FREE ENERGIES AND FLUCTUATIONS FOR THE UNITARY BROWNIAN MOTION

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ABSTRACT. We show that the Laplace transforms of traces of words in independent unitary Brownian motions converge towards an analytic function on a non trivial disc. This results allow to study asymptotics of Wilson loops under the unitary Yang-Mills measure on the plane. The limiting objects obtained are shown to be characterized by equations analog to Schwinger-Dyson’s ones, named here after Makeenko and Migdal.

1. INTRODUCTION

The following paper aims at studying traces of non-commutative polynomials in independent Brownian motions on the group of unitary matrices $U(N)$, as the size $N$ goes to infinity. In [5, 35, 26, 27], it has been shown that for Brownian motions invariant by conjugation, with a proper time-scale, these traces, properly normalized, converge towards a deterministic limit given by the evaluation of the free Brownian motion. We want here to study the Laplace transform of these random variables with normalization analog to one of the mod-$\phi$ convergence ([17]). As a corollary, we obtain the fluctuations around their limit. In [28], the fluctuation of traces in polynomials of one marginal were given, the latter point of the present work gives an extension of their result. Simultaneously to the writing of the present paper, G. Cébron and T. Kemp obtained in [9] a similar result of gaussian fluctuation for diffusions on $GL_N(\mathbb{C})$, among which the $U(N)$ Brownian motion is a special case. A second motivation of our paper is to study the planar Yang-Mills measure for large unitary groups. We are able here to show the convergence to all orders of the Wilson loops and prove that the limiting objects are characterized by analog of Schwinger-Dyson equations. In particular, we prove the existence of a gaussian field indexed by rectifiable loops describing the fluctuation of the convergence towards the master field proved in [27].

Free energies of matricial models: In many random matricial models, asymptotics of $\mathbb{E}[e^{N\text{Tr}(V)}]$, where $V$ is a fixed non-commutative polynomial in a sequence of random matrices of size $N$, have been extensively studied and have

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several applications ranging from theoretical physics, through enumerative combinatorics, free probability and representation theory. A case of study is the Harish-Chandra-Itzykson-Zuber integral ([21, 36])

\[ H(A, B) = \mathbb{E}[e^{NTr(AUBU^*)}], \]

where \( A \) and \( B \) are two deterministic hermitian matrices and \( U \) is a random unitary matrix, distributed according to the Haar measure. When the non-commutative polynomial plays the role of the potential of a Gibbs measure, the normalized logarithm of Laplace tranforms is called the free energy and have been studied in several places, for example in [20, 11, 7]. In the pioneering work [8], formal expansions have been proposed for several physical models. In [10, 12], technics have been developed to study formal expansions for model of random matrices with properties of invariance by conjugation. We shall give here a converging expansion for the following model.

Let \((U_1,t_1,\ldots,U_q,t_q)_{t \in \mathbb{R}^q}^q\) be \( q \) independant Brownian motions invariant by adjunction in \( U(N) \) (see section 2 for a definition) and denotes by Tr the usual non-normalized trace of matrices.

**Theorem 1.1.** For \( t \in \mathbb{R}^q \) and any non-commutative polynomial \( V \) in \( 2q \) variables, there exists \( r_V > 0 \) and analytic functions \( \varphi_{t,V},(\psi_{t,V,N})_{N \geq 1} \) and \( \psi_{t,V} \) on \( D_{r_V} = \{ z \in \mathbb{C} : |z| < r_V \} \), such that

\[ e^{\psi_{t,V,N}(z)} = \mathbb{E}[e^{zNTr(V(U_i,t_i,U_i^*,t_i),i=1..q)})-N^2\varphi_{t,V}(z)] \longrightarrow e^{\psi_{t,V}(z)}, \]

as \( N \to \infty \), where the convergence is uniform on compact subset of \( D_{r_V} \).

For any non-commutative polynomial \( V \) in \( 2q \) variables, whose evaluation is hermitian one unitary matrices, we shall define for any integer \( N \geq 1 \), a probability measure \( \mu_{N,V} \) on \( U(N)^q \) absolutely continuous with respect to the law of \((U_{i,t_i})_{1 \leq i \leq q}\), with density proportional to \( e^{zNTr(V(U_{i,t_i},U_{i,t_i}^*,t_i),i=1..q)}) \). Then, for any \( N \geq 1 \), \((U_{N,1}^V,\ldots,U_{N,q}^V)\) denotes a random variable with law \( \mu_{N,V} \).

**Theorem 1.2.** If \( V,W \in \mathbb{C}\langle X_i,Y_i \rangle_{i=1..q} \) are non-commutative polynomials with small enough coefficients and \( V^* = V \), then, under the probability measure \( \mu_{N,V} \), the random variable \( \frac{1}{N}\text{Tr}(W(U_i^V,U_i^{V*},i=1..q)) \) converges in probability towards a constant \( \Phi_{t,V}(W) \).

**Yang-Mills measure on the plane:** We shall see that this result can be partly extended to the framework of Yang-Mills measure that has been developed in [16, 32, 1, 29, 27]. Therein, we give a recursive way to compute coefficients of \( \varphi_{t,V}(z) \), proving analogs of Schwinger-Dyson equations, called here Makeenko-Migdal equations. The latter equations for the first coefficient in \( z \) appeared in [30] and were first proved rigorously in [27]. The Yang-Mills measure encompasses the different models for all \( q \in \mathbb{N}^* \) and \( t \in \mathbb{R}^q \), into one random object, for which the recursive equations has a simple interpretation. We shall use the approach of
by considering for any $N \geq 1$, a process $(H_l)_l$ indexed by the set $L(\mathbb{R}^2)$ of rectifiable loops in the plane, valued in $U(N)$, whose law will be denoted by $YM_N$.

**Planar master field:** The works \([1, 27]\) proved that under $YM_N$, the random field $((\frac{1}{N} \text{Tr}(H_l))_{l \in L(\mathbb{R}^2)}$ converges in probability towards a deterministic field $(\Phi(l))_{l \in L(\mathbb{R}^2)}$. The statement of this result first appeared in the physics literature, in the study of QCD, with the works \([24, 25, 30]\), and in the mathematical paper \([33]\), as a conjecture. The limiting field was named therein *master field*, following the terminology of \([34]\). This object is the first coefficient of the extension of the functions $\varphi_{l,0}$ in Theorem 1.1. The asymptotic of 2D-Yang-Mills measure on other compact surfaces has also been investigated in the physics literature \([19]\). It won’t be discussed in this text but could lead to future works.

**Fluctuations:** The study of fluctuation of traces of random elements of a compact group of large dimension started with \([15]\), where is was investigated, thanks to representation theory tools, for the Haar measure on the classical compact Lie groups. The Theorem 1.1 allows us in particular to characterize the fluctuations in the convergence of the non-commutative distribution of a $U(N)$-Brownian motion towards the free unitary Brownian motion distribution. We further prove that under $YM_N$ the random field $(\text{Tr}(H_l) - \mathbb{E}[H_l])_{l \in L(\mathbb{R}^2)}$ converges in law towards a gaussian field $(\phi_l)_{l \in L(\mathbb{R}^2)}$, characterized by the Makeenko-Migdal equations. Besides, we observe that when the loops are dilated by a factor $\lambda$, the above fields have the same gaussian behavior as $\lambda \to 0$. Our result extends the work of \([28]\), which study the gaussian fluctuations in the convergence of the empirical measure of a $U(N)$-Brownian motion marginal. The gaussian process obtained therein can be shown to be a deformation the one obtained in \([15]\). The fluctuation results presented in this text are extracted from the PhD thesis of the author, where the case of the orthogonal and symplectic groups have also been addressed. Note also that in \([14, 3]\), fluctuations with another scaling are studied: the one of finite block of a random matrix. For simplicity, we shall restrict here to the study of traces of words in the unitary case.

**Organisation of the paper:** The next section is devoted to the description of the convention we use for the standard Brownian motion on $U(N)$ and the choice of scaling we made. In sections 3 and 4, are obtained the main expressions and estimates needed to get our result. In sections 5 and 6, we give their applications to study respectively the unitary Brownian motion and the Yang-Mills measure. In the last section, we show that the limited object obtained in the paper can be characterized by the recursive equations of Makeenko and Migdal.
2. **Unitary Brownian motion and its large $N$ limit**

2.1. **Definition and time scale of unitary Brownian motion.** For any integer $N$, we shall write $U(N)$ for the group of unitary matrices of $M_N(\mathbb{C})$ and $u(N)$ for its Lie algebra, that is, the set of skew-hermitian matrices. We define a scalar product $\langle \cdot, \cdot \rangle$ on $u(N)$ by setting for any $X, Y \in u(N)$,

$$\langle X, Y \rangle = -N \text{Tr}(XY).$$

Let us write $(K_t)_{t \geq 0}$ the Brownian motion on the Euclidean space $(u(N), \langle \cdot, \cdot \rangle)$ and recall that it is a Gaussian process such that for any $X, Y \in u(N), t, s \geq 0$,

$$\mathbb{E}[\langle X, K_t \rangle \langle Y, K_s \rangle] = \langle X, Y \rangle \min(t, s).$$
Let us define \((U_t)_{t \geq 0}\) as the \(M_N(\mathbb{C})\)-valued solution of the following stochastic differential equation:

\[
(*) \quad dU_t = U_t dK_t - \frac{1}{2} U_t dt \\
U_0 = \text{Id}.
\]

**Lemma 2.1.** i) Almost surely, for all \(t \geq 0\), \(U_t \in U(N)\).

ii) For all \(T \geq 0\), \((U_T^* U_{T+t})_{t \geq 0}\) is independent of the sigma field \(\sigma(U_s, s \leq T)\) and has the same law as \((U_t)_{t \geq 0}\).

iii) For any \(t \geq 0\) and every fixed \(U \in U(N)\), \(UU^*_t U^{-1}\) has the same law as \(U_t\).

**Proof.** Let us prove the first point, the two others are left to the Reader. The processes \((i\sqrt{N}(K_t)_{p,p})_{t \geq 0}\) for \(1 \leq p \leq N\) and \((\sqrt{N}(K_t)_{i,j})_{t \geq 0}\) for \(1 \leq i < j \leq N\) are independent of \(1\)-independent processes, the \(N\) first have the same law as a standard real Brownian motions, whereas the others are distributed as standard complex Brownian motions, so that \(\mathbb{E}[|(K_t)_{1,2}|^2] = 1\). Let us denote by \(\langle \cdot \rangle\) the symbol of quadratic variations, so that

\[
\langle dK_{t}, dK_{t} \rangle = \sum_{1 \leq i < j \leq N} \langle d(K_{t})_{i,j} d(K_{t})_{p,j} \rangle E_{i,j} = -dt \text{Id}.
\]

Itô’s formula then yields

\[
d(U_t U_t^*) = U_t (dK_t + dK_t^*) U_t^* + U_t (\langle dK_t, dK_t^* \rangle - dt \text{Id}) U_t^* = 0.
\]

\[\square\]

We call this process the \(U(N)\)-Brownian motion\(^1\) (see [13, 27] for a similar definition on other classical compact groups). For \(N = 1\), it has the same law as \((e^{t B_t})_{t \geq 0}\), where \((B_t)_{t \geq 0}\) is the standard real Brownian motion. Let us make remarks on the scaling. Let us recall that the scalar product \(\langle \cdot, \cdot \rangle\) on \(u(N)\) induces a Riemannian metric \(d\) on \(U(N)\). On the one hand, this choice of metric yields that the diameter of \(U(N)\) is \(d(\text{Id}, -\text{Id}) = \int_0^1 \|\gamma_t\| dt\), where \(\gamma : t \in [0, 1] \mapsto \exp(t i \pi \text{Id}_N)\), that is, \(\|i \pi \text{Id}\| = N \pi\). On the other hand, the law of large numbers implies that \(\dim(u(N))^{-1} \|K_t\|^2 = N^{-2} \|K_t\|^2\) converges, as \(N \to \infty\), towards \(t\). Heuristically, we may infer that, as \(N \to \infty\), for any \(t > 0\), \(d(\text{Id}, U_t)\) behaves like \(\|K_t\|\) and \(\frac{d(U_t, \text{Id})}{d(\text{Id}, -\text{Id})} \to C_t \in (0, \infty)\). With this scaling, the Brownian motion “has the time to visit”\(^2\) \(U(N)\). Besides, the stochastic differential equation \((*)\) does not depend

\(^1\)It can be shown that it is a diffusion on the Riemannian manifold \(U(N)\) endowed by the left-invariant metric associated to \(\langle \cdot, \cdot \rangle\) and that its generator is the Laplace-Beltrami operator (see [23] or [28], Proposition 2.1, for an elementary proof).

\(^2\)Note that a good scaling to study the convergence of the distance in total variation \(d_{TV}\) between the law of Brownian motion and the Haar measure, is faster than ours. Let \(U\) be a Haar distributed random variable on \(U(N)\). It has been shown in [31] that the function \(t \mapsto d_{TV}(U_{10^{-t}}, U)\) admits a cut-off around the value \(t = 2\).
on $N$ and such an equation makes sense in the context of free stochastic differential equations (see [5]). Let us add a last comment on the time-scale. With the above choice, the U(1)-Brownian motion appears with the same scaling in all U(N)-Brownian motions.

**Lemma 2.2.** For any $N \in \mathbb{N}^*$, let $(U_{t,N})_{t \geq 0}$ be a U(N)-Brownian motion. Then, the process $(\det(U_{t,N}))_{t \geq 0}$ has the same distribution as $(U_{t,1})_{t \geq 0}$.

**Proof.** Observe that for any $N \in \mathbb{N}^*$, $(i \text{Tr}(K_t))_{t \geq 0}$ has the same law as a standard Brownian motion. If $D_2(\det): M_N(\mathbb{C}) \rightarrow \mathbb{C}$ denotes the second derivative of the determinant at a point $M \in M_N(\mathbb{C})$, Itô’s formula yields that

$$d(\det(U_t)) = \det(U_t)d\text{Tr}(K_t) - N/2 \det(U_t)dt + \frac{\det(U_t)}{2} \langle\langle D_2(\det)Id(dK_t, dK_t) \rangle\rangle.$$

What is more,

$$\langle\langle D_2(\det)Id(dK_t, dK_t) \rangle\rangle = \sum_{1 \leq i < j \leq N} \left( \langle\langle (dK_t)_{i,i}, (dK_t)_{j,j} \rangle\rangle - \langle\langle (dK_t)_{j,i}, (dK_t)_{i,j} \rangle\rangle \right) = -\frac{N(N-1)}{2N} dt.$$

Hence,

$$d(\det(U_t)) = \det(U_t)d(\text{Tr}(K_t)) - \frac{1}{2} \det(U_t)dt$$

and $\det(U_0) = 1$. □

2.2. **Free unitary Brownian motion.** Let us recall the first result obtained about the behavior of unitary Brownian motion in large dimension. We shall denote by $(\mu_{t,N})_{t \geq 0}$ the family of random measures given by the empirical measure of eigenvalues of $U_t$: if $\lambda_1, \ldots, \lambda_N \in \mathbb{U}$ are the eigenvalues of $U_t$, $\mu_{t,N} = \frac{1}{N} (\delta_{\lambda_1} + \cdots + \delta_{\lambda_N})$. For any polynomial function $P \in \mathbb{C}[X]$, note that $\text{tr}(P(U_t)) = \int_{\mathbb{U}} P(z) \mu_t(dz)$. The following theorem has first been proved in [5] using harmonic analysis on the unitary group.

**Theorem 2.3 ([5, 35, 26]).** The sequence or random measures $(\mu_{t,N})_{N \geq 0}$ converges weakly in probability\(^3\), towards a deterministic measure $\mu_t$ on $\mathbb{U}$, whose moments are given as follows:

$$\mu_{t,n} = \int_{\mathbb{U}} z^n \mu_t(dz) = e^{-nt} \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} n^{k-1} \binom{n}{k+1}.$$

Using the property of independance and stationarity of multiplicative increments a U(N)-Brownian motion, together with the invariance of its law by adjunction and free probability arguments, the following Theorem was then deduced.

\(^3\)We mean here that for any continuous function $f$, the sequence of random variables $(\int f d\mu_N)_{N \geq 1}$ converges in probability to the constant $\int f d\mu_t$. 


Theorem 2.4 ([5]). For any \( t_1, t_2, \ldots , t_q \geq 0 \) and \( V \) any non-commutative polynomial of \( 2q \)-variables, the random variables \( \frac{1}{N} \text{Tr}(V(U_1, U_1^*, \ldots , U_q, U_q^*)) \) converge in probability towards a constant.

The limiting object is called the non-commutative distribution of the free unitary brownian motion and can be characterized by the family of measures \( (\mu_t)_{t \geq 0} \) together with the asymptotic freeness of the increments. This last theorem was proved in another way in [35, 26, 27] showing directly the convergence for any non-commutative polynomial. Let us recall how the argument of [26, 27] goes to show 2.3. Let us denote by \( \mathfrak{S}_n \) the group of permutations of \( \{1, \ldots , n\} \). For any permutation \( \sigma \in \mathfrak{S}_n \) composed of \( \# \sigma \) cycles, we define a function \( f_\sigma \) on \( U(N) \) by

\[
f_\sigma(U) = N^{-\# \sigma} \text{Tr}(\sigma U^{\otimes n})
\]

and a function on \( \mathfrak{S}_n \), by setting for any \( t > 0, \phi_t^N(\sigma) = \mathbb{E}[f_\sigma(U_{N,t})]. \)

Then, the latter family of functions on the symmetric group is shown to satisfy the following differential system (see [26] or Lemma 2.6).

Lemma 2.5 ([26]). For any permutation \( \sigma \in \mathfrak{S}_n \),

\[
\frac{d}{dt} \phi_t^N(\sigma) = -\frac{n}{2} \phi_t^N(\sigma) - \sum_{1 \leq i < j \leq n} N^{\# \sigma(i,j) - \# \sigma - 1} \phi_t^N(\sigma(i,j)),
\]

\[
\phi_0^N(\sigma) = 1.
\]

The unique solution of this system of ordinary differential equations is a power series in \( \frac{1}{N} \) and converges, as \( N \to \infty \), to a function \( \phi_t \). It can further be shown to satisfy for any \( \sigma \in \mathfrak{S}_n \) with \( a_k \) cycles of length \( k \),

\[
\phi_t(\sigma) = \prod_{k=1}^n \phi_t((1 \cdots k))^{a_k}.
\]

Setting for all \( t \geq 0, n \geq 1, \mu_{t,n} = \phi_t((1 \cdots n)) \), the limit in \( N \) of the former equations takes the following form:

\[
(1) \quad \frac{d}{dt} \mu_{t,n} = -\frac{n}{2} \mu_{t,n} - \frac{n}{2} \sum_{k=1}^{n-1} \mu_{t,k} \mu_{t,n-k},
\]

with initial condition \( \mu_{0,n} = 1 \). This system of equations is then shown to have as unique solution given by the expression of Theorem [5]. It follows that for \( n \in \mathbb{N}, t \geq 0, \)

\[
\mathbb{E}[\int \omega^n \mu_t^N(d\omega)] \to \mu_{t,n}.
\]

To conclude and obtain a convergence in probability, one ultimately needs to estimate the covariances of the complex variables \( \left( \frac{1}{N} \text{Tr}(U_t^n) \right)_{n \in \mathbb{N}, t \geq 0} \) with their
complex conjugate. This latter point together with the Lemma 2.5 can be proved using the following lemma, that allows to study any polynomial in the entries and their conjugate of a unitary Brownian motion.

For any integer \( n \in \mathbb{N}^* \), let us recall the left action of \( S_n \) on \( \mathbb{C}^n \otimes \mathbb{C}^n \), such that for any permutation \( \sigma \in S_n \) and any elementary tensor \( v_1 \otimes v_2 \otimes \ldots \otimes v_n \in (\mathbb{C}^n)^\otimes n \),

\[
\sigma. v_1 \otimes v_2 \otimes \ldots \otimes v_n = v_{\sigma^{-1}(1)} \otimes \ldots v_{\sigma^{-1}(n)}.
\]

The endomorphism of \( (\mathbb{C}^N)^\otimes n \) associated to a permutation \( \sigma \) will be abusively denoted below by the same symbol. For any pair of distinct integers \( i,j \in \{1,\ldots,n\} \), we denote by \( \langle i,j \rangle \) the endomorphism of \( (\mathbb{C}^N)^\otimes n \) which acts like the endomorphism \( \sum_{1 \leq r,s \leq N} E_{r,s} \otimes E_{r,s} \) on the \( i \)th and \( j \)th tensors and trivially on the others.

**Lemma 2.6** ([13, 27]). Let \( U_t \) be a Brownian motion on \( U(N) \). For any positive integers \( a,b \), which add up to \( n \), the following differential equation holds:

\[
\frac{d}{dt} \mathbb{E}[U_t^a \otimes U_t^b] = - \mathbb{E}[U_t^a \otimes U_t^b] \left( \frac{n}{2} + \frac{1}{N} \sum_{i<j \text{ or } a<i<j} \langle i,j \rangle - \frac{1}{N} \sum_{i \leq a<j} \langle i,j \rangle \right).
\]

For any permutation \( \sigma \in S_n \), \( \varphi_t(\sigma) = N^{-\#\sigma} \text{Tr}(\sigma \mathbb{E}[U_t^\otimes n]) \) and the Lemma 2.5 reduces to this more general one.

**Proof.** We shall use the stochastic differential equation (*) and apply Itô formula. First, writing the \( u(N) \)-valued Brownian motion \( (K_t)_t \geq 0 \) as a sum of independent real standard Brownian motions yields that

\[
\langle dK_t \otimes dK_t \rangle = \frac{1}{N} \sum_{1 \leq r,s \leq N} E_{r,s} \otimes E_{s,r} dt = \frac{1}{N}(1 \ 2) dt \in \text{End}((\mathbb{C}^N)^\otimes 2)
\]

and

\[
\langle dK_t \otimes dK_t \rangle = \frac{1}{N} \sum_{1 \leq r,s \leq N} E_{r,s} \otimes E_{r,s} = \frac{1}{N}(1 \ 2) dt.
\]

We can now use the Itô formula to get that the variational-bounded part of the variation of the semi-martingales \( U_t^\otimes 2 \) and \( U_t \otimes U_t \) are respectively \( U_t \otimes U_t \ (-dt + \langle dK_t \otimes dK_t \rangle) = -U_t^\otimes 2 (1 + \frac{1}{N}(1 \ 2)) dt \) and \( U_t \otimes U_t (-dt + \langle dK_t \otimes dK_t \rangle) = -U_t \otimes U_t (1 - \frac{1}{N}(1 \ 2)) dt \). The same analysis yields that the variational-bounded part of the variation of the semi-martingale \( U_t^a \otimes U_t^b \) is

\[
U_t^a \otimes U_t^b \left( -\frac{n}{2} - \frac{1}{N} \sum_{i<j \text{ or } a<i<j} \langle i,j \rangle + \frac{1}{N} \sum_{i \leq a<j} \langle i,j \rangle \right) dt.
\]

\[\square\]
3. Free energy, words in unitary Brownian motions

The convergence of the above paragraph can be considered as a law of large numbers for the traces of words in unitary Brownian motion. We aim at studying their Laplace transform and at deriving from this study a central limit theorem. Note that if $(K_t)_{t \geq 0}$ is a $u(N)$ Brownian motion as defined above, for any $t \geq 0$, $N^{-2} \log \mathbb{E}[e^{N \text{Tr}(K_t)}] = N^{-2} \log \mathbb{E}[e^{K_t \text{id}_N}] = t$ and $N^{-2} \log \mathbb{E}[e^{N \text{Tr}(K_t^2)}] = N^{-2} \log \mathbb{E}[e^{-\|K_t\|^2}] = \log \mathbb{E}[e^{-B_t^2}]$, where $B_t$ is the marginal of a standard real Brownian motion. These two naive examples suggest that the scaling chosen in Theorem 1.1 is the good one. We shall prove it in the following by estimating cumulants.

