Existence and uniqueness of solutions to the damped Navier–Stokes equations with Navier boundary conditions for three dimensional incompressible fluid

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Abstract
In this article, we study the solutions of the damped Navier–Stokes equation with the Navier slip boundary condition in a bounded domain \( \Omega \) in \( \mathbb{R}^3 \) with sufficiently smooth boundary. We employ the Galerkin method to approximate the solutions of the damped Navier–Stokes equations with the Navier-slip boundary conditions. The existence of the solutions is global for \( \beta \geq 1 \). We also established the regularity of the solutions for \( \beta \geq 3 \), and the uniqueness of the solutions for \( \beta \geq 1 \).

Keywords Navier Stokes equation · Galerkin method · Navier boundary condition · Damping term

Mathematics Subject Classification 76D05 · 35A01 · 35Q30 · 76D03

1 Introduction
In this article, we consider the following system of Navier–Stokes (N–S) equations in a simply connected bounded domain \( \Omega \) of \( \mathbb{R}^3 \), with sufficiently smooth boundary,

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \vartheta |u|^\beta - 1 u + \frac{1}{\rho} \nabla p - \mu \Delta u &= f & \text{in} & \ Q_T, \\
\text{div} \ u &= 0 & \text{in} & \ Q_T, \\
u &= u_0 & \text{in} & \ \Omega \times \{0\}, \\
u \cdot v &= 0 & \text{on} & \ \partial \Omega \times (0, T),
\end{align*}
\]
\[ 2D(u)v \cdot \tau + \alpha u \cdot \tau = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \quad (1.5) \]

where \( Q_T = \Omega \times (0, T), \ T > 0. \) Here the unknown function \( p = p(x, t) \) is the pressure and \( u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \) is the velocity of the flow. \( \beta \geq 1 \) and \( \vartheta > 0 \) are two constants in the damping term \( \vartheta |u|^{\beta-1}u. \rho \) is the density, \( f \) is the external force, \( \mu > 0 \) is the kinematic viscosity and \( u_0 \) is the initial velocity of the fluid. The rate of the strain tensor \( D(u) = \frac{1}{2}[\nabla u + (\nabla u)^T] \) and \( \alpha(x) > 0 \) is a continuously differentiable function on the boundary \( \partial\Omega. \ v \) and \( \tau \) are unit exterior normal and the unit tangent vector to the boundary. The balance of momentum of the fluid motion is given by (1.1) and the condition of incompressibility is expressed in (1.2). The Navier slip boundary condition is given by (1.5).

The N–S equations model of important phenomena in fluid mechanics and describe the motion of fluid like water, air, oil under general conditions. The study of Navier–Stokes equations with various boundary conditions has many applications and is also challenging mathematically. The analytical solutions of non-linear N–S equations are complicated to obtain except in some simple situations. Even solutions for simple situations need very high tools of mathematics.

The no-slip boundary condition \( (u = 0 \text{ on the boundary}) \) shows us the breakdown of the traditional macroscopic ideas at small scales between the fluid and solid interface [14]. To hold the no-slip boundary condition at the fluid-solid interface, it is necessary that the fluid is a continuum and, the flow is in thermodynamic equilibrium. The no-slip boundary condition means the fluid velocity relative to the solid boundary is zero. This happens only if the flow is in thermodynamic equilibrium. This needs an infinitely high frequency of collisions between the fluid and the solid surface. So, the tangential velocity must be allowed [15] in the proportion of the tangential component of the stress. Thus the no-slip boundary condition is replaced by the Navier slip boundary condition [14,16,17]. We express the Navier slip boundary condition on \( \partial\Omega \times (0, T) \) as

\[ u \cdot n = 0, \quad (1.6) \]

\[ 2D(u)v \cdot \tau + \alpha u \cdot \tau = 0. \quad (1.7) \]

where \( 2D(u)v \cdot \tau \) denotes the tangential component of \( 2D(u) \cdot n \) and, \( \alpha \) is the coefficient of proportionality. The authors in [1,2,4,5,8–10,24,26] has studied the existence of solutions to N–S equations with the Navier slip boundary condition. Clopeau et al. [7] studied the regular solutions for two-dimensional incompressible Navier–Stokes equations with the Navier boundary conditions. Filho et al. [13] extend the work of Clopeau et al. [7] to study the vorticities in \( L^p \) with \( p > 2. \) Kelliher [11] extends the work of Clopeau et al. [7] and Filho et al. [13] for the bounded domain in \( \mathbb{R}^2. \)

The system (1.1)–(1.5) with \( \vartheta > 0 \) describes the flow with the resistance to the motion, such as porous media flow and drag or friction effects. From a mathematical viewpoint, (1.1) can be viewed as a modification of the Navier–Stokes equation with the regularizing term \( \vartheta |u|^{\beta-1}u. \) Thus it is important to study the regularity property and the uniqueness of the weak solutions [12]. In the back of 1969, J.L.Lions realize the fact that 3D Navier–Stokes equations also well-posed if we replace the viscous diffusion term \(-\Delta\) with a fractional Laplacian \((-\Delta)^{5/4}\) since energy identity gives us
Later Cai and Xiu [6] realize that instead of the fractional Laplacian, sufficiently strong damping will achieve the same goal. A heuristic idea behind this is that the sufficiently strong damping immediately gives us \( u \in L^r_T L^p_x \) for \( p, r \) sufficiently large and essentially a well-known Serrin regularity criteria for the uniqueness of the weak solution adapted to the damped Navier–Stokes equations. We mention that the system (1.1)–(1.5) with the no-slip boundary condition has been studied in [3,6,21,22,25,27,28]. Cai and Jiu [6] established the existence of the weak solution if \( \beta \geq 1 \) and the global strong solutions if \( \beta \geq 7/2 \) of the damped Navier–Stokes equation with the no-slip boundary condition. The uniqueness of the strong solution is proved for \( 5 \leq \beta \geq 7/2 \). Yamazaki [25] established the existence and uniqueness of strong solutions for the four-dimensional Navier–Stokes system with damping in the third and fourth components. Zhang et al. [27] showed that the damped Navier–Stokes equation with no-slip boundary conditions has a global strong solutions when \( \beta > 3 \). They also obtained the uniqueness of the strong solution when \( 5 \geq \beta > 3 \). Further, they proved that the strong solution exists globally for \( \beta = 3 \) and \( \vartheta = \mu = 1 \), and the strong solution is unique among the solutions [28] in \( L^\infty(0, T; L^2(\Omega)^3) \) for \( \beta \geq 1 \).

