IN Variant Four-variable Automorphic Kernel Functions

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Abstract. Let $F$ be a number field, let $\mathbb{A}_F$ be its ring of adeles, and let $g_1, g_2, h_1, h_2 \in \text{GL}_2(\mathbb{A}_F)$. Previously the author provided an absolutely convergent geometric expression for the four variable kernel function

$$\sum_\pi K_\pi(g_1, g_2)K_{\pi^\vee}(h_1, h_2),$$

where the sum is over isomorphism classes of cuspidal automorphic representations $\pi$ of $\text{GL}_2(\mathbb{A}_F)$. Here $K_\pi$ is the typical kernel function representing the action of a test function on the space of the cuspidal automorphic representation $\pi$. In this paper we show how to use ideas from the circle method to provide an alternate expansion for the four variable kernel function that is visibly invariant under the natural action of $\text{GL}_2(F) \times \text{GL}_2(F)$.

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2010 Mathematics Subject Classification. Primary 11F70; Secondary 11F72, 11D85.

The author is thankful for partial support provided by NSF grant DMS-1405708. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.
1. Introduction

Let $F$ be a number field and let $A \leq \text{GL}_2(F_{\infty})$ be the central diagonal copy of $\mathbb{R}_{>0}$. For $f \in C_c^\infty(A\backslash \text{GL}_2(\mathbb{A}_F))$ and cuspidal automorphic representations $\pi$ of $A\backslash \text{GL}_2(\mathbb{A}_F)$ let

$$K_{\pi(f)}(x, y)$$

denote the usual kernel function (for more details on our notational conventions see the introduction to [G2]).

Let $g_\ell = (g'_\ell, g''_\ell), g_r = (g'_r, g''_r) \in \text{GL}_2(\mathbb{A}_F) \times \text{GL}_2(\mathbb{A}_F)$ and let $f_1, f_2 \in C_c^\infty(A\backslash \text{GL}_2(\mathbb{A}_F))$ be test functions unramified outside of a finite set of places $S$. Let

$$\Sigma_{\text{cusp}}(g_\ell, g_r) := \sum_{\pi} K_{\pi(f_1)}(g'_\ell, g'_r)K_{\pi^\vee(f_2)}(g''_\ell, g''_r)\text{Res}_{s=1}L(s, \pi \times \pi^\vee_S).$$

In [G2] the author gave a geometric expression for $\Sigma_{\text{cusp}}(g_\ell, g_r)$. The motivation, as explained in loc. cit., is to integrate this expression over a pair of twisted diagonal subgroups and thereby provide an explicit nonabelian trace formula, that is, a trace formula whose spectral side is a weighted sum over representations invariant under a simple nonabelian subgroup of $\text{Aut}_F(\mathbb{Q})$. This is a step in the author’s program to establish nonsolvable base change for $\text{GL}_2$ (see [G1], [GH], [G2]). Other possible applications are given in §1.2 below.

The defect in the formula for $\Sigma_{\text{cusp}}(g_\ell, g_r)$ given in [G2] is that it is not obviously invariant under $(g_\ell, g_r) \mapsto (\gamma_\ell g_\ell, \gamma_r g_r)$ for $\gamma_\ell, \gamma_r \in \text{GL}_2(F)^{\times 2}$; briefly, it is not invariant under $\text{GL}_2(F)^{\times 2}$. Thus to use it for its intended purpose one seems to be forced to employ some variant of the Rankin-Selberg method.

The root of the lack of invariance in the formula for $\Sigma_{\text{cusp}}(g_\ell, g_r)$ in [G2] is easy to describe. In loc. cit. one investigates a certain limit constructed out of Whittaker coefficients of a product of kernel functions. Taking the Whittaker coefficients introduces integrals over adelic quotients of nilpotent groups and destroys invariance, and it is not so easy to rebuild this invariance on the geometric side of the formula.

In this paper we overcome this difficulty by providing a different geometric formula for $\Sigma_{\text{cusp}}(g_\ell, g_r)$ that is clearly invariant under $\text{GL}_2(F)^{\times 2}$. This will make it easier to integrate the formula over a pair of twisted diagonals. It should also be noted that the approach exposed in this paper should work if we replace $\text{GL}_2$ by an inner form, whereas the approach of [G2] can’t be applied to these groups because it involves integration over nilpotent subgroups.

Moreover, the approach given here is of interest in itself for at least two reasons. First, it is an instance where one can insert the nonstandard test functions of Ngô [N] and Sakellaridis [S] into the trace formula and understand the coarse analytic properties of the result without appealing to known results on automorphic forms. Second, it involves a variant of the circle method in a case where one is not interested in the main term, but in the secondary terms (this is discussed in §1.3 below). Finally, we remark that at this stage in the mathematical
community’s investigation of Langlands functoriality beyond endoscopy, it is vital to develop as many tools and methods as possible in order to broaden our collective understanding.

**Remark.** The approach of [G2] is not without merits. It is a little simpler than the approach exposed here in some respects, and it is unclear which method will generalize easiest to the higher rank case.

1.1. **Statement of the formula.** We assume that $\mathcal{O}_F^S$ has class number 1 and $\mathcal{O}_F/\mathbb{Z}$ is unramified outside of $S$. Let $v \in S - \infty$. Let $k \in \mathbb{Z}_{\geq 0}$. Consider the following assumption on $\Phi_v \in C_c^\infty(\text{GL}_2(F_v))$:

(A) The function $\Phi_v$ is supported in the set of $g$ with valuation $v(\det g) = k$, $\int_{\text{GL}_2(F_v)} \Phi_v(g) dg = 0$, and $\Phi_v \in C_c^\infty(\text{GL}_2(F_v)/\text{GL}_2(\mathcal{O}_v))$.

We introduce our test functions and assumptions, and then comment on them after the statement of our main theorem:

(i) Let $f_1, f_2 \in C_c^\infty(A/\text{GL}_2(F_S))$ and define $f(g', g'') = f_1(g') f_2(g'')$ for $(g', g'') \in \text{GL}_2(\mathbb{A}_F)^{\times 2}$.

(ii) Assume that $f_i = f_i^v \otimes f_i^w$ with $f_i^v, f_i^w \in C_c^\infty(A/\text{GL}_2(F_{S-v}))$, $f_i^v \in C_c^\infty(\text{GL}_2(F_v))$, $f_2^v = 1_{\text{GL}_2(\mathcal{O}_v)}$, and $f_1^v$ satisfying assumption (A) above.

(iii) Assume that the operators

$$R(f_1), R(f_2) : L^2(AGL_2(F)\backslash GL_2(\mathbb{A}_F)) \longrightarrow L^2(AGL_2(F)\backslash GL_2(\mathbb{A}_F))$$

induced by the right regular action and the test functions $f_1$ and $f_2$ respectively have cuspidal image.

(iv) Let $V_1 \in C_c^\infty((0, \infty))$ and define $V \in C^\infty(F_S)$ as in §3 using it.

(v) Let $h(x, y) := W(x) - W(y/x)$ where $W = W_\infty 1_{\mathcal{O}_S^\times} \in C_c^\infty(F_S^\times)$ and $\widetilde{W}(0) = 1$.

(vi) Let $1_{FS}$ be the characteristic function of a fundamental domain for $\mathcal{O}_S^{Sx}$ acting on $F_S^{\times}$ and for $b = (b_1, b_2) \in F_S^{\times} \times F_S^{\times}$ define

$$1_{F}(b) := 1_{FS}(b) 1_{\mathcal{O}_S^{Sx}}(b_1) 1_{\mathcal{O}_S^{Sx}}(b_2).$$

We also abbreviate

$$b \det T := (b_1 \det T_1, b_2 \det T_2)$$

$$P(b, T) := b_1 \det T_1 - b_2 \det T_2$$

$$\text{tr} \gamma T := \text{tr}(\gamma_1 T_1 + \gamma_2 T_2)$$

for $b = (b_1, b_2) \in (\mathbb{A}_F)_{\geq 2}^{\times}$, $\gamma = (\gamma_1, \gamma_2)$, and $T = (T_1, T_2) \in \mathfrak{gl}_2(\mathbb{A}_F)$.

With the notation above in mind, we state the main theorem:
Theorem 1.1. Letting $\widetilde{V}_1$ denote the Mellin transform of $V_1$, one has that $\Sigma_{\text{cusp}}(g_\ell, g_r)$ is equal to
\[
\frac{d_F}{{V}_1(1)} \sum_{0 \neq \gamma \in \text{GL}_2(F)^{\infty}} \sum_{b \in (F^\times)^{\infty}} \int_{\mathbb{A}_F} \frac{1}{t} \mathcal{O}_F^\delta(t) \left( \int_{\text{GL}_2(A_F)^{\infty}} \frac{V(b \det T)}{|b_1 \det T_1|^S} \frac{1}{t} \mathcal{O}_F^\delta(P(b, T)) \right) dt \frac{dt}{|t|^4}.
\]

We complete the proof of Theorem 1.1 in §7 below. We now comment on the various assumptions:

Remarks.

(1) Assumption (A) eliminates the contribution of the nongeneric spectrum; it is no loss of generality for studying the generic spectrum, as we prove in Lemma 4.1.

(2) We assume (iii) only to simplify the spectral side of the formula; it is not used in the analysis of the geometric side which forms the bulk of this paper.

(3) The $V$ function smooths sums over Hecke operators.

(4) The $h$ function comes up in our application of the $\delta$-symbol method (see §2 and §3).

1.2. Possible applications. Our primary motivation for proving Theorem 1.1 is to use it to produce a trace formula isolating representations invariant under a pair of automorphisms $\iota, \tau \in \text{Aut}_Q(F)$. One might then hope to compare this formula with a similar formula over the fixed field of $\langle \iota, \tau \rangle$ acting on $F$ and prove nonsolvable base change for $\text{GL}_2$ (compare [G1], [GH], [G2]). Of course, this is very speculative.

