SMALL POINTS AND FREE ABELIAN GROUPS

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Abstract. Let $F$ be an algebraic extension of the rational numbers and $E$ an elliptic curve defined over some number field contained in $F$. The absolute logarithmic Weil height, respectively the Néron-Tate height, induces a norm on $F^*$ modulo torsion, respectively on $E(F)$ modulo torsion. The groups $F^*$ and $E(F)$ are free abelian modulo torsion if the height function does not attain arbitrarily small positive values. In this paper we prove the failure of the converse to this statement by explicitly constructing counterexamples.

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1. Introduction

Throughout the text we fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and all algebraic extensions of $\mathbb{Q}$ are assumed to be subfields of $\overline{\mathbb{Q}}$. One can ask for which fields $F$ the multiplicative group $F^*$ is free modulo torsion, we call an abelian group $G$ free modulo torsion if $G/G_{\text{tors}}$ is a free abelian group where $G_{\text{tors}}$ denotes the torsion subgroup of $G$. In the rational case $\mathbb{Q}^*$ is free modulo torsion as $\mathbb{Z}$ is a unique factorization domain. More generally, using classical

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ideal factorization theory and Dirichlet’s Unit Theorem one can prove that $K^*$ is free modulo torsion for any number field $K$.

Before we recall more advanced examples, we have to fix some notation. For any subfield $F \subseteq \mathbb{Q}$, let $F^{ab}$ denote the maximal abelian extension of $F$, and $F^{(d)}$ denote the compositum of all extensions of $F$ of degree at most $d$. Moreover, we let $\mathbb{Q}^{tr}$ denote the maximal totally real field extension of $\mathbb{Q}$.

Iwasawa [19] proved that $(K^{ab})^*$ is free modulo torsion. Some years later Schenkman [27] showed that $(\mathbb{Q}^{(d)})^*$ is free modulo torsion for all positive integers $d$. May [23] rediscovered Iwasawa’s result and combined it with Schenkman’s result to show that $(K^{(d)})^{ab})^*$ is free modulo torsion for all number fields $K$ and all positive integers $d$. Another class of fields $F$ such that $F^*$ is free modulo torsion consists of all Galois extensions of $\mathbb{Q}$ which contain only finitely many roots of unity. Horie’s paper [16] contains this result, it appears to be the origin. We immediately see that $(\mathbb{Q}^{tr})^*$ is free modulo torsion.

A related problem is to find algebraic extensions $F/K$ such that the Mordell-Weil group $E(F)$ is free modulo torsion for a given elliptic curve $E$ defined over $K$. This is clearly the case by the Mordell-Weil Theorem if $F$ is a number field. Here too the interest lies in infinite extensions $F/K$.

We now introduce some notation to unify the multiplicative and elliptic cases. Let $\mathcal{G}$ denote either the algebraic torus $\mathbb{G}_m$ or an elliptic curve defined over a number field $K$. We will usually suppose $F \supseteq K$.

If $\mathcal{G} = \mathbb{G}_m$ is a torus, we take the absolute logarithmic Weil height to be the canonical height on $\mathcal{G}$. If $\mathcal{G}$ is an elliptic curve, the canonical height on $\mathcal{G}$ is understood to be the Néron-Tate height $\hat{h} : \mathcal{G}(\mathbb{Q}) \to [0, \infty)$. For the definitions and basic properties of these heights, we refer to Bombieri and Gubler’s book [9]. Our Néron-Tate height is twice the height used by Silverman in §9, Chapter VIII [30]. The canonical height is well-defined modulo torsion, i.e. it factors through to a mapping $\mathcal{G}(\mathbb{Q})/\mathcal{G}(\mathbb{Q})_{\text{tors}} \to [0, \infty)$. We observe that the group $\mathcal{G}(\mathbb{Q})/\mathcal{G}(\mathbb{Q})_{\text{tors}}$ is divisible and torsion free. Thus it carries the structure of a $\mathbb{Q}$-vector space. If $\mathcal{G} = \mathbb{G}_m$, then the canonical height is a norm on this vector space. In the case of an elliptic curve, its square root $\hat{h}^{1/2}$ is one.

The basis of our investigation is the apparent coincidence that, apart from $(K^{(d)})^{ab}$, all fields we discussed above are known to share another property, the Bogomolov property, related to the Weil height. We briefly describe this property.

We say that $F$ has the Bogomolov property with respect to $\mathcal{G}$ if there exists $\epsilon > 0$ such that the canonical height of a non-torsion points of $\mathcal{G}(F)$ is at least $\epsilon$. We recall that torsion points are exactly the points of height
zero. This property was coined by Bombieri and Zannier [10] who worked in the multiplicative setting.

The fields $K^{ab}$, $\mathbb{Q}^{(d)}$ and $\mathbb{Q}^{tr}$ have the Bogomolov property relative to $\mathcal{G}$ [3], [5], [8], [10], [7], [28], [34]. It is an open question, posed in more general form by Amoroso, David, and Zannier [2, Problem 1.4], whether $(K^{(d)})^{ab}$ has the Bogomolov property with respect to $\mathcal{G}$. Recently, the second-named author established another class of fields having the Bogomolov property [15]. If $\mathcal{G}$ is an elliptic curve defined over $\mathbb{Q}$ then the field generated over $\mathbb{Q}$ by $\mathcal{G}(\overline{\mathbb{Q}})_{\text{tors}}$ has the Bogomolov property with respect to the torus and $\mathcal{G}$.

Note that neither the Bogomolov property of a field $F$ nor the property that $\mathcal{G}(F)/\mathcal{G}(F)_{\text{tors}}$ is free abelian is preserved under finite extensions $F'/F$. A counterexample for both properties when $\mathcal{G} = \mathbb{G}_m$ is the extension $\mathbb{Q}^{tr}(i)/\mathbb{Q}^{tr}$. That the stated properties are not preserved in this extension was first observed by Amoroso and Nuccio [4] and May [23, Example 1], respectively. A counterexample in the elliptic curve case is presented by the third-named author, cf. [26, Example 5.7].

Recall that a norm on an abelian group is called discrete if zero is an isolated value of its image. Lawrence [21] and Zorzitto [35] showed that a countable abelian group is free abelian if and only if it admits a discrete norm. The countability condition was later removed by Steprāns [31], but the groups considered in this paper are countable. These results immediately imply the following proposition.

**Proposition 1.1.** If $F$ is a subfield of $\overline{\mathbb{Q}}$ with $F \supseteq K$ that satisfies the Bogomolov property with respect to $\mathcal{G}$, then $\mathcal{G}(F)/\mathcal{G}(F)_{\text{tors}}$ is free abelian.

Our aim is to discuss the failure of the converse of this statement. We will prove that the converse does not hold by explicitly constructing counterexamples in the cases where $\mathcal{G}$ is $\mathbb{G}_m$, an elliptic curve with complex multiplication (CM), and for an arbitrary elliptic curve defined over $\mathbb{Q}$. In other words, in these cases we construct fields $F$ where $\mathcal{G}(F)/\mathcal{G}(F)_{\text{tors}}$ is free abelian, but there are points of arbitrarily small positive canonical height on $\mathcal{G}(F)$. Here and in the rest of this paper, an elliptic curve is said to have CM over $K$ if the ring of endomorphisms of $E$ which are defined over $K$ is strictly larger than $\mathbb{Z}$. The elliptic curve has CM if it has CM over some number field.

Now we can formulate our main result.

