ORBITALLY STABLE STANDING WAVES FOR THE ASYMPTOTICALLY LINEAR ONE-DIMENSIONAL NLS

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Abstract. In this article we study the one-dimensional, asymptotically linear, non-linear Schrödinger equation (NLS). We show the existence of a global smooth curve of standing waves for this problem, and we prove that these standing waves are orbitally stable. As far as we know, this is the first rigorous stability result for the asymptotically linear NLS. We also discuss an application of our results to self-focusing waveguides with a saturable refractive index.

1. Introduction

In this article we study the one-dimensional nonlinear Schrödinger equation
\[ i\partial_t \psi + \partial_{xx}^2 \psi + f(x, |\psi|^2) \psi = 0 \] (NLS)
for $\psi = \psi(t, x) : [0, \infty) \times \mathbb{R} \to \mathbb{C}$. We suppose that $f \in C^1(\mathbb{R} \times \mathbb{R}_+, \mathbb{R})$, and satisfies the following assumptions.

(A0) $f(x, 0) = 0$ for all $x \in \mathbb{R}$, and we have
\[ \lim_{s \to 0} f(x, s) = 0, \text{ uniformly for } x \in \mathbb{R}, \] (1.1)
\[ \lim_{|x| \to \infty} f(x, s) = 0, \text{ uniformly for } s \geq 0. \] (1.2)

(AL) There exists $f_\infty \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that
\[ \lim_{s \to \infty} f(x, s) = f_\infty(x), \text{ uniformly for } x \in \mathbb{R}. \] (1.3)

Equation (NLS) with the assumption (AL) is usually referred to as the asymptotically linear nonlinear Schrödinger equation. The structure of the nonlinearity allows one to look for standing wave solutions
\[ \psi(t, x) = e^{i\lambda t} u(x), \text{ where } \lambda > 0 \text{ and } u : \mathbb{R} \to \mathbb{R}. \]

As we shall see from the stability analysis in Section 3, solutions of this type enjoy remarkable properties with respect to the general dynamics of (NLS). Using this Ansatz to solve (NLS) yields the stationary equation
\[ u'' + f(x, u^2)u = \lambda u. \] (SNLS)

This second order ordinary differential equation will be interpreted as a nonlinear eigenvalue problem, which will be addressed via bifurcation theory.

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By a solution to \((\text{SNLS})\) will be meant a couple \((\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R})\) satisfying \((\text{SNLS})\) in the sense of distribution. Note that, since \(f(x, 0) \equiv 0\), we have a line of trivial solutions, \(\{(\lambda, 0) : \lambda \in \mathbb{R}\} \subset \mathbb{R} \times H^1(\mathbb{R})\). We will prove that there exist non-trivial solutions, bifurcating from the line of trivial solutions. Under appropriate hypotheses on \(f\), in particular assuming that \(f(x, s)\) is positive and even in \(x\), we will obtain a smooth curve of positive even solutions of \((\text{SNLS})\),

\[
\{(\lambda, u(\lambda)) : \lambda \in (0, \lambda_\infty)\} \subset \mathbb{R} \times H^1(\mathbb{R}).
\]  

(1.4)

The number \(\lambda_\infty \in (0, \infty)\) will be characterized as the principal eigenvalue of the linear problem

\[
u'' + f_\infty(x)u = \lambda u,
\]

(1.5)

known as the asymptotic linearization.

Additional properties of the solutions will be proved, in particular their bifurcation behaviour:

\[
\lim_{\lambda \to 0} \|u(\lambda)\|_{H^1(\mathbb{R})} = 0 \quad \text{and} \quad \lim_{\lambda \to \lambda_\infty} \|u(\lambda)\|_{H^1(\mathbb{R})} = \infty.
\]

The curve (1.4) will be obtained, first by a local bifurcation analysis near \(u = 0\), then by analytic continuation and an asymptotic analysis as \(\|u(\lambda)\|_{H^1(\mathbb{R})} \to \infty\). Bifurcation from the line of trivial solutions is difficult in the present context since, under assumption (1.1), the linearization of \((\text{SNLS})\) at \(u = 0\) is

\[
u'' = \lambda u,
\]

(1.6)

and has purely continuous spectrum.

In Subsection 2.1 prescribing the behaviour of \(f(x, s)\) as \(s \to 0\), we will obtain bifurcation of positive solutions of \((\text{SNLS})\), from the bottom of the continuous spectrum of (1.6) (i.e. from \(\lambda = 0\)), by applying a fairly involved perturbation analysis, introduced in 7. This local result is contained in Theorem 2.3.

Under additional assumptions (in particular assuming that \(f(x, s)\) is positive and even in \(x \in \mathbb{R}\)), a global analysis is carried out in Subsection 2.2 where we show that the local branch given by Theorem 2.3 can be extended in a smooth manner to the curve (1.4). The global continuation relies on the non-degeneracy of positive even solutions of \((\text{SNLS})\), given by Lemma 2.9. The asymptotic behaviour of the branch as \(\|u(\lambda)\|_{H^1(\mathbb{R})} \to \infty\) (and \(\lambda \to \lambda_\infty\)) follows by the asymptotic bifurcation analysis that was developed in 9 using topological arguments. Our global result is Theorem 2.10.

In 9 (where we only considered \((\text{SNLS})\) for \(x > 0\)) we obtained global bifurcation from \((\lambda_\infty, \infty)\) in \(\mathbb{R} \times H^2(0, \infty)\). More precisely, we obtained a global connected set of solutions \((\lambda, u)\) such that \(\|u\|_{H^2(\mathbb{R})} \to \infty\) as \(\lambda \to \lambda_\infty\). This was later extended in 10 to a higher dimensional version of \((\text{SNLS})\), under fairly weak hypotheses, providing a strong existence result for positive solutions of \((\text{SNLS})\) with assumption (AL). Previous contributions on this problem — see e.g. 4, 13, 14, 27, 30, 54 and the references in these papers — mostly used variational methods to prove existence of solutions for \((\text{SNLS})\)-(AL), under various assumptions, typically stronger than those required by the topological approach.

However, due to technical restrictions of the method used in 9, we were not able to continue the bifurcating branch down to the line of trivial solutions. Section 2 below completes the discussion initiated in 9 by showing that, under more restrictive assumptions (in particular, symmetry and monotonicity conditions on
f(x, s), as well as a precise asymptotic behaviour as s → 0), there exists a smooth curve of solutions, connecting (0, 0) to (λ∞, ∞) in $\mathbb{R} \times H^1(\mathbb{R})$.

A global solution curve was obtained by Jeanjean and Stuart [15] under similar hypotheses. However, they consider an additional, non-trivial, linear potential in (SNLS). In our notation, this amounts to assuming that $f_0(x) := f(x, 0) \neq 0$ instead of $f(x, 0) \equiv 0$ in (A0). In this case, the linearization at $u = 0$ has the form

$$u'' + f_0(x)u = \lambda u.$$  \hspace{1cm} (1.7)

Under appropriate assumptions on $f_0$ (e.g. $f_0$ is a ‘bump’), the linear problem (1.7) has a principal eigenvalue, from which bifurcation can be obtained via standard bifurcation theory. Global continuation can then be obtained by arguments similar to those of Subsection 2.2. The authors of [15] also discussed the case of an asymptotically linear nonlinearity, as in (1.3). The main difference in the present context is that bifurcation at $u = 0$ occurs from the bottom of the continuous spectrum of the linearization (1.6).

