Projection operator approach to
general constrained systems

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Abstract

We propose a new BRST-like quantization procedure which is applicable to dynamical systems containing both first and second class constraints. It requires no explicit separation into first and second class constraints and therefore no conversion of second class constraints is needed. The basic ingredient is instead an invariant projection operator which projects out the maximal subset of constraints in involution. The hope is that the method will enable a covariant quantization of models for which there is no covariant separation into first and second class constraints. An example of this type is given.

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1 Introduction.

When one wants to quantize theories with constraints it is usually very important to first separate them into first and second class constraints since these classes of constraints usually have to be treated in a different manner. However, very often for relativistic models it is not possible to do this splitting of the original constraints in a covariant way. Examples include the superparticle, the superstring, p-branes and high rank tensor fields. The advantage of the method we propose here is that no such explicit splitting of the constraints are required. Instead it is based on the idea to find the maximal involutive set of the constraints which is a mixture of first class constraints and one-half of the second class ones. We believe that this maximal involutive subset can be covariantly extracted from the full set of constraints in many physical models for which there are no covariant splitting into first and second class constraints. An example is given at the end.

Let us first give the setting for our considerations. We consider a general dynamical theory with finite number of degrees of freedom. (The generalization to infinite degrees of freedom is straightforward.) Given is a phase space of dimension $2n$ spanned by the canonical coordinates $x^I = (q^i, p_i)$, $i = 1, \ldots, n$, with arbitrary Grassmann parities, $\varepsilon(p_i) = \varepsilon(q^i) = \varepsilon_i$. On this phase space we have $m$ constraints

$$T_\alpha(q,p) = 0, \quad \alpha = 1, \ldots, m, \quad \varepsilon(T_\alpha) \equiv \varepsilon_\alpha,$$

which are not required to be irreducible, since the requirement of covariance may force us to use dependent constraints. We have

$$m = m_1 + 2m_2 + q,$$

where $m_1$ is the number of independent first class constraints, $2m_2$ the number of independent second class constraints, and where $q$ is the number of dependent constraints. More precisely

$$\text{rank}\left\{ T_\alpha, T_\beta \right\}_{T=0} = 2m_2, \quad \text{rank} \left. \frac{\partial T_\alpha}{\partial x^I} \right|_{T=0} = m_1 + 2m_2,$$

$$Z_a^\alpha T_\alpha = 0, \quad a = 1, \ldots, q, \quad \text{rank} Z_a^\alpha \bigg|_{T=0} = q,$$

where we have introduced the graded Poisson bracket on the $2n$-dimensional phase space. There are several different procedures to quantize such a system. There is e.g. the method of conversion in which one adds new degrees of freedom by means of which one may convert the second class constraints into first class ones which then may be quantized by the standard procedure for general gauge theories [1]. Another method is to remove half of the second class constraints which in the present case may be stated in the following general form [4]. Find the maximal involutive subset

