TOPOLOGICAL STABLE RANK OF $\mathcal{E}'(\mathbb{R})$

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Abstract. The set $\mathcal{E}'(\mathbb{R})$ of all compactly supported distributions, with the operations of addition, convolution, multiplication by complex scalars, and with the strong dual topology is a topological algebra. In this article, it is shown that the topological stable rank of $\mathcal{E}'(\mathbb{R})$ is 2.

1. Introduction

The aim of this article is to show that the topological stable rank (a notion from topological $K$-theory, recalled below) of $\mathcal{E}'(\mathbb{R})$ is 2, where $\mathcal{E}'(\mathbb{R})$ is the classical topological algebra of compactly supported distributions, with the strong dual topology $\beta(\mathcal{E}, \mathcal{E}')$, pointwise vector space operations, and convolution taken as multiplication.

We recall some key notation and facts about $\mathcal{E}'(\mathbb{R})$ in Section 2 below, including its strong dual topology $\beta(\mathcal{E}, \mathcal{E}')$, and in Section 3 we will recall the notion of topological stable rank of a topological algebra.

We will prove our main result, stated below, in Sections 4 and 5.

Theorem 1.1.

Let $\mathcal{E}'(\mathbb{R})$ be the algebra of all compactly supported distributions on $\mathbb{R}$, with

- pointwise addition, and pointwise multiplication by complex scalars,
- convolution taken as the multiplication in the algebra, and
- the strong dual topology $\beta(\mathcal{E}, \mathcal{E}')$.

Then the topological stable rank of $\mathcal{E}'(\mathbb{R})$ is equal to 2.

2. The Topological Algebra $\mathcal{E}'(\mathbb{R})$

For background on topological vector spaces and distributions, we refer to [2, 5, 9, 10, 11] and [13].

Let $\mathcal{E}(\mathbb{R}) = C^\infty(\mathbb{R})$ be the space of functions $\varphi : \mathbb{R} \to \mathbb{C}$ that are infinitely many times differentiable. We equip $\mathcal{E}(\mathbb{R})$ with the topology of uniform convergence on compact sets for the function and its derivatives. This is defined by the following family of seminorms: for a compact subset $K$ of $\mathbb{R}$, and $M \in \{0, 1, 2, 3 \cdots \} = \mathbb{Z}_{\geq 0}$, we define

$$p_{K,M}(\varphi) = \sup_{0 \leq m \leq M} \sup_{x \in K} |\varphi^{(m)}(x)| \text{ for } \varphi \in \mathcal{E}(\mathbb{R}).$$

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The space \( \mathcal{E}(\mathbb{R}) \) is metrizable, and is a Fréchet and a Montel space [4, Example 3, p.239].

We equip the dual space \( \mathcal{E}'(\mathbb{R}) \) of \( \mathcal{E}(\mathbb{R}) \) with the strong dual topology \( \beta(\mathcal{E}', \mathcal{E}) \), defined by the seminorms

\[
p_B(T) = \sup_{\varphi \in B} |\langle T, \varphi \rangle|,
\]

for bounded subsets \( B \) of \( \mathcal{E}'(\mathbb{R}) \). Then \( \mathcal{E}'(\mathbb{R}) \), being the strong dual of the Montel space \( \mathcal{E}(\mathbb{R}) \), is a Montel space too [10, 5.9, p.147].

Let \( \mathcal{D}'(\mathbb{R}) \) denote the space of all compactly supported functions from \( C^\infty(\mathbb{R}) \), and \( \mathcal{D}'(\mathbb{R}) \) denote, as usual, the space of all distributions. The vector space \( \mathcal{E}'(\mathbb{R}) \) can be identified with the subspace of \( \mathcal{D}'(\mathbb{R}) \) consisting of all distributions having compact support. If \( \mathcal{D}'(\mathbb{R}) \) is also equipped with its strong dual topology, then one has a continuous injection \( \mathcal{E}'(\mathbb{R}) \hookrightarrow \mathcal{D}'(\mathbb{R}) \).

