Electromagnetic coupling effects in natural inflation

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Abstract
In this work we study the effects of the electromagnetic coupling in natural inflation in a systematic manner using the Schwinger–Keldysh formalism. The corresponding influence functional is evaluated in the one-loop level. It can be interpreted as due to a single stochastic force. The equation of motion of the inflaton field is therefore given in the form of a Langevin equation. Lastly, the two-point and the three-point correlation functions of the inflaton field are worked out. They are related to the power spectrum and the nongaussianity of the inflaton field, respectively.

Keywords: natural inflation, in-in formalism, nongaussianity

1. Introduction
In order to remedy some of the problems of the standard big bang theory, an inflationary period has been introduced in the very early Universe before the radiation dominated era [1]. For this inflation paradigm to work, the corresponding inflaton potential must be very flat, or that the inflaton has to roll slowly to have sufficient e-folding expansion and large-scale matter homogeneities. This constitutes a fine-tuning problem. A natural way to avoid this is to take the inflaton to be axion-like, which possesses a naturally flat potential with the shift symmetry [2, 3]. This model of inflation is known as the natural inflation, followed by many of its variant models [4].

One major problem of the natural inflation is that the symmetry breaking scale has to be very close to the Planck scale to have a spectral index for the matter power spectrum compatible with observations [5]. It was noted that this scale could be lowered to the GUT scale in a warm inflation scenario [6–9]. In warm inflation models [10–13] the couplings of the inflaton field...
to other fields are considered during the inflationary period. This is like the coupling to a heat bath in which noise and dissipative effects will be manifest. Specifically the appearance of the dissipative term could prevent the inflaton field from rolling down the potential quickly even for a steep potential, thereby lowering the constraint on the energy scale.

Recently, a lot of attentions have been paid to the study of the phenomenological effects of the axion-vector coupling on inflation [7, 14–27], with particular attentions to the influences of particle production and its associated backreaction to the slow-roll inflation. This leads to very interesting effects such as the generation of non-Gaussian and non-scale-invariant density power spectrum [16, 17], the damping of primordial gravitational waves due to a copious production of free-streaming particles [18], and the generation of stochastic gravitational waves that could be detected at ground-based gravity-wave interferometers [19, 20], and the formation of primordial black holes at density peaks near the end of inflation [21–23].

Despite the above mentioned works, we find that there is lacking a systematic study of the origin of the effects caused by the axion-vector coupling in the natural inflation. In order to understand the effect of the heat bath in more details, one could utilize the tools of the closed time path (CTP) [28, 29], or the Schwinger–Keldysh formalism [30, 31], together with the influence functional method [32]. In this open system approach the degrees of freedom of the heat bath are integrated out. This coarse graining process produces the noise and the dissipation terms [33] that the system is subjected to. In the warm natural inflation model, the axion-like inflaton is coupled to the electromagnetic field. One could therefore integrate out the photon degrees of freedom giving an influence functional with the corresponding noise and dissipation terms. However, we observe that while the CTP formalism assumes a perturbativity of the system, the photon field in the axion inflation becomes tachyonic for a wide range of physical Fourier modes during the course of inflation. This instability is non-perturbative and may invalidate the CTP formalism. In this paper we would explore this possibility, and then we would study the effects of these terms on the evolution of the inflaton field.

In the next section, we shall describe briefly the CTP formalism as applied to the natural inflation model we are considering in this paper. In section 3, we work on the evaluation of the influence functional by integrating out the photon degrees of freedom. This is basically done in a perturbative manner. In section 4, we analyze the noise kernel in the influence functional and define the corresponding probability functional. Subsequently, the equation of motion of the inflaton field is obtained in the form of a Klein–Gordon–Langevin equation. The two-point and the three-point correlators of the inflaton field are presented in section 5. Lastly, we give our conclusions in section 6. In the appendix, the photon mode functions used in our calculation in the relevant approximation are given.

2. CTP formalism

To begin with we consider the following action of axion inflation, with the inflaton field $\phi(x)$, the photon field $A_\mu(x)$, and their interaction,

$$S_{\text{tot}} = \int d^4x \sqrt{-g} \left[ \frac{M_P^2}{2} R - \frac{1}{2} (\partial \phi)^2 - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \alpha \frac{\phi}{f} F_{\mu\nu} F^{\mu\nu} \right] ,$$

where $R$ is the curvature scalar, $M_P$ the reduced Planck mass, $V(\phi)$ the inflaton potential, $f$ the axion decay constant, and $\alpha$ is a dimensionless parameter. The simplest example of the inflaton potential would be

$$V(\phi) = \Lambda^4 \left[ 1 - \cos \left( \frac{\phi}{\Lambda} \right) \right] .$$


with the mass scale \( \Lambda \sim m_{\text{GUT}} \) and \( f < M_p \).

Here we take the spacetime to be conformally flat,
\[
ds^2 = a^2(\tau) (-d\tau^2 + dx^2 + dy^2)
\]
where \( a(\tau) \) is the cosmic scale factor that depends on conformal time. The electromagnetic field is \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( F^\mu\nu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} / \sqrt{-g} \) is its dual, where the potential \( A_\mu \) satisfies the gauge conditions: \( A_\tau = 0 \) and \( \nabla^\mu A_\mu = 0 \). Then one can write \( A_\mu = (0, \vec{A}) \) and expand \( \vec{A} \) in terms of the two physical degrees of freedom,
\[
\vec{A}(x, \tau) = \sum_{\lambda = R, L} \int \frac{d^4 k}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{x}} A_\lambda(\vec{k}, \tau) \vec{\varepsilon}_\lambda(\vec{k}),
\]
where \( \vec{\varepsilon}_\lambda(\vec{k}) \) are the circular polarizations vectors. They have the following properties,
\[
\vec{\varepsilon}_{R,L}(\vec{k}) = \vec{\varepsilon}_{L,R}(\vec{k}) = \vec{\varepsilon}_{R,L}(\vec{k});
\]
\[
\vec{\varepsilon}_\lambda(\vec{k}) \cdot \vec{\varepsilon}_\lambda(\vec{k}) = \delta_{\lambda\lambda'}, \vec{k} \cdot \vec{\varepsilon}_\lambda(\vec{k}) = 0;
\]
\[
\vec{k} \times \vec{\varepsilon}_{R,L}(\vec{k}) = \mp i[\vec{k}] \vec{\varepsilon}_{R,L}.
\]

In terms of the physical modes the free photon action can be expressed as
\[
\int \frac{d^4 x \sqrt{-g}}{4} \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right)
= \frac{1}{2} \int d\tau \sum_\lambda \int d^3 k \left[ \left( \partial_\lambda A_\lambda(\vec{k}, \tau) \right) \left( \partial_\lambda A_\lambda(\vec{k}, \tau) \right) - \vec{k}^2 A_\lambda^2(\vec{k}, \tau) A_\lambda(\vec{k}, \tau) \right].
\]

Note that these two free physical photon modes are equivalent to two independent massless scalar fields, defined by
\[
A_\lambda(x, \tau) = \int \frac{d^3 k}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{x}} A_\lambda(\vec{k}, \tau).
\]
Next, we consider the inflaton photon interaction term. We have
\[
- \int \frac{d^4 x \sqrt{-g}}{4f} \phi F^{\mu\nu} F_{\mu\nu}
= - \frac{\alpha}{f} \int d^4 x \phi(x) \sum_\lambda \int \frac{d^3 k'}{(2\pi)^{3/2}} e^{i\vec{k}' \cdot \vec{x}} \left( \partial_\lambda A_\lambda(\vec{k}', \tau) \right) \left( \vec{\varepsilon}_\lambda(\vec{k}) \right) \times \int \frac{d^3 k}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{x}} \left[ A_\lambda(\vec{k}, \tau) \left( \vec{\varepsilon}_\lambda(\vec{k}) \right) - A_\lambda(\vec{k}, \tau) \left( \vec{\varepsilon}_\lambda(\vec{k}) \right) \right].
\]

To proceed we look more closely at the interaction between the two physical modes of the photon and the inflaton. To do that we expand the inflaton field around a homogeneous background, \( \phi(x, \tau) = \Phi(\tau) + \phi(\vec{x}, \tau) \). First we consider the effect of the homogeneous background on the photon modes. For the homogeneous term, the inflaton photon interaction term becomes
\[
- \int d^4 x \sqrt{-g} \frac{\alpha}{4f} \Phi F^{\mu\nu} F_{\mu\nu}
= \frac{\alpha}{2f} \int d\tau \left( \partial_\tau \Phi(\tau) \right) \int d^3 k |\vec{k}| \left[ A_\lambda^2(\vec{k}, \tau) A_\lambda(\vec{k}, \tau) - A_\lambda^2(\vec{k}, \tau) A_\lambda(\vec{k}, \tau) \right].
\]
where integrations by parts with respect to $\tau$ have been performed. This part can be combined with the free photon action to give

$$\frac{1}{2} \int d\tau \sum_{\lambda=R,L} \int d^3k A_{\lambda}(\vec{k}, \tau) \left[ \partial_\tau^2 + k^2 \mp 2aH\xi|\vec{k}| \right] A_{\lambda}(\vec{k}, \tau),$$  \hspace{1cm} (10)$$

where we have introduced the Hubble parameter and a dimensionless quantity, respectively,

$$H \equiv \frac{1}{a} \frac{da}{d\tau}, \quad \xi \equiv \frac{\alpha}{2fH} \frac{d\Phi}{d\tau},$$  \hspace{1cm} (11)$$

where $t$ is the cosmic time defined by $dt = a(\tau)d\tau$.