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4Among other methods for the quantization of theories with second class constraints there are the conversion in terms of original variables [2, 3], the method of split involution [4], and the generalized BRST proposal in [5].
of $T_\alpha$. The number of such constraints is $m_1 + m_2 + s$ where $m_1 + m_2$ of them are independent and where $s \leq q$ is the number of constraints dependent on the chosen independent set. The procedure of quantization which we propose here is related to the latter idea but will be formulated in a more invariant way. The basic ingredient is a projection matrix, $P_\alpha^\beta$, $\varepsilon(P_\alpha^\beta) = \varepsilon_\alpha + \varepsilon_\beta$, chosen in such a way that
\[
T'_\alpha = P_\alpha^\beta T_\beta, \tag{4}
\]
are constraints in involution, i.e. which satisfy the Poisson algebra
\[
\{T'_\alpha, T'_\beta\} = U'_\alpha^\gamma T'_\gamma. \tag{5}
\]
$P_\alpha^\beta$ is also required to be covariant. The idea is to cast the original theory into a general gauge theory where the covariant constraints $T'_\alpha$ generate the gauge transformations, and where the observables $O$ (including the Hamiltonian) satisfy
\[
\{O, T'_\alpha\} = V'_\alpha^\beta T'_\beta. \tag{6}
\]
In this way the quantization problem of the original theory is reduced to that of an effective theory with the first class constraints $T'_\alpha$ only. In order for this to be possible $T'_\alpha$ must contain a maximal subset of independent $T_\alpha$ in involution. More precisely $P_\alpha^\beta$ must be such that $T'_\alpha$ contains exactly $m_1 + m_2$ independent constraints and $m_2 + q$ dependent ones some of which may be zero identically. (With respect to the original constraints, $T_\alpha$, the independent constraints in $T'_\alpha$ contain $m_1$ independent first class constraints and $m_2$ of the independent second class constraints.) We have
\[
\text{rank} \frac{\partial T'_\alpha}{\partial x^I} \bigg|_{T'_\alpha = 0} = m_1 + m_2, \tag{7}
\]
and there exist a function $Z'^{\alpha}_{\alpha_1}$ with the properties
\[
Z'^{\alpha}_{\alpha_1} T'_\alpha = 0, \quad \text{rank} Z'^{\alpha}_{\alpha_1} \bigg|_{T'_\alpha = 0} = m_2 + q. \tag{8}
\]
Eq.(7) suggests the rank condition
\[
\text{rank} P_\alpha^\beta \bigg|_{T'_\alpha = 0} = m_1 + m_2. \tag{9}
\]
Eq.(8) may e.g. be satisfied by the condition that there should exist a function $Z'^{\alpha}_{\alpha_1}$ with the rank $m_2 + q$ satisfying
\[
Z'^{\alpha}_{\alpha_1} P_\alpha^\beta = 0. \tag{10}
\]
These properties suggest that $P_\alpha^\beta$ may be chosen to be a function satisfying the projection property
\[
P_\alpha^\gamma P_\gamma^\beta = P_\alpha^\beta, \tag{11}
\]
5This does not restrict the generality of the theory as any observable can be brought to the involution \((\mathbb{I})\) by adding combinations of the constraints $T_\alpha$ in \((\mathbb{I})\).
in which case

$$\text{rank} \left( \delta^\beta - P_\alpha^\beta \right) \bigg|_{T^r=0} = m_2 + q. \quad (12)$$

In order to see more explicitly what the condition (10) requires we first notice that we always may write

$$\{T_\alpha, T_\beta\} = C_{\alpha\beta} + U_{\alpha\beta} \gamma T_\gamma, \quad (13)$$

where the separation into the two terms is purely conventional. As we shall see our formalism suggests that there always exist a particular separation and a particular choice of $P_\alpha^\beta$ such that

$$P_\alpha^\gamma C_{\gamma\rho} P_\beta^\rho (-1)^{\varepsilon_\rho(\varepsilon_\beta+1)} = 0. \quad (14)$$

This expression together with the equation obtained by inserting (13) into (5) yields then the conditions

$$\begin{align*}
(P_\alpha^\gamma \{T_\gamma, P_\beta^\rho\} - P_\beta^\gamma \{T_\gamma, P_\alpha^\rho\} (-1)^{\varepsilon_\alpha \varepsilon_\beta} + \{P_\alpha^\rho, P_\beta^\gamma\} T_\gamma (-1)^{\varepsilon_\rho \varepsilon_\beta} + \\
+ P_\beta^\eta P_\alpha^\gamma U_\eta^\rho (-1)^{\varepsilon_\alpha (\varepsilon_\beta + \varepsilon_\rho)} - U_{\alpha\beta}^\gamma P_\rho^\gamma \bigg) T_\rho = 0. \quad (15)
\end{align*}$$

### 2 Invariant formulation: Classical theory.

In order to put the above ideas into a more invariant formulation we have to extend the phase space by ghost variables $C^\alpha$, $\bar{P}_\alpha$ and ghost for ghosts $C^{\alpha_1}$, $\bar{P}_{\alpha_1}$ etc up to a certain stage $L$ satisfying the properties

$$\varepsilon(C^\alpha) = \varepsilon(\bar{P}_\alpha) = \varepsilon_\alpha + 1, \quad \varepsilon(C^{\alpha_r}) = \varepsilon(\bar{P}_{\alpha_r}) = \varepsilon_{\alpha_r} + r + 1, \quad r = 1, \ldots, L,$$

$$\{C^\alpha, \bar{P}_\beta\} = \delta^\alpha_\beta, \quad \{C^{\alpha_r}, \bar{P}_{\beta_r}\} = \delta^{\alpha_r}_{\beta_r}, \quad r = 1, \ldots, L. \quad (16)$$

