SEMIGEOSTRPHIC EQUATIONS IN PHYSICAL SPACE WITH FREE UPPER BOUNDARY

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Abstract. We define various notions of Lagrangian solution in physical space for 3-d incompressible geostrophic system with free upper boundary under different conditions for initial data, then prove their existence via the minimization with respect to a geostrophic functional, generalizing the work of [2] and [7] to the case of free upper boundary. As a byproduct of our proof, we obtain the existence of measure-valued dual space solutions when the initial measure \( \nu_0 \in P_2(\mathbb{R}^3) \) and is supported on \( \{-\frac{1}{\delta} \leq x_3 \leq -\delta\} \)

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1. Introduction

The Semi-Geostrophic system (abbreviated as SG in the following) models large-scale atmospheric-ocean flows, where large scale means the flow is rotation-dominated. In [10], they proved the existence of incompressible SG system in a fixed domain \( \Omega \subset \mathbb{R}^3 \) in the so-called dual space formulation, which is a formal change of variable. Under the dual space formulation, the SG system can be written as a transport
equation coupled with Monge-Ampère equation. Then [3], they considered the free boundary case in 3-D, but with additional assumption that the potential temperature is constant. Under this additional assumption, the system can be rewritten as a 2-D system, the so-called Semi-geostrophic Shallow Water system. They proved the existence of dual-space solutions with initial dual density in $L^p(p > 1)$. The existence of solutions in the original physical variables is first proved in [2], in the Lagrangian formulation of the physical system, for both fixed boundary SG system and shallow water system with the same assumption on the dual density as above, this amounts to some strict convexity condition for the modified pressure. Then in [7], [11], they put forward a more general notion of physical solutions, to deal with the situation when the dual space measure is singular. They also proved a general existence result measure-valued initial data.

In this work, we consider the incompressible SG system in a 3-D domain with free upper boundary, but without the constancy assumption made in [3]. The dual space existence has been proved in [6], using the general theory of Hamiltonian ODE, see [12]. Here we prove the existence of solutions in physical space, generalizing the work of [2] and [7]. The main difficulty involved is the more complicated geostrophic energy in our case since it involves the unknown free boundary profile.

\begin{align}
D_t(u_1^g, u_2^g) + (-u_2, u_1) + (\partial_{x_1} p, \partial_{x_2} p) &= 0 \\
\nabla \cdot u &= 0 \text{ in } \Omega_h \\
D_t \rho &= 0 \\
\nabla p &= (u_2^g, -u_1^g, -\rho)
\end{align}

where $D_t = \partial_t + u \cdot \nabla$, and $\Omega_h = \{(x_1, x_2, x_3) \in \Omega_2 \times [0, \infty) | 0 < x_3 < h(t, x_1, x_2)\}$.

Here $\Omega_2 \subset \mathbb{R}^2$ is a bounded convex region with $C^1$ boundary, $h(t, x_1, x_2) \geq 0$ describes the unknown free upper boundary. In the above $p$ is the pressure, $u$ is the velocity, and $\rho$ is the density.

Of course we need to prescribe suitable boundary and free boundary conditions. We require that no flow can penetrate the fixed boundary, the pressure at the top is a constant which without loss of generality we take to be zero.

\begin{align}
\nabla \cdot n &= 0 \text{ on } \partial \Omega_h - \{x_3 = h\} \\
p(x_1, x_2, x_3) &= 0 \text{ on } \{x_3 = h\} \\
\partial_t h + u_1 \partial_{x_1} h + u_2 \partial_{x_2} h &= u_3 \text{ on } \{x_3 = h\}
\end{align}

We remark that the first+third condition is formally equivalent to

$$\partial_t \sigma_h + \nabla \cdot (u \sigma_h) = 0$$

where

$$\sigma_h(x_1, x_2, x_3) = \chi_{\Omega_h} (x_1, x_2, x_3)$$

Now we put

$$P(t, x) = p(t, x) + \frac{1}{2} (x_1^2 + x_2^2)$$

then the above system can be written as

$$D_t (\nabla P) = J(\nabla P - x)$$
generally, we can define

\begin{equation}
\nabla \cdot u = 0
\end{equation}

In the above

\[
J = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The free boundary condition for \( P \) is

\[
P(x_1, x_2, x_3) = \frac{1}{2}(x_1^2 + x_2^2) \quad \text{on} \quad \{x_3 = h\}
\]

The geostrophic energy is

\[
E = \int_{\Omega_h} \frac{1}{2}(u_1^2 + u_2^2) + \rho x_3 \, dx = \int_{\Omega_h} \frac{1}{2}[(\partial_1 P - x_1)^2 + (\partial_2 P - x_2)^2] - \partial_3 P x_3 \, dx
\]

By Cullen's stability principle, the function \( P \) above should be convex, and \( \nabla P \) should minimize above functional among all possible rearrangement of particles. To make things precise, we are motivated to consider the following.

\[
E_\nu(h, T) = \int_{\Omega} \frac{1}{2}[(x_1 - T_1(x))^2 + (x_2 - T_2(x))^2] - x_3 T_3(x) \, dx
\]

Here \( \nu \in \mathcal{P}(\mathbb{R}^3), T_3 \sigma_h = \nu \) and we require that the actual free boundary profile and \( \nabla P \) minimizes \( E_\nu(h, T) \) among all pairs such that \( T_3 \sigma_h = \nu \).

Also we recall that \( \partial_{x_3} P = -\rho \), so it's reasonable to assume the convex potential \( P \) should satisfy \(-\frac{1}{2} \leq \partial_{x_3} P \leq -\delta \), or we require \( \text{supp} \, \nu \subset \mathbb{R}^2 \times [-\frac{1}{2}, -\delta] \). More generally, we can define

\[
E_\nu(h, \gamma) = \int_{\Omega} \frac{1}{2}[(x_1 - y_1)^2 + (x_2 - y_2)^2] - x_3 y_3 \, d\gamma(x, y)
\]

Here

\[
h \geq 0, \quad \int_{\Omega_2} hd x_1 dx_2 = 1 \quad \gamma \in \Gamma(\sigma_h, \nu)
\]

This will make it easier to find minimizers, just like we did in the usual optimal transport problem. See also [6].

In the following, we will denote

\[
c(x, y) = \frac{1}{2}((x_1 - y_1)^2 + (x_2 - y_2)^2) - x_3 y_3 = \frac{1}{2}(x_1^2 + x_2^2 + y_1^2 + y_2^2) - x \cdot y
\]

In the same way as rigid boundary case, we can derive the dual formulation

\begin{equation}
\partial_t \nu + \nabla \cdot (\nu \mathbf{w}) = 0 \quad \text{in} \quad [0, T] \times \Lambda
\end{equation}

\begin{equation}
\mathbf{w}(t, y) = J(y - \nabla P^*(t, y)) \quad \text{in} \quad [0, T] \times \Lambda
\end{equation}

\begin{equation}
(h, \nabla P) \text{ minimizes } E_\nu(h, T) \text{ among all pairs such that } T_3 \sigma_h = \nu
\end{equation}

Here are some notations and terminology which will be used through this work. In the following, \( \Omega_\infty \) represents the region \( \Omega_2 \times [0, \infty) \), and \( \Omega_H = \Omega_2 \times [0, H] \).

Given \( h : \Omega_2 \to \mathbb{R}_+, \Omega_h = \{x \in \Omega_\infty | 0 < x_3 < h(x_1, x_2)\} \), and \( \sigma_h(x) = \chi_{\Omega_h}(x) \). We indentify an absolute measure (with respect to \( L^3 \)) with its densities. Suppose \( A, B \subset \mathbb{R}^3 \), and we have two functions \( f(x), g(y) \) defined on \( A \) and \( B \) respectively. We say \( f, g \) are convex conjugate to each other, if the following holds.

\[
f(x) = \sup_{y \in B} (x \cdot y - g(y)) \quad x \in A
\]
and \[ g(y) = \sup_{x \in A} (x \cdot y - f(x)) \quad y \in B \]

To conclude this section, we briefly describe the plan of this paper.

In section 2, we study the geostrophic functional and its dual problem in detail, and establish various properties of the optimizers which will be used later on. Then in the case when the dual density \( \nu \in L^q \) for some \( q > 1 \) with compact support, we follow [2] to establish the existence of weak Lagrangian solutions, see Theorem 3.10 in section 3.3. In the case when \( \nu \) is singular, we generalize the notion of weak Lagrangian solutions and prove their existence with suitable initial data, see Theorem 4.14 and theorem 5.2. As a byproduct, we obtain the existence of measure-valued dual space solutions when the initial dual density \( \nu_0 \in P_2(\mathbb{R}^3) \) with support contained in \( \mathbb{R}^2 \times [\frac{1}{\delta}, \delta] \) for some \( \delta > 0 \), see corollary 5.5.

2. The study of the functional \( E_\nu(h, \gamma) \)

2.1. The case when \( \nu \) has bounded support. In this section, we study the functional involved in the geostrophic energy and the associated dual problem, prove basic properties such as unique existence of optimizers. Finally, we give an alternative proof of dual space existence result using time stepping since later on we will need some properties of dual space solutions which are not so clear in Hamiltonian ODE approach as was done in [5].

We study the property of the functional

\[
E_\nu(h, \gamma) = \int_{\Omega \times \Lambda} c(x, y) d\gamma(x, y) = \int_{\Omega \times \Lambda} \left[ \frac{1}{2}(x_1^2 + x_2^2 + y_1^2 + y_2^2) - x \cdot y \right] d\gamma
\]

where \((h, \gamma) \in M_\nu \nu \in P_{ac}(\mathbb{R}^3) \) and \( \text{supp} \nu \subseteq \subseteq \mathbb{R}^2 \times [\frac{1}{\delta}, -\delta] \) and compact. Here let’s choose \( \Lambda = B_D(0) \times [\frac{1}{\delta}, -\delta] \) and assume for technical reasons \( \Omega \subseteq B_D(0) \).

Where

\[ M_\nu = \{ (h, \gamma) | h \geq 0, \int_{\Omega_2} h dx \cdot dx = 1, \gamma \in \Gamma(h, \nu) \} \]

As well as the functional

\[
J^H_\nu(P, R) = \int_{\Omega} \left[ \frac{1}{2}(y_1^2 + y_2^2) - R(y) \right] |\nu(y)| dy + \inf_{0 \leq h \leq H} \int_{\Omega} \left[ \frac{1}{2}(x_1^2 + x_2^2) - P(x) |\sigma_h(x)| dx \right]
\]

where

\[ P(x) + R(y) \geq x \cdot y \quad \forall x \in \Omega_2 \times [0, H] \quad \forall y \in \Lambda \]

Call the collection of all such pairs satisfying the above condition to be \( N_\nu \).

We will also consider the untruncated version, namely

\[
J_\nu(P, R) = \int_{\Omega} \left[ \frac{1}{2}(y_1^2 + y_2^2) - R(y) \right] |\nu(y)| dy + \inf_{h \geq 0} \int_{\Omega} \left[ \frac{1}{2}(x_1^2 + x_2^2) - P(x) |\sigma_h(x)| dx \right]
\]

where we require

\[ R \in L^1(d\nu) \quad P \in L^1(\Omega_K) \forall K > 0, \text{ and } h \in L^1(\Omega_2) \quad P\sigma_h \in L^1(\Omega_\infty) \]

and

\[ P(x) + R(y) \geq x \cdot y \quad \forall x \in \Omega_\infty \quad y \in \Lambda \]

We will see later that \( J_\nu(P, R) \) is dual to \( E_\nu(h, \gamma) \) in the next subsection. Also the study of the dual problem \( J_\nu(P, R) \) will help with proving uniqueness of minimizers of \( E_\nu(h, \gamma) \), since the geostrophic functional \( E_\nu(h, \gamma) \) (in our case) does not seem to have strict convexity as in [3, 4]. This was first noted in [6] and the idea goes back.
Conversely, if $P$ is defined on $\Omega$ then we will show the uniform bound in $L^\infty$ and so it will be equivalent to solving the truncated problem $J_\nu^H(P,R)$ if one takes $H$ large enough depending only on $\delta, \Omega_2$ and $\Lambda$.

Suppose $\partial_{x_3} P(x) \leq -\delta$, let's define

$$\Pi_P(x_1, x_2, s) := \int_0^s \frac{1}{2}(x_1^2 + x_2^2) - P(x_1, x_2, x_3) dx_3$$

It’s easy to see such a function is uniformly convex and so there exists a unique $s_*$ where $\Pi_P$ achieves minimum on $[0, \infty)$. We define this function to be $h_P(x_1, x_2)$. We also define $h_P^H(x_1, x_2)$ to be the unique $s^* \in [0, H]$ where $\Pi_P$ achieves minimum on $[0, H]$. Notice by convexity, one has $h_P^H = \min(h_P, H)$

**Remark 2.3.** Whenever $h_P(x_1, x_2) > 0$, we must have

$$P(x_1, x_2, h_P(x_1, x_2)) = \frac{1}{2}(x_1^2 + x_2^2)$$

Otherwise

$$P(x_1, x_2, 0) \leq \frac{1}{2}(x_1^2 + x_2^2)$$

Conversely, if $P$ is defined on $\Omega_\infty$ and $h$ satisfies above condition, then we will also have $h = h_P$

**Remark 2.4.** It’s easy to see in the situation of $J_\nu(P,R)$

$$\inf_{h \geq 0} \int_{\Omega_\infty} \frac{1}{2}(x_1^2 + x_2^2) - P(x)]\sigma_h(x)dx = \int_{\Omega_\infty} \frac{1}{2}(x_1^2 + x_2^2) - P(x)]\sigma_{h_P}(x)dx$$

and in the situation of $J_\nu^H(P,R)$

$$\inf_{h \geq 0} \int_{\Omega_H^H} \frac{1}{2}(x_1^2 + x_2^2) - P(x)]\sigma_h(x)dx = \int_{\Omega_H^H} \frac{1}{2}(x_1^2 + x_2^2) - P(x)]\sigma_{h_P^H}(x)dx$$

Now we prove the following

**Lemma 2.7.** Suppose there exists a sequence $\partial_{x_1} P_n, \partial_{x_3} P \leq -\delta$, and $P_n \to P$ uniformly on $\Omega_2 \times [0, \infty)$ for each $H > 0$, then we have $h_{P_n}(x_1, x_2) \to h_P(x_1, x_2)$ uniformly on $\Omega_2$. If $P_n, P$ satisfy the same condition but is only defined on $\Omega_2 \times [0, H_0]$, then $h_{P_n}^{h_0} \to h_P^{h_0}$ uniformly on $\Omega_2$

**Proof.** First we show that $\{h_n\}$ is uniformly bounded. Indeed,

$$\frac{1}{2}(x_1^2 + x_2^2) = P_n(x_1, x_2, h_{P_n}(x_1, x_2)) \leq -\delta h_n(x_1, x_2) + P_n(x_1, x_2, 0)$$

Put $\epsilon_n = \sup_{x \in \Omega_2 \times [0, H]} |P_n(x) - P(x)|$. Then we have

If $h_P(x_1, x_2) = 0$, then

$$\frac{1}{2}(x_1^2 + x_2^2) \geq P_n(x_1, x_2, 0) \geq -\epsilon_n + (P_n(x_1, x_2, 0) - P_n(x_1, x_2, h_n(x_1, x_2)) + \frac{1}{2}(x_1^2 + x_2^2)$$

$$\geq -\epsilon_n + \delta h_n(x_1, x_2) + \frac{1}{2}(x_1^2 + x_2^2)$$

If $h_P(x_1, x_2) > 0$, and $h_n \geq h$ then

$$0 = P_n(x_1, x_2, h_n(x_1, x_2)) - P(x_1, x_2, h(x_1, x_2))$$
\[ P_n(x_1, x_2, h_n) - P(x_1, x_2, h_n) + P(x_1, x_2, h_n) - P(x_1, x_2, h) \leq \epsilon_n - \delta(h_n - h) \]
The other case \( h_n \leq h \) can be dealt with similarly. \qed

We also need another lemma which gives control over the absolute bound for the maximizing sequence.

**Lemma 2.8.** Suppose there exists constants \( K > 0 \) and \( P : \Omega_H \to \mathbb{R} \) with \( P(x_1^*, x_2^*, 0) = 0 \) where \((x_1^*, x_2^*) \in \Omega_2 \) and \( P, R \) are convex conjugate over the domain \( \Omega_H \) and \( \Lambda \) respectively, such that for some \( \lambda > 0 \)

\[ -K \leq J^H_{\nu}(P - \lambda, R + \lambda) \]

and

\[ |\nabla P| \leq K - \frac{1}{\delta} \leq \partial_{x_3}P \leq -\delta \]

then

\[ |\lambda| \leq C_1(K, \Lambda, \Omega_2, H, \cdot) \]

as long as \( H \geq \frac{2}{\varepsilon^2(\Omega_2)} \)

**Proof.** First from our assumption on \( P \), we obtain

\[ |P(x)| \leq K\text{diam } \Omega_2 + \frac{H}{\delta} := C_{-1} \quad x \in \Omega_\infty \]

Since \((P, R)\) are convex conjugate, one has

\[ R(y) = \sup_{x \in \Omega_\infty} (x \cdot y - P(x)) \geq x_1 y_1 + x_2 y_2 \geq -\max_{y \in \Lambda} \|y\| \cdot \max_{y \in \Lambda} \|y\| := -C_0 \]

From the definition of \( J^H \), by taking \( h = 0 \), one has

\[ -K \leq \int_{\Lambda} \left[ \frac{1}{2}(y_1^2 + y_2^2) - R(y) - \lambda|\nu(y)|dy \leq \mathcal{L}^3(\Lambda) \max_{y \in \Lambda} (|y_1| + |y_2|)^2 + C_0 - \lambda \right. \]

So

\[ \lambda \leq C_0 + K + \mathcal{L}^3(\Lambda) \max_{y \in \Lambda} (|y_1| + |y_2|)^2 \]

On the other hand, we take \( h = \frac{2}{\varepsilon^2(\Omega_2)} \), we then have

\[ -K \leq \int_{\Lambda} \left[ \frac{1}{2}(y_1^2 + y_2^2) - R(y) - \lambda|\nu(y)|dy + \int_{\Omega_2 \times [0, \frac{1}{\sqrt{2}}]} \left[ \frac{1}{2}(x_1^2 + x_2^2) - P(x) + \lambda|x|^2 \right] dx \right. \]

\[ \leq \mathcal{L}^3(\Lambda) \max_{y \in \Lambda} (|y_1| + |y_2|)^2 + C_{-1} + C_0 + \max_{\Omega_2} |x|^2 + \lambda \]

\[ \lambda \leq \frac{1}{\varepsilon^2(\Omega_2)} \]

\[ \lambda \leq \frac{1}{\varepsilon^2(\Omega_2)} \]

Now we prove the existence of a pair of maximizer of \( J^H \)

**Theorem 2.9.** Suppose \( H \geq \frac{2}{\varepsilon^2(\Omega_2)} \), then the variational problem \( J^H_{\nu} \) has a maximizer which is convex conjugate over \( \Omega_H \) and \( \Lambda \).

**Proof.** We choose a maximizing sequence \((P_n, R_n)\), without loss of generality, we can assume they are convex conjugate on the domain \( \Omega_H \) and \( \Lambda \), then their derivatives are uniformly bounded (with a bound depending on \( \Omega_2, \Lambda, H \)) and \( P \) satisfies \( \frac{1}{\delta} \leq \partial_{x_3}P \leq -\delta \).

Now the functions

\[ \hat{P}_n := P_n - P_n(x_1^*, x_2^*, 0) \quad \hat{R}_n := R_n + P_n(x_1^*, x_2^*, 0) \]
satisfy the assumption of previous lemma with \( \lambda = -P_n(x_1^*, x_2^*, 0) \), we can then conclude
\[
|P_n(x_1^*, x_2^*, 0)| \leq C
\]
Since the derivatives are bounded independent of \( n \), we get \( P_n \) is uniformly bounded independent of \( n \). Also since \( P_n, R_n \) are convex conjugate, we see \( R_n \) are also bounded uniformly independent of \( n \).
Now we can apply Arzela-Ascoli to get a pair \((P, R)\) which is also convex conjugate over \( \Omega_H \) and \( \Lambda \).
Also by lemma 2.6, we also have
\[
\Pi_{P_n} \rightarrow \Pi_P \text{ uniformly on } \Omega_2.
\]
Since we also have \( P_n \rightarrow P \), \( R_n \rightarrow R \) uniformly.
One can see
\[
\nabla P_\delta = \nu
\]
\( h_\delta \) achieves the second infimum above.

**Proof.** Let \( g(y) \in C_b(\Lambda), \delta > 0, \) define
\[
R_\delta(y) := R(y) + \delta g(y) \quad P_\delta(x) := \sup_{y \in \Lambda} (x \cdot y - R_\delta(y))
\]
Since \((P, R)\) a pair of maximizers, we have
\[
J^H(P_\delta, R_\delta) \leq J^H(P, R)
\]
\( \nu \) is the gradient of \( \nu \).

Note that \( P_\delta(x_1, x_2, s) \) achieves minimum over \( [0, H] \) at \( h_\delta \). Therefore \( h_\delta \) achieves the second infimum above.
Then we notice that since \( R_\delta \rightarrow R \) uniformly on \( \Lambda \), we have \( P_\delta \rightarrow P \) uniformly on \( \Omega_H \). Hence by Lemma 0.1, we have
\[
h_\delta(x_1, x_2) \rightarrow h(x_1, x_2) \text{ uniformly}
\]
Suppose \( P \) is differentiable at \( x \), and let \( y_\delta \in \bar{\Lambda} \) be the point such that
\[
P_\delta(x) = x \cdot y_\delta - R_\delta(y_\delta)
\]
then we have
\[ y_5 \rightarrow \bigtriangledown P(x) \]
Also notice that
\[ -g(\bigtriangledown P(x)) \leq \frac{P_5(x) - P(x)}{\delta} \leq -g(y_5) \]
By letting \( \delta \to 0 \), we obtain
\[ -\int_{\lambda} g(y)\nu(y)dy \leq -\int_{\Omega_h} g(\bigtriangledown P(x))dx \]
Replacing \( g \) by \( -g \), we are done. \( \square \)

Now we prove the function \( h^H \) obtained above is Lipschitz.

**Proposition 2.11.** Let \((P,R)\) be a pair of convex conjugate maximizers over the domain \( \Omega_2 \times [0,H] \) and \( \Lambda \) respectively, then \( h^H(x_1, x_2) \) is Lipschitz with a Lipschitz constant depending only on \( \delta, \Lambda, \Omega_2 \)(not \( H \)).

