ON MULTIPLICATIVE DEPENDENCE OF VALUES OF RATIONAL FUNCTIONS AND A GENERALISATION OF THE NORTHCOTT THEOREM

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Abstract. In this paper we study multiplicative dependence of values of polynomials or rational functions over a number field. As an application, we obtain new results on multiplicative dependence in the orbits of a univariate polynomial dynamical system. We also obtain a broad generalisation of the Northcott theorem replacing the finiteness of preperiodic points from a given number field by the finiteness of initial points with two multiplicatively dependent elements in their orbits.

1. Introduction

1.1. Motivation and background. We say that non-zero complex numbers \( \alpha_1, \ldots, \alpha_n \) are \textit{multiplicatively dependent} if there exist integers \( k_1, \ldots, k_n \), not all zero, such that

\[
\alpha_1^{k_1} \cdots \alpha_n^{k_n} = 1.
\]

Consequently, a point in the complex space \( \mathbb{C}^n \) is called \textit{multiplicatively dependent} if its coordinates are all non-zero and are multiplicatively dependent.

The same definition of multiplicative dependence applies to rational functions as well. Moreover, we say that rational functions \( \varphi_1, \ldots, \varphi_s \in \mathbb{C}(X) \) are \textit{multiplicatively independent modulo constants} if there is no non-zero integer vector \( (k_1, \ldots, k_s) \) such that

\[
\varphi_1^{k_1} \cdots \varphi_s^{k_s} \in \mathbb{C}^*.
\]

We also use \( \mathbb{G}_m \) to denote the multiplicative algebraic group, that is \( \mathbb{G}_m = \overline{\mathbb{Q}}^* \) endowed with the multiplicative group law, where as usual \( \overline{\mathbb{Q}} \) denotes the algebraic closure of \( \mathbb{Q} \). The study of intersections of geometrically irreducible algebraic curves \( X \subseteq \mathbb{G}_m^n \), defined over a number field \( \mathbb{K} \), and a union of proper algebraic subgroups of \( \mathbb{G}_m^n \) has been

2010 Mathematics Subject Classification. 11R18, 37F10.

Key words and phrases. Multiplicative dependence, curve of genus zero, rational value, iteration.
initiated by Bombieri, Masser and Zannier [5] (see also [32]). It is well known (see, for example, [4, Corollary 3.2.15]) that each such subgroup of $\mathbb{G}_m^n$ is defined by a finite set of equations of the shape $X_1^{k_1} \cdots X_n^{k_n} = 1$, with integer exponents not all zero. That is, the work [5] is about multiplicative dependence of points on a curve. It has been proved in [5, Theorem 1] that, under the assumption that $\mathcal{X}$ is not contained in any translate of a proper algebraic subgroup of $\mathbb{G}_m^n$, the multiplicatively dependent points on $\mathcal{X}(\mathbb{Q})$ form a set of bounded (absolute logarithmic) Weil height.

Most recently, a new point of view has been introduced in [22]. In particular, the structure of multiplicatively dependent points on $\mathcal{X}(\mathbb{K}^{ab})$ has been established in [22], where $\mathbb{K}^{ab}$ is the maximal abelian extension of a number field $\mathbb{K}$. In turn this implies that the set of such points is finite if $\mathcal{X}$ is of positive genus (see [22, Remark 2.4]). However, other than the structure, this result does not tell us further information about the case when $\mathcal{X}$ is of genus zero. Here we address this issue and give several applications to algebraic dynamical systems.

In algebraic geometry (see, for instance, [16, Section 1.2]), a rational curve defined over a field $\mathbb{F}$ is a curve birationally isomorphic to the projective line $\mathbb{P}^1$. If $\mathbb{F}$ is algebraically closed, this is equivalent to a curve of genus zero. Moreover, each rational curve can be parametrized in the form $(\phi_1(X), \ldots, \phi_s(X))$, where $\phi_i \in \mathbb{F}(X)$ are rational functions, not all constant. Conversely, each curve of this form represents a rational curve. So, when considering multiplicatively dependent points on a curve of genus zero, we essentially study multiplicative dependence in values of some rational functions, which is exactly the topic of this paper. Since we only discuss multiplicative dependence for non-zero complex numbers, this setting automatically excludes poles and zeros of rational functions. See also [7] for the case of translations of algebraic numbers.

This work is also partially motivated by a series of recent results on the distribution of roots of unity and, more generally, of algebraic numbers of bounded house (we refer to Section 1.2 for a precise definition) in orbits of rational functions. Here, we consider a broad generalisation of such problems and in particular study the multiplicative dependence of several consecutive iterations $\phi^{(n+1)}(\alpha), \ldots, \phi^{(n+s)}(\alpha)$.

\[1\text{We take this opportunity to indicate that the justification given in [22, Remark 2.4] mentions the infinitude of the automorphism group of $\mathbb{G}_m$ while in fact it contains only two automorphisms; however the claim itself is correct as the simple comparison of the genera of $\mathbb{G}_m$ and $\mathcal{X}$ shows.}\]
Recall that for a rational function $\varphi \in K(X)$, the $n$-th iterate $\varphi^{(n)}$ of $\varphi$ is recursively defined by

$$\varphi^{(0)} = X, \quad \text{and} \quad \varphi^{(n)} = \varphi\left(\varphi^{(n-1)}\right), \quad n \geq 1.$$ 

For an element $\alpha \in \overline{\mathbb{Q}}$ we define the orbit of $\varphi$ at $\alpha$ as the set

\begin{equation}
\text{Orb}_\varphi(\alpha) = \{u_n : u_0 = \alpha, u_n = \varphi(u_{n-1}), n = 1, 2, \ldots\}.
\end{equation}

**Remark 1.1.** The iterations in the orbit $\text{Orb}_\varphi(\alpha)$ are defined until some point $u_{n-1}$ hits a pole of $\varphi$. Moreover, if some point $u_n$, $n \geq 1$, in (1.1) is defined, then $\alpha$ is not a pole of $\varphi^{(n)}$ and $u_n = \varphi^{(n)}(\alpha)$. However, the converse is not true, and the fact that the evaluation $\varphi^{(n)}(\alpha)$ is defined does not imply the existence of $u_n$, since it is defined if and only if all the previous points $u_0, \ldots, u_{n-1}$ of the orbit (1.1) are defined and $u_{n-1}$ is not a pole of $\varphi$. Clearly, for polynomial systems this distinction does not exist.

**Definition 1.2 (Periodic and preperiodic points).** We say that $\alpha \in \overline{\mathbb{Q}}$ is a periodic point for $\varphi$ if $\varphi^{(n)}(\alpha) = \alpha$ for some positive integer $n$, and we say that $\alpha$ is a preperiodic point for $\varphi$ if $\varphi^{(m)}(\alpha)$ is a periodic point for some positive integer $m$.

The famous Northcott theorem, see [27, Theorem 3.12], gives the finiteness of preperiodic points for $\varphi$ contained in a number field and of bounded Weil height when $\varphi$ is of degree at least 2. Here we obtain an extension of this finiteness result in a new direction, which involves the notion of multiplicative dependence instead of equality.

1.2. **General notation and convention.** Throughout the paper, we use the following notation:

- $\overline{\mathbb{Q}}$: the algebraic closure of the rational numbers $\mathbb{Q}$;
- $\mathbb{U}$: the set of all roots of unity in the complex numbers $\mathbb{C}$;
- $\mathbb{K}$: an algebraic number field;
- $d_{\mathbb{K}}$: the degree over $\mathbb{Q}$ of the number field $\mathbb{K}$;
- $\mathbb{Z}_{\mathbb{K}}$: the ring of integers of $\mathbb{K}$;
- $\mathbb{K}^c = \mathbb{K} (\mathbb{U})$: the maximal cyclotomic extension of $\mathbb{K}$;
- $\mathbb{K}^{ab}$: the maximal abelian extension of $\mathbb{K}$;
- $\varphi = (\varphi_1, \ldots, \varphi_s) \in \mathbb{C}(X)^s$: a vector of rational functions;
- $\varphi(\alpha) = (\varphi_1(\alpha), \ldots, \varphi_s(\alpha)) \in \mathbb{C}^s$ for $\alpha \in \mathbb{C}$.

We recall that by the Kronecker–Weber theorem, $\mathbb{Q}^c = \mathbb{Q}^{ab}$, see [28, Chapter 14]. However, generally we can only claim $\mathbb{K}^c \subseteq \mathbb{K}^{ab}$.

We reserve $|\alpha|$ for the usual absolute value of $\alpha \in \mathbb{C}$ and use $\overline{\alpha}$ for the house of $\alpha$, which is the maximum of absolute values $|\sigma(\alpha)|$ of the conjugates $\sigma(\alpha)$ over $\mathbb{Q}$ of $\alpha \in \overline{\mathbb{Q}}$. 
**Height** always means the absolute logarithmic Weil height which we denote by \( h(\alpha) \) for non-zero \( \alpha \in \overline{\mathbb{Q}} \), see [4, 31].

