LEFT-DEFINITIVE VARIATIONS OF THE CLASSICAL FOURIER EXPANSION THEOREM, PART II

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Abstract. In 2002, Littlejohn and Wellman developed a general left-definite theory for arbitrary self-adjoint operators in a Hilbert space that are bounded below by a positive constant. Zettl and Littlejohn, in 2005, applied this general theory to the classical second-order Fourier operator with periodic boundary boundary conditions. In this paper, we construct sequences of left-definite Hilbert spaces \( \{H_n\}_{n \in \mathbb{N}} \) and left-definite self-adjoint operators \( \{A_n\}_{n \in \mathbb{N}} \) associated with the Fourier operator with semi-periodic boundary conditions. We obtain explicit formulas for the domain of the square root of the self-adjoint operator \( A \) obtained from this boundary value problem as well as explicit representations of the domains \( D(A^{1/2}) \) for all positive integers \( n \). Furthermore, a Fourier expansion theorem is given in each left-definite space \( H_n \).

1. Introduction

For a self-adjoint operator \( A \) in a Hilbert space \( H \), bounded below in \( H \) by a positive constant, Littlejohn and Wellman [6] construct a continuum of Hilbert spaces \( \{H_r\}_{r>0} \) and a continuum of self-adjoint operators \( \{A_r\}_{r>0} \) from the pair \( (H, A) \). For each \( r > 0 \), \( H_r \) is called the \( r \)th left-definite Hilbert space associated with \( (H, A) \) and \( A_r \) is called the \( r \)th left-definite operator associated with \( (H, A) \). These spaces and operators share many properties that the original operator \( A \) and the Hilbert space \( H \) satisfy. Indeed, the spectrum of each \( A_r \) coincides with the spectrum of \( A \), eigenfunctions of \( A \) are also eigenfunctions of each \( A_r \) and, in particular, a complete orthogonal set of eigenfunctions of \( A \) in \( H \) is also a complete set of eigenfunctions of each \( A_r \) in \( H_r \).

This general theory has been applied to several classical singular second-order differential equations, including the Hermite [3], Legendre [4], Jacobi [5], and Laguerre [6] equations. In these papers, the authors construct sequences - but not the full continua - of left-definite spaces and left-definite operators associated with the special self-adjoint operator \( A \) that has the corresponding classical orthogonal polynomials (of Jacobi, Legendre, Hermite, Legendre, and Laguerre, respectively) as eigenfunctions. In [7], the authors further applied this theory to the classical regular second-order Fourier operator endowed with periodic boundary conditions. In this paper, we extend the methods in [7] to explicitly determine the sequences of left-definite spaces \( \{H_n\}_{n \in \mathbb{N}} \) and left-definite operators \( \{A_n\}_{n \in \mathbb{N}} \) associated with the Fourier operator \( A \) in \( H = L^2[a, b] \) determined by semi-periodic boundary conditions. More specifically, we consider the left-definite analysis for the self-adjoint boundary value problem

\[
\begin{align*}
\ell[y](x) &= -y''(x) + ky(x) = \lambda y(x) \quad (x \in [a, b]) \\
y(a) &= -y(b); \quad y'(a) = -y'(b);
\end{align*}
\]

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here \([a, b]\) is a compact interval of the real line and \(k\) is a fixed, positive constant. As in the periodic case, the semi-periodic boundary value problem is well-studied and important in various contexts in mathematics; indeed, the eigenfunction expansion in the semi-periodic case produces a well known Fourier series expansion for \(f\) in \(L^2[a, b]\). We extend this expansion result to each of the left-definite spaces \(H_n\) associated with this self-adjoint boundary value problem. Furthermore, as a consequence of this analysis, for each positive integer \(n\), we obtain explicit characterizations of the domains \(\mathcal{D}(A^{n/2})\) of \(A^{n/2}\) and of the domains \(\mathcal{D}(A_n)\) of each of the left-definite operators \(A_n\) associated with \((H, A)\).

As noted in the previously mentioned papers, the terminology left-definite is due to Schäfke and Schneider [10] but the origins of left-definite theory can be traced back to the work of Hermann Weyl [11] in the early 1900’s. The interest in left-definite theory originated, at least in part, in the study of classical Sturm-Liouville equations with a weight function that changes sign. There is a vast literature for such left-definite problems; we refer to Zettl’s text [12] Chapters 5 and 15 and the recent text of Brown, Bennewitz, and Weikard [2] for excellent discussions of left-definite theory applied to second-order Sturm-Liouville operators as well as references contained in these texts. We note that when the general left-definite theory developed by Littlejohn and Wellman is applied to (self-adjoint) Sturm-Liouville problems, we must assume that these problems are right-definite; of course, this is not the case for the left-definite theory laid out in [2] or [12]. However, except for [6], the existing literature on left-definite Sturm-Liouville theory, including that in [2] and [12], is limited to the study of the first left-definite setting (in the notation of [6]) and does not discuss a continua of left-definite spaces or operators.

The most general self-adjoint operator \(S\), generated by the Fourier expression \(\ell[\cdot]\) in \(L^2[a, b]\), with real coupled boundary conditions is given by

\[
Sf = \ell[f]
\]

(1.2) \(f \in \mathcal{D}(S) = \left\{ f : [a, b] \to \mathbb{C} \mid f, f' \in AC[a, b]; f'' \in L^2[a, b]; \left( \begin{array}{c} f(a) \\ f'(a) \end{array} \right) = K \left( \begin{array}{c} f(b) \\ f'(b) \end{array} \right) \right\},
\]

where

\[
K = \begin{pmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{pmatrix} \in SL_2(\mathbb{R});
\]

that is, \(\det(K) = 1\). The periodic and semi-periodic boundary conditions are, respectively, obtained from (1.2), by letting \(K = I\) or \(K = -I\), where \(I\) is the \(2 \times 2\) identity matrix. These two special cases are among the few cases where the eigenvalues and eigenfunctions are explicitly computable. We are currently working on the general left-definite theory for the general coupled self-adjoint boundary value problem presented in (1.2). The analysis involved in this general case is proving to be considerably more difficult than that given in this manuscript or [7]. Indeed, for non-diagonal \(K\), additional analytic tools are required to determine the sequences \(\{H_n\}_{n=1}^{\infty}\) and \(\{A_n\}_{n=1}^{\infty}\) of the left-definite spaces and left-definite operators associated with \((L^2[a, b], S)\). More specifically, methods developed in this manuscript (and in [7]) appear to only allow us to determine \(\{H_{2n}\}_{n=1}^{\infty}\) and \(\{A_{2n}\}_{n=1}^{\infty}\) in the general non-diagonal case of \(K\).