3.1. Laplace transforms and cumulants of traces.

3.1.1. Scaling of cumulants. For any bounded random variable $X$, the function $\log \mathbb{E}[e^{zX}]$ is analytic on a neighborhood of 0. We denote its analytic expansion

$$
\log \mathbb{E}[e^{zX}] = \sum_{n \geq 1} C_n(X) \frac{z^n}{n!},
$$

the coefficients $(C_n(X))_{n \geq 1}$ are called the cumulants of the random variable $X$. We are interested here in the behavior in $N$ of $N^{-2} \log \mathbb{E}[e^{N \text{Tr}(A_N)}]$, hence of the rescaled cumulant

$$
N^{n-2} C_n(\text{Tr}(A_N)),
$$

where the $A_N$ are bounded random matrices of $M_N(\mathbb{C})$, uniformly bounded in norm.

3.1.2. Cumulants. These coefficients are related to the moments of $X$ via a M"obius inversion formula. For any $n \in \mathbb{N}^*$, the set $\mathcal{P}_n$ of partitions of $\{1, \ldots, n\}$ is endowed with a partial order $\preceq$, such that for $\pi, \nu \in \mathcal{P}_n$, $\pi \preceq \nu$ if the blocks of $\pi$ are included in the one of $\nu$. It has a maximum and a minimum that we denote respectively by $1_n$ and $0_n$. Each partition $\pi$ has $\# \pi$ blocks. For any sequence of complex numbers $(\alpha_A)_{A \subset \{1, \ldots, n\}}$, let us set for any partition $\pi \in \mathcal{P}_n$,

$$
\alpha_\pi = \prod_{A \in \pi} \alpha_A.
$$

Then, there exists a unique sequence $(\beta_{\pi,\nu}(\alpha))_{\pi \preceq \nu}$ such that for any two partitions $\pi \preceq \nu$,

$$
\alpha_\nu = \sum_{\pi \preceq \pi' \preceq \nu} \beta_{\pi,\pi'}(\alpha).
$$

For any $\pi, \nu \in \mathcal{P}_n$ with $\nu \preceq \pi$ and $\pi \neq \nu$, we set $\beta_{\pi,\nu}(\alpha) = 0$. If $X_1, \ldots, X_n$ are bounded complex random variables and for any $A \subset \{1, \ldots, n\}$, $\alpha_A = \mathbb{E}[\prod_{i \in A} X_i]$, let us set $C_n(X_1, \ldots, X_n) = \beta_{0_n,1_n}(\alpha)$ and similarly, for any pair of partitions $\pi, \nu$, ...
\( C_{\pi,\nu}(X_1, \ldots, X_n) = \beta_{\pi,\nu}(\alpha) \). Then, the following expansion holds for any \( z \in \mathbb{C}^n \) in a neighborhood of 0,
\[
\log \mathbb{E}[e^{z_1X_1 + \ldots + z_nX_n}] = \sum_{k \geq 1} \frac{C_k(X_{i_1}, \ldots, X_{i_k}) z_{i_1} \cdots z_{i_k}}{k!}.
\]

If \( Y \) and \( Z \) are bounded random variables coupled with \( X \) and if \( z \in \mathbb{C} \) is in a neighborhood of 0, then
\[
\frac{\mathbb{E}[Ye^{zX}]}{\mathbb{E}[e^{zX}]} = \sum_{k \geq 0} \frac{C_{k+1}(Y, X_{i_1}, \ldots, X_{i_k})}{k!}
\]
and
\[
\frac{\mathbb{E}[YZe^{zX}]}{\mathbb{E}[e^{zX}]} - \frac{\mathbb{E}[Ye^{zX}]\mathbb{E}[Ze^{zX}]}{\mathbb{E}[e^{zX}]^2} = \sum_{k \geq 0} \frac{C_{k+2}(Y, Z, X, \ldots, X)}{k!}.
\]
The coefficient \( C_n(X_1, \ldots, X_n) \) is called a cumulant and is symmetric in the variables \( X_1, \ldots, X_n \). The coefficients \( (C_{\pi,\nu}(X_1, \ldots, X_n))_{\pi \leq \nu} \) are called relative cumulants. For any pair \( \pi \preceq \nu \) and \( A \in \nu \), let us denote by \( A_{\pi} \) the set of blocks of \( \pi \) included in \( A \), then
\[
C_{\pi,\nu}(X_1, \ldots, X_n) = \prod_{A \in \nu} C_{\#A_{\pi}} \left( \prod_{i \in B} X_i, B \subset A \text{ and } B \in \pi \right).
\]

3.1.3. Tensor valued cumulants. Let us fix some notations for tensors. For any finite dimensional vector space \( V \) and any finite set \( A \), let us denote by \( V^{\otimes A} \) the vector space of multilinear map on \((V^*)^A\) and for any \( n \in \mathbb{N}^* \) identify \( V^{\otimes \{1, \ldots, n\}} \) with \( V^{\otimes n} \). Any function \( X : \{1, \ldots, n\} \to V \) defines an elementary element of \( V^{\otimes A} \) that we denote by \( \bigotimes_{i \in A} X_i \). Any partition \( \pi \in \mathcal{P}_n \) defines a multilinear map \( \prod_{A \in \pi} V^{\otimes A} \to V^{\otimes n}, (\alpha_A)_{A \in \pi} \mapsto \bigotimes_{A \in \pi} \alpha_A \), such that for any \( X \in V^n \),
\[
\bigotimes_{A \in \pi} \bigotimes_{i \in A} X_i = \bigotimes_{i \in \{1, \ldots, n\}} X_i.
\]
For instance, for \( v_1, \ldots, v_4 \in V \), if \( \alpha_{\{1,3\}} = v_1 \otimes v_2 \in V^{\otimes 2} \simeq V^{\otimes \{1,3\}} \) and \( \alpha_{\{2,3\}} = v_3 \otimes v_4 \in V^{\otimes 2} \simeq V^{\otimes \{2,4\}} \),
\[
\bigotimes_{A \in \{1,3\}, \{2,4\}} \alpha_A = v_1 \otimes v_3 \otimes v_2 \otimes v_4.
\]
For any sequence \( (\alpha_A)_{A \subset\{1, \ldots, n\}} \) such that for any \( A \subset \{1, \ldots, n\} \), \( \alpha_A \in V^{\otimes A} \), let us set for any partition \( \pi \in \mathcal{P}_n \),
\[
\alpha_{\pi} = \bigotimes_{A \in \pi} \alpha_A \in V^{\otimes n}.
\]
Then, there exists a unique sequence \( (\beta_{\pi,\nu})_{\pi \leq \nu} \) such that for any two partitions \( \pi \preceq \nu \),
\[
\alpha_{\nu} = \sum_{\pi \preceq \pi' \leq \nu} \beta_{\pi,\pi'}(\alpha) \in V^{\otimes n}.
\]
If $X_1, \ldots, X_n$ are bounded random variables valued in $V$ on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and for any $A \subset \{1, \ldots, n\}$, $\alpha_A = \mathbb{E}[\bigotimes_{i \in A} X_i]$, note that for any pair of partitions $\pi, \nu$, $C_{\pi, \nu}(X_1, \ldots, X_n) = \beta_{\pi, \nu}(\alpha)$ is $n$-linear as a function on the space $L_\infty(\Omega, \mathcal{B}, \mathbb{P}) \otimes V$. We then define linear functions on $L_\infty(\Omega, \mathcal{B}, \mathbb{P}) \otimes V^{\otimes n}$ by setting for any random variables $X_1, \ldots, X_n \in L(\Omega, \mathcal{B}, \mathbb{P}) \otimes V$, $C_{\pi, \nu}(X_1 \otimes \ldots \otimes X_n) = C_{\pi, \nu}(X_1, \ldots, X_n)$ and $C_{\pi}(X_1 \otimes \ldots \otimes X_n) = C_{0\pi, 1\nu}(X_1, \ldots, X_n)$. For example, if $A$ and $B$ are two bounded random vectors of $V$,

$$C_2(A \otimes B) = \mathbb{E}[A \otimes B] - \mathbb{E}[A] \otimes \mathbb{E}[B] \in V^{\otimes 2},$$

$$C_{\{(1,2), (3,1)\}}(A \otimes B \otimes C) = \mathbb{E}[A \otimes B \otimes C] - \mathbb{E}[A \otimes B] \otimes \mathbb{E}[C] \in V^{\otimes 3}.$$  

If $A_1, \ldots, A_n$ are random matrices in $M_n(\mathbb{C})$ with bounded operator norms and $\pi, \nu \in \mathcal{P}_n$ is a pair of partitions, then

$$C_{\pi, \nu}(\text{Tr}(A_1), \text{Tr}(A_2), \ldots, \text{Tr}(A_n)) = \text{Tr}(C_{\pi, \nu}(A_1 \otimes \ldots \otimes A_n)).$$

If $\sigma \in \mathfrak{S}_n$ is a permutation whose orbits are included in blocks of the partition $\nu$, then

$$(2) \quad C_{\#\nu}(\prod_{(i_1 \cdots i_k)} \text{Tr}(A_{i_1} \cdots A_{i_k}), B \in \nu) = \text{Tr}(C_{\nu, 1\nu}(\sigma A_1 \otimes A_2 \otimes \ldots \otimes A_n)).$$

3.1.4. Cumulant of exponential tensors. For any $n \in \mathbb{N}^*$, let us denote by $\Psi_n$ the set of subset of $\{1, \ldots, n\}$. For any $A, B \in \Psi_n$, with $B \subset A$, let us write $B^A$ for the smallest partition of $A$ for $\preccurlyeq$, containing $B$ as a block, and set for any endomorphism $T \in \text{End}(V^{\otimes B})$, $\overline{T}^A = \bigotimes_{S \in \{B, A \setminus B\}} \alpha_S \in \text{End}(V^{\otimes A})$, where $\alpha_B = T$ and $\alpha_{A \setminus B} = \text{Id}_{V^{\otimes A \setminus B}}$. If $A = \{1, \ldots, n\}$, we shall write $\overline{T}$ for $\overline{T}^A$.

**Lemma 3.1.** Let $(T_A)_{A \in \Psi_n, \#A \geq 2}$ be a family of endomorphisms such that for any $A \in \Psi_n$, $T_A \in \text{End}(V^{\otimes A})$.

i) If for any $A \in \Psi_n$,

$$\alpha_A = \exp[\sum_{B \subset A, \#B \geq 2} \overline{T}_B^A],$$

then, for any pair of partitions $\pi \preccurlyeq \nu$ in $\mathcal{P}_n$,

$$\beta_{\pi, \nu}(\alpha) = \sum_{k \geq 0} \frac{1}{k!} \sum_{A_1, \ldots, A_k} \overline{T}_{A_1} \ldots \overline{T}_{A_k} \in \text{End}(V^{\otimes n}),$$

where the second sum is over sequences $A_1, \ldots, A_k \in \Psi_n$, with $\#A_i \geq 2$ for any $1 \leq i \leq k$ and $\pi \preccurlyeq A_1 \preccurlyeq A_2 \ldots \preccurlyeq A_k = \nu$.

ii) For any $\mu \in \mathcal{P}_n$, let us set for each block $C \in \mu$, $L_C = \sum_{B \subset C, \#B \geq 2} \overline{T}_B^A$. If for any $t \in \mathbb{R}^\mu$ and $A \in \Psi_n$,

$$\alpha_A(t) = \exp[\sum_{C \in \mu} t_CL_C],$$
then, for any pair of partitions \( \pi, \nu \in \mathcal{P}_n \), \( \beta_{\pi,\nu}(\alpha(\cdot)) \) is a differentiable map and for any \( t \in \mathbb{R}^n \) and \( C \in \mu \),

\[
\frac{d}{dt_C} \beta_{\pi,\nu}(\alpha(t)) = \sum_{B \subset C, \#B \geq 2} \beta_{\pi \lor B,\nu}(\bar{T}_B \alpha(t)).
\]

Proof. i) For any \( \pi, \nu \in \mathcal{P}_n \), let \( \tilde{\beta}_{\pi,\nu} \) the right-hand-side of the formula of the statement. For any \( \nu \in \mathcal{P}_n \), the expansion of tensors of exponential \( \alpha(\nu) \) equals to the sum \( \sum_{k \geq 0} \frac{1}{k!} \sum \bar{T}_{A_1} \ldots \bar{T}_{A_k} \), where the second sum is over sequences \( A_1, \ldots, A_k \in \mathcal{P}_n \), with \( \#A_i \geq 2 \) for any \( 1 \leq i \leq k \) and \( A_1 \lor A_2 \lor \ldots \lor A_k \neq \nu \). This expression is equal to \( \sum_{\pi \leq \nu} \tilde{\beta}_{\pi,\nu} \).

ii) For any \( t \in \mathbb{R}^n \), \( \nu \in \mathcal{P}_n \), and \( B \in \mathcal{B}_n \), with \( \#B \geq 2 \), \( \beta_{\pi \lor B,\nu}(\bar{T}_B \alpha(t)) = \beta_{\pi,\nu}(\alpha(t)) \). Note that the family of endomorphisms \( (L_C)_{C \subseteq \mu} \) is commutative. For any \( C \in \mu \), differentiating with respect to \( t_C \) the expansion of \( \beta_{\pi,\nu}(\alpha(t)) \) given in i) yields the announced formula. \( \square \)

### 3.2. Words in independent unitary Brownian motions and their traces.

We obtain here a differential system for the normalized cumulants in traces of words of unitary brownian motions and show that the latter converge as \( N \to \infty \).

#### 3.2.1. Partitioned and partial words.

For each positive integer \( q \), \( W_q \) denotes the monoid of words in the alphabet made of \( 2q \) symbols \( x_1, \ldots, x_q, x_1^{-1}, \ldots, x_q^{-1} \). An element \( w \) of \( W_q \) writes down uniquely \( x_{i_1}^{e_1} \ldots x_{i_n}^{e_n} \), with \( e_1, \ldots, e_n \in \{-1, 1\} \). We call \( n \) the length of \( w \) and denote it by \( \ell(w) \). Its \( p^{th} \) letter \( x_{i_p}^{e_p} \) is denoted by \( X_p(w) \).

Any subset of \( \{1, \ldots, q\} \), we set \( n_w^+(k) = \#\{ r \in \{1, \ldots, n\} : X_r(w) = x_k^+ \} \) and \( n_w^-(k) = n_w^+(k) + n_w^-(k) \). We call partitioned word every couple \( (S, \pi) \), where \( S \) is a tuple \( (w_1, \ldots, w_m) \) of words of \( W_q \) and \( \pi \in \mathcal{P}_m \). Given such a couple, we set \( w(S) = w_1 w_2 \ldots w_m \) and \( \ell(S, \pi) = \ell(w(S, \pi)) \). We denote the set of partitioned words by \( \mathcal{P}W_q \). The symmetric group \( \mathfrak{S}_m \) acts diagonally on \( \{(S, \pi) : S \in W_q^m, \pi \in \mathcal{P}_m \} \). A partial word \( [S, \pi] \) is the orbit of a partitioned word \( (S, \pi) \in \mathcal{P}W_q \), with \( S \in W_q \), under the diagonal action of \( \mathfrak{S}_m \). We denote the set of partial words by \( \mathcal{P}W_q^\prime \).

#### 3.2.2. Operations on partitioned words.

For any partitioned word \( (S, \pi) \in \mathcal{P}W_q \) with \( S = (w_1, \ldots, w_m) \) and \( w = w(S, \pi) \), let us introduce two transformations of \( (S, \pi) \). For any pair of positive integers \( i, j \) such that \( 1 \leq i < j \leq \ell(S, \pi) \) and \( X_i(w) = X_j(w)^\pm = a \), let us define \( T_{i,j}^\pm(S, \pi) = (T_{i,j}^\pm(S), \pi^\prime) \in \mathcal{P}W_q \) according to two cases.

1. If the \( i^{th} \) and \( j^{th} \) letters of \( w \) belongs to the same word \( w_k = \lambda X_{i}(w) \mu X_{j}(w)^\nu \), then let us set

\[
T_{i,j}^+(S) = (w_1, \ldots, w_{k-1}, \lambda \nu, a \mu, w_{k+1}, \ldots, w_m),
\]

if \( X_i(w) = X_j(w) \),

\[
T_{i,j}^-(S) = (w_1, \ldots, w_{k-1}, \lambda \nu, a \mu^{-1}, w_{k+1}, \ldots, w_m),
\]
Figure 1. Cut transformations

if \( X_i(w) = X_j(w)^{-1} \) and in both cases \( \pi' \in P_{m+1} \) the partition obtained from \( \pi \) by substituting \( l \) with \( l + 1 \) for \( l > k \) and adding \( k + 1 \) to any block of \( \pi \) including \( k \).

2. If the \( i^{th} \) and \( j^{th} \) letters of \( w \) belongs to two words \( w_p = \lambda X_i(w) \mu \) and \( w_q = \nu X_j(w) \chi \), then let us set

\[
T_{i,j}(S) = (w_1, \ldots, \hat{w}_p, \lambda \chi \nu a\mu, \ldots, \hat{w}_q, \ldots, w_m),
\]

if \( X_i(w) = X_j(w) \),

\[
T^{-i,j}(S) = (w_1, \ldots, \hat{w}_p, \lambda \chi \nu^{-1} a\mu, \ldots, \hat{w}_q, \ldots, w_m),
\]

if \( X_i(w) = X_j(w)^{-1} \) and in both cases we let \( \pi' \in P_{m-1} \) be the image of the restriction of \( \pi \) to \( \{1, \ldots, m\} \setminus \{q\} \) by the increasing bijection \( \{1, \ldots, m\} \setminus \{q\} \to \{1, \ldots, m - 1\} \).

Let us make two remarks. In the first case the number of blocks of the partitioned words are constant and the number of words is increased by 1. In the second one, the number of words is decreased by 1, whereas the number of blocks of \( \#\pi' \) is equal to \( \#\pi \), if \( p \) and \( q \) belongs to the same block of \( \pi \) and \( \#\pi - 1 \) otherwise. For any \( f \in \{1, \ldots, q\} \), we define the sets

\[
\mathcal{N}^2_{2,w}(f) = \{(i, j) \in \{1, \ldots, \ell(w)\}^2 : i < j, X_i(f) = X_j^+(f) \in \{x_f, x_f^{-1}\}\},
\]

\[
\mathcal{N}^{0,\pm}_{2,S,\pi}(f) = \{(i, j) \in \mathcal{N}^\pm_{2,w}(f) : \#\pi' = \#\pi - 1 \text{ or } \#S' = \#S + 1\},
\]

\[
\mathcal{N}^{2,\pm}_{2,S,\pi}(f) = \{(i, j) \in \mathcal{N}^\pm_{2,w}(f) : \#\pi' = \#\pi \text{ and } \#S' = \#S - 1\},
\]

\[
\mathcal{S}^{2,\pm}_{2,S,\pi}(f) = \{(i, j) \in \mathcal{N}^\pm_{2,w}(f) : \#\pi' = \#\pi\}.
\]

and

\[
\mathcal{N}^2_{2,w} = \bigcup_{f=1}^q \mathcal{N}^+_2(f) \cup \mathcal{N}^-_{2,w}(f).
\]
3.2.3. A differential system. Let us fix $q$ independent and identically distributed $U(N)$-valued Brownian motions $(U_{1,s})_{s \geq 0}, \ldots, (U_{q,s})_{s \geq 0}$. For any word decomposed as $w = x_{i_1}^{\epsilon_1} \cdots x_{i_n}^{\epsilon_n}$, with $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ and $U_1, \ldots, U_q \in U(N)$, let us set

$$w(U_1, \ldots, U_q) = U_{i_1}^{\epsilon_1} \cdots U_{i_n}^{\epsilon_n}$$

and for any vector $t \in \mathbb{R}_+$,

$$w_t^N = w(U_{1,t_1}, \ldots, U_{q,t_q}).$$

For any partial word $[S, \nu] \in P\mathcal{W}_q$, with $S = (w_1, \ldots, w_m)$, we shall consider for any $t \in \mathbb{R}_+$,

$$K_t(S, \nu) = C_{\#\nu}(\prod_{i \in A} \text{Tr}(w_{i,t}^N), A \in \nu).$$

**Lemma 3.2.** For any $f \in \{1, \ldots, q\}$ and any partitioned word $(S, \nu) \in P\mathcal{W}_q$,

$$\frac{d}{dt} K_t(S, \nu) = -\frac{\pi_w(f)}{2} K_t(S, \nu) - \frac{1}{N} \left( \sum K_t(T_{i,j}^+(S, \nu)) - \sum K_t(T_{i,j}^-(S, \nu)) \right),$$

where the sums are over $(i, j)$ belonging respectively to $N^-_{2,w(S)}(f)$ and $N^+_{2,w(S)}(f)$.

**Proof.** Let us fix a tuple $(w_1, \ldots, w_m)$ of words, set $w = w_1 \cdots w_m = x_{i_1}^{\epsilon_1} \cdots x_{i_n}^{\epsilon_n}$, with $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ and $\iota : \{1, \ldots, n\} \rightarrow \{1, \ldots, m\}$ the map induced by the decomposition of $w$ into $w_1, \ldots, w_m$. For any partition $\nu \in \mathcal{P}_n$, we denote by $\nu^0 \in \mathcal{P}_n$ the biggest partition such that $\iota(\nu^0) = \nu$. For any $\sigma \in \mathcal{S}_n$, let us denote by $S_{\sigma}$ the tuple of words of the form $x_{i_{\iota(1)}}^{\epsilon_{\iota(1)}} \cdots x_{i_{\iota(n)}}^{\epsilon_{\iota(n)}}$, with $(c_1 \ldots c_k)$ cycle of $\sigma$, ordered by the position of their first letter in $w$. We denote $W_q(w) = \{S_{\sigma} : \sigma \in \mathcal{S}_n\}$ and set for any $S \in W_q(w)$, $\sigma_S$ the unique permutation such that $S_{\sigma_S} = S$.

