Symmetry analysis and exact solutions of modified Brans-Dicke cosmological equations

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Abstract

We perform a symmetry analysis of modified Brans-Dicke cosmological equations and present exact solutions. We discuss how the solutions may help to build models of cosmology where, for the early universe, the expansion is linear and the equation of state just changes the expansion velocity but not the linearity. For the late universe the expansion is exponential and the effect of the equation of state on the rate of expansion is just to change the constant Hubble parameter.

1 Introduction

The standard model of cosmology [1] has undergone several modifications in the past twenty years. With the discovery [2] of the cosmic microwave background radiation the early cosmological models [3, 4] developed into the first standard model which had the early radiation dominated and the late dust dominated stages. Observational and theoretical consistency has forced modifications on the standard model. We now believe that the universe is born in a radiation dominated stage, then undergoes an inflationary exponentially expanding stage, then becomes radiation dominated again, and then continues with matter domination, which by today has evolved into a dark energy dominated, slowly but exponentially expanding, stage. The main reason for such a complicated history of domination is that according to Einstein cosmological field equations the rate of expansion of the universe depends on the equation of state of the matter-energy that fills it. One immediate question which arises is whether there is any consistent modification of Einstein’s
equations such that the expansion of the universe is independent of its content. In this paper we would like to study such a model. The model consists of a modified Brans-Dicke-Jordan-Thirry [5] model where the signs of the kinetic and potential terms of the "scalar field" are negative when written to the right hand (wooden) side of the Einstein equations but is positive just as the geometric term containing the square of the time derivative of the scale size of the universe when written to the left hand (marble side). Thus the Friedman equation

\[ 3 \left( \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) = 8\pi G \rho \]  

is changed into

\[ 3 \left( \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) + 2\omega \frac{\dot{\phi}^2}{\phi^2} + \cdots = \frac{4\omega}{\phi^2} \rho \]  

where \( \phi \) is a new geometric field inspired by the Brans-Dicke-Jordan-Thirry model. The reason we call \( \phi \) a geometric field is that its appearance is quite similar to the scale size \( a \) of the universe. The terms \( \cdots \) will be determined by the covariant action. In fact, the analogous equation for a Kaluza-Klein theory with \( n \) internal dimensions is [6]

\[ 3 \left( \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) + \frac{1}{2} n(n-1) \frac{\dot{b}^2}{b^2} + \cdots = 8\pi G \rho \]  

where \( b \) is the scale size of the internal manifold.

In section 2 we present the basic equations of the model as applied to cosmology. In section 3 we perform a symmetry analysis based on dilatational symmetries and show that symmetries exist under the conditions that the mass of the "scalar field" vanishes and/or the curvature of spacelike sections is zero. The solutions are analyzed in sections 4, 5 and 6.

In section 7 we conclude with a model of expansion for the universe where, in the early stages, the mass term of the "scalar field" can be neglected (\( m = 0 \)) but the curvature term is important. For closed (\( k = 1 \)) spacelike sections the universe expands linearly. This linear expansion is independent of the equation of state. However, the rate of change of the newtonian gravitational constant depends on the equation of state. This is in a strong contrast to standard general relativistic cosmology where the rate of expansion strongly depends on the equation of state.

Whereas the curvature term is important in the early stages, we argue that in later stages, after the universe has expanded to a large size, it can be
neglected. The mass term then causes a slow exponential expansion which can be identified with the dark energy phenomenon of the present day universe. This phenomenon is also independent of the equation of state which only slightly influences the exponential rate of expansion.

