On quantum subsystem measurement

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Abstract. It is assumed that an arbitrary composite bipartite pure state in which the two subsystems are entangled is given, and it is investigated how the entanglement transmits the influence of measurement on only one of the subsystems to the state of the opposite subsystem. It is shown that any exact subsystem measurement has the same influence as ideal measurement on the opposite subsystem. In particular, the distant effect of subsystem measurement of a twin observable, i.e., so-called 'distant measurement', is always ideal measurement on the distant subsystem no matter how intricate the direct exact measurement on the opposite subsystem is.

Keywords Entanglement in measurement. Measurement effects due to entanglement. Unitary measurement. Basic dynamics.

1 Introduction

The present article investigates some implications of defining the measuring process by a unitary operator that incorporates the interaction between object and measuring instrument. One deals with so-called nonselective measurement, i.e., measurement short of collapse (if done on an ensemble, this contains all the results). So-called selective measurement is measurement with collapse, when one result is considered (the subensemble of this result is selected). The mechanism of collapse is known to lie outside unitary dynamics [1]. It will not be considered in this study. Most interpretations of collapse are in agreement with the quantum-mechanical formalism, which implies the unitary measurement dynamics presented.

In the literature by measurement one usually means selective measurement. In this article we mean by measurement nonselective measurement unless otherwise stated.

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The terms 'pure state' and 'state vector' (vector of norm one) will be used interchangeably; and so will 'state' and 'density operator', 'observable' and 'Hermitian operator' (with a purely discrete spectrum), event' and 'projector' throughout the paper. The subsystem over which a partial trace is taken will be denoted by index (or indices). Total traces go without indices.

Let subsystem A be the object of measurement, and let

$$O_A = \sum_k o_k E_A^k, \quad k \neq k' \Rightarrow o_k \neq o_{k'}$$

be the measured observable (Hermitian operator with a purely discrete, finite or infinite, spectrum) in its unique spectral form. By 'uniqueness' is meant the non-repetition of the eigenvalues \(\{o_k : \forall k\}\) in (1)). Henceforth, we always mean by 'spectral form' the unique one unless otherwise stated.

Naturally, also the completeness relation \(\sum_k E_A^k = I_A\), \(I_A\) being the identity operator for subsystem A, is valid. Let, further, subsystem B be the measuring instrument equipped with a pointer observable

$$P_B = \sum_k p_k F_B^k,$$

also in its spectral form. The completeness relation \(\sum_k F_B^k = I_B\) is valid too.

The measuring apparatus 'takes cognizance' of the results, eigenvalues \(o_k\) or, equivalently, of the corresponding eigen-events \(E_A^k\), in terms of its 'pointer positions', which are either the eigenvalues \(p_k\) of the pointer observable or, equivalently, the eigen-events \(F_B^k\). (This is stated more precisely below when measurement is defined.)

Finally, let \(U_{AB}\) be the unitary operator incorporating the measurement interaction and mapping any initial composite-system state vector \(|\phi\rangle_A \otimes |\phi\rangle_B^i\) into the final state (at the end of measurement interaction):

$$|\Phi\rangle_{AB}^f \equiv U_{AB} \left( |\phi\rangle_A |\phi\rangle_B^i \right).$$

By \(|\phi\rangle_A\) is denoted an arbitrary state vector of the measured system \(A\), and \(|\phi\rangle_B^i\) is the initial or ready-to-measure state vector of the instrument.

We use the convention that kets and bras denote state vectors.
In this investigation the basic aim is to focus attention on bipartite composite systems in some pure state $|\Phi\rangle_{A_1A_2}$ where $A \equiv A_1 + A_2$ is the object of measurement. We are particularly interested in subsystem measurements on subsystem $A_2$, which we call the nearby subsystem, and on its influence on the opposite, dynamically unaffected subsystem $A_1$, called distant or remote. (The terms are dynamical, not spatial.) The influence is transmitted by the entanglement in the composite state.

