NON-VANISHING OF MIYAWAKI TYPE LIFT

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Abstract. In this paper, we show the non-vanishing of the Miyawaki type lift for $GSpin(2,10)$ constructed in [19], by using the fact that the Fourier coefficient at the identity is closely related to the Rankin-Selberg $L$-function of two elliptic cusp forms. In the case of the original Miyawaki lift of Siegel cusp forms, we reduce the non-vanishing problem to that of the Rankin-Selberg convolution of two Siegel cusp forms.

1. Introduction

Miyawaki type lifts are kinds of Langlands functorial lifts and a special case was first conjectured by Miyawaki [22] and proved by Ikeda for Siegel cusp forms in [13]. Since then, such a lift for Hermitian modular forms was constructed by Atobe and Kojima [2], and for half-integral weight Siegel cusp forms by Hayashida [7], and we constructed Miyawaki type lift for $GSpin(2,10)$ [19]. Recently Ikeda and Yamana [15] generalized Ikeda type construction for Hilbert-Siegel cusp forms in a remarkable way and accordingly Miyawaki type lift for Hilbert-Siegel cusp forms for any level follows. In all these works, a construction of Miyawaki type lift takes two steps as follows: First, construct Ikeda type lift on a bigger group from an elliptic cusp form, and then define a certain integral on a block diagonal element which is an analogue of pull-back formula studied by Garrett [6] for Siegel Eisenstein series. If the integral is non-vanishing, then it is shown that it is a Hecke eigen cusp form, and it is the Miyawaki type lift. The question of non-vanishing of the integral was left open.

In this paper, we show the non-vanishing for certain special cases. The idea is to write the Ikeda type lift as Fourier-Jacobi expansion with matrix index $S$. Then the Fourier coefficients of the Miyawaki type lift become the integral of vector-valued modular forms and theta series. By choosing $S$ carefully, we can show that the Fourier coefficient of index $S$ is non-vanishing.

2010 Mathematics Subject Classification. Primary 11F55; Secondary 11F70, 22E55, 20G41.
Key words and phrases. Miyawaki type lift, Langlands functoriality.

The first author is partially supported by NSERC. The second author is partially supported by JSPS Grant-in-Aid for Scientific Research (C) No.15K04787.
In particular, in Section 2, we show it for the Miyawaki type lift for $GSpin(2,10)$. The Miyawaki type lift in this case is a cusp form on $GSpin(2,10)$ associated to two cusp forms $f \in S_{2k}(SL_2(\mathbb{Z}))$ and $g \in S_{2k+8}(SL_2(\mathbb{Z}))$. In this case, the situation is very nice in that $S = 1_2$ is associated to an even unimodular matrix $E_8$ ($E_8$ denotes the unique even $8 \times 8$ unimodular matrix), and the integral becomes essentially the Rankin-Selberg $L$-function $L(s,f \otimes g)$ at $s = 4$. Therefore it is non-vanishing. Recently, we constructed the Ikeda type lift for the exceptional group of type $E_7$ for any level, and hence the Miyawaki type lift can be generalized in an obvious way. For the non-vanishing for higher level case, we use the adelic language by following [24].

In Section 3, we consider the original Miyawaki lift in [13]. Namely, the Miyawaki lift is a cusp form on $S_{k+n+r}(Sp_{4n+2r}(\mathbb{Z}))$ associated to two cusp forms $f \in S_{2k}(SL_2(\mathbb{Z}))$ and $g \in S_{k+n+r}(Sp_{2r}(\mathbb{Z}))$. Consider the Fourier-Jacobi expansion of the Ikeda lift $F_f \in S_{k+n+r}(Sp_{4n+4r}(\mathbb{Z}))$ with matrix index $S$, where $S$ is a half-integral symmetric matrix of size $2n+r$. If $2n+r$ is divisible by 8, there exists an even unimodular matrix of size $2n+r$, and in that case, the integral becomes the integral of two Siegel cusp forms and theta series. Here we use the Siegel’s formula that says that the linear combination of the theta series is the Eisenstein series. Hence a linear combination of the integrals becomes the Rankin-Selberg convolution of two Siegel cusp forms [25]. It is not known whether the convolution is non-vanishing. If we assume the non-vanishing of the convolution, then the Miyawaki lift is non-vanishing.

In Section 4, we consider the Miyawaki lift for the unitary group in [2]. Namely, let $K$ be an imaginary quadratic field with discriminant $-D$, and $\chi = \chi_D$ be the Dirichlet character corresponding to $K/\mathbb{Q}$. Let $f$ be a normalized Hecke eigen cusp form belonging to \( S_{k+n+r}(Sp_{4n+2r}(\mathbb{Z})) \) if $n$ odd, and \( S_{2k+1}(\Gamma_0(D),\chi) \) if $n$ even. Then given a cusp form $g$ of weight $2k + 2\left\lfloor \frac{n}{2} \right\rfloor + 2r$ on $U(r,r)$ defined over $K/\mathbb{Q}$, the Miyawaki lift $F_{f,g}$ is a cusp form of weight $2k + 2\left\lfloor \frac{n}{2} \right\rfloor + 2r$ on $U(n+r,n+r)$ defined over $K/\mathbb{Q}$. Atobe and Kojima [2] showed that if $F_{f,g}$ is non-vanishing, it is a Hecke eigen form. We consider the special case $4|(n + r)$. In this case, we use classification of even unimodular matrices over imaginary quadratic fields in [4], and use the Siegel-Weil formula for unitary groups [11] in order to obtain the analogue of the Siegel formula for Hermitian theta series. If $r = 1$, the Miyawaki lift $F_{f,g}$ is non-vanishing. If $r > 1$, assuming the non-vanishing of the Rankin-Selberg convolution, we show that the Miyawaki lift is non-vanishing.
In the last section, we give an outline of non-vanishing of the Miyawaki lift for half-integral weight Siegel cusp forms assuming the non-vanishing of a similar integral involving half-integral weight modular forms.

