Research article

Global existence and blow-up of solutions for logarithmic Klein-Gordon equation

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Abstract: This article concerns the initial-boundary value problem for a class of Klein-Gordon equation with logarithmic nonlinearity. By using Galerkin method and compactness criterion, we prove the existence of global solutions to this problem. Meanwhile, the blow-up of solutions in the unstable set is also obtained.

Keywords: Klein-Gordon equation; logarithmic nonlinearity; initial-boundary value problem; global solutions; blow-up

Mathematics Subject Classification: 35L05, 35L10, 35B40

1. Introduction

In this paper, we consider the following problem

\[ u_{tt} - \Delta u + u = u \log |u|, \quad (x, t) \in \Omega \times R^+, \]  
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \]  
\[ u(x, t) = 0, \quad (x, t) \in \partial \Omega \times R^+, \]  

where \( \Omega \subset R^n \) is a bounded domain with smooth boundary \( \partial \Omega \).

The model equation (1.1) arises in logarithmic quantum mechanics and is applied to nuclear physics, optics and geophysics [1–5]. P. Gorka [6] dealt with the equation

\[ u_{tt} - u_{xx} = -u + \varepsilon u \log |u|^2, \quad (x, t) \in O \times (0, T) \]  

with initial datum

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in O \]
and boundary value condition
\[ u(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T), \quad (1.6) \]
where \( \Omega = [a, b] \subset R^1, \varepsilon \in [0, 1] \). By applying the Galerkin method, logarithmic Sobolev inequality and compactness theorem, he established the global weak solutions of the problem (1.4)–(1.6). K. Bartkowski and P. Korka [7] showed the classical solutions and weak solutions to the Cauchy problem of Eq (1.4) for \( \Omega = R^1 \). In [8], T. Cazenave and A. Haraux investigated the local and global solutions for the Cauchy problem of the logarithmic wave equation \( u_{tt} - \Delta u = u \log |u| \).

For the following nonlinear Klein-Gordon equation
\[ u_{tt} - \Delta u + m^2 u = |u|^{p-2}u, \quad x \in \Omega, \quad t > 0, \quad (1.7) \]
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad (1.8) \]
\[ u(x, t) = 0, \quad x \in \partial \Omega, \quad t \geq 0. \quad (1.9) \]
For \( n \geq 3 \), P. Brenner [9] studied \( L_p \)-decay and scattering properties for the Cauchy problem of the Eq (1.7). As \( n = 1 \) and 2, K. Nakanishi [10] showed that the scattering operators for Eq (1.7) are well-defined in whole energy space in \( R^{1+n} \) with \( p > 1 + \frac{4}{n} \). Under the condition of small energy data, such results were known for \( n \geq 3 \) [11–13].

As \( m = 0 \), for sufficiently large initial data, the blow-up results of the problem (1.7)–(1.9) in finite time was proved by H. A. Levine [14] and J. Ball [15]. Furthermore, Y. C. Liu [16], L. E. Payne and D. H. Sattinger [17] and D. H. Sattinger [18] obtained the results of the global existence and nonexistence of weak solutions for the problem (1.7)–(1.9) by establishing the method of potential wells. Also in [16, 19], the authors gave a threshold result of solutions and obtained the vacuum isolating of solutions.

At last we should mention that the logarithmic heat equation was studied by H. Chen and S. Y. Tian [20] and H. Chen, P. Luo and G. W. Liu [21]. Moreover, there were also many researches on the logarithmic Schrödinger equation [22–25].

In this paper, by applying Galerkin method and compactness criterion, we prove the global existence of the problem (1.1)–(1.3). Furthermore, in the sense of \( L^2 \) norm, the blow-up result for this problem is obtained by the concavity method.

2. Preliminaries

2.1. Some lemmas

For the applications through this paper, we list up some known lemmas.

**Definition 2.1** If
\[ u \in C([0, T], H^1_0(\Omega)) , \quad u_t \in C([0, T], L^2(\Omega)) , \quad u_{tt} \in C([0, T], H^{-1}(\Omega)) \]
satisfies
\[ \int_\Omega u_t \varphi dx + \int_\Omega \nabla u \nabla \varphi dx + \int_\Omega u \varphi dx = \int_\Omega u \log |u| \varphi dx, \quad \varphi \in H^1_0(\Omega). \]
Then the function \( u \) is called a weak solution of (1.1)–(1.3) on \([0, T]\).
Lemma 2.1 Assume that $2 \leq r < +\infty$, $n \leq 2$ and $2 \leq \frac{2n}{n-2}$, $n > 2$. Then
\[
||u||_r \leq C||\nabla u||, \ \forall u \in H_0^1(\Omega),
\]
where $C > 0$ is a constant depending on $\Omega$ and $r$.

Lemma 2.2 ([20, 21, 26]) If $u \in H_0^1(\Omega)$, then for each $a > 0$, one has the inequality
\[
\int_\Omega |u|^2 \log |u| dx \leq ||u||^2 \log ||u|| + \frac{a^2}{2\pi}||\nabla u||^2 - \frac{n}{2}(1 + \log a)||u||^2.
\]

Lemma 2.3 Let $u(t)$ be a solution of the problem (1.1)–(1.3), then the energy $E(t)$ is conservation. Namely, $E(t) = E(0), \ \forall t > 0$, where
\[
E(t) = \frac{1}{2}(||u||^2 + ||\nabla u||^2) - \int_\Omega u^2 \log |u| dx + \frac{3}{4}||u||^2,
\]
for $u \in H_0^1(\Omega), \ t \geq 0$ and
\[
E(0) = \frac{1}{2}(||u||^2 + ||\nabla u_0||^2) - \int_\Omega u_0^2 \log |u_0| dx + \frac{3}{4}||u_0||^2
\]
is the initial total energy.

Lemma 2.4 ([27]) Let $X$ be a Banach space, if $f \in L^p(0, T; X)$, $\frac{\partial f}{\partial t} \in L^p(0, T; X)$, then $f$ is a continuous injection from $[0, T]$ on to $X$ when the value is transformed in the set of measure zero in $[0, T]$.

Lemma 2.5 ([28]) Let $u_n(x)$ be a bounded sequence in $L^p(\Omega), \ 1 \leq p < +\infty$ such that $u_n$ almost everywhere converges to $u$. Then $u \in L^p(\Omega)$ and $u_n$ weakly converges in $L^p(\Omega)$ to $u$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain.

The local existence result of the problem (1.1)–(1.3) is described as follows. For its detailed proof process, see references [31–33].

