Research Article

Vibrational analysis of nanoplate with surface irregularity via Kirchhoff plate theory

Mahmoud M Selim¹,² and Taher A Nofal³

Abstract

In this work, an attempt is done to apply the Kirchhoff plate theory to find out the vibrational analyses of a nanoplate incorporating surface irregularity effects. The effects of surface irregularity on natural frequency of vibration of nanomaterials, especially for nanoplates, have not been investigated before, and most of the previous research have been carried for regular nanoplates. Therefore, it must be emphasized that the vibrations of irregular nanoplate are novel and applicable for the nanodevices, in which nanoplates act as the main structure of the nanocomposite. The surface irregularity is assumed in the parabolic form at the surface of the nanoplate. A novel equation of motion and frequency equation is derived. The obtained results provide a better representation of the vibration behavior of irregular nanoplates. It has been observed that the presence of surface irregularity affects considerably on the natural frequency of vibrational nanoplates. In addition, it has been seen that the natural frequency of nanoplate decreases with the increase of surface irregularity parameter. Finally, it can be said, the present results may serve as useful references for designing oscillators and nanoscale devices, in which nanoplates act as a structural component for most prevalent nanocomposites structural element.

Keywords

Vibration analysis, nanoplate, surface irregularity, Kirchhoff plate theory

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Introduction

Due to its outstanding properties, applications of nanoplates in nanodevices are very useful nowadays. The most widely utilized type of nanoplates is graphite, which exist in the form of layered materials.¹ Graphene nanoplates are not only known for an excellent combination of high electrical and thermal conductivity, optical transparency, flexibility, robustness, and environmental stability,²⁻⁵ but also they demonstrate unusual nonlinear dynamics.⁶⁻⁷ To date, various studies are carried out about the vibrational analysis of regular nanoplates, as can be seen from the examples of the literature.⁸⁻¹⁸ Murmu and Pradhan¹⁹ who they have discussed the small-scale impacts on the vibration of nanoplates using the nonlocal elasticity theory. Ansari et al.²⁰ have reported a new results of the free vibrations of single-layered graphene sheets using the nonlocal plate model. Wang and Wang²¹ applied the nonlocal

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continuum model to study the effects of surface energy on the vibration of nano scale plates. Murmu and Adhikari\textsuperscript{22} have investigated the vibration of bonded double-nanoplate-systems using nonlocal continuum theory. Malekzadeh et al.\textsuperscript{23} have reported new results of the effects of the small-scale on free vibration of orthotropic nanoplates. The vibration characteristics of nonlinear nanoplates under thermal conditions have studied by Shooshtari and Rafiee.\textsuperscript{24} Ke et al.\textsuperscript{25} have investigated the free vibration of size-dependent Mindlin microplates using couple stress theory. Wang et al.\textsuperscript{26} have studied the free vibration of size-dependent circular microplates using couple stress theory. Shahrbabaki and Alibeigloo\textsuperscript{27} have studied the free vibration of carbon nanotube-reinforced composite plates using Ritz method. Biswal and Rao\textsuperscript{28} have analyzed the thermo-mechanical vibration of micro-nano circular plate. Ebrahimi and Hosseini\textsuperscript{29} have reported the results of the impacts of thermal stress on nonlinear vibration of viscoelastic nanoplates. Recently, Zur et al.\textsuperscript{30} have studied the free vibration and buckling of magneto-electro-elastic functionally graded material (FGM) nanoplates. Gao et al.\textsuperscript{31} have studied the wave propagation in functionally graded porous plates. Turbulator involving nanomaterial turbulent regime has been investigated by Sheikholeslami et al.\textsuperscript{32} and Sheikholeslami and Farshad.\textsuperscript{33,34} The above studies show that the nanoplate is very important nanomaterial and it is growing rapidly, and numerous kinds of research are done about the mechanical and electrical properties of nanoplates. Irregularities in the constructions of nanoplates may occur as a consequence of a mismatch between the material behaviors of nanoplates. Irregularities in the constructions of nanoplates may occur as consequences of mismatch between the material properties of nanoplates. Therefore, studying the impacts of surface irregularity on the vibration of nanoplates will help the engineers to best designing of the proper nanodevices with irregular shapes. Astonishing mechanical, electrical, and physical properties of nanoplates are suggesting them for a diverse range of applications. Therefore, nanoplates would be suitable nanomaterials for the building block of nanodevices and nanodrives. Some potential application of nanoplates would be in conveyors of nanoscale belts and high-speed nanotapes. For such applications, the effects of surface irregularity on vibration of nanoplates should be realized. In addition, it is very important to conduct research and reveal theoretical results on the vibrational analysis of nanoplates. From our best knowledge, the effects of surface irregularity on the vibrational analysis of nanoplates have not yet been investigated. In this sense, the present work reports on the implantation of Kirchhoff plate model and the effects of surface irregularity on the natural frequency of vibrational nanoplate. The numerical results are plotted in graphs for different values of surface irregularity parameters and have been compared with the case of uniform nanoplate. In this work, the obtained frequency equation and numerical results may give a useful reference to predict the natural frequency of vibrations of nanoplates to estimate the expected experimental values for potential applications of irregular nanoplates in designing of nano-oscillators and nanodevices.