Finally we remark that recently, Atobe \[1\] independently obtained non-vanishing of the original Miyawaki lift by a different method. He assumes Gan-Gross-Prasad conjecture \[5\].

Acknowledgments. We would like to thank H. Atobe, T. Ikeda, S. Hayashida, and M. Tsuzuki for helpful discussions. Special thanks are given to H. Atobe for pointing out some mistakes in an earlier version and to M. Tsuzuki for guiding the second author on the computation in Section 2.2.

2. Miyawaki type lift for \(GSpin(2,10)\)

In this section we prove the non-vanishing of the Miyawaki type lift constructed in \[19\]. We showed that under the assumption of non-vanishing, it is a Hecke eigen cusp form. After these works the authors generalized the main theorems in \[18\] and hence the Miyawaki type lift can be generalized in an obvious way. However we treat the non-vanishing separately for level one and higher level cases because of the nature of the construction.

2.1. Level one. Let \(f \in S_{2k}(SL_2(\mathbb{Z})), g \in S_{2k+8}(SL_2(\mathbb{Z}))\) be Hecke eigen cusp forms. Let \(F_{f,g}\) be the Miyawaki lift constructed in \[19\]. It is defined as an integral:

\[
(2.1) \quad F_{f,g}(Z) = \int_{SL_2(\mathbb{Z}) \setminus \mathbb{H}} F_f \left( \begin{array}{cc} Z & 0 \\ 0 & \tau \end{array} \right) \overline{g(\tau)} (\text{Im} \tau)^{2k+6} d\tau,
\]

where \(F_f\) is the Ikeda type lift constructed in \[18\].

In \[19\] Section 8, we wrote it as

\[
(2.2) \quad F_{f,g}(Z) = \sum_{S} A_S e^{2\pi i Tr(TS)},
\]

where \(S \in J_{2}(\mathbb{Z})_+\) (see \[19\] (8.1)), and

\[
(2.3) \quad A_S = \int_{SL_2(\mathbb{Z}) \setminus \mathbb{H}} F_S(\tau,0) \overline{g(\tau)} \text{Im}(\tau)^{2k+8} d^*\tau,
\]

where \(d^*\tau = \frac{dx dy}{y^2}\) is the invariant measure on \(\mathbb{H}\). (See Section 8 of \[19\].)

If \(S = 1_2 \in J_{2}(\mathbb{Z})_+\), then by \[18\] Appendix] (see also the corrections in Section 2 of \[20\]), the quadratic form \(\sigma_S\) associated to \(S\) is of type \(E_{8}^{\oplus 2}\) and then we can show that \(\Xi(S) = \{0\}\) in \[18\].
Note that for the basis \( \{ \alpha_i \}_{i=0}^7 \) defining the integral Cayley numbers (see Section 2 of [18]), \( \sigma_S = V \perp V \cong E_8 \perp E_8 \) where \( V \) is the quadratic space over \( \mathbb{Z} \) given by

\[
\begin{pmatrix}
2 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\
0 & 2 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 2 & 0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 2 & 1 & 1 & 0 & -1 \\
0 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\
-1 & -1 & 0 & 1 & 0 & 2 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 & 0 & 2 & 0 \\
-1 & 0 & 1 & -1 & 0 & 0 & 0 & 2
\end{pmatrix}
\]

whose determinant is one. Then from the formula [18] (9.4),

\[
F_{S,0}(\tau) = \sum_{N>0} N^{2k-8} \prod_p \tilde{f}_{p,N}(\alpha_p) e^{2\pi i N \tau}.
\]

By the formula in [17], if \( S = 1_2 \),

\[
f_{p,N}^p(X) = \frac{1 - X^{v_p(N)+1}}{1 - X}.
\]

Hence \( \tilde{f}_{S,N}^p = X^{-v_p(N)} + X^{-v_p(N)+2} + \cdots + X^{v_p(N)} \). So \( F_{S,0}(\tau) = f(\tau) \). Therefore,

\[
F_S(\tau, 0) = f(\tau) \theta(\tau),
\]

where \( \theta \) is a theta function in 16 variables, and hence a modular form of weight 8 with respect to \( SL_2(\mathbb{Z}) \). Since \( \dim S_8(SL_2(\mathbb{Z})) = 1 \), \( \theta(\tau) = E_8(\tau) \). Here

\[
E_8(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} j(\gamma, \tau)^{-8},
\]

where \( \Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL_2(\mathbb{Z}) \right\} \), and \( j(\gamma, \tau) = c\tau + d \) for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \). Hence

