Local well-posedness for the nonlinear Schrödinger equation in modulation spaces $M_{p,q}^s(\mathbb{R}^d)$

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Abstract
We show the local well-posedness of the Cauchy problem for the cubic nonlinear Schrödinger equation on modulation spaces $M_{p,q}^s(\mathbb{R}^d)$ for $d \in \mathbb{N}$, $1 \leq p, q \leq \infty$ and $s > d \left(1 - \frac{1}{q}\right)$ for $q > 1$ or $s \geq 0$ for $q = 1$. This improves [4, Theorem 1.1] by Bényi and Okoudjou where only the case $q = 1$ is considered. Our result is based on the algebra property of modulation spaces with indices as above for which we give an elementary proof via a new Hölder-like inequality for modulation spaces.

1. Introduction
We study the Cauchy problem for the cubic nonlinear Schrödinger equation (NLS)
\[
\begin{cases}
\frac{∂u}{∂t}(x,t) + ∆ u(x,t) ± |u|^2 u(x,t) = 0 & (x,t) \in \mathbb{R}^d × \mathbb{R}, \\
u(x,0) = u_0(x) & x \in \mathbb{R}^d,
\end{cases}
\]
where the initial data $u_0$ is in a modulation space $M_{p,q}^s(\mathbb{R}^d)$. A definition of $M_{p,q}^s(\mathbb{R}^d)$ will be given in the next paragraph. As usual, we are interested in mild solutions $u$ of (1), i.e. $u \in C \left([0,T), M_{p,q}^s(\mathbb{R}^d)\right)$ for a $T > 0$ which satisfy the corresponding integral equation
\[
u(·, t) = e^{it∆}u_0 ± i \int_0^t e^{i(t-τ)∆} \left(|u|^2 u(·, τ)\right) dτ \quad (\forall t \in [0,T)).
\]

Modulation spaces $M_{p,q}^s(\mathbb{R}^d)$ were introduced by Feichtinger in [6]. Here, we give a short summary of their definition and properties. (We refer to Section 2 and the literature mentioned there for more information, the notation we use is explained at the end of the introduction.) Fix a so-called window function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. The short-time Fourier transform $V_g f$ of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window $g$ is defined by
\[
V_g f (x, ·) = \mathcal{F} \left(\frac{S_x g f}{S_x g}\right) (·) \in \mathcal{S}'(\mathbb{R}^d) \quad \forall x \in \mathbb{R}^d.
\]
In fact, $V_g f : \mathbb{R}^d × \mathcal{S}(\mathbb{R}^d) → C$ can be represented by a continuous function $\mathbb{R}^d × \mathbb{R}^d → C$. Hence, taking a weighted, mixed $L^p$-norm is possible and we define
\[
M_{p,q}^s(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \left| \|f\|_{M_{p,q}^s(\mathbb{R}^d)} < ∞ \right\} , \text{ where } \|f\|_{M_{p,q}^s(\mathbb{R}^d)} = \left\| ξ ↦ ⟨ξ⟩^s \|V_g f (·, ξ)∥_p \right\|_q
\]
for $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. It can be shown, that the $M_{p,q}^s(\mathbb{R}^d)$ are Banach spaces and that different choices of the window function $g$ lead to equivalent norms.

Our main result is

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Preprint submitted to arXiv.
Theorem 1 (Local well-posedness). Let $d \in \mathbb{N}$ and $1 \leq p, q \leq \infty$. For $q > 1$ let $s > d \left(1 - \frac{1}{q}\right)$ and for $q = 1$ let $s \geq 0$. Assume that $u_0 \in M_{p,q}^{s}(\mathbb{R}^d)$. Then, there exists a unique maximal mild solution $u \in C \left([0, T^*), M_{p,q}^{s}(\mathbb{R}^d)\right)$ of $\left(1\right)$ and the blow-up alternative

$$T^* < \infty \quad \Rightarrow \quad \limsup_{t \to T^*} \|u(\cdot, t)\|_{M_{p,q}^{s}(\mathbb{R}^d)} = \infty$$

holds. Furthermore, for any $0 < T' < T^*$ there exists a neighborhood $V$ of $u_0$ in $M_{p,q}^{s}$, such that the initial data to solution map

$$V \to C \left([0, T'], M_{p,q}^{s}(\mathbb{R}^d)\right), \quad v_0 \mapsto v,$$

is Lipschitz continuous.

Let us remark that the only known local well-posedness results in modulation spaces until now are \cite{Theorem 1.1} by Wang, Zhao and Guo for $M_{0,1}^{0}(\mathbb{R}^d)$ and its generalization \cite{Theorem 1.1} due to Benyi and Okoudjou for $M_{p,1}^{s}(\mathbb{R}^d)$ with $1 \leq p \leq \infty$ and $s \geq 0$. Local well-posedness results without persistence (i.e. initial data in a modulation space, but the solution is not a curve on it) include \cite{Theorem 1.4} for $u_0 \in M_{0,q}^{s}(\mathbb{R}^d)$ with $2 \leq q \leq \infty$.

Theorem 1 generalizes \cite[Theorem 1.1]{Theorem 1} to $q \geq 1$: Although our theorem is stated for the cubic nonlinearity, this is for simplicity of the presentation only. The proof allows for an easy generalization to algebraic nonlinearities considered in \cite{Theorem 1}, which are of the form

$$f(u) = g(|u|^2)u = \sum_{k=0}^{\infty} c_k |u|^{2k} u, \quad \text{where } g \text{ is an entire function.} \quad \left(4\right)$$

Also, Theorems 1.2 and 1.3 in \cite{Theorem 1} which concern the nonlinear wave and the nonlinear Klein-Gordon equation respectively, can be generalized in the same spirit.

