On Exact Reznick, Hilbert-Artin and Putinar’s Representations

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Abstract
We consider the problem of computing exact sums of squares (SOS) decompositions for certain classes of non-negative multivariate polynomials, relying on semidefinite programming (SDP) solvers.

We provide a hybrid numeric-symbolic algorithm computing exact rational SOS decompositions with rational coefficients for polynomials lying in the interior of the SOS cone. The first step of this algorithm computes an approximate SOS decomposition for a perturbation of the input polynomial with an arbitrary-precision SDP solver. Next, an exact SOS decomposition is obtained thanks to the perturbation terms and a compensation phenomenon. We prove that bit complexity estimates on output size and runtime are both singly exponential in the cardinality of the Newton polytope (or doubly exponential in the number of variables). Next, we apply this algorithm to compute exact Reznick, Hilbert-Artin’s representation and Putinar’s representations respectively for positive definite forms and positive polynomials over basic compact semi-algebraic sets. We also report on practical experiments done with the implementation of these algorithms and existing alternatives such as the critical point method and cylindrical algebraic decomposition.

Keywords: Real algebraic geometry, Semidefinite programming, sums of squares decomposition, Reznick’s representation, Hilbert-Artin’s representation, Putinar’s representation, hybrid numeric-symbolic algorithm.

1. Introduction

Let \( \mathbb{Q} \) (resp. \( \mathbb{R} \)) be the field of rational (resp. real) numbers and \( X = (X_1, \ldots, X_n) \) be a sequence of variables. We consider the problem of deciding the non-negativity of \( f \in \mathbb{Q}[X] \) either over \( \mathbb{R}^n \) or over a closed semi-algebraic set \( S \) defined by some constraints \( g_1 \geq 0, \ldots, g_m \geq 0 \) (with \( g_j \in \mathbb{Q}[X] \)). Further, \( d \) denotes the maximum of the total degrees of these polynomials.

This problem is known to be co-NP hard [Blum et al., 2012]. The Cylindrical Algebraic Decomposition algorithm due to Collins (1975) and Wütrich (1976) allows one to solve it in time...
doubly exponential in \( n \) (and polynomial in \( d \)). This has been significantly improved, through the so-called critical point method, starting from \cite{GrigorjevVorob'ev1988} which culminates with \cite{Basu2007} to establish that this decision problem can be solved in time \( (m+1)d^{O(n)} \).

These latter ones have been developed to obtain practically fast implementations which reflect the complexity gain (see e.g. \cite{Bank2001,Bank2005,SafeyElDin2003,SafeyElDin2007,Bank2010,Guo2010,Bank2014,Greuet2014,Greuet2012}). These algorithms are “root finding” ones: they are designed to compute at least one point in each connected component of the set defined by \( f < 0 \). This is done by solving polynomial systems defining critical points of some well-chosen polynomial maps restricted to \( f = -\varepsilon \) for \( \varepsilon \) small enough. Hence the complexity of these algorithms depends on the difficulty of solving these polynomial systems (which can be exponential in \( n \) as the Bézout bound on the number of their solutions is). Moreover, when \( f \) is non-negative, they return an empty list without a certificate that can be checked \textit{a posteriori}. This paper focuses on the computation of such certificates under some favorable situations.

To compute certificates of non-negativity, an approach based on \textit{sums of squares} (SOS) decompositions of polynomials (see \cite{Lasserre2001} and \cite{Parrilo2000}). Many positive polynomials are not sums of squares of polynomials following \cite{Blekherman2006}. However, some variants have been designed to make this approach more general; see e.g. the survey by \cite{Laurent2004} and references therein. In a nutshell, the core and initial idea is as follows.

A polynomial \( f \) is non-negative over \( \mathbb{R}^n \) if it can be written as an SOS \( s_1^2 + \cdots + s_r^2 \) with \( s_i \in \mathbb{R}[x] \) for \( 1 \leq i \leq r \). Also \( f \) is non-negative over the semi-algebraic set \( S \) if it can be written as \( s_1^2 + \cdots + s_r^2 + \sum_{j=1}^m \sigma_j g_j \) where \( \sigma_j \) is a sum of squares in \( \mathbb{R}[x] \) for \( 1 \leq j \leq m \). It turns out that, thanks to the “Gram matrix method” (see e.g. \cite{Choi1995,Lasserre2001,Parrilo2000}), computing such decompositions can be reduced to solving Linear Matrix Inequalities (LMI). This boils down to considering a semidefinite programming (SDP) problem.

For instance, on input \( f \in \mathbb{Q}[x] \) of even degree \( d = 2k \), the decomposition \( f = s_1^2 + \cdots + s_r^2 \) is a by-product of a decomposition of the form \( f = v_k^T L^T D L v_k \), where \( v_k \) is the vector of all monomials of degree \( \leq k \) in \( \mathbb{Q}[x] \), \( L \) is a lower triangular matrix with non-negative real entries on the diagonal and \( D \) is a diagonal matrix with non-negative real entries. The matrices \( L \) and \( D \) are obtained after computing a symmetric matrix \( G \) (the Gram matrix), semidefinite positive, such that \( f = v_k^T G v_k \). Such a matrix \( G \) is found using solvers for LMIs. Such inequalities can be solved symbolically (see \cite{Henrion2016}), but the degrees of the algebraic extensions needed to encode exactly the solutions are prohibitive on large examples \cite{Nie2010}. Besides, there exist fast numerical solvers for solving LMIs implemented in double precision, e.g. SeDuMi by \cite{Sturm1999}, SDPA by \cite{Yamashita2010} as well as arbitrary-precision solvers, e.g. SDPA-GMP by \cite{Nakata2010}, successfully applied in many contexts, including bounds for kissing numbers by \cite{Bachoc2008} or computation of (real) radical ideals by \cite{Lasserre2013}.

But using solely numerical solvers yields “approximate” non-negativity certificates. In our example, the matrices \( L \) and \( D \) (and consequently the polynomials \( s_1, \ldots, s_r \)) are not known exactly.

This raises topical questions. The first one is how to use symbolic computation jointly with these numerical solvers to get \textit{exact} certificates? Since not all positive polynomials are SOS, what to do when SOS certificates do not exist? Also, given inputs with rational coefficients, can we obtain certificates with rational coefficients?

For these questions, we inherit from contributions in the univariate case by \cite{Chevillard2011,Magron2018} as well as in the multivariate case by \cite{Peyrl2008}.
Kaltofen et al. (2008). Note that Kaltofen et al. (2008, 2012) allow us to compute SOS with rational coefficients on some degenerate examples. Moreover, Kaltofen et al. (2012) allows to compute decompositions into sums of squares of rational fractions. Diophantine aspects are considered in Safey El Din and Zhi (2010); Guo et al. (2013). When an SOS decomposition exists with coefficients in a totally real Galois field, Hillar (2009) and Quarez (2010) provide bounds on the total number of squares.

In the univariate (un)-constrained case, given \( f \in \mathbb{Q}[X] \), the algorithm by Chevillard et al. (2011) computes an exact (weighted) SOS decomposition \( f = \sum_{i=1}^{d} c_i g_i^2 \) with \( c_i \in \mathbb{Q} \) and \( g_i \in \mathbb{Q}[X] \). We call such SOS decompositions weighted because the coefficients \( c_i \) are considered outside the square, which helps when one wants to output data with rational coefficients only. To do that, the algorithm considers first a perturbation of \( f \), performs (complex) root isolation to get an approximate SOS decomposition of \( f \). When the isolation is precise enough, the algorithm relies the perturbation terms to recover an exact rational decomposition.

In the multivariate unconstrained case, Parillo and Peyrl designed a rounding-projection algorithm in Peyrl and Parrilo (2008) to compute a weighted rational SOS decomposition of a given polynomial \( f \) in the interior of the SOS cone. The algorithm computes an approximate Gram matrix of \( f \), and rounds it to a rational matrix. With sufficient precision digits, the algorithm performs an orthogonal projection to recover an exact Gram matrix of \( f \). The SOS decomposition is then obtained with an exact LDL\(^T\) procedure. This approach was significantly extended in Kaltofen et al. (2008) to handle rational functions and in Guo et al. (2012) to derive certificates of impossibility for Hilbert-Artin representations of a given degree. In a recent work by Laplagne (2018), the author derives an algorithm based on facial reduction techniques to obtain exact rational decompositions for some sub-classes of non-negative polynomials lying in the border of the SOS cone. Among such degenerate sub-classes, he considers polynomials that can be written as sums of squares of polynomials with coefficients in an algebraic extension of \( \mathbb{Q} \) of odd degree.

**Main contributions.** This work provides an algorithmic framework for computing exact rational weighted SOS decompositions in some favourable situations. The first contribution, given in Section 3, is a hybrid numeric-symbolic algorithm, called \texttt{intsos} , providing rational SOS decompositions for polynomials lying in the interior of the SOS cone. As for the algorithm by Chevillard et al. (2011), the main idea is to perturb the input polynomial, then to obtain an approximate SOS decomposition (through some Gram matrix of the perturbation by solving an SDP problem), and to recover an exact decomposition using the perturbation terms.

In Section 4.1, we rely on \texttt{intsos} to compute decompositions of positive definite forms into SOS of rational functions, based on Reznick’s representations, yielding an algorithm, called \texttt{Reznicksos} . In Section 4.2, we provide another algorithm, called \texttt{Hilbertsos} , to decompose non-negative polynomials into SOS of rational functions, under the assumption that the numerator belongs to the interior of the SOS cone. In Section 5 we rely on \texttt{intsos} to compute weighted SOS decompositions for polynomials positive over basic compact semi-algebraic sets, yielding the \texttt{Putinarsos} algorithm.

When the input is an \( n \)- variate polynomial of degree \( d \) with integer coefficients of maximum bit size \( r \), we prove in Section 5 that Algorithm \texttt{intsos} runs in boolean time \( \tau^d d^{O(n)} \). In the former estimate, the exponent \( d^n \) can be replaced by the cardinality of the Newton polytope of the input polynomial. This also yields bit complexity analysis for Algorithm \texttt{Reznickso} (see Section 4.1) and Algorithm \texttt{Putinarso} (see Section 5). To the best of our knowledge, these are the first complexity estimates for the output of algorithms providing exact multivariate SOS decompositions.

The three algorithms are implemented within a Maple procedure, called \texttt{multiSOS} , integrated
in the RealCertify Maple library by Magron and Safey El Din (2018b). In Section 6, we provide benchmarks to evaluate the performance of multisos. We compare it with previous approaches in Peyrl and Parrilo (2008) as well as with the more general methods based on the critical point method and Cylindrical Algebraic Decomposition.

This paper is the follow-up of our previous contribution (Magron and Safey El Din, 2018a), published in the proceedings of ISSAC’18. The main theoretical and practical novelties are the following: we provide explicit bounds for the bit complexity analyzes of our algorithms. In Section 6, we provide some related numerical comparisons. We also consider benchmarks involving non-negative polynomials which do not belong to the interior of the SOS cone.

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2. Preliminaries

Let $\mathbb{Z}$ be the ring of integers and $X = (X_1, \ldots, X_n)$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, one has $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. For all $k \in \mathbb{N}$, we let $\mathbb{N}_0^k := \{ \alpha \in \mathbb{N}^n : |\alpha| \leq k \}$, whose cardinality is the binomial coefficient $\binom{n+k}{k}$. A polynomial $f \in \mathbb{R}[X]$ of degree $d = 2k$ is written as $f = \sum_{|\alpha| \leq k} f_\alpha X^\alpha$ and we identify $f$ with its vector of coefficients $\mathbf{f} = (f_\alpha)$ in the basis $(X^\alpha)_{\alpha \in \mathbb{N}_0^k}$. When referring to univariate polynomials, we use the indeterminate $E$ and we denote by $\mathbb{Z}[E]$ the set of univariate polynomials with integer coefficients. Let $\Sigma[X]$ be the convex cone of sums of squares in $\mathbb{R}[X]$ and $\Sigma[X]_E$ be the interior of $\Sigma[X]$. We will be interested in those polynomials which lie in $\mathbb{Z}[X] \cap \Sigma[X]$. For instance, the polynomial

$$f = 4X_1^4 + 4X_1^3X_2 - 7X_1^2X_2^2 - 2X_1X_2^3 + 10X_2^4 = (2X_1X_2 + X_2^2)^2 + (2X_1^2 + X_1X_2 - 3X_2^2)^2$$

lies in $\mathbb{Z}[X] \cap \Sigma[X]$.

