HIGHER ORDER CORKS

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Abstract. It is shown that any finite list of smooth, closed, simply-connected 4-manifolds that are homeomorphic to a given one $X$ can be obtained by removing a single compact contractible submanifold (or cork) from $X$, and then regluing it by powers of a boundary diffeomorphism. Furthermore, by allowing the cork to be noncompact, the collection of all the smooth manifolds homeomorphic to $X$ can be obtained in this way. The existence of a universal noncompact cork is also established.

0. Introduction

A cork is a compact contractible 4-manifold $C$ equipped with a diffeomorphism $h: \partial C \to \partial C$. We work throughout in the smooth oriented category, so implicitly assume $C$ is smooth and oriented and $h$ is orientation preserving. The cork $(C, h)$ is finite of order $n$ or infinite, according to whether $h$ is periodic of order $n$ or of infinite order. Corks of order 2 will be called involutory.

If $C$ is embedded in a 4-manifold $X$, then the associated cork twist $X_{C, h} = (X - \text{int}(C)) \cup_h C$ is homeomorphic to $X$ by Freedman [12], but need not be diffeomorphic to $X$ by Akbulut [1]. Thus cork twisting may alter smooth structures on 4-manifolds. In fact it is now known by work of Curtis-Freedman-Hsiang-Stong [10] and Matveyev [21] that any pair of compact simply-connected 4-manifolds that are h-cobordant rel boundary (and thus homeomorphic) are related by a single involutory cork twist, and that for closed manifolds, the cork may be chosen with simply-connected complement; we refer to this as the “Involutory Cork Theorem”. Our first main result extends this to arbitrary finite families of closed 4-manifolds, or more generally compact 4-manifolds bounded by homology spheres, see Theorem 2.5. This demonstrates the ubiquity of nontrivial higher order corks, first shown to exist in [7].

Finite Cork Theorem. Given any finite list $X_i$ $(i \in \mathbb{Z}_n)$ of closed simply-connected 4-manifolds homeomorphic to a given one $X = X_0$, there is a cork $(C, h)$ of order $n$ embedded in $X$ with simply-connected complement whose cork twists $X_{C, h}$ are diffeomorphic to $X_i$ for each $i$.

To address infinite lists $X_i$ $(i \in \mathbb{Z})$ of 4-manifolds homeomorphic to $X$, such as an enumeration of all the exotic smooth structures on $X$, it is tempting to search for a single infinite cork $(C, h)$ embedded in $X$ such that $X_{C, h} \cong X_i$ for all $i$ (here and below, $\cong$ denotes diffeomorphism). Such a cork need not exist, however, as noted by Tange [25]. For example, it follows from the adjunction inequality that knot surgeries on the Kummer surface using any list of knots with unbounded Alexander polynomial degrees cannot be the cork twists associated to a fixed embedding of a single infinite cork (cf. Yasui [28]). Thus the Finite Cork Theorem has no direct infinite analogue. But remarkably, it can be generalized by relaxing the compactness condition on the cork. Define a noncompact cork to be a pair $(C, h)$, where $C$ is a noncompact contractible 4-manifold equipped with a diffeomorphism $h$ of its boundary (when $\partial C \neq \emptyset$) or its ends (when $C$ is open), and define the cork twist associated with an embedding of $C$ as above (see [3] for the precise definitions in the open case). Then we prove (see Theorem 3.2):

† If $X_{C, h}$ and $X$ are diffeomorphic, the embedding $C \subset X$ is said to be trivial. The cork $(C, h)$ is trivial if all its embeddings in all 4-manifolds are trivial, or equivalently if $h$ extends to a diffeomorphism of $C$ [4].
Infinite Cork Theorem. Given any infinite list $X_i$ ($i \in \mathbb{Z}$) of closed simply-connected 4-manifolds homeomorphic to a given one $X = X_0$, there is a noncompact cork $(C, h)$ with an embedding in $X$ whose cork twists $X_{C,h}^i$ are diffeomorphic to $X_i$ for each $i$. The cork $C$ may be chosen to be bounded or open, and in the open case, $C$ may be taken to be homeomorphic to $\mathbb{R}^4$.

At the end of the paper, we show how to construct a single noncompact universal cork.

Universal Cork Theorem. For any doubly infinite list $X_{ij}$ of closed simply-connected 4-manifolds indexed by pairs of integers $i$ and $j$, with all $X_{ij}$ homeomorphic to $X_j := X_{0j}$ for any given $j$, there is a noncompact cork $(C, h)$ with embeddings $C \subset X_j$ for every $j$ so that the cork twists $(X_j)_{C,h}^i$ are diffeomorphic to $X_{ij}$ for each $i$ and $j$. The cork $C$ may be chosen to be bounded or open, and in the open case, $C$ may be taken to be homeomorphic to $\mathbb{R}^4$.

This cork $(C, h)$ is “universal” since the $X_{ij}$ can be chosen to include all closed simply-connected 4-manifolds, letting $j$ index the homeomorphism classes, and $i$ index all exotic smooth structures within each class (with repeats if there are only finitely many distinct smooth structures).

Remark. It is unknown whether compact universal corks exist, i.e. (compact) corks $(C, h)$ for which any pair $X, Y$ of homeomorphic closed simply-connected 4-manifolds are related by a cork twist of $C$, meaning $Y \cong X_{C,h}$ for some embedding $C \subset X$. Note that it can be shown that some nontrivial corks (e.g. the Akbulut-Mazur cork [1]) are not universal.

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1. Preliminaries

AC manifolds

Let $X$ be a 4-manifold that can be built from the 4-ball by adding 1 and 2-handles. Any such handle structure is specified by a link $K \cup L$ in the 3-sphere, where $K$ is a dotted unlink representing the 1-handles and $L$ is the framed link of attaching circles for the 2-handles. We write $X = [K, L]$ to specify this handle structure, which induces a presentation of the fundamental group

$$\pi_1(X) = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_\ell \rangle$$

where $x_1, \ldots, x_k$ are the meridians of the components of $K$, and $r_1, \ldots, r_\ell$ are the words in the free group $F(x_i, \ldots, x_k)$ (determined up to conjugacy) traced out by the components of $L$.

1.1. Definition. Given $X = [K, L]$ as above, suppose that either

1) the 2-handles homotopically cancel some subset of the 1-handles, that is $k \geq \ell$ and (after possibly reordering the generators) $r_i = x_i$ for $i \leq \ell$, or

2) some subset of the 2-handles homotopically cancel the 1-handles, that is $k \leq \ell$ and (after possibly reordering the relations) $r_i = x_i$ for $i \leq k$.

† The 1-handles can be viewed as trivial 2-handles carved out from the 4-ball, see for example [18 Chapter I.2]. These 2-handles are obtained by pushing into $B^4$ a family of disjoint spanning disks in $S^3$ for the dotted circles. Such families of disks are not unique (up to isotopy rel boundary) in $S^3$, but they are unique in $B^4$. 

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Then \([K,L]\) is called an AC structure on \(X\). Note that \(X\) is homotopy equivalent to a wedge of \(k-\ell\) circles in case 1) and to a wedge of \(\ell-k\) two-dimensional spheres in case 2); hence \(X\) is contractible if \(k=\ell\). A 4-manifold \(X\) is AC, or an AC manifold, if it has a handle structure that can be transformed by handle slides into an AC structure. In the contractible case we refer to \(X\) (by abuse of terminology) as an AC cork, even if it is not equipped with a boundary diffeomorphism.

1.2. Remarks. a) The acronym AC refers to the Andrews-Curtis moves on group presentations \([6]\) that correspond to 1 and 2-handle slides:

- replace a generator by its inverse or by its product with another generator, and
- replace a relator by its inverse or by its product with a conjugate of another relator.

It is unknown whether all balanced presentations of the trivial group can be trivialized through Andrews-Curtis moves; this is the Andrews-Curtis Conjecture. Even the weaker “stable” version of this conjecture is unknown, where one allows the addition of new generators along with new relators equal to those generators, corresponding to the introduction of cancelling 1/2-handle pairs.

b) The double \(C \cup -C\) of any AC cork \(C\) is diffeomorphic to \(S^4\). Indeed \(C \cup -C\) is the boundary of \(C \times [0,1] \cong B^5\), since homotopy implies isotopy for curves in 4-manifolds. But if \(C\) is equipped with a boundary diffeomorphism \(h\), then there is no a priori reason to suspect that the associated “twisted doubles” \(C \cup_h -C\) should be diffeomorphic to \(S^4\) (unless of course the smooth 4-dimensional Poincaré Conjecture holds). Indeed every homotopy 4-sphere can be constructed in this way; see Corollary 1.15 below. An AC cork all of whose associated twisted doubles are diffeomorphic to \(S^4\) will be called a simple cork.