Let us denote $\theta : M_N(\mathbb{C}^N) \rightarrow M_N(\mathbb{C}^N), M \mapsto M^t$, set for any integer $i \leq \ell(w)$, $\delta_i = \frac{1}{2w_i}$, and define for any pair of distinct integers $i, j$ two operators $T_{i,j}^+$ and $T_{i,j}^-$ acting on $\text{End}((\mathbb{C}^N)^{\otimes \{i,j\}})$, by setting for any $M \in \text{End}((\mathbb{C}^N)^{\otimes \{i,j\}})$,

$$T_{i,j}^+(M) = M(i \ j),$$

$$T_{i,j}^-(M) = \theta_i^{\delta_i} \circ \theta_j^{\delta_j} \left( \theta_i^{\delta_i} \circ \theta_j^{\delta_j}(M)(i \ j) \right).$$

**Figure 2. Join transformations**
For any collection of words $S \in W_q(w)$, any pair $1 \leq i < j \leq n$ and any $U_1, \ldots, U_n \in U(N)$ with $U_i^{e_i} = U_j^{e_j}$,

$$\text{Tr}(CN)^{\otimes n}(\sigma S \mathcal{T}_{i,j}^\pm(U_1 \otimes \ldots \otimes U_n)) = \text{Tr}(CN)^{\otimes n}(\sigma \mathcal{T}_{i,j}^\pm(S) U_1 \otimes \ldots \otimes U_n).$$

It follows that if $U_1, \ldots, U_n$ are $U(N)$-valued random variables with $U_i^{e_i} = U_j^{e_j}$, then for any partition $\pi \in \mathcal{P}_n$ such that $i$ and $j$ belongs to the same block of $\pi$,

$$\text{Tr}(CN)^{\otimes n}(\sigma S C_{\pi,1,n} \mathcal{T}_{i,j}^\pm(U_1 \otimes \ldots \otimes U_n)) = \text{Tr}(CN)^{\otimes n}(\sigma \mathcal{T}_{i,j}^\pm(S) C_{\pi,1,n}(U_1 \otimes \ldots \otimes U_n)).$$

We shall now apply Lemma 2.6 to the tensors $w_t^{\otimes A} = \bigotimes_{k \in A} U_{i_k}^{e_k}$, for any $A \subset \{1, \ldots, n\}$, that we will simply denote by $w_t^{\otimes}$, for $A = \{1, \ldots, n\}$. Setting $\pi_{w,A}(f) = \# \{i \in A : X_i(w) \in \{x_f, x_f^{-1}\}\}$, Lemma 2.6 yields that for any $f \in \{1, \ldots, q\}$,

$$\frac{d}{dt} \mathbb{E}[w_t^{\otimes A}] = -\frac{\pi_{w,A}(f)}{2} \mathbb{E}[w_t^{\otimes A}] - \frac{1}{N} \left( \sum_{i,j} T_{i,j}^\pm - \sum_{i,j} \mathcal{T}_{i,j}^\pm \right)(\mathbb{E}[w_t^{\otimes A}]),$$

where the first and the second sums are over pairs $(i, j) \in A^2$ belonging respectively to $N_{2,w}(f)$ and $N_{2,w}^c(f)$. Consider now a partitioned word $(S, \nu)$ with $S \in W_q(w)$. According to (2),

$$K_t(S, \nu) = \text{Tr}(CN)^{\otimes n}(C_{\nu,1,n}^{\otimes}(\sigma S w_t^{\otimes})).$$

For any subset $A \subset \{1, \ldots, n\}$, with $\# A \geq 2$, let us set $T_A = 0$, if $\# A \geq 3$ and

$$T_A = \mp \mathcal{T}_{a,b}^\pm,$$

if $A = \{i, j\}$, with $(a, b) \in N_{2,w}$. Applying Lemma 3.1 with the vector space $V = M_N(C)$, the family of tensors $\alpha_A = e^{\frac{1}{2} \sum_{i \leq j \leq q} t_j \pi_{w,A}(f) \mathbb{E}[w_t^{\otimes A}]}$, for $A \subset \{1, \ldots, n\}$, the partition $\{N_{2,w}^c(f) \cup N_{2,w}^c(f) : 1 \leq f \leq q\} \in \mathcal{P}_n$ and considering the evaluation against $\text{Id}_{(CN)^{\otimes n}}$ yield the two following equalities. For any partition $\pi \in \mathcal{P}_n$,

$$C_{\pi,1,n}[w_t^{\otimes}] = e^{-\frac{1}{2} \sum_{i \leq j \leq q} t_j \pi_{w}(f)} \prod_{k \geq 0} \frac{1}{k! N^k} \sum_{t_{i_1} \ldots t_{i_k} \mathcal{T}_{\{a_1,b_1\}} \ldots T_{\{a_k,b_k\}}(\text{Id}_{(CN)^{\otimes n}})},$$

where the second sum is over the sequences $(a_i, b_i)_{1 \leq i \leq k} \in N_{2,w}^k$ and $\pi \lor \{a_1, b_1\} \lor \ldots \lor \{a_k, b_k\} = \pi^{\otimes}$. Moreover, for any $f \in \{1, \ldots, q\}$,

$$\frac{d}{dt} \left( e^{\frac{1}{2} \sum_{i \leq j \leq q} t_j \pi_{w}(f)} C_{\pi,1,n}[w_t^{\otimes}] \right) = \sum_{(a,b) \in N_{2,w}^c(f) \cup N_{2,w}^c(f)} C_{\pi \lor \{a,b\},1,n}(\mathcal{T}_{\{a,b\}}[w_t^{\otimes}]).$$

For any pair $(a, b) \in N_{2,w}$, if $p, q \in \{1, \ldots, m\}$ are such that the $a$-th and the $b$-th letters of $w$ belong respectively to $w_p$ and $w_q$, then according to (5),

$$\text{Tr}(CN)^{\otimes n}(\sigma S C_{\nu \lor \{a,b\},1,n}(\mathcal{T}_{\{a,b\}}[w_t^{\otimes}])) = -\epsilon_{a,b} \text{Tr}(CN)^{\otimes n}(\sigma \mathcal{T}_{a,b}^{e_a \lor e_b}(S) C_{\nu \lor \{a,b\},1,n}(w_t^{\otimes}))$$

and

$$-\epsilon_{a,b} K_t(\mathcal{T}_{a,b}^{e_a \lor e_b}(S, \nu \lor \{p,q\})).$$
The two equations (*) and (**) then imply the announced formula.

3.2.4. Scaling and asymptotic expansion of the cumulants. Let us now introduce a scaling of the above functions that matches the one of section 3.1.1 and that yields a differential system with initial conditions independent of $N$. For any partial word $[S, \nu] \in \overline{PW}_q$, with $S = (w_1, \ldots, w_m)$, the following quantity

$$\varphi_{t,N}([S, \nu]) = N^{2(#\nu-1)-n} C_{#\nu}(\prod_{i \in A} \text{Tr}(w_{i,t}), A \in \nu)$$

satisfies these two conditions. Let us define two operators on $C_{\overline{PW}_q}$, by setting for any function $\varphi \in C_{\overline{PW}_q}$ and $(S, \nu) \in P_{\overline{PW}_q}$, $L_f(\varphi)([S, \nu])$ to be equal to

$$-\frac{n_w(S)}{2} \varphi([S, \nu]) + \sum_{(i,j) \in N_{S,\nu}^0} \varphi([T_{ij}^- (S, \nu)]) - \sum_{(i,j) \in N_{S,\nu}^+} \varphi([T_{ij}^+ (S, \nu)])$$

and

$$D_f(\varphi)([S, \nu]) = \sum_{(i,j) \in N_{S,\nu}^2} \varphi([T_{ij}^- (S, \nu)]) - \sum_{(i,j) \in N_{S,\nu}^{2,+}} \varphi([T_{ij}^+ (S, \nu)]).$$

Proposition 3.3. For any $t \in \mathbb{R}_+^q$, $N \in \mathbb{N}^*$ and $(S, \pi) \in P_{W_q}$,

$$\frac{d}{dt} \varphi_{t,N}([S, \pi]) = (L_f + \frac{1}{N^2} D_f). \varphi_{t,N}([S, \pi])$$

and $\varphi_{0,N}([S, \pi]) = 1$, if $\pi$ has one block and 0 otherwise. As $N \to \infty$, the sequence of functions $\varphi_{t,N}$ converges pointwise towards the unique function $\varphi_t$, such that for any $t \in \mathbb{R}_+^q$ and $(S, \pi) \in P_{W_q}$,

$$\frac{d}{dt} \varphi_t([S, \pi]) = L_f. \varphi_t([S, \pi])$$

and $\varphi_0 = \varphi_{0,1}$.

For any $w \in W_q$, $t \in \mathbb{R}_+^q$ and $N \in \mathbb{N}^*$, we shall use the simpler notation $\varphi_{t,N}(w) = \varphi_{t,N}([([w], 0_1])]$ and $\varphi_{t,N}(w) = \varphi_t([([w], 0_1])].$

Proof. It is a direct consequence of Lemma 3.2.

Corollary 3.4. There exists a sequence of functions $(\psi_{t,q})_{q \geq 1}$ on $P_{W_q}$ such for any $(S, \pi)$, the power series with coefficients $(\psi_{t,q}([S, \pi]))_{q \geq 1}$ has a positive radius of convergence and for $N$ large enough,

$$\varphi_{t,N}([S, \pi]) = \varphi_t([S, \pi]) + \sum_{q \geq 1} N^{-2q} \psi_{t,q}([S, \pi]).$$

Proof. For any fixed $n \in \mathbb{N}^*$, the operators $L_f, D_f$ preserve the finite dimensional space of functions supported on $\{x \in \overline{PW}_q : t(x) \leq n\}$. The above expansion follows then easily from Proposition 3.3.
In particular, for any \( n_1, \ldots, n_m \in \mathbb{Z}^* \), \( N^{m-2} C_m \left( \text{Tr}(U_{n_1}^t), \ldots, \text{Tr}(U_{n_m}^t) \right) \) admits a limit as \( N \to \infty \). Together with the property of independance, stationarity and invariance by unitary adjunction of multiplicative increments of the process \((U_t)_{t \geq 0}\) and the notion of higher order freeness developed in [12], this fact alone implies that for any \((S, \pi) \in \mathcal{P}W_q\), \( \varphi_{t,N}([S, \pi]) \) admits a limit as \( N \to \infty \). Nonetheless, this result does not give easily an expansion in \( N \).

4. TWO ESTIMATES ON THE CUMULANTS

The proof of Theorem 1.1 relies on two estimates on the above cumulants. The first bound gives a bound of their modulus that allows to extend them to broader class of words as we will see in section 6. It is nonetheless too loose to obtain a positive radius of convergence as stated in Theorem 1.1. The second type of estimates gives a much sharper bound that allows to conclude.

4.1. All-order bounds. For any \( t \in \mathbb{R}_q^+ \), let us define a scalar product \( \langle \cdot, \cdot \rangle_t \) on \( \mathbb{R}^m \), by setting for any \( a, b \in \mathbb{R}^m \),

\[
\langle a, b \rangle_t = \sum_{f=1}^q a_f b_f t_f.
\]

For any word \( w \in W_q \), we set

\[
A_t(w) = \langle \pi_w, \pi_w \rangle_t
\]

and for any \((S, \pi) \in \mathcal{P}W_q\) and \( N \in \mathbb{N}^* \), we define inductively a sequence by setting \( \psi_{t,0,N}([S, \pi]) = \varphi_{t,N}([S, \pi]) \) and for any \( g \in \mathbb{N} \),

\[
\psi_{t,g+1,N}([S, \pi]) = N^2 \left( \psi_{t,g,N}([S, \pi]) - \psi_{t,g}([S, \pi]) \right).
\]

Lemma 4.1. For any words \( w_1, \ldots, w_m \in W_q \), \( N \in \mathbb{N}^* \), \( t \in \mathbb{R}_q^+ \), \( \lambda \in [0, 1] \) and \( k \in \mathbb{N} \),

\[
|d^k/d\lambda^k \varphi_{\lambda,N}([w_1, \ldots, w_m], 0_m)| \leq m^k (A_t(w_1) + \cdots + A_t(w_m))^k e^{m(A_t(w_1) + \cdots + A_t(w_m))}
\]

and

\[
|\psi_{t,k,N}([w_1, \ldots, w_m], 0_m)| \leq m^k (A_t(w_1) + \cdots + A_t(w_m))^k e^{m(A_t(w_1) + \cdots + A_t(w_m))}.
\]

Proof. For any \( p \in \mathbb{N} \) and \( M \in \mathcal{M}_p(\mathbb{C}) \), let us set \( \|M\| = \max_{i \in \{1, \ldots, p\}} \sum_{j=1}^p |M_{i,j}| \). Recall that \( \| \cdot \| \) is a sub-multiplicative norm on \( \mathcal{M}_p(\mathbb{C}) \), such that for any matrix \( M \in \mathcal{M}_p(\mathbb{C}) \) and \( v \in \mathbb{C}^p \), \( \max_{i \in \{1, \ldots, p\}} |(Mv)_i| \leq \|M\| \max_{i \in \{1, \ldots, p\}} |v_i| \). Let us fix a word \( w \in W_q \) and denote by \( B_w \) the set of partial words \([S, \pi] \in \mathcal{P}W_q\) with \( n_{w(S)} = n_w \). Note that \( B_w \) is stable by the operations defined in section 3.2.2, so that for any
for any $w \in \{1, \ldots, q\}$, the two operators $L_f$ and $D_f$ preserve the finite dimensional space $\mathcal{F}_w$ of functions on $\overline{P W}_q^w$ with support in $B_w$. For any $F \in \text{End}(\mathcal{F}_w)$, let us set
\[
\|F\| = \max_{y \in B_w} \sum_{x \in B_w} F(\delta_x)(y).
\]
For any $[S', \pi'] \in B_w$, there are at most \(\frac{\alpha_w(f)(\beta_w(f)-1)}{2}\) elements $[S, \pi]$ of $B_w$ such that $(S', \pi')$ is obtained from $(S, \pi)$ by a transformation of the form $T_{i,j}$ with $(i, j) \in N_{2,w(S)}$. It follows that the restriction of $L_f$ and $D_f$ to $\mathcal{F}_w$ satisfy the following inequality
\[
\max\{\|L_f|_{\mathcal{F}_w}\|, \|D_f|_{\mathcal{F}_w}\|, \|L_f|_{\mathcal{F}_w} + \frac{1}{N^2}D_f|_{\mathcal{F}_w}\|\} \leq \frac{\pi_w(f)^2}{2}.
\]
Let us set $L = \sum_{f=1}^q t_f L_f|_{\mathcal{F}_w}$ and $D = \sum_{f=1}^q t_f D_f|_{\mathcal{F}_w}$. Then, according to Proposition 3.3,
\[
\varphi_{t,N}|_{B_w} = e^{L + \frac{1}{N^2}D}(\varphi_0)|_{B_w}
\]
and
\[
\varphi_{t|_{B_w}} = e^L(\varphi_0)|_{B_w}.
\]
Let us recall that for any matrices $A, B \in M_p(\mathbb{C})$,
\[
e^{A+B} - e^A = \int_0^1 e^{s(A+B)}B e^{(1-s)A} ds.
\]
Using iteratively this formula together with the former two equations yields that for any $g \geq 1$,
\[
\psi_{t,g,N}|_{B_w} = \int_{0<s_1<s_2<\ldots<s_g<1} e^{s_1(L + \frac{1}{N^2}D)}D e^{(s_2-s_1)L}D \ldots D e^{(1-s_g)L}ds(\varphi_0|_{B_w}).
\]
Therefore,
\[
\max_{[S, \pi] \in B_w} \{g!|\psi_{t,g,N}([S, \pi])|\} \leq ||D||^q \max_{[S, \pi] \in B_w} \{\varphi_0([S, \pi])\}
\]
\[
\quad \leq 2^{-q} \sum_{f=1}^q t_f \pi_w f^{q/2} e^{\frac{1}{2} \sum_{f=1}^q t_f \pi_w} = 2^{-q} A_t(w)^q e^{\frac{A_t(w)}{2}}.
\]
Besides, for all $\lambda \in [0, 1]$ and $r \in \mathbb{N}$,
\[
\frac{d^r}{d\lambda^r} \varphi_{t,N}|_{B_w} = (L + \frac{1}{N^2}D)^r e^{\lambda(L + \frac{1}{N^2}D)}(\varphi_0)|_{B_w},
\]
so that the left-hand-side is uniformly bounded by $2^{-r} A_t(w)^r e^{\frac{A_t(w)}{2}}$ on $B_w$. Now for any words $w_1 \ldots w_m \in W_q$ with $w_1 \ldots w_m = w$, $\pi_w = \pi_{w_1} + \ldots + \pi_{w_m}$ and we simply use the bound $A_t(w) \leq (2m - 1)(A_t(w_1) + \ldots + A_t(w_m))$ to conclude. \(\square\)
Note that these first estimates are very loose. For example, for \( w = x_n^1 \), the lemma shows that \( |\varphi_t(w, 0_t)| \leq e^{t_1 n^2} \), when we have\(^4\) the simple bound \( |\varphi_t(w, 0_t)| = |\int_U \omega^n \mu_t(d\omega)| \leq 1 \). Furthermore, for this same word, it yields for any \( m \in \mathbb{N}^* \), \( |N^{m-2}C_m(\text{Tr}(U^m_t))| \leq e^{t_1 n^2 m^2} \) and the exponential power series of the sequence on the right-hand-side diverges.

### 4.2. Sharper bounds for the first and second orders

For any positive integer \( m \), let us denote by \( C_m \) the set of Cayley trees on \( m \) vertices. For any \( w_1, \ldots, w_m \in W_q, T \in C_m \), we set

\[
T_t(w_1, \ldots, w_m) = \prod_{\{i,j\} \text{ edge of } T} \langle \pi_{w_i}, \pi_{w_j} \rangle_t
\]

and

\[
\tilde{T}_t(w_1, \ldots, w_m) = \prod_{\{i,j\} \text{ edge of } T} \langle n_{w_i}, n_{w_j} \rangle_t,
\]

if \( m \geq 2 \), and \( T_t(w_1) = \tilde{T}_t(w_1) = 1 \), otherwise.

**Proposition 4.2.** For any words \( w_1, \ldots, w_m \in W_q \) and any \( N \in \mathbb{N}^* \),

\[
|\varphi_{t,N}([w_1, \ldots, w_m, 0_m])| \leq \sum_{T \in C_m} T_t(w_1, \ldots, w_m)
\]

and

\[
\varphi_{t,N}([w_1, \ldots, w_m, 0_m]) = \sum_{T \in C_m} \tilde{T}_t(w_1, \ldots, w_m) + R_{t,N}(w_1, \ldots, w_m),
\]

with \( |R_{t,N}(w_1, \ldots, w_m)| \leq m^m (A_t(1) + \cdots + A_t(w_m)) m e^{m(A_t(w_1) + \cdots + A_t(w_m))} \).

The second estimate shows that it is optimal for tuples of words \( w \in W_q \), such that \( \pi_w = n_w \).

**Proof.** Let us consider \( q \) independant Brownian motions \( (U_{1,t})_{t \geq 0}, \ldots, (U_{q,t})_{t \geq 0} \), on \( U(N) \), fix a tuple of \( m \) words \( S_0 = (w_1, \ldots, w_m) \in W_q^m \), and set \( w = w_1 \ldots w_m = x_{i_1}^1 \cdots x_{i_m}^m, w_i^\otimes = U_{i_1}^{x_1^1} \otimes \cdots \otimes U_{i_m}^{x_m^m}, \sigma_0 = (1 \ldots \ell(w_1)) \times \cdots \times (1 \ldots \ell(w_m)) \) and \( \pi_0 \in \mathcal{P}_n \) the set of its orbits ordered by their first element. Let us further use the same notations as in the first paragraph of proof of Lemma 3.2. According our choice of scaling and (2),

\[
\varphi_{t,N}([w_1, \ldots, w_m, 0_m]) = N^{m-2} \text{Tr}(\mathcal{C}^N \otimes \mathcal{C}^{\otimes m} (\sigma_0 C_{\pi_0, 1_m}(w_1^\otimes))).
\]

The argument goes as follows: we first give an expression of the right-hand-side as an integral over the standard simplex as we did in the proof of Lemma 4.1 but

\^4\) The right decay of this moment sequence is a least polynomial, as the measure \( \mu_t \) is absolutely continuous with respect to the Lebesgue measure with a density that is Hölder continuous. The regularity of the density does depend on \( t_1 \), there are three regimes: \( t_1 < 4, t_1 = 4 \) and \( t_1 > 4 \), see remark 6.8. of [26].
using a different decomposition of the operator $\sum_{f=1}^{q} t_f (L_f + N^{-2} D_f)$. Then, we show that the integrands are normalized traces of unitary matrices.

