ON THE ZEROS OF EPSTEIN ZETA FUNCTIONS NEAR THE CRITICAL LINE

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Abstract. Let $Q$ be a positive definite quadratic form with integral coefficients and let $E(s, Q)$ be the Epstein zeta function associated with $Q$. Assume that the class number of $Q$ is bigger than 1. Then we estimate the number of zeros of $E(s, Q)$ in the region $\Re s > \sigma_T(\theta) := 1/2 + (\log T)^{-\theta}$ and $T < \Im s < 2T$, to provide its asymptotic formula for fixed $0 < \theta < 1$ conditionally. Moreover, it is unconditional if the class number of $Q$ is 2 or 3 and $0 < \theta < 1/13$.

1. Introduction

Let $K = \mathbb{Q}(\sqrt{-D})$ be a quadratic imaginary field of class number $h := h_D$ and let $\chi_1, \ldots, \chi_h$ be its ideal class characters. The Hecke $L$-function attached to $\chi_j$ is defined by

$$L_j(s) := L(s, \chi_j) := \sum_n \frac{\chi_j(n)}{N(n)^s} = \prod_p \left(1 - \frac{\chi_j(p)}{N(p)^s}\right)^{-1}$$

for $\Re s > 1$, where $N$ is the norm. Each $L_j$ has an analytic continuation to $\mathbb{C}$ except for a possible pole at $s = 1$ and it satisfies the functional equation

$$\left(\frac{\sqrt{-D}}{2\pi}\right)^s \Gamma(s) L_j(s) = \left(\frac{\sqrt{-D}}{2\pi}\right)^{1-s} \Gamma(1-s) L_j(1-s).$$

(1.1)

By the Euler product and (1.1), $L_j$ has no zeros in $\Re s > 1$ and the negative integers are the only zeros of $L_j$ in $\Re s < 0$. All the other zeros are on the strip $0 \leq \Re s \leq 1$ and we believe that they are actually on the line $\Re s = 1/2$.

These Hecke $L$-functions have a functional relation with an Epstein zeta function. To be precise, let $Q$ be a positive definite quadratic form with integral coefficients and its fundamental discriminant $D$. The Epstein zeta function $E(s, Q)$ associated with $Q$ is defined by

$$E(s, Q) := \sum_{m,n \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{Q(m,n)^s}$$

for $\Re s > 1$. It satisfies

$$E(s, Q) = \frac{w_D}{h_D} \sum_j \overline{\chi_j(a_Q)} L_j(s),$$

(1.2)

where $w_D$ is the number of roots of unity in $K$ and $a_Q$ is an integer ideal in the ideal class corresponding to the equivalence class of $Q$. If $h_D = 1$, then the Epstein zeta function $E(s, Q) = w_D L_1(s)$ is nothing but a Hecke $L$-function up to a constant factor. Hence we expect $E(s, Q)$ satisfy the generalized Riemann hypothesis. However, if $h_D > 1$, the distribution of zeros of $E(s, Q)$ is different to the Riemann zeta function and indeed $E(s, Q)$

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has zeros off the line $\Re s = 1/2$. Davenport and Heilbronn [8] showed that $E(s, Q)$ has infinitely many zeros on $\Re s > 1$. Voronin [31] showed that the number of zeros of $E(s, Q)$ in the rectangle $\sigma_1 < \Re s < \sigma_2$ and $T < \Im s < 2T$ is

$$N_{E(s, Q)}(\sigma_1, \sigma_2 : T, 2T) \gg T$$

for any fixed $1/2 < \sigma_1 < \sigma_2 < 1$ as a consequence of a joint universality of Hecke $L$-functions. The author in [8] proved that

$$\lim_{T \to \infty} \frac{1}{T} N_{E(s, Q)}(\sigma_1, \sigma_2 : T, 2T) = \int_{\sigma_1}^{\sigma_2} g(\sigma) d\sigma$$

holds for any $1/2 < \sigma_1 \leq \sigma_2$ and some nonnegative continuous function $g(\sigma)$. By a straightforward adaptation of [7], the author in [9] improved the above asymptotic formula to

$$\frac{1}{T} N_{E(s, Q)}(\sigma_1, \sigma_2 : T, 2T) = \int_{\sigma_1}^{\sigma_2} g(\sigma) d\sigma + O\left(\frac{\log \log T}{(\log T)^{\sigma_1/2}}\right)$$

for $h = 2, 3$ and fixed $1/2 < \sigma_1 < \sigma_2 < 1$. The author with Gonek in [4] considered the case $h > 3$ and proved that

$$\frac{1}{T} N_{E(s, Q)}(\sigma_1, \sigma_2 : T, 2T) = \int_{\sigma_1}^{\sigma_2} g(\sigma) d\sigma + O(e^{-b\sqrt{\log \log T}})$$

for fixed $1/2 < \sigma_1 < \sigma_2 < 1$.

Now we examine the zero-density on or near the $1/2$-line for a linear combination of $L$-functions with a same functional equation, which generalizes our Epstein zeta functions. Bombieri and Hejhal in [2] proved that almost all zeros of a linear combination $F(s)$ of inequivalent $L$-functions with a same functional equation are simple and on the $1/2$-line assuming RH and a zero-spacing assumption for each $L$-function. Hejhal in [6] investigated the zeros of $F(s) = (\cos \alpha)e^{iw_1}L_1(s)(\sin \alpha)e^{iw_2}L_2(s)$ near the $1/2$-line and showed that

$$\frac{T \log T}{GV \log \log T} \ll N_{F(s)}(\sigma_1, \infty : T, 2T) \ll \frac{T \log T}{GV \log \log T} \quad (1.3)$$

for $\sigma_1 = 1/2 + G/\log T$ and for almost all $\alpha$, where $(\log \log T)^{\kappa} \leq G \leq (\log T)^{1-\delta}$, $\kappa \in (1, 3)$ and $\delta \in (0, 1/10)$. We expect that it holds for all $\alpha$ except for the cases $\cos \alpha = 0$ and $\sin \alpha = 0$, but it is still an open question whether a given Epstein zeta function with class number 2 or 3 satisfy (1.3). Selberg in [10] sketched his idea which proves (1.3) for almost all linear combinations $F(s)$ of $L$-functions.

The aim of this paper is finding an asymptotic formula for the zero counting function

$$N_{E(s, Q)}(\sigma_T, \infty : T, 2T)$$

of a given Epstein zeta function $E(s, Q)$ as $T \to \infty$, where

$$\sigma_T = \sigma_T(\theta) = \frac{1}{2} + \frac{1}{(\log T)^\theta}$$

for fixed $0 < \theta < 1$. Let $L_1, \ldots, L_J$ be inequivalent Hecke $L$-functions on $K$, i.e., $L_j(s) = L(s, \chi_j)$ and $\chi_j \neq \chi_\ell, \chi_\ell$ for $j \neq \ell$. By the Euler product, we may write

$$\log L(s, \chi_j) = \sum_p \sum_{n=1}^{\infty} \frac{a_j(p^n)}{p^{ns}}$$
for \( \Re s > 1 \), then it is well-known that
\[
\sum_p \frac{a_j(p)a_\ell(p)}{p^{2\sigma T}} = \delta_{j,\ell} \xi_j \theta \log \log T + c_{j,\ell} + O\left(\frac{1}{(\log T)^\theta}\right) \tag{1.4}
\]
for \( j, \ell \leq J \), where \( \delta_{j,\ell} = 1 \) if \( j = \ell \), \( \delta_{j,\ell} = 0 \) if \( j \neq \ell \), \( \xi_j = 4 \) if \( \chi_j \) is real, \( \xi_j = 2 \) if \( \chi_j \) is nonreal and the \( c_{j,\ell} \) are some constants. Consider
\[
F_J(s) := \sum_{j=1}^J b_j L_j(s)
\]
for \( b_1, \ldots, b_J \in \mathbb{C}\ \setminus \{0\} \) satisfying
\[
|b_1|^2 + \cdots + |b_J|^2 = 1.
\]
The Epstein zeta function \( E(s, Q) \) is a special case of \( F_J(s) \) up to a constant factor by (1.2) and the relation \( L(s, \chi) = L(s, \bar{\chi}) \). In this case, \( J \) is the sum of the number of real characters and the half of the number of non-real characters. Hence, \( J = 2 \) if \( h = 2, 3 \) and \( J > 2 \) if \( h > 3 \).

