Strong convergence rates of an explicit scheme for stochastic Cahn–Hilliard equation with additive noise

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Abstract
In this paper, we propose and analyze an explicit time-stepping scheme for a spatial discretization of stochastic Cahn–Hilliard equation with additive noise. The fully discrete approximation combines a spectral Galerkin method in space with a tamed exponential Euler method in time. In contrast to implicit schemes in the literature, the explicit scheme here is easily implementable and produces significant improvement in the computational efficiency. It is shown that the fully discrete approximation converges strongly to the exact solution, with strong convergence rates identified. Different from the tamed time-stepping schemes for stochastic Allen–Cahn equations, essential difficulties arise in the analysis due to the presence of the unbounded linear operator in front of the nonlinearity. To overcome them, new and non-trivial arguments are developed in the present work. To the best of our knowledge, it is the first result concerning an explicit scheme for the stochastic Cahn–Hilliard equation. Numerical experiments are finally performed to confirm the theoretical results.

Keywords
Stochastic Cahn–Hilliard equation · Strong convergence · Spectral Galerkin method · Tamed exponential Euler method

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1 Introduction

Let $\mathcal{D}$ be a bounded convex domain in $\mathbb{R}^d$, $d \in \{1, 2, 3\}$. We denote by $H = L^2(\mathcal{D}, \mathbb{R})$ a real separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and $\dot{H} := \{ v \in H : \int_{\mathcal{D}} v \, dx = 0 \}$. In this paper, we consider the numerical approximation of stochastic Cahn–Hilliard equation (SCHE) in the abstract form

$$\begin{cases}
dX(t) + A(X(t)) + F(X(t))) \, dt = dW(t), & t \in (0, T], \\
X(0) = X_0,
\end{cases}
$$

where $0 < T < \infty$, $-A$ is the Neumann Laplacian and $\{ W(t) \}_{t \geq 0}$ is a $Q$-Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{ \mathcal{F}_t \}_{t \geq 0})$, specified later. The nonlinear term $F$ is assumed to be a Nemytskii operator, given by $F(u)(x) = f(u(x)) = u^3(x) - u(x)$, $x \in \mathcal{D}$. As a phenomenological model from metallurgy and physics, the deterministic version of such equation is used to describe the complicated phase separation and coarsening phenomena in a melted alloy [6, 8] and spinodal decomposition for binary mixture [7]. Adding a noise to the physical model is quite natural as it either represents an external random perturbation or gives a remedy for lack of knowledge of certain involved physical parameters. For example, in [4, 12] and references therein, the authors have expressed the belief that only the stochastic version can correctly describe the whole decomposition process in a binary alloy. The stochastic version (1) has been extensively studied by many authors (see e.g., [1, 4, 13, 14, 16, 18, 20, 27, 30, 31]).

Since the true solution of the problem can not be known explicitly, it is therefore natural to look for reliable numerical solutions. To do the approximation error analysis, one often faces difficulties, raised by the presence of the unbounded operator $A$ in front of the nonlinear term $F$. In the past few years, many authors investigated strong and weak approximations of stochastic Cahn–Hilliard equation [9, 11, 15, 19, 20, 22, 23, 26, 28, 31], where some attempts to address the issue were proposed in literature. For the linearized stochastic Cahn–Hilliard equation, the readers are referred to [11, 26, 28]. In [24], strong convergence rates for the spectral Galerkin spatial approximation of the nonlinear problem in dimension one are proved by combining a general perturbation theory with the exponential integrability properties of the numerical approximation. The authors in [20, 27] derive the strong convergence of the finite element spatial approximation and the backward Euler full discretization of the SCHE driven by spatial regular noise, but with no rates obtained. Very recently, the paper [31] fills the gap left by [20, 27] and recovers the strong convergence rates of the finite element fully discrete scheme. For the space-time white noise, authors in [15] obtained the strong convergence rates of a fully discrete scheme performed by a spatial spectral Galerkin method and a temporal accelerated implicit Euler method. Moreover, the strong convergence rates of an implicit fully discrete mixed finite element method for the SCHE with gradient-type multiplicative noise are derived in [19], where the noise process is a real-valued Wiener process. To the best of our knowledge, the explicit
methods are absent for the SCHE and this paper aims to propose an explicit scheme for the equation and identify its strong convergence rates. As indicated in [2], the fully discrete exponential Euler and fully discrete linear-implicit Euler approximations diverge strongly and numerically weakly in the case of stochastic Allen-Cahn equations. Later, some explicit modified Euler-type schemes have been proposed in [3, 5, 10, 21, 32] to numerically solve the stochastic Allen–Cahn equations. Based on a spectral Galerkin spatial approximation of (1), given by

\[
\begin{aligned}
\frac{dX^N(t) + A(AX^N(t) + P_N F(X^N(t)))dt = P_N dW(t), \quad t \in (0, T]}, \\
X^N(0) = P_N X_0,
\end{aligned}
\]

we propose a tamed exponential Euler scheme in time to obtain the explicit fully discrete method

\[
X_{m+1}^{M,N} = E(t) X_{m}^{M,N} - \int_{t_m}^{t_{m+1}} E(t_{m+1}-s) A P_N F(X_{m}^{M,N}) ds + E(t) P_N \Delta W_m,
\]

where \(\Delta W_m = W(t_{m+1}) - W(t_m), m \in \{0, 1, 2, \ldots, M - 1\}\), \(P_N\) is the projection operator onto \(H_N := \text{span}\{e_1, e_2, \ldots, e_N\}\), \(E(t) = e^{-tA^2}, t \geq 0\) denotes an analytic semigroup on \(H\) generated by \(-A^2\) and \(\tau = \frac{T}{M}\) stands for the time step-size. Compared with existing implicit schemes, the proposed scheme is easy to implement and produces significant improvement in the computational efficiency.

Throughout this article, \(C\) denotes a generic nonnegative constant that is independent of the discretization parameters and may change from line to line. Meanwhile, we use \(\mathbb{N}^+\) to denote the set of all positive integers and \(\mathbb{N} = \{0\} \cup \mathbb{N}^+\). In summary, the contribution of this article to the numerical analysis of stochastic Cahn–Hilliard equation is twofold. On the one hand, the uniform a priori moment bounds of the full discretization are derived based on a certain bootstrap argument. To do this, a key ingredient lies on bounding

\[
\sup_{M, N \in \mathbb{N}^+} \sup_{m \in \{0, 1, \ldots, M\}} \mathbb{E} \left[ \|X_{t_m}^{M,N}\|_{L^6}^p \right] < \infty
\]

by virtue of Gagliardo–Nirenberg inequality in \(d = 1\) and energy estimate in \(d = 2, 3\). On the other hand, as implied by Corollary 16, we identify the strong convergence rate of the fully discrete method:

\[
\sup_{M, N \in \mathbb{N}^+} \sup_{m \in \{0, 1, \ldots, M\}} \|X(t_m) - X(t_m^{M,N})\|_{L^p(\Omega, \dot{H})} \leq C (\nu^2 + \tau^\gamma), \quad \gamma \in (\frac{d}{2}, 4].
\]

where \(\lambda_N\) is the \(N\)-th eigenvalue of \(A\) and \(\gamma\) from Assumption 3 is a parameter used to measure the spatial regularity of the noise process. The above result reveals that the strong convergence rates are essentially governed by the spatial regularity of the noise term. The rates of convergence are optimal. Comparing (2) with the sharp temporal Hölder regularity result in Theorem 6, one can easily observe, for \(\gamma \in (\frac{d}{2}, 2]\), the rate of convergence is in accordance with the temporal Hölder regularity of the mild
solution. For $\gamma \in [2, 4]$, the rate of the convergence can reach 1 and higher than the Hölder continuity of the mild solution due to the fact that the noise is additive. The convergence rates are the same as that obtained for the backward Euler method from the literature [30, 31]. It must be emphasized that the derivation of (2) is not an easy task and requires a variety of delicate error estimates, which are elaborated in Sect. 3.3.

The outline of the article is organized as follows. In the next section, we present some assumptions and give the well-posedness and regularity of the mild solution. Section 3 is devoted to the strong convergence analysis, where spectral Galerkin method is introduced in Sect. 3.1, uniform a priori moment bounds are deduced in Sect. 3.2 and the strong convergence rates are derived in Sect. 3.3. Numerical examples are finally included in Sect. 4 to verify the theoretical findings.

2 Main assumptions and the considered problem

Given another separable Hilbert space $(U, \langle \cdot, \cdot \rangle_U, \| \|_U)$, $\mathcal{L}(U, H)$ represents the space of all bounded linear operators from $U$ to $H$ endowed with the usual operator norm $\| \cdot \|_{\mathcal{L}(U, H)}$ and by $\mathcal{L}_2(U, H) \subset \mathcal{L}(U, H)$ we denote the space consisting of all Hilbert–Schmidt operators from $U$ to $H$. To simplify the notation, we often write $\mathcal{L}(H)$ and $\mathcal{L}_2(H)$ (or $\mathcal{L}_2$ for short) instead of $\mathcal{L}(H, H)$ and $\mathcal{L}_2(H, H)$, respectively. It is easy to prove that $\mathcal{L}_2(U, H)$ is a Hilbert space equipped with the inner product and norm,

$$\langle T_1, T_2 \rangle_{\mathcal{L}_2(U, H)} = \sum_{i \in \mathbb{N}^+} \langle T_1 \phi_i, T_2 \phi_i \rangle, \quad \| T \|_{\mathcal{L}_2(U, H)} = \left( \sum_{i \in \mathbb{N}^+} \| T \phi_i \|^2 \right)^{1/2},$$

independent of the choice of orthonormal basis $\{ \phi_i \}$ of $U$. If $T \in \mathcal{L}_2(U, H)$ and $L \in \mathcal{L}(H, U)$, then $TL \in \mathcal{L}_2(H)$ and $\| TL \|_{\mathcal{L}_2(H)} \leq \| T \|_{\mathcal{L}_2(U, H)} \| L \|_{\mathcal{L}(H, U)}$. Also, $|\langle T_1, T_2 \rangle_{\mathcal{L}_2(U, H)}| \leq \| T_1 \|_{\mathcal{L}_2(U, H)} \| T_2 \|_{\mathcal{L}_2(U, H)}$ holds for $T_1, T_2 \in \mathcal{L}_2(U, H)$. Finally, $V := C(D, \mathbb{R})$ represents the Banach space of all continuous functions from $D$ to $\mathbb{R}$ with supremum norm. Throughout this paper, we define an orthogonal projector $P : H \to \hat{H}$ by

$$Pv = v - |D|^{-1} \int_D v \, dx$$

and then $(I - P)v = |D|^{-1} \int_D v \, dx$ is the average of $v$. Here and below, by $L^r(D, \mathbb{R})$, $r \geq 1$ ($L^r(D)$ or $L^r$ for short) we denote a Banach space consisting of all $r$-times integrable functions.

In the sequel, the main assumptions are made for the abstract model (1).

Assumption 1 Let $D$ be a bounded convex domain of $\mathbb{R}^d$, $d \in \{1, 2, 3\}$ with Lipschitz boundary. Let $-A$ be the Neumann Laplacian, given by $-Au = \Delta u$, with $u \in \text{dom}(A) := \{ v \in H^2(D) \cap \hat{H} : \frac{\partial u}{\partial n} = 0 \text{ on } \partial D \}$. 