Theorem 2.1 (Local existence) Let $u_0 \in H_0^1(\Omega), \ u_1 \in L^2(\Omega)$. Then there exists $T > 0$ such that the problem (1.1)–(1.3) has a unique local solution $u(t)$ satisfying
\[
u \in C([0, T); \ H_0^1(\Omega)), \ u_t \in C([0, T); \ L^2(\Omega)).
\]

2.2. Potential wells

At first, we introduce some useful functionals
\[
\mathcal{J}(u) = \frac{1}{2}(||\nabla u||^2 - \int_\Omega u^2 \log |u| dx) + \frac{3}{4}||u||^2,
\]
\[
\mathcal{K}(u) = ||\nabla u||^2 + ||u||^2 - \int_\Omega u^2 \log |u| dx.
\]

By (2.1), (2.3) and (2.4), we have
\[
\mathcal{J}(u) = \frac{1}{2}\mathcal{K}(u) + \frac{1}{4}||u||^2, \ E(t) = \frac{1}{2}||u_t||^2 + \mathcal{J}(u),
\]
for $u \in H_0^1(\Omega)$. 

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As in [17], the potential well depth is defined as
\[
d = \inf \{ \sup_{\lambda \geq 0} J(\lambda u) : u \in H^1_0(\Omega)/\{0\} \}. \tag{2.6}
\]

Now, we define the Nehari manifold ([29, 30]) by
\[
N = \{ u \in H^1_0(\Omega)/\{0\} : \mathcal{K}(u) = 0 \}.
\]
The stable set \( \mathcal{W} \) and the unstable set \( \mathcal{U} \) can be defined respectively by
\[
\mathcal{W} = \{ u \in H^1_0(\Omega) : \mathcal{K}(\lambda u) > 0, J(\lambda u) < d \} \cap \{ 0 \},
\]
and
\[
\mathcal{U} = \{ u \in H^1_0(\Omega) : \mathcal{K}(\lambda u) < 0, J(\lambda u) < d \},
\]
It is to see that the potential well depth \( d \) may also be described as
\[
d = \inf_{u \in N} J(u) \tag{2.7}
\]

**Lemma 2.6** Let \( u \in H^1_0(\Omega) \) and \( ||u|| \neq 0 \), then we have
\[
(i) \lim_{\lambda \to 0^+} J(\lambda u) = 0, \quad \lim_{\lambda \to +\infty} J(\lambda u) = -\infty;
\]
\[
(ii) \mathcal{K}(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u) \begin{cases} > 0, & 0 < \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda < +\infty, \end{cases} \tag{2.8}
\]
where
\[
\lambda^* = \exp\left( \frac{||\nabla u||^2 + ||u||^2 - \int_\Omega u^2 \log |u| \, dx}{||u||^2} \right).
\]

**Proof.** (i) By \( \lim_{\lambda \to 0^+} \lambda^2 \log \lambda = 0 \), \( \lim_{\lambda \to +\infty} \lambda = +\infty \) and
\[
J(\lambda u) = \frac{\lambda^2}{2} ||\nabla u||^2 + \frac{3}{4} \lambda^2 ||u||^2 - \frac{1}{2} (\lambda^2 \log \lambda) ||u||^2 - \frac{1}{2} \lambda^2 \int_\Omega u^2 \log |u| \, dx,
\]
we get
\[
\lim_{\lambda \to 0^+} J(\lambda u) = 0, \quad \lim_{\lambda \to +\infty} J(\lambda u) = -\infty.
\]
(ii) By direct calculations, we obtain
\[
\frac{d}{d\lambda} J(\lambda u) = \lambda \left[ ||\nabla u||^2 + ||u||^2 - \int_\Omega u^2 \log |u| \, dx \right] - (\lambda \log \lambda) ||u||^2. \tag{2.9}
\]
Let \( \frac{d}{d\lambda} J(\lambda u) = 0 \), then we deduce that
\[
\lambda^* = \exp\left( \frac{||\nabla u||^2 + ||u||^2 - \int_\Omega u^2 \log |u| \, dx}{||u||^2} \right).
\]
From (3.2), we have

$$K(\lambda u) = \lambda^2 [||\nabla u||^2 + ||u||^2 - \int_{\Omega} u^2 \log |u| dx] - (\lambda^2 \log \lambda ||u||^2). \quad (2.10)$$

By (2.9) and (2.10), the equality (2.8) is valid. \hfill \Box

**Lemma 2.7** If \( u \in H_0^1(\Omega) \), then

$$d = \frac{1}{4} (\sqrt{2\pi})^n e^{n/2}. \quad (2.11)$$

**Proof.** By Lemma 2.2, we have

$$K(u) = ||\nabla u||^2 - \int_{\Omega} u^2 \log |u| dx + ||u||^2 \geq \left(1 - \frac{a^2}{2\pi}\right)||\nabla u||^2 + [1 + \frac{n}{2}(1 + \log a) - \log ||u||] \cdot ||u||^2. \quad (2.12)$$

By taking \( a = \sqrt{2\pi} \), we obtain from (2.12) that

$$K(u) \geq [1 + \frac{n}{2}(1 + \log a) - \log ||u||] \cdot ||u||^2. \quad (2.13)$$

Combining Lemma 2.6 and (2.5) yields that

$$\sup_{\lambda \geq 0} J(\lambda u) = J(\lambda^* u) = \frac{1}{2} K(\lambda^* u) + \frac{1}{4} ||\lambda^* u||^2. \quad (2.14)$$

We receive from (2.13) and Lemma 2.6 that

$$0 = K(\lambda^* u) \geq [1 + \frac{n}{2}(1 + \log a) - \log ||\lambda^* u||] \cdot ||\lambda^* u||^2,$$

then

$$||\lambda^* u||^2 \geq a^n e^{n/2}. \quad (2.15)$$

It follows from (2.14) and (2.15) that

$$\sup_{\lambda \geq 0} J(\lambda u) \geq \frac{1}{4} a^n e^{n/2}. \quad (2.16)$$

By (2.6) and (3.16), we have that \( d = \frac{1}{4} (\sqrt{2\pi})^n e^{n/2}. \) \hfill \Box