Let $\chi_1, \chi_2, \chi_3, \chi_4 : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ be a quadruple of characters. A more immediate application of Theorem 1.1 might be studying asymptotics of sums of products of $L$-functions of the form
\[
L\left(\frac{1}{2}, \pi \otimes \chi_1\right)L\left(\frac{1}{2}, \pi \otimes \chi_2\right)
L\left(\frac{1}{2}, \pi \otimes \chi_3\right)L\left(\frac{1}{2}, \pi \otimes \chi_4\right)
\]
as the analytic conductor of $\pi$ increases. W. Zhang has also pointed out to the author the possibility of using the main theorem to prove a new Waldspurger type formula (compare [Z, §4.2]) involving products of $L$-functions as above. In any case, we would like to emphasize that Theorem 1.1 is flexible enough to lead to a variety of applications beyond the primary one motivating the author.

1.3. The method. Let $f \in C^\infty_c((A \setminus \text{GL}_2(A_F))^2)$ and let
\[
\mathbbm{1}_m := \mathbbm{1}_{g \in \text{GL}_2(\mathcal{O}_F^\times)} |\det g|^{-m} |\det g|^{-m}
\]
and for $g = (g', g'') \in \text{GL}_2(A_F^S)^{\times 2}$ let $\mathbbm{1}_m(g) := \mathbbm{1}_m(g') \mathbbm{1}_m(g'')$. We consider
\[
\Sigma(X) = \sum_m \frac{V_1([m]S/X)}{|m|S^X} \sum_{\gamma \in \text{GL}_2(F)^{\times 2}} f \mathbbm{1}_m(g^{-1}_r \gamma g_r)
\]
where the sum on $m$ is over a set of representatives for the ideals of $\mathcal{O}_F$.

The following proposition is proven using an easy modification of the proof of [G2, Proposition 5.1]:

**Proposition 1.2.** If $f = f_1 f_2$ and $R(f_1), R(f_2)$ have cuspidal image, then

$$\tilde{V}_1(1)\Sigma_{\text{cusp}}(g_\ell, g_r) = \zeta_F^S(2) \lim_{X \to \infty} \Sigma(X).$$

This is the only place in the paper where we use the assumption that $R(f_1)$ and $R(f_2)$ have cuspidal image.

The bulk of the paper is devoted to evaluating $\lim_{X \to \infty} \Sigma(X)$ geometrically. To see what is going on, it is perhaps useful to specialize to the case where $S = \infty$, $F = \mathbb{Q}$, $g_\ell = g_r = (I, I)$, and $f \in C_\infty^\infty((A \setminus \text{GL}_2(\mathbb{R}))/\mathbb{Z})^2$ is supported on elements of $\text{GL}_2(\mathbb{R})$ with positive determinant. In this case the sum $\Sigma(X)$ reduces to

$$\sum_{\gamma_1, \gamma_2 \in M_2(\mathbb{Z})} \frac{V_1(\frac{\det \gamma_1}{X})}{X \det \gamma_1} f(\gamma_1, \gamma_2).$$

Thus $\Sigma(X)$ is essentially a smoothed version of the function counting integral points of height at most $X$ on the hypersurface in $M_2 \cong \mathbb{A}^8$ defined by $\det \gamma_1 - \det \gamma_2 = 0$. However, we are not interested in the main term, which comes from the trivial representation, which would have size $X$ if we had not used assumption (ii) of Theorem 1.1 to remove it (compare Lemma 4.2). We are interested in all of the secondary terms. Despite this, the version of the circle method known as the $\delta$-symbol method is still strong enough to give us what we need; a suitable modification of this method is what we use.

**Remarks.**

(1) The hypersurface in question is homogeneous, so in obtaining the main term of $\Sigma(X)$ one could use automorphic techniques as in the work of Duke, Rudnick and Sarnak [DRS]. However, this is of no use to us, for it would just give back the spectral formula for $\Sigma_{\text{cusp}}(g_\ell, g_r)$.

(2) The only other instance that the author knows where secondary terms have been obtained via the circle method is in Vaughan and Wooley [VW] and forthcoming work of Schindler.

1.4. **Outline of the paper.** In §2 we introduce our expansion of the $\delta$-symbol. It is applied to $\Sigma(X)$ in §3. We then apply Poisson summation in $\gamma \in \text{gl}_2(F)^{\oplus 2}$ to the sum and then write $\Sigma(X) = \Sigma_0(X) + \Sigma^0(X)$ where $\Sigma_0(X)$ is the contribution of the $(0, 0)$ term after Poisson summation and $\Sigma^0(X)$ is the contribution of the other terms (see (3.0.2)). We isolate the zeroth term after Poisson summation in §4 and show that it is zero under assumption (ii) in the statement of Theorem 1.1; this is the only place in the paper where this assumption is used.
We are left with analyzing $\Sigma^0(X)$. This requires one more application of Poisson summation (in the multiplicative sense). The computations in the unramified case are contained in §5 and the estimates required to handle the resulting sum are contained in §6. The actual application of Poisson summation in the multiplicative sense comes in §7, and this is where we complete the proof of Theorem 1.1.

1.5. **Notation.** Throughout this paper we use “standard” normalizations of Haar measures (see [GH, §2]). Letting $\psi$ denote the “standard” additive character $\psi$ (see [GH, §3.1]), for $\Phi \in C^\infty_c(\text{gl}_n(A_F))$ we let

$$\hat{\Phi}(Y) := \int_{\text{gl}_n(A_F)} \Phi(X) \psi(\text{tr}(YX)) dX$$

denote the Fourier transform of $\Phi$. The Poisson summation formula then takes the form

$$\sum_{\gamma \in \text{gl}_n(F)} \Phi(\gamma) = \frac{1}{d_F^{n/2}} \sum_{\gamma \in \text{gl}_n(F)} \hat{\Phi}(\gamma)$$

where $d_F \in \mathbb{Z}_{>0}$ is the absolute discriminant of $F$.

**ACKNOWLEDGEMENTS**

The author thanks L. Pierce, D. Schindler and W. Zhang for useful conversations and H. Hahn for her constant encouragement and help with editing.

2. **The $\delta$-symbol**

For $m \in \mathcal{O}_F^S$ let

$$(2.0.1) \quad \delta^S(m) := \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise}. \end{cases}$$

Duke, Friedlander and Iwaniec [DFI] introduced a very useful expression for this simple function in the case where $F = \mathbb{Q}$ which has been used to great effect (see also the work of Heath-Brown [HB]) and generalized to ideals of number fields in work of Browning and Vishe [BV]. We introduce a slight variant of their expression here. It will be applied below in §3.

If $X \in \mathbb{R}_{>0}$ we denote by

$$\Delta(X) \in \mathbb{A}_F^\times$$

the idele that is $X^{[F:\mathbb{Q}]}$ at all places $v|\infty$ and 1 elsewhere. Moreover let $d_F \in \mathbb{Z}_{>0}$ denote the absolute discriminant of $F$.

In this section we prove the following proposition:
Proposition 2.1. Let $W = \prod_{v \in S} W_v \in C_\infty^\infty(F_v^\times)$ be nonnegative and satisfy $\hat{W}(0) = 1$. If $h(x, y) := W(x) - W(x/y)$ then for all sufficiently large $Q \in \mathbb{R}_{>0}$ one has

$$\delta^S(m) = \frac{c_Q}{Q} \sum_{d \in \mathcal{O}_F^S} \mathbb{1}_{d\mathcal{O}_F^S}(m) h \left( \frac{d}{\Delta(Q)}, \frac{m}{\Delta(Q)^2} \right)$$

where for any $N > 0$ one has

$$c_Q = \sqrt{d_F} + O_N(Q^{-N}).$$

Proof. One has

$$\sum_{d \in \mathcal{O}_F^S} \left( W \left( \frac{d}{\Delta(Q)} \right) - W \left( \frac{m}{d\Delta(Q)} \right) \right) = \begin{cases} 0 & \text{if } m \neq 0 \\ \sum_{d \in \mathcal{O}_F^S} W \left( \frac{d}{\Delta(Q)} \right) & \text{if } m = 0 \end{cases}$$

where the first sum is over all $d \in \mathcal{O}_F^S$ dividing $m$. This is an infinite set if $F$ is not $\mathbb{Q}$ or an imaginary quadratic field, but only finitely many of these $d$ yield a nonzero summand for each $m$ and $Q$. Define

$$c_Q := Q \left( \sum_{d \in \mathcal{O}_F^S} W \left( \frac{d}{\Delta(Q)} \right) \right)^{-1}.$$

It is then clear that the stated identity for $\delta^S(m)$ holds. We are left with proving the bound for $c_Q$. By Poisson summation, one has

$$\sum_{d \in \mathcal{O}_F^S} W \left( \frac{d}{\Delta(Q)} \right) = \frac{Q}{\sqrt{d_F}} \sum_{d \in \mathcal{O}_F^S} \hat{W}(\Delta(Q)d).$$

Integration by parts now yields the stated asymptotic for $c_Q$. \qed

For the remainder of the paper we make the assumption that the $W$ in the proposition satisfies

$$W_v = \mathbb{1}_{\mathcal{O}_{F_v}^S}$$

for all $v \in S - \infty$. This makes the considerations of §4 simpler.

