Deformation quantization on the cotangent bundle of a Lie group

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Abstract

We develop a complete theory of non-formal deformation quantization on the cotangent bundle of a weakly exponential Lie group. An appropriate integral formula for the star-product is introduced together with a suitable space of functions on which the star-product is well defined. This space of functions becomes a Fréchet algebra as well as a pre-$C^*$-algebra. Basic properties of the star-product are proved and the extension of the star-product to a Hilbert algebra and an algebra of distributions is given. A $C^*$-algebra of observables and a space of states are constructed. Moreover, an operator representation in position space is presented. Finally, examples of weakly exponential Lie groups are given.

Keywords and phrases: quantum mechanics, deformation quantization, star-product, Lie group

1 Introduction

In the paper we develop a complete theory of quantization of a classical Hamiltonian system whose configuration space is in the form of a Lie group $G$. The phase space $M$ of such system has the form of the cotangent bundle $T^*G$ of $G$. A typical example of such system is a rigid body. The possible configurations of a rigid body can be described by translations and rotations. Thus, as the configuration space of this system we can take the group of translations $(\mathbb{R}^3,+)$, if we are only interested in translational degrees of freedom, or the group of rotations $SO(3)$, if we want to consider only rotational degrees of freedom, or the semi-direct product of these two groups if we want to take into account both translational and rotational degrees of freedom.

Our approach to quantization is based on a deformation quantization theory [1–6]. In this theory we want to describe quantization as a continuous deformation of a classical Hamiltonian system, with respect to some deformation parameter. The deformation parameter is usually interpreted as the Planck’s constant $\hbar$, although in reality we should take as the deformation parameter some dimensionless combination of the Planck’s constant and a parameter characteristic of the physical system under study. In the limit $\hbar \to 0$ we should recover the classical system. In the rest of the paper $\hbar$ will be a fixed non-zero real number. In fact it can be negative as the developed theory still makes sense in such case.

The idea of deformation quantization relies on the fact that all information about a classical Hamiltonian system is encoded in its Poisson algebra $C^\infty(M)$. The algebra $C^\infty(M)$ is the vector space of smooth $\mathbb{C}$-valued functions defined on the phase space $M$, endowed with a point-wise product of functions, a Poisson bracket $\{\cdot,\cdot\}$ and an involution being the complex conjugation. By deforming the algebra $C^\infty(M)$ (or its subalgebra) to a certain noncommutative Poisson algebra we get a quantum Hamiltonian system. The deformation is

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such that the point-wise product is deformed to some noncommutative product $\star$ (the so-called star-product) and the Poisson bracket is deformed to the following Lie bracket

$$[f,g] = \frac{1}{i\hbar}(f \star g - g \star f).$$

(1)

Moreover, there should hold

$$f \star g \to f \cdot g, \quad [f,g] \to \{f,g\} \quad \text{as } h \to 0.$$  

(2)

In the literature can be found many works on the topic of deformation quantization on the cotangent bundle of a Riemannian manifold [7–13], however most of these works do not provide a complete theory of quantization. In [6, 10] authors constructed a quantization scheme with the use of a symbol calculus for pseudodifferential operators on a Riemannian manifold. Then with its help they introduced a star-product on a suitable space of phase space functions. In [11, 12] authors introduced a formal star-product on the cotangent bundle of a Riemannian manifold and constructed a representation of the received star product algebra by means of differential operators. However, in all these papers the received algebra of functions consisted of formal power series in $\hbar$, and therefore did not describe a physical system, or was not big enough to include a pre-$C^\ast$-algebra, which then could be used to define states of the quantum system in a natural way.

Note, that in the approach of strict deformation quantization [14–18] these problems do not appear. This approach is based on a construction of a continuous field of $C^\ast$-algebras by means of an operator representation of phase space functions by compact operators on a suitable Hilbert space. However, in general it is not possible to introduce a non-formal star product on the phase space and it is not easy to extend the $C^\ast$-algebra of observables to a bigger algebra containing all functions interesting from a physical point of view.

For all these reasons we develop a theory of deformation quantization in which we introduce a non-formal star-product on a suitable pre-$C^\ast$-algebra of phase space functions and extend this algebra to include many functions appearing in physics. The developed deformation quantization theory is, in fact, an example of strict deformation quantization. Moreover, we want to introduce quantization without referring to an operator representation and later construct this representation as a byproduct of the developed theory. Such approach seems to be more natural because we then receive quantum theory as a direct deformation of classical mechanics. In the current work we will be dealing only with the case of the phase space in the form of the cotangent bundle of a Lie group, which is much simpler to work with than the cotangent bundle of a general Riemannian manifold.

There are not many works devoted to deformation quantization on the cotangent bundle of a Lie group. Worth noting are papers [19, 20] where authors introduce a star-product on the cotangent bundle of a Lie group in terms of a formal power series in $\hbar$. Also in [21, 22] are studied homogeneous star-products on cotangent bundles of Lie groups, which are characterized in terms of integral formulas. More often star-products are investigated on duals of Lie algebras (see e.g. [23–27]). Worth noting is also the case of a deformation quantization on the cotangent bundle of a homogeneous space (a manifold $Q$ on which a Lie group $G$ acts transitively as a group of symmetries). Such case was studied in [28] from the perspective of strict deformation quantization.

Some ideas of our theory are based on the work [13] where authors consider non-formal deformation quantization on the cotangent bundle of a Riemannian manifold $Q$. In this paper the exponential map $\exp_q$ on $Q$ is required to be a diffeomorphism on the whole tangent space $T_q Q$ onto the whole manifold $Q$. This condition greatly restricts admissible manifolds $Q$. In our approach we replace $Q$ by a Lie group $G$ with the property that its exponential map $\exp$ is a diffeomorphism onto almost the whole group $G$. Many Lie groups interesting from a point of view of physics are of such form.

In practice we do not need the whole Poisson algebra $C^\infty(M)$ to describe a Hamiltonian system but only some subalgebra $\mathcal{F}_0(M)$. For the purpose of quantization we take $\mathcal{F}_0(M)$ as the space of functions on $M$ which momentum Fourier transforms are smooth and compactly supported. The algebra $\mathcal{F}_0(M)$ fully characterizes the classical Hamiltonian system. Indeed, all information about the Poisson manifold $M$ is encoded in $\mathcal{F}_0(M)$. Moreover, states can be defined as appropriate linear functionals on $\mathcal{F}_0(M)$. In fact
\[ \mathcal{F}_0(M) \] is a dense subalgebra of the \( C^* \)-algebra \( \mathcal{A}_0(M) := C_0(M) \) of continuous functions on \( M \) vanishing at infinity, with the standard supremum norm. In other words \( \mathcal{F}_0(M) \) is a pre-\( C^* \)-algebra. States are then defined as continuous linear functionals \( \Lambda \) on \( \mathcal{A}_0(M) \) satisfying

(i) \( \| \Lambda \| = 1 \) (normalization),

(ii) \( \Lambda(\tilde{f} \cdot f) \geq 0 \) for every \( f \in \mathcal{A}_0(M) \) (positive-definiteness).

The Riesz representation theorem provides identification of states \( \Lambda \) with probabilistic Borel measures \( \mu \) through the following formula

\[ \Lambda(f) = \int_M f \, d\mu. \] (3)

The deformation of the Poisson algebra \( \mathcal{F}_0(M) \) will be denoted by \( \mathcal{F}_h(M) \). Quite often we will skip the dependence on \( h \) and just write \( \mathcal{F}(M) \).

The paper is organized as follows. In Section 2 we introduce notation and definitions used throughout the paper. In particular, in Section 2.0 we define the deformed Poisson algebra \( \mathcal{F}(M) \).

In Section 3.1 we introduce on \( \mathcal{F}(M) \) a noncommutative \( \ast \)-product in terms of an integral formula. Basic properties of the \( \ast \)-product are proved and its expansion in \( h \) to the third order is calculated. The algebra \( \mathcal{F}(M) \) is a basis of the whole theory. With its help we can define states of the quantum system. First, in Section 3.2 we extend \( \mathcal{F}(M) \) to a Hilbert algebra \( \mathcal{L}(M) \) and then in Section 3.3 we extend \( \mathcal{L}(M) \) to a \( C^* \)-algebra of observables \( \mathcal{A}(M) \). Quantum states can be defined in a standard way as continuous positively defined linear functionals on \( \mathcal{A}(M) \) normalized to unity. We also provide characterization of states as elements of \( \mathcal{L}(M) \) satisfying certain properties.

On the other hand the algebra \( \mathcal{F}(M) \) can be extended to an algebra of distributions \( \mathcal{F}_\ast(M) \) by treating \( \mathcal{F}(M) \) as the space of test functions (see Section 3.4). The algebra \( \mathcal{F}_\ast(M) \) contains all functions interesting from a physical point of view. In particular we show that all smooth functions polynomial in fiber variables are elements of \( \mathcal{F}_\ast(M) \).

For completeness of the quantization procedure we present in Section 5.3 a short description of the time evolution of states and observables in the language of deformation quantization.

In Section 4 we construct an operator representation of the developed theory. We represent considered in previous sections algebras of functions on the phase space \( M \) by algebras of operators on a Hilbert space \( L^2(G, dm) \) of \( \mathbb{C} \)-valued functions on the Lie group \( G \) square integrable with respect to a Haar measure \( dm \). We give explicit formulas of the constructed representation. The introduced operator representation allows to express a quantum system in the language of the standard Hilbert space approach to quantum mechanics.

The action of the group \( G \) on the phase space \( M = T^* G \) gives rise to a reduced system whose phase space is the dual \( g^* \) of the Lie algebra \( g \) of the Lie group \( G \), endowed with the standard Poisson structure. In Section 5 we describe this reduction operation and show that the introduced quantization procedure respects this reduction. In particular, we give a formula for the \( \ast \)-product on \( g^* \).

The introduced theory of quantization works only on a certain type of Lie groups which we call weakly exponential Lie groups. In Section 2.1 we define the notion of a weakly exponential Lie group and in Section 6 we give examples of such groups.

We end the paper with Section 7 where we give conclusions and final remarks.

## 2 Preliminaries

### 2.1 Lie groups basics

Let \( G \) be an \( n \)-dimensional Lie group. Its Lie algebra will be denoted by \( g \). Moreover, we will denote by \( dm \) a left invariant Haar measure on \( G \).

We will denote by \( L_q \) and \( R_q \) left and right translations in \( G \), i.e.

\[
L_q \colon G \to G, \quad h \mapsto L_q(h) := qh,
R_q \colon G \to G, \quad h \mapsto R_q(h) :=hq.
\] (4)
The derivative of $L_q$ in the unit element $e$ is a linear isomorphism $T_e L_q : g \to T_q G$ which provides a natural identification of a tangent space $T_q G$ with the Lie algebra $g$. The transposition of this map $T^*_q L_q : T^*_q G \to g^*$ provides a natural identification of a cotangent space $T^*_q G$ with the dual of the Lie algebra $g$.

The mappings $L : G \times g \to T G$ and $\theta : G \times g^* \to T^* G$, given by

$$
L_X(q) = T_e L_q X, \quad q \in G, X \in g,
$$

$$
\theta_p(q) = (T^*_q L_q)^{-1} p, \quad q \in G, p \in g^*,
$$

provide natural isomorphisms between tangent and cotangent bundles of $G$, and trivial bundles $G \times g$ and $G \times g^*$, respectively. The diffeomorphism $\theta$ can be used to transfer a natural structure of a symplectic manifold on $T^* G$ to $G \times g^*$. In what follows we will denote the symplectic manifold $G \times g^*$ by $M$.

Recall that the derivative of the exponential map is equal \[29\]

$$
T_X \exp = T_e L_{\exp(X)} \circ \phi(\text{ad}_X) = T_e R_{\exp(X)} \circ \phi(-\text{ad}_X), \quad \phi(x) = \frac{1 - e^{-x}}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} x^n. \tag{6}
$$

Since $\phi(x)$ is analytic on $\mathbb{C}$ the linear operator $\phi(\text{ad}_X)$ is well defined for every $X \in g$ as a convergent series of linear operators. We calculate that

$$
T_X(L_q \circ \exp) = T_{\exp(X)} L_q \circ T_e L_{\exp(X)} \circ \phi(\text{ad}_X) = T_e(L_q \circ L_{\exp(X)}) \circ \phi(\text{ad}_X) = T_e L_{q \exp(X)} \circ \phi(\text{ad}_X), \tag{7a}
$$

$$
T_X(R_q \circ \exp) = T_{\exp(X)} R_q \circ T_e R_{\exp(X)} \circ \phi(-\text{ad}_X) = T_e(R_q \circ R_{\exp(X)}) \circ \phi(-\text{ad}_X) = T_e R_{\exp(X) q} \circ \phi(-\text{ad}_X). \tag{7b}
$$

Let us define the reflection about the unit element $e$ as the following map

$$
s : G \to G, \quad g \mapsto s(g) := g^{-1}. \tag{8}
$$

Moreover, define the reflection about point $q \in G$ as the map

$$
s_q : G \to G, \quad s_q := L_q \circ s \circ L_{q^{-1}} = R_q \circ s \circ R_{q^{-1}}, \quad \text{i.e.} \quad s_q(g) = q(q^{-1}g)^{-1} = qq^{-1}q. \tag{9}
$$

Note, that $s_q^2 = \text{id}$. Denote the derivative of the reflection $s$ in the unit element $e$ by $s'$, i.e.

$$
s' := T_e s, \quad s' : g \to g, \quad s'(X) = -X. \tag{10}
$$

Throughout the paper we will consider Lie groups satisfying the following condition: there exists an open neighborhood $O$ of 0 in $g$, such that:

(i) it is star-shaped and symmetric, i.e. if $X \in O$ then $tX \in O$ for $-1 \leq t \leq 1$,

(ii) the exponential map $\exp : g \to G$ restricted to $O$ is a diffeomorphism onto $U = \exp(O)$,

(iii) $G \setminus U$ is of measure zero.

Lie groups satisfying this property will be called weakly exponential. Note, however, that some authors use the term weakly exponential Lie group in a different sense, namely as such Lie group for which $\exp(g)$ is dense in $G$. Throughout the paper $O$ and $U$ will denote sets, corresponding to a given Lie group $G$, as in the above definition.

Recall that a subset $A$ of a smooth $n$-dimensional manifold $M$ has measure zero if for every smooth chart $(U, \varphi)$ for $M$, the set $\varphi(A \cap U)$ has Lebesgue-measure zero in $\mathbb{R}^n$. \[30\]

Note, that any set of measure zero has dense complement, because if $M \setminus A$ is not dense, then $A$ contains a nonempty open set, which would imply that $\psi(A \cap V)$ contains a nonempty open set of Lebesgue-measure zero for some smooth chart $\psi, V$, which is impossible because the only open set in $\mathbb{R}^n$ of Lebesgue-measure zero is an empty set. Hence if $G$ is weakly exponential then $U$ is dense in $G$.
Observe, that since \( \mathcal{O} \) is reflection invariant (\( \mathcal{O} = -\mathcal{O} \)), then \( s(\mathcal{U}) = \mathcal{U} \) and \( s_q(\mathcal{L}_q(\mathcal{U})) = \mathcal{L}_q(\mathcal{U}) \). Note also that weakly exponential Lie groups are necessarily connected \([31]\).

We will also assume that considered weakly exponential Lie groups are such that \( \exp^{-1}: \mathcal{U} \to \mathcal{O} \) maps precompact sets onto precompact sets, i.e. if \( S \subset \mathcal{U} \) and its closure \( \overline{S} \) is compact in \( G \), then \( \exp^{-1}(\overline{S}) \) is compact in \( \mathcal{O} \).

For weakly exponential Lie groups \( G \) satisfying this property we immediately get that if \( G \) is compact, then the closure \( \overline{\mathcal{O}} \) of \( \mathcal{O} \) is also compact and \( \exp(\overline{\mathcal{O}}) = G \).

### 2.2 Poisson structure on \( M \)

We will give an explicit formula for the canonical Poisson bracket on \( M \). Instead of using Darboux coordinate frame on \( M \), it is convenient to use different frame fields. Let \( X_1, \ldots, X_n \) be a basis in \( g \) and \( X^1, \ldots, X^n \) a dual basis in \( g^* \). Each \( X_j \) defines a left invariant vector field \( L_{X_j} \) on \( G \). The vector fields \( L_{X_j} \) can be lifted to vector fields \( Y_j \) on \( M = G \times g^* \) by means of the identification \( T_{(q,p)}(G \times g^*) = T_q G \oplus g^* \), i.e.

\[
Y_j(q,p) = L_{X_j}(q) \oplus 0. \tag{11}
\]

Similarly covectors \( X^j \) can be identified with constant vector fields \( Z^j \) on \( M = G \times g^* \) according to the prescription

\[
Z^j(q,p) = 0 \oplus X^j. \tag{12}
\]

The \( 2n \) vector fields \( Y_1, \ldots, Y_n, Z^1, \ldots, Z^n \) on \( M \) form at each point \( x \) a basis of tangent vectors. They have the following properties:

\[
\pi_* Z^i = 0, \quad \pi_* X_i = L_{X_i},

\[
[Z^i, Z^j] = [Z^i, Y_j] = 0, \quad [Y_i, Y_j] = C^k_{ij} Y_k,
\]

where \( \pi: G \times g^* \to G \) is a canonical projection and \( C^k_{ij} \) are the structure constants of \( g \) in the basis \( \{X_i\} \) given by

\[
[X_i, X_j] = C^k_{ij} X_k. \tag{14}
\]

Let \( p_j: M \to \mathbb{R} \) be fiber variables defined by \( p_j(q, p) = \langle p, X_j \rangle \). Then \( dp_i(Z^i) = \delta^i_j \) and \( dp_i(Y_j) = 0 \). The Poisson bracket of two functions \( f, g \in C^\infty(M) \) reads:

\[
\{f, g\} = Z^i f Y_i g - Y_i f Z^i g + p_k C^k_{ij} Z^i f Z^j g. \tag{15}
\]

In particular

\[
\{p_i, p_j\} = C^k_{ij} p_k. \tag{16}
\]

The canonical symplectic form \( \omega \) on \( M \) is then given by the formula

\[
\omega = \hat{\alpha}^i \land dp_i + \frac{1}{2} p_k C^k_{ij} \hat{\alpha}^i \land \hat{\alpha}^j, \tag{17}
\]

where \( \hat{\alpha}^i = \pi^* (\alpha^i) \) is a pull-back of a left invariant 1-form \( \alpha^i \) on \( G \) corresponding to \( X_i \), i.e. \( \alpha^i(L_{X_j}) = \delta^i_j \).

Then \( \hat{\alpha}^i(Z^i) = 0 \) and \( \hat{\alpha}^i(Y_j) = \delta^i_j \).

### 2.3 Measures and integration

On the vector space \( g \) we can introduce a measure by means of the Haar measure \( dm \) on \( G \). The Haar measure \( dm \) originates from a left invariant volume form \( \omega_L \) on \( G \), which on the other hand is completely determined by its value in the unit element \( \omega_L(e) \). The \( n \)-form \( \omega_L(e) \) on \( g \) induces, subsequently, a measure \( dX \) on \( g \).