In this article, we use the Galerkin approximation method to show the existence of the solutions to the damped N–S equations with Navier slip boundary conditions in a simply connected domain of \( \mathbb{R}^3 \). It is worth noting that so far there are no results in the literature regarding the existence, uniqueness and regularity of the solutions to damped N–S equations with the Navier slip boundary condition in \( \mathbb{R}^3 \). The article is organised as follows. The preliminaries and assumptions are provided in Sect. 2. We prove the existence of solutions in Sect. 3 and the regularity of solutions in Sect. 4. We also proved the uniqueness of the solution in Sect. 5.

## 2 Preliminaries and assumptions

In this section we collect the notations, Lemmas and solution spaces. For more details, we refer to Temam [23] and Sohr [20]. In the remaining part of the article, \( \Omega \) is a bounded simply connected domain in \( \mathbb{R}^3 \) with sufficiently smooth boundary. We denote the set of all \( C^\infty \) real vector function \( \phi \) with compact support in \( \Omega \) by \( C_0^\infty(\Omega) \). We use the Lebesgue space \( L^p(\Omega) \) (1 \( \leq p \leq \infty \)) and the Sobolev space, \( H^r(\Omega) = \{ \phi \in L^2(\Omega) \mid \text{weak derivative of } \phi \text{ up to order } r \text{ in } L^2(\Omega) \} \). \((\cdot, \cdot)\) denotes \( L^2 \)-inner product and

\[
((u, v)) = \sum_{i=1}^{n} (D_i u, D_i v).
\]

The norm of the above two inner-product will be denoted by \( \| \cdot \| \) with clear subscripts. We define the following function space like in [19]

\[
L^2_0(\Omega) = \{ z \in L^2(\Omega) : \int_{\Omega} z dx = 0 \}, \quad V = \{ v \in H^1(\Omega)^3 : \text{div} \ v = 0 \text{ in } \Omega, \ v \cdot v = 0 \text{ on } \partial \Omega \},
\]
\[
H = \{ v \in L^2(\Omega)^3 : \text{div } v = 0 \text{ in } \Omega, \ v \cdot v = 0 \text{ on } \partial \Omega \},
\]
\[
W = \{ v \in V \cap H^2(\Omega)^3 : 2D(u)v \cdot \tau + \alpha u \cdot \tau = 0 \text{ on } \partial \Omega \}.
\]

We give \( W \) the \( H^2 \)-norm, \( H \) the \( L^2 \)-inner product and norm, which we symbolize by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \|_{L^2(\Omega)} \), and \( V \) the \( H^1 \)-inner product,

\[
\langle (u, v) \rangle = \sum_{i=1}^{n} (D_i u, D_i v)
\]

and associate norm.

We define the trilinear form \( b \) as

\[
b(u, v, w) = \int_{\Omega} (u \cdot \nabla v) \cdot w, \quad \forall u, v, w \in V.
\]

If \( u \in V \), then \( b(u, v, v) = 0 \), \( \forall v \in H^1_0(\Omega) \).

For \( u, v \in V \), we define \( B(u, v) \) by

\[
(B(u, v), w) = b(u, v, w), \quad \forall w \in V,
\]

and we set \( B(u) = B(u, u) \in V' \), \( \forall u \in V \). So,

\[
(Bu, v) = \int_{\Omega} (u \cdot \nabla u) \cdot v.
\]

We shall use the following Lemma. We omit the proof and refer to Temam [23, Lemma III.3.1].

**Lemma 2.1** Assuming dimension of the space \( \leq 4 \) and \( u \in L^2(0, T; V) \). Then the function \( Bu \) defined by

\[
(B(u(t), v) = b(u(t), u(t), v), \quad \forall v \in V
\]

belongs to \( L^1(0, T; V') \). Moreover

\[
\| Bu \|_{V'} \leq C \| u \|^2_{H^1}, \quad \forall u \in V
\]

We will use the following Lemma from [10, Lemma 2.1(ii)].

**Lemma 2.2** For all \( u, v, w \in H^1(\Omega) \), where \( \Omega \in \mathbb{R}^3 \) we have

\[
|b(u, v, w)| \leq C \| u \|^\frac{1}{2} \| H^1(\Omega) \| \| v \|^\frac{3}{2} \| H^1(\Omega) \| \| w \|^\frac{1}{2} \| L^2(\Omega) \| \| w \|^\frac{3}{2} \| H^1(\Omega) \|.
\]
Next we define the operator $A$ as follows

$$(Au, v) = 2((u, v)) + \int_{\partial\Omega} \alpha(u \cdot \tau)(v \cdot \tau), \quad \forall u, v \in V.$$  \hspace{1cm} (2.13)

We have

$$|(Au, v)| \leq \|u\|_V \|v\|_V + L_1 \|u\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} \leq L \|u\|_V \|v\|_V,$$  \hspace{1cm} (2.14)

where $L_1$ (come from $\alpha$) and $L$ are positive constants. So $A : L^2([0, T]; V) \to L^2([0, T]; V')$.

The existence of a complete orthogonal basis for $H$ was proved in [19]. We recall their result as a lemma and use it to prove the main theorem.

**Lemma 2.3** There exists a basis $\{v_n\} \subset H^3(\Omega)^3$, for $V$ which also serves as an orthonormal basis for $H$, that satisfies

$$2D(u) v \cdot \tau + au \cdot \tau = 0 \quad \text{on} \quad \partial\Omega.$$  \hspace{1cm} (2.15)

### 3 Existence of solutions

In this section, we prove the existence of solutions to the damped Navier–Stokes system (1.1)–(1.5) with $\rho = 1$. The idea of the proof is similar to that of [23] for the classical Navier–Stokes equations. The definition of weak solution is given as usual.

**Definition 3.1** The function $u(x, t)$ is called a weak solution of the damped Navier–Stokes system (1.1)–(1.5), if for any $T > 0$ the following conditions are satisfied:

1. $u \in L^2(0, T; V) \cap L^\infty(0, T; H) \cap L^{\beta+1}(0, T; L^{\beta+1}(\Omega))$,

2. for any $v \in V$, we have

$$(u'(t), v) + 2\mu((u(t), v)) + b(u(t), u(t), v) + (\vartheta|u|^{\beta-1}u(t), v) + \mu \int_{\partial\Omega} \alpha(u(t) \cdot \tau)(v \cdot \tau))dS = (f(t), v),$$

where $\partial\Omega$ is sufficiently smooth.

We state the following compactness result. For the proof, we refer to Cai and Jiu [6, Lemma 2.1].

