Super Yangian $Y(osp(1|2))$

and the

Universal $R$-matrix of its Quantum Double

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Abstract

We present the Drinfel’d realisation of the super Yangian $Y(osp(1|2))$, including the explicit expression for the coproduct. We show in particular that it is necessary to introduce supplementary Serre relations. The construction of its quantum double is carried out. This allows us to give the universal $R$-matrix of $DY(osp(1|2))$.

MSC number: 81R50, 17B37
1 Introduction

The Yangian $Y(\mathfrak{a})$ based on a simple Lie algebra $\mathfrak{a}$ is defined as the homogeneous quantisation of the algebra $\mathfrak{a}[u] = \mathfrak{a} \otimes \mathbb{C}[u]$ endowed with its standard bialgebra structure, where $\mathbb{C}[u]$ is the ring of polynomials in the indeterminate $u$. It was introduced by Drinfel’d [1],[2],[3]. The most elegant and concise presentation of the Yangian uses the FRT formalism [4], based on a certain evaluated R-matrix. For unitary algebras, the R-matrix is given by $R(u) = I \otimes I + P/u$ where $P$ is the permutation map. This can be extended to superunitary series by considering the superpermutation instead [5],[6]. This formalism was extended to the orthogonal, symplectic and orthosymplectic cases by taking $R(u) = I \otimes I + P/u - K/(u + \kappa)$ where $K$ is a partial (super)transposition of $P$ and $4\kappa = (\alpha_0 + 2\rho, \alpha_0)$, see for example [7]. Although one can exhibit a suitable R-matrix for any simple Lie algebra or basic simple Lie superalgebra $\mathfrak{a}$ for defining the Yangian $Y(\mathfrak{a})$, the obtained structure is a Hopf algebra, but not a quasi-triangular one. It is well known that the quantum double construction allows one to construct a quasi-triangular Hopf algebra from a Hopf algebra, and that this procedure leads to the universal R-matrix of the algebra under consideration. However, in order to find explicit useful formula for this universal R-matrix, it is necessary to consider another realisation of the Yangian, given in terms of generators and relations similar to the description of a loop algebra as a space of maps. Unfortunately in this realisation no explicit formula for the comultiplication is known in general, except in the $sl(2)$ case [8].

The aim of this paper is to extend this construction to the case of the superalgebra $osp(1|2)$. Let us recall that the super Yangian $Y(osp(1|2))$ is defined by the relations

$$R_{12}(u - v) L_1(u) L_2(v) = L_2(v) L_1(u) R_{12}(u - v)$$

and

$$C(u) = L^t (u - \kappa) L(u) = 1_3$$

where the generators of $Y(osp(1|2))$ are encapsulated into the $3 \times 3$ matrix $L(u)$ and $R_{12}(u)$ is the R-matrix introduced in [7]. Writing a Gauss decomposition of $L(u)$, we then introduce some specific combinations of the generators $L^{ij}(u)$, called $e(u)$, $f(u)$ and $h(u)$, which define a Drinfel’d realisation of the super Yangian $Y(osp(1|2))$. More precisely, we show that the associative algebra $\mathcal{A}$ generated by $e(u)$, $f(u)$ and $h(u)$ subjected to certain relations, and the super Yangian $Y(osp(1|2))$ are isomorphic as bialgebras. At this point, two remarks are in order. First, one is able to find explicit formula for the comultiplication in terms of the Drinfel’d generators $e(u)$, $f(u)$ and $h(u)$, thereby generalising Molev’s formula in the case of $osp(1|2)$. Second, it is necessary to introduce supplementary Serre-type relations among the $e(u)$, $f(u)$, $h(u)$ generators, as in the case of $U_q(osp(1|2))$ [4]. These supplementary Serre-type relations are cubic in the Drinfel’d generators. Indeed, it appears that the quadratic exchange relations among the $e(u)$, $f(u)$, $h(u)$ generators, derived from the RLL relations, lead to a superalgebra which is bigger than $Y(osp(1|2))$. Hence it is necessary to quotient this bigger structure by supplementary relations.

The next step is the construction of the quantum double. The super Yangian $Y(osp(1|2)) = Y^+$ being given in terms of the Drinfel’d generators $e(u)$, $f(u)$, $h(u)$ with positive modes, we introduce another set of generators $e(u)$, $f(u)$, $h(u)$ with negative modes, generating a Hopf algebra $Y^-$.  

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We construct a Hopf pairing between $Y^+$ and $Y^-$, such that $Y^-$ is isomorphic to the dual of $Y^+$ with opposite comultiplication. We prove that this Hopf pairing is not degenerate. This allows us to define the double super Yangian $DY(osp(1|2))$ in a proper way. In particular we are able to give a presentation of the double super Yangian $DY(osp(1|2))$ in terms of the Drinfel’d generators $e(u), f(u), h(u)$, now $\mathbb{Z}$ moded, subjected to suitable quadratic exchange relations, and suitable supplementary Serre-type relations of cubic form.

As we emphasised above, the quantum double procedure allows one to construct explicitly the universal R-matrix. \{$x_i, i \in \mathbb{N}\}$ being a basis of $Y^+$ and \{${x^i, i \in \mathbb{N}}\} \in Y^-$ its dual basis (i.e. \(<x^i, x_j>=\delta^i_j\)), the universal R-matrix of $DY$ is given by $R = \sum x_i \otimes x^i$. Let $\tilde{E}^+$, $\tilde{F}^+$ and $\tilde{H}^+$ denote the unital subalgebras of $Y^+$ generated by the positive modes of $e(u)$, $f(u)$ and $h(u)$, and let $\tilde{F}^-$, $\tilde{E}^-$ and $\tilde{H}^-$ be the dual subalgebras. Following the kind of arguments used in [10], we give Poincaré–Birkhoff–Witt bases for $\tilde{E}^\pm$, $\tilde{F}^\pm$ in terms of the modes of the generators $e(u)$ and $f(u)$. We show that this leads to the usual nice factorised expression of the universal R-matrix of $DY(osp(1|2))$, namely $R = R_E R_H R_F$, where $R_E \in \tilde{E}^+ \otimes \tilde{F}^-$, $R_H \in \tilde{H}^+ \otimes \tilde{H}^-$ and $R_F \in \tilde{F}^+ \otimes \tilde{F}^-$. Finally, considering the action of the universal R-matrix $R$ on evaluation representations of $DY(osp(1|2))$, we obtain the evaluated R-matrix of $DY(osp(1|2))$, which coincides with the one introduced in [7], up to a normalisation factor written as a ratio of $\Gamma_1$ functions of period $2\kappa$.

2 The RTT presentation of super Yangian $Y(osp(1|2))$

We denote by $V$ the 3-dimensional $\mathbb{Z}_2$-graded vector space representation of $osp(1|2)$. The first and the third basis vectors have the grade 0 (mod 2) whereas the second has the grade 1 (mod 2). The same gradation was used by Ding [9] in order to define $U_q(\hat{osp}(1|2))$. The multiplication for the tensor product is defined for $a, b, \alpha, \beta \in Y(osp(1|2))$ by

$$(a \otimes \alpha)(b \otimes \beta) = (-1)^{|b||\alpha|}(ab \otimes \alpha\beta)$$ (2.1)

where $[a] \in \mathbb{Z}_2$ denotes the grade of $a$. Let $E_{ij}$ be the elementary matrix with entry 1 in row $i$ and column $j$ and 0 elsewhere. The “usual” super transposition $^T$ is defined by $A^T = \sum_{i,j,k,l=1}^3 (-1)^{|i||j|+|i||j|} A^{ij} E_{ij}$ for any matrix $A = \sum_{i,j=1}^3 A^{ij} E_{ij}$. The super transposition $^t$ we will use is a conjugation of the previous one:

$$A^t = \sum_{i,j,k,l=1}^3 (-1)^{|i||j|+|i||j|} J^{ij} A^{kj} J^{lk} E_{il} = J A^T J^{-1}$$ where $J = E_{31} + E_{22} - E_{13}$ (2.2)

The super permutation $P_{12}$ (i.e. $X_{21} = P_{12} X_{12} P_{12}$) is defined by $P_{12} = (-1)^{|j|} E_{ij} \otimes E_{ji}$. The super R-matrix $R_{12}(u) \in End(V \otimes V)$ is defined by:
\[ R_{12}(u) = \frac{\rho(u)}{u + 1} \left( \mathbb{I} \otimes \mathbb{I} + \frac{P}{u} - \frac{P^{t_1}}{u + \kappa} \right) \]  

(2.3)

\[ = \frac{\rho(u)}{u + 1} \]

\[
\begin{pmatrix}
\frac{u+1}{u} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{u+1}{u+\kappa - 1} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{u} & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{u+\kappa} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{u+1}{u+\kappa}
\end{pmatrix}
\]

(2.4)

where

\[
\rho(u) = \frac{\Gamma_1(u) \Gamma_1(u + \kappa - 1) \Gamma_1(u + \kappa + 1) \Gamma_1(u + 2\kappa) \Gamma_1(u + 2\kappa + 1)}{\Gamma_1(u + 1) \Gamma_1(u + \kappa) \Gamma_1(u + \kappa + 1) \Gamma_1(u + 2\kappa) \Gamma_1(u + 2\kappa + 1)}, \quad \kappa = \frac{3}{2},
\]

\[ t_1 \] is the super transposition in the first space and \( \mathbb{I} \) is the \( 3 \times 3 \) identity matrix. The function \( \Gamma_1 \) is defined by \( \Gamma_1(x|\omega) = \frac{\omega^{x/\omega}}{\sqrt{2\pi\omega}} \Gamma \left( \frac{x}{\omega} \right) \).