Due to the coupling with the homogeneous background of the inflaton field, $A_{R}(\vec{k}, \tau)$ will grow with $\tau$, while $A_{L}(\vec{k}, \tau)$ will be exponentially suppressed (see appendix A). In short, as long as the parameter $\xi \gg |k\tau|$, $A_{R}(\vec{k}, \tau)$ will be amplified by a factor of $e^{-\xi}$, while the other polarization $A_{L}(\vec{k}, \tau)$ will have no such amplification. In the following consideration we assume that the parameter $\xi$ is indeed large enough for this to happen. We will therefore retain only $A_{R}(\vec{k}, \tau)$ and will treat it as a scalar field $A(\vec{x}, \tau)$ with the above quadratic action (10). Hence we need to deal with the interaction term with $\phi(\vec{x}, \tau)$, which becomes

$$\int d^4x \sqrt{-g} \frac{\alpha}{4f} \phi \tilde{F}^{\mu\nu} F_{\mu\nu}$$

where $\tilde{F}^{\mu\nu}$ is the interaction kernel. The interaction action can be written in a more compact form as

$$S = S_0[A] + S_{\text{int}}[A, \phi]$$

Using the Fourier transform in equation (7), we have the following action for the photon field,

$$S = S_0[A] + S_{\text{int}}[A, \phi]$$

$$= -\frac{1}{2} \int d\tau \int d^3k A^*(\vec{k}, \tau) \left[ \partial_\tau^2 + k^2 - 2aH\xi|\vec{k}| \right] A(\vec{k}, \tau)$$

$$- \frac{\alpha}{f} \int d\tau \int d^3x d^3x' d^3x'' \phi(\vec{x}'', \tau) \left( \partial_\tau A(\vec{x}', \tau) \right) A(\vec{x}, \tau) J(\vec{x}, \vec{x}', \vec{x}''),$$  \hspace{1cm} (13)$$

where

$$J(\vec{x}, \vec{x}', \vec{x}'') = \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} e^{-ik(\vec{x}' - \vec{x}'')} e^{-ik(\vec{x} - \vec{x}'')} |\vec{k}|(\vec{\epsilon}_R(\vec{k}) : \vec{\epsilon}_R(\vec{k}'))$$

$$= \mathcal{J}(\vec{x} - \vec{x}'', \vec{x}' - \vec{x}'').$$  \hspace{1cm} (14)$$

is the interaction kernel. The interaction action can be written in a more compact form as

$$S_{\text{int}}[A, \phi] = \frac{\alpha}{f} \int d^3x d^3x' d^3x'' A(x)A(x')\phi(x'')J(x, x', x''),$$  \hspace{1cm} (15)$$

where

$$J(x, x', x'') = \delta(\tau - \tau'')\partial_\tau \delta(\tau' - \tau'') \mathcal{J}(\vec{x}, \vec{x}', \vec{x}'')$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} e^{-i\omega(\vec{x}' - \vec{x}'')} \epsilon_{\mu'}(\vec{k}) \epsilon_{\mu}(\vec{k}'')$$

$$= \mathcal{J}(\vec{x} - \vec{x}'', \vec{x}' - \vec{x}'').$$  \hspace{1cm} (16)$$

is the four dimensional interaction kernel.
To implement the CTP formalism we use two copies of both the inflaton and the photon fields, and write the CTP action as

$$S = S_A[A_+] - S_A[A_-] + S_{\text{int}}[A_+, \phi_+] - S_{\text{int}}[A_-, \phi_-].$$

The corresponding influence action $S_{\text{IF}}$ is defined by the path integral over the photon degrees of freedom,

$$e^{iS_{\text{IF}}} = \int_{\text{CTP}} DA_+ DA_- e^{i\delta}.$$  \hspace{1cm} (18)

with the CTP boundary conditions. Since we shall follow the perturbative approach, we add sources to the photon action and express the influence functional $e^{iS_{\text{IF}}}$ as

$$\int_{\text{CTP}} DA_+ DA_- e^{i\delta+\int J_+A_+-i\int J_-A_-}$$

$$= e^{i\delta \left[ \frac{1}{2} \frac{\delta^2}{\delta F_+^{\alpha}} + \frac{1}{2} \frac{\delta^2}{\delta F_-^{\alpha}} \right] \int_{\text{CTP}} DA_+ DA_- e^{i(S_0[A_+] - S_0[A_-] + \int J_+A_+ - \int J_-A_-)}}$$ \hspace{1cm} (19)

This CTP integral can be evaluated using the Schwinger–Keldysh Green’s functions of the photon field $A(\vec{x}, \tau)$. As a scalar field $A(\vec{x}, \tau)$ can be written in canonical form,

$$\hat{A}(\vec{x}, \tau) = \int \frac{d^3k}{(2\pi)^3/2} \left[ \hat{a}(\vec{k}) A(\vec{k}, \tau)e^{i\vec{k} \cdot \vec{x}} + \text{h.c.} \right],$$

where the mode function $A(\vec{k}, \tau) \, (20)$ satisfies

$$\left[ \frac{\partial^2}{\partial \tau^2} + \vec{k}^2 - 2aH\xi |\vec{k}| \right] A(\vec{k}, \tau) = 0.$$ \hspace{1cm} (21)

It is chosen in such a way that as $\tau \to -\infty$ it coincides with the Minkowski vacuum mode function (see appendix A). The explicit form of this mode function will be given in the next section. Then the Schwinger–Keldysh Green’s functions can be expressed in terms of $A(\vec{k}, \tau)$. In particular, the Green’s function $G_{++}(x, x')$ can be written as

$$-iG_{++}(x, x')$$

$$= \frac{\delta}{\delta J_+(x)} \left. \frac{\delta}{\delta J_+(x')} \int_{\text{CTP}} DA_+ DA_- e^{i(S_0[A_+] - S_0[A_-] + \int J_+A_+ - \int J_-A_-)} \right|_{J_+ = J_- = 0}$$

$$= -\langle T\hat{A}_+(x)\hat{A}_+(x') \rangle.$$ \hspace{1cm} (22)

Hence, in terms of the mode function $A(\vec{k}, \tau)$, this Green’s function can be expressed as

$$G_{++}(x, x') = -i\theta(\tau' - \tau) \int \frac{d^3k}{(2\pi)^3} A(\vec{k}, \tau)A^*(\vec{k}, \tau') e^{i\vec{k} \cdot (\vec{x} - \vec{x}')}$$

$$- i\theta(\tau - \tau') \int \frac{d^3k}{(2\pi)^3} A(\vec{k}, \tau')A^*(\vec{k}, \tau) e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')}. \hspace{1cm} (23)$$
Similarly for the other Green’s functions, we have

\[ G_{+-}(x, x') = -i \frac{\delta}{\delta J_+(x')} \frac{\delta}{\delta J_-(x)} \int_{\text{CTP}} DA_+ DA_- e^{i(S_0[A_+] - S_0[A_-] + J_+A_+ - J_-A_-)} \bigg|_{J_+ = J_- = 0} \]

\[ = -i(\mathcal{A}_-(x') \mathcal{A}_+(x)) \]

\[ = -i \int \frac{d^3k}{(2\pi)^3} A(k, \tau) A^*(\vec{k}, \tau') e^{i\vec{k} \cdot (x - x')} \]

\[ = G_{+}(x, x'), \tag{24} \]

