Nonconforming virtual element method for $2m$th order partial differential equations in $\mathbb{R}^n$ with $m > n$

Xuehai Huang

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Abstract
The $H^m$-nonconforming virtual elements of any order $k$ on any shape of polytope in $\mathbb{R}^n$ with constraints $m > n$ and $k \geq m$ are constructed in a universal way. A generalized Green’s identity for $H^m$ inner product with $m > n$ is derived, which is essential to devise the $H^m$-nonconforming virtual elements. By means of the local $H^m$ projection and a stabilization term using only the boundary degrees of freedom, the $H^m$-nonconforming virtual element methods are proposed to approximate solutions of the $m$-harmonic equation. The norm equivalence of the stabilization on the kernel of the local $H^m$ projection is proved by using the bubble function technique, the Poincaré inequality and the trace inequality, which implies the well-posedness of the virtual element methods. The optimal error estimates for the $H^m$-nonconforming virtual element methods are achieved from an estimate of the weak continuity and the error estimate of the canonical interpolation. Finally, the implementation of the nonconforming virtual element method is discussed.

Keywords $H^m$-nonconforming virtual element · Generalized Green’s identity · Polyharmonic equation · Norm equivalence · Error analysis

Mathematics Subject Classification 65N30 · 65N12 · 65N22

1 Introduction

The $H^m$-nonconforming virtual elements of order $k \in \mathbb{N}$ on a very general polytope $K \subset \mathbb{R}^n$ in any dimension and any order with constraints $m \leq n$ and $k \geq m$ have been devised in [17]. While an important case $m = 3$ and $n = 2$, i.e. the triharmonic equation in two dimensions is not involved in [17]. To this end, and also for theoretical considerations, we will study the $H^m$-nonconforming virtual element $(K, \mathcal{N}_K, V_K)$ for case $m > n$ in this paper, which can be considered as the second part of the work [17]. Here
$\mathcal{N}_K$ is the set of degrees of freedom, and $V_K$ the finite-dimensional space of shape functions. The virtual element can be defined on polytopes of any shape, and thus allows the division of the domain into different type of polytopes [9, 10], which makes the discrete method easier to capture the singularity of the solution. The key feature of the virtual element method is that it is completely determined by the degrees of freedom, and the virtual element space is only used for the analysis rather than entering the discrete method for elliptic problems.

It is arduous to design $H^m$-conforming or nonconforming finite elements for large $k$ and $m$, especially $m > n$. With the help of the bubble functions, Wu and Xu constructed the minimal $H^m$-nonconforming finite elements on simplices in any dimension with $m = n + 1$ in [31]. Enriching the bilinear form with few interior penalty terms, they proposed a family of interior penalty nonconforming finite element methods for arbitrary $m$ and $n$ in [30], whose shape functions are polynomials of the smallest degree $m$. In two and three dimensions, $H^m$-conforming finite elements for arbitrary $m$ with shape functions being lower order polynomials were devised in [12, 32] for triangular meshes, in [33] for tetrahedral meshes and in [22] for rectangular meshes. And some $H^m$-nonconforming elements of lower degree on triangular meshes for any $m$ were studied in [23]. In addition to standard conforming and nonconforming finite element methods, a $C^0$ interior penalty method in [20] and a cubic $H^1$-nonconforming macro-element method in [24] were developed for a sixth-order elliptic equation in two dimensions, and some mixed finite element methods were advanced in [18, 19, 26] for 2mth-order elliptic equations with $m > n$. When $m \leq n$, We refer to [3, 21, 34, 35] for $H^m$-conforming finite elements, and [27, 28] for $H^m$-nonconforming finite elements.

Constructing the $H^m$-conforming or nonconforming finite elements on general polytopes for arbitrary $k$, $m$ and $n$ in a universal way is extremely difficult, while it is possible for the virtual elements. As a matter of fact, $H^m$-conforming virtual elements in two dimensions with arbitrary $m$ were designed in [2, 11, 15], which were nontrivial to extend to higher dimension $n > 2$. We refer to [8] for the $H^2$-conforming virtual elements in three dimensions. While in [1, 4, 17, 36, 37], $H^m$-nonconforming virtual elements on general polytopes in any dimension $n$ with constraint $m \leq n$ were studied in details. By the way, we refer to [29] for an $H^1$-nonconforming Crouzeix–Raviart type element on polygonal meshes.

In order to construct the $H^m$-nonconforming virtual element in any order on the polytope with any shape in any dimension for $m > n$, by adopting the integration by parts, we first prove the following generalized Green’s identity for the $H^m$ space

$$(\nabla^m u, \nabla^m v)_K = ((-\Delta)^m u, v)_K + \sum_{j=1}^{n-1} \sum_{F \in \mathcal{F}(K)} \sum_{a \in A_j} \left( D^2_{m-j} |\alpha| (u), \frac{\partial |\alpha| v}{\partial v^a_F} \right)_F$$

$$+ \sum_{a \in \mathcal{A}_n} \sum_{\delta \in \mathcal{F}^n(K)} D_{\delta,a}^{2m-n-|\alpha|} (u) \frac{\partial |\alpha| v}{\partial v^\delta} (\delta),$$

where $\mathcal{F}^j(K)$ is the set of all $(n-j)$-dimensional faces of the polytope $K$, $A_j$ the set consisting of all $n$-dimensional multi-indexes $\alpha = (\alpha_1, \ldots, \alpha_n)$ with
The local $H^m$ projection $\Pi^K$ is defined in view of the local $m$-harmonic problem. Adopting the Taylor’s theorem, we prove the inverse inequality of polynomials on the general polytope $K$ by only assuming $K$ is star-shaped, while $K$ admitting a virtual quasi-uniform triangulation is assumed in [16]. According to this inverse inequality, the operator $(-\Delta)^m : \mathbb{P}_k(K) \to \mathbb{P}_{k-2m}(K)$ is shown to be onto and have a continuous right inverse. Based on the fact that the operator $(-\Delta)^m : \mathbb{P}_k(K) \to \mathbb{P}_{k-2m}(K)$ is onto, we propose a stabilization term using only the boundary degrees of freedom, whereas all the degrees of freedom are involved in the stabilization term in [17]. After introducing the discrete right hand side term, we design the $H^m$-nonconforming virtual element methods to approximate solutions of the $m$-harmonic equation. For the case $2m \leq k \leq 3m - 2$, we define the right hand side term as $\langle f, Q^{-1}_h v_h + Q^{k-2m}_h (v_h - \Pi_h v_h) \rangle$, rather than $\langle f, Q^{-1}_h v_h \rangle$ in [17]. The gain of this new right hand side term is that we do not need to modify the virtual element space $V_k(K)$ for $2m \leq k \leq 3m - 2$, whereas the modification of $V_k(K)$ is required in [17].

We analyze the $H^m$-nonconforming virtual element methods under the assumptions that each element in the mesh $\mathcal{T}_h$ is star-shaped, and $\mathcal{T}_h$ admits a virtual quasi-uniform triangulation. Applying the bubble function technique, the generalized Green’s identity, the Poincaré inequality and the trace inequality, the norm equivalence of the stabilization on $\ker(\Pi^K) \cap V_h(K)$ is derived. As in [17], after obtaining a bound on the jump $\| \nabla^2 v_h \|$ and the error estimate of the canonical interpolation, we achieve the optimal error estimates of the $H^m$-nonconforming virtual element methods. We also consider the implementation of the virtual element method.

The rest of this paper is organized as follows. Some notations and mesh conditions are shown in Sect. 2. The generalized Green’s identity and the fully $H^m$...
-nonconforming virtual element are presented in Sect. 3. In Sect. 4, we propose the $H^m$-nonconforming virtual element method, and prove the norm equivalence and the weak continuity. We develop the optimal error analysis for the $H^m$-nonconforming virtual element methods in Sect. 5. Finally we discuss the implementation of the virtual element method in Sect. 6.

2 Preliminaries

2.1 Notation

In this paper we will adopt the same notations as in [17]. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded polytope. For any nonnegative integer $r$ and $1 \leq \ell \leq n$, notation $\mathbb{T}_\ell(r) := (\mathbb{R}^r)^\ell = \prod_{j=1}^{\ell} \mathbb{R}^r$ stands for the set of $r$-tensor spaces over $\mathbb{R}^\ell$. Given a bounded domain $G \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ and a non-negative integer $k$, let $H^k(G;\mathbb{T}_\ell(r))$ be the usual Sobolev space of functions over $G$ taking values in the tensor space $\mathbb{T}_\ell(r)$, whose norm and semi-norm are denoted by $\| \cd \|_{k,G}$ and $| \cd |_{k,G}$ respectively. Set $H^k(G) := H^k(G;\mathbb{T}_\ell(0))$. Define $H^k_0(G)$ as the closure of $C_0^\infty(G)$ with respect to the norm $\| \cd \|_{k,G}$ and define $H^k_0(G;\mathbb{T}_\ell(r))$ in a similar way. Let $(\cd,\cd)_G$ be the standard inner product on $L^2(G;\mathbb{T}_\ell(r))$. If $G$ is $\Omega$, we abbreviate $\| \cd \|_{k,G}, | \cd |_{k,G}$ and $(\cd,\cd)_G$ by $\| \cd \|_k, | \cd |_k$ and $(\cd,\cd)$, respectively. Let $\mathbb{P}_k(G)$ be the set of all polynomials over $G$ with the total degree no more than $k$, whose tensorial version space is denoted by $\mathbb{P}_k(G;\mathbb{T}_\ell(r))$. Let $\mathbb{P}_k(G) := \{0\}$ if $k < 0$. Let $Q^0_k$ be the $L^2$-orthogonal projection onto $\mathbb{P}_k(G;\mathbb{T}_\ell(r))$. For any $F \subset \partial G$, denote by $v_{G,F}$ the unit outward normal to $\partial G$. Without causing any confusion, we will abbreviate $v_{G,F}$ as $v$ for simplicity.

For an $n$ dimensional multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{Z}^+ \cup \{0\}$, define $|\alpha| := \sum_{i=1}^n \alpha_i$. For $0 \leq j \leq n$, let $A_j$ be the set consisting of all multi-indexes $\alpha$ with $\sum_{i=j+1}^n \alpha_i = 0$, i.e., non-zero index only exists for $1 \leq i \leq j$. For any non-negative integer $\ell$, define the scaled monomial $\mathbb{M}_\ell(G)$ on a $j$-dimensional domain $G$

$$ \mathbb{M}_\ell(G) := \left\{ \left( \frac{x - x_G}{h_G} \right)^\alpha \alpha \in A_j, |\alpha| \leq \ell \right\}, $$

where $h_G$ is the diameter of $G$ and $x_G$ is the centroid of $G$. And $\mathbb{M}_\ell(G) := \emptyset$ if $\ell < 0$. For ease of presentation, let $N_{G,\ell} := \#\mathbb{M}_\ell(G)$, and all the functions in $\mathbb{M}_\ell(G)$ be $\{m_{G,i}^{\ell}\}_{i=1}^{N_{G,\ell}}$.

Given $r$-tensors $\tau, \varsigma \in \mathbb{T}_\ell(r)$ and a vector $v \in \mathbb{R}^\ell$, define the scalar product $\tau : \varsigma \in \mathbb{R}$ and the dot product $\tau \cdot v \in \mathbb{T}_\ell(r-1)$ by (cf. [26])

$$ \tau : \varsigma := \sum_{(j_1, \ldots, j_\ell) \in \{1, \ldots, \ell\}^\ell} \tau_{j_1, \ldots, j_\ell} \varsigma_{j_1, \ldots, j_\ell}, $$

$$ (\tau \cdot v)_{j_1, \ldots, j_{\ell-1}} := \sum_{i=1}^{\ell} \tau_{j_1, \ldots, j_{\ell-1}, i} v_i \quad \forall \ (j_1, \ldots, j_{\ell-1}) \in \{1, \ldots, \ell\}^{\ell-1}, $$

which will be abbreviated as $\tau v$. 

\[ Springer\]
Let $\{T_h\}$ be a family of partitions of $\Omega$ into nonoverlapping simple polytopal elements with $h := \max_{K \in T_h} h_K$. Let $\mathcal{F}_h^r$ be the set of all $(n - r)$-dimensional faces of the partition $T_h$ for $r = 1, 2, \ldots, n$, and its boundary part

$$\mathcal{F}_h^{r, \partial} := \{F \in \mathcal{F}_h^r : F \subset \partial \Omega\},$$

and interior part $\mathcal{F}_h^{r, i} := \mathcal{F}_h^r \setminus \mathcal{F}_h^{r, \partial}$. For simplicity, let $\mathcal{F}_h^0 := T_h$. Moreover, we set for each $K \in T_h$

$$\mathcal{F}(K) := \{F \in \mathcal{F}_h^r : F \subset \partial K\}.$$ 

The superscript $r$ in $\mathcal{F}_h^r$ represents the co-dimension of an $(n - r)$-dimensional face $F$. Similarly, we define

$$\mathcal{F}(F) := \{e \in \mathcal{F}_h^{r + j} : e \subset F\}.$$ 

Here $j$ is the co-dimension relative to the face $F$. For any $(n - 2)$-dimensional face $e \in \mathcal{F}_h^2$, denote

$$\partial^{-1} e := \{F \in \mathcal{F}_h^1 : e \subset \partial F\}.$$ 

Similarly for any $(n - 1)$-dimensional face $F \in \mathcal{F}_h^1$, let

$$\partial^{-1} F := \{K \in T_h : F \in \mathcal{F}(K)\}.$$ 

For any $F \in \mathcal{F}_h^r$ with $1 \leq r \leq n - 1$, let $v_{F, 1}, \ldots, v_{F, r}$ be its mutually perpendicular unit normal vectors, and define the surface gradient on $F$ as

$$\nabla_F v := \nabla v - \sum_{i=1}^r \frac{\partial v}{\partial v_{F, i}} v_{F, i},$$

(2.1)

namely the projection of $\nabla v$ to the face $F$, which is independent of the choice of the normal vectors. And denote by $\text{div}_F$ the corresponding surface divergence. For any $\delta \in \mathcal{F}_h^m$ and $i = 1, \ldots, n$, let $v_{\delta, i} := (0, \ldots, 0, 1, 0, \ldots, 0)$ be the $n$-tuple with all components equal to 0, except the $i$th, which is 1. For any $F \in \mathcal{F}_h^r$ and $\alpha \in A_r$ for $r = 1, \ldots, n$, set

$$\frac{\partial^{\alpha}}{\partial v_F^\alpha} := \frac{\partial^{\alpha}}{\partial v_{F, 1}^{\alpha_1} \cdots \partial v_{F, r}^{\alpha_r}}.$$ 

For non-negative integers $m$ and $k$, let

$$H^m(T_h) := \{v \in L^2(\Omega) : v|_K \in H^m(K) \text{ for each } K \in T_h\},$$

$$\mathbb{P}_k(T_h) := \{v \in L^2(\Omega) : v|_K \in \mathbb{P}_k(K) \text{ for each } K \in T_h\}.$$ 

For a function $v \in H^m(T_h)$, equip the usual broken $H^m$-type norm and semi-norm
For any $K \in \mathcal{T}_h$, $\delta \in \mathcal{F}^0(K)$, and any function $v$ defined on $K$, we will rewrite $v(x_\delta)$ as $v(\delta)$ for simplicity.

