A NOTE ON COMMUTING REFLECTION FUNCTORS FOR CALABI-YAU D-FOLDS

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Abstract. We study sets of commuting reflection functors in the derived category of sheaves on Calabi-Yau varieties. We show that such a collection is determined by a set of mutually orthogonal spherical objects. We also show that when the spherical objects are locally-free sheaves then the kernel of the composite transform parametrizes properly torsion-free with zero-dimensional singularity sets and conversely that such a kernel gives rise to a collection of mutually orthogonal spherical vector bundles. We do this using a more detailed analysis of the reason why spherical twists give equivalences.

Introduction

A spherical object $e$ in an exact linear triangulated category $(T, [-])$ is one such that $\dim \text{Hom}(e, e[i]) = \dim H^i(S^d, \mathbb{R})$, the Betti numbers of a $d$-dimensional sphere for some fixed $d$. This concept is especially useful when $T = D^b(X)$ for some $d$-dimensional Calabi-Yau variety $X$ or when $T$ is a $d$-Calabi-Yau category. This is because such objects have, in a suitable sense, the fewest possible derived self-maps. There has been a great deal of interest in them in recent years as they hold the key to understanding the categorical structure of $T$ and its automorphism group $\text{Aut}(T)$. For example, it is conjectured that they give rise to a generating set for $\text{Aut}(T)$ in the case when $T = D(X)$, of a K3 surface. The spherical objects also play a central role in our understanding of Bridgeland stability conditions for some surfaces (see [3]) essentially because of the central role they play in the derived category of the surface. It is likely that they will play a similarly crucial role in our understanding of stability conditions for higher dimensional Calabi-Yau varieties.

In an important paper, [7], Seidel and Thomas show that certain series of spherical objects give rise to actions of the braid group on the derived category. This is done by associating an equivalence $\Phi_a$ of the derived category to each spherical object $a$ (known as a spherical twist). In the $K$-theory of the derived category, these are reflections and so they are sometimes called reflection functors. It was observed in that paper that when two spherical objects $a$ and $b$ are completely orthogonal (in other words, $\text{Hom}(a, b[i]) = 0$ for all integers $i$) then the associated spherical twists commute. This is because $\Phi_a \circ \Phi_b \cong \Phi_{\Phi_b(a)} \circ \Phi_b$ (see [7, Lemma 2.11]) and $\Phi_b(a) \cong a$ as can be checked by direct and easy computation (see [7, Proposition 2.12]). Our first result in this note is to show that the converse also holds: if two spherical twists commute then either they are equal or the associated spherical objects are (completely) orthogonal.

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We then turn our attention to the special case where the spherical objects are actually vector bundles. This is an important class of examples. The associated spherical twists have Fourier-Mukai kernel given by a sheaf parametrizing properly-torsion free sheaves whose singularity set is a single point of $X$. Our second main result is to show that this also has a converse: if $\Phi$ is an exact equivalence of the derived categories of Calabi-Yau $d$-folds such that its Fourier-Mukai kernel is a sheaf parametrizing properly-torsion free sheaves whose singularity set is zero dimensional then it must be a composite of commuting spherical twists. The difficulty in this is to show that the double dual of the kernel (which must be locally-free by assumption) can be reduced essentially to a sum of (completely) orthogonal spherical bundles. To establish this we need to generalise the computations of Ext groups given by Mukai ([5]) and which are so crucial in describing stability conditions for surfaces.

1. Fourier-Mukai Transforms

In this paper we shall take a Fourier-Mukai transform (or FM transform for short) to be an equivalence of categories of the (bounded) derived category of sheaves $D(X)$ and $D(Y)$ on a smooth (complex) projective varieties $X$ and $Y$ given by correspondences of the form $\Phi_F : E \mapsto R\gamma_s(x^*E \otimes F)$, where $F$ is a sheaf on $X \times Y$ called the kernel of the transform. These are discussed in [4] and [1].

Recall that we say that a family of sheaves $M$ is strongly simple if it consists of simple sheaves and if $\text{Ext}^i(E, E') = 0$ for all $i$ and $E \neq E'$ in $M$. (see [2]):

**Theorem 1.1** (Bridgeland). The kernel $F$ gives rise to a FM transform if and only if the restrictions $F$ to $X$ form a strongly simple family and $F_x \otimes K_X \cong F_x$ for all $F_x$ in the family, where $K_X$ is the canonical bundles of $X$.