It is known (see for instance [7]) that:

**Proposition 2.1** The matrix \( R(u) \) satisfies

\[ R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u), \quad \text{(super Yang–Baxter)} \]  

(2.6)

\[ R_{t_1}^{t_1}(-u - \kappa) = R_{12}(u), \quad \text{(crossing symmetry)} \]  

(2.7)

\[ R_{12}(u) R_{12}(-u) = \rho(u) \rho(-u), \quad \text{(unitarity)} \]  

(2.8)

We gave in our previous paper [7] the RTT presentation of super Yangian \( Y(osp(1|2)) \). In the following, 1 will denote the unit of the algebra and \( 1_3 = 1 \mathbb{I} \).

**Theorem 2.2** The super Yangian \( Y(osp(1|2)) \) is isomorphic to the associative superalgebra \( \mathcal{U}(R) \) generated by the elements \( L_{ij}^{(n)} \) (\( 1 \leq i, j \leq 3, n \in \mathbb{Z}_{>0} \)), 1 and the defining relations, given in terms of formal series \( L(u) = 1_3 + \sum_{i,j=1}^{3} \sum_{n \in \mathbb{Z}_{>0}} L_{ij}^{(n)} u^{-n} E_{ij} = \sum_{i,j=1}^{3} L_{ij}^{(n)} E_{ij} :\)

- \[ R_{12} (u - v) \, L_1(u) \, L_2(v) = L_2(v) \, L_1(u) \, R_{12} (u - v) \]  

(2.9)

- \[ C(u) = L^t (u - \kappa) \, L(u) = 1_3 \]  

(2.10)

The Hopf algebra structure of \( \mathcal{U}(R) \) is given by

\[ \Delta(L(u)) = L(u) \hat{\otimes} L(u) \quad \text{i.e.} \quad \Delta(L_{ij}^{(n)}(u)) = \sum_{k=1}^{3} L_{ik}^{(n)}(u) \otimes L_{kj}^{(n)}(u) \]  

(2.11)

\[ S(L(u)) = L(u)^{-1}; \quad \varepsilon(L(u)) = \mathbb{I} \]  

(2.12)
3 The Drinfel’d realisation of $Y(osp(1|2))$

**Definition 3.1** Let $A^+$ be the associative superalgebra generated by the odd elements $e_k, f_k$ ($k \in \mathbb{Z}_{\geq 0}$), the even elements $h_k$ ($k \in \mathbb{Z}_{\geq 0}$), the unit 1 and the defining relations, given in terms of the generating functions $e(u) = \sum_{k=0}^{\infty} e_k u^{-k-1}$, $f(u) = \sum_{k=0}^{\infty} f_k u^{-k-1}$, $h(u) = 1 + \sum_{k=0}^{\infty} h_k u^{-k-1}$:

$$[h(u), h(v)] = 0,$$

$$\{e(u), f(v)\} = \frac{h(u) - h(v)}{u - v},$$

$$[h(u), e(v)] = -\frac{(u - v + \kappa - 1)[h(u), e(u) - e(v)]}{(u - v)(u - v + \kappa)} + \frac{h(u)(e(v) + e(u) - e_0) - (e(u) - e_0)h(u)}{u - v + \kappa},$$

$$[h(u), f(v)] = \frac{(u - v - \kappa + 1)[h(u), f(u) - f(v)]}{(u - v)(u - v - \kappa)} - \frac{h(u)(f(v) + f(u) + f_0) - (f(u) + f_0)h(u)}{u - v - \kappa},$$

$$\{e(u), e(v)\} = \frac{[e(u), e(v)]}{2(u - v)} - \frac{e_0, e(u) - e(v)}{(u - v)} - \frac{(e(u) - e(v))^2}{2(u - v)^2},$$

$$\{f(u), f(v)\} = -\frac{[f(u), f(v)]}{2(u - v)} - \frac{f_0, f(u) - f(v)}{(u - v)} - \frac{(f(u) - f(v))^2}{2(u - v)^2},$$

and the supplementary Serre relations

$$e(u)^3 = e(u)[e(u), e_0] + [e_0, e(u)],$$

$$f(u)^3 = -f(u)[f(u), f_0] + [f_0, f(u)].$$

The relations (3.1)–(3.6) are equivalent to the following commutation relations in terms of the modes $e_k, f_k, h_k$ ($k \geq 0$):

- $h_k$ and $h_l$ : $[h_k, h_l] = 0,$
- $e_k$ and $f_l$ : $\{e_k, f_l\} - h_{k+l} = 0,$
- $h_k$ and $e_l$ : $[h_0, e_l] - e_l = 0,$
- $2 [h_1, e_l] - 2 e_{l+1} = \{h_0, e_l\},$
- $2 [h_{k+2}, e_l] + 2 [h_k, e_{l+2}] - 4 [h_{k+1}, e_{l+1}] = [h_k, e_l] + \{h_{k+1}, e_l\} - \{h_k, e_{l+1}\},$
- $h_k$ and $f_l$ : $[h_0, f_l] + f_l = 0,$
- $2 [h_1, f_l] + 2 f_{l+1} = \{h_0, f_l\},$
- $2 [h_{k+2}, f_l] + 2 [h_k, f_{l+2}] - 4 [h_{k+1}, f_{l+1}] = [h_k, f_l] - \{h_{k+1}, f_l\} + \{h_k, f_{l+1}\}.$
• $e_k$ and $e_l$:
  \[ 2\{e_{k+2}, e_l\} + 2\{e_k, e_{l+2}\} - 4\{e_{k+1}, e_{l+1}\} = \{e_k, e_l\} + [e_{k+1}, e_l] - [e_k, e_{l+1}], \quad (3.17) \]

• $f_k$ and $f_l$:
  \[ 2\{f_{k+2}, f_l\} + 2\{f_k, f_{l+2}\} - 4\{f_{k+1}, f_{l+1}\} = \{f_k, f_l\} - [f_{k+1}, f_l] + [f_k, f_{l+1}], \quad (3.18) \]

The Serre relations (3.7) and (3.8) in terms of modes are conjectured to be (for $k \geq 0$)

\[
\begin{align*}
\{\{e_k, e_{k+1}\}, e_k\} &= 2e^3_k, \\
\{\{e_k, e_{k+1}\}, e_{k+1}\} &= -e_ke_{k+1}e_k - \frac{1}{2}e^2_ke_{k+1} - \frac{1}{2}e_{k+1}e^2_k, \\
\{\{e_k, e_{k+1}\}, e_{k+2}\} &= -2e^2_{k+1}e_k - 2e_{k+1}e_ke_{k+1} - 2e^2_{k+1}, \\
\{\{f_k, f_{k+1}\}, f_k\} &= -2f^3_k, \\
\{\{f_k, f_{k+1}\}, f_{k+1}\} &= f_kf_{k+1}f_k + \frac{1}{2}f^2_kf_{k+1} + \frac{1}{2}f_{k+1}f^2_k, \\
\{\{f_k, f_{k+1}\}, f_{k+2}\} &= 2f^2_{k+1}f_k + 2f_{k+1}f_kf_{k+1} + 2f_kf^2_{k+1}.
\end{align*}
\]

This conjecture is supported by two results: on the one hand we have proved them in the graded algebra (to be defined below); on the other hand we checked explicitly the first nine relations.

**Proposition 3.2** The algebra, $\mathcal{A}^+$ (resp. $\mathcal{U}(R)$), can be equipped with an ascending filtration with the degree of the generators defined by $\deg(e_k) = k$, $\deg(f_k) = k$, $\deg(h_k) = k$ ($\deg(L^{ij}_{(k)}) = k - 1$) and $\deg(xy) = \deg(x) + \deg(y)$, for $x, y \in \mathcal{A}^+$ (resp. $x, y \in \mathcal{U}(R)$).

Let $\text{gr}(\mathcal{A}^+)$ and $\text{gr}(\mathcal{U}(R))$ denote the corresponding graded algebras and $\text{osp}(1|2)[u]$ denote the Lie super algebra of polynomials in an indeterminate $u$ with coefficients in $\text{osp}(1|2)$. The algebras $\text{gr}(\mathcal{A}^+)$, $\text{gr}(\mathcal{U}(R))$ and $\mathcal{U}(\text{osp}(1|2)[u])$ are isomorphic.

