A New Type Warped Product Metric in Contact Geometry

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Abstract

In this study, we show that there is an $\alpha$-Sasakian structure on product manifold $M_1 \times \beta(I)$ where $M_1$ is a Kaehlerian manifold that has exact 1-form and $\beta(I)$ is an open curve. After then, we define a new type warped product metric to study the warped product of almost Hermitian manifolds with almost contact metric manifolds.

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1 Introduction

Contact geometry is a theoretical subject which has so many applications in the fields of science such as physics and engineering. From thermodynamics to optics, from electrics to motion equations, it has an important place in many areas. Many studies have been carried out on contact manifolds and contact geometry which became increasingly important in the 20th century. In addition, the symplectic geometry, Hermitian manifolds and Kählerian manifolds which have serious applications in many fields, are also important topics in mathematics. That’s why, to consider them together, we studied the product of contact and complex manifolds in this study.

The product of contact and complex manifolds have been an interesting area for mathematicians. Pandey (1981) studied the necessary and sufficient condition for a product manifold to be almost complex, Kähler and almost Tachibana manifold. Caprusi (1984) showed that the product of two almost contact manifolds can be an almost Kählerian manifold. In addition, he determined the necessary and sufficient conditions for a product manifold to be a Hermitian, Kähler, almost Kähler and nearly-Kähler. Zegga et al. (2017) investigated the product of Kähler and Sasaki manifolds via the metric

\[ g = g_1 + g_2 + \theta \otimes \theta + \theta \otimes \eta + \eta \otimes \theta \]  

(1)

and obtained a Sasaki structure on this product manifold. Also Gherici et al. (2019) showed that the product of real numbers and a Kähler manifold with a Kähler form has a Sasakian structure via the metric

\[ g(X, Y) = h(X, Y) + \theta(X)\theta(Y), \quad g(X, \partial_r) = \theta(X), \quad g(\partial_r, \partial_r) = 1 \]  

(2)

Also there are valuable studies about warped product which takes an important place in our paper in [1], [2], [10] and [17].

Because metric is fundamental tool to study a product manifolds, we defined the following metric on the product of almost Kählerian manifold \((M_1, J, g_1)\) which has an exact Kählerian form and the open curve \(\beta : I \rightarrow E^n\) to get contact and Sasakian structures on \(M = M_1 \times \beta(I)\) with the structure \((\tilde{\phi}, \tilde{\xi}, \tilde{\eta})\):

\[ g = g_1 + \tilde{\eta} \otimes \tilde{\eta} \]  

(3)

After them, for an almost Kählerian manifold \((M_1, J, g_1)\) with exact 1-form \(\omega\) which satisfies \(d\omega = \Phi_1\) and for an almost contact metric manifold \((M_2, \phi, \xi, \eta, g_2)\), we defined a new type warped product metric on the structure \((M_1 \times M_2, \tilde{\phi}, \tilde{\xi}, \tilde{\eta})\) as following:

\[ g = g_1 + f^2 (g_2 - \eta \otimes \eta - \tilde{\eta} \otimes \tilde{\eta}) \]  

(4)

By (4), we studied the connection on \(M_1 \times M_2\) when \(M_1\) is an almost Kählerian manifold with an exact 1-form and \(M_2\) is almost contact metric manifold, metric manifold and K-contact manifold respectively. Finally, we determined the necessary and sufficient conditions for a curve to be a geodesic in the warped product manifold \(M = M_1 \times M_2\).

2 Preliminaries

Definition 2.1. Let \(M\) be a \((2n + 1)\)-dimensional manifold and \(\phi, \xi, \eta\) be \((1, 1), (1, 0)\) and \((0, 1)\) tensor fields respectively on \(M\). If these tensor fields satisfy the equations (3) and (4) then \((\phi, \xi, \eta)\) is called an almost contact
structure on $M$ and $(M, \phi, \xi, \eta)$ is called an almost contact manifold.

$$\phi^2 X = -X + \eta(X)\xi$$  \hspace{1cm} (5) \\
$$\eta(\xi) = 1$$  \hspace{1cm} (6) \\

From this definition it can be deduced that $\phi(\xi) = 0$ and $\eta \circ \phi = 0$.

**Definition 2.2.** [5] Assume that $(M, \phi, \xi, \eta)$ is an almost contact manifold with dimension $2n + 1$. If $g$ is a Riemannian or Lorentzian metric and satisfies the equation

$$g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y), \ \forall X, Y \in \chi(M)$$  \hspace{1cm} (7) \\

then $(\phi, \xi, \eta, g)$ is called an almost contact metric structure and $(M, \phi, \xi, \eta, g)$ is called an almost contact metric manifold.

**Definition 2.3.** [5] Assume that $(M, \phi, \xi, \eta, g)$ is a $(2n + 1)$-dimensional almost contact metric manifold. If the equation

$$d\eta(X, Y) = g(X, \phi(Y))$$  \hspace{1cm} (8) \\

is satisfied on $M$ then $(M, \phi, \xi, \eta, g)$ is called a contact metric manifold.

**Lemma 2.1.** [4] Let $(M, \phi, \xi, \eta, g)$ be an almost contact metric manifold then for all $X, Y \in \chi(M)$ the followings hold:

i) $g(X, \xi) = \eta(X)$,  

ii) $g(\phi(X), \xi) = 0$,  

iii) $g(\phi(X), Y) + g(\phi(Y), X) = 0$.

**Definition 2.4.** [13] Let $(M, \phi, \xi, \eta, g)$ be an almost contact metric manifold. The fundamental 2-form $\Phi$ of this manifold is defined as following:

$$\Phi : \chi(M) \times \chi(M) \rightarrow C^\infty(M, \mathbb{R}), \quad \Phi(X, Y) = g(X, \phi Y).$$  \hspace{1cm} (9) \\

**Definition 2.5.** [13] Let $(M, J, g)$ be an almost complex manifold. The fundamental 2-form of this manifold is defined as following:

$$\Phi : \chi(M) \times \chi(M) \rightarrow C^\infty(M, \mathbb{R}), \quad \Phi(X, Y) = g(X, JY).$$  \hspace{1cm} (10) \\

**Definition 2.6.** Assume that $(M, \phi, \xi, \eta, g)$ is an almost contact metric manifold. $M$ is called a Sasakian manifold if the structure $(\phi, \xi, \eta, g)$ satisfies the following equation:

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad \forall X, Y \in \chi(M).$$  \hspace{1cm} (11) \\

More generally an $\alpha$–Sasakian structure may be defined by the requirement

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X), \quad \forall X, Y \in \chi(M)$$  \hspace{1cm} (12) \\

where $\alpha$ is a non-zero constant.
Theorem 2.1. Let $(\mathcal{M}, \phi, \xi, \eta)$ be an almost contact manifold then for all $X, Y \in \chi(M)$ the following equation holds:

$$2d\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta[X, Y]$$  \hfill (13)

where $[,]$ stands for Lie bracket.

