Unified first law of black-hole dynamics and relativistic thermodynamics

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Abstract. A unified first law of black-hole dynamics and relativistic thermodynamics is derived in spherically symmetric general relativity. This equation expresses the gradient of the active gravitational energy $E$ according to the Einstein equation, divided into energy-supply and work terms. Projecting the equation along the flow of thermodynamic matter and along the trapping horizon of a black hole yield, respectively, first laws of relativistic thermodynamics and black-hole dynamics. In the black-hole case, this first law has the same form as the first law of black-hole statics, with static perturbations replaced by the derivative along the horizon. In particular, there is the expected term involving the area and surface gravity, where the dynamic surface gravity is defined by substituting the Kodama vector and trapping horizon for the Killing vector and Killing horizon in the standard definition of static surface gravity. The remaining work term is consistent with, for instance, electromagnetic work in special relativity. The dynamic surface gravity vanishes for degenerate trapping horizons and satisfies certain inequalities involving the area and energy which have the same form as for stationary black holes. Turning to the thermodynamic case, the quasi-local first law has the same form, apart from a relativistic factor, as the classical first law of thermodynamics, involving heat supply and hydrodynamic work, but with $E$ replacing the internal energy. Expanding $E$ in the Newtonian limit shows that it incorporates the Newtonian mass, kinetic energy, gravitational potential energy and thermal energy (internal energy with fixed zero). There is also a weak type of unified zeroth law: a Gibbs-like definition of thermal equilibrium requires constancy of an effective temperature, generalising the Tolman condition and the particular case of Hawking radiation, while gravithermal equilibrium further requires constancy of surface gravity. Finally, it is suggested that the energy operator of spherically symmetric quantum gravity is determined by the Kodama vector, which encodes a dynamic time related to $E$.

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I. Introduction

Analogies between the laws of thermodynamics and certain properties of black holes have aroused great interest, particularly since Hawking showed that the analogy becomes actual in at least one respect, namely that the surface gravity determines the effective temperature of a quantum field on a stationary black-hole background. This unexpected link between two hitherto unrelated branches of physics has inspired the hope that they may be unified. It is generally thought that there should be some deeper explanation involving a quantum theory of gravity. However, this work reveals a purely classical connection.

Understanding this requires a paradigm shift from black-hole statics to black-hole dynamics, meaning a theory of dynamically evolving black holes rather than just stationary black holes (and perturbations thereof). This is analogous to the shift from thermostatics to thermodynamics, meaning the theory of dynamically evolving thermal systems rather than just those in thermal equilibrium (and perturbations thereof). Many researchers apparently still believe that the relevant generalisation concerns event horizons, following the textbook definition of black holes by event horizons. However, this definition makes essential use of the global assumption of asymptotic flatness. The real universe is thought not to be asymptotically flat, so that event horizons do not actually exist. Moreover, even theoretically, the global nature of event horizons means that they cannot be physically located by observers. Their location is unknown until the universe has ended. For the same reason, event horizons have no local dynamical significance. An event horizon might be passing through the reader right now, without any physical effect. In short, it seems that event horizons are not the relevant generalisation. According to this view, the relevant second law of black-hole dynamics cannot be what the textbooks state, namely Hawking’s result that the area of an event horizon is non-decreasing. This is reflected by the absence of a corresponding first law for event horizons. The textbook first and zeroth laws concern stationary black holes, so are more accurately described as laws of black-hole statics. The textbook first and second laws do not even involve the same derivative, thereby undermining the analogy with thermodynamics.

A new framework for black holes was introduced in a previous paper [1], referred to as Paper I. This dispensed with global assumptions and proposed a general, dynamical, quasi-local definition of black holes in terms of trapping horizons, hypersurfaces foliated by marginal surfaces. Paper I also derived a second law of black-hole dynamics, expressing the increase of area element along the trapping horizon. This should not be confused with Hawking’s second law for event horizons. In the new framework, black-hole dynamics is formulated as the dynamics of trapping horizons.

A corresponding first law of black-hole dynamics requires a definition of surface gravity for non-stationary black holes. This article gives a new definition of surface gravity for spherically symmetric black holes, with the same form as the usual stationary definition involving the Killing vector on the Killing horizon, but instead using the Kodama vector on the trapping horizon. The Kodama vector encodes a preferred time in spherically symmetric space-times and has physical properties analogous to those of Killing vectors.

Another key concept is energy. In general, there is no agreed definition of gravitational energy in the theory of general relativity. However, in spherical symmetry there is the Misner-Sharp energy \( E \), which does have all the physical properties one might expect of
active gravitational energy, as explained in another recent paper [2], referred to as Paper II. In particular, $E$ has the expected behaviour in various limits: vacuum, small-sphere, large-sphere, Newtonian, test-particle and special-relativistic limits. It is also intimately related to the characteristic strong-gravity phenomena, namely black holes and singularities.

The first law of black-hole dynamics given here just expresses the derivative of $E$ along the trapping horizon, according to the Einstein equation. This turns out to involve the area and surface gravity as expected, with additional terms depending on the matter model, such as an electromagnetic term which has the form of electromagnetic work. This first law has the same form as the first law of black-hole statics, with the perturbation being replaced by the derivative along the horizon. In comparison to most references, it should be noted that this involves the energy on the horizon, rather than the energy at infinity. The new first and second laws involve the same derivative, the dynamical derivative along the trapping horizon.