**Step 1:** Let us remind the definition of $\mathcal{T}_{a,b}$ given in (3) and (7) and that according to (8),

$$e^{\frac{1}{2} \sum_{1 \leq l \leq q} \pi_n(f) C_{\pi,1m}(w_t^{\otimes})} = \sum_{k \geq 0} \frac{1}{k! N_k} \sum t_{i_1} \cdots t_{i_k} \mathcal{T}_{p_1,q_1} \cdots \mathcal{T}_{p_k,q_k} (\text{Id}(\mathbb{C}^N)^{\otimes n}),$$

where the second sum is over the set $\Upsilon_k$ formed by sequences of ordered pairs $(p_l, q_l)_{1 \leq l \leq k} \in \mathcal{N}_{2,w}^k$, with $\vee_{l=1}^k (p_l, q_l) \vee \pi_0 = 1_n$. For any $m \in \mathbb{N}^*$, if $\Delta_m$ denotes the standard simplex $\{ s \in \mathbb{R}^m_+ : s_1 + \cdots + s_m = 1 \}$ and $ds$ the Lebesgue measure on $\Delta_m$, then, for any matrices $A_1, \ldots, A_m \in M_p(\mathbb{C})$,

$$\sum_{k_1, \ldots, k_m \geq 0} \frac{1}{(k_1 + \cdots + k_m + m - 1)!} (A_1 + \cdots + A_m)^{k_m} A_m \cdots (A_1 + A_2)^{k_1} A_2 A_1^{k_1}
= \int_{\Delta_m} e^{s_m(A_1 + \cdots + A_m)} A_m \cdots e^{s_2(A_1 + A_2)} A_1 e^{s_1 A_1} ds.$$

For every $k \geq 1$, each sequence $\gamma = (p_l, q_l)_{1 \leq l \leq k} \in \Upsilon_k$ is the interlacement of a sequence $\gamma \in \Upsilon_m - 1$ with $m$ sequences $\gamma_1 \in \mathcal{N}_{2,w}^{k_1}, \ldots, \gamma_m \in \mathcal{N}_{2,w}^{k_m}$ with $k_1 + \cdots + k_m + m - 1 = k$ and there is a bijection

$$\Upsilon_k \rightarrow \Upsilon_{m-1} \times \bigcup_{k_1, \ldots, k_m = k-1} \mathcal{N}_{2,w}^{k_1}.$$

For any partition $\nu \in \mathcal{P}_m$, let us set $A_{\nu} = \frac{1}{N} \sum_{(a,b) \in \bigcup_{l=1}^{q} \mathcal{N}_{2,w}^{l} \times \{ f \}} \mathcal{T}_{a,b}$. Let $\Psi_m$ be the set of strictly increasing sequences $(\nu_l)_{l=1}^m \in \mathcal{P}_m^m$, with $\nu_1 = 0_m$ and $\nu_m = 1_m$. Any sequence of $\Upsilon_m - 1$ induces an element of $\Psi_m$, so that, rewriting the second sum of (8) thanks to the bijection (1), and then applying (11), yields that $e^{\frac{1}{2} \sum_{1 \leq l \leq q} \pi_n(f) C_{\pi,1m}(w_t^{\otimes})}$ equals to

$$\sum_{(p_k)_{k \in \Psi_m}} \int_{\Delta_m} e^{s_m A_m (A_{1m} - A_{1m-1}) \cdots e^{s_2 A_2 (A_{2m} - A_{2m-1})} e^{s_1 A_1} (\text{Id}(\mathbb{C}^N)^{\otimes n})} ds.$$

**Step 2:** Now, for any partition $\nu \in \mathcal{P}_m$, according to the very first equation on tensors (2.5) that we obtained, for any $v \in \text{End}((\mathbb{C}^N)^{\otimes n})$ and $s \geq 0$,

$$e^{-\frac{1}{2} \sum_{l=1}^{q} s_l n_l(f) + s_{\lambda_{\nu}}(v)} = \bigotimes_{A \in \pi} \mathbb{E}_{\bigotimes_{k \in A} U^{m}_{i_k,s_{t_{i_k}}}} v.$$

For any partition $\mu \in \mathcal{P}_m$, let us denote by $\mu : \{ 1, \ldots, n \} \rightarrow \mu$ the belonging map and introduce $((U_{t,b,f})_{t \geq 0})_{b \in \mu, 1 \leq l \leq q}$ a collection of $q \# \mu$ independent $U(N)$-Brownian motions. Then, let us $(U^{n})_{\mu \in \mathcal{P}_n}$ be a collection of independent random
variables such that for each \( \mu \in \mathcal{P}_n \),
\[
(U^\mu_{t_k,k})_{t \in \mathbb{R}^n_+, 1 \leq k \leq n} \overset{(law)}{=} \left( U^\mu_{t_k,q_l}\right)_{t \in \mathbb{R}^n_+, 1 \leq k \leq n}.
\]

For any element \( x \in \Psi_m \), let us denote by \( \Upsilon(x) \) the set of sequences of pairs of integers in \( \Upsilon_{m-1} \), inducing the sequence of partitions \( x \). Let \( v \in \text{End}((\mathbb{C}^N)^{\otimes n}) \), let us write \( L(v) \) for the endomorphism of left multiplication by \( v \) on \( \text{End}((\mathbb{C}^N)^{\otimes n}) \). It now follows from the last two formulas that \( N^{1-m}C_{\pi,1,m}(w^\otimes_t) \) equals to the image of \( \text{Id}_{(\mathbb{C}^N)^{\otimes n}} \) by

\[
\sum_{\nu \in \Psi_m} \sum_{(p_k,q_k) \in \Upsilon(\nu)} \int_{\Delta^{m-1}} ds E \left[ L(\otimes_{k \in \{1,...,n\}} U^{\nu_{s_m|t_{1,k},k}})T_{p_k,q_k} \cdots T_{p_1,q_1}L(\otimes_{k \in \{1,...,n\}} U^{\nu_{s_1|t_{1,k},k}}) \right].
\]

Notice that for every matrices \( U_1, \ldots, U_n \in U(N) \) and any pair \( (a,b) \in \mathcal{N}_{2,m} \) such that \( U_a = U_b \), \( L(\otimes_{k \in \{1,...,n\}} U_k) \) commutes with \( T_{a,b} \). Therefore, if we set for any sequence \( \nu \in \Psi_m \), \( s \in \Delta^{m-1} \) and \( 1 \leq k \leq n \),
\[
V^\nu_{s,k} = U^{\nu_{s_m|t_{1,k},k}} \cdots U^{\nu_{s_1|t_{1,k},k}} \in U(N),
\]
then, \( N^{1-m} \text{Tr}_{(\mathbb{C}^N)^{\otimes n}}(\sigma_0C_{\pi,1,m}(w^\otimes_t)) \) equals to

\[
\sum_{\nu \in \Psi_m} \sum_{(p_k,q_k) \in \Upsilon(\nu)} \int_{\Delta^{m-1}} ds E[\text{Tr}(\sigma_0T_{p_{m-1},q_{m-1}} \cdots T_{p_1,q_1}(\otimes_{k \in \{1,...,n\}} V^\nu_{s,k})).]
\]

For each \( \gamma = (p_k,q_k) \in \Upsilon_{m-1} \), if \( w_\gamma \) denotes the word so that \( T_{p_1,q_1} \circ \cdots \circ T_{p_{m-1},q_{m-1}}(S,\pi) = ((w_\gamma),1_1) \) and \( t_\gamma = \prod_{k=1}^{m-1} \epsilon_{p_k} t_{p_k} \), then for every tensor \( v \in \text{End}((\mathbb{C}^N)^{\otimes n}) \),
\[
\text{Tr}(\sigma_0T_{p_{m-1},q_{m-1}} \cdots T_{p_1,q_1}(v)) = t_\gamma \text{Tr}(\sigma_{w_\gamma} v).
\]
It remains now to unfold the above notations to get
\begin{equation}
\varphi_{t,N}([(w_1,\ldots,w_m),0_m]) = \sum_{\nu \in \Psi_m} \sum_{\gamma \in \Upsilon(\nu)} t_\gamma \int_{\Delta^{m-1}} E[N^{-1}\text{Tr}(w_\gamma (V^\nu_{s_1},\ldots,V^\nu_{s,n}))] ds.
\end{equation}

To conclude, note that the normalized trace in the integrand of the right-hand-side are bounded by \( 1 \). Let \( \Gamma : \Psi_m \to \mathcal{C}_m \) be the \((m-1)\)!-to-one map that sends every sequence \( \nu = (\sqrt{\chi_{i=1}}(l_t,l_1))_{1 \leq k \leq m} \) to the Cayley tree with edges \( \{(i_k,j_k)\}_{1 \leq k \leq m} \). Then, for every \( \nu \in \Psi_m \), \( \sum_{\gamma \in \Upsilon(\nu)} |t_\gamma| = \Gamma(\nu)(w_1,\ldots,w_m) \) and the first bound of the statement follows. Another consequence of (12), is that, in the Taylor expansion of \( \varphi_{t,N}([(w_1,\ldots,w_m),0_m]) \) around \( 0 \in \mathbb{R}_+^m \), the terms of degree less than \( m-2 \) vanish, whereas, the sum of terms of degree \( m-1 \) is exactly \( \sum_{\nu \in \Psi_m} \sum_{\gamma \in \Upsilon(\nu)} t_\gamma \). But, for any \( T \in \mathcal{C}_m \) and \( \nu \in \Gamma^{-1}(T) \), \( \sum_{\gamma \in \Upsilon(\nu)} |t_\gamma| = \tilde{T}(w_1,\ldots,w_m) \). Hence, applying Proposition 4.1 yields the second estimate. \( \square \)
Remark 4.3. It follows from the definition that
\[ \varphi_{t,1}([\{w_1\}, 0_1]) = e^{-\frac{1}{2}\|\eta w_1\|^2}, \]
whereas the above proof shows that for any \( m \geq 2 \),
\[ \varphi_{t,1}([\{w_1, \ldots, w_m\}, 0_m]) = \sum_{T \in \mathcal{C}_m} \hat{T}(w_1, \ldots, w_m)e^{-\frac{1}{2}\|\eta w_1 + \ldots + \eta w_m\|^2}. \]
For any \( t \in \mathbb{R}_+ \), let us set
\[ \lambda_t = (2t + 1 + 2\sqrt{t(t + 1)})e^{\sqrt{t(t+1)-\frac{1}{2}}} \]
and for any word \( w \in W_q \) and \( t \in \mathbb{R}_+^q \),
\[ \lambda_t(w) = \prod_{f=1}^{q} \lambda_{t,f}(w). \]

Lemma 4.4. For any words \( w_1, \ldots, w_m \in W_q \) with \( m \geq 2 \) and any \( N \in \mathbb{N}^\ast \),
\[ |\psi_{t,1,N}([\{w_1, \ldots, w_m\}, 0_m])| \leq 2^{m-1}m^2 \prod_{i=1}^{m} \lambda_t(w_i) \max_{1 \leq i \leq m} \|\eta w_i\|^2 \sum_{T \in \mathcal{C}_m} T(w_1, \ldots, w_m). \]

Proof. Let us fix \( S_0 = (w_1, \ldots, w_m) \in W_q^m \) with \( m \geq 2 \), \( w = w_1 \ldots w_m \) and consider \( B_0 \) the set of partitioned word obtained from \( x_0 = [S_0, 0_m] \) by a sequence of cut and join transformations. Each such sequence, inducing a partitioned word \( x = [S, \nu] \in \mathcal{P}W_q \), also induces recursively a partition of \( \{1, \ldots, m\} \), with as many blocks as \( \nu \). This partition only depends on \( x \), we denote it by \( \eta_x \in \mathcal{P}_m \) and set for \( \eta \in \mathcal{P}_m, B_\eta = \{ x \in B_0 : \eta_x = \eta \} \). For any \( \eta \in \mathcal{P}_m \), we consider \( \Psi_\eta = \{ (\{p_1, q_1\} \cup \eta)_{0 \leq k \leq m - \#\eta} \in \mathcal{P}^{m - \#\eta + 1}_m : \bigvee_{i=1}^{k} \{p_i, q_i\} = \eta \} \) and \( \Psi_\eta = \{ (\{p_1, q_1\} \cup \eta)_{0 \leq k \leq \#\nu - 1} \in \mathcal{P}^{\#\nu}_m : \bigvee_{i=1}^{k} \{p_i, q_i\} \cup \eta = 1_m \} \). For any \( N \in \mathbb{N}^\ast \), according to Proposition 3.3 and Duhamel’s formula (9),
\[ \psi_{t,1,N}([S_0, 0_m]) = -\int_0^1 e^{(1-s)L_0} D_0 e^{s(L_0 + N^{-2}D_1)}(\varphi_0)(x_0) ds. \]
For any \( t \in \mathbb{R}_\eta \) and \( x, y \in B_0 \), let us set
\[ Q_t(x, y) = e^{L_0} D_1(\delta_y)(x), \]
so that for any \( t \in \mathbb{R}_\eta \),
\[ -\psi_{t,1,N}([S_0, 0_m]) = \sum_{y \in B_0} \int_0^1 Q_t(x_0, y)e^{s(L_0 + N^{-2}D_1)}(\varphi_0)(y) ds \]
\[ = \sum_{\eta \in \mathcal{P}_m} \sum_{y \in B_\eta} \int_0^1 \varphi_{s,N}(y)Q_t(x_0, y) ds. \]
For any $t \in \mathbb{R}^d_+$ and any increasing sequence $\nu \in \mathcal{P}_m'$, induced by a sequence of pairs of integers $(p_i, q_i)_{1 \leq i \leq l}$, let us set

$$
\nu_t(S_0) = \prod_{k=1}^{l} \langle \pi_{w_{y_{kl}}}, \pi_{w_{y_{kl}}} \rangle_t.
$$

Let us fix $\eta \in \mathcal{P}_m$. For any $y \in B_\eta$, a slight modification of the proof of Proposition 4.2 yields that for any $t \in \mathbb{R}^d_+$,

$$
|\varphi_{\nu,N}(y)| \leq \frac{1}{(\# \eta - 1)!} \sum_{\nu \in \Psi_\eta} \nu_t(S_0).
$$

For any $\pi \in \mathcal{P}_m$ and any linear operator $A$ on $\mathbb{C}^{B_0}$, let us define another operator $A^\pi$ by setting for any $\varphi \in \mathbb{C}^{B_0}$ and $x \in B_0$, $A^\pi(\varphi)(x) = \sum_{y \in B_\eta} \varphi(y) A(\delta_y)(x)$. The same argument yields that

$$
\sum_{y \in B_\eta} |Q_t(y)| \leq \sum_{\nu \in \Psi_\eta} \langle \pi_{w_{y_{l}}}, \pi_{w_{y_{l}}} \rangle_t \nu_t(S_0) \int_0^\infty \sum_{y \in B_\eta} |e^{\sum_{i=1}^{q} \tau_{x_i} L_i (\delta_y)(x)}| ds.
$$

To conclude, we shall now expand the exponential in the right-hand-side and use triangular inequality. For each $f \in \{1, \ldots, q\}$, let us define an operator $\tilde{L}_f$ on $\mathbb{C}^{B_0}$, by setting for all $\varphi \in \mathbb{C}^{B_0}$ and $x = (S, \pi) \in B_0$,

$$
\tilde{L}_f(\varphi)(x) = \sum_{(a,b) \in N_{2,S,\pi}^+} \varphi(T_{a,b}^+(x)) + \sum_{(a,b) \in N_{2,S,\pi}^-} \varphi(T_{a,b}^-(x)).
$$

For any $x \in B_0$, $\nu \in \Psi_\eta$ and $s \in \Delta^{\# \eta - 1}$,

$$
\sum_{y \in B_\eta} |e^{\sum_{i=1}^{q} \tau_{x_i} L_i (\delta_y)(x)}| \leq e^{- \frac{1}{2} \Delta^{\# \eta - 1}} \sum_{y \in B_\eta} e^{\sum_{f=1}^{q} \tau_{x_{f}} L_f (\delta_y)(x)}
$$

For any tuple $S$ of words in $W_q$, let us denote, for each $f \in \{1, \ldots, q\}$, by $w_f(S) \in W_q$, the word obtained from $w(S)$ by deleting the letters $x_{f'}$ and $x_{f'}^{-1}$, for $f' \neq f$. Recall that for any $x = [S, \pi] \in \overline{PW}_q$, $\varphi_0(x) = 1$, if $\# \pi = 1$, and 0, otherwise. Then, for any $x = [S, \pi] \in B_\nu$,

$$
\sum_{y \in B_\eta} e^{\sum_{f=1}^{q} \tau_{x_{f}} L_f (\delta_y)(x)} \leq \prod_{f=1}^{q} e^{\sum_{f=1}^{q} \tau_{x_{f}} L_f (\varphi_0)([w_f(S)], 1_{\pi_{w(f)})]).
$$

Let us denote by

$$
\Lambda_t(w) = e^{- \frac{1}{2} \sum_{f=1}^{q} \tau_{x_{f}} \pi_{w(f)}(x)} \sup_{[S, \pi] \in B_0} \prod_{f=1}^{q} e^{\sum_{f=1}^{q} \tau_{x_{f}} L_f (\varphi_0)([w_f(S)], 1_{\pi_{w(f)})]).
$$
Gathering \((\phi)\) with the last four inequalities yields that
\[
|e^{\sum_{j=1}^{q} t_j \pi_w(f) \psi_{t,1,N}(\{S_0, 0_m\})}| \leq \sum_{\eta \in \mathcal{P}_m} \frac{\nu_1(S_0) \nu_2(S_0)}{ (#\eta - 1)! (m - #\eta)!} \Lambda_t(w) \sup_{1 \leq i \leq m} \|\pi_{wi}\|^2_t \\
\leq \Lambda_t(w) \sup_{1 \leq i \leq m} \|\pi_{wi}\|^2_t \sum_{\nu \in \mathcal{V}_{m-1}} \sum_{k=1}^{m} \frac{\nu_i(S_0)}{(k-1)! (m-k)!} \\
\leq 2^{m-1} \Lambda_t(w) \sup_{1 \leq i \leq m} \|\pi_{wi}\|^2_t \sum_{T \in \mathcal{C}_m} T(S_0).
\]

Then, the following lemma implies the announced bound.

**Lemma 4.5.** For any word \(w \in W_q\), \(t \in \mathbb{R}_+^q\),
\[
\Lambda_t(w) \leq \lambda_t(w).
\]

**Proof.** For any \(n \in \mathbb{N}^*\), let \(CW_1\) be the set of finite tuple \(S\) of words in \(W_1\), with \(\ell(w(S)) = n\). For any \(S \in CW_1(n)\), \((a,b) \in N_{2,w(S)}^\pm\), let us define \(T_{a,b}(w)\) as in 1. of section 3.2.2. For any \(f \in \{1, \ldots, q\}\), we define a linear operator \(\hat{L}\) on \(\mathbb{C}^{W_q}\), by setting for any \(\varphi \in \mathbb{C}^{W_q}\) and \(S \in CW_q\),
\[
\hat{L}(\varphi)(S) = \sum_{(a,b) \in N_{2,w(S),1_{n}}^0,1_{n}^1} \varphi(T_{a,b}^{+}(S)) + \varphi(T_{a,b}^{-}(S)).
\]
For any \(n \in \mathbb{N}^*\), let us denote here by \(\varphi_0 \in \mathbb{C}^{W_1}\) the constant function equal to 1 and set for any \(s \in \mathbb{R}_+\), \(\rho_s(n) = e^{s \hat{L}}(\varphi_0)(x^n_1)\). Notice that for any word \(w \in W_1\), with \(\ell(w) = n\), \(e^{s \hat{L}}(\varphi_0)(w) = \rho_s(n)\). Therefore, for any \(t \in \mathbb{R}_+^q\)
\[
(*) \quad \Lambda_t(w) \leq e^{-\frac{1}{2} \sum_{j=1}^{q} t_j \pi_w(f) \prod_{j=1}^{q} \rho_{t_j}(\pi_w(f)).
\]
According to the definition of the operator \(\hat{L}\), the family of functions \((\rho_s(n))_{n \geq 0}\) satisfies the following differential system: for any \(n \in \mathbb{N}^*\) and \(s \geq 0\),
\[
\frac{d}{ds} \rho_s(n) = n(1 + \sum_{p=1}^{n} \rho_s(p) \rho_s(n - p)),
\]
\(\rho_0(n) = 1\). Let us define a formal power series by setting for any \(z\),
\[
\rho_s(z) = 1 + \sum_{n \geq 1} \rho_s(n) z^n.
\]
Then, for any \(s \geq 0\),
\[
\frac{d}{ds} \rho_s(z) = \rho_s(z) + 2z \rho_s(z) \frac{d}{dz} \rho_s(z).
\]
According to the Lemma 13 of [6], \( (\rho_\star(n))_{n \geq 0} \) is the sequence of moments of a hermitian operator (therein, denoted by \( \Lambda_\star \Lambda_\star^* \)) acting on a separable Hilbert space and, according to Proposition 11 of the same article, with spectrum

\[
[\lambda^-_\star, \lambda^+_\star] = [(2s + 1 - 2\sqrt{s(1 + s)})e^{-s(s+1)/2}, (2s + 1 + 2\sqrt{s(s + 1))}e^{s(s+1)}].
\]

It follows that for all \( n \in \mathbb{N}^* \), \( \rho_\star(n) \leq \lambda^+_\star \) and the result then follows from (*). □

5. Applications

5.1. Asymptotics of the free energies. For any function \( V \in \mathbb{C}^{W_q} \), let us set \( \|V\|_1 = \sum_{w \in W_q} |V(w)| \) and \( \|V\|_\infty = \sup_{w \in W_q} |V(w)| \). We define \( \mathcal{F}_{1,q} = \{ V \in \mathbb{C}^{W_q} : \|V\|_1 < \infty \} \) and \( \mathcal{F}_{0,q} \) the set of functions \( V \in \mathbb{C}^{W_q} \), with \( \#\{w \in W_q : V(w) \neq 0\} < \infty \). For any \( N \in \mathbb{N} \), \( U_1, \ldots, U_q \in \mathbb{U}(N) \) and \( V \in \mathcal{F}_{1,q} \) the following sum converges almost surely and defines a random matrix

\[
V(U_1, \ldots, U_q) = \sum_{w \in W_q} V(w)w(U_f, 1 \leq f \leq q) \in M_N(\mathbb{C}).
\]

**Theorem. 1.1** For \( t \in \mathbb{R}^+ \) and \( V \in \mathcal{F}_{0,q} \), there exists \( r_V > 0 \) and analytic functions \( \psi_{t,V}, (\psi_{t,V,N})_{N \geq 1} \) on \( D_{r_V} = \{ z \in \mathbb{C} : |z| < r_V \} \), such that

\[
e^{\psi_{t,V,N}(z)} = \mathbb{E}[e^{zN\text{Tr}(V(U_{1,1}, \ldots, U_{q,q}))} - N^2\psi_{t,V}(z)] \xrightarrow{} e^{\psi_{t,V}(z)},
\]

as \( N \to \infty \), where the convergence is uniform on compact subset of \( D_{r_V} \).

**Proof of Theorem 1.1.** For any function \( V \in \mathcal{F}_{1,q} \), let us define the matrix \( V_{N,t} = V(U_{1,1}, \ldots, U_{q,q}) \) and \( I_{t,V,N}(z) = N^{-2}\log \mathbb{E}[e^{zN^2\text{Tr}(V_{t,N})}] \). The latter analytic function satisfies on a neighborhood of 0,

\[
I_{t,V,N}(z) = \sum_{w \in W_q} V(w)\varphi_{t,N}(w)z + \sum_{m \geq 2} \prod_{i=1}^{m} \frac{V(w_i)}{m!} \varphi_{t,N}([\{w_1, \ldots, w_m\}, 0_m])z^m.
\]

According to Proposition 3.3, the summand of the two sums converge pointwise. The summand of the first sum is bounded by \( |zV(w)| \), so that this sum converges absolutely towards \( \sum_{w \in W_q} V(w)\varphi_{t}(w)z \). Each coefficient of the second power series is bounded by

\[
\sum_{w_1, \ldots, w_m \in W_q} \prod_{i=1}^{m} \frac{|V(w_i)|}{m!} T(w_1, \ldots, w_m) \leq \max_{a \neq b \in \text{supp}(V)} \{ (\pi_a, \pi_b)_t \} m^{-1} m^{m-2} \|V\|_1^m.
\]

It follows that \( I_{t,V,N}(z) = \sum_{m \geq 1, w_1, \ldots, w_m \in W_q} \frac{z^m \prod_{i=1}^{m} V(w_i)}{m!} \varphi_{t,N}([\{w_1, \ldots, w_m\}, 0_m]) \) is well defined on \( D_{r_V} \) with

\[
\frac{1}{r_V} = \max_{a,b \in \text{supp}(V)} \{ (\pi_a, \pi_b)_t \} e\|V\|_1.
\]
and converges uniformly as $N \to \infty$ on its compact subset towards a limit that we denote by $\varphi_{t,V}(z)$. Let us set for any $V \in \mathbb{C}^W_0$,

$$\eta_V = \sup_{a \in \text{supp}(V)} \|\pi_a\|^2 \sum_{w \in W_q} \lambda_l(w)|V(w)|.$$  

For any $m \geq 1$ and $V$ with $\eta_V < \infty$, according to Lemma 4.4, the sum

$$\psi_{N,m}(V) = \sum_{w_1,\ldots,w_m \in W_q} \frac{\prod_{i=1}^m V(w_i)}{m!} \psi_{t,N}((w_1,\ldots,w_m),0_m)$$

is well defined and satisfies

$$|\psi_{N,m}(V)| \leq 2^{m-1} \frac{m^m}{m!} \eta_V^m.$$  

Thanks to Proposition 3.3, $\psi_m(V) = \sum_{w_1,\ldots,w_m \in W_q} \frac{\prod_{i=1}^m V(w_i)}{m!} \psi_t((w_1,\ldots,w_m),0_m)$ is well defined and satisfies $\psi_{N,m}(V) \to \psi_m(V)$. What is more, if $\psi_{t,V,N}(z)$ denotes the power series with coefficients $(\psi_{N,m}(V))_{m \geq 1}$, then $\psi_{t,V,N}$ is well defined on $D_{r'_V}$ and converges uniformly on its compact subset towards a function $\psi_{t,V}$, with

$$\frac{1}{r'_V} = 2e\eta_V.$$  

To conclude, note that $r'_V > r_V$, so that if $|z| < r'_V < \infty$, $\varphi_{t,V}(z)$ is well defined and the analytic function

$$e^{\psi_{t,V,N}(z)} = \mathbb{E}[e^{zN\text{Tr}(V_N)-N^2\varphi_{t,V}(z)}]$$

converges uniformly towards $e^{\psi_{t,V}(z)}$ on $D_{r'_V}$.  