Definition 2.7. On a product manifold such as $M = M_1 \times M_2$, the Lie derivative of two vector fields $X = (X_1, X_2), Y = (Y_1, Y_2) \in \chi(M)$, is defined as following:

$$[ , ] : \chi(M) \times \chi(M) \to \chi(M), \quad [X, Y] = ([X_1, Y_1], [X_2, Y_2])$$  \hfill (14)

Definition 2.8. Let $f_1 : M_1 \to \mathbb{R}$ and $f_2 : M_2 \to \mathbb{R}$ be differentiable functions on the manifolds $M_1$ and $M_2$ respectively. For $X_1 \in \chi(M_1)$ and $X_2 \in \chi(M_2)$ the derivative of $f_1 + f_2$ and $f_1f_2$ in the direction of the vector field $X = (X_1, X_2) \in \chi(M_1 \times M_1)$ are defined as follows:

$$X[f_1 + f_2] = X_1[f_1] + X_2[f_2]$$  \hfill (15)

$$X[f_1f_2] = f_2X_1[f_1] + f_1X_2[f_2]$$  \hfill (16)

Theorem 2.2. Let $(\mathcal{M}, g)$ be a Riemannian manifold and $\nabla$ be the Levi-Civita connection. Then for all $X, Y, Z \in \chi(M)$ the following equation holds:

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y])$$  \hfill (17)

This equation is also called as Koszul Formula.

3 Contact and Sasakian Structures on $M = M_1 \times \beta(I)$

In this section we examine the conditions on $M_1$ and the open curve $\beta : I \to E^n$ to get contact and Sasakian structures on $M = M_1 \times \beta(I)$. We suppose that $(M_1, J, g_1)$ is almost Hermitian manifold of dimension $2n$ which has an exact 1-form $\omega$ which satisfies

$$d\omega(X_1, Y_1) = \Phi_1(X_1, Y_1) = g_1(X_1, JY_1), \quad \forall X_1, Y_1 \in \chi(M_1).$$  \hfill (18)

Therefore, there exists a $(x_1, x_2, ..., x_n, y_1, y_2, ..., y_n)$ coordinate system for any neighborhood $U$ of $M_1$ such that

$$J \partial y_i = \partial x_i,$$  \hfill (19)

$$J \partial x_i = -\partial y_i$$  \hfill (20)

equations hold for $i = 1, 2, ..., n$ where $\frac{\partial}{\partial y_i} = \partial y_i$ and $\frac{\partial}{\partial x_i} = \partial x_i$. We can choose the set

$$\{2\partial x_1, 2\partial x_2, ..., 2\partial x_n, 2\partial y_1, 2\partial y_2, ..., 2\partial y_n\}$$

as an orthonormal base for $M_1$ without lose of generality. Therefore the set $\{\frac{1}{2}dx_1, \frac{1}{2}dx_2, ..., \frac{1}{2}dx_n, \frac{1}{2}dy_1, \frac{1}{2}dy_2, ..., \frac{1}{2}dy_n\}$ is the dual of the chosen base. For the next computations we take the dual base as $\{\frac{1}{2}\omega_1, \frac{1}{2}\omega_2, ..., \frac{1}{2}\omega_n, \frac{1}{2}\omega_{n+1}, \frac{1}{2}\omega_{n+2}, ..., \frac{1}{2}\omega_{2n}\}$

So $\omega$ can be written in the form of dual base vectors such as:
\[ \omega = \sum_{i=1}^{2n} f_i \omega_i \]  

(21)

where \( f_i \in C^\infty(M, \mathbb{R}) \). Thus we have the following equations:

\[ \omega(2\partial x_i) = f_i, \quad \omega(2\partial y_i) = f_{n+i} \quad i = 1, 2, \ldots, n \]  

(22)

Furthermore because \( d\omega = \Phi_1 \) the following equations hold for the coefficient functions of \( \omega \):

\[ \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} = 0, \quad i \neq j \]  

(23)

\[ \frac{\partial f_{n+j}}{\partial y_i} - \frac{\partial f_i}{\partial y_j} = 0, \quad i \neq j \]  

(24)

\[ \frac{\partial f_{n+j}}{\partial x_i} - \frac{\partial f_i}{\partial y_j} = 0, \quad i \neq j \]  

(25)

\[ \frac{\partial f_{n+i}}{\partial x_i} - \frac{\partial f_i}{\partial y_j} = 1, \quad i = j \]  

(26)

**Theorem 3.1.** The triple \( (\bar{\phi}, \bar{\xi}, \bar{\eta}) \) is an almost contact structure on \( M \) where

\[ \bar{\phi} : \chi(M) \to \chi(M), \quad \bar{\phi}(X_1, hT) = (JX_1, -\alpha \omega(JX_1)T) \]  

(27)

\[ \bar{\eta} : \chi(M) \to C^\infty(M, \mathbb{R}), \quad \bar{\eta}(X_1, hT) = \alpha \omega(X_1) + h \]  

(28)

\[ \bar{\xi} = (0, T) \]  

(29)

Here \( T \) stands for the unit tangent vector field of \( \beta \) and \( h \) is a function as \( h : I \to \mathbb{R} \).

**Proof** For \( X = (X_1, hT) \) and \( \bar{\xi} \in \chi(M) \) we have the following equations:

\[ \bar{\phi}^2(X) = (\bar{\phi}(X), \bar{\phi}(Y)) = (-X_1, \alpha \omega(X_1)T) \]  

(30)

and

\[ \bar{\eta}(\bar{\xi}) = \bar{\eta}(0, T) = 1 \]  

(31)

By means of definition 2.1., \( (M, \bar{\phi}, \bar{\xi}, \bar{\eta}) \) is an almost contact manifold.