**Theorem 1.2.** Let $\mathcal{G}$ be $\mathbb{G}_m$ or an elliptic curve. We suppose that $\mathcal{G}$ is defined over a number field $K$. There is an algebraic extension $F/K$ such that $\mathcal{G}(F)/\mathcal{G}(F)_{\text{tors}}$ is free abelian but $F$ does not have the Bogomolov property with respect to $\mathcal{G}$.
(1) if $\mathcal{G}$ is $\mathbb{G}_m$,
(2) if $\mathcal{G}$ is an elliptic curve with CM over $K$, and
(3) if $\mathcal{G}$ is an elliptic curve without CM, and $K = \mathbb{Q}$. In this case $\mathcal{G}(F)_{\text{tors}}$ is finite.

We will give a more precise description of $F$ below. We refer to Proposition 3.1 for part (1) and Proposition 4.3 for parts (2) and (3).

The proof in Theorem 1.2 that the group in question is free abelian is given in Section 2 after providing two criteria for the freeness of abelian groups. Applying the criteria involves investigating the structure of various Galois groups. For example, part (3) requires information on the Galois group of $K(\mathcal{G}(\overline{\mathbb{Q}})_{\text{tors}})/K$. Therefore, Serre’s [29] famous theorem plays an important role in our argument.

In the remaining sections we show that the field $F$ does not satisfy the Bogomolov property with respect to the canonical height on $\mathcal{G}$ in the three cases described in Theorem 1.2. To do this, we explicitly describe how to construct points of arbitrarily small height which are defined over the field $F$. In the multiplicative setting we work with roots of

$$X^n - X - 1 \quad \text{if } n \geq 2. \quad (1.1)$$

These roots have small Weil height as $n$ tends to infinity by basic height inequalities. Moreover, Osada [25] proved that the Galois group attached to the splitting field of (1.1) over the rationals is the full symmetric group $S_n$. This is enough Galois theoretic information to apply one of the two criteria mentioned above. Indeed, we will conclude that our roots are in a common field whose multiplicative group is free modulo torsion.

The basic approach for an elliptic curve $E$ is similar but more involved. The roots now come from a polynomial equality, not unsimilar to (1.1), that involves the multiplication-by-$n$ endomorphism of $E$. The difficulty here lies in determining the Galois group of the associated splitting field. To facilitate this we introduce a new variable $T$ and assume that $n$ is the power of a prime $p$. Moreover, we will suppose that $E$ has supersingular reduction above a place with residue characteristic $p$. The polynomial reduced modulo a place above $p$ lies in $\overline{\mathbb{F}}_p(T)[X]$, with $\overline{\mathbb{F}}_p$ an algebraic closure of $\mathbb{F}_p$, and takes on a particularly simple form, it is a trinomial. We then use a result of Abhyankar [1] to conclude that the reduced polynomial is irreducible. This result also provides sufficient information on its Galois group.

Back in characteristic zero we will use variants of Hilbert’s irreducibility theorem due to Dvornicich-Zannier [11] and Zannier [33]. This allows us to specialize $T$ to a root of unity. This will often preserve irreducibility and the Galois structure of the splitting field without increasing the height too
much. The latter observation is due to the fact that torsion points have canonical height zero. The criteria mentioned above apply again.

In the non-CM case the curve $E$ has supersingular reduction at infinitely many primes by a theorem of Elkies [13]. In the CM case we have infinitely many admissible primes by more classical results. In both cases we obtain a sequence of points with small Néron-Tate height which are sufficient for Theorem 1.2.

As a byproduct of our labor we exhibit infinite extensions of the rationals over which a given elliptic curve without CM has only finitely many points of finite order, cf. part 1 of Lemma 2.9.

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2. Some Group Theory

2.1. Free Abelian Criteria. Recall that a subgroup $H$ of an abelian group $G$ is called pure if $G/H$ is torsion-free. The following version of Pontryagin’s result on free abelian groups is proved (in a stronger form) in [12, Theorem 2.3, Chapter IV].

**Theorem 2.1** (Pontryagin’s Criterion). Let $G$ be a countable abelian group. The following are equivalent:

1. $G$ is free abelian;
2. every finite subset of $G$ is contained in a pure free abelian subgroup of $G$;
3. every finite rank subgroup of $G$ is free abelian.

From this result the following two useful criteria are easily derived. They will be applied in proving Theorem 1.2.
Proposition 2.2 (Criterion A). Let $G$ be $\mathbb{G}_m$ or an elliptic curve. We suppose that $G$ is defined over a number field $K$ and assume that $F$ is a Galois extension of $K$. If $G(F)_{\text{tors}}$ is finite, then $G(F)/G(F)_{\text{tors}}$ is free abelian.

Proof. We refer to [16, Proposition 1] for the $\mathbb{G}_m$ case, and to [24, Proposition 3] for the case of an elliptic curve (or, more generally, an abelian variety).

Proposition 2.3 (Criterion B). Let $G$ be $\mathbb{G}_m$ or an elliptic curve. We suppose that $G$ is defined over a number field $K$, and let $F_0$ and $F$ be algebraic extensions of $K$ with $K \subseteq F_0 \subseteq F$. If every finite subextension of $F/F_0$ is contained in a finite subextension $M/F_0$ such that

(a) $G(M)/G(M)_{\text{tors}}$ is free abelian, and

(b) the torsion subgroup of $G(F)/G(M)$ has finite exponent,

then $G(F)/G(F)_{\text{tors}}$ is free abelian. If, furthermore, $F/F_0$ is Galois, then $G(F)/G(F)_{\text{tors}}$ is free abelian if (1) and (2) are satisfied for all finite Galois extensions $M/F_0$, with $M \subseteq F$.

Proof. The proof of the $\mathbb{G}_m$ case is originally from May (see [23, Lemma 1]). However, his proof applies also if $G$ is an elliptic curve (or even an abelian variety). For convenience we will give the proof in detail here.

We want to use the implication $(2) \implies (1)$ in Pontryagin’s Criterion 2.1. Let $S = \{P_1, \ldots, P_r\}$ be a finite subset of $G(F)/G(F)_{\text{tors}}$ with each $P_i \in G(F)$. The field $F_0(P_1, \ldots, P_r)$ is a finite extension of $F_0$, and thus contained in a finite extension $M/F_0$ with $M \subseteq F$ satisfying (a) and (b). This construction yields $S \subseteq G(M)/G(M)_{\text{tors}}$. Let $m$ be the exponent of $(G(F)/G(M))_{\text{tors}}$. Then

$$H = \{[\overline{P}] \in G(F)/G(F)_{\text{tors}} : [m][\overline{P}] \in G(M)/G(M)_{\text{tors}}\}$$

is a pure subgroup of $G(F)/G(F)_{\text{tors}}$. Moreover $H$ is free abelian, since $G(M)/G(M)_{\text{tors}}$ is free abelian, by assumption (a). Now Pontryagin’s Criterion yields the stated result as $S$ is contained in $H$.

Let $F/F_0$ be Galois and $M/F_0$ a finite extension, with $M \subseteq F$. Then the Galois closure of $M$ over $F_0$ is still contained in $F$. Thus the second statement follows immediately from the first one.

