Abstract: Some probability distributions have moments, and some do not. For example, the normal distribution has power moments of arbitrary order, but the Cauchy distribution does not have power moments. In this paper, by analogy with the renormalization method in quantum field theory, we suggest a renormalization scheme to remove the divergence in divergent moments. We establish more than one renormalization procedure to renormalize the same moment to prove that the renormalized moment is scheme-independent. The power moment is usually a positive-integer-power moment; in this paper, we introduce nonpositive-integer-power moments by a similar treatment of renormalization. An approach to calculating logarithmic moment from power moment is proposed, which can serve as a verification of the validity of the renormalization procedure. The renormalization schemes proposed are the zeta function scheme, the subtraction scheme, the weighted moment scheme, the cut-off scheme, the characteristic function scheme, the Mellin transformation scheme, and the power-logarithmic moment scheme. The probability distributions considered are the Cauchy distribution, the Levy distribution, the $q$-exponential distribution, the $q$-Gaussian distribution, the normal distribution, the Student’s $t$-distribution, and the Laplace distribution.
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1 Introduction

The moment of a distribution with the probability density function $p(x)$, generally, is defined by

$$m_f = \int_{-\infty}^{\infty} p(x) f(x) \, dx. \quad (1.1)$$
Different $f(x)$ define different moments. The most familiar moments are the $n$-th power moment and the logarithmic moment, corresponding to $f(x)$ is a power function or a logarithmic function.

Choosing $f(x) = x^n$ defines the $n$-th power moment:

$$m_n = \int_{-\infty}^{\infty} p(x) x^n dx.$$  \hspace{1cm} (1.2)

The zeroth moment corresponds to the normalization, the first moment is the mean value, the second central moment is the variance, the third central moment is the skewness, and the fourth central moment is the kurtosis [1]. Choosing $f(x) = \ln^n x$ defines the $n$-th logarithmic moment [2]:

$$\tilde{m}_n = \int_{-\infty}^{\infty} p(x) \ln^n x dx.$$  \hspace{1cm} (1.3)

The moment of continuous distributions is usually defined by integrals. If the function $p(x) f(x)$ in Eq. (1.1) is nonintegrable, the moment $m_f$ does not exist. The existence of the $n$-th moment, by definition (1.2), relies on whether $p(x) x^n$ is integrable or not. Similarly, the existence of the logarithmic moment requires $p(x) \ln^n x$ to be integrable.

In various statistical distributions, some have moments, such as the normal distribution, the Laplace distribution, and Student’s $t$-distribution; some have no moments, such as the Cauchy distribution and the Levy distribution; some have moments only under some special values of parameters, such as the $q$-exponential distribution and the $q$-Gaussian distribution [3]. In this paper, by analogy with the renormalization method in quantum field theory, we suggest a renormalization scheme to remove the divergence in the integral definition of moments, Eq. (1.1), to achieve a finite renormalized moment.

For the distribution that has no moment, the integral in the definition (1.1) is divergent. The renormalization treatment aims to achieve a finite moment by removing the divergence in the integral definition (1.1). Roughly speaking, such a treatment is to obtain a finite value by subtracting infinity from infinity. However, this treatment will naturally cause a problem: whether the renormalized result depends on the choice of subtraction scheme. After all, infinity minus infinity can give any value. A correct subtraction treatment must be subtraction-scheme independent. The renormalized results given by the different renormalization schemes must be the same.

In order to show the validity of the renormalization schemes suggested in this paper, we construct a variety of renormalization schemes and renormalize an $n$-th moment with different renormalization schemes. If the renormalized $n$-th moment given by different schemes are the same, then the renormalization-scheme independence is verified.

Constructing more than one renormalization scheme also has a technical reason. The renormalization procedure, practically, depends on exact results, such as exact integrals and exact sums. Nevertheless, the exact result is not always available. We will have more opportunities if we have more than one renormalization scheme. The exact result that cannot be obtained by one scheme may be obtained by other schemes. Therefore, more renormalization schemes mean more operability.

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For the distribution that has no moment, the integral in the definition (1.1) is divergent. The renormalization treatment aims to achieve a finite moment by removing the divergence in the integral definition (1.1). Roughly speaking, such a treatment is to obtain a finite value by subtracting infinity from infinity. However, this treatment will naturally cause a problem: whether the renormalized result depends on the choice of subtraction scheme. After all, infinity minus infinity can give any value. A correct subtraction treatment must be subtraction-scheme independent. The renormalized results given by the different renormalization schemes must be the same.

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Constructing more than one renormalization scheme also has a technical reason. The renormalization procedure, practically, depends on exact results, such as exact integrals and exact sums. Nevertheless, the exact result is not always available. We will have more opportunities if we have more than one renormalization scheme. The exact result that cannot be obtained by one scheme may be obtained by other schemes. Therefore, more renormalization schemes mean more operability.
In this paper, we will construct the following renormalization schemes: zeta function scheme, subtraction scheme, weighted moment scheme, cut-off scheme, characteristic function scheme, Mellin transformation scheme, and power-logarithmic moment scheme.

Using the renormalization scheme suggested in this paper, we will calculate the renormalized $n$-th power moment for the following distributions: Cauchy distribution, Levy distribution, $q$-exponential distribution, and $q$-Gaussian distribution. We use more than one renormalization scheme for each distribution. Whether a renormalization scheme applies to a certain distribution depends on, for example, whether the integral in the renormalization procedure can be exactly worked out.

For power moments, we usually only consider positive-integer power moments. However, in the renormalization treatment, we can obtain arbitrary real-number power moments and even complex-number power moments. By observing a complex-number power moment, we can analyze the singularity structure of the moment function on the complex plane. For example, the Laplace distribution has any positive-integer power moments. However, when we extend the moment function of the Laplace distribution to the complex plane, we will find that the complex moment function of the Laplace distribution is singular at the negative-integer power, and the negative-integer power is the singularity of complex-number moment function. The Laplace distribution does not have negative-integer power moments. In order to obtain the negative-integer power moments of the Laplace distribution, we need to use the renormalization method suggested in this paper.

In addition to renormalization schemes, we also provide a method to calculate logarithmic moments from power moments. The method is suitable for the case that the calculation of logarithmic moments is more complicated than that of power moments. However, some distributions have logarithmic moments but have no power moments. That is, the integral in the logarithmic moment (1.3) converges, but the integral in the power moment (1.2) diverges. If we still want to use the power moment to calculate the logarithmic moment, we can use the renormalized power moment instead of the divergent power moment. In this paper, we give some examples of calculating logarithmic moments from power moments, including Cauchy distribution, Levy distribution, $q$-exponential distribution, $q$-Gaussian distribution, normal distribution, Student’s $t$-distribution, and Laplace distribution. In these examples, some distributions have power moments, and others do not. For distributions that have no power moments, the renormalized power moment needs to be calculated first.

Calculating logarithmic moments from power moments also verifies the validity of the renormalization scheme. For a distribution that has no power moments but has logarithmic moments, we only need to renormalize the power moment, and the logarithmic moment can be obtained directly by definition (1.3). Comparing the logarithmic moment obtained from the renormalized power moment with the logarithmic moment calculated by definition (1.3), we can verify the validity of renormalization.

The Cauchy distribution is widely used in physics and mathematics, such as the phase oscillator systems [4], the fractional generalized Cauchy process [5], many-body localized systems [6], the edge of chaos and avalanches in neural networks [7], the factor models and contingency tables [8], the Dirichlet random probability [9], the statistical depth [10], and
the image processing [11]. The Levy distribution is a special case of the Levy α-stable distribution [12]. The Levy distribution plays important roles in the geometry of multiparameter families of quantum states [13], the efficiency of random target search strategies quantification [14], the nondiffusive suprathermal ion transport [15], and the multimode viscous hydrodynamics for one-dimensional spinless electrons [16]. The q-exponential distribution and the q-Gaussian distribution are examples of the Tsallis distributions arising from the maximization of the Tsallis entropy under appropriate constraints [17]. The q-exponential distribution is a generalization of the exponential distribution, and the q-Gaussian distribution is a generalization of the Gaussian distribution [18]. The q-exponential distribution and the q-Gaussian distribution are popular in complex systems [19], the two-qubit quantum system and the transverse-momentum behavior of hadrons in high-energy proton-proton collisions [20], the quantum Coulomb system in a confining potential [21], financial market [22–25], the clustering algorithm for cryo-electron microscopy [26], the Wasserstein Geometry [27], and the Swarm Quantum-like Particle Optimization algorithms [28].

In section 2, we construct renormalization schemes for power moments. In section 3, we apply the renormalization scheme to calculate the renormalized power moment for various distributions. In section 4, we discuss the generalization of the renormalized power moment. We provide a method to calculate logarithmic moments from power moments in section 5, and give examples in section 6. Discussions and conclusion are in section 7.

2 Renormalization scheme: n-th power moment

In this section, to remove divergences in power moments of distributions that have no well-defined moment, we construct several different renormalization schemes, including the zeta function scheme, the subtraction scheme, the weighted moment scheme, the cut-off scheme, the characteristic function scheme, the Mellin transformation scheme, and the power-logarithmic moment scheme. Moreover, in a renormalization scheme, we introduce the power-logarithmic moment. In the following, we apply these renormalization schemes to various distributions.

2.1 Zeta function scheme

The key step in the renormalization procedure is the analytical continuation. The analytical continuation of the zeta function is well studied in mathematics and has been systematically applied to the renormalization in quantum field theory [29]. In this section, we establish the zeta function renormalization scheme for the distribution that has no power moment.

A spectrum is a series of numbers, discrete or continuous. For example, all eigenvalues of an operator form an eigenvalue spectrum. From a given spectrum, we can define spectral functions, such as spectral zeta functions, one-loop effective actions, vacuum energies, and various thermodynamic quantities [30, 31]. Here, we focus on the spectral zeta function.

Generalized spectral zeta function. For a discrete spectrum \{\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots\}, the spectral zeta function is defined as \( \zeta(s) = \sum_{\lambda_n \geq \lambda_0} (\lambda_n)^{-s} \) [31]. For a continuous spectrum, with the spectral density function \( \rho(\lambda) \), the spectral zeta function is defined as \( \zeta(s) = \int_{\lambda_0}^{\infty} d\lambda \rho(\lambda) \lambda^{-s} \). In physics, however, the physical operators are usually lower
bounded. The above definition of spectral zeta functions applies only to lower bounded spectra. Since the random variables in many distributions are ranged from $-\infty$ to $+\infty$, before we apply the spectral zeta function to statistical distributions, we must generalize the spectral zeta function to the spectrum that is neither lower bounded nor upper bounded.

For the spectrum that is neither lower bounded nor upper bounded, for the discrete spectrum, we generalize the spectral zeta function as

$$
\zeta(s) = \sum_{-\infty < \lambda_n < \infty} \lambda_n^{-s}.
$$

and for the continuous spectrum,

$$
\zeta(s) = \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \lambda^{-s}.
$$

*Moment as zeta function.* To perform renormalization with the zeta function, we regard that all random variables form a spectrum. In this way, each probability distribution has its own zeta function.

Concretely, we regard the random variable $x$ in the distribution function as an element $\lambda$ in a spectrum, and regard the probability density function $p(x)$ as the spectral density function $\rho(\lambda)$. Thus, taking continuous spectra as an example, the spectral zeta function (2.2) becomes

$$
\zeta(s) = \int_{-\infty}^{\infty} dx p(x) x^{-s}.
$$

Comparing with the definition of the $n$-th power moment (1.2) shows that the $n$-th power moment is a spectral zeta function:

$$
m_n^{\text{ren}} = \zeta(-n),
$$

where $m_n^{\text{ren}}$ denotes the $n$-th renormalized power moment.

It should be emphasized that although the integral in the definition of the $n$-th moment (1.2) and the spectral zeta function (2.2) are the same in form, the analytic region of the spectral zeta function is often larger than the integrable region of the $n$-th moment. This is because different representations of the zeta function have different analytic regions. When we express the $n$-th moment with the spectral zeta function, we have indeed made an analytic continuation and extended the analytic region. In this way, the original divergent moment becomes convergent, and the divergent moment no longer diverges.

Although analytic continuation can remove the divergence caused by the integral definition of the moment, the true singularity in the moment function, the obstacle of analytic continuation, can not be removed by analytic continuation. For true singularities, the role of analytical continuation is to expose singularities. These true singularities will be removed by the minimal subtraction discussed later.
2.2 Subtraction method

Consider a divergent integral
\[ m = \int_{-\infty}^{\infty} f(x) \, dx. \]  
(2.5)

We remove divergence by introducing counterterm terms. We consider the divergence of the integral at \( x = 0 \), \( x \to \infty \), \( x \to -\infty \), and \( x \to \pm\infty \), respectively.

2.2.1 Divergence at \( x = 0 \)

First, consider the case that the integral diverges at the lower limit of the integral, \( x = 0 \).

Expand the integrand at \( x = 0 \):
\[ f(x) = \sum_{j=1}^{\infty} a_j x^{\alpha_j} \]
\[ = \sum_{j=1}^{j_{\max}} a_j x^{\alpha_j} + \sum_{j=j_{\min}}^{\infty} a_j x^{\alpha_j}, \]  
(2.6)

where \( j_{\max} \) satisfies \( \text{Re} \alpha_j \leq -1 \) and \( j_{\min} \) satisfies \( \text{Re} \alpha_j > -1 \). The divergence at the lower limit \( x = 0 \) comes from \( \text{Re} \alpha_j \leq -1 \). Thus, we use the method in Ref. [32] to remove the divergence.

Rewrite the integral (2.5) as
\[ m = \left[ \int_{-1}^{0} f(x) \, dx + \int_{0}^{1} f(x) \, dx \right] + \left[ \int_{-\infty}^{-1} f(x) \, dx + \int_{1}^{\infty} f(x) \, dx \right] \]
\[ = I_{\text{div}} + I_{\text{con}}. \]  
(2.7)

The divergence occurs in \( I_{\text{div}} = \int_{-1}^{0} f(x) \, dx + \int_{0}^{1} f(x) \, dx \), and the divergence of \( I_{\text{div}} \) comes from the \( \text{Re} \alpha_j \leq -1 \) term \( \sum_{j=1}^{j_{\max}} a_j x^{\alpha_j} \). Next, we subtract out the divergence term in \( I_{\text{div}} \).