**Theorem 3.2.** \( (M, \bar{\phi}, \bar{\xi}, \bar{\eta}, g) \) is an almost contact manifold with the metric

\[ g : \chi(M) \times \chi(M) \to C^\infty(M, \mathbb{R}) \]  

(32)

\[ g((X_1, h_1 T), (Y_1, h_2 T)) = g_1(X_1, Y_1) + \bar{\eta}(X_1, h_1 T) \bar{\eta}(Y_1, h_2 T) \]

**Proof** For \( X = (X_1, h_1 T) \in \chi(M) \) and \( Y = (Y_1, h_2 T) \in \chi(M) \) we obtain

\[ g(\bar{\phi}(X), \bar{\phi}(Y)) = g_1(JX_1, JY_1) + \bar{\eta}(\bar{\phi}X) \bar{\eta}(\bar{\phi}Y) = g_1(X_1, Y_1) \]

\[ = g(X, Y) - \bar{\eta}(X) \bar{\eta}(Y) \]

From definition 2.2., it’s concluded that \( (M, \bar{\phi}, \bar{\xi}, \bar{\eta}, g) \) is an almost contact metric manifold.
Lemma 3.1. For all \( X = (X_1, h_1 T), Y = (Y_1, h_2 T) \in \chi(M) \) there is a relationship between the fundamental 2-forms of \((M, \bar{\phi}, \bar{\xi}, \bar{\eta}, g)\) and \((M_1, g_1)\) as

\[
\Phi(X, Y) = \Phi_1(X_1, Y_1).
\]

Proof One can easily see by using the definitions of fundamental 2-forms of \((M_1, g_1)\) and \((M, \bar{\phi}, \bar{\xi}, \bar{\eta}, g)\)

Theorem 3.3. \((M, \bar{\phi}, \bar{\xi}, \bar{\eta}, g)\) is a contact metric manifold if and only if \(\alpha = 1\).

Proof For \( X = (X_1, h_1 T) \in \chi(M) \) and \( Y = (Y_1, h_2 T) \in \chi(M) \) we get,

\[
2d\bar{\eta}(X, Y) = X\bar{\eta}(Y) - Y\bar{\eta}(X) - \bar{\eta}[X, Y]
\]

\[
= X_1 [\alpha \omega Y_1] - Y_1 [\alpha \omega X_1] - \omega[X_1, Y_1]
\]

\[
= 2ad\omega(X_1, Y_1)
\]

\[
= 2a\Phi_1(X_1, Y_1)
\]

\[
= 2a\Phi(X, Y)
\]

So \(\bar{\eta}\) is exact 1-form if and only if \(\alpha = 1\). Therefore \((M, \bar{\phi}, \bar{\xi}, \bar{\eta}, g)\) is a contact metric manifold if and only if \(\alpha = 1\).

Now, we determine the conditions for the product manifold \(M\) to be a Sasakian manifold. For this aim we consider the base of \(M\) as \(\{\partial x_1, \partial x_2, ..., \partial x_n, \partial y_1, \partial y_2, ..., \partial y_n, \partial z\}\) where

\[
\partial x_i = (\partial x_i, 0), \quad \partial y_i = (\partial y_i, 0), \quad \partial z = \frac{1}{2} (0, T)
\]

By Theorem 3.2, we can write the followings:

\[
g(2\partial x_i, 2\partial x_j) = g_1(2\partial x_i, 2\partial x_j) + \bar{\eta}(2\partial x_i, 0) \bar{\eta}(2\partial x_j, 0)
\]

\[
= \delta_{ij} + [\alpha \omega(2\partial x_i)] [\alpha \omega(2\partial x_j)]
\]

\[
= \delta_{ij} + \alpha^2 f_i f_j
\]

Similarly others are found as

\[
g(2\partial x_i, 2\partial y_j) = \alpha^2 f_i f_{n+j}
\]

\[
g(2\partial x_i, \partial z) = \alpha f_i
\]

\[
g(2\partial y_i, 2\partial y_j) = \delta_{ij} + \alpha^2 f_{n+i} f_{n+j}
\]

\[
g(2\partial y_i, 2\partial z) = \alpha f_{n+i}
\]

\[
g(2\partial z, 2\partial z) = 1
\]

So the matrix corresponding to the metric \(g\) on the base \(\{\partial x_1, \partial x_2, ..., \partial x_n, \partial y_1, \partial y_2, ..., \partial y_n, \partial z\}\) is founds as

\[
[g_{ij}] = \frac{1}{4} \begin{bmatrix}
\delta_{ij} + \alpha^2 f_i f_j & \alpha^2 f_i f_{n+j} & \alpha f_i \\
\alpha^2 f_j f_{n+i} & \delta_{ij} + \alpha^2 f_{n+i} f_{n+j} & \alpha f_{n+i} \\
\alpha f_i & \alpha f_{n+i} & 1
\end{bmatrix}_{(2n+1) \times (2n+1)}
\]
And inverse of \([g_{ij}]\):

\[
[g_{ij}]^{-1} = [g^{ij}] = 4 \begin{bmatrix}
\delta_{ij} & 0 & -\alpha f_i \\
0 & \delta_{ij} & -\alpha f_{n+j} \\
-\alpha f_i & -\alpha f_{n+j} & 1 + \alpha^2 \sum_{i=1}^{2n} f_i^2
\end{bmatrix}
\]

For the later, we have to calculate the Christoffel symbols for this metric. It is known that 2\(^{nd}\) kind of Christoffel symbols can be calculated by the formula \[6\]

\[
\Gamma^k_{ij} = \frac{1}{2} \sum_{h=1}^{2n+1} g^{kh} (g_{ih,j} + g_{hj,i} - g_{ij,h})
\]

So non-zero of them are calculated as followings:

\[
\Gamma^{n+i}_{ij} = \frac{1}{2} \alpha^2 f_j , \quad \Gamma^{n+j}_{ij} = \frac{1}{2} \alpha^2 f_i , \quad \Gamma^{n+i}_{ii} = \alpha^2 f_i
\]

\[
\Gamma^{2n+1}_{ij} = \frac{1}{2} \frac{\partial f_i}{\partial x_j} - \frac{1}{2} \alpha^3 (f_i f_{n+j} + f_j f_{n+i})
\]

\[
\Gamma^j_{(n+i)(n+j)} = \frac{1}{2} \alpha^2 f_{n+j} , \quad \Gamma^j_{(n+i)(n+j)} = \frac{1}{2} \alpha^2 f_{n+i}
\]