**Proof:** We first recall the notion of graded algebras. We start with an algebra $\mathcal{A}$, equipped with a grading $\deg$, i.e. a morphism from $(\mathcal{A}, \deg)$ to $(\mathbb{N}, +)$. One introduces $\mathcal{A}_k = \{x \in \mathcal{A}, \deg(x) \leq k\}$, $k \geq 0$ and $\text{gr}(\mathcal{A}_k) = \mathcal{A}_k / \mathcal{A}_{k-1}$ for $k \geq 1$, $\text{gr}(\mathcal{A}_0) = \mathcal{A}_0$. Then the graded algebra of $\mathcal{A}$ is $\text{gr}(\mathcal{A}) = \oplus_{k \geq 0} \text{gr}(\mathcal{A}_k)$.

The algebra $\text{gr}(\mathcal{A}^+)$ is the algebra generated by $e_k, f_k, h_k$ ($k \in \mathbb{Z}$) and the relations (3.9)-(3.24) where the right hand side of the equalities is substituted by zero. These equalities are equivalent to:

\[
\begin{align*}
[h_k, h_l] &= 0, \quad \{e_k, f_l\} = h_{k+l}, \quad (3.25) \\
[h_k, e_l] &= e_{k+l}, \quad [h_k, f_l] = -f_{k+l}, \quad (3.26) \\
\{e_n, e_m\} &= \{e_0, e_{n+m}\}, \quad \{f_n, f_m\} = \{f_0, f_{n+m}\}, \quad (3.27) \\
\{e_l, e_n\} &= 0, \quad \{f_l, f_n\} = 0. \quad (3.28)
\end{align*}
\]

which are the relations of $\mathcal{U}(\text{osp}(1|2)[u])$. Then, $\text{gr}(\mathcal{A}^+)$ is isomorphic to $\mathcal{U}(\text{osp}(1|2)[u])$.

The isomorphism between $\text{gr}(\mathcal{U}(R))$ and $\mathcal{U}(\text{osp}(1|2)[u])$ is proved in [7].
Theorem 3.3 The linear map
\[
\phi : \mathcal{A}^+ \rightarrow U(R)
\]
\[
e(-u) \mapsto L^{33}(u)^{-1}L^{23}(u)
\]
\[
f(-u) \mapsto L^{32}(u)L^{33}(u)^{-1}
\]
\[
h(-u) \mapsto L^{22}(u)L^{33}(u)^{-1} + L^{32}(u)L^{33}(u)^{-1}L^{23}(u)L^{33}(u)^{-1}
\]
is an isomorphism of algebra.

Proof: The first step of the proof is to show that \(\phi\) is a morphism of algebra.

\[
\{\phi(e(-u)), \phi(f(-v))\} = \{L^{33}(u)^{-1}L^{23}(u), L^{32}(v)L^{33}(v)^{-1}\}
\]
\[
= -L^{33}(u)^{-1}L^{32}(v) [L^{23}(u), L^{33}(v)^{-1}] + L^{33}(u)^{-1} \{L^{23}(u), L^{32}(v)\} L^{33}(v)^{-1}
\]
\[
= \frac{1}{u-v} (L^{33}(u)^{-1}\phi(h(-v))L^{33}(u) + L^{33}(u)^{-1}\phi(h(-u))L^{33}(u))
\]
\[
= \frac{1}{u-v} (\phi(h(-u)) - \phi(h(-v)))
\]
We used the relations (2.9) and \([\phi(h(-v)), L^{33}(u)] = 0\). For the other relations, the proofs are similar once one remarks, in particular, that

\[
L^{21}(u)L^{33}(u)^{-1} = L^{22}(u)L^{33}(u)^{-1}\phi(f(-u)) - \phi(f(-u)) - [\phi(f_0), L^{22}(u)L^{33}(u)^{-1}],
\]
\[
L^{12}(u)L^{33}(u)^{-1} = \phi(e(-u+1)) - L^{22}(u)L^{33}(u)^{-1}\phi(e(-u+1)) - [\phi(e_0), L^{22}(u)L^{33}(u)^{-1}],
\]
\[
L^{31}(u)L^{33}(u)^{-1} = (\phi(f(-u)))^2 + \{\phi(f_0), \phi(f(-u))\},
\]
\[
L^{13}(u)L^{33}(u)^{-1} = (\phi(e(-u+1)))^2 - \{\phi(e_0), \phi(e(-u+1))\}.
\]
The second step consists in proving the surjectivity of \(\phi\). The relations (3.29)-(3.31), (3.35)-(3.38) and the following particular relations, coming from (2.10),

\[
C^{22}(u) = L^{22}(u-\kappa)L^{22}(u) + L^{32}(u-\kappa)L^{12}(u) - L^{12}(u-\kappa)L^{32}(u) = 1
\]
\[
C^{33}(u) = L^{11}(u-\kappa)L^{33}(u) + L^{21}(u-\kappa)L^{23}(u) - L^{31}(u-\kappa)L^{13}(u) = 1
\]
constitute a system of nine equations. We can show that these equations are independent and allow us to express all the generators of \(U(R)\) in terms of \(\phi(e_n), \phi(h_k)\) and \(\phi(f_l)\) \((n, k, l \geq 0)\). This proves the surjectivity of \(\phi\).

The final step is the proof of the injectivity of \(\phi\). The map \(\phi\) preserves the filtration, therefore defines a surjective morphism between \(gr(\mathcal{A}^+)\) and \(gr(U(R))\). Since the injectivity of the latter morphism is given by the proposition 3.2, the injectivity of \(\phi\) is proved.

Note that the RLL relations encode both the commutation relations and the Serre relations.

Let \(\mathfrak{E}^+\) and \(\mathfrak{F}^+\) be the subalgebras of \(\mathcal{A}^+\), without the unit, generated by \(\{e_k, h_l | k, l \geq 0\}\) and \(\{f_k, h_l | k, l \geq 0\}\), respectively. Let \(\mathfrak{E}^+, \mathfrak{H}^+\) and \(\mathfrak{F}^+\) be the subalgebras of \(\mathcal{A}^+\) generated by \(e_k, h_k\) and
\( f_k \) with \( k \geq 0 \), respectively and \( \tilde{E}^*, \tilde{H}^* \) and \( \tilde{F}^* \) be the same algebras with the unit.

**Proposition 3.4** \( \phi \) provides \( \mathcal{A}^+ \) with a coalgebra structure given by:

- **counit:**
  \[
  \varepsilon(e(u)) = 0, \quad \varepsilon(f(u)) = 0, \quad \varepsilon(h(u)) = 1. \tag{3.41}
  \]

- **coproduct:**
  \[
  \Delta(e(u)) = e(u) \otimes 1 + \left(h(u) \otimes e(u) + [h(u), f_0] \otimes (e(u)^2 - \{e(u), e_0\})\right) \left(1 \otimes 1 + X_{12}(u)\right) \tag{3.42}
  \]
  \[
  X_{12}(u) = \sum_{k>0} (-1)^k \left(f(u-1) \otimes e(u) + (f(u-1)^2 + \{f(u-1), f_0\}) \otimes (e(u)^2 - \{e(u), e_0\})\right)^k
  \]
  \[
  \Delta(f(u)) = 1 \otimes f(u) + \left(f(u) \otimes h(u) + (f(u)^2 + \{f(u), f_0\}) \otimes [h(u), e_0]\right) \left(1 \otimes 1 + Y_{12}(u)\right) \tag{3.43}
  \]
  \[
  Y_{12}(u) = \sum_{k>0} (-1)^k \left(f(u) \otimes e(u+1) + (f(u)^2 + \{f(u), f_0\}) \otimes (e(u+1)^2 - \{e(u+1), e_0\})\right)^k
  \]
  \[
  \Delta(h(u)) = 1 \otimes 1 + \{\Delta(e(u)), f_0 \otimes 1 + 1 \otimes f_0\} \tag{3.44}
  \]

**Proof:** To clarify this proof, we denote \( \Delta_A \) (resp. \( \Delta_U \)) the coproduct of \( \mathcal{A}^+ \) (resp. \( U(R) \)) and \( \varepsilon_A \) (resp. \( \varepsilon_U \)) the counit of \( \mathcal{A}^+ \) (resp. \( U(R) \)).

We construct \( \Delta_A \) thanks to the relation \( \Delta_U \circ \phi = (\phi \otimes \phi) \circ \Delta_A \). At first, we calculate \( \Delta_U(f(u)) \). We begin by

\[
\Delta_U(\phi(f(-u))) = \Delta_U \left(L^{32}(u)L^{33}(u)^{-1}\right) = \Delta_U \left(L^{32}(u)\right) \Delta_U \left(L^{33}(u)^{-1}\right) \tag{3.45}
\]

and

\[
\Delta_U \left(L^{33}(u)^{-1}\right) = \left(\sum_{k=1}^3 \left(L^{k3}(u)L^{33}(u)^{-1} \otimes L^{k3}(u)L^{33}(u)^{-1}\right) L^{33}(u) \otimes L^{33}(u)\right)^{-1}
\]

\[
= \left(L^{33}(u)^{-1} \otimes L^{33}(u)^{-1}\right) \sum_{n \geq 0} \left(\sum_{k=1}^2 L^{k3}(u)L^{33}(u)^{-1} \otimes L^{k3}(u)L^{33}(u)^{-1}\right)^n \tag{3.46}
\]

Using the results of the proof of the previous theorem, we get

\[
\Delta_U(\phi(f(-u))) = (\phi \otimes \phi) \left[1 \otimes f(-u) + (f(-u) \otimes h(-u) + (f(-u)^2 + \{f(-u), f_0\} \otimes [h(-u), e_0]\) \right]
\]

\[
\times \left(1 \otimes 1 + Y_{12}(-u)\right)\]

\[
= (\phi \otimes \phi) \circ \Delta_A(f(-u))
\]

By the injectivity of \( \phi \), we find \( \boxed{3.43} \). For \( \Delta_A(e(u)) \), the proof is similar and for \( \Delta_A(h(u)) \), the equality \( \boxed{3.44} \) is obvious.