In subsequent discussions, embeddings of one AC manifold in another frequently arise. In such cases it may be important to know whether their AC structures can be chosen to be compatible.

1.3. Definition. If an AC manifold \(X\) embeds in the interior of an AC manifold \(Y\) so that any AC structure on \(X\) is a sub-handlebody structure of an AC structure on \(Y\), then we say \(X\) is encased in \(Y\). A simple argument then shows that any “twist” \(Y_{X,h} = (Y - \mathrm{int}(X)) \cup_h X\), where \(h\) is a diffeomorphism of \(\partial X\), is still AC.

AC cobordisms

Let \(X\) be a compact cobordism between closed connected 3-manifolds \(M\) and \(N\) that can be built from \(M \times [0,1]\) by adding 1 and 2-handles along \(M \times \{1\}\). Any such relative handle structure is specified as above by a link \(K \cup L\), but now in \(M \times \{1\}\). This presents \(\pi_1(X)\) as a quotient of the free product \(\pi_1(M) \ast F(x_1, \ldots, x_k)\), where the \(x_i\) correspond to the meridians of the components of \(K\), by the normal subgroup generated by the elements \(r_1, \ldots, r_\ell\) (determined up to conjugacy) traced out by the components of \(L\). To indicate this, we write

\[
\pi_1(X) = (\pi_1(M), x_1, \ldots, x_k | r_1, \ldots, r_\ell).
\]

1.4. Definition. A relative handlebody structure on \(X\) as above is called an AC structure if it satisfies either condition 1) or 2) of Definition 1.1. Thus the quotient space \(X/M\) (crushing \(M\) to a point) is homotopy equivalent to a wedge of \(|k-\ell|\) circles or 2-spheres, in case 1) or 2) respectively. Hence \(X\) is contractible if \(k=\ell\). Furthermore, the map \(\pi_1(M) \rightarrow \pi_1(X)\) induced by inclusion is onto in case 2) since the extra 2-handles simply add relations.

A cobordism \(X\) as above is AC, or an AC cobordism from \(M\) to \(N\), if it has a relative handle structure that can be transformed by handle slides into an AC structure. In particular, if \(X\) can be built from \(M \times [0,1]\) by adding an equal number of homotopically cancelling 1 and 2-handles, so satisfies both conditions 1) and 2), then we call it an AC homology cobordism.
1.5. **Remark.** The “homology cobordism” condition means that the inclusions \( M \) and \( N \) into \( X \) induce isomorphisms on homology, so in particular \( M \) is a homology sphere if and only if \( N \) is. This follows from the exact sequences of \( (X, M) \) and \( (X, N) \), since (by duality) the contractibility of \( X/M \) implies the vanishing of the relative homology of groups of \( (X, M) \), and so also of \( (X, N) \).

**Tight embeddings**

In the arguments below it will be necessary to control the fundamental groups of the complements of certain embedded submanifolds. For this purpose, we formulate the following notion of a “tight” embedding. Let \( C \) and \( X \) be compact simply-connected 4-manifolds. Assume that \( C \) is contractible and that \( X \) is either closed or has nonempty connected boundary.

1.6. **Definition.** A **tight embedding** of \( C \) in \( X \) will mean an embedding \( C \subset \text{int}(X) \) for which \( X - \text{int}(C) \) is either an AC manifold when \( X \) is closed, or an AC cobordism from \( \partial X \) to \( \partial C \) when \( X \) has boundary. Note that since the complement \( X - \text{int}(C) \) has trivial first homology, it must be AC of type 2), and so can be built from either a 4-ball or a collar on \( \partial X \) with only 1 and 2-handles so that (after sliding) some subset of the 2-handles homotopically cancel the 1-handles. When we say \( C \subset X \) is tight, it is understood that the embedding is in \( \text{int}(X) \), and if \( C \) is equipped with a diffeomorphism \( h : \partial C \to \partial C \), we may just say “\((C,h)\) is a tight cork in \( X \)”.

1.7. **Remark.** Let \( C \subset X \) be tight. Then \( X - C \) is simply-connected when \( X \) is closed. If \( X \) has boundary, then as noted in Definition 1.4, the inclusion \( \partial X \hookrightarrow X - C \) induces a surjection on \( \pi_1 \). So, if \( X \) is itself embedded with simply-connected complement in a closed 4-manifold \( Y \), then \( Y - C \) will again be simply-connected (by Van Kampen’s Theorem).

**Multicorks**

For the present purposes, it is useful to broaden the notion of corks and their cork twists.

1.8. **Definition.** A **multicork** is a list \( C = (C_0, C_1, \ldots) \) of compact contractible 4-manifolds equipped with boundary diffeomorphisms \( h_i : \partial C_i \to \partial C_0 \) for each \( i \), with \( h_0 = \text{id} \). The **order** of the multicork is the length of the list, which may be finite or infinite. The \( C_i \) are called the **components** of \( C \). Their boundaries are pairwise diffeomorphic, by definition, but the components themselves need not be diffeomorphic; see [4] for examples. If the components are all AC, then \( C \) is called an AC **multicork**.

If in addition each “twisted double” \( C_0 \cup_{h_i} -C_i \) is diffeomorphic to the 4-sphere, then \( C \) is called a **simple multicork**.

*Multicorks generalize corks.* Indeed, there is an order-preserving (non-surjective) injection

\[ \mu : \text{Corks} \hookrightarrow \text{Multicorks} \]

sending any cork \((C,h)\) to the constant multicork \( \mu(C,h) = (C,C,\ldots) \) of the same order with boundary identifications \( h_i = h^i \). Note that if \((C,h)\) is simple, then so is \( \mu(C,h) \).

1.9. **Definition.** An **embedding** of a multicork \( C = (C_0, C_1, \ldots) \) in a 4-manifold \( X \) is an embedding \( C_0 \subset \text{int}(X) \), and if this embedding is tight, we simply say “\( C \) is a tight multicork in \( X \)”. Such an embedding generates a list of **cork replacements**

\[ X_{C_0, C_i} = (X - \text{int}(C_0)) \cup_{h_i} C_i. \]

These replacements depend of course on the embedding \( C_0 \subset X \) and the boundary identifications \( h_i \), but this dependence is suppressed in the notation since it can often be gleaned from the context. Note that \( \partial X_{C_0, C_i} \) and \( \partial X \) are naturally identified, and will be viewed as being literally equal.
Higher Order Corks

Cork replacements generalize cork twists. Indeed, the cork twists \( X_{C,h} \) of a cork \((C,h)\) associated with an embedding \( C \subset X \) are the cork replacements \( X_{C_0,C_i} \) of the multicork \( \mu(C,h) \) associated with the same embedding. It will be important for what follows to know when we can go in the opposite direction: Given a multicork \( C \) in \( X \), under what conditions is there a cork in \( X \) of the same order whose twists give manifolds diffeomorphic to the cork replacements of \( C \)?

1.10. Definition. An embedded cork \((C,h)\) in a 4-manifold \( X \) is correlated with an embedded multicork \( C = (C_0, C_1, \ldots) \) in \( X \) if the cork twists \( X_{C,h} \) are diffeomorphic rel boundary to the cork replacements \( X_{C_0,C_i} \) for all \( i \) (recall that the boundaries of \( X_{C,h} \) and \( X_{C_0,C_i} \) are naturally identified with \( \partial X \), and thus with each other). Similarly \((C,h)\) is correlated with a list \((A_i, \tau_i)\) of involutory corks embedded in \( X \) if the \( X_{C,h} \) are diffeomorphic rel boundary to \( X_{A_i, \tau_i} \) for each \( i \).

The “pinwheel lemma” below shows that any finite simple multicork \( C \) has an associated higher order cork which has embeddings correlated with any embedding of \( C \). This will be used in §2 to prove a related result – the “consolidation theorem” – for finite lists of simple involutory corks.

