REGULARITY OF STRUCTURE SHEAVES
OF VARIETIES WITH ISOLATED SINGULARITIES

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Abstract. Let \( X \subseteq \mathbb{P}^N \) be a non-degenerate normal projective variety of codimension \( e \) and degree \( d \) with isolated \( \mathbb{Q} \)-Gorenstein singularities. We prove that the Castelnuovo-Mumford regularity \( \text{reg}(\mathcal{O}_X) \leq d - e \), as predicted by the Eisenbud-Goto conjecture. We also classify the extremal and the next to extremal cases.

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Introduction

Given a non-degenerate projective variety \( X \subseteq \mathbb{P}^N \) of codimension \( e \) and degree \( d \), the Eisenbud-Goto conjecture (cf. [4, 8]) predicts that the Castelnuovo-Mumford regularity of the ideal sheaf \( \mathcal{I}_X \) is subject to the Castelnuovo bound
\[
\text{reg}(\mathcal{I}_X) \leq d - e + 1.
\]
The conjecture is equivalent to the surjectivity of the restriction map
\[
H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \to H^0(X, \mathcal{O}_X(k)),
\]
for each \( k \geq d - e \) and that \( \text{reg}(\mathcal{O}_X) \leq d - e \). The latter amounts to the vanishing
\[
H^i(X, \mathcal{O}_X(d - e - i)) = 0 \tag{0.1}
\]
for all \( i \geq 1 \), and is known as the \( \mathcal{O}_X \)-regularity conjecture.

The Eisenbud-Goto conjecture has been shown to be false by the recent work [17] of McCullough-Peeva; in fact, they showed that \( \text{reg}(\mathcal{I}_X) \), in general, cannot be bounded by any polynomial function in \( d \). Nevertheless, it is still highly expected that the conjecture holds for smooth projective varieties and the ones with mild singularities in view of the work in low dimensions, see, e.g. [8, 12, 14, 22, 24, 25] and the references therein. This dichotomy raises the question of which mild singularities class separates projective varieties satisfying the Eisenbud-Goto conjecture from the others.

Concerning the \( \mathcal{O}_X \)-regularity conjecture, Noma [20] gave an affirmative answer for smooth varieties; in his classification, he showed that for a smooth non-degenerate projective variety \( X \) of arbitrary dimension,

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the structure sheaf is \((d - e)\)-regular, except when \(X\) is projectively equivalent to a scroll over a smooth projective curve. This case was proved in [13] among other things. It is natural to understand if the techniques developed by Noma in [20, 21] can be extended to the singular case. Toward this direction, we prove the following theorem:

**Theorem 1.** Let \(X \subseteq \mathbb{P}^N\) be a non-degenerate normal projective variety of codimension \(e\) and degree \(d\) with isolated \(\mathbb{Q}\)-Gorenstein singularities. Then \(\mathcal{O}_X\) is \((d - e)\)-regular.

We will prove a few more precise statements which imply Theorem 1 in §3 (see Theorem 3.1). The main techniques used in this article are Noma’s classification of non-degenerate projective varieties in terms of non-birational centers (cf. §1.1) and Nadel Vanishing for multiplier ideals (cf. §1.2).

Specifically, according to Noma, a projective variety \(X \subseteq \mathbb{P}^N\) of codimension \(e \geq 2\) (the hypersurface case is trivial) is projectively equivalent to one of the following cases:

(i) A scroll over a smooth projective curve;
(ii) A cone over a Veronese surface;
(iii) A birational type divisor of a conical rational scroll;
(iv) A birational type divisor of a conical scroll.

Case (i)-(iii) are achieved via explicit calculation on certain natural resolution of singularities or a prime divisor on a conical rational scroll. Our main contribution in Theorem 1 lies in case (iv). By a result of Noma (cf. Theorem 1.8) and the isolated singularities assumption, each component of the set \(\bar{C}(X)\) (cf. §1.1 for the definition and its role in classification) is either a point or a projective line. Therefore after applying Nadel vanishing, we can reduce the problem to the positivity of some vector bundles on \(\mathbb{P}^1\). We remark that our method also works for smooth varieties except when \(X\) is a smooth scroll over a smooth projective curve and we do not know whether our method can extend to the non-isolated singularities.

We note that a similar result bounding the regularity of \(\mathcal{O}_X\) in dimension 2 and 3 has been obtained in [23, Proposition 2.3], relying on the Eisenbud-Goto conjecture for curves and surfaces.

Following the approach of [13], we can characterize the extremal and the next to extremal cases:

**Theorem 2.** Let \(X \subseteq \mathbb{P}^N\) be a non-degenerate normal projective variety of codimension \(e\) and degree \(d\) with isolated \(\mathbb{Q}\)-Gorenstein singularities.

1. \(\text{reg}(\mathcal{O}_X) = d - e\) if and only if either \(X \subseteq \mathbb{P}^N\) is a hypersurface or a linearly normal projective variety with \(d \leq e + 2\).
2. \(\text{reg}(\mathcal{O}_X) = d - e - 1\) if and only if either \(X \subseteq \mathbb{P}^N\) is an isomorphic projection of a projective variety as in case (1) at one point, a linearly normal variety with \(d = e + 3\) and \(e \geq 2\), or a complete intersection of type \((2, 3)\).

The article is organized as follows: In subsection 1.1, we recall Noma’s classification of projective varieties in terms of exceptional divisors of general inner projections and partial Gauss maps. In subsection 1.2, we briefly recall the definition of multiplier ideals and state a version of Nadel vanishing theorem. In section 2, we prove some properties of inner projections for varieties with mild singularities. In section 3, we prove Theorem 3.1 by treating the cases independently in each subsection. Finally, in section 4, we classify the varieties with the maximal or the next to maximal \(\mathcal{O}_X\)-regularity.

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1. **Preliminaries**

We work over an algebraically closed field \(k\) of characteristic zero.
1.1. **Noma’s classification of projective varieties.** In this subsection, we recall the classification of non-degenerate projective varieties in terms of exceptional divisors on general inner projections and partial Gauss maps. This classification was introduced by Noma in the papers [20,21]. We refer the interested reader to ibid. for details.

**Definition 1.1.** Given a projective variety \( X \subseteq \mathbb{P}^N \), the set \( C(X) \) is defined as
\[
\{ u \in \text{Sm}(X) \mid \text{length}(X \cap (u,x)) \geq 3 \text{ for general } x \in X \}.
\]
And \( \bar{C}(X) \) is defined as the Zariski closure of \( C(X) \) in \( \mathbb{P}^N \).

It’s a fundamental result, due to [26], that if \( X \) is non-degenerate, then \( \bar{C}(X) \) is a union of finitely many linear subspaces of \( \mathbb{P}^{N+1} \).

**Definition 1.2.** Let \( X \subseteq \mathbb{P}^N \) be a non-degenerate variety of dimension \( n \) and codimension \( e \geq 2 \). Let \( 1 \leq m \leq e - 1 \) be a positive integer. For general points \( x_1, \ldots, x_m \in X \), we define \( E_{x_1,\ldots,x_m}(X) \) to be the Zariski closure of
\[
\{ z \in X \setminus \{ x_1, \ldots, x_m \} \mid \text{dim}(x_1, \ldots, x_m, z) \cap X \geq 1 \}
\]
in \( X \), which means that \( E_{x_1,\ldots,x_m}(X) \) is the closure of the set of positive-dimensional fibers of the linear projection
\[
\pi_{\Lambda,X} : X \setminus \Lambda \to \mathbb{P}^{N-m}
\]
from \( \Lambda = (x_1, \ldots, x_m) \). We say that \( X \) satisfies condition \((E_m)\) if
\[
\text{dim}(E_{x_1,\ldots,x_m}(X)) \geq n - 1
\]
for general points \( x_1, \ldots, x_m \) in \( X \), in other words, if the exceptional locus of the general inner projection from a general \((m-1)\)-dimensional linear subspace contains an exceptional divisor.

It directly follows from the definition that if \( X \) satisfies condition \((E_k)\) for some \( 1 \leq k \leq e - 2 \), then it satisfies condition \((E_{k+1})\). If \( X \) satisfies condition \((E_m)\), we denote by \( D_{x_1,\ldots,x_m}(X) \) the union of the irreducible components of \( E_{x_1,\ldots,x_m}(X) \) of dimension \( n - 1 \).

We recall

**Definition 1.3.** Let \( C \) be a smooth projective curve and \( \mathcal{E} \) a vector bundle on \( C \). We denote by \( E^C_{\mathcal{E}} = \mathbb{P}(\mathcal{E}) \) the projectivization of the vector bundle on \( C \) and by \( p : E^C_{\mathcal{E}} \to C \) the canonical projection. Let \( \mu : E^C_{\mathcal{E}} \to \mathbb{P}^N \) be a proper morphism induced by a subsystem of \( |O_{E^C_{\mathcal{E}}}(1)| \) such that \( \mu \) induces a birational morphism between \( E^C_{\mathcal{E}} \) and its image \( X \subseteq \mathbb{P}^N \). Then \( X \) is called a scroll over the smooth projective curve \( C \).

**Theorem 1.4.** Let \( X \subseteq \mathbb{P}^N \) be a non-degenerate projective variety of dimension \( n \geq 2 \) and codimension \( e \geq 2 \). Then one of the following holds:

- Assume that \( X \) satisfies condition \((E_1)\). Then for a general point \( x \in X \) the subvariety \( D_x(X) \) is an \((n-1)\)-dimensional linear space passing through \( x \). Consequently, \( X \) is projective equivalent to a scroll over a curve.
- Assume that \( e \geq 3 \) and \( X \) satisfies condition \((E_2)\) but not condition \((E_1)\). Then \( e = 3 \) and \( X \subseteq \mathbb{P}^{n+3} \) is projectively equivalent to the cone over the Veronese surface \( v_2(\mathbb{P}^2) \subseteq \mathbb{P}^5 \) with an \((n-3)\)-dimensional vertex. In particular, \( X \) is smooth if and only if \( n = 2 \).
- Assume that \( e \geq 4 \). Then \( X \) satisfies condition \((E_m)\) for some \( 3 \leq m \leq e - 1 \) if and only if it satisfies condition \((E_1)\).

