coefficients divided by the number of pixels of the input image) for the subsampled transform is much lower, which leads to a lower memory consumption and faster computation times. While the advantage of the subsampled transform might not be overwhelming in the given example, it gets more pronounced if one uses a larger number of scales. For instance, if we use four scales instead of three, then the redundancy of the fully sampled transform is $41$, while that of the subsampled transform is only $\approx 11.4$. 

