Cramer-Rao Lower Bounds of Joint Delay-Doppler Estimation for an Extended Target

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Abstract—In this paper, the problem on the Cramer-Rao Lower Bound (CRLB) of joint time delay and Doppler stretch estimation for an extended target is considered. The integral representations of the CRLBs for time delay and Doppler shift are derived. For the convenience of computation and analysis, series representations and their approximations are further proposed. According to these series representations, the impact of waveform parameters on the CRLBs is investigated. It is shown that the CRLB of Doppler stretch for a wideband signal is negatively correlated to the effective time-bandwidth product of the waveform, which generalizes the conclusion in the narrowband case. Wideband ambiguity function (WBAF)-based estimator is evaluated in the case of an extended target through numerical experiments, and it is indicated that the WBAF-based estimator is not suitable for the parameter estimation of an extended target.

Keywords—CRLB, time delay, Doppler stretch, wideband, extended target.

I. INTRODUCTION

Joint estimation of time delay and Doppler stretch is a fundamental problem in radar and sonar systems, and has been well addressed for a narrowband signal [1], [2].

In many modern applications, however, wideband signals are adopted and the narrowband model may not hold in these situations. A narrowband model is appropriate when $BT \ll c/2v$ [3], where $B$ and $T$ are the bandwidth and duration of transmitted signal, respectively, $v$ is the relative velocity between the target and sensor, and $c$ is the propagation speed of signal. In an imaging sonar or radar, for instance, since the high range resolution is pursued, signals with large bandwidth $B$ are usually employed. To seek a low interception probability (LPI), as another example, the transmission power should be lowered down, and an effective approach is to choose a signal with large $BT$ such that the energy can be spread over a wider region in the time domain or frequency domain.

There are significant variations in the echoes model when a wideband signal is applied. Firstly, a target can be modeled as a point scatterer under the narrowband assumption. In contrast, when a signal with large bandwidth is utilized, a target may span over several adjacent range units due to the high range resolution (HRR), and should be described with multiple scatterers. In this case, the target is referred to as an extended target. Secondly, under the narrowband model, the Doppler effect on the echoes is approximated by a carrier frequency shift over the transmitted signal. The complex envelop variation of the transmitted waveform is assumed to be negligible. However, as $BT$ approaches $c/2v$, the Doppler effect on the complex envelop should be considered [3], [4].

The Cramer-Rao Lower Bound (CRLB) is an important tool for estimation problems. The CRLB is the minimal variance for any unbiased estimator [5] and thus, it is usually employed to evaluate the performances of various estimators [1]. The CRLB is also a criterion for optimal waveform selections [4], [6], [7]. The CRLB for a narrowband signal is investigated in [1], [2], [8], [9], while studies in [4], [10] concern with the wideband model for a single scatterer. Liu et al. [11] consider the CRLB of HRR profiles and velocity estimate, but the transmission is limited as a step frequency signal. In this paper, we derive the CRLB for an ordinary wideband signal under the extended target model, which is more general and realistic when wideband waveforms are transmitted.

The rest of this paper is organized as follows. Section II establishes the signal model. In Section III the CRLBs of time delay and Doppler stretch are derived and discussed. The integral representations of the CRLBs are presented firstly and, for the convenience of analysis, series representations and their approximations are further proposed. According to the series representations, the influences of the energy, effective bandwidth and effective duration on CRLBs are analyzed. Section IV provides some numerical examples. Section V is dedicated to a brief conclusion.

II. MODELING AND PROBLEM STATEMENT

Consider an extended target which contains multiple scatterers. Assume that the target is moving along the line of sight (LOS) with a radial velocity $v$ relative to the sensor (radar or sonar). The velocity is positive if the target is moving apart from the sensor.

Let $s(t)$ be the transmitted signal which is time-limited to $[0, T]$, that is, $s(t) = 0$ for any $t < 0$ or $t > T$. Thus, the echo can be modeled as

$$y(t) = \sum_{p=1}^{P} x_p s(\gamma(t - \tau_p)) + w(t), \quad (1)$$
where \(\tau_p\) is the time delay of the \(p\)th scatterer, \(\gamma\) represents the Doppler stretch, and the scattering coefficient \(x_p\) accounts for the propagation attenuation and the influence of Doppler stretch on signal energy. Without the loss of generality, the extended target is assumed to consist of \(P\) ideal point scatterers, which are uniformly spread over the space. Thus, the time delay of the \(p\)th scatterer \(\tau_p = \tau + (p - 1)\Delta\), where \(\Delta\) is the sample interval and \(\tau\) is unknown to the sensor.

The size of the target \(L = \frac{1}{\Delta}(\tau_p - \tau)\). The Doppler stretch \(\gamma = (c - v)/(c + v)\), where \(c\) is the propagation speed of transmitted waves in a free space. The noise \(w(t)\) is considered as a white Gaussian noise (WGN) with power spectral density \(N_0\).

Sample the echoes at the rate of \(1/\Delta\), and the original signal \(y\) turns into

\[
y_n = \sum_{p=1}^{P} x_p s(\gamma(n\Delta - \tau_p)) + w_n, \quad n = 0, 1, ..., N - 1,
\]

where \(y_n = y(n\Delta)\), the noise \(w_n = w(n\Delta)\) is distributed as \(\mathbb{C}N(0, \sigma^2)\) and \(\sigma^2\Delta = N_0\). Rewrite (2) in a matrix form as

\[
y = \Phi x + w,
\]

where \(y = [y_0, ..., y_{N-1}]^T \in \mathbb{C}^{N \times 1}\) is the observation vector, \(\Phi \in \mathbb{C}^{N \times P}\) is the complex measurement matrix with \(\Phi_{ij} = s(\gamma((i-1)\Delta - \tau_j))\). Scattering coefficients \(x = [x_1, ..., x_P]^T \in \mathbb{R}^{P \times 1}\) represent the high-range-resolution profile of the target. The noise vector \(w = [w_0, ..., w_{N-1}]^T \in \mathbb{C}^{N \times 1}\) is distributed as \(\mathbb{C}N(0, \sigma^2I)\). We make the following assumptions:

**Assumption 1:** For any \(p\), we have \(\tau_p + \frac{\tau}{\Delta} \leq (N-1)\Delta\), which indicates echoes from the target are completely sampled.

**Assumption 2:** Both \(\frac{\tau}{\Delta}\) and \(\frac{T}{\Delta}\) are considered as integers.

It suggests that the sampling interval \(\Delta\) is small enough.

**Assumption 3:** \(s(t)\) has derivatives of all orders throughout \((-\infty, +\infty)\) and there exist constants \(C_1, C_2 > 0\) such that

\[
M_0^{(k)} \triangleq \int_{-\infty}^{+\infty} \left| s^{(k)}(t) \right|^2 dt < C_1 e^{kC_2}, \quad k = 0, 1, ..., \quad (4)
\]

\[
M_1^{(k)} \triangleq \int_{-\infty}^{+\infty} \left| s^{(k)}(t) \right|^2 dt, \quad (5)
\]

with

\[
\left| M_1^{(k)} \right| < C_1 e^{kC_2}, \quad k = 0, 1, ..., \quad (6)
\]

and

\[
M_2^{(k)} \triangleq \int_{-\infty}^{+\infty} \left| s^{(k)}(t) \right|^2 dt < C_1 e^{kC_2}, \quad k = 0, 1, ..., \quad (7)
\]

where \(s^{(m)}(t)\) is the abbreviation of \(\frac{d^m}{dt} s(t)\). From the assumptions and definitions above, the following theorem can be derived.