3. First manipulations with the geometric side

We now use the notation of §1.1. Recall that $V_1 \in C_\infty^\infty((0, \infty))$. Choose $V_2 \in C_\infty^\infty((0, \infty))$ such that $V_2$ is identically 1 on the support of $V_1$ and $V_3 = \prod_{v \in S} V_{3v} \in C_\infty^\infty(F_v^\times)$ such that $V_{3v}$ is identically 1 on a neighborhood of 1 in $F_v$ for $v|\infty$ and $V_{3v} = \mathbb{1}_{\mathcal{O}_{F_v}^S}$ for $v \in S - \infty$. Write

$$V(x_1, x_2) := V_1(|x_1|) V_2(|x_2|) V_3(x_2/x_1).$$

For $\delta^S$ as in Proposition 2.1 one has

$$\Sigma(X) = \sum_{\gamma = (\gamma_1, \gamma_2)} \sum_{b = (b_1, b_2)} \frac{V(b \det \gamma/X)}{|b_1 \det \gamma_1| X} \delta^S(P(b, \gamma)) \mathbb{1}_F(b \det g e g^{-1}) f_1'(g e g^{-1} (g^{-1} \gamma g))$$
where the sums on $\gamma$ and $b$ are over $\text{GL}_2(F)^{\times 2}$ and $(F^\times)^{\oplus 2}$, respectively. Here $1_F$ is defined as in assumption (vi). We observe that the presence of the $V_2$ and $V_3$ in the definition of $V$ is redundant, but it simplifies matters when we later apply Poisson summation in §7 (compare Proposition 6.1 and Corollary 6.4). We also note that by definition of $V$ the sum on $b$ is finite in a sense depending only on $g_\ell, g_r, V, F$ and $f$.

Applying Proposition 2.1 with $Q = \sqrt{X}$ we obtain

$$
\Sigma(X) = \sum_{\gamma, b} C_{b, \gamma} \left( \frac{\Delta(X)}{\Delta(\sqrt{X})} \right)^s
\sum_{d \in \mathcal{O}_{F}^{2} - 0} \mathbb{1}_{d\mathbb{O}_{F}^{2}}(\gamma, b) h\left( \frac{d}{\Delta(\sqrt{X})}, \frac{P(b, \gamma)}{\Delta(X)} \right)
\times 1_F(b group \hat{g}_1 g_r^{-1}) \int \mathbb{1}_{\mathcal{O}_{F}^{2} \oplus 2}(g_\ell^{-1} \gamma g_r) d\gamma.
$$

Remark. Notice that in the sum above the moduli $d$ satisfy $|d|_{v} \ll (X^{F:Q})^{-1}$ for all $v | \infty$ and the $v$-norm of the entries of $\gamma_1, \gamma_2$ is also bounded by $O((X^{F:Q})^{-1})$ for $v | \infty$. Thus it is reasonable to expect that one can shorten the length of the sum by applying Poisson summation in $\mathfrak{g}l_2(F) \times \mathfrak{g}l_2(F)$. This is indeed the case.

We apply Poisson summation in $\gamma = (\gamma_1, \gamma_2) \in \mathfrak{g}l_2(F)^{\oplus 2}$ (see §1.5) to arrive at

$$
\Sigma(X) = d_{\mathfrak{g}l_2(F)}^{2} \sum_{\gamma \in \mathfrak{g}l_2(F)^{\oplus 2}} \sum_{b} \int \mathbb{1}_{\mathfrak{g}l_2(F)^{\oplus 2}}(b, T) \left( \frac{\Delta(T)}{\Delta(\sqrt{X})} \right)^s
\sum_{d \in \mathcal{O}_{F}^{2} - 0} \mathbb{1}_{d\mathbb{O}_{F}^{2}}(b) h\left( \frac{d}{\Delta(\sqrt{X})}, \frac{P(b, T)}{\Delta(X)} \right)
\times 1_F(b group \hat{g}_1 g_r^{-1}) \int \mathbb{1}_{\mathcal{O}_{F}^{2} \oplus 2}(g_\ell^{-1} \gamma T, g_r) \psi\left( \frac{\text{tr} \gamma T}{d} \right) dT.
$$

Here $T = (T_1, T_2)$, $dT = dt_1 dt_2$ is the Haar measure on $\mathfrak{g}l_2(F)^{\oplus 2}$. It is convenient to write

$$
\Sigma(X) = \Sigma_0(X) + \Sigma^0(X),
$$

where $\Sigma_0(X)$ is the contribution of the $\gamma = (0, 0)$ term and $\Sigma^0(X)$ is the contribution of the terms with $\gamma \neq (0, 0)$. We will show in §4 below that $\Sigma_0(X)$ vanishes under a mild assumption.

To complete our analysis of $\Sigma^0(X)$ we will apply Poisson summation in $d \in F^\times$ in §7. Before doing this we collect the necessary local computations and bounds in §5 and §6, respectively.

4. Vanishing of $\Sigma_0(X)$

Let $v \in S - \infty$ and let $k \in \mathbb{Z}_{>0}$. For $\Phi_v \in C_c^\infty(\text{GL}_2(F_v))$ recall assumption (A) of §1.1:

(A) The function $\Phi_v$ is supported in the set of $g$ with $v(\text{det} g) = k$, $\int_{\text{GL}_2(F_v)} \Phi_v(g) dg = 0$, and $\Phi_v \in C_c^\infty(\text{GL}_2(F_v)/\text{GL}_2(\mathcal{O}_{F_v})).$

It is clear that if $\pi_v$ is an abelian twist of the trivial representation of $\text{GL}_2(F_v)$ and $\Phi_v$ satisfies assumption (A) then $\pi_v(\Phi_v) = 0$. On the other hand, we have the following lemma:
Lemma 4.1. If $\pi_v$ is the local factor of a generic unitary automorphic representation of $GL_2(A_F)$ unramified at $v$ then there exists a $\Phi_v$ satisfying assumption (A) such that $\pi_v(\Phi_v) \neq 0$.

Proof. Let $\varpi_v$ be a uniformizer for $F_v$ and let $q_v := |\varpi_v|^{-1}$. Consider

$$\Phi_v := 1_{\varpi_v}^2 - (q_v^2 + q_v + 1)1_{\varpi_v}GL_2(O_{F_v}).$$

It is a standard result that

$$\bigcup_{g \in gl_2(O_{F_v}): \det g O_v^\times = \varpi_v^2 O_v^\times} GL_2(O_{F_v})gGL_2(O_{F_v})$$

can be written as a disjoint sum of $q_v^2 + q_v + 1$ elements of $GL_2(F_v)/GL_2(O_{F_v})$ [KL, Proof of Proposition 4.4]. Therefore $\Phi_v$ satisfies assumption (A) with $k = 2$.

Let $\alpha, \beta \in \mathbb{C}^\times$ be the Satake parameters of $\pi_v$. Then $\pi_v(\Phi_v)$ projects the space of $\pi_v$ to the spherical vector and acts via the scalar

$$(4.0.1) \quad q_v(\alpha^2 + \alpha \beta + \beta^2) - (q_v^2 + q_v + 1)\alpha \beta$$
on this vector [KL, Proposition 4.4]. If this quantity is zero, then $\alpha/\beta$ is a root of the polynomial $q_v x^2 - (q_v^2 + 1) x + q_v$. But this is impossible, for $|\alpha \beta| = 1$ and $|\alpha|, |\beta| < q_v^{1/2}$ since $\pi_v$ is unitary and generic [JS, (2.5)].

Now assume that our test function $f \in C_c^\infty((A \backslash GL_2(F_S))^\times)$ satisfies $f = f^e \otimes f_1v \tilde{f}_2v$ with $f^e \in C_c^\infty((A \backslash GL_2(F_{S-0}))^\times)$ and $f_1v, f_2v \in C_c^\infty(GL_2(F_v))$. Assume moreover that $f_1v$ satisfies assumption (A) for a given $k > 0$, and that $f_2v = 1_{GL_2(O_{F_v})}$.

Remark. Note that by Lemma 4.1 this assumption is essentially no loss of generality for the purpose of applying our main result, Theorem 1.1.

Lemma 4.2. Under the above assumptions $\Sigma_0(X) = 0$.

Proof. Since $h(x, y) = W(x) - W(y/x)$ it suffices to check that

$$(4.0.2) \quad \int_{gl_2(F_v)^{\otimes 2}} f_1v f_2v (g^{-1}_T g_T) dT = \int_{gl_2(F_v)^{\otimes 2}} f_1v 1_{GL_2(O_{F_v})}(g^{-1}_T g_T) dT$$

and

$$(4.0.3) \quad \int_{gl_2(F_v)^{\otimes 2}} W_v \left( \frac{P(b,T)}{t} \right) 1_F(b \det g_T^{-1}) f_1v f_2v (g^{-1}_T g_T) dT$$

$$= \int_{gl_2(F_v)^{\otimes 2}} 1_{O_{F_v}} \left( \frac{P(b,T)}{t} \right) 1_F(b \det g^{-1}_T) f_1v 1_{GL_2(O_{F_v})}(g^{-1}_T g_T) dT$$

are both zero for any $t \in F_v^\times$. We note that on the support of $f_1v 1_{GL_2(O_{F_v})}$ the measure $dT$ is a scalar multiple of the multiplicative Haar measure $dg$. Thus by assumption (A) it is clear that (4.0.2) vanishes.
On the other hand it not hard to see that the integrand in (4.0.3) is nonzero only if \( t \in \mathcal{O}_{F_\nu}^\times \), and in this case \( 1_{\mathcal{O}_{F_\nu}^\times} \left( \frac{P(b,T)}{t} \right) \mathbb{1}_F(b \det g \gamma r) \) is identically 1 on the support of \( f_{1v} \mathbb{1}_{GL_2(\mathcal{O}_{F_\nu})} \). It follows that (4.0.3) is zero.

\[ \square \]

5. Computation of \( C(\omega) \)

Let \( v \not\in S \) be a nonarchimedean place of \( F \). We work locally in this section and drop the subscript \( v \), writing \( F := F_v \). We let \( \omega \) be a uniformizer for \( F \) and let \( q = |\omega|^{-1} \). We assume \( b_1, b_2 \in \mathcal{O}_F^\times \). We compute

\[
C(t) := |t|^{-8} \int_{\mathfrak{gl}_2(\mathcal{O}_F)^{\oplus 2}} 1_{\mathcal{O}_F}(P(b,T)) \psi \left( \frac{\text{tr} \gamma T}{t} \right) dT;
\]

at least when \( t = \omega \). The main result is Corollary 5.4. Consider the projective scheme

\[
\mathcal{Y}(R) := \{ T = (T_1, T_2) \in \mathbb{P}^2(\mathfrak{gl}_2^\oplus)(R) : P(b,T) = 0 \}.
\]

