Universality of Entanglement Creation in Low-Energy Two-Dimensional Scattering *†

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Abstract
We prove that the entanglement created in the low-energy scattering of two particles in two dimensions is given by a universal coefficient that is independent of the interaction potential. This is strikingly different from the three dimensional case, where it is proportional to the total scattering cross section. Before the collision the state is a product of two normalized Gaussians. We take the purity as the measure of the entanglement after the scattering. We give a rigorous computation, with error bound, of the leading order of the purity at low-energy. For a large class of potentials, that are not-necessarily spherically symmetric, we prove that the low-energy behavior of the purity, \( P \), is universal. It is given by \( P = 1 - \frac{1}{(\ln(\sigma/\hbar))^2} E \), where \( \sigma \) is the variance of the Gaussians and the entanglement coefficient, \( E \), depends only on the masses of the particles and not on the interaction potential. There is a strong dependence of the entanglement in the difference of the masses. The minimum is when the masses are equal, and it increases strongly with the difference of the masses.

Keywords: entanglement; low energy; scattering; purity; gaussian states; two dimensions.

1 Introduction

In this paper we consider the creation of entanglement in the low-energy scattering of two particles without spin in two dimensions. The interaction between the particles is given by a general potential that is not required to be spherically symmetric. Before the scattering the particles are in an incoming asymptotic

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state that is a product of two Gaussians. After the scattering the particles are in an outgoing asymptotic state that is not a product state. The problem that we solve is to compute the loss of purity, due to the entanglement with the other, that is produced by the collision.

In the configuration representation the Hilbert space of states for the two particles is $\mathcal{H} := L^2(\mathbb{R}^4)$. The dynamics of the particles is given by the Schrödinger equation,

$$ i\hbar \frac{\partial}{\partial t} \varphi(x_1, x_2) = H \varphi(x_1, x_2). $$

(1.1)

The Hamiltonian is the following operator,

$$ H = H_0 + V(x_1 - x_2), $$

(1.2)

where $H_0$ is the free Hamiltonian,

$$ H_0 := -\frac{\hbar^2}{2m_1} \Delta_1 - \frac{\hbar^2}{2m_2} \Delta_2. $$

(1.3)

The operator $\Delta_j$, is the Laplacian in the coordinates $x_j, j = 1, 2$, of particle one and two. By $\hbar$ it is denoted Planck’s constant. Furthermore, $m_j, j = 1, 2$, are, respectively, the mass of particle one and two. The interaction potential is a real-valued function, $V(x)$, defined for $x \in \mathbb{R}^2$. We suppose that the interaction depends on the difference of the coordinates $x_1 - x_2$, but we do not require the spherical symmetry of the potential. We consider a general class of potentials that satisfy mild assumptions on its decay at infinity and on its regularity:

**ASSUMPTION 1.1.**

$$(1 + |x|)^\beta V(x) \in L^2(\mathbb{R}^2), \quad \text{for some } \beta > 11. $$

(1.4)

Under this assumption $H$ is a self-adjoint operator.

We also suppose that at zero energy there is neither an eigenvalue nor a resonance (half-bound state), for the Hamiltonian of the relative motion $H_{rel} := -\frac{\hbar^2}{2m} \Delta_x + V(x)$ where $x \in \mathbb{R}^2$ is the relative distance and $m$ is the reduced mass $m := m_1 m_2/(m_1 + m_2)$. A zero-energy resonance (half-bound state) is a bounded solution to $H_{rel} \varphi = 0$ that is not in $L^2(\mathbb{R}^2)$. See [1] for a precise definition. For generic potentials $V$ there is neither a resonance nor an eigenvalue at zero for $H_{rel}$. That is to say, if we consider the potential $\lambda V$ with a coupling constant $\lambda$, zero can be a resonance and/or an eigenvalue for at most a finite or denumerable set of $\lambda$’s without any finite accumulation point.

We study our problem in the center-of-mass frame. We consider an incoming asymptotic state that is a product of two normalized Gaussians, given in the momentum representation by,

$$ \varphi_{in,p_0}(P_1, P_2) := \varphi_{p_0}(P_1) \varphi_{-p_0}(P_2). $$

(1.5)
with,
\[
\varphi_{p_0}(p_1) := \frac{1}{(\sigma^2 \pi)^{1/2}} e^{-(p_1 - p_0)^2/2\sigma^2},
\]
where \( p_i, i = 1, 2 \) are, respectively, the momentum of particles one and two.

In our incoming asymptotic state \( [1.5] \) particle one has mean momentum \( p_0 \) and particle two has mean momentum \(-p_0\). Both particles have the same variance, \( \sigma \), of the momentum distribution. As we suppose that the scattering takes place at the origin at time zero, the average position of both particles is zero in the incoming asymptotic state \( [1.5] \). The outgoing asymptotic state of the two particles, \( \varphi_{\text{out},p_0} \), after the scattering process, is given by,
\[
\varphi_{\text{out},p_0}(p_1, p_2) := \left(S\left(p^2/2m\right)\varphi_{\text{in},p_0}\right)(p_1, p_2).
\]
Here, \( p := \frac{m_2}{m_1 + m_2}p_1 - \frac{m_1}{m_1 + m_2}p_2 \) is the relative momentum, and \( S(p^2/2m) \) is the scattering matrix for the relative motion.

The measure of entanglement of a pure-bipartite state that we use is the purity. Namely, the trace of the square of the reduced density matrix of one of the particles, that is obtained by taking the trace on the other particle of the density matrix of the pure state. Note that the purity of a product state is one.

In the outgoing asymptotic state \( \varphi_{\text{out},p_0} \), the purity is given by,
\[
P(\varphi_{\text{out},p_0}) = \int dp_1 dp_1' dp_2 dp_2' \varphi_{\text{out},p_0}(p_1, p_2) \varphi_{\text{out},p_0}(p_1', p_2) \varphi_{\text{out},p_0}(p_1', p_2') \varphi_{\text{out},p_0}(p_1, p_2').
\]
Remark that \( \varphi_{\text{out},p_0} \) is not a product state, and that its purity is smaller than one, because the relative momentum, \( p \), depends on \( p_1 \) and on \( p_2 \). This implies that the collision has created entanglement between the two particles.

To be in the low-energy regime the following two conditions have to be satisfied. 1. The mean relative momentum \( p_0 \) has to be small. 2. The variance \( \sigma \) has to be small. Note that even if the mean relative momentum \( p_0 \) is small, if \( \sigma \) is large the incoming asymptotic state \( \varphi_{\text{in},p_0} \) has a big probability of having large momentum.

Let us designate by \( \varphi_{\text{in}} \) the incoming asymptotic state with mean relative momentum \( p_0 = 0 \). The corresponding outgoing asymptotic state is \( \varphi_{\text{out}} := S(p^2/2m)\varphi_{\text{in}} \).

Let us designate by
\[
\mu_i := \frac{m_i}{m_1 + m_2}, i = 1, 2,
\]
the fraction of the mass of the \( i \) particle to the total mass.