The last condition is vacuous for Calabi-Yau $d$-folds. The theorem gives us an easy way to recognise when a family of sheaves gives rise to an FM transform.

We aim to study a special class of Fourier-Mukai Transforms which arise from so called spherical bundles. These were first studied by Mukai ([5]) in the case where $X$ is a K3 surface.

Notation: we let $E^\vee = R\text{Hom}(E, \mathcal{O}_X)$ denote the derived dual of an object $E$ of $D(X)$.

2. Commuting Spherical Objects

Throughout this section we assume that $X$ is a smooth Calabi-Yau variety of dimension $d$.

**Definition 2.1.** An object $E$ of $D(X)$ is exceptional if $\text{Ext}^i(E, E)$ is a small as possible (the precise definition depends on $X$ but we will not need to be very definite in what follows). We say that $E$ is spherical if $\text{dim Ext}^i(E, E) = 1$ for $i = 1$ or $i = d$ and is zero otherwise. We say that $E$ is rigid if just $\text{Ext}^1(E, E) = 0$.

Note that a simple rigid sheaf on a Calabi-Yau 2 or 3-fold is automatically exceptional and spherical by Serre duality. To any vector bundle $E$ we can associate a canonical (surjective) map $E \otimes \text{Hom}(E, \mathcal{O}_x) \to \mathcal{O}_x$ given by evaluation. We shall denote the domain
of such maps by $E_H$ for short and the kernel by $E_x$. This extends to a map for any object $E$ of $D(X)$. We shall denote a choice of cone on such a map by $E_x$. Then when $E$ is a bundle, $E_x = F_x[1]$.

In a groundbreaking paper by Seidel and Thomas [7] it is shown (in somewhat greater generality) that when $E$ is a spherical object in $D(X)$, the family of $F_x$ give rise to a Fourier-Mukai transform $D(X) \to D(X)$, denoted $\Phi_E$ or, more usually, $T_E$ (the spherical twist associated to $E$). The kernel of the transform is given by the shift by $-1$ of the cone on the canonical map $R\text{Hom}(\pi_1^* E, \pi_2^* E) \to O_\Delta$ given by adjunction from the composite map

$$\pi_2^* E \xrightarrow{\pi_2^* E \otimes \rho} \pi_2^* E \otimes O_\Delta \sim \pi_2^* E \otimes O_\Delta$$

where $\pi_i : X \times X \to X$ are the two projection maps and $\rho : O_{X \times X} \to O_\Delta$ is the canonical restriction map. We shall denote the functor $R\text{Hom}(\pi_1^* (E \otimes -), \pi_2^* E)$ by $\Psi_E$. So for all $G \in D(X)$ we have a triangle

$$\Phi_E(G) \to \Psi_E(G) \to G$$

which is natural in $G$ (rather unusually for triangles of functors). Their proof that these do give Fourier-Mukai transforms is fairly direct although a somewhat more elegant proof was later given by Ploog ([8]) using a clever choice of spanning class (see [4] for further details). In this paper, we shall give yet another less elegant but more elementary proof in the spirit of Mukai’s original paper ([5]).

The main point of the [7] paper was to show that certain families of spherical objects give rise to a representation of the Braid group on the derived category. As a corollary of the key computational result they also show that if $E$ and $F$ are two spherical objects such that $\text{Hom}(E, F[i]) = 0$ for all $i$ then their FM transforms commute. We can generalise this a little as follows.

**Definition 2.2.** We call a finite collection $E_i, 1 \leq i \leq n$ of objects of $D(X)$ strongly spherical if

$$\dim \text{Hom}(E_i, E_j[k]) = \begin{cases} 1 & \text{if } i = j \text{ and } (k = 0 \text{ or } k = d) \\ 0 & \text{otherwise} \end{cases}$$

In other words, each of the numbers $\dim \text{Hom}(E_i, E_j[k])$ are as small as possible.