The measure \( dX \) on \( g \), in turn, determines a measure \( dp \) on \( g^* \) by means of the Fourier transform. To be more precise, if we define the Fourier transform of functions on \( g \) by the formula

\[
\hat{f}(p) = \int_g f(X) e^{-i\langle p, X \rangle} \, dX, \quad p \in g^*, \tag{18}
\]
then we choose the measure \( dp \) on \( \mathfrak{g}^* \) so that the inverse Fourier transform will be given by the equation

\[
f(X) = \frac{1}{(2\pi\hbar)^n} \int_{\mathfrak{g}^*} \hat{f}(p)e^{i(p,X)} \, dp.
\]

On the symplectic manifold \( M = G \times \mathfrak{g}^* \) we have a distinguished measure \( dx \) induced by the Liouville volume form \( \Omega \) on \( M \). The measure \( dx \) is the product measure and can be written in the form

\[
dx = dm(q) \times dp.
\]

Indeed, using (17) we calculate that

\[
\Omega = \frac{1}{n!} \omega \wedge \cdots \wedge \omega = \alpha^1 \wedge \cdots \wedge \alpha^n \wedge dp_n = (-1)^{(n+1)/2} \alpha^1 \wedge \cdots \wedge \alpha^n \wedge dp_1 \wedge \cdots \wedge dp_n.
\]

We have that

\[
\alpha^1 \wedge \cdots \wedge \alpha^n = \pi_1^*(\alpha^1) \wedge \cdots \wedge \pi_1^*(\alpha^n) = \pi_1^*(\alpha^1 \wedge \cdots \wedge \alpha^n),
\]

\[
dp_1 \wedge \cdots \wedge dp_n = \pi_2^*(X_1) \wedge \cdots \wedge \pi_2^*(X_n) = \pi_2^*(X_1 \wedge \cdots \wedge X_n),
\]

where \( X_i \) are treated as constant 1-forms on \( \mathfrak{g}^* \) and \( \pi_1: G \times \mathfrak{g}^* \to G, \pi_2: G \times \mathfrak{g}^* \to \mathfrak{g}^* \) are projections onto the first and second component of the product manifold \( G \times \mathfrak{g}^* \). Thus \( \Omega \) is a wedge product of a pull-back of a left invariant volume form \( \omega^1 \wedge \cdots \wedge \omega^n \) on \( G \) and a pull-back of a constant volume form \( X_1 \wedge \cdots \wedge X_n \) on \( \mathfrak{g}^* \). Therefore, the measure corresponding to \( \Omega \) will be a product of a Haar measure on \( G \) and a canonical measure on \( \mathfrak{g}^* \). Note, that the Liouville measure \( dx \) is independent on the normalization of the Haar measure \( dm \) and the measure \( dp \) corresponding to it, since \( dp \) scales inversely proportionally to \( dm \).

On the manifold \( G \times \mathfrak{g} \) we will introduce the following measure

\[
dn(q, X) = dm(q) \times dX.
\]

We will often use a normalized Liouville measure \( dl = \frac{dx}{|2\pi\hbar|^n} \).

For function \( f \) defined on \( M = G \times \mathfrak{g}^* \) we define the Fourier transform of \( f \) in the momentum variable by

\[
\hat{f}(q, X) = \frac{1}{|2\pi\hbar|^n} \int_{\mathfrak{g}^*} f(q, p)e^{i(p,X)} \, dp
\]

and the inverse Fourier transform by

\[
f(q, p) = \int_{\mathfrak{g}^*} \hat{f}(q, X)e^{-i(p,X)} \, dX.
\]

Quite often we will consider integrals of functions \( f \in L^2(M, dl) \) whose momentum Fourier transforms \( \hat{f} \) are compactly supported, bounded and smooth on \( G \times \mathcal{O} \). Such functions may not be Lebesgue integrable, however, we can define their integrals as improper integrals. Let \( \{K_j \mid j = 1, 2, \ldots\} \) be a sequence of compact subsets of \( \mathfrak{g}^* \) such that \( K_1 \subset K_2 \subset \cdots \subset \mathfrak{g}^* \) and \( \bigcup_{j=1}^{\infty} K_j = \mathfrak{g}^* \). Then the improper integral of the function \( f \) is defined as the following limit of integrals:

\[
\int_M f(x) \, dl(x) = \lim_{j \to \infty} \int_{G \times K_j} f(x) \, dl(x).
\]

Note, that because of the equality

\[
\int_{K_j} f(q, p) \, dp = \int_{\mathfrak{g}^*} \chi_{K_j}(p)f(q, p) \, dp = |2\pi\hbar|^n (\hat{\chi}_{K_j} \ast \hat{f})(q, 0),
\]

(27)
where $\chi_{K_j}$ is the characteristic function of the set $K_j$ and $*$ denotes the usual convolution of functions, we get that
\[
\int_M f(x) \, dl(x) = \lim_{j \to \infty} \int_G (\chi_{K_j} * \tilde{f})(q, 0) = \int_G (\delta * \tilde{f})(q, 0) = \int_G \tilde{f}(q, 0).
\] (28)
This shows that the improper integral of functions in the above form always exists, is finite and does not depend on the sequence of compact sets $\{K_j\}$.

2.4 Baker-Campbell-Hausdorff product

On the Lie algebra $\mathfrak{g}$ we define operation $\diamond$ by the formula
\[
X \diamond Y = \exp^{-1}(\exp(X) \exp(Y)).
\] (29)
It is well defined on $V = \{(X, Y) \in O \times O \mid \exp(X) \exp(Y) \in U\}$. Clearly $X \diamond Y \in O$ for $(X, Y) \in V$ and $(O \times O) \setminus V$ is of measure zero. The operation $\diamond$ has the following easy to verify properties:

(i) $X \diamond (Y \diamond Z) = (X \diamond Y) \diamond Z$ whenever $(X, Y \diamond Z), (X \diamond Y, Z) \in V$ (associativity),

(ii) $0 \diamond X = X \diamond 0 = X$ (0 is a neutral element),

(iii) $X \diamond (-X) = (-X) \diamond X = 0$ ($-X$ is an inverse element to $X$),

(iv) $(-X) \diamond (-Y) = -(Y \diamond X)$,

(v) $\exp(X \diamond Y) = \exp(X) \exp(Y)$.

For a fixed element $Y \in O$ let $O_Y = \{X \in O \mid (X, Y) \in V\}$. The maps $R_Y: O_Y \to O_Y$ and $L_Y: O_Y \to O_Y$ given by
\[
R_Y(X) = X \diamond Y, \quad L_Y(X) = Y \diamond X
\] (30)
are one-to-one with inverses equal $R_{-Y}: O_{-Y} \to O_Y$ and $L_{-Y}: O_Y \to O_{-Y}$ respectively. Note, that $O \setminus O_Y$ is of measure zero.

In general $\diamond$ is not commutative and bilinear. By virtue of the Baker-Campbell-Hausdorff formula we get the following expansion of $X \diamond Y$ around $(0, 0)$ up to third order
\[
X \diamond Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) - \frac{1}{24}[Y, [X, [X, Y]]] + \cdots .
\] (31)
We will call the operation $\diamond$ a Baker-Campbell-Hausdorff product.

2.5 Notation

For any $q \in G$, we use the notations:

- $V_q = (L_q \circ \exp|_O)^{-1}$, so that $V_q(a) = \exp^{-1}(L_q^{-1}(a)) = \exp^{-1}(q^{-1}a)$ for $a \in L_q(U)$,

- $\frac{dm(s_q)}{dm}$ is the Jacobian obtained by transforming the measure $dm$ under the diffeomorphism $s_q$,

- $\frac{dm(\exp)}{dX}$ is the Jacobian obtained by transforming the measure $dm$ under the diffeomorphism $\exp|_O$,

- $\chi_O$ is the characteristic function of the set $O$, i.e.
\[
\chi_O(X) = \begin{cases} 
1 & \text{for } X \in O \\
0 & \text{for } X \notin O
\end{cases}
\]
\begin{itemize}
  \item $j_q(a) = |\det \psi(\text{ad}_{V_q(a)})/\chi_{\partial}(2V_q(a))|$ for $a \in L_q(U)$, where $\psi(x) = \frac{x}{2} \coth \frac{x}{2} = \sum_{n=0}^{\infty} \frac{B_{2n}x^{2n}}{(2n)!}$ and $B_n$ is the $n$-th Bernoulli number,
  \item $J_q(a) = j_q(a) \frac{dm(s_q)}{dm} \bigg|_a$ for $a \in L_q(U)$,
  \item $\Phi(X) = \sqrt{|\det \phi(\text{ad}_X) \det \text{Ad}_{\exp(\frac{1}{2}X)}|} = \sqrt{|\det \lambda(\text{ad}_X)|}$ for $X \in \mathfrak{g}$, where $\lambda(x) = \phi(x)e^{\frac{x}{2}} = \frac{2}{x} \sinh \frac{x}{2} = \sum_{n=0}^{\infty} \frac{4^n(2n+1)!}{x^{2n}}$.
  \item $L(X,Y) = \begin{cases} F(X)F(Y) & \text{for } (X,Y) \in \mathcal{V} \\ F(X \circ Y) & \text{for } (X,Y) \notin \mathcal{V} \end{cases}$
\end{itemize}

Note, that
\begin{equation}
V_q(a) = -V_q(s_q(a)) = -V_a(q) \tag{32}
\end{equation}

since
\begin{equation}
V_q \circ s_q = (L_q \circ \exp)^{-1} \circ s_q = (s_q \circ L_q \circ \exp)^{-1} = (L_q \circ s \circ \exp)^{-1} = (L_q \circ \exp \circ s')^{-1} = s' \circ (L_q \circ \exp)^{-1} = s' \circ V_q \tag{33}
\end{equation}

and
\begin{equation}
V_q(s_q(a)) = \exp^{-1}(q^{-1}s_q(a)) = \exp^{-1}(a^{-1}q) = V_a(q). \tag{34}
\end{equation}

Hence, $j_q$ is invariant under the reflection $s_q$.

We have that
\begin{equation}
\frac{dm(\exp)}{dX} \bigg|_X = \frac{dm(L_q \circ \exp)}{dX} \bigg|_X = |\det \phi(\text{ad}_X)|, \tag{35}
\end{equation}
\begin{equation}
\frac{dm(s)}{dm} \bigg|_q = \det \text{Ad}_q, \tag{36}
\end{equation}
\begin{equation}
\frac{dm(s_q)}{dm} \bigg|_a = \frac{dm(s)}{dm} \bigg|_{q^{-1}a} = \det \text{Ad}_{q^{-1}a}. \tag{37}
\end{equation}

Indeed, if we choose a basis $X^1, \ldots, X^n$ in $\mathfrak{g}^*$, then the left invariant volume form $\omega_L$ on $G$ corresponding to the Haar measure $dm$ can be expressed by the formula: $\omega_L = c \alpha^1 \wedge \cdots \wedge \alpha^n$, where $\alpha^i(a) = (T^*_c L_a)^{-1}X^i$ is a left invariant 1-form induced by $X^i$ and $c \in \mathbb{R}$ is some constant. The volume form on $G$ corresponding to the measure $dX$ takes then the form $cX^1 \wedge \cdots \wedge X^n$. Let $\Phi = L_q \circ \exp$. The pull-back of the 1-form $\alpha^i$ by the map $\Phi$ is equal
\begin{equation}
\Phi^* \alpha^i(X) = \alpha(\Phi(X)) \circ \Phi_*(X) = \alpha(q \exp(X)) \circ T_X(L_q \circ \exp) = (T^*_c L_q \exp(X))^{-1}X^i \circ T_X(L_q \circ \exp)
= \left(T^*_c (L_q \circ \exp) \circ (T^*_c L_q \exp(X))^{-1}\right)X^i = \left((T^*_c L_q \exp(X))^{-1} \circ T_X(L_q \circ \exp)\right)^*X^i
= \phi(\text{ad}_X)^*X^i. \tag{38}
\end{equation}

Thus, the pull-back of the volume form $\omega_L$ is equal
\begin{equation}
\Phi^* \omega_L(X) = c(\Phi^* \alpha^1(X)) \wedge \cdots \wedge (\Phi^* \alpha^n(X)) = c\phi(\text{ad}_X)^*X^1 \wedge \cdots \wedge \phi(\text{ad}_X)^*X^n
= \det \phi(\text{ad}_X)cX^1 \wedge \cdots \wedge X^n. \tag{39}
\end{equation}

This proves \cite{55}.
The function $F(X)$ is symmetric: $F(X) = F(-X)$. Note also, that for unimodular groups (so in particular for all compact groups)

$$\frac{dm(s)}{dn} \bigg|_q = \det \Ad_q = 1$$

(40)

for every $q \in G$, so that $F(X) = \sqrt{\det \phi(\text{ad}_X)}$.

### 2.6 Spaces $\mathcal{F}(M)$ and $\mathcal{L}(M)$

Let $L^2(G \times O, dn)$ denote the Hilbert space of $\mathbb{C}$-valued square integrable functions on $G \times \mathfrak{g}$ with support in $G \times \overline{O}$ (by $\overline{O}$ we denote the closure of $O$). The scalar product in this space is given by

$$(f, g) = \int_{G \times \mathfrak{g}} f(q, X)g(q, X) \, dn(q, X).$$

(41)

Moreover, let $\tilde{\mathcal{F}}(M)$ be a subspace of $L^2(G \times O, dn)$ consisting of functions $\tilde{f}$ such that the functions

$$f(a, b) = \tilde{f}(a \exp(\frac{i}{\hbar}V_a(b)), V_a(b))F(V_a(b))^{-1},$$

(42)

defined on a dense subset $\{(a, b) \in G \times \mathfrak{g} \mid a^{-1}b \in U\}$ of $G \times \mathfrak{g}$, extend to smooth functions with compact support defined on the whole space $G \times G$. Note, that since $F(X)$ is continuous on $\mathfrak{g}$ and smooth on $O$ functions in $\tilde{\mathcal{F}}(M)$ are bounded, have compact support and are smooth on $G \times O$. If a function $\tilde{f} \in \tilde{\mathcal{F}}(M)$ has a support in $G \times \overline{O}$, then it will be smooth on the whole set $G \times \mathfrak{g}$. However, if the support of $\tilde{f}$ is not in $G \times O$ but only in its closure $G \times \overline{O}$, then $\tilde{f}$ may not be smooth (or even continuous) on $G \times \mathfrak{g}$. As we will see in Section 4, the functions (42) are the integral kernels of operators in the corresponding operator representation. So that, we define $\tilde{\mathcal{F}}(M)$ in such a way that the corresponding space of operators will consist of all integral operators whose integral kernels are smooth and compactly supported.

Denote by $\mathcal{L}(M)$ and $\mathcal{F}(M)$ the images of $L^2(G \times O, dn)$ and $\tilde{\mathcal{F}}(M)$ with respect to the inverse Fourier transform in momentum variable. The set $\mathcal{L}(M)$ obtains the structure of a Hilbert space from $L^2(G \times O, dn)$, where the scalar product on $\mathcal{L}(M)$ takes the form

$$(f, g) = \int_M \overline{f(x)}g(x) \, dl(x).$$

(43)

The space $\mathcal{L}(M)$ is a Hilbert subspace of the space $L^2(M, dl)$ of $\mathbb{C}$-valued square integrable functions on $M$. Since functions in $\mathcal{F}(M)$ are inverse momentum Fourier transforms of bounded functions with compact support, then it is easy to show that $\mathcal{F}(M)$ consists of smooth functions. Note, that elements of $\mathcal{F}(M)$ may not be Lebesgue integrable. Therefore, integrals of these functions will be considered as improper integrals defined as in Section 2.3.

On the space $\mathcal{F}(M)$ we define a trace functional $\text{tr}$ by the formula

$$\text{tr}(f) = \int_M f(x) \, dl(x).$$

(44)

By $\text{Tr}$ we will denote the usual operator trace.

Inverting formula (42) we can see that elements of $\mathcal{F}(M)$ are all functions $f$ of the form

$$f(q, p) = \int_O f(q \exp(-\frac{i}{\hbar}X), q \exp(\frac{i}{\hbar}X))e^{-\frac{i}{\hbar}(p, X)}F(X) \, dX$$

(45)

for functions $f \in C_c^\infty(G \times \mathfrak{g})$. Therefore, we have a one-to-one correspondence between elements of $\mathcal{F}(M)$ and $C_c^\infty(G \times \mathfrak{g})$. In the case $G$ is a compact group we will topologize $\mathcal{F}(M)$ by the following family of semi-norms

$$\|f\|_{k,l} = \sup_{a, b \in G} |D^{k,l}f(a, b)|,$$

(46)
where \( k, l \in \mathbb{N}^n \) are multi-indices and \( D^{k,l} \) are differential operators expressed by left-invariant vector fields \( L_{X_1}, \ldots, L_{X_n} \) corresponding to a basis \( X_1, \ldots, X_n \) in \( \mathfrak{g} \), and defined by the formula

\[
D^{k,l} = (L_{X_1})^{k_1} \cdots (L_{X_n})^{k_n} (L_{X_1})^{l_1} \cdots (L_{X_n})^{l_n},
\]

(47)

where \( L_{X_i}^{(1)} \) and \( L_{X_i}^{(2)} \) are vector fields on \( G \times G \) defined by

\[
L_{X_i}^{(1)}(a, b) = L_{X_i}(a) \oplus 0, \quad L_{X_i}^{(2)}(a, b) = 0 \oplus L_{X_i}(b)
\]

(48)

according to the identification \( T_{(a, b)}(G \times G) = T_a G \oplus T_b G \). The introduced topology on \( \mathcal{F}(M) \) is independent on the choice of a basis \( \{ X_i \} \) in \( \mathfrak{g} \). It makes from \( \mathcal{F}(M) \) a Fréchet space.

Note, that since \( C_c^\infty(G \times G) \subset L^2(G \times G, dm \times dm) \) we can introduce for \( f, g \in C_c^\infty(G \times G) \) their scalar product \( (f, g) \). A similar consideration takes place for the space \( \mathcal{F}(M) \subset \mathcal{L}(M) \). We will prove a useful lemma.