In order to further study the problem (1.1)–(1.3), for \( 0 < \varepsilon < 1 \) and \( u \in H_0^1(\Omega) \), we define some functionals as follows

$$J_\varepsilon(u) = \frac{\varepsilon}{2} ||\nabla u||^2 - \frac{1}{2} \int_{\Omega} u^2 \log |u| dx + \frac{3}{4} ||u||^2, \quad (2.17)$$

$$K_\varepsilon(u) = \varepsilon ||\nabla u||^2 + ||u||^2 - \int_{\Omega} u^2 \log |u| dx. \quad (2.18)$$

Let

$$N_\varepsilon(u) = \{ u \in H_0^1(\Omega); J_\varepsilon(u) = 0, ||\nabla u|| \neq 0 \} \quad (2.19)$$
then we define $d(\varepsilon)$ as

$$
d(\varepsilon) = \inf_{u \in N_\varepsilon(u)} J(u). \tag{2.20}
$$

**Proposition 2.1** If $d(\varepsilon)$ is defined by (2.20), then

$$
d(\varepsilon) = 2\lambda_1 de(1 - \varepsilon)e^{\frac{\varepsilon}{2}}, \tag{2.21}
$$

where $\lambda_1$ is the first eigenvalue of the following boundary value problem

$$
\begin{cases}
-\Delta u = \lambda u, & x \in \Omega, \\
\quad u = 0, & x \in \partial \Omega.
\end{cases} \tag{2.22}
$$

**Proof.** By $u \in N_\varepsilon(u)$, we get

$$
J(u) = \frac{1 - \varepsilon}{2} ||\nabla u||^2 + J_\varepsilon(u) = \frac{1 - \varepsilon}{2} ||\nabla u||^2. \tag{2.23}
$$

From (2.17) and Lemma 2.2 that

$$
\varepsilon ||\nabla u||^2 = \int_\Omega u^2 \log |u| dx - \frac{3}{2} ||u||^2 \\
\leq \frac{a^2}{2\pi} ||\nabla u||^2 + ||u||^2 \log ||u|| - \frac{n}{2}(1 + \log a)||u||^2 - \frac{3}{2} ||u||^2. \tag{2.24}
$$

By taking $a^2 = 2\pi \varepsilon$ in (2.24), we obtain

$$
[\log ||u||^2 - n(1 + \log a) - 3]||u||^2 \geq 0,
$$

which implies that

$$
||u||^2 \geq a^n e^{n+3}.
$$

Since the eigenvalue $\lambda_1$ satisfies the problem (2.22), so we gain

$$
||\nabla u||^2 \geq \lambda_1 a^n e^{n+3} = 4\lambda_1 de e^{\frac{\varepsilon}{2}}. \tag{2.25}
$$

It follows from (2.23) and (2.25) that

$$
J(u) \geq 2\lambda_1 de(1 - \varepsilon)e^{\frac{\varepsilon}{2}}.
$$

Thus, by (2.20), we have

$$
d(\varepsilon) = 2\lambda_1 de(1 - \varepsilon)e^{\frac{\varepsilon}{2}}. \tag{2.26}
$$

\hfill \Box

**Proposition 2.2** As a function of $\varepsilon$, $d(\varepsilon)$ have the following properties for $\varepsilon \in [0, 1]$: 

(a) $d(0) = d(1) = 0$.

(b) $d(\varepsilon)$ is increasing on $[0, \varepsilon_0]$ and decreasing on $[\varepsilon_0, 1]$. Thus, $d(\varepsilon)$ gets the maximum at $\varepsilon_0 = \frac{n}{n+2}$, and $d(\varepsilon_0) = 4\lambda_1 de \left(\frac{n}{n+2}\right)^{\frac{n}{n+2}}$.

(c) For $\forall h \in (0, d(\varepsilon_0))$, the equation $d(\varepsilon) = h$ has two roots $\varepsilon_1$ and $\varepsilon_2$ in the interval $(0, \varepsilon_0)$ and $(\varepsilon_0, 1)$, respectively.
Proof. (a) is easy to be proved. Here we omit the proof of it.

(b) By calculation, we have

$$d'(\varepsilon) = 2\lambda_1d\varepsilon \left[-\varepsilon^2 + \frac{n}{2}(1-\varepsilon)e^{\varepsilon-1}\right] = \lambda_1dee^{\varepsilon-1}[n-(n+2)\varepsilon]. \quad (2.26)$$

From $d'(\varepsilon) = 0$, we conclude that $\varepsilon_0 = \frac{n}{n+2}$. In addition, by (2.26), we get $d'(\varepsilon) > 0$ on $(0, \varepsilon_0)$ and $d'(\varepsilon) < 0$ on $(\varepsilon_0, 1)$. Therefore, $d(\varepsilon)$ takes the maximum value at $\varepsilon_0 = \frac{n}{n+2}$, and

$$d(\varepsilon_0) = 2\lambda_1d(1-\varepsilon_0)e^{\varepsilon_0} = \frac{4\lambda_1d}{n+2} \left(\frac{n}{n+2}\right)^2.$$

(c) Let $f(\varepsilon) = d(\varepsilon) - h$, then $f(0) = -h < 0$, $f(\varepsilon_0) = d(\varepsilon_0) - h > 0$, $f(1) = -h < 0$. According to the continuity of function $f(\varepsilon)$ on interval $[0, 1]$, the equation $f(\varepsilon) = 0$ i.e. $d(\varepsilon) = h$ has two roots $\varepsilon_1 \in (0, \delta_0)$ and $\varepsilon_2 \in (\delta_0, 1)$.

\[\square\]

Proposition 2.3 Let $r(\varepsilon) = 4\lambda_1dee^{\varepsilon^2}$, then

(1) If $J(u) \leq d(\varepsilon)$, then $0 < \|\nabla u\|^2 \leq r(\varepsilon)$ if only and if $J_\varepsilon(u) \geq 0$.

(2) If $J_\varepsilon(u) < 0$, then $\|\nabla u\|^2 > r(\varepsilon)$. Proof. (1) For $a^2 = 2\pi\varepsilon$, we have

$$J_\varepsilon(u) = \frac{\varepsilon}{2} \|\nabla u\|^2 - \frac{1}{2} \int_\Omega u^2 \log |u| dx + \frac{3}{4} \|u\|^2$$

$$\geq \left(\frac{\varepsilon}{2} - \frac{a^2}{4\pi}\right) \|\nabla u\|^2 - \frac{1}{2} \|u\|^2 \log \|u\| + \frac{n}{4}(1 + \log a)\|u\|^2 + \frac{3}{4} \|u\|^2$$

$$= \frac{1}{4} \|u\|^2 [-2 \log \|u\| + n(1 + \log a) + 3]$$

$$= \frac{1}{4} \|u\|^2 \log \frac{\lambda_1d^2e^{n+3}}{\|\nabla u\|^2} = \frac{1}{4} \|u\|^2 \log \frac{4\lambda_1dee^{\varepsilon^2}}{\|\nabla u\|^2}. \quad (2.27)$$

By $0 < \|\nabla u\|^2 \leq r(\varepsilon)$, we see that $\log \frac{4\lambda_1dee^{\varepsilon^2}}{\|\nabla u\|^2} \geq 0$. Therefore, we conclude from (2.27) that $J_\varepsilon(u) \geq 0$.