Pinwheels

1.11. Definition. The pinwheel of a finite multicork \( C = (C_0, \ldots, C_{n-1}) \) is the finite cork \((P, \pi)\), where \( P \) is the boundary connected sum \( C_0 \natural \cdots \natural C_{n-1} \) and \( \pi \) is the “obvious” boundary rotation shifting each \( \partial C_i \) to \( \partial C_{i-1} \) (here subscripts are taken mod \( n \)). More precisely, fix a linear rotation \( \pi \) of the 3-sphere of order \( n \) and a principal orbit \( x_0, \ldots, x_{n-1} \) of the action of the cyclic group generated by \( \pi \), with \( x_i = \pi^{-i}(x_0) \) for each \( i \). Choose \( y_0 \in \partial C_0 \), and set \( y_i = h_i^{-1}(y_0) \in \partial C_i \), where \( h_i: \partial C_i \to \partial C_0 \) are the boundary identifications of \( C \). Now build \( C_0 \natural \cdots \natural C_{n-1} \) from the disjoint union \( B^4 \sqcup C_0 \sqcup \cdots \sqcup C_{n-1} \) by adding 1-handles joining \( x_i \in S^3 \) to \( y_i \in \partial C_i \) for each \( i \). Then \( \pi \) extends to a rotation of \( \partial P = \partial C_0 \# \cdots \# \partial C_{n-1} \cong \partial C_0 \# \cdots \# \partial C_0 \) of order \( n \), sending each \( \partial C_i \) to \( \partial C_{i-1} \) via \( h_{i-1}^{-1}h_i \), as shown in Figure 1a for the case \( n = 4 \).

1.12. Remark. The pinwheel construction preserves many of the properties discussed above. For example, it follows easily from Definition 1.1 that the pinwheel of an AC multicork is an AC cork, and from Definition 1.8 that the pinwheel of a simple multicork is simple.

![Figure 1. Pinwheels](image.png)

Given an embedding of a finite simple multicork \( C \), the “Pinwheel Lemma” below produces an embedding of the pinwheel of \( C \) whose cork twists are diffeomorphic (rel boundary) to the cork
replacements of \( C \), that is, the embeddings of \( C \) and its pinwheel are correlated (see Definition 1.10). The proof of this lemma generalizes an argument of Matveyev [21] (see also Kirby [19]) that produces involutory cork embeddings correlated with embeddings of order 2 multicorks; a similar generalization was used in the construction of equivariant corks in [7].

1.13. Pinwheel Lemma. For any embedding of a finite simple multicork \( C \) in a 4-manifold \( X \), there is a correlated embedding of its pinwheel \( (P, \pi) \) in \( X \). If the embedding of \( C \) is tight, then the embedding of \( P \) can be chosen to be tight.

Proof. Let \( C = (C_0, \ldots, C_{n-1}) \) with \( C_0 \subset X \). Since \( C \) is simple, \( C_0 \cup_{\partial X} -C_i \cong S^4 \) for each \( i \). Thus the \( C_i \) embed in disjoint 4-balls \( B_i \) in \( X - C_0 \) with \( B_i - C_i \) diffeomorphic rel boundary to a punctured copy of \( \text{int}(-C_0) \). Note that these embeddings are tight since \( C_0 \) is AC. Combining them with the original embedding \( C_0 \subset X \) gives an embedding of \( \cup C_i = C_0 \cup \cdots \cup C_{n-1} \) in \( X \), which then extends to an embedding of \( P = C_0 \natural \cdots \natural C_{n-1} \), guided by a collection of disjoint arcs in \( X - \cup C_i \) as illustrated in Figure 1a.

For each \( j \), the map \( \pi^j \) rotates \( P \) clockwise by \( (j/n) \)th turn, and it follows that the cork twist \( X_{P,\pi^j} \) is diffeomorphic rel boundary to the 4-manifold \( X_j \) obtained from \( X \) by replacing each \( C_i \) by \( C_{i+j} \) (with subscripts mod \( n \)). These replacements have no effect except when \( i = 0 \) since \( C \) is simple (and all diffeomorphisms of \( \partial B^4 \) extend to \( B^4 \) by Cerf [9]) so \( X_j \) is diffeomorphic rel boundary to \( X_{C_0,C_j} \). Therefore the embedding \( P \subset X \) is correlated with the embedding of \( C \) in \( X \).

Now assume \( C_0 \subset X \) is tight. Recall that this reduces that \( X \) is compact and simply-connected with connected, possibly empty, boundary. In fact we can assume that \( \partial X \) is nonempty since the closed case follows by removing a 4-ball. Then, the embedding of \( C = C_0 \natural B_1 \natural \cdots \natural B_{n-1} \) in \( X \) is tight, since \( C \) is an isotopic deformation of \( C_0 \). But \( X - \text{int}(P) \) is obtained from \( X - \text{int}(C) \) by attaching only homotopically cancelling 1 and 2-handles, since each \( B_i - \text{int}(C_i) \) is an AC homology cobordism. Thus \( X - \text{int}(P) \) is an AC cobordism from \( \partial X \) to \( \partial P \). That is, \( P \subset X \) is tight. \( \square \)

The Involutory Cork Theorem

The Involutory Cork Theorem, due to Curtis-Freedman-Hsiang-Stong [10] and Matveyev [21], states that any two (smooth) closed simply-connected 4-manifolds that are homotopy equivalent, and thus homeomorphic, are related by a single cork replacement (see Definition 1.9). It was shown in [10], based on earlier work of Stong [24], that the common complement of the corks can be chosen to be simply-connected. Moreover, as demonstrated in [21], this cork replacement can be accomplished by an involutory cork twist. A nice exposition of the full theorem is given in [19].

We now state a relative version of the theorem in a form that will be convenient for our purposes; it immediately implies the absolute version by removing a 4-ball. The proof can be gleaned from a careful reading of [10] and [19], but for the reader’s convenience we include a sketch.

1.14. Relative Involutory Cork Theorem. If \( X_0 \) and \( X_1 \) are homeomorphic simply-connected compact 4-manifolds with homology sphere boundaries, then any diffeomorphism \( f: \partial X_0 \to \partial X_1 \) extends to a diffeomorphism \( (X_0)_P,\tau \to X_1 \) for some tight simple involutory cork \( (P, \tau) \) in \( X_0 \).

Proof. First construct a relative h-cobordism \( W \) from \( X_0 \) to \( X_1 \) with the mapping cylinder \( M_f \) on the lateral boundary, as follows: The closed manifold \( X_1 \cup_{f} -X_0 \) has zero signature, hence bounds a 5-manifold which, appropriately surgered, has a relative handlebody structure with only 2 and 3-handles, as in Wall’s foundational paper [27]. Splitting this handlebody along its middle level (between the 2 and 3-handles) and re-gluing using Wall’s Theorem 2 [26, pg. 136] (note that it may be necessery to introduce an extra cancelling 2/3-handle pair to apply Wall’s result) gives the desired h-cobordism \( W \), built from \( X_0 \times [0, \varepsilon] \) by adding only 2 and 3-handles.
The middle level $X_{1/2}$ of the $h$-cobordism $W$ contains two sets of embedded 2-spheres, the belt spheres $S_-$ of the 2-handles and the attaching spheres $S_+$ of the 3-handles of $W$. These spheres can be assumed to meet transversely, to pair algebraically, and (after a suitable sequence of finger moves as pioneered by Casson [8] in the 1970s) to have simply-connected complement $X_{1/2} - (S_- \cup S_+)$. Let $U_{1/2}$ be a regular neighborhood of the union $S_- \cup S_+$ in $X_{1/2}$. Surger $U_{1/2}$ along $S_-$ and $S_+$ in turn to obtain AC submanifolds $U_0 \subset X_0$ and $U_1 \subset X_1$ of 1) with $\pi_1(X_i - U_i) = 1$. These submanifolds are joined by a cobordism $U \subset W$ built from a thickening $U_{1/2} \times [a, b]$ (where $[a, b]$ is a short interval centered at 1/2) by adding the 2-handles of $W$ (viewed upside down) as 3-handles attached along $S_- \times \{a\}$, and the 3-handles of $W$, attached along $S_+ \times \{b\}$.

Now extend the standard height function on $M_f$ to a Morse function $h : W \to [0, 1]$ compatible with the handle structure on $W$, with $X_i = h^{-1}(i)$ for $i = 0, 1/2$ and 1. As all the handles in $W$ lie in $U$, the complementary cobordism $W - U$ acquires a product structure from the gradient flow of $h$ (with respect to a suitable metric) which gives a diffeomorphism $G : X_0 - U_0 \to X_1 - U_1$ extending $f$ on the boundary $\partial X_0$. Since $U_0$ is of type 1) and $\pi_1(X_0 - U_0) = 1$, the handlebody techniques in [24] can be used to encase $U_0$ (as in Definition [1.3]) in a tightly embedded AC cork $C_0$ in $X_0$. In particular this follows from the Encasement Lemma [2.2] below, as noted in Remark 2.3.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Proof of the Relative Involutory Cork Theorem}
\end{figure}

Set $V_0 = C_0 - \text{int}(U_0)$, and let $V$ be the product cobordism swept out by $V_0$ under the gradient flow of $h$. Set $C = U \cup V$ and $C_i = C \cap X_i$ for $i = 1/2$ or 1. Then $C_1$ is an AC cork in $X_1$ encasing $U_1$ (since $C_0$ encases $U_0$, and $U_1$ is AC) whose boundary is identified with $\partial C_0$ via the restriction $g = G|_{\partial C_0}$, as shown in Figure 2. This makes $(C_0, C_1)$ into an AC multicork in $X_0$, such that the restriction $X_0 - C_0 \to X_1 - C_1$ of $G$ extends to a diffeomorphism $(X_0)_{C_0, C_1} \to X_1$ that restricts to $f$ on the boundary.