By Littlewood’s lemma, the zero counting function
\[
N_{F_J}(\sigma : T) := N_{F_J}(\sigma, \infty : T, 2T)
\]
for \( \sigma > 1/2 \) is essentially a derivative of the integral
\[
\frac{1}{2\pi} \int_T^{2T} \log |F_J(\sigma + it)| dt.
\]
Moreover, we believe that the following conjecture is true.

**Conjecture 1.1.** Let \( J > 1 \), \( 0 < \theta < 1 \) and \( \sigma_T = 1/2 + (\log T)^{-\theta} \). Then there exists \( \eta > 0 \) such that
\[
\frac{1}{T} \int_T^{2T} \log |F_J(\sigma_T + it)| dt = \mathbb{E}[\log |F_J(\sigma_T : X)|] + O\left(\frac{1}{(\log T)^\eta}\right) \tag{1.5}
\]
as \( T \to \infty \), where
\[
F_J(s : X) := \sum_{j=1}^J b_j L_j(s : X)
\]
and
\[
L_j(s : X) := \prod_p \left(1 - \frac{\chi_j(p)X(N(p))}{N(p)^s}\right)^{-1}
\]
are the random models of \( F_J(s) \) and \( L_j(s) \) for \( j = 1, \ldots, J \). Here, the \( X(p) \) are uniformly and independently distributed on the unit circle \( \mathbb{T} \) and \( X(p^\ell) := X(p)\ell \).

Conjecture [1.1] for \( J > 2 \) is technically more difficult than the estimates in [4] for fixed \( \sigma > 1/2 \), since there are more logarithmic singularities near the 1/2-line. However, if \( J = 2 \), it is possible to prove Conjecture [1.1] for a small \( \theta \). One sees that
\[
\int_T^{2T} \log |F_2(\sigma_T + it)| dt = \int_T^{2T} \log |b_1 L_2(\sigma_T + it)| dt + \int_T^{2T} \log \left|\frac{L_1}{L_2}(\sigma_T + it) + \frac{b_2}{b_1}\right| dt.
\]
The first integral on the right hand side can be estimated by an usual Dirichlet polynomial approximation for \( \log |L_2(\sigma_T + it)| \) and the second integral can be estimated by adapting [5]. Since its proof is straightforward from [5], we state it without a proof as follows.

**Theorem 1.2.** Conjecture 1.1 holds for \( J = 2 \) and \( 0 < \theta < 1/13 \) with \( \eta < (1 - 13\theta)/4 \).

The main feature of this paper is our estimation of

\[
\mathbb{E}[\log |F_j(\sigma_T : X)|]
\]

as \( T \to \infty \) for \( J > 1 \).

**Theorem 1.3.** Let \( J > 1 \), \( 0 < \theta < 1 \) and \( \sigma_T = 1/2 + (\log T)^{-\theta} \). Then

\[
\mathbb{E}[\log |F_j(\sigma_T : X)|] = \frac{\sqrt{\theta} \log \log T}{\sqrt{\pi} j} \sum_{\ell=1}^{J} \int_{\mathcal{R}_\ell} u_\ell e^{-\sum_j \xi_j^2 / \xi_j} du
\]

\[
+ \sum_{\ell=1}^{J} \frac{\log |b_j|}{\sqrt{\pi} j} \int_{\mathcal{R}_\ell} e^{-\sum_j \xi_j^2 / \xi_j} du + O\left(\frac{1}{(\log \log T)^{1/4}}\right)
\]

as \( T \to \infty \), where

\[
\xi := \prod_j \xi_j,
\]

\[
\mathcal{R}_\ell := \{(u_1, \ldots, u_J) : u_\ell = \max\{u_1, \ldots, u_J\}\}
\]

for \( \ell = 1, 2, \ldots, J \) and \( du = du_1 \cdots du_J \).

One expects that Littlewood’s lemma, Conjecture 1.1 and Theorem 1.3 imply an asymptotic of \( N_{F_j}(\sigma_T : T) \) as \( T \to \infty \), but the \( O \)-term in Theorem 1.3 is too big. Instead, we estimate the difference

\[
\mathbb{E}[\log |F_j(\sigma_T(\theta_1) : X)|] - \mathbb{E}[\log |F_j(\sigma_T(\theta_2) : X)|]
\]

for a small \( |\theta_1 - \theta_2| \) to prove our main theorem.

**Theorem 1.4.** Let \( J \geq 2 \) and \( 0 < \theta < 1 \) and assume Conjecture 1.1, then

\[
N_{F_j}(\sigma_T(\theta) : T) = \frac{1}{4\pi^{1+j/2}} \frac{(\log T)^{\theta}}{\sqrt{\xi} \log \log T} \sum_{\ell=1}^{J} \int_{\mathcal{R}_\ell} u_\ell e^{-\sum_j \xi_j^2 / \xi_j} du + O\left(\frac{(\log T)^{\theta}}{(\log \log T)^{5/4}}\right)
\]

as \( T \to \infty \), where \( \xi \) and \( \mathcal{R}_\ell \) are defined in Theorem 1.3.

By Theorem 1.2 we see that Theorem 1.4 holds for \( J = 2 \) and \( 0 < \theta < 1/13 \) unconditionally. When \( F_j(s) \) is the Epstein zeta function \( E(s, Q) \) up to a constant factor, it is interesting to see that

\[
\xi = 2^{3J - h}.
\]

### 2. Proof of Theorems 1.3 and 1.4

Let

\[
L(\sigma : X) := (|\log L_1(\sigma : X)|, \ldots, |\log L_J(\sigma : X)|, \Im L_1(\sigma : X), \ldots, \Im L_J(\sigma : X)).
\]

Define

\[
\Psi_{\theta,T}(B) := \mathbb{P}(L(\sigma_T : X) \in B) = \text{meas}\{X \in \mathbb{T}^\infty : L(\sigma_T : X) \in B\}
\] (2.1)
for a Borel set $B$ in $\mathbb{R}^{2J}$. It is known that the measure $\Psi_{\theta,T}$ has a density function $G_{\theta,T}$, so that for

$$\Psi_{\theta,T}(B) = \int_B G_{\theta,T}(u,v)dudv,$$

where $u = (u_1, \ldots, u_J), v = (v_1, \ldots, v_J) \in \mathbb{R}^J$. (For instance, see the proof of Lemma 3.3 in [1]). Then we see that

$$\mathbb{E}[\log |F_J(\sigma_T : X)|] = \int_{\mathbb{R}^{2J}} \log |\sum_{j \leq J} b_je^{uj+ivj}| d\Psi_{\theta,T}(u,v)$$

$$= \int_{\mathbb{R}^{2J}} \log |\sum_{j \leq J} b_je^{uj+ivj}| G_{\theta,T}(u,v)dudv \quad (2.2)$$

and

$$\hat{\Psi}_{\theta,T}(x,y) = \int_{\mathbb{R}^J} e^{2\pi i(u \cdot x + v \cdot y)} d\Psi_{\theta,T}(u,v) = \int_{\mathbb{R}^J} e^{2\pi i(u \cdot x + v \cdot y)} G_{\theta,T}(u,v)dudv,$$

where $u = (u_1, \ldots, u_J), v = (v_1, \ldots, v_J), x = (x_1, \ldots, x_J), y = (y_1, \ldots, y_J) \in \mathbb{R}^J, u \cdot x = u_1x_1 + \cdots + u_Jx_J$ and $v \cdot y = v_1y_1 + \cdots + v_Jy_J$. Since $G_{\theta,T}(u,v)$ is the inverse Fourier transform of $\hat{\Psi}_{\theta,T}(x,y)$

$$G_{\theta,T}(u,v) = \int_{\mathbb{R}^J} e^{-2\pi i(u \cdot x + v \cdot y)} \hat{\Psi}_{\theta,T}(x,y)dudv,$$

we examine various properties of $\hat{\Psi}_{\theta,T}(x,y)$ to study $G_{\theta,T}(u,v)$.

