E₁-DEGENERATION AND d′d''-LEMMA

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Abstract. For a double complex \((A, d', d'')\), we show that if it satisfies the \(d'd''\)-lemma and the spectral sequence \(\{E^p_{r,q}\}\) induced by \(A\) does not degenerate at \(E_0\), then it degenerates at \(E_1\). We apply this result to prove the degeneration at \(E_1\) of a Hodge-de Rham spectral sequence on compact bi-generalized Hermitian manifolds that satisfy a version of \(d'd''\)-lemma.

Keywords: \(\partial\bar{\partial}\)-lemma, Hodge-de Rham spectral sequence, \(E_1\)-degeneration, bi-generalized Hermitian manifold.

1. Introduction

Complex manifolds that satisfy the \(\partial\bar{\partial}\)-lemma enjoy some nice properties such as they are formal manifolds ([DGMS]), their Bott-Chern cohomology, Aeppli cohomology and Dolbeault cohomology are all isomorphic. Compact Kähler manifolds are examples of such manifolds. The Hodge-de Rham spectral sequence \(E^{r,*}_{r,*}\) of a complex manifold \(M\) is built from the double complex \((\Omega^{r,*}(M), \partial, \bar{\partial})\) of complex differential forms which relates the Dolbeault cohomology of \(M\) to the de Rham cohomology of \(M\). It is well known that \(E^{p,q}_1\) is isomorphic to \(H^p(M, \Omega^q)\) and the spectral sequence \(E^{r,*}_{r,*}\) converges to \(H^*(M, \mathbb{C})\). The goal of this paper is to prove an algebraic version of the result that the \(\partial\bar{\partial}\)-lemma implies the \(E_1\)-degeneration of a Hodge-de Rham spectral sequence. The following is our main result.

Theorem 1.1. If a double complex \((A, d', d'')\) satisfies the \(d'd''\)-lemma and the spectral sequence \(\{E^p_{r,q}\}\) induced by \(A\) does not degenerate at \(E_0\), then it degenerates at \(E_1\).

We define a spectral sequence that is analogous to the Hodge-de Rham spectral sequence of complex manifolds for bi-generalized Hermitian manifolds. Applying result above, we are able to show that for compact bi-generalized Hermitian manifolds that satisfy a version of \(\partial\bar{\partial}\)-lemma, the sequence degenerates at \(E_1\).

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2. Degeneration of a Hodge-de Rham spectral sequence

Definition 2.1. A spectral sequence is a sequence of differential bi-graded modules \(\{(E^{r,*}_{r,*}, d_r)\}\) such that \(d_r\) is of degree \((r, 1-r)\) and \(E^{p,q}_{r+1}\) is isomorphic to \(H^{p,q}(E^{r,*}_{r,*}, d_r)\).

Definition 2.2. A filtered differential graded module is a \(\mathbb{N}\)-graded module \(A = \bigoplus_{k=0}^{\infty} A^k\), endowed with a filtration \(F\) and a linear map \(d : A \rightarrow A\) satisfying

1. \(d\) is of degree 1: \(d(A^k) \subset A^{k+1}\);
2. \(d \circ d = 0\);

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(3) the filtered structure is descending:
\[ A = F^0A \supseteq F^1A \supseteq \cdots \supseteq F^kA \supseteq F^{k+1}A \supseteq \cdots; \]

(4) the map \( d \) preserves the filtered structure: \( d(F^kA) \subseteq F^kA \) for all \( k \).