**Theorem 1:** For any \(p, q \in \mathbb{N}^+\) and \(s(t)\) satisfying Assumption 3, following equations hold:

\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} s^{(p)}(t) s^{(q)}(t) dt \right\} = \begin{cases} (-1)^{p+k} M_0^{(k)}, & p + q = 2k, \\ 0, & p + q = 2k + 1, \end{cases} \quad (8)
\]

\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} t s^{(p)}(t) s^{(q)}(t) dt \right\} = \begin{cases} (-1)^{p+k} M_1^{(k)}, & p + q = 2k, \\ (-1)^{p+1} (p - k - \frac{1}{2}) M_0^{(k)}, & p + q = 2k + 1, \end{cases} \quad (9)
\]

\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} t^2 s^{(p)}(t) s^{(q)}(t) dt \right\} = \begin{cases} (-1)^k M_2^{(k)} + (-1)^{k+1} 2k M_0^{(k-1)}, & q = 2k, \\ (-1)^{k+1} (2k+1) M_1^{(k)}, & q = 2k + 1, \end{cases} \quad (10)
\]

where \(^*\) denotes the complex conjugate.

**Proof:** See Appendix.

The parameter \(M_0^{(0)}\) is the energy of the transmitted waveform, and \(M_0^{(1)}\) and \(M_2^{(1)}\) can be considered as the measurement of bandwidth and duration, respectively. The **effective bandwidth** of \(s(t)\) is defined by [12]

\[
B = \left( \frac{M_0^{(0)}}{M_0^{(1)}} \right)^{\frac{1}{2}} = \left( \frac{\int_{-\infty}^{+\infty} |s^{(1)}(t)|^2 dt}{\int_{-\infty}^{+\infty} |s^{(0)}(t)|^2 dt} \right)^{\frac{1}{2}}. \quad (11)
\]

According to the Fourier transformation, \(s(t)\) can be rewritten as

\[
B^2 = \frac{\int_{-\infty}^{+\infty} (2\pi f)^2 |S(f)|^2 df}{\int_{-\infty}^{+\infty} |S(f)|^2 df}. \quad (12)
\]

where \(S(f)\) is the frequency spectrum of \(s(t)\). Since \(\hat{B}\) measures the spread of signal \(s(t)\) in the frequency domain in an root mean square sense, we also refer to it as the root mean square bandwidth. The **effective duration** can be defined by [4], [12]

\[
\hat{T} = \left( \frac{M_2^{(1)}}{M_0^{(1)}} \right)^{\frac{1}{2}} = \left( \frac{\int_{-\infty}^{+\infty} |s^{(1)}(t)|^2 t^2 dt}{\int_{-\infty}^{+\infty} |s^{(1)}(t)|^2 dt} \right)^{\frac{1}{2}}. \quad (13)
\]

It is also called as root mean square duration. To avoid confusing, we refer to \(\hat{B} \hat{T}\) as effective time-bandwidth product. For a narrowband signal

\[
s(t) = \exp]\left(j2\pi f_c t \right) (u(t) - u(t - T)), \quad (14)
\]
where \( u(t) \) is the unit step function, we have \( \bar{B} = f_c \) and \( \bar{T} = \sqrt{\frac{\pi}{2}} T \).

In addition, for notational conciseness, some matrices are introduced. We define \( \mathbf{A} = [\Lambda_{ij}] \in \mathbb{R}^{P \times P}, 1 \leq i, j \leq P \), where

\[
\Lambda_{ij} = \text{Re} \left\{ \int_{-\infty}^{+\infty} s^*(t) s(t + \gamma(\tau_i - \tau_j)) dt \right\},
\]

(15)

and \( \mathbf{F}^{(k)} = \left[ \Gamma^{(k)}_{ij} \right] = \mathbb{R}^{P \times P}, 1 \leq i, j \leq P; k = 0, 1, 2, \ldots \), where

\[
\Gamma^{(k)}_{ij} = (\tau_i - \tau_j)^k = (i - j)^k \Delta^k.
\]

(16)

Particularly, \( \mathbf{F}^{(0)} = \mathbf{1}_{P \times P} = \mathbf{11}^T \), that is, all of the elements in \( \mathbf{F}^{(0)} \) equal to 1.

### III. DERIVATIONS OF THE CRLBs

#### A. The integral representations of CRLBs

The CRLB is the minimal variance of any unbiased estimator and is usually used as a benchmark to evaluate the performance of estimators. Let \( p(y; \theta) \) be the probability density function of the observations \( y \) in (3). Then, the covariance matrix of any unbiased estimator \( \hat{\theta} \) satisfies

\[
\mathbf{C}_\theta \triangleq E \left\{ (\hat{\theta} - \theta) \cdot (\hat{\theta} - \theta)^T \right\} \geq \text{FIM}^{-1},
\]

(17)

where \( \text{FIM} \in \mathbb{R}^{(P+2) \times (P+2)} \) is the Fisher information matrix defined by

\[
\text{FIM} = E \left\{ \nabla_\theta (\ln p(y; \theta)) \cdot \nabla_\theta^T (\ln p(y; \theta)) \right\}
\]

(18)

with

\[
\nabla_\theta \triangleq \left[ \frac{\partial}{\partial \tau^1}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_P} \right]^T.
\]

(19)

Inequality between matrices \( \mathbf{A} \geq \mathbf{B} \), means \( \mathbf{A} - \mathbf{B} \) is positive semidefinite. The CRLB of \( \theta \) is given by the diagonal elements of \( \text{FIM}^{-1} \).

Note that \( y \sim \mathcal{C}_N(\mu(\theta), \sigma^2 I) \), where \( \mu(\theta) = \Phi x \). Thus, the Fisher information matrix can be calculated by (11)

\[
[\text{FIM}]_{ij} = 2 \sigma^2 \text{Re} \left\{ \frac{\partial \mu^H(\theta)}{\partial \theta_i} \frac{\partial \mu(\theta)}{\partial \theta_j} \right\}.
\]

(20)

Partition \( \text{FIM} \) as

\[
\text{FIM} = \begin{bmatrix}
F_{11} & F_{12} \\
F_{12} & F_{22}
\end{bmatrix} = \begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix},
\]

(21)

where \( F_{ij} \in \mathbb{R}^{1 \times 1}, i = 1, 2 \); \( F_{3i} \in \mathbb{R}^{P \times 1}, i = 1, 2 \); \( F_{33} \in \mathbb{R}^{P \times P} \); \( J_{11} = [F_{1j}], i = 1, 2 \); \( J_{21} = [F_{31}, F_{32}] \) \in \mathbb{R}^{P \times 2} \).