\[
A_{12} = \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} f(\tau) g(\tau) E_8(\tau) Im(\tau)^{2k+8} \ d^* \tau.
\]
Then by the usual unfolding method,
\[
A_{12} = \int_{SL_2(\mathbb{Z})/\mathbb{H}} \sum_{\gamma \in \Gamma_0 \backslash SL_2(\mathbb{Z})} f(\gamma \tau) \overline{g(\gamma \tau)} \text{Im}(\gamma \tau)^{2k+8} d^* \tau \\
= \int_{\Gamma_0 \backslash SL_2(\mathbb{Z})} f(\tau) \overline{g(\tau)} y^{2k+8} d^* \tau \\
= \int_0^\infty y^{2k+6} \left( \int_0^1 f(x + iy) \overline{g(x + iy)} dx \right) dy.
\]
Let \( f(\tau) = \sum_{n=1}^\infty a(n) e^{-2\pi in\tau} \) and \( g(\tau) = \sum_{n=1}^\infty b(n) e^{-2\pi in\tau} \). Then
\[
\int_0^1 f(x + iy) \overline{g(x + iy)} dx = \sum_{n=1}^\infty a(n) \overline{b(n)} e^{-4\pi ny}.
\]
Therefore,
\[
A_{12} = \sum_{n=1}^\infty a(n) \overline{b(n)} \int_0^\infty y^{2k+6} e^{-4\pi ny} dy = (4\pi)^{-2k-7} \Gamma(2k + 7) \sum_{n=1}^\infty \frac{a(n) \overline{b(n)}}{n^{2k+7}}.
\]
Let \( L(s, f \otimes \bar{g}) \) be the Rankin-Selberg \( L \)-function:
\[
L(s, f \otimes \bar{g}) = \zeta(2s) \sum_{n=1}^\infty \frac{a(n) \overline{b(n)}}{n^{s+2k+3}} = \prod_{p} \prod_{i=1}^2 \prod_{j=1}^2 (1 - \alpha_{f,i}(p) \alpha_{g,j}(p)p^{-s})^{-1},
\]
where \( \alpha_{f,i}(p), \alpha_{g,j}(p) \) are roots of \( X^2 - \frac{a(p)}{p^{k+\frac{3}{2}}} X + 1 = 0, \quad X^2 - \frac{b(p)}{p^{k+\frac{3}{2}}} X + 1 = 0, \) resp.
Hence
\[
L(4, f \otimes \bar{g}) = \zeta(8) \sum_{n=1}^\infty \frac{a(n) \overline{b(n)}}{n^{2k+7}}.
\]
Now \( L(s, f \otimes \bar{g}) \) converges absolutely for \( \text{Re}(s) > 1 \) and it has the Euler product. Hence it is non-vanishing for \( \text{Re}(s) > 1 \). Therefore \( A_{12} \) is non-vanishing. Hence we have proved

**Theorem 2.1.** Let \( \mathcal{F}_{f,g} \) be the Miyawaki type lift as above. Then it is non-zero.

2.2. **Higher level.** Let \( N, k \) be positive integers and \( \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\} \).

We denote by \( S_k(\Gamma_0(N)) \) the space of elliptic cusp forms of weight \( k \) with respect to \( \Gamma_0(N) \).

Throughout this section we keep this notation.

Let \( f \in S_{2k}(\Gamma_0(N)), g \in S_{2k+8}(\Gamma_0(N)) \) be two newforms. Let \( \varphi_f = \otimes_p' f_p, \varphi_g = \otimes_p' g_p \) be the decomposition of cuspidal automorphic forms associated to \( f, g \) where each component at \( p \) is chosen as a local newform defined in [23] so that it takes the value 1 at the identity \( I_2 \) when \( p \) is an unramified place. Let \( \pi_f = \otimes_p' \pi_{f,p}, \pi_g = \otimes_p' \pi_{g,p} \) be the cuspidal automorphic representations
generated by \( \varphi_f, \varphi_g \) respectively. Applying Theorem 1.1 of [20] with \( \otimes_{p < \infty} \pi_{f,p} \) and using the classical interpretation (cf. Section 5.1 of [18]), we have the Ikeda type lift \( F_f \). As in the case of level one, we can also define the Miyawaki type lift \( F_{f,g} \) by means of the integral given in [19, (1.1)] as follows:

\[
F_{f,g}(Z) = \int_{\Gamma_0(N) \backslash \mathbb{H}} F_f \left( \begin{array}{cc} Z & 0 \\ 0 & \tau \end{array} \right) g(\tau)(\text{Im}(\tau))^{2k+6} d\tau.
\]

By the definition it is easy to see that \( F_{f,g} \) is a modular form for a congruence subgroup of \( GSpin(2,10)(\mathbb{Z}) \).

**Theorem 2.2.** Keep the notations as above. Suppose that \( N \) is square free. Then the Miyawaki type lift \( F_{f,g} \) is non-vanishing.

**Remark 2.3.** As in [19], we may show that \( F_{f,g} \) is a Hecke eigen form and hence gives rise to a cuspidal representation of \( GSpin(2,10) \).

