The effect of density turbulent diffusion intensity on magnetohydrodynamic wave processes in a spherical layer of electrically conducting liquid

S Peregudin¹, E Peregudina² and S Kholodova³

¹ Saint Petersburg State University, Universitetskaya Emb. 7-9, Saint Petersburg, 199034 Russia
² Saint Petersburg Mining University, 21-st Line 2, Saint Petersburg, 199106 Russia
³ Saint Petersburg University ITMO, Kronverksky Pr. 49, Saint Petersburg, 197101 Russia

E-mail: s.peregudin@spbu.ru

Abstract. The paper considers large-scale 3D wave motion of weakly compressible and perfectly conducting inhomogeneous inviscid liquid subject to a magnetic field and the Coriolis force, with due account the spherical topography of bounding surfaces. The results of the studies can be used in geophysics and astrophysics, and in particular, in the study of dynamical processes occurring in the Earth liquid core and in stellar interiors. Moreover, they can be also employed in evaluation of magnetohydrodynamic characteristics of a liquid medium and for estimation of the parameters of the wave disturbance in the electromagnetic field induced by the corresponding hydrodynamic excitement. In addition, the above results can be used in the study of magnetohydrodynamic processes in engineering devices and in the study of foreground energy materials.

1. Introduction

Advances in computational technology paved the way for direct numerical solution of kinematic dynamical problems for model flows in various geometric topologies, and in particular, on the sphere [1, 2]. The principal issue of the above approach is that calculations can be carried out only for magnetic Reynolds numbers which do not satisfy the practical requirements of astrophysics and geophysics. Besides, for small Ekman numbers, boundary layers appear near the liquid boundary, which cannot be calculated numerically with required accuracy and which should be analyzed with the help of analytical. The paper [3] puts forward an analytical solution capable of describing the effect of the boundary surfaces relief on the magnetohydrodynamic characteristics of a wave process in the liquid layer. In [4], the system of nonlinear partial differential equations, which model perturbations in a perfect electrically conducting rotating liquid, was reduced to a scalar equation with due regard for the inertia, gravity, Coriolis, and Lorentz forces, and with consideration of the available density inhomogeneities. Moreover, in [4] a conclusion was made that the solution to the small perturbation problem in the liquid under consideration assumes an analytical representation. The analysis of the solution thus obtained shows the existence of a steady-state regime of oscillations for large time, which supports the important role of the liquid layer density stratification that controls its principal dynamics in a large number of cases and which is an important evolution factor of the model under consideration.
According to [5], magnetic field dynamics depends substantially on the motion of the liquid under consideration in the thin near-boundary layer. This conclusion was made for perturbations of magnetohydrodynamic quantities whose horizontal scale of variation is much smaller than the layer radius.

Of special interest are the large-scale motions, whose horizontal scale is comparable with the layer radius. Namely, it would be interesting to find the propagation of density and all magnetohydrodynamic quantities consequent on thermal changes near its boundary.

In the present study, we consider a model with variable stratification without the use of magnetostrophic and Boussinesq approximations.

2. Fundamental equations and boundary conditions

So, we shall deal with wave equations near the spherical layer boundary of a stably stratified liquid. In magnetic hydrodynamics, the system of equations for the description of the motion of a perfect inviscid electrically conducting weakly compressible liquid rotating with angular rate \( \omega \) reads as [5, 6]

\[
\frac{d \rho}{dt} + \rho \left( \frac{\partial v_r}{\partial r} + \frac{2v_r}{r} \right) = 0,
\]

\[
\frac{d v_r}{dt} + \frac{v_r}{r} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( v_r \sin \theta \right) + \frac{1}{r \sin \theta} \frac{\partial v_\lambda}{\partial \lambda} = 0,
\]

\[
\frac{d v_\theta}{dt} + \frac{v_\theta}{r} = -2 \frac{\alpha_{\lambda}}{r} \cos \theta = - \frac{1}{\rho r \sin \theta} \frac{\partial p}{\partial \lambda} + \frac{1}{\mu r} \left( b_\theta W_r - b_r W_\theta \right),
\]

\[
\frac{d v_\lambda}{dt} + \frac{v_\lambda}{r} = -2 \frac{\alpha_{\lambda}}{r} \sin \theta = - \frac{1}{\rho r} \frac{\partial p}{\partial \theta} - g - \frac{1}{\mu r} \left( b_\lambda W_\theta - b_\theta W_\lambda \right),
\]