In fact the multicork $(C_0, C_1)$ is simple, that is, $\Sigma = C_1 \cup_g C_0$ is diffeomorphic to a 4-sphere. Indeed $\Sigma = \partial C$, and $C$ is easily seen to be a 5-ball (cf. Kirby [19]): It is built from $C_{1/2} \times [a, b]$ by adding 3-handles along the spheres $S = (S_- \times \{a\}) \cup (S_+ \times \{b\})$. As $C_{1/2}$ is AC of type 2), its thickening $C_{1/2} \times [a, b]$ can be built from the 5-ball by adding only 2-handles. But the core of each 2-handle lies in one of the spheres in $S$, and so the 2-handle is cancelled by the 3-handle attached to that sphere.

Since the multicork $(C_0, C_1)$ is simple and tightly embedded, its involutory pinwheel $(P, \pi)$ is also simple and has a tight embedding $P \subset X_0$ correlated with the multicork embedding $C_0 \subset X_0$, by Lemma 1.13 and Remark 1.12. Hence $f$ extends to a diffeomorphism $(X_0)_{P, \pi} \to X_1$ as desired. \end{proof}
Remark. Theorem 1.14 need not hold when $\partial X_0$ is not a homology sphere. For example, setting $X = X_0 = X_1 = S^2 \times D^2$, the Gluck twist $\tau$ on $\partial X$ does not extend across $X$, even after a cork twist, as $X \cup_{\text{id}} -X = S^2 \times S^2$ and $X \cup_{\tau} -X = S^2 \times S^2$ are not even homotopy equivalent.

1.15. Corollary. Every homotopy 4-sphere $\Sigma$ is diffeomorphic to a twisted double $W \cup_h -W$ for some AC cork $(W, h)$.

Proof. Decompose $\Sigma$ as the union of a homotopy 4-ball $X$ and a 4-ball $B$ meeting in their common 3-sphere boundary $S$. By Theorem 1.14, the identity map $\partial X \to \partial B$ extends to a diffeomorphism $f : X_{P, \tau} \to B$ for some simple involutory cork $(P, \tau)$ tightly embedded in $X$. Enlarging the corks $P \subset X$ and $Q = f(P) \subset B$ along arcs to the boundary, we may assume that $P$ and $Q$ (shown in green on the left side of Figure 3) meet $S$ in disjoint 3-balls. Set $P' = \text{cl}(X - P)$ and $Q' = \text{cl}(B - Q)$ (shown in purple and blue on the left side).

Then $P'$ and $Q'$ are diffeomorphic AC corks, each built from a 4-ball (the collar on a punctured copy of $S$) by adding homotopically cancelling 1 and 2-handles. Hence $\Sigma = X \cup B = W \cup Z$, where $W = P' \sharp Q$ and $Z = Q' \sharp P$ (shown in purple and blue on the right side of the figure) are AC corks. But $f$ induces a diffeomorphism $g : -W \to Z$, and so $\Sigma = W \cup_h -W$ where $h = g|_{\partial W}$. \hfill \square

Remark. This result extends to any homeomorphic pair of closed simply-connected 4-manifolds $X$ and $Y$, showing that $X \# -Y$ is diffeomorphic to a twisted double $W \cup_h -W$, where the honest double $W \cup -W$ is diffeomorphic to a connected sum of $S^2$-bundles over $S^2$.

# 2. Finite Cork Theorems

The main tool for proving the cork theorems in the introduction is the following “consolidation theorem”, showing that in suitable 4-manifolds $X$, any finite collection of simple involutory corks in $X$ is correlated with some higher order cork in $X$:

2.1. Consolidation Theorem. Let $(A_1, \tau_1), \ldots, (A_{n-1}, \tau_{n-1})$ be simple involutory corks embedded in a compact simply-connected 4-manifold $X$ whose boundary is either empty or a homology sphere. Then the $A_i$ can be isotoped so that they lie in a single simple cork $(C, h)$ of order $n$ in $X$ whose twists $X_{C,h,i}$ are diffeomorphic rel boundary to $X_{A_i, \tau_i}$. If $X$ is closed and the $A_i$ have simply-connected complements, then $C \subset X$ may be chosen with simply-connected complement.

The proof will require careful manipulation of a handle structure for $X$. We begin with generalities about such structures, and establish two preliminary results, the “encasement” and “finger” lemmas.

Consider the case when $\partial X$ is a homology sphere, or more generally nonempty and connected, and fix a handle structure on $X$ with a single 0-handle and no 4-handles. By convention, assume...
that no collars are added between the handles, except for a final collar after all of the handles are attached. The \( k \)-skeleton \( X^k \) of \( X \) consists of all the handles of index \( \leq k \). Inverting \( X \) produces the dual handlebody \( X^* \) with dual \( k \)-skeleton \( X^{k*} \), consisting of a collar on \( \partial X \) with all the dual handles of index \( \leq k \) (upside down handles of index \( \geq 4-k \)) attached.

The middle level of \( X \) is the bounded 3-manifold \( M = \partial X' \cap \partial X^{*1} \). Because of our collar convention, \( M \) can be seen as the complement in \( \partial X' \) of the 2-handle attaching regions, or equivalently, the complement in \( \partial X^{*1} \) of \( \partial X \) of the dual 2-handle attaching regions. By general position, it follows that the inclusion induced maps

\[
\pi_1(X^1) \leftarrow \pi_1(M) \rightarrow \pi_1(X^{*1})
\]

are surjective, where a base point in \( M \) is implicit. Note that \( \pi_1(X^1) \) is the free group \( F(x_1, \ldots, x_k) \), where the \( x_i \) are given by the 1-handles of \( X \), and \( \pi_1(X^{*1}) \) is the free product of \( \pi_1(\partial X) \) with the free group \( F(x_1^*, \ldots, x_m^*) \) on generators \( x_i^* \) given by the dual 1-handles of \( X \).

The meridians and framed longitudes of the attaching circles of the 2-handles \( H_i \) of \( X \) are unbased loops in \( M \), and thus correspond to conjugacy classes \( m_i \) and \( \ell_i \) in \( \pi_1(M) \). It is geometrically evident that the projected conjugacy classes \( p(m_i) \) and \( p^*(\ell_i) \) are trivial, while the classes

\[
r_i = p(\ell_i) \quad \text{and} \quad r_i^* = p^*(m_i)
\]

recording the attaching circles of the \( H_i \) and their duals \( H_i^* \) are typically nontrivial. These classes (or their representatives) are called the relators and dual relators of \( X \). Since \( \pi_1(X) \) is trivial, they normally generate \( \pi_1(X^1) \) and \( \pi_1(X^{*1}) \), respectively, and it follows that the map

\[
p \times p^* : \pi_1(M) \rightarrow \pi_1(X^1) \times \pi_1(X^{*1}) = F(x_1, \ldots, x_k) \times (\pi_1(\partial X) \ast F(x_1^*, \ldots, x_m^*))
\]

is surjective; cf. Stong’s Lemma [24, p. 500] in the closed case (the proof is the same here). This was a key observation from [24] used to produce corks with simply-connected complements in closed 4-manifolds [10], and is essential to our argument below.

The other input from [24] is an analysis of the effect on the relators and dual relators of sliding one 2-handle \( H_i \) over another \( H_j \) along a path \( \gamma \) joining their attaching circles [24, pp. 499-500]. Stong notes that in the dual picture, \( H_j^* \) slides over \( -H_i^* \) along the reverse of \( \gamma \); this can be seen for example by comparing longitudes and meridians of the attaching circles before and after the slide (as in Figure 4).

