Some remarks about Fibonacci elements in an arbitrary algebra

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Abstract. In this paper, we prove some relations between Fibonacci elements in an arbitrary algebra. Moreover, we define imaginary Fibonacci quaternions and imaginary Fibonacci octonions and we prove that always three arbitrary imaginary Fibonacci quaternions are linear independents and the mixed product of three arbitrary imaginary Fibonacci octonions is zero.

Keywords: Fibonacci quaternions, Fibonacci octonions, Fibonacci elements.

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1. Introduction

Fibonacci elements over some special algebras were intensively studied in the last time in various papers, as for example: [Akk; ], [Fl, Sa; 15], [Fl, Sh; 13(1)], [Fl, Sh; 13(2)], [Ha; ],[Ha1; ],[Ho; 61],[Ho; 63],[Kc; ]. All these papers studied properties of Fibonacci quaternions, Fibonacci octonions in Quaternion or Octonion algebras or in generalized Quaternion or Octonion algebras or studied dual vectors or dual Fibonacci quaternions ( see [Gu;], [Nu; ]).

In this paper, we will prove that some of the obtained identities can be obtained over an arbitrary algebras. We introduce the notions of imaginary Fibonacci quaternions and imaginary Fibonacci octonions and we prove, using the structure of the quaternion algebras and octonion algebras, that always arbitrary three of such elements are linear dependents. For other details, properties and applications regarding quaternion algebras and octonion algebras, the reader is referred, for example, to [Sc; 54], [Sc; 66], [Fl, St; 09], [Sa, Fl, Ci; 09].

2. Fibonacci elements in an arbitrary algebra
Let $A$ be a unitary algebra over $K \ (K = \mathbb{R}, \mathbb{C})$ with a basis $\{e_0 = 1, e_1, e_2, ..., e_n\}$. Let $\{f_n\}_{n \in \mathbb{N}}$ be the Fibonacci sequence

\[ f_n = f_{n-1} + f_{n-2}, \ n \geq 2, f_0 = 0, f_1 = 1. \]

In algebra $A$, we define the Fibonacci element as follows:

\[ F_m = \sum_{k=0}^{n} f_{m+k}e_k. \]

**Proposition 2.1.** With the above notations, the following relations hold:

1) $F_{m+2} = F_{m+1} + F_m$;

2) $\sum_{i=0}^{p} F_i = F_{p+2} - F_1$.

**Proof.** 1) $F_{m+1} + F_m = \sum_{k=0}^{n} f_{m+k+1}e_k + \sum_{k=0}^{m} f_{m+k}e_k = \sum_{k=0}^{n} (f_{m+k+1} + f_{m+k})e_k = \sum_{k=0}^{n} f_{m+k+2}e_k = F_{m+2}$.

2) $\sum_{i=0}^{p} F_i = F_1 + F_2 + ... + F_p = \sum_{k=0}^{n} f_{k+1}e_k + \sum_{k=0}^{n} f_{k+2}e_k + ... + \sum_{k=0}^{n} f_{k+p}e_k = e_0 (f_1 + ... + f_p) + e_1 (f_2 + ... + f_{p+1}) + e_2 (f_3 + ... + f_{p+2}) + ... + e_n (f_{k+n} + ... + f_{p+n}) = e_0 (f_{p+1} - 1) + e_1 (f_{p+2} - f_1) + e_2 (f_{p+3} - f_2 - f_1) + e_3 (f_{p+4} - f_3 - f_2 - f_1) + ... + e_n (f_{p+n+1} - f_2 - ... - f_n) = F_{p+1} - 2$. We used the identity $\sum_{i=1}^{p} f_i = f_{p+2} - 1$ (for usual Fibonacci numbers) and $1 + f_1 + f_2 + ... + f_n = f_{n+2}$.

**Remark 2.2.** The equalities 1, 2 from the above proposition generalize the corresponding formulae from [Ke; | [Ha; | [Nu; | [Ha1; ]).

**Proposition 2.3.** We have the following formula (Binet’s formula):

\[ F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}, \]

where $\alpha = \frac{1 + \sqrt{5}}{2}, \beta = \frac{1 - \sqrt{5}}{2}, \alpha^* = \sum_{k=0}^{n} \alpha^k e_k, \ \beta^* = \sum_{k=0}^{n} \beta^k e_k$.