2 Definition and Basic Dynamical Property of Measurement

Exact measurement is defined by requiring the validity of the so-called calibration condition. It reads: If the initial state of the object has a definite value of the measured observable, then the final composite-system state has the corresponding definite value of the pointer observable. ‘Corresponding’ we write as ‘having the same index value’ (cf (1) and (2)).

Since approximate measurements are not studied in this article, henceforth we drop the term ‘exact’.

All quantum-mechanical relations have a statistical meaning and are tested on ensembles of equally prepared systems. The precise statistical form of the calibration condition is expressed in terms of the usual probability formulae:

$$\forall k : \langle \phi | A E_A^k | \phi \rangle_A = 1 \Rightarrow \langle \Phi | F_B^k | \Phi \rangle_{AB} = 1,$$

where $\Rightarrow$ denotes logical implication, and the final state $|\Phi\rangle_{AB}$ is given by (3).

To derive an equivalent, more practical, form of (4), we need a useful general and known, but perhaps not well known, auxiliary claim (proved in Appendix A for the reader’s convenience).

An event $E$ is certain, i.e., has probability one, in a pure state $|\psi\rangle$ if and only if the former, acting on the latter, does not change it:

$$\langle \psi | E | \psi \rangle = 1 \iff E | \psi \rangle = | \psi \rangle.$$

(The symbol ”$\iff$” denotes logical implication in both directions.)
Equivalence (5) makes it obvious that the calibration condition can be equivalently expressed in the more practical form:

\[ \forall k : |\phi\rangle_A = E_k^A |\phi\rangle_A \Rightarrow |\Phi\rangle_{AB} = F_k^B |\Phi\rangle_{AB} \quad (6) \]

(cf (1)-(3)).

Now we state and prove the basic dynamical property of measurement. Actually, it is a necessary and sufficient condition for the calibration condition, or otherwise put, it is another definition of measurement. (We call it "dynamical" because it involves the unitary evolution operator explicitly.) The claim goes as follows.

One has measurement if and only if

\[ \forall |\phi\rangle_A, \forall k : \left( U_{AB}^{k} \right) \left( |\phi\rangle_A |\phi\rangle_B^i \right) = \left( U_{AB}^{k} E_k^A \right) \left( |\phi\rangle_A |\phi\rangle_B^i \right) \quad (7) \]

is valid.

One proves necessity as follows. The completeness relation \[ \sum_{k'} E_{A}^{k'} = I_A \], repeated use of the calibration condition (6), and orthogonality and idempotency of the \[ F_{B}^{k} \] projectors enable one to write for each \[ k \] value (we shall put \[ \times \] after a number whenever a term in an expansion begins by that number):

\[ F_{B}^{k} U_{AB} |\phi\rangle_A |\phi\rangle_B^i = \sum_{k'} |E_{A}^{k'} |\phi\rangle_A|| E_{B}^{k'} |\phi\rangle_A/ ||E_{A}^{k'} |\phi\rangle_A|| |\phi\rangle_B^i \]

\[ = \sum_{k'} |E_{A}^{k'} |\phi\rangle_A|| F_{B}^{k} E_{B}^{k'} U_{AB} \left( E_{A}^{k'} |\phi\rangle_A/ ||E_{A}^{k'} |\phi\rangle_A|| \right) |\phi\rangle_B^i \]

\[ + |E_{A}^{k} |\phi\rangle_A|| F_{B}^{k} U_{AB} \left( E_{A}^{k} |\phi\rangle_A/ ||E_{A}^{k} |\phi\rangle_A|| \right) |\phi\rangle_B^i \]

Finally, on account of (6) the auxiliary claim (5) allows one to omit \[ F_{B}^{k} \], so that, after cancelation, one obtains:

\[ lhs = U_{AB} E_{A}^{k} |\phi\rangle_A |\phi\rangle_B^i \]

To prove sufficiency, let

\[ \left( U_{AB} E_{A}^{k} \right) \left( |\phi\rangle_A |\phi\rangle_B^i \right) = \left( F_{B}^{k} U_{AB} \right) \left( |\phi\rangle_A |\phi\rangle_B^i \right) \]
be valid for all $k$ values, and let $|\phi\rangle_A = E_A^{k'} |\phi\rangle_A$ be satisfied for a fixed value $k \equiv k'$. Then, one has in particular
\[
(U_{AB} E_A^{k'}) (|\phi\rangle_A |\phi\rangle_B) = (F_B^{k'} U_{AB}) (|\phi\rangle_A |\phi\rangle_B).
\]
One can here omit $E_A^{k'}$ due to the assumed definite value using (5), and thus the calibration condition (6) is obtained. This ends the proof.