We introduce jumps on $(n-1)$-dimensional faces. Consider two adjacent elements $K^+$ and $K^-$ sharing an interior $(n-1)$-dimensional face $F$. Denote by $n^+$ and $n^-$ the unit outward normals to the common face $F$ of the elements $K^+$ and $K^-$, respectively.

For a scalar-valued or tensor-valued function $v$, write $v^+ := v|_{K^+}$ and $v^- := v|_{K^-}$. Then define the jump on $F$ as follows:

$$\left[ v \right] := v^+ n^+ \cdot v^- n^-.$$

On a face $F$ lying on the boundary $\partial \Omega$, the above term is defined by

$$\left[ v \right] := v n_F \cdot v.$$

### 2.2 Mesh conditions

We impose the following conditions on the mesh $\mathcal{T}_h$.

(A1) Each element $K \in \mathcal{T}_h$ and each face $F \in \mathcal{F}_h^r$ for $1 \leq r \leq n-1$ is star-shaped with a uniformly bounded chunkiness parameter.

(A2) There exists a quasi-uniform simplicial mesh $\mathcal{T}_h^*$ such that each $K \in \mathcal{T}_h$ is a union of some simplexes in $\mathcal{T}_h^*$.

Throughout this paper, we also use “$\lesssim \cdots$” to mean that “$\lesssim C \cdots$”, where $C$ is a generic positive constant independent of mesh size $h$, but may depend on the chunkiness parameter of the polytope, the degree of polynomials $k$, the order of differentiation $m$, the dimension of space $n$, and the shape regularity and quasi-uniform constants of the virtual triangulation $\mathcal{T}_h^*$, which may take different values at different appearances. And $A \asymp B$ means $A \lesssim B$ and $B \lesssim A$. Hereafter, we always assume $k \geq m$.

Note that (A1) and (A2) imply $\text{diam}(F) \asymp \text{diam}(K)$ for all $F \in \mathcal{F}(K)$, $1 \leq r \leq n-1$.

For a star-shaped domain $D$, the following trace inequality of $H^1(D)$ holds [14, (2.18)]

$$\|v\|_{0,\partial D}^2 \lesssim h_D^{-1} \|v\|_{0,D}^2 + h_D \|v\|_{1,D}^2 \quad \forall \ v \in H^1(D). \quad (2.2)$$

When $D \subset \mathbb{R}$, the notation $\|v\|_{0,\partial D}$ means $\|v\|_{L^\infty(\partial D)}$.

### 3 $H^m$-nonconforming virtual element with $m > n$

In this section, we will construct the $H^m$-nonconforming virtual element with integer $m > n \geq 1$. For any scalar or tensor-valued smooth function $v$, nonnegative integer $j$, $F \in \mathcal{F}_h^r$ with $1 \leq r \leq n$, and $\alpha \in \mathbb{A}_r$, we use $D^j_{F,\alpha}(v)$ to denote some $j$th order derivative of $v$ restricted on $F$, which may take different expressions at different appearances.
For \( n = 1 \) and any \( e \in \mathcal{T}_h \), applying the integration by parts in one dimension, we have for any \( u \in H^2(e) \) and \( v \in H^m(e) \)
\[
(u^{(m)}, v^{(m)})_e = (-1)^m (u^{(2m)}, v)_e + \sum_{\delta \in \mathcal{F}^1(e)} \sum_{i=0}^{m-1} (-1)^i u^{(2m-1-i)}(\delta) \frac{\partial v^{(i)}(\delta)}{\partial \nu_e} v_{e,\delta}.
\]
This is just the Green’s identity in one dimension. Here \( v^{(i)} \) means the \( i \)th order derivative of \( v \), and
\[
v_{e,\delta} = \begin{cases} 1, & \text{if } \delta \text{ is the right end point of } e, \\ -1, & \text{if } \delta \text{ is the left end point of } e. \end{cases}
\]

### 3.1 Generalized Green’s identity in two dimensions

Then consider the generalized Green’s identity in two dimensions in this subsection, i.e. \( n = 2 \).

For each \( e \in \mathcal{F}^1_h \), denote by \( t_e \) the unit tangent vector, which will be also represented by \( v_{e,2} \) for ease of presentation. Let \( \EuScript{G} \) be the set of all permutations of \( (1, 2, \ldots, \ell) \) for each positive integer \( \ell \). For \( i = 0, 1, \ldots, \ell \), define a set
\[
\mathfrak{G}(\ell, i) := \{(j_1, \ldots, j_\ell) : \text{there exists } \sigma \in \mathfrak{S}_\ell \text{ such that } j_{\sigma(1)} = \cdots = j_{\sigma(i)} = 1, \ j_{\sigma(i+1)} = \cdots = j_{\sigma(\ell)} = 2\}.
\]

Apparently
\[
\mathfrak{G}(\ell, 0) = \{(2, \ldots, 2)\}, \quad \mathfrak{G}(\ell, \ell) = \{(1, \ldots, 1)\}.
\]

For \( 0 \leq i \leq \ell \leq r \), any \( \tau \in \mathbb{T}_2(r) \) and \( \sigma = (j_1, \ldots, j_\ell) \in \mathfrak{G}(\ell, i) \), define \( \tau v^\sigma_e \in \mathbb{T}_2(r - \ell) \) as
\[
\tau v^\sigma_e := \tau_{1,j_1} \cdots \tau_{\ell,j_\ell} v^\sigma_e.
\]

We also use \( \tau v^\sigma_e \) to mean \( \tau \) when \( \ell = 0 \) for ease of presentation.

**Lemma 1** Let \( K \in \mathcal{T}_h \), \( e \in \mathcal{F}^3(K) \) and \( s \) be a positive integer. It holds for any \( \tau \in H^s(e; \mathbb{T}_2(s)) \) and \( (\nabla^s v)|_e \in L^2(e; \mathbb{T}_2(s)) \)
\[
(\tau, \nabla^s v)_e = \sum_{j=0}^{s} \sum_{\sigma \in \mathfrak{S}_j(s)} (-1)^{s-j} \left( \frac{\partial^{s-j}(\tau v^\sigma_e)}{\partial \tau^{s-j}_e}, \frac{\partial v}{\partial \nu^{j}_e} \right)_e
+ \sum_{j=0}^{s-1} \sum_{\ell=0}^{\tau} \sum_{\sigma \in \mathfrak{S}_j(\ell)} \sum_{\delta \in \partial e} v_{e,\delta} (-1)^{s-1-j} \frac{\partial^{s-1-j}(\tau v^\sigma_e)}{\partial \tau^{s-1-j}_e} (\delta) \frac{\partial v}{\partial \nu^{j}_e} \frac{\partial v}{\partial \nu^{j}_{e,1}} (\delta).
\]

**Proof** It follows from the integration by parts.
Due to the integration by parts, we get

\[ (\tau, \nabla^s v)_e = (\tau, \left( v_{e,1} \frac{\partial}{\partial v_{e,1}} + t_e \frac{\partial}{\partial t_e} \right)^s v)_e = \sum_{\ell=0}^s \sum_{\sigma \in \mathcal{S}(s, \ell)} \left( \tau v_{e,\sigma}^\ell, \frac{\partial^{s-\ell} v}{\partial t_e^{s-\ell} \partial v_{e,1}^\ell} \right)_e \]

Applying the integration by parts to the second term of the right hand side, we get

\[ (\tau, \nabla^s v)_e = \sum_{j=s-2}^s \sum_{\sigma \in \mathcal{S}(s,j)} (-1)^{s-j} \left( \frac{\partial^{s-j} (\tau v_{e,\sigma}^j)}{\partial t_e^{s-j}}, \frac{\partial^j v}{\partial v_{e,1}^j} \right)_e \]

Along this way, we can finish the proof by applying the integration by parts recursively. □

**Lemma 2** Let \( K \in \mathcal{T}_h \) and integer \( s \geq n = 2 \). There exist differential operators \( D_{e,\alpha}^{s-1-|\alpha|} \) for \( E \in \mathcal{F}^1(K) \) and \( \alpha \in A_1 \) with \(|\alpha| \leq s-1\), and \( D_{\delta,\alpha}^{s-2-|\alpha|} \) for \( \delta \in \mathcal{F}^2(K) \) and \( \alpha \in A_2 \) with \(|\alpha| \leq s-2\) such that for any \( \tau \in H^s(K; \mathbb{T}_2(s)) \) and \( v \in H^s(K) \), it holds

\[ (\tau, \nabla^s v)_K = - (\text{div} \tau, \nabla^{s-1} v)_K + \sum_{e \in \mathcal{F}^1(K)} \sum_{\alpha \in A_1} \left( D_{e,\alpha}^{s-1-|\alpha|}(\tau), \frac{\partial |\alpha| v}{\partial v_{e,1}^\alpha} \right)_e \]

\[ + \sum_{\delta \in \mathcal{F}^2(K)} \sum_{\alpha \in A_2} D_{\delta,\alpha}^{s-2-|\alpha|}(\tau) \frac{\partial |\alpha| v}{\partial v_{\delta}^\alpha}(\delta). \]

**Proof** Due to the integration by parts, we get

\[ (\tau, \nabla^s v)_K = - (\text{div} \tau, \nabla^{s-1} v)_K + \sum_{e \in \mathcal{F}^1(K)} (\tau v_{e,\times}, \nabla^{s-1} v)_e \]

\[ = (\text{div}^s \tau, v)_K + \sum_{e \in \mathcal{F}^1(K)} \sum_{l=0}^{s-1} (-1)^{s-1-l} \left( \text{div}^{s-1-l} \tau v_{e,\times}, \nabla^l v \right)_e. \]

Then it follows from Lemma 1
\[(\tau, \nabla^s v)_K - ((-\text{div})^s \tau, v)_K \]
\[
= \sum_{e \in \mathcal{F}(K)} \sum_{i=1}^{s-1} \sum_{j=0}^{i-1} \sum_{\ell=0}^{j} \sum_{\sigma \in \Omega(i, \ell)} \sum_{\delta \in \partial e} D_{s-2-j}^{e-1} (\tau) \frac{\partial^j v}{\partial t_e^{i-\ell} \partial v_e^{j}} (\delta) \\
+ \sum_{e \in \mathcal{F}(K)} \sum_{j=0}^{i-1} \sum_{\sigma \in \Omega(i, j)} \left( D_{s-1-j}^{e-1} (\tau), \frac{\partial^j v}{\partial v_e^{j}} \right) \\
= \sum_{e \in \mathcal{F}(K)} \sum_{j=0}^{i-1} \sum_{\sigma \in \Omega(i, j)} \sum_{\delta \in \partial e} D_{s-2-j}^{e-1} (\tau) \frac{\partial^j v}{\partial t_e^{i-\ell} \partial v_e^{j}} (\delta) \\
+ \sum_{e \in \mathcal{F}(K)} \sum_{j=0}^{i-1} \sum_{\sigma \in \Omega(i, j)} \left( D_{s-1-j}^{e-1} (\tau), \frac{\partial^j v}{\partial v_e^{j}} \right),
\]

where

\[
D_{s-2-j}^{e-1} (\tau) = (-1)^{s-2-j} \frac{\partial^j ((-\text{div})^{s-1-i} \tau) v_K e^i}{\partial t_e^{i-\ell} v_e^j}, \\
D_{s-1-j}^{e-1} (\tau) = (-1)^{s-1-j} \frac{\partial^j ((-\text{div})^{s-1-i} \tau) v_K e^i}{\partial v_e^j}.
\]

This indicates (3.1). \(\square\)

As an immediate result of (3.1), we achieve the generalized Green’s identity in two dimensions as follows.