\textbf{Remark 5.1.} It follows from (12) that if $x_1^{[m]}$ denotes the tuple composed with $m$ copies of $x_1$, and $(U_t)_{t \geq 0}$ a $U(N)$-Brownian motion, then

$$\varphi_{t,N}([x_1^{[m]},0_m]) = (m-1)!t_1^{m-1} \int_{\Delta_{m-1}} \mathbb{E}[\text{Tr}(U_{s_{m-1}t_1}U_{s_{m-2}t_1} \cdots U_{s_{1}t_1})]ds = t_1^{m-1}e^{-\frac{t_1}{2}}.$$  

Therefore, if $V = \delta_{x_1}$, for any $N \in \mathbb{N}^*$,

$$\varphi_{t,N,V}(z) = \varphi_{t,V}(z) = \frac{(e^{zt_1} - 1)e^{-\frac{t_1}{2}}}{t_1}.$$  

For any function $V \in \mathbb{C}^W_0$ and any word $w = x_{i_1}^{\epsilon_1} \cdots x_{i_n}^{\epsilon_n} \in W_q$, with $\epsilon_1,\ldots,\epsilon_n \in \{-1,1\}$, let us set $V^*(w) = V(x_{i_n}^{-\epsilon_n} \cdots x_{i_1}^{-\epsilon_1})$. We say that $V \in \mathbb{C}^W_0$ is symmetric if $V^* = V$. For any $N \in \mathbb{N}^*$, let $(U_{t_1},t_2,\ldots,\tq,U_{t_2},t_3,\ldots,\tq)$ be $q$ independant $U(N)$-Brownian motions. For any symmetric function $V \in \mathcal{F}_{l,q}$ and $t \in \mathbb{R}^*_+$, the random matrix

$$V_{l,N} = V(U_{l,t_1},\ldots,U_{l,t_q})$$

is hermitian and its operator norm is bounded by $\|V\|_1$. In particular, it satisfies $0 < \mathbb{E}[e^{N\text{Tr}(V_{l,N})}] < \infty$. Let $\mu_{t,V}$ be the probability measure on $U(N)^q$, whose
Remark 5.2. For any Cayley tree $T$ in $C_m$, let us write $(d_T(i))_{1 \leq i \leq m}$ for the degree distribution of $T$. For $m \geq 2$, Cauchy-Schwarz inequality yields that for $w_1, \ldots, w_m \in W_q$, \[
abla \sum_{i=1}^{m} \frac{|V(w_i)|}{m!} T_i(w_1, \ldots, w_m) \leq \sum_{w_1, \ldots, w_m \in W_q} \frac{1}{m!} \prod_{i=1}^{m} |V(w_i)| \|\| \pi_{w_i} \|^2 d_T(i).
\]

For any $V \in \mathbb{C}^{W_q}$, according to formula (1) of [4], \[
\sum_{w_1, \ldots, w_m \in W_q} \prod_{i=1}^{m} |V(w_i)| T_i(w_1, \ldots, w_m) \leq \sum_{w_1, \ldots, w_m \in W_q} \prod_{i=1}^{m} \|\| \pi_{w_i} \| \| V(w_i) \| (\sum_{i=1}^{m} \|\| \pi_{w_i} \|)^{m-2}.
\]

In particular, $J_{1,V}(W), J_{2,V}(W) < \infty$, if $V, W \in F_{0,q}$, with \[
\|V\|_{\infty}, \sup_{a \in W_q} \|\| \pi_a \| \| \leq \frac{1}{e}.
\]

Proof. Let $W, V \in F_{1,q}$, with $J(V) < \infty$, the sum of random matrices $W_T V = \sum_{a \in W_q} W(a) w(U_{1,T}, \ldots, U_{q,T})$, converges almost surely. Then, $\mathbb{E}[\frac{1}{2} \text{Tr}(W_T V)]$ and $\text{Var}(\text{Tr}(W_T V))$ are respectively equal to the following absolutely convergent sums, \[
\sum_{w \in W_q} W(w) \varphi_{T,N}(w) + \sum_{m \geq 1, w_1, w_2, \ldots, w_m \in W_q} \frac{1}{m!} W(w) \prod_{k=1}^{m} V(w_k) \varphi_{T,N}([(w, w_1, \ldots, w_m), 0_{m+1}])
\]
and
\[ \sum_{a,b \in W_q} W(a)W(b)\varphi_{t,N}(([a, b], 0_2]) \]
\[ + \sum_{m \geq 1, w_1, \ldots, w_m \in W_q} \frac{1}{m!} W(a)W(b) \prod_{k=1}^{m} V(w_k)\varphi_{t,N}(([a, b, w_1, \ldots, w_m], 0_{m+2})). \]

According to Proposition 4.2, dominated convergence implies that these two sequences have a limit as \( N \to \infty \).

5.2. Central limit theorem. As a consequence of Proposition 3.3, we get the

**Proposition 5.3.** For any \( t \in \mathbb{R}_+^q \), the family \((\text{Tr}(w_{t,N}) - N\varphi_1(w))_{w \in W_q}\) converges towards the centered gaussian field \((\phi_w)_{w \in W_q}\), such that for any \( a, b \in W_q\),
\[ \text{cov}(\phi_w, \phi_{w'}) = \varphi_t(([w, w'], 0_2)). \]

**Proof.** For any word \( w \in W_q \), \( E[N^{-1}\text{Tr}(w_{t,N})] = \varphi_t(w) + O(N^{-2}) \), whereas for any \( m \geq 2 \), \( C_m(\text{Tr}(w_{1.t,N}), \ldots, \text{Tr}(w_{m.t,N})) = N^{2-m}\varphi_{t,N}([w_1, \ldots, w_m], 0_m) \) converges, as \( N \to \infty \), towards \( \varphi_t(([w_1, w_2], 0_2]) \), if \( m = 2 \), and 0, if \( m \geq 3 \).

Let us remark that in the proof of Proposition 4.2, we obtained in formula (12) an expression of the function \( \varphi_{t,N} \) for any \( N \in \mathbb{N}^* \), in terms of its restriction to single words. Specialized to partial words with two blocks, this gives an expression of the covariance of the above field. Let us define a family of \( 3q \) independant \( U(N)\)-Brownian motions \((U_{1.t}, V_{1.t})_{t \geq 0}, (V_{2.t})_{t \geq 0}, \ldots, (U_{q.t}, V_{q.t})_{t \geq 0}, (V_{q+1.t})_{t \geq 0}, (V_{q+2.t})_{t \geq 0}\). For any \( w_1, w_2 \in W_q \) and \( (a, b) \in \mathcal{N}_2,(w_1, w_2) f) \setminus \mathcal{S}_2(w_1, w_2, 0_2) \), let \( \chi : \{1, \ldots, |w_1| \} \to \{1, 2\} \) be the function such that the \( i \)-th letter of \( T_{a,b}((w_1, w_2)) \) belongs to \( w_{\chi(i)} \). If \( T_{a,b}((w_1, w_2)) = x_{i_1}^{e_1} \ldots x_{i_n}^{e_n} \), let us set for any \( r, s \in \mathbb{R}_+^q \),
\[ w_{a,b}(r, s) = \left(U_{i_1} V_{i_1}^{\chi(1)}ight)^{e_1} \ldots \left(U_{i_n} V_{i_1}^{\chi(n)}ight)^{e_n}. \]

According to formula (12),
\[ \text{cov}(\text{Tr}(w_{1.t,N}), \text{Tr}(w_{2.t,N})) = \sum_{1 \leq f \leq q} e_{f} \int_0^1 \sum_{s \in \{-1, 1\}} \frac{1}{N} E[\text{Tr}(w_{a,b}(st, (1-s)t))]ds, \]
where the second sum is over pairs \((a, b) \in \mathcal{N}_2,(w_1, w_2) f) \setminus \mathcal{S}_2(w, w'), 0_2\).

6. Planar Yang-Mills measure

We shall see in this section that representing words as loops in the plane allows to define and study the previous models in one same framework. In the next section, we shall then give analog of Schwinger-Dyson’s equations.
6.1. Paths of finite length. Let us first specify the family of loops we are considering. Let us call parametrized path any Lipschitz function from \([0,1]\) to \(\mathbb{R}^2\), that are either constant or with speed bounded by below. We denote by \(P(\mathbb{R}^2)\) the set of parametrized paths up to bi-Lipschitz increasing reparametrization and call its elements paths. For any path \(\gamma \in P(\mathbb{R}^2)\) with parametrization \(c : [0,1] \rightarrow \mathbb{R}^2\), let us denote its endpoints \(c(0)\) and \(c(1)\) by \(\gamma\) and \(\overline{\gamma}\), and by \(\gamma^{-1}\) the reverse path parametrized by \(t \in [0,1] \mapsto c(1-t)\). For any \(x \in \mathbb{R}^2\), we denote by \(L_x(\mathbb{R}^2)\) the set of paths \(\gamma \in P(\mathbb{R}^2)\) such that \(\gamma = x = \overline{\gamma}\) and call elements of \(L_x(\mathbb{R}^2)\) loops based at \(x\). We set \(L(\mathbb{R}^2) = \bigcup_{x \in \mathbb{R}^2} L_x(\mathbb{R}^2)\). For any loop \(l\) based at some point \(x \in \mathbb{R}^2\) and parametrized by the Lipschitz-continuous map \(\tilde{t} : [0,1] \rightarrow \mathbb{R}^2\), we call non-based loop the induced map \(U \rightarrow \mathbb{R}^2\) up to bi-Lipschitz order preserving one-to-one mapping of \(U\). If \(a\) and \(b\) are two paths such that \(\overline{a} = b\), we denote by \(ab\) the path of \(P(\mathbb{R}^2)\) obtained by concatenation.

6.2. Embedded graphs. We call here embedded graph in the plane the data of a triple of finite sets \(G = (V, E, F_b)\), where \(F_b\) are simply connected domains of the plane with disjoint interior, which boundary is the image of a non-based loops, \(E\) is a set of paths of \(P(\mathbb{R}^2)\) such that the union of their image is the union of boundaries of elements of \(F_b\), \(V\) is the set of endpoints of \(E\) and the graph induced by \(E\) on \(V\) is connected. With this convention, any edge \(e \in E\) is either a simple loop or an injective path of finite length. We denote by \(F_{\infty,G}\) (or simply \(F_{\infty}\)) the interior of \(\mathbb{R}^2 \setminus (\bigcup_{F \in F_b} F)\). We set \(F = F_b \cup \{F_{\infty}\}\) and denote by \(|F|\) the area of any element of \(F \in F_b\) and by \(\partial F\) the non-based loop whose image is the boundary of \(F\) oriented counterclockwise. We shall write \(P(G)\) for the set of paths that are concatenation of elements of \(E\) and \(L(G)\) (and resp. for any \(v \in V\), \(L_v(G)\)) for the set of loops (resp. loops based at \(v\)) in \(P(G)\).

6.3. A free group: reduced loops of an embedded graph. Let us fix an embedded graph \(G\). For any pair of paths \(\gamma_1\) and \(\gamma_2\) of \(P(G)\), let us write \(\gamma_1 \sim \gamma_2\) and say that \(\gamma_1\) and \(\gamma_2\) are equivalent if one can get \(\gamma_1\) from \(\gamma_2\) or vice-versa by adding or erasing paths of the form \(e.e^{-1}\), with \(e \in E\). For any path \(\gamma\), there is a unique element of minimal length in its equivalence class, that the call the reduction of \(\gamma\). The set of reduced paths endowed with the operation of concatenation and reduction forms a groupoid that we denote by \(RP(G)\). For any \(v \in V\), we denote by \(RL_v(G)\) the set of reduced paths that are loops based at \(v\). Endowed with the above multiplication, \(RL_v(G)\) is a free group of rank \(#F_b\) (we shall highlight specific free basis in section 6.6).

6.3.1. Multiplicative functions. For any \(N \in \mathbb{N}^*\), we call a function \(h : L_v(\mathbb{R}^2) \rightarrow U(N)\), multiplicative if for any paths \(a, b \in L(\mathbb{R}^2)\), with \(a = \overline{b}\),

\[
    h(ab) = h(a)h(b).
\]

We denote the space of multiplicative functions by \(\mathcal{M}(L(\mathbb{R}^2), U(N))\) and by \(\mathcal{C}\) the smallest \(\sigma\)-fields such that for any \(l \in L(\mathbb{R}^2)\), \(h \in \mathcal{M}(L(\mathbb{R}^2), U(N)) \mapsto h(l) \in \mathcal{C}\).
$U(N)$ is measurable, where $U(N)$ is endowed with its Borel $\sigma$-fields. We define as well, for any $v \in \mathcal{V}$, the set $\mathcal{M}(L_v(G), U(N))$ of multiplicative functions on $L_v(G)$. For any choice of of basis $\Lambda$ of $\text{RL}_v(G)$, there is a bijection
\[ \Theta_{\Lambda} : \mathcal{M}(L_v(G), U(N)) \rightarrow U(N)^{\#F_b} \]
\[ h \mapsto (h(\lambda))_{\lambda \in \Lambda}. \]

It is easy to see that the preimage $\sigma$-field of the Borel $\sigma$-field by this map does not depend on this choice and we denote it by $\mathcal{C}_G$.

6.3.2. Lassos basis and discrete Yang-Mills measure. For any loop $v \in \mathcal{V}$ and $l \in L_v(G)$, we say that $l$ is a lasso based at $v$, if $l = a \partial_\lambda F a^{-1}$ where $a \in P(\mathbb{R}^2)$, with $a = v$, $\lambda$ is a vertex in the image of $\partial F$ and $\partial_\lambda F$ is the rooting of $\partial F$ at $\lambda$. We shall see in the next section that there exists basis of $\text{RL}_v(G)$ formed with lassos. Let $\Lambda = (\lambda_F)_{F \in F_b}$ be a lassos free basis of $\text{RL}_v(G)$, and let $YM_\Lambda^G$ be the law of $\Theta_{\Lambda}^{-1}((U_{(F_1)_{F \in F_b}}))$ on $(\mathcal{M}(L_v(G), U(N)), \mathcal{C}_G)$, where $((U_{(F_1)_{F \in F_b}}))$ is a family of independant Brownian motions on $U(N)$.

Lemma 6.1 ([27, 18]). i) For any lasso free basis $\Lambda, \Lambda'$ of $\text{RL}_v(G)$, $YM_\Lambda^G = YM_{\Lambda'}^G$. We denote this law by $YM^v$.

ii) If $G'$ is another embedded graph with $P(G') \subset P(G)$ and $R_{G'}^G : \mathcal{M}(L_v(G)) \rightarrow \mathcal{M}(L_v(G'))$ denotes the restriction map, then
\[ R_{G'}^G YM^v_G = YM^v_{G'}. \]

We denote by $(H_t)_{t \in L_v(G)}$ the canonical process on $\mathcal{M}(L_v(G), U(N))$ with law $YM_G$. The first point follows from the invariance of the law of the $U(N)$-Brownian motion by adjunction and the following result.

Theorem ([22]). Let $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ be two free basis for the free group $F_n$, such that $x_i$ is conjugated to $y_i$ for all $i \in \{1, \ldots, n\}$. Then $X$ can be obtained from $Y$ by a sequence of transformations of the kind $(u_1, \ldots, u_n) \mapsto (u'_1, \ldots, u'_n)$ where, for some $i, j$, $u'_i = u_i u_j u_j^{-1}$ or $u_j^{-1} u_i u_j$ and $u'_k = u_k$ for $k \neq i$.

The second point of Lemma 6.1 requires a proof (see [27, 18]) that we won't reproduce here; an argument goes as follows. Let $G' = (\mathcal{V}', \mathcal{E}', \mathbb{F}'_b)$ be an embedded graph with $v \in \mathcal{V}'$ and $P(G') \subset P(G)$. Assume that there exists $F \in \mathbb{F}_b$ and $F_1, F_2 \in \mathbb{F}_b$ with $\mathcal{F} = F_1 \cup F_2$. If $\lambda$ and $\lambda', \lambda''$ are lassos in $L_v(G')$ and $L_v(G)$, with faces $F, F_1, F_2$ such that $\lambda = \lambda_1 \lambda_2$, then under $YM_G$, $H_\lambda$ has the same law as $U_{(1_{F_1} U_{2_{F_2}})}$, where $(U_{(1_{F_1} U_{2_{F_2}})})$ are two independant $U(N)$ brownian motions. Hence, it has the same law as $U_{(1_{F_1} U_{2_{F_2}})}$, that is the law of $H_\lambda$ under $YM_G$.

6.4. Yang-Mills measure. Let $d_1$ and $d_b$ be the two distances on $P(\mathbb{R}^2)$ defined in the following way: for any path of paths $\gamma_1, \gamma_2 \in P(\mathbb{R}^2)$, parametrized by $c_1, c_2 : [0, 1] \rightarrow \mathbb{R}_2$,
\[ d_1(\gamma_1, \gamma_2) = |\gamma_1 - \gamma_2| + \int_0^1 |c_1'(t) - c_2'(t)| \, dt \]

and

\[ d_\ell(\gamma_1, \gamma_2) = \inf_{\phi, \psi} \sup_{r, s \in [0, 1]} \{ |c_1 \circ \phi(r) - c_2 \circ \psi(s)| \} + |\ell(c_1) - \ell(c_2)|, \]

where we have denoted by \( \ell(c) \) the length of a path \( \gamma \in \mathcal{P}(\mathbb{R}^2) \) and the infimum is taken over all increasing bijections of \([0, 1]\). It has been proved in [29] that \( d_1 \) and \( d_{\ell} \) induce the same topology on \( \mathcal{P}(\mathbb{R}^2) \), though \( \mathcal{P}(\mathbb{R}^2), d_1 \) is complete and \( \mathcal{P}(\mathbb{R}^2), d_\ell \) is not. In the following, we shall only use this topology on closed space \( L_0(\mathbb{R}^2) \) and say that a sequence of paths \( (l_n)_{n \geq 0} \) converges to \( l \) if \( d_{\ell}(l_n, l) \to 0 \) and \( l_n = \ell \) for every \( n \in \mathbb{N}^* \). For any embedded graph \( G \), with \( v \) as a vertex, let us denote by \( \mathcal{R}_G^v : \mathcal{M}(L(\mathbb{R}^2), U(N)) \to \mathcal{M}(L_n(\mathbb{G}), U(N)) \) the restriction mapping. This application is measurable with respect to the \( \sigma \)-fields \( \mathcal{C} \) and \( \mathcal{C}_G \). It is shown in [29] that the measures \( YM_G \), with \( G \) embedded graphs, can be extended in the following way.

**Theorem 6.2.** There exists a probability measure \( YM_N \) on \((\mathcal{M}(L(\mathbb{R}^2), U(N)), \mathcal{C})\) such that, for any embedded graph \( G, v \in \mathbb{R}^2 \),

\[ \mathcal{R}_G^v, YM_N = YM_G. \]

Let \((H_l)_{l \in L(\mathbb{R}^2)}\) be a random multiplicative function with law \( YM_N \). If \((l_n)_{n \geq 0}\) is a sequence of paths in \( L(\mathbb{R}^2) \) that converges to \( l \), then, under \( YM, H_{l_n} \) converges in probability towards \( H_l \). If \( h \) is an area preserving diffeomorphism of \( \mathbb{R}^2 \), then the process \((H_{h(l)})_{l \in L(\mathbb{R}^2)}\) and \((H_l)_{l \in L(\mathbb{R}^2)}\) have the same law.

For any \( l \in L(\mathbb{R}^2) \) and \( N \in \mathbb{N}^* \), the random variable \( \frac{1}{N} \text{Tr}(H_l) \) is called a Wilson loop.

### 6.5. \( U(1) \)-Yang-Mills measure.

Let us consider the commutative case, \( N = 1 \). Let \( G \) be an embedded graph in the plane, \( v \) a vertex of \( G \). For any loop \( l \in RL_v(\mathbb{G}) \), its winding number function defines an element \( n_l \in L^2(\mathbb{R}^2) \). Let us fix a family of lassos \((\lambda_F)_{F \in \mathcal{F}}\), of \( G \). Under \( U(1) \)-YM measure, \((H_{\lambda_F})_{F \in \mathcal{F}}\) has the same law as \#\( \mathcal{F}_v \) independent marginals of \( U(1) \)-Brownian motion \((U_{F_i})_{F_i \in \mathcal{F}}\). Let \( W \) be a white noise on the plane with intensity given by the Lebesgue measure. The random family \((H_l)_{l \in RL_v(\mathbb{G})}\) is equal to \((\prod_{F \in \mathcal{F}} H_{\lambda_F}^{\gamma(F)})_{l \in RL_v(\mathbb{G})}\) and has the same law as \((\text{exp}(iW(n_l)))_{l \in RL_v(\mathbb{G})}\). For any loop \( l \in L(\mathbb{R}^2) \), according to Banachoff-Pohl inequality (see Lemma 6.7), its winding number defines an element \( n_l \in L^2(\mathbb{R}^2) \). Moreover, according to Theorem 3.3.1. of [29], the map \( l \in L_0(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \) is continuous, so that, if \((l_n)_{n \geq 0}\) is a sequence of \( L_0(\mathbb{R}^2) \) that converges for \( d_1 \) topology to a loop \( l \in L_0(\mathbb{R}^2) \), the sequence of random variables \( \text{exp}(iW(n_{l_n})) \) converges to \( \text{exp}(iW(n_l)) \) in distribution. Hence, the process \((H_l)_{l \in L(\mathbb{R}^2)}\) introduced in Theorem 6.2 has the same law as \((\text{exp}(iW(n_l)))_{l \in L(\mathbb{R}^2)}\). Moreover, the same argument and Lemma 2.2 yield the following Lemma.
Lemma 6.3. For any integer \( N \in \mathbb{N}^* \), under \( Y M_N \), the law of \( (\det(H_1))_{l \in \text{La}(\mathbb{R}^2)} \) and \( (\exp(iW(n_l)))_{l \in \text{La}(\mathbb{R}^2)} \) is \( Y M_1 \).