If $J_\varepsilon(u) \geq 0$, then from (2.20) and

$$J(u) = \frac{1 - \varepsilon}{2} \|\nabla u\|^2 + J_\varepsilon(u) \leq d(\varepsilon), \quad (2.28)$$

we have

$$\frac{1 - \varepsilon}{2} \|\nabla u\|^2 \leq 2\lambda_1d(1-\varepsilon)e^{\varepsilon^2},$$

which implies that $\|\nabla u\|^2 \leq r(\varepsilon)$.

(b) By (2.24) and $J_\varepsilon(u) < 0$, we have $\log \frac{4\lambda_1dee^{\varepsilon^2}}{\|\nabla u\|^2} < 0$, which implies that $\|\nabla u\|^2 > r(\varepsilon).$ \[\square\]

On the basis of Proposition 2.1 and Proposition 2.2, we define a family of potential wells by

$$\mathcal{W}_\varepsilon = \{u \in H_0^1(\Omega) : J_\varepsilon(u) > 0, J(u) < d(\varepsilon)\} \cap \{0\},$$

and

$$\mathcal{U}_\varepsilon = \{u \in H_0^1(\Omega) : J_\varepsilon(u) < 0, J(u) < d(\varepsilon)\},$$
for $\varepsilon \in (0, 1)$.

**Remark** From $\mathcal{F}_{\varepsilon}(u) > 0$ and

$$\mathcal{F}(u) = \frac{1 - \varepsilon}{2} \|\nabla u\|^2 + \mathcal{F}_{\varepsilon}(u),$$

we have $\mathcal{F}(u) > 0$.

### 3. Global existence of solutions

In this section, by applying Galerkin method and the compactness principle, we study the global solutions of the problem (1.1)–(1.3).

**Theorem 3.1** If $u_0 \in \mathcal{W}, u_1 \in L^2(\Omega)$ satisfy $0 < \varepsilon(0) < d$, then the problem (1.1)–(1.3) admits a global solution $u(x, t)$ such that $u(x, t) \in L^\infty([0, +\infty); H^1_0(\Omega)), u(t, x) \in L^\infty([0, +\infty); L^2(\Omega))$.

**Proof.** Let $\{\omega_j\}_{j=1}^\infty$ be a basis for $H^1_0(\Omega)$. We are going to find out the approximate solution $u_m(t)$ in the form $u_m(t) = \sum_{j=1}^m g_{jm}(t)\omega_j$ with $g_{jm}(t) \in C^2[0, T], \forall T > 0$, where the unknown functions $g_{jm}(t)$ are determined by the following ordinary differential equation

$$(u_{mj}(t), \omega_j) + (\nabla u_m(t), \nabla \omega_j) + (u_m(t), \omega_j) = (u_m(t) \log |u_m(t)|, \omega_j), \ j = 1, 2, \cdots, m \tag{3.1}$$

with initial data

$$u_m(0) = u_{0m}, \ u_m(0) = u_{1m}. \tag{3.2}$$

By the density of $H^1_0(\Omega)$ in $L^2(\Omega)$, there exist $\alpha_{jm}$ and $\beta_{jm}, \ j = 1, 2, \cdots, m$ such that

$$u_{0m} = \sum_{j=1}^m \alpha_{jm}\omega_j \to u_0(x) \text{ strongly in } H^1_0(\Omega), \ m \to \infty, \tag{3.3}$$

$$u_{1m} = \sum_{j=1}^m \beta_{jm}\omega_j \to u_1(x) \text{ strongly in } L^2(\Omega), \ m \to \infty. \tag{3.4}$$

By a Picard’s iteration method, there exists solution $g_{jm}(t)$ of the problem (3.1) and (3.2) in interval $[0, t_m^1)$ for some $t_m^1 \leq T$. From the uniformly boundedness of function $g_{jm}(t)$ and the extension theorem, we can extend this solution to the whole interval $[0, T]$ for any given $T > 0$ by making use of the a priori estimates below.

Multiplying both sides of (3.1) by $g'_{jm}(t)$ and summing with respect to $j$ from 1 to $m$, and integrating over $[0, t]$, we have from (2.1) and (2.3) that

$$\mathcal{E}_m(t) = \frac{1}{2} \|u_m(t)\|^2 + \mathcal{F}(u_m(t)) = \frac{1}{2} \|u_m(0)\|^2 + \mathcal{F}(u_m(0)) = \mathcal{E}_m(0) < d. \tag{3.5}$$

By (3.5), we can verify

$$u_m(t) \in \mathcal{W}, \ \forall t \in [0, T]. \tag{3.6}$$

In fact, suppose that (3.6) is false and let $\tau$ be the smallest time for that $u_m(\tau) \notin \mathcal{W}$. Then in virtue of the continuity of $u_m(t)$, we see $u_m(\tau) \in \partial\mathcal{W}$. From the continuity $\mathcal{F}(u(t))$ and $\mathcal{K}(u(t))$ with respect to $t$, AIMS Mathematics Volume 6, Issue 7, 6898–6914.
we have either $\mathcal{F}(u_m(\tau)) = d$ or $\mathcal{K}(u_m(\tau)) = 0$. By (3.5), we get $\mathcal{F}(u_m(\tau)) < d$. So, the former case is impossible. Assume that $\mathcal{K}(u_m(\tau)) = 0$ is valid, then $u_m(\tau) \in \mathcal{N}$. From (2.7), we obtain $\mathcal{F}(u_m(\tau)) \geq d$ which is contradictory with (3.5). Therefore, the latter case is impossible as well.