**Proof.** Let’s denote \( h^H(x_1, x_2) = h \). The proof is based on the following
\[ \text{(i)} P(x_1, x_2, h(x_1, x_2)) = \frac{1}{2}(x_1^2 + x_2^2) \text{ whenever } 0 < h < H \]
\[ \text{(ii)} \partial_{x_1} P \leq -\delta, \text{ and } \partial_{x_2} P \leq C. \text{ Here } C \text{ depend only on } \Lambda \]

Pick \((x_1, x_2), (z_1, z_2) \in \Omega_2\), without loss of generality, we can assume \( h(z_1, z_2) < h(x_1, x_2) \)

If \( 0 < h(z_1, z_2) < h(x_1, x_2) < H \), then we have
\[ P(x_1, x_2, h(x_1, x_2)) = \frac{1}{2}(x_1^2 + x_2^2) \]
Thus
\[ -C(|x_1 - z_1| + |x_2 - z_2|) \leq P(x_1, x_2, h(x_1, x_2)) - P(x_1, x_2, h(z_1, z_2)) + P(x_1, x_2, h(z_1, z_2)) + P(z_1, z_2, h(z_1, z_2)) \]
\[ \leq -\delta(h(x_1, x_2) - h(z_1, z_2)) + C(|x_1 - z_1| + |x_2 - z_2|) \]

Next consider \( 0 < h(z_1, z_2) < h(x_1, x_2) = H \), then we have
\[ P(x_1, x_2, H) \geq \frac{1}{2}(x_1^2 + x_2^2) \]
Thus
\[ -C(|x_1 - z_1| + |x_2 - z_2|) \leq P(x_1, x_2, H) - P(z_1, z_2, h(z_1, z_2)) \]
\[ \leq C(|x_1 - z_1| + |x_2 - z_2|) - \delta(H - h(z_1, z_2)) \]

Notice that if \( h(z_1, z_2) = 0 \), then \( P(z_1, z_2, 0) \leq \frac{1}{2}(z_1^2 + z_2^2) \), so other cases can be dealt with similarly. \( \square \)

**Corollary 2.12.** Suppose \( H \geq \frac{2}{C(\Omega_2)} \)(such that \( J^H(P, R) \) has maximizers by thm 2.8), and let \((P, R)\) be convex conjugate maximizers, then there exists a constant \( C = C(\text{diam } \Omega_2, \delta, \Lambda) \), such that
\[ h^H(x_1, x_2) \leq C \]

**Proof.** By lemma 0.4, we have
\[ \int_{\Omega_2} h = 1 \]
so there exists \((z_1, z_2) \in \Omega_2\), such that \( h(z_1, z_2) \leq \frac{2}{C(\Omega_2)} \).
By above cor, we know \( h \) is Lipschitz, and since
\[ P(x_1, x_2, h) = \frac{1}{2}(x_1^2 + x_2^2) \text{ whenever } h > 0 \]
so
\[ \nabla h(x_1, x_2) = \frac{1}{\partial x_3 P}(x_1 - \partial_1 P, x_2 - \partial_2 P) \]
Thus
\[ |\nabla h| \leq \frac{\max_{\Omega_2} |x| + \max_{\Lambda}(|y_1| + |y_2|)}{\delta} \]
So
\[ |h| \leq \frac{2}{C^2(\Omega_2)} + \frac{2\max_{\Omega_2} |x| + \max_{\Lambda}(|y_1| + |y_2|)}{\delta} \cdot \text{diam } \Omega_2 \]  
(2.13)

We deduce a trivial result from above corollary.

**Corollary 2.14.** Suppose \( H > C \), here the \( C \) is the constant the right hand side of (2.13), if \((P, R)\) is a convex conjugate maximizer of \( J^H(P, R) \), then \( 0 \leq h^H_P < H \).

**Corollary 2.15.** Suppose \( H > 0 \) satisfy the same condition as in previous corollary, then there exists constant \( C = C(\delta, \Omega_2, \Lambda, H) \) such that if \((P, R)\) is a convex conjugate maximizer of \( J^H_P(P, R) \), we have \( |P(x)| \leq C, \forall x \in \Omega_H \).

**Proof.** Fix \((x_1, x_2) \in \Omega_2\), if \( h(x_1, x_2) > 0 \), we then have
\[ \frac{1}{2}(x_1^2 + x_2^2) \leq P(x_1, x_2, 0) \]
On the other hand
\[ \frac{1}{2}(x_1^2 + x_2^2) = P(x_1, x_2, h(x_1, x_2)) = P(x_1, x_2, 0) + \int_0^h \partial x_3 P(x_1, x_2, s) ds \]
\[ \geq -\frac{1}{\delta}h(x_1, x_2) + P(x_1, x_2, 0) \]
Thus by corollary 2.11, we know
\[ |P(x_1, x_2, 0)| \leq C \text{ such that } h(x_1, x_2) > 0 \]
where \( C \) has the said dependence.
Now notice that since by assumption \((P, R)\) are convex conjugate over the domain \( \Omega_H \) and \( \Lambda \) respectively, we know
\[ \partial P(\Omega_H) \subset \Lambda \]
Hence
\[ |\nabla P(x)| \leq C(\Lambda, \delta) \text{ a.e} \]
Fix \((z_1, z_2)\) such that \( h(z_1, z_2) > 0 \), then for \( x \in \Omega_H \), we have
\[ P(x_1, x_2, x_3) = P(z_1, z_2, 0) + \int_0^1 \nabla P(t(z_1, z_2, 0) + (1-t)(x_1, x_2, x_3))(x_1-z_1, x_2-z_2, x_3) dt \]
The result follows easily. \( \square \)

In the sequel, we will always assume
\[ H > \frac{2}{C^2(\Omega_2)} + \frac{2\max_{\Omega_2} |x| + \max_{\Lambda}(|y_1| + |y_2|)}{\delta} \cdot \text{diam } \Omega_2 \]  
(2.16)
unless otherwise stated.
Theorem 2.17. Suppose $H$ is as in (2.16), $\nu \in \mathcal{P}(\mathbb{R}^3)$, with $\text{supp} \ \nu \subset \Lambda$ then we have

(i) $E_\nu(h, \gamma) \geq J^H(P,R)$, for any $(h, \gamma) \in \mathcal{M}$ and $(P,R)$ satisfying $P(x) + R(y) \geq x \cdot y \ \forall x \in \Omega_H \ \forall y \in \Lambda$

(ii) Suppose $(P,R)$ is a convex conjugate maximizer of $J^H(P,R)$, then we have equality above iff $\gamma = (id \times \nabla P)_2 \sigma_h$ and $h = h^H_P$

(iii) In the situation of (ii), and if $\nu << \mathcal{L}^3$, we also have

$$\nabla R_2 \nu = \sigma_h$$

Proof. First we prove (i).

Without loss of generality, we can assume $(P, R)$ be a pair of convex dual maximizers of $J^H(P, R)$, and we can naturally extend $P$ to be defined on $\Omega_\infty$ such that $(P, R)$ are again convex conjugate. Indeed,

$$P(x) = \sup_{y \in \Lambda} (x \cdot y - R(y))$$

and we define $P(x)$ for $x \in \Omega_\infty$ by the same formula. Then one can check

$$R(y) = \sup_{x \in \Omega_H} (x \cdot y - P(x)) = \sup_{x \in \Omega_\infty} (x \cdot y - P(x))$$

This implies in particular

$$P(x) + R(y) \geq x \cdot y \ \forall x, y \in \Omega_\infty \ \forall \Lambda$$

Therefore, assuming $(h, \gamma) \in \mathcal{M}$, we can write

$$\int c(x, y) d\gamma(x, y) \geq \int_{\Lambda} \left[ \frac{1}{2}(y_1^2 + y_2^2) - R(y) \right] \nu(y) dy + \int_{\Omega_\infty} \left[ \frac{1}{2}(x_1^2 + x_2^2) - P(x) \right] \sigma_h(x) dx$$

$$\geq \int_{\Lambda} \left[ \frac{1}{2}(y_1^2 + y_2^2) - R(y) \right] \nu(y) dy + \int_{\Omega_\infty} \left[ \frac{1}{2}(x_1^2 + x_2^2) - P(x) \right] \sigma_h^H(x) dx = J^H(P, R) \hspace{1cm} (*)$$

Here only the second inequality requires some explanation.

By corollary 2.13 and by strict convexity of $\Pi_p(x_1, x_2, s)$ in $s$, we see that $s \mapsto \Pi_p(x_1, x_2, s)$ attains minimum over $[0, \infty)$ (not just $[0, H]$) at $h^H_p(x_1, x_2)$. So we have

$$\int_{\Omega_\infty} \left[ \frac{1}{2}(x_1^2 + x_2^2) - P(x) \right] \sigma_h(x) dx = \int_{\Omega_2} \Pi_p(x_1, x_2, h^H(p, x_1, x_2)) dx_1 dx_2$$

$$\geq \int_{\Omega_2} \Pi_p(x_1, x_2, h^H(p, x_1, x_2)) dx_1 dx_2 = \int_{\Omega_\infty} \left[ \frac{1}{2}(x_1^2 + x_2^2) - P(x) \right] \sigma_h^H(x) dx$$

It’s easy to see above inequality takes equality iff $h(x_1, x_2) = h^H(p, x_1, x_2)$.

Up to now, we proved (i).

Now we can prove (ii). Suppose we have equality, then both inequality in (*) must be equality. Thus we know $h = h^H_P$ and we know from corollary 2.12 that $\text{supp} \ \sigma_h \subset \Omega_H$. The first inequality takes equality iff $P(x) + R(y) = x \cdot y \ \gamma = a.e \hspace{0.2cm} (x, y)$, or $y \in \partial P(x) \gamma = a.e \hspace{0.2cm} (x, y)$, so we obtain $\gamma = (id \times \nabla P)_2 \sigma_h$. Conversely, if both inequalities are equalities, then we have $E_\nu(h, \gamma) = J^H(P, R)$.

To see (iii), note that by (ii), we have

$$\nabla P_2 \sigma_h = \nu$$

And since $\nu << \mathcal{L}^3$, we know

$$\nabla R \circ \nabla P(x) = x \ \sigma_h \ a.e$$
Indeed put

\[ E = \{ y \in \Lambda | \nabla R \text{ is not defined} \} \]

then

\[ \nu(E) = \sigma_h ((\nabla P)^{-1}(E)) = 0 \]

So

\[ \sigma_h = \nabla R_{\sharp} \nabla P_{\sharp} \sigma_h = \nabla R_{\sharp} \nu \]

\[ \square \]

Now we can prove the unique existence of the above variational problem

**Corollary 2.18.** Suppose \( H \) is as (2.16), then \( E_{\nu}(h, \gamma) \) has a unique minimizer \( (h, \gamma) \) and \( J^H(P, R) \) has a unique convex conjugate maximizer in the sense that if \( (P_0, R_0), (P_1, R_1) \) are both maximizers, convex conjugate over \( \Omega_{\infty} \) and \( \Lambda \), we have \( P_0 = P_1 \sigma_{\nu_0} \text{ a.e.} \)

**Proof.** Existence of at least one maximizer has been proved in theorem 2.8. Theorem 2.16 (ii) gives the existence of at least one minimizer of \( E_{\nu}(h, \gamma) \). Now we show uniqueness.

First we fix a maximizer of \( J^H(P, R) \), say \( (P_0, R_0) \). If \( (h_0, \gamma_0), (h_1, \gamma_1) \) are both minimizers of \( E_{\nu}(h, \gamma) \), then we have by above theorem

\[ h_0 = h_1 = h_{P_0}^H(x_1, x_2) := h_0 \text{, and } \gamma_1 = \gamma_0 = (id \times \nabla P_0)_{\sharp} \sigma_h \]

This proves the uniqueness of the minimizer of \( E_{\nu}(h, \gamma) \).

To see the uniqueness of \( J^H(P, R) \), say both \( (P_0, R_0), (P_1, R_1) \) are maximizers. Let \( h_0 = h_{P_0}^H \) and \( \gamma_0 = (id \times \nabla P_0)_{\sharp} \sigma_{\nu_0} \). Then we know by prop 2.10 that \( h_0 \) is Lipschitz, also we know by thm 1.10 (ii) that \( h = h_{P_1}^H \)

Also by uniqueness of minimizer for \( E_{\nu}(h, \gamma) \), we know from above theorem (ii)

\[ (id \times \nabla P_0)_{\sharp} \sigma_{\nu_0} = (id \times \nabla P_1)_{\sharp} \sigma_{\nu_0} \]

This implies

\[ \nabla P_0 = \nabla P_1 \sigma_{\nu_0} \text{ a.e.} \]

Let \( U \subset \Omega_2 \) to be a connected component of the open set \( \{ (x_1, x_2) \in \Omega_2 | h_0(x_1, x_2) > 0 \} \), then the set \( U_h := \{ (x_1, x_2, x_3) | (x_1, x_2) \in U, 0 < x_3 < h_0(x_1, x_2) \} \) is connected and so we have

\[ P_0 = P_1 + C_U \text{ on } U_h \]

But we also have

\[ P_0(x_1, x_2, h_0(x_1, x_2)) = P_1(x_1, x_2, h_0(x_1, x_2)) = \frac{1}{2} (x_1^2 + x_2^2) \]

for \( (x_1, x_2) \in U \), hence \( C_U = 0 \) So we have

\[ P_0 = P_1 \sigma_{\nu_0} \text{ a.e.} \]

\[ \square \]

**Corollary 2.19.** Let \( \Lambda = B_D(0) \times [-\frac{1}{\delta}, -\delta] \) and assume also that \( \Omega_2 \subset B_D(0), H \) is chosen as (2.16). Then there exists a unique maximizer \( (P_2, R_2) \) of \( J^H(P, R) \) which has following properties:

(i) \( (P_2, R_2) \) are convex conjugate over both \( \Omega_{\nu_0} \cup \{ x_3 = 0 \}, \Lambda \) and \( \Omega_{H}, \Lambda \).

(ii) \( P_2(x_1, x_2, 0) = \frac{1}{2} (x_1^2 + x_2^2) \) whenever \( h_0(x_1, x_2) = 0 \)
Proof. The uniqueness is easy to see, since by (i), we must have
\[ R_2(y) = \sup_{x \in \Omega_h \cup \{x_3 = 0\}} (x \cdot y - P_2(x)) \]
The value of \(P_2\) on \(\Omega_h \cup \{x_3 = 0\}\) is uniquely defined by previous corollary on \(\Omega_h\) and (ii) of this corollary on \(\{x_3 = 0\}\).
And
\[ P_2(x) = \sup_{y \in \Lambda} (x \cdot y - R_2(y)) \quad x \in \Omega_H \]
because \((P_2, R_2)\) are assumed to be convex conjugate over \(\Omega_\infty, \Lambda\).
The existence of such a maximizer is more technical and is implied by corollary 6.3 in the appendix. \(\square\)

Remark 2.20. We observe here that above two properties are preserved in the limit. Namely, if \((P_n, R_n)\) are maximizers of \(J^H_{\nu_n}(P, R)\) with above properties \(h_0\) replaced by \(h_n\) and \(P_n \to P, R_n \to R\) uniformly on \(\Omega_H\) and \(\Lambda, h_n \to h\) uniformly on \(\Omega_2\), then the limit \((P, R)\) satisfies above properties with \(h_0\) replaced by \(h\).
Indeed, \(P, R\) is easily seen to be convex conjugate over \(\Omega_H\) and \(\Lambda\). To see they are also convex conjugate over \(\Omega_h \cup \{x_3 = 0\}\), we notice that
\[
clos(\Omega_n \cup \{x_3 = 0\}) = \{x \in \Omega_H | 0 \leq x_3 \leq h_n(x_1, x_2)\} \to \{x \in \Omega_H | 0 \leq x_3 \leq h(x_1, x_2)\}
= clos(\Omega \cup \{x_3 = 0\})
\]
since \(h_n\) converges uniformly.
Therefore
\[
R_n(y) = \sup_{clos(\Omega_n \cup \{x_3 = 0\})} (x \cdot y - P_n(x)) \to \sup_{clos(\Omega_h \cup \{x_3 = 0\})} (x \cdot y - P(x))
= R(y)
\]
since all the functions involved are continuous.
The property (ii) for \(P_n\) means \(P_n(x_1, x_2, h_n(x_1, x_2)) = \frac{1}{2}(x_1^2 + x_2^2)\) and is preserved in the uniform limit.

To conclude this section, we prove a stability result. We start with some lemmas

Lemma 2.21. Let \(\nu_n \to \nu\) narrowly, then we have
\[
\inf_{(h, \gamma) \in \mathcal{M}_\nu} E_\nu(h, \gamma) \leq \lim_{n \to \infty} \inf_{(h, \gamma) \in \mathcal{M}_{\nu_n}} E_{\nu_n}(h, \gamma)
\]
Proof. We choose \((h_n, \gamma_n)\) to be the minimizer of \(E_{\nu_n}(h, \gamma)\). Then we have shown that \(||h_n||_{W^{1, \infty}(\Omega_2)} \leq C\) for some universal constant \(C\). So by Arzela-Ascoli, we can take a subsequence (not relabeled) \(h_n \to h_0\) uniformly on \(\Omega_2\) and thus \(\sigma_{h_n} \to \sigma_{h_0}\) narrowly. Now since \(\{\gamma_n\}_{n \geq 1}\) is tight, we can take a subsequence \(\gamma_n \to \gamma_0\). It’s easy to see \((h_0, \gamma_0) \in \mathcal{M}_\nu\). And so
\[
\inf_{(h, \gamma) \in \mathcal{M}_\nu} E_\nu(h, \gamma) \leq \inf_{(h, \gamma) \in \mathcal{M}_\nu} E_\nu(h_0, \gamma_0) = \int c(x, y) d\gamma_0(x, y) = \lim_{n \to \infty} \int c(x, y) d\gamma_n(x, y) 
\leq \lim_{n \to \infty} \inf_{(h, \gamma) \in \mathcal{M}_{\nu_n}} E_{\nu_n}(h, \gamma)
\]
\(\square\)
Lemma 2.22. Let \( \nu_n \to \nu \) narrowly, we then have
\[
\sup_{(P,R) \in \mathcal{N}} J^H_{\nu}(P,R) \geq \lim_{n \to \infty} \sup_{(P,R) \in \mathcal{N}_{\nu_n}} J^H_{\nu_n}(P,R)
\]

Proof. For each \( n \geq 1 \), take \((P_n, R_n)\) to be maximizer of \( J^H_{\nu_n}(P,R) \), so that they are convex conjugate over \( \Omega_H \) and \( \Lambda \) respectively. Then we have \( \partial(P_n) \subset \Lambda \) in particular \( \{P_n\}_{n \geq 1} \) are equitriangular. Also we proved in corollary 2.14 they are uniformly bounded. So by Arzela-Ascoli, we can take a subsequence (not relabeled) \( P_n \) converges to \( P_0 \) uniformly on \( \Omega_H \). Since \( R_n \) is convex dual of \( P_n \), we know \( R_n \) converges uniformly in \( \Lambda \) to \( R_0 \) as well and the limit \((P_0, R_0)\) will also be convex conjugate, namely \((P_0, R_0) \in \mathcal{N}\). But then
\[
\sup_{(P,R) \in \mathcal{N}} J^H_{\nu}(P,R) \geq J^H_{\nu}(P_0, R_0) = \lim_{n \to \infty} J^H_{\nu_n}(P_n, R_n) = \lim_{n \to \infty} \sup_{(P,R) \in \mathcal{N}} J^H_{\nu_n}(P_n, R_n)
\]

Corollary 2.23. Suppose \( \nu_n \to \nu \) narrowly, then
\[
\lim_{n \to \infty} \inf_{(h, \gamma) \in \mathcal{M}_{\nu_n}} E_{\nu_n}(h, \gamma) = \inf_{(h, \gamma) \in \mathcal{M}_{\nu}} E_{\nu}(h, \gamma)
\]

Proof. ” \( \geq \) ” follows from lemma 2.21. ” \( \leq \) ” follows from Lemma 2.20. Recall that by thm 2.16(ii), we have
\[
\inf_{(h, \gamma) \in \mathcal{M}_{\nu}} E_{\nu}(h, \gamma) = \sup_{(P,R) \in \mathcal{N}} J^H_{\nu}(P,R)
\]

Next we prove a stability result of the optimizers under narrow convergence. It will be useful when one proves the continuity in time of certain quantities.

Theorem 2.24. Suppose \( \nu_n, \nu \in \mathcal{P}(\Lambda) \) and \( \nu_n \to \nu \) narrowly. Let \((P_n, R_n), (P, R)\) be the unique maximizers of \( J^H_{\nu_n}, J^H_{\nu} \) given by cor 2.11. \((h_n, \gamma_n)\) be the maximizers of \( E_{\nu_n}, E_{\nu} \) respectively, then the following holds.
(i) \( h_n \to h \) uniformly.
(ii) \( P_n \to P \) in \( W^{1,r}(\Omega_H) \) for any \( r < \infty \)
(iii) \( R_n \to R \) in \( W^{1,r}(\Lambda) \) for any \( r < \infty \)

Proof. First we prove (i).
We proved in proposition 2.10 and corollary 2.18 that \( h_n \) is bounded in \( W^{1,\infty}(\Omega) \). Hence by Arzela-Ascoli, for any subsequence (not relabeled) \( h_n \), there is a further subsequence \( h_n \) which converges uniformly to \( h_0 \in W^{1,\infty}(\Omega) \). This implies \( \sigma_{h_n} \to \sigma_{h_0} \) narrowly.

Also since \( \{\gamma_n\}_{n \geq 1} \) have uniformly bounded support, they are tight and so we can take a subsequence \( \gamma_n \) which converges to \( \gamma_0 \) narrowly. Now since \( \gamma_n \in \Gamma(\sigma_{h_n}, \nu_n) \). It's easy to see by passing to limit that
\[
\int c(x,y)d\gamma_n(x,y) \to \int c(x,y)d\gamma_0(x,y)
\]
and
\[
\gamma_0 \in \Gamma(h_0, \nu)
\]
By previous corollary, we know
\[
E_{\nu}(h_0, \gamma_0) = \lim_{n \to \infty} \int c(x,y)d\gamma_n(x,y) = \lim_{n \to \infty} \inf_{(h, \gamma) \in \mathcal{M}_{\nu_n}} E_{\nu_n}(h, \gamma) = \inf_{(h, \gamma) \in \mathcal{M}_{\nu}} E_{\nu}(h, \gamma)
\]
Therefore \((h_0, \gamma_0)\) is the minimizer of \(E_\nu(h, \gamma)\)
By the uniqueness of minimizer, we know any subsequence \(h_n\) must have the same limit, therefore the whole sequence must converge.

Next we prove \((ii)\).
The same argument as in \((i)\) shows that any subsequence \((P_n, R_n)\) has a further subsequence which converges uniformly on \(\Omega_H, \Lambda\) respectively. Let’s denote the limit to be \((P_0, R_0)\) then \((P_0, R_0)\) satisfies properties \((i), (ii)\) of cor 2.11 due to the remark after 2.11. Since \(P_n \to P_0\), we have by lemma 2.6 \(h^H_{P_n} \to h^H_{P_0}\) uniformly and it’s easy to see \(J^H_{\nu_0}(P_n, R_n) \to J^H_{\nu_0}(P_0, R_0)\). Again by corollary 2.21, 2.22, \((P_0, R_0)\) is a maximizer by recalling the duality proved in theorem 2.16. Now by the uniqueness of maximizer with properties \((i), (ii)\), the whole sequence must converges.

\(\square\)

Now we define
\begin{equation}
\lambda_0 = B_{D_0}(0) \times [-\frac{1}{\delta}, -\delta] \text{ and } \Lambda = B_D(0) \times [-\frac{1}{\delta}, -\delta]
\end{equation}

Based on what we proved above about the properties of the functional \(E_\nu(h, \gamma)\) and \(J^H_\nu(P, R)\) in theorem 2.16, the dual space problem can be reformulated as
\[
\partial_t \nu + \nabla \cdot (\nu w) = 0
\]
\[
w = J(y - \nabla P^s(t, y)) \in \mathbb{R}^3
\]
\[(h, \nu \times \nabla P)\] is a minimizer of \(E(h, \gamma)\) among \(M_\nu(t, \cdot)\).

In the above, \(P^s(t, y) = \sup_{x \in \Omega_H} (x \cdot y - P(x))\) for some large enough \(H\)
Next we apply the above obtained results to give an alternative proof for the existence of weak solutions in dual spaces with absolute continuous initial data, which was first done in [6], using Hamiltonian ODE approach. Our approach here is more straightforward.