For the counit, the proof is similar by using \( \varepsilon_A = \varepsilon_U \circ \phi \). \( \blacksquare \)
In the following, we replace the notations $D$ and $Y$ to $DY$.

To prove (3.49), we need to calculate the anticommutator of the relation (3.44).

\[ \Delta(e(u)) = e(u) \otimes 1 + h(u) \otimes e(u) + \text{mod}(F^{+} \otimes E^{+}E^{+}), \]
\[ \Delta(f(u)) = 1 \otimes f(u) + f(u) \otimes h(u) + \text{mod}(F^{+}F^{+} \otimes F^{+}), \]
\[ \Delta(h(u)) = h(u) \otimes h(u) + \text{mod}(E^{+} \otimes F^{+}) \]

To prove (3.49), we need to calculate the anticommutator of the relation (3.44).

\[ \text{Proposition 4.2} \]

**The Hopf algebra structure of** 

\[ \text{Definition 4.1} \]

To prove (3.49), we need to calculate the anticommutator of the relation (3.44).

\[ \text{4 The construction of the double } DY(osp(1|2)) \]

In the following, we replace the notations $L(u), L^{ij}(u), e(u), f(u)$ and $h(u)$ by $L^{+}(u), L^{ij}(u), e^{+}(u), f^{+}(u)$ and $h^{+}(u)$ respectively.

\[ \text{4.1 RTT presentation} \]

**Proposition 4.2** The bilinear form $<,>$ between the two subalgebras of $DU(R), U^{-}(R) = \{ L^{ij}_{(n)} | n \in \mathbb{Z}_{<0} \}$ with opposite coproduct and $U^{+}(R) = U(R) = \{ L^{ij}_{(n)} | n \in \mathbb{Z}_{>0} \}$ given by:

\[ < L^{-}_{1}(u), L^{+}_{2}(v) >= R_{i1}^{-1}(v - u) \text{ i.e. } < L^{-ij}_{1}(u), L^{+kl}_{1}(v) >= (R^{-1}(v - u))^{ij}_{kl} \]
\[ < L^{-}(u), 1_{3} >=< 1_{3}, L^{+}(v) >=< 1_{3}, 1_{3} >= 1 \]

is a Hopf pairing i.e. satisfies the conditions for $a, b \in U^{-}(R)$ and $\alpha, \beta \in U^{+}(R)$:

\[ < a, \alpha \beta >= (-1)^{[\alpha][\beta]} < \Delta(a), \beta \otimes \alpha >, \ < ab, \alpha >=< a \otimes b, \Delta(\alpha) > \]
\[ < a, 1_1 >=< 1_1, a >, \ < S^{-1}(a), \alpha >=< a, S(\alpha) > \]
\[ < a \otimes b, \alpha \otimes \beta >= (-1)^{[b][\alpha]} < a, \alpha > < b, \beta > \]
**Definition 4.3** Let $DA$ be the associative superalgebra generated by the unit $1$, the even elements $h_k$ ($k \in \mathbb{Z}$) and the odd elements $e_k, f_k$ ($k \in \mathbb{Z}$), gathered in the generating functions

\[
\begin{align*}
    e^+(u) &= \sum_{k=0}^{\infty} e_k u^{-k-1}, \quad e^-(u) = -\sum_{k=-\infty}^{-1} e_k u^{-k-1}, \quad e(u) = e^+(u) - e^-(u), \\
f^+(u) &= \sum_{k=0}^{\infty} f_k u^{-k-1}, \quad f^-(u) = -\sum_{k=-\infty}^{-1} f_k u^{-k-1}, \quad f(u) = f^+(u) - f^-(u), \\
h^+(u) &= 1 + \sum_{k=0}^{\infty} h_k u^{-k-1}, \quad h^-(u) = 1 - \sum_{k=-\infty}^{-1} h_k u^{-k-1}
\end{align*}
\]

satisfying the relations

\[
\begin{align*}
h^\alpha (u) h^\beta (v) &= h^\beta (v) h^\alpha (u) \text{ where } \alpha, \beta = \pm, \\
\{e(u), f(v)\} &= \delta(u - v) (h^+(u) - h^-(u)) \text{ with } \delta(u - v) = \sum_{k=-\infty}^{\infty} u^k v^{-k-1}, \\
(u - v - 1)(2u - 2v + 1) e(u) h^\pm (v) &= (u - v + 1)(2u - 2v - 1) h^\pm (v) e(u), \\
(u - v + 1)(2u - 2v - 1) f(u) h^\pm (v) &= (u - v - 1)(2u - 2v + 1) h^\pm (v) f(u), \\
(u - v - 1)(2u - 2v + 1) e(u) e(v) &= -(u - v + 1)(2u - 2v - 1) e(v) e(u), \\
(u - v + 1)(2u - 2v - 1) f(u) f(v) &= -(u - v - 1)(2u - 2v + 1) f(v) f(u)
\end{align*}
\]

and the supplementary Serre relations

\[
\begin{align*}
\{e_0, e^\pm (u)\} e^\pm (u) &= \frac{5}{2} \{e_0^2, e^\pm (u)\} - 2u [e_0^2, e^\pm (u)] + e_0 e^\pm (u) e_0 - \{e_0, e_1\} e^\pm (u), \\
\{f_0, f^\pm (u)\} f^\pm (u) &= -\frac{5}{2} \{f_0^2, f^\pm (u)\} - 2u [f_0^2, f^\pm (u)] - f_0 f^\pm (u) f_0 - \{f_0, f_1\} f^\pm (u).
\end{align*}
\]

The bialgebra structure of $DA$ is given by (3.42), (3.43), (3.44) and (3.41), adding superscripts $\pm$ to $e(u), f(u)$ and $h(u)$.
Remark 4.1 $DA$ could be alternatively defined by the relations (3.1)–(3.6), adding a superscript $\epsilon$ to the generating functions with parameter $u$, and a superscript $\epsilon'$ to the generating functions with parameter $v$, where $\epsilon, \epsilon' = \pm$. The Serre relations (3.7) and (3.8) with $\epsilon$ are also valid in $DA$, but not sufficient, because they do not couple enough positive modes with the negative ones.

The commutation relations in $DA$ between $e_k, f_k$ and $h_k$ ($k \in \mathbb{Z}$) are the same as the relations (3.9)–(3.18) with, in this case, $k, l \in \mathbb{Z}$ and with the following additional relations, for $k \in \mathbb{Z}$:

$$2[h_{-1}, e_k] - 2e_{k-1} = [h_{-1}, e_{k-2}] - \{h_{-1}, e_{k-1}\},$$  \hspace{1cm} (4.23)

$$2[h_{-1}, f_k] - 2f_{k-1} = [h_{-1}, f_{k-2}] + \{h_{-1}, f_{k-1}\}$$ \hspace{1cm} (4.24)

Similarly to section 3, we conjecture that the Serre relations in terms of modes are of the form (3.19)–(3.24) with now $k \in \mathbb{Z}$. As before, this conjecture is supported by explicit computations on the first orders. Moreover, we can define a graded algebra for $DA$, $\text{grad}(DA)$, as in proposition 3.2. Then the Serre relations in $\text{grad}(DA)$ are the relations (3.19)–(3.24), for $k \in \mathbb{Z}$, substituting the right hand side of the equalities by zero. Beside this, the expansion of (4.21), (4.22) in terms of modes shows that the remaining terms of the Serre relations in $DA$ have strictly lower degree and are also cubic. These results are sufficient to prove the next theorems.

Let $A^+$ and $A^-$ be the subalgebras of $DA$ generated respectively by $\{e_n, f_n, h_n|n \in \mathbb{Z}_{\geq 0}\}$ and $\{e_k, f_k, h_k, E_{-1}, F_{-1}|k \in \mathbb{Z}_{<0}\}$.

Theorem 4.4 The linear maps $\Phi : DA \rightarrow DU(R)$ and $\phi^\pm : A^\pm \rightarrow U^\pm(R)$ given by

$$e^\pm(-u) \mapsto L^{\pm 33}(u)^{-1}L^{23}(u)$$

$$f^\pm(-u) \mapsto L^{\pm 32}(u)L^{\pm 33}(u)^{-1}$$

$$h^\pm(-u) \mapsto L^{\pm 22}(u)L^{\pm 33}(u)^{-1} + L^{\pm 32}(u)L^{\pm 33}(u)^{-1}L^{\pm 23}(u)L^{\pm 33}(u)^{-1}$$

are isomorphisms of bialgebra.

Proof: The proof is similar to the one of theorem 3.3 and proposition 3.4.