For \( T, S \in \mathcal{E}'(\mathbb{R}) \), we define their convolution \( T * S \in \mathcal{E}'(\mathbb{R}) \) by

\[
\langle T * S, \varphi \rangle = \left\langle T, \left[ x \mapsto \langle S, \varphi(x + \cdot) \rangle \right] \right\rangle, \quad \varphi \in \mathcal{E}(\mathbb{R}).
\]

The map \( * : \mathcal{E}'(\mathbb{R}) \times \mathcal{E}'(\mathbb{R}) \to \mathcal{E}'(\mathbb{R}) \) is (jointly) continuous [11, Chapter VI, §3, Theorem IV, p. 157].

Thus \( \mathcal{E}'(\mathbb{R}) \), endowed with the strong dual topology, forms a topological algebra with pointwise vector space operations, and with convolution taken as multiplication. The multiplicative unit is \( \delta_0 \), the Dirac delta distribution supported at 0. In general, we will denote by \( \delta_a \) the Dirac delta distribution supported at \( a \in \mathbb{R} \).

We also recall (see e.g. [14, Proposition 29.1, p.307]) that the Fourier-Laplace transform of a compactly supported distribution \( T \in \mathcal{E}'(\mathbb{R}) \) is an entire function, given by

\[
\hat{T}(z) = \left\langle T, \left( x \mapsto e^{-2\pi i \xi x} \right) \right\rangle \quad (z \in \mathbb{C}).
\]

3. **Topological stable rank**

An analogue of the Bass stable rank (useful in algebraic K-theory) for topological rings, was introduced in the seminal article [8].

**Definition 3.1.** Let \( \mathcal{A} \) be a commutative unital topological algebra endowed with a topology \( \mathcal{T} \), making the algebraic operations of addition, scalar multiplication and multiplication continuous with the product topologies on the appropriate spaces.

- (Unimodular \( n \)-tuple) Let \( n \in \mathbb{N} := \{1, 2, 3, \ldots \} \). We call an \( n \)-tuple \( (a_1, \ldots, a_n) \in \mathcal{A}^n := \mathcal{A} \times \cdots \times \mathcal{A} \) (\( n \) times) unimodular if there exists \( (b_1, \ldots, b_n) \in \mathcal{A}^n \) such that the Bézout equation \( a_1 b_1 + \cdots + a_n b_n = 1 \) is satisfied. The set of all unimodular \( n \)-tuples is denoted by \( U_n(\mathcal{A}) \).

Note that \( U_1(\mathcal{A}) \) is the group of invertible elements of \( \mathcal{A} \). An element from \( U_2(\mathcal{A}) \) is referred to as a coprime pair. It can be seen that if \( U_n(\mathcal{A}) \) is dense in \( \mathcal{A}^n \), then \( U_{n+1}(\mathcal{A}) \) is dense in \( \mathcal{A}^{n+1} \).
(Topological stable rank) If there exists a least natural number \( n \in \mathbb{N} \) for which \( U_n(A) \) is dense in \( A^n \), then that \( n \) is called the topological stable rank of \( A \), denoted by \( \text{tsr} \ A \). If if no such \( n \) exists, then \( \text{tsr} \ A \) is said to be infinite.

While the notion of topological stable rank was introduced in the context of Banach algebras, and the topological stable rank of many concrete Banach algebras has been determined previously in several works (e.g. [4], [12], [13]), in this article we find the topological stable rank of the classical algebra \( E'(\mathbb{R}) \) from Schwartz’s distribution theory, and show that \( \text{tsr}(E'(\mathbb{R})) = 2 \).

4. \( \text{tsr}(E'(\mathbb{R})) = 2 \)

The idea is that if the \( \text{tsr} \) were 1, then we could approximate any \( T \) from \( E'(\mathbb{R}) \) by compactly supported distributions whose Fourier transform would be zero-free, and by an application of Hurwitz Theorem, \( \hat{T} \) would need to be zero-free too, which gives a contradiction, since we can easily choose \( T \) at the outset to not allow this.

