UPPER AND LOWER DENSITIES OF GABOR GAUSSIAN SYSTEMS

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Abstract. We study the upper and the lower densities of complete and minimal Gabor Gaussians systems. In contrast to the classical lattice case when they are both equal to 1, we prove that the lower density may reach 0 while the upper density may vary at least from $\frac{1}{\pi}$ to $e$. In the case when the upper density exceeds 1, we establish a sharp inequality relating the upper and the lower densities.

1. Introduction

Given a function $f$ in $L^2(\mathbb{R})$ and two real numbers $t, \omega$, we consider its time-frequency shift

$$\rho_{t,\omega}f(x) = e^{2i\pi \omega x} f(x - t).$$

Given a set $\Lambda$ of points in $\mathbb{R}^2$ (which we identify with $\mathbb{C}$) we define the corresponding Gabor system $\{\rho_{t,\omega}f : (t, \omega) \in \Lambda\}$. In this note we are interested in the Gaussian Gabor systems with $f = \varphi$, $\varphi(x) = 2^{1/4} e^{-\pi x^2}$. Denote

$$\mathcal{G}_\Lambda = \{\rho_{t,\omega}\varphi : (t, \omega) \in \Lambda\}.$$

It is well-known that if $\Lambda$ is a separated sequence in $\mathbb{R}^2$, then the system $\mathcal{G}_\Lambda$ is a frame in $L^2(\mathbb{R})$ if and only if its Beurling–Landau density exceeds 1 [3, 5, 6]. If we just require that $\mathcal{G}_\Lambda$ is a complete and minimal system in $L^2(\mathbb{R})$, the situation changes dramatically. In 2009 Ascenzi, Lyubarskii, and Seip [1] proved that if $\Lambda = \{(-1,0),(1,0),(0,\pm\sqrt{2n}), (\pm\sqrt{2n},0) : n \geq 1\}$, then the system $\mathcal{G}_\Lambda$ is complete and minimal. The density of such $\Lambda$ is $\frac{2}{\pi} < 1$. Furthermore, they proved that if $\mathcal{G}_\Lambda$ is complete and minimal and if $\Lambda$ is

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a regularly distributed set (that is, $\Lambda$ has the angular density

$$
\lim_{r \to \infty} \frac{\text{card}(\Lambda \cap \{\lambda : |\lambda| < r, \theta_1 < \arg \lambda \leq \theta_2\})}{\pi r^2}
$$

for all $\theta_1, \theta_2$, except, possibly, for a countable set of $\theta_1, \theta_2$, and the finite limit $\lim_{r \to \infty} \sum_{|\lambda| < r, \lambda \in \Lambda} \lambda^{-2}$ exists), then the density of $\Lambda$ is between $\frac{2}{\pi}$ and $1$.

In this note, we study what is happening when one does not impose the regular distribution condition on $\Lambda$, completing thus the work of Ascenzi–Lyubarskii–Seip.

First of all, we prove (Theorem 2.1 (b)) that if $G_\Lambda$ is a complete system, then the upper density of $\Lambda$

$$
\mathcal{D}_+(\Lambda) = \limsup_{r \to \infty} \frac{\text{card}(\Lambda \cap B(0, r))}{\pi r^2}
$$

is at least $\frac{1}{3\pi}$. Here and later on, $B(z, r)$ is the open disk of center $z$ and radius $r$. Next we give (Theorem 2.1 (a)) an example of a complete and minimal system $G_\Lambda$ such that $\mathcal{D}_+(\Lambda) = \frac{1}{\pi}$. Thus, removing the regularity condition permits us to halve the upper density. It remains an open question to find the precise lower bound for $\mathcal{D}_+(\Lambda)$ for such $\Lambda$.

A simple argument shows that the lower density of $\Lambda$ for a complete and minimal system $G_\Lambda$,

$$
\mathcal{D}_-(\Lambda) = \liminf_{r \to \infty} \frac{\text{card}(\Lambda \cap B(0, r))}{\pi r^2}
$$

could be as small as $0$. It suffices to consider $\Lambda$ consisting of densely packed points on rapidly increasing circles. On the other hand, we prove that the upper density cannot be too large for fixed lower density (see Theorem 2.2 below) under the minimality condition on the system $G_\Lambda$. In particular, $\mathcal{D}_+(\Lambda)$ does not exceed $e$.

Finally, let us note that possible upper densities of $\Lambda$ for complete and minimal systems $G_\Lambda$ may vary at least from $\frac{1}{\pi}$ to $e$. 
1.1. The Fock space. The Bargmann transform is defined by the following formula:

\[ Bf(z) = e^{-\pi xy e^{\frac{y^2}{2}}} \int_{\mathbb{R}} f(t)(\rho_{x-y} \varphi)(t) dt = 2^{1/4} \int_{\mathbb{R}} f(t)e^{-\pi t^2}e^{2\pi tz}e^{-\frac{\pi}{2}z^2} dt, \]

with \( z = x + iy \).