2 Basic equations of the model

The action is the following:

\[
S = \int d^4x \sqrt{g} \left[ -\frac{1}{8\omega} \phi^2 R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 + L_M \right], \tag{2.1}
\]

where \( \phi \) represents the Brans-Dicke scalar field, \( \omega \) denotes the dimensionless Brans-Dicke parameter taken to be much larger than 1, \( \omega \gg 1 \). The scalar field has a kinetic term \(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \) with the wrong sign and the potential of the scalar field contains only a mass term with the wrong sign \( \frac{1}{2} m^2 \phi^2 \). \( L_M \), on the other hand, is the matter Lagrangian such that the scalar field \( \phi \) does not couple with it. \( R \) is the Ricci scalar. For simplicity we also restrict our analysis to the Robertson Walker metric to emphasize that \( \phi \) is necessarily spatially homogeneous:

\[
ds^2 = dt^2 - a^2(t) \frac{d\vec{x}^2}{1 + k \frac{\vec{x}^2}{4}}, \tag{2.2}
\]

where \( k \) is the curvature parameter with \( k = -1, 0, 1 \) corresponding to open, flat, closed universes respectively and \( a(t) \) is the scale factor of the universe. After applying the variational procedure to the action and assuming \( \phi = \phi(t) \) and energy momentum tensor of matter and radiation excluding \( \phi \) is in the perfect fluid form of \( T^\mu_\nu = \text{diag} (\rho, -p, -p, -p) \) where \( \rho \) is the energy density and \( p \) is the pressure and also noting that the right hand side of the \( \phi \) equation must be zero in accordance with our previous argument on \( L_M \) being independent of \( \phi \), the field equations reduce to (dots denote \( \frac{d}{dt} \))

\[
\frac{3}{4\omega} \phi^2 \left( \frac{\ddot{a}}{a^2} + \frac{k}{a^2} \right) + \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m^2 \phi^2 + \frac{3}{2\omega} \dot{\phi} \ddot{a} = \rho \tag{2.3}
\]

\[
\frac{1}{4\omega} \phi^2 \left( 2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) - \frac{1}{\omega a} \ddot{\phi} - \frac{1}{2\omega} \dot{\phi} \ddot{a} + \left( \frac{1}{2} - \frac{1}{2\omega} \right) \dot{\phi}^2 - \frac{1}{2} m^2 \phi^2 = p \tag{2.4}
\]
\[
\ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} + \left[ m^2 + \frac{3}{2 \omega} \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \phi = 0. \tag{2.5}
\]

The following identity shows that the continuity equation could be considered as a consequence of these three equations:

\[- \left[ \frac{d}{dt} (2.3) + 3 \frac{\dot{a}}{a} \times ((2.3) + (2.4)) - \dot{\phi} \times (2.5) \right] \equiv \dot{\rho} + 3 \frac{\dot{a}}{a} (p + \rho) = 0. \tag{2.6}\]

Instead, we prefer to consider more complicated pressure equation (2.4) as an algebraic consequence of \(\frac{d}{dt} (2.3)\), (2.5), (2.3) and the continuity equation in (2.6).

We assume the power law for the density

\[ \rho = C a^\nu \implies \dot{\rho} = \nu \rho H \tag{2.7} \]

which implies that the continuity equation (2.6) becomes the equation of state:

\[ p = -\left( \frac{\nu + 3}{3} \right) \rho, \tag{2.8} \]

so that for dust we have \(\nu = -3, p = 0\) and for radiation \(\nu = -4, p = \rho/3\).

Now define new unknowns \(F(a)\) and \(H(a)\) by the formulas

\[ F = \frac{\dot{\phi}}{\phi}, \quad H = \frac{\dot{a}}{a} \implies \frac{\ddot{a}}{a} = aHH' + H^2, \quad \frac{\ddot{\phi}}{\phi} = aHF' + F^2, \tag{2.9} \]

where prime denotes derivative of a function of \(a\) and the relations \(\dot{H} = H'(a) \dot{a} = aHH', \quad F = F'(a) \dot{a} = aHF'\) were used. Equations (2.3)–(2.5) become the equations for unknowns \(H(a)\) and \(F(a)\)

\[
H^2 + 2HF + \frac{2\omega}{3} (F^2 + m^2) + \frac{k}{a^2} = \left( \frac{4\omega}{3} \right) \frac{\rho}{\phi^2} \equiv \left( \frac{4\omega}{3} \right) \frac{Ca^\nu}{\phi^2} \tag{2.10}
\]

