Large cycles in random generalized Johnson graphs
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Abstract

This paper studies thresholds in random generalized Johnson graphs for containing large cycles, i.e. cycles of variable length growing with the size of the graph. Thresholds are obtained for different growth rates.

Keywords: random graphs, Johnson graph, Kneser graph, large cycles, threshold

1 Introduction and new results

A simple graph $G(n,r,s) = (V,E)$ is called a generalized Johnson graph if

1. $0 \leq s < r < n,$
2. $V = \binom{[n]}{r}$ is the set of all $r$-subsets of the set $[n] = 1, \ldots, n,$
3. $\forall x, y \in V : \{x, y\} \in E \iff |x \cap y| = s.$

Note that the special case $G(n,r,r-1)$ is known as a Johnson graph and $G(n,r,0)$ is known as a Kneser graph.

Throughout the article it is assumed that $r$ and $s$ are constant and $n$ approaches $+\infty$. The total number of vertices in this graph is denoted by $N$:

$$N = |V| = \binom{n}{r} \sim \frac{n^r}{r!}.$$

From the symmetry of the definition it is evident that $G(n,r,s)$ is a regular graph. Let $N_1$ denote the degree of its vertex:

$$N_1 = \binom{r}{s} \binom{n-r}{r-s} \sim \frac{n^{r-s}}{(r-s)!}.$$

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A random generalized Johnson graph $G_p(n, r, s)$ is a binomial random subgraph of $G(n, r, s)$. It is obtained from $G(n, r, s)$ by independent removal of each edge with probability $(1 - p)$. This is a generalization of the classical Erdős-Rényi model [1, 2] of a random graph, corresponding to the case $r = 1, s = 0$, in which $G(n, 1, 0) \cong K_n$.

Let $A = A(n)$ be some graph property, formally defined as a set of spanning subgraphs of $G(n, r, s)$. For a spanning subgraph $G \subset G(n, r, s)$ the fact $G \in A$ is usually denoted as $G \models A$. The function $\hat{p} = \hat{p}(n)$ is called a threshold for the property $A$ if

$$
P(G_p(n, r, s) \models A) \to \begin{cases} 0, p = o(\hat{p}) \\
1, p = \omega(\hat{p}) \end{cases}
$$

where $p = \omega(\hat{p})$ means $\hat{p} = o(p)$. A threshold $\hat{p}$ is called sharp if $\forall \varepsilon > 0$:

$$
P(G_p(n, r, s) \models A) \to \begin{cases} 0, p > (1 + \varepsilon)\hat{p} \\
1, p < (1 - \varepsilon)\hat{p} \end{cases}
$$

The problem of thresholds for various properties and, more generally, the asymptotic behavior of the probability of a random graph to possess a certain property has been widely addressed in literature [3, 4, 5, 6]. Particularly well studied is the property of subgraph containment [1, 7, 8, 9, 10, 11]. In more recent works some results concerning the asymptotic properties of random Kneser graphs [12, 13, 14] and of $G_p(n, r, s)$ [15, 16, 17, 18, 19, 20] can be found.

This work is focused on thresholds for the property of cycle containment in $G_p(n, r, s)$. In [18] Burkin found the threshold for containment in $G_p(n, r, s)$ of a subgraph isomorphic to a fixed graph (under certain assumptions). Applied to a simple cycle $C_t$ of a fixed length $t$, his theorem can be stated as follows.

**Theorem 1 (Burkin, 2016, [18])** Let $0 \leq s < r$ and $t$ be fixed integers. Then the threshold for containment of $C_t$ in $G_p(n, r, s)$ is

$$
\hat{p} = n^{-(r-s)/t}.
$$

In this paper we generalize this result to growing cycles, i.e. $t \to +\infty$ as $n \to +\infty$. Moreover, if the growth rate of $t$ is fast enough, the threshold is shown to be sharp. These results are summarized in the following theorem.
Theorem 2 Let $0 \leq s < r$ be fixed integers. If $t = t(n)$ satisfies the condition
\[ t = o\left(\sqrt{N_1}\right), \tag{2} \]
then the threshold for containment of $C_t$ in $G_p(n, r, s)$ is
\[ \hat{p} = \frac{n^{-s/t}}{N_1}. \tag{3} \]
Moreover, if $s = 0$ and $t \to +\infty$ or $s$ is arbitrary and $t = \omega(\ln n)$, then the threshold \( \hat{p} \) is sharp.

