Reverse Lebesgue and Gaussian isoperimetric inequalities for parallel sets with applications

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June 2020

Abstract

The $r$-parallel set of a measurable set $A \subseteq \mathbb{R}^d$ is the set of all points whose distance from $A$ is at most $r$. In this paper, we show that the surface area of an $r$-parallel set in $\mathbb{R}^d$ with volume at most $V$ is upper-bounded by $e^{\Theta(d)} V/r$. We also show that the Gaussian surface area of any $r$-parallel set in $\mathbb{R}^d$ is upper-bounded by $\max(e^{\Theta(d)} / r, e^{\Theta(d)}/r)$. We apply our results to two problems in theoretical machine learning: (1) bounding the computational complexity of learning $r$-parallel sets under a Gaussian distribution; and (2) bounding the sample complexity of estimating robust risk, which is a notion of risk in the adversarial machine learning literature that is analogous to the Bayes risk in hypothesis testing.

1 Introduction

The isoperimetric problem in $\mathbb{R}^d$ poses the following question: What is the minimum surface area of a set in $\mathbb{R}^n$ with a given volume? Equivalently, what is the maximum volume of a set in $\mathbb{R}^n$ with a given surface area? It is well known that Euclidean balls are the unique extremal sets for both formulations; i.e., the following inequality holds for all sets with surface area $S$ and volume $V$, with equality if and only if the set is a Euclidean ball:

$$S^d \geq d^d \omega_d V^{d-1},$$

where $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$. Although intuitive, this inequality is non-trivial to prove and holds in astonishing generality. Indeed, with the right definition of “surface area,” the isoperimetric inequality holds for all measurable sets, with no regularity conditions on the boundary [1]. The volume of a measurable set $A \subseteq \mathbb{R}^n$ is its Lebesgue measure $\lambda(A)$. In this paper, we use the notion of Minkowski surface area, defined as follows:

**Definition 1.** Let $A$ be a measurable set. For $x \in \mathbb{R}^n$, let the distance of $x$ from $A$ be denoted by $d(x, A) = \inf_{y \in A} d(x, y)$, where $d(x, y)$ is the usual Euclidean distance. For $\delta > 0$, denote the $\delta$-neighborhood of $A$ by $A_\delta = \{x \in \mathbb{R}^n : d(x, A) \leq \delta\}$. Define the following:

$$\lambda(\partial A) := \liminf_{\delta \to 0} \frac{\lambda(A_\delta \setminus A)}{\delta}, \quad (1)$$

$$\bar{\lambda}(\partial A) := \limsup_{\delta \to 0} \frac{\lambda(A_\delta \setminus A)}{\delta}. \quad (2)$$

We call the former the lower-Lebesgue surface area and the later the upper-Lebesgue surface area. If $\lambda(\partial A) = \bar{\lambda}(\partial A)$, we refer to the quantity as the Lebesgue surface area and denote it by $\lambda(\partial A)$.
The set $A_\delta$ can also be thought of as the Minkowski sum of $A$ with the closed ball of radius $\delta$; i.e., $A_\delta = A \oplus B_d(\delta)$. With this notation, the isoperimetric inequality is written as 

$$\lambda(\partial A)^d \geq d^d \omega_d \lambda(A)^{d-1}. \quad (3)$$

Since the “volume” of a set is taken to be its Lebesgue measure, we shall refer to inequality (3) as the Lebesgue isoperimetric inequality.

As hinted above, it is possible to interpret “volume” using a measure other than the Lebesgue measure. A popular alternative is the standard Gaussian measure, which we denote by $\gamma$.

**Definition 2.** The lower- and upper-Gaussian surface areas of a measurable set $A$ are defined as follows:

$$\gamma(\partial A) := \liminf_{\delta \to 0} \frac{\gamma(A_\delta \setminus A)}{\delta}, \quad (4)$$

$$\check{\gamma}(\partial A) := \liminf_{\delta \to 0} \frac{\gamma(A_\delta \setminus A)}{\delta}. \quad (5)$$

If $\gamma(\partial A) = \check{\gamma}(\partial A)$, we refer to the quantity as the Gaussian surface area and denote it by $\gamma(\partial A)$.

Analogous to the Lebesgue isoperimetric inequality, Sudakov and Tsirel’son [2] and Borel [3] established the Gaussian isoperimetric inequality. This inequality states that among all sets with a given Gaussian volume, halfspaces have the minimum possible Gaussian surface areas. It is worth noting that a halfspace has infinite Lebesgue surface area but its Gaussian surface area is bounded above by the constant $\sqrt{1/2\pi}$.

The reverse isoperimetric problem is the following: What is the maximum surface area of a set in $\mathbb{R}^d$ with fixed volume? Equivalently, what is the minimum volume of a set in $\mathbb{R}^d$ with a fixed surface area? A little reflection reveals that this question does not make sense as posed, for we can have sets such as spheres that have zero volume but arbitrarily large surface area. To make sense of the reverse isoperimetric problem, it is necessary to impose some regularity conditions—to prevent “wiggliness” of the boundary—on the class of sets being considered.

Interestingly, reverse isoperimetric inequalities are more easily described in the Gaussian setting than the Lebesgue setting. Ball [4] established a reverse Gaussian isoperimetric inequality for convex sets: The Gaussian surface area of any convex set $A \subseteq \mathbb{R}^d$ is bounded above by $4d^{1/4}$. Nazarov [5] further refined Ball’s bound and also showed that the bound is essentially tight by constructing a set with $\Theta(d^{1/4})$ Gaussian surface area. Generalizations of Ball [4] and Nazarov [5] for log-concave measures were obtained in Livshyts [6, 7]. Klivans, O’Donnell, and Servedio [8] established a link between the Gaussian surface areas of sets and the ability to learn them efficiently under the probably-approximately-correct (PAC) and agnostic learning models. Klivans et al. showed that sets with small Gaussian surface areas can be learned efficiently under the Gaussian distribution. Klivans et al. obtained bounds on the Gaussian surface areas of cones and balls, and Kane [9] bounded the surface areas sets obtained from thresholded polynomials of a fixed degree.

In the Lebesgue setting, most existing work focuses on sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ with some kind of curvature constraint on the boundary of the sets. Howard and Treibergs [10] showed that if the average curvature $\kappa$ of a curve in $\mathbb{R}^2$ satisfies $|\kappa| \leq 1$, and if the area enclosed by the curve is small enough, then a certain “peanut shape” has the largest perimeter for a fixed area. Gard [11] extended this result to surfaces of revolution in $\mathbb{R}^3$. Pan, Tang, and Wang [12] obtained a version of the reverse isoperimetric inequality for sets in $\mathbb{R}^2$ by lower-bounding the perimeter in terms of the area of the set as well as the the area of the locus of its curvature centers. The one result we were able to find that holds in higher dimensions is that of Chernov, Drach, and Tatarko [13],
where the authors showed that for convex sets satisfying a weak notion of curvature constraint called \( \lambda \)-concavity, the sausage body (Minkowski sum of a line segment and a ball) has the largest surface area for a fixed volume. Another result that holds for convex sets in arbitrary dimensions is that of Ball [14]; however, it involves transforming the set via a volume-preserving linear map and thus cannot be compared to the above results.

In this paper, we take a different approach towards imposing regularity conditions. Our goal will be to study reverse isoperimetric inequalities for \( r \)-parallel sets, which are defined as follows:

**Definition 3.** Let \( r > 0 \) and \( d \geq 1 \). A set \( \tilde{A} \subseteq \mathbb{R}^d \) is called an \( r \)-parallel set if \( \tilde{A} = A_r \) for some measurable set \( A \).

Throughout this paper, an \( r \)-parallel set will always be represented by \( A_r \) for some measurable set \( A \). We remark that it is possible to assume \( A \) is closed since \( A_r = (\text{cl}(A))_r \), where \( \text{cl}(A) \) is the closure of \( A \). Since \( A_r = A \oplus B_d(r) \), the intuition is that even if \( A \) has a very wiggly boundary, the smoothed set \( A_r \) will have a better-behaved boundary. It is clear, though, that the boundary of \( A_r \) need not be twice-continuously differentiable, or even a union of finitely many such pieces. Moreover, the sets \( A_r \) need not be convex. These observations preclude the possibility of directly using any of the reverse isoperimetric inequalities known in the literature.