Write \( I_{\text{div}} \) as
\[ I_{\text{div}} = \int_{-1}^{0} \left[ f(x) - \sum_{j=1}^{j_{\max}} a_j x^{\alpha_j} \right] \, dx + \int_{0}^{1} \left[ f(x) - \sum_{j=1}^{j_{\max}} a_j x^{\alpha_j} \right] \, dx \]
\[ + \int_{-1}^{0} \sum_{j=1}^{j_{\max}} a_j x^{\alpha_j} \, dx + \int_{0}^{1} \sum_{j=1}^{j_{\max}} a_j x^{\alpha_j} \, dx. \]  
(2.8)

Here, \( I_{\text{div}} \) consists of four integrals. The first two integrals are already convergent because the divergent part (\( \text{Re} a_j \leq -1 \)) has been subtracted out. The divergence of \( I_{\text{div}} \) is in the last two integrals.
In the divergent integral, a negative-power term \( a_{-1} x^{-1} \) may appear and the integral of this term is a logarithmic function. We write this term separately as follows:

\[
\int_{-1}^{0} \left[ \sum_{j=1}^{j_{\text{max}}} a_j x^{\alpha_j} \right] \, dx = \left[ \sum_{j=1}^{j_{\text{max}}} \frac{a_j}{\alpha_j + 1} x^{\alpha_j+1} + a_{-1} \ln x \right]_{x=0^{-}} - \left[ \sum_{j=1}^{j_{\text{max}}} \frac{a_j}{\alpha_j + 1} x^{\alpha_j+1} + a_{-1} \ln x \right]_{x=-1}, \tag{2.9}
\]

and

\[
\int_{0}^{1} \left[ \sum_{j=1}^{j_{\text{max}}} a_j x^{\alpha_j} \right] \, dx = \left[ \sum_{j=1}^{j_{\text{max}}} \frac{a_j}{\alpha_j + 1} x^{\alpha_j+1} + a_{-1} \ln x \right]_{x=1} - \left[ \sum_{j=1}^{j_{\text{max}}} \frac{a_j}{\alpha_j + 1} x^{\alpha_j+1} + a_{-1} \ln x \right]_{x=0^{+}}, \tag{2.10}
\]

where \( \sum' \) denotes that the negative-power term is not included in the sum, i.e., \( \alpha_j \neq -1 \). Obviously, these two integrals converge at \( x = -1 \) and \( x = 1 \), and the divergence comes from \( x = 0 \). After removing the divergent term \( \left[ \sum'_{j=1} \frac{a_j}{\alpha_j + 1} x^{\alpha_j+1} + a_{-1} \ln x \right]_{x=0^{\pm}} \), we arrive at a renormalized result,

\[
I_{\text{ren}}^{\text{div}} = \int_{-1}^{0} \left[ f(x) - \sum_{j=1}^{j_{\text{max}}} a_j x^{\alpha_j} \right] \, dx + \int_{0}^{1} \left[ f(x) - \sum_{j=1}^{j_{\text{max}}} a_j x^{\alpha_j} \right] \, dx - \sum_{j=1}^{j_{\text{max}}} \frac{(-1)^{\alpha_j+1} a_j}{\alpha_j + 1} + i\pi a_{-1} + \sum_{j=1}^{j_{\text{max}'} \alpha_j + 1}. \tag{2.11}
\]

The renormalized \( n \)-th moment, by Eqs. (2.7) and (2.11), reads

\[
m_{\text{ren}}^{(n)} = \int_{-\infty}^{-1} f(x) \, dx + \int_{-1}^{0} \left[ f(x) - \sum_{j=1}^{j_{\text{max}}} a_j x^{\alpha_j} \right] \, dx + \int_{0}^{1} \left[ f(x) - \sum_{j=1}^{j_{\text{max}}} a_j x^{\alpha_j} \right] \, dx - \sum_{j=1}^{j_{\text{max}}} \frac{(-1)^{\alpha_j+1} a_j}{\alpha_j + 1} + i\pi a_{-1} + \sum_{j=1}^{j_{\text{max}'} \alpha_j + 1} a_j + \int_{1}^{\infty} f(x) \, dx. \tag{2.12}
\]

### 2.2.2 Divergence at \( x \to \infty \)

If the integral diverges at \( x \to \infty \), we expand the integrand at \( x \to \infty \):

\[
f(x) = \sum_{j=1}^{\infty} b_j x^{-j}. \tag{2.13}
\]

The terms of \( \text{Re} \beta_j \leq 1 \) will diverge. We use the treatment above to remove the divergence.
Write the integral (2.5) in two parts:

\[
m = \int_{-\infty}^{1} f(x) \, dx + \int_{1}^{\infty} f(x) \, dx
\]

\[
= I_{\text{con}} + I_{\text{div}}. \tag{2.14}
\]

\( I_{\text{con}} \) is convergent and \( I_{\text{div}} \) diverges.

Write the divergent integral \( I_{\text{div}} \) as

\[
I_{\text{div}} = \int_{1}^{\infty} \left[ f(x) - \sum_{j=1}^{j_{\text{max}}} b_j \frac{1}{x^{\beta_j}} \right] \, dx + \left[ \sum_{j=1}^{j_{\text{max}}} \frac{b_j}{1 - \beta_j x^{\beta_j}} + b_{-1} \ln x \right]_{x=1} \rightarrow \infty
\]

where the sum \( \sum' \) does not contain the negative-power term, i.e., \( \beta_j \neq 1 \), and \( j_{\text{max}} \) satisfies \( \text{Re} \beta_j \leq 1 \). After removing the divergent term \( \left[ \sum_{j=1}^{j_{\text{max}}} \frac{b_j}{1 - \beta_j x^{\beta_j}} + b_{-1} \ln x \right]_{x=1} \rightarrow \infty \), we arrive at a renormalized result

\[
I_{\text{div}}^{\text{ren}} = \int_{1}^{\infty} \left[ f(x) - \sum_{j=1}^{j_{\text{max}}} b_j \frac{1}{x^{\beta_j}} \right] \, dx - \sum_{j=1}^{j_{\text{max}}} \frac{b_j}{1 - \beta_j}. \tag{2.16}
\]

The renormalized \( n \)-th moment, by Eqs. (2.14) and (2.16), reads

\[
m^{\text{ren}} = \int_{-\infty}^{1} f(x) \, dx + \int_{1}^{\infty} \left[ f(x) - \sum_{j=1}^{j_{\text{max}}} b_j \frac{1}{x^{\beta_j}} \right] \, dx - \sum_{j=1}^{j_{\text{max}}} \frac{b_j}{1 - \beta_j}. \tag{2.17}
\]

\subsection*{2.2.3 Divergence at \( x \rightarrow -\infty \)}

If the integral diverges at \( x \rightarrow -\infty \), we expand the integrand at \( x \rightarrow -\infty \):

\[
f(x) = \sum_{j=1}^{\infty} c_j \frac{1}{x^{\gamma_j}}. \tag{2.18}
\]

The terms of \( \text{Re} \gamma_j \leq 1 \) will diverge.

Write the integral (2.5) in two parts:

\[
m = \int_{-\infty}^{-1} f(x) \, dx + \int_{-1}^{\infty} f(x) \, dx \tag{2.19}
\]

\[
= I_{\text{div}} + I_{\text{con}}. \tag{2.20}
\]

\( I_{\text{con}} \) is convergent and \( I_{\text{div}} \) diverges.
Write the divergent integral $I_{\text{div}}$ as

$$I_{\text{div}} = \int_{-\infty}^{-1} \left[ f(x) - \sum_{j=1}^{\text{max}} c_j \frac{1}{x^{\gamma_j}} \right] dx + \int_{-\infty}^{1} \left[ \sum_{j=1}^{\text{max}} c_j \frac{1}{x^{\gamma_j}} \right] dx$$

$$= \int_{-\infty}^{-1} \left[ f(x) - \sum_{j=1}^{\text{max}} c_j \frac{1}{x^{\gamma_j}} \right] dx + \int_{-\infty}^{1} \left[ \sum_{j=1}^{\text{max}} c_j \frac{1}{1 - \gamma_j x^{\gamma_j - 1}} + c_{-1} \ln x \right]_{x\to-1}$$

$$- \left[ \sum_{j=1}^{\text{max}}' \frac{1}{1 - \gamma_j x^{\gamma_j - 1}} + c_{-1} \ln x \right]_{x=-\infty},$$

where the sum $\sum'$ does not contain the negative-power term, i.e., $\gamma_j \neq 1$, and $j_{\text{max}}$ satisfies $\text{Re}\, \gamma_j \leq 1$. After removing the divergent term $\left[ \sum_{j=1}^{\text{max}} \frac{c_j}{\gamma_j} \left( \frac{1}{2} \right)^{\gamma_j - 1} + c_{-1} \ln x \right]_{x=-\infty}$, we arrive at a renormalized result

$$I_{\text{div}}^{\text{ren}} = \int_{-\infty}^{-1} \left[ f(x) - \sum_{j=1}^{\text{max}} c_j \frac{1}{x^{\gamma_j}} \right] dx + \int_{-\infty}^{1} \left[ \sum_{j=1}^{\text{max}}' \frac{(-1)^{\gamma_j - 1} c_j}{1 - \gamma_j} + i\pi c_{-1} \right].$$

The renormalized $n$-th moment, by Eqs. (2.14) and (2.16), reads

$$m_{\text{ren}} = \int_{-1}^{\infty} f(x) dx + \int_{-\infty}^{-1} \left[ f(x) - \sum_{j=1}^{\text{max}} c_j \frac{1}{x^{\gamma_j}} \right] dx + \int_{-\infty}^{1} \left[ \sum_{j=1}^{\text{max}}' \frac{(-1)^{\gamma_j - 1} c_j}{1 - \gamma_j} + i\pi c_{-1} \right].$$

### 2.2.4 Divergence at $x \to \pm \infty$

If the integral diverges at both $x \to \pm \infty$, we expand the integrand at $x \to \pm \infty$, respectively:

$$f(x) = \sum_{j=1}^{\infty} d_j \frac{1}{x^{\delta_j}}.$$  

The terms of $\text{Re}\, \delta_j \leq 1$ will diverge.

Write the integral (2.5) in two parts:

$$m = \int_{-1}^{1} f(x) dx + \left[ \int_{-\infty}^{-1} f(x) dx + \int_{1}^{\infty} f(x) dx \right]$$

$$= I_{\text{con}} + I_{\text{div}}.$$  

$I_{\text{con}}$ is convergent and $I_{\text{div}}$ diverges.

Write the divergent integral $I_{\text{div}}$ diverges as

$$I_{\text{div}} = \int_{-\infty}^{-1} \left[ f(x) - \sum_{j=1}^{\text{max}} d_j \frac{1}{x^{\delta_j}} \right] dx + \int_{-\infty}^{1} \left[ \sum_{j=1}^{\text{max}} d_j \frac{1}{x^{\delta_j}} \right] dx$$

$$+ \int_{1}^{\infty} \left[ f(x) - \sum_{j=1}^{\text{max}} d_j \frac{1}{x^{\delta_j}} \right] dx + \int_{1}^{\infty} \left[ \sum_{j=1}^{\text{max}} d_j \frac{1}{x^{\delta_j}} \right] dx.$$  

(2.26)
where \( j_{\text{max}} \) satisfies \( \text{Re} \delta_j \leq 1 \). Working out the integral gives
\[
I_{\text{div}} = \int_{-\infty}^{-1} \left[ f(x) - \sum_{j=1}^{j_{\text{max}}} d_j \frac{1}{x^{\delta_j}} \right] dx + \sum_{j=1}^{j_{\text{max}}} \frac{d_j}{1-\delta_j} x^{1-\delta_j} \bigg|_{x=-1}^{x=-\infty} + d_{-1} \ln x \bigg|_{x=-1}^{x=-\infty}
+ \int_{1}^{\infty} \left[ f(x) - \sum_{j=1}^{j_{\text{max}}} d_j \frac{1}{x^{\delta_j}} \right] dx + \sum_{j=1}^{j_{\text{max}}} \frac{d_j}{1-\delta_j} x^{1-\delta_j} \bigg|_{x=1}^{x=\infty} + d_{-1} \ln x \bigg|_{x=1}^{x=\infty},
\]
where the sum \( \sum' \) does not contain the negative-power term, i.e., \( \delta_j \neq 1 \). After dropping the divergent terms, \( \sum' x^{1-\delta_j} \bigg|_{x=-\infty}^{x=-1}, \sum_{j=1}^{j_{\text{max}}} \frac{d_j}{1-\delta_j} x^{1-\delta_j} \bigg|_{x=1}^{x=\infty}, \) and \( d_{-1} \ln x \bigg|_{x=\infty} \), we arrive at a renormalized result
\[
I_{\text{div}}^{\text{ren}} = \int_{-\infty}^{-1} \left[ f(x) - \sum_{j=1}^{j_{\text{max}}} d_j \frac{1}{x^{\delta_j}} \right] dx + \sum_{j=1}^{j_{\text{max}}} \frac{(-1)^{1-\delta_j} d_j}{1-\delta_j} + id_{-1} \pi
+ \int_{1}^{\infty} \left[ f(x) - \sum_{j=1}^{j_{\text{max}}} d_j \frac{1}{x^{\delta_j}} \right] dx - \sum_{j=1}^{j_{\text{max}}} \frac{d_j}{1-\delta_j}.
\]

The renormalized \( n \)-th moment, by Eqs. (2.25) and (2.28), reads
\[
m^{\text{ren}} = \int_{-1}^{1} f(x) dx + \int_{-\infty}^{-1} \left[ f(x) - \sum_{j=1}^{j_{\text{max}}} d_j \frac{1}{x^{\delta_j}} \right] dx + \sum_{j=1}^{j_{\text{max}}} \frac{(-1)^{1-\delta_j} d_j}{1-\delta_j} + id_{-1} \pi
+ \int_{1}^{\infty} \left[ f(x) - \sum_{j=1}^{j_{\text{max}}} d_j \frac{1}{x^{\delta_j}} \right] dx - \sum_{j=1}^{j_{\text{max}}} \frac{d_j}{1-\delta_j}.
\]

### 2.3 Weight function scheme

In this section, we establish a weighted moment scheme to renormalize power moments. In the weighted moment scheme, we introduce a weighted moment and then remove the divergence in the weighted moment by a renormalization procedure. The weighted moment, of course, depends on the choice of weight functions [24], but the renormalized moment must be independent of the weight function, i.e., the weighted moment scheme must be scheme-independent.

For the divergent \( n \)-th moment (1.2), we choose the weight function \( g_s(x) \) satisfying
\[
g_s(x) \bigg|_{s=0} = 1,
\]
so that the following weighted moment
\[
m_n(s) = \int_{-\infty}^{\infty} g_s(x) p(x) x^n dx
\]
converges. The weighted moment recovers the moment when \( s = 0 \):
\[
m_n(s) \bigg|_{s=0} = m_n.
\]
If the $n$-th moment does not exist, then $s = 0$ must be a singularity of $m_n(s)$. In order to expose the singularity, we expand the weighted moment $m_n(s)$ at $s = 0$:

$$m_n(s) = \sum_{i=-N}^{\infty} a_i s^i = \frac{a_{-N}}{s^N} + \frac{a_{-N+1}}{s^{N+1}} + \cdots + a_0 + a_1 s + a_2 s^2 + \cdots \quad (2.33)$$

The divergence appears in negative-power terms.

In order to obtain a finite $n$-th moment $m_n$, we use the minimal subtraction in quantum field theory. When taking $s = 0$ to recover the $n$-th moment, the negative-power term diverges, and the positive-power term vanishes. Dropping the divergent negative-power terms, we arrive at a renormalized moment,

$$m_n^{\text{ren}} = a_0 \quad (2.34)$$

which is the zeroth-order term in the expansion (2.33).

The weighted moment scheme introduces a parameter $s$ so that the moment is a function of the parameter and the divergence is a singularity of this function.

### 2.4 Cut-off scheme

Some divergent integrals may become finite after cutting off the integral limit. For example, when the integral in the moment (1.1) diverges at the upper limit, replacing the upper limit with a finite value will give a cut-off-dependent finite moment. Cutting off the upper limit of an integral is equivalent to adding a step function as the weight function:

$$g_\Lambda(x) = \theta(\Lambda - x) \quad (2.35)$$

Then by Eq. (2.31) we arrive at a cutting-off moment

$$m_n(\Lambda) = \int_{-\infty}^{\infty} \theta(\Lambda - x) p(x) x^n dx = \int_{-\infty}^{\Lambda} p(x) x^n dx. \quad (2.36)$$

In this treatment, by introducing a cut-off parameter $\Lambda$, we obtain a finite cut-off-dependent moment. The same treatment also applies to the cases where the divergence appears at lower limits and at both upper and lower limits. The cut-off scheme is essentially a special case of the weighted moment scheme. Thus, the weighted moment method in the previous section (2.3) is also applicable. If the moment diverges at $\infty$, then $\Lambda \to \infty$ is a singular point of $m_n(\Lambda)$. We expand $m_n(\Lambda)$ at $\Lambda \to \infty$, or equivalently, at $1/\Lambda = 0$. The zero-power term of the expansion of $m_n(\Lambda)$, as shown in Eq. (2.34), is the renormalized moment. A similar treatment has been used to deal with the divergence in the scattering of quantum mechanics and proved to be valid [33].
2.5 Characteristic function scheme

The characteristic function is defined as

\[ f(k) = \int_{-\infty}^{\infty} p(x) e^{ikx} dx, \quad (2.37) \]

which is the Fourier transform of the probability density function \( p(x) \) [3]. In this section, we suggest a renormalization scheme based on the characteristic function. The characteristic function scheme can also be viewed as a special weight function scheme with the weight function,

\[ g_k(x) = e^{ikx}. \quad (2.38) \]

The weighted \( n \)-th moment with the weight function (2.38) is

\[ m_n(k) = \int_{-\infty}^{\infty} p(x) x^n e^{ikx} dx. \quad (2.39) \]

It can be seen that the moment (2.39) is a Fourier transform of the function \( p(x) x^n \). This allows us to perform analytical continuation with the help of the Fourier transform. For example, a Fourier-type integral \( \int_{-\infty}^{\infty} \frac{1}{x} e^{ikx} dx \) is not integrable, but by analytical continuation the Fourier transform of \( \frac{1}{x} \) is \( i \text{sgn}(k) \sqrt{\pi} \).