\[
\frac{2}{3} aH (H' + F') + H^2 + \frac{4}{3} HF + \frac{2}{3} (2 - \omega) F^2 + \frac{k}{3a^2} + \frac{2\omega}{3} m^2
\]

\[
= \left( - \frac{4\omega}{3} \right) \frac{p}{\phi^2}, \tag{2.11}
\]

\[
aH \left( \frac{1}{2} H' + \frac{\omega}{3} F' \right) + H^2 + \omega HF + \frac{\omega}{3} (F^2 + m^2) + \frac{k}{2a^2} = 0. \tag{2.12}
\]
The more lengthy equation (2.11) is still an algebraic consequence of \(\frac{d(2.10)}{dt}\), (2.12), (2.10) and the continuity equation in (2.6) due to the linear relation:

\[
\frac{d(2.10)}{dt} + 2F \times (2.10) + 3H \times ((2.10) - (2.11)) + 4F \times (2.12) = 0,
\]

so we choose (2.10) and (2.12) together with the state equation (2.8) as independent dynamic equations. However, these equations do not form a closed system for determining \(H(a)\) and \(F(a)\) because of \(\phi\) in the right-hand side of (2.10). To eliminate \(\phi\) from the \(\rho\)-equation (2.10), we differentiate (2.10) with respect to time, use

\[
\dot{a} = aH, \quad \dot{\phi} = F\phi, \quad \dot{H} = aHH'(a), \quad \dot{F} = aHF'(a)
\]

and eliminate \(\phi\) with the aid of (2.10) with the result

\[
(H^2 + HF) H' + \left( H^2 + \frac{2}{3} \omega HF \right) F' = \frac{\nu}{2a} H^3 + \frac{(\nu - 1)}{a} H^2 F
\]

\[
+ \frac{(\nu \omega - 6)}{3a} HF^2 - \frac{2\omega}{3a} F^3 + \frac{k}{a^3} \left[ \frac{(\nu + 2)}{2} H - F \right] + \frac{m^2 \omega}{3a} (\nu H - 2F).
\]

Solving algebraically equations (2.13) and (2.12) with respect to \(H'\) and \(F'\), we obtain the closed system of two first order equations in normal form that determines two unknown functions \(H(a)\) and \(F(a)\):

\[
(2\omega - 3)aH \frac{dH}{da} = (\nu \omega + 6)H^2 + 2(\nu + 4)\omega HF + \frac{2\omega}{3} [(\nu + 6)\omega - 3] F^2
\]

\[
+ \frac{2\omega}{3} (\nu \omega + 3)m^2 + \frac{k}{a^2} [ (\nu + 2) \omega + 3],
\]

\[
-(2\omega - 3)aH \frac{dF}{da} = \frac{3}{2} (\nu + 4)H^2 + 3(2\omega + \nu + 1)HF + [(\nu + 8)\omega - 6] F^2
\]

\[
+ (\nu + 2)\omega m^2 + \frac{3k(\nu + 4)}{2a^2}.
\]

Equation (2.10) serves as initial condition for this dynamical system.

3 Symmetry group analysis of basic dynamical equations

We will use classical Lie group analysis [7] for symmetries of the system (2.14), (2.15) as a tool for finding exact solutions of these equations. Since
the equations are of first order, it is impossible to find all Lie groups of point symmetries that are admitted by this system (a symmetry condition cannot be split in the derivatives of unknowns to generate an overdetermined system, that can be solved and yield all the point symmetries, because these derivatives are expressed from (2.14) and (2.15)). Therefore, we make an ansatz for the form of symmetries that could be admitted by these equations under certain conditions. We note that equations (2.14) and (2.15)) are very close to those ones that admit scaling (or dilatational) symmetries and an obstacle to this is the mass term and/or curvature term (the one with \( k \)) in each equation. Indeed, a scaling symmetry group of transformations of independent and dependent variables is defined by

\[
\tilde{a} = \lambda^\alpha a, \quad \tilde{F} = \lambda^\beta F, \quad \tilde{H} = \lambda^\gamma H.
\]  

(3.1)