\[
W_r = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( b_\lambda \sin \theta \right) - \frac{\partial b_\theta}{\partial \lambda} \right], \quad W_\theta = \frac{1}{r \sin \theta} \left[ \frac{\partial b_r}{\partial \theta} - \frac{\partial (rb_\lambda \sin \theta)}{\partial \lambda} \right], \quad W_\lambda = \frac{1}{r} \left[ \frac{\partial}{\partial r} \left( rb_\theta \right) - \frac{\partial b_r}{\partial \theta} \right],
\]

\[
\frac{d \rho}{dt} = \kappa \Delta \rho - \frac{\alpha \rho}{c_p} \frac{Q}{\rho}, \quad \rho = \rho_0 \left( 1 - \alpha (T - T_0) \right),
\]

\[
\frac{\partial b_r}{\partial t} = \frac{\partial b_\theta}{\partial r} \frac{\partial v_r}{\partial \theta} + \frac{b_\lambda}{\rho r \sin \theta} \frac{\partial b_\lambda}{\partial \lambda} - \frac{\partial b_\theta}{\partial \theta} - \frac{b_r}{\rho} \frac{\partial \rho}{\partial t},
\]

\[
\frac{\partial b_\theta}{\partial t} = \frac{\partial b_\lambda}{\partial r} \frac{\partial v_\theta}{\partial \theta} + \frac{b_\lambda}{\rho r \sin \theta} \frac{\partial b_\lambda}{\partial \lambda} - \frac{\partial b_\lambda}{\partial \theta} - \frac{b_\theta}{\rho} \frac{\partial \rho}{\partial t},
\]

\[
\frac{\partial b_\lambda}{\partial t} = \frac{\partial b_r}{\partial r} \frac{\partial v_\lambda}{\partial \theta} + \frac{b_\lambda}{\rho r \sin \theta} \frac{\partial b_\lambda}{\partial \lambda} - \frac{\partial b_\lambda}{\partial \theta} - \frac{b_\lambda}{\rho} \frac{\partial \rho}{\partial t},
\]

\[
\frac{1}{r \sin \theta} \frac{\partial}{\partial r} \left( r^2 b_r \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( b_\theta \sin \theta \right) + \frac{\partial b_\lambda}{\partial \lambda} = 0,
\]

here \( b \) is the magnetic induction vector, \( v \) is the liquid velocity in the frame rotating with angular velocity \( \omega \), \( \rho \) is the pressure, \( \rho \) is the density, \( g \) is the acceleration of gravity, \( T \) is the temperature, \( \kappa \) is the thermometric conductivity, \( Q \) is the rate of heat addition per unit mass by external heat sources, \( \rho_0 \) and \( T_0 \) are, respectively, the mean liquid density and the mean temperature, and \( \alpha \) is the coefficient of thermal expansion, \( c_p \) is the specific heat at constant pressure. The magnetic permeability and the electrical conductivity are assumed to be constant.
In consideration of large-scale motions, whose horizontal scale comparable with the Earth radius, the principal equations of quasi-geostrophic motion undergo several changes. The reduction of the problem geometry to the planar case is not a natural approximation anymore. One also cannot neglect the horizontal variability of the principal density field. So, for an adequate description of the motion of such scales, one needs to consider the fundamental equations of magnetic hydrodynamics with due account of the corresponding relations between the scales. We shall assume that the vertical scale of the density is sufficiently large in comparison with the vertical scale of motion.

We study the dynamics under the Ekman layer adjacent to the liquid layer boundary. Since the viscosity is insignificant in the domain under consideration, it suffices to specify the normal velocity component on the upper boundary. So, \( v_z = v_z(\theta, \lambda, 1) \). On the upper boundary of the domain under consideration, the density should be in accord with the given gravity field near the boundary, which varies insignificantly in the narrow Ekman layer near the liquid layer boundary. Using the state equation, this condition can be written in the form:

\[
\rho(\theta, \lambda, 1) = -\frac{\rho g D^2 F_0}{2\omega W_0^2} T_2(\theta, \lambda),
\]

where \( F_0 \) is the characteristic magnitude of the surface temperature variation. The function \( T_2(\theta, \lambda) \) controls the horizontal structure of the surface field temperature. In addition to the above boundary conditions, we require that

\[
b_\theta = 0, \quad b_\lambda = 0, \quad z \to 0, \quad v_\theta \to 0, \quad v_\lambda \to 0, \quad b_\theta \to 0, \quad b_\lambda \to 0, \quad z \to -\infty.
\]