Here \( \mathbb{P}(\mathfrak{gl}_2^\oplus) \cong \mathbb{P}^7 \) is the projective space attached to the free \( \mathbb{Z} \)-module \( \mathfrak{gl}_2^\oplus \). We also define

\[
\mathcal{W}(R) := \{ T = (T_1, T_2) \in \mathcal{Y}(R) : \text{tr} \gamma T = 0 \}.
\]

We have

\[
C(\omega) = 1 + \sum_{\alpha \in (\mathcal{O}_F/\omega)^\times} \sum_{T \in \mathcal{Y}((\mathcal{O}_F/\omega)} \psi \left( \frac{\alpha \text{tr} \gamma T}{q} \right)
\]

\[
= q|\mathcal{W}(\mathcal{O}_F/\omega)| - |\mathcal{Y}(\mathcal{O}_F/\omega)| + 1.
\]

Lemma 5.1. One has

\[
|\mathcal{Y}(\mathcal{O}_F/\omega)| = q^6 + q^5 + q^4 + 2q^3 + q^2 + q + 1.
\]

Proof. Upon taking an appropriate change of variables we see that we may assume \( b_1 = b_2 = 1 \). The number of pairs of matrices \( T_1, T_2 \) with determinant \( \beta \) is independent of \( \beta \) provided that \( \beta \neq 0 \). Thus we obtain

\[
|\mathcal{Y}(\mathcal{O}_F/\omega)| = |\text{SL}_2(\mathcal{O}_F/\omega)|^2 + (q - 1)^{-1}(|\{T_1, T_2 \in M_2(\mathcal{O}_F/\omega) : \det T_1 = \det T_2 = 0\}| - 1).
\]

One computes that \( |\text{SL}_2(\mathcal{O}_F/\omega)| = (q^2 - 1)q \) and

\[
|\{T_1, T_2 \in M_2(\mathcal{O}_F/\omega) : \det T_1 = \det T_2 = 0\}| - 1 = (q^3 - q + q^2)^2 - 1.
\]

Altogether this yields

\[
|\mathcal{Y}(\mathcal{O}_F/\omega)| = q^6 + q^5 + q^4 + 2q^3 + q^2 + q + 1.
\]

\[ \square \]
Lemma 5.2. Assume that $\gamma_1, \gamma_2 \in \text{GL}_2(\mathcal{O}_F/\varpi)$. Then

$$|\mathcal{W}(\mathcal{O}_F/\varpi)| = q^5 + q^4 + q^3 + q^2 + q + 1 + \begin{cases} q^3 & \text{if } b_1 \det \gamma_2 = b_2 \det \gamma_1 \\ 0 & \text{if } b_1 \det \gamma_2 \neq b_2 \det \gamma_1. \end{cases}$$

Proof. Assume first that $\gamma_1 = \gamma_2 = I$. Writing

$$T_1 = \left( \begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array} \right), T_2 = \left( \begin{array}{cc} t_1 & t_2 \\ t_3 & t_4 \end{array} \right)$$

we see that we are to count projective solutions to the equation

$$b_1((-x_4 - t_1 - t_4)x_4 - x_2x_3) - b_2(t_1t_4 - t_2t_3) = 0.$$ 

There are $q^5$ solutions when $x_2 \neq 0$, and $q^4$ solutions with $x_2 = 0$, $t_2 \neq 0$. Thus $|\mathcal{W}(\mathcal{O}_F/\varpi)|$ is equal to $q^5 + q^4$ plus

$$\sum_{(t_1,t_3,t_4)\in\mathbb{P}(\mathcal{O}_F/\varpi)} 1 = \sum_{(t_1,t_3,t_4)\in\mathbb{P}^3(\mathcal{O}_F/\varpi)} 1 + \sum_{(t_1,t_3,t_4)\in\mathbb{P}^4(\mathcal{O}_F/\varpi)} 1$$

$$= 2q^2 + q + 1 + q^2 \sum_{(t_1,t_4)\in(\mathcal{O}_F/\varpi)\otimes^2} 1.$$ 

If $-b_1 - b_2t_4 \neq 0$ for some $t_4$ then the equation $b_1(-1 - t_1 - t_4) - b_2t_1t_4 = 0$ has a unique solution $t_1$. Thus the above is equal to

$$2q^2 + q + 1 + q^2(q - 1) + q^2 \sum_{(t_1,t_4)\in(\mathcal{O}_F/\varpi)\otimes^2} 1.$$ 

The last summand is 0 unless $b_1 = b_2$, in which case it is $q$.

Now consider the case of general $(\gamma_1, \gamma_2) \in \text{GL}_2(\mathcal{O}_F) \times \text{GL}_2(\mathcal{O}_F)$. Taking a change of variables $(T_1, T_2) \mapsto (\gamma_1^{-1}T_1, \gamma_2^{-1}T_2)$ we arrive at

$$|\mathcal{W}(\mathcal{O}_F/\varpi)| = \{(T_1, T_2) \in \mathbb{P}(\text{GL}_2^2(\mathcal{O}_F/\varpi) : \text{tr} T = 0, b_1 \det \gamma_1^{-1}T_1 + b_2 \det \gamma_2^{-1}T_2 = 0\}.$$ 

Thus the lemma in general follows from the special case where $\gamma_1 = \gamma_2 = I$. □

The following lemma handles the remaining case:

Lemma 5.3. Assume that $\det \gamma_1 = 0$ or $\det \gamma_2 = 0$ and that $(\gamma_1, \gamma_2) \neq (0, 0) \pmod{\varpi}$. Then

$$|\mathcal{W}(\mathcal{O}_F/\varpi)| = q^5 + q^4 + q^3 + q^2 + q + 1 + \begin{cases} q^3 & \text{if } b_1 \det \gamma_2 = b_2 \det \gamma_1 \\ 0 & \text{if } b_1 \det \gamma_2 \neq b_2 \det \gamma_1. \end{cases}$$

Proof. Assume first that $\gamma_1 = \left( \begin{array}{cc} a_1 & 0 \\ 0 & 0 \end{array} \right)$ where $\varpi \nmid a_1$. Write

$$(5.0.3) \quad T_1 = \left( \begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array} \right), T_2 = \left( \begin{array}{cc} t_1 & t_2 \\ t_3 & t_4 \end{array} \right).$$
We see that we are to compute projective solutions to the equation
\[ b_1(-a_1^{-1} \text{tr} (\gamma_2 T_2)x_4 - x_2 x_3) - b_2(t_1 t_4 - t_2 t_3) = 0. \]

There are \( q^5 \) solutions with \( x_2 \neq 0 \), so we have
\[
|\mathcal{W} (\mathcal{O}_F)| = q^5 + |\{(x_4, t_1, t_2, t_3, t_4) \in (\mathcal{O}_F/\mathcal{O})^{\oplus 5} : -b_1 a_1^{-1} \text{tr} (\gamma_2 T_2)x_4 - b_2(t_1 t_4 - t_2 t_3) = 0\}| \\
+ |\{(x_4, t_1, t_2, t_3, t_4) \in \mathbb{P}^4 (\mathcal{O}_F/\mathcal{O}) : -b_1 a_1^{-1} \text{tr} (\gamma_2 T_2)x_4 - b_2(t_1 t_4 - t_2 t_3) = 0\}|.
\]

Assume that \( \gamma_2 = \left( \begin{smallmatrix} a_1 & 0 \\ 0 & 0 \end{smallmatrix} \right) \). Then we are to count solutions to
\[ -b_1 a_1^{-1} t_1 x_4 - b_2(t_1 t_4 - t_2 t_3) = 0 \]
in \( (\mathcal{O}_F/\mathcal{O})^{\oplus 5} \). There are \((q - 1)q^3\) solutions with \( t_2 \neq 0 \). There are \((q - 1)q^2\) solutions with \( t_2 = 0 \) and \( t_1 \neq 0 \), and \( q^3 \) solutions with \( t_1 = t_2 = 0 \). This implies that \((5.0.4)\)
\[ |\mathcal{W} (\mathcal{O}_F/\mathcal{O})| = q^5 + (q - 1)q^3 + (q - 1)q^2 + q^3 + q^2 + q^2 + q + 1 \\
= q^5 + q^4 + 2q^3 + q^2 + q + 1. \]

in this case. Taking an appropriate change of variables we can reduce ourselves to this computation provided that \( \det \gamma_1 = \det \gamma_2 = 0 \).

Now assume that \( \gamma_2 = I \). Then we are to count solutions to
\[ -b_1 a_1^{-1}(t_1 + t_4)x_4 - b_2(t_1 t_4 - t_2 t_3) = 0 \]
in \( (\mathcal{O}_F/\mathcal{O})^{\oplus 5} \). We take a change of variables \( t_1 \mapsto t_1 - t_4 \) to arrive at
\[ -b_1 a_1^{-1} t_1 x_4 - b_2(t_1 - t_4)t_4 + b_2 t_2 t_3 = 0. \]
There are \((q - 1)q^3\) solutions with \( t_2 \neq 0 \), \((q - 1)q^2\) solutions with \( t_2 = 0 \) and \( t_1 \neq 0 \) and \( q^2 \) solutions with \( t_1 = t_2 = 0 \). Thus altogether we arrive at \((5.0.5)\)
\[ |\mathcal{W} (\mathcal{O}_F/\mathcal{O})| = q^5 + (q - 1)q^3 + (q - 1)q^2 + q^3 + q^2 + q + 1 \\
= q^5 + q^4 + q^3 + q^2 + q + 1. \]

We can employ a change of variables to reduce ourselves to this case whenever one of \( \gamma_1 \), \( \gamma_2 \) is invertible and the other is nonzero but has determinant zero. The lemma follows. \( \square \)