**Lemma 3.2** Let $X_0, X$ be Hilbert space satisfying a compact imbedding $X_0 \hookrightarrow X$. Let $0 < \gamma \leq 1$ and $(v_j)_{j=1}^\infty$ be a sequence in $L^2(R, X_0)$ satisfying

$$\sup_j \left( \int_{-\infty}^{+\infty} \|v_j\|^2_{X_0} dt \right) < \infty, \quad \sup_j \left( \int_{-\infty}^{+\infty} |\xi|^{2\gamma} \|\hat{v}_j\|^2_X d\xi \right) < \infty.$$
where
\[
\hat{v}_j(\xi) = \int_{-\infty}^{\infty} v_j(t) \exp(-2\pi i \xi t) dt
\]
is the Fourier transformation of \(v_j(t)\) on the time variable. Then there exists a subsequence of \((v_j)_{j=1}^{\infty}\) which converges strongly in \(L^2(R; X)\) to some \(v \in L^2(R; X)\).

Our main result of this section is given as follows.

**Theorem 3.3** Suppose that \(\beta \geq 1\), \(u_0 \in H\) and \(f \in L^2(0, T; H)\). Then for any given \(T > 0\), there exists a function \(u(x, t)\) such that
\[
u \in L^2(0, T; V) \cap L^\infty(0, T; H) \cap L^{\beta+1}(0, T; L^{\beta+1}(\Omega)).
\]
and \(u(x, t)\) will satisfy
\[
(u'(t), w) + 2\mu((u(t), w)) + b(u(t), u(t), w) + (\vartheta |u|^{\beta-1}u(t), w)
\]
\[
+ \mu \int_{\partial\Omega} \alpha(u(t) \cdot \tau)(w \cdot \tau)dS = (f(t), w), \quad \forall w \in V,
\]
where \(\partial\Omega\) is sufficiently smooth.

**Proof** We use the Galerkin approximations to prove the theorem. It follows from Lemma 2.3 that there exists an orthonormal basis for \(W\). We denote the basis as \(\{w_i \in H^2(\Omega)^3\}\). We note that \(\{w_i\}\) also forms a basis for \(V\). For each positive integer \(m\), we define an approximate solution \(u_m\) to (3.16) as follows:
\[
u_m(t) = \sum_{i=1}^{m} g_{im}(t)w_i
\]
Substituting \(u_m\) in (3.16), we obtain
\[
(u'_m(t), w_j) + 2\mu((u_m(t), w_j)) + b(u_m(t), u_m(t), w_j) + (\vartheta |u_m|^{\beta-1}u_m(t), w_j)
\]
\[
+ \mu \int_{\partial\Omega} \alpha(u_m(t) \cdot \tau)(w_j \cdot \tau)dS = (f(t), w_j) \quad j = 1, \ldots, m
\]
\[
u_m(0) = \sum_{j=1}^{m} (u_0, w_j)w_j = u_{0m}.
\]
Further, the equations (3.20)–(3.21) can be rewritten as
\[
\sum_{i=1}^{m} (w_i, w_j)g_{im}'(t) + 2\mu \sum_{i=1}^{m} ((w_i, w_j))g_{im}(t) + \sum_{i,j=1}^{m} b(w_i, w_j)g_{im}(t)g_{jm}(t)
\]
+ \sum_{i=1}^{m} \vartheta |u_m|^{\beta-1}(w_i, w_j)g_{im}(t) + \mu \int_{\partial\Omega} \alpha(u_m(t) \cdot \tau)(w_j \cdot \tau) dS = (f(t), w_j), \quad j = 1, \ldots, m. \tag{3.22}

u_m(0) = \sum_{j=1}^{m} (u_0, w_j)w_j = u_0m. \tag{3.23}

The equations (3.22)–(3.23) form a nonlinear system of differential equations in the function $g_{1m}, \ldots, g_{mm}$. Now we apply Picard’s theorem, which ensures us an existence of a unique local solution of (3.22) in some interval $[0, T_m] \subset [0, T]$.

We have a priori estimates on the approximate solutions $u_m$ as follows which ensure us the existence of solution for all $T$.

Lemma 3.4 Suppose that $u_0 \in H$. Then for any given $T > 0$ and any $\beta \geq 1$, we have

$$
\sup_{0 \leq t \leq T} \|u_m(t)\|_{L^2}^2 + 2\mu \int_0^T \|u_m(t)\|_{H^1}^2 dt + 2\vartheta \int_0^T \|u_m(t)\|_{L^{\beta+1}}^{\beta+1} dt \leq \|u_0\|_{L^2}^2 + \frac{2}{\mu} \int_0^T \|f(t)\|_{L^2}^2 dt.
$$

Proof We multiply (3.20) by $g_{jm}(t)$ and add these equation for $j = 1, \ldots, m$. We get

$$(u_m'(t), u_m(t)) + 2\mu \|u_m(t)\|_{H^1}^2 + \vartheta \|u_m(t)\|_{L^{\beta+1}}^{\beta+1} + \mu \int_{\partial\Omega} \alpha(u_m(t) \cdot \tau)(u_m(t) \cdot \tau) dS = (f(t), u_m(t)), \tag{3.24}$$

where we have used the fact that $b(u, v, v) = 0$, $\forall v \in H^1_0(\Omega)$, $u \in V$.

$$
\frac{1}{2} \frac{d}{dt} \|u_m(t)\|_{L^2}^2 + 2\mu \|u_m(t)\|_{H^1}^2 + \vartheta \|u_m(t)\|_{L^{\beta+1}}^{\beta+1} + \mu \int_{\partial\Omega} \alpha(u_m(t) \cdot \tau)^2 dS = (f(t), u_m(t)). \tag{3.25}
$$

The right-hand side of (3.25) is estimated as by

$$
|(f(t), u_m(t))| \leq \|f(t)\|_{V'} \|u_m(t)\|_{H^1} \leq \mu \|u_m(t)\|_{H^1}^2 + \frac{1}{\mu} \|f(t)\|_{V'}^2. \tag{3.26}
$$

Using (3.26) in (3.25), we obtain

$$
\frac{1}{2} \frac{d}{dt} \|u_m(t)\|_{L^2}^2 + \mu \|u_m(t)\|_{H^1}^2 + \vartheta \|u_m(t)\|_{L^{\beta+1}}^{\beta+1} \leq \frac{2}{\mu} \|f(t)\|_{V'}^2. \tag{3.27}
$$
We integrate (3.27) from 0 to $T$ and obtain
\[
\sup_{0 \leq t \leq T} \|u_m(t)\|_{L^2}^2 + 2\mu \int_0^T \|u_m(t)\|_{H^1}^2 dt + 2\vartheta \int_0^T \|u_m(t)\|_{L^\beta+1}^\beta dt \\
\leq \|u_0\|_{L^2}^2 + 2\vartheta \int_0^T \|f(t)\|_{L^2}^2 dt.
\]