2.2. Some Facts on Matrix Groups. We collect some facts on matrix groups $G = \text{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ where $p$ will denote a prime number and $n \geq 1$ an integer. For any finite group $H$, we denote its exponent by $\text{ex}(H)$. In what follows we repeatedly use the classical Jordan-Hölder Decomposition Theorem without mentioning it directly.
The group $G$ lies in the short exact sequence

$$1 \to U^{(1)} \to G \to \text{GL}_2(\mathbb{F}_p) \to 1$$

where

$$U^{(1)} = 1 + p\text{Mat}_2(\mathbb{Z}/p^n\mathbb{Z}).$$

So $U^{(1)}$ is a $p$-group of order $p^{4(n-1)}$ or trivial. Therefore

$$|G| = p^{4(n-1)}|\text{GL}_2(\mathbb{F}_p)| = (p^2 - 1)(p^2 - p)p^{4(n-1)}. \quad (2.1)$$

We generalize $U^{(1)}$ by setting

$$U^{(k)} = \left\{ 1 + p^kB : B \in \text{Mat}_2(\mathbb{Z}/p^n\mathbb{Z}) \right\}$$

for $1 \leq k \leq n$. Each $U^{(k)}$ is a normal subgroup of $G$ of order $p^{4(n-k)}$ and lies in the short exact sequence

$$1 \to U^{(k)} \to G \to \text{GL}_2(\mathbb{Z}/p^k\mathbb{Z}) \to 1. \quad (2.2)$$

Over the prime field we find the exact sequences

$$1 \to \text{SL}_2(\mathbb{F}_p) \to \text{GL}_2(\mathbb{F}_p) \to \mathbb{F}_p^* \to 1, \quad \text{and}$$

$$1 \to \{\pm 1\} \to \text{SL}_2(\mathbb{F}_p) \to \text{PSL}_2(\mathbb{F}_p) \to 1. \quad (2.3)$$

Recall that a $p$-group only has $\mathbb{Z}/p\mathbb{Z}$ as composition factors and that $\text{PSL}_2(\mathbb{F}_p)$ is simple if $p \geq 5$. We recall that $\text{PSL}_2(\mathbb{F}_p)$ are solvable if $p \leq 3$. So if $p \geq 5$, then $\text{PSL}_2(\mathbb{F}_p)$ is a composition factor of $G$ and all other composition factors are abelian.

What about the exponent of $G$? The determinant is a homomorphism

$$G \to (\mathbb{Z}/p^n\mathbb{Z})^\times$$

onto the group of units of $\mathbb{Z}/p^n\mathbb{Z}$. The unit group is cyclic of order $p^{n-1}(p-1)$ if $p \geq 3$. In this case, the exponent $\text{ex}(G)$ is a multiple of $p^{n-1}(p-1)$. If $p = 2$ and $n \geq 2$, then $(\mathbb{Z}/2^n\mathbb{Z})^\times$ contains a cyclic subgroup of order $2^{n-2}$, and $\text{ex}(G)$ is a multiple of $p^{n-2} = 2^{n-2}$.

**Lemma 2.4.** If $H$ is a subgroup of $G$, then $|H| < p^{4+8\text{ord}_p\text{ex}(H)} \leq p^4\text{ex}(H)^8$.

**Proof.** We factor $|H| = p^em$, where $p \nmid m$ and $e \geq 0$. As $p^em$ divides $|G| = p^{4(n-1)}(p^2 - 1)(p^2 - p)$, we see that $m$ divides $(p^2 - 1)(p-1)$, and therefore

$$m < p^3.$$

To bound $p^e$ we may assume $e \geq 1$. Let $H'$ be a $p$-Sylow subgroup of $H$, so for some $f \geq 1$ we have $p^f = \text{ex}(H') \mid \text{ex}(H)$.

If $p^f = 2$ we set $k = 2$ and otherwise we take $k = f$; note that $k \geq f$. We claim that $U^{(k)} \cap H'$ contains at most $p^4f$ elements.
Indeed, say $1 + p^kB$ lies in this intersection where $B \in \text{Mat}_2(\mathbb{Z}/p^n\mathbb{Z})$. Then expanding the right-hand side of $1 = (1 + p^kB)^{p^f}$ and subtracting 1 yields

$$0 = \sum_{l=1}^{p^f} \binom{p^f}{l} p^{kl} B^l = p^{k+f} B \left( 1 + \sum_{l=2}^{p^f} \binom{p^f}{l} p^{kl-k-f} B^{l-1} \right); \quad (2.4)$$

we observe $kl-k-f \geq 2k-k-f \geq 0$ in the sum. If $l > 2$, then $kl-k-f > 0$, so

$$p \text{ divides } \binom{p^f}{l} p^{kl-k-f}$$

for $l \in \{3, \ldots, p^f\}$ and even if $l = 2$. Therefore, the matrix in brackets on the right of (2.4) lies in $U^{(1)} \subseteq \text{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$, and we have

$$p^{k+f} B = 0.$$

If $k \geq n$, then $1 + p^kB$ was trivial to start out with. Otherwise, we may represent the entries of $B$ with integers in $[0, p^{n-k})$. If $k + f \leq n$ then these entries are divisible by $p^{n-k-f}$, so there are at most $p^{4f}$ possibilities for $B$. If finally $k < n < k + f$, then the number of possible $B$ is at most $p^{4(n-k)} < p^{4f}$. Our claim holds true.

According to the exact sequence (2.2) the quotient $H'/(U^{(k)} \cap H')$ is isomorphic to a subgroup of $\text{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ and it is a $p$-group or trivial. We recall (2.1) and the bound for $|U^{(k)} \cap H'|$ from above to find

$$|H'| \leq |U^{(k)} \cap H'| \cdot p^{4(k-1)+1} \leq p^{4f+4(k-1)+1} \leq p^{4(k+f)-3}.$$

As $k \leq f + 1$ we conclude $|H'| \leq p^{8f+1} \leq p \cdot \text{ex}(H)^8$.

We have $|H'| = p^e$. Taking the product of the bounds for $m$ and $p^e$ yields the lemma. \qed

**Lemma 2.5.**

1. Let $H$ be a subgroup of $G$. Then any non-abelian composition factor of $H$ is isomorphic to $\text{PSL}_2(\mathbb{F}_p)$ for some prime $p \geq 5$.

2. If $n \geq 6$ is an integer and $p \geq 5$ is a prime, then $A_n$ and $\text{PSL}_2(\mathbb{F}_p)$ are not isomorphic.

**Proof.** For the first claim let $C$ be a non-abelian composition factor of $H$. The kernel of $\text{GL}_2(\mathbb{Z}/p^n\mathbb{Z}) \rightarrow \text{GL}_2(\mathbb{F}_p)$ is trivial or a $p$-group. As $C$ is not abelian it is also a composition factor of the image of $H$ in $\text{GL}_2(\mathbb{F}_p)$. The two exact sequences (2.3) indicate that $C$ is a composition factor of a subgroup of $\text{PSL}_2(\mathbb{F}_p)$. So $p \geq 5$ because $\text{PSL}_2(\mathbb{F}_2)$ and $\text{PSL}_2(\mathbb{F}_3)$ are both solvable. Of course, $C$ could be isomorphic to the full group $\text{PSL}_2(\mathbb{F}_p)$, which is simple. This case is covered in the first part of the lemma. Otherwise, $C$ is a quotient of a proper subgroup of $\text{PSL}_2(\mathbb{F}_p)$. Dickson classified all possible subgroups
of $\text{PSL}_2(\mathbb{F}_p)$, see [17] Hauptsatz II.8.27. According to this classification a proper subgroup is either solvable or isomorphic to $A_5$. As $C$ is simple and non-abelian it must be isomorphic to $A_5 \cong \text{PSL}_2(\mathbb{F}_5)$. We have established the first claim in the lemma.

