Aut($F_2$) puzzles

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Abstract This paper defines a tiling problem related to the automorphism group of $F_2$. Our main result is that the corresponding tilings admit a complete, concrete classification.

Keywords Tilings · Automorphisms of free groups · Flat closing conjecture

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In this paper we introduce a class of tilings of the Euclidean plane by regular polygons. These tilings appear naturally as flat planes in certain random groups (in the sense of [4]) associated to the group Aut($F_2$) of automorphisms of the free group $F_2$. The main goal of this paper is to show that these tilings admit a complete classification.

The tilings are defined as follows.

Definition 1 A ring puzzle is a tessellation of the Euclidean plane $\mathbb{R}^2$ using planar polygons with coloured angles. The ring at a vertex of the puzzle is the coloured circle of small radius around that vertex.
By *ring puzzle problem* we mean the problem of classifying all puzzles whose coloured polygons and rings belong to prescribed sets. The classification is understood up to ring puzzle isomorphisms, which are affine transformations of $\mathbb{R}^2$ which preserve both the tiling structure and the colour of every shape angle. Similarly, the rings are considered up to a similarity preserving the coloration. Our convention is that a ring is a circle of length $2\pi$.

In the case of $\text{Aut}(F_2)$, the two prescribed sets of shapes and rings are defined as follows:

Shapes := \[ \begin{array}{c}
\begin{array}{c}
\text{an equilateral triangle and a lozenge}
\end{array}
\end{array} \]

(an equilateral triangle and a lozenge) and

Rings := \[ \begin{array}{c}
\begin{array}{c}
\text{the yellow arcs have length } 2\pi/3, \text{ the others } \pi/3.
\end{array}
\end{array} \]

This paper shows that the corresponding tiling problem admit a solution, and that these tilings can be completely classified. We shall call $\text{Aut}(F_2)$ puzzle a ring puzzle associated with these two prescribed sets. Thus, the definition of $\text{Aut}(F_2)$ is completely elementary, involving only simple local construction rules, that are not directly reminiscent of the group $\text{Aut}(F_2)$.

The study of tilings of $\mathbb{R}^2$ is a classical subject with a rich history, which we shall not attempt to review here (we refer to [7], for instance, for more information on tilings). A tiling of the plane is *periodic* if its symmetry group acts sufficiently transitively (compare e.g. [7, §1.3], where these tilings are called regular). Besides periodic tilings, which can be entirely classified, a lot of attention has been devoted to the study of aperiodic tilings, one famous example of which being the Penrose tilings.

Since the relation to $\text{Aut}(F_2)$ is entirely local, the classification of $\text{Aut}(F_2)$ puzzles can be applied to study other groups whose local geometry is related (but possibly different) to that $\text{Aut}(F_2)$. For example, the random groups associated to $\text{Aut}(F_2)$ by the model defined in [4] are such groups, and the result presented below give structural constraints on the structure of flat planes in these groups. Section 8 discusses flat plane problems for this type of groups.

The main result of this paper is a more precise version of the following statement.

**Theorem 2** There is an explicit classification of all $\text{Aut}(F_2)$ puzzles.

The classification finds:

- 4 infinite families of puzzles
- 9 exceptional puzzles not belonging to the families

It relies on a series of preliminary results on the local-global geometry of these puzzles. The final classification will be obtained in Theorem 23.

For example, here is a permitted (bi-infinite) portion of an $\text{Aut}(F_2)$ puzzle, which we shall call the 2-strip.

Although the full algebraic structure of $\text{Aut}(F_2)$ is not used directly in these constructions, these ring puzzles still give informative “snapshots” of the nature of $\text{Aut}(F_2)$ itself. The group $\text{Aut}(F_2)$ acts properly with compact quotient on a 2-dimensional CAT(0) space, called the
Brady complex, which is essentially uniquely determined by Aut($F_2$) itself, by results of Crisp and Paoluzzi [5]. As mentioned above, the space of Aut($F_2$) puzzles is related to the space of flats in the Brady complex (see Remark 24). We note that (like for the Penrose tilings), large portions of a puzzle do not determine the puzzle in general. The ring puzzles associated with Aut($F_2$) are “bifurcating spaces” with an interesting (pointed) Gromov–Hausdorff topology. We shall not pursue the general study of ring puzzles in this paper. The ring condition is in line with some earlier definitions we made in intermediate rank geometry, which aim to restrict the possibilities for the local rank of the given complex (see e.g. [2,3]).

The proof of Theorem 23 is given in Sects. 1–7. The last section of the paper discusses ring complexes, and the relation between the ring puzzle problem (in the case of Aut($F_2$) puzzles) and flat plane problems in geometric group theory.

1 ◦-components

The first part of the proof concentrates on the connected component built exclusively using the lozenge shapes. We first establish that the geometry of these components is restricted.