Parallel sets in convex geometry appear prominently in the context of quermassintegrals, intrinsic volumes, and Steiner’s formula for the volume of the Minkowski sum of a convex set with a ball [15]. Over the years, parallel sets of arbitrary closed sets have also been investigated and some regularity properties have been established in the process. The work most relevant to ours is Stacho [16], and we shall utilize several results and techniques from that paper in the course of our proofs. For now, we point out that for \( r > 0 \), Stacho [16] showed that the Minkowski surface area of a bounded set \( A_r \) can be calculated as the limit

\[
\lambda(\partial A_r) = \lim_{\delta \to 0} \frac{\lambda(A_r + \delta \setminus A_r)}{\delta},
\]

that is \( \delta(\partial A_r) = \bar{\lambda}(\partial A_r) \). Recent work by Hug, Last, and Weil [17] and Rataj and Winter [18] has strengthened the results from Stacho [16]. Hug et al. [17] showed a local version of Steiner’s formula for arbitrary closed sets, whereas Rataj and Winter [18] proved results concerning rectifiability of parallel sets and established relations between various notions of surface areas of parallel sets, including the Hausdorff measure of the boundary, the lower and upper Minkowski contents of the boundary, and Minkowski’s surface area from Definition 1.

Another motivation for considering \( r \)-parallel sets comes from information theory. The information theoretic concepts of entropy and Fisher information have often been compared to the geometric concepts of volume and surface area [19]. A striking similarity exists between the definition of surface area in equation (1) and de Brujin’s identity from information theory: Given a random vector \( X \) on \( \mathbb{R}^d \) and a standard normal random variable \( Z \) that is independent of \( X \), the Fisher information of \( X \), denoted by \( J(X) \), satisfies the relation

\[
\frac{d}{dt} h(X + \sqrt{t}Z) \bigg|_{t=0} = \frac{J(X)}{2}.
\]

This means that

\[
\frac{d}{dt} e^{h(X + \sqrt{t}Z)} \bigg|_{t=0} = \frac{e^{h(X)}J(X)}{2}.
\]

Thus, the Minkowski sum with a ball is replaced by a sum with independent Gaussian noise; volume is replaced by the exponential of the entropy; and surface area is replaced by a scaled version of
the Fisher information. The analogous notion of \( r \)-parallel sets in information theory would be the set of all random vectors \( X_r := X + \sqrt{r} Z \), which we call \( r \)-smoothed random variables. A version of the reverse isoperimetric inequality in information theory could be stated as: Given an \( r \)-smoothed random variable of a fixed entropy \( h_0 \), how large can its scaled-Fisher information be? Surprisingly, it is very easy to obtain such an upper bound. It is a well-known fact that Fisher information is a convex functional on the space of distributions \([20]\), so \( J(X_r) = J(X + \sqrt{r} Z) \leq J(\sqrt{r} Z) = d/r \). Thus, we conclude that \( e^{h(X)} J(X)/2 \leq e^{h_0} d/2r \). Does a version of the reverse isoperimetric inequality exist for \( r \)-parallel sets in geometry?

This is precisely the question addressed in our paper. We study two problems of interest: (i) Is it possible to upper bound the Lebesgue surface area of an \( r \)-parallel set given a bound on its volume?; and (ii) is there a version of the reverse Gaussian isoperimetric inequality for \( r \)-parallel sets? Our result concerning (i) may be informally stated as follows:

**Result 1** (Formal statement in Theorem 1). Let \( r > 0 \) and let \( A_r \subseteq \mathbb{R}^d \) satisfy \( \lambda(A_r) \leq V \). Then the surface area of \( A_r \) satisfies the inequality

\[
\lambda(\partial A) \leq C_d \frac{V}{r},
\]

where \( C_d = e^{\Theta(d)} \) is a dimension-dependent constant.

Observe that the bound increases as \( r \) decreases, which is to be expected, since the sets \( A_r \) have fewer restrictions on their boundaries. It is interesting to note that the \( 1/r \) dependence is the same as in the information theoretic reverse isoperimetric inequality in equation (6). Having proved the reverse Lebesgue isoperimetric inequality, we use a proof technique from Ball \([4]\) to establish its analog for the Gaussian measure. Our result can be informally stated as follows:

**Result 2** (Formal statement in Theorem 2). Let \( r > 0 \) and let \( A_r \) be the \( r \)-parallel set of an arbitrary measurable set \( A \). Then the following inequality holds:

\[
\gamma(\partial A) \leq \max \left( C_d, \frac{C_d}{r} \right),
\]

where \( C_d = e^{\Theta(d)} \) is a dimension-dependent constant.

Just as in the reverse isoperimetric inequality for convex sets in Ball \([4]\) and Nazarov \([5]\), we do not need to impose any boundedness assumptions on \( A_r \). Note that halfspaces are \( r \)-parallel sets for any \( r > 0 \) and their Gaussian surface areas are lower-bounded by a constant. This implies that the upper bound cannot keep decreasing with increasing \( r \), leading to the max term.

We also provide two applications of the reverse Gaussian isoperimetric inequality to learning theory. First, we show that the machinery in Klivans et al. \([8]\) provides computational complexity bounds for learning \( r \)-parallel sets under the Gaussian distribution. Our second application concerns adversarial machine learning. Notions of robust risk, analogous to Bayes risk in standard hypothesis testing, have recently been proposed in the machine learning literature. Some recent work by Bhagoji, Cullina, and Mittal \([21]\) and Pydi and Jog \([22]\) characterizes robust risk in terms of an optimal transport cost between the data distributions of two classes in a binary classification setting. We show that for sufficiently smooth data distributions, the Gaussian reverse isoperimetric inequality can be used to provide sample complexity bounds for estimating robust risk.

The structure of this paper is as follows: In Section 2, we present a puzzle in \( \mathbb{R}^2 \) whose solution captures the essence of our proof. In Section 3 and Section 4, we prove the reverse isoperimetric inequalities for the Lebesgue and Gaussian measures, respectively. In Section 5, we describe applications to learning theory. Finally, we conclude the paper in Section 6.
Notation:

- The Euclidean ball in $\mathbb{R}^d$ with center $x$ and radius $r$ is denoted by $B_d(x; r)$ (or $B(x; r)$, if the dimension is clear from context). The ball centered at the origin is denoted by $B_d(r)$ or $B(r)$.

- Given two measurable sets $A, B \subseteq \mathbb{R}^d$, their Minkowski sum is given by
  \[ A \oplus B = \{ a + b \mid a \in A, b \in B \}. \]

- The volume of the $d$-dimensional unit ball is denoted by $\omega_d$ and its surface area is denoted by $\Omega_d$.

- The solid angle subtended by a set $S$ towards a point $x$ is denoted by $\Omega(S; x)$.

- The distance of a point $x$ from a set $A$ is $d(x, A) = \inf_{a \in A} d(x, a)$, where $d$ is the usual Euclidean distance in $\mathbb{R}^d$.

- $\|x\|$ indicates the usual Euclidean norm in $\mathbb{R}^d$.

- $1\{x \in A\}$ is the indicator function for the event $x \in A$.

- The $(d - 1)$-dimensional Hausdorff measure is denoted by $\mathcal{H}^{d-1}$.

- Given functions $f, g : \mathbb{N} \to \mathbb{R}$, we say that $f = \Theta(g)$ if there exist constants $c_1, c_2 > 0$ such that $c_1g(n) \leq f(n) \leq c_2g(n)$.

2 A puzzle in $\mathbb{R}^2$

In this section, we present a puzzle in two dimensions, whose solution neatly captures the main ideas in our approach.

**Puzzle:** Consider $N \geq 1$ points $x_i \in B(x_0; 1)$, for $1 \leq i \leq N$. Let $A = \{x_i \in \mathbb{R}^2 \mid 0 \leq i \leq N\}$. Show that the perimeter of $A_1 = A \oplus B(1)$ is no more than that of $B(x_0; 2)$; i.e.,

\[ \lambda(\partial A_1) \leq \lambda(\partial B(2)) = 4\pi. \]  

(7)

Figure 1 shows an example of the set $A_1$.

**Solution:** A simple upper bound on the perimeter of $A_1$ is $\sum_{i=0}^{N} \lambda(\partial(B(x_i; 1))) = 2\pi(N + 1)$; however, this bound becomes progressively weaker with increasing $N$. One may wonder whether equality is ever achieved in inequality (7), and a little reflection reveals that almost any arrangement of $N$ points on the circumference of $B(x_0; 1)$ gives equality. The only condition needed for the arrangement is that $\partial A_1$ contains no contribution from $\partial B(x_0; 1)$. For instance, three points equally spaced on the perimeter of $B(x_0; 1)$ suffice. We make two observations:

1. The set $A_1$ is *star-shaped* from the point of view of $x_0$; i.e., any ray starting from $x_0$ intersects the boundary of $A_1$ at most once. Suppose this were not the case and a ray from $x_0$ were to intersect the boundary of $A_1$ in two points $y_1$ and $y_2$, where we assume that $y_1$ is closer to $x_0$ than $y_2$. For $i \in \{1, 2\}$, the very fact that $y_i$ lies on the boundary of $A_1$ means that $B(y_i; 1) \cap B(x_0; 1)$ cannot contain any points $x_i$, except for one or more that lie on $\partial B(y_i; 1) \cap B(x_0; 1)$. This immediately leads to a contradiction, since any point $x_i$ that lies on $\partial B(y_2; 1) \cap B(x_0; 1)$ will lie within $B(y_1; 1) \cap B(x_0; 1)$, but not be on $\partial B(y_1; 1) \cap B(x_0; 1)$. 