Then for a strongly spherical collection $\Gamma = \{E_i\}_{i=1}^n$ we have a finite cone (in the sense of limits) $E_i \boxtimes E_i' \to O_\Delta$. This has a limit (up to shift) constructed explicitly as the cone on $\bigoplus_{i=1}^n E_i \boxtimes E_i' \to O_\Delta$. Denote the limit by $E_{1,2,...,n}$ and its associated integral transform by $\Phi_\Gamma$. Then the following is an easy exercise

**Proposition 2.3 ([7]).** For any strongly spherical collection $\Gamma$ of objects on a Calabi-Yau $d$-fold, $\Phi_\Gamma = \Phi_{E_1} \circ \Phi_{E_2} \circ \cdots \circ \Phi_{E_n}$

In fact, there is a converse:
Theorem 2.4. Suppose $E$ and $F$ are two spherical objects in $D(X)$ such that $\Phi_E$ and $\Phi_F$ are distinct. Then $\Phi_E \circ \Phi_F \cong \Phi_F \circ \Phi_E$ implies that $F \in E^\perp$.

Before proving this we prove a technical lemma first proposed by David Ploog in his thesis ([6 Question 1.23]). We let $\langle E \rangle$ denote the smallest triangulated category containing $E$ in $D(X)$. This means that each object has a filtration whose factors are all shifts of isomorphic copies of $E$.

Lemma 2.5. Suppose $E$ is a spherical object of $D(X)$ and $d = \dim X \geq 2$. Then, for any object $G \in D(X)$, $\Phi_E(G) = G[-d]$ if and only if $G \in \langle E \rangle$.

Proof. Recall that $G \in E^\perp$ if and only if $\Phi_E(G) = G$ (see [6, Lemma 1.22]). The reverse implication of our lemma was also proved in [6, Lemma 1.22]. So suppose $\Phi_E(G) = G[-d]$. Define

$$d_E(G) = \sum_{i=-\infty}^{\infty} \dim \text{Hom}(E, G[i]).$$

We induct on $d_E(G)$. If $d_E(G) = 0$ then $G \in E^\perp$ and so $\Phi_E(G) = G$ and hence $G = 0$. If $d_E(G) = 1$ (wlog $\text{Hom}(E, G) \neq 0$) then $G[-d]$ fits in a triangle

$$G[-d] \xrightarrow{f} E \xrightarrow{f} G,$$

where the unique maps (up to scale) are Serre dual to each other. But then $f \circ f^\vee : G[-d] \to G$ must be Serre dual to the identity $G \to G$ and so cannot vanish. But $f \circ f^\vee = 0$ as the composite of two consecutive maps of a triangle must always vanish. The contradiction shows that $d_E(G)$ cannot equal 1. Now assume that for all $n < d_E(G)$ we know that if $d_E(G') = n$ and $\Phi_E(G') = G'[−d]$ then $G' \in \langle E \rangle$. Pick any $f \in \bigoplus \text{Hom}(E, G[i])$ and again without loss of generality assume $i = 0$. Let $C$ be a cone on $f : E \to G$. Then $\Phi_E(C) = C[−d]$ because $\Phi_E(f) = f[−d]$. But we also have that $d_E(C) = d_E(G) − 2$ by applying $\text{Hom}(E, −)$ to the triangle defining $C$ and because $\dim X > 1$. Then by induction $d_E(G)$ must be even and $C \in \langle E \rangle$. Hence, $G \in \langle E \rangle$ as it is an extension of $C$ by $E$. □

Remark 2.6. We can extract a bit more from the proof by observing that it shows that if $G \in \langle E \rangle$ has $d_E(G) = 2$ then $G \cong E[i]$ for some integer $i$. In fact, we can go further to observe that $d_E(G)/2$ is the length of a filtration of $G \in \langle E \rangle$ with factors given by shifts of $E$ (always under the assumption that $d > 1$). It follows that the length of such a filtration is well defined as a function of $G$.

We shall use this in the following way: if $F \in \langle E \rangle$ is spherical then applying $F[i] \to$ to the triangle $F[-d] \to \Psi_E(F) \to F$ implies that $d_E(F) = 2$ and so $F \cong E[i]$ for some integer $i$.

Lemma 2.7. Suppose $E$ and $F$ are two spherical objects such that $\Phi_E$ and $\Phi_F$ commute. Then $G \in \langle E \rangle$ if and only if $\Phi_F(G) \in \langle E \rangle$.