It maps \( L^2(\mathbb{R}) \) isometrically onto the (Hilbert) Fock space \( \mathcal{F} \) of entire functions:

\[ \mathcal{F} = \left\{ F \in \text{Hol}(\mathbb{C}) : \|F\|^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi |z|^2} dm_2(z) < \infty \right\}, \]

where \( m_2 \) is planar Lebesgue measure (see, for instance, [2, Section 3.4] for this fact and some other properties of the Bargmann transform).

The reproducing kernel in \( \mathcal{F} \) is \( k_{\lambda}(z) = \exp(\pi \overline{\lambda}z) \),

\[ \langle f, k_{\lambda} \rangle = f(\lambda), \quad f \in \mathcal{F}, \lambda \in \mathbb{C}. \]

Furthermore, \( B\varphi_{\rho_{x}, \xi} = e^{-\pi |\lambda|^2/2k_{\lambda}}, \lambda \in \mathbb{C} \).

Now a standard duality argument and the symmetry of \( \mathcal{F} \) show that for \( \Lambda \subset \mathbb{C} \), \( \mathcal{G}_\Lambda \) is a complete and minimal system in \( L^2(\mathbb{R}) \) if and only if the system of reproducing kernels \( \{k_{\lambda}\}_{\lambda \in \Lambda} \) is a complete and minimal system in \( \mathcal{F} \) if and only if \( \Lambda \) is a uniqueness set for \( \mathcal{F} \) and for every \( \lambda \in \Lambda \), the set \( \Lambda \setminus \{\lambda\} \) is not a uniqueness set for \( \mathcal{F} \).

2. Main results

We start with two results on the density of complete and minimal Gabor Gaussian systems.

**Theorem 2.1.**

(a) There exists \( \Lambda \subset \mathbb{C} \) such that \( D_+(\Lambda) = \frac{1}{\pi} \) and \( \mathcal{G}_\Lambda \) is a complete and minimal system in \( L^2(\mathbb{R}) \).

(b) Let \( \Lambda \subset \mathbb{C} \). If \( \mathcal{G}_\Lambda \) is a complete system in \( L^2(\mathbb{R}) \), then \( D_+(\Lambda) \geq \frac{1}{3\pi} \).

**Theorem 2.2.**

(a) Given \( 0 \leq \beta < 1 \), there exists \( \Lambda \subset \mathbb{C} \) such that \( D_+(\Lambda) > 1 \),

\[ \beta = D_-(\Lambda) = D_+(\Lambda) \log \frac{e}{D_+(\Lambda)} \]

and \( \mathcal{G}_\Lambda \) is a complete and minimal system in \( L^2(\mathbb{R}) \).
On the other hand, if $G_{\Lambda}$ is a minimal system in $L^2(\mathbb{R})$ and $D^+_{\Lambda} > 1$, then
\[ D^-(\Lambda) \leq D^+(\Lambda) \log \frac{e}{D^+(\Lambda)}. \]

In particular, if $G_{\Lambda}$ is a (complete and) minimal system in $L^2(\mathbb{R})$, then $D^+(\Lambda) \leq e$, and this estimate is sharp.

Applying the Bargmann transform, we can reformulate these results in the language of the Fock space.

**Theorem 2.3.**
(a) There exists $\Lambda \subset \mathbb{C}$ such that $D^+(\Lambda) = 1/\pi$ and $\{k_{\lambda}\}_{\lambda \in \Lambda}$ is a complete and minimal system in $\mathcal{F}$.
(b) Let $\Lambda \subset \mathbb{C}$. If $\{k_{\lambda}\}_{\lambda \in \Lambda}$ is a complete system in $\mathcal{F}$, then $D^+(\Lambda) \geq \frac{1}{3\pi}$.

**Theorem 2.4.**
(a) Given $0 \leq \beta < 1$, there exists $\Lambda \subset \mathbb{C}$ such that $D^+(\Lambda) > 1$,
\[ \beta = D^-(\Lambda) = D^+(\Lambda) \log \frac{e}{D^+(\Lambda)}, \]
and $\{k_{\lambda}\}_{\lambda \in \Lambda}$ is a complete and minimal system in $\mathcal{F}$.
(b) On the other hand, if $\{k_{\lambda}\}_{\lambda \in \Lambda}$ is a minimal system in $\mathcal{F}$ and $D^+(\Lambda) > 1$, then
\[ D^-(\Lambda) \leq D^+(\Lambda) \log \frac{e}{D^+(\Lambda)}. \]

3. PROOFS

**Proof of Theorem 2.3.** Part (a). Denote by $\mathcal{E}$ the set of all entire functions and by $\mathcal{F}_0$ the set of all functions $F$ analytic in $\mathbb{C} \setminus B(0,1)$ and such that
\[ \int_{|z| > 1} |F(z)|^2 e^{-|z|^2} \, dm_2(z) < \infty. \]
Given a function $F$ in $\mathcal{E}$, denote by $Z_F$ its zero set.