In Section 3 we will consider the standing wave solutions $\psi_\lambda(t, x) = e^{i\lambda t}u(\lambda)(x)$ of (NLS) corresponding to the solutions (1.4) of (SNLS). We will prove that they are orbitally stable amongst the set of solutions $\psi(t, x) \in C([0, \infty), H^1(\mathbb{R}, \mathbb{C}))$. This result, Theorem 3.8, is based on the general theory of orbital stability for Hamiltonian systems, see [12, 26]. Given a standing wave $\psi_{\lambda_0}$, it follows from the theory that, under appropriate conditions on the spectrum of the linearization of (NLS) at $\psi_{\lambda_0}$, this standing wave is orbitally stable if the mapping $\lambda \rightarrow \|u(\lambda)\|^2_{L^2}$ is increasing at the point $\lambda = \lambda_0$. Since we have a smooth curve of solutions, this can be obtained by checking that

$$\frac{d}{d\lambda}\bigg|_{\lambda=\lambda_0} \|u(\lambda)\|^2_{L^2} > 0.$$  \hspace{1cm} (1.8)

Our bifurcation analysis in Section 2 shows that $\|u(\lambda)\|_{L^2} \to 0$ as $\lambda \to 0$, and so (1.8) must hold for some $\lambda_0 \in (0, \lambda_\infty)$. Hence, by continuity, we need only check that

$$\frac{d}{d\lambda}\|u(\lambda)\|^2_{L^2} \neq 0, \quad \text{for all } \lambda \in (0, \lambda_\infty).$$

This is done in Subsection 3.2 using an integral identity that was first derived in [16] to study orbital stability along the solution curve obtained in [15]. However, the authors of [16] were not able to deal with the asymptotically linear case. In fact, a careful inspection of the proof of [16, Theorem 2.1] shows that the non-trivial potential $f_0$ in (1.7) obstructs the argument under assumption (1.3). In the context of (A0) (i.e. with $f_0 \equiv 0$), we can prove that condition (1.8) is verified under assumption (1.3), for all $\lambda_0 \in (0, \lambda_\infty)$. This result (Proposition 3.3) and the spectral conditions (Proposition 3.3) yield the stability of the standing waves $\psi_\lambda(t, x) = e^{i\lambda t}u(\lambda)(x)$, for all $\lambda \in (0, \lambda_\infty)$ (Theorem 3.3). To the best of our knowledge, Theorem 3.3 is the first rigorous stability result for the asymptotically linear Schrödinger equation.

Lastly, Section 4 is devoted to an application in nonlinear optics. Following our study in [8] of self-focusing planar waveguides in Kerr media, the results of Sections 2 and 3 under assumption (AL) now allow us to discuss the existence and stability of TE travelling waves in materials having a saturable dielectric response.
1.1. **Open problems.** Various problems remain unsolved in the higher dimensional setting. For instance, the global analytic continuation carried out in Subsection 2.2 makes use of the uniqueness of positive even solutions to (SNLS) (for each fixed \( \lambda \)). As far as we know, the uniqueness problem in higher dimension is still open. Moreover, due to a lack of compactness coming from the unboundedness of the domain, the analytic continuation seems hard to obtain without uniqueness.

In dimension \( N \geq 3 \), the orbital stability of standing waves along a local branch of solutions — such as that obtained in Theorem 2.3 — follows from [7, Theorem 1 (b)]. Nonetheless, even if a smooth global curve of positive radial solutions existed in the radial case, our method to prove stability along the whole curve might not work in dimension \( N > 1 \) because of an extra term of the form \( \frac{N-1}{r} u'(r) \) in the radial (higher dimensional) version of (SNLS), coming from the expression of Laplacian in polar coordinates. We have previously failed to handle this problem in the simpler case of the power-type nonlinearity considered in [6].

1.2. **The prototype.** The function \( f \) will be required to satisfy numerous structural and technical assumptions in order to establish our results. It may be helpful to keep in mind the following typical example.

**Example 1.1.** Under appropriate conditions on \( V : \mathbb{R} \to \mathbb{R} \) and \( \alpha > 0 \), the function \( f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \) defined by

\[
f(x, s) := V(x) \frac{s^\alpha}{1 + s^\alpha}
\]

will satisfy all of our assumptions. We will state these conditions in due course, to illustrate the general case.

**Terminology and notation.** For brevity, we will often refer to properties of solutions \((\lambda, u)\) of (SNLS), e.g. positivity, evenness etc., while actually meaning that \( u \) possesses these properties.

We will work in both the real and the complex Sobolev spaces \( H^1(\mathbb{R}, \mathbb{C}) \) and \( H^1(\mathbb{R}, \mathbb{R}) \), depending on whether we consider (NLS) or (SNLS), respectively. When no confusion is possible, we will merely write \( H^1 (\mathbb{R}) \), and similarly for \( L^q(\mathbb{R}) \), \( H^2 (\mathbb{R}) \), etc. All of these spaces will be regarded as real Banach spaces, endowed with their usual inner products and norms.

The symbol \( C \) will denote various positive constants, the exact value of which does not play an essential role in the analysis.

2. **A global curve of solutions**

Our approach in this section will be based on previous results [6, 8] about bifurcation for semilinear equations in \( \mathbb{R}^N \). Firstly, in Subsection 2.1 we will show that a local smooth branch of solutions of (SNLS) bifurcates from the line of trivial solutions at the point \((0, 0) \in \mathbb{R} \times H^1 (\mathbb{R})\). This will be based on similar results to those of [7], where local bifurcation and stability results are established for standing waves of the NLS in dimension \( N \geq 3 \). Under appropriate symmetry and monotonicity assumptions, we will then show in Subsection 2.2 that a version of the implicit function theorem can be applied at any positive even solution of (SNLS), thereby ensuring global continuation of the local branch. The asymptotic bifurcation results of [6] will then allow us to discuss the asymptotic behaviour as \( \lambda \to \lambda_\infty \).
2.1. Bifurcation of small solutions. In \cite{7} we proved local bifurcation and stability results for the NLS in dimension \( N \geq 3 \). The nonlinearities we considered in \cite{7} can be written as perturbations — in a sense that will be made more precise below — of the signed-power nonlinearity

\[
g(x, s) := V(x)|s|^{p-1}s, \quad p > 1, \tag{2.1}
\]

with \( V \in C^1(\mathbb{R}^N) \). The main hypotheses about \( g \) involve a parameter \( b \in (0, 2) \). Roughly speaking, it is required that \( V(x) \sim |x|^{-b} \) as \( |x| \to \infty \) and that the problem be ‘subcritical’, in the sense that \( p < 1 + \frac{4-2b}{N-2} \). We will formulate the exact hypotheses in the one-dimensional setting below, but let us already mention two differences from the case where \( N \geq 3 \). Firstly, if \( N = 1 \), we must impose \( b \in (0, 1) \). This is a requirement of the variational formulation of a limit problem involving the coefficient \( |x|^{-b} \). Secondly, in dimensions \( N = 1, 2 \), the problem is ‘subcritical’ for all \( p > 1 \), so we can dispose of the above upper bound. However, we will only be interested here in the case of bifurcation from the line of trivial solutions, while more general situations are considered in \cite{7}, allowing for asymptotic bifurcation (i.e. bifurcation from \((0, \infty) \) in \( \mathbb{R} \times H^1(\mathbb{R}^N) \)) as well. This restriction will impose another upper bound on \( p > 1 \), namely \( p < 5 - 2b \). As can be seen from Example 3.4, this condition is also essential to the stability of the standing waves of (NLS).

We will now state the one-dimensional version of the bifurcation result of \cite{7}. It is convenient to define

\[
\tilde{f}(x, s) := f(x, s^2)s, \quad x, s \in \mathbb{R}. \tag{2.2}
\]

We then suppose that

\[
\tilde{f}(x, s) = g(x, s) + r(x, s), \tag{2.3}
\]

where \( g \) is defined in \( 2.4 \), \( V \in C^1(\mathbb{R}) \) satisfies

\[
\lim_{|x| \to \infty} |x|^b V(x) = 1 \quad \text{and} \quad \lim_{|x| \to \infty} |x|^b x V'(x) = -b, \tag{2.4}
\]

for some \( b \in (0, 1) \), and the rest \( r \), defined by \( 2.3 \), is ‘small’ in a precise, technical sense. As in \cite{7}, we are dealing here with situations where the linearization of (SNLS) at \( u = 0 \) has purely continuous spectrum. The method we used in \cite{7} to get bifurcation from the continuous spectrum is by perturbation of the model nonlinearity \( g \) that was considered earlier in \cite{6}.

A fairly technical method was developed in \cite{6}, based on a rescaling and a perturbative argument, using a limit equation involving the nonlinearity \( g \) with \( V(x) = |x|^{-b} \). Thus, via continuation from this limit problem, the asymptotic behaviour of \( V \) as \( |x| \to \infty \) turns out to govern the local bifurcation from \( \lambda = 0 \). Hypotheses about the rest \( r \) are formulated in \cite{7} — see \cite{7} (r1)-(r5) —, ensuring that the perturbed nonlinearity retains the main properties of \( g \) for small \( |s| \), and the same asymptotic behaviour under scaling, in the limit \( \lambda \to 0 \).