**Proposition 4.1.** \( \text{tsr}(E'(\mathbb{R})) \geq 2 \).

*Proof.* Suppose on the contrary that \( \text{tsr}(E'(\mathbb{R})) = 1 \). Let

\[
T = \frac{\delta_{-1} - \delta_{1}}{2i} \in E'(\mathbb{R}).
\]

By our assumption, \( U_1(E'(\mathbb{R})) \) is dense in \( E'(\mathbb{R}) \). But then \( U_1(E'(\mathbb{R})) \) is also sequentially dense: This follows from the fact that a subset \( F \) of \( E'(\mathbb{R}) \) is closed in \( \beta(E',E) \) if and only if it is sequentially closed. (See [7] Satz 3.5, p.231], which says that \( E' \), with the \( \beta(E',E) \)-topology, is sequential whenever \( E \) is Fréchet-Montel. A locally convex space \( F \) is sequential if any subset of \( F \) is closed if and only if it is sequentially closed. If \( F \) has this property, then the closure of any subset equals its sequential closure and therefore being dense is the same as being sequentially dense. In our case, \( E = E(\mathbb{R}) \) is Fréchet-Montel, and so \( E'(\mathbb{R}) \) is sequential. In fact in the remark following [7] Satz 3.5] the case of \( E'(\mathbb{R}) \) is mentioned as a corollary.)

Thus there exists a sequence \((T_n)_{n \in \mathbb{N}} \) in \( U_1(E'(\mathbb{R})) \) such that \( T_n \xrightarrow{n \to \infty} T \) in \( E'(\mathbb{R}) \). But since each \( T_n \) is invertible in \( E'(\mathbb{R}) \), there exists a sequence \((S_n)_{n \in \mathbb{N}} \) in \( E'(\mathbb{R}) \) such that

\[
T_n * S_n = \delta_0 \quad \text{for all } n \in \mathbb{N}.
\]

Taking the Fourier-Laplace transform, we obtain

\[
\hat{T}_n(z) \cdot \hat{S}_n(z) = 1 \quad \text{for all } z \in \mathbb{C} \text{ and all } n \in \mathbb{N}.
\]

In particular, the entire functions \( \hat{T}_n \) are all zero-free.

But as \( T_n \xrightarrow{n \to \infty} T \) in \( E'(\mathbb{R}) \), we now show that \( (\hat{T}_n)_{n \in \mathbb{N}} \) converges to \( \hat{T} \) uniformly on compact subsets of \( \mathbb{C} \) as \( n \to \infty \). The pointwise convergence of \( (\hat{T}_n)_{n \in \mathbb{N}} \) to \( \hat{T} \) is clear by taking the test function \( x \mapsto e^{-2\pi iz} \):

\[
\hat{T}_n(z) = \langle T_n, e^{-2\pi iz} \rangle \xrightarrow{n \to \infty} \langle T, e^{-2\pi iz} \rangle = \hat{T}(z).
\]
Now for any $\varphi \in \mathcal{E}(\mathbb{R})$, we know that the sequence $\langle T_n, \varphi \rangle_{n \in \mathbb{N}}$ converges to $\langle T, \varphi \rangle$, and in particular, the set
\[
\Gamma(\varphi) := \{ \langle T_n, \varphi \rangle : n \in \mathbb{N} \}
\]
is bounded, for every $\varphi \in \mathcal{E}(\mathbb{R})$. By the Banach-Steinhauss Theorem for Fréchet spaces (see for example [9, Theorem 2.6, p.45]), applied in our case to the Fréchet space $\mathcal{E}(\mathbb{R})$, we conclude that
\[
\Gamma = \{ T_n : n \in \mathbb{N} \}
\]
is equicontinuous. Thus for every $\varepsilon > 0$, there exists a neighbourhood $V$ of 0 in $\mathcal{E}(\mathbb{R})$ such that $T_n(V) \subset B(0, \varepsilon) := \{ z \in \mathbb{C} : |z| < \varepsilon \}$ for all $n \in \mathbb{N}$. From here it follows that there exist $M \in \mathbb{Z}_{\geq 0}$, $R > 0$ and $C > 0$ such that
\[
|\langle T_n, \varphi \rangle| \leq C \left( 1 + \sup_{0 \leq m \leq M} \sup_{|x| \leq R} |\varphi^{(m)}(x)| \right).
\]
By taking $\varphi = (x \mapsto e^{-2\pi ixz})$ in the above, we obtain
\[
|\hat{T}_n(z)| \leq C'(1 + |z|)^{M'}e^{R'|z|}, \quad z \in \mathbb{C}, \; n \in \mathbb{N}.
\]
Also, by the Paley-Wiener-Schwartz Theorem (see e.g. [2, Theorem 4.12, p.139]) for $T \in \mathcal{E}'(\mathbb{R})$, we have
\[
|\hat{T}(z)| \leq C''(1 + |z|)^{M''}e^{R''|z|}, \quad z \in \mathbb{C}, \; n \in \mathbb{N}.
\]
It now follows that for some constants $C_*, M_*, R_*$ that
\[
|\hat{T}_n(z) - \hat{T}(z)| \leq C_*(1 + |z|)^{M_*/e^{R_*/|z|}}, \quad z \in \mathbb{C}, \; n \in \mathbb{N}.
\]
But this means that the pointwise convergent sequence $(\hat{T}_n)_{n \in \mathbb{N}}$ of entire functions is uniformly bounded on compact subsets of $\mathbb{C}$ (that is, the sequence constitutes a normal family). It follows from Montel’s Theorem (see e.g. [15, Exercise 9.4, p.157]) that $(\hat{T}_n)_{n \in \mathbb{N}}$ converges to $\hat{T}$ uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$.