Figure 1: Given \( \{x_1, \ldots, x_n\} \subseteq B(x_0; 1) \), the shaded region is the union of unit balls with centers at \( x_i \), for \( 0 \leq i \leq N \). The problem is to upper-bound the perimeter of the shaded region with that of \( B(x_0; 2) \).

(2) The boundary of \( A_1 \) can be partitioned as \( \partial A_1 = \bigcup_{i=0}^{N} (\partial A_1)^i \), where \( (\partial A_1)^i = \bigcup_{i=0}^{N} \partial B(x_i; 1) \cap \partial A_1 \) is the arc of the circle \( B(x_i; 1) \) that lies on \( \partial A_1 \). (Note that several of the sets \( (\partial A_1)^i \) may be empty.) The perimeter of \( A_1 \) may be expressed as \( \lambda(\partial A_1) = \sum_{i=1}^{N} \lambda((\partial A_1)^i) \). Without loss of generality, suppose \( (\partial A_1)^1 = \partial B(x_1; 1) \cap \partial A_1 \neq \emptyset \). Clearly, we have \( \lambda((\partial A_1)^1) = \Omega((\partial A_1)^1; x_1) \); i.e., the perimeter of the arc \( (\partial A_1)^1 \) is simply the central angle of the arc \( (\partial A_1)^1 \). Now comes our key observation: The angle subtended by the arc to \( (\partial A_1)^1 \) to \( x_0 \), which is denoted by \( \Omega((\partial A_1)^1; x_0) \), is at least as large as \( \Omega((\partial A_1)^1; x_1) / 2 \); i.e.,

\[
\Omega((\partial A_1)^1; x_0) \geq \frac{\Omega((\partial A_1)^1; x_1)}{2}.
\]

If \( x_0 \) lay on the circumference of \( B(x_1; 1) \), this would be an exact equality by the inscribed angle theorem from geometry. In this case, the point \( x_0 \) might lie in the interior of \( B(x_1, 1) \), but it may be easily verified that the the angle subtended by the arc at \( x_0 \) would be at least as large as the inscribed angle of the arc.

Observations (1) and (2) are illustrated in Figure 2. We now combine observations (1) and (2). Since \( A_1 \) is star-shaped from the point of view of \( x_0 \), we have

\[
2\pi = \sum_{i=0}^{N} \Omega((\partial A_1)^i; x_0).
\]

Using the inequality from observation (2), we obtain

\[
2\pi = \sum_{i=0}^{N} \Omega((\partial A_1)^i; x_0) \geq \frac{1}{2} \sum_{i=0}^{N} \Omega((\partial A_1)^i; x_i) = \frac{\lambda(\partial A_1)}{2}.
\]

This leads to \( 2\pi \geq \frac{\lambda(\partial A_1)}{2} \), which completes the solution to the puzzle.
Figure 2: In (a), the boundary of $A_1$ is shown to be a union of arcs of various circles. In (b), it follows from the inscribed angle theorem that $\phi \geq \theta/2$.

3 Reverse Lebesgue isoperimetric inequality for parallel sets

We first prove a version of the puzzle in Section 2 in $d$ dimensions.

**Proposition 1.** Let $r, \delta > 0$. Consider $N \geq 1$ points $x_i \in B_d(x_0;r)$, for $1 \leq i \leq N$. Let $A = \{x_i \in \mathbb{R}^d \mid 0 \leq i \leq N\}$. Then

$$\lambda(A_{r+\delta} \setminus A_r) \leq 2^{d-1}\Omega_d \frac{(r+\delta)^d - r^d}{d}.$$  \hspace{1cm} (8)

**Corollary 3.1.** The surface area of $A_r$ satisfies the following inequality:

$$\lambda(\partial A_r) \leq 2^{d-1}\Omega_d r^{d-1}.$$ \hspace{1cm} (9)

**Proof.** Corollary 3.1 follows immediately from Proposition 1 by taking the limit as $\delta \to 0$. The proof strategy employed is adapted from Stacho [16]. Let the Voronoi region $D_i$ associated to each $x_i$ be defined as

$$D_i = \{x \in \mathbb{R}^d \mid d(x, x_j) \geq d(x, x_i) \text{ if } j \leq i \text{ and } d(x, x_j) > d(x, x_i) \text{ if } j > i\}.$$ 

Note that the $D_i$'s are pairwise disjoint convex regions, not necessarily bounded, which cover all of $\mathbb{R}^d$. Thus, we may write

$$\lambda(A_{r+\delta} \setminus A_r) = \sum_{i=0}^{n} \lambda((A_{r+\delta} \setminus A_r) \cap D_i) \hspace{1cm} (10)$$

$$= \sum_{i=0}^{N} \lambda((B(x_i;r+\delta) \setminus B(x_i;r)) \cap D_i) \hspace{1cm} (11)$$

$$\leq \sum_{i=0}^{N} \int_{r}^{r+\delta} \lambda(\partial B(x_i;t) \cap D_i)dt. \hspace{1cm} (12)$$
Step (a) is reasoned as follows: A point \(x \in A_{r+\delta} \setminus A_r\) iff it satisfies \(r + \delta \geq d(x, A) > r\). If \(x\) lies in \(D_i\) as well, then \(d(x, A) = d(x, x_i)\), giving the relation \(r + \delta \geq d(x, A) = d(x, x_i) > r\), so that \(x \in B(x_i; r + \delta) \setminus B(x_i; r)\). This shows that \((A_{r+\delta} \setminus A_r) \cap D_i \subseteq (B(x_i; r + \delta) \setminus B(x_i; r)) \cap D_i\). A similar argument gives \((B(x_i; r + \delta) \setminus B(x_i; r)) \cap D_i \subseteq (A_{r+\delta} \setminus A_r) \cap D_i\). Step (b) follows from the co-area formula.

The following observation appears within a proof in Stacho [16]. We give a short proof for completeness.

**Lemma 3.1.** The solid angle \(\Omega(\partial(B(x_i; t) \cap D_i); x_i)\) is a monotonically decreasing function for \(t > 0\).

**Proof.** Since \(D_i\) is a convex set containing \(x_i\), it is star-shaped from the point of view of \(x_i\). This means that rays coming from \(x_i\) either continue forever within \(D_i\) or hit the boundary of \(D_i\) and terminate. As \(t\) increases, the solid angle of the terminated rays monotonically increases, so the solid angle of the unterminated rays which comprise \(\partial B(x_i; t) \cap D_i\) monotonically decreases. \(\square\)

Lemma 3.1 implies that for \(t \geq r\),

\[
\lambda(\partial B(x_i; t) \cap D_i) = t^{d-1} \Omega(\partial B(x_i; t) \cap D_i; x_i) \leq t^{d-1} \Omega(\partial B(x_i; r) \cap D_i; x_i).
\]

Substituting this into inequality (12), we obtain

\[
\lambda(A_{r+\delta} \setminus A_r) \leq \sum_{i=0}^{N} \int_{r}^{r+\delta} t^{d-1} \Omega(\partial B(x_i; r) \cap D_i; x_i) dt
\]

\[
= \frac{(r + \delta)^d - r^d}{d} \sum_{i=0}^{N} \Omega(\partial B(x_i; r) \cap D_i; x_i).
\]

**Lemma 3.2.** The set \(A_r\) is star-shaped from the point of view of \(x_0\).

**Proof.** The proof is essentially identical to observation (1) from Section 2, so we omit it. \(\square\)

**Lemma 3.3.** The solid angle subtended by \(\partial B(x_i; r) \cap D_i\) at \(x_0\) satisfies the bound

\[
\Omega(\partial B(x_i; r) \cap D_i; x_0) \geq \frac{\Omega(\partial B(x_i; r) \cap D_i; x_i)}{2^{d-1}}.
\]

**Proof.** Unfortunately, for \(d > 2\), the inscribed angle theorem no longer holds; i.e., the solid angle subtended by a region on the sphere to an arbitrary point on the sphere is not a fixed fraction of the solid angle subtended to the center of the sphere.