$C^\alpha$ and $\bar{P}_\alpha$ have ghost number one and minus one respectively, i.e. $gh(C^\alpha) = 1$ and $gh(\bar{P}_\alpha) = -1$, and $gh(C^{\alpha_r}) = r + 1$ ($gh(\bar{P}_{\alpha_r}) = -r - 1$). The ghost number $gh(f)$ of a function $f$ is defined by

$$\{G, f\} = gh(f) f, \quad G \equiv C^\alpha \bar{P}_\alpha (-1)^{\varepsilon_\alpha} + \sum_{r=1}^L (r + 1) C^{\alpha_r} \bar{P}_{\alpha_r} (-1)^{\varepsilon_{\alpha_r} + r}, \quad (17)$$

where $G$ is the ghost charge. In terms of the ghost variables (16) we have an odd, real function $\Omega$ with ghost number one containing the terms

$$\begin{align*}
\Omega = C^\alpha T_\alpha + C^{\alpha_1} Z_{\alpha_1} \bar{P}_\alpha (-1)^{\varepsilon_\alpha} + \sum_{r=2}^L C^{\alpha_r} Z^{\alpha_{r-1}} \bar{P}_{\alpha_{r-1}} (-1)^{\varepsilon_{\alpha_{r-1}}} + \\
+ (-1)^{\varepsilon_\rho} \frac{1}{2} C^{\beta} C^\alpha U^\gamma_{\alpha \beta} \bar{P}_\gamma (-1)^{\varepsilon_\gamma} + (-1)^{\varepsilon_\alpha} C^\alpha C^{\beta_1} U^1_{\beta_1 \alpha} \bar{P}_{\alpha_1} (-1)^{\varepsilon_{\alpha_1}} + \\
+ (-1)^{\varepsilon_\delta + \varepsilon_{\alpha_1}} \frac{1}{6} C^{\gamma} C^{\beta} C^\alpha U_{\alpha \beta \gamma} \bar{P}_{\alpha_1} (-1)^{\varepsilon_{\alpha_1}} + \ldots, \quad (18)
\end{align*}$$
which apart from the first term is a general ansatz. In the first line we have the lowest order terms, while the second and third lines explicitly represent the terms linear in $\bar{P}_\alpha$ and $\bar{P}_{\alpha r}$, and the dots mean the remaining terms allowed by the conditions $\varepsilon(\Omega) = 1$, $gh(\Omega) = 1$. All Grassmann parities are determined by $\varepsilon(\Omega) = 1$. The functions $Z$ e.g. have the Grassmann parity $\varepsilon(Z_{\alpha r}^{-1}) = \varepsilon_{\alpha r} + \varepsilon_{\alpha r-1}$. $\Omega$ is a BRST-like charge. However, in the case when $T_\alpha$ contains second class constraints $\Omega$ may not be required to satisfy the condition $\{\Omega, \Omega\} = 0$ as in the standard BFV-prescription. (The nonzero matrix $C_{\alpha \beta}$ in (13) causes the obstruction.) In the presence of second class constraints we need therefore a new principle which tells us how to choose the terms in (18) apart from the first one which is a boundary term. We propose here such a principle by means of which we may also extract a conventional BFV-BRST charge for the given theory. This principle requires us first to introduce an even, real function $\Pi$ with ghost number zero given by the ansatz

$$\Pi = C^\alpha P_\beta (\bar{P}_\beta(-1)^{\varepsilon_{\beta r}} + \sum r=1^L (-1)^{r} C^\alpha r P_{\alpha r} (\bar{P}_{\beta r}(-1)^{\varepsilon_{\beta r}} + \ldots), \tag{19}$$

where the last dots indicates terms containing higher powers in the ghosts. The matrix functions $P^\beta_{\alpha r}$, $r = 1, \ldots, L$, have the Grassmann parity $\varepsilon(P^\beta_{\alpha r}) = \varepsilon_{\alpha r} + \varepsilon_{\beta r}$, and the matrix function entering in the first term will be the one mentioned in the introduction, $\varepsilon(P^\beta_{\alpha r}) = \varepsilon_{\alpha r} + \varepsilon_{\beta r}$. The terms in (18) and (19) are then required to satisfy the following two invariant conditions

$$\{\Pi, \{\Pi, \Omega\}\} = \{\Pi, \Omega\}, \tag{20}$$

and

$$\{\Pi, \{\Pi, \{\Omega, \Omega\}\}\} = \{\Pi, \{\Omega, \Omega\}\}. \tag{21}$$

These two conditions imply that

$$\Omega' \equiv \{\Pi, \Omega\} \tag{22}$$

satisfies the property

$$\{\Omega', \Omega'\} = 0. \tag{23}$$

Notice that conditions (20) and (21) are equivalent to (20) and (23). The odd function $\Omega'$ should be a conventional BFV-BRST charge for the projected constraints $T'_\alpha$. $\Omega'$ will then be used in the quantization of the original constrained theory.