In Section 3 we rigorously prove the following results on the leading order of the purity at low energy.
\[ \mathcal{P}(\phi_{\text{out}}, p_0) = \mathcal{P}(\phi_{\text{out}}) + \frac{|p_0|}{\sigma} O \left( \frac{1}{|\ln(\sigma/h + |p_0|/h)|^2} \right), \quad \text{as } \sigma/h + |p_0|/h \to 0, \]  
\[ \mathcal{P}(\phi_{\text{out}}) = 1 - \frac{1}{(\ln(\sigma/h))^2} \mathcal{E}(\mu_1) + O \left( \frac{1}{|\ln(\sigma/h)|^3} \right), \quad \text{as } \sigma/h \to 0, \]  
where \( \mathcal{E}(\mu_1) \) is the entanglement coefficient,

\[ \mathcal{E}(\mu_1) := \frac{2\pi^2}{1 + (2\mu_1 - 1)^2} \left[ 1 + \sqrt{1 + (2\mu_1 - 1)^2} \right] - \frac{2}{\pi} [J(\mu_1, 1 - \mu_1) + J(1 - \mu_1, \mu_1)], \]

with

\[ J(\mu_1, \mu_2) := \int dq_2 \int dq_1 \, \text{Exp} \left\{ -\frac{1}{2} (\mu_1^2 + \mu_2^2) (q_1 + q_2)^2 - (\mu_2 q_1 - \mu_1 q_2)^2 - q_2^2/2 \right\}, \]

\[ I_0(|\mu_1 - \mu_2| |q_1 + q_2| |\mu_2 q_1 - \mu_1 q_2|)^2, \]

where \( I_0 \) is the modified Bessel function \[2\].

We have that \( \mathcal{E}(\mu_1) = \mathcal{E}(1 - \mu_1) \). This is a consequence of the invariance of \( \mathcal{P}(\phi_{\text{out}}) \) under the exchange of particles one and two.

Note that \( J(1/2, 1/2) = \pi^3 \). In the appendix we prove explicitly that \( J(1, 0) = 16.6377 \). For \( \mu_1 \in [0, 1] \setminus \{1/2, 1\} \) we compute \( J(\mu_1, 1 - \mu_1) \) numerically using Gaussian quadratures.

The entanglement coefficient \( \mathcal{E}(\mu_1) \) is universal in the sense that is independent of the interaction potential. Of course, this is only true if the potential is not identically zero, see the low-energy estimate for the scattering matrix given in \[2.11\]. It follows from \[1.11\] that the creation of entanglement is a second-order effect. Note that in the second-order term in \[2.11\] it appears the scattering length, \( a \), that depends on the potential. However, the contributions to the leading order of the purity that depend on the scattering length cancel each other.

The universality of the entanglement at low-energy in two dimensions is strikingly different from the three dimensional case that we previously studied in \[3\], where the entanglement created by the collision is proportional to the total scattering cross section. It is a natural question to ask what are the physical reasons why the results are so different in two and in three dimensions. This is certainly a non trivial issue. I propose the following answer. First note that constraining particles to live in two dimensions is a strong requirement. It dramatically changes the kinematics for low energy. This can be seen, for example, in the well known logarithmic divergence of the free Green’s function, that is absent in three dimensions. This difference in the kinematics also affects the dynamics of the particles, specially at low energy, where the asymptotics of the scattering matrix is fundamentally different in two and in three dimensions. Both in two and in three dimensions the creation of entanglement is due to kinematical effects that depend on the masses of the particles and to dynamical effects that depend on the potential of interaction. In three dimensions these effects are of the same order at low energy. However, when the particles are constrained
to two dimensions, also the dimension of the phase space is reduced, and the kinematical effects play a dominant role at low energy, and, in consequence, the details of the potential do not play a role in the leading order of the entanglement. In intuitive physical terms this is the main reason for the universality of the low-energy entanglement in two dimensions.

Table 1 and Figure 1 show that -as in three dimensions [3]- the entanglement coefficient depends strongly in the difference of the masses. The minimum is taken for $\mu_1 = 0.5$, i.e. for equal the masses, and it strongly increases with the difference of the masses, when $\mu_1$ tends to one. For a physical interpretation of this fact see [3].

Universal results, like the one of this paper, are certainly of independent interest. They point out to deep fundamental physical issues and they do not need to be justified by applications. However, our result has important potential applications. For example, in the creation of entanglement in two-dimensional systems where scattering is essential, like ultracold particles, or solid state devices. In this situation, our result shows that it is possible to produce entanglement in experiments where the particles interact very weakly, provided that the difference in the masses is large. Furthermore, we provide a formula for the created entanglement that can be verified experimentally. This, of course, requires experiments where it is possible to experimentally measure the entanglement, what is an issue on itself.

There are many other reasons to study the entanglement creation in scattering processes. Entanglement is a central issue in quantum information and scattering is fundamental in all areas of physics. For a detailed physical motivation and for other possible applications see [3].

For previous results in the generation of entanglement in scattering processes in one dimension, mainly for potentials with explicit solution, see [4], [5], and the references quoted in these papers. Actually these papers do not obtain low-energy estimates of the creation of entanglement in one dimension that can be compared to our results in the three dimensional case in [3] or to our two-dimensional results in this paper. In fact, a precise analysis of the low-energy entanglement creation in one dimension is an open problem that we intent to study in future investigations. Furthermore, [6], [7], [8], and the references quoted in these papers, study a system consisting of heavy and light particles. They study the asymptotic dynamics and the decoherence that is produced on the heavy particles by the collision with the light particles in the limit when the mass ratio is small. Note that this is a different problem from the one that we discuss here and in [3]. Furthermore, the loss of quantum coherence that is induced on heavy particles by the interaction with light ones has attracted a great deal of attention. For example, see [9], and [10].

The paper is organized as follows. In Section 2 we study the low-energy asymptotics of the scattering
matrix for the relative motion of the particles. In Section 3 we give the proof of the results in the creation of entanglement. In Section 4 we give our conclusions. In the Appendix we explicitly evaluate integrals that we need in Section 3. Along the paper we denote by \( C \) a generic positive constant that does not necessarily have the same value in different appearances.

### 2 Scattering at Low-Energy in Two Dimensions

We denote by \( \hat{\mathcal{H}} := L^2(\mathbb{R}^4) \) the state space in the momentum representation. The momentum of the particles one and two are, respectively, \( p_1, p_2 \). It is convenient to take as coordinates in the momentum representation the momentum of the center of mass and the relative momentum,

\[
\begin{align*}
    p_{\text{cm}} &:= p_1 + p_2, \\
    p &:= \frac{m_2 p_1 - m_1 p_2}{m_1 + m_2}.
\end{align*}
\]

The state space in the momentum representation factorizes as a tensor product,

\[
\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\text{cm}} \otimes \hat{\mathcal{H}}_{\text{rel}},
\]

where \( \hat{\mathcal{H}}_{\text{cm}} = L^2(\mathbb{R}^2), \hat{\mathcal{H}}_{\text{rel}} := L^2(\mathbb{R}^2) \) are, respectively, the state spaces in the momentum representation for the center-of-mass motion and the relative motion.

Since the potential depends on the difference of the coordinates of particles one and two the scattering matrix for the system decomposes as the tensor product \( I_{\text{cm}} \otimes S(p^2/2m) \) of the identity on \( \hat{\mathcal{H}}_{\text{cm}} \) times the scattering matrix for the relative motion, \( S(p^2/2m) \), in \( \hat{\mathcal{H}}_{\text{rel}} \), where \( m \) is the relative mass,

\[
m := \frac{m_1 m_2}{m_1 + m_2}.
\]

The scattering matrix \( S(p^2/2m) \) is a unitary operator in \( L^2(S^1) \) for each \( p^2/2m \in (0, \infty) \), where we denote by \( S^1 \) the unit circle in \( \mathbb{R}^2 \).