Let \(S := \partial B(x_i; r) \cap D_i \subseteq \partial B(x_i; r)\), and assume that \(S \neq \emptyset\) without loss of generality. Note that \(x_0 \notin S\), but \(x_0 \in B(x_i; r)\). Consider a small surface area element \(dS\) in \(S\) around a point \(x \in S\). Note that \(\Omega(dS; x_i) = \lambda(dS)/(r^{d-1} \Omega_d)\). As shown in Figure 3, let \(\angle x_ix_0 = \theta\). Extend the line joining \(x\) and \(x_0\) to \(\tilde{x}_0\) on \(B(x_i; r)\). Using trigonometry, we can check that

\[d(x, \tilde{x}_0) = 2r \cos \theta.\]
Figure 3: Comparison between the angle subtended by a small area element $dS$ to $x_0$ and $x_i$.

Also, it is not hard to check that

\[
\Omega(dS; x_0) = \frac{\lambda(dS) \cos \theta}{\Omega_d(r, x_0)^{d-1}} \geq \frac{\lambda(dS) \cos \theta}{\Omega_d(r, \tilde{x}_0)^{d-1}} \geq \frac{\lambda(dS) \cos \theta}{\Omega_d(2r \cos \theta)^{d-1}} = \Omega(dS; x_i) \cdot \frac{1}{2^{d-1} \cos^{d-2} \theta} \geq \frac{\Omega(dS; x_i)}{2^{d-1}}. \tag{19}
\]

Noting that $\Omega(S; x_0) = \int_S \Omega(dS; x_0)$ and $\Omega(S; x_i) = \int_S \Omega(dS; x_i)$, we may integrate the inequality in (19) to conclude the desired result. □

Continuing from inequality (14), we have

\[
\lambda(A_{r+\delta} \setminus A_r) \leq \frac{(r + \delta)^d - r^d}{d} \sum_{i=0}^N \Omega(\partial B(x_i; r) \cap D_i; x_i)
\leq \frac{(r + \delta)^d - r^d}{d} 2^{d-1} \Omega(\partial B(x_i; r) \cap D_i; x_0)
\leq \frac{(r + \delta)^d - r^d}{d} \cdot 2^{d-1} \cdot \Omega_d.
\]

Here, (a) follows from Lemma 3.3 and (b) follows from Lemma 3.2. This concludes the proof of Proposition 1. □
Before stating our next proposition, we define the packing number of a set in $\mathbb{R}^d$.

**Definition 4.** Let $A \subseteq \mathbb{R}^d$ be a measurable set and let $\epsilon > 0$. A collection of points denoted by $\text{Packing}(A; \epsilon) := \{x_i \mid 1 \leq i \leq N\}$ is said to be an $\epsilon$-packing of $A$ if for every $x, y \in \text{Packing}(A; \epsilon)$, we have $d(x, y) > \epsilon$. The $\epsilon$-packing number of $A$, denoted by $N(A; \epsilon)$, is the largest size of an $\epsilon$-packing of $A$.

**Proposition 2.** Let $R > r$, and let $\delta > 0$. Consider $N \geq 1$ arbitrary points $x_1, \ldots, x_N$ in $\mathbb{R}^d$, and let $A = \{x_1, \ldots, x_N\}$. Then

$$
\lambda(A_{r+\delta} \setminus A_r) \leq N(A; r) \cdot \left(2^{d-1} \Omega_d \frac{(r + \delta)^d - r^d}{d}\right),
$$

(20)

where $N(A; r)$ is the $r$-packing number of $A$.

**Corollary 3.2.** The surface area of $A_r$ satisfies

$$
\lambda(\partial A_r) \leq N(A; r) \cdot \left(2^{d-1} \Omega_d r^{d-1}\right).
$$

**Proof.** The proof of Corollary 3.2 follows by taking the limit as $\delta \to 0$, so we shall only prove the bound (20).

Let $\widehat{A} = \{\widehat{x}_1, \ldots, \widehat{x}_k\} \subseteq A$ be a maximal $r$-packing of the set $A$; i.e., if $i \neq j$, then $d(\widehat{x}_i; \widehat{x}_j) > r$, but $d(x, \widehat{A}) \leq r$ for every $x \in A$. Note that $k = N(A; r)$. Let $\widehat{A}^i = \{x \in A \mid d(x, \widehat{x}_i) \leq r\}$. Observe that $\cup_{i=1}^k \widehat{A}^i = A$, but the $\widehat{A}^i$’s need not be mutually exclusive. This means that

$$
A_{r+\delta} \setminus A_r = \cup_{i=1}^k \left((\widehat{A}^i)_{r+\delta}\right) \setminus A_r
= \cup_{i=1}^k \left((\widehat{A}^i)_{r+\delta}\right) \setminus A_r
\subseteq \cup_{i=1}^k \left((\widehat{A}^i)_{r+\delta}\right) \setminus (\widehat{A}^i)_r,
$$

so

$$
\lambda(A_{r+\delta} \setminus A_r) \leq \lambda\left(\cup_{i=1}^k \left((\widehat{A}^i)_{r+\delta}\right) \setminus (\widehat{A}^i)_r\right)
\leq \sum_{i=1}^k \lambda\left((\widehat{A}^i)_{r+\delta}\right) \setminus (\widehat{A}^i)_r
\leq k \left(2^{d-1} \Omega_d \frac{(r + \delta)^d - r^d}{d}\right) \tag{22}
\leq N(A; r) \cdot \left(2^{d-1} \Omega_d \frac{(r + \delta)^d - r^d}{d}\right). \tag{23}
\leq N(A; r) \cdot \left(2^{d-1} \Omega_d \frac{(r + \delta)^d - r^d}{d}\right) \cdot \tag{24}
\leq \lambda(A_{r+\delta} \setminus A_r) \leq \lambda(\partial A_r)
\leq \frac{V}{\omega_d(r/2)^d} \cdot \left(2^{d-1} \Omega_d \frac{(r + \delta)^d - r^d}{d}\right) = \frac{V}{r^d} \cdot \left(2^{2d-1}((r + \delta)^d - r^d)\right), \tag{21}
$$

Here, inequality (a) follows directly from Proposition 1, and (b) follows from the maximal property of the packing.

**Theorem 1.** Let $r, \delta > 0$, and let $A_r$ be an $r$-parallel set in $\mathbb{R}^d$ satisfying $\lambda(A_r) \leq V$. Then the following inequalities hold:

$$
\lambda(A_{r+\delta} \setminus A_r) \leq \frac{V}{\omega_d(r/2)^d} \cdot \left(2^{d-1} \Omega_d \frac{(r + \delta)^d - r^d}{d}\right) = \frac{V}{r^d} \cdot \left(2^{2d-1}((r + \delta)^d - r^d)\right), \quad \text{and}
\lambda(\partial A_r) \leq \frac{V}{\omega_d(r/2)^d} \cdot \left(2^{d-1} \Omega_d r^{d-1}\right) = \frac{V}{r} \cdot \left(2^{2d-1}d\right).
$$

10
Proof. Since $A$ is compact (closed and bounded), for each $n \geq 1$, there exists a finite set $A^n \subseteq A$ such that $A \subseteq (A^n)_{1/n}$. Clearly, the sets $A^n$ converge to $A$ in the Hausdorff metric, since $d_{\text{Hausdorff}}(A, A^n) \leq 1/n$. For any $t > 0$, we have $(A^n)_t \subseteq A_t$ and $A_t \subseteq (A^n)_{t+1/n}$. Equivalently, for all large enough $n$, we have the inclusion

$$A_{t-1/n} \subseteq (A^n)_t \subseteq A_t,$$

which implies

$$\lambda(A_{t-1/n}) \leq \lambda((A^n)_t) \leq \lambda(A_t).$$

From Stacho [16], the volume function $t \to \lambda(A_t)$ is continuous, so

$$\lambda((A^n)_t) \to \lambda(A_t).$$

Note that

$$\lambda((A^n)_{r+\delta} \setminus (A^n)_r) = \lambda((A^n)_{r+\delta}) - \lambda((A^n)_r) \leq N(A^n; r) \cdot \left(2^{d-1} \Omega_d \frac{(r + \delta)^d - r^d}{d}\right)$$

where $(a)$ is true because each of the sets $A^n$ consists of finitely many points and Proposition 2 may be applied; and $(b)$ is true since $A^n \subseteq A$. Taking the limit as $n \to \infty$ and using the volume convergence from equation (25), we conclude that

$$\lambda(A_{r+\delta} \setminus A_r) \leq N(A; r) \cdot \left(2^{d-1} \Omega_d \frac{(r + \delta)^d - r^d}{d}\right).$$