Let us isolate an important property of the two sets of shapes and rings presented in the introduction. It reflects the fact that Aut($F_2$) puzzles are “nonpositively curved”:

**Definition 3** A ring puzzle problem has the $\theta_0$-extension property if the prescribed sets satisfy the following condition: whenever a coloured segment $[0, \theta]$ of length $\theta \in [0, 2\pi]$ can be isometrically embedded in distinct coloured rings respecting the coloration, then $\theta \leq \theta_0$.

For Aut($F_2$) puzzles, we have that $\theta_0 = \pi$, which corresponds to nonpositive curvature. The following consequence is straightforward:

**Lemma 4** Assuming the $\theta_0$-extension property, every ring puzzle is such that if consecutive pieces meet at a vertex contributing to an angle $> \theta_0$, then the remaining pieces at this vertex are uniquely determined (up to colour preserving isometry).

Let $P$ be an Aut($F_2$) puzzle.

**Definition 5** A ◦-gallery in $P$ is a sequence of consecutive (= intersecting along an edge) lozenges. Two lozenges are connected if they belong to a ◦-gallery. The connected components of lozenges in $P$ are called the ◦-components of $P$.

**Lemma 6** Let $C$ be a ◦-component. The inner angle at every vertex of $\partial C$ is equal to either $\pi/3$, $2\pi/3$ or $\pi$.

**Proof** The inner angle is a multiple of $\pi/3$. Assume that it is $> \pi$; since we are on the boundary, it can only be $4\pi/3$ or $5\pi/3$. The value $4\pi/3$ would single out the first ring, however, the remaining colour (namely yellow) corresponds to a lozenge, contradicting the fact that we are on the boundary. The value $5\pi/3$ is prohibited by the fact that they is no yellow–blue simplicial path of length $5\pi/3$ inside in the prescribed set of rings (by simplicial we mean both of whose end points are vertices of the ring).

**Lemma 7** Every finite ◦-component of $P$ is a parallelogram of size $m \times n$ such that $\min(m, n) \leq 2$. 
Proof The fact that finite $\diamond$-components are parallelograms follow from Lemma 6. If $m \times n$ denotes the size of such a component with $m \leq n$, then we claim that $m \leq 2$. This is a combinatorial problem which can be solved in the 3-neighbourhood of the component. Note that the 3rd ring has to be used at every point in the boundary, except at the two obtuse angles, where it is the 2nd ring that has to be used.

The solution of the problem, assuming $m \geq 3$ towards a contradiction, is summarized in the following drawing:

The drawing explicits one of several similar cases, for $m = 3$. In the given configuration, the rightmost part does not extend in $P$ given the prescribed sets of types. Indeed, the unique extension property forces two distinct shapes on the open edge of the triangle on the bottom right. It is not hard to see that a similar argument works when $m > 3$ as well.

2 $w$-blocks

In general, the prescribed sets of shapes and rings in a ring puzzle problem will determine a few basic blocks, which fit together to build larger portions of the puzzle using the unique extension property.

An interesting such block in the case of Aut($F_2$) puzzles is the following $w$-block:

Note that the $w$-block appeared already in the figure given in the proof of Lemma 7. We observe that $w$-blocks are “forward analytic” in the following sense:

Lemma 8 Every $w$-block in $P$ extends to a unique $w$-strip in $P$.

By $w$-strip we mean the strip (infinite at least on the right hand side):
Lemma 8 is a consequence of Lemma 4.
Note that two $w$-strips cannot intersect transversally in the forward direction.

**Lemma 9** The $\diamond$-components of type $2 \times 2$ in $P$ extend uniquely into a bi-infinite strip of height 4 of the form

![Diagram]

**Proof** This is a consequence of Lemma 8. □

We call this strip the double $w$-strip.

**Lemma 10** If $n \geq 3$, the finite $\diamond$-component of type $2 \times n$ are contained in a unique puzzle.

**Proof** A $\diamond$-component $C$ of size $2 \times n$ with $n \geq 3$

![Diagram]

admits a unique extension of the form

![Diagram]
by Lemma 8.
Furthermore using the extension property we have a unique upward extension (and by symmetry a unique downward extension) of the strip. This in turn exhibits a \( w\)-strip in the north-east direction (and by symmetry in the south-west direction) that bounds an acute sector together with the horizontal \( w\)-strip. But this implies that the horizontal strip belongs to at most one puzzle. A construction by induction then shows that the puzzle exists. Such a construction is shown in the figure below in the case \( n = 4 \).

\[ \square \]

**Definition 11** We will call the puzzle appearing the proof above the **puzzle of type** \( 2 \times n \).

Here is a portion of the puzzle of type \( 2 \times 4 \):

![Puzzle of type 2 x 4](image.png)

**Remark 12** Roughly speaking, Lemma 9 can be understood as a limiting case “\( n \to 2 \)” of Lemma 10.

### 3 Finite \( \diamond \)-components of type \( 1 \times n \)

**Lemma 13** Assume that \( P \) contains a finite \( \diamond \)-component \( C \) of type \( 1 \times n \) where \( n \geq 3 \). Then \( C \) is included in a strip \( S \) of height \( n \) in \( P \) (infinite both to the left and to the right) of the form:

\[ \square \] Springer
namely, alternating $\diamond$-components and 2-strip components of height $n$ (in the figure $n = 4$). Furthermore, the following two conditions are satisfied:

(a) the $\diamond$-components of $S$ parallels to $C$ are all of type $1 \times n$, except for at most one component of type $2 \times n$. If there is such a component, then $P$ is the puzzle of type $2 \times n$, otherwise $S$ is as indicated in the figure above.