The proof of Lemma 3.4 is finished. \hfill \Box

Applying Lemma 3.4, we obtain the global existence of the approximate solutions $u_m \in L^2(0, T; V) \cap L^\infty(0, T; H) \cap L^{\beta+1}(0, T; L^{\beta+1}(\Omega))$. Next, we will use Lemma 3.2 to prove the strong convergence of $u_m$ or its subsequence in $L^2 \cap L^\beta([0, T] \times \Omega)$. Let $\tilde{u}_m$ denote the function from $\mathbb{R}$ to $V$, defined as follows
\[
\tilde{u}_m = \begin{cases} u_m & \text{on } [0, T] \\
0 & \text{otherwise.}
\end{cases}
\]

Similarly, we define $\tilde{g}_{im}(t)$ to $\mathbb{R}$ by defining
\[
\tilde{g}_{im}(t) = \begin{cases} g_{im} & \text{on } [0, T] \\
0 & \text{otherwise.}
\end{cases}
\]

The Fourier transformations on time variable of $\tilde{u}_m$ and $\tilde{g}_{im}$ are denoted by $\hat{\tilde{u}}_m$ and $\hat{\tilde{g}}_{im}$ respectively. We want to show that
\[
\int_{-\infty}^{+\infty} |\xi|^{2\gamma} \|\hat{\tilde{u}}_m(\xi)\|_{L^2}^2 d\xi \leq C, \quad \text{for some } \gamma > 0 \tag{3.28}
\]
where $C$ is some positive constant. Note that approximate solutions $\tilde{u}_m$ satisfy
\[
\frac{d}{dt}(\hat{\tilde{u}}_m(t), w_j) = (\hat{\tilde{f}}_m(t), w_j) + (\vartheta |\hat{\tilde{u}}_m|^{\beta-1} \hat{\tilde{u}}_m(t), w_j) + (u_0, w_j)\delta_0 \\
- (u_m(T), w_j)\delta_T, \quad j = 1, \ldots, m \tag{3.29}
\]
where $\delta_0, \delta_T$ are Dirac distributions at 0 and $T$ and $f_m = f - \mu A u_m - B u_m$, $\hat{\tilde{f}}_m = f$ on $[0, T]$, 0 outside this interval. We already define $A$ and $B$ in (2.13) and (2.10) respectively.

Taking the Fourier transformation about the time variable, (3.29) gives
\[
2\pi i \xi (\hat{\tilde{u}}_m(\xi), w_j) = (\hat{\tilde{f}}_m(\xi), w_j) + \vartheta (|\hat{\tilde{u}}_m|^{\beta-1} \hat{\tilde{u}}_m(\xi), w_j) + (u_0, w_j) \\
- (u_m(T), w_j) \exp(-2\pi i T \xi), \tag{3.30}
\]
where $\hat{\tilde{f}}_m$ denotes the Fourier transformation of $\tilde{f}_m$. \hfill \Box
We multiply (3.30) by $\hat{g}_{jm}(\xi)$ and add the resulting equations for $j = 1, \ldots, m$, we get

$$2\pi i \xi \|\hat{u}_m(\xi)\|_{L^2}^2 = (\hat{f}_m(\xi), \hat{u}_m(\xi)) + \vartheta (|u_m|^{\beta-1}\hat{u}_m(\xi), \hat{u}_m(\xi)) + (u_m(0), \hat{u}_m(\xi)) \exp(-2\pi i T\xi).$$

(3.31)

Because of (2.11) and (2.14), for any $v \in L^2(0, T; V) \cap L^{\beta+1}(0, T; L^{\beta+1})$, we have

$$(f_m(t), v) = (f(t), v) - \mu (A u_m(t), v) - (B u_m(t), v) \leq C (\|f(t)\|_{V'} + \mu \|u_m(t)\|_{H^1} + c_1 \|u_m(t)\|_{H^{1+1}}^2) \|v\|_{H^1}.$$ 

It follows that for any given $T > 0$

$$\int_0^T \|f_m(t)\|_{V'} dt \leq \int_0^T C (\|f(t)\|_{V'} + \mu \|u_m(t)\|_{H^1} + c_1 \|u_m(t)\|_{H^{1+1}}^2) \|v\|_{H^1} \leq C,$$

hence

$$\sup_{\xi \in \mathbb{R}} \|\hat{f}_m(\xi)\|_{V'} \leq \int_0^T \|f_m(t)\|_{V'} dt \leq C. \quad (3.32)$$ 

Also, it follows from Lemma 3.4 that

$$\int_0^T \|u_m|^{\beta-1} u_m(t)\|_{\frac{\beta+1}{p}} dt \leq \int_0^T \|u_m(t)\|_{\frac{\beta+1}{p+1}} dt \leq C,$$

which implies that

$$\sup_{\xi \in \mathbb{R}} \|u_m|^{\beta-1} u(\xi)\|_{\frac{\beta+1}{p}} \leq C. \quad (3.33)$$ 

From Lemma 3.4, we get

$$\|u_m(0)\|_{L^2} \leq C, \quad \|u_m(T)\|_{L^2} \leq C. \quad (3.34)$$ 

We deduce from (3.31)–(3.34) that

$$|\xi| \|\hat{u}_m(\xi)\|_{L^2}^2 \leq C (\|\hat{u}_m(\xi)\|_{H^1} + \|\hat{u}_m(\xi)\|_{\beta+1})$$

For any $\gamma$ fixed, $0 < \gamma < \frac{1}{4}$, we observe that

$$|\xi|^{2\gamma} \leq C \frac{1 + |\xi|}{1 + |\xi|^{1-2\gamma}}, \quad \forall \xi \in \mathbb{R}.$$ 