For a rational function \( \varphi \in \mathbb{K}(X) \) with \( \varphi = f/g \), \( f, g \in \mathbb{K}[X] \) and \( \gcd(f, g) = 1 \), we define the degree of \( \varphi \), denoted by \( \deg \varphi \), to be \( \max\{\deg f, \deg g\} \). We say that \( \varphi \) is monic, if both \( f \) and \( g \) are monic.

**Definition 1.3** (Special rational functions). We say that a rational function \( \varphi \in \mathbb{K}(X) \) of degree \( d \) is special if \( \varphi \) is a conjugate, with respect to the conjugation action given by \( \text{PGL}_2(\mathbb{K}) \) on \( \mathbb{K}(X) \), either to \( \pm X^d \) or to \( \pm T_d(X) \).

Here, we use \( T_d \) to denote the Chebyshev polynomial of degree \( d \) which is uniquely defined by the functional equation \( T_d(X + X^{-1}) = X^d + X^{-d} \).

Throughout the paper, we use the Landau symbol \( O \) and the Vinogradov symbol \( \ll \). Recall that the assertions \( U = O(V) \) and \( U \ll V \) are both equivalent to the inequality \( |U| \leq cV \) with some absolute constant \( c > 0 \). To emphasise the dependence of the implied constant \( c \) on some parameter (or a list of parameters) \( \rho \), we write \( U = O_{\rho}(V) \) or \( U \ll_{\rho} V \).

**1.3. Main results and methods.** First, we show that under rather natural conditions on \( \varphi \in \mathbb{K}(X)^s \) the point \( \varphi(\alpha) \) is multiplicatively independent for all but finitely many elements \( \alpha \in \mathbb{K}^{ab} \), see Theorem 4.2 below.

Then, in Section 4.2 we establish several results about multiplicative independence of \( s \geq 1 \) consecutive elements in an orbit of a polynomial for all but finitely many initial values \( \alpha \in \mathbb{K}^{c} \). These give a very broad generalisation of previously known results on roots of unity in orbits (which corresponds to \( s = 1 \)).

In Section 4.3 we investigate the question of multiplicative dependence of pairs of elements (not necessary consecutive) in orbits. More precisely, we prove in Theorem 4.11 that for a polynomial \( f \in \mathbb{K}[X] \) of degree at least two and without multiple roots, there are only finitely many elements \( \alpha \in \mathbb{K} \) such that for some distinct integers \( m, n \geq 0 \) the values \( f^{(m)}(\alpha) \) and \( f^{(n)}(\alpha) \) are multiplicatively dependent. In particular, this can be considered as an extension of the Northcott theorem, see [27, Theorem 3.12], about the finiteness of preperiodic points (clearly \( f^{(m)}(\alpha) = f^{(n)}(\alpha) \) can be considered as a very special instance of multiplicative dependence).

The proofs of the above results rest on Section 2, where we collect several general statements about polynomials and their iterations, and
on Section 3 where we present more specialised auxiliary results, some of which are new and maybe of independent interest.

The results of Section 4.2 are based on a combination of several ideas. First we need to record a rather precise description of the structure of multiplicatively dependent values of rational functions in elements from $\mathbb{K}^{ab}$, see Lemma 3.9 below, complementing [22, Theorem 2.1]. The proof is based on the ideas of Bombieri, Masser and Zannier [5] and is similar to those in [22]. This description is then combined with an argument of Dvornicich and Zannier [8], which has recently also been used in [21]. However, compared to the orginial scheme of [8] now somewhat simpler argument is possible, thanks to the results of Fuchs and Zannier [10], which in turn extend previous results of Zannier [30] from Laurent polynomials to arbitrary rational functions.

The main result of Section 4.3, that is, Theorem 4.11 is based on some classical Diophantine techniques. More precisely, we use results about the finiteness of perfect powers amongst polynomial values, which are due to Bérczes, Evertse and Győry [3], combined with the celebrated result of Faltings [9] on the finiteness of rational points on a plane curve of genus $g > 1$.

1.4. Further generalisations. It is easy to see that some of our results can be extended to any field with the Bogomolov property, that is, fields $L \subseteq \overline{Q}$ for which there exists a constant $c_L > 0$, such that for any non-zero $\alpha \in L \setminus \mathbb{U}$ we have $h(\alpha) \geq c_L$. In particular, from [2, Theorem 1.2] it follows that $\mathbb{K}^{ab}$ has the Bogomolov property, see [1, 11, 13, 14] for non-abelian examples of such fields and further references.

2. Preliminaries

2.1. Bounding exponents in multiplicative relations. One of our main tools is the following result obtained in the proof of [22, Theorem 2.1], which follows the same approach as in [5, Theorem 1]. It can be seen as a generalisation of Loxton and van der Poorten’s result [20, 24] on bounding exponents of multiplicative relations of algebraic numbers.

**Lemma 2.1.** Let $\varphi = (\varphi_1, \ldots, \varphi_s) \in \mathbb{K}(X)^s$ whose components are multiplicatively independent modulo constants. Then, for any $\alpha \in \mathbb{K}^{ab}$ such that the point $\varphi(\alpha)$ is multiplicatively dependent, there exist integers $k_1, \ldots, k_s$, not all zero, satisfying

$$\max |k_i| \ll_{d_{\varphi}, \varphi} 1,$$
and such that
\[ \varphi_1(\alpha)^{k_1} \cdots \varphi_s(\alpha)^{k_s} = \zeta \]
for some root of unity \( \zeta \in \mathbb{U} \cap \mathbb{K}(\alpha) \).

We remark that the necessary condition “multiplicatively independent modulo constants” in Lemma 2.1 comes from [5, Theorem 1 and Theorem 1’]; see also [22, Remark 2.6]. Here, we present a simple example. Take \( \varphi_1 = X + 1, \varphi_2 = X - 1, \varphi_3 = 2(X^2 - 1) \), then \( \varphi_1, \varphi_2, \varphi_3 \) are multiplicatively independent but they are not multiplicatively independent modulo constants. For any integer \( m \geq 2 \), let \( \alpha_m = 2^m - 1 \). Then, we have the multiplicative dependence relation
\[ \varphi_1(\alpha_m)^{-(m+1)} \varphi_2(\alpha_m)^{-m} \varphi_3(\alpha_m)^m = 1, \]
where the absolute values of the exponents go to infinity as \( m \) goes to infinity.

2.2. Hilbert’s Irreducibility Theorem over \( \mathbb{K}^c \). We need the following result due to Dvornicich and Zannier [8, Corollary 1]. We present it however in a weaker form as in [21, Lemma 2.1] that is needed for our purpose, but the proof is given within the proof of [8, Corollary 1].

**Lemma 2.2.** Let \( f \in \mathbb{K}^c[X,Y] \) be such that \( f(X,Y^m) \) as a polynomial in \( X \) does not have a root in \( \mathbb{K}^c(Y) \) for all positive integers \( m \leq \deg_X f \). Then, \( f(X,\zeta) \) has a root in \( \mathbb{K}^c \) for only finitely many roots of unity \( \zeta \).

2.3. Representations via linear combinations of roots of unity. Loxton [19, Theorem 1] has proved that any algebraic integer \( \alpha \) contained in some cyclotomic field has a short representation as a sum of roots of unity, that is, \( \alpha = \sum_{i=1}^{b} \zeta_i \), where \( \zeta_1, \ldots, \zeta_b \in \mathbb{U} \), and the integer \( b \) depends only on \( \alpha \).

Dvornicich and Zannier [8, Theorem L] extended the result of Loxton [19, Theorem 1] to algebraic integers contained in a cyclotomic extension of a given number field. Here we present a simplified version.

**Lemma 2.3.** There exists a finite set \( \mathcal{E} \subseteq \mathbb{K} \) depending on \( \mathbb{K} \) such that any algebraic integer \( \alpha \in \mathbb{K}^c \) can be written as \( \alpha = \sum_{i=1}^{b} c_i \xi_i \), where \( c_i \in \mathcal{E}, \xi_i \in \mathbb{U}, i = 1, \ldots, b \), and the integer \( b \) depends only on \( \mathbb{K} \) and \([\alpha]\).

2.4. Multiplicative independence of polynomial iterates. We need the following special case of the result of Young [29, Corollary 1.2], which generalises the previous result of Gao [12, Theorem 1.4] to multiplicative independence of consecutive iterations of polynomials over fields of characteristic zero.
Lemma 2.4. Let $\mathbb{F}$ be an arbitrary field of characteristic zero, and let $f \in \mathbb{F}[X]$ be a polynomial of degree at least 2 which is not a monomial. Then, for any fixed integer $n \geq 1$, the polynomials $f^{(1)}(X), \ldots, f^{(n)}(X)$ are multiplicatively independent modulo constants.

We also note that the result of [29, Corollary 1.2] applies to rational functions as well, under some mild conditions.

3. Arithmetic properties of polynomials and their iterations

3.1. Growth of the number of terms in iterates of rational functions. Another important tool for our main results is a bound of Fuchs and Zannier [10, Corollary], on the number of terms in the iterates of a rational function. First we introduce the following:

Definition 3.1 (Sparsity of rational functions). We define the sparsity $S(\varphi)$ of a rational function $\varphi \in \mathbb{C}(X)$ as the smallest number of total terms (in both numerator and denominator) in any representation $\varphi = f/g$ with $f, g \in \mathbb{C}[X]$.