In particular, the zeroth moment, corresponding to \( n = 0 \) in Eq. (2.39),

\[ m_0(k) = \int_{-\infty}^{\infty} p(x) e^{ikx} dx \equiv f(k) \quad (2.40) \]

is just the characteristic function.

The characteristic function also plays the role of the moment generating function, i.e., the \( n \)-th moment can be expressed as the derivatives of the characteristic function, or the coefficient of the expansion of the characteristic function:

\[ m_n^{\text{ren}} = (-i)^n \left. \frac{d^n f(k)}{dk^n} \right|_{k=0}. \quad (2.41) \]

For the distribution that has no moment, the Fourier transform of the probability density function \( p(x) \) is equivalent to performing the analytical continuation. Then by Eq. (2.41), we obtain a renormalized moment.

If a distribution does not have moments, the derivative of the characteristic function in Eq. (2.41) is not well-defined. We need to deal with such derivatives by analytical continuation. To do this, we use an integral representation of the derivation [34]:

\[ \frac{d^n f(k)}{dk^n} = \frac{1}{\Gamma(-n)} \int_0^k (k-s)^{-n-1} f(s) ds. \quad (2.42) \]

Then by Eqs. (2.41) and (2.42), we have

\[ m_n^{\text{ren}} = (-i)^n \left. \frac{1}{\Gamma(-n)} \int_0^k (k-s)^{-n-1} f(s) ds \right|_{k=0}. \quad (2.43) \]
2.6 Mellin transform scheme

In this section, we establish a Mellin transform scheme to renormalize power moments. In this approach, the analytical continuation is performed through the Mellin transform.

2.6.1 Characteristic function approach

According to Eq. (2.43), the $n$-th moment is generated by the characteristic function. We rewrite Eq. (2.43) as

$$m_{\text{ren}}^n = -(-i)^n \frac{1}{\Gamma(-n)} \left. \frac{1}{k} (k-s)^{-n-1} f(s) \right|_{k=0} ds$$

$$= -(-i)^n (-1)^{-n-1} \frac{1}{\Gamma(-n)} \int_0^{\infty} s^{-n-1} f(s) ds$$

$$= \frac{i^n}{\Gamma(-n)} \int_0^{\infty} s^{-n-1} f(s) ds. \quad (2.44)$$

By the Mellin transform [35],

$$\mathcal{M}_\sigma [g(x)] = \int_0^{\infty} g(x) x^{\sigma-1} dx, \quad (2.45)$$

we express the $n$-th moment (2.44) as the Mellin transform of the characteristic function:

$$m_{\text{ren}}^n = \frac{i^n}{\Gamma(-n)} \mathcal{M}_{-n} [f(k)]. \quad (2.46)$$

2.6.2 Density function approach

We can also express the $n$-th moment as the Mellin transform of the probability density function $p(x)$:

$$m_n = \int_{-\infty}^{\infty} p(x) x^n dx$$

$$= \int_{-\infty}^{0} p(x) x^n dx + \int_{0}^{\infty} p(x) x^n dx$$

$$= \int_{-\infty}^{0} p(-x) (-x)^n dx + \int_{0}^{\infty} p(x) x^n dx$$

$$= (-1)^n \int_{-\infty}^{0} p(-x) x^{n+1-1} dx + \int_{0}^{\infty} p(x) x^n dx. \quad (2.47)$$

Compared with the definition of the Mellin transform (2.45), we have

$$m_{\text{ren}}^n = (-1)^n \mathcal{M}_{n+1} [p(-x)] + \mathcal{M}_{n+1} [p(x)]. \quad (2.48)$$

For $x > 0$, we have

$$m_{\text{ren}}^n = \mathcal{M}_{n+1} [p(x)]. \quad (2.49)$$
2.7 Power-logarithmic moment scheme

In this scheme we introduce the power-logarithmic moment:

\[ M_{n,m} = \int_{-\infty}^{\infty} p(x) x^n \ln^m x \, dx. \]  

(2.50)

If the \( n \)-th power moment

\[ m_n = \int_{-\infty}^{\infty} p(x) x^n \, dx \]  

(2.51)

is divergent, we can first work out the following power-logarithmic moment

\[ M_{n-1,m} = \int_{-\infty}^{\infty} p(x) x^{n-1} \ln^m x \, dx, \]  

(2.52)

and then achieve the power moment from the power-logarithmic moment by their relation given below. This approach applies to the distribution with which \( p(x) x^n \) is not integrable, but \( p(x) x^{n-1} \ln^m x \) is integrable.

Generally speaking, compared with \( p(x) x^n \), \( p(x) x^{n-1} \ln^m x \) is easier to satisfy the integrability condition. Since \( x \) diverges faster than \( \ln x \), \( [p(x) x^{n-1}] \ln x \) is easier to satisfy the integrability condition than \( [p(x) x^{n-1}] x = p(x) x^n \). Thus, when the \( n \)-th moment (2.51) does not exist, the integral \( \int_{-\infty}^{\infty} p(x) x^{n-1} \ln x \, dx \), even the integral \( \int_{-\infty}^{\infty} p(x) x^{n-1} \ln^m x \, dx \), may possibly satisfy the integrability condition. In the power-logarithmic moment scheme, if \( \int_{-\infty}^{\infty} p(x) x^{n-1} \ln^m x \, dx \) is integrable, instead of the integral in (2.51), we turn to work out the power-logarithmic moment (2.52), and then obtain the \( n \)-th moment \( m_n \) from the power-logarithmic moment \( M_{n-1,m} \).

A probability density function \( p(x) \), with which \( p(x) x^n \) is not integrable but \( p(x) x^{n-1} \ln^m x \) is integrable, has no power moment (2.51) but has power-logarithmic moment (2.52). By

\[ \ln^m x = \frac{1}{x^{n-1}} \frac{d^n x^{n-1}}{dn^m} \]  

(2.53)

we have

\[ M_{n-1,m} = \frac{d^n}{dn^m} \int_{-\infty}^{\infty} p(x) x^{n-1} \, dx \]  

\[ = \frac{d^n}{dn^m} m_{n-1}. \]  

(2.54)

Thus, when the power-logarithmic moment \( M_{n-1,m} \) is known, the \( n \)-th moment, by Eq. (2.54), can be solved from the following differential equation:

\[ \frac{d^n}{dn^m} m_n = M_{n,m}. \]  

(2.55)

In particular, for \( m = 1 \), denoting \( M_{n,1} \equiv M_n \), we have

\[ \frac{d}{dn} m_n = M_n. \]  

(2.56)

The \( n \)-th moment, by solving the differential equation (2.56), reads

\[ m_n^{ren} = \int M_n \, dn. \]  

(2.57)

In this way, we can obtain the renormalized \( n \)-th moment \( m_n^{ren} \) if the power-logarithmic moment \( M_n \) exists.
3 Renormalized power moment: examples

In this section, we use the renormalization scheme suggested in section 2 to calculate the renormalized $n$-th power moment for the Cauchy distribution, the Levy distribution, the $q$-exponential distribution, and the $q$-Gaussian distribution. In order to verify that the renormalization treatment is scheme-independent, for each distribution, we use more than one renormalization scheme.

3.1 Cauchy distribution

The probability density function of the Cauchy distribution is [3]

$$p(x) = \frac{1}{\pi (1 + x^2)}.$$  (3.1)

The $n$-th moment of the Cauchy distribution, by definition (1.2), is

$$m_n = \int_{-\infty}^{\infty} \frac{x^n}{\pi (1 + x^2)} dx.$$  (3.2)

The $n$-th moment of the Cauchy distribution does not exist, for the integral diverges when $n = 1, 2, 3, \cdots$. Next, we use different renormalization treatments to renormalize the divergent moments.

3.1.1 Zeta function method

We first use the zeta function method in section 2.1 to calculate the renormalized $n$-th moment.

The spectral zeta function of the Cauchy distribution, by Eq. (2.2), is

$$\zeta(s) = \int_{-\infty}^{\infty} \frac{x^{-s}}{\pi (1 + x^2)} dx = e^{-is\pi/2}.$$  (3.3)

By Eq. (2.4), the $n$-th moment of the Cauchy distribution is

$$m_n^{\text{ren}} = \zeta(-n) = e^{in\pi/2}.$$  (3.4)

We list the first several moments:

$$m_n^{\text{ren}} = i, m_2^{\text{ren}} = -1, m_3^{\text{ren}} = -i, m_4^{\text{ren}} = 1, \cdots.$$  (3.5)

It can be seen that for integer-order moments, we have

$$m_{2n}^{\text{ren}} = (-1)^n, \quad n = \text{even},$$

$$m_{2n-1}^{\text{ren}} = (-1)^n i, \quad n = \text{odd}.$$  (3.6)

It can be seen that some of the renormalized moments are less than zero, and some are pure imaginary numbers.
3.1.2 Subtraction method

We use the subtraction method in section 2.2 to calculate the renormalized $n$-th moment.

The $n$-th moment of the Cauchy distribution (3.2) diverges at $x \to \pm\infty$. Expand the integrand in Eq. (3.2) at $x \to -\infty$ and $x \to \infty$:

$$\frac{x^n}{\pi (1 + x^2)} = -\frac{1}{\pi} \sum_{j=1}^{\infty} \cos \left( \frac{j\pi}{2} \right) \frac{1}{x^{j-n}}. \tag{3.7}$$

The divergence is caused by the term $j - n \leq 1$, i.e.,

$$j \leq n + 1. \tag{3.8}$$

By Eq. (2.29), we have

$$m_{\text{ren}}^n = \int_{-\infty}^{-1} \left[ \frac{x^n}{\pi (1 + x^2)} + \frac{1}{\pi} \sum_{j=1}^{j \leq n+1} \cos \left( \frac{j\pi}{2} \right) \frac{1}{x^{j-n}} \right] dx + i\pi c_{-1} - \sum_{j=1}^{j \leq n+1} \cos \left( \frac{j\pi}{2} \right) \frac{1}{1 - (j-n)}.$$

For example, when $n = 1$, the first-order moment is

$$m_{\text{ren}}^1 = \int_{-\infty}^{-1} \left[ \frac{x}{\pi (1 + x^2)} - \frac{1}{\pi x} \right] dx + i\pi c_{-1} + \int_{-1}^{1} \frac{x}{\pi (1 + x^2)} dx + \int_{1}^{\infty} \left[ \frac{x^n}{\pi (1 + x^2)} - \frac{1}{\pi x} \right] dx \tag{3.9}$$

where the coefficient of $\frac{1}{x}$ in the expansion is $c_{-1} = \frac{1}{\pi} m$.

3.1.3 Weight function method (1)

We use the weighted moment method in section 2.3 to calculate the renormalized $n$-th moment. In order to exemplify that the renormalized moment is independent of the choice of weight functions, we choose two weight functions.

Taking the weight function as

$$g_s (x) = e^{-sx} \tag{3.11}$$

gives the weighted moment

$$m_n (s) = \int_{-\infty}^{\infty} \frac{1}{\pi (1 + x^2)} x^n e^{-sx} dx$$

$$= e^{is\pi/2} \left[ \cos s + \text{sgn}(s) \sin s - (\text{sgn}(s) + i) \frac{i^{-n}s^{-n+1}}{\Gamma(2-n)} \frac{\sum_{j=1}^{j \leq n+1} \cos \left( \frac{j\pi}{2} \right) \frac{1}{x^{j-n}}}{1} \right]. \tag{3.12}$$
According to Eq. (2.34), we expand the weighted moment \( m_n(s) \) at \( s = 0 \):

\[
e^{\im \pi n/2} \left[ 1 + s - \frac{(1 + i)s(i s)^{-n}}{\Gamma(2 - n)} + \cdots \right],
\]

where the zero-power term is the renormalized moment

\[
m_n^{\text{ren}}(0) = e^{\im \pi n/2}.
\]

### 3.1.4 Weight function method (2)

Taking another the weight function,

\[
g_s(x) = x^{s-1},
\]

the weighted moment is

\[
m_n(s) = \int_{-\infty}^{\infty} \frac{1}{\pi(1 + x^2)} x^n x^{s-1} dx = \frac{1}{2} \left[ (-1)^{n+s+1} + 1 \right] \csc \left( \frac{1}{2} \pi(n + s) \right).
\]

According to Eq. (2.34), we expand the weighted moment \( m_n(s) \) at \( s = 0 \):

\[
e^{\im \pi n/2} \left[ 1 + \frac{i\pi(s - 1)}{2} + \cdots \right],
\]

where the zero-power term is the renormalized moment

\[
m_n^{\text{ren}} = e^{\im \pi n/2}.
\]

### 3.1.5 Cut-off method

We use the cut-off method in section 2.4 to calculate the renormalized \( n \)-th moment.

Cutting off the upper and lower integration limit of the \( n \)-th moment (3.2) gives

\[
m_n(\Lambda_1, \Lambda_2) = \int_{-\Lambda_2}^{\Lambda_1} \frac{1}{\pi(1 + x^2)} x^n dx.
\]

When \( \Lambda_1 \to \infty \) and \( \Lambda_2 \to \infty \), we obtain the \( n \)-th moment: \( m_n = m_n(\infty, \infty) \). Working out the integral in Eq. (3.19) gives

\[
m_n(\Lambda_1, \Lambda_2) = \frac{\Lambda_1^{n+1} 2F_1 \left( 1, \frac{n+1}{2}; \frac{n+3}{2}; -\Lambda_1^2 \right) - (-\Lambda_2)^{n+1} 2F_1 \left( 1, \frac{n+1}{2}; \frac{n+3}{2}; -\Lambda_2^2 \right)}{\pi(n+1)}.
\]

In order to obtain the renormalized \( n \)-th moment, we expand \( m_n(\Lambda) \) at \( \Lambda_1, \Lambda_2 \to \infty \):

\[
m_n(\Lambda) = e^{\im \pi n/2} + \frac{\Lambda_1^{n-1}}{\pi(n-1)} - \frac{(-\Lambda_2)^{n-1}}{\pi(n-1)} + \cdots.
\]

Dropping the divergent terms, we obtain the renormalized \( n \)-th moment,

\[
m_n^{\text{ren}} = e^{\im \pi n/2}.
\]
3.1.6 Characteristic function method

We use the characteristic function method in section 2.5 to calculate the renormalized $n$-th moment.

The characteristic function of the Cauchy distribution by Eq. (2.37) is

$$f(k) = \int_{-\infty}^{\infty} p(x) e^{ikx} dx = e^{-|k|}. \quad (3.23)$$

By the integral representation of the derivation, Eq. (2.42), we obtain

$$\frac{d^n f(k)}{dk^n} = e^{-k+i\pi n} \frac{\Gamma(-n) - \Gamma(-n,-k)}{\Gamma(-n)}. \quad (3.24)$$

By Eq. (2.41), the renormalized $n$-th moment is

$$m_{n}^{\text{ren}} = e^{i\pi n/2} \frac{\Gamma(-n) - \Gamma(-n,0)}{\Gamma(-n)}. \quad (3.25)$$

When $n$ is a positive integer, the renormalized $n$-th moment is

$$m_{n}^{\text{ren}} = e^{in\pi/2}.$$

3.1.7 Mellin transform scheme: characteristic function method

We use the Mellin transform method for the characteristic function in section 2.6.1 to calculate the renormalized $n$-th moment.

