On some novel features of the Kerr-Newman-NUT Spacetime

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July 9, 2018

Abstract

In this work we have presented a special class of Kerr-Newman-NUT black hole, having its horizon located precisely at \( r = 2M \), for \( Q^2 = l^2 - a^2 \), where \( M, l, a \) and \( Q \) are mass, NUT charge, rotation parameter and electric charge of the black hole respectively. Unlike any other black hole in this family, the one under consideration has spacelike singularity and hence it is in conformity with the cosmic censorship conjecture. The separability of the Hamilton-Jacobi equations further facilitates to write down the radial and angular geodesics using the Carter constant. Study of null and timelike geodesics in both equatorial and off-equatorial planes has been performed with several distinct features in comparison to the other members of the Kerr-Newman-NUT family. Similar discerning features are also present in the energy extraction mechanisms, e.g., the black hole under study radiates less energy through the super-radiant modes and Penrose process than other black holes in this family. Thermodynamical features of this special class of black holes are also discussed.

1 Introduction

Duality of Maxwell’s equations in the presence of magnetic monopole has far reaching consequences. Therefore, it seems legitimate to explore whether there can exist any such duality for gravitational dynamics as well. Surprisingly, it turns out that there is indeed such a duality in the realm of gravitational field of a Kerr-Newman black hole. For asymptotically flat spacetimes, the unique solutions of the Einstein-Maxwell field equations are the black holes in Kerr-Newman family [1]. However if the condition of asymptotic flatness is dropped, then one can have an additional hair on the black hole, known as the NUT charge and the black holes are referred to as the Kerr-NUT solutions [2, 3]. We also refer our readers to Refs. [4–8] for a descriptive overview on NUT solutions and their various implications in Einstein as well as alternative theories of gravity. Besides admitting separable Hamilton-Jacobi and Klein-Gordon equations [9], the above family also shares a very intriguing duality property: the spacetime structure is invariant under the transformation mass ↔ NUT charge and radius ↔ angular coordinate [10]. Given this duality transformation one can associate a physical significance to the NUT parameter, namely a measure

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of gravitational magnetic charge. Even though it is possible to arrive at a NUT solution without having rotation, the above duality only works if the rotation parameter is non-zero [11]. Thus in order to have a concrete theoretical understanding of the present scenario it is necessary to include the rotation parameter in the spacetime structure.

On the other hand, even though there is no observational evidence whatsoever for the existence of gravitomagnetic mass, investigation of the geodesics in Kerr-Newman-NUT spacetime has significance from both theoretical as well as conceptual points of view. The observational avenues to search for the gravitomagnetic monopole includes, understanding the spectra of supernovae, quasars and active galactic nuclei [12]. All of these scenarios require presence of thin accretion disk [13] and can be modelled if circular geodesics in the spacetime are known. Thus a proper understanding of the geodesic motion in the presence of NUT parameter is essential. Following such implications in mind, there have been attempts to study circular timelike geodesics in presence of NUT parameter [14,15] as well as motion of charged particles in this spacetime [16]. Various weak field tests, e.g., perihelion precession, Lense-Thirring effect has also been discussed [17] (for a taste of these weak field tests in theories beyond general relativity, see [18–22]). We would like to emphasize that all these studies on the geodesic motion crucially hinges on the equatorial plane, however there can also be interesting phenomenon when off-the-equatorial plane motion is considered.

Besides understanding the geodesic structure, it is of utmost importance to explore the possibilities of energy extraction from black holes as well. In particular, the phenomenon of Penrose process [23], superradiance [24,25] and the Bañados-Silk-West effect are well studied in the context of Kerr spacetime [26]. Implications and modifications to these energy extraction processes in presence of NUT charge is another important issue to address. It will be interesting to see how the efficiency of energy extraction in the Penrose process depends on the gravitomagnetic charge inherited by the spacetime. Further, being asymptotically non-flat, whether some non-trivial corrections to the energy extraction process appear is something to wonder about. Moreover, it is expected that the phenomenon of superradiance and the counter-intuitive Banados-Silk-West effect will inherit modifications over and above the Kerr spacetime due to presence of the NUT charge. In particular, for what values of angular momentum the center-of-mass energy of a system of particles diverges is an interesting question in itself.

Finally we should say a word about our choice of the metric for this investigation. In the Kerr-Newman-NUT metric, rotation, NUT charge and the electric charge parameters, \(a^2, l^2\) and \(Q^2\), appear in \(\Delta = r^2 - 2Mr - l^2 + a^2 + Q^2\) linearly. Thus if we make the following unusual choice: \(Q^2 + a^2 = l^2\) [27], \(\Delta\) becomes simply \(r^2 - 2Mr\) and thereby black hole horizon is entirely determined by mass alone and coincides with that of the Schwarzschild's. Despite this, the black hole is having both electric and NUT charge as well as a non-zero rotation parameter. This happens because the electric charge appears only in \(\Delta\) and nowhere else in the metric, while NUT and rotation parameters also define geometrical symmetry of the spacetime. This is why it could be simply added or subtracted, i.e., in \(\Delta\) of Kerr-NUT metric, simply add \(Q^2\) to obtain the Kerr-Newman-NUT solution of the Einstein-Maxwell equations. It is noteworthy that despite presence of rotation and electric charge, the singularity is not time-like but rather space-like, as of the Schwarzschild black hole. That is, the choice of \(Q^2 + a^2 = l^2\) indicates that repulsive effect due to charge and rotation is fully balanced by attractive effect due to the NUT parameter. This is why the causal structure of spacetime has been radically altered [28]. This is therefore a very interesting special case of the Kerr-Newman-NUT family of spacetimes, whose structure we wish to understand in this paper for studying its various interesting properties.

The paper is organized as follows: In Section 2 we have elaborated the spacetime structure we will be considering in this work. Subsequently in Section 3 the trajectory of a particle in both equatorial and non-equatorial plane has been presented. Various energy extraction processes in this spacetime have
been illustrated in Section 4 and finally thermodynamics of black holes in presence of NUT charge has been jotted down in Section 5. We finish with a discussion on the results obtained in Section 6. Detailed calculations of energy extraction processes have been deferred to Appendix A and Appendix B respectively.

Notations and Conventions: Throughout this paper we have set the fundamental constants $c = 1 = G$. All the Greek indices run over four dimensional spacetime coordinates, while the roman indices run over spatial three dimensional coordinates.

2 The spacetime structure

The Kerr-Newman-NUT solution in general involves mass of the black hole $M$, rotation parameter $a$, electric charge $Q$ and NUT charge $l$ and the spacetime metric is given by

$$ds^2 = -\frac{\Delta}{\rho^2} (dt - Pd\phi)^2 + \frac{\sin^2\theta}{\rho^2} \left\{ \left( r^2 + a^2 + l^2 \right) d\phi - adt \right\}^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2. \quad (1)$$

where the quantities $\Delta$, $P$ and $\rho^2$ have the following explicit expressions,

$$\Delta = r^2 - 2Mr + a^2 + Q^2 - l^2; \quad P = a \sin^2\theta - 2l \cos\theta; \quad \rho^2 = r^2 + (l + a \cos\theta)^2. \quad (2)$$

As evident for $l = 0$, we get back the Kerr-Newman solution, while for $Q = 0$ it is the Kerr-NUT solution. The location of the horizons can be obtained by solving for $\Delta = 0$, which in general will have two distinct roots and this is in contrast to the one horizon of Schwarzschild black hole. Furthermore, the singularity in this context is timelike and located at $\rho^2 = 0$. This corresponds to a ring having $r = 0$ and $\cos\theta = -\ell/a$. For $\ell = 0$, we get back the ring singularity of Kerr (or, Kerr-Newman) located at $r = 0$ and $\theta = \pi/2$.

Surprisingly, there exists one subclass of the solution for which the location of the event horizon coincides with that of the Schwarzschild, located at $r = 2M$. As evident from the expression for $\Delta$ it is clear that this will happen when $Q^2 = l^2 - a^2$, which in turn demands $l \geq a$. Thus whenever the NUT charge is larger than (or, equal to) the black hole rotation parameter, a suitable choice for the electric charge will lead to such a configuration. Given its structural simplicity and special location of the event horizon, it will be very interesting to understand various intriguing features this spacetime has to offer.

Let us now specialize the metric presented in Eq. (1) with $Q^2 = l^2 - a^2$, which leads to,

$$ds^2 = -\frac{\Delta}{\rho^2} (dt - Pd\phi)^2 + \frac{\sin^2\theta}{\rho^2} \left\{ \left( r^2 + a^2 + l^2 \right) d\phi - adt \right\}^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \quad (3)$$

with $\Delta = r^2 - 2Mr$. Though the horizon is only determined by the mass $M$, and coincides with that of the Schwarzschild’s, yet the black hole has non-zero rotation parameter as well as inherits both electric and NUT charges. As Eq. (3) suggests, the above black hole geometry can be described by essentially three hairs, namely, the black hole mass $M$, rotation parameter $a$ and the NUT parameter $l$. However there is indeed an electric charge $Q^2 = l^2 - a^2$, but is not manifest in the metric structure. The duality between the gravitational mass $M$ and the gravitomagnetic mass $l$ will exist in this spacetime as well. The line element is still given by Eq. (3), with only $\Delta$ being modified to $\Delta = r(r - 2M)$. As emphasized earlier, this is in stark contrast to a generic Kerr-Newman-NUT spacetime, since in this case there is a single black hole horizon located at $r = 2M$, without any possibility of having naked singularity for any values of $a$ or $l$ whatsoever. Furthermore, the singularity associated with the spacetime only exists for $l = a$ at $r = 0, \theta = \pi$ and is spacelike unlike the situation with general Kerr-Newman-NUT spacetimes.
Surprisingly, the singularity does not exist for $l > a$, as the equation $\cos \theta = -l/a$ does not have any solution for $l > a$. Hence the above spacetime with $Q^2 = l^2 - a^2$ with $l > a$ represents a regular black hole solution, with an event horizon, but without any singularity. Interestingly enough, the above spacetime turns out to be an exact solution of the Einstein-Maxwell field equations, with the Maxwell field tensor having the following form,

$$F = \sqrt{l^2 - a^2} \left\{ \frac{1}{\rho^4} \left\{ r^2 + l^2 - a^2 \cos^2 \theta \right\} \left( 1 + \frac{4al \cos \theta \rho^2}{(r^2 + l^2 - a^2 \cos^2 \theta)^2} \right)^{1/2} dr \wedge \left[ dt - a \sin^2 \theta d\phi \right] \right. + \left. \frac{2ar \sin \theta \cos \theta}{\rho^4} d\theta \wedge \left[ (r^2 + a^2 + l^2) d\phi - adt \right] \right\}. \tag{4}$$

Where, ‘$\wedge$’ is defined as the outer product. This suggests several interesting points regarding this spacetime. First of all, the NUT parameter cannot be set to zero, because $l^2 \geq a^2$, without setting $a = 0$, resulting into Schwarzschild spacetime. On the other hand when $a = 0$, we end up with the Reissner-Nordström-NUT black hole.

Due to presence of the rotation parameter it follows that, the spacetime consists of an ergoregion alongside the usual event horizon at $r = 2M$. The boundary surface of the ergoregion can be obtained by solving the equation $g_{tt} = 0$, which in this case translates into $\Delta - a^2 \sin^2 \theta = 0$. This being a quadratic equation in the radial coordinate $r$, yields two solutions related to the outer and the inner ergoregion boundaries respectively,

$$r_{\text{ergo-outer}} = M + \sqrt{M^2 + a^2 \sin^2 \theta}, \quad \text{and} \quad r_{\text{ergo-inner}} = M - \sqrt{M^2 + a^2 \sin^2 \theta}. \tag{5}$$

It is easy to see that the inner boundary is unphysical as it is situated at $r < 0$ and hence we will only refer to the outer boundary henceforth. The location of the ergoregion and the horizon are presented in Fig. 1. As evident, for $a = 0$, the ergoregion ceases to exist and the horizon is located at $r = 2M$. With the increase in the angular momentum of the black hole, the ergoregion starts to surround the event horizon. The width of the ergoregion increases with the increase of angular momentum. Note that neither the event horizon nor the ergoregion depends on the value of the NUT charge $l$ because of the following relation: $Q^2 + a^2 = l^2$. In this case, the ergoregion is bounded as $2M \leq r \leq M(1 + \sqrt{1 + a^2/M^2})$ while for the Kerr, it is $M(1 + \sqrt{1 - a^2/M^2}) \leq r \leq 2M$. That is, the lower limit for the ergoregion of a Kerr-Newman-NUT black hole with $l = a$ is the upper limit for the ergoregion of a Kerr black hole.