The next observation is that, for a perfect fluid, the derivative of $E$ along the fluid flow is given by the hydrodynamic work, as shown in Paper II. This therefore reads as a relativistic version of the classical first law of thermodynamics, albeit for the thermodynamically trivial case with no thermal flux. To include such thermodynamic effects, it is necessary to generalise the model of a perfect fluid. A generally relativistic theory of thermodynamics has been developed in an accompanying paper [3], referred to as Paper III. This theory models heat relativistically by an energy tensor, called the thermal energy tensor, whose various components are thermal energy density, flux and stress. (Thermal energy is internal energy with fixed zero). This turns out to add a heat-supply term to the first law, consistent with the classical first law of thermodynamics. Actually, this requires no thermodynamics other than the form of the energy tensor given by Eckart [4] and the definition of heat supply as an integral of thermal flux. Thus we have a quasi-local first law of relativistic thermodynamics. This shows that the gravitational energy $E$ incorporates the thermal energy, as may be verified explicitly in the Newtonian limit. Then the unified first law is just the equation expressing the gradient of $E$ according to the Einstein equation. Projecting the unified first law along the material flow or along a trapping horizon yields the first law of thermodynamics or black-hole dynamics, respectively.

One might also expect there to be a unified zeroth law, concerning gravithermal equilibrium. The Gibbs definition of thermal equilibrium is constancy of the partial derivative of entropy with respect to thermal energy. Replacing thermal energy with gravitational energy $E$ in this definition yields constancy of an effective temperature, differing from the actual temperature by a red-shift factor. This generalises the Tolman condition [6] for gravithermal equilibrium and agrees with the case of Hawking radiation [5] on a stationary background. Surface gravity is also constant for stationary black holes, thereby linking the two zeroth laws.

The article is organised as follows. Section II reviews the relevant properties of spherically symmetric space-times, defining $E$ and deriving the unified first law. Section III considers work in the case of electromagnetism and Section IV reviews black-hole statics. Section V describes the Kodama vector or dynamic time, Section VI introduces the dynamic surface gravity and Section VII derives the first law of black-hole dynamics. Section VIII derives the corresponding first law of relativistic thermodynamics and shows
II. Unified first law

In spherical symmetry, the area $A$ of the spheres of symmetry is a geometrical invariant. It is convenient to use the areal radius $r = \sqrt{A/4\pi}$, giving

$$A = 4\pi r^2. \quad (2.1)$$

Following Papers I and II, a sphere is said to be untrapped, marginal or trapped as $\nabla^\sharp r$ is spatial, null or temporal respectively, $\nabla^\sharp r = 0$ being degenerate. Here $\nabla$ denotes the covariant derivative operator and $\sharp$ denotes the contravariant dual (index raising) with respect to the space-time metric $g$. Similarly, $\flat$ will denote the covariant dual (index lowering). If the space-time is time-orientable and $\nabla^\sharp r$ is future (respectively past) causal, then the sphere is said to be future (respectively past) trapped or marginal. Untrapped or marginal spheres have a spatial orientation given by the direction of $\nabla^\sharp r$: a spatial or null direction is said to be outward (respectively inward) if $r$ is increasing (respectively decreasing) in that direction. A hypersurface foliated by marginal spheres is called a trapping horizon. A trapping horizon is said to be outer, degenerate or inner as $\nabla^2 r > 0$, $\nabla^2 r = 0$ or $\nabla^2 r < 0$ respectively.

Paper II proposed that future (respectively past) outer trapping horizons be taken as the dynamical definition of the outer boundaries of black (respectively white) holes. Black holes are thereby characterised quasi-locally, actually locally in spherical symmetry. This definition should not be confused with those of event horizons or apparent horizons [7], both of which require the global assumption of asymptotic flatness. It is also unnecessary to make assumptions about the global nature of the trapping horizon, for instance that it remains of the future outer type. Such questions are still of interest, but depend on the dynamics of the chosen matter model. The results of this paper are essentially local, independent of the matter model except for energy conditions and, with minor exceptions for degenerate cases, apply to any trapping horizon.

The Misner-Sharp energy [8] may be defined by

$$E = \frac{1}{2} r \left(1 - \nabla r \cdot \nabla^\sharp r\right) \quad (2.2)$$

where the dot denotes contraction, the sign convention is that spatial metrics are positive definite and the Newtonian gravitational constant is unity. Paper II investigated the physical properties of $E$ in detail, establishing that it represents active gravitational energy. This point deserves emphasis, since the literature contains many definitions of so-called energy which do not have the relevant properties in physically understood limits; a list of references may be found in [9]. In comparison, $E$ does behave as active gravitational energy in the vacuum, small-sphere, large-sphere, Newtonian, test-particle and special-relativistic limits.

Two invariants of the (contravariant) energy tensor $T$ will be useful: the function

$$w = -\frac{1}{2} \text{trace} T \quad (2.3)$$
and the vector
\[ \psi = T \cdot \nabla r + w \nabla^2 r \] (2.4)
where the trace refers to the two-dimensional space normal to the spheres of symmetry. It is also convenient to use the areal volume
\[ V = \frac{4}{3} \pi r^3 \] (2.5)
due to the relation
\[ \nabla V = A \nabla r. \] (2.6)
Then the gradient of \( E \) is determined by the Einstein equations as
\[ \nabla E = A \psi^\flat + w \nabla V. \] (2.7)
This equation turns out to be the unified first law, as explained subsequently. This may be generalised beyond spherical symmetry in terms of the Hawking energy [10].

One may physically interpret \( w \) as an energy density and \( \psi \) as an energy flux or momentum density. Actually, assuming the null energy condition, \( w \) is a lower bound for the energy density measured by an observer. Assuming the dominant energy condition, \( w \geq 0 \). Assuming the null energy condition, \( \psi \) is past (respectively future) causal in future (respectively past) trapped regions, and outward spatial or null in untrapped regions. If the space-time is asymptotically flat, one finds that \( \psi \) becomes future (respectively past) null at future (respectively past) null infinity, reducing to the Bondi flux. This can be seen from the first law (2.7), since \( w \nabla V \) tends to zero at infinity. Thus \( \psi \) is a local version of the Bondi flux, actually the outward flux minus the inward flux. Then the first term \( A \psi \) in the unified first law may be interpreted as an energy-supply term, analogous to the heat-supply term in the classical first law of thermodynamics, while the second term \( w \nabla V \) may be interpreted as a work term.

