Completion of Newton’s Iterations  
Initialized at a Quasi-Universal Set  

Victor Y. Pan  

Lehman College of CUNY  
Bronx, NY 10468 USA and  
Ph.D. Programs in Math. and Computer Science  
The Graduate Center of CUNY  
New York, NY 10036 USA*  
victor.pan@lehman.cuny.edu  
http://comet.lehman.cuny.edu/vpan/  

Abstract  

Recently Schleicher and Stoll proposed efficient initialization of Newton’s iterations. Given a black box subroutine for the evaluation of the Newton’s ratio of a polynomial and its derivative, their algorithm very fast approximates all roots of a univariate polynomial except for a small fraction of them. Our recipes for the approximation of the remaining roots within the same asymptotic computational cost should answer the authors’ challenge. Our work can be of independent interest as an example of synergistic variation and combination of polynomial root-finding methods towards enhancing their power.

Key Words: Polynomial roots; Newton iteration; Universal initial set; Quasi-universal set; Deflation; Maps of the variable; Functional iterations

2000 Math. Subject Classification: 26C10, 30C15, 65H05

1 Introduction: the Problem

The paper [HSS01] proposed an ingenious initialization of Newton’s iterations

$$y_0 = c, \ y^{(h+1)} = y^{(h)} - p(y^{(h)})/p'(y^{(h)}), \ h = 0, 1, \ldots,$$

*Supported by NSF Grant CCF–1116736 and PSC CUNY Award 68862–00 46
for the approximation of all $d$ roots of a univariate polynomial
\[
p(x) = \sum_{i=0}^{d} p_i x^i = p_d \prod_{j=1}^{d} (x - x_j), \quad p_d \neq 0, \quad |x_j| \leq 1, \text{ for all } j,
\]
given by a subroutine for the evaluation of the Newton’s ratio
\[
N_p(x) = \frac{p(x)}{p’(x)}
\]
at a complex point $x$; one can ensure that all roots $x_j$ lie in the unit disc $D(0, 1) = \{x: |x| \leq 1\}$ by means of shifting and/or scaling the variable $x$.

The authors define a universal set $\mathbb{U}_d$ of at most $1.1d \log^2(d)$ points on the complex plane such that Newton’s iterations initiated at these points converge to all roots of any polynomial $p(x)$ of (1).

The subsequent papers \[\text{BAS15, BLS13, S02, S08, and S13}\] decreased the cardinality of the set $\mathbb{S}_d$ to $O(d \log \log(d)^2)$, and it was proved in \[\text{BAS15}\] that one can approximate all roots of $p(x)$ within relative errors of at most $1/2^k$ by applying an expected overall number $O(d^2 \log^4(d) + db)$ of Newton’s iterations initialized at the points of that set.

In the paper \[\text{SSa}\] Newton’s iterations initialized at just $m_d = O(d)$ points of a quasi-universal set $\mathbb{Q}_d$ and applied to any polynomial $p(x)$ of (1) converged to almost all its roots, except for a small fraction of $w = w(p)$ roots (\(w < 0.001 d\) for $d < 2^{17}$ and $w < 0.01 d$ for $d < 2^{20}$). We call these roots wild and w.l.o.g. denote them $x_1, \ldots, x_w$ (otherwise we would re-enumerate the roots); we call the other roots $x_{w+1}, \ldots, x_d$ tame.

We propose three directions to the approximation of the wild roots without increasing the overall asymptotic computational cost. Our recipes for enhancing the power of Newton’s and/or other functional iterations for polynomial root-finding by means of combining them successively or concurrently under various maps of the variable can be of independent interest.

## 2 Taming the Wild Roots by Means of Deflation

Our first recipe is deflation: we apply Newton’s iterations to the polynomial
\[
q(x) = \sum_{i=0}^{w} q_i x^i = p_d \prod_{j=1}^{w} (x - x_j), \quad p_d \neq 0.
\]
In our first algorithm the deflation is explicit: we compute the coefficients of $q(x)$ and then apply to it Newton’s iteration. Unless the number of wild roots is small, this technique can be prone to numerical problems, but our second algorithm avoids computation of the coefficients of $q(x)$: it applies implicit deflation by exploiting the well-known identity
\[
1/N_p(x) = \sum_{j=1}^{n} \frac{1}{x - x_j}.
\]
Algorithm 1. Explicit Deflation.