The article is organized as follows: the “Formulation of the problem” section describes the mathematical formulation, the geometry and coordinate systems of the problem, the vibration equation for the irregular nanoplate, the governing equation of the nanoplate, and the stress–strain relations of the nanoplate; in the “Solution method” section, the solution of governing equation is presented, the angular frequency and the natural frequency of the nanoplate are analyzed and discussed; in the “Results and discussion” section, the numerical results for vibration of the simply supported nanoplate under the effects of surface irregularity are carried out using the simulation parameters and the results obtained are analyzed and discussed; the “Conclusions” section summarizes the main results of the present work and provides the useful information for future studies.

Formulation of the problem

The geometry and coordinate systems used for describing the effect of surface irregularity on vibration analysis of the irregular nanoplate are shown in Figure 1. The displacements of the nanoplate are indicated by $u$, $v$, and $w$ in the direction of $x$, $y$, and $z$ axes, respectively. The nanoplate has a thickness $h$, length $a$, and width $b$.

In case of parabolic surface irregularity, the boundary surface may be described by:

\[ z = -h + \varepsilon \delta(x), \quad \delta(x) = \begin{cases} \frac{2s(1 - \frac{x^2}{s^2})}{s^2} & \text{for } |x| < s \\ 0 & \text{for } |x| \geq s \end{cases} \quad (1) \]

where $\varepsilon = \frac{h^2}{s^2} << 1$. 

Figure 1. Geometry and coordinate systems of an irregular nanoplate.
The density, Poisson’s ratio and elastic modulus of the considered nanoplate are indicated by \( \rho \), \( \nu \), and \( E \), respectively.

To investigate the vibrational analysis of nanoplates based on the Kirchhoff model, we could write the displacement components as a function of the transverse displacement as follows\(^{36,37}\)

\[
u = -z \frac{\partial w}{\partial x}, \quad \nu = -z \frac{\partial w}{\partial y} \tag{2}\]

where \( w \) is the transverse displacement.