If we define the vector fields \(e_i = 2\partial y_i - \alpha f_{n+i} \xi\) on \(M\), we get \(\tilde{\phi}(e_i) = 2\partial x_i - \alpha f_i \xi\), \(\tilde{\eta}(e_i) = 0\) and also \(\tilde{\eta}(\tilde{\phi}(e_i)) = 0\). Thus we obtain

\[
g \left( e_i, e_j \right) = g_1 (2\partial y_i, 2\partial y_j) + \tilde{\eta}(e_i)\tilde{\eta}(e_j)
\]

\[
= \delta_{ij}
\]

Similarly we can obtain the following equations:

\[
g \left( \tilde{\phi}(e_i), \tilde{\phi}(e_j) \right) = \delta_{ij}
\]

\[
g \left( e_i, \tilde{\phi}(e_j) \right) = 0
\]

\[
g \left( e_i, \tilde{\xi} \right) = 0
\]

\[
g \left( \tilde{\phi}(e_i), \tilde{\xi} \right) = 0
\]

Therefore the set \(\{e_1, e_2, ..., e_n, \tilde{\phi}(e_1), \tilde{\phi}(e_2), ..., \tilde{\phi}(e_n), \tilde{\xi}\}\) is an orthonormal base for \(M\).

**Lemma 3.2.** For the set \(\{e_1, e_2, ..., e_n, \tilde{\phi}(e_1), \tilde{\phi}(e_2), ..., \tilde{\phi}(e_n), \tilde{\xi}\}\), we have the followings:

\[
\nabla_{e_i} e_j = \nabla_{\tilde{\phi}(e_i)} \tilde{\phi}(e_j) = \nabla_{\tilde{\xi}} \tilde{\xi} = 0
\]

\[
\nabla_{e_i} \tilde{\phi}(e_j) = \alpha \delta_{ij} \tilde{\xi} = -\nabla_{\tilde{\phi}(e_i)} e_j
\]

\[
\nabla_{\tilde{\xi}} e_i = -\alpha \tilde{\phi}(e_i) = \nabla_{e_i} \tilde{\xi}
\]

\[
\nabla_{\tilde{\xi}} \tilde{\phi}(e_i) = \alpha e_i = \nabla_{\tilde{\phi}(e_i)} \tilde{\xi}
\]
For $e_i = 2\partial\bar{y}_i - \alpha f_n\bar{t}_i$, we have

$$\nabla_{\bar{t}_i} e_i = \sum_{k=1}^{2n+1} \Gamma^{k}_{(2n+1)(n+i)} \partial k - a \left( \bar{t}_i [f_n\bar{t}_i] + f_n \nabla_{\bar{t}_i} \bar{t}_i \right)$$

$$= -a (2\partial\bar{y}_i - \alpha f_n)$$

$$= -a\phi(e_i).$$

Others can be shown similarly.

**Theorem 3.4.** Let $(M_1, J, g_1)$ be a Kahlerian manifold with the exact 1-form. Then $(M, \bar{\phi}, \bar{\xi}, \bar{\eta}, g)$ is a Sasakian manifold if and only if $\bar{\eta} = 1$.

**Proof.** Let $\tilde{\nabla}$ be the Levi-Civita connection on $M_1$. Using Lemma 3.3 for $X = (X_1, h_1 T) \in \chi(M)$ and $Y = (Y_1, h_2 T) \in \chi(M)$ one can find the Levi-Civita connection on $M$ as following:

$$\nabla_X Y = \left( \begin{array}{c} \tilde{\nabla}_{X_1} Y_1 - a\bar{\eta}(Y)JX_1 - a\bar{\eta}(X)JY_1, \\
(\bar{\eta}(Y) - a\omega(\tilde{\nabla}_{X_1} Y_1) + g_1 JX_1, Y_1 + a^2\bar{\eta}(Y)\omega(JX_1) + a^2\bar{\eta}(X)\omega(JY_1)) T \end{array} \right).$$

(38)

So $\nabla_X (\bar{\phi}Y)$ is calculated as

$$\nabla_X (\bar{\phi}Y) = \left( \begin{array}{c} \tilde{\nabla}_{X_1} JY_1 - a\bar{\eta}(\bar{\phi}Y)JX_1 - a\bar{\eta}(X)J^2 Y_1, \\
(\bar{\eta}(\bar{\phi}Y) - a\omega(\tilde{\nabla}_{X_1} JY_1) + g_1 JX_1, JY_1 + a^2\bar{\eta}(\bar{\phi}Y)\omega(JX_1) + a^2\bar{\eta}(X)\omega(J^2 Y_1)) T \end{array} \right).$$

Similarly $\bar{\phi}(\nabla_X Y)$ can be obtained as

$$\bar{\phi}(\nabla_X Y) = \left( \begin{array}{c} J\tilde{\nabla}_{X_1} Y_1 + a\bar{\eta}(Y)X_1 + a\bar{\eta}(X)Y_1, \\
(\omega(J\tilde{\nabla}_{X_1} Y_1) - a\bar{\eta}(Y)\omega X_1 - a\bar{\eta}(X)\omega Y_1) T \end{array} \right).$$

(39)

So we get

$$(\nabla_X \bar{\phi}) Y = \nabla_X (\bar{\phi}Y) - \bar{\phi}(\nabla_X Y)$$

$$= \left( \begin{array}{c} \tilde{\nabla}_{X_1} J Y_1 - a\bar{\eta}(X)Y_1, \\
an(\omega(\tilde{\nabla}_{X_1} J Y_1) + g_1 (X_1, Y_1) + \bar{\eta}(Y)\omega X_1) T \end{array} \right).$$

(40)

Because $M_1$ is Kahlerian $\tilde{\nabla}J = 0$ and

$$g_1(X_1, Y_1) = g(X, Y) - a^2\omega(X_1)\omega(Y_1) - a\omega(X_1)h_2 - a\omega(Y_1)h_1 - h_1 h_2$$