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For \( v \in \mathcal{H} \), we extend the definition as \( A v = A P v \). Then there exists a family of orthonormal eigenbasis \( \{ e_j \}_{j \in \mathbb{N}} \) with corresponding eigenvalues \( \{ \lambda_j \}_{j \in \mathbb{N}} \) such that

\[
A e_j = \lambda_j e_j, \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to \infty, \text{ as } j \to \infty
\]

where \( e_0 = |D|^{-\frac{1}{2}} \) and \( \{ e_j \}_{j \in \mathbb{N}^+} \) forms an orthonormal basis of \( \dot{\mathcal{H}} \). We define the fractional powers of \( A \) on \( \dot{\mathcal{H}} \) by the spectral theory, e.g., \( A^\alpha v = \sum_{j=1}^{\infty} \lambda_j^\alpha (v, e_j)e_j \), \( \alpha \in \mathbb{R} \). The space \( \dot{\mathcal{H}}^\alpha := \text{dom}(A^{\frac{\alpha}{2}}) \) is a Hilbert space with the inner product \( \langle \cdot, \cdot \rangle_\alpha \) and the associated norm \( \cdot \parallel_{\alpha} \) defined by

\[
\langle v, w \rangle_\alpha = \sum_{j=1}^{\infty} \lambda_j^\alpha (v, e_j)(w, e_j), \quad \|v\|_\alpha = \|A^{\frac{\alpha}{2}} v\| = \left( \sum_{j=1}^{\infty} \lambda_j^\alpha \|v, e_j\|^2 \right)^{\frac{1}{2}}, \quad \alpha \in \mathbb{R}.
\]

Note that for integer \( k \geq 0 \), \( \dot{\mathcal{H}}^k \) is a subspace of \( H^k(D) \cap \dot{\mathcal{H}} \) characterized by certain boundary conditions. Let us recall the following results concerning the spaces \( \dot{\mathcal{H}}^\alpha \) and the associated norms \( \cdot \parallel_{\alpha} \) for \( \alpha \in [0, 2] \), see \([20, 25, 33]\) for more details. For \( \alpha \in [0, \frac{3}{2}) \), one has

\[
\dot{\mathcal{H}}^\alpha = H^\alpha(D),
\]

and for \( \alpha \in (\frac{3}{2}, 2) \), one has

\[
\dot{\mathcal{H}}^\alpha = H^\alpha_N(D) := \{ v \in H^\alpha(D) : \frac{\partial v}{\partial n} = 0 \text{ on } \partial D \} \subset H^\alpha(D).
\]

Additionally, for \( \alpha \in [0, \frac{3}{2}) \cup (\frac{3}{2}, 2] \), the norm \( \cdot \parallel_{\alpha} \) is equivalent on \( \dot{\mathcal{H}}^\alpha \) to the standard Sobolev norm \( \| \cdot \parallel_{H^\alpha(D)} \). Since \( H^2(D) \) is an algebra, one can deduce that for any \( f, g \in \dot{\mathcal{H}}^2 \),

\[
\|fg\parallel_{H^2(D)} \leq C\|f\parallel_{H^2(D)}\|g\parallel_{H^2(D)} \leq C|f|_2|g|_2. \tag{3}
\]

Additionally, the operator \(-A^2\) generates an analytic semigroup \( E(t) = e^{-tA^2} \) on \( \mathcal{H} \), given by

\[
E(t)v = e^{-tA^2}v = \sum_{j=0}^{\infty} e^{-t\lambda_j^2} (v, e_j)e_j = \sum_{j=1}^{\infty} e^{-t\lambda_j^2} (v, e_j)e_j + (v, e_0)e_0 = Pe^{-tA^2}v + (I - P)v, \quad v \in \mathcal{H}.
\]

At last, the properties of \( E(t) \) are obtained by expansion in terms of the eigenbasis of \( A \) and using Parseval’s identity,

\[
\|A^\mu E(t)\parallel_{\mathcal{L}(\dot{\mathcal{H}})} \leq Ct^{-\frac{\mu}{2}}, \quad t > 0, \quad \mu \geq 0, \tag{4}
\]

\[
\|A^{-\nu}(I - E(t))\parallel_{\mathcal{L}(\dot{\mathcal{H}})} \leq Ct^{\frac{\nu}{2}}, \quad t \geq 0, \quad \nu \in [0, 2], \tag{5}
\]
\[
\int_{t_1}^{t_2} \| A^{\rho} E(s) v \|^2 ds \leq C |t_2 - t_1|^{1-\rho} \| v \|^2, \quad \forall v \in \dot{H}, \rho \in [0, 1], 0 \leq t_1 \leq t_2. \tag{6}
\]
\[
\left\| A^{2\rho} \int_{t_1}^{t_2} E(t_2 - s) v ds \right\| \leq C |t_2 - t_1|^{1-\rho} \| v \|, \quad \forall v \in \dot{H}, \rho \in [0, 1], 0 \leq t_1 \leq t_2. \tag{7}
\]

**Assumption 2** Let \( F : L^6(D, \mathbb{R}) \to H \) be the Nemytskii operator given by
\[
F(v)(x) = f(v(x)) = v^3(x) - v(x), \quad x \in D, v \in L^6(D, \mathbb{R}).
\]
Then, for \( v, \zeta, \zeta_1, \zeta_2 \in L^6(D, \mathbb{R}) \), we have
\[
\begin{align*}
(F'(v)(\zeta))(x) &= f'(v(x))\zeta(x) = (3v^2(x) - 1)\zeta(x), \quad x \in D, \\
(F''(v)(\zeta_1, \zeta_2))(x) &= f''(v(x))\zeta_1(x)\zeta_2(x) = 6v(x)\zeta_1(x)\zeta_2(x), \quad x \in D,
\end{align*}
\]
where the above derivatives can be understood as Gateaux derivatives in Banach spaces. As a result, there exists a constant \( C > 0 \) such that
\[
\begin{align*}
&-\langle F(u) - F(v), u - v \rangle \leq \| u - v \|^2, \quad u, v \in L^6(D). \tag{8} \\
&\| F'(v)u \| \leq C(1 + \| v \|_V^2) \| u \|, \quad u, v \in V. \tag{9} \\
&\| F(u) - F(v) \| \leq C(1 + \| u \|_V^2 + \| v \|_V^2) \| u - v \|, \quad u, v \in V. \tag{10}
\end{align*}
\]
To simplify the presentation, we assume the average of the Wiener process to be zero so that the covariance operator \( Q \) of the \( Q \)-Wiener process belongs to \( L(\dot{H}) \).

**Assumption 3** Let \( \{W(t)\}_{t \in [0,T]} \) be a \( \dot{H} \)-valued (possibly cylindrical) \( Q \)-Wiener process with the covariance operator \( Q \in \mathcal{L}(\dot{H}) \) satisfying
\[
\left\| A^{\gamma-2} Q^2 \right\|_{\mathcal{L}_2} < \infty \quad \text{for some} \quad \gamma \in \left( \frac{d}{2}, 4 \right). \tag{11}
\]

**Assumption 4** Let \( X_0 : \Omega \to \dot{H} \) be \( \mathcal{F}_0/B(\dot{H}) \)-measurable and satisfy that for a sufficiently large number \( p_0 \in \mathbb{N} \),
\[
\mathbb{E}[|X_0|_{\gamma p_0}^p] < \infty,
\]
where \( \gamma \) is the parameter from (11).

Before moving on, similar to [15, Equations (2.5), (2.7)], we give the following lemma concerning the spatio-temporal regularity results of stochastic convolution
\[
\mathcal{O}_t := \int_0^t E(t - s)dW(s).
\]
Lemma 5 Suppose Assumptions 1 and 3 hold. Then for any \( p \geq 1 \), the stochastic convolution \( O_t \) satisfies

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \| O_t \|_{V}^p \right] + \mathbb{E} \left[ \sup_{t \in [0,T]} | O_t |_{V}^p \right] < \infty,
\]

and for \( \alpha \in [0, \gamma] \) and \( 0 \leq s \leq t \leq T \),

\[
\| O_t - O_s \|_{L^p(\Omega, \dot{H}^\alpha)} \leq C (t-s)^{\min\left\{ \frac{1}{2}, \frac{\gamma - \alpha}{4} \right\}}.
\]

**Proof** Applying the factorization method used in [17, Theorem 5.10] yields that, for \( \alpha \in (0, 1) \),

\[
O_t = \frac{\sin(\alpha \pi)}{\pi} \int_0^t (t-s)^{\alpha-1} E(t-s) Y_\alpha(s) ds,
\]

where

\[
Y_\alpha(s) := \int_0^s (s-r)^{-\alpha} E(s-r) dW(r).
\]

Then, by the Burkholder–Davis–Gundy inequality and Hölder’s inequality, for a sufficiently large \( p > 1 \) and \( \frac{d}{2} < \theta < \min\{\gamma, 2\} \) such that \( \frac{1}{p} + \frac{\theta}{2} < \alpha < \min\{\frac{\gamma}{4}, \frac{1}{2}\} \), we obtain

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} | O_t |_{\dot{H}^\theta}^p \right] = \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \frac{\sin(\alpha \pi)}{\pi} \int_0^t (t-s)^{\alpha-1} E(t-s) Y_\alpha(s) ds \right|_{\dot{H}^\theta}^p \right]
\]

\[
\leq C \mathbb{E} \left[ \sup_{t \in [0,T]} \left( \int_0^t (t-s)^{\alpha-1 - \frac{\theta}{4}} | Y_\alpha(s) | ds \right)^p \right]
\]

\[
\leq C \mathbb{E} \left[ \sup_{t \in [0,T]} \left( \int_0^t (t-s)^{\alpha-1 - \frac{\theta}{4}} ds \right)^{\frac{p}{q}} \cdot \int_0^t | Y_\alpha(s) |^p ds \right]
\]

\[
\leq C \int_0^T \mathbb{E} \left[ | Y_\alpha(s) |^p \right] ds
\]

\[
\leq C \int_0^T \left( \int_0^s (s-r)^{-2\alpha} \| E(s-r) Q_1^2 \|_{L_2^2}^2 \right)^{\frac{p}{2}} ds
\]

\[
\leq C \int_0^T \left( \int_0^s (s-r)^{-2\alpha + \min\left\{ \frac{\gamma-2}{2}, 0 \right\}} \| A^{\frac{\gamma-2}{2} Q_1^2 \|_{L_2^2}^2 \right)^{\frac{p}{2}} ds
\]

\[
< \infty.
\]

With aid of the Sobolev embedding inequality \( \dot{H}^\theta \subset V, \theta > \frac{d}{2} \), we arrive at

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \| O_t \|_{V}^p \right] < \infty.
\]
By means of the Burkholder–Davis–Gundy–type inequality, (6) and (11), we have
\[
\| \mathcal{O}_t \|_{L^p(\Omega, \dot{H}^{\gamma})} \leq C \left( \int_0^t \| A^{\frac{\gamma}{2}} E(t-s) \|_2 \| A^{\frac{\gamma}{2}} E(t-s) \|_2 ds \right)^{\frac{1}{2}}
\]
\[
= C \left( \int_0^t \| AE(t-s)A^{\frac{\gamma}{2}} \|_2 \| A^{\frac{\gamma}{2}} E(t-s) \|_2 ds \right)^{\frac{1}{2}}
\]
\[
\leq C \| A^{\frac{\gamma}{2}} Q \|_2 < \infty.
\]
Similarly, for \( \alpha \in [0, \gamma] \),
\[
\| \mathcal{O}_t - \mathcal{O}_s \|_{L^p(\Omega, \dot{H}^{\alpha})}
\leq C \left( \int_s^t \| A^{\frac{\gamma}{2}} (E(t-r) - E(s-r)) \|_2 ds \right)^{\frac{1}{2}} + C \left( \int_s^t \| A^{\frac{\gamma}{2}} E(t-s) \|_2 \| A^{\frac{\gamma}{2}} E(t-s) \|_2 ds \right)^{\frac{1}{2}}
\]
\[
= C \left( \int_s^t \| AE(s-r)A^{\frac{\gamma}{2}} (E(t-s) - I)A^{\frac{\gamma}{2}} \|_2 ds \right)^{\frac{1}{2}}
\]
\[
+ C \left( \int_s^t \| A^{2-\frac{\gamma}{2}} E(t-r)A^{\frac{\gamma}{2}} \|_2 ds \right)^{\frac{1}{2}}
\]
\[
\leq C(t-s)^{\frac{\gamma}{4}} + C(t-s)^{\min\left( \frac{1}{2}, \frac{\gamma}{4} \right)}
\]
\[
\leq C(t-s)^{\min\left( \frac{1}{2}, \frac{\gamma}{4} \right)}.
\]
Hence, we complete the proof. \( \square \)

At last, we consider the mild solution of (1) by following a semigroup approach proposed in [17]. As already proved in [15, Proposition 6 & Proposition 7], the above assumptions are sufficient to establish well-posedness of the model (1) and spatio-temporal regularity of the mild solution for \( \gamma \in (\frac{1}{4}, 4), d = 1 \) and \( \gamma \in [3, 4], d \in \{2, 3\} \). Later, we have extended the results to the case \( \gamma \in (\frac{d}{2}, 4), d \in \{1, 2, 3\} \), which are shown in [30, Theorem 3.6]. The relevant results are presented in the following theorem.

**Theorem 6** (Well-posedness and regularity results) Under Assumptions 1–4, there is a unique mild solution \( X : [0, T] \times \Omega \to \dot{H} \) to (1) given by
\[
X(t) = E(t)X_0 - \int_0^t E(t-s)APF(X(s)) \, ds + \int_0^t E(t-s) \, dW(s), \; t \in [0, T].
\]
Furthermore, for \( \gamma \in (\frac{d}{2}, 4) \) and \( p \geq 1 \),
\[
\sup_{t \in [0, T]} X(t) \|_{L^p(\Omega, \dot{H}^{\gamma})} < \infty,
\]
and for \( \alpha \in [0, \gamma] \),
\[
\| X(t) - X(s) \|_{L^p(\Omega, \dot{H}^{\alpha})} \leq C(t-s)^{\min\left( \frac{1}{2}, \frac{\gamma}{4} \right)}, \; 0 \leq s \leq t \leq T.
\]
3 Strong convergence analysis of numerical approximation

This section aims to derive strong convergence rates of the numerical discretization, done by a tamed exponential Euler method based on the spectral Galerkin approximation.

3.1 The spectral Galerkin spatial discretization

We start this part by introducing a finite dimension space spanned by the first \( N \) eigenvectors of the dominant linear operator \( A \), i.e., \( H_N = \text{span}\{e_1, \ldots, e_N\} \) and the projection operator \( P_N : \dot{H}^\beta \to H_N \) is defined by \( P_N x = \sum_{i=1}^{N} \langle x, e_i \rangle e_i \) for \( \forall x \in \dot{H}^\beta, \beta \geq -2 \). Given the identity mapping \( I \in L(\dot{H}) \), one can easily obtain that

\[
\| (P_N - I) A^{-\alpha} \|_{L(\dot{H})} \leq C \lambda_N^{-\alpha}, \quad \forall \alpha \geq 0.
\]

Then applying the spectral Galerkin method to (1) results in the finite dimensional stochastic differential equation, given by

\[
\begin{cases}
    dX^N(t) + A(AX^N(t) + PNF(X^N(t)))dt = PNdW(t), & t \in (0, T], \\
    X^N(0) = PNX_0,
\end{cases}
\]

whose unique mild solution is adapted and satisfies

\[
X^N(t) = E(t)PNX_0 - \int_0^t E(t-s)APNF(X^N(s))ds + \int_0^t E(t-s)PNdW(s). \quad (13)
\]

The following theorem, concerning the strong convergence rate of the spectral Galerkin method, is an immediate consequence of [30, Theorem 3.5].