Applying \((5.0.2)\) we obtain

**Corollary 5.4.** Assume that \((\gamma_1, \gamma_2) \neq (0, 0) \pmod{\mathcal{O}} \). Then
\[
\mathcal{C}(\mathcal{O}) = \begin{cases} 
q^4 - q^3 & \text{if } b_1 \det \gamma_2 = b_2 \det \gamma_1 \\
-q^3 & \text{if } b_1 \det \gamma_2 \neq b_2 \det \gamma_2.
\end{cases}
\]
\( \square \)
6. Bounds for local integrals

In this section we collect the rough bounds on local integrals we require to analyze $\Sigma(X)$ in §7 below. For the remainder of this paper, if $v$ is a place of $F$, 
\[ |(\gamma_1, \gamma_2)|_v := \max(|\gamma_i|_v). \]

6.1. Archimedean integrals. Fix an archimedian place $v|\infty$ and omit it from notation, writing $F := F_v$. In this subsection we prove the following proposition:

**Proposition 6.1.** Let $h_0(x, y)$ be either $W(x)$ or $W(y/x)$ for $W \in C^\infty_c(F^\times)$ and let $f \in C^\infty_c(GL_2(F)^\times)$. Let $\gamma \in gl_2(F) - (0, 0)$, let $b_1, b_2 \in F^\times$ and assume $|b_1| > |b_2| > 1$. For any $N \in \mathbb{Z}_{\geq 0}$, $1 > \varepsilon > 0$, $\lambda > \varepsilon - 4$, (unitary) character $\chi : F^\times \to \mathbb{C}^\times$ and $s \in \mathbb{C}$ with $\lambda > \text{Re}(s) > \varepsilon - 4$ the integral
\[
\int_{F^\times} \left( \int_{gl_2(F)^\oplus 2} h_0(t, P(b, T)) f(T) \psi \left( \frac{\text{tr} \gamma T}{t} \right) dT \right) \chi(t)|t|^s dt^x
\]
is bounded by a constant depending on $h_0, f, N, \lambda, \varepsilon$ and the bounds on $b_1, b_2$ times 
\[
\max(|\gamma_1|, |\gamma_2|, 1)^{-N} C(\chi, \text{Im}(s))^{-N} \max(|\gamma_1|, |\gamma_2|)^{6 - 32/\varepsilon}.
\]

In the proposition $C(\chi, t)$ is the analytic conductor of $\chi$ normalized as in [B, §1].

**Proof.** The assertions when $h_0(x, y)$ is $W(x)$ are clear, for $W(t)f(T)$ is smooth and compactly supported as a function of $(t, T) \in F^\times \times gl_2(F)^\oplus 2$. If $h_0(x, y) = W(y/x)$ then by Fourier inversion we obtain
\[
\int_{F^\times} \left( \int_{gl_2(F)^\oplus 2} W \left( \frac{P(b, T)}{t} \right) f(T) \psi \left( \frac{\text{tr} \gamma T}{t} \right) dT \right) \chi(t)|t|^s dt^x
\]
\[
= \int_{F^\times} \left( \int_{F \times gl_2(F)^\oplus 2} \hat{W}(x) f(T) \psi \left( \frac{\text{tr} \gamma T - x P(b, T)}{t} \right) dx dT \right) \chi(t)|t|^s dt^x.
\]

Let $D = t \frac{\partial}{\partial t}$ (and if $v$ is complex, $\overline{D} = t \frac{\partial}{\partial \overline{t}}$), viewed as differential operators on $F^\times$. We claim that for any $i \geq 0$ (and $j \geq 0$ if $v$ is complex), $\varepsilon > 0$ and $N \geq 0$ one has
\[
D^i \overline{D}^j \left( \int_{F \times gl_2(F)^\oplus 2} \hat{W}(x) f(T) \psi \left( \frac{\text{tr} \gamma T - x P(b, T)}{t} \right) dx dT \right)
\]
\[
\ll i, j, \varepsilon, N \max(|\gamma_1|, |\gamma_2|, 1)^{-N} \min(|t|^{4-\varepsilon}, |t|^{-N}) \max(|\gamma_1|, |\gamma_2|)^{6 - 32/\varepsilon}.
\]

Here and in the remainder of the proof all implied constants are allowed to depend on $f, W$ and the bounds on $|b_1|, |b_2|$. Assuming the claim, repeated application of integration by parts in $t$ (and $\overline{t}$ if $v$ is complex) implies the proposition. On the other hand, since $W$ and $f$ are arbitrary it is not hard to see that the claim follows from the special case where $i = j = 0$. Thus we are to bound
\[
(6.1.1) \quad \int_{F \times gl_2(F)^\oplus 2} \hat{W}(x) f(T) \psi \left( \frac{\text{tr} \gamma T - x P(b, T)}{t} \right) dx dT.
\]
Consider first the case where $|t| \gg 1$. In this case we apply repeated integration by parts in $x$ (integrating $\hat{W}$ and differentiating $\psi$) to arrive at a bound of $O_N(|t|^{-N})$, where the implied constant is independent of $\gamma$. This establishes the claim in this case.

We henceforth assume $|t| \ll 1$. Write $T_1 := (t_{1,ij})$ and $T_2 := (t_{2,ij})$ and similarly for $\gamma = (\gamma_1, \gamma_2)$. Assume that $|\gamma_{1,11}| = \max(|\gamma_1|, |\gamma_2|, 1)$. We apply integration by parts in $t_{1,11}, t_{1,12}, t_{2,11}, t_{2,12}$ to see that for any $\varepsilon > 0$ and $N \in \mathbb{Z}_{\geq 0}$ the integral (6.1.1) is equal to $O_{N, \varepsilon}(|t|^{|N\varepsilon/4|\gamma_{1,11}|^{-N/2}})$ plus

\begin{equation}
O \left( \int |\hat{W}(x)f(T)|dxdT \right)
\end{equation}

where the integral is over the set of $x, T$ such that

\begin{equation}
|\gamma_{1,11} - xb_1 t_{1,22}|, |\gamma_{1,21} + xb_1 t_{2,21}|, |\gamma_{2,11} + xb_2 t_{2,22}|, |\gamma_{2,22} - xb_2 t_{1,21}| \leq |t|^{1-\varepsilon/4}|\gamma_{1,11}|^{1/2}.
\end{equation}

Since $f$ is compactly supported as a function of $t_{1,22} \in F$, if $|\gamma_{1,11}|$ is sufficiently large, then the first inequality implies that $|\gamma_{1,11}| \ll |x|$. If, on the other hand, $\gamma_{1,11}$ lies in a compact set and $|t|$ is sufficiently small then the first inequality implies $1 \ll |x|$ where the implied constant depends on the compact set. Thus bounding the integrals in (6.1.2) trivially and using the fact that $\hat{W}$ is rapidly decreasing we see that for any $N' \in \mathbb{Z}_{\geq 0}$ the integral (6.1.1) is bounded by a constant times $|t|^{4-\varepsilon}|\gamma_{1,11}|^{-N'}$.

An analogous argument handles the cases where $\max(|\gamma_1|, |\gamma_2|, 1)$ is the norm of another matrix entry of $\gamma_1$ or $\gamma_2$.

Assume now that $\max(|\gamma_1|, |\gamma_2|, 1) = 1$. Assume moveover that $|\gamma_{1,11}| = \max(|\gamma_1|, |\gamma_2|)$; this is bigger than zero since $\gamma \neq (0, 0)$.

Applying integration by parts as above we see that for any $\varepsilon > 0$ and $N' \in \mathbb{Z}_{\geq 0}$ the integral (6.1.1) is equal to $O_{N', \varepsilon}(|t|^{N\varepsilon/4}|\gamma_{1,11}|^{-2N'})$ plus the integral (6.1.2) taken over the set of $x, T$ such that

\begin{equation}
|\gamma_{1,11} - xb_1 t_{1,22}|, |\gamma_{1,21} - xb_1 t_{2,21}|, |\gamma_{2,11} + xb_2 t_{2,22}|, |\gamma_{2,21} - xb_2 t_{1,21}| \leq |t|^{1-\varepsilon/4}|\gamma_{1,11}|^2.
\end{equation}

Using a partition of unity in the $x$ variable, one can separate the integral into the contribution of $|x| \gg 1$ and $|x| \ll 1$. The $|x| \gg 1$ contribution can be bounded via a minor modification of the previous argument, yielding a bound of $|t|^{4-\varepsilon}|\gamma_{1,11}|^8$ for this contribution. If $|x| \ll 1$, the first inequality in (6.1.4) implies that for $|\gamma_{1,11}|$ sufficiently small we must have $|x| \gg |\gamma_{1,11}|$, and for $|\gamma_{1,11}|$ bounded away from zero and $|t|$ sufficiently small we have $|x| \gg 1$, where the implied constant depends on the bound on $|\gamma_{1,11}|$. Hence the integral in (6.1.2) taken over the domain (6.1.4) is bounded by a constant times $|t|^{4-\varepsilon}|\gamma_{1,11}|^4$. 
Taking $N' = \left\lfloor \frac{16}{3} - 4 \right\rfloor$ and noting that $0 > -2\left\lfloor \frac{16}{3} - 4 \right\rfloor \geq 6 - \frac{32}{5}$ we arrive at a bound of $O(\lvert t \lvert^{1-\varepsilon} \lvert \gamma_{1,1} \lvert^{\varepsilon - \frac{32}{5}})$ for (6.1.1) in this case. An analogous argument establishes the same bound if $\max(\lvert \gamma_{1}, \lvert \gamma_{2} \lvert)$ is less than 1 and equal to the norm of some other matrix entry of $\gamma_{1}$ or $\gamma_{2}$. □

6.2. **Nonarchimedean integrals.** In this section we assume that $v$ is a nonarchimedean place of $F$ and omit it from notation, writing $F := F_{v}$, etc. We denote by $\varpi$ a uniformizer for $F$, let $q = |\varpi|^{-1}$, and let $D_{F}$ be the absolute different of $F$.