(41)

then equation (38) turns to

$$(\nabla_X \bar{\phi}) Y = a(\bar{\eta}(X)Y_1, (g_1(X_1, Y_1) + \bar{\eta}(Y)\omega X_1) T)$$

$$= a(g(X, Y)\bar{t}_1 - \bar{\eta}(Y)X) + (-a^2a\omega(Y_1)h_1 + \bar{\eta}(Y)h_1 - ah_1 h_2)\bar{t}_1.$$

(42)
By forward calculations it is seen that

\[-a^2 \omega(Y_1) h_1 + \bar{\eta}(Y) h_1 - ah_1 h_2 = 0\]  \hspace{1cm} (43)

So \((\nabla_X \bar{\phi}) Y\) is found as

\[(\nabla_X \bar{\phi}) Y = a(g(X,Y)\bar{\xi} - \bar{\eta}(Y)X).\]  \hspace{1cm} (44)

Equation (42) tells that \(M = M_1 \times \beta(I)\) is an \(a\)-Sasakian manifold with the given structure. And it is a Sasakian manifold if and only if \(\alpha = 1\)

**Example** Consider the Kaehlerian manifold \((\mathbb{R}^4, J, g_1)\) where the metric is given as

\[g_1 = \frac{1}{2} dx_1^2 + \frac{1}{2} dx_2^2 + \frac{1}{2} dy_1^2 + \frac{1}{2} dy_2^2\]  \hspace{1cm} (45)

and the almost complex structure is given as

\[J(\partial x_i) = \partial y_i, \quad J(\partial y_i) = -\partial x_i, i = 1, 2.\]  \hspace{1cm} (46)

Taking \(\omega = -\frac{1}{2} x_1 dy_1 - \frac{1}{2} x_2 dy_2\) results \(d\omega = \frac{1}{2} (dy_1 \wedge dx_1 + dy_2 \wedge dx_2)\). One can see that \(d\omega = \Phi_1\). We consider the curve \(\beta : (0, 2\pi) \to E^2, \beta(t) = (\cos t, \sin t)\) with tangent \(T = (-\sin t, \cos t) = 2 \frac{d}{dt}\). So for \(M = \mathbb{R}^4 \times \beta(I)\), the structure \((M, \bar{\phi}, \bar{\xi}, \bar{\eta}, g)\) is an \(a\)-Sasakian manifold with the followings:

\[\bar{\phi} : \chi(M) \to \chi(M), \quad \bar{\phi} \left( X_1, 2h \frac{d}{dt} \right) = \left( JX_1, -2a\omega(JX_1) \frac{d}{dt} \right)\]  \hspace{1cm} (47)

\[\bar{\eta} : \chi(M) \to C^\infty(M, \mathbb{R}), \quad \bar{\eta} \left( X_1, 2h \frac{d}{dt} \right) = a\omega(X_1) + h\]  \hspace{1cm} (48)

\[g = \frac{1}{4} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 + a^2 x_1^2 & a^2 x_1 x_2 & -ax_1 \\
0 & 0 & a^2 x_1 x_2 & 1 + a^2 x_2^2 & -ax_2 \\
0 & 0 & -ax_1 & -ax_2 & 1
\end{bmatrix}\]  \hspace{1cm} (49)

### 4 A New Type Warped Product Metric

In this section, we define a new type warped product metric and examine some product manifolds by using this metric.

**Theorem 4.1.** Let \((M_1, J, g_1)\) be an almost Hermitian manifold with exact 1-form \(\omega\) which satisfies \(d\omega = \Phi_1\) and \((M_2, \phi, \xi, \eta, g_2)\) be an almost contact metric manifold. Then \((M_1 \times M_2, \bar{\phi}, \bar{\xi}, \bar{\eta}, g)\) is an almost contact metric manifold with the following tensor fields;
\[
\begin{align*}
\ddot{\phi}(X_1, X_2) &= (JX_1, \phi X_2 - a\omega(JX_1)\xi), \\
\ddot{\eta}(X_1, X_2) &= a\omega(X_1) + \eta(X_2), \\
\ddot{\xi} &= (0, \xi) \\
g((X_1, X_2), (Y_1, Y_2)) &= g_1(X_1, X_2) + f^2(g_2(X_2, Y_2) - \eta(X_2)\eta(Y_2) - \ddot{\eta}(X)\ddot{\eta}(Y))
\end{align*}
\]

where \( X = (X_1, X_2) \in \chi(M), Y = (Y_1, Y_2) \in \chi(M), a \in \mathbb{R} \) and \( f \in C^\infty(M_2, \mathbb{R}) \) is a positive function.

**Proof** By forward calculation it can be found that

\[
\ddot{\phi}(X_1, X_2) = (J^2X_1, \phi^2(X_2) - a\omega(JX_1)\phi(\xi) - a\omega(J^2X_1)\xi) = (-X_1, -X_2 + \eta(X_2)\xi + a\omega((X_1)\xi) = -X + \ddot{\eta}(X)\ddot{\xi}
\]

Then it’s understood that \((M_1 \times M_2, \ddot{\phi}, \ddot{\xi}, \ddot{\eta})\) is an almost contact manifold. Also, because

\[
g(\ddot{\phi}(X), \ddot{\phi}(Y)) = g(JX_1, \phi X_2 - a\omega(JX_1)\xi, JY_1, \phi Y_2 - a\omega(JY_1)\xi) = g_1(X_1, Y_1) + f^2g_2(\phi(X_2), \phi(Y_2)) = g_1(X_1, Y_1) + f^2(g_2(X_2, Y_2) - \eta(X_2)\eta(Y_2)) = g(X, Y) - \ddot{\eta}(X)\ddot{\eta}(Y)
\]

then \((M_1 \times M_2, \ddot{\phi}, \ddot{\xi}, \ddot{\eta}, g)\) is an almost contact metric manifold.