\(^1\)Interestingly, the linearity of \( \bar{C}(X) \) fails in positive characteristic by Furukawa [7].
When $X$ is smooth, a famous result due to Zak cf. [15] says the Gauss map $\gamma : X \to \mathbb{G}(n, N)$, which sends a point to its embedded projective tangent space in a Grassmannian is finite, and hence is non-constant on any subvariety of positive dimension. In general, Gauss map $\gamma$ is defined only on the smooth locus $\text{Sm}(X)$ of $X$.

Let $\Lambda$ be one irreducible component of $\mathcal{C}(X)$, and $\gamma|_{\Lambda}$ denote the restriction of the Gauss map to $\Lambda \cap \text{Sm}(X)$. This morphism is called a partial Gauss map. The partial Gauss maps give further classification of projective varieties, as we will see as follows.

**Definition 1.5.** Let $\Lambda \subseteq \mathbb{P}^N$ be a linear subspace of dimension $l$ and consider $E$ be an ample vector bundle of rank $n-l \geq 1$ on $\mathbb{P}^1$. The conical rational scroll $E^\Lambda_2$ with vertex $\Lambda$ is the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus E)$ with the projection $p : E^\Lambda_2 \to \mathbb{P}^1$. Observe that the conical scroll rational has a natural birational embedding $\psi : E^\Lambda_2 \to \mathbb{P}^N$ defined by a linear subsystem of $|\mathcal{O}_{E^\Lambda_2}(1)|$ such that the subbundle $E_{\mathbb{P}^1} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1))$ maps onto $\Lambda$ by $\psi$. A projective variety $X \subseteq \mathbb{P}^N$ is called a birational divisor of the rational conical scroll $E^\Lambda_2$ if $X$ is the birational image of a prime divisor $\tilde{X}$ on $E^\Lambda_2$ by the birational embedding $\psi$. Moreover, we will say that the divisor $X$ is of type $(\mu, b)$ if $\tilde{X} \in |\mathcal{O}_{E^\Lambda_2}(\mu) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(b)|$ with $\mu, b \in \mathbb{Z}$.

**Theorem 1.6.** Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension $n \geq 1$ and codimension $e \geq 2$. Let $\Lambda \subseteq \mathbb{P}^N$ be a subspace of dimension $l$ with $n-1 \geq l \geq 0$. Suppose that $X$ is non-degenerate, $\Lambda$ is an irreducible component of $\mathcal{C}(X)$, and $\gamma|_{\Lambda}$ is non-constant. Then $X$ is a birational divisor of type $(\mu, 1)$ with $\mu \geq 2$ on a conical rational scroll $E^\Lambda_2$ with vertex $\Lambda$. In particular, $\dim(\Lambda) \leq \dim(\text{Sing}(X)) + 2$.

**Definition 1.7.** Let $\Lambda \subseteq \mathbb{P}^N$ be a linear subspace of dimension $l$ and let $\mathbb{P}^{\Lambda} \subseteq \mathbb{P}^N$ be a linear subspace disjoint from $\Lambda$ with $\mathbb{P}^{\Lambda} = N - l - 1$. Consider the linear inner projection
\[
\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{\Lambda}
\]
from $\Lambda$. Let $\pi_{\Lambda,X} : X \setminus \Lambda \to \mathbb{P}^{\Lambda}$ be its restriction to $X$. We will consider the incidence variety
\[
\mathbb{F}^\Lambda = \{ (x, w) \mid x \in \langle \Lambda, w \rangle \} \subseteq \mathbb{P}^N \times \mathbb{P}^{\Lambda}.
\]
The projection $\tau : \mathbb{F}^\Lambda \to \mathbb{P}^{\Lambda}$ on the second coordinate gives $\mathbb{F}^\Lambda$ a $(\mathbb{P}^1)$-bundle structure over $\mathbb{P}^{\Lambda}$. Given a birational embedding $\nu : Y \to \mathbb{P}^{\Lambda}$ of a smooth projective variety $Y$ we define a conical scroll with vertex $\Lambda$ over $Y$ to be the pull-back
\[
\tau_Y : \mathbb{F}^\Lambda_Y \to Y
\]
of $\tau$ by $\nu$. Observe that $\mathbb{F}^\Lambda_Y$ has a birational embedding $\phi_Y$ into $\mathbb{P}^N$ induced by the first projection of $\mathbb{P}^N \times Y$. We say that the projective variety $X \subseteq \mathbb{P}^N$ is a birational divisor of the conical scroll $\mathbb{F}^\Lambda_Y$ if $X$ is birational to some prime divisor $\tilde{X}$ on $\mathbb{F}^\Lambda_Y$ by $\phi_Y$. Moreover, we will say that the divisor $X$ is of type $(\mu, \mathcal{L})$ if $\tilde{X} \in |\mathcal{O}_{\mathbb{F}^\Lambda_Y}(\mu) \otimes \tau_Y^*(-\mathcal{L})|$ for $\mu \in \mathbb{Z}$ and $\mathcal{L} \in \text{Pic}(Y)$.

**Theorem 1.8.** Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension $n \geq 1$ and codimension $e \geq 2$. Let $\Lambda \subseteq \mathbb{P}^N$ be a subspace of dimension $l$ with $n-1 \geq l \geq 0$. Suppose that $X$ is nondegenerate, $\Lambda$ is an irreducible component of $\mathcal{C}(X)$, $\Lambda \cap \text{Sm}(X) \neq \emptyset$ and $\gamma|_{\Lambda}$ is constant. Then $X$ is a birational divisor of type $(\mu, \mathcal{L})$ with $\mu \geq 2$ and $\mathcal{L} \in \text{Pic}(Y)$ on a conical scroll $\mathbb{F}^\Lambda_Y$ with vertex $\Lambda$ over an $(n-\dim(\text{Sing}(X)))$-dimensional smooth projective variety $Y$ with a non-degenerate birational embedding $\nu : Y \to \mathbb{P}^N$. In particular, $\dim(\Lambda) \leq \dim(\text{Sing}(X))+1$.

The following corollary is a summary of Theorem 1.4, 1.6, and 1.8.

**Corollary 1.9.** Let $X \subseteq \mathbb{P}^N$ be a non-degenerate projective variety of dimension $n \geq 2$ and codimension $e \geq 2$. Then one of the following holds:

1. $X$ is projectively equivalent to a scroll over a smooth curve,
(2) $X$ is projectively equivalent to a cone over the Veronese surface,
(3) $X$ does not satisfy condition $(E_{c-1})$, and
- The partial Gauss map is non-constant on some irreducible component of $\tilde{C}(X)$. Therefore, $X$ is a birational type divisor of a conical rational scroll $E^2_C$, or
- The partial Gauss map is constant on every irreducible component of $\tilde{C}(X)$. Therefore, $X$ is a birational type divisor of a conical scroll $F^1_C$.

1.2. Nadel Vanishing. In this subsection, we recall a version of Nadel vanishing Theorem for linear systems that will be used in the proof of the main theorem.

**Definition 1.10.** A log pair $(X, \Delta)$ is a normal projective variety $X$ with an effective divisor $\Delta$ on $X$ such that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor on $X$. Observe that if $X$ is $\mathbb{Q}$-Gorenstein, then $(X, \Delta)$ is a log pair for every effective $\mathbb{Q}$-Cartier divisor $\Delta$ on $X$.

**Definition 1.11.** Let $(X, \Delta)$ be a log pair, a log resolution of $(X, \Delta)$ is a projective birational morphism $\pi: Y \to X$ from a smooth projective variety $Y$ such that $\text{Ex}(\pi)$ is purely of codimension one and $\text{Ex}(\pi) \cup \pi^{-1}_*(\Delta)$ is a divisor with simple normal crossing support on $Y$.

**Definition 1.12.** Let $(X, \Delta)$ be a log pair and $\pi: Y \to X$ be a log resolution of the pair. Then the multiplier ideal $J(X, \Delta)$ of $(X, \Delta)$ is

$$\pi_*\mathcal{O}_Y([K_Y - \pi^*(K_X + \Delta)]) \subseteq \mathcal{O}_X$$

The multiplier ideal $J(X, \Delta) \subseteq \mathcal{O}_X$ is an ideal sheaf which is independent of the chosen log resolution.

**Remark 1.13.** Multiplier ideals are integrally closed ideal sheaves (see, e.g. [16, Cor. 9.6.13]), therefore they admit a primary decomposition.

The following theorem is Nadel vanishing for pairs (see, e.g. [16, Theorem 9.4.17] or [6, Theorem 3.2]).

**Theorem 1.14.** Let $(X, \Delta)$ be a log pair and $L$ a Cartier divisor on $X$ such that $L - (K_X + \Delta)$ is a big and nef $\mathbb{Q}$-Cartier divisor. Then

$$H^i(X, \mathcal{O}_X(L) \otimes J(X, \Delta)) = 0,$$

for $i > 0$.

**Definition 1.15.** Let $X$ be a $\mathbb{Q}$-Gorenstein projective variety and $M$ be a $\mathbb{Q}$-Cartier divisor with a non-empty associated linear system. Consider a projective birational morphism $\pi: Y \to X$ such that $Y$ is a smooth projective variety, $\pi^*[M] = [W] + F$, where $F + \text{Ex}(\pi)$ is a divisor with simple normal crossing support, and $[W]$ is a base point free linear system. For a rational number $c > 0$, we define the multiplier ideal $J(X, c \cdot |M|)$ associated to the linear system $|M|$ to be

$$\pi_*\mathcal{O}_Y([K_Y - \pi^*(K_X + cF)]) \subseteq \mathcal{O}_X$$

The multiplier ideal $J(X, c \cdot |M|) \subseteq \mathcal{O}_X$ is an ideal sheaf which is independent of the chosen log resolution.