For \( p, q, r \in \mathbb{Z} \), let
\[
Z^{p,q}_r = \left\{ \xi \in F^pA^{p+q} \bigg| d\xi \in F^{p+r}A^{p+q+1} \right\}, \quad Z^{p,q}_\infty = F^pA^{p+q} \cap \ker d
\]
\[
B^{p,q}_r = F^pA^{p+q} \cap dF^{p-r}A^{p+q-1}, \quad B^{p,q}_\infty = F^pA^{p+q} \cap \text{Im} d
\]
\[
E^{p,q}_r = \frac{Z^{p,q}_r}{Z^{p+1,q-1}_r + B^{p,q}_r}, \quad E^{p,q}_\infty = \frac{F^pA^{p+q} \cap \ker d + F^pA^{p+q} \cap \text{Im} d}{F^{p+1}A^{p+q} \cap \ker d}
\]
with the convention \( F^{-k}A^{p+q} = A^{p+q} \) and \( A^{-k} = \{0\} \) for \( k \geq 0 \). Let \( d_r : E^{p,q}_r \to E^{p+r,q-r+1}_r \) be the differential induced by \( d : Z^{p,q}_r \to Z^{p+r,q-r+1}_r \).

Throughout this paper, we always assume that \( A = \bigoplus_{p,q \geq 0} A^{p,q} \) is a double complex of vector spaces over some field with two maps \( d'_{p,q} : A^{p,q} \to A^{p+1,q} \) and \( d''_{p,q} : A^{p,q} \to A^{p,q+1} \) satisfying
\[
d'_{p+1,q}d''_{p,q} = 0, \quad d''_{p,q+1}d'_{p,q} = 0, \quad \text{and} \quad d''_{p,q+1}d''_{p,q} + d''_{p+1,q}d'_{p,q} = 0 \quad \text{for all} \quad p, q \geq 0.
\]
To make notation cleaner, we allow \( p, q \) to be any integers by defining \( A^{p,q} = 0 \) for \( p < 0 \) or \( q < 0 \).

Let \( A^k = \bigoplus_{p+q = k} A^{p,q} \). Define
\[
F^pA^k = \bigoplus_{s=0}^k A^{s,k-s}
\]
For \( p > k \), define \( F^pA^k = \{0\} \). This gives a descending filtration on \( A^k \).

Let \( d = d' + d'' \). The double complex \( (A, d', d'') \) then defines a filtered differential graded module \((A, d, F)\). Let \( \{E^{p,q}_r\} \) be the corresponding spectral sequence. We are interested in the convergence of \( E^{p,q}_r \).

Definition 2.3. Let \( \{E^{p,q}_r\} \) be the spectral sequence associated to the double complex \((A, d', d'')\). If \( d_s = 0 \) for all \( s \geq r \), then we say that \( \{E^{p,q}_r\} \) or \( A \) degenerates at \( E_r \).

The following simple lemmas will be used frequently.

Lemma 2.4. If \( G' \) is a vector space and \( H < G, H < H' \) are subspaces of \( G' \), the natural map \( \varphi : \frac{G}{H} \to \frac{G'}{H'} \) is injective if and only if \( G \cap H' = H \), and surjective if and only if \( G' = G + H' \).

Lemma 2.5. Let \( p, q, r \in \mathbb{Z} \). There are inclusions
\[
\cdots \subset B^{p,q}_0 \subset B^{p,q}_1 \subset \cdots \subset B^{p,q}_\infty \subset Z^{p,q}_\infty \subset \cdots \subset Z^{p,q}_1 \subset Z^{p,q}_0 \subset \cdots , \quad Z^{p,q}_{r-1} \subset Z^{p,q}_r \quad \text{and} \quad Z^{p,q}_{r} \subset Z^{p,q}_{r+1} \subset \cdots , \quad d(Z^{p,q}_{r-r+q-r}) = B^{p,q}_r
\]

Definition 2.6. Let \( \alpha^{p,q}_{r+1} : E^{p,q}_{r+1} \to \frac{Z^{p,q}_{r+1}}{Z^{p,q}_{r+1} \cap B^{p,q}_{r+1}} \) be the map induced by the composition of inclusion and projection, and \( \beta^{p,q}_{r} : E^{p,q}_r \to \frac{Z^{p,q}_{r}}{Z^{p,q}_{r} \cap B^{p,q}_{r}} \) be the map induced by the projection.