Substituting the characteristic function of the Cauchy distribution (3.23) into Eq. (2.46), we obtain

$$m_{n}^{\text{ren}} = \frac{i^n}{\Gamma(-n)} \mathcal{M}_{-n} \left[ e^{-|k|} \right] = \frac{i^n}{\Gamma(-n)} \times \Gamma(-n) = i^n \quad (3.26)$$

3.1.8 Mellin transform scheme: probability density function

We use the Mellin transform method for the probability density function in section 2.6.2 to calculate the renormalized $n$-th moment.

Substituting the probability density function of the Cauchy distribution (3.1) into Eq. (2.48), we obtain

$$m_{n}^{\text{ren}} = (-1)^n \mathcal{M}_{n+1} [p(-x)] + \mathcal{M}_{n+1} [p(x)]$$

$$= (-1)^n \left( \frac{1}{2 \sec \frac{n\pi}{2}} \right) + \frac{1}{2 \sec \frac{n\pi}{2}}$$

$$= e^{in\pi/2}. \quad (3.27)$$
3.1.9 Power-logarithmic moment method

The power-logarithmic moment of the Cauchy distribution, by Eq. (2.52), is

\[ M_{n-1} = \int_{-\infty}^{\infty} \frac{1}{\pi (1 + x^2)} x^{n-1} \ln x \, dx = \frac{\pi}{2} e^{in\pi/2}. \] (3.28)

By the relation between the \(n\)-th moment \(m_n\) and the power-logarithmic moment \(M_{n-1}\), Eq. (2.57), we have

\[ m_n^{\text{ren}} = e^{in\pi/2}. \] (3.29)

The renormalized \(n\)-th moment given by various renormalization schemes is the same. This verifies that the renormalization treatment is scheme-independent.

3.2 Levy distribution

The probability density function of the Levy distribution is [3]

\[ p(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x^{3/2}} e^{-1/(2x)}, \quad 0 \leq x < \infty. \] (3.30)

The \(n\)-th moment of the Levy distribution, by definition (1.2), is

\[ m_n = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{x^{3/2}} e^{-1/(2x)} x^n \, dx. \] (3.31)

The \(n\)-th moment of the Levy distribution does not exist, for the integral diverges when \(n = 1, 2, 3, \ldots\). In order to verify that the renormalization treatment is scheme-independent, we use different renormalization treatments to renormalize the divergent moments.

3.2.1 Zeta function method

We first use the zeta function method in section 2.1 to calculate the renormalized \(n\)-th moment.

The spectral zeta function of the Levy distribution, by Eq. (2.2), is

\[ \zeta(s) = \int_{0}^{\infty} e^{-1/(2x)} \frac{1}{\sqrt{2\pi}} \frac{1}{x^{3/2}} x^{-s} \, dx = \frac{2^s}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + s\right). \] (3.32)

By Eq. (2.4), the \(n\)-th moment of the Levy distribution is

\[ m_n^{\text{ren}} = \zeta(-n) = \frac{1}{2^n \sqrt{\pi}} \Gamma\left(\frac{1}{2} - n\right). \] (3.33)

The first several renormalized \(n\)-th moments can be obtained directly,

\[ m_1^{\text{ren}} = -1, \ m_2^{\text{ren}} = \frac{1}{3}, \ m_3^{\text{ren}} = -\frac{1}{15}, \ m_4^{\text{ren}} = \frac{1}{105}, \ldots. \] (3.34)
3.2.2 Subtraction method

We use the subtraction method in section 2.2.2 to calculate the renormalized \( n \)-th moment of the Levy distribution.

The \( n \)-th moment of the Levy distribution (3.31) diverges at \( x \to \infty \). Expand the integrand at \( x \to \infty \):

\[
e^{-\frac{1}{2x}} \frac{1}{\sqrt{2\pi x^{3/2}}} x^n = \sum_{j=0}^{\infty} \frac{(-1)^j 2^{-\frac{1}{2}-j} 1}{\sqrt{\pi j!}} x^{-n+3/2+j}.
\]

(3.35)

The divergence is caused by the term \(-n+3/2+j \leq 1\), i.e., \( j \leq n - \frac{1}{2} \). Since \( j \) is an integer, the divergent term is

\[
j < n.
\]

(3.36)

By Eq. (2.17), we obtain the renormalized \( n \)-th moment

\[
m_{n}^{\text{ren}} = \int_{0}^{1} \left( e^{-\frac{1}{2x}} \frac{1}{\sqrt{2\pi x^{3/2}}} x^n \right) dx + \int_{1}^{\infty} \left( e^{-\frac{1}{2x}} \frac{1}{\sqrt{2\pi x^{3/2}}} x^n - \sum_{j=0}^{j<n} \frac{(-1)^j 2^{-\frac{1}{2}-j} 1}{\sqrt{\pi j!}} x^{-n+3/2+j} \right) dx
\]

\[
- \sum_{j=0}^{j<n} \frac{(-1)^j 2^{-\frac{1}{2}-j} 1}{\sqrt{\pi j!}} \left( \frac{1}{x^{n-\frac{3}{2}+j}} \right)\]  

(3.37)

In particular, the first-order moment is

\[
m_{1}^{\text{ren}} = \int_{0}^{1} \left( e^{-\frac{1}{2x}} \frac{1}{\sqrt{2\pi x^{3/2}}} x \right) dx + \int_{1}^{\infty} \left( e^{-\frac{1}{2x}} \frac{1}{\sqrt{2\pi x^{3/2}}} - \frac{1}{\sqrt{2\pi x^{1/2}}} \right) dx - \frac{1}{\sqrt{2\pi}}
\]

\[
= \sqrt{\frac{2}{e\pi}} - \text{erfc} \left( \frac{1}{\sqrt{2}} \right) + \left( \sqrt{\frac{2}{\pi}} - \sqrt{\frac{2}{e\pi}} \right) - \sqrt{\frac{2}{\pi}}
\]

\[
= -1.
\]

(3.38)

3.2.3 Weight function method (1)

We use the weighted moment method in section 2.3 to calculate the renormalized \( n \)-th moment of the Levy distribution. In order to exemplify that the renormalized moment is independent of the choice of weight functions, we use two weight functions.

Taking the weight function as

\[
g_{s}(x) = e^{-sx}
\]

(3.39)

gives the weighted moment

\[
m_{n}(s) = \int_{0}^{\infty} e^{-\frac{1}{2x}} \frac{1}{\sqrt{2\pi x^{3/2}}} x^n e^{-sx} dx
\]

\[
= \frac{1}{\sqrt{\pi}} 2^{\frac{1}{2} + \frac{1}{2}} e^{-\frac{1}{2} - s} K_{n+\frac{1}{2}} \left( \sqrt{2s} \right),
\]

(3.40)

where \( K_{n}(z) \) is the modified Bessel function of the second kind [36]. According to Eq. (2.34), we expand the weighted moment \( m_{n}(s) \) at \( s = 0 \):

\[
= \frac{1}{2^n \sqrt{\pi}} \Gamma \left( \frac{1}{2} - n \right) + \frac{2^{-n} \Gamma \left( \frac{3}{2} - n \right)}{\sqrt{\pi(2n+1)}} s + \frac{\Gamma \left( n - \frac{1}{2} \right)}{\sqrt{2\pi}} s^{\frac{1}{2}-n} + \cdots,
\]

(3.41)
where the zero-order term is the renormalized moment:

\[ m_{n}^{\text{ren}}(0) = \frac{1}{2^n \sqrt{\pi}} \Gamma \left( \frac{1}{2} - n \right). \]  

(3.42)

In particular, the first moment is \( m_{1}^{\text{ren}} = -1 \).

### 3.2.4 Weight function method (2)

Taking another the weight function,

\[ g_s(x) = x^{s-1}, \]  

(3.43)

the weight moment is

\[ m_n(s) = \int_{0}^{\infty} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi} x^{3/2}} x^n x^{s-1} dx \]

\[ = \frac{2^{-n-s+1}}{\sqrt{\pi}} \Gamma \left( -n - s + \frac{3}{2} \right). \]  

(3.44)

According to (2.34), we expand the weighted moment \( m_n(s) \) at \( s = 0 \):

\[ \frac{1}{2^n \sqrt{\pi}} \Gamma \left( \frac{1}{2} - n \right) - \frac{1}{2^n \sqrt{\pi}} (s-1) \Gamma \left( \frac{1}{2} - n \right) \left( \psi^{(0)} \left( \frac{1}{2} - n \right) + \ln 2 \right) + \cdots, \]  

(3.45)

where the zero-order term is the renormalization \( n \)-th moment:

\[ m_{n}^{\text{ren}}(1) = \frac{1}{2^n \sqrt{\pi}} \Gamma \left( \frac{1}{2} - n \right). \]  

(3.46)

### 3.2.5 Cut-off method

We use the cut-off method in section 2.4 to calculate the renormalized \( n \)-th moment.

Cutting off the upper integration limit of the \( n \)-th moment gives

\[ m_n(\Lambda) = \int_{0}^{\Lambda} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi} x^{3/2}} x^n dx. \]  

(3.47)

When \( \Lambda \to \infty \), we obtain the \( n \)-th moment: \( m_n = m_n(\infty) \). Working out the integral in Eq. (3.47) gives

\[ m_n(\Lambda) = \frac{1}{2^n \sqrt{\pi}} \Gamma \left( \frac{1}{2} - n, \frac{1}{2\Lambda} \right). \]  

(3.48)

In order to obtain the renormalized \( n \)-th moment, we expand \( m_n(\Lambda) \) at \( \Lambda \to \infty \):

\[ m_n(\Lambda) = \frac{1}{2^n \sqrt{\pi}} \Gamma \left( \frac{1}{2} - n \right) + \frac{\sqrt{2}}{\sqrt{\pi} (2n - 1)} \Lambda^{n-1/2} + \cdots. \]  

(3.49)

The zero-order term is the renormalized \( n \)-th moment,

\[ m_{n}^{\text{ren}} = \frac{1}{2^n \sqrt{\pi}} \Gamma \left( \frac{1}{2} - n \right). \]  

(3.50)
3.2.6 Characteristic function method

We use the characteristic function method in section 2.5 to calculate the renormalized $n$-th moment of the Levy distribution.

The characteristic function of the Levy distribution by Eq. (2.37) is

$$f(k) = e^{-\sqrt{|k|}} (\cos \sqrt{|k|} + i \text{sgn}(k) \sin \sqrt{|k|}) .$$

By the integral representation of the derivation, Eq. (2.42), we obtain

$$\frac{d^n f(k)}{dk^n} = \pi^{3/2} 2^{n-3/2} k^{-n} \left[ 8 \tilde{F}_4 \left( 1; 1 \frac{3}{4}, -1 - n \frac{3}{2}, -n \frac{2}{2}, -\frac{k^2}{64} \right) \right.
\left. - ik \tilde{F}_4 \left( 1; 1 \frac{5}{4}, -1 - n - n \frac{3}{2}, -n \frac{2}{2}, -\frac{k^2}{64} \right) \right] + (-1)^{5/8} \sqrt{\pi} 2^{-2n-1/2} e^{-\frac{i}{4} \pi n} k^{2n-3/2} \left[ \text{bei}_{1/2-n} \sqrt{2k} + i \text{ber}_{1/2-n} \sqrt{2k} \right] ,$$

where $\tilde{F}_q (a; b; z)$ is regularized hypergeometric function, given by $p F_q (a; b; z) / (\Gamma(b_1) \cdots \Gamma(b_q))$, bei$_{\nu}(z)$ and ber$_{\nu}(z)$ are the Kelvin function and bei$_{\nu}(z) + i \text{ber}_{\nu}(z) = J_{\nu} \left( xe^{3\pi i / 4} \right)$, and $J_{\nu}(x)$ is a Bessel function of the first kind [36].

By Eqs. (2.41) and (3.52), we have

$$m_1^{\text{ren}} = (-i) \frac{df(k)}{dk} \bigg|_{k=0} = -1 ,
$$

$$m_2^{\text{ren}} = (-i)^2 \frac{d^2 f(k)}{dk^2} \bigg|_{k=0} = \frac{1}{3} ,
$$

$$\ldots$$

(3.53)

3.2.7 Mellin transform method: characteristic function

We use the Mellin transform method of the characteristic function in section 2.6.1 to calculate the renormalized $n$-th moment of the Levy distribution.

Substituting the characteristic function of the Levy distribution, Eq. (3.51), into Eq. (2.46), we obtain

$$m_n^{\text{ren}} = \frac{i^n}{\Gamma(-n)} M_{-n} [f(k)]
= \frac{i^n}{\Gamma(-n)} 2^{n+1} (-i)^n \Gamma(-2n)
= \frac{1}{2^n \sqrt{\pi}} \Gamma \left( \frac{1}{2} - n \right) .$$

(3.54)

3.2.8 Mellin transform method: probability density function

We use the Mellin transform method of the probability density function in section 2.6.2 to calculate the renormalized $n$-th moment of the Levy distribution.
Substituting the probability density function of the Levy distribution, Eq. (3.30) into Eq. (2.48), we obtain

\[ m_n^{\text{ren}} = M_{n+1} [p(x)] = \frac{1}{2^n \sqrt{\pi}} \Gamma \left( \frac{1}{2} - n \right). \] (3.55)

### 3.2.9 Power-logarithmic moment method

We use the power-logarithmic moment method in section 2.7 to calculate the renormalized \( n \)th moment of the Levy distribution. The power-logarithmic moment of the Levy distribution, by Eq. (2.52), is

\[ M_{n-1} = \int_{-\infty}^{\infty} e^{-\frac{1}{2x^2}} \frac{1}{\sqrt{2\pi}} x^{n-1} \ln x dx \]

\[ = -\frac{1}{\sqrt{\pi}} 2^{1-n} \Gamma \left( \frac{3}{2} - n \right) \left( \psi^{(0)} \left( \frac{3}{2} - n \right) + \ln 2 \right), \] (3.56)

where \( \psi^{(n)} (z) \) is the \( n \)th derivative of the digamma function, \( \psi^{(n)} (z) = \frac{d^n \psi(z)}{dz^n} \), and \( \psi (z) \) is the logarithmic derivative of the gamma function, \( \psi (z) = \frac{\Gamma'(z)}{\Gamma(z)} \) [36]. Substituting Eq. (3.56) into the relation between the \( n \)th moment and the power-logarithmic moment \( M_{n-1} \), Eq. (2.57), we have

\[ m_n^{\text{ren}} = \frac{1}{2^n \sqrt{\pi}} \Gamma \left( \frac{1}{2} - n \right). \] (3.57)

The renormalized \( n \)th moment given by the different renormalization schemes is the same. This verifies that the renormalization treatment is scheme-independent.

### 3.3 \( q \)-exponential distribution

The \( q \)-exponential distribution has power moments for some values of the parameter \( q \) and no power moments for others. We only need to renormalize the case having no power moment. The self-consistency requires that the renormalized power moment be valid for the \( q \) with which the power moment is well-defined. That is, the renormalized \( n \)th moment is valid for all \( q \).

The probability density function of the \( q \)-exponential distribution is [37]

\[ p(x) = \begin{cases} \lambda(2 - q)(1 - \lambda(1 - q)x)^{1-q}, & q < 1, \ 0 \leq x < \frac{1}{\lambda(1-q)}, \\ \lambda e^{-\lambda x}, & q = 1, \ x \geq 0, \\ \lambda(2 - q)(1 - \lambda(1 - q)x)^{1-q}, & q > 1, \ x \geq 0. \end{cases} \] (3.58)

For \( q = 1 \), the \( q \)-exponential distribution returns to the exponential distribution.