We prove the following proposition:

**Proposition 6.2.** Let $b_{1}, b_{2} \in F^{\times}$, $f \in C_{c}^{\infty}(gl_{2}(F)^{\oplus 2})$. The integral

$$
\int_{F^{\times}} \mathbb{1}_{\mathcal{O}_{F}}(t) \int_{gl_{2}(F)^{\oplus 2}} \mathbb{1}_{t\mathcal{O}_{F}}(P(b, T)) f(T) \psi \left( \frac{tr \gamma T}{t} \right) dT \chi(t) \lvert t \rvert^{s} dt
$$

converges absolutely for $\text{Re}(s) > 0$. It vanishes if $\lvert \gamma_{1}, \lvert \gamma_{2} \lvert$, or the absolute norm of the conductor of $\chi$ is sufficiently large in a sense depending on $f$ and $\lvert b_{1}, \lvert b_{2} \rvert$.

For $t \in \mathcal{O}_{F} - 0$ consider the integral

$$
(6.2.1) \quad \int_{gl_{2}(F)^{\oplus 2}} \mathbb{1}_{t\mathcal{O}_{F}}(P(b, T)) f(T) \psi \left( \frac{tr \gamma T}{t} \right) dT.
$$

If $(\gamma_{1}, \gamma_{2}) \neq (0, 0)$ and the valuation $\nu(t) > 1$ then (6.2.1) is bounded in absolute value by a constant depending on $f$, $\lvert b_{1}, \lvert b_{2} \rvert$ times

$$
q^{4 \min(\nu(\gamma_{1,i}), \nu(\gamma_{2,i,j}))} \lvert t \rvert^{4}.
$$

**Proof.** We use a Fourier transform to rewrite the integral as

$$
\int_{F^{\times}} \mathbb{1}_{\mathcal{O}_{F}}(t) \left( \int_{D_{F}^{-1} \times gl_{2}(F)^{\oplus 2}} f(T) \psi \left( \frac{tr \gamma T - xP(b, T)}{t} \right) dT dx \right) \chi(t) \lvert t \rvert^{s} dt.
$$

We assume without loss of generality that

$$
f = \mathbb{1}_{\varpi^{m} + \varpi^{k}gl_{2}(\mathcal{O}_{F})^{\oplus 2}}
$$

for some $\beta = (\beta_{1}, \beta_{2}) \in gl_{2}(\mathcal{O}_{F})^{\oplus 2}$ and $m, k \geq 0$; thus the above becomes

$$
(6.2.2) \quad \int_{F^{\times}} \mathbb{1}_{\mathcal{O}_{F}}(t) \left( \int_{D_{F}^{-1} \times gl_{2}(F)^{\oplus 2}} \mathbb{1}_{\varpi^{k}gl_{2}(\mathcal{O}_{F})^{\oplus 2}}(T - \beta \varpi^{m}) \psi \left( \frac{tr \gamma T - xP(b, T)}{t} \right) dT dx \right) \chi(t) \lvert t \rvert^{s} dt.
$$

It is clear that the multiple integral over $F^{\times} \times D_{F}^{-1} \times gl_{2}(F)^{\oplus 2}$ converges absolutely for $\text{Re}(s) > 0$. We therefore assume that $\text{Re}(s) > 0$ until otherwise stated to justify our manipulations. The integral (6.2.2) is equal to $q^{8m}$ times

$$
\int_{F^{\times}} \mathbb{1}_{\mathcal{O}_{F}}(t) \left( \int_{D_{F}^{-1} \times gl_{2}(F)^{\oplus 2}} \mathbb{1}_{\varpi^{k+m}gl_{2}(\mathcal{O}_{F})^{\oplus 2}}(T - \beta) \psi \left( \frac{tr \gamma T - xP(b, T)}{\varpi^{m}t} - \frac{xP(b, T)}{\varpi^{2m}t} \right) dT dx \right) \chi(t) \lvert t \rvert^{s} dt.
$$
If we multiply the integral over $D_F^{-1} \times \mathfrak{g}_2(F)^{\otimes 2}$ by $\chi(t)$ the resulting function of $t$ is invariant under $t \mapsto ut$ for $u \in O_F^\times(\mathbb{O}^{k+m})$. Therefore the integral vanishes if the absolute norm of the conductor of $\chi$ is sufficiently large in a sense depending only on $k, m$.

Letting $\ell \geq 0$ be the smallest integer such that $b_1\omega^\ell$ and $b_2\omega^\ell$ are both integral we see that the above is

\[(6.2.3)\int_{F^\times} 1_{O_F}(t) \left( \int_{D_F^{-1} \times \mathfrak{g}_2(F)^{\otimes 2}} 1_{\omega^{k+m}\mathfrak{g}_2(O_F)^{\otimes 2}}(T - \beta) \psi\left(\frac{\text{tr} \gamma T}{\omega^m t} - \frac{xP(\omega^\ell b, T)}{\omega^{2m+\ell} t}\right) dT dx \right) \chi(t)|t|^s dt^x.
\]

This is zero unless $\gamma_1, \gamma_2 \in \omega^{-(m+\ell+k)} D_F^{-1} \mathfrak{g}_2(O_F)$.

With the vanishing statements claimed in the proposition proven, we are left with establishing a bound for the integrand (6.2.1). We now assume that $(\gamma_1, \gamma_2) \neq (0, 0)$ and $v(t) > 1$. Taking a change of variables $T \mapsto \omega^{k+m} T + \beta$ we see that (6.2.1) is equal to $q^{-8(k+m)}$ times

\[
\int_{D_F^{-1} \times \mathfrak{g}_2(O_F)^{\otimes 2}} 1_{\mathfrak{g}_2(O_F)^{\otimes 2}}(T_1, T_2) \psi\left(\frac{\omega^{m+\ell} \text{tr} \gamma(\omega^{k+m} T + \beta) - xP(\omega^\ell b, \omega^{k+m} T + \beta)}{\omega^{2m+\ell} t}\right) dT dx.
\]

We apply the the $p$-adic stationary phase method of Dabrowski and Fisher [DF] to estimate this integral. Choose a generator $\delta \in O_F$ for the ideal $D_F$ (this $\delta$ has no relation to the $\delta^S$-function from §2). For $\ell = 1, 2$, let

\[F_{i,x}(T_i) := \delta\left(x(-1)^{i-1} b_i \omega^\ell \det(\omega^{k+m} T_i + \beta_i) + \omega^{\ell + m} \text{tr}(\gamma_i(\omega^{k+m} T_i + \beta_i))\right)
\]

and let $F_x(T_1, T_2) = F_{1,x}(T_1) + F_{2,x}(T_2)$. Let $p$ be the rational prime below $v$ and let

\[D_x := \text{Res}_{O_F/[p]} \text{Spec}(O_F[T_1, T_2]/(\nabla F_x)) \subseteq \mathbb{A}^8_{\mathbb{Z}_p} \quad (\text{affine } 8[F: \mathbb{Q}_p] \text{ space}.
\]

Here $(\nabla F_x) \leq O_F[T_1, T_2]$ is the ideal generated by the entries of $\nabla F_x$. Writing $T_i = (t_{i,j})$, $\gamma_i = (\gamma_{i,j})$, and $\beta_i = (\beta_{i,j})$ one has

\[
\nabla F_{i,x}(T_i) = \delta \omega^{k+m+\ell} \begin{pmatrix} \omega^m \gamma_{i,11} - (-1)^{i-1} x b_i(\omega^{k+m} t_{i,22} + \beta_{i,22}) \\ \omega^m \gamma_{i,12} + (-1)^{i-1} x b_i(\omega^{k+m} t_{i,12} + \beta_{i,12}) \\ \omega^m \gamma_{i,21} + (-1)^{i-1} x b_i(\omega^{k+m} t_{i,21} + \beta_{i,21}) \\ \omega^m \gamma_{i,22} - (-1)^{i-1} x b_i(\omega^{k+m} t_{i,11} + \beta_{i,11}) \end{pmatrix}
\]

and thus the Hessian is

\[
H_x(T_i) = (-1)^{i-1} x b_i \omega^{2k+\ell+2m} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}
\]

which has valuation $v(x^4 b_i^4 \delta^4) + 8k + 4\ell + 8m$. Thus the valuation of the Hessian of $\nabla F_x(T_1, T_2)$ is $v(x^8 b_1^4 b_2^4 \delta^8) + 16k + 8\ell + 16m$. Assume that $(\gamma_1, \gamma_2) \neq (0, 0)$. In this case at any point where $\nabla F_x$ vanishes we must have $x \neq 0$. It follows that $|D_x(\mathbb{Z}_p)| \leq 1$ and
\[ v(x) \leq \min(v(\gamma_{1,ij}), v(\gamma_{2,i'j'})) + \kappa' \] for some real number \( \kappa' \) depending only on \( k, \ell, m \). Applying [DF, Theorem 1.8] we see that (6.2.4) is bounded by
\[
(6.2.4) \quad |t|^{8} |t|^{-8/2} \left( \max_{x \in \mathcal{D}_{F}, \alpha \in D_{k}(\mathbb{Z}_{p})} |H_{x}^{\pm}(\alpha)|^{-1} \right)^{1/2} \leq |t|^{4} q^{4 \min(v(\gamma_{1,ij}), v(\gamma_{2,i'j'}))}
\]
for some real number \( \kappa \) depending only on \( k, \ell, m \). This completes the proof of the proposition.

We can extract from the proof the following refinement:

**Corollary 6.3.** Assume that \( b_{1}, b_{2} \in \mathcal{O}_{F}^{\times} \) and that \( F \) is absolutely unramified. Then if \(|t| > 1 \) and \( \gamma \not= (0,0) \) the integral
\[
\left| \int_{\mathfrak{gl}_{2}(F)^{\oplus 2}} 1_{\mathcal{O}_{F}}(P(b, T)) 1_{\mathfrak{gl}_{2}(\mathcal{O}_{F})^{\oplus 2}}(T) \psi\left( \frac{\text{tr} \gamma T}{t} \right) dT \right|
\]
is bounded by \(|t|^{4} q^{4 \min(v(\gamma_{1,ij}), v(\gamma_{2,i'j'}))} \) and vanishes if \( \gamma_{1}, \gamma_{2} \not\in \mathfrak{gl}_{2}(\mathcal{O}_{F}^{\times}) \) or \( \chi \) is ramified.