**Proof.** Using the formula for the real quaternions, $f_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}$, we obtain

\[ F_m = \sum_{k=0}^{n} f_{m+k}e_k = \frac{\alpha^m}{\alpha - \beta} e_0 + \frac{\alpha^m - \beta^m}{\alpha - \beta} e_1 + \frac{\alpha^{m+2} - \beta^{m+2}}{\alpha - \beta} e_2 + ... + \frac{\alpha^{m+n} - \beta^{m+n}}{\alpha - \beta} e_n = \frac{\alpha^m}{\alpha - \beta} (e_0 + \alpha e_1 + \alpha^2 e_2 + ... + \alpha^n e_n) + \frac{\beta^m}{\alpha - \beta} (e_0 + \beta e_1 + \beta^2 e_2 + ... + \beta^n e_n) = \frac{\alpha^m - \beta^m}{\alpha - \beta}. \]
Remark 2.4. The above result generalizes the Binet formulae from the papers [Gu;] [Akk;] [Ke;] [Ha;] [Nu;] [Ha1;].

Theorem 2.5. The generating function for the Fibonacci number over an algebra is of the form

\[ G(t) = \frac{F_0 + (F_1 - F_0)t}{1 - t - t^2}. \]

Proof. We consider the generating function of the form

\[ G(t) = \sum_{m=0}^{\infty} F_m t^m. \]

We consider the product

\[ G(t) (1 - t - t^2) = \sum_{m=0}^{\infty} F_m t^m = \sum_{m=0}^{\infty} F_m t^m - \sum_{m=0}^{\infty} F_m t^{m+1} - \sum_{m=0}^{\infty} F_m t^{m+2} = \]

\[ = F_0 t + F_1 t + F_2 t^2 + F_3 t^3 + \ldots - F_0 t^2 - F_1 t^3 - F_2 t^4 - \ldots = F_0 + (F_1 - F_0) t. \]

Remark 2.6. The above Theorem generalizes results from the papers [Gu;], [Akk;] [Ke;] [Ha;] [Nu;].

The Cassini identity

First, we obtain the following identity.

Proposition 2.7.

\[ F_{m-1} = (-1)^{m+1} f_m F_1 + (-1)^m f_{m+1} F_0. \] (2.1)

Proof. We use induction. For \( m = 1 \), we obtain \( F_{-1} = f_1 F_1 - f_2 F_0 \), which is true. Now, we assume that it is true for an arbitrary integer \( k \)

\[ F_{-k} = (-1)^{k+1} f_k F_1 + (-1)^k f_{k+1} F_0 \]

For \( k + 1 \), we obtain

\[ F_{-(k+1)} = (-1)^{k+2} f_{k+1} F_1 + (-1)^{k+1} f_{k+2} F_0 = \]

\[ = (-1)^k f_k F_1 + (-1)^k f_{k-1} F_1 + (-1)^{k-1} f_{k+1} F_0 + \]

\[ + (-1)^{k-1} f_k F_0 = F_{-(n-1)} - F_{-n}. \] Therefore, this statement is true. \( \square \)

Theorem 2.8. (Cassini’s identity) With the above notations, we have the following formula

\[ F_{m-1} F_{m+1} - F_m^2 = (-1)^m (F_{-1} F_1 - F_0^2). \]

Proof. We consider

\[ F_{m-1} = f_{m-1} e_0 + f_m e_1 + f_{m+1} e_2 + f_{m+2} e_3 + \ldots + f_{m+n-1} e_n, \]
\( F_{m+1} = f_{m+1}\epsilon_0 + f_{m+2}\epsilon_1 + f_{m+3}\epsilon_2 + f_{m+4}\epsilon_3 + \ldots + f_{m+n+1}\epsilon_n, \)

\( F_m = f_m\epsilon_0 + f_{m+1}\epsilon_1 + f_{m+2}\epsilon_2 + f_{m+3}\epsilon_3 + \ldots + f_{m+n}\epsilon_n. \)