The \( n \)th moment of the \( q \)-exponential distribution, by definition (1.2), is

\[ m_n = \int_0^\infty \lambda(2 - q) [1 - \lambda(1 - q)x]^{1-q} x^n dx. \] (3.59)

For \( q \leq 1 \), the \( n \)th moment \( m_n \) exists. For \( q > 1 \), the \( n \)th moment \( m_n \) exists when \( q < \frac{n+2}{n+1} \). For example, only for \( q < \frac{3}{2} \), there exists the first moment, \( m_1 = \frac{1}{3\lambda(1-2q)} \); only for
\( q < \frac{4}{3}, \) there exists the second moment, \( m_2 = \frac{2}{\lambda^2(6q^2 - 17q + 12)}; \) only for \( q < \frac{5}{4}, \) there exists the third moment, \( m_3 = -\frac{6}{\lambda^3(24q^3 - 98q^2 + 135q - 60)}; \) only for \( q < \frac{6}{5}, \) there exists the fourth moment, \( m_4 = \frac{24\Gamma \left( \frac{1}{q} - 1 \right)}{\lambda^4q(q-1)^4}; \) and so on; otherwise the \( n \)-th moment does not exist.

Next, we calculate the renormalized \( n \)-th moment for the case having no power moments.

### 3.3.1 Zeta function method

We first use the zeta function method in section 2.1 to calculate the renormalized \( n \)-th moment.

The spectral zeta function of the \( q \)-exponential distribution, by (2.2), is

\[
\zeta(s) = \int_0^\infty \lambda(2 - q)(1 - \lambda(1 - q)x)^{-s}x^{-s}dx = \Gamma(1 - s)\Gamma \left( \frac{-s - q(1 - s) + 2}{q - 1} \right) = \Gamma \left( \frac{1}{q - 1} - 1 \right) |\lambda(q - 1)|^s. \tag{3.60}
\]

The \( n \)-th moment of the \( q \)-exponential distribution, by Eq. (2.4), is

\[
m_{\text{ren}}^n = \zeta(-n) = \frac{\Gamma(n + 1)\Gamma \left( -n + \frac{1}{q - 1} - 1 \right)}{\Gamma \left( \frac{1}{q - 1} - 1 \right) |\lambda(q - 1)|^n}. \tag{3.61}
\]

For integer \( n \), \( m_{\text{ren}}^n \) can be written as

\[
m_{\text{ren}}^n = (-1)^n \frac{n!}{\lambda^n \prod_{l=1}^{n} [(l + 1)q - (l + 2)]}. \tag{3.62}
\]

Thus, the first several moments are

\[
m_{\text{ren}}^1 = \zeta(-1) = -\frac{1}{\lambda(2q - 3)},
\]
\[
m_{\text{ren}}^2 = \zeta(-2) = \frac{2}{\lambda^2(2q - 3)(3q - 4)},
\]
\[
m_{\text{ren}}^3 = \zeta(-3) = -\frac{6}{\lambda^3(2q - 3)(3q - 4)(4q - 5)},
\]
\[
m_{\text{ren}}^4 = \zeta(-4) = \frac{24}{\lambda^4(2q - 3)(3q - 4)(4q - 5)(5q - 6)}. \tag{3.63}
\]

### 3.3.2 Subtraction method

We use the subtraction method in section 2.2 to calculate the renormalized \( n \)-th moment of the \( q \)-exponential distribution.
The $n$-th moment of the $q$-exponential distribution (3.59) diverges at $x \to \infty$. Expand the integrand at $x \to \infty$:

\[
\lambda(2-q)[1+\lambda(q-1)x]^{\frac{1}{1-q}}x^n = \left[\frac{1}{x^\frac{1}{1-q}} + \lambda(q-1)\right]^\frac{1}{1-q}x^n
\]

\[
= \lambda(2-q)\sum_{j=0}^{\infty} \left(\frac{1}{j}\right) [\lambda(q-1)]^{\frac{1}{1-q}-j} \frac{1}{x^j-n+\frac{1}{q-1}} \quad (3.64)
\]

where \(\binom{m}{n} = \frac{\Gamma(m+1)}{\Gamma(n+1)\Gamma(m-n+1)}\) is the binomial expansion coefficient. Following section 2.2.2, divergence is caused by the term $j-n+\frac{1}{q-1} \leq 1$, i.e.,

\[
j \leq 1+n-\frac{1}{q-1}. \quad (3.65)
\]

By Eq. (2.17), we arrive at the renormalized $n$-th moment,

\[
m_n^{\text{ren}} = \int_0^1 \lambda(2-q)[1+\lambda(q-1)x]^{\frac{1}{1-q}}x^n \, dx + \int_1^\infty \left[\lambda(2-q)[1+\lambda(q-1)x]^{\frac{1}{1-q}}x^n \right. \left. - \lambda(2-q)\sum_{j=0}^{j\leq1+n-\frac{1}{q-1}} \left(\frac{1}{j}\right) [\lambda(q-1)]^{\frac{1}{1-q}-j} \frac{1}{x^j-n+\frac{1}{q-1}} \right] \, dx
\]

\[
- \lambda(2-q)\sum_{j=0}^{j\leq1+n-\frac{1}{q-1}} \left(\frac{1}{j}\right) \frac{[\lambda(q-1)]^{\frac{1}{1-q}-j}}{1-(j-n+\frac{1}{q-1})} \quad (3.66)
\]

This gives

\[
m_n^{\text{ren}} = (2-q)\lambda \Gamma(n+1) \quad \tilde{F}_1 \left( n+1, \frac{1}{q-1}, n+2, -\lambda(q-1) \right)
\]

\[
+ (-1)^{-\frac{1}{1-q}} \frac{(q-2)}{\lambda^n (1-q)^n} B \left( \frac{1}{\lambda(1-q)} \frac{1}{q-1} - n - 1, \frac{q-2}{q-1} \right), \quad (3.67)
\]

where $\tilde{F}_1 (\alpha, \beta, \gamma, z)$ is the regularized hypergeometric function and $B (\alpha, \beta, z)$ is the Beta function.

For example, the first-order moment, by (3.67), is

\[
m_1^{\text{ren}} = \frac{1}{(3-2q)\lambda}. \quad (3.68)
\]

3.3.3 Weight function method

We use the weighted moment method in section 2.3 to calculate the renormalized $n$-th moment of the $q$-exponential distribution. In order to exemplify that the renormalized moment is independent of the choice of weight functions, we choose two weight functions.

Taking the weight function as

\[
g_s (x) = e^{-sx} \quad (3.69)
\]
gives the weighted moment
\[ m_n(s) = \int_0^\infty \lambda (2 - q)(1 - \lambda (1 - q)x)^{1/q} x^n e^{-sx} \, dx \]
\[ = \frac{\Gamma(n + 1) \Gamma\left(\frac{1}{q-1} - n - 1\right)}{\Gamma\left(\frac{1}{q-1} - 1\right)} \left[ \lambda(q - 1) \right]^{n} \Gamma\left(\frac{1}{q-1} - 1\right) \]
\[ - \lambda(q - 2) \left[ \lambda(q - 1) \right]^{\frac{1}{q-1} - 1} \Gamma\left(\frac{n + 1 + \frac{1}{q} - 1}{q - 1} - n; \frac{s}{(q - 1)\lambda}\right). \]

According to Eq. (2.34), we expand the weighted moment \( m_n(s) \) at \( s = 0 \):
\[ m_n(s) = \frac{\Gamma(n + 1) \Gamma\left(-n + \frac{1}{q-1} - 1\right)}{\Gamma\left(\frac{1}{q-1} - 1\right)} \left[ \lambda(q - 1) \right]^{n} \]
\[ + s^{\frac{1}{q-1} - 1} \Gamma\left(\frac{n + 1 - \frac{1}{q-1}}{q - 1} - n\right) \left[ \lambda(q - 1) \right]^{\frac{1}{q-1} - 1} \Gamma\left(\frac{1}{q - 1} + n + 1\right) \]
\[ + \frac{(n + 1)s \Gamma(n + 1) \Gamma\left(\frac{1}{q-1} - 1 - n\right)}{\left[ \lambda(q - 1) \right]^{n} \lambda(nq - n + 2q - 3) \Gamma\left(\frac{1}{q-1} - 1\right)} + \cdots, \]

where the zero-order term is the renormalized moment:
\[ m_n^{\text{ren}} = \frac{\Gamma(n + 1) \Gamma\left(-n + \frac{1}{q-1} - 1\right)}{\Gamma\left(\frac{1}{q-1} - 1\right)} \left[ \lambda(q - 1) \right]^{n}. \]

### 3.3.4 Weight function method (2)

Taking another weight function as
\[ g(x) = x^{s-1} \]
the weight moment is
\[ m_n(s) = \int_0^\infty \lambda (2 - q)[1 - \lambda (1 - q)x]^{1/q} x^n x^{s-1} \, dx \]
\[ = \frac{\Gamma(n + s) \Gamma\left(\frac{1}{q-1} - n - s\right)}{\Gamma\left(\frac{1}{q-1} - 1\right) \left[ \lambda(q - 1) \right]^{n+s-1}.} \]

The weighted moment at \( s = 1 \) is the renormalized \( n \)-th moment of the \( q \)-exponential distribution:
\[ m_n^{\text{ren}} = m_n(1) = \frac{\Gamma(n + 1) \Gamma\left(\frac{1}{q-1} - n - 1\right)}{\Gamma\left(\frac{1}{q-1} - 1\right) \left[ \lambda(q - 1) \right]^{n}}. \]
3.3.5 Cut-off method

We use the cut-off method in section 2.4 to calculate the renormalized $n$-th moment of the $q$-exponential distribution.

Cutting off the upper integration limit of the $n$-th moment gives

$$ m_n (\Lambda) = \int_0^\Lambda \lambda (2 - q) [1 - \lambda (1 - q)x] \frac{1}{1 - q} x^n dx. \quad (3.76) $$

When $\Lambda \to \infty$, we obtain the $n$-th moment: $m_n = m_n (\infty)$. Working out the integral in Eq. (3.76) gives

$$ m_n (\Lambda) = \lambda (2 - q) \Lambda^{n+1} \Gamma (n + 1) \tilde{F}_1 \left( n + 1, \frac{1}{q - 1}; n + 2; (1 - q)\lambda \Lambda \right). \quad (3.77) $$

In order to obtain the renormalized $n$-th moment, we expand $m_n (\Lambda)$ at $\Lambda \to \infty$:

$$ m_n^{\text{ren}} = \frac{\Gamma (n + 1) \Gamma \left( \frac{1}{q - 1} - 1 - n \right)}{\Gamma \left( \frac{1}{q - 1} - 1 \right) \left[ \lambda (q - 1) \right]^n} \frac{\lambda (q - 2) \left[ \lambda (q - 1) \right]^{n+1}}{\left[ n (q - 1) - 1 \right] \Gamma \left( 1 + \frac{q - 1}{n (q - 1) - 1} \right)} - \ldots. \quad (3.78) $$

Dropping the divergent terms at $\Lambda \to \infty$, we obtain the renormalized $n$-th moment:

$$ m_n^{\text{ren}} = \frac{\Gamma (n + 1) \Gamma \left( \frac{1}{q - 1} - 1 - n \right)}{\Gamma \left( \frac{1}{q - 1} - 1 \right) \left[ \lambda (q - 1) \right]^n}. \quad (3.79) $$

3.3.6 Characteristic function method

We use the characteristic function method in section 2.5 to calculate the renormalized $n$-th moment of the $q$-exponential distribution.

The characteristic function of the $q$-exponential distribution, by Eq. (2.37), is

$$ f \left( k \right) = -\frac{q - 2}{q - 1} e^{\frac{ik}{\lambda - q \lambda}} \frac{\Gamma \left( i k \right)}{\Gamma \left( \frac{q - 1}{\lambda - q \lambda} \right)}, \quad (3.80) $$

where $E_n (z) = \int_1^\infty \frac{e^{-zt}}{t^n} dt$ is the exponential integral function.

By the integral representation of the derivation, Eq. (2.42), we obtain

$$ \frac{d^n f \left( k \right)}{dk^n} = -\frac{q - 2}{q - 1} k^{n-1} \left[ k \Gamma \left( 1 + \frac{1}{1 - q} \right) 2 \tilde{F}_2 \left( 1, 1 - n, 2 + \frac{1}{1 - q}; \frac{i k}{\lambda - q \lambda} \right) - i \lambda (q - 2) \Gamma \left( \frac{1}{q - 1} - 1 \right) \Gamma \left( \frac{q - 2}{q - 1} \right) \left( \frac{i k}{\lambda - q \lambda} \right)^{n+1} \tilde{F}_1 \left( \frac{1}{q - 1}; \frac{1}{q - 1} - n; \frac{i k}{\lambda - q \lambda} \right) \right], \quad (3.81) $$

where $_p \tilde{F}_q \left( a; b; z \right)$ is regularized hypergeometric function, given by $_p \tilde{F}_q \left( a; b; z \right) / \left( \Gamma (b_1) \cdots \Gamma (b_q) \right)$. 

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By (2.41), the renormalized $n$-th moment is
\[
m_{1}^{{\text{ren}}} = (-i) \left. \frac{df(k)}{dk} \right|_{k=0} = -\frac{1}{\lambda(2q-3)},
\]
\[
m_{2}^{{\text{ren}}} = (-i)^{2} \left. \frac{d^{2}f(k)}{dk^{2}} \right|_{k=0} = \frac{2}{\lambda^{2}(2q-3)(3q-4)},
\]
\[\ldots .\] (3.82)

3.3.7 Mellin transform method: characteristic function

We use the Mellin transform method of the characteristic function in section 2.6.1 to calculate the renormalized $n$-th moment of the $q$-exponential distribution.

Substituting the characteristic function of the $q$-exponential distribution (3.80) into Eq. (2.46), we obtain
\[
m_{n}^{{\text{ren}}} = i^{n} \frac{\Gamma(n)}{\Gamma(-n)} \mathcal{M}_{-n}[f(k)]
= -\frac{\pi \csc(\pi n)}{\Gamma(-n)} \frac{\Gamma\left( \frac{1}{q-1} - n - 1 \right)}{\Gamma\left( \frac{1}{q-1} - 1 \right)} [\lambda(q-1)]^{n}.
\]
By $\Gamma(n+1) = -\pi \csc(\pi n) / \Gamma(-n)$, we have
\[
m_{n}^{{\text{ren}}} = \frac{\Gamma(n+1)\Gamma\left( \frac{1}{q-1} - n - 1 \right)}{\Gamma\left( \frac{1}{q-1} - 1 \right)} [\lambda(q-1)]^{n}.
\] (3.84)

3.3.8 Mellin transform method: probability density function

We use the Mellin transform method of the probability density function in section 2.6.2 to calculate the renormalized $n$-th moment of the $q$-exponential distribution.

Substituting the probability density function of the $q$-exponential distribution, Eq. (3.58), into Eq. (2.48), we obtain
\[
m_{n}^{{\text{ren}}} = \mathcal{M}_{n+1}[p(x)] = \frac{\Gamma(n+1)\Gamma\left( \frac{1}{q-1} - n - 1 \right)}{\Gamma\left( \frac{1}{q-1} - 1 \right)} [\lambda(q-1)]^{n}.
\] (3.85)

3.3.9 Power-logarithmic moment method

We use the power-logarithmic moment method in section 2.7 to calculate the renormalized $n$-th moment of the $q$-exponential distribution.

