Matrix product is many-sorted algebra

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Abstract

It is known that a category of many-sorted algebras on pure sets of similarity type is “concretely equivalent” to a category of single-sorted algebras. In this paper, we characterize a single-sorted variety that corresponds to a many-sorted variety. Such variety is also characterized by the condition that is decomposable with respect to matrix product.

1 Introduction

Recent paper [7] proves that a category of many-sorted algebras on pure sets (Definition 2.8 below) is categorically equivalent to a category of single-sorted algebras. In this paper, we consider on the case of varieties. We characterize a single-sorted variety that corresponds to a many-sorted variety. Moreover, we present one to one correspondence between many-sorted varieties and single-sorted varieties “with a diagonal pair”. We also exhibit this correspondence preserves underlying sets in the sense that is compatible with the functor described in Proposition 2.10.

On the other hand, there is a notion, studied in clone theory and study on category of algebras, called matrix product of algebras (e.g.[4],[6]). Matrix product is a construction of a new algebra and the constructed algebra always has a diagonal pair. We also prove that an algebra (or a variety) is decomposable with respect to the notion of matrix product if and only if the algebra (or the variety) has a diagonal pair.

2 Preliminaries

2.1 Many-sorted variety and clone

As single-sorted case, there is a natural correspondence between many-sorted varieties and algebras of terms; clones of many-sorted varieties.

In this section, we quickly introduce the concept of many-sorted variety and many-sorted clone, then we explain the correspondence between varieties and clones.

Through this paper, we explain an algebra that has (possibly) infinitely many-sorts corresponds to an infinitary single-sorted algebra. Therefore, we consider (possibly) infinitary single-sorted/many-sorted algebras in this paper. However, almost all results of this paper seem new even if we consider only on the case that all (classes of) algebras are finitary and finitely many-sorted. If the reader is interested only in finitary, finitely many-sorted algebras, please read the arity bound k in the paragraph the countable cardinal ℵ₀ and the number of sorts S a finite cardinal.

In this paper, we use ⊆ for the subset relation includes equality, namely, A ⊆ B means x ∈ A ⇒ x ∈ B holds for all x. We write A ⊊ B the condition A ⊆ B and A ̸= B.

First, we describe the usual type-based definition of algebras and varieties.

Definition 2.1. Let S be a cardinal.

1. An S-sorted type is a tuple (F, ar, dom, cod) that satisfies the following conditions:
   • F is a set.

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Definition 2.2. Let essentially includes type-based description as the case that the clone is freely generated.

For an S-sorted type F, F-algebra is a pair (A, τ) that satisfies the following conditions:

- A = (A_s)_{s \in S} is an S-indexed family of sets.
- τ is a function defined on F and τ(f) is a mapping \( \prod_{i \in \text{ar}(f)} A_{\text{dom}(f)(i)} \rightarrow A_{\text{cod}(f)} \) for each \( f \in F \).

For an infinite cardinal \( \kappa \), F is said \( <\kappa \)-ary if \( \text{ar}(f) < \kappa \) hold for all \( f \in F \).

The height of \( F \) is defined as

\[
\text{Ht}(F) := \begin{cases} 
\omega_0 & \text{if } F \text{ is finitary,} \\
\min \{ \kappa \in \text{Ord} \mid \text{cf}(\kappa) > \text{ar}(f) \text{ for all } f \in F \} & \text{if } F \text{ is infintary,}
\end{cases}
\]

where \( \omega_0 \) is the minimum infinite ordinal, \text{Ord} is the class of all ordinals, and \text{cf}(\kappa) \) is the cofinality of \( \kappa \), i.e., the minimum cardinal \( \alpha \) such that there exists an \( \alpha \)-indexed family \( (\lambda_i)_{i \in \alpha} \) of cardinals that satisfies \( \lambda_i < \kappa \) (for all \( i \in \alpha \)) and \( \sup_{i \in \alpha} \lambda_i = \kappa \).

For an S-sorted type F, an ordinal \( \alpha \), a cardinal \( \lambda \), a mapping \( v : \lambda \rightarrow S \) and an element \( s \in S \), the set \( T_{\alpha,(\lambda,v,s)} \) of all \( (\lambda,v) \)-ary \( s \)-valued terms of \( F \) with complexity less than \( \alpha \) is inductively defined as follows:

- \( T_{0,(\lambda,v,s)} := \{ x_i \mid v(i) = s, i \in \lambda \} \). (The set of variable symbols.)
- \( T_{\alpha+1,(\lambda,v,s)} := T_{\alpha,(\lambda,v,s)} \cup \{(f,(t_i)_{i \in \text{ar}(F)}) \mid f \in F, t_i \in T_{\alpha,(\lambda,v,\text{dom}(f)(i))}, \text{cod}(f) = s\} \).
- \( T_{\alpha,(\lambda,v,s)} := \bigcup_{\beta < \alpha} T_{\beta,(\lambda,v,s)} \) if \( \alpha \) is a limit ordinal.

The set of all \( (\lambda,v) \)-ary \( s \)-valued terms of \( F \) is \( T_{\text{Ht}(F),(\lambda,v,s)} \).

The action of terms to an algebra \( (A,\tau) \) is inductively defined as follows:

- For \( x_i \in T_{0,(\lambda,v,\text{ar}(f))} \), \( \tau(x_i) : \prod_{j \in \lambda} A_{\text{ar}(f)} \rightarrow A_{\text{ar}(f)} \) is defined as \( (a_j)_{j \in \lambda} \mapsto a_i \).
- If \( t = (f,(t_i)_{i \in \text{ar}(F)}), \tau(t) : \prod_{j \in \lambda} A_{\text{ar}(f)} \rightarrow A_{\text{cod}(f)} \) is defined as \( (a_j)_{j \in \lambda} \mapsto \tau(f)(\tau(t_i)(a_j))_{j \in \text{ar}(F)} \).

For \( s \in S \), An \( s \)-sorted identity is a pair of two \( s \)-valued terms. The tuple \( (t_1, t_2) \) is usually denoted by \( t_1 = t_2 \) when it is considered as an identity. A relation \( (A,\tau) \models t_1 = t_2 \) between an algebra \( (A,\tau) \) and an equation \( t_1 = t_2 \) is defined by \( \tau(t_1) = \tau(t_2) \).

An equational theory is a set of identities. An algebra \( A \) and an equational theory \( E \), \( A \models E \) means \( A \models e \) for all \( e \in E \). A pair \( (F,E) \) of type \( F \) and equational theory \( E \) is called a type with equational theory.

A class \( \mathcal{V} \) of \( F \)-algebras is said a variety if there exists an equational theory \( E \) such that \( \mathcal{V} = \{ A \mid A \models E \} \).

The variety of \( F \)-algebras defined by \( E \) is denoted by \( \mathcal{V}(F,E) \).

The above is traditional style of the definition of algebras. On the other hand, as the single-sorted finitary case, we can define essentially the same notion of algebras by the following clone-based description.

(At least in the author’s opinion,) clone-based definition is simpler than type-based definition. By this reason, we use the clone-based description through this paper. Note that, the clone-based description essentially includes type-based description as the case that the clone is freely generated.

**Definition 2.2.** Let \( S \) and \( \kappa \) be cardinals. An \( S \)-sorted \( <\kappa \)-ary clone \( M \) is a many-sorted algebra satisfying the following conditions:
• (Sort) The set of all sorts of $M$ consists of tuples $(\lambda, v, s)$, where $\lambda < \kappa$ is a cardinal, $v : \lambda \to S$ and $s \in S$.

• (Operation) $M$ has the following two types of operations:
  
  - (Projection) For $\lambda < \kappa$, $v : \lambda \to S$ and $i \in \lambda$, $M$ has a nullary operation $\pi_{(\lambda,v,i)} \in M_{(\lambda,v,s(i))}$.
  
  - (Composition) For $\lambda_k < \kappa$, $v_k : \lambda_k \to S$ ($k = 1, 2$) and $s \in S$, $M$ has an operation
    
    $$c(\lambda_1,v_1,s),(\lambda_2,v_2) : M(\lambda_1,v_1,s) \times \prod_{i \in \lambda_1} M(\lambda_2,v_2,v_1(i)) \to M(\lambda_2,v_2,s)$$

• (Axiom) $M$ satisfying the following equations:
  
  - (Associativity) For $\lambda_k < \kappa, v_k : \lambda_k \to S$ ($k = 1, 2, 3$) and $s \in S$,
    
    $$c(\lambda_1,v_1,s),(\lambda_3,v_3)(x,(c(\lambda_2,v_2,v_1(i)),(\lambda_2,v_2)(y_i),(\zeta_j)_{j \in \lambda_2}))_{i \in \lambda_1} = c(\lambda_2,v_2,s),(\lambda_3,v_3)(c(\lambda_1,v_1,s),(\lambda_2,v_2)(x,y_i)_{i \in \lambda_1},(\zeta_j)_{j \in \lambda_2})$$
  
  - (Outer identity law) For $\lambda_k < \kappa, v_k : \lambda_k \to S$ ($k = 1, 2$) and $i_0 \in \lambda_1$
    
    $$c(\lambda_1,v_1,s_0),(\lambda_2,v_2) \tau_{(\lambda_1,v_1),(\lambda,\nu)}(x_{i_0})_{i \in \lambda_1} = x_{i_0}$$
  
  - (Inner identity law) For $\lambda < \kappa, v : \lambda \to S$ and $s \in S$,
    
    $$c(\lambda,v,s),(\lambda,v) \tau_{(\lambda,v)}(x,\pi_{(\lambda,v,i)})_{i \in \lambda} = x.$$

If there are no possibility of confusion, we omit the subscript of $M$ or $c$.

**Definition 2.3.** Let $S$ and $\kappa$ be a cardinals, $M$ be an $S$-sorted $<\kappa$-ary clone. A tuple $(A, \tau)$ is said an $M$-algebra if the following conditions hold:

• $A = (A_s)_{s \in S}$ is an $S$-indexed family of sets.

• $\tau = (\tau_{(\lambda,v,s)})$ is a family of mappings, where $(\lambda,v,s)$ runs all tuples that satisfy $\lambda < \kappa, v : \lambda \to S$ and $s \in S$, and $\tau_{(\lambda,v,s)}$ is defined on $M(\lambda,v,s)$ and $\tau_{(\lambda,v,s)}(f)$ is a mapping $\prod_{i \in \lambda} A_{v(i)} \to A_s$ for each $f \in M(\lambda,v,s)$.

• For $f \in M_{(\lambda_1,v_1,s_1)}$, $(g_i)_{i \in \lambda_1} \in \prod_{i \in \lambda_1} M_{(\lambda_2,v_2,v_1(i))}$ and $(a_j)_{j \in \lambda_2} \in \prod_{j \in \lambda_2} A_{v_2(j)}$, the following equation holds:
  
  $$\tau_{(\lambda_2,v_2,s)}(c_{(\lambda_1,v_1,s),(\lambda_2,v_2)}(f,(g_i)_{i \in \lambda_1}))(a_j)_{j \in \lambda_2} = \tau_{(\lambda_1,v_1,s)}(f)(\tau_{(\lambda_2,v_2,v_1(i))}(g_i)(a_j)_{j \in \lambda_2})_{i \in \lambda_1}.$$

The class of all $M$-algebras is denoted by $\mathcal{V}(M)$. The category of $M$-algebras, that is, the class of objects is $\mathcal{V}(M)$ and the set of morphisms $A \to B$ is the set of all homomorphisms $A \to B$, is denoted by $\text{Cat}(\mathcal{V}(M))$.

In this paper, all classes of algebras appear in the text consists of algebras of a common clone.

The case $S$ is a singleton, $S$-sorted clone is simply called clone, or single-sorted clone. In this case, the sort $(\lambda,v,s)$ is simply denoted by $\lambda$, $\lambda$-ary $i$-th projection is denoted by $\pi_{(\lambda,i)}$.

Next, we quickly explain connection between type-based definition and clone-based definition.

**Definition 2.4.** Let $S$ be a cardinal, $(F,E)$ be an $S$-sorted type with equational theory. Let $\kappa > S$ be an infinite cardinal. We define the $S$-sorted $<\kappa$-ary clone of $F$ modulo $E$ as follows:

• The underlying set of a sort $(\lambda,v,s)$ is $T_{(\lambda,v,s)}/\sim_E$ where $T_{(\lambda,v,s)}$ is the set of all $(\lambda,v)$-ary $s$-valued terms of $F$. The equivalence $t_1 \sim_E t_2$ is defined by the condition
  
  $"A \models E \Rightarrow A \models t_1 \equiv t_2$ for all $F$-algebra $A"$. 

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Proposition 2.6. The varieties \( V \) are said to be definitionally equivalent to each other if \( M(\mathcal{V}) \) is isomorphic to \( M(\mathcal{V}) \) by using the variety \( \mathcal{V}(F,E) \) defined by the equational theory \( E \).

In this paper, we consider only on varieties. By this reason we can define the notion of definitional equivalence by a term of the corresponding clone.

Definition 2.5. Let \( S \) be a cardinal, \( \kappa \) be an infinite cardinal and \( \mathcal{V}_1, \mathcal{V}_2 \) be \( S \)-sorted \( <\kappa \)-ary varieties. The varieties \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are said to be definitionally equivalent to each other if \( M(\mathcal{V}_1) \) is isomorphic to \( M(\mathcal{V}_2) \).

Proposition 2.6. Let \( S \) be a non-zero cardinal, \( \kappa > S \) be an infinite cardinal.