**Lemma 1.** Let \( f, g \in \mathcal{F}(M) \) and \( f, g \in C_c^\infty(G \times G) \) be the corresponding integral kernels. Then,

\[
(f, g) = (f, g).
\]

(49)

**Proof.** We calculate that

\[
(f, g) = \int_{G \times G} \overline{f(a, b)} g(a, b) \, dm(a) \, dm(b)
\]

\[
= \int_G \left( \int_G \overline{f(a \exp(\frac{i}{\hbar} V_a(b)), V_a(b))} \, \bar{g}(a \exp(\frac{i}{\hbar} V_a(b)), V_a(b)) \, F(V_a(b))^{-2} \, dm(b) \right) \, dm(a)
\]

\[
= \int_G \left( \int_G \overline{f(a \exp(\frac{i}{\hbar} X, X))} \, \bar{g}(a \exp(\frac{i}{\hbar} X, X)) \, \det \exp(-\frac{i}{\hbar} X) \, dm(a) \right) \, dX
\]

\[
= \int_{G \times G} \overline{f(a, X)} \bar{g}(a, X) \, dm(a) \, dX = (\bar{f}, \bar{g}) = (f, g).
\]

(50)

\[
\square
\]

### 3 Deformation quantization of the classical system

#### 3.1 Star-product on \( \mathcal{F}(M) \)

On the space \( \mathcal{F}(M) \) we introduce the following star-product

\[
(f \star g)(q, p) = \int_{G \times G} \bar{f}(q \exp(-\frac{i}{\hbar} (X \circ Y)) \exp(\frac{i}{\hbar} X, X)) \, \bar{g}(q \exp(\frac{i}{\hbar} (X \circ Y)) \exp(-\frac{i}{\hbar} Y, Y) e^{-\frac{i}{\hbar} \langle p, X \circ Y \rangle} \\
\times L(X, Y) \, dX \, dY.
\]

(51)

Such star-product was already considered in \( \cite{22} \) from the perspective of formal deformation quantization. Note, that since \( \bar{f} \) and \( \bar{g} \) have supports in \( G \times \mathcal{O} \) the \( \star \)-product of \( f \) and \( g \) is a well defined function on \( M \). In what follows we will show that \( f \star g \in \mathcal{F}(M) \) so that the space \( \mathcal{F}(M) \) together with the \( \star \)-product becomes an algebra.

For \( Y \in \mathcal{O} \) let \( \mathcal{R}_Y \) and \( \mathcal{L}_Y \) be maps defined in \( \cite{30} \). With the use of \( \cite{71} \) we can calculate Jacobians of these transformations:

\[
|\det T_X \mathcal{R}_Y| = \left( \frac{F(X)}{F(X \circ Y)} \right)^2 \det \exp(-\frac{i}{\hbar} Y),
\]

(52a)

\[
|\det T_X \mathcal{L}_Y| = \left( \frac{F(X)}{F(Y \circ X)} \right)^2 \det \exp(\frac{i}{\hbar} Y).
\]

(52b)
If in (51) under the integral with respect to $X$ we perform the following change of variables: $X \rightarrow \mathcal{R}_Y(X)$, then we can write the $\ast$-product (51) in the form

$$(f \ast g)(q, p) = \int_{g \times g} \tilde{f}(q \exp(-\frac{1}{2}X) \exp(\frac{1}{2}(X \circ Y)), X \circ Y) \tilde{g}(q \exp(\frac{1}{2}X) \exp(\frac{1}{2}Y), -Y) e^{-\frac{1}{2}(p, X)}$$

$$\times L(X, Y) \det \text{Ad}_\exp(-\frac{1}{2}Y) \, dX \, dY. \quad (53)$$

We can introduce a twisted convolution of functions $\tilde{f}, \tilde{g} \in \tilde{\mathcal{F}}(M)$ by the formula

$$(\tilde{f} \circ \tilde{g})(q, X) = \int_{g} \tilde{f}(q \exp(-\frac{1}{2}X) \exp(\frac{1}{2}(X \circ Y)), X \circ Y) \tilde{g}(q \exp(\frac{1}{2}X) \exp(\frac{1}{2}Y), -Y)$$

$$\times L(X, Y) \det \text{Ad}_\exp(-\frac{1}{2}Y) \, dY. \quad (54)$$

Then from (53) we immediately get that

$$(f \ast g)^\sim = \tilde{f} \circ \tilde{g}. \quad (55)$$

The following theorem gathers basic properties of the $\ast$-product.

**Theorem 1.** The $\ast$-product on $\mathcal{F}(M)$ is a bilinear operation with the following properties:

(i) $f \ast g \in \mathcal{F}(M)$ (the space $\mathcal{F}(M)$ is closed with respect to the $\ast$-product),

(ii) $(f \ast g) \ast h = f \ast (g \ast h)$ (associativity),

(iii) $f \ast g = g \ast f$ (complex-conjugation is an involution),

(iv) $\int_{M} f \ast g \, dx = \int_{M} f(x)g(x) \, dx,$

for $f, g, h \in \mathcal{F}(M)$.

**Proof.** If $f, g \in C_c^\infty(G \times G)$ are functions corresponding to $\tilde{f}$ and $\tilde{g}$ according to (42), then by virtue of the equality

$$V_a(b) \circ Y = V_a(b \exp(Y)), \quad (56)$$

we get the following expression for a function $h$ corresponding to $\tilde{f} \circ \tilde{g}$

$$h(a, b) = (\tilde{f} \circ \tilde{g})(a \exp(\frac{1}{2}V_a(b)), V_a(b)) F(V_a(b))^{-1} = \int_{g} \tilde{f}(a \exp(\frac{1}{2}V_a(b \exp(Y))), V_a(b \exp(Y)))$$

$$\times \tilde{g}(b \exp(\frac{1}{2}Y), -Y) F(V_a(b \exp(Y)))^{-1} F(Y)^{-1} \frac{dm(L_a \circ \exp)}{dY} \bigg|_{Y} \, dY$$

$$= \int_{G} \tilde{f}(a \exp(\frac{1}{2}V_a(c)), V_a(c)) F(V_a(c))^{-1} \tilde{g}(c \exp(\frac{1}{2}V_c(b)), V_c(b)) F(V_c(b))^{-1} \, dm(c)$$

$$= \int_{G} f(a, c)g(c, b) \, dm(c). \quad (57)$$

The function

$$h(a, b) = \int_{G} f(a, c)g(c, b) \, dm(c) \quad (58)$$

is smooth and compactly supported. Indeed, $\text{supp}(f) \subset \pi_1(\text{supp}(f)) \times \pi_2(\text{supp}(f))$ and similarly $\text{supp}(g) \subset \pi_1(\text{supp}(g)) \times \pi_2(\text{supp}(g))$, where $\pi_i: G \times G \rightarrow G$, $i = 1, 2$, are canonical projections onto the first and second component of the product manifold $G \times G$. Thus $\text{supp}(h) \subset \pi_1(\text{supp}(f)) \times \pi_2(\text{supp}(g))$ and since $\pi_1$ and $\pi_2$ are continuous the support of $h$ is a subset of a compact set and as such is also compact. The value of the integral in (58) does not change if we restrict the integration to a compact set $\pi_2(\text{supp}(f)) \cap \pi_1(\text{supp}(g))$. Using this fact we can interchange the integration with a limiting operation and easily show that partial derivatives of
any order of \( h \) exist and are jointly continuous as functions on \( G \times G \). Therefore, \( h \in C_c^\infty(G \times G) \) which further implies that \( \hat{f} \circ \hat{g} \in \hat{F}(M) \) and consequently \( f \ast g \in F(M) \).

We have that

\[
((\hat{f} \circ \hat{g}) \circ \hat{h})(q, X) = \int_g \left( \int_g (\tilde{f}(q \exp(-\frac{1}{2}X) \exp(\frac{1}{2}(X \circ Y \circ Z)), X \circ Y \circ Z) \times \tilde{g}(q \exp(\frac{1}{2}X) \exp(\frac{1}{2}Z), -Z) \tilde{h}(q \exp(\frac{1}{2}X) \exp(\frac{1}{2}Y), -Y) \times L(X, Y)L(X \circ Y, Z) \det Ad_{\exp(-\frac{1}{2}Y)} \exp(-\frac{1}{2}Z) \right) dY \\
= \int_O \left( \int_{\mathcal{O}_Y} (\tilde{f}(q \exp(-\frac{1}{2}X) \exp(\frac{1}{2}(X \circ L_Y(Z))), X \circ L_Y(Z)) \times \tilde{g}(q \exp(\frac{1}{2}X) \exp(\frac{1}{2}(L_Y(Z) \circ Y)), (L_Y(Z) \circ Y)) \times \tilde{h}(q \exp(\frac{1}{2}X) \exp(\frac{1}{2}(L_Y(Z) \circ Y)), (L_Y(Z) \circ Y)) \times \det Ad_{\exp(-\frac{1}{2}Y)} \exp(-\frac{1}{2}L_Y(Z)) \right) dY \\
= \int_g (f \circ (\hat{g} \circ \hat{h}))(q, X).
\]  

Applying (55) yields the associativity of \( \ast \).

This property follows directly from the definition (51) of the \( \ast \)-product.

We calculate that

\[
\int_M f \ast g \, dx = \int_G (f \ast g)(q, 0) \, dm(q) = \int_G (\tilde{f} \circ \tilde{g})(q, 0) \, dm(q) \\
= \int_G \left( \int_g (f(q \exp(\frac{1}{2}Y))Y) \tilde{g}(q \exp(\frac{1}{2}Y), -Y) \det Ad_{\exp(-\frac{1}{2}Y)} \, dY \right) \, dm(q) \\
= \int_G \left( \int_g (\tilde{f}(q, Y)Y) \, dY \right) \, dm(q) = \int_G (\tilde{f} \circ \tilde{g})(q, 0) \, dm(q) = \int_M f(x)g(x) \, dx,
\]

where \( \ast \) denotes the usual convolution.

From (57) we can see that the integral kernel \( h \in C_c^\infty(G \times G) \) corresponding to the product \( f \ast g \) is expressed by the integral kernels \( f, g \in C_c^\infty(G \times G) \) of \( f \) and \( g \) by the formula

\[
h(a, b) = \int_G f(a, c)g(c, b) \, dm(c).
\]

From this, in the case of a compact group \( G \), we get the following estimate on the semi-norms of the product \( f \ast g \):

\[
||f \ast g||_{k,l} \leq m(G)||f||_{k,0}||g||_{0,l},
\]

where \( m(G) \) is the measure of the group \( G \). Therefore, the \( \ast \)-product as a map \( F(M) \times F(M) \to F(M) \) will be a continuous operation on the Fréchet space \( F(M) \).

At this point a remark should be made concerning the dependence of the quantization on the global topological and geometrical structure of the Lie group \( G \). It might seem that we are neglecting all global issues by working only on an open subset \( \mathcal{U} \) of the Lie group \( G \), which we then map to an open subset \( \mathcal{O} \) of the Lie algebra \( \mathfrak{g} \). However, the global topological and geometrical structure of \( G \) is encoded in the definition
of the algebra $\mathcal{F}(M)$. This may be readily seen from the fact that there is a one-to-one correspondence between elements of $\mathcal{F}(M)$ and $C^\infty_c(G \times G)$, where to the $*$-product of functions corresponds a composition of integral kernels given by (61). Clearly the space $C^\infty_c(G \times G)$ depends on the global structure of $G$ and so is $\mathcal{F}(M)$.

In what follows we will investigate the dependence of $\mathcal{F}(M)$ on $\hbar$. For this purpose we will change, for a moment, the definition of the Fourier transform in the momentum variable in the following way

$$
\hat{f}(q, X) = \frac{1}{(2\pi)^n} \int f(q, p) e^{i(p, X) \hbar} \, dp,
$$

so that it will not depend on $\hbar$. Then we also have to redefine the space $\mathcal{F}(M)$ as the space of square integrable functions $f$ with support in $G \times h^{-1} \mathcal{F}$ such that the functions

$$
f(a, b) = |\hbar|^{-n} \hat{f}(a \exp(2iV_a(b)), h^{-1}V_a(b)) F(V_a(b))^{-1}
$$

extend to smooth functions on $G \times G$ with compact support. We will explicitly denote the dependence of the spaces $\mathcal{F}(M)$ and $\mathcal{F}_h(M)$ on $\hbar$ by writing $\mathcal{F}(M)$ and $\mathcal{F}_h(M)$. Moreover, we will denote by $\mathcal{F}_h(M)$ the space $C^\infty_c(G \times g)$ of smooth functions with compact support and by $\mathcal{F}_0(M)$ its inverse momentum Fourier transform. Then for any $f \in \mathcal{F}_0(M)$ there exists a sufficiently small $\hbar_0 > 0$ for which $f \in \mathcal{F}_h(M)$ for every $\hbar \in \mathbb{R}$ such that $|\hbar| < \hbar_0$.

Introducing a Baker-Campbell-Hausdorff product $\circ_h$ on the Lie algebra $g$ corresponding to a Lie bracket $[\cdot, \cdot]_h = \hbar[\cdot, \cdot]$, we can write the *-product in the following way

$$
(f *_h g)(q, p) = \int_{\mathfrak{g}} \hat{f}(q \exp(-\frac{1}{\hbar} h(X \circ_h Y)) \exp(\frac{1}{\hbar} h(X)), X)\hat{g}(q \exp(\frac{1}{\hbar} h(X \circ_h Y)) \exp(-\frac{1}{\hbar} h(Y)), Y)e^{-i(p, X \circ_h Y)}
$$

$$
\times L_h(X, Y) \, dX \, dY,
$$

where $L_h(X, Y) = F_h(X \circ_h Y)^{-1} F_h(X) F_h(Y)$ and $F_h(X) = F(hX)$. From this presentation of the *-product it can be easily seen that for $f, g \in \mathcal{F}_0(M)$ and a sufficiently small $\hbar_0 > 0$ the functions $f, g \in \mathcal{F}_h(M)$ for every $\hbar \in \mathbb{R}$ such that $|\hbar| < \hbar_0$, and for any $x \in M$ the function $\hbar \mapsto (f *_h g)(x)$ is smooth on $(-\hbar_0, \hbar_0)$ and $(f *_h g)(x) \to f(x)g(x)$ as $\hbar \to 0$. From the following theorem we also get that $[f, g]_h(x) \to [f, g](x)$ as $\hbar \to 0$, where $\{\cdot, \cdot\}$ is the canonical Poisson bracket on $M$.

**Theorem 2.** The *-product enjoys the following power series expansion in $\hbar$ around $\hbar = 0$ up to third order:

$$
f *_h g = fg + \frac{i\hbar}{2}\{f, g\} + \frac{1}{2!} \left(\frac{i\hbar}{2}\right)^2 \left( B_2(f, g) + B_2(g, f) \right) + \frac{1}{3!} \left(\frac{i\hbar}{2}\right)^3 \left( B_3(f, g) - B_3(g, f) \right) + O(\hbar^4),
$$

where

$$
B_2(f, g) = Z^i Z^j f Y_i Y_j g - Z^i Y_j f Z^i Y_i g + C_{ijk} Z^j f Z^i Y_i g - 2p_{ik} C^i_{jk} Z^i Y_i f Z^j Z^i g - \frac{1}{6} C_{ijk} C^i_{jk} Z^i f Z^j Z^k g
$$

$$
+ \frac{1}{2} p_{ik} p_{lm} C^i_{jk} C^l_{km} Z^i f Z^k Z^l g + \frac{2}{3} p_{ik} C^i_{jk} Z^j f Z^i Z^l g,
$$

$$
B_3(f, g) = Z^i Z^j Z^k f Y_i Y_j Y_k g - 3 Z^i Z^j Y_k f Z^i Y_j Y_k g + 3 C_{ijk} Z^i f Z^j Y_k g - 3 C_{ijk} Z^i Y_i f Z^j Z^k g
$$

$$
+ C_{ijk} C^i_{jk} Z^j f Z^i Z^j g + 3 p_{ik} C^i_{jk} Z^j Y_k f Z^i Z^j g - 3 p_{ik} C^i_{jk} Z^j Y_i f Z^i Z^k g
$$

$$
- 3 p_{ik} C^i_{jk} Z^j Y_k f Z^i Z^j g - 2 p_{ik} C^i_{jk} Z^j f Z^i Y_k g + 2 p_{ik} C^i_{jk} Z^j Y_i f Z^i Y_k g
$$

$$
- 3 p_{ik} C^i_{jk} C^l_{km} Z^i f Z^j Z^k g - \frac{1}{2} p_{ik} p_{lm} C^i_{jk} C^l_{km} Z^i f Z^j Z^k g + \frac{1}{2} p_{ik} p_{lm} C^i_{jk} C^l_{km} Z^i f Z^j Z^k g
$$

$$
+ \frac{1}{2} p_{ik} p_{lm} C^i_{jk} C^l_{km} Z^i f Z^j Z^k g + \frac{1}{2} p_{ik} p_{lm} C^i_{jk} C^l_{km} Z^i f Z^j Z^k g + 2 p_{ik} p_{lm} C^i_{jk} C^l_{km} Z^i f Z^j Z^k g
$$

$$
+ \frac{1}{2} p_{ik} p_{lm} C^i_{jk} C^l_{km} Z^i f Z^j Z^k g + \frac{1}{2} p_{ik} p_{lm} C^i_{jk} C^l_{km} Z^i f Z^j Z^k g.
$$
Moreover, we calculate that
\[
(f \ast_h g)(x) = f(x)g(x) + \frac{d}{dh}(f \ast_h g)(x) \bigg|_{h=0} + \frac{1}{2!} \frac{d^2}{dh^2}(f \ast_h g)(x) \bigg|_{h=0} h^2 \\
+ \frac{1}{3!} \frac{d^3}{dh^3}(f \ast_h g)(x) \bigg|_{h=0} h^3 + O(h^4).
\]

Therefore, we have to calculate derivatives with respect to \(h\) of \((f \ast_h g)(x)\) at \(h = 0\).

For \(\varphi \in C^\infty(G)\) and a smooth function \(A : \mathbb{R} \to \mathfrak{g}\) such that \(A(0) = 0\) we get
\[
\frac{d}{dh}\varphi(q \exp(A(h))) = T_q \exp(A(h)) \varphi(T_e L_q \exp(A(h)) (\text{ad}_{A(h)} A'(h))) = T_q \exp(A(h)) \varphi(T_e L_q \exp(A(h)) B(h))
\]
\[
= T_q \exp(A(h)) \varphi(L_B(h)(q \exp(A(h)))) = L_B(h) \varphi(q \exp(A(h))),
\]
where \(B(h) = \phi(\text{ad}_{A(h)}) A'(h)\) and \(L_B(h)\) denotes a left-invariant vector field corresponding to \(B(h) \in \mathfrak{g}\).