The second claim follows since $\text{PSL}_2(\mathbb{F}_p)$ is not isomorphic to an alternating group if $p \geq 7$; Artin [6] gives a simple proof by comparing cardinalities. □

We conclude with the following observation about the commutator subgroup $[H,H]$ of an arbitrary subgroup $H$ of $G$.

**Lemma 2.6.** Let $H$ be a subgroup of $G$, and set $d = [G : H]!$. Then $|[H,H]| \geq p^{n/3}/d$.

**Proof.** Since $d = [G : H]!$, the $d$-th power of any element in $G$ lies in $H$. We define

$$A = \begin{pmatrix} 1 & dx \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ dx & 1 \end{pmatrix}$$

for $x \in \mathbb{Z}/p^n\mathbb{Z}$. Then $A, B \in H$ and

$$ABA^{-1}B^{-1} = \begin{pmatrix} 1 + (dx)^2 + (dx)^4 & -(dx)^3 \\ (dx)^3 & 1 - (dx)^2 \end{pmatrix} \in [H,H].$$

If $x$ runs over the elements represented by integers in $[0, p^{n/3}d^{-1})$ then $ABA^{-1}B^{-1}$ runs over a set of distinct elements in $[H,H]$, and the lemma follows. □

### 2.3. Applications

If $F$ is a subfield of $\overline{\mathbb{Q}}$, then $F^{sa}$ denotes the field obtained by adjoining to $F$ all roots of irreducible polynomials in $F[X]$ with symmetric or alternating Galois groups, of any degree.

**Proposition 2.7.** Let $G$ be $\mathbb{G}_m$ or an elliptic curve. We suppose that $G$ is defined over a number field $K$. Let $d$ be a positive integer, and let $F = ((K^{ab})^{sa})^{(d)}$. Then $G(F)/G(F)_{\text{tors}}$ is free abelian

1. if $G$ is $\mathbb{G}_m$,
2. if $G$ is an elliptic curve with CM over $K$, and
3. if $G$ is an elliptic curve without CM. In this case $G(F)_{\text{tors}}$ is finite.

For a more general version of (3) regarding elliptic curves without CM we refer to part (1) of Lemma 2.9 below.

**Lemma 2.8.** Let $G$ be $\mathbb{G}_m$ or an elliptic curve. We suppose that $G$ is defined over a number field $K$. Let $m$ be a positive integer and $F/K$ an algebraic extension such that all $m$-torsion points of $G$ are defined over $F$. If $P$ is an algebraic point of $G$ whose image in $\overline{G}/G(F)$ has order $m$, then $F(P)/F$
is a finite abelian Galois extension, and $\text{Gal}(F(P)/F)$ contains an element of order $m$.

**Proof.** Let $[m]$ denote the multiplication-by-$m$ endomorphism of $G$. If $[m]P = Q \in G(F)$, then the conjugates of $P$ over $F$ are other solutions of this equation, and they differ from $P$ by $m$-torsion points. In particular, since all of these torsion points are defined over $F$, we know that $F(P)/F$ is Galois. We have a homomorphism $\Phi : \text{Gal}(F(P)/F) \to G[m]$ given by $\sigma \mapsto \sigma(P) - P$ with target the points $G[m]$ in $G(\mathbb{Q})$ of order dividing $m$. The image of $\Phi$ is not contained in $G[n]$ for any $n < m$. If it were, then $[n]P$ would be fixed by each element of $\text{Gal}(F(P)/F)$, meaning $[n]P \in G(F)$, which is contrary to our assumption that $P$ has order $m$ in $G(\mathbb{Q})/G(F)$. By the Elementary Divisor Theorem the image of $\Phi$, which is a finite abelian group, contains an element of order $m$. The lemma follows as $\Phi$ is injective. \qed

We call a finite group *admissible* if it has no composition factor which is abelian or isomorphic to $\text{PSL}_2(F_p)$ for any $p \geq 5$. We call a field extension *admissible* if it is Galois and its Galois group is admissible. Note that the compositum of two admissible extensions of a common base field is again admissible over the base field. A Galois subextension of an admissible extension is also admissible.

The exponent of a finite Galois extension is the exponent of its Galois group. The compositum of two finite Galois extensions of exponent dividing some $e \in \mathbb{N}$ is again a finite Galois extension with exponent dividing $e$.

**Lemma 2.9.** Let $G$ be $G_m$ or an elliptic curve. Assume that $G$ is defined over a number field $K$, and let $k, e \in \mathbb{N}$. Suppose

$$K^{eb} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{2k-1} \subseteq F_{2k} \subseteq \mathbb{Q}$$

is a tower of field extensions with the following property: If $1 \leq i \leq 2k$ then $F_i$ is generated by all admissible extensions of $F_{i-1}$ in $\mathbb{Q}$ for even $i$, and $F_i$ is generated by all finite Galois extensions of $F_{i-1}$ in $\mathbb{Q}$ with exponent dividing $e$ for odd $i$. Then the following properties hold.

1. If $G = E$ is an elliptic curve without CM, then $E(F_{2k})_{\text{tors}}$ is finite and $E(F_{2k})/E(F_{2k})_{\text{tors}}$ is free abelian.
2. If $M$ is a finite Galois subextension of $F_{2k}/F_0$ and if $G_{\text{tors}} \subseteq G(F_0)$, then the exponent of $(G(F_{2k})/G(M))_{\text{tors}}$ divides $e^k$ and $G(F_{2k})/G(F_{2k})_{\text{tors}}$ is free abelian.

**Proof.** All extensions $F_i/F_{i-1}$ are Galois and $F_0/K$ is Galois too. We show by induction on $i$ that $F_i/K$ is Galois. If $i \geq 1$ and if $\sigma : F_i \to \mathbb{Q}$ is the identity on $K$ then $\sigma(F_{i-1}) = F_{i-1}$ and $\sigma(F_i)/F_{i-1}$ is generated by Galois
extensions with certain group-theoretic properties. Hence $\sigma(F_i) = F_i$ and thus $F_i/K$ is Galois.

The second claim in part (1) follows from the first one and Criterion A, Proposition 2.3. We now prove the first claim in (1).

Let $P \in E(F_{2k})$ be a point of finite order. It suffices to show that the said order is bounded independently of $P$. Without loss of generality we may assume that the order is $p^n$ for some prime $p$ and some integer $n \geq 1$.

We construct inductively an auxiliary tower of number fields by first taking $K_2$ to be the normal closure of $K(P)/K$ in $\overline{\mathbb{Q}}$, so $K_2 \subseteq F_{2k}$. As $K(E[p^n])/K$ is normal we have $K_{2k} \subseteq K(E[p^n])$. For any $1 \leq i \leq 2k$ we construct $K_i$ using the diagram

\[
\begin{array}{c}
F_i \\
\downarrow \\
K_i \\
\downarrow \\
F_{i-1} \\
\downarrow \\
K_{i-1} = K_i \cap F_{i-1} \\
\downarrow \\
K
\end{array}
\]

(2.5)

By induction we find that $K_{i-1}/K$ is Galois. We will repeatedly use the fact that $K_i/K_{i-1}$ is Galois and that the restriction induces an isomorphism between $\text{Gal}(K_i/F_{i-1})$ and $\text{Gal}(K_i/K_{i-1})$.