Note that if $t = \text{const}$, then there is no sharp threshold, which follows from the fact that in this case for $p = c \cdot n^{-(r-s)-s/t}$, where $c > 0$ is some constant, the number of cycles has asymptotically Poisson distribution \([18]\).

The proof of Theorem 2 presented in section 2 uses the classical methods of the first and the second moments. However, the case $t \to +\infty$ requires more careful analysis of the expectation and the variance of the number of cycles, than the case of constant $t$.

2 Proofs

The proof of Theorem 1 provided in \([18]\) by Burkin is also based on the methods of the first and the second moments. Let $X$ and $c_t$ be the numbers of copies of $C_t$ in $G_p(n, r, s)$ and in $G(n, r, s)$ respectively. To calculate the expectation $E[X] = c_t p^t$, Burkin finds the asymptotics of $c_t$. For this purpose he effectively shows that $c_t$ is asymptotically equivalent to the number of cycles entirely contained in the neighborhood of a single vertex, which is easy to compute explicitly. However, this is not generally true if $t \to +\infty$, in which case the estimation of $c_t$ should be more subtle. This problem is resolved in section 2.1 which adapts Burkin’s approach to the case of unbounded $t$.

Estimation of the variance $\text{Var} X$ in case $t = \text{const}$ is practically the same as in the Erdős-Rényi model for any balanced subgraph. If, however, $t \to +\infty$, then this problem becomes non-trivial. In the section 2.2 an upper bound for $\text{Var} X$ is obtained using the specificities of $C_t$ topology.

Finally, using the bounds for $E X$ and $\text{Var} X$, Theorem 2 is proved in section 2.3.

2.1 The number of cycles

Counting simple cycles in $G(n, r, s)$ can be reduced to counting simple paths due the following lemma.
Lemma 1 Let \( \{x, y\} \in E \) be an arbitrary edge in \( G(n, r, s) \). Then the number \( p_t \) of simple paths on \( t \) vertices, whose first vertex is \( x \) and last vertex is \( y \), does not depend on \( x \) and \( y \) and

\[
c_t = \frac{1}{2^t} N \cdot N_1 \cdot p_t
\]

(4)

Proof. The fact that \( p_t \) does not depend on \( x \) and \( y \) is evident from the symmetry of \( G(n, r, s) \) definition.

An isomorphism between \( C_t \) and a subgraph of \( G(n, r, s) \) can be chosen as follows. First, choose an arbitrary vertex \( y \in V \) in \( N \) ways. Next, choose its arbitrary neighbor \( x \) in \( N_1 \) ways. Then, choose a simple path on \( t \) vertices with \( x \) as its first vertex and \( y \) as its last vertex in \( p_t \) ways. Finally, considering that the number of automorphisms of \( C_t \) is \( 2^t \), the equality (4) is obtained. \( \square \)

As will be shown below, the estimation of \( p_t \) can be facilitated by introducing vertex sets

\[
V_j(u) = \{v \in V \mid |v \cap u| = j\}, j \in \mathbb{0}, r
\]

for \( u \in V \), which form a partition of \( V \). Let \( i, j \in \mathbb{0}, r \), \( y \in V \), and \( x \in V_i(y) \). Following Burkin [18], let’s introduce values

\[
A^j_i = |V_j(y) \cap V_i(x)|,
\]

which do not depend on the choice of \( x \) and \( y \) due to the symmetry of \( G(n, r, s) \) definition.