\[ G_{-+}(x, x') = -i \frac{\delta}{\delta J_+(x')} \frac{\delta}{\delta J_-(x)} \int_{\text{CTP}} DA_+ DA_- e^{i(S_0[A_+] - S_0[A_-] + J_+A_+ - J_-A_-)} \bigg|_{J_+ = J_- = 0} \]

\[ = -i(\mathcal{A}_-(x') \mathcal{A}_+(x)) \]

\[ = -i \int \frac{d^3k}{(2\pi)^3} A(k, \tau) A^*(\vec{k}, \tau') e^{i\vec{k} \cdot (x - x')} \]

\[ = G_{+}(x', x), \tag{25} \]

\[ G_{--}(x, x') = i \frac{\delta}{\delta J_+(x')} \frac{\delta}{\delta J_-(x)} \int_{\text{CTP}} DA_+ DA_- e^{i(S_0[A_+] - S_0[A_-] + J_+A_+ - J_-A_-)} \bigg|_{J_+ = J_- = 0} \]

\[ = -i(T \mathcal{A}_-(x) \mathcal{A}_+(x')) \]

\[ = -i \Theta(\tau - \tau') \int \frac{d^3k}{(2\pi)^3} A(k, \tau') A^*(\vec{k}, \tau) e^{i\vec{k} \cdot (x - x')} \]

\[ - i \Theta(\tau' - \tau) \int \frac{d^3k}{(2\pi)^3} A(k, \tau) A^*(\vec{k}, \tau') e^{i\vec{k} \cdot (x - x')} \]

\[ = -G_{+}(x, x'), \tag{26} \]

where \( T \) is for the anti-time ordered operation.

From the representations of these Green’s functions in terms of the functional derivatives with respect to the sources, it is possible to express the zeroth order CTP integral in equation (19) as

\[ \int_{\text{CTP}} DA_+ DA_- e^{i(S_0[A_+] - S_0[A_-] + J_+A_+ - J_-A_-)} \]

\[ = e^{-i \frac{1}{2} \int (J_+ G_+ + J_- G_- - J_+ G_- - J_- G_+) \}

\[ \tag{27} \]

Putting this result back into equations (18) and (19) the influence functional \( S_{IF} \) can be evaluated in a perturbative manner in powers of \( \alpha/f \), the parameter in the interaction action. Later, we will show that the dimensionless expansion parameter in our calculation is actually \( \alpha H/f \), as evident from the results in equations (67) and (70). In terms of the slow-roll parameter defined by \( \epsilon = (d\phi/d\tau)^2/(2H^2M_p^2) \), we have \( \xi = \alpha M_P / f \sqrt{\epsilon}/2 \). In realistic inflation models [16, 17, 19, 22, 25, 26], assuming \( f = M_P \), the Hubble parameter is \( H \sim 10^{-5} M_P \). The Planck CMB measurement of the \( \epsilon \) parameter implies that \( \alpha/f < 32 \) or \( \xi < 2.2 \) on CMB scales. If \( \xi = 2.2 \) in the beginning of inflation, \( \xi \) will grow to values of about 10 at later stage of inflation. Since we
have $\alpha H/\mathcal{f} < 10^{-3}$, our perturbative expansion approach is justified. The first-order term gives the in-in expectation value of the interaction. Up to the second order in the inflaton field it is possible to identify the corresponding noise and dissipation kernels due to the inflaton–photon interaction. We shall work this out in detail in the next two sections. Subsequently one can also obtain the equation of motion of the inflaton field by taking functional variation on the influence action. This equation of motion will be in the form of a Langevin equation with a stochastic force of which the correlation function is given by the noise kernel.

### 3. Influence functional

In this section we shall detail the evaluation of the influence action $S_{\text{IF}}$ in powers of the parameter $\alpha/\mathcal{f}$. As shown in the last section this is achieved by taking functional derivatives on the zeroth order CTP integral which is given in terms of the various Schwinger–Keldysh Green’s functions. These functions are constructed using the modefunction of the corresponding quantum field (equations (23)–(26)). In the case where $\xi$ is large as compared to $|\kappa\mathcal{f}|$, the right-handed photon $A_R(k, \tau)$ is enhanced while the left-handed one is not. Therefore we can use the large $\xi$ approximation of $A_R(k, \tau)$ (equation (A6)) for mode function $A(k, \tau)$. Dropping the constant angular phase in $A_R(k, \tau)$, we have

$$A(k, \tau) = \frac{1}{\sqrt{2k}} \left( -\frac{k\tau}{2\xi} \right)^{1/4} e^{k\xi} e^{-2\sqrt{-2k\xi\tau}}. \quad (28)$$

Note that $A(k, \tau)$ is real and depends only on $k$. As a consequence, all the Schwinger–Keldysh Green’s functions are equal:

$$G_{++}(x, x') = G_{--}(x, x') = G_{+-}(x', x) = G_{-+}(x, x') \equiv G(x, x');$$

$$G(x, x') = -i \int \frac{d^3k}{(2\pi)^3} A(k, \tau)A(k, \tau')e^{i\vec{k}(\vec{x}' - \vec{x})}. \quad (29)$$

Now we are ready to expand the influence functional $e^{\delta S_{\text{IF}}}$ (19) in powers of $\alpha/\mathcal{f}$ in $S_{\text{int}}$ (15),

$$S_{\text{IF}} = \sum_{i=1}^{\infty} \delta S_{\text{IF}}^{(i)}. \quad (30)$$

Following equation (19) the first order $\delta S_{\text{IF}}^{(1)}$ is obtained by second functional derivatives on the zeroth order CTP integral (equation (27)).

$$\delta S_{\text{IF}}^{(1)} = \frac{\alpha}{\mathcal{f}} \int d^4x'd^4x''[\frac{1}{i} \frac{\delta}{\delta J_+}(x) \frac{\delta}{\delta J_+}(x') \phi_+(x'') - \frac{1}{i} \frac{\delta}{\delta J_-}(x) \frac{\delta}{\delta J_-}(x') \phi_-(x'')] \times \mathcal{F}(x, x', x'').$$

$$= \frac{i\alpha}{\mathcal{f}} \int d^4x'd^4x'' \mathcal{F}(x, x', x'')G(x, x')\Delta \phi(x''), \quad (31)$$

where we have defined $\Delta \phi(x) \equiv \phi_+(x) - \phi_-(x)$. Using equations (5), (16) and (29), it is straightforward to show that

$$\int d^4x' \mathcal{F}(x, x', x'')G(x, x') = \frac{i}{2} \partial_x \int \frac{d^3k}{(2\pi)^3} k A(k, \tau'')A(k, \tau''). \quad (32)$$
where $\Delta(k, \tau)$ is given by the large $\xi$ approximation in equation (28). In the leading order of the asymptotic expansion of $1/\xi$, this integral can be evaluated readily to give

$$
\int d^4x d^4x' \mathcal{F}(x, x', x'')G(x, x') = i \left( \frac{135}{65536\pi^2} \right) \frac{e^{2\pi i \xi}}{\xi^2 \tau}.
$$

(33)

Hence, we reach the final form of the first-order term as

$$
\delta \mathcal{S}_{1}^{(1)} = - \alpha \frac{e^{2\pi i \xi}}{f} \int d^4x \Delta \phi(x) \frac{135}{65536\pi^2\xi^2 \tau}.
$$

(34)

For the second-order term, we have the following functional derivatives.

$$
\mathcal{F}(x, x', x'') = e^{\frac{-i}{f}} (J_{\phi} + J_{\phi}^{(2)})^2 e^{\frac{-i}{f}} (J_{\phi} + J_{\phi}^{(2)})^2 e^{\frac{-i}{f}} (J_{\phi} + J_{\phi}^{(2)})^2.
$$

$$
\Delta \mathcal{F}(x, x', x'') = \mathcal{F}(x_1, x_2, x_3) \mathcal{F}(x_1, x_2, x_3) + \mathcal{F}(x_1, x_2, x_3) \mathcal{F}(x_1, x_2, x_3)
$$

(35)

The first term in the brackets above represents the disconnected diagram which we shall ignore. For the other two terms we follow the asymptotic analysis in [34] to obtain the leading contribution in $1/\xi$ in the evaluation of the corresponding integrals. We then again arrived at the following asymptotic expansions:

$$
\int d^4x d^4x' d^4x d^4x' \mathcal{F}(x_1, x_2, x_3)G(x_1, x_3)G(x_2, x_4) \mathcal{F}(x_1, x_2, x_3)
$$

$$
= \int \frac{dk}{(2\pi)^3} e^{i k (\xi - \phi)} \frac{1}{(\sqrt{\tau_5} + \sqrt{\tau_6})} e^{-2\sqrt{2k_0}(\sqrt{\tau_5} + \sqrt{\tau_6}) + \ldots}.
$$