But now by Hurwitz Theorem (see e.g. [15, Exercise 5.6, p.85]), and considering, say, the compact set $K = \{ z \in \mathbb{C} : |z| \leq 1 \}$, we conclude that $\hat{T}$ must be either be identically zero on $K$ or that it must be zero-free in $K$. But $\hat{T}$ is neither:
\[
\hat{T}(z) = \frac{e^{2\pi iz} - e^{-2\pi iz}}{2i} = \sin(2\pi z),
\]
a contradiction. Hence $\text{tsr}(\mathcal{E}'(\mathbb{R})) \geq 2$.

5. $\text{tsr}(\mathcal{E}'(\mathbb{R})) \leq 2$

The idea is to reduce the tsr determination to $\text{tsr} \mathbb{C}[z]$ of the polynomial ring $\mathbb{C}[z]$, by mollifying the given pair from $\mathcal{E}'(\mathbb{R})$ (to make a pair in $\mathcal{D}(\mathbb{R})$, and then by approximating these smooth functions by a linear combination of Dirac distributions with uniform spacing (the uniform spacing affords the identification of the linear combination of Dirac deltas with the ring of polynomials).
For \( n \in \mathbb{N} \), we define the collection \( D_n \) of all “finitely supported Dirac delta combs” with spacing \( 1/n \) by
\[
D_n := \text{span}\{\delta_{k/n} : k \in \mathbb{Z}\}.
\]
Here “span” denotes taking all (finite) linear combinations.

**Lemma 5.1** (Approximating a pair of Dirac combs by a *unimodular* pair). Let \( T, S \in D_n \). Then there exist sequences \((T_k)_{k \in \mathbb{N}}\) and \((S_k)_{k \in \mathbb{N}}\) in \( D_n \), which converge to \( T, S \), respectively, in \( \mathcal{E}'(\mathbb{R}) \), and such that for each \( k \), \((T_k, S_k) \in U_2(\mathcal{E}'(\mathbb{R}))\).