The following is another corollary:

**Corollary 6.4.** Let \( h_{0}(x, y) = 1_{\mathcal{O}_{F}^{\times}}(x) \) or \( 1_{\mathcal{O}_{F}^{\times}}(x/y) \) and \( f \in C_{c}^{\infty}(\text{GL}_{2}(F)^{\times 2}) \). Assume that \(|b_{1}| \geq |b_{2}| \geq 1 \). For Re\( (s) > 0 \) the integral
\[
\int_{F^{\times} \times \mathfrak{gl}_{2}(F)^{\oplus 2}} 1_{\mathcal{O}_{F}}(t) h_{0}(t, P(b, T)) f(T) \psi\left( \frac{\text{tr} \gamma T}{t} \right) \chi(t) |t|^{s} dT dt^{\times}
\]
converges absolutely. It vanishes if \(|\gamma_{1}|, |\gamma_{2}| \) or the conductor of \( \chi \) is sufficiently large in a sense depending only on \( f \) and the bounds on \( b_{1}, b_{2} \). Finally,
\[
\int_{F^{\times}} 1_{\mathcal{O}_{F}}(t) \left| \int_{\mathfrak{gl}_{2}(F)^{\oplus 2}} h_{0}(t, P(b, T)) f(T) \psi\left( \frac{\text{tr} \gamma T}{t} \right) dT \right| |t|^{s} dt^{\times}
\]
is bounded in the half-plane Re\( (s) > -4 \) by \( q^{4 \min(v(\gamma_{1,ij}), v(\gamma_{2,i'j'}))} \) times a constant depending only on \( f \).

**Proof.** The assertions when \( h_{0}(x, y) = 1_{\mathcal{O}_{F}^{\times}}(x) \) are clear, for \( 1_{\mathcal{O}_{F}^{\times}}(t) f(T) \) is smooth and compactly supported as a function of \((t, T) \in F^{\times} \times \mathfrak{gl}_{2}(F)^{\oplus 2} \).

If \( h_{0}(x, y) = 1_{\mathcal{O}_{F}^{\times}}(y/x) \) then the fact that \( f \in C_{c}^{\infty}(\text{GL}_{2}(F)^{\times 2}) \) implies that
\[
(6.2.5) \quad \int_{\mathfrak{gl}_{2}(F)^{\oplus 2}} 1_{\mathcal{O}_{F}^{\times}}\left( \frac{P(b, T)}{t} \right) f(T) \psi\left( \frac{\text{tr} \gamma T}{t} \right) dT
\]
vanishes if \(|t| \) is sufficiently large in a sense depending on \( f \) and the bounds on \(|b_{1}|, |b_{2}| \). Thus there is an \( \ell \in \mathbb{Z} \) depending on \( f \) and the bounds on \( b_{1}, b_{2} \) such that
\[
\int_{F^{\times}} \left( \int_{\mathfrak{gl}_{2}(F)^{\oplus 2}} 1_{\mathcal{O}_{F}^{\times}}\left( \frac{P(b, T)}{t} \right) f(T) \psi\left( \frac{\text{tr} \gamma T}{t} \right) dT \right) \chi(t) |t|^{s} dt^{\times}
\]
\[ \int_{E} \left( \int_{\GL_2(F)^{\oplus 2}} 1_{\mathcal{O}_F}(t) 1_{\mathcal{O}_F} \left( \varpi^{-\ell} P(b, T) \right) f(T) \psi \left( \frac{\text{tr}(\varpi^{-\ell} \gamma T)}{t} \right) dT \right) \chi(\varpi^T t) |\varpi^T t|^s dt^x \]

\[ - \int_{E} \left( \int_{\GL_2(F)^{\oplus 2}} 1_{\mathcal{O}_F}(t) 1_{\mathcal{O}_F} \left( \varpi^{-\ell-1} P(b, T) \right) f(T) \psi \left( \frac{\text{tr}(\varpi^{-\ell} \gamma T)}{t} \right) dT \right) \chi(\varpi^T t) |\varpi^T t|^s dt^x. \]

We can now apply Proposition 6.2 and trivial bounds when \( 0 \leq v(t) \leq 1 \) to both summands to deduce the corollary.

\[ \square \]

7. Poisson summation in \( d \in E \)

In this section we prove the following theorem:

**Theorem 7.1.** The limit \( \lim_{X \to \infty} \Sigma^0(X) \) exists and is equal to

\[
d_{d}^{1/2} \sum_{0 \neq \gamma \in \GL_2(F)^{\oplus 2}} \sum_{G \in (E)^{\oplus 2}} \sum_{d \in \mathcal{O}_F^*} \int_{\mathcal{A}_F} \frac{V(b \det T)}{|b_1 \det T_1|^|} \left( \int_{\GL_2(A_F)^{\oplus 2}} h(t, P(b, T)) 1_{\mathcal{F}}(b \det g_t g_r^{-1}) f \mathbf{1}_{\GL_2(\mathcal{O}_F)^{\oplus 2}}(g_t^{-1} T g_r) \psi \left( \frac{\text{tr} \gamma T}{d} \right) dT \right) dt \bigg|_{|t|^4}.
\]

The sum over \( b, \gamma \) is absolutely convergent.

Theorem 7.1 together with Proposition 1.2 and Lemma 4.2 yield Theorem 1.1, our main theorem.

**Proof.** In (3.0.1) we found that \( \Sigma^0(X) \) was equal to

\[
d_{d}^{1/2} \sum_{0 \neq \gamma \in \GL_2(F)^{\oplus 2}} \sum_{G \in (E)^{\oplus 2}} \sum_{d \in \mathcal{O}_F^*} \int_{\mathcal{A}_F} \frac{V(b \det T/X)}{|b_1 \det T_1|^X \mathcal{X}} \mathbf{1}_{\mathcal{O}_F^{\oplus 2}}(P(b, T)) \]

\[
\times h \left( \frac{d}{\Delta(\sqrt{X})}, \frac{P(b, T)}{\Delta(X)} \right) 1_{\mathcal{F}}(b \det g_t g_r^{-1}) f \mathbf{1}_{\GL_2(\mathcal{O}_F)^{\oplus 2}}(g_t^{-1} T g_r) \psi \left( \frac{\text{tr} \gamma T}{d} \right) dT.
\]

Applying propositions 6.1 and 6.2 and corollaries 5.4 and 6.4 we see that it is permissible to apply Poisson summation in \( d \in E \) to this expression, which implies that it is equal to

\[
\frac{d_{d}^{1/2}}{2 \pi i \text{Res}_{s=1} \mathcal{L}_F(s)} \sum_{0 \neq \gamma \in \GL_2(F)^{\oplus 2}} \sum_{G \in (E)^{\oplus 2}} \chi_{\mathcal{A}_F}(s) \int_{\text{Re}(s)=\sigma} \frac{V(b \det T/X)}{|b_1 \det T_1|^X \mathcal{X}} \mathbf{1}_{\mathcal{O}_F^{\oplus 2}}(P(b, T)) \]

\[
\times h \left( \frac{t}{\Delta(\sqrt{X})}, \frac{P(b, T)}{\Delta(X)} \right) 1_{\mathcal{F}}(b \det g_t g_r^{-1}) f \mathbf{1}_{\GL_2(\mathcal{O}_F)^{\oplus 2}}(g_t^{-1} T g_r) \psi \left( \frac{\text{tr} \gamma T}{t} \right) dT \chi(t) |t|^s dt^x ds.
\]
Here we take $\sigma$ sufficiently large (trivial bounds imply that $\sigma > 1$ is sufficient). A possible reference for this application of Poisson summation is [BB, §2], bearing in mind that our measure differs from theirs by a factor of $\zeta_F(1) d_F^{-1/2}$.

Let $S_0 \supseteq S$ be a finite set of places large enough that $g_{\ell}^{S_0}, g_r^{S_0} \in \text{GL}_2(\hat{O}_F^{S_0})$. For $\chi \in (AF^x \backslash A_F^x / \hat{O}_F^{S_x})$ define

\begin{equation}
D_{\gamma,b,\chi}(s) := \int_{A_F} \mathbb{1}_{\hat{O}_F^{S_0}}(t) \left( \int_{\mathfrak{gl}_2(A_F)^{\otimes 2}} \mathbb{1}_{\hat{O}_F^{S_0}}(P(b,T)) \mathbb{1}_{\mathfrak{gl}_2(\hat{O}_F^{S_0})^{\otimes 2}}(T) \psi \left( \frac{\text{tr} \gamma T}{t} \right) dT \right) \chi(t) |t|^{-s} dt^{\times}.
\end{equation}

After a change of variables $(t,T) \mapsto \sqrt{X}(t,T)$ we then see that $\Sigma^0(X)$ is equal to

\begin{equation}
\frac{d_f^{\gamma/2}}{2\pi i \text{Res}_{s=1} \zeta_F(s)} \sum_{\gamma,b,\chi} \text{Res}_{s=-1} D_{\gamma,b,\chi}(s) \int_{F_{S_0}} \mathbb{1}_{\hat{O}_F^{S_0}}(t) \left( \int_{\mathfrak{gl}_2(F_{S_0})^{\otimes 2}} V(b \det T) |b_1 \det T_1|_S \right) \times c_{\sqrt{X}} \mathbb{1}_{\hat{O}_F^{S_0}}(P(b,T)) h(t,P(b,T)) \mathbb{1}_F(b \det g_{\ell}^{-1} g_r) \frac{\text{tr} \gamma T}{t} dT \chi(t) X^{(3+s)/2} |t|^{-s} dt^{\times}.
\end{equation}

We note that by definition of $\mathbb{1}_F$ and $V$ in §1.1 and §3, respectively, the sums over $b_1$ and $b_2$ in this expression can be taken to run over a finite set independent of $\chi$, $T$ and $t$.