(36)

$$
\int d^4x d^4x' d^4x d^4x' \mathcal{F}(x_1, x_2, x_3)G(x_1, x_3)G(x_2, x_4) \mathcal{F}(x_1, x_2, x_3)
$$

$$
= \int \frac{dk}{(2\pi)^3} e^{i k (\xi - \phi)} \frac{1}{(\sqrt{\tau_5} + \sqrt{\tau_6})} e^{-2\sqrt{2k_0}(\sqrt{\tau_5} + \sqrt{\tau_6}) + \ldots}.
$$

(37)

With these expansions the second order term in equation (30) can be expressed as

$$
\delta \mathcal{S}_{1}^{(2)} = i \left( \frac{\alpha^2}{2f^2} e^{2\pi i \xi} \right) \int d^4x d^4x' \Delta \phi(x) \Delta \phi(x') \int \frac{dk}{(2\pi)^3} e^{i k (\xi - \phi)} e^{-2\sqrt{2k_0}(\sqrt{\tau_5} + \sqrt{\tau_6}) + \ldots}
$$

$$
\left\{ \left( \frac{15 k^{3/2}}{256\sqrt{2\pi^2 \xi^{3/2}}} \right) + \left( \frac{105 k}{1024\pi^2 \xi^4} \right) \right\}.
$$

(38)
where the ellipsis represents terms higher order in $1/\xi$ in the asymptotic expansion.

Similarly, omitting the disconnected terms, we can write the third-order term as

\[
\frac{\alpha^3}{6 f^3} \left\{ \int d^4 x d^4 x' d^4 x'' \left[ \frac{\delta}{i \delta J_+(x)} \frac{\delta}{i \delta J_+(x')} \phi_+(x') - \frac{\delta}{-i \delta J_-(x)} \frac{\delta}{-i \delta J_-(x')} \phi_-(x') \right] \right\}
\]

\[
F(x, x', x'') \right|_{J^+ = J^- = 0} = - \frac{\alpha^3}{6 f^3} \int \Delta^3(x) \Delta^3(x') \Delta^3(x'') \Delta^3(0) \frac{\phi_+(x') \phi_+(x'') \phi_-(x') \phi_-(x'')}{F(x_1, x_2, x_3) F(x_4, x_5, x_6, x_7) \cdots}
\]

\[
G(x_1, x_2, x_3) G(x_4, x_5, x_6) + G(x_1, x_3) G(x_4, x_6, x_7) + G(x_1, x_4) G(x_2, x_3, x_5) + G(x_1, x_5) G(x_2, x_4, x_6) + G(x_1, x_6) G(x_2, x_3, x_4) \cdots
\]

\[
+ G(x_1, x_4) G(x_2, x_5) G(x_3, x_6) + G(x_1, x_5) G(x_2, x_3) G(x_4, x_5) \right).
\]

(39)

From the symmetry of the terms we can separate them into two groups, one being with the first six terms and the other with the last two terms. Note that in the first group, two of the coordinates, $x_2, x_4,$ and $x_6,$ are in one Green’s function. In the last two terms, these coordinates are present in the three different Green’s functions.

Let us evaluate the integral involving the first term. Following similar steps in dealing with the terms in $\delta S_{\text{IW}}^2,$ we obtain

\[
\int d^4 x_1 d^4 x_2 d^4 x_3 d^4 x_4 d^4 x_5 d^4 x_6 \Delta^3(x_1, x_2, x_3) \Delta^3(x_4, x_5, x_6) \Delta^3(0)
\]

\[
G(x_1, x_3) G(x_2, x_5) G(x_4, x_6)
\]

\[
= \frac{\alpha^3}{6 f^3} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} \epsilon^{\hat{k} \cdot (\hat{\xi} - \hat{x})} \epsilon^{\hat{k}' \cdot (\hat{\xi} - \hat{x}')} \epsilon^{\hat{k} \cdot (\hat{\xi} - \hat{0})} \epsilon^{\hat{k}' \cdot (\hat{\xi} - \hat{0})} \epsilon^{\hat{k} \cdot (\hat{\xi} - \hat{0})} \epsilon^{\hat{k}' \cdot (\hat{\xi} - \hat{0})}
\]

\[
\left[ -i \left( \frac{105 k^3/2 k'^{1/2}}{8192 \pi^2 \xi^4} \right) \left( 1 + \frac{\hat{k} \cdot \hat{k}'}{kk'} \right) \right]
\]

\[
\frac{e^{-2 \sqrt{2} \xi \overline{\xi} + \sqrt{\overline{\xi} \xi} + \sqrt{\overline{\xi} + \overline{\xi}} \xi + \sqrt{\overline{\xi} + \overline{\xi}} \xi}}{2 \xi^8} + \ldots
\]

(40)

The other five integrals in the same group can be easily worked out by making permutations of $\hat{x}_1, \hat{x}_2,$ and $\hat{x}_3.$ The last two integrals in equation (39) can be similarly evaluated. Hence, the third-order term in equation (30) is

\[
\delta S_{\text{IW}}^{(3)}
\]

\[
= \left( \frac{\alpha^3}{6 f^3} \right) \int d^4 x_1 d^4 x_2 d^4 x_3 \Delta^3(x_1) \Delta^3(x_2) \Delta^3(x_3)
\]

\[
\int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} \epsilon^{\hat{\xi} \cdot \hat{x}_1} \epsilon^{\hat{\xi} \cdot \hat{x}_2} \epsilon^{\hat{\xi} \cdot \hat{x}_3}
\]
Let us summarize what we have obtained for the perturbative expansion of the influence action $S_{\text{IF}}$. Up to the third order, we have

$$S_{\text{IF}}$$

$$= - \int d^4x \sqrt{-g(x)} \delta V(x) \Delta \phi(x)$$

$$+ \frac{i}{2} \int d^4x_1 d^4x_2 \sqrt{-g(x_1)} \sqrt{-g(x_2)} \Delta \phi(x_1) N_2(x_1, x_2) \Delta \phi(x_2)$$

$$+ \frac{1}{6} \int d^4x_1 d^4x_2 d^4x_3 \sqrt{-g(x_1)} \sqrt{-g(x_2)} \sqrt{-g(x_3)} \Delta \phi(x_1) \Delta \phi(x_2) \Delta \phi(x_3) N_3(x_1, x_2, x_3)$$

$$+ \ldots$$

(42)

where

$$\delta V(x) = \frac{1}{a(r)} \left( \frac{\alpha}{f(x)} e^{2\pi \xi} \right) \left( \frac{135}{6536\pi^2} \right) \frac{1}{\xi^4}$$

(43)

$$N_2(x_1, x_2)$$

$$= \frac{1}{a^4(r_1) a^4(r_2)} \left( \frac{\alpha^2}{f^2} e^{4\pi \xi} \right) \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} e^{-2\sqrt{\xi(k_1^2 + k_2^2)}}$$

$$\left\{ \left[ \frac{15 k^{3/2}}{256 \sqrt{2} \pi^2 \xi^{7/2}} \right] \frac{1}{(\sqrt{-\tau_1} + \sqrt{-\tau_2})^2} + \left[ \frac{105 k}{1024 \pi^2 \xi^4} \right] \frac{1}{(\sqrt{-\tau_1} + \sqrt{-\tau_2})^6} \right\} + \ldots$$

(44)

$$N_3(x_1, x_2, x_3)$$

$$= \frac{1}{a^4(r_1) a^4(r_2) a^4(r_3)} \left( \frac{\alpha^3}{f^3} e^{6\pi \xi} \right) \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} e^{i\vec{k'} \cdot (\vec{x}_2 - \vec{x}_3)}$$

$$\times \left\{ \left[ \frac{105 (k^{3/2} k'^{1/2} + kk')}{8192 \pi^2 \xi^4} \right] \left( 1 + \frac{\vec{k} \cdot \vec{k'}}{kk'} \right) \frac{e^{-2\sqrt{\xi(k_1^2 + k_2^2)} \sqrt{\xi(k_2^2 + k_3^2)} \sqrt{\xi(k_3^2 + k_1^2)}}}{(\sqrt{-\tau_1} + \sqrt{-\tau_2})^2} \right\} + 5 \text{ permutations of } (x_1, x_2, x_3) + \ldots$$

(45)
4. Noise kernel and the Langevin equation

The influence action $S_{IF}$ obtained in the previous section has a stochastic interpretation in analogy to the quantum Brownian motion model [33]. In particular, the terms quadratic and cubic in $\Delta \phi$ in $S_{IF}$ can be represented as the effect of a stochastic force.