Since we have a non-trivial level, we should be careful with the fact that a priori we do not have an explicit form as in the case of level one. However by virtue of Lemma 7.1 and Lemma 7.3 of [20], we see that

\[
F_{1_2}(\tau,0) = f(\tau)\theta(\tau)
\]

where \( \theta(\tau) = E_8(\tau) \) is as before and the left hand side is defined similarly as in (2.2), (2.3). As in level one case, we have

\[
A_{1_2} = \int_{\Gamma_0(N) \backslash \mathbb{H}} f(\tau)g(\tau)E_8(\tau)\text{Im}(\tau)^{2k+8} d^*\tau.
\]

Theorem 2.2 will follow from the following:

**Theorem 2.4.** There exists a non-zero constant \( C \) such that

\[
A_{1_2} = C \cdot L(4, \pi_f \times \pi_g) \neq 0,
\]

where \( L(s, \pi_f \otimes \pi_g) \) is the Rankin-Selberg convolution for cuspidal representations \( \pi_f, \pi_g \) attached to \( f, g \) respectively.

**Proof.** We work on adelic forms. Let \( K_p = GL_2(\mathbb{Z}_p) \) and \( K_0(p^r) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p : c \equiv 0(\text{mod } p^r) \right\} \).
Let $\Phi_p$ be the local section in the principal series $\pi_p(| \cdot |^{\frac{7}{2}}, | \cdot |^{\frac{-7}{2}})$ defined by, if $p \neq \infty$,

$$\Phi_p(g) = \begin{cases} |ad^{-1}|_p^{\frac{1}{4}} & \text{if } g \in \left( \begin{array}{cc} a & * \\ 0 & d \end{array} \right) K_0(p^{\text{ord}_p(N)}) \\ 0 & \text{otherwise.} \end{cases}$$

Similarly we also define a spherical vector $\Phi_{ur}^p$ by replacing $K_0(p^{\text{ord}_p(N)})$ with $K_p$. If $p = \infty$, we define $\Phi_{ur}^\infty(g) = |ad^{-1}|_e^{4\sqrt{-1}e^k}$ if $g = pk_\theta \in \left( \begin{array}{c} a \\ d \end{array} \right) K_\infty$, where $k_\theta = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) \in K_\infty \simeq SO(2)$.

Put $\Phi_{ur} = \otimes'_p \Phi_{ur}^p$ and $\Phi = \otimes'_p \Phi_p$. The global section $\Phi_{ur}$ gives rise to $E_8(\tau)$, the Eisenstein series of level one, while the global section $\Phi$ gives rise to the Eisenstein series $E_8^{(N)}(\tau) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} j(\gamma, \tau)^{-8}$ for $\Gamma_0(N)$ with respect to the cusp $\infty$.

By the usual unfolding method, there exists a non-zero constant $C'$ depending on the normalization at all Steinberg places and at the infinite place such that

$$A_{12} = C' \int_{Z(\mathbb{Q})GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{Q})} \sum_{\gamma \in B(\mathbb{Q}) \backslash GL_2(\mathbb{Q})} \Phi_{ur}(\gamma g) \varphi_f(g) \overline{\varphi_g(g)} dg$$

$$= C' \prod_p \int_{PGL_2(\mathbb{Q}_p)} \Phi_{ur}^p(g) \varphi_{f,p}(g) \overline{\varphi_{g,p}(g)} dg.$$

For any unramified prime $p$, by Proposition 3.8.1 of [3] we have

$$\int_{PGL_2(\mathbb{Q}_p)} \Phi_{ur}^p(g) \varphi_{f,p}(g) \overline{\varphi_{g,p}(g)} dg = \zeta_p(8)^{-1} L(4, \pi_{f,p} \times \pi_{g,p}).$$

For a bad place $p|N$ (hence $p||N$ by the assumption), by applying computation in the proof of Lemma 2.14 of [24] (see line -7 through the bottom in page 22 of loc.cit.), we have

$$\int_{GL_2(\mathbb{Q}_p)} \Phi_{ur}^p(g) \varphi_{f,p}(g) \overline{\varphi_{g,p}(g)} dg$$

$$= \int_{PGL_2(\mathbb{Q}_p)} \Phi_{p}(g) \varphi_{f,p}(g) \overline{\varphi_{g,p}(g)} dg + \int_{PGL_2(\mathbb{Q}_p)} \Phi_{p} \left( \begin{array}{cc} p \\ 0 \\ 0 \\ 1 \end{array} \right) g \varphi_{f,p}(g) \overline{\varphi_{g,p}(g)} dg.$$

$$= [K_p, K_0(p)]^{-1} p^4 \zeta_p(8)^{-1} L(4, \pi_{f,p} \times \pi_{g,p}).$$
For \( p = \infty \), it is well-known (cf. p.136 of [21]) that
\[
C_\infty := \int_{PGL_2(\mathbb{R})} \Phi_\infty(g) \varphi_{f,\infty}(g) \varphi_{g,\infty}(g) dg = \zeta_\infty(8)^{-1} \Gamma(8) \Gamma(2k + 7) = \Gamma(2k + 7).
\]

Put \( C = C'C_\infty \zeta_N(8)^{-1} \left[ SL_2(\mathbb{Z}) : \Gamma_0(N) \right]^{-1} \) \( N^4 \) where \( \zeta_N(s) = \prod_{p \nmid N} \zeta_p(s) \) is the partial Riemann zeta function outside \( N \). Then we have \( A_{12} = C \cdot L(4, \pi_f \times \pi_g) \). Now \( L(s, \pi_f \times \pi_g) \) converges absolutely for \( \text{Re}(s) > 1 \) and it has the Euler product. Hence it is non-vanishing for \( \text{Re}(s) > 1 \).