The reality conditions of $\Omega$ and $\Pi$ may be met by the following choices: $T_\alpha$, $C^\alpha$, and $C^\alpha r$, $r = 1, \ldots, L$, are real, and $\bar{P}_\alpha^* = -\bar{P}_\alpha (-1)^{\varepsilon_{\alpha r}}$, $\bar{P}_{\alpha r}^* = -\bar{P}_{\alpha r} (-1)^{\varepsilon_{\alpha r} + r}$,

$$\begin{align*}
(P^\beta_{\alpha r})^* &= P^\beta_{\alpha r} (-1)^{\varepsilon_{\beta r}(\varepsilon_{\alpha r}+1)}, \\
(P^\beta_{\alpha r})^* &= P^\beta_{\alpha r} (-1)^{\varepsilon_{\beta r}+r}(\varepsilon_{\alpha r}+r+1), \\
(Z^\alpha_{\alpha r})^* &= Z^\alpha_{\alpha r} (-1)^{\varepsilon_{\alpha r}+1}(\varepsilon_{\alpha r}+1), \\
(Z^\alpha_{\alpha r})^* &= Z^\alpha_{\alpha r} (-1)^{\varepsilon_{\alpha r}+r}(\varepsilon_{\alpha r}+r), \quad r = 1, \ldots, L.
\end{align*}$$

\footnote{Such a generalized BRST-charge have been used for second class constraints for irreducible $T_\alpha$ but without ghost for ghosts in [3].}
Let us to start with the pure abelian case when \( \{ T_\alpha, T_\beta \} = C_{\alpha \beta} \) is a constant and all matrix functions, \( P \), as well as all \( Z \)-functions are constants, and when \( \{ T'_\alpha, T'_\beta \} = 0 \). \( \Omega \) and \( \Pi \) are given by (18) and (19) up to quadratic terms in the ghosts. In this case we find

\[
\Omega' \equiv \{ \Pi, \Omega \} = C^\alpha T'_\alpha + C^\alpha Z'_{\alpha r} \tilde{\rho}_\alpha (-1)^{\varepsilon_\alpha} + \sum_{r=2}^{L} C^\alpha Z'_{\alpha r-1} \tilde{\rho}_{r-1} (-1)^{\varepsilon_{r-1}}, \tag{24}
\]

where

\[
T'_\alpha = P^\beta_\alpha T'_\beta, \quad Z'_{\alpha r-1} = P^\beta_\alpha Z'_{\beta r-1} - Z'_{\alpha r-1} P^\beta_\alpha, \quad r = 1, \ldots, L. \tag{25}
\]

Furthermore, we get

\[
\Omega'' \equiv \{ \Pi, \Omega' \} = C^\alpha T''_\alpha + C^\alpha Z''_{\alpha r} \tilde{\rho}_\alpha (-1)^{\varepsilon_\alpha} + \sum_{r=2}^{L} C^\alpha Z''_{\alpha r-1} \tilde{\rho}_{r-1} (-1)^{\varepsilon_{r-1}}, \tag{26}
\]

where

\[
T''_\alpha = P^\beta_\alpha T''_\beta, \quad Z''_{\alpha r-1} = P^\beta_\alpha Z''_{\beta r-1} - Z''_{\alpha r-1} P^\beta_\alpha, \quad r = 1, \ldots, L. \tag{27}
\]

The condition (20), i.e. \( \Omega'' = \Omega' \), requires then the property

\[
T''_\alpha = T'_\alpha \iff \left( P^\gamma_\alpha P^\beta_\gamma - P^\beta_\alpha \right) T'_\beta = 0, \tag{28}
\]

and

\[
Z''_{\alpha r-1} = Z'_{\alpha r-1}, \quad r = 1, \ldots, L. \tag{29}
\]

The condition (21) or (23), i.e. \( \{ \Omega', \Omega' \} = 0 \), requires on the other hand

\[
P^\gamma_\alpha C^\gamma_\rho P^\rho_\beta (-1)^{\varepsilon_\rho (\varepsilon_\beta + 1)} = 0, \quad Z''_{\alpha r-1} T'_\beta = 0, \quad Z''_{\alpha r-1} Z''_{\beta r-2} = 0, \quad r = 2, \ldots, L. \tag{30}
\]

