PUSHING DOWN THE RUMIN COMPLEX TO CONFORMALLY SYMPLECTIC QUOTIENTS

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Abstract. Given a contact manifold $M_\#$ together with a transversal infinitesimal automorphism $\xi$, we show that any local leaf space $M$ for the foliation determined by $\xi$ naturally carries a conformally symplectic (cs–) structure. Then we show that the Rumin complex on $M_\#$ descends to a complex of differential operators on $M$, whose cohomology can be computed. Applying this construction locally, one obtains a complex intrinsically associated to any manifold endowed with a cs–structure, which recovers the generalization of the so–called Rumin–Seshadri complex to the conformally symplectic setting. The cohomology of this more general complex can be computed using the push–down construction.

1. Introduction

This article is motivated by the results [8] of M. Eastwood and H. Goldschmidt on integral geometry and the subsequent work [9] of M. Eastwood and J. Slovák on conformally Fedosov structures. The main tool used in [8] is a family of complexes of differential operators on complex projective space $\mathbb{C} P^n$. The results on integral geometry are deduced from vanishing of some cohomology groups of these complexes. The form and length of these complexes is rather intriguing and the article [9] takes steps towards an explanation. The main notion introduced there is the one of a conformally Fedosov structure, which combines a conformally symplectic structure and a projective structure, which satisfy a suitable compatibility condition. Given these data, the authors construct a tractor bundle endowed with a (linear) tractor connection which is naturally associated to the conformally Fedosov structure. This should open the possibility to construct sequences and complexes of differential operators following the ideas of the Bernstein–Gelfand–Gelfand (BGG) machinery as introduced in [6] and [4] in the setting of parabolic geometries.

The tractor bundle associated to a conformally Fedosov structure looks similar to the standard tractor bundle associated to a contact projective structure (see [10]). This is an instance of a so–called parabolic contact structure, the best know example of which are (hypersurface type) CR structures. It is known that the

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homogeneous models of parabolic contact structure are, via forming quotients by transversal infinitesimal automorphisms, related to special symplectic connections, see [3] and Sections 5.2.18 and 5.2.19 of [5] for an exposition in the language of parabolic geometries.

The starting point for our considerations is the hope to obtain complexes like the ones constructed in [8] from BGG sequences associated to parabolic contact structures via similar quotient constructions. In this article, we show that this can indeed be done in the special case of the BGG sequence associated to the trivial representation. It is shown in [2] that in this case one obtains the Rumin complex (see [11]), which can be naturally constructed for any contact structure, see [2] for a simple direct construction. From either construction it follows that the Rumin complex is a fine resolution of the constant sheaf $\mathbb{R}$, so in particular it computes the de–Rham cohomology of a contact manifold. Following this, our article also works in the setting of general contact manifolds and does not use parabolic geometry techniques.

We first prove that the quotient of a contact structure by a transversal infinitesimal automorphism naturally inherits a symplectic structure, and thus in particular a conformally symplectic structure (or cs–structure). In contrast to the traditional approach to defining such a structure via a specific two–form, we just view it as an appropriate line subbundle in the bundle of two–forms, which simplifies matters in several respects.

Next, we show that the Rumin complex can be pushed down to a complex of differential operators on the quotient space, which coincides with the complex on a symplectic manifold constructed in [13, 14] and in [2], where it is called Rumin–Seshadri complex, see also [12]. The push–down construction easily leads to a long exact sequence relating the cohomology of this complex to de–Rham cohomology. In particular, one can immediately read off (in this very simple special case) the cohomological information needed in the applications in [8].

To complete the picture, we prove that the push down construction can be used to construct a version of this complex and an analog of the long exact sequence on any smooth manifold endowed with a cs–structure. We prove that any such manifold can be locally represented as the quotient of a contact structure by a transversal infinitesimal automorphism. Moreover, any local isomorphism of cs–structures can be lifted to a contactomorphism “upstairs”. Naturality of the Rumin complex under contactomorphisms then implies that one can use the local contactifications to obtain a complex and a long exact sequence on the whole cs–manifold and that they are intrinsic to the cs–structure.

The extension of the construction of the Rumin–Seshadri complex from symplectic manifolds to cs–manifolds is not a new result in its own right, a direct construction is available in [7]. The main advantage of our approach is not the result itself, but the potential for generalizations.

2. A QUOTIENT OF THE RUMIN COMPLEX

We start by discussing local quotients of contact manifolds by transverse infinitesimal automorphisms.
2.1. **Contact manifolds and differential forms.** By a contact manifold, we will always mean a manifold $M_\#$ of odd dimension $2n + 1$ endowed with a maximally non–integrable distribution $H \subset TM$ of rank $2n$. While this implies that locally $H$ can be written as the kernel of a one–form $\alpha \in \Omega^1(M)$ such that $\alpha \wedge (d\alpha)^n$ is nowhere vanishing, we do not initially assume that such a contact form exists globally or that there locally is a preferred choice.

The best way to conceptually formulate the condition of maximal non–integrability is via the **Levi–bracket**. Defining $Q := TM_\#/H$, which clearly is a line bundle naturally associated to any corank one subbundle in the tangent bundle, the Levi–bracket is the bilinear bundle map $\mathcal{L} : H \times H \to Q$ induced by the Lie bracket of vector fields. The condition that $H$ defines a contact structure is than exactly that $\mathcal{L}$ is non–degenerate in each point. In this case, $\mathcal{L}$ induces an isomorphism $H \cong H^* \otimes Q$ of vector bundles, whose inverse can be also viewed as a non–degenerate bilinear bundle map $\mathcal{L}^{-1} : H^* \times H^* \to Q^*$ respectively as an element of $\Lambda^2 H \otimes Q^*$.

We can use the Levi–bracket and its inverse to construct subbundles in the exterior powers of $H^*$. Indeed, wedging with $\mathcal{L}$ can be viewed as a bundle map $\Lambda^k H^* \to \Lambda^{k+2} H^* \otimes Q$, while insertion of $\mathcal{L}^{-1}$ defines a map $\Lambda^k H^* \to \Lambda^{k-2} H^* \otimes Q^*$. Non–degeneracy implies that the first map is injective for $k \leq n - 1$ and surjective for $k \geq n - 1$ (when $\dim(M_\#) = 2n + 1$) while the second map is surjective for $k \leq n + 1$ and injective for $k \geq n + 1$. Thus we can define $\Lambda^k_0 H^* \subset \Lambda^k H^*$ for $k \leq n$ as the kernel of the insertion of $\mathcal{L}^{-1}$ and for $k > n$ as the kernel of wedging with $\mathcal{L}$.

