Buckling and post-buckling of FRC laminated beams in thermal environment using a generalized higher-order shear deformation zig-zag beam model

Qiduo Jin1,2 and Yiru Ren1,2*

1State Key Laboratory of Advanced Design and Manufacturing for Vehicle Body, Hunan University, Changsha, Hunan 410082, China
2College of Mechanical and Vehicle Engineering, Hunan University, Changsha, Hunan 410082, China
*Corresponding author’s e-mail: renyiru@hnu.edu.cn

Abstract. The thermal buckling and post-buckling of fibre-reinforced composite (FRC) beams are analyzed based on a generalized higher-order shear deformation zig-zag beam theory (RHDZT). The current zig-zag beam model satisfies the stress boundary conditions on the upper and lower surfaces and the reality that the shear strain is discontinuous at the layer interfaces. The material properties of FRCs are assumed to be temperature independent and the in-plane boundary conditions supposed immovable. By using the Hamilton variation principle, the governing equations are established based on the mid-plane of the beam. A two-step perturbation method is employed to obtain the approximate analytical solutions of the critical thermal buckling loads and post-buckling behaviors with clamped-clamped boundary conditions. The critical buckling results based on the zig-zag model and its degenerate equivalent single-layer models are compared and verified with those in the literatures. The post-buckling results of the FRC beams with different slenderness ratios, elastic modulus ratios and ply angles are investigated. Numerical results show the remarkable accuracy of the current model and also demonstrate the significant roles the above-mentioned effects play in the buckling and post-buckling problem of FRC laminated beams.

1. Introduction

In some extreme service environments of materials, great changes in temperature will cause considerable thermal stress, which will lead to buckling instability of structures. Therefore, many researchers have devoted themselves to the accurate prediction of buckling and post-buckling equilibrium paths of composite beams in thermal environment. Emam and Eltaheer [1] studied the buckling and post-buckling of composite beams under hydrothermal environment using a high-order shear deformation theory. They found the contribution of the moisture concentration on the critical buckling loads is insignificant. Vosoughi et al. [2] adopted the differential quadrature method analyze the thermal buckling and post-buckling of laminated composite beams with temperature-dependent material properties.

There are many related papers using classical theory, first-order shear deformation theory or higher-order shear deformation theory to analyze the static and dynamic characteristics of structure. Actually, these theories belong to the category of equivalent single-layer theories, but for laminated plates and shells with large transverse shear deformation or thickness, the results obtained by the equivalent single-layer theory are quite different from those obtained by the three-dimensional elastic theory. Xie et al. [3] added a zig-zag function to the displacement field on the basis of the equivalent
single-layer theory mentioned above to satisfy the shear strain discontinuity at the interface. The results show that for structures with large transverse shear deformation, the accuracy of the results can be effectively improved by considering the strain discontinuity at the interface. Because of the large number of generalized displacements of this theory, the finite element method is the common method to program and solve problems.

In this paper, a two-step perturbation method is used for the first time to obtain an approximate analytical solution to the buckling and post-buckling problem of the laminated beams with clamped-clamped boundaries based on this higher-order shear deformation zig-zag beam theory. The solution results based on the zig-zag model and its degenerated equivalent single-layer models are verified and compared in detail. Effects of different ply modes, elasticity modulus ratio and slenderness ratio on the post-buckling behaviors are studied and intuitively presented.

2. Governing Equations

A generalized high-order shear deformation zig-zag beam model can be established as [3-4]

\[ \ddot{u}(x,z,t) = u(x,t) + f(z)w(x,t) + \frac{\partial w(x,t)}{\partial x} + g(z)\varphi(x,t) + \varphi(z,k) \eta(x,t) \]

(1)

where \( f, g \) are the higher-order shape functions, defined by

\[ f(z) = -\frac{4z^3}{3h^2}, \quad g(z) = z - \frac{4z^3}{3h^2} \]

(2)

and \( \varphi \) is the zig-zag function, defined by

\[ \varphi = (-1)^k \left\{ \frac{2}{h_k} \left[ z - \frac{1}{2} (z_{k+1} + z_k) \right] - \frac{8z^3}{3h_k h^2} \right\} \]