The last step is to bound $N(A; r)$ in terms of the volume of $A_r$. Consider any $r$-packing of $A$ given by $\text{Packing}(A; r) = \{x_i \mid 1 \leq i \leq N\}$. Clearly, we have $\bigcup_{i=1}^N B(x_i; r/2) \subseteq A_{r/2} \subseteq A_r$. Comparing volumes and noting that $B(x_i; r/2)$ are disjoint, we conclude that

$$N(A; r) \leq \frac{V}{\lambda(B(r/2))} = \frac{V}{\omega_d(r/2)^d}.$$

To prove the bound on $\partial(A_r)$, we can divide both sides in inequality (26) by $\delta$ and take the limit as $\delta \to 0$. By the results in Stacho [16], the limit of the left hand side exists, and we conclude that

$$\lambda(\partial A_r) \leq \frac{V}{\omega_d(r/2)^d} \cdot \left(2^{d-1} \Omega_d r^{d-1}\right).$$

Corollary 3.3. Let $R \geq r > 0$, and let $\delta > 0$. Let $A$ be an arbitrary closed set contained in $B(R)$. Then the following bounds hold for the $r$-parallel set $A_r$:

$$\lambda(A_{r+\delta} \setminus A_r) \leq \frac{(R + r/2)^d}{(r/2)^d} \cdot \left(2^{d-1} \Omega_d \frac{(r + \delta)^d - r^d}{d}\right),$$

and

$$\lambda(\partial A_r) \leq \frac{(R + r/2)^d}{(r/2)^d} \cdot \left(2^{d-1} \Omega_d r^{d-1}\right).$$

Proof. The proof is identical to that of Theorem 1, with the only change being that the $r$-packing number of $A$ is bounded as

$$N(A; r) \leq \frac{(R + r/2)^d}{(r/2)^d}.$$

\qed
4 Reverse Gaussian isoperimetric inequality for parallel sets

We now prove a reverse isoperimetry inequality for parallel sets under the Gaussian measure in $\mathbb{R}^d$. In the Gaussian isoperimetric inequality, the notion of Gaussian surface area used is the lower-Gaussian surface area. For reverse isoperimetric inequalities, it makes more sense to use the upper-Gaussian surface, since the aim is to provide upper bounds on the Gaussian surface area. For many well-behaved sets such as convex sets or sets with twice-continuously differentiable boundaries, the two notions of surface areas are identical [5]. Moreover, the Gaussian surface area is obtained by integrating the Gaussian distribution with respect to the Hausdorff measure on the boundary of the set; i.e.,

$$
\gamma(\partial A) = \lim_{\delta \to 0} \frac{\gamma(A_\delta \setminus A)}{\delta} = \int_{\partial A} \phi(x) d\mathcal{H}^{d-1}(x).
$$

We do not investigate whether the Gaussian surface area of parallel sets is also given by such a surface integral. Our main result is as follows:

**Theorem 2.** Let $A \subseteq \mathbb{R}^d$ be an arbitrary closed set, and for $r > 0$, consider the $r$-parallel set $A_r$. Let $\delta > 0$. Then the upper-Gaussian surface area of $A_r$ satisfies the bound

$$
\bar{\gamma}(A_r) \leq \max(C, C \frac{r}{d}),
$$

where $C$ is a dimension-dependent constant that grows like $e^{\Theta(d)}$.

**Proof.** Our proof relies on an observation in Ball [4] which leads to a (loose) upper bound of $\sqrt{d}$ on the Gaussian surface area of arbitrary convex sets in $\mathbb{R}^d$. (The tight upper-bound is $\Theta(d^{1/4})$.)

The observation is simple:

$$
e^{-\frac{1}{2}||x||^2} = \int_{\tau=0}^{\infty} \tau e^{-\tau^2/2} \mathbb{1}\{x \in B(\tau)\} d\tau.
$$

Using this, we rewrite $\gamma(A_{r+\delta} \setminus A_r)$ as

$$
\gamma(A_{r+\delta} \setminus A_r) = \frac{1}{(2\pi)^{d/2}} \int_{A_{r+\delta} \setminus A_r} e^{-\frac{1}{2}||x||^2} dx
$$

$$
= \frac{1}{(2\pi)^{d/2}} \int_{A_{r+\delta} \setminus A_r} \int_{\tau=0}^{\infty} \tau e^{-\tau^2/2} \mathbb{1}\{x \in B(\tau)\} d\tau dx
$$

$$
= \frac{1}{(2\pi)^{d/2}} \int_{\tau=0}^{\infty} \tau e^{-\tau^2/2} \left( \int_{A_{r+\delta} \setminus A_r} \mathbb{1}\{x \in B(\tau)\} dx \right) d\tau
$$

$$
= \frac{1}{(2\pi)^{d/2}} \int_{\tau=0}^{\infty} \tau e^{-\tau^2/2} \lambda((A_{r+\delta} \setminus A_r) \cap B(\tau)) d\tau
$$

$$
\leq \frac{1}{(2\pi)^{d/2}} \int_{\tau=0}^{\infty} \tau e^{-\tau^2/2} \lambda((A^r \setminus (A^r)_r) \cap B(\tau)) d\tau.
$$

In (a), we let $A^r := A \cap B(\tau + r + \delta)$. Since points in $A \setminus A^r$ are more than $r + \delta$ distance away from $B(\tau)$, we may think of $A^r$ as the part of $A$ that is relevant to $B(\tau)$. In particular, the set

$$
\lambda((A^r \setminus (A^r)_r) \cap B(\tau))
$$

is a subset of $\lambda((A^r \setminus (A^r)_r) \cap B(\tau + r + \delta))$, which is bounded from above by $\lambda((A_{r+\delta} \setminus A_r) \cap B(\tau))$. Therefore, we can conclude that

$$
\gamma(A_{r+\delta} \setminus A_r) \leq \frac{1}{(2\pi)^{d/2}} \int_{\tau=0}^{\infty} \tau e^{-\tau^2/2} \lambda((A^r \setminus (A^r)_r) \cap B(\tau)) d\tau.
$$

Hence, we obtain

$$
\bar{\gamma}(A_r) \leq \max(C, C \frac{r}{d}).
$$

This completes the proof of Theorem 2.
We may expand \((\tau + \delta + 3r/2)^d\). Using the closed-form expression of the integral, we obtain
\[
\int_{\tau=0}^{\infty} e^{-\tau^2/2} \tau^{i+1} d\tau = 2^{i/2} \Gamma(1 + i/2),
\]
we arrive at
\[
\gamma(A_{r+\delta} \setminus A_r) \leq \frac{1}{(2\pi)^{d/2}} \left(2^{d-1} \Omega_d \frac{(r + \delta)^d - r^d}{d}\right) \cdot \frac{1}{(r/2)^d} \sum_{i=0}^{d} \binom{d}{i} 2^{i/2} \Gamma(1 + i/2)(3r/2)^{d-i}.
\](33)

Dividing both sides by \(\delta\) and taking the lim sup as \(\delta \to 0\),
\[
\bar{\gamma}(A_r) \leq \frac{1}{(2\pi)^{d/2}} \left(2^{d-1} \Omega_d \right) \cdot \sum_{i=0}^{d} C_i r^{i-1},
\](34)
where \(C_i := \binom{d}{i} 2^{(d-i)/2} \Gamma(1 + (d - i)/2)(3/2)^i\). For simplicity, let us assume that \(r \leq 1\). Note that for \(r > 1\), the bound for \(r = 1\) continues to be valid. This is because any \(r\)-parallel set \(A_r\) for \(r > 1\) is also a 1-parallel set for some set \(A\). Denote the bound for \(r = 1\) by \(C\); i.e.,
\[
C := \frac{1}{(2\pi)^{d/2}} \left(2^{d-1} \Omega_d \right) \cdot \sum_{i=0}^{d} C_i.
\]

For \(r \leq 1\), we conclude that
\[
\bar{\gamma}(A_r) \leq \frac{1}{(2\pi)^{d/2}} \left(2^{d-1} \Omega_d \right) \cdot \sum_{i=0}^{d} C_i r^{i-1}
\leq \frac{1}{(2\pi)^{d/2}} \left(2^{d-1} \Omega_d \right) \cdot \sum_{i=0}^{d} C_i \cdot \frac{1}{r}
= \frac{C}{r}.
\]

This leads to our final bound,
\[
\bar{\gamma}(A_r) \leq \max \left(C, \frac{C}{r}\right).
\]
To obtain a rough upper bound on $C$, note that each $C_i$ can be upper-bounded by $2^d \cdot 2^{d/2} \cdot \Gamma(1 + d/2)(3/2)^d$, which leads to

$$C = \frac{2^{2d-1} d \pi^{d/2}}{(2\pi)^{d/2} \Gamma(1 + d/2)} \cdot \sum_{i=0}^{d} C_i \leq \frac{2^{2d-1} \cdot d}{2^{d/2} \Gamma(1 + d/2)} \cdot d 2^{d/2} \Gamma(1 + d/2) 3^d = 2^{2d-1} d 3^d.$$

Moreover,

$$C \geq \frac{2^{2d-1} d \pi^{d/2}}{(2\pi)^{d/2} \Gamma(1 + d/2)} \cdot C_0 = \frac{2^{2d-1} d \pi^{d/2}}{(2\pi)^{d/2} \Gamma(1 + d/2)} \cdot 2^{d/2} \Gamma(1 + d/2) 2^d = 2^{2d-1} d.$$

This gives $C = e^{\Theta(d)}$. We make no claims about the tightness of this bound with regards to the dimension $d$. As we shall note in Remark 5, there conflicting arguments whether our analysis can be strengthened to derive a bound with $C = \Theta(1)$.