Therefore \( A_{12} \) is non-vanishing. \( \square \)

3. Miyawaki lift for Siegel cusp forms

In this section we assume that the readers are familiar with notations and results in [12] and [13].

Let \( h(\tau) \in S_{k+1}^+(\Gamma_0(4)) \) be a Hecke eigenform in Kohnen’s plus space corresponding to a Hecke eigenform \( f(\tau) \in S_{2k}(SL_2(\mathbb{Z})). \) Let \( n, r \) be positive integers such that \( n + r \equiv k \mod 2 \). Then we have the Ikeda lift \( F_f \in S_{k+n+r}(Sp_{4n+4r}(\mathbb{Z})) \) whose standard \( L \)-function is
\[
\zeta(s) \prod_{k=1}^{2n+2r} L(s + k + n + r - i, f).
\]

Now for \( g \in S_{k+n+r}(Sp_{2r}(\mathbb{Z})) \), the Miyawaki lift is given by
\[
F_{f,g}(Z) = \int_{Sp_{2r}(\mathbb{Z}) \backslash \mathbb{H}_{2n+r}} F_f \left( \begin{array}{cc} Z & 0 \\ 0 & W \end{array} \right) g^c(W) \det(\text{Im}W)^{k+n-1} dW,
\]
where \( Z \in \mathbb{H}_{2n+r} \), and \( g^c(W) = g(-W) \). Ikeda [13] showed that if the integral is non-vanishing, \( F_{f,g} \) is a Hecke eigenform in \( S_{k+n+r}(Sp_{4n+2r}(\mathbb{Z})) \). Now we have the Fourier-Jacobi expansion of \( F_f \):
\[
F_f \left( \begin{array}{cc} Z & u \\ 1 & W \end{array} \right) = \sum_S F_S(W, u) e^{2\pi i \text{Tr}(ZS)},
\]
where \( S \in S'_{2n+r}(\mathbb{Z}) \), and \( F_S \) is a Fourier-Jacobi coefficient of index \( S \). Then
\[
F_S(W, u) = \sum_{\lambda \in \Lambda} \theta_{[\lambda]}(S; W, u) F_{S,\lambda}(W),
\]
where \( \Lambda = (2S)^{-1} \mathbb{Z}^{2n+r} / \mathbb{Z}^{2n+r} \), and \( \theta_{[\lambda]}(S; W, u) \) is a theta series of \( 2n + r \) variables.

Then
\[
F_{f,g}(Z) = \int_{Sp_{2r}(\mathbb{Z}) \backslash \mathbb{H}_{2n+r}} \left( \sum_S F_S(W, 0) e^{2\pi i \text{Tr}(ZS)} \right) g^c(W) \det(\text{Im}W)^{k+n-1} dW = \sum_S A_S e^{2\pi i \text{Tr}(ZS)},
\]
where

\[ A_S = \int_{Sp_{2r}(\mathbb{Z})\backslash \mathbb{H}_r} \mathcal{F}_S(W,0)g^c(W) \text{det}(ImW)^{k+n-1} dW. \]

Now in order to interchange the sum and the integral, we need to show that the integral is absolutely convergent. We follow Lemma 7.1 of [18]: We have

\[ \mathcal{F}_S(W,0) e^{-2\pi Tr(YS)} = \int_X F_f\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} e^{-2\pi Tr(XS)} dX. \]

Setting \( Y = \frac{1}{Tr(S)} I_{2n+r} \), we have \( |\mathcal{F}_S(W,0)| \ll (ImW)^{-(k+n+r)} Tr(S)^2(k+n+r) \). Then for a fixed \( Z \),

\[ \sum_S |\mathcal{F}_S(W,0)e^{2\pi i Tr(ZS)}| \leq (ImW)^{-(k+n+r)} \sum_S Tr(S)^2(k+n+r)e^{-2\pi Tr(YS)}. \]

Now \( Tr(YS) \geq c_Y Tr(S) \) for a constant \( c_Y > 0 \). Hence

\[ \sum_S Tr(S)^2(k+n+r)e^{-2\pi Tr(YS)} \leq \sum_S Tr(S)^2(k+n+r)e^{-2\pi c_Y Tr(S)}, \]

which is bounded. Therefore,

\[ \int_{Sp_{2r}(\mathbb{Z})\backslash \mathbb{H}_r} \left( \sum_S \mathcal{F}_S(W,0)e^{2\pi i Tr(ZS)} \right) g^c(W) \text{det}(ImW)^{k+n-1} dW, \]

converges absolutely.

Now we consider the special case: \( r \) is even and \( 2n+r \) is a multiple of 8. Choose \( S \) so that \( 2S \) is an even unimodular matrix. Then \( \Lambda \) is trivial, and \( \mathcal{F}_S(W,0) = H(W)\theta_S(W) \), where \( H(W) \) is a Siegel cusp form of weight \( k + \frac{r}{2} \) and \( \theta_S(W) := \theta_{[0]}(S; W, 0) \) is the theta function for \( S \) and the trivial class \([0]\), which is a Siegel modular form of weight \( n + \frac{r}{2} \) and level one. By using Lemma 7.1 and 7.2 (see also Corollary 7.3-(2),(3)) and Lemma 7.7 of [15], one can show that \( H(W) \) is the Ikeda lift \( F_f^{(r)} \) of \( f \) to \( Sp_{2r} \).