### 3 Trajectory of massive and massless particles

Understanding the geometry of any spacetime requires a thorough analysis of the trajectory of a massive as well as massless particles. Keeping this in mind, in this section we shall derive the basic equations describing the trajectory of a test particle in the Kerr-Newman-NUT spacetime presented above. In the spirit of Kerr geometry, the geodesic motion is completely integrable in a Kerr-Newman-NUT spacetime as well due to the presence of the Carter constant [29]. This can be easily shown by investigating the Hamilton-Jacobi equation and hence establishing the separability of the radial and the angular part. Due to invariance of the metric elements under time translation and rotation with $\phi$ as the rotation angle, we can treat the energy $E$ and the angular momentum $L$ as conserved quantities. This demands to write down the action associated with the motion of the particle in the following manner,

$$\mathcal{A} = -Et + L\phi + \mathcal{A}_r(r) + \mathcal{A}_\theta(\theta) \tag{6}$$
Figure 1: The event horizon (black curve) and the ergoregion is shown (red outer curve) in a Kerr-Newman-NUT spacetime. Event horizon is always located at $r_{eh} = 2M$ and it is independent of the angular momentum as well as NUT parameter. The ergoregion is a function of ‘a’ and independent of the NUT parameter ‘l’.

Here $A_r(r)$ and $A_\theta(\theta)$ correspond to the parts of the action dependent on radial coordinate $r$ and angular coordinate $\theta$ respectively. Given the above structure of the action associated with a particle moving in the Kerr-Newman-NUT spacetime, the Hamilton-Jacobi equation becomes separable. Thus one can obtain separate equations for $A_r(r)$ and $A_\theta(\theta)$ respectively, having the following structures,

$$\left( \frac{dA_\theta}{d\theta} \right)^2 + \frac{(EP - L)^2}{\sin^2 \theta} + m^2 (l + a \cos \theta)^2 = K \tag{7}$$

$$\Delta \left( \frac{dA_r}{dr} \right)^2 + m^2 r^2 - \frac{1}{\Delta} \{E(r^2 + a^2 + l^2) - aL\}^2 = -K \tag{8}$$

In the above expressions the quantity $K$ acts as the separation constant and $m$ is the mass of the orbiting particle. Among others $l$ is the NUT charge, $a$ is the rotation parameter and $P = a \sin^2 \theta - 2l \cos \theta$. Since the components of the momentum four vector of the orbiting particle correspond to, $\partial A/\partial x^\mu$, it is possible to rewrite Eq. (7) and Eq. (8) in a more explicit form. This involves writing the separation constant $K$ appearing in both Eq. (7) and Eq. (8), such that $K = \lambda + (L - aE)^2$, with $\lambda$ as the Carter constant. Thus we obtain the following geodesic equations,

$$m^2 \rho^4 \left( \frac{dr}{d\tau} \right)^2 = \left\{ E(r^2 + a^2 + l^2) - aL \right\}^2 - (\lambda + m^2 r^2) \Delta - \Delta (L - aE)^2 \tag{9}$$

$$m^2 \rho^4 \sin^2 \theta \left( \frac{d\theta}{d\tau} \right)^2 = \lambda \sin^2 \theta + (L - aE)^2 \sin^2 \theta - [EP - L]^2 - m^2 \sin^2 \theta (l + a \cos \theta)^2 \tag{10}$$
where \( \tau \) represents the affine parameter for a timelike geodesic. Similar to the above case with massive particles, for photons with zero rest mass, the geodesic equations become,

\[
\rho^4 \left( \frac{dr}{d\nu} \right)^2 = \left\{ E(r^2 + a^2 + l^2) - aL \right\}^2 - \lambda\Delta - \Delta (L - aE)^2
\] (11)

\[
\rho^4 \sin^2 \theta \left( \frac{d\theta}{d\nu} \right)^2 = \lambda \sin^2 \theta + (L - aE)^2 \sin^2 \theta - (EP - L)^2
\] (12)

where \( \nu \) stands for affine parameter along the null geodesic. Given the above two geodesic equations one can proceed towards solving these equations, which will ultimately lead to the trajectory of a particle moving in the Kerr-Newman-NUT spacetime. With the complicated structure of the geodesic equations as presented above, it is very difficult to solve them in general circumstances. However in some specific situations, e.g., in the equatorial plane (located at \( \theta = \pi/2 \)) it is indeed possible to solve the above equations analytically, which we will discuss next. Having grasped the analytical structure of the trajectory associated with the equatorial plane, we will consider the general scenario later on.

3.1 Equatorial Plane

On the equatorial plane, we have \( \theta = \pi/2 \) and hence the angular equation does not play any role, while the radial equation is the one that matters. In this case as well, it is possible to divide the trajectory into two classes, one for massless particles and the other for massive particles. We first carry out the discussion on the trajectory of massless particles on the equatorial plane, before taking up the motion of massive particles.

3.1.1 The massless particles

For a massless particle (i.e., with \( m = 0 \)), the Carter constant \( \lambda \) identically vanishes. This is because on the equatorial plane \( P = a \) and the left hand side of Eq. (10) becomes zero, while the right hand side becomes proportional to the Carter’s constant. Thus the effective radial potential associated with the motion of a massless particle on the equatorial plane become,

\[
V_{\text{eff}}(r) = \left\{ E(r^2 + a^2 + l^2) - aL \right\}^2 - \Delta (L - aE)^2.
\] (13)

Thus the effective potential depends on both the energy of the particle and its angular momentum. Let us mention one interesting situation in passing, which corresponds to the following relation between angular momentum and energy: \( L = aE \). In this case the impact parameter, defined as the ratio of angular momentum and energy, turns out to be \( D_{\text{sp}} \equiv L/E = a \). Thus one can establish the following results regarding the trajectory of the photon,

\[
\dot{r} = \pm E, \quad \dot{t} = \frac{E(r^2 + a^2 + l^2)}{\Delta}, \quad \dot{\phi} = \frac{aE}{\Delta},
\] (14)

where ‘dot’ defines derivative with respect to some affine parameter \( \nu \) associated with null geodesics. Note that as the rotation parameter \( a \) vanishes, the angular momentum also goes to zero. Thus in the case of vanishing rotation parameter, the photon with the above impact parameter moves along radial geodesics. Hence by analogy, the null geodesics in the present context as well are dubbed as radial-like geodesics. Further, the \( \pm \) sign in the expression for \( \dot{r} \) denotes both inward as well as outward motion. Having dealt with this special case, we now concentrate on a more important aspect of these null geodesics, namely the location of circular photon orbits.
Circular photon orbits: Let us now describe the structure of photon circular orbits in the Kerr-Newman-NUT spacetime with \( \Delta = r^2 - 2Mr \). The necessary and sufficient conditions for the existence of a circular orbit are given by,

\[
\dot{r} = \mathcal{V}_{\text{eff}}(r) = 0, \quad \text{and} \quad \ddot{r} = \frac{1}{2} \frac{dV_{\text{eff}}(r)}{dr} = 0 .
\] (15)

Here \( \mathcal{V}_{\text{eff}}(r) \) corresponds to the effective potential presented in Eq. (13). In order to satisfy Eq. (15) it is necessary to solve the equation \( \mathcal{V}_{\text{eff}}(r) = 0 \), which yields a quadratic equation for the impact parameter \( D \equiv L/E \). The quadratic equation can be solved in a straightforward manner to arrive at the following expression for \( D \), which will be denoted as \( D_{\text{ph}} \),

\[
D_{\text{ph}} = a\left( l^2 + a^2 + 2Mr \right) \pm \sqrt{\Delta} \frac{aL - \Delta}{a^2 - \Delta}.
\] (16)

Finally, the location of the photon sphere can be obtained by solving the other equation as in Eq. (15), i.e., \( dV_{\text{eff}}/dr = 0 \). This in turn yields an algebraic equation depending on the impact parameter \( D \). By substituting for the same from Eq. (16), we obtain the following algebraic equation that the radius of the photon circular orbit must satisfy,

\[
a^2 \left\{ r^2(3r - 5M) + l^2(r - M) \right\} - \Delta \left\{ r^2(r - 3M) + l^2(M - r) \right\} \pm 2a\sqrt{\Delta} \left( Mr^2 + l^2(r - M) + ra^2 \right) = 0.
\] (17)

In principle it is possible to solve this equation and hence obtain the corresponding solutions for the radius of the photon circular orbit \( r_{\text{ph}} \). However in practice it is difficult to present an analytic solution to the above algebraic equation and hence we have demonstrated the corresponding solution numerically. In particular the variation of the radius of the photon circular orbits with the NUT parameter has been explicitly presented in Fig. 2.

3.1.2 The timelike geodesics

Having demonstrated the photon circular orbit in the context of massless particles, let us take up the case for massive particles. Unlike the case of massless particles, one can demonstrate using Eq. (10) that a massive particle on the equatorial plane has a nonzero Carter constant, \( \lambda = m^2l^2 \). Substituting this value of ‘\( \lambda \)’ into the radial equation given by Eq. (9), the effective radial potential for a unit mass particle looks like:

\[
V_{\text{eff}}(r) = \left[ E(r^2 + a^2 + l^2) - aL \right]^2 - \Delta \left( L - aE \right)^2 - \Delta (r^2 + l^2) .
\] (18)

In this case as well, for the specific choices of angular momentum and energy, such that, \( L = aE \) we obtain,

\[
\dot{r} = \pm \sqrt{\frac{\Delta}{(r^2 + l^2)}}, \quad \dot{t} = \frac{E(r^2 + a^2 + l^2)}{\Delta}, \quad \dot{\phi} = \frac{aE}{\Delta} .
\] (19)

Here, the ‘dot’ defines the derivative with respect to the proper time \( \tau \). As the rotation parameter vanishes, the angular velocity will vanish as well, which in turn depicts the motion along radial geodesics. Following this analogy, the above trajectory of massive particles depict radial-like geodesics. Further note that the expression for \( \dot{r} \) associated with the geodesic of a massive particle differs from the one for massless particles, by the factor proportional to \( \Delta \). Thus on the black hole horizon, \( \Delta \) vanishes and hence the geodesic becomes null as expected.

Having described this particular case in which the impact parameter is directly related to black hole spin, let us now concentrate on the motion of massive particles moving in circular orbits.
Circular orbits for timelike particles: In this section, we shall discuss the possible existence of timelike circular orbits and hence compute the exact expressions for conserved quantities associated with it. In this case as well, the conditions for circular orbit are given by $V_{\text{eff}} = 0$ and $V'_{\text{eff}} = 0$, see Eq. (15). To see the existence of circular orbits in this spacetime in a direct manner, we have plotted the effective potential $V_{\text{eff}}$, rescaled by $(r^2 + l^2)^{-1}$, for specific choices of various constants of motion and parameters appearing in the problem in Fig. 3. In particular, we have considered only marginally bound orbits, i.e., orbits with $E = 1$ and the situation when the rotation parameter and NUT charge of the black hole coincide, implying $l = a$. As evident from Fig. 3 the effective potential indeed exhibits a minima for various choices of the $(a/M)$ as well as $(L/M)$ ratio and hence allows existence of stable circular orbits in the spacetime. Thus we have explicitly demonstrated the existence of stable circular orbits in this spacetime, which we will now explore analytically.