We now prove $K_i = K_{i-1}$ for even $i$. According to the short exact sequence

\[1 \to \text{Gal}(K_i/K_{i-1}) \to \text{Gal}(K_i/K) \to \text{Gal}(K_{i-1}/K) \to 1\]

any composition factor of $\text{Gal}(K_i/K_{i-1})$ is a composition factor of $\text{Gal}(K_i/K)$. It is thus a composition factor of $H = \text{Gal}(K(E[p^n])/K)$, because the latter group restricts onto $\text{Gal}(K_i/K)$. By (2.5) and the hypothesis of this lemma $\text{Gal}(K_i/K_{i-1})$ is admissible. We can identify $H$ with a subgroup of $\text{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$, and so we have $K_i = K_{i-1}$ by Lemma 2.5, part (1).

We are left with a contracted tower $K_0 \subseteq K_2 \subseteq K_4 \subseteq \cdots \subseteq K_{2k}$ such that $K_i/K_{i-2}$ has exponent dividing $e$ for even $i$. So $\text{ex}(\text{Gal}(K_{2k}/K)) \mid e^k$.

We abbreviate $\Gamma = \text{Gal}(K_{2k}/K)$, which is a quotient of $H$. Serre’s Theorem [29] implies that there is a constant $p_0$ depending only on $E$ such that $\Gamma = \text{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ if $p > p_0$. Moreover, if $p \leq p_0$ then $[\text{GL}_2(\mathbb{Z}/p^n\mathbb{Z}) : H]$ and $|\ker(H \to \Gamma)|$ are bounded independently of $n$. 


Let us assume $p > \max\{4, p_0, e^k\}$. We have a short exact sequence

$$1 \to \Gal(K_{2k}/K_0) \to \Gamma = \GL_2(\mathbb{Z}/p^n\mathbb{Z}) \to \Gal(K_0/K) \to 1.$$ 

The group $\Gal(K_0/K)$ is abelian since $K_0 \subseteq F_0 = K^{ab}$. So $\PSL_2(F_p)$ is a composition factor of $\Gal(K_{2k}/K_0)$ by the discussion in Section 2.2. We conclude $\ex(\PSL_2(F_p)) \leq \ex(\Gal(K_{2k}/K_0)) \leq e^k$. As

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has order $p$ as an element of $\PSL_2(F_p)$ we find $p \leq e^k$, contradicting our assumption on $p$. Therefore, $p \ll 1$, where the implied constant here and below only depends on $E, K, e$ and $k$.

It remains to show $n \ll 1$. Let $H'$ denote the preimage of $\Gal(K_{2k}/K_0)$ under the quotient map $H \to \Gamma$. We use the conclusion of Serre’s Theorem together with

$$\ex(H') \leq |\ker(H \to \Gamma)|\ex(\Gal(K_{2k}/K_0)) \leq |\ker(H \to \Gamma)|e^k$$

to conclude $\ex(H') \ll 1$. As $p$ is bounded too, Lemma 2.4 implies $|H'| \ll 1$. Recall that $H/H' \cong \Gamma/\Gal(K_{2k}/K_0) \cong \Gal(K_0/K)$ is abelian. So $[H, H] \subseteq H'$ and in particular $[H, H] \leq |H'| \ll 1$. We apply Lemma 2.6 to $H$ and recall $|\GL_2(\mathbb{Z}/p^n\mathbb{Z}) : H| \ll 1$, to conclude $n \ll 1$. This concludes the proof of (1).

For part (2) we introduce the auxiliary fields $F'_i = F_iM$ for $0 \leq i \leq 2k$. Since $M/F_0$ is a finite Galois extension, so is $F'_i/F_i$. As $F_i/F_{i-1}$ is Galois if $i \geq 1$, we conclude the same for $F'_i/F'_{i-1}$. Moreover, $\Gal(F'_i/F'_{i-1})$ is isomorphic to $\Gal(F_i/F_{i-1} \cap F_i)$, and $F'_{i-1} \cap F_i/F_{i-1}$ is finite and Galois. By hypothesis, a finite Galois extension of $F'_{i-1}$ inside $F'_i$ is admissible if $2 \mid i$ and has exponent dividing $e$ if $2 \nmid i$. We will apply this observation in just a moment.

Now suppose $m$ is the order of an element in $(\mathcal{G}(F_{2k})/\mathcal{G}(M))_{\text{tors}}$ which is represented by $P \in \mathcal{G}(F_{2k})$. We abbreviate $L = M(P)$; this is a subfield of
$F_{2k}$ and it fits into the diagram

$$
\begin{array}{c}
F'_i \\
\downarrow \\
F'_{i-1} (L \cap F'_i) \\
\downarrow \\
F_{i-1} \\
\downarrow \\
L \cap F'_i \\
\downarrow \\
L \cap F'_{i-1} \\
\downarrow \\
M
\end{array}
$$

By Lemma 2.8 the extension $L/M$ is finite abelian, and its exponent is a multiple of $m$; here we have used our assumption that $G(F_0)$ contains all torsion points of $G$. Therefore, $L \cap F'_i/M$ is abelian and so is $F'_{i-1}(L \cap F'_i)/F'_{i-1}$ by the diagram. We recall the observation above. If $2 \mid i$, then $F'_{i-1}(L \cap F'_i)/F'_{i-1}$ is also admissible. But only the trivial group is admissible and abelian. Therefore, the extension is trivial and so $L \cap F'_i = L \cap F'_{i-1}$.

As in the proof of (1) our tower of fields contracts, i.e.

$$M = L \cap F'_0 \subseteq L \cap F'_2 \subseteq \cdots \subseteq L \cap F'_{2k} = L \cap F_{2k} = L.$$ 

If on the other hand we have $2 \nmid i$, then $L \cap F'_i / L \cap F'_{i-1}$ has exponent dividing $e$. We conclude that $L/M$ has exponent dividing $e^k$ and thus $m \mid e^k$ by the conclusion of Lemma 2.8.

We want to apply Criterion B, Proposition 2.3, in order to prove that $G(F_{2k})/G(F_0)^\text{tors}$ is free abelian. We have already verified part (b) of the hypothesis. The extension $F_{2k}/F_0$ is Galois. Therefore, we are left to show that $G(M)/G(M)^\text{tors}$ is free abelian for every finite Galois extension $M/F_0$, with $M \subseteq F_{2k}$. We claim that $M$ satisfies the Bogomolov property with respect to $G$, then Proposition 1.1 concludes the proof.

Let $M = K^{ab}(\alpha)$, then $M$ is contained in an abelian extension of the number field $K(\alpha)$. In the case of $G_m$, the Bogomolov property of $M$ follows from [5, Theorem 1.1]; for general abelian varieties the result was proved in [8, Theorem 0.1].

Note that part (2) of the previous lemma remains true on replacing admissible by the weaker condition that all composition factors in the respective Galois groups are non-abelian.

Proof of Proposition 2.7. We will construct a tower $K^{ab} = F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4$ of fields as in Lemma 2.9 with $e = 60d!$, such that $F \subseteq F_4$. 

□
Let $F'$ be a Galois extension of $K^{ab}$ with Galois group isomorphic to $A_n$ or $S_n$. If $n \leq 5$, then $\text{ex}(\text{Gal}(F'/K^{ab})) | 60$ and so $F' \subseteq F_1$. If $n \geq 6$, then $\text{Gal}(F'/K^{ab})$ is either isomorphic to $A_n$, which is admissible by the second part of Lemma 2.5, or $K^{ab}$ has a quadratic extension contained in $F' \cap F_1$. In either case we find that $F'$ is a subfield of $F_2$. We have showed $(K^{ab})_{sa} \subseteq F_2$.