Now we apply the Siegel formula.

**Theorem 3.1 (Siegel).** For \( 8|m, C_m \) be the set of classes (up to isomorphism) of even unimodular lattices of \( m \) variables, and let \( g_S \in C_m \) be the order of the automorphism group of \( S \in C_m \). Let \( E_{2k}(Z) = \sum_{(C,D)} \text{det}(CZ + D)^{-2k} \) be the usual Siegel Eisenstein series of weight \( 2k \). Then

\[ \sum_{S \in C_m} \frac{1}{g_S} \theta_S = M_m E_{2k}, \quad \text{where} \quad M_m = \sum_{S \in C_m} \frac{1}{g_S}. \]
Now for $C_{2n+r}$, consider the sum
\begin{equation}
(3.1) \sum_{S \in C_{2n+r}} \frac{1}{g_S} A_S.
\end{equation}
It is
\begin{equation}
(3.2) M_{2n+r} \int_{Sp_{2r}(\mathbb{Z}) \backslash H_r} H(W)g^r(W)E_{n+\frac{r}{2}}(W) \det(ImW)^{k+n+r} d^*W,
\end{equation}
where $d^*W = \det(ImW)^{-(r+1)}dW$. Let
\[ H(W) = F_f^{(r)}(W) = \sum_{T \in S_r(\mathbb{Z})^+} a_{F_f^{(r)}}(T)e^{2\pi iT r(TW)}, \quad g(W) = \sum_{T \in S_r(\mathbb{Z})^+} a_g(T)e^{2\pi iT r(TW)}. \]
Then by [16], the integral (3.2) is exactly a scalar multiple of the Rankin convolution
\[ R(k+n+\frac{r-1}{2}, H, \bar{g}) = \sum_{T \in \bar{S}_r(\mathbb{Z})^+} \frac{a_{F_f^{(r)}}(T)a_g(T)}{\epsilon(T) \det(T)^{k+n+r-1}}, \]
where $\bar{S}_r(\mathbb{Z})^+$ is the set of $GL_r(\mathbb{Z})$-equivalence classes of matrices $T \in S_r(\mathbb{Z})^+$, and $\epsilon(T) = \# \{ U \in GL_r(\mathbb{Z}) | UT^U = T \}$. By [25, Lemma 3.1], the above series converges absolutely if $2n > r + 2$. However, we do not know the non-vanishing. We assume

**Conjecture 3.2.** $R(k+n+\frac{r-1}{2}, H, \bar{g})$ is non-vanishing.

Under Conjecture 3.2, (3.1) is non-vanishing. Then one of $A_S$ is non-vanishing. Hence we have proved:

**Theorem 3.3.** Let $f(\tau) \in S_{2k}(SL_2(\mathbb{Z}))$, and $g \in S_{k+n+r}(Sp_{2r}(\mathbb{Z}))$ such that $2n+r$ is a multiple of 8. Then under Conjecture 3.2 the Miyawaki lift $F_{f,g}$ is non-vanishing.

4. **Non-vanishing of Miyawaki Lifts for $U(n,n)$**

We review the Miyawaki lift for the unitary group in [2]. Let $K$ be an imaginary quadratic field with discriminant $-D$ and let $O$ be the ring of integers. Let $\chi = \chi_D$ be the Dirichlet character corresponding to $K/\mathbb{Q}$. Let $f$ be a normalized Hecke eigen cusp form belonging to
\[ \begin{cases} 
S_{2k}(SL_2(\mathbb{Z})), & \text{if } n \text{ odd} \\
S_{2k+1}(\Gamma_0(D), \chi), & \text{if } n \text{ even}
\end{cases} \]

Ikeda [14] constructed a lift $I^{(n+2r)}(f) \in S_l(\Gamma_K^{(n+2r)}, \det^{-\frac{1}{2}})$, where $l = 2k + 2[\frac{n}{2}] + 2r$. It is a Hecke eigenform on the Hermitian upper half space $\mathcal{H}_{n+2r}$ of degree $n+2r$. 
Now for a Hecke eigen cusp form \( g \in S_l(\Gamma_K^{(r)}, \det^{-\frac{l}{2}}) \), define

\[
\mathcal{F}_{f,g}(Z) = \int_{\Gamma_K^{(r)} \backslash \mathcal{H}_r} I^{(n+2r)}(f) \left( \begin{array}{cc} Z & 0 \\ 0 & W \end{array} \right) g^{c}(W) \det(I MW)^{l-2r} \, dW,
\]

for \( Z \in \mathcal{H}_{n+r} \). (Here if \( 4 \mid l \), \( h \in S_l(SL_2(\mathbb{Z})) \) can be regarded as \( h \in S_l(\Gamma_K^{(1)}, \det^{-\frac{l}{2}}) \). Note that \( \Gamma_K^{(1)} = SL_2(\mathbb{Z}) \cdot \{ \alpha \cdot 1_2 | \alpha \in O^{\times} \} \). [14, page 1111]) Then \( \mathcal{F}_{f,g} \in S_l(\Gamma_{K}^{(n+r)}, \det^{-\frac{l}{2}}) \). Atobe and Kojima [2] showed that if \( \mathcal{F}_{f,g} \) is not identically zero, it is a Hecke eigen form, and its standard \( L \)-function is given by

\[
L(s, \mathcal{F}_{f,g}, St) = L(s, g, St) \prod_{i=1}^{n} L(s + \frac{n-1}{2} - i, \pi_f) L(s + \frac{n-1}{2} - i, \pi_f \otimes \chi).
\]