INPUT: The tame roots $x_w, \ldots, x_d$ of a polynomial $p(x)$ of (2); the quasi-universal set $Q_w = \{z_1, \ldots, z_m\}$, and a black-box program EVAL$_p$ that evaluates a polynomial $p(x)$ of (2) at a complex point $x$.

INITIALIZATION: Fix the integer $h = \lceil \log_2 w \rceil$, such that $w < 2^h \leq 2w$, write $m = 2^h$ and $\theta_m = \exp(2\pi \sqrt{-1}/m)$ (so that $\theta_m$ is a primitive $m$th root of unity), and compute all $m$th roots of unity, $1, \theta_m, \theta_m^2, \ldots, \theta_m^{m-1}$. Write

$$s(x) = \prod_{j=w+1}^{d} (x - x_j) = p(x)/q(x).$$

(6)

COMPUTATIONS:
1. Compute the values $p(\theta_m^k)$ and $s(\theta_m^k), k = 0, 1, \ldots, m-1$.
2. Compute the ratios $p(\theta_m^k)/s(\theta_m^k), k = 0, 1, \ldots, m-1$, for $q(x)$ of (1).
3. Interpolate to the polynomial $q(x) = p(x)/s(x)$.
4. Compute and output the values $N_q(z_j) = q(z_j)/q'(z_j)$, for $j = 1, \ldots, m_w$.

Complexity of Algorithm 1: At Stage 1 we call the program EVAL$_p$ $m$ times and in addition perform $(2d - 2w - 1)m$ arithmetic operations; Stage 2 involves $m$ divisions; at Stage 3 we perform $(1.5 \log_2(m) + 1)m$ arithmetic operations (by using Inverse FFT), and Stage 4 involves $(4w - 2)m_w$ arithmetic operations.

Algorithm 2. Implicit Deflation.

INPUT: The tame roots $x_w, \ldots, x_d$ of a polynomial $p(x)$ of (2); the quasi-universal set $Q_w = \{z_1, \ldots, z_m\}$, and a black-box program EVAL$_{p'/p}$ that evaluates the ratio $p'(x)/p(x)$ at a complex point $x$.

COMPUTATIONS:
1. Compute the values $r_k = p'(z_k)/p(z_k), k = 1, \ldots, m_w$.
2. Compute the values $s_k = \sum_{j=w+1}^{d} \frac{1}{z_k - z_j},$ for $k = 1, \ldots, m_w$.
3. Compute and output the values $N_q(z_k) = \frac{q(z_k)}{q'(z_k)} = \frac{1}{r_k - s_k},$ for $k = 1, \ldots, m_w$.

At Stage 1 we call the program EVAL$_{p'/p}$ $m_w$ times.
Stage 2 involves $(d - m_w)m_w$ divisions and $(2d - 2m_w - 1)m_w$ additions and subtractions.
At Stage 3 we perform $m_w$ subtractions and $m_w$ divisions.

\[\text{D. A. Bini proposed to improve numerical stability of this iteration by scaling this expression as follows:}\]
\[N_q(z_k) = \frac{1}{r_k} \frac{1}{1 - s_k / r_k} = \frac{N_q(z_k)}{1 - s_k N_q(z_k)}.\]
3 Taming the Wild Roots by Means of Functional Iterations

Another natural approach is to combine Newton’s method with other functional iterations for root-finding. A number of iterative processes, most notably the Weierstrass’s \[W03\] (frequently attributed to its later rediscoveries, e.g., by Durand \[D60\] and Kerner \[K66\]) and Ehrlich–Aberth’s \[E67\], \[A73\], globally converge very fast in practice, although this empirical observation has no adequate formal support.

One can extend the progress in \[SSa\] by substituting the tame roots into the Weierstrass’s and Ehrlich–Aberth’s iterations, which would greatly simplify their computations. Let us specify this *implicit deflation* for the latter iterations:

\[
z_{j}^{(k+1)} = z_{j}^{(k)} \frac{1}{u_{j}(z_{j}^{(k)})} \quad \text{for} \quad u_{j}(x) = \frac{1}{N_{p}(x) - \sum_{h \neq j} \frac{1}{x_{h} - x}} \quad \text{and} \quad j = 1, \ldots, d. \quad (7)
\]

Given the tame roots \(x_{j}\) of \(p(x)\), for \(j = w + 1, \ldots, d\), we only need to apply this iteration for the approximation of the remaining wild roots \(x_{1}, \ldots, x_{w}\), that is, we only need to apply (7) for \(j = 1, \ldots, w\), and we can adjust accordingly the initialization of these iterations in \[B96\], \[BF00\], and \[BR14\].