Proof. For any $G \in \langle E \rangle$ we have

$$\Phi_E(\Phi_F(G)) \cong \Phi_F(\Phi_E(G)) \cong \Phi_F(G[−d]).$$
So $\Phi_F(G) \in \langle E \rangle$ by Lemma 2.5. Applying this to $G = \Phi_F^{-1}(G')$ gives us the converse as well. □

Proof of Theorem 2.4. Assume that that $\Phi_E$ and $\Phi_F$ commute and suppose that $E$ and $F$ are not orthogonal. Then $\Phi_E(F) \in \langle F' \rangle$ by Lemma 2.7. But $\Phi_E(F)$ is spherical and so by the remark above, $\Phi_E(F) = F[i]$ for some $i$. By assumption, we have a non-zero map $E \to F$ (replacing $F$ by a suitable shift if necessary). Applying the composite functor $\Phi_E^n[i + d]$, for any positive integer $n$ to this gives a non-zero map $E \to F[n(i + d)]$. But $D(X)$ has bounded cohomology and exts and so $i = -d$.

So $\Phi_E(F) \in \langle E \rangle$ by Lemma 2.5 again. Then $F \in \langle E \rangle$ by Lemma 2.7. By the remark, $F = E[i]$ for some $i$ and that implies that $\Phi_E = \Phi_F$ contradicting our assumption. □

3. Spherical Bundles

We shall now restrict our attention to the case of spherical bundles on complex Calabi-Yau $d$-folds. We shall see that this case can be tackled more directly in the spirit of Mukai’s paper.

We first assume that $E$ is a simple rigid bundle and consider the double exact complex associated to the bi-functor $\text{Ext}^*(-, -)$ applied to the short exact sequence

$$0 \to E_x \to E_H \to O_x \to 0.$$ 

Using the fact that $\text{Ext}^i(E_H, O_x) = 0$ for all $i > 0$ and $\text{Ext}^i(O_x, E_H)$ vanishes for all $i < d$, we have $\dim \text{Ext}^1(O_x, F) = 1$, $\dim \text{Ext}^1(F, O_x) = \text{rk}(E)^2 - 1 + d$, $\dim \text{Hom}(F, E_H) = \dim \text{Hom}(E_H, E_H) = \text{rk}(E)^2$ and, crucially, $\text{Ext}^1(F, E_H) = 0$ (using the fact that $d > 2$ for this: the case $d = 2$ is much simpler and is left to the reader). From this we have

$$\dim \text{Ext}^1(F, F) = d - 1 + \dim \text{Hom}(F, F)$$

Since $\text{Ext}^1(E_H, O_x) = 0$, we have that the map

$$\text{Ext}^2(O_x, F) \to \text{Ext}^2(E, F)$$

vanishes and so $\text{Ext}^2(F, F) \to \text{Ext}^2(O_x, F)$ surjects. The map

$$\text{Hom}(F, F) \to \text{Ext}^1(O_x, F) \cong \mathbb{C}$$

is the boundary map and must be non-zero as the identity map is contained in the domain. Hence, this map also surjects. We can conclude

$$\dim \text{Hom}(F, F) = \dim \text{Hom}(E_H, F) + 1$$

The following result is a stronger version of [5], Prop 3.9.

Lemma 3.1. The map $\text{Hom}(E_H, E_H) \to \text{Hom}(E_H, O_x)$ injects

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1 The reader is urged to write a large part of this double complex out on a large piece of paper before proceeding!
Proof. Consider a map \( f : E_H \to E_H \). If we fix a basis for \( \text{Hom}(E, \mathcal{O}_x) \), then \( f \) is given by an \( r \times r \) matrix with scalar entries (since \( E \) is simple). The image of \( f \) is given by a subspace \( V \) of \( \text{Hom}(E, \mathcal{O}_x) \) and \( f \) is zero if and only if this subspace is zero. But if it is not zero then the image of \( E \otimes V \) in \( \mathcal{O}_x \) is non-zero and so the image of \( f \) in \( \text{Hom}(E_H, \mathcal{O}_x) \) is also non-zero.

We deduce that \( \text{Hom}(E_H, F) = 0 \) and hence \( \dim \text{Hom}(F, F) = 1 \). Now we can conclude that \( \dim \text{Ext}^1(F, F) = d \).