**Remark 1.16.** It is known that we have a natural inclusion $J(X, |M|) \supseteq b(|M|)$ where $b(|M|)$ is the base scheme of the linear system associated to the divisor $M$ (see, e.g. [16, §9]).

**Theorem 1.17.** Let $X$ be a normal $\mathbb{Q}$-Gorenstein projective variety, $c \in \mathbb{Q}_+$, and $M$ a Cartier divisor. Let $L$ be a Cartier divisor on $X$. Suppose that $L - (K_X + cM)$ is big and nef. Then

$$H^i(X, \mathcal{O}_X(L) \otimes J(X, c \cdot |M|)) = 0,$$

for $i > 0$, where $J(X, c \cdot |M|)$ is the multiplier ideal of the linear system $c \cdot |M|$ with respect to $X$.

**Proof.** Similar to the proof of [16, Prop 9.2.26], we can find $k$ large enough such that for general elements $A_1, \ldots, A_k \in |M|$ the effective $\mathbb{Q}$-divisor $\Delta = \frac{k}{k}(A_1 + \cdots + A_k)$ holds that $J(X, c \cdot |M|) = J(X, \Delta)$. Therefore the assertion follows from Theorem 1.14. □
2. INNER PROJECTION FOR VARIETIES WITH MILD SINGULARITIES

In this section we prove Theorem 2.7. This theorem is proved in the smooth case in [20, Theorem 1]. In the singular case, the proof is similar by using a double point formula for normal varieties (see Lemma 2.3). First, we introduce some notation and recall some lemmas which are proved in [20, §1 and §2].

Notation 2.1. Consider a non-degenerate projective variety $X \subseteq \mathbb{P}^N$ of dimension $n$ and codimension $e$. Suppose that $X$ has dimension $n \geq 2$ and let $1 \leq m \leq e - 1$ be an integer number. Consider $x_1, \ldots, x_m \in X$ be general points on $X$ and $\Lambda = \langle x_1, \ldots, x_m \rangle$ its linear span. Let $\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{N-m}$ be the linear projection from $\Lambda$, and let $\pi_{X,\Lambda} : X \setminus \Lambda \to \mathbb{P}^{N-m}$ be the restriction of this morphism to $X$, we denote the closure of the image of $\pi_{X,\Lambda}$ by $\overline{\Lambda}$. The induced morphism $X \setminus \Lambda \to \overline{X}_\Lambda$ will be denoted by $\pi_{\Lambda,X}$. In [20, Lemma 1.2], the author proves that the equality $X \cap \Lambda = \{x_1, \ldots, x_m\}$ holds scheme-theoretically, therefore if we blow-up the points $x_1, \ldots, x_m$ we obtain a projective morphism $\pi_{\Lambda,X} : \tilde{X} \to X$. The rational map $\pi_{\Lambda,X}$ extends to a morphism

$$\hat{\pi}_{\Lambda,X} : \tilde{X} \to \mathbb{P}^{N-m}$$

such that $\hat{\pi}_{\Lambda,X} = \pi_{\Lambda,X} \circ \sigma$ as rational maps. The map $\pi_{\Lambda,X}$ will be called the induced projection and $\hat{\pi}_{\Lambda,X}$ is the extended projection.

Lemma 2.2. Let $X \subseteq \mathbb{P}^N$ be a nondegenerate projective variety of codimension $e \geq 2$. Let $m$ be a positive integer with $1 \leq m \leq e - 1$. Let $x$ be a point of $\text{Sm}(X) \setminus C(X)$. Then for general points $x_1, \ldots, x_m$ of $X$ and $\Lambda = \langle x_1, \ldots, x_m \rangle$, the induced projection $\pi_{\Lambda,X}^*$ is an isomorphism at $x$.

Now, we turn to a version of the birational double-point formula for projective normal varieties.

Lemma 2.3. Consider a morphism

$$f : X \to \mathbb{P}^{n+1}$$

from a normal projective variety $X$ of dimension $n$. Assume that $f$ maps $X$ birationally onto an hypersurface $Y \subseteq \mathbb{P}^{n+1}$. Then there exist effective Weil divisors $D$ and $E$ on $X$, where $E$ is $f$-exceptional, such that

$$f^*(K_{\mathbb{P}^{n+1} + Y}) - K_X \sim D - E.$$  

Moreover, if $f : X \to Y$ is isomorphic at $x \in X$ and $x$ is in the smooth locus of $X$, then one can choose $D$ such that $x \not\in \text{supp}(D - E)$.

Proof. Consider a smooth point $x \in X$ such that the birational morphism $f : X \to Y$ is an isomorphism at $x$. Take $\pi : X' \to X$ a resolution of singularities of $X$ which does not blow-up centers containing the point $x \in X$. Observe that if $\phi : X \to Y$ is an isomorphism at $x \in X$, then $\phi \circ f : X' \to Y$ is an isomorphism at the pre-image $x' = \pi^{-1}(x)$ too. Moreover, we can choose a canonical divisor $K_{X'}$ on $X'$ such that $\pi_*(K_{X'}) = K_X$. Hence applying the birational double point formula [16, Lemma 10.2.8] for the morphism

$$f \circ \pi : X' \to \mathbb{P}^{n+1},$$

we deduce that there exist effective divisors $D'$ and $E'$ on $X'$, where $E'$ is $(f \circ \pi)$-exceptional, such that

$$(f \circ \pi)^*(K_{\mathbb{P}^{n+1} + Y}) - \pi_* K_{X'} \sim D' - E',$$

and the support of $D' - E'$ does not contain the point $x'$. Therefore, pushing forward the above linear equivalence via $\pi$ we obtain

$$f^*(K_{\mathbb{P}^{n+1} + Y}) - \pi_* (K_{X'}) \sim \pi_*(D') - \pi_*(E').$$

Since the divisor $E = \pi_*(E')$ is $f$-exceptional and $D = \pi_*(D')$ does not contain $x$ in its support, we conclude that

$$f^*(K_{\mathbb{P}^{n+1} + Y}) - K_X \sim D - E,$$

where $D - E$ is a divisor which does not contain $x \in X$ in its support. \qed
Next we collect a few lemmas from [20], which are proved for possibly singular non-degenerate projective varieties.

**Lemma 2.4.** Let $2 \leq m \leq e$ be an integer. For general points $x_1, \ldots, x_m \in X \subseteq \mathbb{P}^N$ and its linear span $\Lambda \subseteq \mathbb{P}^N$, we have that

$$\text{length}(X \cap \Lambda) = m,$$

which means that we have a scheme-theoretic equality $X \cap \Lambda = \{x_1, \ldots, x_m\}$, where the latter set is considered with the reduced scheme structure.

**Lemma 2.5.** Assume $n \geq 2$ and let $1 \leq m \leq e - 1$ be a positive integer. Let $x_1, \ldots, x_m \in X$ be general points and $\Lambda = \langle x_1, \ldots, x_m \rangle$ be the linear span. The set

$$\tilde{E}_{x_1,\ldots,x_m} := \{ z \in \tilde{X} \mid \dim(\tilde{\pi}_{\Lambda,X}^{-1}(\tilde{\pi}_{\Lambda,X}(z))) \geq 1\}$$

is a closed subset of $\tilde{X}$, and we have

$$\sigma(\tilde{E}_{x_1,\ldots,x_m}) \supseteq E_{x_1,\ldots,x_m}(X) \supseteq \sigma(\tilde{E}_{x_1,\ldots,x_m} \setminus \{x_1, \ldots, x_m\}).$$

In particular, $\dim(E_{x_1,\ldots,x_m}(X)) \geq n - 1$ if and only if $\dim(\tilde{E}_{x_1,\ldots,x_m}) \geq n - 1$.

Now, we turn to state and prove the main theorem of this section. In order to do so, we define the linear system of a Weil divisor on a normal variety.

**Definition 2.6.** Let $X$ be a normal projective variety and $D$ a Weil divisor on $X$. We define the linear system associated to $D$, denoted by $|D|$, to be the set of all effective Weil divisors on $X$ which are linearly equivalent to the divisor $D$. If $D$ is a Cartier divisor, then the linear system $|D|$ can be endowed with the structure of a projective space, more precisely $|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D)))$. We define the base locus of $|D|$ to be the subset $b(|D|) = \bigcap_{E \in |D|} E$ of $X$.

**Theorem 2.7.** Let $X \subseteq \mathbb{P}^N$ be a non-degenerate normal projective variety of dimension $n \geq 2$, codimension $e \geq 2$, and degree $d$. Let $1 \leq m \leq e - 1$ and assume that $X$ does not satisfy $(E_m)$. Then the base locus of the linear system $|\mathcal{O}_X(d-m-n-2)-K_X|$ is contained in $\mathcal{C}(X) \cup \text{Sing}(X)$.

**Proof.** Let $x \in X$ such that $x \notin \mathcal{C}(X) \cup \text{Sing}(X)$. Choose general points $x_1, \ldots, x_m \in X$ and put $\Lambda = \langle x_1, \cdots, x_m \rangle$. Consider the linear projection $\pi_{\Lambda,X} : X \setminus \Lambda \to \mathbb{P}^{N-m}$. Let $X_\Lambda$ be the Zariski closure of the image of $\pi_{\Lambda,X}$. We have

- $X \cap \Lambda = \{x_1, \ldots, x_m\}$ holds scheme-theoretically by Lemma 2.4,
- $\dim E_{x_1,\ldots,x_m}(X) \leq n - 2$ by Lemma 2.5,
- the induced morphism $\pi_{\Lambda,X} : X \setminus \Lambda \to X_\Lambda$ is isomorphic at $x$ by Lemma 2.2.