To check the above properties, one may use double-null coordinates \((\xi^+, \xi^-)\), in terms of which the line-element is
\[ ds^2 = r^2 d\Omega^2 + 2g_{+-}d\xi^+ d\xi^- \] (2.8)
where \( d\Omega^2 \) refers to the unit sphere and \( r \geq 0 \) and \( g_{+-} < 0 \) are functions of \( \xi^\pm \). Then the coordinate forms of (2.2–4) and (2.7) are
\[ E = r \left( \frac{1}{2} - g^{+-} \partial_+ r \partial_- r \right) \] (2.9a)
\[ w = -g_{+-} T^{+-} \] (2.9b)
\[ \psi = T^{++} \partial_+ r \partial_+ r + T^{--} \partial_- r \partial_- r \] (2.9c)
\[ \partial_\pm E = A \psi_\pm + w \partial_\pm V = -Ag^{+-} (T_{+-} \partial_+ r - T_{\pm\pm} \partial_\pm r). \] (2.9d)

The last expression was derived from the Einstein equations in Paper II. In terms of these coordinates, a sphere is trapped if \( \partial_+ r \partial_- r > 0 \), being future trapped if \( \partial_\pm r < 0 \) and past trapped if \( \partial_\pm r > 0 \), taking \( \partial_\pm \) to be future pointing. In an untrapped region, \( \partial_+ r \partial_- r < 0 \) and one may locally choose the orientation \( \partial_+ r > 0, \partial_- r < 0 \), meaning that
∂_+ is outward and ∂_- is inward. As in Papers I and II, note that the null energy condition requires $T_{±±} ≥ 0$ and that the dominant energy condition requires $T_{+-} ≥ 0$. The Einstein equations translated from Paper II are

\[
\begin{align*}
\partial_± \partial_± r - \partial_± \log (-g_{++}) \partial_± r &= -4\pi r r_{±±} \\
r \partial_+ \partial_- r + \partial_+ r \partial_- r - \frac{1}{2} g_{++} &= 4\pi r^2 T_{+-} \\
r^2 \partial_+ \partial_- \log (-g_{++}) - 2\partial_+ r \partial_- r + g_{++} &= 8\pi r^2 (g_{+-} T_θ^θ - T_{+-})
\end{align*}
\]

(2.10a, 2.10b, 2.10c)

where θ is a standard latitude.

III. Electromagnetic work

Since the desired first law of black-hole dynamics will involve work terms, it is necessary to identify the correct definition of work for the relevant matter fields. This is done as follows for the Maxwell electromagnetic field (excluding magnetic monopoles). The only non-zero components of a spherically symmetric electromagnetic field tensor $F$ are given by

\[
E = -g^{+-} F_{+-} = g^{+-} F_{-+}
\]

(3.1)

which may be interpreted as the electric field strength, the magnetic field vanishing. A standard definition of charge \[11\] is

\[
e = \frac{1}{4\pi} \oint E = r^2 E
\]

(3.2)

where the integral is over a sphere of symmetry, the area form being implicit. Thus the field may be characterised by either $E$ or $e$. The electromagnetic energy tensor $T_e$ has one independent component, yielding

\[
w_e = E^2 / 8\pi
\]

(3.3a)

\[
ψ_e = 0
\]

(3.3b)

where the notation is that of (2.3–4) with the subscript $e$ referring to the electromagnetic field. Therefore the unified first law (2.7) reads

\[
\nabla E = \frac{E^2 \nabla V}{8\pi} + w_o \nabla V + Aψ_o^b
\]

(3.4)

where the subscript $o$ refers to other (non-electromagnetic) fields, i.e. $T = T_e + T_o$ and so on. In the absence of other fields, the Reissner-Nordström case is recovered. This identifies the electromagnetic work as $E^2 \nabla V / 8\pi$. This agrees with the standard expression for electric work \[12\] in special relativity, which is

\[
W = \frac{1}{8\pi} \int_Σ E^2 dV
\]

(3.5)
where \( dV \) is the volume form of a flat spatial hypersurface containing the region \( \Sigma \). The integrand is just the local energy density \( w_e \) of the electromagnetic field. Some authors accept this expression for work only for infinite \( \Sigma \), allowing it to be rewritten in various ways involving boundary terms which vanish at infinity. However, such terms are generally non-zero at a regular centre. Indeed, typical suggestions become infinite.

It is also possible to write the electromagnetic work in terms of the electric field covector

\[
\hat{E} = F \cdot k
\]

where \( k \) is the Kodama vector defined subsequently by (5.1). Then \( \hat{E} = \mathcal{E} \nabla r \), so that the electromagnetic work is

\[
\frac{\mathcal{E}^2 \nabla V}{8\pi} = \frac{1}{2} e \hat{E}.
\]

The second expression has the same form as the work \( e_0 \hat{E} \) done on a test charge \( e_0 \) by the electric field \( \hat{E} \) in special relativity, with the half arising to avoid counting the self-interaction twice. In other words, the work is that done on the charge distribution by its own electric field.