3 Subsystem Measurement in Composite State

In this section we assume that an arbitrary composite bipartite system $A \equiv A_1 + A_2$ in an arbitrary pure state $|\psi\rangle_{A_1,A_2}$ and an arbitrary subsystem observable $O_{A_2} = \sum_k \sigma_k E_{A_2}^k$ for the nearby subsystem are given. We investigate the consequences of the basic dynamical characterization of measurement (7) for this case to find out how entanglement transmits the subsystem measurement dynamics on the nearby subsystem $A_2$ onto the state of the remote opposite subsystem $A_1$.

To begin with, it is known that any unitary change to subsystem $A_2$, with or without an ancilla $A_3$, does not have any influence on the state of subsystem $A_1$.

More precisely, the claim is that, if there is no interaction between subsystems $A_1$ and $A_2 + A_3$, i.e., if the composite unitary evolution operator can be factorized $U_{A_1,A_2,A_3} = U_{A_1} \otimes U_{A_2,A_3}$, then the final remote subsystem state reads
\[
\rho_{A_1}' \equiv \text{tr}_{A_2,A_3} \left( U_{A_1,A_2,A_3} |\phi\rangle_{A_1,A_2,A_3} \langle \phi|_{A_1,A_2,A_3} U_{A_1,A_2,A_3}^\dagger \right) = U_{A_1} \rho_{A_1}^i U_{A_1}^\dagger, \tag{8}
\]
where $\rho_{A_1}^i \equiv \text{tr}_{A_2,A_3} \langle \phi|_{A_1,A_2,A_3} \langle \phi|_{A_1,A_2,A_3}$ is the initial state of subsystem $A_1$ in the composite-system state $|\phi\rangle_{A_1,A_2,A_3} = |\phi\rangle_{A_1,A_2} |\phi\rangle_A$.

Note that what makes the ancilla $A_3$ an auxiliary system is the fact that it is initially uncorrelated with the system $A_1 + A_2$ that is considered. Further, one should note that if there is no interaction with the ancilla, then the ancilla evolves independently, and it can be disregarded.

Though claim (8) is known, for the reason of completeness, we sketch the proof. But for this we need a general auxiliary claim, which will be referred to as the 'under-the-partial-trace commutativity' (it will be used again below). It reads:
\[
O_A \equiv \text{tr}_B \left( Y_B X_{AB} \right) = \text{tr}_B \left( X_{AB} Y_B \right), \tag{9}
\]
where $Y_B$ and $X_{AB}$ are arbitrary subsystem and composite-system operators respectively. This general claim is proved in Appendix B.

Proof for claim (8) follows immediately from the definition in (8) when one takes into account the facts (i) that opposite-subsystem operators can be taken out of the partial trace preserving the order of the operators ($U_{A_1}$ and $U_{A_1}^\dagger$ in this case), (ii) that one has the under-the-partial-trace commutativity (9), which concerns $U_{A_2,A_3}$ with the rest, and finally, (iii) that a unitary operator ($U_{A_2,A_3}$ in this case) multiplied by its inverse gives the identity operator. This ends the proof.

Since a measurement instrument $B$ qualifies for an ancilla (cf (3)), though its role is far from auxiliary, it is clear from claim (8) that nonselective measurement of any nearby subsystem observable $O_{A_2}$ in any pure state of a composite system $A_1 + A_2$ cannot influence the state of the distant subsystem $A_1$.

Next, we are interested in selective subsystem measurement. The general claim, a consequence of the basic dynamical relation (7), goes as follows.