Next we consider two distinct points \( x \) and \( y \) of \( X \) and the two associated kernels \( F_x \) and \( F_y \). Since \( \text{Ext}^i(\mathcal{O}_x, \mathcal{O}_y) = 0 \) for all \( i \) and \( F_y \) is locally-free away from \( x \) we can conclude from the double exact sequence associated to the two sequences for \( F_x \) and \( F_y \), that \( \text{Hom}(F_x, F_y) \cong \text{Hom}(E_H, F_y) = 0 \) and \( \text{Ext}^1(F_x, F_y) \cong \text{Ext}^1(E_H, F_y) \) which is also zero.

The following generalises Corollary 2.12 of [5].

**Proposition 3.2.** If \( E \) is a simple rigid vector bundle and \( d > 3 \) then there are natural isomorphisms

\[
\text{Ext}^i(F_x, F_y) \cong \text{Ext}^i(\mathcal{O}_x, \mathcal{O}_y) \oplus \text{Ext}^i(E_H, E_H)
\]

for all \( x, y \in X \) (not necessarily distinct) and \( 1 < i < d - 1 \).

**Proof.** The proof uses the double exact sequence we considered above. Start at \( i = 2 \) and observe that \( \text{Ext}^n(E_H, F_y) \cong \text{Ext}^n(E_H, E_H) \) for \( 1 \leq n < d \) (the case \( n = 1 \) follows because \( E \) is rigid) and there is a natural injection of \( \text{Ext}^n(E_H, E_H) \) into \( \text{Ext}^n(F_x, E_H) \). We also have \( \text{Ext}^n(F_x, \mathcal{O}_y) \cong \text{Ext}^{n+1}(\mathcal{O}_x, \mathcal{O}_y) \) and so the map \( g : \text{Ext}^n(F_x, E_H) \to \text{Ext}^n(F_x, \mathcal{O}_y) \) is given by the composite

\[
\text{Ext}^n(F_x, E_H) \to \text{Ext}^{n+1}(\mathcal{O}_x, E_H) \to \text{Ext}^{n+1}(\mathcal{O}_x, \mathcal{O}_y) \to \text{Ext}^n(F_x, E_H),
\]

But \( \text{Ext}^{n+1}(\mathcal{O}_x, E_H) = 0 \) and so the composite vanishes for \( n = 1, \ldots, d - 1 \). Moreover, the surjection \( \text{Ext}^n(F_x, F_y) \to \text{Ext}^n(F_x, E_H) \) splits naturally since the image is

\[
\text{Ext}^n(E_H, E_H) \cong \text{Ext}^n(E_H, F_y)
\]

and the image of this in \( \text{Ext}^n(F_x, F_y) \to \text{Ext}^n(E_H, E_H) \) is the identity. \( \square \)

This shows that \( \{ F_y \} \) is a strongly simple family. Using Theorem 2.1, we have an alternative proof of

**Theorem 3.3 ([7], [6]).** If \( E \) is a spherical bundle on a Calabi-Yau \( d \)-fold \( X \) then the moduli space of sheaves \( \{ F_x \} \) constructed above is naturally isomorphic to \( X \) and gives rise to a non-trivial Fourier-Mukai transform \( D(X) \to D(X) \).

4. Recovering the Strongly Spherical Collection

We shall now consider the reverse process: given a Fourier-Mukai transform determined by a family of non-locally-free torsion-free sheaves \( \{ F_y \} \) with dimension 0 singularity sets, can we find a strongly spherical collection of bundles \( \Gamma = \{ E_i \}_{i=0}^n \) such that \( F_x \) is the kernel of the canonical map \( \bigoplus_{i=0}^n E_i \otimes \text{Hom}(E_i, \mathcal{O}_x) \to \mathcal{O}_x \)? We shall see that this is indeed possible. The first observation we need to make is that the parameter space \( \{ F_y \} \) is
naturally (isomorphic to) $X$. This is immediate since the map $F_y \to F_y^{**}$ has quotient $\mathcal{O}_T$ and we see that the parameter space $Y$ sits inside a space of kernels $F_y^{**} \to \mathcal{O}_T$ as $T$ varies in $\text{Hilb}^{[T]}(X)$. Since the moduli space must be complete we see that the map $Y \to X$ given by the singularity of $F_y$ is an isomorphism. We also see that $F_y^{**} = F_y^{**}$ for any pair $y$ and $y'$. We shall write $F$ for $F_y^{**}$. Since $F$ is locally-free away from $x$ and from $y$ we see that $F$ is locally-free over the whole of $X$. Without loss of generality we assume in what follows that the isomorphism $Y \cong X$ is the identity.