Let $\sigma : \tilde{X} \to X$ be the blowup of $X$ at the points $x_1, \ldots, x_m$ and $E_i$ the exceptional divisor over $x_i$. The extended projection $\hat{\pi}_{\Lambda,X} : \tilde{X} \to \mathbb{P}^{N-m}$ has no exceptional divisor. Let $\tilde{x} = \pi_{\Lambda,X}(x)$. If $m = e - 1$, then $N - m = n + 1$. Otherwise if $m < e - 1$, take a general $(N - m - n - 2)$-plane $\Lambda'$ in $\mathbb{P}^{N-m}$ such that

$$\Lambda' \cap X_\Lambda = \emptyset \quad \text{and} \quad \Lambda' \cap (T_{\tilde{x}}(\tilde{X}_\Lambda) \cup \text{Cone}(x, \tilde{X}_\Lambda)) = \emptyset.$$

Here we have used the assumption that $\dim T_x(X) \leq n$. Consider the projection $\pi_{\Lambda'} : \mathbb{P}^{N-m} \setminus \Lambda' \to \mathbb{P}^{n+1}$. Let $\tilde{X}$ be the image $\pi_{\Lambda'}(\tilde{X}_\Lambda)$. Since $\Lambda' \cap \tilde{X}_\Lambda = \emptyset$, the induced morphism $\pi_{\Lambda',\tilde{X}_\Lambda} : \tilde{X}_\Lambda \to \tilde{X}$ is finite.

The composite map $\hat{\pi} : \tilde{X} \to \tilde{X}$ has no exceptional divisor and $\hat{\pi}$ is isomorphic at $\hat{x} = \sigma^{-1}(x)$. Then by Lemma 2.3, there exists an effective divisor $\hat{D}$ on $\tilde{X}$ such that $\hat{D} \sim \hat{\pi}^*K_{\tilde{X}} - K_{\tilde{X}}$ and $\hat{x} \notin \text{supp}(\hat{D})$. Let $D$ be the closure of $\sigma(\hat{D})_{\tilde{X} \setminus \cup_i E_i}$. Then $D$ is an effective Weil divisor on $X$ that does not contain $x$. We have that on $X \setminus \{x_1, \cdots, x_m\}$ it holds that $\pi^*K_X - K_X - D \sim 0$. Because of the condition $n \geq 2$ and the normality of
$X$, $\pi^*K_X$ extends to $\mathcal{O}_X(d - m - n - 2)$, therefore $D \sim \mathcal{O}_X(d - m - n - 2) - K_X$ with $x \notin \text{Supp}(D)$. This shows that $\text{Bs}|\mathcal{O}_X(d - m - n - 2) - K_X| \subseteq C(X) \cup \text{Sing}(X)$. \hfill $\square$

3. Regularity of structure sheaves: Proof of Theorem 1

This section aims to establish the following vanishing theorem.

**Theorem 3.1.** Let $X \subseteq \mathbb{P}^N$ be a non-degenerate projective variety of dimension $n$, codimension $e$, and degree $d$ with normal isolated $\mathbb{Q}$-Gorenstein singularities. If $X$ is not projectively equivalent to a smooth scroll over a smooth projective curve, then we have that

$$H^i(X, \mathcal{O}_X(k)) = 0$$

for every $i > 0$ and every $k \geq d - e - n$.

If $X$ is projectively equivalent to a smooth scroll over a smooth projective curve, then we know that $\text{reg}(\mathcal{O}_X) \leq d - e$ by [13, Proposition 3.6]. Thus the above vanishing theorem implies Theorem 1.

Since the curve case is known by [8] (see also [13, Proposition 3.3]) and the hypersurface case is trivial, we assume from now on that $n, e \geq 2$. To prove Theorem 3.1, it is sufficient to show that

$$H^1(X, \mathcal{O}_X(k)) = 0$$

for every $k \geq d - e - n$.

To see this, consider a general hyperplane section $Y \subseteq \mathbb{P}^{N-1}$ of $X \subseteq \mathbb{P}^N$ which is a non-degenerate smooth projective variety of dimension $n - 1$, codimension $e$, and degree $d$. We have an exact sequence

$$0 \to \mathcal{O}_X(k) \to \mathcal{O}_X(k + 1) \to \mathcal{O}_Y(k + 1) \to 0$$

for any integer $k$. If $Y$ is not projectively equivalent to a scroll over a smooth projective curve, then by induction, we have that

$$H^i(Y, \mathcal{O}_Y(k + 1)) = 0$$

for every $i \geq 1$ and every $k + 1 \geq d - e - (n - 1)$ (equivalently, $k \geq d - e - n$). If $Y$ is projectively equivalent to a scroll over a smooth projective curve (in particular, $n \geq 3$), then it is a rational scroll by Lemma 3.4 below. In this case, the same cohomology vanishing (3.2) hold. In any case, we obtain that $H^i(X, \mathcal{O}_X(k)) = H^i(X, \mathcal{O}_X(k + 1))$ for every $i \geq 2$ and every $k \geq d - e - n$. By Serre vanishing, we obtain that $H^i(X, \mathcal{O}_X(k)) = 0$ for every $i \geq 2$ and $k \geq d - e - n$. So in the remaining of this section, we will focus on showing that (3.1) holds.

Before the following discussion, we recall a useful lemma, which will be used frequently in the sequel.

**Lemma 3.2 ([18, Theorem 2]).** Let $X$ be a normal projective variety of dimension $\geq 2$ and $L$ be a nef and big line bundle on $X$. Then

$$H^1(X, L^{-1}) = 0.$$

**Proof.** Let $\mu : X' \to X$ be a resolution of singularities. Since $X$ is normal, $\mu_*\mathcal{O}_{X'} \cong \mathcal{O}_X$. One has the exact sequence induced from the Leray spectral sequence

$$0 \to H^1(X, L^{-1}) \to H^1(X', \mu^*L^{-1}) \to H^0(X, L^{-1} \otimes R^1\mu_*\mathcal{O}_{X'}) \to H^2(X, L^{-1}) \to H^2(X', \mu^*L^{-1}).$$

By Kawamata-Viehweg vanishing for $\mu^*L^{-1}$, we obtain $H^1(X', \mu^*L^{-1}) = 0$; so the assertion follows. \hfill $\square$
3.1. **Scroll over a smooth projective curve.** In this subsection, we prove the vanishing (3.1) in the case that $X$ is a scroll over a smooth projective curve.

**Proposition 3.3.** Let $X \subseteq \mathbb{P}^N$ be a projective variety of codimension $c$ and degree $d$. Assume that $X$ is a singular scroll over a smooth projective curve $C$ and $X$ has normal, isolated, $\mathbb{Q}$-Gorenstein singularities. Then $H^1(X, \mathcal{O}_X(k)) = 0$ for every $k \in \mathbb{Z}$.

We begin with the following observation.

**Lemma 3.4.** Keep the assumption of Proposition 3.3 and suppose that $\dim X \geq 3$. Then $X$ has canonical singularities and $C \cong \mathbb{P}^1$.

**Proof.** For each $x \in X$, $\dim \mu^{-1}(x) \leq 1$. Since $X$ is $\mathbb{Q}$-Gorenstein, we have linear equivalence

$$K_{\mathbb{P}^c_X} = \mu^* K_X + \sum a_i E_i,$$

where $E_i$ are $\mu$-exceptional divisors and $a_i \in \mathbb{Q}$. Restricting the above to the open set $U = \mathbb{P}^c_X \setminus \mu^{-1}(X_{\text{sing}})$ and observing that $\text{codim}(\mu^{-1}(X_{\text{sing}}), \mathbb{P}^c_X) \geq 2$, we deduce that $a_i \geq 0$, for all $i$. Now, let $Z \to \mathbb{P}^c_X \to X$ be a log resolution of singularities of $X$ which is obtained by blowing-up a sequence of smooth centers starting on $\mathbb{P}^c_X$. Denote by $\mu_Z: Z \to \mathbb{P}^c_X$ the induced morphism. Then we have that

$$K_Z = \mu_Z^* (K_{\mathbb{P}^c_X}) + F,$$

for some effective $\mu_Z$-exceptional divisor $F$. Thus

$$K_Z = (\mu \circ \mu_Z)^* (K_X) + \sum a_i \mu_Z^* (E_i) + F,$$

implying that all the discrepancies of $Z \to X$ are non-negative, so we conclude that $X$ has canonical singularities. Moreover it follows from for instance [9, Corollary 1.5] that $\mu^{-1}(x)$ is rationally chain connected for every $x \in X$, and therefore $C$ is isomorphic to $\mathbb{P}^1$. $\square$

**Proof of Proposition 3.3.** First consider the case that $\dim X = 2$. We know that $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor. Indeed, the Picard rank of $X$ is one being a singular ruled surface. By Kodaira vanishing, we obtain that $H^1(X, \mathcal{O}_X(k)) = 0$ for every $k \in \mathbb{Z}$. We assume now that $\dim X \geq 3$. By Lemma 3.4, we know that $C \cong \mathbb{P}^1$ and that $X$ has canonical singularities, and hence has rational singularities (see e.g., [11, Theorem 5.22]). It follows that

$$H^1(X, \mathcal{O}_X(k)) \simeq H^1 \left( \mathbb{P}^c_X, \mathcal{O}_{\mathbb{P}^c_X}(k) \right).$$

Therefore we are reduced to proving

$$H^1 \left( \mathbb{P}^c_X, \mathcal{O}_{\mathbb{P}^c_X}(k) \right) = 0.$$

If $k < 0$, then the above vanishing follows from Kawamata-Viehweg vanishing theorem. If $k \geq 0$, then $R^j \pi_* \mathcal{O}_{\mathbb{P}^c_X}(k) = 0$ for $j > 0$, and hence

$$H^1 \left( \mathbb{P}^c_X, \mathcal{O}_{\mathbb{P}^c_X}(k) \right) \simeq H^1(C, S^k \mathcal{E}) \simeq H^1(\mathbb{P}^1, S^k \mathcal{E}).$$

Since $\mathcal{O}_{\mathbb{P}^c}(1)$ is base point free, any symmetric power $S^k \mathcal{E}$ is a nef vector bundle on $\mathbb{P}^1$, and hence splits into a direct sum of $\mathcal{O}_{\mathbb{P}^1}(a_l)$ with $a_l \geq 0$. So the $H^1$ vanishing holds. $\square$
3.2. Cones over the Veronese surface. Cones over the Veronese surface are of minimal degree, i.e. they satisfy \( d = e + 1 \), according to a classification by de Pezzo-Bertini, cf. [5]. In this subsection, we will show that they have at worst rational singularities and obtain the vanishing

**Proposition 3.5.** Let \( X \subseteq \mathbb{P}^N \) be a cone over the Veronese surface. Then \( H^1(X, \mathcal{O}_X(k)) = 0 \) for every \( k \in \mathbb{Z} \).