Consider the Dirichlet series $D_{\gamma,b,\chi}(s)$ defined by (7.0.1). If $b_1$ det $\gamma_2 \neq b_2$ det $\gamma_1$, then upon combining corollaries 5.4 and 6.3 we see that $D_{\gamma,b,\chi}(s)$ converges absolutely in the half plane $\text{Re}(s) > -\frac{7}{2} = -\frac{1}{2} - 3$. Similarly, if $b_1$ det $\gamma_2 = b_2$ det $\gamma_1$ and $\gamma \neq (0,0)$ then $D_{\gamma,b,\chi}(s)L(s+4, \chi^{S_0})^{-1}$ converges absolutely in the half-plane $\text{Re}(s) > -\frac{7}{2}$.

Moving all the contours in (7.0.2) to the line $\text{Re}(s) = -\frac{13}{4} = -\frac{1}{4} - 3$, we see therefore see that it is equal to the sum of the contribution of the residues at $s = -3$:

\begin{equation}
\frac{d_f^{\gamma/2}}{2\pi i \text{Res}_{s=1} \zeta_F(s)} \sum_{\gamma \neq \epsilon \mathfrak{gl}_2(\mathbb{F})^{\otimes 2}} \sum_{\epsilon \mathfrak{gl}_2(\mathbb{F})^{\otimes 2}} \text{Res}_{s=-13/4} D_{\gamma,b,1}(s) \int_{F_{S_0}} \mathbb{1}_{\hat{O}_F^{S_0}}(t) \left( \int_{\mathfrak{gl}_2(F_{S_0})^{\otimes 2}} V(b \det T) |b_1 \det T_1|_S \right) \times c_{\sqrt{X}} \mathbb{1}_{\hat{O}_F^{S_0}}(P(b,T)) h(t,P(b,T)) \mathbb{1}_F(b \det g_{\ell}^{-1} g_r) \frac{\text{tr} \gamma T}{t} dT \chi(t) X^{(3+s)/2} |t|^{-s} dt^{\times}.
\end{equation}

plus

\begin{equation}
\frac{d_f^{\gamma/2}}{2\pi i \text{Res}_{s=1} \zeta_F(s)} \sum_{\gamma \neq \epsilon \mathfrak{gl}_2(\mathbb{F})^{\otimes 2}} \sum_{\epsilon \mathfrak{gl}_2(\mathbb{F})^{\otimes 2}} \sum_{\chi} \text{Res}_{s=-13/4} D_{\gamma,b,\chi}(s) \times \int_{F_{S_0}} \mathbb{1}_{\hat{O}_F^{S_0}}(t) \left( \int_{\mathfrak{gl}_2(F_{S_0})^{\otimes 2}} V(b \det T) |b_1 \det T_1|_S c_{\sqrt{X}} \mathbb{1}_{\hat{O}_F^{S_0}}(P(b,T)) h(t,P(b,T)) \mathbb{1}_F(b \det g_{\ell}^{-1} g_r) \frac{\text{tr} \gamma T}{t} dT \chi(t) X^{(3+s)/2} |t|^{-s} dt^{\times}.
\end{equation}
× f^\gamma_{\ell,T}^{-1}(g_T^{-1}T_g)\psi_S \left( \frac{\text{tr} \gamma T}{t} \right) dT \chi(t)^{X^{(3+s)/2}} |t|^s dt^x ds.

If \gamma \in \mathfrak{gl}_2(\mathcal{O}_F^S) - (0,0) let \mathcal{S}_0(\gamma) be the union of \mathcal{S}_0 and the set of places dividing all nonzero principal divisors of \gamma_1 and \gamma_2. Let \mathcal{D}_{\gamma,b,\chi}(s) be the product of the Euler factors of \mathcal{D}_{\gamma,b,\chi}(s) at the places away from \mathcal{S}_0(\gamma).

By corollaries 5.4 and 6.3 if \text{det} b_1 \neq b_2 \text{det} \gamma_1 then \mathcal{D}_{\gamma,b,\chi}(s) is bounded independently of \gamma_1, b, \chi on the line Re(s) = -\frac{13}{4}, and if \text{det} \gamma_1 = b_2 \text{det} \gamma_1 then \mathcal{D}_{\gamma,b,\chi}(s)L(s+4, \chi^{\mathcal{S}_0(\gamma)^{-1}}) is bounded independently of \gamma_1, b, \chi on the line Re(s) = -\frac{13}{4}. Therefore, in the latter case, \mathcal{D}_{\gamma,b,\chi}(s) is bounded by a constant times \mathcal{L}(s+4, \chi^{\mathcal{S}_0(\gamma)}) on the line Re(s) = -\frac{13}{4}. By standard preconvex bounds [B, (10)], there is a \beta > 0 such that \mathcal{L}(s+4, \chi^{\mathcal{S}_0(\gamma)}) is bounded by \mathcal{C}(\gamma, \text{Im}(s+4))^\beta on the line Re(s) = -\frac{13}{4}. Thus (7.0.4) is bounded by a constant times

\sum_{0 \neq \gamma \in \mathfrak{gl}_2(F)^{\otimes 2}} \sum_{b \in (F^x)^{\otimes 2}} \sum_{\chi} \int_{\text{Re}(s) = -\frac{13}{4}} \mathcal{D}_{\gamma,b,\chi,\mathcal{S}_0(\gamma)}(s) \mathcal{C}(\gamma, \text{Im}(s+4))^\beta \times \int_{\mathfrak{gl}_2(F_S^{\otimes 2})} \mathcal{1}_{O_{F_S^{\otimes 2}}}(t) \left( \int_{\mathfrak{gl}_2(F_S^{\otimes 2})} \frac{V(b \text{det} T)}{|b_1 \text{det} T_1|} \mathcal{1}_{O_{F_S^{\otimes 2}}}(P(b, T)) h(t, P(b, T)) \mathcal{1}_F(b \text{det} g_T g_r^{-1}) \times f^\gamma_{\ell,T}^{-1}(g_T^{-1}T_g)\psi_S \left( \frac{\text{tr} \gamma T}{t} \right) dT \chi(t)^{X^{(3+s)/2}} |t|^s dt^x ds.

Applying corollaries 5.4, 6.3 and 6.4 and propositions 6.1 and 6.2 we see that the above is

\mathcal{O}(X^{-1/8}).

We now consider (7.0.3). Taking the limit as \text{X} goes to infinity and applying Proposition 2.1 we see that it is equal to

\frac{d^4}{\text{Res}_{s=1} \zeta_F}(s) \sum_{0 \neq \gamma \in \mathfrak{gl}_2(F)^{\otimes 2}} \sum_{b \in (F^x)^{\otimes 2}} \text{Res}_{s=3} \mathcal{D}_{\gamma,b,1}(s) \int_{\mathfrak{gl}_2(F_S^{\otimes 2})} \mathcal{1}_{O_{F_S^{\otimes 2}}}(t) \left( \int_{\mathfrak{gl}_2(F_S^{\otimes 2})} \frac{V(b \text{det} T)}{|b_1 \text{det} T_1|} \right) dt^x |t|^3.

Notice that

(7.0.5) \quad \frac{\text{Res}_{s=3} \mathcal{D}_{\gamma,b,1}(s)}{\text{Res}_{s=1} \zeta_F}(s) = \lim_{s \to -3} \frac{\mathcal{D}_{\gamma,b,1}(s)}{\zeta_F(s+4)}

where the limit is taken over s \in \mathbb{R}_{> -3} (say). Since \text{dt^x} = \zeta_v(1)\frac{dt_v}{|t|^v} for each v \uparrow \infty we conclude using the definition (7.0.1) of \mathcal{D}_{\gamma,b,1}(s) together with Corollaries 5.4 and 6.3, that

(7.0.6) \quad \frac{\text{Res}_{s=3} \mathcal{D}_{\gamma,b,1}(s)}{\text{Res}_{s=1} \zeta_F}(s)
\[ = \int_{A^0_{F_p}} 1_{\mathcal{O}_F^{S_0}}(t) \left( \int_{\mathfrak{gl}_2(\mathbb{A}_{F})^{\otimes 2}} 1_{\mathfrak{o}_F^{S_0}(P(b,T))} \frac{V(b\det T)}{b_1 \det T_1 S} dt \int_{\mathfrak{gl}_2(\mathbb{A}_{F})^{\otimes 2}} 1_{\mathfrak{o}_F^{S_0}(g_{\ell}^{-1} T g_r)} \psi \left( \frac{\text{tr} \gamma T}{t} \right) dT \right) \left( \int_{\mathfrak{gl}_2(\mathbb{A}_{F})^{\otimes 2}} 1_{\mathfrak{o}_F^{S_0}(\mathfrak{o}_F^{S_0})^{\otimes 2}}(T) \psi \left( \frac{\text{tr} \gamma T}{t} \right) dT \right) \frac{dt}{|t|^4}. \]

We conclude that (7.0.3) is equal to

\[ (7.0.7) \quad d_F^4 \sum_{b \neq \gamma \in \mathfrak{gl}_2(F)^{\otimes 2}} \sum_{b_2 \det \gamma_1 = b_1 \det \gamma_2} \int_{A^0_F} 1_{\mathcal{O}_F^{S_0}}(t) \left( \int_{\mathfrak{gl}_2(\mathbb{A}_{F})^{\otimes 2}} V(b\det T) \frac{1}{|b_1 \det T_1 S|} dt \right) \left( \int_{\mathfrak{gl}_2(\mathbb{A}_{F})^{\otimes 2}} 1_{\mathfrak{o}_F^{S_0}(\mathfrak{o}_F^{S_0})^{\otimes 2}}(g_{\ell}^{-1} T g_r) \psi \left( \frac{\text{tr} \gamma T}{t} \right) dT \right) \frac{dt}{|t|^4}. \]

The statement on the absolute convergence of the sum over \( \gamma, b \) follows from an easier analogue of the absolute convergence statements we have already proven. \( \square \)

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