It is generated by the infinitesimal generator

\[
X = \alpha a \frac{\partial}{\partial a} + \beta F \frac{\partial}{\partial F} + \gamma H \frac{\partial}{\partial H}.
\]  

(3.2)

Under transformations (3.1), equations (2.14) and (2.15)) become

\[
(2\omega - 3) a H \frac{dH}{da} = (\nu \omega + 6) H^2 + \lambda^{\beta - \gamma} 2(\nu + 4) \omega F
\]

\[
+ \lambda^{2(\beta - \gamma)} \frac{2\omega}{3} [(\nu + 6) \omega - 3] F^2 + \lambda^{-2\gamma} \frac{2\omega}{3} (\nu \omega + 3) m^2
\]

\[
+ \lambda^{-2(\alpha + \gamma)} \frac{k}{a^2} [(\nu + 2) \omega + 3],
\]

\[-(2\omega - 3) a H \frac{dF}{da} = \lambda^{\gamma - \beta} \frac{3}{2} (\nu + 4) H^2 + 3(2\omega + \nu + 1) F
\]

\[
+ \lambda^{-\gamma} [(\nu + 8) \omega - 6] F^2 + \lambda^{-(\beta + \gamma)} (\nu + 2) \omega m^2
\]

\[
+ \lambda^{-(2\alpha + \beta + \gamma)} \frac{3k(\nu + 4)}{2a^2}.
\]  

(3.3)

The condition of invariance of equations (2.14) and (2.15)) under the Lie group of transformations (3.1) is that the transformed equations (3.3) and (3.4) should coincide with equations (2.14) and (2.15)), that is, all the \( \lambda \)-dependence should vanish.

The first obvious condition is

\[
\gamma = \beta,
\]  

(3.5)
because it alone eliminates $\lambda$ in three terms, so that $\lambda$ will be present only in the terms with $k$ and $m$. We have to consider several cases of constraints on $k$ and $m$ for vanishing the remaining dependence on $\lambda$.

1. **Case 1 (generic).**

   $\alpha + \beta = 0$ and $\alpha = 0$, so that $\alpha = \beta = \gamma = 0$ and hence the symmetry generator $X = 0$ in (3.2). There are no symmetries in the generic case.

2. **Case 2.**

   $k = 0$, arbitrary $m$. Then $\beta = \gamma = 0$ and $\alpha$ is arbitrary, so it can be set to 1. The symmetry (3.2) becomes

   $$X = a \frac{\partial}{\partial a}.$$  

3. **Case 3.**

   $m = 0$, arbitrary $k$. Then $\alpha + \beta = 0$ and, since $\gamma = \beta$, we can set $\beta = \gamma = 1$ and $\alpha = -1$ in (3.2), so that the symmetry becomes

   $$X = H \frac{\partial}{\partial H} + F \frac{\partial}{\partial F} - a \frac{\partial}{\partial a}.$$  

4. **Case 4.**

   $k = m = 0$. Then $\beta = \gamma$ is the only condition and we have two inequivalent choices:

   (a) $\beta = \gamma = 0$, $\alpha = 1$ in (3.2) with the resulting symmetry

   $$X_1 = a \frac{\partial}{\partial a},$$  

   (b) $\beta = \gamma = 1$, $\alpha = 0$ and the symmetry is

   $$X_2 = H \frac{\partial}{\partial H} + F \frac{\partial}{\partial F}.$$  

To summarize, we have only one symmetry in cases 2 and 3 and two symmetries in case 4. One symmetry is not enough for integrating a system of two first order equations, so in cases 2 and 3 we can only find invariant solutions [7], particular solutions of our equations. In case 4 with two symmetries we can find all solutions of our equations, that is, integrate our system in quadratures.
4 Case of flat spacelike sections: $k = 0$