This is due to Benyi’s and Okoudjou’s and our proofs being based on the well-known Banach’s contraction principle, an estimate for the norm of the Schrödinger propagator and the fact that the considered modulation spaces $M_{p,q}^{s}(\mathbb{R}^d)$ are Banach ⋆-algebras\cite{Theorem 1} with respect to pointwise multiplication. Let us state the two latter ingredients formally and comment on them.

The first is given by

Proposition 2 (Algebra property). Let $d \in \mathbb{N}$ and $1 \leq p, q \leq \infty$. For $q > 1$ let $s > d \left(1 - \frac{1}{q}\right)$ and for $q = 1$ let $s \geq 0$. Then $M_{p,q}^{s}(\mathbb{R}^d)$ is a Banach ⋆-algebra with respect to pointwise multiplication and complex conjugation. These operations are well-defined due to the following embedding

$$M_{p,q}^{s}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d) := \{ f \in C(\mathbb{R}^d) | f \text{ bounded} \}.$$

Proposition 2 had been observed already in 1983 by Feichtinger in his pioneering work on modulation spaces, cf. \cite[Proposition 6.9]{Theorem 2} where he proves it using a rather abstract approach via Banach convolution triples. This might explain why the algebra property seems to be not well-known in the PDE community. In \cite[Corollary 2.6]{Theorem 1} Proposition 2 for $q = 1$ is stated without referring to Feichtinger and a proof via the theory of pseudodifferential operators is said to be along the lines of \cite[Theorem 3.1]{Theorem 2}. In contrast to these approaches, our proof of the algebra property is elementary. It follows from the new Hölder-like inequality stated in

Theorem 3 (Hölder-like inequality). Let $d \in \mathbb{N}$ and $1 \leq p, p_1, p_2, q \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. For $q > 1$ let $s > d \left(1 - \frac{1}{q}\right)$ and for $q = 1$ let $s \geq 0$. Then there exists a constant $C = C(d, s, q) > 0$ such that

$$\|fg\|_{M_{p,q}^{s}(\mathbb{R}^d)} \leq C \|f\|_{M_{p_1,q}^{s}(\mathbb{R}^d)} \|g\|_{M_{p_2,q}^{s}(\mathbb{R}^d)} \cdot \quad \left(5\right)$$

1For us a Banach ⋆-algebra $X$ is a Banach algebra over $\mathbb{C}$ on which a continuous involution $*$ is defined, i.e. $(x+y)^* = x^*+y^*$, $(\lambda x)^* = \overline{\lambda} x^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for any $x, y \in X$ and $\lambda \in \mathbb{C}$. We neither require $X$ to have a unit nor $C = 1$. In the estimates $\|x \cdot y\| \leq C \|x\| \|y\|$, $\|x^*\| \leq C \|x\|$. 2
for all \( f \in M^s_{p,q}(\mathbb{R}^d) \), \( g \in M^s_{p,q}(\mathbb{R}^d) \). The pointwise multiplication is well-defined due to the embedding formulated in Proposition 3.

Crucial for the proof of Theorem 3 is the algebra property of the sequence spaces \( l^s_p(\mathbb{Z}^d) \) stated in Lemma 3 (\( s, q \) and \( d \) are as in Theorem 3, \( l^s_p(\mathbb{Z}^d) \) is defined at the end of the introduction).

The second crucial ingredient for the proof of Theorem 1 is the boundedness of the Schrödinger propagator \( e^{it\Delta} \) on all modulation spaces \( M^s_{p,q}(\mathbb{R}^d) \). Let us fix the window function \( x \mapsto e^{-|x|^2} \) in the definition of the modulation space norm. Then we have (notation is explained at the end of the introduction)

**Theorem 4 (Schrödinger propagator bound).** There is a constant \( C > 0 \) such that for any \( d \in \mathbb{N} \), \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R} \) the inequality

\[
\| e^{it\Delta} \|_{L^p(M^s_{p,q}(\mathbb{R}^d))} \leq C^d (1 + |t|)^{d/2 - 1/2}
\]

holds for all \( t \in \mathbb{R} \). Furthermore, the exponent of the time dependence is sharp.

The boundeness has been obtained e.g. in [3, Theorem 1] whereas the sharpness was proven in [3, Proposition 4.1]. We sketch a simple proof of Theorem 4 in Section 2.

The remainder of our paper is structured as follows. We start with Section 2 providing an overview over modulation spaces, showing that Proposition 2 follows from Theorem 3 and sketching a simple proof of Theorem 4. In Section 3 we prove an algebra property of the weighted sequence spaces \( l^s_p(\mathbb{Z}^d) \) for sufficiently large \( s \). In the subsequent Section 4 we prove the Hölder-like inequality from Theorem 3. Finally, we prove Theorem 4 on the local well-posedness in Section 5.