**IV. First law of black-hole statics**

This section reviews the theory of black-hole statics, or equilibrium mechanics, which concerns stationary black holes; see e.g. Carter [13] or Wald [14]. Stationary space-times are defined by a Killing vector \( \xi \) generating asymptotic time translations. That is, \( \xi \) satisfies the Killing equation

\[
\nabla \otimes \xi^b = 0
\]

where \( \otimes \) denotes the symmetric tensor product. A Killing horizon is a null hypersurface such that \( \xi \) is tangent to the null generators. Killing horizons define the boundaries of stationary black holes, white holes and cosmological regions. It follows from the definition that \( \xi \cdot (\nabla \wedge \xi^b) \) is parallel to \( \xi^b \) on a Killing horizon, where \( \wedge \) denotes the antisymmetric tensor product. The proportionality constant defines the surface gravity \( \kappa \):

\[
\xi \cdot (\nabla \wedge \xi^b) = \kappa \xi^b \quad \text{on a Killing horizon.}
\]

Taking the standard example of the source-free electromagnetic field, the only spherically symmetric solution to the Einstein-Maxwell equations is the Reissner-Nordström solution, parametrised by two constants, the charge \( e \) and asymptotic energy \( m \); these provide an analogue of thermostatic state space for stationary black holes. For \( m \geq |e| \) the solution describes a black hole with surface gravity \( \kappa = r^{-2} \sqrt{m^2 - e^2} \) on the Killing horizon at radius \( r = m + \sqrt{m^2 - e^2} \). Evaluating these quantities on the horizon yields

\[
dm = \frac{\kappa dA}{8\pi} + \frac{e de}{r}
\]

where \( d \) denotes the differential in state space \((m, e)\). This is the form usually given for the first law in this case, though it may be generalised to non-stationary perturbations [15] (of
stationary solutions). However, this form is not suitable for generalisation to space-times which are not asymptotically flat, due to the appearance of the asymptotic energy $m$. Instead, one may use the quasi-local energy $E$, which for the Reissner-Nordström solution is

$$E = m - \frac{e^2}{2r}.$$  

(4.4)

Rewriting (4.3) in terms of $E$ yields

$$\text{d}E = \frac{\kappa \text{d}A + \mathcal{E}^2 \text{d}V}{8\pi}$$  

(4.5)

where $\mathcal{E} = e/r^2$ as in (3.2). This is a physically acceptable form for the first law of black-hole statics in the electromagnetic case, since the last term may be interpreted as electromagnetic work, in agreement with special relativity (3.5) and the general expression (3.7) for electromagnetic work. Thus the analogue of internal energy for a black hole is the energy $E$ on the horizon, rather than the energy at infinity; put another way, the energy of the black hole rather than the energy of the space-time. This is exactly as one would have expected by physical intuition and provides further evidence for the physical correctness of $E$ as a definition of active gravitational energy.

V. Dynamic time

In black-hole dynamics, unlike black-hole statics, there is no Killing vector to define such quantities as surface gravity. However, in spherical symmetry there is a natural analogue, the vector

$$k = \text{curl } r$$  

(5.1)

where the curl refers to the two-dimensional space normal to the spheres of symmetry. In double-null coordinates,

$$k = -g^+ (\partial_+ r \partial_- - \partial_- r \partial_+)$$  

(5.2)

up to orientation, which in an untrapped region may be locally fixed such that $k$ is future-pointing. Kodama [16] introduced $k$ and Paper II discussed some properties. It follows that

$$k \cdot \nabla r = 0$$  

(5.3a)

$$k \cdot k^b = \frac{2E}{r} - 1$$  

(5.3b)

which equivalently defines $k$. So $k$ is spatial, null or temporal for trapped, marginal or untrapped spheres respectively. In particular, the definition of the boundary of stationary black holes, a hypersurface where the Killing vector is null, may be generalised to a hypersurface where the Kodama vector is null, which is equivalent to the definition of trapping horizon. Unlike a Killing horizon, a trapping horizon is generally not null.

The Kodama and Killing vectors agree for the Reissner-Nordström black hole. More generally, they agree in a stationary, spherically symmetric space-time if $k$ commutes with
\[ \nabla^2 r. \] This condition implies that one may introduce coordinates \((r,t)\) where \(k = \partial/\partial t\), for which the line-element becomes

\[
ds^2 = r^2d\Omega^2 + \left(1 - \frac{2E}{r}\right)^{-1}dr^2 - \left(1 - \frac{2E}{r}\right)dt^2.
\]  \tag{5.4}

This is a Schwarzschild-like form, with \(E\) generally a function of \((r,t)\). Then examine the various components of the Killing equation (4.1):

\[
\xi^c \partial_c g_{ab} + 2g_{c(a} \partial_{b)} \xi^c = 0
\]  \tag{5.5}

for a Killing vector \(\xi\) normal to the spheres of symmetry. The components tangential to the spheres of symmetry yield \(\xi \cdot \nabla r = 0\), so that \(\xi\) is parallel to \(k\): \(\xi = \xi^0 k\). The \(rr\) component yields \(\xi \cdot \nabla E = 0\), the \(tt\) component yields \(k \cdot \nabla \xi^0 = 0\), and the \(tr\) component yields \(\nabla^2 r \cdot \nabla \xi^0 = 0\), so that \(\xi^0\) is constant. If the space-time is asymptotically flat and the Killing vector is normalised as usual by \(\xi \cdot \xi \rightarrow -1\) as \(r \rightarrow \infty\), then \(\xi^0 = 1\) since \(k \cdot k^b \rightarrow -1\) as \(r \rightarrow \infty\). Thus the Kodama vector reduces to the Killing vector, \(\xi = k\).

