Optimal estimates for harmonic functions in the unit ball

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Abstract We find the sharp constants $C_p$ and the sharp functions $C_p = C_p(x)$ in the inequality

$$|u(x)| \leq \frac{C_p}{(1 - |x|^2)^{(n-1)/p}} \|u\|_{h^p(B^n)}, u \in h^p(B^n), x \in B^n,$$

in terms of Gauss hypergeometric and Euler functions. This extends and improves some results of Axler et al. (Harmonic function theory, New York, 1992), where they obtained similar results which are sharp only in the cases $p = 2$ and $p = 1$.

Keywords Harmonic functions · Hardy spaces

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1 Introduction and statement of the results

Let $n \geq 2$ and let $h^p(B^n)$, $1 \leq p \leq \infty$, be the harmonic Hardy spaces on the unit $n$-dimensional ball $B^n$ in the Euclidean space $\mathbb{R}^n$, the space of all harmonic functions $u$ satisfying growth condition...
\[ \|u\|_p^p := \|u\|_{h^p(B^n)}^p = \sup_{0<r<1} \int_S |u(r\zeta)|^p d\sigma(\zeta) < \infty \]

where \( S = S^{n-1} \) is the unit sphere and \( \sigma \) is the unique normalized rotation invariant Borel measure on \( S \). It is well known that a harmonic function \( u \in h^p(B^n) \) possesses radial (angular) limit \( u(\zeta) \) in almost all points on sphere \( \zeta \in S^{n-1} \) and that for \( p > 1 \) it is possible to express in the form

\[ u(x) = \int_S P(x, \zeta) u(\zeta) d\sigma(\zeta), \quad (1.1) \]

where

\[ P(x, \zeta) = \frac{1 - |x|^2}{|x - \zeta|^n}, \quad \zeta \in S \]

is Poisson kernel.

The maximum principle implies that, if \( u \in h^\infty(B^n) \), then \( |u(x)| \leq \|u\|_\infty \). On the other hand, it follows from the Poisson representation formula (1.1) that, if \( u \in h^1(B^n) \), then

\[ |u(x)| \leq \sup_{\zeta \in S} P(x, \zeta) \|u\|_1. \]

Then

\[ \sup_{\zeta \in S} P(x, \zeta) = \frac{(1 + |x|^n)}{(1 - |x|^2)^{n-1}}. \]

In this work we find a representation for the sharp constants \( C_p \) and the sharp functions \( C_p = C_p(x) \) in the inequality

\[ |u(x)| \leq \frac{C_p}{(1 - |x|^2)^{(n-1)/p}} \|u\|_p \]

where \( x \) is an arbitrary point in the unit ball \( B^n \).

It is well known that \( C_p(x) \) is a bounded function in \( B^n \) for \( 1 \leq p \leq \infty \), and the power \( (n-1)/p \) is optimal. See [2, Proposition 6.16] for the case \( n \geq 2 \) and \( 1 \leq p \leq \infty \) and [9, Lemma 5.1.1] for the case of analytic functions \( n = 2 \) and \( 0 < p < \infty \).

In the case when \( h^p(B^n) \) is Hilbert space, that is for \( p = q = 2 \) we have next sharp point estimate

\[ |u(x)| \leq \sqrt{\frac{1 + |x|^2}{(1 - |x|^2)^{n-1}} \|u\|_{h^2(B^n)}}. \quad (1.2) \]