6.6. Two free basis of the group of reduced loops. We shall present two families of free basis of \( \text{RL}_n(\mathbb{G}) \). Let \( \mathbb{E}^+ \) be an orientation of \( \mathbb{G} \), that is a subset of \( \mathbb{E} \) such that for any \( e \in \mathbb{E}, e \) or \( e^{-1} \in \mathbb{E}^+ \). Let us also fix a spanning tree \( T \) of the graph \( \mathbb{G} \) and set \( T^+ \) the collection of positively oriented edges of \( T \). We denote by \( e : \mathbb{F}_b \rightarrow \mathbb{E}^+ \setminus T^+ \) the unique bijection such that for any face \( F \in \mathbb{F}_b \), \( e(F) \) is bounding the face \( F \). For any \( e \in \mathbb{E}, \) bounding a face \( F \), we denote by \( \partial_e F \) the loop starting with \( e \) and bounding \( F \). For any \( x, y \in \mathbb{V} \), we denote by \( [x, y]_T \) the unique path in \( T \) going from \( x \) to \( y \). Let us now define two families of loops by setting for any edge \( e \in \mathbb{E}, \)

\[
\beta_e = [v, e]_T, e, e^\prime \in \mathbb{E}^+ \setminus T^+.
\]

and for any face \( F \in \mathbb{F}_b, \)

\[
\lambda_F = [v, e(F)]_T \partial_e(F) F[e(F), v]_T.
\]

It is easy to see that \( \text{RL}_n(\mathbb{G}) \) is a free group of rank \#\( \mathbb{F}_b \) with free basis \( (\beta_e)_{e \in \mathbb{E}^+ \setminus T^+} \). For any loop \( l \in \text{L}(\mathbb{G}) \),

\[
(13) \quad l \sim \beta_{e_1} \beta_{e_2} \cdots \beta_{e_n},
\]

where \( e_1, \ldots, e_n \) are the edges in \( \mathbb{E} \setminus T \), used by the loop \( l \) in this order. In [27], it is proved that the second family of loops is another free basis of \( \text{RL}_n(\mathbb{G}) \).

Lemma 6.4 ([27]). The family \( (\lambda)_{F \in \mathbb{F}_b} \) is a free basis of \( \text{RL}_{n_0}(\mathbb{G}) \)

Let us give the change of basis between the two basis of \( \text{RL}_n(\mathbb{G}) \) defined above. Denote by \( \hat{\mathbb{G}} = (\hat{\mathbb{V}}, \hat{\mathbb{E}}) \) the graph dual to \( \mathbb{G} \): its vertices are indexed by the set of faces \( \mathbb{F} \), whereas its edges are indexed by the set \( \mathbb{E} \) such that two faces \( F_1 \) and \( F_2 \) of \( \mathbb{G} \) are neighbours in \( \hat{\mathbb{G}} \), if their boundaries share a common edge. For any edge \( e \in \mathbb{E}, \) we denote by \( F_L(e) \) and \( F_R(e) \) the edges on the left and on the right of \( e \) and denote by \( \hat{e} \) the edge \((F_L(e), F_R(e)) \in \hat{\mathbb{E}} \). Let \( \hat{T} = \mathbb{E} \setminus T \) be the dual spanning tree of \( T \), considered as rooted at the infinite face \( F_\infty \). We fix an orientation \( \mathbb{E}^+ \) of \( \mathbb{G} \), such that for any edge \( e \in \mathbb{E}^+ \setminus T \), the distance in \( \hat{T} \) to the root \( F_\infty \) decreases along \( \hat{e} \). Note that for any bounded face \( F, F_L(e(F)) = F \). For any face \( F \), we denote by \( \hat{T}_F \) the subtree of \( \hat{T} \) with root \( F \) and with vertices the set of childs of \( F \) in \( \hat{T} \), that we denote by \( C_F \). For any edge \( e \in \mathbb{E}^+ \setminus T^+ \), denote by \( \preceq_e \) the order on \( \hat{T}_{F_L(e)} \) induced by the time of the first visit by the clockwise contour process boarding the dual tree \( \hat{T} \) starting along the left of \( \hat{e}^{-1} \), as is displayed with an example in figure 3. Then, for any edge \( e \in \mathbb{E}^+ \setminus T, \)

\[
\lambda_{F_L(e)} = \beta_e \left( \prod_{F \in C_{F_L(e)}} \beta_{e(F)} \right)^{-1}
\]
Figure 3. We represent with black lines a spanning tree of a square grid together with its dual with dashed red lines. We also display an edge \( e \) of \( \mathbb{E}^+ \setminus T \) in blue together with the order \( \preceq_e \) on \( C_{F_e(e)} \), by numbering its elements and drawing in black the clockwise contour process around \( \hat{T}_{F_e(e)} \).

and

\[
\beta_e = \prod_{F \in \hat{T}_{F_e(e)}} \lambda_F,
\]

where \( \prod \) denotes the product of terms increasing for \( \preceq_e \), from the left to the right. For any loop \( l \in L(\mathbb{G}) \), we denote by \( w_l \) the word with letters \( (\lambda_F)_{F \in \mathbb{F}_b} \) and their inverse such that \( l \sim w \), given by the decomposition (13) and the inversion formula (14). Using notation of section 3.2.1, for any face \( F \in \mathbb{F}_b \) and any complex number \( z \in F \), \( n_{w_l}(F) = n_l(z) \).

6.6.1. Complexity of lassos decompositions: We can now give an estimate on the complexity of the decomposition of a loop in \( \mathbb{G} \) in a word of lassos associated to a spanning tree \( T \). We display in that section results of [27] in a slightly different form adapted to our purpose. For any subset \( E \subseteq \mathbb{E} \) and any loop \( l \in L(\mathbb{G}) \), denote by \( \mathcal{L}_E(l) \) the number of times that \( l \) uses the edges of \( E \) or \( E^{-1} \).

**Lemma 6.5.** Let \( l \in L(\mathbb{G}) \) be a loop of \( \mathbb{G} \). Then, for any face \( F \in \mathbb{F}_b \),

\[
\tilde{n}_{\lambda_F}(w_l) = \mathcal{L}_{[F,F_{\infty}]_T}(l).
\]
Proof. The decomposition (13) and (14) yield the equality. \qed

From now on, we shall choose the spanning tree in the following way.

**Lemma 6.6.** There exists a spanning tree $T$ of $G$ such that for any face $F \in \mathcal{F}_b$, $$d_T(F, F_\infty) = d(F, F_\infty).$$

For any loop $l$ of an embedded graph, we shall control the maximal amperean area $A_t(w_l) = \sum_{F \in \mathcal{F}} |F| \bar{n}_{w_l}(F)^2$ with the length of the loop $\ell(l)$. For any loop $l \in \mathcal{L}(\mathbb{R}^2)$, denote by $n_l$ the winding number function of $l$. The Amperean area of $l$ is the integral

$$A(l) = \int_{\mathbb{R}^2} n_l(x)^2 dx.$$

**Lemma 6.7** (Banchoff-Pohl inequality, [2]). For any loop of finite length $l \in \mathcal{L}(\mathbb{R}^2)$, $$A(l) \leq \pi \ell(l)^2.$$ Note that if $\bar{n}_l = \pm n_l \in \mathbb{Z}^\ell$, that is, if $l$ winds only to the left or only to the right, then the Banchoff-Pohl inequality gives the expected bound. To treat more general loops, we need the following lemma.

**Lemma 6.8.** There exists a loop $\bar{l} \in \mathcal{L}(G)$, which does not use any edge twice, such that for any face $F \in \mathcal{F}$ and $z \in F$, $$n_l(z) = d(F, F_\infty).$$

**Lemma 6.9.** Fix $p \in \mathbb{N}^*$ and loops $l_1, \ldots, l_m \in \mathcal{L}(G)$ in an embedded graph $G$ such that the union of the edges of $l_1, \ldots, l_m$ is $\mathcal{E}$. Let $l \in \mathcal{L}(\mathbb{R}^2)$ be a loop that uses each edge at most $p$ times. Then $$A(w_l) \leq \pi p^2 (\ell(l_1) + \ldots + \ell(l_m))^2.$$ Proof. The assumptions together with Lemma 6.5 yield that for any face $F \in \mathcal{F}_b$, $$\bar{n}_{w_l}(F) \leq p d(F, F_\infty).$$ Let us now choose a loop $\bar{l}$ as in Lemma 6.8. Then, $A_t(w_l) \leq p^2 A(\bar{l})$, $\ell(\bar{l}) \leq \ell(l_1) + \ldots + \ell(l_m)$ and Banchoff-Pohl inequality yields the expected bound. \qed

6.7. Asymptotics of Wilson loops as $N \to \infty$. We call a finite family of loops of $\mathcal{L}(\mathbb{R}^2)$ a skein. We denote the set of skein by $\mathcal{S}(\mathbb{R}^2)$ and we endow it with the topology associated to the product topology. If $\mathcal{S} = \{l_1, \ldots, l_m\}$ is a skein, let us set $\mathcal{S}^* = \{l_1^{-1}, \ldots, l_m^{-1}\}$ and for any $N \in \mathbb{N}^*$, $$\Phi_N(\mathcal{S}) = N^{m-2} C_m(\text{Tr}(H_{l_1}), \ldots, \text{Tr}(H_{l_m})), $$ where the cumulants are with respect to the measure $YM_N$. Observe that the law of the unitary Brownian motion is invariant under complex conjugation. Hence, for any skein $\mathcal{S} \in \mathcal{Sk}(\mathbb{R}^2)$ $$\Phi(\mathcal{S}) = \overline{\Phi(\mathcal{S})} = \Phi_m(\mathcal{S}^*)$$
is real-valued. Let us denote by \( \mathcal{E}_A \) the set of skeins of piecewise affine loops of \( \mathbb{R}^2 \) with transverse intersections of multiplicity at most 2. We call elements of \( \mathcal{E}_A \) affine skeins.

**Proposition 6.10.** For any affine skein \( S \in \mathcal{E}_A \), the sequence \( \Phi_N(S) \) converges as \( N \to \infty \). We denote its limit by \( \Phi(S) \).

**Proof.** For any affine skein \( S \), there exists an embedded graph \( G \) such that the element of \( S \) belongs to \( L(G) \). Choosing an arbitrary base point \( v \in V \) and decomposing each loops in a lassos basis yields that under \( Y M_N \), the random family \( (H_i)_{i \in S} \) has the same law a collection of words in marginals of independant \( U(N) \) Brownian motions. Therefore, the Proposition 3.3 implies the result. \( \square \)

**Proposition 6.11.** Let us fix a constant \( K > 0 \). For any skein \( S \in \mathcal{E}_A \) of loops of length smaller than \( K > 0 \) and taking their values in a ball center of radius \( K \),

\[
|\Phi_N(S) - \Phi(S)| \leq \frac{36m^2K^2}{N^2}e^{36\pi m^2K^2}.
\]

**Proof.** Let us assume that \( S = \{l_1, \ldots, l_m\} \) is a family of loops in \( \mathcal{E}_A \) all based at 0. Let \( G_S = (V_S,E_S,F_S) \) be the embedded graph with vertices the set of intersection points of the elements of \( S \) and with edges the restriction of elements of \( S \) between points of intersection. Lemma 6.9 and the second inequality of Lemma 4.1, for \( k = 2 \), imply that

\[
|\Phi_N(S) - \Phi(S)| \leq \frac{4\pi}{N^2}(\ell(l_1) + \cdots + \ell(l_m))^2 e^{4\pi m(\ell(l_1) + \cdots + \ell(l_m))}. \tag{15}
\]

Consider now \( (l_1, \ldots, l_m) \in \mathcal{E}_A \) satisfying the assumption of the Theorem. For any \( \epsilon > 0 \), let us choose piecewise affine paths \( \gamma_1, \ldots, \gamma_m \) such that \( \gamma_i = l_i \), \( \gamma_i = 0 \), \( \ell(\gamma_i) \leq K(1 + \epsilon) \) for any \( i \{1, \ldots, m\} \) and \( \{\gamma_i l_i \gamma_i^{-1}, i \in \{1, \ldots, m\}\} \) is an affine skein. The application of (15) to \( \{\gamma_i l_i \gamma_i^{-1}, i \in \{1, \ldots, m\}\} \) yields the announced inequality.

This result allows then to extend the function \( \Phi \) to all of \( \text{Sk}(\mathbb{R}^2) \). Surprisingly, an argument analog to the proof Theorem 5.14. of [27] applies as well to the higher order case.

**Theorem 6.12.** For any skein \( S \in \text{Sk}(\mathbb{R}^2) \), the sequence \( \Phi_N(S) \) converges as \( N \to \infty \). We denote its limit by \( \Phi(S) \). The function \( \Phi \) is a real-valued continuous function on \( \text{Sk}(\mathbb{R}^2) \). If \( h \) is an area-preserving diffeomorphism of \( \mathbb{R}^2 \), for any \( S \in \text{Sk}(\mathbb{R}^2) \), \( \Phi(h(S)) = \Phi(S) \).

We shall also call the function \( \Phi : \text{Sk}(\mathbb{R}^2) \to \mathbb{R} \) planar master field.

**Proof.** For any \( K > 0 \), let \( \text{Sk}_K \) (resp. \( \mathcal{E}_K \)) be the set of affine skeins with elements included in the ball of radius 0 and with length less than \( K \). As \( \cup_{K > 0} \text{Sk}_K = \text{Sk}(\mathbb{R}^2) \), it is enough to prove the result on \( \text{Sk}_K \). The set \( \mathcal{E}_K \) is dense in \( \text{Sk}_K \). Indeed, any
loops $L(\mathbb{R}^2)$ can be approximated by its linear interpolation, which itself can be approached by affine loop with a finite number of intersections. According to Theorem 6.2, for any $N \geq 1$, the function $\Phi_N$ is continuous on $Sk_K$. Moreover, Proposition 6.11 shows that $\Phi_N$ converges uniformly towards $\Phi$ on $E_K$. Therefore, $\Phi_N$ converges uniformly towards the unique continuous extension $\tilde{\Phi}$ to $Sk_K$ of $\Phi$.

A consequence of this Theorem is that for any $m \geq 3$, and any loops $l_1, \ldots, l_m$ in $L(\mathbb{R}^2)$, under $YM_N$, 

$$C_m(\text{Tr}(H_{l_1}), \ldots, \text{Tr}(H_{l_m})) = N^{2-m} \Phi_N(l_1, \ldots, l_m) \to 0,$$

as $N \to \infty$. The following theorem follows.

**Theorem 6.13.** Under $YM_N$, the random family $(\text{Tr}(H_l) - \mathbb{E}[\text{Tr}(H_l)])_{l \in L(\mathbb{R}^2)}$ converges weakly as $N \to \infty$, towards a Gaussian field $(\phi_l)_{l \in L(\mathbb{R}^2)}$, such that for any $a, b \in L(\mathbb{R}^2)$, $\text{cov}(\phi_a, \phi_b) = \Phi(\{a, b\})$. If $(l_n)_{n \geq 0}$ is a fixed sequence of loops in $L(\mathbb{R}^2)$ that converges towards $l \in L(\mathbb{R}^2)$, then $\phi_{l_n} \to \phi_l$ in distribution. If $h$ is an area preserving diffeomorphism of $\mathbb{R}^2$, the process $(\phi_{h(l)})_{l \in L(\mathbb{R}^2)}$ has the same law as $(\phi_l)_{l \in L(\mathbb{R}^2)}$.

It is very tempting to apply the result of section 5.1 to study the Yang-Mills measure with a potential. One can obtain an improvement of Proposition 6.11 thanks to Lemma 4.4. Though, at this point, arguing as in Theorem 6.12, the present form 4.4 would only allow to study a very restricted class loops and potentials.

### 6.8. Small area limit

For any $\alpha > 0$ and any loop $l \in L(\mathbb{R}^2)$, denote by $\alpha.l$ the image of $l$ by the dilatation of rate $\alpha$, centered at 0. If $S = \{l_1, \ldots, l_m\}$ is a skein, $\alpha.S = \{\alpha.l_1, \ldots, \alpha.l_m\}$. The following proposition shows that, as $\alpha \to 0$, all the quantities defined above have the same behavior as $\alpha \to 0$.

**Proposition 6.14.** The following Taylor expansions are true for any $N \in \mathbb{N}^*$. As $\alpha \to 0$, for any loop $l \in L(\mathbb{R}^2)$,

$$\Phi_N(\sqrt{\alpha}.l) = 1 - \frac{\alpha}{2} \int_{\mathbb{R}^2} n_l^2(x) dx + O(\alpha^2) = \Phi(\sqrt{\alpha}.l) + O(\alpha^2)$$

and for any skein $S$ with at least two loops,

$$\Phi_N(\sqrt{\alpha}.S) = (-\alpha)^{#S-1} \sum_{T_s} \prod_{\{l_1, l_2\} \in T_s} \int_{\mathbb{R}^2} n_{l_1}(x)n_{l_2}(x) dx + O(\alpha^{#S})$$

$$= \Phi(\sqrt{\alpha}.S) + O(\alpha^{#S}),$$

where the sum is over connected graph with vertices $S$ and $#S - 1$ edges. In both cases, there exists a positive continuous function $b$, independant of $N$, such that $O(\alpha^{#S}) \leq \alpha^{#S} b(\sum_{l \in S} l(l))$. 
Proof. If $S \in \mathcal{E}_A$, the assertion is a direct consequence of Proposition 4.2. Continuity of the functions $\Phi_N, \Phi$ and $b$ allows then to conclude. \qed

A direct consequence is the following

**Corollary 6.15.** Let $W$ be a white noise on $\mathbb{R}^2$, with intensity given by the Lebesgue measure. As $t \to 0$, the Gaussian field $(t^{-1/2} \text{Tr}(H_l) - N\Phi(t,l))_{l \in \mathbb{L}(\mathbb{R}^2)}$, under $YM_N$, converge in distribution towards the gaussian field $(iW(n))_{n \in \mathbb{L}(\mathbb{R}^2)}$.

7. **Makeenko-Migdal equations**

We shall now address the problem of the computation and characterization of the master field. We say that $S \in \text{Sk}(\mathbb{R}^2)$ is a regular skein if its elements are smooth loops with transverse intersection of order 2, denote by $\text{Sk}_r(\mathbb{R}^2)$ the set of regular skeins and by $\overline{\text{Sk}}_r(\mathbb{R}^2)$ its quotient under diffeomorphisms. For any integer $n$, the set of equivalence classes of skeins with less than $n$ intersections is finite. Thanks to its invariance property under area-preserving diffeomorphisms and to its continuity, the master field is characterized by its value on $\overline{\text{Sk}}_r(\mathbb{R}^2)$ and yields functions indexed by $\overline{\text{Sk}}_r(\mathbb{R}^2)$, that we show how to compute it inductively.

7.1. **Makeenko-Migdal equations for the master field on skeins.** For any skein $S$, let us denote by $W_S$ the expectation $E_{YM_N} \left[ \prod_{l \in S} \text{Tr}(H_l) \right]$, we call this function a Wilson skein and say it is regular whenever the associated skein is. In view of the definition of discrete Yang-Mills measure, one may try to compute the master field of higher order of a regular skein $S$ using Itô formula to yield a first order differential system for the family $(W_S)_{S \in \text{Sk}_r(\mathbb{R}^2)}$, with areas of the faces of a graph $G$ containing $S$ as variables. However, this differential system yields at first sight non-regular Wilson skeins $W_S$ on $\mathcal{M}(\mathcal{P}(G), U(N))$, as features the example 7.1.

**Example 7.1.** Consider a loop $l$ that winds three times around the origin. Let us name the faces $A,B$ and $C$ and choose a lasso basis $(l_A,l_B,l_C)$ according to a spanning tree as illustrated in figure 4 in dashed lines. In this basis, the loop is decomposed as $l = l_C l_B l_A$.

Using Itô formula as described in Lemma 2.6 and differentiating with respect to the area of the faces $C$ and $B$ yields $\frac{d}{d|B|} W_l = -\frac{W_l}{2}$ and

$$\mathcal{N} \left( \frac{d}{d|B|} (W_l) + W_l \right) = -W_{l_B l_A} W_{l_C l_B l_A} = -W_{l_A l_B l_A} W_{l_C l_B l_A}.$$

These first two derivatives can be expressed in terms of regular Wilson skeins. However, the derivative with respect to the face of index 3 yields terms that do not

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\(^5\text{We warn the reader that these functions are not normalized as they can be in the literature, so that, with this conventions, }W_{\text{cst}} = N.\)
seem to be polynomials of regular Wilson skeins:

\[ N \left( \frac{d}{d|C|} (W_i) + \frac{3}{2} W_i \right) = -W_{l_Bl_A} W_{l_Cl_A} - W_{l_Cl_B} - W_{l_A} W_{l_B} W_{l_Cl_B}. \]

For any regular skein \( S \), one must therefore face the problem of finding a closed system of Wilson skeins containing \( W_S \). The system of equations given by Lemma 2.6 gives such a system, but its size happens to grow exponentially with the number of faces of the original skein (see section 6.8 of [27], where the smallest closed system obtained is made of what is called Wilson garland). The Makeenko-Migdal equation solves this problem and gives linear combinations of area derivatives operators that preserve the set of function indexed by skeins, so that the size of the system grows as a polynomial in the number of faces. Let \( S \) be a skein and \( x \) be a point of intersection of its elements (between themself or each other). Let us denotes by \( S_x \) the skein composed with the same loops as \( S \) except for the loop or the pair of loops containing \( x \) that is replaced respectively by the pair of loops or the loop that instead of going straight along the same strand of \( S \), turns at the point \( x \) using the other strand with the same orientation (see figure 5).
The following proposition is proved in [27], in a more general framework and relies on integration by parts applied to the product of a function on \( \mathcal{M}(\text{P}(G), U(N)) \) with the density of the discrete Yang-Mills measure. We provide here another proof relying on the decomposition in lassos described in section 6.6 and on the invariance of the Brownian motion by adjunction. Let us fix a regular skein \( \mathcal{S} \) and an embedded graph \( G \) such that elements of \( \mathcal{S} \) belongs to \( \text{P}(G) \).

**Proposition 7.2** (Makeenko-Migdal equation). Let \( F_1, \ldots, F_4 \) be the four faces of \( G \) around a point of intersection \( x \in \mathcal{V} \) of \( \mathcal{S} \) in a cyclic order and such that \( F_1 \) is the face bounded by the two edges of \( \mathcal{S} \) leaving \( x \). Then,

\[
(16) \quad \left( \frac{d}{d|F_1|} - \frac{d}{d|F_2|} + \frac{d}{d|F_3|} - \frac{d}{d|F_4|} \right) \mathcal{E}_{YM,N} \left[ \prod_{l \in \mathcal{S}} \text{Tr}(H_l) \right] = \frac{1}{N} \mathcal{E}_{YM,N} \left[ \prod_{l \in \mathcal{S}_x} \text{Tr}(H_l) \right].
\]

For any skein \( \mathcal{S} \), let us set \( n_{\mathcal{S}} = \sum_{l \in \mathcal{S}} n_l \). Notice that for \( N = 1 \), the equality (16) is equivalent to the fact

\[
n_{\mathcal{S}}(F_1)^2 - n_{\mathcal{S}}(F_2)^2 + n_{\mathcal{S}}(F_3)^2 - n_{\mathcal{S}}(F_4)^2 = -2.
\]

**Proof.** Let us denote by \( \mu_x \) the operator \( \frac{d}{d|F_1|} - \frac{d}{d|F_2|} + \frac{d}{d|F_3|} - \frac{d}{d|F_4|} \), where the faces are numbered as in the Proposition. The strategy of our proof is to choose a spanning tree of \( G \) to decompose the loops of the skein as described in section 6.6 and to use Lemma 3.2 in such a way that terms on the left-hand-side of (16) cancel themselves so that a single cut and join transformation contributes.