**Proposition 2.1.** There exist constants $c_1, c_2 > 0$ such that

$$|\hat{\Psi}_{\theta,T}(x,y)| \leq \exp\left(-c_1 \frac{(\sum_{j \leq J} x_j^2 + y_j^2)^{1/2}}{\log(\sum_{j \leq J} x_j^2 + y_j^2)}\right)$$

for $\sum_{j \leq J} x_j^2 + y_j^2 \geq c_2$,

$$|\hat{\Psi}_{\theta,T}(x,y)| \leq \exp\left(-\frac{\pi^2 t}{2} \log \log T + O(1) \right) \sum_{j \leq J} (x_j^2 + y_j^2)$$

for $\sum_{j \leq J} (x_j^2 + y_j^2) \leq e^{(\log T)^{3/2}}$, and

$$\hat{\Psi}_{\theta,T}(x,y) = e^{-\pi^2 \theta \log \log T \sum_{j \leq J} \xi_j(x_j^2 + y_j^2)} \left( P(x,y) + O\left( \sum_{j=1}^{J} (x_j^2 + y_j^2)^{3} + \frac{1}{(\log T)^{3/2}} \right) \right)$$

for $\sum_{j} x_j^2 + y_j^2 \leq c_2$, where

$$P(x,y) := \sum_{k, \ell \in (\mathbb{Z}_{\geq 0})^J} \tilde{B}_{k, \ell} x_k y_{\ell}$$

and the coefficients $\tilde{B}_{k, \ell}$ are independent to $\theta$ and $T$ satisfying

$$\tilde{B}_{0,0} = 1$$

and

$$\tilde{B}_{k, \ell} = 0$$
if \( K(k + \ell) = 1 \) or \( > 5 \). Here, \( x = (x_1, \ldots, x_J), y = (y_1, \ldots, y_J) \in \mathbb{R}^J, k = (k_1, \ldots, k_J), \ell = (\ell_1, \ldots, \ell_J) \in (\mathbb{Z}_{\geq 0})^J \) and

\[
K(k) := k_1 + \cdots + k_J, \quad x^k := \prod_{j \leq J} x_j^{k_j}.
\]

**Proposition 2.2.** Let \( k, \ell, m, n \) be vectors in \((\mathbb{Z}_{\geq 0})^J\) and

\[
q_{k,\ell,m,n} = \frac{B_{2k+m,2\ell+n}}{i^{k(m+n)}2\ell+K(2k+2\ell+m+n)\prod_j \Gamma((k_j+1/2)\Gamma((\ell_j+1/2)/\xi_j^k+\ell_j+m_j+n_j+1))}{(2k+m)!(2\ell+n)!}.
\]

Then, we have

\[
G_{\theta,T}(u,v) = \exp \left( -\sum_j u_j^2 + v_j^2 \right) \sum_{k,\ell,m,n \in (\mathbb{Z}_{\geq 0})^J} \frac{q_{k,\ell,m,n}}{(\theta \log \log T)^{J+K(k+\ell+m+n)}} u^m v^n
\]

\[
+ O\left( \frac{1}{(\log \log T)^{J+3}} \right)
\]

for all \( u = (u_1, \ldots, u_j), v = (v_1, \ldots, v_j) \in \mathbb{R}^J, \) and there exists a constant \( c > 0 \) such that

\[
G_{\theta,T}(u,v) \ll \exp \left( -c \sum_{j \leq J} u_j^2 + v_j^2 \right)
\]

for all \( u = (u_1, \ldots, u_j), v = (v_1, \ldots, v_j) \in \mathbb{R}^J. \)

Note that

\[
q_{0,0,0,0} = \pi^{-J} \prod_j \xi_j^{-1} = \pi^{-J} \xi^{-1}
\]

and

\[
q_{k,\ell,m,n} = 0
\]

if \( K(2k + 2\ell + m + n) = 1 \) or \( > 5 \). We also need the following lemma.

**Lemma 2.3.** Let \( k \) be a positive integer, \( M \geq 1 \) and \( b_j \in \mathbb{C} \) for \( j \leq J \). Then there exists an absolute constant \( C > 0 \) such that

\[
\int_{\mathbb{R}^{2J}} \log \left| \sum_{j \leq J} b_j e^{u_j + iv_j} \right|^2 e^{-\sum_{j \leq J} (u_j^2 + v_j^2)/M} dudv \ll M^{J+k}(Ck)^k + M^J(Ck)^{2k}.
\]

We prove the propositions and the lemma in \[3\]. Now we shall estimate the integral in \[2.2\]. By the Cauchy-Schwarz inequality, Proposition \[2.2\] and Lemma \[2.3\], we find that

\[
\mathbb{E}[\log |F_J(\sigma_T : X)|] = \int_{[-M_1,M_1]^{2J}} \log \left| \sum_{j \leq J} b_j e^{u_j + iv_j} \right| G_{\theta,T}(u,v)dudv + O\left( \frac{1}{(\log \log T)^\eta} \right)
\]

for any \( \eta > 0 \), where \( M_1 = \eta' \sqrt{\log \log T \log \log \log T} \) with \( \eta' > 0 \) depending on \( \eta \). By the Cauchy-Schwarz inequality and Lemma \[2.3\], we also have

\[
\left( \int_{[-M_1,M_1]^{2J}} \log \left| \sum_{j \leq J} b_j e^{u_j + iv_j} \right| dudv \right)^2 \leq (2M_1)^2 \int_{[-M_1,M_1]^{2J}} \left( \log \left| \sum_{j \leq J} b_j e^{u_j + iv_j} \right| \right)^2 dudv
\]
Once again, by the Cauchy-Schwarz inequality, Proposition 2.2 and Lemma 2.3, we find that
\[ \sum_{j \leq J} b_j e^{u_j + iv_j} \leq M_1^{2J} \int_{\mathbb{R}^{2J}} \left( \log \left| \sum_{j \leq J} b_j e^{u_j + iv_j} \right| \right)^2 e^{-\sum_j (u_j^2 + v_j^2)/M_1^2} dudv \]
\[ \leq M_1^{4J+2}. \]

Hence, by Proposition 2.2
\[ \mathbb{E}[\log |F_j(\sigma_T : X)|] = \sum_{k, \ell, m, n \in (\mathbb{Z}_{\geq 0})^J} \frac{q_{k, \ell, m, n}}{(\theta \log \log T)^{J+K(k+\ell+m+n)}} \int_{[-M_1, M_1]^{2J}} \log \left| \sum_{j \leq J} b_j e^{u_j + iv_j} \right| \exp \left( -\sum_j \frac{u_j^2 + v_j^2}{\theta \xi_j \log T} \right) u^m v^n dudv \]
\[ + O\left( \frac{(\log \log T)^{J+1/2}}{(\log T)^{5/2}} \right). \]

Once again, by the Cauchy-Schwarz inequality, Proposition 2.2 and Lemma 2.3, we find that
\[ \mathbb{E}[\log |F_j(\sigma_T : X)|] = \sum_{k, \ell, m, n \in (\mathbb{Z}_{\geq 0})^J} \frac{q_{k, \ell, m, n}}{(\theta \log \log T)^{J+K(k+\ell+m+n)}} I_{m,n}(\theta, T) + O\left( \frac{(\log \log T)^{J+1/2}}{(\log T)^{5/2}} \right), \]
where
\[ I_{m,n}(\theta, T) := \int_{\mathbb{R}^{2J}} \log \left| \sum_{j \leq J} b_j e^{(u_j + iv_j)\sqrt{\theta \log T}} \right| e^{-\sum_j (u_j^2 + v_j^2)/\xi_j} u^m v^n dudv. \]

The logarithm is dominated by the biggest term in the \( J \)-sum, so that we divide \( \mathbb{R}^{2J} \) into \( J \) pieces
\[ I_{m,n}(\theta, T) = \sum_{\ell=1}^J \int_{\mathcal{R}_\ell} \log \left| \sum_{j \leq J} b_j e^{(u_j + iv_j)\sqrt{\theta \log T}} \right| e^{-\sum_j (u_j^2 + v_j^2)/\xi_j} u^m v^n dudv, \]
where \( \mathcal{R}_\ell \) is defined in (1.6). By symmetry, it is enough to consider \( \mathcal{R}_1 \). Then
\[ \int_{\mathcal{R}_1} \log \left| \sum_{j \leq J} b_j e^{(u_j + iv_j)\sqrt{\theta \log T}} \right| e^{-\sum_j (u_j^2 + v_j^2)/\xi_j} u^m v^n dudv \]
\[ = \int_{\mathcal{R}_1} \log |b_1 e^{(u_1 + iv_1)\sqrt{\theta \log T}}| e^{-\sum_j (u_j^2 + v_j^2)/\xi_j} u^m v^n dudv + \mathcal{E}_{m,n,1}(\theta, T) \]
\[ = d_n \int_{\mathcal{R}_1} (\sqrt{\theta \log T} u_1 + \log |b_1|) e^{-\sum_j v_j^2/\xi_j} u^m du + \mathcal{E}_{m,n,1}(\theta, T), \]
where

\[ \mathcal{E}_{m,1}(\theta, T) := \int_{\mathbb{R}} \int_{\mathcal{R}_1} \log \left| 1 + \sum_{j=2}^{J} \frac{b_j}{b_1} e^{(u_j-u_1+i(v_j-v_1))\sqrt{\theta \log \log T}} e^{-\sum_j (u_j^2/v_j) \xi_j} u_m v^n \, du \, dv \right| \]

and

\[ d_n := \int_{\mathbb{R}} e^{-\sum_j v_j^2/\xi_j} v^n \, dv = \prod_{j \leq J} \left( \frac{\pi (n_j+1)}{2} \int_{\mathbb{R}} v^{n_j} e^{-v^2} \, dv \right). \tag{2.4} \]