The power-logarithmic moment of the $q$-exponential distribution, by Eq. (2.52), is
\[
M_{n-1} = \int_{0}^{\infty} \lambda(2-q)(1-\lambda(1-q)x)^{\frac{1}{q-1}} x^{n-1} \ln x dx
= \frac{\lambda(q-2)\Gamma(n)\Gamma\left( \frac{1}{q-1} - n \right) \left[ \psi^{(0)}\left( \frac{1}{q-1} - n \right) - \psi^{(0)}(n) + \ln(\lambda(q-1)) \right]}{\Gamma\left( \frac{1}{q-1} \right)} [\lambda(q-1)]^{n}.
\] (3.86)
By the relation between the \( n \)-th moment \( m_n \) and the power-logarithmic moment \( M_{n-1} \), Eq. (2.57), we have

\[
m^n_{\text{ren}} = \frac{\Gamma(n + 1)\Gamma\left(\frac{1}{q-1} - n - 1\right)}{\Gamma\left(\frac{1}{q-1} - 1\right) [\lambda(q - 1)]^n}.
\] (3.87)

### 3.3.10 Removing singularity from renormalized \( n \)-th moment

Before renormalization, the \( n \)-th moment (3.59) of the \( q \)-exponential distribution (3.58) only exists when

\[
q < \frac{n + 2}{n + 1}.
\] (3.88)

After renormalization, we obtain the renormalized \( n \)-th moment (3.61). However, the renormalized moment (3.61) still has singularities at

\[
q = 1,
\]

\[
q = \frac{n + 2}{n + 1}, \quad n \geq 1.
\] (3.90)

Here \( q = 1 \) is the removable singularity and \( q = \frac{n + 2}{n + 1} \) is the essential singularity.

\( q = 1 \) is a removable singularity. The zero-order term of the expansion at \( q = 1 \) of the renormalized moment (3.61) (i.e., the limit of \( q \to 1 \)) is the renormalized \( n \)-th moment:

\[
m^n_{\text{ren}} (q = 1) = \frac{1}{\lambda^n} \Gamma(n + 1).
\] (3.91)

When \( q = 1 \), the \( q \)-exponential distribution returns to the exponential distribution \( p(x) = \lambda e^{-\lambda x} \). Eq. (3.91) just gives the \( n \)-th moment of the exponential distribution.

For the moment at the essential singularity \( q = \frac{n + 2}{n + 1} \), we need further renormalization. Expanding the renormalized \( n \)-th moment (3.61) at \( q = \frac{n + 2}{n + 1} \), we arrive at

\[
m^n_{\text{ren}} = \frac{1}{\lambda^n} \left( \frac{1}{1 + n} \right)^{2-n} \frac{1}{q - \frac{n + 2}{n + 1}} \Gamma(n + 1)
\]

\[
+ \frac{n}{\lambda^n} \left( \frac{1}{n + 1} \right)^{1-n} \left[ (n + 1) \left( \psi^{(0)}(n) + \gamma_E \right) + 1 - n \right] + \cdots,
\] (3.92)

where \( \gamma_E = 0.577216 \) is Euler’s constant. By the minimal subtraction, dropping the divergent terms at \( q = \frac{n + 2}{n + 1} \), we obtain the renormalized \( n \)-th moment

\[
m^n_{\text{ren}} \bigg|_{q = \frac{n + 2}{n + 1}} = \frac{n}{\lambda^n} \left( \frac{1}{n + 1} \right)^{1-n} \left[ (n + 1) \left( \psi^{(0)}(n) + \gamma_E \right) + 1 - n \right].
\] (3.93)

### 3.4 \( q \)-Gaussian distribution

The probability density function of the \( q \)-Gaussian distribution is [18]

\[
p(x) = \begin{cases} 
\frac{\sqrt{1-q}}{2\pi} \left[ \frac{(q-1)(\mu-x)^2}{2\beta^2} + 1 \right]^{\frac{1}{q-2}} \frac{1}{\Gamma\left(\frac{1}{q-1} + \frac{3}{2}\right)} & , \quad q < 1, \quad -1 < \sqrt{1-q} \sqrt{\frac{2}{\beta^2}} (x - \mu) < 1, \\
\frac{1}{\sqrt{2\pi} \beta} \exp \left( -\frac{(x-\mu)^2}{2\beta^2} \right) , & , \quad q = 1, \\
\frac{\sqrt{1-q}}{2\pi} \left[ \frac{(q-1)(\mu-x)^2}{2\beta^2} + 1 \right]^{\frac{1}{q}} \frac{1}{\Gamma\left(\frac{1}{q-1} + 1\right)} , & , \quad 1 < q < 3, \\
0, & , \quad q > 3.
\end{cases}
\] (3.94)
When \( q = 1 \), the \( q \)-Gaussian distribution recovers the Gaussian distribution, and the \( n \)-th moment of the Gaussian distribution exists. When \( q = 2 \), the \( q \)-Gaussian distribution recovers the Cauchy distribution, and though the \( n \)-th moment of the Cauchy distribution does not exist, we have obtained the renormalized \( n \)-th moment in section 3.1. Thus, what we need to renormalize is the \( n \)-th moment of \( 1 < q < 3 \),

\[
m_n = \int_{-\infty}^{\infty} \frac{\sqrt{q-1}}{\sqrt{2\pi}} \left[ (q-1)(\mu-x)^2 + 1 \right]^{\frac{1}{q-1}} \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1} + \frac{1}{2}\right)} x^n dx. \quad (3.95)
\]

The first several moments of the \( q \)-Gaussian distribution, by Eq. (3.94), is

\[
m_1 = \begin{cases} 
\mu, & 1 < q < 2 \\
none, & 2 < q < 3 
\end{cases}, \quad m_2 = \begin{cases} 
\mu^2 + \frac{2\beta^2}{3-3q}, & 3q < 5 \\
none, & \frac{3}{2} < q < 3 
\end{cases}, \quad m_3 = \begin{cases} 
\mu^3 + \frac{6\beta^2\mu}{3-3q}, & 2q < 3 \\
none, & \frac{3}{2} < q < 3 
\end{cases}, \quad m_4 = \begin{cases} 
\mu^4 + \frac{12\beta^4}{15q^2-45q+35} + \frac{12\beta^2\mu^2}{5-5q}, & 5q < 7 \\
none, & \frac{7}{5} < q < 3 
\end{cases}. \quad (3.96)
\]

It can be seen that when \( q > \frac{n+3}{n+1} \), the \( n \)-th moment does not exist. Next, we use the renormalization method to give renormalized moments for \( q > \frac{n+3}{n+1} \).

### 3.4.1 Zeta function method

For simplicity, we only consider \( \mu = 0 \).

The spectral zeta function of the \( q \)-Gaussian distribution of \( 1 < q < 3 \), given by Eqs. (3.94) and (2.2), is

\[
\zeta(s) = \int_{-\infty}^{\infty} \frac{\sqrt{q-1}}{\sqrt{2\pi}} \left[ (q-1)(\frac{1}{q-1}) \right]^{\frac{1}{q-1}} \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)} x^{-s} dx \\
= \frac{[(-1)^{-s} + 1] (q-1)^{s/2} \Gamma\left(\frac{1-s}{2}\right) \Gamma\left(\frac{1}{q-1} - \frac{1-s}{2}\right) \Gamma\left(\frac{1}{q-1} + \frac{1-s}{2}\right)}{2^{s/2+1} \sqrt{\pi} \beta^s \Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}. \quad (3.97)
\]

We are concerned about the \( n \)-th moment. By Eq. (2.4), the renormalized \( n \)-th moment of the \( q \)-Gaussian distribution is

\[
m_n^{\text{ren}} = \zeta(-n) = \frac{2^{n/2-1} [(-1)^n + 1] \beta^n \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{1}{q-1} - \frac{1+n}{2}\right)}{\sqrt{\pi} (q-1)^{n/2} \Gamma\left(\frac{1}{q-1} - \frac{n}{2}\right)}. \quad (3.98)
\]
Thus, the first several renormalized moments are

\[
\begin{align*}
  m_{1}^{\text{ren}} &= \zeta (-1) = 0, \\
  m_{2}^{\text{ren}} &= \zeta (-2) = -\frac{2\beta^2}{5 - 3q}, \\
  m_{3}^{\text{ren}} &= \zeta (-3) = 0, \\
  m_{4}^{\text{ren}} &= \zeta (-4) = \frac{12\beta^4}{15q^2 - 46q + 35}.
\end{align*}
\] (3.99)

### 3.4.2 Subtraction method

The \(n\)-th moment of the \(q\)-Gaussian distribution exists at \(q \leq 1\), but it does not exist at \(1 < q < 3\) and \(q > \frac{n+3}{n+1}\). Thus, we only need to deal with the latter. We use the subtraction method in section 2.2.2 to calculate the renormalized \(n\)-th moment. For simplicity, we only consider \(\mu = 0\).

The \(n\)-th moment of the \(q\)-Gaussian distribution (3.59) diverges at \(x \to \pm \infty\). Expand the integrand at \(x \to \pm \infty\):

\[
\frac{\sqrt{q-q}\Gamma(\frac{1}{q-1})}{2\pi\beta^2(\frac{3-q}{2(q-1)})} x^{\frac{q}{2} - \frac{1}{2}} \left( \frac{q - 1}{2\beta^2} + \frac{1}{x^2} \right)^{\frac{1}{q-1}}
\]

\[
= \frac{\sqrt{q-q}\Gamma(\frac{1}{q-1})}{2\pi\beta^2(\frac{3-q}{2(q-1)})} \sum_{j=0}^{\infty} \left( \frac{1}{j} \right) \left( \frac{q - 1}{2\beta^2} \right)^{\frac{1}{q-1} - j} \left( \frac{1}{x^2} \right)^{j + \frac{1}{q-1}} \frac{1}{x^n}.
\] (3.100)

where \(\binom{m}{n} = \frac{\Gamma(m+1)}{\Gamma(n+1)\Gamma(m-n+1)}\) is the binomial expansion coefficient. Following section 2.2.2, the divergence is caused by the term \(2j - n + \frac{2}{q-1} \leq 1\), i.e.,

\[
j \leq 1 + \frac{n}{2} - \frac{1}{q-1}.
\] (3.101)

By Eq. (2.17), we obtain the renormalized \(n\)-th moment

\[
m_{n}^{\text{ren}} = \int_{-1}^{1} f_{q}(x) \, dx + \int_{-\infty}^{-1} \left[ f_{q}(x) - \sum_{j=0}^{\frac{q}{2} + \frac{n}{q-1}} c_{j} \left( \frac{1}{x^2} \right)^{j + \frac{1}{q-1}} \frac{1}{x^n} \right] \, dx
\]

\[+ \sum_{j=0}^{\frac{q}{2} + \frac{n}{q-1}} (-1)^{-n-1} c_{j} \frac{1}{1 - (2j - n + \frac{2}{q-1})} + i\pi c_{-1} + \int_{1}^{\infty} \left[ f_{q}(x) - \sum_{j=0}^{\frac{q}{2} + \frac{n}{q-1}} c_{j} \left( \frac{1}{x^2} \right)^{j + \frac{1}{q-1}} \frac{1}{x^n} \right] \, dx
\]

\[+ \sum_{j=0}^{\frac{q}{2} + \frac{n}{q-1}} c_{j} \frac{1}{1 - (2j - n + \frac{2}{q-1})};
\] (3.102)

\[\text{– 31 –}
\]
where

\[
f_q(x) = \frac{\sqrt{q - 1} \Gamma \left( \frac{1}{q-1} \right) \frac{(q-1)x^2}{2\beta^2} + 1}{\sqrt{2\pi}\beta \Gamma \left( \frac{3-q}{2(q-1)} \right)} x^n,
\]

\[
c_j = \frac{\sqrt{q - 1} \Gamma \left( \frac{1}{q-1} \right) \frac{1}{j} \left( \frac{q - 1}{2\beta^2} \right)^{\frac{1}{q-1}-j}}{\sqrt{2\pi}\beta \Gamma \left( \frac{3-q}{2(q-1)} \right)}.
\]  

(3.104)

Working out the integral gives

\[
m_{\text{ren}}^n = \frac{2}{\sqrt{2\pi}} \left[ (\frac{-1}{q})^n + 1 \right] \sqrt{q - 1} \Gamma \left( \frac{1}{q-1} \right) 2F_1 \left( \frac{n+1}{2}, \frac{1}{q-1}; \frac{n+3}{2}, \frac{1-q}{2\beta^2} \right)
\]

\[
- \frac{2\beta^2}{q - 3 + n(q-1)} 2F_1 \left( -\frac{1}{2} - \frac{n}{2}, -\frac{1}{2}; -\frac{1}{2} - \frac{n}{2}, -\frac{1}{2}; \frac{\Lambda^2}{2\pi^2} \frac{2\beta^2}{q-1} \right) \right].
\]  

(3.105)

It can be verified that this renormalized $n$-th moment is the same as that given by the zeta function method.

From Eq. (3.105), for example, the first and second moments are

\[
m_{\text{ren}}^1 = 0,
\]

\[
m_{\text{ren}}^2 = -\frac{2\beta^2}{5 - 3q}.
\]  

(3.106)

### 3.4.3 Cut-off method

We use the cut-off method in section 2.4 to calculate the renormalized $n$-th moment of the $q$-Gaussian distribution.

Cutting off the upper integration limit of the $n$-th moment gives

\[
m_n(\Lambda) = \int_{-\Lambda}^{\Lambda} \frac{\sqrt{q - 1} \Gamma \left( \frac{1}{q-1} \right) \frac{(q-1)x^2}{2\beta^2} + 1}{\sqrt{2\pi}\beta \Gamma \left( \frac{3-q}{2(q-1)} \right)} x^n dx.
\]  

(3.107)

When $\Lambda \to \infty$, we obtain the $n$-th moment: $m_n = m_n(\infty)$. Working out the integral in Eq. (3.107) gives

\[
m_n(\Lambda) = \frac{\Lambda \sqrt{q - 1} \Gamma(\Lambda^n + (-\Lambda)^n) \Gamma \left( \frac{1}{q-1} \right) \frac{n+1}{2}, \frac{1}{q-1}; \frac{n+3}{2}, \frac{-\frac{(q-1)\Lambda^2}{2\beta^2}}{q-1} - \frac{1}{2}}{\sqrt{2\pi}\beta(n+1) \Gamma \left( \frac{1}{q-1} \right) - \frac{1}{2}}.
\]  

(3.108)
In order to obtain the renormalized $n$-th moment, we expand $m_n(\Lambda)$ at $\Lambda \to \infty$:

$$m_n(\Lambda) = \frac{2^{n/2-1}(-1)^n + 1}{\sqrt{\pi} \Gamma \left( \frac{1}{q-1} - \frac{1}{2} \right)} \left( q - 1 \right)^{n/2}$$

$$+ \Lambda^{n+1-\frac{2}{q-1}} \frac{2^{1-\frac{1}{q-1}} [1 + (-1)^n] \Gamma \left( \frac{1}{q-1} \right) \beta^{2-q} \left( \Gamma \left( \frac{3}{2} \right) - \Gamma \left( \frac{1}{q-1} \right) \right)}{(q - 1)^{3/2(q-1)} (n-1) [n(q-1) + q - 3]} 2^{\sqrt{\pi} \Gamma \left( \frac{1}{q-1} - \frac{1}{2} \right)} + \ldots.$$ (3.109)

Dropping the divergent terms at $\Lambda \to \infty$, we obtain the renormalized $n$-th moment,

$$m^n_{\text{ren}} = \frac{2^{n/2-1}(-1)^n + 1}{\sqrt{\pi} \Gamma \left( \frac{1}{q-1} - \frac{1}{2} \right)} \left( q - 1 \right)^{n/2}$$ (3.110)

### 3.4.4 Characteristic function method

We use the characteristic function method in section 2.5 to calculate the renormalized $n$-th moment of the $q$-Gaussian distribution.