We introduce some notation that we need. We denote \( v := \sqrt{|V(x)|} \). Let \( P, Q \) be the projector operators in \( L^2(\mathbb{R}^2) \),

\[
P := \frac{1}{\alpha} v(x) \langle \cdot, v \rangle, \quad Q := 1 - P,
\]

where,

\[
\alpha := \int_{\mathbb{R}^2} |V(x)| \, dx.
\]

Furthermore,

\[
U(x) := \begin{cases} 
1, & \text{if } V(x) \geq 0, \\
-1, & \text{if } V(x) < 0.
\end{cases}
\]
By $M_{00}$ we denote the integral operator with kernel,

$$M_{00}(x, y) := U(x) \delta(x - y) - \frac{1}{2\pi} \frac{2m}{\hbar^2} v(x) \ln \left( \frac{e^\gamma |x - y|}{2} \right) v(y), \quad (2.7)$$

where $\gamma$ is Euler’s constant. Moreover, by $N_{00}$ we denote the integral operator with kernel,

$$N_{0,0}(x, y) := U(x) \delta(x - y) - \frac{1}{2\pi} \frac{2m}{\hbar^2} v(x) \ln (|x - y|) v(y), \quad (2.8)$$

and

$$D_0 := (QM_{00}Q)^{-1}, \text{ a bounded operator, } QL^2(\mathbb{R}^2) \to QL^2(\mathbb{R}^2). \quad (2.9)$$

The assumption that 0 is neither a resonance nor an eigenvalue for $H_{rel}$ precisely means that $(QM_{00}Q)$ is invertible on $QL^2(\mathbb{R}^2)$ with bounded inverse.

Finally, we designate,

$$Y_0(\nu) := \frac{1}{\sqrt{2\pi}}, \nu \in S^1. \quad (2.10)$$

For $X, Y$ Banach spaces we denote by $B(X; Y)$ the Banach space of all bounded linear operators from $X$, into $Y$. In the case $X = Y$ we use the notation $B(X)$. By $\text{Tr} A$ we designate the trace of the operator $A$.

**THEOREM 2.1.** Suppose that Assumption [1.1] is satisfied and that at zero $H_{rel}$ has neither a resonance (half-bound state) nor an eigenvalue. Then, in the norm of $B\left(L^2(S^1)\right)$ we have for $|p/\hbar| \to 0$ the expansion,

$$S(p^2/2m) = I + i\pi \frac{1}{\ln |p/\hbar|} \Sigma + \left(i\pi \ln 2 - \frac{\pi^2}{2} \right) \frac{1}{\ln (|p/\hbar|^2)} \Sigma + O\left(\frac{1}{|\ln (|p/\hbar|^2)|^3}\right), \quad (2.11)$$

where $I$ is the identity operator on $L^2(S^1)$,

$$\Sigma := (\cdot, Y_0) Y_0, \quad (2.12)$$

and $a$ is the scattering length defined by

$$\frac{1}{a} := \frac{2\pi}{\alpha} \text{Tr} \left[ P N_{00} P - P M_{00} Q D_{00} M_{00} + P M_{00} Q \right]. \quad (2.13)$$

**Proof:** Let us denote by $S_1(\lambda)$ the scattering matrix for the Hamiltonian $H_1 := -\Delta + \frac{2m}{\hbar^2} V(x)$. It follows from an elementary argument that,

$$S(p^2/2m) = S_1((p/\hbar)^2). \quad (2.14)$$

Furthermore [11],

$$S_1(\lambda) = I - 2\pi i \frac{2m}{\hbar^2} \Gamma(\lambda) v (M(\lambda))^{-1} v \Gamma^*(\lambda), \quad (2.15)$$

where $\Gamma(\lambda)$ is the trace operator,

$$(\Gamma(\lambda) \varphi)(\nu) := \frac{1}{\sqrt{2}} \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i\sqrt{\nu} \cdot x} \varphi(x) \, dx, \quad \nu \in S^1, \quad (2.16)$$
and
\[ M(\lambda) := \left( U + \frac{2m}{\hbar^2} v(-\Delta - \lambda - i0)^{-1} v \right) \]  
(2.17)

has a bounded inverse in \( L^2(\mathbb{R}^2) \).

Moreover, for all \( f \in L^2(S^1) \),
\[ \| v(x) \left( \Gamma^* - \frac{1}{\sqrt{2}} \Sigma \right) f \|_{L^2(\mathbb{R}^2)} \leq \sqrt{\lambda} \frac{1}{2\sqrt{\pi}} \| v(x) \|_{L^2(\mathbb{R}^2)} \| f \|_{L^2(\mathbb{S}^1)}. \]  
(2.18)

Furthermore, by Schwarz's inequality,
\[ \| v(x) |x| \|_{L^2(\mathbb{R}^2)}^2 = \| V(x) |x| ^2 \|_{L^1(\mathbb{R}^2)} \leq \| (1 + |x|)^\beta V(x) \|_{L^2(\mathbb{R}^2)} \| (1 + |x|)^{-\beta+2} \|_{L^2(\mathbb{R}^2)} \leq C, \]
and it follows that,
\[ v(x) \Gamma^*(\lambda) = \frac{1}{\sqrt{2}} v(x) \Sigma + O(\sqrt{\lambda}), \quad \lambda \to 0, \]  
(2.19)
in the operator norm in \( B(L^2(S^1), L^2(\mathbb{R}^2)) \). Taking the adjoint in both sides of (2.19) we obtain that,
\[ \Gamma(\lambda) v(x) = \frac{1}{\sqrt{2} \pi} (\cdot, v) + O(\sqrt{\lambda}), \quad \lambda \to 0, \]  
(2.20)
in the operator norm in \( B(L^2(\mathbb{R}^2), L^2(S^1)) \). By (2.15), (2.19), (2.20),
\[ S_1(\lambda) = I - \frac{i}{2} \frac{2m}{\hbar^2} (M(\lambda))^{-1} v, v \Sigma + O(\sqrt{\lambda}), \quad \lambda \to 0, \]  
(2.21)
in the norm of \( B(L^2(S^1)) \). Equation (2.11) follows from (2.21) and Theorem 6.2 of [1].

The low-energy expansion (2.11) was previously proved by [12] in the case of exponentially decreasing potentials such that \( \int V(x) \neq 0 \).

3 The Creation of Entanglement at Low-Energy

As mentioned in the introduction, we consider a pure state of the two-particle system. The wave function in the momentum representation is given by \( \varphi(p_1, p_2) \). We designate by \( \rho(\varphi) \) the one-particle reduced density matrix with integral kernel,
\[ \rho(\varphi)(p_1, p'_1) := \int \varphi(p_1, p_2) \overline{\varphi(p'_1, p_2)} \, dp_2. \]

The purity, \( \mathcal{P}(\varphi) \), is given by,
\[ \mathcal{P}(\varphi) := \text{Tr}(\rho^2) = \int dp_1 dp'_1 dp_2 dp'_2 \varphi(p_1, p_2) \overline{\varphi(p'_1, p_2)} \varphi(p'_1, p_2') \overline{\varphi(p_1, p'_2)}. \]  
(3.1)
As is well known [13, 14, 15], the purity is a measure of entanglement that is related closely to the Rényi entropy of order 2, $-\ln \Tr(\rho^2)$. Furthermore, it has a trivial relation with the linear entropy, $S_L$, given by $S_L = 1 - P$. Clearly, it satisfies $0 \leq P \leq 1$ if $\varphi$ is normalized to one. Furthermore, it is equal to one for a product state, $\varphi = \varphi_1(p_1) \varphi_2(p_2)$. As we will show, the purity can be directly computed in terms of the scattering matrix. For this reason it is a measure of entanglement that is convenient for the study of entanglement creation in scattering processes.

