EXACT S-MATRICES FOR THE NONSIMPLY-LACED AFFINE TODA THEORIES $a_{2n-1}^{(2)}$

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ABSTRACT

We derive the exact, factorized, purely elastic scattering matrices for the $a_{2n-1}^{(2)}$ family of nonsimply-laced affine Toda theories. The derivation takes into account the distortion of the classical mass spectrum by radiative corrections, as well as modifications of the usual bootstrap assumptions since for these theories anomalous threshold singularities lead to a displacement of some single particle poles.
Affine Today theories are massive two-dimensional field theories represented by lagrangians of the form

$$\mathcal{L} = -\frac{1}{2\beta^2} \bar{\phi} \Box \phi - \sum q_i \beta^2 \bar{\alpha}_i \cdot \phi$$  \hspace{1cm} (1)

Here the $\bar{\alpha}_i$ are the simple roots of a rank $r$ Lie algebra augmented by (the negative of) a maximal root and $\phi = (\phi_1, \phi_2, ..., \phi_r)$ are bosonic fields describing $r$ massive particles, with classical masses rigidly controlled by properties of the Lie algebra. The constants $q_i$ are such that $\sum q_i \bar{\alpha}_i = 0$ and $\beta$ is the coupling constant. These theories are classically integrable: the lagrangian above admits an infinite number of symmetries described by conserved currents $J_{\pm}^{(s)}$ of increasing spin $s$. If at least one of these symmetries with $s > 2$ survives quantization, the $n$-particle scattering matrices of these theories factorize into a product of elastic two-particle S-matrices. Assumptions of analyticity, unitarity and a bootstrap principle \cite{1} should allow the full S-matrix to be determined exactly. Indeed, exact S-matrices have been proposed for all affine Toda theories based on simply-laced Lie algebras and perturbative calculations based on the corresponding lagrangians have verified the correctness of these proposals \cite{2, 3, 4}. On the other hand arguments have been put forward to suggest that a corresponding construction would fail for affine Toda theories based on nonsimply-laced algebras and indeed, so far, no satisfactory S-matrices have been found for these theories.

There are several reasons for the belief that nonsimply-laced Toda theories might not have exact factorizable S-matrices: a) Usually the bootstrap procedure leads to restrictions on the mass spectrum of the particles. These restrictions are consistent with the classical masses, and indeed for simply-laced theories the mass spectrum does not seem to be affected by radiative corrections (except for an overall, irrelevant rescaling). On the other hand, for the nonsimply-laced theories radiative corrections distort the mass spectrum in a manner that at first sight seems incompatible with bootstrap restrictions \cite{2, 3, 4}. b) Exact S-matrices contain higher-order poles which must be explained as anomalous threshold singularities of corresponding Feynman diagrams (or, in a pure S-matrix approach, of extended unitarity-analyticity diagrams \cite{6}). However, for the nonsimply-laced theories such an explanation has failed to account for all the singularities of proposed S-matrices \cite{2}. c) We have recently pointed out that certain anomalous threshold singularities may lead to a breakdown of some of the bootstrap assumptions \cite{7}.

In view of these facts it may be reasonable to believe that quantum effects destroy the classical integrability of these theories: the classical conservation laws which imply the factorization and elasticity of the scattering matrix do not survive quantization. However this is not the case. A study of the renormalization of the higher spin currents has revealed that for example in the $a_3^{(2)}$ nonsimply-laced Toda theory a non-anomalous higher spin symmetry does exist \cite{8} and this implies that at least in this case it should be possible to construct a factorized S-matrix.

In fact, the problems mentioned above concerning the mass spectrum, anomalous threshold structure and breakdown of the bootstrap provide each other's solution. We have been able to construct an exact factorized S-matrix for the $a_{2n-1}^{(2)}$ family of
nonsimply-laced Toda theories and we outline this construction here; further details, the perturbative verification, and the construction of quantum-conserved currents will be presented elsewhere [8, 9]. For general information about the techniques used the reader is referred to the considerable literature that has built up in recent years; for a review and references see [10].