The contents of this paper are as follows. In Section 2, we review the left-definite theory developed by Littlejohn and Wellman. Section 3 deals with the semi-periodic self-adjoint operator \(A\) generated from [11] and its properties, including information about its spectrum, its eigenfunctions and the fact that \(A\) is bounded below in \(L^2[a, b]\) by \(kI\), where \(k\) is the constant appearing in the differential expression in (1.1). The left-definite analysis of \(A\) - specifically, the construction of the sequence of left-definite spaces \(\{H_n\}_{n \in \mathbb{N}}\) and left-definite operators \(\{A_n\}_{n \in \mathbb{N}}\) - is developed in Section 4. In addition, we develop a Fourier expansion theorem valid in each left-definite space \(H_n\) in Section 4.
Lastly, in Section 5, some special cases of these left-definite spaces and left-definite operators are discussed.

2. A Review of Left-Definite Theory

Let $V$ denote a vector space (over the complex field $\mathbb{C}$) and suppose that $(\cdot, \cdot)$ is an inner product with norm $\|\cdot\|$ generated from $(\cdot, \cdot)$ such that $H = (V, (\cdot, \cdot))$ is a Hilbert space. Suppose $V_r$ (the subscripts will be made clear shortly) is a linear manifold (vector subspace) of the vector space $V$ and let $(\cdot, \cdot)_r$ and $\|\cdot\|_r$ denote an inner product and its associated norm, respectively, over $V_r$ (quite possibly different from $(\cdot, \cdot)$ and $\|\cdot\|$). We denote the resulting inner product space by $H_r = (V_r, (\cdot, \cdot)_r)$.

Throughout this section, we assume that $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by $kI$ for some $k > 0$; that is,

$$(Ax, x) \geq k(x, x) \quad (x \in \mathcal{D}(A)).$$

It follows that $A^r$, for each $r > 0$, is a self-adjoint operator that is bounded below in $H$ by $krI$.

We now define an $r^{th}$ left-definite space associated with $(H, A)$.

**Definition 2.1.** Let $r > 0$ and suppose $V_r$ is a linear manifold of the Hilbert space $H = (H, (\cdot, \cdot))$ and $(\cdot, \cdot)_r$ is an inner product on $V_r$. Let $H_r = (V_r, (\cdot, \cdot)_r)$. We say that $H_r$ is an $r^{th}$ left-definite space associated with the pair $(H, A)$ if each of the following conditions hold:

1. $H_r$ is a Hilbert space,
2. $\mathcal{D}(A^r)$ is a linear manifold of $V_r$,
3. $\mathcal{D}(A^r)$ is dense in $H_r$,
4. $(x, x)_r \geq kr^r (x, x)$ $(x \in V_r)$, and
5. $(x, y)_r = (A^r x, y)$ $(x \in \mathcal{D}(A^r), y \in V_r)$.

It is not clear, from the definition, if such a self-adjoint operator $A$ generates a left-definite space for a given $r > 0$. However, in [6], the authors prove the following theorem; the Hilbert space spectral theorem plays a critical role in establishing this result.

**Theorem 2.1.** (see [6] Theorems 3.1 and 3.4) Suppose $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by $kI$, for some $k > 0$. For $r > 0$, define $H_r = (V_r, (\cdot, \cdot)_r)$ by

$$(V_r, (\cdot, \cdot)_r) = \mathcal{D}(A^{r/2}),$$

and

$$(x, y)_r = (A^{r/2} x, A^{r/2} y) \quad (x, y \in V_r).$$

Then $H_r$ is a left-definite space associated with the pair $(H, A)$. Moreover, suppose $H'_r := (V'_r, (\cdot, \cdot)'_r)$ is another $r^{th}$ left-definite space associated with the pair $(H, A)$. Then $V_r = V'_r$ and $(x, y)_r = (x, y)'_r$ for all $x, y \in V_r = V'_r$; i.e. $H_r = H'_r$. That is to say, $H_r = (V_r, (\cdot, \cdot)_r)$ is the unique left-definite space associated with $(H, A)$. Moreover,

(a) suppose $A$ is bounded. Then, for each $r > 0$,

i. $V = V_r$;

ii. the inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_r$ are equivalent.

(b) suppose $A$ is unbounded. Then, for each $r, s > 0$,

i. $V_r$ is a proper subspace of $V$;

ii. $V_s$ is a proper subspace of $V_r$ whenever $0 < r < s$;

iii. the inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_r$ are not equivalent for any $r > 0$;

iv. the inner products $(\cdot, \cdot)_r$ and $(\cdot, \cdot)_s$ are not equivalent for any $r, s > 0$, $r \neq s$. 

Theorem 2.1.
Remark 2.1. Although all five conditions in Definition 2.1 are necessary in the proof of Theorem 2.1, the most important property, in a sense, is the one given in (5). Indeed, this property asserts that the \( r \)th left-definite inner product is generated from the \( r \)th power of \( A \). If \( A \) is generated from a Lagrangian symmetric differential expression \( \ell[\cdot] \), we see that the \( r \)th powers of \( A \) are then determined by the \( r \)th powers of \( \ell[\cdot] \). Consequently, in this case, it is possible to explicitly obtain these powers only when \( r \) is a positive integer. We refer the reader to [6] where, however, an example of a self-adjoint operator \( A \) in \( \ell^2(\mathbb{N}) \) is discussed in which the entire continuum of left-definite spaces is explicitly obtained.