For later convenience, we introduce the following combinations in $DA$ (for $k \in \mathbb{Z}$):

$$E_{2k+1} = \{e_k, e_{k+1}\} - \frac{e_k^2}{4}$$  \hspace{1cm} (4.28)

$$F_{2k+1} = \{f_k, f_{k+1}\} - \frac{f_k^2}{4}.$$  \hspace{1cm} (4.29)

In particular, $E_{-1}$ and $F_{-1}$ will be essential as well as their image in $U^-(R)$:

$$\Phi : E_{-1} \mapsto -\frac{5}{4} \left( \sum_{k \geq 0} (-L^{33}_{(0)})^k L^{23}_{(0)} \right)^2 - \sum_{k \geq 0} (-L^{33}_{(0)})^k L^{13}_{(0)}$$  \hspace{1cm} (4.30)

$$F_{-1} \mapsto -\frac{5}{4} \left( L^{32}_{(0)} \sum_{k \geq 0} (-L^{33}_{(0)})^k \right)^2 + L^{31}_{(0)} \sum_{k \geq 0} (-L^{33}_{(0)})^k$$  \hspace{1cm} (4.31)
Proposition 4.5 The Hopf pairing $\langle , \rangle$ between $\mathcal{A}^-$ and $\mathcal{A}^+$ is given by:

\[
\langle f^-(u), e^+(v) \rangle = \frac{1}{u - v}, \quad \langle e^-(u), f^+(v) \rangle = \frac{-1}{u - v}, \quad \text{(4.32)}
\]
\[
\langle h^-(u), h^+(v) \rangle = \frac{(u - v - 1)(2u - 2v + 1)}{(u - v + 1)(2u - 2v - 1)}, \quad \text{(4.33)}
\]
\[
\langle F_{-1}, e_0^2 \rangle = 1, \quad \langle E_{-1}, f_0^2 \rangle = 1 \quad \text{(4.34)}
\]

or in terms of generators $(n, k \geq 0)$:

\[
\begin{align*}
\langle 1, 1 \rangle &= 1, \quad \langle f_{-k-1}, e_n \rangle = -\delta_{n,k}, \quad \langle e_{-k-1}, f_n \rangle = \delta_{n,k}, \quad \text{(4.35)} \\
\langle h_{-k-1}, h_n \rangle &= -\frac{1}{3} \left(2 + \left(-\frac{1}{2}\right)^{n-k}\right) \binom{n}{k}, \quad \text{(4.36)}
\end{align*}
\]

Proof: We use the theorem [4.4] to prove this proposition, for example

\[
\langle e^-(u), f^+(v) \rangle = \langle L^{-33}(u)^{-1}L^{-23}(u), L^{+32}(v)L^{+33}(v)^{-1} \rangle = \langle L^{-33}(u), L^{+32}(v)\rangle \Delta(L^{+32}(v))\Delta(L^{+33}(v))^{-1} = \langle L^{-23}(u), L^{+32}(v) \rangle > \frac{1}{u - v} \quad \text{(4.40)}
\]

The difficult point lies in the step between equalities (4.38) and (4.39). It is done using the explicit form (3.46) and showing that only the first term of the sum contributes to the pairing.

For $\langle f^-(u), e^+(v) \rangle$, the proof is similar.

The identity $\langle h^-(u), h^+(v) \rangle = \langle h^-(u), \{e^+(v), f_0\} + 1 \rangle$ and the two previous results allow us to obtain the relation (4.33).

The pairings $\langle F_{-1}, e_0^2 \rangle$ and $\langle E_{-1}, f_0^2 \rangle$ are calculated thanks to the explicit expressions (4.30) and (4.31).

To find the explicit form (4.36), we remark that $(n \geq 0)$:

\[
\begin{align*}
\langle h^-(u), h_0 \rangle &= 1, \quad \langle h^-(u), h_1 \rangle = u + \frac{1}{2} \quad \text{(4.41)} \\
\langle h^-(u), h_{n+2} \rangle &= \left(2u + \frac{1}{2}\right) - \langle h^-(u), h_{n+1} \rangle + (u + 1) \left(u - \frac{1}{2}\right) < h^-(u), h_n \rangle = 0. \quad \text{(4.42)}
\end{align*}
\]

A trivial induction shows that $\langle h^-(u), h_{n+2} \rangle = \frac{1}{3} (u - \frac{1}{2})^{n+2} + \frac{2}{3} (u + \frac{1}{2}) (u + 1)^{n+1} + \frac{1}{3} (u + 1)^{n+1} \quad \text{which gives the result.}$

\[\square\]

### 4.3 Dual bases

Now, we look for bases of $\mathcal{A}^+$ and $\mathcal{A}^-$. Let $\mathcal{E}^-$ and $\mathcal{F}^-$ be the subalgebras of $\mathcal{A}^-$, generated by $\{e_k, E_{-1}, h_l | k, l < 0\}$ and $\{f_k, F_{-1}, h_l | k, l < 0\}$ respectively. Let $\mathcal{E}^-, \mathcal{H}^-$ and $\mathcal{F}^-$ be the subalgebras of $\mathcal{A}^-$, without the unit, generated by $\{e_k, E_{-1} | k < 0\}$, $\{h_k | k < 0\}$ and $\{f_k, F_{-1} | k < 0\}$, respectively and $\tilde{\mathcal{E}}^-, \tilde{\mathcal{H}}^-$ and $\tilde{\mathcal{F}}^-$ be the same algebras with the unit.
Proposition 4.6 (i) Let \( a_\pm \in A^\pm \), then \( a_+ \in \tilde{\mathcal{E}}^+ \tilde{\mathcal{H}}^+ \tilde{\mathcal{F}}^+ \) and \( a_- \in \tilde{\mathcal{F}}^- \tilde{\mathcal{H}}^- \tilde{\mathcal{E}}^- \).

(ii) \( \forall e_\pm \in \tilde{\mathcal{E}}^\pm, h_\pm \in \tilde{\mathcal{H}}^\pm, f_\pm \in \tilde{\mathcal{F}}^\pm, \)

\[
< f_-h_-e_-, e_+h_+f_+ > = (-1)^{|e_+||e_-|} < f_-, e_+ < h_-, h_+ > < e_-, f_+ > \quad (4.43)
\]

Proof: (i) We use a proof similar to the one given in [11]. We first consider \( A^+ \). As \( \{e_k, h_i, f_m | k, l, m \geq 0 \} \) is a generating set of \( A^+ \), it is enough to prove that any monomial \( \prod_{i=0}^{j} e_k^{\alpha_i} h_i^{\beta_i} f_m^{\gamma_i} \), with \( j, k, \alpha_i, \beta_i, m_i, \gamma_i \in \mathbb{Z}_{\geq 0} \), is a linear combination of elements belonging to \( \tilde{\mathcal{E}}^+ \tilde{\mathcal{H}}^+ \tilde{\mathcal{F}}^+ \). We make an induction on the degree \( p = \sum_{i=0}^{j} (\alpha_i + \beta_i + \gamma_i) \) of such monomial.

For \( p = 1 \), the assertion is obvious.

Let assume the assertion is true for \( p \geq 1 \) and consider an element \( \prod_{i=0}^{j} e_k^{\alpha_i} h_i^{\beta_i} f_m^{\gamma_i} \) such that \( \sum_{i=0}^{j} (\alpha_i + \beta_i + \gamma_i) = p + 1 \). The last \( p \) generators can be ordered using the induction hypothesis on \( p \). Then, three cases are possible depending on the first element:

- It belongs to \( \tilde{\mathcal{E}}^+ \). Then, the element is ordered.
- It belongs to \( \tilde{\mathcal{H}}^+ \). If the second generator belongs to \( \tilde{\mathcal{H}}^+ \) or \( \tilde{\mathcal{F}}^+ \), the assertion for \( p + 1 \) is proven. It remains the case where the second generator (say \( e_k \)) belongs to \( \tilde{\mathcal{E}}^+ \). We make an induction on the index, \( l \), of the first generator \( h_i \). Let \( a_{l-1} \) be the ordered product of the last \( p - 1 \) generators.

  If \( l = 0 \) then

\[
h_0 e_k a_{l-1} = e_k h_0 a_{l-1} + e_k a_{l-1} \quad \text{due to (3.11)}
\]

As the induction on \( p \) allows us to order \( h_0 a_{l-1} \), we can order this element.

  If \( l = 1 \) then

\[
h_1 e_k a_{l-1} = e_k h_1 a_{l-1} + e_{k+1} a_{l-1} + h_0 e_k a_{l-1} + e_k h_0 a_{l-1} \quad \text{due to (3.12)}
\]

Using the case \( l = 0 \) for \( h_0 e_k a_{l-1} \) and the induction hypothesis on \( p \) for \( h_1 a_{l-1} \) and \( h_0 a_{l-1} \), we can order this element.