(b) every $\diamond$-component of $S$ remains a $\diamond$-component in $P$ (otherwise said, it is adjacent to triangles in $P \setminus S$ on both sides, as indicated in the figure).

The strip described in the figure above is called the bi-infinite strip of type $1 \times n$ (all $\diamond$-component of the strip are of type $1 \times n$).

**Proof** Starting with a $\diamond$-component in $P$ of size $1 \times n$

(where we have rotated the picture for drawing convenience) there is a unique extension of the form

which by induction form a half strip $S_\uparrow \supset C$ in the upward direction, together with the right 1-neighbourhood of it. The same argument applies in downward direction by symmetry, which exhibits the indicated strip $S = S_\uparrow \cup S_\downarrow \supset C$ together with a 1-neighbourhood of it on both sides.

Inspection of these neighbourhoods shows that the $\diamond$-components parallel to $C$ are $\diamond$-component in $P$, which therefore are of the form $m \times n$ for $m = 1, 2$. 

```text
Proof
```
We have to prove that a component of type $2 \times n$ can only appear once. This can be done by observing that the puzzle of type $2 \times n$ actually contains only one $\diamond$-component of type $2 \times n$, or by the following direct proof. Suppose we have two components, without loss of generality we may assume that they are separated only by components of type $1 \times n$.

Note that the eastern neighbourhood of any $\diamond$-component of type $2 \times n$ contains two $w$-blocks whose forward direction intersect transversally.

By Lemma 8, $w$-blocks extend uniquely in the forward direction. This contradicts the fact that $w$-strips do not intersect transversally, and proves the result.

The figure above illustrates this argument. We have indicated the completion in the case of two $\diamond$-components of type $2 \times n$ separated by precisely one $\diamond$-component of type $1 \times n$. In this case a direct local contradiction appears in the 1-neighbourhood of the upmost component of type $2 \times n$. \hfill \qed

**Lemma 14** Assume that every component is finite and of type $1 \times n$ with $n = 1, 2$. Then every finite component is of type $1 \times 2$. The resulting puzzle is unique and called the puzzle of type $1 \times 2$.

**Proof** It is easy to check that there is no puzzle in which every $\diamond$-component is of type $1 \times 1$. The proof of the lemma, starting from a component of type $1 \times 2$, is then in the same spirit as that of Lemma 13, given the fact that every component has type $1 \times n$ with $n \leq 2$ (so that the neighbouring components are of type $1 \times 2$ as well). The puzzle of type $1 \times 2$ is
4 Semi-infinite $\diamond$-strips

In view of Sect. 3, it remains to classify the puzzles which contain an infinite $\diamond$-components.

**Lemma 15** An infinite $\diamond$-component is either

(a) the tessellation of the plane by lozenges (called the $\diamond$-plane) or half of it (called the half $\diamond$-plane)
(b) a bi-infinite strip of lozenges ($\diamond$-strip) or half of it (semi-infinite $\diamond$-strip)
(c) a $\diamond$-sector (with either an acute or an obtuse angle)

**Proof** This follows from Lemma 6.

Every (semi-infinite) $\diamond$-strip of lozenges is bounded by a (semi-infinite) 2-strip. In particular half $\diamond$-planes and bi-infinite $\diamond$-strips can be combined with the 2-strip to form various alternating 2-strip/$\diamond$-strip $\text{Aut}(F_2)$ puzzles.

Note that:

**Lemma 16** If $P$ contains a semi-infinite $\diamond$-strip $S$ of height $n$, then $n = 1$ or $n = 2$.

The proof is similar to that of Lemma 7.
Lemma 17  The semi-infinite $\diamond$-strips of type $2 \times \infty$ can be extended in precisely two ways:

1. a puzzle called the star puzzle of type $2 \times \infty$, or
2. a half-puzzle whose boundary is the following strip of height 3:

We refer to the above strip as “the 3-strip”, and the half-puzzle containing it, defined in (2), will be called the half puzzle of type $2 \times \infty$. (By half puzzle we mean a half plane in a puzzle with singular boundary.)

Proof  Take a semi-infinite (horizontal) $\diamond$-strip $S$ of height 2. The proof of Lemma 13 shows that there exits a unique sector extending $S$ “in the upper direction”. This sector contains a double $w$-strip as follows:

There are two extensions of this section, corresponding to the two ways a lozenge can be oriented with respect to adjacent the south-east triangles in the figure.