Consider the Fourier-Jacobi expansion

\[
I^{(n+2r)}(f) \left( \begin{array}{cc} Z & 0 \\ 0 & W \end{array} \right) = \sum_{S} \mathcal{F}_S(W, 0) e^{2\pi i Tr(Z S)},
\]

where \( S \in S_{n+r}^{\prime}(O), (n+r) \times (n+r) \) positive definite semi-integral Hermitian matrices, and \( \mathcal{F}_S \) is a Fourier-Jacobi coefficient of index \( S \). Then

\[
\mathcal{F}_S(W, 0) = \sum_{\lambda \in \Lambda} \theta_{[\lambda]}(S; W, 0) \mathcal{F}_{S, \lambda}(W),
\]

where \( \Lambda = (2S)^{-1}O^{n+r}/O^{n+r} \), and \( \theta_{[\lambda]}(S; W, 0) \) is a theta series of \( n+r \) variables. Then

\[
\mathcal{F}_{f,g}(Z) = \sum_{S} A_S e^{2\pi i Tr(Z S)},
\]

where

\[
A_S = \int_{\Gamma_K^{(r)} \backslash \mathcal{H}_r} \mathcal{F}_S(W, 0) g^{c}(W) \det(I MW)^{2k+2[\frac{n}{2}]} \, dW.
\]

Now we consider the special case: \( 4 \mid (n+r) \). Choose \( S \) so that \( 2S \) is an even unimodular Hermitian matrix. Then \( \Lambda \) is trivial, and \( \mathcal{F}_S(W, 0) = H(W) \theta_S(W) \), where \( H(W) \) is a Hermitian cusp form of weight \( 2k + 2[\frac{n}{2}] - n + r \) and \( \theta_S(W) := \theta_{[0]}(S; W, 0) \) is the theta function for \( S \) and the trivial class \( [0] \), which is a Hermitian modular form of weight \( n+r \) and level one.

For \( l > 2m \), let \( E_l(Z) := \sum_{\Gamma_{\infty} \backslash \Gamma_K^{(m)}} (\det g)^{\frac{l}{2}} \det(CZ + D)^{-l} \). By applying the Siegel-Weil formula for unitary groups \([11]\), we obtain the following analogue of the Siegel formula for Hermitian lattices:
**Theorem 4.1.** For $4|m$, let $C_m$ be the set of classes (up to isomorphism) of even unimodular Hermitian lattices of $m$ variables, and let $g_S \in C_m$ be the order of the automorphism group of $S \in C_m$. Then

$$
\sum_{S \in C_m} \frac{1}{g_S} \theta_S = M_m E_m, \quad \text{where } M_m = \sum_{S \in C_m} \frac{1}{g_S}.
$$

Now for $C_{n+r}$, consider the sum

$$
\sum_{S \in C_{2n+r}} \frac{1}{g_S} A_S.
$$

It is

$$(4.1) \quad M_{2n+r} \int_{\Gamma_0^r(K) \backslash \mathcal{H}} H(W) g(W) E_{n+r}(W) \det(ImW)^{2k+2[\frac{n}{2}]+2} \frac{dW}{\epsilon(T) det(T)^{2k+2[\frac{n}{2}]+2+r}}.
$$

Let

$$
H(W) = \sum_{T \in S_r^+(O)} a_H(T) e^{2\pi i Tr(TW)}, \quad g(W) = \sum_{T \in S_r^+(O)} a_g(T) e^{2\pi i Tr(TW)}.
$$

Now the integral (4.1) is exactly a scalar multiple of the Rankin convolution

$$
R(k+n+r, H, g) = \sum_{T \in S_r^+(O)^+} \frac{a_H(T)a_g(T)}{\epsilon(T) det(T)^{2k+2[\frac{n}{2}]+2+r}},
$$

where $S_r^+(O)^+$ is the set of $GL_r(O)$-equivalence classes of matrices $T \in S_r(O)^+$, and $\epsilon(T) = \# \{ U \in GL_r(O) | UT^t U = T \}$.

Now as in the Siegel case, we can show that $H(W)$ is the Ikeda lift of $f$ to $U(r, r)$. When $r = 1$, $n$ is of the form $4m+3$, and the above integral is related to the Rankin-Selberg $L$-function $L(\frac{n+1}{2}, H \otimes \tilde{g})$ as in Section 2. Hence it is non-vanishing.