Thus, a contact structure induces a filtration on the bundles of differential forms, as well as a finer decomposition of the bundles occurring as subquotients of this filtration. The passage from the de–Rham complex to the Rumin complex can be viewed as a passage from the bundles of differential forms to some of this filtration bundles, which does not change the cohomology. The price one has to pay for restricting to simpler bundles, is that (at least in one instance) the exterior derivative has to be replaced by a higher order operator. Let us collect the information on differential forms on a contact manifold we will need.

**Proposition 2.1.** Let $(M_\#, H)$ be a contact manifold and put $Q := TM_\#/H$. Then for each $k = 1, \ldots, 2n$ we get an exact sequence

$$0 \to \Lambda^{k-1} H^* \otimes Q^* \to \Lambda^k T^*M_\# \to \Lambda^k H^* \to 0$$

as well as decompositions

$$\Lambda^k H^* = \Lambda^k_0 H^* \oplus (\Lambda^{k-2}_0 H^* \otimes Q^*) \oplus \cdots \oplus (\Lambda_0^{k-2i} H^* \otimes \otimes^i Q^*) \quad \text{for } k \leq n.$$  

$$\Lambda^k H^* = \Lambda^k_0 H^* \oplus (\Lambda^{k+2}_0 H^* \otimes Q) \oplus \cdots \oplus (\Lambda^{k+2j}_0 H^* \otimes \otimes^j Q) \quad \text{for } k > n.$$  

Here $i$ is the largest integer such that $2i \leq k$ and $j$ is the largest integer such that $k + 2j \leq 2n$.

**Proof.** The first statement follows immediately from dualizing the exact sequence defining $Q$ and then taking exterior powers. For the first decomposition, we use insertion of $\mathcal{L}^{-1}$ to obtain a surjection from $\Lambda^k H^*$ onto $\Lambda^{k-2} H^* \otimes Q^*$ with kernel...
\( \Lambda^k H^* \). The wedge product with \( \mathcal{L} \) defines a splitting of the corresponding exact sequence, which shows that \( \Lambda^k H^* \cong \Lambda^k_0 H^* \oplus (\Lambda^{k-2} H^* \otimes Q^*) \). From this the statement follows by induction. For the second decomposition, one argues in the same way exchanging the roles of the two maps.

2.2. Cs–quotients of contact manifolds.

**Definition 2.2.** Let \((M_\#, H)\) be a contact manifold. A transversal infinitesimal automorphism of \(M\) is a vector field \(\xi \in \mathfrak{X}(M)\) such that \(\xi(x) \notin H_x\) for all \(x \in M_\#\) and such that the local flow of \(\xi\) preserves the contact distribution.

Note that in particular \(\xi\) is nowhere vanishing on \(M_\#\), thus defining a foliation of \(M_\#\) with one–dimensional leaves. On the other hand, the condition that the flow of \(\xi\) preserves the contact distribution is easily seen to be equivalent to the fact that for any \(\eta \in \Gamma(H) \subset \mathfrak{X}(M_\#)\) the Lie bracket \([\xi, \eta]\) also lies in \(\Gamma(H)\).

Next, we define a quotient of a contact manifold \(M_\#\) by a transversal infinitesimal automorphism \(\xi \in \mathfrak{X}(M_\#)\) to be a surjective submersion \(q : M_\# \to M\) of dimension \(2n\) such that for each \(x \in M_\#\) the kernel of \(T_xq : T_x M_\# \to T_{q(x)} M\) is spanned by \(\xi(x)\) and such that the fibers of \(q\) are connected.

This simply means that \(M\) is a (global) space of leaves for the foliation defined by \(\xi\). Local existence of such leaf spaces is a consequence of the Frobenius theorem, and this is all we will formally need in the sequel, since our results are local in nature. A global notion is only used in order to be able to work with larger leaf space than the ones provided by the Frobenius theorem in case that they are available. First, we show that the quotient space naturally inherits a symplectic structure. For later use it will be more useful to emphasize the conformally symplectic aspects of the construction, where we view a conformally symplectic structure as a line subbundle \(\ell\) in the bundle of two–forms, whose non–zero elements are all non–degenerate and which locally admits sections which are closed as two–forms.

**Proposition 2.2.** Let \(q : M_\# \to M\) be a quotient of a contact manifold \((M_\#, H)\) by a transversal infinitesimal automorphism \(\xi\). Then we have:

1. For each \(x \in M_\#\), the tangent map \(T_xq\) restricts to a linear isomorphism \(H_x \to T_{q(x)} M\). Via this isomorphism, \(\mathcal{L}\) determines a line subbundle \(\ell \subset \Lambda^2 T^* M\) such that each non–zero element of \(\ell\) is non–degenerate on the corresponding tangent space.

2. There is a unique contact form \(\alpha\) on \(M_\#\) such that \(\alpha(\xi) = 1\) and a unique symplectic form \(\omega\) on \(M\) such that \(q^* \omega = d\alpha\). This form \(\omega\) is a section of \(\ell \subset \Lambda^2 T^* M\), so \(\ell\) defines a conformally symplectic structure on \(M\).

**Proof.** (1) By definition, the kernel of \(T_xq\) is the line spanned by \(\xi(x)\), which has zero intersection with \(H_x\) so the first claim follows. Fixing \(x\), \(\mathcal{L}_x : \Lambda^2 H_x \to Q_x\) can be viewed as a non–degenerate bilinear form on \(H_x\) determined up to scale. Via the isomorphism \(H_x \to T_{q(x)} M\), it thus determines a line in \(\Lambda^2 T^*_{q(x)} M\) whose non–zero elements are non–degenerate. Since \(\xi\) is an infinitesimal automorphism, its flow preserves the contact distribution. Hence the tangent maps of its flow induce bundle maps \(H \to H\) and \(Q \to Q\). Naturality of the Levi–bracket implies...
that these bundle maps are compatible with $\mathcal{L}$, and since the flow lines of $\xi$ are the fibers of $q$, we conclude that the line in $\Lambda^2T^*_{q(x)}M$ constructed above only depends on $q(x)$ and not on $x$. Choosing a local section of $q$, one easily concludes that this defines a smooth line subbundle $\ell \subset \Lambda^2T^*M$.

