Opportunities for Making Production-Related Decisions on the basis of Shadow Prices

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Abstract

This article describes how shadow prices can be used as active constraints (in this case constraints of mine production capacity) to address and support production-related decision-making. This is an algorithm from a post-optimal analysis developed by the author as part of a method for rationalising production decisions for a formal group (PGG, a company) of hard coal mines. Opportunities for using shadow prices are presented using examples of actual mines. The developed algorithm provides a quick way of obtaining information, with no need to solve the problem again, about possible gains or losses resulting from an increase or a decrease in a selected production limit, to determine how changes to such constraints will affect the profits and production and sales structures for specific coal sizes.

Keywords: shadow prices, post-optimal analysis, simplex algorithm

Introduction

The method presented here relies on what is known as the simplex tableau, generated on the basis of an optimisation model developed by the author. It constitutes an optimum solution for the production and sales of coal by mines [2, 3, 4, 5].

The tableau contains a complete set of balance-sheet equations and coefficients representing objective function sensitivity to changes in decision variables. In addition, it is a starting point for the post-optimal analysis, where shadow prices play a very important role, since they help assess the extent and direction of changes to the optimum value of the criterion function as constraints change. Shadow prices make it possible to predict, with no need to solve the problem again, the possible gains or losses resulting from an increase or a decrease in a selected (production or sales) limit [7]. This is particularly important for hard-coal mining, where mines frequently have spare production capacities. The practical example of using shadow prices to make rational production decisions, as presented here, shows how specific decisions could affect companies’ or mines’ optimum coal production and sales plans.

This method can be successfully used in the decision making process for planning mining works in hard coal mines [1].

The nature of shadow prices

The coal production and sales optimisation model, as developed by the author, has the following canonical form [2, 3, 4, 5]:

\[ A \cdot X = B, \]  
\[ J = c^T \cdot X, \]  
\[ X \geq 0, \]

where:
A – constraint matrix (simplex tableau),  
X – vector of decision variable values together with inequality constraint slack variables (1),  
B – vector of the right-hand sides of the equation;  
c – vector of objective function coefficients and zero coefficients for slack variables,  
J – objective function (quality coefficient).

The solution to the problem (1-3) is vector X divided into sub-vectors of basic variables with positive values, and vector of the other variables, referred to as non-basic, whose optimum values are zeroes. This solution is a simplex tableau.

The optimal solution, which – in relation to the basic and nonbasic variables and the quality coefficient – is represented by the following equations [4, 5, 6]:

\[ x^B = \left[ A^B \right]^{-1} \cdot B - \left[ A^B \right]^{-1} \cdot A^N \cdot x^N \]  
\[ J = c^B \cdot \left[ A^B \right]^{-1} \cdot B - \left[ c^B \cdot \left[ A^B \right]^{-1} \cdot A^N \right] \cdot x^N - c^N \cdot x^N \]

where:
\[ A^B, A^N \] submatrices of the A matrix (A – matrix of the constraint coefficients);  
\[ c^B, c^N \] subvectors of objective-function coefficients;  
\[ x^B, x^N \] vectors of basic and nonbasic decision variables, respectively.

In the index maximisation task (2) optimum coefficient values in formula (5) with non-basic variables are non-positive.

With \( x^{B0} \) and \( x^{N0} \) representing, respectively, optimum values of basic and non-basic decision variables, and the following symbols:

\[ A^0 = \left[ A^B \right]^{-1} \cdot A^N \]  
\[ c^0 = \left[ c^B \cdot \left[ A^B \right]^{-1} \cdot A^N \right] \cdot x^N - c^N \]

and disregarding all non-basic variables \( x^{N0} \) (as in the opti-
Tab. 2. Comparison of selected values characterising calculation variants [Source: own study]