In the present context, having \( 2.1 \) to \( 2.4 \) in mind, we will formulate these assumptions in the one-dimensional setting directly in terms of \( \tilde{f} \). Note that the bifurcation analysis for the model nonlinearity \( g \) was carried out in \cite{7} in dimension \( N = 1 \), similarly to the higher dimensional problem treated in \cite{7}.

Let us finally remark that singularities at \( x = 0 \) were allowed in \cite{6–8}. We will not need to handle singularities here, and so the present hypotheses are formulated in a slightly different manner.
Remark 2.1. Note that the conditions (A), (A1), (A2) and (A3) as stated in Theorem 2.3. \( \lambda, u \) bifurcation occurs from the line of trivial solutions — this is analogous to the results for \( N = 1 \) with the model nonlinearity \( g \) in [8], rather than those of [6] dealing with \( N \geq 3 \). Note that the condition \( p < 5 - 2b \) in (A2) ensures that bifurcation occurs from the line of trivial solutions — this is analogous to the condition \( p < 1 + \frac{4 - 2b}{N} \) in Theorem 1 of [7].

Remark 2.4. Note that no symmetry or sign assumptions on the nonlinearity are required for Theorem 2.3. In fact, the solutions inherit their positivity from the sign properties of the limit nonlinearity [2.1], and the local analysis as \( \lambda \to 0 \).

This follows from the assumption made in the introduction that \( f \in C^1(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}) \), but we state it here for completeness.
2.2. Global continuation. We will now prove that, under appropriate assumptions, the local branch of solutions of SNLS given by Theorem 2.3 can be extended to a global $C^1$ curve. In particular, we will now suppose that the problem is symmetric with respect to $x = 0$ — which will allow us to restrict the discussion to the half-line, $x \in (0, \infty)$ — and that the nonlinearity satisfies some monotonicity conditions. Our precise hypotheses are the following:

(A4) $f(-x, s) = f(x, s)$ for all $(x, s) \in \mathbb{R} \times \mathbb{R}_+$;

(A5) $\partial_1 f(x, s) < 0$ and $\partial_2 f(x, s) > 0$ for all $x, s > 0$;

(A6) (i) $\partial_1 f(\cdot, s) \in L^\infty(\mathbb{R})$ for all $s \geq 0$ and $\|\partial_1 f(\cdot, s)\|_{L^\infty(\mathbb{R})}$ is uniformly bounded for $s$ in compact subsets of $\mathbb{R}_+$;

(ii) $\partial_2 f(\cdot, s) \in L^\infty(\mathbb{R})$ for all $s \geq 0$ and $\{\partial_2 f(x, \cdot)\}_{x \in \mathbb{R}}$ is equicontinuous.

(A7) (AL) holds with $f_\infty(0) > \lim_{|x| \to \infty} f_\infty(x)$.

For brevity, we will refer from now on to the assumptions (A), (A4) to (A7) as assumption (A').

Example 2.5. In addition to the hypotheses made in Example 2.2, we take $\alpha \geq 1$, and we suppose that $V$ is even, $V > 0$ on $\mathbb{R}$, and $V'(x) < 0$ for $x > 0$. Then the function $f$ defined by (1.9) satisfies assumption (A').

Let us now collect some important consequences of (A').

Remark 2.6.

(a) From (A0), (A5) and (A7), there exists $M > 0$ such that

$$0 \leq f(x, s) \leq f_\infty(x) \leq M \quad \text{for all } (x, s) \in \mathbb{R}^2_+,$$

(2.7)

(b) From (A0), (A4) and (A5), $f_\infty$ is even and non-increasing on $[0, \infty)$, with

$$\lim_{|x| \to \infty} f_\infty(x) = 0.$$

(2.8)

(c) (A6)(ii) implies that $\|\partial_2 f(\cdot, s)\|_{L^\infty(\mathbb{R})}$ is uniformly bounded and $\{\partial_2 f(x, \cdot)\}_{x \in \mathbb{R}}$ is uniformly equicontinuous on the compact subsets of $\mathbb{R}_+$ — see e.g. [17] Lemma 5.1. Furthermore, it follows by integration that $\{f(x, \cdot)\}_{x \in \mathbb{R}}$ is also uniformly equicontinuous on the compact subsets of $\mathbb{R}_+$.

(d) The asymptotic linearization (1.5) has a principal eigenvalue. Indeed, setting

$$-\lambda_\infty := \inf_{u \in H^1(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} (u')^2 - f_\infty(x) u^2 \, dx}{\int_{\mathbb{R}} u^2 \, dx},$$

(A7) and (2.3) imply $\lambda_\infty \in (0, \infty)$. Furthermore, it follows from the spectral theory of Schrödinger operators (see e.g. [22]) that $\lambda_\infty$ is the supremum of the spectrum of (1.5). Since $\lim_{|x| \to \infty} f_\infty(x) = 0$, we have $\sigma_{\text{ess}} = (-\infty, 0]$, where $\sigma_{\text{ess}}$ denotes the essential spectrum of (1.5). Hence, $\lambda_\infty > 0$ is the principal eigenvalue of (1.5).

In order to discuss global continuation, it is convenient to introduce the function $F : \mathbb{R} \times H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ defined by

$$F(\lambda, u)(x) := u''(x) + f(x, u(x)^2)u(x) - \lambda u(x),$$

(2.9)

where $H^{-1}(\mathbb{R})$ denotes the topological dual of $H^1(\mathbb{R})$, and the right-hand side of (2.12) is interpreted as an element of $H^{-1}(\mathbb{R})$ via the canonical identifications:

$$\langle \varphi, v \rangle_{H^{-1} \times H^1} \equiv \int_{\mathbb{R}} \varphi v \, dx \quad \text{for all } \varphi \in L^2(\mathbb{R}), \; v \in H^1(\mathbb{R});$$

(2.10)
For any solutions, and it follows from (A1) that they are C.

Proof. First, it is easily seen that weak solutions of (SNLS) are in fact classic al

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obtain positive even solutions of (SNLS) by solving the problem

Lemma 2.7. Let f satisfy (A1), (A4) and (A5), and (λ, u) ∈ R × H(1)(R) be a solution of (2.12). Then u(−x) = u(x) for all x ≥ 0, u ∈ C3(R) with u′(0) = 0 and u′(x) < 0 for all x > 0, and (λ, u) is a classical solution of (SNLS).

Proof. First, it is easily seen that weak solutions of (SNLS) are in fact classical solutions, and it follows from (A1) that they are C3. The remaining statements then follow by standard arguments — see e.g. the proof of [12, Lemma 2]. □

The following lemma establishes further properties of the solutions, in particular their exponential decay. We will suppose that (A') holds throughout the rest of this section.

Lemma 2.8. Let (λ, u) ∈ R × H(1)(R) be a solution of (2.12).

(i) 0 < λ < λ∞.

(ii) For any ε ∈ (0, λ), let η = λ − ε. Then there exists rε > 0 such that

Furthermore,

|u(x)| ≤ ∥u∥∞ e−√π|x−rε|, for all x ∈ R.

(2.13)

Proof. To the principal eigenvalue λ∞ of (15) corresponds an eigenfunction φ∞ > 0. The proof of (i) then follows in a similar way to that of [3, Proposition 14 (iv)].

Property (ii) was stated in [2, Proposition 14 (iii)] in the context of (SNLS) on the half-line, but we did not give the proof explicitly there, so we present it here for completeness. By (A0), for any ε ∈ (0, λ), there exists rε > 0 such that

|2| ≥ rε ⇒ f(x, s) ≤ ε < λ ∀ s ≥ 0.

Define a function

z(x) := ∥u∥∞ e−√π|x−rε|, x ∈ R,

and a set Ωε := {x ∈ R : |x| ≥ rε, z(x) < 0}. For all x ∈ Ωε we have

u′′(x) = [λ − f(x, u(x)2)]u(x) ≥ [λ − ε]u(x) = ηu(x).

Hence,

z′′(x) = η∥u∥∞ e−√π|x−rε| − u′′(x)

≤ η(∥u∥∞ e−√π|x−rε| − u(x)) = ηz(x) ∀ x ∈ Ωε.