**Theorem 7** Let \( X(t) \) be the mild solution of (1) and let \( X^N(t) \) be the solution of (12). Suppose Assumptions 1–4 are valid, then for any \( p \in [1, \infty) \), it holds that

\[
\sup_{t \in [0, T]} \| X(t) - X^N(t) \|_{L^p(\Omega, \dot{H})} \leq C \lambda_N^{-\frac{2}{p}}.
\]

3.2 An explicit fully discrete scheme and its a priori moment bounds

This subsection concerns the a priori moment bounds of a spatio-temporal full discretization based on the spatial spectral Galerkin approximation. In order to introduce the fully discrete scheme, we define the nodes \( t_m = m\tau \) with a uniform time step-size \( \tau = \frac{T}{M} \) for \( m \in \{0, 1, \ldots, M\} \), \( M \in \mathbb{N}^+ \) and introduce a notation \([t]_\tau := t_i\) for \( t \in [t_i, t_{i+1}), i \in \{0, 1, \ldots, M - 1\} \). It is worthwhile to mention that the fully discrete exponential Euler and fully discrete linear-implicit Euler approximations diverge...
strongly and numerically weakly in the case of stochastic Allen-Cahn equations [2].
Thus, we apply the tamed exponential Euler scheme to (12) and get
\[
X_{m+1}^M = E(\tau)X_m^M - \frac{A^{-1}(I-E(\tau))P_N F(X_m^M)}{1+\tau\|P_N F(X_m^M)\|} + \int_{tm}^{tm+1} E(tm+1 - \lfloor s \rfloor \tau) P_N dW(s).
\] (14)

Particularly, the following continuous version of (14) will be used frequently,
\[
X_t^M = E(t)P_N X_0 - \int_0^t E(t-s)AP_N F(X_s^M) \, ds + O_t^M, \tag{15}
\]
which is \( \mathcal{F}_t \)-adapted. Here for simplicity of presentation we denote
\[
O_t^M := \int_0^t E(t - \lfloor s \rfloor \tau) P_N dW(s),
\]
which satisfies the following regularity result.

**Lemma 8** Suppose Assumptions 1 and 3 hold. Then for all \( p \geq 1 \) and \( \theta \in [0, \min\{\gamma, 2\}) \), the discrete stochastic convolution \( O_t^M \) satisfies
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} |O_t^M|^p_{\theta} \right] < \infty.
\]

**Proof** Following a similar approach used in [17, Theorem 5.10], we can rewrite \( O_t^M \) as
\[
O_t^M = \frac{\sin(\alpha \pi)}{\pi} \int_0^t (t-s)^{\alpha-1} E(t-s)Y_\alpha(s) \, ds, \quad \alpha \in (0, 1)
\]
with
\[
Y_\alpha(s) := \int_0^s (s-r)^{-\alpha} E(s - \lfloor r \rfloor) P_N dW(r).
\]

Indeed, by stochastic Fubini theorem, we get
\[
\frac{\sin(\alpha \pi)}{\pi} \int_0^t (t-s)^{\alpha-1} E(t-s)Y_\alpha(s) \, ds \\
= \frac{\sin(\alpha \pi)}{\pi} \int_0^t (t-s)^{\alpha-1} E(t-s) \left[ \int_0^s (s-r)^{-\alpha} E(s - \lfloor r \rfloor) P_N dW(r) \right] \, ds \\
= \frac{\sin(\alpha \pi)}{\pi} \int_0^t \left[ \int_r^t (t-s)^{\alpha-1}(s-r)^{-\alpha} \, ds \right] E(t - \lfloor r \rfloor) P_N dW(r) \\
= \int_0^t E(t - \lfloor r \rfloor \tau) P_N dW(r),
\]
where a basic fact
\[
\int_r^t (t-s)^{\alpha-1}(s-r)^{-\alpha}ds = \frac{\pi}{\sin(\alpha\pi)}, \quad 0 \leq r \leq t, \quad \alpha \in (0, 1)
\]
was invoked in the last equality. As a result, using the Burkholder–Davis–Gundy inequality and Hölder’s inequality leads to
\[
E\left[ \sup_{t \in [0,T]} |O_{i}^{M,N}|_{\rho}^{p} \right] = E\left[ \sup_{t \in [0,T]} \frac{\sin(\alpha\pi)}{\pi} \int_0^t (t-s)^{\alpha-1} E(t-s)Y_\alpha(s)ds \right]^{p}_{\rho}
\leq C E\left[ \sup_{t \in [0,T]} \left( \int_0^t (t-s)^{\alpha-1-\frac{\theta}{4}}|Y_\alpha(s)|ds \right)^{\frac{p}{\theta}} \right]
\leq C E\left[ \sup_{t \in [0,T]} \left( \int_0^t (t-s)^{(\alpha-1-\frac{\theta}{4})q}ds \right)^{\frac{p}{\theta}} \cdot \int_0^t |Y_\alpha(s)|^{p}ds \right]
\leq C \int_0^T E\left[ |Y_\alpha(s)|^{p} \right]ds
\leq C \int_0^T \left( \int_0^s (s-r)^{-2\alpha+\min\{\frac{\gamma}{2}, 0\}} \|A^{\frac{\gamma-2}{2}}Q^{\frac{1}{2}}\|_{L^2}^2 dr \right)^{\theta} ds
\leq C \int_0^T \left( \int_0^s (s-r)^{-2\alpha+\min\{\frac{\gamma}{2}, 0\}} \|A^{\frac{\gamma-2}{2}}Q^{\frac{1}{2}}\|_{L^2}^2 dr \right)^{\frac{\theta}{2}} ds
< \infty,
\]
where \( \alpha > \frac{1}{p} + \frac{\theta}{4} \) was used in the third inequality and \( \alpha \in (0, \min\{\frac{\gamma}{4}, \frac{1}{2}\}) \) was used in the last inequality. Finally, choosing sufficiently large \( p > 1 \) and \( \theta < \min\{\gamma, 2\} \) completes the proof.

The forthcoming lemma is a direct consequence of [30, Lemma 3.2], which is crucial to the moment bound and convergence analysis.

**Lemma 9** Let \( F : L^6 \to H \) be the Nemytskii operator in Assumption 2. Then it holds for any \( \iota \in (\frac{1}{2}, 1) \) and \( d = 1, \)
\[
|F'(u)v|_1 \leq C\left( 1 + |u|_2^2 \right)|v|_1, \quad u \in \dot{H}^\iota, \quad v \in \dot{H}^1,
\]
and for any \( \iota \in (\frac{d}{2}, 2), \) \( d = 2, 3, \)
\[
|F'(u)v|_1 \leq C\left( 1 + |u|_2^2 \right)|v|_1, \quad u \in \dot{H}^\iota, \quad v \in \dot{H}^1.
\]

Next we construct a sequence of decreasing subevents
\[
\Omega_{R,i} = \left\{ \omega \in \Omega : \sup_{j \in [0,1,\ldots,i]} \|X_{ij}^{M,N}(\omega)\|_{L^6} \leq R \right\}, \quad R \in (0, \infty), \quad i \in \{0, 1, \ldots, M\}.
\]
By $\Omega^c$ and $\chi_\Omega$ we denote the complement and indicator function of a set $\Omega$, respectively. It is easy to see that $\chi_{\Omega_{R,t_i}}$ is $\mathcal{F}_{t_i}$-adapted and $\chi_{\Omega_{R,t_i}} \leq \chi_{\Omega_{R,t_j}}$ for $t_i \geq t_j$.

Besides, we introduce a process $Y_t^{M,N}$ by

$$Y_t^{M,N} := X_t^{M,N} - \mathcal{O}_t^{M,N} = E(t)P_NX_0 - \int_0^t \frac{E(t-s)AP_NF\left(X_{s|\tau}^{M,N}\right)}{1+\tau\|P_NF\left(X_{s|\tau}^{M,N}\right)\|}ds,$$

which can be rewritten as

$$Y_t^{M,N} = E(t)P_NX_0 - \int_0^t E(t-s)AP_NF\left(X_{s|\tau}^{M,N}\right)ds + \int_0^t E(t-s)AP_NZ_s^{M,N}ds,$$

with $Z_s^{M,N} := F\left(X_t^{M,N}\right) - \frac{F\left(X_{s|\tau}^{M,N}\right)}{1+\tau\|P_NF\left(X_{s|\tau}^{M,N}\right)\|}$. Then, for $t \in (0, T]$, $Y_t^{M,N}$ satisfies

$$\frac{d}{dt}Y_t^{M,N} + A^2Y_t^{M,N} + AP_NY_t^{M,N} + \mathcal{O}_t^{M,N} = AP_NZ_t^{M,N}. \quad (16)$$

Equipped with the above preparations, we are ready to present the following two lemmas, which aim to bound the numerical approximations on the well-chosen subevents $\Omega_{R_t,t_{i-1}}$.

**Lemma 10** Suppose Assumptions 1–4 are valid. Let $p \in [1, \infty)$ and $R_\tau = \tau^{-\min\left(\frac{d}{d_H}, \frac{d}{d_T}\right)}$ for $\gamma \in (\frac{d}{2}, 4]$ coming from (11). Then for $i \in \{0, 1, \ldots, M\}$,

$$\|\sup_{s \in [0,t_i]} \chi_{\Omega_{R_t,t_{i-1}}} Y_s^{M,N}\|_{L^2_p(\Omega, \mathcal{H})}^2 + \int_0^{t_i} \chi_{\Omega_{R_t,t_{i-1}}} \|AY_s^{M,N}\|_{L^2_p(\Omega, \mathcal{H})}^2 ds \leq L_p(\Omega, \mathbb{R})$$

$$+ \int_0^{t_i} \chi_{\Omega_{R_t,t_{i-1}}} \|\nabla[(Y_s^{M,N})^2]\|_{L^2_p(\Omega, \mathbb{R})}^2 ds < \infty. \quad (17)$$

**Proof** One can easily derive from (16) that

$$\langle \frac{d}{dt}Y_t^{M,N}, A^{-1}Y_t^{M,N} \rangle + \langle A^2Y_t^{M,N}, A^{-1}Y_t^{M,N} \rangle + \langle AP_NY_t^{M,N} + \mathcal{O}_t^{M,N}, A^{-1}Y_t^{M,N} \rangle = \langle AP_NZ_t^{M,N}, A^{-1}Y_t^{M,N} \rangle,$$

which can be rewritten as

$$\frac{1}{2} \frac{d}{dt} |Y_t^{M,N}|_{L^2_{-1}}^2 + |Y_t^{M,N}|_{L^2_1}^2 + \langle F(Y_t^{M,N} + \mathcal{O}_t^{M,N}), Y_t^{M,N} \rangle = \langle Z_t^{M,N}, Y_t^{M,N} \rangle.$$

\[ \square \] Springer
Integrating over $[0, t_i]$ and then using Young’s inequality yield

\[
\begin{aligned}
|Y_{M,N}^{t_i}|^2_{-1} - |Y_{M,N}^0|^2_{-1} &= -2 \int_0^{t_i} |Y_s^{M,N}|^2 ds - 2 \int_0^{t_i} \langle F(Y_s^{M,N} + O_s^{M,N}), Y_s^{M,N} \rangle ds \\
&\quad + 2 \int_0^{t_i} \langle Z_s^{M,N}, Y_s^{M,N} \rangle ds \\
&= -2 \int_0^{t_i} |Y_s^{M,N}|^2 ds - 2 \int_0^{t_i} \|Y_s^{M,N}\|_{L^4}^4 ds + 2 \int_0^{t_i} \|Y_s^{M,N}\|^2 ds \\
&\quad - 2 \int_0^{t_i} \langle 3(Y_s^{M,N})^2 O_s^{M,N} + 3Y_s^{M,N}(O_s^{M,N})^2 + (O_s^{M,N})^3 - O_s^{M,N}, Y_s^{M,N} \rangle ds \\
&\quad + 2 \int_0^{t_i} \langle Z_s^{M,N}, Y_s^{M,N} \rangle ds \\
&\leq - \int_0^{t_i} |Y_s^{M,N}|^2 ds - \int_0^{t_i} \|Y_s^{M,N}\|_{L^4}^4 ds + C \int_0^{t_i} |Y_s^{M,N}|^2_{-1} ds \\
&\quad + C \int_0^{t_i} \|Z_s^{M,N}\|_{-1}^2 ds + C \int_0^{t_i} (1 + \|O_s^{M,N}\|_{L^4}^4) ds.
\end{aligned}
\]