Definition 2.2. For \( r > 0 \), let \( H_r = (V_r, (\cdot, \cdot)_r) \) denote the \( r \)th left-definite space associated with \((H, A)\). If there exists a self-adjoint operator \( A_r : \mathcal{D}(A_r) \subset H_r \to H_r \) that is a restriction of \( A \); that is, \[ A_r f = Af \quad (f \in \mathcal{D}(A_r) \subset \mathcal{D}(A)), \] we call such an operator an \( r \)th left-definite operator associated with \((H, A)\).

Again, it is not immediately clear that such an \( A_r \) exists for a given \( r > 0 \); in fact, however, as the next theorem shows, \( A_r \) exists and is unique for each \( r > 0 \).

Theorem 2.2. (see [6, Theorems 3.2 and 3.4]) Suppose \( A \) is a self-adjoint operator in a Hilbert space \( H \) that is bounded below by \( kI \) for some \( k > 0 \). For any \( r > 0 \), let \( H_r = (V_r, (\cdot, \cdot)_r) \) be the \( r \)th left-definite space associated with \((H, A)\). Then there exists a unique left-definite operator \( A_r \) in \( H_r \) associated with \((H, A)\); in fact, \[ \mathcal{D}(A_r) = V_{r+2}. \]

Each \( A_r \) is bounded below in \( H_r \) by \( kI \). Moreover, from Theorem 2.1, we have the following results:

(a) Suppose \( A \) is bounded. Then, for each \( r > 0 \), \( A = A_r \).

(b) Suppose \( A \) is unbounded. Then, for each \( r, s > 0 \),

(i) \( \mathcal{D}(A_r) \) is a proper subspace of \( \mathcal{D}(A) \) for each \( r > 0 \);

(ii) \( \mathcal{D}(A_s) \) is a proper subspace of \( \mathcal{D}(A_r) \) whenever \( 0 < r < s \).

The last theorem that we state in this section shows that the point spectrum, continuous spectrum, and resolvent set of a self-adjoint operator \( A \) and each of its associated left-definite operators \( A_r \) (\( r > 0 \)) are identical.

Theorem 2.3. (see [6, Theorem 3.6]) For each \( r > 0 \), let \( A_r \) denote the \( r \)th left-definite operator associated with the self-adjoint operator \( A \) that is bounded below by \( kI \), where \( k > 0 \). Then

(a) the point spectra of \( A \) and \( A_r \) coincide; i.e. \( \sigma_p(A_r) = \sigma_p(A) \);

(b) the continuous spectra of \( A \) and \( A_r \) coincide; i.e. \( \sigma_c(A_r) = \sigma_c(A) \);

(c) the resolvent sets of \( A \) and \( A_r \) are equal; i.e. \( \rho(A_r) = \rho(A) \).

We refer the reader to [6] for additional theorems, and examples, associated with the general left-definite theory of self-adjoint operators \( A \) that are bounded below; see also [3], [4], [5], [6], and [7].

3. The Fourier Operator \( A \) with semi-periodic boundary conditions

From here on, we let

\[
(3.1) \quad H := L^2[a, b],
\]
for \(-\infty < a < b < \infty\), denote the classical Hilbert space of Lebesgue measurable functions \(f : [a, b] \to \mathbb{C}\) satisfying \(\int_a^b |f(x)|^2 \, dx < \infty\) with inner product

\[
(f, g)_{L^2[a,b]} := \int_a^b f(x)\overline{g}(x) \, dx \quad (f, g \in H),
\]

and associated norm

\[
\|f\|_{L^2[a,b]} = (f, f)_{L^2[a,b]}^{1/2} \quad (f \in H).
\]

Fix \(k > 0\) and let \(\ell[\cdot]\) denote the regular differential (Fourier) expression defined by

\[
\ell[f](x) := -f''(x) + kf(x) \quad (x \in [a, b]).
\]

The operator \(A\) that we deal with in this paper is defined as

\[
\begin{cases}
Af = \ell[f] & (f \in D(A)) \\
D(A) = \{f : [a, b] \to \mathbb{C} \mid f, f' \in AC[a, b]; f'' \in H; f(a) = -f(b); f'(a) = -f'(b)\};
\end{cases}
\]

It is well known (see, for example, [8] or [12]) that \(A\) is self-adjoint in \(H\) and has a discrete spectrum \(\sigma(A)\). A calculation shows that the eigenvalues of \(A\) are given by

\[
\lambda_m := \left( \frac{(2m-1)\pi}{b-a} \right)^2 + k \quad (m \in \mathbb{N}).
\]

For \(m \in \mathbb{N}\), the general solution of \(\ell[f](x) = \lambda_m f(x)\) on \([a, b]\) is

\[
f_m(x) = c_{m,1} \cos \left( \frac{(2m-1)\pi}{b-a} x \right) + c_{m,2} \sin \left( \frac{(2m-1)\pi}{b-a} x \right).
\]

Furthermore, with

\[
\begin{cases}
y_{m,1}(x) = \cos \left( \frac{(2m-1)\pi}{b-a} x \right) & (m \in \mathbb{N}) \\
y_{m,2}(x) = \sin \left( \frac{(2m-1)\pi}{b-a} x \right) & (m \in \mathbb{N}),
\end{cases}
\]

calculations show that

\[
\|y_{m,j}\|_{L^2[a,b]} = \sqrt{\frac{b-a}{2}} \quad (m \in \mathbb{N}; \ j = 1, 2).
\]

Consequently,

\[
E = \{c_m \mid m \in \mathbb{N}\} = \{z_{m,1} \mid m \in \mathbb{N} \cup \{z_{m,2} \mid m \in \mathbb{N},
\]

where

\[
z_{m,1}(x) = \sqrt{\frac{2}{b-a}} \cos \left( \frac{(2m-1)\pi}{b-a} x \right) \quad (m \in \mathbb{N})
\]

\[
z_{m,2}(x) = \sqrt{\frac{2}{b-a}} \sin \left( \frac{(2m-1)\pi}{b-a} x \right) \quad (m \in \mathbb{N})
\]

is a complete orthonormal basis in \(L^2[a,b]\).