(2) By definition, $\xi$ defines a nowhere–vanishing section of the bundle $Q = TM_\# / H$, thus trivializing this line bundle. Hence there is a unique section of $Q^*$ pairing to one with that section. Since $Q^*$ can be identified with the annihilator of $H$ in $T^*M_\#$ this section can be viewed as a contact form $\alpha$ such that $\alpha(\xi) = 1$. Of course, this uniquely determines $\alpha$ among all contact forms. Since $\xi$ is an infinitesimal automorphism, we must have $0 = d\alpha([\xi, \eta]) = d\alpha(\xi, \eta)$ for all $\eta \in \Gamma(H)$. Of course $d\alpha(\xi, \xi) = 0$, so $i_\xi d\alpha = 0$. This also shows that $L_\xi d\alpha = 0$, where $L_\xi$ denotes the Lie derivative along $\xi$. By Corollary 2.3 in [1] the last two properties imply that there is a two–form $\omega$ on $M$ such that $d\alpha = q^*\omega$. In particular, this shows that $0 = dq^*\omega = q^*d\omega$ and since $q$ is a surjective submersion, this implies that $\omega$ is closed and uniquely determined.

Finally, the restriction of $d\alpha$ to $\Lambda^2H$ can be written as the composition of the trivialization of $Q^*$ defined by evaluating elements on $\xi$ with the Levi bracket. This implies that $\omega$ is a section of $\ell$, which completes the proof. □

Having the cs–structure $\ell$ on $M$, we get decompositions of the spaces of differential forms similarly to the ones in Proposition 2.1. We first observe that the inclusion $\ell \hookrightarrow \Lambda^2T^*M$ can be viewed as a canonical section $\Omega$ of $\Lambda^2T^*M \otimes \ell^*$. Non–degeneracy implies that $\Omega$ defines an isomorphism $TM \rightarrow T^*M \otimes \ell^*$, whose inverse can be viewed as a canonical section $\Omega^{-1}$ of the bundle $\Lambda^2 TM \otimes \ell$. In particular, wedging by $\Omega$ defines a bundle map $\Lambda^k T^*M \rightarrow \Lambda^{k+2} T^*M \otimes \ell^*$ while insertion of $\Omega^{-1}$ induces a bundle map $\Lambda^k T^*M \rightarrow \Lambda^{k-2} T^*M \otimes \ell$. As before, non–degeneracy implies that the first map is injective for $k \leq n - 1$ and surjective for $k \geq n - 1$ while the second map is surjective for $k \leq n + 1$ and injective for $k \geq n + 1$. Similarly to 2.1 we define $\Lambda^k T^*M$ to be the kernel of the insertion of $\Omega^{-1}$ for $k \leq n$ and the kernel of wedging by $\Omega$ for $k > n$.

Corollary 2.2. The conformally symplectic structure $\ell$ on $M$ gives rise to decompositions

$$\Lambda^k T^*M = \Lambda^k_0 T^*M \oplus (\Lambda^k_0 T^*M \otimes \ell) \oplus \cdots \oplus (\Lambda^k_{-2i} T^*M \otimes \ell^i) \quad \text{for } k \leq n.$$  
$$\Lambda^k T^*M = \Lambda^k_0 T^*M \oplus (\Lambda^k_0 T^*M \otimes \ell) \oplus \cdots \oplus (\Lambda^k_{-2j} T^*M \otimes \ell^j) \quad \text{for } k > n.$$  

Here $i$ is the largest integer such that $2i \leq k$ and $j$ is the largest integer such that $k + 2j \leq 2n$.

Moreover, for each $r$, the linear maps induced by $T_x q$ restrict to linear isomorphisms $\Lambda^*_k H_x^r \cong \Lambda^*_0 T^*_{q(x)}M$.

Proof. The decompositions are proved exactly as in the proof of Proposition 2.1. The last statement follows readily from the proof of part (1) of Proposition 2.2. □

2.3. Pushing down the Rumin complex. For any contact manifold $(M_\#, H)$, the filtration of the cotangent bundle from Proposition 2.3 makes the de–Rham complex $(\Omega^*(M_\#), d)$ into a (2–step) filtered complex. This is determined by the
subspaces $\mathcal{F}_k \subset \Omega^k(M_\#)$ formed by those $k$–forms which vanish if all their entries are from $H \subset TM_\#$. This means that $\mathcal{F}_k = \Gamma(\Lambda^{k-1}H^* \otimes Q^*)$ and $\Omega^k(M_\#) / \mathcal{F}_k \cong \Gamma(\Lambda^k H^*)$, with the last isomorphism induced by restricting $k$–forms to multilinear maps defined on $H$.

The simplest construction of the Rumin complex comes from the spectral sequence associated to this filtration of the de–Rham complex, see [2]. It is easy to see that restricting the exterior derivative to $\mathcal{F}_k$ and then composing with the projection $\Omega^{k+1}(M_\#) / \mathcal{F}_{k+1}$ one obtains a tensorial operator induced by the bundle map

$$\partial : \Lambda^{k-1}H^* \otimes Q^* \to \Lambda^{k+1}H^*,$$

which (up to a non–zero factor) is just the composition of the alternation with $\mathcal{L}^\ast : Q^* \to \Lambda^2 H^*$ tensorized by $\text{id}_{\Lambda^{k-1}H^*}$. Hence linear algebra implies that if $\dim(M_\#) = 2n + 1$ the bundle map $\partial$ in (1) is injective if $k \leq n$ and surjective if $k \geq n$. In particular, from Proposition 2.2 we see that the cohomology bundle $\mathcal{H}^k_\# := \ker(\partial) / \text{im}(\partial)$ of $\partial$ in degree $k$ is $\Lambda^k_0 H^*$ for $k \leq n$ and $\Lambda^{k-1}_0 H^* \otimes Q^*$ for $k > n$.

Now the standard construction of spectral sequences provides operators $D_k : \Gamma(\mathcal{H}^k_\#) \to \Gamma(\mathcal{H}^{k+1}_\#)$, which are easily seen to be differential operators forming a complex. This is the Rumin complex, which by construction computes the de–Rham cohomology of $M_\#$.

Now assume that we have given quotient $q : M_\# \to M$ of $M_\#$ by a transversal infinitesimal automorphism $\xi \in \mathfrak{X}(M_\#)$. Then we know from [2,2] that $M$ inherits a conformally symplectic structure $\ell \subset \Lambda^2 T^* M$. The idea for pushing down the Rumin complex to a complex of differential operators on $M$ is easy. One just restricts to the subsheaves of forms in the kernel of the Lie derivative $L_\xi$.