In this case we have only one symmetry (3.6) with the basis of invariants \{H, F\}. The equation expressed only in terms of invariants with solutions of the form $F = F(H)$ can be obtained by dividing one of equations (2.14), (2.15) over the other one, with $k = 0$, with the result of the form $dF/dH = G(H, F)$ which is a first order equation with no more known symmetries for $m \neq 0$ and hence it cannot be integrated. Though we cannot find the general solution in case 2, we still can search for invariant solutions with respect to the symmetry (3.6), which satisfy the conditions

$$\frac{dH}{da} = 0, \quad \frac{dF}{da} = 0 \quad (4.1)$$

and produce a "static" solution \{H, F\} = constant. These H and F are roots of the algebraic equations

$$(\nu \omega + 6)H^2 + 2(\nu + 4) \omega HF + \frac{2 \omega}{3} [(\nu + 6) \omega - 3] F^2 + \frac{2 \omega}{3} (\nu \omega + 3)m^2 = 0, \quad (4.2)$$

$$\frac{3}{2} (\nu + 4)H^2 + 3(2 \omega + \nu + 1)HF + [(\nu + 8) \omega - 6]F^2 + (\nu + 2) \omega m^2 = 0.$$

There are four roots of this system of two quadratic equations. Two of them are real:

$$H = \pm \frac{2m \sqrt{\omega}}{\sqrt{-\omega(\nu^2 + 6 \nu) - 12}}, \quad F = \pm \frac{m \nu \sqrt{\omega}}{\sqrt{-\omega(\nu^2 + 6 \nu) - 12}} \implies F = \frac{\nu}{2} H, \quad (4.3)$$

where the expression under the square root in the denominators is positive for $-6 \leq \nu_2 < \nu < \nu_1 \leq 0$ with the roots of the denominator $\nu_{1,2} = 3 \left( -1 \pm \sqrt{1 - 4/(3 \omega)} \right)$ and $\omega > 10^4$. The latter range of values of $\nu$ includes the interesting values $\nu_{dust} = -3$ and $\nu_{radiation} = -4$. For these $\nu$ and $\omega$, two other roots of (4.2) are imaginary and will not be considered

$$H = \pm \frac{2im(\omega - 1) \sqrt{\omega}}{\sqrt{6\omega^2 - 17 \omega + 12}}, \quad F = \mp \frac{im \sqrt{\omega}}{\sqrt{6\omega^2 - 17 \omega + 12}}, \quad (4.4)$$

where the roots of the denominator are $\omega_1 = 3/2$ and $\omega_2 = 4/3$, so that $6\omega^2 - 17 \omega + 12 > 0$. 
The equations that define $H$ and $F$

\[ \frac{\dot{a}}{a} = H = \text{constant}, \quad \frac{\dot{\phi}}{\phi} = F = \frac{\nu}{2} H \]

are integrated to give time dependence of the solution

\[ a = a_0 e^{Ht}, \quad \phi = \phi_0 e^{\frac{\nu}{2} Ht}. \]  

(4.5)

The solution (4.5) satisfies the initial condition (2.10) if arbitrary constants of integration $\phi_0$ and $a_0$ are related by the formula

\[ \phi_0 = \pm a_0^{\nu/2} m \sqrt{\frac{C}{3}} \sqrt{\frac{(\nu^2 + 6\nu)\omega + 12}{\nu(\omega - 1) + 1}}, \]  

(4.6)

where $C$ is the coefficient of power law for the density (2.7).

5 **Massless case: $m = 0$**

In this case we also have only one symmetry (3.7) with the basis of invariants \{\( \varphi = aF, \psi = aH \) \} (indeed, \( X\varphi = 0, X\psi = 0 \)). Substituting \( F = \varphi/a \) and \( H = \psi/a \) into (2.14) and (2.15) with \( m = 0 \), we obtain equations with the new unknowns $\varphi$ and $\psi$:

\[ (2\omega - 3)\alpha \psi \frac{d\psi}{da} = [(\nu + 2)\omega + 3](\psi^2 + k) + 2(\nu + 4)\omega \varphi \psi \]

\[ + \frac{2\omega}{3} [(\nu + 6)\omega - 3]\varphi^2, \]  

(5.1)

\[-(2\omega - 3)\alpha \psi \frac{d\varphi}{da} = \frac{3}{2} (\nu + 4)(\psi^2 + k) + [4\omega + 3(\nu + 2)]\varphi \psi \]

\[ + [(\nu + 8)\omega - 6]\varphi^2. \]  