The Toda theory based on the nonsimply-laced twisted affine Lie algebras $a_{2n-1}^{(2)}$ is described by the lagrangian

\[ \beta^2 \mathcal{L} = -\frac{1}{2} \sum_{a=1}^{n-1} \phi_a \square \phi_a - \frac{1}{2} \phi_n \square \phi_n \]

\[ - \sum_{k=2}^{n-1} 2 \exp \left( -\sqrt{\frac{2}{h}} \sum_{a} m_a \cos \left( \frac{(2k-1)a\pi}{h} \right) \phi_a \right) \exp \left( -\sqrt{\frac{2}{h}} \sum_{a} (-1)^a m_a \phi_a \right) \]

\[ - \exp \left( -\sqrt{\frac{2}{h}} \sum_{a} m_a \cos \left( \frac{a\pi}{h} \right) \phi_a \phi_n \right) \exp \left( -\sqrt{\frac{2}{h}} \sum_{a} (-1)^a m_a \phi_a \phi_n \right) \]

with masses and three-point couplings given by

\[ m_a^2 = 8 \sin^2 \left( \frac{a\pi}{h} \right) \quad a = 1, \ldots, n-1, \quad m_n^2 = 2 \]

\[ c_{abc} = -\frac{1}{\sqrt{2}h} m_a m_b m_c \quad \text{if} \quad a + b + c = 2n - 1 \]

\[ c_{abc} = \frac{1}{\sqrt{2}h} m_a m_b m_c \quad \text{if} \quad a \pm b \pm c = 0 \]

\[ c_{ann} = \frac{1}{\sqrt{2}h} m_n^2 m_a \cos \frac{a\pi}{h} \]

All other three-point couplings are zero. We have introduced the Coxeter number $h = 2n - 1$ and set the overall mass-scale to 1.

At the one-loop level radiative corrections lead to mass shifts that have been computed in Ref. [5]. We absorb the shift in $m_n$ into an overall mass rescaling, in which case the results in eqs.(20,22) of the above reference become

\[ \delta m_a^2 = -\frac{2a}{h^2} \sin \frac{2a\pi}{h}, \quad \delta m_n^2 = 0 \]

In contrast, for a simply-laced theory the mass corrections vanish after the above rescaling.

For the specific case $n = 2$, i.e. the $a_3^{(2)}$ theory, the lagrangian is

\[ \beta^2 \mathcal{L} = -\frac{1}{2} \phi_1 \square \phi_1 - \frac{1}{2} \phi_2 \square \phi_2 - e^{-\phi_1 - \phi_2} - e^{-\phi_1 + \phi_2} - e^{2\phi_1} \]

Using the classical field equations it is straightforward to check the conservation $\partial_- J_+^{(s)} + \partial_+ J_-^{(s)} = 0$ of spin 2 and spin 4 currents, where the spin 2 current is the stress tensor, while

\[ J_+^{(4)} \equiv J_{+++} = (\partial_+ \phi_1)^2 (\partial_+ \phi_2)^2 + (\partial_+^2 \phi_2)^2 + 2 \partial_+ \phi_1 \partial_+ \phi_2 \partial_+^2 \phi_2 \]
with a suitable expression for \( J^{(4)} \). At the quantum level we have shown \[8\] that the conservation laws still hold with a renormalized current

\[
J^{(4)}_+ = (1 + \frac{h}{2})(\partial_+ \phi_1)^2(\partial_+ \phi_2)^2 - \frac{h}{12}(1 + 3h + h^2)(\partial_+^2 \phi_1)^2 - \frac{h}{12}(\partial_+ \phi_1)^4 - \frac{h}{12}(\partial_+ \phi_2)^4
+ (1 + \frac{23}{12}h + h^2 + \frac{1}{6}h^3)(\partial_+^2 \phi_2)^2 + (2 + 3h + h^2)\partial_+ \phi_1 \partial_+ \phi_2 \partial_+^2 \phi_2
\]

(8)