If $F'/((K^{ab})^{sa})$ is an extension of degree at most $d$, then its normal closure is an extension of $(K^{ab})^{sa}$ of exponent dividing $d!$. Thus $F' \subseteq F_3$ because $d! \mid e$.

In case (1) of the proposition the torsion $G_{\text{tors}}$ is defined over $K^{ab}$, which contains all roots of unity. The same is true in case (2), since we assume that the elliptic curve has CM over $K$. These two cases of the proposition follow from Lemma 2.9, part (2). The remaining case (3) follows from part (1) of the same lemma. $\square$

In order to prove Theorem 1.2 it remains to show that the field $F$ does not have the Bogomolov property in any of the three cases described in this theorem. The proof of this is the content of the next two sections.

3. Small Points on $\mathbb{G}_m$

Let $\kappa$ be a field with a fixed algebraic closure $\overline{\kappa}$. If $f \in \kappa[X]$ is a polynomial, then we denote with $\kappa(f)$ the splitting field of $f$ over $\kappa$ inside $\overline{\kappa}$.

In this section we prove the following more precise version of Theorem 1.2 (1).

**Proposition 3.1.** Let $K$ be a number field, $d$ a positive integer, and let $F = ((K^{ab})^{sa})^d$. Then $\mathbb{G}_m(F)/\mathbb{G}_m(F)_{\text{tors}}$ is free abelian, but $F$ does not satisfy the Bogomolov property with respect to the Weil height.

**Proof.** We already know that $\mathbb{G}_m(F)/\mathbb{G}_m(F)_{\text{tors}}$ is free abelian from Proposition 2.7.

Let $n \geq 2$ be an integer. A result of Osada, cf. [25, Corollary 3], states that the polynomial

$$f_n = X^n - X - 1$$

is irreducible over $\mathbb{Q}$ and has symmetric Galois group. Moreover, the discriminant of $f_n$ is given by

$$\text{disc}(f) = \pm (n^n + (-1)^n(n - 1)^{(n-1)}),$$

as explained in [22].

Let $\Delta$ be the discriminant of $K$, and let $n \geq 5$ be an integer with $\Delta \mid n$. Assume that there is a common prime divisor $p$ of $\Delta$ and the discriminant of $\mathbb{Q}(f_n)$. Then $p$ also divides $\text{disc}(f_n)$ and $n$. By [23] it follows that $p$ divides $n - 1$ as well. This is not possible, and therefore $\Delta$ and the discriminant of
$\mathbb{Q}(f_n)$ are coprime. In particular $K$ and $\mathbb{Q}(f_n)$ are linearly disjoint over $\mathbb{Q}$. Hence,

$$\text{Gal}(K(f_n)/K) \cong \text{Gal}(\mathbb{Q}(f_n)/\mathbb{Q}) \cong S_n.$$ 

The group $\text{Gal}(K^{ab}(f_n)/K^{ab}) \cong \text{Gal}(K(f_n)/(K(f_n) \cap K^{ab}))$ is a normal subgroup of $S_n \cong \text{Gal}(K(f_n)/K)$, since $K^{ab}$ and $K(f_n)$ are Galois extensions of $K$. It is non trivial and therefore isomorphic to $A_n$ or $S_n$. In both cases we find that

$$K(f_n) \subseteq (K^{ab})^{sa}.$$ 

Let $\alpha$ be a root of $f_n$. Since $\text{Gal}(K^{ab}(f_n)/K^{ab}) \cong A_n$ or $S_n$, we know that $\alpha \not\in K^{ab}$. In particular, $\alpha$ is not a root of unity, and so $h(\alpha) > 0$ by Kronecker’s Theorem. Using basic height properties we have

$$n \cdot h(\alpha) = h(\alpha^n) = h(\alpha + 1) \leq \log 2 + h(\alpha) + h(1) = \log 2 + h(\alpha),$$ 

which yields

$$h(\alpha) \leq \frac{\log 2}{n - 1}.$$ 

Since we can take $n$ to be arbitrarily large, this gives us that $F$ does not have the Bogomolov property with respect to the Weil height, and our proof is complete. \qed

4. Small Points on Elliptic Curves

4.1. Irreducibility. For any number field $K$, we denote by $\mathcal{O}_K$ the ring of integers in $K$. We recall that all number fields lie in a fixed algebraic closure of $\mathbb{Q}$.

Proposition 4.1. Let $K$ be a number field and let $p$ be a maximal ideal in $\mathcal{O}_K$ with residue characteristic $p \geq 3$. If $f \in \mathcal{O}_K[T, X]$ is monic in $X$, such that $f \equiv X^n - X^2 + T^s \pmod{p}$, where $n \geq 5$ is odd, $p \mid n$, and $2 \mid s$, then the following properties hold true.

1. The polynomial $f(T^m, X)$ is irreducible as an element of $\mathbb{Q}(T)[X]$ for all integers $m \geq 1$.
2. The group $\text{Gal}(F(T)(f)/F(T))$ is isomorphic to $S_n$ or $A_n$ for all number fields $F \supseteq K$.

Proof. Let $\mathbb{F}_p$ be an algebraic closure of $\mathbb{F}_p$. Then $X^n - X^2 + T^s$ is irreducible as a polynomial in $\mathbb{F}_p(T)[X]$ and separable. For a proof of the former fact see the first few paragraphs of Section 20 in [1]. Our assumptions on $n$ and $s$ assure that the splitting field of $X^n - X^2 + T^s \in \mathbb{F}_p(T)[X]$ is a Galois extension of $\mathbb{F}_p(T)[X]$ with Galois group isomorphic to $A_n$; cf. (II).1 Section 20 in [1]. This polynomial is one of Abhyankar’s “tilde polynomials,” and the proof of the claimed statement does not require the full classification theorem for finite simple groups.
The polynomial $f$ from the hypothesis reduces to one of Abhyankar’s tilde polynomials, but so do twists by taking powers of $T$. So $f(T^m, X) \pmod{\mathfrak{p}}$ is irreducible as an element of $\overline{F}_p[T, X]$ for all $m \geq 1$. It follows from the Gauss Lemma that $f(T^m, X)$ is irreducible as an element of $\overline{\mathbb{Q}}[T, X]$. This yields part (1).

To prove part (2) we set the stage to apply [14, Lemma 6.1.1] and use some of the notation introduced before its statement. Let $F/K$ be a finite extension. Since $\mathcal{O}_F$ is integrally closed in $F$, the ring $\mathcal{O}_F[T]$ is integrally closed in $F(T)$. Let $\mathfrak{P} \subseteq \mathcal{O}_F$ be any prime ideal above $\mathfrak{p}$. Then $\mathfrak{P}[T]$ is a prime ideal of $\mathcal{O}_F[T]$. The quotient $\mathcal{O}_F[T]/\mathfrak{P}[T]$ is a polynomial ring over a finite field $\mathbb{F}_q$, where $q$ is the ideal norm of $\mathfrak{P}$. Therefore, the quotient field of $\mathcal{O}_F[T]/\mathfrak{P}[T]$ is $\tilde{F} = \mathbb{F}_q(T)$.