Proposition 2.7. Let \( r \in \mathbb{Z} \). Then

(1) \( d_r = 0 \) if and only if \( \beta^{p,q}_{r} \) is an isomorphism for all \( p, q \in \mathbb{Z} \).

(2) \( d_r = 0 \) implies that \( \alpha^{p,q}_{r+1} \) is an isomorphism for all \( p, q \in \mathbb{Z} \).
Proof. (1) We first note that the map $\beta_{p,q,r}$ is always surjective. By Lemma 2.4, $\beta_{p,q,r}$ is an isomorphism if and only if $Z^p,q \cap (Z^{p+1,q-1}_{r-1} + B^p,q_r) = Z^{p+1,q-1}_{r-1} + B^p,q_r$, or equivalently, $B^p,q_r \subseteq Z^{p+1,q-1}_{r-1} + B^p,q_r$. The map $d_{p,q,r+1}^r : E^p,q_r \to E^p,q_{r+1}$ is the zero map if and only if $\text{Im}d_{p,q,r+1}^r = \{0\}$. This is equivalent to $d(Z^p,q_{r,q+1-r}) = B^p,q_r \subseteq Z^{p+1,q-1}_{r-1} + B^p,q_r$, which is equivalent to $\beta_{p,q,r}$ being an isomorphism.

(2) We recall that the isomorphism $E^p,q_{r+1} \cong H^p,q, (E_{r+1}, d_r)$ (see [M] Proof of Theorem 2.6) is induced from some canonical projections and inclusions. If $d_r = 0$, $H^p,q, (E_{r+1}, d_r) \cong E^p,q_r$ and we have a commutative diagram

\[
\begin{array}{ccc}
E^p,q_{r+1} & \cong & E^p,q_r \\
\downarrow \beta_{p,q,r} & & \downarrow \alpha_{p,q,r} \\
Z^p,q_{r-1+1} & \cong & B^p,q_r ^{p,q}
\end{array}
\]

By (1), $\beta_{p,q,r}$ is an isomorphism and hence $\alpha_{p,q,r}$ is an isomorphism.

\[\square\]

Definition 2.8. Fix a pair of integers $(p, q)$. For nonzero $\xi = \sum_i \xi_i \in \bigoplus_{i \geq 0} A^{p+i,q-i}$ where $\xi_i \in A^{p+i,q-i}$, let $i_0 = \min\{\xi_i \neq 0\}$. We call $\xi_{i_0}$ the leading term of $\xi$ and denote it as $\ell^p,q(\xi)$. We define $\ell^p,q(0) = 0$. For $r \geq 1$, $p, q \in \mathbb{Z}$, let

\[E^p,q_r := \left\{ \xi = \xi_0 + \xi_1 + \cdots + \xi_{r-1} \mid \xi_i \in A^{p+i,q-i}, d\xi = d'\xi_{r-1} \notin \text{Im}d', \ell^p,q(\eta) \neq \xi_0 \text{ for all } d\text{-closed } \eta \right\}\]

and

\[E^{p,q}_{r-1} := B^p,q_r - (Z^{p+1,q-1}_{r-1} + B^p,q_r)\]

Lemma 2.9. Fix $r_0 \geq 1$.

1. If the map $\alpha_{p,q,r}$ is an isomorphism for all $p, q \in \mathbb{Z}$, $r \geq r_0$, then $E^p,q_r = \emptyset$ for all $p, q \in \mathbb{Z}$, $r \geq r_0$.