For later purposes, we observe that

\[
z_{m,j}^{(r)}(a) = -z_{m,j}^{(r)}(b) \quad (j = 1, 2; \ r = 0, 1, \ldots).
\]

We remind the reader of the classical expansion theorem (see [11] Chapter 4) for functions \(f \in L^2[a, b]\) in terms of the eigenfunctions of \(A\); that is, the classical Fourier series expansion theorem in \(L^2[a, b]\).
Theorem 3.1. Let $f \in L^2[a, b]$; for each $N \in \mathbb{N}$, define the partial sums
\[
s_N(f)(x) = \sum_{m=1}^{N} a_m(f) \cos \left( \frac{(2m-1)\pi}{b-a} x \right) + \sum_{m=1}^{N} b_m(f) \sin \left( \frac{(2m-1)\pi}{b-a} x \right) \quad (x \in [a, b]),
\]
where
\[
a_m(f) := (f, z_{m,1}) = \sqrt{\frac{2}{b-a}} \int_a^b f(x) \cos \left( \frac{(2m-1)\pi}{b-a} x \right) \, dx \quad (m \in \mathbb{N}),
\]
and
\[
b_m(f) := (f, z_{m,2}) = \sqrt{\frac{2}{b-a}} \int_a^b f(x) \sin \left( \frac{(2m-1)\pi}{b-a} x \right) \, dx \quad (m \in \mathbb{N})
\]
are the Fourier coefficients of $f$ corresponding to the orthonormal basis $E$ defined in (3.3). Then
\[
\|f - s_N(f)\|_{L^2[a, b]} \to 0 \quad \text{as} \quad N \to \infty,
\]
and
\[
\|f\|_{L^2[a, b]}^2 = \sum_{m=0}^{\infty} |a_m(f)|^2 + \sum_{m=1}^{\infty} |b_m(f)|^2.
\]
For $f \in \mathcal{D}(A)$, we see from integration by parts and the boundary conditions in (3.3) that
\[
(Af, f)_{L^2[a, b]} = \int_a^b \left[ -f''(x) + kf(x) \right] \overline{f}(x) \, dx
\]
\[
= -f'(x) \overline{f}(x) \bigg|_a^b + \int_a^b \left[ |f'(x)|^2 + k |f(x)|^2 \right] \, dx
\]
\[
= \int_a^b |f'(x)|^2 + k |f(x)|^2 \, dx
\]
\[
\geq k \int_a^b |f(x)|^2 \, dx = k(f, f)_{L^2[a, b]};
\]
that is, $A$ is bounded below by $kI$ in $H$. Consequently, the left-definite theory discussed in the last section can be applied to this operator $A$. This is done in the next section.

4. The Left-Definite Spaces and Operators Associated with $(L^2[a, b], A)$

Let the self-adjoint differential operator $A$ in $H = L^2[a, b]$ be defined by (3.3). In this section we use the theory given in Section 3 to explicitly construct the left-definite spaces $H_n$ and the left-definite operators $A_n$, associated with the pair $(H, A)$, for all positive integer values of $n$. As can be easily established (see (7)), we note for each $n \in \mathbb{N}$ that
\[
\ell^n[y] = \ell[\ell^{n-1}[y]] = \sum_{j=0}^{n} (-1)^j \binom{n}{j} k^{n-j} y^{(2j)}.
\]

Definition 4.1. For $n \in \mathbb{N}$, define
(i) $V_n := \{ f : [a, b] \to \mathbb{C} \mid f^{(j)} \in AC[a, b] \ (j = 0, 1, \ldots, n - 1); f^{(j)}(a) = -f^{(j)}(b) \ (j = 0, 1, \ldots, n - 1); f^{(n)} \in L^2[a, b] \};$
(ii) $(f, g)_n := \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \int_a^b f^{(j)}(x) \overline{g^{(j)}}(x) \, dx \quad (f, g \in V_n);$
(iii) $\|f\|_n := (f, f)_n^{1/2};$
(iv) $H_n := (V_n, (\cdot, \cdot)_n).$
Remark 4.1. We note that \( V_n \) is a vector subspace of \( L^2[a, b] \). Furthermore, since \( k > 0 \), it is clear that \((\cdot, \cdot)_n\) is an inner product on \( V_n \times V_n \).

Remark 4.2. Notice that the inner product \((\cdot, \cdot)_n\) is generated by the \( n^{th} \) integral power \( f^n[\cdot] \) of the differential expression \( \ell[\cdot] \); indeed, see (4.1) and item (5) in Definition 4.1.

Remark 4.3. From (3.10), we see that \( E \), the set of orthonormal eigenfunctions of \( A \) given in (3.7), is contained in \( H_n \) for each \( n \in \mathbb{N} \). In Theorem 4.5 below we show that \( E \) is a complete orthogonal set in each space \( H_n \). Theorem 4.3 shows that \( H_n \) is the \( n^{th} \) left-definite space associated with the pair \( (H, A) \).

Theorem 4.1. For each \( n \in \mathbb{N} \), the space \( H_n \), defined in Definition 4.1, is a Hilbert space.