We deduce from (2.5), (3.5) and (3.6) that

$$d > J(u_m(t)) = \frac{1}{4}\|u_m(t)\|^2 + \frac{1}{2}K(u_m(t)) > \frac{1}{4}\|u_m(t)\|^2,$$

(3.7)

which implies that

$$\|u_m(t)\|^2 < 4d.$$

(3.8)

From (2.1), (3.5) and Lemma 2.2, we obtain

$$\|u_m(t)\|^2 + \|\nabla u_m(t)\|^2 + \frac{3}{2}\|u_m(t)\|^2 \leq 2d + \int_\Omega u_m^2(t) \log |u_m(t)| dx$$

$$\leq 2d + \|u_m\|^2 \log \|u_m(t)\| + \frac{a^2}{2\pi} \|\nabla u_m(t)\|^2 - \frac{n}{2}(1 + \log a)\|u_m(t)\|^2. $$

(3.9)

Let $a = \sqrt{\pi}$, then we have from (3.8) and (3.9)

$$2\|u_m(t)\|^2 + \|\nabla u_m(t)\|^2 \leq 4d + (\log \|u_m(t)\|^2 - \log(\sqrt{\pi})^n - 3)\|u_m(t)\|^2$$

$$\leq 4d[1 + \log 4d - \log(\sqrt{\pi})^n e^{n+3}] = 2nd \log 2,$$

(3.10)

which implies

$$\|u_m(t)\| < \sqrt{nd \log 2}, \quad \|\nabla u_m(t)\| \leq \sqrt{2nd \log 2}. $$

(3.11)

We know that $u_{m\tau}(t)$ is uniformly bounded in $L^\infty(0,T;H^{-1}(\Omega))$ by a standard discussion. Then, there exists a function $u(t)$ and a convergent subsequence of $\{u_{m}\}$, still denoted by $\{u_m\}$. As $m \to \infty$, we obtain

$$u_m \to u \text{ weakly star in } L^\infty(0,T;H_0^1(\Omega)),$$

(3.12)

$$u_{m\tau} \to u_\tau \text{ weakly star in } L^\infty(0,T;L^2(\Omega)),$$

(3.13)

$$u_{m\tau} \to u_\tau \text{ weakly star in } L^\infty(0,T;H^{-1}(\Omega)).$$

(3.14)

From (3.12)–(3.14) and Aubin-Lions lemma, we have

$$u_m \to u \text{ strongly in } L^2(0,T;L^2(\Omega)),$$

(3.15)

which implies

$$u_m \to u \text{ a.e. in } (0,T) \times \Omega.$$

(3.16)

By (3.16), we can infer that

$$u_m \log |u_m| \to u \log |u| \text{ a.e. in } (0,T) \times \Omega.$$
Let $\Omega_1 = \{x \in \Omega; |u_m(x)| \leq 1\}$ and $\Omega_2 = \{x \in \Omega; |u_m(x)| > 1\}$, then by direct calculation, we get from (3.11)
\[
\int_{\Omega} |u_m(t)| \log |u_m(t)| dx = \int_{\Omega_1} |u_m(t)|^2 (\log |u_m(t)|)^2 dx + \int_{\Omega_2} |u_m(t)|^2 (\log |u_m(t)|)^2 dx \\
\leq e^{-2|\Omega|} + \left(\frac{n-2}{2}\right)^2 \int_{\Omega_2} |u_m(t)|^{\frac{2n}{n-2}} dx \\
\leq e^{-2|\Omega|} + \left(\frac{n-2}{2}\right)^2 C_1^{\frac{n}{n-2}} \|
abla u_m(t)\|^{\frac{2n}{n-2}} \\
\leq e^{-2|\Omega|} + \left(\frac{n-2}{2}\right)^2 (2ndC^2 \log 2)^{\frac{n}{n-2}}.
\]
The estimate (3.18) indicates that $u_m \log |u_m|$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$. Thus there exists a function $\chi$ such that
\[
u_m \log |u_m| \rightharpoonup \chi \text{ weakly star in } L^\infty(0, T; L^2(\Omega)).
\]
From (3.17), (3.18) and Lemma 2.5, we have
\[
\nu_m \log |u_m| \rightharpoonup u \log |u| \text{ weakly in } L^\infty(0, T; L^2(\Omega)).
\]
It follows from (3.19) and (3.20) that
\[
\chi = u \log |u|.
\]
Let $m \to \infty$ in (3.1), by using (3.12), (3.14), (3.19) and (3.20), we obtain
\[
(u_t, \omega_j) + (\nabla u, \nabla \omega_j) + (u, \omega_j) = (u \log |u|, \omega_j), \ \forall j.
\]
By the density of the system $\{\omega_j\}_{j=1}^\infty$ in $H^1_0(\Omega)$, we deduce that
\[
(u_t, \varphi) - (\Delta u, \varphi) + (u, v) = (u \log |u|, \varphi)
\]
for $\forall \varphi \in H^1_0(\Omega)$. That is to say $u$ satisfies the Eq (1.1) in the weak sense.

Next, we prove that $u(0) = u_0, u_t(0) = u_1$ are held. It follows from (3.12), (3.13) and Lemma 2.4 that $u(t) : [0, T] \to L^2(\Omega)$ is continuous. Hence, we gain that $u(0)$ is valid and $u_m(0) \to u(0)$ weakly in $L^2(\Omega)$. By (3.3), we obtain $u(0) = u_0$.

To prove $u_t(0) = u_1$, we note that
\[
\int_0^T (u_{mt}, \xi \omega_j) dt = -\int_0^T (u_{mt}, \xi \omega_j) dt - (u_{mt}(0), \omega_j),
\]
where $\xi(t)$ is a smooth function with $\xi(0) = 1, \xi(T) = 0$.

For given $j$, as $m \to \infty$, in the distribution sense, we have
\[
\int_0^T (u_t, \xi \omega_j) dt = -\int_0^T (u_t, \xi \omega_j) dt - (u_t(0), \omega_j)
\]
in \( D'(\{0, T\}) \). On the other hand, by (3.1), we get
\[
\int_0^T (u_{mt}, \xi \omega_j) dt = \int_0^T [(\Delta u_m, \xi \omega_j) - (u_m, \xi \omega_j) + (u_m \log |u_m|, \xi \omega_j)] dt. \tag{3.23}
\]
Taking the limitation on both sides of (3.23) as \( m \to \infty \), we obtain
\[
\int_0^T (u_t, \xi \omega_j) dt = \int_0^T [(\Delta u, \xi \omega_j) - (u, \xi \omega_j) + (u \log |u|, \xi \omega_j)] dt.
\]
Therefore,
\[
\int_0^T (u_t, \xi \omega_j) dt = - \int_0^T (u_t, \xi \omega_j) dt - (u_1, \omega_j). \tag{3.24}
\]
It follows from (3.22) and (3.24) that \((u_t(0), \omega_j) = (u_1, \omega_j)\). By the density of \( \{\omega_j\}_{j=1}^m \) in \( L^2(\Omega) \), we get
\[ u_t(0) = u_1. \]