**Notation**

We denote generic constants by \( C \). To emphasize on which quantities a constant depends we write e.g. \( C = C(d) \) or \( C = C(d, s) \). Sometimes we omit a constant from an inequality by writing \( \cdot \lesssim \cdot \), e.g. \( A \lesssim B \) instead of \( A \leq C(d)B \). Special constants are \( d \in \mathbb{N} \) for the dimension, \( 1 \leq p, q \leq \infty \) for the Lebesgue exponents and \( s \in \mathbb{R} \) for the regularity exponent. By \( p' \) we mean the dual exponent of \( p \), that is the number satisfying \( \frac{1}{p} + \frac{1}{p'} = 1 \). To simplify the subsequent claims we shall call a regularity exponent \( s \) sufficiently large, if

\[
\begin{cases}
  > \frac{d}{q} & \text{for } q > 1, \\
  \geq 0 & \text{for } q = 1.
\end{cases}
\]

We denote by \( \mathcal{S}(\mathbb{R}^d) \) the set of **Schwartz functions** and by \( \mathcal{S}'(\mathbb{R}^d) \) the space of **tempered distributions**. Furthermore, we denote the **Bessel potential spaces** or simply \( L^2 \)-based **Sobolev spaces** by \( H^s = H^s(\mathbb{R}^d) \) or by \( H^s(\mathbb{T}^d) \), if we are on the \( d \)-dimensional Torus \( \mathbb{T}^d \). For the space of bounded continuous functions we write \( C_b \) and for the space of smooth functions with compact support we write \( C_c^\infty \). The letters \( f, g, h \) denote either generic functions \( \mathbb{R}^d \to \mathbb{C} \) or generic tempered distributions. Whereas \( (a_k)_{k \in \mathbb{Z}^d}, (b_k)_{k \in \mathbb{Z}^d}, (c_k)_{k \in \mathbb{Z}^d} \) or \( (a_k)_k, (b_k)_k, (c_k)_k \) denote generic complex-valued sequences. By \( \langle \cdot \rangle = \sqrt{1 + |\cdot|^2} \) we denote the **Japanese bracket**.

For a Banach space \( X \) we write \( X^* \) for its dual and \( \| \cdot \|_X \) for the norm it is canonically equipped with. By \( \mathcal{L}(X) \) we denote the space of all bounded linear maps on \( X \). By \( [X,Y]_p \) we mean complex interpolation between \( X \) and another Banach space \( Y \). For brevity we write \( \| \cdot \|_p \) for the \( p \)-norm on the Lebesgue space \( L^p(\mathbb{R}^d) \), the sequence space \( l^p = l^p(\mathbb{Z}^d) \) or \( l^p = l^p(\mathbb{N}_0) \) and \( \|(a_k)\|_{q,s} := \|\langle k \rangle^s a_k\|_{q} \) for the norm on \( \langle \cdot \rangle^s \)-weighted sequence spaces \( l^s_q = l^s_q(\mathbb{Z}^d) \). Also, we shorten the notation for modulation spaces: \( M^s_{p,q}(\mathbb{R}^d) \) and even \( M^0_{p,q}(\mathbb{R}^d) \). If the norm is clear from the context, we write \( B_r(x) \) for a ball of radius \( r \) around \( x \in X \) and set \( B_r = B_r(0) \).

Furthermore, we denote the **Fourier transform** by \( \mathcal{F} \) and the inverse Fourier transform by \( \mathcal{F}^{-1} \), where we use the symmetric choice of constants and write also

\[
\hat{f}(\xi) := (\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx, \quad \hat{g}(x) := (\mathcal{F}^{-1} g)(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi \cdot x} g(\xi) d\xi.
\]
Finally, we introduce the operations $S_x f(y) = f(y - x)$ of translation by $x \in \mathbb{R}^d$, $(M_k f)(y) = e^{ik \cdot y} f(y)$ of modulation by $k \in \mathbb{R}^d$ and $\overline{f}$ of complex conjugation.

2. Modulation spaces

As already mentioned in the introduction, modulation spaces were introduced by Feichtinger in the setting of locally compact Abelian groups. The textbook [8] by Gröchenig gives a thorough introduction, although it lacks the characterization of modulation spaces via isometric decomposition operators defined below. A presentation incorporating these operators is contained in the paper [12, Section 2.3] by Wang and Hudzik. A survey on modulation spaces and nonlinear evolution equations is given in [10].

A convenient equivalent norm on modulation spaces which we are going to use is constructed as follows: Let $Q_0 := \left[\frac{-1}{2}, \frac{1}{2}\right]^d$ and $Q_k := Q_0 + k$ for all $k \in \mathbb{Z}^d$.Consider a smooth partition of unity $(\sigma_k)_{k \in \mathbb{Z}^d} \in \left(C^\infty(\mathbb{R}^d)\right)^{\mathbb{Z}^d}$ satisfying

(i) $\exists c > 0 : \forall k \in \mathbb{Z}^d : \forall \eta \in Q_k : |\sigma_k(\eta)| \geq c$,
(ii) $\forall k \in \mathbb{Z}^d : \text{supp}(\sigma_k) \subseteq B_{\sqrt{d}}(k)$,
(iii) $\sum_{k \in \mathbb{Z}^d} \sigma_k = 1$,
(iv) $\forall m \in \mathbb{N}_0 : \exists C_m > 0 : \forall k \in \mathbb{Z}^d : \forall \alpha \in \mathbb{N}_0^d : |\alpha| \leq m \Rightarrow \|D^\alpha \sigma_k\|_\infty \leq C_m$

and define the isometric decomposition operators $\square_k := \mathcal{F}^{-1} \sigma_k \mathcal{F}$. Let us mention the fact that $\square_k f \in C^\infty(\mathbb{R}^d)$ for $f \in S'(\mathbb{R}^d)$ by [7, Theorem 2.3.1]. We cite from [12, Proposition 1.9] the following often used

**Lemma 5 (Bernstein multiplier estimate).** Let $d \in \mathbb{N}$, $1 \leq p \leq \infty$, $s > \frac{d}{2}$ and $\sigma \in H^s(\mathbb{R}^d)$. Then the multiplier operator $T_{\sigma} = \mathcal{F}^{-1} \sigma \mathcal{F} : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ corresponding to the symbol $\sigma$ is bounded on $L^p(\mathbb{R}^d)$. More precisely, there is a constant $C = C(s, d) > 0$ such that

$$
\|T_{\sigma}\|_{L^p(L^p(\mathbb{R}^d))} \leq C \|\sigma\|_{H^s(\mathbb{R}^d)}.
$$

By Lemma 5 the family $(\square_k)_{k \in \mathbb{Z}^d}$ is bounded in $L^p(L^p(\mathbb{R}^d))$ independently of $p$. The aforementioned equivalent norm for the modulation space $M^s_{p, q}$ is given by

$$
\|f\|_{M^s_{p, q}} \equiv \left\| \left( \|\square_k f\|_p \right)_{k \in \mathbb{Z}^d} \right\|_{l^q(\mathbb{Z}^d)}.
$$

Choosing a different partition of unity $(\sigma_k)$ yields yet another equivalent norm.