1. If \( \mathcal{V} \) is an \( S \)-sorted \( <\kappa \)-ary variety, then the clone \( M(\mathcal{V}) \) of terms of \( \mathcal{V} \) is an \( S \)-sorted \( <\kappa \)-ary clone.
2. For an \( S \)-sorted \( <\kappa \)-ary clone \( M \), there exists an \( S \)-sorted \( <\kappa \)-ary variety \( \mathcal{V} \) such that \( M(\mathcal{V}) \) is isomorphic to \( M \).
3. Let \( \kappa' > \kappa \) be a cardinal, \( \mathcal{V}_1, \mathcal{V}_2 \) be \( S \)-sorted \( <\kappa \)-ary varieties. Then \( M(\mathcal{V}_1) \cong M(\mathcal{V}_2) \) if and only if \( M(\mathcal{V}_1) \cong M(\mathcal{V}_2) \).

Remark 2.7. For simplification, we use, for example, the following variable notations through this paper:

- In the case \( \lambda \) is finite, the sort \( (\lambda, v, s) \) of many-sorted clone is denoted by \( (v(0), \ldots, v(\lambda - 1)) \rightarrow s \). The case \( \lambda = 1 \), \( (1, v, s) \) is denoted by \( v(0) \rightarrow s \). The case \( v \) is constant mapping, \( v(i) = t \) for all \( i \), \( (n, v, s) \) is denoted by \( nt \rightarrow s \).
- The composition operation \( c \) of a clone is sometimes written as a composition of mappings. For example, \( c(x, (y_i)_{i \in \lambda}) \) is denoted by \( x \circ (y_i)_{i \in \lambda} \) or \( x(y_i)_{i \in \lambda} \).
- We write \( (b, (a_s)_{s \in S \setminus \{a_0\}}) \) the tuple \( (x_s)_{s \in S} \) such that \( x_{s_0} = b \) and \( x_s = a_s \) for \( s \in S \). The notation \( ((a_s)_{s \in T}, (b_s)_{s \in S \setminus T}) \) means the tuple \( (x_s)_{s \in S} \) such that \( x_s = a_s \) if \( s \in T \) and \( x_s = b_s \) if \( s \in S \setminus T \).
- If we write \( \pi_{(X,v,x)} \in M_{(X,v,s)} \), where \( M \) is an \( S \)-sorted clone, \( X \) is a set, \( v : X \rightarrow S \), \( s \in S \) and \( x \in X \), we implicitly fix a bijection \( \varphi : X \rightarrow |X| \) and \( M_{(X,v,s)} \) is identified with \( M_{(|X|,v \circ \varphi^{-1},s)} \) and \( \pi_{(X,v,x)} \) means \( \pi_{(|X|,v \circ \varphi^{-1},\varphi(x))} \).

2.2 Pure set and diagonal algebra

Let \( \mathcal{V} \) be a class of many-sorted algebras. If there is a class of single-sorted algebras naturally corresponding to \( \mathcal{V} \), then all algebras in \( \mathcal{V} \) have the property that the underlying sets of each sorts of the algebra are simultaneously empty or simultaneously non-empty. Here, “naturally corresponding to \( \mathcal{V} \)” precisely means there is a natural equivalence between \( \mathcal{V} \) and a single-sorted variety that is compatible with the natural equivalence described in Proposition 2.10.

Through this paper, \( S \) be a fixed non-zero cardinal and \( \kappa \) be a fixed infinite cardinal that \( \kappa > S \).

Definition 2.8 (cf. [7] Definition 2.1)].

1. A family \( (A_s)_{s \in S} \) of sets said to be pure if one of the following conditions holds: One condition is \( A_s = \emptyset \) for all \( s \in S \). The other is \( A_s \neq \emptyset \) for all \( s \in S \).
2. Let \( A = (A_s)_{s \in S} \) and \( B = (B_s)_{s \in S} \) be pure sets. A morphism \( f : A \rightarrow B \) of pure sets is a tuple \( f = (f_s)_{s \in S} \) such that \( f_s : A_s \rightarrow B_s \) for all \( s \in S \).
3. A category \( S \)-sorted pure sets is the category that the class of objects consists of all \( S \)-sorted pure sets and the set of morphisms from a pure set \( A \) to a pure set \( B \) is the set of all morphisms of pure sets \( A \rightarrow B \).
Definition 2.9 (cf. [7, Definition 3.1]). The following single-sorted $<\kappa$-ary clone $C_D$ is called the clone corresponds to the variety of diagonal algebras of degree $S$:

- (Generators) $C_D$ is generated by an element $d \in (C_D)_S$ of sort $S$. (Namely, $d$ is an $S$-ary term.)
- (Relations) The set of fundamental relations consists of the following only one equation, called diagonal identity:
  $$d(d(\pi(s \times S, (s, t)))_{t \in S})_{s \in S} = d(\pi(s \times S, (s, s)))_{s \in S}.$$  

A $C_D$-algebra is called a diagonal algebra of degree $S$. The variety of diagonal algebras of degree $S$ is denoted by $D_S$.

Proposition 2.10 ([7, Theorem 3.4]). The category $Set^S$ is categorically equivalent to the category $D_S$ via the following categorical equivalence $\phi$:

- (Objects correspondence) For $A = (A_s)_{s \in S} \in Set^S$, $\phi(A)$ is the following diagonal algebra:
  - The underlying set is the product set $\prod_{s \in S} A_s$.
  - For $(x, t)_{t \in S} \in \prod_{t \in S} A_t$ ($s \in S$), the value of $d$ is defined as
    $$d((x, t))_{t \in S} := (x_s)_{s \in S}.$$  

- (Morphism correspondence) For a morphism $f = (f_s)_{s \in S} : (A_s)_{s \in S} \to (B_s)_{s \in S}$ of pure set, the corresponding homomorphism $\phi(f) : \phi((A_s)_{s \in S}) \to \phi((B_s)_{s \in S})$ is defined by $(x_s)_{s \in S} \mapsto (f_s(x_s))_{s \in S}$.

3 Homogenization of many-sorted variety

Let $M$ be an $S$-sorted clone and consider the following condition: The variety of all $M$-algebras is concretely equivalent to a single-sorted variety, that is, categorically equivalent via a categorical equivalence that compatible with the categorical equivalence described in Proposition 2.10.

This condition contains the condition that “the underlying family of sets of an $M$-algebra is pure”. The quoted condition is equivalent to that $M$ satisfies the next definition.

Definition 3.1. An $S$-sorted clone $M$ is said to be pure if $M_{s_1 \to s_2} \neq \emptyset$ for all $s_1, s_2 \in S$.

In this section, we prove this condition also be sufficient that there exists a corresponding single-sorted variety.

Next, we explain the construction of the corresponding single-sorted clone and variety from a manysorted (not necessarily pure) clone.

Definition 3.2. Let $M$ be an $S$-sorted $<\kappa$-ary clone. A single-sorted $<\kappa$-ary clone $H(M)$ defined as follows is said a homogenization of $M$:

- (Underlying set) For a cardinal $\lambda < \kappa$, the underlying set $H_\lambda(M)$ of a sort $\lambda$ (the sort corresponding to all $\lambda$-ary terms) consists of all tuple $(f_s)_{s \in S}$, where $f_s \in M_{\lambda \times S, p_2, s}$. Here, and through this paper, the symbol $p_2$ is used for the second projection of a product set. (In this case, $p_2$ is the mapping $(i, s') \mapsto s'$ from $\lambda \times S$ to $S$.)
- (Projection) The nullary operation $\pi(\lambda, i) \in H_\lambda(M)$ corresponds to $\lambda$-ary $i$-th projection is defined as
  $$\pi(\lambda, i) := (\pi(\lambda \times S, p_2(i, s)))_{s \in S}.$$  

- (Composition) For $f = (f_s)_{s \in S} \in H_{\lambda_1}(M)$ and $g_i = (g_i(s))_{s \in S} \in H_{\lambda_2}(M)$ ($i \in \lambda_1$) the composition is defined as
  $$c(f, (g_i)_{i \in \lambda_1}) := c(f_s, (g_i(i, t))_{(i, t) \in \lambda_1 \times S})_{s \in S}.$$  

Definition 3.3. Let $M$ be an $S$-sorted $<\kappa$-ary clone. Let $A = (A_s)_{s \in S}$ be an $M$-algebra. We define the homogenization $H(A)$ of $A$ as the following $H(M)$-algebra:
• The underlying set is \( \prod_{s \in S} A_s \).

• The action of \( f = (f_s)_{s \in S} \in H_\lambda(M) \) is
  \[
  ((a_{i,t})_{i \in \lambda})_{i \in \lambda} \mapsto (f_s(a_{i,t})_{(i,t) \in \lambda \times S})_{s \in S}.
  \]

**Proposition 3.4.** Let \( M \) and \( A \) be as Definition 3.3. Then \( H(A) \) is actually \( H(M) \)-algebra. Namely, the compatibility condition

\[
 f(g_i(a_j)_{j \in \lambda_2})_{i \in \lambda_1} = c(f_i(g_i)_{i \in \lambda})(a_j)_{j \in \lambda_2}
\]

holds for \( \lambda_1, \lambda_2 < \kappa, f \in H_{\lambda_1}(M), g_i \in H_{\lambda_2}(M) \) \( (i \in \lambda_1) \) and \( a_j \in H(A) \) \( (j \in \lambda_2) \).

**Proof.** Let \( f = (f_s)_{s \in S}, g_i = (g_i,s)_{s \in S} \) and \( a_j = (a,j,s)_{s \in S} \). Then

\[
 f(g_i(a_j)_{j \in \lambda_2})_{i \in \lambda_1} = f((g_i,s(a_j,u)_{(j,u) \in \lambda_2 \times S})_{s \in S})_{i \in \lambda_1}
 = (f_s(g_i,s(a_j,u)_{(j,u) \in \lambda_2 \times S})_{(i,t) \in \lambda_1 \times S})_{s \in S}
 = (c(f_s,g_i,s)(a_j,u)_{(j,u) \in \lambda_2 \times S})_{s \in S}
 = c(f,g_i,s)(a_j)_{j \in \lambda_2}.
\]

\[\square\]

Homogenization is extended as a functor from a category of many-sorted algebras to the category of corresponded single-sorted algebras.

**Proposition 3.5.** Let \( M \) be an \( S \)-sorted \( \prec \kappa \)-ary clone, \( A, B \) be \( M \)-algebras, and \( \varphi : A \rightarrow B \) be a homomorphism of \( M \)-algebras. Then \( H(\varphi) : (a_s)_{s \in S} \mapsto (\varphi_s(a_s))_{s \in S} \) is a homomorphism \( H(A) \rightarrow H(B) \) of \( H(M) \)-algebras. Furthermore, the following correspondence \( H \) is a functor from the category \( \text{Cat}(\mathcal{V}(M)) \) of \( M \)-algebras to the category \( \text{Cat}(\mathcal{V}(H(M))) \) of \( H(M) \)-algebras:

• (Object correspondence) \( A \mapsto H(A) \).

• (Morphism correspondence) \( \varphi \mapsto H(\varphi) \).

**Proof.** Let \( f = (f_s)_{s \in S} \in H_\lambda(M) \) and \( a_i = (a_i,s)_{s \in S} \) for \( i \in \lambda \). Then

\[
 H(\varphi)(f(a_i)_{i \in \lambda}) = H(\varphi)(f_s(a_{i,t})_{(i,t) \in \lambda \times S})_{s \in S}
 = (\varphi_s f_s(a_{i,t})_{(i,t) \in \lambda \times S})_{s \in S}
 = (f_s(\varphi_i(a_{i,t}))_{(i,t) \in \lambda \times S})_{s \in S}
 = (f(H(\varphi)(a_i))_{i \in \lambda}.
\]

The correspondence \( H \) being a functor means

\[
 H(id_A) = id_{H(A)}
\]

holds for all \( A \in \mathcal{V}(M) \), and

\[
 H(\psi) \circ H(\varphi) = H(\psi \circ \varphi)
\]

hold for all homomorphisms \( \psi, \varphi \) that \( \text{cod}(\varphi) = \text{dom}(\psi) \). It is easily verified. \[\square\]

Homogenization of a pure clone always has terms that satisfy the following identities. In section 5 we prove the converse, i.e., single-sorted clone that has terms satisfy these identities is isomorphic to homogenization of a many-sorted pure clone.

**Proposition 3.6.** Let \( M \) be an \( S \)-sorted \( \prec \kappa \)-ary pure clone. Assume \( d \in H_\Sigma(M) \) and \( e_s \in H_1(M) \) \( (s \in S) \) be terms satisfying the following conditions:

• \( d \) is defined as \( d = (\pi_{(S \times S, \rho_{2(s,s)})})_{s \in S} \). (Intuitively, the term operation of \( d \) acts as

\[
 A^S \ni ((a_{s,t})_{i \in S})_{s \in S} \mapsto (a_{s,t})_{s \in S} \in A
\]

for each \( H(M) \)-algebra \( A \).)
• Each term $e_s$ satisfies follows:

- $e_s \circ (\pi_{(S, \text{id}_S, s)})_u \in S \setminus \{s\} = e_s$ (Intuitively, the value of term operation $e_s(a_u)_{u \in S}$ depends only on $s$-component $a_s$.)
- If $e_s = (e_{s,t})_{t \in S}$, where $e_{s,t} \in M_{(S, \text{id}_S, t)}$, then $e_{s,u} = \pi_{(S, \text{id}_S, s, t)}$. (Intuitively, the value $e_{s,u}(a_u)_{u \in S}$ of $s$-component is $a_s$.)

Then the following equations hold in $H(M)$.