Let \( l \geq 2 \) and assume that for \( l - 2 \) and \( l - 1 \) the elements can be ordered. Then, using the commutation relations (3.13), one gets

\[
h_l e_k a_{l-1} = ( -h_{l-2} e_k + h_{l-1} e_k/2 - h_{l-2} e_{k+1}/2 + h_{l-2} e_k/2 + 2h_{l-1} e_{k+1}) a_{l-1} + b_{l+1}
\]

where \( b_{l+1} \) can be ordered thanks to the induction on \( p \). The other elements are ordered by the induction on \( l - 2 \) and \( l - 1 \).

This ends the induction on \( l \).

- It belongs to \( \tilde{\mathcal{F}}^+ \). We denote it \( f_m \). If the second element belongs to \( \tilde{\mathcal{F}}^+ \), the assertion for \( p + 1 \) is proven. If the second element belongs to \( \tilde{\mathcal{H}}^+ \), we prove the assertion for \( p + 1 \) analogously to the previous case, using (3.14), (3.13) and (3.10). Finally, if the second element \( e_k \) belongs to \( \tilde{\mathcal{E}}^+ \), then

\[
f_m e_k a_{l-1} = -e_k f_m a_{l-1} + h_{m+k} a_{l-1} \quad \text{due to (3.10)}
\]

and \( f_m a_{l-1}, h_{m+k} a_{l-1} \) are ordered thanks to the hypothesis on \( p \).
This ends the induction on $p$ and (i) is proven for $\mathcal{A}^+$.

For $\mathcal{A}^-$, the proof is almost similar. However, an additional difficulty appears because of exchange relations between $e_{-k}$ and $h_{-l}$. The relation $(4.23)$ allows us to order $\hat{e}_{-l} = 2e_{-l} - e_{-l-1} - e_{-l-2}$ ($l \geq 1$) and $h_{-l}$. Fortunately, $e_{-k}$ can be expressed in terms of $\hat{e}_{-l}$:

$$\forall k \geq 1, e_{-k} = \frac{1}{3} \sum_{l=k}^{+\infty} \left( 1 - \left( -\frac{1}{2} \right)^{l-k+1} \right) \hat{e}_{-l}$$  \hspace{1cm} (4.48)

Therefore, we can order $e_{-k}$ and $h_{-l}$. Likewise, $e_{-k}$ and $h_{-n}$ can be ordered.

(ii) Let $\bar{f}_- \in \mathcal{F}^- \mathcal{H}^-, \bar{e}_+ \in \mathcal{E}^+ \mathcal{H}^+, e_- \in \mathcal{E}^-$ and $f_+ \in \mathcal{F}^+$. One computes

$$\langle \bar{f}_- e_-, \bar{e}_+ f_+ \rangle = \langle \bar{f}_- \otimes e_-, \Delta(\bar{e}_+) \Delta(f_+) \rangle$$

$$= \langle \bar{f}_- \otimes e_-, (\bar{e}_+ \otimes 1 + mod(\mathcal{A}_- \otimes \mathcal{E}_+))(1 \otimes f_+ + mod(\mathcal{F}_- \otimes \mathcal{A}_+)) \rangle$$

$$= (-1)^{[\bar{e}_+][e_-]} \langle \bar{f}_-, \bar{e}_+ \rangle < e_-, f_+ >$$ \hspace{1cm} (4.51)

using the following identities:

$$\langle e_-, \mathcal{E}_+ f_+ \rangle = (-1)^{[f_+][\mathcal{E}_+]} \langle \Delta(e_-), f_+ \otimes \mathcal{E}_+ \rangle$$

$$\langle \bar{f}_-, \bar{e}_+ \mathcal{F}_+ \rangle = (-1)^{[\bar{e}_+][\mathcal{F}_+]} \langle \Delta(\bar{f}_-), \mathcal{F}_+ \otimes \bar{e}_+ \rangle$$

$$\langle \bar{f}_-, \mathcal{A}_+ \mathcal{F}_+ \rangle = (-1)^{[\mathcal{F}_+][\mathcal{A}_+]} \langle \Delta(\bar{f}_-), \mathcal{F}_+ \otimes \mathcal{A}_+ \rangle$$

$$\langle \bar{f}_-, \mathcal{F}_+ \mathcal{A}_+ \rangle = (-1)^{[\mathcal{F}_+][\mathcal{A}_+]} < 1 \otimes \bar{f}_- + mod(\mathcal{F}_- \otimes \mathcal{A}_+), \mathcal{F}_+ \otimes \bar{e}_+ > = 0$$ \hspace{1cm} (4.52)

$$\langle \bar{f}_-, \mathcal{H}_+ \mathcal{F}_+ \rangle = (-1)^{[\mathcal{F}_+][\mathcal{H}_+]} < 1 \otimes \bar{f}_- + mod(\mathcal{F}_- \otimes \mathcal{H}_+), \mathcal{F}_+ \otimes \bar{e}_+ > = 0$$ \hspace{1cm} (4.53)

Let $f_- \in \mathcal{F}^-, h_- \in \mathcal{H}^-, e_+ \in \mathcal{E}^+$ and $h_+ \in \mathcal{H}^+$ such that $\bar{f}_- = f_- h_-$ and $\bar{e}_+ = e_+ h_+$. Then, we prove analogously that $\langle \bar{f}_-, \bar{e}_+ \rangle = \langle f_- h_-, e_+ h_+ \rangle = \langle f_-, e_+ \rangle < h_- h_+ >$ using $\Delta(h_+) = 1 \otimes h_+ + \sum_i h_i \otimes h'_i + mod(\mathcal{E}_+ \mathcal{H}_+ \otimes \mathcal{H}_+ \mathcal{F}_+)$ where $h_i$ and $h'_i \in \mathcal{H}^+$. Remarking that $[\bar{e}_+] = [e_+]$ and that for the unit the theorem is obvious, we prove the second assertion of the theorem.

**Remark 4.2** The point (ii) of the previous theorem shows that the dual of $\mathcal{E}^-$ (resp. $\mathcal{H}^-$, $\mathcal{F}^-$) is $\mathcal{F}^+$ (resp. $\mathcal{H}^+$, $\mathcal{E}^+$). In particular, one has $\langle \mathcal{E}^-, \mathcal{E}^+ \rangle = 0 = \langle \mathcal{E}^-, \mathcal{H}^+ \rangle$, and the same relations changing $\mathcal{E}$ by $\mathcal{F}$.

**Theorem 4.7** Bases of $\mathcal{E}^+$, $\mathcal{F}^-$, $\mathcal{F}^+$ and $\mathcal{E}^-$ are respectively

$$\mathcal{B}_E^+ = \{ e_0 a_0 E_1 b_0 e_1 a_1 E_3 b_1 \cdots e_k a_k E_{2k+1} b_k \cdots | a_0, a_1, \ldots, b_0, b_1 \in \mathbb{Z}_{\geq 0} \}$$ \hspace{1cm} (4.55)

$$\mathcal{B}_F^- = \{ F_1 b_0 f_0 a_0 F_3 b_1 f_2 a_1 \cdots F_{2k-1} b_k f_{k-1} a_k \cdots | a_0, a_1, \ldots, b_0, b_1 \in \mathbb{Z}_{\geq 0} \}$$ \hspace{1cm} (4.56)

$$\mathcal{B}_F^+ = \{ F_{2k+1} b_k f_k a_k \cdots F_3 b_1 f_1 a_1 F_1 b_0 f_0 a_0 \cdots | a_0, a_1, \ldots, b_0, b_1 \in \mathbb{Z}_{\geq 0} \}$$ \hspace{1cm} (4.57)

$$\mathcal{B}_E^- = \{ e_{-k-1} a_k E_{-2k-1} b_k \cdots e_{-2} a_1 E_{-3} b_1 e_{-1} a_0 E_{-1} b_0 \cdots | a_0, a_1, \ldots, b_0, b_1 \in \mathbb{Z}_{\geq 0} \}$$ \hspace{1cm} (4.58)
Proof: To prove that $B_{E+}$, $B_{F-}$, $B_{F+}$ and $B_{E-}$ generate $\tilde{E}^+$, $\tilde{F}^-$, $\tilde{F}^+$ and $\tilde{E}^-$, respectively, the methods are the same as the ones used in the proof of the assertion $(i)$ of proposition 4.6, using the commutation relations between $e_k$ and $e_l$ as well as the proved results on the Serre relations. The independence of the set of generators is given by their independence in the corresponding graded algebras which is obvious.

Theorem 4.8 For any element $b_+$ of $B_{E+}$ (resp. $B_{F+}$), there is one and only one element $b_-$ in $B_{F-}$ (resp. $B_{E-}$) such that the pairing $< b_-, b_+ >$ does not vanish. They are given by:

\[
< \prod_{n=0}^{k} F_{2n-1} b_n, \prod_{m=0}^{k} E_{2m+1} a_m > = (-1)^{\sum_{0 \leq n < m \leq k} c_n c_m} \prod_{l=0}^{k} (-1)^i a_l! b_l!
\]

\[
< \prod_{n=0}^{k} e_{-n} 2a_n, \prod_{m=0}^{k} F_{2m+1} a_m > = (-1)^{\sum_{0 \leq n < m \leq k} c_n c_m} \prod_{l=0}^{k} a_l! b_l!
\]

where $k \in \mathbb{Z}_{\geq 0}$, $a_k, b_k \in \mathbb{Z}_{\geq 0}$, $c_k \in \{0, 1\}$, $\prod_{n=0}^{k} e_n = e_0 e_1 \ldots e_{k-1} e_k$ and $\prod_{n=0}^{k} e_n = e_k e_{k-1} \ldots e_1 e_0$.