Furthermore, z(x) = ∥u∥∞ − u(x) ≥ 0 for |x| = rε and lim|x|→∞ z(x) = 0. Therefore, if Ωε ≠ ∅, it follows by the weak maximum principle [11, Theorem 8.1] that z ≥ 0 in Ωε, a contradiction. Hence Ωε = ∅ and so

u(x) ≤ ∥u∥∞ e−√π|x−rε|, |x| ≥ rε.
A similar argument applied to \(-u\) yields
\[
-u(x) \leq \|u\|_{L^\infty} e^{-\sqrt{\pi}(|x|-r_\epsilon)}, \quad |x| \geq r_\epsilon.
\]
Since we clearly have \(|u(x)| \leq \|u\|_{L^\infty} e^{-\sqrt{\pi}(|x|-r_\epsilon)}\) for \(|x| \leq r_\epsilon\), (2.13) is proved.

Finally, by de l’Hospital’s rule,
\[
\lim_{|x| \to \infty} \frac{u'(x)^2}{u(x)^2} = \lim_{|x| \to \infty} \frac{2u'(x)u''(x)}{2u(x)u'(x)} = \lim_{|x| \to \infty} \frac{u''(x)}{u(x)} = \lim_{|x| \to \infty} \lambda - f(x, u(x)^2) = \lambda,
\]
where we have used (A0) and (A6)(ii) in the last equality. Since we know from Lemma 2.7 that \(u\) is even with \(u' < 0\) on \((0, \infty)\), (2.13) follows. \(\square\)

Let us now prove that the solutions of (2.12) are non-degenerate. This will allow us to extend the local curve of solutions obtained in Theorem 2.3 in a smooth manner. We will denote by \(D_2F\) the Fréchet derivative of \(F\) with respect to its second argument.

**Lemma 2.9.** Let \((\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R})\) be a solution of (2.12). Then the linear mapping \(D_2F(\lambda, u) : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})\) is an isomorphism.

**Proof.** The linear operator \(D_2F(\lambda, u) : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})\) is explicitly given by
\[
D_2F(\lambda, u)v = v'' + [2\partial_2 f(x, u^2)u^2 + f(x, u^2)]v - \lambda v.
\]
Note that we can write it as \(D_2F(\lambda, u) = R_\lambda + C\), where \(R_\lambda v := v'' - \lambda v\) and \(Cv := [2\partial_2 f(x, u^2)u^2 + f(x, u^2)]v\), \(v \in H^1(\mathbb{R})\). Since \(u \in H^1(\mathbb{R})\), it follows from (A0) and (A6)(ii) that
\[
\lim_{|x| \to \infty} 2\partial_2 f(x, u(x)^2)u(x)^2 + f(x, u(x)^2) = 0.
\]
(2.15)
It is then easily seen that \(C : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})\) is a compact linear operator. Consequently, since \(R_\lambda v : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})\) is an isomorphism for all \(\lambda > 0\), we need only show that \(D_2F(\lambda, u)\) is injective.

To prove this by contradiction, let us suppose that there exists \(v \in H^1(\mathbb{R}) \setminus \{0\}\) such that
\[
 v'' + [2\partial_2 f(x, u^2)u^2 + f(x, u^2)]v = \lambda v \quad \text{in } H^{-1}(\mathbb{R}). \tag{2.16}
\]
Clearly, it follows that \(v \in C^2(\mathbb{R}) \cap H^2(\mathbb{R})\) and that the equation holds in the classical sense. We will first prove that \(v\) is even. Let \(w \in H^1(\mathbb{R})\) be the odd part of \(v\), \(w(x) := \frac{1}{2}(v(x) - v(-x))\), \(x \in \mathbb{R}\). We need to show that \(w \equiv 0\). Suppose instead that \(w \not\equiv 0\). We have that \(w \in C^2(\mathbb{R})\), \(w(0) = 0\) and \(w\) is also a solution of the linear equation (2.16). Without loss of generality, we can then suppose that \(w'(0) > 0\). Hence, \(w > 0\) in a right neighbourhood of \(x = 0\). Let \(x_0 > 0\) be the first positive zero of \(w\), or \(x_0 = \infty\) if \(w > 0\) on \((0, \infty)\). In case \(x_0 < \infty\) we have \(w(x_0) = 0\) and \(w'(x_0) < 0\).

We now let \(z := u'\). By Lemma 2.7, \(z < 0\), \(z \in C^2(\mathbb{R})\), and \(z\) satisfies
\[
 z'' + [2\partial_2 f(x, u^2)u^2 + f(x, u^2)]z + \partial_1 f(x, u^2)u = \lambda z. \tag{2.17}
\]
In case \(x_0 < \infty\), integrating the Lagrange identity for \(w\) and \(z\) between \(x = 0\) and \(x = x_0\), and using \(w(0) = 0\), yields
\[
z(x_0)w'(x_0) = \int_0^{x_0} \partial_1 f(x, u^2)wu \, dx.
\]
If \( x_0 = \infty \), we get
\[
0 = \lim_{x \to \infty} z(x)w'(x) - w(x)z'(x) = \int_0^\infty \partial_1 f(x, u^2)wu \, dx.
\]

It follows from the previous discussion that \( z(x_0)w'(x_0) > 0 \). However, in both cases the integral on the right-hand side is \( < 0 \) by (A5). This contradiction shows that we must have \( w \equiv 0 \) indeed. Hence, \( v \) is even and \( v'(0) = 0 \).

Consequently, integrating the Lagrange identity for \( u \) and \( v \) over \((0, \infty)\) yields
\[
\int_0^\infty \partial_2 f(x, u^2)u^3v \, dx = 0.
\]

Since \( \partial_2 f(x, u(x)^2)u(x)^3 > 0 \) for all \( x > 0 \) by (A5), it follows that \( v \) must have at least one zero in \((0, \infty)\). Furthermore, for any \( x > 0 \), multiplying \( (2.10) \) by \( v \) and integrating over \((x, \infty)\) yields
\[
\int_x^\infty vv'' \, dy + \int_x^\infty [2\partial_2 f(y, u^2)u^2 + f(y, u^2)]v^2 \, dy = \lambda \int_x^\infty v^2 \, dy.
\]

Hence, integrating by parts,
\[
v(x)v'(x) = -\int_x^\infty v'(y)^2 \, dy + \int_x^\infty [2\partial_2 f(y, u^2)u^2 + f(y, u^2) - \lambda]v^2 \, dy, \quad \text{for all } x > 0.
\]

But it follows by \( (2.13) \) that there exists \( r > 0 \) such that
\[
2\partial_2 f(y, u(y)^2)u^2 + f(y, u(y)^2) - \lambda \leq -\lambda/2 \quad \text{for all } y \geq r,
\]

and so \( v'(x)v(x) < 0 \) for all \( x \geq r \). In particular, there exists \( x_1 > 0 \) such that \( v(x_1) = 0 \) and \( v(x) \neq 0 \) for all \( x > x_1 \). Without loss of generality, we can suppose that \( v'(x_1) > 0 \) and \( v(x) > 0 \) for all \( x > x_1 \). Integrating the Lagrange identity for \( v \) and \( z \) over \((x_1, \infty)\) yields
\[
zv' - vz' \bigg|_{x_1}^\infty = \int_{x_1}^\infty \partial_1 f(x, u^2)vu \, dx.
\]

In view of (A6), it follows easily from \( (2.17) \) that \( z \in H^2(\mathbb{R}) \), and so
\[
\lim_{x \to \infty} z(x)v'(x) - v(x)z'(x) = 0.
\]

Therefore, \( -z(x_1)v'(x_1) < 0 \) by (A5), so that \( v'(x_1) < 0 \). This contradiction finishes the proof.

We are now in a position to prove the main result of this section.

**Theorem 2.10.** Let assumption (A') hold. There exists \( u \in C^1((0, \lambda_{\infty}), H^1(\mathbb{R})) \) such that, for all \( \lambda \in (0, \lambda_{\infty}) \), \( (\lambda, u(\lambda)) \) is the unique positive solution of \((\text{SNLS})\), \( u(\lambda) \in C^2(\mathbb{R}) \cap H^2(\mathbb{R}) \), \( u(\lambda) \) is even, and satisfies Lemmas 2.4 and 2.5.