**Proof.** Write \( T = \sum_{\ell=-L}^{L} t_{\ell} \delta_{\ell/n} \), and \( S = \sum_{\ell=-L}^{L} s_{\ell} \delta_{\ell/n} \), for some \( L \in \mathbb{N} \), \( t_{\ell}, s_{\ell} \in \mathbb{C} \).
Define
\[
p_T := t_{-L} + t_{-L+1}z + \cdots + t_{L}z^{2L},
p_S := s_{-L} + s_{-L+1}z + \cdots + s_{L}z^{2L}.
\]
Let \( \epsilon = 1/(2^k \cdot 2L) > 0 \). Then we can perturb the coefficients of the polynomials \( p_T, p_S \) within a distance of \( \epsilon \) to make them have no common zeros, that is after perturbation of coefficients they are coprime in the ring \( \mathbb{C}[z] \). Indeed any polynomial \( p_T, p_S \) can be factorized as
\[
p_T = C \prod (z - \alpha_{\ell}), \quad p_S = C' \prod (z - \beta_{\ell}),
\]
and if there is some common zero \( \alpha_\ell = \beta_\ell \), we simply replace \( \beta_\ell \) by \( \beta_\ell + \epsilon' \) with an \( \epsilon' \) small enough so that the final coefficients (of this new perturbed polynomial obtained from \( p_S \)), which are polynomial functions of the zeros, lie within the desired \( \epsilon \) distance of the coefficients of \( p_S \). So we can choose \( \tilde{t}_{-L,k}, \cdots, \tilde{t}_{L,k} \) and \( \tilde{s}_{-L,k}, \cdots, \tilde{s}_{L,k} \) such that for all \( \ell = -L, \cdots, L \), we have
\[
|t_\ell - \tilde{t}_\ell| < \frac{1}{2} \cdot \frac{1}{2L} \quad \text{and} \quad |s_\ell - \tilde{s}_\ell| < \frac{1}{2} \cdot \frac{1}{2L},
\]
and so that
\[
\tilde{p}_T, \tilde{p}_S \quad \text{have no common zeros. Thus \( \tilde{p}_T, \tilde{p}_S \) are coprime in \( \mathbb{C}[z] \), and hence there exist polynomials \( q_T, q_S \in \mathbb{C}[z] \) (\( \Box \) Corollary 8.5, p.374) such that}
\[
\tilde{p}_T q_T + \tilde{p}_S q_S = 1.
\]
Set \( Q_{T,k} := z^L q_{T,k} \) and \( Q_{S,k} := z^L q_{S,k} \), and
\[
P_T := \tilde{t}_{-L,k}z^{-L} + \tilde{t}_{-L+1,k}z^{-L+1} + \cdots + \tilde{t}_{L,k}z^{L},
P_S := \tilde{s}_{-L,k}z^{-L} + \tilde{s}_{-L+1,k}z^{-L+1} + \cdots + \tilde{s}_{L,k}z^{L}.
\]
Then in the ring $\mathbb{C}[z, z^{-1}]$ of linear combinations of monomials $z^n$, where $n \in \mathbb{Z}$ (i.e. the Laurent polynomial ring $\mathbb{C}[z, z^{-1}] = \mathbb{C}[z, w]/(zw - 1)$; see for example [11, p.367]), we have

$$P_{T,k}Q_{T,k} + P_{S,k}Q_{S,k} = 1. \quad (1)$$

Let

$$Q_{T,k} := \tau_{L',k}z^{L+L'} + \tau_{L'-1,k}z^{L+L'-1} + \cdots + \tau_{0,k}z^L,$$
$$Q_{S,k} := \sigma_{L',k}z^{L+L'} + \sigma_{L'-1,k}z^{L+L'-1} + \cdots + \sigma_{0,k}z^L.$$

Finally, set

$$T_k := \sum_{\ell=-L}^L \tilde{t}_{\ell,k} \delta_{\ell/n}, \quad S_k := \sum_{\ell=-L}^L \tilde{s}_{\ell,k} \delta_{\ell/n},$$
and

$$U_k := \tau_{L',k} \delta_{(L+L')/n} + \tau_{L'-1,k} \delta_{(L+L'-1)/n} + \cdots + \tau_{0,k} \delta_{L/n},$$
$$V_k := \sigma_{L',k} \delta_{(L+L')/n} + \sigma_{L'-1,k} \delta_{(L+L'-1)/n} + \cdots + \sigma_{0,k} \delta_{L/n}.$$

Then it follows from (1) that

$$T_k * U_k + S_k * V_k = \delta_0. \quad (2)$$

To see this, we note that $\Phi : \mathbb{C}[z, z^{-1}] \to D_n$ given by $\Phi(z) := \delta_{1/n}$ and $\Phi(1) = \delta_0$ defines a ring homomorphism, and then (2) above follows by applying $\Phi$ on both sides of (1). Hence $(T_k, U_k) \in U_2(\mathcal{E}'(\mathbb{R}))$. Also, for any $\varphi \in \mathcal{E}(\mathbb{R})$, we have

$$\left| \langle (T - T_k), \varphi \rangle \right| = \left| \sum_{\ell=-L}^L (t_{\ell,k} - \tilde{t}_{\ell,k}) \langle \delta_{\ell/n}, \varphi \rangle \right|$$
$$= \frac{1}{2k} \sup_{x \in [-L/n, L/n]} |\varphi(x)|$$
$$= \frac{1}{2k} \sup_{x \in [-L/n, L/n]} |\varphi(x)| \xrightarrow{k \to \infty} 0.$$