**Remark 1.** The $1/r$ dependence of the Gaussian surface area upper bound is optimal. To see this, let the set $S^r$ be a maximal packing arrangement of radius-$r$ balls in $B_d(1)$, and let $N^r$ be the number of balls that are packed. Clearly, the Gaussian surface area of $S^r$ is at most $N^r \Omega_{d^2} r^{-d} = C_d M^r r^{-d} - 1$, where $C_d$ is a dimension-dependent constant. It is now easy to check that $N^r = \Omega(1/r^d)$, so that $\gamma(\partial S^r) = \Omega(1/r)$.

**Remark 2.** We fixed $r \leq 1$ in our proof, which may seem arbitrary. Indeed, a tighter bound on the constant $C$ can be obtained if $r \leq r^*$, where $r^*$ minimizes the bound in expression (34). However, by examining the coefficient $C_1$, it may be verified that the tighter constant is still $e^{\Theta(d)}$.

**Remark 3.** For the scaled Gaussian distribution $\mathcal{N}(0, \sigma^2 I_d)$, the Gaussian surface area upper bound changes to $\max\left(\frac{C}{\sigma}, \frac{C}{\pi}\right)$, where $C$ is as in Theorem 2.

**Remark 4.** Theorem 2 also holds for distributions that have been smoothed by convolving with the Gaussian density. Specifically, if $\mu = \tilde{\mu} * \mathcal{N}(0, \sigma^2 I_d)$ then the $\mu$-surface area of $r$-parallel sets is upper-bounded by $\max\left(\frac{C}{\sigma}, \frac{C}{\pi}\right)$.

## 5 Applications to machine learning

We now describe applications of the preceding results to problems in machine learning.

### 5.1 Computational complexity for learning parallel sets

The sample complexity of learning a class $\mathcal{C}$ of Boolean functions on $\mathbb{R}^d$ under an unknown distribution is characterized by the Vapnik-Chervonenkis (VC) dimension of $\mathcal{C}$ [23]. It is not hard to check that the VC dimension of indicator functions on $r$-parallel sets is infinite (even if we consider “bounded” $r$-parallel sets within a ball $B(R)$ for $R > r$), so this class is not learnable without making some assumptions on the data distribution. Klivans et al. [8] suggested the Gaussian data
distribution as a natural setting in which to study the learnability of indicator functions of subsets of \( \mathbb{R}^d \). In particular, the authors examined the computational complexity of learning such functions for a variety of subsets, such as halfspaces, convex sets, Euclidean balls, and intersections of halfspaces. The authors proposed the Gaussian surface area of a set as a useful “complexity measure” for determining the difficulty of learning, and provided three reasons for doing so: (1) Every measurable set can be assigned a complexity measure; (2) it is a natural geometric notion; and (3) sets with “wiggly” boundaries are harder to learn, which is captured by their larger Gaussian surface area. The main result from Klivans et al. is as follows:

**Theorem 3** (Theorems 9, 10, and 15 from Klivans et al. [8]). Let \( \mathcal{C} \) be a class of measurable sets in \( \mathbb{R}^d \) such that the Gaussian surface area of all sets in \( \mathcal{C} \) is upper-bounded by \( S \). We shall denote the set of indicator functions of sets in \( \mathcal{C} \) by \( \mathcal{C} \), as well. Under the standard Gaussian distribution, the following results hold for learning \( \mathcal{C} \) up to an accuracy of \( \epsilon \) and confidence of \( 1 - \delta \):

1. **Agnostic learning:** There exists an algorithm that runs in time \( \text{poly} \left( d^{O(S^2/\epsilon^4)}, \frac{1}{\epsilon}, \log \frac{1}{\delta} \right) \) and agnostically learns \( \mathcal{C} \).

2. **PAC learning:** There exists an algorithm that runs in time \( \text{poly} \left( d^{O(S^2/\epsilon^2)}, \frac{1}{\epsilon}, \log \frac{1}{\delta} \right) \) and PAC learns \( \mathcal{C} \).

An example of an application of the above result is for learning convex sets: Klivans et al. showed that, the class of all convex sets—despite having infinite VC dimension—is efficiently learnable under the Gaussian distribution, by exploiting the fact that the Gaussian surface areas of convex sets in \( \mathbb{R}^d \) is bounded above by \( \Theta(d^{1/4}) \). If one is able to bound the Gaussian surface areas of sets in \( \mathcal{C} \), then Theorem 3 may be directly applied to bound the computational complexity of learning \( \mathcal{C} \). Since Theorem 2 provides bounds on the Gaussian surface areas of \( r \)-parallel sets, we may directly apply Theorem 3 to conclude the following result concerning the computational complexity of learning \( r \)-parallel sets:

**Theorem 4.** Let \( r > 0 \). Let \( \mathcal{C}_r = \{ A_r \mid A \subset \mathbb{R}^d \text{ is measurable} \} \). Under the standard Gaussian distribution, the following results hold for learning \( \mathcal{C}_r \) up to an accuracy of \( \epsilon \) and confidence of \( 1 - \delta \):

1. **Agnostic learning:** There exists an algorithm that runs in time

   \[
   \text{poly} \left( d^{O(\max(C^2,C^2/r^2)/\epsilon^4)}, \frac{1}{\epsilon}, \log \frac{1}{\delta} \right)
   \]

   and agnostically learns \( \mathcal{C} \).

2. **PAC learning:** There exists an algorithm that runs in time

   \[
   \text{poly} \left( d^{O(\max(C^2,C^2/r^2)/\epsilon^2)}, \frac{1}{\epsilon}, \log \frac{1}{\delta} \right)
   \]

   and PAC learns \( \mathcal{C} \).

Klivans et al. noted that for two sets \( K_1 \) and \( K_2 \), we have the inequality \( \gamma(\partial(K_1 \cup K_2)) \leq \gamma(\partial K_1) + \gamma(\partial K_2) \). They also showed that the Gaussian surface areas of Euclidean balls (of any radius) are upper-bounded by a constant. Applying this result to the union of balls, we may derive upper bounds on the computational complexity of learning a union of \( O(1) \) ball. However, since \( r \)-parallel sets are the union of (possibly) uncountably many balls, the results from Klivans et al. cannot be applied directly. Observe also that as \( r \) decreases, the boundaries of sets in \( \mathcal{C}_r \) become
more “wiggly”, and the increased difficulty of learning is reflected in the larger exponent \( C^2/r^2 \). Lastly, our analysis reveals that \( C = e^{\Theta(d)} \), so for a fixed \( r < 1 \), the exponent of \( d \) in the learning time bounds is \( d^{\Theta(\max(d))/r^2} \) for agnostic learning and \( d^{\exp(\Theta(d))/r^2} \) for PAC learning. As noted earlier, it may be possible to derive a stronger upper bound where \( C = \Theta(1) \); if so, the corresponding computational complexity bounds would also be strengthened.

**Remark 5.** One reason to believe \( C \) could be made \( \Theta(1) \) is as follows. By taking \( r \to \infty \) in Theorem 2, the upper bound is simply \( C \). Intuitively, a parallel set when \( r \) is very large resembles a halfspace whose Gaussian surface area is known to be bounded by \( \Theta(1) \). A reason to believe \( C \) cannot be made \( \Theta(1) \) is because it would lead to the surprising result that 1-parallel sets are essentially as hard to learn as halfspaces. This runs counter to intuition since 1-parallel sets appear to be far more expressive than halfspaces.

### 5.2 Sample complexity for estimating robust risk

Adversarial machine learning has been the focus of much research in the recent past, owing to the observed fragility of deep neural networks under adversarial perturbations. A brief description of the underlying mathematical problem phrased in the language of hypothesis testing is provided below.