We compute

\[
F_{m+1} - F_m = \left[ f_{m+1} - f_m \right] \epsilon_0 + \left[ f_{m+2} - f_m \right] \epsilon_1 + \left[ f_{m+3} - f_m \right] \epsilon_2 + \left[ f_{m+4} - f_m \right] \epsilon_3 + \ldots + \left[ f_{m+n+1} - f_m \right] \epsilon_n
\]

Using Proposition 2.7, we have

\[
\left[ f_{m+1} - f_m \right] \epsilon_0 + \left[ f_{m+2} - f_m \right] \epsilon_1 + \left[ f_{m+3} - f_m \right] \epsilon_2 + \left[ f_{m+4} - f_m \right] \epsilon_3 + \ldots + \left[ f_{m+n+1} - f_m \right] \epsilon_n
\]

Now, we compute

\[
F_n^2 = \left[ f_m^2 \epsilon_0^2 + f_m f_{m+1} \epsilon_0 \epsilon_1 + f_m f_{m+2} \epsilon_0 \epsilon_2 + f_m f_{m+3} \epsilon_0 \epsilon_3 + \ldots + f_m f_{m+n} \epsilon_0 \epsilon_n \right] + \left[ f_m f_{m+1} \epsilon_1 \epsilon_0 + f_m f_{m+2} \epsilon_1 \epsilon_2 + f_m f_{m+3} \epsilon_1 \epsilon_3 + \ldots + f_m f_{m+n} \epsilon_1 \epsilon_n \right] + \left[ f_m f_{m+2} \epsilon_2 \epsilon_0 + f_m f_{m+3} \epsilon_2 \epsilon_2 + f_m f_{m+4} \epsilon_2 \epsilon_3 + \ldots + f_m f_{m+n} \epsilon_2 \epsilon_n \right] + \left[ f_m f_{m+3} \epsilon_3 \epsilon_0 + f_m f_{m+4} \epsilon_3 \epsilon_2 + f_m f_{m+5} \epsilon_3 \epsilon_3 + \ldots + f_m f_{m+n} \epsilon_3 \epsilon_n \right] + \ldots + \left[ f_m f_{m+n} \epsilon_n \epsilon_0 + f_m f_{m+n+1} \epsilon_n \epsilon_0 + f_m f_{m+n+2} \epsilon_n \epsilon_2 + f_m f_{m+n+3} \epsilon_n \epsilon_3 + \ldots + f_m f_{m+n} \epsilon_n \epsilon_n \right].
\]

Consider the difference

\[
F_{m+1} - F_m = e_n \left[ f_m f_{m+n+1} - f_{m+1} f_m \right] + e_1 \left( f_{m+1} f_{m+2} - f_m f_{m+1} \right) + \ldots + e_n \left( f_{m+1} f_{m+n} - f_m f_{m+n} \right)
\]

Using the formula \( f_i f_j = f_{i+k} f_{j-k} = (-1)^{j-k} f_i f_j \) (see Koshy, p. 87, formula 2) and the identities \( f_1 = 1, f_m = (-1)^{m+1} f_m \) (see Koshy, p. 84), we obtain

\[
F_{m+1} - F_m = e_0 \left[ f_0 f_1 - f_1 f_0 \right] + e_1 \left[ f_0 f_2 - f_2 f_0 \right] + \ldots + e_n \left[ f_0 f_n - f_n f_0 \right] + e_1 \left[ f_1 f_2 - f_2 f_1 \right] + \ldots + e_n \left[ f_1 f_n - f_n f_1 \right] + \ldots + e_n \left[ f_n f_1 - f_1 f_n \right]
\]

Using Proposition 2.7, we have

\[
F_{m+1} - F_m = (-1)^{m+1} \left[ e_0 f_1 - e_1 f_0 + e_2 f_2 - e_3 f_3 + \ldots + e_n f_n \right] + e_1 \left[ f_0 f_2 - f_2 f_0 + e_2 f_3 - e_3 f_4 + \ldots + e_n f_{n+1} \right] + \ldots + e_n \left[ f_n f_{n+1} - f_{n+1} f_n \right]
\]

Using Proposition 2.7, we have

\[
F_{m+1} - F_m = \left[ (-1)^n f_n f_{n+1} + \left( (-1)^{n-1} f_{n-1} F_1 - (-1)^{m+1} e_0 f_1 - e_1 f_0 + e_2 f_2 - \ldots + e_n f_n \right) \right]
\]

\[
= \left[ (-1)^m \left( e_0 f_1 - e_1 f_0 + e_2 f_2 - \ldots + e_n f_n \right) \right]
\]