Another physically relevant property of \(k\) is that it can be used to define \(E\) as a Noether charge, as shown in Paper II. Quoting this result, both \(k\) and the corresponding energy-momentum density

\[
j = -T \cdot k^b
\]  \tag{5.6}

are conserved:

\[
\nabla \cdot k = 0 \quad (5.7a)
\]

\[
\nabla \cdot j = 0. \quad (5.7b)
\]

Therefore the Gauss theorem yields conserved charges which turn out to be just the areal volume and energy:

\[
V = -\int_\Sigma *u^b \cdot k \quad (5.8a)
\]

\[
E = -\int_\Sigma *u^b \cdot j = \int_\Sigma *u \cdot T^b \cdot k \quad (5.8b)
\]

where \(\ast\) denotes the volume form and \(u\) the unit future normal vector of an arbitrary spatial hypersurface \(\Sigma\) with regular centre. Paper II also showed how the Hájíček [17] energy \(\tilde{E}\) of a spherical shell of test particles of rest mass \(m\) and velocity (unit future-temporal vector) \(u\) may be defined analogously by

\[
\tilde{E} = \int_\Sigma *u \cdot \tilde{T}^b \cdot k
\]  \tag{5.9}

where the energy tensor \(\tilde{T}\) of the shell is

\[
\tilde{T} = m\delta u \otimes u
\]  \tag{5.10}
and $\delta$ is the Dirac distribution with support on the shell. Then
\[ \tilde{E} = -mu^b \cdot k. \] (5.11)

This has the same form as the standard definition of energy $-mu^b \cdot \xi$ for test particles in the stationary case [14].

This allows a definition of ergoregion as a region where test particles may have positive or negative energy $\tilde{E}$ at each point, depending on their velocity $u$ [18]. The overall sign of $\tilde{E}$ is conventional, as in the stationary case, with $\tilde{E} > 0$ for future pointing $k$ in untrapped regions. Ergoregions are physically important because they allow energy extraction by the Penrose process [14], whereby discarding negative-energy particles yields a net gain in energy. It follows from (5.11) that ergoregions are regions where $k$ is spatial. Again this is analogous to the stationary case. In spherical symmetry, ergoregions therefore coincide with trapped regions.

In summary, the Kodama vector generates a preferred flow of time in spherically symmetric space-times, analogous to the Killing vector of stationary space-times and sharing its relevant physical properties. Moreover, it generates active gravitational energy as a conserved charge, suggesting the name dynamic time. This further suggests that dynamic gravitational entropy might be defined by analogy with the method of Wald [15].

VI. Dynamic surface gravity

The above properties suggest defining dynamic surface gravity by replacing the Killing horizon with the trapping horizon and the Killing vector with the Kodama vector. To see whether this is possible, note firstly that there appears to be an ambiguity in such an analogy, since the Killing equation (4.1) allows the definition of surface gravity (4.2) to be rewritten using different linear combinations of $\xi \cdot (\nabla \wedge \xi^b)$ and $\xi \cdot (\nabla \otimes \xi^b)$. So one needs to calculate both
\[ k \cdot (\nabla \wedge k^b) = \left( E/r^2 - 4\pi rw \right) \nabla r \] (6.1a)
\[ k \cdot (\nabla \otimes k^b) = 4\pi r \psi^b. \] (6.1b)

The term proportional to $\nabla r$ is what is needed, since $\nabla r = \pm k^b$ on a trapping horizon $\partial_{\pm} r = 0$, but the term proportional to $\psi$ is generally not parallel to $k$. This resolves the ambiguity, leading to the definition of dynamic surface gravity
\[ \kappa = E/r^2 - 4\pi rw. \] (6.2)

When $w = 0$, this has the same form as Newtonian surface gravity, or indeed Newtonian gravitational acceleration anywhere, with $E$ replacing the Newtonian mass. Explicitly, the definition satisfies
\[ k \cdot (\nabla \wedge k^b) = \kappa \nabla r \] (6.3)
and therefore
\[ k \cdot (\nabla \wedge k^b) = \pm \kappa k^b \quad \text{on a trapping horizon } \partial_{\pm} r = 0. \] (6.4)
This has the same form as the standard stationary definition of surface gravity (4.2). The new definition therefore recovers the Reissner-Nordström surface gravity, but generally differs from the definitions of both Paper I (evaluated in spherical symmetry) and Fodor et al. [19]. Based on the stationary case, one might conjecture that a dynamic spherically symmetric quantum field has a local Hawking temperature \( \hbar \kappa / 2\pi \), but that remains an open question.

The dynamic surface gravity may equivalently be expressed as

\[
\kappa = \frac{1}{2} \text{div} \text{ grad} \, r
\]

where the divergence and gradient refer to the two-dimensional space normal to the spheres of symmetry. This may be regarded as the more fundamental, purely geometrical definition, with the form (6.2) following from the Einstein equations. Then outer, degenerate and inner trapping horizons, as defined in Section II or in Papers I or II, respectively have \( \kappa > 0 \), \( \kappa = 0 \) and \( \kappa < 0 \). The overall sign is conventional, but this confirms the desired property that surface gravity should vanish for degenerate black holes.

The dynamic surface gravity satisfies certain inequalities involving the area and energy, assuming the dominant energy condition. Directly from the definition (6.2),

\[
A\kappa \leq 4\pi E.
\]

This holds anywhere in the space-time, though the interpretation of \( \kappa \) as surface gravity is usually intended only on a horizon. Denoting values on the trapping horizon by a subscripted zero and recalling \( E_0 = \frac{1}{2} r_0 \),

\[
\kappa_0 \leq \sqrt{\frac{\pi}{A_0}}
\]

or equivalently \( \kappa_0 \leq 1/2 r_0 \). This has the same form as an inequality of Visser [20] for stationary black holes. Thus for a black hole of given area, the surface gravity has an upper bound.