The proof of Theorem 3.1 is completed. \( \square \)

For the case \( K(u_0) \geq 0 \) and \( E(0) = d \), the global existence result of the problem (1.1)–(1.3) reads as follows:

**Theorem 3.2** Given that \( u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega) \). If \( E(0) = d \) and \( K(u_0) \geq 0 \), then there exists a global weak solution \( u(x, t) \) for the problem (1.1)–(1.3) such that \( u(x, t) \in L^\infty([0, +\infty); H_0^1(\Omega)), u_t(x, t) \in L^\infty([0, +\infty); L^2(\Omega)). \)

**Proof.** Let \( \rho_0 = 1 - \frac{1}{k} \) and \( u_{0k} = \rho_k u_0 \) for \( k \geq 2 \). We consider the following problem
\[
\begin{aligned}
&u_t + \Delta u + u = u \log |u|, \quad (x, t) \in \Omega \times R^+,
&u(x, 0) = u_{0k}(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,
&u(x, t) = 0, \quad (x, t) \in \partial \Omega \times R^+.
\end{aligned}
\tag{3.25}
\]
By \( K(u_0) \geq 0 \) and Lemma 2.6, we have \( \lambda^* = \lambda^*(u_0) \geq 1 \). Therefore, we conclude that \( K(u_{0k}) > 0 \). Thus, we have
\[ F(u_{0k}) = \frac{1}{4} \|u_{0k}\|^2 + \frac{1}{2} K(u_{0k}) > 0 \]
and \( F(u_{0k}) = F(\mu_k u_0) < F(u_0) \). Therefore,
\[ 0 < E_k(0) = \frac{1}{2} \|u_t\|^2 + J(u_{0k}) < \frac{1}{2} \|u_t\|^2 + F(u_0) = E(0) = d. \]
So, we obtain \( u_{0k} \in \mathcal{W} \). For each \( k \), by Theorem 3.1, the problem (3.25) admits a global weak solution \( u_k(t) \) which satisfies that \( u_k(t) \in L^\infty([0, +\infty); H_0^1(\Omega)), u_{0k}(t) \in L^\infty([0, +\infty); L^2(\Omega)) \) and
\[
(u_{k1}, \varphi) + \int_0^T [(\Delta u_k, \varphi) + (u_k, \varphi)] ds = (u_1, \varphi) + \int_0^T (u_k \log |u_k|, \varphi) ds \tag{3.26}
\]
for any \( \varphi \in H_0^1(\Omega) \). In addition,
\[
E_k(t) = \frac{1}{2} \|u_k(t)\|^2 + J(u_k) = \frac{1}{2} \|u_t\|^2 + F(u_{0k}) = E_k(0) < d. \tag{3.27}
\]
By using the formula (3.27) and the same argument as (3.6), we may verify \( u_k(t) \in \mathcal{W} \).

The remainder proof for Theorem 3.2 is the same process as Theorem 3.1. Here, we omit it. \( \square \)
Next, we study the global existence of solution to the problem (1.1)–(1.3) in a family of potential wells \( \mathcal{W}_\varepsilon \). For this purpose, we need the following lemmas

**Lemma 3.1** Suppose that \( u_0 \in H_0^1(\Omega) \), \( u_1 \in L^2(\Omega) \) and \( 0 < \mathcal{E}(0) < d(\varepsilon_0) \). \( \varepsilon_1, \varepsilon_2 \) are the two roots of the equation \( d(\varepsilon) = \mathcal{E}(0), u(x, t) \) is a solution of the problem (1.1)–(1.3). Then

(i) If \( J_{\varepsilon_0}(u_0) > 0 \), then \( u(t) \in \mathcal{W}_\varepsilon \) for \( \forall \varepsilon \in (\varepsilon_1, \varepsilon_2) \).

(ii) If \( J_{\varepsilon_0}(u_0) < 0 \), then \( u(t) \in \mathcal{U}_\varepsilon \) for \( \forall \varepsilon \in (\varepsilon_1, \varepsilon_2) \).

**Proof.** Firstly, under the conditions in Lemma 4.1, we prove the sign of \( J_\varepsilon(u) \) is invariant on the interval \((\varepsilon_1, \varepsilon_2)\).

Multiplying both sides of the Eq (1.1) by \( u \), then we get from integrating over \( \Omega \times [0, t] \) that

\[
\mathcal{E}(t) = \frac{1}{2} \|u\|^2 + J(u) = \frac{1}{2} \|u_1\|^2 + J(u_0) = \mathcal{E}(0) = d(\varepsilon). \tag{3.28}
\]

By (3.28) and \( 0 < \mathcal{E}(0) < d(\varepsilon_0) \), it is easy to see that \( 0 < J(u) < d(\varepsilon_0) \). Namely, \( \|\nabla u\| \neq 0 \).

By contradiction, we suppose that the sign of \( J_\varepsilon(u) \) is variable on \((\varepsilon_1, \varepsilon_2)\), then there exists \( \varepsilon' \in (\varepsilon_1, \varepsilon_2) \) such that \( J_{\varepsilon'}(u) = 0 \). From (3.28), (2.20) and Proposition 2.2, we gain that \( \mathcal{E}(0) \geq J(u) \geq d(\varepsilon_1) = d(\varepsilon_2) \), which is contradictive with \( \mathcal{E}(0) = d(\varepsilon_1) = d(\varepsilon_2) \).

(i) Because \( J_{\varepsilon_0}(u_0) > 0 \) and the sign of \( J_\varepsilon(u) \) is not changed for \((\varepsilon_1, \varepsilon_2)\), we have \( \|\nabla u_0\| \neq 0 \) and \( J_\varepsilon(u_0) > 0 \), \( \forall \varepsilon \in (\varepsilon_1, \varepsilon_2) \). From (3.28), we get \( J(u_0) \leq \mathcal{E}(0) < d(\varepsilon) \). Thus, we obtain \( u_0 \in \mathcal{W}_\varepsilon, \forall \varepsilon \in (\varepsilon_1, \varepsilon_2) \).

