On group properties and reality conditions
of UOSp(1|2) gauge transformations

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For osp(1|2; C) graded Lie algebra, which proper Lie subalgebra is su(2), we con-
sider the Baker-Campbell-Hausdorff formula and formulate a reality condition for
the Grassmann-odd transformation parameters that multiply the pair of odd gen-
erators of the graded Lie algebra. Utilization of su(2)-spinors clarifies the nature
of Grassmann-odd transformation parameters and allow us an investigation of the
 corresponding infinitesimal gauge transformations. We also explore action of the cor-
responding group element of UOSp(1|2) on an appropriately graded representation
space and find that the graded generalization of hermitian conjugation is compatible
with the Dirac adjoint. Consistency of generalized (graded) unitary condition with
the proposed reality condition is shown.

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I. INTRODUCTION

A natural extension of the Lie algebras, which underlie the modern gauge theory, are graded Lie algebras introduced and studied to some extent, for example, in the articles [1, 2, 3]. In this paper we study those properties of the graded extension, $osp(1|2; \mathbb{C})$, of $su(2)$ Lie algebra, which are pivotal for the purposes of constructing a meaningful gauge theory of the Yang-Mills type (see, e.g., [4, 5, 6]). The explicit form of $osp(1|2; \mathbb{C})$ defining relations (written as in the articles [5, 6, 7]) utilizes the Pauli matrices and strongly suggests a relation to spinors. Exponentiating the algebra to obtain the graded Lie group $UOSp(1|2)$, we observe the necessity of introduction of anticommuting (Grassmann-odd) spinors, which multiply the odd generators of the graded Lie algebra. We study some of the infinitesimal properties of composition law of the group transformations and show how to formulate a reality condition for the Grassmann-odd spinors. Action of the corresponding group element of $UOSp(1|2)$ on an appropriately graded representation space is explored and consistency of the graded generalization of hermitian conjugation with the Dirac adjoint is demonstrated. Finally, for the example of Grassmann algebra on two generators (generalization to the case of even or infinite number of generators of Grassmann algebra is straightforward), we show that the reality condition is compatible with the proper generalization of unitary condition [7], thus, making necessary preparations for an investigation of the gauge invariance of the proposed field strength [5, 6] for such a graded Yang-Mills theory.

II. GRADED LIE ALGEBRA $OSP(1|2; \mathbb{C})$

The algebra $osp(1|2; \mathbb{C})$ is a graded extension of $su(2)$ by a pair of odd generators, $\tau_A$, which anticommute with one another and commute with the three even generators, $T_a$, of $su(2)$. It is customary to assign a degree, $\text{deg} T_a$, to the even ($\text{deg} T_a = 0$) and odd ($\text{deg} \tau_A = 1$) generators. We use the square brackets to denote the commutator and the curly ones to denote the anticommutator. The defining relations have the form [5, 6, 7] (also cf. [3, 8, 9]):

$$[T_a, T_b] = i\varepsilon_{abc} T^c, \quad [T_a, \tau_A] = \frac{1}{2} (\sigma^a)_A^B \tau_B, \quad \{\tau_A, \tau_B\} = \frac{i}{2} (\sigma_a)_{AB} T_a.$$  \hspace{1cm} (2.1)
Summation is assumed over all repeated indices. Lowercase Roman indices from the beginning of the alphabet run from 1 to 3; uppercase Roman indices run over 1 and 2; \( \delta_{ab} = \delta^{ab} \) (\( \delta_{ab} = \delta_{ba} \)), \( \varepsilon_{abc} \) (\( \varepsilon_{123} = \varepsilon^{123} = 1 \)) and \( \epsilon_{AB} \) (\( \epsilon_{12} = \epsilon^{12} = 1 \)) are the three dimensional identity matrix and the Levi-Civita totally antisymmetric symbols in three and two dimensions, respectively; the matrices \( (\sigma_a)_{AB} = (\sigma^a)_{BA} = \delta^{ab}(\sigma_b)_{AB} = \delta^{ab}(\sigma_b)_{A}^{C} \epsilon_{CB} \) are just the usual Pauli matrices:

\[
(\sigma^a)_{A}^{B} = (\sigma_a)_{A}^{B} = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & i \\
i & 0
\end{pmatrix},
\]

\[
(\sigma^a)_{AB} = (\sigma_a)_{AB} = \begin{pmatrix}
-1 & 0 \\
0 & 1 \\
i & 0 \\
0 & -i
\end{pmatrix},
\]