Similarly one can implicitly deflate other functional iterations for roots of \(p(x)\) such as the Gauss-Seidel’s and Werner’s accelerated variations of the Ehrlich–Aberth’s and Weierstrass’s iterations (cf. \[BR14\] and \[W82\]).

4 Taming the Wild Roots by Means of Mapping the Variable

Our next techniques can complement or replace the iterative algorithms of the previous sections in order to approximate all wild roots of a polynomial \(p(x)\) of (2). Namely we propose to apply Newton’s iterations to the polynomials

\[v(z) = v_{a,b,c}(z) = (z + c)^{d}p\left(a + \frac{b}{z + c}\right)\]

for various triples of complex scalars \(a, b \neq 0,\) and \(c\), whose number, however, must be limited in order to control the overall computational cost.

**Algorithm 3.**

**Initializations:** Fix a triple \(a, b \neq 0,\) and \(c\) defining a polynomial \(v(z) = v_{a,b,c}(z)\).

**Computations:**

1. Shift and/or scale the variable \(z\) in order to move all roots of this polynomial into the unit disc \(D(0, 1) = \{z: |z| = 1\}\).

2. Apply Newton’s iteration to the resulting polynomial by using initialization at quasi-universal set of \([SSa]\). (In order to simplify the notation, assume that this is still the same polynomial \(v(z)\).)
3. Avoid the computation of its coefficients by expressing the Newton’s ratios
by applying the following equations:
\[ \frac{1}{N_v(z)} = \frac{v'(z)}{v(z)} = \frac{d}{z + c} - \frac{b}{(z + c)^2 N(z)} \text{ for } N(z) = N_p(a+b/(z+c))(z). \tag{8} \]

4. Having approximated a root \( z_j = \frac{b}{x_j - a} - c \) of \( v(z) \) for any \( j \), readily recover
the root of \( p(x) \),
\[ x_j = a + \frac{b}{z_j + c}. \tag{9} \]

One can accelerate this algorithm by combining it with Algorithm 1 or 2.

In the particular case where \( a = c = 0 \) and \( b = 1 \), the above expressions are
simplified as follows: \( z = 1/x; v(z) \) turns into the reverse polynomial of \( p(x) \),
\[ v(z) = p_{rev}(z) = \sum_{i=0}^{d} p_{d−i}z^{i} = z^{d}p(1/z), \tag{10} \]
such that \( p_{rev}(x) = p_0 \prod_{j=1}^{d} (x - 1/x_j) \) if \( p_0 \neq 0 \), and
\[ \frac{1}{N_v(z)} = \frac{v'(z)}{v(z)} = \frac{d}{z} - \frac{1}{z^2 N(z)} \text{ for } N(z) = N_p(1/z)(z). \]

We can hope to obtain all roots of \( p(x) \) by applying Newton’s iterations for a
reasonable number of triples of \( a, b \) and \( c \), but we can also extend our approach
by using more general rational maps \( y = r(x) \) (cf., e.g., [MP00]).

For a simple example, consider the Dandelin’s root-squaring map [H59]:
\[ u(y) = (-1)^d p(\sqrt{y})p(-\sqrt{y}) = \prod_{j=1}^{d} (y - x_j^2). \tag{11} \]

In this case one should either assume that a polynomial \( p(x) \) of \([2]\) is monic or
scale it to make it monic; then one can express the Newton’s ratio as follows:
\[ N_u(y) = \frac{1}{2}(-1)^d y^{-1/2} \left( \frac{1}{N_{p(\sqrt{y})}} - \frac{1}{N_{p(-\sqrt{y})}} \right). \]

Note that under map (11) the roots lying in the unit disc \( D(0,1) \) stay in it.

Having approximated the \( n \) roots \( y_1, \ldots, y_n \) of the polynomial \( u(y) \), we can
readily recover the the \( n \) roots \( x_1, \ldots, x_n \) of the polynomial \( p(x) \) from the \( 2n \)
values \( \pm \sqrt{y_j}, j = 1, \ldots, n \).