**Lemma 6 (Continuous embeddings).** Let $s_1 \geq s_2$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$. Then

(a) $M^{s_1}_{p_1, q_1}(\mathbb{R}^d) \subseteq M^{s_2}_{p_2, q_2}(\mathbb{R}^d)$ and the embedding is continuous,
(b) $M_{p, 1}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d)$.

**Lemma 6** is well-known (cf. [12, Proposition 2.5, 2.7]). For convenience we sketch a

**Proof.** (a) One can change indices one by one. The inclusion for “$s$” is by monotonicity and the inclusion for “$q$” is by the embeddings of the $l^q$ spaces. For the “$p$”-embedding consider $\tau \in C^\infty(\mathbb{R}^d)$ such that $\tau|_{B_{\sqrt{d}}} \equiv 1$ and $\text{supp}(\tau) \subseteq B_d$. Define the shifted $\tau_k = \tau_k \tau$ and the corresponding multiplier operators $\square_k = \mathcal{F}^{-1} \tau_k \mathcal{F}$. Clearly, $\square_k \square_k = \square_k$ and $\square_k f = \frac{1}{(2\pi)^{\frac{d}{2}}} \left( M_k \tau \right) * f$. Hence

$$
\|\square_k f\|_p = \left\| \left( \square_k \square_k f \right) \right\|_{l^p(\mathbb{Z}^d)} \leq \left( \frac{1}{(2\pi)^{\frac{d}{2}}} \right) \left( \left\| M_k \tau \right\|_p \right) \left\| f \right\|_{l^p(\mathbb{Z}^d)}.
$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

Recalling (8) finishes the proof.
(b) By part (a) it is enough to show \( M_{\infty, 1} \hookrightarrow C_b \). For any \( f \in M_{\infty, 1} \) we have
\[
\sum_{|k| \leq N} \Box_k f \to f \text{ in } S' \text{ as } N \to \infty. \]
But simultaneously
\[
\left\| \sum_{N_1 \leq |k| \leq N_2} \Box_k f \right\|_{L^\infty} \leq \sum_{N_1 \leq |k| \leq N_2} \| \Box_k f \|_{L^\infty} \leq \sum_{k \in \mathbb{Z}^d} \| \Box_k f \|_{L^\infty} < \infty.
\]
So \( f \in C_b \) and \( \sum_{|k| \leq N} \Box_k f \to f \in C_b \) as \( N \to \infty \).

We are now ready to give a

**Proof of Proposition 2.** We have \( l_2^q \hookrightarrow l^1 \) for sufficiently large \( s \), since
\[
\sum_{k \in \mathbb{Z}^d} |a_k| = \sum_{k \in \mathbb{Z}^d} \frac{1}{(k)^s} |a_k| \text{ Hölder} \leq \left( \sum_{k \in \mathbb{Z}^d} \frac{1}{(k)^{sq'}} \right)^{\frac{1}{q'}} \left( \sum_{i \in \mathbb{Z}^d} (i)^s |a_i|^q \right)^{\frac{1}{q}} < \infty \text{ for } s > d.
\]

Then \( 3 \) yields \( M_{p, q}^s \hookrightarrow M_{p, 1} \) and by Lemma 4 \( 3 \) we have \( M_{p, 1} \hookrightarrow C_b \). This proves the claimed embedding.

Choosing \( \sigma_k \) real-valued in \( 3 \) shows that complex conjugation does not change the modulation space norm.

Choosing \( p_1 = p_2 = 2p \) in Theorem 3 and applying Lemma 3 (a) shows the estimate for the continuity of pointwise multiplication and finishes the proof.

**Lemma 7 (Dual space).** For \( s \in \mathbb{R}, 1 \leq p, q < \infty \) we have
\[
(M_{p, q}^s)^* = M_{p', q'}^{-s}
\]
(see \( 4 \), Theorem 3.1].)

**Theorem 8 (Complex interpolation).** For \( 1 \leq p_1, q_1 < \infty, 1 \leq p_2, q_2 \leq \infty \), \( s_1, s_2 \in \mathbb{R} \) and \( \theta \in (0, 1) \) one has
\[
[M_{p_1, q_1}^{s_1}(\mathbb{R}^d), M_{p_2, q_2}^{s_2}(\mathbb{R}^d)]_{\theta} = M_{p, q}^{s}(\mathbb{R}^d),
\]
with
\[
\frac{1}{p} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1 - \theta}{q_1} + \frac{\theta}{q_2}, \quad s = (1 - \theta)s_1 + \theta s_2
\]
(see \( 4 \), Theorem 6.1 (D)].)