**Lemma 3** Let \( K \in T_h \) and integer \( m > n = 2 \). There exist differential operators \( D_{e,a}^{2m-1-|a|} \) for \( e \in \mathcal{F}(K) \) and \( a \in A_1 \) with \( |a| \leq m - 1 \), and \( D_{s,a}^{2m-2-|a|} \) for \( \delta \in \mathcal{F}^2(K) \) and \( a \in A_2 \) with \( |a| \leq m - 2 \) such that for any \( u \in H^m(K) \) and \( v \in H^n(K) \), it holds

\[
(\nabla^m u, \nabla^m v)_K = ((-\Delta)^m u, v)_K + \sum_{e \in \mathcal{F}(K)} \sum_{|a| \leq m-1} \left( D_{e,a}^{2m-1-|a|} (u), \frac{\partial |a| v}{\partial v_e^a} \right) \\
+ \sum_{\delta \in \mathcal{F}^2(K)} \sum_{|a| \leq m-2} D_{s,a}^{2m-2-|a|} (u) \frac{\partial |a| v}{\partial v_\delta^a} (\delta).
\] (3.3)

### 3.2 Generalized Green’s identity in \( n \) dimensions

Now we extend Lemmas 1 and 3 to any dimension. To this end, we recall two results in [17].
Lemma 4 (Lemma 3.1 in [17]) Let $K \in T_h$, $F \in F^r(K)$ with $1 \leq r \leq n-1$, and $s$ be a positive integer satisfying $s \leq n-r$. There exist differential operators $D_{e,\alpha}^{s-j-|\alpha|}$ for $j = 0, \ldots, s$, $e \in F^j(F)$ and $\alpha \in A_{r+j}$ with $|\alpha| \leq s-j$ such that for any $\tau \in H^s(F; \mathbb{T}_n(s))$ and $(\nabla^s \tau)|_F \in L^2(F; \mathbb{T}_n(s))$, it holds

$$
(\tau, \nabla^s \tau)_F = \sum_{j=0}^{s} \sum_{e \in F^j(F)} \sum_{\alpha \in A_{r+j}} \left( D_{e,\alpha}^{s-j-|\alpha|}(\tau), \frac{\partial |\alpha|}{\partial v_e} \right)_e. \tag{3.4}
$$

Another one is the recurrence relation derived in the proof of Lemma 3.1 in [17]

$$
(\tau, \nabla^{\ell+1} \tau)_F = \sum_{i=1}^{\ell} (\tau v_{F,i}, \nabla^{\ell-i} \tau)_F - (\text{div}_F \tau, \nabla^{\ell} \tau)_F + \sum_{e \in F^j(F)} (\tau v_{F,e}, \nabla^{\ell} \tau)_e
$$

for any positive integer $\ell$, and $F \in F^r(K)$ with $1 \leq r \leq n-1$.

Lemma 5 Let $K \in T_h$, $F \in F^r(K)$ with $1 \leq r \leq n-1$, and positive integer $s \geq n-r$. There exist differential operators $D_{e,\alpha}^{s-j-|\alpha|}$ for $j = 0, \ldots, n-r-1$, $e \in F^j(F)$ and $\alpha \in A_{r+j}$ with $|\alpha| \leq s-j$, and differential operators $D_{\delta,\alpha}^{s+r-n-|\alpha|}$ for $\delta \in F^{n-r}(F)$ and $\alpha \in A_{n}$ with $|\alpha| \leq s+r-n$ such that for any $\tau \in H^s(F; \mathbb{T}_n(s))$ and $(\nabla^s \tau)|_F \in L^2(F; \mathbb{T}_n(s))$, it holds

$$
(\tau, \nabla^s \tau)_F = \sum_{j=0}^{n-r-1} \sum_{e \in F^j(F)} \sum_{\alpha \in A_{r+j}} \left( D_{e,\alpha}^{s-j-|\alpha|}(\tau), \frac{\partial |\alpha|}{\partial v_e} \right)_e + \sum_{\delta \in F^{n-r}(F)} \sum_{\alpha \in A_{n}} D_{\delta,\alpha}^{s+r-n-|\alpha|}(\tau) \frac{\partial |\alpha|}{\partial v_{\delta}}(\delta). \tag{3.6}
$$

Proof The identities (3.6) and (3.4) are same for $s = n-r$. Assume the identity (3.6) holds for $s = \ell$ with integer $\ell \geq n-r$, then let us prove it is also true for $s = \ell + 1$. Applying (3.6) with $s = \ell$ to each term in the right hand side of (3.5), we have
\[ (\tau v_{F,i}, \nabla^\ell \frac{\partial v}{\partial v_{F,i}})_F = \sum_{j=0}^{n-r-1} \sum_{e \in \mathcal{F}(F)} \sum_{a \in A_{j+i}} \left( D_{e,a}^{\ell-j-|a|} (\tau v_{F,i}), \frac{\partial |a|}{\partial v_{e}} \left( \frac{\partial v}{\partial v_{F,i}} \right)_e \right) \]

\[ + \sum_{\delta \in \mathcal{F}^\ell(F)} \sum_{a \in \Lambda_n} \left( D_{\delta,a}^{\ell+r-n-|a|} (\tau v_{F,i}), \frac{\partial |a|}{\partial v_{\delta}} \left( \frac{\partial v}{\partial v_{F,i}} \right)_\delta \right), \]

\[ (\text{div}_F \tau, \nabla^\ell v)_F = \sum_{j=0}^{n-r-1} \sum_{e \in \mathcal{F}(F)} \sum_{a \in A_{j+i}} \left( D_{e,a}^{\ell-j-|a|} (\text{div}_F \tau), \frac{\partial |a|}{\partial v_{e}} \right)_e \]

\[ + \sum_{\delta \in \mathcal{F}^\ell(F)} \sum_{a \in \Lambda_n} \left( D_{\delta,a}^{\ell+r-n-|a|} (\text{div}_F \tau), \frac{\partial |a|}{\partial v_{\delta}} \right)_\delta, \]

\[ (\tau v_{F,e}, \nabla^\ell v)_e = \sum_{j=0}^{n-r-2} \sum_{e \in \mathcal{F}(e)} \sum_{a \in A_{j+i}} \left( D_{e,a}^{\ell-j-|a|} (\tau v_{F,e}), \frac{\partial |a|}{\partial v_{e}} \right)_e \]

\[ + \sum_{\delta \in \mathcal{F}_{e-1}^\ell(e)} \sum_{a \in \Lambda_n} \left( D_{\delta,a}^{\ell+r+1-n-|a|} (\tau v_{F,e}), \frac{\partial |a|}{\partial v_{\delta}} \right)_\delta. \]

Hence we conclude (3.6) for \( s = \ell + 1 \) by combining the last four equations. Finally we ends the proof based on the mathematical induction. \( \square \)

**Lemma 6** Let \( K \in \mathcal{T}_h \) and positive integer \( m > n \). There exist differential operators \( D_{F,a}^{2m-j-|a|} \) for \( j = 1, \ldots, n-1, F \in \mathcal{F}(K) \) and \( a \in A_j \) with \( |a| \leq m - j \), and differential operators \( D_{\delta,a}^{2m-n-|a|} \) for \( \delta \in \mathcal{F}^m(K) \) and \( a \in A_n \) with \( |a| \leq m - n \) such that for any \( u \in H^2(K) \) and \( v \in H^m(K) \), it holds

\[ (\nabla^m u, \nabla^m v)_K = (-\Delta)^m u, v)_K + \sum_{j=0}^{n-1} \sum_{F \in \mathcal{F}(K)} \sum_{a \in A_j} \left( D_{F,a}^{2m-j-|a|} (u), \frac{\partial |a|}{\partial v_{a}} \right)_F, \]

\[ + \sum_{\delta \in \mathcal{F}^m(K)} \sum_{a \in \Lambda_n} \left( D_{\delta,a}^{2m-n-|a|} (u), \frac{\partial |a|}{\partial v_{\delta}} \right)_\delta. \]  

**Proof** It is sufficient to prove that there exist differential operators \( D_{F,a}^{m-j-|a|} \) for \( j = 1, \ldots, n-1, F \in \mathcal{F}(K) \) and \( a \in A_j \) with \( |a| \leq m - j \), and differential operators \( D_{\delta,a}^{m-n-|a|} \) for \( \delta \in \mathcal{F}^m(K) \) and \( a \in A_n \) with \( |a| \leq m - n \) such that for any \( \tau \in H^m(K; \mathbb{T}_h(m)) \) and \( v \in H^m(K) \), it holds
\[(\tau, \nabla^m v)_K = ((-\text{div})^m \tau, v)_K + \sum_{j=1}^{n-1} \sum_{F \in \mathcal{F}^j(K)} \sum_{\alpha \in A_j} \left( R_{F,\alpha}^{m-j-|\alpha|} (\tau), \frac{\partial |\alpha|_{\nu}}{\partial v_\alpha} \right)_F \]
\[+ \sum_{\delta \in \mathcal{F}^j(K)} \sum_{\alpha \in A_n} D_{\delta,\alpha}^{m-n-|\alpha|} (\tau) \frac{\partial |\alpha|_{\nu}}{\partial v_\delta} (\delta).\]
\[(3.8)\]

As (3.2), we get from the integration by parts
\[(\tau, \nabla^m v)_K = ((-\text{div})^m \tau, v)_K + \sum_{F \in \mathcal{F}^j(K)} \sum_{j=1}^{m} \left( ((-\text{div})^{j-1} \tau)_F, \nabla^{m-j} v_F \right).\]

Therefore (3.8) follows from (3.4) and (3.6).

\[\square\]

### 3.3 Virtual element space

Inspired by identity (3.7), for any element \(K \in \mathcal{T}_h\) and integer \(k \geq m\), the local degrees of freedom \(N_K^j(K)\) are given as follows:
\[(h^j_K(\nabla^j v)(\delta)), \forall \delta \in \mathcal{F}^j(K), j = 0, 1, \ldots, m-n,\]
\[(3.9)\]
\[\frac{1}{|F|^{n-j-|\alpha|/(n-j)}} \left( \frac{\partial |\alpha|_{\nu}}{\partial v_\alpha}, q \right)_F, \forall q \in \mathbb{H}_{k-(2m-j-|\alpha|)}(F), F \in \mathcal{F}^j(K),\]
\[(3.10)\]
\[j = 1, \ldots, n-1, \alpha \in A_j \text{ with } |\alpha| \leq m-j,\]
\[\frac{1}{|K|} (v, q)_K, \forall q \in \mathbb{H}_{k-2m}(K).\]
\[(3.11)\]

We will use \(\chi_{j,i}^{F,\alpha}\) to denote the degrees of freedom (3.10) for simplicity, where \(i = 1, \ldots, N_{F,k-(2m-j-|\alpha|)}\).

According to the first terms in the inner products of the right hand side of (3.7), and the degrees of freedom (3.10)–(3.11), it is inherent to define the local space of the \(H^m\)-nonconforming virtual element as
\[V^j_K : = \{ u \in H^m(K) : (-\Delta)^m u \in \mathbb{P}_{k-2m}(K),\]
\[D_{F,\alpha}^{2m-j-|\alpha|}(u)|_F \in \mathbb{P}_{k-(2m-j-|\alpha|)}(F) \forall F \in \mathcal{F}(K),\]
\[j = 1, \ldots, n-1, \alpha \in A_j \text{ and } |\alpha| \leq m-j\}.

Combining Lemma 4, (3.6) and the definition of the degrees of freedom (3.10) yields the following property.
Lemma 7 Let \( K \in \mathcal{T}_h, F \in \mathcal{F}(K) \) with \( 1 \leq r \leq n - 1, s = n - r, \ldots, m - r \) satisfying \( k \geq 2m - (r + s) \). For any \( \tau \in \mathbb{P}_{k-(2m-r-s)}(F; \mathbb{T}_n(s)) \) and \( (\nabla^sv)_F \in L^2(F; \mathbb{T}_n(s)) \), the term
\[
(\tau, \nabla^sv)_F
\]
is uniquely determined by the degrees of freedom (3.10) for all nonnegative integer \( j \leq n - r - 1, e \in \mathcal{F}(F), \alpha \in A_{r+j} \) with \( |\alpha| \leq s - j \), and (3.9) for all \( \delta \in \mathcal{F}^{n-r}(F) \) and nonnegative integer \( j \leq s + r - n \). When \( s < n - r \), the term
\[
(\tau, \nabla^sv)_F
\]
is uniquely determined by the degrees of freedom (3.10) for all nonnegative integer \( j \leq n - s \), \( e \in \mathcal{F}(F), \alpha \in A_{r+j} \) with \( |\alpha| \leq s - j \).

Employing the same argument as in the proof of Lemma 3.5 in [17], we get from the generalized Green’s identity (3.7) and Lemma 7 that the degrees of freedom (3.9)–(3.11) are unisolvent for the local virtual element space \( V_k(K) \).

Remark 1 When \( n = 1 \), for any element \( e \in \mathcal{T}_h \) and integer \( k \geq m \), the local degrees of freedom (3.9)–(3.11) will be reduced to
\[
\frac{1}{|e|}(v, q)_e, \forall q \in M_{k-2m}(e).
\]
And the shape function space will be
\[
V_k(e) = \left\{ v \in H^m(e) : v^{(2m)} \in \mathbb{P}_{k-2m}(e) \right\} = \begin{cases} \mathbb{P}_k(e), & k \geq 2m, \\ \mathbb{P}_{2m-1}(e), & k < 2m. \end{cases}
\]
Thus the \( H^m \)-nonconforming virtual element of order \( k \) in one dimension is exactly the \( C^{m-1} \)-continuous finite element, whose shape functions are polynomials of degree \( \max\{k, 2m-1\} \).

Remark 2 When \( n = 2 \), for any element \( K \in \mathcal{T}_h \) and integer \( k < 2m \), the local degrees of freedom (3.9)–(3.11) will be reduced to
\[
|e|^{-1}(\frac{\partial^j v}{\partial v_{e,1}}, q)_e, \forall q \in M_{j+1-(2m-k)}(e), e \in \mathcal{F}(K), j = 0, 1, \ldots, m - 1.
\] (3.13)
And the shape function space will be
If each element \( K \in T_h \) is a simplex and \( k = m > 2 \), the degrees of freedom (3.12)–(3.13) are same as those mentioned in [23, p. 268].

Hereafter we always assume \( n \geq 2 \).