Since \( \Omega' \) should be a standard BFV-BRST charge for a reducible theory [7, 8], we have also the standard rank conditions: For the ranges of the indices, \( \alpha_r = 1, \ldots, k_r \)

we have \( (\alpha_0 = \alpha, k_0 = m_1 + 2m_2 + q) \):

\[
\text{rank} Z''_{\alpha r-1} \bigg|_{T'_r=0} = \gamma_r, \quad \gamma_r \equiv \sum_{r'=r}^{L} k_{r'} (-1)^{r'-r}, \tag{31}
\]

where

\[
\gamma_0 = \text{rank} \frac{\partial T'_\alpha}{\partial x^I} \bigg|_{T'_r=0} = m_1 + m_2, \tag{32}
\]
which serves as a restriction on the ranges \( k_r \). Notice that \( \gamma_1 = m_2 + q \). There are several ways in which these conditions may be met by appropriate choices of the functions in \( \Omega \) and \( \Pi \). One simple choice is

\[
P_{\alpha r}^\beta = r\delta_{\alpha r}^{\beta r}, \quad r = 2, \ldots, L, \quad \Rightarrow \quad Z_{\alpha r}^{\alpha r - 1} = \alpha r^{\alpha r - 1} = Z_{\alpha r}^{\alpha r - 1}, \quad r = 2, \ldots, L,
\]

(33)

where \( Z_{\alpha r}^{\alpha r - 1}, \quad r = 2, \ldots, p \), must be chosen to satisfy (30) and (31). The condition (29) for \( r = 1 \) may then be solved by imposing the projection property (11) on \( P_{\alpha r}^\beta \), in which case we have

\[
Z_{\alpha 1}^{\alpha 1} = Z_{\alpha 1}^{\beta 1} = \delta_{\alpha 1}^{\beta 1} - P_{\alpha 1}^\beta,
\]

(34)

which in turn implies that \( Z_{\alpha 1}^{\alpha 1} \) automatically satisfies the second condition in (30).

In the case of a general first stage theory \( (L = 1) \) we have

\[
\Omega' = \{\Pi, \Omega'\} = C^\alpha T_\alpha' + C^{\alpha 1} Z_{\alpha 1}^{\alpha 1} \bar{P}_\alpha (1 - 1)^{\epsilon_\alpha} + \frac{1}{2} C^{\beta} C^{\alpha} U_{\alpha 1}^{\gamma} \bar{P}_\gamma (-1)^{\epsilon_\alpha + \epsilon_\beta} + \ldots,
\]

(35)

where \( T_\alpha' \) and \( Z_{\alpha 1}^{\alpha 1} \) are given by (25) and

\[
U_{\alpha 1}^{\gamma} = \{T_\alpha, P_\beta^\gamma\} - \{T_\beta, P_\alpha^\gamma\} (1 - 1)^{\epsilon_\alpha + \epsilon_\beta} + P_\alpha^\beta U_{\rho 1}^\gamma - P_\beta^\rho U_{\rho 1}^\gamma (1 - 1)^{\epsilon_\alpha + \epsilon_\beta} - U_{\alpha 1}^\rho P_\rho^\gamma.
\]

(36)

The condition (20), i.e. \( \Omega'' \equiv \{\Pi, \Omega'\} = \Omega' \) requires e.g. (28) and (29) for \( r = 1 \), and

\[
U_{\alpha 1}^{\gamma} = U_{\alpha 1}^{\gamma}.
\]

(37)

Again (29) is satisfied for the choice (33), \( P_{\alpha 1}^{\alpha 1} = \delta_{\alpha 1}^{\beta 1} \), and if (11) is valid. In this case we have

\[
Z_{\alpha 1}^{\alpha 1} \equiv Z_{\alpha 1}^{\beta 1} = \delta_{\alpha 1}^{\beta 1} - P_{\alpha 1}^\alpha.
\]

(38)

Property (11) implies furthermore that (37) requires (33) to be satisfied. Thus, \( U_{\alpha 1}^{\gamma} \) in \( \Omega \) may be identified with \( U_{\alpha 1}^{\gamma} \) in (13) when (11) is satisfied. In principle \( U_{\alpha 1}^{\gamma} \) in (33) may contain a term \( U_{\alpha 1}^{\gamma} Z_{\alpha 1}^{\alpha 1} \) coming from the following term in \( \Pi \):

\[
\frac{1}{2} (-1)^{\epsilon_\alpha + \epsilon_\beta} C^\alpha C^{\alpha 1} U_{\alpha 1}^{\gamma} \bar{P}_\alpha (1 - 1)^{\epsilon_\alpha 1}.
\]

(39)