The complexity estimates in this paper rely on the bit complexity model. The bit size of an integer $b$ is denoted by $\tau(b) := \lceil \log_2(|b|) \rceil + 1$ with $\tau(0) := 1$. For $f = \sum_{|\alpha| \leq k} f_\alpha X^\alpha \in \mathbb{Z}[X]$ of degree $d$, we denote $\|f\|_{\infty} := \max_{|\alpha| \leq k} |f_\alpha|$ and $\tau(f) := \tau(\|f\|_{\infty})$ with slight abuse of notation. Given $b \in \mathbb{Z}$ and $c \in \mathbb{Z} \setminus \{0\}$ with $\gcd(b, c) = 1$, we define $\tau(b/c) := \max\{\tau(b), \tau(c)\}$. For two
mappings \( g, h : \mathbb{N}^p \to \mathbb{R} \), we use the notation “\( g(v) = O(h(v)) \)” to state the existence of \( b \in \mathbb{N} \) such that \( g(v) \leq bh(v) \), for all \( v \in \mathbb{N}^p \).

The Newton polytope or cage \( C(f) \) is the convex hull of the vectors of exponents of monomials that occur in \( f \in \mathbb{R}[X] \). For the above example, \( C(f) = \{(4,0), (3,1), (2,2), (1,3), (0,4)\} \).

For a symmetric real matrix \( G \), we denote \( G \succeq 0 \) (resp. \( G > 0 \)) when \( G \) has only non-negative (resp. positive) eigenvalues and we say that \( G \) is positive semidefinite (SDP) (resp. positive definite). The minimal eigenvalue of a real symmetric matrix \( G \) is denoted by \( \lambda_{\text{min}}(G) \). For a given Newton polytope \( P \), let \( \Sigma_\rho[X] \) be the convex cone of sums of squares whose Newton polytope is contained in \( P \). Since the Newton polytope \( P \) is often clear from the context, we suppress the index \( P \).

With \( f \in \mathbb{R}[X] \) of degree \( d = 2k \), we consider the SDP feasibility program:

\[
\begin{align*}
\text{Find } G \succeq 0 & \quad \text{s.t.} \quad \text{Tr} \left( GB_{\gamma} \right) = f_{\gamma}, \quad \forall \gamma \in \mathbb{N}^q, \\
\end{align*}
\]

where \( \text{Tr}(M) \) (for a given matrix \( M \)) denotes the trace of \( M \), \( B_{\gamma} \) has rows (resp. columns) indexed by \( \mathbb{N}^q \) with \( (\alpha, \beta) \) entry equal to 1 if \( \alpha + \beta = \gamma \) and 0 otherwise. When \( f \in \Sigma[X] \), SDP (1) has a feasible solution \( G = \sum_{i=1}^{r} A_i q_i q_i^T \), with the \( q_i \) being the eigenvectors of \( G \) corresponding to the non-negative eigenvalues \( \lambda_i \), for all \( i = 1, \ldots, r \), and \( f = \sum_{i=1}^{r} A_i q_i^2 \).

For more details, see, e.g., [P. A. Parrilo, 2000; Lasserre, 2001]. For the sake of efficiency, one reduces the size of the matrix \( G \) by indexing its rows and columns by half of \( C(f) \):

**Theorem 1.** [Reznick 1978] *Theorem 1* Let \( f \in \Sigma[X] \) with \( f = \sum_{i=1}^{r} s_i^2 \), \( P := C(f) \) and \( Q := P/2 \cap \mathbb{N}^p \). Then for all \( i = 1, \ldots, r \), \( C(s_i) \subseteq Q \).

Then from now on, we consider SDP (1) with \( G \) (resp. each matrix \( B_{\gamma} \)) having its rows and columns indexed by \( Q \).

Given \( f \in \mathbb{R}[X] \), one can theoretically certify that \( f \) lies in \( \Sigma[X] \) by solving SDP (1). However, available SDP solvers are typically implemented in finite-precision and require the existence of a strictly feasible solution \( G \succeq 0 \) to converge. This is equivalent for \( f \) to lie in \( \Sigma[X] \) as stated in [Choi et al., 1995, Proposition 5.5]:

**Theorem 2.** Let \( f \in \mathbb{Z}[X] \) with \( P := C(f) \), \( Q := P/2 \cap \mathbb{N}^p \) and \( v_k \) be the vector of all monomials with support in \( Q \). Then \( f \in \Sigma[X] \) if and only if there exists a positive definite matrix \( G \) such that \( f = v_k^T G v_k \).

Eventually, we will rely on the following bound for the roots of polynomials with integer coefficients:

**Lemma 3.** [Mignotte 1992] *Theorem 4.2 (ii)* Let \( f \in \mathbb{Z}[E] \) of degree \( d \), with coefficient bit size bounded from above by \( \tau \). If \( f(e) = 0 \) and \( e \neq 0 \), then \( \frac{1}{2^{\tau + 1}} \leq |e| \leq 2^{\tau} + 1 \).

### 3. Exact SOS representations

The aim of this section is to state and analyze a hybrid numeric-symbolic algorithm, called int sos, computing weighted SOS decompositions of polynomials in \( \mathbb{Z}[X] \cap \Sigma[X] \). This algorithm relies on perturbations of such polynomials. We first establish the following preliminary results.
Proposition 4. Let \( f \in \mathbb{Z}[X] \cap \mathbb{S}[X] \) of degree \( d = 2k \), with \( \tau = \tau(f) \), \( P = C(f) \) and \( Q := P/2\cap \mathbb{N}^\alpha \). Then, there exists a feasible \( G > 0 \) for SDP \( \Pi \) and positive integers \( R, N \) such that \( \sqrt{\text{Tr}(G^2)} \leq R \) and \( \lambda_{\min}(G) > 2^{-N} \), with \( \tau(R), N \leq \tau d^{O(n)} \).

Proof. Let \( v_k \) be the vector of all monomials \( X^\alpha \), with \( \alpha \) in \( Q \). Note that each monomial in \( v_k \) has degree \( \leq k \) and that \( v_k^T v_k = \sum_{\alpha \in Q} X^{2\alpha} \). Since \( f \in \mathbb{S}[X] \), there exists by Theorem 2 a matrix \( G > 0 \) such that \( f = v_k^T G v_k \), with positive smallest eigenvalue. Let us note \( n_{\text{dop}} \) the size of \( G \) and \( m_{\text{dop}} \) the number of equality constraints of SDP \( \Pi \). Then (Porokhov and Khachiyan, 1997, Theorem 3.1 (i)) implies that \( \sqrt{\text{Tr}(G^2)} \leq R \), with \( \tau(R) = \tau n_{\text{dop}}^{O(n_{\text{dop}})} \). Using that for all \( k \geq 2 \),
\[
n_{\text{dop}} \leq \binom{n+k}{n} \leq \frac{(n+k)\cdots(k+1)}{n!} = \frac{(1+\frac{k}{n})(1+\frac{k}{n-1})\cdots(1+\frac{k}{n-k+1})}{n} \leq \frac{k^{n-1}(1+k)}{2^n} \leq d^n,
\]
and \( m_{\text{dop}} \leq \binom{n}{\frac{n}{2}} \leq 2d^n \), we obtain the desired bit estimate for \( R \), as well for entries of \( G \). Then note that the minimal eigenvalue is a root of the univariate polynomial \( \det(G-\epsilon I) \in \mathbb{Q}[\epsilon] \). The coefficients of this latter polynomial are obtained by \( n_{\text{dop}}! \) products of numbers of bit size upper bounded by \( \tau d^{O(n)} \), thus their bit size is also upper bounded by \( \tau d^{O(n)} \). Then Lemma 5 yields the desired result.

\( \square \)

Proposition 5. Let \( f \) as above. Then, there exists \( N \in \mathbb{N} - \{0\} \) such that for \( \varepsilon := \frac{1}{2^\tau} \), \( f - \varepsilon \sum_{\alpha \in Q} X^{2\alpha} \in \mathbb{S}[X] \), with \( N \leq \tau(\varepsilon) \leq d^{O(n)} \).

Proof. As in the proof of Proposition 4 since \( f \in \mathbb{S}[X] \), there exists by Theorem 2 a matrix \( G > 0 \) such that \( f = v_k^T G v_k \), with positive smallest eigenvalue \( \lambda \). Let us define \( N := \lceil \log_2 \frac{1}{\varepsilon} \rceil + 1 \), i.e. the smallest integer such that \( \varepsilon = \frac{1}{2^\lambda} \leq \frac{1}{2} \). Then, \( \lambda > \varepsilon \) and the matrix \( G - \varepsilon I \) has only positive eigenvalues. Hence, one has
\[
f_\varepsilon := f - \varepsilon \sum_{\alpha \in Q} X^{2\alpha} = v_k^T G v_k - \varepsilon v_k^T I v_k = v_k^T (G - \varepsilon I) v_k,
\]
yielding \( f_\varepsilon \in \mathbb{S}[X] \). The upper bound on the bit size on \( N \) follows directly from Proposition 4.

\( \square \)

The following can be found in Bai et al., 1989, Lemma 2.1) and (Bai et al., 1989, Theorem 3.2).

Proposition 6. Let \( G > 0 \) be a matrix with rational entries indexed on \( \mathbb{N}^\alpha \). Let \( L \) be the factor of \( G \) computed using Cholesky’s decomposition with finite precision \( \delta \). Then \( LL^T = G + F \) where
\[
|F_{\alpha,\beta}| \leq \frac{(r+1)2^{-\delta}|G_{\alpha,\alpha} G_{\beta,\beta}|}{1 - (r+1)2^{-\delta}}.
\]
In addition, if the smallest eigenvalue \( \lambda \) of \( G \) satisfies the inequality
\[
2^{-\delta} \leq \frac{\lambda}{r^2 + r + (r-1)\lambda},
\]
Cholesky’s decomposition returns a rational nonsingular factor \( L \).

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Algorithm 1 \texttt{intsos}

\begin{itemize}
\item \textbf{Input:} $f \in \mathbb{Z}[X]$, positive $\varepsilon \in \mathbb{Q}$, precision parameters $\delta, R \in \mathbb{N}$ for the SDP solver, precision $\delta_c \in \mathbb{N}$ for the Cholesky’s decomposition
\item \textbf{Output:} list $c_{\text{list}}$ of numbers in $\mathbb{Q}$ and list $s_{\text{list}}$ of polynomials in $\mathbb{Q}[X]
\end{itemize}

1: $P := C(f), Q := P/2 \cap \mathbb{N}^n$
2: $t := \sum_{\alpha \in Q} X^{2\alpha}, f_e := f - \varepsilon t$
3: while $f_e \notin \sum[X] \subset \mathbb{R}^n$ do $\varepsilon := \frac{\varepsilon}{2}, f_e := f - \varepsilon t$
4: done
5: ok := false
6: while not ok do
7: $(\tilde{G}, \tilde{\lambda}) := \text{sdp}(f_e, \delta, R)$
8: $(s_1, \ldots, s_r) := \text{cholesky}(\tilde{G}, \tilde{\lambda}, \delta_c)$ \hspace{1cm} $\ast f_e \approx \sum_{i=1}^{r} s_i^2$
9: $c_{\text{list}} := [1, \ldots, 1], s_{\text{list}} := [s_1, \ldots, s_r]$
10: for $\alpha \in Q$ do $\varepsilon_\alpha := \varepsilon$
11: done
12: $c_{\text{list}}, s_{\text{list}}, (\varepsilon_\alpha) := \text{absorb}(u, Q, (\varepsilon_\alpha), c_{\text{list}}, s_{\text{list}})$
13: if $\min_{\alpha \in Q} |\varepsilon_\alpha| \geq 0$ then ok := true
14: else $\delta := 2\delta, R := 2R, \delta_c := 2\delta_c$
15: end
16: done
17: for $\alpha \in Q$ do $c_{\text{list}} := c_{\text{list}} \cup \{\varepsilon_\alpha\}, s_{\text{list}} := s_{\text{list}} \cup [X^\alpha]$
18: done
19: return $c_{\text{list}}, s_{\text{list}}$

Algorithm 2 \texttt{absorb}

\begin{itemize}
\item \textbf{Input:} $u \in \mathbb{Q}[X]$, multi-index set $Q$, lists $(\varepsilon_\alpha)$ and $c_{\text{list}}$ of numbers in $\mathbb{Q}$, list $s_{\text{list}}$ of polynomials in $\mathbb{Q}[X]$
\item \textbf{Output:} lists $(\varepsilon_\alpha)$ and $c_{\text{list}}$ of numbers in $\mathbb{Q}$, list $s_{\text{list}}$ of polynomials in $\mathbb{Q}[X]$
\end{itemize}

1: for $\gamma \in \text{supp}(u)$ do
2: if $\gamma \in (2\mathbb{N})^n$ then $\alpha := \frac{\gamma}{2}, \varepsilon_\alpha := \varepsilon_\alpha + u_\gamma$
3: else
4: Find $\alpha, \beta \in Q$ such that $\gamma = I - \alpha$
5: $\varepsilon_\alpha := \varepsilon_\alpha - \frac{|\gamma|}{2}, \varepsilon_\beta := \varepsilon_\beta - \frac{|\gamma|}{2}$
6: $c_{\text{list}} := c_{\text{list}} \cup \{|\varepsilon_i|\}$
7: $s_{\text{list}} := s_{\text{list}} \cup [X^\alpha + \text{sgn}(u_\gamma)X^\beta]$
8: end
9: done
3.1. Algorithm \texttt{intsos}

We present our algorithm \texttt{intsos} computing exact weighted rational SOS decompositions for polynomials in \(\mathbb{Z}[X] \cap \Sigma[X]\).