We use the Levi-Civita symbols in two dimensions to raise and lower uppercase Roman indices paying attention to their antisymmetric properties:

\[
\Sigma = \|\epsilon^{AB}\| = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} = \|\epsilon_{AB}\| = -\Sigma^{-1}.
\]

Note that, as concerned to these indices, we are working with two-component spinors and adopt Penrose’s conventions of the book [10]. We follow those conventions even when complex conjugation of spinor and pseudo-conjugation of Grassmann quantities are involved.

In the adjoint representation [11] the matrices \( T_a \) and \( \tau_A \) can be written as follows (solid lines are drawn to emphasize their block structure):

\[
T_1 = \begin{pmatrix}
0 & i & 0 & 0 \\
0 & 0 & 0 & 0 \\
-i & 0 & 0 & 0 \\
0 & 0 & 1/2 & 0
\end{pmatrix},
\]

\[
T_2 = \begin{pmatrix}
0 & -i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i/2
\end{pmatrix},
\]

\[
T_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2
\end{pmatrix},
\]

\[
T_4 \equiv \tau_1 = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -10 & 0 \\
0 & 0 & -i & 0 \\
-i & 0 & 0 & 0
\end{pmatrix},
\]

\[
T_5 \equiv \tau_2 = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
i & 0 & 0 & 0
\end{pmatrix}.
\]
Let us denote $T_4 = \tau_1$, $T_5 = \tau_2$ and employ lowercase Greek indices from the beginning of the alphabet ($\alpha$, $\beta$, etc.) to run over the whole set, $T_\alpha$, of the generators of $osp(2/1; \mathbb{C})$.

We then find \cite{11} that the non-degenerate super-Killing form, $B(T_\alpha, T_\beta)$, is given by

\begin{equation}
B(T_\alpha, T_\beta) = \frac{2}{3} \text{str}(T_\alpha T_\beta) = \begin{pmatrix}
\delta_{ab} & 0 \\
0 & i\epsilon_{AB}
\end{pmatrix},
\end{equation}

where the supertrace operation is adopted from \cite{12, pp. 18-19, 42}.

It turns out that not all of the $osp(1|2; \mathbb{C})$ algebra generators are hermitian. A proper generalization of the hermitian conjugation \cite{13} (graded adjoint) is denoted by $(\dagger)$: on the even generators the operation coincides with ordinary hermitian conjugation $(\dagger)$ while the odd ones obey more complicated relations (see below and at the end of Sec. V). Following the paper \cite{14}, we call them the grade star hermiticity conditions:

\begin{equation}
\tau^\dagger_{\pm} = \pm \tau^\mp,
\end{equation}

where we denoted $\tau_{\pm} = \tau_1 \pm i\tau_2$.

Let us consider complex-valued matrices divided into blocks according to the scheme (cf. (2.2) and (2.3)):

\begin{equation}
M_{\text{even}} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \quad \text{and} \quad M_{\text{odd}} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},
\end{equation}

where, for the purposes of this paper, $B$ and $C$ are $2\times3$ rectangular blocks and $A$ and $D$ are $3\times3$ and $2\times2$ square blocks, respectively. On these matrices the supertrace operation gives $\text{str}M_{\text{even}} = \text{tr}A - \text{tr}D$ and $\text{str}M_{\text{odd}} = 0$ (here “tr” denotes the ordinary trace) while the grade star hermiticity condition reads

\begin{equation}
M^\dagger_{\text{even}} = \begin{pmatrix} A^\dagger & 0 \\ 0 & D^\dagger \end{pmatrix} \quad \text{and} \quad M^\dagger_{\text{odd}} = \begin{pmatrix} 0 & C^\dagger \\ B^\dagger & 0 \end{pmatrix}.
\end{equation}