(3)

This beam model can be degenerated to different equivalent single-layer models:

(1) Reddy’s higher order deformation theory (RHDT)

\[ f(z) = -\frac{4z^3}{3h^2}, \quad g(z) = z - \frac{4z^3}{3h^2}, \varphi(z,k) = 0 \]

(4)

(2) Timoshenko first order deformation theory (FDT)

\[ f(z) = 0, \quad g(z) = z, \varphi(z,k) = 0 \]

(5)

(3) Euler-Bernoulli classical theory (CT)

\[ f(z) = -z, \quad g(z) = 0, \varphi(z,k) = 0 \]

(6)

The von Kármán strain-displacement relationship is employed to obtain the nonlinear strains of the beam [1]. Based on the in-plane bending assumption, the simplified constitutive equations can be obtained [5]

\[ \sigma_{xx}^{(k)} = \tilde{Q}_{11}^{(k)} \varepsilon_{xx}^{(k)} - \tilde{Q}_{12}^{(k)} \tilde{Q}_{16}^{(k)} \left[ \begin{array}{c} \alpha_{xx} \\ \alpha_{xy} \end{array} \right] \Delta T \]

(7)

\[ \tau_{zx}^{(k)} = k_t \tilde{Q}_{35}^{(k)} \varepsilon_{zx}^{(k)} \]


where

\[
\begin{align*}
\bar{Q}_i^{(k)} &= \tilde{Q}_i^{(k)} - \left[ \tilde{Q}_i^{(k)} \right]^T \left[ \begin{array}{ccc}
\tilde{Q}^{(k)}_{12} & \tilde{Q}^{(k)}_{22} & \tilde{Q}^{(k)}_{26} \\
\tilde{Q}^{(k)}_{26} & \tilde{Q}^{(k)}_{66} & \tilde{Q}^{(k)}_{66} \\
\end{array} \right]^{-1} \left[ \tilde{Q}^{(k)}_{i6} \\
\end{align*}
\]

\[
\bar{Q}_i^{(k)} = \left[ \tilde{Q}_i^{(k)} - \frac{1}{\tilde{Q}^{(k)}_{44}} (\tilde{Q}^{(k)}_{44})^2 \right]
\]

(8)

The governing equations of the FRC laminated beam can be derived using the Hamilton’s principle, whose expression is given

\[
\int_0^L \delta T \, dt = \int_0^L \left( \delta T - \delta U + \delta W \right) dt = 0
\]

(9)

where \(\delta T\) denotes the virtual kinetic energy, \(\delta U\) donates the virtual strain energy of composite laminated beam, and \(\delta W\) the virtual work done by external forces. The governing equations of FRC laminated beams can be derived and be expressed by

\[
\begin{align*}
\frac{\partial N_i}{\partial x} &= 0 \\
\frac{\partial Q_{ix}}{\partial x} &= \frac{\partial^2 M_i}{\partial x^2} + N_i \frac{\partial^2 w}{\partial x^2} = 0 \\
\frac{\partial P_i}{\partial x} &= 0 \\
\frac{\partial T_i}{\partial x} &= 0
\end{align*}
\]

(10)

where \(N_i\) and \(Q_{ix}\) are axial positive and transverse shear forces, \(M_i\) is bending moment per unit length, \(P_i\) and \(T_i\) are higher-order bending moments, and \(P_{ix}\) and \(T_{ix}\) are higher-order shear forces, respectively. By introducing the constitutive relations, these generalized internal forces can be expressed by the displacements, referred to [3]. Further, by using the immovable in-plane displacement boundary conditions, the following governing equations expressed by the displacements can be written as follows

\[
\begin{align*}
&\left( D_{11} - B_{11} \frac{E_{11}}{A_{11}} \right) \frac{\partial^4 w}{\partial x^4} + \left( G_{11} - B_{11} E_{11} \frac{1}{A_{11}} \right) \frac{\partial^3 \vartheta}{\partial x^3} + \left( H_{11} - \frac{B_{11} F_{11}}{A_{11}} \right) \frac{\partial^2 \eta}{\partial x^2} - D_{55} \frac{\partial^2 w}{\partial x^2} - G_{55} \frac{\partial \vartheta}{\partial x} - H_{55} \frac{\partial \eta}{\partial x} \\
&- \left[ \int_0^L A_{11} \left( \frac{\partial w}{\partial x} \right)^2 \right] dx = A_{11} \Delta T = 0
\end{align*}
\]