We start with the following elementary statement.

**Lemma 3.1.** Let $F$ be an entire function with simple zeros such that for some $m, n \in \mathbb{Z}$ we have $z^m F \in \mathcal{F}_0$, $z^n F \mathcal{E} \cap \mathcal{F}_0 = \{0\}$. Then, adding to $Z_F$ or removing from $Z_F$ a finite set of points, we obtain a set $\Lambda$ such that $\{k_{\lambda}\}_{\lambda \in \Lambda}$ is a complete and minimal system in $\mathcal{F}$. 
Proof. Without loss of generality, we can assume \( m = n - 1 \). If \( n \geq 0 \), set \( F_1(z) = F(z)(z - \lambda_1)...(z - \lambda_n) \) for some distinct \( \lambda_j \in \mathbb{C} \setminus \mathcal{Z}_F \), \( 1 \leq j \leq n \). Otherwise, set \( F_1(z) = F(z)/((z - \lambda_1)...(z - \lambda_{n-1})) \) for some zeros \( \lambda_1, ..., \lambda_{n-1} \) of \( F \). Then \( F_1 \) is an entire function, for every zero \( \lambda \) of \( F_1 \) we have \( F_1(z)/(z - \lambda) \in \mathcal{F} \), and for every entire function \( G \neq 0 \), \( F_1 G \notin \mathcal{F} \). Hence, the system \( \{ k_\lambda \}_{\lambda \in \Lambda} \) is complete and minimal in \( \mathcal{F} \), where \( \Lambda = \mathcal{Z}(F_1) \).

First we construct an auxiliary subharmonic function of “rotating” growth and then approximate it by the logarithm of an entire function. This entire function \( F \) satisfies the conditions of Lemma 3.1 and we have \( D_+(\mathcal{Z}_F) = 1/\pi \).

Fix a large integer \( K \), set \( R = \exp \exp(\pi K) \), and for \( |z| > R \) define

\[
\begin{align*}
\theta(z) &= \log \log |z|, \\
g_1(z) &= \cos(2 \arg(z)), \\
g_2(z) &= \cos(2 \arg(z) - 2\theta(z)), \\
g_3(z) &= \cos(2 \arg(z) - \theta(z)) \cos(\theta(z)).
\end{align*}
\]

Then \( g_3(z) = (g_1(z) + g_2(z))/2 \). Next we set

\[
S_n = \{ z \in \mathbb{C} : \theta(z) = \pi n \}, \quad n \geq K,
\]

\[
\gamma_k = \left\{ z \in \mathbb{C} : \arg z = \frac{\theta(z)}{2} + \frac{\pi k}{2} \pmod{2\pi}, |z| \geq R \right\}, \quad 1 \leq k \leq 4.
\]

Furthermore, set

\[
S = \left( \bigcup_{k=1}^4 \gamma_k \right) \cup \left( \bigcup_{n \geq K} S_n \right).
\]

If \( z \in S \), then \( g_1(z) = g_2(z) = g_3(z) \). Now set

\[
L_n = \{ z \in \mathbb{C} : \pi n < \theta(z) < \pi(n + 1) \}, \quad n \geq K.
\]

The curves \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \) divide \( L_n \) into four disjoint open domains. We denote by \( L_n^k \) the domain between \( \gamma_k \) and \( \gamma_{k+1} \), \( 1 \leq k \leq 4 \) (with the notation \( \gamma_5 = \gamma_1 \)). Finally, set \( \ell_1 = \mathbb{R}_+ \) and

\[
\ell_2 = \{ z \in \mathbb{C} : \arg(z) \equiv \theta(z) \pmod{2\pi}, |z| \geq R \},
\]

see Figure 1.

For every \( n \geq K \), we have \( \ell_1 \cap L_n = \ell_1 \cap L_n^{u(n,1)} \) with \( u(n, 1) \equiv 3 - n \pmod{4} \) and \( \ell_2 \cap L_n = \ell_2 \cap L_n^{u(n,2)} \) with \( u(n, 2) \equiv n \pmod{4} \).
Denote
\[ P_1 = \bigcup_{n \geq K} L_n^{u(n,1)}, \]
\[ P_2 = \bigcup_{n \geq K} L_n^{u(n,2)}, \]
\[ P_3 = \bigcup_{n \geq K} L_n \setminus (P_1 \cup P_2 \cup S). \]

Consider a continuous function \( g \) defined on \( P = P_1 \cup P_2 \cup P_3 \) in the following way:
\[
g(z) = \begin{cases} 
g_3(z), & z \in P \setminus (P_1 \cup P_2), 
g_1(z), & z \in P_1, 
g_2(z), & z \in P_2. \end{cases}
\]