Given \(f \in \mathbb{Z}[X]\) of degree \(d = 2k\), one first computes its Newton polytope \(P := C(f)\) (see line [1] and \(Q := P/2 \cap \mathbb{N}^d\) using standard algorithms such as quickhull by [1996]. The loop going from line [3] to line [4] finds a positive \(\varepsilon \in \mathbb{Q}\) such that the perturbed polynomial \(f_\varepsilon := f - \varepsilon \sum_{\alpha \in Q} X^{2\alpha}\) is also in \(\Sigma[X]\). This is done thanks to any external oracle deciding the non-negativity of a polynomial. Even if this oracle is able to decide non-negativity, we would like to emphasize that our algorithm outputs an SOS certificate in order to certify the non-negativity of the input. In practice, we often choose the value of \(\varepsilon\) while relying on a heuristic technique rather than this external oracle, for the sake of efficiency (see Section [5] for more details).

Next, we enter in the loop starting from line [5]. Given \(f_\varepsilon \in \mathbb{Z}[X]\), positive integers \(\delta\) and \(R\), the \texttt{adp} function calls an SDP solver and tries to compute a rational approximation \(\tilde{G}\) of the Gram matrix associated to \(f_\varepsilon\) together with a rational approximation \(\lambda\) of its smallest eigenvalue.

In order to analyse the complexity of the procedure (see Remark [7]), we assume that \texttt{adp} relies on the ellipsoid algorithm by [1993].

\textbf{Remark 7.} In [2016], the authors analyze the complexity of the short step, primal interior point method, used in SDP solvers. Within fixed accuracy, they obtain a polynomial complexity, as for the ellipsoid method, but the exact value of the exponents is not provided.

Also, in practice, we use an arbitrary-precision SDP solver implemented with an interior-point method.

SDP problems are solved with this latter algorithm in polynomial-time within a given accuracy \(\delta\) and a radius bound \(R\) on the Frobenius norm of \(\tilde{G}\). The first step consists of solving SDP (1) by computing an approximate Gram matrix \(\tilde{G} \succeq 2^{-\delta}I\) such that

\[
|\text{Tr}(\tilde{G}B_r) - (f_\varepsilon)_r| = |\sum_{\alpha \neq \gamma} \tilde{G}_{\alpha,\beta} - (f_\varepsilon)_\gamma| \leq 2^{-\delta}
\]

and \(\sqrt{\text{Tr}(\tilde{G}^2)} \leq R\). We pick large enough integers \(\delta\) and \(R\) to obtain \(\tilde{G} > 0\) and \(\lambda > 0\) when \(f_\varepsilon \in \Sigma[X]\).

The \texttt{cholesky} function computes the approximate Cholesky’s decomposition \(LL^T\) of \(\tilde{G}\) with precision \(\delta_c\). In order to guarantee that \(L\) will be a rational nonsingular matrix, a preliminary step consists of verifying that the inequality (4) holds, which happens when \(\delta_c\) is large enough. Otherwise, \texttt{cholesky} selects the smallest \(\delta_c\) such as (4) holds. Let \(v_k\) be the size \(r\) vector of all monomials \(X^\alpha\) with \(\alpha\) belonging to \(Q\). The output is a list of rational polynomials \([s_1, \ldots, s_r]\) such that for all \(i = 1, \ldots, r\), \(s_i\) is the inner product of the \(i\)-th row of \(L\) by \(v_k\). One would expect to have \(f_\varepsilon = \sum_{i=1}^r s_i^2\) with \(s_i \in \mathbb{R}[X]\) after using exact SDP and Cholesky’s decomposition. Here, we have to consider the remainder \(u = f - \varepsilon \sum_{\alpha \in Q} X^{2\alpha} - \sum_{i=1}^r s_i^2\), with \(s_i \in \mathbb{Q}[X]\).

After these steps, the algorithm starts to perform symbolic computation with the \texttt{absorb} subroutine at line [10]. The loop from \texttt{absorb} is designed to obtain an exact weighted SOS decomposition of \(et + u = \varepsilon \sum_{\alpha \in Q} X^{2\alpha} + \sum_{\alpha} u_\alpha X^{\alpha}\), yielding in turn an exact decomposition of \(f\). Each term \(u_\alpha X^{\alpha}\) can be written either \(u_\alpha X^{2\alpha}\) or \(u_\alpha X^{\alpha-\beta}\), for \(\alpha, \beta \in Q\). In the former case (line [10]), one has

\[
\varepsilon X^{2\alpha} + u_\alpha X^{2\alpha} = (\varepsilon + u_\alpha)X^{2\alpha}.
\]
In the latter case (line \[\text{4}\]), one has
\[\epsilon(X^{2\alpha} + X^{2\beta}) + u, X^{\alpha+\beta} = |u|/2(X^\alpha + \text{sgn}(u)X^\beta)^2 + (\epsilon - |u|/2)(X^{2\alpha} + X^{2\beta}).\]

If the positivity test of line \[\text{3}\] fails, then the coefficients of \(u\) are too large and one cannot ensure that \(et + u\) is SOS. So we repeat the same procedure after increasing the precision of the SDP solver and Cholesky’s decomposition.

In prior work [Magron et al. 2018], the authors and Schweighofer formalized and analyzed an algorithm called \(\text{univsos2}\), initially provided in [Chevillard et al. 2011]. Given a univariate polynomial \(f > 0\) of degree \(d = 2k\), this algorithm computes weighted SOS decompositions of \(f\). With \(\ell := \sum_{i=0}^{k} X^{2i}\), the first numeric step of \(\text{univsos2}\) is to find \(\epsilon\) such that the perturbed polynomial \(f_\ell := f - \epsilon t > 0\) and to compute its complex roots, yielding an approximate SOS decomposition \(s_1^2 + s_2^2\). The second symbolic step is very similar to the loop from line \[\text{4}\] to line \[\text{5}\] in \(\text{intsos}\): one considers the remainder polynomial \(u := f_\ell - s_1^2 - s_2^2\) and tries to computes an exact SOS decomposition of \(et + u\). This succeeds for large enough precision of the root isolation procedure. Therefore, \(\text{intsos}\) can be seen as an extension of \(\text{univsos2}\) in the multivariate case by replacing the numeric step of root isolation by SDP and keeping the same symbolic step.

**Example 8.** We apply Algorithm \(\text{intsos}\) on
\[f = 4X_1^4 + 4X_1^3X_2 - 7X_1^2X_2^2 - 2X_1X_2^3 + 10X_2^4,\]
with \(\epsilon = 1\), \(\delta = R = 60\) and \(\delta_e = 10\). Then
\[Q := C(f)/2 \cap \mathbb{N}^n = \{(2, 0), (1, 1), (0, 2)\}\]
(line \[\text{4}\]). The loop from line \[\text{4}\] to line \[\text{5}\] ends and we get \(f - \epsilon t = f - (X_1^4 + X_1^3X_2 + X_2^2) \in \Sigma[X]\).

The \(\text{sdp}\) (line \[\text{4}\]) and \(\text{cholesky}\) (line \[\text{5}\]) procedures yield
\[s_1 = 2X_1^2 + X_1X_2 - \frac{8}{3}X_2^2, \quad s_2 = \frac{4}{3}X_1X_2 + \frac{3}{2}X_2^2 \quad \text{and} \quad s_3 = \frac{2}{7}X_2^2.\]
The remainder polynomial is \(u = f - \epsilon t - s_1^2 - s_2^2 - s_3^2 = -X_1^4 + \frac{2}{3}X_1^3X_2 - \frac{8}{3}X_1X_2^3 - \frac{781}{768}X_2^4\).
At the end of the loop from line \[\text{4}\] to line \[\text{5}\] we obtain \(\epsilon_{(2, 0)} = (\epsilon = X_1^4 = 0, \text{which is the coefficient of } X_1^4 \text{ in } et + u).\)
Then,
\[\epsilon(X_1^4X_2^2 + X_2^4) = \frac{2}{3}X_1X_2^3 = \frac{1}{3}(X_1X_2 - X_2^3)^2 + (\epsilon - \frac{1}{3})(X_1^2X_2^2 + X_2^4).\]
In the polynomial \(et + u\), the coefficient of \(X_1^2X_2^2\) is \(\epsilon_{(1, 1)} = \epsilon = \frac{1}{3} = \frac{2}{7}\) and the coefficient of \(X_1^4\) is \(\epsilon_{(0, 2)} = \epsilon - \frac{1}{7} = \frac{8}{7} = \frac{395}{1764}\).

Eventually, we obtain the weighted rational SOS decomposition:
\[4X_1^4 + 4X_1^3X_2 - 7X_1^2X_2^2 - 2X_1X_2^3 + 10X_2^4 = \frac{1}{3}(X_1X_2 - X_2^3)^2 + \frac{5}{9}(X_1X_2)^2 + \frac{395}{1764}X_2^4\]
\[+ (2X_1^2 + X_1X_2 - \frac{8}{3}X_2^2)^2 + (\frac{4}{3}X_1X_2 + \frac{3}{2}X_2^2)^2 + (\frac{2}{7}X_2^2)^2).\]
3.2. Correctness and bit size of the output

Let \( f \in \mathbb{Z}[X] \cap \Sigma[X] \) of degree \( d = 2k \), \( \tau := \tau(f) \) and \( Q := C(f)/2 \cap \mathbb{N}^n \).

**Proposition 9.** Let \( G \) be a positive definite Gram matrix associated to \( f \) and take \( 0 < \varepsilon \in \mathbb{Q} \) as in Proposition 5 so that \( f_{\varepsilon} = f - \varepsilon \sum_{a \in Q} X^a \in \Sigma[X] \). Then, there exist positive integers \( \delta, R \) such that \( G - \varepsilon I \) is a Gram matrix associated to \( f_{\varepsilon} \), satisfies \( G - \varepsilon I \succeq 2^{-\delta}I \) and \( \sqrt{\text{Tr}((G - 2^{-\delta}I)^2)} \leq R \). Also, the maximal bit sizes of \( \delta \) and \( R \) are upper bounded by \( \tau d^{d(\varepsilon)} \).

**Proof.** Let \( \lambda = \lambda_{\text{min}}(G) \) be the smallest eigenvalue of \( G \). By Proposition 5, \( G \succeq \varepsilon I \) for \( \varepsilon = \frac{1}{2^\delta} \leq \frac{1}{2} \) with \( N \leq \tau d^{d(\varepsilon)} \). By defining \( \delta := N + 1, 2^{-\delta} = \frac{1}{2N+1} \leq \frac{1}{4} \), yielding \( G - \varepsilon I \succeq 2^{-\delta}I \). As \( N \leq \tau d^{d(\varepsilon)} \), one has \( \delta \leq \tau d^{d(\varepsilon)} \). The bound on \( R \) follows directly from Proposition 4. \( \square \)

**Proposition 10.** Let \( f \) be as above. When applying Algorithm \texttt{intsos} to \( f \), the procedure always terminates and outputs a weighted SOS decomposition of \( f \) with rational coefficients. The maximum bit size of the coefficients involved in this SOS decomposition is upper bounded by \( \tau d^{d(\varepsilon)} \).

**Proof.** Let us first consider the loop of Algorithm \texttt{intsos} defined from line 3 to line 4. From Proposition 5, this loop terminates when \( f_{\varepsilon} \in \Sigma[X] \) for \( \varepsilon = \frac{1}{2^\delta} \) and \( N \leq \tau d^{d(\varepsilon)} \).