We shall also use multiplication of algebra generators by scalars. Such an operation must take into account that Grassmann-odd scalars anticommute with the odd algebra generators while commute with complex numbers and the even algebra generators \cite{15}. The following construction possesses all of these properties. Let $a$ be a scalar and $\text{deg} a$ be its degree ($0$ or $1$ depending on whether it is Grassmann-even or Grassmann-odd, respectively). Then multiplication by $a$ is defined as follows:
\[ aM_{\text{odd}} = \left( \begin{array}{cc} a & 0 \\ 0 & (-1)^{\deg a}a \end{array} \right) \left( \begin{array}{c} B \\ C \end{array} \right) = (-1)^{\deg a} \left( \begin{array}{cc} a & 0 \\ 0 & (-1)^{\deg a}a \end{array} \right) = \]

\[ aM_{\text{even}} = \left( \begin{array}{cc} a & 0 \\ 0 & (-1)^{\deg a}a \end{array} \right) \left( \begin{array}{c} A \\ D \end{array} \right) = \left( \begin{array}{cc} A & 0 \\ 0 & D \end{array} \right) \left( \begin{array}{cc} a & 0 \\ 0 & (-1)^{\deg a}a \end{array} \right) = M_{\text{even}}a. \]

### III. THE GROUP PROPERTY

Given a Lie algebra one can turn over to a Lie group by exponentiating the generators multiplied by transformation parameters. This, in a usual fashion, gives us the gauge transformations. In the case of a graded Lie algebra we are faced with a problem: anticommutators seem to rule out the application of the Baker-Campbell-Hausdorff formula, which is necessary to prove that subsequent transformations do not leave the group manifold. This problem is solved via introduction of Grassmann-odd parameters (cf. [1]). In the case under consideration these are Grassmann-odd \( su(2) \)-spinors \( \xi^A, \theta^A \), etc., which multiply the odd generators. They are included on equal footing with ordinary (Grassmann-even) parameters \( \epsilon^a \) multiplying the even generators (hopefully, there will not be confusion about use the same kernel letter, \( \epsilon \), to denote a Grassmann-even transformation parameter and the Levi-Civita totally antisymmetric symbol in three dimensions). By definition, \( \xi^A, \theta^A \), etc. satisfy

\[ [\epsilon^a, \theta^A] = 0, \{\xi^A, \xi^B\} = \{\theta^A, \theta^B\} = 0, \{\xi^A, \theta^B\} = 0. \]

Then, the necessary relations can be given in terms of commutators only:

\[ [\xi^A \tau_A, \theta^B \tau_B] = -\frac{i}{2} \{\xi^A \theta^B\} (\sigma^a)_{AB} T_a, \]

where \( \xi^A \theta^B = 1/2(\xi^A \theta^B + \xi^B \theta^A) \) and \( \xi^A \theta^B = 1/2(\xi^A \theta^B - \xi^B \theta^A) \) are convenient shorthand notations. This result was obtained using anticommutator for odd generators in definition (2.1). Using a fundamental fact of spinor algebra, \( \epsilon_{AB} \epsilon_{CD} + \epsilon_{AC} \epsilon_{DB} + \epsilon_{AD} \epsilon_{BC} = 0 \), one can calculate

\[ \xi^A \theta^B = \frac{1}{2}(\xi^A \theta^B), \epsilon_{AB} \]

From symmetry of \( (\sigma^a)_{AB} \) in the uppercase indices, it then follows that

\[ [\xi^A \tau_A, \theta^B \tau_B] = -\frac{i}{2} \xi^A \theta^B (\sigma^a)_{AB} T_a \equiv -\frac{i}{2} \xi^A \theta^B (\sigma^a)_{AB} T_a \quad (3.1) \]
and, in particular, the commutator $[\theta^A \tau_A, \theta^B \tau_B]$ vanishes identically. One can also calculate

$$[\epsilon^a T_a, \epsilon^b T_b] = i \kappa^a \epsilon^b \epsilon_{abc} T_c \quad \text{and} \quad [\epsilon^a T_a, \theta^A \tau_A] = \tilde{\theta}^B \tau_B,$$

where $2\tilde{\theta}^B = \epsilon^a \theta^A (\sigma_a)_A^B$ is again a Grassmann-odd transformation parameter.