Note that \( \int_{\mathbb{R}} v^{n_j} e^{-v^2} \, dv = 0 \) if \( n_j \) is odd, and \( = \Gamma((n_j + 1)/2) \) otherwise. Therefore,

\[ I_{m,n}(\theta, T) = \sum_{\ell=1}^{J} d_n \int_{\mathcal{R}_\ell} (\sqrt{\theta \log \log T} u_{\ell} + \log |b_{\ell}|) e^{-\sum_j v_j^2/\xi_j} u_m \, du + \mathcal{E}_{m,n}(\theta, T) \]

\[ = \sqrt{\theta \log \log T} \, d_n \sum_{\ell=1}^{J} \int_{\mathcal{R}_\ell} u_{\ell} e^{-\sum_j v_j^2/\xi_j} u_m \, du \]

\[ + d_n \sum_{\ell=1}^{J} \int_{\mathcal{R}_\ell} \log |b_{\ell}| e^{-\sum_j v_j^2/\xi_j} u_m \, du + \mathcal{E}_{m,n}(\theta, T), \tag{2.5} \]

where

\[ \mathcal{E}_{m,n}(\theta, T) := \sum_{\ell=1}^{J} \mathcal{E}_{m,n,\ell}(\theta, T). \]

By estimating \( \mathcal{E}_{m,n}(\theta, T) \) in (3.4), we prove the following proposition.

**Proposition 2.4.** Let \( J \geq 2 \). Then we have

\[ \mathcal{E}_{m,n}(\theta, T) = O\left( \frac{1}{(\log \log T)^{1/4}} \right). \]

Therefore, by (2.3), (2.5) and Propositions 2.3 and 2.2, we find that

\[ \mathbb{E}[\log |F_j(\sigma_T : X)|] \]

\[ = q_{0,0,0} d_0 \left( \sqrt{\theta \log \log T} \sum_{\ell=1}^{J} \int_{\mathcal{R}_\ell} u_{\ell} e^{-\sum_j v_j^2/\xi_j} \, du + \sum_{\ell=1}^{J} \int_{\mathcal{R}_\ell} \log |b_{\ell}| e^{-\sum_j v_j^2/\xi_j} \, du \right) \]

\[ + \mathcal{E}_{0,0,\ell}(\theta, T) = \pi^{-J/2} \prod_{j} \xi_j^{-1/2} \sqrt{\theta \log \log T} \sum_{\ell=1}^{J} \int_{\mathcal{R}_\ell} u_{\ell} e^{-\sum_j v_j^2/\xi_j} \, du \]

\[ + \pi^{-J/2} \prod_{j} \xi_j^{-1/2} \sum_{\ell=1}^{J} \log |b_{\ell}| \int_{\mathcal{R}_\ell} e^{-\sum_j v_j^2/\xi_j} \, du + O\left( \frac{1}{(\log \log T)^{1/4}} \right). \tag{2.6} \]

This proves Theorem 1.3.
Next we prove Theorem 1.4 assuming Conjecture 1.1. By Littlewood’s lemma and (2.3) we see that
\[
\int_{\sigma_T(\theta_2)}^{\sigma_T(\theta_1)} N_{F_j}(w : T) dw \\
= \frac{1}{2\pi} \int_T^{2T} \log |F_j(\sigma_T(\theta_1) + it)| dt \\
- \frac{1}{2\pi} \int_T^{2T} \log |F_j(\sigma_T(\theta_2) + it)| dt + O\left( \frac{T}{(\log T)^{\theta_2}} \right) \\
= \frac{T}{2\pi} \left( \mathbb{E}[\log |F_j(\sigma_T(\theta_1) : X)|] - \mathbb{E}[\log |F_j(\sigma_T(\theta_2) : X)|] \right) + O\left( \frac{T}{(\log T)^{\eta}} \right),
\]
where \( \sigma_T(\theta) = 1/2 + (\log T)^{-\theta} \) and \( 0 < \theta_2 < \theta_1 \). We need the following lemma.

**Lemma 2.5.** Let \( \alpha \) be a real number, \( \theta_1 > \theta_2 > 0 \) and \( H_T = \theta_1 - \theta_2 \). Suppose that \( H_T \to 0 \) as \( T \to \infty \). Then for each \( i = 1, 2 \) we have
\[
\theta_1^\alpha I_{m,n}(\theta_1, T) - \theta_2^\alpha I_{m,n}(\theta_2, T) \\
= H_T \sqrt{\log \log T} d_n \left( \alpha + \frac{1}{2} \right) \theta_i^{\alpha - 1/2} \sum_{\ell = 1}^{J} u_{\ell} e^{-\sum_{j} u_j^{2}/\xi_j} u^m \frac{d}{d\ell} \\
+ H_T d_n \alpha \theta_i^{\alpha - 1} \sum_{\ell = 1}^{J} \int_{R_{\ell}} \log |b_{\ell}| e^{-\sum_{j} u_j^{2}/\xi_j} u^m \frac{d}{d\ell} \phi \\
+ O\left( \frac{H_T}{(\log \log T)^{1/4}} + H_T^2 \sqrt{\log \log T} \right).
\]

We prove it in §3.5. Suppose that \( H_T \log \log T = o(1) \), then
\[
\int_{\sigma_T(\theta - H_T)}^{\sigma_T(\theta)} N_{F_j}(w : T) dw \leq \left( \sigma_T(\theta - H_T) - \sigma_T(\theta) \right) N_{F_j}(\sigma_T(\theta) : T) \\
= \frac{H_T \log \log T}{(\log T)^{\theta}} \left( 1 + O(H_T \log \log T) \right) N_{F_j}(\sigma_T(\theta) : T)
\]
and
\[
\int_{\sigma_T(\theta + H_T)}^{\sigma_T(\theta)} N_{F_j}(w : T) dw \geq \left( \sigma_T(\theta) - \sigma_T(\theta + H_T) \right) N_{F_j}(\sigma_T(\theta) : T) \\
= \frac{H_T \log \log T}{(\log T)^{\theta}} \left( 1 + O(H_T \log \log T) \right) N_{F_j}(\sigma_T(\theta) : T).
\]

By (2.7), (2.3) and Lemma 2.5 we find that
\[
\frac{H_T \log \log T}{(\log T)^{\theta}} \left( 1 + O(H_T \log \log T) \right) N_{F_j}(\sigma_T(\theta) : T) \\
= \frac{H_T T}{2\pi} \sum_{k, \ell, m, n \in \mathbb{Z}_{\geq 0}} \frac{q_{k, \ell, m, n}}{\left( \theta \log \log T \right)^{k(\ell + k) + k(m + n)}/2}
\]
\[
\left(\sqrt{\log \log T} d_n \left( - \mathcal{K}(k + \ell) + \frac{1 - \mathcal{K}(m + n)}{2} \right) \right)^{\theta - 1/2} \sum_{\ell = 1}^{J} \int_{\mathbb{R}_\ell} u_\ell e^{-\sum_j u_j^2/\xi_j} u^m du
\]
\[
+ d_n \left( - \mathcal{K}(k + \ell) - \frac{K(m + n)}{2} \right) \theta^{-1} \sum_{\ell = 1}^{J} \int_{\mathbb{R}_\ell} \log |b_\ell| e^{-\sum_j u_j^2/\xi_j} u^m du
\]
\[
+ O \left( \frac{T (\log \log T)^{J+1/2}}{(\log T)^{5/2}} + \frac{H_T T}{(\log T)^{1/4}} + H_T^2 T^{1/2} \right).
\]