Moreover, Paper II established the Penrose inequality

\[
\sqrt{\pi A_0} \leq 4\pi E
\]

where \( E \) is evaluated anywhere in the untrapped region outward from the trapping horizon. (Recall that the outward orientation is defined for spatial and null directions). Therefore

\[
A_0 \kappa_0 \leq 4\pi E
\]

under the same conditions. If the untrapped region is asymptotically flat, the same inequality holds for the asymptotic (Bondi and ADM) energies, which are limits of \( E \) at null and spatial infinity respectively. This has the same form as an inequality of Heusler [21] for stationary black holes. Thus the total surface gravity of a black hole provides a lower bound for the energy measured outside the black hole.
VII. First law of black-hole dynamics

The desired first law of black-hole dynamics may now be given as

$$E' = \frac{\kappa A'}{8\pi} + wV'$$  \hspace{1cm} (7.1)

where the prime denotes the derivative along the trapping horizon, $f' = z \cdot \nabla f$, where $z$ is a vector tangent to the trapping horizon, the normalisation of $z$ being irrelevant. This follows by projecting the unified first law (2.7) along $z$, as follows. The non-trivial part is to show that $A\psi^b \cdot z = \kappa A'/8\pi$. Writing $z$ in double-null coordinates, $z = z^+ \partial_+ + z^- \partial_-$, and taking the horizon to be given by $\partial_+ r = 0$, one has $0 = (\partial_+ r)' = z^+ \partial_+ r + z^- \partial_- r$, which expands to $0 = (\partial_+ r)' = g^{+}_{+} \kappa z^- - 4\pi r T_{++} z^+$, using the Einstein equations (2.10) and the definition of $\kappa$ (6.2). Therefore $A\psi^b \cdot z = Ag^{+}_{+} T_{++} z^+ \partial_- r = r \kappa z^- \partial_- r = r \kappa r' = \kappa A'/8\pi$.

The term $\kappa A'/8\pi$ is exactly as expected for a first law of black-hole dynamics, with the derivative along the horizon replacing the state-space differential of the first law of black-hole statics (4.5). To complete the desired physical interpretation, the last term $wV'$ should be the work done along the horizon. This can be seen for the electromagnetic field, for which the first law becomes

$$E' = \frac{\kappa A' + \mathcal{E}^2 V'}{8\pi} + w_0 V'$$  \hspace{1cm} (7.2)

as follows from (3.3). The term $\mathcal{E}^2 V'/8\pi$ is the electromagnetic work done along the horizon, in agreement with the stationary (4.5) and special-relativistic (3.5) cases.

VIII. Quasi-local first law of relativistic thermodynamics

Relativistic thermodynamics, as described for instance in Paper III, involves a conserved material current $\rho u$, where $\rho$ is the density and $u$ the unit velocity vector,

$$u \cdot u^b = -1$$  \hspace{1cm} (8.1)

with the physical interpretation as the flow vector of a fluid, or as giving the centre-of-momentum frame of a solid. The various components of the energy tensor $T$ of the matter define the thermal energy density

$$\varepsilon = u \cdot T^b \cdot u - \rho$$  \hspace{1cm} (8.2)

the thermal flux

$$q = -\bot (T \cdot u^b)$$  \hspace{1cm} (8.3)

and the thermal stress

$$\tau = \bot T$$  \hspace{1cm} (8.4)

where $\bot$ denotes projection by the spatial metric $g + u^b \otimes u^b$, which is orthogonal to the material flow. Therefore the energy tensor of the thermodynamic matter is

$$T = (\rho + \varepsilon)u \otimes u + 2u \otimes q + \tau.$$  \hspace{1cm} (8.5)
Since specific thermal energy is $\varepsilon/\rho$, this is effectively the energy tensor of Eckart [4], who defined specific internal energy as $u \cdot T^\phi \cdot u/\rho$ plus an undetermined constant.

The vorticity $\perp (\nabla \wedge u^\phi)$ vanishes in spherical symmetry, so there exist hypersurfaces orthogonal to the flow vector $u$, labelled by a time coordinate $t$. Take another coordinate $x$ on these hypersurfaces, orthogonal to the spheres of symmetry. Choosing vanishing shift vector, the line-element (2.8) becomes

$$ds^2 = r^2d\Omega^2 + e^{2\phi}dx^2 - e^{2\phi}dt^2$$

where $(r, \phi, \zeta)$ are functions of $(t, x)$. These are comoving coordinates adapted to the fluid. The volume form of the hypersurfaces is given by

$$\int_{\Sigma} \ast f = \int \frac{f \partial_x V dx}{\alpha}$$

(8.7)

where $\Sigma$ is a region of one of the hypersurfaces and

$$\alpha = e^{-\zeta} \partial_x r = \pm \left(1 + r^2 - \frac{2E}{r}\right)^{1/2}$$

(8.8)

where the dot denotes the material (or comoving) derivative, $\dot{f} = u \cdot \nabla f$, and the sign is that of $\partial_x r$. The second expression gives the definition of $E$ in comoving coordinates [8]. The red-shift factor $\alpha$ tends to one in the Newtonian limit.