**Remark 3** When \( k = m \), the local degrees of freedom (3.9)–(3.11) will be reduced as follows:

\[
(\nabla^j v)(\delta) \quad \forall \ \delta \in \mathcal{F}^n(K), j = 0, \ldots, m - n, \tag{3.14}
\]

\[
(\frac{\partial |\alpha|}{\partial v_{\alpha}}, 1)_F \quad \forall \ F \in \mathcal{F}(K), j = 1, \ldots, n - 1, \alpha \in A_j, |\alpha| = m - j. \tag{3.15}
\]

If each element \( K \in T_h \) is a simplex and \( k = m = n + 1 \), the degrees of freedom (3.14)–(3.15) coincide with those of the nonconforming finite element in [31].

### 3.4 Local projections

To design the virtual element method, we first need a local \( H^m \) projection. We define a local \( H^m \) projection \( \Pi^K_K : H^m(K) \to P_k(K) \) for each \( K \in T_h \) as follows: given \( v \in H^m(K) \), let \( \Pi^K_K v \in P_k(K) \) be the solution of the problem

\[
(V^m \Pi^K_K v, \nabla^m q)_K = (V^m v, \nabla^m q)_K \quad \forall \ q \in P_k(K), \tag{3.16}
\]

\[
\sum_{F \in \mathcal{F}^r(K)} Q_F^F(V^{m-r} \Pi^K_K v) = \sum_{F \in \mathcal{F}^r(K)} Q_F^F(V^{m-r} v), \quad r = 1, \ldots, n - 1, \tag{3.17}
\]

\[
\sum_{\delta \in \mathcal{F}^n(K)} (\nabla^j \Pi^K_K v)(\delta) = \sum_{\delta \in \mathcal{F}^n(K)} (\nabla^j v)(\delta), \quad j = 0, 1, \ldots, m - n. \tag{3.18}
\]

The number of equations in (3.17)–(3.18) is

\[
\sum_{r=1}^{m} C_{n+m-r}^{n-1} = C_{n+m-1}^n = \dim(\mathbb{P}_{m-1}(K)).
\]

Without causing any confusion, we will write \( \Pi^K_K \) as \( \Pi^K \) for simplicity. By the similar argument as in Section 3.3 in [17], we have the following results on \( \Pi^K \) from the generalized Green’s identity (3.7) and Lemma 7.

**Lemma 8** The operator \( \Pi^K : H^m(K) \to P_k(K) \) is an \( H^m \)-stable projector, i.e.
\[ \Pi^K q = q \quad \forall \, q \in \mathbb{P}_k(K), \quad (3.19) \]
\[ |\Pi^K v|_{m,K} \leq |v|_{m,K} \quad \forall \, v \in H^m(K). \quad (3.20) \]

And the projector \( \Pi^K \) can be computed using only the degrees of freedom (3.9)–(3.11).

Denote by \( I^*_K : H^m(K) \rightarrow V_k(K) \) the canonical interpolation operator based on the degrees of freedom in (3.9)–(3.11). Due to the last statement in Lemma 8, we have
\[ \Pi^K v = \Pi^K (I^*_K v) \quad \forall \, v \in H^m(K). \quad (3.21) \]

### 3.5 Serendipity virtual element

Following the ideas in [5, 6, 25], we will give a short discussion on the reduction of the virtual element \((K, \mathcal{N}_k(K), V_k(K))\) by the serendipity approach in this subsection.

For ease of presentation, all the degrees of freedom (3.9)–(3.11) are denoted by \( \chi_1, \chi_2, \ldots, \chi_{N_K} \) in order, where \( N_K \) is the dimension of \( V_k(K) \). Assume there exist some positive integer \( N_s \leq N_K \), nonnegative integer \( k_s \leq k \), and permutation \( \sigma \) of \( (1, 2, \ldots, N_K) \) such that

(S) for any \( q \in \mathbb{P}_{k_s}(K) \) satisfying \( \chi_{\sigma(1)}(q) = \chi_{\sigma(2)}(q) = \cdots = \chi_{\sigma(N_s)}(q) = 0 \), it holds \( q = 0 \).

Define an operator \( \Pi^s_k : V_k(K) \rightarrow \mathbb{P}_{k_s}(K) \) for each \( K \in \mathcal{T}_h \) as
\[
\sum_{i=1}^{N_s} \chi_{\sigma(i)}(\Pi^s_k v) \chi_{\sigma(i)}(q) = \sum_{i=1}^{N_s} \chi_{\sigma(i)}(v) \chi_{\sigma(i)}(q) \quad \forall \, v \in V_k(K), \, q \in \mathbb{P}_{k_s}(K).
\]

The assumption (S) ensures the well-posedness of the operator \( \Pi^s_k \), and
\[ \Pi^s_k q = q \quad \forall \, q \in \mathbb{P}_{k_s}(K). \quad (3.22) \]

Define the space of the serendipity shape functions
\[ V^s_k(K) := \{ v \in V_k(K) : \chi_{\sigma(i)}(v) = \chi_{\sigma(i)}(\Pi^s_k v) \quad \text{for} \, i = N_s + 1, \ldots, N_K \}. \]

Due to (3.22), it holds \( \mathbb{P}_{k_s}(K) \subseteq V^s_k(K) \). Let \( \mathcal{N}^s_k(K) := \{ \chi_{\sigma(1)}, \chi_{\sigma(2)}, \ldots, \chi_{\sigma(N_s)} \} \), then we obtain the serendipity virtual element \((K, \mathcal{N}^s_k(K), V^s_k(K))\). The well-posedness of the serendipity virtual element \((K, \mathcal{N}^s_k(K), V^s_k(K))\) follows from (3.22) and the well-posedness of the virtual element \((K, \mathcal{N}_k(K), V_k(K))\).

Now we give an example to illustrate the previous process. Let \( n = 2, \, m = 3, \, k = 5 \) and \( K \) be a triangle, then the local degrees of freedom (3.9)–(3.11) will be reduced to
Take $k_s = 4$, the reduced virtual element space $V_s^4(K) = \mathbb{P}_4(K)$ and the following reduced freedoms of freedom $N_s^4(K)$:

$$v(\delta), \nabla v(\delta), \quad \forall \delta \in \mathcal{F}^2(K),$$
$$\left( \frac{\partial^j v}{\partial x_k}, q \right)_{e}, \quad \forall q \in \mathbb{M}_j(e), e \in \mathcal{F}^1(K), j = 0, 1, 2.$$

The assumption (S) for $N_s^4(K)$ and $\mathbb{P}_4(K)$ holds due to Lemma 3.1 in [23]. Indeed the serendipity virtual element $(K, N_s^4(K), V_s^4(K))$ is exactly the $H^3$-nonconforming finite element of the first case (2.2) in [23].

Therefore, with a suitable choice of the degrees of freedom $N_s^4(K)$, the serendipity virtual element $(K, N_s^4(K), V_s^4(K))$ may reduce to an $H_m$-nonconforming finite element, i.e. $V_s^4(K) = \mathbb{P}_{k_s}(K)$. We point out that it is not easy to verify the assumption (S). However, it gives a hint to recover some existing finite elements and construct new $H_m$-nonconforming finite elements.

### 4 Discrete method

We will present the $H^m$-nonconforming virtual element method for the polyharmonic equation based on the virtual element $(K, N_k^h(K), V_k(K))$ in this section.

#### 4.1 Discretization

Consider the polyharmonic equation with homogeneous Dirichlet boundary condition

\[
\begin{cases}
(-\Delta)^m u &= f \\
u &= \partial u / \partial \nu = \cdots = \partial^{m-1} u / \partial \nu^{m-1} = 0
\end{cases}
\quad \text{in } \Omega,
\]
\[
\left( \nabla^m u, \nabla^m v \right) = (f, v) \quad \forall \ v \in H^m_0(\Omega).
\]

where $f \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^n$ with $m > n \geq 2$. The variational formulation of the polyharmonic equation (4.1) is to find $u \in H^m_0(\Omega)$ such that

Let the global $H^m$-nonconforming virtual element space be

$$V_h := \{ v_h \in L^2(\Omega) : v_h|_K \in V_k(K) \text{ for each } K \in T_h \}; \text{ all the degrees of freedom (3.9)–(3.10) are continuous across each } F \in \mathcal{F}^r_h, \text{ and vanish on each } F \in \mathcal{F}^r_{h,\partial} \text{ for } r = 1, \ldots, n \}.$$
To introduce the bilinear form, let the stabilization

\[ S_K(w, v) := h_K^{m-2m} \sum_{j=1}^{n-1} \sum_{F \in \mathcal{F}(K)} \sum_{a \in \mathcal{A}_j} \sum_{i=1}^{N_{F,k-j-\vert a \vert}} X_{j,i}^{F,a}(w) X_{j,i}^{F,a}(v) + h_K^{m-2m} \sum_{\delta \in \mathcal{F}(K)} \sum_{j=0}^{m-n} h_{Kj}^j (\nabla^j w)(\delta) : (\nabla^j v)(\delta). \]

The stabilization term \( S_K(\cdot, \cdot) \) only includes the boundary degrees of freedom, whereas all the degrees of freedom are involved in the stabilization term in [17]. Define the local bilinear form \( a_{h,K}(\cdot, \cdot) : V_h(K) \times V_h(K) \to \mathbb{R} \) as

\[ a_{h,K}(w, v) := (\nabla^m \Pi^K w, \nabla^m \Pi^K v)_K + S_K(w - \Pi^K w, v - \Pi^K v), \]

and the global bilinear form \( a_h(\cdot, \cdot) : V_h \times V_h \to \mathbb{R} \) as

\[ a_h(w_h, v_h) := \sum_{K \in T_h} a_{h,K}(w_h, v_h). \]

To present the right hand side, for any nonnegative integer \( \ell \), denote by \( Q_h^\ell \) the \( L^2 \)-orthogonal projection onto \( \mathcal{P}_\ell(T_h) \). Define \( \Pi_h : H^m(T_h) \to \mathcal{P}_k(T_h) \) as follows: given \( v \in H^m(T_h) \),

\[ (\Pi_h v)|_K := \Pi^K(v|_K) \quad \forall \ K \in T_h. \]

Then the right hand side is given by

\[ \langle f, v_h \rangle := \begin{cases} (f, \Pi_h v_h), & m \leq k \leq 2m - 1, \\ (f, Q_h^{m-1}\Pi_h v_h + Q_h^{k-2m}(v_h - \Pi_h v_h)), & 2m \leq k \leq 3m - 2, \\ (f, Q_h^{m-n}\Pi_h v_h), & 3m - 1 \leq k. \end{cases} \]

Notice that when \( 2m \leq k \leq 3m - 2 \), it holds

\[ (Q_h^{m-1}\Pi_h v_h + Q_h^{k-2m}(v_h - \Pi_h v_h), q) = (v_h, q) \quad \forall \ q \in \mathcal{P}_{k-2m}(T_h). \quad (4.2) \]

Combining previous components leads to the \( H^m \)-nonconforming virtual element method for the polyharmonic equation (4.1) in any dimension: find \( u_h \in V_h \) such that

\[ a_h(u_h, v_h) = \langle f, v_h \rangle \quad \forall \ v_h \in V_h. \quad (4.3) \]

**Remark 4** The virtual element method (4.3) is completely determined by the degrees of freedom \( \mathcal{N}_h(K) \), i.e. (3.9)–(3.11). The space of shape functions \( V_k(K) \) is not necessary in the definition and thus the implementation of the virtual element method (4.3). Introducing \( V_k(K) \) is merely for the purpose of analysis, thus the space of shape functions \( V_k(K) \) is virtual.
Remark 5 Differently from [17], the global virtual element space $V_h$ in this paper is defined directly from $V_K$ rather than some modification of $V_K$, since indeed we do not need the computable $L^2$-projection $Q_h^{-1} v_h$ for any $v_h \in V_h$.

4.2 Inverse inequality and Poincaré inequality

For any $K \in T_h$, let $B_K$ be the maximal ball with respect to which $K$ is star-shaped, and $K_s \subset \mathbb{R}^n$ be the regular inscribed simplex of $B_K$, where all the edges of $K_s$ share the common length.

Lemma 9 Assume the mesh $T_h$ satisfies condition (A1) and $K \in T_h$. It holds for any nonnegative integer $\ell$ that

$$
\| q \|_{0,K} \approx \| q \|_{0,K_s} \quad \forall \; q \in \mathbb{P}_{\ell}(K). \tag{4.4}
$$

Proof Taking any $x \in K$, let

$$
v(t) := q(x_{K_s} + t(x - x_{K_s})) \quad \forall \; t \in [0, 1].
$$

Then $v(t)$ is a polynomial of degree $\ell$ on the interval $[0, 1]$. By the Taylor’s theorem, it follows

$$
q(x) = v(1) = \sum_{i=0}^{\ell} \frac{v^{(i)}(0)}{i!}.
$$

Since $v^{(i)}(0) = \nabla^i q(x_{K_s})(x - x_{K_s}, \ldots, x - x_{K_s})$, we get from the inverse inequality of polynomials on $K_s$ that

$$
|v^{(i)}(0)| \lesssim h_K^{i} \| \nabla^i q \|_{L^\infty(K_s)} \lesssim \| q \|_{L^\infty(K_s)} \lesssim h_K^{-\ell/2} \| q \|_{0,K_s}.
$$

Thus we have

$$
|q(x)| \lesssim h_K^{-\ell/2} \| q \|_{0,K_s} \quad \forall \; x \in K,
$$

which implies

$$
\| q \|_{0,K} \lesssim h_K^{\ell/2} \| q \|_{L^\infty(K)} \lesssim \| q \|_{0,K_s}.
$$

The another side of (4.4) is clear. □

Lemma 10 Assume the mesh $T_h$ satisfies condition (A1) and $K \in T_h$. It holds for any nonnegative integers $\ell$ and $i$ that

$$
\| q \|_{0,K} \lesssim h_K^{\ell-i} \| q \|_{-i,K} \quad \forall \; q \in \mathbb{P}_{\ell}(K). \tag{4.5}
$$

Proof Applying (4.4) and the inverse inequality of polynomials on simplices, it follows
\[ \|q\|_{0,K} \lesssim \|q\|_{0,K} \lesssim h^{-\frac{m}{2}} \|q\|_{-\frac{m}{2},K}, \]

which yields (4.5).