We expect that a similar conserved current exists for all the \( a_{2n-1}^{(2)} \) Toda theories. Together with the renormalized stress tensor this defines corresponding spin 1 and spin 3 charges whose presence should guarantee the existence of factorizable elastic S-matrices. The bootstrap procedure should then determine them. However, there are some subtleties \[7\]:

The current conservation implies that in a scattering process the sum of the charges of incoming particles equals that of the outgoing particles. The bootstrap principle, which asserts that in the two-body S-matrix a simple pole is to be associated with a particle in the spectrum via the process \( a + b \rightarrow c \rightarrow a + b \) extends this to the vertex function \( <a, b, c> \) with three on-shell particles. In particular, in the \( n = 2 \) theory described above, the existence of the conserved spin 4 current would imply that either the spin 3 charge of particle \( \phi_1 \) is zero, or else the corresponding vertex function \( <1, 1, 1> \) vanishes. An examination of the currents above reveals that at the classical level the charge does indeed vanish, which is consistent with the presence of a \( \phi_1^3 \) coupling, but loop calculations reveal an apparent inconsistency: the charge is not zero \[4, 8\], and the vertex function does not vanish. These results can be explained in fact by the presence of an anomalous threshold singularity in a triangle graph which renders the vertex function infinite on shell, but they do indicate that one has to be careful in applying bootstrap ideas to processes where the \( <a, b, h - a - b> \) vertex enters.

We shall construct the S-matrix by following procedures similar to those used in the simply-laced case, but with two important differences: we will admit that it has simple particle poles at positions shifted away from the classical mass values, and we shall relax the bootstrap principle since some simple poles are shifted away from their single-particle positions due to anomalous threshold effects. We will have to prove however that all the singularities we find can be accounted for in this fashion.

Assuming that higher-spin quantum-conserved currents similar to those in eq.\(8\) exist for all the \( a_{2n-1}^{(2)} \) theories, we postulate the existence of purely elastic two-body amplitudes. Imposing unitarity and real analyticity on the S-matrix for the process \( a + b \rightarrow a + b \) restricts \( S_{ab}(\theta) \) to be a product of fundamental building blocks

\[
S_{ab}(\theta) = \prod_{x \in A_{ab}} (x) \quad \text{where} \quad (x) \equiv \frac{\text{sh} \left( \frac{\theta}{2} + i\pi x \right)}{\text{sh} \left( \frac{\theta}{2} - i\pi x \right)}
\]

(9)

and \( \theta = \theta_a - \theta_b \) is the relative rapidity. We are using the notation of reference \[4\]. This notation is related to that of reference \[3, 4\] by \( f_\alpha \equiv (h\alpha) \). Crossing symmetry acts on the blocks by \( (x) \rightarrow -(h - x) \).
We start by determining the element $S_{nn}$. The three-point couplings in eq.(11) suggest that all the particles $\phi_1 \cdots \phi_{n-1}$ appear as intermediate particles in this process, both in the direct and in the crossed channel. They lead to poles $((p + q)^2 - \tilde{m}_a^2)^{-1}$ and $((p - q)^2 - \tilde{m}_a^2)^{-1}$ in $S_{nn}$. Here $\tilde{m}_a$ are the radiatively corrected masses. Without loss of generality we write, with reference to eq.(3)

$$\tilde{m}_a^2 = 4\tilde{m}_a^2 \sin^2 \left(\frac{\pi}{h} (a + \epsilon_a(\beta))\right)$$

In the rapidity plane these poles are located at:

- **s-channel:** $2\tilde{m}_a^2 (1 + \mathrm{ch} \theta) - \tilde{m}_a^2 = 0 \Rightarrow \theta = \frac{i\pi}{h} (h - 2a - 2\epsilon_a)$
- **u-channel:** $2\tilde{m}_a^2 (1 - \mathrm{ch} \theta) - \tilde{m}_a^2 = 0 \Rightarrow \theta = \frac{i\pi}{h} (2a + 2\epsilon_a)$