Since \( g_1(z) - g_2(z) = 2 \sin(\theta(z)) \sin(\theta(z) - 2 \arg z) \), we have \( g_1 \geq g_2 \) on \( \ell_1 \cap L_n^{u(n,1)} \), \( g_1 = g_2 \) on \( \partial L_n^{u(n,1)} \), and, hence, \( g_1 \geq g_2 \) on \( L_n^{u(n,1)} \), \( n \geq K \). Analogously, \( g_2 \geq g_1 \) on \( L_n^{u(n,2)} \), \( n \geq K \). Therefore, \( g \geq g_3 \) on \( P \).
Given $r > 0$, we have $\int_0^{2\pi} g_1(re^{i\varphi})d\varphi = \int_0^{2\pi} g_2(re^{i\varphi})d\varphi = 0$, and, hence,

$$\int_0^{2\pi} g(re^{i\varphi})d\varphi = \frac{1}{2} \int_{re^{i\varphi} \in P_1} (g_1 - g_2)(re^{i\varphi})d\varphi + \frac{1}{2} \int_{re^{i\varphi} \in P_2} (g_2 - g_1)(re^{i\varphi})d\varphi = 2|\sin \theta(r)|.$$

Next we define the functions

$$h(re^{i\varphi}) = \left(\frac{\pi r^2}{2} - \frac{4r^2}{\log r}\right) g(re^{i\varphi}) + \frac{4r^2}{\log r},$$

$$h_j(re^{i\varphi}) = \left(\frac{\pi r^2}{2} - \frac{4r^2}{\log r}\right) g_j(re^{i\varphi}) + \frac{4r^2}{\log r}, \quad j = 1, 2, 3.$$

Direct calculation shows that the functions $h_j$, $1 \leq j \leq 3$, are subharmonic on $P$ if $K$ is sufficiently large. Fix such $K$.

We have $h = h_3$ on $S$ and $h \geq h_3$ on $P$. Let $\tilde{h}$ be the harmonic extension of $h$ into $B(0, \exp \exp(\pi K))$. For a sufficiently large $L$, the function

$$f(z) = \begin{cases} h(z), & z \in P, \\ \tilde{h}(z) - L \log \frac{|z|}{\exp \exp(\pi K)}, & z \in \mathbb{C} \setminus P, \end{cases}$$

is subharmonic in $\mathbb{C} \setminus \{0\}$. Fix such $L$. Furthermore, $f(z) = h(z)$ for sufficiently large $|z|$ and $f$ is a subharmonic function of order 2.

Next, we are going to use the following approximation result of Yulmukhametov in [7].

**Theorem 3.2.** Let $f$ be a subharmonic function in the complex plane of finite order $\rho$. Then there exists an entire function $F$ such that for every $\alpha \geq \rho$,

$$|\log |F(z)| - f(z)| \leq C_\alpha \log |z|, \quad z \in \mathbb{C} \setminus E(f, \alpha),$$

where $E(f, \alpha)$ is covered by a family of disks $B(z_j, t_j)$ such that

$$\sum_{|z_j| > R} t_j = o(R^{\rho - \alpha}), \quad R \to \infty.$$

Without loss of generality, we can assume that such a function $F$ has simple zeros and $F(0) \neq 0$. 
We apply this theorem to the function \( f_1(z) = f(z) + L \log |z| \) with \( \rho = 2 \) and \( \alpha = 4 \) and denote \( E = E(f_1, 4) \).

It remains to verify that \( F \) satisfies the assumptions of Lemma 3.1.

Denote by \( n \) the counting function of the zeros of the function \( F \), \( n(t) = \text{card}(B(0, t) \cap \mathcal{Z}_F) \). Set \( U(r) = \partial B(0, r) \). For sufficiently large \( r \) there exist \( r_1 \in (r - 1/r, r) \) and \( r_2 \in (r, r + 1/r) \) such that the circles \( U(r_1) \) and \( U(r_2) \) do not intersect \( E \). Therefore, \( \log |F(z)| - f(z) \leq c \log |z| \) for \( z \in U(r_1) \cup U(r_2) \). Furthermore, by construction, \( |h(re^{i\varphi}) - h(r_je^{i\varphi})| \leq c \) for \( j = 1, 2 \) and some constant \( c \), and, hence, \( |f_1(re^{i\varphi}) - f_1(r_je^{i\varphi})| \leq c, 0 \leq \varphi < 2\pi \). Applying the Jensen formula, we obtain that

\[
2\pi \int_0^r \frac{n(t)}{t} dt = \int_0^{2\pi} \log |F(re^{i\varphi})| d\varphi - \log |F(0)|
= \left( \pi r^2 - \frac{8r^2}{\log r} \right) |\sin(\theta(r))| + \frac{8\pi r^2}{\log r} + O(\log r), \quad r \to \infty.
\]

Therefore, for \( \varepsilon > 0 \) we have

\[
2\pi \int_r^{r(1+\varepsilon)} \frac{n(t)}{t} dt = (2\varepsilon + \varepsilon^2)\pi r^2 |\sin(\theta(r))| + O\left( \frac{r^2}{\log r} \right), \quad r \to \infty,
\]

and, hence,

\[
n(r) = \left( |\sin(\theta(r))| + o(1) \right) r^2, \quad r \to \infty.
\]

Thus,

\[
(3.1) \quad \limsup_{r \to \infty} \frac{n(r)}{\pi r^2} = \frac{1}{\pi},
\]

and the upper density of the zero set of \( F \) is equal to \( 1/\pi \).