Applying the trace inequality (2.2) and the same argument used in the proof of Lemma A.5 in [17], we get the Poincaré inequality on the kernel of the local \( H^m \) projection \( \Pi^K \) under the mesh conditions (A1)–(A2)

\[ \sum_{j=0}^{n-m} \sum_{s=0}^{m-j} h_{K}^{s+j/2} \| \nabla^s v \|_{0,F} \lesssim h_{K}^{m} \| \nabla^m v \|_{0,K} \quad \forall \ v \in \ker(\Pi^K), \ K \in T_h, \quad (4.6) \]

where \( \ker(\Pi^K) := \{ v \in H^m(K) : \Pi^K v = 0 \} \).

### 4.3 Norm equivalence

**Lemma 11** Assume the mesh \( T_h \) satisfies conditions (A1) and (A2). For any \( K \in T_h \), we have

\[ S_K(v, v) \lesssim \sum_{j=0}^{m} h_{K}^{2(j-m)} |v|_{j,K}^2 \quad \forall \ v \in V_k(K), \quad (4.7) \]

\[ S_K(v, v) \lesssim \| \nabla^m v \|_{0,K}^2 \quad \forall \ v \in V_k(K) \cap \ker(\Pi^K). \quad (4.8) \]

**Proof** We get from the proof of Lemma A.6 in [17]

\[ h_{K}^{m-2m} \sum_{j=1}^{n-1} \sum_{F \in F^m(K)} \sum_{a \in A_j |a| \leq m-j} \sum_{i=1}^{N_{F,K-j-|a|}} \left( \chi_{j,i}^{F,a} \right)^2 (v) \]

\[ \lesssim \sum_{j=1}^{n-1} \sum_{F \in F^m(K)} \sum_{a \in A_j |a| \leq m-j} \sum_{i=1}^{N_{F,K-j-|a|}} h_{K}^{2|a|-2m+j} \| \nabla |a| v \|_{0,F}^2. \]

Due to the trace inequality (2.2), it follows

\[ \sum_{\delta \in \mathbb{P}^m(K)} \sum_{i=0}^{m-n} h_{K}^{n-2m+2i} | \nabla^i v(\delta) |^2 \lesssim \sum_{e \in \mathbb{P}^{m-1}(K)} \sum_{i=0}^{m-n} h_{K}^{n-2m+2i-1} \| \nabla^i v \|_{0,e}^2 \]

\[ + \sum_{e \in \mathbb{P}^{m-1}(K)} \sum_{i=0}^{m-n} h_{K}^{n-2m+2i+1} \| \nabla^{i+1} v \|_{0,e}^2. \]

Then we have
\[ S_K(v, v) \lesssim \sum_{j=1}^{n-1} \sum_{F \in \mathcal{T}_h} \sum_{i=0}^{m-j} h_K^{2i-2m+j} \| \nabla^j v \|^2_{0,F}. \]

Hence we acquire (4.7) by applying the trace inequality (2.2) recursively. Finally we conclude (4.8) from (4.7) and the Poincaré inequality (4.6). \( \square \)

We then consider another side of the norm equivalence. Take an element \( K \in \mathcal{T}_h \). Define a bubble function

\[ b_\delta(x) := \prod_{\delta' \in \mathcal{F}^n(K) \setminus \{ \delta \}} \frac{(x_\delta - x_{\delta'})^T(x - x_{\delta'})}{|x_\delta - x_{\delta'}|^2}, \]

for each \( \delta \in \mathcal{F}^n(K) \). Apparently we have \( b_\delta(\delta) = 1 \), and \( b_\delta(\delta') = 0 \) for each \( \delta' \in \mathcal{F}^n(K) \setminus \{ \delta \} \).

**Lemma 12** Assume the mesh \( \mathcal{T}_h \) satisfies condition (A1). Take any \( F \in \mathcal{T}_h \) with \( 1 \leq j < n \). The following norm equivalence holds

\[ \| q \|^2_{0,F} \approx h_F^{n-j} \sum_{i=1}^{N_{F,i}} q_i^2 \quad \forall q := \sum_{i=1}^{N_{F,i}} q_i m_{F,i} \in \mathbb{P}_k(F), \quad (4.9) \]

where \( q_i \in \mathbb{R} \).

**Proof** Noticing that \( F_s \) is a simplex, it follows from (4.4), the scaling argument and the norm equivalence of the finite-dimensional space

\[ \| q \|^2_{0,F} \approx \| q \|^2_{0,F_s} \approx h_F^{n-j} \sum_{i=1}^{N_{F,i}} q_i^2, \]

which gives (4.9). \( \square \)

**Lemma 13** Assume the mesh \( \mathcal{T}_h \) satisfies conditions (A1) and (A2). Let \( K \in \mathcal{T}_h \). We have for any \( v \in V_h(K) \) that

\[ h_K^{m-n} \| (-\Delta)^m v \|_{0,K} + \sum_{\delta \in \mathcal{F}^n(K)} \sum_{\alpha \in A_\delta} h_K^{m-|\alpha|-n/2} |D_{\alpha}^{2m-n-|\alpha|}(v)| \]

\[ + \sum_{j=1}^{n-1} \sum_{F \in \mathcal{T}_h} \sum_{|\alpha| \leq m-j} h_K^{m-|\alpha|-j/2} \| D_{F,\alpha}^{2m-j-|\alpha|}(v) \|_{0,F} \lesssim \| \nabla^m v \|_{0,K}. \quad (4.10) \]

**Proof** Adopting the same argument as in the proof of Lemmas A.1–A.2 in [17], we get
\[
\sum_{j=1}^{n-1} \sum_{F \in \mathcal{F}(K)} \sum_{a \in \Lambda_j} h_K^{m-|a|-j/2} \|D_{F,a}^{2m-j-|a|}(v)\|_{0,F} + h_K^m \|(-\Delta)^m v\|_{0,K} \lesssim \|\nabla^m v\|_{0,K}.
\]  

(4.11)

Now consider \(\delta \in \mathcal{F}^n(K)\) and \(\alpha \in A_n\) with \(|\alpha| \leq m - n\). Notice that \(D_{\delta,a}^{2m-n-|a|}(v)\) is a constant, which can be regarded as the constant function in \(\mathbb{R}^n\). Let

\[
\phi_\delta(x) := \frac{1}{a!} b_{\delta}^{2m} D_{\delta,a}^{2m-n-|a|}(v) \prod_{i=1}^n (v_D^\top (x - x_\delta))^a,
\]

where \(a = a_1 \ldots a_n\), then we have

\[
\|\phi_\delta\|_{0,K} \lesssim h_K^{|a|+n/2} |D_{\delta,a}^{2m-n-|a|}(v)|,
\]

\[
\frac{\partial^{|\alpha|} \phi_\delta}{\partial \nu_\delta^\alpha}(\delta) = (b_\delta(\delta))^{2m} D_{\delta,a}^{2m-n-|a|}(v) = D_{\delta,a}^{2m-n-|a|}(v). \tag{4.12}
\]

Hence

\[
\left| D_{\delta,a}^{2m-n-|a|}(v) \right|^2 = D_{\delta,a}^{2m-n-|a|}(v) \frac{\partial^{|\alpha|} \phi_\delta}{\partial \nu_\delta^\alpha}(\delta). \tag{4.13}
\]

Noticing that \(b_\delta(\delta') = 0\) for all \(\delta' \in \mathcal{F}^n(K) \setminus \{\delta\}\), we have for each \(\delta' \in \mathcal{F}^n(K) \setminus \{\delta\}\)

\[
\frac{\partial^{|\beta|} \phi_\delta}{\partial \nu_{\delta'}^\beta}(\delta') = 0 \quad \forall \ \beta \in A_n \text{ with } |\beta| \leq m - n.
\]

For any \(\beta \in A_n\), \(|\beta| < |\alpha|\), since \(\frac{\partial^{|\beta|}}{\partial \nu_{\delta'}^\beta} \left( \prod_{i=1}^n (v_D^\top (x - x_\delta))^a \right)(\delta) = 0\), it yields

\[
\frac{\partial^{|\beta|} \phi_\delta}{\partial \nu_\delta^\beta}(\delta) = 0.
\]

For any \(\beta \in A_n\), \(|\beta| = |\alpha|\), but \(\beta \neq \alpha\), noting that \(\frac{\partial (v_D^\top (x - x_\delta))}{\partial \nu_{\delta',\ell'}} = 0\) for \(i \neq \ell\), we also have \(\frac{\partial^{|\beta|} \phi_\delta}{\partial \nu_\delta^\beta}(\delta) = 0\). Based on the previous discussion, we obtain from (4.13), the generalized Green’s identity (3.7) and the density argument

\[
\left| D_{\delta,a}^{2m-n-|a|}(v) \right|^2 = (\nabla^m v, \nabla^m \phi_{\delta})_{K} - ((-\Delta)^m v, \phi_{\delta})_{K}
\]

\[
- \sum_{j=1}^{n-1} \sum_{F \in \mathcal{F}(K)} \sum_{|\beta| \leq m-j} \left( D_{F,a}^{2m-j-|\beta|}(v), \frac{\partial^{|\beta|} \phi_\delta}{\partial \nu_F^\beta}(\delta) \right)_F
\]

\[
- \sum_{|\alpha| < |\beta| \leq m-n} D_{\delta,a^\beta}^{2m-n-|\beta|}(v) \frac{\partial^{|\beta|} \phi_\delta}{\partial \nu_\delta^\beta}(\delta).
\]
Employing the Cauchy–Schwarz inequality, the inverse inequality (4.5) and (4.4), it follows
\[
\left| D_{\delta,\alpha}^{2m-n-|\alpha|} (v) \right|^2 
\lesssim h_K^{-m} \| \nabla^m v \|_{0,K} \| \phi_{\delta} \|_{0,K} + \| (-\Delta)^m v \|_{0,K} \| \phi_{\delta} \|_{0,K} 
+ \sum_{j=1}^{n-1} \sum_{F \in \mathcal{F}(K)} \sum_{\beta \in A_j} h_K^{-|\beta|/2} \left\| D_{F,\beta}^{2m-j-|\beta|} (v) \right\|_{0,F} \| \phi_{\delta} \|_{0,K} 
+ \sum_{\beta \in A_n} h_K^{-|\beta|-n/2} \left| D_{\delta,\beta}^{2m-n-|\beta|} (v) \right| \| \phi_{\delta} \|_{0,K},
\]
which combined with (4.12) yields
\[
h_K^{-m-|\alpha|-n/2} \left| D_{\delta,\alpha}^{2m-n-|\alpha|} (v) \right| \lesssim \| \nabla^m v \|_{0,K} + h_K^{-m} \| (-\Delta)^m v \|_{0,K} 
+ \sum_{j=1}^{n-1} \sum_{F \in \mathcal{F}(K)} \sum_{\beta \in A_j} |\beta| \leq m - j h_K^{-m-|\beta|-j/2} \left\| D_{F,\beta}^{2m-j-|\beta|} (v) \right\|_{0,F} 
+ \sum_{\beta \in A_n} h_K^{-m-|\beta|-n/2} \left| D_{\delta,\beta}^{2m-n-|\beta|} (v) \right|.
\]
Then we get from (4.11)
\[
h_K^{-m-|\alpha|-n/2} \left| D_{\delta,\alpha}^{2m-n-|\alpha|} (v) \right| \lesssim \| \nabla^m v \|_{0,K} + \sum_{\beta \in A_n} h_K^{-m-|\beta|-n/2} \left| D_{\delta,\beta}^{2m-n-|\beta|} (v) \right|.
\]
Finally applying this inequality recursively gives
\[
h_K^{-m-|\alpha|-n/2} \left| D_{\delta,\alpha}^{2m-n-|\alpha|} (v) \right| \lesssim \| \nabla^m v \|_{0,K},
\]
which together with (4.11) implies (4.10).

**Lemma 14** Assume the mesh $\mathcal{T}_h$ satisfies condition (A1) and $K \in \mathcal{T}_h$. For any $q \in \mathbb{P}_{k-2m}(K)$, there exists $p \in \mathbb{P}_{k}(K)$ satisfying
\[
(-\Delta)^m p = q, \quad \text{and} \quad \| \nabla^i p \|_{0,K} \lesssim h_K^{2m-i} \| q \|_{0,K}
\]
for any nonnegative integer $i$.