Note that if \( i = r \), then \( |x \cap y| = r \), which means that \( x = y \). Therefore, \( A^j_i = \delta_{s,j} \cdot N_1 \), where \( \delta_{s,j} \) is the Kronecker symbol. If \( j = r \), then \( |v \cap y| = r \), which means that \( v = y \) and, therefore, that \( A^r_i = \delta_{i,s} \). In the general case, the following formula holds:

\[
A^j_i = \sum_{m=0}^{s} \binom{i}{m} \binom{r-i}{s-m} \binom{r-i}{j-m} \binom{n-2r+i}{r-s-j+m} = \sum_{m=m_{\text{min}}}^{m_{\text{max}}} \binom{i}{m} \binom{r-i}{s-m} \binom{r-i}{j-m} \binom{n-2r+i}{r-s-j+m}
\]

(5)

assuming that \( \binom{0}{0} = 1, \binom{k}{k} \equiv \binom{k}{<0} \equiv \binom{<0}{k} \equiv 0 \) for any \( k \in \mathbb{Z} \). \( m_{\text{min}} \) and \( m_{\text{max}} \) are respectively the minimum and the maximum values of \( m \) for which the corresponding term in the sum is nonzero.

It is clear that the \( m \)th term in the sum (5) is nonzero iff

\[
\begin{align*}
\max\{0, \max\{i, j\} - (r-s), i + j - r\} & \leq m \leq \min\{s, i, j\} \\
m & \leq n - 2r + i + j - (r-s)
\end{align*}
\]
therefore, for sufficiently large $n$:
\[
\begin{align*}
    m_{\text{min}} &= \max\{0, \max\{i, j\} - (r - s), i + j - r\} \\
    m_{\text{max}} &= \min\{s, i, j\}
\end{align*}
\]  

(6)

The necessary and sufficient condition on $i$ and $j$ under which $A_i^j \neq 0$ (for sufficiently large $n$) is $m_{\text{min}} \leq m_{\text{max}}$, or, equivalently,
\[
\begin{align*}
    |i - j| &\leq r - s \\
    i + j &\leq r + s
\end{align*}
\]  

(7)

Note that for $i, j$ satisfying (7) the value $m_{\text{max}}$ defines the asymptotic order of $A_i^j$, namely,

\[
A_i^j \sim \text{const} \cdot n^{r - s - j + m_{\text{max}}} = \text{const} \cdot n^{r - s - j + \min\{s, i, j\}}.
\]  

(8)

**Lemma 2** If $t = o\left(\sqrt{N_1}\right)$, then
\[
\begin{align*}
    c_t &= O\left(\frac{n^t}{N_1^t}\right) \\
    c_t &= \Omega\left(\frac{1}{2t} N_1^t\right) \\
    c_t &= \Omega\left(\frac{n^t}{2t} (N_1/(r^t))^{\frac{t}{s}}\right)
\end{align*}
\]  

(9)

*Proof.* In this proof it will be assumed for simplicity that $t \geq 2s + 2$ for all $n$. The proof can be routinely generalized to the case in which $t < 2s + 2$ for some $n$.

The upper bound follows from Formula (4) and the obvious fact that $p_t \leq N_1^{t-2}$.

To obtain the lower bounds, it is enough to estimate $p_t$ from below. Let’s fix some adjacent vertices $x$ and $y$ and estimate the number of ways to choose a simple path $x_1, \ldots, x_t$, where $x_1 = x$ and $x_t = y$. The vertices $x_2, \ldots, x_{t-s-1}$ can be chosen in no less than $(N_1 - t)^{t-s-2}$ ways. Let $i = |x_{t-s-1} \cap y|$ so that $x_{t-s-1} \in V_i(y)$. Let’s restrict the choice of the rest of the vertices to the following vertex sets:

\[
x_{t-s} \in V_{\min\{i+1,s\}}(y), x_{t-s+1} \in V_{\min\{i+2,s\}}(y), \ldots, x_{t-1} \in V_{\min\{i+s,s\}}(y) = V_s(y).
\]

Then these vertices can be chosen in no less than

\[
\left(\binom{N_1}{\min\{i+1,s\}} - t\right) \cdot \left(\binom{N_1}{\min\{i+2,s\}} - t\right) \cdot \ldots \cdot \left(\binom{N_1}{\min\{i+s,s\}} - t\right) = \Omega\left(n^{r-s-1} s\right)
\]
ways if all the multiplicands are positive, which is true if \( r - s \geq 2 \) because for \( j \in 0, s - 1 \):
\[
A_j^{j+1} = \Theta \left( n^{r-s-1} \right) = \Omega \left( n^{(r-s)/2} \right) = \omega(t)
\]
and
\[
A_s = \Theta \left( n^{r-s} \right) = \omega(t).
\]
Thus, for \( r - s \geq 2 \):
\[
pt = \Omega \left( N^t_{1-s-2} \cdot n^{(r-s-1):s} \right) = \Omega \left( N^t_{1-2} \cdot n^{-s} \right).
\]