Using these results we sketch a

**Proof of Theorem 4.** We have \( V_g(e^{it \Delta} f) = V_{e^{-it \Delta} g} f \) by duality, i.e. the Schrödinger time evolution of the initial data can be interpreted as the backwards time evolution of the window function. The price for changing from window \( g_0 \) to window \( g_1 \) is \( \| V_{g_0} g_1 \|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} \) by \( 3 \), Proposition 11.3.2 (c)]. For \( g(x) = e^{-|x|^2} \) one explicitly calculates
\[
\| V_{e^{-it \Delta} g} \|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = C d (1 + |t|)^{\frac{d}{2}},
\]
which proves the claimed bound for \( p \in \{1, \infty\} \). Conservation for \( p = 2 \) is easily seen from \( 3 \). Complex interpolation between the cases \( p = 2 \) and \( p = \infty \) yields \( 4 \) for \( 2 \leq p \leq \infty \). The remaining case \( 1 < p < 2 \) is covered by duality.

Optimality in the case \( 1 \leq p \leq 2 \) is proven by choosing the window \( g \) and the argument \( f \) to be a Gaussian and explicitly calculating \( \| e^{it \Delta} f \|_{M_{p, q}} \approx (1 + |t|)^{d\left(\frac{1}{p} - \frac{1}{2}\right)} \). This implies the optimality for \( 2 < p \leq \infty \) by duality.
3. Algebra property of some weighted sequence spaces

Let us recall the definition of the $(\cdot)^s$-weighted sequence spaces

\[
l^{s}_q(\mathbb{Z}^d) = \left\{ (a_k) \in C^{(2^d)} \left| \|(a_k)\|_{q,s} < \infty \right. \right\}, \quad \text{where} \quad \|(a_k)\|_{q,s} = \left\{ \left( \sup_{k \in \mathbb{Z}^d} |k|^q |a_k| \right)^{\frac{1}{q}} \right\} \quad \text{for} \quad 1 \leq q < \infty,
\]

\[
\|(a_k)\|_{q,s} = \left\{ \left( \sum_{k \in \mathbb{Z}^d} |k|^q |a_k| \right)^{\frac{1}{q}} \right\} \quad \text{for} \quad q = \infty,
\]

and $s \in \mathbb{R}, d \in \mathbb{N}$. We have

**Lemma 9** (Algebra property). Let $1 \leq q \leq \infty$. For $q > 1$ let $s > d \left( 1 - \frac{1}{q} \right)$ and for $q = 1$ let $s \geq 0$. Then $l^{s}_q(\mathbb{Z}^d)$ is a Banach algebra with respect to convolution

\[
(a_l) * (b_m) = \left( \sum_{m \in \mathbb{Z}^d} a_{l-m} b_m \right)_{k \in \mathbb{Z}^d}, \quad (9)
\]

which is well-defined, as the series above always converge absolutely.

This result is most likely not new. For the sake of self-containedness of the presentation, and because we could not come up with any suitable reference, we will give a proof. The inspiration for Lemma 9 comes from the fact that $H^s(\mathbb{R}^d)$ for $s > \frac{d}{2}$ is a Banach algebra with respect to pointwise multiplication and $l^2(\mathbb{Z}^d) = \mathcal{F}(H^s(\mathbb{R}^d))$. A proof for the algebra property of $H^s(\mathbb{R}^d)$ can be given using the Littlewood-Paley decomposition, see e.g. [1, Proposition II.A.2.1.1 (ii)]. We were able to adapt that proof to the $l^s_q(\mathbb{Z}^d)$ case, even for $q \neq 2$, by noting that we are already on the Fourier side.

Let us recall that the Littlewood-Paley decomposition of a tempered distribution is a series essentially such that the Fourier transform of $l$-th summand has its support in the annulus with radii comparable to $2^l$. In the same spirit we formulate

**Lemma 10** (Discrete Littlewood-Paley characterization). Let $1 \leq q \leq \infty$ and $s \in \mathbb{R}$. Define $C(s) = 2^{s l}$,

\[
A_0 := \{0\} \subseteq \mathbb{Z}^d, \quad \text{and} \quad A_l := \left\{ k \in \mathbb{Z}^d \big| |k| \leq 2^l \right\} \quad \forall l \in \mathbb{N}.
\]

(a) (Necessary condition) For any $(a_k) \in l^s_q(\mathbb{Z}^d)$ there is a sequence $(C_l) \in l^q(\mathbb{N}_0)$ such that $\|C_l\|_q = 1$ and

\[
\| \mathcal{A}_l (k) a_k \|_q \leq C(s) 2^{-ls} C_l \| (a_k) \|_{q,s} \quad \forall l \in \mathbb{N}_0.
\]

(b) (Sufficient condition) Conversely, if for some $N \geq 0$ and $(C_l) \in l^q(\mathbb{N}_0)$ with $\| (C_l) \|_q \leq 1$ the estimate

\[
\| \mathcal{A}_l (k) a_k \|_q \leq \frac{1}{C(s)} 2^{-ls} C_l N \quad \forall l \in \mathbb{N}_0
\]

holds, then $(a_k) \in l^s_q(\mathbb{Z}^d)$ and $\| (a_k) \|_{q,s} \leq N$.

**Proof.** Observe that $2^{l-1} \leq |k| < 2^{l+1}$ so $(k)^t \leq 2^l 2^t = C(t) 2^t$ for each $l \in \mathbb{N}_0, k \in A_l$ and $t \in \mathbb{R}$.