**Theorem 2.26.** Let \(1 < q < \infty, T > 0\) be given and \(\nu_0 \in L^q(\Lambda_0)\) with \(\nu_0 \in \mathcal{P}_{ac}(\mathbb{R}^3)\).
Now we define
\begin{equation}
D > D_0 + \max_{\Omega_2} |x| \cdot (T + 1)
\end{equation}

\begin{equation}
H > \frac{2}{\mathcal{L}^2(\Omega_2)} + \frac{2\max_{\Omega_2} |x| + 2D}{\delta} \cdot \text{diam } \Omega_2
\end{equation}

Then there exists a weak solution in dual space \((h, P, R)\) with \(\nu(t, \cdot) := \nabla P_{t_2} \sigma_{h(t, \cdot)}\)
such that
\begin{enumerate}
\item \(t \mapsto \nu(t, \cdot) \in \mathcal{P}_{ac}(\mathbb{R}^3)\) narrowly continuous with \(\text{supp } \nu \subset \Lambda\)
\item \(\nu(t, \cdot) \in L^q(\Lambda)\), and \(|\nu(t, \cdot)\|_{L^q(\Lambda)} \leq |\nu_0(\cdot)\|_{L^q(\Lambda)} \forall t \in [0, T]\)
\item \((P(t, \cdot), R(t, \cdot))\) are unique maximizers of \(J^H_{\nu(t, \cdot)}(P, R)\) with properties \((i), (ii)\) in corollary 2.18
\item \(\mathcal{W}_1(\nu(t_1, \cdot), \nu(t_2, \cdot)) \leq C|t_1 - t_2|
\item \(P(t, \cdot) \in L^\infty([0, T], W^{1,\infty}(\Omega_\infty)) \cap C([0, T], W^{1, r}(\Omega_\infty))\)
\item \(R \in L^\infty([0, T], W^{1,\infty}(\Lambda)) \cap C([0, T], W^{1, r}(\Lambda)) \forall r < \infty
\item Let \(\gamma(t, \cdot) = (id \times \nabla P)\sigma_{h(t, \cdot)}\) \((h, \gamma)\) are minimizers of \(E_\nu(h, \gamma)\) and
\end{enumerate}
Define

\[ \nu^0(y) = j_s \ast \nu_0(y) \]

Then for \( 0 < s < \frac{3}{2} \), we have

\[ \text{supp } \nu^0_s \subset B_{R_0}(0) \times (-\frac{2}{\delta}, -\frac{\delta}{2}) \]

Next we construct approximate solutions, here we need to control the speed of propagation of the support of approximate solutions. Define

\[ D^k_s = D_0 + ks \max_{t \in [0, T]} |x| \quad 0 \leq k \leq \frac{T}{s} \]

Given \( \nu^k_s := \nu_\sigma(k s, \cdot) \) with \( \text{supp } \nu^k_s \subset B^2_{P_k}(0) \times [-\frac{1}{2}, -\frac{1}{s}] \), we obtain \( \nu^{k+1}_s \) such that \( \text{supp } \nu^{k+1}_s \subset B^2_{D^{k+1}_s}(0) \times [-\frac{1}{2}, -\frac{1}{s}] \) in the following way.

Let \((h^k_s, \nu^k_s)\) be the minimizer of \( E_{\nu^k_s}(h, \gamma) \). Let \((P^k_s, R^k_s)\) a maximizer of \( J^H_{\nu^k_s}(P, R) \) which are convex conjugate in \( \Omega_H, \Lambda \) and satisfies properties (i), (ii) in corollary 2.18. This is possible by our assumption on \( H \) and \( D \). We also see that \( \gamma^k_s = (\text{id} \times \nabla P^k_s)_{\sigma h^k_s} \) and \( \nabla R^k_s \) is the optimal map of \( \sigma h^k_s \) to \( \nu^k_s \) under cost \( c(x, y) \) and \( \nabla R^k_s \) is the optimal map of \( \nu^k_s \) to \( \sigma h^k_s \) under cost \( c(x, y) \) (Note \( c(x, y) = c(x, y) \)).

Define

\[ w^k_s(y) = J(y - \nabla R^k_s(y)) \]
\[ u^k_s(y) = J(y - \nabla (j_s \ast R^k_s(y))) \]

Observe that \( u^k_s(y) \) is \( C^\infty \) and divergence free and we obtain \( \nu^{k+1}_s \) by using the transport equation

\[ \partial_t \nu^k_s = -\nabla \cdot (u^k_s \nu^k_s) \quad \text{in } [ks, (k+1)s) \times \mathbb{R}^3 \]
\[ \nu^k_s(ks, y) = \nu^k_s(y) \quad \text{in } \mathbb{R}^3 \]

and set \( \nu^{k+1}_s = \nu_s((k+1)s, y) \).

In more detail, we solve the ODE

\[ \frac{d\Phi^k_s(t, y)}{dt} = u^k_s(\Phi^k_s(t, y)) \quad t \in [ks, (k+1)s], y \in \mathbb{R}^3 \]
\[ \Phi^k_s(ks, y) = y \]

Then

\[ \nu^k_s(t, y) = \nu^k_s((\Phi^k_s)^{-1}(t, y)) \quad t \in [ks, (k+1)s] \]

Therefore

\[ \text{supp } \nu^k_s(t, \cdot) \subset \Phi^k_s(t, \cdot)(\text{supp } \nu^k_s) \]

Now take \( y_0 \in \text{supp } \nu^k_s \), we know

\[ \frac{d|\Phi^k_s(t, y_0)|}{dt} = \frac{\Phi^k_s \frac{d\Phi^k_s}{dt}}{|\Phi^k_s|} = \frac{\Phi^k_s \cdot J(\Phi^k_s - \nabla (j_s \ast R^k_s))}{|\Phi^k_s|} \]

Now notice that

\[ \Phi^k_s \cdot J\Phi^k_s = 0 \quad |J(\nabla (j_s \ast R^k_s))|_3 = 0 \]
Hence
\[ |\Phi^k_s \cdot J(\Phi^k_s - \nabla (j_s \ast R^k_s))| \leq |\Phi^k_s| |\nabla_2 R^k_s| \leq |\Phi^k_s| \max_{x \in \Omega_2} |x| \]

Here \( \nabla_2 \) stands for the gradient with respect to the first two variables only. It follows that
\[ \frac{d|\Phi^k_s(t, y_0)|}{dt} \leq \max_{\Omega_2} |x| \quad t \in [ks, (k+1)s] \]

hence
\[ |\Phi^k_s((k+1)s, y_0)| \leq D^k_s + s \max_{\Omega_2} |x| \leq D^{k+1}_s \]

So we obtained
\[ \text{supp } \nu^{k+1}_s \subset B^2_{D^{k+1}_s} \times [-\frac{1}{\delta}, \delta] \]

Also we can define \( P^{k+1}_s, R^{k+1}_s, h^{k+1}_s, w^{k+1}_s, u^{k+1}_s \) in the same way as above. Denote
\( P_s(t, \cdot), R_s(t, \cdot), \tilde{R}_s(t, \cdot), \gamma_s(t, \cdot) \) be above defined piecewise in time constant function, i.e., \( \tilde{P}_s(t, \cdot) \equiv P^k_s \) \( t \in [ks, (k+1)s] \), similar for others.

For \( t \in [0, T] \) define
\[
(h_s(t, \cdot), \gamma_s(t, \cdot)) = \text{the minimizer of } E_{\nu_s(t, \cdot)}(h, \gamma)
\]

\[
(P_s(t, \cdot), R_s(t, \cdot)) = \text{the maximizer of } J^H_{\nu_s(t, \cdot)}(P, R)
\]

obtained in a way similar as described above. Also define
\[
w_s(t, y) = J(y - \nabla R_s(t, y))
\]

\[
u_s(t, y) = J(y - \nabla (j_s \ast R_s)(t, y))
\]

Recall the definition of \( D \), we have shown that if \( s < \frac{\delta}{2} \)
\[ \text{supp } \nu_s(t, \cdot) \subset B_{D}(0) \times [-\frac{1}{\delta}, \delta] \]

Similar to the proof in Cullen-Gangbo, one has
\[ W_1(\nu_s(t_1, \cdot), \nu_s(t_2, \cdot)) \leq (D + \max_{x \in \Omega_2} |x|) ||\nu_0||_{L^1(\mathbb{R}^3)} |t_1 - t_2| \]

where \( W_1(\cdot, \cdot) \) is the 1-Wasserstein distance. Follow the proof of [3] thm 5.3, we can get up to a subsequence
\[
\nu_{s_j} \rightharpoonup \nu \text{ weakly in } L^r([0, T] \times \mathbb{R}^3)
\]

and
\[
\nu_{s_j}(t, \cdot) \rightharpoonup \nu(t, \cdot) \text{ weakly in } L^r(\Lambda)
\]

Also
\[
\nu(t, \cdot) \in L^r([0, T] \times \Lambda) \quad ||\nu(t, \cdot)||_{L^r(\Lambda)} \leq ||\nu_0||_{L^r(\Lambda)}
\]

\[
W_1(\nu(t_1, \cdot), \nu(t_2, \cdot)) \leq (D + \max_{x \in \Omega_2} |x|) ||\nu_0||_{L^1(\mathbb{R}^3)} |t_1 - t_2| \]

Since \( \nu_{s_j}(t, \cdot) \rightharpoonup \nu(t, \cdot) \) narrowly as measures, we conclude by theorem 2.24 that
\( \nabla \tilde{R}_s(t, \cdot) \rightharpoonup \nabla R(t, \cdot) \) in \( L^r(\Lambda) \) and hence \( \tilde{u}_{s_j} \rightharpoonup \tilde{w} \) in \( L^r(\Lambda) \).

\[
\tilde{u}_{s_j} \nu_{s_j}(t, \cdot) \rightharpoonup w \nu(t, \cdot) \text{ weakly in } L^r(\Lambda, \mathbb{R}^3)
\]

so we conclude and \( \nu \) satisfies the equation
\[
\partial_t \nu + \nabla \cdot (w \nu) = 0 ; \nu(0, \cdot) = \nu_0
\]

in the sense of distribution. where \( w = J(y - \nabla R(t, y)) \) and \( (P(t, \cdot), R(t, \cdot)) \) is the minimizer of \( J^H_{\nu(t, \cdot)}(P, R) \) The property of \( P, h \) follows from the stability result proved above and the narrow continuity of \( \nu_t \) \( \Box \)
2.2. Generalization to \( \nu \) with unbounded support. In this subsection, we will consider the case when \( \nu \) may have unbounded support and generalize the properties obtained in previous subsection. The result of this section will be used only in section 5, when the initial data has only \( L^2 \) instead of \( L^\infty \) gradient. The ideas are quite similar, but certain complications arise.

We will always take in this subsection \( \Lambda = \mathbb{R}^2 \times [-\frac{1}{3}, -\delta] \), and we assume \( \nu \in \mathcal{P}_2(\mathbb{R}^3) \) with \( \text{supp} \ \nu \subset \Lambda \). In this setting, \( E_\nu(\mathcal{h}, \gamma) \) is defined the same way as (2.1). \( J_\nu(P, R) \) is defined as in (2.2). Trivial examples as \( P(x) = \frac{1}{2} (x_1^2 + x_2^2) \), \( R(y) = \frac{1}{2} (y_1^2 + y_2^2) \) shows

\[
(2.29) \quad \sup_{(P, R) \in \mathcal{N}} J_\nu(P, R) \geq 0
\]

Recall \( \mathcal{N} \) is the set of pairs \((P, R)\) such that \( P(x) + R(y) \geq x \cdot y \) with suitable integrability condition. See subsection 2.1.

Suppose \((P, R) \in \mathcal{N} \), we can then use the double convexification trick to define

\[
R_0(y) = \sup_{x \in \Omega_\infty} (x \cdot y - P(x))
\]

and

\[
P_0(x) = \sup_{y \in \Lambda} (x \cdot y - R_0(y))
\]

we have

\[
P_0 \leq P \text{ and } R_0 \leq R
\]

Therefore

\[
J_\nu(P, R) \leq J_\nu(P_0, R_0)
\]

then set \( \hat{P}_0(x) = \max(P_0(x), \frac{1}{2} (x_1^2 + x_2^2)) \) and set

\[
R_1(y) = \sup_{x \in \Omega_\infty} (x \cdot y - \hat{P}_0(x))
\]

\[
P_1(x) = \sup_{y \in \Lambda} (x \cdot y - R_1(y))
\]

Since \( \hat{P}_0(x) \geq \frac{1}{2} (x_1^2 + x_2^2) \), it’s easy to see \( R_1(y) \leq \frac{1}{2} (y_1^2 + y_2^2) \). Since \( \hat{P}_0(x) + R_1(y) \geq x \cdot y \), we see \( P_1(x) \leq \hat{P}_0(x) \) and the definition of \( P_1 \) shows \( P_1(x) + R_1(y) \geq x \cdot y \). The proof of lemma 6.1(ii) below shows \( P_1(x_1, x_2, 0) \geq \frac{1}{2} (x_1^2 + x_2^2) \). Since \( R_1(y) \leq R(y) \), we have

\[
\int_{\Omega_\infty} \left[ \frac{1}{2} (y_1^2 + y_2^2) - R(y) \right] d\nu(y) \leq \int_{\Omega_\infty} \left[ \frac{1}{2} (y_1^2 + y_2^2) - R_1(y) \right] d\nu(y)
\]

Put \( h_0(x) = h_{P_0}(x) \), which is well defined since \( -\frac{1}{3} \leq \partial_{x_i} P \leq -\delta \) (this follows from conjugation), then we have

\[
\inf_{h \geq 0} \int_{\Omega_\infty} \left[ \frac{1}{2} (x_1^2 + x_2^2) - P_0(x) \right] dx = \int_{\Omega_h} \left[ \frac{1}{2} (x_1^2 + x_2^2) - P_0(x) \right] dx = \int_{\Omega_h} \left[ \frac{1}{2} (x_1^2 + x_2^2) - \hat{P}_0(x) \right] dx
\]

\[
= \inf_{h \geq 0} \int_{\Omega_h} \left[ \frac{1}{2} (x_1^2 + x_2^2) - \hat{P}_0(x) \right] dx \leq \inf_{h \geq 0} \int_{\Omega_h} \left[ \frac{1}{2} (x_1^2 + x_2^2) - P_1(x) \right] dx
\]

Since \( P_1(x) \leq \hat{P}_0(x) \), which gives \( J_\nu(P_0, R_0) = J_\nu(\hat{P}_0, R_0) \leq J_\nu(P_1, R_1) \). Summarizing above discussion, we get the following lemma.
Lemma 2.30. Let \((P, R) \in \mathcal{N}\), then there exists \((P_1, R_1) \in \mathcal{N}\) such that

(i) \(J_\nu(P, R) \leq J_\nu(P_1, R_1)\)

(ii) \(P_1(x_1, x_2, 0) \geq \frac{1}{2}(x_1^2 + x_2^2)\) and \(R_1(y) \leq \frac{1}{2}(y_1^2 + y_2^2)\)

(iii) \(P_1(x), R_1(y)\) are convex conjugate over \(\Omega_\infty, \Lambda\)

Let \((P_n, R_n)\) be a maximizing sequence of \(J_\nu(P, R)\). By the above lemma, one can assume \((P_n, R_n)\) has additional properties (ii), (iii) above. Besides, we can also assume \(J_\nu(P_n, R_n) \geq 0\) by (2.28). Next we will derive some bound for the maximizing sequence, which allows us to pass to limit and prove the existence of at least one maximizer.

Lemma 2.31. Let \((P_n, R_n) \in \mathcal{N}\) be a maximizing sequence of \(J_\nu(P, R)\) with the properties (ii), (iii) in previous lemma, Suppose \(J_\nu(P_n, R_n) \geq 0\), and put \(h_n = h_{P_n}\), then there exists a constant \(C = C(M_2(\nu), \delta, \Omega_2)\), such that

\[
\|[h_n]\|_{L^2(\Omega_2)}, \|[P_n(\cdot, 0)]\|_{L^2(\Omega_2)} \leq C
\]

and constant \(C_K = C(K, M_2(\nu), \delta, \Omega_2)\), such that

\[
\|P_n\|_{L^2(\Omega_2 \times [0, K])} \leq C_K
\]

We have also

\[
-\max_{\Omega_2} |x||y| - \frac{2C_0}{\mathcal{L}^2(\Omega_2)} \leq R_n(y) \leq \frac{1}{2}(y_1^2 + y_2^2)
\]

where \(C_0\) has the same dependence as \(C\) and in particular \(\|R_n\|_{L^1(d\nu)} \leq C\)

Proof. We start with the following estimate

\[
0 \leq J_\nu(P_n, R_n) = \int [\frac{1}{2}(y_1^2 + y_2^2) - R_n(y)]d\nu(y) + \inf_{h \geq 0} \int [\frac{1}{2}(x_1^2 + x_2^2) - P_n(x)]\sigma_h(x)dx
\]

By taking \(h = \frac{2}{\mathcal{L}^2(\Omega_2)}\), and noticing that \(-\frac{1}{\delta} \leq \partial_{x_3} P_n \leq -\delta\), we see

above \leq \int [\frac{1}{2}(y_1^2 + y_2^2) - R_n(y)]d\nu(y) + \int_{\Omega_2 \times [0, \frac{2}{\mathcal{L}^2(\Omega_2)}]} [\frac{1}{2}(x_1^2 + x_2^2) - P_n(x)]d\nu(x)dx

\leq \int [\frac{1}{2}(y_2^2 + y_2^2) - R_n(y)]d\nu(y) + \int_{\Omega_2 \times [0, \frac{2}{\mathcal{L}^2(\Omega_2)}]} [\frac{1}{2}(x_1^2 + x_2^2) - P_n(x, x_2, 0) + \frac{1}{3}x_3]d\nu(x)dx

(2.35)

\leq \frac{1}{2} \left[M_2(\nu) + 2 \max_{\Omega_2} |x|^2 + \frac{4}{\delta \mathcal{L}^2(\Omega_2)} \right] - \int R_n(y)d\nu(y) - \frac{2}{\mathcal{L}^2(\Omega_2)} \int_{\Omega_2} P_n(x_1, x_2, 0)d\nu(x_1, x_2, 0)d\nu(x_2)

Now we notice

\[P_n(x_1, x_2, 0) + R_n(y) \geq x_1 y_1 + x_2 y_2 \geq -\frac{1}{2}(x_1^2 + x_2^2 + y_1^2 + y_2^2)\]

Therefore integrating above inequality against \(\chi_{\Omega_2 \times d\nu}\), we obtain

\[
\int_{\Omega_2} P_n(x_1, x_2, 0)d\nu(x_1, x_2) + \mathcal{L}^2(\Omega_2) \int R_n(y)d\nu(y) \geq -\frac{1}{2} M_2(\nu) \mathcal{L}^2(\Omega_2) - \frac{1}{2} \mathcal{L}^2(\Omega_2) \max_{\Omega_2} |x|^2
\]

So we deduce from (2.34)

\[
0 \leq \frac{1}{2} [M_2(\nu) + 2 \max_{\Omega_2} |x|^2 + \frac{4}{\delta \mathcal{L}^2(\Omega_2)}] + \frac{\mathcal{L}^2(\Omega_2)}{2} (M_2(\nu) + \max_{\Omega_2} |x|^2) - \frac{1}{2} \mathcal{L}^2(\Omega_2) \int_{\Omega_2} P_n(x_1, x_2, 0)
\]
That is
\begin{equation}
(2.36)
\int_{\Omega_2} P_n(x_1, x_2, 0) \leq \frac{L^2(\Omega_2)}{2}[M_2(\nu)+2\max_{\Omega_2} |x|^2 + \frac{4}{\delta L^2(\Omega_2)}] + \frac{(L^2(\Omega_2))^2}{2}(M_2(\nu)+\max_{\Omega_2} |x|^2) := C_0
\end{equation}

Next we want to derive a bound for $R_n$. By the $L^1$ bound derived above, we can find $(x_1^n, x_2^n) \in \Omega_2$, such that $P_n(x_1^n, x_2^n, 0) \leq \frac{2C_0}{L^2(\Omega_2)}$, so
\[ R_n(y) = \sup_{x \in \Omega_\infty} (x \cdot y - P_n(x)) \geq x_1^n y_1 + x_2^n y_2 - P_n(x_1^n, x_2^n, 0) \]
\[ \geq -\max_{\Omega_2} |x||y| - \frac{2C_0}{L^2(\Omega_2)} \]

To summarize
\[ -\max_{\Omega_2} |x||y| - \frac{2C_0}{L^2(\Omega_2)} \leq R_n(y) \leq \frac{1}{2}(y_1^2 + y_2^2) \]

So we have
\[ \int \frac{1}{2}(y_1^2 + y_2^2) - R_n(y) d\nu(y) \leq \int \frac{1}{2}(y_1^2 + y_2^2) + \max_{\Omega_2} |x||y| + \frac{2C_0}{L^2(\Omega_2)} d\nu(y) \]
\[ \leq M_2(\nu) + \max_{\Omega_2} |x|^2 + \frac{2C_0}{L^2(\Omega_2)} := C_1 \]

Next we proceed to derive an $L^2$ bound for $P_n(\cdot, 0)$. Put $h_n = h_{P_n}$. This is well defined since $\partial_x P_n \leq -\delta$. Thus
\[ -C_1 \leq \inf_{h \geq 0} \int \frac{1}{2}(x_1^2 + x_2^2) - P_n(x)|\sigma_h(x) dx = \int \frac{1}{2}(x_1^2 + x_2^2) - P_n(x)|\sigma_{h_n}(x) dx \]

To proceed further, we notice that since $P_n(x_1, x_2, h_n(x_1, x_2)) = \frac{1}{2}(x_1^2 + x_2^2)$, and for $0 \leq x_3 \leq h_n$, one has
\[ \frac{1}{2}(x_1^2 + x_2^2) \leq P_n(x) - \delta(h_n(x_1, x_2) - x_3) \]

So we conclude
\[ -C_1 \leq -\delta \int [h_n(x_1, x_2) - x_3] \sigma_{h_n}(x) dx = -\frac{\delta}{2} \int_{\Omega_2} h_n^2 \]

That is
\[ \int_{\Omega_2} h_n^2 \leq \frac{2C_1}{\delta} := C_2 \]

Finally we notice
\[ \frac{1}{2}(x_1^2 + x_2^2) \leq P_n(x_1, x_2, 0) \leq \frac{1}{\delta} h_n + P_n(x_1, x_2, h_n) = \frac{1}{\delta} h_n + \frac{1}{2}(x_1^2 + x_2^2) \]

So we have $||P_n(\cdot, 0)||_{L^2(\Omega_2)} \leq C_3$

Since $-\frac{\delta}{2} x_3 + P_n(x_1, x_2, 0) \leq P_n(x) \leq P_n(x_1, x_2, 0) - \delta x_3$, we have $||P_n||_{L^2(\Omega_2 \times [0, K])} \leq C(K, M_2(\nu), \Omega_2) \forall n \geq 1, K \geq 0$

The bounds derived in previous lemma allow us to pass to limit. By Lemma 6.5 in the appendix, we know $P_n(\cdot, 0)$ is uniformly bounded on each compact set $F$ of $\Omega_2$ and notice $-\frac{\delta}{2} \leq \partial_x P_n \leq -\delta$, $P_n$ are uniformly bounded on each set of the form $F \times [0, K]$. The argument in Lemma 6.5 shows they are equicontinuous on such a set and we can use Arzela-Ascoli.

Therefore, we can take a subsequence (not relabeled) $P_n, R_n, h_n, \sigma_n$ such that

□
\(P_n \to P\) uniformly on each compact subset of \(\Omega_\infty\) and in \(L^r(\Omega_\infty \times [0,K])\), for any \(r \in [1,2)\) and \(K \geq 0\)

\(h_n \to h\) locally uniformly in \(\Omega_\infty\) and in \(L^r(\Omega_\infty)\) for any \(r \in [1,2)\)

\(R_n \to R\) locally uniformly in \(\Lambda\) and in \(L^1(d\nu)\) (by dominated convergence)

Since for each \(n\), we have \(P_n(x_1, x_2, h_n) = \frac{1}{2}(x_1^2 + x_2^2)\), it’s easy to see in the limit \(P(x_1, x_2, h) = \frac{1}{2}(x_1^2 + x_2^2)\). So \(h = h_P\)

We now need to show the limit \((P, R)\) is a maximizer.