We must reproduce these poles with the building blocks in eq.(3). Noting that $(x)$ has a pole at $\theta = \frac{i\pi}{h} x$ we choose

$$S_{nn} = \prod_{a=0}^{n-1} \frac{(2a + 2\epsilon_a) (h - 2a - 2\epsilon_a)}{(2a + 2\eta_a) (h - 2a - 2\eta_a)}$$

where $\epsilon_a$ and $\eta_a$ are both zero when $\beta = 0$ so that the S-matrix reduces to the identity matrix. This $S_{nn}$ is crossing symmetric. We let the product start at $a = 0$ for generality. None of the extra blocks which we introduced in eq.(12) should produce any additional poles on the physical sheet (i.e., for $0 < \theta < i\pi$) and this requires

$$\frac{h}{2} - a \geq \eta_a \geq -a \quad \text{and} \quad \epsilon_0 \leq 0.$$  

Eventually, comparison to perturbation theory will justify this Ansatz.

We now turn to the determination of the S-matrix elements $S_{an}$, $a = 1 \cdots n - 1$. These are determined by $S_{nn}$ through the bootstrap principle [1]

$$S_{an} = S_{nn}(\theta + \frac{1}{2} \theta_{an}^a) S_{nn}(\theta - \frac{1}{2} \theta_{an}^a)$$

where $\theta_{an}^a$ is the relative rapidity at which $S_{nn}$ has the s-channel pole corresponding to particle $a$, i.e., $\theta_{nn}^a = \frac{i\pi}{h} (h - 2a - 2\epsilon_a)$, see eq.(11). We find

$$S_{an} = \prod_{p=0}^{n-1} \frac{(2p + 2\epsilon_p - \frac{h}{2} + a + \epsilon_a) (h - 2p - 2\epsilon_p - \frac{h}{2} + a + \epsilon_a)}{(2p + 2\eta_p - \frac{h}{2} + a + \epsilon_a) (h - 2p - 2\eta_p - \frac{h}{2} + a + \epsilon_a)} \times \frac{(2p + 2\epsilon_p + \frac{h}{2} - a - \epsilon_a) (h - 2p - 2\epsilon_p + \frac{h}{2} - a - \epsilon_a)}{(2p + 2\eta_p + \frac{h}{2} - a - \epsilon_a) (h - 2p - 2\eta_p + \frac{h}{2} - a - \epsilon_a)}$$

This expression has a large number of poles, many more than perturbation theory can be expected to explain. However for special values of $\epsilon_a$ and $\eta_a$ many of the building blocks cancel each other. We can not choose $\epsilon_a = \eta_a$ because that would
cancel the wanted poles in eq.(12). One allowed choice which cancels many poles in eq.(13) is \( \epsilon_a = 0 \) and \( \eta_a = \frac{\beta}{2} \) for all \( a \) and some \( B = B(\beta) \). This choice corresponds to unrenormalized mass ratios. We note however that there is a choice which has even fewer poles\(^1\):

\[
\epsilon_a = a \epsilon, \quad \eta_a = \left( a - h \right) \epsilon, \quad a = 0, \ldots, n - 1
\]

\( \epsilon \) depends on \( \beta \) and satisfies \(-\frac{1}{2n} \leq \epsilon \leq 0\) in order to fulfill eq.(13). This reduces eq.(15) to

\[
S_{an} = \prod_{p=1}^{a-1} \frac{\left( \frac{h}{2} + (-a + 2p)(1 + \epsilon) \right)^2}{\left( \frac{h}{2} + (-a + 2p)(1 + \epsilon) - 2h\epsilon \right) \left( \frac{h}{2} + (-a + 2p)(1 + \epsilon) + 2h\epsilon \right)} \times \frac{\left( \frac{h}{2} - a(1 + \epsilon) \right) \left( \frac{h}{2} + a(1 + \epsilon) \right)}{\left( \frac{h}{2} - a(1 + \epsilon) - 2h\epsilon \right) \left( \frac{h}{2} + a(1 + \epsilon) + 2h\epsilon \right)}
\]