When calling the \texttt{sdp} function at line 7 to solve SDP 1 with precision parameters \( \delta \) and \( R \), we compute an approximate Gram matrix \( \tilde{G} \) of \( f_{\varepsilon} \) such that \( \tilde{G} \succeq 2^{\delta}I \) and \( \text{Tr}(\tilde{G}^2) \leq R^2 \). From Proposition 5, this procedure succeeds for large enough values of \( \delta \) and \( R \) of bit size upper bounded by \( \tau d^{d(\varepsilon)} \). In this case, we obtain a positive rational approximation \( \tilde{\lambda} \geq 2^{-\delta} \) of the smallest eigenvalue of \( \tilde{G} \).

Then the Cholesky decomposition of \( \tilde{G} \) is computed when calling the \texttt{cholesky} function at line 8. The decomposition is guaranteed to succeed by selecting a large enough \( \delta \), such that (3) holds. Let \( r \) be the size of \( \tilde{G} \) and \( \delta_r \) be the smallest integer such that \( 2^{-\delta} \leq \frac{1}{r+1} \). Since the function \( x \mapsto \frac{1}{r+1} \) is increasing on \([0, \infty)\) and \( \tilde{\lambda} \geq 2^{-\delta} \), (3) holds. We obtain an approximate weighted SOS decomposition \( \sum_{i=1}^s \gamma_i^2 \) of \( f_{\varepsilon} \) with rational coefficients.

Let us now consider the remainder polynomial \( u = f - \sum_{i=1}^s \gamma_i^2 \). The second loop of Algorithm \texttt{intsos} defined from line 9 to line 17 terminates when for all \( \alpha \in Q, \varepsilon - \sum_{\beta \in Q} |b_{\alpha \beta}|/2 + u_\alpha \geq 0 \). This condition is fulfilled when for all \( \alpha \in Q, \varepsilon - \sum_{\beta \in Q} |b_{\alpha \beta}|/2 + u_\alpha \geq 0 \). This latter condition holds when for all \( \gamma \in \text{supp}(u), |u_\gamma| \leq \frac{\varepsilon}{2} \).

Next, we show that this happens when the precisions \( \delta \) of \texttt{sdp} and \( \delta_r \) of \texttt{cholesky} are both large enough. From the definition of \( u \), one has for all \( \gamma \in \text{supp}(u), u_\gamma = f_\gamma - \varepsilon_\gamma - \sum_{\alpha \in \gamma} G_{\alpha \beta} \), where \( \varepsilon_\gamma = \varepsilon \) when \( \gamma \in (2\mathbb{N})^n \) and \( \varepsilon_\gamma = 0 \) otherwise. The positive definite matrix \( \tilde{G} \) computed by the SDP solver is an approximation of an exact Gram matrix of \( f_{\varepsilon} \). At precision \( \delta \), one has for all \( \gamma \in \text{supp}(f), \tilde{G} \succeq 2^{-\delta}I \) and

\[
|f_\gamma - \varepsilon_\gamma - \text{Tr}(\tilde{G}_{\gamma \gamma})| = |f_\gamma - \varepsilon_\gamma - \sum_{\alpha \in \gamma} G_{\alpha \beta}| \leq 2^{-\delta}.
\]

In addition, it follows from (2) that the approximated Cholesky decomposition \( LL^T = \tilde{G} + F \) with

\[
|F_{\alpha \beta}| \leq \frac{(r+1)^2 - 2^{-\delta} \cdot |\tilde{G}_{\alpha \alpha} \tilde{G}_{\beta \beta}|}{10}.
\]
for all \( \alpha, \beta \in Q \). Moreover, by using Cauchy-Schwartz inequality, one has

\[
\sum_{\alpha \in Q} \tilde{G}_{\alpha, \alpha} = \text{Tr} \tilde{G} \leq \sqrt{\text{Tr} I} \sqrt{\text{Tr} \tilde{G}^2} \leq \sqrt{r} R.
\]

For all \( \gamma \in \text{supp}(u) \), this yields

\[
| \sum_{\alpha \neq \beta, \gamma} \tilde{G}_{\alpha, \alpha} \tilde{G}_{\beta, \gamma} |^2 \leq \sum_{\alpha \neq \beta, \gamma} \frac{\tilde{G}_{\alpha, \alpha}}{2} + \frac{\tilde{G}_{\beta, \gamma}}{2} \leq \text{Tr} \tilde{G} \leq \sqrt{r} R,
\]

where the first inequality comes again from Cauchy-Schwartz inequality.

Thus, for all \( \gamma \in \text{supp}(u) \), one has

\[
| \sum_{\alpha \neq \beta, \gamma} \tilde{G}_{\alpha, \alpha} \tilde{G}_{\beta, \gamma} | \leq \sum_{\alpha \neq \beta, \gamma} \frac{\tilde{G}_{\alpha, \alpha}}{2} + \frac{\tilde{G}_{\beta, \gamma}}{2} \leq \sqrt{r} R.
\]

Now, let us take the smallest \( \delta \) such that \( 2^{-\delta} \leq \frac{r}{2r} \) as well as the smallest \( \delta_c \) such that

\[
2^{-\delta_c} \leq \frac{1}{1 - (r + 1)2^{-\delta}}.
\]

From the previous inequalities, for all \( \gamma \in \text{supp}(u) \), it holds that

\[
|\bar{u}| = |f_r - e_r - (\sum_{i=1}^{r} s_i^2)\gamma| \leq |f_r - e_r - \tilde{G}_{\alpha, \alpha}\gamma| + |\sum_{\alpha \neq \beta, \gamma} \tilde{G}_{\alpha, \alpha} \bar{G}_{\beta, \gamma}| + |\sum_{\alpha \neq \beta, \gamma} \tilde{G}_{\alpha, \alpha} - (\sum_{i=1}^{r} s_i^2)\gamma| \leq \frac{\varepsilon}{2r} + \frac{\varepsilon}{2r} = \frac{\varepsilon}{r}.
\]

This ensures that Algorithm \textsc{intsos} terminates.

Let us note

\[
\Delta(u) := \{(\alpha, \beta) : \alpha + \beta \in \text{supp}(u), \alpha, \beta \in Q, \alpha \neq \beta\}.
\]

When terminating, the first output \( s_{\text{list}} \) of Algorithm \textsc{intsos} is a list of non-negative rational numbers containing the list \( \{1, \ldots, 1\} \) of length \( r \), the list \( \{\frac{1}{2}, \ldots, \frac{1}{2} : (\alpha, \beta) \in \Delta(u)\} \) and the list \( \{e_\alpha : \alpha \in Q\} \). The second output \( \alpha_{\text{list}} \) of Algorithm \textsc{intsos} is a list of polynomials containing the list \( \{s_1, \ldots, s_r\} \), the list \( \{X^\alpha + \text{sgn}(u_{\alpha, \beta})X^\beta : (\alpha, \beta) \in \Delta(u)\} \) and the list \( \{X^\alpha : \alpha \in Q\} \). From the output, we obtain the following weighted SOS decomposition

\[
f = \sum_{i=1}^{r} s_i^2 + \sum_{\alpha, \beta \neq \alpha, \beta} \frac{|u_{\alpha, \beta}|}{2} (X^\alpha + \text{sgn}(u_{\alpha, \beta})X^\beta)^2 + \sum_{\alpha \in Q} e_\alpha X^{2\alpha}.
\]

Now, we bound the bit size of the coefficients.

Since \( r \leq \binom{n+k}{n} \leq d^n \) and \( N \leq \tau d^{\phi_Q} \), one has \( \delta \leq \tau d^{\phi_Q} \). Similarly, \( \delta_c \leq \tau d^{\phi_Q} \).

This bounds not only the maximal bit size of the coefficients involved in the approximate decomposition \( \sum_{i=1}^{r} s_i^2 \) but also the coefficients of \( u \). In the worst case, the coefficient \( e_\alpha \) involved in the exact SOS decomposition is equal to \( e - \sum_{\beta \in Q} |u_{\alpha, \beta}|/2 + u_\alpha \) for some \( \alpha \in Q \). Using again that the cardinal \( r \) of \( Q \) is less than \( \binom{n+k}{n} \leq d^n \), we obtain a maximum bit size upper bounded by \( \tau d^{\phi_Q} \). \qed
3.3. Bit complexity analysis

**Theorem 11.** For $f$ as above, there exist $\varepsilon, \delta, R, \delta_1$ of bit sizes upper bounded by $\tau d^{\Theta(\varepsilon)}$ such that $\text{intsos}(f, \varepsilon, R, \delta_1)$ runs in boolean time $\tau^2 d^{\Theta(\varepsilon)}$.

**Proof.** We consider $\varepsilon, \delta, R$ and $\delta_1$ as in the proof of Proposition 10 so that Algorithm $\text{intsos}$ only performs a single iteration within the two while loops before terminating. Thus, the bit size of each input parameter is upper bounded by $\tau d^{\Theta(\varepsilon)}$.

Computing $C(f)$ with the quickhull algorithm runs in boolean time $O(V^2)$ for a polytope with $V$ vertices. In our case $V \leq (n+d) \leq 2d^\gamma$, so that this procedure runs in boolean time $O(d^{2\gamma})$. Next, we investigate the computational cost of the call to $\text{sdp}$ at line 8. Let us note $n_{\text{sdp}} = r$ (resp. $m_{\text{sdp}}$) the size (resp. number of entries) of $G$. This step consists of solving SDP (1), which is performed in $O(n_{\text{sdp}}^2 \log(2^\delta n_{\text{sdp}} R 2^\beta))$ iterations of the ellipsoid method, where each iteration requires $O(n_{\text{sdp}}^2 (m_{\text{sdp}} + n_{\text{sdp}}))$ arithmetic operations over $\log_2(2^\delta n_{\text{sdp}} R 2^\varepsilon)$-bit numbers (see e.g. (Grötschel et al., 1993; Porkolab and Khachiyan, 1997)). Since $m_{\text{sdp}}, n_{\text{sdp}}$ are both bounded above by $(n+d) \leq 2d^\gamma$, one has

$$\log_2(2^\delta n_{\text{sdp}} R 2^\beta) \leq \tau d^{\Theta(\varepsilon)},$$

$$n_{\text{sdp}}^2 (m_{\text{sdp}} + n_{\text{sdp}}) \leq O(d^{2\Theta(\varepsilon)}),$$

$$n_{\text{sdp}}^4 \log_2(2^\delta n_{\text{sdp}} R 2^\varepsilon) \leq \tau d^{\Theta(\varepsilon)}.$$  

Overall, the ellipsoid algorithm runs in boolean time $\tau^2 d^{\Theta(\varepsilon)}$ to compute the approximate Gram matrix $\tilde{G}$. We end with the cost of the call to Cholesky at line 3. Cholesky’s decomposition is performed in $O(n_{\text{sdp}}^3)$ arithmetic operations over $\delta$-bit numbers. Since $\delta \leq \tau d^{\Theta(\varepsilon)}$, the function runs in boolean time $\tau d^{\Theta(\varepsilon)}$. The other elementary arithmetic operations performed while running Algorithm $\text{intsos}$ have a negligible cost w.r.t. to the $\text{sdp}$ procedure. \qed

3.4. Comparison with the rounding-projection algorithm of Peyrl and Parrilo

We recall the algorithm designed in Peyrl and Parrilo (2008). We denote this rounding-projection algorithm by RoundProject.

The first main step in Line 5 consists of rounding the approximation $\tilde{G}$ of a Gram matrix associated to $f$ up to precision $\delta_1$. The second main step in Line 6 consists of computing the orthogonal projection $G$ of $\tilde{G}$ on an adequate affine subspace in such a way that $\sum_{\alpha \in \text{supp}(f)} G_{\alpha, \alpha} = f_\gamma$, for all $\gamma \in \text{supp}(f)$. For more details on this orthogonal projection, we refer to Peyrl and Parrilo, 2008, Proposition 7). The algorithm then performs in (10) an exact diagonalization of the matrix $G$ via the $\text{LDL}^T$ decomposition (see e.g. (Golub and Loan, 1996 § 4.1)). It is proved in Peyrl and Parrilo (2008, Proposition 8) that for $f \in \Sigma[X]$, Algorithm RoundProject returns a weighted SOS decomposition of $f$ with rational coefficients when the precision of the rounding and SDP solving steps are large enough.

The main differences w.r.t. Algorithm $\text{intsos}$ are that RoundProject does not perform a perturbation of the input polynomial $f$ and computes an exact $\text{LDL}^T$ decomposition of a Gram matrix $G$. In our case, we compute an approximate Cholesky’s decomposition of $\tilde{G}$ instead of a projection, then perform an exact compensation of the error terms, thanks to the initial perturbation.