**Lemma 2.37.** Let \(P, R, h\) be as in previous paragraph, then we have

\[J_\nu(P, R) \geq \lim_{n \to \infty} \sup_{n} J_\nu(P_n, R_n)\]

in particular, \((P, R)\) is a maximizer. Besides,

\[P(x_1, x_2, 0) \geq \frac{1}{2}(x_1^2 + x_2^2)\]

\[P(x) = \sup_{y \in \Lambda} (x \cdot y - R(y))\]

**Proof.** First since \(R_n\) converges in \(L^1(d\nu)\), we have

\[\int \frac{1}{2}(y_1^2 + y_2^2) - R_n(y)\,d\nu(y) \to \int \frac{1}{2}(y_1^2 + y_2^2) - R(y)\,d\nu(y)\]

Next observe that on \(\Omega_{h_n}\), we have \(P_n(x) \geq \frac{1}{2}(x_1^2 + x_2^2)\), so \(0 \leq P_n(x)\sigma_{h_n}(x) \to P(x)\sigma_h(x)\) pointwise. So we can apply Fatou to see

\[\int P(x)\sigma_h(x) \leq \lim_{n \to \infty} \inf \int P_n(x)\sigma_{h_n}(x)\,dx\]

Combining the \(L^1\) convergence of \(h_n\), we have

\[\int \frac{1}{2}(x_1^2 + x_2^2) - P(x)\sigma_h(x)\,dx \geq \lim_{n \to \infty} \int \frac{1}{2}(x_1^2 + x_2^2) - P_n(x)\sigma_{h_n}(x)\,dx\]

Therefore we see

\[J_\nu(P, R) \geq \lim_{n \to \infty} \sup_{n} J_\nu(P_n, R_n)\]

Since for each \(n\), \(P_n(x_1, x_2, 0) \geq \frac{1}{2}(x_1^2 + x_2^2)\), one easily sees this is preserved in the limit.

To see \(P(x) = \sup_{y \in \Lambda} (x \cdot y - R(y))\), we first observe for each \(n\), \(P_n(x) = \sup_{y \in \Lambda} (x \cdot y - R_n(y))\). So \(P_n(x) + R_n(y) \geq x \cdot y\), and the sup is achieved on \(\partial P_n(x)\). In the limit we have \(P(x) + R(y) \geq x \cdot y\). Therefore

\[P(x) \geq \sup_{y \in \Lambda} (x \cdot y - R(y))\]

To see the reverse inequality, fix \(x^0 \in \Omega_\infty\). Find a compact set \(K\) such that \(x^0 \in K \subseteq \Omega_\infty\). Now \(\partial P_n(K)\) must be uniformly bounded in \(\Lambda\), so we can assume for some \(F\) compact \(\partial P_n(K) \subseteq F \subseteq \Lambda\), for any \(n\). Given \(\epsilon > 0\), by uniform convergence in compact sets we can take \(n = n_0\) such that

\[|P(x) - P_n(x)| \leq \epsilon \ \forall \ x \in K \ and \ |R_n(y) - R(y)| \leq \epsilon \ \forall \ y \in F\]

Let \(p \in \partial P_n(x^0) \subseteq F\), we then have

\[P(x^0) \leq P_n(x^0) + \epsilon = x^0 \cdot p - R_n(p) + \epsilon \leq x^0 \cdot p - R(p) + 2\epsilon \leq \sup_{y \in \Lambda} (x \cdot y - R(y)) + 2\epsilon\]

\(\square\)
On the other hand, if we define \( \hat{R}(y) = \sup_{x \in \Omega_\infty} (x \cdot y - P(x)) \), then \( \hat{R}(y) \leq R(y) \leq \frac{1}{2} \left( y_1^2 + y_2^2 \right) \), and so \( J_P(P, R) \leq J_P(P, \hat{R}) \) and \( (P, \hat{R}) \) is also a maximizer. Also because of the bound (2.35), this bound will also be satisfied by \( P \), so we can conclude \( R \) satisfies the bound (2.33). To summarize, we get

**Corollary 2.38.** There exists a maximizer \((P_0, R_0)\) of \( J_\nu(P, R) \), such that

(i) \((P_0, R_0)\) convex conjugate over \( \Omega_\infty \) and \( \Lambda \)

(ii) \( P_0(x_1, x_2, 0) \geq \frac{1}{2} (x_1^2 + x_2^2) \)

(iii) Put \( h_0 = h_{P_0} \), then the bound in (2.31), (2.32), (2.33) hold true for \( P_0, h_0, R_0 \)

But the most crucial properties are the following

\[
(2.39) \quad P(x) = \sup_{y \in \Lambda} (x \cdot y - R(y)) \quad x \in \Omega_\infty
\]

and

\[
(2.40) \quad P(x_1, x_2, 0) \geq \frac{1}{2} (x_1^2 + x_2^2)
\]

The first condition ensures \(-\frac{1}{\delta} \leq \partial_x P \leq -\delta\) so that \( h_P \) is a well-defined function on \( \Omega_2 \). The second condition ensures \( P(x_1, x_2, h) = \frac{1}{2} (x_1^2 + x_2^2) \). Besides they can be preserved in the limit.

As before, we can prove the following property of maximizers. The argument is the same as corollary, so we will omit the proof.

**Lemma 2.41.** Let \((P, R) \in \mathcal{N}\) be a maximizer of \( J_\nu(P, R) \) and such that (i) of above corollary is satisfied. Let \( h(x) = h_P(x) \), then \( \nabla P \sigma_h \nu = \nu \). In particular, \( \int_{\Omega_2} h = 1 \)

Now we consider \( E_\nu(h, \gamma) \). Similar to what we proved in section 2, we have the following.

**Lemma 2.42.** Let \( \nu \in \mathcal{P}_2(\mathbb{R}^3) \) with \( \text{supp} \nu \subset \mathbb{R}^2 \times [-\frac{1}{\delta}, -\delta] \), then

(i) \( E_\nu(h, \gamma) \geq J_\nu(P, R) \) for any \((h, \gamma) \in \mathcal{M}\)

(ii) Suppose \((P_0, R_0)\) be a maximizer of \( J_\nu(P, R) \), and satisfies properties (2.38), (2.39) then

(i) takes equality iff \( h = h_{P_0} \) and \( \gamma = (id \times \nabla P_0)_\nu \sigma_h \)

(iii) \( E_\nu(h, \gamma) \) has a unique minimizer \((h_0, \gamma_0)\) and there is a universal bound on \( h \)

\[
(2.43) \quad \|h_0\|_{L^2(\Omega_2)} \leq C(M_2(\nu), \delta, \Omega_2)
\]

(iv) If \((P_0, R_0), (P_1, R_1)\) are two maximizers of \( J_\nu(P, R) \), both satisfy (2.38), (2.39), then we have \( P_0 = P_1 \) on \( \Omega_{h_0} \cup \{x_3 = 0\}\)

**Proof.** The proof for theorem 2.17 and corollary 2.18 works here. The \( L^2 \) bound on \( h \) is a consequence of corollary 2.36. Also recall that by corollary 2.36 (ii) whenever \( h(x_1, x_2) = 0 \) for some \( (x_1, x_2) \in \Omega_2 \), we have exactly \( P(x_1, x_2, 0) = \frac{1}{2} (x_1^2 + x_2^2) \)

Next we study the stability property of these optimizers under narrow convergence of \( \nu \) which remain bounded in \( \mathcal{P}_2(\mathbb{R}_3) \)

Suppose \( \nu^n, \nu \in \mathcal{P}_2(\mathbb{R}_3), \text{supp} \nu^n, \nu \subset \Lambda \), and \( \nu^n \rightarrow \nu \) narrowly with \( \sup_{n \geq 1} M_2(\nu^n) < \infty \). Let \( (h^n, \gamma^n) \) be the minimizer of \( E_{\nu^n}(h, \gamma), (P^n, R^n) \) a maximizer of \( J_{\nu^n}(P, R) \) with properties as in corollary 2.36. Then \( \sup_{n \geq 1} W_2(\nu^n, \nu) < \infty \). Now noticing (iii) of corollary 2.36, which gives various bounds on the sequence independent of \( n \), we can take a subsequence such that \( P^n \rightarrow \bar{P} \) uniformly on each compact subset of \( \Omega_\infty \) and in \( L^r(\Omega_2 \times [0, K]) \), for any \( r \in [1, 2] \) and \( K \geq 0 \)

\( h^n \rightarrow \bar{h} \) locally uniformly in \( \Omega_2 \) and in \( L^r(\Omega_2) \), for any \( r \in [1, 2] \).
$R^n \to \tilde{\hat{R}}$ locally uniformly in $\Lambda$

Put $\gamma^n = (id \times \nabla P^n )_{\hat{\gamma}^n}$ and $\tilde{\gamma} = (id \times \nabla P)_{\hat{\gamma}}$, then above convergence implies $\gamma^n \to \tilde{\gamma}$ narrowly. Indeed for any $g \in C_b (\mathbb{R}^3 \times \mathbb{R}^3)$, we have

$$\int g(x, y) d\gamma^n(x, y) = \int g(x, \nabla P^n(x)) \sigma_{\hat{h}^n}(x) dx \to \int g(x, \nabla \tilde{P}(x)) \sigma_{\hat{h}}(x) dx$$

$$= \int g(x, y) d\tilde{\gamma}$$

The above convergence is due to $L^1$ convergence of $h^n$ and pointwise convergence of $\nabla P^n$.

Similar as before, the following semi-continuity result holds.

**Lemma 2.44.** Let $h^n, \gamma^n, \tilde{h}, \tilde{\gamma}$ be defined as above, then we have

$$E_{\nu}(\tilde{h}, \tilde{\gamma}) \leq \lim_{n \to \infty} \inf E_{\nu}(h^n, \gamma^n)$$

**Proof.** This follows from the narrow convergence of $\gamma^n$ since the integrand is non-negative. □

**Lemma 2.45.** The following statements are true.

(i) $\tilde{P}(x) = \sup_{y \in \Lambda} (x \cdot y - R(y))$, and $\tilde{P}(x_1, x_2, 0) \geq \frac{1}{2} (x_1^2 + x_2^2)$

(ii) $(\tilde{h}, \tilde{\gamma})$ is the unique minimizer of $E_{\nu}(h, \gamma)$.

(iii) Define $P_1(x) = \max (\tilde{P}(x), \frac{1}{2} (x_1^2 + x_2^2))$, and put $R_1(y) = \sup_{x \in \Omega_{\infty}} (x \cdot y - P_1(x))$, then $(P_1, R_1)$ is a maximizer.

**Proof.** We first show $\tilde{P}(x) = \sup_{y \in \Lambda} (x \cdot y - R(y))$. The argument is the same as lemma 2.36.

To see $(\tilde{h}, \tilde{\gamma})$ is the minimizer, and $(P_1, R_1)$ is a maximizer, we prove it by showing $E_{\nu}(\tilde{h}, \tilde{\gamma}) = J_{\nu}(P_1, R_1)$

Indeed, we have by our definition of $R_1$ and the convexity of $P_1$ that

$$P_1(x) + R_1(\nabla P_1(x)) = x \cdot \nabla P_1(x) \cdot \mathcal{L}^3 \quad a.e \ x \in \Omega_{\infty}$$

On the other hand, by passing to limit, we see easily $\tilde{P}(x_1, x_2, \tilde{h}) = \frac{1}{2} (x_1^2 + x_2^2)$ on $\tilde{\Omega}$

Since $P_1 = \tilde{P}$ on $\tilde{\Omega}$ and by our definition of $\tilde{\gamma}$, we see

$$P_1(x) + R_1(y) = x \cdot y \tilde{\gamma} - a.e$$

Hence

$$E_{\nu}(\tilde{h}, \tilde{\gamma}) = \int \left[ \frac{1}{2} (x_1^2 + x_2^2) - y_1^2 + y_2^2 \right] dx \cdot y d\tilde{\gamma}(x, y)$$

$$= \int \left[ \frac{1}{2} (y_1^2 + y_2^2) - R_1(y) \right] dx \cdot y + \int \left[ \frac{1}{2} (x_1^2 + x_2^2) - P_1(x) \right] \sigma_{\hat{h}}(x) dx$$

Since $\tilde{h} = h_{\tilde{\rho}}$

$$\int \left[ \frac{1}{2} (x_1^2 + x_2^2) - \tilde{P}(x) \right] \sigma_{\hat{h}}(x) dx = \inf_{h \geq 0} \int \left[ \frac{1}{2} (x_1^2 + x_2^2) - \tilde{P}(x) \right] \sigma_{\hat{h}}(x) dx$$

We obtain $E_{\nu}(\tilde{h}, \tilde{\gamma}) = J_{\nu}(P_1, R_1)$ □

We are now ready to prove the following stability result.
Theorem 2.46. Let $\nu^n, \nu \in \mathcal{P}_2(\mathbb{R}^3)$, $\text{supp } \nu^n, \nu \subset \Lambda$, with $\nu^n \to \nu$ narrowly and $\sup_{n \geq 1} M_2(\nu^n) < \infty$. Let $(h^n, \gamma^n), (\bar{h}, \bar{\gamma})$ be the unique minimizer of $E_\nu(h, \gamma), E_\nu(h, \gamma)$ respectively, let $(P^n, R^n), (\bar{P}, \bar{R})$ be a maximizer of $J_{\nu^n}(P, R), J_\nu(P, R)$ respectively which satisfies (2.39), (2.40), then the following convergence are true:

(i) $h^n \to \bar{h}$ in $L^r(\Omega_2)$, for any $r \in [1, 2)$.

(ii) $\gamma^n \to \bar{\gamma}$ narrowly

(iii) $\xi(P^n, \nabla P^n)\sigma_{h^n} \to \xi(\bar{P}, \nabla \bar{P})\sigma_{\bar{h}}$ in $L^1(\Omega_\infty)$, for any $\xi \in C_b(\mathbb{R} \times \mathbb{R}^3)$

If one further assumes $W_2(\nu^n, \nu) \to 0$, then (i) can be improved to $h^n \to \bar{h}$ in $L^2(\Omega_2)$

Proof. We can see from previous lemma that for any subsequence of $(h^n, \gamma^n)$, there is a further subsequence, say $(h^{n_j}, \gamma^{n_j})$, such that $h^{n_j}$ converges in $L^r(\Omega_2)$, for any $r \in [1, 2)$, $\gamma^{n_j}$ converges narrowly, and the limit is the unique minimizer of $E_\nu(h, \gamma)$. This is sufficient to conclude the whole sequence must converge. This proves (i), (ii).

The argument for (iii) is similar. We first show any sequence has a further subsequence which converges $L^3 - a.e.$ to $\xi(\bar{P}, \nabla \bar{P})\sigma_{\bar{h}}$. Indeed, let $(P^n, R^n)$ be maximizers of $J_{\nu^n}(P, R)$ given by corollary 2.38. Let $\xi(P^n, \nabla P^n)\sigma_{h^n}$ be a subsequence (not relabeled), by what has been discussed, we can take a further subsequence, say $(P^{n_j}, R^{n_j})$ which converges locally uniformly on $\Omega_\infty$. A respectively to a maximizer $(\bar{P}, \bar{R})$, this maximizer also satisfies (2.39), (2.40), also $\xi(P^{n_j}, \nabla P^{n_j})\sigma_{h^{n_j}}$ will converge $L^3 - a.e.$ to $\xi(\bar{P}, \nabla \bar{P})\sigma_{\bar{h}}$, but $\xi(P^{n_j}, \nabla P^{n_j})\sigma_{h^{n_j}} = \xi(P^n, \nabla P^n)\sigma_{h^n}, \xi(\bar{P}, \nabla \bar{P})\sigma_{\bar{h}} = \xi(\bar{P}, \nabla \bar{P})\sigma_{\bar{h}}$ because of the uniqueness property proved in lemma 2.41.

To see such a convergence is in $L^1$, just need to observe that $\sup_{n \geq 1} \int_{\{x \geq K\}} |\sigma_{h^n}| dx \to 0$ as $K \to \infty$. Indeed

$$\int_{\{x \geq K\}} |\sigma_{h^n}| dx = \int_{\Omega_2} (h_n - K)^+ dx_1 dx_2 \leq \frac{||h_n||_{L^2(\Omega_2)}}{K}$$

Recall the universal bound on $||h^n||_{L^2(\Omega_2)}$ asserted in lemma 2.41, we proved (iii).

Now we assume $W_2(\nu^n, \nu) \to 0$. Let $(P^n, R^n)$ be the above chosen maximizers of $J_{\nu^n}(P, R)$ given by Corollary 2.38 and $\gamma^n$ is the above chosen subsequence which converges. Because of the bound (2.34), and the assumed convergence $W_2(\nu^n, \nu) \to 0$, we have

$$\int \left[ \frac{1}{2} (y_1^2 + y_2^2) - R_n(y) |\nu^n(y)| \right] dy \to \int \left[ \frac{1}{2} (y_1^2 + y_2^2) - R(y) |\nu(y)| \right] dy$$

while by Fatou

$$\int \left[ \frac{1}{2} (x_1^2 + x_2^2) - \bar{P}(x) |\sigma_{\bar{h}}(x)| \right] dx \geq \lim_{n \to \infty} \sup \int \left[ \frac{1}{2} (x_1^2 + x_2^2) - \bar{P}(x) |\sigma_{h^n}(x)| \right] dx$$

Hence

$$J_{\nu}(\bar{P}, \bar{R}) \geq \lim_{n \to \infty} \sup J_{\nu^n}(P^n, R^n)$$

Recall lemma 2.44 and lemma 2.42 (i), (ii), we actually have under the stronger convergence of $\nu^n$

$$E_\nu(\bar{h}, \bar{\gamma}) = \lim_{n \to \infty} E_{\nu^n}(h^n, \gamma^n)$$

As a result of this, and noticing that $|x_1 y_1 + x_2 y_2| \leq \frac{1}{2} (x_1^2 + x_2^2 + y_1^2 + y_2^2)$, we have

$$\int (-y_3) x_3 d\nu^n(x, y) \to \int (-y_3) x_3 \sigma_{\bar{h}} d\bar{\gamma}(x, y)$$
Recall $0 \leq \delta x_3 \leq (-y_3)x_3$, we obtain
\[
\int x_3\sigma_h(x)dx \to \int x_3\sigma_h^+(x)dx
\]
which is just
\[
\int_{\Omega_2}(h^n)^2 \to \int_{\Omega_2} (\tilde{h})^2
\]
Combined with (i), we get $L^2$ convergence of $h^n$. \hfill \Box

To conclude this section, we observe that for the maximizers found for $J_H^H(P,R)$ in subsection 2.1 are also maximizers for $J_\nu(P,R)$ upon suitable extension. This fact is contained in the following lemma.

**Lemma 2.47.** Let $\nu \in P(\mathbb{R}^3)$ with supp $\nu \subset \Lambda$, where $\Lambda$ is as in (2.25). Let $H$ be chosen as (2.28). Let $(P,R)$ be the unique pair of maximizer given by corollary 2.19. Define $\hat{P}(x) = \sup_{y \in \Lambda} (x \cdot y - R(y))$ for $x \in \Omega_\infty$ to be the extension of $P$ to $\Omega_\infty$. Then
(i) $(P,R)$ are convex conjugate over $\Omega_\infty, \Lambda$
(ii) $(\hat{P}, R)$ is a maximizer for $J_\nu(P,R)$

**Proof.** To prove (i), we just need to check
\[
R(y) = \sup_{x \in \Omega_\infty} (x \cdot y - \hat{P}(x))
\]
But since $(P,R)$ is assumed to be convex conjugate over $\Omega_H, \Lambda$, one has $R(y) \leq RHS$
The other side is obvious from the definition of $\hat{P}$
To see $(\hat{P}, R)$ is a maximizer for $J_\nu(P,R)$. Since $(P,R)$ is assumed to be a maximizer of $J_H^H(P,R), E_\nu(P,R) = J_H^H(P,R)$ by theorem 2.17. But we have $J_\nu^H(P,R) = J_\nu(\hat{P}, R)$ because $h_p = h^n_H < H$ as guaranteed by the choice of sufficiently large $H$. \hfill \Box

3. Existence of Lagrangian solutions

In this section, we prove the existence of weak Lagrangian solutions when the initial dual density is absolutely continuous, using the properties of geostrophic functional $E_\nu(h, \gamma)$ already proved in subsection 2.1. The proof here is similar to [2]

3.1. Basic definitions. First let’s define the notion of admissible initial data and weak Lagrangian solution.

Now let’s define the notion of admissible initial data. Fix $H_0, \delta > 0$.

**Definition 3.1.** Given $(h_0, P_0)$ with $h_0 \in W^{1,\infty}(\Omega_2)$ and $P_0 \in W^{1,\infty}(\Omega_{H_0})$, with $H_0 \geq \frac{\max_{x \in \Omega_2} P_0(x_1,x_2,0)}{\delta}$, we say it is an admissible initial data if the following holds true.
(i) $0 \leq h_0 \leq H_0$, $h_0dx_1dx_2 = 1$,
(ii) $P_0$ is convex, and $\partial P_0(\Omega_2 \times [0,H_0]) \subset \mathbb{R}^2 \times [-\frac{1}{\delta}, -\delta]$ and bounded.
(iii) $P(x_1,x_2, h_0(x_1,x_2)) = \frac{1}{2}(x_1^2 + x_2^2)$.

**Remark 3.2.** Above assumption $H_0 > \frac{\max_{x \in \Omega_2} P_0(x_1,x_2,0)}{\delta}$ actually guarantees $h_0 < H_0$.

The above definition guarantees at least on $\Omega_{h_0}, P_0$ is the restriction of some maximizer of $J_{\nu_0}(P,R)$, where $\nu_0 = \nabla P_0 \sigma_{h_0}$
Proposition 3.3. Let $D_0 = ||\nabla P_0||_{L^\infty} + 1$. Choose $\Lambda, \Lambda_0, H$ as in (2.25), (2.27), (2.28). Let $(h_0, P_0)$ be admissible initial data. Let $\nu_0 = \nabla P_0 \sigma_{h_0}$ so that $\text{supp } \nu_0 \subset B_{D_0} \times [-\frac{1}{2}, 0]$), Let $(P_0, \hat{R}_0)$ be the unique maximizer of $J^{h}_0$ which satisfies the properties (i), (ii) in corollary 2.19, $(h_0, \gamma)$ be the unique minimizer of $E_{v_0}(h, \gamma)$, then $P_0 = \hat{P}_0$ on $\Omega_{h_0} \cup \{x_3 = 0\}$, and $\hat{h}_0 = h_0, \gamma = (id \times \nabla P_0)^2 \sigma_{h_0}$

Proof. Define

$$\hat{R}_0(y) = \sup_{x \in \Omega_H} (x \cdot y - P_0(x))$$

Then one has

$$P_0(x) + \hat{R}_0(y) \geq x \cdot y \quad x \in \Omega_H, y \in \Lambda$$

Also observe that

$$h_0(x_1, x_2) = h^H_{P_0}(x_1, x_2)$$

By (iii) of above definition.

Recall $h^H_{P_0}(x_1, x_2)$ is the unique $s^* \in [0, H]$ where $s \longmapsto \int_0^s \frac{1}{2}(x_1^2 + x_2^2) - P_0(x) dx_3$ has a minimum. See section 2. Let $\gamma_0 = (id \times \nabla P_0)^2 \sigma_{h_0}$, we will show $J^{H}_{0}(P_0, \hat{R}_0) = E_{v_0}(h_0, \gamma_0)$.

Indeed, by definition

$$J^{H}_{0}(P_0, \hat{R}_0) = \int_\Lambda \frac{1}{2}(y_1^2 + y_2^2) - \hat{R}_0(y) |\nu_0(y) dy + \inf_{0 \leq h \leq H} \int_{\Omega_\infty} \frac{1}{2}(x_1^2 + x_2^2) - P_0(x) |\sigma_h(x) dx$$

$$= \int_{\Lambda} \frac{1}{2}(y_1^2 + y_2^2) - \hat{R}_0(y) |\nu_0(y) dy + \int_{\Omega_\infty} \frac{1}{2}(x_1^2 + x_2^2) - P_0(x) |\sigma_h(x) dx$$

$$= \int_{\Omega_\infty \times \Lambda} \frac{1}{2}(x_1^2 + x_2^2) - P_0(x) + \frac{1}{2}(y_1^2 + y_2^2) - \hat{R}_0(y) |\nu_0(x, y) xy$$

The second equality is due to above observation, and the third equality is because $\gamma_0$ has $\sigma_{h_0}$ and $\nu_0$ as marginals.