Furthermore, there is bifurcation from the line of trivial solutions at \( \lambda = 0 \), in the sense of \( (2.10) \), and asymptotic bifurcation at \( \lambda = \lambda_{\infty} \), in the following sense: if \( \lambda_n \to \lambda \in (0, \lambda_{\infty}) \) as \( n \to \infty \) then
\[
\lim_{n \to \infty} \|u(\lambda_n)\|_{H^2(\mathbb{R})} = \lim_{n \to \infty} \|u(\lambda_n)\|_{L^\infty(\mathbb{R})} = \infty \iff \lambda = \lambda_{\infty}.
\]

**Proof.** Global asymptotic bifurcation for \((\text{SNLS})\) on \((0, \infty)\) was established in \((3)\) by degree theoretic arguments. Corollary 2 of \((3)\) holds under the present hypotheses — restricted to the problem on the half-line in the obvious manner. One needs only remark that, in hypothesis (f2) of \((3)\), \( f_0 \) can be assumed to be zero, without
any change to the proof of Corollary 2. Moreover, under hypothesis (A5), the assumption \( \lambda := \limsup_{\tau \to \infty} f_\tau(x) > f_0 \) can be relaxed to an equality, allowing for \( f = \infty \). By even extension to \( \mathbb{R} \), Corollary 2 of \( \ref{cor:orbital} \) then yields a continuous curve of solutions of \( \ref{eq:SNLS} \), \((0, \lambda_\infty) \ni \lambda \to u(\lambda) \), with the asymptotic behaviour \( \ref{eq:asymptotic} \).

Under hypotheses \((A4)\) and \((A5)\), it can be proved by the method of ‘separation of graphs’, as presented in \( \cite{32} \), that the positive solution of \((\text{SNLS})\) is unique, for any \( \lambda \in (0, \lambda_\infty) \). Therefore, the local branch obtained in Theorem \( \ref{th:local} \) lies on this curve. Thus, the bifurcation behaviour at \( \lambda = 0 \) follows from Theorem \( \ref{th:local} \), whereas the asymptotic behaviour as \( \lambda \to \lambda_\infty \) follows from Corollary 2 of \( \cite{9} \). Finally, Lemma \( \ref{lem:global} \) shows that \( u \in C^1((0, \lambda_\infty), H^1(\mathbb{R})) \), concluding the proof. \( \square \)

3. Orbital stability of standing waves

Using the function \( u \in C^1((0, \lambda_\infty), H^1(\mathbb{R}, \mathbb{C})) \) given by Theorem \( \ref{th:local} \), standing wave solutions of \((\text{NLS})\) are constructed as

\[
\psi_\lambda(t, x) := e^{it\lambda} u(\lambda)(x), \quad \lambda \in (0, \lambda_\infty).
\]

The mapping \( \lambda \to \psi_\lambda \) defines a smooth curve of solutions of \((\text{NLS})\) in the space \( C\left([0, \infty), H^1(\mathbb{R}, \mathbb{C})\right) \cap C^1\left((0, \infty), H^{-1}(\mathbb{R}, \mathbb{C})\right) \). General solutions of \((\text{NLS})\) are functions \( \varphi \in C\left([0, T), H^1(\mathbb{R}, \mathbb{C})\right) \cap C^1\left((0, T), H^{-1}(\mathbb{R}, \mathbb{C})\right) \) satisfying \((\text{NLS})\) in the weak sense (with the identifications \( \ref{eq:identification1}, \ref{eq:identification2} \)), for all \( t \in (0, T) \). Here, \( T > 0 \) determines the maximal interval of existence of the solution \( \varphi \). If \( T = \infty \), the solution is called global — this is obviously the case for standing waves.

A prerequisite for the stability analysis of \((\text{NLS})\) is the global well-posedness of the Cauchy problem. This is thoroughly investigated in \( \cite{2} \), for very general nonlinearities. We will only need the following result here, which is proved in Section 3.5 of \( \cite{2} \).

**Theorem 3.1.** Let \( \bar{f} \) be defined by \( \ref{eq:barf} \), with \( f \in C^0(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}) \), and suppose that there exist \( C > 0 \) and \( \sigma \in [0, 4) \) such that

\[
|\bar{f}(x, s)| \leq C|1 + |s|^{\sigma}|s|, \quad \text{for all } (x, s) \in \mathbb{R}^2.
\]

For any \( \varphi_0 \in H^1(\mathbb{R}, \mathbb{C}) \), there is a unique global solution \( \varphi \in C\left([0, \infty), H^1(\mathbb{R}, \mathbb{C})\right) \) of \((\text{NLS})\), with initial condition \( \varphi(0, \cdot) = \varphi_0 \).

Under assumption \((A')\), we can take \( \sigma = 0 \) in \( \ref{th:existence} \) (see \( \ref{eq:existence} \)), hence the conclusion of Theorem \( \ref{th:existence} \) holds.

**Definition 3.2.** We say that the standing wave \( \psi_\lambda \) in \( \ref{eq:stationary} \) is orbitally stable if

\[
\forall \varepsilon > 0 \exists \delta > 0 \text{ such that }
\]

for any (global) solution \( \varphi(t, x) \) of \((\text{NLS})\) with initial data \( \varphi_0 \in H^1(\mathbb{R}) \) we have

\[
\|\varphi_0 - u(\lambda)\|_{H^1} \leq \delta \implies \inf_{\theta \in \mathbb{R}} \|\varphi(t, \cdot) - e^{it\theta} u(\lambda)\|_{H^1} \leq \varepsilon \quad \forall t \geq 0.
\]

A general theory of orbital stability for infinite-dimensional Hamiltonian systems was established in \( \cite{12} \) — see also \( \cite{26} \), where this issue was revisited in great detail, and applied to \((\text{NLS})\). The stability of a standing wave \( \psi_\lambda \) is related to spectral properties of the linearization of \((\text{NLS})\) at \( \psi_\lambda \). When the spectral conditions are
satisfied, it can be inferred from \[26\] that, in the present context\footnote{Note that the parameter \(\lambda\) in \[26\] Section 7 has an opposite sign from ours.} 

\[ \psi_\lambda(t, x) = e^{i\lambda t} u_\lambda(x) \]

is orbitally stable if \( \frac{d}{d\lambda} \int_{\mathbb{R}} u(\lambda)(x)^2 \, dx > 0. \) \hspace{1cm} (3.3)

This condition is often referred to as the slope condition.

3.1. **The spectral conditions.** Following the discussion in \[26\] (see in particular part (5) of the summary in \[26\] Section 7.4), the spectral conditions pertain to the linear operators \( L_1^\lambda, L_2^\lambda : H^2(\mathbb{R}, \mathbb{R}) \subset L^2(\mathbb{R}, \mathbb{R}) \rightarrow L^2(\mathbb{R}, \mathbb{R}) \) defined by

\[ L_1^\lambda v := -v'' - \partial_2 \bar{f}(x, u(\lambda))v + \lambda v, \quad L_2^\lambda v := -v'' - \frac{\bar{f}(x, u(\lambda))}{u(\lambda)}v + \lambda v. \] \hspace{1cm} (3.4)

Let us denote by \( M(A) \) the Morse index of a self-adjoint operator \( A : D(A) \subset L^2 \rightarrow L^2, \) defined by

\[ M(A) := \sum_{E \in \mathcal{E}} \dim E, \]

where \( \mathcal{E} \) is the collection of all eigenspaces corresponding to negative eigenvalues of \( A \) (and \( M(A) := 0 \) if \( \mathcal{E} = \emptyset \)). We will also denote by \( \sigma(A) \) and \( \sigma_e(A) \) the spectrum of \( A \) and the essential spectrum of \( A, \) respectively. Then the spectral conditions required by the stability analysis are the following:

- **(S1)** \( \inf \sigma_e(L_1^\lambda) > 0, \ M(L_1^\lambda) = 1 \) and \( \ker L_1^\lambda = \{0\}; \)
- **(S2)** \( \inf \sigma_e(L_2^\lambda) > 0, \ 0 = \inf \sigma(L_2^\lambda) \) and \( \ker L_2^\lambda = \text{span}\{u(\lambda)\}. \)

**Proposition 3.3.** Let \((A')\) hold, \( \bar{f} \) be defined by \[22\], and \( u \in C^1((0, \lambda_\infty), H^1(\mathbb{R})) \) be given by Theorem \[2, 10\]. Then the operators \( L_1^\lambda, L_2^\lambda : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) defined in \[3, 4\] satisfy (S1) and (S2), respectively.