The next result gives upper bounds on the bit size of the coefficients involved in the SOS decomposition returned by RoundProject as well as upper bounds on the boolean running time.
the outputs a rational SOS decomposition of \( f \) with rational coefficients. The maximum bit size of the coefficients involved in this SOS decomposition is upper bounded by \( \tau d^{O(n)} \) and the boolean running time is \( \tau^2 d^{O(n)} \).

Proof. Let us assume that Algorithm RoundProject returns a matrix \( G \succ 0 \) associated to \( f \) with smallest eigenvalue \( \lambda \) and let \( N \in \mathbb{N} \) be the smallest integer such that \( 2^{-N} \leq \lambda \). As in Proposition[9] one proves that the bit size of \( N \) is upper bounded by \( \tau d^{O(n)} \). By Peyrl and Parrilo, 2008, Proposition 8), Algorithm RoundProject terminates and outputs such a matrix \( G \) together with a weighted rational SOS decomposition of \( f \) if \( 2^{-\delta} + 2^{-\delta'} \leq 2^{-N} \), where \( \delta' \) stands for the euclidean distance between \( G' \) and \( G \), yielding

\[
\sqrt{\sum_{a,\beta \in Q} (G_{a,\beta} - G'_{a,\beta})^2} \leq 2^{-\delta'}.
\]

For all \( a, \beta \in Q \), one has \( |G'_{a,\beta} - G_{a,\beta}| \leq 2^{-\delta} \). As in the proof of Proposition[10] at SDP precision \( \delta \), one has for all \( \gamma \in \text{supp}(f) \), \( \hat{G} \geq 2^{-I} I \) and

\[
|f_{\gamma} - \sum_{a,\beta : \gamma} \hat{G}_{a,\beta}| \leq 2^{-\delta}.
\]

For all \( a, \beta \in Q \), let us define \( e_{a,\beta} := \sum_{a' \neq a, \beta' \neq \beta} G'(a', \beta') - f_{a+\beta} \) and note that

\[
|e_{a,\beta}| \leq \sum_{a' \neq a, \beta' \neq \beta} |G'(a', \beta') - G(a', \beta')| + \sum_{a' \neq a, \beta' \neq \beta} |G(a', \beta') - f_{a+\beta}| \leq \eta(a + \beta) 2^{-\delta} + 2^{-\delta'}.
\]
For all $\alpha, \beta \in Q$, we use the fact that $\eta(\alpha + \beta) \geq 1$ and that the cardinal of $Q$ is less than the size $r$ of $G$, with $r \leq d^r$, to obtain
\[
2^{-\delta} = \sum_{\alpha, \beta \in Q} \frac{e^{\alpha \beta}}{\eta(\alpha + \beta)} \leq d^{2n}(2^{-\delta_i} + 2^{-\delta}).
\]
To ensure that $2^{-\delta_i} + 2^{-\delta} \leq 2^{-N}$, it is sufficient to have $(d^{2n} + 1)2^{-\delta_i} + d^{2n}2^{-\delta} \leq 2^{-N}$, which is obtained with $\delta_i$ and $\delta$ with bit size upper bounded by $\tau d^{\delta^{r+\alpha}}$. The bit size of the coefficients involved in the weighted SOS decomposition is upper bounded by the output bit size of the $LDL^T$ decomposition of the matrix $G$, that is $O(\delta, \tau^r) = \tau d^{\delta^{r+\alpha}}$.

The bound on the running time is obtained exactly as in Theorem 10.

4. Exact Reznick and Hilbert-Artin’s representations

Next, we show how to apply Algorithm intosos to decompose positive definite forms and positive polynomials into SOS of rational functions.

4.1. Exact Reznick’s representations

Let $G_n := \sum_{i=1}^n X_i^2$ and $S^{n-1} := \{x \in \mathbb{R}^n : G_n(x) = 1\}$ be the unit $(n-1)$-sphere. A positive definite form $f \in \mathbb{R}[X]$ is a homogeneous polynomial which is positive over $S^{n-1}$. For such a form, we set
\[
\epsilon(f) := \frac{\min_{x \in S^{n-1}} f(x)}{\max_{x \in S^{n-1}} f(x)},
\]
which measures how close $f$ is to having a zero in $S^{n-1}$. While there is no guarantee that $f \in \Sigma[X]$, Reznick [1995] proved that for large enough $D \in \mathbb{N}$, $fG_n^0 \in \Sigma[X]$. Such SOS decompositions are called Reznick’s representations and $D$ is called the Reznick’s degree. The next result states that for large enough $D \in \mathbb{N}$, $fG_n^0 \in \Sigma[X]$, as a direct consequence of Reznick [1995].

Lemma 13. Let $f$ be a positive definite form of degree $d = 2k$ in $\mathbb{Z}[X]$ and $D \geq \frac{n(d-1)}{4 \log 2 \epsilon(f)} - \frac{nD}{2} + 1$. Then $f G_n^0 \in \Sigma[X]$.

Proof. First, for any positive $e < \min_{x \in S^{n-1}} f(x)$, the form $(f - eG_n^0)$ is positive on $S^{n-1}$. Then, for any nonzero $x \in \mathbb{R}^n$, one has
\[
f(x) - eG_n(x)^{k} = G_n(x)^{k}(f\left(\frac{x}{G_n(x)}\right) - e) > 0,
\]
implying that $(f - eG_n^0)$ is positive definite. Next, Reznick [1995] Theorem 3.12) implies that for any positive integer $D_e$ such that
\[
D_e \geq D_e := \frac{nd(d-1)}{4 \log 2 \epsilon(f - eG_n^0)} - \frac{n + d}{2},
\]
one has $(f - eG_n^0) G_n^{D_e} \in \Sigma[X]$. One has $G_n^{D_e} = \sum_{i=k+d_e} \left(\begin{array}{c} k+d_e \\ \varepsilon_i \end{array}\right) X_2^{d_e}$. Let $v_{k+d_e}(X)$ be the vector of monomials with exponents in $\mathbb{N}^n_{k+d_e}$. Then, one can write $G_n^{D_e} = v_{k+d_e}^T A v_{k+d_e}$ with $A$ being a diagonal matrix with positive entries $\left(\begin{array}{c} k+d_e \\ \varepsilon_i \end{array}\right)$, thus $A > 0$. Next we select $e$ small
This will yield the desired \( e \). Theorem 2 implies that the Newton polytope of \( f - eG_n^k \) is equal to \( \mathbb{N}^n_{D} \). This in turn implies that the Newton polytope of \( f - eG_n^k \) is equal to \( \mathbb{N}^n_{D} \). Since \( (f - eG_n^k) \in \Sigma[X] \), there exists \( A' \geq 0 \) indexed by \( \mathbb{N}^n_{D} \) such that \( f - eG_n^k = v_{k+D}^T A' v_{k+D} \). This yields \( fG_n^k = v_{k+D}^T (eA + A') v_{k+D} \). Since \( eA + A' > 0 \),

\[ (fG_n^k = v_{k+D}^T A' v_{k+D} \) is equal to \( \mathbb{N}^n_{D} \).

Next, with \( D := \frac{i_2}{i_2} \), we prove that there exists a large enough \( N \in \mathbb{N} \) such that for \( e = \frac{1}{2} \), \( D_e \leq \frac{2}{2} \). Since \( fG_n^k \in \Sigma[X] \) for all \( D_e \), this will yield the desired result. For any \( x \in \mathbb{Q}^n \), one has \( \|G_n(x)\| = 1 \), thus

\[ \min_{x \in \mathbb{Q}^n} (f(x) - eG_n(x)^k) = \min_{x \in \mathbb{Q}^n} f(x) - e, \quad \max_{x \in \mathbb{Q}^n} (f(x) - eG_n(x)^k) = \max_{x \in \mathbb{Q}^n} f(x) - e. \]

Hence,

\[ \varepsilon(f - eG_n^k) = \frac{\min_{x \in \mathbb{Q}^n} f(x)[1 - 1/N]}{\min_{x \in \mathbb{Q}^n} f(x)[1/N]} = \frac{\varepsilon(f)(N-1)}{N-\varepsilon(f)}. \]

Therefore, one has \( D_e = \frac{N-\varepsilon(f)}{N-1} \frac{i_2}{i_2} \), yielding \( D_e = \frac{D}{N-1} \frac{i_2}{i_2} \). By choosing \( N > \frac{i_2}{i_2} \), one ensures that \( D_e - \frac{2}{2} \leq 1 \), which concludes the proof.

Algorithm Reznicksos takes as input \( f \in \mathbb{Z}[X] \), finds the smallest \( D \in \mathbb{N} \) such that \( fG_n^k \in \Sigma[X] \), thanks to an oracle which decides if some given polynomial is a positive definite form. Further, we denote by interiorSOScone a routine which takes as input \( f, G_n \) and \( D \) and returns true if and only if \( fG_n^k \in \Sigma[X] \), else it returns false. Then, interiorSOScone is applied on \( fG_n^k \).

Algorithm 4 Reznicksos

Input: \( f \in \mathbb{Z}[X] \), positive \( e \in \mathbb{Q} \), precision parameters \( \delta, R \in \mathbb{N} \) for the SDP solver, precision \( \delta_e \in \mathbb{N} \) for the Cholesky’s decomposition

Output: list c_list of numbers in \( \mathbb{Q} \) and list s_list of polynomials in \( \mathbb{Q}[X] \)

1: \( D := 0 \)
2: while interiorSOScone(f G_n, D) = false do \( D := D + 1 \)
3: done
4: return interiorSOScone(f G_n^k, e, \delta, R, \delta_e)

Example 14. Let us apply Reznicksos on the perturbed Motzkin polynomial

\[ f = (1 + 2^{20})(X_1^2 + X_1^2 X_2^2 + X_1^2 X_2^2) - 3X_1^2 X_2^2 X_3^2. \]

With \( D = 1 \), one has \( f = (X_1^2 + X_2^2 + X_3^2) f \in \Sigma[X] \) and interiorSOScone yields an SOS decomposition of \( f \) with \( e = 2^{-20} \), \( \delta = R = 60 \), \( \delta_e = 10 \).

Theorem 15. Let \( f \in \mathbb{Z}[X] \) be a positive definite form of degree \( d \), coefficients of bit size at most \( \tau \). On input \( f \), Algorithm Reznicksos terminates and outputs a weighted SOS decomposition for \( f \). The maximum bit size of the coefficients involved in the decomposition and the boolean running time of the procedure are both upper bounded by \( 2^{O(d \tau \log^{(e)} d)} \).

The proof of this result relies on the following technical statement whose proof is postponed in the Appendix.
Proposition 16. Let $g \in \mathbb{Z}[X]$ of degree $d$ and let $\tau = \tau(g)$. Assume that the algebraic set $V(g) \subset \mathbb{C}^n$ defined by $g = 0$ is smooth. Then, there exists a polynomial $w \in \mathbb{Z}[X_1]$ of degree $\leq d^6$ with coefficients of bit size $\leq \tau \cdot (4d + 2)^{3n}$ such that its set of real roots contains the critical values of the restriction of the projection on the $X_1$-axis to $V(g)$.

Proof of Theorem 15. By Lemma 3, the while loop from line 2 to 3 is ensured to terminate for a positive integer $D \geq \frac{\log d - 1}{4 \log 2^{n+1}} - \frac{\epsilon_2 d}{4}$. By Proposition 10 when applying \texttt{int}$\rightarrow$\texttt{reals} to $f G^d_{\mathbb{Z}}$, the procedure always terminates. The outputs are a list of non-negative rational numbers $[c_1, \ldots, c_r]$ and a list of rational polynomials $[s_1, \ldots, s_r]$ providing the weighted SOS decomposition $f G^d_{\mathbb{Z}} = \sum_{i=1}^r c_is_i^2$. Thus, we obtain $f = \sum_{i=1}^r \frac{c_i}{s_i}$, yielding the first claim.