2. If the map $\alpha_{p,q,r_0}$ is not an isomorphism, then $E^{p,q}_{r_0} \neq \emptyset$.

Proof. Note that by Lemma 2.4, the surjectivity of $\alpha_{p,q,r}$ is equivalent to the condition

\[Z^p,q_r = Z^p,q_{r+1} + Z^{p+1,q-1}_{r-1} + B^p,q_r = Z^{p+1,q-1}_{r-1} + Z^p,q_{r+1} + Z^p,q_r.\]

(1) Suppose that $\alpha_{p,q,r}$ is an isomorphism for all $r \geq r_0$. Then $Z^p,q_i = Z^p,q_{i+1} + Z^{p+1,q-1}_{i-1}$ for all $i \geq r_0$. Assume that $E^{p,q}_{r_0} \neq \emptyset$ for some $r \geq r_0$, $p, q \in \mathbb{Z}$. Let $\xi \in E^{p,q}_{r_0}$. By definition, $Z^p,q_{q+2} = Z^p,q_{q+3} = \cdots = Z^p,q_{\infty}$. So we may take $j > r$ such that $Z^p,q_j = Z^p,q_{\infty}$. Note that $\xi \in Z^p,q_j$. Using the relation above, we may write $\xi = \eta_1 + \eta_2$ where $\eta_1 \in Z^p,q_j$, $\eta_2 \in Z^{p+1,q-1}_{j-2} + \cdots + Z^{p+1,q-1}_{r_0-1}$. Since $\ell^p,q(\xi) \neq 0$, by comparing the degrees of both sides of $\xi = \eta_1 + \eta_2$, we have $\ell^p,q(\xi) = \ell^p,q(\eta_1)$. But $d\eta_1 = 0$ which contradicts to the fact that $\ell^p,q(\xi)$ is not the leading term of any $d$-closed element.

(2) Fix $r \geq 1$. Suppose that $\alpha_{p,q,r}$ is not an isomorphism, then $Z^p,q_r + Z^{p+1,q-1}_{r-1} \subseteq Z^p,q_r$.

Let $\xi = \xi_0 + \cdots + \xi_k \in Z^p,q_r - (Z^{p,q}_{r+1} + Z^{p+1,q-1}_{r-1})$ where $\xi_i \in A^{p+i,q-i}$.
If \( k > r - 1 \), let \( \xi' = \xi_r + \xi_{r+1} + \cdots + \xi_k \in F^{p+r}A^{p+q} \subset F^{p+1}A^{p+q} \). We have
\[
d\xi' = d\xi_r + \cdots + d\xi_k \in F^{p+r}A^{p+q+1} = F^{(p+1)+(r-1)}A^{(p+1)+(q-1)+1}\]
which means that \( \xi' \in \mathbb{Z}_{r+1}^{p+q+1} \). Let \( \xi'' \in \mathbb{Z}_{r+1}^{p+q+1} \). If \( \xi'' \in \mathbb{Z}_{r+1}^{p+1,q-1} + \mathbb{Z}_{r+1}^{p,q-1} \), then \( \xi = \xi' + \xi'' \in \mathbb{Z}_{r+1}^{p+q} + \mathbb{Z}_{r+1}^{p+1,q-1} \) which contradicts to our assumption. Therefore \( \xi'' = \xi_0 + \cdots + \xi_{r-1} \in \mathbb{Z}_r^{q} - (\mathbb{Z}_{r+1}^{p,q} + \mathbb{Z}_{r+1}^{p+1,q-1}) \). Hence we may assume \( \xi = \xi_0 + \cdots + \xi_{r-1} \).

(i) Since \( \xi \in \mathbb{Z}_0^{q} \), by definition, \( d\xi \in F^{p+r}A^{p+q+1} \). But \( d(\xi_0 + \cdots + \xi_{r-2}) + d''\xi_{r-1} \in A^{p,q+1} \cap A^{p,q+1} \cap \cdots \cap A^{p+1,q+r-2} \). This forces \( d(\xi_0 + \cdots + \xi_{r-2}) + d''\xi_{r-1} = 0 \) and hence \( d\xi = d'\xi_{r-1} \).