**Notation 3.6.** Given a linear subspace \( \Lambda \subseteq \mathbb{P}^N \) of dimension \( l \geq 1 \), we denote by \( \mathbb{P}^N_\Lambda \subseteq \mathbb{P}^N \) a disjoint linear subspace with \( N = N - l - 1 \). Given a smooth subvariety \( Y \subseteq \mathbb{P}^N \) we denote by \( \mathbb{P}^N_\Lambda \) the conical scroll over \( Y \) with vertex \( \Lambda \).

There is a natural resolution of singularities of \( \mathbb{P}^N_\Lambda \) obtained by blowing up \( \mathbb{P}^N_\Lambda \) at the vertex \( \Lambda \). Indeed, let \( \mathcal{E} = \mathcal{O}^{(l+1)} \oplus \mathcal{O}(1) \) on \( \mathbb{P}^N \) and \( \mathcal{E}_Y := \mathcal{E}|_Y \). We have the following commutative diagram:

\[
\begin{array}{ccccccccc}
Y \times \Lambda^C & \xrightarrow{\pi_Y} & \mathbb{P}(\mathcal{E}_Y) & \xrightarrow{t} & \mathbb{P}^N \\
\downarrow & & \downarrow & & \downarrow \\
Y^C & \xrightarrow{\pi} & \mathbb{P}^N_\Lambda & \xrightarrow{p_1} & \mathbb{P}^N \\
\end{array}
\]

The scheme theoretic image of \( t \) is \( \mathbb{P}^N_\Lambda \). Moreover, we have the following diagram with Cartesian squares

\[
\begin{array}{cccccc}
F_x^C & \xrightarrow{\{x\}} & \Phi = Y \times \Lambda^C & \xrightarrow{\pi_Y} & \mathbb{P}(\mathcal{E}_Y) \\
\downarrow & & \downarrow & & \downarrow \\
\{x\} & \xrightarrow{\Lambda} & \mathbb{P}^N_\Lambda & \xrightarrow{t} & \mathbb{P}^N \\
\end{array}
\]

Clearly for each \( x \in \Lambda \), it holds that \( F_x \cong Y \). Since \( \Phi \) is a divisor in \( \mathbb{P}(\mathcal{E}_Y) \), we have

\[ \mathcal{O}_{\mathbb{P}(\mathcal{E}_Y)}(\Phi) \cong \pi_Y^* \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_Y)}(m) \]

for some line bundle \( \mathcal{L} \) on \( Y \) and some \( m \in \mathbb{Z} \). Pushing down the exact sequence

\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E}_Y)}(\Phi) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_Y)}(1) \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E}_Y)}(1) \rightarrow \mathcal{O}_{\Phi}(1) \rightarrow 0 \]

by \( \pi_Y \) yields the exact sequence

\[ 0 \rightarrow \mathcal{L} \otimes S^{m+1} \mathcal{E}_Y \rightarrow \mathcal{E}_Y \rightarrow \mathcal{O}_Y^{(l+1)} \]

Because of rank \( m = -1 \). So \( \mathcal{L} \cong \mathcal{O}_Y(1) \), and it follows that

\[ N^*_{\Phi/\mathcal{E}_Y}(\Phi) \cong \mathcal{O}_{\mathcal{E}_Y}(\Phi) \otimes \mathcal{O}_{\Phi} \cong \pi_Y^* \mathcal{O}_Y(1) \otimes \pi_\Lambda^* \mathcal{O}_\Lambda(-1). \]

Consider the exact sequence

\[ 0 \rightarrow N_{F_x/\Phi} \rightarrow N_{F_x/\mathcal{E}_Y} \rightarrow N_{\Phi/\mathcal{E}_Y}|_{F_x} \rightarrow 0 \]

(3.3) where by (3.3) \( N_{\Phi/\mathcal{E}_Y}|_{F_x} \cong \mathcal{O}_Y(-1) \), and since \( \pi_\Lambda \) is flat, we have an isomorphism \( N_{F_x/\Phi} \cong \oplus \mathcal{O}_{F_x} \).

**Lemma 3.7.** Suppose that \( H^1(Y, \mathcal{O}_Y(1)) = 0 \). Then we have an isomorphism

\[ N^*_{F_x/\mathcal{E}_Y} \cong \oplus \mathcal{O}_Y(1). \]

**Proof.** Since \( \text{Ext}^1_Y(N_{\Phi/\mathcal{E}_Y}|_{F_x}, N_{F_x/\Phi}) \cong \oplus \mathcal{O}_Y(1) \), the sequence (3.4) splits. So the assertion follows. \( \Box \)
Proposition 3.8. Suppose \( Y \subseteq \mathbb{P}^5 \) is the Veronese embedding \( \mathbb{P}^2 \to [\mathcal{O}(2)] \to \mathbb{P}^5 \). Then the cone \( \mathbb{F}^\Lambda_Y \) has rational singularities, where \( \Lambda \cong \mathbb{P}^{N-6} \).

Proof. To begin with, we shall show that \( \mathbb{F}^\Lambda_Y \) is normal. In view of the commutative diagram

\[
\begin{array}{c}
\mathcal{O}_{\mathbb{P}^N} \\
\downarrow \alpha \\
\mathcal{O}_{\mathbb{P}^N, x} \rightarrow t_* \mathcal{O}(\xi_Y)
\end{array}
\]

it suffices to show that the natural map \( \mathcal{O}_{\mathbb{P}^N} \to t_* \mathcal{O}(\xi_Y) \) is surjective. Let \( x \in \Lambda \) be a closed point and \( m \) the maximal ideal of the local ring \( \mathcal{O}_{\mathbb{P}^N, x} \). Let \( F \) denote the fibre of \( t \) over \( x \) and \( \mathcal{I} \) denote the ideal sheaf of \( F \) in \( \mathbb{P}(\xi_Y) \). Passing to the completions and using the theorem of formal functions [10, III. 11], we are reduced to show the surjectivity of the induced map

\[
\mathcal{O}_{\mathbb{P}^N, x}^\Lambda = \lim_n \mathcal{O}_{\mathbb{P}^N, x}/m^n \to \lim_n H^n(\mathcal{O}_{\mathcal{F}F}) \simeq (t_* \mathcal{O}(\xi_Y))_x^\Lambda.
\]

To this end, consider the commutative diagram with exact rows

\[
\begin{array}{c}
0 \to m^n/m^{n+1} \to \mathcal{O}_{\mathbb{P}^N, x}/m^{n+1} \to \mathcal{O}_{\mathbb{P}^N, x}/m^n \to 0 \\
\downarrow \alpha_n \downarrow \beta_{n+1} \downarrow \beta_n \\
0 \to H^0(\mathcal{I}/\mathcal{I}^{n+1}) \to H^0(\mathcal{O}_{(n+1)F}) \to H^0(\mathcal{O}_F) \to 0.
\end{array}
\]

Assuming that \( \alpha_n \) is surjective for all \( n \) for the moment, then by the snake Lemma and induction on \( n \), we deduce that \( \beta_n \) is surjective (the case \( n = 1 \) is straightforward), and hence the map between the completions surjects too.

When \( n = 1 \), the linear map \( \alpha_1 : T_{\mathbb{P}^N, x}^* \to H^0(N_{\mathcal{F}F/\mathbb{P}(\xi_Y)}^*) \) is injective. Since the spaces have the same dimension, \( \alpha_1 \) is an isomorphism. For \( n > 1 \), consider the commutative diagram

\[
\begin{array}{c}
S^n(m/m^2) \cong m^n/m^{n+1} \\
\downarrow \alpha_n \\
S^n(H^0(\mathcal{I}/\mathcal{I}^2)) \to H^0(\mathcal{I}/\mathcal{I}^{n+1})
\end{array}
\]

By Lemma 3.7 and the cohomology of \( \mathbb{P}^2 \), we deduce that the sheaf \( \mathcal{I}/\mathcal{I}^2 = N_{\mathcal{F}F/\mathbb{P}(\xi_Y)}^* \) is 0-regular. Therefore the bottom horizontal map is surjective, and hence the right vertical map \( \alpha_n \) is surjective. Thus we have proved that \( \mathbb{F}^\Lambda_Y \) is normal. The vanishing \( R^i t_* \mathcal{O}(\xi_Y) = 0 \) for \( i > 0 \) follows from [27, Thm 2.9]. Hence, we conclude that \( \mathbb{F}^\Lambda_Y \) has rational singularities.

Proof of Proposition 3.5. By Proposition 3.8, \( X \) is normal. So Lemma 3.2 implies that \( H^1(\mathcal{O}_X(k)) = 0 \) for \( k < 0 \). The \( H^1 \) vanishing for \( k \geq 0 \) is a direct consequence of the fact that \( \mathcal{O}_X \) is \( d - e = 1 \) regular.

Remark 3.9. With the fact \( X \) has rational singularities, one can easily obtain that \( H^i(\mathcal{O}_X(k)) = 0 \) for any \( 0 < i < \dim X \) and \( k \in \mathbb{Z} \) by Serre duality for rational singularities.