Remark 2.9.
i) Similarly, we can prove an analogue of Cassini’s formula:

\[ F_{m+1}F_{m-1} - F_m^2 = (-1)^m \left[ F_1F_{-1} - F_0^2 \right]. \]

ii) Theorem 2.8 generalizes Cassini’s formula for all real algebras.

iii) If the algebra \( A \) is algebra of the real numbers \( \mathbb{R} \), in this case, we have \( F_m = f_m \). From the above theorem, it results that

\[ f_{m+1}f_{m-1} - f_m^2 = (-1)^m \left[ f_1f_{-1} - f_0^2 \right] = (-1)^m, \]

which it is the classical Cassini’s identity.

3. Imaginary Fibonacci quaternions and imaginary Fibonacci octonions

In the following, we will consider a field \( K \) with \( \text{char}K \not= 2, 3 \), \( V \) a finite dimensional vector space and \( A \) a finite dimensional unitary algebra over a field \( K \), associative or nonassociative.

Let \( \mathbb{H}(\alpha, \beta) \) be the generalized real quaternion algebra, the algebra of the elements of the form \( a = a_0 \cdot 1 + a_1 i + a_2 j + a_3 k \), where \( a_i \in \mathbb{R}, i^2 = -\alpha, j^2 = -\beta, k = ij = ji \). We denote by \( t(a) \) and \( n(a) \) the trace and the norm of a real quaternion \( a \). The norm of a generalized quaternion has the following expression \( n(a) = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \gamma a_3^2 \) and \( t(a) = 2a_1 \). It is known that for \( a \in \mathbb{H}(\alpha, \beta) \), we have \( a^2 - t(a)a + n(a) = 0 \). The quaternion algebra \( \mathbb{H}(\alpha, \beta) \) is a division algebra if for all \( a \in \mathbb{H}(\alpha, \beta), a \neq 0 \) we have \( n(a) \neq 0 \), otherwise \( \mathbb{H}(\alpha, \beta) \) is called a split algebra.

Let \( \mathcal{O}(\alpha, \beta, \gamma) \) be a generalized octonion algebra over \( \mathbb{R} \), with basis \( \{1, e_1, ..., e_7\} \), the algebra of the elements of the form \( a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7 \) and the multiplication given in the following table:

| \cdot | 1 | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
|-------|---|-----|-----|-----|-----|-----|-----|-----|
| 1     | 1 | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
| e_1   | e_1 | -\alpha | e_3 | -\alpha e_2 | e_5 | -\alpha e_4 | -\alpha e_7 | \alpha e_6 |
| e_2   | e_2 | -e_3 | -\beta | \beta e_1 | e_4 | e_7 | -\beta e_4 | -\beta e_5 |
| e_3   | e_3 | \alpha e_2 | -\beta e_1 | -\alpha \beta | e_6 | e_7 | -\alpha e_6 | \beta e_5 | -\alpha \beta e_4 |
| e_4   | e_4 | -e_5 | -e_6 | -e_7 | -\gamma | \gamma e_1 | \gamma e_2 | \gamma e_3 |
| e_5   | e_5 | \alpha e_4 | -e_7 | \alpha \gamma | -\gamma e_1 | -\gamma e_2 | -\gamma e_3 | \alpha \gamma e_2 |
| e_6   | e_6 | e_7 | \beta e_4 | -\gamma e_1 | -\gamma e_2 | -\gamma e_3 | -\beta \gamma e_1 | -\beta \gamma |
| e_7   | e_7 | -\alpha e_6 | \beta e_5 | \alpha \beta e_4 | -\gamma e_3 | -\alpha \gamma e_2 | \beta \gamma e_1 | -\alpha \beta \gamma |