**Theorem 4.2.** Let \((M_1, J, g_1)\) be an almost Hermitian manifold with exact 1-form \(\omega\) which satisfies \(d\omega = \Phi_1\) and \((M_2, \phi, \xi, \eta, g_2)\) be an almost contact metric manifold. The following equations hold for all \(X, Y \in \chi(M_1)\) and \(U, V, W \in \chi(M_2)\) and for the Levi-Civita the Levi-Civita connections \(\nabla, \bar{\nabla}\) and \(\nabla\) on \(M_1, M_2\) and \(M = M_1 \times M_2\) respectively:

i. \(\nabla_{(X,0)}(Y, 0) = (\nabla_X Y - \alpha^2\omega(X)J(Y) - \alpha^2\omega(Y)J(X), \alpha (-\omega(\nabla_X Y) + \alpha^2\omega(X)\omega(Y) + \alpha^2\omega(Y)\omega(JX) + \frac{1}{2}\xi)\)](\)

where \(\alpha = \lambda(X, Y) = X\omega(Y) + Y\omega(X) + \omega([X, Y])\)

ii. \(g(\nabla_{(X,0)}(0, U), (Y, W)) = fX[f]g_2(\phi(U), \phi(W) + 2a\eta(U)d\omega(X, Y) + 2a\omega(X)d\eta(U, W))\)

iii. \(g(\nabla_{(0, U)}(0, V), (Y, W)) = f^2g_2(\bar{\nabla}_U V, W) + (1 - f^2)\eta(V)d\eta(U, W) + (1 - f^2)\eta(U)d\eta(V, W) + \frac{1}{2}\ddot{\eta}(Y, W) - \frac{1}{2}f^2\eta(W) - fY[f]g_2(\phi(U), \phi(V))\)

where \(\theta = \theta(U, V) = U\eta(V) + V\eta(U) + \eta[U, V]\)

**Proof** i) Using Koszul formula for \((Z, W) \in \chi(M)\) we get
\[ 2g \left( \nabla_{(X,0)}(Y,0), (Z, W) \right) = 2g_1 \left( \nabla_X Y, Z \right) + 2\alpha^2 \omega(X) d\omega(Y, Z) + 2\alpha^2 \omega(Y) d\omega(X, Z) + \alpha (\alpha \omega(Z) + \eta(W)) (X \omega(Y) + Y \omega(X) + \omega([X, Y])) \]

\[ = 2g_1 \left( \nabla_X Y, Z \right) - 2\alpha^2 \omega(X) g_1(JY, Z) - 2\alpha^2 \omega(Y) g_1(JX, Z) + \alpha \eta(Z) W \lambda \]

\[ = 2g_1 \left( \nabla_X Y - \alpha^2 \omega(X) JY - \alpha^2 \omega(Y) JX, Z \right) + \alpha \lambda g((0, \xi), (Z, W)) \]

On the other by putting \( \nabla_X Y - \alpha^2 \omega(X) JY - \alpha^2 \omega(Y) JX = \mu \) one can see that

\[ 2g_1 \left( \nabla_X Y - \alpha^2 \omega(X) JY - \alpha^2 \omega(Y) JX, Z \right) = 2g \left( (\mu, -\alpha \omega(\mu) \xi), (Z, W) \right) \]

So we get

\[ g \left( \nabla_{(X,0)}(Y,0), (Z, W) \right) = g \left( (\mu, -\alpha \omega(\mu) \xi), (Z, W) \right) + \frac{\alpha}{2} \lambda g((0, \xi), (Z, W)) \]

\[ = g \left( (\mu, -\alpha \omega(\mu) \xi) + \frac{\alpha}{2} \lambda(0, \xi), (Z, W) \right) \]

Because the metric is positive definite it’s concluded that

\[ \nabla_{(X,0)}(Y,0) = (\mu, -\alpha \omega(\mu) \xi) + \frac{\alpha}{2} \lambda(0, \xi) \]

\[ = \left( \mu, \alpha \left( -\alpha \omega(\mu) + \frac{\lambda}{2} \right) \xi \right) \]

\[ = \left( \nabla_X Y - \alpha^2 \omega(X) JY - \alpha^2 \omega(Y) JX, \alpha \left( -\omega(\nabla_X Y) + \alpha^2 \omega(X) \omega(JY) + \alpha^2 \omega(Y) \omega(JX) + \frac{\lambda}{2} \right) \xi \right) \]

ii) Using Koszul formula for \((Y, W) \in \chi(M)\) we get

\[ 2g \left( \nabla_{(X,0)}(0, U), (Y, W) \right) = (X, 0) g((0, U), (Y, W)) + (0, U) g((X, 0), (Y, W)) + (Y, W) g((X, 0), (0, U)) - g((0, U), [Y, W]) - g((0, Y), [X, U]) + g((Y, W), [X, 0]) \]

\[ = 2f X[f] g_2(\phi(U), \phi(W)) + \alpha \eta(U) X \omega(Y) + \omega(X) U \eta(W) - \alpha \eta(U) Y \omega(X) - \omega(X) W \eta(U) - \alpha \omega(X) \eta[U, W] + \alpha \eta(U) \omega[Y, W] \]

\[ = 2f X[f] g_2(\phi(U), \phi(W) + 2\alpha \eta(U) d\omega(X, Y) + 2\omega(X) d\eta(U, W)) \]

Then it’s found that

\[ g \left( \nabla_{(X,0)}(0, U), (Y, W) \right) = f X[f] g_2(\phi(U), \phi(W)) + \alpha \eta(U) d\omega(X, Y) + \alpha \omega(X) d\eta(U, W)) \]

iii) Again using Koszul formula for \((Y, W) \in \chi(M)\) we get

\[ 2g \left( \nabla_{(0, U)}(0, V), (Y, W) \right) = (0, U) g((0, V), (Y, W)) + (0, V) g((0, U), (Y, W)) - (Y, W) g((0, U), (0, V)) - g((0, U), [0, V]) + g((0, V), [0, U]) + g((Y, W), [0, [U, V]]) \]

\[ = f^2 g_2(\nabla_U V, W) + 2(1 - f^2) \eta(U, W) + 2(1 - f^2) \eta(U) d\eta(V, W) + (1 - f^2) \theta(\eta(W) + \alpha \omega(X) \theta - 2Y f [Y] g_2(\phi(U), \phi(V)) \]

\[ = 10 \]
Because $\bar{\eta}(Y, W) = \alpha \omega(X) + \eta(W)$ it’s calculated that

$$g \left( \nabla_{(0, U)}(0, V), (Y, W) \right) = f^2 g_2 \left( \bar{\nabla}_U V, W \right) + (1 - f^2) \eta(V) d\eta(U, W) + (1 - f^2) \eta(U) d\eta(V, W)$$

$$+ \frac{\theta}{2} \bar{\eta}(Y, W) - \frac{f^2}{2} \theta \eta(W) - fY f g_2(\phi(U), \phi(V))$$