As we already said, we consider an incoming asymptotic state, in the center-of-mass frame, that is a product of two normalized Gaussian wave functions,

$$\varphi_{in,p_0}(p_1, p_2) := \varphi_{p_0}(p_1) \varphi_{-p_0}(p_2), (3.2)$$

where

$$\varphi_{p_0}(p_1) := \frac{1}{(\sigma^2 \pi)^{1/4}} e^{-|p_1-p_0|^2/2\sigma^2}. (3.3)$$

Observe that by (2.1) the mean value of the relative momentum in the state (3.2) is equal to $p_0$.

Since the incoming asymptotic state $\varphi_{in,p_0}$ is a product state its purity is one,

$$P(\varphi_{in,p_0}) = 1. (3.4)$$

The outgoing asymptotic state of the two particles, $\varphi_{out,p_0}$ -after the scattering process is over- is given by

$$\varphi_{out,p_0}(p_1, p_2) := (S(p^2/2m)\varphi_{in,p_0})(p_1, p_2). (3.5)$$

As the relative momentum $p$ depends on $p_1$ and on $p_2$, $\varphi_{out,p_0}$ is not a product state, and then it has purity smaller than one. This implies that the scattering process has created entanglement between the two particles.

Let us introduce some notations that we use later.

Let us designate by $\varphi_{in}$ the incoming asymptotic state with mean value of the relative momentum zero,

$$\varphi_{in}(p_1, p_2) := \varphi(p_1) \varphi(p_2), (3.6)$$

with,

$$\varphi(p) := \frac{1}{(\sigma^2 \pi)^{1/4}} e^{-p^2/2\sigma^2}, (3.7)$$

and by $\varphi_{out}$ the outgoing asymptotic state with incoming asymptotic state $\varphi_{in}$,

$$\varphi_{out}(p_1, p_2) := (S(p^2/2m)\varphi_{in})(p_1, p_2). (3.8)$$

Recall that,

$$\mu_i = \frac{m_i}{m_1 + m_2}, i = 1, 2, (3.9)$$
is the ratio of the mass of the $i$ particle to the total mass.

It follows from (2.1) that,

$$ p_1 = \mu_1 p_{cm} + p, \quad (3.10) $$

$$ p_2 = \mu_2 p_{cm} - p. \quad (3.11) $$

We prepare some results that we will use. It is a consequence of (3.2, 3.3, 3.10, 3.11) that

$$ \varphi_{in,p_0} = \frac{1}{\sigma^2} e^{-\frac{1}{\sigma^2}(\mu_1^2 + \mu_2^2)p_{cm}^2/2\sigma^2} e^{-\frac{1}{\sigma^2}(p-p_0)^2/\sigma^2} e^{-\frac{1}{\sigma^2}(\mu_1 - \mu_2)p_{cm}(p-p_0)}. \quad (3.12) $$

**REMARK 3.1.** For some positive constant $\delta$,

$$ |\varphi_{in}| \leq \frac{1}{\sigma^2} e^{-\delta(p_{cm}^2 + p^2)/2\sigma^2}. \quad (3.13) $$

**Proof:** Note that $\mu_1^2 + \mu_2^2 = \frac{1}{2} (1 + (\mu_1 - \mu_2)^2)$. Then, for $\alpha \geq 0$,

$$ (\mu_1^2 + \mu_2^2)p_{cm}^2/2\sigma^2 + p^2/\sigma^2 + (\mu_1 - \mu_2)p_{cm} \cdot p \geq$$

$$ \left( \frac{1}{4} + (\mu_1 - \mu_2)^2 \frac{1}{4} - \frac{\alpha}{2} \right) \frac{p_{cm}^2}{\sigma^2} + \frac{p^2}{\sigma^2} \left( 1 - \frac{1}{2\alpha} \right) \geq \delta(p_{cm}^2 + p^2)/2\sigma^2, \quad (3.14) $$

provided that we choose $\alpha$ so that $0 < \delta/2 \leq \min[\frac{1}{4} + (\mu_1 - \mu_2)^2(\frac{1}{4} - \frac{\alpha}{2}), (1 - \frac{1}{2\alpha})]$. The remark follows from (3.12) and (3.14).

**PROPOSITION 3.2.** For any $\alpha, \beta \geq 0$,

$$ \left\| \frac{(\ln(2 + |p|/\hbar))^{\alpha}}{(1 + \ln(|p|/\hbar))^{\beta}} \varphi_{in,p_0} \right\| = O \left( \frac{1}{|\ln(\sigma/\hbar + |p_0|/\hbar)|^\beta} \right), \quad \text{as } \sigma/\hbar + |p_0|/\hbar \to 0, \quad (3.15) $$

$$ \left\| \frac{(\ln(2 + |p|/\hbar))^{\alpha}}{(1 + \ln(|p|/\hbar))^{\beta}} \varphi_{in} \right\| = O \left( \frac{1}{|\ln(\sigma/\hbar)|^\beta} \right), \quad \text{as } \sigma/\hbar \to 0. \quad (3.16) $$

Furthermore, uniformly for $|p_0|/\sigma$ in bounded sets,

$$ \left\| \frac{(\ln(2 + |p|/\hbar))^{\alpha}}{(1 + \ln(|p|/\hbar))^{\beta}} (\varphi_{in,p_0} - \varphi_{in}) \right\| \leq \frac{|p_0|}{\sigma} O \left( \frac{1}{|\ln(\sigma/\hbar + |p_0|/\hbar)|^{\beta}} \right), \quad \text{as } \sigma/\hbar + |p_0|/\hbar \to 0. \quad (3.17) $$

**Proof:** By (3.13) we have that,

$$ \left\| \frac{(\ln(2 + |p|/\hbar))^{\alpha}}{(1 + \ln(|p|/\hbar))^{\beta}} \varphi_{in,p_0} \right\|^2 \leq I_1 + I_2, \quad (3.18) $$

where, for some $1 > \gamma > 0$,

$$ I_1 := \int_{|p|/\sigma \geq 1/(\sigma/\hbar + |p_0|/\hbar)^\gamma} \left( \frac{(\ln(2 + |p + p_0|/\hbar))^{\alpha}}{(1 + \ln(|p + p_0|/\hbar))^{\beta}} \right)^2 \frac{1}{\sigma^{\gamma\beta}} e^{-\delta(p_{cm}^2 + p^2)/\sigma^2} d\rho_{cm} d\rho \leq C_N (\sigma/\hbar + |p_0|/\hbar)^N. \quad (3.19) $$
Moreover, for $|p|/\sigma \leq 1/(\sigma/h + |p_0|/h)^\gamma$ and $(\sigma/h + |p_0|/h)^{1-\gamma} \leq 1/2$,

\[
\frac{1}{1 + |\ln(p + p_0)/|h|)} \leq \frac{1}{1 + (1 - \gamma)|\ln(\sigma/h + |p_0|/\sigma)| 2^{1/(1 - \gamma)}},
\]

and then,

\[
I_2 = O \left( \frac{1}{|\ln(\sigma/h + |p_0|/\sigma)|^\sigma} \varphi_{in} \right)^2 \leq I_1 + I_2, \tag{3.22}
\]

where, for some $1 > \gamma > 0$,

\[
I_1 := \int_{|p|/\sigma \geq 1/(\sigma/h)^\gamma} \left( \frac{\ln(2 + |p|/h)}{|\ln(p)/h)|^\beta} \right)^2 \frac{1}{\sigma^4 \pi^2} e^{-\delta(p_{cm}^2 + p^2)/\sigma^2} dp_{cm} dp \leq C N(\sigma/h)^N, \quad N = 1, 2, \cdots \tag{3.23}
\]

and

\[
I_2 := \int_{|p|/\sigma \leq 1/(\sigma/h)^\gamma} \left( \frac{\ln(2 + |p|/h)}{|\ln(p)/h)|^\beta} \right)^2 \frac{1}{\sigma^4 \pi^2} e^{-\delta(p_{cm}^2 + p^2)/\sigma^2} dp_{cm} dp. \tag{3.24}
\]

As above, for $|p|/\sigma \leq 1/(\sigma/h)^\gamma$ and $(\sigma/h)^{1-\gamma} \leq 1$,

\[
\frac{1}{|\ln(p)/h|} \leq \frac{1}{(1 - \gamma)|\ln(\sigma/h)|^\gamma},
\]

and then,

\[
I_2 = O \left( \frac{1}{|\ln(\sigma/h)|^\sigma} \right), \quad \sigma/h \to 0. \tag{3.25}
\]

Equation (3.16) follows from (3.23) and (3.25).