The elements of \( \text{FIM} \) are calculated as following

\[
F_{11} = \frac{2}{\sigma^2} x^H \text{Re} \left\{ \frac{\partial \Phi^H}{\partial \theta} \frac{\partial \Phi}{\partial \theta} \right\} x
\]

\[
= \frac{2}{\sigma^2} \sum_{i=1}^{P} \sum_{j=1}^{P} \sum_{n=0}^{P-1} x_i x_j x_n
\]

\[
\text{Re} \left\{ \frac{\partial s^*(\gamma(n\Delta - \tau_i))}{\partial \tau} \frac{\partial s(\gamma(n\Delta - \tau_j))}{\partial \tau} \right\}
\]

\[
= \frac{2}{\sigma^2} \sum_{i=1}^{P} \sum_{j=1}^{P} \sum_{n=0}^{P-1} x_i x_j x_n
\]

\[
\text{Re} \left\{ s^*(1) (\gamma(n\Delta - \tau_i)) s(1) (\gamma(n\Delta - \tau_j)) \right\} \gamma^2
\]

\[
\approx \frac{2\gamma^2}{N_0} \sum_{i=1}^{P} \sum_{j=1}^{P} x_i x_j x_n
\]

\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} s^*(1) (t + \gamma(\tau_i - \tau_j)) dt \right\},
\]

(22)

where in the third line of the equation, we use the fact that \( s(t) \) is equal to zero for any \( t \notin [0, T] \). In the last line, the summation is approximated with an integral by letting \( \Delta \to 0 \), which is based on the assumption that the sampling interval \( \Delta \) is small enough. Similarly, we have

\[
F_{12} = \frac{2}{\sigma^2} x^H \text{Re} \left\{ \frac{\partial \Phi^H}{\partial \gamma} \frac{\partial \Phi}{\partial \gamma} \right\} x
\]

\[
\approx \frac{2}{\gamma N_0} \sum_{i=1}^{P} \sum_{j=1}^{P} x_i x_j x_n
\]

\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} t s^*(1) (t + \gamma(\tau_i - \tau_j)) s(1) (t + \gamma(\tau_i - \tau_j)) dt \right\},
\]

(23)

\[
F_{22} = \frac{2}{\sigma^2} x^H \text{Re} \left\{ \frac{\partial \Phi^H}{\partial \gamma} \frac{\partial \Phi}{\partial \gamma} \right\} x
\]

\[
\approx \frac{2\gamma^2}{\gamma^2} \sum_{i=1}^{P} \sum_{j=1}^{P} x_i x_j x_n
\]

\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} t s^*(1) (t + \gamma(\tau_i - \tau_j)) s(1) (t + \gamma(\tau_i - \tau_j)) dt \right\},
\]

(24)

\[
[F_{31}]_{11} = \frac{2}{\sigma^2} \text{Re} \left\{ \frac{\partial \Phi^H}{\partial \tau} \right\} x
\]

\[
\approx \frac{2}{\gamma N_0} \sum_{i=1}^{P} \sum_{j=1}^{P} x_i x_j x_n
\]

\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} s^*(1) (t + \gamma(\tau_i - \tau_j)) dt \right\},
\]

(25)
\[ [\mathbf{F}_{33}]_{ij} = \frac{2}{\sigma^2} \text{Re} \{ \mathbf{\Phi}^H a_{ij} \mathbf{\Phi} \} \mathbf{x}_j \]
\[ \approx \frac{2}{\gamma^2 N_0} \sum_{j=1}^P x_j \text{Re} \left\{ \int_{-\infty}^{+\infty} ts^*(t + \gamma(t_j - t_i)) s^{(1)}(t) dt \right\}, \]  
(26)

\[ \mathbf{F}_{33} = \frac{2}{\sigma^2} \text{Re} \{ \mathbf{\Phi}^H a_{ij} \mathbf{\Phi} \} \approx \frac{2}{\gamma^2 N_0} \mathbf{\Delta}. \]  
(27)

In applications of radar and sonar, parameters \( \tau \) and \( \gamma \), which represent the range and velocity of target, respectively, are of special interest. This paper, only the CRLBs of \( \tau \) and \( \gamma \) are calculated in details. Apply the formula for inverting partitioned matrices and it yields the CRLB matrix for \( \tau \) and \( \gamma \) as

\[ [\mathbf{FIM}^{-1}]_{1:2,1:2} = (\mathbf{J}_{11} - \mathbf{J}_{21}^T \mathbf{F}_{33}^{-1} \mathbf{J}_{21})^{-1} \]  
(28)

and then elements [\( [\mathbf{FIM}^{-1}]_{1:1} \) and [\( [\mathbf{FIM}^{-1}]_{2:2} \) give the CRLBs of time delay and Doppler stretch, respectively,

\[ \text{var}(\tau) \geq \text{CRLB}_\tau = \frac{a_{22}}{a_{11} a_{22} - a_{12}^2}, \]  
(29)

\[ \text{var}(\gamma) \geq \text{CRLB}_\gamma = \frac{a_{11}}{a_{11} a_{22} - a_{12}^2}. \]  
(30)

where

\[ a_{ij} = [\mathbf{J}_{11} - \mathbf{J}_{21}^T \mathbf{F}_{33}^{-1} \mathbf{J}_{21}]_{ij} = F_{ij} - \mathbf{F}_{3i}^T \mathbf{F}_{3j}^{-1} \mathbf{F}_{3j}, i, j = 1, 2. \]  
(31)

Given the waveform \( s(t) \) and target parameters, the CRLBs are obtained with (22)-(27) and (29)-(31). Notice that integral operations on \( s(t + \gamma(t_j - t_i)) \) and \( s^{(1)}(t + \gamma(t_j - t_i)) \) are involved. Therefore, for different delays \( \tau_j - \tau_i \) and Doppler stretch \( \gamma \), the integrals are required to calculate correspondingly.

**B. The series representations of CRLBs**

In the previous subsection, the integral representations of CRLBs (22)-(27) and (29)-(31) are derived. However, it is not convenient to apply these formulas to compute and analyze the CRLBs. There might be two problems. Firstly, the integrals involved in (22)-(27) should be computed correspondingly for different time delays and Doppler stretch, causing the computational load large. Secondly, due to these complicated integrals, it is difficult to analyze the influences of waveform parameters on the CRLBs with the integral representations of CRLBs. To relieve these two drawbacks, we replace functions \( s(t + \gamma(t_j - t_i)) \) and \( s^{(1)}(t + \gamma(t_j - t_i)) \) with their Taylor series, respectively, then rewrite the CRLBs in the form of series. Unlike the integral representations that involve many complex integrals, these series representations only consist of integrals \( M^k, i = 1, 2, 3, k \in \mathbb{N} \), which can be calculated and stored previously for a specific waveform, e.g. chirp or phase-coded signal. Since it is avoided to compute the integrals that involve various time delays and Doppler stretch, the computational load is significantly reduced. Meanwhile, the influences of waveforms on CRLBs can be easier to analyze by employing the series representation. Owing to these advantages, the series representations are valuable. In this subsection, the series representations of CRLBs are derived and then, a kind of approximations on CRLBs are presented.

Under Assumption 3, for any \( t \in \mathbb{R} \), following equations on Taylor series hold (13):

\[ s(t + \gamma(t_i - t_j)) = \sum_{m=0}^{+\infty} \frac{\gamma^m (t_i - t_j)^m}{m!} s^{(m)}(t), \]  
(32)

\[ s^{(1)}(t + \gamma(t_i - t_j)) = \sum_{m=1}^{+\infty} \frac{\gamma^{m-1} (t_i - t_j)^{m-1}}{(m-1)!} s^{(m)}(t), \]  
(33)

\[ (t + \gamma(t_i - t_j)) s^{(1)}(t + \gamma(t_i - t_j)) = \sum_{m=0}^{+\infty} \frac{\gamma^n (t_i - t_j)^m}{m!} (m s^{(m)}(t) + ts^{(m+1)}(t)), \]  
(34)

where \( 0! \triangleq 1 \). Substituting (33) into (22) and applying the Lebesgue’s Dominated Convergence Theorem which is one of the rules on the interchangeability between integral and limit (14), we have

\[ F_{11} = \frac{2\gamma^2}{N_0} \sum_{i=1}^P \sum_{j=1}^P \sum_{m=1}^{+\infty} x_i x_j \frac{\gamma^{m-1} (t_i - t_j)^{m-1}}{(m-1)!} \times \]
\[ \text{Re} \left\{ \int_{-\infty}^{+\infty} s^{(1)}(t) s^{(m)}(t) dt \right\} \]
\[ = \sum_{k=0}^{+\infty} \frac{(-1)^k 2\gamma^2 k^{2k+1}}{(2k)! N_0} M_0^{(k+1)} x^T \Gamma^{(2k)} x \]
\[ = \lim_{K \rightarrow +\infty} F_{11}^{(K)} \]  
(35)

where

\[ F_{11}^{(K)} = \sum_{0 \leq k \leq K} \frac{(-1)^k 2\gamma^2 k^{2k+1}}{(2k)! N_0} M_0^{(k+1)} x^T \Gamma^{(2k)} x. \]  
(36)