Proof. Suppose \( \{f_m\} \subset H_n \) is Cauchy so

\[
\|f_m - f_r\|_n^2 = (f_m - f_r, f_m - f_r)_n^2 \\
= \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \int_a^b |f_m^{(n-j)}(t) - f_r^{(n-j)}(t)|^2 dt \\
= \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \|f_m^{(j)} - f_r^{(j)}\|_{L^2[a,b]}^2 \\
\rightarrow 0 \quad \text{as} \ m, r \rightarrow \infty.
\]

For \( 0 \leq j \leq n \), \( \|f_m^{(j)} - f_r^{(j)}\|_{L^2[a,b]}^2 \leq \|f_m - f_r\|_n^2 \), so \( \{f_m^{(j)}\} \) is Cauchy in \( L^2[a, b] \). From the completeness of \( L^2[a, b] \), there exists \( g_j \in L^2[a, b] \) such that

\[
f_m^{(j)} \rightarrow g_{n-j} \text{ in } L^2[a, b] \quad (j = 0, 1, \ldots, n).
\]

In particular,

\[
f_m^{(n)} \rightarrow g_0 \text{ in } L^2[a, b].
\]

By Hölder’s inequality, \( f_m^{(n)} \rightarrow g_0 \) in \( L^1[a, b] \); moreover,

\[
f_m^{(n-1)}(x) - f_m^{(n-1)}(a) = \int_a^x f_m^{(n)}(t) dt \rightarrow \int_a^x g_0(t) dt \quad (x \in [a, b])
\]

and, in particular,

\[
f_m^{(n-1)}(b) - f_m^{(n-1)}(a) = \int_a^b f_m^{(n)}(t) dt \rightarrow \int_a^b g_0(t) dt.
\]

Since \( f_m^{(n-1)}(b) = -f_m^{(n-1)}(a) \), the above identity is rewritten as \(-2f_m^{(n-1)}(a) \rightarrow \int_a^b g_0(t) dt \) so that

\[
f_m^{(n-1)}(a) \rightarrow -\frac{1}{2} \int_a^b g_0(t) dt := A_0.
\]

Combining (4.4) and (4.5), we see that

\[
f_m^{(n-1)}(x) \rightarrow A_0 + \int_a^x g_0(t) dt \quad (x \in [a, b]).
\]

From (4.3), with \( j = n - 1 \), we find

\[
f_m^{(n-1)} \rightarrow g_1 \text{ in } L^2[a, b].
\]
Combining (4.6) and (4.7), we see that

\[ g_1(x) = A_0 + \int_a^x g_0(t)dt \quad (x \in [a,b]). \tag{4.8} \]

Observe that

(i) \( g_1 \in AC[a,b], \)
(ii) \( g' = g_0, \)
(iii) \( g_1(a) = A_0, \)
(iv) \( g_1(b) = A_0 + \int_a^b g_0(t)dt = A_0 - 2A_0 = -A_0 = -g_1(a). \)

It is helpful to consider the next iteration in the construction of the limit function \( f \) of the Cauchy sequence \( \{f_m\} \) in \( H_2. \) Repeating the above analysis, we see that

\[ g_2(x) = A_1 + \int_a^x g_1(t)dt, \tag{4.9} \]

where

\[ A_1 = -\frac{1}{2} \int_a^b g_1(t)dt. \tag{4.10} \]

From (4.9), (4.10), and the above calculations, we see that

(I) \( g_2 \in AC[a,b] \)
(II) \( g'_2 = g_1 \in AC[a,b] \)
(III) \( g''_2 = g'_1 = g_0 \)
(IV) \( g_2(a) = A_1 \)
(V) \( g_2(b) = A_1 + \int_a^b g_1(t)dt = -A_1 = -g_2(a); \)
(VI) \( g'_2(b) = g_1(b) = -g_1(a) = -g'_2(a). \)

Repeating this argument \( i \) times, where \( 1 \leq i \leq n, \) the above procedure shows that

(a) \( g_i(x) = A_{i-1} + \int_a^x g_{i-1}(t)dt \)
(b) \( g_i \in AC[a,b] \)
(c) \( g'_i = g_{i-1}, \ g''_i = g_{i-2}, \ldots, g^{(i)}_i = g_0 \)
(d) \( g_i, g'_{i-1}, \ldots, g^{(i-1)}_i \in AC[a,b] \)
(e) \( g_i(b) = -g_i(a); \ g'_i(b) = -g'_i(a); \ldots, g^{(i)}_i(b) = -g^{(i)}_i(a). \)

Choosing \( i = n \) above, we see that the function \( f := g_n \in H_n. \) Moreover, from (a)-(e) and (4.3), we see that

\[
\|f_m - f\|_n^2 = \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \left\| f^{(j)}_m - f^{(j)} \right\|^2_{L^2[a,b]} \\
= \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \left\| f^{(j)}_m - g_{n-j} \right\|^2_{L^2[a,b]} \\
\rightarrow 0 \text{ as } m \rightarrow \infty.
\]

Thus, \( H_n = (V_n, (\cdot, \cdot)_n) \) is a Hilbert space. \( \square \)

Observe, for \( f \in H_n, \)

\[ \|f\|_n \geq k^n \|f\|_{L^2[a,b]} . \tag{4.11} \]
Recall that, for $j = 1, 2$ and $m \in \mathbb{N}$, the trigonometric functions $z_{m,j}$, defined in (3.8) and (3.9), are orthonormal eigenfunctions in $L^2[a, b]$ associated with the eigenvalue $\lambda_m$ defined in (3.4). From (3.10), (4.1), integration by parts, and the definition of $V_n$, we see that, for $j = 1, 2$ and $f \in H_n$,

\begin{equation}
\chi_n^m(z_{m,j}, f)_{L^2[a,b]} = (A^n z_{m,j}, f)_{L^2[a,b]} = \int_a^b f^n [z_{m,j}](x) \overline{f}(x) \, dx
\end{equation}

\begin{align*}
&= \sum_{r=0}^n (-1)^r \binom{n}{r} k^{n-r} \int_a^b z_{m,j}^{(2r)}(x) \overline{f}(x) \, dx \\
&= \sum_{r=0}^n (-1)^r \binom{n}{r} k^{n-r} \left[ \sum_{s=0}^{r-1} (-1)^s z_{m,j}^{(2r-1-s)}(x) \overline{f}^{(s)}(x) \right]_a^b + (-1)^r \int_a^b z_{m,j}^{(r)}(x) \overline{f}^{(r)}(x) \, dx \\
&= \sum_{r=0}^n \binom{n}{r} k^{n-r} \int_a^b \overline{f}^{(r)}(x) \, dx
\end{align*}

(4.12)

In particular, if we set $f = z_{r,i}$ in (4.12), we see that

\begin{equation}
(z_{m,j}, z_{r,i})_n = \chi_n^m(z_{m,j}, z_{r,i})_{L^2[a,b]} = \lambda_n^m \delta_{m,r} \quad (i, j = 1, 2; m, r \in \mathbb{N}).
\end{equation}

From Theorem 3.1, the orthonormality in $L^2[a, b]$ of the functions $E = \{z_{m,1}\}_{m=1}^\infty \cup \{z_{m,2}\}_{m=1}^\infty$, and (4.13), we obtain the following theorem.