Group elements of UOSp(1|2) are obtained by exponentiating the algebra

$$U(\varepsilon, \theta) = \exp(i(\epsilon^a T_a + \theta^A \tau_A)) \quad (3.3)$$

and the Baker-Campbell-Hausdorff formula,

$$\exp(M) \exp(N) = \exp(M + N + \frac{1}{2}[M, N] + \ldots), \quad (3.4)$$

may be applied to determine motion in the parameter space under a (left) multiplication with a group element $U(\kappa, \xi)$:

$$U(\varepsilon', \theta') = U(\kappa, \xi) U(\varepsilon, \theta).$$

**IV. INFINITESIMAL TRANSFORMATIONS AND REALITY CONDITIONS**

Let us examine expression (3.4) restricting ourselves by taking into account the first non-trivial contribution, i.e. the two-fold commutator $[M, N]$. Writing $M = i(\epsilon^a T_a + \theta^A \tau_A)$ and $N = i(\kappa^a T_a + \xi^A \tau_A)$, we have

$$i(\epsilon'^a T_a + \theta'^A \tau_A) = M + N + \frac{1}{2}[M, N] + \ldots,$$

where dots denote the sum of linear combinations of $k$-fold ($k > 2$, $k \in \mathbb{N}$) commutators of $M$ and $N$ [16]. Substituting expressions for $M$, $N$ and using (3.1), we obtain after some algebra

$$\epsilon'^a = \epsilon^a + \kappa^a - \frac{1}{2} \kappa_b \epsilon_c \epsilon^{bca} + \frac{1}{4} [\xi^A, \theta^B] (\sigma^a)_{AB} + \ldots,$$

$$\theta'^A = \theta^A + \xi^A + \frac{i}{4} (\kappa_b \theta^B - \epsilon_b \xi^B) (\sigma^b)_A^B + \ldots \quad (4.1)$$

Here again dots denote the contribution from the sum of linear combinations of $k$-fold ($k > 2$, $k \in \mathbb{N}$) commutators. The first three summands in the first row of formula (4.1) reflect the non-commutative character of the proper Lie subalgebra, $su(2)$, of $osp(1|2; \mathbb{C})$ the last one being contribution from the odd part of the graded Lie algebra. The last summand in
the second row of the formula is obviously a Grassmann-odd quantity, and it reflects the non-commutative property of the even and odd parts of the graded Lie algebra.

In the view of intended applications, contribution from Grassmann-odd part of the algebra into the law of composition of Grassmann-even parameters needs to be investigated in more detail. First, let us calculate that

\[ 2[\xi^A, \theta^B](\sigma^a)_{AB} = \xi_A \epsilon^{AB} (\sigma^a)_B C \theta^C - \theta_A \epsilon^{AB} (\sigma^a)_B C \xi^C = \xi^t \Sigma \sigma^a \theta - \theta^t \Sigma \sigma^a \xi, \quad (4.2) \]

where we employed some self-evident matrix notations; the superscript (t) denotes transpose. Comparing the result (4.2) and a description of su(2)-spinors of 3D Euclidean space in the book [17, p. 48], one immediately realizes that the last term of the first equation in system (4.1) is, in general, a complex vector of 3D Euclidean space. Second, the representation (4.2) tells us that components of this vector vanish if \( \xi_A = \theta_A \) as required by a property of a one-parameter subgroup of transformations (3.3). Finally, this vector also has all components equal to zero if \( \xi_A = -\theta_A \). This shows that the inverse of the group element \( U(\varepsilon, \theta) \) has the form

\[ U^{-1}(\varepsilon, \theta) = \exp[-i(\varepsilon^a T_a + \theta^A \tau_A)]. \quad (4.3) \]