It is important to note that in Definition 3.1 we do not impose the coprimality condition on the polynomials $f$ and $g$.

We present next the result of Fuchs and Zannier [10, Corollary] in the form that is needed for this paper, that is, for iterates of polynomials. We note that their result is proven for iterates of rational functions of degree $d \geq 3$. Although it is very likely that [10, Corollary] can be extended to rational functions of degree $d = 2$ as well, see the comments after [10, Corollary], here we choose a simpler path which is quite sufficient for our purpose. First we need the following simple statement about the Chebyshev polynomial $T_4$ of degree 4.

Lemma 3.2. Let $\mathbb{F}$ be a field of characteristic zero. If $T_4(X) = f^{(2)}(X)$ for some polynomial $f \in \mathbb{F}[X]$, then $f(X) = T_2(X)$.

Proof. Recall the explicit form $T_4(X) = X^4 - 4X^2 + 2$. We can assume that $f(X) = aX^2 + bX + c, a \neq 0$. Clearly, the coefficient of $X^3$ in $f^{(2)}(X)$ comes only from the term $af(X)^2$ and is equal to $2a^2b$. Since $a \neq 0$, we have $b = 0$. Thus, $f(X) = aX^2 + c$ and $f^{(2)}(X) = a^3X^4 + 2a^2cX^2 + ac^2 + c$. Therefore,

$$a^3 = 1, \quad 2a^2c = -4, \quad ac^2 + c = 2.$$

From the second relation above, we derive $2a^3c = -4a$, and since $a^3 = 1$ we obtain $c = -2a$. Hence,

$$ac^2 + c = 4a^3 - 2a = 4 - 2a,$$
and recalling the third relation, we obtain \( a = 1 \) and thus \( c = -2 \). Therefore, \( f(X) = X^2 - 2 = T_2(X) \).

Now, we are ready to present the following slight variation of the result of Fuchs and Zannier in [10, Corollary].

**Lemma 3.3.** Let \( \mathbb{F} \) be a field of characteristic zero. Let \( q \in \mathbb{F}(X) \) be a non-constant rational function, and let \( f \in \mathbb{F}[X] \) be of degree \( d \geq 2 \). Assume that \( f \) is not special. Then, for any \( n \geq 1 \) we have

\[
S(f^{(n)}(q(X))) \geq \frac{(n-5) \log d - \log 2016}{\log 5}.
\]

**Proof.** If \( d \geq 3 \), then the claimed result is included in [10, Corollary]. In the following, we assume that \( d = 2 \).

Since \( S(f^{(n)}(q(X))) \geq 1 \) we can obviously assume that \( n > 5 \).

If \( n = 2m \) is even, then setting \( g(X) = f^{(2)}(X) \), we see from Lemma 3.2 that \( g \) is also not special, and then for \( f^{(m)}(q(X)) = g^{(m)}(q(X)) \), by [10, Corollary] we have

\[
S(f^{(n)}(q(X))) = S(g^{(m)}(q(X))) \\
\geq \frac{(m-2) \log 4 - \log 2016}{\log 5} \\
= \frac{(n/2 - 2) \log 4 - \log 2016}{\log 5} \\
= \frac{(n - 4) \log 2 - \log 2016}{\log 5},
\]

which is better than the claimed result.

Now, assume that \( n = 2m - 1 \) is odd. Since \( n > 5 \), we have \( m > 3 \). We rewrite

\[
S(f^{(n)}(q(X))) = S(f^{(2(m-1))}(f(q(X)))),
\]

then as the previous case, we obtain

\[
S(f^{(n)}(q(X))) \geq \frac{(m-3) \log 4 - \log 2016}{\log 5} \\
= \frac{(n - 5) \log 2 - \log 2016}{\log 5}.
\]

This completes the proof. \( \square \)

### 3.2. On the size of elements in orbits.

We make use of the following simple facts about the growth of elements in orbits, which are given in different forms or follow exactly the same proof as in [6, Lemma 3.5] and [21, Lemma 2.5, Lemma 2.7 and Corollary 2.8].

We separate them into Archimedean and non-Archimedean cases.

**Lemma 3.4.** Let \( f(X) = \sum_{i=0}^{d} a_i X^i \in \mathbb{K}[X] \) of degree \( d \geq 2 \) and let

\[
L = \max_{\sigma} \left( 1 + |\sigma(a_d^{-1})| + \sum_{i=0}^{d-1} |\sigma(a_i/a_d)| \right), \tag{3.1}
\]
where the maximum is taken over all embeddings $\sigma$ of $K$ into $\mathbb{C}$. Then, if $\alpha \in \mathbb{Q}$ is such that

(i) $|\alpha| > L$, then $\{|\sigma(f\alpha)|\}_{n \in \mathbb{N}}$ is a strictly increasing sequence for any embedding $\sigma$ of $K$ into $\mathbb{C}$;

(ii) $f^{(n)}(\alpha) \leq A$ for some positive real number $A$ and some integer $n \geq 1$, then

\[
\left| f^{(r)}(\alpha) \right| \leq \max(A, L), \quad \text{for all } 0 \leq r \leq n - 1.
\]

**Lemma 3.5.** Let $f(X) = \sum_{i=0}^{d} a_i X^i \in K[X]$ of degree $d \geq 2$ and let $\alpha \in \mathbb{Q}$ be such that

(3.2) $|\alpha|_v > \max_{j=0,\ldots,d-1} \{1, |a_j/a_d|_v, |a_d|_v^{-1}\}$

for a non-archimedean absolute value $|\cdot|_v$ of $K$ (normalised in some way and extended to $\mathbb{Q}$). Then, $\{|f^{(n)}(\alpha)|_v\}_{n \in \mathbb{N}}$ is a strictly increasing sequence.

However, we also need a different form of Lemmas 3.4 and 3.5, which we present below.

**Lemma 3.6.** Let $f(X) = \sum_{i=0}^{d} a_i X^i \in K[X]$ of degree $d \geq 2$. Then, there exists a real number $L > 0$ and an integer $E$, both depending only on $f$, such that, for any non-zero $\alpha, \beta \in \mathbb{Q}$ with $|\beta|_v \ll 1$, we have:

(i) If $|\beta f^{(n)}(\alpha)| \leq A$ for some constant $A$ and some $n \geq 1$, then

\[
|\beta f^{(r)}(\alpha)| \ll \max(A, L)
\]

for any $0 \leq r \leq n - 1$.

(ii) If $\beta$ is an algebraic integer and $|\beta f^{(n)}(\alpha)|$ is also an algebraic integer for some $n \geq 1$, then $E\beta f^{(r)}(\alpha)$ is also an algebraic integer for any $0 \leq r \leq n - 1$.

**Proof.** (i) Let $g(X) = \beta f(\beta^{-1}X)$ and let $L$ be defined by (3.1). Then, for any $k \geq 1$, one has

\[
g^{(k)}(X) = \beta f^{(k)}(\beta^{-1}X).
\]

From the hypothesis, we know that $|g^{(n)}(\beta \alpha)| \leq A$ for some $n \geq 1$. Now, applying Lemma 3.4 with the polynomial $g$ and the initial point $\beta \alpha$, we obtain that

\[
|\beta f^{(r)}(\alpha)| = |g^{(r)}(\beta \alpha)| \leq \max(A, L_g)
\]

for all $0 \leq r \leq n - 1$, where as in (3.1)

\[
L_g = \max_{\sigma} \left( 1 + |\sigma(\beta)|^{d-1} |\sigma(a_d^{-1})| + \sum_{i=0}^{d-1} |\sigma(\beta)|^{d-i} |\sigma(a_i/a_d)| \right),
\]
where the maximum is taken over all embeddings $\sigma$ of $K$ into $\mathbb{C}$. However, since $|\bar{\beta}| \ll 1$, we have $L_g \ll L$, and thus
\[
|\beta f^{(r)}(\alpha)| \ll \max(A, L)
\]
for all $0 \leq r \leq n - 1$. This concludes the part (i).

(ii) We enlarge the field $K$ if needed such that it contains $\alpha, \beta$. Let $|\cdot|_v$ be any non-archimedean absolute value of $K$. From the hypothesis we know that $g^{(n)}(\beta \alpha)$ is an algebraic integer, and so $|g^{(n)}(\beta \alpha)|_v \leq 1$. Then, by Lemma 3.5, applied again to the polynomial $g$ and the initial points $g^{(r)}(\beta \alpha), r = 0, 1, \ldots, n - 1$, we see that the analogue of the condition (3.2) fails and we obtain that
\[
|g^{(r)}(\beta \alpha)|_v \leq \max_{i=0, \ldots, d-1} \{1, |\beta|_v^{d-i} |a_i/a_d|_v, |\beta|_v^{d-1} |a_d|_v^{-1}\}.
\]
Taking now a sufficiently large integer $E$ such that $E a_i/a_d$ and $E/a_d$ are algebraic integers and so $|E a_i/a_d|_v, |E/a_d|_v^{-1} \leq 1$, for $i = 0, \ldots, d-1$, we have $|E \beta f^{(r)}(\alpha)|_v \leq 1$ and conclude that $E \beta f^{(r)}(\alpha)$ is an algebraic integer for any integer $r$ with $0 \leq r \leq n - 1$.