**A differential equation indexed by skeins:** Let us fix a vertex \( v \in \mathcal{V} \), a spanning tree \( T \) of \( G \) and choose the lassos basis \( \Lambda \) of \( \text{RL}_v(G) \) associated to \( T \) and \( v \), as in section 6.6. Decompose each loop \( l \in \mathcal{S} \) as in (13) and use the inversion formula (14) \(^6\) as is done in the example 7.1 for a single loop. Let us order arbitrarily the elements of \( \mathcal{S} \) and denote by \( \mathcal{S} = (w_1, \ldots, w_m) \) the family of words in lassos associated to element of \( \mathcal{S} \), set \( w = w_1 \ldots w_m \) and \( n = t(w) \). Let us denote by \( t \in \mathbb{R}^+_x \), the vector of bounded faces area. According to the definition of the discrete Yang-Mills measure given in section 6.3.2 and recalling the notation of section 3.2.3,

\[
W_{\mathcal{S}} = K_t(S, 1_{\#S}).
\]

By assumption, each edge \( e \in \mathcal{E}^+ \) is used at most once by a loop of \( \mathcal{S} \), we set \( \epsilon_e = -1 \), if a loop of \( \mathcal{S} \) goes backwards along \( e \) and \( \epsilon(e) = 1 \), otherwise (in the example 7.1, \( \epsilon_e = 1 \) for any \( e \in \mathcal{E}^+ \)). Therefore, in the decompositions of \( w_1, \ldots, w_m \), each letter corresponds to a face \( F \) and an edge \( e \in [F, F_\infty]_T \) visited by a loop of \( \mathcal{S} \). Let us denote by \( \Gamma_T \) the set \( \{(e, F) \in (\mathcal{E}^+ \setminus T^+) \times \mathcal{F} : e \in [F, F_\infty] \text{ visited by } \mathcal{S} \} \).

\(^6\)Note that the words obtained may be non reduced, for example, if \( A \) is the only child of \( B \) and is a leaf in \( T \), then this decomposition is \( \lambda_B = \beta_{e(B)} \beta_{e(A)}^{-1} = \lambda_B \lambda_A \lambda_A^{-1} \).
The ordering of the elements of $\mathcal{S}$ and their decomposition in the basis $\Lambda$ yield an order $\preceq$ on $\Gamma_T$:

$$w = \prod_{(e,F) \in \Gamma_T} \lambda_{F}^{\epsilon(e)},$$

where the product is from left to right according to $\preceq$. The element $(e,F) \in \Gamma_T$ are ordered with a lexicographic order $\preceq$ taking into account first the number of the loop visiting $e$, second the time of visit by a loop of $\mathcal{S}$ and thirdly the order $\succeq_e$ on vertices of $\hat{T}_e$ induced by the clockwise contour if $\epsilon(e) = 1$ and its symmetric if $\epsilon(e) = -1$. For any skein $\mathcal{S}'$, whose elements are obtain by permutation of the letters of elements $\mathcal{S}$ and any two distinct elements $a,b \in \Gamma_T$, with $a = (e_1,F)$, $b = (e_2,F)$, let us denote by $T^e_{a,b}(\mathcal{S})$ or simply by $T_{a,b}(\mathcal{S})$ the cut and join transformation of $\mathcal{S}$, as defined in section 3.2.2. Then, Lemma 3.2 implies that

$$\frac{d}{d|F|} W_{\mathcal{S}} = -\frac{\mathcal{L}[F,F_\infty](\mathcal{S})}{2} W_{\mathcal{S}} - \frac{1}{N} \sum_{\dot{e} \neq \dot{e}' \text{ traversed by } \mathcal{S}} \epsilon(e)\epsilon(e') W_{T^{(e_1),\dot{e}}_{(e),\dot{e}',(e'),F}(\mathcal{S})}.$$  

**General compensations:** Consider two faces $A$ and $B$ such that $A$ is the first child of $B$ in $\hat{T}_B$ for $\preceq_{e(B)}$. Observe that if the letter $\lambda_B$ (resp. $\lambda_B^{-1}$) occurs in the words $w_1, \ldots, w_m$, then it is always on the left (resp. on the right) of $\lambda_A$ (resp. $\lambda_A^{-1}$). It follows that for any pair of distinct edges $e_1, e_2$ such that $\hat{e}_1, \hat{e}_2 \in [B,F_\infty]_T$, the reduction of the elements of $T^{(e_1),\dot{e}}_{(e_1,A),\dot{e}',(e_2_A)}(\mathcal{S})$ and $T^{(e_1),\dot{e}}_{(e_1,B),\dot{e}',(e_2_B)}(\mathcal{S})$ are equal. Therefore, equation (17) implies that

$$\frac{d}{d|A|} - \frac{d}{d|B|} W_{\mathcal{S}} = -\frac{1}{2} W_{\mathcal{S}} - \frac{1}{N} \sum_{\dot{e}' \text{ traversed by } \mathcal{S}} \epsilon(e(A))\epsilon(e') W_{T^{(e(A),A),\dot{e}'},(e',A)(\mathcal{S})}.$$  

**Compensations for a specific spanning tree:** Consider now four faces satisfying the assumptions of the Proposition. Thanks to the “locality” of the relation (16) and to the consistency of the discrete Yang-Mills measures, one may further assume that the four faces around $x$ are distinct, not equal to $F_\infty,G$ and that they do not disconnect $\hat{G}$, that is, the graph $(\hat{V} \setminus \{F_1, \ldots, F_4\}, \hat{E} \setminus \cup_{i=1}^4 (e, F_i \in e))$ is connected. Let us order the faces $F_1, \ldots, F_4$ counterclockwisely and choose a spanning tree $T$ of $G$ such that $F_2$ and $F_3$ are leaves of $\hat{T}$ and such that $(F_3,F_4,F_1)$ and $(F_2,F_1)$ are paths of $\hat{T}$, as is displayed in figure 6. Observe that the orientation of the three edges having $x$ as endpoints and belonging to $\hat{T}$, that is induced by the loops of $\mathcal{S}$, is the same as the one induced by such a spanning tree. Moreover, $F_2$ (resp. $F_3$) is the successor of $F_1$ (resp. $F_4$) for the orders $\preceq_e$ with $e$ an edge such that $F_1,F_2 \in T_{F_e,F} (\text{resp. } F_3,F_4 \in T_{F_e,F})$. One would like now to apply twice
Figure 6. A spanning tree of $\hat{G}$ that induces an orientation on the edges around $x$ that matches the one induced by the loops crossing at $x$ and such that $F_2$ (resp. $F_3$) is the successor of $F_1$ (resp. $F_4$) for the orders $\preceq_e$, with $e$ an edge such that $F_1, F_2 \in \hat{T}_{F_L(e)}$ (resp. $F_3, F_4 \in \hat{T}_{F_L(e)}$).

formula (18) and compare \( \left( \frac{d}{d|F_3|} - \frac{d}{d|F_4|} \right) W_S \) with \( \left( \frac{d}{d|F_2|} - \frac{d}{d|F_4|} \right) W_S \). Instead, let us remark the following.

For any face $A, B \in F_b$, let us denote by $\Theta_{A,B}$ the automorphism of the free group $RL_v(G)$ that maps $\lambda_B$ to $\lambda_A \lambda_B \lambda_A^{-1}$ and fixes $\lambda_F$ for $F \neq B$. According to Lemma 6.1, for any skein $S'$ with elements in $L_v(G)$,

$$W_{\Theta_{A,B}(S')} = W_{S'}.$$ 

The automorphism $\Theta_{A,B}$ transposes the letters $\lambda_A$ and $\lambda_B$ in the decomposition of $l_1, \ldots, l_m$ in $\Lambda$. In particular, when $A$ and $B$ are two faces such that $A$ is the first child of $B$ in $T_B$ for $\preceq_{e(B)}$ formula 18 also applies with $\Theta_{A,B}(S)$ in place of $S$. Notice that the letters $\lambda_{F_2}$ and $\lambda_{F_3}$ are consecutives in the decomposition of elements $\Theta_{F_3,F_4}(S)$ in $\Lambda$ (this is not true for $S$). Therefore, for any edge $e' \in [F_2, F_\infty]$ used by $S$,

$$\mathcal{T}_{(e(F_3), F_3),(e', F_4)}(\Theta_{F_3,F_4}(S)) = \mathcal{T}_{(e(F_2), F_2),(e', F_2)}(\Theta_{F_3,F_4}(S)).$$
The single cut transformation contributing to Makeenko-Migdal equation, when $x$ is a self-intersection point of an element of $\mathcal{S}$. Here $e \in \mathcal{E}$ denotes the edge with $\overline{e} = x$, that does not belong to $T$.

Now, applying twice formula (18) to respectively $(\frac{d}{d|F_3|} - \frac{d}{d|F_4|}) W_{\Theta_{F_3,F_4}(\mathcal{S})}$ and $(\frac{d}{d|F_2|} - \frac{d}{d|F_1|}) W_{\Theta_{F_3,F_4}(\mathcal{S})}$ yields that

\begin{equation}
N_{\mu_x} W_{\mathcal{S}} = \sum_{e' \in [F_2,F_\infty]_F} e(e(F_2))e(e') W_{\mathcal{T}(e(F_2),p_2),(e',p_2)(\Theta_{F_3,F_4}(\mathcal{S}))}
- \sum_{e' \in [F_3,F_\infty]_F} e(e(F_3))e(e') W_{\mathcal{T}(e(F_3),p_3),(e',p_3)(\Theta_{F_3,F_4}(\mathcal{S}))}.
\end{equation}

The equalities (19), together with $\epsilon(e(F_3)) = \epsilon(e(F_2)) = 1$, yield that the equation (20) gets simplified into

$$\mu_x W_{\mathcal{S}} = \frac{1}{N} W_{\mathcal{T}(e(F_3),p_3),(e(F_4),p_4)(\Theta_{F_3,F_4}(\mathcal{S}))}.$$

Let us observe that $\mathcal{T}(e(F_3),p_3),(e(F_4),p_4)(\Theta_{F_3,F_4}(\mathcal{S})) = \Theta_{F_3,F_4}(\mathcal{S})$, (see figure 7.1 when $x$ is a self-intersection point of an element of $\mathcal{S}$). Using once again the invariance of the discrete Yang-Mills measure under $\Theta_{F_3,F_4}$ implies the result.

If $x \in \mathcal{V}$ is a point of intersection point of one loop $l \in \mathcal{S}$, we denote respectively by $\tilde{l}_x^L$ and $\tilde{l}_x^R$ the loop that turns left and the loop that turns right at $x$. If $x$ is the
Theorem 7.3. Let \( S = \{l_1, \ldots, l_m\} \) be a regular skein, \( x \in \mathbb{N}_S \) a point of intersection and \( F_1, \ldots, F_4 \) faces around \( x \) numbered as in Proposition 7.2. If \( x \) is the intersection point of two different loops \( l_1 \) and \( l_2 \),

\[
(*) \quad \left( \frac{d}{d|F_1|} - \frac{d}{d|F_2|} + \frac{d}{d|F_3|} - \frac{d}{d|F_4|} \right) \Phi_N(S) = \Phi_N(l_1 \circ x l_2, l_3, \ldots, l_m).
\]

If \( x \) is an intersection point of the loop \( l_1 \), then

\[
(**) \quad \sum_{S_x^L \cup S_x^R = S_x} \Phi_N(S_x^L) \Phi_N(S_x^R) + \frac{1}{N^2} \Phi_N(\tilde{R}, \tilde{l}_x, l_2, \ldots, l_m).
\]

Moreover, if \( F \in \mathbb{N}_b \) is a neighbour face of the unbounded face, then

\[
(***) \quad \frac{d}{d|F|} \Phi_N(S) = -\frac{1}{2} \Phi_N(S).
\]

Proof. For any skein \( \{l_1, \ldots, l_m\} \) of \( m \) loops and any \( \pi, \nu \in \mathcal{P}_m \) with \( \pi \leq \nu \), let us set \( E_\pi[l_1, \ldots, l_m] = \prod_{B \in \pi} \mathbb{E}[\prod_{i \in B} \text{Tr}(H_i)], C_\nu[l_1, \ldots, l_m] = \prod_{i \in \nu} C_{\#B}(\text{Tr}(H_i), i \in B) \) and \( C_{\pi, \nu} = C_{\pi, \nu}(\text{Tr}(H_i), i \in \{1, \ldots, m\}) \). We shall consider the normalized cumulant

\[
\Phi_\pi(l_1, \ldots, l_m) = \prod_{B \in \pi} \Phi(l_1, i \in B) = N^{m-2\#\pi} C_\pi(l_1, \ldots, l_m).
\]

Assume that \( x \) is an intersection point of \( l_1 \). For any partition \( \nu \) of \( \{1, \ldots, m + 1\} \) connecting 1 with 2, denote by \( \nu' \) the partition of \( \{1, \ldots, m\} \) obtained by identifying 1 with 2. Then for any partition \( \nu \in \mathcal{P}_{m+1} \) connecting 1 with 2,

\[
\sum_{\nu' \leq \nu} N_{\#\nu} C_{\nu'}(l_1, l_2, \ldots, l_m) = \sum_{\pi \leq \nu'} N_{\#\pi} C_{\nu'}(l_1, l_2, \ldots, l_m)
\]

\[
= N_{\#\pi} \mathbb{E}_\nu[l_1, l_2, \ldots, l_m] = \mathbb{E}_\nu[\tilde{l}_x^{L}, \tilde{l}_x^{R}, l_2, \ldots, l_m]
\]

\[
= \mathbb{E}_\nu[\tilde{l}_x^{L}, \tilde{l}_x^{R}, l_2, \ldots, l_m].
\]

Therefore, for any partition \( \pi \in \mathcal{P}_{m+1} \) connecting 1 and 2,

\[
C_{1(1,2), \pi}(\tilde{l}_x^{L}, \tilde{l}_x^{R}, 1, \ldots, l_m) = N_{\#\pi} C_{\pi'}(l_1, l_2, \ldots, l_m).
\]

In particular,
\[\mu_x \Phi_N(l_1, \ldots, l_m) = N^{m-3} C_{1(1,2),1_{m+1}}(i_{1,x}, i_{1,x}, \ldots, l_m) \]
\[= \sum_{\pi \in \mathcal{P}_{m+1}} N^{m-3} C_{\pi}(i_{1,x}, i_{1,x}, \ldots, l_m) \]
\[= \sum_{\pi \in \mathcal{P}_{m+1}} N^{2|\pi|} \Phi_{\pi}(i_{1,x}, i_{1,x}, \ldots, l_m) \]

If \(\pi \in \mathcal{P}_{m+1}\) satisfies \(\pi \lor 1_{\{1,2\}} = 1_{m+1}\), then, whether \(|\pi| = 1\) or \(|\pi| = 2\) and the equation (***) follows. Assume now that \(x\) is an intersection point of \(l_1\) with \(l_2\). Then, for any partition \(\pi \in \mathcal{P}_m\),
\[\mu_x E_\pi(l_1, l_2, \ldots, l_m) = \begin{cases} \frac{1}{N} E_\pi(l_1 \circ_x l_2, \ldots, l_m), & \text{if } 1, 2 \text{ in the same block of } \pi, \\ 0, & \text{otherwise}. \end{cases}\]

Note that for any partition \(\pi \in \mathcal{P}_m\), such that 1 and 2 are not in the same block of \(\pi\), \(\mu_x C_{\pi}(l_1, l_2, \ldots, l_m) = 0\). For any partition \(\pi \in \mathcal{P}_{m-1}\), denote by \(\tilde{\pi} \in \mathcal{P}_m\) the partition obtained by shifting \(\pi\) by 1 and adding 1 to the block containing 2. For any partition \(\nu \in \mathcal{P}_{m-1}\),
\[\sum_{\pi \leq \nu} N \mu_x C_{\tilde{\pi}}(l_1, l_2, l_3, \ldots, l_m) = \sum_{W \leq \tilde{\nu}} N \mu_x C_W(l_1, l_2, l_3, \ldots, l_m) \]
\[= N \mu_x E_\nu[l_1, l_2, \ldots, l_m] = E_\nu[l_1 \circ_x l_2, l_3, \ldots, l_m].\]

It follows that for any partition \(\pi \in \mathcal{P}_{m-1}\),
\[\mu_x C_{\tilde{\pi}}(l_1, l_2, \ldots, l_m) = \frac{1}{N} C_{\pi}(l_1 \circ_x l_2, l_3, \ldots, l_m).\]

For \(\pi = 1_{m-1}\), the latter equality yields
\[\mu_x \Phi_N(l_1, l_2, \ldots, l_m) = \Phi_N(l_1 \circ_x l_2, l_3, \ldots, l_m).\]

Observe that equations (*) and (***) on \(\Phi_N\) do not depend on \(N\).

### 7.2. Generalized Kazakov basis

For any regular skein \(S\), denote by \(G_S = (S, \mathbb{E}_S, \mathbb{F}_S)\) an embedded graph such that \(S \subset \mathcal{P}(G_S)\) minimizing \#\(\mathbb{E}_S\) and \#\(\mathbb{V}_S\). We warn the Reader that the image of edges of \(G_S\) are not necessarily contained in the image of loops of \(S\) and that \(G_S\) is not always unique. Consider \(G_S = (S, E_S)\) the graph with vertices indexed by \(S\), such that two loops are connected in \(G_S\) if and only if they intersect each other. Then, images of edges of \(G_S\) are included in the images of loops of \(S\) if and only if \(G_S\) is connected. In that case, \(G_S\) is the unique finest embedded graph such that \(S \subset \mathcal{P}(G_S)\). In this section, we shall fix a skein \(S\) such that \(G_S\) is connected and set \(G = G_S\). Let \(E^+\) and \(\lambda\)
be respectively the orientation and the permutation of the edges $E$ induced by $S$. We want now to determine whether area-derivative operators can be obtained by linear combinations of the operators appearing on the left-hand-side of Theorem 7.3. To that purpose, let us set

$$\mu : C^F \longrightarrow C^{E+}$$

$$u \mapsto \left( e \mapsto u(F_L(e)) - u(F_R(e)) - u(F_L(\lambda^{-1}(e))) + u(F_R(\lambda^{-1}(e))) \right).$$

For any loop $l \in L(G)$, denote respectively by $n_l \in C^F$ and $\delta_l \in C^{E+}$ the winding number function of $l$ and the function $\sum_{l \text{ traverses } e} \delta_e$. For any $v \in V$, set

$$*v = \sum_{e \in \text{Out}(v)} \delta_e.$$

The following lemma is proved in [27](Lemma 6.28.).

**Lemma 7.4.**

i) The kernel of $\mu$ is spanned by $\{n_{l_1}, n_{l_2}, \ldots, n_{l_m}, 1_E\}$.

ii) The image of $\mu$ is the orthogonal space to $\{*v, v \in V\} \cup \{\delta_l, l \in S\}$.

iii) The intersection of linear spaces spanned by $\{*v, v \in V\}$ and $\{\delta_l, l \in S\}$ is $C^{1+E}$.

Lemma 7.4 yields that $\dim(\text{Im}(\mu)) = \#F - m - 1 = \#F_b - m$. We have a first answer to our question: for any regular skein $S$, the vector space spanned by the operators $\{\mu(\nabla a)(e), e \in E\}$ is not span($\{\frac{d}{dF}, F \in F_b\}$). Nonetheless, for some skeins, the third condition (***) allows to complete the lacking information.

**Lemma 7.5.** Suppose that there exists $m$ distinct faces $F_1, \ldots, F_m$ of $G$, neighbours of the unbounded face such that for any $i \in \{1, \ldots, m\}$, $l_i$ is bounding $F_i$. Let $F_{\infty, 1} = \{F_\infty, F_1, \ldots, F_m\}$. Then, the application

$$\overline{\mu} : C^F \longrightarrow \text{Im}(\mu) \oplus C^{F_{\infty, 1}}$$

$$u \mapsto (\mu(u), u|_{F_{\infty, 1}})$$

is an isomorphism.

**Proof.** Thanks to Lemma 7.4, the source and the target of $\mu$ have the same dimension. If $u \in \ker(\mu)$, then there exists $\alpha \in C^{m+1}$ such that $u = \alpha_{n+1}1_F + \sum_{i=1}^m \alpha_i n_{l_i}$. Moreover, $u(F_\infty) = \alpha_{n+1}$ and for any $i \in \{1, \ldots, m\}$ $u(F_i) = \alpha_i$. It follows that $\ker(\overline{\mu}) = \{0\}$. \hfill $\square$

We call a skein satisfying the condition of the Lemma a **skein based at infinity**. Note that each vertex of $G$ has degree 4, hence $\#E = 2\#V$ and Euler relation implies $\#F = \#V + 2$. Therefore $\dim(\text{Im}(\mu)) = \#V - m + 1$. For any vertex $v \in V$, let $e_1(v)$ and $e_2(v)$ be the two outgoing edges at $v$ ordered clockwise and set

$$\alpha_v = \delta_{e_1(v)} - \delta_{e_2(v)} \in C^{E+}.$$
Let us denote respectively by $V_s$ and $V_f$ the set of self-intersection points of each loops and the set of intersection points of pair of distinct loops of $S$. The type of crossing induces on $V_f$ an equivalence relation $\sim$ such two points $x$ and $y$ are equivalent if they belong to the same loops. For any pair of distinct points $x, y \in V_f$ such that $x \sim y$, denote by $\beta_{x,y}$ the function on $\mathbb{E}^+$ that is $\alpha_x + \epsilon_{x,y} \alpha_y$, where $\epsilon_{x,y} = 1$, if $e_1(x)$ and $e_1(y)$ belong to different loops and $-1$ otherwise.

**Lemma 7.6.** i) For any $v \in V_s$ and any pair of vertices $x, y \in V_f$ belonging to the the same pair of loops, $\alpha_v, \beta_{x,y} \in \text{Im}(\mu)$.

ii) Let $T_f$ be a spanning acyclic directed subgraph of the complete graph on $V_f$ such that connected components of $T_f$ are the equivalence class of $\sim$ on $V_f$. The family $\{\alpha_v : v \in V_s\} \cup \{\beta_{x,y} : (x, y) \in T_f\}$ is a free family of $\text{Im}(\mu)$ and is a basis if and only $G_S$ is a tree.

**Proof.** i) Indeed, for any $v \in V_s$ and $x, y \in V_f$ belonging to the same two of loops, the vectors $\alpha_v$ and $\beta_{x,y}$ are orthogonal to $\{v, v \in V\}$ and $\{\delta_l, l \in S\}$.

ii) It is easy to see that thanks to the acyclicity of the graph $T_f$, the family $\{\alpha_v : v \in V_s\} \cup \{\beta_{x,y} : (x, y) \in T_f\}$ is free. Moreover, its cardinality is $\#V - m'$, where $m'$ is the number of connected components of $T_f$. The latter are in bijection with edges of $G_S$. Recall that $G_S$ is connected so that $m' = m - 1$ if and only if $G_S$ is a tree. To conclude, recall that $\dim(\text{Im}(\mu)) = \#V - m + 1$. $\square$

If $S$ is made of a pair of loops that intersect each other, then the conclusion of ii) of Lemma 7.6 trivially holds, we give an example of such a basis in figure 8. There is a choice of directed acyclic graph that makes the decomposition in the basis easier, namely in each equivalence class of $V_f$, choose a base point and connect any other point towards it. If $R_S$ is a set of class representatives for $\sim$, for any $v \in V_f$, we denote by $v$ the unique element of $R_S$ equivalent to $v$.