1. $e_s \circ d = e_s \circ \pi_{(S, s)}$.
2. $d \circ (e_s \circ \pi_{(S, s)})_{s \in S} = d$.
3. $d \circ \pi_{(S, 0)}_{s \in S} = \pi_{(S, 0)}$, where $0 \in S$.

Proof. 1.

\[
\begin{align*}
  e_s \circ d &= e_s \circ (\pi^M_{(S \times S, p_2, (t, t))})_{t \in S} \\
  &= e_s \circ (\pi^M_{(S \times S, p_2, (s, t))})_{t \in S} \\
  &= e_s \circ \pi^H_{(S, s)}.
\end{align*}
\]

Here and through the proof of this proposition, $\pi^M$ and $\pi^H(M)$ denote projection constants of $M$ and $H(M)$ respectively.

2.

\[
\begin{align*}
  d \circ (e_s \circ \pi^H_{(S, s)})_{s \in S} &= (\pi^M_{(S \times S, p_2, (s, u))})_{s \in S} \circ ((e_s)_{t \in S} \circ (\pi^M_{(S \times S, p_2, (s, u))})_{u \in S})_{s \in S} \\
  &= (e_{s, u} \circ (\pi^M_{(S \times S, p_2, (s, u))})_{u \in S})_{s \in S} \\
  &= (\pi^M_{(S \times S, p_2, (s, u))})_{s \in S} \\
  &= d.
\end{align*}
\]

3.

\[
\begin{align*}
  d \circ \pi^H_{(S, 0)}_{s \in S} &= (\pi^M_{(S \times S, p_2, (s, u))})_{s \in S} \circ (\pi^M_{(S \times S, p_2, (0, t))})_{t \in S} \\
  &= (\pi^M_{(S \times S, p_2, (s, u))})_{s \in S} \\
  &= \pi^H_{(S, 0)}.
\end{align*}
\]

We give a name for a tuple of these terms.

**Definition 3.7.** Let $C$ be a single-sorted $<\kappa$-ary clone. The tuple $(d, (e_s)_{s \in S}) \in C_S \times C_1^S$ is said an $S$-ary diagonal pair of $C$ if the following equations holds:

1. $e_s \circ d = \pi_{(S, s)}$ for all $s \in S$.
2. $d(e_s \circ \pi_{(S, s)})_{s \in S} = d$.
3. $d(\pi_{(S, 0)}_{s \in S} = \pi_{(S, 0)}$ for $0 \in S$.

**Remark 3.8.** If $(d, (e_s)_{s \in S})$ is a diagonal pair of a clone $C$, then $d$ satisfies diagonal identity

\[
d(\pi_{(S \times S, (s, t))})_{t \in S} = d(\pi_{(S \times S, (s, s))})_{s \in S}.
\]

Namely, $C$-algebras have diagonal algebras reduct.

Proof.

\[
\begin{align*}
  d(\pi_{(S \times S, (s, t))})_{t \in S} &= d(e_s(d(\pi_{(S \times S, (s, t))})_{t \in S}))_{s \in S} \\
  &= d(e_s(\pi_{(S \times S, (s, s))})_{s \in S} \\
  &= d(\pi_{(S \times S, (s, s))})_{s \in S}.
\end{align*}
\]
4 Matrix product

A homogenization $H(M)$ of an $S$-sorted pure clone has a diagonal pair. A clone that has a diagonal pair is also characterized as being decomposable with respect to matrix product. In this section, we explain these conditions are equivalent. In the next section, we explain these conditions also be equivalent to the following condition: There exists a many-sorted pure clone $M$ such that the homogenization of $M$ is isomorphic to the single-sorted clone.

We start from the definition of matrix product.

Definition 4.1. Let $C$ be a single-sorted $<\kappa$-clone.

1. Let $e \in C_1$ be an idempotent element, namely, $e$ satisfies $e \circ e = e$. An idempotent retract, denoted by $e(C)$, of $C$ by $e$ is the following $<\kappa$-ary clone.

   - (Underlying set) The underlying set $e(C)_\lambda$ of the sort $\lambda < \kappa$ is \{ $f \in C_\lambda$ | $e \circ f = f$ $\} / _{\sim_\lambda}$, where $\sim_\lambda$ is the equivalence relation defined as $f \sim_\lambda g :\iff f \circ (e \pi_{(\lambda,i)})_{i} = g \circ (e \pi_{(\lambda,i)})_{i}$.  

   - (Projection) The $\lambda$-ary $i$-th projection $\pi^{(C)}_{(\lambda,i)}$ is $e \circ \pi^{(C)}_{(\lambda,i)} / _{\sim_\lambda}$, where $\pi^{(C)}_{(\lambda,i)}$ is the $\lambda$-ary $i$-th projection of $C$.

   - (Composition) For $f / _{\sim_\lambda} \in e(C)_\lambda$ and $g / _{\sim_\lambda} \in C(C)_\lambda$ ($i \in \lambda_1$), we define
     \[ e^{(C)}(f / _{\sim_\lambda}, (g_i / _{\sim_\lambda})_{i} \in \lambda_1) := e^{(C)}(f, (g_i)_{i} \in \lambda_1) / _{\sim_\lambda}, \]
     where $e^{(C)}$ is the composition operation of $C$.

2. Let $(e_s)_{s \in S} \in C_1^{S}$ be a family of idempotent elements. The matrix product $\boxtimes_{s \in S} e_s(C)$ of a family of idempotent retracts $(e_s(C))_{s \in S}$ is defined as the following $<\kappa$-ary clone:

   - (Underlying set) The underlying set $(\boxtimes_{s \in S} e_s(C))_\lambda$ of the sort $\lambda < \kappa$ is
     \[ \{(f_s / _{\sim_\lambda})_{s \in S} \in (C_{\lambda \times S}) / _{\sim_\lambda} \} \mid \forall s \in S; e_s f_s = f_s, \]
     where the equivalence $_{\sim_\lambda}$ is defined as
     \[ f \sim_\lambda g :\iff f \circ (e \pi_{(\lambda \times S,(i,t))})_{(i,t)} \in \lambda \times S = g \circ (e \pi_{(\lambda \times S,(i,t))})_{(i,t)} \in \lambda \times S. \]

   - (Projection) The $\lambda \times S$-ary $i$-th projection $\pi^{(C)}_{(\lambda \times S,(i,t))}$ is defined by $(e_s \circ \pi^{(C)}_{(\lambda \times S,(i,t))})_{(i,t)} / _{\sim_\lambda}$, where $\pi^{(C)}_{(\lambda \times S,(i,t))}$ is the $\lambda \times S$-ary $(i,s)$-th projection constant of $C$.

   - (Composition) For $f = (f_s / _{\sim_\lambda})_{s \in S} \in (\boxtimes_{s \in S} e_s(C))_{\lambda_1}$ and $g = (g_i / _{\sim_\lambda})_{i} \in S \in (\boxtimes_{s \in S} e_s(C))_{\lambda_2}$ ($i \in \lambda_1$), the composition is defined by
     \[ e^{(C)}(f, (g_i)_{i} \in \lambda_1) := e^{(C)}(f_s, (g_i)_{(i,t)} \in \lambda_1 \times S) / _{\sim_\lambda} \in S, \]
     where $e^{(C)}$ is the composition operation of $C$.

Definition 4.2 (cf. [3] Definition 2.4, Lemma 3.5]). Let $C$ be a $<\kappa$-ary clone, $A$ be a $C$-algebra.

1. Let $e \in C_1$ be an idempotent element. The idempotent retract (in literature e.g. [4],[6], this is referred as a neighbourhood) of $A$ by $e$, denoted by $e(A)$, is defined as the following $e(C)$-algebra:

   - The underlying set is $e(A) = \{ e(a) \mid a \in A \}$.

   - The action of $f / _{\sim_\lambda} \in e(C)_\lambda$ is $(a_i)_{i} \mapsto f(a_i)_{i} \in \lambda_\lambda$.

2. Let $(e_s)_{s \in S}$ be a family of idempotent elements of $C$. The matrix product $\boxtimes_{s \in S} e_s(A)$ is defined as the following $\boxtimes_{s \in S} e_s(C)$-algebra:

   - The underlying set is $\prod_{s \in S} e_s(A)$. 

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Proof. Let $d\in \mathbb{E}_{\alpha_i} e_s(A)$ be a homomorphism of $\mathbb{E}_{\alpha_i} e_s(A)$ and $\varphi \mapsto \tilde{\varphi}$ is a functor $\mathbb{V}(C) \rightarrow \mathbb{V}(\mathbb{E}_{\alpha_i} e_s(A))$.

**Proposition 4.3.** Let $C$ be a $<\kappa$-ary clone and $(e_s)_{s\in S}$ be a family of idempotent elements of $C$. Let $A$ and $B$ be $C$-algebras, $\varphi : A \rightarrow B$ be a homomorphism of $C$-algebras. Then the mapping $\tilde{\varphi} : \mathbb{E}_{\alpha_i} e_s(A) \rightarrow \mathbb{E}_{\alpha_i} e_s(B)$ defined as $(a_s)_{s\in S} \mapsto (\varphi(a_s))_{s\in S}$ is a homomorphism of $\mathbb{E}_{\alpha_i} e_s(A)$-algebras. Furthermore, the (pair of) correspondence $A \mapsto \mathbb{E}_{\alpha_i} e_s(A)$ and $\varphi \mapsto \tilde{\varphi}$ is a functor $\mathbb{V}(C) \rightarrow \mathbb{V}(\mathbb{E}_{\alpha_i} e_s(A))$.

**Proof.** Let $f = (f_u/\sim\lambda)_{\lambda\in S} \in (\mathbb{E}_{\alpha_i} e_s(C))_\lambda$ and $a_i = (a_i, a_{i\in S}) \in \mathbb{E}_{\alpha_i} e_s(A)$ for $i \in \lambda$. Then

\[
\tilde{\varphi}(f_{\alpha_i})_{i\in \lambda} = \left(\varphi\left(f_u(\alpha_i, (\alpha_i)_{\lambda\in S})\right)_{\lambda\in S}\right)_{i\in \lambda} = \left(f_u(\varphi(\alpha_i, (\alpha_i)_{\lambda\in S}))_{\lambda\in S}\right)_{i\in \lambda} = f(\tilde{\varphi}(a_i))_{i\in \lambda}.
\]

The correspondence $\varphi \mapsto \tilde{\varphi}$ clearly preserves identity morphisms and composition. Thus the correspondence is a functor.

Matrix product has a diagonal pair. A diagonal pair of a matrix product is, for example, constructed as in the next proposition.

**Proposition 4.4.** Let $C$ be a single-sorted $<\kappa$-ary clone, $(e_s)_{s\in S}$ be a family of idempotent elements of $C$. Let $d \in (\mathbb{E}_{\alpha_i} e_s(C))_S$ and $\tilde{e}_s \in (\mathbb{E}_{\alpha_i} e_s(C))_1$ be the terms defined as follows:

\[
d := (e_s \circ \pi_{(S \times S,(s,t))}/\sim S)_{s\in S},
\]

\[
\tilde{e}_s := (e_t \circ \pi_{(S,s)}/\sim_1)_{t\in S}.
\]

Then $(d, (\tilde{e}_s)_{s\in S})$ is a diagonal pair of $\mathbb{E}_{\alpha_i} e_s(C)$.

**Proof.** $d, \tilde{e}_s \in \mathbb{E}_{t\in S} e_1$ immediately follows from the definition. In the proof of this proposition, we write $\pi_C/\pi_{\text{Ee}(C)}$ the projection terms of $C/\mathbb{E}_{\alpha_i} e_s(C)$ respectively.

1. $\tilde{e}_s \circ d = \tilde{e}_s \circ (e_t \circ \pi_{(S \times S,(s,t))})_{t\in S}$

\[=	ilde{e}_s \circ (e_t \circ \pi_{(S \times S,(s,t))})_{t\in S} \quad (\text{\tilde{e}_s \ depends \ only \ on \ s-component})
\]

2. $d \circ (\tilde{e}_s \circ \pi_{(S,s)})_{s\in S} = (e_t \circ \pi_{(S \times S,(t,s))})_{t\in S} \circ ((e_t \circ \pi_{(S \times S,(s,t))})_{t\in S} \circ (\pi_{(S \times S,(s,t))})_{t\in S} = (e_t \circ \pi_{(S \times S,(t,s))})_{t\in S} \circ ((e_t \circ \pi_{(S \times S,(s,t))})_{t\in S} \circ (\pi_{(S \times S,(s,t))})_{t\in S} = (e_t \circ \pi_{(S \times S,(t,s))})_{t\in S} = (e_t \circ \pi_{(S \times S,(t,s))})_{t\in S} = d.
\]

3. $d \circ (\pi_{(S,0)})_{s\in S} = (e_t \circ \pi_{(S \times S,(t,0))})_{t\in S} \circ ((e_t \circ \pi_{(S \times S,(0,t))})_{t\in S} = (e_t \circ \pi_{(S \times S,(t,0))})_{t\in S} = (e_t \circ \pi_{(S \times S,(0,t))})_{t\in S} = (e_t \circ \pi_{(S \times S,(0,t))})_{t\in S} = \pi_{\text{Ee}(C)}.$

\[= \pi_{(S,0)}.
\]
Conversely, a clone that has a diagonal pair is decomposed with respect to matrix product. In this sense, matrix product is characterized by existence of a diagonal pair.

**Theorem 4.5.** Let $C$ be a single-sorted clone, $(d, (e_s)_{s \in S})$ be a diagonal pair. Then the clone $C$ is isomorphic to $\mathbb{E}_s \in S e_s(C)$.