Given an entire function \( T \), define \( M_T(r) = \max_{\varphi \in [0, 2\pi]} |T(re^{i\varphi})| \). By the maximum modulus principle, \( M_T \) is an increasing function. For sufficiently large \( r \) choose \( r_1 \) such that the circle \( U(r_1) \) does not intersect \( E \) and \( r < r_1 \). For some \( \varphi_0 \in [0, 2\pi] \) we have

\[
\log(M_F(r_1)) = \log |F(r_1 e^{i\varphi_0})| \leq f(r_1 e^{i\varphi_0}) + c \log(r)
\leq \frac{\pi r_1^2}{2} + c \log(r) \leq \frac{\pi r^2}{2} + c \log(r) + 2\pi.
\]

Therefore,

\[
M_F(r) \leq M_F(r_1) \leq e^{(\pi r^2/2 + 2\pi r c)},
\]
and $z^n F \in F_0$ if $n < -c - 1$.

Let $G$ be an entire function, $G(0) = 1$. Since $\mathbb{C} \setminus (\ell_1 \cup \ell_2)$ is a union of relatively compact components, by the maximum principle we can find a sequence of points $\zeta_k \in \ell_1 \cup \ell_2$, $k \geq 1$, with $|\zeta_k| \to \infty$ as $k \to \infty$, such that $|G(\zeta_k)| \geq 1$. Furthermore, $g(\zeta_k) = 1$ and $h(z) = \frac{\pi}{2} |z|^2 + O(1)$, $z \in B(\zeta_k, 1/|\zeta_k|)$, $k \to \infty$. Hence, $\log |F(z)| \geq \frac{\pi}{2} |z|^2 - O(\log |z|)$, $z \in B(\zeta_k, 1/|\zeta_k|) \setminus E$, $k \to \infty$. Applying the mean value inequality to the analytic function $G^2$ on the set $\Omega_k = \bigcup_{0 < s < 1/|\zeta_k|} \partial B(\zeta_k, s) \cap \partial B(\zeta_k, s)$, for some $c > 0$, $c_1 \in \mathbb{N}$ we obtain that

$$\int_{\Omega_k} |F(s)G(s)|^2 e^{-\pi |s|^2} dm_2(s) \geq c |\zeta_k|^{-2c_1}.$$ 

Thus, $z^{c_1} F(z)G(z) \notin F_0$. Applying Lemma 3.1 we complete the proof of part (a).

Part (b). Let $\{k_\lambda\}_{\lambda \in \Lambda}$ be a complete system in $F$, and let $D_+(\Lambda) < C/\pi$. Denote the elements of $\Lambda$ by $a_1, a_2, \ldots$ in such a way that $|a_1| \leq |a_2| \leq \ldots$. Then

$$|a_n| \geq \sqrt{n/C}, \quad n \geq n_0.$$ 

Set $\Lambda_1 = \Lambda \cup i\Lambda$. Note that $G_{\Lambda_1}$ is also complete. We are going to show that $G_{\Lambda_1}$ cannot be complete if $C < 1/3$.

Since the series $\sum \frac{1}{|a_j|^3}$ converges, the following Weierstrass canonical product is an entire function vanishing on $\Lambda_1$:

$$F(z) = \prod_{j=1}^{\infty} \left( 1 - \frac{z}{a_j} \right) e^{\frac{z}{a_j} + \frac{2}{2a_j}} \left( 1 - \frac{z}{ia_j} \right) e^{\frac{z}{ia_j} + \frac{2}{2ia_j}}.$$ 

We have

$$\log |F(z)| = \sum_{j=1}^{\infty} \left( \log \left| 1 - \frac{z}{a_j} \right| + \log \left| 1 - \frac{z}{ia_j} \right| + \Re \left( \frac{z}{a_j} + \frac{z}{ia_j} \right) \right).$$ 

Denote

$$f(z) = \log \left| 1 - z \right| + \log \left| 1 - iz \right| + \Re \left( z + iz \right).$$ 

We have $\log |F(z)| = \sum_{j=1}^{\infty} f(-iz/a_j)$. 