Using the double exact sequence from the previous section we can immediately conclude that $\dim \text{Hom}(F, F) = \text{rk}(F)$ and $\text{Hom}(F, F) \cong \text{Hom}(F, \mathcal{O}_x)$, for any $x \in X$. We can also conclude that $\text{Ext}^i(F, F) = 0$ for $i = 1, \ldots, d-1$. If $\text{rk}(F) = 1$ then $F$ must be exceptional.

Assume now that $\text{rk}(F) > 1$. We observe also that the kernels of a suitable family of maps $\lambda_x : F \to \mathcal{O}_x$, as $x$ varies, generate the family $\{F_x\}$. Since $\dim \text{Hom}(F, F) > 1$ we can find an endomorphism of $F$ which has rank less than $r$ and so we have a sheaf $P$ which factors such an endomorphism. We can assume $P$ is reflexive by factoring the torsion out of $F/P = Q$, say. We now consider the double exact sequences associated to pairs of short exact sequences taken from

$$
0 \to F_x \to F \to \mathcal{O}_x \to 0,
$$

$$
0 \to P \to F \to Q \to 0
$$

and

$$
0 \to K \to F \to P \to 0.
$$

From these it follows that $\text{Hom}(P, F_x) = 0$ and $\text{Hom}(Q, F_x) = 0$. It follows from this that $\text{Ext}^1(Q, F_x) = 0$ and, crucially, $\text{Hom}(Q, F) = \text{Hom}(Q, \mathcal{O}_x)$ and $\text{Hom}(P, F) = \text{Hom}(P, \mathcal{O}_x)$. These imply that both $P$ and $Q$ are locally-free.

We now appeal to the following useful technical result (true in much greater generality for suitable objects in any noetherian abelian category).

**Lemma 4.1.** If $E$ is a torsion-free sheaf which is not simple then there exists a simple sheaf $G$ (not necessarily unique) and an injection $\alpha : G \to E$ and a surjection $\beta : E \to G$ such that either $\beta \alpha$ is zero or the identity. Moreover, if $G \to E$ is any non-zero map then it must inject.

**Proof.** Since $E$ is not simple, we can consider the set of sheaves $G$ which factor non-isomorphisms $E \to E$. Such a sheaf $G$ is automatically torsion-free and gives rise to maps $\alpha$ and $\beta$. The set is partially ordered by compositions $E \to G \to G' \to E$. Since $r(G') < r(G)$ (otherwise the kernel of $G \to G'$ would be a torsion sheaf), we can pick (using Zorn’s Lemma) a minimal element with respect to this order. Call it $G$. Then $G$ is simple since otherwise we could factor a map $G \to G$ via $G'$ which would be strictly smaller than $G$ in the order. Now the composite $\beta \alpha$ is either zero or a multiple of the identity (in which case we replace $\beta$ with a suitable multiple).

The last statement follows because if such a map is not injective then the image would be strictly smaller in the order. \qed

7
Applying this to our current situation we may assume $P$ is simple and is minimal with respect to the ordering of the proof above. Moreover, any (non-zero) map $P \to F$ must inject.

We now repeat this construction in a family. Suppose, as in the previous section, that $E$ is the universal sheaf corresponding to the family $\{F_y\}$ and consider $S = E^{**}/E$. Since, $F_y$ is singular only at $y$ we have that $S|_{X \times \{y\}} = \mathcal{O}_y$ and so (wlog) $S$ is supported on the diagonal $\Delta \subset X \times X$ and is locally-free there. If we twist by $\pi_2^*(\pi_2^*S)^*$ then we may assume without further loss of generality that $S = \mathcal{O}_\Delta$.