However, condition (37) together with (11) excludes such a term. The nilpotency of \( \Omega' \) requires (30). The second condition in (30) is satisfied since (11) follows when (11) is satisfied. It is clear that conditions (20) and (23) require the property (3) together with all its Jacobi identities. This first stage reducible treatment is sufficient if \( \alpha_1 = 1, \ldots, m_2 + r \) since the rank of \( Z_{\alpha 1}^{\alpha 1} \) is \( m_2 + r \). However, in order for \( \alpha_1 \) to be a covariant index it might not be possible to satisfy this range condition in which case one is forced to consider a higher stage reducible treatment.
In the case when the original constraints are first class ones, \(i.e.\) when

\[
\text{rank}\{T_\alpha, T_\beta\} \bigg|_{T=0} = 0, \quad (40)
\]

it is possible to choose \(\Omega\) to satisfy \(\{\Omega, \Omega\} = 0\). \(\Omega\) is then determined. \(\Pi\) may then be chosen to be the ghost charge \(G\) in (17) in which case \(\Omega' = \Omega\). However, it may also be possible to find a \(\Pi\) different from \(G\) which satisfy (20) and (21). In this case \(\Omega' \neq \Omega\) but \(\Omega'\) will then be canonically equivalent to \(\Omega\).

### 3 Invariant formulation: Quantum theory.

At the quantum level all canonical variables above are turned into operators. We have then the nonzero fundamental commutation relations (denoting the operators by the same symbols as above)

\[
[x^i, p_j] = i\hbar \delta^i_j, \quad [C^\alpha, \bar{P}_\beta] = i\hbar \delta^\alpha_\beta, \quad [C^\alpha_1, \bar{P}_\beta_1] = i\hbar \delta^\alpha_1 \beta_1, \ldots \quad (41)
\]

These operators have the same Grassmann parities as the corresponding classical variables and all commutators are graded ones. All real functions in the classical theory are then turned into hermitian operators which we give in a Weyl-ordered form below. The ghost charge operator \(G\) is (cf (17))

\[
G = \frac{1}{2} \left( C^\alpha \bar{P}_\alpha (-1)^{\varepsilon_\alpha} - \bar{P}_\alpha C^\alpha \right) + \]

\[
+ \sum_{r=1}^{L} \frac{(r+1)}{2} \left( C^{\alpha_r} \bar{P}_{\alpha_r} (-1)^{\varepsilon_{\alpha_r}+r} - \bar{P}_{\alpha_r} C^{\alpha_r} \right). \quad (42)
\]

The odd, hermitian BRST-like charge \(\Omega\) with ghost number one is of the form (cf the classical expression (18))

\[
\Omega = C^\alpha T_\alpha + C^{\alpha_1} Z_1^\alpha \bar{P}_\alpha (-1)^{\varepsilon_\alpha} + \sum_{r=2}^{L} C^{\alpha_r} Z_1^{\alpha_{r-1}} \bar{P}_{\alpha_{r-1}} (-1)^{\varepsilon_{\alpha_{r-1}}} + \ldots,
\]

\[
(i\hbar)^{-1}[G, \Omega] = \Omega, \quad (43)
\]

where the dots indicate terms of higher powers in the ghosts. The hermitian projection operator \(\Pi\) of ghost number zero is of the form (cf the classical expression (19))

\[
\Pi = \frac{1}{2} \left( C^\alpha P^\beta_\alpha \bar{P}_\beta (-1)^{\varepsilon_\beta} - \bar{P}_\beta (-1)^{\varepsilon_\beta} P^\beta_\alpha (1)^{\varepsilon_\alpha \varepsilon_\beta} \right) - 
\]

\[
+ \frac{1}{2} \sum_{r=1}^{L} (-1)^r \left( C^{\alpha_r} P^\beta_\alpha \bar{P}_{\beta_r} (-1)^{\varepsilon_{\beta_r}} + \bar{P}_{\beta_r} (-1)^{\varepsilon_{\beta_r}} P^\beta_\alpha C^{\alpha_r} (1)^{\varepsilon_{\alpha_r+1} \varepsilon_{\beta_r+1}} \right) + \ldots,
\]

\[
(i\hbar)^{-1}[G, \Pi] = 0. \quad (44)
\]
The explicit terms in (43) and (44) are boundary terms chosen in accordance with the requirements of the corresponding classical theory. Hermiticity may be obtained if we choose $T$, the requirements of the corresponding classical theory. Hermiticity may be obtained but (47) is only satisfied if $\Omega$ is nilpotent in which case $\Omega'$ and $\Omega''$ may be obtained only if the original constraints are purely first class ones. In the latter case we also have finite number of degrees of freedom, at least if we relax the requirement of covariance which might lead to obstructions. The reason is that we may solve the conditions for the abelian case treated classically in the previous section, and that we expect that the general case may be obtained by a unitary transformation of this abelian case at least locally. The latter property is true in the standard BFV treatment [9].