**Theorem 4.2.** For each $n \in \mathbb{N}$, the set

\begin{equation}
E_n := \{Z_{m,n,1}\}_{m \in \mathbb{N}} \cup \{Z_{m,n,2}\}_{m \in \mathbb{N}},
\end{equation}

where

\begin{align*}
Z_{m,n,1}(x) &= \sqrt{\frac{2}{b-a}} \frac{1}{\sqrt{\left( \frac{(2m-1)\pi}{b-a} \right)^2 + k}} \cos \left( \frac{(2m-1)\pi}{b-a} x \right) = \lambda_m^{-n/2} z_{m,1}(x) \\
Z_{m,n,2}(x) &= \sqrt{\frac{2}{b-a}} \frac{1}{\sqrt{\left( \frac{(2m-1)\pi}{b-a} \right)^2 + k}} \sin \left( \frac{(2m-1)\pi}{b-a} x \right) = \lambda_m^{-n/2} z_{m,2}(x)
\end{align*}

(4.15)

forms an orthonormal set in $H_n$.

Later in this section (see Theorem 4.3 part (3)), we prove that $E_n$ is, in fact, a complete orthonormal set in $H_n$ for each $n \in \mathbb{N}$.

For later purposes, we need the following equality involving finite linear combinations of eigenfunctions of $A$ - the so-called trigonometric polynomials. Let $N_1, M_1, N, M \in \mathbb{N}$ with $N_1 \leq N$ and $M_1 \leq M$ and let $\alpha_m, \beta_r, \in \mathbb{C}$ ($m = N_1, \ldots, N; r = M_1, \ldots, M$). Suppose

\begin{equation}
p(x) = \sum_{m=N_1}^N \alpha_m e_m(x), \quad q(x) = \sum_{r=M_1}^M \beta_r e_r(x),
\end{equation}
where each \( e_m \in E \), defined in (3.7). Then \( p, q \in H_n \) for all \( n \in \mathbb{N} \) and, by (4.12) and linearity, we see that
\[
(A^n p, q)_{L^2[a,b]} = \sum_{m=N_1}^{N} \sum_{r=M_1}^{M} \alpha_m \beta_r (A^n e_m, e_r)_{L^2[a,b]}
\]
(4.16)
\[
= \sum_{m=N_1}^{N} \sum_{r=M_1}^{M} \alpha_m \beta_r (e_m, e_r)_n
\]
\[
= (p, q)_n.
\]

We are now in position to prove the following main theorem.

**Theorem 4.3.** For each \( n \in \mathbb{N} \), let
\[
H_n = (V_n, (\cdot, \cdot)_n),
\]
where
\[
V_n := \{ f : [a, b] \to \mathbb{C} \mid f^{(j)} \in AC[a, b], \quad f^{(j)}(a) = -f^{(j)}(b) \quad (j = 0, 1, \ldots, n - 1); f^{(n)} \in L^2[a, b] \},
\]
and
\[
(f, g)_n := \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \int_{a}^{b} f^{(j)}(x)g^{(j)}(x)dx \quad (f, g \in V_n).
\]

Then \( H_n \) is the \( n \)th left-definite space associated with the pair \((H, A)\).

**Proof.** Let \( n \in \mathbb{N} \). We need to establish properties (1)-(5) in Definition 2.1
(i) \( H_n \) is a Hilbert space
This is proved in Theorem 4.1.
(ii) \( \mathcal{D}(A^n) \subset V_n \)
Let \( f \in \mathcal{D}(A^n) \). Since the set \( E = \{ e_m \mid m \in \mathbb{N} \} = \{ z_{m,1} \}_{m=1}^{\infty} \cup \{ z_{m,2} \}_{m=1}^{\infty} \) of eigenfunctions of \( A \) form a complete orthonormal set in \( L^2[a, b] \), we see that
\[
p_j := \sum_{m=0}^{j} c_m e_m \to f \quad \text{as} \quad j \to \infty \quad \text{in} \quad L^2[a, b],
\]
where \( \{ c_m \} \) are the Fourier coefficients of \( f \) in \( L^2[a, b] \), defined by
\[
c_m := \int_{a}^{b} f(t)e_m(t)dt = (f, e_m)_{L^2[a,b]} \quad (m \in \mathbb{N}_0).
\]
Since \( A^n f \in L^2[a, b] \), we also have
\[
\sum_{m=0}^{j} d_m e_m \to A^n f \quad \text{as} \quad j \to \infty \quad \text{in} \quad L^2[a, b],
\]
where
\[
d_m = (A^n f, e_m)_{L^2[a,b]} \quad (m \in \mathbb{N}).
\]
With $\tilde{\lambda}_m$ denoting the eigenvalue of $A$ associated with $e_m$, we see from the self-adjointness of $A$ that
\[ d_m = (A^n f, e_m)_{L^2[a,b]} = (f, A^n e_m)_{L^2[a,b]} = \tilde{\lambda}_m^n (f, e_m)_{L^2[a,b]} = \tilde{\lambda}_m^n e_m. \]
Substituting this identity into (4.21), and using the linearity of $A^n$, we obtain (4.22)
\[ A^n p_j \rightarrow A^n f \text{ as } j \rightarrow \infty \text{ in } L^2[a,b], \]
where $p_j$ is defined in (4.20). From (4.16), (4.20), and (4.22), it follows that
\[
\|p_j - p_r\|_n^2 = (A^n(p_j - p_r), p_j - p_r)_{L^2[a,b]}
\rightarrow 0 \text{ as } j, r \rightarrow \infty;
\]
that is to say, $\{p_j\}_{j \in \mathbb{N}}$ is Cauchy in $H_n$. From the completeness of $H_n$, there exists $g \in V_n \subset L^2[a,b]$ such that
\[ p_j \rightarrow g \text{ in } H_n. \]
Furthermore, (4.11) shows us that
\[
\|p_j - g\|_n^2 \geq k^n \|p_j - g\|^2_{L^2[a,b]},
\]
so (4.23)
\[ p_j \rightarrow g \text{ as } j \rightarrow \infty \text{ in } L^2[a,b]. \]
Comparing (4.20) and (4.23), we see that $f = g \in V_n$; consequently, $D(A^n) \subset V_n$ as required.