It follows from Gronwall’s inequality that

\[
|Y_{M,N}^{t_i}|^2_{-1} \leq C \left( |Y_{M,N}^0|^2_{-1} + \int_0^{t_i} |Z_s^{M,N}|^2_{-1} ds + \int_0^{t_i} (1 + \|O_s^{M,N}\|_{L^4}^4) ds \right),
\]

which implies that

\[
\begin{aligned}
\int_0^{t_i} |Y_s^{M,N}|^2 ds + \int_0^{t_i} \|Y_s^{M,N}\|_{L^4}^4 ds \\
&\leq C \left( |Y_0^{M,N}|^2_{-1} + \int_0^{t_i} |Z_s^{M,N}|^2_{-1} ds + \int_0^{t_i} (1 + \|O_s^{M,N}\|_{L^4}^4) ds \right).
\end{aligned}
\]

(18)

Since $\|Y_0^{M,N}\|_{L^p(\Omega, H^\gamma)} < \infty$ and $\|O_t^{M,N}\|_{L^p(\Omega, H^\gamma)} < \infty$, we deduce

\[
\| \int_0^{t_i} |Y_s^{M,N}|^2 ds \|_{L^p(\Omega, \mathbb{R})} + \| \int_0^{t_i} \|Y_s^{M,N}\|_{L^4}^4 ds \|_{L^p(\Omega, \mathbb{R})} \\
\leq C \left( 1 + \int_0^{t_i} \|Z_s^{M,N}\|_{L^2(\Omega, H^{-1})}^2 ds \right).
\]
Taking inner product of (16) by $Y_{t}^{M,N}$ and integrating from 0 to $t_i$ lead to
\[
\|Y_{t_i}^{M,N}\|^2 - \|Y_{0}^{M,N}\|^2
= -2\int_{0}^{t_i} \langle AY_{s}^{M,N}, Y_{s}^{M,N} \rangle \, ds - 2\int_{0}^{t_i} \langle AF(\bar{Y}_{s}^{M,N} + \mathcal{O}_{s}^{M,N}), Y_{s}^{M,N} \rangle \, ds + 2\int_{0}^{t_i} \langle Z_{s}^{M,N}, AY_{s}^{M,N} \rangle \, ds
\leq -\int_{0}^{t_i} \|AY_{s}^{M,N}\|^2 \, ds - 6\int_{0}^{t_i} \|Y_{s}^{M,N}\| \|\nabla Y_{s}^{M,N}\| \, ds + 2\int_{0}^{t_i} |Y_{s}^{M,N}|^2 \, ds + \int_{0}^{t_i} |Z_{s}^{M,N}|^2 \, ds
- 2\int_{0}^{t_i} \langle 3(Y_{s}^{M,N})^2\mathcal{O}_{s}^{M,N} + 3\mathcal{O}_{s}^{M,N} (\mathcal{O}_{s}^{M,N})^2 + (\mathcal{O}_{s}^{M,N})^3 - \mathcal{O}_{s}^{M,N}, AY_{s}^{M,N} \rangle \, ds
\leq -\int_{0}^{t_i} \|AY_{s}^{M,N}\|^2 \, ds - \frac{3}{2}\int_{0}^{t_i} \|\nabla(Y_{s}^{M,N})^2\|^2 \, ds + 2\int_{0}^{t_i} |Y_{s}^{M,N}|^2 \, ds + \int_{0}^{t_i} |Z_{s}^{M,N}|^2 \, ds
+ C \int_{0}^{t_i} \|Y_{s}^{M,N}\|_L^2 \mathcal{O}_{s}^{M,N}_r \|_V^2 + \|Y_{s}^{M,N}\|_V \|\mathcal{O}_{s}^{M,N}_r\|_V + \|\mathcal{O}_{s}^{M,N}_r\|_V^2 \|\mathcal{O}_{s}^{M,N}_r\|_V^2 \, ds.
\]
Again, the use of (18) gives
\[
\|Y_{t_i}^{M,N}\|^2 + \int_{0}^{t_i} \|AY_{s}^{M,N}\|^2 \, ds + \int_{0}^{t_i} \|\nabla(Y_{s}^{M,N})^2\|^2 \, ds
\leq C \left( 1 + \sup_{s \in [0,T]} \|\mathcal{O}_{s}^{M,N}_r\|_V \right) \left( 1 + \|Y_{0}^{M,N}\|^2 + \int_{0}^{t_i} \|Z_{s}^{M,N}\|^2 \, ds \right).
\]
At the moment, we turn to the estimate $E[\chi_{\Omega_{R_i - t_i - 1}} \|Z_{s}^{M,N}\|_V^2]$. For $s \in [0, t_i]$, we have
\[
\chi_{\Omega_{R_i - t_i - 1}} \|Z_{s}^{M,N}\|_V
\leq \chi_{\Omega_{R_i - t_i - 1}} \|F(\bar{X}_{s}^{M,N}) - F(\bar{X}_{s}^{M,N})_\tau\|_V
+ \chi_{\Omega_{R_i - t_i - 1}} \|F(\bar{X}_{s}^{M,N}) - F(\bar{X}_{s}^{M,N})_\tau\|_V
\leq C \chi_{\Omega_{R_i - t_i - 1}} \left( 1 + \|X_{s}^{M,N}\|_V + \|X_{s}^{M,N}\|_V \right) \left( \|X_{s}^{M,N} - X_{s}^{M,N}_\tau\|_V \right)
+ C \chi_{\Omega_{R_i - t_i - 1}} \tau \|F(\bar{X}_{s}^{M,N})\|_V^2.
\]
Before further proof, we claim that
\[
\chi_{\Omega_{R_i - t_i - 1}} \|X_{s}^{M,N}\|_V \leq C \left( 1 + \|X_0\|_V + R_\tau^3 + \|\mathcal{O}_{s}^{M,N}_r\|_V \right), \forall s \in [0, t_i).
\]
Indeed, by stability of the semigroup $E(t)$ in $V$ and Sobolev embedding inequality $\mathcal{H}^d \subset V, \delta > \frac{d}{2}$,
\[
\chi_{\Omega_{R_i - t_i - 1}} \|X_{s}^{M,N}\|_V \leq \chi_{\Omega_{R_i - t_i - 1}} \left( \|E(s)P_N X_0\|_V + \int_{0}^{s} \|E(s - r)AF(\bar{X}_{s}^{M,N})_\tau\|_V \, dr + \|\mathcal{O}_{s}^{M,N}_r\|_V \right)
\leq \chi_{\Omega_{R_i - t_i - 1}} \left( \|X_0\|_V + \int_{0}^{s} (s - r)^{-\frac{2 + \delta}{4}} \|PF(\bar{X}_{s}^{M,N})\|_V \, dr + \|\mathcal{O}_{s}^{M,N}_r\|_V \right)
\leq C \left( 1 + \|X_0\|_V + R_\tau^3 + \|\mathcal{O}_{s}^{M,N}_r\|_V \right).
\]
Next, owing to (15), one can write

\[ X^{M,N}_s - X^{M,N}_{[s]\tau} = \left[ E(s) - E([s]\tau) \right] P_N X_0 - \int_0^s \frac{E(s-u)AP_N \chi_{\Omega_1} F(X^{M,N}_{[u]\tau})}{1+\tau\|P_N F(X^{M,N}_{[u]\tau})\|} \, du \]

\[ + \int_0^{[s]\tau} \frac{E([s]\tau-u)AP_N \chi_{\Omega_1} F(X^{M,N}_{[u]\tau})}{1+\tau\|P_N F(X^{M,N}_{[u]\tau})\|} \, du + O^{M,N}_s - O^{M,N}_{[s]\tau}, \]

which implies that

\[ \chi_{\Omega_{\tau^{-1}},t^{-1}} \| X^{M,N}_s - X^{M,N}_{[s]\tau} \| \]

\[ \leq \tau^{\frac{4}{9}} \| X_0 \|_Y + \chi_{\Omega_{\tau^{-1}},t^{-1}} \left\| \int_0^{[s]\tau} E([s]\tau - u)(E(s - [s]\tau) - I) \frac{AP_N \chi_{\Omega_1} F(X^{M,N}_{[u]\tau})}{1+\tau\|P_N F(X^{M,N}_{[u]\tau})\|} \, du \right\| \]

\[ + \chi_{\Omega_{\tau^{-1}},t^{-1}} \int_0^{[s]\tau} \left\| \frac{E(s-u)AP_N \chi_{\Omega_1} F(X^{M,N}_{[u]\tau})}{1+\tau\|P_N F(X^{M,N}_{[u]\tau})\|} \right\| \, du + \chi_{\Omega_{\tau^{-1}},t^{-1}} \| O^{M,N}_s - O^{M,N}_{[s]\tau} \| \]

\[ \leq \tau^{\frac{4}{9}} \| X_0 \|_Y + \frac{1}{9} \left( 1 + R_5^3 \right) \| F \|_{\tau} \]

\[ + \frac{1}{18} \| O^{M,N}_s - O^{M,N}_{[s]\tau} \| \]

\[ \leq \tau^{\frac{4}{9}} \| X_0 \|_Y + \frac{3}{18} \left( 1 + R_5^3 \right) \| F \|_{\tau} + \| O^{M,N}_s - O^{M,N}_{[s]\tau} \|. \]

Therefore,

\[ \chi_{\Omega_{\tau^{-1}},t^{-1}} \| Z_s^{M,N} \| \leq C \tau \left( 1 + R_5^6 \right) \]

\[ + C \left( 1 + \| X_0 \|_Y \right) R_5^6 + \| O^{M,N}_s \|_V^2 + \| O^{M,N}_{[s]\tau} \|_V^2 \]

\[ \times \left( \tau^{\frac{4}{9}} \| X_0 \|_Y + \left( 1 + R_5^3 \right) \| F \|_{\tau} + \| O^{M,N}_s - O^{M,N}_{[s]\tau} \| \right). \]

Note that

\[ \mathbb{E} \left[ \| O^{M,N}_s - O^{M,N}_{[s]\tau} \|_V^p \right] \leq C \tau^{\min \{ \frac{4}{9}, \frac{\gamma}{4} \} p}. \]

Then taking \( R_5 = \tau^{-\min \{ \frac{4}{9}, \frac{\gamma}{4} \}} \) leads to

\[ \mathbb{E} \left[ \chi_{\Omega_{\tau^{-1}},t^{-1}} \| Z_s^{M,N} \|^{2p} \right] < \infty. \]

As a result, we can deduce that

\[ \sup_{s \in [0,t]} \chi_{\Omega_{\tau^{-1}},t^{-1}} Y_s^{M,N} \]

\[ \left\| \int_0^{t_i} \chi_{\Omega_{\tau^{-1}},t^{-1}} \| AY_s^{M,N} \|_{L^p(\Omega,\mathbb{R})} \right\| \]

\[ + \left\| \int_0^{t_i} \chi_{\Omega_{\tau^{-1}},t^{-1}} \| \nabla \left( Y_s^{M,N} \right) \|_{L^p(\Omega,\mathbb{R})} \right\| < \infty. \]
This completes the proof. \[\square\]

With the aid of Lemma 10, we are able to obtain \(p\)-th moment bounds of \(\|X_{t_i}^{M,N}\|_{L^6}\) on the subevents \(\Omega_{R_t, t_{i-1}}\).

**Lemma 11** Let \(p \in [1, \infty)\), \(R_t = \tau - \min\{\frac{4}{81}, \frac{\gamma}{3}\}\) for \(d = 1\) and \(R_t = \tau - \min\{\frac{1}{54}, \frac{\gamma - 1}{36}\}\) for \(d = 2, 3\). Under Assumptions 1–4, the fully discrete solution \(X_{t_i}^{M,N}\) satisfies

\[
\sup_{M,N \in \mathbb{N}^+} \sup_{i \in \{0, 1, \ldots, M\}} \mathbb{E}\left[\chi_{\Omega_{R_t, t_{i-1}}} \|X_{t_i}^{M,N}\|_{L^6}^p\right] < \infty,
\]

where with the convention, we set \(\chi_{\Omega_{R_t, t_{i-1}}} = 1\).