In the usual approach, where the cubic term is absent, the influence action can be rewritten as a path integral over a stochastic force $\zeta$,

$$e^{-\frac{1}{2} \int \Delta \phi N_2 \Delta \phi} \equiv \int D\zeta \, P_g[\zeta] e^{-i \int \zeta \Delta \phi},$$

where $P_g[\zeta]$ is a Gaussian probability functional

$$P_g[\zeta] = \det(2\pi N_2)^{-1/2} e^{-\frac{1}{2} \int \zeta N_2^{-1} \zeta}.$$  (47)

With this probability functional one obtain the correlation function of the stochastic force $\zeta$

$$\langle \zeta(x_1) \zeta(x_2) \rangle = \int D\zeta \, P_g[\zeta] \zeta(x_1) \zeta(x_2) = N_2(x_1, x_2).$$

(48)

Therefore, $N_2(x_1, x_2)$ is referred to as the noise kernel characterizing the quantum fluctuations of the environment, that is, the photon field.

Now with the quadratic and the cubic $\Delta \phi$ terms in equation (42), it is still possible to rewrite them as the effect of a single stochastic force.

$$e^{-\frac{1}{2} \int \Delta \phi N_2 \Delta \phi + \frac{i}{6} \int \Delta \phi \Delta \phi \Delta \phi N_3} = \int D\zeta \, P[\zeta] e^{-i \int \zeta \Delta \phi},$$

where $P[\zeta]$ is again a probability distribution functional normalized to $\langle 1 \rangle_{\zeta} \equiv \int D\zeta P[\zeta] = 1$. Evaluating the various correlation functions of $\zeta$, we have

$$\langle \zeta(x) \rangle = \int D\zeta P[\zeta] \zeta(x) = 0,$$

(50)

$$\langle \zeta(x_1) \zeta(x_2) \rangle = \int D\zeta P[\zeta] \zeta(x_1) \zeta(x_2) = N_2(x_1, x_2),$$

(51)

$$\langle \zeta(x_1) \zeta(x_2) \zeta(x_3) \rangle = \int D\zeta P[\zeta] \zeta(x_1) \zeta(x_2) \zeta(x_3) = N_3(x_1, x_2, x_3).$$

(52)

The outstanding feature of these correlators is that the three-point correlation function is non-vanishing. The probability functional $P[\zeta]$ of the stochastic force is therefore nongaussian. As we shall see in the next section, this will be the source of nongaussianity for the fluctuations of the inflaton field.

As in the usual case one can identify $N_2(x_1, x_2)$ as the noise kernel originated from the electromagnetic interaction. According to the dissipation–fluctuation theorem, one would expect a corresponding dissipation kernel. However, it is absent from the influence action in equation (42). In fact, the dissipation and noise kernels are derived from the real and the imaginary parts of the Green’s function of the quantum field, respectively. In the large $\xi$ limit, the mode function in equation (28) is actually real, and the corresponding Green’s function in equation (29) is therefore purely imaginary. As given in equation (44), the noise kernel in $N_2(x_1, x_2) \sim e^{4\xi^2}$. This indicates that the instability of the photon enhances its quantum fluctuations, while the dissipation effects are not enhanced in comparison to that. This could be a general phenomenon which is worth further investigation.
Next, we investigate the corresponding equation of motion for the inflaton field with the effects of the noise kernel included. From the action for the inflaton field $\phi$ in equation (1) up to the quadratic order,

$$S_\phi = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} V''(\Phi) \phi^2 \right].$$

(53)

Hence, doing the variation

$$\frac{\delta}{\delta \phi} \left( S_{\phi}^{\phi} - S_{\phi}^{\phi-} + S_{IF}^{\phi+} \right) \big|_{\phi_{-} = \phi_{+} = 0} = 0,$$

(54)

the equation of motion for $\phi$ can be obtained as, with $S_{IF}$ given by equations (18) and (49),

$$\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu \nu} \partial_{\nu} \phi \right) - V''_{\text{ren}}(\Phi) \phi = \zeta,$$

(55)

which is in the form of a Klein–Gordon–Langevin equation with the stochastic force term given by $\zeta$. Note that we have defined $V''_{\text{ren}}(\Phi) = V''(\Phi) + \delta V$. In the metric (3), it reads

$$\partial_{\tau}^2 \phi + 2H \partial_{\tau} \phi - \nabla^2 \phi + a^2 V''_{\text{ren}}(\Phi) \phi = -a^2 \zeta.$$

(56)

We can see that the effect of the photon field, other than the renormalization of the potential, is the production of the stochastic force $\zeta$. The mode function of this equation without the stochastic force term can be solved in the usual manner by choosing the Bunch–Davies vacuum. One can construct the retarded Green’s function from this mode function. With this Green’s function, the mode function with the stochastic force in the inhomogeneous equation can be obtained. Then we can derive the correlation functions of the inflaton field by taking the expectation value of the inflaton field as well as the stochastic average. However, one can directly work with the path integral by putting in source terms. We shall derive the correlation functions in this approach in the next section.

Most literature has suggested that the backreaction due to the photon production would give rise to additional friction in the Hubble expansion term, $3\beta H \partial_{\tau} \phi$, in equation (56) [7, 19, 22, 26], where

$$\beta \equiv 1 - 2\pi \xi \frac{\langle \vec{E} \cdot \vec{B} \rangle}{J (3H d\Phi/dt)},$$

(57)

which can be much larger than one in the regime of strong backreaction. When $\xi < 4$, the backreaction is negligible, so $\beta \approx 1$; for $\xi \gtrsim 6$, the strong backreaction leads to $\beta \approx 100$, which can efficiently dampen the inflaton fluctuations [22]. This additional friction can be considered to be playing a role as dissipation in the axion–photon system. In the Schwinger–Keldysh formalism, at one-loop level the dissipation manifests as a non-local term in the Langevin equation which can be approximated as a local friction term, $3(\hat{H} + \Gamma) \partial_{\tau} \phi$, where the $\Gamma$ term should be related to the imaginary part of the one-loop correction. However, in the present work we have shown that in the one-loop level the additional friction term is absent and we have $\beta = 1$. This seems to contradict the fluctuation–dissipation theorem. One should be reminded that the theorem holds only in the consideration of perturbative processes. It does not apply to the present case, because there is the spinodal instability in which photon obtains a negative effective mass and becomes tachyonic, thus making all the Schwinger–Keldysh photon Green’s functions equal. As a result, the photon production is a perfect energy sink rather than a dissipating process. Our result of $\beta = 1$ would then call for a re-examination of the condition...
for the energy conservation in the axion–photon system. It would be interesting to investigate
the relation between the growth of the inflaton fluctuations and the noise term. Could the noise
term serve as the backreaction to limit the growth of the inflaton fluctuations in a self-consistent
manner? Otherwise, one should take into account the effects of gravitation, which could
presumably dampen the high energy density peaks resulted from large inflaton fluctuations to a
consistent level [24].

5. Correlation functions

Now we work on the correlation functions of the perturbation \( \phi \). To do that we go back to
the path integral. We add source terms so that the correlation functions can be obtained by
taking functional derivatives with respect to these sources. We shall consider both the two-
point and three-point correlation functions. The two-point correlation function is related to the
power spectrum of the CMB anisotropy, while the three-point one to the nongaussianity of the
spectrum.

To start with we consider the CTP path integral of the effective action \( \Gamma \), including the
influence action in equation (42) as well as source terms for \( \phi_+ \) and \( \phi_- \).

\[
e^{i\Gamma[J_+,J_-]}
\]

\[
= \int D\phi_+ D\phi_- \int D\zeta P[\zeta] \frac{1}{e^{\frac{1}{2} \left[ \frac{i}{2} (\phi_+)^2 - \frac{i}{2} \partial \phi \right] \zeta}} + i \int J_+ \phi_+ \\
\]

\[
= \int D\zeta P[\zeta] e^{\frac{1}{2} \left[ (U_+ - \zeta \delta G_{+\phi}^0 U_+ - \zeta) - (U_+ - \zeta \delta G_{+\phi}^0 U_+ - \zeta) - (U_- - \zeta \delta G_{-\phi}^0 U_- - \zeta) + (U_- - \zeta \delta G_{-\phi}^0 U_- - \zeta) \right]},
\]

(58)

where we have imposed the CTP boundary conditions to obtain the various
Schwinger–Keldysh Green’s functions. For the inflaton field these Green’s functions are

\[
G_{+\phi}^0(x,x') = -i(T\phi^{(0)}(x)\phi^{(0)}(x')), \quad G_{+\phi}^0(x,x') = -i(\phi^{(0)}(x)\phi^{(0)}(x)), \\
G_{-\phi}^0(x,x') = -i(\phi^{(0)}(x)\phi^{(0)}(x)'), \quad G_{-\phi}^0(x,x') = -i(T\phi^{(0)}(x)\phi^{(0)}(x)'),
\]

(59)

where \( \phi^{(0)}(x) \) is the free inflaton field, and \( T \) is for anti-time ordered.