We now prove (3.17). We first consider the case when $|p_0|/\sigma \leq 1$. By (3.12, 3.13),

\[
|\varphi_{in, p_0} - \varphi_{in}| \leq \frac{1}{\sigma^2 \pi} e^{-\delta(p_{cm}^2 + p^2)/2\sigma^2} e^{-(p_0^2 + 2p_0 + (\mu_1 - \mu_2)|p_{cm}|p_0)/\sigma^2} - 1 \leq \frac{1}{\sigma^2 \pi} e^{-\delta(p_{cm}^2 + p^2)/2\sigma^2} e^{(p_0^2 + 2p_0 + (\mu_1 - \mu_2)|p_{cm}|p_0)/\sigma^2} \leq \frac{1}{\sigma^2 \pi} e^{-\delta(p_{cm}^2 + p^2)/2\sigma^2} e^{(1 + 2|p|/\sigma + |p_{cm}|/\sigma)} \frac{p_0}{\sigma} (1 + 2|p|/\sigma + |p_{cm}|/\sigma). \tag{3.26}
\]
Then,
\[ \left\| \frac{(\ln(2+|p|/\hbar))^{p_0}}{(1 + |\ln(|p|/\hbar)|)^p} \ (\varphi_{in,p_0} - \varphi_{in}) \right\|^2 \leq \frac{1}{\sigma^{\pi^2}} \int \frac{(\ln(2+|p|/\hbar))^{p_0}}{(1 + |\ln(|p|/\hbar)|)^p} e^{-\delta(p_0^2 + p^2)/\sigma^2} e^{2(1+2|p|/\sigma + |p_{cm}|/\sigma)} \ |p_0/\sigma|^2 \ (1 + 2|p|/\sigma + |p_{cm}|/\sigma)^2 \ dp_{cm} \ dp. \]

Estimating as in equations \[3.18\] \[3.21\] with \( p_0 = 0 \), we prove that for \( |p_0|/\sigma \leq 1 \),
\[ \left\| \frac{(\ln(2+|p|/\hbar))^{p_0}}{(1 + |\ln(|p|/\hbar)|)^p} \ (\varphi_{in,p_0} - \varphi_{in}) \right\| \leq |p_0/\sigma| \ O \left( \frac{1}{|\ln(|p_0|/\hbar)|^{\pi^2}} \right) \ |p_0/\sigma| \ O \left( \frac{1}{|\ln(|p_0|/\hbar)|^{\pi^2}} \right), \ as \ \sigma/\hbar + |p_0|/\hbar \to 0. \]

In the case \( |p_0/\sigma| \geq 1 \) the estimate is immediate from \[3.15\], because,
\[ \left\| \frac{(\ln(2+|p|/\hbar))^{p_0}}{(1 + |\ln(|p|/\hbar)|)^p} \ (\varphi_{in,p_0} - \varphi_{in}) \right\| \leq \left\| \frac{(\ln(2+|p|/\hbar))^{p_0}}{(1 + |\ln(|p|/\hbar)|)^p} \varphi_{in,p_0} \right\| + \left\| \frac{(\ln(2+|p|/\hbar))^{p_0}}{(1 + |\ln(|p|/\hbar)|)^p} \varphi_{in} \right\| = O \left( \frac{1}{|\ln(|p_0|/\hbar)|^{\pi^2}} \right), \ as \ \sigma/\hbar + |p_0|/\hbar \to 0. \]

We define,
\[ T(p^2/2m) := S(p^2/m) - I + i\pi \frac{1}{1 + |\ln(|p|/\hbar)|}, \] (3.27)
where \( I \) is the identity operator on \( L^2(S^1) \). It follows from \[2.11\] and since \( \|S(p^2/2m)\|_{B(L^2(S^1))} = 1 \), that
\[ \|T(p^2/2m)\|_{B(L^2(S^1))} \leq C \left( \frac{(\ln(2+|p|/\hbar))^2}{(1 + |\ln(|p|/\hbar)|)^2} \right). \] (3.28)
Hence by \[3.15\],
\[ \|T(p^2/2m)\varphi_{in,p_0}\| = O \left( \frac{1}{|\ln(|p_0|/\hbar)|^{\pi^2}} \right), \ as \ \sigma/\hbar + |p_0|/\hbar \to 0. \] (3.29)

Let us denote,
\[ \mathcal{L}(\phi_1, \phi_2, \phi_3, \phi_4) := \int d\phi_1 \ d\phi_1' \ d\phi_2 \ d\phi_2' \ \varphi_1(\phi_1, \phi_2) \varphi_2(\phi_1', \phi_2') \ \overline{\varphi_3(\phi_1, \phi_2)} \ \overline{\varphi_4(\phi_1', \phi_2')} \] (3.30)
We have that,
\[ \mathcal{P}(\phi) = \mathcal{L}(\phi, \phi, \phi, \phi). \]
The Schwarz inequality implies that,
\[ |\mathcal{L}(\phi_1, \phi_2, \phi_3, \phi_4)| \leq \Pi_{j=1}^4 \| \phi_j \|. \] (3.31)
We state below our first result in the low-energy behavior of the purity.

**THEOREM 3.3.** Suppose that Assumption \[1.1\] is satisfied and that at zero \( H_{\text{rel}} \) has neither a resonance (half-bound state) nor an eigenvalue. Then, uniformly for \( |p_0|/\sigma \) in bounded sets,
\[ \mathcal{P}(\varphi_{\text{out}, p_0}) = \mathcal{P}(\varphi_{\text{out}}) + \frac{|p_0|}{\sigma} \ O \left( \frac{1}{|\ln(|p_0|/\hbar)|^{\pi^2}} \right), \ as \ \sigma/\hbar + |p_0|/\hbar \to 0. \] (3.32)
Proof: Writing $\varphi_{\text{out.p}_0}$ as,

$$\varphi_{\text{out.p}_0} := S(p^2/2m)\varphi_{\text{in.p}_0} = \varphi_{\text{in.p}_0} - i\pi \frac{1}{1 + |\ln(|p|/\hbar)|} \varphi_{\text{in.p}_0} + T(p^2/2m)\varphi_{\text{in.p}_0},$$

and using (3.4), we see that we can write $P(\varphi_{\text{out.p}_0})$ as follows,

$$P(\varphi_{\text{out.p}_0}) = 1 + \sum_{i=1}^{4} L_{1,i}(p_0, \psi_1, \psi_2, \psi_3, \psi_4) + R(p_0), \quad (3.33)$$

where

$$L_{1,i}(p_0, \psi_1, \psi_2, \psi_3, \psi_4) = L(\psi_1, \psi_2, \psi_3, \psi_4), \quad (3.34)$$