Thus

$$\int_{-\infty}^{+\infty} |\xi|^{2\gamma} \|\hat{u}_m(\xi)\|_{L^2}^2 d\xi \leq C \int_{-\infty}^{+\infty} \frac{1 + |\xi|}{1 + |\xi|^{1-2\gamma}} \|\hat{u}_m(\xi)\|_{L^2}^2 d\xi.$$
\[
\begin{align*}
&\leq C \int_{-\infty}^{+\infty} \left\| \hat{u}_m(\xi) \right\|_{L^2}^2 d\xi + C \int_{-\infty}^{+\infty} \frac{\left\| \hat{u}_m(\xi) \right\|_{H^1}}{1 + |\xi|^{1-2\gamma}} d\xi \\
&+ C \int_{-\infty}^{+\infty} \left\| \hat{u}_m(\xi) \right\|_{\beta+1} \frac{1}{1 + |\xi|^{1-2\gamma}} d\xi. 
\end{align*}
\]

(3.35)

Using the Parseval equality and Lemma 3.4, the first integral on the right hand side of (3.35) is bounded as \( m \to \infty \). By the Schwartz inequality, the Parseval equality and Lemma 3.4, we get

\[
\int_{-\infty}^{+\infty} \left\| \hat{u}_m(\xi) \right\|_{H^1} \frac{1}{1 + |\xi|^{1-2\gamma}} d\xi \leq \left( \int_{-\infty}^{+\infty}\frac{d\xi}{(1 + |\xi|^{1-2\gamma})^2} \right)^{\frac{1}{2}} \left( \int_{0}^{T} \left\| u_m(\xi) \right\|_{H^1}^2 \right)^{\frac{1}{2}} \leq C
\]

for \( 0 < \gamma < \frac{1}{4} \).

Similarly, when \( 0 < \gamma < \frac{1}{2} (\beta+1) \), we get

\[
\int_{-\infty}^{+\infty} \left\| \hat{u}_m(\xi) \right\|_{\beta+1} \frac{1}{1 + |\xi|^{1-2\gamma}} d\xi \leq \left( \int_{-\infty}^{+\infty}\frac{d\xi}{(1 + |\xi|^{1-2\gamma})^{\beta+1}} \right)^{\frac{1}{\beta+1}} \left( \int_{-\infty}^{+\infty} \left\| \hat{u}_m(\xi) \right\|_{\beta+1} d\xi \right)^{\frac{1}{\beta+1}} \leq C (\int_{-\infty}^{+\infty} \left\| \hat{u}_m(\xi) \right\|_{\beta+1} d\xi)^{\frac{1}{\beta+1}} \]

\[
\leq C T^{\frac{\beta-1}{\beta+1}} \left( \int_{0}^{T} \left\| u_m(\xi) \right\|_{\beta+1} d\xi \right)^{\frac{1}{\beta+1}} \]

(3.37)

It follows from (3.35) that

\[
\int_{-\infty}^{+\infty} |\xi|^{2\gamma} \left\| \hat{u}_m(\xi) \right\|_{L^2}^2 d\xi \leq C.
\]

(3.38)

From lemma 3.4, we can say that, there exists a function \( u(x, t) \) such that

\[
u \in L^2(0, T; V) \cap L^\infty(0, T; H) \cap L^{\beta+1}(0, T; L^{\beta+1}(\Omega))
\]

(3.39)

and there exists a subsequence of \( \{u_m\}_{m=1}^{\infty} \), still denoted by \( u_m \), such that

\[
u_m \rightharpoonup u \text{ weakly in } L^2(0, T; V),
\]

\[
u_m \to u \text{ weak-star topology of } L^\infty(0, T; H),
\]

and \( u_m \to u \text{ weakly in } L^{\beta+1}(0, T; L^{\beta+1}(\Omega)) \).

Moreover, we choose \( \Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots \) with smooth boundary, satisfying \( \bigcup_{i=1}^{\infty} \Omega_i = \Omega \). For any fixed \( i = 1, 2, \ldots \), we take \( X_0 = V, \ X = L^2(\Omega_i) \) in Lemma 3.2. From Lemma 3.2, Lemma 3.4 and (3.38), we obtain that there exists
a subsequence of \(\{u_m\}_{m=1}^{\infty}\), still denoted by itself, such that \(u_m \to u\) strongly in \(L^2(0, T; L^2(\Omega_i))\). Also, \(\int_0^T \int_{\Omega_i} |u_m|^\beta + 1 \, dx \, dt \leq C\), we obtain that \(u_m \to u\) strongly in \(L^p(0, T; L^p_{\text{loc}}(\Omega))\) for \(2 \leq p < \beta + 1\) if \(\beta > 1\).

Let \(\phi\) be a continuously differentiable function on \([0, T]\) with \(\phi(t) = 0\). We multiply (3.20) by \(\phi(t)\), and then integrate by parts. We obtain

\[
- \int_0^T (u_m(t), \phi'(t)w_j) dt + \int_0^T \phi(t) \left(2\mu((u_m(t), w_j)) + b(u_m(t), u_m(t), w_j)\right) dt + (\vartheta |u_m|^{\beta-1}u_m(t), w_j) + \mu \int_{\partial\Omega} \alpha(u_m(t) \cdot \tau)(w_j \cdot \tau) dS - (f(t), w_j) dt = (u_0, w_j) \phi(0) \quad j = 1, \ldots, m.
\]

We already have \(u_m \to u\) weakly in \(L^2(0, T; H)\). Let \(v_m = Tu_m\) and \(v = Tu\), where \(T : W^{1,2}(\Omega) \to L^2(\partial\Omega)\) trace operator. Since \(T\) is linear and continuous \(v_m \to v\) weakly in \(L^2(0, T; H)\). Also we know that \(W^{1-1/p,p}(\Omega)\) compactly embedded in \(L^p(\partial\Omega)\). Then the weak convergence of \(v_m\) imply strong convergence of \(v_m\). By compactness result [23, Theorem III.2.2] and [18, Theorem II.6.2], we have \((u_m(t) \cdot \tau) \to (u \cdot \tau)\) strongly in \(L^2(\partial\Omega)\). Passing limit to the (3.40), we obtain the equation

\[
- \int_0^T (u(t), \phi'(t)v) dt + \int_0^T \phi(t) \left(2\mu((u(t), v)) + b(u(t), u(t), v)\right) dt + (\vartheta |u|^{\beta-1}u(t), v) + \mu \int_{\partial\Omega} \alpha(u(t) \cdot \tau)(v \cdot \tau) dS - (f(t), v) = (u_0, v) \phi(0),
\]

holds for \(v = w_1, w_2, \ldots\) by linearity this equation holds for \(v = \) any finite linear combination of the \(w_j\), and by continuity argument (3.41) is true for any \(v \in V\). So, \(u(x, t)\) is solution of damped Navier–Stokes system. The proof of Theorem (3.3) is finished.