(iii) $D(A^n)$ is dense in $H_n$
Since $E$ is contained in $D(A^n)$, it suffices to show that $E$ is a complete orthogonal set in $H_n$. From this, it will follow (see [9] Chapter 4) that the vector subspace $T \subset D(A^n)$ of all trigonometric polynomials (that is, all finite linear combinations of elements from the set $E$ is dense in $H_n$ and, consequently, $D(A^n)$ is dense in $H_n$. To this end, suppose
\[
(e_m, f)_n = 0 \quad (m \in \mathbb{N}_0)
\]
for some $f \in H_n$. From (4.12), we see that
\[ 0 = (e_m, f)_{L^2[a,b]} = (A^n e_m, f)_{L^2[a,b]} = \tilde{\lambda}_m^n (e_m, f)_{L^2[a,b]}, \]
where $\tilde{\lambda}_m > 0$ is the eigenvalue associated with $e_m$. It follows that (4.24)
\[
(e_m, f)_{L^2[a,b]} = 0 \quad (m \in \mathbb{N}_0).
\]
As remarked in Section 3, $E$ is a complete orthonormal set in $L^2[a,b]$; consequently, (4.24) implies that $f = 0$ in $L^2[a,b]$. From this, it is clear that $f = 0$ in $H_n$, thereby completing the proof that $E$ is a complete orthogonal set in $H_n$. Consequently, we see that $E_n$, defined in (4.14), is a complete orthonormal set in $H_n$.

(iv) $(f, f)_n \geq k^n (f, f)_{L^2[a,b]}$ for all $f \in V_n$
This is clear from the definition of $(\cdot, \cdot)_n$:
\[
(f, f)_n = \sum_{j=0}^n \binom{n}{j} k^{n-j} \int_a^b |f^{(j)}(x)|^2 \, dx
\geq k^n \int_a^b |f^{(j)}(x)|^2 \, dx = k^n (f, f)_{L^2[a,b]};
\]
see also (4.11).
(v) \((A^n f, g)_{L^2[a,b]} = (f, g)_n\) for all \(f \in \mathcal{D}(A^n)\) and \(g \in V_n\)

Let \(f \in \mathcal{D}(A^n)\) and \(g \in V_n\). From (4.16), we see that

\[
(A^n p, q)_{L^2[a,b]} = (p, q)_n
\]

for all trigonometric polynomials \(p\) and \(q\) of the form

\[
p = \sum_{m=1}^{N} \alpha_m e_m, \quad q = \sum_{m=1}^{M} \beta_m e_m.
\]

From part (iii) of this proof, we know that the space \(T\) of all trigonometric polynomials is dense in \(H_n\). Hence there exists \(\{p_j\}_{j \in \mathbb{N}}, \{q_j\}_{j \in \mathbb{N}} \subset T\) such that

\[
p_j \to f, \quad q_j \to g \quad \text{as} \quad j \to \infty \quad \text{in} \quad H_n.
\]

Since convergence in \(H_n\) implies convergence in \(L^2[a,b]\) (from part (iii)), we see that

\[
p_j \to f, \quad q_j \to g \quad \text{as} \quad j \to \infty \quad \text{in} \quad L^2[a,b].
\]

Moreover, from part (ii) of this proof, we see that

\[
A^n p_j \to A^n f \quad \text{as} \quad j \to \infty \quad \text{in} \quad L^2[a,b].
\]

Consequently, from (4.25), (4.26), (4.27), and (4.28), we see that

\[
(A^n f, g)_{L^2[a,b]} = \lim_{j \to \infty} (A^n p_j, q_j)_{L^2[a,b]} = \lim_{j \to \infty} (p_j, q_j)_n = (f, g)_n.
\]

This completes the proof of (v) and the proof of the theorem. \(\square\)

The following result, part of which is proved in step (iii) of the above theorem, is the analogous result in each left-definite space \(H_n\) of the classical Fourier expansion theorem in \(L^2[a,b]\) stated in Theorem 3.1. Note the identities in (4.31) and (4.32); these formulae relate the Fourier coefficients of \(f\) relative to the orthonormal basis \(E_n\) of \(H_n \subset L^2[a,b]\) to the Fourier coefficients of \(f\) relative to the orthonormal basis \(E\) of \(L^2[a,b]\).

**Theorem 4.4. (Fourier Expansion Theorem in Left-Definite Spaces)** For each \(n \in \mathbb{N}\), let

\[
E_n = \{Z_{m,n,1}\}_{m \in \mathbb{N}} \cup \{Z_{m,n,2}\}_{m \in \mathbb{N}}
\]

be as in (4.14) and (4.15). Then \(E_n\) is a complete orthonormal set in \(H_n\). Furthermore, for \(f \in H_n \subset L^2[a,b]\) and \(N \in \mathbb{N}\), define the partial sums

\[
S_{N,n}(f)(x) = \sum_{m=1}^{N} A_{m,n}(f) \cos \left( \frac{2m-1}{b-a} \pi x \right) + \sum_{m=1}^{N} B_{m,n}(f) \sin \left( \frac{2m-1}{b-a} \pi x \right) \quad (x \in [a,b]),
\]

where \(\{A_{m,n}(f)\}_{m \in \mathbb{N}}\) and \(\{B_{m,n}(f)\}_{m \in \mathbb{N}}\) are the Fourier coefficients of \(f\) relative to \(E_n\) defined by

\[
A_{m,n}(f) := (f, Z_{m,n,1})_n \quad (m \in \mathbb{N})
\]

and

\[
B_{m,n}(f) := (f, Z_{m,n,2})_n \quad (m \in \mathbb{N}).
\]