If one intends, as customarily done in a meaningful Yang-Mills theory, to treat \( \varepsilon^a, \kappa^a \), etc. as real-valued Grassmann-even transformation parameters, then it is necessary to impose some conditions on the su(2)-spinors \( \xi_A, \theta_A \), etc. in order to ensure that (4.2) will be a real 3D Euclidean vector. Such a condition must be compatible with transformation properties of the corresponding space of su(2)-spinors, \( \xi_A \), and take into account that its members are also Grassmann-odd quantities. In fact, this condition should involve a passage from an su(2)-spinor to its conjugate and, thus, rely on the definition of an anti-involution in the space of spinors (see, e.g. [17, p. 100]). Let us observe first that for a Grassmann algebra on one generator the last term in the first relation in (4.1) vanishes identically. This is a somewhat trivial situation. The next non-trivial one arises when all su(2)-spinors under consideration take values in a Grassmann algebra on two odd generators (more generally on even or infinite number of odd generators, cf. [7]), \( \beta_1 \) and \( \beta_2 \): \( \beta_1^2 = \beta_2^2 = 0, \beta_1 \beta_2 = -\beta_2 \beta_1 \) (see, e.g. [12, p. 7]). We employ lowercase Roman indices from the middle of the alphabet running over 1 and 2 to enumerate the decompositions of various quantities in the corresponding basis of the Grassmann algebra. Decomposing \( \xi_A \) and \( \theta_A \) into this basis one
obtains
\[ \xi_i A = \xi A \beta_i \] and \[ \theta_j B = \theta B \beta_j, \]
where \( \xi_i A \) and \( \theta_j B \) are ordinary, i.e. complex-valued Grassmann-even, \( su(2) \)-spinors of 3D Euclidean space, and summation over repeated indices is assumed. In this case we can write
\[ [\xi A, \theta B] (\sigma^a)_{AB} = 2 \beta_1 \beta_2 (\xi^i \Sigma^a \theta - \theta^i \Sigma^a \xi). \] (4.4)

Now we impose some additional conditions on \( su(2) \)-spinors \( \xi i A, \theta j B, \) etc. to ensure that (4.4) gives a real Grassmann-even 3D Euclidean vector. One way of doing so in a manner preserving all the spinor transformations properties (Majorana conditions) is to define \[ \xi_1 A = i C_A B' \xi_2 B', \theta_1 A = i C_A B' \theta_2 B', \] etc., (4.5)
where the ‘charge conjugation’ matrix \( C \) (\( \overline{CC} = -I \)) is given by
\[ C = ||C_A B' || = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) = ||\overline{C_A B'} || = \overline{C}. \]

In (4.5) a bar over the spinors in the left-hand sides of the relations and primes over the indices denote complex conjugation (\( \overline{\xi'} = (\xi A)^\ast \)). The ‘charge conjugation’ matrix, \( C_A B' \), is responsible for invariant preservation of spinor properties (for details see, e.g., the review article [18, pp. 108 – 109], where this object is denoted by \( \Pi_{\lambda\mu} \); also compare with treatment in [17]). Note that definitions (4.5) are essentially the proper generalization of reality conditions from complex numbers to spinors. As also seen from that definitions, each Grassmann-odd \( su(2) \)-spinor \( \xi A, \theta A, \) etc. is defined by a single ordinary \( su(2) \)-spinor. For the sake of notations denoting
\[ \eta_A = \xi_{\frac{1}{2}} A \] and \[ \vartheta_B = \theta_{\frac{1}{2}} B, \]
respectively, we write
\[ v^a \equiv \xi^i \Sigma^a \theta - \theta^i \Sigma^a \xi = i(\overline{\eta} C^t \Sigma^a \vartheta - \overline{\vartheta} C^t \Sigma^a \eta). \] (4.6)