**Proof** Combining (17) with Sobolev embedding inequality and Gagliardo–Nirenberg inequality implies that for \(d = 1\),

\[
\begin{align*}
\mathbb{E}\left[\chi_{\Omega_{R_t, t_{i-1}}} \|Y_{t_i}^{M,N}\|_{L^6}^p\right] & \leq \mathbb{E}\left[\chi_{\Omega_{R_t, t_{i-1}}} \|E(t_i)X_0\|_{L^6}^p\right] \\
& + \mathbb{E}\left[\left\| \int_0^{t_i} \chi_{\Omega_{R_t, t_{i-1}}} E(t_i - s) A P_N(Z_s^{M,N} - F(X_s^{M,N})) ds \right\|_{L^6}^p\right] \\
& \leq C + C \mathbb{E}\left[\int_0^{t_i} (t_i - s)^{-\frac{7}{12}} \right. \\
& \times \left(1 + \|O_s^{M,N}\|_{L^6}^3 + \chi_{\Omega_{R_t, t_{i-1}}} \|Y_s^{M,N}\|_{L^6}^3 + \chi_{\Omega_{R_t, t_{i-1}}} \|Z_s^{M,N}\|_{L^6}\right)^p ds] \\
& \leq C + C \mathbb{E}\left[\int_0^{t_i} (t_i - s)^{-\frac{7}{12}} \chi_{\Omega_{R_t, t_{i-1}}} A Y_s^{M,N} \left\|\frac{3}{2} Y_s^{M,N} \|_{L^6}^2 \right\| ds\right] \\
& \leq C + C \left(\int_0^{t_i} (t_i - s)^{-\frac{7}{9}} ds\right)^\frac{p}{2} \mathbb{E}\left[\sup_{s \in [0, t_i]} \chi_{\Omega_{R_t, t_{i-1}}} \|Y_s^{M,N}\|_{L^6}^{10p}\right] \\
& + C \mathbb{E}\left[\int_0^{t_i} \chi_{\Omega_{R_t, t_{i-1}}} \|A Y_s^{M,N}\|_{L^6}^2 ds\right]^p \\
& < \infty.
\end{align*}
\]

For \(d = 2, 3\), we define a Lyapunov functional \(J(u)\) by

\[
J(u) = \frac{1}{2} \|\nabla u\|^2 + \int_D \Phi(u) dx,
\]

where \(\Phi\) is the primitive of \(F\). Multiplying (16) by \(A^{-1}Y_t^{M,N}\) yields

\[
\left|\dot{Y}_t^{M,N}\right|_{-1}^2 + \frac{1}{2} \frac{d|Y_t^{M,N}|^2_{-1}}{dt} + \langle F(Y_t^{M,N} + C_t^{M,N}, \dot{Y}_t^{M,N}) , \dot{Y}_t^{M,N} \rangle = \langle Z_t^{M,N}, \dot{Y}_t^{M,N} \rangle.
\]
To proceed further, owing to Hölder’s inequality, for $\theta > \frac{d}{2}$, we have

$$\|(Y_t^{M,N})^2 \nabla O_t^{M,N}\| \leq \|(Y_t^{M,N})^2\|_{L^{2(d+\delta)}} \|\nabla O_t^{M,N}\|_{L^{2+\delta}} \leq C\|(Y_t^{M,N})^2\|_1|O_t^{M,N}|_{\theta},$$

where the Sobolev embedding inequality $H^{\frac{d}{2}-\frac{d}{p}} \subset L^p$ for $p \geq 2$ was used in the last inequality and we take sufficiently small $\delta > 0$ for $d = 2$ and $\delta = 1$ for $d = 3$. Therefore,

$$|P(3(Y_t^{M,N})^2O_t^{M,N} + 3Y_t^{M,N}(O_t^{M,N})^2 + (O_t^{M,N})^3 - O_t^{M,N})|_1 \leq C\|(Y_t^{M,N})^2\|_1\|O_t^{M,N}\|_V + \|(Y_t^{M,N})^2 \nabla O_t^{M,N}\| + \|\nabla Y_t^{M,N}\|\|O_t^{M,N}\|_V^2$$

$$+ |Y_t^{M,N}|_V \|\nabla O_t^{M,N}\| + \|\nabla O_t^{M,N}\|\|O_t^{M,N}\|_V^2 + \|\nabla O_t^{M,N}\|) \leq C(1 + |(Y_t^{M,N})^2|_1 + \|AY_t^{M,N}\|)(1 + |O_t^{M,N}|_\theta^3).$$

Combining it with the fact $\Phi'(t) = F(t)$ and Cauchy–Schwartz inequality infers that for $\theta > \frac{d}{2}$,

$$-\langle F(Y_t^{M,N} + O_t^{M,N}), \dot{Y}_t^{M,N}\rangle + \langle Z_t^{M,N}, \dot{Y}_t^{M,N}\rangle$$

$$= -\langle F(Y_t^{M,N}), \dot{Y}_t^{M,N}\rangle + \langle Z_t^{M,N}, \dot{Y}_t^{M,N}\rangle$$

$$- \langle 3(Y_t^{M,N})^2O_t^{M,N} + 3Y_t^{M,N}(O_t^{M,N})^2 + (O_t^{M,N})^3 - O_t^{M,N}, \dot{Y}_t^{M,N}\rangle$$

$$\leq -\frac{d}{dt} \int_D \Phi(Y_t^{M,N}) \, dx + |Z_t^{M,N}|_1|\dot{Y}_t^{M,N}|_{-1}$$

$$+ |P(3(Y_t^{M,N})^2O_t^{M,N} + 3Y_t^{M,N}(O_t^{M,N})^2 + (O_t^{M,N})^3 - O_t^{M,N})|_1|\dot{Y}_t^{M,N}|_{-1}$$

$$\leq -\frac{d}{dt} \int_D \Phi(Y_t^{M,N}) \, dx + |\dot{Y}_t^{M,N}|_{-1}^2 + \frac{1}{2}|Z_t^{M,N}|_{1}^2$$

$$+ C(1 + |(Y_t^{M,N})^2|_1 + \|AY_t^{M,N}\|)(1 + |O_t^{M,N}|_{\theta}^6).$$

Therefore,

$$J(Y_t^{M,N}) \leq J(Y_0^{M,N}) + C \int_0^t |Z_s^{M,N}|_{1}^2 \, ds$$

$$+ C\left(1 + \int_0^t (|(Y_s^{M,N})^2|_1 + \|AY_s^{M,N}\|_1^2) \, ds\right)(1 + \sup_{s \in [0,T]} |O_s^{M,N}|_{\theta}^6).$$

Applying (17) and Lemma 8 infers that

$$\mathbb{E}[\langle \chi_{\Omega_{R_{1/\epsilon}^{t-1}}} J(Y_t^{M,N}) \rangle^p] \leq C\left(1 + \mathbb{E}\left[\int_0^t \chi_{\Omega_{R_{1/\epsilon}^{t-1}}} |Z_s^{M,N}|_{1}^2 \, ds\right]^p\right).$$
Further, we adapt similar arguments used in the proof of (19) to get for \( s \in [0, t_i) \) and \( \kappa \in \left( \frac{d}{2}, \min\{\gamma, 2\} \right) \),

\[
\| \chi_{\Omega_{R_t,t_i-1}} X^{M,N}_s \|_{L^p(\Omega, \dot{H}^\kappa)} \leq C \left( \| X_0 \|_{L^p(\Omega, \dot{H}^\kappa)} + \| O_s^{M,N} \|_{L^p(\Omega, \dot{H}^\kappa)} \right) + \frac{\kappa}{2} \int_0^s (s - r)^{-\frac{\kappa+2}{4}} \| \chi_{\Omega_{R_t,t_i-1}}^r \| \| P F(X_{[s]\tau}^M,N) \|_{L^p(\Omega, \mathbb{R})} \, dr \\
\leq C(1 + R_t^2).
\]

Using (9) and the Sobolev embedding inequality \( \dot{H}^\kappa \subset V \) yields

\[
\| \chi_{\Omega_{R_t,t_i-1}} F(X^{M,N}_{[s]\tau}) \|_{L^p(\Omega, \dot{H}^1)} = \| \chi_{\Omega_{R_t,t_i-1}} F'(X^{M,N}_{[s]\tau}) \nabla X^{M,N}_{[s]\tau} \|_{L^p(\Omega, \dot{H})} \\
\leq C \left( 1 + \| \chi_{\Omega_{R_t,t_i-1}} X^{M,N}_{[s]\tau} \|_{L^p(\Omega, \dot{H}^\kappa)} \right) \\
\leq C(1 + R_t^3).
\]

Similarly, employing (4), (7) and Assumption 4 yields

\[
\| \chi_{\Omega_{R_t,t_i-1}} (X^{M,N}_s - X^{M,N}_{[s]\tau}) \|_{L^p(\Omega, \dot{H}^1)} \\
\leq C \left( \frac{\tau^{\gamma_1-1}}{\gamma_1-1} \| X_0 \|_{L^p(\Omega, \dot{H}^\gamma)} + \tau^{\min\{\frac{1}{\gamma_1}, \frac{\gamma_1-1}{\gamma_1-1}\}} \right) \\
+ \frac{\kappa}{2} \int_0^s (s - u)^{-\frac{\kappa+2}{4}} \| \chi_{\Omega_{R_t,t_i-1}}^r \| \| P N F(X_{[s]\tau}^{M,N}) \|_{L^p(\Omega, \mathbb{R})} \, du \\
+ \frac{3}{2} \| E(s-u) A^{3/2} P N F(X_{[s]\tau}^{M,N}) \|_{L^p(\Omega, \mathbb{R})} \right) \\
\leq C \tau^{\min\{\frac{1}{2}, \frac{\gamma_1-1}{\gamma_1-1}\}} + C \left( 1 + R_t^3 \right) (\tau^{\frac{1}{6}} + \frac{\kappa}{2} \tau^{\frac{1}{7}}) \\
\leq C \tau^{\min\{\frac{1}{2}, \frac{\gamma_1-1}{\gamma_1-1}\}} (1 + R_t^3).
\]

Finally, it suffices to bound \( \mathbb{E} \left[ \| \chi_{\Omega_{R_t,t_i-1}} | Z_s^{M,N} |^p \|_1 \right] \) for \( s \in [0, t_i] \). Taking \( R_t = \tau^{-\min\{\frac{1}{2}, \frac{\gamma_1-1}{\gamma_1-1}\}} \) yields

\[
\| \chi_{\Omega_{R_t,t_i-1}} Z_s^{M,N} \|_{L^p(\Omega, \dot{H}^1)} \leq \| \chi_{\Omega_{R_t,t_i-1}} (F(X_s^{M,N}) - F(X_{[s]\tau}^{M,N})) \|_{L^p(\Omega, \dot{H}^1)} \\
+ \frac{\kappa}{2} \int_0^s (s - u)^{-\frac{\kappa+2}{4}} \| \chi_{\Omega_{R_t,t_i-1}}^r \| \| P N F(X_{[s]\tau}^{M,N}) \|_{L^p(\Omega, \mathbb{R})} \, du \\
\leq C \left( \| \chi_{\Omega_{R_t,t_i-1}} (X_s^{M,N} - X_{[s]\tau}^{M,N}) \|_{L^2p(\Omega, \dot{H}^1)} \left( 1 + \| \chi_{\Omega_{R_t,t_i-1}} X_{[s]\tau}^{M,N} \|_{L^4p(\Omega, \dot{H}^\kappa)} \right) \\
+ \| \chi_{\Omega_{R_t,t_i-1}} X_s^{M,N} \|_{L^4p(\Omega, \dot{H}^\kappa)} \right) \\
+ C \tau \| \chi_{\Omega_{R_t,t_i-1}} F(X_{[s]\tau}^{M,N}) \|_{L^2p(\Omega, \dot{H}^1)} \| \chi_{\Omega_{R_t,t_i-1}} F(X_{[s]\tau}^{M,N}) \|_{L^2p(\Omega, \dot{H})}
\]
\[
\leq C \tau^{\min\left(\frac{1}{6}, \frac{1}{4}\right)} (1 + R_t^9) + C \tau (1 + R_t^{12}) < \infty.
\]

The above estimates in combination with the fact \( |Y_{t_i}^{M,N}|^2 \leq 2 J(Y_{t_i}^{M,N}) \) yield

\[
\|X_{\Omega_{R,t_i-1}} Y_{t_i}^{M,N}\|_{L^p(\Omega, H^1)} < \infty, \quad d = 2, 3. \tag{21}
\]

Gathering (20), (21), Sobolev embedding inequality \( H^1 \subset L^6 \), \( d = 2, 3 \) and Lemma 8 together completes the proof. \( \square \)

By adopting similar arguments in [32, Theorem 4.6], we can obtain a priori moment bound of \( \|X_{t_m}^{M,N}\|_{L^6} \) via Markov’s inequality.

**Theorem 12** Under Assumptions 1–4, it holds that for any \( p \geq 1 \),

\[
\sup_{M,N \in \mathbb{N}^+} \sup_{i \in \{0, 1, \ldots, M\}} \mathbb{E}\left[ \|X_{t_i}^{M,N}\|_{L^p}^p \right] < \infty.
\]

**Proof** By virtue of Lemma 11 and the fact that \( \Omega_{R,t_i} \subset \Omega_{R,t_i-1} \), it suffices to bound

\[
\sup_{M,N \in \mathbb{N}^+} \sup_{i \in \{0, 1, \ldots, M\}} \mathbb{E}\left[ \|X_{\Omega_{R,t_i}} X_{t_i}^{M,N}\|_{L^p}^p \right]
\]

The case \( i = 0 \) is trivial and we only consider \( i \in \{1, \ldots, M\} \). Following a standard argument and using the Sobolev embedding inequality \( H^1 \subset L^6 \) yield

\[
\|X_{t_i}^{M,N}\|_{L^6} \leq \|E(t_i) P_N X_0\|_{L^6} + \left\| \int_0^{t_i} E(t_i - [s]_\tau) \frac{A_{M,N}^{i,s} F(X_{t_i}^{M,N})}{1 + \tau \|P_N F(X_{t_i}^{M,N})\|} \right\|_{L^6}
\]

\[
\leq C \left( \|X_0\|_{L^6} + \|O_{t_i}^{M,N}\|_{L^p} \right)
\]

\[
+ \frac{1}{\tau} \left\| A_{M,N}^{i,s} E(t_i - s) \right\|_{L^p(H)} \left\| P_N F(X_{t_i}^{M,N}) \right\|_{L^p((\Omega, L^r))} \left( t_i - s \right)^{-\frac{3}{2}} ds
\]

\[
\leq C \left( \|X_0\|_{L^6} + \|O_{t_i}^{M,N}\|_{L^6} + \left\| \frac{A_{M,N}^{i,s} E(t_i - s)}{1 + \tau \|P_N F(X_{t_i}^{M,N})\|} \right\|_{L^p((\Omega, L^r))} \right)
\]