Proof: We first consider the pairing $< a_-, e_k >$ for $a_- \in B_{F-}$. If $a_-$ has degree at least 2, there are $a', a'' \in B_{F-}$ such that $a_- = a' a''$. Using \((4.35)\), we get

\[
< a_-, e_k > = < a'_- \otimes a''_-, \Delta(e_k) >
\]

\[
= < a'_- \otimes a''_-, e_k \otimes 1 + 1 \otimes e_k + \sum_{l=0}^{k-1} h_l \otimes e_{k-l-1} + \text{mod}(\tilde{F}^+ \otimes \tilde{E}^+ \tilde{E}^+) >
\]

\[
= < a'_- \otimes a''_-, e_k \otimes 1 + 1 \otimes e_k + \sum_{l=0}^{k-1} h_l \otimes e_{k-l-1} > = 0,
\]

This shows that the pairing of $< a_-, e_k >$ is different from zero only for $a_- = f_{-k-1}$ and given by \((4.33)\).

Consider now $< a_-, e_k^2 >$. If $a_- = f_{-m}$ has degree 1, one computes:

\[
< f_{-m}, e_k^2 > = < \Delta(f_{-m}), e_k \otimes e_k >= < f_{-m} \otimes 1 + 1 \otimes f_{-m} + \sum_{l=0}^{k-1} h_l \otimes f_{-m-l-1}, e_k \otimes e_k >= 0
\]

\[(4.61)\]

For $a_- = a'_- a''_-$ of degree at least 2, one has

\[
< a_-, e_k^2 > = < a'_- \otimes a''_-, \Delta(e_k)^2 > = < a'_- \otimes a''_-, \sum_{l=0}^{k-1} [e_k, h_l] \otimes e_{k-l-1} >
\]

\[
= < a'_- \otimes a''_-, -e_k \otimes e_{k-1} + \sum_{l=1}^{k-1} \Psi_l(e_k, e_{k+1}, \ldots, e_{k+l}) \otimes e_{-l-1} >
\]

\[(4.63)\]
where \( \Psi_l(e_k, e_{k+1}, \ldots, e_{k+l}) \) is a linear combination of \( e_k, e_{k+1}, \ldots, e_{k+l} \). Then, for \( k \geq 1 \), \( \langle a_-, e_k^2 \rangle \) is equal 1 for \( a'_- = f_{-k-1} \) and \( a''_- = f_{-k} \) (i.e. for \( a_- = F_{2k-1} \)) and 0 otherwise. For \( k = 0 \), by (4.34) and the previous calculation, we know that the pairing of \( e_0^2 \) does not vanish only with \( F_{-1} \). Similarly, \( \langle f_{-k-1}^{-1}, a_+ \rangle \) is equal to \( 1 \) if \( a_+ = F_{2k+1} \), and to 0 in the other cases.

Now, we show by induction that \( \langle F_{-2k-1}, e_0^2 \rangle = b! \) and \( \langle F_{-2k-1} f_{-k-1}, e_0^{2b+1} \rangle = -b! \) and that the other pairings with \( e_0^{2b} \) or \( e_0^{2b+1} \) are zero. We assume these assertions for \( b < b_0 \).

\[
\langle a_-, e_k^{2b_0} \rangle = \langle a'_- \otimes a''_-, \left( e_k^2 \otimes 1 + 1 \otimes e_k^2 + \sum_{l=0}^{k-1} \Psi_l(e_k, e_{k+1}, \ldots, e_{k+l}) \otimes e_{k-l-1} \right) \rangle^{b_0} = \langle a'_- \otimes a''_-, \sum_{p=0}^{b_0} \left( \binom{b_0}{p} e_k^{2p} \otimes e_k^{2b_0-p} \right) \rangle
\]

(4.64)

Due to the hypothesis, this pairing is non zero only if \( a' = F_{-2k-1}^p \) and \( a'' = F_{-2k-1}^{b_0-p} \) and, in this case, is equal to

\[
\left( \binom{b_0}{p} \right) \langle F_{-2k-1}^p, e_k^{2p} \rangle = \langle F_{-2k-1}^{b_0-p}, e_k^{2(b_0-p)} \rangle = \left( \binom{b_0}{p} \right) p! (b_0 - p)! = b_0!
\]

(4.65)

A similar result is proven for \( e_k^{2b_0+1} \) and then by induction on \( b \), the assertions are proven.

Similarly, \( \langle f_{-k-1}^{2a}, E_{2k+1} e \rangle = a! \) and \( \langle f_{-k-1}^{2a+1}, e_k E_{2k+1} e \rangle = -a! \).

We can sum up all these results by

\[
\forall a, b \in \mathbb{Z}_{\geq 0}, c \in \{0, 1\}, \quad F_{-2k-1}^- f_{-k-1}^{-2a+c}, e_k^{2b+c} E_{2k+1} e = (-1)^c a! b!
\]

(4.66)

and all other pairings with \( e_k^{2b+c} E_{2k+1} e \) are zero. Similarly, we show that all other pairings with \( F_{-2k-1}^{-1} f_{-k-1}^{-2a+c} \) are also zero.

\[
\langle \prod_{n=0}^{l} F_{-2n+1}^{-1} f_{-n+1}^{-2a'_n+c'_n}, \prod_{m=0}^{k} e_m^{2b_m+c_m} E_{2m+1} a_m \rangle
\]

\[
= \langle F_{-1}^{-b_0} f_{-1}^{-2a'_0+c'_0} \otimes \prod_{n=1}^{l} F_{-2l-1}^{-1} f_{-l+1}^{-2a'_l+c'_l} \Delta(e_0^{2b_0+c_0} E_1 a_0) \Delta(\prod_{m=1}^{k} e_m^{2b_m+c_m} E_{2m+1} a_m) \rangle
\]

\[
= (-1)^{c_0(c'_1 + \cdots + c'_l)} \delta_{a'_0, a_0} \delta_{b'_0, b_0} \delta_{c'_0, c_0} (-1)^{c_0} a_0! b_0! \langle \prod_{n=1}^{l} F_{-2l-1}^{-1} f_{-l+1}^{-2a'_l+c'_l}, \prod_{m=1}^{k} e_m^{2b_m+c_m} E_{2m+1} a_m \rangle
\]

Repeating this calculus \( k \) times, we prove (4.59) of the theorem.

(4.60) is proven along the same lines.

**Remark 4.3** Since \( \tilde{\mathcal{H}}^+ \) is Abelian, one of its basis is \( \{ h_{0}^{a_0} h_{1}^{a_1} \ldots h_{k}^{a_k} \ldots | a_0, a_1, \ldots \in \mathbb{Z}_{\geq 0} \} \). In addition, the pairing restricted of the subalgebras \( \tilde{\mathcal{H}}^- \) and \( \tilde{\mathcal{H}}^+ \) is not degenerated.
Remark 4.4 A corollary of the previous results is that the pairing between $\mathcal{A}^-$ and $\mathcal{A}^+$ is not degenerated. Then, thanks to the isomorphisms $\phi^\pm$ and $\Phi$, neither is the pairing between $\mathcal{U}^-(R)$ and $\mathcal{U}^+(R)$.

Theorem 4.9 $\mathcal{U}^+(R) \otimes \mathcal{U}^-(R)$ is the quantum double of $\mathcal{U}^+(R)$ with the multiplication between $\mathcal{U}^+(R)$ and $\mathcal{U}^-(R)$ defined by (4.2). Thus, it is isomorphic, as a Hopf algebra, to the quantum double of $Y(osp(1|2))$, denoted $DY(osp(1|2))$.

Similarly, $\mathcal{A}^+(R) \otimes \mathcal{A}^-(R)$ is the quantum double of $\mathcal{A}^+(R)$.

Proof: From $(\Delta \otimes 1)\Delta(L^\pm(u)) = L^\pm(u) \otimes L^\pm(u) \otimes L^\pm(u)$, the cross-multiplication in a quantum double is defined by

\[
L_2^-(v)L_1^+(u) = < S(L_2^-(v)), L_1^-(u) > L_1^+(u)L_2^+(v) < L_2^-(v), L_1^+(u) >
\]

which is equivalent to (4.2).

The other assertions are obvious.

5 Universal R-matrix

5.1 Construction of the universal R-matrix

We express the universal R-matrix of double super Yangian $DY(osp(1|2))$ according to the generators of Drinfel’d basis. Since $DA$ is the quantum double of $A^+$, it admits a canonical universal R-matrix given by $\mathcal{R} = \sum x_i \otimes x^i$ where $\{x_i, i \in \mathbb{N}\}$ is the basis of $A^+$ and $\{x^i, i \in \mathbb{N}\} \in A^-$ is the dual basis (i.e. $< x^i, x_j > = \delta_j^i$). Therefore, thanks to the explicit expression of the pairing, we have the following result.