Then

(a) \(\|f - S_{N,n}(f)\|_n \to 0\) as \(N \to \infty\);

(b) \(\|f\|_n^2 = \sum_{m=0}^{\infty} |A_{m,n}(f)|^2 + \sum_{m=1}^{\infty} |B_{m,n}(f)|^2\).
\[(c)\]
\[A_{m,n}(f) = \lambda_m^{n/2} a_m(f) \quad (m \in \mathbb{N})\]
\[B_{m,n}(f) = \lambda_m^{n/2} b_m(f) \quad (m \in \mathbb{N}),\]

where \(\{a_m(f)\}_{m \in \mathbb{N}}\) and \(\{b_m(f)\}_{m \in \mathbb{N}}\) are the Fourier coefficients of \(f\), defined respectively in (3.11), and (3.12), relative to the orthonormal basis \(E\), given in (3.4), in \(L^2[a,b]\) and where \(\{\lambda_m\}_{m \in \mathbb{N}}\) are the eigenvalues of \(A\) defined in (3.4).

**Proof.** The fact that \(E_n\) is a complete orthonormal set in \(H_n\) is given in the proof of part (iii) in Theorem 4.3. Let \(f \in H_n\). The proofs of parts (a) and (b) are standard for any complete orthonormal set in a Hilbert space; see [9, Theorem 4.18]. For \(m \in \mathbb{N}\), we see from (4.13) that
\[(4.33)\]
\[A_{m,n}(f) = (f, Z_{m,n,1})_n = (f, \lambda_m^{-n/2} z_{m,1})_n = \lambda_m^{-n/2} (f, z_{m,1})_n = \lambda_m^{-n/2} (f, A^n z_{m,1})_{L^2[a,b]} = \lambda_m^{-n/2} \lambda_n^2 (f, z_{m,1})_{L^2[a,b]} = \lambda_m^{n/2} (f, z_{m,1})_{L^2[a,b]} = \lambda_m^{n/2} a_m(f);\]

this proves (4.31). A similar calculation establishes (4.32). \(\square\)

By combining Theorem 4.3 with Theorems 2.2 and 2.3, we obtain the following result concerning the sequence of left-definite operators \(\{A_n\}_{n \in \mathbb{N}}\) associated with the pair \((H, A)\).

**Theorem 4.5.** Let \(n \in \mathbb{N}\) and let \(H_n = (V_n, \langle \cdot, \cdot \rangle_n)\) be the \(n\)th left-definite operator associated with the pair \((L^2[a,b], A)\). Define the operator \(A_n : \mathcal{D}(A_n) \subset H_n \to H_n\) by
\[
\mathcal{D}(A_n) := V_{n+2} = \{ f : [a,b] \to \mathbb{C} \mid f^{(j)} \in AC[a,b], f^{(j)}(a) = -f^{(j)}(b) \ (j = 0, 1, \ldots, n+1); f^{(n+2)} \in L^2[a,b] \}.
\]
\[(A_nf)(x) := \ell[f](x) = -y^n(x) + ky(x) \quad (f \in \mathcal{D}(A_n)).\]

Then \(A_n\) is the \(n\)th left-definite operator associated with the pair \((H, A)\). In particular, \(A_n\) is self-adjoint in \(H_n\) and the spectrum \(\sigma(A_n)\) is a purely discrete point spectrum given explicitly by
\[
\sigma(A_n) = \sigma(A) = \left\{ \left( \frac{2m-1}{b-a} \right)^2 + k \mid m \in \mathbb{N} \right\}.
\]

5. **Concluding Remarks**

In this last section, we focus on some special left-definite spaces and operators for the semi-periodic operator \(A\).

**Remark 5.1.** For an arbitrary self-adjoint operator \(A\) in a Hilbert space that is bounded below by a positive constant we see, from Theorem 2.4, that the domain \(\mathcal{D}(A^{1/2})\) of its positive square root \(A^{1/2}\) is given by the first left-definite vector space \(V_1\). For our specific operator \(A\), defined in (3.3), we have the explicit characterization of this domain:
\[(5.1) \quad \mathcal{D}(A^{1/2}) = \{ f : [a,b] \to \mathbb{C} \mid f \in AC[a,b]; f(a) = -f(b); f' \in L^2[a,b] \}.\]
Remark 5.2. From Theorem 4.3, the domain of the first left-definite operator $A_1$, which is a self-adjoint operator in the first left-definite space $H_1$, is given by

$$\mathcal{D}(A_1) = V_3 = \{ f : [a, b] \to \mathbb{C} \mid f^{(j)}(a) \in AC[a, b] \text{ and } f^{(j)}(a) = -f^{(j)}(b) \ (j = 0, 1, 2); \ f^{(3)} \in L^2[a, b] \}. $$

Notice that $\mathcal{D}(A_1)$ is also the domain of $A^{3/2}$. The domain of the second left-definite operator $A_2$, which is self-adjoint in the Hilbert space $H_2 = (V_2, \langle \cdot, \cdot \rangle_2)$, where

$$V_2 = \{ f : [a, b] \to \mathbb{C} \mid f^{(j)}(a) \in AC[a, b] \text{ and } f^{(j)}(a) = -f^{(j)}(b) \ (j = 0, 1); \ f'' \in L^2[a, b] \}
\begin{align*}
(f, g)_2 &= \int_a^b \left( f''(x) \overline{g''(x)} + 2kf'(x) \overline{g'(x)} + k^2 f(x) \overline{g(x)} \right) \, dx
\end{align*}
$$

is given by

$$\mathcal{D}(A_2) = V_4 = \{ f : [a, b] \to \mathbb{C} \mid f^{(j)}(a) \in AC[a, b] \text{ and } f^{(j)}(a) = -f^{(j)}(b) \ (j = 0, 1, 2, 3); \ f^{(4)} \in L^2[a, b] \}. $$

Observe that $V_2 = \mathcal{D}(A)$.

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