**Proof.** First of all, an argument similar to the proof of \[24\] Lemma 3.4 (i) shows that all eigenvalues of \( L_1^\lambda \) and \( L_2^\lambda \) are simple. Also,

\[ \lim_{|x| \rightarrow \infty} \partial_2 \bar{f}(x, u(\lambda)) = \lim_{|x| \rightarrow \infty} \frac{\bar{f}(x, u(\lambda))}{u(\lambda)} = 0 \implies \inf \sigma_e(L_i^\lambda) = \lambda > 0, \ i = 1, 2. \]

We now complete the proof of (S1). First, arguments similar to the proof of \[3\] Lemma 13 show that \( M(L_1^\lambda) = 1 \) for \( \lambda > 0 \) small enough. Since \( \|\partial_2 \bar{f}(x, u(\lambda))\|_{L^\infty} \) depends continuously on \( \lambda \in (0, \lambda_\infty) \), it follows from the min-max characterization of eigenvalues (see e.g. \[18\] Section XIII.1)), that the (isolated) eigenvalues of \( L_1^\lambda \) depend continuously on \( \lambda \in (0, \lambda_\infty). \) But \( \ker L_1^\lambda = \{0\} \) for all \( \lambda \in (0, \lambda_\infty) \) by the proof of Lemma \[2, 9\] which prevents eigenvalues from ‘crossing zero’ as \( \lambda \) varies. Hence \( M(L_1^\lambda) = 1 \) for all \( \lambda \in (0, \lambda_\infty). \)

Regarding (S2), noting that

\[ L_2^\lambda v = -v'' - f(x, u(\lambda)^2)v + \lambda v, \ v \in H^2(\mathbb{R}), \]

it follows from Theorem \[2, 10\] that \( u(\lambda) \in \ker L_2^\lambda. \) The eigenvalues of \( L_2^\lambda \) being simple, \( \ker L_2^\lambda = \text{span}\{u(\lambda)\}, \ \lambda \in (0, \lambda_\infty). \) Finally, since \( \inf \sigma_e(L_2^\lambda) > 0, \) \( 0 \) is an isolated eigenvalue, with corresponding eigenfunction \( u(\lambda) > 0. \) It follows that \( 0 = \inf \sigma(L_2^\lambda), \) which completes the proof. \( \square \)
3.2. The slope condition. In order to show that the slope condition is verified, we will need the following additional assumption.

**H** The function

\[(x, s) \rightarrow \frac{2f(x, s) + x\partial_1 f(x, s)}{\partial_2 f(x, s)} - 1, \quad x > 0, \quad s > 0,\]

is positive and non-increasing as \(x\) increases and \(s\) decreases.

**Example 3.4.** Under the hypotheses of Examples 2.2 and 2.5, it is easily seen that the function \(f\) defined by (1.9) also satisfies assumption **H**, provided that 

\[-b \leq xV'(x)/V(x) < 0 \quad \text{for} \quad x > 0, \quad \text{and that the mapping} \quad x \rightarrow xV'(x)/V(x) \quad \text{is non-increasing for} \quad x > 0.

Let us now summarize our hypotheses on \(V: \mathbb{R} \rightarrow \mathbb{R}\) and \(\alpha > 0\) ensuring that the function \(f\) defined in Example 1.1 satisfies (A') and (H):

(a) \(V \in C^1(\mathbb{R})\) is even, \(V > 0\) on \(\mathbb{R}\), and \(V'(x) < 0\) for \(x > 0); \(\Phi : (0, \infty) \rightarrow \mathbb{R}\) is non-increasing, and \(0 < \Phi(s) \leq \alpha\) for all \(s > 0;\)

(b) \(\lim_{s \to 0^+} \Phi(s) = 1.\)

In particular, conditions (c) and (\(\phi_2\)) (simply (c) and (d) in the special case (1.9)) ensure that (H) is satisfied.

In view of Proposition 3.3 and [26], the orbital stability of the standing waves \(\psi_{\lambda}\) in (3.1) will be established if we prove the following result.

**Proposition 3.5.** Let hypotheses (A') and (H) hold, and \(u \in C^1((0, \lambda_\infty), H^1(\mathbb{R}))\) be given by Theorem 2.10. Then

\[
\frac{d}{d\lambda} \int_{\mathbb{R}} u(\lambda)(x)^2 \, dx > 0 \quad (3.5)
\]

for all \(\lambda \in (0, \lambda_\infty)\).

Our proof of Proposition 3.5 is very similar to that of (1.5) in [8, Theorem 1.7], where a power nonlinearity is considered. Since the structure of the nonlinearity is quite different in the present case, one needs to check carefully that every step of the proof carried out in Section 5.2 of [8] works under assumption (A'). Therefore, we will present the whole argument here.
We will suppose that assumption (A') holds for the rest of this section. We already know from Theorem 2.10 — more precisely from bifurcation at \( \lambda = 0 \) — that there is some \( \lambda > 0 \) for which \( 3.5 \) holds. Therefore, by continuity, we need only show that
\[
\frac{d}{d\lambda} \int_{\mathbb{R}} u(\lambda)(x)^2 \, dx = 2 \frac{d}{d\lambda} \int_{0}^{\infty} u(\lambda)(x)^2 \, dx \neq 0 \quad \forall \lambda \in (0, \lambda_\infty).
\] (3.6)

Now,
\[
\frac{d}{d\lambda} \int_{0}^{\infty} u(\lambda)(x)^2 \, dx = 2 \int_{0}^{\infty} u(\lambda)(x)\xi(\lambda)(x) \, dx,
\]
where
\[
\xi(\lambda) := \frac{du}{d\lambda}(\lambda) \in H^1(\mathbb{R}), \quad \text{for all } \lambda \in (0, \lambda_\infty).
\]
We will use a fairly involved integral indentity derived from the equations satisfied by \( u(\lambda) \) and \( \xi(\lambda) \), namely \( \text{(SNLS)} \) and
\[
\xi'' + f(x, u^2)\xi + 2\partial_2 f(x, u^2)u^2\xi = \lambda\xi + u. \quad (3.7)
\]
(We will omit the variables \( \lambda \) and/or \( x \) when no confusion is possible.) The equation for \( \xi \) is easily obtained by differentiation of the identity \( F(\lambda, u(\lambda)) = 0 \) with respect to \( \lambda \), where \( F \) is defined in \( 2.10 \). We will first prove two lemmas establishing the required integral identity and some useful properties of \( \xi \). Proposition 3.5 will then be proved using these results.

The strategy of proof applied here was first used in \( 16 \), where the authors considered a similar problem, however with a power-type nonlinearity (although their proof allows for more general situations — albeit not the asymptotically linear case), and a non-trivial linear potential.

**Lemma 3.6.** For \( u, \xi \in H^1(\mathbb{R}) \) as above, the following identity holds:
\[
\int_{0}^{\infty} [2f(x, u^2) + x\partial_1 f(x, u^2) - \partial_2 f(x, u^2)u^2]u\xi \, dx = 2\lambda \int_{0}^{\infty} u^2 \xi \, dx. \quad (3.8)
\]

**Proof.** The proof follows that of \( 8 \), Lemma 5.2. Let us first remark that, using the properties of \( u \), it follows from \( 3.1 \) that \( \xi \in C^2(\mathbb{R}) \cap H^2(\mathbb{R}) \), and \( \xi \) is even with \( \xi'(0) = 0. \) Integrating the Lagrange identity for \( \text{SNLS} \) and \( 3.7 \) then yields
\[
\int_{0}^{\infty} u^2 \, dx = 2 \int_{0}^{\infty} \partial_2 f(x, u^2)u^3 \xi \, dx. \quad (3.9)
\]
On the other hand, multiplying \( \text{SNLS} \) by \( u \) and integrating gives
\[
\int_{0}^{\infty} f(x, u^2)u^2 - (u')^2 \, dx = \lambda \int_{0}^{\infty} u^2 \, dx. \quad (3.10)
\]
Now multiplying \( \text{SNLS} \) by \( xu' \) and integrating by parts (using the exponential decay of \( u' \)) yields
\[
\int_{0}^{\infty} xf(x, u^2)uu' - (u')^2 - xu'u'' \, dx = \lambda \int_{0}^{\infty} xu' \, dx.
\]
Using \( \text{SNLS} \) and integrating by parts, it follows that
\[
\int_{0}^{\infty} 2xf(x, u^2)uu' - (u')^2 \, dx = 2\lambda \int_{0}^{\infty} xu' \, dx = 2\lambda \int_{0}^{\infty} x(\frac{1}{2}u^2)' \, dx
\]
\[
= \lambda \left[ xu^2 \bigg|_{0}^{\infty} - \int_{0}^{\infty} u^2 \, dx \right] = -\lambda \int_{0}^{\infty} u^2 \, dx. \quad (3.11)
\]
Furthermore, computing
\[ \frac{d}{dx} \left( x \int_0^x f(x, s) \, ds \right) = \int_0^x f(x, s) \, ds + x f(x, u^2) 2u' + x \int_0^x \partial_1 f(x, s) \, ds, \]
we can substitute the first term of the LHS of (3.11) and get
\[ \int_0^\infty \left\{ \frac{d}{dx} \left( x \int_0^x f(x, s) \, ds \right) - \int_0^x [f(x, s) + x \partial_1 f(x, s)] \, ds - (u')^2 \right\} \, dx = -\lambda \int_0^\infty u^2 \, dx. \quad (3.12) \]