(11)

where the stiffness coefficients are derived by
\[
(A_1, B_1, D_1, E_1, F_1, G_1, H_1, I_1, J_1, R_1) = \sum_{k=1}^{N_1} \left( A_{1k}, B_{1k}, D_{1k}, E_{1k}, F_{1k}, G_{1k}, H_{1k}, I_{1k}, J_{1k}, R_{1k} \right) \int_{z} \left( 1, f, f^2, g, \varphi, f \varphi, f^2 \varphi, g \varphi, \varphi^3 \right) Q_{1k}^{(i)} \, dz
\]

(12)

\[
(D_{55}, G_{55}, H_{55}, I_{55}, J_{55}, R_{55}) = k \sum_{k=1}^{N_1} \int \left( f^2, \overline{f}_g, \overline{f}, \overline{g}, \overline{g}_\varphi, \overline{\varphi} \right) Q_{55}^{(i)} \, dz
\]

and

\[
A_{1k}^T = \sum_{k=1}^{N_1} \left[ \overline{Q}_{1k}^{(i)} \overline{Q}_{12}^{(i)} \overline{Q}_{13}^{(i)} \right] \begin{bmatrix} \alpha_s \\ \alpha_y \\ \alpha_{\varphi} \\ \alpha_{\varphi^3} \end{bmatrix} dz
\]

(13)

To facilitate the mathematical treatment of the governing equations, the following dimensionless parameters are defined

\[
X = \pi \frac{x}{L}, W = \frac{w}{L}, \Theta = \frac{\theta}{\Theta}, \varphi = \frac{\varphi}{\varphi}, \phi = \frac{\phi}{\phi}, \gamma = \frac{\gamma}{\gamma}, \eta = \frac{\eta}{\eta}, \lambda = \frac{\lambda}{\lambda}, \nu = \frac{\nu}{\nu}, \kappa = \frac{\kappa}{\kappa}, \sigma = \frac{\sigma}{\sigma},
\]

(14)

\[
\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9 = \frac{1}{D_0} \left( s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9 \right)
\]

\[
\gamma_{10}, \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15} = \frac{L^2}{\pi^2 D_0} \left( D_{55}, G_{55}, H_{55}, I_{55}, J_{55}, R_{55} \right)
\]

in which

\[
s_1 = D_{11} - \frac{B_{11}^2}{A_{11}}, s_2 = G_{11} - \frac{B_{11} E_{11}}{A_{11}}, s_3 = H_{11} - \frac{B_{11} F_{11}}{A_{11}}, s_4 = G_{11} - \frac{B_{11} E_{11}}{A_{11}}, s_5 = G_{11} - \frac{B_{11} E_{11}}{A_{11}}, s_6 = J_{11} - \frac{E_{11} F_{11}}{A_{11}}, s_7 = J_{11} - \frac{E_{11} F_{11}}{A_{11}}, s_8 = R_{11} - \frac{F_{11}^2}{A_{11}}, s_9 = R_{11} - \frac{F_{11}^2}{A_{11}}
\]

(15)