We can combine the above maps recursively (a limited number of times,
in order to control the overall computational cost) and then recover the roots
from their images in these rational maps by extending the lifting/descending
techniques of [P95], [P02].

The approach can be applied independently of the algorithms of the previous
sections or can be combined with them, particularly with implicit deflation, in
order to decrease the computational cost or to extend their power to a larger domain of inputs.

For practical benefits, the selected functional iterations for the selected polynomials \( p(x), u(y), \) and \( v(z) \) can be implemented concurrently, with minimal need for processor communication and synchronization, as long as sufficiently many processors are available.

### 5 Conclusions

Some of our techniques may be of independent interest. Combination of various functional root-finding iterations towards their faster convergence in wider input domains is a well-known challenge. In order to meet this challenge, one can first transform an input polynomial \( p(x) \) by means of the maps \( x = a + \frac{b}{z^c} \) for various triples of \( a, b, \) and \( c \) or by means of more general rational maps \( x = x(y) \), then successively or concurrently apply the selected functional iterations to the resulting polynomials, and finally recover the roots of \( p(x) \) from the same roots output by these iterations.

For a specific application of such maps, consider another important challenge, of the approximation of the roots isolated in the unit disc \( D(0, 1) \) on the complex plane.

Both Weierstrass’s and Ehrlich–Aberth’s iterations do not seem to do well for this task when the number, \( \nu \), of the roots in the disc is much smaller than \( d \). Indeed in this case Newton’s and other functional iterations (cf. [MP07] and [MP13 Chapter 9]) promise to accelerate the solution by a factor of \( d/\nu \). For a formal support of their global convergence, however, we cannot bluntly apply the techniques of the papers [HSS01], [BAS15], [BLS13], [S02], [S08], [S13], and [SSa], because they exploit the large size of the areas about the orbits connecting the roots to the infinity. The following algorithm may alleviate this difficulty.

**Algorithm 4.**

**Computations:**

1. Apply a limited number of Dandelin’s root squaring steps. [Then all superfluous roots of \( p(x) \), lying in the exterior of that disc (denote them \( x_{\nu+1}, \ldots, x_d \)) are moved farther from the disc \( D(0, 1) \), while the other roots, \( x_1, \ldots, x_\nu, \) stay in this disc.]

2. Apply the map \( y = 1/z \). [It moves all the superfluous roots of \( p(x) \) into a small disc inside \( D(0, 1) \), while moving the roots \( x_1, \ldots, x_\nu \) into the exterior of the disc \( D(0, 1) \).]

3. Move all roots into the unit disc \( D(0, 1) \) by applying shift and/or scaling of the variable. [This keeps the images of the roots \( x_1, \ldots, x_\nu \) strongly isolated in a small disc inside the disc \( D(0, 1) \).]

4. Apply Newton’s iteration to the resulting polynomial by using initialization at the universal or quasi-universal set.

5. Recover the roots of \( p(x) \) from their images computed by these iterations.
We can conjecture that the iterations would first converge to the images of the roots \(x_1, \ldots, x_\nu\) of the polynomial \(p(x)\), thus achieving a desired acceleration by a factor of \(d/\nu\) or perhaps \(d/\nu^2\). This conjecture can be tested empirically, but one can also try to yield its formal support by extending the study in [HSS01], [BAS15], [BLS13], [S02], [S08], [S13], and [SSa].

References

[A73] O. Aberth. Iteration Methods For Finding All Zeros of a Polynomial Simultaneously, *Mathematics of Computation*, 27, 122, 339–344, 1973.

[B96] D. A. Bini. Numerical Computation of Polynomial Zeros by Means of Aberth’s Method, *Numerical Algorithms*, 13, 3–4, 179–200, 1996.

[BAS15] T. Bilarev, M. Aspenberg, D. Schleicher. On the speed of convergence of Newton’s method for complex polynomials. *Mathematics of Computation*, 2015. DOI: [http://dx.doi.org/10.1090/mcom/2985](http://dx.doi.org/10.1090/mcom/2985). Also in arXiv:1202.2475, 2012.