3.3. Birational type divisor of a rational scroll. In this subsection, we prove the main theorem in the case that \( X \) is a birational type divisor of a rational scroll \( \mathbb{F}^\Lambda_{\xi} \). It is worth to mention that only normality is needed for this case.
Proposition 3.10. Let $X$ be a non-degenerate normal projective variety of codimension $e$ and degree $d$. Assume that $X$ is a birational type divisor of a rational scroll $E^3_k$. Then $H^1(X, \mathcal{O}_X(k)) = 0$ for every $k \in \mathbb{Z}$.

Lemma 3.11. Using notation of Definition 1.5. Let $V \subseteq H^0(E^3_k, \mathcal{O}_{E^3_k}(1))$ be a base point free subsystem such that the induced morphism $\psi: E^3_k \rightarrow \mathbb{P}^N$ induces a birational morphism $\psi: E^3_k \rightarrow X$. Then the exceptional locus of $\psi$ is contained in $\Lambda$.

Proof. Consider a point $x \in X$ such that $\dim \psi^{-1}(x) \geq 1$. Since $\mathcal{O}_{E^3_k}(1)$ is $p$-very ample, the induced map $p^{-1}(x) \rightarrow C$ is bijective. Let $\Gamma = (p^{-1}(x))_{\text{red}}$. Then $\pi_\Gamma : \Gamma \rightarrow C$ is a birational morphism, in particular $C$ is isomorphic to the normalization of $\Gamma$.

The closed immersion $\sigma : \Gamma \hookrightarrow E^3_k$ is induced by a rank one quotient
\[ \mathcal{O}_E^\oplus(l+1) \oplus \pi_\Gamma^* \mathcal{E} \rightarrow \mathcal{L}. \]

Then $\deg(\mathcal{L}) = \sigma(\Gamma) \cdot c_1(\mathcal{O}_{E^3_k}(1)) = \sigma(\Gamma) \cdot \psi^* (c_1(\mathcal{O}_{\mathbb{P}^N}(1))) = 0$, because $\Gamma$ is contracted by $\psi$. Since $\mathcal{E}$ is ample, the pullback $\pi_\Gamma^* \mathcal{E}$ by a finite map is ample, and consequently any of its quotients is ample. In particular, there is no nonzero map $\pi_\Gamma^* \mathcal{E} \rightarrow \mathcal{L}$. Therefore $\mathcal{O}_E^\oplus(l+1) \oplus \pi_\Gamma^* \mathcal{E} \rightarrow \mathcal{L}$ factors through a quotient $\mathcal{O}_E^\oplus(l+1) \rightarrow \mathcal{L}$, which induces an embedding
\[ \Gamma = \mathbb{P}(\mathcal{L}) \hookrightarrow \mathbb{P} \left( \mathcal{O}_E^\oplus(l+1) \right) \rightarrow \tilde{\Lambda}. \]

This completes the proof. \[ \square \]

Proof of Proposition 3.10. By Lemma 3.11, the birational morphism $\tilde{X} \rightarrow X$ is isomorphic outside of $\Lambda$. So according to the local computation in [21, Proposition 5.2 (2)], the morphism $\tilde{X} \rightarrow X$ is finite. Since $X$ is normal, Zariski main theorem asserts that this is indeed an isomorphism. Therefore we may carry out the exactly same computations as in [20, p. 4620] to conclude the proof in the following.

Assume that $\tilde{X}$ is a divisor of type $(\mu, 1)$ with $\mu \geq 2$. In order to prove
\[ H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(k)) = 0 \]
by the exact sequence
\[ 0 \rightarrow \mathcal{O}_{E^3_k}(k - \mu) \otimes p^* \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{E^3_k}(k) \rightarrow \mathcal{O}_{\tilde{X}}(k) \rightarrow 0, \]
it suffices to show the vanishing
\[ H^1(\mathcal{O}_{E^3_k}(k)) \simeq 0, \quad H^2(\mathcal{O}_{E^3_k}(k - \mu) \otimes p^* \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq 0. \]
Since $\mathcal{O}_{E^3_k}(1)$ is nef and big, the left vanishing for $k < 0$ and the right vanishing for $k - \mu < 0$ follow from Kawamata-Viehweg vanishing. Note that for any $j \geq 0$, we have $R^i_p \mathcal{O}_{E^3_k}(j)$ for $i > 0$. By the Leray spectral sequence, we find that
\[ H^1(\mathcal{O}_{E^3_k}(k)) \simeq H^1(\mathbb{P}^1, \text{Sym}^k(\mathcal{O}_{\mathbb{P}^1}^{l+1} \oplus \mathcal{E})) = 0 \]
in the case that $k \geq 0$, and
\[ H^2(\mathcal{O}_{E^3_k}(k - \mu) \otimes p^* \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq H^2(\mathbb{P}^1, \text{Sym}^{k-\mu}(\mathcal{O}_{\mathbb{P}^1}^{l+1} \oplus \mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = 0 \]
in the case that $k - \mu \geq 0$. \[ \square \]
3.4. Birational type divisor of a scroll $\mathbb{F}_3^N$. In this subsection, we prove the main theorem in the most delicate case that $X$ is a birational type divisor of a scroll $\mathbb{F}_3^N$.

**Proposition 3.12.** Let $X \subseteq \mathbb{P}^N$ be a non-degenerate projective variety of dimension $n \geq 2$, codimension $e \geq 2$, and degree $d$. Assume that $X$ is a birational type divisor of a scroll $\mathbb{F}_3^N$ with $\Lambda \subseteq \mathbb{P}^N$ a linear subspace of dimension at most one, and that $X$ has normal, isolated $\mathbb{Q}$-Gorenstein singularities. Suppose that $X$ does not satisfy condition $(E_{c-1})$. Then $H^1(X, \mathcal{O}_X(k)) = 0$ for every $k \geq d - e - n$.

**Proof.** For each $i \geq 0$, we consider the $\mathbb{Q}$-Cartier divisor
\[ M_i = (d - e - n + i - 1)H - K_X \]
and its corresponding multiplier ideal $J_i = \mathcal{J}(X, \frac{1}{c} \cdot |cM_i|)$, where $c$ is a fixed positive integer such that $cK_X$ is Cartier. By Theorem 1.8 and Theorem 2.7, we know that the base scheme $b(|cM_i|)$ has support contained in a finite union of one-dimensional subspaces of $\mathbb{P}^N$ and isolated points. Moreover, one has the inclusions of ideals
\[ \mathcal{J}_i = \mathcal{J}(X, \frac{1}{c} \cdot |cM_i|) \supseteq \mathcal{J}(X, |cM_i|) \supseteq b(|cM_i|), \]
therefore the scheme $Z_i = V(\mathcal{J}_i)$ is also contained in a finite union of points and projective lines. By Theorem 1.17, applied to the Cartier divisor $(d - e - n + i)H$, we conclude that
\[ H^1(X, \mathcal{O}_X((d - e - n + i)H) \otimes \mathcal{J}_i) = 0. \]
Hence, in order to prove the vanishing
\[ H^1(X, \mathcal{O}_X((d - e - n + i)H)) = 0, \]
it suffices to show that
\[ H^1(Z_i, \mathcal{O}_{Z_i}((d - e - n + i)H)) = 0. \]
By the above considerations $\dim Z_i \leq 1$ for every $i \geq 0$. In what follows, we will denote $\mathcal{J} = \bigcap_{i=1}^{r} \mathcal{J}_i$ and $Z = Z_i$ in order to shorten notation. Thus, we need to prove that the following vanishing holds
\[ H^1(Z, \mathcal{O}_{Z_i}((d - e - n + i)H)) = 0. \]
Recall from Remark 1.13, that the multiplier ideal $\mathcal{J}$ is integrally closed, therefore it admits a minimal primary decomposition
\[ \mathcal{J} = \bigcap_{j=1}^{r} \mathcal{I}_j, \]
where $\mathcal{I}_j$ is a $p_j$-primary ideal and $p_j \neq p_j'$ for $j \neq j'$. We may assume that the prime ideals $p_j$, with $j \in \{1, \ldots, s\}$ define one-dimensional schemes and the prime ideals $p_j'$, with $j' \in \{s + 1, \ldots, r\}$, define zero-dimensional schemes. For $j \in \{1, \ldots, s\}$, we know that the ideal $p_j$ equals $\mathcal{I}_{\ell_j}$, where $\ell_j$ is a projective line. Observe that we have an exact sequence
\[ 0 \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{Z'} \oplus \mathcal{O}_{Z''} \rightarrow \mathcal{O}_{Z' \cap Z''} \rightarrow 0, \]
where $Z'$ is the scheme defined by $\bigcap_{j=1}^{s} \mathcal{I}_j$ and $Z''$ is the scheme defined by $\bigcap_{j'=s+1}^{r} \mathcal{I}_j'$. Tensor the above exact sequence with $\mathcal{O}_X((d - e - n + i)H)$ and take the induced long exact sequence of cohomology groups. Observe that being $\mathcal{O}_{Z'}$ zero-dimensional, we get the homomorphism
\[ H^0(Z'', \mathcal{O}_{Z''}((d - e - n + i)H)) \rightarrow H^0(Z' \cap Z'', \mathcal{O}_{Z' \cap Z''}((d - e - n + i)H)) \]
is surjective. Hence, in order to establish the vanishing (3.5) it suffices to prove that
\[ H^1(Z', \mathcal{O}_{Z'}((d - e - n + i)H)) = 0. \]
Replacing $Z$ by $Z'$, we may assume that $Z$ has no zero-dimensional embedded components. In what follows we will adopt the notation
\[ \mathcal{I}_{k_1, \ldots, k_s} = \mathcal{I}_{l_1}^{k_1} \cap \cdots \cap \mathcal{I}_{l_s}^{k_s}, \]
where $k_1, \ldots, k_s \geq 0$, and denote the corresponding subscheme by $Z_{k_1, \ldots, k_s}$. Observe that for each $j$ we can choose $r_j$ large enough so that $\mathcal{I}_{l_j}^{k_j} \subseteq \mathcal{I}_j$, hence we have that $\mathcal{I}_{r_1, \ldots, r_s} \subseteq \mathcal{I}$. We have an exact sequence of $\mathcal{O}_X$-modules
\[ 0 \to \mathcal{I}_{r_1, \ldots, r_s} / \mathcal{I} \to \mathcal{O}_Z \to \mathcal{O}_{Z_{r_1, \ldots, r_s}} \to 0. \]
Then it suffices to prove that
\[ H^1(X, \mathcal{O}_X((d - e - n + i)H) \otimes \mathcal{I}_{r_1, \ldots, r_s} / \mathcal{I}) = 0. \]
Moreover, from the exact sequence of $\mathcal{O}_{Z_{r_1, \ldots, r_s}}$-modules
\[ 0 \to \mathcal{I} / \mathcal{I}_{r_1, \ldots, r_s} \to \mathcal{I}_{r_1, \ldots, r_s} / \mathcal{I} \to \mathcal{I}_{r_1, \ldots, r_s} / \mathcal{I} \to 0, \]
we conclude that it is enough to show the vanishing
\[ H^1(X, \mathcal{O}_X((d - e - n + i)H) \otimes \mathcal{I}_{r_1, \ldots, r_s} / \mathcal{I}_{r_1, \ldots, r_s}) = 0. \]
Observe that we have a natural filtration of ideal sheaves
\[ \mathcal{I}_{1, \ldots, 1, 2} \supseteq \mathcal{I}_{1, \ldots, 2, r_2, \ldots, r_s} \supseteq \mathcal{I}_{r_1, 1, r_2, \ldots, r_s} \supseteq \mathcal{I}_{r_1, \ldots, r_s}. \]
Hence, we are reduced to proving that
\[ H^1(X, \mathcal{O}_X((d - e - n + i)H) \otimes \mathcal{I}_{1, \ldots, 1, 2} / \mathcal{I}_{1, \ldots, 1, 2}) = 0, \]
for arbitrary positive integers $k_1, \ldots, k_s$. Furthermore, observe that the sheaf
\[ \mathcal{I}_{k_1, \ldots, k_s} / \mathcal{I}_{k_1, \ldots, k_{j-1}, k_j + 1, k_{j+1}, \ldots, k_s} \]
of $\mathcal{O}_X$-modules is isomorphic to the sheaf $\mathcal{I}_{l_j}^{k_j} / \mathcal{I}_{l_j}^{k_j+1}$ as $\mathcal{O}_{\ell_j}$-modules. Thus we shall prove that
\[ H^1(\ell_j, \mathcal{O}_X((d - e - n + i)H) \otimes \mathcal{I}_{l_j}^{k_j} / \mathcal{I}_{l_j}^{k_j+1}) = 0, \]
for each $k \geq 1$ and $j \in \{1, \ldots, s\}$. Since $\ell_j$ is arbitrary, we denote it by $\ell$. Denote by $\tau_k$ the torsion subsheaf of the $\mathcal{O}_\ell$-module $\mathcal{I}_{l_j}^{k_j} / \mathcal{I}_{l_j}^{k_j+1}$ and by $\mathcal{E}_k$ the quotient by its torsion subsheaf.