**Proof** Since $\Delta : \mathbb{P}_\ell(K_s) \to \mathbb{P}_\ell(K_s)$ is surjective for any nonnegative integer $\ell$, the operator $(-\Delta)^m : \mathbb{P}_k(K_s) \to \mathbb{P}_{k-2m}(K_s)$ is surjective. Thus the operator $(-\Delta)^m : \mathbb{P}_k(K_s)/\ker((-\Delta)^m) \to \mathbb{P}_{k-2m}(K_s)$ is an isomorphism. Then there exists $p \in \mathbb{P}_k(K_s)$ such that
\[ (-\Delta)^m p = q |_{K_i}, \quad (4.14) \]

and by the scaling argument,

\[ \| p \|_{0,K_i} \leq h_K^{2m} \| q \|_{0,K_i}. \]

Notice that \( p \in \mathbb{P}_k(K_i) \) is spontaneously regarded as a polynomial in \( \mathbb{P}_k(K) \). Applying the inverse inequality (4.5) and (4.4), we have

\[ \| \nabla^j p \|_{0,K} \leq h_K^{-j} \| p \|_{0,K} \leq h_K^{-m} \| p \|_{0,K} \leq h_K^{2m} \| q \|_{0,K_i} \leq h_K^{2m-j} \| q \|_{0,K_i}. \]

The identity (4.14) implies \( ((-\Delta)^m p - q) |_{K_i} = 0 \), which together with the fact \((-\Delta)^m p - q \in \mathbb{P}_{k-2m}(K)\) ends the proof. \( \square \)

**Lemma 15** Assume the mesh \( T_h \) satisfies conditions (A1) and (A2). For any \( K \in T_h \), it holds

\[ \| \nabla^m v \|_{0,K}^2 \leq S_K(v,v) \quad \forall \ v \in V_k(K) \cap \ker(\Pi^K). \quad (4.15) \]

**Proof** Employing Lemma 14, there exists \( p \in \mathbb{P}_k(K) \) satisfying

\[ (-\Delta)^m p = (-\Delta)^m v, \quad (4.16) \]

\[ \| \nabla^m p \|_{0,K} \leq h_K^m \| (-\Delta)^m v \|_{0,K} \leq \| \nabla^m v \|_{0,K}, \quad (4.17) \]

in which we have used (4.10). By the definition of \( \Pi^K \), it follows

\[ \| \nabla^m v \|_{0,K}^2 = (\nabla^m v, \nabla^m v)_K = (\nabla^m (v - p), \nabla^m v)_K. \]

And exploiting the generalized Green’s identity (3.7) and (4.16), we have

\[ \| \nabla^m v \|_{0,K}^2 = \sum_{j=1}^{n-1} \sum_{F \in \mathcal{F}(K)} \sum_{a \in \Lambda_j} \left( D_{F,a}^{2m-j-|a|} (v - p), \frac{\partial |a|_v}{\partial F} \right)_F 
+ \sum_{\delta \in \mathcal{J}(K)} \sum_{a \in \Lambda_n} D_{\delta,a}^{2m-n-|a|} (v - p) \frac{\partial |a|_v}{\partial \delta}, \quad (4.18) \]

Since \( v \in V_k(K) \), we have \( D_{F,a}^{2m-j-|a|} (v - p) \big|_F \in \mathbb{P}_{k-(2m-j-|a|)}(F) \) for any \( F \in \mathcal{F}(K) \). Then there exist constants \( c_i, i = 1, \ldots, N_{F,k-(2m-j-|a|)} \) such that

\[ \left( D_{F,a}^{2m-j-|a|} (v - p), \frac{\partial |a|_v}{\partial F} \right)_F = h_K^{n-j-|a|} \sum_{i=1}^{N_{F,k-(2m-j-|a|)}} c_i \chi_j^{F,a}(v). \]

Applying the norm equivalence on the polynomial space \( \mathbb{P}_{k-(2m-j-|a|)}(F) \), cf. (4.9), we get
\[
\| D_{F,\alpha}^{2m-j-|\alpha|} (v - p) \|_{0,F}^2 \approx h_K^{n-j} \sum_{i=1}^{N_{F,K-2m-j-|\alpha|}} c_i^2.
\]

Hence it follows from (4.10) and (4.17)

\[
\left( D_{F,\alpha}^{m-j-|\alpha|} (v - p), \frac{\partial^{|\alpha|} v}{\partial \nu_F} \right)_F \\
\lesssim h_K^{(n-j)/2-|\alpha|} \| D_{F,\alpha}^{2m-j-|\alpha|} (v - p) \|_{0,F} \sqrt{\sum_{i=1}^{N_{F,K-2m-j-|\alpha|}} \left( \chi_{j,i}^{F,\alpha} \right)^2 (v)} \\
\lesssim \| \nabla^m (v - p) \|_{0,K} \sqrt{S_K(v,v)} \lesssim \| \nabla^m v \|_{0,K} \sqrt{S_K(v,v)}.
\]

Applying (4.10) and (4.17) again, it holds for each \( \delta \in \mathcal{F}^n(K) \), and \( \alpha \in A_n \) with \( |\alpha| \leq m - n \)

\[
D_{\delta,\alpha}^{m-n-|\alpha|} (v - p) \frac{\partial^{|\alpha|} v}{\partial \nu_\delta} \lesssim \| \nabla^m (v - p) \|_{0,K} \sqrt{S_K(v,v)} \lesssim \| \nabla^m v \|_{0,K} \sqrt{S_K(v,v)}.
\]

Therefore we conclude (4.15) from (4.18) and the last two inequalities. \( \square \)

**Remark 6** The reason of the stabilization term \( S_K(\cdot, \cdot) \) only involving the boundary degrees of freedom is that the operator \( (-\Delta)^m : P_k(K) \to P_{k-2m}(K) \) is onto and has a continuous right inverse (cf. Lemma 14).

At last, combining (4.8) and (4.15) gives the norm equivalence (4.19).

**Theorem 1** Assume the mesh \( T_h \) satisfies conditions (A1) and (A2). For any \( K \in T_h \), the following norm equivalence holds

\[
S_K(v,v) \simeq \| \nabla^m v \|_{0,K}^2 \quad \forall \ v \in V_k(K) \cap \ker(\Pi^K),
\]

where the constant is independent of \( h_K \), but may depend on the chunkiness parameter \( \rho_K \), the degree of polynomials \( k \), the order of differentiation \( m \), the dimension of space \( n \), and the shape regularity and quasi-uniform constants of the virtual triangulation \( T_h \).

From now on, we always assume the mesh \( T_h \) satisfies conditions (A1) and (A2). By the Cauchy–Schwarz inequality and the norm equivalence (4.19), we have

\[
S_K(w,v) \lesssim |w|_{m,K} |v|_{m,K} \quad \forall \ w, v \in V_k(K) \cap \ker(\Pi^K).
\]

which implies the continuity of \( a_h(\cdot, \cdot) \)

\[
a_h(w_h,v_h) \lesssim |w_h|_{m,h} |v_h|_{m,h} \quad \forall \ w_h, v_h \in V_h \oplus P_k(T_h).
\]

Next we verify the coercivity of \( a_h(\cdot, \cdot) \).
Lemma 16  For any \( v_h \in V_h + \mathbb{P}_k(T_h) \), it holds
\[
|v_h|_{m,h}^2 \leq a_h(v_h, v_h). \tag{4.22}
\]

Proof  Since \( \Pi^K \) is the \( H^m \)-orthogonal projection,
\[
|v_h|_{m,K}^2 = \left| \Pi^K(v_h|_K) \right|_{m,K}^2 + |v_h - \Pi^K(v_h|_K)|_{m,K}^2.
\]
Applying (4.19), we have
\[
|v_h|_{m,K}^2 \lesssim \left| \Pi^K(v_h|_K) \right|_{m,K}^2 + S_K(v_h - \Pi^K(v_h|_K), v_h - \Pi^K(v_h|_K)) \tag{4.23}
\]
which implies (4.22). \( \square \)

Therefore the nonconforming virtual element method (4.3) is uniquely solvable by the Lax–Milgram lemma.

4.4 Weak continuity

Based on Lemma 7, for any \( F \in \mathcal{F}_h^1 \), \( v_h \in V_h \), we have the weak continuity
\[
\left( \nabla_h^s v_h, \tau \right)_F = 0 \quad \forall \tau \in \mathbb{P}_{k-(2m-1-s)}(F; \mathcal{T}_h(s)) \tag{4.24}
\]
for \( s = 0, 1, \ldots, m - 1 \), and
\[
Q_0^s(\nabla_h^s v_h) = 0 \quad \forall e \in \mathcal{F}^{m-s-1}(F) \tag{4.25}
\]
for \( s = m - n, \ldots, m - 1 \), where \( \nabla_h \) is the elementwise gradient with respect to the partition \( T_h \).

Recall the following error estimates of the \( L^2 \) projection and the Bramble–Hilbert Lemma (cf. [13, Lemma 4.3.8]).

Lemma 17  Let \( \ell' \geq 0 \). For each \( F \in \mathcal{F}_h^r \) with \( r = 0, 1, \ldots, n - 1 \), and \( e \in \mathcal{F}^1(F) \), we have for any \( v \in H^{\ell'+1}(F) \) that
\[
\| v - Q_{\ell'}^e v \|_{0,F} \lesssim h_F^{\ell'+1} |v|_{\ell'+1,F}, \tag{4.26}
\]
\[
\| v - Q_{\ell'}^e v \|_{0,e} \lesssim h_F^{\ell'+1/2} |v|_{\ell'+1,F}. \tag{4.27}
\]
For each \( K \in \mathcal{T}_h \cup \mathcal{T}_h^* \), there exists a linear operator \( T_{\ell'}^K : L^1(K) \to \mathbb{P}_{\ell'}(K) \) such that for any \( v \in H^{\ell'+1}(K) \),
\[
\| v - T_{\ell'}^K v \|_{j,K} \lesssim h_K^{\ell'+1-j} |v|_{\ell'+1,K} \quad \text{for} \quad 0 \leq j \leq \ell' + 1. \tag{4.28}
\]
Define \( T_h : L^2(\Omega) \to \mathbb{P}_k(T_h) \) as
\[
(T_h v)|_K := T^K_h(v|_K) \quad \forall K \in T_h.
\]

**Lemma 18** Given \( F \in \mathcal{F}_h^d \) and nonnegative integer \( s < m - n \), it holds for any \( v_h \in V_h \)
\[
\sum_{j=0}^{n-2} \sum_{e_j \in F(F)} h^{j/2}_F \| \nabla^s v_h \|_{0,e_j} \lesssim \sum_{j=0}^{n-2} \sum_{e_j \in F(F)} h^{j/2 + m-n-s}_F \| \nabla^{m-n} v_h \|_{0,e_j}.
\]  
\[(4.29)\]

**Proof** For \( j = 1, \ldots, n - 2 \), applying the trace inequality (2.2), it follows
\[
\sum_{e_j \in F(F)} h^{j/2}_F \| \nabla^s v_h \|_{0,e_j} \lesssim \sum_{e_{j-1} \in F(F)} h^{(j-1)/2}_F \| \nabla^s v_h \|_{0,e_{j-1}} + \sum_{e_{j-1} \in F(F)} h^{(j+1)/2}_F \| \nabla^{s+1} v_h \|_{0,e_{j-1}}.
\]

Then employing this inequality recursively, we obtain
\[
\sum_{e_j \in F(F)} h^{j/2}_F \| \nabla^s v_h \|_{0,e_j} \lesssim \| \nabla^s v_h \|_{0,F} + \sum_{i=0}^{j-1} \sum_{e_i \in F(F)} h^{(i+2)/2}_F \| \nabla^{s+1} v_h \|_{0,e_i}.
\]

Hence we get from (4.26)
\[
\sum_{j=0}^{n-2} \sum_{e_j \in F(F)} h^{j/2}_F \| \nabla^s v_h \|_{0,e_j} \lesssim \| \nabla^s v_h \|_{0,F} + \sum_{i=0}^{n-3} \sum_{e_i \in F(F)} h^{(i+2)/2}_F \| \nabla^{s+1} v_h \|_{0,e_i}.
\]
\[(4.30)\]

Adopting the trace inequality (2.2), it follows from (4.30)
Nonconforming virtual element method for 2nd order partial…

\[ \sum_{\delta \in \mathcal{F}^{n-1}(F)} h_F^{(n-1)/2} \| \nabla^s v_h \|_{0,F} - Q_0^F(\nabla^s v_h) \|_{0,e} \leq \sum_{e \in \mathcal{F}^{n-1}(F)} h_F^{(n-2)/2} \| \nabla^s v_h \|_{0,e} + h_F^{n/2} \| \nabla^{s+1} v_h \|_{0,e}. \]

Proof

Take some \( \delta \in \mathcal{F}^{n-1}(F) \). Due to the degrees of freedom (3.9), we have \( \| \nabla^s v_h \|_{0,F} = 0 \). Then for \( j = 0, 1, \ldots, n-2 \) and any \( e_j \in \mathcal{F}(F) \), it follows from (4.26)

\[
\begin{align*}
\sum_{j=0}^{n-2} \sum_{e_j \in \mathcal{F}(F)} h_F^{j/2} \| \nabla^s v_h \|_{0,e_j} & \leq \sum_{j=0}^{n-2} \sum_{e_j \in \mathcal{F}(F)} h_F^{(j+2)/2} \| \nabla^{s+1} v_h \|_{0,e_j},
\end{align*}
\]

which indicates (4.29). \( \square \)

Lemma 19

For each \( F \in \mathcal{F}_h^1 \) and nonnegative integer \( s < m \), it holds

\[ \| \nabla^s v_h \|_{0,F} \leq \sum_{K \in \partial^{s-1}F} h_K^{m-s-1/2} |v_h|_{m,K} \quad \forall \ v_h \in V_h. \] (4.31)

Proof

According to (4.25) and the proof of Lemmas 4.5–4.6 in [17], we get for \( s = m-1, m-2, \ldots, m-n \) and any \( e \in \mathcal{F}(F) \) with \( j = 0, 1, \ldots, m-1-s \)

\[ \| \nabla^s v_h \|_{0,e} \leq \sum_{K \in \partial^{s-1}F} h_K^{m-s-(j+1)/2} |v_h|_{m,K}. \] (4.32)

For \( s < m-n \), it follows from (4.29) and (4.32) that

\[ \| \nabla^s v_h \|_{0,F} \leq \sum_{j=0}^{n-2} \sum_{e_j \in \mathcal{F}(F)} h_F^{j/2} \| \nabla^{m-n} v_h \|_{0,e_j} \leq \sum_{K \in \partial^{s-1}F} h_K^{m-s-1/2} |v_h|_{m,K}, \]

with together with (4.32) with \( j = 0 \) again implies (4.31). \( \square \)
Given the virtual triangulation $T^*_h$, for each nonnegative integer $r < m$, define the tensorial $(m - r)$th order Lagrange element space associated with $T^*_h$

$$S^r_h := \{ \tau_h \in H^1_0(\Omega; \mathbb{T}_n(r)): \tau_h|_K \in \mathbb{P}_{m-r}(G; \mathbb{T}_n(r)) \quad \forall K \in T^*_h \}.$$ According to Lemma 4.7 in [17], (4.28) and (4.31), for $r = 0, 1, \ldots, m - 1$ and any $v_h \in V_h$, there exists $\tau_r = \tau_r(v_h) \in S^r_h$ such that

$$|\nabla h v_h|_{r,j} \lesssim h^{m-r-j}|v_h|_{m,h} \quad \text{for } j = 0, 1, \ldots, m - r. \quad (4.33)$$