Moreover, we calculate that
\[
\frac{d^2}{dh^2}\varphi(q \exp(A(h))) = L_B(h) L_B(h) \varphi(q \exp(A(h))) + L_{B'(h)} \varphi(q \exp(A(h)));
\]
\[
\frac{d^3}{dh^3}\varphi(q \exp(A(h))) = L_B(h) L_B(h) L_B(h) \varphi(q \exp(A(h))) + 2 L_B(h) L_{B'(h)} \varphi(q \exp(A(h)))
\]
\[
+ L_{B'(h)} L_{B(h)} \varphi(q \exp(A(h))) + L_{B''(h)} \varphi(q \exp(A(h))).
\]

At \(h = 0\) the function \(B(h)\) and its derivatives are equal
\[
B(0) = A'(0), \quad B'(0) = A''(0), \quad B''(0) = A'''(0) = \frac{1}{2}[A'(0), A''(0)].
\]

Therefore,
\[
\frac{d}{dh}\varphi(q \exp(A(h))) \bigg|_{h=0} = L_A'(0) \varphi(q),
\]
\[
\frac{d^2}{dh^2}\varphi(q \exp(A(h))) \bigg|_{h=0} = L_A''(0) L_A(0) \varphi(q) + L_{A'(0)} \varphi(q),
\]
\[
\frac{d^3}{dh^3}\varphi(q \exp(A(h))) \bigg|_{h=0} = L_A''(0) L_A'(0) L_A(0) \varphi(q) + 3 L_A''(0) L_A(0) \varphi(q)
\]
\[
- \frac{3}{2} L_{[A'(0), A''(0)]} \varphi(q) + L_{A'''(0)} \varphi(q).
\]

For \(A(h) = h((-\frac{1}{2} X \circ_h Y)) \circ_h (\frac{1}{2} X)\) with the help of the expansion \([71]\) we can calculate that
\[
A'(0) = -\frac{1}{2} Y, \quad A''(0) = -\frac{1}{4}[X, Y], \quad A'''(0) = -\frac{3}{16}[Y, [Y, X]].
\]

Hence,
\[
\frac{d}{dh} \tilde{f}(q \exp(-\frac{1}{2} h(X \circ_h Y))) \exp(\frac{1}{2} hX), X) \bigg|_{h=0} = -\frac{1}{2} (L_Y f)\sim(q, X),
\]
\[
\frac{d^2}{dh^2} \tilde{f}(q \exp(-\frac{1}{2} h(X \circ_h Y))) \exp(\frac{1}{2} hX), X) \bigg|_{h=0} = \frac{1}{4} (L_Y L_Y f)\sim(q, X) - \frac{1}{4} (L_{[X,Y]} f)\sim(q, X),
\]
\[
\frac{d^3}{dh^3} \tilde{f}(q \exp(-\frac{1}{2} h(X \circ_h Y))) \exp(\frac{1}{2} hX), X) \bigg|_{h=0} = -\frac{1}{8} (L_Y L_Y L_Y f)\sim(q, X) + \frac{3}{8} (L_Y L_{[X,Y]} f)\sim(q, X),
\]

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Therefore,

\[
\frac{d}{dh} \hat{g}(q \exp(\frac{1}{2}h(X \circ_h Y)) \exp(-\frac{1}{2}hY), Y) \bigg|_{h=0} = \frac{1}{2} (L_X g)^{(q, Y)},
\]
\[
\frac{d^2}{dh^2} \hat{g}(q \exp(\frac{1}{2}h(X \circ_h Y)) \exp(-\frac{1}{2}hY), Y) \bigg|_{h=0} = \frac{1}{4} (L_X L_X g)^{(q, Y)} + \frac{1}{4} (L_{[X,Y]} g)^{(q, Y)}, \quad (75)
\]
\[
\frac{d^3}{dh^3} \hat{g}(q \exp(\frac{1}{2}h(X \circ_h Y)) \exp(-\frac{1}{2}hY), Y) \bigg|_{h=0} = \frac{1}{8} (L_X L_X L_X g)^{(q, Y)} + \frac{3}{8} (L_X L_{[X,Y]} g)^{(q, Y)}.
\]

The function \( F(X) \) can be written in the form

\[
F(X) = \sqrt{\text{det} \lambda(ad_X)} = \sqrt{\text{det}(\exp(\ln(\lambda(ad_X))))} = \exp \left( \frac{1}{2} \text{Tr} (\ln(\lambda(ad_X))) \right).
\] (76)

Therefore,

\[
L(X, Y) = \exp \left( \frac{1}{2} \text{Tr} (\ln(\lambda(ad_X)) + \ln(\lambda(ad_Y)) - \ln(\lambda(ad_{[X,Y]}))) \right).
\] (77)

Using the above formula the derivatives of the function \( L_h(X, Y) = L(hX, hY) \) can be easily calculated:

\[
\frac{d}{dh} L_h(X, Y) \bigg|_{h=0} = 0, \quad \frac{d^2}{dh^2} L_h(X, Y) \bigg|_{h=0} = -\frac{1}{12} \text{Tr}(ad_X \circ ad_Y),
\]
\[
\frac{d^3}{dh^3} L_h(X, Y) \bigg|_{h=0} = -\frac{1}{8} \text{Tr}(ad_{X+Y} \circ ad_{[X,Y]}).
\] (78)

With the help of the expansion (31) of the product \( \circ_h \) we get the following derivatives of the exponent:

\[
\frac{d}{dh} e^{-i(p, X \circ_h Y)} \bigg|_{h=0} = \frac{1}{2} (p, [X, Y]) e^{-i(p, X+Y)},
\]
\[
\frac{d^2}{dh^2} e^{-i(p, X \circ_h Y)} \bigg|_{h=0} = \left( -\frac{1}{4} (p, [X, Y]) (p, [X, Y]) - \frac{1}{6} i (p, [X - Y, [X, Y]]) \right) e^{-i(p, X+Y)},
\]
\[
\frac{d^3}{dh^3} e^{-i(p, X \circ_h Y)} \bigg|_{h=0} = \left( \frac{i}{8} (p, [X, Y]) (p, [X, Y]) (p, [X, Y]) - \frac{1}{4} (p, [X, Y]) (p, [X - Y, [X, Y]])
\]
\[+ \frac{1}{4} (p, [Y, [X, Y]]) \right) e^{-i(p, X+Y)}. \] (79)

Let \( X_1, \ldots, X_n \) be a basis in \( g \) and \( Y_1, \ldots, Y_n, Z^1, \ldots, Z^n \) a corresponding frame fields on \( M \) as in Section 2.2. Moreover, let \( p_j : M \to \mathbb{R} \) be fiber variables defined by \( p_j(q, p) = (p, X_j) \) and \( C^k_{ij} \) the structure constants of \( g \) in the basis \( \{X_i\} \). After expanding \( X, Y \) in the basis \( \{X_i\} \) and using formulas (74), (75), (78), and (79) we can calculate the derivatives with respect to \( h \) of the \( \ast \)-product. Formula (69) then follows. \( \square \)

### 3.2 Extension of the \( \ast \)-product to the Hilbert space \( L(M) \)

In what follows we will extend the \( \ast \)-product to the Hilbert space \( L(M) \) introduced in Section 2.6. The space \( \mathcal{F}(M) \) inherits from \( L(M) \) the scalar product (43). The norm corresponding to this scalar product will be denoted by \( \| \cdot \|_2 \).

**Theorem 3.** For \( f, g \in \mathcal{F}(M) \) the following inequality holds

\[
\|f \ast g\|_2 \leq \|f\|_2 \|g\|_2.
\] (80)
Proof. Let \( f, g \in C_c^\infty(G \times G) \) correspond to \( f \) and \( g \) as in (12). Then, in accordance to (57), a function \( h \in C_c^\infty(G \times G) \) corresponding to the product \( f \star g \) will be given by the formula (61). The functions \( f, g, h \) are elements of the Hilbert space \( L^2(G \times G, dm \times dm) \). By virtue of Lemma 1 their \( L^2 \)-norms are equal to the \( L^2 \)-norms of \( f, g \) and \( f \star g \). Therefore, the inequality (80) is equivalent to the following inequality
\[
\|h\|_2 \leq \|f\|_2 \|g\|_2. \tag{81}
\]

For fixed \( a, b \in G \) we can apply the Schwartz inequality to functions \( f(a, \cdot), g(\cdot, b) \in L^2(G, dm) \) receiving the following inequality
\[
\left| \int_G f(a,c)g(c,b) \, dm(c) \right|^2 \leq \int_G |f(a,c)|^2 \, dm(c) \int_G |g(d,b)|^2 \, dm(d). \tag{82}
\]

With its use we calculate that
\[
\|h\|_2^2 = \int_G \int_G |h(a,b)|^2 \, dm(a) \, dm(b) = \int_G \int_G \left| \int_G f(a,c)g(c,b) \, dm(c) \right|^2 \, dm(a) \, dm(b) \\
\leq \int_G \int_G \left( \int_G |f(a,c)|^2 \, dm(c) \int_G |g(d,b)|^2 \, dm(d) \right) \, dm(a) \, dm(b) \\
= \int_G \int_G |f(a,c)|^2 \, dm(a) \, dm(c) \int_G \int_G |g(d,b)|^2 \, dm(d) \, dm(b) = \|f\|_2^2 \|g\|_2^2, \tag{83}
\]

which proves (81) and consequently (80). \( \square \)

The consequence of the inequality (80) is continuity of the \( \star \)-product in the \( L^2 \)-norm. From this property and the fact that \( \mathcal{F}(M) \) is dense in \( \mathcal{L}(M) \) we can continuously extend the \( \star \)-product to the whole Hilbert space \( \mathcal{L}(M) \). Note, that since complex-conjugation is continuous in the \( L^2 \)-norm it will remain an involution for the extended \( \star \)-product. Hence, \( \mathcal{L}(M) \) is an involutive algebra with an inner product \((\cdot, \cdot)\), which satisfies the following properties:

(i) for every \( f \in \mathcal{L}(M) \) the maps \( g \mapsto f \star g \) and \( g \mapsto g \star f \) are bounded on \( \mathcal{L}(M) \) (multiplication is a bounded operator),

(ii) \( (f \star g, h) = (g, f \star h) \) and \( (g \star f, h) = (g, h \star f) \) (involution is the adjoint),

(iii) \( (f, g) = (\overline{g}, f) \) (involution is an antilinear isometry),

(iv) \( \mathcal{L}(M) \star \mathcal{L}(M) \) is linearly dense in \( \mathcal{L}(M) \).

Therefore, \( \mathcal{L}(M) \) is a Hilbert algebra. Observe, that for \( f, g \in \mathcal{F}(M) \), by virtue of the property (iv) of Theorem 1, we can write the scalar product of \( f, g \) in the form
\[
(f, g) = \int_M \overline{f} \star g \, dl. \tag{84}
\]

We can also extend the trace functional \( \text{tr} \) introduced in (14) to the whole space \( \mathcal{L}^2(M) = \mathcal{L}(M) \star \mathcal{L}(M) \). It then follows that
\[
(f, g) = \text{tr}(\overline{f} \star g), \quad f, g \in \mathcal{L}(M). \tag{85}
\]

### 3.3 \( C^* \)-algebra of observables and states

In Section 3.2 we presented an extension of the algebra \( \mathcal{F}(M) \) to the Hilbert algebra \( \mathcal{L}(M) \). In what follows we will extend \( \mathcal{L}(M) \) to a \( C^* \)-algebra. Let us introduce on the space \( \mathcal{L}(M) \) the following norm:
\[
\|f\| = \sup\{ \|f \star h\|_2 \mid h \in \mathcal{F}(M), \|h\|_2 = 1 \}. \tag{86}
\]

This is a \( C^* \)-norm, i.e. it satisfies
Theorem 5. If \( \rho \in L(M) \) satisfies

\[
\|f \ast g\| \leq \|f\|\|g\|,
\]

(ii) \( \|f\| = \|f\| \),

(iii) \( \|f \ast f\| = \|f\|^2 \),

for \( f, g \in L(M) \). Indeed, this follows directly from the fact that the map \( h \mapsto f \ast h \) is a bounded linear operator on the Hilbert space \( L(M) \) defined on a dense domain \( F(M) \), and the norm \( \|f\| \) is just the operator norm of this operator.

Since \( \|f\| \) is the smallest constant \( C \) satisfying the inequality

\[
\|f \ast h\|_2 \leq C\|h\|_2 \text{ for all } h \in F(M), \tag{87}
\]

it is clear from (80) that \( \|f\| \leq \|f\|_2 \), and so convergence in \( L^2 \)-norm implies convergence in the norm \( \| \cdot \| \).

The spaces \( F(M) \) and \( L(M) \) are not complete with respect to the \( C^* \)-norm \( \| \cdot \| \), thus \( F(M) \) and \( L(M) \) are only pre-\( C^* \)-algebras. However, they can be completed to a \( C^* \)-algebra. This completion will be denoted by \( A(M) \). The algebra \( A(M) \) is a \( C^* \)-algebra of observables.

In what follows we will explicitly denote the dependence of the algebra \( A(M) \) on \( h \) by writing \( A_h(M) \). Moreover, we will denote by \( A_0(M) \) the commutative \( C^* \)-algebra of \( C \)-valued continuous functions on \( M \) which vanish at infinity with the usual supremum norm \( \|f\|_0 = \sup_{x \in M} |f(x)| \). The space \( F_0(M) \) introduced in Section 3.1 is dense in \( A_0(M) \). It happens that the field of \( C^* \)-algebras, \( h \mapsto A_h(M) \), is a strict deformation quantization of the symplectic manifold \( M \). In fact, there holds:

**Theorem 4.** For any \( f, g \in F_0(M) \) there exists \( h_0 > 0 \) such that for every \( h \in \mathbb{R}, |h| < h_0 \) the functions \( f, g \in A_h(M) \) and

(i) the map \( h \mapsto \|f\|_h \) is continuous on \(( -h_0, h_0) \),

(ii) \( \|f \ast h g - f \cdot g\|_h \to 0 \) as \( h \to 0 \),

(iii) \( \|[f, g]_h - \{f, g\}\|_h \to 0 \) as \( h \to 0 \).

The above theorem can be proved with the help of the operator representation introduced in Section 4 and through similar considerations as in [28, 32].

The \( C^* \)-algebra of observables \( A(M) \) can be used to define states of the system. From definition a state is a continuous linear functional \( \Lambda: A(M) \to \mathbb{C} \), which is positively defined and normalized to unity, i.e.

(i) \( \|\Lambda\| = 1 \),

(ii) \( \Lambda(f \ast f) \geq 0 \) for every \( f \in A(M) \).

The set of all states is convex. Extreme points of this set are called pure states. These are states which cannot be written as convex linear combinations of some other states. In other words \( \Lambda_{\text{pure}} \) is a pure state if and only if there do not exist two different states \( \Lambda_1 \) and \( \Lambda_2 \) such that \( \Lambda_{\text{pure}} = p\Lambda_1 + (1 - p)\Lambda_2 \) for some \( p \in (0, 1) \).

The expectation value of an observable \( f \in A(M) \) in a state \( \Lambda \) is from definition equal

\[
\langle f \rangle_\Lambda = \Lambda(f). \tag{88}
\]

If \( f \) is self-adjoint, i.e. \( \check{f} = f \), then \( \langle f \rangle_\Lambda \in \mathbb{R} \).

Similarly as in classical mechanics, where states can be characterized in terms of probabilistic distribution functions on phase space, in quantum mechanics we can also associate with states certain phase-space functions called quasi-probabilistic distribution functions. The next two theorems provide such characterization. Their proofs follow directly from the operator representation introduced in Section 4.

**Theorem 5.** If \( \rho \in L(M) \) satisfies
are continuous, uniformly for $h$.

The consistency of the above definition is guaranteed by the property (iv) of Theorem 1. Note, that the maps which proves the continuity of the functionals (92). Thus we may write

improper integral from Section 2.3, is well defined for every test function.

(92) we can also identify other functions with distributions provided that the above integral, treated as an

These functionals are continuous on $F$ and $h$ to $M$.

(ii) $\int_M \rho \, dl = 1$,

(iii) $\int_M f \ast f \ast \rho \, dl \geq 0$ for every $f \in F(M)$,

then the functional

\[ \Lambda_\rho(f) = \int_M f \ast \rho \, dl \equiv \int_M f \cdot \rho \, dl \]  

is a state. Vice versa, for every state $\Lambda$ there exists a unique function $\rho \in L_1(M)$ satisfying properties $[\otimes] - [\otimes]$ such that $\Lambda = \Lambda_\rho$.

**Theorem 6.** A state $\Lambda_\rho$ is pure if and only if the corresponding function $\rho$ is idempotent, i.e.

\[ \rho \ast \rho = \rho. \]  

From (93) the expectation value of an observable $f \in F(M)$ in a state $\Lambda_\rho$ can be written in a form

\[ \langle f \rangle_\rho = \int_M f(x) \rho(x) \, dl(x). \]  

In the next section we will extend $F(M)$ to an algebra of distributions $F_*(M)$. The above formula will then extend, in a direct way, to general observables $f \in F_*(M)$ and states $\rho \in F(M)$.

### 3.4 Extension of the $\ast$-product to an algebra of distributions

In what follows we will extend the $\ast$-product to a suitable space of distributions. The algebra $F(M)$ will play the role of the space of test functions. The following construction is based on [13, 33]. We will assume that the group $G$ is compact. In such case $F(M)$ is a Fréchet algebra. The following considerations will also hold for non-compact groups, although, then we do not have a topology on $F(M)$ so we would not be dealing with any continuity issues.

We will denote by $F(M)$ the space of continuous linear functionals on $F(M)$, i.e. distributions. The dual space $F^\prime(M)$ is endowed with the strong dual topology, that of uniform convergence on bounded subsets of $F(M)$. For $f \in F^\prime(M)$ we will denote by $\langle f, h \rangle$ the value of the functional $f$ at $h \in F(M)$. We will identify functions $f \in F(M)$ with the following functionals

\[ h \mapsto \int_M f(x) h(x) \, dx. \]  

These functionals are continuous on $F(M)$. Indeed, if $f, h \in C^\infty_c(G \times G)$ are integral kernels corresponding to $f$ and $h$, then by Lemma 4

\[ \left| \int_M f(x) h(x) \, dx \right| \leq |2\pi h|^n \int_{G \times G} \left| f(a, b) h(b, a) \, dm(a) \, dm(b) \right| \leq |2\pi h|^n \int_{G \times G} \left| f(a, b) \right| \, dm(a) \, dm(b) \|h\|_{0,0}, \]  

which proves the continuity of the functionals (92). Thus we may write $F(M) \subset F^\prime(M)$. By the formula (92) we can also identify other functions with distributions provided that the above integral, treated as an improper integral from Section 2.3, is well defined for every test function.

For $f \in F^\prime(M)$ and $g \in F(M)$ we define $f \ast g \in F^\prime(M)$ and $g \ast f \in F^\prime(M)$ by the formulas

\[ \langle f \ast g, h \rangle = \langle f, g \ast h \rangle, \quad \langle g \ast f, h \rangle = \langle f, h \ast g \rangle \]  

for every $h \in F(M)$.