Thanks to Lemma 8 and Assumption 4, we have for \( p \geq 2 \),

\[
\|X_{t_i}^{M,N}\|_{L^p(\Omega, L^6)} \leq C (1 + \tau^{-1}), \quad i \in \{0, 1, \ldots, M\}. \tag{22}
\]

Note that

\[
\Omega_{R,t_i}^{\tau_0} = \Omega_{R,t_i-1}^{\tau_0} \cup \left( \Omega_{R,t_i} \cap \left\{ \omega \in \Omega : \|X_{t_i}^{M,N}\|_{L^6} > R_\tau \right\} \right).
\]

\( \odot \) Springer
Meanwhile, we recall \( \chi_{\Omega^c_{R, t-1}} = 0 \) and then derive

\[
\chi_{\Omega^c_{R, t_i}} = \chi_{\Omega^c_{R, t_{i-1}}} + \chi_{\Omega_{R, t_{i-1}}} \cdot \chi_{\{X_{t_j}^M, N \| L_6 > R_t \}} \\
= \sum_{j=0}^i \chi_{\Omega_{R, t_{j-1}}} \cdot \chi_{\{X_{t_j}^M, N \| L_6 > R_t \}}.
\]

Combining (22) with Markov’s inequality and Hölder’s inequality shows that

\[
\mathbb{E} \left[ \chi_{\Omega^c_{R, t_i}} \| X_{t_j}^M, N \|_{L_6}^p \right] \\
= \sum_{j=0}^i \mathbb{E} \left[ \| X_{t_j}^M, N \|_{L_6}^p \cdot \chi_{\Omega_{R, t_{j-1}}} \cdot \chi_{\{X_{t_j}^M, N \| L_6 > R_t \}} \right] \\
\leq \sum_{j=0}^i \left( \mathbb{E} \left[ \| X_{t_j}^M, N \|_{L_6}^{2p} \right] \right)^{\frac{1}{2}} \cdot \left( \mathbb{E} \left[ \chi_{\Omega_{R, t_{j-1}}} \cdot \chi_{\{X_{t_j}^M, N \| L_6 > R_t \}} \right] \right)^{\frac{1}{2}} \\
\leq \sum_{j=0}^i C (1 + \tau^{-p}) \cdot \left( \mathbb{E} \left[ \chi_{\Omega_{R, t_{j-1}}} \| X_{t_j}^M, N \|_{L_6} \right] \right)^{\frac{1}{2}} \\
\leq C (1 + \tau^{-p}) \sum_{j=0}^i \left( \mathbb{E} \left[ \chi_{\Omega_{R, t_{j-1}}} \| X_{t_j}^M, N \|_{L_6}^{2(p+1)} / (R_t^{\tau}) \right] \right)^{\frac{1}{2}} \\
\leq C (1 + \tau^{-p}) \sum_{j=0}^i \tau^{p+1} \left( \mathbb{E} \left[ \chi_{\Omega_{R, t_{j-1}}} \| X_{t_j}^M, N \|_{L_6}^{2(p+1)} \right] \right)^{\frac{1}{2}} < \infty,
\]

where \( \varrho(\gamma) = \min\{\frac{4}{81}, \frac{2}{24} \} \) for \( d = 1 \) and \( \varrho(\gamma) = \min\{\frac{1}{34}, \frac{2-1}{36} \} \) for \( d = 2, 3 \). The proof is now completed. \( \square \)

With Theorem 12 at hand, it is trivial to verify the regularity of \( X_{t_i}^M, N \) in the next corollaries.

**Corollary 13** Let Assumptions 1–4 be fulfilled. Then for any \( p \geq 1 \), we have

\[
\sup_{M, N} \sup_{t \in [0, T]} \mathbb{E}[|X_{t_i}^M, N|_p] < \infty.
\]

**Proof** It follows from the Burkholder–Davis–Gundy–type inequality, (6) and (11) that

\[
\mathbb{E} \left[ \int_0^t E(t - [s \tau]) P_{N} dW(s) \right]_{L^p(\Omega, \mathcal{H}_v)} \\
\leq C \left( \int_0^t \| A^{\frac{v}{2}} E(t - [s \tau]) \|_{L^2}^2 ds \right)^{\frac{1}{2}}.
\]

\( \square \) Springer
Further, by taking any fixed number $\delta \in (\frac{3}{2}, 2)$, we first consider the case $\gamma \in [1, \delta)$.

$$
\|X_t^{M,N}\|_{L^p(\Omega, H^\gamma)} 
\leq \|E(t)X_0^{M,N}\|_{L^p(\Omega, H^\gamma)} + \left\| \int_0^t E(t-s)AP_NF(X_{[s]_T}^{M,N}) \frac{ds}{1+\tau\|P_NF(X_{[s]_T}^{M,N})\|} \right\|_{L^p(\Omega, H^\gamma)}
+ \left\| \int_0^t E(t-[s]_T)P_NdW(s) \right\|_{L^p(\Omega, H^\gamma)}
\leq C + \left\| \int_0^t E(t-s)AP_NF(X_{[s]_T}^{M,N}) \frac{ds}{1+\tau\|P_NF(X_{[s]_T}^{M,N})\|} \right\|_{L^p(\Omega, H^\gamma)}.
$$

This together with Assumption 4 yields that

$$
\|X_t^{M,N}\|_{L^p(\Omega, H^\gamma)} \leq \|E(t)X_0^{M,N}\|_{L^p(\Omega, H^\gamma)} + \int_0^t \|E(t-s)AP_NF(X_{[s]_T}^{M,N})\|_{L^p(\Omega, H^\gamma)} ds
+ \int_0^t \|E(t-[s]_T)P_NdW(s)\|_{L^p(\Omega, H^\gamma)} ds
\leq C + \int_0^t \|E(t-s)AP_NF(X_{[s]_T}^{M,N})\|_{L^p(\Omega, H^\gamma)} ds
\leq C + \int_0^t \|E(t-s)AP_NF(X_{[s]_T}^{M,N})\|_{L^p(\Omega, H^\gamma)} ds
\leq C\left(1 + \sup_{i \in \{0,1,...,M\}} \|X_{t_i}^{M,N}\|_{L^3(\Omega, L^6)}^3\right) < \infty,
$$

where (4) and Theorem 12 were used. Next, we turn to the case $\gamma \in (\delta, 3)$. By using the similar approach and the Sobolev embedding inequality $\dot{H}^\delta \subset V$, we obtain

$$
\|X_t^{M,N}\|_{L^p(\Omega, \dot{H}^\gamma)} \leq \|E(t)X_0^{M,N}\|_{L^p(\Omega, \dot{H}^\gamma)} + \int_0^t \|E(t-s)AP_NF(X_{[s]_T}^{M,N})\|_{L^p(\Omega, \dot{H}^\gamma)} ds
+ \int_0^t \|E(t-[s]_T)P_NdW(s)\|_{L^p(\Omega, \dot{H}^\gamma)} ds
\leq C\left(1 + \sup_{i \in \{0,1,...,M\}} \|X_{t_i}^{M,N}\|_{L^3(\Omega, \dot{H}^\gamma)}^3\right) < \infty.
$$
Before proceeding further, one uses (3) to derive
\[
\sup_{t \in [0, T]} \| P F(M, N) \|_{L^p(\Omega, \mathcal{H}^2)} \leq C \sup_{t \in [0, T]} \| P F(M, N) \|_{L^p(\Omega, \mathcal{H}^2(D))} \\
\leq C \left( 1 + \sup_{t \in [0, T]} \| X^M, N \|_{L^3_p(\Omega, \mathcal{H}^2)} \right) < \infty
\]

Bearing this in mind and repeating the same lines of (23) we can prove for \( \gamma \in [3, 4) \),
\[
\left\| \int_0^t E(t - s) A \frac{\gamma}{4} \int_0^t \| P F(M, N) \|_{L^p(\Omega, \mathcal{H}^2)} ds \right\|_{L^p(\Omega, \mathcal{H}^2)} \\
\leq C \left( 1 + \sup_{t \in [0, T]} \| X^M, N \|_{L^3_p(\Omega, \mathcal{H}^2)} \right) < \infty.
\]

When \( \gamma = 4 \), we only need the boundedness of \( \| X^M, N \|_{L^p(\Omega, \mathcal{H}^{\frac{d}{2} + 2\epsilon})} \) for small enough \( \epsilon > 0 \) due to the Sobolev embedding theorem. This is guaranteed by the regularity estimate in \( \mathcal{H}^\gamma \), \( \gamma \in [3, 4) \). Thus, the proof is finished. \hfill \Box

**Corollary 14** Let Assumptions 1–4 be fulfilled, then for any \( p \geq 1 \) and \( \beta \in [0, \gamma] \), there exists a constant \( C > 0 \) such that
\[
\sup_{M, N \in \mathbb{N}^+} \sup_{t \in [0, T]} \left\| X^M, N(t) - X^M, N(s) \right\|_{L^p(\Omega, \mathcal{H}^\beta)} \leq C (t - s)^{\frac{\gamma - \beta}{4}}, \quad 0 \leq s < t \leq T.
\]

### 3.3 Strong convergence rate of the fully discrete scheme

In this subsection, we are well prepared to analyze the strong convergence rate of the tamed exponential Euler method.

**Theorem 15** (Strong convergence rate of temporal semi-discretization) Suppose Assumptions 1–4 are valid. Let \( X^N(t) \) and \( X^M, N(t) \) be given by (13) and (15), respectively. Then for all \( p \geq 1 \) we have
\[
\sup_{M, N \in \mathbb{N}^+} \sup_{t \in [0, T]} \left\| X^N(t) - X^M, N \right\|_{L^p(\Omega, \mathcal{H})} \leq C \tau^\frac{\gamma}{4}.
\]

**Proof** Firstly, we introduce an auxiliary process,
\[
\tilde{X}^M, N(t) = E(t) P_N X_0 - \int_0^t E(t - s) A \frac{\gamma}{4} \int_0^t \| P F(M, N) \|_{L^p(\Omega, \mathcal{H}^2)} ds + \int_0^t E(t - s) P_N dW(s).
\]
According to the uniform moment bounds of $X^{M,N}_s$, we can follow a standard approach to obtain $\| \tilde{X}_t^{M,N} \|_{L^P(\Omega, \tilde{H})} < \infty$ for any $t \in [0, T]$. Then we can separate $\| X^N(t) - X^M(t) \|_{L^P(\Omega, \tilde{H})}$ into two terms:

$$\| X^N(t) - X^M(t) \|_{L^P(\Omega, \tilde{H})} \leq \| \tilde{X}_t^{M,N} - X^M_t \|_{L^P(\Omega, \tilde{H})} + \| X^N(t) - \tilde{X}_t^{M,N} \|_{L^P(\Omega, \tilde{H})}. $$

Next, we split the proof into two parts.

**Step 1:** Estimate of $\| \tilde{X}_t^{M,N} - X^M_t \|_{L^P(\Omega, \tilde{H})}$.

We decompose the error $\| \tilde{X}_t^{M,N} - X^M_t \|_{L^P(\Omega, \tilde{H})}$ into three further parts,

$$\| \tilde{X}_t^{M,N} - X^M_t \|_{L^P(\Omega, \tilde{H})} = \left\| \int_{0}^{t} E(t-s)APN F(X^M_{x,s}, t) \right\|_{L^P(\Omega, \tilde{H})} - \int_{0}^{t} E(t-s)APN F(X^M_{x,s}, t)ds \right\|_{L^P(\Omega, \tilde{H})} + \left\| \int_{0}^{t} E(t-s)P_N dW(s) - \int_{0}^{t} E(t-[s] \tau) P_N dW(s) \right\|_{L^P(\Omega, \tilde{H})} \leq \left\| \int_{0}^{t} E(t-s)APN (F(X^M_{x,s}, t) - F(X^M_{x,s}, t)) \right\|_{L^P(\Omega, \tilde{H})} + \left\| \int_{0}^{t} E(t-s)APN F(X^M_{x,s}, t)ds \right\|_{L^P(\Omega, \tilde{H})} + \left\| \int_{0}^{t} E(t-s)P_N dW(s) - \int_{0}^{t} E(t-[s] \tau) P_N dW(s) \right\|_{L^P(\Omega, \tilde{H})} =: J_1 + J_2 + J_3.

By using Taylor’s formula and mild form of $X^M_t$, we divide $J_1$ into four terms,

$$J_1 \leq \left\| \int_{0}^{t} E(t-s)APN F’(X^M_{x,s}, t)(E(s-[s] \tau) - I)X^M_{x,s} \right\|_{L^P(\Omega, \tilde{H})} + \left\| \int_{0}^{t} E(t-s)APN F’(X^M_{x,s}, t) \int_{[s] \tau}^{S} E(s-r)APN F(X^M_{x,r \tau})dr \right\|_{L^P(\Omega, \tilde{H})} + \left\| \int_{0}^{t} E(t-s)APN \int_{[s] \tau}^{S} E(s-[r] \tau) P_N dW(r)ds \right\|_{L^P(\Omega, \tilde{H})} + \left\| \int_{0}^{t} E(t-s)APN F’(X^M_{x,s}, t) \left( X^M_{x,s} - X^M_{x,[s] \tau} \right) \left( X^M_{x,s} - X^M_{x,[s] \tau} \right) \right\|_{L^P(\Omega, \tilde{H})} =: J_{11} + J_{12} + J_{13} + J_{14},

where $\lambda(X^M_{x,s}, X^M_{x,[s] \tau}) := X^M_{x,[s] \tau} + \lambda(X^M_{x,s} - X^M_{x,[s] \tau})$. 