(a) For $(a_k) = 0$ there is nothing to show, so assume $\| (a_k) \|_{q,s} > 0$. Then for any $l \in \mathbb{N}_0$

\[
\| \mathcal{A}_l (k) a_k \|_q = \left\| \left( \mathcal{A}_l (k) \frac{k^s a_k}{|k|^s} \right) \right\|_q \leq \frac{C(s)}{2^{ls}} \| (\mathcal{A}_l (k) a_k) \|_{q,s} = C(s) 2^{-ls} C_l \| (a_k) \|_{q,s},
\]

where $C_l := \| (\mathcal{A}_l (k) a_k) \|_{q,s}$.
(b) We have \( (a_k) = (\sum_{l=0}^{\infty} I_{A_l}(k)a_k) \). Thus, for \( q < \infty \),

\[
\|(a_k)\|_{q,s}^q = \sum_{l=0}^{\infty} \|(I^s k I_{A_l}(k)a_k)\|_q^q \leq C(s)^q \sum_{l=0}^{\infty} 2^{lsq} \|(I_{A_l}(k)a_k)\|_q^q \leq N^q \sum_{l=0}^{\infty} C_l^q \leq N^q.
\]

Similarly, for \( q = \infty \), we have

\[
\|(a_k)\|_{\infty,s} = \sup_{t \in \mathbb{N}_0} \max(k)^s |a_k| \leq \sup_{t \in \mathbb{N}_0} C(s)2^{ls} \|(I_{A_l}(k)a_k)\|_{\infty} \leq N \sup_{t \in \mathbb{N}_0} C_t \leq N.
\]

For the proof of Lemma \( \textbf{10} \) we will require yet another sufficient condition. The discrete Littlewood-Paley decomposition in Lemma \( \textbf{11} \) consisted of sequences having their supports in disjoint dyadic annuli. We now consider non-disjoint dyadic balls \( B_m \).

**Lemma 11 (Sufficient condition for balls).** Let \( 1 \leq q \leq \infty \) and \( s > 0 \). Define \( C(s) = \frac{2^s}{1-2^{-s}} \) and \( B_m := \{ k \in \mathbb{Z}^d \mid |k| < 2^m \} \quad \forall m \in \mathbb{N}_0 \).

For each \( m \in \mathbb{N}_0 \) let \( (a_{k,m})_{k \in \mathbb{Z}^d} \) be such that \( \text{supp} ((a_{k,m})_{k \in \mathbb{Z}^d}) \subseteq B_m \). If for some \( N \geq 0 \) and \( (C_m) \in l^q(\mathbb{N}_0) \) with \( \|(C_m)\|_q \leq 1 \) the estimate

\[
\|(a_{k,m})_{k \in \mathbb{Z}^d}\|_q \leq \frac{1}{C(s)}2^{-ms}C_mN \quad \forall m \in \mathbb{N}_0
\]

holds, then

\[
(a_k) := \left( \sum_{m=0}^{\infty} a_{k,m} \right) \in l^q_s(\mathbb{Z}^d) \quad \text{and} \quad \|(a_k)\|_{q,s} \leq N.
\]

**Proof.** We want to apply the sufficient condition for annuli. Observe, that \( A_l \cap B_m = \emptyset \) if \( l > m \). Hence

\[
\|(I_{A_l}(k)a_k)\|_q = \left\| \left( \sum_{m=0}^{\infty} I_{A_l \cap B_m}(k)a_{k,m} \right) \right\|_q \leq \sum_{m=0}^{\infty} \|(a_{k,m})\|_q \leq \frac{1}{C(s)}N2^{-ls} \sum_{m=l}^{\infty} 2^{-(m-l)s}C_m
\]

for all \( l \in \mathbb{N}_0 \). It remains to show that \( (\tilde{C}_l) \in l^q(\mathbb{N}_0) \) and \( \|(\tilde{C}_l)\|_q \leq \frac{1}{1-2^{-s}} \). We can assume \( 1 < q < \infty \), as the proof for the other cases is easier and follows the same lines. We have

\[
\tilde{C}_l = \sum_{m=l}^{\infty} 2^{-l+1} C_m \times 2^{-(m-l)s} C_m \quad \text{Hölder \leq} \quad \left( \sum_{m=0}^{\infty} 2^{-ms} \right)^{\frac{1}{2}} \times \left( \sum_{m=l}^{\infty} 2^{-(m-l)s} C_m \right)^{\frac{1}{2}}
\]

for all \( l \in \mathbb{N}_0 \). Using the geometric series formula we recognize

\[
\sum_{m=0}^{\infty} 2^{-ms} = \frac{1}{1-2^{-s}} \quad \text{and}
\]

\[
\sum_{l=0}^{\infty} \sum_{m=l}^{\infty} 2^{-(m-l)s} C_m = \sum_{m=0}^{\infty} C_m 2^{-ms} \sum_{l=0}^{\infty} 2^{ls} = \sum_{m=0}^{\infty} C_m 2^{-ms} \left( \frac{2^{(m+1)s} - 1}{2^s - 1} \right) \leq \frac{1}{1-2^{-s}} \sum_{m=0}^{\infty} C_m.
\]

Recalling \( \|(C_m)\|_q \leq 1 \) and \( \frac{1}{q} + \frac{1}{q} = 1 \) finishes the proof.

We are now ready to give a
Proof of Lemma 9. As already mentioned in the proof of Proposition 2 (see Section 2), \( l^p \hookrightarrow l^1 \) for sufficiently large \( s \) (recall (7)). Hence, by Young’s inequality, the series in (9) is absolutely convergent and the case \( s = 0 \) is obvious. Consider now the case \( s > 0 \).