This expression still has two simple poles at \( \theta = \frac{i\pi}{h} \left( \frac{h}{2} + a(1 + \epsilon) \right) \) and at \( \theta = \frac{i\pi}{h} \left( \frac{h}{2} - a(1 + \epsilon) \right) \) as well as several double poles. We discuss the double poles later on and note here that the single poles correspond to particle \( \phi_n \) in the intermediate \( s- \) and \( u- \) channels. Indeed

\[
s - \tilde{m}_n^2 = \tilde{m}_n^2 + \tilde{m}_a^2 + 2\tilde{m}_n\tilde{m}_a \text{ch} \theta - \tilde{m}_n^2 = 0 \quad \text{at} \quad \theta = \frac{i\pi}{h} \left( \frac{h}{2} + a(1 + \epsilon) \right)
\]

and similarly for the \( u- \) channel.

Let us introduce some simplifying notation:

\[
H = \frac{h}{1 + \epsilon}, \quad B = -2h\frac{\epsilon}{1 + \epsilon}
\]

\[
\left( x \right)_H = \frac{\text{sh} \left( \frac{a}{2} + \frac{i\pi}{2H} x \right)}{\text{sh} \left( \frac{a}{2} - \frac{i\pi}{2H} x \right)}, \quad \left\{ x \right\}_H = \frac{\left( x - 1 \right)_H \left( x + 1 \right)_H}{\left( x - 1 + B \right)_H \left( x + 1 - B \right)_H}
\]

and write eq.(12) and eq.(17) as

\[
S_{nn} = \prod_{a=0}^{n-1} \frac{\left( 2a \right)_H \left( H - 2a \right)_H}{\left( 2a + B \right)_H \left( H - 2a - B \right)_H}
\]

\[
S_{an} = \prod_{p=1}^{a} \left\{ \frac{H}{2} + 2p - a - 1 \right\}_H = S_{na}.
\]

In terms of this notation eq.(11) implies that the renormalized masses have the same form as the classical masses, but again with \( h \) replaced by \( H \).

\(^1\)Choosing \( \eta_0 = 0 \) instead of \( \eta_0 = -h\epsilon \) leads to the S-matrix for the \( A^{(1)}(0,2n) \) Toda theory, as we discuss at the end of the paper.
The remaining S-matrix elements $S_{ab}, a, b = 1 \cdots n - 1$ are obtained by another application of the bootstrap

$$S_{ab}(\theta) = S_{nb}(\theta + \frac{1}{2} \theta^a_{mn}) S_{nb}(\theta - \frac{1}{2} \theta^a_{mn})$$

$$= \prod_{p=1}^{b} \{2p - b - 1 + a\} H \{H + 2p - b - 1 - a\} H$$

(23)

This expression is symmetric in $a, b$ as one can verify by using relations such as $\prod_{x}^{p=1} \{2p - x\} H = 1$. Crossing symmetry can be easily checked by a change of variable $p \rightarrow -p + b + 1$ in one of the terms in eq.(23).

$S_{ab}$ has four simple poles. For $a > b$, these are located at $\frac{H}{i\pi} \theta = (a - b), (H - a + b), (H - a - b)$ and $(a + b)$. Let us check whether we can identify these poles as single particle poles. If the pole at $\theta = i\pi H(a - b)$ corresponds to a particle in the $u$-channel then this particle has mass

$$\tilde{m}^2 = \tilde{m}_a^2 + \tilde{m}_b^2 - 2\tilde{m}_a\tilde{m}_b \cos \frac{\pi}{H}(a - b) = 4\tilde{m}_n^2 \sin^2 \frac{(a - b)\pi}{H}$$

(24)

and this identifies it as particle $\phi_{(a-b)}$. Similarly the pole at $\theta = i\pi H(h - a + b)$ is identified as the $s$-channel pole of the same particle. In the case $a = b$ the pole is located at the edge of the physical sheet and does not correspond to a single particle pole.