Hence $T_k \xrightarrow{k \to \infty} T$ in $\mathcal{E}'(\mathbb{R})$ as $k \to \infty$. Similarly, $S_k \xrightarrow{k \to \infty} S$ in $\mathcal{E}'(\mathbb{R})$ as $k \to \infty$. This completes the proof. \hfill \Box

**Lemma 5.2** (Approximation in $\mathcal{E}'(\mathbb{R})$ by Dirac combs).

*Let $T \in \mathcal{E}'(\mathbb{R})$. Then there exists a sequence $(T_n)_{n \in \mathbb{N}}$ such that

- for all $n \in \mathbb{N}$, $T_n \in D_n$, and
- $T_n \xrightarrow{n \to \infty} T$ in $\mathcal{E}'(\mathbb{R})$."

**Proof.** Let $k \in \mathbb{N}$ be such that the support of $T$ is contained in $(-k, k)$. We first produce a mollified approximating sequence for $T$. Let $\varphi : \mathbb{R} \to (0, \infty)$ be any test function in $\mathcal{D}(\mathbb{R})$ with support in $[-a, a]$ for some $a > 0$. Then we know that if we define

$$\varphi_m(x) := m \varphi(mx) \quad (m \in \mathbb{N}),$$
then for each $m$, $f_m := T \ast \varphi_m$ is a smooth function having a compact support, and moreover, $T \ast \varphi_m \overset{m \to \infty}{\to} T$ in $\mathcal{E}'(\mathbb{R})$; see for example [22, Theorem 3.3, p.97]. Moreover, as the support of $f_m = T \ast \varphi_m$ is contained in the sum of the supports of $\varphi_m$ and of $T$, for all $m$ large enough, say $m \geq M$, we have

$$\text{supp}(T \ast \varphi_m) \subseteq \text{supp}(T) + \text{supp}(\varphi_m) \subseteq \text{supp}(T) + [-a/m, a/m] \subseteq [-k, k].$$

From now on, we will assume that $m \geq M$, so that $\text{supp}(f_m) \subset [-k, k]$. Now we will approximate $f_m$ by Dirac comb elements. To this end, we define

$$T_{m,n} := \sum_{\ell=0}^{n-1} \frac{2k}{n} f_m(-k + \frac{2k}{n} \ell) \delta_{-k + \frac{2k}{n} \ell} \in \mathcal{D}_n.$$ 

We will show that $T_{m,n} \overset{n \to \infty}{\to} f_m$ in $\mathcal{E}'(\mathbb{R})$. Let $\psi \in \mathcal{E}(\mathbb{R})$. Then

$$\langle T_{m,n}, \psi \rangle = \left\langle \sum_{\ell=0}^{n-1} \frac{2k}{n} f_m(-k + \frac{2k}{n} \ell) \delta_{-k + \frac{2k}{n} \ell}, \psi \right\rangle \overset{n \to \infty}{\to} \langle f_m, \psi \rangle.$$

Thus $\langle T_{m,n}, \psi \rangle$ gives a Riemann sum for the integral of the continuous function $f_m \psi$ with compact support contained in $[-k, k]$, giving

$$\left| \langle T_{m,n}, \psi \rangle - \langle f_m, \psi \rangle \right| \overset{n \to \infty}{\to} 0.$$ 

Hence $T_{m,n} \overset{n \to \infty}{\to} f_m$ in $\mathcal{E}'(\mathbb{R})$. This completes the proof.

\textbf{Proposition 5.3.} $\text{tsr}(\mathcal{E}'(\mathbb{R})) \leq 2$.