Note that formula (8) is employed to deduce the second equality in (35). Similarly, substitute (32)-(34) into (23)-(27) and apply formula (8)-(10) in Theorem 1, then other elements of \( \mathbf{FIM} \) can be written as

\[ F_{12} = \lim_{K \rightarrow +\infty} F_{12}^{(K)}, i = 1, 2, \]
\[ F_{3i} = \lim_{K \rightarrow +\infty} F_{3i}^{(K)}, i = 1, 2, 3, \]  
(37)

where

\[ F_{12}^{(K)} = \sum_{0 \leq k \leq K} \frac{(-1)^{k+1} 2\gamma^2 k^{2k-1}}{(2k)! N_0} M_1^{(k+1)} x^T \Gamma^{(2k)} x, \]  
(38)
\[
\begin{align*}
F_{22}^{(K)} &= \sum_{1 \leq 2k \leq K} \frac{(-1)^k(k-1)\gamma 2^{k-3}}{(2k-1)!N_0} M_0^{(k)} x^T \Gamma^{(2k)} x + \\
&\sum_{0 \leq 2k \leq K} \frac{(-1)^k2^{k-3}M_0^{(k)} x^T \Gamma^{(2k)} x}{(2k)!N_0},
\end{align*}
\]
(39)

\[
\begin{align*}
F_{31}^{(K)} &= \sum_{0 \leq 2k-1 \leq K} \frac{(-1)^{k+1}2^{k-1}}{(2k-1)!N_0} M_0^{(k)} \Gamma^{(2k-1)} x,
\end{align*}
\]
(40)

\[
\begin{align*}
F_{32}^{(K)} &= \sum_{0 \leq 2k \leq K} \frac{(-1)^k2^{k-1}}{(2k)!N_0} M_0^{(k)} \Gamma^{(2k)} x + \\
&\sum_{0 \leq 2k+1 \leq K} \frac{(-1)^{k+1}2^{k-1}}{(2k+1)!N_0} M_0^{(k)} \Gamma^{(2k+1)} x,
\end{align*}
\]
(41)

\[
\begin{align*}
F_{33}^{(K)} &= \sum_{0 \leq 2k \leq K} \frac{(-1)^k 2^{k-1}}{(2k)!N_0} M_0^{(k)} \Gamma^{(2k)}. 
\end{align*}
\]
(42)

Notice that in (38), \( x^T \Gamma^{(2k+1)} x = 0, \forall k \in \mathbb{N}, \) is applied.

Define the following matrices

\[
\begin{align*}
\text{FIM}^{(K)} &\triangleq \begin{bmatrix}
F_{11}^{(K)} & F_{12}^{(K)} & \left(F_{31}^{(K)}\right)^T \\
F_{12}^{(K)} & F_{22}^{(K)} & \left(F_{32}^{(K)}\right)^T \\
F_{31}^{(K)} & F_{32}^{(K)} & F_{33}^{(K)}
\end{bmatrix},
\end{align*}
\]
(43)

\[
\begin{align*}
\text{E}^{(K)} = \text{FIM} - \text{FIM}^{(K)}.
\end{align*}
\]
(44)

Note that

\[
\begin{align*}
\text{FIM} = \lim_{K \to +\infty} \text{FIM}^{(K)}.
\end{align*}
\]
(45)

and therefore, the \( \text{FIM}^{(K)} \) can be considered as an approximation of \( \text{FIM} \) and \( \text{E}^{(K)} \) is the approximation error. From the results shown above, approximation representations of CRLBs can be derived if the following assumption holds:

**Assumption 5** For any \( t \in [0, T] \), it holds

\[
\begin{align*}
s \left(t + \frac{2L}{c} \gamma\right) &\approx \sum_{k=0}^{K} \frac{s^{(k)}(t)}{k!} \left(\frac{2L}{c}\right)^k \gamma^k, \\
\bar{s}^{(1)} \left(t + \frac{2L}{c} \gamma\right) &\approx \sum_{k=0}^{K} \frac{s^{(k+1)}(t)}{k!} \left(\frac{2L}{c}\right)^k \gamma^k.
\end{align*}
\]
(46)

(47)

More strictly, given the size of target \( L \), the integer \( K \) is chosen large enough such that

\[
\begin{align*}
\|\text{FIM}^{(K)} - \text{FIM}\| < \|\text{FIM}^{-1}\|^{-1},
\end{align*}
\]
(48)

where \( \| \cdot \| \) is a norm of a matrix. Inequality (48) means that \( \text{FIM}^{(K)} \) is close enough to \( \text{FIM} \) and, thus, makes the approximations reasonable. According to (45), a larger \( K \) is required as the size of target increases or the more accurate results are needed. The inverse of Fisher information matrix \( \text{FIM}^{-1} \) can be written as (13)

\[
\text{FIM}^{-1} = \left(\text{FIM}^{(K)}\right)^{-1} + O\left(\text{E}^{(K)}\right) \approx \left(\text{FIM}^{(K)}\right)^{-1},
\]
(49)

as \( K \to +\infty \), where

\[
\begin{align*}
O\left(\text{E}^{(K)}\right) &= -\sum_{n=1}^{+\infty} \left(\text{FIM}^{-1}\text{E}^{(K)}\right)^n \text{FIM}^{-1}.
\end{align*}
\]
(50)

Notation \( f(x) = O(g(x)) \), as \( x \to x_0 \) means that there exists a constant \( C_3 \neq 0 \), such that \( f(x)/g(x) \to C_3 \), as \( x \to x_0 \).

The error on the inverse \( \text{FIM} \) due to approximation (49) is calculated by

\[
\begin{align*}
\left\|\left(\text{FIM}^{(K)}\right)^{-1} - \text{FIM}^{-1}\right\| < \frac{\left\|\text{FIM}^{-1}\right\|^2 \left\|\text{E}^{(K)}\right\|}{1 - \left\|\text{FIM}^{-1}\text{E}^{(K)}\right\|},
\end{align*}
\]
(51)

which indicates that the error decreases if a larger \( K \) is used. The approximations of CRLBs are given by

\[
\begin{align*}
\text{var}(\tau) \geq \text{CRLB}_{\tau}^{(K)} &= \frac{a_{22}^{(K)}}{a_{11}^{(K)} a_{12}^{(K)} - a_{11}^{(K)} a_{12}^{(K)}},
\end{align*}
\]
(52)

\[
\begin{align*}
\text{var}(g) \geq \text{CRLB}_{g}^{(K)} &= \frac{a_{11}^{(K)} - a_{11}^{(K)} a_{22}^{(K)} - a_{12}^{(K)}}{a_{11}^{(K)} a_{12}^{(K)} - a_{11}^{(K)} a_{12}^{(K)}},
\end{align*}
\]
(53)

where

\[
\begin{align*}
a_{ij}^{(K)} &= F_{ij}^{(K)} - \left(F_{31}^{(K)}\right)^T \left(F_{33}^{(K)}\right)^{-1} F_{3j}^{(K)}, \ i, j = 1, 2.
\end{align*}
\]
(54)