| Problem | Kopania | XM0301 | XM040107 | XM040113 | XM040116 | X050116 | Q501 | Q505 |
|---------|---------|--------|----------|----------|----------|---------|------|------|
| OBJ     | 3,158.08 | 54,332 | 2,206    | 17,496   | 14,946   | 0.0     | 3,429 | 20,791 |
| XM0101  | 160,022.5| 0.0    | 0.0      | 0.0      | 0.0      | 0.0     | 0.11  | 0.0  |
| XM0201  | 21,821.3 | 0.0    | 0.0      | 0.0      | 0.0      | 0.0     | 0.015 | 0.0  |
| X050113 | 264,764.2| 1.0    | 0.0      | 0.0      | 0.0      | 0.0     | 0.182 | 0.0  |
| X040109 | 0.0     | 0.0    | 1.0      | 1.0      | 1.0      | 0.0     | 0.0   | 0.0  |
| XM0401  | 675,000.4| 0.0    | 0.0      | 0.0      | 0.0      | 0.0     | 0.464 | 0.0  |
| X050113 | 317,135.2| 0.0    | 0.0      | 0.0      | 0.0      | 1.0     | 0.218 | 0.0  |
| X060115 | 16,002.3 | 0.0    | 0.0      | 0.0      | 0.0      | 0.0     | 0.011 | 0.0  |
| X010214 | 112,672.7| 0.022  | 0.0      | 0.0      | 0.0      | 0.0     | 0.005 | 0.011|
| XM0202  | 250,787.5| 0.027  | 0.0      | 0.0      | 0.0      | 0.0     | 0.011 | 0.024|

Tab. 2. Porównanie wybranych wielkości charakteryzujących warianty obliczeniowe [Źródło: opracowanie własne]

mum solution these take zero values) the result is:

- optimum values of basic variables in the form:
  \[ x^{BO} = \left[ A^B \right]^{-1} \cdot B \]  
  \[ (8) \]

- optimum value of quality indicator:
  \[ J^O = c^{OT} \left[ A^B \right]^{-1} \cdot B = W^T \cdot B \]  
  \[ (9) \]

where:

- \( W \) - weight vector, which represents the impact of individual constraint values \( B \) on the quality indicator.

When correlations (4) and (5) are substituted to equations (6-9), we obtain formulas directly used in post-optimal analysis [5, 6]:

\[ x^B = x^{BO} = x^{BO} - A^O \cdot x^N \]  
\[ (10) \]

and

\[ J = J^O - c^{OT} \cdot x^N \]  
\[ (11) \]

where:

- \( c^i \) - non-basic variable shadow prices, \( \geq 0 \) for the maximisation of the quality indicator,
- \( A^O \) - optimum solution coefficient matrix,
- \( J^O \) - optimum value of the quality indicator.

Let us determine, using \( J(B) \), the optimum value of the objective function for the primal problem, as produced by the constraint \( B = [B^i] \) adopted for the purposes of these calculations. The optimum value of shadow price \( c_j \) is a partial derivative of function \( J(B) \) in relation to limit \( B_j \) [7]:

\[ c_j = \frac{\Delta J(x^*)}{\Delta B_j} \]  
\[ \text{przy} \quad \Delta B_j = 1 \]  
\[ (12) \]

If in the \( i \)-th constraint of the problem the absolute term increases by a unit (while not causing the constraints to be not satisfied), then the optimum value of the objective function of the primal problem \( f(x^*) \) increases by \( c_j \) units, i.e., to \( J(x^*) + c_j \) and vice versa [7].

Generally, shadow price is treated as the measure of how efficiently a limited resource \( B_j \) of the \( i \)-th production factor is used. When shadow price is interpreted on the basis of the partial derivative in formula 12, shadow price \( c_j \) can be considered to be the marginal productivity of the \( i \)-th factor. The above interpretations are correct if they refer to certain production factors with predefined limits (\( \geq \) model constraints) [7].

Shadow prices for active inequality constraints (constraints on the productive capacity of the mine) directly show how changes to these constraints affect the quality indicator [6].