**Proof.** Suppose $(d, (e_s)_{s \in S})$ is a diagonal pair of $C$. We define $\varphi_\lambda : C_\lambda \to (\mathbb{E}_s \in S e_s(C))_\lambda$ as

$$\varphi_\lambda : f \mapsto (e_s f \circ (d \circ (\pi_{(\lambda \times S, (i,t))}))_{i \in \lambda})_{s \in \lambda}.$$  

Note that $\varphi_\lambda(f) \in (\mathbb{E}_s \in S e_s(C))_\lambda$ follows from $e_s f = e_s$.

We prove injectivity of $\varphi_\lambda$. Let $f, g \in C_\lambda$ and assume $\varphi_\lambda(f) = \varphi_\lambda(g)$, namely, assume the equality

$$e_s f \circ (d \circ (\pi_{(\lambda \times S, (i,t))}))_{i \in \lambda} = e_s g \circ (d \circ (\pi_{(\lambda \times S, (i,t))}))_{i \in \lambda} \circ (e_t \pi_{(\lambda \times S, (i,t))})(t, t) \in \lambda \times S$$

$$= e_s g \circ (d \circ (\pi_{(\lambda \times S, (i,t))}))_{i \in \lambda} \circ (e_t \pi_{(\lambda \times S, (i,t))})(t, t) \in \lambda \times S$$

$$= e_s g \circ (d \circ (e_t \pi_{(\lambda \times S, (i,t))}))_{i \in \lambda} \in \lambda$$

hold for all $s \in S$. Then

$$f = d \circ (e_s)_{s \in S} \circ f \circ (\pi_{(\lambda, i)})_{i \in \lambda}$$

$$= d \circ (e_s)_{s \in S} \circ f \circ (d \circ (\pi_{(\lambda, i)}))_{i \in \lambda}$$

$$= d \circ (e_s)_{s \in S} \circ f \circ (d \circ (\pi_{(\lambda \times S, (i,t))}))_{i \in \lambda} \circ (e_t \pi_{(\lambda \times S, (i,t))})(t, t) \in \lambda \times S$$

$$= d \circ (e_s)_{s \in S} \circ f \circ (d \circ (e_t \pi_{(\lambda \times S, (i,t))}))_{i \in \lambda} \circ (\pi_{(\lambda, i)})(t, t) \in \lambda \times S$$

$$= d \circ (e_s)_{s \in S} \circ g \circ (d \circ (e_t \pi_{(\lambda \times S, (i,t))}))_{i \in \lambda} \circ (\pi_{(\lambda, i)})(t, t) \in \lambda \times S$$

$$= g.$$  

Next, we prove surjectivity of $\varphi_\lambda$. Let $(f_s/\sim_\lambda)_{s \in S} \in (\mathbb{E}_s \in S e_s(C))_\lambda$. We define $\hat{f} \in C_\lambda$ as

$$\hat{f} := d \circ (f_s)_{s \in S} \circ (e_t \pi_{(\lambda, i)})(t, t) \in \lambda \times S.$$  

Then

$$\varphi_\lambda(\hat{f}) = (e_s d \circ (f_t)_{t \in S} \circ (e_t \pi_{(\lambda, i)})(t, t) \in \lambda \times S) \circ (d \circ (\pi_{(\lambda \times S, (i,u))})_{u \in S})_{i \in \lambda} = (e_s d \circ (e_t \pi_{(\lambda \times S, (i,u))})(t, t) \in \lambda \times S)_{i \in \lambda}$$

$$= (f_s \circ (e_t \pi_{(\lambda \times S, (i,u))})(t, t) \in \lambda \times S)_{i \in \lambda} = (f_s / \sim_\lambda)_{s \in S}.$$  

Finally, we prove $\varphi = (\varphi_\lambda)_{\lambda \in \kappa}$ is a homomorphism. Let $\lambda_1, \lambda_2 < \kappa$ and $f \in C_{\lambda_1}$, $g_i \in C_{\lambda_2}$ for $i \in \lambda_1$. Then

$$\varphi_{\lambda_2}(f \circ (g_i)_{i \in \lambda_1})$$

$$= (e_s f \circ (g_i)_{i \in \lambda_1} \circ (d \circ (\pi_{(\lambda_2 \times S, (j,t))}))_{j \in \lambda_2} / \sim_{\lambda_2})_{s \in S}$$

$$= (e_s f \circ (d \circ (g_i)_{u \in S}) \circ (d \circ (\pi_{(\lambda_2 \times S, (i,u))}))_{i \in \lambda_1} \circ (d \circ (\pi_{(\lambda \times S, (j,t))}))_{j \in \lambda_2} / \sim_{\lambda_2})_{s \in S}$$

$$= (e_s f \circ (d \circ (e_u g_i)_{u \in S}) \circ (d \circ (\pi_{(\lambda_2 \times S, (i,u))}))_{i \in \lambda_1} \circ (d \circ (\pi_{(\lambda \times S, (j,t))}))_{j \in \lambda_2} / \sim_{\lambda_2})_{s \in S}$$

$$= (e_s f \circ (d \circ (\pi_{(\lambda_1 \times S, (i,u))}))_{i \in \lambda_1} / \sim_{\lambda_1} \circ (e_u g_i) \circ (d \circ (\pi_{(\lambda_2 \times S, (j,t))}))_{j \in \lambda_2} / \sim_{\lambda_2})_{(i,u) \in \lambda_1 \times S}$$

Finally, we prove $\varphi = (\varphi_\lambda)_{\lambda \in \kappa}$ is a homomorphism. Let $\lambda_1, \lambda_2 < \kappa$ and $f \in C_{\lambda_1}$, $g_i \in C_{\lambda_2}$ for $i \in \lambda_1$. Then

$$\varphi_{\lambda_2}(f \circ (g_i)_{i \in \lambda_1})$$

$$= (e_s f \circ (g_i)_{i \in \lambda_1} \circ (d \circ (\pi_{(\lambda_2 \times S, (j,t))}))_{j \in \lambda_2} / \sim_{\lambda_2})_{s \in S}$$

$$= (e_s f \circ (d \circ (g_i)_{u \in S}) \circ (d \circ (\pi_{(\lambda_2 \times S, (i,u))}))_{i \in \lambda_1} \circ (d \circ (\pi_{(\lambda \times S, (j,t))}))_{j \in \lambda_2} / \sim_{\lambda_2})_{s \in S}$$

$$= (e_s f \circ (d \circ (e_u g_i)_{u \in S}) \circ (d \circ (\pi_{(\lambda_2 \times S, (i,u))}))_{i \in \lambda_1} \circ (d \circ (\pi_{(\lambda \times S, (j,t))}))_{j \in \lambda_2} / \sim_{\lambda_2})_{s \in S}$$

$$= (e_s f \circ (d \circ (\pi_{(\lambda_1 \times S, (i,u))}))_{i \in \lambda_1} / \sim_{\lambda_1} \circ (e_u g_i) \circ (d \circ (\pi_{(\lambda_2 \times S, (j,t))}))_{j \in \lambda_2} / \sim_{\lambda_2})_{(i,u) \in \lambda_1 \times S}$$

$$= \varphi_{\lambda_1}(f) \circ (\varphi_{\lambda_2}(g_i))_{i \in \lambda_1}.$$  

$\square$
5 Heterogenization

Let $M$ be an $S$-sorted pure clone. Then the homogenization $H(M)$ has an $S$-ary diagonal pair. The converse, if a clone $C$ has an $S$-ary diagonal pair then there exists an $S$-sorted pure clone $M$ such that $H(M)$ is isomorphic to $C$, is also true. In this section, we explain the construction of the many-sorted clone $M$ from a single-sorted clone $C$. The isomorphism is proved in the next section.

Definition 5.1. Let $C$ be a single-sorted $<\kappa$-ary clone. Assume $C$ has an $S$-ary diagonal pair $(d, (e_s)_{s\in S})$. We define an $S$-sorted $<\kappa$-ary clone $C_{(d, (e_s)_{s\in S})}$ as follows:

- (Underlying set) For $\lambda < \kappa$, $v : \lambda \to S$ and $s \in S$, the underlying set of the sort $(\lambda, v, t)$ is
  \[
  C_{(d, (e_s)_{s\in S})}(\lambda, v, t) := \{ f \in C_{\lambda} \mid e_t f = f \}/\sim_{\lambda, v},
  \]
  where the equivalence $\sim_{\lambda, v}$ is defined as
  \[
  f \sim_{\lambda, v} g :\iff f \circ (e_{v(i)} \circ \pi_{(\lambda,i)})_{i\in \lambda} = g \circ (e_{v(i)} \circ \pi_{(\lambda,i)})_{i\in \lambda}.
  \]
- (Projection) For $\lambda < \kappa$, $v : \lambda \to S$ and $i \in \lambda$, the projection constant $\pi_{(\lambda, v, i)}$ is defined as $e_{v(i)} \circ \pi_{(\lambda,i)} /\sim_{\lambda, v}$.
- (Composition) For $\lambda_k < \kappa$, $v : \lambda_k \to S$ ($k = 1, 2$) and $s \in S$, the composition operation
  \[
  c_{(\lambda_1, v_1, s), (\lambda_2, v_2)} : C_{(d, (e_s)_{s\in S})}(\lambda_1, v_1, s) \times \prod_{i\in \lambda_1} C_{(d, (e_s)_{s\in S})}(\lambda_2, v_2, v_1(i)) \to C_{(d, (e_s)_{s\in S})}(\lambda_2, v_2, s)
  \]
  is defined by $(f /\sim_{\lambda_1, v_1}, (g_i /\sim_{\lambda_2, v_2})_{i\in \lambda_1}) \mapsto (f \circ (g_i)_{i\in \lambda_1}) /\sim_{\lambda_2, v_2}$.

Definition 5.2. Let $C$ be a $<\kappa$-ary clone that has a diagonal pair $(d, (e_s)_{s\in S})$. Let $A$ be a $C$-algebra. We define a $C_{(d, (e_s)_{s\in S})}$-algebra $A_{(d, (e_s)_{s\in S})}$ as follows:

- (Underlying set) The underlying set $A_{(d, (e_s)_{s\in S})}$ of the sort $s \in S$ is $e_s(A)$.
- (Action of $C_{(d, (e_s)_{s\in S})}$) A term $f /\sim_{\lambda, v} \in C_{(d, (e_s)_{s\in S})}(\lambda, v, s)$ acts as
  \[
  \prod_{i\in \lambda} A_{(d, (e_s)_{s\in S})}(v(i)) \ni (a_i)_{i\in \lambda} \mapsto f(a_i)_{i\in \lambda} \in A_{(d, (e_s)_{s\in S})}(s).
  \]

Remark 5.3. In the Definition 5.2, $g := f \circ (e_{v(i)} \circ \pi_{(\lambda,i)})_{i\in \lambda} /\sim_{\lambda, v}$ $f$ holds for $f \in C_{\lambda}$. Moreover $g$ satisfies $g \circ (e_{v(i)} \circ \pi_{(\lambda,i)})_{i\in \lambda} = g$. Thus for any element $\tilde{f}$ of $C_{(d, (e_s)_{s\in S})}(\lambda, v, t)$, we can choose a term $g \in C_{\lambda}$ that satisfies $g \circ (e_{v(i)} \circ \pi_{(\lambda,i)})_{i\in \lambda} = \tilde{f}$ as a representative for $\tilde{f}$.

The structure of the $S$-sorted clone $C_{(d, (e_s)_{s\in S})}$ and the algebra $A_{(d, (e_s)_{s\in S})}$ defined above do not depend on the family of projection idempotents $(e_s)_{s\in S}$. Next, we prove this fact. We start from the following lemma.

Lemma 5.4. Let $C$ be a single-sorted $<\kappa$-ary clone and $(d, (e_s)_{s\in S})$, $(d, (e_s')_{s\in S})$ be diagonal pairs of $C$. Then $e_s e_s' = e_s$ holds for $s \in S$.

Proof.

\[
\begin{align*}
  e_s e_s' &= e_s d(e'_s \pi_{(\{0\}, 0)})_{t\in S} = e_s d(\pi_{(\{0\}, 0)})_{t\in S} = e_s.
\end{align*}
\]

Proposition 5.5. Let $C$ be a $<\kappa$-ary clone and $(d, (e_s)_{s\in S})$, $(d, (e'_s)_{s\in S})$ be diagonal pairs of $C$. Then $C_{(d, (e_s)_{s\in S})}$ is isomorphic to $C_{(d, (e'_s)_{s\in S})}$ via an isomorphism $\varphi = (\varphi_{(\lambda, v, s)})$ where

\[

\varphi_{(\lambda, v, s)} : C_{(d, (e_s)_{s\in S})}(\lambda, v, s) \ni f /\sim \mapsto e'_s f(e_{v(i)} \pi_{(\lambda,i)})_{i\in \lambda} /\sim' \in C_{(d, (e'_s)_{s\in S})}(\lambda, v, s).

\]

Here, $\sim$ and $\sim'$ are equivalences defined as

\[
\begin{align*}
  f \sim g &\iff f \circ (e_{v(i)} \circ \pi_{(\lambda,i)})_{i\in \lambda} = g \circ (e_{v(i)} \circ \pi_{(\lambda,i)})_{i\in \lambda}, \\
  f \sim' g &\iff f \circ (e'_{v(i)} \circ \pi_{(\lambda,i)})_{i\in \lambda} = g \circ (e'_{v(i)} \circ \pi_{(\lambda,i)})_{i\in \lambda}.
\end{align*}
\]
This equation means

By Lemma 5.4, Proposition 5.6.