The dimensionless governing equations can be written as

\[
\gamma_1 \frac{\partial^4 W}{\partial X^4} + \gamma_2 \frac{\partial^3 \Theta}{\partial X^3} + \gamma_3 \frac{\partial^3 H}{\partial X^3} - \left[ \frac{\pi \gamma_0}{2} \left( \frac{\partial W}{\partial X} \right)^2 \right] \frac{\partial^2 \Theta}{\partial X^2} + \gamma_4 \frac{\partial^2 H}{\partial X^2} = 0
\]

\[
\gamma_5 \frac{\partial^4 W}{\partial X^4} + \gamma_6 \frac{\partial^3 \Theta}{\partial X^2} + \gamma_7 \frac{\partial^3 H}{\partial X^2} \left( \frac{\gamma_{11}}{\gamma_{12}} \frac{\partial W}{\partial X} + \frac{\gamma_{13}}{\gamma_{14}} \frac{\partial H}{\partial X} + \frac{\gamma_{15}}{\gamma_{16}} \frac{\partial \Theta}{\partial X} \right) = 0
\]

\[
\gamma_9 \frac{\partial^4 W}{\partial X^4} + \gamma_8 \frac{\partial^3 \Theta}{\partial X^2} + \gamma_9 \frac{\partial^3 H}{\partial X^2} \left( \frac{\gamma_{11}}{\gamma_{12}} \frac{\partial W}{\partial X} + \frac{\gamma_{13}}{\gamma_{14}} \frac{\partial H}{\partial X} + \frac{\gamma_{15}}{\gamma_{16}} \frac{\partial \Theta}{\partial X} \right) = 0
\]

(16)

3. Solution Methodology

A two-step perturbation method is used to solve the dimensionless governing equations. In the present case, we assume that

\[
\lambda(X, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \lambda_k^{(i)}(X) W(X, \varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k w_k(X)
\]

\[
\Theta(X, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \Theta_k(X) H(X, \varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k \eta_k(X)
\]

(17)

By substituting Eq. (17) into (16), a series of perturbation equations will be obtained [6]. The trial functions of the first order equations are assumed to be
The critical buckling load can be obtained by letting $W_m = 0$

$$\lambda_{cr} = \lambda^{(0)} \eta + \varepsilon^2 \lambda^{(2)} + O(\varepsilon^4)$$

$$= \left[ \gamma_{11} (2m) + \gamma_{12} (2m)^2 \right] R + \left[ \gamma_{12} (2m) + \gamma_{33} (2m)^3 \right] S$$

$$= \frac{(2m)^2 \gamma_r}{\gamma_r} + \frac{(2m)^2 \gamma_{11} + \gamma_{10}}{\gamma_r} + \frac{\gamma_{10} (m \pi W_m)^2}{4 \gamma_r}$$

The critical buckling load can be obtained by letting $W_m = 0$

$$\lambda_{cr} = \frac{\left[ \gamma_{11} (2m) + \gamma_{12} (2m)^2 \right] R + \left[ \gamma_{12} (2m) + \gamma_{33} (2m)^3 \right] S}{(2m)^2 \gamma_r} + \frac{(2m)^2 \gamma_{11} + \gamma_{10}}{\gamma_r}$$

where

$$R = \begin{bmatrix}
-2m^3 \gamma_{11} + 2m \gamma_{12} & (2m)^2 \gamma_{12} + \gamma_{13} \\
(2m)^2 \gamma_{12} & (2m)^2 \gamma_{13} + \gamma_{14}
\end{bmatrix},
S = \begin{bmatrix}
2m^2 \gamma_{11} + \gamma_{13} & (2m)^2 \gamma_{13} + \gamma_{14} \\
(2m)^2 \gamma_{13} & (2m)^2 \gamma_{14} + \gamma_{15}
\end{bmatrix},
$$

in which $m$ is the order of buckling mode, but only the minimum buckling load and the equilibrium path are concerned for the buckling problem of beams unless studying the second buckling or the jumping of buckling mode. In this case, letting $m=1$ yields the critical buckling load and the post-buckling equilibrium path.