We have already encountered the first extension. It can be derived from the proof of Lemmas 10 or 13 and appears as “a half of” the limit (in the Gromov–Hausdorff sense, say) of the puzzle of type $2 \times n$ as $n \to \infty$: 
The lowermost 3-strip is the 3-strip.
The second extension, which leads to a unique puzzle, is

That it can be extended into at most one puzzle follows by Lemma 8 (and Lemma 4), and it is easy to show that this puzzle (the star puzzle of type $2 \times \infty$) exists by induction. \qed

In particular, it appears that the semi-infinite $\triangledown$-strip of type $2 \times \infty$ embeds at most twice in a single $\text{Aut}(F_2)$ puzzle.

**Lemma 18** If $P$ contains a semi-infinite $\triangledown$-strip of type $1 \times \infty$ then one of the following holds:

- $P$ is the puzzle of type $2 \times n$ where $n \geq 3$
- $P$ contains the half-puzzle of type $2 \times \infty$
- $P$ is the star puzzle of type $2 \times \infty$
- $P$ contains the double $w$-strip (see Lemma 9)
- $P$ contains the half-puzzle of type $1 \times n$ (which will be defined in the proof below) whose boundary strip of type $1 \times n$ for some $n \geq 3$
- $P$ is the $V$-puzzle:

**Proof** Take a semi-infinite (horizontal) $\Diamond$-strip $S$ of height 1. Again as in the proof of Lemma 13 one can construct a quarter puzzle $P_\uparrow$ extending $S$ “in the upper direction” which contains parallel (semi-infinite) 2-strips and (by assumption) semi-infinite $\Diamond$-strip $S$ of height 1.

The lower part of $P_\uparrow$ admits a unique extension as follows:
where again for drawing convenience, we are now representing the semi-infinite strips horizontally (the initial strip $S$, which forms the lower part of $P^\uparrow$, is the strip drawn in the figure).

If the piece at $X$ is a triangle, then we have a $\diamond$-component of type $2 \times n$ below it, heading south-east. There are 4 cases:

- If $n = 1$ then $P^\uparrow$ extends uniquely east using a 3-strip. The resulting half-puzzle $P^\uparrow \rightarrow$ is uniquely defined (it piles up horizontal 3-strips).
- If $n = 2$ then by Lemma 9 the quarter puzzle $P^\uparrow$ extends uniquely east using the double $w$-strip (compare Lemma 22).
- If $3 \leq n < \infty$ then $P$ is the puzzle of type $2 \times n$ (see Lemma 10).
- Otherwise, $n = \infty$, and Lemma 17 shows that either $P$ is the star of type $2 \times \infty$ or contains the half-puzzle of type $2 \times \infty$.

We now assume that the piece at $X$ is a losenge and extend the portion of the puzzle uniquely as follows:

If the piece at $Y$ is a triangle, then by Lemma 7 the puzzle contains a $\diamond$-component of type $m \times 3$ with $m = 1, 2$, and we can apply Lemma 13 to find either that $P$ is the puzzle of type $2 \times 3$, or if $n = 1$ that $P$ contains the half-puzzle of type $1 \times 3$.

The half-puzzle of type $1 \times n$ (shown here horizontally for $n = 4$), with an infinite south extension.

The conclusion is identical for every $n = 3, 4, \ldots$, writing $n$ for the height of the $\diamond$-component bounded by $Y$. If $n = \infty$ we obtain a sector in $P$ which is part of the $V$-puzzle.
(and corresponds to the limit $n \to \infty$). Since the horizontal strips alternate 2-strips and $\diamond$-strips of height 1, this sector in $P$ extends uniquely to the $V$-puzzle.

5 $\diamond$-sectors

Let us now turn to the case of $\diamond$-sectors. Both of them (acute and obtuse) can be viewed as geometric limits of half $\diamond$-strips, as the height increases, but we recall that the $\diamond$-strips can only have height 1 or 2 in a puzzle.

Lemma 19 If the puzzle contains a sector with acute angle, then it contains a component of type $2 \times n$ for some $n = 1, \ldots, \infty$ (if $n = \infty$ this is the semi-infinite $\diamond$-strip of type 2).

Proof Acute sectors extend uniquely (up to symmetry along the bisector) into the following portion of a puzzle:

This puzzle contains (at the bottom-left) a $\diamond$-component of type $2 \times n$ with $n = 1, \ldots, \infty$. If $n = 1$ then the sector extends uniquely to a half plane containing the 3-strip. If $n = 2$ then it extends uniquely to a half-plane containing the double $w$-strip. Otherwise Lemma 10 gives the conclusion.

Lemma 20 There exists a unique puzzle containing a sector with obtuse angle.

Proof Obtuse sectors extend to:
which creates a \( w \)-strip. Existence and uniqueness of the puzzle is proved by induction using Lemma 4 as earlier.

\[
\Box
\]

### 6 Strips of height 3 and 4

Let us then classify extensions of the 3-strips and the double \( w \)-strips.