Proof. Well-definedness of \( \varphi(\lambda,v,s) \) directly follows from the definition.
Let \( \psi(\lambda,v,s) : C(d,(e_s')_{s \in S}),((\lambda,v,s)) \to C(d,(e_s')_{s \in S}),((\lambda,v,s)) \) be the mapping \( g/\sim \mapsto e_s g(e_s' \pi(\lambda,i)) \in \lambda/\sim \)
Then by the previous lemma,

\[
\psi(\lambda,v,s)(f/\sim) = \left( e_s e' f(e_s(i) \pi(\lambda,i)) \right) /\sim \\
= (e_s f(e_s(i) \pi(\lambda,i)) \in \lambda /\sim \\
= f/\sim.
\]

By the same way, \( \varphi(\lambda,v,s) \circ \psi(\lambda,v,s) = \text{id}_{C(d,(e_s')_{s \in S}),((\lambda,v,s))} \) holds. Thus \( \psi(\lambda,v,s) \) is the inverse of \( \varphi(\lambda,v,s) \).

We prove \( \varphi \) is compatible with \( S \)-sorted clone operations. Let \( \pi(\lambda,v,i) = e_v(i) \pi(\lambda,i) /\sim \) be the \( (\lambda,v) \)-ary \( i \)-th projection of \( C(d,(e_s)_{s \in S}) \).

Then \( \varphi(\lambda,v,i)(\pi(\lambda,v,i)) = (e_v(i) e_v(j) \pi(\lambda,j) /\sim) /\sim' \)
by the previous lemma. It is the \( (\lambda,v) \)-ary \( i \)-th projection of \( C(d,(e_s)_{s \in S}) \).

Next, let \( \lambda_k < \kappa, v_k : \lambda_k \to S \) \((k = 1,2, s \in S) \)
and

\[
f/\sim \in C(d,(e_s)_{s \in S}),((\lambda_1,v_1,s)),\psi f/\sim \in C(d,(e_s')_{s \in S}),((\lambda_2,v_1,v_2,i))
\]
for \( i \in \lambda_1 \). Then

\[
\varphi(\lambda_1,v_1,i)(f/\sim) \circ (\varphi(\lambda_2,v_2,v_2(i)) (g_l/\sim)) \in \lambda_1 \\
= (e_v(i) e_v(j) \pi(\lambda,i) /\sim) \circ (e_v(i) e_v(j) \pi(\lambda,j) /\sim) /\sim /\sim' \\
= (e_v(i) e_v(j) \pi(\lambda,j) /\sim) /\sim' \\
= (e_v(i) e_v(j) \pi(\lambda,j) /\sim) /\sim' \\
= \varphi(\lambda_2,v_2,s) (f \circ (g_l) /\sim) /\sim' \text{ (Lemma 5.4)}
\]

Under the identification by this isomorphism, the \( S \)-sorted algebras by \( (d,(e_s)_{s \in S}) \) and \( (d,(e_s')_{s \in S}) \)
are isomorphic.

Proposition 5.6. Let \( C \) be a single-sorted \( \kappa \)-ary clone, \( (d,(e_s)_{s \in S}) \), \( (d,(e_s')_{s \in S}) \) be diagonal pairs of \( C \) and \( A = (A_s)_{s \in S} \) be a \( C \)-algebra. Then \( A(d,(e_s)_{s \in S}) \) and \( A(d,(e_s')_{s \in S}) \)
are isomorphic via \( \varphi = (\varphi_t)_{t \in S} \)
where

\[
\varphi_t : A(d,(e_s)_{s \in S}),t \ni a \mapsto e_t(a) \in A(d,(e_s')_{s \in S}),t.
\]

Proof. By Lemma 5.4

\[
a = e_t e_t(a), a' = e_t e_t'(a')
\]
hold for \( a \in A(d,(e_s)_{s \in S}),t \) and \( a' \in A(d,(e_s')_{s \in S}),t \). Thus \( a' \mapsto e_t(a') \) is the inverse of \( \varphi_t \) and \( \varphi_t \) is bijective.

Let \( f/\sim \in C(d,(e_s)_{s \in S}),((\lambda,v,t)) \) and \( a_t \in A(d,(e_s)_{s \in S}),((\lambda,v,t)) \).

Then \( \varphi_t(f/\sim)(a_t) = e_t f(a_t) /\sim \)
\[
= e_t f(e_v(i)(a_t)) /\sim \\
= e_t f(e_v(i) e_v'(i)(a_t)) /\sim \\
= e_t f(e_v(i) \pi(\lambda,i)(a_t)) /\sim \\
= \varphi(\lambda,v,t)(f/\sim)(\varphi_t(a_t)) /\sim.
\]

This equation means \( \varphi = (\varphi_t)_{t \in S} \) is a homomorphism \( A(d,(e_s)_{s \in S}) \) to \( A(d,(e_s')_{s \in S}) \).

\[
\square
\]
In the following paragraph, we omit projection idempotents for writing heterogenization. Namely, we simply write $C_d$ the $S$-sorted clone $C_{d,(e_s)_{s \in S}}$.

We end this section to prove the correspondence $A \mapsto A_d$ above is extended to a functor.

**Proposition 5.7.** Let $C$ be a single-sorted $\kappa$-ary clone, $(d, (e_s)_{s \in S})$ be a diagonal pair of $C$, $\varphi : A \to B$ be a homomorphism of $C$-algebras. Then $\varphi : A_d \to B_d$ defined as $\varphi_{d,s}(a) := \varphi(a)$ for $s \in S, a \in A_d$ is a homomorphism of $C_d$-algebras. Further, the correspondence $d (A \mapsto A_d$ for $A \in \mathcal{V}(C)$ and $\varphi \mapsto \varphi_d$ for morphisms of $C$-algebras) is a functor $\text{Cat}(\mathcal{V}(C)) \to \text{Cat}(\mathcal{V}(C_d))$.

**Proof.** At first, notice that $a \in A_{d,s} = e_s(A)$ implies $\varphi(a) = \varphi(e_s(a)) = e_s(\varphi(a)) \in e_s(B) = B_s$. Thus $\varphi_{d,s}$ is actually a mapping $A_s \to B_s$.

Let $f \in C_{d,((\lambda,v,s))}, a_t \in A_{d,v(t)}$. Then

$$\varphi_{d,s}(f/\sim)(a_t)_{i \in \lambda} = \varphi f(a_t)_{i \in \lambda} = f(\varphi(a_t))_{i \in \lambda} = (f/\sim)(\varphi_{d,s}(a_t))_{i \in \lambda}.$$ 

This equation means nothing but $\varphi_d$ is a homomorphism.

It is easily verified that $d$ preserves identity and composition, thus $d$ is a functor. $\square$

6 Concrete equivalence

The aim of this section is proving that the functors described in Proposition 3.5 and 5.7 are mutually inverse categorical equivalence.

**Proposition 6.1.** Let $C$ be a single-sorted $\kappa$-ary clone that has a diagonal pair $(d, (e_s)_{s \in S})$. Then $H(C_d)$ and $C$ are isomorphic via an isomorphism $\nu = \{\nu_\lambda\}_{\lambda \in \kappa}$

$$\nu_\lambda : C_\lambda \ni f \mapsto (e_s f \circ (d \circ \pi_{(\lambda \times S,(i,t))})_{i \in \lambda})_{s \in S} \in H(C_d),$$

where $\sim$ is the following equivalence relation on $C_{\lambda \times S}$:

$$f_s \sim g_s \iff f_s \circ (e_t \circ \pi_{(\lambda \times S,(i,t))})_{i \in \lambda \times S} = g_s \circ (e_t \circ \pi_{(\lambda \times S,(i,t))})_{i \in \lambda \times S}.$$ 

**Proof.** (Injectivity) Let $f, g \in C_\lambda$ and assume $\nu_\lambda(f) = \nu_\lambda(g)$, namely

$$e_s f \circ (d \circ \pi_{(\lambda \times S,(i,t))})_{i \in \lambda} \circ (e_t \circ \pi_{(\lambda \times S,(i,t))})_{i \in \lambda \times S}$$

$$= e_s g \circ (d \circ \pi_{(\lambda \times S,(i,t))})_{i \in \lambda} \circ (e_t \circ \pi_{(\lambda \times S,(i,t))})_{i \in \lambda \times S}$$

holds for all $s \in S$. Then

$$f = d \circ (e_s f)_{s \in S} = d \circ (e_s f)_{s \in S} \circ (d \circ \pi_{(\lambda,i)})_{i \in \lambda}$$

$$= d \circ (e_s f)_{s \in S} \circ (d \circ \pi_{(\lambda,i)})_{i \in \lambda} \circ (e_t \circ \pi_{(\lambda,i)})_{i \times S}$$

$$= d \circ (e_s f)_{s \in S} \circ (d \circ \pi_{(\lambda,i)})_{i \in \lambda} \circ (e_t \circ \pi_{(\lambda,i)})_{i \times S}$$

$$= d \circ (e_s g)_{s \in S} \circ (d \circ \pi_{(\lambda,i)})_{i \in \lambda} \circ (e_t \circ \pi_{(\lambda,i)})_{i \times S}$$

$$= g.$$

Thus, $\nu_\lambda$ is injective.

(Surjectivity) Suppose $f \in H(C_d)$. By definition of $H(C_d)$, $f$ is described as $f = (f_s/\sim)_{s \in S}$ where $f_s \in C_{\lambda \times S}$ and each $f_s$ satisfies $e_s f_s = f_s$. Let us define

$$\tilde{f} := d \circ (f_s)_{s \in S} \circ (e_t \pi_{(\lambda,i)})_{i \in \lambda \times S}.$$
Then $\nu_{\lambda}(\tilde{f})$ is calculated as follows:

$$
\nu_{\lambda}(\tilde{f}) = (e, f \circ (d \circ (\pi_{(\lambda \times S,(i,u)\,)}))_s \in \lambda)_{s \in S}
\quad = (e, d \circ (\nu_{\lambda}) \circ (d \circ (\pi_{(\lambda \times S,(i,u)\,)}))_s \in \lambda)_{s \in S}
\quad = (f \circ (e, \pi_{(\lambda \times S,(i,u)\,)}))_s \in \lambda \sim s_{\in S}
\quad = (f \circ (\nu_{\lambda \times S,(i,u)\,}))_s \in \lambda \sim s_{\in S}
\quad = f.$$

(Compatibility) Let $f \in C_{\lambda_1}$, $g_i \in C_{\lambda_2}$ $(i \in \lambda_1)$. Then

$$
\nu_{\lambda_2}(f \circ (g_i)_{i \in \lambda_1})
\quad = (e, f \circ (g_i)_{i \in \lambda_1} \circ (d \circ (\pi_{(\lambda_2 \times S,(j,t)\,)}))_s \in \lambda_2)_{s \in S}
\quad = (e, f \circ (d \circ (\pi_{(\lambda_2 \times S,(j,t)\,)}))_s \in \lambda_2 \sim s_{\in S}
\quad = (e, f \circ (d \circ (\pi_{(\lambda_2 \times S,(j,t)\,)}))_s \in \lambda_2 \sim s_{\in S}
\quad = \nu_{\lambda_1}(f) \circ (\nu_{\lambda_2}(g_i))_{i \in \lambda_1}.
$$

By the isomorphism $\nu$ in this proposition, an $H(C_d)$-algebra naturally has $C$-algebra structure. The following isomorphism exists through this identification.

**Proposition 6.2.** Let $C$ be a $\kappa$-ary clone with a diagonal pair $(d, (e_s)_{s \in S})$. Let $A$ be a $C$-algebra. Then $\nu_A : A \ni a \mapsto (e_s(a))_{s \in S} \in H(C_d)$ is an isomorphism of $C$-algebras. Furthermore, $(\nu_A)_{A \in \mathcal{V}(C)}$ is a natural transformation from id$_{\mathcal{C}(\mathcal{V}(C))}$ to the functor $[A \mapsto H(A_d)]$.

**Proof.** By definition of diagonal pair, $(a_s)_{s \in S} \mapsto d(a_s)_{s \in S}$ is the inverse mapping of $\nu_A$. Thus $\nu_A$ is bijective.

We prove $\nu_A$ is a homomorphism of $C$-algebras. Let $f \in C_{\lambda}$, $\lambda < \kappa$ and $(a_i)_{i \in \lambda} \in A^\lambda$. Then

$$
\nu_{\lambda_2}(f)\nu_{\lambda_2}(a_i)_{i \in \lambda} = (e, f \circ (d \circ (e_t(a_i)))_s \in \lambda)_{s \in S}
\quad = (e, f \circ (a_i)_{i \in \lambda})_{s \in S}
\quad = \nu_{\lambda_2}(f \circ (a_i)_{i \in \lambda}).
$$

This equation is nothing but the property we should prove.

Next, we prove naturalness of $(\nu_A)_{A \in \mathcal{V}(C)}$. Let $A, B \in \mathcal{V}(C)$ and $\varphi : A \to B$ be a homomorphism. We should the diagram

$$
A \quad \overset{\nu_A}{\longrightarrow} \quad H(A_d)
\quad \downarrow \varphi
\quad \overset{\nu_B}{\longrightarrow} \quad H(B_d)
B
$$

commute. For $a \in A$,

$$
H(\varphi_d)(\nu_A(a)) = H(\varphi_d)(e_s(a))_{s \in S}
\quad = (\varphi(e_s(a)))_{s \in S}
\quad = (e_s(\varphi(a)))_{s \in S}
\quad = \nu_B(\varphi(a))
$$

holds. It means $(\nu_A)_{A \in \mathcal{V}(C)}$ is a natural transformation.
We complete to prove that the variety corresponding to the clone $H(C_d)$ is definitionally equivalent to the variety corresponding to $C$ via the natural equivalence $(\nu_A)_{A \in V}$. Next, we prove a natural equivalence of the converse direction.