In order to determine the location of the circular orbits, we need to solve for the conditions $V_{\text{eff}} = 0$ and $V'_{\text{eff}} = 0$ analytically. This can be achieved by substituting $L = aE + x$ and hence computing $x$ from the circular orbit conditions as given above, resulting into [30],

$$V_{\text{eff}}(r) = E^2 (r^2 + l^2)^2 + 2Mr^3 - r^4 - 2aE (r^2 + l^2) x + a^2 x^2 + 2Mr^2 - r^2 x^2 - l^2 \Delta = 0$$

$$V'_{\text{eff}}(r) = 4E^2 r (l^2 + r^2) + 6Mr^2 - 4r^3 - 4aErx + 2Mx^2 - 2rx^2 - l^2 (2M - 2r) = 0. \tag{20}$$

Here, the ‘prime’ denotes a derivative with respect to the radial coordinate and $\Delta = r^2 - 2Mr$. Given the above equations one can easily solve for the energy $E$ by recombining both the above equations such that the quadratic terms cancel away. This can be achieved by considering the following combination: $4rV_{\text{eff}}(r) - (r^2 + l^2)V'_{\text{eff}}(r) = 0$. This will yield,

$$E = \left[ l^4 (-M + r) + l^2 \left\{ r^3 - Mx^2 + rx^2 \right\} + r \left\{ 2a^2 x^2 + M \left( r^3 + 3rx^2 \right) - r^2 x^2 \right\} \right] \left\{ 2ar(l^2 + r^2)x \right\}^{-1} \tag{21}$$
Figure 3: The effective potential \( V_{\text{eff}} \), rescaled using \( (r^2 + l^2)^{-2} \), is presented in the above figure for marginally bound orbits (i.e., \( E = 1 \)) and various choices of angular momentum of the black hole. We discuss two particular cases of interest, one is \( L = 3.6M \) and another corresponds to \( L = 3.7M \). In both these plots, the NUT and black hole rotation parameters are taken to be identical. The plots explicitly depict presence of circular orbits, given the existence of minima of the potential.

However the above expression for energy involves the unknown quantity \( x \), determination of which is essential for an estimation of angular momentum as well. This can be achieved by substituting the expression for energy as in Eq. (21) into the equation, \( 2 r V_{\text{eff}}(r) - (r^2 + l^2) V'_{\text{eff}}(r) = 0 \). This results into the following quartic equation for \( x \), which reads,

\[
A x^4 + B x^2 + C = 0. \tag{22}
\]

The coefficients appearing in the above expression are functions of the radial distance as well as mass angular momentum and NUT charge of the black hole. Introducing \( u = 1/r \), we finally obtain,

\[
A = M^2 u^2 \left( l^2 u^2 - 3 \right)^2 + \left\{ u^2 l^2 - 1 \right\}^2 - 4 l^2 u^2 - 2 M u \left[ 3 + u^2 \left\{ -4 a^2 + l^2 (-4 + u^2 l^2) \right\} \right] = Z_+ Z_- \]
\[
B = M^2 u \left( 3 - 4 l^2 u^2 + l^4 u^4 \right) + u \left[ l^2 \left\{ -1 + u^2 l^2 \right\} - 2 a^2 \right] + M \left[ -1 + u^2 \left\{ 4 a^2 + l^2 (5 - 2 l^2 u^2) \right\} \right] \]
\[
C = (1 + l^2 u^2)^2 \left\{ M \left( 1 - l^2 u^2 \right) + u l^2 \right\}^2. \tag{23}
\]

In the above expression the quantity \( A \) can be written as a product of two quantities denoted as \( Z_{\pm} \) with the following expressions for each of them,

\[
Z_{\pm} = \left( 1 - 3 M u - 2 a^2 u^2 - l^2 u^2 + M l^2 u^3 \right) \pm 2 a u^{3/2} \sqrt{M \left( 1 - l^2 u^2 \right) + u (a^2 + l^2)} \tag{24}
\]

Using the above expressions for the three quantities, namely \( A, B \) and \( C \) it is certainly possible to determine a solution for \( x \) given Eq. (22). The solution turns out to be much simpler by defining \( G_u = 1 - 2 M u, \)
resulting into,
\[ x^2 u^2 = \frac{1 + l^2 u^2}{Z_+ Z_-} \left\{ G_u Z_+ - Z_+ Z_- \right\}. \]

It turns out that the term within square bracket can be written in a nice form, such that the solution for \( x \) itself can be presented in a simplified manner as,
\[ x_{\pm} = \frac{1}{\sqrt{u Z_{\pm}}} \left\{ a \sqrt{u} \mp \sqrt{M (1 - l^2 u^2) + u (a^2 + l^2)} \right\}. \]

Therefore we have succeeded in determining the quantity \( x \) and hence the energy and angular momentum associated with the circular motion can be easily obtained. In particular the expression for energy of a particle moving in a circular orbit of radius \( r = r_c \) takes the following form,
\[ E_c^{\pm} = \left(1 - 2M u_c - a^2 u_c^2 \right) \pm a u_c \left\{ M u_c + a^2 (a^2 + l^2) - M l^2 u_c^3 \right\}^{1/2} \left\{ Z_{\pm} (1 + l^2 u_c^2) \right\}^{-1/2}. \]

Here all the quantities have been evaluated at the circular orbit radius \( r = r_c \). The two signs present in the above expression denotes energy for direct and retrograde orbits respectively. Given the above expression for energy several interesting quantities can be determined. For example the condition for photon circular orbit is located at the solution of the following algebraic equation,
\[ G_u - u \left\{ a \sqrt{u} \mp \sqrt{M (1 - l^2 u^2) + u (a^2 + l^2)} \right\}^2 = 0. \]

Even though, analytical solutions to the above equation seem to be a formidable task, one can numerically confirm that the photon orbits given by Eq. (17) exactly match with the solution space of Eq. (28). Among the others, the location of the marginally bound and marginally stable orbits are of importance. The location of these radii can be obtained in a straightforward manner as explained below.

- The marginally bound orbits correspond to \( E = 1 \). Given Eq. (27) the above condition demands,
\[ Z_{\pm} (1 + l^2 u^2) = \left\{ (1 - 2M u - a^2 u^2) \pm a u \sqrt{M u + (a^2 + l^2) u^2 - M l^2 u^3} \right\}^2. \]

In Fig. 4, the marginal orbits are shown for a variety of NUTcharges both in Kerr and Kerr-Newman-NUT black hole.

- For innermost stable circular orbits, one sets \( V''_{\text{eff}}(r) = 0 \). The computation is simplified if the rotation parameter \( a \) and the NUT charge \( l \) are assumed to be proportional to each other, i.e., \( l = \chi a \), where \( \chi \) is a constant. In this case the condition for innermost stable circular orbit becomes,
\[ Mr^4 (r - 6M) + 2Ma^2 r^2 \left\{ 2M \chi^2 - 3r (2 + \chi^2) \right\} - a^4 \left\{ 6M^2 \chi^4 - 3M r \chi^2 (4 + 3 \chi^2) + 4r^2 (2 + 3 \chi^2 + \chi^4) \right\} \pm \left\{ M r^2 - a^2 (M \chi^2 - r - r \chi^2) \right\}^{1/2} \left( 8a M r^{5/2} + 8 a^3 r^{3/2} (1 + \chi^2) - 8 a^3 M r^{1/2} \chi^2 \right) = 0. \]
Figure 4: The variation of the location of the marginally bound orbits on the equatorial plane with black hole rotation parameter are presented for different NUT charges.

Figure 5: The location of the innermost stable circular orbits on the equatorial plane are presented for a variety of NUT charges and the black hole rotation parameter.
This finishes our discussion on trajectory of a massive as well as massless particle on the equatorial plane. We will next take up the computation on the motion of a particle on an arbitrary plane.

### 3.2 Non-Equatorial Plane

In the previous section we have described the motion of both massive and massless particles in the equatorial plane of a Kerr-Newman-NUT black hole. However to understand some other subtle features associated with this spacetime it is important that we consider motion in non-equatorial plane as well. In this case, both for massive and massless particles the Carter constant will be non-zero and will play a significant role in determining various properties of the trajectory.

Due to the complicated nature of the geodesic equations it will be worthwhile if we briefly recapitulate the non-equatorial trajectories of a massless particle for the Kerr-Newman black hole, which would be identical to Kerr geometry. In this case, the angular equation is given by:

\[
\rho^4 \left( P^\theta \right)^2 + \cos^2 \theta \left( \frac{L^2}{\sin^2 \theta} - a^2 E^2 \right) = \lambda. \tag{31}
\]

where \( \lambda \) has the usual meaning of Carter constant and \( P^\theta = d\theta/d\nu \) is the momentum in the \( \theta \) direction with \( \nu \) being the affine parameter along the null geodesic. From the above equation, one can easily read off the potential \( V_{\text{ang}}(\theta) \) associated with the angular motion as the second term on the left hand side of Eq. (31). Substituting a new variable \( \mu = \cos \theta \), the angular potential can be written as:

\[
V_{\text{ang}}(\mu) = \mu^2 \left\{ \frac{L^2}{a^2 E^2} - \frac{a^2 E^2}{1 - \mu^2} (1 - \mu^2) \right\}. \tag{32}
\]

Note that for \( \mu = 0 \), or equivalently for \( \theta_0 = \pi/2 \) the potential \( V_{\text{ang}} \) identically vanishes and the motion remains on the equatorial plane. The potential can be further sub-categorized by investigating the behaviour of the term, \( \frac{L}{aE} \). For \( L < aE \), it only vanishes on the equatorial plane. In what follows we will consider the case for which \( L/aE < 1 \). In this case the above result suggests to rewrite the angular equation by introducing the two angles \( \theta_1 \) and \( \theta_2 \), such that:

\[
\left( \rho^2 P^\theta \right)^2 = \lambda - \frac{a^2 E^2 (\cos \theta - \cos \theta_0)^2}{1 - \cos^2 \theta} \left\{ (\cos \theta - \cos \theta_1)(\cos \theta - \cos \theta_2) \right\}. \tag{34}
\]

It is obvious that for the momentum to have any real solution we must have the right hand side to be positive. With the following redefinitions: \( \xi = (L/E) \), \( \eta = (\lambda/E^2) \) and \( \Theta = \rho^4(\rho^\theta)^2 \), we arrive at the following expression for angular motion,

\[
\left( \frac{\Theta}{E^2} \right) = \eta - \frac{a^2 \cos^2 \theta}{1 - \cos^2 \theta} \left\{ (\cos \theta - \cos \theta_1)(\cos \theta - \cos \theta_2) \right\}. \tag{35}
\]

Let us now discuss three situations depending on the value of \( \eta \), i.e., redefined Carter constant. We list below these three choices and associated physical implications.
(A) For $\eta = 0$: The first case corresponds to a vanishing Carter constant. Since the Carter constant identically vanishes the momentum along the angular direction is negative of the angular potential. Thus for physical motion the angular potential should either be zero or negative. In this case the following orbits are possible:

\[ \theta_0, \theta_1, \theta_2 \]

Figure 6: Angular potential and momentum for vanishing Carter constant (i.e., $\lambda = 0$) has been presented for the following choice of the black hole rotation parameter: $a = M$.

- $P^\theta$ vanishes for $\theta = \theta_0, \theta_1$ and $\theta_2$ which are referred to as the turning points in the $\theta$ direction. In addition, the potential attains its local maximum value on the equatorial plane. Hence, a particle would remain on the equatorial plane unless acted upon by an external perturbation (see Fig. 6a), even though it is not a stable equilibrium point.

- Along with the confined motion on the equatorial plane, a massless particle can have off equatorial trajectories whenever $V(\theta)$ is negative. This is only possible if it follows, $\theta_2 < \theta < \theta_0$ and $\theta_0 < \theta < \theta_1$. The particle never reach the equator as it asymptotically approaching the equatorial plane. This is shown in Fig. 6a.