Choose \( H_T = (\log \log T)^{-2} \) to optimize it, we see that
\[
N_{F_j}(\sigma_T(\theta) : T) = \frac{T (\log T)^\theta}{2\pi} \sum_{k, \ell, m, n \in (\mathbb{Z}_{\geq 0})^J} q(k, \ell, m, n) d_n \left( \frac{\theta \log \log T}{(\log \log T)^{1+K(k+\ell)+K(m+n)/2}} \right)
\]
\[
\left( \sqrt{\theta \log \log T} \left( - \mathcal{K}(k + \ell) + \frac{1 - \mathcal{K}(m + n)}{2} \right) \right)^{J} \sum_{\ell = 1}^{J} \int_{\mathbb{R}_\ell} u_\ell e^{-\sum_j u_j^2/\xi_j} u^m du
\]
\[
+ \left( - \mathcal{K}(k + \ell) - \frac{K(m + n)}{2} \right) \sum_{\ell = 1}^{J} \int_{\mathbb{R}_\ell} \log |b_\ell| e^{-\sum_j u_j^2/\xi_j} u^m du
\]
\[
+ O \left( \frac{T (\log T)^\theta}{(\log \log T)^{5/4}} \right).
\]

We see that the summands are smaller than the \( O \)-term unless \( k = \ell = 0 \) and \( \mathcal{K}(m + n) = 0,1 \). Moreover, \( q_{0,0,m,n} = 0 \) if \( \mathcal{K}(m + n) = 1 \). Hence,
\[
N_{F_j}(\sigma_T(\theta) : T) = \frac{T (\log T)^\theta}{\sqrt{\theta \log \log T}} \sum_{j = 1}^{J} \int_{\mathbb{R}_\ell} u_\ell e^{-\sum_j u_j^2/\xi_j} u^m du + O \left( \frac{T (\log T)^\theta}{(\log \log T)^{5/4}} \right).
\]

Since
\[
q_{0,0,0,0} = \pi^{-J/2} \prod_j \xi_j^{-1/2},
\]
we prove the theorem.

**3. Proof of propositions and lemmas**

3.1. **Proof of Proposition 2.1.** Let \( z_j = \pi(x_j + iy_j) \) for \( j = 1, \ldots, J \), then
\[
\hat{\Psi}_{\theta,T}(\mathbf{x}, \mathbf{y}) = \mathbb{E} \left[ \exp \left( 2\pi i \sum_{j \leq J} (x_j \mathbb{R} \log L_j(\sigma_T : X) + y_j \mathbb{I} \log L_j(\sigma_T : X)) \right) \right]
\]
\[
= \mathbb{E} \left[ \exp \left( 2\pi i \sum_{j \leq J} \mathbb{R} \left( (x_j - iy_j) \log L_j(\sigma_T : X) \right) \right) \right]
\]
\[
= \mathbb{E} \left[ \exp \left( i \sum_{j \leq J} z_j \log L_j(\sigma_T : X) + z_j \log L_j(\sigma_T : \bar{X}) \right) \right].
\]

Write
\[
\log L_j(\sigma : X) := \sum_p g_j(p, \sigma : X), \quad g_j(p, \sigma : X) := \sum_{n=1}^{\infty} \frac{a_j(p^n) X(p)^n}{p^{n\sigma}},
\]
Then

$$
\hat{\Psi}_{\theta,T}(x, y) = \prod_p \mathbb{E} \left[ \exp \left( i \sum_{j \leq J} \bar{z}_j g_j(p, \sigma_T : X) + z_j g_j(p, \sigma_T : \bar{X}) \right) \right]
$$

$$
= \prod_p \mathbb{E} \left[ \prod_{j \leq J} \exp \left( i (\bar{z}_j g_j(p, \sigma_T : X) + z_j g_j(p, \sigma_T : \bar{X})) \right) \right]
$$

and we see that

$$
\left| \mathbb{E} \left[ \prod_{j \leq J} \exp \left( i (\bar{z}_j g_j(p, \sigma_T : X) + z_j g_j(p, \sigma_T : \bar{X})) \right) \right] \right| \leq 1. \quad (3.1)
$$

By Lemma 2.5 in [8] and the argument to justify the equation (3.28) in [8, p. 1828–1829], there is a constant $C_1 > 0$ such that

$$
\left| \mathbb{E} \left[ \prod_{j \leq J} \exp \left( i (\bar{z}_j g_j(p, \sigma_T : X) + z_j g_j(p, \sigma_T : \bar{X})) \right) \right] \right| \leq C_1 \frac{p^{\sigma_T/2}}{(\sum_j x_j^2 + y_j^2)^{1/4}}
$$

for $p^{-\sigma_T} \sqrt{\sum_j x_j^2 + y_j^2} \geq 1$. Hence if $p^{\sigma_T} \leq C_2 \sqrt{\sum_j x_j^2 + y_j^2}$ with $C_2 = \min \{1, C_1^{-2} e^{-1} \}$, then

$$
\left| \mathbb{E} \left[ \prod_{j \leq J} \exp \left( i (\bar{z}_j g_j(p, \sigma_T : X) + z_j g_j(p, \sigma_T : \bar{X})) \right) \right] \right| \leq e^{-1/2}. \quad (3.2)
$$

Thus, by (3.1), (3.2) and the prime number theorem, we have

$$
|\hat{\Psi}_{\theta,T}(x, y)| \leq \prod_{p^{\sigma_T} \leq C_2 \sqrt{\sum_j x_j^2 + y_j^2}} e^{-1/2} \leq \exp \left( -C_3 \frac{(\sum_j x_j^2 + y_j^2)^{1/2 \sigma_T}}{\log(\sum_j x_j^2 + y_j^2)} \right)
$$

for $\sum_j x_j^2 + y_j^2 \geq C_4$ and for some $C_3, C_4 > 0$. This proves the first inequality in Proposition 2.

Let

$$
A_{k,\ell}(p, \sigma) := \mathbb{E} \left[ \prod_{j \leq J} g_j(p, \sigma : X)^{k_j} \bar{g}_j(p, \sigma : \bar{X})^{\ell_j} \right] \quad (3.3)
$$

for $k = (k_1, \ldots, k_J)$ and $\ell = (\ell_1, \ldots, \ell_J)$, then each factor of $\hat{\Psi}_{\theta,T}(x, y)$ is

$$
\sum_{k, \ell \geq 0} \frac{i^{K(k+\ell)}}{k! \ell!} \mathbb{E} \left[ \prod_{j \leq J} g_j(p, \sigma_T : X)^{k_j} \bar{g}_j(p, \sigma_T : \bar{X})^{\ell_j} \right] \bar{z}^k z^\ell
$$

$$
= \sum_{k, \ell \geq 0} \frac{i^{K(k+\ell)}}{k! \ell!} A_{k,\ell}(p, \sigma_T) \bar{z}^k z^\ell,
$$

where the sums are over all $k = (k_1, \ldots, k_J), \ell = (\ell_1, \ldots, \ell_J) \in (\mathbb{Z}_{\geq 0})^J$ and $K(k) = k_1 + \cdots + k_J$, $k! = k_1! \cdots k_J!$ and $z^k = \prod_{j \leq J} z_j^{k_j}$. Since $A_{0,0}(p, \sigma) = 1$ and $A_{0,k}(p, \sigma) = A_{k,0}(p, \sigma) = 0$ for $k \neq 0$, the above sum equals

$$
1 + \sum_{k, \ell} \frac{i^{K(k+\ell)}}{k! \ell!} A_{k,\ell}(p, \sigma_T) \bar{z}^k z^\ell =: 1 + A_{\theta,T,x,y}(p),
$$

and we have

$$
A_{\theta,T,x,y}(p) \leq \exp \left( \frac{-C_3 \sqrt{\sum_j x_j^2 + y_j^2}}{\log(\sum_j x_j^2 + y_j^2)} \right)
$$