Observe that $E^{**}$ is flat over both projections and has the property that $E^{**}|_{X \times \{y\}} = F$ for all $y \in X$ and so is locally-free. Observe we have a diagram of natural transformations of functors

$$\Phi_E \longrightarrow \Phi_{E^{**}} \longrightarrow \text{Id}$$

This diagram has the property that for any object $G \in D(X)$ there is a distinguished triangle

$$(2) \quad \Phi_E G \longrightarrow \Phi_{E^{**}} G \longrightarrow G.$$ 

which is natural in $G$. Since $\Phi_E F = F[-d]$ we see that $F \to \Phi_E F[1]$ is zero and so $\Phi_{E^{**}}(F^*) \cong F^* \oplus F^*[-d]$. Hence, $E^{**}|_{\{x\} \times X} \cong F^*$.

**Lemma 4.2.** In the given situation, $\Phi_E^{**}(P^*) \cong P^*$.

**Proof.** By the semi-continuity of direct images $\Phi_E^{**}(P^*)$ is locally-free of rank $r(P)$. We also have $\text{Hom}(P, F_y) = 0 = \text{Ext}^1(P, F_y)$ and so $\Phi_E^{**}(P^*) = 0 = \Phi_E^{1}(P^*)$. The the cohomology of the triangle (2) provides the required isomorphism. \[\square\]

If we use the Leray-Serre spectral sequence for $\pi_2$ we see that

$$H^0((P^* \boxtimes P) \otimes E^{**}) \cong H^0(R^0\pi_2^*(\pi_1^* P^* \otimes E^{**}) \otimes P)$$

$$\cong H^0(R^0\Phi_{E^{**}}(P^*) \otimes P)$$

$$\cong H^0(P^* \otimes P).$$

So we have natural isomorphisms $H^0((P^* \boxtimes P) \otimes E) \cong \text{Hom}(P^*, P^*) \cong \mathbb{C}\langle \text{id} \rangle$ and dually we also have $H^0((P \boxtimes P^*) \otimes E) \cong \text{Hom}(P^*, P^*)$. We can conclude that there are unique maps (up to scalars) $\alpha : P \boxtimes P^* \to E$ and $\beta : E \to P \boxtimes P^*$. If we apply $R^0\pi_2^*(P^* \boxtimes P) \otimes (-)$ to these maps we obtain the maps $\alpha'$ and $\beta' : P^* \otimes P \to P^* \otimes P$. But $\alpha'|_O$ has image $O$ and $\beta'$ is non-zero on this copy of $O$ (corresponding to the identity element in $P^* \otimes P$). Hence, $\beta' \cdot \alpha'$ is not zero and so $\beta' \cdot \alpha$ is also not zero. But $P$ is simple and thus $P \boxtimes P^*$ is also simple (using the Leray-Serre spectral sequence again). Consequently, $\beta \cdot \alpha$ is the identity map. This implies that $E^{**} = (P \boxtimes P^*) \oplus Q$ for some vector bundle $Q$. It also follows that $P$ is spherical as it is a direct summand of $F$.

But now, $Q$ enjoys the same properties as $E^{**}$ and again we can choose a simple $P'$ such that $Q = (P' \boxtimes P') \oplus Q'$. Repeating, we have $E^{**} = \bigoplus_{i=1}^n E_i$, where $E_i \cong P_i \boxtimes P_i^*$ and $P_i$ are spherical bundles. Observe that the uniqueness of $\alpha$ and $\beta$ imply that $\text{Ext}^k(P_i, P_j) = 0$ for all $k$ and $i \neq j$.

We have thus proved:
Theorem 4.3. Let X and Y be (smooth) Calabi-Yau d-folds. If $F \rightarrow X \times Y$ is a family of properly torsion-free sheaves over X parametrized by Y with 0-dimensional singularity sets and $\Phi_F$ is a Fourier-Mukai Transform then

1. there is an isomorphism $\phi : Y \rightarrow X$ and
2. there exists a unique strongly spherical collection of bundles $\Gamma = \{P_i\}_{i=0}^n$ on X such that $(1 \times \phi)^* \Phi_\Gamma = \Phi_F$.

In the case of a K3 surface, if $\text{Pic} X = \mathbb{Z}\langle h \rangle$ then strongly spherical collections can only have cardinality 1. This can be easily seen from the numerical invariants of such a collection. In that case, we recover Yoshioka's result (8) that a family of properly torsion-free sheaves giving rise to an FM transform arise from a spherical object. But in general, this will not be the case. For example, if $L$ is a line bundle on a K3 surface whose sheaf cohomology vanishes in every degree then $\{\mathcal{O}_X, L\}$ is a strongly spherical collection.

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