The path \( \gamma \) determines an element in \( \pi_1(M) \) (also denoted \( \gamma \) by abuse of notation) via a choice of base paths from its endpoints to the base point in \( M \). Set \( (c, c^*) = (p(\gamma), p^*(\gamma)) \). Note that the effect of the handleslide on the relators and dual relators depends only on \( c \) and \( c^* \), and so we simply say “slide \( H_i \) over \( H_j \) along \( (c, c^*) \)” to specify its relevant features. In particular, the slide changes \( r_i \) to \( r_i c r_j c \), and thus \( r_i^* \) to \( r_j^* c r_i^* c^* \) by Stong’s observation, fixing all the other relators and dual relators; here and below overbars denote inverses. This is demonstrated in Figure 4. Since \( p \times p^* \) is onto, 2-handles can be slid along arbitrary elements in \( \pi_1(X^1) \times \pi_1(X^{*1}) \).

The following two operations suffice for our purposes:

**Single slide**: Slide \( H_i \) over \( H_j \) along \( (c, 1) \). This changes \( r_i \) to \( r_i c r_j c \) and \( r_j^* \) to \( r_j^* c r_i^* \) while fixing all other relators and dual relators.

**Double slide**: Slide \( H_i \) over \( H_j \) along \( (1, 1) \), and then over \( -H_j \) along \( (1, 1) \). This changes \( r_i \) to \( r_i c r_j c \) and \( r_j^* \) while fixing all other relators, and all the dual relators.

Observe that single slides multiply a relator by a conjugate of another, while double slides multiply a relator by a commutator of another. It follows by a variation on Stong’s argument [24] (used to
control the fundamental group of the cork complement in the Involutary Cork Theorem \[\text{[10]}\] that AC submanifolds can often be encased in tightly embedded corks.

2.2. Encasement Lemma. Let $U$ be an AC manifold of type 1) embedded in the interior of a compact simply-connected 4-manifold $X$ that is either closed or bounded by a homology sphere. Then $U$ can be encased in an AC cork $C$ in $X$ (in the sense of Definition 1.3). If $X - U$ is simply-connected, then the embedding $C \subset X$ can be chosen to be tight (see Definition 1.6).

2.3. Remark. This lemma generalizes Stong’s Structure Theorem \[\text{[24, Theorem 2]}\] (which is the special case when $U$ is a 4-ball and $X$ is closed) and was a key ingredient in the proof above of the Relative Involutory Cork Theorem 1.14, where $U_0$ played the role of $U$.

Proof. The closed case follows from the bounded case by removing a 4-ball, so we assume that $\partial X$ is a homology sphere. By hypothesis there is an AC structure $U = [K_0 \cup K, L_0]$, where $K_0$ is a dotted unlink given by the 1-handles homotopically cancelled by the 2-handles attached along $L_0$ (below, matching subscripts will indicate homotopically cancelling handles), and $K$ is the unlink of dotted circles given by all the remaining 1-handles. The handles in $U$ will not be moved in subsequent constructions. See the left side of Figure 5 for a concrete example, where $K_0$ and $L_0$ are shown in black and $K$ in red.

To construct $C$, extend $[K_0 \cup K, L_0]$ to a handle structure on all of $X$ with no new 0-handles or 4-handles. This augments the dotted unlink $K$ to $K_1$ and the framed link $L_0$ to $L_0 \cup L$, so $X = [K_0 \cup K_1, L_0 \cup L]$ with some 3-handles added. Since $\pi_1(X) = 1$, we can arrange for some sublink $L_1$ of $L$ to homotopically cancel $K_1$, after first sliding the 2-handles (attached along) $L$.\[\text{Figure 5. Handlebodies } U = [K_0 \cup K, L_0] \text{ (left) and } C = [K_0 \cup K_1, L_0 \cup L_1] \text{ (right).}\]
over one another and over $L_0$, and possibly introducing cancelling 2/3-handle pairs along the way to avoid Andrews-Curtis issues. Now provisionally set $C = [K_0 \cup K_1, L_0 \cup L_1]$, consisting of all the 1-handles in $X$ and an equal number of 2-handles that homotopically cancel those 1-handles. Thus $C$ is an AC cork encasing $U$, proving the first statement in the lemma. See the right side of Figure [3] where the link $K_0 \cup L_0$ is shown in black, and $K_1 \cup L_1$ in red.

Now assume $\pi_1(X - U) = 1$. We will show how to modify $C$ to make it tight, leaving $U$ fixed. In fact the entire handlebody $[K_0 \cup K_1, L_0]$ will remain untouched except that $K_1$ may expand to include some new 1-handles, whereas $L$ (and thus $L_1$) may change.

To begin the argument, first note that the relative handle structure on $X - \text{int}(U)$, built from $\partial U$ by attaching 2-handles (along $L$) and 3-handles, gives a dual handle structure on the same manifold built from $\partial X$ by attaching dual 1-handles (upside down 3-handles) and dual 2-handles (along $L^*$, the meridians of $L$). This dual structure yields a presentation of the trivial group

$$1 = \pi_1(X - U) = (\pi_1(\partial X), x_1^*, \ldots, x_m^* | r_1^*, \ldots, r_n^*)$$

as in the preamble to Definition [1.4] where the $x_i^*$ correspond to the dual 1-handles and the $r_i^*$ are the relators given by the dual 2-handles $L^*$.

Now introduce $m$ cancelling dual 2/3-handle pairs, one for each dual 1-handle, and slide these new dual 2-handles over $L^*$ to homotopically cancel the dual 1-handles. This can be achieved by single slides as defined above since each dual generator $x_i^*$ is a product of conjugates of the $r_i^*$ (indeed the $r_i^*$ normally generate the free product $\pi_1(\partial X) * F(x_1^*, \ldots, x_m^*)$ since $X - U$ is simply-connected). Dually (i.e. rightside up) the effect on $X$ is to introduce $m$ cancelling 1/2-pairs with 2-handles, called the upper 2-handles, attached along a link $J$, and then to slide $L$ over $J$ so that the dual upper 2-handles homotopically cancel the dual 1-handles.

Next extend $K_1$ to include all the new 1-handles, and consider the subhandlebody

$$W = [K_0 \cup K_1, L_0 \cup L] \subset X.$$ 

We claim that $H_1(W)$ is trivial. Indeed $X - \text{int}(W)$ is an AC homology cobordism from $\partial X$ to $\partial W$ (see Definition [1.4] obtained from a collar on $\partial X$ by adding all the dual 1-handles and their homotopically cancelling dual 2-handles, the duals of the upper 2-handles. Since $\partial X$ is a homology sphere, so is $\partial W$ by Remark [1.5], and it follows from a Mayer-Vietoris argument that $H_1(W) = 0$.

Observe also that the fundamental group of the subhandlebody $[K_0 \cup K_1, L_0] \subset W$ is free on the generators $x_1, \ldots, x_k$ given by $K_1$, since $L_0$ homotopically cancels $K_0$. Since $H_1(W) = 0$, the 2-handles $L$ can be slid among themselves so that some subset homologically pairs with $K_1$. The handle structure after these handleslides yields a presentation of the trivial group

$$\pi_1(X) = (x_1, \ldots, x_k | r_1, \ldots, r_n, s_1, \ldots, s_m) = 1$$

where the $x_i$ are given by the 1-handles $K_1$, the $r_i$ and $s_i$ are the relators given by $L$ and $J$, respectively, and $r_i = x_i \prod_{j=1}^{\ell_i} [a_{ij}, b_{ij}]$ for $i = 1, \ldots, k$ and suitable $a_{ij}, b_{ij} \in F(x_1, \ldots, x_k)$.

Finally introduce $\ell_1 + \cdots + \ell_k$ cancelling 2/3-handle pairs indexed by pairs $(i, j)$ for $i = 1, \ldots, k$ and $j = 1, \ldots, \ell_i$, called the extra handles. Each $b_{ij}$ is a product of conjugates of the relators $r_1, \ldots, r_n, s_1, \ldots, s_m$ given by the 2-handles $L \cup J$, so we can single slide the extra 2-handles over the 2-handles $L \cup J$ until the $i^{th}$ one $L_{ij}$ has attaching word $b_{ij}$. Then for each $i = 1, \ldots, k$, double slide the $i^{th}$ 2-handle in $L$ over $L_{ij}$ along $(a_{ij}, 1)$ successively for $j = 1, \ldots, \ell_i$ until the corresponding relator is $r_i = x_i$. The result is a presentation

$$\pi_1(X) = (x_1, \ldots, x_k | x_1, \ldots, x_k, r_{k+1}^*, \ldots, r_n, s_1, \ldots, s_m, b_{11}, \ldots, b_{kk}) = 1$$
where the first \( k \) relators correspond to a sublink \( L_1 \) of \( L \) that homotopically cancels \( K_1 \). Note that the property that each dual extra 1-handle is homotopically cancelled by its dual extra 2-handle is maintained while performing these single and double slides.