Selective measurement does, in general, influence the state of the remote subsystem $A_1$. More precisely, if a nearby-subsystem observable $O_{A_2} = \sum_k o_k E^{k}_{A_2}$ is measured selectively with the result $o_k$ in a bipartite pure state $|\phi\rangle_{A_1,A_2}$ in which one has positive probability $\langle \phi |_{A_1,A_2} E^{k}_{A_2} | \phi \rangle_{A_1,A_2} > 0$, then the final selective distant-subsystem state

$$\rho^{f,k}_{A_1} \equiv \text{tr}_{A_2,B}\left[\left| F^k_B |\Phi\rangle_{A_1,A_2,B} / ||F^k_B |\Phi\rangle_{A_1,A_2,B}||\right| \left(\langle \Phi |_{A_1,A_2,B} F^k_B / ||F^k_B |\Phi\rangle_{A_1,A_2,B}||\right)\right]$$

has the form:

$$\rho^{f,k}_{A_1} = U_{A_1} \left(\rho_{A_1}(E^{k}_{A_2})\right) U_{A_1}^\dagger,$$

(11a)

where by

$$\rho_{A_1}(G_{A_2}) \equiv \text{tr}_{A_2}\left((|\phi\rangle_{A_1,A_2} \langle \phi |_{A_1,A_2} G_{A_2}) / \text{tr}\left((|\phi\rangle_{A_1,A_2} \langle \phi |_{A_1,A_2} G_{A_2})\right)\right)$$

(11b)

($G_{A_2}$ being any projector in the state space $\mathcal{H}_{A_2}$) is denoted the conditional state of the remote subsystem $A_1$ under the condition of the
occurrence of the event $G_{A_2}$ in the composite-system state $|\phi\rangle_{A_1,A_2}$, and $U_{A_1}$ is the unitary evolution operator of the remote subsystem.

To prove (11a), we evaluate $\rho^{f,k}_{A_1}$ from its definition (10). By this we utilize the following equalities, which are a consequence of (7) and (3), of the fact that a unitary operator does not change the norm, and finally of the fact that the norm of a tensor product is the product of the norms.

\[
||F^k_B | \Phi\rangle_{A_1,A_2,B}|| = ||E_{A_2} | \phi\rangle_{A_1,A_2}|| = \left(\langle \phi | A_1,A_2 E_{A_2}^k | \phi\rangle_{A_1,A_2}\right)^{1/2} = \left[\text{tr}\left( |\phi\rangle_{A_1,A_2} \langle \phi | A_1,A_2 E_{A_2}^k\right)\right]^{1/2}. \tag{12}
\]

Besides (12), we take again resort to (7), take into account the partial-trace property that opposite-subsystem operators can be taken out of the partial trace (preserving the order of the operators as factors), as well as the 'under-the-partial-trace commutativity' (9) twice:

\[
\rho^{f,k}_{A_1} = \left(\langle \phi | A_1,A_2 E_{A_2}^k | \phi\rangle_{A_1,A_2}\right)^{-1} \times \\
\text{tr}_{A_2,B}\left[U_{A_1}U_{A_2,B}E_{A_2}^k (|\phi\rangle_{A_1,A_2} | \phi\rangle_B) \left(\langle \phi | A_1,A_2 \langle \phi | B \rangle E_{A_2}^k U^\dagger_{A_1} U_{A_2,B}\right)\right] = \\
U_{A_1}\left\{\text{tr}_{A_2,B}\left[E_{A_2}^k (|\phi\rangle_{A_1,A_2} | \phi\rangle_B \langle \phi | A_1,A_2 \langle \phi | B \rangle E_{A_2}^k U^\dagger_{A_2,B} U_{A_2,B}\right]\right\}U^\dagger_{A_1} / \\
\text{tr}\left(\langle \phi | A_1,A_2 \langle \phi | A_1,A_2 E_{A_2}^k\right) = \\
U_{A_1}\left\{\text{tr}_{A_2}\left[E_{A_2}^k (|\phi\rangle_{A_1,A_2} \langle \phi | A_1,A_2 E_{A_2}^k\right)\right\}\text{tr}_{B}\left( |\phi\rangle_B \langle \phi | B \rangle \right)\right\}U^\dagger_{A_1} / \\
\text{tr}\left(\langle \phi | A_1,A_2 \langle \phi | A_1,A_2 E_{A_2}^k\right) = \\
U_{A_1}\left[\text{tr}_{A_2}\left( |\phi\rangle_{A_1,A_2} \langle \phi | A_1,A_2 E_{A_2}^k\right)\right]U^\dagger_{A_1} / \text{tr}\left(\langle \phi | A_1,A_2 \langle \phi | A_1,A_2 E_{A_2}^k\right) = \\
U_{A_1}\rho_{A_1}(E_{A_2}^k)U^\dagger_{A_1}.
\]