for $\sum_j x_j^2 + y_j^2 \geq C_4$ and for some $C_3, C_4 > 0$. This proves the second inequality in Proposition 2.
where the \( k, \ell \in (\mathbb{Z}_{\geq 0})^J \). By estimating (3.3) one can show that
\[
|A_{k, \ell}(p, \sigma_T)| \leq C_5 p^{-\sigma_T K(k + \ell)}
\]
for some \( C_5 > 0 \) and all nonzero \( k, \ell \in (\mathbb{Z}_{\geq 0})^J \). Thus,
\[
|A_{\theta, T, x, y}(p)| \leq C_5 \sum_{k, \ell} k! \prod_{j \leq J} \left( \frac{|z_j|}{p^{\sigma_T}} \right)^{k_j + \ell_j} = C_5 \left( \exp \left( \frac{\sum_{j=1}^J |z_j|}{p^{\sigma_T}} \right) - 1 \right)^2. \tag{3.4}
\]
Let \( Y = e^{(\log T)^{9/2}} \), then there exists a constant \( C_6 > 0 \) such that
\[
|A_{\theta, T, x, y}(p)| \leq C_6 \sum_{j=1}^J |z_j|^2 \leq C_6
\]
for \( \sum_{j \leq J} |z_j|^2 \leq Y \) and \( p \geq Y \). Thus, by (3.1)
\[
|\hat{\Psi}_{\theta, T}(x, y)| \leq \left| \prod_{p \geq Y} (1 - A_{\theta, T, x, y}(p)) \right|
= \left| \prod_{p \geq Y} \exp \left( A_{\theta, T, x, y}(p) + O \left( \frac{\left( \sum_{j=1}^J |z_j|^2 \right)^2}{p^{4 \sigma_T}} \right) \right) \right|
= \left| \exp \left( \sum_{p \geq Y} A_{\theta, T, x, y}(p) + O \left( \sum_{j=1}^J |z_j|^2 \right) \right) \right|
\]
The \( p \)-sum is
\[
\sum_{p \geq Y} A_{\theta, T, x, y}(p) = \sum_{p \geq Y} \sum_{k, \ell} k! \ell! \prod_{j \leq J} \left( \frac{|z_j|}{p^{\sigma_T}} \right)^{k_j + \ell_j} A_{k, \ell}(p, \sigma_T) z^k z^\ell
\]
\[
= - \sum_{p \geq Y} \sum_{j_1, j_2 \leq J} \mathbb{E} \left[ g_{j_1}(p, \sigma_T : X) g_{j_2}(p, \sigma_T : X) \right] z_{j_1} z_{j_2} + O \left( \sum_{j=1}^J |z_j|^2 \right)
\]
\[
= - \sum_{j_1, j_2 \leq J} \sum_{p \geq Y} \frac{\alpha_{j_1}(p) \alpha_{j_2}(p)}{p^{2 \sigma_T}} z_{j_1} z_{j_2} + O \left( \sum_{j=1}^J |z_j|^2 \right)
\]
\[
= - \sum_{j \leq J} |z_j|^2 \left( \sum_{p \geq Y} \frac{\alpha_j(p)^2}{p^{2 \sigma_T}} + O(1) \right)
\]
\[
\leq - \left( \frac{\pi^2 \theta}{2} \log \log T + O(1) \right) \sum_{j \leq J} (x_j^2 + y_j^2).
\]
Therefore,
\[
|\hat{\Psi}_{\theta, T}(x, y)| \leq \exp \left( - \left( \frac{\pi^2 \theta}{2} \log \log T + O(1) \right) \sum_{j \leq J} (x_j^2 + y_j^2) \right)
\]
holds for \( \sum_{j \leq J} (x_j^2 + y_j^2) \leq e^{(\log T)^{9/2}} \), which proves the second inequality in Proposition 2.1.
Next, we find an asymptotic of \( \hat{\Psi}_{\theta,T}(x, y) \) for \( \sum_{j \leq J} (x_j^2 + y_j^2) \leq C_7 \). By (3.4) and choosing \( C_7 > 0 \) sufficiently small, we have that \( |A_{\theta,T,x,y}(p)| \leq 1/2 \) for every prime \( p \). Thus,

\[
\hat{\Psi}_{\theta,T}(x, y) = \prod_p (1 + A_{\theta,T,x,y}(p))
\]

\[
= \prod_p \exp \left( A_{\theta,T,x,y}(p) - \frac{1}{2} (A_{\theta,T,x,y}(p))^2 + O\left( \frac{\sum_{j=1}^J (x_j^2 + y_j^2)^3}{p^{6\sigma_T}} \right) \right)
\]

\[
= \exp \left( \sum_p A_{\theta,T,x,y}(p) - \frac{1}{2} \sum_p (A_{\theta,T,x,y}(p))^2 + O\left( \sum_{j=1}^J (x_j^2 + y_j^2)^3 \right) \right).
\]

The sum

\[
\sum_p A_{\theta,T,x,y}(p) - \frac{1}{2} \sum_p (A_{\theta,T,x,y}(p))^2
\]

has a power series representation in \( z_1, \bar{z}_1, \ldots, z_J, \bar{z}_J \), so let it be

\[
\sum_{k, \ell}^\infty B_{k,\ell}(\sigma_T)z^k \bar{z}^\ell.
\]

For \( K(k + \ell) \geq 3 \), we have

\[
B_{k,\ell}(\sigma_T) = B_{k,\ell}(1/2) + O\left( \frac{1}{(\log T)^\theta} \right).
\]

For \( K(k) = K(\ell) = 1 \), we have

\[
\sum_{K(k) = K(\ell) = 1} B_{k,\ell}(\sigma_T) = \sum_p \sum_{K(k) = K(\ell) = 1} \frac{i^{K(k+\ell)}}{k! \ell!} A_{k,\ell}(p, \sigma_T)z^k \bar{z}^\ell
\]

\[
= - \sum_p \sum_{j_1, j_2 \leq J} \mathbb{E} g_{j_1}(p, \sigma_T : X)g_{j_2}(p, \sigma_T : \bar{X}) \bar{z}_{j_1} z_{j_2}.
\]

By (1.4), we find that

\[
\mathbb{E} \left[ g_{j_1}(p, \sigma_T : X)g_{j_2}(p, \sigma_T : \bar{X}) \right] = C_{j_1, j_2} + O\left( \frac{1}{(\log T)^\theta} \right)
\]

for \( j_1 \neq j_2 \) and

\[
\mathbb{E} \left[ g_j(p, \sigma_T : X)g_j(p, \sigma_T : \bar{X}) \right] = \sum_p \frac{\alpha_j(p)^2}{p^{2\sigma_T}} + C_j' + O\left( \frac{1}{(\log T)^\theta} \right)
\]

\[
= \xi_j \theta \log \log T + C_{j,j} + O\left( \frac{1}{(\log T)^\theta} \right)
\]

for some constants \( C_{j_1, j_2}, C_{j,j}, C_j' \) independent to \( \theta \). Thus,

\[
\sum_{K(k) = K(\ell) = 1} B_{k,\ell}(\sigma_T) = -\theta \log \log T \sum_{j \leq J} \xi_j |z_j|^2 + \sum_{j_1, j_2 \leq J} C_{j_1, j_2} \bar{z}_{j_1} z_{j_2} + O\left( \frac{1}{(\log T)^\theta} \right).
\]
Therefore, we have
\[
\hat{\Psi}_{\theta,T}(x, y) = e^{-\theta \log \log T \sum_{j \leq J} \xi_j |z_j|^2} \exp \left( \sum_{j_1, j_2 \leq J} C_{j_1, j_2} \bar{z}_{j_1} z_{j_2} + \sum_{K(k+\ell) = 3, 4, 5} B_{k, \ell} (1/2) z^k \bar{z}^\ell \right)
\]
\[
\times \exp \left( O \left( \sum_{j=1}^J (x_j^2 + y_j^2)^3 + \frac{1}{(\log T)^6} \right) \right)
\]
\[
= e^{-\pi^2 \theta \log \log T \sum_{j \leq J} \xi_j (x_j^2 + y_j^2)} \left( P(x, y) + O \left( \sum_{j=1}^J (x_j^2 + y_j^2)^3 + \frac{1}{(\log T)^6} \right) \right)
\]
for \( \sum_{j \leq J} (x_j^2 + y_j^2) \leq C_7 \), where \( P(x, y) \) is a polynomial of degree \( \leq 5 \) and may be written as
\[
1 + \sum_{K(k+\ell) = 2, 3, 4, 5} \hat{B}_{k, \ell} x^k y^\ell.
\]
This completes the proof of Proposition \ref{prop:main}