If, however, \( r - s = 1 \), then \( A_j^{j+1} = \text{const.} \) Let’s adopt another approach in this case choosing the vertices only from the following vertex sets:
\[
x_2 \in V_{s-1}(y), x_3 \in V_{s-2}(y), \ldots, x_{s+1} \in V_0(y),
\]
\[
x_{s+2}, \ldots, x_{t-s-1} \in V_0(y),
\]
\[
x_{t-s} \in V_1(y), x_{t-s+1} \in V_2(y), \ldots, x_{t-1} \in V_s(y),
\]
which can be done in no less than
\[
A_s^{s-1} \cdot A_{s-1}^{s-2} \cdot \ldots \cdot A_0^0 \cdot (A_0^0 - t)^{t-2s-2} \cdot (A_1^1 - 1) \cdot (A_2^2 - 1) \cdot \ldots \cdot (A_s^s - 1 - 1) \geq
\]
\[
\geq \Theta(N_s^s) \cdot N_1^{t-2s-2} = \Theta(N_1^{t-2} N_1^{-s}) = \Theta(N_1^{t-2} \cdot n^{-s})
\]
ways.
Thus, in any case, as follows from Formula (11),
\[
c_t = \Omega \left( \frac{1}{2t} N \cdot N_1 \cdot N_1^{-t-2} \cdot n^{-s} \right) = \Omega \left( \frac{1}{2t} N_1^t \right).
\]

The last bound follows from Formula (11) and the fact that
\[
p_t \geq (A_s^s - t)^{t-2} \sim \left( \frac{N_1}{t^s} \right)^{t-2},
\]
which can be obtained by choosing only \( x_2, \ldots, x_{t-1} \in V_s(y) \). □

### 2.2 Variance of the number of cycles

**Lemma 3** If if there exists such function \( M = M(n) \) that for some \( \varepsilon > 0 \) the following conditions are satisfied:
\[
\begin{cases}
M \geq N_1 \\
M_p > 1 + \varepsilon \\
t^2/M = O(1) \\
\frac{t^2}{M_p} \to 0
\end{cases}
\]
then for any $\delta > 0$:

$$
\frac{\text{Var} \ X}{(\mathbb{E}X)^2} \leq \frac{1}{\mathbb{E}X} + o\left(\frac{M^t}{2t \mathbb{E}X} \cdot \left(\frac{t^2/M}{M^o}\right)^{\frac{1}{2} - \delta}\right).
$$

**Proof.** Let $\gamma_1, \ldots, \gamma_{c_t}$ be all copies of $C_t$ in $G(n, r, s)$ and $X_i = [\gamma_i \subset G_p(n, r, s)]$, where $[ \cdot ]$ is the Iverson bracket. Then

$$
\text{Var} \ X = \sum_{i=1}^{c_t} \text{Var} \ X_i + \sum_{i \neq j} \text{cov}(X_i, X_j) = \sum_{i=1}^{c_t} \text{Var} \ X_i + \sum_{(i,j) \in I} \text{cov}(X_i, X_j)
$$

$$
\leq \sum_{i=1}^{c_t} \mathbb{E}(X_i^2) + \sum_{(i,j) \in I} \mathbb{E}(X_iX_j) = \sum_{i=1}^{c_t} \mathbb{E} X_i + \sum_{(i,j) \in I} \mathbb{E}(X_iX_j) = \mathbb{E} X + \sum_{(i,j) \in I} \mathbb{E}(X_iX_j),
$$

where $I$ is the set of ordered pairs $(i, j)$ of such indices that $\gamma_i$ and $\gamma_j$ have at least one common edge.