3.3. Compositions of rational functions and monomials. We need the following result which claims that the compositions of rational functions usually cannot give a monomial.

Lemma 3.7. Let $\varphi \in \overline{\mathbb{Q}}(X)$ be a rational function. Assume that $\varphi$ is not a power of a linear fractional function. Then, for
\[
R(X, Y) = \varphi(X) - b Y, \quad b \in \overline{\mathbb{Q}},
\]
there exists no rational function $S \in \overline{\mathbb{Q}}(Y)$ such that
\[
R(S(Y), Y^m) = 0 \quad \text{for some } m \geq 1.
\]

Proof. Without loss of generality, we can assume that $\varphi$ is non-constant. By contradiction, suppose that there exists a non-constant rational function $S(Y) = S_1(Y)/S_2(Y) \in \overline{\mathbb{Q}}(Y)$, where $S_1, S_2 \in \overline{\mathbb{Q}}[X]$ with $\gcd(S_1, S_2) = 1$, and a positive integer $m \geq 1$ such that
\[
R(S(Y), Y^m) = 0.
\]
Let $\beta_1, \ldots, \beta_t \in \overline{\mathbb{Q}}$ be all the distinct zeros and poles of $\varphi$. Then
\[
R(X, Y) = \frac{a \prod_{j: D_j > 0} (X - \beta_j)^{D_j}}{\prod_{j: D_j < 0} (X - \beta_j)^{-D_j}} - b Y;
\]
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for some \( a \in \mathbb{Q} \) and some non-zero integers \( D_1, \ldots, D_t \). If \( \varphi \) has no poles, then the factor \( \prod_{j:D_j<0} (X - \beta_j)^{-D_j} \) automatically becomes 1.

We set a similar convention when \( \varphi \) has no zeros.

We denote \( D = D_1 + \ldots + D_t \).

Clearing the denominators, we can rewrite (3.3) as

\[
a \prod_{j:D_j>0} (S_1(Y) - \beta_jS_2(Y))^{D_j} = bY^mS_2(Y)^D \prod_{j:D_j<0} (S_1(Y) - \beta_jS_2(Y))^{-D_j}.
\]

Since \( \gcd(S_1, S_2) = 1 \) and all \( \beta_i, i = 1, \ldots, t \), are pairwise distinct, we know that for any \( i \neq j \) we have

\[
\begin{align*}
\gcd(S_1(Y) - \beta_iS_2(Y), S_1(Y) - \beta_jS_2(Y)) &= 1, \\
\gcd(S_2(Y), S_1(Y) - \beta_jS_2(Y)) &= 1.
\end{align*}
\]

We remark that if \( S_2 \) is constant then the factors \( S_1 - \beta_jS_2 \) are not constant and that, otherwise, at least one of \( S_1 - \beta_iS_2, S_1 - \beta_jS_2 \) is not constant for \( i \neq j \). Then, since \( Y^mS_2^D \) is not constant, it is easy to see from (3.4), (3.5) and from this remark, that if \( D \neq 0 \) then \( S_2 \) is constant and \( \varphi \) has only one zero of multiplicity \( m \). Similarly, if \( D = 0 \) then \( \varphi \) has exactly one zero and one pole both of multiplicity \( m \). This completes the proof by noticing the choice of \( \varphi \).

\[ \square \]

Remark 3.8. The assumption in Lemma 3.7 is necessary. For example, if \( \varphi(X) = (aX + b)^m/(cX + d)^m \) with \( ad - bc \neq 0 \) and \( m \geq 1 \), then one can take \( S(Y) = (-dY + b)/(cY - a) \) to conclude that \( \varphi(S(Y)) - Y^m = 0 \).

3.4. Structure of multiplicatively dependent values. We start by supplementing the result of Bombieri, Masser and Zannier [5, Theorem 1] and also its generalisation in [22, Theorem 2.1] with a more explicit description of the multiplicatively dependent values of rational functions.

Lemma 3.9. Let \( \varphi = (\varphi_1, \ldots, \varphi_s) \in \mathbb{K}(X)^s \) whose components are multiplicatively independent modulo constants. Then, there exist an integer \( A \geq 1 \) and a finite set \( S \subseteq \mathbb{K}^* \) both depending only on \( d_{\varphi} \) and \( \varphi \), such that each element \( \alpha \in \mathbb{K}_{ab}^* \), for which \( \varphi(\alpha) \) is a multiplicatively dependent point, satisfies \( \mathbb{K}^*(\alpha) : \mathbb{K}^* \leq A \) and

\[
\alpha = \frac{\gamma}{a - \eta},
\]
where $a \in S$, $\eta \in U \cap \mathbb{K}(\alpha)$, and $\gamma \in \mathbb{K}(\alpha)$ with $\gamma \leq A$ and $A\gamma \in \mathbb{Z}_{\mathbb{K}(\alpha)}$.

In particular, if $\varphi_1, \ldots, \varphi_s$ are all monic, one can choose $a = 1$.

**Proof.** Let $\alpha \in \mathbb{K}^{ab}$ be such that $\varphi(\alpha)$ is multiplicatively dependent. Then, by Lemma 2.1, we know that there exist $k_1, \ldots, k_s \in \mathbb{Z}$, not all zero, satisfying

$$\max |k_i| \ll_{d_{\mathbb{K}}, \varphi} 1,$$

and such that

$$\varphi_1(\alpha)^{k_1} \cdots \varphi_s(\alpha)^{k_s} = \zeta$$

for some root of unity $\zeta \in U \cap \mathbb{K}(\alpha)$.

For each $i = 1, \ldots, s$, write $\varphi_i = f_i / g_i$, where $f_i, g_i \in \mathbb{K}[X]$ with $\gcd(f_i, g_i) = 1$. Then, from (3.7) we obtain that $\alpha$ is a root of the polynomial

$$\Psi(X) = \prod_{i=1}^{s} f_i(X)^{k_i} \prod_{i=1}^{s} g_i(X)^{-k_i}$$

$$- \zeta \prod_{i=1}^{s} g_i(X)^{k_i} \prod_{i=1}^{s} f_i(X)^{-k_i}$$

with coefficients of absolute value upper bounded only in terms of $d_{\mathbb{K}}$ and $\varphi$. Since $\varphi_1, \ldots, \varphi_s$ are multiplicatively independent, $\Psi$ is a non-zero polynomial.

In view of (3.8), we can find a positive integer $D$ depending only on $d_{\mathbb{K}}, \varphi$ such that $\deg \Psi \leq D$, which implies that

$$[\mathbb{K}^c(\alpha) : \mathbb{K}^c] \leq D.$$ 

Note that we have two cases. One is that the leading coefficient of $\Psi(X)$ is $b$ or $b\zeta$ for some $b \in \mathbb{K}^*$ ($b$ depends only on $d_{\mathbb{K}}, \varphi$), and so in this case, it is easy to see that $E_1 \alpha$ is an algebraic integer for some large integer $E_1$ depending only on $d_{\mathbb{K}}, \varphi$, and

$$\gamma \ll_{d_{\mathbb{K}}, \varphi} 1.$$ 

Thus, the claimed form of $\alpha$ follows by choosing $a = 1, \eta = -1$ and putting $\gamma = \alpha(1 - \eta) = 2\alpha$, where we still have

$$\gamma \ll_{d_{\mathbb{K}}, \varphi} 1 \quad \text{and} \quad E_1 \gamma \text{ is an algebraic integer.}$$

The other case is that the leading coefficient of $\Psi(X)$ is $c - b\zeta$ for some $b, c \in \mathbb{K}^*$, where $b, c$ depend only on $d_{\mathbb{K}}, \varphi$. In particular, if $\varphi_1, \ldots, \varphi_s$
are all monic, we have \( b = c = 1 \). Extending the products in \( \Psi(X) \), we obtain that \( \alpha \) satisfies an equation of the form
\[
(c - b\zeta)\alpha^e + \sum_{i=0}^{e-1} b_i\alpha^i = 0,
\]
or equivalently, if we denote \( \beta = \alpha(c - b\zeta) \),
\[
\beta^e + \sum_{i=0}^{e-1} (c - b\zeta)^{e-1-i}b_i\beta^i = 0,
\]
for some integer \( e \geq 1 \) depending only on \( d_K, \varphi \) and with coefficients \( b_i, i = 0, 1, \ldots, e - 1 \), with
\[
|b_i| \ll d_K, \varphi \quad \text{and} \quad |(c - b\zeta)^{e-1-i}| \ll d_K, \varphi.
\] (3.10)

So, it is easy to see that we can choose a large integer \( E_2 \) depending only on \( d_K, \varphi \) such that \( E_2\beta \) is an algebraic integer, and by (3.10) we also have
\[
\beta \ll d_K, \varphi.
\]

From \( \beta = \alpha(c - b\zeta) \), we have
\[
\alpha = \frac{\beta/b}{c/b - \zeta}.
\]
Then, let \( a = c/b, \gamma = \beta/b \) and \( \eta = \zeta \). Based on the choices of \( b, c \), we know that \( a \in K^* \) depends only on \( d_K, \varphi \), and thus the finiteness of choices of \( a \) follows from (3.6). We can also enlarge \( E_2 \) if needed such that \( E_2\gamma \) is also an algebraic integer (\( E_2 \) still depends only on \( d_K, \varphi \)), and also
\[
|\gamma| \ll d_K, \varphi.
\] (3.11)

Thus, from (3.9) and (3.11) we have proved that there exists an integer \( B \geq 1 \) that depends only on \( d_K, \varphi \) such that we always have \( |\gamma| \leq B \).