(ii) If \( d'\xi_{r-1} = d''\eta_r \) for some \( \eta_r \in A^{p+r,q-r} \), then \( d(\xi - \eta_r) = d'\xi_{r-1} - d'\eta_r - d''\eta_r = -d''\eta_r \in F^{(p+1)(r+1)}A^{p+q+1} \). Since \( \eta_r \in F^{p}A^{p+q} \) and \( d\eta_r \in A^{p+r,q+r-1} \cap \mathbb{Z}_r^{p,q} \cap \mathbb{Z}_r^{p+1,q-1} \), we have \( \eta_r \in \mathbb{Z}_r^{p,q} \). Therefore \( \xi = (\xi - \eta_r) + \eta_r \in \mathbb{Z}_r^{p,q} + \mathbb{Z}_r^{p+1,q-1} \) which is a contradiction. Hence \( d'\xi_{r-1} \notin \text{Im}d'' \).

(iii) If \( \xi_0 \) is the leading term of a \( d \)-closed form \( \tau \in F^{p}A^{p+q} \), then \( \xi - \tau \in F^{p+1}A^{p+q} \) and \( d(\xi - \tau) = d\xi \in F^{p+r+1}A^{p+q+1} = F^{(p+1)+(r-1)}A^{p+q+1} \). Hence \( \xi - \tau \in \mathbb{Z}_r^{p+1,q-1} \). Then \( \xi = \tau + (\xi - \tau) \in \mathbb{Z}_r^{p+1,q-1} \) which is a contradiction.

Hence \( \xi \in \mathbb{E}_r^{p,q} \).

\[ \square \]

**Lemma 2.10.**

1. \( \mathbb{E}_0^{p,q} = \emptyset \) if and only if \( \beta_{p,q,0} \) is an isomorphism.
2. For \( r \geq 1 \), if \( \mathbb{E}_r^{p-r,q+r-1} = \emptyset \), then \( \beta_{p,q,r} \) is an isomorphism.
3. For \( r \geq 1 \), if \( \mathbb{E}_r^{p-r,q+r-1} \neq \emptyset \), then \( \beta_{p,q,j} \) is not an isomorphism for \( j = 1, 2, \ldots, r \).

**Proof.** We note that \( \beta_{p,q,r} \) is an isomorphism if and only if \( B_0^{p,q} \subset \mathbb{Z}_r^{p+1,q-1} + B_{r-1}^{p,q} \).

(1) This follows from the definition.

(2) Assume that \( \beta_{p,q,r} \) is not an isomorphism. Then there exists \( \xi \in B_r^{p,q} = (\mathbb{Z}_r^{p+1,q-1} + B_{r-1}^{p,q}) \). So \( \xi = d\eta \) for some \( \eta \in F^{p-r}A^{p+q} \). Let
\[ \eta = \eta_0 + \eta_1 + \cdots + \eta_k \]
where \( \eta_i \in A^{p-r+i,q+r-1} \).

If \( k \geq r \), let \( \eta' = \eta_r + \cdots + \eta_k \in F^pA^{p+q} \subset F^{(p-r)}A^{p+q+1} \). Then \( d\eta' \in F^pA^{p+q} \cap d(F^pA^{p+q} \cap d(F^{p-r}A^{p+q+1}) = B_r^{p,q} \). If \( d(\eta - \eta') \in \mathbb{Z}_r^{p,q} + B_{r-1}^{p,q} \), then \( \xi = d(\eta - \eta') + d\eta' \in \mathbb{Z}_r^{p,q} + B_{r-1}^{p,q} \). Hence we may assume \( \xi = d\eta \) where \( \eta = \eta_0 + \cdots + \eta_{r-1} \).

(i) Comparing the degrees of \( \xi \) and \( d\eta \), we see that \( d\eta = d'\eta_{r-1} \).

(ii) If \( \eta_0 = 0 \), then \( \xi = d(\eta_1 + \cdots + \eta_{r-1}) \in F^pA^{p+q} \cap d(F^{p-r}A^{p+q+1}) = B_r^{p,q} \) which is a contradiction. So \( \eta_0 \neq 0 \).