The two-point correlator of the inflaton field, including the backreaction of the electromagnetic
interaction, can be obtained by taking functional derivatives on the effective action with respect to
the sources.

\[
\langle \phi(x)\phi(x') \rangle = \langle \phi_-(x)\phi_+(x') \rangle \bigg|_{\phi_+ = \phi_- = 0} = \frac{1}{\sqrt{-g(x)}} \frac{\delta}{\delta J_-(x)} \frac{1}{\sqrt{-g(x')}} \frac{\delta}{\delta J_+(x')} e^{i\Gamma[J_+,J_-]} \bigg|_{J_+ = J_- = 0}
\]
quantization the inflaton field, The first term is the usual two-point correlator of the inflaton vacuum fluctuations. Under

\[ \left. \frac{d}{dx_1} \frac{d^4}{d^4 x_2} g(x_1) \right] \int d^4 x_2 \sqrt{-g(x_2)} \]

\[ \times \left[ G_{-+}^0 (x, x_1) - G_{-+}^0 (x, x_1) \right] \zeta (x_1) \zeta (x_2) \left[ -G_{++}^0 (x_2, x') + G_{++}^0 (x_2, x') \right] + \cdots \right\}. \]

(60)

Let us recall that the combination of the Green’s functions

\[ G_{-+}^0 (x, x_1) - G_{-+}^0 (x, x_1) = -i \langle \phi (0) \phi (0) (x_1) \rangle + i \langle \hat{T} \phi (0) (x_1) \phi (0) (x_1) \rangle \]

\[ = -i \theta (\tau - \tau_1) \left( \phi (0), \phi (0) (x_1) \right) \]

\[ = \mathcal{G}_{\text{ret}}^0 (x, x_1), \]

(61)

gives exactly the retarded Green function \( G_{\text{ret}}^0 (x, x') \). The appearance of this retarded Green’s function is related to the fact that the equation of motion derived from the in-in or Schwinger–Keldysh formalism respects causality explicitly. In a similar fashion, the other combination of the Green’s functions also gives the retarded Green’s function

\[ -G_{++}^0 (x_2, x') + G_{++}^0 (x_2, x') = i (T \phi (0) (x_2) \phi (0) (x')) - i \langle \phi (0) (x_2) \phi (0) (x') \rangle \]

\[ = G_{\text{ret}}^0 (x', x_2), \]

(62)

Moreover, according to the definition of the probability density in equation (51), the stochastic average of two \( \zeta \)'s is just the noise kernel \( N_2 (x_1, x_2) \). Therefore, the two-point correlator in equation (60) becomes

\[ \langle \phi (x) \phi (x') \rangle = i G_{-+}^0 (x, x') + \int d^4 x_1 \sqrt{-g(x_1)} \int d^4 x_2 \sqrt{-g(x_2)} G_{\text{ret}}^0 (x, x_1) G_{\text{ret}}^0 (x', x_2) N_2 (x_1, x_2) + \cdots \]

(63)

The first term is the usual two-point correlator of the inflaton vacuum fluctuations. Under quantization the inflaton field,

\[ \phi (0) (x) = \int d^3 k [ a_\xi \varphi_n (\tau, \vec{x}) + a_\xi^\dagger \varphi_n^* (\tau, \vec{x})] \]

(64)

where \( \varphi_n (\tau, \vec{x}) \) is the mode function. During the inflationary epoch, the mode function can simply be represented by

\[ \varphi_n (\tau, \vec{x}) = \left( -\frac{H}{2^{3/2} \pi k^{3/2}} \right) (-k \tau)^{3/2} H_{3/2}^{(1)} (-k \tau) e^{i \vec{k} \cdot \vec{x}} \]

(65)

when the Bunch–Davies vacuum is chosen. With this mode function the first term above at equal conformal time can be readily evaluated to be

\[ i G_{-+}^0 \bigg| _{\tau_1 = \tau} = \int \frac{d^3 k}{(2 \pi)^{3/2}} \int \frac{d^3 k'}{(2 \pi)^{3/2}} e^{i \vec{k} \cdot \vec{x}} e^{i \vec{k}'} \delta (\vec{k} + \vec{k}') \left( \frac{2 \pi^2}{k^3} \right) P_{\phi}^{(0)}, \]

(66)

where

\[ P_{\phi}^{(0)} = \frac{H^2}{4 \pi^2} (1 + k^2 \tau^2) \]

(67)
is the power spectrum of the free inflaton field. As $\tau \to 0$ at the end of inflation, we recover the scale-invariant power spectrum, $P^0(0) = H^2/(4\pi^2)$.

The second term in equation (63) gives the backreaction effect due to the electromagnetic interaction. It involves the retarded Green’s function of the free inflaton field. In terms of the mode function in equation (65),

$$G_\text{ret}(x, x') = i\theta(\tau - \tau') \int d^3k \left[ \varphi_k(\tau, \vec{x})\varphi^*_k(\tau', \vec{x}') - \varphi^*_k(\tau, \vec{x})\varphi_k(\tau', \vec{x}') \right]$$  \hspace{1cm} (68)

Together with the noise kernel in equation (44), one can work on the integrals in this second term. As in the evaluation of the influence action in section 3, we shall again develop an asymptotic expansion in $1/\xi$. In this manner, at equal conformal time, the second term gives

$$\int d^4x_1 \sqrt{-g(x_1)} \int d^4x_2 \sqrt{-g(x_2)} G_\text{ret}(x, x_1)G_\text{ret}(x', x_2)N_2(x_1, x_2) \bigg|_{\tau' \to \tau}$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d^3k'}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} e^{i\vec{k}'\cdot\vec{x}'} \left( \frac{\alpha^2}{f^2} e^{4\pi\xi} \right) \left( \frac{H^4}{4\pi^2k^3\xi^8} \right) e^{-4\sqrt{2}\rho} \left[ \frac{105}{262144}\rho^2 + \frac{15}{32768\sqrt{2}}\rho^{5/2} \right] + \ldots \hspace{1cm} (69)$$

where we have defined $\rho \equiv -\xi k\tau$. From this result one can obtain the correction to the power spectrum [16],

$$P^{(1)}(0) = \left( \frac{\alpha^2}{f^2} \right) \left( \frac{H^2}{4\pi^2} \right)^2 \left( \frac{e^{4\pi\xi}}{\xi^8} \right) h_2(\rho)$$ \hspace{1cm} (70)

where the function

$$h_2(\rho) = e^{-4\sqrt{2}\rho} \left[ \frac{105}{131072}\rho^2 + \frac{15}{16384\sqrt{2}}\rho^{5/2} \right]. \hspace{1cm} (71)$$

Before taking $\tau \to 0$ at the end of inflation to find the value of $h_2(\rho)$, the reader should be reminded that the expression in equation (28) is an approximated solution to the mode function $A_k(k, \tau)$ in the range,

$$\frac{1}{8\xi} \leq -k\tau \leq 2\xi, \hspace{1cm} (72)$$

which corresponds to the period during which the right-handed gauge field is maximally amplified [7, 16]. In order for the approximation to work, the value of $\rho$ must be at least $1/8$, so we have

$$h_2|_{\rho=1/8} \sim 2.18 \times 10^{-6}. \hspace{1cm} (73)$$

We have also found the maximum value,

$$h_2|_{\text{max}} \sim 5.86 \times 10^{-6} \hspace{1cm} (74)$$

at $\rho \sim 0.6$. 