Note that $f$ is a subharmonic function. Set

$$M_f(r) = \max_{\varphi \in [0, 2\pi]} f(re^{i\varphi}), \quad r > 0.$$
By the maximum modulus principle, the function \( M_f \) is nondecreasing. Therefore,

\[
(3.4) \quad \log |F(z)| \leq \sum_{n=1}^{\infty} M_f \left( \sqrt{C/n} |z| \right) + O(|z|), \quad |z| \to \infty.
\]

Denote \( w = e^{i\pi/4} \). Then

\[
f(wz) = \log |(1 - wz)(1 - wz)e^{wz + wz}| = \log |(1 - \sqrt{2}z + z^2)e^{\sqrt{2}z}|.
\]

The Taylor series of the entire function

\[
g(z) = (1 - \sqrt{2}z + z^2)e^{\sqrt{2}z} = 1 + \sum_{k \geq 3} \frac{2^{(k-2)/2}}{k(k-3)!} z^k
\]

has non-negative coefficients. Therefore, for fixed \( r \), \( |g(re^{i\varphi})| \) attains its maximum at \( \varphi = 0 \). Thus,

\[
(3.5) \quad M_f(r) = \log(1 - \sqrt{2}r + r^2) + \sqrt{2}r.
\]

Now, \( (3.4) \) and \( (3.5) \) give that

\[
\frac{\log M_F(R)}{R^2} \leq \frac{1}{R^2} \sum_{n=1}^{\infty} \left[ \log \left( 1 - \sqrt{\frac{2C}{t} R + \frac{C R^2}{t}} \right) + \sqrt{\frac{2C}{t} R} \right] + O(R^{-1})
\]

(by the monotonicity of \( M_f \) on \( \mathbb{R}_+ \))

\[
\leq \frac{1}{R^2} \int_{1}^{\infty} \left[ \log \left( 1 - \sqrt{\frac{2C}{t} R + \frac{C R^2}{t}} \right) + \sqrt{\frac{2C}{t} R} \right] dt + O(R^{-1}), \quad R \to \infty.
\]

Using the substitution \( s = R \sqrt{\frac{C}{t}} \), we obtain that

\[
\frac{\log M_F(R)}{R^2} \leq 2C \int_{0}^{R \sqrt{C}} \left( \log(1 - \sqrt{2}s + s^2) + \sqrt{2}s \right) \frac{ds}{s^3} + O(R^{-1})
\]

\[
< 2C \int_{0}^{\infty} \left( \log(1 - \sqrt{2}s + s^2) + \sqrt{2}s \right) \frac{ds}{s^3} + O(R^{-1}), \quad R \to \infty.
\]
Integrating by parts, we get
\[
2 \int_0^\infty \left( \log(1 - \sqrt{2}s + s^2) + \sqrt{2}s \right) \frac{ds}{s^2}
= -\frac{1}{s^2} \left( \log(1 - \sqrt{2}s + s^2) + \sqrt{2}s \right) \bigg|_0^\infty + \int_0^\infty \left( \frac{-\sqrt{2} + 2s}{1 - \sqrt{2}s + s^2} + \frac{\sqrt{2}}{s^2} \right) \frac{ds}{s^2}
= \int_0^\infty \frac{\sqrt{2}}{1 - \sqrt{2}s + s^2} ds = 2 \int_0^\infty \frac{dt}{2 - 2t + t^2} = 2 \int_0^\infty \frac{dx}{x^2 + 1} = \frac{3\pi}{2}.
\]

Therefore, \( F \in \mathcal{F} \) if \( C < \frac{1}{3} \). Thus, if \( G \Lambda \) is complete, then \( D_\Lambda(\Lambda) \geq \frac{1}{3\pi} \).

**Proof of Theorem 2.4.** Part (a). Set

(3.6) \( \tau(t) = t \log \frac{e}{t} \).

Then \( \tau(1) = 1 \), \( \tau'(t) = \log \frac{1}{t} \) is strictly negative and \( \tau(t) < 1 \) for \( t \in (1, \infty) \). Let \( h \) be a radially symmetric subharmonic function in the complex plane such that \( h(0) = 0 \). Denote \( \Delta h(re^{i\varphi}) = 2\pi r \, d\nu(r) \otimes d\varphi \). Then \( \Delta h(B(0, r)) = 4\pi^2 \int_0^r s \, d\nu(s) \). Furthermore, Green’s formula says that

\[
\Delta h(r) = \frac{1}{2\pi} \int_{B(0,r)} \log \frac{r}{s} \, \Delta h(se^{i\varphi}) = 2\pi \int_0^r s \log \frac{r}{s} \, d\nu(s).
\]

Choose \( a > 1 \) such that \( \tau(a^2) = \beta \) and set \( \delta = a/\sqrt{\beta} > 1 \). Next, given a rapidly growing sequence of real numbers \( \{R_k\}_{k \geq 1} \), consider the positive measure

\[
d\nu(r) = \beta dr + \sum_{k \geq 1} \left( \frac{1}{2} (a^2 - \beta) R_k \delta R_k - \beta \chi_{[R_k, \delta R_k]} dr \right).
\]

and the corresponding subharmonic function \( h \) vanishing at the origin such that \( \Delta h(re^{i\varphi}) = 2\pi r \, d\nu(r) \otimes d\varphi \).