\section*{4 Regularity of solution}

In this section we discuss about the regularity of solution obtained in Sect. 3 of damped Navier–Stokes system (1.1)–(1.5).

\begin{Theorem}
Let \(u_0 \in \mathcal{W}, f \in L^\infty(0, T; H), f' \in L^1(0, T; H)\) and if \(\beta \geq 3\) and \(\mu\) is large enough or if \(f\) and \(u_0\) are small enough (\(||f|| < \epsilon_1, ||u_0|| < \epsilon_2\), for some arbitrary small \(\epsilon_1, \epsilon_2 > 0\)) with sufficient smooth boundary \(\partial\Omega\), then the solution \(u\) from Theorem (3.3) will satisfy

\[u' \in L^2(0, T; V) \cap L^\infty(0, T; H).\]
\end{Theorem}
Proof  We show that the approximate solution defined in (3.19) also satisfies an a priori estimate:

\[ u'_m \in L^2(0, T; V) \cap L^\infty(0, T; H). \]  

(4.43)

In the limit (4.43) implies (4.42).

Since \( u_0 \in \mathcal{W} \), we choose \( u_{0m} \) as the orthogonal projection in \( \mathcal{W} \) of \( u_0 \) onto the space spanned by \( w_1, \ldots, w_m \), then

\[ u_{0m} \to u_0 \text{ in } H^2(\Omega), \quad \text{and} \quad \|u_{0m}\|_{H^2} \leq \|u_0\|_{H^2}. \]  

(4.44)

**Lemma 4.2**  Let the hypotheses of the Theorem 4.1 hold and \( u_m \) be the approximate solution defined by (3.19). Then \( \{u'_m(t)\} \) belongs to bounded subset of \( H \).

Proof  we multiply (3.20) by \( g'_{jm}(t) \) and then add resulting equation for \( j = 1, \ldots, m \).

We obtain

\[
\|u'_m(t)\|^2_{L^2} + 2\mu((u_m(t), u'_m(t))) + b(u_m(t), u'_m(t)) + (\vartheta |u_m|^{\beta-1} u_m(t), u'_m(t)) \\
+ \mu \int_{\partial \Omega} \alpha(u_m(t) \cdot \tau)(u'_m(t) \cdot \tau) dS = (f(t), u'_m(t)).
\]  

(4.45)

Considering time \( t = 0 \), we get

\[
\|u'_m(0)\|^2_{L^2} + 2\mu((u_m(0), u'_m(0))) + b(u_m(0), u'_m(0)) \\
+ (\vartheta |u_m(0)|^{\beta-1} u_m(0), u'_m(0)) \\
+ \mu \int_{\partial \Omega} \alpha(u_m(0) \cdot \tau)(u'_m(0) \cdot \tau) dS = (f(0), u'_m(0)).
\]  

(4.46)

We note that

\[
2\mu((u_m(0), u'_m(0))) + \mu \int_{\partial \Omega} \alpha(u_m(0) \cdot \tau)(u'_m(0) \cdot \tau) dS \\
= -\mu \int_{\Omega} \Delta u_m(0) u'_m(0) dx \\
= (-\mu \Delta u_m(0), u'_m(0)).
\]  

(4.47)

From the definition of \( w_j \) in \( u_m \), we get the following estimate

\[ \|\Delta u_m(0)\|_{L^2} \leq c_1 \|u_m(0)\|_{H^2} \leq c_1 \|u(0)\|_{H^2} \]  

(4.48)

for some positive constant \( c_1 \). Moreover, for some positive constant \( c_2 \) we get from Lemma 2.2 is that

\[ |b(u_m(0), u_m(0), u'_m(0))| \leq c_2 \|u_m(0)\|_{H^2}^2 \|u'_m(0)\|_{L^2}. \]  

(4.49)
Using (4.47) and (4.48) in (4.46), we obtain
\[ \|u'_m(0)\|_{L^2}^2 = (f(0), u'_m(0)) + \mu(\Delta u_m(0), u'_m(0)) - b(u_m(0), u_m(0), u'_m(0)) \]
\[ - (\vartheta |u_m|^{\beta-1} u_m(0), u'_m(0)) \leq \left( \|f(0)\|_{L^2} + \mu c_1 \|u_m(0)\|_{H^2} + c_2 \|u_m(0)\|_{H^2}^2 + \vartheta |u_m|^{\beta-1} \|u_m(0)\|_{L^2} \right) \|u'_m(0)\|_{L^2}. \]

It follows that
\[ \|u'_m(0)\|_{L^2} \leq \|f(0)\|_{L^2} + \mu c_1 \|u_m(0)\|_{H^2} + c_2 \|u_m(0)\|_{H^2}^2 + \vartheta |u_m|^{\beta-1} \|u_m(0)\|_{L^2} = d_1, \]
where \( d_1 \) is a finite positive constant. Hence \( \{u'_m(0)\} \) belongs to bounded subset of \( H \). The proof of Lemma 4.2 is finished. \( \square \)

**Lemma 4.3** Let the hypotheses of the Theorem 4.1 hold and \( u_m \) be the approximate solution defined by (3.19). Then \( \{u'_m\} \) belongs to a bounded subset of \( L^\infty(0, T; H) \) and \( L^2(0, T; V) \).

**Proof** As \( f' \in L^1(0, T; H) \), we differentiate (3.20) in the \( t \) variable, we obtain
\[
(u''_m(t), w_j) + 2\mu((u'_m(t), w_j)) + b(u'_m(t), u_m(t), w_j) + b(u_m(t), u'_m(t), w_j) \\
+ (\vartheta |u_m|^{\beta-1} u'_m(t), w_j) + (\vartheta (\beta - 1) |u_m|^{\beta-3} u_m(t) \cdot u'_m(t) u_m(t), w_j) \\
+ \mu \int_{\partial \Omega} \alpha(u'_m(t) \cdot \tau)(w_j \cdot \tau) dS = (f'(t), w_j) \quad j = 1, \ldots, m.
\]

We multiply (4.51) by \( g'_{jm}(t) \) and adding these equations for \( j = 1, \ldots, m \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u'_m(t)\|_{L^2}^2 + 2\mu \|u'_m(t)\|_{H^1}^2 + b(u'_m(t), u_m(t), u'_m(t)) + (\vartheta |u_m|^{\beta-1} u'_m(t), u'_m(t)) \\
+ (\vartheta (\beta - 1) |u_m|^{\beta-3} u_m(t) \cdot u'_m(t) u_m(t), u'_m(t)) + \mu \int_{\partial \Omega} \alpha(u'_m(t) \cdot \tau)^2 dS = (f'(t), u'_m(t)).
\]