- For $\xi = (L/E) = M$, one has $L/aE > 1$ and hence in this case the potential has a minima and it vanishes only on the equatorial plane. Since it never becomes negative, motion is only allowed on the equatorial plane with vanishing momentum along the angular direction (see Fig. 6b), i.e, $P^\theta = 0$.

(B) For $\eta > 0$: In this case, the Carter constant is positive and hence there can be two possibilities as far as the angular potential is concerned. These include:

- For $(L/aE) < 1$, the potential $V_{\text{ang}}$ is negative within the region $\theta_2 < \theta < \theta_1$. Since $\eta > 0$ it follows that the momentum $P^\theta$ has two turning points located at $\theta^\pm$, satisfying: $\theta^- < \theta_2$ and $\theta^+ > \theta_1$ respectively. As evident from Fig. 7a the particle oscillates about the equatorial plane.

- For $(L/aE) > 1$, unlike the previous case, here the particle can travel beyond the equatorial plane upto the point where $\{P^\theta\}^2$ vanishes. Thus in this case the particle can travel away from the equatorial plane. This is depicted in Fig. 7b.
We set $\xi = 0.5M$.

We set $\xi = M$.

Figure 7: The angular potential and momentum associated with the angular motion of a massless particle in Kerr-Newman spacetime has been presented with non-zero Carter constant: $\lambda = M^2$ and rotation parameter: $a = M$.

(C) For $\eta < 0$: If the Carter constant becomes negative, then in order to ensure that $\{P^\theta\}^2$ is positive one must have negative $V_{\text{ang}}$. In this case the following results are obtained:

- In this case with $(L/aE) < 1$, not only $V_{\text{ang}}$ has to be negative, one has to ensure that $|V_{\text{ang}}| > |\lambda|$. Thus on the equatorial plane $V_{\text{ang}}$ vanishes and hence there can be no physical motion on the equatorial plane. Thus in this case the particle has to travel off the equatorial plane. In particular, the particle has to obey either of these two conditions — (a) $\theta_2 < \theta^- < \theta < \theta^-_{\pi/2} < (\pi/2)$ or, (b) $(\pi/2) < \theta^+_{\pi/2} < \theta < \theta^+ < \theta_1$, where $\theta^\pm_{\pi/2}$ and $\theta^\pm$ are the turning points of the momentum $P^\theta$. A qualitative description can be found in Fig. 8a.

- On the other hand, for $(L/aE) > 1$ it follows that $V_{\text{ang}}$ is always positive. Thus in this case there is absolutely no phase space available for the particle. Hence this corresponds to a unphysical situation, as presented in Fig. 8b.

3.2.1 Massless Particles in Kerr-Newman-NUT black holes

Having described the trajectories of a massless particle in the context of Kerr-Newmann black hole as a warm up exercise, let us now present the orbits of a massless particle in the Kerr-Newman-NUT spacetime. Through this exercise we can easily read off the differences appearing due to the presence of NUT charge in the present context. In this case, by defining $R = \rho^k \{P^r\}^2$ and $\Theta = \rho^l \{P^\theta\}^2$, with $P^r = dr/d\nu$ and $P^\theta = d\theta/d\nu$, the following expressions for the radial and angular equations are obtained,

\[
\left(\frac{R}{E^2}\right) = \{(r^2 + a^2 + l^2) - a\xi\}^2 - \Delta(\xi - a)^2 - \Delta\eta,
\]

\[
\left(\frac{\Theta}{E^2}\right) = \eta - \{(a \sin \theta - \xi \csc \theta - 2l \cot \theta)^2 - (\xi - a)^2\}.
\]
Figure 8: The momentum and potential associated with the angular motion of a massless particle in the Kerr-Newman spacetime is depicted with negative Carter constant, i.e., with $\lambda = -0.1M^2$ and rotation parameter: $a = M$.

Here we have introduced the following notations, namely $\xi = L/E$ and $\eta = \lambda/E^2$. We will now consider the angular part of the geodesic equation before considering the radial part of the same.

(A) angular motion: Similar to the previous case with Kerr-Newman black hole, given the angular equation one can write down the associated potential by substituting $\mu = \cos \theta$, resulting into

$$V_{\text{ang, gen}}(\mu) = \frac{\mu}{1-\mu^2} \left\{ 4l\xi + 4l^2\mu + \xi^2\mu + 4al(\mu^2 - 1) + a^2\mu(\mu^2 - 1) \right\}. \quad (38)$$

Thus the angular potential depends on quartic powers of $\mu$. Hence the angular coordinates where the potential vanishes correspond to a quartic equation for $\mu$. As evident from Eq. (38) the solution $\mu = 0$ (or, $\theta = \pi/2$) is a trivial solution and hence even in this context the potential in the angular direction vanishes on the equatorial plane. Hence, one can rewrite the angular potential as,

$$V(\mu) = \frac{a^2\mu}{1-\mu^2}(\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3). \quad (39)$$

Whether all the three solutions will be real or not depends on the choices of the parameter space spanned by $a$, $l$ and $\xi$ respectively. It turns out that one can have the two following possibilities.

- If the rotation parameter $a$, NUT charge $l$ and the specific angular momentum $\xi$ are such that the following condition is satisfied

$$G \equiv \{ a^4 - 44a^2l^2 + 2a^3\xi - (4l^2 + \xi^2)^2 + a^2(56l^2\xi - 2\xi^3) \} > 0 \quad (40)$$

the solutions to Eq. (38) can be written as,

$$\mu_1 = -\frac{4l}{3a} + \frac{(3a^2 + 4l^2 - 3\xi^2)^{1/2}}{3a}\{2\cos \alpha\} \quad (41)$$

$$\mu_{2,3} = -\frac{4l}{3a} - \frac{(3a^2 + 4l^2 - 3\xi^2)^{1/2}}{3a}(\cos \alpha \mp \sqrt{3}\sin \alpha). \quad (42)$$
Here $\alpha$ is an angle within the range $(0, 2\pi)$ and can be explicitly written as

$$\alpha = \frac{1}{3} \arctan \left( \frac{3\sqrt{3} B}{A} \right), \quad (43)$$

with $A$ and $B$ defined as, $A = 36a^5l - 54a^4\xi + 2a^3(4l^3 + 9\xi^2)$ and $B = a^3(a - \xi)\sqrt{G}$. For $l > 0$, the order of the solutions for the angular variables are, $\theta_3 > \pi > \theta_2 > \theta_0 (= \pi/2) > \theta_1$. Here, $\theta_i = \cos^{-1}(\mu_i)$ with $i$ running from 0 to 3. Thus depending on the value of the Carter constant, one can have different motion in the angular direction following Eq. (37). Note that for $l = 0$, one arrives at $\alpha = \pi/6$ and hence one gets three solutions such as, $\mu = 0, \pm \sqrt{1 - (\xi/a)^2}$. This is consistent with the corresponding result for Kerr black hole. (Note that the angular motion is independent of the choice $Q^2 = l^2 - a^2$ hence the results derived above will be applicable even in the general situation. This is why we have discussed the $l = 0$ limit in the context of angular motion.)

- On the other hand if we have, $G < 0$ there will be two solutions. One of them corresponds to the usual equatorial plane while the other one is at $\mu = \mu'$ and given by

$$\mu' = -\frac{4l}{3a} + \frac{1}{3a^2} \left\{ (A + 3\sqrt{3}B')^{1/3} + \frac{a^2(3a^2 + 4l^2 - 3\xi^2)}{(A + 3\sqrt{3}B')^{1/3}} \right\}, \quad (44)$$

where, $A$ is already mentioned earlier and $B'$ is given as, $B' = a^3(a - \xi)\sqrt{-G}$. Interestingly, for $l = 0$, $\mu'$ become zero with the constraint $\xi > a$. This matches with the results discussed earlier in the context of Kerr-Newman black hole.

Having determined the angular coordinates marking the vanishing of the angular potential $V_{\text{ang,gen}}$, we can comfortably describe the trajectories related to positive, negative and vanishing Carter constant. This is what we discuss next.

1. **For $\eta = 0$:** In the case of vanishing Carter constant, as evident from Eq. (37), the potential $V_{\text{ang,gen}}$ has to be negative. This results into the following behaviours:

   - For $G > 0$, the angular potential can be negative only if $\theta_1 < \theta < \theta_0$ or, $\theta_2 < \theta < \pi$ (see Fig. 9a).
     
   - Unlike the angular motion in the context of Kerr-Newman black hole, $V_{\text{ang,gen}}$ has neither a maxima nor a minima on the equatorial plane located at $\theta = \theta_0 = \pi/2$. The motion is depicted in Fig. 9 for a particular set of parameters.

2. **For $\eta > 0$:** In the case of positive Carter constant, the potential can take both positive and negative values. In the case of positive potential, motion along angular direction is possible only if numerical value of Carter constant is larger than the potential.

   - For positive values of $V(\theta)$ with $G > 0$, the particle has to be within the angular range: $\theta^- < \theta < \theta_1$ or, $\theta_0 < \theta < \theta_2$ (see Fig. 10a). While for $G < 0$, orbits with exist, provided the angular coordinate satisfy: $\theta^- < \theta < \theta'$ and $\theta_0 < \theta < \theta^+$ (see Fig. 10b). Here $\theta^\pm$ are two turning points in the presence of positive Carter constant $\lambda$.

   - $V(\theta)$ can also take negative values. In this case for $G > 0$, the only possibilities are: $\theta_1 < \theta < \theta_0$ and $\theta_2 < \theta < \pi$ (see Fig. 10a). Otherwise, with $G < 0$, one must have $\theta' < \theta < \theta_0$ (see Fig. 10b).
Figure 9: The angular potential and angular momentum for the Kerr-Newman-NUT black hole has been presented with vanishing Carter constant. Among other relevant quantities the impact parameter is taken as \( L/E \equiv \xi = 0.5M \) and finally the rotation parameter being \( a = M \).

(a) The above figure is for: \( l = M/4, G > 0 \).

(b) In the above figure we have: \( l = M, G < 0 \).

Figure 10: The angular motion of a particle in Kerr-Newman-NUT spacetime has been presented with a non-zero Carter constant \( \lambda = M^2 \), having the impact parameter \( L/E = \xi = 0.5M \) and the rotation parameter set to black hole mass.

(a) In the above figure we have the following choices for the parameters: \( l = M/4, G > 0 \).

(b) The above figure is for \( G < 0 \) with \( l = M \).
3. **For** $\eta < 0$: With negative Carter constant the potential can only be negative and also should have a magnitude larger than the Carter constant. This results into the following situation:

- In this case, for $G > 0$, one has to ensure either $\theta_1 < \theta^- < \theta < \theta^+_{\pi/2} < \theta_0$ or $\theta_2 < \theta^+_{\pi/2} < \theta < \pi$.
  
  Here $\theta^\pm$ and $\theta^\pm_{\pi/2}$ are turning points of the momentum along the angular direction (see Fig. 11a).

- For $G < 0$, we need to have $\theta' < \theta^- < \theta < \theta^+ < \theta_0(\pi/2)$. The corresponding situation is depicted in Fig. 11b.

![Potential and Momentum for Angular Motion](image)

(a) In the above we have set $l = M/4$, $G > 0$.

(b) The above figure is for the following parameter choice: $l = M$ along with $G < 0$.

Figure 11: We have presented the potential and momentum responsible for angular motion of a massless particle with $L/E \equiv \xi = 0.5M$, $a = M$ and $\lambda = -M^2/10$.

**B) Radial Equation:** The radial equation has already been presented in Eq. (36) involving $\Delta$. In the present context of Kerr-Newman-NUT spacetime, the corresponding metric elements read, $\Delta = r^2 - 2Mr$.