where one of the $\psi_j$ is equal to $T(p^2/2m)\varphi_{\text{in.p}_0}$ and the remaining 3 are equal to $\varphi_{\text{in.p}_0}$. Similarly,

$$R(p_0) := \sum_{i=1}^{A} L_{2,i}(p_0, \psi_1, \psi_2, \psi_3, \psi_4), \quad (3.35)$$

for some integer $A$, and where each of the $L_{2,i}(p_0, \psi_1, \psi_2, \psi_3, \psi_4)$ is equal to,

$$L_{2,i}(p_0, \psi_1, \psi_2, \psi_3, \psi_4) = L(\psi_1, \psi_2, \psi_3, \psi_4), \quad (3.36)$$

where for some $2 \leq k \leq 4$, $k$ of the $\psi_j$ are equal either to $-i\pi \frac{1}{1 + |\ln(|p|/\hbar)|} \varphi_{\text{in.p}_0}$ or to $T(p^2/2m)\varphi_{\text{in.p}_0}$ and the remaining $4 - k$ are equal to $\varphi_{\text{in.p}_0}$. Similarly,

$$P(\varphi_{\text{out}}) = 1 + \sum_{i=1}^{4} L_{1,i}(0, \psi_1, \psi_2, \psi_3, \psi_4) + R(0), \quad (3.37)$$

with

$$R(0) := \sum_{i=1}^{A} L_i(0, \psi_1, \psi_2, \psi_3, \psi_4). \quad (3.38)$$

Below we prove that,

$$L_{1,i}(p_0, \psi_1, \psi_2, \psi_3, \psi_4) = L_{1,i}(0, \psi_1, \psi_2, \psi_3, \psi_4) + \frac{|p_0|}{\sigma} O \left( \frac{1}{|\ln(|\sigma/\hbar + |p_0|/\hbar)|^2} \right), \quad i = 1, 2, 3, 4, \quad (3.39)$$

$$R(p_0) = R(0) + \frac{|p_0|}{\sigma} O \left( \frac{1}{|\ln(|\sigma/\hbar + |p_0|/\hbar)|^2} \right), \quad (3.40)$$

what proves the theorem in view of (3.33, 3.37).

We proceed to prove (3.39). Without losing generality we can assume that,

$$L_{1,1}(p_0, \psi_1, \psi_2, \psi_3, \psi_4) = L(T(p^2/2m)\varphi_{\text{in.p}_0}, \varphi_{\text{in.p}_0}, \varphi_{\text{in.p}_0}, \varphi_{\text{in.p}_0}). \quad (3.41)$$

We have that,

$$L_{1,1}(p_0, \psi_1, \psi_2, \psi_3, \psi_4) = L(T(p^2/2m)\varphi_{\text{in.p}_0}, \varphi_{\text{in.p}_0}, \varphi_{\text{in.p}_0}, \varphi_{\text{in.p}_0}) +$$

$$L(T(p^2/2m)(\varphi_{\text{in.p}_0} - \varphi_{\text{in}}), \varphi_{\text{in.p}_0}, \varphi_{\text{in.p}_0}, \varphi_{\text{in.p}_0}). \quad (3.42)$$

By (3.17, 3.28, 3.31, 3.42),

$$L_{1,1}(p_0, \psi_1, \psi_2, \psi_3, \psi_4) = L(T(p^2/2m)\varphi_{\text{in.p}_0}, \varphi_{\text{in.p}_0}, \varphi_{\text{in.p}_0}, \varphi_{\text{in.p}_0}) + \frac{|p_0|}{\sigma} O \left( \frac{1}{|\ln(|\sigma/\hbar + |p_0|/\hbar)|^2} \right). \quad (3.43)$$
In the same way, using \([3.17][3.29][3.43]\), we prove that,

\[
\mathcal{L}_{1,1}(p_0, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}(T(p^2/2m)\varphi_{in}, \varphi_{in}, \varphi_{in,p_0}, \varphi_{in,p_0}) + \frac{|p_0|}{\sigma} O\left(\frac{1}{|\ln(\sigma/\hbar + |p_0|/\hbar)|^2}\right).
\] (3.44)

Repeating this argument two more times we obtain that,

\[
\mathcal{L}_{1,1}(p_0, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}_{1,1}(0, \psi_1, \psi_2, \psi_3, \psi_4) + \frac{|p_0|}{\sigma} O\left(\frac{1}{|\ln(\sigma/\hbar + |p_0|/\hbar)|^2}\right).
\] (3.45)

We prove in the same way that \([3.39]\) holds for \(i = 2, 3, 4\). Furthermore, \([3.40]\) is proven by the same argument.

The next theorem gives us the leading order of the purity of \(\varphi_{out}\) at low-energy.

**Theorem 3.4.** Suppose that Assumption \([1.1]\) is satisfied and that at zero \(H_{ref}\) has neither a resonance (half-bound state) nor an eigenvalue. Then, as \(\sigma/\hbar \to 0\).

\[
\mathcal{P}(\varphi_{out}) = \mathcal{P}\left[I + i\pi \frac{1}{\ln|p/\hbar|} \Sigma + \left(i\pi(\ln 2 - \gamma + \frac{1}{a}) - \frac{\pi^2}{2}\right) \frac{1}{(\ln|p/\hbar|)^2} \varphi_{in}\right] + O\left(\frac{1}{|\ln(\sigma/\hbar)|^3}\right).
\] (3.46)

**Proof:** We write \(\varphi_{out}\) as follows,

\[
\varphi_{out} = \varphi_{out,1} + \mathcal{T}_1(p^2/2m)\varphi_{in},
\]

where,

\[
\varphi_{out,1} := \left[I + i\pi \frac{1}{\ln|p/\hbar|} \Sigma + \left(i\pi(\ln 2 - \gamma + \frac{1}{a}) - \frac{\pi^2}{2}\right) \frac{1}{(\ln|p/\hbar|)^2} \varphi_{in}\right],
\] (3.47)

and

\[
\mathcal{T}_1 := \mathcal{S}(p^2/2m) - I - i\pi \frac{1}{\ln|p/\hbar|} \Sigma - \left(i\pi(\ln 2 - \gamma + \frac{1}{a}) - \frac{\pi^2}{2}\right) \frac{1}{(\ln|p/\hbar|)^2} \Sigma.
\]

By \([2.11]\) and since \(\|\mathcal{S}(p^2/2m)\|_{\mathcal{B}(L^2(\Sigma^1))} = 1\),

\[
\|\mathcal{T}_1(p^2/2m)\|_{\mathcal{B}(L^2(\Sigma^1))} \leq C(\frac{\ln(2 + |p|/\hbar)^3}{|\ln|p/\hbar|^3}).
\] (3.48)

Using this decomposition we write \(\mathcal{P}(\varphi_{out})\) as follows,

\[
\mathcal{P}(\varphi_{out}) = \mathcal{P}(\varphi_{out,1}) + \mathcal{R}_1(\sigma),
\] (3.49)

where \(\mathcal{R}_1(\sigma)\) is given by,

\[
\mathcal{R}_1(\sigma) := \frac{D}{\mathcal{L}_i(\sigma, \psi_1, \psi_2, \psi_3, \psi_4),}
\] (3.50)

for some integer \(D\), and where each of the \(\mathcal{L}_i(\sigma, \psi_1, \psi_2, \psi_3, \psi_4)\) is equal to,

\[
\mathcal{L}_i(\sigma, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}(\psi_1, \psi_2, \psi_3, \psi_4),
\] (3.51)

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where for some $1 \leq k \leq 4$, $k$ of the $\psi_j$ are equal to $\varphi_{\text{out},1}$ and the remaining $4 - k$ are equal to $T_1(p^2/2m)\varphi_{\text{in}}$.