Observe first, that the fact that $\xi$ is an infinitesimal automorphism of the contact manifold $(M_\#, H)$ immediately implies that $L_\xi(\mathcal{F}_k) \subset \mathcal{F}_k$ for all $k$. We have noted already that one obtains a well defined Lie derivative on the bundles $H$ and $Q$ and thus on their duals and all bundles obtained by tensorial constructions from them. Naturality of the Lie derivative then implies that, for each $k$, we get a commutative diagram with exact rows

$$
\begin{array}{cccc}
0 & \longrightarrow & \Gamma(\Lambda^{k-1}H^* \otimes Q^*) & \longrightarrow & \Omega^k(M_\#) & \longrightarrow & \Gamma(\Lambda^k H^*) & \longrightarrow & 0 \\
& & \downarrow L_\xi & & \downarrow L_\xi & & \downarrow L_\xi & & \\
0 & \longrightarrow & \Gamma(\Lambda^{k-1}H^* \otimes Q^*) & \longrightarrow & \Omega^k(M_\#) & \longrightarrow & \Gamma(\Lambda^k H^*) & \longrightarrow & 0
\end{array}
$$

Since $L_\xi$ commutes with the exterior derivative, we conclude from the above construction that also $L_\xi \circ \partial = \partial \circ L_\xi$ and $D_k \circ L_\xi = L_\xi \circ D_k$. Hence we conclude that $D_k$ induces an operator mapping $\ker(L_\xi) \subset \Gamma(\mathcal{H}^k_\#)$ to $\ker(L_\xi) \subset \Gamma(\mathcal{H}^{k+1}_\#)$. Now we can determine explicitly, what these kernels look like. For $k \geq 0$, we denote the space of sections of $\Lambda^k_0 T^* M$ by $\Omega^k_0(M)$.

**Proposition 2.3.** Let $q : M_\# \to M$ be a quotient of a contact manifold $(M_\#, H)$ by a transversal infinitesimal automorphism $\xi \in \mathfrak{X}(M)$. Then for any $k$ we have the following isomorphisms
| Space | $\ker(L_\xi)$ isomorphic to | Isomorphism |
|-------|-----------------------------|-------------|
| $\Gamma(\Lambda^k H^*)$ | $\Omega^k(M)$ | $\varphi \mapsto q^*\varphi|_{\Lambda^k H}$ |
| $\Gamma(\Lambda_0^k H^*)$ | $\Omega^k_0(M)$ | $\varphi \mapsto q^*\varphi|_{\Lambda_0^k H}$ |
| $\Gamma(\Lambda^k H^* \otimes Q^*)$ | $\Omega^k(M)$ | $\varphi \mapsto \alpha \wedge q^*\varphi$ |
| $\Gamma(\Lambda_0^k H^* \otimes Q^*)$ | $\Omega^k_0(M)$ | $\varphi \mapsto \alpha \wedge q^*\varphi$ |
| $\Omega^k(M_\#)$ | $\Omega^k(M) \oplus \Omega^{k-1}(M)$ | $(\varphi_1, \varphi_2) \mapsto q^*\varphi_1 + \alpha \wedge q^*\varphi_2$ |

In this table, $\alpha$ denotes the unique contact form on $M_\#$ such that $\alpha(\xi) = 1$ and the isomorphisms map the spaces in the second column to the kernel of $L_\xi$ contained in the space in the first column.

**Proof.** The quickest way to prove this is via the well known characterization of pullback forms on $M_\#$, see Corollary 2.3 in [1], which applies by connectedness of the fibers of $q$. In our situation this says that a form $\psi \in \Omega^k(M_\#)$ is of the form $q^*\varphi$ for some $\varphi \in \Omega^k(M)$ if and only if $i_\xi \psi = 0$ and $L_\xi \psi = 0$. Since $q$ is a surjective submersion, the form $\varphi$ is then uniquely determined by $\psi$.

Starting with $\psi \in \Omega^k(M_\#)$ such that $L_\xi \psi = 0$ and taking into account that $i_\xi$ and $L_\xi$ commute, we can use this to conclude that $i_\xi \psi \in \Omega^k(M)$ is a section $\sigma$ such that $\psi - \alpha \wedge q^*\varphi_2$ for a unique $\varphi_2 \in \Omega^k(M)$. Then $\psi - \alpha \wedge q^*\varphi_2$ by construction lies in the kernel of $i_\xi$ and since $L_\xi \alpha = 0$, it is of the form $q^*\varphi_1$ for a unique form $\varphi_1 \in \Omega^k(M)$. Since the converse inclusion is obvious this establishes the last isomorphism.

Next, given $\varphi \in \Omega^k(M_\#)$ we can form $q^*\varphi \in \Omega^k(M_\#)$ and project it to the quotient $\Gamma(\Lambda^k H^*)$ to obtain a section $\sigma$, which by construction satisfies $L_\xi \sigma = 0$. Conversely, given such a section $\sigma$, we can extend it arbitrarily to a smooth form $\tilde{\psi} \in \Omega^k(M_\#)$ and then define $\psi := \tilde{\psi} - \alpha \wedge i_\xi \tilde{\psi}$. By construction, this is still an extension of $\sigma$ and $i_\xi \psi = 0$. (Indeed, it is uniquely determined by these two properties.) Moreover, we also get $i_\xi L_\xi \psi = 0$. But since $\psi$ projects to $\sigma$, naturality of the Lie derivative implies that $L_\xi \psi$ projects to $L_\xi \sigma = 0$ in $\Gamma(\Lambda^k H^*)$. Thus we conclude that $L_\xi \psi = 0$, so $\psi = q^*\varphi$ for some $\varphi \in \Omega^k(M)$ and the first isomorphism is established.

From the construction and the definitions it is clear, that in the above we can restrict to $\varphi \in \Gamma(\Lambda_0^k T^*M)$ and then the restriction of $q^*\varphi$ will have values in $\Gamma(\Lambda_0^k H^*)$. Conversely, given $\sigma \in \Gamma(\Lambda_0^k H^*)$, the above construction leads to $\varphi \in \Gamma(\Lambda_0^k T^*M)$, so the second isomorphism follows as above.

Finally, $\Gamma(\Lambda^k H^* \otimes Q^*) \subset \Omega^{k+1}(M_\#)$ consists of those forms which vanish if all their entries are from $H$. If $\psi$ is such a form, then $\psi = \alpha \wedge i_\xi \psi$. As above, $L_\xi \psi = 0$ implies $L_\xi i_\xi \psi = 0$, so $i_\xi \psi = q^*\varphi$ for some $\varphi \in \Omega^k(M)$. Since the converse inclusion is obvious, the third isomorphism is proved. The passage to $\Lambda_0^k H^* \otimes Q^*$ upstairs and $\Lambda_0^k T^*M$ downstairs needed to obtain the fourth isomorphism is again straightforward.