3.2. Proof of Proposition \ref{prop:main2} By Proposition \ref{prop:main}, we have
\[
G_{\theta,T}(u, v) = \int_{\mathbb{R}^{2J}} \hat{\Psi}_{\theta,T}(x, y) e^{-2\pi i (x \cdot u + y \cdot v)} dx dy
\]
\[
= \int_{\sum_j (x_j^2 + y_j^2) \leq C_7} e^{-\pi^2 \theta \log \log T \sum_{j \leq J} \xi_j (x_j^2 + y_j^2) - 2\pi i (x \cdot u + y \cdot v)} P(x, y) dx dy
\]
\[
+ O \left( \frac{1}{(\log \log T)^{J+3}} \right),
\]
where \( P(x, y) \) is the polynomial defined in Proposition \ref{prop:main}. By the change of variables
\[
x_j = \frac{\bar{x}_j}{\pi \sqrt{\theta \xi_j \log \log T}} - \frac{iu_j}{\pi \theta \xi_j \log \log T}
\]
and
\[
y_j = \frac{\bar{y}_j}{\pi \sqrt{\theta \xi_j \log \log T}} - \frac{iv_j}{\pi \theta \xi_j \log \log T},
\]
one finds that
\[
P(x, y) = \sum_{k, \ell, m, n \in \mathbb{Z}_{\geq 0}^J} \frac{P_{k, \ell, m, n}}{(\theta \log \log T)^{K(k+\ell)/2 + K(m+n)}} \bar{x}^k \bar{y}^\ell u^m v^n,
\]
where
\[
P_{k, \ell, m, n} = \frac{\hat{B}_{k+m, \ell+n}}{\pi^{K(k+\ell+m+n)/2+K(m+n)}} \frac{(k+m)!(\ell+n)!}{k!m!\ell!n!} \prod_j \xi^{-(k_j+\bar{\ell}_j)/2 - m_j - n_j}.
\]
Then we see that
\[
G_{\theta,T}(u, v) = \exp \left( - \sum_j \frac{u_j^2 + v_j^2}{\theta \xi_j \log \log T} \right) \prod_j \frac{1}{(\pi^2 \theta \xi_j \log \log T)}
\]
\[ \int_{\mathbb{R}^2} e^{-\sum_j (\tilde{x}_j + \tilde{y}_j^2)} \sum_{k, \ell, m, n \in \mathbb{Z}_{\geq 0}} \frac{p_{k, \ell, m, n}}{(\theta \log \log T)^{k+(k+\ell)/2+m+n}} \tilde{x}_j^{k} \tilde{y}_j^\ell u^m v^n d\tilde{x} d\tilde{y} \]

\[ + O\left( \frac{1}{(\log \log T)^{J+3}} \right) \]

\[ = \exp \left( -\sum_j \frac{u_j^2 + v_j^2}{\theta \xi_j \log \log T} \right) \sum_{k, \ell, m, n \in \mathbb{Z}_{\geq 0}} \frac{c_{k, \ell} p_{k, \ell, m, n}}{(\theta \log \log T)^{J+k+(k+\ell)/2+m+n}} u^m v^n \]

\[ + O\left( \frac{1}{(\log \log T)^{J+3}} \right), \]

where

\[ c_{k, \ell} := \pi^{-2J} \left( \prod_j \xi_j^{-1} \right) \int_{\mathbb{R}^2} e^{-\sum_j (\tilde{x}_j + \tilde{y}_j^2)} \tilde{x}_j^{k} \tilde{y}_j^\ell d\tilde{x} d\tilde{y} \]

\[ = \pi^{-2J} \prod_j \left( \xi_j^{-1} \int_{\mathbb{R}} e^{-x^2} x^{k_j} dx \int_{\mathbb{R}} e^{-y^2} y^{\ell_j} dy \right). \]

Thus, if there is odd \( k_j \) or odd \( \ell_j \), then \( c_{k, \ell} = 0 \). Otherwise,

\[ c_{k, \ell} = \pi^{-2J} \prod_j \left( \xi_j^{-1} \Gamma \left( \frac{k_j}{2} \right) \Gamma \left( \frac{\ell_j}{2} \right) \right). \]

Hence, we have

\[ G_{\theta, T}(u, v) = \exp \left( -\sum_j \frac{u_j^2 + v_j^2}{\theta \xi_j \log \log T} \right) \sum_{k, \ell, m, n \in \mathbb{Z}_{\geq 0}} \frac{c_{2k, 2\ell} p_{2k, 2\ell, m, n}}{(\theta \log \log T)^{J+k+(k+\ell)/2+m+n}} u^m v^n \]

\[ + O\left( \frac{1}{(\log \log T)^{J+3}} \right). \]

Letting \( q_{k, \ell, m, n} = c_{2k, 2\ell} p_{2k, 2\ell, m, n} \), we prove the first identity of the proposition. The second one can be deduced by modifying the proof of Theorem 6 in [1].

3.3. **Proof of Lemma [2.3]** Our proof is basically the same as the proof of Lemma 3.3 in [4], but we need the dependency on \( M \). We first see that

\[ \int_{\mathbb{R}^2} \left| \log \left( \sum_{j \leq J} b_j e^{u_j+i v_j} \right) \right|^{2k} e^{-\sum_j (u_j^2+v_j^2)/M} dudv \]

\[ = \int_{\mathbb{R}^J} \int_{[0,2\pi]^J} \left| \log \left( \sum_{k \in \mathbb{Z}^J} b_k e^{u_j+i (v_j+2\pi k_j)} \right) \right|^{2k} e^{-\sum_j (u_j^2+(v_j+2\pi k_j)^2)/M} dudv \]

\[ = \int_{\mathbb{R}^J} \int_{[0,2\pi]^J} \left| \sum_{j \leq J} b_j e^{u_j+i v_j} \right|^{2k} \sum_{k \in \mathbb{Z}^J} e^{-\sum_j (v_j+2\pi k_j^2)/M} dv - \sum_j u_j^2/M du \]

\[ \ll M^{J/2} \int_{\mathbb{R}^J} \int_{[0,2\pi]^J} \left| \sum_{j \leq J} b_j e^{u_j+i v_j} \right|^{2k} dv - \sum_j u_j^2/M du. \]
Next we need the inequality
\[
\int_0^{2\pi} \left( \log |a - be^{iv}| \right)^{2k} dv \ll (C_1 \log |a|)^{2k} + (C_1 \log |b|)^{2k} + (C_1 k)^{2k}
\]
for some constant $C_1 > 0$. (See Lemma 2.1 in [4] for a proof.) Hence, we see that
\[
\int_{\mathbb{R}^J} \int_{[0,2\pi]^J} \left| \log \left| \sum_{j \leq J} b_j e^{uj + iv} \right| \right|^{2k} dv e^{-\sum_j u_j^2/M} du
\]
\[
\ll \int_{\mathbb{R}^J} \left( \sum_{j \leq J} (C_2 \log |b_j e^{uj}|)^{2k} + (C_2 k)^{2k} \right) e^{-\sum_j u_j^2/M} du
\]
\[
\ll \sum_{j \leq J} \int_{\mathbb{R}^J} ((C_3 u_j)^{2k} + C_3^{2k}) e^{-\sum_j u_j^2/M} du + M^{J/2} (C_2 k)^{2k}
\]
\[
\ll M^{J/2} \sum_{j \leq J} \int_{\mathbb{R}^J} ((C_3 u_j)^{2k} + C_3^{2k}) e^{-\sum_j u_j^2/M} du + M^{J/2} (C_2 k)^{2k}
\]
\[
\ll M^{J/2} (C_4 k M)^k + M^{J/2} (C_2 k)^{2k}.
\]
Thus,
\[
\int_{\mathbb{R}^J} \left| \log \left| \sum_{j \leq J} b_j e^{uj + iv} \right| \right|^{2k} dv e^{-\sum_j (u_j^2 + v_j^2)/M} du dv \ll M^{J+k}(Ck)^k + M^J(Ck)^{2k}.
\]

3.4. Proof of Proposition 2.7.4. By symmetry, it is enough to estimate
\[
\mathcal{E}_{m,n,1}(\theta, T) := \int_{\mathbb{R}^J} \int_{\mathcal{R}_1} \log \left| \frac{1 + \sum_{j=2}^J b_j e^{(uj - u_1 + i(v_j - v_1))\sqrt{\theta \log \log T}}}{b_1} \right| e^{-\sum_j (u_j^2 + v_j^2)/\xi_j} u^m v^n du dv.
\]
Let $A_T = (\log \log \log T)/4$. We divide $\mathcal{R}_1$ into a disjoint union of the sets:
\[
\mathcal{R}_{1,S} := \{(u_1, \ldots, u_J) \in \mathcal{R}_1 : \frac{A_T}{\sqrt{\theta \log \log T}} < u_\ell - u_1 \leq 0 \text{ for } \ell \in S, \quad \frac{A_T}{\sqrt{\theta \log \log T}} \text{ for } j \in \{2, \ldots, J\} \setminus S \}
\]
for $S \subset \{2, \ldots, J\}$. Let
\[
\mathcal{E}_S := \mathcal{E}_{m,n,1,S}(\theta, T)
\]
\[
:= \int_{\mathbb{R}^J} \int_{\mathcal{R}_{1,S}} \log \left| \frac{1 + \sum_{j=2}^J b_j e^{(uj - u_1 + i(v_j - v_1))\sqrt{\theta \log \log T}}}{b_1} \right| e^{-\sum_j (u_j^2 + v_j^2)/\xi_j} u^m v^n du dv,
\]
so that
\[
\mathcal{E}_{m,n,1}(\theta, T) = \sum_{S \subset \{2, \ldots, J\}} \mathcal{E}_S.
\]
First consider $\mathcal{E}_\emptyset$. In this case it is easy to see that
\[
\mathcal{E}_\emptyset = \int_{\mathbb{R}^J} \int_{\mathcal{R}_{1,\emptyset}} O(e^{-A_T}) e^{-\sum_j (u_j^2 + v_j^2)/\xi_j} u^m v^n du dv = O(e^{-A_T}).
\]
Next consider $S \neq \emptyset$, then there is at least one element $\ell \in S$. We first observe the $u_\ell$ integral:

$$
\int_{u_1}^{u_1} \exp \left[ - \log \log \frac{u_\ell}{\sqrt{\log \log T}} \right] \left( 1 + \sum_{j=2}^{J} \frac{b_j}{b_1} e^{(u_j-u_1+i\nu_j-i\nu_1)\sqrt{\log \log T}} \right) e^{-u_\ell^2/\xi \mu u_\ell^2} du_\ell
\ll \int_{-A_T}^{0} \log \left| \frac{b_j}{b_1} e^{(u_j+i\nu_j-i\nu_1)\sqrt{\log \log T}} + \sum_{j \neq \ell} \frac{b_j}{b_1} e^{(u_j-u_1+i\nu_j-i\nu_1)\sqrt{\log \log T}} \right| du_\ell
\ll \frac{1}{\sqrt{\log \log T}} \int_{e^{-A_T}}^{1} \log \left| \frac{b_j}{b_1} e^{i(\nu_j-i\nu_1)\sqrt{\log \log T}} + \sum_{j \neq \ell} \frac{b_j}{b_1} e^{(u_j-u_1+i\nu_j-i\nu_1)\sqrt{\log \log T}} \right| \frac{dw}{w}
\ll \frac{A_T}{\sqrt{\log \log T}} + \frac{1}{\sqrt{\log \log T}} \int_{e^{-A_T}}^{1} \log \left| w + \sum_{j \neq \ell} \frac{b_j}{b_\ell} e^{(u_j-u_1+i\nu_j-i\nu_1)\sqrt{\log \log T}} \right| \frac{dw}{w}
\ll \frac{e^{A_T}}{\sqrt{\log \log T}}
$$

by the substitution $w = e^{u_\ell \sqrt{\log \log T}}$. Here, the last inequality holds by the following lemma.

**Lemma 3.1.** Let $B$ be a fixed positive real number and let $\xi_T > 0$ be a decreasing function to $0$ as $T \to \infty$. Then we have

$$
\int_{\xi_T}^{1} \log |u + z| \frac{du}{u} = O \left( \frac{1}{\xi_T} \right)
$$

as $T \to \infty$ uniformly for all $|z| \leq B$.

**Proof.** We first observe that it is enough to prove that

$$
\int_{0}^{1} \log |u + z| du = O(1)
$$

uniformly for bounded $z = \alpha + i\beta$. By the inequality

$$
-|\log |u + \alpha|| \leq \log |u + \alpha| \leq \log |u + z| = \log \sqrt{(u + \alpha)^2 + \beta^2} \leq \log \sqrt{2 \max \{ (u + \alpha)^2, \beta^2 \}}
$$

we see that

$$
\int_{0}^{1} \log |u + z| du \leq \int_{0}^{1} \log |u + \alpha| du + O(1).
$$

If $|\alpha| \geq 2$, then it is easy to see that

$$
\int_{0}^{1} \log |u + \alpha| du = \int_{0}^{1} \log |\alpha| + O(1) du = O(1).
$$

If $|\alpha| < 2$, then we split the interval into two intervals depending on the condition $\log |u + \alpha| \geq 0$. Thus,

$$
\int_{0}^{1} \log |u + \alpha| du = \int_{|u + \alpha| \geq 1} \log |u + \alpha| du + \int_{|u + \alpha| < 1} -\log |u + \alpha| du.
$$

$$
It is easy to see that
\[
0 \leq \int_{[0,1] \cap \{|u+\alpha| \geq 1\}} \log |u + \alpha| du \leq \log(1 + B)
\]
and
\[
0 \leq \int_{[0,1] \cap \{|u+\alpha| < 1\}} -\log |u + \alpha| du \leq 2 \int_0^1 -\log u du \leq 2.
\]

Hence, we find that
\[
E_S = O\left(\frac{e^{AT}}{\sqrt{\log \log T}}\right)
\]
for \(S \neq \emptyset\) and
\[
E_{m,n}(\theta, T) = O(e^{-AT}) + O\left(\frac{e^{AT}}{\sqrt{\log \log T}}\right) = O\left(\frac{1}{(\log \log T)^{1/4}}\right).
\]

3.5. **Proof of Lemma 2.5.** We see that for a fixed real \(\beta\) and for each \(i = 1, 2\)
\[
\theta_1^{\beta} - \theta_2^{\beta} = H_T(\beta \theta_i^{\beta-1} + O(H_T)).
\]

Thus, by (2.5)
\[
\theta_1^\alpha I_{m,n}(\theta_1, T) - \theta_2^\alpha I_{m,n}(\theta_2, T)
= H_T \sqrt{\log \log T} d_n \left( \alpha + \frac{1}{2} \right) \theta_1^{\alpha-1/2} \sum_{\ell=1}^J \int_{R_\ell} u e^{-\sum_j u_j^2/\xi_j} u^m du
+ H_T d_n \alpha \theta_1^{\alpha-1} \sum_{\ell=1}^J \int_{R_\ell} \log |b| e^{-\sum_j u_j^2/\xi_j} u^m du
+ \theta_1^\alpha E_{m,n}(\theta_1, T) - \theta_2^\alpha E_{m,n}(\theta_2, T) + O(H_T^2 \sqrt{\log \log T})
\]
for each \(i = 1, 2\). Recall that \(E_{m,n}(\theta, T) := \sum_{\ell=1}^J E_{m,n,\ell}(\theta, T)\). Hence, without loss of generality, we consider
\[
\theta_1^\alpha E_{m,n,1}(\theta_1, T) - \theta_2^\alpha E_{m,n,1}(\theta_2, T)
\]
\[
= \int_{R_1} \int_{R_1} \log \left| 1 + \sum_{j=2}^J \frac{b_j}{b_1} e^{(u_j-u_1+i(u_j-v_1)) \sqrt{\log \log T}} \right|
\times \left( \theta_1^{\alpha'} e^{-\sum_j (u_j^2+v_j^2)/(\theta_1 \xi_j)} - \theta_2^{\alpha'} e^{-\sum_j (u_j^2+v_j^2)/(\theta_2 \xi_j)} \right) u^m v^n dudv,
\]
where
\[
\alpha' = \alpha - J - \frac{K(m+n)}{2}.
\]
We see that
\[
\theta_1^{\alpha'} e^{-\sum_j (u_j^2+v_j^2)/(\theta_1 \xi_j)} - \theta_2^{\alpha'} e^{-\sum_j (u_j^2+v_j^2)/(\theta_2 \xi_j)}
= \int_{\theta_2}^{\theta_1} \frac{\partial}{\partial w} \left( w^{\alpha'} e^{-\sum_j (u_j^2+v_j^2)/(w \xi_j)} \right) dw
\]
\begin{align*}
&= \int_{\theta_2}^{\theta_1} \left( \sum_{j \leq J} \frac{u_j^2 + v_j^2}{w^2 \xi_j} + \frac{\alpha'}{w} \right) w^{\alpha'} e^{-\sum_j (u_j^2 + v_j^2)/(w \xi_j)} \, dw \\
& \ll H_T \left( \sum_{j \leq J} (u_j^2 + v_j^2) \right) e^{-\sum_j (u_j^2 + v_j^2)/(\theta_2 \xi_j)} \cdot
\end{align*}

Thus, by adapting the proof of Proposition 2.4 we find that

\[ \theta_1^a \mathcal{E}_{m,n,1}(\theta_1, T) - \theta_2^a \mathcal{E}_{m,n,1}(\theta_2, T) \]
\[ \ll H_T \int_{\mathbb{R}^J} \int_{\mathbb{R}^1} \left| \log \left| 1 + \sum_{j=2}^{J} \frac{b_j}{b_1} e^{(u_j-u_1+i(v_j-v_1))\sqrt{\log \log T}} \right| e^{-\sum_j (u_j^2 + v_j^2)/(\theta_2 \xi_j)} \right| \]
\[ \left( \sum_{j \leq J} (u_j^2 + v_j^2) + 1 \right) |u^m v^n| dudv \]

\[ \ll \frac{H_T}{(\log \log T)^{1/4}} \]

and this completes the proof of the lemma.

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