Given a particular target (can be an extended target) with size \( L \), we can calculate the approximate CRLBs with following the two steps:

**Step 1** Select a \( K \), which relies on \( L \), such that Assumption 5 holds.

**Step 2** Substitute the selected \( K \) into (52)-(53).

We consider a special case where \( K = 1 \), which means the extension of the target \( L \) is not large, then

\[
\begin{align*}
a_{11}^{(1)} &= \frac{2\gamma M_0^{(1)}}{N_0} g_1,
\end{align*}
\]
(55)

\[
\begin{align*}
a_{12}^{(1)} &= -\frac{2M_1^{(1)}}{N_0} g_1 - \frac{M_0^{(1)} M_0^{(1)}}{N_0} x^T \Gamma^{(1)} \Lambda^{-1} 1_p x,
\end{align*}
\]
(56)

\[
\begin{align*}
a_{22}^{(1)} &= \frac{2M_1^{(1)}}{\gamma^3 N_0} \bar{x}^2 - \frac{1}{2\gamma^3 N_0} x^T \Gamma^{(1)} \Lambda^{-1} g_2.
\end{align*}
\]
(57)

where \( g_1 = \bar{x}^2 + \gamma^2 M_0^{(1)} x^T \Gamma^{(1)} \Lambda^{-1} 1_p x \), \( \bar{x} = \sum_{p=1}^P x_p \) and \( g_2 = M_0^{(1)} 1_p x + 2\gamma M_1^{(1)} \Gamma^{(1)} x \).
Compared with the results in the integral form (53)–(57) and (29)–(31), formulas (53)–(57) are easier to calculate. The integrals required to compute are only \( M_i^{(0)} \), \( M_i^{(1)} \) and \( M_2^{(1)} \) that do not depend on the time delays and Doppler stretch.

C. A special case: \( P = 1 \)

In previous subsections, we derive the CRLBs for an extended target (29)–(31) in the integral or series representations, respectively. In addition, formulas (53)–(54) can be used to approximate the theoretical CRLBs. When \( P = 1 \), the extended target reduces to a single scatterer, and the scattering coefficients \( x \) reduce to a scalar \( x \). Substitute \( P = 1 \) into (29)–(31), or equivalently into (35)–(42), and then the CRLBs of a single scatterer target are obtained as

\[
\text{var}_{\rho = 1}(\tau) \geq \text{CRLB}_{\tau, \rho = 1} = \frac{\tilde{a}_{22}}{\tilde{a}_{11} \tilde{a}_{22} - \tilde{a}_{12}^2}, \quad (58)
\]

\[
\text{var}_{\rho = 1}(\gamma) \geq \text{CRLB}_{\gamma, \rho = 1} = \frac{\tilde{a}_{11}}{\tilde{a}_{11} \tilde{a}_{22} - \tilde{a}_{12}^2}, \quad (59)
\]

where

\[
\tilde{a}_{22} = \frac{2\gamma x}{\gamma N_0} \left( M_2^{(1)} - \frac{M_0^{(0)}}{4} \right), \quad (60)
\]

\[
\tilde{a}_{12} = \frac{2\gamma^2 M_0^{(1)}}{\gamma N_0}, \quad (61)
\]

\[
\tilde{a}_{11} = \frac{2\gamma x^2 M_0^{(1)}}{N_0}, \quad (62)
\]

These results accord with the previous ones in [4], where the CRLBs are calculated in the case that the amplitude \( x \) of signal is assumed to be known. The only difference lies in the representation of \( \tilde{a}_{22} \), which is \( \frac{2\gamma x}{\gamma N_0} M_2^{(1)} \) under the assumption of a known amplitude [4].

According to (53)–(57), waveform parameters that have influences on CRLBs are \( M_0^{(0)} \), \( M_0^{(1)} \), \( M_1^{(1)} \) and \( M_2^{(1)} \), which relate to the energy, effective bandwidth and effective duration of transmitted signal. To investigate the influences of those parameters on CRLBs, we assume that \( \bar{B}, \bar{T} \) are independent of \( M_0^{(0)} \), which holds if the alteration of \( M_0^{(0)} \) results from changing the amplitude of transmitted signal. According to the Cauchy inequality [14] and \( s(t) = 0, t \notin [0, T] \), we have

\[
\left( \frac{1}{T} M_2^{(1)} \right)^2 \leq \left( M_1^{(1)} \right)^2 \leq M_0^{(1)} M_2^{(1)}, \quad (63)
\]

which implies

\[
\frac{\bar{T}^2}{\bar{T}^2} \leq \left( M_1^{(1)} \right)^2 \leq M_0^{(1)} M_2^{(1)} \leq 1. \quad (64)
\]

From (64), it is reasonable to consider

\[
\left( M_1^{(1)} \right)^2 = C_4 M_0^{(1)} M_2^{(1)}, \quad (65)
\]

where \( C_4 > 0 \) is a constant. Notice that \( M_1^{(1)} = M_0^{(0)} \bar{B}^2 \), \( M_2^{(1)} = M_0^{(0)} \bar{B}^2 \bar{T}^2 \). Substitute (60), (62) and (65) into (58)–(59), and when \( M_0^{(0)} \), \( \bar{B}, \bar{T} \rightarrow +\infty \), we have

\[
\text{CRLB}_{\tau, \rho = 1} = \frac{N_0}{2\gamma x^2} \left( \frac{M_2^{(1)} - M_0^{(0)}}{M_0^{(0)} - \left( M_1^{(1)} \right)^2} \right)^2 \left( \frac{O(M_0^{(0)} \bar{B}^2 \bar{T}^2)}{O(M_0^{(0)} \bar{B}^2)} \right) = O \left( \left( M_0^{(0)} \right)^{-1} \bar{B}^{-2} \right). \quad (66)
\]

\[
\text{CRLB}_{\gamma, \rho = 1} = \frac{\gamma^2 N_0}{2\bar{T}^2} \left( \frac{M_0^{(1)}}{M_0^{(0)} - \left( M_1^{(1)} \right)^2} \right)^2 \left( \frac{O(M_0^{(0)} \bar{B}^2 \bar{T}^2)}{O(M_0^{(0)} \bar{B}^2)} \right) = O \left( \left( M_0^{(0)} \right)^{-1} \bar{B}^{-2} \bar{T}^{-2} \right). \quad (67)
\]

From the definition of the \( y = O(x) \) mentioned in Subsection III-B [66] and (67) imply that there exists a constant \( C_5 > 0 \), such that

\[
0 < M_0^{(0)} \bar{B}^2 \cdot \text{CRLB}_{\tau, \rho = 1} < C_5, \quad (68)
\]

\[
0 < M_0^{(0)} \bar{B}^2 \bar{T}^2 \cdot \text{CRLB}_{\gamma, \rho = 1} < C_5, \quad (69)
\]

as \( M_0^{(0)}, \bar{B}, \bar{T} \rightarrow +\infty \). Note that both \( \text{CRLB}_{\tau, \rho = 1} \) and \( \text{CRLB}_{\gamma, \rho = 1} \) are positive due to the positive definite property of a CRLB matrix. From (68) and (69), the following conclusions are indicated:

1) The waveform parameters that have influences on CRLBs of time delay and Doppler stretch are the energy, effective bandwidth and effective duration.

2) The energy of transmitted signal is negatively correlated to the CRLBs of time delay and Doppler stretch.

3) There exists a negative correlation between the effective bandwidth and the CRLB of time delay.

4) There exists a negative correlation between the effective time-bandwidth product \( \bar{B} \bar{T} \) and the CRLB of Doppler stretch.