3. Solution of the magnetic hydrodynamics problem

By analyzing the above mathematical model capable of calculating three-dimensional motions with large time scale and spatial horizontal scale comparable with the layer radius and by constructing a successive approximation scheme in which the geostrophic approximation is a first step, we get an analytical solution to the system of nonlinear partial differential equations that models the stationary geostrophic motion in a layer of perfectly electrically conducting rotating liquid. So, the distribution of magnetohydrodynamic quantities appearing in the liquid as a result of thermodynamic variations near the layer boundary reads as

\[
\rho(\theta, \lambda, z) = \frac{T_2(\theta, \lambda)}{k(\theta)} \exp(kz), \quad \rho(\theta, \lambda, z) = -T_2(\theta, \lambda) \exp(kz),
\]

\[
v_\phi(\theta, \lambda, z) = \frac{1}{k \cos^2 \theta} \left( \frac{\partial T_2}{\partial \theta} - \frac{T_2}{k^2} \frac{d}{d\theta} + \frac{T_2}{k^2} \frac{d}{d\theta} \right) \exp(kz), \quad v_\theta(\theta, \lambda, z) = \frac{1}{k \cos \theta \sin \theta} \frac{\partial T_2}{\partial \lambda} \exp(kz),
\]

\[
v_z(\theta, \lambda, z) = v_{zo}(\theta, \lambda) + \frac{1}{k^2 \cos^2 \theta} \frac{\partial T_2}{\partial \lambda} (\exp(kz) - 1),
\]

the asymptotic value of the velocity is \( v_{zo} = \frac{Cv}{\cos \theta} \), as controlled with fixed \( C \) by the parameter \( v \), which characterizes the density turbulent diffusion intensity, and the dissipation is responsible for the lifting of liquid from the lower boundary of the liquid layer. Hence, the expressions for the magnetic field component read as
where $k(\theta) = \frac{T_s(\theta)}{C + C_0 \sin \theta}$, $J_1$ is the Bessel function of the first kind.

So, if the turbulent density diffusion is negligible, then the vertical velocity component tends to zero away from the outer boundary, and hence, the principal structure of the hydrodynamic fields can be found as in the case of absent dissipation, whose principal role is in the lifting of liquid from the lower boundary — this differs from the time-varying model of [7], in which the vertical velocity component of a liquid particle away from the outer boundary depends explicitly not only on the intensity parameter of the density turbulent diffusion, but also on the distribution of the surface temperature function, and the density diffusion has an effect on all magnetohydrodynamic characteristics inside the liquid volume.

Conclusions
The main result of the above study is the construction of an analytical solution of the system of partial differential equations that models the geostrophic motion in a spherical layer of electrically conducting inhomogeneous rotating liquid with due account of the effects of turbulent density diffusion. From the structure of the fields of magnetohydrodynamic quantities one can judge about their qualitative dynamics in a thin layer adjacent to the boundary surface. The problem of wave motions caused by oscillations of a flat wall was considered in article [8].

References
[1] Plunian F and Radler K-H 2002 Magnetohydrodynamics 38 95–106
[2] Zheligovsky V A, Podvigina O M and Frisch U 2001 Geophys. Astrophys. Fluid Dyn. 95 227–68
[3] Kholodova S E 2009 J. Appl. Mech. and Tech. Phys. 50 25–34
[4] Kholodova S E 2008 Comp. Math. and Math Phys. 48 882–98
[5] Alfvén H and Fälthammar C-G 1963 Cosmical electrodynamics. Fundamental principles (Clarendon Press: Oxford) p 228
[6] Kholodova S E 2009 Vestn. St. Petersburg. Univ. Ser. 10: Prikl. Mat. Inf. Protsessy Upr. 1 118–33
[7] Peregudin S I, Peregudina E S and Kholodova S E 2018 J. Comp. Tech. 1. Part. 2 177–83
[8] Peregudin S I and Kholodova S E 2010 J. Mining Institute 187 113–6 (in Russian)