For the existence of a circular photon orbit, the necessary condition being $R = dR/dr = 0$. Solving for $\eta = (\lambda/E^2)$ and $\xi = (L/E)$, we arrive at two distinct solutions for the pair as,

$$\\xi_c^{(1)} = \frac{r^2 + l^2 + a^2}{a}; \quad \eta_c^{(1)} = -\frac{(r^2 + l^2)^2}{a^2}. \quad (45)$$

and,

$$\\xi_c^{(2)} = \frac{a^2(M - r) + l^2(M - r) + r^2(r - 3M)}{a(M - r)};$$

$$\\eta_c^{(2)} = -\frac{1}{a^2(r - M)^2}\left\{r^4(r - M)^2 - 2l^2r^2(r^2 - 4Mr + 3M^2)
+ r^3\left(r(r - 3M)^2 - 4a^2(r - 2M)\right)\right\}. \quad (46)$$

Considering the two solutions presented above for the parameters $\xi_c$ and $\eta_c$, one can explicitly demonstrate that in the first case (presented in Eq. (45)), $\eta_c + (\xi_c - a)^2$ become null and hence the angular equation will lead to $\Theta < 0$. Thus for the first choice ($\xi_c^{(1)}$, $\eta_c^{(1)}$) no angular motion is possible and hence we shall take Eq. (46) as the parameters associated with photon circular orbits.
3.2.2 Massive Particles in Kerr-Newman-NUT black holes

(A) Angular Equations: For massive particles the geodesic equation associated with angular motion takes the following form,

$$\frac{\Theta_m}{E^2} = \eta - \left\{ (a \sin \theta - \xi \csc \theta - 2l \cot \theta)^2 + \frac{m^2}{E^2} (l + a \cos \theta)^2 - (\xi - a)^2 \right\}$$  \hspace{1cm} (47)

Here \( \Theta_m = \rho^4 m^2 (d\theta/d\tau)^2 \) and the various quantities used in the above equation has the following definitions:

- \( \eta \equiv \lambda/E \),
- \( \xi \equiv L/E \), where \( \lambda \) is the Carter constant. Due to complicated nature of the angular equations let us consider a simple situation with \( E = m \). This corresponds to a marginally bound orbit and it is possible to analytically compute the nature of trajectories in the angular direction. In this context the above equation for angular motion reads,

$$\frac{\Theta_m}{E^2} = \eta - \left\{ (a \sin \theta - \xi \csc \theta - 2l \cot \theta)^2 + (l + a \cos \theta)^2 - (\xi - a)^2 \right\}$$  \hspace{1cm} (48)

Similar to the case for massless particles, there can be two possibilities, which will be discussed below.

- In the first case, where we have all the solutions for \( \theta \) originating from setting \( \Theta_m = 0 \) in Eq. (48), it is necessary that the parameters associated with the black hole satisfies the following identity,

$$\mathcal{F} = \left\{ 16a^4 l^2 - 96a^3 l^2 \xi - 3 (l^2 - \xi^2) (3l^2 + \xi^2)^2 \right.$$

$$\left. + a^2 (-72l^4 + 180l^2 \xi^2 + \xi^4) + 4a (18l^4 \xi - 29l^2 \xi^3 - \xi^5) \right\} > 0$$  \hspace{1cm} (49)

Unlike the case for massless particles, in the present context the angular potential does not vanish at \( \theta = \pi/2 \), as evident from Eq. (48). However the angular potential being a cubic expression in terms of \( \cos \theta \equiv \mu \), there will be at most three solutions for which the potential vanishes. The first solution takes the following form,

$$\mu_1 = -\frac{3l^2 + \xi^2}{6al} + \frac{\chi}{3al} \cos \beta.$$  \hspace{1cm} (50)

where \( \chi^2 \equiv 12a^2 l^2 - 24al^2 \xi + (3l^2 + \xi^2)^2 \) and \( \beta \) is an angle taking values within the range \((0, 2\pi)\) whose explicit expression can be given by,

$$\beta = \frac{1}{3} \arctan \left( \frac{6\sqrt{3}D}{C} \right),$$  \hspace{1cm} (51)

Here \( D \) and \( C \) are mathematical quantities having the following expressions,

$$C = -27l^6 - 27l^4 \xi^2 - 9l^2 \xi^4 - \xi^6 + 36al^2 \xi(3l^2 + \xi^2) - 18a^2 l^2 (6l^2 + \xi^2), \quad \text{and} \quad D = al^2 \sqrt{\mathcal{F}}.$$  \hspace{1cm} (52)

The equation \( \Theta_m = 0 \) is a cubic equation for \( \mu = \cos \theta \) and there should be three independent solutions at most. One of the solution is given above by \( \mu = \mu_1 \), while the other two solutions take the following form,

$$\mu_{2,3} = -\frac{3l^2 + \xi^2}{6al} - \frac{\chi}{6al} \left( \cos \beta \pm \sqrt{3} \sin \beta \right).$$  \hspace{1cm} (53)

The parameters introduced above have their usual meanings.
been several explorations to model such phenomena \[32, 33\], it can be theoretically intriguing if it has its

In passing, we would like to point out that for \(l = 0\) and \(E = m\), the timelike geodesic has a vanishing angular potential only in the equatorial plane of a Kerr-Newman black hole. However, in our context with \(l = 0\), we have \(\beta = (\pi/3)\) and substituting it back into Eq. (50), we easily get \(\mu_1 = 0\). On the other hand, solutions like \(\mu_2, \mu_3\) and \(\mu'\) are only possible for a nonzero NUT charge and are discerning features of the Kerr-Newman-NUT spacetime.

### 4 Energy extraction from Kerr-Newman-NUT black hole

The origin of high energy particles in the universe is a long standing problem. Even though, there have been several explorations to model such phenomena \[32, 33\], it can be theoretically intriguing if it has its
roots back to some exotic objects, such as black hole or neutron star. Historically, many high energetic events in the universe has their connections one way or another into black hole or stars, such as formation of jets from rotating objects as a result of gamma ray burst [34] or active galactic nuclei [35]. More recently, it has been proposed that black holes could also be used as a system to accelerate particles, giving rise to arbitrary large energy Debris [36–38], which in principle dictates a modified version of the Penrose process. This idea was originally suggested by Penrose and Floyed in the late seventies [23], concerning energy extraction from a black hole in the presence of a ergoregion. Since then, many investigations have been carried out in many aspects to examine the implications of Penrose process in various astrophysical domains [39–48].

In the present context, we would reconsider the possibilities of energy extraction from a Kerr-Newman-NUT black hole constrained with ∆ = r^2 – 2Mr and Q^2 = l^2 – a^2. We start with the original Penrose process and study the bounds from Wald inequality. Afterward, we investigate the implication of collisional Penrose process followed by a survey of recent Bañados-Silk-West effect regarding the divergence of collisional energy in the center of mass frame. Finally, we will address the superradiance phenomenon in the Kerr-Newman-NUT black hole and discuss the advantages over other spacetimes such as Kerr or Kerr-Newman.

4.1 The original Penrose process

In the original Penrose process, the idea is to send a particle that breaks up into two in the ergoregion, one of which crosses the horizon with a negative energy while the other comes out with energy more than the initial energies. By this means, rotational energy of a black hole could be extracted. For this purpose, the energy of the initial particle plays a crucial role. Thus we would like to write the energy of the particle in terms of angular momentum. This can be achieved by setting either \( p_r = 0 \) or by substituting all the momentum components in the on-shell condition, i.e., \( p_\mu p^\mu = -m^2 \). They both should give identical result. Thus substituting \( dr/d\tau = 0 \), we arrived at

\[
E^2(r^2 + a^2 + l^2)^2 + a^2L^2 - 2aEL(r^2 + a^2 + l^2) - (\lambda + m^2r^2) \Delta - \Delta (L^2 + a^2E^2 - 2aEL) = 0. \tag{57}
\]

Choosing \( \Delta = r^2 - 2Mr \) and \( \lambda = m^2l^2 \), the above algebraic equation can be reduced to,

\[
E^2 (r^4 + a^4 + l^4 + r^2a^2 + 2a^2l^2 + 2a^2l^2 + 2Mr a^2) - L^2 (r^2 - 2Mr - a^2) - 2aEL(a^2 + l^2 + 2Mr)
- \Delta m^2 (r^2 + l^2) = 0. \tag{58}
\]

From which one can solve for the energy per mass \( \tilde{E} \) in terms of the angular momentum per mass \( \tilde{L} \) and radius \( r \) as,

\[
\tilde{E} = \left\{ (r^2 + l^2)^2 + a^4 + a^2(r^2 + 2Mr + 2l^2) \right\}^{-1} \left[ a (a^2 + l^2 + 2Mr) \tilde{L} \pm \sqrt{(r^2 - 2Mr)(r^2 + l^2)} \left\{ a^4 + (r^2 + l^2)(r^2 + l^2 + \tilde{L}^2) + a^2(r^2 + 2Mr + 2l^2) \right\} \right]^{1/2} \tag{59}\]

From the above equation, we only consider the expression with ‘+’ sign. This is because, for the limit corresponds to \( r \to \infty \), only the positive sign produces \( \tilde{E} = 1 \), while the other gives \( \tilde{E} = -1 \). So, if we consider a timelike particle arriving from spatial infinity, only the positive sign suits the present analysis.

Similar to this, the expression of angular momentum can also be derived from Eq. (58) in terms of energy and radial distance,

\[
\tilde{L} = \left\{ a^2 - (r^2 - 2Mr) \right\}^{-1} \left[ (a^2 + l^2 + 2Mr) \mp \sqrt{(r^2 - 2Mr)(r^2 + l^2)} \right] \left\{ \tilde{E}^2(r^2 + l^2) + a^2 - (r^2 - 2Mr) \right\}^{1/2} \tag{60}\]
In this case, the ‘∓’ sign corresponds to co-rotating and counter-rotating orbits respectively. Using an identical recipe as massive particle, the angular momentum for massless particles can be written directly proportional to energy as

\[ L = E \left\{ a^2 - (r^2 - 2Mr) \right\}^{-1} \left[ a \left( a^2 + l^2 + 2Mr \right) ± (r^2 + l^2) \sqrt{(r^2 - 2Mr)} \right] \]  \hspace{1cm} (61)

If one now considers a particle coming from infinity with \( E = m \), breaks into two parts, then the amount of energy extracted in the process becomes,

\[ E_{\text{ext}} = \frac{1}{2} \left[ \sqrt{\left( \frac{a^2 + 2Mr + l^2}{r^2 + l^2} \right)} - 1 \right] \]  \hspace{1cm} (62)

Thus whenever the breaking point is located close to the event horizon, i.e., \( r \to 2M \), we end up getting the extracted energy to be

\[ E_{\text{ext}} = \frac{1}{2} \left[ \sqrt{1 + \frac{a^2}{l^2 + 4M^2}} - 1 \right] \]  \hspace{1cm} (63)

It is interesting to note that by fixing the NUTcharge \( l \) to its minimum value, i.e, \( l = a \), there is no upper bound on the rotation parameter \( a \), while the electric charge parameter identically vanishes. This way, extracted energy become

\[ E_{\text{ext}} = \frac{1}{2} \left[ \sqrt{1 + \frac{a^2}{a^2 + 4M^2}} - 1 \right] \]  \hspace{1cm} (64)

For a large momentum parameter, \( E_{\text{ext}} \) reaches a similar bound as the Kerr black hole, otherwise it is always less than that. This clearly suggests that the original Penrose process in a Kerr-Newman-NUT spacetime is less efficient than its Kerr counterpart.