Table 1

The algebra \( \mathcal{O}(\alpha, \beta, \gamma) \) is non-commutative and non-associative.
If \( a \in \mathbb{O}(\alpha, \beta, \gamma) \), \( a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7 \) then 
\( \bar{a} = a_0 - a_1e_1 - a_2e_2 - a_3e_3 - a_4e_4 - a_5e_5 - a_6e_6 - a_7e_7 \) is called the \textit{conjugate} of the element \( a \). The scalars \( t(a) = a + \bar{a} \in \mathbb{R} \) and

\[
\mathbf{n}(a) = a\bar{a} = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \gamma a_3^2 + \alpha\beta a_4^2 + \alpha\gamma a_5^2 + \beta\gamma a_6^2 + \alpha\beta\gamma a_7^2 \in \mathbb{R},
\]

are called the \textit{trace}, respectively, the \textit{norm} of the element \( a \in A \). It follows that 
\( a^2 - t(a) = 0 \), \( \forall a \in A \).The octonion algebra \( \mathbb{O}(\alpha, \beta, \gamma) \) is a \textit{division algebra} if for all \( a \in \mathbb{O}(\alpha, \beta, \gamma) \), \( a \neq 0 \) we have \( \mathbf{n}(a) \neq 0 \), otherwise \( \mathbb{O}(\alpha, \beta, \gamma) \) is called a \textit{split algebra}.

Let \( V \) be a real vector space of dimension \( n \) and \( <,> \) be the inner product. The \textit{cross product} on \( V \) is a continuos map

\[
X : V^s \rightarrow V, s \in \{1, 2, ..., n\}
\]

with the following properties:

1) \( <X(x_1, ...x_s), x_i> = 0, i \in \{1, 2, ..., s\}; \)
2) \( <X(x_1, ...x_s), X(x_1, ...x_s)> = \text{det} (<x_i, x_j>) \). (see [Br; ])

In [Ro; 96], was proved that if \( d = \text{dim}_\mathbb{R} V \), therefore \( d \in \{0, 1, 3, 7\} \). (see [Ro; 96], Proposition 3)

The values 0, 1, 3 and 7 for dimensions are obtained from Hurwitz’s theorem, since the real Hurwitz division algebras \( \mathcal{H} \) exist only for dimensions 1, 2, 4 and 8 dimensions. In this situations, the cross product is obtained from the product of the normed division algebra, restricting it to imaginary subspace of the algebra \( \mathcal{H} \), which can be of dimension 0, 1, 3 or 7. (see [Ja; 74]) It is known that the real Hurwitz division algebras are only: the real numbers, the complex numbers, the quaternions and the octonions.

In \( \mathbb{R}^3 \) with the canonical basis \( \{i_1, i_2, i_3\} \), the cross product of two linearly independent vectors \( x = x_1i_1 + x_2i_2 + x_3i_3 \) and \( y = y_1i_1 + y_2i_2 + y_3i_3 \) is a vector, denoted by \( x \times y \) and can be expressed computing the following formal determinant

\[
x \times y = \begin{vmatrix} i_1 & i_2 & i_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.
\]

The cross product can also be described using the quaternions and the basis \( \{i_1, i_2, i_3\} \) as a standard basis for \( \mathbb{R}^3 \). If a vector \( x \in \mathbb{R}^3 \) has the form \( x = x_1i_1 + x_2i_2 + x_3i_3 \) and is represented as the quaternion \( x = x_1i + x_2j + x_3k \), therefore the cross product of two vectors has the form \( x \times y = xy + <x, y> \), where \( <x, y> = x_1y_1 + x_2y_2 + x_3y_3 \) is the inner product.

A cross product for 7-dimensional vectors can be obtained in the same way by using the octonions instead of the quaternions. If \( x = \sum_{i=0}^{7} x_ie_i \) and \( y = \sum_{i=0}^{7} y_ie_i \),
are two imaginary octonions, therefore
\[ x \times y = (x_2y_4 - x_4y_2 + x_3y_7 - x_7y_3 + x_5y_6 - x_6y_5)e_1 + \\
+ (x_3y_5 - x_5y_3 + x_4y_1 - x_1y_4 + x_6y_7 - x_7y_6)e_2 + \\
+ (x_4y_6 - x_6y_4 + x_5y_2 - x_2y_5 + x_7y_1 - x_1y_7)e_3 + \\
+ (x_5y_7 - x_7y_5 + x_6y_3 - x_3y_6 + x_1y_2 - x_2y_1)e_4 + \\
+ (x_6y_1 - x_1y_6 + x_7y_4 - x_4y_7 + x_2y_3 - x_3y_2)e_5 + \\
+ (x_7y_2 - x_2y_7 + x_1y_5 - x_5y_1 + x_3y_4 - x_4y_3)e_6 + \\
+ (x_1y_4 - x_4y_1 + x_2y_6 - x_6y_2 + x_3y_5 - x_5y_3)e_7, \]
(3.3.)

see [Si; 02].