We complete the proof of the theorem proving that,

$$\mathcal{R}_1(\sigma) = O\left(\frac{1}{\ln|\sigma/h|^3}\right), \quad \text{as } \sigma/h \to 0. \quad (3.52)$$

We can assume that,

$$\mathcal{L}_1(\sigma, \psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}(\varphi_{\text{out},1}, \varphi_{\text{out},1}, \varphi_{\text{out},1}, T_1(p^2/2m)\varphi_{\text{in}}). \quad (3.53)$$

By (3.16) we have that,

$$\mathcal{L}_1(\sigma, \psi_1, \psi_2, \psi_3, \psi_4) = O\left(\frac{1}{\ln|\sigma/h|^3}\right), \quad \text{as } \sigma/h \to 0. \quad (3.54)$$

We estimate the remaining terms in (3.52) in the same way.

Let us denote,

$$\psi(q) := \frac{1}{\pi^{1/2}} e^{-q^2/2}, q \in \mathbb{R}^2,$$

$$\psi_{\text{in}}(q_1, q_2) := \psi(q_1) \psi(q_2).$$

**Proposition 3.5.** For any $\alpha, \beta \geq 0$,

$$\left\| \frac{(\ln|q|)^{\alpha}}{\ln|\sigma|q/h|^{3/2}} \psi_{\text{in}} \right\| = O\left(\frac{1}{\ln|\sigma/h|^3}\right), \quad \text{as } \sigma/h \to 0. \quad (3.55)$$

**Proof:** We follow the proof of (3.16). By (3.13) with $\sigma = 1$,

$$\left\| \frac{(\ln|q|)^{\alpha}}{\ln|\sigma|q/h|^{3/2}} \psi_{\text{in}} \right\|^2 \leq I_1 + I_2, \quad (3.56)$$

where, for some $1 > \gamma > 0$,

$$I_1 := \int_{|q| \geq 1/(\sigma/h)^{\gamma}} \left(\frac{|\ln|q||^{2\alpha}}{\ln|\sigma|q/h|^{3/2}}\right)^{1/2} \frac{1}{\pi^{1/2}} e^{-\delta(q_{\text{cm}}^2 + q^2)}/ dq_{\text{cm}} dq \leq C_N(\sigma/h)^N, \quad N = 1, 2, \ldots, \quad (3.57)$$

and

$$I_2 := \int_{|q| \leq 1/(\sigma/h)^{\gamma}} \left(\frac{(\ln|q|)^{\alpha}}{\ln|\sigma|q/h|^{3/2}}\right)^{2} \frac{1}{\pi^{1/2}} e^{-\delta(q_{\text{cm}}^2 + q^2)} dq_{\text{cm}} dq. \quad (3.58)$$

Furthermore, for $|q| \leq 1/(\sigma/h)^{\gamma}$ and $(\sigma/h)^{1-\gamma} \leq 1$,

$$\frac{1}{\ln|\sigma|q/h|} \leq \frac{1}{(1-\gamma)|\ln(\sigma/h)|}. \quad (3.59)$$
and then,

\[ I_2 = O\left(\frac{1}{\ln(\sigma/h)^2}\right), \quad \sigma/h \to 0. \]  

Equation (3.55) follows from (3.57) and (3.59).

\[ \square \]

A straightforward computation with the help of (3.55) shows that,

\[ \mathcal{P}\left[I + i\pi \frac{1}{\ln|\mathbf{p}/h|} \Sigma + \left(i\pi(\ln 2 - \gamma + \frac{1}{\pi}) - \frac{\sigma}{2\mathcal{P}}\right) \frac{1}{\ln|\mathbf{p}/h|^2} \Sigma \right] \varphi_{in} = 1 - \frac{1}{\ln(\sigma/h)^2} (\mathcal{P}_1(\psi_{in}) + \mathcal{P}_2(\psi_{in})) + O\left(\frac{1}{\ln(\sigma/h)^2}\right), \quad \text{as } \sigma/h \to 0, \]

where,

\[ \mathcal{P}_1(\psi_{in}) = \sum_{j=1}^{3} \mathcal{P}_{1,j}(\psi_{in}), \]

with

\[ \mathcal{P}_{1,1}(\psi_{in}) = -2\pi^2 \int dq_1 dq_2 dq_3 \left(\Sigma \psi_{in}(q_1, q_2) \right) \left(\Sigma \psi_{in}(q_3, q_2) \right) \psi_{in}(q_1, q_3), \]

\[ \mathcal{P}_{1,2}(\psi_{in}) = -2\pi^2 \int dq_1 dq_2 dq_3 \left(\Sigma \psi_{in}(q_1, q_2) \right) \left(\Sigma \psi_{in}(q_1, q_3) \right) \psi_{in}(q_2, q_3), \]

\[ \mathcal{P}_{1,3}(\psi_{in}) = 2\pi^2 \left[ \int dq_1 dq_2 \left(\Sigma \psi_{in}(q_1, q_2) \right) \psi_{in}(q_1, q_2) \right]^2, \]

and

\[ \mathcal{P}_2(\psi_{in}) = 2\pi^2 \int dq_1 dq_2 \left(\Sigma \psi_{in}(q_1, q_2) \right) \psi_{in}(q_1, q_2). \]

Explicitly evaluating the integrals in (3.62), (3.63) and (3.64) using (3.12), we prove that,

\[ \mathcal{P}_{1,1}(\psi_{in}) = -\frac{2}{\pi} J(\mu_1, \mu_2), \]

\[ \mathcal{P}_{1,2}(\psi_{in}) = -\frac{2}{\pi} J(\mu_2, \mu_1), \]

\[ \mathcal{P}_{1,3}(\psi_{in}) = 2\pi^2 \left(\mathcal{L}(\mu_1, \mu_2)\right)^2, \]

\[ \mathcal{P}_2(\psi_{in}) = 2\pi^2 \mathcal{L}(\mu_1, \mu_2), \]

where,

\[ J(\mu_1, \mu_2) := \int dq_2 \left[ \int dq_1 \exp\left[-\frac{1}{2}(\mu_1^2 + \mu_2^2)q_1^2 + \mu_2 q_2^2 - \mu_1 q_2 q_2 - q_1^2/2\right] \right] \cdot I_0(\mu_1 - \mu_2 |q_1 + q_2| |\mu_2 q_1 - \mu_1 q_2|)^2. \]

Here \( I_0 \) is the modified Bessel function \( [2] \), and

\[ \mathcal{L}(\mu_1, \mu_2) := \int_0^\infty \int_0^\infty d\lambda d\rho e^{-2\lambda} e^{-(\mu_1^2 + \mu_2^2)\rho} \left( I_0(\mu_1 - \mu_2 |\sqrt{\lambda} \rho|) \right)^2. \]

We prove in the appendix that,

\[ L(\mu_1, 1 - \mu_1) = \frac{1}{\sqrt{1 + (2\mu_1 - 1)^2}}. \]
We denote by \( E(\mu) \) the entanglement coefficient,

\[
E(\mu) := 2\pi^2 \frac{L(\mu_1, 1 - \mu_1) (1 + L(\mu_1, 1 - \mu_1)) - \frac{2}{\pi} [J(\mu_1, 1 - \mu_1) + J(1 - \mu_1, \mu_1)]}{1 + (2\mu_1 - 1)^2} \left[ 1 + \sqrt{1 + 4\mu_1 - 1} \right] - \frac{2}{\pi} [J(\mu_1, 1 - \mu_1) + J(1 - \mu_1, \mu_1)].
\] (3.73)

The next theorem is our main result.