The same calculation for the pole at $\theta = i\pi H(a + b)$ leads to a mass

$$\tilde{m}^2 = \tilde{m}_a^2 + \tilde{m}_b^2 + 2\tilde{m}_a\tilde{m}_b \cos \frac{\pi}{H}(a + b) = 4\tilde{m}_n^2 \sin^2 \frac{(a + b)\pi}{H}$$

(25)

If $a + b < \frac{H}{2}$ then this is the mass of particle $\phi_{(a+b)}$. But if $a + b > \frac{H}{2}$ the pole does not appear at the expected position. The particle $\phi_{(h-a-b)}$ has a mass

$$\tilde{m}^2_{(h-a-b)} = 4\tilde{m}_n^2 \sin^2 \frac{(h - a - b)\pi}{H}$$

(26)

which is not equal to eq.(25) due to the fact that $h \neq H$. As we will now explain, this displacement of the pole is due to the presence of an anomalous threshold singularity.

In perturbation theory (or in a pure analyticity-unitarity S-matrix approach [3]) using renormalized masses, the amplitude $S_{ab}$ for $a + b > n$ has not only a simple pole corresponding to the particle $\phi_{h-a-b}$ but also neighboring poles produced as anomalous threshold singularities from the various diagrams in Fig. (1.b,c,d,e,f). Indeed, using the value of the renormalized masses it is straightforward to check by means of a dual diagram analysis that the triangle diagrams produce pole singularities located at $\theta = i\pi H(a + b)$ and the crossed box in Fig. (1.d) has a double pole at the same position. Using standard formulas the coefficients of these poles are

$$T_{ab} = \frac{1}{8} \frac{\sin \frac{\pi}{H}(a + b)}{\cos \frac{\pi}{H} a \cos \frac{\pi}{H} b}$$

$$R_{ab} = 8T_{ab}^2 \sin \frac{2\pi}{H}(a + b)$$

(27)
respectively, multiplied by appropriate coupling constants. Specifically, denoting by \( \delta_{ab} \) the shift of the S-matrix pole from its expected position, i.e.

\[
\delta_{ab} = 4\tilde{m}_n^2 \left( \sin^2 \frac{\pi}{H} (a + b) - \sin^2 \frac{\pi}{H} (h - a - b) \right)
\]  

(28)

and \( \sigma = s - \tilde{m}_{h-a-b}^2 \) we have the contributions from the six diagrams

(a) : \[
\frac{1}{\sigma} c_{ab,h-a-b}^2
\]

(b) : \[
\frac{T_{ab}}{\sigma - \delta_{ab}} c_{ann} c_{bnn} c_{abnn}
\]

(c) : \[
\frac{T_{ab}}{\sigma(\sigma - \delta_{ab})} c_{ann} c_{bnn} c_{ab,h-a-b} c_{nn,h-a-b}
\]

(d) : \[
\frac{R_{ab}}{(\sigma - \delta_{ab})^2 c_{ann}^2 c_{bnn}^2}
\]

(e) : \[
\frac{T_{ab}^2}{(\sigma - \delta_{ab})^2 c_{ann}^2 c_{bnn}^2 c_{nnn}}
\]

(f) : \[
\frac{T_{ab}^2}{\sigma(\sigma - \delta_{ab})^2 c_{ann}^2 c_{bnn}^2 c_{nn,h-a-b}}
\]

(29)

Here the coupling constants can be obtained directly from the lagrangian in a lowest order computation, but for a complete comparison higher-order corrections should be included, as well as contributions from the subleading parts of the triangle and crossed box diagrams.

We have checked that in the sum of the above terms the pole at \( \sigma = 0 \) cancels, leaving a simple pole at \( \sigma = \delta \) with the correct \( O(\beta^2) \) residue and indeed reproducing the result obtained from the exact S-matrix. This explains the shift of the simple pole in the S-matrix from the expected location at the (renormalized) mass. We emphasize that the cancellation of the pole at \( \sigma = 0 \) is due to a subtle interplay between the location of the anomalous threshold poles and their residues.

It is interesting to contrast the above discussion of the anomalous poles with that in the \( A^{(2)}(0, 2n - 1) \) theory \[11\] which differs from our theory by the addition of one fermion: there, the bosonic field \( \phi_n \) gives rise to the same anomalous threshold structure, but this is precisely cancelled by an identical structure from the fermion, so that the simple particle pole ends up in the position predicted by the mass formula (of the classical theory).