We will have shown $J^{H}_{0}(P_0, \hat{R}_0) = E_{v_0}(h_0, \gamma_0)$ provided we can show $P_0(x) + \hat{R}_0(y) = x \cdot y \gamma_0 \text{ a.e.}$

Indeed, let’s show for $x_0 \in \Omega_{h_0}$ such that $P_0$ is differentiable at $x_0$, we have

$$P_0(x_0) + \hat{R}_0(\nabla P_0(x_0)) = x_0 \cdot \nabla P_0(x_0)$$

This is implied by

$$P_0(x) \geq P_0(x_0) + \nabla P_0(x_0)(x - x_0) \quad x \in \Omega_\infty$$

Since $P_0$ is convex in $\Omega_\infty$, So we have shown $J^{H}_{0}(P_0, \hat{R}_0) = E_{v_0}(h_0, \gamma_0)$. So $(h_0, \gamma_0)$ defined above is the unique minimizer.

Now let $(P_0, \hat{R}_0)$ be the unique maximizer of $J^{H}_{0}(P, R)$, then we have

$$\gamma_0 = (id \times \nabla P_0)^2 \sigma_{h_0} = (id \times \nabla \hat{P}_0)^2 \sigma_{h_0}$$

and

$$h_0 = h^H_{P_0} = h^H_{\hat{P}_0}$$

The same argument as in cor 2.10 implies $P_0 = \hat{P}_0$ on $\Omega_{h_0}$. Also for $(x_1, x_2) \in \Omega_2$ such that $h_0(x_1, x_2) = 0$, we have $P(x_1, x_2, 0) = \frac{1}{2}(x_1^2 + x_2^2)$

Next we define Eulerian solutions for the system in physical space.
Definition 3.4. Let \( T > 0, H > 0 \). Let \( P(t, x) \in L^\infty([0, T], W^{1,\infty}(\Omega_\infty)) \), such that 
\[
\forall t \mapsto P(t, \cdot) \text{ is convex } \forall t \in [0, T]
\]
Let \( u \in L^1([0, T] \times \Omega_\infty) \).
Let \( h \in L^\infty([0, T], W^{1,\infty}(\Omega_2)) \) be such that 
\[
0 \leq h < H \int_{\Omega_2} h = 1
\]
Then we say the triple \((P, u, h)\) is a weak Eulerian solution if the following is satisfied.
(i) 
\[
\int_0^T \int_{\Omega_{h(t, \cdot)}} u \cdot \nabla \phi(x, t) \, dx \, dt + \int_0^T \int_{\Omega_{h(t, \cdot)}} \partial_s \phi(t, x) \, dx \, dt + \int_{\Omega_{h_0}} \phi(0, x) \, dx = 0
\]
\( \forall \phi \in C^1_c((0, T) \times \Omega_H) \)
(ii) \( P(t, x_1, x_2, h) = \frac{1}{2}(x_1^2 + x_2^2) \)
(iii) 
\[
\int_0^T \int_{\Omega_{h(t, \cdot)}} \nabla P(t, x) \partial_s \psi(t, x) \, dx \, dt + \int_0^T \int_{\Omega_{h(t, \cdot)}} \nabla P(t, x) \cdot (u \cdot \nabla) \psi(t, x) \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega_\infty} |J(\nabla P(t, x) - x)| \cdot \psi(t, x) \, dx \, dt + \int_{\Omega_{h_0}} \nabla P_0(x) \psi(0, x) \, dx = 0
\]
\( \forall \psi \in C^1_c((0, T) \times \Omega_H; \mathbb{R}^3) \).

Remark 3.5. As we will see in the following, (i) implies \( u \) is divergence free, the boundary condition on the fixed boundary and the free boundary condition.

In the following we show that a weak Eulerian solution with sufficient regularity gives a classical solution. This justifies our definition.

Proposition 3.6. Suppose \( u \in C^1([0, T] \times \Omega_\infty), h \in C^1([0, T] \times \Omega_2), P \in C^2([0, T] \times \Omega_\infty) \) and \((P, u, h)\) is a weak Eulerian solution, then they solve the equation in the classical sense.

Proof. First we wish to deduce from (i) above the divergence free of \( u \) and the free boundary condition. Indeed under our regularity assumption
\[
\int_0^T \int_{\Omega_h} u \cdot \nabla \phi(t, x) \, dx \, dt \, dx
\]
\[
= \int_0^T \int_{\partial \Omega_h \setminus \{x_3 = h\}} (u \cdot n) \phi(t, x) \, dSdt + \int_0^T \int_{\{x_3 = h\}} (u \cdot n) \phi(t, x) \, dSdt
\]
\[
- \int_0^T \int_{\Omega_h} (\nabla \cdot u) \phi(t, x) \, dx \, dt
\]
The second term above can be written as
\[
\int_0^T \int_{\{x_3 = h\}} (u \cdot n) \phi(t, x) \, dSdt = \int_0^T \int_{\Omega_2} (-u_1 \partial_{x_1} h - u_2 \partial_{x_2} h + u_3) \phi(x, t) \big|_{x_3 = h} \, dx \, dt
\]
For the rest of the terms
\[
\int_0^T \int_{\Omega_h} \partial_t \phi(t, x) \, dx \, dt
\]
In the above, we used divergence free of \( u \) is exactly what we want.

Finally we recover the equation

\[
D_t(\nabla P) = J(\nabla P - x)
\]

First observe that

\[
\nabla \cdot \nabla (\nabla P(t,x) \partial_t \psi(t,x)) dx dt
\]

Combining terms to obtain

\[
\int_0^T \int_{\Omega_h} (\nabla \cdot u) \phi(t,x) dS dt - \int_0^T \int_{\Omega_h} (\nabla \cdot u) \phi(t,x) dx dt \\
+ \int_0^T \int_{\Omega_2} (-\partial_t h - u_1 \partial_{x_1} - u_2 \partial_{x_2} h + u_3) \phi(x,t)|_{x_3 = h} dx dt = 0
\]

By choosing appropriate test functions we see that

\[
\nabla \cdot u = 0 \text{ in } \Omega_h
\]

As well as

\[
u \cdot n = 0 \text{ on } \partial \Omega_h - \{x_3 = h\}
\]

and

\[
\partial_t h + u_1 \partial_{x_1} h + u_2 \partial_{x_2} h = u_3 \text{ on } \{x_3 = h\}
\]

Finally we recover the equation

\[
D_t(\nabla P) = J(\nabla P - x)
\]

while the term

\[
\int_0^T \int_{\Omega_h} \nabla \cdot \left[ (\nabla P \cdot \psi) u \right] - (u \cdot \nabla) (\nabla P(t,x)) \cdot \psi(t,x) dx dt
\]

In the above, we used divergence free of \( u \) as well as the boundary condition \( \mathbf{u} \cdot \mathbf{n} = 0 \) on \( \partial \Omega_h - \{x_3 = h\} \)

Now collect terms and use the free boundary condition to see the resulting equation is exactly what we want.
Now we define the notion of weak Lagrangian solution with admissible initial data.

**Definition 3.7.** Let $T > 0, q \in (1, \infty)$, let $h$ be such that

$$h(t, x_1, x_2) \in L^\infty([0, T), W^{1,\infty}(\Omega_2)) \cap C^0([0, T] \times \bar{\Omega}_2)$$

Let $P(t, x_1, x_2, x_3)$ defined on $\Omega_H$ be such that

$$t \mapsto P(t, \cdot) \text{ is convex } \forall t \in [0, T]$$

and additionally

$$P \in L^\infty([0, T], W^{1,\infty}(\Omega_H)) \cap C([0, T] \times W^{1,\infty}(\Omega_H))$$

which assumes initial data $(h, P_0)$ on $\Omega_{h_0} \bigcup \{x_3 = 0\}$, i.e.

$$h(0, x) = h_0(x), \quad x \in \Omega_2$$

and $P(0, x) = P_0(x)$ $x \in \Omega_{h_0} \bigcup \{x_3 = 0\}$

Let $F : [0, T] \times \Omega_{h_0} \rightarrow \Omega_H$ be a Borel map and such that

$$F \in C([0, T], L^r(\sigma_{h_0} d{\mathcal{L}}^3, \mathbb{R}^3))$$

Then we say the triple $(h, P, F)$ is a weak Lagrangian solution with initial data $(h_0, P_0)$ (admissible in the sense above) if the following holds:

(i) For each $t \in [0, T]$,

$$0 \leq h(t, x_1, x_2) < H \text{ and } \int_{\Omega_2} h = 1$$

(ii) $P(t, x_1, x_2, h(t, x_1, x_2)) = \frac{1}{2}(x_1^2 + x_2^2)$

(iii) $F(0, x) = x \sigma_{h_0} d\mathcal{L}^3 \text{ a.e and } F_t \sigma_{h_0} = \sigma_h$

(iv) There exists Borel map

$$F^* : \bigcup_{t \in [0, T]} \{t\} \times \Omega_{h(t, \cdot)} \rightarrow \Omega_H$$

such that

$$F_t(F_t^*(x)) = x \sigma_h \text{ a.e and } F_t^*(F_t(x)) = x \sigma_{h_0} \text{ a.e}$$

(v) Put

$$Z(t, x) = \nabla P_t(F_t(x))$$

then the equation

$$\partial_t Z(t, x) = J(Z(t, x) - F(t, x)) \text{ in } \Omega_{h_0}$$

is satisfied in the weak sense, i.e.

$$\int_0^T \int_{\Omega_{h_0}} Z(t, x) \partial_t \phi(t, x) dx dt + \int_0^T \int_{\Omega_{h_0}} J(Z(t, x) - F(t, x)) \phi(t, x) dx dt$$

$$+ \int_{\Omega_{h_0}} Z(0, x) \phi(0, x) dx = 0$$

$$\forall \phi(t, x) \in C^1_c([0, T) \times \Omega_H)$$

**Remark 3.8.** (i) In the (v) of above definition, it is possible to choose test function $\phi$ such that $\partial_t \phi \in L^\infty([0, T) \times \Omega_H)$ without assuming $\nabla_x \phi$ exists. Indeed, we may define $\phi(-t, x) \equiv \phi(0, x)$ for $t > 0$ and convolve with $J_s(t, x)$ with $s$ small such that $\phi_s := \phi * j_s$ is a legitimate test function. Then $\partial_t \phi_s, \phi_s$ converges a.e in $[0, T] \times \Omega_{\infty}$.
to $\partial_t\phi, \phi$ as $s \to 0$ then the result follows from dominated convergence.

(ii) In the (iv) above, note that

$$\bigcup_{t \in [0,T]} \{t\} \times \Omega_{h(t,\cdot)} = \{(t, x) \in [0, T] \times [0, H] | 0 < x_3 < h(t, x_1, x_2)\}$$

Of course we must show above definition does not lose any information, i.e. we need to show with additional regularity assumption, weak Lagrangian solution gives weak Eulerian solution.

**Proposition 3.9.** Suppose $(h, P, F)$ is a weak Lagrangian solution, suppose also that $\partial_t F_t \in L_\infty([0, T] \times \Omega_H)$. Define $u(t, x) = \partial_t F_t(F_t^*(x))$, then $(P, u, h)$ gives a weak Eulerian solution.

**Proof.** We only need to check that (i) and (iii) of the defn of Eulerian solution are satisfied.

First notice that $F_t \sigma_{h_0} = \sigma_h$. Thus for $\phi \in C^1_c([0, T] \times \Omega_\infty)$, we have

$$\int_0^T \int_{\Omega_h} u \cdot \nabla \phi(x, t) dx dt = \int_0^T \int_{\Omega_h} \partial_t F_t(F_t^*(x)) \nabla \phi(x, t) dx dt$$

$$= \int_0^T \int_{\Omega_{h_0}} \partial_t F_t(x) \nabla \phi(F_t(x), t) dx dt$$

$$= \int_0^T \int_{\Omega_{h_0}} \partial_t(\phi(F_t(x), t)) - \partial_t \phi(F_t(x), t) dx dt$$

$$= -\int_{\Omega_{h_0}} \phi(x, 0) dx - \int_0^T \int_{\Omega_{h_0}} \partial_t \phi(F_t(x), t) dx dt$$

This verifies (i).

In the second line above, we used $F_t \sigma_{h_0} = \sigma_h(t, \cdot)$ and also $F_t^*(F_t(x)) = x \sigma_{h_0}$ a.e.

In the third line above, we used

$$\text{for } \sigma_{h_0} \text{ a.e. } x, \partial_t(\phi(F_t(x), t)) = \partial_t \phi(F_t(x), t) + \nabla \phi(F_t(x), t) \partial_t F_t(x)$$

since $t \mapsto F_t(x)$ is Lipschitz.

Now we verify (iii). Indeed

$$\int_0^T \int_{\Omega_h} \nabla P(t, x) \partial_t \psi(t, x) dx dt = \int_0^T \int_{\Omega_{h_0}} Z(t, x) \partial_t \psi(t, F_t(x)) dx dt$$

$$= \int_0^T \int_{\Omega_{h_0}} Z(t, x)[\partial_t(\psi(t, F_t(x)) - \nabla \psi(t, F_t(x)) \partial_t F_t(x)] dx dt$$

$$= \int_0^T \int_{\Omega_{h_0}} Z(t, x) \partial_t(\psi(t, F_t(x))) dx dt - \int_0^T \int_{\Omega_h} (u \cdot \nabla) \psi(t, x) \nabla P(t, x) dx dt$$

In the second line above, we used the fact that since $\partial_t F_t(x) \in L_\infty([0, T] \times \Omega_H), \partial_t(\psi(t, F_t(x))) \in L_\infty([0, T] \times \Omega_H)$ and usual chain rule holds.
Now we choose test function as \( \psi(t, F_t(x)) \) in the definition \( (v) \) above to get the following. This is justified because of the remark after the definition above.

\[
\int_0^T \int_{\Omega_0} Z(t, x) \partial_t (\psi(t, F_t(x))) dx dt \\
= - \int_0^T \int_{\Omega_0} J(Z(t, x) - F(t, x)) \psi(t, F_t(x)) dx dt - \int_{\Omega_0} Z(0, x) \psi(0, x) dx \\
= - \int_0^T \int_{\Omega_0} J(\nabla P(t, x) - x) \psi(t, x) dx dt - \int_{\Omega_0} \nabla P_0(x) \psi(0, x) dx
\]

Put things together, we get \( (iii) \).

Now we can state the existence result of weak Lagrangian solutions.

**Theorem 3.10.** Let \( T > 0 \), \( 1 < q < \infty \), and admissible initial data \((h_0, P_0)\) be given, suppose also \( \nu_0 := \nabla P_0 \sigma_{h_0} \in L^q(\mathbb{R}^3) \).

Suppose also that

\[
H > \frac{2}{\mathcal{L}^2(\Omega_2)} + \frac{2 \text{diam } \Omega_2}{\delta} || P_0 ||_{L^\infty} T + \max_{\Omega_2} |x|(T + 2) + 2
\]

then there exists a weak Lagrangian solution \((h, P, F)\) on \([0, T] \times \Omega_H\). Moreover the function \( Z(\cdot, x) \in W^{1,\infty}(\mathbb{R}^3) \) for \( a.e \ x \in \Omega_{h_0} \) and the equations are satisfied in the following sense:

\[
\partial_t Z(t, x) = J(Z(t, x) - F(t, x)) \text{ for } L^4 - a.e \text{ in } (t, x) \in [0, T] \times \Omega_{h_0} \\
Z(0, x) = \nabla P_0(x) \text{ for } L^3 - a.e \text{ in } x \in \Omega_{h_0}
\]

3.2. **Lagrangian flow in dual space.** Next we study the Lagrangian flow in dual space. It is similar to Cullen-Feldman section 2.3.

Recall that \( \nu_0 = \nabla P_0 \sigma_{h_0} \), so

\[
(3.12) \quad \text{supp } \nu_0 \subset B_{D_0}(0) \times [-\frac{1}{\delta}, -\delta] := \Lambda_0
\]

where

\[
(3.13) \quad D_0 = || \nabla P_0 ||_{L^\infty} + 1
\]

Put

\[
(3.14) \quad D = D_0 + T \max_{\Omega_2} |x| + 1
\]

and define

\[
(3.15) \quad \Lambda = B_D(0) \times [-\frac{1}{\delta}, -\delta]
\]

Choose \( H \) as in (3.11). Let \( h, P, R, \nu \) be the dual space solution given by theorem 2.26 where we have chosen the parameters \( T, H, D, D_0 \) as here, recall that \( \nu(t, \cdot) \) satisfies the transport equation

\[
\partial_t \nu + w \cdot \nabla \nu = 0 \quad \nu(0, \cdot) = \nu_0
\]

where \( w = J(y - \nabla R(t, y)) \) is divergence free.

Then it follows from thm 1.15 that

\[
w \in L^\infty_{loc}([0, T] \times \mathbb{R}^3) \quad w \in L^\infty([0, T], BV(B(0, R))) \quad \forall R > 0
\]
Here we naturally extend $w$ to $\mathbb{R}^3$ by the same formula above. Since $\nu$ is supported in $B_D(0) \times [-\frac{1}{\delta}, \delta] \times [0, T]$ we can modify $w$ outside $B_D(0) \times [-\frac{1}{\delta}, \delta]$ such that the modified $\tilde{w} \in L^\infty([0, T] \times \mathbb{R}^3)$ and $\tilde{w} \in L^\infty([0, T], BV(B(0, R))) \forall R > 0$ and $\nabla \cdot (\tilde{w}(t, \cdot)) = 0$ To construct $\tilde{w}$, let $\zeta \in C^\infty(\mathbb{R}^3)$ be a cut-off function such that
\[
\zeta(s) \equiv 1 \text{ for } |s| \leq D + \frac{1}{\delta} \quad \zeta(s) \equiv 0 \text{ for } |s| \geq D + \frac{1}{\delta} + 1 \quad 0 \leq \zeta \leq 1
\]
Then define
\[
H(y) = (\zeta(|y_1|)y_1, \zeta(|y_2|)y_2, \zeta(|y_3|)y_3)
\]
Then we take
\[
\tilde{w} = J(H(y) - \nabla R(t, y)) \quad y \in \mathbb{R}^3
\]
As in [2], we can apply Ambrosio’s theory [8] to $\tilde{w}$ to get a Lagrangian flow $\Phi$ in dual space and establish the following lemma.

**Lemma 3.16.** Let $\tilde{w}$ be defined as above. Then there exists a unique locally bounded Borel measurable map $\Phi : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$, satisfying

(i) $\Phi(\cdot, y) \in W^{1, \infty}([0, T]; \mathbb{R}^3)$ for a.e $y \in \mathbb{R}^3$

(ii) $\Phi(0, y) = y$ for $L^1$-a.e $y \in \mathbb{R}^3$

(iii) For a.e $(t, y) \in \mathbb{R}^3 \times (0, T)$
\[
\partial_t \Phi(t, y) = \tilde{w}(t, \Phi(t, y))
\]

(iv) $\Phi(t, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ is a $L^3$ measure preserving map for every $t \in [0, T]$

**Remark 3.17.** Notice that by our definition of $\tilde{w}$, we have $\tilde{w}_3 = 0$. By (i) and (iii) above, one sees that
\[
\Phi_3(t, y) = y \quad \forall t \in [0, T]
\]

**Lemma 3.18.** Let $\tilde{w}$ be defined as above, and let $\Phi$ be the flow in previous lemma, then

(i) $\Phi(t, y) \subset B_D^2(0)) \times [-\frac{1}{\delta}, \delta]$ for a.e $(t, y) \in [0, T] \times \nabla P_0(\Omega_\infty)$

In particular,
\[
\partial_t \Phi(t, y) = w(t, \Phi(t, y)) \quad \text{for a.e } (t, y) \in [0, T] \times \nabla P_0(\Omega_\infty)
\]

(ii) There exists a Borel map $\Phi^* : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that for every $t \in [0, T]$ the map $\Phi^* : \mathbb{R}^3 \to \mathbb{R}^3$ is $L^3$ measure preserving, and such that $\Phi^* \circ \Phi(t, y) = y$ and $\Phi_t \circ \Phi^*(y) = y$ for a.e $y \in \mathbb{R}^3$

The proof of this lemma is almost identical to the lemma 2.9 of [2]

**Proposition 3.19.** Let $\Omega_H, T, \nu$ be as in theorem 2.26. Let $h, P, R, \nu$ be the dual space solution obtained in that theorem, Let $\tilde{w}, \Phi$ be as in above definition, Let $D = \|\nabla P_0\|_L^\infty + (T + 1) \max_{\Omega_\infty} |x| + 1$, then for $t \in [0, T]$
\[
\nu(t, \cdot) = \Phi_t^* \nu_0
\]

Moreover, for $t \in [0, T]$, we have
\[
\nu(t, y) = \nu_0(\Phi_t^*(y)) \quad \text{a.e } y
\]

The proof is again similar to Proposition 2.11 [2]. Keep in mind that we have strong convergence of the transporting vector $w_{\epsilon_j}$ to $w$ according to the proof of Theorem 2.26. This enables us to use the stability result Theorem 6.5 of [8].
3.3. Lagrangian flow in physical space. We want to define a Lagrangian flow in the physical space $F : [0, T] \times \Omega_{ho} \to \Omega_{\infty}$ by defining $F_t : \Omega_{ho} \to \Omega_H$ by

$$F_t := \nabla R_t \circ \Phi_t \circ \nabla P_0$$

Of course, we need to check above formula is well defined.

**Lemma 3.20.** The right hand side above is defined $\mathcal{L}^4$–a.e. For any $t \in [0, T]$, above is defined $\mathcal{L}^3$–a.e. and $F : [0, T] \times \Omega_{ho} \to \Omega_H$ is a Borel map.

This is checked in the same way as Lemma 2.12 of [2]. So we omit the proof. In the following, we collect some results which can be proved in the same way as [2], they correspond to proposition 2.13–2.16 in that paper.

**Lemma 3.21.**

$$F(0, x) = x \text{ a.e } x \in \Omega_{ho}$$

**Lemma 3.22.** For every $t \in [0, T]$, we have

$$F_t \sigma_{ho} = \sigma_{h(t, \cdot)}$$

**Proposition 3.23.** For any $t_0 \in [0, T]$, any $r \in [1, \infty)$, one has

$$\lim_{t \to t_0, t \in [0, T]} \int_{\Omega_{ho}} |F_t(x) - F_{t_0}(x)|^r \, dx = 0$$

**Proposition 3.24.** There exists a Borel map

$$F^* : \bigcup_{t \in [0, T]} \{t\} \times \Omega_{h(t, \cdot)} \to \Omega_H$$

such that for any $t \in [0, T]$

$$F_t(F_t^*(x)) = x \text{ a.e } a.e \text{ and } F_t^*(F_t(x)) = x \text{ a.e}$$

**Proposition 3.25.** Put $Z(t, x) = \nabla P_t(F_t(x))$, then $Z(t, x)$ satisfies

$$\int_0^T \int_{\Omega_{ho}} Z(t, x) \partial_t \phi(t, x) \, dx \, dt + \int_0^T \int_{\Omega_{ho}} J(Z(t, x) - F(t, x)) \phi(t, x) \, dx \, dt$$

$$+ \int_{\Omega_{ho}} Z(0, x) \phi(0, x) \, dx = 0$$

$\forall \phi(t, x) \in C^1([0, T] \times \Omega_{ho})$

Moreover, possibly after changing $Z(t, x)$ on a negligible subset of $[0, T] \times \Omega_{ho}$, we have $Z(\cdot, x) \in W^{1, \infty}([0, T]; \mathbb{R}^3)$ for a.e $x \in \Omega_{ho}$ and

$$\partial_t Z(t, x) = J(Z(t, x) - F(t, x)) \text{ for } \mathcal{L}^4 \text{–a.e in } (t, x) \in [0, T] \times \Omega_{ho}$$

$$Z(0, x) = \nabla P_0(x) \text{ for } \mathcal{L}^3 \text{–a.e in } x \in \Omega_{ho}$$

**Proof.** By Lemma 3.20, we have

$$Z(t, x) = \nabla P_t \circ F_t(x) = \Phi_t \circ \nabla P_0(x)$$

Except for a $\mathcal{L}^4$ negligible set.