By (4.33), we have the discrete Poincaré inequality (cf. Lemma 4.8 in [17])

$$\|v_h\|_{m,h} \lesssim |v_h|_{m,h} \quad \forall v_h \in V_h, \quad (4.34)$$

and thus

$$\|v_h\|_{m,h} \approx |v_h|_{m,h} \quad \forall v_h \in V_h.$$ 

5 Error analysis

In this section, we will analyze the nonconforming virtual element method (4.3). Denote by $I_h : H^m_0(\Omega) \rightarrow V_h$ the standard canonical interpolation operator based on the degrees of freedom in (3.9)–(3.11). Adopting the same argument as in [17], we have the following error estimate for the interpolation operator $I_h$

$$|v - I_h v|_{m,K} \lesssim h^{k+1-m}|v|_{k+1,K} \quad \forall v \in H^{k+1}(\Omega), K \in T_h. \quad (5.1)$$

Due to (3.19) and (3.16), we have the following $k$-consistency

$$a_{h,K}(p, v) = (\nabla^m p, \nabla^m v)|_K \quad \forall p \in \mathbb{P}_k(K), v \in V^K_h. \quad (5.2)$$

5.1 Consistency error estimate

**Lemma 20** Let $u \in H^m_0(\Omega) \cap H^{k+1}(\Omega)$ be the solution of the polyharmonic equation (4.1). For $i = 0, 1, \ldots, \min\{m - 1, k - m\}$, it holds for any $v_h \in V_h$ that

$$\left| (\text{div}^i \nabla u, \nabla^m v_h) + (\text{div}^{i+1} \nabla u, \nabla^{m-(i+1)} v_h) \right| \lesssim h^{k+1-m}|u|_{k+1,v_h}|_{m,h}. \quad (5.3)$$

**Proof** It follows from the weak continuity (4.24) with $s = m - (i + 1)$ that the projection $Q^{F}_{k-(m+i)}(\nabla^{m-(i+1)} v_h) = 0$ for each $F \in \mathcal{F}^i_h$ and $i = 0, \ldots, m - 1$. Applying integration by parts, we get
Nonconforming virtual element method for 2mth order partial…

\[
\text{(div}^i \nabla^m u, \nabla^{m-i} v_h) + (\text{div}^{i+1} \nabla^m u, \nabla^{m-(i+1)} v_h) = \sum_{K \in \mathcal{T}_h} \left(\left(\text{div}^i \nabla^m u\right)_K, \nabla^{m-i} v_h\right)_{0K} \\
= \sum_{F \in \mathcal{F}_h} \left(\left(\text{div}^i \nabla^m u\right)_F, \left[\nabla^{m-i} v_h\right]_F\right)
\]

with together with (4.27) and (4.31) gives (5.3).

\[\text{Lemma 21} \quad \text{Let } u \in H^m_0(\Omega) \cap H^{2m-1}(\Omega) \text{ be the solution of the polyharmonic equation (4.1). Assume } m \leq k < 2m - 1. \text{ For } i = k - m + 1, k - m + 2, \ldots, m - 2, \text{ it holds for any } v_h \in V_h \text{ that}
\]

\[
\left| (\text{div}^i \nabla^m u, \nabla^{m-i} v_h) + (\text{div}^{i+1} \nabla^m u, \nabla^{m-(i+1)} v_h) \right| \leq \left( h^i |u|_{m+i} + h^{i+1} |u|_{m+i+1} \right) |v_h|_{m,h}, \tag{5.4}
\]

\[
((\text{div})^{m-1} \nabla^m u, \nabla v_h) - (f, v_h) \leq (h^{m-1} |u|_{2m-1} + h^m |f|_0) |v_h|_{m,h}. \tag{5.5}
\]

**Proof** Thanks to (4.33), for \( i = k - m + 1, k - m + 2, \ldots, m - 1, \) there exists \( \tau_{m-(i+1)} \in W_h^{m-(i+1)} \) such that

\[
\left| \nabla^{m-(i+1)} v_h - \tau_{m-(i+1)} \right|_{j,h} \leq h^{i+1-j} |v_h|_{m,h} \quad \text{for } j = 0, 1. \tag{5.6}
\]

Since \( \tau_{m-(i+1)} \in H^1_0(\Omega; \mathbb{T}_h(m - (i + 1))) \), we get for \( i = k - m + 1, \ldots, m - 2 \)

\[
(\text{div}^i \nabla^m u, \nabla \tau_{m-(i+1)}) + (\text{div}^{i+1} \nabla^m u, \tau_{m-(i+1)}) = 0,
\]

\[
((-\text{div})^{m-1} \nabla^m u, \nabla \tau_0) - (f, \tau_0) = 0.
\]

Then we have for \( i = k - m + 1, \ldots, m - 2 \)

\[
(\text{div}^i \nabla^m u, \nabla^{m-i} v_h) + (\text{div}^{i+1} \nabla^m u, \nabla^{m-(i+1)} v_h) = (\text{div}^i \nabla^m u, \nabla_h(\nabla^{m-(i+1)} v_h - \tau_{m-(i+1)}))
\]

\[
+ (\text{div}^{i+1} \nabla^m u, \nabla^{m-(i+1)} v_h - \tau_{m-(i+1)}),
\]

\[((-\text{div})^{m-1} \nabla^m u, \nabla_h v_h) - (f, v_h) = ((-\text{div})^{m-1} \nabla^m u, \nabla_h(v_h - \tau_0)) - (f, v_h - \tau_0).
\]

Hence we conclude (5.4)–(5.5) from (5.6).

Next consider the perturbation of the right hand side.
Lemma 22 Assume \( f \in H^\ell(T_h) \) with \( \ell = \max\{0, k + 1 - 2m\} \), then it holds for any \( v_h \in V_h \) that
\[
(f, v_h) - \langle f, v_h \rangle \lesssim h^{k+1-m+\max\{0,2m-k-1\}}|f|_{\ell,h}|v_h|_{m,h}.
\] (5.7)

Proof For \( m \leq k \leq 2m - 1 \), we get from the local Poincaré inequality (4.6)
\[
(f, v_h) - \langle f, v_h \rangle = (f, v_h - \Pi_h v_h) \lesssim h^m \|f\|_0|v_h|_{m,h}.
\]

For \( 2m \leq k \leq 3m - 2 \), it follows from (4.2), (4.26) and (4.6)
\[
(f, v_h) - \langle f, v_h \rangle = (f - Q_h^k f, v_h - Q_h^m v_h + (Q_h^m - Q_h^{k-2m})(v_h - \Pi_h v_h))
\]
\[
\lesssim \|f - Q_h^k f\|_0 \|v_h - Q_h^m v_h\|_0 + \|v_h - \Pi_h v_h\|_0
\]
\[
\lesssim h^{k+1-m\max\{0,2m-k-1\}}|f|_{k+1-2m,h}|v_h|_{m,h}.
\]

For \( k \geq 3m - 1 \), it holds from (4.26)
\[
(f, v_h) - \langle f, v_h \rangle = (f, v_h - Q_h^k f) = (f - Q_h^k f, v_h - Q_h^{k-2m} v_h)
\]
\[
\lesssim \|f - Q_h^k f\|_0 \|v_h - Q_h^{k-2m} v_h\|_0
\]
\[
\lesssim h^{k+1-m\max\{0,2m-k-1\}}|f|_{k+1-2m,h}|v_h|_{m,h}.
\]

Combining the last three inequalities indicates (5.7). \( \Box \)

Lemma 23 Let \( u \in H^m_0(\Omega) \cap H^r(\Omega) \) with \( r = \max\{k + 1, 2m - 1\} \) be the solution of the polyharmonic equation (4.1). Assume \( f \in H^\ell(T_h) \) with \( \ell = \max\{0, k + 1 - 2m\} \). It holds for any \( v_h \in V_h \) that
\[
(\nabla^m u, \nabla^m v_h) - \langle f, v_h \rangle \lesssim h^{k+1-m\max\{0,2m-k-1\}}|f|_{\ell,h}|v_h|_{m,h}.
\] (5.8)

Proof Notice that
\[
(\nabla^m u, \nabla^m v_h) - (f, v_h)
\]
\[
= \sum_{i=0}^{m-2} (-1)^i \left( (\text{div}^i \nabla^m u, \nabla^{m-i} v_h) + (\text{div}^{i+1} \nabla^m u, \nabla^{m-(i+1)} v_h) \right)
\]
\[
+ (\text{div}^{m-i} \nabla^m u, \nabla^i v_h) - (f, v_h).
\]

Then it follows from (5.3)–(5.5)
\[
(\nabla^m u, \nabla^m v_h) - (f, v_h) \lesssim h^{k+1-m\max\{0,2m-k-1\}}|f|_{\ell,h}|v_h|_{m,h},
\]
which together with (5.7) yields (5.8). \( \Box \)
5.2 Error estimate

With previous preparation, we can show the optimal order convergence of the nonconforming virtual element method (4.3).

**Theorem 2** Let \( u \in H^m_0(\Omega) \cap H^r(\Omega) \) with \( r = \max\{k + 1, 2m - 1\} \) be the solution of the polyharmonic equation (4.1), and \( u_h \in V_h \) be the nonconforming virtual element method (4.3). Assume the mesh \( T_h \) satisfies conditions (A1) and (A2). Assume \( f \in H^\ell(T_h) \) with \( \ell = \max\{0, k + 1 - 2m\} \). Then it holds

\[
|u - u_h|_{m,h} \lesssim h^{k+1-m} \|u\|_r + h||f||_0 + h^{\max(0.2m-k-1)} |f|_\ell,h). \tag{5.9}
\]

**Proof** Let \( v_h = I_h u - u_h \). It follows from (5.2), (4.21), (5.1) and (4.28)

\[
a_h(I_h u, v_h) - (\nabla^m u, \nabla^m v_h) = a_h(I_h u - T_h u, v_h) + a_h(T_h u, v_h) - (\nabla^m u, \nabla^m v_h)
\]

\[
= a_h(I_h u - T_h u, v_h) + (\nabla^m(T_h u - u), \nabla^m v_h)
\]

\[
\lesssim |I_h u - T_h u|_{m,h} |v_h|_{m,h} + |u - T_h u|_{m,h} |v_h|_{m,h}
\]

\[
\lesssim (|u - I_h u|_{m,h} + |u - T_h u|_{m,h}) |v_h|_{m,h}
\]

\[
\lesssim h^{k+1-m} |u|_{k+1} |v_h|_{m,h}. \tag{5.10}
\]

Notice that

\[
a_h(I_h u, v_h) - \langle f, v_h \rangle = a_h(I_h u, v_h) - (\nabla^m u, \nabla^m v_h) + (\nabla^m u, \nabla^m v_h) - \langle f, v_h \rangle.
\]

We get from (4.22), (4.3), (5.10) and (5.8)

\[
|I_h u - u_h|_{m,h}^2 \lesssim a_h(I_h u - u_h, v_h) = a_h(I_h u, v_h) - \langle f, v_h \rangle
\]

\[
\lesssim h^{k+1-m} \|u\|_r + h||f||_0 + h^{\max(0.2m-k-1)} |f|_\ell,h),
\]

which together with (5.1) implies (5.9).

\( \square \)

6 Implementation of the virtual element method

In this section, we will discuss the implementation of the nonconforming virtual element method (4.3). The implementation of the virtual element method of second order problems can be found in [7, 10].