(iii) If \( \eta_0 \) is the leading term of a \( d \)-closed form \( \eta' \), then \( \eta - \eta' \in F^{p-r+1}A^{p+q} \) and \( \xi = d\eta = d(\eta - \eta') \) for some \( \eta \in A^{p,q-1} \), then \( \xi = d'\eta \) which is a contradiction. Hence \( \eta_0 \) is not the leading term of any \( d \)-closed form.

(iv) Assume that \( \mathbb{E}_r^{p-r,q+r-1} \neq \emptyset \). Let \( \eta = \eta_0 + \cdots + \eta_{r-1} \in \mathbb{E}_r^{p-r,q+r-1} \). Since \( d\eta \in B_r^{p,q} \), if \( d\eta \notin \mathbb{Z}_r^{p+1,q-1} + B_{r-1}^{p,q} \), \( \beta_{p,q,r} \) is not an isomorphism. So we may assume \( d\eta = d'\eta_{r-1} = \xi' + d\eta' \) where \( \xi' \in \mathbb{Z}_r^{p+1,q-1} \) and \( d\eta' \in B_{r-1}^{p,q} \). Let \( \eta' = \eta_1 + \eta_2 + \cdots + \eta_r \), where
$\eta' \in \mathbb{A}^{n-r+i,g+r-1-i}$. The degree of $d'\eta_{r-1}$ is $(p,q)$, so by comparing degrees of both sides of $d'\eta_{r-1} = \xi + d\eta'$, we get

$$d'\eta_{r-1} = d'\eta'_{r-1} + d''\eta'_{r-1} + d''\eta_{r-1} = 0.$$ 

If $d'\eta'_{r-1} \in \text{Im}d''$, then $d'\eta_{r-1} \in \text{Im}d''$ which contradicts to the fact that $\eta \in \mathcal{E}_{p,q}^{r,q+r-1}$. So $d'\eta'_{r-1} \notin \text{Im}d''$. Note that if $\eta'_{r-1}$ is the leading term of a $d$-closed element $\tau$, we may write $\tau = \eta'_{r-1} + \tau_r + \cdots + \tau_k$ for some $k > r - 1$ and each $\tau_i \in \mathbb{A}^{p,q+r-1-i}$. Then comparing the degrees of $d'\tau = -d''\tau$, we get $d'\eta_{r-1} = -d''\tau$, which contradicts to the fact that $d'\eta_{r-1} \notin \text{Im}d''$.

From the above verification, we see that $\eta'_{r-1} \in \mathcal{E}_{p}^{r-1,q}$. Assume that $d\eta'_{r-1} \in Z_0^{p+1,q-1} + B_0^{p,q}$. Write $d\eta'_{r-1} = \gamma + d\sigma$ where $\gamma = \gamma_1 + \cdots \in Z_0^{p+1,q-1}$, $\gamma_1 \in \mathbb{A}^{p+1,q-1}$, $\sigma = \sigma_0 + \sigma_1 + \cdots \in B_0^{p,q}$. Since the degree of $d\eta'_{r-1}$ is $(p,q)$, comparing the degrees of both sides of $d\eta'_{r-1} = \gamma + d\sigma$, we get $d\eta'_{r-1} = d''\sigma_0$ which contradicts to the fact that $\eta'_{r-1} \in \mathcal{E}_{p}^{r-1,q}$. Therefore $d\eta'_{r-1} \notin Z_0^{p+1,q-1} + B_0^{p,q}$ and hence $\beta_{p,q,1}$ is not an isomorphism.

\[\square\]

**Theorem 2.11.** Suppose that $(A = \oplus_{p,q \geq 0} \mathbb{A}^{p,q}, d', d'')$ is a double complex and $r \geq 1$. The spectral sequence $(\mathcal{E}^{p,q}_{r})$ induced by $A$ degenerates at $E_r$ but not at $E_{r-1}$ if and only if the following conditions hold:

1. $E^{p,q}_{r} = 0$ for all $p,q \in \mathbb{Z}, k \geq r$ and
2. $E^{p,q}_{r-1} \neq 0$ for some $p,q$.