Having discussed the implications of original Penrose process in a Kerr-Newman-NUT spacetime, let us now describe some upper bounds on the extracted energy known as the Wald inequality [49, 50]. It explicitly depends on the geometry of the spacetime as well as the velocity components of fragments. Suppose, a particle with initial energy \( E \) and four velocity \( U^a \) breaks into two parts and one of them acquire a negative energy giving the process of energy extraction to take place. Considering the energy of this segment is \( E \) and the spatial velocity is \( V^a \), the Lorentz factor is given as,

\[ \gamma = \frac{1}{\sqrt{1 - |V|^2}} \]  \hspace{1cm} (65)

Following the formalism developed in S. Chandrasekhar’s book [30], the relation to constraint the energy becomes,

\[ \gamma \left( E - |V|(E^2 + g_{tt})^{1/2} \right) < E < \gamma \left( E - |V|(E^2 + g_{tt})^{1/2} \right). \]  \hspace{1cm} (66)

(Note the difference in sign of \( g_{tt} \) from Ref. [30] due to opposite sign convention) To have any process of energy extraction to take place, we have to have \( E < 0 \), which suggests

\[ |V| > \frac{E}{(E^2 - g_{tt})^{1/2}} = \frac{1}{(1 + E^{-2}g_{tt})^{1/2}}. \]  \hspace{1cm} (67)
Now in case of a Kerr black hole, the lower bound of $g_{tt}$ would be 1, which exists on the horizon for a maximally rotating Kerr black hole. In addition to that, stable co-rotating orbits could also exist up to the horizon with an energy of $E_{\text{isco}} = 1/\sqrt{3}$ for $a = M$ limit. Taking these facts into account, one can establish that the minimum velocity of the fragment has to be 0.5, i.e., the fragments have to possess relativistic energies to extract energy from a black hole. Turns out, no astrophysical event could ever produce such high velocity particles instantaneously, and this rules out the idea of original Penrose process from a pragmatic standpoint.

With the above results stored in the backdrop of our mind, we would now investigate a similar situation in the present context. The ‘tt’ component of the metric is given as

$$g_{tt} = \frac{a^2 - r(r - 2M)}{r^2 + l^2}. \quad (68)$$

Unlike the familiar prescription used in the Kerr case, here we would carry out a detail discussion to maximize the term $E^{-2}g_{tt}$ in Eq. (67) which would provide $|V_{\min}|$. Employing Eq. (27) and Eq. (68), one can derive an expression for $E^{-2}g_{tt}$ and maximize it with respect to $r$, various quantities are shown in Table 1.

Table 1: The numerical value of the minimum velocity $|V_{\min}|$ introduced above is of importance to the energy extraction process. In this table we have provided numerical estimates for the minimum velocity $|V_{\min}|$ with $l = a$ and various choices of the ratio $a/M$. As evident for larger values of $a/M$, the minimum velocity decreases from being unity. See text for more discussion.

| $a/M$ | Break up radius (in units of $M$) | Radius of the ergoregion (in units of $M$) | Energy of the Circular orbit | $|V_{\min}|$ |
|-------|----------------------------------|---------------------------------|----------------------------|----------|
| 1.0   | 2.41421                          | 2.41421                         | $\infty$                  | 1.00000  |
| 3.0   | 3.07096                          | 4.16228                         | 1.26112                   | 0.91484  |
| 5.0   | 4.09834                          | 6.09902                         | 1.28703                   | 0.89775  |
| 7.0   | 5.23256                          | 8.07107                         | 1.31377                   | 0.89680  |
| 9.0   | 6.45706                          | 10.0554                         | 1.33204                   | 0.89807  |

4.2 Bañados-Silk-West Process

It is recently proposed by Bañados et. al. that the collisional energy between two particles computed in the center of mass frame, $E_{\text{cm}}$, can diverge in a rotating spacetime [26]. Since then, this proposal has been investigated in many aspects along with different models, which only strengthened its validity as a more general phenomenon [51, 52]. In the present purpose, we investigate the same in the Kerr-Newman-NUT spacetime with $\Delta = r^2 - 2Mr$. Here we will only state the important findings while the detail discussions are given in Appendix A.
The computed energy in the center of mass frame $E_{cm}$ is given by

$$E_{cm}^2 = \frac{2m_0^2}{(r^2 + l^2)} \left\{ (r^2 - a^2 + l^2) - L_1 L_2 + a (L_1 + L_2) \right\}$$

$$+ \frac{1}{2} \left\{ (r^2 + a^2 + l^2) - aL_2 \right\} \left\{ (r^2 + a^2 + l^2) - 2aL_1 + L_1^2 \right\}$$

$$+ \frac{1}{2} \left\{ (r^2 + a^2 + l^2) - aL_1 \right\} \left\{ (r^2 + a^2 + l^2) - 2aL_2 + L_2^2 \right\}$$

(69)

It is straightforward to note that $E_{cm}$ diverges, whenever one of the colliding particle has an angular momentum $L$, equal to $L_1, L_2 \equiv (r_H^2 + a^2 + l^2)/a$. We would like to emphasize that in our case, the black hole horizon is located at $r_H = 2M$, independent of the rotation parameter or the NUT charge. Thus, unlike the case for Kerr black hole, in this situation the spectrum of angular momentum related to divergent center-of-mass energy, $E_{cm}$ is wider, which may lead to not-so-rare occurrence of this ultra-high energy particle accelerator in the context of Kerr-Newman-NUT black hole.

### 4.3 Superradiance in Kerr-Newman-NUT spacetime

Superradiance is another way of extracting energy from a rotating object originally proposed by Zel’dovich in the early seventies. He suggested that in a particular limit, the amplitude of the reflected wave scattered by a rotating object can be larger than the amplitude of the incident wave. However, this rotating object has to have a well defined boundary and Zel’dovich had conducted his experiment with a rotating cylinder [24]. Afterwards, the idea to include the model of a black hole spacetime to explain the superradiance was investigated in Refs. [53–55] and investigated by many others [56–59]. In the present context, we use the Kerr-Newman-NUT spacetime with $\Delta = r^2 - 2Mr$ and explore the possibilities of energy extraction via superradiance. The detail calculations can be found in Appendix B.

To complete the task, we shall assume a scalar field $\Phi$ defined as,

$$\Box \Phi = \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \Phi \right) = 0$$

(70)

We consider that the incident wave is scattered by the event horizon and a part of this gets transmitted across the horizon, while the other is reflected and travels to spatial infinity. Following the standard text books formalism with assuming an ansatz given as,

$$\Phi = e^{-i\omega t} e^{im\phi} \Theta(\theta) R(r)$$

(71)

we arrive at the flux of energy through the horizon,

$$\frac{dE}{dt} = \omega (\omega - m \Omega_H) \left( r_H^2 + a^2 + l^2 \right) \int d\theta d\phi \rho_H^2 \sin^2 \theta \frac{\Theta^2}{\rho_H}$$

$$= \omega (\omega - m \Omega_H) (2Mr_H + a^2 + l^2) \text{ constant}$$

(72)

At this outset we would like to make a few remarks, which will help to understand the implications of the above expression for energy loss due to superradiance. Firstly, with $\omega < m \Omega_H$, the energy flux has a negative sign which essentially indicates that there is a nonzero amount of energy carried out to infinity
and superradiance is possible. On the other hand, for \( \omega > m\Omega_H \), the rate of energy loss is positive and hence superradiance does not take place. Secondly, in a Kerr and Kerr-Newman spacetime, the pre-factor of the above equation becomes \( 2Mt_H \) and \( 2Mt_H - Q^2 \) respectively, which is smaller compared to the Kerr-Newman-NUT black hole discussed in the present context. However the value of \( \Omega_H \) is less in the context of Kerr-Newman-NUT black hole and hence the frequency range upto which super-radiance can occur is smaller. Thus the total energy radiated by the superradiant modes in the present Kerr-Newman-NUT black hole is smaller compared to the Kerr or more general Kerr-Newman-NUT black holes.

5 Thermodynamics of Kerr-Newman-NUT black hole

In this section, we will try to understand some thermodynamical aspects of the Kerr-Newman-NUT black hole. The first in the list corresponds to the computation of area associated with the black hole horizon, which can be obtained by setting \( \Delta \rightarrow 0 \). This leads to,

\[
\text{Area}_H = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{g_{\theta\theta} g_{\phi\phi}} = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin \theta (r_H^2 + a^2 + l^2),
\]

\[
= 4\pi (r_H^2 + a^2 + l^2). \tag{73}
\]

From the above expression, the entropy of the horizon can be written as,

\[
S_H = \frac{\text{Area}_H}{4} = \pi (r_H^2 + a^2 + l^2). \tag{74}
\]

The other most important thing corresponds to the surface gravity, or the temperature associated with the black hole \cite{60}. This originates from the Killing vector field,

\[
\xi^\mu = t^\mu + \Omega_H \phi^\mu \tag{75}
\]

where, \( t^\mu = (\partial/\partial t)^\mu = (1, 0, 0, 0) \) is the Killing vector ensuring stationarity and \( \phi^\mu = (\partial/\partial \phi)^\mu = (0, 0, 0, 1) \) is the Killing vector from axi-symmetry. Thus norm of the vector \( \xi^\mu \) yields,

\[
\xi^\mu \xi_\mu = g_{\mu\nu} (t^\mu + \Omega_H \phi^\mu) (t^\nu + \Omega_H \phi^\nu) = g_{tt} + 2\Omega_H g_{t\phi} + \Omega_H^2 g_{\phi\phi}
\]

\[
= g_{tt} - 2g_{t\phi} \left( \frac{g_{t\phi}}{g_{\phi\phi}} \right) g_{\phi\phi} = g_{tt} - \Omega_H^2 g_{\phi\phi}
\]

\[
= -\Delta + a^2 \sin^2 \theta \rho^2 - \frac{a^2 \sin^2 \theta (r^2 + a^2 + l^2)^2 - \Delta P^2}{(r^2 + a^2 + l^2)^2} = \frac{\Delta \{a^2 P^2 - (r^2 + a^2 + l^2)^2\}}{(r^2 + a^2 + l^2)^2 \rho^2}. \tag{76}
\]

Thus taking derivative of the same, only the radial part will contribute and that also with derivative of \( \Delta \), since ultimately we are interested in the \( \Delta \rightarrow 0 \) limit. Thus the surface gravity can be read off from the result \( \nabla_\alpha (\xi^\mu \xi_\mu) = -2\kappa \xi_\alpha \), to yield,

\[
\kappa = \frac{r_H - M}{(r_H^2 + a^2 + l^2)}. \tag{77}
\]

For \( \Delta = r_H^2 - 2Mr_H = 0 \), we obtain the associated temperature to be,

\[
T = \frac{\kappa}{2\pi} = \frac{M}{2\pi (2Mr_H + a^2 + l^2)}. \tag{78}
\]
It is interesting to note one particular difference from the usual Kerr black hole regarding the extremal condition, i.e., at $a = M$ limit. Even though surface gravity or temperature identically vanishes in this limit for a Kerr spacetime, in the present context with $\Delta = r^2 - 2Mr$, both of them have nonzero contribution.

6 Concluding remarks

The duality between gravitational mass and NUT charge makes the Kerr-Newman-NUT spacetime an interesting testbed to understand gravitational physics. In generic situations the Kerr-Newman-NUT spacetime inherits two horizons and a timelike singularity. However, for a particular choice of the charge parameter, namely, $Q^2 + a^2 = l^2$, where $l$ and $a$ are the NUT charge and the black hole rotation parameter respectively, the horizon sits at $r = 2M$. This particular relation between the black hole hairs, namely the electric charge $Q$, NUT charge $l$ and rotation parameter $a$ ensures that the amount of repulsion offered by $a$ and $Q$ is being exactly balanced by the attraction due to the NUT charge and hence as a consequence the horizon is located at a position as if none of these hairs are present. Recall that the NUT parameter is essentially gravitomagnetic charge [12] and the rotation parameter also produces gravitomagnetic effects. It is therefore not out of consonance to seek a relation between them and that can be achieved by the choice $l = a$. It turns out that for this case with $l^2 = a^2$, the spacetime singularity indeed exists and is spacelike in nature, in stark contrast to the generic Kerr-Newman-NUT spacetimes. On the other hand, for $l^2 > a^2$ it follows that $r^2 + (l + a \cos \theta)^2 \neq 0$ for any real value of $r$ and hence remarkably we have a singularity free regular black hole solution.