On comparison with [17], one can check that \( v^a \) is indeed a real 3D Euclidean vector [11]. In components it reads:
\[ v^1 = i(\overline{\eta}_1 \vartheta_2 - \overline{\vartheta}_2 \eta_1 + \overline{\eta}_2 \vartheta_1 - \overline{\vartheta}_1 \eta_2), \]
\[ v^2 = \overline{\eta}_2 \vartheta_1 + \overline{\vartheta}_1 \eta_2 - \overline{\eta}_1 \vartheta_2 - \overline{\vartheta}_2 \eta_1, \]
\[ v^3 = i(\overline{\eta}_1 \vartheta_1 + \overline{\vartheta}_1 \eta_1 - \overline{\eta}_2 \vartheta_2 + \overline{\vartheta}_2 \eta_2). \] (4.7)
These are obviously real quantities and the vector $v^a$ vanishes if and only if $\eta_A = \pm \vartheta_A$ as required.

V. ACTION ON A REPRESENTATION SPACE

Having formulated meaningful reality conditions, we are in position to explore action of the group element (3.3) on a suitable representation vector space.

First, let us observe that because of definition of the matrix $U$ by its Taylor’s expansion, the fact that the generators $T_a$ and $\tau_A$ are block and off-block diagonal, respectively, their multiplication properties and those of the Grassmann-even and Grassmann-odd transformation parameters, it is easy to see that any matrix $U$ has a specific decomposition

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A$ is a $(r \times p)$ sub-matrix, $B$ is a $(s \times p)$ sub-matrix, $C$ is a $(r \times q)$ sub-matrix and $D$ is a $(s \times q)$ sub-matrix. Following nomenclature of the book [12], we call the matrix $U$ a $(p/q \times r/s)$ super-matrix. Moreover, the sub-matrices $A$ and $D$ have contributions only from an even number of $\tau s'$ multipliers and, hence, only even multipliers of Grassmann-odd transformation parameters $\theta s'$ are present there. Thus, elements of those sub-matrices are in the even subspace, $\mathbb{C}B_{L0}$, of the complex Grassmann algebra (for more details see the book [12, pp. 10–11]). The sub-matrices $B$ and $C$ by an analogous argument include an odd number of $\tau s'$ and $\theta s'$ multipliers and, hence, are in the odd subspace, $\mathbb{C}B_{L1}$, of the complex Grassmann algebra. Therefore, any such a super-matrix $U$ is an even super-matrix and by the results of the previous section such matrices form a supergroup. Furthermore, by construction any such a super-matrix is invertible.

Second, consider even super-column $\Psi$ ($(p/q \times 0/1)$ super-matrices) and super-row $\Phi$ ($(1/0 \times r/s)$ super-matrices) vectors:

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix},$$

where $\Psi_1$ and $\Phi_1$ are $(1 \times p)$ and $(r \times 1)$ sub-matrices, $\Psi_2$ and $\Phi_2$ are $(1 \times q)$ and $(s \times 1)$ sub-matrices, respectively. The elements of $\Psi_1$ and $\Phi_1$ are Grassmann-even and those of
\( \Psi_2 \) and \( \Phi_2 \) are Grassmann-odd entities. *Action of even super-matrices \( U \) on such even super-column(row) vectors transform them again into even super-column(row) vectors.*

Third, since the sub-matrices \( B \) and \( C \) are in the odd sub-space of \( \mathbb{C}B_{L1} \), one needs to modify the very notion of complex conjugation. Instead, the operation called pseudo-conjugation is to be used \( [7, 13] \), which is one of at least two inequivalent generalizations of complex conjugation to supernumbers (cf. \( [19] \)); it coincides with ordinary complex conjugation (denoted above by the asterisk (\( * \))) on Grassmann-even quantities (e.g. ordinary complex numbers) being there an involution and is an anti-involution on the Grassmann-odd ones. Following \( [7] \), it will be denoted by a superscript diamond (\( \diamond \)).