Taking now \( A = \max(B, D, E_1, E_2) \), we conclude the proof. \( \square \)

**Remark 3.10.** We see from Lemma 3.9 that if the components of \( \varphi \) are all monic, then for each such \( \alpha \in K^{ab} \), there exists an algebraic integer \( \delta \in K^{ab} \) (that is, \( \delta = A(1 - \eta) \) with \( A \) and \( \eta \) as in Lemma 3.9) with \( |\delta| \leq 2A \) such that \( a\delta \) is an algebraic integer, which roughly means that the “denominator” of \( \alpha \) is uniformly bounded.

**Remark 3.11.** In Lemma 3.9, even if we assume that such \( \alpha \) are algebraic integers, we cannot have that the house \( |\alpha| \) of such \( \alpha \) is uniformly upper bounded (that is, only in terms of \( d_K, \varphi \)); see Example 3.12 below.
Example 3.12. Choose $\varphi_1(X) = X$ and $\varphi_2(X) = X + 1$. Certainly, $\varphi_1$ and $\varphi_2$ are multiplicatively independent modulo constants. However, for any $n$-th root of unity $\zeta_n \neq 1$, let $\alpha = 1/(\zeta_n - 1)$. It is easy to see that $\varphi_1(\alpha)$ and $\varphi_2(\alpha)$ are multiplicatively dependent. In addition, it is well-known (see [28, Proposition 2.8]) that if $n$ has at least two distinct prime factors, then $\zeta_n - 1$ is a unit of $\mathbb{Z}[\zeta_n]$, and thus $1/(\zeta_n - 1)$ is an algebraic integer.

Remark 3.13. There are also some cases where one can claim the finiteness of the set of $\alpha$ in Lemma 3.9. Here are some examples.

1. If $\varphi \in (\mathbb{K}(X) \cap \mathbb{R}(X))^s$, then one immediately obtains the finiteness of such $\alpha \in \mathbb{K}_{ab} \cap \mathbb{R}$ in Lemma 3.9. This follows directly from the proof of Lemma 3.9, since in this case in (3.7) we have $\zeta = \pm 1$ because the left-hand side is real, and then in (3.8) the polynomial $\Psi(X)$ is defined over $\mathbb{K}$ and has bounded degree. So, such elements $\alpha \in \mathbb{K}_{ab} \cap \mathbb{R}$ are of bounded degree and bounded height, which leads to the finiteness.

2. Similarly, if $\mathbb{K}$ is a totally real number field, then there are only finitely many elements $\alpha \in \mathbb{Q}_{tr}$, where $\mathbb{Q}_{tr}$ is the field of all totally real algebraic numbers, such that the point $\varphi(\alpha)$ is multiplicatively dependent. This conclusion follows since Lemma 2.1 still holds when we replace $\mathbb{K}_{ab}$ by $\mathbb{Q}_{tr}$. Indeed, the proof of Lemma 2.1 for $\mathbb{Q}_{tr}$ follows the same as in the proof of [22, Theorem 2.1], where instead of [2, Theorem 1.2] one uses an early result due to Schinzel [26] on the heights of totally real algebraic numbers. So, in this case, the inequality (3.6) still holds, and in (3.7) we also have $\zeta = \pm 1$.

For the two examples in Remark 3.13, we want to indicate that $\mathbb{Q}_{ab} \cap \mathbb{R} \subset \mathbb{Q}_{tr}$. This is based on two facts: one is that any abelian extension of $\mathbb{Q}$ is either totally real, or contains a totally real subfield over which it has degree two; and the other is that there exist totally real fields which are not abelian over $\mathbb{Q}$.

4. Main Results

4.1. Finiteness of multiplicatively dependent values. Now, we give a stronger version of Lemma 3.9 by proving finiteness of $\alpha \in \mathbb{K}_{ab}$ for which $\varphi(\alpha)$ is a multiplicatively dependent point for a class of $\varphi \in \mathbb{K}(X)^s$. First, we introduce a definition.

Definition 4.1. We say that the rational functions $\varphi_1, \ldots, \varphi_s \in \mathbb{K}(X)$ multiplicatively generate a power of a linear fractional function if there
exists integers $k_1, \ldots, k_s$, not all zero, such that $\varphi_1^{k_1} \cdots \varphi_s^{k_s}$ is a power of a linear fractional function.

Note that a linear fractional function can be a constant function, and the zero power of a linear fractional function is 1 by convention. So, if $\varphi_1, \ldots, \varphi_s$ cannot multiplicatively generate a power of a linear fractional function, then they are automatically multiplicatively independent modulo constants.

The possibility of the following result has been indicated in [22, Remark 4.2], here we present it in full detail.

**Theorem 4.2.** Let $\varphi = (\varphi_1, \ldots, \varphi_s) \in \mathbb{K}(X)^s$ whose components cannot multiplicatively generate a power of a linear fractional function. Then, there are only finitely many elements $\alpha \in \mathbb{K}^{ab}$ such that $\varphi(\alpha)$ is a multiplicatively dependent point.

**Proof.** First, we remark that it is enough to prove the finiteness of the elements $\alpha \in \mathbb{K}^c$ such that $\varphi(\alpha)$ is multiplicatively dependent. Indeed, if $\mathcal{X}$ is the rational curve parametrized by $\varphi$, then proving the theorem is equivalent to showing that there are only finitely many multiplicatively dependent points in $\mathcal{X}(\mathbb{K}^{ab})$. By [22, Theorem 2.1 and Remark 2.2], the set of dependent points in $\mathcal{X}(\mathbb{K}^{ab})$ is the union of a finite set with the dependent ones in $\mathcal{X}(\mathbb{K}^c)$.

Let now $\alpha \in \mathbb{K}^c$ such that $\varphi(\alpha)$ is a multiplicatively dependent point. By Lemma 2.1, one can find integers $k_1, \ldots, k_s$, not all zero, that are uniformly bounded only in terms of $d_\mathcal{X}$ and $\varphi$, such that

$$\varphi_1(\alpha)^{k_1} \cdots \varphi_s(\alpha)^{k_s} = \zeta,$$

for some root of unity $\zeta \in \mathbb{U}$.

Let

$$R(X, Y) = \varphi_1(X)^{k_1} \cdots \varphi_s(X)^{k_s} - Y.$$

By assumption, $\varphi_1(X)^{k_1} \cdots \varphi_s(X)^{k_s}$ is not a power of a linear fractional function. Then, from Lemma 3.7, we conclude that there is no rational function $S(Y) \in \mathbb{K}^c(Y)$ such that $R(S(Y), Y^m) = 0$ for any $m \geq 1$. Applying now Lemma 2.2 to the numerator of $R(X, Y)$, we obtain that there are only finitely many roots of unity $\zeta \in \mathbb{U}$ such that $R(X, \zeta)$ has a zero in $\mathbb{K}^c$. Noticing again that $k_1, \ldots, k_s$ are all uniformly bounded only in terms of $d_\mathcal{X}$ and $\varphi$, we have that there are only finitely many equations of the form (4.1), and thus we conclude the proof.

**Remark 4.3.** We remark that in the above finiteness result, the number of such $\alpha$ depends on $\mathbb{K}, \varphi$. Besides, Example 3.12 suggests that the assumption in Theorem 4.2 is indeed necessary.
Remark 4.4. Theorem 4.2 can be interpreted that under a rather weak condition, the rational curve $X$ parametrized by $\varphi$ has only finitely many multiplicatively dependent points defined over $\mathbb{K}^{ab}$.

4.2. Finiteness of consecutive multiplicatively dependent elements in orbits. First we consider compositions of polynomial iterations with several multiplicatively independent rational functions. These results generalise that of [21] on roots of unity in orbits of a wide class of rational functions. For this we recall the definition of a special rational function as in Definition 1.3.

Theorem 4.5. Let $\varphi \in \mathbb{K}(X)^*$ whose components are multiplicatively independent modulo constants, and let $f \in \mathbb{K}[X]$ be a non-special polynomial of degree at least 2. Then, the following hold:

(i) there exists a non-negative integer $n_0$ depending only on $f, \varphi$ and $\mathbb{K}$ such that there are at most finitely many elements $\alpha \in \mathbb{K}^c$ for which $\varphi(f^{(n)}(\alpha))$ is a multiplicatively dependent point for some integer $n \geq n_0$;

(ii) if furthermore the components of $\varphi$ cannot multiplicatively generate a power of a linear fractional function, then (i) holds with $n_0 = 0$.