Let \( k \) denote constraint number in system (10) corresponding to the production capacity of the \( k \)-th mine, and \( j \) the number of the corresponding slack variable. Matrix \( A^N \) (Formula 4) has in its \( j \)-th column zero elements, except a single one in the \( k \)-th row (this results directly from the definition of the slack variable). Based on this and on correlation (5), it is possible to demonstrate that weight \( W \) in formula (9) is equal to the shadow price for variable \( j \), i.e., \( W = c_j \). Assuming a negative value of the \( j \)-th non-basic variable, one can calculate the increase in the quality indicator obtained by increasing the \( k \)-th constraint by the module of this value. The acceptable value of this increase and its impact on basic variables are determined in the same way as for positive increases in non-basic variables.

Analysis of shadow prices

Having an optimum solution in the form of a simplex tableau, one can quickly see the measurable effect produced by tapping into the unused production capacities in mines. Moreover, it is possible to analyse the consequences (to the optimum production and sales plans) to which such an increase by a specific value could lead. This was presented on the basis of the coal company Alfa [5].

Shadow prices for active inequality constraints (constraints on the productive capacity of the mine) directly show how changes to these constraints affect quality indicator levels. In Table 1, production capacity constraints correspond to non-basic variables Q501 and Q505, (for mines „A” and „E”...
| Company „Alpha” | Max. Extraction: 15,949,350 ton | Profit: 336,778,099 PLN |
|----------------|---------------------------------|------------------------|
| Sold: 11,423,865 ton | Company reserves: 1,896,203 ton | |

| Mine „A” | Max. Extraction: 1,454,750 ton | Profit: 4,843,298 PLN |
|----------|---------------------------------|------------------------|
| Sold: 597,902 ton | Mine reserves: 0 ton | |

| Name of consumer group | Coal size | adjusted amount of sales | The basic amount of sales | Difference |
|-----------------------|-----------|--------------------------|---------------------------|------------|
|                       | grade     | [ton]                    | [ton]                     | + increase | – decrease |
| Dust kettles          | fine coal I | 264,764                  | 264,764                   | 0          |            |
| Dust kettles          | fine coal II | 317,135                 | 317,135                   | 0          |            |
| Grates 4              | slurry     | 16,002                   | 16,002                    | 0          |            |
| Dumping coal          | cobbles    | 160,023                  | 160,023                   | 0          |            |
| Dumping coal          | nut coal   | 21,821                   | 21,821                    | 0          |            |
| Dumping coal          | fine coal II A | 675,004                 | 675,004                   | 0          |            |

| Mine „B” | Max. Extraction: 793,500 ton | Loss: -8,873,886 PLN |
|----------|---------------------------------|------------------------|
| Sold: 112,783 ton | Mine reserves: 429,690 ton | |

| Name of consumer group | Coal size | adjusted amount of sales | The basic amount of sales | Difference |
|-----------------------|-----------|--------------------------|---------------------------|------------|
|                       | grade     | [ton]                    | [ton]                     | + increase | – decrease |
| Grates 3              | fine coal II | 112,783                   | 112,673                  | 110        |            |
| Dumping coal          | cooking coal | 251,027                   | 250,787                 | 240        |            |

| Mine „C” | Max. Extraction: 1,110,900 ton | Profit: 49,931,963 PLN |
|----------|---------------------------------|------------------------|
| Sold: 996,546 ton | Mine reserves: 114,354 ton | |

| Name of consumer group | Coal size | adjusted amount of sales | The basic amount of sales | Difference |
|-----------------------|-----------|--------------------------|---------------------------|------------|
|                       | grade     | [ton]                    | [ton]                     | + increase | – decrease |
| Export 5              | cooking coal | 130,74                    | 130,74                   | 0          |            |
| Coking plants 3       | cooking coal | 865,806                   | 865,806                  | 0          |            |

| Mine „D” | Max. Extraction: 3,174,000 ton | Profit: 74,459,137 PLN |
|----------|---------------------------------|------------------------|
| Sold: 1,801,825 ton | Mine reserves: 1,352,159 ton | |