Let \( \mathbb{H} \) be the real division quaternion algebra and \( \mathbb{H}_0 = \{ x \in \mathbb{H} | t(x) = 0 \} \).
An element \( F_n \in \mathbb{H}_0 \) is called an imaginary Fibonacci quaternion element if it is of the form
\( F_n = f_n1 + f_{n+1}j + f_{n+2}k \), where \( (f_n)_{n \in \mathbb{N}} \) is the Fibonacci numbers sequence. Let \( F_k, F_m, F_n \) be three imaginary Fibonacci quaternions. Therefore, we have the following result.

In the proof of the following results, we will use some relations between Fibonacci numbers, namely:

**D’Ocagne’s identity**
\[ f_m f_{n+1} - f_n f_{m+1} = (-1)^n f_{m-n} \]  
(3.4.)

see relation (33) from [Wo], and

**Johnson’s identity**
\[ f_a f_b - f_c f_d = (-1)^r (f_{a-r} f_{b-r} - f_{c-r} f_{d-r}), \]  
(3.5.)

for arbitrary integers \( a, b, c, d, \) and \( r \) with \( a + b = c + d \), see relation (36) from [Wo].

**Proposition 3.1.** With the above notations, for three arbitrary Fibonacci imaginary quaternions, \( F_k, F_m, F_n \) are linear dependents.

The above result is similar with the result for dual Fibonacci vectors obtained in [Gu;], Theorem 11.

Let \( \mathbb{O} \) be the real division octonion algebra and \( \mathbb{O}_0 = \{ x \in \mathbb{O} | t(x) = 0 \} \). An element \( F_n \in \mathbb{O}_0 \) is called an imaginary Fibonacci octonion element if it is of the form
\( F_n = f_ne_1 + f_{n+1}e_1 + f_{n+2}e_1 + f_{n+3}e_1 + f_{n+4}e_1 + f_{n+5}e_1 + f_{n+6}e_1 \), where \( (f_n)_{n \in \mathbb{N}} \) is the Fibonacci numbers sequence. Let \( F_k, F_m, F_n \) be three imaginary Fibonacci octonions.

**Proposition 3.2.** With the above notations, for three arbitrary Fibonacci imaginary octonions, \( F_k, F_m, F_n \) are linear dependents.

\[ < F_k \times F_m, F_n > = 0. \]
**Proof.** Using formulae (3.3), (3.4) and (3.5), we will compute $F_k \times F_m$.

The coefficient of $e_1$ is

$$f_{m+2}f_{k+4} - f_{k+2}f_{m+4} + f_{m+3}f_{k+7} - f_k + 3f_{m+7} + f_{m+5}f_{k+6} - f_k + 5f_{m+6} =$$

$$= f_m f_{k+2} - f_m f_{k+1} + f_m f_{k+4} - f_k f_{m+1} + f_k f_{m+1} =$$

$$= f_m (f_{k+2} - f_{k+4} - f_{k+1}) + f_k (f_{m+2} + f_{m+4} + f_{m+1}) =$$

$$= f_m (f_k - f_k) + f_k (f_{m+4} - f_m) =$$

$$= -f_m (3f_{k+1} + f_k) + f_k (3f_{m+1} + f_m) =$$

$$= -3 (f_{m} f_{k+1} - f_{k} f_{m+1}) = -3 (-1)^k f_{m-k}.$$ 

The coefficient of $e_2$ is

$$f_{m+3}f_{k+5} - f_k + 3f_{m+5} + f_{m+4}f_{k+1} - f_{k+4}f_{m+1} + f_{m+6}f_{k+7} - f_k + 6f_{m+7} =$$

$$= -f_m f_{k+2} + f_k f_{m+2} - f_m f_{k+3} + f_k f_{m+3} + f_m f_{k+1} - f_k f_{m+1} =$$