**Proof.** Suppose that $(E^{p,q}_{r})$ degenerates at $E_r$ but not at $E_{r-1}$ for some $r \geq 1$. By Proposition 2.7, $\alpha_{p,q,i}$ is an isomorphism for all $p,q \in \mathbb{Z}, i \geq r$. Then by Lemma 2.9 $\mathcal{E}^{p,q} = 0$ for all $p,q \in \mathbb{Z}, i \geq r$. Since $d'_{r-1} \neq 0$, by Proposition 2.7, there are some $p,q \in \mathbb{Z}$ such that $\beta_{p,q,r-1}$ is not an isomorphism. Then by Lemma 2.10 $E^{p,q}_{r-1} \neq 0$.

Conversely, suppose that (1) and (2) hold. By Lemma 2.10 $\beta_{p,q,i}$ is an isomorphism for all $p,q \in \mathbb{Z}, k \geq r$. Then by Proposition 2.7 $d_k = 0$ for $k \geq r$. For the case $r = 1$, by definition, $\mathcal{E}^{p,q} = 0$ implies that $\beta_{p,q,1}$ is not an isomorphism. And hence by Proposition 2.7 $d_0 \neq 0$. For the case $r \geq 2$, if $\beta_{p,q,r-1}$ is an isomorphism for all $p,q \in \mathbb{Z}$, by Proposition 2.7 $d_{r-1} \neq 0$. Then we have $d_k = 0$ for $k \geq r - 1$. By the proof above, $E^{p,q}_{r-1} = 0$ for $k \geq r - 1$. In particular, $\mathcal{E}^{p,q}_{r-1} = 0$ for all $p,q \in \mathbb{Z}$ which contradicts to our assumption (2). Therefore there exist some $p_0, q_0$ such that $\beta_{p_0,q_0,1}$ is not an isomorphism. By Proposition 2.7, $d_{r-1} \neq 0$.

\[\square\]

**Definition 2.12.** We say that a double complex $(A, d', d'')$ satisfies the d"{d}'-lemma at $(p,q)$ if

$$\text{Im}d' \cap \ker d'' \cap A^{p,q} = \ker d' \cap \text{Im}d'' \cap A^{p,q} = \text{Im}d' \cap \ker d'' \cap A^{p,q}$$

and $A$ satisfies the d"{d}'-lemma if $A$ satisfies the d"{d}'-lemma at $(p,q)$ for all $(p,q)$.

Now we can give a proof of the main result Theorem 1.1.

**Proof.** Note that by definition, d"{d}'-lemma implies that $\text{Im}d' \cap \ker d'' \cap A^{p,q} = \text{Im}d' \cap \text{Im}d'' \cap A^{p,q}$ for all $p,q$. Since $(E^{p,q}_{r})$ does not degenerate at $E_0$, $\beta_{p,q,0}$ is not an isomorphism for some $p,q$, hence by Lemma 2.10 $E^{p,q}_{0} \neq 0$. Assume that $E^{p,q}_{r-1} \neq 0$ for some $p,q \in \mathbb{Z}, r \geq 1$. Then there is $a = \sum_{i=0}^{r-1} \alpha_i \in E^{p,q}_{i}$ where $\alpha_i \in A^{p,i+q-1}$. From the condition $d\alpha = d'\alpha_{r-1}$, we have $d''\alpha_{r-1} = -d'\alpha_{r-2}$ and hence $d''d\alpha = -d''d'd\alpha_{r-1} = 0$. So $d\alpha = d'\alpha_{r-1} \in \text{Im}d' \cap \ker d'' \cap A^{p,q} = (\text{Im}d' \cap \text{Im}d'') \cap A^{p,q}$. But by the definition of $E^{p,q}_{r}$, $d'\alpha_{r-1} \notin \text{Im}d''$ which leads to a contradiction. Therefore by Theorem 2.11 $(E^{p,q}_{r})$ degenerates at $E_1$.