**Proposition 6.3.** Let $M$ be an $S$-sorted $<\kappa$-ary pure clone. Let $d \in H_S(M)$ and $e_s \in H_1(M)$ ($s \in S$) be terms that satisfy the following properties:

- The term $d$ is defined as $d := (\pi_{(S \times S,(s,s))})_{s \in S}$.
- Each $e_s$ can be written as in the form $e_s = (e_{s,t} \pi_{(S, id_s)})_{t \in S}$, where $e_{s,t} \in M_{s \rightarrow t}$ and satisfies $e_{s,s} = \pi_{(s, id_s)}$.

Then $(d, (e_s)_{s \in S})$ is a diagonal pair of $(H(M))_d$ isomorphic to $M$ via an isomorphism

$$\mu_{(\lambda, v, a)} : M(\lambda, v, a) \ni f \mapsto (e_{s,t})_{t \in S} \circ f \circ (\pi_{(\lambda \times S, p_2, (i,v(i)))})_{t \in \lambda \sim} \in (H(M))_d(\lambda, v, a),$$

where $\sim$ is the equivalence

$$\begin{align*}
(f_s)_{s \in S} &\sim (g_s)_{s \in S} \iff (f_s)_{s \in S} \circ \pi_{(S \times S, (i,j))} = (g_s)_{s \in S} \circ \pi_{(S \times S, (i,j))} \forall s \in S; \quad (f_s \circ \pi_{(\lambda \times S, p_2, (i,v(i)))})_{t \in \lambda \times S} = (g_s \circ \pi_{\lambda \times S, p_2, (i,v(i)))})_{t \in \lambda \times S},
\end{align*}$$

where $\pi_{H(M)}$ denotes the projection constant of $H(M)$.

**Proof.** First, we prove $(d, (e_s)_{s \in S})$ is a diagonal pair of $(H(M))_d$.

1. $e_s \circ d = e_d = e_s \circ (\pi_{(S, s\times S, p_2, (s,t))})_{t \in S} = e_s \circ (\pi_{(S, S\times S, p_2, (s,t))})_{t \in S} = e_s \circ \pi_{H(M)}$.

Here and through the proof of this proposition, $\pi_M$ and $\pi_{H(M)}$ denote projection constants of $M$ and $H(M)$ respectively.

2. $d \circ (e_s \circ \pi_{(S, s)}) = d$ is proved as follows:

$$\begin{align*}
d(e_s \circ \pi_{(S, s)})_{s \in S} &= (\pi_{(S \times S, p_2, (s, s))})_{s \in S} \circ (e_s \circ \pi_{(S, S, p_2, (s, t))})_{t \in S} \in S \\
&= (e_s \circ \pi_{(S, S, p_2, (s, t))})_{t \in S} \in S \\
&= (\pi_{(S \times S, p_2, (s, s))})_{s \in S} \\
&= d.
\end{align*}$$

3. $d \circ (\pi_{(S, 0)})_{s \in S} = \pi_{(S, 0)}$ is proved as follows:

$$\begin{align*}
d \circ (\pi_{(S, 0)})_{s \in S} &= (\pi_{(S \times S, p_2, (s, 0))})_{s \in S} \circ (\pi_{(S \times S, p_2, (0, t))})_{t \in S} \in S \\
&= (\pi_{(S \times S, p_2, (0, t))})_{t \in S} \in S \\
&= \pi_{(S, 0)}.
\end{align*}$$

We complete the proof that $(d, (e_s)_{s \in S})$ is a diagonal pair of $(H(M))_d$.

(Injectivity of $\mu_{(\lambda, v, a)}$) Let $f, g \in M(\lambda, v, a)$ and assume $\mu_{(\lambda, v, a)}(f) = \mu_{(\lambda, v, a)}(g)$, namely,

$$\begin{align*}
e_{s,t} f \circ (\pi^M_{(\lambda \times S, p_2, (i,v(i)))})_{i \in \lambda} \circ (e_{v(i), u} \pi_{(\lambda \times S, p_2, (i,v(i)))})_{i \in \lambda \times S} \\
&= e_{s,t} g \circ (\pi^M_{(\lambda \times S, p_2, (i,v(i)))})_{i \in \lambda} \circ (e_{v(i), u} \pi_{(\lambda \times S, p_2, (i,v(i)))})_{i \in \lambda \times S}
\end{align*}$$

hold for all $t \in S$. Notice that the following two equations hold: The first equation is

$$\begin{align*}
\pi^M_{(\lambda, v, v(i))} &= e_{v(i), v(i)} \circ \pi^M_{(\lambda, v, v(i))} \\
&= \pi^M_{(\lambda \times S, p_2, (i,v(i)))} \circ (e_{v(j), u} \circ \pi^M_{(\lambda, v, v(j))})_{i \in \lambda \times S}
\end{align*}$$
The second equation is
\[ f = \operatorname{id}_A \circ f \circ (\pi^M_{(\lambda,v,v(i))})i\in\lambda. \]

By these equations injectivity of \( \mu_{(\lambda,v,v)} \) is proved as follows:

\[
\begin{align*}
\mu &= \operatorname{id}_A \circ f \circ (\pi^M_{(\lambda,v,v(i))})i\in\lambda \\
&= e_{s,t} \pi^M_{(\lambda,S,p_2(i,v(i)))}f_{\lambda}(e_{s,t}) \in S \circ f_{\lambda} \circ (\pi^M_{(\lambda,v,v(i))})i\in\lambda \\
&= e_{s,t} \pi^M_{(\lambda,S,p_2(i,v(i)))}f_{\lambda}(e_{s,t}) \in S \circ g_{\lambda} \circ (\pi^M_{(\lambda,S,p_2,i,v(i)))}i\in\lambda \\
&= g.
\end{align*}
\]

(Surjectivity of \( \mu_{(\lambda,v,v)} \)) Let \( f/\sim \in (H(M))_{d,(\lambda,v,v)} \). Define \( \tilde{f} \) by \( \tilde{f} = \pi^M_{(\lambda,S,p_2(i,v(i)))}f_{\lambda} \circ (\pi^M_{(\lambda,v,v(i))}i,u) \in \lambda \times S \). Then the following formula holds.

\[
\begin{align*}
\mu_{(\lambda,v,v)}(\tilde{f}) &= (e_{s,t})_{\lambda} \in S \circ \tilde{f} \circ (\pi^M_{(\lambda,S,p_2,i,v(i)))}i\in\lambda) \in S \\
&= (e_{s,t})_{\lambda} \in S \circ \pi^M_{(\lambda,S,p_2,j,v(i)))}f_{\lambda}(e_{s,t}) \in S \circ (\pi^M_{(\lambda,v,v(i))}i,u) \in \lambda \times S \\
&= (f \circ (e_{v(i),u} \circ (\pi^M_{(\lambda,v,v(i))}i,u) \in \lambda \times S) \in S \\
&= f/\sim.
\end{align*}
\]

(Compatibility of \( \mu \)) Let \( f \in M_{(\lambda,v,v,i)} \) and \( g_i \in M_{(\lambda,v,v,i)} \) for \( i \in \lambda_1 \). Then

\[
\begin{align*}
\mu_{(\lambda_2,v,v,i)}(f \circ (g_i)_{\lambda_1}) &= \mu_{(\lambda_2,v,v,i)}(f \circ (g_i)_{\lambda_1}) \\
&= (e_{s,t})_{\lambda_1} \in S \circ f \circ (g_i)_{\lambda_1} \circ (\pi^M_{(\lambda_2,v,v(j))}j)_{\lambda_2} \\
&= (e_{s,t})_{\lambda_1} \in S \circ f \circ (e_{v_1(i),v_1(i)}g_i)_{\lambda_1} \circ (\pi^M_{(\lambda_2,v,v(j))}j)_{\lambda_2} \\
&= (e_{s,t})_{\lambda_1} \in S \circ f \circ (\pi^M_{(\lambda_1,v,v(j))}j)_{\lambda_1} \circ (e_{v_1(i),u}(i,u) \in \lambda_1 \times S \circ (g_i)_{\lambda_1} \circ (\pi^M_{(\lambda_2,v,v(j))}j)_{\lambda_2} \\
&= \mu_{(\lambda_1,v,v)}(f) \circ (\mu_{(\lambda_2,v,v,i)}(g_i))_{\lambda_1}.
\end{align*}
\]

We complete the proof of the proposition. \( \square \)

By this proposition, an \((H(M))_d\)-algebra is identified with an \( M \)-algebra. Under this identification, the following holds.

**Proposition 6.4.** Let \( M, d, e, e_s, e_t \) be as in the previous proposition, \( A = (A_s)_{s\in S} \) be an \( M \)-algebra. Then \((H(A))_d\) is isomorphic to \( A \) via an isomorphism \( \mu_A \), where

\[
\mu_{A,s} : A_s \ni a \mapsto (e_{s,t}(a))_{t\in S} \in (H(A))_{d,s}.
\]

Moreover, \( (\mu_A)_{A\in V(M)} \) is a natural transformation \( \operatorname{id}_{\operatorname{Cat}(V(M))} \) to \([A \mapsto (H(A))_d]\).

**Proof.** Let \( f \in M_{(\lambda,v,v)} \) and \( a_i \in A_{v(i)} \) for \( i \in \lambda \). Then

\[
\begin{align*}
\mu_{(\lambda,v,v)}(f)(\mu_{A,v(v)}(a_i))_{i\in\lambda} &= (e_{s,t})_{\lambda} \in S \circ f \circ (\pi^M_{(\lambda,v,v(i))}(e_{s,t})) \in \lambda \times S \\
&= (e_{s,t})_{\lambda} \in S \circ f(e_{v(i),v(i)}(a_i)) \in \lambda \\
&= (e_{s,t})_{\lambda} \in S \circ f(a_i) \in \lambda \\
&= \mu_{A,s}(f(a_i))_{\lambda}.
\end{align*}
\]

This equation is nothing but the proposition stated.
We prove \((\mu_A)_{A \in V(M)}\) is a natural transformation. Let \(A = (A_s)_{s \in S}, B = (B_s)_{s \in S} \in V(M)\) and \(\varphi = (\varphi_s)_{s \in S}\) be a homomorphism \(A \to B\). We prove should the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\mu_A} & (H(A))_d \\
\varphi \downarrow & & \downarrow (H(\varphi))_d \\
B & \xrightarrow{\mu_B} & (H(B))_d
\end{array}
\]

commute. For \(s \in S\) and \(a \in A_s\), the equation

\[
\mu_{B,s}(\varphi_s(a)) = (\epsilon_{s,t}(\varphi(t,s)))_{t \in S} \\
= H(\varphi)(\epsilon_{s,t}(a))_{t \in S} \\
= (H(\varphi))_{d,s}(\mu_{A,s}(a))
\]

holds. This is nothing but \(\mu_B \circ \varphi = (H(\varphi))_d \circ \mu_A\), thus it is proved that \((\mu_A)_{A \in V(M)}\) is a natural transformation.

By Proposition 6.1 and 6.3, there is natural correspondence between many-sorted pure clones and single-sorted clones with diagonal operations (that can be extended to diagonal pair). Proposition 3.5, 5.7, 6.2, and 6.4, many-sorted varieties are essentially equivalent to single-sorted varieties that have diagonal pairs. We state these facts explicitly.

**Theorem 6.5.** Let \(\kappa\) be infinite cardinal and \(S\) be a cardinal that \(0 < S < \kappa\). Then \((C, d) \mapsto C_d\) is one to one correspondence between (isomorphism classes of) single-sorted \(<\kappa\text{-ary}\) clones with \(S\text{-ary}\) diagonal operations and (isomorphism classes of) \(S\text{-sorted}\) \(<\kappa\text{-ary}\) pure clones. The converse correspondence is given by \(M \mapsto (H(M), (\pi_{S \times S, p_2, (s, s)})_{s \in S})\).

Here, a \(<\kappa\text{-ary}\) clone with an \(S\text{-ary}\) diagonal operation means a tuple \((C, d)\) that \(C\) is a \(<\kappa\text{-ary}\) clone, \(d \in C_S\) and there exists \((\epsilon_s)_{s \in S}\) such that \((d, (\epsilon_s)_{s \in S})\) is a diagonal pair of \(C\). Clones with diagonal operations \((C, d), (C', d')\) are said to be isomorphic if there is an clone isomorphism \(\varphi : C \to C'\) such that \(\varphi(d) = d'\).

**Theorem 6.6.** Let \(\kappa\) be infinite cardinal, \(S\) be a cardinal that \(0 < S < \kappa\), \(M\) be an \(S\text{-sorted}\) \(<\kappa\text{-ary}\) clone. Then there exists a categorical equivalence \(\nu : V(M) \to V(H(M))\) such that the following diagram commute.

\[
\begin{array}{ccc}
V(M) & \xrightarrow{\nu} & V(H(M)) \\
F \downarrow & & \downarrow G \\
Set^S & \xrightarrow{\varphi} & DS
\end{array}
\]

The vertical arrow \(F\) is the forgetful functors attaining the underlying family of sets. \(G\) is the forgetful functor determined by the clone homomorphism \(d \in C_{D_S}\) maps to \((\pi_{S \times S, p_2, (s, s)})_{s \in S}\). The lower horizontal arrow \(\varphi\) is the functor described in Proposition 2.10.

## 7 Applications

By the correspondence between many-sorted algebras and single-sorted algebras, many results on single-sorted algebras are generalized to many-sorted algebras. At the end of this paper, we show few such examples.