The characteristic function of the $q$-Gaussian distribution, by Eq. (2.37), is

$$f(k) = 0 F_1 \left( \frac{3}{2} - \frac{1}{q-1}, \cdot \frac{k^2 \beta^2 q}{2(q-1)} \right) - \frac{\beta^{3-q} (q-1)^{3q-5}}{2^{\frac{3-q}{2}}} \Gamma \left( \frac{3}{2} + \frac{1}{q-1} \right)$$

$$\times \Gamma \left( \frac{q}{q-1} \right) \left( \sqrt{\frac{2}{\pi}} - \frac{\beta^{3-q}}{2^{\frac{3-q}{2}} \sqrt{\pi}} \right) 0 F_1 \left( \frac{1}{2} + \frac{1}{q-1}, \cdot \frac{k^2 \beta^2 q}{2(q-1)} \right) \frac{3}{q-1}.$$ (3.111)

By the integral representation of the derivation, Eq. (2.42), we obtain

$$\frac{d^n f(k)}{dk^n} = 2^n \sqrt{\pi} \tilde{F}_3 \left( \frac{3}{2}, 1; \frac{1-n}{2}, 1 - \frac{n}{2}; \frac{3}{2} + 1 - q \right) \frac{k^n - q^n}{2(q-1)}$$

$$\times \frac{\Gamma \left( \frac{q}{q-1} \right) \left( \sqrt{\frac{2}{\pi}} - \frac{\beta^{3-q}}{2^{\frac{3-q}{2}} \sqrt{\pi}} \right) \Gamma \left( \frac{3}{2} + \frac{1}{q-1} \right)}{\Gamma \left( \frac{q}{q-1} \right)} \frac{3}{q-1}.$$ (3.112)

$q = \frac{2}{3}$ and $q = \frac{4}{3}$ are not singularities, and the moment can be obtained directly from Eq. (3.112): at $q = \frac{2}{3}$

$$m^n_{\text{ren}} = (-i) \left. \frac{df(k)}{dk} \right|_{k=0} = 0;$$ (3.113)

at $q = \frac{4}{3}$

$$m^n_{\text{ren}} = (-i) \left. \frac{d^2 f(k)}{dk^2} \right|_{k=0} = -2 \beta^2.$$ (3.114)

This means that $q = \frac{2}{3}$ and $q = \frac{4}{3}$ are not true singularities, and the renormalized $n$-th moment can be obtained by analytical continuation directly.
$q = 2$ and $q = \frac{5}{3}$ are singularities, which need to be renormalized. For $q = 2$,

$$m^\text{ren}_1 = (-i) \frac{df(k)}{dk} \bigg|_{k=0} = i \sqrt{2} \beta; \quad (3.115)$$

for $q = \frac{5}{3}$,

$$m^\text{ren}_2 = (-i)^2 \frac{d^2 f(k)}{dk^2} \bigg|_{k=0} = 3 \beta^2 \left( \ln \frac{2}{\sqrt{3} \beta} - 1 - i \pi \frac{1}{2} \right). \quad (3.116)$$

This means that $q = 2$ and $q = \frac{5}{3}$ are true singularities, and the finite moments cannot be obtained through analytical continuation.

**3.4.5 Mellin transform method: characteristic function**

We use the Mellin transform method of the characteristic function in section 2.6.1 to calculate the renormalized $n$-th moment of the $q$-Gaussian distribution.

Substituting the characteristic function of the $q$-Gaussian distribution (3.111) into Eq. (2.46) gives immediately

$$m^\text{ren}_n = \frac{i^n}{\Gamma(-n)} \mathcal{M}_{-n} [f(k)]$$

$$= \frac{2^{n/2-1}((-1)^n + 1) \beta^n \Gamma \left( \frac{n+1}{2} \right) \Gamma \left( \frac{1}{q-1} - \frac{1+n}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{1}{q-1} - \frac{1}{2} \right) (q-1)^{n/2}}. \quad (3.117)$$

**3.4.6 Mellin transform method: probability density function**

We use the Mellin transform method of the probability density function in section 2.6.2 to calculate the renormalized $n$-th moment of the $q$-Gaussian distribution.

Substituting the probability density function of the $q$-Gaussian distribution (3.94) into Eq. (2.48) gives

$$m^\text{ren}_n = (-1)^n \mathcal{M}_{n+1} [p(-x)] + \mathcal{M}_{n+1} [p(x)]$$

$$= \frac{2^{n/2-1}((-1)^n + 1) \beta^n \Gamma \left( \frac{n+1}{2} \right) \Gamma \left( \frac{1}{q-1} - \frac{1+n}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{1}{q-1} - \frac{1}{2} \right) (q-1)^{n/2}}. \quad (3.118)$$

**3.4.7 Removing singularity from renormalized $n$-th moment**

Before renormalization, the $n$-th moment (3.59) of the $q$-exponential distribution (3.94) only exists when

$$q < \frac{n+3}{n+1}. \quad (3.119)$$

After renormalization, we obtain the renormalized $n$-th moment (3.4). However, the renormalized moment (3.98) still has singularities at

$$q = 1, \quad (3.120)$$

$$q = \frac{n+3}{n+1}, \quad n \geq 1. \quad (3.121)$$
Here \( q = 1 \) is the removable singularity and \( q = \frac{n+3}{n+1} \) is the essential singularity. 

\( q = 1 \) is a removable singularity, the zero-order term of the expansion at \( q = 1 \) of the renormalized moment \((3.98)\) (i.e., the limit of \( q \to 1 \)) is the renormalized \( n \)-th moment:

\[
m_n^{\text{ren}} |_{q=1} = \frac{2^{\frac{n}{2}-1}}{\sqrt{\pi}} \left[ (-1)^n + 1 \right] \beta^n \Gamma \left( \frac{1}{2} + \frac{n}{2} \right). \tag{3.122}\]

When \( q = 1 \), the \( q \)-exponential distribution returns to the exponential distribution \( p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \). Eq. \((3.122)\) just gives the \( n \)-th moment of the normal distribution.

For the moment at the essential singularity \( q = \frac{n+3}{n+1} \), we need further renormalization. Expanding the renormalized \( n \)-th moment \((3.98)\) at \( q = \frac{n+3}{n+1} \), we arrive at

\[
m_n^{\text{ren}} |_{q=\frac{n+3}{n+1}} = \frac{2 \left[ (-1)^n + 1 \right] \left( \frac{n+1}{n+3} \right)^{2-n} \beta^n \Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi} \left( q - \frac{n+3}{n+1} \right)} \Gamma \left( \frac{n+1}{2} \right) \left[ (n+1) \left( \psi(0) \left( \frac{n}{2} \right) + \gamma_E \right) + 2 - n \right] + \cdots. \tag{3.123}\]

By the minimal subtraction, dropping the divergent terms at \( q = \frac{n+3}{n+1} \), we obtain the renormalized \( n \)-th moment:

\[
m_n^{\text{ren}} |_{q=\frac{n+3}{n+1}} = - \frac{\left[ (-1)^n + 1 \right] \left( \frac{n+1}{n+3} \right)^{1-n} \beta^n \Gamma \left( \frac{n+1}{2} \right) \left[ (n+1) \left( \psi(0) \left( \frac{n}{2} \right) + \gamma_E \right) + 2 - n \right]}{2 \sqrt{\pi} \Gamma \left( \frac{n}{2} \right)}. \tag{3.124}\]

It can be directly verified that this result is consistent with other renormalization methods.

4 Nonpositive integer power moment

The \( n \)-th moment is the positive integer power moment. The renormalization schemes are essentially based on analytical continuation. Moreover, the analytical continuation can give not only positive integer power moments but also real and even complex moments. For example, the complex moments of the Cauchy distribution, Levy distribution, \( q \)-exponential distribution, and \( q \)-Gaussian distribution are

\[
m_z^{\text{Cauchy}} = e^{i z \pi/2},
\]

\[
m_z^{\text{Levy}} = \frac{1}{2z \sqrt{\pi}} \Gamma \left( \frac{1}{2} - z \right),
\]

\[
m_z^{q\text{-exp}} = \frac{1}{[\lambda (q-1)]^{z^2}} \frac{\Gamma(z+1) \Gamma \left( \frac{1}{q-1} - 1 - z \right)}{\Gamma \left( \frac{1}{q-1} - 1 \right)},
\]

\[
m_z^{q\text{-Gauss}} = \frac{2z^{2-1} \left( 1 + e^{i \pi z} \right) \Gamma \left( \frac{z+1}{2} \right) \beta^z \Gamma \left( \frac{1}{q-1} - \frac{1+z}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{1}{q-1} - \frac{1}{2} \right) (q-1)^{z/2}}. \tag{4.1}\]
The renormalization treatment, besides giving moments for the distributions that have no moment, also gives the non-positive integer moment of the distribution that has the positive integer power moment.

Normal distribution \( p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \). The standard normal distribution does not have negative \( n \)-th moments. However, the complex moment after analytical continuation is

\[
m_z^{\text{Normal}} = \frac{2^{z/2-1}}{\sqrt{\pi}} \left(1 + e^{i\pi z}\right) \Gamma \left(\frac{z+1}{2}\right). \tag{4.2}
\]

\( m_z^{\text{Normal}} \) is ill-defined at negative integers. In order to obtain the negative \( n \)-th moment, we use the renormalization treatment. Expanding the complex moment (4.2) at \( z = -1, -2, -3, \) and \( 4 \), respectively, and dropping the divergent term, we obtain the following negative \( n \)-th moment:

\[
m_{-1}^{\text{Normal}} = -i \sqrt{\frac{\pi}{2}}, \quad m_{-2}^{\text{Normal}} = -1, \quad m_{-3}^{\text{Normal}} = i \frac{\sqrt{\pi}}{2}, \quad m_{-4}^{\text{Normal}} = \frac{1}{3} \ldots . \tag{4.3}
\]

Student’s t-distribution \( p(x) = \frac{\nu}{\sqrt{\nu\pi}B(\frac{\nu}{2}, \frac{1}{2})} x^{\nu/2-1} (1 + x^2/\nu)^{-\nu/2} \). The complex moment after analytical continuation is

\[
m_z = \frac{(-1)^z \left(1 + e^{-i\pi z}\right) \nu^{z/2} \Gamma \left(\frac{\nu+z}{2}\right) \Gamma \left(\frac{\nu-z}{2}\right)}{2 \sqrt{\pi} \Gamma \left(\frac{\nu}{2}\right)}. \tag{4.4}
\]

This result still can not give the negative \( n \)-th moment. Expanding the complex moment (4.4) at \( z = -1, -2, -3, \) and \( 4 \), respectively, and dropping the divergence term, we obtain the negative \( n \)-th moment:

\[
m_{-1}^{\text{T}} = -\frac{i\pi}{\sqrt{\nu} B \left(\frac{\nu}{2}, \frac{1}{2}\right)}, \quad m_{-2}^{\text{T}} = -1, \quad m_{-3}^{\text{T}} = i \sqrt{\frac{\pi}{\nu}} \Gamma \left(\frac{\nu+3}{2}\right) \Gamma \left(\frac{\nu-3}{2}\right), \quad m_{-4}^{\text{T}} = \frac{2+\nu}{3}, \ldots . \tag{4.5}
\]

Laplace distribution \( p(x) = \frac{\lambda}{2} e^{-\lambda|x-\mu|} \). For \( \mu = 0 \), the complex order moment after analytical continuation is

\[
m_z = \frac{\lambda e^{izx} + 1}{2\lambda^z} \Gamma(1+z). \tag{4.6}
\]

This result can not give the negative \( n \)-th moment of the Laplace distribution. Expanding the complex moment (4.4) at \( z = -1, -2, -3, \) and \( 4 \), respectively, and dropping the divergent term, we obtain the negative \( n \)-th moment,

\[
m_{-1}^{\text{Laplace}} = -\frac{i\pi}{2\lambda}, \quad m_{-2}^{\text{Laplace}} = \lambda^2 \left(\ln \lambda - i\frac{\pi}{2} + \gamma_E - 1\right),
\]

\[
m_{-3}^{\text{Laplace}} = -\frac{i\pi}{4\lambda^3}, \quad m_{-4}^{\text{Laplace}} = \frac{\lambda^4}{6} \left(\ln \lambda - i\frac{\pi}{2} + \gamma_E - \frac{11}{6}\right). \ldots \tag{4.7}
\]

5 Calculating logarithmic moment from power moment

In this section, we provide a method to calculate the logarithmic moment (1.3) from the power moment. However, for some distributions, the logarithmic moment exists, but the power moment does not. In such cases, we can use the renormalized power moment given in
this paper to calculate the logarithmic moment. In particular, this method can directly verify the validity of the renormalized power moment by comparing the logarithmic moments calculated by definition (1.3) with the logarithmic moments calculated by the renormalized power moment.

5.1 Power moment scheme

If a distribution has power moments, it also has logarithmic moments. If the power moment does not exist, we can first obtain the renormalized power moment by the renormalization procedure in section 2 and then calculate the logarithmic moment from the renormalized power moment. Moreover, calculating the logarithmic moment from the power moment provides an approach for calculating the logarithmic moment.

The second characteristic function of distribution of the probability density function $p(x)$,

$$\phi(\sigma) = \int_{-\infty}^{\infty} p(x) x^{\sigma - 1} dx,$$

(5.1)
is a moment generating function of logarithmic moment [2]. The power moment can be expressed as

$$m_n = \int_{-\infty}^{\infty} p(x) x^n dx = \phi(n + 1).$$

(5.2)

By the relation (2.53) and

$$\frac{d^n x^\sigma}{d\sigma^n} \bigg|_{\sigma=0} = x^\sigma \ln^n x \bigg|_{\sigma=0} = \ln^n x,$$

(5.3)

we have

$$\tilde{m}_n = \int_{-\infty}^{\infty} p(x) \ln^n x dx$$

$$= \int_{-\infty}^{\infty} p(x) \frac{d^n x^{\sigma - 1}}{d\sigma^n} \bigg|_{\sigma=0} dx = \frac{d^n}{d\sigma^n} \int_{-\infty}^{\infty} p(x) x^{\sigma - 1} dx \bigg|_{\sigma=1}$$

$$= \frac{d^n \phi(\sigma)}{d\sigma^n} \bigg|_{\sigma=1}.$$ 

(5.4)

Compared with Eq. (5.2), we have

$$\tilde{m}_n = \frac{d^n m_{\sigma-1}}{d\sigma^n} \bigg|_{\sigma=1}.$$ 

(5.5)

By the relation (5.5), the logarithmic moment can be obtained from the renormalized power moment:

$$\tilde{m}_{ren} = \frac{d^n m_{ren}^{\sigma-1}}{d\sigma^n} \bigg|_{\sigma=1}.$$ 

(5.6)
5.2 Zeta function method

Observing the expression of the zeta function, Eq. (2.2), we can see that the second characteristic function (5.1) can be expressed by the zeta function:

$$\phi (\sigma) = \zeta (1 - \sigma). \quad (5.7)$$

By Eq. (5.5), the logarithmic moment can be expressed as the zeta function:

$$\tilde{m}_n = \frac{d^n \zeta (1 - \sigma)}{d\sigma^n} \bigg|_{\sigma = 1}. \quad (5.8)$$

In this way, the zeta function renormalization scheme can be directly applied to calculate the logarithmic moment.

This result shows that the zeta function (n is substituted by \(-n\)) is the moment generating function of the logarithmic moment.

5.3 Characteristic function method

The first characteristic function is the moment generating function of power moments, and the second characteristic function is the moment generating function of logarithmic moment [2]. In this section, we present a method of obtaining the logarithmic moment from the first characteristic function.

By Eqs. (2.41) and (2.41), we directly obtain the renormalized logarithmic moment,

$$\tilde{m}_n^{\text{ren}} = \frac{d^n m_{\sigma - 1}^{\text{ren}}}{d\sigma^n} \bigg|_{\sigma = 1} = \frac{d^n}{d\sigma^n} \left[ (-i)^{\sigma - 1} \frac{d^{\sigma - 1} f (k)}{d k^{\sigma - 1}} \bigg|_{k = 0} \right] \bigg|_{\sigma = 1}. \quad (5.9)$$

The logarithmic moment can be obtained from the first characteristic function. It can be seen that \((-i)^{\sigma - 1} \frac{d^{\sigma - 1} f (k)}{d k^{\sigma - 1}} \big|_{k = 0} = \phi (\sigma)\) in Eq. (5.9) is the second characteristic function, which is also the generating function of logarithmic moment.

6 Calculating logarithmic moment from power moment: example

In section 5, we provide a method for calculating logarithmic moments from power moments. Below we provide some examples.