Proof. (i) Let $\alpha \in \mathbb{K}^c$ be such that $\varphi(f^{(n)}(\alpha))$ is multiplicatively dependent for some integer $n \geq 0$. Then, applying Lemma 3.9 we have

$$f^{(n)}(\alpha) = \frac{\gamma_{a,n}}{a_{a,n} - \eta_{a,n}},$$

where $a_{a,n}$ is in a finite set $S \subseteq \mathbb{K}^*$, $\eta_{a,n} \in U \cap \mathbb{K}(\alpha)$, and $\gamma_{a,n} \in \mathbb{K}(\alpha)$ such that

$$[(a_{a,n} - \eta_{a,n})f^{(n)}(\alpha)] \ll d_{\mathbb{K}, \varphi} 1,$$

where $d_{\mathbb{K}} = [\mathbb{K} : \mathbb{Q}]$, and there exists a sufficiently large integer

$$A \ll d_{\mathbb{K}, \varphi} 1$$

such that $A(a_{a,n} - \eta_{a,n})f^{(n)}(\alpha)$ is an algebraic integer.

We apply now the method of [8, Theorem 2] and [21, Theorem 1.2]. Indeed, let $M$ be a sufficiently large positive integer, chosen to satisfy

$$M > \frac{(B+2)\log 5 + \log 2016}{\log d} + 5,$$

where $B$ defined below is a constant depending only on $f, \varphi$ and $\mathbb{K}$.

We also denote by $S_{f, \varphi}(M)$ the set of $\alpha \in \mathbb{K}^c$ such that the point $\varphi(f^{(n)}(\alpha))$ is multiplicatively dependent for some integer $n > M$. Our purpose is to show that $S_{f, \varphi}(M)$ is a finite set.
Let now $\alpha \in S_{f, \varphi}(M)$. Using (4.2), we apply Lemma 3.6 to conclude, since $\left[a_{\alpha,n} - \eta_{\alpha,n}\right] \ll_{d, K, \varphi, \kappa} 1$ and $A(a_{\alpha,n} - \eta_{\alpha,n})f^{(n)}(\alpha)$ is an algebraic integer, that

$$\left[a_{\alpha,n} - \eta_{\alpha,n}\right]f^{(r)}(\alpha) \ll_{d, K, \varphi, \kappa} 1, \quad 0 \leq r \leq M,$$

and there exists a positive integer $E \ll_{d, K, \varphi, \kappa} 1$ with the property that $E(a_{\alpha,n} - \eta_{\alpha,n})f^{(r)}(\alpha)$ is an algebraic integer for any $0 \leq r \leq M$.

Applying now Lemma 2.3 for $E(a_{\alpha,n} - \eta_{\alpha,n})f^{(r)}(\alpha), r = 0, 1, \ldots, M$, there exist a positive integer $B$ (depending only on $f, \varphi, \kappa$) and a finite set $E$ (depending only on $\kappa$) such that we can write $E(a_{\alpha,n} - \eta_{\alpha,n})f^{(r)}(\alpha)$ in the form

$$(4.4) \quad E(a_{\alpha,n} - \eta_{\alpha,n})f^{(r)}(\alpha) = c_{\alpha,r,1}\xi_{\alpha,r,1} + \cdots + c_{\alpha,r,B}\xi_{\alpha,r,B},$$

where $c_{\alpha,r,i} \in E$ and $\xi_{\alpha,r,i} \in \mathbb{U}$.

By contradiction, suppose that $S_{f, \varphi}(M)$ is an infinite set. Then, since both $S$ and $E$ are finite sets, we can choose an infinite subset $T_{f, \varphi}(M)$ of $S_{f, \varphi}(M)$ such that for any $\alpha \in T_{f, \varphi}(M)$, the coefficients $a_{\alpha,n}$ and $c_{\alpha,r,i} \in E$ in (4.4) are all fixed (independent of $\alpha$) for $r = 0, 1, \ldots, M$ and $i = 1, 2, \ldots, B$. For these fixed coefficients, to simplify the notation from now on, we denote

$$a = a_{\alpha,n} \quad \text{and} \quad c_{r,i} = c_{\alpha,r,i}. $$

So, it suffices to consider the elements in $T_{f, \varphi}(M)$.

We use the first equation corresponding to $r = 0$ to replace $\alpha \in T_{f, \varphi}(M)$ on the left-hand side of (4.4) and thus consider the equations with unknowns $Y, X_{r,i}, r = 0, 1, \ldots, M, i = 1, 2, \ldots, B$:

$$(4.5) \quad E(a - Y)f^{(r)}\left(\frac{c_{0,1}X_{0,1} + \cdots + c_{0,B}X_{0,B}}{E(a - Y)}\right) = c_{r,1}X_{r,1} + \cdots + c_{r,B}X_{r,B}, \quad r = 1, \ldots, M.$$ 

Then, for any $\alpha \in T_{f, \varphi}(M)$, the points $(\eta_{\alpha,n}, \xi_{\alpha,r,i}), r = 0, 1, \ldots, M, i = 1, 2, \ldots, B$, are torsion points on the variety $Z \subseteq \mathbb{G}^{B(M+1)+1}_{m}$ defined by the equations in (4.5). Since $T_{f, \varphi}(M)$ is an infinite set and in view of (4.4), there are infinitely many such points.

By the toric analogue of the Manin–Mumford conjecture (also called the torsion points theorem) proved by Laurent [18] and also more elementarily by Sarnak and Adams [25], there exists a 1-dimensional torsion coset of $\mathbb{G}^{B(M+1)+1}_{m}$ contained in the Zariski closure of the torsion points in the variety $Z$. We can parametrize this coset by $X_{r,i} = \beta_{r,i}t^{e_{r,i}}$ and $Y = \tau t^\ell$ with a parameter $t$, where $\beta_{r,i}, \tau$ are roots of unity and $e_{r,i}, \ell$
are integers, not all zero, and so we obtain the following identities
\begin{equation}
(4.6) \quad f(r) \left( \frac{\sum_{j=1}^B c_{0,j} \beta_{0,j} t^{e_{0,j}}}{E(a - \tau t^\ell)} \right) = \frac{\sum_{j=1}^B c_{r,j} \beta_{r,j} t^{e_{r,j}}}{E(a - \tau t^\ell)}, \quad r = 1, \ldots, M.
\end{equation}

Now, for
\[ q(t) = \frac{\sum_{j=1}^B c_{0,j} \beta_{0,j} t^{e_{0,j}}}{E(a - \tau t^\ell)}, \]
the equation (4.6) shows that \( f(r)(q(t)), r = 1, \ldots, M \), is a rational function having all together at most \( B + 2 \) terms. Since \( f \) is a non-special polynomial of degree at least 2, we directly apply Lemma 3.3 to conclude that we must have
\[ B + 2 \geq \frac{1}{\log 5} ((M - 5) \log d - \log 2016), \]
which contradicts the choice of \( M \) as in (4.3). Thus, \( S_{f,\varphi}(M) \) is a finite set. Taking \( n_0 = M + 1 \), this concludes the proof of the first part (i).

(ii) From Theorem 4.2, we know that there are only finitely many elements \( \beta \in \mathbb{K}^c \) such that \( \varphi(\beta) \) is multiplicatively dependent. We can fix one such \( \beta \in \mathbb{K}^c \), and we are thus left to prove that there are only finitely many \( \alpha \in \mathbb{K}^c \) such that \( f^{(n)}(\alpha) = \beta \) for some \( n \geq 0 \).

Let \( M \) be a positive integer chosen to satisfy (4.3). It has been proved in the above that \( S_{f,\varphi}(M) \) is a finite set. So, we only need to consider \( n \leq M \). Since \( \beta \in \mathbb{K}^c \) is fixed, we conclude that there are only finitely many \( \alpha \in \mathbb{K}^c \) satisfying \( f^{(n)}(\alpha) = \beta \) for any \( 0 \leq n \leq M \). This completes the proof. \( \square \)

Remark 4.6. It is plausible that the result in Theorem 4.5 and the rest of results of this section also hold when we replace the polynomial \( f \) by a non-special rational function \( g/h \in \mathbb{K}(X) \) with \( g, h \in \mathbb{K}[X] \) and \( \deg g - \deg h > 1 \); see [6, 21].

Remark 4.7. We note that a weaker statement of Theorem 4.5 (ii) may also follow from [6, Theorems 1.5 and 2.5], under several other restrictions on the polynomial \( f \). Indeed, from the proof of Theorem 4.5 (ii), we can fix an element \( \beta \in \mathbb{K}^c \) and reduce the problem to proving that there are only finitely many \( \alpha \in \mathbb{K}^c \) such that \( f^{(n)}(\alpha) = \beta \) for some \( n \geq 1 \). In particular this means \( f^{(n)}(\alpha) \ll 1 \). We apply now [6, Theorem 1.5] to conclude that there are only finitely many such \( \alpha \in \mathbb{K}^c \), under the condition that there does not exist a rational function \( S(X) \in \mathbb{K}^c(X) \) such that \( f(S(X)) \) is a “short” Laurent polynomial.
Remark 4.8. From Theorem 4.5 (ii), taking \( s = 1 \) and \( \varphi = f \), one can recover the main result in [21, Theorem 1.2]: there are only finitely many \( \alpha \in \mathbb{K}^c \) such that \( f^{(n)}(\alpha) \in U \) for some integer \( n > 0 \).