| Name of consumer group | Coal size | adjusted amount of sales | The basic amount of sales | Difference |
|-----------------------|-----------|--------------------------|---------------------------|------------|
|                       | grade     | [ton]                    | [ton]                     | + increase | – decrease |
| Export 1              | cooking coal | 24,324                    | 24,324                   | 0          |            |
| Export 2              | cooking coal | 287,249                   | 287,359                 | –110       |            |
| Export 3              | cooking coal | 233,299                   | 233,299                 | 0          |            |
| Indv. consumers 2     | cobbles    | 40,042                   | 40,512                   | –470       |            |
| Indv. consumers 3     | fine coal II A | 695,179                 | 703,929                 | –8,750     |            |
| Grates 3              | fine coal II | 9,097                    | 9,207                   | –110       |            |
| Coking plants 2       | cooking coal | 78,136                    | 78,136                  | 0          |            |
| Coking plants 1       | cooking coal | 387,109                   | 398,899                 | –11,790    |            |
| Chamber grates 1      | fine coal II A | 47,39                    | 47,39                   | 0          |            |
| Dumping coal          | fine coal I | 20,016                    | 20,256                  | –240       |            |
respectively). An increase in the production capacity of any of these mines causes its profit to increase by an amount equal to the value of the increased production capacity multiplied by the shadow price for that variable. As can be seen, the most beneficial option is to increase the production capacity in mine „E”, as this generates an additional PLN 20.8 for every additional tonne [5].

For example, we calculated the increase in the quality indicator that can be achieved as a result of increasing the production capacity in mine „E” by 10,000 tonnes. The results of these calculations are presented as adjusted coal production and sales plans for company Alfa in Table 2 and Figure 1.
Impact assessment of the conducted dual price analysis

The applied adjustment has caused the following changes in comparison to the optimal plan [3]:
1. The company’s profit earned owing to the applied adjustment increased by 0.06% (increase by PLN 208,000); sales remained unchanged; production reserves increased by 1.1%.
2. As for the „A” and „C” mines, the optimal plan remained unchanged.
3. In the “B” mine, the loss rate decreased slightly by 0.13%. The volume of coal sales by 0.1%, the mine’s production reserves decreased by 0.08%.
4. The profit of „D” mine decreased by 4.8%. Sales decreased by 1.16% and production reserves decreased by 1.6%.
5. In the “E” mine, as a result of the increase in production capacity, the profit increased by 1.1% and the sales volume increased by 0.32%.
6. In the “F” mine there was an increase in profit by 2.08% as a result of an increase in the sales volume by 0.39%.
7. In the “G” mine, profit decreased by 1.6%, and the sales volume by almost 0.02%.

Summary

The presented adjustment and the solutions described in [5] make it possible to adjust production plans to reflect the actual internal and external conditions in which mines are expected to function. This can help significantly reduce computation time while not requiring the optimisation procedure to be run again with new constraints.

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Możliwości podejmowania decyzji produkcyjnych z wykorzystaniem cen dualnych

W artykule zaprezentowano sposób wykorzystania cen dualnych aktywnych ograniczeń (w tym przypadku ograniczeń zdolności wydobywczych kopalń) do rozwiązywania i wspomagania decyzji produkcyjnych. Jest to jeden z algorytmów opracowanych przez autora analizy postępowalnej opracowanej w ramach metody racjonalizacji decyzji produkcyjnych dla sformalizowanej grupy (spółki, PGG) kopalń węgła kamiennego. Możliwości wykorzystania cen dualnych przedstawiono na rzeczywistym przykładzie kopalni. Dzięki opracowanemu algorytmowi można szybko uzyskać informację, bez konieczności ponownego rozwiązywania zadania, co uzyskaliśmy lub stratiliśmy zwiększając lub zmniejszając wybrany limit produkcji, i określić wpływ zmian tych ograniczeń na osiągany zysk i strukturę wielkości produkcji i sprzedaży poszczególnych sortymentów węgla.

Słowa kluczowe: ceny dualne, analiza postępowalna, algorytm Simpleks