6.1 Cauchy distribution

In principle, the logarithmic moment of the Cauchy distribution can be directly calculated by definition (1.3), but here we calculate the logarithmic moment from the renormalized power moment. The renormalized power moment of the Cauchy distribution is given by Eq. (3.4).
By the relation between power moment and logarithmic moment, Eq. (5.9), we can calculate the logarithmic moment from the \( n \)-th power moment:

\[
\tilde{m}_n = \left. \frac{d^n m_{\sigma-1}}{d\sigma^n} \right|_{\sigma=1} = \left( \frac{i\pi}{2} \right)^n e^{i\pi/2}
\]

which is consistent with the result obtained from the definition of the logarithmic moment, Eq. (1.3).

The first several logarithmic moments then read

\[
\tilde{m}_1 = \frac{i}{2}, \quad \tilde{m}_2 = -\frac{\pi^2}{4}, \quad \tilde{m}_3 = -\frac{i}{8}\pi^3, \quad \text{and} \quad \tilde{m}_4 = \frac{\pi^4}{16}.
\]

### 6.2 Levy distribution

The renormalized power moment of the Levy distribution (3.30) is given by Eq. (3.33).

By the relation between \( n \)-th power moments and logarithmic moments, Eq. (5.9), we obtain the first-order logarithmic moment:

\[
\tilde{m}_1 = \left. \frac{d m_{\sigma-1}}{d\sigma} \right|_{\sigma=1} = \gamma_E + \ln 2,
\]

the second-order logarithmic moment

\[
\tilde{m}_2 = \left. \frac{d^2 m_{\sigma-1}}{d\sigma^2} \right|_{\sigma=1} = \frac{\pi^2}{2} + \ln^2 2 + \gamma_E (\gamma_E + 2 \ln 2),
\]

the third-order logarithmic moment

\[
\tilde{m}_3 = \left. \frac{d^3 m_{\sigma-1}}{d\sigma^3} \right|_{\sigma=1} = 14\zeta(3) + (\gamma_E + \ln 2)^3 + \frac{3}{2}\pi^2 (\gamma_E + \ln 2),
\]

and the fourth logarithmic moment

\[
\tilde{m}_4 = \left. \frac{d^4 m_{\sigma-1}}{d\sigma^4} \right|_{\sigma=1} = (\gamma_E + \ln 2)^4 + 3\pi^2(\gamma_E + \ln 2)^2 + 56\zeta(3)(\gamma_E + \ln 2) + \frac{7\pi^4}{4},
\]

where \( \zeta(s) \) is the Riemann zeta function.

These results are consistent with the result obtained by definition (1.3).

### 6.3 \( q \)-exponential distribution

The renormalized \( n \)-th power moment of the \( q \)-exponential distribution (3.58) for \( q > 1 \) and \( x \geq 0 \) is given by Eq. (3.61).

By the relation between the \( n \)-th power moment and the logarithmic moment (5.9), we obtain the first-order logarithmic moment:

\[
\tilde{m}_1 = \left. \frac{d m_{\sigma-1}}{d\sigma} \right|_{\sigma=1} = -H_{\frac{1}{q-1}} - \ln(\lambda(q-1))
\]

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with \( H_n = \sum_{i=1}^{n} \frac{1}{i} \) the \( n \)-th harmonic number \([36]\), the second-order logarithmic moment

\[
\tilde{m}_2 = \left. \frac{d^2 m_{\sigma-1}}{d\sigma^2} \right|_{\sigma=1} = \left[ H_{\frac{1}{q-1} \cdot 2} + \ln(\lambda(q-1)) \right]^2 + \psi^{(1)}\left( \frac{1}{q-1} - 1 \right) + \frac{\pi^2}{6} \quad (6.7)
\]

with \( \psi^{(n)}(z) = \frac{d^n \psi(z)}{dz^n} \) the \( n \)-th derivative of the digamma function, \( \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \) the logarithmic derivative of the gamma function, and the third logarithmic moment

\[
\tilde{m}_3 = \left. \frac{d^3 m_{\sigma-1}}{d\sigma^3} \right|_{\sigma=1} = -\frac{1}{2} \left[ \ln(\lambda(q-1)) + \gamma_E \right] \left\{ 2 \ln(\lambda(q-1)) \left[ \ln(\lambda(q-1)) + 2\gamma_E \right] + \pi^2 + 2\gamma_E^2 \right\} - 2\zeta(3)
\]

\[
- \frac{1}{2} \psi^{(0)}\left( \frac{1}{q-1} - 1 \right) \left\{ 6 \ln(\lambda(q-1)) + 2\gamma_E \ln(\lambda(q-1)) + \pi^2 + 6\gamma_E^2 \right\}
\]

\[
- 3\psi^{(0)}\left( \frac{1}{q-1} - 1 \right)^2 \left[ \ln(\lambda(q-1)) + \gamma_E \right] - \psi^{(0)}\left( \frac{1}{q-1} - 1 \right)^3
\]

\[
- 3\psi^{(1)}\left( \frac{1}{q-1} - 1 \right) \left[ H_{\frac{1}{q-1} \cdot 2} + \ln(\lambda(q-1)) \right] - \psi^{(2)}\left( \frac{1}{q-1} - 1 \right). \quad (6.8)
\]

The above result is consistent with the result obtained from the definition (1.3).

6.4 \( q \)-Gaussian distribution

The renormalized \( n \)-th power moment of the \( q \)-Gaussian distribution (3.94) is Eq. (3.98), which is valid for all values of \( q \).

By the relation between \( n \)-th power moment and logarithmic moment (5.9), we obtain the first-order logarithmic moment:

\[
\tilde{m}_1 = \left. \frac{dm_{\sigma-1}}{d\sigma} \right|_{\sigma=1} = -\frac{1}{2} \left[ H_{\frac{1}{q-1} \cdot \frac{3}{2}} + \ln \frac{2(q-1)}{\beta^2} - i\pi \right], \quad (6.9)
\]

the second-order logarithmic moment

\[
\tilde{m}_2 = \left. \frac{d^2 m_{\sigma-1}}{d\sigma^2} \right|_{\sigma=1} = \frac{1}{4} \psi^{(0)}\left( \frac{1}{q-1} - \frac{1}{2} \right) \left( H_{\frac{1}{q-1} \cdot \frac{3}{2}} + 2 \ln \frac{q-1}{\beta^2} - 2i\pi + \gamma_E + \ln 4 \right)
\]

\[
+ \frac{1}{2} \left( \gamma_E - i\pi \right) \ln \frac{2(q-1)}{\beta^2} + \frac{1}{4} \ln \frac{q-1}{\beta^2} \ln \frac{4(q-1)}{\beta^2}
\]

\[
+ \frac{1}{4} \psi^{(1)}\left( \frac{1}{q-1} - \frac{1}{2} \right) \left[ 3\pi^2 - \frac{1}{2} i\gamma_E \pi + \frac{1}{4} \gamma_E^2 + \frac{1}{4} \ln^2 2, \quad (6.10)
\]

and so on.

The above result is consistent with the result obtained from the definition (1.3).
6.5 Normal distribution

The normal distribution

\[ p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \]  

(6.11)

has \( n \)-th power moments and does not need renormalization. The \( n \)-th power moment of the standard normal distribution can be directly calculated by definition (1.2):

\[ m_n = \frac{1}{\sqrt{\pi}} 2^{\frac{n}{2} - 1} [(-1)^n + 1] \Gamma \left( \frac{n + 1}{2} \right). \]  

(6.12)

By the relation between \( n \)-th power moments and logarithmic moments, Eq. (5.9), we obtain the first-order logarithmic moment:

\[ \tilde{m}_1 = \left. \frac{d m_{\sigma^{-1}}}{d \sigma} \right|_{\sigma=1} = -\frac{1}{2} (\ln 2 + \gamma_E - i\pi) \]  

(6.13)

and the second-order logarithmic moment

\[ \tilde{m}_2 = \left. \frac{d^2 m_{\sigma^{-1}}}{d \sigma^2} \right|_{\sigma=1} = \frac{1}{4} (\ln 2 + \gamma_E - i\pi)^2 - \frac{\pi^2}{8}. \]  

(6.14)

The expressions of the third-order and fourth-order logarithmic moments are too long to be listed here.

The above result is consistent with the result obtained from the definition (1.3).

6.6 Student’s t-distribution

The Student’s t-distribution

\[ p(x) = \frac{\nu^{\nu/2} \Gamma \left( \frac{\nu+1}{2} \right)}{\sqrt{\nu\pi} \Gamma \left( \frac{\nu}{2} \right)} \]  

(6.15)

has the \( n \)-th power moment, and does not need renormalization. The \( n \)-th power moment can be directly calculated by definition (1.2):

\[ m_n = \frac{[(-1)^n + 1] \nu^{n/2} \Gamma \left( \frac{n+1}{2} \right) \Gamma \left( \frac{\nu-n}{2} \right)}{2\sqrt{\pi\Gamma \left( \frac{\nu}{2} \right)}}. \]  

(6.16)

By the relation between \( n \)-th power moments and logarithmic moments, Eq. (5.9), we obtain the first logarithmic moment,

\[ \tilde{m}_1 = \left. \frac{d m_{\sigma^{-1}}}{d \sigma} \right|_{\sigma=1} = \frac{1}{2} \left( \ln \frac{\nu}{4} + i\pi - H_{\frac{\nu}{2} - 1} \right) \]  

(6.17)

and the second logarithmic moment

\[ \tilde{m}_2 = \left. \frac{d^2 m_{\sigma^{-1}}}{d \sigma^2} \right|_{\sigma=1} = \frac{1}{8} \left[ 2\ln^2 \nu + 4i\pi \ln \nu + 2\psi^{(1)} \left( \frac{\nu}{2} \right) - 3\pi^2 \right. \]

\[ \left. + 2 \left( \psi^{(0)} \left( \frac{\nu}{2} \right) + \gamma_E + 2\ln 2 \right) \left( 2\ln 2 - 2\ln \nu + \psi^{(0)} \left( \frac{\nu}{2} \right) - 2i\pi + \gamma_E \right) \right]. \]  

(6.18)

The expressions of the third-order and fourth-order logarithmic moments are too long to be listed here.

The above result is consistent with the result obtained from the definition (1.3).
6.7 Laplace distribution

The probability density function of the Laplace distribution is

\[ p(x) = \begin{cases} \frac{\lambda}{2} e^{-\lambda (x - \mu)}, & x > \mu \\ \frac{\lambda}{2} e^{-\lambda (-x + \mu)}, & x \leq \mu \end{cases}. \] (6.19)

The \( n \)-th power moment of the Laplace distribution by the \( n \)-th moment definition (1.2) is

\[ m_n = (-1)^n \frac{e^{-\lambda \mu} \Gamma(n + 1, -\lambda \mu) + e^{\lambda \mu} \Gamma(n + 1, \lambda \mu)}{2^{n+1}}. \] (6.20)

By the relation between \( n \)-th power moments and logarithmic moments, Eq. (5.9), we obtain the first logarithmic moment,

\[ \tilde{m}_1 = \frac{d m_{\sigma-1}}{d \sigma} \bigg|_{\sigma=1} = \frac{1}{2} \left( e^{-\lambda \mu} \Gamma(0, -\lambda \mu) + e^{\lambda \mu} \Gamma(0, \lambda \mu) \right) + \ln \mu + i \pi, \] (6.21)

and the second logarithmic moment

\[ \tilde{m}_2 = \frac{d^2 m_{\sigma-1}}{d \sigma^2} \bigg|_{\sigma=1} = e^{-\lambda \mu} G^{3,0}_{2,3} \left( -\lambda \mu \left| \begin{array}{c} 1, 1 \\ 0, 0, 0 \end{array} \right. \right) + e^{\lambda \mu} G^{3,0}_{2,3} \left( \lambda \mu \left| \begin{array}{c} 1, 1 \\ 0, 0, 0 \end{array} \right. \right) \]

\[ + e^{\lambda \mu} \Gamma(0, \lambda \mu) \ln \mu + e^{-\lambda \mu} (\ln \mu + 2i \pi) \Gamma(0, -\lambda \mu) + (\ln \mu + i \pi)^2 - \pi^2 \] (6.22)

where \( G_{p,q}^{m,n}(z \left| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right. ) \) is the Meijer \( G \) function and \( P_{F_q}(a; b; z) \) is the generalized hypergeometric function ([36]). The expressions of the third-order and fourth-order logarithmic moments are too long to be listed here.

The above result is consistent with the result obtained from the definition (1.3).

7 Discussion and conclusion

In this paper, we suggest renormalization schemes for seeking moments for distributions that have no moments. The key to renormalization is that the renormalized result must be renormalization scheme-independent. In order to show that the renormalized moment is scheme-independent, we construct a variety of renormalization schemes. For each distribution, we use different renormalization schemes to calculate the same renormalized power moment to exemplify scheme independence.

The renormalization schemes in this paper can be divided into the following cases.

1) The renormalized moment can be obtained by directly analytically continuing the moment definition. In this case, the divergence is caused by the form of the definition. The moment of the distribution is usually defined by integrals. If it is not integrable, then the distribution has no moment. It is known that different representations of the same function
usually have different domains. As an analogy, we take the gamma function as an example. The integral definition of the gamma function is

$$\Gamma (z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad \text{Re} \ z > 0. \quad (7.1)$$

In this integral definition, the domain of the gamma function is \( \text{Re} \ z > 0 \). If we define the gamma function by the functional equation,

$$\Gamma (z + 1) = z \Gamma (z), \quad (7.2)$$

then the gamma function is analytic throughout the whole complex plane except for isolated singularities. That is to say, under the functional equation definition, the gamma function is only undefined on some isolated points. The integral definition and the function equation definition of the gamma function are equivalent for \( \text{Re} \ z > 0 \). However, the domains in these two definitions are different, and the function equation definition gives a larger domain. That is, through analytical continuation, we can use one definition to replace another and extend the domain of the gamma function. In addition to the integral definition and the function equation definition, the gamma function also has other definitions. These different definitions, which are analytic continuations to each other, will give the same value in the overlapping area of their domains. In practice, working out the integral or the sum in the definition is just to do analytic continuation. The uniqueness of analytical continuation ensures that the renormalized result is scheme-independent.

2) If analytic continuation of the moment definition still cannot give a finite moment, which shows that the divergence is caused by true singularities — the obstacle of analytic continuation, it requires minimal subtraction to remove the singularity. The analytical continuation can only remove the singularities caused by the form of definition, but not the true singularities. Taking the gamma function as an example, by replacing the integral definition with the function equation definition, we can extend the domain of the gamma function to the entire complex plane except for several isolated singularities. These isolated singularities are true singularities, which are obstacles to the analytical continuation and cannot be removed by an analytical continuation. In fact, it is because of these true singularities that the gamma function, such as integral definition and series definition, is limited to a certain local area of the complex plane. This means that even if we extend the domain of moments through analytical continuation, we still cannot define moments on these true singularities. The classification of isolated singularities is done through series. Furthermore, minimal subtraction is used to remove these singularities and give the renormalized moments on these true singularities.

3) A new parameter is introduced, making the moment a function of this parameter. The divergence of the moment then becomes the singularity of this function and then is removed by renormalization treatments. In other words, the integral in the moment definition is divergent. After introducing a new parameter, the integral only diverges at some special values of this parameter but is well-defined at other parameter values. This turns the divergence problem into a problem that can be handled by renormalization. Both the weighted moment method and the cut-off method belong to this case.
We develop a method to express logarithmic moments by power moments. In this way, the renormalization treatment of power moments can be applied to calculating logarithmic moments. This method can also be used to verify the validity of the renormalization scheme by comparing the logarithmic moment obtained directly from the definition and from the renormalized power moment.

In addition, technically, the renormalization procedure often relies on the explicit result of an integral. In different renormalization schemes, we encounter different integrals. If an integral cannot be worked out in a renormalization scheme, we can turn to another renormalization scheme. The more renormalization schemes are, the greater the possibility of successful renormalization is.

We will discuss the various random processes with renormalized moments in further consideration.

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