This ends the proof.
It is important to note that claim (11a) implies that it is irrelevant what kind of measurement is performed on the nearby subsystem, the effect on the distant subsystem is **one and the same**, and the influence of the measurement goes only in terms of the **eigen-projectors** of the measured observable. Another way to express this fact is to say that any measurement on the nearby subsystem acts on the distant subsystem equally as the simplest, i. e., **ideal measurement**.

Consistency of no change in nonselective measurement on the one hand, and of the evaluated change in selective measurement on the other, i. e., of (8) and (11a), is seen in the following decomposition.

\[
ρ_{A_1}^{j} = \sum_k \langle φ|_{A_1,A_2} E_{A_2}^k | φ⟩_{A_1,A_2} ρ_{A_1}(E_{A_2}^k).
\]  

(13)

To prove decomposition (13), we make use of the completeness relation \(\sum_k E_{A_2}^k = I_{A_2}\) and of (12):

\[
ρ_{A_1}^{j} = \sum_k \left( \langle φ|_{A_1,A_2} E_{A_2}^k | φ⟩_{A_1,A_2} \right) \times \left\{ \text{tr}_{A_2} \left( | φ⟩_{A_1,A_2} ⟨ φ|_{A_1,A_2} E_{A_2}^k \right) \right\} / \left[ \text{tr} \left( | φ⟩_{A_1,A_2} ⟨ φ|_{A_1,A_2} E_{A_2}^k \right) \right].
\]

In view of (11b), this ends the proof.

One should note that any orthogonal projector decomposition of the identity operator \(I_{A_2}\) induces likewise a decomposition of \(ρ_{A_1}^{j}\) (displays the density operator as an improper mixture [3]). For the measurement of \(O_{A_2} = \sum_k o_k E_{A_2}^k\), one of this mixtures, particularly (13), is relevant.

Relation (11a) tells us that all that selective nearby-subsystem measurement with the result \(o_k\) accomplishes on the remote subsystem is that it picks the state \(ρ_{A_1}(E_{A_2}^k)\) in the corresponding mixture (13). In view of (8), the state \(ρ_{A_1}(E_{A_2}^k)\) then evolves according to the dynamics of the remote subsystem with no regard to the chosen measurement on the nearby system.

This insight might be useful for any theory of collapse, i. e., of selective measurement.
4 Subsystem Measurement of Twin Observable; Distant Measurement

Now we assume that, for a given bipartite pure state $|\phi\rangle_{A_1,A_2}$, a pair of (opposite subsystem) twin observables $O_{A_1}$ and $O_{A_2}$ are given. By definition, they can be written as

$$O_{A_q} = \sum_k o_k^{(q)} E_{A_q}^k + O_{A_q}', \quad q = 1, 2, \quad (14a, b)$$

where the sums are written as unique spectral forms, and also

$$\forall k : E_{A_1}^k |\phi\rangle_{A_1,A_2} = E_{A_2}^k |\phi\rangle_{A_1,A_2}; \quad (14c)$$

$$O_{A_q}' |\phi\rangle_{A_1,A_2} = 0, \quad q = 1, 2 \quad (14d)$$

are valid (cf [4]).