**Theorem 4.3.** Let $(M_1, J, g_1)$ be an almost Hermitian manifold with exact 1-form $\omega$ which satisfies $d\omega = \Phi_1$ and $(M_2, \phi, \xi, \eta, g_2)$ be a contact metric manifold. For all $X \in \chi(M_1)$ and $U \in \chi(M_2)$ the following equation holds:

$$\nabla_{(X, 0)}(0, U) = \nabla_{(0, U)}(X, 0)$$

$$= \left( -\alpha \eta(U) JX, \frac{X[f]}{f} \phi^2(U) - \alpha \frac{\omega(X) \phi(U)}{f^2} + \alpha^2 \eta(U) \omega(JX) \xi \right)$$

**Proof** If $(M_2, \phi, \xi, \eta, g_2)$ is a contact metric manifold then it’s also an almost contact manifold. Then (ii) of Theorem 4.2. holds. Besides, because $(M_2, \phi, \xi, \eta, g_2)$ is a contact metric manifold, then for all $U, W \in \chi(M_2)$

$$d\eta(U, W) = g_2(U, \phi(W)).$$

Putting $d\eta(U, W) = g_2(U, \phi(W))$ and $d\omega(X, Y) = g_1(X, JY)$ in (ii)

$$g \left( \nabla_{(X, 0)}(0, U), (Y, W) \right) = fX[f]g_2(\phi(U), \phi(W)) + a\eta(U)g_1(X, JY) + a\omega(X)g_2(U, \phi(W)).$$

On the other hand using the definiton of the metric $g$ we find

$$g(\phi(X, 0), (Y, W)) = -g_1(X, JY)$$

$$g((0, U), \phi(Y, W)) = f^2 g_2(U, \phi(W)).$$

Because $g$ is self adjoint we write

$$g((0, U), \phi(Y, W)) = -g(\phi(0, U), (Y, W))$$

$$= -g((0, \phi U), (Y, W))$$

$$= -f^2 g_2(U, \phi(W))$$

Replacing $\phi U$ instead $U$ it’s seen that

$$g_2(\phi(U), \phi(W)) = -\frac{1}{f^2} g((0, \phi^2(U)), (Y, W)).$$

Applying (56), (57) and (58) in (55) we conclude

$$g \left( \nabla_{(X, 0)}(0, U), (Y, W) \right) = g \left( \left( -\frac{X[f]}{f}(0, \phi^2(U)) - \frac{\alpha}{f^2} \omega(X)(0, \phi(U)) - a\eta(U)\phi(X, 0) \right), (Y, W) \right)$$
Since this equation is provided for all \((Y, W) \in \chi(M)\) then
\[
\nabla_{(X, 0)}(0, U) = \left(-\alpha\eta(U)JX, \frac{X[f]}{\gamma^2(U) - \frac{\alpha}{f^2}\omega(X)\phi(U) + \alpha^2\eta(U)\omega(JX)\xi}\right)
\]
Besides, using \([X, 0], [0, U] = (0, 0)\) and \([X, 0], [0, U] = \nabla_{(X, 0)}(0, U) - \nabla_{(0, U)}(X, 0)\) we get
\[
\nabla_{(X, 0)}(0, U) = \nabla_{(0, U)}(X, 0)
\]

**Theorem 4.4.** Let \((M_1, J, g_1)\) be an almost Hermitian manifold with exact 1-form \(\omega\) which satisfies \(d\omega = \Phi_1\) and \((M_2, \phi, \xi, \eta, g_2)\) be a K-contact manifold. The following equation holds for all \(U, V \in \chi(M_2)\) and \(U, V, W \in \chi(M_2)\) and for the Levi-Civita connections \(\nabla, \tilde{\nabla}\) and \(\nabla\) on \(M_1, M_2\) and \(M = M_1 \times M_2\) respectively:
\[
\nabla_{(0, U)}(0, V) = \left(\begin{array}{c}
\nabla_U V + \frac{f}{\gamma^2}(\eta(V)\phi(U) + \eta(U)\phi(V) + \alpha f g_2(\phi(U), \phi(V)))\omega(\text{grad} f)\xi
\end{array}\right)
\]

**Proof** Since \((M_2, \phi, \xi, \eta, g_2)\) is a K-contact manifold it’s also a contact metric manifold and satisfies
\[
d\eta(U, W) = g_2(U, \phi(W))
\]
\[
d\eta(V, W) = g_2(V, \phi(W))
\]
Because the preconditions in (iii) of Theorem 4.2. are provided, using (61) and (622) in (iii), it’s found that:
\[
2g(\nabla_{(0, U)}(0, V), (Y, W)) = g_2(2f^2\nabla_U V + 2(f^2 - 1)\eta(V)\phi(U) + 2(f^2 - 1)\eta(U)\phi(V) - f^2\xi, W) - 2fg_1(\text{grad} f, Y)g_2(\phi(U), \phi(V)) + \theta\eta(Y, W)
\]
In this equation if we put \(2f^2\nabla_U V + 2(f^2 - 1)\eta(V)\phi(U) + 2(f^2 - 1)\eta(U)\phi(V) - f^2\xi = \mu\) then it’s seen that
\[
g_2(\mu, W) = \frac{1}{f^2}g((0, \mu), (Y, W))
\]
On the other hand it’s seen that
\[
g((\text{grad} f, 0), (Y, W)) = g_1(\text{grad} f, Y) + \alpha \omega(\text{grad} f) g((0, \xi), (Y, W))
\]
So we find
\[
Y[f] = g_1(\text{grad} f, Y) = g((\text{grad} f, -\alpha \omega(\text{grad} f) \xi), ((Y, W)))
\]
Writing \(\tilde{\eta}(Y, W) = g((0, \xi), (Y, W))\) and applying equations (64),(65) and (66) in (63) we conclude that
\[
\nabla_{(0, U)}(0, V) = \left(\begin{array}{c}
\nabla_U V + \frac{f}{\gamma^2}(\eta(V)\phi(U) + \eta(U)\phi(V) + \alpha f g_2(\phi(U), \phi(V)))\omega(\text{grad} f)\xi
\end{array}\right)
\]

