REFINING THE MOND INTERPOLATING FUNCTION AND TeVeS LAGRANGIAN

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ABSTRACT

The phenomena customarily described with dark matter or modified Newtonian dynamics (MOND) have been argued by Bekenstein to be the consequences of a covariant scalar field, controlled by a free function [related to the MOND interpolating function \( \tilde{\mu}(g/a_0) \)] in its Lagrangian density. In the context of this relativistic MOND theory (TeVeS), we examine critically the interpolating function in the transition zone between weak and strong gravity. Bekenstein’s toy model produces a \( \tilde{\mu} \) that varies too gradually, and it fits rotation curves less well than the standard MOND interpolating function \( \tilde{\mu}(x) = x/(1 + x^2)^{1/2} \). However, the latter varies too sharply and implies an implausible external field effect. These constraints on opposite sides have not yet excluded TeVeS, but they have made the zone of acceptable interpolating functions narrower. An acceptable “toy” Lagrangian density function with simple analytical properties is singled out for future studies of TeVeS in galaxies. We also suggest how to extend the model to solar system dynamics and cosmology.

Subject headings: dark matter — galaxies: kinematics and dynamics — gravitation

1. INTRODUCTION

On galaxy scales, dark matter generally dominates over baryons (stars plus gas) at large radii. At intermediate radii in a galaxy where dark matter and baryons overlap with comparable amounts, the two mass profiles are not uncorrelated (McGaugh 2005). The correlation between the Newtonian gravity of the baryons \( g_B \) and the overall gravity \( g \) (baryons plus dark matter) can be loosely parameterized by Milgrom’s (1983) empirical relation

\[ \tilde{\mu}(g/a_0)g = g_B, \quad (1) \]

where the interpolating function \( \tilde{\mu}(x) \) is a function that runs smoothly from \( \tilde{\mu}(x) = x \) at \( x \ll 1 \) to \( \tilde{\mu}(x) = 1 \) at \( x \gg 1 \) with a dividing gravity scale \( a_0 \sim 10^{-8} \) cm s\(^{-2}\) \( \sim cH_0/6 \) at the transition. This simple correlation was taken as the basis for the MOND theory (or more precisely, the quadraclitic Lagrangian theory) by Bekenstein & Milgrom (1984, hereafter BM84), in which one modifies the Newtonian gravity of a baryonic galaxy to eliminate the need for dark matter.

Recently, interest in the subject of MOND has been further stimulated since Bekenstein (2004, hereafter B04) provided a Lorentz-covariant theory (dubbed TeVeS), which passes standard tests to check general relativity and allows for rigorous modeling of Hubble expansion and gravitational lensing (e.g., Zhao et al. 2006). In TeVeS the MONDian behavior originates from a scalar field, the dynamics of which is controlled by a Lagrangian density involving a free function that yields the expected dynamics in the low-acceleration limit (although BM84 theory is not precisely recovered). This freedom of the Lagrangian density, which echoes the freedom in the choice of the interpolating function \( \tilde{\mu} \) in MOND, means that every choice of the free function defines a distinct theory. As this class of theories do not at present derive from any basic principle and are purely phenomenological, the only constraints on the free function must come from phenomenological grounds. A refinement of the function studied by B04 as a toy model is surely needed. In this Letter, we differentiate popular choices of the MOND \( \tilde{\mu} \)-function by fitting a benchmark rotation curve, and we argue that many of those functions are likely unphysical in the TeVeS context. We then propose a new free function for TeVeS in the domain relevant for galaxies, with a possible extension to solar system dynamics and cosmology.

2. WARMING UP TO TeVeS

TeVeS is a tensor-vector-scalar Lorentz-covariant field theory in which the tensor is the Einstein metric \( g_{\mu\nu} \), out of which is built the usual Einstein-Hilbert action, \( U_g \), is a dynamical normalized vector field, and \( \phi \) is a dynamical scalar field. The action is the sum of the Einstein-Hilbert action for the tensor \( g_{\mu\nu} \), the matter action, the action of the vector field \( U_a \), and the action of the scalar field \( \phi \). Einstein-like equations are obtained for each of these fields by varying the action with respect to each of them. The action of the scalar field \( \phi \) involves a dimensionless parameter \( k \) (order a few percent), a length-scale parameter \( l \sim (3k)^{-1/2}c^2/4\pi G \), and a free dimensionless function linking \( k l^2 |\nabla\phi|^2 \propto y \) with the auxiliary nondynamical scalar field \( \mu \).