(5.2)

We can obtain only one equation expressed solely in terms of invariants by dividing one of these equations over another with the result of the form $d\psi/d\varphi = G(\varphi, \psi)$, but this equation has no more known symmetries for $k \neq 0$ and hence it cannot be integrated. Though we cannot find the general solution in case 3, again we can search for invariant solutions with respect to the symmetry (3.7) now, which satisfy the conditions

\[ \frac{d\varphi}{da} = 0, \quad \frac{d\psi}{da} = 0, \]  

(5.3)
so that now $\varphi$ and $\psi$ are constants independent of $a$. With these conditions, equations (5.1) form an algebraic system:

$$
[(\nu + 2)\omega + 3](\psi^2 + k) + 2(\nu + 4)\omega \varphi \psi + \frac{2\omega}{3} [(\nu + 6)\omega - 3]\varphi^2 = 0,
$$

(5.4)

$$
3(\nu + 4)(\psi^2 + k) + 2[4\omega + 3(\nu + 2)]\varphi \psi + 2[(\nu + 8)\omega - 6]\varphi^2 = 0,
$$

which again has four roots. Two real roots are

$$
\varphi = \pm \frac{(\nu + 2)\sqrt{3k}}{\sqrt{-2[(\nu^2 + 8\nu)\omega + 12\omega + 6]}}, \quad \psi = \pm \frac{\sqrt{6k}}{\sqrt{-2[(\nu^2 + 8\nu)\omega + 12\omega + 6]}},
$$

(5.5)

that also yields $F = \varphi/a$ and $H = \psi/a$. Formulas (5.5) imply

$$
\varphi = \frac{(\nu + 2)}{2} \psi \implies F = \frac{(\nu + 2)}{2} H.
$$

(5.6)

The expression under the square root in the denominators is positive for the values of $\nu$ satisfying $-6 \leq \nu_2 < \nu < \nu_1 \leq -2$ where the roots of the denominator $\nu_{1,2} = -4 \pm \sqrt{4 - 6/\omega}$ and $\omega > 10^4$. Physically interesting values $\nu_{\text{dust}} = -3$ and $\nu_{\text{radiation}} = -4$ again lie in this range, so the roots (5.5) are indeed real. Two other roots are $\varphi = 0 \implies F = 0 \implies \phi = \phi_0 = \text{constant}$, $\psi = \pm \sqrt{-k}$. Thus, if $k = 0$, $H = F = 0$; if $k = 1$, $H$ is imaginary; if $k = -1$, $\psi = \pm 1$, $H = \pm 1/a$ and $a = a_0 \pm t$, that is a trivial solution.

For the first two real roots we have $\dot{a} = \psi = \text{constant}$ and so $a = \psi t + a_0$. Then the relation $\dot{\phi}/\phi = \varphi/a$ yields time dependence of our solution

$$
\phi = \phi_0(\psi t + a_0)^{\varphi/\psi} = \phi_0(\psi t + a_0)^{(\nu+2)/2}, \quad a = \psi t + a_0,
$$

(5.7)

where we have used (5.6). The solution (5.7) satisfies the initial condition (2.10) if $\phi_0$ is expressed in terms of other constants as follows

$$
\phi_0 = \pm \frac{2\sqrt{C\omega}}{\sqrt{3(\psi^2 + k) + 6\varphi \psi + 2\omega \varphi^2}} = \pm \frac{4\sqrt{C\omega}}{\sqrt{(\nu + 2)[(\nu + 2)(2\omega + 3) + 12\psi^2 + 12k]}},
$$

(5.8)

where $C$ is again the coefficient of power law for the density (2.7).
6 Massless case with flat spacelike sections: 
\(k = m = 0\)