Now set \( C = [K_0 \cup K_1, L_0 \cup L_1] \), which is the union of \( U \) with all of the 1-handles in \( X \) and their homotopically cancelling 2-handles. The embedding \( C \subset X \) is tight since every dual 1-handle in \( X - C \) is homotopically cancelled by a dual 2-handle. □

The Encasement Lemma can be used to prove the Consolidation Theorem, and thence the higher order cork Theorems 2.5 and 3.2, if we can find a way to reposition a collection of corks with simply-connected complements so that their union also has simply-connected complement. The precise result we need is the following, proved by a standard argument using finger moves:

**2.4. Finger Lemma.** Any finite collection of AC corks \( A_1, \ldots, A_n \) embedded in a 4-manifold \( X \) can be repositioned by isotopies so that their union \( \cup A_i \) is an AC manifold of type 1) encasing each of the \( A_i \), and so that \( X - \cup A_i \) is simply-connected provided each \( X - A_i \) is simply-connected.

Proof. Fix AC structures \( A_i = [K_i, L_i] \) for \( i = 1, \ldots, n \). Pull the \( A_i \) apart so that their 1-skeleta lie in disjoint 4-balls and their core 2-skeleta intersect transversely, then push the 0-handles of the \( A_i \) back together into a single 0-handle whose boundary contains the links \( K_i \cup L_i \), embedded in disjoint 3-balls. In this position, the union \( \cup A_i \) is obtained from the boundary sum \( \sum A_i \) by plumbing the 2-handles together according to the intersection of their cores, each intersection point having the effect of clasping the relevant components of \( \cup L_i \) and encasing the clasp in a dotted circle. This changes \( \cup L_i \) into a more complicated framed link \( L \), while the dotted circles about the clasps form an unlink \( J \) disjoint from \( K = \cup K_i \) and homotopically unlinked from \( L \). The result is an AC handle structure of type 1)

\[
\cup A_i = [J \cup K, L]
\]

encasing each \( A_i \), built in a canonical way from \( \sum A_i \). This is illustrated in Figure 6 for the case \( n = 2 \) where \( A_1 \) (in green) is the Mazur manifold and \( A_2 \) (in blue) is the Akbulut-Matveyev “positron” [5], embedded so that their 2-handles are plumbed three times geometrically, and once algebraically, and the components of \( J \) (in red) are the small dotted circles around the clasps.

To arrange for \( \cup A_i \) to have simply-connected complement in \( X \) when the \( A_i \) do, the belt circle of each 2-handle (say in \( A_j \)) should bound an immersed disk in \( X - \cup A_i \). Since \( \pi_1(X - A_j) \) is trivial, such a disk \( \Sigma \) can be found in \( X - A_j \), but \( \Sigma \) might (transversely) intersect the core disk \( D \) of some 2-handle \( H \) in another \( A_k \), as shown in Figure 7a; the (red) dot in the middle represents the 1-handle \( h \) homotopically cancelled by \( H \), while all the edges in the figure are cores of 2-handles. In fact, \( \Sigma \) can be assumed to have zero algebraic intersection number with \( D \), since any point in \( \Sigma \cap D \) can be exchanged for some number of cancelling pairs of intersection points between \( \Sigma \) and the 2-handle cores of \( A_k \), by a finger move of \( \Sigma \) along \( D \) across \( h \) (see Figure 7b).
If the geometric intersection number of $\Sigma$ with the core $E$ of some 2-handle (say in $A_k$) is still nonzero, then choose a cancelling pair $p, q$ of points in $\Sigma \cap E$ and associated Whitney circle $\delta$. This circle bounds an immersed disk $\Delta$ in the complement of $A_k$, since $\pi_1(X - A_k) = 1$ (see Figure 7b again). Pushing the core $F$ of each 2-handle that intersects $\Delta$ across $E$ by a finger move, we can arrange for $\Delta$ to lie in the complement of $\cup A_i$ (see Figure 7c). Note that here we are moving the corks rather than $\Sigma$, and introducing extra intersection points between the cores of their 2-handles; however, this does not alter the property that each $A_i$ is encased in $\cup A_i$.

Now, a trivialization of the normal bundle of $\Delta$ induces a framing of $\delta$ which can be made to match the Whitney framing by “boundary twisting” $\Delta$ around $\Sigma$, at the cost of introducing additional intersection points between $\Delta$ and $\Sigma$ (see [13, §1.3–1.4]). The result is an immersed Whitney disk $\Delta$ for $\delta$, giving rise to a regular homotopy of $\Sigma$ that eliminates the intersection points $p$ and $q$, without adding any additional intersections between $\Sigma$ and $\cup A_i$ (see Figure 7d). Repeating this process removes all the intersections of $\Sigma$ with $\cup A_i$. Since finger moves are supported near arcs, which miss disks in $X$ by general position, this process can be iterated to produce null-homotopies for all the belt circles of the 2-handles in $\cup A_i$, completing the proof.

**Proof of the Consolidation Theorem 2.1**

Isotop the $A_i$ (for $i = 1, \ldots, n - 1$) by the Finger Lemma 2.4 so that each is encased in their union $\cup A_i$, which (as in the proof of the lemma) is equipped with an $AC$ structure extending ones $[K_i, L_i]$ on the $A_i$. The Encasement Lemma 2.2 extends this structure to one $[K, L]$ on an $AC$ cork $C_0$ in $X$ containing $\cup A_i$, and thus also encasing each $A_i$. Let $C_i = (C_0)_{A_i, \Gamma_i}$, which is $AC$ by the last remark in Definition 1.3. By definition, the cork replacements $X_{C_0, C_i}$ are diffeomorphic to the cork twists $X_{A_i, \Gamma_i}$ rel boundary.

We argue that the multicork $C = (C_0, C_1, \ldots, C_{n-1})$ is simple. A handlebody structure for the double $C_0 \cup -C_0$ is obtained from $[K, L]$ by attaching “dual” 2-handles $L^*$ along the belt circles of the 2-handles of $C_0$ (i.e. the meridians of $L$) and “dual” 3-handles along the belt 2-spheres of the 1-handles of $C_0$, and then capping off with a 4-ball. Thus, as $A_i$ is encased in $C_0$, the complement $(C_0 - A_i) \cup -C_0$ of each $A_i$ can be built from $\partial A_i \times [0, \varepsilon]$ by attaching the 1-handles $K - K_i$ and 2-handles $(L - L_i) \cup L^*$, plus some 3-handles and a single 4-handle.

Since the 2-handles $L - L_i$ homotopically cancel the 1-handles $K - K_i$ in the double $C_0 \cup -C_0$, each 2-handle in $L - L_i$ can be slid over its dual 2-handle in $L^*$ in $(C_0 - A_i) \cup -C_0$ until it geometrically cancels its corresponding 1-handle in $K - K_i$. The 2-handles attached along $L^* - L_i^*$ are then free to be cancelled with 3-handles. The handlebody structure for $(C_0 - A_i) \cup -C_0$ now consists solely of the dual 2-handles attached along $L_i^*$ and the 3-handles dual to the 1-handles $K_i$. But, this is
exactly $-A_i$. Thus each $C_i \cup -C_0$ is diffeomorphic to the twisted double $A_i \cup \tau_i - A_i$ which is in turn diffeomorphic to the 4-sphere, since $A_i$ is simple.

Now since $C$ is simple, by Lemma 1.13 and Remark 1.12 the pinwheel $P$ of $C$ is simple and has an embedding in $X$ correlated with the embedding $C_0 \subset X$. Note that if $X$ is closed and each $A_i$ has simply-connected complement, the Finger Lemma 2.4 gives an isotopy of the $A_i$ so that their union $\cup A_i$ has simply-connected complement. Hence, the embedding $C_0 \subset X$ and therefore the embedding $P \subset X$ can be made tight (by the Encasement Lemma 2.2 and the Pinwheel Lemma 1.13). Since $X$ is closed, this implies that $X - P$ is simply-connected. Thus the pinwheel of $C$ is a simple order $n$ cork with the desired properties.

The Consolidation Theorem implies the following version of the Finite Cork Theorem, extending the one stated in the introduction that treats the closed case.

2.5. Finite Cork Theorem. Let $X_i$ ($i \in \mathbb{Z}_n$) be any finite list of compact simply-connected 4-manifolds all homeomorphic to a given one $X = X_0$, and assume that $X$ is either closed or bounded by a homology sphere. Then there is a simple cork $(C,h)$ of order $n$ embedded in $X$, with simply-connected complement in the closed case, whose cork twists $X_{C,h}$ are diffeomorphic to $X_i$ for each $i$. Furthermore, in the bounded case these diffeomorphisms can be chosen to extend any given boundary identifications $f_i: \partial X \to \partial X_i$.