Since, $(X_1^2 + \cdots + X_n^2)^D = \sum_{|\alpha| = D} D_{\alpha} X^{2\alpha}$, each coefficient of $G^d_{\mathbb{Z}}$ is upper bounded by $\sum_{|\alpha| = D} D_{\alpha} = n^D$. Thus, $\tau f G^d_{\mathbb{Z}} \leq \tau + D \log n$. Using again Proposition 10 the maximum bit size of the coefficients involved in the weighted SOS decomposition of $f G^d_{\mathbb{Z}}$ is upper bounded by $(\tau + D \log n)(d + D)^{(d+D)^{m\cdot n}}$. Now, we derive an upper bound on $D$. Since $f$ is a positive form of degree $d$, one has

$$\min_{x \in \mathbb{S}^{n-1}} f(x) = \max \{ |e| : \forall x \in \mathbb{R}^n, f(x) - e G_n(x)^d \geq 0 \}.$$

We rely on Proposition 16 to show that $\min_{x \in \mathbb{S}^{n-1}} f(x) \geq 2^{-\tau \cdot (4d+6)^{m\cdot n}}$. For this, let us define the polynomial $g(X,E) := f - E G_n^d$, and consider the algebraic set $V$ defined by:

$$V := \left\{ (x, e) \in \mathbb{R}^{n+1} : g(x, e) = \frac{\partial g}{\partial x_1} = \cdots = \frac{\partial g}{\partial x_n} = 0 \right\}.$$

First, note that the minimum of $f$ on the sphere belongs to the projection of $V$ on the $E$-axis. Using Proposition 16 there exists a polynomial in $\mathbb{Z}[E]$ of degree less than $(d + 1)^{n+1}$ with coefficients of bit size less than $\tau \cdot (4d+6)^{m\cdot n}$ such that its set of real roots contains the projection of $V$ on the $E$-axis. By Lemma 3, the bit size of the minimum of $f$ on the sphere is upper bounded by $\tau \cdot (4d+6)^{m\cdot n}$.

Similarly, we obtain $\max_{x \in \mathbb{S}^{n-1}} f(x) \leq 2^{-\tau \cdot (4d+6)^{m\cdot n}}$ and thus $\frac{1}{nf} \leq 2^{2^{\tau \cdot (4d+6)^{m\cdot n}}}$. Overall, we obtain

$$\frac{nd(d-1)}{4 \log 2^{n+1}} + \frac{n + d}{2} + 1 \leq D \leq 2^{\tau \cdot (4d+6)^{m\cdot n}}.$$

This implies that

$$(\tau + D \log n) \cdot (d + D)^{(d+D)^{m\cdot n}} \leq 2^{2^{\tau \cdot (4d+6)^{m\cdot n}}}.$$

From Theorem 11 the running time is upper bounded by $(\tau + D \log n)^2 \cdot (d + D)^{(d+D)^{m\cdot n}}$, which ends the proof.

The bit complexity of \texttt{Reznick}$\rightarrow$\texttt{real} is polynomial in the \texttt{Reznick}’s degree $D$ of the representation. In all the examples we tackled, this degree was rather small as shown in Section 6.

4.2. Exact Hilbert- Artin’s representations

Here, we focus on the subclass of non-negative polynomials in $\mathbb{Z}[X]$ which admit an Hilbert-Artin’s representation of the form $f = \frac{\sigma}{h}$, with $h$ being a nonzero polynomial in $\mathbb{R}[X]$ and $\sigma \in \Sigma[X]$. We start to recall the famous result by Artin, providing a general solution to Hilbert’s 17th problem:
Theorem 17. [Artin, 1927] Theorem 4) Let \( f \in \mathbb{R}[X] \) be a polynomial non-negative over the reals. Then, \( f \) can be decomposed as a sum of squares of rational functions with rational coefficients and there exist a nonzero \( h \in \mathbb{Q}[X] \) and \( \sigma \in \mathbb{Q}[X] \) such that \( f = \frac{h^2}{\sigma} \).

Given \( f \in \mathbb{R}[X] \) non-negative over the reals, let us note \( \deg f = d = 2k \), and \( \tau = \tau(f) \). Given \( D \in \mathbb{N} \), we denote by \( S_D \) the convex hull of the set

\[
\text{supp}(f) + N_{2D}^\mathbb{N} = \{ \alpha + \beta \mid \alpha \in \text{supp}(f), \beta \in N_{2D}^\mathbb{N} \} \subseteq N_{d+2D}^\mathbb{N}.
\]

Finally, we set \( Q_D := S_D / 2 \cap N_{d+D}^\mathbb{N} \).

To compute Hilbert-Artin’s representation, one can solve the following SDP program:

\[
\begin{align*}
\sup_{G \geq 0} \ & \text{Tr } G \\
\text{s.t.} \ & \text{Tr } (H F_\gamma) = \text{Tr } (G B_\gamma), \quad \forall \gamma \in Q_D, \\
\ & \text{Tr } (H) = 1.
\end{align*}
\]

where \( B_\gamma \) is as for SDP (4), with rows (resp. columns) indexed by \( Q_D \), and \( F_\gamma \) has rows (resp. columns) indexed by \( N_D^\mathbb{N} \) with (\( \alpha, \beta \)) entry equal to \( \sum_{\gamma \in \mathbb{N}^\alpha} f_\gamma \). Let us now provide the rationale behind SDP (4). The first set of trace equality constraints allows one to find a Gram matrix \( H \) associated to \( h^2 \), with rows (resp. columns) indexed by \( N_D^\mathbb{N} \), as well as a Gram matrix \( G \) associated to \( \sigma \), with rows (resp. columns) indexed by \( Q_D \). The last trace equality constraint allows one to ensure that \( H \) is not the zero matrix. Note that we are only interested in finding a strictly feasible solution for SDP (4), thus we can choose any objective function. Here, we maximize the trace, as we would like to obtain a full rank matrix for \( G \).

Proposition 18. Let \( f \in \mathbb{Z}[X] \) be a polynomial non-negative over the reals, with \( \deg f = d = 2k \). Let us assume that \( f \) admits the Hilbert-Artin’s representation \( f = \frac{h^2}{\sigma} \), with \( \sigma \in \hat{\Sigma}[X], h \in \mathbb{Q}[X], \deg h = D \in \mathbb{N} \) and \( \deg \sigma = 2(D+k) \). Let \( Q_D \) be defined as above. Then, there exist \( \tilde{\sigma}_D, \hat{\sigma} \in \hat{\Sigma}[X] \) such that

\[
\tilde{\sigma}_D f = \hat{\sigma},
\]

ensuring the existence of a strictly feasible solution \( G, H > 0 \) for SDP (4).

Proof. By applying Proposition 10 to \( h^2 f \), there exists \( \epsilon > 0 \) such that \( \hat{\sigma} := h^2 f - \epsilon \sum_{\gamma \in Q_D} X^{2\gamma} \in \hat{\Sigma}[X] \). In addition, for all \( \lambda > 0 \), one has

\[
h^2 f = h^2 f + \lambda f \sum_{\alpha \in N_D^\mathbb{N}} X^{2\alpha} = \left( h^2 + \lambda \sum_{\alpha \in N_D^\mathbb{N}} X^{2\alpha} \right) f - \lambda f \sum_{\alpha \in N_D^\mathbb{N}} X^{2\alpha} = \hat{\sigma} + \epsilon \sum_{\alpha \in N_D^\mathbb{N}} X^{2\alpha}.
\]

Let us define \( u_\lambda := \lambda f \sum_{\alpha \in N_D^\mathbb{N}} X^{2\alpha} \). As in the proof of Proposition 10, we show that for small enough \( \lambda \), the polynomial \( \epsilon \sum_{\alpha \in Q_D} X^{2\alpha} + u_\lambda \) belongs to \( \Sigma[X] \). Fix such a \( \lambda \), and define \( \tilde{\sigma} := \hat{\sigma} + \epsilon \sum_{\alpha \in Q_D} X^{2\alpha} + u_\lambda \) and \( \tilde{\sigma}_D := h^2 + \lambda \sum_{\alpha \in N_D^\mathbb{N}} X^{2\alpha} \). Since \( \tilde{\sigma} \in \Sigma[X] \), there exists a positive definite Gram matrix \( G \) associated to \( \tilde{\sigma} \). Similarly, there exists a positive definite Gram matrix \( H \) associated to \( \tilde{\sigma}_D \). By Theorem 2, this implies that \( \tilde{\sigma}, \tilde{\sigma}_D \in \Sigma[X] \), showing the claim.

To find such representations in practice, we consider a perturbation of the trace equality constraints of SDP (4) where we replace the matrix \( G \) by the matrix \( G - \epsilon I \):

\[
P^\epsilon : \sup_{G \geq 0} \text{Tr } G
\]

\[
\text{s.t.} \quad \text{Tr } (H F_\gamma) = \text{Tr } (G B_\gamma) - \epsilon \text{Tr } (B_\gamma), \quad \forall \gamma \in Q_D,
\]

\[
\text{Tr } (H) = 1.
\]
For $D \in \mathbb{N}$, let us note $\tilde{\Sigma}_D(X) := \{ \frac{\sigma}{\sigma_D} : \sigma \in \tilde{\Sigma}[X], \sigma_D \in \Sigma[X] \text{ with } \deg \sigma_D \leq 2D \}$.

Algorithm \texttt{Hilbertsos} takes as input $f \in \mathbb{Z}[X]$, finds $\sigma_D \in \Sigma[X]$ of smallest degree $2D$ such that $f \sigma_D \in \tilde{\Sigma}[X]$, thanks to an oracle as in \texttt{intsos} (i.e., the smallest $D$ for which $f \in \tilde{\Sigma}_D(X)$).

Then, the algorithm finds the largest rational $\varepsilon > 0$ such that Problem $P^\varepsilon$ has a strictly feasible solution. Problem $P^\varepsilon$ is solved by calling the \texttt{sdp} function, relying on an SDP solver. Eventually, the algorithm calls the procedure \texttt{absorb}, as in \texttt{intsos}, to recover an exact rational SOS decomposition.

\begin{algorithm}
\textbf{Algorithm 5 Hilbertsos}
\begin{itemize}
    \item \textbf{Input:} $f \in \mathbb{Z}[X]$ of degree $d = 2k$, positive $\varepsilon \in \mathbb{Q}$, precision parameters $\delta, R \in \mathbb{N}$ for the SDP solver, precision $\delta_\epsilon \in \mathbb{N}$ for the Cholesky's decomposition lists $c\_list_1, c\_list_2$ of numbers in $\mathbb{Q}$ and lists $s\_list_1, s\_list_2$ of polynomials in $\mathbb{Q}[X]$.
    \item $D := 1$
    \item while $f \not\in \tilde{\Sigma}[X]/\tilde{\Sigma}_D[X]$ do $D := D + 1$
    \item done
    \item Compute the convex hull $\mathbb{S}_D$ of $\text{supp}(f) + \mathbb{N}_d^{2D}$
    \item $Q_D := \mathbb{S}_D / 2 \cap \mathbb{N}_d^{2D}$
    \item $t := \sum_{i \in Q_D} X_1^{2i}$
    \item while Problem $P^\varepsilon$ has no strictly feasible solution do $\varepsilon := \frac{\varepsilon}{2}$
    \item done
    \item $\text{ok := false}$
    \item while not $\text{ok}$ do
        \item $(\tilde{G}, \tilde{H}, \tilde{\lambda}_1, \tilde{\lambda}_2) := \text{sdp}(f, \varepsilon, \delta, R)$
        \item $(s_1, \ldots, s_{r_1}) := \text{cholesky}(\tilde{G}, \tilde{\lambda}_1, \delta_\epsilon)$
        \item $(s_2, \ldots, s_{r_2}) := \text{cholesky}(\tilde{H}, \tilde{\lambda}_2, \delta_\epsilon)$
        \item $\tilde{\delta} := \sum_{i=1}^{r_1} s_{i1}^2, \tilde{\delta}_D := \sum_{i=1}^{r_2} s_{i2}^2$
        \item $u := \tilde{\delta}_D - \tilde{\delta} - \varepsilon t$
        \item $c\_list_1 := [1, \ldots, 1], s\_list_1 := [s_1, \ldots, s_{r_1}]$
        \item $c\_list_2 := [1, \ldots, 1], s\_list_2 := [s_2, \ldots, s_{r_2}]$
        \item for $\alpha \in Q_D$ do $c\_\alpha := \varepsilon$
        \item done
        \item $c\_list_1, s\_list_1, c\_\alpha := \text{absorb}(u, Q_D, (\varepsilon_\alpha), c\_list_1, s\_list_1)$
        \item if $\min_{\alpha \in Q_D} \varepsilon_\alpha \geq 0$ then $\text{ok := true}$
        \item else $\delta := 2\delta_\epsilon, R := 2R, \delta_\epsilon := 2\delta_\epsilon$
        \item end
    \item done
    \item for $\alpha \in Q_D$ do $c\_list_1 := c\_list_1 \cup \{\varepsilon_\alpha\}, s\_list_1 := s\_list_1 \cup \{X_\alpha\}$
    \item done
    \item return $c\_list_1, c\_list_2, s\_list_1, s\_list_2$
\end{itemize}
\end{algorithm}

\textbf{Theorem 19.} Let $f \in \mathbb{Z}[X] \cap \tilde{\Sigma}_D(X)$ and assume that the SOS polynomials involved in the denominator of $f$ have coefficients of bit size at most $\tau_D \geq \tau$. On input $f$, Algorithm $\texttt{Hilbertsos}$ terminates and outputs a weighted SOS decomposition for $f$. There exist $\varepsilon, \delta, R, \delta_\epsilon$, of bit sizes upper bounded by $\tau_D (d+D)^{d+D \delta_\epsilon^{1/2}}$ such that $\texttt{Hilbertsos}(f, \varepsilon, \delta, R, \delta_\epsilon)$ runs in boolean running time $\tau_D^2 \cdot (d+D)^{d+D \delta_\epsilon^{1/2}}$.