Denote by $L$ the splitting field of $f$ over $F(T)$, and by $G$ the Galois group of $L/F(T)$. The integral closure of $\mathcal{O}_F[T]$ in $L$ contains a prime ideal above $\mathfrak{P}[T]$. We write $\tilde{L}$ for the quotient field of the said integral closure modulo the said prime ideal. By construction $L$ contains all roots of $f$. They are integral over $\mathcal{O}_F[T]$ as $f$ is monic in $X$. Therefore, the reduction $\tilde{f} \in \mathbb{F}_q(T)[X]$ factors completely in $\tilde{L}[X]$. In particular, $\tilde{L}$ contains the splitting field $\tilde{F}(\tilde{f})$.

By [14 Lemma 6.1.1(a)] the extension $\tilde{L}/\tilde{F}$ is normal, and a subgroup of $G$ surjects onto its automorphism group $\text{Aut}(\tilde{L}/\tilde{F})$. So we have

$$|G| \geq |\text{Aut}(\tilde{L}/\tilde{F})| \geq |\text{Gal}(\tilde{F}(\tilde{f})/\tilde{F})|.$$

On the other hand, $A_n \cong \text{Gal}(\mathbb{F}_p(T)(\tilde{f})/\mathbb{F}_p(T))$ is isomorphic to a subgroup of $\text{Gal}(\tilde{F}(\tilde{f})/\tilde{F})$, and therefore $|G| \geq n!/2$. We have proven that $G$ is isomorphic to $S_n$ or $A_n$, concluding our proof. □

4.2. Hilbert Irreducibility Theorem for Algebraic Groups. We will use a special case of Dvornicich and Zannier’s Hilbert Irreducibility Theorem for algebraic groups to prove the following proposition.

For a subfield $F \subseteq \overline{\mathbb{Q}}$ we let $F^{\text{cyc}}$ be the subfield of $\overline{\mathbb{Q}}$ obtained by adjoining all roots of unity to $F$.

**Proposition 4.2.** Let $K$ be a number field. Suppose $f \in K[T, X]$ has degree $n \geq 5$ as a polynomial in $X$ and satisfies the following properties.

1. The polynomial $f(T^m, X)$ is irreducible as an element of $\overline{\mathbb{Q}}(T)[X]$ for all integers $m \geq 1$.
2. The group $\text{Gal}(F(T)(f)/F(T))$ is isomorphic to $A_n$ or $S_n$ for all number fields $F \supseteq K$. 
Then for all but finitely many roots of unity $\zeta \in \overline{\mathbb{Q}}$, the specialization $g = f(\zeta, X)$ is irreducible of degree $n$ as an element of $K^{cyc}[X]$, and $\text{Gal}(K^{cyc}(g)/K^{cyc})$ is isomorphic to $A_n$ or $S_n$.

\textbf{Proof.} We notice that the hypotheses assure that $\text{Gal}(\overline{\mathbb{Q}}(T)(f)/\overline{\mathbb{Q}}(T))$ is isomorphic to $A_n$ or $S_n$. This means that the inclusion $\overline{\mathbb{Q}}(T) \subseteq \overline{\mathbb{Q}}(T)(f)$ of function fields induces a covering $\pi : Y \to \mathbb{G}_m$ of degree $n!/2$ or $n!$, where $Y$ is a geometrically irreducible curve defined over a finite extension $K^{cyc}(\alpha)$ of $K^{cyc}$. By [33, Proposition 2.1] we can factor

\[ \pi = [N] \circ \rho, \]

where $\rho : Y \to \mathbb{G}_m$ is a rational map that satisfies Zannier’s property (PB) from [33], and $N \geq 1$ is an integer.

Now $[N]$ comes from a function field $L = K^{cyc}(\alpha)(T) \supseteq K^{cyc}(\alpha)(T^N)$. This extension is Galois with group $\mathbb{Z}/N\mathbb{Z}$ and also a quotient of $A_n$ or $S_n$. As $n \geq 5$ we must have $N \leq 2$ and again $\text{Gal}(L(f)/L)$ is isomorphic to $A_n$ or $S_n$.

Let us fix a primitive element $U$ with $L(U) = L(f)$ and an irreducible polynomial $B \in K^{cyc}(\alpha)[T, X]$ with $B(T, U) = 0$. We apply [33, Theorem 2.1] to $\rho : Y \to \mathbb{G}_m$. The specialization $B(\zeta, X)$ is thus irreducible in $K^{cyc}(\alpha)[X]$ of degree $[L(f) : L]$ for all but finitely many roots of unity $\zeta$.

Next we use a general specialization principle. More precisely, we apply [32, Lemma 1.5] to specialize the first variable in $B$ to $\zeta$. In the reference’s notation we take $R = K^{cyc}(\alpha)[T]$ and $A$ to be all roots of $B \in L[X]$, together with all roots of $f \in K(T)[X]$. Let $u \in \overline{\mathbb{Q}}$ denote a root of $B(\zeta, X)$. After omitting finitely many $\zeta$, the extension $K^{cyc}(\alpha)(u)/K^{cyc}(\alpha)$ is also Galois with Galois group isomorphic to $\text{Gal}(L(f)/L)$. Moreover, the specialization $g = f(\zeta, X) \in K^{cyc}(\alpha)[X]$ splits in $K^{cyc}(\alpha)(u)$. So $\text{Gal}(K^{cyc}(\alpha)(g)/K^{cyc}(\alpha))$ is a quotient of $\text{Gal}(L(f)/L)$ and thus isomorphic to a quotient of $A_n$ or $S_n$. As $n \geq 5$, the only possibilities are $A_n$, $S_n$, and the trivial group. However, by [11, Corollary 1] and hypothesis (1) the polynomial $g \in K^{cyc}(\alpha)[X]$ is irreducible of degree $n$ after omitting finitely many $\zeta$. We may thus rule out the trivial group. The Galois group of $K^{cyc}(g)/K^{cyc}$ is isomorphic to a subgroup of $S_n$ and contains a subgroup isomorphic to $A_n$ or $S_n$, and therefore $\text{Gal}(K^{cyc}(g)/K^{cyc})$ must also be of this type. \hfill \Box

4.3. \textbf{Proof of Theorem 1.2 for elliptic curves.} Here we will prove the following result which implies parts (2) and (3) of Theorem 1.2.

\textbf{Proposition 4.3.} Let $E$ be an elliptic curve defined over a number field $K$. We suppose
(1) that $E$ has CM over $K$
(2) or that $E$ does not have CM and $K = \mathbb{Q}$.

Let $d \geq 2$ be a positive integer, and let $F = ((K^{ab})^{sa})^{(d)}$. Then $E(F)/E(F)_{tors}$ is a free abelian group that does not satisfy the Bogomolov property with respect to the Néron-Tate height.

**Proof.** Let $E, K,$ and $F$ be as in the theorem. We will work implicitly with a fixed short Weierstrass equation for $E$ with coefficients in $\mathcal{O}_K$.