For the sake of argument let \( U \) be \((1/1 \times 1/1)\) matrices (see \( (5.1) \)), the actual size can be easily treated the same way, and let also \( \Psi \) be a \((1/1 \times 0/1)\) even super-column vector as regarded to the linear transformations defined below. The entries \( \Psi_1 \) and \( \Psi_2 \) themselves could be, for example, Dirac bispinors. Consider a linear transformation

\[
\begin{pmatrix}
\Psi'_1 \\
\Psi'_2
\end{pmatrix} = \begin{pmatrix}
A \Psi_1 + B \Psi_2 \\
C \Psi_1 + D \Psi_2
\end{pmatrix} \equiv U \Psi.
\]

Taking transposition of each line in \( (5.2) \) (it acts on \( \Psi \)'s) and pseudo-conjugate as well as introducing the (modified to supernumbers) Dirac conjugation by \( \bar{\Psi}'_i = (\Psi_i^t)^{\gamma_0} \), we obtain:

\[
\begin{pmatrix}
\bar{\Psi}'_1 & \bar{\Psi}'_2
\end{pmatrix} = \begin{pmatrix}
\bar{\Psi}_1 A^\diamond - \bar{\Psi}_2 B^\diamond \\
C^\diamond \bar{\Psi}_1 + D^\diamond \bar{\Psi}_2
\end{pmatrix},
\]

where the Grassmann character of the involved quantities has been taken into account. Recall that for any super-matrix \( U \) partitioned as in \( (5.1) \) the super-transpose is defined by

\[
U^{st} = \begin{pmatrix}
A^t & \mathbf{(-1)^{deg} U C^t} \\
\mathbf{(-1)^{deg} U B^t} & D^t
\end{pmatrix},
\]

where \((^t)\) denotes the ordinary transposition; for *even* super-column(row) vectors this implies:

\[
\Psi^{st} = \begin{pmatrix}
\Psi'_1 \\
\Psi'_2
\end{pmatrix} \quad \text{and} \quad \Phi^{st} = \begin{pmatrix}
\Phi'_1 \\
-\Phi'_2
\end{pmatrix}.
\]

It then follows that

\[
\begin{pmatrix}
A^\diamond & C^\diamond \\
-B^\diamond & D^\diamond
\end{pmatrix} = \begin{pmatrix}
A & C \\
-B & D
\end{pmatrix}^\diamond = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^{st\diamond},
\]

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thus, generalizing the corresponding result in the Yang-Mills theory. As one can easily check, the *pseudo-conjugate super-transpose* possesses all the properties of the graded adjoint (†).  

**VI. PROOF OF GRADED UNITARY PROPERTY**

One needs to verify that $U^\dagger = U^{-1}$ with the given definition of (†). If $U = \exp(M)$ then the Backer-Campbell-Hausdorff formula implies that one should have $M^\dagger = -M$, where $M = i(\varepsilon^a T_a + \theta^A \tau_A)$.

Considering the example with a Grassmann algebra on two odd generators (generalization to Grassmann algebras with any even or infinite number of odd generators is straightforward), we have

$$\beta_1^\circ = -\beta_2, \quad \beta_2^\circ = \beta_1. \quad (6.1)$$

Recall that for $\theta_A = j A \beta_j$ we defined in (4.5)

$$\theta_1 A = i C_A^{B'} \tilde{\theta}_B^1, \quad \theta_2 A = -i C_A^{B'} \tilde{\theta}_B^2, \quad \tilde{\theta}_1 A' = -i C_{A'}^B \theta_B^2, \quad \tilde{\theta}_2 A' = i C_{A'}^B \theta_B^1,$$

and that on $osp(1\mid 2; \mathbb{C})$ generators $T_a$ and $\tau_A$ the pseudo-conjugation coincides with complex conjugation. A direct calculation shows that

$$(T_a)^\dagger \equiv (T^\dagger)_a = T_a, \text{ and } (\tau_A)^\dagger \equiv (\tau^\dagger)_{A'} = -i C_{A'}^B \tau_B. \quad (6.2)$$