Next, we prove \( u(t) \in \mathcal{W}_\varepsilon \) for \( \forall \varepsilon \in (\varepsilon_1, \varepsilon_2) \) and \( 0 < t < T \), where \( T \) is the existence time of \( u(t) \). Assume that there exists a number \( t_1 \in (0, T) \) such that \( u(t_1) \notin \mathcal{W}_\varepsilon \). Then, in virtue of the continuity of \( u(t) \), we see \( u(t_1) \in \partial \mathcal{W}_\varepsilon, \forall \varepsilon \in (\varepsilon_1, \varepsilon_2) \). From the definition of \( \mathcal{W}_\varepsilon \) and the continuity of \( J(u(t)) \) and \( J_\varepsilon(u(t)) \) with respect to \( t \), we have

\[
J_\varepsilon(u(t_1)) = 0, \quad \|\nabla u(t_1)\| \neq 0, \tag{3.29}
\]

or

\[
J(u(t_1)) = d(\varepsilon). \tag{3.30}
\]

It follows from (3.28) that

\[
J(u(t)) < \mathcal{E}(0) = d(\varepsilon), \quad t \in (0, T). \tag{3.31}
\]

Thus, the case (3.30) is impossible. If (3.29) holds, then, by (3.17), we have \( J(u(t_1)) \geq d(\varepsilon) \) which is contradictive with (3.31). Consequently, the case (3.29) is also impossible. Thus, we conclude that \( u(t) \in \mathcal{W}_\varepsilon, \forall \varepsilon \in (\varepsilon_1, \varepsilon_2) \).

(ii) Since the sign of \( J_\varepsilon(u) \) is not changed for \((\varepsilon_1, \varepsilon_2)\), by \( J_{\varepsilon_0}(u_0) < 0 \), we get \( J_\varepsilon(u_0) < 0 \) for \( \forall \varepsilon \in (\varepsilon_1, \varepsilon_2) \). Thus, we have \( u_0 \in \mathcal{U}_\varepsilon \) from \( J(u_0) < \mathcal{E}(0) = d(\varepsilon) \). Now we prove \( u(t) \in \mathcal{U}_\varepsilon \) for each \( \varepsilon \in (\varepsilon_1, \varepsilon_2) \), \( 0 < t < T \). If it is not right, then there exists \( t_2 \in (0, T) \) with \( u(t_2) \in \partial \mathcal{U}_\varepsilon \), \( \forall \varepsilon \in (\varepsilon_1, \varepsilon_2) \), i.e. either \( J_\varepsilon(u(t_2)) = 0 \) or \( J(u(t_2)) = d(\varepsilon) \). By (3.31), \( J(u(t_2)) = d(\varepsilon) \) is impossible. Moreover, let \( t_2 \) be the first time such that \( J_\varepsilon(u(t_2)) = 0 \), then \( J_\varepsilon(u(t)) < 0 \) for \( 0 \leq t < t_2 \). Combining (3.28) and Proposition 2.3, we get \( \|\nabla u(t)\| > r(\varepsilon) \) for \( t \in [0, t_2) \). Hence, we obtain \( \|\nabla u(t_2)\| \geq r(\varepsilon) \). From (3.17), it follows that \( J(u(t_2)) \geq d(\varepsilon) \) which is contradictive with (3.31). This implies that \( J_\varepsilon(u(t_2)) = 0 \) is also impossible. Therefore, we have \( u(t) \in \mathcal{U}_\varepsilon, \forall \varepsilon \in (\varepsilon_1, \varepsilon_2) \).
Theorem 3.1. We get $u(x, t)$ are solutions of the problem (1.1)--(1.3). Then 
(i) If $J_{e_0}(u_0) > 0$, then $u(t) \in W_{e}$ for $\forall e \in (e_1, e_2)$. 
(ii) If $J_{e_0}(u_0) < 0$, then $u(t) \in U_{e}$ for $\forall e \in (e_1, e_2)$.

We can prove Lemma 3.2 by means of the similar method shown in the proof of Lemma 3.1. Here, we omit it.

Theorem 3.3 Suppose that $e_1, e_2$ are the two roots of the equation $d(e) = E(0)$ and $J_{e}(u_0) > 0$. If $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ and $0 < E(0) < d(e)$, then the problem (1.1)--(1.3) admits a global weak solution $u(x, t)$ such that 
\[ u(x, t) \in L^m(0, T; H^1_0(\Omega)), \quad u_t(x, t) \in L^\infty(0, T; L^2(\Omega)), \] 
for any $T > 0$.

Proof: By using the similar argument as Theorem 3.1, we are going to prove Theorem 3.3. Under the conditions in Theorem 3.3, by Lemma 3.1, we have $u_0 \in W_{e}$ for $e \in (e_1, e_2)$. For any given $e_1 < e < e_2$, we derive $J_{e}(u_m(0)) > 0$ and $E_m(0) < d(e)$, which implies that $u_m(0) \in W_{e}$. Once again, we get $u_m(t) \in W_{e}$ by Lemma 3.1. Here, the approximate solutions $u_m(t)$ are given in the proof of Theorem 3.1.

Multiplying both sides of (3.1) by $g'_m(t)$, summing over $j$ from 1 to $m$ and integrating with respect to $t$, we obtain 
\[ E_m(t) = \frac{1}{2}||u_m(t)||^2 + J(u_m(t)) = \frac{1}{2}||u_m(0)||^2 + J(u_m(0)) = E_m(0) < d(e). \] 
(3.32)

From (2.3) and (2.17), we deduce 
\[ J(u_m(t)) = \frac{1 - e}{2}||\nabla u_m(t)||^2 + J_{e}(u_m(t)). \] 
(3.33)

Combining (2.21), (3.32), (3.33), by $u_m(t) \in W_{e}$, we get $J(u_m(t)) > 0$ and the following estimates 
\[ ||\nabla u_m(t)|| < 2 \sqrt{\lambda_1 \epsilon^2}. \] 
(3.34)

From (2.21) and (3.32), we find that 
\[ ||u_m(t)|| < 2 \sqrt{\lambda_1 \epsilon^2 (1 - \epsilon) \epsilon^2}. \] 
(3.35)

By means of the same procedure as the estimates (3.18), we obtain 
\[ \int_\Omega |u_m(t) \log |u_m(t)||^2 dx \leq e^{-2|\Omega| + \left(\frac{n-2}{2}\right)^2 (4 \lambda_1 \epsilon C^2 \epsilon^3) \frac{n-1}{n}}. \] 
(3.36)

The remainder of the proof for Theorem 3.3 is the same as those of Theorem 3.1. Here, we omit them. 

4. Blow-up of solution

In this section, we establish the blow-up property of solution for the problem (1.1)–(1.3).