The consistency of the above definition is guaranteed by the property $[\otimes]$ of Theorem 1. Note, that the maps $g \mapsto f \ast g$ and $g \mapsto g \ast f$ are continuous from $F(M)$ to $F^\prime(M)$, since the maps $g \mapsto \langle f, g \ast h \rangle$ and $g \mapsto \langle f, h \ast g \rangle$ are continuous, uniformly for $h$ in a bounded subset of $F(M)$ (by the joint continuity of $g$ and $h$).
Denote by $\mathcal{F}_*(M)$ the following subspace of distributions:

$$\mathcal{F}_*(M) = \{ f \in \mathcal{F}(M) \mid f * g \text{ and } g * f \in \mathcal{F}(M) \text{ for every } g \in \mathcal{F}(M) \}. \tag{95}$$

In particular, $\mathcal{F}(M) \subset \mathcal{F}_*(M)$. For $f \in \mathcal{F}_*(M)$ the maps $g \mapsto f * g$ and $g \mapsto g * f$ are continuous from $\mathcal{F}(M)$ to $\mathcal{F}(M)$ by the closed graph theorem. Thus, for $f, g \in \mathcal{F}_*(M)$ we may define their $*$-product by the formula

$$\langle f * g, h \rangle = \langle f, g * h \rangle = \langle g, h * f \rangle \quad \text{for every } h \in \mathcal{F}(M). \tag{96}$$

Straightforward calculations with the use of (94) and (96) verify that this functional is continuous. We calculate that

$$\langle f * g, h \rangle = \langle f, g * h \rangle = \langle g, h * f \rangle \quad \text{for every } h \in \mathcal{F}(M). \tag{97}$$

Thus, $\mathcal{F}_*(M)$ is an involutive algebra with unity, being a natural extension of the algebra $\mathcal{F}(M)$.

In what follows we will show that all smooth functions polynomial in fiber variables $p_j$, i.e. functions of the form

$$f(q, p) = \sum_{k \geq 0}^k f^{i_1i_2...i_k}(q) p_{i_1} p_{i_2} ... p_{i_k} \tag{98}$$

for $k \geq 0$ and $f^{i_1i_2...i_k} \in C^\infty(G)$, are in $\mathcal{F}_*(M)$.

**Theorem 7.** If $f(q)$ is a smooth function on $G$, then $f \in \mathcal{F}_*(M)$.

*Proof.* Let $h \in \mathcal{F}(M)$. Since $f(q)\hat{h}(q, 0)$ is a smooth compactly supported function on $G$ the integral in (92) will be finite and define a proper linear functional on $\mathcal{F}(M)$ (even if $G$ is not compact). We will show that this functional is continuous. We calculate that

$$\begin{align*}
\langle \hat{f}, h \rangle = |\int_M f(q)h(q, p) \, d\mu(q) \, dp| &\leq |\int_G f(q)h(q, 0) \, d\mu(q)| = |\int_G f(q)h(q, 0) \, d\mu(q)| \\
&\leq |\int_G |f(q)| \, d\mu(q)||h||_{0, 0}. \tag{99}
\end{align*}$$

where $h \in C^\infty_c(G \times G)$ is an integral kernel corresponding to $h$. The integral in the last term is finite because $G$ is compact. Therefore, the functional $\langle f, \cdot \rangle$ will be continuous.

Now, we will show that $f \in \mathcal{F}_*(M)$, also in the case when $G$ is not compact. For $g, h \in \mathcal{F}(M)$ we have

$$\begin{align*}
\langle f * g, h \rangle = \langle f, g * h \rangle = &\int_G \left( \int_B f(q)\hat{g}(q \exp(\frac{i}{2}Y), Y) \hat{h}(q \exp(\frac{i}{2}Y), -Y) \det \text{Ad}_{\exp(-\frac{i}{2}Y)} \, dY \right) \, d\mu(q) \\
&= |2\pi h|^n \int_G \left( \int_B f(q)\hat{g}(q \exp(\frac{i}{2}Y), Y) \hat{h}(q \exp(\frac{i}{2}Y), -Y) \det \text{Ad}_{\exp(-\frac{i}{2}Y)} \, dY \right) \, d\mu(q) \\
&= |2\pi h|^n \int_G \left( \int_B f(q)\hat{g}(q, Y) \hat{h}(q, -Y) \, d\mu(q) \right) \, dY \\
&= \int_{G \times g^*} \left( \int_B f(q)\hat{g}(q, Y) e^{-\frac{i}{2}(p \cdot Y)} \, dY \right) \, h(q, p) \, d\mu(q). \tag{100}
\end{align*}$$

Hence

$$f(q)(q, p) = \int_{G \times g^*} f(q)\hat{g}(q, X) e^{-\frac{i}{2}(p \cdot X)} \, dX. \tag{101}$$

If $g \in C^\infty_c(G \times G)$ corresponds to $\tilde{g}$ as in (12), then

$$f * g = b^*(a \exp(\frac{i}{2}V_a(b)), V_a(b)) F(V_a(b))^{-1} = f(a)g(a, b) \tag{102}$$

is a smooth compactly supported function on $G \times G$. Thus $f * g \in \mathcal{F}(M)$. Similarly we can prove that $g * f \in \mathcal{F}(M)$. Therefore $f \in \mathcal{F}_*(M)$. 

\[\square\]
Theorem 8. If $p_j$ is a fiber variable corresponding to a basis $\{X_i\}$ in $\mathfrak{g}$, then $p_j \in \mathcal{F}_*(M)$.

Proof. Let $h \in \mathcal{F}(M)$. Since for a fixed $q \in G$ the function $X \mapsto h(q, X)$ is smooth in the neighborhood of 0, we have

$$\langle p_j, h \rangle = \int_G \left( \int_{q^*} p_j(q, p) h(q, p) \, dp \right) \, dq = -ih[2\pi \hbar]^n \int_G \frac{\partial}{\partial X_j} h(q, X) \bigg|_{X=0} \, dq. \quad (103)$$

The function $\frac{\partial}{\partial X_j} h(q, X)\big|_{X=0}$ is smooth and compactly supported on $G$, therefore its integral will be finite. Thus, the functional $\langle p_j, \cdot \rangle$ is well defined (even when $G$ is not compact). We will show that this functional is continuous. Let $h \in C^\infty_c(G \times G)$ correspond to $\hat{h}$ as in (102). Then we calculate that

$$\langle p_j, h \rangle = ih[2\pi \hbar]^n \int_G \frac{\partial}{\partial X_j} h(q, X) \bigg|_{X=0} \, dq = \int_G h(q \exp(\frac{i}{\hbar} X), q \exp(-\frac{i}{\hbar} X)) F(X) \, dq \bigg|_{X=0} \, dq \bigg|_{X=0}. \quad (104)$$

For $\varphi \in C^\infty(G)$ we have that

$$\frac{\partial}{\partial X_j} \varphi(q \exp(X)) \bigg|_{X=0} = T_q \varphi(T_e L_q X_j) = T_q \varphi(L_{X_j}(q)) = L_{X_j} \varphi(q), \quad (105)$$

where $L_{X_j}$ is a left-invariant vector field corresponding to $X_j$. If $X = u^i X_i$ is an expansion of $X$ in the basis $\{X_i\}$ and $\{X^i\}$ is a dual basis to $\{X_i\}$, then

$$\text{Tr}(\text{ad}_X) = u^i \text{Tr}(\text{ad}_{X_i}) = u^i \langle X^j, \text{ad}_{X_i}, X_j \rangle = u^i \langle X^j, C^k_{ij} X_k \rangle = C^k_{ij} u^i \delta^j_k = C^k_{ij} u^i. \quad (106)$$

Hence

$$\frac{\partial}{\partial X_j} \det \text{Ad}_{\exp(-\frac{1}{\hbar} X)} \bigg|_{X=0} = \frac{\partial}{\partial X_j} e^{-\frac{i}{\hbar} \text{Tr}(\text{ad}_X)} \bigg|_{X=0} = \frac{\partial}{\partial u^j} e^{-\frac{i}{\hbar} C^k_{ij} u^i} \bigg|_{u=0} = -\frac{1}{2} C^k_{ij}. \quad (107)$$

Since $F'(0) = 0$ we get, with the use of (105) and (107), that

$$\frac{\partial}{\partial X_j} h(q \exp(X), q) F(X) \det \text{Ad}_{\exp(-\frac{1}{\hbar} X)} \bigg|_{X=0} = L^{(1)}_{X_j} h(q, q) - \frac{1}{2} C^k_{ij} h(q, q), \quad (108)$$

where $L^{(1)}_{X_j}$ is a vector fields on $G \times G$ defined as in (108). Because compact groups are unimodular $C^k_{ij} = 0$ and we get from (104) and (108) that

$$|\langle p_j, h \rangle| \leq (2\pi)^n |h|^{n+1} m(G) \|h\|_{k,0} \quad (109)$$

for $k = (0, \ldots, 0, 1, 0, \ldots, 0)$ where there is 1 on the $j$-th place. This shows the continuity of the functional $\langle p_j, \cdot \rangle$.

Now, we will show that $p_j \in \mathcal{F}_*(M)$, also in the case when $G$ is not compact. For $g, h \in \mathcal{F}(M)$ we have

$$\langle p_j * g, h \rangle = \langle p_j, g * h \rangle = \int_G \left( \int_{g^*} p_j(q, p)(g * h)(q, p) \, dp \right) \, dq = |2\pi \hbar|^n \int_G ih \frac{\partial}{\partial X_j} (\tilde{g} \circ \tilde{h})(q, -X) \bigg|_{X=0} \, dq. \quad (110)$$

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Let \( g, h \in C_c^\infty(G \times G) \) correspond to \( \tilde{g} \) and \( \tilde{h} \) as in \[42\]. Then we can write

\[
\langle p_j \ast g, h \rangle = i\hbar |2\pi\hbar|^n \frac{\partial}{\partial X_j} \int_G g(q \exp(\frac{1}{\hbar}X), c)h(c, q \exp(-\frac{1}{\hbar}X))F(X) \, dm(c) \, dm(q) \bigg|_{X=0} = i\hbar |2\pi\hbar|^n \frac{\partial}{\partial X_j} \int_G g(q \exp(X), c)h(c, q)F(X) \det \text{Ad}_{\exp(-\frac{1}{\hbar}X)} \, dm(c) \, dm(q) \bigg|_{X=0} = |2\pi\hbar|^n \int_G f(q, c)h(c, q) \, dm(c) \, dm(q),
\]

where in accordance to \[108\]

\[
f(q, c) = i\hbar \frac{\partial}{\partial X_j} g(q \exp(X), c)F(X) \det \text{Ad}_{\exp(-\frac{1}{\hbar}X)} \bigg|_{X=0} = i\hbar \left( L^{(1)}_{X_j}g(q, c) - \frac{1}{2} C^L_{jk}g(q, c) \right),
\]

so that \( f \in C_c^\infty(G \times G) \). By Lemma \[1\] we get that

\[
\langle p_j \ast g, h \rangle = \int_M f(x)h(x) \, dx,
\]

where \( f \in \mathcal{F}(M) \) is a function corresponding to \( f \). Therefore, \( f = p_j \ast g \) which shows that \( p_j \ast g \in \mathcal{F}(M) \). Analogically we can prove that \( g \ast p_j \in \mathcal{F}(M) \). Thus \( p_j \in \mathcal{F}_*(M) \).

**Theorem 9.** If \( f(g) \) is a smooth function on \( G \), then for every \( g \in \mathcal{F}(M) \) the following expansion in \( \hbar \) holds

\[
f \ast g = \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\frac{i\hbar}{2} \right)^k L_{X_1} \ldots L_{X_k} f \,
\]

where \( L_{X_i} \) are left-invariant vector fields corresponding to a basis \( \{X_i\} \) in \( g \).

**Proof.** First observe that the following equality holds

\[
\frac{d^k}{dt^k} f(q \exp(tX)) \bigg|_{t=0} = L_X \ldots L_X f(q).
\]

Expanding function \( t \mapsto f(q \exp(tX)) \) in a Taylor series and using the above equality results in

\[
f(q \exp(tX)) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dt^k} f(q \exp(tX)) \bigg|_{t=0} = \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\frac{i\hbar}{2} \right)^k L_{X_1} \ldots L_{X_k} f(q).\]

For \( X = u^iX_i \) and \( t = -\frac{1}{2} \) this gives

\[
f(q \exp(-\frac{1}{2}X)) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\frac{1}{2} \right)^k L_{X_1} \ldots L_{X_k} f(q)u^{i_1}u^{i_2} \ldots u^{i_k}.
\]

From the above result and the integral formula \[101\] for the product \( f \ast g \) we get \[114\].

**Theorem 10.** If \( p_j \) is a fiber variable corresponding to a basis \( \{X_i\} \) in \( g \), then for every \( g \in \mathcal{F}(M) \) the following expansion in \( \hbar \) holds

\[
p_j \ast g = p_j g + \sum_{k=1}^{\infty} \frac{(i\hbar)^k}{k!} \left( \frac{1}{2} - 2^{k-1} \right) B_k C_{i_1 j_1} C_{i_2 j_2} \cdots C_{i_k j_k} Z^{i_1} Z^{i_2} \cdots Z^{i_k} Y_{j_k} g
\]

\[
+ B_k C_{i_1 j_1} C_{i_2 j_1} \cdots C_{i_k j_{k-1}} p_{j_k}Z^{i_1} Z^{i_2} \cdots Z^{i_k} g
\]

\[
- \frac{1}{2} \sum_{l=1}^{[k/2]} \binom{k}{2l} B_{k-2l} B_{2l} C_{j_1 j_2} C_{i_1 i_2} \cdots C_{j_{2l-1} j_{2l-1}} C_{i_{2l+1} i_{2l+1}} C_{j_{2l} j_{2l+2}} \cdots C_{i_{2l+2} i_{2l+2}} Z^{i_1} Z^{i_2} \cdots Z^{i_k} g,
\]

where \( B_k \) is the \( k \)-th Bernoulli number and \([k/2]\) denotes the nearest integer number smaller or equal to \( k/2 \).
Proof. From the analysis of the proof of Theorem [8] we can deduce that the integral kernel $f$ of $p_j \ast g$ will be given by the formula

$$f(a, b) = i\hbar \frac{\partial}{\partial X_j} \hat{g}(a \exp(X) \exp(\frac{1}{2}V_a \exp(X)(b)), V_a \exp(X)(b)) F(V_a \exp(X)(b))^{-1} F(X) \det \text{Ad}_{\exp(-\frac{1}{2}X)} \bigg|_{X=0}.$$  

Therefore, from the equality

$$V_q \exp(-\frac{1}{2}Y) \exp(X)(q \exp(\frac{1}{2}Y)) = (-X) \circ Y,$$

we get

$$\langle \partial X_j \rangle (q, p) = \int_0^\infty i\hbar \frac{\partial}{\partial X_j} \hat{g}(q \exp(-\frac{1}{2}Y) \exp(X) \exp(\frac{1}{2}((-X) \circ Y)), (-X) \circ Y) F((-X) \circ Y)^{-1} F(X) \times \det \text{Ad}_{\exp(-\frac{1}{2}X)} \bigg|_{X=0} e^{-\frac{i}{\hbar}(p, Y)} F(Y) dY$$

$$= \int_0^\infty i\hbar \frac{\partial}{\partial X_j} \hat{g}(q \exp(\frac{1}{2}(X \circ Y))) \exp(-\frac{1}{2}Y), Y) e^{-\frac{i}{\hbar}(p, Y \circ Y)} L(X, Y) \bigg|_{X=0} dY,$$

where $\frac{\partial}{\partial X_j}$ refers to a differentiation with respect to $X$ variable.

Observe, that from the integral version of the Baker-Campbell-Hausdorff formula

$$X \circ Y = Y + \left( \int_0^1 \varphi(e^{s \text{ad}_X e^{\text{adv}}} \, ds \right) X,$$

(122)

where $\varphi(x) = \frac{\ln x}{x - 1}$ Using this formula we can calculate that

$$\frac{\partial}{\partial X_j} (X \circ Y) = \frac{d}{dt} ((X + t X_j) \circ Y) \bigg|_{t=0} = \frac{d}{dt} \left( Y + \left( \int_0^1 \varphi(e^{s \text{ad}_X + t X_j} e^{\text{adv}}) \, ds \right) (X + t X_j) \right) \bigg|_{t=0}$$

$$= \frac{d}{dt} \left( Y + \left( \int_0^1 \varphi(e^{s \text{ad}_X + t X_j} e^{\text{adv}}) \, ds \right) X + t \left( \int_0^1 \varphi(e^{s \text{ad}_X + t X_j} e^{\text{adv}}) \, ds \right) X_j \right) \bigg|_{t=0}$$

$$= \left( \frac{d}{dt} \int_0^1 \varphi(e^{s \text{ad}_X + t X_j} e^{\text{adv}}) \, ds \bigg|_{t=0} \right) X + \left( \int_0^1 \varphi(e^{s \text{ad}_X e^{\text{adv}}}) \, ds \right) X_j.$$  

(123)

From this we get that

$$\frac{\partial}{\partial X_j} (X \circ Y) \bigg|_{X=0} = \varphi(X) X_j = \sum_{k=0}^{\infty} \frac{B_k}{k!} \text{ad}_X^k X_j,$$

(124)

since $\varphi(x) = \frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$ is a generating function for the Bernoulli numbers. If $Y = v^i X_i$ is an expansion of $Y$ in the basis $\{X_i\}$, then by (123) we get

$$\frac{\partial}{\partial X_j} e^{-\frac{i}{\hbar}(p, Y \circ Y)} \bigg|_{X=0} = -i \hbar \sum_{k=0}^{\infty} \frac{B_k}{k!} C_{i_1 j_1}^{i_2 j_2} C_{i_2 j_2}^{i_3 j_3} \cdots C_{i_k j_k}^{i_{k+1} j_{k+1}} p_{i_1} v^{i_2} v^{i_3} \cdots v^{i_k} e^{-\frac{i}{\hbar}(p, Y)}.$$

We have that

$$F(X) = \exp \left( \frac{1}{2} \text{Tr} f(\text{ad}_X) \right) \quad \text{for} \quad f(x) = \ln(\lambda(x)) = \ln \left( \frac{2}{x} \sinh \frac{x}{2} \right),$$

(126)

and

$$\frac{\partial}{\partial X_j} F(X) = \frac{1}{2} F(X) \frac{\partial}{\partial X_j} \text{Tr} f(\text{ad}_X) = \frac{1}{2} \left( \frac{1}{2} F(X) \text{Tr} (f'(\text{ad}_X) \circ \text{ad}_X) \right).$$