So after redefining $Z$ on a negligible set, we may redefine $Z(t, x) = \Phi_t \circ \nabla P_0(x)$ and we will prove this version of $Z$ has all the properties claimed.

By Lemma 3.12, we know that $\Phi(\cdot, y) \in W^{1, \infty}([0, T]; \mathbb{R}^3)$ for a.e $y$. Since $\nabla P_0 \sigma_{ho} << \mathcal{L}^3$, one has

$$t \mapsto \Phi_t(\nabla P_0(x)) \in W^{1, \infty}([0, T]; \mathbb{R}^3) \text{ for } \mathcal{L}^4 \text{–a.e } x \in \Omega_{ho}$$
Let $\tilde{N}_3$ be such that $\mathcal{L}^3(\tilde{N}_3) = 0$ and for $y \in \tilde{N}_3^c,(i),(ii)$ of lemma 3.16 holds and such that for $a.e \ t \in [0,T],\partial_t \Phi(t,y) = \tilde{w}(t,\Phi(t,y))$ holds.Then for such $y$,one has

$$\Phi(t,y) = y + \int_0^t \tilde{w}(s,\Phi(s,y))ds \ \forall t \in [0,T]$$

Due to the same reason as above and note that $|\nabla P_0(x)| \leq D \ \forall x \in \Omega_{ho}$ by our choice of $D$,one can conclude for $\mathcal{L}^3 - a.e \ x \in \Omega_{ho}$

$$\Phi(t,\nabla P_0(x)) = \nabla P_0(x) + \int_0^t \mathbf{w}(s,\Phi(s,\nabla P_0(x)))ds \ \forall t \in [0,T]$$

Therefore

$$Z(0,x) = x \ \mathcal{L}^3 - a.e \ x \in \Omega_{ho}$$

Also for such $x$,one has

$$\partial_t Z(t,x) = \partial_t (Z(t,x) - F(t,x)) \ \text{for} \mathcal{L}^4 - a.e \ \text{in} \ (t,x) \in [0,T] \times \Omega_{ho}$$

The distributional identity is obtained by multiplying a test function to above $a.e$ identity and integrate by parts in $t$.

\[ \square \]

**Proof of theorem 3.10:**

Let $D_0 = \|\nabla P_0\|_{L^\infty} + 1$,set $A_0 = B_{D_0}(0) \times [-\frac{1}{2\delta},-\frac{4}{\delta}]$,and put $\nu_0 := \nabla P_0 \sigma_{ho}$.Then $supp \ \nu_0 \subset A_0$.Also put $D = D_0 + (T+1)\max_{\Omega_2}|x| + 1$ and $\Lambda = B_D(0) \times [-\frac{1}{\delta},-\delta]$.Choose $H$ as stated in thm 2.3.Then such $H$ also satisfies

$$H > \frac{2}{\mathcal{L}^2(\Omega_2)} + \frac{2\max_{\Omega_2}|x| + 2D}{\delta} \cdot \text{diam } \Omega_2$$

as required by thm 2.16.

Now let $(h,P,R)$ be the dual space solution corresponding to the initial data $\nu_0$ given by thm 1.15,and let $F$ be as defined as in the beginning of this subsection.Then $(h,P,F)$ gives a weak Lagrangian solution with initial data $(h_0,P_0)$. Indeed,$(h,P)$ assumes initial data $(h_0,P_0)$ on $\Omega_{ho} \bigcup \{x_3 = 0\}$ is guaranteed by propoposition 3.3,property $(i),(ii)$ comes from our construction of dual space solution,other properties follows from the lemmas and propopositions proved in this subsection.

4. **Existence of relaxed Lagrangian solution**

4.1. **Basic definition.** In this section,we prove the existence of relaxed Lagrangian solution in a similar way as was done in [7].The relaxed Lagrangian solution is an even weaker notion than weak Lagrangian solutions defined in previous sections,but it will allow for more general initial data $(h_0,P_0)$,in particular,we will no longer require $\nabla P_0 \sigma_{ho} \ll \mathcal{L}^3$.To motivate the definition,recall that the weak Lagrangian solution given by thm 3.10 will satisfy the additional property

$$\partial_t Z(t,x) = \partial_t (Z(t,x) - F(t,x)) \ \text{for} \mathcal{L}^4 - a.e \ \text{in} \ (t,x) \in [0,T] \times \Omega_{ho}$$

$$Z(0,x) = \nabla P_0(x) \ \text{for} \mathcal{L}^3 - a.e \ \text{in} \ x \in \Omega_{ho}$$
Recalling the definition of $Z(t,x)$, this implies for $\xi \in C^1(\mathbb{R}^3)$, we have
\[
\partial_t \xi(\nabla P_t(F_t(x))) = \nabla \xi(\nabla P_t(F_t(x))) - J(\nabla P_t(F_t(x)) - F(t,x)) \quad \text{for } L^4 - \text{a.e. in } (t,x) \in [0,T] \times \Omega_{h_0}
\]
and has initial data $\xi(\nabla P_0(x))$.

Thus if we define a Borel family of measures $[0,T] \ni t \mapsto \mathrm{d} \alpha_t(x,\hat{x}) := (id \times F_t)_*\sigma_{h_0}$ and put $\mathrm{d} \alpha = \mathrm{d} \alpha_t dt$, then at least formally we can obtain from above almost everywhere defined equality that
\[
(4.1) \int_0^T \int_{\Omega_{h_0}^2} \xi(\nabla P_t(x)) \partial_t \psi(t,x) d\alpha_t(x,\hat{x}) dt + \int_0^T \int_{\Omega_{h_0}^2} \nabla \xi(\nabla P_t(x)) \cdot J(\nabla P_t(x) - \hat{x}) \psi(t,x) d\alpha_t(x,\hat{x}) dt + \int_{\Omega_{h_0}^2} \xi(\nabla P_0(x)) \psi(0,x) dx = 0 \quad \forall \xi \in C^1_c(\mathbb{R}^3)
\]

Above discussion motivates the following definition of relaxed Lagrangian solutions.

**Definition 4.2.** $(h_0, P_0)$ be admissible initial data. Consider a Borel function $P : \Omega_\infty \times [0,T] \to \mathbb{R}$ such that $P(t,\cdot)$ is convex for any $t \in [0,T]$ and $h : \Omega_2 \times [0,T] \to \mathbb{R}$ such that $h(t,\cdot) \in L^2(\Omega_2)$ and a family of Borel measures $[0,T] \ni t \mapsto \alpha_t(x,\hat{x}) \in \mathcal{P}(\Omega_\infty \times \Omega_\infty)$, let $\alpha$ be given by $\mathrm{d} \alpha = \mathrm{d} \alpha_t dt$. We say that the triple $(P,h,\alpha)$ is a relaxed Lagrangian solution if the following holds

(i) $h \geq 0 \int_{\Omega_2} \mathrm{d}x_1 \mathrm{d}x_2 = 1 \forall t \in [0,T]
(ii) P_t(x_1,x_2,h) = \frac{1}{2}(x_1^2 + x_2^2), \nabla P_t \in L^2(\Omega_{h_0})$, and $-\frac{1}{\delta} \leq \partial_{x_2} P_t \leq -\delta$ for any $t \in [0,T]
(iii) \exists \Omega_{\infty} \ni \partial_{x_2} P_t = \sigma_{h_0} \quad \exists \Omega_{\infty} \ni \partial_{x_2} P_t = \sigma_{h(t,\cdot)}
(iv) (4.1) above holds true.
(v) $P(0,\cdot) = P_0$ on $\Omega_{h_0}$, $h(0,\cdot) = h_0$

Since we derived (4.1) only formally, we need to check (4.1) makes sense.

**Lemma 4.3.** Suppose (iii) above holds, then the left hand side of (4.1) is well-defined for any $d\alpha(t,x,\hat{x}) = d\alpha_t(x,\hat{x}) dt$ with $\alpha_t$ satisfies (iii) in definition 4.2.

**Proof.** The main issue comes from the fact that for each fixed $t$, $\nabla P_t$ is not an honest function but is defined only a.e. We have to check that different choice of Borel representative of $\nabla P_t$ does not affect the integral in the left hand side. Actually, we will show for each fixed time slice $t$, the inner integral is well-defined. Fix $t \in [0,T]$, let $\Psi_1, \Psi_2$ be two Borel representatives of $\nabla P_t$, i.e $\Psi_1, \Psi_2$ are Borel functions and $\forall \in \nabla P_t L^3 - \text{a.e.}. Concerning the first term, consider the set
\[
E = \{(x,\hat{x}) \in \Omega_{\infty}^2 \mid |\xi(\Psi_1(\hat{x})) \partial_t \psi(t,x) \neq \xi(\Psi_2(\hat{x})) \partial_t \psi(t,x)|\}
\]
Then $\alpha_t(E) = \sigma_{h}(\{(\hat{x} \in \Omega_{\infty} \mid \Psi_1(\hat{x}) \neq \Psi_2(\hat{x}))\}) = 0$. The argument for other terms are the same.

Next we show that the notion of relaxed Lagrangian solution defined here is consistent with weak Lagrangian solution when the measure $\alpha_t$ is induced by some physical flow map, except possibly the inverse map may not exist.

**Lemma 4.4.** Let $(P,h,\alpha)$ be relaxed Lagrangian solution such that $P \in C([0,T];H^1(\Omega_\infty))$ with admissible initial data. Suppose there exists a Borel map $F : [0,T] \times \Omega_{h_0} \to \Omega_{h_0}$ which is weakly continuous in the sense that
\[
\int_{\Omega_{h_0}} F_t(x) \psi(x) dx \to \int_{\Omega_{h_0}} F_0(x) \psi(x) dx \quad \forall \psi \in C_0(\mathbb{R}^3 ; \mathbb{R}^3) \text{ as } t \to t_0 \in [0,T]
\]
such that $\alpha_t = (id \times F_t)\sigma_{h_0}$. Then $(P, h, F)$ is a weak Lagrangian solution except possibly $(iv)$

Proof. First we observe that the assumption $P \in L^\infty([0,T], W^{1,\infty}(\Omega_\infty)) \cap C([0,T], H^1(\Omega_\infty))$ implies that $P \in C([0,T], W^{1,r}(\Omega_\infty)) \forall r < \infty$. Also observe the following

$$\int_{\Omega_{h_0}} |F_t(x)|^2 dx = \int_{\Omega_{h(t,\cdot)}} |\dot{x}|^2 d\tilde{x} = \int_{\Omega_{h(t_0,\cdot)}} |\dot{x}|^2 dx = \int_{\Omega_{h_0}} |F_{t_0}(x)|^2 dx$$

The equality above is because $\sigma_{h(t,\cdot)} = \text{proj}_x \alpha_t = F_{t_0}\sigma_{h_0}$ and the limit happens is because of the assumption $h \in C([0,T] \times \bar{\Omega}_2)$. Since $F_t$ is always bounded, this combines with weak continuity implies $F \in C([0,T], L'(\sigma_{h_0}, L^1))$, $\forall r < \infty$.

Now it only remains to check that for $Z(t,x) = \bigtriangledown P_t(F_t(x))(\text{well-defined thanks to the above observed } F_{t_0}\sigma_{h_0} = \sigma_{h(t,\cdot)})$, it satisfies $(v)$, but this is seen by taking $\xi(y) = y_t$ in (4.1).

\hfill \Box

4.2. Measure valued solution in the dual space. Recall that when $\nu_0 \in \mathcal{P}_{ac}(\mathbb{R}^3)$, the system in dual space can be written as

$$\partial_t \nu + \bigtriangledown \cdot (\nu w) = 0$$

$$w = J(y - \bigtriangledown P^*(t,y)) \in \mathbb{R}^3$$

$$(h, (id \times \bigtriangledown P)\sigma_h) \text{ minimizes } E_\nu(h, \gamma) \text{ with } P(t,\cdot) \text{ convex}$$

$${\nu}|_{t=0} = \nu_0$$

See the end of section 2.

Since the second equation involves $\bigtriangledown P^*(t,y)$, it is not well-defined if $\nu$ is singular, we need to find an substitute for $\bigtriangledown P^*$

Definition 4.5. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^3), \lambda \in \Gamma(\mu, \nu), \gamma : \mathbb{R}^3 \to \mathbb{R}^3$ is the barycentric projection of $\lambda$ to $\nu$ if the following is true

$$\int_{\mathbb{R}^3} \xi(y) \cdot \gamma(y) d\mu(y) = \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \xi(y) \cdot x d\lambda(x,y) \forall \xi \in C_b(\mathbb{R}^3; \mathbb{R}^3)$$

Now if $P_t$ be a family of convex functions and set $\nu_t = \bigtriangledown P_t\sigma_{h(t,\cdot)}$, then we denote $\gamma_t(y)$ to be the barycentric projection of $(id \times \bigtriangledown P_t)\sigma_h$ onto $\nu_t$, or equivalently

$$\int_{\mathbb{R}^3} \xi(y) \cdot \gamma_t(y) d\nu_t(y) = \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \xi(y) \cdot x d\sigma_h(x,y) \forall \xi \in C_b(\mathbb{R}^3; \mathbb{R}^3)$$

It’s easy to see when $\nu_t \ll \mathcal{L}^3$, then $\bigtriangledown P_t^*(y) = \gamma_t(y)$, $\nu_t - a.e$

Thus in the general case when $\nu_t$ is not necessarily absolutely continuous, a natural way to write the system in the dual space is (where $[0,T] \ni t \mapsto \nu_t$ is a Borel family of measures)

$$\partial_t \nu + \bigtriangledown \cdot (\nu w) = 0 \text{ in } [0,T] \times \mathbb{R}^3$$

$$w(t,y) = J(y - \gamma_t(y)) \text{ in } [0,T] \times \mathbb{R}^3$$

$$(h, (id \times \bigtriangledown P_t)\sigma_h) \text{ minimizes } E_\nu(h, \gamma) \text{ with } P(t,\cdot) \text{ convex}$$

$${\nu}|_{t=0} = \nu_0$$

For immediate use, we first prove a lemma
Lemma 4.6. Let $\nu$ be a solution of above system, fix $t \in [0,T]$ and $\gamma := (id \times \nabla P_t)_{t} \sigma_t$ be the optimal measure with marginals $\sigma_t$ and $\nu_t$ under quadratic cost, let $\gamma \overline{\gamma}$ be defined as above then

$$
\|J[id - \overline{\gamma}]\|_{L^2(\nu_t)} \leq 2(\max_{\Omega} |x| + W_2(\nu_t, \delta_0))
$$

Proof. Observe that

$$
\|J[id - \overline{\gamma}]\|_{L^2(\nu_t)} = \sup_{\phi \in C_{c}(\mathbb{R}^3), \|\phi\|_{L^2} \leq 1} \int J(y - \overline{\gamma}(y)) \phi(y) d\nu_t(y)
$$

But

$$
\int J(y - \overline{\gamma}(y)) \phi(y) d\nu_t(y) = \int J(\nabla P_t(x) - x) \phi(\nabla P_t(x)) d\sigma_h(t, \cdot)(x)
$$

$$
\leq (\int (x_1 - \partial_1 P_t(x))^2 + (x_2 - \partial_2 P_t(x))^2 d\sigma_h(t, \cdot)(x))^{\frac{1}{2}} \left( \int \phi^2(\nabla P_t(x)) d\sigma_h(t, \cdot) \right)^{\frac{1}{2}}
$$

$$
\leq [2(\max_{\Omega} |x|^2 + \int_{\Omega_h(t, \cdot)} |\nabla P_t(x)|^2 dx)]^{\frac{1}{2}} \|\phi\|_{L^2(\nu_t)}
$$

But since $\nabla P_t \sigma_h(t, \cdot) = \nu_t$, we know

$$
\int_{\Omega_h(t, \cdot)} |\nabla P_t(x)|^2 dx = W_2^2(\nu_t, \delta_0)
$$

We denote by $AC^p(0, T; \mathcal{P}_2(\mathbb{R}^3))$ (See also [5], chapter 8) to be the set of all paths $\mu : [0, T] \ni t \rightarrow \mu_t \in \mathcal{P}_2(\mathbb{R}^3)$ for which there exists $\beta \in L^p(0, T)$ such that

$$
W_2(\mu_s, \mu_t) \leq \int_s^t \beta(\tau) d\tau \ \forall s, t \in [0, T]
$$

First we want to study the stability of the measure valued dual space solution under perturbations of initial data

Proposition 4.7. Let $\Lambda, \Lambda, H, T$ be as in theorem 2.26 $\{\nu_0\} \cup \{\nu_0^n\}_{n=1}^\infty \subset \mathcal{P}_2(\mathbb{R}^3)$ with supp $\nu_0, \nu_0^n \subset \Lambda$ be such that

$$
\nu_0^n \rightarrow \nu_0 \text{ narrowly and } \sup_{n \geq 1} W_2(\nu_0^n, \nu_0) < \infty
$$

Let $\nu^n \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$ with supp $\nu^n \subset \Lambda$ be solutions to the dual space system corresponding to initial data $\nu^n_0$. Then a subsequence of $\nu^n$ converges to a solution $\nu \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$ with initial data $\nu_0$, i.e

(i) $W_2(\nu^n_t, \nu_t) \rightarrow 0$ for all $t \in [0, T]$ and all $p \in [1, 2]$

(ii) $\nu \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$ is a solution to dual space system with initial data $\nu_0$.

Proof. Let $P_n(t, \cdot), R_n(t, \cdot)$ be the convex conjugate maximizers of $J_{\nu^n_0}$ given by cor 2.19, $\langle h_n, \gamma_n \rangle$ be the unique minimizer of $E_{\nu(t, \cdot)}(h, \gamma)$ given by theorem 2.17, they also satisfy the uniform bound (depending only on $\Lambda, H, \Omega_2, \delta$, see section 2)

$$
C_0 := \sup_{t, n} ||h^n(t, \cdot)||_{W^{1, \infty}(\Omega_2)} < \infty
$$

$$
C_1 := \sup_{t, n} ||P^n(t, \cdot)||_{W^{1, \infty}(\Omega_H)} < \infty
$$

$$
C_2 := \sup_{t, n} ||R_n(t, \cdot)||_{W^{1, \infty}(\Lambda)} < \infty
$$
First we have for each \( n \), and \( 0 \leq s \leq t \leq T \)
\[
W_2(\nu^n_s, \nu^n_t) \leq \int_s^t ||J[id - \tilde{\gamma}^n_t]||_{L^2(\nu^n_t;\mathbb{R}^3)} d\tau \leq 2 \int_s^t W_2(\nu^n_\tau, \delta_0) d\tau + 2 \max_{\Omega_2} |x|(t - s)
\]
The first inequality used [5] Theorem 8.3.1. The second inequality is by Lemma 4.6

Now take \( s = 0 \), we then have
\[
W_2(\delta_0, \nu^n_0) \leq \sup_{n \geq 1} W_2(\delta_0, \nu^n_0) + W_2(\nu^n_0, \nu^n_t)
\]
\[
\leq \sup_{n \geq 1} W_2(\delta_0, \nu^n_0) + 2 \int_0^t W_2(\nu^n_\tau, \delta_0) d\tau + 2 \max_{\Omega_2} |x|t
\]

Then we can apply Gronwall to obtain
\[
W_2(\delta_0, \nu^n_t) \leq \sup_{n \geq 1} W_2(\delta_0, \nu^n_0)e^{2T} + (e^{2T} - 1)T \max_{\Omega_2} |x| 0 \leq t \leq T
\]

We can then conclude
\[
W_2(\nu^n_s, \nu^n_t) \leq 2e^{2T}[\sup_{n \geq 1} W_2(\delta_0, \nu^n_0) + (T + 1) \max_{\Omega_2} |x||t - s|
\]

By assumption, \( \text{supp} \nu^n \subset \Lambda \), so for each \( t \in [0, T] \), the sequence of measures \( \{\nu^n\}_{n \geq 1} \) is tight. Now we take a subsequence (not relabeled), such that
\[
\nu^n_t \to \nu_t \text{ narrowly } \forall t \in [0, T] \bigcap Q
\]

Since our measures are supported on a bounded set \( \Lambda \), we know that
\[
\int_{\Lambda} |x|^2 d\nu^n_t \to \int_{\Lambda} |x|^2 d\nu_t \forall t \in [0, T] \bigcap Q
\]

Therefore, we can conclude by [5] proposition 7.1.5
\[
W_2(\nu^n_t, \nu_t) \to 0 \forall t \in [0, T] \bigcap Q
\]

But the convergence happens for all \( t \in [0, T] \), thanks to the bound \( \dagger \) we established above
\[
W_2(\nu^n_t, \nu^n_m) \leq W_2(\nu^n_s, \nu^n_m) + 4e^{2T}[\sup_{n \geq 1} W_2(\nu^n_0, \delta_0) + \max_{\Omega_2} |x| + C_0]|t - s|
\]

So far we proved \( (i) \). Now we prove \( (ii) \).

That \( \nu_t \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3)) \) is easy to see by (4.9). Also we have \( \text{supp} \nu_t \subset \Lambda \). Now let \( (P, R), (h, \gamma) \) be optimizers associated to \( \nu \). Then for each \( t, (h(t, \cdot), (id \times \nabla R)_t) \sigma_h \) minimizes \( E_{\nu_t}(h, \gamma) \). It only remains to check \( \nu \) solves the equation.

By the narrow convergence proved above, and by the stability result theorem 2.24, we know for each fixed \( t \in [0, T] \), we have
\[
h^n(t, \cdot) \to h(t, \cdot) \text{ uniformly } P^n(t, \cdot) \to P(t, \cdot) \text{ in } H^1(\Omega_{\infty}) \ R_n \to R \text{ in } H^1(\Lambda)
\]

Since each \( \nu_n \) is a solution, by taking the test function \( \phi(t, y) = \chi(t)\psi(y), \) where \( \chi(t) \in C^2_c([0, T]), \psi \in C^2_0(\mathbb{R}^3) \), we obtain
\[
\int_0^T \int_{\mathbb{R}^3} \chi'(t)\psi(y) d\nu^n_t(y) dt + \int_0^T \int_{\mathbb{R}^3} [J(y - \tilde{\gamma}^n_t(y))] \cdot \nabla \psi(y) \chi(t) d\nu^n_t(y) dt + \int_{\mathbb{R}^3} \chi(0)\psi(y) d\nu^n_0(y) = 0
\]
By sending $n \to \infty$, we obtain by narrow convergence proved above that
\[
\int_0^T \int_{\mathbb{R}^3} \chi'(t)\psi(y)\,d\nu_{\alpha}^n(y)dt \to \int_0^T \int_{\mathbb{R}^3} \chi'(t)\psi(y)\,d\nu(y)
\]
\[
\int_0^T \int_{\mathbb{R}^3} J_y \cdot \nabla \psi(y)\chi(t)\,d\nu_{\alpha}^n(y)dt \to \int_0^T \int_{\mathbb{R}^3} J_y \cdot \nabla \psi(y)\chi(t)\,d\nu(y)
\]
\[
\int_{\mathbb{R}^3} \chi(0)\psi(y)\,d\nu_{\alpha}^n(y) \to \int_{\mathbb{R}^3} \chi(0)\psi(y)\,d\nu(y)
\]
For the remaining term, one has
\[
\int_0^T \int_{\mathbb{R}^3} J_y \cdot \nabla \psi(y)\chi(t)\,d\nu_{\alpha}^n(y)dt \to \int_0^T \int_{\Omega_{\infty}} J_x \cdot \nabla \psi(\nabla P_t^n(x))\chi(t)\,d\sigma_{h^\alpha(t,\cdot)}(x)dt
\]
\[
\to \int_0^T \int_{\Omega_{\infty}} J_x \cdot \nabla \psi(\nabla P_t(x))\chi(t)\,d\sigma_{h^\alpha(t,\cdot)}(x)dt = \int_0^T \int_{\mathbb{R}^3} J_\gamma^n(t) \cdot \nabla \psi(y)\chi(t)\,d\nu_{\alpha}(y)dt
\]
The equality above used the definition of barycentric projection, and the convergence happens because of strong convergence of $h^n$ and $\nabla P^n$.