D. Discussions on waveform parameters

In this subsection, discussions about influences of waveform parameters on CRLBs in the case of a single scatterer are generalized to the extended target situation. It is worth mentioning that the alteration of effective bandwidth or effective duration results in changes of \( M_i^{(k)} \), \( i = 0, 1, 2, k \geq 2 \), which also affect the CRLBs. Therefore, the influences of \( B \) and \( T \) on CRLBs are partly reflected in other waveform parameters. Observing the leading terms (53)–(57) in the series representations of CRLBs, we find that these terms only
contain \( M_i^{(0)}, M_i^{(1)} \) and have no immediate relations with \( M_i^{(k)}, k \geq 2 \). Thus, it is believable that for an extended target, the energy, effective bandwidth and effective duration have more significant effects on the CRLBs than \( M_i^{(k)} \) or \( M_i^{(k)}/M_i^{(1)}, k \geq 2 \). To simplify the discussion, we assume that 1) \( M_i^{(k)}/M_i^{(0)} \) are independent of \( M_i^{(0)}, i = 0, 1, 2 \) and \( k \in \mathbb{N}^+ \), 2) \( (M_i^{(1)})^2 = O(M_i^{(1)} M_2^{(1)}) \), 3) the influences of \( M_i^{(k)}/M_i^{(1)} \) on CRLBs are negligible, \( i = 0, 1, 2 \) and \( k \geq 2 \), 4) \( T \) and \( B \) are mutually independent. Note that the first two assumptions are similar to those mentioned in Subsection III-C. Since these assumptions are not strictly examined, the discussions presented in this subsection are qualitative, but can still give some meaningful insight.

Referring to (35)-(42), we have

\[
F_{11} = M_0^{(1)} \sum_{k=0}^{4} \frac{(-1)^k k^{2k+1} M_0^{(k+1)}}{(2k)! N_0 M_0^{(1)}} x T^{2k+1} \mathbf{1},
\]

\[
= O(M_0^{(1)}) = O(M_0^{(0)}/B^2), \quad (70)
\]

\[
F_{12} = O(M_1^{(1)}) = O(M_0^{(0)}/B^2 \tilde{1}), \quad (71)
\]

\[
F_{22} = O(M_2^{(1)}) = O(M_0^{(0)}/B^2 \tilde{1}^2), \quad (72)
\]

\[
F_{31} = O(M_0^{(1)}/B^2 \mathbf{1}) = O(M_0^{(0)}/B^2 \mathbf{1}), \quad (73)
\]

\[
F_{32} = O(M_0^{(1)}/B^2 \mathbf{1}), \quad (74)
\]

\[
F_{33} = O(M_0^{(1)}/B^2 \mathbf{1}), \quad (75)
\]

Substitute (70)-(75) into (31), and we have

\[
a_{11} = O(M_0^{(0)}/B^2) - O(M_0^{(0)}/B^2 \mathbf{1}^T (\mathbf{1}^T)^T) \times
\]

\[
O\left(\frac{1}{M_0^{(0)/B^2}} \bar{A}^{-1}\right) O\left(M_0^{(0)/B^2} \mathbf{1} \right)
\]

\[
= O(M_0^{(0)/B^2}) + O(M_0^{(0)/B^2} \mathbf{1}^T \bar{A}^{-1} \mathbf{1})
\]

\[
= O(M_0^{(0)/B^2}), \quad (76)
\]

\[
a_{12} = O(M_0^{(0)/B^2} \bar{1}) - O(M_0^{(0)/B^2} \mathbf{1}^T (\mathbf{1}^T)^T) \times
\]

\[
O\left(\frac{1}{M_0^{(0)/B^2}} \bar{A}^{-1}\right) O\left(M_0^{(0)/B^2} \bar{1} \mathbf{1} \right)
\]

\[
= O(M_0^{(0)/B^2} \bar{1}) + O(M_0^{(0)/B^2} \mathbf{1}^T \bar{A}^{-1} \mathbf{1})
\]

\[
= O(M_0^{(0)/B^2} \bar{1}), \quad (77)
\]

Similarly,

\[
a_{22} = O(M_0^{(0)}/B^2 \tilde{1}^2). \quad (78)
\]

Substituting (76)-(78) into (29) and (30), we obtain

\[
\text{CRLB}_{\tilde{r}} = O\left(M_0^{(0)}/B^2 - \tilde{1}^2\right), \quad (79)
\]

\[
\text{CRLB}_{\gamma} = O\left(M_0^{(0)}/B^2 - \tilde{1}^2\tilde{1}^2\right), \quad (80)
\]

which are the same as (60) and (61), respectively. Thus, we obtain similar arguments:

1) The energy of transmitted signal is negatively correlated to the CRLBs of time delay and Doppler stretch.

2) Qualitatively, there exists a negative correlation between the CRLB of time delay and the effective bandwidth.

3) Qualitatively, the CRLB of Doppler stretch is negatively correlated to the effective time-bandwidth product.

For the narrowband signal \(\tilde{f} = f_c\) and \(\tilde{T} = \sqrt{\gamma} \tilde{T}\), we have

\[
\text{CRLB}_{\tilde{r}} = O\left(M_0^{(0)}/B^2 - f_c^2\right), \quad (81)
\]

\[
\text{CRLB}_{\gamma} = O\left(M_0^{(0)}/B^2 - f_c^2 \tilde{T} \tilde{T}\right). \quad (82)
\]

The Doppler shift is defined by \( f_d = \gamma f_c - f_c \). According to (11), the CRLB of Doppler shift is calculated by

\[
\text{CRLB}_{f_d} = f_c^2 \text{CRLB}_{\gamma} = O\left(M_0^{(0)}/B^2 - f_c^2 \tilde{T} \tilde{T}\right). \quad (83)
\]

It indicates that under the narrowband assumption, there exists a negative correlation between the CRLB of Doppler shift and duration.

### IV. Simulation

In this section, we compare the performances of estimators with the derived CRLBs and give several numerical examples on the properties of CRLBs.

In the case where a narrowband signal is transmitted, the standard method to estimate time delay and Doppler stretch is using the ambiguity function (AF) [1], [2], which is asymptotically efficient, that is, the estimator is unbiased and reaches the CRLB when the number of independent observations approaches infinity [5]. For a wide-band model, when the observed target has only a single scatterer, the wide-band ambiguity function (WBAF) applies [3], [16]. It is shown in [4] that under high SNRs, the WBAF estimator is asymptotically unbiased and the variances are close to the CRLBs for a large variety of signals. In this paper, we continue to use the WBAF-based estimator for an extended target. Numerical results are presented in this subsection and theoretical analyses on the achievability of CRLBs are remained for future work.

The WBAF, suggested by [8], is

\[
W_{s,s_d}(\tau, \gamma) = \sqrt{\gamma} \int_{-\infty}^{+\infty} s_r(t)^{s_d}(\gamma(t-\tau))dt \quad (84)
\]
where $s_r$ and $s_d$ are the received and reference signals, respectively.

The received signal $s_r$ is modeled as $\{1\}$, and the reference $s_d$ is chosen separately for different estimators.

Oracle matched filter:

$$[\hat{\tau}_s, \hat{\gamma}_s] = \arg\max_{\tau,\gamma} W_{s_r,s_d}$$

with $s_d = \sum_{p=1}^P x_p s(\gamma(t - \tau_p))$.

WBAF estimator:

$$[\hat{\tau}, \hat{\gamma}] = \arg\max_{\tau,\gamma} W_{x_r,x_d}$$

with $s_d = s(\gamma(t - \tau_p))$, where $[x, y] = \arg\max f(x, y)$ means the pair $(x, y)$ is the maximizer of function $f(x, y)$.