(127)
Since
\[ f'(x) = \frac{1}{2} \coth \frac{x}{2} - \frac{1}{x} = \sum_{k=1}^{\infty} \frac{B_{2k} x^{2k-1}}{(2k)!} \]
we get
\[ \frac{\partial}{\partial X_j} F(X) = \frac{1}{2} F(X) \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \text{Tr}(\text{ad}_X^{2k-1} \circ \text{ad}_{X_j}) \]
\[ = \frac{1}{2} F(X) \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} C_{j2k}^{i1} C_{i2j}^{j2} C_{i3j}^{j3} \cdots C_{i2k,j2k-1}^{j2k} u^{2i} u^{i3} \cdots u^{2k} \] (129)
for \( X = u'X_j \). We calculate that
\[ \left. \frac{\partial}{\partial X_j} L(X, Y) \right|_{X=0} = \left. \frac{\partial}{\partial X_j} F(X) F(Y) F(X \circ Y)^{-1} \right|_{X=0} = \left. F(Y)^{-1} \frac{\partial}{\partial X_j} F(X \circ Y)^{-1} \right|_{X=0} \]
\[ = -F(Y)^{-1} \frac{\partial}{\partial X_j} F(X \circ Y) \left|_{X=0} = -F(Y)^{-1} \frac{\partial}{\partial X_i} (Y) \frac{\partial}{\partial X_j} (X \circ Y)^i \right|_{X=0} \] (130)
where \((X \circ Y)^i\) denotes the \(i\)-th component of \(X \circ Y\) in the expansion with respect to the basis \(\{X_i\}\). Combining (124) and (129) gives
\[ \left. \frac{\partial}{\partial X_j} L(X, Y) \right|_{X=0} = -\frac{1}{2} \sum_{k=0}^{[k/2]} \frac{B_{k-2l}}{(k-2l)!} \frac{B_{2l}}{(2l)!} C_{j2k}^{i1} C_{i2j}^{j2} C_{i3j}^{j3} \cdots C_{i2k,j2k-1}^{j2k} \]
\[ \times C_{i2l+1j}^{i2l+1} C_{i2l+2j2l+1}^{i2l+2} C_{i2l+3j2l+2}^{i2l+3} \cdots C_{i2k+1j,k-1}^{i2k+1} u^{2i} u^{i3} \cdots u^{2k} \] (131)
For \( \varphi \in C^\infty(G) \) we have
\[ \left. \frac{\partial}{\partial X_j} \varphi(q \exp(\frac{1}{2}(X \circ Y)) \exp(-\frac{1}{2}Y)) \right|_{X=0} = \left. \frac{\partial}{\partial X_j} \varphi(q \exp(-\frac{1}{2}Y) \exp(\frac{1}{2}Y) \exp(\frac{1}{2}(X \circ Y)) \exp(-\frac{1}{2}Y)) \right|_{X=0} \]
\[ = \left. \frac{\partial}{\partial X_j} \varphi \left( q \exp \left( \frac{1}{2} Y \right) \exp \left( \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{1}{2} \right)^k \text{ad}^k_X (X \circ Y) \right) \right) \right|_{X=0} \] (132)
By (127) and (124) we get
\[ \left. \frac{\partial}{\partial X_j} \varphi(q \exp(\frac{1}{2}(X \circ Y)) \exp(-\frac{1}{2}Y)) \right|_{X=0} = T_q \varphi \left( T_e L_q \left( \phi(\text{ad}^k_Y) \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{1}{2} \right)^k \frac{B_l}{l!} \text{ad}^{k+l}_Y X_j \right) \right) \]
\[ = T_q \varphi \left( T_e L_q \left( \phi(\text{ad}^k_Y) \frac{1}{2} \sum_{r=0}^{l} \frac{1}{r!} B_r \left( \frac{1}{2} \text{ad}^r_Y X_j \right) \right) \right) \] (133)
where
\[ B_r(\frac{1}{2}) = \sum_{l=0}^{r} \binom{r}{l} \left( \frac{1}{2} \right)^{r-l} B_l = \left( \frac{1}{2^{r-1}} - 1 \right) B_r \] (134)
is the value of the Bernoulli polynomial at \( \frac{1}{2} \). An expansion of the function \( \phi \) in a Taylor series results in
\[ \left. \frac{\partial}{\partial X_j} \varphi(q \exp(\frac{1}{2}(X \circ Y)) \exp(-\frac{1}{2}Y)) \right|_{X=0} = \]
\[ = -\sum_{k=0}^{\infty} \frac{1}{(k+1)!} \sum_{r=0}^{k} \binom{k+1}{r} \left( \frac{1}{2} \right)^{k+1-r} B_r \left( \frac{1}{2} \right) T_q \varphi \left( T_e L_q \left( \text{ad}^{k+1-r}_Y X_j \right) \right) \]
\[ = \sum_{k=1}^{\infty} \frac{1}{k!} \left( B_k \left( \frac{1}{2} \right) - B_k(0) \right) T_q \varphi \left( T_e L_q \left( \text{ad}^{k-1}_Y X_j \right) \right) \] (135)
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where we have used the translation property of Bernoulli polynomials. Using the above result we get

\[
\frac{\partial}{\partial X_j} \tilde{g}(q \exp(\frac{i}{\hbar} (X \circ Y)) \exp(-\frac{i}{\hbar} Y), Y) \bigg|_{X=0} = \\
= \sum_{k=1}^{\infty} \frac{1}{k!} \frac{1 - 2^k}{2^{k-1}} B_k C_{ij}^{j_2} C_{ik}^{j_3} \cdots C_{ik}^{j_k} (Y_{jk} g) \sim (q, Y) v^{i_2} v^{i_3} \cdots v^{i_k}.
\] (136)

Applying (125), (131) and (136) to (121) we receive (118).

Now we can show that all smooth functions polynomial in fiber variables \(p_j\) belong to \(\mathcal{F}_s(M)\). First we will prove that polynomials in \(p_j\) are in \(\mathcal{F}_s(M)\). Assume that for a given \(k \geq 1\) all monomials \(p_i p_2 \cdots p_{i_k}\) of order \(k\) are in \(\mathcal{F}_s(M)\). Then, the expansion (118) will also hold for these monomials in place of \(g\) and we can see that the product \(p_{i_{k+1}} \ast p_{i_1} p_{i_2} \cdots p_{i_k}\) will be of the form of a monomial \(p_{i_1} p_{i_2} \cdots p_{i_k} p_{i_{k+1}}\) plus some polynomial of order \(k\). Thus monomials \(p_{i_1} p_{i_2} \cdots p_{i_k} p_{i_{k+1}}\) of order \(k + 1\) will also belong to \(\mathcal{F}_s(M)\). Since in Theorem 8 we showed that all \(p_j\) are in \(\mathcal{F}_s(M)\), then by induction we see that all monomials, and consequently all polynomials, are in \(\mathcal{F}_s(M)\). Next, assume that for a given \(k \geq 0\) all smooth functions which are polynomial of order \(k\) in \(p_j\) are in \(\mathcal{F}_s(M)\). The expansion (118) also holds for \(g\) replaced by an arbitrary polynomial in \(p_j\). Then, we can see that the \(\ast\)-product of an arbitrary smooth function \(f(q)\) on \(G\) and a monomial \(p_{i_1} p_{i_2} \cdots p_{i_k}\) of order \(k\) will be in the form of a function \(f(q) p_{i_1} p_{i_2} \cdots p_{i_k+1}\) plus some smooth function polynomial in \(p_j\) of order \(k\). Thus smooth functions polynomial in \(p_j\) of order \(k + 1\), and by induction of all orders, will be in \(\mathcal{F}_s(M)\).

Note, that from (118) we get in particular that

\[
p_i \ast p_j = p_i p_j + \frac{1}{2} \hbar C_{ij}^k p_k + \frac{1}{24} \hbar^2 C_{ij}^k C_{ij}^l C_{ij}^l.
\] (137)

From this we receive the following commutation relation

\[
[p_i, p_j] = C_{ij}^k p_k
\] (138)

being an analog of its classical counterpart (10).

### 3.5 Time evolution

For completeness of the quantization procedure we present a short description of the time evolution of a quantum system. The time evolution is governed by a Hamiltonian function \(H \in \mathcal{F}_s(M)\) which is, similarly as in classical mechanics, some distinguished observable. The equation describing the time evolution of an observable \(A \in \mathcal{F}_s(M)\) can be received from its classical counterpart by replacing the Poisson bracket \(\{ \cdot, \cdot \}\) with its deformation \(\lfloor \cdot, \cdot \rfloor\):

\[
\frac{dA}{dt}(t) - \lfloor A(t), H \rfloor = 0, \quad A(0) = A.
\] (139)

If \(A \in \mathcal{A}(M)\) and \(A(t)\) is its time development, then the time evolution of a state \(\Lambda\) can be given by the formula

\[
\Lambda(t)(A) = \Lambda(A(t)).
\] (140)

In particular, if \(\Lambda = \Lambda_\rho\) then we receive the following formula for the time development of the pseudo-probabilistic distribution function \(\rho\):

\[
\frac{d\rho}{dt}(t) - [H, \rho(t)] = 0, \quad \rho(0) = \rho.
\] (141)
4 Operator representation

4.1 Representation of the algebra of observables $A(M)$

By virtue of Gelfand-Naimark theorem the $C^*$-algebra of observables $A(M)$ can be isometrically represented as a subalgebra of the $C^*$-algebra $B(H)$ of bounded linear operators on a certain Hilbert space $H$. In what follows we will present an explicit construction of this representation for $H = L^2(G, dm)$. We will in fact receive a position representation of a quantum system, where the Hilbert space $H$ will play the role of the space of wave functions.

Let $H = L^2(G, dm)$ be a Hilbert space of $\mathbb{C}$-valued square integrable functions on $G$ with a scalar product given by

$$(\varphi, \psi) = \int_G \varphi(q) \psi(q) \, dm(q),$$

(142)

where $dm$ is the left-invariant Haar measure on $G$. For a function $f \in F(M)$ we define an operator $\hat{f}$ acting in $H$ as an integral operator given by an integral kernel $\hat{f}$ corresponding to $f$ according to the formula (42):

$$\hat{f} \psi(a) = \int_G f(a, b) \psi(b) \, dm(b).$$

(143)

Using (42) and performing the following change of variables under the integral sign: $b \mapsto X = V_a(b)$, formula (143) takes the form

$$\hat{f} \psi(q) = \int_0 \hat{f} \left( q \exp\left( \frac{1}{2}X \right), X \right) \psi(q \exp(X)) F(X) \det Ad_{\exp(-\frac{1}{2}X)} \, dX.$$  

(144)

**Theorem 11.** For $f, g \in F(M)$

(i) the map $f \mapsto \hat{f}$ is a linear isomorphism of $F(M)$ onto the space of integral operators whose integral kernels $f \in C^\infty_c(G \times G)$,

(ii) $\hat{f} \ast g = \hat{f} \hat{g}$,

(iii) $\hat{f} = \hat{f}^\dagger$,

(iv) $\hat{f}$ is a trace class operator and $\text{Tr}(\hat{f}) = \text{tr}(f)$,

(v) the Hilbert-Schmidt scalar product of operators $\hat{f}$ and $\hat{g}$ is equal $(\hat{f}, \hat{g}) = \text{Tr}(\hat{f}^\dagger \hat{g}) = (f, g)$,

(vi) $\|\hat{f}\| = \|f\|$.

**Proof.**

(i) This property follows immediately from the definition of the space $F(M)$.

(ii) From the proof of Theorem 4 it follows that the integral kernel corresponding to the product $f \ast g$ is exactly equal to the integral kernel of $\hat{f} \hat{g}$.

(iii) Let $f(a, b)$ be an integral kernel corresponding to $f$. Then the integral kernel of $\hat{f}^\dagger$ will be equal

$$\tilde{f}(b, a) = \hat{f}(b \exp(\frac{1}{2}V_b(a)), V_b(a)) F(V_b(a))^{-1} = \hat{f}(b \exp(V_b(a)) \exp(-\frac{1}{2}V_b(a)), -V_b(a)) F(V_b(a))^{-1} = \hat{f}(a \exp(\frac{1}{2}V_a(b)), V_a(b)) F(V_a(b))^{-1},$$

(145)

which is just the integral kernel corresponding to $\hat{f}$.

(iv) Since $\hat{f}$ is an integral operator whose integral kernel is smooth and compactly supported it will be of trace class and its trace will be expressed by the formula

$$\text{Tr}(\hat{f}) = \int_G f(q, q) \, dm(q) = \int_G \hat{f}(q, 0) \, dm(q) = \frac{1}{|2\pi \hbar|^n} \int_G \int_{\mathbb{R}^n} f(q, p) \, dm(q) \, dp = \frac{1}{|2\pi \hbar|^n} \int_{\mathbb{R}^n} f(x) \, dx = \text{tr}(f).$$

(146)
Theorem 12 we get that the quantum system. For \( \phi, \psi \in \mathcal{H} \) the following properties of the functions \( W(\psi, \phi) \) thus by (45) the Wigner function \( W(\psi, \phi) \) will be expressed by the formula

\[
W(\psi, \phi)(q, p) = \int_{\mathbb{C}} \varphi(q \exp(iX)) \psi(q \exp(-iX)) e^{-\frac{i}{\hbar}(p, X)} F(X) \, dX.
\]  

(148)

The following properties of the functions \( W(\psi, \phi) \) are an immediate consequence of Theorem 11 and formula (148).

**Theorem 12.** For \( \varphi, \psi, \phi, \chi \in \mathcal{H} \) and \( f \in \mathcal{F}(\mathcal{M}) \)

(i) \( W(\varphi, \psi) = W(\psi, \varphi) \),

(ii) \( \int_M W(\varphi, \psi) \, dl = (\varphi, \psi) \),

(iii) \( (W(\varphi, \psi), W(\phi, \chi)) = (\varphi, \phi)(\psi, \chi) \),

(iv) \( W(\varphi, \psi) \ast W(\phi, \chi) = (\varphi, \chi)W(\psi, \phi) \),

(v) \( f \ast W(\varphi, \psi) = W(\varphi, f \psi) \) and \( W(\varphi, \psi) \ast f = W(f^\dagger \varphi, \psi) \),

(vi) \( \frac{1}{|2\pi \hbar|^n} \int_{\mathbb{R}^n} W(\varphi, \psi)(q, p) \, dp = |\varphi(q)|^2 \).

We can define a tensor product of the Hilbert space \( \mathcal{H} \) and its dual \( \mathcal{H}^* \) in terms of the Wigner transform \( \mathcal{W} \):

\[
\varphi^* \otimes \psi = \mathcal{W}(\varphi, \psi),
\]

(149)

where \( \varphi \mapsto \varphi^* \) is an anti-linear isomorphism of \( \mathcal{H} \) onto \( \mathcal{H}^* \) appearing in the Riesz representation theorem. The map \( \otimes : \mathcal{H}^* \times \mathcal{H} \to \mathcal{L}(\mathcal{M}) \) is clearly bilinear and from property (iii) from Theorem 12 it satisfies

\[
(\varphi^* \otimes \psi, \phi^* \otimes \chi) = (\varphi^*, \phi^*)(\psi, \chi).
\]

(150)

Moreover, since the set of generalized Wigner functions \( \mathcal{W}(\varphi, \psi) \) is linearly dense in \( \mathcal{L}(\mathcal{M}) \) the map \( \otimes \) indeed defines a tensor product of \( \mathcal{H}^* \) and \( \mathcal{H} \) equal to \( \mathcal{L}(\mathcal{M}) \).

For \( f \in \mathcal{F}(\mathcal{M}) \) we can treat \( f \ast \) as an operator on the Hilbert space \( \mathcal{L}(\mathcal{M}) \). Then, by property (v) from Theorem 12 we get that

\[
f \ast = \mathbb{1} \otimes f.
\]

(151)

Note, that the operator representation of the algebra \( \mathcal{L}(\mathcal{M}) \) gives a one to one correspondence between states \( \rho \in \mathcal{L}(\mathcal{M}) \) and density operators \( \hat{\rho} \), i.e. trace class operators satisfying
(i) \( \hat{\rho}^\dagger = \hat{\rho} \),

(ii) \( \text{Tr}(\hat{\rho}) = 1 \),

(iii) \( (\varphi, \hat{\rho} \varphi) \geq 0 \) for every \( \varphi \in \mathcal{H} \).

Indeed, every density operator \( \hat{\rho} \) gives rise to a continuous linear functional on the \( C^* \)-algebra \( \mathcal{K}(\mathcal{H}) \) of compact operators, which is positively defined and normalized to unity. Such functional will be given by the formula \( f \mapsto \text{Tr}(f \hat{\rho}) \), and every continuous positive linear functional on \( \mathcal{K}(\mathcal{H}) \) normalized to unity will be of such form for some density operator \( \hat{\rho} \). From this observation we also immediately get Theorems 5 and 6 characterizing states in terms of quasi-probabilistic distribution functions.

From the correspondence between states and density operators we can see that a function \( \rho \in \mathcal{L}(M) \) is a state if and only if it is in the form

\[
\rho = \sum_k p_k \mathcal{W}(\varphi_k, \varphi_k),
\]

where \( \varphi_k \in \mathcal{H} \), \( \|\varphi_k\| = 1 \), \( p_k \geq 0 \), and \( \sum_k p_k = 1 \). In particular, \( \rho \in \mathcal{L}(M) \) is a pure state if and only if

\[
\rho = \mathcal{W}(\varphi, \varphi)
\]

for some normalized vector \( \varphi \in \mathcal{H} \).

The operator representation can be extended to the space of distributions \( \mathcal{F}(M) \) and to the algebra \( \mathcal{F}_*(M) \) in the following manner. For \( \varphi, \psi \in C_c^\infty(G) \) the Wigner function \( \mathcal{W}(\varphi, \psi) \in \mathcal{F}(M) \). Let \( f \in \mathcal{F}(M) \), then we can define the following bilinear form

\[
(\varphi, \psi) \mapsto (f, \mathcal{W}(\varphi, \psi)), \quad \varphi, \psi \in C_c^\infty(G).
\]

If this form happens to be continuous with respect to the first variable, then it uniquely defines (possibly unbounded) operator \( \hat{f} \) with a dense domain \( C_c^\infty(G) \) by the formula

\[
(\varphi, \hat{f}\psi) = \frac{1}{\sqrt{2\pi\hbar^2}} (f, \mathcal{W}(\varphi, \psi)), \quad \varphi, \psi \in C_c^\infty(G).
\]

In the particular case, when \( f \in \mathcal{F}(M) \), this definition of the operator \( \hat{f} \) coincides with the definition (143).

In what follows we will show that for every \( f \in \mathcal{F}_*(M) \) the bilinear form (154) is continuous with respect to the first variable. From this will immediately follow that the whole algebra \( \mathcal{F}_*(M) \) can be represented as an algebra of (possibly unbounded) operators on \( \mathcal{H} \). From properties (iii) and (iv) from Theorem 12 we get that

\[
\mathcal{W}(\varphi, \psi) = \frac{1}{\|\psi\|^2} \mathcal{W}(\psi, \psi) * \mathcal{W}(\varphi, \psi), \quad \|\mathcal{W}(\varphi, \psi)\|_2 = \|\varphi\|\|\psi\|.
\]

Using these equalities and Schwartz inequality we calculate that for \( f \in \mathcal{F}_*(M) \) and \( \varphi, \psi \in C_c^\infty(G) \)

\[
\frac{1}{\sqrt{2\pi\hbar^2}} \|f, \mathcal{W}(\varphi, \psi)\| = \frac{1}{\sqrt{2\pi\hbar^2}} \frac{1}{\|\psi\|^2} \|f, \mathcal{W}(\psi, \psi) * \mathcal{W}(\varphi, \psi)\| = \frac{1}{\sqrt{2\pi\hbar^2}} \frac{1}{\|\psi\|^2} \|f * \mathcal{W}(\psi, \psi), \mathcal{W}(\varphi, \psi)\|
\]

\[
= \frac{1}{\|\psi\|^2} \left| \int_M (f * \mathcal{W}(\psi, \psi))(x) \mathcal{W}(\varphi, \psi)(x) \, dx \right| = \frac{1}{\|\psi\|^2} \left| \int (f * \mathcal{W}(\psi, \psi))(x) \mathcal{W}(\varphi, \psi)(x) \, dx \right|
\]

\[
\leq \frac{1}{\|\psi\|^2} \|f * \mathcal{W}(\psi, \psi)\|_2 \|\mathcal{W}(\varphi, \psi)\|_2 = \|f * \mathcal{W}(\psi, \psi)\|_2 \|\varphi\|,
\]

where we have used the fact that since \( f \in \mathcal{F}_*(M) \) then \( f * \mathcal{W}(\psi, \psi) \in \mathcal{F}(M) \). This proves continuity of the bilinear form (154) with respect to the variable \( \varphi \). Note, that the operator \( \hat{f} \) corresponding to \( f \in \mathcal{F}_*(M) \) takes values in \( C_c^\infty(G) \) and the property (iv) from Theorem 12 still holds for \( f \in \mathcal{F}_*(M) \) and \( \varphi, \psi \in C_c^\infty(G) \).
4.3 Examples of operators

In what follows we will derive formulas for operators corresponding to couple particular functions on phase space.