Moreover, combining Theorem 4.5 with Lemma 2.4, we have the following corollary.

Corollary 4.9. Let \( f \in \mathbb{K}[X] \) be non-special and of degree at least 2. Then, for any integer \( s \geq 1 \), the following hold:

(i) there are only finitely many \( \alpha \in \mathbb{K}^c \) such that the \( s \) consecutive iterations \( f^{(n+1)}(\alpha), \ldots, f^{(n+s)}(\alpha) \) are multiplicatively dependent for infinitely many integers \( n \geq 0 \);

(ii) if furthermore the iterations of \( f \) cannot multiplicatively generate a power of a linear fractional function, then there are only finitely many \( \alpha \in \mathbb{K}^c \) such that the \( s \) consecutive iterations \( f^{(n+1)}(\alpha), \ldots, f^{(n+s)}(\alpha) \) are multiplicatively dependent for some integer \( n \geq 0 \).

Remark 4.10. The condition of Corollary 4.9 (ii) on the iterations is not very restrictive. For example, it is easy to see that if 0 is a not a periodic point of \( f \), then all the iterations of \( f \) are relatively prime.

4.3. Finiteness of multiplicatively dependent points in orbits at arbitrary positions. Recall that \( \mathbb{Z}_K \) is the ring of integers of \( K \). Our main result below relies on some finiteness results on the number of perfect powers amongst polynomial values (see [3]), as well as the result of Faltings [9] on the finiteness of \( K \)-rational points on curves of genus greater than one.

Theorem 4.11. Let \( f \in \mathbb{K}[X] \) be a polynomial without multiple roots, of degree \( d \geq 3 \) or, if \( d = 2 \), we also assume that \( f^{(2)} \) has no multiple roots. Then, there are only finitely many elements \( \alpha \in \mathbb{K} \) such that for some distinct integers \( m, n \geq 0 \) the values \( f^{(m)}(\alpha) \) and \( f^{(n)}(\alpha) \) are multiplicatively dependent.

Proof. Let \( d_\mathbb{K} = [\mathbb{K} : \mathbb{Q}] \), and write the \( d_\mathbb{K} \) embeddings of \( \mathbb{K} \) into \( \mathbb{C} \) as \( \sigma_1, \ldots, \sigma_{d_\mathbb{K}} \). For \( i = 1, \ldots, d_\mathbb{K} \), let \( f_i = \sigma_i(f) \).

Choose now a sufficiently large number \( N > L \) (see the end of the proof), where \( L \) is defined by (3.1). Denote by \( \mathcal{T}(N, \mathbb{K}) \) the set of elements \( \alpha \in \mathbb{K} \) with \( \left\lfloor \sqrt[\mathbb{K}]{\alpha} \right\rfloor \leq N \). Then, \( \mathcal{T}(N, \mathbb{K}) \) is a set of bounded height contained in \( \mathbb{K} \), and so it is a finite set. Thus, we only need to consider elements in the set \( \mathbb{K} \setminus \mathcal{T}(N, \mathbb{K}) \).

Now, for \( \alpha \in \mathbb{K} \setminus \mathcal{T}(N, \mathbb{K}) \), assume that there exists a pair of non-negative integers \( (m, n) \) with \( m > n \) such that for some integers \( k \) and
\[\ell\text{ with }(k, \ell) \neq (0,0),\text{ we have}\]
\[(f^{(m)}(\alpha))^k = (f^{(n)}(\alpha))^\ell.\]  

Note that there is an embedding, say \(\sigma_j\), such that \(|\sigma_j(\alpha)| = |\alpha|\) and so \(|\sigma_j(\alpha)| > N\). By (4.7) we have
\[
(f_j^{(m)}(\sigma_j(\alpha)))^k = (f_j^{(n)}(\sigma_j(\alpha)))^\ell.
\]

By the choices of \(\sigma_j(\alpha), m\) and \(n\), using Lemma 3.4 (applied with \(\sigma_j(\alpha)\) instead of \(\alpha\)) we obtain
\[(4.8)\]
\[|f_j^{(m)}(\sigma_j(\alpha))| > |f_j^{(n)}(\sigma_j(\alpha))| \geq |\sigma_j(\alpha)| > N.\]

So, we cannot have \(k = 0\) or \(\ell = 0\). Then, we must have
\[1 \leq k < \ell.
\]
Let
\[r = \frac{k}{\gcd(k, \ell)}, \quad t = \frac{\ell}{\gcd(k, \ell)}.
\]
Clearly,
\[1 \leq r < t, \quad \gcd(r, t) = 1.
\]
From (4.7), we have
\[(4.9)\]
\[(f^{(m)}(\alpha))^r = \eta (f^{(n)}(\alpha))^t\]
for some \(\eta \in \mathbb{U} \cap \mathbb{K}\).

Since \(\gcd(r, t) = 1\), we can now find some integers \(a\) and \(b\) with \(ar + bt = 1\). Hence, using (4.9) we get
\[(4.10)\]
\[f^{(m)}(\alpha) = \eta^a \left((f^{(n)}(\alpha))^a (f^{(m)}(\alpha))^b\right)^t.
\]

Now, given a polynomial \(g \in \mathbb{K}[X]\) we can write it as
\[g(X) = \frac{1}{D}G(X)
\]
with \(G \in \mathbb{Z}_K[X]\) and a positive integer \(D\), both are uniquely defined by the minimality condition on \(D\). Then we use \(\mathcal{S}_g\) to denote the subset of places of \(\mathbb{K}\), which consists of all the infinite places and all the finite places corresponding to the prime ideal divisors of \(D\) and of the leading coefficient of \(G\).

One can easily verify that
\[(4.11)\]
\[\mathcal{S}_{f^{(m)}} \subseteq \mathcal{S}_f, \quad m = 1, 2, \ldots.
\]

Furthermore, for a prime ideal \(\mathfrak{p}\) of \(\mathbb{Z}_\mathbb{K}\) we use \(\nu_\mathfrak{p}(\vartheta)\) to denote the (additive) valuation of \(\vartheta \in \mathbb{K}\) at the place corresponding to \(\mathfrak{p}\). We denote by \(\mathbb{Z}_{\mathbb{K}, \mathcal{S}_f}\) the set of \(\mathcal{S}_f\)-integers in \(\mathbb{K}\), that is, the set of \(\vartheta \in \mathbb{K}\)
with \(v_p(\vartheta) \geq 0\) for any \(p \not\in S_f\) (alternatively, \(v_p(\vartheta) < 0\) implies \(p \in S_f\)). Note that \(f(\alpha)\) is in \(\mathbb{Z}_{K,S_f}\) for \(\alpha \in \mathbb{Z}_{K,S_f}\).

Now, we first assume that \(m = n + 1\). From (4.9), we have

\[
(4.12) \quad \left(f \left( f^{(m-1)}(\alpha) \right) \right)^r = \eta \left( f^{(m-1)}(\alpha) \right)^t.
\]

Clearly, the polynomials \(f(X)\) and \(X\) are multiplicatively independent modulo constants. Applying Lemma 2.1 to (4.12) and noticing that there are only finitely many roots of unity in \(K\), we obtain that the exponents \(r\) and \(t\) are upper bounded in terms of \(f\) and \(K\) only. So, there are only finitely many possible values of \(f^{(m-1)}(\alpha)\) satisfying (4.12). Hence, we have

\[
(4.13) \quad \left[f^{(m-1)}(\alpha)\right] \ll_{f,K} 1.
\]

We now assume that \(m = n + 2\). From (4.9), we have

\[
(4.14) \quad \left(f^{(2)} \left( f^{(m-2)}(\alpha) \right) \right)^r = \eta \left( f^{(m-2)}(\alpha) \right)^t.
\]

Since \(f\) only has simple roots and its degree \(d \geq 2\), the polynomials \(f^{(2)}(X)\) and \(X\) are multiplicatively independent modulo constants. As the above, using Lemma 2.1, we deduce that there are only finitely many possible values of \(f^{(m-2)}(\alpha)\) satisfying (4.14). Hence, we have

\[
\left[f^{(m-2)}(\alpha)\right] \ll_{f,K} 1,
\]

which ensures that (4.13) holds in this case too.

In the following, we assume that \(m > n + 2\).

We distinguish between two cases: one when (4.10) holds for \(\alpha \in \mathbb{Z}_{K,S_f}\) and one for \(\alpha \not\in \mathbb{Z}_{K,S_f}\).

**Case I: \(\alpha \in \mathbb{Z}_{K,S_f}\).** In this case, since \(f^{(m)}(\alpha) \in \mathbb{Z}_{K,S_f}\) and \(\eta \in U \cap \mathbb{Z}_{K}\), by (4.10) we also have

\[
\left(f^{(m)}(\alpha)\right)^a \left(f^{(m)}(\alpha)\right)^b \in \mathbb{Z}_{K,S_f},
\]

(because it is in \(K\) and its \(t\)-th power is in \(\mathbb{Z}_{K,S_f}\).)