**Claim 3.13.** For each $k \geq 1$, the coherent sheaf $\mathcal{E}_k$ is a nef vector bundle on $\ell$.

**Proof of Claim 3.13.** To begin with, we denote the coherent sheaf $\mathcal{E}_1$ by $\mathcal{E}$. We will first prove the case $k = 1$. Consider the commutative diagram
\[
\begin{array}{ccc}
P_l(\mathcal{E}) & \longrightarrow & E \\
\downarrow & & \downarrow \\
\hat{X} & \longrightarrow & \text{Bl}_l(\mathbb{P}_N) \xrightarrow{\pi} \mathbb{P}^N \\
\downarrow & & \downarrow \\
E & \longrightarrow & X \\
\end{array}
\]
where $\hat{X}$ is the blowup of $X$ along $\ell$, $E = \text{Proj}(\oplus_{j \geq 0} \mathcal{I}_j^{j+1})$ is the exceptional divisor, and $\pi$ is the projective bundle structure map of $\text{Bl}_l(\mathbb{P}_N)$ over $\mathbb{P}_N$. Let the composite map $P_l(\mathcal{E}) \to \mathbb{P}^N$ be $h$. Then
\[ \mathcal{O}_{P_l(\mathcal{E})}(1) \simeq h^*\mathcal{O}_{\mathbb{P}_N}(1), \]
which is nef. Therefore $\mathcal{E}$ is a nef bundle on $\ell$. For $k \geq 2$, observe that $\mathcal{E}_k \simeq S^k(\mathcal{E})$ is a nef vector bundle on $\ell$ as well. \(\square\)
Now, being $X$ smooth at the generic point of $\ell$, the support of the torsion part of $I_{k}/I_{k+1}$ is zero-dimensional for every $k \geq 1$, so we have that
\[ H^1(\ell, O_{\ell}(d - e - n + i)) \otimes I_{k}/I_{k+1} \simeq H^1(\ell, O_{\ell}(d - e - n + i)) \otimes E_k. \]
If $d - e - n + i \geq 0$, then we obtain $H^1(\ell, O_{\ell}(d - e - n + i)) \otimes E_k) = 0$.

It only remains to consider the case that $d - e - n + i < 0$. In this case, by Lemma 3.2, we have more generally that $H^1(X, O_X(l)) = 0$ for any integer $l < 0$. This completes the proof. \qed

Theorem 3.1 now follows from Corollary 1.9, and Propositions 3.3, 3.5, 3.10, and 3.12.

4. Classification of the extremal cases: Proof of Theorem 2

In this section, we prove Theorem 2, which characterizes projective varieties in Theorem 1 with the maximal and the next to maximal $O_X$-regularity.

Proof of Theorem 2. Let $X \subseteq \mathbb{P}^N$ be a non-degenerate projective variety of dimension $n$, codimension $e$, and degree $d$ with normal isolated $\mathbb{Q}$-Gorenstein singularities, and $H$ be its hyperplane section. Since the theorem is known for smooth varieties by [13, Theorem B], we may assume that $X$ is a singular variety. We can also assume that $n, e \geq 2$. Let $S \subseteq \mathbb{P}^{2+e}$ and $C \subseteq \mathbb{P}^{1+e}$ be a general surface section and a general curve section, respectively, and $g$ be the genus of $C$.

(1) We first prove the ‘if’ part. If $d = e + 1$, by Theorem 1, $\text{reg}(O_X) \leq d - e = 1$. Since it is well known that $\text{reg}(O_Z) \geq 1$ for any variety $Z$, we obtain $\text{reg}(O_X) = 1 = d - e$. If $d = e + 2$, by Theorem 1, $\text{reg}(O_X) \leq d - e = 2$. Suppose that $\text{reg}(O_X) = 1$. Since $X \subseteq \mathbb{P}^N$ is linearly normal, it follows that $\text{reg}(X) = 2$, which implies that $\text{reg}(C) = 2$. Then $C \subseteq \mathbb{P}^{1+e}$ is a rational normal curve so that $d = e + 1$, which is a contradiction. Thus $\text{reg}(O_X) = 2 = d - e$.

Now, we prove the ‘only if’ part. Suppose that $\text{reg}(O_X) = d - e$. Then $X \subseteq \mathbb{P}^N$ must be linearly normal since if not, then it is obtained by an isomorphic projection of $X \subseteq \mathbb{P}^{N+1}$ so that $\text{reg}(O_X) \leq d - (e + 1) = d - e - 1$. By Theorem 3.1, we know that
\[ H^1(X, O_X(d - e - 1 - i)) = 0 \]
for $1 \leq i \leq n - 1$. It then follows that
\[ H^n(X, O_X(d - e - 1 - n)) \neq 0. \]

By considering general hyperplane sections successively, we see that $H^1(C, O_C(d - e - 2)) \neq 0$ (see [13, Proof of Theorem B]), and hence, by [13, Theorem B], we have that $d \leq e + 2$. This completes the proof for (1).

(2) As before, we first prove the ‘if’ part. It is enough to consider the case that $d = e + 3$ and $X \subseteq \mathbb{P}^N$ is linearly normal. By (1), we know that $\text{reg}(O_X) \leq d - e - 1 = 2$. If $\text{reg}(O_X) = 1$, then $\text{reg}(X) = 2$ so that $\text{reg}(C) = 2$. As in (1), we then obtain $d = e + 1$, which is a contradiction. Thus we have that $\text{reg}(O_X) = 2 = d - e - 1$.

Now, we prove the ‘only if’ part. Suppose that $\text{reg}(O_X) = d - e - 1$. By (1), we may assume that $X \subseteq \mathbb{P}^N$ is linearly normal and $d \geq e + 3$. By Theorem 3.1, we know that
\[ H^1(X, O_X(d - e - 2 - i)) = 0 \]
for $1 \leq i \leq n - 2$ under the assumption that $n \geq 3$. It then follows that
\[ H^n(X, O_X(d - e - 2 - n)) \neq 0 \quad \text{or} \quad H^{n-1}(X, O_X(d - e - 1 - n)) \neq 0. \]

By considering general hyperplane section successively (see [13, Proof of Theorem B]), we see that
\[ H^2(S, O_S(d - e - 4)) \neq 0 \quad \text{or} \quad H^1(S, O_S(d - e - 3)) \neq 0. \]
In particular, we have that \( \text{reg}(\mathcal{O}_S) = d - e - 1 \). If \( n \geq 3 \), then \( S \) is smooth. In this case, the assertion follows from [13, Theorem B]. Thus we suppose that \( n = 2 \) and \( X = S \) is a normal singular surface. Keep in mind that \( S \subseteq \mathbb{P}^{2+e} \) is linearly normal. If \( H^2(S, \mathcal{O}_S(d - e - 4)) \neq 0 \), then \( H^1(C, \mathcal{O}_S(d - e - 3)) \neq 0 \). In this case, by [13, Theorem B], either \( d = e + 3 \) or \( C \subseteq \mathbb{P}^{1+e} \) is a complete intersection of type \((2,3)\). Thus the assertion holds. It only remains to consider the case that

\[
H^2(S, \mathcal{O}_S(d - e - 4)) = 0 \quad \text{and} \quad H^1(S, \mathcal{O}_S(d - e - 3)) \neq 0.
\]

We may assume that \( S \) does not satisfy condition \((E_{e-1})\) because if not, then \( H^1(S, \mathcal{O}_S(k)) = 0 \) for any \( k \in \mathbb{Z} \) by Propositions 3.3, 3.5, 3.10. In particular, \( S \) is not projectively equivalent to a scroll over a smooth projective curve.