One may notice that if we choose $\Pi = G$ then (46) is satisfied by construction but (47) is only satisfied if $\Omega$ is nilpotent in which case $\Omega' = \Omega$. This is possible only if the original constraints are purely first class ones. In the latter case we also obtain an $\Omega'$ which is unitary equivalent to $\Omega$ if there exists a $\Pi \neq G$.

We need also a BRST invariant Hamiltonian. The original Hamiltonian may always be turned into a Hamiltonian $H_0$ satisfying the observability condition (50) by simply adding linear combinations of the constraints $T_\alpha$, i.e. $\{H_0, T'_\alpha\} = V_\alpha^\beta T'_\alpha$. We have then after quantization ($(V_\alpha^\beta)^\dagger = V_\alpha^\beta(-1)^{\xi_\beta(\xi_{\alpha+1})}$)

$$(ih)^{-1}[H, \Omega'] = 0, \quad H = H_0 + \frac{1}{2}(C^\alpha V_\alpha^\beta \bar{P}_\beta(-1)^{\xi_\beta} - \bar{P}_\beta(-1)^{\xi_\beta} V_\alpha^\beta C^\alpha(-1)^{\xi_\alpha\xi_\beta}) + \ldots \tag{51}$$
Due to the condition (46) this implies $[\mathcal{H}', \Omega'] = 0$ where

$$
\mathcal{H}' \equiv (i\hbar)^{-1}[\Pi, \mathcal{H}] = \frac{1}{2} \left( C^\alpha (P_\alpha^\beta V_\beta^\rho - V_\alpha^\beta P_\beta^\rho) \tilde{\mathcal{P}}_\rho(-1)^{s_\rho} - \tilde{\mathcal{P}}_\rho(-1)^{s_\rho}(V_\beta^\rho P_\alpha^\beta - P_\beta^\rho V_\alpha^\beta) C^\alpha (-1)^{s_{\beta}(s_{\alpha} + s_{\alpha} + 1)} \right) + \ldots \tag{52}
$$

We expect that $\mathcal{H}' = (i\hbar)^{-1}[\rho, \Omega']$. Notice in this connection that $\mathcal{H} \rightarrow \mathcal{H} + (i\hbar)^{-1}[\psi, \Omega']$ implies $\mathcal{H}' \rightarrow \mathcal{H}' + (i\hbar)^{-1}[\psi + \psi', \Omega']$ where $\psi' \equiv (i\hbar)^{-1}[\Pi, \psi]$.

$\Omega'$ constitutes the BRST charge in the minimal sector for the original theory. A complete BRST quantization requires the introduction of antighosts and Lagrange multipliers and their conjugate momenta up to a certain extension required by the prescription given in [7]. The total BRST charge $Q'$ has then the form

$$
Q' = \Omega' + \sum_{s'=0}^{L} \sum_{s=s'} \pi_{s'}^{s'} P_{s'}, \tag{53}
$$

and the total Hamiltonian $H$ is

$$
H = \mathcal{H} + (i\hbar)^{-1}[\Psi, Q'], \quad (i\hbar)^{-1}[H, Q'] = 0, \tag{54}
$$

where

$$
\Psi = \sum_{s'=0}^{L} \sum_{s=s'} \left( \tilde{C}^{s'}_{s} \chi^{s'}_{s} + \tilde{\chi}^{s'}_{s} \lambda^{s'}_{s} \right) \tag{55}
$$

is an odd gauge-fixing fermion. In (53) and (55) we have used a short-hand notation which may be understood by a comparison with [7]. $\tilde{C}^{s'}_{s}$ and $\lambda^{s'}_{s}$ represents antighosts and Lagrange-multipliers (extra ghosts), and $P_{s'}^{s'}$ and $\pi_{s'}^{s'}$ their conjugate momenta. $\chi^{s'}_{s}$ and $\tilde{\chi}^{s'}_{s}$ are gauge fixing functions (see [7]). $Q'$ is always nilpotent when $\Omega'$ is nilpotent. Physical states are determined by the condition

$$
Q'|_{phys} = 0. \tag{56}
$$

It is clear that there also exist an extended $\Omega$ and an extended projection operator $\Pi$, denoted $Q$ and $\tilde{\Pi}$ respectively, satisfying

$$
Q'' \equiv (i\hbar)^{-1}[\tilde{\Pi}, Q'] = Q', \quad Q' \equiv (i\hbar)^{-1}[\tilde{\Pi}, Q]. \tag{57}
$$