Given all these distinguishing features associated with the above black hole spacetime, we have studied the trajectories of massive as well as massless particles in this spacetime, in both equatorial as well as non-equatorial planes to understand its physical properties in a more comprehensive manner. In particular, we have studied the photon circular orbits as well as marginally bound and marginally stable circular orbits for massive particles. The results obtained thereafter explicitly demonstrate the departure of the present context from the usual Kerr-Newman-NUT scenario, e.g., the innermost stable circular orbits are located at a different location and for large NUT parameter they can actually become larger even compared to the corresponding radius in Schwarzschild spacetime. This may have interesting astrophysical implications, e.g., this will affect the structure of accretion disk around the black hole, which in turn will affect the observed luminosity from the accretion disk. Besides the above, we have also studied various energy extraction processes in this spacetime. It turns out that in both the Penrose process and super-radiance the amount of energy extracted is less in comparison to the corresponding situation with Kerr black hole.

On the other hand, in the black hole spacetime under consideration, the center of mass energy of a pair of colliding particles can be very large (the Banados-Silk-West effect) for a much wider class of angular momentum of the incoming particles. This is also in sharp contrast with the corresponding scenario for Kerr spacetime. Finally, we have also commented on the thermodynamical aspects, in which case unlike the general Kerr-Newman-NUT spacetime, the black hole temperature does not vanish for any parameter space of the NUT charge and the rotation parameter.

Finally, we would like to point out that for this particular relation among black hole hairs, namely, $Q^2 + a^2 = l^2$, the ratio $(a/M)$ is completely free and can even take values larger than unity, while at the same time if $l > a$, the black hole will be free of any singularity. Thus any observational evidence that indicates the possibility of having a super-rotating black hole with $(a/M) > 1$, need not necessarily be a signature of naked singularity but instead it could as well be the case that the black hole has a non-trivial NUT charge. This, as well as all other features mentioned above would indeed make a good case for studying the role of NUT parameter in high energy astrophysical setting and phenomena.
Acknowledgement

Research of S.C. is supported by the SERB-NPDF grant (PDF/2016/001589) from SERB, Government of India. The authors acknowledge the warm hospitality provided by the Albert Einstein Institute, Golm, Germany as well as the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, India where parts of this work were carried out during short visits.

A Calculations for the Bañados-Silk-West effect

In this appendix we will explicitly demonstrate the derivation of the Bañados-Silk-West effect in the present context, which have been used in the main text. Keeping this in mind, we start with the equatorial plane, located at $\theta = \pi/2$, on which the radial equation is given by

$$ (r^2 + l^2)^2 \left( \frac{dr}{d\tau} \right)^2 = \{E(r^2 + a^2 + l^2) - aL\}^2 - (r^2 + l^2)\Delta - \Delta(L - aE)^2. \tag{79} $$

While the temporal equation and the azimuthal equation read as,

$$ \frac{dt}{d\tau} = \frac{(r^2 + a^2 + l^2)^2 - \Delta a^2}{\Delta(r^2 + l^2)} E - \frac{a(r^2 + a^2 + l^2 - \Delta)}{\Delta(r^2 + l^2)} L, \tag{80} $$

$$ \frac{d\phi}{d\tau} = \frac{a(r^2 + a^2 + l^2 - \Delta)}{\Delta(r^2 + l^2)} E + \frac{\Delta - a^2}{\Delta(r^2 + l^2)} L. \tag{81} $$

The collisional energy, computed in the center of mass frame, can be written as

$$ E_{\text{cm}}^2 = -\{(P_1)_\mu + (P_2)_\mu\} \{(P_1)_\mu + (P_2)_\mu\}, $$

$$ = (m_1)^2 + (m_2)^2 - 2m_1 m_2 (u_1)^\mu (u_2)_\mu. \tag{82} $$

For simplicity, we consider $m_1 = m_2 = m_0$ and the above equation can be written more conveniently

$$ E_{\text{cm}}^2 = 2m_0^2 \{1 - g_{\mu\nu} u_1^{\mu} u_2^{\nu} \}. \tag{83} $$
Hence the coefficient of $\mathcal{C}$ diverge. This can be written as

\[
g_{\mu\nu}u_1^\mu u_2^\nu = g_{tt}u_1^t u_2^t + g_{t\phi}u_1^t u_2^\phi + g_{\phi t}u_1^\phi u_2^t + g_{\phi\phi}u_1^\phi u_2^\phi + g_{rr}u_1^r u_2^r
\]

\[
= -\frac{\Delta - a^2}{(r^2 + l^2)^2}\left\{\frac{(r^2 + a^2 + l^2)^2 - \Delta a^2}{\Delta(r^2 + l^2)} E_1 - \frac{a(r^2 + a^2 + l^2 - \Delta)}{\Delta(r^2 + l^2)} L_1\right\}
\]

\[
\times\left\{\frac{(r^2 + a^2 + l^2)^2 - \Delta a^2}{\Delta(r^2 + l^2)} E_2 - \frac{a(r^2 + a^2 + l^2 - \Delta)}{\Delta(r^2 + l^2)} L_2\right\}
\]

\[
- \frac{a(r^2 + a^2 + l^2 - \Delta)}{(r^2 + l^2)^2}\left\{\frac{(r^2 + a^2 + l^2)^2 - \Delta a^2}{\Delta(r^2 + l^2)} E_1 - \frac{a(r^2 + a^2 + l^2 - \Delta)}{\Delta(r^2 + l^2)} L_1\right\}
\]

\[
\times\left\{\frac{a(r^2 + a^2 + l^2 - \Delta)}{\Delta(r^2 + l^2)} E_2 + \frac{\Delta - a^2}{\Delta(r^2 + l^2)} L_2\right\}
\]

\[
+ \frac{(r^2 + a^2 + l^2)^2 - \Delta a^2}{(r^2 + l^2)^2}\left\{\frac{a(r^2 + a^2 + l^2 - \Delta)}{\Delta(r^2 + l^2)} E_1 + \frac{\Delta - a^2}{\Delta(r^2 + l^2)} L_1\right\}
\]

\[
\times\left\{\frac{a(r^2 + a^2 + l^2 - \Delta)}{\Delta(r^2 + l^2)} E_2 + \frac{\Delta - a^2}{\Delta(r^2 + l^2)} L_2\right\}
\]

\[
+ \frac{1}{\Delta(r^2 + l^2)}\sqrt{E_1(r^2 + a^2 + l^2) - aL_1}^2 - (r^2 + l^2)\Delta - \Delta(L_1 - aE_1)^2
\]

\[
\times\sqrt{E_2(r^2 + a^2 + l^2) - aL_2}^2 - (r^2 + l^2)\Delta - \Delta(L_2 - aE_2)^2.
\]

Hence the coefficient of $E_1 E_2$ becomes,

\[
C_{E_1 E_2} = \frac{\Delta - a^2}{(r^2 + l^2)^2}\left\{\frac{(r^2 + a^2 + l^2)^2 - \Delta a^2}{\Delta(r^2 + l^2)} - \frac{2a^2(r^2 + a^2 + l^2 - \Delta)^2}{\Delta^2(r^2 + l^2)^2}\right\}
\]

\[
+ \frac{(r^2 + a^2 + l^2)^2 - \Delta a^2}{(r^2 + l^2)^2}\frac{a^2(r^2 + a^2 + l^2 - \Delta)}{\Delta^2(r^2 + l^2)^2}
\]

\[
= \left\{\frac{(r^2 + a^2 + l^2)^2 - \Delta a^2}{(r^2 + l^2)^2}\right\}\left[-a^2(r^2 + a^2 + l^2 - \Delta)^2 - (\Delta - a^2) \{r^2 + a^2 + l^2\}^2 - \Delta a^2\right]
\]

\[
= -\frac{(r^2 + a^2 + l^2)^2 - \Delta a^2}{(r^2 + l^2)^2}\Delta a^2.
\]

(84)

(85)
Then coefficient of $L_1 L_2$ becomes,

$$C_{L_1L_2} = -\frac{\Delta - a^2}{(r^2 + l^2)^3 \Delta^2} a^2(r^2 + a^2 + l^2 - \Delta)^2 + \frac{2a^2(r^2 + a^2 + l^2 - \Delta)^2}{\Delta^2(r^2 + l^2)^3} (\Delta - a^2)$$

$$+ \frac{(r^2 + a^2 + l^2 - \Delta)^2 - \Delta^2}{(r^2 + l^2)^3 \Delta^2} (\Delta - a^2)^2$$

$$= \frac{\Delta - a^2}{(r^2 + l^2)^3} \left[a^2(r^2 + a^2 + l^2 - \Delta)^2 + (\Delta - a^2) \left((r^2 + a^2 + l^2)^2 - \Delta a^2\right)\right]$$

$$= \frac{\Delta - a^2}{\Delta(r^2 + l^2)}. \quad (86)$$

Coefficient of $E_1 L_2$ becomes,

$$C_{E_1L_2} = \frac{\Delta - a^2}{(r^2 + l^2)^3 \Delta^2} a^2(r^2 + a^2 + l^2 - \Delta) \left((r^2 + a^2 + l^2)^2 - \Delta a^2\right)$$

$$- \frac{(r^2 + a^2 + l^2 - \Delta)}{(r^2 + l^2)^3 \Delta^2} a(r^2 + a^2 + l^2 - \Delta) \left((r^2 + a^2 + l^2)^2 - \Delta a^2\right)$$

$$+ \frac{(r^2 + a^2 + l^2 - \Delta)^3 a^3}{(r^2 + l^2)^3 \Delta^2} + \frac{(r^2 + a^2 + l^2 - \Delta)^3}{(r^2 + l^2)^3 \Delta^2} a(r^2 + a^2 + l^2 - \Delta) \left((r^2 + a^2 + l^2)^2 - \Delta a^2\right)$$

$$= \frac{a(r^2 + a^2 + l^2 - \Delta)}{(r^2 + l^2)^3 \Delta^2} \left[(r^2 + a^2 + l^2 - \Delta)^2 a^2 + \left((r^2 + a^2 + l^2)^2 - \Delta a^2\right) (\Delta - a^2)\right]$$

$$= \frac{a(r^2 + a^2 + l^2 - \Delta)}{(r^2 + l^2)} \left(87\right).$$

which would be identical to the coefficient of $E_2 L_1$. Before evaluating the square root, it is assumed that the particle are coming from infinity and hence $E_1 = 1 = E_2$. Further, the term within square root becomes,

$$C_{\sqrt{r}} = \left\{(r^2 + a^2 + l^2 - aL_1)^2 - (r^2 + l^2) \Delta - \Delta(L_1 - a)^2\right\} - 2aL_1 \left((r^2 + a^2 + l^2 - \Delta) - a^2\right) + L_1^2 (a^2 - \Delta). \quad (88)$$

Thus in total, the center of mass energy is given by,

$$\frac{1}{2m_0^2} E_{cm}^2 = 1 - g_{\mu\nu} u'^{\mu}_1 u'_2$$

$$= 1 + \left\{\frac{(r^2 + a^2 + l^2 - \Delta a^2)}{(r^2 + l^2) \Delta} \right\} - \frac{\Delta - a^2}{\Delta (r^2 + l^2)} L_1 L_2 - \frac{a(r^2 + a^2 + l^2 - \Delta)}{(r^2 + l^2) \Delta} \left(89\right).$$

$$- \frac{1}{\Delta (r^2 + l^2)} \sqrt{(r^2 + a^2 + l^2) \left((r^2 + a^2 + l^2) - \Delta\right) - 2aL_1 \left((r^2 + a^2 + l^2) - \Delta\right) + L_1^2 (a^2 - \Delta)}$$

$$\times \sqrt{(r^2 + a^2 + l^2) \left((r^2 + a^2 + l^2) - \Delta\right) - 2aL_1 \left((r^2 + a^2 + l^2) - \Delta\right) + L_1^2 (a^2 - \Delta)}. \quad (89)$$
Otherwise, we will have,

\[ E_{cm}^2 = \frac{2m_0^2}{\Delta (r^2 + l^2)} \left\{ \left( r^2 + a^2 + l^2 \right)^2 - a^2 L_1 L_2 - a(r^2 + a^2 + l^2)(L_1 + L_2) \right\} \]

\[ + \Delta \left\{ \left( r^2 - a^2 + l^2 \right) - L_1 L_2 + a(L_1 + L_2) \right\} \]

\[ - \sqrt{\left\{ \left( r^2 + a^2 + l^2 \right) - aL_1 \right\}^2 - \Delta \left\{ \left( r^2 + a^2 + l^2 \right) - 2aL_1 + L_2^2 \right\} \}
\]

\[ \times \sqrt{\left\{ \left( r^2 + a^2 + l^2 \right) - aL_2 \right\}^2 - \Delta \left\{ \left( r^2 + a^2 + l^2 \right) - 2aL_2 + L_2^2 \right\} \}. \] (90)