**Lemma 7.7.** Let $R_S$ be set of class representatives of $\sim$, the family $\{\alpha_v, v \in V_s\} \cup \{\beta_{v,\pi}, v \in V_f \setminus R_S\}$ is a free family and for any function $\varphi$ belonging to its span,

$$
\varphi = \sum_{v \in V_s} \varphi(e_1(v)) \alpha_v + \sum_{v \in V_f \setminus R_S} \varphi(e_1(v)) \beta_{v,\pi}.
$$

These free-families have the following pre-image under the function $\mu$. For any $v \in V_s$, let $l_v$ be the loop that starts with $e_1(v)$ and stops at its first return at $v$. For any pair $(x, y)$ of distinct vertices in $V_f$ that are intersection of the same pair of loops, let $l_{x,y}$ be the loop that starts with $e_1(x)$, uses the same loop until it reaches $y$ and then goes back to $x$ using the second loop. See figure 9, where we draw the pre-image of the family described in figure 8.

**Lemma 7.8.** For any $v \in V_s$ and any pair $(x, y)$ of distinct equivalent vertices of $V_f$,

$$
\mu(l_v) = \alpha_v
$$

(22)
Figure 8. Here is the basis of $\text{Im} (\mu)$ associated to the spanning tree on $\mathcal{V}_f$ drawn with dashed lines. In this example, if the supports of two basis functions intersect, then the two functions take the same value on this intersection that we print on the edge.

and

$$\mu(n_{l,x,y}) = \beta_{x,y}. \quad (23)$$

Figure 9. The application $\mu$ maps the winding number functions of loops drawn on this figure to the basis represented in figure 8. These family of winding number functions completed by the winding number functions of elements of $\mathcal{S}$ is a basis of $\mathbb{C}^g$.

If $\mathcal{G}_S$ have cycles, we shall complete the basis given by Lemma 7.6 in the following way. For each edge $(l_1, l_2) \in E_S$, we denote by $v_{l_1,l_2}$ the element of $R_S$ at the intersections of $l_1$ and $l_2$. For each edge $(l, l') \in E_S \setminus T_S$, consider the unique

Note that such considerations are not useful to compute the correlation functions of the Gaussian master field thanks to Makeenko-Migdal relations.
path \((l_1, l_2, \ldots, l_k)\) in \(T_S\) from \(l\) to \(l'\) and for any \(i \in \{1, \ldots, k\}\), let \(c_i\) be the regular path that is the restriction of \(l_{i+1}\) such that \(c_{i} = v_{i,i+1}, \overline{c_i} = v_{i+1,i+2}\) (where loops are indexed by \(\mathbb{Z}/k\mathbb{Z}\)). The concatenation \(c_1c_2 \ldots c_k\) is a loop of \(G\), that follows elements of the cycle \((l_1, l_2, \ldots, l_k, l_1)\) and change from one strand to another at the points of \(R_S\). The application \(\mu\) maps the loop \(\tilde{l}_{l,l'}\) to

\[
\gamma_{l,l'} = \mu(\tilde{l}_{l,l'}) = \epsilon_1\alpha_{v_1,i_2} + \epsilon_2\alpha_{v_2,i_3} + \ldots + \epsilon_k\alpha_{v_k,i_1},
\]

where for any \(i \in \mathbb{Z}/k\mathbb{Z}\), \(\epsilon_i = 1\), if \(e_1(x_i) \in l_{i+1}\) and \(-1\) if \(e_1(x_i) \in l_i\) (with the above notation, \(\epsilon_i = \epsilon_{v_{i-1},i,v_{i,i+1}}\)). Let us write \(\epsilon_{i,l} = \epsilon_k\).

**Lemma 7.9.** The family \(\{\alpha_v : v \in V_s\} \cup \{\beta_v, \pi : v \in V_f \setminus R_S\} \cup \{\gamma_{l,l'} : (l, l') \in E_S \setminus T_S\}\) is a basis of \(\text{Im}(\mu)\). Moreover for any function \(\varphi \in \mathbb{C}^\mathbb{Z}\) belonging to \(\text{Im}(\mu)\),

\[
(K) \quad \varphi = \sum_{v \in V_s} \varphi(e_1(v))\alpha_v + \sum_{v \in V_f \setminus R_S} \varphi(e_1(v))\beta_v, \pi
\]

\[+ \sum_{(l,l') \in E_S \setminus T_S} \epsilon_{l,l'} \left( \varphi(e_1(\overline{l_{l,l'}})) - \sum_{\pi = v_{l,l'}} \epsilon_{v,v_{l,l'}} \varphi(e_1(v)) \right) \gamma_{l,l'}.\]

**Proof.** First, we should notice that \#\(R_S\) = \#\(E_S\) = \(m - 1 + \#(E_S \setminus T_S)\) and that this family has the good cardinality:

\[
\#V_s + (\#V_f - (m - 1) - \#(E_S \setminus T_S)) + \#(E_S \setminus T_S) = \#V - m + 1 = \dim(\text{Im}(\mu)).
\]

To conclude, we check that the relation \((K)\) holds true for any function \(\varphi\) in the span of this family. We denote by \(R'\) the subset of \(R_S\) representing classes indexed by \(E_S \setminus T_S\). Let \(c \in \mathbb{C}^{V \setminus R_S} \cup R'\) be a vector such that \(\varphi = \sum_{v \in V_s} c_v\alpha_v + \sum_{v \in V_f \setminus R_S} c_v\beta_v, \pi + \sum_{(l,l') \in E_S \setminus T_S} c_{v_{l,l'}} \gamma_{l,l'}\). Let \(r\) belong to \(R'\) and let \((l, l')\) be the edge \(E_S \setminus T_S\) such that \(r \in l \cap l'\). Then, \(\gamma_{l,l'}(e_1(r)) = \epsilon_{l,l'}\) and for any \(v \in V_f \setminus R_S\), such that \(v = r\), \(\beta_v, \pi e_1(r) = \epsilon_{v,v}\), whereas \(e_1(r)\) cancels any other element of the family. What is more, for any \(v \in V \setminus R_S\), \(\gamma_{l,l'}(e_1(v)) = 0\). This computation immediately yields the formula \((K)\). \(\square\)

Let us give the simplest example, where \(G_S\) is not a tree, that is when \(S\) is made of three cycles that intersect each others twice. The graph \(G_S\) is a triangle, we choose as set of representatives the three points of intersections lying on the boundary of \(F_{\infty}\). We draw on figure the families of loops one get when the circles have the same orientation.

Let us see how the above construction answer our initial question. For any loop \(l \in S\), \(\overline{p}(n_l) = e_1(F_l) \in \mathbb{C}^{\mathbb{Z} \times (m-1)}\), where \(e_l = n_l(F_l) \in \{-1, 1\}\). Hence, for any set of class representatives \(R_S\) for \(\sim\) and \(T_S\) a spanning tree of \(G_S\), the family \(\{n_v, v \in V_s \setminus R_S\} \cup \{n_v, v \in V_s\} \cup \{n_{i,l'} : (l, l') \in E_S \setminus T_S\} \cup S \cup \{1_F\}\) is a
Figure 10. Here is the loops that are mapped to the basis $\beta$ and $\gamma$, for three circles counterclockwisely oriented. The class representative of $\sim$ are taken on the boundary of $F_\infty$. In this case, the loop $\tilde{l}_{l,l'}$ does not depend on the choice of the spanning tree of $G_S$ and is drawn with dashed lines, whereas the family $\beta$ has three elements.

basis of $\mathbb{C}F$ and its image under $\overline{\mu}$ is the free family $\{\alpha_v, v \in V_s\} \cup \{\beta_{x,y}, (x, y) \in T_f\} \cup \{\gamma_{l,l'}, (l, l') \in E_S \setminus T_S\}$ completed by the canonical basis of $\mathbb{C}_F^{\infty_1}$. Moreover, for any $\varphi \in \text{Im}\ (\mu) \oplus \mathbb{C}^{\infty_1} \subset \mathbb{C}E^+ \oplus \mathbb{C}^{\infty_1}$,

\[
(\star) \quad \varphi = \sum_{v \in V_s} \varphi(e_1(v)) \alpha_v + \sum_{v \in V_f \setminus R_S} \varphi(e_1(v)) \beta_{x,y} + \sum_{F \in F_\infty} \varphi(F) \delta_F
\]

\[+ \sum_{(l,l') \in E_S \setminus T_S} \epsilon_{l,l'} \left( \varphi(e_1(v_{l,l'})) - \sum_{v = v_{l,l'}} \epsilon_{v,v_{l,l'}} \varphi(e_1(v)) \right) \gamma_{l,l'}
\]

and

\[
\overline{\mu}^{-1}(\varphi) = \sum_{v \in V_s} \varphi(e_1(v)) n_v + \sum_{v \in V_f \setminus R_S} \varphi(e_1(v)) n_{v, \pi} + \sum_{l \in S} \epsilon_l \varphi(F_l) n_l
\]

\[+ \sum_{(l,l') \in E_S \setminus T_S} \epsilon_{l,l'} \left( \varphi(e_1(v_{l,l'})) - \sum_{v \in V_f, v = v_{l,l'}} \epsilon_{v,v_{l,l'}} \varphi(e_1(v)) \right) n_{l,l'}
\]
If a skein is based at infinity, considering \( \varphi = \bar{\mu}(\nabla_a) \) yields a complete answer to our initial question. For any face \( F \in \mathcal{F} \),

\[
\frac{d}{d|F|} = \sum_{v \in V_s} n_v(F) \mu(\nabla_a)(e_1(v)) + \sum_{v \in V_f \setminus R_S} n_v(F) \mu(\nabla_a)(e_1(v)) + \sum_{l \in S} \epsilon_{n}(F) \frac{d}{d|F|} \]

\[
\left( \varphi \right) + \sum_{(l, l') \in E_S \setminus T_S} \epsilon_{l, l'} n_{l, l'}(F) \left( \mu(\nabla_a)(e_1(v_{l, l'})) - \sum_{v \in V_f \setminus \{v_{l, l'} \}} \epsilon_{v, v_{l, l'}} \mu(\nabla_a)(e_1(v)) \right).
\]

Using Theorem 7.3, we have an expression for all area derivatives of \( \Phi(\mathcal{S}) \), for any skein based at infinity and such that \( \mathcal{G}_S \) is connected. What is more, note that if \( \mathcal{S} \) is based at infinity and \( \mathcal{G}_S \) is not connected, then \( \Phi_N(\mathcal{S}) = 0 \). Nonetheless, observe that if \( v \in V_s \), it may happen, as in example of figure 9 at the only point of \( V_s \), that among the two skeins \( \mathcal{S}^L_v \) and \( \mathcal{S}^R_v \) obtained by splitting \( \mathcal{S} \) at \( v \), one of them is not based at infinity. To solve this problem and compute the master field against all skeins, we could enlarge the type of loops families. Instead, observe that any skein can be obtained by putting some areas of a skein based at infinity to zero. We must now prove that these equations characterize the higher-order master field \( \Phi \).

7.3. Complexity of skeins. We shall consider in this last section a slightly different notion of embedded graph. We call a multi-embedded graph in the plane a triplet \( \mathcal{G} = (V, E, F_R) \) satisfying the same conditions as an embedded graph as defined at the beginning of section 6.2 but where the element of \( F_R \) are allowed to be non-simply connected.

Let us fix a regular skein \( \mathcal{S} \). We let \( \mathcal{G}_S = (V_S, E_S, F_S) \) be the finest multi-connected embedded graph such that \( \mathcal{S} \subset P(\mathcal{G}_S) \). The graph \( \mathcal{G}_S \) is connected if and only if \( \mathcal{G}_S \) is connected. For any \( l \in P(\mathcal{G}_S) \), we consider

\[
d_{\infty, \mathcal{S}}(l) = \inf\{d_{\mathcal{G}_S}(F, F_\infty) - 1 : F \in \mathcal{G}_S, F \cap F_\infty, \mathcal{G}_S(l) = \emptyset \}.
\]

Let \( I(\mathcal{S}) \) be the number of intersections of \( \mathcal{S} \) and for any loop \( l \in \mathcal{S} \) denote by \( I_S(l) \) the number of intersections of \( l \) with itself and other loops of \( \mathcal{S} \). We define the complexity of \( \mathcal{S} \) to be the number

\[
\mathcal{C}(\mathcal{S}) = I(\mathcal{S}) + 2 \sum_{l \in \mathcal{S}} d_{\infty, \mathcal{S}}(l).
\]

Example 7.10. A skein \( \mathcal{S} \) is based at infinity if and only if \( \mathcal{C}(\mathcal{S}) = I(\mathcal{S}) \).

Example 7.11. If \( \mathcal{C}(\mathcal{S}) = 0 \), then \( \mathcal{S} \) is an union of closed Jordan curved bounding disjoints domains. Therefore, \( \Phi(\mathcal{S}) = 0 \), if \( \#\mathcal{S} \geq 2 \) and \( e^{-\frac{|D|}{2}} \), if \( \mathcal{S} \) has a single loop bounding a simply connected domain \( D \).
Lemma 7.12.  
i) For any point of intersection $v$ of a skein $S$, if $v \in V_f$, 
$$C(\mathcal{S}_v) < C(S)$$ 
and if $v \in V_s$, for any partition $S^L_v \cup S^R_v = S_v$, 
$$\max\{C(S^L_v), C(S^R_v)\} < C(S).$$

ii) For any regular skein $S$, there exists a family $(S^\epsilon)_{\epsilon > 0}$ of skeins based at infinity with $C(S^\epsilon) = C(S)$ for any $\epsilon > 0$ and a family of smooth paths $(r_1)_{r \in S}$ such that for any $l \in S$, $r_1 \in l$ and $S^\epsilon$ converges to $\{r_1lr_1^{-1}, l \in S\}$, as $\epsilon \to 0$.

Proof. i) Assume that $v \in V_f$. Then, for any loop $l \in S$, that does not contain $v$, $l \in \mathcal{S}_v$ and $d_{\infty, \mathcal{S}_v}(l) \leq d_{\infty, S}(l)$. If $l_1$ and $l_2$ are the two loops crossing at $v$, then $d_{\infty, \mathcal{S}_v}(l_1 \cup v, l_2) \leq \min\{d_{\infty, S}(l_1), d_{\infty, S}(l_2)\}$. Moreover, $V_s(\mathcal{S}_v) = V_s(S)$ and for any $w \in V_s(S)$, $d_{\infty, \mathcal{S}_v}(l_v) \leq d_{\infty, S}(l_v)$. Therefore, the fact that $I(\mathcal{S}_v) = I(S) - 1$ yields the expected inequality.

Assume now that $v \in V_s$. Let $l \in S$ be the loop of $S$ crossing at $v$. Recall that $l_{v,L}$ and $l_{v,R}$ are the loops that turns respectively to the left and to right at $v$ (so that they use respectively the edge $e_1(v)$ and $e_2(v)$). Fix a partition $S^L_v \cup S^R_v$ of $S_v$ separating these two loops. For any loop $l' \in S_v \setminus \{l^L_v, l^R_v\}$, $d_{\infty, \mathcal{S}_v}(l') \leq d_{\infty, S}(l')$. Suppose that there exists a path $c \in P(\mathcal{G}_S)$ such that $F_{\infty, \mathcal{G}_S} \cap c = \emptyset$ and $\ell(c) - 1 = d_{\infty, \mathcal{S}_v}(l_v) = d_{\infty, S}(l)$. Then, on the left side, $I(S^L_v) \leq I(S_v) - 1$ and $C(S^L_v) \leq C(S) - 1$. The right side needs more caution. Consider the paths $c^+ \in P(\mathcal{G}_S)$ that start with $(F^R(e_1(v)), F^L(e_1(v)))$, board $l^L_v$ respectively on the right and on the left, such that one path follows the orientation of $l_{v,L}$ and the other goes in the reverse direction, until they cross $c$ and then follow $c$ up to $F_{\infty, \mathcal{G}_S}$. Their combinatorial length satisfies 

$$\ell(c^+) + \ell(c^-) \leq 2 + I_{\mathcal{S}_v}(l^L_v) + 2 \ell(c).$$

Therefore, 

$$d_{\infty, \mathcal{S}_v}(l^R_v) \leq \min\{\ell(c^+), \ell(c^-)\} - 1 \leq \frac{I_{\mathcal{S}_v}(l^L_v)}{2} + d_{\infty, S}(l).$$

The number of intersections of the right skein $S^R_v$ is bounded by $I(S \setminus \{l\}) + I_{\mathcal{S}_v}(l^R_v)$. Moreover, for any loop $l' \in S^R_v \setminus \{l^R_v\}$, $d_{\infty, \mathcal{S}_v}(l') \leq d_{\infty, S}(l')$. It follows that 

$$C(S^R_v) \leq I(S \setminus \{l\}) + I_{\mathcal{S}_v}(l^R_v) + I_{\mathcal{S}_v}(l_{v,L}) + 2 \sum_{l' \in S \setminus \{l\}} d_{\infty, S}(l').$$

The equality $I_{\mathcal{S}_v}(l^R_v) + I_{\mathcal{S}_v}(l^L_v) + I(S \setminus \{l\}) = I(S_v) = I(S) - 1$ concludes.

ii) For each $l \in S$, such that $d_{\infty, \mathcal{S}_v}(l) > 0$, consider a self-avoiding path $c_l \in \mathcal{P}(\mathcal{G}_S)$ such that $F_{\infty, \mathcal{G}_S} \cap c_l = \emptyset$ and $c_l = F_{\infty}(\mathcal{G}_S)$. Choose such a family $(c_l)_{l \in S}$ of loops that do not cross each other but may be merged with one another. Deform each loop $l$ along $c_l$ into $l$ so that the deformation intersects exactly twice each
Each path $c_l \in \mathcal{P}(\tilde{\mathcal{G}})_{\tilde{S}}$ induces a path $\tilde{c}_l \in \mathcal{P}(\tilde{\mathcal{G}})_{\tilde{S}}$ such that faces belonging to $\tilde{c}_l$ are boarded by $\tilde{c}_l$. When the areas $(|F|)_{F \in \tilde{S}}$ go to zero, the vector of random variables $(H_l)_{l \in \tilde{S}}$ converges in distribution to $(H_r l r^{-1})$, where $r_l$ are smooth paths such that $r_l \in l$. For any loop $l \in S$, $H_r l r^{-1} = H^{-1}_r H_l$ and $\text{Tr}(H_r l r^{-1}) = \text{Tr}(H_l)$.

Considering $\Phi_N(\tilde{S})$ and $\Phi_N(S)$ as functions of $(|F|)_{F \in \mathcal{P}_{\tilde{S}}}$, it follows that
\[
\Phi_N(\tilde{S}) \to \Phi_N(S),
\]
as $(|F|)_{F \in \mathcal{P}_{\tilde{S}}} \to 0$. \hfill \square

We can now solve our differential system recursively ordering skeins by their complexity. Recall that if $x$ is a point of intersection of a skein $S$, then $\mu_x = \frac{d}{dF_1} - \frac{d}{dF_2} + \frac{d}{dF_3} - \frac{d}{dF_4}$, where $F_1, F_2, F_3$ and $F_4$ are faces around the vertex $v$ in cyclic order and $F_1$ is the face bounded by the two outgoing edges of $x$. For any pair of skeins $S$ and $S'$, let us say that $S$ and $S'$ are equivalent and write $S \sim S'$ if there exists a family $\{c_l, l \in S\}$ of paths of $\mathcal{P}(\mathbb{R}^2)$ such that $c_l \in l$ for any $l$ and $S' = \{c_l c_l^{-1}, l \in S\}$.

**Theorem 7.13.** There exists a unique function $\Phi$ on $\text{Sk}(\mathbb{R}^2)$ satisfying the following equations.

1. $\Phi(\{1\}) = 1$.
2. If $S^-$ and $S^+$ are two skeins that are separated by a closed Jordan curve, $\Phi(S^- \cup S^+) = 0$.
3. $\Phi$ is continuous for the topology of 1-variation.
4. If $S \sim S'$, then $\Phi(S') = \Phi(S)$.
5. For any area-preserving diffeomorphism $g$ of the plane, $\Phi \circ g = \Phi$.
6. For any regular skein $S$, $\Phi$ is differentiable with respect to $(|F|)_{F \in \mathcal{P}_S}$ and satisfy the following differential equations. If $x$ is the intersection of two different loops,
\[
\mu_x \Phi(S) = \Phi(S_x).
\]
If $x$ is the intersection of a loop of $S$ with itself,
\[
\mu_x \Phi(S) = \sum_{S^l_x \cup S^R_x = S_x} \Phi_N(S^l_x) \Phi_N(S^R_x).
\]
For any face $F \in \mathcal{P}_S$, neighbour of $F_\infty$,
\[
\frac{d}{d|F|} \Phi(S) = -\frac{1}{2} \Phi(S).
\]
Proof. The function $\Phi_N$ satisfies by construction the point 1.-4.. For any regular skein $S$, $\Phi_N(S)$ is analytic in $(|F|)_{F \in F_S}$; satisfies $(\ast)$, $(\ast\ast)$ and $(\ast\ast\ast)$ and converges uniformly on every compact set of $\mathbb{R}_+^S$. Therefore, $\Phi(S)$ is analytic and satisfies the equations of point 5. Let us now consider the question of uniqueness. Let $\Psi$ be a function on finite skeins satisfying point 1 to 5. Using points 3 and 4, it is enough to prove that $\Psi(S) = \Phi(S)$ for any regular skein $S$. For any integer $n$, denote by $Sk_n$ the set of regular skeins of complexity less than $n$. Let us prove inductively that $\Psi|_{Sk_n} = \Phi|_{Sk_n}$. Thanks to points 1 and 2, the equality holds for $n = 0$. Assume that it is true for $n \in \mathbb{N}$ and consider a regular skein $S \in Sk_{n+1}$. Suppose that $S$ is based at infinity. According to $(\diamond)$ and to point 6, for any face $F \in F_S$, $\frac{d}{d|F|}\Phi$ and $\frac{d}{d|F|}\Psi$ are a linear combination of terms of the form $\Phi(S'), \Psi(S')$ or $\Phi(S_L^L)\Phi(S_R^R), \Psi(S_L^L)\Psi(S_R^R)$, with $C(S'), C(S_L^L), C(S_R^R) < n$. Hence, by induction hypothesis, $\Psi(S) = \Phi(S)$. Assume now that $S$ is not based at infinity. Let $(S')_{\epsilon > 0}$ and $\{r_l, l \in S\}$ be given as in Lemma 7.12. Then, for any $\epsilon > 0$, $S'$ is based at infinity and $\Psi(S') = \Phi(S')$. The points 3 and 4 yield that $\Psi(S) = \Phi(S)$. □

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