\[\square\]
In the following, we apply the main result to prove the $E_1$-degeneration of a spectral sequence of bi-generalized Hermitian manifolds. We refer the reader to [G1, C] for generalized complex geometry, and to [CHT] for bi-generalized complex manifolds. We give a brief recall here. A bi-generalized complex structure on a smooth manifold $M$ is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ where $\mathcal{J}_1, \mathcal{J}_2$ are commuting generalized complex structures on $M$. A bi-generalized complex manifold is a smooth manifold $M$ with a bi-generalized complex structure. A bi-generalized Hermitian manifold $(M, \mathcal{J}_1, \mathcal{J}_2, \mathcal{G})$ is an oriented bi-generalized complex manifold $(M, \mathcal{J}_1, \mathcal{J}_2)$ with a generalized metric $\mathcal{G}$ which commutes with $\mathcal{J}_1$ and $\mathcal{J}_2$. We define

$$U^{p,q} := U^p_1 \cap U^q_2$$

where $U^p_1, U^q_2 \subset \Gamma(\Lambda^p \mathcal{T}M \otimes \mathbb{C})$ are eigenspaces of $\mathcal{J}_1, \mathcal{J}_2$ associated to the eigenvalues $ip$ and $iq$ respectively and $\mathcal{T}M = TM \oplus T^* M$ is the generalized tangent space. It can be shown that the exterior derivative $d$ is an operator from $U^{p,q}$ to $U^{p+1,q+1} \oplus U^{p+1,q-1} \oplus U^{p-1,q+1} \oplus U^{p-1,q-1}$ and we write

$$\delta_+ : U^{p,q} \to U^{p+1,q+1}, \delta_- : U^{p,q} \to U^{p+1,q-1}$$

for the projection of $d$ into corresponding spaces.

**Definition 2.13.** On a bi-generalized Hermitian manifold $M$, there is a double complex $\{(A, d', d'')\}$ given by

$$A^{p,q} := U^{p+q,p-q}, d' = \delta_+, d'' = \delta_-$$

We call the spectral sequence $(E^*_+)$ associated to this double complex the $\partial_1$-Hodge-de Rham spectral sequence.

By Theorem [1.1] we have the following result.

**Theorem 2.14.** Suppose that $M$ is a compact bi-generalized Hermitian manifold which satisfies the $\delta_+\delta_-$-lemma and has positive dimension. Then the $\partial_1$-Hodge-de Rham spectral sequence degenerates at $E_1$.

Now we give a proof of the $E_1$-degeneration of the $\partial_1$-Hodge-de Rham spectral sequence.

**Proof.** Since $\bigoplus_{p,q} U^{p,q} = \Omega^*(M) \otimes \mathbb{C}$ (see [Ca07], pg 36) where $\Omega^*(M)$ is the collection of smooth forms on $M$, some $U^{p,q}$ is not empty. The space $U^{p,q}$ is a $C^\infty(M, \mathbb{C})$-module where $C^\infty(M, \mathbb{C})$ is the ring of complex-valued smooth functions on $M$, and $M$ has positive dimension, therefore $U^{p,q}$ is an infinite dimensional complex vector space. If $\delta_-$ is a zero map, we have $H^{p,q}_{\delta_-}(M) = U^{p,q}$ for all $p, q$. But $M$ is compact, this contradicts to the fact that $H^{p,q}_{\delta_-}(M)$ is finite dimensional([CHT, Theorem 2.14, Corollary 3.11]). Hence $\delta_-$ is not the zero map. and the spectral sequence does not degenerate at $E_0$. Since we assume that $M$ satisfies the $\delta_+\delta_-$-lemma, by Theorem [1.1] the spectral sequence degenerates at $E_1$. 

\[\square\]

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