In this case we have two commuting symmetries (3.8) and (3.9) which is enough for the complete integration of our equations in quadratures that will yield their general solution. We apply first the symmetry generator 
\[X_1 = a \partial / \partial a\]
with the invariants \(H\) and \(F\). A solution expressed solely in terms of invariants has the form 
\[F = F(H)\]. Dividing equation (2.15) over (2.14) with \(k = m = 0\), we obtain a single first order equation for \(F(H)\):

\[
-\frac{dF}{dH} = \frac{[(\nu + 8)\omega - 6]F^2 + 3(2\omega + \nu + 1)HF + (3/2)(\nu + 4)H^2}{(2\omega/3)[(\nu + 6)\omega - 3]F^2 + 2(\nu + 4)\omega HF + (\nu \omega + 6)H^2}.
\]

(6.1)

This equation still admits one symmetry 
\[X_2 = F \frac{\partial}{\partial F} + H \frac{\partial}{\partial H}\]
with the invariant \(G = F/H\) (and also \(a\), not contained explicitly in (6.1)), so that a new independent variable is \(H\) and a new unknown is \(G = G(H)\). In the new variables \(\{H, G\}\) equation (6.1) becomes the one with separated variables \(H\) and \(G\)

\[-\frac{dH}{H} = 2 \frac{2\omega[(\nu + 6)\omega - 3]G^2 + 6(\nu + 4)\omega G + 3(\nu \omega + 6)}{\{2[(\nu + 6)\omega - 3]G + 3(\nu + 4)\}(2\omega G^2 + 6G + 3)} dG,\]

(6.2)

where the cubic polynomial in \(G\) in the denominator was factorized. Integrating both sides of this equation in \(H\) and \(G\) and getting rid of logarithms, we obtain explicitly the function \(H(G)\)

\[H = H_0 \left\{ \frac{2\omega G^2 + 6G + 3}{2[(\nu + 6)\omega - 3]G + 3(\nu + 4)} \right\}^q \exp \left\{ (\nu + 3)l \tan^{-1} \left[ \frac{2\omega G + 3}{\sqrt{3(2\omega - 3)}} \right] \right\},\]

(6.3)

where we have introduced the following constants

\[b = \frac{(\nu + 6)\omega - 3}{(\nu + 6)^2 \omega - 6(\nu + 5)}, \quad l = \frac{2\sqrt{3(2\omega - 3)}}{(\nu + 6)^2 \omega - 6(\nu + 5)}, \quad q = \frac{3(\nu + 4)}{(\nu + 6)^2 \omega - 6(\nu + 5)}.
\]

(6.4)

Using (6.3), from the definition of \(G\) we also have \(F\) as a function of \(G\): 
\[F = GH(G)\].
The dependence $G(a)$ can be determined from (2.14) with $F = GH(G)$ at $k = 0$ and $m = 0$:

$$\frac{da}{a} = \frac{3(2\omega - 3)}{2\omega[(\nu + 6)\omega - 3]G^2 + 6(\nu + 4)\omega G + 3(\nu\omega + 6)} \times \frac{dH}{H},$$

where $dH/H$ is expressed from (6.2). The resulting equation is

$$\frac{da}{a} = \frac{-6(2\omega - 3)dG}{\{2[(\nu + 6)\omega - 3]G + 3(\nu + 4)\}(2\omega G^2 + 6G + 3)}. \quad (6.5)$$

Integrating both sides of (6.5) and eliminating logarithms, we obtain an explicit dependence $a(G)$

$$a = a_0 \left\langle \frac{2\omega G^2 + 6G + 3}{\{2[(\nu + 6)\omega - 3]G + 3(\nu + 4)\]^2} \right\rangle^b \times \exp \left\{ l \tan^{-1} \left[ \frac{2\omega G + 3}{\sqrt{3(2\omega - 3)}} \right] \right\}. \quad (6.6)$$

Formulas (6.3) and (6.6) together with $F = GH(G)$ yield a parametric representation, with a parameter $G$, of the general solution $H(a), F(a)$ to equations (2.14) and (2.15) with $k = m = 0$.