Let us verify that

(i) \( h(aR_k) = \frac{\pi}{2} a^2 R_k^2 + O(\log R_k), \quad k \to \infty \),

(ii) \( h(x) \leq \frac{\pi}{2} x^2 + O(\log x), \quad x \to \infty \),

(iii) \( \liminf_{R \to \infty} \frac{(2\pi)^{-1} \Delta h(B(0, R))}{\pi R^2} = \beta \),

(iv) \( \limsup_{R \to \infty} \frac{(2\pi)^{-1} \Delta h(B(0, R))}{\pi R^2} = a^2 \).
To prove (i), we use that

\[
\begin{align*}
  h(aR_k) &= 2\pi \int_0^{aR_k} s \log \frac{aR_k}{s} \, d\nu(s) \\
  &= 2\pi \int_0^{R_k} \beta s \log \frac{aR_k}{s} \, ds + \pi R_k^2 (a^2 - \beta) \log a \\
  &\quad + \sum_{1 \leq j < k} \left( \pi R_j^2 (a^2 - \beta) \log \frac{aR_k}{R_j} - 2\pi \beta \int_{R_j}^{R_k} \log \frac{aR_k}{s} \, ds \right) \\
  &= 2\pi \int_0^{R_k} \beta s \log \frac{aR_k}{s} \, ds + \pi R_k^2 (a^2 - \beta) \log a \\
  &\quad + \sum_{1 \leq j < k} \left( \pi R_j^2 (a^2 - \beta) \log \frac{a}{R_j} - 2\pi \beta \int_{R_j}^{R_k} \log \frac{a}{s} \, ds \right) \\
  &= 2\pi \int_0^{R_k} \beta s \log \frac{aR_k}{s} \, ds + \pi R_k^2 (a^2 - \beta) \log a \, O(\log R_k) \\
  &= \frac{\pi}{2} a^2 R_k^2 + O(\log R_k), \quad k \to \infty,
\end{align*}
\]

if \( R_k \gg R_{k-1}, \ k \geq 2. \)

Since

\[
\frac{1}{2\pi} \Delta h(B(0, t)) = \frac{2}{t^2} \int_0^t s \, d\nu(s),
\]

we prove (iii), we need to verify that

\[
(3.7) \quad \lim \inf_{t \to \infty} \frac{2}{t^2} \int_0^t s \, d\nu(s) = \beta.
\]

Set

\[
H(t) = 2 \int_0^t s \, d\nu(s) - \beta t^2.
\]

The function \( H \) is continuous on \( \mathbb{R}_+ \setminus \{R_k\}_{k \geq 1}, \ H = 0 \) outside \( \bigcup_{k \geq 1} [R_k, \delta R_k] \), \( H(R_k + 0) > 0, \ H'(\delta R_k - 0) < 0, \) and \( H'' = -2\beta \) on \( \bigcup_{k \geq 1} (R_k, \delta R_k) \). Therefore,

\[
(3.8) \quad H(t) \geq 0, \quad t \geq 0.
\]

This gives (3.7) and, hence, (iii).
To prove (ii), we note that

\[ h'(r) = \frac{2\pi}{r} \int_0^r s \, d\nu(s), \]

\[ h''(r) = 2\pi \frac{d\nu}{ds}(r) - \frac{2\pi}{r^2} \int_0^r s \, d\nu(s), \]

and, hence, by (3.8),

\[ h'' \leq \pi \beta \quad \text{on the intervals } (R_k, R_{k+1}), \quad k \geq 1. \]

Thus, the function \( F(t) = \frac{\pi t^2}{2} - h(t) \) is convex on the intervals \( (R_k, R_{k+1}) \), \( k \geq 1 \), and

\[ (3.9) \quad F'' \geq \pi(1 - \beta) > 0 \]

there.

By (i),

\[ (3.10) \quad F(aR_k) = O(\log R_k), \quad k \to \infty. \]

Furthermore,

\[ F'(aR_k) = \pi aR_k - \frac{2\pi}{aR_k} \int_0^{aR_k} s \, d\nu(s) \]

\[ = \pi aR_k - \frac{2\pi}{aR_k} \int_{R_{k-1}}^{R_k} s \, d\nu(s) + o(1) \]

\[ = \pi aR_k - \frac{2\pi}{aR_k} \int_0^{R_k} \beta s \, ds - \frac{2\pi}{aR_k} \frac{a^2 - \beta}{2} R_k^2 + o(1) \]

\[ = o(1), \quad k \to \infty. \]

Therefore, taking into account (3.9) and (3.10), we conclude that

\[ F(t) \geq -O(\log R_k), \quad R_k \leq t \leq R_{k+1}, \quad k \to \infty, \]

that gives (ii).