More relevant to us, the physical metric in TeVeS near a quasi-static galaxy or the solar system is identical to that of general relativity, with a potential

\[ \Phi = \Xi \Phi_N + \phi, \quad (2) \]

where \( \Xi \approx 1 \). This means that the scalar field \( \phi \) plays the role of the dark matter gravitational potential. It is related to the Newtonian potential \( \Phi_N \) (generated by the baryonic density \( \rho \)) through the equation

\[ \nabla \cdot (\mu \nabla \phi) = \nabla^2 \Phi_N = 4\pi G\rho \quad (3) \]
Fig. 1.—Fits to the rotation curve of the “benchmark” galaxy NGC 3198 using different $\mu$-functions: “simple” (eq. [7]; solid line) and Bekenstein “toy” (eq. [5]; dot-dashed line) functions.

(similar to the field equation for the full $\Phi$ in BM84), where $\mu_s$ is a function of the scalar field strength $g_s = |\nabla \phi|$. It is related to the $\mu$-function of B04 and the interpolating function $\tilde{\mu}$ of MOND (in spherical symmetry) by

$$\mu_s \equiv \mu/k' = \tilde{\mu}/(1 - \tilde{\mu}), \quad k' \equiv k/4\pi. \quad (4)$$

In the intermediate- to deep-MOND regime, the toy free function in the scalar action of B04 gives rise to the following interpolating function:

$$\tilde{\mu}(x) = \frac{x(1 + 4x) - 1}{x^2 + 4x + 1}. \quad (5)$$

3. ROTATION CURVES

Milgrom’s formula (eq. [1]) is a good approximation of the BM84 and TeVeS theories, since for most disk galaxies the curl field is negligible or exactly zero when solving equation (3) (Brada & Milgrom 1995). It provides a plausible unified picture for the phenomenology of individual galaxies, which is more challenging to understand in the dark matter framework. However, while the precise functional form of the interpolating function is not necessary to make many fundamental predic-

ions (see Sanders & McGaugh 2002), it is nevertheless important in order to fit the rotation curves of galaxies. Indeed, one of the most striking successes of MOND is the ability of Milgrom’s formula to fit the rotation curves of a wide range of galaxies with virtually the same value $a_0 = 1.2 \times 10^{-8}$ cm s$^{-2}$ and the same “standard” interpolating function:

$$\tilde{\mu}(x) = x/\sqrt{1 + x^2}. \quad (6)$$

This “standard” interpolating function was originally put in by hand and does not derive from any physical principle. However, with the number of galaxies with good data on the rise, the freedom of this function should be restrained by the observations. Famaey & Binney (2005, hereafter FB05) have, for example, found that the “simple” interpolating function

$$\tilde{\mu}(x) = x/(1 + x) \quad (7)$$
gives a better fit to the terminal velocity curve of the Milky Way than equation (6), while yielding an extremely good fit to the rotation curve of the standard external galaxy NGC 3198 (see Fig. 1). In comparison, the toy model in B04 gives rise to equation (5) in spherical symmetry in the intermediate- to deep-MOND regime. This interpolating function triggers too slow a transition from the MONDian to the Newtonian regime in the benchmark rotation curve of NGC 3198 (cf. Fig. 1); the
same conclusion is reached for the terminal velocity curve of the Milky Way (FB05).

In short, from the analysis of rotation curves, equations (6) and (7) are preferred over equation (5). FB05 noted that equation (7) is preferred over equation (6) to fit the terminal velocity curve of the Milky Way. They also derived the best MONDian fit (in the strong and intermediate regimes only) to the Milky Way and found the interpolating function to transition smoothly from equation (7) at $x \leq 1$ to equation (6) at $x \geq 10$. Of course, it is not obvious that all the other galaxies will give the same answer as the Milky Way and NGC 3198. The observational constraints presented in this section should thus be considered as an indication more than a rigorous constraint.