The strain components in Cartesian coordinate are expressed as \(^{38}\)

\[
\varepsilon_{xx} = -2z \frac{\partial^2 w}{\partial x^2}; \quad \varepsilon_{yy} = -2z \frac{\partial^2 w}{\partial y^2}; \quad \varepsilon_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y} \tag{3}\]

The vibration equation for the nanoplates can be expressed as \(^{39,40}\)

\[
\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = \rho h \frac{\partial^2 w}{\partial t^2} \tag{4}\]

In which \( M_{xx}, M_{yy}, M_{xy} \) represent the resultants of the bending moment acting on the nanoplate and are given by the integral relations

\[
M_{xx} = \int_{-h/2}^{h/2} z \tau_{xx} \, dz \tag{5}\]

\[
M_{yy} = \int_{-h/2}^{h/2} z \tau_{yy} \, dz \tag{6}\]

\[
M_{xy} = \int_{-h/2}^{h/2} z \tau_{xy} \, dz \tag{7}\]

Based on Hook’s law, the constitutive relations of the nanoplate can be written as:

\[
\tau_{xx} = \frac{E}{1 - \nu^2} (\varepsilon_{xx} + \nu \varepsilon_{yy}) = -\frac{E(\varepsilon(x) - h)}{1 - \nu^2} \frac{\partial^2 w}{\partial x^2} - \nu \frac{\partial^2 w}{\partial y^2} \tag{8}\]

\[
\tau_{yy} = \frac{E}{1 - \nu^2} (\varepsilon_{yy} + \nu \varepsilon_{xx}) = -\frac{E(\varepsilon(x) - h)}{1 - \nu^2} \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \tag{9}\]

\[
\tau_{xy} = \frac{E}{2(1 + \nu)} \varepsilon_{xy} = -\frac{E(\varepsilon(x) - h)}{1 + \nu} \frac{\partial^2 w}{\partial x \partial y} \tag{10}\]

where \( \tau_{xx}, \tau_{yy}, \tau_{xy} \) are the stress components.

From equations (5), (6), (7), (8), (9) and (10) the bending moment, can be expressed as:

\[
M_{xx} = -\frac{3\beta(\varepsilon(x) - h)}{2h} \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \tag{11}\]

\[
M_{yy} = -\frac{3\beta(\varepsilon(x) - h)}{2h} \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \tag{12}\]

\[
M_{xy} = -\frac{3\beta(1 - \nu)(\varepsilon(x) - h)}{2h} \frac{\partial^2 w}{\partial x \partial y} \tag{13}\]

where \( \beta = \frac{E h^3}{12(1 - \nu^2)} \) is the bending stiffness of the nanoplate.

Using equations (11) to (13), the vibration equation of the nanoplate (4) becomes

\[
-\frac{3\beta(\varepsilon(x) - h)}{2h} \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = \rho h \frac{\partial^2 w}{\partial t^2} \tag{14}\]

**Solution method**

The problem, in which mathematical formulation is expressed in the previous subsection, is a governing equation for the nanoplate. In the investigation of this problem, for the simply supported boundary condition, the analytical solution of equation (14) can be given in the form\(^{13}\)

\[
w_{mn}(x, y, t) = W_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) e^{-i\omega t} \tag{15}\]

where \( i = \sqrt{-1} \), and \( W_{mn} \) is unknown coefficient that must be determined to obtain the real-time response of the nanoplate. Based on the derivation presented in Appendix 1, the dispersion relation of frequency equation of the irregular nanoplate at the given \( m \) and \( n \) half-wave numbers can be yielded from the solution of equation (14) as follows

\[
\omega_{mn} = \sqrt{\frac{3\beta(h - \varepsilon(x))}{2\rho h^2}} \left( \frac{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2}{2} \right) \tag{16}\]

In the present study, we suppose that all four edges of the plate are assumed to be simply supported with no inplane displacements, so the axial wave number \( k_x = \frac{m\pi}{a} \) and the half-wave number in \( y \) direction are constant \( k_y = \frac{n\pi}{b} \).

The natural frequency of the nanoplate is given from the relation \( f_{mn} = \omega_{mn} / 2\pi \), where \( m \) and \( n \) denote the half-wave numbers for \( x \) and \( y \) directions, respectively.