**Lemma 4.1** Let \(u(t)\) be a solution of (1.1)–(1.3). If \(u_0 \in \mathcal{U}\) and \(\mathcal{E}(0) < d\), then \(u(t) \in \mathcal{U}\) and \(\mathcal{E}(t) < d\), for all \(t \geq 0\).

*Proof.* It follows from Lemma 2.3 that

\[
\mathcal{E}(t) = \mathcal{E}(0) < d, \quad \forall t \geq 0.
\]

By (2.5), we obtain

\[
\mathcal{J}(u) \leq \mathcal{E}(t) < d, \quad \forall t \geq 0.
\]

By contradiction, we assume that there exists \(t^* \in [0, +\infty)\) such that \(u(t^*) \notin \mathcal{U}\), then, from the continuity of \(\mathcal{K}(u(t))\) on \(t\), we have \(\mathcal{K}(u(t^*)) = 0\). This implies that \(u(t^*) \notin \mathcal{N}\). We get from (2.7) that \(\mathcal{J}(u(t^*)) \geq d\), which is contradiction with (4.1). Consequently, Lemma 4.1 is valid. \(\square\)

**Lemma 4.2** Suppose that \(u \in \mathcal{U}\), then \(\mathcal{K}(u(t)) < 2[\mathcal{J}(u(t)) - d]\).

*Proof.* If \(u \in \mathcal{U}\), then it follows from Lemma 2.6 that there exists a \(\lambda^*\) such that \(0 < \lambda^* < 1\) and \(\mathcal{K}(\lambda^* u) = 0\). By the definition of \(d\) in (2.6), we get

\[
d < \mathcal{J}(\lambda^* u) = \frac{1}{2} \mathcal{K}(\lambda^* u) + \frac{1}{4} ||\lambda^* u||^2 = \frac{1}{4} ||\lambda^* u||^2 < \frac{1}{4} ||u||^2.
\]

We have from (2.5) that \(d < \mathcal{J}(u(t)) - \frac{1}{2} \mathcal{K}(u(t))\), which implies that \(\mathcal{K}(u(t)) < 2[\mathcal{J}(u(t)) - d]\). \(\square\)

**Theorem 4.1** If the initial datum \(u_0 \in \mathcal{U}\), \(u_1 \in L^2(\Omega)\) satisfy that \(\mathcal{E}(0) < d\) and \(\int_\Omega u_0 u_1 dx > 0\), then the solution \(u(t)\) in Theorem 2.1 of the problem (1.1)–(1.3) blows up as time \(t\) goes to infinity, which means that

\[
\lim_{t \to +\infty} ||u(t)||^2 = +\infty.
\]

*Proof.* Let \(P(t) = ||u(t)||^2\), then \(P(t) > 0, \quad t \geq 0\). Direct computations show that

\[
P'(t) = 2(u, u_t).
\]

From (1.1) and (2.4), we get

\[
P''(t) = 2||u_t||^2 + 2 \int_\Omega uu_t dx
\]

\[
= 2||u_t||^2 - 2(||\nabla u||^2 + ||u||^2 - \int_\Omega u^2 \log |u| dx)
\]

\[
= 2||u_t||^2 - 2\mathcal{K}(u).
\]

It follows from Cauchy-Schwarz inequality and (4.2) that

\[
||P'(t)||^2 \leq 4P(t)||u_t||^2, \quad t \geq 0.
\]
Then we have from (2.5) and Lemma 2.3 that
\[ P''(t)P(t) - [P'(t)]^2 \geq 2P(t)[||u||^2 - \mathcal{K}(u(t))] - 4P(t)||u||^2 \]
\[ = -2P(t)[||u||^2 + \mathcal{K}(u(t))] \]
\[ \geq -2P(t)[2\mathcal{E}(t) - 2\mathcal{J}(u(t)) + \mathcal{K}(u(t))]. \]  

From \( u_0 \in \mathcal{U}, \mathcal{E}(0) < d \) and Lemma 4.1, we have \( u \in \mathcal{U}, \mathcal{E}(t) < d \). Hence by Lemma 4.2, we obtain that
\[ 2\mathcal{E}(t) - 2\mathcal{J}(u(t)) + \mathcal{K}(u(t)) < 2d - 2\mathcal{J}(u(t)) + 2(\mathcal{J}(u(t)) - d) = 0. \]
We conclude from (4.5) and (4.6) that
\[ P''(t)P(t) - [P'(t)]^2 > 0. \]  

Furthermore, by direct calculation, it is easy to see that
\[ (\log |P(t)|)' = \frac{P'(t)}{P(t)}. \]  
\[ (\log |P(t)|)'' = \left( \frac{P'(t)}{P(t)} \right)' = \frac{P''(t)P(t) - [P'(t)]^2}{P^2(t)} > 0. \]  

We know from (4.9) that the function \( (\log |P(t)|)' = \frac{P'(t)}{P(t)} \) is increasing on time \( t \). By integrating both sides of (4.8) from 0 to \( t \), we get
\[ \log |P(t)| - \log |P(0)| = \int_0^t (\log |P(s)|)'ds = \int_0^t \frac{P'(s)}{P(s)}ds \geq \frac{P'(0)}{P(0)}t, \]
for \( t > 0 \). Therefore,
\[ P(t) \geq P(t_0)\exp \left( \frac{P'(t_0)}{P(t_0)}(t - t_0) \right). \]  
From the definition of \( P(t) \), (4.10) means that
\[ \lim_{t \to +\infty} ||u(t)||^2 = +\infty. \]

This finishes the proof of Theorem 4.1. \( \square \)

5. Conclusions

By applying logarithmic Sobolev inequality, the Galerkin method and compactness theorem, we prove the global existence results of the problem (1.1)–(1.3) under the conditions that the initial values \( u_0 \in \mathcal{W}, u_1 \in L^2(\Omega) \) satisfy (i) \( 0 < \mathcal{E}(0) < d \) or (ii) \( \mathcal{K}(u_0) \geq 0 \) and \( \mathcal{E}(0) = d \). Meanwhile, under the condition of positive initial energy, by using the concavity analysis method, we establish the finite time blow-up result of solutions in the sense of \( L^2 \) norm. On the other hand, the global existence of solution for this problem is also obtained in a family of potential wells \( \mathcal{W}_\varepsilon \). Our result implies that the polynomial nonlinearity is important for the solutions of such kinds of Klein-Gordon equation to be blow-up in finite time.
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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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