Theorem 5.1 The universal R-matrix can be factorised as

\[
\mathcal{R} = \mathcal{R}_E \mathcal{R}_H \mathcal{R}_F
\]

where $\mathcal{R}_E \in \tilde{\mathcal{E}}^+ \otimes \tilde{\mathcal{F}}^-$, $\mathcal{R}_H \in \tilde{\mathcal{H}}^+ \otimes \tilde{\mathcal{H}}^-$ and $\mathcal{R}_F \in \tilde{\mathcal{F}}^+ \otimes \tilde{\mathcal{E}}^-$. The explicit expressions of the universal factors $\mathcal{R}_E$ and $\mathcal{R}_F$ are

\[
\mathcal{R}_E = \prod_{i \geq 0} \left[ \exp \left( e_i^2 \otimes F_{-2i-1} \right) (1 \otimes 1 - e_i \otimes f_{-i-1}) \exp \left( E_{2i+1} \otimes f_{-i-1}^2 \right) \right]
\]

\[
\mathcal{R}_F = \prod_{i \geq 0} \left[ \exp \left( F_{2i+1} \otimes e_{-i-1}^2 \right) (1 \otimes 1 + f_i \otimes e_{-i-1}) \exp \left( f_i^2 \otimes E_{-2i-1} \right) \right]
\]
Proof: The factorisation of the the universal R-matrix is involved by the relation (ii) of the proposition (1.6). In addition, to prove the expression of \( R_E \), we expand the exponentials and the products

\[
R_E = \prod_{i \geq 0} \sum_{a_i, b_i \geq 0} \frac{1}{a_i! b_i!} (e_i^{2a_i} \otimes F_{-2i-1}^{a_i}) (1 \otimes 1 - e_i \otimes f_{-i-1}) (E_{2i+1}^{b_i} \otimes f_{-i-1}^{2b_i}) \quad (5.4)
\]

\[
= \prod_{i \geq 0} \sum_{a_i, b_i, c_i \geq 0} \frac{(-1)^{c_i}}{a_i! b_i!} (e_i^{2a_i+c_i} E_{2i+1}^{b_i} \otimes F_{-2i-1}^{a_i} f_{-i-1}^{2b_i+c_i}) \quad (5.5)
\]

Therefore, \( R_E \) can be written as \( \sum_{x_i \in E^+} x_i \otimes x^i \) and \( < x^i, x_j > = \delta^j_i \) due to (1.59). The proof to find the explicit form of \( R_F \) is similar.

Theorem 5.2 The factor \( R_H \) of the universal R-matrix of \( DY(osp(1|2)) \) is given by

\[
R_H = \prod_{n \geq 0} \exp \left\{ \sum_{i \geq 0} \left( \frac{d}{d u} K_-(u) \right)_i \otimes \left( C(T^{1/2}) K_+(v + 3n + \frac{3}{2}) \right)_{i-1} \right\} \quad (5.7)
\]

where \( K_\pm(u) = \ln h^\pm(u) \), \( C(q) = q + 1 + q^{-1} \), \( T \) is the shift operator: \( Tf(u) = f(u + 1) \) and \( (\psi(u))_i = \psi_i \) for any function \( \psi(u) = \sum_i \psi_i u^{-i+1} \).

Proof: The proof is inspired by the results exposed in [12]. Starting from the pairing (4.33), a direct calculation shows that

\[
<K_-(u), K_+(v)> = \ln \frac{(u - v - 1)(2u - 2v + 1)}{(u - v + 1)(2u - 2v - 1)} \quad (5.8)
\]

from which it follows

\[
<\frac{d}{du} K_-(u), K_+(v)> = \frac{1}{u - v - 1} + \frac{1}{u - v + 1/2} - \frac{1}{u - v + 1} - \frac{1}{u - v - 1/2} \quad (5.9)
\]

\[
= (T^{-1} + T^{1/2} - T - T^{-1/2}) \frac{1}{u - v} \quad (5.10)
\]

Therefore one obtains

\[
<\frac{d}{du} K_-(u), (T^{-1} + T^{1/2} - T - T^{-1/2})^{-1} K_+(v)> = \frac{1}{u - v} \quad (5.11)
\]

The formal inversion of the operator \( (T^{-1} + T^{1/2} - T - T^{-1/2})^{-1} \) is given by

\[
(T^{-1} + T^{1/2} - T - T^{-1/2})^{-1} = \sum_{n \geq 0} T^{3n+2} + T^{3n+3/2} + T^{3n+1} \quad (5.12)
\]
Let \( B(q) \) be the \( q \)-analog of the symmetrised Cartan matrix of \( osp(1|2) \). We define \( C(q) \) by the relation
\[
B(q)^{-1} = \frac{1}{[2\kappa]_q^{1/2}} C(q),
\]
\( C(q) \) is a matrix with polynomial entries in \( q \) and \( q^{-1} \) and positive coefficients. One gets \( C(q) = q + 1 + q^{-1} \). It follows that
\[
\sum_{n \geq 0} < \frac{d}{du} K_-(u), C(T^{1/2}) K_+(v + 3n + 3/2) > = \frac{1}{u - v}
\]
(5.13)

Since the pairing (5.13) exhibits a duality relation in diagonal form, one gets immediately the expression (5.7) for the universal factor \( R_H \).

5.2 Evaluated \( R \)-matrix

**Proposition 5.3** Let \( \pi \) be the fundamental 3-dimensional representation of \( osp(1|2) \) with representation space \( V \) and \( V_z \) a \( \mathbb{C} \)-module. Then \( \pi_z \) such that
\[
\pi_z: DA \rightarrow V_z \otimes V
\]
\[
e_n \mapsto z^n E_{12} + z'^n E_{13}
\]
\[
f_n \mapsto z^n E_{21} - z'^n E_{23}
\]
\[
h_n \mapsto z^n E_{11} + (z^n - z'^n) E_{22} - z'^n E_{33}
\]
is an evaluation representation of the double super Yangian \( \text{DY}(osp(1|2)) \) for \( z' = z + \frac{1}{2} \).

**Proof:** The image by \( \pi_z \) of the elements of \( DA \) have to satisfy the commutation relations (4.15)-(4.22). For example, we prove for (4.19). We use that \( \pi_z(e(u)) = \delta(z - u) E_{12} + \delta(z' - u) E_{23} \). Then, one gets :
\[
(u - v - 1)(2u - 2v + 1) \pi_z(e(u)) \pi_z(e(v)) + (u - v + 1)(2u - 2v - 1) \pi_z(e(v)) \pi_z(e(u))
\]
\[
= [(u - v - 1)(2u - 2v + 1)\delta(z - u)\delta(z' - v) + (u - v + 1)(2u - 2v - 1)\delta(z - v)\delta(z' - u)] E_{13}
\]
\[
= [(z - z' - 1)(2z - 2z' + 1) + (z' - z + 1)(2z' - 2z - 1)] E_{13} = 0
\]
(5.17)
The other commutation relations are proven analogously.

**Theorem 5.4** Let \( \pi_z \) and \( \pi_w \) be two fundamental evaluation representations, then
\[
(\pi_z \otimes \pi_w) R = R_{12}(z - w)
\]
(5.18)
where the \( R \)-matrix \( R_{12}(z) \) is given by (2.4).

\(^{1}\)The presence of \( q^{1/2} \) instead of \( q \) in the definition of \( C(q) \) is due to the normalisation of the fermionic simple root.
Proof:

\[
(\pi_z \otimes \pi_w)R_E = 1_3 \otimes 1_3 + \sum_{j > i \geq 0} \pi_z(e_i e_j) \otimes \pi_w(f_{-i-1} f_{-j-1})
\]

\[
+ \sum_{i \geq 0} \left[ \pi_z(e_i^2) \otimes \pi_w(F_{-2i-1}) - \pi_z(e_i) \otimes \pi_w(f_{-i-1}) + \pi_z(E_{2i+1}) \otimes \pi_w(f_{-i-1}^{-2}) \right]
\]

\[
= \frac{E_{12} \otimes E_{21}}{z-w} - \frac{E_{23} \otimes E_{32}}{z-w} - \frac{E_{12} \otimes E_{32}}{z-w - \frac{1}{2}} + \frac{E_{23} \otimes E_{21}}{z-w + \frac{1}{2}} + \frac{4(z-w)+3}{(z-w)(2(z-w)+1)}E_{13} \otimes E_{31}
\]

The explicit form of \( R_F \) is proven analogously.

The calculation of \( R_H \) is standard, but one has to use the following formula introduced in [12]:

\[
\prod_{n \geq 0} \exp \left\{ \sum_{i \geq 0} \left( \frac{1}{u - \gamma} \right)_i \left( \ln \frac{x - \alpha + Nn + 1}{x - \beta + Nn + 1} \right)_{-i-1} \right\} = \frac{\Gamma(\gamma - \beta + 1)}{\Gamma(\gamma - \alpha + 1)}
\]

(5.19)

where \( (\psi(u))_i = \psi_i \) is defined as in theorem 5.2.

Acknowledgements: We warmfully thank J. Avan and A. Molev for discussions and advices. Some preliminary computations were done using the symbolic manipulation program FORM, by J. Vermaseren [13].

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