Proof. The closed case follows from the bounded case by removing the interior of a 4-ball and noting that any orientation preserving diffeomorphism of the 3-sphere is isotopic to the identity. In the bounded case, the Relative Involutory Cork Theorem 1.14 provides simple involutory corks $(A_1, \tau_1), \ldots, (A_{n-1}, \tau_{n-1})$ with tight embeddings $A_i \subset X$ such that $f_i$ extends to a diffeomorphism $X_{A_i, \tau_i} \to X_i$ for each $i$. Thus, the theorem follows as a direct corollary of the Consolidation Theorem 2.1.

2.6. Remarks. a) When $X$ is closed, both the cork $(C,h)$ in the Finite Cork Theorem and its complement can be made Stein (or PC, borrowing the terminology from [3]) by extending Akbulut and Matveyev’s proof of Theorem 5 in [3] to the finite order case. In outline, first the complement of the multicork from the proof of the Consolidation Theorem 2.1 is made Stein, followed by each component of the multicork. This involves adding only homotopically cancelling 1,2-handle pairs to each manifold. So, the multicork remains simple, and its complement simply-connected. Stein structures for the cork $(C,h)$ and its complement are then obtained by arguing that the pinwheel of an embedded simple order $n$ multicork with Stein components and complement is Stein (as shown in Part (3) of the proof of Theorem 5 in [3] when $n = 2$) as is the complement of the correlated embedding.

b) The Finite Cork Theorem allows one to generalize many conclusions that can be drawn from the Involutory Cork Theorem. The relative version in particular helps to localize corks. For instance, suppose $E_i$ (for $0 < i < n$) is any list of 4-manifolds homeomorphic to some minimal elliptic surface $E = E(m)$, and obtained from it by log transforms along regular fibers. These fibers can be taken inside a Gompf nucleus $N$ of $E$, so we obtain a list of manifolds $N_i \subset E_i$, homeomorphic to $N$, with diffeomorphic complements $E_i - N_i$. Since $N$ is simply-connected with homology sphere boundary, the Finite Cork Theorem produces a cork $(C,h)$ of order $n$ in $N$ with $N_{C,h}$ is diffeomorphic to $N_i$ rel boundary for each $i$. Since all self-diffeomorphisms of $\partial N$ extend over $E - N$, by [16] Lemma 3.7, the cork twist $E_{C,h}$ is diffeomorphic to $E_i$ for each $i$. Thus the 4-manifolds $E_i$ are all obtained by twisting a single order $n$ cork inside a nucleus in $E$. 


3. Infinite Cork Theorems

Throughout this section, \( X \) will denote a fixed closed simply-connected 4-manifold, and \( X_i \) will be a countably infinite list of (not necessarily distinct) 4-manifolds homeomorphic to \( X = X_0 \), indexed by the integers. We first combine the Consolidation Theorem \[2.7\] with the Relative Involutory Cork Theorem \[1.14\] to show how to pull apart (rather than consolidate) infinite collections of corks in \( X \).

### 3.1. Separation Theorem

There are simple involutory corks \((A_i, \tau_i)\) disjointly embedded in \( X \) with simply-connected complements whose cork twists \( X_{A_i, \tau_i} \) are diffeomorphic to \( X_i \) for each \( i \).

**Proof.** For convenience, reindex the \( X_i \) by the nonnegative integers, maintaining \( X_0 = X \). The absolute version of the Involutory Cork Theorem gives a list of corks \((A_i, \tau_i)\) satisfying all the conditions in the lemma except the disjointness of the embeddings. To arrange for the corks to be disjoint, we modify this list inductively.

Suppose the first \( n - 1 \) corks \( A_1, \ldots, A_{n-1} \) are already disjoint. By the Consolidation Theorem, there exists a cork \((C, h)\) of order \( n + 1 \) embedded with simply-connected complement in \( X \) containing \( A_0 \cup \cdots \cup A_{n-1} \), and diffeomorphisms \( X_{C,h^i} \to X_i \) for \( i = 1, \ldots, n \). Although the proof of the Consolidation Theorem in general requires an initial isotopy of the \( A_i \), in this case we need only isotop \( A_n \) since the corks \( A_1, \ldots, A_{n-1} \) are already disjoint.

By the Relative Involutory Cork Theorem \[1.14\] there exists a simple involutory cork \((A'_n, \tau'_n)\) tightly embedded in \( Y = X - \text{int}(C) \) with a diffeomorphism \( Y_{A'_n, \tau'_n} \to Y \) that extends the map \( h^n : \partial C \to \partial C \) on the boundary. Extending by the identity across \( C \) gives a diffeomorphism \( X_{A'_n, h'_n} \to X_{C,h^n} \), which in turn gives a diffeomorphism \( X_{C,h^n} \to X_n \). Furthermore, since \( A'_n \subset Y \) is tight, \( \tau_1(\partial C) \) maps onto \( \tau_1(Y - A'_n) \) and so \( X - A'_n \) is simply-connected. Substituting \( A'_n \) for \( A_n \), the first \( n \) corks in the list are now disjoint. \( \square \)

**Remark.** This proof fails when \( X \) has nonempty boundary, as we cannot guarantee the existence of a cork \( C \) such that \( X - C \) is simply-connected, and so cannot apply the Relative Involutory Cork Theorem in the inductive step.

The Infinite Cork Theorem stated in the Introduction now follows from the Separation Theorem \[3.1\] and an infinite analogue of the pinwheel construction introduced in §1. The argument producing open corks also uses Freedman’s disk theorem \[12\], and in this case a few preliminary words are in order to explain the meaning of cork twisting\[1\].

So consider a contractible open 4-manifold \( C \) with a self-diffeomorphism \( h \) of \( C - \text{int}(K_h) \) for some compact, codimension 0 submanifold \( K_h \subset C \). Two such diffeomorphisms \( h \) and \( g \) are considered equivalent if they agree off of some compact set \( K \subset R \) containing both \( K_h \) and \( K_g \). As in \[17\] Definition 2.1, a **diffeomorphism of the ends** of \( C \) is an equivalence class of diffeomorphisms under this relation; by abuse of notation we also write \( h \) for the equivalence class of \( h \), and refer to the pair \((C, h)\) as an open cork. Now given an embedding \( C \subset X \), consider the open cork twist \( X_{C,h} = (C - \text{int}(K_h)) \cup_h K_h \), which is evidently well-defined up to diffeomorphism, independent of the choice of representative of the class \( h \).

### 3.2. Infinite Cork Theorem

Given any list \( X_i \) \((i \in \mathbb{Z})\) of closed simply-connected 4-manifolds homeomorphic to \( X = X_0 \), there is a simple noncompact cork \((C, h)\) in \( X \) whose cork twists \( X_{C,h^i} \) are diffeomorphic to \( X_i \) for each \( i \). The cork \( C \) may be chosen to be bounded or open, and in the open case, it may be taken to be homeomorphic to \( \mathbb{R}^4 \), i.e. an exotic \( \mathbb{R}^4 \).

\[1\] We are grateful to Bob Gompf for suggesting the possibility of proving this version of the theorem; his results in \[17\] are closely related.
The separation theorem provides simple involutory corks \((A_i, \tau_i)\) with disjoint embeddings \(A_i \subset X\) such that \(X_{A_i, \tau_i}\) is diffeomorphic to \(X_i\) for each \(i\). Since \(X = X_0\), we may take \(A_0 = B^4\). At any finite stage in the inductive proof of the separation theorem (done in the order \(A_0, A_1, A_{-1}, \ldots\) for instance), 1-handles can be added so that in the limit, we produce an embedding of the infinite boundary sum \(C_0 = \ldots \# A_{-1} \# A_0 \# A_1 \# \ldots\) with two ends into \(X\), extending the embeddings of the \(A_i\). Let \(C_i\) be the result of cork twisting \(C_0\) along a shrunken copy of \((A_i, \tau_i)\) in \(A_i\).

Since each \(A_i\) is simple, the multicork \((C_0, C_1, C_{-1}, \ldots)\) is also simple. Removing a boundary collar from \(C_0\) if necessary, there is enough room in the complement of \(C_0\) to trivially embed the \(C_i\) for \(i \neq 0\) in disjoint 4-balls \(B_i\), so the replacements \((B_i)_{C_i, C_j}\) are all 4-balls. The cork we seek will be built from these pieces together with one additional contractible 4-manifold \(C_\infty\), a 4-ball with one point removed from its boundary, by an infinite analog of the finite pinwheel construction in Definition 1.11. Throughout, \(\partial C_\infty\) will be identified with \(\mathbb{R}^3\), and \(h: \partial C_\infty \to \partial C_\infty\) will denote the translation \(h(x, y, z) = (x - 1, y, z)\).