**Remark 2.3.** In view of potential generalizations, we want to point out that the first four isomorphisms claimed in the theorem can actually be proved directly without passing through differential forms (and in more general situations such proofs will be simpler). The fact that passing through forms is more efficient is only due to the simplicity of the filtration of the involved bundles.
From this result it is clear that the Rumin complex on $M_\#$ descends to a complex on $M$. To understand which cohomology this complex computes, we only have to interpret what we have done in terms of sheaf theory. Of course, we can view the Lie derivative $L_\xi$ as an endomorphism of the sheaf of $k$–forms on $M_\#$ and thus its kernel as a subsheaf $\mathcal{A}^k$ of the sheaf of $k$–forms on $M_\#$. Via the map $q : M_\# \to M$, one forms a sheaf $q_*\mathcal{A}^k$ on $M$, which by definition to an open subset $U$ of $M$ associates $\mathcal{A}^k(q^{-1}(U))$. Since the exterior derivative (on $M_\#$) is a morphism $\mathcal{A}^k \to \mathcal{A}^{k+1}$, we obtain a complex of sheaves $(q_*\mathcal{A}^k, d)$ on $M$.

Corollary 2.3. Via the isomorphisms from Proposition 2.3, the Rumin complex on $M_\#$ descends to a complex $(\mathcal{H}^i, D_i)$ of differential operators where $\mathcal{H}^i = \Omega^i_0(M)$ for $i \leq n$ and $\mathcal{H}^i = \Omega^{i-1}_0(M)$ for $i > n$. The operators $D_i$ are all of first order, except for $D_n : \Omega^0_0(M) \to \Omega^1_0(M)$ which has order two.

The cohomology of this complex coincides with the sheaf cohomology of the complex $(q_*\mathcal{A}^k, d)$ of sheaves on $M$.

Proof. The first part follows immediately from Proposition 2.3 and the corresponding results for the Rumin complex. For the last part, recall that the Rumin complex arises from the de–Rham resolution on $M_\#$ via the spectral sequence associated to a (very simple) filtration. This filtration restricts to $\mathcal{A}^*$ and we can again use the corresponding spectral sequence to compute the cohomology. But Proposition 2.3 shows that in the first step of this spectral sequence one obtains sheaves that descend to $M$, and the operators on these sheaves produced by the spectral sequence form exactly the complex $(\mathcal{H}^i, D_i)$.

2.4. Cohomology of the descended complex. To compute the cohomology of the descended complex, we have to understand the complex $(q_*\mathcal{A}^k, d)$ of sheaves. This again is easy using Proposition 2.3.

Theorem 2.4. For any open subset $U \subset M$ let us denote by $H^k(U)$ the $k$–th de–Rham cohomology of $U$. Then the cohomology groups of the complex $(\mathcal{H}^k|_U, D)$ over $U$ fit into a long exact sequence

$$
\cdots \to H^k(U) \to H^k(\mathcal{H}^k|_U, D) \to H^{k-1}(U) \to H^{k+1}(U) \to \cdots
$$

In this sequence, the connecting homomorphism $H^{k-1}(U) \to H^{k+1}(U)$ is given by the wedge product with the cohomology class of $\omega|_U$, where $\omega$ is the symplectic form on $M$ from Proposition 2.3.

In particular, for contractible $U$, the cohomology groups of $(\mathcal{H}^k, D)$ are isomorphic to $\mathbb{R}$ in degrees 0 and 1, while all higher cohomologies vanish. On the other hand, if $\omega$ is not exact on $M$, then $H^k(\mathcal{H}^k, D) \cong H^k(M)$ for $k = 0, 1$.

Proof. By Corollary 2.3, the cohomology of $(\mathcal{H}^k|_U, D)$ coincides with the cohomology of $(\mathcal{A}^*(q^{-1}(U)), d)$. Now on $U_\# := q^{-1}(U)$, let us denote by $i_\xi$ the insertion operator defined by the vector field $\xi$. Applying Proposition 2.3 to $q|_{U_\#} : U_\# \to U$, we get a short exact sequence of complexes

$$
0 \to (\Omega^*(U), d) \to (\mathcal{A}^*(q^{-1}(U)), d) \xrightarrow{i_\xi} (\Omega^{\ast-1}(U), d) \to 0.
$$
The corresponding long exact sequence in cohomology has the form claimed in the theorem. To describe the connecting homomorphism, consider a closed \((k - 1)\)–form \(\varphi\) on \(U\). As a preimage in \(\mathcal{A}^k(q^{-1}(U))\) under \(i_\xi\), we can use \(\alpha \wedge q^*\varphi\). The exterior derivative of this is \(d\alpha \wedge q^*\varphi = q^*(\omega \wedge \varphi)\) by Proposition 2.1.

In particular, if \(\omega\) is exact, then all connecting homomorphisms vanish, and one obtains a short exact sequence \(0 \rightarrow H^k(U) \rightarrow H^k(\mathcal{H}^\ast|U, D) \rightarrow H^{k-1}(U) \rightarrow 0\) for each \(k\). From this the claim on contractible \(U\) follows.

Next, the long exact sequence immediately implies that \(H^0(U) \cong H^0(\mathcal{H}^\ast|U, D)\). If \(\omega\) is non–exact on \(M\), then wedging with \(\omega\) is an injection \(H^0(M) \rightarrow H^2(M)\), so the long exact sequence shows that \(H^1(M) \cong H^1(\mathcal{H}^\ast, D)\). □

**Remark 2.4.** Of course, in the case of non–exact \(\omega\), one can get more out of the long exact sequence than just the final claim of the theorem. We have emphasized that last result only, because it is (for a much more general family of complexes, but only in the case of the simply connected space \(\mathbb{C}P^n\)) exactly what is need to prove the integral geometry results of [8].

A natural context to further exploit the long exact sequence in the theorem would for example be the case of symplectic manifolds which satisfy the hard Lefschetz theorem, like \(\mathbb{K}\)ähler manifolds. In this case, the wedge product by the cohomology class of \(\omega\) is injective below half the dimension and surjective above, and the cohomology of \((\mathcal{H}^\ast, D)\) nicely relates to primitive cohomology.