Take any \( K \in T_h \). Let \( n_K := \dim \mathbb{P}_k(K) \), and denote all the functions in \( \mathbb{M}_k(K) \) by \( m_1, m_2, \ldots, m_{n_K} \). Let the bases of \( V_h(K) \) be \( \phi_1, \phi_2, \ldots, \phi_{N_K} \), which are dual to \( \chi_1, \chi_2, \ldots, \chi_{N_K} \), i.e.

\[
\chi_i(\phi_j) = \delta_{ij} \quad i, j = 1, 2, \ldots, N_K.
\]

Here \( \delta_{ij} \) is Kronecker delta.
6.1 Local $H^m$ projection

Since $\Pi^K \phi_j \in \mathbb{P}_k(K)$ for $j = 1, 2, \ldots, N_K$, we can write

$$
\Pi^K \phi_j = \sum_{i=1}^{n_k} \pi_{ij} m_i.
$$

(6.1)

Denote the matrix representation $(\pi_{ij})_{n_k \times N_K}$ of $\Pi^K$ by $\Pi^K$, then

$$(\Pi^K \phi_1, \Pi^K \phi_2, \ldots, \Pi^K \phi_{N_K}) = (m_1, m_2, \ldots, m_{n_k}) \Pi^K.$$

(6.2)

Let $G := \begin{pmatrix} G_{11} & G_{12} \\ O & G_{22} \end{pmatrix}$, where $O \in \mathbb{R}^{(n_k-n_m-1) \times n_m-1}$ is the zero matrix, and matrices $G_{11} \in \mathbb{R}^{n_m \times n_m}$, $G_{12} \in \mathbb{R}^{n_m \times (n_k-n_m-1)}$ and $G_{22} \in \mathbb{R}^{(n_k-n_m-1) \times (n_k-n_m-1)}$ are given by

$$
G_{11} := \begin{pmatrix}
\sum_{\delta \in F^2(K)} m_1(\delta) & \cdots & \sum_{\delta \in F^2(K)} m_{n_m-1}(\delta) \\
\sum_{\delta \in F^2(K)} (\nabla m_1)(\delta) & \cdots & \sum_{\delta \in F^2(K)} (\nabla m_{n_m-1})(\delta) \\
\vdots & \ddots & \vdots \\
\sum_{F \in F^{n-1}(K)} Q_0^F(\nabla^{m-1} m_1) & \cdots & \sum_{F \in F^{n-1}(K)} Q_0^F(\nabla^{m-1} m_{n_m-1})
\end{pmatrix},
$$

$$
G_{12} := \begin{pmatrix}
\sum_{\delta \in F^2(K)} m_{n_m-1+1}(\delta) & \cdots & \sum_{\delta \in F^2(K)} m_{n_k}(\delta) \\
\sum_{\delta \in F^2(K)} (\nabla m_{n_m-1+1})(\delta) & \cdots & \sum_{\delta \in F^2(K)} (\nabla m_{n_k})(\delta) \\
\vdots & \ddots & \vdots \\
\sum_{F \in F^{n-1}(K)} Q_0^F(\nabla^{m-1} m_{n_m-1+1}) & \cdots & \sum_{F \in F^{n-1}(K)} Q_0^F(\nabla^{m-1} m_{n_k})
\end{pmatrix},
$$

$$
G_{22} := \begin{pmatrix}
(\nabla^m m_{n_m-1+1}, \nabla^m m_{n_m-1+1})_K & \cdots & (\nabla^m m_{n_m-1+1}, \nabla^m m_{n_k})_K \\
(\nabla^m m_{n_k}, \nabla^m m_{n_m-1+1})_K & \cdots & (\nabla^m m_{n_k}, \nabla^m m_{n_k})_K
\end{pmatrix}.
$$

Noting that

$$
\nabla^i m_i = 0 \quad \text{for } i = 1, 2, \ldots, n_{j-1}, j = 1, 2, \ldots, m-1,
$$

there are many zero entries in the submatrix $G_{11}$. Let $B := (b_1, b_2, \ldots, b_{N_K})_{n_k \times N_K}$ with

$$
\mathbb{C} \text{ Springer}
$$
Then the linear system of the problem (3.16)–(3.18) with \(v = \phi_1, \phi_2, \ldots, \phi_{N_K}\) is

\[
\mathbf{G}\Pi^K = \mathbf{B}.
\]

Hence we can compute \(\Pi^K\) as follows

\[
\Pi^K = \mathbf{G}^{-1}\mathbf{B}. \tag{6.3}
\]

Define matrix

\[
\mathbf{D} := (D_{ij})_{N_K \times n_k} = \begin{pmatrix}
\chi_1(m_1) & \chi_1(m_2) & \cdots & \chi_1(m_{n_k}) \\
\chi_2(m_1) & \chi_2(m_2) & \cdots & \chi_2(m_{n_k}) \\
\vdots & \vdots & \ddots & \vdots \\
\chi_{N_K}(m_1) & \chi_{N_K}(m_2) & \cdots & \chi_{N_K}(m_{n_k})
\end{pmatrix}.
\]

Then

\[
(m_1, m_2, \ldots, m_{n_k}) = (\phi_1, \phi_2, \ldots, \phi_{N_K})\mathbf{D}. \tag{6.4}
\]

It follows from (6.2) and (6.3) that

\[
(\Pi^K \phi_1, \Pi^K \phi_2, \ldots, \Pi^K \phi_{N_K}) = (\phi_1, \phi_2, \ldots, \phi_{N_K})\mathbf{D}\Pi^K = (\phi_1, \phi_2, \ldots, \phi_{N_K})\mathbf{D}\mathbf{G}^{-1}\mathbf{B}.
\]

**Lemma 24** It holds

\[
\mathbf{G} = \mathbf{B}\mathbf{D}.
\]

This provides another way to compute \(\mathbf{G}\).

**Proof** Applying (6.4), (6.2) and (6.3), we get
\[(m_1, m_2, \ldots, m_{n_k}) = (\Pi^K m_1, \Pi^K m_2, \ldots, \Pi^K m_{n_k}) = (\Pi^K \phi_1, \Pi^K \phi_2, \ldots, \Pi^K \phi_{N_k}) D = (m_1, m_2, \ldots, m_{n_k}) \Pi^K D = (m_1, m_2, \ldots, m_{n_k}) G^{-1} B D,\]

as required. \qed

### 6.2 Local stiffness matrix

Denote the local stiffness matrix by \(A_K := ((A_K)_{ij}) \in \mathbb{R}^{N_K \times N_K}\), where

\[
(A_K)_{ij} := a_{h,K}(\phi_j, \phi_i) = (\nabla m_1 \Pi^K \phi_j, \nabla m_2 \Pi^K \phi_i)_K + S_K(\phi_j - \Pi^K \phi_j, \phi_i - \Pi^K \phi_i).
\]

Using (6.1), the consistency term

\[
\nabla m_1 \Pi^K \phi_j, \nabla m_2 \Pi^K \phi_i)_K = \sum_{s, r=1}^{N_K} \pi_{si} (\nabla m_r, \nabla m_s)_K \pi_{rj}.
\]

Hence the matrix representation of the consistency term is

\[
A^c_K = (\Pi^K)^T \begin{pmatrix}
O_{n_{m-1} \times n_{m-1}} & O_{n_{m-1} \times (n_k - n_{m-1})} \\
O_{(n_k - n_{m-1}) \times n_{m-1}} & G_{22}
\end{pmatrix} \Pi^K.
\]

Next consider the stability term. Let matrix

\[
S := h_K^{n-2m} \begin{pmatrix}
I_{(N_K - n_k - 2m) \times (N_K - n_k - 2m)} & O_{(N_K - n_k - 2m) \times n_k - 2m} \\
O_{n_k - 2m \times (N_K - n_k - 2m)} & O_{n_k - 2m \times n_k - 2m}
\end{pmatrix}.
\]

The right-bottom zero submatrix in \(S\) reflects the fact that the stabilization \(S_K(\cdot, \cdot)\) only involves the boundary degrees of freedom. The stability term

\[
S_K(\phi_j - \Pi^K \phi_j, \phi_i - \Pi^K \phi_i) = \sum_{s, r=1}^{N_K} (I - D \Pi^K)_{si} S_K(\phi_r, \phi_s)(I - D \Pi^K)_{rj}.
\]

Thus the matrix representation of the stability term is

\[
A^s_K = (I - D \Pi^K)^T S (I - D \Pi^K).
\]

Therefore the local stiffness matrix

\[
A_K = A^c_K + A^s_K = (\Pi^K)^T \begin{pmatrix}
O & O \\
O & G_{22}
\end{pmatrix} \Pi^K + (I - D \Pi^K)^T S (I - D \Pi^K).
\]
6.3 Right hand side term

Finally we discuss the implementation of the right hand side term. The vector representation of the right hand side term restricted on $K$ is $b := (b_1, b_2, \ldots, b_{N_K})^\top$ with

$$b_i := \begin{cases} (f, \Pi^K \phi_i)_K, & m \leq k \leq 2m - 1, \\ (f, Q_{k-2m}^K \Pi^K \phi_i + Q_{k-2m}^K (\phi_i - \Pi^K \phi_i))_K, & 2m \leq k \leq 3m - 2, \\ (f, Q_{k-2m}^K \phi_i)_K, & 3m - 1 \leq k. \end{cases}$$

Set

$$F := ((f, m_1)_K, (f, m_2)_K, \ldots, (f, m_{N_j})_K)^\top,$$

$$\tilde{F} := ((f, m_1)_K, (f, m_2)_K, \ldots, (f, m_{n_{k-2m}})_K)^\top,$$

$$\bar{F} := ((f, m_1)_K, (f, m_2)_K, \ldots, (f, m_{n_{k-2m}})_K)^\top.$$

Let

$$M := \begin{pmatrix} (m_1, m_1)_K & (m_1, m_2)_K & \cdots & (m_1, m_{N_j})_K \\ (m_2, m_1)_K & (m_2, m_2)_K & \cdots & (m_2, m_{N_j})_K \\ \vdots & \vdots & \ddots & \vdots \\ (m_{n_{k-2m}}, m_1)_K & (m_{n_{k-2m}}, m_2)_K & \cdots & (m_{n_{k-2m}}, m_{N_j})_K \end{pmatrix},$$

$$\tilde{M} := \begin{pmatrix} (m_1, m_1)_K & (m_1, m_2)_K & \cdots & (m_1, m_{n_{k-2m}})_K \\ (m_2, m_1)_K & (m_2, m_2)_K & \cdots & (m_2, m_{n_{k-2m}})_K \\ \vdots & \vdots & \ddots & \vdots \\ (m_{n_{k-2m}}, m_1)_K & (m_{n_{k-2m}}, m_2)_K & \cdots & (m_{n_{k-2m}}, m_{n_{k-2m}})_K \end{pmatrix},$$

$$\bar{M} := \begin{pmatrix} (m_1, m_1)_K & (m_1, m_2)_K & \cdots & (m_1, m_{n_{k-2m}})_K \\ (m_2, m_1)_K & (m_2, m_2)_K & \cdots & (m_2, m_{n_{k-2m}})_K \\ \vdots & \vdots & \ddots & \vdots \\ (m_{n_{k-2m}}, m_1)_K & (m_{n_{k-2m}}, m_2)_K & \cdots & (m_{n_{k-2m}}, m_{n_{k-2m}})_K \end{pmatrix}.$$  

For $m \leq k \leq 2m - 1$, it follows from (6.1) that

$$b_i = (f, \Pi^K \phi_i) = \sum_{j=1}^{n_j} \pi_{ji} (f, m_j)$$

Thus we have

$$b = (\Pi^K)^\top F.$$  

For $k \geq 3m - 1$, since it holds for positive integer $j \leq n_{k-2m}$ that

$$(Q_{k-2m}^K \phi_i, m_j)_K = (\phi_i, m_j)_K = |K| \chi_{N_K-n_{k-2m}+j}(\phi_i) = |K| \delta_{N_K-n_{k-2m}+j,i},$$

we get
\((Q_{k-2m}^K \phi_1, Q_{k-2m}^K \phi_2, \ldots, Q_{k-2m}^K \phi_N)\) \\
\quad = |K|(m_1, m_2, \ldots, m_{n_{k-2m}})\tilde{\mathbf{M}}^{-1}(O_{n_{k-2m} \times (N_k - n_{k-2m})}, I_{n_{k-2m} \times n_{k-2m}}).

Hence it follows \\
\(b = (f, Q_{k-2m}^K \phi_1, f, Q_{k-2m}^K \phi_2, \ldots, f, Q_{k-2m}^K \phi_N)')^T\) \\
\quad = |K|(O_{n_{k-2m} \times (N_k - n_{k-2m})}, \tilde{\mathbf{M}}^{-1})^T\mathbf{F} = |K|(O_{(N_k - n_{k-2m}) \times 1}^{N_k - n_{k-2m}})\mathbf{M}^{-1}\mathbf{F}.

Now consider the case \(2m \leq k \leq 3m - 2\). Noting that \\
\((\phi_1 - \Pi^K \phi_1, \phi_2 - \Pi^K \phi_2, \ldots, \phi_N - \Pi^K \phi_N) = (\phi_1, \phi_2, \ldots, \phi_N)(I - \mathbf{DII}^K),\) we obtain \\
\(\begin{pmatrix} (f, Q_{k-2m}^K (\phi_1 - \Pi^K \phi_1)) \\ \vdots \\ (f, Q_{k-2m}^K (\phi_N - \Pi^K \phi_N)) \end{pmatrix} = |K|(I - \mathbf{DII}^K)^T(O_{(N_k - n_{k-2m}) \times 1}^{N_k - n_{k-2m}})\mathbf{M}^{-1}\mathbf{F}.\)

On the other side, we have \\
\((Q_{m-1}^K m_1, \ldots, Q_{m-1}^K m_N) = (m_1, \ldots, m_{n_{m-1}})\tilde{\mathbf{M}}^{-1}\mathbf{M},\) and thus \\
\((f, Q_{m-1}^K m_1), \ldots, (f, Q_{m-1}^K m_N) = \mathbf{F}^T\tilde{\mathbf{M}}^{-1}\mathbf{M}.\)

It follows from (6.2) that \\
\(\begin{pmatrix} (f, Q_{m-1}^K \Pi^K \phi_1) \\ \vdots \\ (f, Q_{m-1}^K \Pi^K \phi_N) \end{pmatrix} = (\Pi^K)^T(f, Q_{m-1}^K m_1) \\ \vdots \\ (f, Q_{m-1}^K m_N) = (\Pi^K)^T\mathbf{M}^T\tilde{\mathbf{M}}^{-1}\mathbf{F}.\)

Therefore in this case we achieve \\
\(b = (\Pi^K)^T\mathbf{M}^T\tilde{\mathbf{M}}^{-1}\mathbf{F} + |K|(I - \mathbf{DII}^K)^T(O_{(N_k - n_{k-2m}) \times 1}^{N_k - n_{k-2m}})\mathbf{M}^{-1}\mathbf{F}.\)

7 Conclusion

Based on a generalized Green’s identity for \(H^m\) inner product \(m > n\), we present the \(H^m\)-nonconforming virtual element method of any order \(k\) on any shape of polytope in \(\mathbb{R}^n\) with constraints \(m > n\) and \(k \geq m\) in a universal way to continue the work in [17]. We improve the discrete method in [17] as follows:
(1) The stabilization term involves only the boundary degrees of freedom, whereas all the degrees of freedom are involved in the stabilization term in [17];

(2) For the case \(2m \leq k \leq 3m - 2\), we define the right hand side term as \((f, Q_h^{m-1} \Pi_h v_h + Q_h^{k-2m}(v_h - \Pi_h v_h))\), rather than \((f, Q_h^{m-1} v_h)\) in [17], as a result of which the modification of the space of shape functions is not required.

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