$$= f_m (-f_{k+2} + f_{k+3} + f_{k+1}) + f_k (f_{m+2} - f_{m+3} - f_{m+1}) =$$

$$= 2 (f_m f_{k+1} - f_k f_{m+1}) = 2 (-1)^k f_{m-k}.$$ 

The coefficient of $e_3$ is

$$f_{m+4}f_{k+6} - f_m f_{k+5} + f_{m+5}f_{k+2} - f_{m+4}f_{k+5} + f_{m+7}f_{k+1} - f_{k+7}f_{m+1} =$$

$$= f_m f_{k+2} - f_m f_{k+3} + f_m f_{k+4} - f_m f_{k+6} + f_m f_{k+6} =$$

$$= f_m (f_{k+2} - f_{k+3} + f_{k+6}) + f_k (-f_{m+2} + f_{m+3} - f_{m+6}) =$$

$$= 7 (f_m f_{k+1} - f_k f_{m+1}) = 7 (-1)^k f_{m-k}.$$ 

The coefficient of $e_4$ is

$$f_{m+5}f_{k+7} - f_{m+5}f_{m+7} - f_{m+6}f_{k+3} - f_{k+6}f_{m+3} + f_{m+1}f_{k+2} - f_{m+2}f_{k+1} =$$

$$= -f_m f_{k+2} + f_k f_{m+2} - f_m f_{k+3} + f_k f_{m+3} - f_m f_{k+1} + f_k f_{m+1} =$$

$$= f_m (-f_{k+2} + f_{k+3} - f_{k+1}) = 0.$$ 

The coefficient of $e_5$ is

$$f_{m+6}f_{k+1} - f_{k+6}f_{m+1} + f_{m+7}f_{k+4} - f_{k+7}f_{m+4} + f_{m+2}f_{k+3} - f_{k+2}f_{m+3} =$$

$$= -f_m f_{k+4} + f_k f_{m+4} + f_{m+3}f_{k+4} - f_k + 3f_{m+4} =$$

$$= f_m (-f_k + 5f_{m} + f_{m+3}f_{k} - f_k + 3f_{m} + f_m f_{k+1} - f_k f_{m+1}) =$$

$$= f_m (f_k - f_k + 3f_{m} + f_k + 1) + f_k (-f_{m+5} + f_{m+3} - f_{m+1}) =$$

$$= 4 (f_m f_{k+1} - f_k f_{m+1}) = 4 (-1)^k f_{m-k}.$$ 

The coefficient of $e_6$ is

$$f_{m+7}f_{k+2} - f_{k+7}f_{m+2} + f_{m+4}f_{k+5} - f_k + 4f_{m+4} + f_{m+3}f_{k+4} - f_k + 3f_{m+4} =$$

$$= f_m f_{k+2} + f_k f_{m+2} - f_m f_{k+3} + f_k f_{m+3} - f_m f_{k+1} + f_k f_{m+1} =$$

$$= f_m (-f_{k+2} + f_{k+3} + f_{k+1}) + f_k (f_{m+2} - f_{m+3} - f_{m+1}) =$$

$$= -9 (f_m f_{k+1} - f_k f_{m+1}) = -9 (-1)^k f_{m-k}.$$ 

The coefficient of $e_7$ is

$$f_{m+1}f_{k+3} - f_k + 3f_{m+3} + f_{m+2}f_{k+6} - f_k + 2f_{m+6} + f_{m+4}f_{k+5} - f_k + 4f_{m+5} =$$

$$= f_m (-f_k + 2 + f_{k+4} + f_{k+1}) + f_k (f_{m+2} - f_{m+4} - f_{m+1}) =$$

$$= 3 (f_m f_{k+1} - f_k f_{m+1}) = 3 (-1)^k f_{m-k}.$$ 

We obtain that

$$F_k \times F_m = (-1)^k f_{m-k} (-3e_1 + 2e_2 + 7e_3 + 4e_5 - 9e_6 + 3e_7).$$

Therefore

$$<F_k \times F_m, F_n> = (-1)^k f_{m-k} (-3f_{n+1} + 2f_{n+2} + 7f_{n+3} + 4f_{n+5} - 9f_{n+6} + 3f_{n+7}) =$$

$$= -2f_{n+2} + 2f_{n+1} + 2f_n = 0.$$
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