But the first term in the LHS of (3.12) can be integrated and yields
\[ x \int_0^\infty f(x, s) \, ds \bigg|_0^\infty = \lim_{x \to \infty} x \int_0^x f(x, s) \, ds = 0 \]
since, by (2.7) and the exponential decay of \( u \),
\[ \left| x \int_0^{u(x)^2} f(x, s) \, ds \right| \leq Mxu(x)^2 \to 0 \quad \text{as } x \to \infty. \]

Hence we finally have
\[ \int_0^\infty \int_0^x [f(x, s) + x \partial_1 f(x, s)] \, ds \, dx + (u')^2 \int_0^\infty u^2 \, dx = -\lambda \int_0^\infty u^2 \, dx. \quad (3.13) \]

Now adding (3.11) and (3.13) yields
\[ \int_0^\infty f(x, u^2)u^2 + \int_0^\infty [f(x, s) + x \partial_1 f(x, s)] \, ds \, dx = 2\lambda \int_0^\infty u^2 \, dx. \quad (3.14) \]

Next, differentiating (3.14) with respect to \( \lambda \), we obtain
\[ \int_0^\infty [2f(x, u^2) + x \partial_1 f(x, u^2) + \partial_2 f(x, u^2)u^2]u\xi \, dx = \int_0^\infty u^2 \, dx + 2\lambda \int_0^\infty u\xi \, dx. \quad (3.15) \]

Finally, using (3.14) to substitute the first term of the RHS of (3.13), we get (3.8). \( \square \)

The following lemma establishes useful properties of \( \xi \).

**Lemma 3.7.** For all \( \lambda \in (0, \lambda_\infty) \), the function \( \xi = \xi(\lambda) \) has the following properties: \( \xi(0) > 0 \) and there exists a unique \( x_0 = x_0(\lambda) \in (0, \infty) \) such that \( \xi(x_0) = 0 \), \( \xi(x) \geq 0 \) for all \( x \in (0, x_0) \), and \( \xi(x) \leq 0 \) for all \( x \in (x_0, \infty) \).

**Proof.** We fix \( \lambda \in (0, \lambda_\infty) \), and we simply write \( \xi(x) \) for \( \xi(\lambda)(x) \), \( x \in \mathbb{R} \). The first part of the proof — showing that \( \xi(0) > 0 \) — is easily adapted from that of [8, Lemma 5.3], using the sign of \( \partial_1 f(x, u^2) \) given by (A5).

To prove the existence of an \( x_0 \in (0, \infty) \) \((x_0 = x_0(\lambda))\) with the asserted properties, it is convenient to establish first that there is no \( a > 0 \) such that \( \xi(x) \geq 0 \) for all \( x \geq a \). This is the part of the proof where some amendments to that of [8, Lemma 5.3] need to be mentioned. We will prove this by contradiction, so we assume that such an \( a \) exists. Integrating the Lagrange identity for (2.11) and (3.7) from \( x \geq a \) to \( \infty \) yields
\[ -z(x)^2 \left( \frac{\xi'}{z'} \right)'(x) = \xi(x)z'(x) - \xi'(x)z(x) = \int_x^\infty uz + \partial_1 f(y, u^2)u\xi \, dy < 0, \quad x \geq a. \]
Hence \((\xi/z)' > 0\) on \([a, \infty)\) and \(\xi/z\) is increasing on this interval. But \(\xi/z \leq 0\) on \([a, \infty)\) and so there exists a number \(L\) such that
\[
\lim_{x \to \infty} \frac{\xi(x)}{z(x)} = L \leq 0.
\]
Since \(\xi \in H^2(\mathbb{R})\), we have \(\xi'(x) \to 0\) as \(x \to \infty\). Furthermore, \(z'(x) \to 0\) as \(x \to \infty\) by \([\text{SNLS}]\). It then follows from de l’Hospital’s rule that
\[
L = \lim_{x \to \infty} \frac{\xi(x)}{z(x)} = \lim_{x \to \infty} \frac{\xi'(x)}{z'(x)} = \lim_{x \to \infty} \frac{\xi''(x)}{z''(x)},
\]
as long as the last limit exists. Now
\[
\frac{\xi''}{z''} = \frac{\Lambda \xi + u - f(x, u^2)\xi - 2\partial_2 f(x, u^2)u^2\xi}{\lambda z - f(x, u^2)z - \partial_1 f(x, u^2)u - 2\partial_2 f(x, u^2)u^2z}
\]
where, as \(x \to \infty\):
\[
\frac{\xi}{z} \to L, \quad \frac{u}{z} \to -\lambda^{-1/2},
\]
\(f(x, u^2) \to 0\) by (A0), \(\partial_2 f(x, u^2)u \to 0\) by (A6)(ii),
and
\[\partial_1 f(x, u^2) \to 0\] by \([2.5]\).
Hence
\[
L = \lim_{x \to \infty} \frac{\xi''(x)}{z''(x)} = \frac{\Lambda L - \lambda^{-1/2}}{\lambda} = L - \lambda^{-3/2},
\]
a contradiction. Therefore, there is no \(a > 0\) such that \(\xi(x) \geq 0\) for all \(x \geq a\). The remainder of the proof is easily adapted from that of \([8\], Lemma 5.3\], using again \(\partial_1 f(x, u^2) < 0\). We leave the details to the reader. \(\square\)

**Proof of Proposition 3.5.** We prove \([3.3]\) by contradiction. Hence we suppose that
\[
\int_{0}^{\infty} u(\lambda)(x) \xi(\lambda)(x) \, dx = 0
\]
for some \(\lambda \in (0, \lambda_{\infty})\). We omit the dependence on \(\lambda\) from \(u\) and \(\xi\) for the rest of the proof. It follows from \([3.8]\) and our assumption that
\[
\int_{0}^{\infty} \left\{ 2f(x, u^2) + x\partial_1 f(x, u^2) \frac{\partial_2 f(x, u^2)u^2}{\partial_2 f(x, u^2)u^2} - 1 \right\} \partial_2 f(x, u^2)u^3 \xi \, dx = 0. \tag{3.16}
\]
Letting
\[
\zeta(x) := \frac{2f(x, u^2) + x\partial_1 f(x, u^2) \frac{\partial_2 f(x, u^2)u^2}{\partial_2 f(x, u^2)u^2} - 1}{x}, \quad x > 0,
\]
it follows from (H) that \(\zeta\) is positive and non-increasing on \((0, \infty)\). Using the zero \(x_0\) of \(\xi\) given by Lemma \([3.7]\) we can rewrite the identity \([3.10]\) as
\[
\int_{0}^{\infty} [\zeta(x) - \zeta(x_0)] \partial_2 f(x, u^2)u^3 \xi \, dx + \zeta(x_0) \int_{0}^{\infty} \partial_2 f(x, u^2)u^3 \xi \, dx = 0.
\]
By \([3.10]\), this becomes
\[
\int_{0}^{\infty} [\zeta(x) - \zeta(x_0)] \partial_2 f(x, u^2)u^3 \xi \, dx + \frac{\zeta(x_0)}{2} \int_{0}^{\infty} u^2 \, dx = 0.
\]
Since \(\int_{0}^{\infty} u^2 \, dx > 0\), the properties of \(\zeta\) yield the desired contradiction. The proposition is proved. \(\square\)
Theorem 3.8. Suppose that hypotheses (A’) and (H) hold. The standing waves of (NLS) defined by (3.1) are orbitally stable.

Proof. In view of the discussion of orbital stability for (NLS) in [26] (see part (5) of the summary in Section 7.4 of [26]), the result follows immediately from Propositions 3.3 and 3.5.