Let $(i, j) \in I$. Let $\alpha(i, j)$ denote the number of all inclusion maximal non-degenerate (with at least one edge) simple paths in $\gamma_i \cap \gamma_j$. Let $x(i, j)$ denote the total number of edges in $\gamma_i \cap \gamma_j$. Clearly, $\alpha(i, j) \leq x(i, j) \leq t - \alpha(i, j)$ and $\alpha(i, j) \leq t/2$. So, given $\alpha \leq [t/2]$, $x \leq t - \alpha$, let’s estimate from above the number of such pairs $(i, j) \in I$ that $\alpha(i, j) = \alpha$ and $x(i, j) = x$. First, let’s fix the index $i$, which can be done in $c_t$ ways, and count the number of ways to choose $j$. An isomorphism between $C_t$ and $\gamma_j$ for any $j$ can be written as follows:

$$(v_1, \ldots, v_\xi, b_1, l_1, u_1, \ldots, u_{r_1},$$

$$v_\xi, \ldots, v_{\xi+b_2+l_2}, u_{r_1+1}, \ldots, u_{r_1+r_2},$$

$$\ldots,$$

$$v_{\xi-a_1-1}, \ldots, v_{\xi-a_1+b-a_1-l_{a-1}}, u_{r_1+\ldots+r_{a-2}+1}, \ldots, u_{r_1+\ldots+r_{a-1}},$$

$$v_{\xi}, \ldots, v_{\xi+a-b-a_l}, u_{r_1+\ldots+r_{a-1}+1}, \ldots, u_{t-(x+\alpha)}),$$

where $v_1, \ldots, v_t$ are the vertices of $\gamma_i$ such that $\{v_i, v_{i+1}\} \in E$, $(v_{-(t-1)}, \ldots, v_0) = (v_1, \ldots, v_t)$, $u_1, \ldots, u_{t-(x+\alpha)} \in V$ and for $m \in I, \alpha$:

$$
\begin{cases}
1 \leq \xi_m \leq t \\
b_m \in \{-1, 1\} \\
1 \leq l_m \leq x \\
0 \leq r_m \leq t - (x + \alpha) \\
l_1 + \ldots + l_\alpha = x \\
r_1 + \ldots + r_\alpha = t - (x + \alpha)
\end{cases}
$$
Thus, such isomorphism is uniquely defined by the tuple

\[(\xi_1, \ldots, \xi_\alpha, b_1, \ldots, b_\alpha, l_1, \ldots, l_\alpha, r_1, \ldots, r_\alpha, u_1, \ldots, u_{t-(x+\delta)})\]

the number of ways to choose which, considering the constraints, is bounded from above by

\[t^{\alpha}2^\alpha \binom{x-1}{\alpha-1} \binom{t-x-1}{\alpha-1} N_1^{t-x-\alpha}.

Since the probability of appearance of two fixed cycles with \(x\) common edges is \(p^{2t-x}\), the sum \(\sum_{(i,j)\in I} E(X_i X_j)\) can be estimated from above as follows:

\[
\sum_{(i,j)\in I} E(X_i X_j) \leq \sum_{\alpha=1}^{t/2} \sum_{x=\alpha}^{t-\alpha} c_t(2t)^\alpha \binom{x-1}{\alpha-1} \binom{t-x-1}{\alpha-1} N_1^{t-x-\alpha} p^{2t-x} \leq \\
\leq \frac{1}{t} c_t p^t \sum_{\alpha=1}^{t/2} \sum_{x=\alpha}^{t-\alpha} t(2t)^\alpha \binom{x}{\alpha} \binom{t-x}{\alpha-1} M^{t-x-\alpha} p^{t-x} = \\
= \frac{1}{t} c_t p^t (Mp)^t \sum_{\alpha=1}^{t/2} \sum_{x=\alpha}^{t-\alpha} (Mp)^{-x} \left(\frac{2t}{M}\right)^\alpha \frac{t\alpha}{t-x-\alpha+1} \left(\frac{t-x}{\alpha}\right)^\alpha \leq \\
\leq \frac{1}{t} c_t p^t D^t \sum_{a,x}^{+\infty} D^{-x} \frac{t}{\max\{1, t-x-\alpha+1\}} \left(\alpha^{1/\alpha} 2e^{2t^2} \frac{x}{M^2}\right)^\alpha \leq \\
\leq \frac{1}{t} c_t p^t D^t \sum_{a,x}^{+\infty} \frac{t(1+\varepsilon)^{-\delta(x+\alpha)}}{\max\{1, t-x-\alpha+1\}} D^{-x(1-\delta)} \left(1+\varepsilon\right)^{-\delta(x+\alpha)} \left(\alpha^{1/\alpha} 2e^{2t^2} \frac{x}{M^2}\right)^\alpha = \\
\leq \frac{1}{t} c_t p^t D^t \sum_{a,x}^{+\infty} \left[\frac{x+\alpha \leq \frac{r}{\delta}}{\max\{1, t-x-\alpha+1\}} \frac{t(1+\varepsilon)^{-\delta(x+\alpha)}}{\max\{1, t-x-\alpha+1\}} D^{-x(1-\delta)} \left(\Theta \left(\frac{t^2}{M} \frac{x}{\alpha^2}\right)\right)^\alpha \leq \\
\leq \frac{1}{t} c_t p^t D^t \sum_{a,x}^{+\infty} \left(2 + \frac{2}{\delta \ln(1+\varepsilon)} \right) D^{-x(1-\delta)} \left(\Theta \left(\frac{t^2}{M} \frac{x}{\alpha^2}\right)\right)^\alpha \leq 
\]
\[ \leq \frac{1}{t} c_1 \rho^t D^t \sum_{\alpha,x=1}^{+\infty} (1 + \varepsilon)^{-\delta x} D^{-x(1-2\delta)} \left( \frac{d x}{\alpha^2} \right)^\alpha \]