#### 5.2.1 Problem setting and background

Consider two equally likely hypotheses, denoted by \( \{0, 1\} \). For \( i \in \{0, 1\} \), under hypothesis \( i \), a sample drawn from distribution \( \mu_i \) is observed. To minimize the error probability, it is well known that the optimal decision rule is the maximum likelihood rule and the resulting error (called the Bayes risk) is given by \( 1 - d_{TV}(\mu_0, \mu_1)^2 \), where \( d_{TV} \) is the total variation distance. Hypothesis testing under adversarial contamination considers an identical setting with one modification: The adversary is allowed to arbitrarily perturb the observed sample within a Euclidean ball of a certain radius, say \( r > 0 \). The radius \( r \) is the adversary’s budget.

Finding the optimal decision region for hypothesis testing with an adversary has been studied recently in Bhagoji, Cullina, and Mittal [21] and Pydi and Jog [22]. Suppose \( A \) is the (measurable) set where hypothesis 1 is declared. Then the robust risk for this decision region is given by

\[
\mathcal{E}(A) = \frac{\mu_0(A_r) + \mu_1((A^c)_r)}{2} \leq \frac{\mu_0(A_r) + 1 - \mu_1(((A^c)_r))^c}{2} \leq \frac{1}{2} - \frac{\mu_1(A - r) - \mu_0(A_r)}{2},
\]

where in \((a)\), we use the notation \( A - r = ((A^c)_r)^c \). As shown in Pydi and Jog [22], the optimal robust risk may also be expressed as

\[
\mathcal{E}^* = \frac{1}{2} - \sup_A \frac{\mu_1(A_r) - \mu_0(A_r)}{2} = \frac{1}{2} - \sup_A \frac{\mu_1(A) - \mu_0(A_{2r})}{2}.
\]

The main result of Bhagoji et al. and Pydi and Jog connects the optimal robust risk to an optimal transport cost between the two data distributions. To be precise, the following result was established:
Theorem 5 (Bhagoji et al. [21] and Pydi and Jog [22]). Define the cost function \( c_r : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) as
\[
c(x, y) = 1\{d(x, y) > 2r\},
\]
where \( d \) is the usual Euclidean distance. Define the optimal transport cost \( D_r(\mu_0, \mu_1) \) between two distributions \( \mu_0 \) and \( \mu_1 \) on \( \mathbb{R}^d \) as
\[
D_r(\mu_0, \mu_1) = \inf_{X \sim \mu_0, Y \sim \mu_1} \mathbb{E} c_r(X, Y),
\]
where the infimum is taken over all couplings of \( X \) and \( Y \) with marginals \( \mu_0 \) and \( \mu_1 \), respectively.

Then for an adversarial budget \( r \), the optimal robust risk for a binary hypothesis testing problem with equal priors and data distributions \( \mu_0 \) and \( \mu_1 \) satisfies the equality
\[
\mathcal{E}^* = \frac{1 - D_r(\mu_0, \mu_1)}{2}.
\]

The above result follows from Strassen’s theorem [24], which gives the equality
\[
\sup_A \left( \frac{\mu_1(A) - \mu_0(A_{2r})}{2} \right) = D_r(\mu_0, \mu_1).
\]

5.2.2 Estimating \( D_r(\mu_0, \mu_1) \)

Theorem 5 is useful because it provides a fundamental lower bound for robust risk that holds for all hypothesis testing rules. One may evaluate a testing rule based on the closeness of its performance to this optimal value. However, this is not possible in practice, since the data distributions \( \mu_0 \) and \( \mu_1 \) are unknown; one only has access to the “empirical distribution” derived from a data set composed of independent draws from the data distribution.

In Bhagoji et al. and Pydi and Jog, the authors calculate \( D_r(\cdot, \cdot) \) between the empirical distributions (based on a finite data set), instead. However, neither work addresses the proximity of the empirically calculated \( D_r \) to the true \( D_r \). Indeed, it is not even clear if \( D_r \) calculated from the empirical distribution is a consistent estimator of the true \( D_r \).

Some intuition about \( D_r \) can be obtained by observing that when \( r = 0 \), it equals the total variation distance. As noted in Pydi and Jog, however, \( D_r \) is neither a metric nor a pseudo-metric on the space of probability distributions. Moreover, estimating the total variation distance between \( \mu_0 \) and \( \mu_1 \) by calculating the total variation distance between the the empirical distributions is bound to fail, since the latter will always yield a value of 1 for continuous \( \mu_0 \) and \( \mu_1 \). Interestingly, this is not the case for \( D_r \) when \( r > 0 \).

In what follows, we show that under suitable smoothness conditions, the plug-in estimator is a consistent estimator of the true \( D_r \). We also provide bounds on the number of samples necessary to approximate \( D_r(\mu_0, \mu_1) \) up to an error of \( \epsilon \) with a probability of \( 1 - \delta \). The main technical ingredient is the reverse Gaussian isoperimetric inequality from Theorem 2.

5.2.3 Sample complexity bounds for estimating \( D_r(\mu_0, \mu_1) \)

We make the following assumptions on \( \mu_0 \) and \( \mu_1 \):

(A1) Each \( \mu_i \) is a Gaussian-smoothed version of some \( \tilde{\mu}_i \); i.e., \( \mu_i = \tilde{\mu}_i * \mathcal{N}(0, \sigma^2 I_d) \).

(A2) Each \( \tilde{\mu}_i \) has bounded support on \( \mathbb{R}^d \).
Assumption (a) is easy to satisfy in practice by simply adding Gaussian noise to the observed data samples. Assumption (b) makes our analysis simpler, but we note that it can be considerably relaxed. We use the notation $\mu^n_i$ for the empirical distribution with $n$ samples, and use the random variables $X_0 \sim \mu_0$, $X_1 \sim \mu_1$, $X^n_0 \sim \mu^n_0$, and $X^n_1 \sim \mu^n_1$. If $X \sim \mu_X$ and $Y \sim \mu_Y$, we shall use the notation $D_r(\mu_X, \mu_Y)$ and $D_r(X, Y)$ interchangeably. We denote $C(\sigma, r) := \max \left( \frac{\sigma}{\sigma}, \frac{\sigma}{\tau} \right)$, where $C$ is as in Theorem 2.

**Lemma 5.1.** ([Corollary 3.1 from Pydi and Jog [22]]) The following inequality holds:

$$D_r(\mu_0, \mu_1) \leq \frac{W_1(\mu_0, \mu_1)}{2r},$$

where $W_1$ is the 1-Wasserstein distance.

Lemma 5.1 is a straightforward consequence of applying Markov’s inequality to the equality $D_r(X, Y) = \inf_{\Pi(X, Y)} P(d(X, Y) > 2r)$, where $\Pi(X, Y)$ is the set of couplings of $X$ and $Y$.

**Lemma 5.2.** Let $\eta \in (0, r/3)$. The following inequalities hold:

\[
\begin{align*}
D_{r+2\eta}(X_0, X_1) &\leq D_r(X^n_0, X^n_1) + D_\eta(X_0, X^n_0) + D_\eta(X_1, X^n_1), \quad \text{and} \\
D_{r-2\eta}(X_0, X_1) &\geq D_r(X^n_0, X^n_1) - D_\eta(X_0, X^n_0) - D_\eta(X_1, X^n_1).
\end{align*}
\]

**Proof.** Consider a coupling of $(X_0, X_1, X^n_0, X^n_1)$ such that the Markov chain $X_0 \rightarrow X^n_0 \rightarrow X^n_1 \rightarrow X_1$ holds. The joint distributions of adjacent links in the chain are as follows: $(X_0, X^n_0) \sim \pi_0$, which is optimal for $D_\eta; (X_1, X^n_1) \sim \pi_1$, which is optimal for $D_r; (X_0, X_1) \sim \pi_0$, which is optimal for $D_r$. The Markov chain induces a coupling on $(X_0, X_1)$ that is not necessarily optimal for the $D_{r+2\eta}$ cost. This means that

\[
D_{r+2\eta}(X_0, X_1) \leq \mathbb{P}(d(X_0, X_1) > 2r + 4\eta) \\
\leq \mathbb{P}(d(X_0, X^n_0) > 2\eta) + \mathbb{P}(d(X_1, X^n_1) > 2\eta) + \mathbb{P}(d(X^n_0, X^n_1) > 2r) \\
= D_\eta(X_0, X^n_0) + D_\eta(X_1, X^n_1) + D_r(X^n_0, X^n_1).
\]