The estimations $[\hat{\tau}_s, \hat{\gamma}_s]$ are ideal but untractable in practice, because the number of scatterers $P$ and the scattering coefficients $x$ are unknown. The oracle matched filter is introduced as a comparison to discuss the properties of CRLBs. In actual scenarios, the WBAF estimator $[\hat{\tau}, \hat{\gamma}]$ is often applied.

The CRLBs and mean square errors (MSEs) of $\hat{\tau}_s$ and $\hat{\gamma}_s$ versus various SNRs are shown in Fig.1 and Fig.2. The number of scatterers are selected as $P = 4$ and 16 for Fig.1 and Fig.2 respectively. All of the scattering coefficients $x_p$ are assumed to equal 1. The time delay $\tau = 2 \times 10^{-4} \text{s}$ and the Doppler stretch $\gamma = 1/1.06$. The source signal $s(t)$ is a monopulse Chirp signal, time-limited to $[0, 5 \times 10^{-5} \text{s}]$ and band-limited to $[0, 1.28 \times 10^5 \text{Hz}]$, that is,

$$s(t) = \cos(2\pi at^2)[u(t) - u(t - T)],$$

where $a = 2.56 \times 10^9 \text{Hz/s}$, $T = 5 \times 10^{-5} \text{s}$ and $u(t)$ is the unit step function. The SNR is defined as

$$\text{SNR} = \frac{1}{N_0} \int_{-\infty}^{+\infty} \left( \sum_{p=1}^P x_p s(\gamma(t - \tau_p)) \right)^2 dt = \frac{1}{\gamma N_0} x^T \Lambda x$$

and is changed by altering $N_0$. The sampling interval $\Delta = 6.25 \times 10^{-8} \text{s}$. The CRLBs are calculated by (22)-(27). The MSEs are computed with 100 independent Monte Carlo trials.

As presented in Fig.1 and Fig.2, the MSEs of $\hat{\tau}_s$ are smaller than the corresponding CRLBs when the SNR is relatively large (e.g. larger than 26dB when $P = 4$) and the reason is that the Oracle matched filter $[\hat{\tau}_s, \hat{\gamma}_s]$ assumes that all of the $x_p$ are known and thus the number of unknown parameters decreases. In addition, the MSEs of $\hat{\tau}$ gradually deviate from the corresponding CRLBs, indicating that the WBAF estimator $[\hat{\tau}, \hat{\gamma}]$ is not appropriate under high SNRs. Comparing Fig.1 and Fig.2, we find that under high SNRs, the performances of estimators are significantly affected by the number of scatterers.

The approximation formulas of CRLBs (52)-(54) are compared with the theoretical CRLBs (29)-(31). The comparisons are presented in Fig.3 with $P = 4$ and 100, respectively. The approximate CRLBs are calculated by (55)-(57) which is a special case of (52)-(54) where $K = 1$. Other parameters are the same as those for Fig.1. It is also indicated that the approximate CRLBs are accurate in the case of small target ($P = 4$) and become invalid when the target is large ($P = 100$). This is because the Assumption 5 does not hold for $K = 1$ in the case of a large target. According to (45), a larger $K$ is required as the size of target increases.

The influences of the size of target on the CRLBs are shown in Fig.4 and Fig.5 where $P = 1, 4, 16$ and 100, the other parameters are the same as those for Fig.1. The CRLBs are calculated with (22)-(27). It indicates that the CRLBs are higher and therefore, the performance limits of estimators become worse, when the size of target increases.

The influences of effective bandwidth on the CRLBs of time delay are shown in Fig.6 where $a$ changes from $0.256 \times 10^9 \text{Hz/s}$ to $2.560 \times 10^9 \text{Hz/s}$ and other parameters are the same as those for Fig.1. The effective bandwidth $B$ increases from $0.7604 \times 10^5 \text{Hz}$ to $9.0884 \times 10^5 \text{Hz}$. The effective duration $T$ increases from $0.3549 \times 10^{-4} \text{s}$ to $0.3893 \times 10^{-4} \text{s}$ and can be
considered as almost unchanged. The CRLBs are calculated with (22)-(27). These numerical results demonstrate that the CRLB of time delay is negatively correlated to the effective bandwidth of transmitted signal.

Two experiments are implemented to demonstrate the relation between the time-bandwidth product and CRLBs of Doppler stretch. In the first one, $BT$ changes and $T$ keeps constant. In the second one, $BT$ keeps constant and $T$ varies. The results are depicted in Fig.7 and Fig.8, respectively. Note that the effective time-bandwidth product $\bar{B}\bar{T}$ is proportional to $aT^2$ for a Chirp signal. In Fig.4, $a$ changes from $0.256 \times 10^6$Hz/s to $2.560 \times 10^6$Hz/s and other parameters are the same as those for Fig.1. The effective time-bandwidth product $\bar{B}\bar{T}$ increases from $2.6988 \times 10^{-5}s$ to $35.3786$. The effective duration $T$ increases from $3.549 \times 10^{-5}s$ to $3.893 \times 10^{-5}s$ and can be considered as almost unchanged. These parameters are designed similarly to those for Fig.6. In Fig.8 $aT^2 \equiv 6.4$, $T$ increases from $1.5 \times 10^{-5}s$ to $5 \times 10^{-5}s$ and other parameters are the same as those for Fig.1 implying that $\bar{T}$ increases from $1.1678 \times 10^{-5}s$ to $3.8927 \times 10^{-5}s$ and $\bar{B}\bar{T} \equiv 35.3786$. The CRLBs in both figures are calculated with (22)-(27). Combining Fig.7 with Fig.8 we find 1) there exists a negative correlation between the CRLB of Doppler stretch and the effective time-bandwidth product, 2) the relation between the CRLB of Doppler stretch and the effective duration is not apparent.

V. CONCLUSION

In this paper, the integral and series representations of CRLBs for an extended target are derived. According to the series formulas, approximations of CRLBs are obtained. Both theoretical analyses and numerical examples indicate that the CRLBs of time delay and Doppler stretch are negatively
correlated to the effective bandwidth and the effective time-bandwidth product, respectively. In addition, compared with a single scatterer, an extended target consisting of multiple scatterers leads to higher CRLBs under the same SNR level.

**APPENDIX**

**Proof of Theorem 1**

**Lemma 1:** For real-valued functions \(a(t)\) and \(b(t)\) satisfying Assumption 3, the following equation holds:

\[
\int_{-\infty}^{+\infty} a(t)b^{(1)}(t)dt = -\int_{-\infty}^{+\infty} a^{(1)}(t)b(t)dt. \tag{89}
\]

**Proof:** Integrating by parts, we have

\[
\int_{-\infty}^{+\infty} a(t)b^{(1)}(t)dt = \lim_{t \to +\infty} a(t)b(t) - \lim_{t \to -\infty} a(t)b(t)
\]

\[
-\int_{-\infty}^{-\infty} a^{(1)}(t)b(t)dt + \int_{-\infty}^{+\infty} a^{(1)}(t)b(t)dt
\]

\[
= -\int_{-\infty}^{+\infty} a^{(1)}(t)b(t)dt. \tag{90}
\]

Note that \(\int_{-\infty}^{+\infty} f(t)dt < +\infty\) implies \(\lim_{t \to \infty} f(t) = 0\), which is employed in the second equality in (90).

By using Lemma 1, Theorem 1 is proved as following.