\textbf{Proof.} Since $f \in \tilde{\Sigma}_D(X)$, the first loop of Algorithm $\texttt{Hilbertsos}$ terminates and there exists
a strictly feasible solution for SDP (4), by Proposition (15). Thus, there exists a small enough \( \varepsilon > 0 \) such that Problem \( \mathbf{P}^\varepsilon \) has also a strictly feasible solution. This ensures that the second loop of Algorithm \( \text{Hilbert\text{-}sos} \) terminates. Then, one shows as for Algorithm \( \text{intsos} \) that the absorption procedure succeeds, yielding termination of the third loop. Let us note

\[
\Delta_D(u) := \{ (\alpha, \beta) : \alpha + \beta \in \text{supp}(u), \alpha, \beta \in Q_D, \alpha \neq \beta \}.
\]

The first output \( c_\text{list}_1 \) of Algorithm \( \text{Hilbert\text{-}sos} \) is a list of non-negative rational numbers containing the list \([1, \ldots, 1]\) of length \( r_1 \), the list \([\frac{1}{2}, \ldots, \frac{1}{2}] \) of \( |\Delta_D(u)| \) and the list \([e_{\alpha} : \alpha \in Q_D] \). The second output \( s_\text{list}_1 \) of Algorithm \( \text{Hilbert\text{-}sos} \) is a list of polynomials containing the list \([s_{11}, \ldots, s_{r_11}] \), the list \([X^{\alpha} + \text{sgn}(u_{\alpha+\beta})X^{\beta} : (\alpha, \beta) \in \Delta_D(u)] \) and the list \([X^{\alpha} : \alpha \in Q_D] \). From these two outputs, one reconstructs the weighted SOS decomposition of the numerator \( f \) and the fourth output is a list of polynomials \([2, \ldots, 11] \), the list \([1, \ldots, 1] \) of degrees less than \( D \). In Lombardi et al. (2018), the degree \( D \) of the denominator, this degree can be rather large. In Lombardi et al. (2018), the authors provide an upper bound expressed with a tower of five exponentials for the degrees of the denominators involved in Hilbert-Artin’s representations.

Remark 20. Note that even if the bit complexity of \( \text{Hilbert\text{-}sos} \) is singly exponential in the degree \( D \) of the denominator, this degree can be rather large. In Lombardi et al. (2018), the authors provide an upper bound expressed with a tower of five exponentials for the degrees of the denominators involved in Hilbert-Artin’s representations.

5. Exact Putinar’s representations

We let \( f, g_1, \ldots, g_m \in \mathbb{Z}[X] \) of degrees less than \( d \in \mathbb{N} \) and \( r \in \mathbb{N} \) be a bound on the bit size of their coefficients. Assume that \( f \) is positive over \( S := \{ x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \} \) and reaches its infimum with \( f^* := \min_{x \in S} f(x) > 0 \). With \( \bar{f} = \sum_{|\sigma| \leq D} f_{\sigma}X^{\sigma} \), we set \( \|f\| := \max_{|\sigma| \leq D} \frac{f_{\sigma}}{\bar{f}^{\frac{|\sigma|}{D}}} \) and \( g_0 := 1 \).

We consider the quadratic module \( Q(S) := \{ \sum_{j=0}^m \sigma_j g_j : \sigma_j \in \Sigma[X] \} \) and, for \( D \in \mathbb{N} \), the \( D \)-truncated quadratic module \( Q_D(S) := \{ \sum_{j=0}^m \sigma_j g_j : \sigma_j \in \Sigma[X], \deg(\sigma_j g_j) \leq D \} \) generated by \( g_1, \ldots, g_m \). We say that \( Q(S) \) is archimedean if \( N - G_n(Q(S)) \) for some \( N \in \mathbb{N} \) (recall that \( G_n := \sum_{i=1}^n \mathbb{X}^n \)). We also assume in this section:

Assumption 21. The set \( S \) is a basic compact semi-algebraic set with nonempty interior, included in \([-1, 1]^n \) and \( Q(S) \) is archimedean.

Under Assumption 21 \( f \) is positive over \( S \) only if \( f \in Q_D(S) \) for some \( D \in 2\mathbb{N} \) (see Putinar (1993)). In this case, there exists a Putinar’s representation \( f = \sum_{i=0}^m \sigma_j g_j \) with \( \sigma_j \in \Sigma[X] \) for \( 0 \leq j \leq m \). One can certify that if \( f \in Q_D(S) \) for \( D = 2k \) by solving the next SDP with \( k \geq \lceil d/2 \rceil \):

\[
\inf_{G_0, G_1, \ldots, G_m \geq 0} \text{Tr}(G_0 B_0) + \sum_{i=1}^m g_i(0) \text{Tr}(G_i C_{i0}) \quad \text{subject to:} \quad \text{Tr}(G_0 B_r) + \sum_{j=1}^m \text{Tr}(G_j C_{rj}) = f_r, \quad \forall r \in \mathbb{N}_D - \{0\},
\]
where \( B_j \) is as for SDP (1) and \( C_{ij} \) has rows (resp. columns) indexed by \( N^m_{k-w_j} \) with \((\alpha,\beta)\) entry equal to \( \sum_{\alpha+i\beta+\gamma} g_{\beta} \). SDP (5) is a reformulation of the problem

\[
f_D^* := \sup \{ b : f - b \in Q_D(S) \}.
\]

Thus \( f_D^* \) is also the optimal value of SDP (5). The next result follows from [Lasserre, 2001, Theorem 4.2):

**Theorem 22.** We use the notation and assumptions introduced above. For \( D \in 2\mathbb{N} \) large enough, one has

\[
0 < f_D^* \leq f^*.
\]

In addition, SDP (5) has an optimal solution \((G_0,G_1,\ldots,G_m)\), yielding the following Putinar’s representation:

\[
f - f_D^* = \sum_{i=1}^{r} \lambda_i q_i^2 + \sum_{i=1}^{m} g_j \sum_{i=1}^{r_j} \lambda_i q_i^2,
\]

where the vectors of coefficients of the polynomials \( q_{ij} \) are the eigenvectors of \( G_j \) with respective eigenvalues \( \lambda_j \), for all \( j = 0,\ldots,m \).

The complexity of Putinar’s Positivstellensätze was analyzed by [Nie and Schweighofer, 2007]:

**Theorem 23.** With the notation and assumptions introduced above, there exists a real \( \chi_S > 0 \) depending on \( S \) such that

(i) for all even \( D \geq \chi_S \exp(d^2n^{d(1+\frac{1}{2d})}) \), \( f \in Q_D(S) \).

(ii) for all even \( D \geq \chi_S \exp(2d^2n^d) \), \( 0 \leq f^* - f_D^* \leq \frac{6d^2n^d}{\sqrt{\log D}} \).

From a computational viewpoint, one can certify that \( f \) lies in \( Q_D(S) \) for \( D = 2k \) large enough, by solving SDP (5). Next, we show how to ensure the existence of a strictly feasible solution for SDP (5) after replacing the set defined by our initial constraints \( S \) by the following one

\[
S' := \{ x \in S : 1 - x^{2\alpha} \geq 0, \forall \alpha \in N^m_{k} \}.
\]

### 5.1. Preliminary results

We first give a lower bound for \( f^* \).

**Proposition 24.** With the above notation and assumptions, one has

\[
f^* \geq 2^{-r(d+\log_2(n+1)d+1)} - O(d^{\frac{d}{2}}) \geq 2^{-r(d^{d+2})}.
\]

**Proof.** Let \( Y = (Y_1,\ldots,Y_n) \) and \( \bar{f} \in \mathbb{R}[Y] \) be the polynomial obtained by replacing \( Y_i \) by \( 2nY_i - 1 \) in \( f \). Note that if \( x = (x_1,\ldots,x_n) \in S \subseteq [-1,1]^n \), then \( y = \left( \left( \frac{x_1}{2n} \right) \right)_{1 \leq i \leq n} \) lies in the standard simplex \( \Delta_n \), so the polynomial \( \bar{f} \) takes only positive values over \( \Delta_n \). Since \( x_t = 2ny_t - 1 \) and \( (2n-1)d \leq (2n)^d \), the polynomial \( \bar{f} \) has coefficients of bit size at most \( r + d + d \log_2 n \). Then, the inequality follows from [Jeronimo and Perrucci, 2010, Theorem 1], stating that

\[
\min_{y \in \Delta_n} \bar{f}(y) > 2^{-r(d+1)n^d} d^{-(n+1)d+1}.
\]

We obtain the second inequality after noticing that for all \( d \geq 2 \), one has \( d \log_2(n^d) \leq d^2n^d \leq d^{d+1}n^d \leq d^{2d+1}n^d \leq 2^{d^2}d^{d^2} \). \( \square \)
Theorem 25. We use the notation and assumptions introduced above. There exists $D \in 2\mathbb{N}$ such that:
(i) $f \in Q_D(S)$ with the representation
\[ f = f_D^* + \sum_{j=0}^{m} \sigma_j g_j, \]
for $f_D^* > 0$, $\sigma_j \in \Sigma[X]$ with $\deg(\sigma_j g_j) \leq D$ for all $j = 0, \ldots, m$.
(ii) $f \in Q_D(S')$ with the representation
\[ f = \sum_{j=0}^{m} \sigma_j g_j + \sum_{|\alpha| \leq k} c_{\alpha}(1-X^{2\alpha}), \]
for $\sigma_j \in \hat{\Sigma}[X]$ with $\deg(\sigma_j g_j) \leq D$, for all $j = 0, \ldots, m$, and some sequence of positive numbers $(c_{\alpha})_{|\alpha| \leq k}$.
(iii) There exists a real $C_S > 0$ depending on $S$ and $\varepsilon = \frac{\varepsilon}{2^n}$ with positive $N \in \mathbb{N}$ such that
\[ f - \varepsilon \sum_{|\alpha| \leq k} X^{2\alpha} \in Q_D(S'), \quad N \leq 2C_S\varepsilon^{2n+2}, \]
where $\tau$ is the maximal bit size of the coefficients of $f, g_1, \ldots, g_m$.