where
\[ \delta = \text{const} \in (0, 1/4), \]
\[ d = \Theta \left( \frac{t^2}{M} \right) = O(1) \leq d_0 = \text{const}, \]
\[ D = N_1 p > 1 + \varepsilon. \]

Let \( \sigma = \sqrt{d/D} \to 0. \) Then
\[ D^{-x(1-2\delta)} \left( \frac{d x}{\alpha^2} \right)^\alpha = ([d \leq \sigma] + [d > \sigma]) D^{-x(1-2\delta)} \left( \frac{d x}{\alpha^2} \right)^\alpha \leq \]
\[ \leq (1 + \varepsilon)^{-x(1-2\delta)} \sigma \left( \frac{x}{\alpha^2} \right)^\alpha + \sigma x(1-2\delta) \left( d_0 \frac{x}{\alpha^2} \right)^\alpha = e^{f(x,(1+\varepsilon)^{-1-2\delta},\sigma)} + e^{f(x,\sigma^{1-2\delta},d_0)}, \]

where
\[ f(x, y, z) = \ln \left( y^x \left( \frac{z x}{\alpha^2} \right)^\alpha \right) \]

for \( x > 0, 0 < y < 1, z > 0, \alpha \geq 1. \) From the equation \( \partial f / \partial x = 0 \) the maximum point \( x_{\text{max}} \) of \( f(x, y, z) \) is obtained:
\[ x_{\text{max}} = -\frac{\alpha}{\ln y}, \]
\[ f(x_{\text{max}}, y, z) = \alpha \ln \left( \frac{z}{e\alpha y \ln(1/y)} \right) \leq \alpha \ln \left( \frac{z}{y \ln(1/y)} \right). \]

Thus, for small enough \( \sigma \) (i.e. for large enough \( n \)):
\[ f(x, (1 + \varepsilon)^{-1-2\delta}, \sigma) \leq \alpha \ln \left( \frac{\sigma(1 + \varepsilon)^{1-2\delta}}{(1-2\delta) \ln(1+\varepsilon)} \right) \leq \alpha(1 - 3\delta) \ln \sigma. \]