To prove (iv) we use that

\[ \frac{1}{2\pi} \frac{\Delta h(B(0, t))}{t^2} = \beta + \frac{H(t)}{t^2}, \]

and \( H = 0 \) on \( \mathbb{R}_+ \setminus \cup_{k \geq 1} [R_k, \delta R_k] \), \( H > 0 \) on \( \cup_{k \geq 1} [R_k, \delta R_k] \). Hence,

\[ \limsup_{t \to \infty} \frac{1}{\pi t^2} \Delta h(B(0, t)) = \limsup_{t \in \cup_{k \geq 1} [R_k, \delta R_k], t \to \infty} \frac{2}{t^2} \int_0^t s \, d\nu(s). \]
Since \( \int_0^t s \, d\nu(s) \) is constant on every interval \((R_k, \delta R_k), \ k \geq 1\), we obtain that
\[
\limsup_{t \to \infty} \frac{1}{2\pi R^2} \Delta h(B(0, t)) = \limsup_{k \to \infty} \frac{2}{R_k^2} \int_0^{R_k+0} s \, d\nu(s) = a^2.
\]

It remains to apply to \( h \) the approximation result by Yulmukhametov (Theorem 3.2) to obtain an entire function \( F \) with simple zeros such that
\[
(i') \quad \log |F(b_k)| = \frac{\pi}{2} b_k^2 + O(\log b_k), \quad k \to \infty,
\]
\[
(ii') \quad \log |F(z)| \leq \frac{\pi}{2} |z|^2 + O(\log |z|), \quad |z| \to \infty,
\]
for some \( b_k \in (aR_k, aR_k + 1/R_k) \).

By the Jensen formula, for every \( \varepsilon > 0 \),
\[
\text{card}(Z_F \cap B(0, t)) - o(t^2) \leq \frac{1}{2\pi} \Delta h(B(0, (1 + \varepsilon)t)) \leq \text{card}(Z_F \cap B(0, (1 + \varepsilon)^2t)) + o(t^2), \quad t \to \infty.
\]

Therefore, we have
\[
(iii') \quad \liminf_{R \to \infty} \frac{\text{card}(Z_F \cap B(0, R))}{\pi R^2} = \beta,
\]
\[
(iv') \quad \limsup_{R \to \infty} \frac{\text{card}(Z_F \cap B(0, R))}{\pi R^2} = a^2.
\]

Using Lemma 3.1 as in the proof of Theorem 2.3 (a), we complete the proof of part (a).

Part (b). Let \( \{k_\lambda\}_{\lambda \in \Lambda} \) be a minimal system in \( \mathcal{F} \) such that \( \mathcal{D}_+(\Lambda) > 1 \). Suppose that \( \tau(\mathcal{D}_+(\Lambda)) < \mathcal{D}_-(\Lambda) \) and choose \( \alpha \) and \( \beta \) such that
\[
\mathcal{D}_+(\Lambda) > \alpha > 1
\]
and
\[
\mathcal{D}_-(\Lambda) > \beta > \tau(\alpha),
\]
where \( \tau \) is defined by (3.6). Then
\[
\frac{\text{card}(\Lambda \cap B(0, R))}{\pi R^2} \geq \beta, \quad R > R_0.
\]
Furthermore, let \( \{R_k\}_{k \geq 1} \) be a sequence of positive numbers such that \( \lim_{k \to \infty} R_k = \infty \) and
\[
\frac{\text{card}(\Lambda \cap B(0, R_k))}{\pi R_k^2} \geq \alpha, \quad k \geq 1.
\]
Denote by \( n = n_\Lambda \) the counting function of the sequence \( \Lambda \). By the Jensen inequality for every \( \gamma > 1 \) we have

\[
\int_0^{\gamma R_k} \frac{n(t)}{t} \, dt \leq \frac{\pi}{2} (\gamma R_k)^2 + \log R_k + O(1), \quad k \to \infty.
\]

Furthermore,

\[
\int_0^{\gamma R_k} \frac{n(t)}{t} \, dt \geq \int_0^{\gamma R_k} \frac{n(t)}{t} \, dt + \int_{R_0}^{R_k} \frac{n(t)}{t} \, dt + O(1)
\geq \int_{R_k}^{\gamma R_k} \pi \alpha R_k^2 \frac{dt}{t} + \int_{R_0}^{R_k} \pi \beta t^2 \frac{dt}{t} + O(1)
= \pi \alpha R_k^2 \log \gamma + \frac{\pi \beta}{2} R_k^2 + O(1), \quad k \to \infty.
\]

Hence, \( \alpha \log \gamma^2 + \beta \leq \gamma^2 \). Choose \( \gamma = \sqrt{\alpha} \). Since \( \beta > \tau(\alpha) \), we obtain

\[
\alpha \log \alpha + \alpha \log \frac{e}{\alpha} < \alpha.
\]

This contradiction shows that \( \tau(D_+(\Lambda)) \geq D_-(\Lambda) \). \( \square \)

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