[BF00] D. A. Bini, G. Fiorentino. Design, Analysis, and Implementation of a Multiprecision Polynomial Rootfinder, *Numerical Algorithms*, 23, 127–173, 2000.

[BLS13] B. Bollobás, M. Lackmann, D. Schleicher. A Small Probabilistic Universal Set of Starting Points for Finding Roots of Complex Polynomials by Newton’s Method, *Math. of Computation*, 82, 443–457, 2013.

[BP94] D. Bini, V. Y. Pan, *Polynomial and Matrix Computations, Volume 1: Fundamental Algorithms*, Birkhäuser, Boston, 1994.

[BR14] D. A. Bini, L. Robol. Solving secular and polynomial equations: a multiprecision algorithm, *Journal of Computational and Applied Mathematics*, 272, 276–292, 2014.

[D60] E. Durand. Solutions numériques des équations algébriques, *Tome 1: Équations du type F(X)=0; Racines d’un polynôme*, Masson, Paris, 1960.

[E67] L. W. Ehrlich. A Modified Newton Method for Polynomials, *Comm. of ACM*, 10, 107–108, 1967.

[HSS01] J. Habbard, D. Schleicher, S. Sutherland. How to Find All Roots of Complex Polynomials by Newton’s Method, *Invent. Math.*, 146, 1–33, 2001.

[H59] A. S. Householder. Dandelin, Lobachevskii, or Graeffe, *American Mathematical Monthly*, 66, 464–466, 1959.
[K66] I. O. Kerner. Ein Gesamtschrittverfahren zur Berechnung der Nullstellen von Polynomen, *Numerische Math.* 8, 290–294, 1966.

[MP07] J.M. McNamee. *Numerical Methods for Roots of Polynomials*, Part I, Elsevier, 2007.

[MP00] B. Mourrain, V. Y. Pan. Lifting/Descending Processes for Polynomial Zeros and Applications, *J. of Complexity*, 16, 1, 265–273, 2000.

[MP13] J.M. McNamee, V.Y. Pan. *Numerical Methods for Roots of Polynomials*, Part II, Elsevier, 2013.

[P95] V. Y. Pan. Optimal (up to Polylog Factors) Sequential and Parallel Algorithms for Approximating Complex Polynomial Zeros, *Proc. 27th Ann. ACM Symp. on Theory of Computing (STOC’95)*, 741–750, ACM Press, New York, 1995.

[P01] V. Y. Pan. *Structured Matrices and Polynomials: Unified Superfast Algorithms*, Birkhäuser/Springer, Boston/New York, 2001.

[P02] V. Y. Pan. Univariate Polynomials: Nearly Optimal Algorithms for Factorization and Rootfinding, *Journal of Symbolic Computations*, 33, 5, 701–733, 2002.

[S02] D. Schleicher. On the number of iterations of Newton’s method for complex polynomials, *Ergodic Theory and Dynamical Systems*, 22, 935–945, 2002.

[S08] D. Schleicher. Newton’s method as a dynamical system: efficient In: *Holomorphic dynamics and renormalization*. A volume in honour of John Milnor’s 75th birthday, ed. Mikhail Lyubich and Michael Yampolsky, Fields Institute Communications, 53, 213–224, 2008.

[S13] D. Schleicher. On the Efficient Global Dynamics of Newton’s Method for Complex Polynomials, 2013. [arXiv:1108.5773](https://arxiv.org/abs/1108.5773)

[SSa] D. Schleicher, R. Stoll. Newton’s method in practice: finding all roots of polynomials of degree one million efficiently, *Theoretical Computer Science*, Special Issue on Symbolic–Numerical Algorithms (Stephen Watt, Jan Verschelde, and Lihong Zhi, editors), in print.

[T63] A. L. Toom. The Complexity of a Scheme of Functional Elements Realizing the Multiplication of Integers, *Soviet Mathematics Doklady*, 3, 714–716, 1963.

[W03] K. Weierstrass. Neuer Beweis des Fundamentalsatzes der Algebra, *Mathematische Werke*, Tome III, 251–269, Mayer und Mueller, Berlin, 1903.
[W82] W. Werner. Some improvements of classical iterative methods for the solution of nonlinear equations, in *Numerical Solution of Nonlinear Equations, Proc. Bremen 1980 (L.N.M. 878)*, ed. E.L. Allgower et al, Springer, Berlin, 427–440, 1982.