**Remark.** Alternatively, \(\partial C_\infty\) can be viewed as hyperbolic 3-space \(\mathbb{H}^3\), and the map \(h\) as a parabolic transformation (with one fixed point at infinity)\(^\dagger\) We call this the “hyperbolic model” to distinguish it from the “euclidean model” where \(\partial C_\infty = \mathbb{R}^3\).

The cork \(C\) will be an infinite boundary connected sum, built from the disjoint union

\[
C_\infty \sqcup (\cdots \sqcup C_{-1} \sqcup C_0 \sqcup C_1 \cdots)
\]

by adding 1-handles joining \((i, 0, 0) \in \partial C_\infty\) to \(y_i \in \partial C_i\) for each integer \(i\), where the \(y_i\) are chosen to correspond under the natural boundary identification of the \(\partial C_i\). Note that \(C\) is a contractible 4-manifold with infinitely many ends, two for each \(C_i\) for \(i \neq \infty\) and one in \(C_\infty\). Moreover, \(h\) extends to an automorphism of \(\partial C\) fixing \(\partial C_\infty\) setwise and sending each \(\partial C_i\) to \(\partial C_{i-1}\). The pair \((C, h)\) is the desired noncompact bounded cork, shown schematically in Figure 8a. Figure 8b depicts the hyperbolic model, where the fixed point is the bottom point of the horocircle along which the \(C_i\) are attached.

![Diagram of corks and embeddings](image)

**Figure 8.** Two schematics of the infinite order cork \((C, h)\).

The embeddings of the \(C_i\) (all of which are trivial, except for \(C_0 \subset X\)) and \(C_\infty\) in \(X\) give a correlated embedding of \(C \subset X\) as in Definition 1.10 that is, each cork twist \(X_{C, h_{i+1}}\) is diffeomorphic to \(X_{A_i, \tau_i}\). This follows since each \((B_j)_{C_j, C_{i+j}}\) is a 4-ball, and so by Cerf’s theorem \([9]\), there

\[^\dagger\] See Roice Nelson’s website \([22]\) for a marvelous, dynamic illustration of parabolic transformations.
are diffeomorphisms \((B_j)_{C_j,C_{i+j}} \rightarrow B_j\) extending the diffeomorphism from the cork replacement \(X_{C_0,C_i} - \sqcup_j B_j\) to the cork twist \(X_{A_i,\tau^i} - \sqcup_j B_j\). This embedding is shown schematically in Figure 9 using the hyperbolic model.

\[\text{Figure 9. The embedding of the infinite order cork } (C, h) \text{ in } X.\]

To obtain an open cork with the desired properties one can just remove the boundary of \(C\), but a little more work is required to arrange for the cork to be an exotic \(\mathbb{R}^4\). For \((A_i, \tau_i)\) as in the preceding argument, the argument in [11, Section 3] shows that there are smooth open manifolds \(R_i \subset A_i\) homeomorphic to \(\mathbb{R}^4\) with involutions \(\sigma_i\) on their ends such that the diffeomorphism \(\tau_i\) on \(\partial A_i\) extends across \((A_i)_{R_i,\sigma_i}\); for the reader’s convenience, we outline the construction of \(R_i\) described in [11], borrowing all notation from our proof of the Relative Involutory Cork Theorem 1.14:

First, take a relative h-cobordism built with only 2 and 3-handles from \(A_i\) to itself with mapping cylinder \(M_{\tau_i}\) on its lateral boundary. Recall that the middle level of this h-cobordism (between the 2 and 3-handles) contains a neighborhood \(U_{1/2}\) of the union of the ascending spheres of the 2-handles and descending spheres of the 3-handles of the h-cobordism, which may be assumed to have simply-connected complement. Hence, Freedman’s disk theorem [12] (see also [13]) may be applied to obtain a collection of disjointly embedded topological Whitney disks in the complement of \(U_{1/2}\) for any pairing of the cancelling intersection points between the spheres.

Let \(R_{1/2}\) be the union of the interior of \(U_{1/2}\) with open (and hence smooth) regular neighborhoods of these disks. Choosing diffeomorphic open neighborhoods around each disk and pairing the intersection points appropriately, as in [11], there is an involution of \(R_{1/2}\) that swaps the ascending and descending spheres. Thus, surgering either the ascending or descending spheres yields the same contractible manifold \(R_i\). Furthermore, there is a diffeomorphism from \((A_i)_{R_i,\sigma_i}\) to \(A_i\) that extends \(\tau_i\) on \(\partial A_i\), where \(\sigma_i\) is the restriction of the involution of \(R_{1/2}\) to its end (also the end of \(R_i\)).

Hence, each embedding \(A_i \subset X\) gives an embedding \(R_i \subset X\) such that \(X_{R_i,\sigma_i}\) is diffeomorphic to the cork twist \(X_{A_i,\tau^i}\). Using the open manifolds \(R_i\) instead of the corks \(A_i\) at each step of the construction of the bounded cork described above yields an open cork, also denoted \(C\) by abuse of notation, with a diffeomorphism \(h\) on its end. First, “end sum” the \(R_i\) to form \(C\) (this operation is the analog of boundary-summing for open manifolds, first defined in [14] and [15]). Then define the map \(h\) to rotate the end of \(C\), mimicking the map \(h\) in the previous proof. This is the desired open cork \((C, h)\), with \(C\) an exotic \(\mathbb{R}^4\). \(\square\)
We conclude by showing how the proof of the previous theorem can be expanded to establish the existence of a universal noncompact cork:

3.3. **Universal Cork Theorem.** For any doubly infinite collection $X_{ij}$ of closed simply-connected 4-manifolds indexed by pairs of integers $i$ and $j$, with all $X_{ij}$ homeomorphic to $X_j := X_{0j}$ for any given $j$, there is a simple noncompact cork $(C, h)$ with embeddings $C \subset X_j$ for every $j$ so that the cork twists $(X_j)_C, h_i$ are diffeomorphic to $X_{ij}$ for each $i$ and $j$. The cork $C$ may be chosen to be bounded or open, and in the open case, $C$ may be taken to be homeomorphic to $\mathbb{R}^4$.

**Proof.** The proof is transparent from Figure 10, but for the skeptical reader, here are the details: For each $j \in \mathbb{Z}$, let $C_j = (C_{0j}, C_{1j}, C_{-1j}, \ldots)$ be a simple multicork embedded in $X_j$ such that $X_{ij} = (X_j)_C_{0j}, C_{ij}$, as in the proof of Theorem 3.2, and $y_{ij}$ be points in $\partial C_{ij}$ that correspond for each $j$ under the boundary identifications. Thus $C_{0j} = z_i A_{ij}$ for corks $(A_{ij}, \tau_{ij})$ (with $A_{0j} = B^4$) and $C_{ij} = C_{0j} A_{ij}, \tau_{ij}$.

As before, $C_\infty$ will denote a 4-ball minus a boundary point, equipped with the unit translation $h$ in the $x$-direction on $\partial C_\infty = \mathbb{R}^3$. Now, add a doubly infinite family of 1-handles to the disjoint union $C_\infty \sqcup (\sqcup_{i,j} C_{ij})$ joining the lattice points $(i, 0, j) \in \partial C_\infty$ to $y_{ij} \in \partial C_{ij}$, as shown in Figure 8b. This gives the desired (bounded) noncompact universal cork $C$. Note that $h$ extends to an automorphism of $\partial C$ fixing $\partial C_\infty$ setwise and sending each $\partial C_{ij}$ to $\partial C_{(i-1)j}$. To find an embedding $C \subset X_j$ so that the cork twists $(X_j)_C, h_i$ are diffeomorphic to $X_{ij}$, note that the embedding $C_{0j} \subset X_j$ given by the proof of Theorem 3.2 and trivial embeddings of all other $C_{ij}$ can be extended to the desired embedding $C \subset X_j$. As in that proof, a modified construction using exotic 4-spaces $R_{ij}$ initially in place of the $A_{ij}$, yields an open universal cork homeomorphic to $\mathbb{R}^4$. \qed

![Diagram of a universal cork](https://via.placeholder.com/150)

**Figure 10.** Two schematics of a universal cork, evoking a spectacular swarm of birds.

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