From the above equation, it can be observed that, if the surface irregularity is ignored (\( \varepsilon = 0 \)), the frequency equation will be taken in the following form

\[
\omega_{mn} = \sqrt{\frac{3\beta \pi^2}{2\rho h^2}} \left( \frac{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2}{2} \right) \tag{17}\]

which agrees with the result of Kitipornchai et al.\(^{41}\)
Results and discussion

In this section, numerical results for vibration of the simply supported nanoplate under the effects of surface irregularity are computed using the parameters presented in Table 1. The natural frequency (Hz) as a function of half-wave numbers and surface irregularly parameter $\varepsilon$ have been calculated using relation (16) and are compared with the results of uniform nanoplates ($\varepsilon = 0$).

Firstly, to show the effectiveness of surface irregularity on the natural frequency of the nanoplates, the results in case of uniform nanoplate ($\varepsilon = 0$) are presented. The Kirchhoff plate model, which is given by Assadi,16 is considered. To compute the natural frequency of vibrations of nanoplates, the computer software MATLAB R203a is used. It is pointed out that with minor substitution, the present results can be used to analyze the impacts of surface irregularities on vibration of nanoplates. The numerical calculations are carried for different values of the surface irregularity parameter ($\varepsilon = 0.28$, 0.56). The results are plotted and shown in Figures 2 to 4. The length and width of the nanoplate are chosen to be $a = b = 10$ nm.

Figure 2 shows the change of natural frequency of the uniform ($\varepsilon = 0$) nanoplates due to the change of half wavenumber in $x$-direction. It is clearly seen from the figure the natural frequencies of the nanoplates change in the range $0 - 4 \times 10^8$ Hz. In addition, it can be seen that, the values of natural frequency are different over any arbitrary thickness. Also, it can be observed that, the natural frequencies of the nanoplate at the thickness ($h = 15$ nm) are more than that of the nanoplate with thicknesses ($h = 5$ or 10 nm). Moreover, the sensitivity of natural frequency to the thickness change of the nanoplates is much higher for near the higher values of half wave number.

Next, it is tried to show the vibration behavior of nanoplate with different values of the surface irregularity parameter ($\varepsilon$). The numerical results in Figure 3 are given for the vibration at the surface irregularly parameter ($\varepsilon = 0.28$). From Figure 3, it can be observed that the natural frequencies of the nanoplates with different thicknesses ($h = 5$, 10, and 15 nm) are affected by the surface irregularity present in the nanoplates. It can be observed that the natural frequencies are quite different compared with the results obtained for the uniform nanoplate ($\varepsilon = 0$) at the same half-wave number, as shown in Figure 2. The values of the natural frequencies change from 0 to $3.5 \times 10^8$ Hz compared with $0 - 4 \times 10^8$ Hz at the same half-wave number. The reason is very clear that the change is attributed to the irregularity present in the surface of the nanoplate. In addition, it can be concluded form Figure 3 that, the influences of surface irregularities on the vibration become obvious at different thicknesses ($h = 5$, 10, and 15 nm) of the nanoplates. Further observation can be seen; the presence of the surface irregularity decreases the natural frequency of vibrations.

| $E$ (TPa) | $\nu$ | $a = b$ (nm) | $\rho$ (kg/m$^3$) |
|----------|-------|-------------|---------------|
| 1.06     | 0.25  | 10          | 2250          |

Figure 2. Variations of natural frequency of irregular nanoplate versus axial half-wave number at ($\varepsilon = 0$).

Figure 3. Variations of natural frequency of irregular nanoplate versus half-wave number at ($\varepsilon = 0.28$).

Figure 4. Variations of natural frequency of irregular nanoplate versus half-wave number at ($\varepsilon = 0.84$).
The effects of surface irregularity become obvious at the higher values of irregularity’s parameter \( (\varepsilon = 0.84) \), as shown in Figure 4. In this figure, the natural frequencies are quite different compared with the values obtained at \( (\varepsilon = 0) \) and \( (\varepsilon = 0.28) \). The values of the natural frequency are decreased to \( 0.3 \times 10^6 \) Hz compared with \( 3.5 \times 10^6 \) Hz at \( (\varepsilon = 0.28) \). Once the reason is very clear due to the increase of surface irregularity parameter. This means that the natural frequency of the irregular nanoplates is more affected by the increase of surface irregularity parameter \( (\varepsilon) \). In addition, it can be seen from Figure 4 that the effects of surface irregularity depend on the thickness of the nanoplate at both cases \( (\varepsilon = 0.28) \) and \( (\varepsilon = 0.84) \).