**Lemma 21** If \( P \) contains the 3-strip \( S \), then \( P \) contains only parallel 3-strips and at most two copies of the half puzzle of type \( 2 \times \infty \).

**Proof** This is an easy consequence Lemma 17.

The 3-strip puzzle is the puzzle made only of parallels 3-strips. Every other puzzle appearing in the lemma is called the 3-strip puzzle of height \( h \) where \( h \geq 1 \) is the number of parallels 3-strips it contains. We call half 3-strip puzzle the puzzle containing exactly one half-puzzle of type \( 2 \times \infty \). Every 3-strip puzzle of height \( h < \infty \) contains two such puzzles.

**Lemma 22** There are precisely two puzzles containing the double \( w \)-strip.

**Proof** Note that, using the symmetry along the vertical axis, the double \( w \)-strip extends uniquely as follows:
By Lemma 8 this can be extended north in a unique way. □

The two puzzles in Lemma 22 will be called the opposite acute sector puzzle and the adjacent acute sector puzzle. The corresponding half puzzle is:

7 The classification

Theorem 23 The $\text{Aut}(F_2)$ puzzles come in 4 infinite series and 9 exceptional puzzles not belonging to the series.

The 9 exceptional puzzles are:

(a) the $\bigodot$-puzzle
(b) the $2 \times 1$-puzzle (see Lemma 14)
(c) the star puzzle of type $2 \times \infty$
(d) the 3-strip puzzle and the half 3-strip puzzle
(e) the opposite acute sector puzzle and the adjacent acute sector puzzle (see Lemmas 9 and 22).
(f) the obtuse sector puzzle (see Lemma 20)
(g) the $V$-puzzle

The 4 infinite series are:

(A) the alternating 2-strip/$\bigodot$-strip series. This contains three subseries:

- the bounded 2-strip/$\bigodot$-strip series. They are parametrized by biinfinite sequences $(\ldots, n_{-1}, n_0, n_1, \ldots)$ recording the heights of the $\bigodot$-strips in the given order, where $n_i \geq 1$ is arbitrary. There are uncountably many such puzzles.
• The sided 2-strip/⋄-strip series. The 1-sided series is parametrized by right infinite sequences \((n_0, n_1, \ldots)\) of heights of ⋄-strips, where \(n_i \geq 1\) is arbitrary, giving uncountably many puzzles. The 2-sided series is parametrized by finite sequences \((n_0, n_1, n_r)\) where \(r\) is finite but arbitrary. This gives countably many puzzles.

(B) the 1 \(\times n\) series. They are constructed using the strip of type 1 \(\times n\) (Lemma 13) and can be encoded by the biinfinite sequence \((\ldots, n_{-1}, n_0, n_1, \ldots)\) of heights here \(n_i \geq 1\) is arbitrary, giving uncountably many such puzzles. This series also splits into 3 subseries, the bounded 1 \(\times n\) series and the sided 1 \(\times \infty\) series, which can either be one-sided or two-sided.

(C) the 2 \(\times n\) series, where \(n = 3, 4, \ldots (n < \infty)\). There is a unique puzzle for every \(n \times 2\) (Lemma 10).

(D) the 3-strip series of height \(h \geq 1\) (Lemma 21).

Several of the exceptional puzzles appear as limits of elements of the infinite series. For example, a), c), and d) can be seen as limiting cases of A), C), and D) respectively, and b) can be seen as a limit of B when \(n \to 2\). The V-puzzle g) can also be seen as a limiting case for B), and D) can be seen as a limiting case for C) when \(n \to \infty\). The obtuse sector puzzle, on the other hand, does not appear to be a limit, and similarly for e).

Proof Let \(P\) be an Aut\((F_2)\) puzzle. We assume that \(P\) is not one of the exceptional puzzles and prove that it belongs to one of the series.

Note that \(P\) does not contain an obtuse sector. If it contains an acute sector then it contains a strip of type 2 \(\times n\). We first deal with infinite strips.

If \(P\) contains a semi-infinite strip of type 2 \(\times \infty\) and since it is not the star puzzle of type 2 \(\times \infty\) it contains the 3-strip. Not being in d) it belongs to D).

If \(P\) contains a semi-infinite strip of type 1 \(\times \infty\) but no semi-infinite strip of type 2 \(\times \infty\), then it belongs to B) by Lemma 18.

If \(P\) contains a bi-infinite ⋄-strip then it belongs to A.