\textbf{Proof.} Let $T, S \in \mathcal{E}'(\mathbb{R})$. Let $V$ be a neighbourhood of $(T, S)$ in $(\mathcal{E}'(\mathbb{R}))^2$. By Lemma [5,2], it follows that that $\mathcal{D}_n \times \mathcal{D}_n$ is sequentially dense in $(\mathcal{E}'(\mathbb{R}))^2$. As $\mathcal{E}'(\mathbb{R})$ with its strong dual topology $\beta(\mathcal{E}, \mathcal{E}')$ is sequential [7, Satz 3.5, p.231], we conclude that $\mathcal{D}_n \times \mathcal{D}_n$ is dense in $(\mathcal{E}'(\mathbb{R}))^2$. Hence there exists a pair $(T_*, S_*) \in V \cap (\mathcal{D}_n \times \mathcal{D}_n)$. By Lemma [5,1] there exists a sequence $(T_n, S_n)_{n \in \mathbb{N}}$ in $U_2(\mathcal{D}_n)$ that converges to $(T_*, S_*)$ in $(\mathcal{E}'(\mathbb{R}))^2$. Since $V$ is also a neighbourhood of $(T_*, S_*)$ in $(\mathcal{E}'(\mathbb{R}))^2$, there exists an index $N$ large enough so that for all $n > N$, $(T_n, S_n) \in V$. Thus $U_2(\mathcal{D}_n)$ is sequentially dense in $(\mathcal{E}'(\mathbb{R}))^2$. But as $U_2(\mathcal{D}_n) \subset U_2(\mathcal{E}'(\mathbb{R}))$, we have that $U_2(\mathcal{E}'(\mathbb{R}))$ is sequentially dense in $(\mathcal{E}'(\mathbb{R}))^2$. But again, as $\mathcal{E}'(\mathbb{R})$ with its strong dual topology $\beta(\mathcal{E}, \mathcal{E}')$ is sequential, we conclude that $U_2(\mathcal{E}'(\mathbb{R}))$ is dense in $(\mathcal{E}'(\mathbb{R}))^2$. \qed
From Propositions 4.1 and 5.3 it follows that \( \text{tsr}(E'(\mathbb{R})) = 2 \), proving Theorem 1.1.

**Remark 5.4** (Some open questions). We remark that in higher dimensions, with a similar analysis, it should be possible to show that \( \text{tsr}(E'(\mathbb{R}^d)) = d+1 \).

The Bass stable rank, a notion from algebraic \( K \)-theory, recalled below, of \( E'(\mathbb{R}) \) is also not known. Recall that if \( A \) is a commutative unital algebra, then an \( (n+1) \)-tuple \( (a_1, \ldots, a_n, b) \in U_{n+1}(A) \) is called reducible if there exists \( (\alpha_1, \ldots, \alpha_n) \in A^n \) such that \( (a_1 + \alpha_1 b, \ldots, a_n + \alpha_n b) \in U_n(A) \). It can be seen that if every \( (n+1) \)-tuple in \( U_{n+1}(A) \) is reducible, then every \( (n+2) \)-tuple from \( U_{n+2}(A) \) is reducible too. The Bass stable rank of \( A \), denoted by \( \text{bsr} A \), is the smallest \( n \in \mathbb{N} \) such that every element in \( U_{n+1}(A) \) is reducible. If no such \( n \) exists, then \( \text{bsr} A := \infty \). It is known that for commutative unital Banach algebras \( A \), \( \text{bsr} A \leq \text{tsr} A \) [3, Theorem 3]. We conjecture that \( \text{bsr} E'(\mathbb{R}) \) is 2.

There are also several other natural convolution algebras of distributions on \( \mathbb{R} \), for example

\[
\begin{align*}
\mathcal{D}'_{\geq}(\mathbb{R}) & := \{ T \in \mathcal{D}'(\mathbb{R}) : \text{supp}(T) \text{ is bounded on the left} \}, \\
\mathcal{D}'_{\geq 0}(\mathbb{R}) & := \{ T \in \mathcal{D}'(\mathbb{R}) : \text{supp}(T) \subset [0, \infty) \},
\end{align*}
\]

and we leave the determination of the stable ranks of these algebras as open questions.

**Remark 5.5** (Corona-type pointwise condition for coprimeness). It is known [6, Corollary 3.1] that \( T_1, T_2 \in U_2(E'(\mathbb{R})) \) if and only if there exist positive \( C, N, M \) such that for all \( z \in \mathbb{C} \),

\[
|\hat{T}_1(z)| + |\hat{T}_2(z)| \geq C(1 + |z|^2)^{-N} e^{-M|\text{Im}(z)|}.
\]

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