**Theorem 13.** Let \( f \in C^\infty(G) \), then \( \hat{f} \) is an operator of multiplication by the function \( f \), i.e.
\[
\hat{f}\psi(q) = f(q)\psi(q), \quad \psi \in C^\infty_c(G).
\] (158)

**Proof.** For \( \varphi, \psi \in C^\infty_c(G) \)
\[
\frac{1}{|2\pi\hbar|^n} \langle f, \mathcal{W}(\varphi, \psi) \rangle = \frac{1}{|2\pi\hbar|^n} \int_G \left( \int_{\mathfrak{g}^*} f(q)\mathcal{W}(\varphi, \psi)(q, p) \, dp \right) \, dm(q) = \int_G f(q)\mathcal{W}(\varphi, \psi)(q, 0) \, dm(q)
\]
\[
= \int_G f(q)\varphi(q)\psi(q) \, dm(q) = \langle \varphi, f\psi \rangle.
\] (159)

**Theorem 14.** Let \( p_j \) be a fiber variable corresponding to a basis \( \{X_i\} \) in \( \mathfrak{g} \), then
\[
\hat{p}_j = i\hbar \left( L_{X_j} - \frac{1}{2} C^k_{jk} \right),
\] (160)

where \( L_{X_j} \) is a left-invariant vector field corresponding to \( X_j \). In particular, for unimodular group \( G \)
\[
\hat{p}_j = i\hbar L_{X_j}.
\] (161)

**Proof.** For \( \varphi, \psi \in C^\infty_c(G) \)
\[
\frac{1}{|2\pi\hbar|^n} \langle p_j, \mathcal{W}(\varphi, \psi) \rangle = \frac{1}{|2\pi\hbar|^n} \int_G \left( \int_{\mathfrak{g}^*} p_j(q, p)\mathcal{W}(\varphi, \psi)(q, p) \, dp \right) \, dm(q)
\]
\[
= -i\hbar \int_G \left. \frac{\partial}{\partial X_j} \mathcal{W}(\varphi, \psi)(q, X) \right|_{X=0} \, dm(q)
\]
\[
= i\hbar \int_G \left. \frac{\partial}{\partial X_j} \varphi(q \exp(-\frac{1}{2}X))\psi(q \exp(\frac{1}{2}X))F(X) \, dm(q) \right|_{X=0}
\]
\[
= i\hbar \int_G \left. \frac{\partial}{\partial X_j} \varphi(q)\psi(q \exp(X))F(X) \right|_{X=0} \, dm(q) \, \det \text{Ad}_{\exp(-\frac{1}{2}X)} \, dm(q) \right|_{X=0}.
\] (162)

Using (105) and (107), and the fact that \( F'(0) = 0 \) we get
\[
\frac{1}{|2\pi\hbar|^n} \langle p_j, \mathcal{W}(\varphi, \psi) \rangle = \int_G \varphi(q) i\hbar \left( L_{X_j}\psi(q) - \frac{1}{2} C^k_{jk} \psi(q) \right) \, dm(q),
\] (163)

which proves (160). □

Note, that the operators \( \hat{p}_j \) satisfy the following commutation relations
\[
[\hat{p}_i, \hat{p}_j] = i\hbar C^k_{ij} \hat{p}_k.
\] (164)

To a function \( f(q, p) = f^i(q)p_i \) \( (f^i \in C^\infty(G)) \) linear in fiber variables corresponds the following operator
\[
\hat{f} = \frac{1}{2} f^i \hat{p}_i + \frac{1}{2} \hat{p}_i f^i,
\] (165)
and to a function \( f(q, p) = f^{ij}(q)p_ip_j \ (f^{ij} \in C^\infty(G) \) being symmetric with respect to indices \( i, j \) \) quadratic in fiber variables corresponds the operator

\[
\hat{f} = \frac{1}{4} f^{ij} \hat{p}_i \hat{p}_j + \frac{1}{2} \hat{p}_i f^{ij} \hat{p}_j + \frac{1}{4} \hat{p}_i \hat{p}_j f^{ij} - \frac{1}{24} \hbar^2 C^k_{il} C^l_{jk} f^{ij}. \tag{166}
\]

Indeed, with the help of the expansion (118) we can write

\[
f^{ij} p_i = \frac{1}{2} f^i \ast p_i + \frac{1}{2} p_i \ast f^i,
\]

\[
f^{ij} p_j = \frac{1}{2} f^{ij} \ast p_j + \frac{1}{2} p_j \ast f^{ij} + \frac{1}{4} p_i \ast p_j \ast f^{ij} - \frac{1}{24} \hbar^2 C^k_{il} C^l_{jk} f^{ij}, \tag{167}
\]

from which follow (165) and (166). Observe, that (165) is a symmetrically ordered function of operators \( f^i \) and \( \hat{p}_i \), and (166) is a symmetrically ordered function of operators \( f^{ij} \) and \( \hat{p}_i \) plus an additional term \(- \frac{1}{24} \hbar^2 C^k_{il} C^l_{jk} f^{ij}\) which can be treated as a quantum correction to the potential.

### 4.4 Quantizer

The operator representation can be expressed in terms of a family of operators \( \{\Delta_x \ | \ x \in M\} \) on the Hilbert space \( \mathcal{H} \) called a quantizer. Let us define

\[
\Delta_x \psi(a) = \frac{1}{|\pi \hbar|^n} \sqrt{J_q(a)} e^{-\frac{\pi}{\hbar} (p, V_q(a))} \psi(s_q(a)), \quad x = (q, p), \ q \in G, \ p \in \mathfrak{g}^*,
\]

where \( s_q \) is the reflection about point \( q \) defined in (9) and \( J_q \) was defined in Section 2.5. The function \( \Delta_x \psi(a) \) is well defined by the formula (168) for \( a \in L_q(\mathcal{U}) \) and, since \( G \setminus L_q(\mathcal{U}) \) is of measure zero, it can be uniquely extended to an element of \( L^2(G, dm) \). Therefore, the operator \( \Delta_x \) is well defined.

**Theorem 15.** The operators \( \Delta_x \) are self-adjoint on domains

\[
\mathcal{D}(\Delta_x) = \{ \psi \in L^2(G, dm) \mid \int_G j_q |\psi|^2 \, dm < \infty \},
\]

i.e. \( \Delta_x^\dagger = \Delta_x \).

**Proof.** The domain \( \mathcal{D}(\Delta_x) \) is a dense vector subspace of \( L^2(G, dm) \). It is the biggest domain on which operator \( \Delta_x \) is well defined. Indeed, we calculate that for \( \psi \in L^2(G, dm) \)

\[
\|\Delta_x \psi\|^2 = \int_G |\Delta_x \psi(a)|^2 \, dm(a) = \frac{1}{|\pi \hbar|^n} \int_G |\psi(s_q(a))|^2 j_q(a) \frac{dm(s_q)}{dm} \bigg|_a \, dm(a)
\]

\[
= \frac{1}{|\pi \hbar|^n} \int_G |\psi(a)|^2 j_q(a) \, dm(a). \tag{170}
\]

Thus \( \Delta_x \psi \in L^2(G, dm) \) if and only if \( \psi \in \mathcal{D}(\Delta_x) \).

Now we will prove that \( \Delta_x \) are self-adjoint on \( \mathcal{D}(\Delta_x) \). Let \( \varphi, \psi \in \mathcal{D}(\Delta_x) \). We get

\[
(\varphi, \Delta_x \psi) = \int_G \varphi(a) \Delta_x \psi(a) \, dm(a) = \frac{1}{|\pi \hbar|^n} \int_G \varphi(a) \psi(s_q(a)) e^{-\frac{\pi}{\hbar} (p, V_q(a))} \sqrt{J_q(a)} \, dm(a). \tag{171}
\]

Since

\[
\frac{dm(s_q)}{dm} \bigg|_a = \sqrt{J_q(a)} \frac{dm(s_q)}{dm} \bigg|_{s_q(a)} = 1 \tag{172}
\]

we get that

\[
\sqrt{J_q(a)} = \sqrt{J_q(s_q(a))} \frac{dm(s_q)}{dm} \bigg|_a. \tag{173}
\]
From the above identity and the fact that \( V_q(a) = -V_q(s_q(a)) \) we receive

\[
(\varphi, \Delta_x \psi) = \frac{1}{|\pi \hbar|^n} \int_G \overline{\varphi(a)} \psi(s_q(a)) e^{\frac{2i}{\hbar} (p, V_q(s_q(a)))} \sqrt{J_q(s_q(a))} \frac{dm(s_q)}{dm} \left| \frac{dm(a)}{dm} \right| \, dm(a)
\]

\[
= \frac{1}{|\pi \hbar|^n} \int_G \overline{\varphi(s_q(a))} \psi(a) e^{\frac{2i}{\hbar} (p, V_q(a))} \sqrt{J_q(a)} \, dm(a) = (\Delta_x \varphi, \psi).
\]

(174)

Hence operators \( \Delta_x \) are symmetric. To prove that they are self-adjoint we have to show that \( D(\Delta_x) = D(\Delta_x^\dagger) \). Clearly \( D(\Delta_x) \subset D(\Delta_x^\dagger) \) since \( \Delta_x \) are symmetric, so let \( \psi \in D(\Delta_x^\dagger) \). For every \( \varphi \in D(\Delta_x) \) from the definition of adjointness: \( (\Delta_x \varphi, \psi) = (\varphi, \Delta_x^\dagger \psi) \) and previous calculations we get

\[
\int_G \overline{\varphi(a)} \left( \frac{1}{|\pi \hbar|^n} \psi(s_q(a)) e^{-\frac{2i}{\hbar} (p, V_q(a))} \sqrt{J_q(a)} - \Delta_x^\dagger \psi(a) \right) \, dm(a) = 0.
\]

(175)

The above equality holds for every \( \varphi \in D(\Delta_x) \), hence

\[
\Delta_x^\dagger \psi(a) = \frac{1}{|\pi \hbar|^n} \psi(s_q(a)) e^{-\frac{2i}{\hbar} (p, V_q(a))} \sqrt{J_q(a)}.
\]

(176)

Since \( \Delta_x^\dagger \psi \in L^2(G, dm) \) also the right hand side of the above equality has to be square integrable. Thus \( \psi \in D(\Delta_x) \).

\textbf{Theorem 16.} Let \( f \in \mathcal{F}(M) \) and \( \hat{f} \) be the corresponding operator on \( \mathcal{H} \). Then

\[
\hat{f} = \int_M f(x) \Delta_x \, dx, \quad f(x) = |2\pi \hbar|^n \text{Tr}(\Delta_x \hat{f}).
\]

(177)

\textbf{Proof.} For \( \psi \in C_c^\infty(G) \) and \( a \in G \)

\[
\int_M f(x) \Delta_x \psi(a) \, dx = \frac{1}{|\pi \hbar|^n} \int_M f(q, p) \sqrt{J_q(a)} e^{-\frac{2i}{\hbar} (p, V_q(a))} \psi(s_q(a)) \, dx
\]

\[
= \frac{1}{|\pi \hbar|^n} \int_G \left( \int_{q^*} f(q, p) e^{-\frac{2i}{\hbar} (p, V_q(a))} \, dp \right) \psi(s_q(a)) \sqrt{J_q(a)} \, dm(q)
\]

\[
= 2^n \int_{L_a(U)} \hat{f}(q, -2V_q(a)) \psi(s_q(a)) \sqrt{J_q(a)} \, dm(q).
\]

(178)

Performing the following change of variables under the integral sign \( X \mapsto q = a \exp(\frac{i}{\hbar} X) \) we get

\[
\int_M f(x) \Delta_x \psi(a) \, dx = \int_{2\mathbb{O}} \hat{f}(a \exp(\frac{i}{\hbar} X), X) \psi(\exp(\frac{i}{\hbar} X)) \sqrt{J_a \exp(\frac{i}{\hbar} X)}(a) \frac{dm(\exp)}{dX} \, \frac{dX}{\frac{i}{\hbar} X}.
\]

(179)

We calculate that

\[
\sqrt{J_a \exp(\frac{i}{\hbar} X)}(a) \left| \frac{dm(\exp)}{dX} \right| \frac{dX}{\frac{i}{\hbar} X} = |\text{det} \psi(\text{ad} \frac{i}{\hbar} X) \, \text{det} \text{Ad}_{\exp(\frac{i}{\hbar} X)}(a) \, \text{det} \phi(\text{ad} \frac{i}{\hbar} X) | \chi(\mathbb{O}(X))
\]

\[
= |\text{det} \phi(\text{ad} X) \, \text{det} \text{Ad}_{\exp(\frac{i}{\hbar} X)}(X) | \sqrt{\text{det} \text{Ad}_{\exp(\frac{i}{\hbar} X)}(X) \chi(\mathbb{O}(X))}
\]

\[
= F(X) \, \text{det} \text{Ad}_{\exp(\frac{i}{\hbar} X)}(X) \chi(\mathbb{O}(X))
\]

(180)

since \( \psi(x) \phi^2(x) = \phi(2x) \). Thus, because of the fact that \( \hat{f} \) has support in \( G \times \mathbb{O} \), we get

\[
\int_M f(x) \Delta_x \psi(a) \, dx = \int_{\mathbb{O}} \hat{f}(a \exp(\frac{i}{\hbar} X), X) \psi(\exp(\frac{i}{\hbar} X)) F(X) \, \text{det} \text{Ad}_{\exp(\frac{i}{\hbar} X)}(X) \, dX = \hat{f}(a)
\]

(181)
which proves the first equality in (177).

Let \( f \in C^\infty_c(G \times G) \) be an integral kernel of \( \hat{f} \). Then an integral kernel of the operator \( \Delta_x \hat{f} \) is equal

\[
(a, b) \mapsto \frac{1}{|\pi \hbar|^n} \sqrt{J_q(a)} e^{-\frac{i}{\hbar} \langle p, V_q(a) \rangle} f(s_q(a), b).
\]

Therefore,

\[
|2\pi \hbar|^n \text{Tr} (\Delta_x \hat{f}) = 2^n \int_G \sqrt{J_q(a)} e^{-\frac{i}{\hbar} \langle p, V_q(a) \rangle} f(s_q(a), a) \, dm(a).
\]

Performing the following change of variables under the integral sign \( X \mapsto a = q \exp(\frac{i}{2} X) \) we get

\[
|2\pi \hbar|^n \text{Tr} (\Delta_x \hat{f}) = \int_{\mathcal{O}} f(q \exp(-\frac{i}{2} X), q \exp(\frac{i}{2} X)) e^{-\frac{i}{\hbar} \langle p, X \rangle} \sqrt{J_q(q \exp(\frac{i}{2} X))} \frac{d\text{m}(\exp(\frac{i}{2} X))}{\partial X} \, dX
\]

since

\[
\left. \frac{\sqrt{J_q(q \exp(\frac{i}{2} X))} \frac{d\text{m}(\exp(\frac{i}{2} X))}{\partial X}}{\frac{i}{2} X} \right|_{\frac{i}{2} X} = \sqrt{\det \psi(\text{ad}_{\frac{i}{2} X}) \det \text{Ad}_{\exp(\frac{i}{2} X)} | \det \phi(\text{ad}_{\frac{i}{2} X}) | | \chi_{\mathcal{O}}(X) = F(X) \chi_{\mathcal{O}}(X)\right).
\]

This proves the second equality in (177).

\[\square\]

**Corollary 1.** For \( \varphi \in L^2(G, dm) \) and \( \psi \in \mathcal{D}(\Delta_x) \) there holds

\[
(\varphi, \Delta_x \psi) = \frac{1}{|2\pi \hbar|^n} \mathcal{W}(\varphi, \psi)(x).
\]

From (186) it follows that the operator corresponding to the Dirac delta distribution \( \delta_x \in \mathcal{F}'(M) \) is equal to \( \Delta_x \):

\[
\hat{\delta_x} = \Delta_x.
\]

### 5 Phase space reduction

The Lie group \( G \) acts naturally on \( M \), where the action of \( G \) on \( M \) is given by the formula

\[
\Phi: G \times M \to M, \quad \Phi_g(q, p) = (gq, p), \quad g \in G, \quad (q, p) \in M.
\]

If \( G \) is a symmetry group of the system, then it will give rise to a reduced system. The action \( \Phi \) is free and proper, and for every \( g \in G \) the map \( \Phi_g \) preserves the Poisson structure on \( M \). Hence, the quotient space \( M/G \) is a smooth manifold with a natural Poisson structure. This Poisson manifold is diffeomorphic to the dual of the Lie algebra \( g^* \) endowed with the following Poisson bracket

\[
\{f, g\} = p_k \epsilon_{ijk} Z^j f Z^i g, \quad f, g \in C^\infty(g^*).
\]

Moreover, \( G \)-invariant functions in \( C^\infty(M) \) are exactly those functions which depend only on momentum variables, i.e. they are elements of \( C^\infty(g^*) \). Therefore, the space \( C^\infty(g^*) \) is a classical Poisson algebra of the reduced system and \( g^* \) is its phase space. Note, that the Poisson bracket (189) agrees with (15) for functions depending only on momentum variables.