The former equation in (6.2) is just a restatement of hermiticity of the even generators while the later once again exhibits a strong connection between Grassmann-odd sector of the compact graded Lie algebra $osp(1\mid 2; \mathbb{C})$ on one side and 3D Euclidean spinors on the other (cf. (2.4)). It then follows that

$$M^\dagger = ((i(\varepsilon^a T_a + \theta^A \tau_A))^\dagger)^\circ = -i(\varepsilon^a (T_a)^\dagger + (\theta^A)^\circ (\tau_A)^\dagger) =$$

$$= -i \varepsilon^a T_a - (\theta^A_j \beta_j)^\circ C_{A'}^B \tau_B = -i \varepsilon^a T_a - (\tilde{\theta}_j A' \beta_j^\circ C_{A'}^B \tau_B \quad (6.3)$$
(it is easy to see that for a generic $\varepsilon^a = \tilde{\varepsilon}^a + \tilde{\varepsilon}^a_1 \beta_1 \beta_2$ with coefficients $\tilde{\varepsilon}^a$ and $\tilde{\varepsilon}^a$ real in the ordinary sense the property $(\varepsilon^a)^\diamond = \varepsilon^a$ holds because of (6.1) and the Grassmann-odd nature of generators $\beta_j$). Here we expand with the use of (6.1)

$$(\bar{\theta}^A \beta_j^2 \bar{C}^B A' = (\varepsilon^{A'D'}(\bar{\theta}^D A' / \beta_1^2 + \bar{\theta}^D A' / \beta_2^2)) \bar{C}^B A' = i\epsilon^{A'D'} \bar{C}^D B (\theta C \beta_2 + \theta C \beta_1) \bar{C}^A_B =$$

$$= i\bar{C}^A_B \epsilon^{A'D'} \bar{C}^D B (\theta C \beta_j) = i\bar{C}^A_B \epsilon^{A'D'} \bar{C}^D B \theta C,$$

where in the first summand in the parenthesis in the last equality of the first line an important cancellation of minus signs happens both because of definition of reality condition on spinors [11] (see also [17, 20]) and pseudo-conjugation on Grassmann numbers adopted from [7]. Using the identity $C^C \Sigma = \Sigma$ or via direct calculation, one can also check that the relation

$$\bar{C}^A_B \varepsilon^{A'D'} \bar{C}^D B \theta C = \epsilon^{BC}$$

holds. From (6.3) with the aid of (6.4) and (6.5), we, finally, obtain

$$M^\dagger = -i\varepsilon^a T_a - i\epsilon^{BC} \theta C \tau B \equiv -i(\varepsilon^a T_a + \theta A \tau A) = -M,$$

thus showing that the (Grassmann-valued) matrix $M$ is graded anti-hermitian. It then immediately follows that the group element (3.3) of UOSp(1|2) is graded unitary

$$U^\dagger = U^{-1}$$

proving that the corresponding group is a graded unitary orthosymplectic group.

VII. CONCLUSIONS AND OUTLOOK

We have accomplished algebraic preliminaries necessary to check the gauge invariance of the proposed field strength [3, 6] for UOSp(1|2) graded Yang-Mills theory on 4D Minkowski space-time and to develop an analogue of ‘non-commutative electrodynamics’ with massive matter fields. These required utilization of 3D Euclidean spinors, reality conditions on them [11] (see also [17, 20]), and notion of pseudo-conjugation on Grassmann-valued quantities [7, 13].

In the conventional Yang-Mills theory the number of generators of the underlying Lie algebra correspond to the number of gauge bosons. In this respect we shall be interested
in exploring the role of Grassmann-odd generators of $osp(1|2; \mathbb{C})$ in such a graded Yang-Mills theory. It will be also interesting to investigate the properties of the Grassmann-odd sector of the representation space for such a graded generalization of ‘non-commutative electrodynamics’ with massive matter content. Another important question is whether there exists an analogous connection between Euclidean spinors and other compact graded Lie algebras (cf. [12]).

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