If $r > 1$, we assume the analogue of Conjecture 3.2. Then

**Theorem 4.2.** Let $K$ be an imaginary quadratic field with discriminant $-D$. Let $\chi = \chi_D$ be the Dirichlet character corresponding to $K/\mathbb{Q}$. Let $f$ be a normalized Hecke eigen cusp form belonging to

$$
\begin{cases}
S_{2k}(SL_2(\mathbb{Z})), & \text{if } n \text{ odd} \\
S_{2k+1}(\Gamma_0(D), \chi), & \text{if } n \text{ even}
\end{cases}
$$

let $l = 2k + 2[\frac{n}{2}] + 2r$, and $g \in S_l(\Gamma_K^{(r)}, det^{-\frac{1}{2}})$, and $F_{f,g}$ be the Miyawaki lift in $S_l(\Gamma_K^{(n+r)}, det^{-\frac{1}{2}})$. Let $4|(n+r)$. If $r = 1$, $F_{f,g}$ is non-vanishing. If $r > 1$, assuming the analogue of Conjecture 3.2, it is non-vanishing.
In this section, assuming the non-vanishing of the integral \((5.1)\), we give an outline of non-vanishing of the Miyawaki lift for half-integral Siegel cusp forms given in \([7]\). Let \(I^{(2n)}(f)\) be the Ikeda lift as in Section\(3\) for \(f \in S_{2k}(SL_2(\mathbb{Z})).\) Consider its Fourier-Jacobi expansion with integer index:

\[
I^{(2n)}(f) \begin{pmatrix} Z_1 & u \\ t & \tau \end{pmatrix} = \sum_{m=1}^{\infty} \psi_m(Z_1, u) e^{2\pi i m \tau},
\]

where \(Z_1 \in \mathbb{H}_{2n-1},\) and \(\tau \in \mathbb{H}.\) Here \(\psi_1(Z_1, u)\) is a Jacobi cusp form of weight \(k+n\) and index 1 of degree \(2n-1.\) By the Eichler-Zagier-Ibukiyama correspondence \([10]\), there exists a Siegel cusp form \(F_f \in S_{k+n-1}^{+}(\Gamma_0(2n-1)(4))\) which corresponds to \(\psi_1.\) For \(g \in S_{k+n-1}^{+}(\Gamma_0(4))\), we put

\[
F_{f,g}(Z) = \int_{\Gamma_0(4)\backslash \mathbb{H}} F_f \begin{pmatrix} Z & 0 \\ 0 & \tau \end{pmatrix} g(\tau) Im(\tau)^{k+n-\frac{5}{2}} d\tau,
\]

for \(Z \in \mathbb{H}_{2n-2}.\) Then \(F_{f,g}\) is a cusp form in \(S_{k+n-1}^{+}(\Gamma_0(2n-2)(4)).\) Hayashida \([7]\) proved that if \(F_{f,g}\) is not identically zero, it is an eigenform with the standard \(L\)-function

\[
L(s, F_{f,g}, St) = L(s, g) \prod_{i=1}^{2n-3} L(s - i, h).
\]

Consider the Fourier-Jacobi expansion of \(F_f\) with matrix index

\[
F_f \begin{pmatrix} Z & 0 \\ 0 & \tau \end{pmatrix} = \sum S \mathcal{F}_S(\tau, 0) e^{2\pi i Tr(ZS)},
\]

where \(S \in S'_{2n-2}(\mathbb{Z}),\) and \(\mathcal{F}_S\) is a Fourier-Jacobi coefficient of index \(S.\) Then

\[
\mathcal{F}_S(\tau, 0) = \sum \lambda \in \Lambda \theta_{[\lambda]}(S; \tau, 0) \mathcal{F}_{S, \lambda}(\tau),
\]

where \(\Lambda = (2S)^{-1}Z^{2n-2}/Z^{2n-2},\) and \(\theta_{[\lambda]}(S; \tau, 0)\) is a theta series of \(2n-2\) variables. Then

\[
F_{f,g}(Z) = \int_{\Gamma_0(4)\backslash \mathbb{H}} \left( \sum S \mathcal{F}_S(\tau, 0) e^{2\pi i Tr(ZS)} \right) g(\tau) Im(\tau)^{k+n-\frac{5}{2}} d\tau = \sum A_S e^{2\pi i Tr(ZS)},
\]

where

\[
A_S = \int_{\Gamma_0(4)\backslash \mathbb{H}} \mathcal{F}_S(\tau, 0) g(\tau) Im(\tau)^{k+n-\frac{5}{2}} d\tau.
\]

Now we consider the special case: \(2n-2\) is a multiple of \(8,\) and choose \(S\) so that \(2S\) is an even unimodular matrix. Then \(\Lambda\) is trivial, and \(\mathcal{F}_S(\tau, 0) = H(\tau) \theta_S(\tau),\) where \(H(\tau)\) is a cusp form of weight \(k + \frac{1}{2}.\)
Now we apply the Siegel formula. For $8|(2n - 2)$, let $C_{2n-2}$ be the set of classes (up to isomorphism) of even unimodular lattices of $2n - 2$ variables, and let $g_S \in C_{2n-2}$ be the order of the automorphism group of $S \in C_{2n-2}$. Then
\[
\sum_{S \in C_{2n-2}} \frac{1}{g_S} \theta_S = M_{2n-2} E_{\frac{2n-2}{2}}, \quad \text{where} \quad M_{2n-2} = \sum_{S \in C_{2n-2}} \frac{1}{g_S}.
\]

Now consider the sum
\[
\sum_{S \in C_{2n-2}} \frac{1}{g_S} A_S.
\]

It is
\[
(5.1) \quad M_{2n-2} \int_{\Gamma_0(4) \backslash \mathbb{H}} H(\tau) g(\tau) E_{n-1}(\tau)(\text{Im}\tau)^{k+n-\frac{5}{2}} d\tau.
\]

It may be possible to show that it is non-vanishing. Then one of $A_S$ is non-vanishing, and $F_{f,g}$ is non-vanishing.

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