### 7.1 Variety theorem

The first example is a many-sorted version of Birkhoff’s characterization of varieties. (See also [1]. Another version of many-sorted variety theorem.) We start from compatibility of homogenization and generating operators homomorphic images \(H\), subalgebras \(S\) and direct products \(P\) of variety.
Definition 7.1. Let $A = (A_s)_{s \in S}$ be an $S$-sorted algebra.

1. A subalgebra of $A$ is an $S$-sorted family $B = (B_s)_{s \in S}$ that satisfies the following conditions:
   - $B_s \subseteq A_s$ for all $s \in S$.
   - For $(\lambda, v)$-ary $s$-valued term operation $f$ of $A$ and $(b_i)_{i \in \lambda} \in \prod_{i \in \lambda} B_{v(i)}$, $f(b_i)_{i \in \lambda} \in B_s$ holds.

2. A congruence of $A$ is a subalgebra $(\theta_s)_{s \in S}$ of $A^2 = (A_s^2)_{s \in S}$ such that $\theta_s$ is an equivalence relation on $A_s$ for each $s \in S$. The set of all congruences of $A$ is denoted by $\text{Con}(A)$.

Proposition 7.2. Let $M$ be an $S$-sorted clone, $A$ be an $M$-algebra.

1. If $B = (B_s)_{s \in S} \subseteq \text{Sub}(A)$, then $H(B) := \prod_{s \in S} B_s \in \text{Sub}(H(A))$. Conversely, any subalgebra of $H(A)$ is described as in the form $H(B)$ by $B \in \text{Sub}(A)$.

2. If $\theta = (\theta_s)_{s \in S} \subseteq \text{Con}(A)$, then
   
   $$H(\theta) := \prod_{s \in S} \theta_s := \{(a_s)_{s \in S}, (b_s)_{s \in S} \in (H(A))^2 \mid \forall s \in S; (a_s, b_s) \in \theta_s \} \subseteq \text{Con}(H(A)).$$

   Conversely, any congruence of $H(A)$ is described as in the form $H(\theta)$ by a congruence $\theta \in \text{Con}(A)$.

3. If $\theta \in \text{Con}(A)$, then $H(A/\theta) \cong H(A)/H(\theta)$.

4. If $(A_i)_{i \in I}$ is a family of $M$-algebras, then
   
   $$\prod_{i \in I} H(A_i) \cong H \left( \prod_{i \in I} A_i \right).$$

Proof. We only show that any subalgebra of $H(A)$ is in the form $H(B)$ by a subalgebra $B$ of $A$. Other parts are easily verified or proved similarly.

If $H(A) = \emptyset$, then the assertion is trivial. Thus, we assume $A_s \neq \emptyset$ for all $s \in S$. Let $B'$ be a subalgebra of $H(A)$. Define

$$B_s := \{b \in A_s \mid \exists (a_t)_{t \in S \setminus \{s\}} \in \prod_{t \in S \setminus \{s\}} A_t; (b, (a_t)_{t \in S \setminus \{s\}}) \in B'\}.$$

Clearly, $B' \subseteq \prod_{s \in S} B_s$ holds. We show $\prod_{s \in S} B_s \subseteq B'$. Let $(b_s)_{s \in S} \in \prod_{s \in S} B_s$. By definition of $B_s$, there exist $(c_{st})_{t \in S} \in B'$ such that $c_{ss} = b_s$. Let $d = (\pi_{(s \times S, p_2, (s, a))})_{s \in S}$ be the diagonal term. Then

$$B' \ni d(c_{st})_{t \in S} = (c_{ss})_{s \in S} = (b_s)_{s \in S}.$$  \hfill \Box

Remark 7.3. If $M$ is a pure clone, then the underlying family of sets of any subalgebra of an $M$-algebra $A$ is pure. In such case, the correspondence $B \mapsto H(B)$ between $\text{Sub}(A)$ and $\text{Sub}(H(A))$ is bijective. On the other hand, this correspondence may not be bijective if $M$ is not pure. For a simple example, $S = 2 = \{0, 1\}$, $M$ be the trivial 2-sorted clone (that is, the 2-sorted clone only consists of projection constants) and $A = (A_i)_{i \in 2} = (A_0, A_1)$, then the both $(\emptyset, A_1)$ and $(A_0, \emptyset)$ of members of $\text{Sub}(A)$ correspond to $\emptyset = \emptyset \times A_1 = A_0 \times \emptyset$.

Theorem 7.4. Let $M$ be an $S$-sorted pure clone, $K$ be a class of $M$-algebras. Then the variety $\mathcal{V}(K)$ generated by $K$ coincides with $HSP(K)$.

Proof. We show

$$H(\mathcal{V}(K)) = \mathcal{V}(H(K)) = HSP(H(K)) = H(HSP(K)).$$

The second equality is Birkhoff’s variety theorem for classes of single-sorted algebras. The last equality follows from Proposition 7.2.
The first equality is proved as follows. For \( f, g \in H_\lambda(M) \), where \( f = (f_s)_{s \in S} \), \( g = (g_s)_{s \in S} \), the following equivalence holds:

\[
H(V(K)) \models f = g \iff \forall s \in S; \ V(K) \models f_s = g_s
\]

\[
\iff \forall s \in S; \ K \models f_s = g_s
\]

\[
\iff H(K) \models f = g
\]

\[
\iff V(H(K)) \models f = g.
\]

It means the equational theories defining \( V(H(K)) \) and \( H(V(K)) \) coincide with each other. Therefore, \( H(V(K)) = V(H(K)) \) holds. By \([1]\) and Theorem \([3]\), \( V(K) = HSP(K) \).

**Remark 7.5.** As noted in preliminary section, the above clone-based formulation includes type-based formulation as the case that \( M \) is a free clone. Thus, Theorem \([7,4]\) above is essentially equivalent to usual formulation of variety theorem.

### 7.2 Relational clone and categorical equivalence

Denecke and Lüders proved in \([3]\) that two finite single-sorted algebras are categorically equivalent if and only if their relational clones are isomorphic. In this subsection, we introduce the notion of a relational clone of a many-sorted algebra, and we show that a relational clone of an algebra such that the clone of its term operations is pure is isomorphic to the relational clone of its homogenization. As a consequence of this fact, we obtain a characterization of categorical equivalence by isomorphism relation of corresponding relational clones for many-sorted algebras.

**Definition 7.6.** Let \( A \) be an \( S \)-sorted \( <\kappa \)-ary algebra, \( \kappa \) be a cardinal. A \( <\kappa' \)-ary relational clone of \( A \) is the family \( Inv_{<\kappa'}(A) = (Inv_\mu(A))_{\mu < \kappa'} \), where \( Inv_\mu(A) = \{H(B) \mid B \in Sub(A)\} \), namely, \( Inv_\mu(A) \) is the set of all subsets \( r \) of \( (\prod_{s \in S} A_s)^{\mu} \) that satisfy

\[
f_s \in Cl_{\lambda \times S, \mu}(A), \lambda, (a_i, j) \in S \times \mu \in r \ (\text{for } i \in \lambda)
\]

\[
\implies (f_s(a_i, j), (i, j) \in \lambda \times S)(s, j) \in S \times \mu \in r.
\]

Here, \( Cl_{\lambda \times S, \mu}(A) \) is the set of all \( (\lambda, v) \)-ary \( s \)-valued term operations of \( A \). A member of \( Inv_\mu(A) \) is said an \( \mu \)-ary invariant set of \( A \).

**Remark 7.7.** We can consider the set of all closed subsets for component-wise operations. Here, a set \( r \subset \prod_{s \in \mu} A_{(s)} \) (\( \mu \) is a cardinal, \( v : \mu \to S \)) is said closed if the following condition holds:

\[
f_j \in Cl_{\lambda \times \mu, \mu}(A), \lambda, (a_i, j) \in \mu \in r \ (\text{for } i \in \lambda)
\]

\[
\implies (f_j(a_i, j), (i, j) \in \lambda \in \mu \in r.
\]

However, this condition considers only on operations related at most one sort. Thus, it should be said a concept of the non-indexed product of an one-sorted algebra, rather than a concept of a many-sorted algebra.

Next, we define the structure of relational clone and define the notion of isomorphism. As the single-sorted case, invariant sets are closed under primitive-positive definitions. It is easily verified as the same way as the single-sorted case.

**Proposition 7.8.** Let \( A \) be an \( S \)-sorted \( <\kappa \)-ary algebra and \( r_k \in Inv_{\mu_k}(A) \) for \( k \in K \). Assume a relation \( r \subset (\prod_{s \in S} A_s)^{\mu} \) is defined by the following form

\[
r = \left\{ (a_{s, j}^i)_{(s, j) \in S \times \mu} \in \left( \prod_{s \in S} A_s \right)^{\mu} \mid \exists (a_{s, j}^i)_{(s, j) \in S \times \nu} \in \left( \prod_{s \in S} A_s \right)^{\nu} \right.
\]

\[
\left. \left( \bigwedge_{u \in U} (a_{s, f(u), j})_{(s, j) \in \mu \cap \nu} \in r_u(u) \right) \right\},
\]

where \( g : U \to K \) and \( f : \bigcup_{u \in U} \{u\} \times \mu_g(u) \to \mu \cap \nu \). Then \( r \in Inv_\mu(A) \) holds. Here, \( \mu \cap \nu \) denotes the disjoint union of \( \mu \) and \( \nu \).
Definition 7.9. Let $\kappa'$, $S_A$ and $S_B$ be cardinals, $A$ and $B$ be $S_A$-sorted, $S_B$-sorted algebra respectively. A $<\kappa'$-ary relational clone homomorphism $\text{Inv}_{<\kappa'}(A) \rightarrow \text{Inv}_{<\kappa'}(B)$ is a family $(\varphi_\mu)$ of mappings that $\varphi_\mu : \text{Inv}_\mu(A) \rightarrow \text{Inv}_\mu(B)$ and preserves primitive-positive definitions, namely, if $r_\mu \in \text{Inv}_{\mu}(A)$ and
\[ r = \left\{ (a_{s,j'})_{(s,j') \in S \times \mu} \in \prod_{s \in S} A_s \mid \exists (a_{s,j})_{(s,j) \in S \times \nu} \in \prod_{s \in S} A_s \ ; \bigwedge_{u \in U} (a_{s,f(u)},j)_{(s,j) \in \mu(u)} \in r_g(u) \right\}, \]
then
\[ \varphi_\mu(r) = \left\{ (b_{s,j'})_{(s,j') \in S \times \mu} \in \prod_{s \in S} B_s \mid \exists (b_{s,j})_{(s,j) \in S \times \nu} \in \prod_{s \in S} B_s \ ; \bigwedge_{u \in U} (b_{s,f(u)},j)_{(s,j) \in \mu(u)} \in \varphi_\mu(u)(r_g(u)) \right\}. \]

The next is the main theorem about relational clones of many-sorted algebras. However, the proof is extremely easy; it directly follows from the definition of homogenization and relational clone.

Theorem 7.10. Let $M$ be an $S$-sorted $<\kappa$-ary pure clone, $A$ be an $M$-algebra, and $\kappa'$ be an arbitrary cardinal. Then the $<\kappa'$-ary relational clone $\text{Inv}_{<\kappa'}(A)$ is isomorphic to $\text{Inv}_{<\kappa'}(\text{H}(A))$.

Proof. By the definition of homogenization and relational clone,
\[ \text{Inv}_{\mu}(A) \ni r \mapsto \{(a_{j,s})_{s \in S} \mid (a_{j,s})_{(j,s) \in \mu \times S} \in r \} \in \text{Inv}_{\mu}(\text{H}(A)) \]
is an isomorphism of relational clone. Note that the bijectivity of this correspondence follows from pureness of $M$ (Remark 7.3).

Using the result on single-sorted algebras, we obtain a characterization of categorical equivalence of many-sorted algebras.

Corollary 7.11. Let $S_k$ be a non-zero cardinal, $\kappa$ be an infinite cardinal that $S_k < \kappa$ for $k = 1, 2$. Let $A_k$ be an $S_k$-sorted $<\kappa$-ary algebra that the clones of term operations are pure for $k = 1, 2$. Let $\kappa'$ be an infinite cardinal that satisfies
\[ \lambda < \kappa \implies |A_1|^\lambda, |A_2|^\lambda < \kappa', \]
where $|A_k|$ is the product cardinal $\prod_{s \in S_k} |A_{k,s}|$. Then the following assertions are equivalent.

1. There exists a categorical equivalence $\mathcal{V}(A_1) \rightarrow \mathcal{V}(A_2)$ that maps $A_1$ to $A_2$.

2. The relational clones $\text{Inv}_{<\kappa'}(A_1)$ is isomorphic to $\text{Inv}_{<\kappa'}(A_2)$.

Proof. This corollary follows from Theorem 6.6 and Theorem 7.10 and the following theorem.

Theorem 7.12 [5]. Let $\kappa$ be an infinite cardinal and $A_k$ a single-sorted $<\kappa$-ary algebra for $k = 1, 2$. Let $\kappa'$ be an infinite cardinal that satisfies $\lambda < \kappa \implies |A_1|^\lambda, |A_2|^\lambda < \kappa'$. Then the following assertions are equivalent.

1. There exists a categorical equivalence $\mathcal{V}(A_1) \rightarrow \mathcal{V}(A_2)$ that maps $A_1$ to $A_2$.

2. The relational clones $\text{Inv}_{<\kappa'}(A_1)$ is isomorphic to $\text{Inv}_{<\kappa'}(A_2)$.
7.3 Mal’cev type characterization

As a special case of invariant relations, there are bijective correspondences between subalgebras or congruences of a many-sorted algebra and subalgebras or congruences of its homogenization. This fact implies various many-sorted generalization of results on single-sorted algebras. As an example, we show characterization theorems of congruence properties by term existence conditions.