Subsequently, we treat the above four terms separately. It follows from (4), (5), (9) and Corollary 13 that

\[ J_{11} \leq \int_0^T \left\| E(t-s)A P_N F'(X_{[s],r}^M,N)(E(s-[s]_r)-I)X_{[s],r}^M,N \right\|_{L^p(\Omega,\dot{H})} \, ds \]

\[ \leq C \int_0^T (t-s)^{-\frac{1}{2}} \left\| F'(X_{[s],r}^M,N)(E(s-[s]_r)-I)X_{[s],r}^M,N \right\|_{L^p(\Omega,\dot{H})} \, ds \]

\[ \leq C \int_0^T (t-s)^{-\frac{1}{2}} \left( 1 + \left\| X_{[s],r}^M,N \right\|_{L^4(\Omega,\dot{H}Y)} \right)^2 \, ds \]

\[ \times \left\| (E(s-[s]_r)-I)A^{-\frac{\gamma}{2}} A^\frac{\gamma}{2} X_{[s],r}^M,N \right\|_{L^2(\Omega,\dot{H})} \, ds \]

\[ \leq C \tau^{\frac{\gamma}{2}}. \]

For the error term \( J_{12} \), we apply (4), (9) and Corollary 13 to deduce that for \( \gamma \in \left( \frac{d}{2}, 2 \right) \),

\[ J_{12} \leq \int_0^T \left\| E(t-s)A P_N F'(X_{[s],r}^M,N) \int_{[s],r} E(s-r)A P_N F(X_{[r],r}^M,N) \right\|_{L^p(\Omega,\dot{H})} \, ds \]

\[ \leq \int_0^T \int_{[s],r} (t-s)^{-\frac{1}{2}} \left\| F'(X_{[s],r}^M,N) \right\|_{L^p(\Omega,\dot{H})} \, dr \, ds \]

\[ \leq \int_0^T \int_{[s],r} (t-s)^{-\frac{1}{2}} \left( 1 + \left\| X_{[s],r}^M,N \right\|_{L^4(\Omega,\dot{H}Y)} \right)^2 \, dr \, ds \]

\[ \times \left\| (s-r)^{-\frac{1}{2}} \left\| P F(X_{[r],r}^M,N) \right\|_{L^2(\Omega,\dot{H})} \right\|_{L^2(\Omega,\dot{H})} \, ds \]

\[ \leq C \tau^{\frac{1}{2}} \int_0^T (t-s)^{-\frac{1}{2}} \, ds \left( 1 + \sup_{s \in [0,T]} \left\| X_{[s],r}^M,N \right\|_{L^4(\Omega,\dot{H}Y)}^2 \right) \]

\[ \times \sup_{r \in [0,T]} \left\| P F(X_{[r],r}^M,N) \right\|_{L^2(\Omega,\dot{H})} \]

\[ \leq C \tau^{\frac{1}{2}}. \]

and for \( \gamma \in [2, 4) \),

\[ J_{12} \leq \int_0^T \left\| E(t-s)A P_N F'(X_{[s],r}^M,N) \int_{[s],r} E(s-r)A P_N F(X_{[r],r}^M,N) \right\|_{L^p(\Omega,\dot{H})} \, ds \]

\[ \leq \int_0^T \int_{[s],r} (t-s)^{-\frac{1}{2}} \left\| F'(X_{[s],r}^M,N) \right\|_{L^p(\Omega,\dot{H})} \, dr \, ds \]

\[ \leq \int_0^T \int_{[s],r} (t-s)^{-\frac{1}{2}} \left( 1 + \left\| X_{[s],r}^M,N \right\|_{L^4(\Omega,\dot{H}Y)} \right)^2 \, dr \, ds \]

\[ \times \left\| (s-r)^{-\frac{1}{2}} \left\| P F(X_{[r],r}^M,N) \right\|_{L^2(\Omega,\dot{H})} \right\|_{L^2(\Omega,\dot{H})} \, ds \]

\[ \leq C \tau \int_0^T (t-s)^{-\frac{1}{2}} \, ds \left( 1 + \sup_{s \in [0,T]} \left\| X_{[s],r}^M,N \right\|_{L^4(\Omega,\dot{H}Y)}^2 \right) \]

\[ \times \sup_{r \in [0,T]} \left\| P F(X_{[r],r}^M,N) \right\|_{L^2(\Omega,\dot{H})} \]

\[ \leq C \tau^{\frac{1}{2}}. \]
Concerning \( J_{13} \), we have

\[
J_{13} = \left\| \int_0^t E(t-s)A P_N F'(X_{[s],t}^{M,N}) \int_{[s],t} E(s-[r]_\tau) P_N dW(r) \, ds \right\|_{L^p(\Omega,\mathcal{H})}
\]

\[
\leq \left\| \int_0^t E(t-s)A P_N F'(X_{[s],t}^{M,N}) \int_{[s],t} E(s-[r]_\tau) P_N dW(r) \, ds \right\|_{L^p(\Omega,\mathcal{H})}
\]

\[
+ \left\| \int_0^t E(t-s)A P_N F'(X_{[s],t}^{M,N}) \int_{[s],t} E(s-[r]_\tau) P_N dW(r) \, ds \right\|_{L^p(\Omega,\mathcal{H})}
\]

\[
= J_{131} + J_{132}.
\]

Without loss of generality, we assume that there exists an integer \( m \in \mathbb{N} \) such that \( \lfloor t \rfloor_\tau \leq t_m \) and then apply the stochastic Fubini theorem, Burkholder–Davis–Gundy-type inequality and the Hölder inequality to obtain

\[
J_{131} = \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \left\| \int_{t_k}^{t_{k+1}} E(t-s)A P_N F'(X_{t_k}^{M,N}) \int_{t_k}^{t_{k+1}} E(s-[r]_\tau) P_N dW(r) \, ds \right\|_{L^p(\Omega,\mathcal{H})}
\]

\[
= \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \left\| \int_{t_k}^{t_{k+1}} X_{t_k,s}(r) E(t-s)A P_N F'(X_{t_k}^{M,N}) E(s-[r]_\tau) \, ds \right\|_{L^p(\Omega,\mathcal{H})}
\]

\[
\leq \left( \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \left\| \int_{t_k}^{t_{k+1}} X_{t_k,s}(r) E(t-s)A P_N F'(X_{t_k}^{M,N}) E(s-[r]_\tau) \, ds \right\|^2_{L^p(\Omega,\mathcal{H})} \right)^{1/2}
\]

\[
\leq C \tau \left( \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \int_{l=1}^{\infty} \|E(t-s)A F'(X_{t_k}^{M,N}) E(s-[r]_\tau) \|_{L^p(\Omega,\mathcal{H})} \, ds \right)^{1/2}
\]

Further, by using (4), Lemma 9, Corollary 13 and (11), one can find that for \( \gamma \in (1, 4] \) and \( \kappa = \frac{3}{4}, d = 1 \) and \( \kappa = 1, d = 2, 3, \)

\[
J_{131} \leq C \tau \left( \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \int_{l=1}^{\infty} (t-s)^{-\frac{2-\kappa}{2}} \|A^{\frac{\kappa}{2}} F'(X_{t_k}^{M,N}) \|_{L^p(\Omega,\mathcal{H})} \, ds \right)^{1/2}
\]

\[
\leq C \tau \left( \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t-s)^{-\frac{2-\kappa}{2}} (1 + \|X_{t_k}^{M,N}\|_{L^p(\Omega,\mathcal{H})}^4) \right)^{1/2}
\]

\[
\times \left( \sum_{l=1}^{\infty} \|A^{\frac{\gamma}{2}} A^{\frac{\gamma}{2}} E(s-[r]_\tau) \|_{L^p(\Omega,\mathcal{H})}^4 \right)^{1/2}
\]

\[
\leq C \tau \left( \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t-s)^{-\frac{2-\kappa}{2}} ds \right)^{1/2} \|A^{\frac{\gamma}{2}} Q^\frac{\gamma}{2} \|_{L^p(\Omega,\mathcal{H})}^2
\]

\[
\leq C \tau \left( \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t-s)^{-\frac{2-\kappa}{2}} ds \right)^{1/2} \|A^{\frac{\gamma}{2}} Q^\frac{\gamma}{2} \|_{L^p(\Omega,\mathcal{H})}^2
\]
For $\gamma \in (\frac{1}{\tau}, 1]$, from $[t]_{\tau} = t_m$, (4), (6), (9), (11), Corollary 13 and Burkholder–Davis–Gundy–type inequality, we deduce

$$J_{131} = \left\| \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} E(t-s) A P_N F'(X_{t_k}^{M,N}) \int_{t_k}^{s} E(s-[r]_{\tau}) dW(r) ds \right\|_{L^p(\Omega,\mathcal{H})}$$

$$\leq \left\| \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t-s)^{-\frac{1}{2}} \left( 1 + \|X_{t_k}^{M,N}\|_{L^4p(\Omega,V)}^2 \right) \int_{t_k}^{s} \|E(s-[r]_{\tau})\|_{L^2_t}^{\frac{1}{2}} ds \right\|_{L^2(\Omega,\mathcal{H})} ds$$

$$\leq C \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t-s)^{-\frac{1}{2}} \left( \int_{t_k}^{s} \|E(s-[r]_{\tau})\|_{L^2_t}^{\frac{1}{2}} ds \right)^{\frac{1}{2}} ds$$

$$\leq C \tau^{\frac{\gamma}{2}}.$$

In what follows, we use the same argument to estimate $J_{132}$,

$$J_{132} \leq \int_{[t]_{\tau}}^{t} \left\| E(t-s) A P_N F'(X_{[s]_{\tau}}^{M,N}) \int_{[s]_{\tau}}^{s} E(s-[r]_{\tau}) P_N dW(r) \right\|_{L^p(\Omega,\mathcal{H})} ds$$

$$\leq C \int_{[s]_{\tau}}^{t} (t-s)^{-\frac{1}{2}} ds \left( 1 + \sup_{s \in [0,T]} \|X_{s}^{M,N}\|_{L^2p(\Omega,\mathcal{H})} \right) \tau^{\min[\gamma,2]}$$

$$\leq C \tau^{\frac{2+\min[\gamma,2]}{4}}.$$

Owing to the fact $L^1 \subset \mathcal{H}^{-\delta_0}$ with $\delta_0 \in (\frac{3}{2}, 2)$ and the regularity of $X_{t}^{M,N}$, we obtain

$$J_{14} \leq C \int_{0}^{t} (t-s)^{-\frac{2+\delta_0}{4}} \int_{0}^{1} F''(\lambda(X_{s}^{M,N}, X_{[s]_{\tau}}^{M,N}))$$

$$\times (X_{s}^{M,N} - X_{[s]_{\tau}}^{M,N}, X_{s}^{M,N} - X_{[s]_{\tau}}^{M,N}) (1 - \lambda) d\lambda \left\| \right\|_{L^p(\Omega,\mathcal{H})} ds$$

$$\leq C \int_{0}^{t} \int_{0}^{1} (t-s)^{-\frac{2+\delta_0}{4}} \|X_{s}^{M,N} - X_{[s]_{\tau}}^{M,N}\|_{L^4(\Omega,\mathcal{H})}^2$$

$$\times \|\lambda(X_{s}^{M,N}, X_{[s]_{\tau}}^{M,N})\|_{L^{2p}(\Omega,V)} d\lambda \ ds$$

$$\leq C \tau^{\min[1,\frac{\gamma}{2}]} \left( 1 + \sup_{s \in [0,T]} \|X_{s}^{M,N}\|_{L^2p(\Omega,\mathcal{H})} \right) \int_{0}^{t} (t-s)^{-\frac{2+\delta_0}{4}} ds$$

$$\leq C \tau^{\min[1,\frac{\gamma}{2}]}.$$

Combining the above estimates together leads to

$$J_1 \leq C \tau^{\frac{\gamma}{4}}.$$
Due to the regularity of $X_t^{M,N}$ in Corollary 13 and properties of nonlinear term $F$, we obtain

$$J_2 = \left\| \int_0^t E(t-s)APNF(X_{[s]\tau}^{M,N})ds - \int_0^t \frac{E(t-s)APNF(X_{[s]\tau}^{M,N})}{1+\tau\|P_NF(X_{[s]\tau}^{M,N})\|^2_{L^P(\Omega,\mathcal{H})}}\right\|_{L^P(\Omega,\mathcal{H})} \leq C \tau \int_0^t (t-s)^{-\frac{1}{2}}ds \sup_{s\in[0,T]}\|PF(X_s^{M,N})\|^2_{L^{2P}(\Omega,\mathcal{H})} \leq C \tau.$$

It remains to estimate $J_3$ by virtue of (5), (6) and (11),

$$J_3 = \left\| \int_0^t E(t-s)(I - E(s-[s]\tau))dW(s)\right\|_{L^P(\Omega,\mathcal{H})} = \left(\int_0^t \|AE(t-s)A^{-\frac{1}{2}}(I - E(s-[s]\tau))A^{\frac{1}{2}}Q^{\frac{1}{2}}\right)^{\frac{1}{2}} ds \leq C \tau^\frac{2}{3}.$$