From $\dot{\phi}/\phi = F = GH$ we have

$$\frac{d\phi}{\phi} = GHdt = G\frac{da}{a} = \frac{-6(2\omega - 3)dG}{\{2[(\nu + 6)\omega - 3]G + 3(\nu + 4)\}(2\omega G^2 + 6G + 3)},$$

where we have expressed $da/a$ from (6.5). Integrating both sides of this equation and eliminating logarithms, we obtain the explicit dependence $\phi(G)$

$$\phi = \phi_0 \left\langle \frac{2[(\nu + 6)\omega - 3]G + 3(\nu + 4)}{\sqrt{2\omega G^2 + 6G + 3}} \right\rangle^q \times \exp \left\{ \frac{-(\nu + 6)}{2} \tan^{-1} \left[ \frac{2\omega G + 3}{\sqrt{3(2\omega - 3)}} \right] \right\}. \quad (6.7)$$

and because of the known dependence $a(G)$ from (6.6) we have determined $\phi(a)$ in a parametric representation. In order to satisfy the initial condition (2.10), the constants in the formulas (6.3), (6.6), and (6.7) should be related as follows

$$\phi_0 = \pm 2\sqrt{C\omega} \frac{a_0^{\nu/2}}{H_0}. \quad (6.8)$$
To determine time dependence of the solution, we use $\dot{a}/a = H$ to obtain

$$t - t_0 = \int \frac{da}{aH}$$

and then express $da/a$ from (6.5) and $H$ from (6.3). The integrand simplifies if we introduce the new parameter $g = (2\omega G + 3)/\sqrt{3(2\omega - 3)}$ with the resulting integral

$$t - t_0 = -\frac{3(2\omega - 3)}{H_0} \left( \frac{2}{\omega} \right)^{(\nu+3)b} \int \frac{\left\{ [ (\nu + 6)\omega - 3]g - \sqrt{3(2\omega - 3)} \right\}^{2(\nu+3)b-1}}{(g^2 + 1)^{(\nu+3)b}} \times \exp \left[ - (\nu + 3)\tan^{-1} g \right] dg,$$

that gives us explicitly functions $t(g)$ and hence $t(G)$ and implicitly the time dependence $G(t)$. Then from the functions $a(G)$ and $\phi(G)$, determined by (6.6) and (6.7) respectively, we know the time dependencies $a(t)$ and $\phi(t)$ in a parametric representation with the parameter $g$ in terms of the quadrature (6.9).

For dust, $\nu = \nu_{dust} = -3$ and the integral (6.9) simplifies substantially with the result

$$t - t_0 = -\frac{(2\omega - 3)}{(\omega - 1)H_0} \ln \left[ 3(\omega - 1)g - \sqrt{3(2\omega - 3)} \right],$$

that can be inverted easily to yield $g(t)$ and so an explicit expression for $G(t)$:

$$G = \frac{1}{2\omega(\omega - 1)} \left\{ \sqrt{\frac{2\omega - 3}{3}} \exp \left[ -\frac{(\omega - 1)}{(2\omega - 3)} H_0(t - t_0) \right] - \omega \right\}.$$  

For radiation, $\nu = \nu_{radiation} = -4$ and the simplification is not enough for evaluating the integral in (6.9).

7 Conclusion

We have shown that, in the model of cosmology we are considering, the equation of state does not affect the expansion law of the universe. Thus, for the $m = 0$ case the expansion of the universe, following the invariant solution (5.7), is always linear. What is interesting is that according to (5.5) this solution is nontrivial only for $k = 1$ i.e. a closed universe. It is also interesting
to note that the equation of state which determines the coefficient $\nu$ affects the rate of expansion. This case may be relevant for the early universe where the curvature term cannot be neglected. (5.7) shows that with $a_0 = 0$, for $t << m^{-1}$

$$\frac{\dot{\phi}}{\phi} = \frac{\nu + 2}{2} \frac{1}{t} >> \frac{\nu + 2}{2} m,$$

(7.1)

so that $m|\phi| << |\dot{\phi}|$ and for early times the mass term can indeed be neglected. After the universe expands to a large size, the curvature term can be neglected and in this case the $k=0$ solution becomes important. (4.3), (4.5) show that the universe expands exponentially for all equations of state. The equation of state is important just for the value of the Hubble constant. This behavior we identify with the dark energy or cosmological constant which is observed today.

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