The first example is Mal’cev’s characterization of congruence permutability. To make precise, we start from a definition.

**Definition 7.13.** Let \( A \) be an \( S \)-sorted algebra.

1. Let \( \theta, \eta \subseteq A^2 \). A relational product \( \theta \circ \eta \) of \( \theta = (\theta_s)_s \in S \) and \( \eta = (\eta_s)_S \) is defined by sort-wise relational product \( (\theta_s \circ \eta_s)_s \in S \).

2. An algebra \( A \) is said congruence permutable if \( \theta \circ \eta = \eta \circ \theta \) for all \( \theta, \eta \in \text{Con}(A) \).

3. A class of algebras \( \mathcal{K} \) is said congruence permutable if all members of \( \mathcal{K} \) are congruence permutable.

**Theorem 7.14.** Let \( M \) be an \( S \)-sorted pure clone. Then the following conditions are equivalent.

1. \( \mathcal{V}(M) \) is congruence permutabe.
2. \( \mathcal{V}(H(M)) \) is congruence permutabe.
3. There exists \( p \in H_3(M) \), called Mal’cev term, that satisfies
   \[ p((\pi(3,0), \pi(3,0), \pi(3,1)), \pi(3,1), \pi(3,1)) = \pi(3,0), \]
   where 3 is a three elements set \( \{0, 1, 2\} \).
4. For each \( s \in S \), there exists \( p_s \in M_{(3 \times S, p_2, s)} \) such that
   \[ p_s((\pi(3 \times S, p_2, (0,t)))_{t \in S}, (\pi(3 \times S, p_2, (0,t)))_{t \in S}, (\pi(3 \times S, p_2, (1,t)))_{t \in S}) = \pi(3 \times S, p_2, (1,t)), \]
   \[ p_s((\pi(3 \times S, p_2, (0,t)))_{t \in S}, (\pi(3 \times S, p_2, (1,t)))_{t \in S}, (\pi(3 \times S, p_2, (1,t)))_{t \in S}) = \pi(3 \times S, p_2, (0,t)). \]
5. For each \( s \in S \), there exists \( p_s \in M_{3 S \rightarrow s} \) such that
   \[ p_s(\pi(3 S \rightarrow s, 0), \pi(3 S \rightarrow s, 0), \pi(3 S \rightarrow s, 1)) = \pi(3 S \rightarrow s, 1), \]
   \[ p_s(\pi(3 S \rightarrow s, 0), \pi(3 S \rightarrow s, 1), \pi(3 S \rightarrow s, 1)) = \pi(3 S \rightarrow s, 0). \]

**Proof.** 1 \( \iff \) 2 is an easy consequence of Proposition 7.2.

2 \( \iff \) 3 is well known Mal’cev’s theorem (See e.g. [2, Theorem 12.2]).

3 \( \iff \) 4 follows from the definition of homogenization.

5 \( \iff \) 4 is trivial.

1 \( \Rightarrow \) 5. Let \( F \) be a rank \((3 S S)_{t \in S}\) free algebra, namely, there are three elements \( a, b, c \) of sort \( s \) such that \( \{a, b, c\} \) freely generates \( F \). Let \( \theta \) and \( \eta \) be congruences of \( F \) generated by \( (a, b) \) and \( (b, c) \) respectively. Then \( (a, c) \in \theta \circ \eta = \eta \circ \theta \), namely, there is \( x \in F_s \) such that \( (a, x) \in \eta, (x, c) \in \theta \). Let \( p_s \) be the term corresponding to \( x \), then equations stated in Condition 5 hold.

**Corollary 7.15.** Let \( C \) be a \( <\kappa\)-ary single-sorted clone, \( (d, (e_s))_{s \in S} \) be a diagonal pair of \( C \). Then the following conditions are equivalent.

1. The variety \( \mathcal{V}(C) \) is congruence permutabe.
2. The variety \( \mathcal{V}(e_s(C)) \) is congruence permutabe for all \( s \in S \).

**Proof.** Condition 1 of this corollary is equivalent to that \( \mathcal{V}_{\ell} \) satisfy the condition 1 of the previous theorem.

Condition 2 of this corollary is equivalent to that \( \mathcal{V}_{\ell} \) satisfy the condition 5 of the previous theorem. Thus, these conditions are equivalent.
Similar results hold about congruence distributivity, modularity, etc. However, many results on Mal’cev condition that does not fixed the number (or the form of equation) depend on finiteness of arity. In such case, the corresponding generalized results are limited. Moreover, we can construct some counter examples of infinitary or infinitely many-sorted classes by using the correspondence of many-sorted and single-sorted algebras.

We only show a generalization of characterization of congruence distributivity.

**Theorem 7.16.** Let $S$ be a finite set, $M$ be an $S$-sorted finitary pure clone. Then the following conditions are equivalent:

1. $\mathcal{V}(M)$ is congruence distributive.
2. $\mathcal{V}(\mathcal{H}(M))$ is congruence distributive.

3. There are an integer $n \geq 0$ and $d_0, \ldots, d_{2n} \in H_3(M)$, we call Jónsson term of length $n$ in this paper, that satisfy the following equations:

   \[
   d_0 = \pi_{(3,0)}, d_{2n} = \pi_{(3,2)}
   \]

   \[
   d_{i-1}(\pi_{(3,0)}, \pi_{(3,1)}), \pi_{(3,0)}) = d_i(\pi_{(3,0)}, \pi_{(3,1)}), \pi_{(3,0)}) \quad (1 \leq i \leq 2n)
   \]

   \[
   d_{2i}(\pi_{(3,0)}, \pi_{(3,1)}), \pi_{(3,2)}) = d_{2i+1}(\pi_{(3,0)}, \pi_{(3,1)}), \pi_{(3,2)}) \quad (0 \leq i \leq n - 1)
   \]

   \[
   d_{2i-1}(\pi_{(3,0)}, \pi_{(3,1)}), \pi_{(3,2)}) = d_{2i}(\pi_{(3,0)}, \pi_{(3,2)}), \pi_{(3,2)}) \quad (1 \leq i \leq n)
   \]

4. There exists an integer $n \geq 0$ such that, for each $s \in S$, there exist $d_{s,i} \in M_{(3 \times S, p_2,s)} \ (0 \leq i \leq 2n)$ that satisfy the following equations:

   \[
   d_{s,0} = \pi_{(3 \times S, p_2,(0,s))}, d_{s,2n} = \pi_{(3 \times S, p_2,(2,s))},
   \]

   \[
   d_{s,i-1}((\pi_{(3 \times S, p_2,(0,t))})_{t \in S}, (\pi_{(3 \times S, p_2,(1,t))})_{t \in S}, (\pi_{(3 \times S, p_2,(0,t))})_{t \in S}) \quad (1 \leq i \leq 2n)
   \]

   \[
   d_{s,2i}((\pi_{(3 \times S, p_2,(0,t))})_{t \in S}, (\pi_{(3 \times S, p_2,(1,t))})_{t \in S}, (\pi_{(3 \times S, p_2,(2,t))})_{t \in S}) \quad (0 \leq i \leq n - 1)
   \]

   \[
   d_{s,2i-1}((\pi_{(3 \times S, p_2,(0,t))})_{t \in S}, (\pi_{(3 \times S, p_2,(1,t))})_{t \in S}, (\pi_{(3 \times S, p_2,(2,t))})_{t \in S}) \quad (1 \leq i \leq n)
   \]

5. For each $s \in S$, there exist an integer $n \geq 0$ and $d_{s,i} \in M_{(3 \times S, p_2,s)} \ (0 \leq i \leq 2n)$ that satisfy the same equations displayed in the previous condition.

6. For each $s \in S$, there exist an integer $n \geq 0$ and $d_{s,i} \in M_{(3 \times S, p_2,s)} \ (0 \leq i \leq 2n)$ that satisfy the following equations:

   \[
   d_{s,0} = \pi_{(3 \times S, p_2),(0,s)}, d_{s,2n} = \pi_{(3 \times S, p_2),(2,s)},
   \]

   \[
   d_{s,i-1}((\pi_{(3 \times S, p_2,(0,t))})_{t \in S}, (\pi_{(3 \times S, p_2,(1,t))})_{t \in S}, (\pi_{(3 \times S, p_2,(0,t))})_{t \in S}) \quad (1 \leq i \leq 2n)
   \]

   \[
   d_{s,2i}((\pi_{(3 \times S, p_2,(0,t))})_{t \in S}, (\pi_{(3 \times S, p_2,(1,t))})_{t \in S}, (\pi_{(3 \times S, p_2,(2,t))})_{t \in S}) \quad (0 \leq i \leq n - 1)
   \]

   \[
   d_{s,2i-1}((\pi_{(3 \times S, p_2,(0,t))})_{t \in S}, (\pi_{(3 \times S, p_2,(1,t))})_{t \in S}, (\pi_{(3 \times S, p_2,(2,t))})_{t \in S}) \quad (1 \leq i \leq n)
   \]

**Proof.** Equivalence of 1, 2, 3 and 4 is similar to corresponding equivalence of Theorem 7.14. 2 $\Rightarrow$ 3 is known as Jónsson’s theorem (See e.g. [2] Theorem 12.6). 5 $\Rightarrow$ 4 easily follows from finiteness of $S$. 6 $\Rightarrow$ 5 is trivial.

1 $\Rightarrow$ 6. Let $F$ be a rank $(3S)_n \in S$ free algebra of $\mathcal{V}(M)$, \{a, b, c\} be a set of free generators of $F$, and $\theta, \eta, \zeta$ be the congruences of $F$ generated by (a, c), (a, b), (b, c) respectively. Then

   \[
   (a, c) \in \theta \land (\eta \lor \zeta) = (\theta \land \eta) \lor (\theta \land \zeta).
   \]

By finiteness of arity of $M$, the join in congruence lattice is described as $\alpha \lor \beta = \bigcup_{n \in \mathbb{N}} (\alpha \lor \beta)^n$, where $\alpha \lor \beta)^n$ denotes the $k$-th relational power of $\alpha \lor \beta$. Thus, there are $x_0, \ldots, x_{2n}$ such that

\[
\begin{align*}
    x_0 = a, \\
x_{2n} = c, \\
(x_{2i}, x_{2i+1}) \in \theta \land \eta, \\
(x_{2i-1}, x_{2i}) \in \theta \land \zeta.
\end{align*}
\]

The terms $d_{s,i}$ corresponding to $x_i$ satisfy the equations stated in Condition 6. □
Next example is a counter example of infinitary variety for Jónsson’s Theorem.

**Example 7.17.** Let $S = N = \mathbb{N} = \aleph_0$ and $C_n$ be the $<\aleph_1$-ary clone defined by the following presentation:

- The set of generators consists of 3-ary elements $d_0, \ldots, d_{2n}$ and a nullary element $u$.
- The set of fundamental relations consists of Jónsson equations, that is, equations displayed in Condition 3 of the previous theorem.

Let $C$ be the direct product clone $\prod_{n \in \mathbb{N}} C_n$ and define

$$d = (\pi_{(n,i)})_{n \in \mathbb{N}} \in C_{\mathbb{N}},
e n = (\pi_{(0,i)}, (u)_{i \in \mathbb{N} \setminus \{n\}}) \in C_1$$

for $n \in \mathbb{N}$. (Intuitively, $d$ and $e_n$ are the operations such that $d : ((a_{ij})_{j \in \mathbb{N}}) \mapsto (a_{ij})_{i \in \mathbb{N}}$ and $e_n : (a_i)_{i \in \mathbb{N}} \mapsto (a_n, (u)_{i \in \mathbb{N} \setminus \{n\}}).$) Then $(d, (e_n)_{n \in \mathbb{N}})$ is a diagonal pair of $C$. Moreover, the following assertions hold.

1. $V(C)$ and $V(C_d)$ are congruence distributive.
2. $C$ does not have Jónsson term.
3. $C_d$ has terms that satisfy the equations stated in 6 of the previous theorem.
4. $C_d$ is essentially finitary.

**Proof.** 1. Each $C_n$ has Jónsson term, thus $V(C_n)$ is congruence distributive. Thus,

$$V(C) = \bigvee_{n \in \mathbb{N}} V(C_n) \cong \bigotimes_{n \in \mathbb{N}} V(C_n)$$

is congruence distributive. Here, $\otimes_{n \in \mathbb{N}} V(C_n)$ denotes non-indexed product of the family $\{V(C_n)\}_{n \in \mathbb{N}}$ of varieties. By Proposition [2], $V(C_d)$ also be congruence distributive.

2. If $C$ has Jónsson term of length $n$, then $C_{n+1}$ also has Jónsson term of length $n$. It is verified by the induction on $n$ that this is impossible.

3. It is easily verified that $e_n(C)$ is isomorphic to $C_n$. Particularly, $e_n(C)$ has Jónsson term. These are nothing but the terms of $C_d$ satisfying Condition 6 of the theorem.

4. Let $f \in C_{d,(\mathbb{N},n,n)}$, where $v : \mathbb{N} \to \mathbb{N}$ and $n \in \mathbb{N}$. By the definition of heterogenization, $f$ is represented by $\tilde{f} \in C_{\mathbb{N}}$ such that $e_n \circ \tilde{f} = \tilde{f}$. This condition implies that the $n$-th component of $\tilde{f}$ is an element of $C_n$ (it is finitary), and other components are $u$ (depend no variables). Therefore, $f$, particularly $f$, is finitary. \[\square\]

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