By Lemma 15 we may now assume that every ⋄-component of \(P\) is finite. If has a component of type 2 \(\times n\) then we are in case C). Otherwise every component is finite of type 1 \(\times n\). Then this is case B).

\[\square\]

Remark 24 As with other types of planar tilings (such as the Penrose tilings), one can organize the puzzles into a “space of puzzles” as follows. A marked puzzle is a parametrization \(f : \mathbb{R}^2 \to P\) where \(f\) is isometric. Define a valuation \(v\) on the set \textbf{Puzzles}' of all marked puzzles (with prescribed shapes and rings) by

\[v(f, f') = \sup\{r \mid f(B(0, r)) \sim f'(B(0, r))\}\]

(where the isometry preserves the rings) with corresponding metric

\[d(f, f') = e^{-v(f, f')}\]

The space of marked puzzles is

\[\textbf{Puzzles} := \textbf{Puzzles}' / \{d = 0\}\]

Some of the statements regarding convergence of puzzles can be interpreted in the space \textbf{Puzzles}. For example the obtuse sector puzzle “is not a limit” because it corresponds to an isolated point in \textbf{Puzzles}. Note that the space of puzzles is compact if the sets of shapes and rings are finite. It is endowed with an action of \(\mathbb{R}^2\) given by

\[t \cdot f := f \circ t^{-1}\]
for $t$ a translation of $\mathbb{R}^2$ turning puzzles into a lamination whose leaves correspond to puzzles with the given set of shapes and rings.

8 Ring complexes and flat plane problems

Ring puzzles, or portions thereof, can be glued together to form “nonplanar ring puzzles”, which are more complicated objects to understand.

Let us start with a 2-complex $X$ whose faces are flat polygons in the Euclidean plane with coloured angles. The rings at a vertex of $X$ is set of all coloured circles of length $2\pi$ of small radius around that vertex. We refer to such an $X$ as a ring complex. The type of a ring complex is the set of coloured faces and coloured rings (considered up to coloured isometry). In this section we study ring complexes using the prescribed sets considered in the previous section:

$$T := \{ \begin{array}{c} \includegraphics[width=2cm]{triangle.png} \end{array} \}$$

Note that ring complexes are naturally endowed with a global metric, namely the length metric. They are not necessarily nonpositively curved.

A flat plane in $X$ is an isometric embedding $\mathbb{R}^2 \hookrightarrow X$. Every flat plane in a ring complex is a ring puzzle of the same type.

The $\mathbb{Z}^2$ embedding problem for $X$ is the question of whether the existence of a flat plane in $X$ imply that of a copy of $\mathbb{Z}^2$ in $\Gamma$:

$$\mathbb{R}^2 \hookrightarrow X \Rightarrow \mathbb{Z}^2 \hookrightarrow \Gamma$$

whenever $\Gamma \curvearrowright X$ properly with $X/\Gamma$ compact. Gromov asks for example if this is true for $X$ simplicial and nonpositively curved, say, of dimension 2 (see [6]). When $X$ is nonpositively curved the converse is known

$$\mathbb{Z}^2 \hookrightarrow \Gamma \Rightarrow \mathbb{R}^2 \hookrightarrow X$$

by the flat torus theorem and the “flat closing conjecture” asserts the $\mathbb{Z}^2$ embedding problem has a positive answer (i.e., if there exists a flat then there exists a periodic flat plane). As mentioned in the introduction, our original motivation for studying ring puzzles was to establish the flat closing conjecture in certain models of random groups associated with $\text{Aut}(F_2)$. In particular, it can be shown that there are many groups for which the assumptions in Theorem 26 are satisfied. We show here how the classification theorem can be used to prove the existence of $\mathbb{Z}^2$ in the acting group $\Gamma$.

Definition 25

Consider a simplicial strip $S \simeq \mathbb{R} \times [a, b]$ and a simplicial subset $L \subset S$. The strip $S$ is said to be uniquely $L$-embeddable in $X$ if for any two simplicial embeddings $f, f': S \hookrightarrow X$ and any translation $g : S \rightarrow S$

$$f' = f \circ g \text{ on } L \Rightarrow f' = f \circ g \text{ on } S.$$ 

(Note that $g$ is then necessarily a simplicial map.)

Consider the following two 1-strips in a ring complex of type $T$:

- the $\Diamond$-strip:
• the $\triangle$-strip:

These strips are said to be *uniquely $\diamond$-embeddable* if they are uniquely $L$-embeddable with respect to a lozenge $L$.

We say that $X$ is acylindrical if there is no embedding

$$S^1 \times \mathbb{R} \rightarrow X$$

of a bi-infinite cylinder into $X$.

**Theorem 26** If in an acylindrical ring complex of type $T$ every $\diamond$-strip and every $\triangle$-strip is uniquely $\diamond$-embeddable, then the $\mathbb{Z}^2$-embedding problem has a positive solution.

The proof of Theorem 26 relies on two lemmas. The first lemma uses the classification.

**Lemma 27** Suppose that $X$ is a ring complex $X$ of type $T$ and fix $\Gamma \curvearrowright X$ with $X/\Gamma$ compact. If $X$ contains a flat, then it contains a flat which contains infinitely many parallel $\diamond$-strips, or a flat that contains infinitely many parallel $\triangle$-strips.