The unified first law (2.7), written in comoving coordinates, is

$$\partial_t E = Ae^{-2\zeta}(T_{tx} \partial_x r - T_{xx} \partial_t r)$$

(8.9a)

$$\partial_x E = Ae^{-2\phi}(T_{tt} \partial_x r - T_{tx} \partial_t r).$$

(8.9b)

For the thermodynamic matter, the relevant energy components are

$$T_{tt} = e^{2\phi}(\rho + \varepsilon)$$

(8.10a)

$$T_{tx} = -e^{\phi+\zeta} \hat{q}$$

(8.10b)

$$T_{xx} = e^{2\zeta} p$$

(8.10c)

where $p = \tau^x_x$ is the radial pressure and $\hat{q}$ is the strength of the thermal flux $q^\phi = \hat{q} \nabla x / |\nabla x|$. The heat supply as defined in Paper III is

$$Q = -\int_{\hat{\Sigma}} \hat{q}$$

(8.11)

where $\hat{\Sigma}$ is a region of a hypersurface generated by flowlines of $u$, with volume form given by

$$\int_{\hat{\Sigma}} \hat{* f} = \int f Ae^\phi dt.$$
Thus
\[ \Delta Q = -c^\phi A \hat{q} \]  
(8.13)

where \( \Delta f = \partial_t f \). Equivalently in spherical symmetry, \( \dot{Q} = -A \hat{q} \). Then the component (8.9a) of the unified first law along the flow reads
\[ \Delta E = \alpha \Delta Q - p \Delta V. \]  
(8.14)

This is the desired quasi-local first law of relativistic thermodynamics. Since the quantities involved are local in spherical symmetry, this first law may also be written in terms of the material derivative as
\[ \dot{E} = \alpha \dot{Q} - p \dot{V}. \]  
(8.15)

The first term is a relativistic modification involving \( \alpha \) of the heat-supply term \( \dot{Q} \) in the classical first law of thermodynamics. The last term has the same form as the hydrodynamic work term of the classical case, with the relevant radial pressure \( p \) and with the areal volume \( V = \int_{\Sigma} *\alpha \) replacing the proper volume \( \int_{\Sigma} *1 \), again differing by the relativistic factor \( \alpha \). Recalling the role of \( E \) in black-hole dynamics, this establishes that the analogy between the two first laws is more than an analogy: the two first laws are different projections of the unified first law. Both express derivatives of the active gravitational energy according to the Einstein equation.

Unlike the classical first law of thermodynamics, which is an assumed principle, the quasi-local first law has been derived from the Einstein equations and the material model. Thus heat is included in general relativity as a form of energy, the quasi-local first law being a consequence of the relativistic definition of energy. Comparing with the non-relativistic first law, this indicates that the active gravitational energy \( E \) must incorporate the thermal energy, thereby linking thermodynamics and gravity. This may be seen explicitly as follows. The other component (8.9b) of the unified first law reads
\[ \partial_x E = (\rho + \varepsilon + \dot{r} \hat{q} / \alpha) \partial_x V \]  
(8.16)

which integrates to
\[ E = \int_{\Sigma} *(\alpha(\rho + \varepsilon) + \dot{r} \hat{q}). \]  
(8.17)

The integrand provides the definition of active gravitational energy density. In this form, \( E \) manifestly involves both thermal energy and thermal flux, though the latter disappears in the Newtonian limit, as follows.

The Newtonian limit may be described as in Paper II or III by inserting factors of the speed \( c \) of light by the formal replacements
\[ (*, r) \mapsto (*, r) \quad \dot{r} \mapsto c^{-1} \dot{r} \quad \rho \mapsto c^{-2} \rho \quad (\varepsilon, \tau, E) \mapsto c^{-4}(\varepsilon, \tau, E) \quad q \mapsto c^{-5} q \]  
(8.18)

then taking the limit \( c^{-1} \rightarrow 0 \). Then
\[ E = Mc^2 + H + K + U + O(c^{-2}) \]  
(8.19)
where the mass \( M \), thermal energy or heat \( H \), kinetic energy \( K \) and gravitational potential energy \( U \) are defined by

\[
M = \int_{\Sigma} *\rho
\]
\[
H = \int_{\Sigma} *\varepsilon
\]
\[
K = \int_{\Sigma} *\frac{1}{2} \rho v^2
\]
\[
U = -\int_{\Sigma} *\frac{M \rho}{r}.
\]

All four quantities have the same form as in Newtonian theory. Thus \( E \) incorporates all the energies and mass present in Newtonian theory. Again this confirms the correctness of \( E \) as a definition of active gravitational energy.

**IX. Zeroth laws**

The zeroth law of black-hole statics states that for a stationary black hole, the surface gravity \( \kappa \) is constant [13,14], a trivial property in spherical symmetry. Since a stationary space-time may be taken as the definition of gravitational equilibrium, this is analogous to the classical zeroth law of thermodynamics, which states that the temperature is constant in thermal equilibrium. In the Gibbs theory of thermostatics, often called equilibrium thermodynamics or reversible thermodynamics, this is expressed in terms of the inverse temperature \( \beta = \partial S/\partial H \).

To see whether there is a genuine connection between these zeroth laws, restrict from thermodynamics to thermostatics, in which case the second law of thermodynamics

\[
\Delta S \geq \Delta S_\circ
\]

becomes an equality. Here \( S \) is the entropy and \( S_\circ \) the entropy supply, as defined in Paper III. The classic relation between temperature \( \vartheta \), entropy flux and thermal flux yields

\[
\Delta Q = \vartheta \Delta S_\circ
\]

in spherical symmetry. Alternatively, one may regard this as defining absolute temperature \( \vartheta \), as in Gibbsian thermostatics. Then the quasi-local first law (8.14) implies a quasi-local Gibbs equation*

\[
\Delta E = \alpha \vartheta \Delta S - p \Delta V.
\]

If \( S \) is a function of \( V \) and \( E \), the time derivatives may be replaced with partial derivatives and *thermal equilibrium* may be defined by constancy of

\[
\beta = \frac{\partial S}{\partial E}.
\]

* Authors less familiar with thermodynamics often confuse the Gibbs equation, which involves temperature and entropy, with the first law of thermodynamics, which involves neither. The two are not equivalent except in thermostatics.
Then
\[ \beta^{-1} = \alpha \vartheta. \]  
(9.5)

Therefore temperature \( \vartheta \) is not uniform in thermal equilibrium in a gravitational field. This was originally predicted by Tolman [6], who argued on different grounds that gravitational red-shift leads to a non-uniform temperature. In the stationary case, \( \dot{r} \) vanishes and \( \alpha \) reduces to \( (1 - 2E/r)^{1/2} \), which coincides remarkably with the Tolman red-shift factor. The above condition is more general since it allows thermal equilibrium even in a non-stationary space-time. Such a possibility is also predicted by relativistic kinetic theory of gases [22].