Expanding to leading orders in \( \Delta \), we obtain,

\[ E_{cm}^2 = \frac{2m_0^2}{\Delta (r^2 + l^2)} \left\{ \left( r^2 + a^2 + l^2 \right)^2 - a^2 L_1 L_2 - a(r^2 + a^2 + l^2)(L_1 + L_2) \right\} \]

\[ + \Delta \left\{ \left( r^2 - a^2 + l^2 \right) - L_1 L_2 + a(L_1 + L_2) \right\} \]

\[ - \left\{ \left( r^2 + a^2 + l^2 \right) - aL_2 \right\} \left\{ \left( r^2 + a^2 + l^2 \right) - aL_1 \right\} \left\{ 1 - \frac{\Delta}{2} \left\{ \left( r^2 + a^2 + l^2 \right) - 2aL_1 + L_2^2 \right\} \right\} + \mathcal{O}(\Delta^2) \]

\[ \times \left\{ 1 - \frac{\Delta}{2} \frac{\left\{ \left( r^2 + a^2 + l^2 \right) - 2aL_2 + L_2^2 \right\}}{\left\{ \left( r^2 + a^2 + l^2 \right) - aL_2 \right\}^2} + \mathcal{O}(\Delta^2) \right\} \] (91)

Further simplification of this result has been presented in Eq. (69), which we have used in the main text.

**B Detailed derivations of superradiance**

In this appendix we present the derivation of super-radiant modes associated with the particular Kerr-Newman-NUT spacetime under consideration. The results presented here will be used directly in the main body of this work. We start with the basic equation for the scalar field as presented in Eq. (71). Using the result \( \sqrt{-g} = \rho^2 \sin \theta \) and the expressions for \( g^{\mu \nu} \) we obtain,

\[ \frac{1}{\rho^2 \sin \theta} \partial_\rho \left[ \rho^2 \sin \theta g^{\rho \rho} \partial_\rho \Phi \right] + \frac{1}{\rho^2 \sin \theta} \partial_\rho \left[ \rho^2 \sin \theta g^{\phi \phi} \partial_\rho \Phi \right] + \frac{1}{\rho^2 \sin \theta} \partial_\phi \left[ \rho^2 \sin \theta g^{\phi \phi} \partial_\phi \Phi \right] \]

\[ + \frac{1}{\rho^2 \sin \theta} \partial_\phi \left[ \rho^2 \sin \theta g^{\phi \phi} \partial_\phi \Phi \right] + \frac{1}{\rho^2 \sin \theta} \partial_r \left[ \rho^2 \sin \theta g^{r r} \partial_r \Phi \right] + \frac{1}{\rho^2 \sin \theta} \partial_\theta \left[ \rho^2 \sin \theta g^{\theta \theta} \partial_\theta \Phi \right] = 0. \] (92)

Substituting all the inverse metric elements we obtain,

\[ \left[ \frac{\rho^2}{\sin^2 \theta} - \frac{(r^2 + a^2 + l^2)^2}{\Delta} \right] \partial_r^2 \Phi + \partial_r \left[ \Delta \partial_r \Phi \right] - 2 \frac{(r^2 + a^2 + l^2) a \sin^2 \theta - \Delta P}{\Delta \sin^2 \theta} \partial_r \partial_\theta \Phi \]

\[ + \frac{\Delta - a^2 \sin^2 \theta}{\Delta \sin^2 \theta} \partial_\phi^2 \Phi + \frac{1}{\sin \theta} \partial_\theta \left[ \sin \theta \partial_\theta \Phi \right] = 0. \] (93)
Let us now write down the scalar field in the following manner,

$$\Phi = e^{-i\omega t} e^{im\phi} \Theta(\theta) R(r)$$  \hspace{1cm} (94)

Substitution in the previous expression with using inverse metric components and dividing by $e^{-i\omega t} e^{im\phi}$ yields,

$$\omega^2 \left[ -\frac{P^2}{\sin^2 \theta} + \frac{(r^2 + a^2 + l^2)^2}{\Delta} \right] \Theta(\theta) R(r) + \Theta(\theta) \frac{d}{dr} \left[ \Delta \frac{dR(r)}{dr} \right]$$

$$+ 2m\omega \left[ \frac{(r^2 + a^2 + l^2)\Delta}{\Delta} + a - 2l \cos \theta \right] \Theta(\theta) R(r) - m^2 \left[ \frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right] \Theta(\theta) R(r)$$

$$+ \frac{R(r)}{\sin \theta} \frac{d}{d\theta} \left[ \sin \theta \frac{d\Theta(\theta)}{d\theta} \right] = 0$$  \hspace{1cm} (95)

Let us now use separation of variables and obtain, the following equation satisfied by $\Theta(\theta)$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left[ \sin \theta \frac{d\Theta(\theta)}{d\theta} \right] - \frac{m^2}{\sin^2 \theta} \Theta(\theta) - 4m\omega \frac{l \cos \theta}{\sin^2 \theta} \Theta(\theta) - \omega^2 \frac{(a \sin^2 \theta - 2l \cos \theta)^2}{\sin^2 \theta} \Theta(\theta) = -C \Theta(\theta)$$  \hspace{1cm} (96)

Therefore, the radial equation becomes,

$$\frac{d}{dr} \left[ \Delta \frac{dR(r)}{dr} \right] + \omega^2 \frac{(r^2 + a^2 + l^2)^2}{\Delta} R(r) + 2m\omega \left[ \frac{(r^2 + a^2 + l^2)a}{\Delta} + a \right] R(r) + \frac{m^2a^2}{\Delta} R(r) = CR(r)$$  \hspace{1cm} (97)

which can be further simplified to,

$$\frac{d}{dr} \left[ \Delta \frac{dR(r)}{dr} \right] + \omega^2 \frac{(r^2 + a^2 + l^2)^2}{\Delta} R(r) - 2m\omega a \frac{(r^2 + a^2 + l^2 - \Delta)}{\Delta} R(r) + \frac{m^2a^2}{\Delta} R(r) = CR(r)$$  \hspace{1cm} (98)

where $C$ is a constant of separation. Let us now introduce the tortoise coordinate by the following coordinate transformation,

$$\frac{dr^*}{dr} = \frac{r^2 + a^2 + l^2}{\Delta}$$  \hspace{1cm} (99)

This suggests,

$$\frac{d}{dr} \left[ \Delta \frac{dR(r)}{dr} \right] = \frac{dr^*}{dr} \frac{d}{dr^*} \left[ \Delta \left( \frac{dr^*}{dr} \right) \frac{dR}{dr^*} \right]$$

$$= \left( \frac{dr^*}{dr} \right)^2 \Delta \frac{d^2R}{dr^*} + \frac{dR}{dr^*} \frac{d}{dr^*} \left( r^2 + a^2 + l^2 \right)$$

$$= \left( \frac{dr^*}{dr} \right)^2 \Delta \frac{d^2R}{dr^*} + 2r \frac{dR}{dr^*}.$$  \hspace{1cm} (100)

Thus substitution of the above results in the radial equation yields,

$$\frac{(r^2 + a^2 + l^2)^2}{\Delta} \frac{d^2R}{dr^*} + 2r \frac{dR}{dr^*} + \omega^2 \frac{(r^2 + a^2 + l^2)^2}{\Delta} R - 2m\omega a \frac{(r^2 + a^2 + l^2 - \Delta)}{\Delta} R + \frac{m^2a^2}{\Delta} R = CR(r).$$  \hspace{1cm} (101)
Subsequently multiplying by $\Delta/(r^2 + a^2 + l^2)^2$, we obtain,

$$\frac{d^2 R}{dr^*^2} + \frac{2r\Delta}{(r^2 + a^2 + l^2)^2} \frac{dR}{dr^*} + \omega^2 R - 2m\omega a \frac{(r^2 + a^2 + l^2 - \Delta)}{(r^2 + a^2 + l^2)^2} R + \frac{m^2 a^2}{(r^2 + a^2 + l^2)^2} R = C \frac{\Delta}{(r^2 + a^2 + l^2)^2} R.$$

(102)

or,

$$\frac{d^2 R}{dr^*^2} + \frac{2r\Delta}{(r^2 + a^2 + l^2)^2} \frac{dR}{dr^*} + \left[ \omega^2 + \frac{-2m\omega a (r^2 + a^2 + l^2 - \Delta) + m^2 a^2 - C\Delta}{(r^2 + a^2 + l^2)^2} \right] R = 0.$$  (103)

Therefore, in the $r \to \infty$ limit, the above equation becomes,

$$\frac{d^2 R}{dr^*^2} + \frac{2r\Delta}{r \, dr^*} + \omega^2 R = 0.$$  (104)

with $r$ and $r^*$ are identified. Therefore, the solution of $R$ becomes,

$$R_\infty \sim \frac{1}{r} \exp(\pm i\omega r^*)$$  (105)

While, in the near horizon region we have $\Delta \to 0$ and hence the equation gets modified to,

$$\frac{d^2 R}{dr^*^2} + \left[ \omega - \frac{m\Omega_H}{r} \right]^2 R = 0.$$  (106)

The angular velocity in general corresponds to,

$$\omega = \frac{\partial_{\phi} g}{\partial_{\phi\phi} g} = \frac{-\Delta P + a(r^2 + a^2 + l^2) \sin^2 \theta}{\sin^2 \theta (r^2 + a^2 + l^2)^2 - \Delta P^2}.$$  (107)

such that angular velocity of the horizon becomes,

$$\Omega_H = \frac{a}{r^2 + a^2 + l^2}.$$  (108)

Also note that on the horizon $r^2 + a^2 + l^2 = 2Mr + 2l^2$. Therefore, the above differential equation becomes,

$$\frac{d^2 R}{dr^*^2} + [\omega - m\Omega_H]^2 R = 0.$$  (109)

and hence,

$$R_H \sim \exp \left[ \pm i \left( \omega - m\Omega_H \right) r^* \right]$$  (110)

Therefore the energy momentum tensor of the scalar field yields,

$$T_{\mu\nu} = \text{Re} \left[ \partial_{\rho} \phi \partial_{\rho^*} \phi^* - \frac{1}{2} \theta_{\mu\nu} \left( \partial_{\rho} \phi \partial_{\rho^*} \phi^* \right) \right].$$  (111)
Thus we obtain the \((t,r)\) component of the above energy momentum tensor to yield,

\[
T^t_r = \text{Re} \left[ \partial_t \Phi \partial^r \Phi^* \right] \\
= \text{Re} \left[ g^{tt} \partial_t \Phi \partial^r \Phi^* \right] \\
= \text{Re} \left[ \frac{\Delta}{\rho^2} \left( r^2 + a^2 + l^2 \right) \left( -i\omega \right) \left( \omega - m\Omega_H \right) \Theta(\theta)^2 \right] \\
= \omega (\omega - m\Omega_H) \left( \frac{r^2 + a^2 + l^2}{\rho^2} \right) \Theta^2.
\]

Here the scalar field is being evaluated near the black hole horizon. This is the expression which have been used to compute the rate of energy change of the scalar field system and appears in Eq. (72).

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