Note that it is enough to show that \( d \leq e + 3 \). To derive a contradiction, suppose that \( d \geq e + 4 \). Suppose furthermore that \( g \leq 1 \). Since \( H^1(S, \mathcal{O}_S(-1)) = 0 \) by Kodaira vanishing, it follows that

\[
h^1(S, \mathcal{O}_S) \leq h^1(C, \mathcal{O}_C) \leq 1.
\]

We then have that

\[
d - g + 1 = h^0(C, \mathcal{O}_C(1)) \leq h^0(S, \mathcal{O}_S(1)) - 1 + h^1(S, \mathcal{O}_S) \leq e + 3
\]

so that

\[
d \leq e + 2 + g \leq e + 3,
\]

which is a contradiction. Thus we should have that \( g \geq 2 \).

Recall that \( H^1(S, \mathcal{O}_S(d - e - 2)) = 0 \) and \( H^1(S, \mathcal{O}_S(d - e - 3)) \neq 0 \). Thus the natural restriction map

\[
H^0(S, \mathcal{O}_S(d - e - 2)) \to H^0(C, \mathcal{O}_C(d - e - 2))
\]

is not surjective. In particular, \( C \subseteq \mathbb{P}^{1+e} \) is not \((d - e - 2)\)-normal. First, consider the case that \( e \geq 3 \). By applying [19, Theorem 1] for \( l = 2 \), we see that \( C \subseteq \mathbb{P}^{1+e} \) is \((d - e - 2)\)-normal, which is a contradiction. Next, consider the only remaining case that \( e = 2 \), i.e., we have a linearly normal surface \( S \subseteq \mathbb{P}^4 \) of degree \( d \geq 6 \) such that \( \text{reg}(\mathcal{O}_S) = d - 3 \) and \( H^1(S, \mathcal{O}_S(d - 5)) \neq 0 \). Recall that \( C \subseteq \mathbb{P}^3 \) is not \((d - 4)\)-normal and \( d \geq 6, g \geq 2 \). By [3, Théorème 0.1], \( C \subseteq \mathbb{P}^3 \) admits a \((d - 2)\)-secant line \( \ell \). By taking the projection of \( C \subseteq \mathbb{P}^3 \) centered at \( \ell \), we see that \( C \) is a hyperelliptic curve. In particular, we have \( H^1(C, \mathcal{O}_C(k)) = 0 \) for \( k \geq 1 \), and \( d \geq g + 3 \) by Riemann-Roch formula.

Let \( \pi : S' \to S \) be the minimal resolution, and \( H' := \pi^*H \). Then \((S', H')\) is a generically polarized smooth surface which is a-minimal in the sense of [2]. Since we assume that \( S \) is not a scroll over a curve and \( H^1(S, \mathcal{O}_S(d - 5)) \neq 0 \), it follows from [2, Theorem 2.5] that \( K_{S'} + H' \) is nef and big. Furthermore, by [2, Theorem 2.7], \( K_{S'} + H' \) is nef and big. Recall that \( H^1(C, \mathcal{O}_C(k)) = 0 \) for \( k \geq 1 \). Thus

\[
H^2(S, \mathcal{O}(k)) = 0
\]

for \( k \geq 0 \). This implies that

\[
H^2(S', \mathcal{O}_{S'}(k)) = 0
\]

for \( k \geq 0 \). In particular, \( p_g(S') = 0 \). If \( d \geq 10 \), then \( |K_S + H| \) gives a birational map by [1, Theorem 1.1]. However, since \( (K_S + H)|_C = K_C \) and \( |K_C| \) gives a 2-to-1 map, we get a contradiction. Thus \( d \leq 9 \), and \( g \leq d - 3 \leq 6 \). By [1, Corollary 2.5], \( h^1(S', \mathcal{O}_{S'}(2)) \leq 1 \), so we obtain \( h^1(S, \mathcal{O}_S) \leq 1 \). Suppose that \( H^1(S, \mathcal{O}_S) = 0 \). Then \( H^1(S, \mathcal{O}_S(1)) = 0 \) since \( H^1(C, \mathcal{O}_C(1)) = 0 \). Thus

\[
d - 3 = d - e - 1 = \text{reg}(\mathcal{O}_S) \leq 2,
\]

so \( d \leq 5 \), which is a contradiction. This means that

\[
h^1(S, \mathcal{O}_S) = h^1(S', \mathcal{O}_{S'}) = 1.
\]
If $C \subseteq \mathbb{P}^3$ is not linearly normal, then $H^1(S, \mathcal{O}_S(1)) = 0$ so that
d $3 = \text{reg}(\mathcal{O}_S) \leq 2$,
which is a contradiction. Thus $C \subseteq \mathbb{P}^3$ is linearly normal. Since the natural restriction map
$H^0(S, \mathcal{O}_S(d - 4)) \to H^0(C, \mathcal{O}_C(d - 4))$
is not surjective and $H^1(C, \mathcal{O}_C(k)) = 0$ for $k \geq 1$, it follows that $h^1(S, \mathcal{O}_S(l)) = 1$
for $0 \leq l \leq d - 5$ and $h^1(S, \mathcal{O}_S(l')) = 0$ for $l' \leq d - 4$. By the same reasoning, we also see that
$h^1(S', \mathcal{O}_{S'}((d'H')) = 1$ for $0 \leq l \leq d - 5$ and $h^1(S', \mathcal{O}_{S'}(l'H')) = 0$ for $l' \leq d - 4$. It follows that $S$
has only rational singularities.
Now, recall that $S$ does not satisfy condition $(E_{-1})$. Thus the base locus of the linear system $| - K_S +$
$(d - 5)H|$ is contained in $\tilde{C}(S) \cup \text{Sing}(S)$ by Theorem 2.7. We know that $\dim \tilde{C}(S) \leq 1$. Suppose first that
$\dim \tilde{C}(S) = 1$. Then $S$ is the birational image of an effective divisor $D \in |\mathcal{O}_E(\mu) \otimes \pi^*L|$ on
a projectivized bundle $\mathbb{P}(E)$ for some $\mu \geq 2$, where $E = \mathcal{O}_{\mathbb{P}^2} \oplus L'$ is a rank 3 vector bundle on
a smooth projective curve $Y$ with $\mathcal{L}, \mathcal{L}' \in \text{Pic}(Y)$ and $\tau : \mathbb{P}(E) \to Y$ is the natural projection. Recall that the birational image of $\mathbb{P}(E)$
given by $|\mathcal{O}_E(\mu)|$ is a conical scroll with vertex $\Lambda$, where $\Lambda$ is a projective line and an irreducible component
of $\tilde{C}(S)$, and both $\mathcal{L}$ and $\mathcal{L}'$ are ample. Notice that $\mathcal{O}_E(\mu)$ is nef and big. By Kawamata-Viehweg vanishing,
$H^1(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(-D)) = 0$ and $H^1(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(-\mu + 1) \otimes \pi^*(-\mathcal{L})) = 0$
for $i = 1, 2$, so we obtain
$H^1(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}) = H^1(D, \mathcal{O}_D)$ and $H^1(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)) = H^1(D, \mathcal{O}_D(\mathcal{O}_{\mathbb{P}(E)}(1)|_D))$.
Since $S$ has only rational singularities, it follows that
$h^1(Y, \mathcal{O}_Y) = h^1(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}) = h^1(D, \mathcal{O}_D) = h^1(S, \mathcal{O}_S) = 1$
and
$2 = h^1(Y, \mathcal{E}) = h^1(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)) = h^1(D, \mathcal{O}_D(\mathcal{O}_{\mathbb{P}(E)}(1)|_D)) = h^1(S, \mathcal{O}_S(1)) = 1$,
which is a contradiction. Thus $\dim \tilde{C}(S) = 0$, so $-K_S + (d - 5)H$ is nef. Since $K_S + H'$ is nef and
$\pi^*(K_S + H) = K_{S'} + H' + E'$, where $E'$ is an effective $\pi$-exceptional divisor, it follows that $\pi^*K_S + H' =$
$\pi^*(K_S + H)$ is also nef. Now let $F = (d - 4)H' + G = \pi^*K_S + H'$. Both are nef $\mathbb{Q}$-divisors on $S'$. Note
that $F^2 > 2FG$ is equivalent to
$$(d - 4)d > 2g - 4.$$
Recall that $d = g + 3 \geq 6$. Thus the inequality is equivalent to
$$(g - 1)^2 > 0,$$
which is true since $g \geq 3$. Thus we have that $F^2 > 2FG$. By [15, Theorem 2.2.15] we have that $F - G =$
$\pi^*(-K_S + (d - 5)H)$ is nef and big, and so is $-K_S + (d - 5)H$. By Kawamata-Viehweg vanishing, we obtain
$H^1(S, \mathcal{O}_S(d - 5)) = 0$,
which is a contradiction. Therefore, we finish the proof. \qed

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