As an aid to further discussion we note that many of the results and expressions derived in Ref. \[11\] for some of the amplitudes and the dual diagram constructions in the \( A^{(2)}(0, 2n - 1) \) theory can be taken over to our theory. Also, the exact S-matrix elements \( S_{an} \) and \( S_{ab} \) have the same form in the two theories, provided the blocks are reinterpreted according to our definitions above i.e. with \( h \to H \).

We consider now the double poles in the amplitude \( S_{an} \) which occur at

\[
\theta = i\frac{\pi}{H} \left( \frac{H}{2} + 2p - a \right) \quad p = 1, 2, \ldots a - 1
\]  

(30)
It is straightforward to check that indeed these can be accounted for by both uncrossed and crossed box diagrams with dual diagrams similar to the ones of Fig. 12 in Ref. [11]. We emphasize that although the construction is done with the renormalized masses, so that the lengths of the lines are different from what they were in the above reference, nonetheless the dual diagrams exist for both the crossed and the uncrossed boxes. The only change is in the location of the double poles, and this agrees with their location in the exact S-matrix. To one-loop order the coefficients of these double poles are the same as for the superalgebra case (where these contributions are also from bosons only), and since our $S_{an}$ has the same structure there is no need for any further checks at the one-loop level.

A new feature appears when we examine the double poles of the $S_{ab}$ amplitudes. For example, in the superalgebra case one obtained double poles from both an uncrossed and a crossed box, cf. Fig. 13 of reference [11] for the case of the $S_{n-1,n-1}$ amplitude. Now however, using the renormalized masses, the dual diagram for the uncrossed box can no longer be drawn in the plane and it would seem that although a double pole is still produced by the crossed box its coefficient would not match that of the exact S-matrix; the contribution from the uncrossed box is needed. The explanation is provided by realizing that in addition to the box diagram there are seemingly higher-order diagrams where one of the internal lines is replaced by the set of diagrams appearing in Fig. 1 which are responsible for the displacement of the single particle pole. In going through the Landau-Cutkoski analysis for the location and nature of singularities, one realizes then that a double pole is indeed produced, with the correct residue. Equivalently one can perform the dual diagram analysis using not the actual particle mass $\tilde{m}^2_{a-b}$ but that corresponding to the actual pole of the S-matrix, namely $\tilde{m}^2_{a-b} + \delta_{ab}$. In this manner all the double poles of the exact S-matrix can be accounted for, and aside from verifying that the higher-order coefficients are correctly given we claim to have checked the self-consistency of the S-matrix we have constructed.

To make contact with the lagrangian of the $a^{(2)}_{2n-1}$ theory we have performed some further perturbation theory checks. We assert that our S-matrix agrees at tree level with that computed from the lagrangian in eq.(2). This is obvious for the amplitudes $S_{an}$ and $S_{ab}$ provided we choose $\epsilon(\beta) = -\frac{\beta^2}{4\pi h} + O(\beta^4)$, since the corresponding amplitudes of the $A^{(2)}(0,2n-1)$ theory were checked to agree with the tree level amplitudes and the bosonic lagrangian of that theory is identical to our lagrangian. On the other hand our $S_{nn}$ amplitude has a rather different form, and a separate check was necessary. In this respect, extending the product in eq.(12) to include the $a = 0$ term was crucial.