To that end, let us study what happens to the parts of the Littlewood-Paley decompositions of \((a_l)\) and \((b_m)\) under convolution. Let the annuli \( A_i \) and the balls \( B_j, (i, j \in \mathbb{N}_0) \) be defined as in the Lemmas 10 and 11. By the preceding remark, all of the occurring series are absolutely convergent and hence the following manipulations are justified:

\[
(a_l) \ast (b_m) = \left( \sum_{i=0}^{\infty} \mathbb{1}_{A_i}(l)a_l \right) \ast \left( \sum_{j=0}^{\infty} \mathbb{1}_{A_j}(m)b_m \right) = \sum_{i=0}^{\infty} \mathbb{1}_{A_i}(l)a_l \ast \left( \sum_{j=0}^{\infty} \mathbb{1}_{A_j}(m)b_m \right)_{m} + \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \left( \mathbb{1}_{A_i}(l)a_l \ast \left( \mathbb{1}_{A_j}(m)b_m \right)_{m} \right)
\]

\[
= \sum_{i=0}^{\infty} \mathbb{1}_{A_i}(l)a_l \ast \left( \mathbb{1}_{B_i}(m)b_m \right)_{m} + \sum_{i=0}^{\infty} \left( \sum_{j=1}^{j-1} \mathbb{1}_{A_i}(l)a_l \ast \left( \mathbb{1}_{A_j}(m)b_m \right)_{m} \right) + \sum_{i=0}^{\infty} \left( \mathbb{1}_{B_i}(l)a_l \ast \left( \mathbb{1}_{A_{i+1}}(m)b_m \right)_{m} \right)
\]

Observe that \( \text{supp}((a_k,i)_{k}) \subseteq B_{i+1} \) and \( \text{supp}((b_k,j)_{k}) \subseteq B_{j+2} \) by the properties of convolution and so the sufficient condition for balls could be applied. Indeed we have

\[
\| (a_k,i)_{k} \|_q \leq \| (B_i(m)b_m)_{m} \|_q \| (A_i(l)a_l)_{l} \|_q \lesssim 2^{-iC} C_j \| (b_m) \|_q \| (a_l) \|_q,
\]

where we used Young’s inequality, the embedding \( l^p \hookrightarrow l^1 \) and the necessary condition for \((a_l) \in l^2 \) from Lemma 10 (\( C_i \) was called \( C_l \) there). Hence, \( \sum_{i=0}^{\infty} (a_k,i)_{k} \in l^2 \) with \( \sum_{i=0}^{\infty} (a_k,i)_{k} \|_{q,s} \lesssim \| (a_l) \|_{q,s} \| (b_m) \|_{q,s} \) by Lemma 11. The same argument applies to \( \sum_{j=0}^{\infty} (b_k,j)_{k} \) and finishes the proof.

4. Proof of the Hölder-like inequality, Theorem 3.

We have already shown \( M_{p,q} \hookrightarrow C_k \) in the proof of Proposition 2 in Section 2 so it remains to prove (5). Fix a \( k \in \mathbb{Z}^d \). By the definition of the operator \( \square_k \) we have

\[
\square_k(fg) = \frac{1}{(2\pi)^d} F^{-1} \left( \sigma_k(\hat{f} \ast \hat{g}) \right) = \frac{1}{(2\pi)^d} \sum_{l,m \in \mathbb{Z}^d} F^{-1} \left( \sigma_k(\sigma_l \ast \sigma_m \hat{g}) \right)
\]

As the supports of the partition of unity are compact, many summands vanish. Indeed, for any \( k,l,m \in \mathbb{Z}^d \)

\[
\text{supp} \left( \sigma_k \left( (\sigma_l \ast (\sigma_m \hat{g})) \right) \right) \subseteq \text{supp}(\sigma_k) \cap \left( \text{supp}(\sigma_l) \cup \text{supp}(\sigma_m) \right) \subseteq B_{\sqrt{d}}(k) \cap B_{2\sqrt{d}}(l + m)
\]

and so \( \sigma_k \left( (\sigma_l \ast (\sigma_m \hat{g})) \right) \equiv 0 \) if \(|(k - l) - m| > 3\sqrt{d} \). Hence, the double series over \( l, m \in \mathbb{Z}^d \) boils down to a finite sum of discrete convolutions

\[
\square_k(fg) = \frac{1}{(2\pi)^d} F^{-1} \left( \sigma_k \sum_{m \in M} \sum_{l \in \mathbb{Z}^d} \sigma_l \ast (\sigma_k \ast (\sigma_{k-l} \hat{g})) \right) = \square_k \sum_{m \in M} \sum_{l \in \mathbb{Z}^d} \left( \square_l f \right) \cdot \left( \square_{k+m-t} g \right),
\]

where \( M = \left\{ m \in \mathbb{Z}^d \mid |m| \leq 3\sqrt{d} \right\} \) and \( \#M \leq (6\sqrt{d} + 1)^d < \infty \). That was the job of \( \square_k \) and we now get rid of it,

\[
\| \square_k(fg) \|_p \lesssim \sum_{m \in M} \sum_{l \in \mathbb{Z}^d} \| \left( \square_l f \right) \cdot \left( \square_{k+m-t} g \right) \|_p,
\]

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using the Bernstein multiplier estimate from Lemma 5.

Invoking Hölder’s inequality we further estimate

\[
\left\| \Box_k(fg) \right\|_{L_t^p} \lesssim \sum_{m \in \mathbb{M}} \left( \left\| \Box_l(f) \right\|_{L_t^p} \right)^* \left( \left\| \Box_{m+l}(g) \right\|_{L_t^p} \right)_{m+n}
\]

pointwise in \( k \) and hence

\[
\left\| fg \right\|_{M_{p,q}^s} \lesssim \left( \left\| \Box_l(f) \right\|_{L_t^p} \right)_{l} \left( \left\| \sum_{m \in \mathbb{M}} \left( \left\| \Box_{m+l}(g) \right\|_{L_t^p} \right)_{m+n} \right)_{q,s}
\]

by the algebra property of \( l_t^q \) from Lemma 9. Finally, we remove the sum over \( m \)

\[
\sum_{m \in \mathbb{M}} \left( \left\| \Box_{m+l}(g) \right\|_{L_t^p} \right)_{m+n} \lesssim \left\| g \right\|_{M_{p,q}^s}
\]

applying Peetre’s inequality \( \langle k + l \rangle^s \lesssim 2^{|\ell|} \langle k \rangle^s \langle \ell \rangle^s \). See e.g. [11, Proposition 3.3.31].