(4.52) can be written as follow
\[
\frac{1}{2} \frac{d}{dt} \|u'_m(t)\|_{L^2}^2 + 2\mu \|u'_m(t)\|_{H^1}^2 + b(u'_m(t), u_m(t), u'_m(t)) + \vartheta |u_m|^{\beta-1} \|u'_m(t)\|_{L^2}^2 \\
+ \vartheta (\beta - 1) |u_m|^{\beta-3} (u_m(t) \cdot u'_m(t)) u_m(t), u'_m(t)) + \mu \int_{\partial \Omega} \alpha(u'_m(t) \cdot \tau)^2 dS = (f'(t), u'_m(t)).
\]
Note that \( \partial (\beta - 1)(|u_m|^{\beta - 3}(u_m \cdot u'_m)u_m(t), u'_m(t)) \geq 0 \) when \( \beta \geq 3 \). Since
\[
\partial (\beta - 1)(|u_m|^{\beta - 3}(u_m(t) \cdot u'_m(t))u_m(t), u'_m(t)) = \partial (\beta - 1) \int_{\Omega} (|u_m|^{\beta - 3}(u_m(t) \cdot u'_m(t))u_m(t) \cdot u'_m(t))\,dx
\]
= \( \partial (\beta - 1) \int_{\Omega} |u_m|^{\beta - 3}(u_m(t) \cdot u'_m(t))^2\,dx \)

Now (4.53) becomes
\[
\frac{1}{2} \frac{d}{dt} \|u'_m(t)\|^2_{L^2} + 2\mu \|u'_m(t)\|^2_{H^1} + b(u'_m(t), u_m(t), u'_m(t)) \leq (f'(t), u'_m(t)) \quad (4.54)
\]

It follows from (2.12) that
\[
|b(u'_m(t), u_m(t), u'_m(t))| \leq c_3 \|u'_m(t)\|_{H^1} \|u_m(t)\|_{H^1} \|u'_m(t)\|_{H^1}
\]
= \( c_3 \|u'_m(t)\|^2_{H^1} \|u_m(t)\|_{H^1} \).

Using the above estimates in (4.54), we get
\[
\frac{d}{dt} \|u'_m(t)\|^2_{L^2} + 2(2\mu - c_3 \|u_m(t)\|_{H^1}) \|u'_m(t)\|_{H^1}^2 \leq 2(f'(t), u'_m(t)). \quad (4.55)
\]

From (3.27) we get
\[
\mu \|u_m(t)\|^2_{H^1} \leq \frac{1}{\mu} \|f(t)\|^2_{V'} - (u_m(t), u'_m(t)) - \partial \|u_m(t)\|_{L^{\beta + 1}}^{\beta + 1}
\]
\[
\leq \frac{1}{\mu} \|f(t)\|^2_{V'} + 2\|u_m(t)\|_{L^2} \|u'_m(t)\|_{L^2} + \partial \|u_m(t)\|_{L^{\beta + 1}}^{\beta + 1}
\]
\[
\leq \frac{d_2}{\mu} + 2 \left( \|u_0\|^2_{L^2} + \frac{T d_2}{\mu} \right)^{1/2} \|u'_m(t)\|_{L^2} + \partial \|u_m(t)\|_{L^{\beta + 1}}^{\beta + 1}, \quad (4.56)
\]

where \( d_2 = \|f\|^2_{L^\infty(0,T;V')} \). So,
\[
\mu \|u_m(0)\|^2_{H^1} \leq \frac{d_2}{\mu} + 2(\|u_0\|^2_{L^2} + \frac{T d_2}{\mu})^{1/2} d_1 + \partial \|u_0\|_{L^{\beta + 1}}^{\beta + 1} = d_3. \quad (4.57)
\]

For \( \mu \) large enough or \( f \) and \( u_0 \) are small enough, we define \( d_4 \) such that
\[
d_4 = \frac{d_2}{\mu} + (1 + d_1^2)(\|u_0\|^2_{L^2} + \frac{T d_2}{\mu})^{1/2} \exp(\int_0^T \|f'(s)\|_{L^2} ds) + \partial \|u_m(t)\|_{L^{\beta + 1}}^{\beta + 1} < \frac{\mu^3}{c_3^2} \quad (4.58)
\]

for some positive constant \( c_3 \). Then
\[
d_3 \leq d_4.
\]
Thus from (4.57)–(4.58), it follows that
\[
\mu \| u_m(0) \|^2_{H^1} \leq d_3 \leq d_4 < \frac{\mu^3}{c_3^2},
\]
and
\[
\| u_m(0) \|^2_{H^1} < \frac{\mu^2}{c_3^2}
\]
\[
\Rightarrow \mu^2 > c_3^2 \| u_m(0) \|^2_{H^1}
\]
\[
\Rightarrow 2\mu - c_3 \| u_m(0) \|_{H^1} > 0.
\]
That is \( 2\mu - c_3 \| u_m(t) \|_{H^1} > 0, \quad t \in I \), where \( I \) is an interval containing 0. Let \( T_m \) be the first time \( t \) such that \( t \leq T \) and
\[
2\mu - c_3 \| u_m(T_m) \|_{H^1} = 0.
\]
If this is not the case, then we must have \( T_m = T \).
\[
2\mu - c_3 \| u_m(t) \|_{H^1} \geq 0, \quad 0 \leq t \leq T_m.
\]
(4.59)
Now from (4.55) using (4.59) we obtain,
\[
\frac{d}{dt} \| u'_m(t) \|^2_{L^2} \leq 2 \| f'(t) \|_{L^2} \| u'_m(t) \|_{L^2}.
\]
Further, we have the following estimate
\[
\frac{d}{dt} (1 + \| u'_m(t) \|^2_{L^2}) \leq \| f'(t) \|_{L^2} (1 + \| u'_m(t) \|^2_{L^2})
\]
(4.60)
Applying Gronwall’s lemma and using (4.50), we obtain
\[
1 + \| u'_m(t) \|^2_{L^2} \leq (1 + \| u'_m(0) \|^2_{L^2}) \exp(\int_0^t \| f'(s) \|_{L^2} ds)
\]
\[
\leq (1 + d_2^2) \exp(\int_0^t \| f'(s) \|_{L^2} ds).
\]
(4.61)
From (4.56), we get
\[
\mu \| u_m(t) \|^2_{H^1} \leq \frac{d_2}{\mu} + 2(\| u_0 \|^2_{L^2} + \frac{T d_2}{\mu})^2 \| u'_m(t) \|_{L^2} + \vartheta \| u_m(t) \|_{L^{\beta+1}}
\]
\[
\leq \frac{d_2}{\mu} + (\| u_0 \|^2_{L^2} + \frac{T d_2}{\mu})^2 (1 + \| u'_m(t) \|^2_{L^2}) + \vartheta \| u_m(t) \|_{L^{\beta+1}}
\]
\[ \leq \frac{d_2}{\mu} + (\|u_0\|_{L^2}^2 + \frac{T d_2}{\mu}) \frac{1}{2} (1 + d_1^2 \mu) \exp\left( \int_0^t \|f'(s)\|_{L^2} ds \right) + \vartheta \|u_m(t)\|_{L^{\beta+1}}^{\beta+1} \]
\[ = d_4. \] (4.62)