**Theorem 3.6.** Suppose that Assumption 1.1 is satisfied and that at zero \( H_{rel} \) has neither a resonance (half-bound state) nor an eigenvalue. Then,

\[
P(\varphi_{out}) = 1 - \frac{1}{(\ln(\sigma/h))^2} E(\mu) + O\left(\frac{1}{(\ln(\sigma/h))^3}\right), \quad \text{as } \sigma/h \to 0,
\] (3.74)

where the entanglement coefficient \( E(\mu) \) is given by (3.73).

**Proof:** The theorem follows from (3.46, 3.60, 3.61, 3.66-3.69, 3.72).

\[ \square \]

Note that \( E(\mu) = E(1 - \mu) \), as it should be, because \( P(\varphi_{out}) \) is invariant under the exchange of particles one and two.

Observe that,

\[
J(1/2, 1/2) = \pi^3.
\]

By (3.73) for \( \mu = 1/2 \), when the masses are equal, the entanglement coefficient is zero, \( E(1/2) = 0 \). Of course, this only means that in this case the purity is one at leading order.

We explicitly evaluate in the appendix \( J(1, 0) \),

\[
J(1, 0) = 16.6377.\) (3.75)

For \( \mu \in [0, 1] \setminus \{1/2, 1\} \) we compute \( J(\mu_1, 1 - \mu_1) \) numerically using Gaussian quadratures. In Table 1 and in Figure 1 we give values of \( E(\mu_1) \) for \( 0.5 \leq \mu_1 := m_1/(m_1 + m_2) \leq 1 \).

### 4 Conclusions

In this paper we give a rigorous computation, with error bound, of the entanglement created in the low-energy scattering of two particles in two dimensions. The interaction between the particles is given by potentials that are not required to be spherically symmetric. Before the scattering the particles are in a pure state that is a product of two normalized Gaussians with the same variance \( \sigma \). After the collision the particles are in a outgoing asymptotic state that is not a product state. The measure of the entanglement
created by the collision is the purity, $\mathcal{P}$, in the state after the collision. Before the collision the purity is one.

We prove that $\mathcal{P} = 1 - \frac{1}{(\ln(\sigma/\hbar))^2} \mathcal{E} + O\left(\frac{1}{(\ln(\sigma/\hbar))^3}\right)$, as $\sigma/\hbar \to 0$, where $\sigma$ is the variance of the Gaussians and the entanglement coefficient, $\mathcal{E}$, depends only on the masses of the particles and not on the interaction potential. This proves that the entanglement created at low-energy in two dimensions is universal, in the sense that it is independent on the interaction potential between the particles. This is strikingly different with the three dimensional case, that we considered in [3], where the entanglement created at low-energy is proportional to the total scattering cross section. However, the entanglement depends strongly in the difference of the masses. As in three dimensions [3] the minimum is taken when the masses are equal, and it rapidly increases with the difference of the masses.

## Appendix

By (3.71) we have that [2]

$$L(\mu_1, 1 - \mu_1) = \int_0^\infty d\lambda e^{-2\lambda} \frac{2}{1+(2\mu_1-1)^2} I_0\left(\frac{(2\mu_1-1)^2\lambda}{1+(2\mu_1-1)^2}\right) \exp\left(\frac{(2\mu_1-1)^2\lambda}{1+(2\mu_1-1)^2}\right) = \frac{1}{\sqrt{1+(2\mu_1-1)^2}}.$$  

(5.1)

Moreover, by (3.70) and denoting $q_{\text{cm}} := q_1 + q_2$,

$$J(1,0) = \int d\mathbf{q_2} e^{-3q_2^2} \left[ \int d\mathbf{q_{\text{cm}}} \exp\left(-q_{\text{cm}}^2 + \mathbf{q_{\text{cm}}} \cdot \mathbf{q_2}\right) I_0(|q_{\text{cm}}||q_2|)|^2 \right].$$  

(5.2)

Furthermore, using polar coordinates, [2],

$$J(1,0) = \pi^3 \int_0^\infty d\lambda e^{-3\lambda} \left[ \int_0^{\infty} d\rho e^{-\rho} \left(I_0(\sqrt{\lambda})\right)^2 \right]^2 = \pi^3 \int_0^\infty d\lambda e^{-2\lambda} \left(I_0(\lambda/2)\right)^2 = \pi^2 K(0.25) = 16.6377,$$

(5.3)

where $K(x)$ is the complete elliptic integral.

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## References

[1] A. Jensen and G. Nenciu, Rev. Math. Phys. 13 (2001) 717-754.
[2] I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals, Series and Products, Seventh Edition, Amsterdam, 2007.

[3] R. Weder, Phys. Rev. A **84** (2011) 06230, 11 pp.

[4] F. Schmüser, D. Janzing, Phys. Rev. A, **73**, (2006) 052313, 7 pp.

[5] N. L. Harshman, P. Singh, J. Phys. A: Math. Theor., **41** (2008) 155304, 12 pp.

[6] G. Dell’Antonio, Int. J. Mod. Phys. B, **18** (2004) 643–654.

[7] D. Dürr, R. Figari, A. Teta, J. Math. Phys, **45** (2004) 1291–1309.

[8] R. Adami, R. Figari, D. Finco, A. Teta, Commun. Math. Phys., **268** (2006) 819–852.

[9] E. Joos, H.D. Zeh, Z. Phys. B, **59** (1985) 223–243.

[10] M. R. Gallis, G. N. Fleming, Phys. Rev. A, **42** (1990) 38–48.

[11] D. R. Yafaev, Mathematical Scattering Theory: Analytic Theory, AMS, Providence, Rhode Island, 2010.

[12] D. Bollé, F. Gesztesy, G. Dannels, Ann. Inst. H. Poincaré, A **48** (1988) 175–204.

[13] G. Adesso, F. Illuminati, J. Phys. A: Math. Theor., **40** (2007) 7821–7880.

[14] R.S. Ingarden, A. Kossakowski and M. Ohya, Information Dynamics and Open Systems: Classical and Quantum Approach, No. 86, in Fundamental Theories of Physics, Kluwer, Dordrecht, 1997.

[15] D.A. Lidar, A. Shabani, R. Alicki, Chemical Physics, **322** (2006) 82–86.
Table 1: The Entanglement Coefficient $\mathcal{E}(\mu_1)$

| $\mu_1 := m_1/(m_1 + m_2)$ | $\mathcal{E}(\mu_1)$ |
|-----------------------------|-----------------------|
| 0.5                         | 0.000                 |
| 0.525                       | 0.0001                |
| 0.55                        | 0.0012                |
| 0.575                       | 0.0057                |
| 0.6                         | 0.0174                |
| 0.625                       | 0.0408                |
| 0.65                        | 0.0806                |
| 0.675                       | 0.1410                |
| 0.7                         | 0.2253                |
| 0.725                       | 0.3357                |
| 0.75                        | 0.4725                |
| 0.775                       | 0.6348                |
| 0.8                         | 0.8203                |
| 0.825                       | 1.0255                |
| 0.85                        | 1.2462                |
| 0.875                       | 1.4776                |
| 0.9                         | 1.7151                |
| 0.925                       | 1.9542                |
| 0.95                        | 2.1909                |
| 0.975                       | 2.4216                |
| 1                           | 2.6436                |

Figure 1: The entanglement coefficient $y = \mathcal{E}(\mu_1)$, as a function of $x = \mu_1 = m_1/(m_1 + m_2)$, for $0.5 \leq \mu_1 \leq 1$. 