Let us suppose for the moment that $p$ is a prime ideal in $\mathcal{O}_K$ of norm $q = p^k$, with $p \geq 5$, where $E$ has good, supersingular reduction $\tilde{E}$. By [13, Chapter 13, Theorem 6.3] a power $\text{Fr}_q^{\nu}$ of the Frobenius endomorphism $\text{Fr}_q$ of $\tilde{E}$ equals $[p^\nu]$, the multiplication-by-$p^\mu$ endomorphism, where $\nu, \mu \in \mathbb{N}$. Taking the degree yields $q^\nu = p^{2\mu}$, and so

$$\text{Fr}_q^{2\nu} = [p^{2\mu}] = [q^\nu].$$

The first coordinate in the multiplication-by-$q^\nu$ morphism of $E$ is represented by a quotient $a/b$ of polynomials $a = X^{q^{2\nu}} + \cdots, b = q^{2\nu}X^{q^{2\nu} - 1} + \cdots \in \mathcal{O}_K[X]$. By the previous paragraph we have $a/b \equiv X^{q^{2\nu}} \pmod{p}$. Hence

$$a \equiv X^{q^{2\nu}} \pmod{p} \quad \text{and} \quad b \equiv 1 \pmod{p},$$

as $\mathbb{F}_q[X]$ is factorial. We define the auxiliary polynomial

$$f = X^{2p}a - (X^2 - T^2)b \in \mathcal{O}_K[T, X].$$

It is monic in $X$ of degree $q^{2\nu} + 2p$ as $\deg(b) < q^{2\nu}$ and

$$f \equiv X^{q^{2\nu} + 2p} - (X^2 - T^2) \pmod{p}.$$

Thus $f$ satisfies the hypothesis of Proposition 4.1. By Proposition 4.2 $g(X) = f(\zeta, X)$ is irreducible in $K^{cyc}[X]$ for all but finitely many roots of unity $\zeta$, and moreover $\text{Gal}(K^{cyc}(g)/K^{cyc})$ is isomorphic to $S_n$ or $A_n$ with $n = q^{2\nu} + 2p$.

Let $\alpha$ be a root of $g$ and let $\beta \in \overline{\mathbb{Q}}$ such that $Q = (\alpha, \beta)$ lies on $E$. By properties of $g$ we have $\alpha \in (K^{cyc})^{sa}$. Now $K^{cyc} \subseteq K^{ab}$, and the Galois group of $K^{cyc}(g)/K^{ab} \cap K^{cyc}(g)$ is normal in $A_n$ or $S_n$ with abelian quotient, so it is again $A_n$ or $S_n$, as $n = q^{2\nu} + 2p \geq 5$. We conclude $\alpha \in K^{ab}(g) \subseteq (K^{ab})^{sa}$. Solving for $\beta$ merely involves taking a square root, so $\beta \in ((K^{ab})^{sa})^{(2)} \subseteq F$ with $F$ as in the hypothesis. Hence $Q \in E(F)$.

Since $\alpha$ has degree $q^{2\nu} + 2p$ over $K^{cyc}$, we have $b(\alpha) \neq 0$ and $\alpha \neq 0$. Set $[q^\nu]Q = (\alpha', \beta')$. Then the choice of $f$ yields

$$\alpha'^2 = \alpha^2 - \zeta^2.$$
Fundamental properties of the Néron-Tate height $\hat{h}$ on $E$ and of the absolute logarithmic Weil height $h$ imply
\[
q^{2\nu}\hat{h}(Q) = \hat{h}((\alpha', \beta')) \leq h(\alpha') + c \\
= h(\alpha^{2-2p} - \zeta^2\alpha^{-2p}) + c \leq h(\alpha^{2p-2}) + h(\alpha^{2p}) + c + \log 2 \\
= (4p - 2)h(\alpha) + c + \log 2 \leq (4p - 2)(\hat{h}(Q) + c) + c + \log 2 \\
\leq 4p\hat{h}(Q) + 4cp + \log 2
\]
where $c$ depends only on $E$ and compares the Néron-Tate height to the height of the $x$-coordinate. As $q^{2\nu} > 2p$ this gives the inequality
\[
\hat{h}(Q) \leq \frac{4cp + \log 2}{q^{2\nu} - 4p}
\]
Observe that the right-hand side tends to $0$ as $p \to \infty$.

In case (2), where the elliptic curve $E$ does not have CM, but is defined over $\mathbb{Q}$, we know that $E$ has supersingular reduction at infinitely many primes by Elkies’s Theorem [13]. In case (1), infinitude follows from more classical considerations; cf. [20, Chapter 13, Theorem 12]. Hence the construction above yields a sequence of points $Q_1, Q_2, \ldots \in E(F)$ with Néron-Tate height tending to $0$. This sequence contains infinitely many pair-wise distinct members as the lower bound in $[K_{cyc}(Q) : K_{cyc}] \geq q^{2\nu} + 2p$ tends to $+\infty$ in $p$. To prove that $F$ does not have the Bogomolov property relative to $\hat{h}_E$ it remains to show that there are infinitely many non-torsion points among the constructed points $Q_1, Q_2, \ldots$

This is quite easy in case (1), where $E$ has CM. In fact, none of the $Q = (\alpha, \beta)$ constructed above have finite order. Indeed, let us assume the contrary. The field $K$, and a fortiori $K_{cyc}$, contains the CM field of $E$. So $K_{cyc}(\alpha)$ is an abelian extension of $K_{cyc}$. On the other hand $K_{cyc}(\alpha) \subseteq K_{cyc}(g)$, and the latter is a Galois extension of $K_{cyc}$ with Galois group isomorphic to $A_n$ or $S_n$. We use $n \geq 5$ again to find $[K_{cyc}(\alpha) : K_{cyc}] \leq 2$ as any abelian quotient of $A_n$ or $S_n$ has order at most 2. This contradicts $[K_{cyc}(\alpha) : K_{cyc}] = q^{2\nu} + 2p > 2$, so $Q$ has infinite order and thus $\hat{h}(Q) > 0$ by Kronecker’s Theorem.

Finally, if $E$ does not have CM, we already know that $E(F)$ contains only finitely many torsion points, by Proposition 2.7, completing our proof. □

4.4. Open problems. As usual in this paper let $G$ be $\mathbb{G}_m$ or an elliptic curve defined over a number field $K$. In proving Propositions 4.3 and 4.1 we have seen that Criteria A and B (Propositions 2.2 and 2.3) are not valid if we replace the phrase is free abelian with satisfies the Bogomolov property with respect to $G$. Inspired by results of May [23], we state the following open questions, where the second one is a generalization of [2, Problem 1.4].
Question 4.4. Let $F$ be an algebraic extension of $K$ such that every finite extension of $F$ has the Bogomolov property with respect to $\mathcal{G}$.

(1) Does $F^{(d)}$ satisfy the Bogomolov property with respect to $\mathcal{G}$ for every integer $d$?

(2) Does $F^{ab}$ satisfy the Bogomolov property with respect to $\mathcal{G}$ if in addition every finite extension of $F$ contains only finitely many roots of unity?

The Bogomolov property is not preserved under finite field extension. But perhaps it holds under additional restrictions.

Question 4.5. Let $F_0/K$ be an algebraic extension such that $F_0$ has the Bogomolov property with respect to $\mathcal{G}$. Does a finite extension field $F/F_0$, with $F$ Galois over $K$ and with $\mathcal{G}(F)_{\text{tors}} \setminus \mathcal{G}(F_0)_{\text{tors}}$ a finite set necessarily satisfy the Bogomolov property with respect to $\mathcal{G}$?

Note that none of the assumptions can be removed. The necessity of the finiteness of $\mathcal{G}(F)_{\text{tors}} \setminus \mathcal{G}(F_0)_{\text{tors}}$ is due to the failure of the Bogomolov property in the extension $\mathbb{Q}^{tr}(i)/\mathbb{Q}^{tr}$. The necessity of a Galois extension $F/K$ comes from a variation of this extension; cf. [23, Example 1].

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