### 3. **The Rumin–Seshadri complex on general conformally symplectic manifolds**

To conclude this article, we want to show how the quotient construction from section 2 can be used to obtain a complex on general cs–manifolds and prove that this complex is intrinsic to the conformally symplectic structure. Moreover, we again obtain an exact sequence, which nicely relates its cohomology to de–Rham cohomology.

#### 3.1. **Local contactifications.**

As indicated in section 2 already, we view a cs–manifold as a manifold \(M\) of even dimension \(2n \geq 4\) together with a line subbundle \(\ell \subset \Lambda^2 T^*M\) such that all non–zero elements of \(\ell\) are (point–wise) non–degenerate as bilinear forms and such that \(\ell\) admits local non–vanishing sections which are closed as two–forms on \(M\). Observe that if \(\Omega\) is a local closed non–vanishing section of \(\ell\) then any other local section of \(\ell\) is of the form \(f\Omega\) for a smooth function \(f\). But then \(d(f\Omega) = df \wedge \Omega\) so since \(\dim(M) > 2\), this vanishes only if \(df = 0\) by non–degeneracy of \(\Omega\). Hence these local non–vanishing closed sections are determined up to a constant factor.

Given such a structure, each point \(x \in M\) has an open neighborhood \(U\) in \(M\) such that there are local sections of \(\ell|_U\) which are exact as two forms on \(M\). Then it is well known that such neighborhoods can be realized as cs–quotients:

**Lemma 3.1.** Let \((M, \ell)\) be a conformally symplectic manifold and let \(U \subset M\) be an open subset and let \(\beta \in \Omega^1(U)\) be such that \(d\beta\) is a section of \(\ell\) which is nowhere vanishing on \(U\). Put \(U_\# := U \times \mathbb{R}\) and let \(t\) be the standard coordinate on the second factor.


Given \( \tilde{\alpha} \) is a contact form on \( \tilde{U} \), such that \( q := \text{pr}_1: \tilde{U} \to U \) is a cs–quotient with respect to the Reeb field \( \xi := \partial_t \) of \( \alpha \).

**Proof.** We have \( d\alpha = q^*d\beta \) so this is non–degenerate on \( TU \subset \tilde{TU} \), so clearly \( dt \wedge (q^*d\beta)^n = \alpha \wedge (d\alpha)^n \) is a volume form on \( \tilde{U} \). Thus \( \alpha \) is a contact form, and it is obvious that its Reeb field \( \xi \) equals \( \partial_t \). This shows that \( \xi \) spans the vertical subbundle of \( q = \text{pr}_1 \) and by construction \( d\alpha \) descends to the section \( d\beta \) of \( \ell \). \( \square \)

There also is a local uniqueness result:

**Proposition 3.1.** Let \( q: M_\# \to M \) and \( \tilde{q}: \tilde{M}_\# \to \tilde{M} \) be cs–quotients with respect to infinitesimal automorphisms \( \xi \in \mathfrak{x} (M_\#) \) and \( \tilde{\xi} \in \mathfrak{x} (\tilde{M}_\#) \). Suppose that \( \varphi: \tilde{M} \to M \) is a cs–diffeomorphism, i.e. \( \varphi^*(\ell) = \ell \).

Then for each point \( \tilde{x} \in \tilde{M} \) and each point \( u \in M_\# \) with \( \tilde{q}(\tilde{u}) = \tilde{x} \), there are open neighborhoods \( \tilde{U} \) of \( \tilde{x} \) in \( \tilde{M} \) and \( U \) of \( u \) in \( M_\# \) such that \( \tilde{q} \) restricts to a surjective submersion \( \tilde{U}_\# \to \tilde{U} \) with connected fibers and an immersion \( \Phi: \tilde{U}_\# \to M_\# \) which restricts to a contactomorphism onto its image and satisfies \( q \circ \Phi = \varphi \circ \tilde{q} \) and \( \Phi^*\xi = \lambda \tilde{\xi} \) for some non–zero constant \( \lambda \).

**Proof.** Given \( \tilde{x} \), choose a contractible neighborhood \( U \) of \( x = \varphi(\tilde{x}) \) in \( M \) such that there is \( \beta \in \Omega^1(U) \) for which \( d\beta \) is a section of \( \ell \), which is nowhere vanishing on \( U \). Further, take any point \( u \in q^{-1}(x) \) and let \( \alpha \in \Omega^1(M_\#) \) be the unique contact form such that \( \alpha(\xi) = 1 \). Then by part (2) of Proposition 2.2 \( d\alpha \) descends to a locally non–vanishing closed section of \( \ell \). Replacing \( \beta \) by a constant multiple if necessary, we may thus assume that \( d\alpha \) descends to \( d\beta \). Since we also know that \( i_\xi d\alpha = i_\xi dq^* \beta = 0 \), we conclude that \( d(\alpha - q^* \beta) = 0 \).

Possibly shrinking \( U \), we can choose an open neighborhood \( U_\# \) of \( u \) in \( M_\# \) such that \( q: U_\# \to U \) is a submersion with connected fibers and such that \( \alpha - q^* \beta = df \) for some smooth \( f : U_\# \to \mathbb{R} \) with \( f(u) = 0 \). By construction \( df(\xi) = 1 \), so \( \ker(df) \) is always transversal to \( \ker(Tq) = \mathbb{R} \cdot \xi \). This implies that \( (q, f) : U_\# \to U \times \mathbb{R} \) is a local diffeomorphism, so possibly again shrinking \( U \) and \( U_\# \) we may assume that it is a diffeomorphism \( U_\# \to U \times (-\epsilon, \epsilon) \) for some \( \epsilon > 0 \).

Now we apply the same construction to the neighborhood \( \varphi^{-1}(U) \) of \( \tilde{x} \) in \( \tilde{M} \), the form \( \tilde{\beta} = \varphi^* \beta \in \Omega^1(U) \), and the point \( \tilde{u} \in \tilde{q}^{-1}(\tilde{x}) \). Note however, that we are not allowed to rescale \( \tilde{\beta} \) any more, so we can only obtain \( \lambda \tilde{\alpha} = \tilde{q}^* \tilde{\beta} + df \) for some non–zero constant \( \lambda \). Again we can arrange things in such a way that \( (\tilde{q}, \tilde{f}) \) is a diffeomorphism \( \tilde{U}_\# \to U \times (-\epsilon, \epsilon) \).