As a corollary, we can deduce the existence result of dual space solution with a general measure-valued initial data (with compact support).

**Corollary 4.10.** Let $\nu_0 \in \mathcal{P}(\mathbb{R}^3)$ and such that $\text{supp} \nu_0 \subset \mathbb{R}^2 \times [-\frac{1}{\delta}, \delta]$ and is compact. Then there exists a solution $\nu \in AC_{\infty}(0, T; \mathcal{P}_2(\mathbb{R}^3))$ with $\text{supp} \nu \subset \mathbb{R}^2 \times [-\frac{2}{\delta}, \delta]$ which satisfies the dual space equations and has $\nu_0$ as initial data.

**Proof.** Choose $\Lambda_{0,\text{H}} = B_{D_0} \times [-\frac{1}{\delta}, \delta]$ such that $\text{supp} \nu_0 \subset \Lambda_0$. Choose $\Lambda, H$ as in theorem 2.24. Let $\nu_0^n \in \mathcal{P}_{\text{ac}}(\mathbb{R}^3)$ and such that $\text{supp} \nu_0^n \subset \Lambda_0, \nu_0^n \in L^q(\Lambda)$ for some $q \in (1, \infty)$ and $\nu_0^n \to \nu_0$ narrowly as measure. Let $(h^n, P^n, \sigma^n)$ be the dual space solution given by theorem 2.24 with initial data $\nu_0^n$. That theorem also gives $\text{supp} \nu_0^n \subset \Lambda$. The same argument which was used to derive (4.9) above can be used to show $\nu^n \in AC_{\infty}(0, T; \mathcal{P}_2(\mathbb{R}^3))$. Now we are in a position to apply above proposition to obtain a solution $\nu \in AC_{\infty}(0, T; \mathcal{P}_2(\mathbb{R}^3))$.

4.3. **Relaxed Lagrangian solutions in the physical space ($\nabla P_t$ bounded).** In this section, we will prove the existence of renormalized solutions with admissible initial data. First we observe that, similar to the fixed boundary case, the renormalized Lagrangian solution will give rise to a measure valued solution in dual space.

**Proposition 4.11.** Let $(P, h, \alpha)$ be a renormalized Lagrangian solution and let $\nu_t = \nabla P_t\sigma_{h(t, \cdot)}$, then

(i) $\nu$ solves the dual space system with initial data $\nu_0 = \nabla P_0\sigma_{h_0}$,

(ii) $\nu_t \in AC_{\infty}(0, T; \mathcal{P}_2(\mathbb{R}^3))$

**Proof.** We choose test function as $\phi(t, y) = \chi(t)\psi(y)$, then we can compute
\[
\int_0^T \int_{\mathbb{R}^3} \chi'(t)\psi(y)\,d\nu_{\alpha}(y)dt = \int_0^T \int_{\Omega_{\infty}} \chi'(t)\psi(\nabla P_t(x))\,d\sigma_{h(t, \cdot)}(x)dt = \int_0^T \int_{\Omega_{\infty}} \psi(\nabla P_t(x))\chi'(t)\,d\alpha_t(x, \hat{x})
\]
\[
= -\int_0^T \int_{\Omega_{\infty}} \nabla \psi(\nabla P_t(x)) \cdot J(\nabla P_t(x) - \hat{x})\chi(t)\,d\alpha_t(x, \hat{x})dt - \int_{\Omega_{\infty}} \psi(\nabla P_0(x))\chi(0)dx
\]
In the above, we have used (4.1) and the marginal properties of \( d\alpha_t(x, \dot{x}) \). Now
\[
\int_0^T \int_{\Omega^\infty} \nabla\psi(\nabla P_t(\dot{x})) \cdot J(\nabla P_t(\dot{x})) \chi(t) d\alpha_t(x, \dot{x}) = \int_0^T \int_{\Omega_{\Lambda(t, \cdot)}} \nabla\psi(\nabla P_t(\dot{x})) \cdot J(\nabla P_t(\dot{x})) \chi(t) dx dt
\]
while the term
\[
\int_0^T \int_{\Omega^\infty} \nabla\psi(\nabla P_t(\dot{x})) \cdot J\dot{x} \chi(t) d\alpha_t(x, \dot{x}) dt = \int_0^T \int_{\Omega^\infty} \nabla\psi(\nabla P_t(x)) \cdot Jx \chi(t) d\sigma_{\Lambda(t, \cdot)}(x) dt
\]
The last term
\[
\int_{\Omega_{\Lambda}} \psi(\nabla P_0(x)) \chi(0) dx = \int_{\mathbb{R}^3} \psi(y) \chi(0) d\nu_0(y)
\]
So we proved (i). To see (ii) is true, we notice
\[
W_2(\nu_s, \nu_t) \leq \int_s^t ||J(id - \tilde{\gamma})||_{L^2(\nu_r)} d\tau \leq \int_s^t 2(\max_{\Omega^2} |x| + W_2(\nu_r, \delta_0)) d\tau
\]
In the above, we used lemma 4.6. Notice that \( \text{supp} \nu_t \subset \Lambda \), so \( \nu_t \) is bounded in \( \mathcal{P}_2(\mathbb{R}^3) \). The same argument used in deriving (4.9) shows \( \nu \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3)) \) \( \square \)

Now we can prove the existence of renormalized solutions, the idea is to use compactness of a sequence of renormalized solutions.

**Lemma 4.12.** Let \((P^n_0, h^n_0)\) be a sequence of admissible initial data defined on \( \Omega_{\Lambda_0} \), suppose
\[
\sup_{n \geq 1} \max_{x \in \Omega^2} P^n_0(x_1, x_2, 0) < \delta H_0, \ \forall n
\]
\[
\partial P^n_0(\Omega^2 \times [0, H_0]) \subset \mathbb{R}^2 \times [-\frac{1}{\delta}, -\delta] \text{ and uniformly bounded } \forall n
\]
Suppose also that
\[
P^n_0 \to P_0 \text{ uniformly on } \Omega_{\Lambda_0}
\]
Let \( D_0 = \sup_{n \geq 1} ||\nabla P^n_0||_{L^\infty(\Omega_{\Lambda_0})} + 1 \), define \( \Lambda_0, \Lambda, H \) as in (3.12)-(3.15). Let \((P^n, h^n, \alpha^n)\) be relaxed Lagrangian solution with initial data \((P^n_0, h^n_0)\) such that \( \text{supp} \nabla P^n_0 \sigma_{h^n(t, \cdot)}(\cdot) \subset \Lambda \). Then possibly up to a sequence, \((P^n, h^n, \alpha^n)\) will converge to a renormalized solution \((P, h, \alpha)\) with initial data \((P_0, h_0)\). More specifically, we have
(i) \( P^n_t \to P_t \) in \( W^{1, r}(\Omega_H) \) \( \forall r < \infty, \ t \in [0, T] \)
(ii) \( h^n(t, \cdot) \to h(t, \cdot) \) uniformly \( \forall t \in [0, T] \)
(iii) \( \alpha^n \to \alpha \) narrowly as measures.

**Proof.** First we recall by lemma 2.7, \( P^n_0 \to P_0 \) uniformly implies \( h^n_0 \to h_0 \) uniformly, where \( h_0 = h^{H_0}_0 \). Also notice by our assumption, \((P_0, h_0)\) will also be admissible initial data. Denote \( \nu^n_t = \nabla P^n_0 \sigma_{h^n(t, \cdot)} \). Then by previous proposition, \( \nu^n \) solves the dual space system with initial data \( \nu^n_0 := \nabla P^n_0 \sigma_{h^n_0} \) and \( \nu^n_t \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3)) \). Since \( P^n_0 \to P_0 \) uniformly, \( \nabla P^n_0 \to \nabla P_0 \) in \( L^\infty(\Omega_H) \), for any \( r < \infty, h^n_0 \to h_0 \) uniformly, so
we can conclude by applying dominated convergence and noticing that they have support contained in $\Lambda_0$

$$\nu_0^n \to \nu_0 \text{ narrowly and } \sup_{n \geq 1} W_2(\nu_0^n, \nu_0) < \infty$$

Now we are in a position to apply proposition 4.7 to conclude for some subsequence $\nu_t^n$ converges to $\nu_t$ narrowly \( \forall t \in [0, T] \) and $\nu_t \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$

Let \((P_t^n, R_t^n)\) be the unique maximizers of $J_t^n(P, R)$ given by corollary 2.19, \((h^n_t, \gamma^n_t)\) be the unique minimizer of $E_{\nu_t^n}(h, \gamma)$ given by theorem 2.17. Then \((i), (ii)\) is given by theorem 2.24.

To see \((iii)\), note that \(\text{supp } \alpha^n \subset [0, T] \times \Omega \times \Omega_H\), so $\alpha^n$ is tight and one can assume up to a subsequence

$$\alpha^n \to \alpha \text{ narrowly}$$

and $\alpha$ has disintegration $d\alpha(t, x, \hat{x}) = d\alpha_i(x, \hat{x})dt$. Since $d\alpha^n_i(x, \hat{x})$ has marginals $\sigma_{h^n_0}, \sigma_{h^n(t, \cdot)}$, then $d\alpha_i(x, \hat{x})$ has marginals $\sigma_{h_0}, \sigma_{h(t, \cdot)}$

Now we need to check the limit gives a relaxed Lagrangian solution with initial data $(P_0, h_0)$. \((v)\) is again implied by proposition 3.3. The only thing nontrivial to check is \((iv)\). For each fixed $n$, we have

$$\int_0^T \int_{\Omega_H^n} \xi(\nabla P_t^n(\hat{x})) \partial_t \psi(t, x) d\alpha^n(t, x, \hat{x}) dt + \int_0^T \int_{\Omega_H^n} \Delta(\nabla P_t^n(\hat{x}) - \dot{x}) \psi(t, x) d\alpha^n(t, x, \hat{x}) dt$$

$$+ \int_{\Omega_H^n} \xi(\nabla P_0^n(x)) \psi(0, x) dx = 0 \quad \forall \xi \in C^1_0(\mathbb{R}^3)$$

We wish to pass each term to the limit. First it's obvious that

$$\int_{\Omega_{h^n_0}} \xi(\nabla P_0^n(x)) \psi(0, x) dx \to \int_{\Omega_{h_0}} \xi(\nabla P_0(x)) \psi(0, x) dx$$

, since this convergence happens $\mathcal{L}^3$-a.e on $\Omega_{h_0}$ and $h^n_0 \to h_0$ uniformly as noted above. Next we look at first term. We choose $g \in C_b(\mathbb{R}^3, \mathbb{R}^3)$, such that

$$\mathcal{L}^4(\{(t, x, \hat{x}) \in [0, T] \times \Omega \times \Omega_H | \nabla P_t(\hat{x}) \neq g(t, \hat{x})\}) < \epsilon$$

$$\|g\|_{L^\infty} \leq M := \sup_{t, n} \| \nabla P_t^n(x) \|_{L^\infty}$$

Then we can write

$$\int_0^T \int_{\Omega_H^n} \xi(\nabla P_t^n(\hat{x})) \partial_t \psi(t, x) d\alpha^n(t, x, \hat{x}) - \int_0^T \int_{\Omega_H^n} \xi(\nabla P_t(\hat{x})) \partial_t \psi(t, x) d\alpha(t, x, \hat{x})$$

$$= \int_0^T \int_{\Omega_H^n} \xi(\nabla P_t^n(\hat{x})) - \xi(\nabla P_t(\hat{x})) \partial_t \psi(t, x) d\alpha^n(t, x, \hat{x})$$

$$+ \int_0^T \int_{\Omega_H^n} \xi(\nabla P_t^n(\hat{x}) - \dot{x}) \partial_t \psi(t, x) d\alpha^n(t, x, \hat{x})$$

$$+ \int_0^T \int_{\Omega_H^n} \xi(g(t, \hat{x})) \partial_t \psi(t, x) d\alpha^n(t, x, \hat{x}) - \alpha(t, x, \hat{x})$$

$$+ \int_0^T \int_{\Omega_H^n} \xi(g(t, \hat{x})) \partial_t \psi(t, x) d\alpha(t, x, \hat{x}) = A + B + C + D$$
Now straightforward estimate gives
\[ |A| \leq ||\partial_t \psi||_{L^\infty} \int_0^T \int_{\Omega_\infty} |\xi(\nabla P^t_i(\hat{x})) - \xi(\nabla P_t(\hat{x}))|dx \to 0 \]
This is by dominated convergence.
\[ |B|, |D| \leq 2 ||\partial_t \psi||_{L^\infty} ||\xi||_{L^\infty} \epsilon \]
while \( C \to 0 \) by narrow convergence.
So we can pass to limit and this verifies \( (P, h, \alpha) \) is a renormalized solution.

\[ \square \]

Now we can prove the existence of renormalized solutions

**Theorem 4.14.** Let \( (P_0, h_0) \) be admissible initial data. Let \( \Lambda_0, \Lambda, H \) be as in (3.12-3.15). Then there exists a renormalized Lagrangian solution \( (P, h, \alpha) \) with initial data \( (P_0, h_0) \). Besides, we also have
\( i \) \( P \in L^\infty(0, T; W^{1,\infty}(\Omega_\infty)) \cap C([0, T]; H^1(\Omega_\infty)) \)
\( ii \) \( h \in L^\infty(0, T; W^{1,\infty}(\Omega_2)) \cap C([0, T] \times \Omega_2) \)

**Proof.** Let \( v_0 = \nabla P_0 \sigma_{h_0} \), then \( \text{supp} \ v_0 \subset \Lambda_0 \). Choose a sequence \( v^n_0 \in \mathcal{P}_{ac}(\mathbb{R}^3) \), such that \( \text{supp} \ v^n_0 \subset \Lambda_0, v^n_0 \in L^2(\Lambda_0) \), and \( v^n_0 \to v_0 \) narrowly. Let \( (P^n_0, R^n_0) \) be the unique maximizers of \( J^n_{h_0}(P, R) \) given by corollary 2.19. Then by putting \( h^n_0 = h^n_{P^n_0, (P^n_0, h^n_0)} \) gives admissible initial data and \( v^n_0 = \nabla P^n_{00} \sigma_{h^n_0} \in L^2(\mathbb{R}^3) \). Let \( (h^n, P^n, \alpha^n) \) be the weak Lagrangian solution given by theorem 3.10. Put \( \alpha^n = (id \times \nabla x^n)_2 \sigma_{h^n_0} \) and \( \text{supp} \ (t, x, \dot{x}) = \text{supp} \alpha^n(x, \dot{x})dt \). Then \( (h^n, P^n, \alpha^n) \) are relaxed Lagrangian solutions due to the additional properties proved by theorem 3.10. Also we have by theorem 2.24 that \( \text{supp} \ v^n_0 \subset \Lambda \). Then the previous lemma gives us a renormalized solution with initial data \( (P_0, h_0) \).

\[ \square \]

5. Existence of renormalized solution with more general initial data \( \nabla P_0 \in L^2 \)

5.1. **definition of generalized data and main result.** In this section, we generalize the result of previous section to more general initial data, namely we no longer require \( P_0 \in W^{1,\infty} \). Instead, we only require \( \nabla P_0 \in L^2(\Omega_\infty) \). The proofs are quite similar as before but several complications come up.
First we define a more general class of initial data, and state the existence result of renormalized solutions in such a case. To prove the existence, we need to study the functional \( E_\alpha(h, \gamma) \) and \( J_\alpha(P, R) \) with more general \( \nu \), this is done in subsection 2.2. As a byproduct of our proof, we get the dual space existence result with a general measure valued initial data whose support is contained in \( \mathbb{R}^2 \times [-\frac{1}{\delta}, \frac{\delta}{\delta}] \) for some \( \delta > 0 \)

**Definition 5.1.** Let \( P_0 : \Omega_\infty \to \mathbb{R} \) be a convex function, and \( h_0 \in L^2(\Omega_2) \). Then we say \( (P_0, h_0) \) is a generalized data if the following holds:
\( i \) \( \nabla P_0 \in L^2(\Omega_{h_0}) \) and \(-\frac{1}{\delta} \leq \partial_{x_3} P_0 \leq -\delta \)
\( ii \) \( h_0 \geq 0 \) and \( \int h_0 dx_1 dx_2 = 1 \)
\( iii \) \( P_0(x_1, x_2, h_0(x_1, x_2)) = \frac{1}{2}(x_1^2 + x_2^2) \forall (x_1, x_2) \in \Omega_2 \)

Here is the main result of this section.
Theorem 5.2. Let \((P_0,h_0)\) be generalized data in the sense of above definition, then there exists a relaxed Lagrangian solution \((P,h,\alpha)\) having \((P_0,h_0)\) as initial data. Besides, we have the following continuity in time:

(i) \(h \in C([0,T]; L^2(\Omega_2))\)

(ii) \(\xi(P,\nabla P)\sigma_\eta \in C([0,T]; L^1(\Omega_\infty)) \forall \xi \in C_0(\mathbb{R} \times \mathbb{R}^3)\)

5.2. Measure-valued solution in dual space with unbounded support.

First we need to prove a generalization of proposition 4.7.

Proposition 5.3. Suppose \(\{\nu^n_0, \nu^n_1\}_{n \geq 1} \subset \mathcal{P}_2(\mathbb{R}^3)\) with \(\text{supp} \nu^n_0, \nu^n_1 \subset \mathbb{R}^2 \times [-\frac{1}{3}, \delta]\) and such that

\[ \nu^n_0 \to \nu_0 \text{ narrowly and } \sup_{n \geq 1} M_2(\nu^n_0) < \infty \]

Let \(\nu^n \in AC^\infty(0,T; \mathcal{P}_2(\mathbb{R}^3))\) be solutions to the dual space system corresponding to initial data \(\nu^n_0\), and also that \(\text{supp} \nu^n \subset \mathbb{R}^2 \times [-\frac{1}{3}, \delta]\). Then a subsequence \(\nu^n\) converges to a solution \(\nu \in AC^\infty(0,T; \mathcal{P}_2(\mathbb{R}^3))\) with initial data \(\nu_0\), i.e.

(i) \(W_p(\nu^n, \nu_1) \to 0\) for all \(t \in [0,T]\), and all \(p \in [1,2]\)

(ii) \(\nu \in AC^\infty(0,T; \mathcal{P}_2(\mathbb{R}^3))\) is a solution to dual space system with initial data \(\nu_0\)

Proof. The same argument as lead to (4.8) proves the following

\[
W_2(\delta_0, \nu^n_1) \leq \sup_{n \geq 1} W_2(\delta_0, \nu^n_0) e^{2T} + (e^{2T} - 1) T \max_{\Omega_2} |x| 0 \leq t \leq T
\]

which means \(\{\nu^n_1\}_{t \in [0,T], n \geq 1}\) are uniformly bounded in \(\mathcal{P}_2(\mathbb{R}^3)\), so it is tight. Also the same argument as lead to (4.9) we have

\[
W_2(\nu^n_1, \nu^n_0) \leq 2 e^{2T} \left[ \sup_{n \geq 1} W_2(\delta_0, \nu^n_0) + (T + 1) \max_{\Omega_2} |x||t - s| \right]
\]

which means \(\{\nu^n_1\}_{n \geq 1}\) are equicontinuous in \(t\). The argument in that prop allows us to take a subsequence, such that

\[ \nu^n_1 \to \nu_1 \text{ narrowly } \forall t \in [0,T] \]

Since \(\sup_{n \geq 1} \int |y|^2 d\nu^n_0 < \infty\), we see

\[ \int |y|^p d\nu^n_0 \to \int |y|^p d\nu_0 \forall p \in [1,2] \]

this implies \(W_p(\nu^n_1, \nu_1) \to 0\) by [3] proposition 7.1.5. Also it is easy to see \(\nu_1 \in AC^\infty(0,T; \mathcal{P}_2(\mathbb{R}^3))\) by (5.4). Finally let’s check \(\nu_1\) solve the dual space system. Take \(\phi(y,t) = \chi(t)\eta(y)\) be test function such that \(\chi(T) = 0\), then for each \(n\), we have

\[
\int_0^T \int_{\mathbb{R}^3} \chi'(t) \eta(y) d\nu^n_1(y) + \int_0^T \int_{\mathbb{R}^3} J(y - \tilde{\gamma}^n_t(y)) \cdot \nabla \eta(y) d\nu^n_1(y) \\
+ \int_{\mathbb{R}^3} \chi(0) \eta(y) d\nu^n_0(y) = 0
\]

where \(\tilde{\gamma}^n_t(y)\) is the barycentric projection of \((id \times \nabla P^n_1)\sigma_{h^n_0} \nu^n_0\) onto \(\nu^n_1\). Because of narrow convergence and \(W_p\) convergence for \(p \in [1,2]\) already noted, we see

\[
\int_0^T \int_{\mathbb{R}^3} \chi'(t) \eta(y) d\nu^n_1(y), \int_0^T \int_{\mathbb{R}^3} \chi(0) \eta(y) d\nu^n_0(y), \int_0^T \int_{\mathbb{R}^3} J(y \cdot \nabla \eta(y) d\nu^n_1(y) \text{ will converge to the right limit, while}
\]

\[
\int_0^T \int_{\mathbb{R}^3} J\tilde{\gamma}^n_t(y) \cdot \nabla \eta(y) d\nu^n_1(y) = \int_0^T \int_{\Omega_\infty} Jx \cdot \nabla \eta(\nabla P^n_1(x)) \sigma_{h^n_0}(x) dx
\]
Notice that above integrand does not involve $x_3$, by the definition of $J$, and $x_1, x_2$
are bounded, we can conclude by using thm 5.10 that
\[
\int_0^T \int_{\Omega_\infty} Jx \cdot \nabla \eta(\nabla P_t^\delta(x)) \sigma_{h_t}(x) dx = \int_0^T \int_{\Omega_\infty} Jx \cdot \nabla \eta(\nabla P_t(x)) \sigma_{h_t}(x) dx = \int_0^T \int_{\mathbb{R}^3} J\tau(y) \cdot \nabla \eta(y) d\nu_t^n(y)
\]
\[\square\]

As a corollary to above proposition, we can deduce the existence of measure-valued solution in dual space with possibly unbounded support.

**Corollary 5.5.** Let $\nu_0 \in \mathcal{P}_2(\mathbb{R}^3)$ and such that $\text{supp} \nu_0 \subset \mathbb{R}^2 \times [-\frac{1}{2}, -\delta]$, then there exists a family of measures $[0, T] \ni t \mapsto \nu_t$ such that $\nu \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$, $\text{supp} \nu_t \subset \mathbb{R}^2 \times [-\frac{1}{2}, -\delta]$ and solves the dual space system.

**Proof.** Define $\nu_0^n = \frac{\delta(0, \cdot)}{\nu_0(\delta(0, \cdot))} \nu_0$, then $\nu_0^n \in \mathcal{P}_2(\mathbb{R}^3)$, $\text{supp} \nu_0^n \subset \mathbb{R}^2 \times [-\frac{1}{2}, -\delta]$ and compact. Also we have $\nu_0^n \rightarrow \nu_0$ narrowly. Then we can use corollary 4.10 to get a dual space solution $\nu_t^n \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$ and has $\nu_0^n$ as initial data with $\text{supp} \nu_t^n \subset \mathbb{R}^2 \times [-\frac{1}{2}, -\delta]$. The previous proposition gives us a solution $\nu_t$ with initial data $\nu_0$.

\[\square\]

### 5.3. Existence of relaxed Lagrangian solution with generalized data.

First we show that for the generalized data $(P_0, h_0)$ defined previously, $P_0 |_{\Omega_{h_0}}$ gives a maximizer. More precisely, we have

**Lemma 5.6.** Let $(P_0, h_0)$ be a generalized data, put $\nu_0 = \nabla P_0 \sigma_{h_0}$. Let $(\tilde{P}_0, \tilde{h}_0)$ be a pair of maximizer of $J_{\nu_0}(P, R)$, which satisfies (3.39), (3.40), then $P_0 = \tilde{P}_0$ on $\Omega_{h_0}$ and $(h_0, (id \times \nabla P_0)_t \sigma_{h_0})$ is the minimizer of $E_{\nu_0}(h, \gamma)$.