### Conclusion

In this work, a novel frequency equation of irregular nanoplates based on Kirchhoff plate model has been reported. The worth mentioning components of the present analysis are to examine the effects of surface irregularity on the natural frequency of vibration of nanoplates. The numerical results have been shown graphically and the dependence of the vibration characteristics of the nanoplate on the surface irregularity parameter is investigated. It has been detected that the presence of surface irregularity in the nanoplate has notable effects on the natural frequency of vibration. The obtained numerical and analytical results show that the increase of surface irregularity parameter decreases the natural frequency of the vibration of nanoplates. Thus, it can be concluded that the investigation presented may provide useful information for the next generation studies and accurate designs of nanodevices, nano-oscillators, and nanosensors.

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### Supplemental material

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**Appendix 1**

\[
\frac{\partial w_{mn}(x,y,t)}{\partial x} = \left(\frac{m\pi}{a}\right) W_{mn} \cos \left(\frac{m\pi x}{a}\right) \sin \left(\frac{n\pi y}{b}\right) e^{-i\omega t}
\]

\[
\frac{\partial w_{mn}(x,y,t)}{\partial y} = \left(\frac{n\pi}{b}\right) W_{mn} \sin \left(\frac{m\pi x}{a}\right) \cos \left(\frac{n\pi y}{b}\right) e^{-i\omega t}
\]

\[
\frac{\partial^2 w_{mn}(x,y,t)}{\partial x^2} = -\left(\frac{m^2}{a^2}\right) W_{mn}
\]

\[
\frac{\partial^2 w_{mn}(x,y,t)}{\partial y^2} = -\left(\frac{n^2}{b^2}\right) W_{mn}
\]

\[
\frac{\partial^3 w_{mn}(x,y,t)}{\partial x^3} = -\left(\frac{m^3}{a^3}\right) W_{mn} \cos \left(\frac{m\pi x}{a}\right) \sin \left(\frac{n\pi y}{b}\right) e^{-i\omega t}
\]

\[
\frac{\partial^3 w_{mn}(x,y,t)}{\partial y^3} = -\left(\frac{n^3}{b^3}\right) W_{mn} \sin \left(\frac{m\pi x}{a}\right) \cos \left(\frac{n\pi y}{b}\right) e^{-i\omega t}
\]

\[
\frac{\partial^4 w_{mn}(x,y,t)}{\partial x^4} = \left(\frac{m^4}{a^4}\right) W_{mn}
\]

\[
\frac{\partial^4 w_{mn}(x,y,t)}{\partial y^4} = \left(\frac{n^4}{b^4}\right) W_{mn}
\]
\[
\frac{\partial^4 w_{mn}(x, y, t)}{\partial x^2 \partial y^2} = \left( \frac{\partial^2 w_{mn}(x, y, t)}{\partial y^2} \right) = \frac{\partial^2}{\partial x^2} \left[ \left( \frac{n\pi}{b} \right)^2 w_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) e^{-i\omega t} \right] = \left( \frac{m\pi}{a} \right)^2 \left( \frac{n\pi}{b} \right)^2 W_{mn}
\]

\[
\frac{\partial w_{mn}(x, y, t)}{\partial t} = (-i\omega) W_{mn} \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) e^{-i\omega t}
\]

\[
\frac{\partial^2 w_{mn}(x, y, t)}{\partial t^2} = (-\omega^2) W_{mn} \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) e^{-i\omega t}
\]