To obtain the second inequality, consider a different coupling between $(X_0, X_1, X^n_0, X^n_1)$ such that the Markov chain $X^n_0 \rightarrow X_0 \rightarrow X_1 \rightarrow X^n_1$ holds. The joint distributions of adjacent links in the chain are as follows: $(X_0, X^n_0) \sim \pi_0$, which is optimal for $D_\eta; (X_1, X^n_1) \sim \pi_1$, which is optimal for $D_\eta; (X_0, X_1) \sim \pi_0$, which is optimal for $D_{r-2\eta}$. The Markov chain induces a coupling on $(X^n_0, X^n_1)$ that is not necessarily optimal for the $D_r$ cost. This means that

\[
D_r(X^n_0, X^n_1) \leq \mathbb{P}(d(X^n_0, X^n_1) > 2r) \\
\leq \mathbb{P}(d(X_0, X^n_0) > 2\eta) + \mathbb{P}(d(X_1, X^n_1) > 2\eta) + \mathbb{P}(d(X_0, X_1) > 2r - 4\eta) \\
= D_\eta(X_0, X^n_0) + D_\eta(X_1, X^n_1) + D_{r-2\eta}(X_0, X_1).
\]

**Lemma 5.3.** Let $0 < r_1 < r_2$. Then the following inequality holds:

$$0 \leq D_{r_1}(\mu_0, \mu_1) - D_{r_2}(\mu_0, \mu_1) \leq 2C(\sigma, 2r_1)(r_2 - r_1).$$

**Proof.** By the definition of $D_r$, it follows immediately that $D_{r_1}(\mu_0, \mu_1) \geq D_{r_2}(\mu_0, \mu_1)$. Let $A^*$ be the set that achieves the equality \footnote{If $A^*$ does not exist, the proof goes through by considering a sequence of sets $(A^n)^*$ such that $\mu_0((A^n)^*) - \mu_1((A^n)^*) \rightarrow \sup \mu_0(A) - \mu_1(A_{2r_1})$.} $D_{r_1}(\mu_0, \mu_1) = \mu_0(A^*) - \mu_1(A^*_{2r_1})$. 

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We have the sequence of inequalities
\[
D_{r_2}(\mu_0, \mu_1) = \sup_A \mu_0(A) - \mu_1(A_{2r_2}) \\
\geq \mu_0(A^*) - \mu_1(A_{2r_2}^*) \\
= \mu_0(A^*) - \mu_1(A_{2r_1}^*) - \mu_1(A_{2r_2}^* \setminus A_{2r_1}^*) \\
= D_{r_1}(\mu_0, \mu_1) - \mu_1(A_{2r_2}^* \setminus A_{2r_1}^*).
\]

We also have the equality
\[
\mu_1(A_{2r_2}^* \setminus A_{2r_1}^*) = \int_{\tau=0}^{2(r_2-r_1)} \mu_1(\partial A_{2r_1}^* + \tau)d\tau \\
\leq 2C(\sigma, 2r_1)(r_2-r_1).
\]

Here, (a) follows from Remarks 3 and 4.

**Theorem 6.** Let \(\mu_0\) and \(\mu_1\) satisfy the assumptions (A1) and (A2). Also assume that \(d \geq 3\). Let \(r, \epsilon > 0\) and \(\delta \in (0, 1)\). Then for \(n \geq N_0 = \Theta\left(C(\sigma, 2r_1)^d \log(1/\delta)^{1/d}\right)\), the following inequality holds with probability at least \(1 - \delta\):
\[
|D_r(\mu_0, \mu_1) - D_r(\mu_0^n, \mu_1^n)| \leq \epsilon.
\]

**Proof.** Let \(\eta \in (0, r/3)\). Let \(N_0\) be such that for \(n \geq N_0\), the following holds with probability \(1 - \delta/2\), for \(i \in \{0, 1\}\):
\[
W_1(X_i, X_i^n) \leq 2\eta \cdot \frac{\epsilon}{4}.
\]

Numerous results exist concerning the convergence of the empirical measure in terms of the Wasserstein metric; we use here a result from Fournier and Guillin [25, Theorem 2], which states that for all \(N \geq 1\) and all small enough \(x\),
\[
\mathbb{P}(W_1(\mu_i, \mu_i^n) > x) \leq c_0 e^{-c_1 N x^d},
\]
where \(c_0\) and \(c_1\) are constants that depend on \(d\). Substituting \(x\) to be \(\eta\epsilon/2\) and
\[
N \geq \frac{\log(2/\delta) + \log c_0}{c_1 (\eta\epsilon/2)^d} =: N_0,
\]
inequality (35) is satisfied with probability \(1 - \delta/2\). When \(N \geq N_0\), Lemma 5.2 implies that the following bound holds with probability \(1 - \delta\):
\[
D_{r+2\eta}(X_0, X_1) \leq D_r(X_0^n, X_1^n) + D_\eta(X_0, X_0^n) + D_\eta(X_1, X_1^n) \\
\leq D_r(X_0^n, X_1^n) + \frac{W_1(X_0, X_0^n)}{2\eta} + \frac{W_1(X_1, X_1^n)}{2\eta} \\
\leq D_r(X_0^n, X_1^n) + \frac{\epsilon}{2}.
\]

Lemma 5.3 gives the inequality
\[
D_r(X_0, X_1) \leq D_{r+2\eta}(X_0, X_1) + 4\eta C(\sigma, 2r),
\]

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implying that
\[ D_r(X_0, X_1) \leq D_r(X_0^n, X_1^n) + \frac{\epsilon}{2} + 4\eta C(\sigma, 2r). \] (36)

Similarly, with probability \(1 - \delta\),
\[ D_{r-2\eta}(X_0, X_1) \geq D_r(X_0^n, X_1^n) - D_{\eta}(X_0, X_0^n) - D_{\eta}(X_1, X_1^n) \geq D_r(X_0^n, X_1^n) - \frac{\epsilon}{2}. \]

Lemma 5.3 gives the inequality
\[ D_r(X_0, X_1) \geq D_{r-2\eta}(X_0, X_1) - 4\eta C(\sigma, 2(r-2\eta)), \]
implying
\[ D_r(X_0, X_1) \geq D_r(X_0^n, X_1^n) - \frac{\epsilon}{2} - 4\eta C(\sigma, 2(r-2\eta)). \] (37)

Combining inequalities (36) and (37), we see that with probability \(1 - \delta\),
\[ |D_r(X_0, X_1) - D_r(X_0^n, X_1^n)| \leq \frac{\epsilon}{2} + 4\eta C(\sigma, 2(r-2\eta)) \leq \frac{\epsilon}{2} + 4\eta C(\sigma, 2r/3). \]

Now pick \( \eta = \frac{\epsilon}{2C(\sigma, 2r/3)} \) to conclude that
\[ |D_r(X_0, X_1) - D_r(X_0^n, X_1^n)| \leq \epsilon. \]

Note that
\[ N_0 = \frac{\log(2/\delta) + \log c_0}{c_1(\eta/2)^d} = \Theta \left( \frac{C(\sigma, 2r/3)^d \log(1/\delta)}{\epsilon^{2d}} \right). \]

This concludes the proof. \( \square \)

Remark 6. Observe that smaller values of \(r\) correspond to a larger sample size requirement. Also, the smaller the variance \(\sigma\) used for smoothing the distributions, the more samples are required. Both observations align with intuition.

6 Conclusion

Convolving with a small Gaussian noise is a common technique used in analysis to smooth probability distributions. The natural counterpart to such a procedure in geometry is to take the parallel set of any measurable set. It is intuitive that parallel sets are “more smooth,” since they cannot have arbitrarily wiggly boundaries. In this paper, we showed that bounded parallel sets in \(\mathbb{R}^d\) have bounded surface areas under the Lebesgue measure and arbitrary parallel sets have bounded surface areas under the Gaussian measure. We showed that our reverse isoperimetric inequalities have applications in machine learning.

We mention a few open problems that are worth exploring. The dependence of \(C\) on the dimension \(d\) in Theorem 2 is \(e^{\Theta(d)}\). It is unclear whether a tighter analysis could yield a bound
where $C$ is replaced by a dimension-independent constant. An interesting but challenging problem is identifying the Gaussian surface area of an optimally dense packing of unit balls in $\mathbb{R}^d$, and use this to get a lower bound for the dimension dependence of $C$.

Our analysis relied heavily on geometric properties of Euclidean balls and is therefore restricted to $r$-parallel sets in the Euclidean norm. In adversarial machine learning, one is interested in $\ell_p$-norms for all values of $p \geq 1$, especially the $\ell_\infty$-norm. It would be very interesting to extend our bounds to parallel sets in $\ell_p$-norms and more general Minkowski-sum-based smoothing procedures. A possible approach would be to establish analogs of Theorems 1 and 2 for Minkowski smoothing by convex polytopes, and taking the limit to generalize these results for arbitrary convex sets.

Acknowledgements

The author is grateful to the National Science Foundation for funding his research through the grants CCF-1907786 and CCF-1942134, and to Ankit Pensia and Muni Sreenivas Pydi for helpful discussions. The figures in Section 2 were created using GeoGebra, a free online tool for geometry.

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