4. THE EXTERNAL FIELD EFFECT

In TeVeS, the potential consists of two parts (eq. [2]): the Newtonian potential, and the scalar field, which satisfies the BM84 formulation (eq. [3]). One of the key features of the BM84 theory is the existence of the dilatation effect of an external field, which is why MOND does not satisfy the strong equivalence principle. Consider the perturbation generated by a body of low mass $m$ inside a dominating uniform external scalar field of strength $g_z$: equation (3) for $\phi$ becomes

$$\nabla \cdot (\mu \nabla \phi) \approx \mu_s (\Delta_1 \partial_z^2 + \partial_y^2 + \partial_x^2) \phi = 4\pi Gm \delta(r),$$

(8)

which differs from the Newtonian Poisson equation by the dilatation factor $\Delta_1$, given by

$$\Delta_1 = 1 + \frac{d \ln \mu_s}{d \ln g_s}, \quad \mu_s \equiv \mu_s(g_s)$$

(BM84; Zhao 2005). As in BM84, the solution is an equipotential surface given by

$$\phi(x, y, z) = -g_z z - \frac{Gm}{\mu_s \tilde{r}}, \quad \tilde{r} = \sqrt{x^2 + \Delta_1 (y^2 + z^2)},$$

(10)

where $m / \mu_s$ is the effective mass of the satellite and $\tilde{r}$ is the effective distance from the center of the satellite. Hence the perturbation scalar field is stretched in the $z$-direction by a factor $1 / \Delta_1^{1/2}$. To exclude an imaginary dilation, a theory should have

$$\Delta_1 = \frac{d \ln \mu_s g_s}{d \ln g_s} > 0.$$

(11)

This requires models where $g_s$ is an increasing function of $g_s = \mu g_s$. The factor $\Delta_1$ also determines the shape of the Roche lobe of a rotating satellite, which will have a ratio of middle to long axes $[2 / (3 \Delta_1)]^{1/4}$ (Zhao 2005). In fact, the stretching factor $\Delta_1^{1/2}$ enters almost all processes involving satellites (see, e.g., Brada & Milgrom 2000). For these reasons we consider models with a negative $\Delta_1$, unphysical. In the original BM84 theory this is not a problem, because $1 \leq \Delta_1 < 2$ thanks to a monotonically increasing $\tilde{\mu}(g)$ (not $\mu_s(g)$ as we are concerned with), so the external field effect is a mild curiosity. This, however, is not the case if the MOND effect is created by a scalar field. From equation (4), we have that models with the standard $\tilde{\mu}$-function (eq. [6]) yield a $g_s$ that increases with $g_s$ to some point and then starts decreasing in the intermediate regime. At the same scalar field strength $g_s$, there are two different (spherical) Newtonian gravity strengths, that is, the scalar function $\mu_s(g_s)$ becomes multivalued for the same $g_s$ and hence is ill-defined (see Fig. 2). This is a general feature of any sharply increasing MOND $\tilde{\mu}$-function [e.g., the “exponential” function $\tilde{\mu} = 1 - \exp(-x)$, and the best fit of FB05]. This type of behavior is undesirable in a physical model for the scalar field. On the other hand, the simple function (eq. [7]) and B04 toy model (eq. [5]) give physical, monotonic $\mu_s(g_s)$’s and hence a positive $\Delta_1$.

5. NEW FREE FUNCTION

As a toy model, B04 chose for the $\mu$-function (see eq. [4] above) the implicit, discontinuous formulation

$$\frac{y(\mu)}{3k^2} = \frac{h^{\prime} \phi_0 \phi_0}{a_0} = \frac{\mu^2 (\mu - 2)^2}{k^2 (1 - \mu)^4},$$

(12)

where the notation is as in B04. Here cosmology ($\mu > 2$) and galaxies ($0 < \mu < 1$) are completely detached from each other, while the fit to galactic rotation curves is poor.

Our aim here is to propose a new explicit, monotonic, and continuous interpolating function $\mu_s(s) = \mu k^n$. A simple choice could be

$$s = \frac{g_s}{a_0} = \sqrt{\frac{|y|}{3k^2}} = \frac{\mu}{(k + \mu)(1 - \mu)^n}, \quad n = 0, 1.$$

(13)

Here $y > 0$ with $0 < \mu < 1$ corresponds to quasi-static systems as in B04, and $y < 0$ corresponds to cosmology. This toy function is given by two root branches of the essentially fourth-order polynomial equation $s = \mu / ((1 + \mu_0)(1 + 2\mu)^{1/2})$ and is a very lengthy multivalued function.
tion is consistent with rotation curves, since it reduces to the simple interpolating function (eq. [7]) in the range for galaxies, that is, \( \mu = k' \mu, \sim (0-10)k' \ll 1 \), where we have the explicit relations

\[
\mu_s \approx g_s/(a_0 - g_s), \quad s = \mu_s/(1 + \mu_s),
\]

which are ready to be fed into a solver for equation (3).