We observe that our S-matrix predicts a very specific manner in which the masses of the theory renormalize:

$$\tilde{m}_a^2 = 4\tilde{m}_n^2 \sin^2 \left( \frac{\pi a}{h} \left( 1 + \epsilon \right) \right)$$

This is consistent with the one-loop mass corrections in eq.(3), and already provides a nontrivial check of our S-matrix and of the restrictions that follow from the bootstrap. Of course it would be interesting to perform a direct two-loop calculation of the mass shifts and compare with those predicted here.
As already discussed, there is agreement at the one-loop level between the coefficients of the double poles in the exact S-matrix and those computed perturbatively. However, one-loop checks also allow us to determine the coupling constant dependence of the S-matrix. We have calculated from the lagrangian the one-loop corrections to the three-point functions for the $a^{(2)}_{3}$ theory. These corrections arise from three sources: genuine vertex corrections, wave-function renormalization effects, and a rescaling of the mass scale in the lagrangian (which rescales all the couplings) that we performed in order to keep the mass of the particle $\phi_n$ at its classical value. The result, when substituted in the perturbative S-matrix, can be compared with that from the exact S-matrix through order $\beta^4$ and $\epsilon^2$ respectively. We find agreement provided we choose

$$\epsilon = -\frac{\beta^2}{4\pi h} + (h + 1) \left(\frac{\beta^2}{4\pi h}\right)^2 + O(\beta^6)$$

(32)

and therefore, to this order

$$B \equiv -\frac{2h\epsilon}{1 + \epsilon} = \frac{1}{2\pi} \frac{\beta^2}{1 + \frac{\beta^2}{4\pi}}$$

(33)

which is the standard dependence on the coupling constant that one has found in simply-laced theories [12].

Finally, as a further perturbative check, we have computed for the $a^{(2)}_{3}$ case, up to one-loop order, the spin 3 charges of the two particles of the theory, and found agreement with those predicted by the exact S-matrix [8].

Let us summarize: we have used very little information from the lagrangian of the $a^{(2)}_{2n-1}$ Toda theory to determine the S-matrix in eqs.(21,22,23). We assumed the existence of a higher spin symmetry but did not need its explicit form. We admitted the fact that the classical masses renormalize in a nontrivial manner but did not make any assumptions about the form of this renormalization; and we used the existence of three-point couplings $c_{nna}$ but did not use their values. This information was almost sufficient to determine the S-matrix using unitarity, real analyticity and the bootstrap principle. We chose not to include more CDD poles and zeros in $S_{nn}$ and we chose the smallest number of poles in $S_{an}$. We obtained an S-matrix which satisfies all the bootstrap consistency conditions except those arising from certain displaced poles in $S_{ab}$. We explained the displaced simple poles and all the higher-order poles as arising from anomalous threshold singularities. Finally, by comparing with perturbative calculations, we demonstrated that we are indeed dealing with the S-matrix of the $a^{(2)}_{2n-1}$ Toda theory. In particular, we obtained the relation between the parameter $B$ and the coupling constant $\beta$ of the Toda theory. Obviously, given the present state of the technology, this could only be done to low orders of perturbation theory, but in this respect the situation is similar to that of most simply-laced Toda theories.

We comment on two other features: first, application of the thermodynamic Bethe Ansatz [13] leads to the prediction of the central charge in the conformal limit, $c = n$, as one might expect. Second, it does not appear that a minimal S-matrix exists for this theory. One cannot drop the blocks involving the dependence
on the coupling constant through the parameter $B$ since this would lead to a larger number of poles in eq. (13) than can be accounted for.

Finally, it appears that with a minor modification our S-matrix also describes scattering for the Toda theory based on the Lie superalgebra $A^{(4)}(0, 2n - 1)$. We recall that this theory can be obtained from the $A^{(2)}(0, 2n - 1)$ theory by dropping the boson $\phi_n$ instead of the fermion $\psi$. Consequently the particle masses are again shifted away from their classical values, this time by an amount which is just the negative of the one in eq. (5). Defining for $S_{\psi\psi}$ the same expression as for $S_{nn}$ but choosing $\eta_0 = 0$ in eq. (16) and replacing $\epsilon$ by $-\epsilon$ leads by bootstrap to a $S_{\psi a}$ and $S_{ab}$ which have the same form as the previous $S_{na}$ and $S_{ab}$ and provide a consistent description of the S-matrix for this Toda theory.

Details of our work and extension to other nonsimply-laced theories will be presented in separate publications [8, 9].

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Figure 1: Contributions to the $S_{ab}$ amplitude which are responsible for the shift in the particle pole position.

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