**Proof** We claim that if $X$ contains a flat, then it contains a flat which is isomorphic to either a $\diamond$-puzzle, a puzzle of uniformly bounded type $1 \times n$ (including $n = 2$), or a puzzle of uniformly bounded alternating 2-strip/$\diamond$-strip type. The lemma follows easily from this.

The proof of the claim is a standard sort of compactness argument, taking geometric limits of flats, which relies on the classification. Let $\Pi$ be a flat and $K$ be a relatively compact fundamental domain for $\Gamma \curvearrowright X$.

Suppose for example that $\Pi$ contains a $\diamond$-sector $S$. Choose vertices $x_n$ in $S$ moving apart from the boundary of $S$, and $s_n \in \Gamma$ such that $s_n x_n \in K$. For every $r > 0$ we can find infinitely many $n$ for which the flats $s_n \Pi$ coincide on $r$-neighbourhood of $K$. A diagonal argument delivers a flat

$$\Pi_\infty := \lim_{n \text{ in a subsequence}} s_n \Pi$$

with $x \in \Pi_\infty$ such that for every $r$ and every $n$ large enough (depending on $r$) the ball of radius $r$ and center $x$ in $\Pi$ coincide with the ball of radius $r$ and center $s_n^{-1} x$ in $\Pi_n$. This shows that $\Pi_\infty$ is the $\diamond$-flat.

This covers cases a), d), e), f) and C) and D) of the classification. The same argument works if $\Pi$ contains $\diamond$-strips of arbitrary large height, showing that case B) of the classification leads to either a $\diamond$-flat, or $\Pi$ a uniformly bounded flat of type $1 \times n$ (including $n = 2$). This covers cases b) and B).

In cases c) and g) it is easy to choose $x_n$ so that $\Pi_\infty$ is the alternating 2-strip/$\diamond$-strip flat where the $\diamond$-strip has height 1. Finally in case A), either the height is uniformly bounded, or we can choose $x_n$ so that $\Pi_\infty$ is the $\diamond$-flat. □

**Remark 28** For the proof of Theorem 26, the conclusion of Lemma 27 (that if $X$ contains a flat, then it contains a flat which contains infinitely many parallel $\diamond$-strips, or a flat that
contains infinitely many parallel $\triangle$-strips) will be the only information extracted from the classification theorem. Thus, some of the steps in the complete classification of puzzles are irrelevant (for example, one does not to actually classify all the flats with a $\diamond$-sector) if one is only interested in Theorem 26.

The second lemma is more general.

**Lemma 29** Suppose that the 2-complex $X$ is acylindrical and uniformly locally finite, and fix a proper action $\Gamma \curvearrowright X$ with $X/\Gamma$ compact. Assume furthermore that

- $X$ contains a flat that contains infinitely many parallel copies of a simplicial strip $S$
- $S$ is uniquely $L$-embeddable in $X$ for some non empty compact simplicial set $L$
- there are infinitely many distinct simplicial translates of $L$ in $S$.

Then $\mathbb{Z}^2 \hookrightarrow \Gamma$.

**Proof** For the proof we will (1) produce a first element of infinite order acting on a flat as in the first point, (2) produce a second element also of infinite order, and (3) show that the first element commutes to a power of the second element.

Let $S$ denote the strip given with simplicial set $L \subset S$.

Let $\Pi$ be a flat in $X$ which contains infinitely many parallel copies of $S$, say

$$f_k : S \hookrightarrow X \text{ with } \text{ran } f_k \parallel \text{ran } f_l, \quad f_k \neq f_l \quad (k \neq l \geq 1)$$

Note that $f_k \neq f_l$ on $L$ if $k \neq l$. (By ran $f$ we mean the range of the map $f$, i.e., the geometric strip in $\Pi$.)

Since $X/\Gamma$ is compact there exist an index $k_0$ and elements $s_k \in \Gamma$ such that

$$s_k \circ f_k = s_{k_0} \circ f_{k_0} \text{ on } L$$

for infinitely many $k \geq 1$.

Since $S$ is uniquely $L$-embeddable we have that

$$s_k \circ f_k = s_{k_0} \circ f_{k_0} \text{ on } S$$

for infinitely many $k \geq 1$.

Choose such an index $k$ so that $\text{ran } f_k \cap \text{ran } f_{k_0} = \emptyset$ and write $f = f_{k_0}$, $f' = f_k$ and $s = s_k^{-1} \circ s_{k_0}$ so that $f' = s \circ f$. The element $s$ is our first element.

By assumption we can find infinitely many distinct translates of $L$ in $S$. Let $g_n : S \rightarrow S$ denote the corresponding translations ($n \geq 1$). Since $X/\Gamma$ is compact, there exist an index $n_0 \geq 1$ and elements $t_n \in \Gamma$ such that

$$t_n \circ f \circ g_n = f \circ g_{n_0} \text{ on } L$$

for infinitely many $n \geq 1$.

Since $S$ is uniquely $L$-extendable we have that

$$t_n \circ f \circ g_n = f \circ g_{n_0} \text{ on } S$$

for infinitely many $n \geq 1$.