**Proof.** The argument is similar to proposition 3.3. The difference is that one need to define
\[
\tilde{P}_0(x) = \max(P_0(x), \frac{1}{2}(x_1^2 + x_2^2))
\]
and
\[
\tilde{h}_0(y) = \sup_{x \in \Omega_\infty} (x \cdot y - \tilde{P}_0(x))
\]
This definition ensures $\tilde{h}_0$ is finite, then one can show $J_{\nu_0}(\tilde{P}_0, \tilde{h}_0) = E_{\nu_0}(h_0, \gamma_0)$ in the same way as lemma 2.45. The rest of the argument goes in the same way as proposition 3.3.

\[\square\]

**Remark 5.7.** Recall definition 4.2, the same argument as above shows that for each fixed $t \in [0, T]$, if we put $\nu_t = \nabla P_t \sigma_{h_t}$, then $(h_t, (id \times \nabla P_t)_t \sigma_{h_t})$ is the minimizer of $E_{\nu_t}(h, \gamma)$, $\tilde{P}_t$ is the restriction of some maximizer of $J_{\nu_t}(P, R)$ restricted on $\Omega_{h_t}$.

**Lemma 5.8.** Let $(P_0^n, h_0^n), (P_0, h_0)$ be generalized initial data, Suppose also that
\[
\sup_{n \geq 1} \| \nabla P_0^n \|_{L^2(\Omega_{h_0^n})} < \infty
\]
and
\[
h_0^n \rightharpoonup h_0 in \ L^1(\Omega_2) \quad \xi(P_0^n, \nabla P_0^n) \sigma_{h_0^n} \rightharpoonup \xi(P_0, \nabla P_0) \sigma_{h_0} in \ L^1(\Omega_\infty)
\]
\[ \forall \xi \in C_b(R \times R^3) \]

Let \((P^n_t, h^n, \alpha^n)\) be relaxed Lagrangian solution with initial data \((P^n_0, h^n_0, \alpha^n)\), then there exists a subsequence which converges to a relaxed solution with initial data \((P_0, h_0)\). More precisely, for any \( t \in [0, T] \)

(i) \( h^n_t \to h_t \) in \( L^r(\Omega) \) for all \( r \in [1, 2) \)

(ii) \( \alpha^n \to \alpha \) narrowly

(iii) \( \xi(P^n_t, \nabla P^n_t)\sigma_h \to \xi(\bar{P}_t, \nabla \bar{P}_t)\sigma_h \) in \( L^1(\Omega) \) \( \forall \xi \in C_b(R \times R^3) \)

**Proof.** Define \( \nu^n = \nabla P^n_t^\# h^n \) then by proposition 4.11, we know \( \nu^n \) solves the dual space system with \( \nu^n \in AC^\infty(0, T; \mathcal{P}_2(R^3)) \). Since \(-\frac{1}{3} \leq \partial_{x_i} P_t \leq -\delta \), we also know that sup sup \( \nu^n \subset R^3 \times [-\frac{1}{3}, -\delta] \).

Define \( \nu_0 = \nabla P_{0t}^\# h_0 \). By assumption, one has sup sup \( \nu_0 \subset R^3 \times [-\frac{1}{3}, -\delta] \). Also

\[
\int_{R^3} \eta(y) d\nu_0^n (y) = \int_{\Omega} \eta(\nabla P_0^n(x)) \sigma^n_{h_0}(x) dx \to \int_{\Omega} \eta(\nabla P_0(x)) \sigma_{h_0}(x) dx
\]

\[
= \int_{R^3} \eta(y) d\nu_0 (y) \forall \eta \in C_b(R^3)
\]

So \( \nu_0^n \to \nu_0 \) narrowly. Also we have sup sup \( M_2(\nu^n) < \infty \), because we assumed sup \( \|\nabla P_0^n\|_{L^2(\Omega_3)} < \infty \).

We are now in a position to apply proposition 5.3 to get a subsequence of \( \nu^n \) which converges to a solution \( \nu \in AC^\infty(0, T; \mathcal{P}_2(R^3)) \) which solves the dual space system with initial data \( \nu_0 \).

By proposition 5.3, we know \{\( \nu^n \)\}_{t \in [0, T], n \geq 1} is uniformly bounded in \( \mathcal{P}_2(R^3) \), hence \{\( \|\nabla P_t^n\|_{L^2(\Omega_3)}\)\}_{t \in [0, T], n \geq 1} is uniformly bounded. Remark 5.7 shows for each \( t \in [0, T] \), \( (h_t^n, (d_\nu \nabla P^n_t)^\# \sigma_h) \) is the unique minimizer of \( E_{\nu^n_t}(h, \gamma) \). Now let \( (\bar{P}_t^n, \bar{R}_t^n, (\bar{P}, \bar{R}) \) be maximizers of \( J_{\nu^n_t}(P, R, J_{\nu^n_t}(P, R) \) respectively satisfying (2.39), (2.40), then by theorem 2.46, we conclude for each \( t \in [0, T] \)

\[
h^n_t \to h_t \text{ in } L^r(\Omega) \text{ for any } r \in [1, 2]
\]

\[
\xi(\bar{P}_t^n, \nabla \bar{P}_t^n)\sigma_h \to \xi(\bar{P}_t, \nabla \bar{P}_t)\sigma_h \text{ in } L^1(\Omega) \text{ } \forall \xi \in C_b(R \times R^3)
\]

Also we know that \{\( \|h^n_t\|_{L^2(\Omega_3)}\}_{t \in [0, T], n \geq 1} is bounded by a universal constant by lemma 2.42 Again by previous remark, we actually have \( \bar{P}_t^n = P_t^n \) on \( \Omega_{h_t^n} \). so we really have

\[
\xi(P_t^n, \nabla P_t^n)\sigma_{h_t^n} \to \xi(\bar{P}_t, \nabla \bar{P}_t)\sigma_{h_t} \text{ in } L^1(\Omega) \text{ } \forall \xi \in C_b(R \times R^3)
\]

Finally since for each \( n, \text{proj}_{x} \alpha^n = \sigma_{h^n_t}, \text{proj}_{\hat{x}} \alpha^n = \sigma_{h^n_t} \). Because of the \( L^2 \) bound of \( h^n_t \), we know \{\( \alpha^n \)\} as a measure on \([0, T] \times \Omega_3 \times \Omega_3\) is tight. Therefore up to a subsequence, we can assume \( \alpha^n \to \alpha \) narrowly, and the limit \( \alpha \) must disintegrate as \( d\alpha(x, \hat{x}) = d\alpha dt, \text{with proj}_{x} \alpha_t = \sigma_h \) and \( \text{proj}_{\hat{x}} \alpha_t = \sigma_{h_t} \). It only remains to check (\( \bar{P}, h, \alpha \)) is a relaxed solution with initial data \((P_0, h_0)\). To see the initial data is assumed, we use the previous lemma. To see it satisfy the equation in the sense of distribution, we argue in the same way as lemma 4.12. The only difference is that when we estimate the term (4.13) in the proof of lemma 4.12, we do it in the following way:

\[
\int_0^T \int_{\Omega} (\xi(\nabla P^n_t(\hat{x})) - \xi(\nabla P_t(\hat{x}))) \partial_t \psi(t, x) d\alpha^n(t, x, \hat{x}) \leq
\]

\[
\|\partial_t \psi\|_{L^\infty} \int_0^T \int_{\Omega} |\xi(\nabla P^n_t(\hat{x})) - \xi(\nabla P_t(\hat{x})))\sigma_{h^n_t}(\hat{x}) d\hat{x} dt
\]
Then we can write
\[ |\xi(\nabla P^n_t(x)) - \xi(\nabla \hat{P}_t(\hat{x}))|_{\sigma_h^n(x)} \leq |\xi(\nabla P^n_t(x)) - \xi(\nabla \hat{P}_t(\hat{x}))|_{\sigma_h^n(x)} + ||\xi||_{L^P} |\sigma_h^n(x) - \sigma_h^n(\hat{x})| \]
By the convergence already noted, their integral goes to zero. The rest is the same as lem 4.12

Now we are ready to prove theorem 5.2, the existence of relaxed Lagrangian solutions.

**Proof.** Let \( \nu_0 = \nabla P_0 \sigma_h \), then \( \nu_0 \in \mathcal{P}_2(\mathbb{R}^3) \) and \( \text{supp } \nu_0 \subset \mathbb{R}^2 \times [-\frac{1}{\delta}, -\delta] \). Define \( \nu_0^n = \chi_{B(0,n)} \nu_0 \). Then \( \nu_0^n \) has compact support and \( \nu_0^n \to \nu_0 \) narrowly. Let \( (P^n_0, R^n_0) \) be the maximizer of \( J^H_{\nu^n} (P, R) \) with \( H_n \) taken large enough, \( (h^n_0, \gamma^n_0) \) be the minimizer of \( E_{\nu^n} (h, \gamma) \), then \( (P^n_0, h^n_0) \) are admissible initial data. We know by lemma 2.47 that it is also the maximizer of \( J_{\nu^n} (P, R) \) after suitable extension of \( P^n_0 \). Also by theorem 2.46, we have \( h^n_0 \to h_0 \) in \( L^r(\Omega_2) \), for any \( r \in [1, 2] \), and \( \xi(P^n_0, \nabla P^n_0) \sigma_h^n \to \xi(P_0, \nabla P_0) \sigma_h \) in \( L^r(\Omega_\infty) \). Let \( (P^n, h^n, \alpha^n) \) be the relaxed solution with initial data \( (P^n_0, h^n_0) \) given by theorem 4.14. Then previous lemma gives us a relaxed solution with initial data \( (P_0, h_0) \).

The continuity property in time is given by theorem 2.46 upon noticing that \( \nu \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3)) \) \( \square \)

## 6. Appendix

Here we prove that

**Lemma 6.1.** Given \( (P_0, R_0) \) be a pair of convex conjugate maximizer of \( J^H_{\nu} (P, R) \), let \( h_0 = h^H_{P_0} \) be such that \( 0 \leq h_0 < H \), suppose also \( \Omega_2 \subset B_D(0) \), where \( \Lambda = B_D(0) \times [-\frac{1}{\delta}, -\delta] \), define
\[
R_1(y) = \sup_{x \in \Omega_H} \{ x \cdot y - \max(P_0(x), \frac{1}{2}(x_1^2 + x_2^2)) \}
\]
\[
P_1(x) = \sup_{y \in \Lambda} (x \cdot y - R_1(y))
\]

Then the following holds
(i) \( (P_1, R_1) \) are convex conjugate over \( \Omega_H, \Lambda \)
(ii) \( P_1(x_1, x_2, 0) \geq \frac{1}{2}(x_1^2 + x_2^2) \quad \forall (x_1, x_2) \in \Omega_2 \)
(iii) \( P_1(x_1, x_2, 0) \leq \frac{1}{2}(x_1^2 + x_2^2) \quad \text{whenever } h_0(x_1, x_2) = 0 \)
(iv) \( (P_1, R_1) \) is a maximizer of \( J^H_{\nu} (P, R) \) and \( h^H_{P_1} = h_0 \)

**Proof.** First we prove (i), denote \( \tilde{P}_0(x) = \max(P_0(x), \frac{1}{2}(x_1^2 + x_2^2)) \), we need to show
\[
R_1(y) = \sup_{x \in \Omega_H} (x \cdot y - P_1(x))
\]
By definition of \( R_1(y) \), we know
\[
R_1(y) \geq \tilde{P}_0(x) \quad \forall x \in \Omega_H, \forall y \in \Lambda
\]
Hence
\[
\tilde{P}_0(x) \geq P_1(x) \quad x \in \Omega_H
\]
So
\[ R_1(y) \leq \sup_{x \in \Omega_H} (x \cdot y - P_1(x)) \]
By the definition of \( P_1 \), we have
\[ P_1(x) + R_1(y) \geq x \cdot y \quad \forall x \in \Omega_H, \forall y \in \Lambda \]
So
\[ R_1(y) \geq \sup_{x \in \Omega_H} (x \cdot y - P_1(x)) \]
(i) is proved. Now we prove (ii). We can observe that
\[ R_1(y) \leq \sup_{x \in \Omega_H} (x \cdot y - \frac{1}{2}(x_1^2 + x_2^2)) \]
Fix \((x_0^1, x_0^2) \in \Omega_2\), by assumption, we can find \( z_0 \in [-\delta, \delta] \), such that \((x_1^0, x_2^0, z_0) \in \Lambda\), then
\[ R_1(x_1^0, x_2^0, z_0) \leq \sup_{x \in \Omega_H} (x \cdot (x_1^0, x_2^0, z_0) - \frac{1}{2}(x_1^2 + x_2^2)) \]
\[ = \sup_{x \in \Omega_2} (x_1^0 x_1 + x_2^0 x_2 - \frac{1}{2}(x_1^2 + x_2^2)) \leq \frac{1}{2}[(x_1^0)^2 + (x_2^0)^2] \]
In the equality above, we noticed \( x_3 z_0 \leq 0 \). Therefore,
\[ P_1(x_1^0, x_2^0, 0) = \sup_{y \in \Lambda} (x_1^0 y_1 + x_2^0 y_2 - R_1(y)) \geq (x_1^0)^2 + (x_2^0)^2 - R_1(x_1^0, x_2^0, z_0) \geq \frac{1}{2}[(x_1^0)^2 + (x_2^0)^2] \]
(ii) is proved. Now we prove (iii). Fix \((x_0^1, x_0^2) \in \Omega_2\) such that \( h_0(x_1^0, x_2^0) = 0 \), then
\((x_1^0, x_2^0, 0) \notin \Omega_{h_0}\)
But
\[ R_1(y) = \max[ \sup_{x \in \Omega_{h_0}} (x \cdot y - \hat{P}_0(x)), \sup_{x \notin \Omega_{h_0}} (x \cdot y - \hat{P}_0(x))] \]
On \(\Omega_{h_0}\), \(P_0(x) \geq \frac{1}{2}(x_1^2 + x_2^2)\), so \(\hat{P}_0(x) = P_0(x)\), otherwise \(\hat{P}_0(x) = \frac{1}{2}(x_1^2 + x_2^2)\)
Hence
\[ R_1(y) \geq \sup_{x \notin \Omega_{h_0}} (x \cdot y - \frac{1}{2}(x_1^2 + x_2^2)) \]
Then
\[ (x_1^0, x_2^0, 0) \cdot y - R_1(y) \leq \frac{1}{2}((x_1^0)^2 + (x_2^0)^2) \quad y \in \Lambda \]
Taking supremum over \(y\), we obtain
\[ P_1(x_1^0, x_2^0, 0) \leq \frac{1}{2}((x_1^0)^2 + (x_2^0)^2) \]
(iii) is proved. Finally we prove (iv). We start by showing that \( P_1 = P_0 \) on \(\Omega_{h_0}\)
Indeed
\[ R_1(y) = \sup_{x \in \Omega_H} (x \cdot y - \hat{P}_0(x)) \leq \sup_{x \in \Omega_H} (x \cdot y - P_0(x)) = R_0(y) \]
So
\[ P_1(x) = \sup_{y \in \Lambda} (x \cdot y - R_1(y)) \geq \sup_{y \in \Lambda} (x \cdot y - R_0(y)) = P_0(x) \]
On the other hand
\[ R_1(y) \geq \sup_{x \in \Omega_{h_0}} (x \cdot y - P_0(x)) \]
So if \(x \in \Omega_{h_0}\)
\[ P_1(x) = \sup_{y \in \Lambda} (x \cdot y - R_1(y)) \leq P_0(x) \]
Finally we only need to see $J_{\nu}^H(h,\gamma)$. So

$$J_{\nu}^H(P_1, R_1) = \int_{\Lambda} \left[ \frac{1}{2} (y_1^2 + y_2^2) - R_1(y) \right] dv(y) + \int_{\Omega_{h_0}} \left[ \frac{1}{2} (x_1^2 + x_2^2) - P_1(x) \right] dx$$

$$= \int_{\Omega_H \times \Lambda} \left[ \frac{1}{2} (y_1^2 + y_2^2) + \frac{1}{2} (x_1^2 + x_2^2) - R_1(y) - P_1(x) \right] d\gamma_0 = E_{\nu}(h_0, \gamma_0)$$

Since $(P_1, R_1)$ convex conjugate, and $P_0 = P_1$ on $\Omega_{h_0}, R_1(y) + P_1(x) = x \cdot y \gamma_0 - a.e.$

**Remark 6.2.** We notice that above argument still works even if $H = \infty$. 

**Lemma 6.3.** Suppose $(P_1, R_1)$ are convex conjugate (over $\Omega_H$ and $\Lambda$) maximizers of $J_{\nu}^H(P, R)$, suppose also that $P_1(x_1, x_2, 0) = \frac{1}{2} (x_1^2 + x_2^2)$ whenever $h_1(x_1, x_2) = 0$, where $h_1 = h_{P_1}^H < H$. We define

$$R_2(y) = \sup_{x \in \Omega_{h_1} \cup \{ x_3 = 0 \}} (x \cdot y - P_1(x))$$

and

$$P_2(x) = \sup_{y \in \Lambda} (x \cdot y - R_2(y))$$

then

(i) $P_1 = P_2$ on $\Omega_{h_1} \cup \{ x_3 = 0 \}$

(ii) $(P_2, R_2)$ are convex conjugate over $\Omega_H$ and $\Lambda$.

(iii) $(P_2, R_2)$ are also maximizers and $h_{P_2}^H = h_1$.

**Proof.** Take $x \in \Omega_H \cup \{ x_3 = 0 \}$, then $\forall y \in \Lambda$, we have

$$x \cdot y - R_2(y) = x \cdot y - \sup_{\bar{x} \in \Omega_H \cup \{ x_3 = 0 \}} (\bar{x} \cdot y - P_1(\bar{x})) \leq P_1(x)$$

Take supremum over $y$ to get

$$P_2(x) \leq P_1(x) \ (x \in \Omega_H \cup \{ x_3 = 0 \})$$

On the other hand, by defn $R_2(y) \leq R_1(y)$ since $(P_1, R_1)$ conjugate, we obtain

$$P_1(x) \leq P_2(x) \ (x \in \Omega_H)$$

(i) is proved.

To see $(P_2, R_2)$ also convex conjugate over $\Omega_H$ and $\Lambda$, we only need to show

$$R_2(y) = \sup_{x \in \Omega_H} (x \cdot y - P_2(x))$$

Obviously $LHS \leq RHS$ by (i)

On the other hand, for all $x \in \Omega_H$

$$x \cdot y - P_2(x) = x \cdot y - \sup_{\bar{y} \in \Lambda} (x \cdot \bar{y} - R_2(\bar{y})) \leq R_2(y)$$

So (ii) is proved.

Finally we only need to see $J_{\nu}^H(P_1, R_1) = J_{\nu}^H(P_2, R_2)$. By assumption, we know $(h_1, (id \times \nabla P_1)_1 \sigma_{h_0})$ is the minimizer of $E_{\nu}(h, \gamma)$

By (i), we know that

$$P_2(x_1, x_2, 0) = \frac{1}{2} (x_1^2 + x_2^2) \ \text{whenever} \ h_1(x_1, x_2) = 0$$
Combining the fact that $P_1 = P_2$ on $\Omega_{h_i}$ it's easy to see $h_{P_{2}}^H = h_1$. The same argument as previous lemma (iv) shows that $(P_2, R_2)$ is a maximizer. \hfill \square

We derive the following corollary as an easy consequence of previous two lemmas.

**Corollary 6.4.** Let $\Lambda = B_D(0) \times [-\frac{1}{\gamma}, -\delta]$ be such that $\Omega_2 \subset B_D(0)$, and $H$ is chosen such that $J^H\nu(P, R)$ has convex conjugate maximizers (over $\Omega_H$ and $\Lambda$), say $(P_0, R_0)$, and $h_0 := h_{P_0}^H < H$. Then there exists a maximizer $(P_2, R_2)$ of $J^H\nu(P, R)$ which satisfies the following conditions:

(i) $(P_2, R_2)$ are convex conjugate over both $\Omega_{h_0} \cup \{x_3 = 0\}$, $\Lambda$ and $\Omega_H$, $\Lambda$.

(ii) $P_2(x_1, x_2, 0) = \frac{1}{2}(x_1^2 + x_2^2)$ whenever $h(x_1, x_2) = 0$.

Proof. Let $(P_1, R_1)$ be the pair given by lemma 5.1. The conclusion of lemma 5.1 shows $(P_1, R_1)$ satisfies the assumptions of lem 5.2. Let $(P_2, R_2)$ be the pair given by lemma 5.2. Then such a pair is a maximizer by lemma 5.2 (iii). They are convex conjugate over $\Omega_{h_0} \cup \{x_3 = 0\}$ by their very definition. They are convex conjugate over $\Omega_\infty, \Lambda$ by lemma 5.2 (ii). $P_2$ satisfies (ii) because of lemma 5.2(i) and lemma 5.1 (ii), (iii). \hfill \square

Here we prove a technical result about convex functions. It can be found in [9] section 6.3 Theorem 1.

**Lemma 6.5.** Let $\Omega$ be a convex domain in $\mathbb{R}^d$, let $P : \Omega \rightarrow \mathbb{R}$ be a convex function such that $\|P\|_{L^1(\Omega)} < \infty$. Let $\Omega_1 = \{x \in \Omega | \text{dist}(x, \Omega^c) > r\}$. Then there exists a constant $C = C(\|P\|_{L^1(\Omega)}, r, d)$ such that

\[
\|P\|_{L^\infty(\Omega_1)} \leq C \|\nabla P\|_{L^\infty(\Omega_1)} \leq C
\]

Proof. First let's see $\|P\|_{L^\infty(\Omega_1)} \leq C$.

Take $x^0 \in \Omega_1$, then $B_r(x^0) \subset \Omega$. Since $P$ is convex, it is subharmonic, hence

\[
P(x^0) \leq \frac{1}{\mathcal{L}^d(B_r(x^0))} \int_{B_r(x^0)} P(y) dy \leq \frac{1}{\alpha(d) r^d} ||P||_{L^1(\Omega)}
\]

Here $\alpha(d)$ is the volume of the unit ball in $\mathbb{R}^d$.

To estimate from below, notice that $\forall y \in B_{2r}(x^0)$, we have by convexity

\[2P(y) \leq P(x^0) + P(2y - x^0)
\]

Integrate above inequality on $B_{2r}(x^0)$ and change variable on the last term, we obtain

\[2 \int_{B_{2r}(x^0)} P(y) dy \leq \alpha(d) \left(\frac{r}{2}\right)^d P(x^0) + \frac{1}{2^d} \int_{B_r(x^0)} P(x) dx
\]

Combining (6.7), (6.8), we get $|P(x^0)| \leq C(d, r) \|P\|_{L^1(\Omega)}$. Let $\Omega_3 = \{x \in \Omega | \text{dist}(x, \Omega^c) > \frac{r}{2}\}$, then we can bound $\|P\|_{L^1(\Omega_3)}$.

To estimate the gradient, we assume in addition that $P$ is differentiable at $x^0$, fix $\mathbf{e}$ a unit vector and define $g(t) = P(x^0 + t\mathbf{e})$. Also take $t_1 < 0$, $t_2 > 0$ such that $x + t_i \mathbf{e} \in \partial \Omega_3$, $i = 1, 2$. Then $|t_1|, |t_2| \geq \frac{r}{2}$. Finally we notice that by convexity of $g$

\[
g(0) - g(t_1) \leq \frac{g(t) - g(0)}{t} \leq g(t_2) - g(0) \leq \frac{r}{2}
\]

∀ $0 < t < t_1$

Sending $t \rightarrow 0$, the middle term goes to $\nabla P(x^0) \cdot \mathbf{e}$ and both ends can be bounded by a constant depending only on $\|P\|_{L^\infty(\Omega_3), r}$. \hfill \square
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