Next we work on the three-point correlator. There are four types of functional derivatives:

\[- \frac{1}{\sqrt{-g(x)}} \frac{\delta}{\delta J_+(x)} \frac{1}{\sqrt{-g(x')}} \frac{\delta}{\delta J_+(x')} \frac{1}{\sqrt{-g(x'')}} \frac{\delta}{\delta J_+(x'')} e^{i\Gamma[J_+J_-]} | J_+=J_-=0 \]

\[= i \langle \phi(x) \phi(x') \phi(x'') \rangle \]  

(75)

\[= i \langle \phi(x) T(\phi(x') \phi(x'')) \rangle \]  

(76)

\[= i \langle \tilde{T}(\phi(x) \phi(x')) \phi(x'') \rangle \]  

(77)

\[= i \langle \tilde{T}(\phi(x) \phi(x') \phi(x'')) \rangle \]  

(78)

It is obvious that each of them would converge to the same equal-time three-point correlation function \( \langle \phi(\tau, \vec{x}) \phi(\tau, \vec{x}') \phi(\tau, \vec{x}'') \rangle \). For example, it is straightforward to show that

\[ \langle \phi(\tau, \vec{x}) \phi(\tau, \vec{x}') \phi(\tau, \vec{x}'') \rangle \]

\[= \langle T(\phi(x) \phi(x') \phi(x'')) \rangle |_{\tau_1, \tau_2, \tau_3 \to \tau, \tau, \tau} \]

\[= \left[ \left[ \left[ \frac{1}{\sqrt{-g(x)}} \frac{\delta}{\delta J_+(x)} \frac{1}{\sqrt{-g(x')}} \frac{\delta}{\delta J_+(x')} \frac{1}{\sqrt{-g(x'')}} \frac{\delta}{\delta J_+(x'')} e^{i\Gamma[J_+J_-]} | J_+=J_-=0, \tau_1, \tau_2, \tau_3 \to \tau, \tau, \tau \right] \right] \right. \]

\[- \int d^4x_1 \sqrt{-g(x_1)} \int d^4x_2 \sqrt{-g(x_2)} \int d^4x_3 \sqrt{-g(x_3)} \times \]

\[G_{\mathrm{eq}}^{(x_1, x_1)} G_{\mathrm{eq}}^{(x_2, x_2)} G_{\mathrm{eq}}^{(x_3, x_3)} N_3(x_1, x_2, x_3) |_{\tau_1, \tau_2, \tau_3 \to \tau, \tau, \tau} \]

(79)

where \( N_3 \) is given by the stochastic force three-point function in equation (45).

With the expression in equation (79), we can now calculate the three point correlation function at equal conformal time as the two point correlator above. Again we are concerned with the asymptotic behavior of this three point correlator for large \( \xi \). After evaluating the 6 terms due to permutations of \( x_1, x_2, \) and \( x_3 \) in \( N_3(x_1, x_2, x_3) \), we arrive at

\[ \langle \phi(\tau, \vec{x}) \phi(\tau, \vec{x}') \phi(\tau, \vec{x}'') \rangle \]

\[= \int \frac{d^3k_1}{(2\pi)^{3/2}} \int \frac{d^3k_2}{(2\pi)^{3/2}} \int \frac{d^3k_3}{(2\pi)^{3/2}} e^{i\vec{k}_1 \cdot \vec{x}} e^{i\vec{k}_2 \cdot \vec{x}'} e^{i\vec{k}_3 \cdot \vec{x}''} \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \left( \frac{1}{k^6} \right) \]

\[\times \left( \frac{-3}{10} \right) \left( \frac{2\pi^{5/2}}{10} \right) \left( \frac{\alpha^3}{f^3} \right) \left( \frac{\xi}{4\xi^2} \right) \left( \frac{H^2}{4f^2} \right) ^3 \left( \frac{e^\xi}{\xi^{12}} \right) \left( \frac{1 + \frac{x_2^2 + x_3^2}{x_1^2x_3} - \frac{x_1^2}{x_2^2} \frac{x_3^2}{x_1^2}}{x_2^2x_3} \right) h_3(\rho, x_2, x_3) \]

(80)
where we have parametrized the magnitudes of the vectors, \( \vec{k}_1, \vec{k}_2, \) and \( \vec{k}_3, \) constituting a triangle as \( |\vec{k}_1| = k, |\vec{k}_2| = x_2 k, \) and \( |\vec{k}_3| = x_3 k. \) The function \( h_3(\rho; x_2, x_3) \) is given by

\[
h_3(\rho; x_2, x_3) = \left( \frac{175}{25 \times 165824} \right) \rho^5 \frac{x_3^2 x_2^2 (1 + x_2 + x_3)^2}{x_2^2 (1 + \sqrt{x_2})^2} \times \left[ \frac{(3 + 2 \sqrt{x_2} + 3 x_2) (1 + x_2 - x_3)^2}{x_3^2 (1 + \sqrt{x_3})^2} e^{-4 \sqrt{\rho} (1 + \sqrt{x_3})} + \frac{(3 + 2 \sqrt{x_3} + 3 x_3) (1 - x_2 + x_3)^2}{x_2^2 (1 + \sqrt{x_2})^2} e^{-4 \sqrt{\rho} (1 + \sqrt{x_2})} + \frac{(3 x_2 + 2 \sqrt{x_2} \sqrt{x_3} + 3 x_3) (1 - x_2 - x_3)^2}{x_3^2 x_2^2 (\sqrt{x_2} + \sqrt{x_3})^2} e^{-4 \sqrt{\rho} (\sqrt{x_2} + \sqrt{x_3})} \right] .
\]

(81)

To get a feeling on the magnitude of \( h_3, \) we consider the case with the shape of an equilateral triangle where \( x_2 = x_3 = 1. \) Here,

\[
h_3(\rho; 1, 1) = \left( \frac{525}{4194304} \right) \rho^5 e^{-8 \sqrt{\rho}} .
\]

(82)

The maximum value of this function

\[
h_3|_{\text{max}} \sim 1.65 \times 10^{-9}
\]

at \( \rho \sim 0.78. \) If we take \( \rho = 1/8, \)

\[
h_3(1/8; 1, 1) = 7.0 \times 10^{-11}.
\]

(84)

According to the way that equation (80) is laid out, one can identify the nonlinear parameter of nongaussianity \( f_{\text{NL}} \) of the inflaton field [16] as

\[
f_{\text{NL}} = - \left( \frac{\alpha^3}{H^3} \right) \left( \frac{e^{6 \pi \xi}}{\xi^{12}} \right) h_3(\rho; x_2, x_3).
\]

(85)

6. Conclusion

We have studied the electromagnetic coupling effects in natural inflation in the Schwinger–Keldysh formalism. The axion–photon-like coupling renders the spinodal instability that leads to a copious production of helical electromagnetic fields during inflation. By tracing out the electromagnetic field, we have obtained the corresponding influence functional in the one-loop level, up to cubic order in the inflaton field. It is possible to interpret this functional as due to the effects of a single stochastic force. In this respect, the resulting probability density for the stochastic force will no long be Gaussian. The two-point correlation function of the stochastic force is still given by the noise kernel. On the other hand, the three-point correlator will be nonvanishing. In turn, this will give rise to the nongaussianity effect of the inflaton field correlators. In the present work, we have used a more systematic approach that is in general useful in computing equal-time correlations up to arbitrary orders.
We have also derived the equation of motion for the inflaton perturbation at the one-loop level. It is in the form of a Langevin equation with the noise term originated from the fluctuations of the photon production. However, we have found that the dissipation term is absent in the Langevin equation, contrary to the usual treatment by adding a frictional force in the Hubble expansion term. This result may invite a reassessment of the dissipation in the axion–photon system. The phenomenological implications of the photon production and the associated backreaction effects on cosmological observables in cosmic microwave background, stochastic gravitational waves, and large-scale structures have attracted a lot of attentions in recent years. In particular, the dissipation term was considered as a stopper for the growth of inflaton fluctuations in the strong backreaction regime to avoid an excessive production of primordial black holes near the end of inflation. The absence of the dissipation term would warrant reconsideration of the phenomenology of the axion–photon model. Furthermore, it could have important implications for the viability of the model in the strong backreaction regime in which one may take into account gravitational effects to consider an axion–photon–graviton system.

The two-point and the three-point correlation functions are worked out explicitly here. It is interesting to note that only the retarded Green’s function is involved since the in-in formalism is explicitly causal. The two-point correlator gives the correction to the usual power spectrum of the inflaton field, whereas the three-point correlator gives the nongaussianity effect of the inflaton field. Both these results mostly agree with those in the previous work [7, 16, 17].