Now define \( \Phi : \tilde{U}_\# \to U_\# \) as \( (q, f)^{-1} \circ (\varphi, \text{id}) \circ (\tilde{q}, \tilde{f}) \). Then of course \( \Phi \) is a diffeomorphism such that \( \Phi^* df = d\tilde{f} \) and \( q \circ \Phi = \varphi \circ \tilde{q} \) and thus \( \Phi^* q^* \beta = \tilde{q}^* \tilde{\beta} \).

This shows that \( \Phi^* \alpha = \lambda \tilde{\alpha} \), so \( \Phi \) is a contactomorphism and the Reeb field \( \xi \) of \( \alpha \) is pulled back to the Reeb field \( \lambda \tilde{\xi} \), which is \( \frac{1}{\lambda} \tilde{\xi} \). \( \square \)

3.2. **An intrinsic complex via local push downs.** The local existence and uniqueness results from 3.1 are not quite enough to construct an intrinsic sequence of differential operators from Corollary 2.3. The point is that in the uniqueness result Proposition 3.1, we do not get true compatibility between the infinitesimal
automorphism, but only compatibility up to a non–zero constant multiple. Looking at the first four isomorphisms in Proposition 2.3 which are used to construct the descended complex, we see that the first two of them remain unchanged if $\xi$ is rescaled by a non–zero constant. The other two isomorphisms, however, involve the contact form $\alpha$ characterized by $\alpha(\xi) = 1$, so these change if $\xi$ changes.

We can easily correct this by using a slightly modified version of the construction from 2.3. Consider the line bundle $\ell \subset \Lambda^2 T^* \overline{M}$ defining the conformally symplectic structure. This can be viewed as an abstract line bundle on $\overline{M}$, and then we can consider the spaces $\Omega^k(\overline{M}, \ell)$ of $\ell$–valued differential forms on $\overline{M}$. Moreover, we have seen above that $\ell$ admits local sections which are closed as two–forms on $\overline{M}$, and these are uniquely determined up to a constant multiple. Thus we can define a linear connection $\nabla$ on $\ell$ by requiring that these sections are parallel for $\ell$, which of course implies that $\nabla$ is flat. In particular, twisting the exterior derivative by $\nabla$ we obtain the twisted de–Rham complex $(\Omega^*(\overline{M}, \ell), d\nabla)$.

Now suppose that $q : M_\# \to M$ is a cs–quotient with respect to a transversal infinitesimal automorphism $\xi$ of $M_\#$. Then from the proof of Proposition 2.2 we see that if $\alpha$ is a contact form on $\overline{M}$ such that $\alpha(\xi)$ is constant, then $d\alpha$ descends to a non–vanishing section of $\ell$. Any nonzero element in (a fiber of) $\ell$ can be obtained in this way, so we conclude that $\xi$ gives rise to a section $\sigma_\xi$ of the dual bundle $\ell^* \to M$ and thus also to a (tensorial) map $\Omega^k(\overline{M}, \ell) \to \Omega^k(M)$ which we denote by the same symbol. Now we can modify Proposition 2.3 and construct an isomorphism between $\Omega^k(\overline{M}, \ell)$ and $\ker(L_\xi) \subset \Gamma(\Lambda^k H^* \otimes Q^*)$ by mapping $\varphi$ to $\alpha \wedge q^*(\sigma_\xi(\varphi))$, where again $\alpha$ is the unique contact form such that $\alpha(\xi) = 1$. Evidently, if we multiply $\xi$ by a non–zero constant, then $\sigma_\xi$ gets multiplied by the same constant while $\alpha$ gets divided by that constant, so the resulting isomorphism stays the same.

There is also a natural analog $\Omega^k_0(\overline{M}, \ell)$ of $\Lambda^2 T^* \overline{M} \otimes \ell^*$. Wedging with $\Omega$ can also be interpreted as a bundle map $\Lambda^k T^* \overline{M} \otimes \ell \to \Lambda^{k+2} T^* \overline{M}$ for each $k \geq 1$, which is surjective for $k \geq n – 1$. For $k \geq n$, we denote by $(\Lambda^k T^* \overline{M} \otimes \ell)_0$ the kernel of this bundle map and by $\Omega^k_0(\overline{M}, \ell)$ the space of sections of this bundle. As in Proposition 2.3, it is clear that the isomorphism between $\Omega^k(\overline{M}, \ell)$ and $\ker(L_\xi) \subset \Gamma(\Lambda^k H^* \otimes Q^*)$ from above restricts to an isomorphism between $\Omega^k_0(\overline{M}, \ell)$ and the kernel of $L_\xi$ in the subspace $\Gamma(\Lambda^k H^* \otimes Q^*)$ for $k \geq n$.

Having all that in hand, we can construct a complex intrinsic to a conformally symplectic structure:

**Theorem 3.2.** Let $(M, \ell)$ be a cs–manifold of dimension $2n$. Then there is a differential complex $(\mathcal{H}^*, D)$ which is intrinsically associated to the cs–structure such that $\mathcal{H}^i = \Omega^i_0(M)$ for $i \leq n$ and $\mathcal{H}^i = \Omega^{n-i}_0(M, \ell)$ for $i > n$. The operators $D$ are all of first order except for $D : \Omega^n_0(M) \to \Omega^n_0(M, \ell)$, which has order two.

**Proof.** Given $\varphi \in \mathcal{H}^i$ we consider the restriction $\varphi|_U$ for an open subset $U \subset M$ over which we find a local contactification $q : M_\# \to U$. Over $U$ we can use the push down construction from 2.3 to obtain a section $D(\varphi|_U) \in \mathcal{H}^{i+1}|_U$. Proposition 3.1 and naturality of the Rumin complex under contactomorphisms imply that this
section is independent of the choice of contactification and that the resulting local sections piece together to a well defined element \( D(\varphi) \in H^{i+1} \). By construction, this defines a complex which is intrinsic to the cs–structure. \( \square \)

Of course, if we start with a symplectic structure, we can use the symplectic form to trivialize the bundle \( \ell \) and thus identity \((\Omega^*(M, \ell), d^\nabla)\) with \((\Omega^*(M), d)\). It is then easy to check directly that the complex constructed here coincides with the one from the last section of [2], where it is called the Rumin–Seshadri complex, and we keep this name in the more general setting of lcs–structures. That reference also contains some information on the history of this complex and discusses the relations of this complex to [13, 14]. In the situation of a general conformally symplectic structure, one can fix a section of \( \ell \) symplectic structure, one can use the symplectic contactification and that resulting local trivialize the bundle \( \ell \), this defines a complex which is intrinsic to the cs–structure. □

To conclude our article, we show how the push–down methods can be used to analyze the cohomology of the Rumin–Seshadri complex on conformally symplectic manifolds. The result is essentially contained in Theorem 2 of [7], apart from the fact that we do not work with a fixed section of \( \ell \). Our proof via the quotient construction is different, however, and we include it for completeness.