**Theorem 4.5.** Let \((M_1, J, g_1)\) be an almost Hermitian manifold with exact 1-form \(\omega\) which satisfies \(d\omega = \Phi_1\) and \((M_2, \phi, \xi, \eta, g_2)\) be a K-contact manifold. Let \(\gamma : I \rightarrow M_1\) and \(\beta : I \rightarrow M_2\) geodesic curves on \(M_1\) and \(M_2\) respectively. For \(X = \gamma'\) and \(V = \beta'\) vector fields, the curve \((\gamma, \beta) : I \rightarrow M_1 \times M_2\) is a geodesic on \(M_1 \times M_2\) if and only if...
i. \( 2\alpha \tilde{\eta}(X, V)JX + fg_2(\phi(V), \phi(V)) \) grad \( f \) = 0

ii. \(- \frac{2X[f]}{f} \phi^2 V + 2 \left( -\frac{1}{f^2} \tilde{\eta}(X, V) + \eta(V) \right) \phi(V) + \alpha(2\alpha \tilde{\eta}(X, V) + X\omega(JX) + X\omega(X) + fg_2(\phi(V), \phi(V))\omega(\text{grad} f)) \) \( \xi \) = 0

**Proof** Let \( \tilde{\nabla}, \hat{\nabla} \) and \( \nabla \) be the Levi-Civita connections on \( M_1, M_2 \) and \( M = M_1 \times M_2 \) respectively. Because \( \gamma : I \to M_1 \) and \( \beta : I \to M_2 \) geodesic curves, for \( X = \gamma' \) and \( V = \beta' \) vector fields \( \tilde{\nabla}_X X = 0 \) and \( \hat{\nabla}_V V = 0 \). For \( (X, V) = (\gamma', \beta') = (\gamma, \beta)' \)

\[
\nabla_{(X,V)}(X, V) = \nabla_{(X,0)}(X, 0) + \nabla_{(X,0)}(0, V) + \nabla_{(0,V)}(X, 0) + \nabla_{(0,V)}(0, V)
\]

\[
= \nabla_{(X,0)}(X, 0) + 2\nabla_{(X,0)}(0, V) + \nabla_{(0,V)}(0, V)
\]

\[
= \left( \nabla_X X - \alpha^2 \omega(X)J(X) - \alpha^2 \omega(X)J(X), \alpha \left( \omega(\nabla_X X) + \alpha^2 \omega(X)\omega(JX) + \alpha^2 \omega(X)\omega(JX) + \frac{\lambda}{2} \right) \right)
\]

\[
+ 2 \left( -\alpha \eta(V)JX, \frac{X[f]}{f} \phi^2 V - \frac{\alpha^2}{f} \omega(X)\phi(V) + \alpha^2 \eta(V)\omega(JX) \right)
\]

\[
+ \left( \begin{array}{c}
-2\alpha \tilde{\eta}(X, V)JX - fg_2(\phi(V), \phi(V)) \text{grad} f,
-\frac{2X[f]}{f} \phi^2 V + \left( -\frac{\alpha}{f^2} \tilde{\eta}(X, V) + 2\eta(V) \right) \phi(V)
+ 2\alpha^2 \tilde{\eta}(X, V) + X\omega(JX) + X\omega(X) + fg_2(\phi(V), \phi(V))\omega(\text{grad} f) \end{array} \right) \xi
\]

So to be \( \nabla_{(X,V)}(X, V) = 0 \), (i) and (ii) are the necessary and sufficient conditions for the curve \((\gamma, \beta)\) to be a geodesic curve in \( M = M_1 \times M_2 \)

**References**

[1] Arslan K., Ezenta s r., Mihai I., Murathan C. (2005). Contact CR-warped product sunmanifolds in Kenmotsu space forms. J. Korean Math. Soc., 42(5), 1101-1110

[2] Atçeken M. (2011) Contact CR-warped product sunmanifolds in cosymlectic space forms. Collect. Math, 62, 1726-1741

[3] Belkhelfa, M. Hırıca, I.E., Rosca, R. ve Verstraelen, L. (2002). On Legendre curves in Riemannian and Lorentzian Sasaki spaces. Soochow J.Math, 28, 81-91

[4] Blair, D. E. (2002). Riemannian geometry of contact and symplectic manifolds, Progressiv Mathematics 203. Birkhauser Boston, Inc., Boston, MA .

[5] Blair, D.E. (1976). Contact Manifolds in Riemannian Geometry, Lecture Notes in Math. Vol. 509, Springer-Verlag.

[6] Blair, D. E. and Oubina J. A. (1990). Conformal and related changes of metric on the product of two almost contact metric manifolds. Publicacions Matemátiques, 34,199-207
[7] Blair, D.E. (2013). D-homothetic warping. Publications De L’institut Mathématique, 94(108), 47-54

[8] Boothby, W.M. (1986). An Introduction to differentiable manifolds and Riemannian geometry. Academic Press.

[9] Capruci M. (1984), Some remarks on the product of two almost contact manifolds, Al. I.Cuza, XXX, 75-79

[10] Chen B. Y. (2017). Differential geometry of warped product manifolds and submanifolds. World Scientific, Singapore

[11] Gherici B., Cherif A. M. ve Zegga K. (2019). Sasakian structures on products of real line and Kahlerian manifold. The Korean Journal of Mathematics, 27(4), 1061-1075

[12] Gieges H. (2001), A Brief History of Contact Geometry and Topology. Expositiones Mathematicae, 19, 25-53.

[13] K. Yano and M. Kon (1984), Structures on Manifolds, World Scientific.

[14] Lie, S. (1880). Theorie der Transformationsgruppen I. Math. Ann. 16, 441–528.

[15] Pandey H. B. (1981), Cartesian Product of Two Manifolds, Indian J. Pure Appl. Math, 12(1): 55-60

[16] Sasaki, S. and Y. Hatakeyama, On differentiable manifolds with contact metric structures, J. Math. Soc. Japan, 14- 249-271, 1962.

[17] Sular, S. ve Özgür, C. (2011). Doubly warped product submanifolds of (k, μ)- contact metric manifolds, Ann. Polon. Math. 100, 223–236.

[18] Zegga K., Gherici B. ve Cherif A. M. (2017). Sasakian Structure on the product of Sasakian and Kahlerian manifolds. Journal of Geometry and Topology, 20(4), 409-425