Let us finish the proof remarking that the only estimate involving “p”s we used was Hölder’s inequality and thus indeed \( C = C(d, s, q) \).

\[\Box\]

5. Proof of the local well-posedness, Theorem 1

For \( T > 0 \) let \( X(T) = C([0, T] \times M_{p,q}^s(\mathbb{R}^d)) \). Proposition 2 immediately implies that \( X \) is a Banach *-algebra, i.e.,

\[
\left\| uv \right\|_X = \sup_{0 \leq t \leq T} \left\| uv(\cdot, t) \right\|_{M_{p,q}^s} \lesssim \left( \sup_{0 \leq t \leq T} \left\| u(\cdot, s) \right\|_{M_{p,q}^s} \right) \left( \sup_{0 \leq t \leq T} \left\| v(\cdot, t) \right\|_{M_{p,q}^s} \right) = \left\| u \right\|_X \left\| v \right\|_X.
\]

For \( R > 0 \) we denote by \( M(R, T) = \{ u \in X \mid \| u \|_{X(T)} \leq R \} \) the closed ball of radius \( R \) in \( X(T) \) centered at the origin. We show that for some \( T, R > 0 \) the right-hand side of (2),

\[
(Tu)(\cdot, t) := e^{it\Delta}u_0 + \text{i} \int_0^t e^{i(t-\tau)\Delta} \left( \left| u \right|^2 u(\cdot, \tau) \right) d\tau \quad (\forall t \in [0, T]), \tag{10}
\]

defines a contractive self-mapping \( T = T(u_0) : M_{R,T} \to M_{R,T} \).

To that end, let us observe that Theorem 3 implies the homogeneous estimate

\[
\left\| \left( e^{it\Delta}v \right) \right\|_X \lesssim (1 + T)^{\frac{3}{2}} \left\| v \right\|_{M_{p,q}^s} \quad (\forall v \in M_{p,q}^s),
\]

which, together with the algebra property of \( X(T) \), proves the inhomogeneous estimate

\[
\left\| \int_0^t e^{i(t-\tau)\Delta} \left( \left| u \right|^2 u(\cdot, \tau) \right) d\tau \right\|_{M_{p,q}^s} \lesssim (1 + T)^{\frac{3}{2}} \int_0^t \left\| \left| u \right|^2 u(\cdot, \tau) \right\|_{M_{p,q}^s} d\tau \lesssim T(1 + T)^{\frac{3}{2}} \left\| u \right\|_X^3,
\]

holding for \( 0 \leq t \leq T \) and \( u \in X \).

Applying the triangle inequality in (10) yields \( \| Tu \|_X \leq C(1 + T)^{\frac{3}{2}} (\| u_0 \|_{M_{p,q}^s} + TR^3) \) for any \( u \in M(R, T) \). Thus, \( T \) maps \( M(R, T) \) onto itself for \( R = 2C \| u_0 \|_{M_{p,q}^s} \) and \( T \) small enough. Furthermore,

\[
\left| u \right|^2 u - \left| v \right|^2 v = (u - v) \left| u - v \right|^2 + (\overline{u}u - \overline{v}v)v = (u - v)(\left| u \right|^2 + \overline{u}) + (\overline{u} - \overline{v})v^2
\]

and hence

\[
\left\| Tu - Tv \right\|_X \lesssim T(1 + T)^{\frac{3}{2}} R^2 \left\| u - v \right\|_X
\]
for $u, v \in M(R, T)$, where we additionally used the algebra property of $X$ and the homogeneous estimate. Taking $T$ sufficiently small makes $T$ a contraction.

Banach’s fixed-point theorem implies the existence and uniqueness of a mild solution up to the minimal time of existence $T_\ast = T_\ast\left(\|u_0\|_{M_{p,q}^r}\right) \approx \|u_0\|_{M_{p,q}^r}^2 > 0$. Uniqueness of the maximal solution and the blow-up alternative now follow easily by the usual contradiction argument.

For the proof of the Lipschitz continuity let us notice that for any $r > \|u_0\|_{M_{p,q}^r}$, $v_0 \in B_r$ and $0 < T \leq T_\ast(r)$ we have

$$\|u - v\|_{X(T)} = \|T(u_0) - T(v_0)\|_{X(T)} \lesssim (1 + T)^{\frac{d}{p}r} \|u_0 - v_0\|_{M_{p,q}^r} + T(1 + T)^{\frac{d}{p}r} R^2 \|u - v\|_{X(T)},$$

where $v$ is the mild solution corresponding to the initial data $v_0$ and $R = 2Cr$ as above. Collecting terms containing $\|u - v\|_{X(T)}$ shows Lipschitz continuity with constant $L = L(r)$ for sufficiently small $T$, say $T_i = T_i(r)$. For arbitrary $0 < T^* < T_\ast$ put $r = 2 \|u\|_{X(T^*)}$ and divide $[0, T^*)$ into $n$ subintervals of length $\leq T_i$. The claim follows for $V = B_\delta(u_0)$ where $\delta = \frac{\|u\|_{X(T^*)}}{2}$ by iteration. This concludes the proof. □

Acknowledgements

We gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173. Dirk Hundertmark also thanks Alfried Krupp von Bohlen und Halbach Foundation for their financial support.

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