Further, for \( x \geq \alpha \) and large enough \( n \):
\[ \frac{\partial f(x, \sigma^{1-2\delta}, d_0)}{\partial x} = (1 - 2\delta) \ln \sigma + \frac{\alpha}{x} < 0 \]
and therefore
\[ f(x, \sigma^{1-2\delta}, d_0) \leq f(\alpha, \sigma^{1-2\delta}, d_0) = \alpha \ln \left( \sigma^{1-2\delta} d_0 \frac{1}{\alpha} \right) \leq \alpha(1 - 3\delta) \ln \sigma. \]
Thus,
\[
D^{-x(1-2\delta)} \left( \frac{d^x}{\alpha^2} \right) \leq 2e^{a(1-3\delta) \ln \sigma} = 2\sigma^{a(1-3\delta)},
\]

\[
\sum_{(i,j) \in I} E(X_i X_j) \leq \frac{1}{t} c_i p^t D^t \sum_{\alpha,x=1}^{+\infty} 2(1+\varepsilon)^{-\delta x} \sigma^{a(1-3\delta)} \leq \frac{1}{t} c_i p^t D^t \left( \frac{2\sigma^{a(1-3\delta)} \sum_{x=1}^{+\infty} (1+\varepsilon)^{-\delta x}}{1-\sigma^{a(1-3\delta)}} \right) = o \left( \frac{1}{t} c_i p^t D^t \left( \frac{d}{D} \right)^{\frac{1-a}{2}} \right)
\]

and, finally, for any \( \delta > 0 \) letting \( \tilde{\delta} = \delta/2 \) yields:

\[
\frac{\text{Var} X}{(E X)^2} \leq \frac{E X + o \left( \frac{1}{t} c_i p^t D^t \left( \frac{d}{D} \right)^{\frac{1-a}{2}} \right)}{(E X)^2} = \frac{1}{E X} + o \left( \frac{M^t \cdot \left( \frac{t^2/M}{M^t} \right)^{\frac{1}{2}}}{2 t c_i} \right) \rightarrow 0.
\]

\[\square\]

### 2.3 Proof of Theorem 2

Let \( p = o(\hat{p}) \). Then by Markov’s inequality and Lemma 2

\[
P(X \geq 1) \leq E X = c_i p^t = O \left( \frac{1}{2t} (N_1 p n^{s/t})^t \right) = O \left( \frac{1}{2t} (p/\hat{p})^t \right) \rightarrow 0,
\]

where \( X \) is the number of copies of \( C_t \) in \( G_p(n, r, s) \).

Let \( p = \omega(\hat{p}) \). Then Lemma 2 and Lemma 3 for \( M = N_1 \cdot \max \{ 1, n^{s/t}/(r/s) \} \) yield

\[
E X = \Omega \left( \frac{1}{2t} \left( N_1 p n^{s/t} / (r/s) \right)^t \right) = \Omega \left( \frac{1}{2t} (\omega(1))^t \right) \rightarrow +\infty,
\]

\[
\frac{\text{Var} X}{(E X)^2} \leq \frac{1}{E X} + o \left( \frac{N_1^t \cdot \max \left\{ 1, n^{s/t} / (r/s) \right\}^t}{2 t \cdot \max \left\{ \frac{1}{2t} N_1^t, \frac{n^s}{2t} (N_1 / (r/s))^t \right\}} \right) \rightarrow 0.
\]

By Chebyshev’s inequality:

\[
P(X = 0) \leq P(|X - E X| \geq E X) \leq \frac{\text{Var} X}{(E X)^2} \rightarrow 0.
\]
Thus, it is proved that $\hat{p}$ is a threshold.

If $t \to +\infty$, then for $EX \to 0$ it is enough that $p \leq (1 - \varepsilon)\hat{p}$ for some $\varepsilon > 0$:

$$EX = c_t p^t = O \left( \frac{1}{2^t} (N_1 p^{n/t})^t \right) = O \left( \frac{1}{2^t} (1 - \varepsilon)^t \right) \to 0.$$  

If, in addition, $s = 0$ or $t = \omega(\ln n)$, then for $p \geq (1 + \varepsilon)\hat{p}$:

$$N_1 p \geq (1 + \varepsilon)n^{-s/t} \sim 1 + \varepsilon > 1 + \frac{\varepsilon}{2}.$$ 

Therefore, Lemma 2 and Lemma 3 for $M = N_1$ yield

$$EX = c_t p^t = \Omega \left( \frac{1}{2^t} (N_1 p)^t \right) = \Omega \left( \frac{1}{2^t} \left( 1 + \frac{\varepsilon}{2} \right)^t \right) \to +\infty,$$

$$\frac{\text{Var } X}{(EX)^2} \leq \frac{1}{EX} + o(1) \to 0$$

which means that the threshold is sharp. □

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