The quantization procedure introduced in this paper respects the reduction operation described above. For simplicity let us assume that \( G \) is compact and that the Haar measure \( dm \) is normalized to unity. Denote
by $\mathcal{F}(\mathfrak{g})^*$ the subspace of $\mathcal{F}(M)$ consisting of functions depending only on momentum variables. Then $\mathcal{F}(\mathfrak{g})^*$ is a Fréchet subspace of $\mathcal{F}(M)$. The $*$-product \([51]\) of functions $f, g \in \mathcal{F}(\mathfrak{g})^*$ takes the form

$$
(f \ast g)(p) = \int_{\mathfrak{g} \times \mathfrak{g}} \tilde{f}(X)\tilde{g}(Y)e^{-\frac{i}{\hbar} (p \cdot X + Y)} L(X, Y) \, dX \, dY.
$$

(190)

Observe, that $f \ast g$ is only a function of $p$, therefore it is an element of $\mathcal{F}(\mathfrak{g})^*$. Thus, $\mathcal{F}(\mathfrak{g})^*$ is a Fréchet subalgebra of $\mathcal{F}(M)$ describing a reduced quantum system. The extension of $\mathcal{F}(\mathfrak{g})^*$ to a Hilbert algebra $\mathcal{L}(\mathfrak{g})^*$ is a Hilbert subspace of $\mathcal{L}(M)$ consisting of those functions in $\mathcal{L}(M)$ which depend only on momentum variables. Denote by $\mathcal{A}(\mathfrak{g})^*$ the extension of $\mathcal{L}(\mathfrak{g})^*$ to a $C^*$-algebra. The algebra $\mathcal{A}(\mathfrak{g})^*$ is an algebra of observables of the reduced quantum system and defines admissible states of the system. Moreover, the operator representation restricted to the $C^*$-algebra $\mathcal{A}(\mathfrak{g})^*$ can be reduced to an isometric $*$-representation on some Hilbert subspace of $\mathcal{H} = L^2(G, dm)$.

6 Examples

6.1 The group $\mathbb{R}^n$

The simplest example of a Lie group $G$ is an Abelian group $(\mathbb{R}^n, +)$ where the group operation is the addition of vectors. Its Lie algebra can be identified with the vector space $\mathbb{R}^n$ equipped with a trivial Lie bracket $[\cdot, \cdot] = 0$. This group is trivially weakly exponential for which $O = \mathbb{R}^n$ and $U = \mathbb{R}^n$. The group $\mathbb{R}^n$ can be used to describe translational degrees of freedom of a body or a system of $N$ bodies for $n = 3N$. In this case the $*$-product \([51]\) takes the form of a Moyal product

$$
(f \ast g)(q, p) = \int_{\mathbb{R}^{2n}} \tilde{f}(q - \frac{1}{2} Y, X)\tilde{g}(q + \frac{1}{2} X, Y)e^{-\frac{i}{\hbar} (p \cdot X + Y)} \, dX \, dY.
$$

(191)

6.2 The rotation group $SO(3)$

A non-trivial example of a Lie group $G$ which is weakly exponential is the rotation group $SO(3)$, i.e. the group of $3 \times 3$ real orthogonal matrices with determinant equal 1 ($A^T A = I$, $\det A = 1$). The Lie algebra of $SO(3)$ is denoted by $\mathfrak{so}(3)$ and consists of all real skew-symmetric $3 \times 3$ matrices ($X^T = -X$). The general element $X$ in $\mathfrak{so}(3)$ is of the form

$$
X = \begin{bmatrix}
0 & -z & y \\
z & 0 & -x \\
-y & x & 0
\end{bmatrix},
$$

(192)

where $\omega = (x, y, z)$ is the Euler vector which represents the axis of rotation and whose length is equal to the angle of rotation. For $X, Y \in \mathfrak{so}(3)$ the Euler vector corresponding to the Lie bracket $[X, Y]$ is equal to the vector product $\omega_1 \times \omega_2$ of the Euler vectors corresponding to $X$ and $Y$. On $\mathfrak{so}(3)$ we can introduce a norm $\|X\|$ of element $X$ as the length $|\omega|$ of the corresponding Euler vector $\omega$. Note, that $\|\cdot\|$ is compatible with the Lie bracket on $\mathfrak{so}(3)$: $\|[X, Y]\| \leq \|X\|\|Y\|$. In what follows we will show that the rotation group $SO(3)$ is weakly exponential.

Any rotation can be represented by a unique angle $\theta$ in the range $0 \leq \theta \leq \pi$ (rotation angle) and a unit vector $n$ (rotation axis). If $0 < \theta < \pi$ the vector $n$ is also unique, $\theta = 0$ corresponds to the identity matrix, and for $\theta = \pi$ vectors $\pm n$ correspond to the same rotation. Thus, as the set $O$ on which the exponential map exp is diffeomorphic we can take the set of all matrices $X$ which corresponding Euler vectors $\omega$ have length in the range $0 \leq |\omega| < \pi$. In other words $O = \{X \in \mathfrak{so}(3) \mid \|X\| < \pi\}$ is an open ball centered at 0 and of radius $\pi$. Then $U = \exp(O)$ consists of all rotations except ones with angle $\pi$. Note, that rotations with angle $\pi$ are exactly those matrices $R \in SO(3)$ for which $\text{Tr} R = -1$. Indeed, any matrix $R \in SO(3)$ can be written in a form $e^X$ for some $X \in \mathfrak{so}(3)$. From Rodrigues’ formula

$$
R = e^X = I + \frac{\sin \theta}{\theta} X + 2 \frac{\sin^2 \theta}{\theta^2} X^2,
$$

(193)
Yet another example of a weakly exponential Lie group is the special linear group \( \text{SL}(2, \mathbb{C}) \), therefore, \( \text{SO}(3) \) is complex, traceless 2 matrices. The eigenvalues of \( X \) are equal to square roots of \( \pm \lambda \). In a basis
\[
E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]
of the Lie algebra \( \mathfrak{so}(3) \) a matrix corresponding to the operator \( \text{ad}_X \) is just equal \( X \). Thus \( \text{ad}_X \) and \( X \) have the same set of eigenvalues. The eigenvalues of \( X \in \mathfrak{so}(3) \) are equal \( 0, i\|X\|, -i\|X\| \), so it is readily seen that for \( X \in \mathcal{O} \) its eigenvalues will satisfy condition \( \lambda_j \neq 2k\pi i, k = \pm 1, \pm 2, \ldots \). Thus we proved that \( \exp^{-1} \) is smooth.

In what follows we will prove that \( \text{SO}(3) \setminus \mathcal{U} \) has measure zero. Define function \( \Phi: \text{SO}(3) \rightarrow \mathbb{R} \) by the formula
\[
\Phi(A) = \text{Tr} A + 1.
\]
This function is smooth and its differential \( d\Phi(A) \) is nonzero for every \( A \in \text{SO}(3) \). Thus the level set \( \Phi^{-1}(0) = \text{SO}(3) \setminus \mathcal{U} \) will be a submanifold of \( \text{SO}(3) \) of dimension \( \dim \text{SO}(3) - 1 \). Therefore, \( \text{SO}(3) \setminus \mathcal{U} \) will be of measure zero [30, Corollary 5.14].

### 6.3 The group \( SL(2, \mathbb{C}) \)

Yet another example of a weakly exponential Lie group is the special linear group \( SL(2, \mathbb{C}) \). It is the group of \( 2 \times 2 \) complex matrices with determinant equal 1. The Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) of \( SL(2, \mathbb{C}) \) consists of all complex, traceless \( 2 \times 2 \) matrices. The eigenvalues \( \pm \lambda \) of a matrix \( X \in \mathfrak{sl}(2, \mathbb{C}) \) are equal to square roots of \( -\det X \) and the following equality holds: \( X^2 = \lambda^2 I \). It is then not difficult to see, that the exponential map on \( SL(2, \mathbb{C}) \) takes the form
\[
\exp(X) = \cosh \lambda I + \frac{\sinh \lambda}{\lambda} X.
\]
The eigenvalues of \( \exp(X) \) are then equal \( e^{\pm \lambda} \). Let
\[
\mathcal{O} = \{ X \in \mathfrak{sl}(2, \mathbb{C}) \mid |\text{Im} \lambda| < \pi \text{ where } \lambda \text{ is one of the eigenvalues of } X \}
\]
and
\[
\mathcal{U} = \{ A \in SL(2, \mathbb{C}) \mid \text{Tr} A \in \mathbb{C} \setminus (-\infty, -2) \}.
\]

In what follows we will show that \( \exp: \mathcal{O} \rightarrow \mathcal{U} \) is a diffeomorphism. First observe that for \( A = e^X \)
\[
\text{Tr} A = 2 \cosh \lambda,
\]
where \( \lambda \) is an eigenvalue of \( X \). On the other hand eigenvalues \( \alpha_1, \alpha_2 \) of \( A \in SL(2, \mathbb{C}) \) can be expressed by \( \text{Tr} A \) according to the formulas
\[
\alpha_1 = \frac{\text{Tr} A - \sqrt{(\text{Tr} A)^2 - 4}}{2}, \quad \alpha_2 = \frac{\text{Tr} A + \sqrt{(\text{Tr} A)^2 - 4}}{2}.
\]
Note, that exp is a bijection of \( \{ z \in \mathbb{C} \mid |\text{Im} z| < \pi \} \) onto \( \mathbb{C} \setminus (-\infty, 0] \) and its inverse \( \exp^{-1} = \ln \) is a principal value of the logarithmic function. Because \( z \pm \sqrt{z^2 - 1} \neq 0 \) for every \( z \in \mathbb{C} \), \( z \pm \sqrt{z^2 - 1} < 0 \) only for \( z \in (-\infty, -1] \) and \( z \pm \sqrt{z^2 - 1} = (z \mp \sqrt{z^2 - 1})^{-1} \) the function
\[
    f(z) = \frac{\ln(z \pm \sqrt{z^2 - 1})}{\pm \sqrt{z^2 - 1}}
\]
is well defined for \( z \in \mathbb{C} \setminus (-\infty, -1] \) and independent on the choice of the square root \( \pm \sqrt{z^2 - 1} \) of \( z^2 - 1 \) used in its definition. If we define for \( A \in \mathcal{U} \)
\[
    \exp^{-1}(A) = f\left(\frac{1}{2} \text{Tr} A \right) \left( A - \frac{1}{2}(\text{Tr} A)I \right),
\]
then it is easy to check that \( \exp^{-1} \) given by the above formula is indeed an inverse function to \( \exp : \mathcal{O} \rightarrow \mathcal{U} \). This proves bijectivity of \( \exp \).

Note, that \( \exp : \mathfrak{sl}(2, \mathbb{C}) \rightarrow SL(2, \mathbb{C}) \) is not surjective \cite{29}. In fact the set of matrices not in the image of \( \exp \) is equal
\[
    SL(2, \mathbb{C}) \setminus \exp(\mathfrak{sl}(2, \mathbb{C})) = \{ A \in SL(2, \mathbb{C}) \mid \text{Tr} A = -2, \ A \neq -I \}. \quad (204)
\]
Remains to check that \( \exp^{-1} : \mathcal{U} \rightarrow \mathcal{O} \) is smooth. Similarly as in Section 6.2 this will be the case if for all \( X \in \mathcal{O} \) eigenvalues \( \lambda_k \) of \( \text{ad}_X \) will satisfy: \( \lambda_k \neq 2k\pi i, \ k = \pm 1, \pm 2, \ldots \). Observe that if \( \pm \lambda \) are eigenvalues of \( X \), then eigenvalues of \( \text{ad}_X \) are equal 0, \( -2\lambda, 2\lambda \). Since \( |\text{Im} \lambda| < \pi \) it follows that \( \exp^{-1} \) is smooth.

Now we will show that \( SL(2, \mathbb{C}) \setminus \mathcal{U} \) is of measure zero. Let \( \Sigma = \{ A \in SL(2, \mathbb{C}) \mid \text{Tr} A \in \mathbb{R} \} \). Define function \( \Phi : SL(2, \mathbb{C}) \rightarrow \mathbb{R} \) by the formula
\[
    \Phi(A) = \text{Im} \text{Tr} A.
\]
This function is smooth and its differential \( d\Phi(A) \) is nonzero for every \( A \in SL(2, \mathbb{C}) \). Thus the level set \( \Phi^{-1}(0) = \Sigma \) will be a submanifold of \( SL(2, \mathbb{C}) \) of dimension \( \dim \Sigma = \dim SL(2, \mathbb{C}) - 1 \) \cite[Corollary 5.14]{31}. Therefore, \( \Sigma \) will be of measure zero \cite[Corollary 6.12]{31} and consequently \( SL(2, \mathbb{C}) \setminus \mathcal{U} \subset \Sigma \) as well.

The special linear group \( SL(2, \mathbb{C}) \) is an example of a non-compact simply connected Lie group. Note, however, that its subgroup \( SL(2, \mathbb{R}) \) is not weakly exponential. The sets \( \mathcal{O} \) and \( \mathcal{U} \) can be defined in an analogous way and \( \exp : \mathcal{O} \rightarrow \mathcal{U} \) will be a diffeomorphism, but \( SL(2, \mathbb{R}) \setminus \mathcal{U} \) will not have measure zero. In fact \( \{ A \in SL(2, \mathbb{R}) \mid \text{Tr} A < -2 \} \subset SL(2, \mathbb{R}) \setminus \mathcal{U} \) is a non-empty open subset and as such has non-zero measure. On the other hand \( SU(2) \) is also a subgroup of \( SL(2, \mathbb{C}) \), which as we will see, is weakly exponential.

### 6.4 The group \( SU(2) \)

Another example of a weakly exponential Lie group is the special unitary group \( SU(2) \). It is the group of \( 2 \times 2 \) complex unitary matrices with determinant equal 1 \( (U^\dagger U = I, \ \det U = 1) \). The Lie algebra \( \mathfrak{su}(2) \) of \( SU(2) \) consists of all complex skew-Hermitian \( 2 \times 2 \) matrices with trace zero \( (X^\dagger = -X, \ \text{Tr} X = 0) \). On \( \mathfrak{su}(2) \) we can introduce a norm by the following formula: \( \|X\| = 2\sqrt{\text{det} X} \). Note, that \( \| \cdot \| \) is compatible with the Lie bracket on \( \mathfrak{su}(2) \): \( \|[X,Y]\| \leq \|X\|\|Y\| \).

The exponential map \( \exp \) maps an open ball \( \mathcal{O} = \{ X \in \mathfrak{su}(2) \mid \|X\| < 2\pi \} \) diffeomorphically onto \( \mathcal{U} = SU(2) \setminus \{-I\} \), see \cite{29}. Indeed, eigenvalues of \( X \in \mathfrak{su}(2) \) are equal \( \pm \frac{1}{2} \|X\| \) and for \( U \in SU(2) \) we have that \( \text{Tr} U \in \mathbb{R} \) and \( |\text{Tr} U| \leq 2 \). Thus the sets \( \mathcal{O} \) and \( \mathcal{U} \) corresponding to the group \( SL(2, \mathbb{C}) \) and intersected with \( \mathfrak{su}(2) \) and \( SU(2) \), respectively, are equal to the sets \( \mathcal{O} \) and \( \mathcal{U} \) defined above. The exponential map \( \exp : \mathcal{O} \rightarrow \mathcal{U} \) will be then a diffeomorphism. Clearly \( SU(2) \setminus \mathcal{U} = \{-I\} \) is of measure zero. The special unitary group \( SU(2) \) is an example of a compact simply connected Lie group.

### 7 Conclusions and final remarks

In the paper was presented a complete theory of quantization of a classical Hamiltonian system whose configuration space is in the form of a Lie group. The received theory can be now used to quantize particular
systems, like a rigid body. The configuration space of this system is the rotation group $SO(3)$ representing rotational degrees of freedom. The translational degrees of freedom can be included by taking as the configuration space the semi-direct product of $\mathbb{R}^3$ with $SO(3)$, which is equal to a special Euclidean group $E^+(3)$.

The use of deformation quantization approach to quantize a rigid body might help in developing in quantum mechanics techniques from a classical theory. Particular examples of rigid bodies are tops describing a precession of a body under the influence of gravity. Especially interesting are Euler, Lagrange, and Kovalevskaya tops, which are integrable Hamiltonian systems. It would be interesting to use the developed quantization theory to construct quantum versions of these systems together with a theory of quantum integrability.

Worth noting are papers [31] where authors determined for the exponential function $\exp: \mathfrak{g} \to G$ maximal open domains in $\mathfrak{g}$ on which $\exp$ is injective regardless of the structure of $\mathfrak{g}$ or $G$. In [34] authors showed that if we choose a norm on $\mathfrak{g}$ such that $\| [X,Y] \| \leq \| X \| \| Y \|$, then $\exp$ will be injective on an open ball $B_\pi = \{ X \in \mathfrak{g} \mid \| X \| < \pi \}$ of radius $\pi$ (and if $G$ is simply connected, then $\exp$ will be injective on an open ball $B_{2\pi}$ of radius $2\pi$). The presented examples of the groups $SO(3)$ and $SU(2)$ agree with this result. In [31] authors introduced a function $\sigma: \mathfrak{g} \to \mathbb{R}^+$ defined by the formula

$$\sigma(X) = \max\{ |\text{Im}\lambda| \mid \lambda \in \text{Spec}(\text{ad}_X) \}.$$  

(206)

Then they showed that $\exp$ is a diffeomorphism of $\mathcal{O} = \{ X \in \mathfrak{g} \mid \sigma(X) < \pi \}$ onto $\mathcal{U} = \exp(\mathcal{O})$. This result agrees with the presented examples of the groups $SO(3)$, $SU(2)$, and $SL(2, \mathbb{C})$. In fact for the groups $SU(2)$ and $SL(2, \mathbb{C})$ the exponential function is diffeomorphic on a larger set equal $\mathcal{O} = \{ X \in \mathfrak{g} \mid \sigma(X) < 2\pi \}$, which is caused by the fact that these groups are simply connected.

In the paper we introduced a topology on the space $\mathcal{F}(M)$ in the case of a compact group $G$. This was done by establishing an isomorphism between $\mathcal{F}(M)$ and $C^\infty(G \times G)$ and with its help transferring a natural Fréchet space topology on $C^\infty(G \times G)$ onto $\mathcal{F}(M)$. To extend this construction in a meaningful way to non-compact groups, so that the $*$-product would remain continuous and the results of Section 3.4 would still hold, we would have to extend the space $\mathcal{F}(M)$ by extending the corresponding space $C^\infty_c(G \times G)$ of integral kernels to some bigger space with nicer properties. One of the candidates worth investigating is a Harish-Chandra’s Schwartz space of functions on $G \times G$ whose derivatives are rapidly decreasing [35].

The presented approach to quantization can be reformulated and applied to systems which configuration spaces are in the form of a Riemann-Cartan manifold, i.e. a manifold endowed with a metric tensor and a metric affine connection. An analog of the weakly exponential Lie group will be a weakly geodesically simply connected Riemann-Cartan manifold, i.e. a Riemann-Cartan manifold $\mathcal{Q}$ such that for every point $q \in \mathcal{Q}$ there exists a neighborhood $\mathcal{U}_q \subset \mathcal{Q}$ of $q$ for which $\mathcal{Q} \setminus \mathcal{U}_q$ is of measure zero and every point in $\mathcal{U}_q$ can be connected with $q$ by a unique geodesics.

Then the exponential map $\exp_q$ will diffeomorphically some open neighborhood $\mathcal{O}_q \subset T_q \mathcal{Q}$ of $0$ onto $\mathcal{U}_q$. In fact a semi-simple Lie group is an example of a Riemann-Cartan manifold which metric tensor is given by a Killing-Cartan form and which affine connection is such that the exponential map at unit element is equal to the exponential map on the Lie group. Such Riemann-Cartan manifold has vanishing curvature tensor and non-vanishing torsion tensor in the non-Abelian case.

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