Thus for \( 0 < t \leq T_m \), we have
\[ \mu \|u_m(t)\|_{H^1}^2 \leq d_4, \]
\[ \Rightarrow \|u_m(t)\|_{H^1} \leq \sqrt{\frac{d_4}{\mu}}. \]

Now
\[ 2\mu - c_3 \|u_m(t)\|_{H^1} \geq 2\mu - c_3 \sqrt{\frac{d_4}{\mu}} \] (4.63)
\[ > \mu, \quad 0 \leq t \leq T_m. \] (4.64)

Then \( T_m = T \), and (4.55) implies
\[ \frac{d}{dt} \|u'_m(t)\|_{L^2}^2 + 2(2\mu - c_3 \|u_m(t)\|_{H^1}) \|u'_m(t)\|_{H^1}^2 \]
\[ \leq 2 \|f'(t)\|_{L^2} \|u'_m(t)\|_{L^2}, \quad 0 \leq t \leq T. \] (4.65)

Using Gronwall’s Lemma, we can conclude that \( u'_m \) is a bounded set of \( L^2(0, T; V) \cap L^\infty(0, T; H) \). The proof of Lemma 4.3 is finished. \( \square \)

It follows from Lemma 4.2 and Lemma 4.3, the proof of the theorem is completed. \( \square \)

## 5 Uniqueness of solution

In this section, we prove the uniqueness theorem for the solution obtained in Sect. 4 of the system (1.1)–(1.5).

**Theorem 5.1** Let \( u_0 \in \mathcal{W}, \ f \in L^\infty(0, T; H), \ f' \in L^1(0, T; H) \). Then there is atmost one solution \( u \) of the damped Navier–Stokes system (1.1)–(1.5) with sufficient smooth boundary \( \partial \Omega \) such that
\[ u \in L^\infty(0, T; L^4(\Omega)), \quad u' \in L^2(0, T; V) \cap L^\infty(0, T; H). \]

**Proof** Let us assume that \( u_1 \) and \( u_2 \) are two solutions, and let \( u(t) = u_1(t) - u_2(t) \).

The difference \( u(t) = u_1(t) - u_2(t) \) satisfies
\[ u'(t) + \vartheta |u_1|^{\beta-1} u_1(t) - \vartheta |u_2|^{\beta-1} u_2(t) = -Bu_1(t) + Bu_2(t), \]
\[ u(0) = 0. \] (5.66)
Taking the scalar product of (5.66) with \( u(t) \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + 2\mu \|u(t)\|_{H^1}^2 + \int_{\Omega} \left( |u_1|^{\beta-1}u_1(t) - |u_2|^{\beta-1}u_2(t) \right) (u_1(t) - u_2(t)) dx \\
+ \mu \int_{\partial \Omega} \alpha(u(t) \cdot \mathbf{n})^2 dS = b(u_2(t), u_2(t), u(t)) - b(u_1(t), u_1(t), u(t)).
\]

(5.67)

Note that

\[
\int_{\Omega} \left( |u_1|^{\beta-1}u_1(t) - |u_2|^{\beta-1}u_2(t) \right) (u_1(t) - u_2(t)) dx \\
\geq \int_{\Omega} \left( |u_1|^{\beta+1} - |u_2|^{\beta} + |u_1|^\beta |u_2| + |u_2|^{\beta+1} \right) dx \\
= \int_{\Omega} (|u_1|^{\beta} - |u_2|^{\beta})(|u_1| - |u_2|) dx \geq 0.
\]

Also

\[
b(u_2, u_2, u_1 - u_2) - b(u_1, u_1, u_1 - u_2) = b(u_2, u_2, u_2) + b(u_1, u_1, u_2) - b(u_1, u_2, u_2) \\
= -b(u_1, u_2, u_1) + b(u_2, u_2, u_1) \\
+ b(u_1, u_2, u_2) - b(u_2, u_2, u_2) \\
= -b(u_1 - u_2, u_2, u_1) + b(u_1 - u_2, u_2, u_2) \\
= b(u_1 - u_2, u_2, u_1 - u_2) \\
= b(u, u, u_2).
\]

(5.68)

Now (5.67) becomes

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + 2\mu \|u(t)\|_{H^1}^2 \leq b(u(t), u(t), u_2(t)).
\]

(5.69)

By the Hölder inequality and (2.12)

\[
|b(u(t), u(t), v(t))| \leq k_1 \|u(t)\|_{L^2}^{\frac{1}{2}} \|u(t)\|_{H^1}^{\frac{3}{2}} \|v(t)\|_{L^4(\Omega)}.
\]

(5.70)

Now (5.69) becomes

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + 2\mu \|u(t)\|_{H^1}^2 \leq 2k_1 \|u(t)\|_{L^2}^{\frac{1}{2}} \|u(t)\|_{H^1}^{\frac{3}{2}} \|u_2(t)\|_{L^4(\Omega)} \\
\leq 2\mu \|u(t)\|_{H^1}^2 + k_2 \|u(t)\|_{L^2}^2 \|u_2(t)\|_{L^4(\Omega)}^8.
\]

So we get

\[
\frac{d}{dt} \|u(t)\|_{L^2}^2 \leq k \|u(t)\|_{L^2}^2 \|u_2(t)\|_{L^4(\Omega)}^8.
\]
Applying Gronwall’s Lemma, we conclude that

\[
\|u(t)\|_{L^2}^2 \leq \|u(0)\|_{L^2} \exp \left( \int_0^t k \|u_2(t)\|_{L^4(\Omega)}^4 \, dt \right)
\]

\[= 0.\]

Thus \(u_1(t) = u_2(t)\). The proof of the Theorem 5.1 is finished. \(\square\)

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