Choose $n$ large enough so that $f \circ g_n(L) \cap f \circ g_{n_0}(L) = \emptyset$ and write $g = g_{n_0} \circ g^{-1}_n$ and $t = t_n$ so that we have the intertwining relation

$$t \circ f = f \circ g$$
on $S$. Note that $g$ is a simplicial translation. Furthermore $t^m \circ f = f \circ g^m$ on $S$ for every $m \in \mathbb{Z}$. The element $t$ is our second element.

Denote by $f \lor f': [a, b] \times \mathbb{R} \to \Pi \subseteq X$ the simplicial strip lying “between $f$ and $f'$” in $\Pi$ (whose range is the convex closure of both strips in $\Pi$). Thus there is $a < a' < b' < b$ such that $S$ is simplicially isomorphic to both $[a, a'] \times \mathbb{R}$ and $[b', b] \times \mathbb{R}$, and the restriction of $f \lor f'$ to these strips is $f$ and $f'$, respectively.

Since $X$ is locally finite, there are only finitely many strips in $X$ isometric to $\text{ran } f \lor f'$. Therefore there exists $m$ and a translation $h: [a, b] \times \mathbb{R} \to [a, b] \times \mathbb{R}$ such that

$$t^m \circ (f \lor f') = (f \lor f') \circ h.$$

Note that $h$ coincides with $g^m$ on $[a, a'] \times \mathbb{R}$, so we will replace $t$ with $t^m$ to obtain the relations

$$t \circ f = f \circ h \ 	ext{and} \ t \circ (f \lor f') = (f \lor f') \circ h.$$

on $S \simeq [a, a'] \times \mathbb{R}$ and $[a, b] \times \mathbb{R}$. This implies that

$$t^q \circ f = f \circ h^q \ 	ext{and} \ t^q \circ (f \lor f') = (f \lor f') \circ h^q.$$

on $S \simeq [a, a'] \times \mathbb{R}$ and $[a, b] \times \mathbb{R}$, respectively, for every $q \in \mathbb{Z}$, and in particular that

$$t^q \circ f' = f' \circ h^q$$

on $S \simeq [b', b] \times \mathbb{R}$, for every $q \in \mathbb{Z}$.

Then

$$t^q \circ s \circ f = t^q \circ f' = f' \circ h^q = s \circ f \circ h^q = s \circ t^q \circ f.$$

Choose a point $x \in \text{ran } f$. The relation shows that for every $q \in \mathbb{Z}$ the commutator

$$[t^q, s] \in \text{Stab}_\Gamma(x),$$

which is a finite group by assumption. Therefore we can find $q \neq q'$ such that

$$[t^q, s] = [t^{q'}, s]$$

so

$$[t^{q-q'}, s] = e.$$

Since

$$t^q \circ f = f \circ h^q$$

for every $q$, it follows that $t$ has infinite order in $\Gamma$.

Assume towards a contradiction that $s$ has finite order in $\Gamma$. Then the subset of $X$

$$\bigcup_{k=0}^{\text{order of } s} \text{ran } s^k \circ (f \lor f')$$

is a simplicial cylinder in $X$, whose existence is precluded by assumption.

Thus, $(t^{q-q'}, s) \simeq \mathbb{Z}^2$ in $\Gamma$. \qed
Remark 30  The space of marked flats $\text{Flats}(X)$ is the set

$$\text{Flats}(X) = \{\mathbb{R}^2 \hookrightarrow X\}$$

of all isometric embeddings $\mathbb{R}^2 \hookrightarrow X$. This is a classical object associated with $X$; it has been studied in particular by Pansu and is useful when studying the rigidity of symmetric spaces (cf. [8, §14]).

The space $\text{Flats}(X)$ has two commuting actions of $\mathbb{R}^2$ and $\Gamma$

$$t \cdot f := f \circ t^{-1} \quad \text{and} \quad s \cdot f := s \circ f$$

respectively at the source and the range, for $t$ a translation of $\mathbb{R}^2$, $s \in \Gamma$, and $f : \mathbb{R}^2 \hookrightarrow X$ an embedding.

If $X$ is locally finite and $X/\Gamma$ is compact then $\text{Flats}(X)/\Gamma$ is a compact space. It is not empty if $X$ is not hyperbolic, and it is endowed with a structure of a lamination whose leaves are associated with flat planes in $X$ and given by the action of $\mathbb{R}^2$.

If $X$ is a ring complex of type $T$ we have an obvious map

$$\text{Flats}(X) \rightarrow \text{Puzzles}(\text{Aut}(F_2))$$

which is continuous and open.

Note that in the case of ring complexes of type $T$, the subset $\text{Flats}_x(X)$ corresponding to flats in $X$ of type $x = a, b, \ldots$ is closed whenever flats of type $x$ are geometrically periodic (i.e. the simplicial isometry group acts with compact quotient). This holds for flats of type a) or b), for example.

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