The effective temperature \( \alpha \vartheta \) also agrees with that used for stationary black holes by Carter [13] and references therein. Moreover, for the case of Hawking radiation for the Schwarzschild black hole, the Hawking temperature

\[ \beta^{-1} = \frac{\hbar \kappa}{2\pi} \]  
(9.6)

is also an effective temperature, differing by the same factor from the temperature \( \vartheta = (1 - 2E/r)^{-1/2}\beta^{-1} \) measured by constant-radius detectors [5]. Here units are such that the Boltzmann constant is unity. This confirms that active gravitational energy plays the role that Gibbsian thermostatics assigns to thermal energy, strengthening the link between gravity and thermodynamics.

The factor \( \kappa \beta = 2\pi/\hbar \) is not determined by the above argument alone. However, it does show that if gravithermal equilibrium is defined by a stationary space-time with the above condition for thermal equilibrium, then both \( \kappa \) and \( \beta \) are constant, which may be interpreted as a weak type of unified zeroth law.

X. Conclusion

In summary, the unified first law is just the equation (2.7) expressing the gradient of the active gravitational energy according to the Einstein equation, divided into energy-supply and work terms. Projecting this equation along the trapping horizon and along the flow of thermodynamic matter yield, respectively, first laws of black-hole dynamics (7.1) and relativistic thermodynamics (8.15), reproduced here:

\[ E' = \frac{\kappa A'}{8\pi} + wV' \]

\[ \dot{E} = \alpha \dot{Q} - p\dot{V}. \]

In the thermodynamic case, this generalises from classical thermodynamics to general relativity. This shows that active gravitational energy incorporates thermal energy, as seen explicitly in the Newtonian limit in (8.19), thereby revealing a physical connection between gravity and thermodynamics.

In the black-hole case, this shifts the paradigm from statics to dynamics, that is, from stationary to non-stationary black holes. The standard first law of black-hole statics, involving a state-space differential, has been replaced by a first law of black-hole dynamics,
involving a horizon derivative. This is ironically analogous to the paradigm shift from thermo-
statics to thermodynamics that has occurred this century, replacing state-space differ-
entials with material derivatives and formulating thermodynamic field equations including
local first and second laws. The reader should be warned that this shift to genuine ther-
odynamics has been ignored by most authors attempting to draw analogies with black
holes, as well as by most authors of introductory thermodynamics texts. More enlightened
references may be found in Paper III.

Put another way, the paradigm shift is from Killing horizons and event horizons to
trapping horizons. For instance, Hawking’s second law for event horizons is replaced by
the second law of black-hole dynamics of Paper I, which states that the area element of
a future outer trapping horizon is non-decreasing, assuming the null energy condition. In
spherical symmetry this reduces to

\[ A' \geq 0. \]

Note that the new first and second laws of black-hole dynamics, unlike the textbook ver-
sions, involve the same derivative.

Gibbsian definitions of thermal equilibrium and gravithermal equilibrium have been
given, yielding a connection between the zeroth laws, expressing constancy of surface
gravity and an effective temperature. In contrast, the conventional second law of relativistic
thermodynamics as described in Paper III, that the divergence of the entropy current be
non-negative, is unrelated to the above second law of black-hole dynamics. Similarly,
unlike the case of a quantum field on a stationary background, there will be no general
agreement between the temperature and surface gravity, since one would expect to describe
gravitational collapse of matter with arbitrary temperature to black holes with unrelated
surface gravity. Moreover, it is unclear whether the gravitational entropy should agree
with the area for non-stationary black holes.

The analogies between black-hole dynamics and thermodynamics have been known
to be genuine in only one respect, namely Hawking’s connection between surface gravity and
effective temperature for quantum fields in the stationary case. This has remained
so despite much effort to relate the area of a black hole to some plausible definition of
gravitational entropy in general relativity, or indeed, in quite different theories of gravity.
(However, see Ashtekar et al. [23] and references therein). The unified first law has now
provided another genuine connection, neither restricted to the stationary case nor depen-
dent on quantum theory. With hindsight, this may not seem so surprising, since both
textbook first laws effectively express conservation of energy. However, this was just an
analogy in the absence of any obvious link between the internal energy of classical thermo-
dynamics and the total mass of a black-hole space-time, and indeed the latter turns out to
be not quite right. The missing link has now been revealed as active gravitational energy.

This provides further evidence for the physical importance of gravitational energy. In
particular, generalising the results of this paper beyond spherical symmetry would require
a more general definition of active gravitational energy in general relativity, on which there
is currently no consensus. Since energy is also fundamental in quantum theory, one would
expect gravitational energy to be important for quantum gravity. Moreover, since energy
and time share a quantum-mechanical duality, as expressed by the Heisenberg uncertainty
principle, this is relevant to the oft-discussed problem of time in quantum gravity. This
problem is neatly soluble in spherical symmetry, where the Kodama vector \( k \) encodes a dynamic time, related to \( E \). This suggests that

\[
\hat{E} = i\hbar k \cdot \nabla
\]

should be the energy operator of spherically symmetric quantum gravity [18]. Indeed, it may be argued that the correspondence principle requires this choice of time. Along with an area operator such as that provided by the loop theory [23], a basis for spherically symmetric quantum gravity may thereby be laid.

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