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Appendix A. Photon mode functions

Henceforth we assume that the spacetime is quasi de Sitter, meaning that $H$ and $d\Phi/d\tau$ are approximately constant. Therefore, we can treat $\xi$ as a constant parameter and that $a = e^{H\tau} = -1/(H\tau)$. The photon mode equation can be obtained from the quadratic term in the action (10) as

$$\left[ \partial_\tau^2 + k^2 \pm \frac{2\xi k}{\tau} \right] A_{R,\ell}(\vec{k}, \tau) = 0, \quad (A1)$$

where $k = |\vec{k}|$. To solve the mode equation, let us begin with the Whittaker equation:

$$\frac{d^2 y}{dz^2} + \left[ -\frac{1}{4} + \frac{\kappa}{z} + \frac{1 - \mu^2}{4z^2} \right] y = 0. \quad (A2)$$

The independent solutions of this second-order differential equation are the Whittaker functions, $W_{\kappa,\mu/2}(z)$ and $W_{-\kappa,\mu/2}(-z)$, composed of the confluent hypergeometric functions [35]. If
we take \( z = -2k\tau, \kappa = \pm i\xi, \) and \( \mu = 1, \) then the photon mode equation will be the Whittaker equation with solutions,

\[
A_{R,L}(k, \tau) = \frac{e^{\pm\xi\pi/2}}{\sqrt{2k}} W_{\pm\xi,1/2}(2ik\tau),
\]

(A3)

which are normalized such that as \( k\tau \to -\infty, \) they reduce to the adiabatic vacuum solutions, \( A_{R,L}(k, \tau) \to e^{-ik\tau} / \sqrt{2k}. \) Here in taking the limit, we have used the asymptotic form of the Whittaker functions [36]:

\[
W_{\kappa,\mu/2}(z) \sim z^{\kappa} e^{-z/2} \left[ 1 + O\left(\frac{1}{z}\right)\right],
\]

(A4)

as \( |z| \to \infty \) with \( |\arg(z)| < 3\pi/2. \) Next let us consider the behavior of the mode functions in limiting values of \( \xi. \) As \( \xi \to 0, \) they resumethe plane wave solutions,

\[
A_{R}(k, \tau) \sim e^{-i(k\tau / \sqrt{2k})}.
\]

This is expected from the conformal invariance of a photon field in a conformally flat space-time that prohibits the growth of the photon field. As \( \xi \to \infty, \) making use of the asymptotic forms of the Whittaker functions for large \( \kappa \) with \( \text{Im}(\kappa) > 0 \) and \( \text{Im}(\kappa) < 0 \)[36]:

\[
W_{\kappa,\mu/2}(z) \sim z^{\kappa} e^{-z/2} (\kappa\ln \kappa - \kappa \pm i(\pi\kappa - \pi/4 - 2\sqrt{z\kappa})),
\]

(A5)

we obtain that

\[
A_{R}(k, \tau) \sim \frac{1}{\sqrt{2k}} \left(\frac{k|\tau|}{2\xi}\right)^{1/4} e^{\xi \ln \kappa - \kappa \ln \xi - \kappa / 2\sqrt{\xi\kappa}},
\]

(A6)

\[
A_{L}(k, \tau) \sim \frac{1}{\sqrt{2k}} \left(\frac{k|\tau|}{2\xi}\right)^{1/4} e^{\xi \ln \xi - \kappa \ln \xi + 2\xi / 2\sqrt{\xi\kappa}}.
\]

(A7)

Note that \( A_{R}(k, \tau) \) is amplified by a factor of \( \exp(\pi\xi) \) as compared to \( A_{L}(k, \tau). \) We thus conclude that the effect of photon particle production on inflation comes mainly from the \( A_{R} \) modes. This justifies our treatment of neglecting the \( A_{L} \) modes in the CTP calculation.

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References

[1] Guth A H 1981 *Phys. Rev. D* 23 347
[2] Freese K, Frieman J A and Olinto A V 1990 *Phys. Rev. Lett.* 65 3233
[3] Adams F, Bond J R, Freese K, Frieman J and Olinto A 1993 *Phys. Rev. D* 47 426
[4] Arkani-Hamed N, Cheng H-C, Creminelli P and Randall L 2003 *Phys. Rev. Lett.* 90 221302
[5] Dimopoulos S, Kachru S, McGreevy J and Wacker J G 2008 *J. Cosmol. Astropart. Phys.* JCAP08(2007)003
[6] McAllister L, Silverstein E and Westphal A 2010 *Phys. Rev. D* 82 046003
[7] Kaloper N and Sorbo L 2009 *Phys. Rev. Lett.* 102 121301
[8] Savage C, Freese K and Kinney W H 2006 *Phys. Rev. D* 74 123511
[9] Mohanty S and Nautiyal A 2008 *Phys. Rev. D* 78 123515
[10] Anber M M and Sorbo L 2010 *Phys. Rev. D* 81 043534
[11] Visinelli L 2011 *J. Cosmol. Astropart. Phys.* JCAP09(2007)013
[9] Mishra H, Mohanty S and Nautiyal A 2012 Phys. Lett. B 710 245
[10] Berera A and Fang L Z 1995 Phys. Rev. Lett. 74 1912
[11] Berera A 1995 Phys. Rev. Lett. 75 3218
[12] Lee W and Fang L Z 1997 Int. J. Mod. Phys. D 6 305

Lee W and Fang L Z 1999 Phys. Rev. D 59 083503 (Engl transl.)
[13] Berera A, Moss I G and Ramos R O 2009 Rep. Prog. Phys. 72 026901
[14] Anber M M and Sorbo L 2006 J. Cosmol. Astropart. Phys. JCAP10(2007)018
[15] Durrer R, Hollenstein L and Jain R K 2011 J. Cosmol. Astropart. Phys. JCAP03(2007)037
[16] Barnaby N and Peloso M 2011 Phys. Rev. Lett. 106 181301

Barnaby N, Namba R and Peloso M 2011 J. Cosmol. Astropart. Phys. JCAP04(2007)009
[17] Meerburg P D and Pajer E 2013 J. Cosmol. Astropart. Phys. JCAP02(2007)017
[18] Ng K-W 2012 Phys. Rev. D 86 103510
[19] Barnaby N, Pajer E and Peloso M 2012 Phys. Rev. D 85 023525

Cook J L and Sorbo L 2012 Phys. Rev. D 85 023534
Cook J L and Sorbo L 2012 Phys. Rev. D 86 069901(E) (erratum)
[20] Anber M M and Sorbo L 2012 Phys. Rev. D 85 123537
Barnaby N, Moxon J, Namba R, Peloso M, Shiu G and Zhou P 2012 Phys. Rev. D 86 103508
[21] Lin C-M and Ng K-W 2013 Phys. Lett. B 718 1181
Linde A, Mooij S and Pajer E 2013 Phys. Rev. D 87 103506

Bugaev E and Klimai P 2014 Phys. Rev. D 90 103501
[22] Fujita T, Namba R, Tada Y, Takeda N and Tashiro H 2015 J. Cosmol. Astropart. Phys. JCAP05(2007)054
[23] Adshead P, Giblin J T, Scully T R and Sfakianakis E I 2015 J. Cosmol. Astropart. Phys. JCAP12(2007)034
[24] Cheng S-L, Lee W and Ng K-W 2016 Phys. Rev. D 93 063510
Ferreira R Z, Ganc J, Noreña J and Sloth M S 2016 J. Cosmol. Astropart. Phys. JCAP04(2007)039

Ferreira R Z, Ganc J, Noreña J and Sloth M S 2016 J. Cosmol. Astropart. Phys. JCAP10(2007)E01 (erratum)
[25] Jordan R D 1986 Phys. Rev. D 33 444
[26] Calzetta E and Hu B L 2008 Nonequilibrium Quantum Field Theory (Cambridge: Cambridge University Press)

[27] Calzetta E and Hu B L 1987 Phys. Rev. D 35 495
[28] Schwinger J S 1961 J. Math. Phys. 2 407
[29] Keldysh L V 1964 Zh. Eksp. Teor. Fiz. 47 1515
Keldysh L V 1965 Sov. Phys. JETP 20 1018
[30] Feynman R and Vernon F 1963 Ann. Phys. 24 118
[31] Calzetta E and Hu B L 2008 Nonequilibrium Quantum Field Theory (Cambridge: Cambridge University Press)

[32] Bender C M and Orszag S A 1978 Advanced Mathematical Methods for Scientists and Engineers (New York: McGraw-Hill)
[33] Mathews J and Walker R L 1971 Mathematical Methods of Physics 2nd edn (Reading, MA: Addison-Wesley)
[34] Buchholz H 1969 The Confluent Hypergeometric Function (Berlin: Springer)