**Proof of formula (89).**

**Proof:** Write \(s(t)\) in the form of \(u(t) + iv(t)\), where \(i^2 = -1\). Then, applying (90), for any \(m = 2k, k \in \mathbb{N}^+\), we have

\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} s^{(0)}(t)s^{(m)}(t)dt \right\}
\]

\[
= \int_{-\infty}^{+\infty} u(t)u^{(2k)}(t) + v(t)v^{(2k)}(t)dt
\]

\[
= (-1) \int_{-\infty}^{+\infty} u^{(1)}(t)u^{(2k-1)}(t) + v^{(1)}(t)v^{(2k-1)}(t)dt
\]

\[
= (-1)^2 \int_{-\infty}^{+\infty} u^{(2)}(t)u^{(2k-2)}(t) + v^{(2)}(t)v^{(2k-2)}(t)dt
\]

\[
... = (-1)^k \int_{-\infty}^{+\infty} u^{(k)}(t)u^{(k)}(t) + v^{(k)}(t)v^{(k)}(t)dt
\]

\[
= (-1)^k M_0^{(k)}. \tag{91}
\]

Similarly, for any \(m = 2k + 1, k \in \mathbb{N},\)

\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} s^{(0)}(t)s^{(m)}(t)dt \right\}
\]

\[
= (-1)\text{Re} \left\{ \int_{-\infty}^{+\infty} s^{(1)}(t)s^{(2k+1)}(t)dt \right\}
\]

\[
= (-1)^{2k+1} \text{Re} \left\{ \int_{-\infty}^{+\infty} s^{(2k+1)}(t)s^{(0)}(t)dt \right\}
\]

\[
= -\text{Re} \left\{ \int_{-\infty}^{+\infty} s^{(0)}(t)s^{(m)}dt \right\}, \tag{92}
\]

which implies

\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} s^{(0)}(t)s^{(2k+1)}(t)dt \right\} = 0. \tag{93}
\]
Similarly to the technique used in (91), for any \( p, q \in \mathbb{N} \), the formula (8) is derived as
\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} s^{(p)}(t)s^{(q)}(t)dt \right\} = \begin{cases} (-1)^p \text{Re} \left\{ \int_{-\infty}^{+\infty} s^{(p)}(t)s^{(q)}(t)dt \right\} & \text{for } p + q = 2k. \\ 0, & \text{for } p + q = 2k + 1. \end{cases}
\]
(94)

**Proof of formula (9).**

**Proof:** For \( m, n \in \mathbb{N}^+ \) and \( n \leq m \), using (89) and (94), we have
\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} ts^{(p)}(t)s^{(m)}(t)dt \right\} = (-1)^n \text{Re} \left\{ \int_{-\infty}^{+\infty} ts^{(p)}(t)s^{(m-n)}(t)dt \right\} + \sum_{l=0}^{n-1} (-1)^{l+1} \text{Re} \left\{ \int_{-\infty}^{+\infty} s^{(l)}(t)s^{(m-l-1)}(t)dt \right\}.
\]
(95)

By making use of formula (223), the second term in the last line of equation (95) becomes
\[
\sum_{l=0}^{n-1} (-1)^{l+1} \text{Re} \left\{ \int_{-\infty}^{+\infty} s^{(l)}(t)s^{(m-l-1)}(t)dt \right\} = \left\{ \begin{array}{ll} (-1)^{k+1} n M_0^{(k)}, & m - 1 = 2k. \\ 0, & m - 1 = 2k + 1. \end{array} \right.
\]
(96)

Thus, if \( m = 2k \), let \( n = k \), and (95) becomes
\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} ts^{(0)}(t)s^{(2k)}(t)dt \right\} = (-1)^k M_1^{(k)},
\]
(97)

if \( m = 2k + 1 \), let \( n = m \), and (95) becomes
\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} ts^{(0)}(t)s^{(2k+1)}(t)dt \right\} = (-1)^{(2k+1)} \text{Re} \left\{ \int_{-\infty}^{+\infty} ts^{(2k+1)}(t)s^{(0)}(t)dt \right\} + (-1)^{k+1}(2k + 1) M_0^{(k)},
\]
(98)

which implies
\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} ts^{(0)}(t)s^{(2k+1)}(t)dt \right\} = (-1)^{k+1}(k + \frac{1}{2}) M_0^{(k)}.
\]
(99)

Apply (94), (97) and (99). For any \( p, q \in \mathbb{N} \), formula (9) is derived as
\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} ts^{(p)}(t)s^{(q)}(t)dt \right\} = (-1)^{p} \text{Re} \left\{ \int_{-\infty}^{+\infty} ts^{(p-1)}(t)s^{(q+1)}(t)dt \right\} + (-1)^{p} \text{Re} \left\{ \int_{-\infty}^{+\infty} ts^{(p)}(t)s^{(p+1)}(t)dt \right\}
\]
\[
= (-1)^{p} \text{Re} \left\{ \int_{-\infty}^{+\infty} ts^{(p-1)}(t)s^{(q+1)}(t)dt \right\} + (-1)^{p} \text{Re} \left\{ \int_{-\infty}^{+\infty} ts^{(p)}(t)s^{(p+1)}(t)dt \right\}
\]
\[
= \left\{ \begin{array}{ll} (-1)^{p+k} M_1^{(k)}, & m - 1 = 2k, \\ (-1)^{p+k}(p - k - \frac{1}{2}) M_0^{(k)}, & m - 1 = 2k + 1. \end{array} \right.
\]
(100)

**Proof of formula (10).**

**Proof:** For \( m, n \in \mathbb{N}^+ \) and \( n \leq m \), using (89) and (100), we have
\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} t^2 s^{(0)}(t)s^{(m)}(t)dt \right\} = (-1)\text{Re} \left\{ \int_{-\infty}^{+\infty} t^2 s^{(1)}(t)s^{(m-1)}(t)dt \right\} + (-1)^2 \text{Re} \left\{ \int_{-\infty}^{+\infty} t^2 s^{(2)}(t)s^{(m-2)}(t)dt \right\}
\]
\[
= \left\{ \begin{array}{ll} (-1)^{k+1} n M_0^{(k)}, & m - 1 = 2k, \\ 0, & m - 1 = 2k + 1. \end{array} \right.
\]
(101)
By making use of formula (100), the second term in the last line of equation (101) becomes
\[
\sum_{l=0}^{n-1} (-1)^{l+1} 2 \text{Re} \left\{ \int_{-\infty}^{+\infty} t s^{(l)}(t) s^{(m-l-1)}(t) dt \right\} = \begin{cases} 
(1)^{k+1} 2 n M_{1}^{(k)}, & m - 1 = 2k, \\
(1)^{k+1} (n^2 - 2(k + 1)n) M_{0}^{(k)}, & m - 1 = 2k + 1.
\end{cases}
\] (102)

Thus, if \( m = 2k \), let \( n = k \), and (101) becomes
\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} t^2 s^{(0)}(t) s^{(m)}(t) dt \right\} = (-1)^{k} M_{2}^{(k)} + (-1)^{k+1} k^2 M_{0}^{(k-1)},
\] (103)
if \( m = 2k + 1 \), let \( n = m \), and (101) becomes
\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} t^2 s^{(0)}(t) s^{(m)}(t) dt \right\} = (-1)^{2k+1} \text{Re} \left\{ \int_{-\infty}^{+\infty} t^2 s^{(m)}(t) s^{(0)}(t) dt \right\} + (-1)^{k+1} 2(2k + 1) M_{1}^{(k)},
\] (104)
which implies
\[
\text{Re} \left\{ \int_{-\infty}^{+\infty} t^2 s^{(0)}(t) s^{(m)}(t) dt \right\} = (-1)^{k+1} (2k + 1) M_{1}^{(k)}.
\] (105)

Combining (103) and (105) gives formula (10).

\[\blacksquare\]

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