6. SOLAR SYSTEM

Our toy function also recovers solar system dynamics at a level similar to B04, if not better. For example, in the \( n = 1 \) case (the default), the scalar field goes to infinity when \( \mu \) approaches 1. This new “toy” function is easier to use than that in B04 because it allows for an analytic formulation of the scalar-field interpolating function \( \mu_s(s) = \mu/k' = \mu/(1 - \tilde{\mu}) \) in both strong and weak gravity, namely,

\[
k'\mu_s(s) = 1 - k' + \frac{-1 + \sqrt{1 + k's^2 + 1^2 - 4s}}{2s}.
\]

The dilation factor \( \Delta_y = 1 + d \ln \mu_s/d \ln s = [2 + (1 - k')\mu_s]/(1 + k'\mu_s^2) \sim 1-4 \) is in the plausible range for \( k' = 0.03-1 \). The new function yields a monotonic increasing scalar field strength \( g_s \), and the correction for the solar system dynamics is less than in the B04 toy model (when adopting \( k = 0.01 \) the variation from the Sun to Saturn is smaller by an order of magnitude), which may or may not be relevant to the Pioneer anomaly.

7. COSMOLOGY

Although the B04 discontinuous cosmological branch of the function \( y(\mu) \) could as well be appended to our function in the range \( \mu > 2 \), it is more desirable to be in a universe where the Lagrangian density has a smooth transition between weakly gravitating quasi-static systems \( y \propto (\nabla \phi)^2/a_0^2 > 0 \) and cosmology \( y \propto -2(\partial \phi)\phi^2 < 0 \). A possible way, as done here (eq. [15]), is to copy our function in the \( y > 0 \) regime into the \( y < 0 \) regime as a simple mirror image (cf. Fig. 3). This means that the outer parts of galaxies would connect smoothly into the cosmological expansion as \( y \) passed from \( 0^+ \) to \( 0^- \). While a mirror image without fine-tuning is attractive, it is not a necessary condition: any positive continuous function \( \mu(y) \) going through \( \mu(0^-) = \mu(0^+) \sim 0 \) is worth exploring. The next question is whether this kind of cosmology can produce realistic Hubble expansion and the cosmic microwave background.

8. CONCLUSION

Part of the amazing successes of the nonrelativistic version of MOND in explaining galaxy dynamics is due to its “standard” interpolating function \( \tilde{\mu} = x/(1 + x^2)^{1/2} \). When exploring a range of other empirical functions in the context of TeVeS, we see that the fit to rotation curves becomes poorer for a more gradual \( \tilde{\mu} \)-function, while an external field effect (EFE) with imaginary dilation happens for more rapid changes of the \( \tilde{\mu} \)-function. These two independent constraints from opposite sides suggest a fairly narrow range of TeVeS free functions. Among these, we propose a simple expression that works for both very weak and very strong gravity (eq. [15]), with a possible extension to cosmology. Unlike that in B04, the new function has the nice feature that it links quantities of TeVeS and of MOND easily and hence facilitates future examinations using galaxy dynamics and solar system data. The explicit simple monotonic interpolating function \( \mu_s(g_s/a_0) \) could be easily fed into a numerical solver for equation (3) (e.g., as developed by Ciotti et al. 2006) and could allow for the modeling of realistic galaxy geometries (the curl field of BM84, neglected here following conventional wisdom, could be put back with a realistic amount). Combined with a galaxy for which sensitive kinematic data exist, this may confirm or falsify our “toy” function and hence further establish or squeeze the parameter space of the TeVeS theory. As two final remarks, we note (1) that multiply imaged gravitational systems present a challenge to all MOND/TeVeS interpolating functions (cf. Zhao et al. 2006) and (2) that the dark matter potential is fundamentally different from the scalar field, although the two are sometimes degenerate in fitting rotation curves. Indeed, there is no equivalent of EFE in dark matter; hence, the dark matter potential enjoys more freedom.

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