Therefore, the estimates of $J_1$, $J_2$ and $J_3$ imply

$$\|\tilde{X}_t^{M,N} - X_t^{M,N}\|_{L^P(\Omega,\mathcal{H})} \leq C \tau^\frac{2}{3}. \quad (24)$$

**Step 2**: Estimate of $\|X_t^N(t) - \tilde{X}_t^{M,N}\|_{L^P(\Omega,\mathcal{H})}$.

For short, by $e^{M,N}(t)$ we denote $X_t^N(t) - \tilde{X}_t^{M,N}$, which satisfies

$$\frac{d}{dt}e^{M,N}(t) + A^2e^{M,N}(t) = APNF(X_t^{M,N}) - F(X_t^N(t)).$$

Multiplying $A^{-1}e^{M,N}(t)$ on both sides and using (8), (10) and Hölder’s inequality lead to

$$\frac{1}{2} \frac{d}{dt}[|e^{M,N}(t)|^2_{-1} + |e^{M,N}(t)|^2_1] = \langle e^{M,N}(t), F(X_t^{M,N}) - F(X_t^N(t)) \rangle$$

$$= \langle e^{M,N}(t), F(\tilde{X}_t^{M,N}) - F(X_t^N(t)) \rangle + \langle e^{M,N}(t), F(X_t^{M,N}) - F(\tilde{X}_t^{M,N}) \rangle$$

$$\leq \frac{3}{2} \|e^{M,N}(t)\|^2 + \frac{1}{2} \|F(X_t^{M,N}) - F(\tilde{X}_t^{M,N})\|^2$$

$$\leq \frac{1}{2} \|e^{M,N}(t)\|^2_1 + \frac{9}{8} \|e^{M,N}(t)\|^2_{-1}$$

$$+ C \|X_t^{M,N} - \tilde{X}_t^{M,N}\|^2(1 + \|X_t^{M,N}\|^4_V + \|\tilde{X}_t^{M,N}\|^4_V).$$

Based on the Gronwall inequality and taking expectation, we achieve

$$\left\| \int_0^t |e^{M,N}(s)|^2_{1}ds \right\|_{L^P(\Omega,\mathbb{R})} \leq C \tau^\frac{2}{3},$$
where the regularity of \( \tilde{X}^{M,N}_t \) and \( X^{M,N}_t \) and (24) were used. We decompose \( \|e^{M,N}(t)\|_{L^p(\Omega,\mathcal{H})} \) as follows,

\[
\|e^{M,N}(t)\|_{L^p(\Omega,\mathcal{H})} = \left\| \int_0^t E(t-s) A \left( P_N F(X^{M,N}_s) - P_N F(X^N(s)) \right) \right\|_{L^p(\Omega,\mathcal{H})} ds \\
\leq \int_0^t \| E(t-s) A \left( F(X^{M,N}_s) - F(\tilde{X}^{M,N}_s) \right) \|_{L^p(\Omega,\mathcal{H})} ds \\
+ \left\| \int_0^t E(t-s) A \left( F(\tilde{X}^{M,N}_s) - F(X^N(s)) \right) ds \right\|_{L^p(\Omega,\mathcal{H})} \\
=: K_1 + K_2.
\]

Thanks to the regularity of \( \tilde{X}^{M,N}_t \) and \( X^{M,N}_t \), (4) and (24), one can show

\[
K_1 \leq \int_0^t (t-s)^{-\frac{1}{2}} \| X^{M,N}_s - \tilde{X}^{M,N}_s \|_{L^2 p(\Omega,\mathcal{H})} ds \\
\leq \int_0^t (t-s)^{-\frac{1}{2}} \| X^{M,N}_s - \tilde{X}^{M,N}_s \|_{L^2 p(\Omega,\mathcal{H})} ds \\
\times \left( 1 + \| \tilde{X}^{M,N}_s \|_{L^4 p(\Omega,V)}^2 + \| X^{M,N}_s \|_{L^4 p(\Omega,V)}^2 \right) ds \\
\leq C \tau^{\frac{3}{4}}.
\]

Resorting to (4), Lemma 9, the regularity of \( \tilde{X}^{M,N}_t \) and \( X^N(t) \) and Hölder’s inequality, we acquire for \( \eta = \min\{\gamma, \frac{3}{4}\} \),

\[
K_2 \leq C \left\| \int_0^t (t-s)^{-\frac{2-\eta}{4}} \| e^{M,N}(s) \|_{L^\infty(\Omega,\mathcal{H})} \right\|_{L^p(\Omega,\mathcal{H})} ds \\
\leq C \left\| \int_0^t (t-s)^{-\frac{2-\eta}{4}} \| e^{M,N}(s) \|_{L^\infty(\Omega,\mathcal{H})} ds \right\|_{L^p(\Omega,\mathcal{H})} \\
\leq C \left\| \left( \int_0^t | e^{M,N}(s) |^2 ds \right)^{\frac{1}{2}} \right\|_{L^p(\Omega,\mathcal{H})} \\
\times \left( \int_0^t (t-s)^{-\frac{2-\eta}{4}} \left( 1 + | X^N(s) |^2_{\gamma} + | \tilde{X}^{M,N}_s |^2_{\gamma} \right) ds \right)^{\frac{1}{2}} \\
\leq C \left\| \int_0^t | e^{M,N}(s) |^2 ds \right\|_{L^p(\Omega,\mathcal{H})}^{\frac{1}{2}} \\
\leq C \tau^{\frac{3}{4}}.
\]

Collecting all the estimates obtained so far finishes the proof. \hfill \Box

At last, gathering Theorem 15 with Theorem 7, we get the strong convergence rates of the fully discrete scheme (14).

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Corollary 16 (Strong convergence rates of the full discretization) Let Assumptions 1–4 be satisfied. Then for $p \geq 1$ it holds that

$$\sup_{M, N \in \mathbb{N}^+} \sup_{m \in \{0, 1, \ldots, M\}} \|X(t_m) - X_{t_m}^{M,N}\|_{L^p(\Omega, \dot{H})} \leq C(\lambda_N^{\frac{\gamma}{2}} + \tau^\gamma), \quad \gamma \in \left(\frac{d}{2}, 4\right].$$

4 Numerical experiments

In this section, we include some numerical results to confirm the above assertions. Consider the following one-dimensional stochastic Cahn–Hilliard equation

$$\begin{cases}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \dot{W}, \\
w = -\frac{\partial^2 u}{\partial x^2} + u^3 - u,
\end{cases} \quad (t, x) \in (0, T] \times (0, 1),$$

\begin{align*}
\frac{\partial u}{\partial x} |_{x=0} &= \frac{\partial u}{\partial x} |_{x=1} = 0, \\
\frac{\partial w}{\partial x} |_{x=0} &= \frac{\partial w}{\partial x} |_{x=1} = 0,
\end{align*}

where $\{W(t)\}_{t \in [0,T]}$ is a $Q$-Wiener process and the orthonormal eigensystem $\{\lambda_j, e_j\}_{j \in \mathbb{N}^+}$ of the Neumann Laplacian on $\dot{H}$ is

$$\lambda_j = j^2 \pi^2, \quad e_j(x) = \sqrt{2} \cos(j \pi x), \quad j \geq 1.$$ 

Firstly, we approximate (25) by using the (non-tamed) exponential Euler method, given by

$$X_{t_{m+1}}^{M,N} = E(\tau) X_{t_m}^{M,N} - \int_{t_m}^{t_{m+1}} E(t_{m+1} - s) A P_N F(X_{t_m}^{M,N}) ds + E(\tau) P_N \Delta W_m.$$  

(26)

Table 1 shows Monte Carlo simulations of the first moment $\mathbb{E}[\|X_T^{M,N}\|]$ of the exponential Euler approximation (26) with the initial value $u(0, x) = 20\sqrt{2} \cos(\pi x)$, $x \in (0, 1)$ and $N = 100$, where one can observe that $\mathbb{E}[\|X_T^{M,N}\|]$ tends to positive infinity rapidly as $M$ increases. Here the value ‘Inf’ represents positive infinity and ‘NaN’ represents ‘not-a-number’ because of an operation ‘Inf-Inf’. On the contrary, the tamed exponential Euler method works well and does not explode for all $M$.

Next, we will show the convergence rates of the tamed exponential Euler method as obtained in Theorem 15. For this purpose, we use the fully discrete method (14) to solve (25) with $u(0, x) = \sqrt{2} \cos(\pi x)$, $x \in (0, 1)$. The error bounds are measured in the mean-square sense at the endpoint $T = 1$. Note that the expectations are approximated by computing averages over 1000 samples. Since the exact solutions are not available at hand, fixing $N = 500$, the reference solution is identified with a very small time stepsize $\tau_{exact} = 2^{-16}$. Four different time stepsizes $\tau = 2^{-j}, j = 9, 10, 11, 12$ are then used to carry out the numerical simulations.
Table 1 Simulations of the first absolute moment $E[\|X_{M,N}^T\|]$ with $M \in \{1, 2, \ldots, 20\}$

| $M$   | $E[\|X_{M,N}^T\|]$ | $M$   | $E[\|X_{M,N}^T\|]$ |
|-------|--------------------|-------|--------------------|
| $M = 1$ | 22.2175            | $M = 7$ | 3.7510e+73         |
| $M = 2$ | 34.1425            | $M = 8$ | Inf               |
| $M = 3$ | 132.9797           | $M = 9$ | NaN               |
| $M = 4$ | 8.1205e+03         | $M = 10$| NaN               |
| $M = 5$ | 1.8550e+09         | $M = 10$| NaN               |
| $M = 6$ | 2.2128e+25         | $M = 20$| NaN               |

Fig. 1 Strong convergence rate of the tamed Euler method (white noise)

We are now ready to make some explanations on the numerical results. For the white noise case (i.e., $Q = I$), the condition (11) in Assumption 3 is then fulfilled with $\gamma$ closing to $\frac{3}{2}$ and the convergence order obtained in Theorem 15 is almost $\frac{3}{8}$. The mean-square errors are depicted in Fig. 1, against $\tau$ on a log-log scale, where one can observe that the resulting numerical errors decrease at a slope close to $\frac{3}{8}$. This coincides with the theoretical result. For the trace-class noise case, we choose $Q$ such that

$$Q e_1 = 0, \quad Q e_i = \frac{1}{i \log(i)^2} e_i, \quad \forall i \geq 2.$$ (27)

Obviously, (27) guarantees $\text{Tr}(Q) < \infty$ and thus the condition (11) is satisfied with $\gamma = 2$. As expected, the convergence rate of order is detected in Fig. 2, which is consistent with the finding in Theorem 15. For the smoother noise, $Q$ is then chosen to satisfy
Fig. 2  Strong convergence rate of the tamed Euler method (trace-class noise)

Fig. 3  Strong convergence rate of the tamed Euler method (smoother noise)

\[ Qe_1 = 0, \quad Qe_i = \frac{1}{i^5 \log(i)^2} e_i, \quad \forall i \geq 2. \]

In this case, condition (11) holds with \( \gamma = 4 \) and the obtained convergence rate in theory is 1. From Fig. 3, it is obvious to find that the approximation errors decrease with order 1, which agrees with the theoretical result.
Table 2  Comparison of tamed exponential Euler method (TEEM) and backward Euler method (BEM)

| Stepsizes | Error (TEEM) | Error (BEM) |
|-----------|--------------|--------------|
| $\tau = 2^{-9}$ | 0.0195 | 0.0155 |
| $\tau = 2^{-10}$ | 0.0145 | 0.0116 |
| $\tau = 2^{-11}$ | 0.0101 | 0.0083 |
| $\tau = 2^{-12}$ | 0.0069 | 0.0058 |
| $\tau = 2^{-13}$ | 0.0042 | 0.0037 |

Fig. 4  Strong convergence rate of the tamed Euler method for $d = 2$ (trace-class noise)

Moreover, we compare the error between the tamed exponential Euler method (14) and the backward Euler method. Based on the simulations over 1000 samples, Table 2 lists the approximation errors of these two schemes for five different temporal stepsizes. Clearly, both schemes give satisfactory accuracy. However, the backward Euler method needs to solve a large nonlinear algebraic system by certain iteration and thus costs more computational efforts than the tamed exponential Euler method.

For the multi-dimensional case $d = 2$, the resulting errors for trace-class noise (27) are plotted in Fig.4 on a log-log scale, where one can also detect the expected convergence rate. Note that the above numerical experiments are performed under the commutativity of $A$ and $Q$. Next, by choosing

$$Q\eta_1 = 0, \quad Q\eta_i = \frac{1}{i \log(i)^2} e_i, \quad \eta_k(x) = \sqrt{2}\sin(k\pi x), \quad \forall i \geq 2, \quad k \geq 1,$$

we attempt to illustrate the error bounds for the fully discrete scheme (14) without the commutative condition of $A$ and $Q$. From Fig.5, one can observe the expected convergence rate of order $\frac{1}{2}$, which agrees with that indicated in Theorem 15. Finally,
we mention an interesting circulant embedding approach to the noise sampling recently proposed by [29]. We leave it a future work together with some necessary analysis.

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Declarations

Conflict of interest This work does not have any conflicts of interest.

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