Write \(f^{(m)}(\alpha) = f \left( f^{(m-1)}(\alpha) \right)\),

or as

\( f^{(m)}(\alpha) = f^{(2)} \left( f^{(m-2)}(\alpha) \right) \)

if \((d,t) = (2,2)\). By a result of Bérczes, Evertse and Györy [3, Theorem 2.3], we obtain that the exponent \(t \geq 2\) in (4.10) is upper bounded in terms of \(f\) and \(K\). Applying now [3, Theorems 2.1 and 2.2] to (4.10), we conclude that

\[
h \left( f^{(m-1)}(\alpha) \right) \ll_{f,K} 1.
\]
Since $f^{(m-1)}(\alpha) \in \mathbb{K}$, by the Northcott theorem we conclude that there are only finitely many possible values of $f^{(m-1)}(\alpha)$, and thus the inequality (4.13) holds again.

**Case II: $\alpha \notin \mathbb{Z}_K, S_f$.** We choose a prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ with $\mathfrak{p} \notin S_f$ and $v_\mathfrak{p}(\alpha) < 0$. Then, using (4.11), we see that $\mathfrak{p} \notin S_{f^{(m)}}$ and thus we easily derive

\begin{equation}
 v_\mathfrak{p} \left( f^{(m)}(\alpha) \right) = d^m v_\mathfrak{p}(\alpha).
\end{equation}

Indeed, let

\[ f^{(m)}(X) = \frac{1}{E} \sum_{j=0}^{d^m} A_j X^j \]

where $E, A_{d^m}, \ldots, A_0 \in \mathbb{Z}_K$ with $v_\mathfrak{p}(A_{d^m} E) = 0$. Then

\begin{equation}
 v_\mathfrak{p} \left( \alpha^{-d^m} f^{(m)}(\alpha) \right) = v_\mathfrak{p} \left( E^{-1} \left( A_{d^m} + \sum_{j=0}^{d^m-1} A_j \alpha^{j-d^m} \right) \right)
 = v_\mathfrak{p} \left( A_{d^m} + \sum_{j=0}^{d^m-1} A_j \alpha^{j-d^m} \right) = 0,
\end{equation}

since $v_\mathfrak{p}(A_{d^m}) = 0$, while

\begin{align*}
 v_\mathfrak{p} \left( \sum_{j=0}^{d^m-1} A_j \alpha^{j-d^m} \right) &= v_\mathfrak{p}(\alpha^{-1}) + v_\mathfrak{p} \left( \sum_{j=0}^{d^m-1} A_j \alpha^{j-d^m+1} \right) \\
 &\geq v_\mathfrak{p}(\alpha^{-1}) + \min_{0 \leq j \leq d^m-1} v_\mathfrak{p}(A_j \alpha^{j-d^m+1}) \\
 &\geq v_\mathfrak{p}(\alpha^{-1}) > 0.
\end{align*}

Clearly (4.16) implies (4.15).

Hence, recalling (4.9), we conclude that

\[ rd^m v_\mathfrak{p}(\alpha) = td^m v_\mathfrak{p}(\alpha). \]

Since $1 \leq r < t$, $\gcd(r, t) = 1$ and $v_\mathfrak{p}(\alpha) \neq 0$, we conclude that $r = 1$ and $t = d^m - n$, and moreover

\begin{equation}
 f^{(m)}(\alpha) = \eta \left( f^{(n)}(\alpha) \right)^{d^m-n}.
\end{equation}

We enlarge $\mathbb{K}$ to a field $\mathbb{L}$ such that it contains all $d^3$-th roots of roots of unity in $\mathbb{K}$. Noticing $m > n + 2$, we obtain from (4.17)

\[ f^{(m)}(\alpha) = w^{d^3} \]

for some $w \in \mathbb{L}$. Since $f$ is of degree $d \geq 2$ and has only simple roots, the curve $f(X) = Y^{d^3}$ is smooth of genus greater than 1, see [15,
By the celebrated result of Faltings [9], we know that the curve \( f(X) = Y^{d^3} \) has only finitely many \( \mathbb{L} \)-rational points, and in particular this implies that we have the inequality (4.13) again.

Therefore, we can always choose the constant \( N \) above to be sufficiently large (depending on \( f, \mathbb{K} \)) such that (4.13) can be written as

\[
\left| f^{(m-1)}(\alpha) \right| < N.
\]

This together with Lemma 3.4 (ii) implies that \( \alpha \leq N \), which however contradicts the assumption \( \alpha > N \). Therefore, we see that there is no \( \alpha \in \mathbb{K} \setminus \mathcal{T}(N, \mathbb{K}) \) which satisfies (4.7). This completes the proof.  \( \square \)

**Remark 4.12.** We note that the assumption that \( f^{(2)} \) has no multiple roots when \( d = 2 \) in Theorem 4.11 is equivalent to imposing that the critical point of \( f \) is not a root of \( f^{(2)} \).

We remark that since Faltings’ theorem is ineffective, the result in Theorem 4.11 is also ineffective.

We note that Theorem 4.11 also implies that if we fix a non-preperiodic point \( \alpha \in \mathbb{K} \), then there are only finitely many \( n \geq 1 \) such that \( \alpha \) and \( f^{(n)}(\alpha) \) are multiplicatively dependent. However, such a conclusion can be easily obtained in much greater generality as in the next result.

We also note that if \( \varphi \in \mathbb{K}[X] \) is of degree at least 2 and not special, then by [8, Theorem 2] there are only finitely many preperiodic points \( \alpha \in \mathbb{K^c} \) of \( \varphi \). Thus, we look at multiplicative relations in the orbits of non-preperiodic points.

**Theorem 4.13.** Let \( \varphi \in \mathbb{K}(X) \) be of degree \( d \geq 2 \) and not of the form \( \beta X^d \) with \( \beta \in \mathbb{K}^* \) and let \( \alpha \in \overline{\mathbb{Q}} \setminus \mathbb{U} \) be non-preperiodic for \( \varphi \). Then, there are only finitely many positive integers \( n \) such that \( \alpha \) and \( \varphi^{(n)}(\alpha) \) are multiplicatively dependent.

**Proof.** First, we can extend the field \( \mathbb{K} \) to include also the element \( \alpha \), and thus also all elements \( \varphi^{(n)}(\alpha), n \geq 1 \).

Assume that there is a multiplicative relation, that is, there exists integers \( k_1, k_2 \), not both zero, such that

\[
(4.18) \quad \alpha^{k_1} \varphi^{(n)}(\alpha)^{k_2} = 1
\]

for some \( n \geq 1 \). Since \( \alpha \) is not a root of unity, we have \( k_2 \neq 0 \) in (4.18).

If we denote by \( \Gamma \) the group generated by \( \alpha \) in \( \mathbb{K}^* \), the equation (4.18) is equivalent to \( \varphi^{(n)}(\alpha) \in \overline{\mathbb{Q}} \setminus \mathbb{U} \), where \( \overline{\mathbb{Q}} \) is the so-called division group of \( \Gamma \) defined by

\[
\overline{\Gamma} = \{ a \in \overline{\mathbb{Q}}^* : a^m \in \Gamma \text{ for some positive integer } m \}.
\]
We now apply a version \cite{23}, Corollary 2.3 (which is based on \cite{17}) to conclude that there are finitely many values $\varphi^{(n)}(\alpha)$ satisfying (4.18) when $n$ varies. Note that although \cite{23}, Corollary 2.3 is stated only for polynomials, the same argument works for rational functions satisfying our condition as well, and also under a more relaxed condition. Indeed, let $S$ be a finite set of places of $\mathbb{K}$, including the Archimedean ones, such that $|\alpha|_v = 1$ for any $v \notin S$. So, $|\beta|_v = 1$ for any $\beta \in \Gamma$ and $v \notin S$. Now, for any $\gamma \in \Gamma$, that is $\gamma^m \in \Gamma$ for some positive integer $m$, we also have $|\gamma|_v = 1$ for any $v \notin S$. Thus, $\varphi^{(n)}(\alpha) \in R_S^*$, where $R_S^*$ is the ring of $S$-units in $\mathbb{K}$, and now the finiteness of the corresponding set of values $\varphi^{(n)}(\alpha)$ follows from \cite{17}, Proposition 1.6, (a).

Since $\alpha$ is non-preperiodic, the points in the orbit $\text{Orb}_{\varphi}(\alpha)$ are all distinct. This implies that there are finitely many $n \geq 1$ satisfying (4.18). So, we complete the proof. □

Acknowledgement

The authors are grateful to David Masser for valuable discussions around Lemma 3.9 and to Gabriel Dill for pointing out some errors in an early version of the paper. They also would like to thank the referee for a careful reading and valuable comments.

The first, the third and the fourth authors are also grateful to the Fields Institute for the hospitality and generous support during the Thematic Program on Unlikely Intersections, Heights, and Efficient Congruencing, where some parts of this paper were developed.

During the preparation of this work, A. O. was supported by the UNSW FRG Grant PS43704 and by the ARC Grant DP180100201, M. S. was supported by the Macquarie University Research Fellowship, and I. S. was supported by the ARC Grants DP170100786 and DP180100201.

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