The following claim holds true. If only $O_{A_2}$ of the above pair of twin observables is measured selectively on the nearby subsystem with the result $o_{k}^{(2)}$, then the final state of the remote subsystem is

$$\rho_{A_1}^{f,k} = U_{A_1} \{ E_{A_1}^k \rho_{A_1}^{i} E_{A_1}^k / \left[ tr\left( \rho_{A_1}^{i} E_{A_1}^k \right) \right] \} U_{A_1}^\dagger, \quad (15)$$

and this is valid for every value of $k$.

To prove claim (15), we make use of (11b), of idempotency, of under-the-partial-trace commutativity, of the twin-observables definition (14c), and finally of the possibility to take out opposite-subsystem operators from the partial trace:

$$\rho_{A_1}(E_{A_2}^k) \equiv tr_{A_2} \left[ (|\phi\rangle_{A_1,A_2} \langle \phi|_{A_1,A_2} E_{A_2}^k) / \left[ tr\left( |\phi\rangle_{A_1,A_2} \langle \phi|_{A_1,A_2} E_{A_2}^k \right) \right] \right] =$$

$$tr_{A_2} \left[ (E_{A_2}^k |\phi\rangle_{A_1,A_2} \langle \phi|_{A_1,A_2} E_{A_2}^k) / \left[ tr\left( |\phi\rangle_{A_1,A_2} \langle \phi|_{A_1,A_2} E_{A_2}^k \right) \right] \right] =$$

$$tr_{A_2} \left[ (E_{A_1}^k |\phi\rangle_{A_1,A_2} \langle \phi|_{A_1,A_2} E_{A_1}^k) / \left[ tr\left( |\phi\rangle_{A_1,A_2} \langle \phi|_{A_1,A_2} E_{A_1}^k \right) \right] \right] = E_{A_1} \rho_{A_1}^{i} E_{A_1}^k / \left[ tr\left( \rho_{A_1}^{i} E_{A_1}^k \right) \right].$$

In view of (11a), this ends the proof.
The change of state
\[ \rho_{A_1}^i \rightarrow E_{A_1}^k \rho_{A_1}^i E_{A_1}^k/\text{tr}(\rho_{A_1}^i E_{A_1}^k) \quad (16a) \]
is the well-known Lüders selective change-of-state formula (cf [5], [6], [7]), which characterizes ideal selective measurement.

One should note that \( \text{tr}(\rho_{A_1}^i E_{A_1}^k) = \langle \phi |_{A_1,A_2} E_{A_2}^k | \phi \rangle_{A_1,A_2} \) (cf (12)) is the probability of the result \( o_k^{(2)} \). Hence, the nonselective version of the same subsystem measurement on the nearby subsystem \( A_2 \) gives rise to
\[ \sum_k \text{tr}(\rho_{A_1}^i E_{A_1}^k) \left[ E_{A_1}^k \rho_{A_1}^i E_{A_1}^k/\text{tr}(\rho_{A_1}^i E_{A_1}^k) \right] = \sum_k E_{A_1}^k \rho_{A_1}^i E_{A_1}^k, \quad (16b) \]

This is not distinct from \( \rho_{A_1}^i \) because the completeness relation \( \sum_k E_{A_1}^k = I_{A_1} \) implies \( \rho_{A_1}^i = \sum_k E_{A_1}^k \rho_{A_1}^i E_{A_1}^{k'} \), and, for \( k \neq k' \), one has on account of the twin relation (14c), under-the-partial-trace commutativity, and orthogonality of the eigen-projectors:
\[ E_{A_1}^k \rho_{A_1}^i E_{A_1}^{k'} = \text{tr}_{A_2} \left( E_{A_1}^k | \phi \rangle_{A_1,A_2} \langle \phi |_{A_1,A_2} E_{A_2}^{k'} \right) = \text{tr}_{A_2} \left( | \phi \rangle_{A_1,A_2} \langle \phi |_{A_1,A_2} (E_{A_2}^{k'} E_{A_2}^k) \right) = 0. \]

Naturally, the fact that nonselective subsystem measurement of a twin observable on the nearby subsystem causes no change in the state of the distant subsystem is a special case of the general statement that every nearby subsystem measurement behaves in this way (that is proved in claim (8)).