4. Self-focusing planar waveguides

In this last section we will briefly present an important application of our results to nonlinear optics. Self-focusing planar waveguides have been thoroughly investigated, both from the physical and the mathematical standpoints. Amongst the wide literature about nonlinear waveguides, let us mention [1, 3, 19, 21, 31, 33] regarding the physics, and [8, 20, 22, 25, 27, 29] for some mathematical results. Historically, the mathematical study of (NLS) has grown in parallel to — and was largely motivated by — the development of nonlinear waveguide theory. A striking example of this close interaction is the early paper [33], where the slope condition (3.3) was first introduced to discuss the stability of travelling waves in a cylindrical waveguide with a saturable dielectric response. (We will precisely be interested in saturable materials here, see (4.2) below.)

The mathematical modelling of a planar self-focusing waveguide is summarized in [8], where we used bifurcation and stability results similar to Theorem 2.10 and Theorem 3.8 to discuss the behaviour of TE travelling waves in a waveguide with a power-type dielectric response. The planar waveguide is idealized as a slab of dielectric material parallel to the $xz$-plane, having infinite extension in the $x$-direction, and semi-infinite extension along $z$, in the $z > 0$ direction. We will look for electromagnetic waves in the optical regime, travelling along the $z > 0$ half-axis, and so $x$ will be transverse to the direction of propagation. According to Maxwell’s equations, the propagation of a light beam depends on the material, characterized by a dielectric response $\varepsilon > 0$ (or, alternatively, the refractive index $n = \sqrt{\varepsilon}$). In a nonlinear medium, the dielectric response can be decomposed as

$$\varepsilon(x, s) = \varepsilon_L(x) + \varepsilon_{NL}(x, s),$$

where $\varepsilon_L(x)$ and $\varepsilon_{NL}(x, s)$ respectively denote the linear and nonlinear contributions to the dielectric response. As we shall see below, the variable $s$ is proportional to the squared modulus of the electric field of the light beam. The dependence on the variable $x$ accounts for a medium which is inhomogeneous in the transverse direction. We will consider the case where the linear contribution, defined by $\varepsilon_L \equiv \varepsilon(x, 0)$, is a positive constant, i.e. the material is homogeneous when the beam is switched off. On the other hand, we will suppose that $\varepsilon_{NL}(-x, s) \equiv \varepsilon_{NL}(x, s)$, and $\varepsilon_{NL}$ decreases away from $x = 0$. This behaviour helps to focus the waves around $x = 0$ since, according to Snell’s law, the light beam bends towards regions with a higher refractive index. A dielectric medium is called self-focusing when $\varepsilon_{NL}(x, s)$ is an increasing function of the variable $s$ (hence of the beam’s intensity).

In [8] we treated the case of a Kerr medium, where

$$\varepsilon_{NL}(x, s) = a(x)s$$  \hspace{1cm} (4.1)

for some positive even function $a \in C^1(\mathbb{R})$, decreasing for $x > 0$. Assumption (4.1), leading to a cubic nonlinearity, gives a good approximation, in the low power regime, of so-called ‘Kerr materials’. However, other materials — e.g. photorefractive
materials (see [3]) — present a *saturation* phenomenon as the power of the beam becomes large. Namely,

$$\varepsilon_{NL}(x, s) \to \varepsilon_\infty(x) \quad \text{as } s \to \infty,$$

(4.2)

where $\varepsilon_\infty(x)$ adds up to $\varepsilon_L$, yielding the asymptotic dielectric response $\varepsilon_L + \varepsilon_\infty(x)$, in the limit of high power beams. As can be seen from the equations below, this leads to an asymptotically linear nonlinearity (sometimes referred to as a ‘saturable nonlinearity’). We are now able to deal with this case.

A TE travelling wave is a special solution of Maxwell’s equations in the waveguide, having an electric field transverse to the $xz$-plane, of the form

$$E(x, y, z, t) = \text{Re} \left(0, U(x)e^{i(kz-\omega t)}, 0\right).$$

(4.3)

The envelope $U$ of the electric field is such that $U \in H^1(\mathbb{R}, \mathbb{R})$ — so as to verify the ‘guidance conditions’, namely decay of the electromagnetic field at infinity and finiteness of the energy density —, and satisfies the Helmholtz equation

$$U'' + \left(\frac{\omega c}{c}\right)^2[\varepsilon_L + \varepsilon_{NL}(x, \frac{1}{2}\psi^2)]U = k^2U, \quad x \in \mathbb{R}.$$

(4.4)

Since we only consider monochromatic waves, i.e. with $\omega > 0$ a fixed frequency in the optical regime, the time variable $t$ in (4.2) does not play an essential role in the analysis. The stability of the TE travelling waves (4.3) is usually discussed in the context of the *paraxial approximation*, with respect to the larger class of TE modes, which are more general solutions of Maxwell’s equations than (4.3) (still with an electric field transverse to the $xz$-plane). After a rescaling of the variables, this approximation leads to the following equation, governing the behaviour of TE modes in the waveguide:

$$i\partial_z \psi + \partial_{xx} \psi + \left(\frac{\omega}{c}\right)^2\varepsilon_{NL}(x, \frac{1}{2}\psi^2)\psi = 0, \quad z > 0, \quad x \in \mathbb{R}.$$

(4.5)

A detailed discussion of the paraxial approximation is given in [8], where we also explain the reduction of the problem to the canonical form (4.5), from which the constant $\varepsilon_L$ has disappeared. In this approach, the TE travelling waves (4.3) can be approximated by standing wave solutions of (4.5). Since (4.3) has the form of (NLS), and (4.4) that of (SNLS), we can apply the results of the previous sections, yielding existence of solutions for (4.4), hence of TE travelling waves, and orbital stability of standing waves of (4.5). As explained in [8, Section 6.3], a global bifurcation result such as Theorem 2.10 enables one to make the paraxial approximation as accurate as desired. Hence stability of the travelling waves (4.3) can be inferred from Theorem 3.8 — the physical meaning of ‘stability’ for the TE travelling waves (4.3) is discussed in Section 6.2 of [25], where Kerr media are studied using the results in [15, 16, 24].

The rigorous proof of stability of the travelling waves (4.3), thus obtained from the analysis of (NLS), puts on firm mathematical grounds the phenomenon of *self-trapping*, known to physicists since the early 70’s (first predicted theoretically and later observed experimentally — see [21]). This phenomenon is briefly described as follows. In a self-focusing medium, the nonlinear contribution $\varepsilon_{NL}(x, s)$ is an increasing function of $s > 0$, typically larger near to $x = 0$. By increasing locally, around $x = 0$, the dielectric response, the light beam induces its own waveguide, forcing the waves to focus in the $x = 0$ region, along the $z$-axis. When this focusing effect balances the dispersion due to the term $\partial_{xx}^2 \psi$ in (4.5), self-trapping occurs, yielding stable guided waves.
Our main results for self-focusing planar waveguides with a saturable dielectric response (4.2) follow from a similar discussion to that in [8, Section 6.3] for the Kerr medium. They can be summarized as follows. Letting

\[ f(x, s) = \left( \frac{\omega}{c} \right)^2 \varepsilon_{NL}(x, \frac{1}{2}s), \]

we suppose that assumptions (A') and (H) are satisfied. Then, applying Theorem 2.10 and Theorem 3.8 to (4.4) and (4.5), we get the following results.

1. There exist guided TE travelling waves (4.3), corresponding to solutions \( U = U_k \) of (4.4), for all wave numbers \( k \in (k_1, k_3) \), where

\[ k_1 := \frac{\omega}{c} \sqrt{\varepsilon_L} \quad \text{and} \quad k_3 := \left( \left( \frac{\omega}{c} \right)^2 \varepsilon_L + \lambda_{\infty} \right)^{1/2}. \]

2. The TE travelling waves (4.3) are stable, for all \( k \in (k_1, k_3) \).

3. Defining the power of the beam associated with \( U_k \) as

\[ P(k) := \frac{c^2k}{2\omega} \int_R U_k(x)^2 \, dx, \]

it follows from our bifurcation results in Theorem 2.10 that

\[ \lim_{k \to k_1} P(k) = 0 \quad \text{and} \quad \lim_{k \to k_3} P(k) = \infty. \]

Hence, we obtain guided waves from arbitrary low to arbitrary high power.

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