The last property implies

$$
\tilde{\Pi}|_{phys} = |_{phys}'. \tag{58}
$$

Since $\tilde{\Pi}$ contains a ghost dependence which is close to an extended ghost charge it seems as if we may impose the condition

$$
\tilde{\Pi}|_{phys} = 0 \tag{59}
$$

without affecting the true physical degrees of freedom that contribute to the BRST cohomology. Maybe it is even possible to impose $Q|_{phys} = 0$. However, in this case the BRST invariant Hamiltonian $H$ satisfying (54) is required to satisfy the stronger conditions $[H, Q] = 0$ and $[H, \tilde{\Pi}] = 0$. 

9
4 An example

Our method may e.g. be applied to the Ferber-Shirafuji ”twistorized” particle model [10, 11]. Its Hamiltonian constraint analysis was given in [12], where the separation into first and second class constraints only was made in a translation noninvariant way. The simplest model of this type is the $d = 4$ twistor model for the massless particle. (For a detailed review of the twistorized models, see [13].) The action is

$$S = \frac{1}{2} \int d\tau \sigma_{\mu a} \dot{x}^\mu \lambda^a \dot{\lambda}^\dot{a},$$

(60)

where $\mu = 0, 1, 2, 3; a, \dot{a} = 1, 2$. $\sigma_{\mu a}$ is a Pauli matrix, and $\lambda, \dot{\lambda}$ are bosonic SL(2,C) spinors. The indices $a, \dot{a}$ are raised and lowered by $\varepsilon^{ab}, \varepsilon_{ab}$ ($\varepsilon^{12} = \varepsilon_{21} = 1$). The complete set of independent constraints is

$$p_\mu - \sigma_{\mu a} \lambda^a \dot{\lambda}^\dot{a} = 0, \quad p_\dot{a} = 0, \quad \bar{p}_\dot{a} = 0,$$

(61)

where $p_\mu, p_\dot{a}$, and $\bar{p}_\dot{a}$ are conjugate momenta to $x^\mu, \lambda^a$, and $\dot{\lambda}^\dot{a}$. This set contains two first class constraints and six second class ones. The explicit separation given in [12] involved the combinations $\mu_\dot{a} = \sigma_{\mu a} \dot{x}^\mu$ and $\bar{p}_\dot{a} = \sigma_{\mu a} \lambda^a x^\mu$ which violate manifest translation invariance. According to our approach we have not to separate the constraints into first and second class ones, but to find a set of constraints in involution containing the maximal independent set which here has five elements. There are two covariant choices to pick five such constraints. The first choice is

$$p_\mu - \sigma_{\mu a} \lambda^a \dot{\lambda}^\dot{a} = 0, \quad i(\lambda^a p_\dot{a} - \dot{\lambda}^\dot{a} \bar{p}_\dot{a}) = 0.$$

(62)

The second option is

$$\bar{p}_\dot{a} = 0, \quad p_\dot{a} = 0, \quad p^\mu p_\mu = 0.$$

(63)

It seems very difficult to find a projection matrix $P_\alpha^\beta$ which takes us from the set (61) to the set (62) or (63) in a covariant manner. Even if it would be possible we would have to deal with a higher reducible situation due to the difficulty to find a covariant $Z_{\alpha_1}$ with $\alpha_1 = 1, 2, 3$. However, if we from the very beginning consider the following reducible set

$$T_\alpha \equiv \left( p_\mu - \sigma_{\mu a} \lambda^a \dot{\lambda}^\dot{a}, \quad \bar{p}_\dot{a}, \quad p_\dot{a}, \quad p^\mu p_\mu, \quad i(\lambda^a p_\dot{a} - \dot{\lambda}^\dot{a} \bar{p}_\dot{a}) \right),$$

(64)

which constitutes ten covariant constraints out of which only eight are independent, then it is possible to project out the subset (62) or (63) in a trivial manner with a covariant projection matrix $P_\alpha^\beta$. Furthermore, the resulting theory will be a first stage reducible theory, and both $\Omega$ and $\Pi$ and thereby $\Omega'$ will be manifestly invariant. Notice that $T_\alpha = 0$, where $T_\alpha$ is given by (64), is completely equivalent to (61).

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