Result (15) can be read in the following manner: An instantaneous ideal measurement of \( O_{A_1} \) appears to be performed on the initial distant-subsystem state \( \rho_{A_1}^i \), and then the state evolves in its unitary way till the end of the measurement of \( O_{A_2} \) on the nearby subsystem. The defining relations (11c) immediately implied this statement for ideal measurement on subsystem \( A_2 \). Now, on account of the claim (11a), which covers all measurements on the nearby subsystem, we have the general validity of the statement.

The notion of distant measurement, introduced in [8], covered only the case when ideal subsystem measurement was performed on the nearby subsystem and it gave rise to ideal measurement on the remote subsystem (without interaction, only due to the entanglement). Since one rarely succeeds to perform ideal measurement in direct interaction, the distant-measurement
concept was thus on feet of clay. Now the notion of **distant measurement** is on firm ground: Any measurement of a twin observable \( O_{A_2} \) (cf (14a-d)) on the nearby subsystem brings about **distant**, i.e., interaction free, **ideal measurement** of its twin observable \( O_{A_1} \) on the opposite, remote subsystem.

**Appendix A. Relation of certainty in a pure state**

We *prove* now the general claim that the following equivalence is valid for a pure state \(|\psi\rangle\) and an event \( E \):

\[
\langle \psi | E | \psi \rangle = 1 \iff |\psi\rangle = E |\psi\rangle.
\]

One can write

\[
\langle \psi | E | \psi \rangle = 1 \implies \langle \psi | E^c | \psi \rangle = 0,
\]

where \( E^c \equiv I - E \) is the ortho-complementary projector and \( I \) is the identity operator. Further, one has \(||E^c | \psi\rangle|| = 0\), \( E^c | \psi\rangle = 0\), and \( E | \psi\rangle = |\psi\rangle\) as claimed.

**Appendix B. Under-the-partial-trace commutativity**

We prove now the general relation

\[
\text{tr}_B \left( Y_B X_{AB} \right) = \text{tr}_B \left( X_{AB} Y_B \right)
\]

(cf (9)) by straightforward evaluation of both sides in an arbitrary pair of complete orthonormal bases \( \{|k\rangle_A : \forall k\}\), \( \{|n\rangle_B : \forall n\}\).

\[
\langle k | \text{lhs} | k' \rangle_A = \sum_n \langle k | A \langle n | B (Y_B X_{AB}) | k' \rangle_A | n \rangle_B =
\]

\[
\sum_n \sum_{k''} \sum_{n'} \langle k | A \langle n | B (I_A \otimes Y_B) | k'' \rangle_A \langle n' | B (X_{AB}) | k' \rangle_A | n \rangle_B =
\]

\[
\sum_n \sum_{n'} \langle n | B Y_B \langle n' | B (X_{AB}) | k' \rangle_A \langle k | A | n \rangle_B =
\]

\[
\langle k | \text{rhs} | k' \rangle_A = \sum_n \langle k | A \langle n | B (X_{AB} Y_B) | k' \rangle_A | n \rangle_B =
\]

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\[
\sum_n \sum_{k'} \sum_{n'} \langle k|_A \langle n|_B X_{AB} | k''\rangle_A | n'|_B \rangle \times \langle k''|_A \langle n'|_B (I_A \otimes Y_B) | k'\rangle_A | n\rangle_B = \\
\sum_n \sum_{n'} \langle k|_A \langle n|_B X_{AB} | k'\rangle_A | n'|_B \rangle \times \langle n'|_B Y_B | n\rangle_B.
\]

Finally, we exchange the order of the two factors and the two mute indices \( n \) and \( n' \) to obtain

\[
\langle k|_A \text{rhs} | k'\rangle_A = \sum_{n'} \sum_n \langle n|_B Y_B | n'|_B \rangle \times \langle k|_A \langle n'|_B (X_{AB}) | k'\rangle_A | n\rangle_B.
\]

Thus, we see that \( \text{lhs} = \text{rhs} \) as claimed.

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