Consider the (tensorial) map \( \Omega^k(M, \ell) \to \Omega^{k+2}(M) \) defined by wedging with the canonical section \( \Omega \in \Omega^2(M, \ell^*) \). If \( \sigma_0 \) is a locally non–vanishing section of \( \ell \) which is parallel for \( \nabla \), then by definition \( d(\Omega \wedge \sigma_0) = 0 \), while for \( \varphi \in \Omega^k(M) \) we have \( d^\nabla (\varphi \otimes \sigma_0) = d\varphi \otimes \sigma_0 \). Using these facts, one immediately verifies that \( d(\Omega \wedge \psi) = \Omega \wedge d^\nabla \psi \) holds for all \( \psi \in \Omega^k(M, \ell) \). Otherwise put \( d^\nabla^* \Omega = 0 \), where \( \nabla^* \) is the connection on \( \ell^* \) induced by \( \nabla \). In particular, we can form the class \( [\Omega] \) in the twisted de–Rham cohomology group \( H^2(M, \ell^*) \) and wedging with this class defines a map \( H^k(M, \ell) \to H^{k+2}(M) \). Using this, we can formulate:

**Theorem 3.3.** Let \( (M, \ell) \) be a conformally symplectic manifold. Then the cohomology groups of the Rumin–Seshadri complex \((\mathcal{H}^*, D)\) fit into a long exact sequence of the form

\[
\cdots \to H^k(M) \to H^k(\mathcal{H}^*, D) \to H^{k-1}(M, \ell) \to H^{k+1}(M) \to \cdots,
\]

in which the connecting homomorphism \( H^{k-1}(M, \ell) \to H^{k+1}(M) \) is given by the wedge product with \([\Omega] \in H^2(M, \ell^*)\).

**Proof.** We continue modifying the results from Proposition 2.3 as started in 3.2. Given an open subset \( U \subset M \) over which there is a contactification, we can define an isomorphism \( \Omega^k(U) \oplus \Omega^{k-1}(U, \ell) \to \mathcal{A}^k(q^{-1}(U)) \) by \((\varphi, \psi) \mapsto q^* \varphi + \alpha \wedge \sigma_\xi(\psi)\) (with notation as in 3.2). This remains unchanged if \( \xi \) is replaced by a constant multiple. One immediately verifies that under this isomorphism, the action of the exterior derivative on \( \mathcal{A}^k(q^{-1}(U)) \) corresponds to \((\varphi, \psi) \mapsto (d\varphi + \Omega \wedge \psi, -d^\nabla \psi)\), and this defines a differential \( \tilde{D} \) on \( \Omega^*(U) \oplus \Omega^{*-1}(U, \ell) \).

Combining this with the constructions of 2.3 we get a filtration of the complex \((\Omega^*(U) \oplus \Omega^{*-1}(U, \ell), \tilde{D})\) such that the associated spectral sequence produces the...
Pushing down the Rumin complex $(\mathcal{H}^*|_U, D)$, which thus computes the same cohomology. By naturality of all the involved constructions, these fit together to define a filtration on

$$(\Omega^*(M) \oplus \Omega^{*-1}(M, \ell), \tilde{D})$$

such that the associated spectral sequence produces $(\mathcal{H}^*, D)$, which thus computes the same cohomology. From this the result follows, since there is an evident short exact sequence

$$0 \to (\Omega^*(M), d) \to (\Omega^*(M) \oplus \Omega^{*-1}(M, \ell), \tilde{D}) \to (\Omega^{*-1}(M, \ell), -d\nabla) \to 0$$

for which the connecting homomorphism in the resulting long exact sequence has the claimed form. □

REFERENCES

[1] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths, Exterior differential systems, Mathematical Sciences Research Institute Publications, vol. 18, Springer-Verlag, New York, 1991. MR1083148 (92h:58007)

[2] R. Bryant, M. Eastwood, A. R. Gover, and K. Neusser, Some differential complexes within and beyond parabolic geometry, available at arXiv:1112.2142

[3] M. Cahen and L. J. Schwachhöfer, Special symplectic connections, J. Differential Geom. 83 (2009), no. 2, 229–271. MR2577468 (2011b:53045)

[4] D. M. J. Calderbank and T. Diemer, Differential invariants and curved Bernstein-Gelfand-Gelfand sequences, J. Reine Angew. Math. 537 (2001), 67–103. MR1856258 (2002k:58048)

[5] A. Čap and J. Slovák, Parabolic geometries. I. Background and general theory, Mathematical Surveys and Monographs, vol. 154, American Mathematical Society, Providence, RI, 2009. MR2532439 (2010j:53037)

[6] A. Čap, J. Slovák, and V. Souček, Bernstein-Gelfand-Gelfand sequences, Ann. of Math. 154 (2001), no. 1, 97–113. MR1847589 (2002h:58034)

[7] M. Eastwood, Extensions of the coeffective complex, Illinois J. Math., to appear, available at arXiv:1203.6714

[8] M. Eastwood and H. Goldschmidt, Zero-energy fields on complex projective space, J. Differential Geom. 94 (2013), no. 1, 129–157. MR3031862

[9] M. Eastwood and J. Slovák, Conformally Fedosov manifolds, available at arXiv:1210.5597

[10] D. J. F. Fox, Contact projective structures, Indiana Univ. Math. J. 54 (2005), no. 6, 1547–1598. MR2189678 (2007b:53163)

[11] M. Rumin, Un complexe de formes différentielles sur les variétés de contact, C. R. Acad. Sci. Paris Sér. I Math. 310 (1990), no. 6, 401–404. MR1046521 (91a:58004)

[12] M. Rumin and N. Seshadri, Analytic torsions on contact manifolds, Ann. Inst. Fourier (Grenoble) 62 (2012), no. 2, 727–782. MR2985515

[13] L.-S. Tseng and S.-T. Yau, Cohomology and Hodge theory on symplectic manifolds: I, J. Differential Geom. 91 (2012), no. 3, 383–416. MR2981843

[14] L.-S. Tseng and S.-T. Yau, Cohomology and Hodge theory on symplectic manifolds: II, J. Differential Geom. 91 (2012), no. 3, 417–443. MR2981844

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