4. Numerical Examples
4.1 Validation and comparison
To verify the correctness of the models and methods used in this paper, this section calculates the dimensionless critical buckling temperature rise under different slenderness ratio based on HDZT and its degradation models. The results are compared with those using different equivalent single layer theories and other numerical methods in the literatures, and the causes for errors are analyzed. The material characteristics are presented as follows [7]

* $E_1 / E_2 = open, E_2 = E_3, G_{12} = G_{23} = 0.6 E_2, G_{13} = 0.5 E_2,$
* $\mu_{12} = \mu_{13} = 0.25, L / h = open, \alpha_2 / \alpha_1 = open$

The dimensionless buckling temperature rise is defined by
\[
\Delta T_{cr} = \Delta T_{cr} \alpha_1 \left( \frac{L}{h} \right)^2
\]  

(23)

### Table 1  Dimensionless critical buckling temperature rise of \((0^\circ/90^\circ)\) FRC beams \(\frac{E_1}{E_2} = 20, \frac{\alpha_2}{\alpha_1} = 3\).

| BCs  | Reference                  | \(L/h\)     |
|------|----------------------------|-------------|
|      |                            | 5           | 10          | 20           | 30           | 50           | 100          |
| c-c  | Present RHDZT              | 0.511       | 0.850       | 1.029        | 1.071        | 1.094        | 1.108        |
|      | Present RHDT               | 0.557       | 0.885       | 1.042        | 1.078        | 1.097        | 1.108        |
|      | Present FDT                | 0.512       | 0.858       | 1.033        | 1.073        | 1.095        | 1.108        |
|      | Present CT                 | 1.108       | 1.108       | 1.108        | 1.108        | 1.108        | 1.108        |
|      | Aydogdu [8] HDT            | 0.557       | 0.885       | 1.092        | -            | 1.098        | -            |
|      | Khdeir [9] HDT             | 0.583       | 0.926       | 1.090        | -            | 1.148        | -            |
|      | Ngoc-Duong Nguyen [10] HDT | 0.558       | 0.887       | 1.045        | 1.081        | 1.100        | -            |

As can be found from the table, the results based on RHDT are almost the same as those based on different higher-order shear deformation models in the literatures, which verifies the correctness of the method in this paper. More concretely, when the slenderness ratio is large, the results based on different theories are similar, but when the slenderness ratio is small, the results obtained by the models neglecting the zig-zag effects are relatively higher. This is because the transverse shear deformation is larger for short and thick beams, and in this case the difference between the results of the models considering the zig-zag effect and neglecting the zig-zag effect are relatively large.

4.2 Parameter analysis

In this section, the parameters are analyzed according to the numerical results. Figure 1-3 respectively shows the effect of the ratio of elastic modulus, different ply angles and slenderness ratio on the post-buckling equilibrium path.
Figure 1. Effect of elasticity modulus ratio on the post-buckling paths of the (0°/90°) FRC laminated beams.

Figure 2. Effect of the ply angles on the post-buckling paths of the FRC laminated beams.
As can be presented from Figure 1-3, the following conclusions can be intuitively drawn by comparing the results of different models and material and geometric parameters:

1. (0°/90°) laminated beams have the best post-buckling bearing capacity than (45°/-45°) and (30°/60°), and the larger slenderness ratio is, the smaller the critical buckling value is, and the worse the post buckling capacity is.

2. For the case when the slenderness ratio is small or elastic modulus ratio is large, the results obtained by using the equivalent single layer theories in the literature are overestimated. Conversely, the results based on different models are close.

3. (0°/90°) laminated beams have the best sensitivity of the zig-zag effect, while for (45°/-45°) and (30°/60°) the results based on RHDZT and its degeneration models are closer.

4. In the initial post buckling, the bearing capacity of post-buckling is better when the elastic modulus is smaller, but in the deep post buckling stage, there will be opposite results.

5. **Conclusion**

In this paper, considering the geometric nonlinearity and zig-zag effects, the governing equations are derived based on a refined zig-zag model of the laminated beams. The governing equations of composite beams are solved by a two-step perturbation technique, and the critical buckling temperature rise and post-buckling paths are obtained. The effects of elastic modulus ratio, ply angles, and slenderness ratio on the post-buckling equilibrium paths are obtained. Numerical results reveal that when the laminated beams are not symmetrically or anti-symmetrically laid, the difference between the zig-zag model and its degradation models are relatively large. Also, when the slenderness ratio is small or the elastic modulus ratio is large, the zig-zag effects cannot be neglected.

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