Proof. Let $\chi_S$ be as in Theorem 23 and $D = 2k$ be the smallest integer larger than
\[ D := \max \{ \chi_S \exp \left( \frac{12d^2n^2(\|f\|)}{f^*} \right), \chi_S(2d^2n^2)^{\frac{1}{2}} \}. \]

Theorem 23 implies that $f \in Q_D(S)$ and $f^* - f_D^* \leq \frac{6d^2n^2(\|f\|)}{\chi_S \log \frac{N}{\tau}} \leq f^*$. For each term $t_j : = \sum_{|\alpha| \leq k} X^{2\alpha}$, let us define
\[ t_0 := \sum_{|\alpha| \leq k} X^{2\alpha}, \quad t := \sum_{j=0}^{m} t_j g_j. \]

For a given $\nu > 0$, we use the perturbation polynomial $-\nu t = -\nu \sum_{|\alpha| \leq D} t_\gamma X^{\gamma}$. For each term $-t_\gamma X^{\gamma}$, one has $\gamma = \alpha + \beta$ with $\alpha, \beta \in \mathbb{N}_k$, thus
\[ -t_\gamma X^{\gamma} = |t_\gamma|(-1 + \frac{1}{2}(1-X^{2\alpha}) + \frac{1}{2}(1-X^{2\beta}) + \frac{1}{2}(X^{\gamma} - \text{sgn}(t_\gamma)X^{\gamma^2})). \]

As in the proof of Proposition 10 let us note
\[ \Delta(t) := \{(\alpha, \beta) : \alpha + \beta \in \text{supp}(t), \alpha, \beta \in \mathbb{N}_k, \alpha \neq \beta \}. \]

Hence, for all $\alpha \in \mathbb{N}_k$, there exists $d_\alpha \geq 0$ such that
\[ f = f - \nu t + \nu t = f_D^* - \sum_{|\gamma| \leq D} \nu |t_\gamma| + \sum_{j=0}^{m} \sigma_j g_j + \nu \sum_{|\alpha| \leq k} d_\alpha (1-X^{2\alpha}) + \nu \sum_{(\alpha, \beta) \in \text{supp}(t)} \frac{|t_\alpha \beta|}{2}(X^{\alpha} - \text{sgn}(t_\alpha \beta)X^{\beta^2}). \]
Since one has not necessarily $d_\alpha > 0$ for all $\alpha \in \mathbb{N}_k^n$, we now explain how to handle the case when $d_\alpha = 0$ for $\alpha \in \mathbb{N}_k^n$. We write

$$- \sum_{|\nu| \leq D} \nu |\nu| + \sum_{|\nu| \leq k} d_\nu(1 - X^{2\nu}) = - \sum_{|\nu| \leq D} \nu |\nu| - \sum_{|\nu| = 0} \nu + \sum_{|\nu| = 0} \nu(1 - X^{2\nu}) + \sum_{|\nu| = 0} \nu X^{2\nu} + \sum_{|\nu| = 0} d_\nu(1 - X^{2\nu}) + \sum_{|\nu| = 0} d_\nu(1 - X^{2\nu}).$$

For $\alpha \in \mathbb{N}_k^n$, we define $c_\alpha := \nu$ if $d_\alpha = 0$ and $c_\alpha := d_\alpha$ otherwise, $a := \sum_{|\nu| \leq D} |\nu| + \sum_{|\alpha|, d_\alpha = 0} 1$, $\hat{\sigma}_j := \sigma_\nu + \nu \tau_\alpha$, for each $j = 1, \ldots, m$ and

$$\sigma_0 := f^*_D - \nu a + \nu t_\alpha + \nu \sum_{|\alpha|, d_\alpha = 0} \frac{|\nu|}{2} \left( X^\alpha - \text{sgn}(t_{\alpha+\beta})X^\beta \right) + \sum_{|\nu| = 0} \nu X^{2\nu}.$$

So, there exists a sequence of positive numbers $(c_\alpha)_{|\alpha| \leq k}$ such that

$$f = \sum_{j=0}^m \hat{\sigma}_j g_j + \sum_{|\alpha| \leq k} c_\alpha(1 - X^{2\nu}).$$

Now, let us select $\nu := \frac{1}{2^m}$ with $M$ being the smallest positive integer such that $0 < \nu \leq \frac{\frac{1}{2^m}}{M}$. This implies the existence of a positive definite Gram matrix for $\hat{\sigma}_0$, thus by Theorem 2, $\hat{\sigma}_0 \in \Sigma[X]$. Similarly, for $1 \leq j \leq m$, $\hat{\sigma}_j$ belongs to $\tilde{\Sigma}[X]$, which proves the second claim.

(iii) Let $N := M + 1$ and $\nu := \frac{1}{2^m} = \frac{1}{2^m}$. One has

$$f - \nu t_\alpha X^{2\nu} = f - \nu t_\alpha = \hat{\sigma}_0 - \nu t_\alpha + \sum_{j=1}^m \hat{\sigma}_j g_j + \sum_{|\alpha| \leq k} c_\alpha(1 - X^{2\nu}).$$

Thus, $\sigma_0 + (\nu - \nu) t_\alpha \in \tilde{\Sigma}[X]$. This implies that $\sigma_0 - \nu t_\alpha \in \tilde{\Sigma}[X]$ and $f - \nu t_\alpha \in Q_D(S')$. Next, we derive a lower bound of $\frac{f^*_*}{\alpha}$. Since

$$t = \sum_{|\alpha| \leq k} X^{2\nu} + \sum_{j=1}^m g_j \sum_{|\alpha| \leq k-w_j} X^{2\nu},$$

one has

$$\sum_{|\nu| \leq D} |\nu| \leq 2^m (m + 1) \binom{n + D}{n}.$$

This implies that

$$\alpha \leq 2^m (m + 1) \binom{n + D}{n} + \binom{n + k}{k} \leq 2^m (m + 1) \binom{n + D}{n}. $$

Recall that $\frac{f^*}{\alpha} \leq f^*$, implying

$$\frac{f^*}{\alpha} \geq \frac{f^*}{2^m + 1 (m + 2) \binom{n + D}{n}} \geq \frac{1}{(m + 2) 2^{-1 \text{d}^{\text{min}}}}.$$
where the last inequality follows from Theorem 24. Let us now give an upper bound of \( \log_2 D \). First, note that for all \( \alpha \in \mathbb{N}^d \), \( \frac{\|t\|}{\alpha_1 \ldots \alpha_d} \geq 1 \), thus \( \|f\| \leq 2^\tau \). Since \( D \) is the smallest even integer larger than \( D \), one has

\[
\log_2 D \leq 1 + \log_2 D \leq 1 + \log_\chi N + (12d^3n^{2d+2\tau}2^{O(d^{\alpha d})})^{1+\tau}.
\]

Next, since \( N \) is the smallest integer such that \( \epsilon = \frac{1}{N} = \frac{2}{2^\tau} \), it is enough to take

\[
N \leq 1 + \log_2(m + 2) + 2^{O(n^{\alpha d})} + n\log_2 D \leq 2C_\alpha r^{\alpha d},
\]

for some real \( C_\alpha > 0 \) depending on \( S \), the desired result. \( \Box \)

5.2. Algorithm \( \texttt{PutinarSos} \)

We can now present Algorithm \( \texttt{PutinarSos} \).

For \( f \in \mathbb{Z}[X] \) positive over a basic compact semi-algebraic set \( S \) satisfying Assumption 21, the first loop outputs the smallest positive integer \( D = 2k \) such that \( f \in \mathcal{Q}_D(S) \).

Then the procedure is similar to \( \texttt{intsos} \). As for the first loop of \( \texttt{intsos} \), the loop from line 6 to line 8 allows us to obtain a perturbed polynomial \( f_\epsilon \in \mathcal{Q}_D(S') \), with \( S' := \{ x \in S : 1 - x^{2\alpha} \geq 0, \forall \alpha \in \mathbb{N}_f \} \).

Then one solves SDP (5) with the \( \texttt{sdp} \) procedure and performs Cholesky’s decomposition to obtain an approximate Putinar’s representation of \( f_\epsilon = f - \epsilon \tau \) and a remainder \( \tau \).

Next, we apply the \( \texttt{absorb} \) subroutine as in \( \texttt{intsos} \). The rationale is that with large enough precision parameters for the procedures \( \texttt{sdp} \) and \( \texttt{cholesky} \), one finds an exact weighted SOS decomposition of \( u + \tau \), which yields in turn an exact Putinar’s representation of \( f \) in \( \mathcal{Q}_D(S') \) with rational coefficients.

Example 26. Let us apply \( \texttt{PutinarSos} \) to \( f = -X_1^2 - 2X_1X_2 - 2X_2^2 + 6 \), \( S := \{ (x_1, x_2) \in \mathbb{R}^2 : 1 - x_1^2 \geq 0, 1 - x_2^2 \geq 0 \} \) and the same precision parameters as in Example 8. The first and second loop yield \( D = 2 \) and \( \epsilon = 1 \). After running \( \texttt{absorb} \), we obtain the exact Putinar’s representation

\[
f = \frac{23853407}{292204836} + \frac{23}{49}X_1^2 + \frac{130657269}{291009481}X_2^2 + \frac{12442^2}{2437^2}(X_1 - X_2)^2 + \frac{13}{7}(1 - X_1^2 + \frac{13}{7}(1 - X_2^2).
\]

5.3. Bit complexity analysis

Theorem 27. We use the notation and assumptions introduced above. For some constant \( C_r > 0 \) depending on \( S \), there exist \( \epsilon \) and \( D = 2k \) of bit sizes less than \( O(2C_r r^{\alpha d}) \), and \( \delta, R, \delta_i \) bit sizes less than \( (m + 1)D^n \) for which \( \texttt{PutinarSos}(f, S, \epsilon, \delta, R, \delta_i) \) terminates and outputs an exact Putinar’s representation with rational coefficients of \( f \in \mathcal{Q}(S') \), with \( S' := \{ x \in S : 1 - x^{2\alpha} \geq 0, \forall \alpha \in \mathbb{N}_f \} \). The maximum bit size of these coefficients and the boolean running time are both upper bounded by \( (m + 1)D^n \).

Proof. The loops going from line 3 to line 4 and from line 9 to line 10 always terminate as respective consequences of Theorem 25(i) and Theorem 25(iii) with \( \log_2 D \leq 2C_r r^{\alpha d} \), \( \epsilon = \frac{1}{2^\tau} \), with \( N \leq 2C_\alpha r^{\alpha d} \), for some real \( C_\alpha > 0 \) depending on \( S \). What remains to prove is similar to Proposition 10 and Theorem 11. Let \( \nu, \sigma_0, \ldots, \sigma_m, (c_n)_{0 \leq n < k} \) be as in the proof of Theorem 25. Note that \( \nu \) (resp. \( \sigma_i \) for \( 1 \leq j \leq m \).
Algorithm 6 Putinarsos.

Input: \( f, g_1, \ldots, g_m \in \mathbb{Z}[X] \) of degrees less than \( d \in \mathbb{N}, S := \{ x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \} \), positive \( \varepsilon \in \mathbb{Q} \), precision parameters \( \delta, R \in \mathbb{N} \) for the SDP solver, precision \( \delta_c \in \mathbb{N} \) for the Cholesky’s decomposition

Output: lists \( c_{\text{list}0}, \ldots, c_{\text{list}m}, c_{\text{alpha}} \) of numbers in \( \mathbb{Q} \) and lists \( s_{\text{list}0}, \ldots, s_{\text{list}m} \) of polynomials in \( \mathbb{Q}[X] \)

1: \( k := \left\lfloor d/2 \right\rfloor, D := 2k, g_0 := 1 \)
2: \( \text{while } f \notin Q_D(S) \text{ do } k := k + 1, D := D + 2 \)
3: \( \text{done} \)
4: \( P := \mathbb{N}^d_D, Q := \mathbb{N}^p_D, S' := \{ x \in S : 1 - x^{2\alpha} \geq 0, \forall \alpha \in \mathbb{N}^p \} \)
5: \( t := \sum_{x \in Q} X^{2\alpha}, f_x := f - \varepsilon t \)
6: \( \text{while } f_x \notin Q_D(S') \text{ do } \varepsilon := \frac{1}{2}, f_x := f - \varepsilon t \)
7: \( \text{done} \)
8: \( \text{ok := false} \)
9: \( \text{while } \text{not ok do} \)
10: \( \{ \tilde{G}_0, \ldots, \tilde{G}_m, \tilde{\lambda}_0, \ldots, \tilde{\lambda}_m, (\tilde{c}_u)_{|u| \leq k} \}, := \text{sdp}(f_x, \delta, R, S') \)
11: \( c_{\text{alpha}} := (\tilde{c}_u)_{|u| \leq k} \)
12: \( \text{for } j \in \{0, \ldots, m\} \text{ do} \)
13: \( (s_{ij}, \ldots, s_{r_{ij}}) := \text{cholesky}(\tilde{G}_j, \tilde{\lambda}_j, \delta_c), \tilde{\sigma}_j := \sum_{i=1}^{r_{ij}} s_{ij}^2 \)
14: \( c_{\text{list}j} := [1, \ldots, 1], s_{\text{list}j} := [s_{ij}, \ldots, s_{r_{ij}}] \)
15: \( \text{done} \)
16: \( u := f_x - \sum_{j=0}^{m} \tilde{\sigma}_j g_j - \sum_{|u| \leq k} \tilde{c}_u (1 - X^{2\alpha}) \)
17: \( \text{for } \alpha \in Q \text{ do } \varepsilon_\alpha := \varepsilon \)
18: \( \text{done} \)
19: \( (c_{\text{list}}, s_{\text{list}}, (\varepsilon_\alpha)) := \text{absorb}(u, Q, (\varepsilon_\alpha), c_{\text{list}}, s_{\text{list}}) \)
20: \( \text{if } \min_{\alpha \in Q}(\varepsilon_\alpha) \geq 0 \text{ then ok := true} \)
21: \( \text{else } \delta := 2\delta, R := 2R, \delta_c := 2\delta_c \)
22: \( \text{end} \)
23: \( \text{done} \)
24: \( \text{for } \alpha \in Q \text{ do} \)
25: \( c_{\text{list}0} := c_{\text{list}0} \cup \{ \varepsilon_\alpha \}, s_{\text{list}0} := s_{\text{list}0} \cup \{ x^\alpha \} \)
26: \( \text{done} \)
27: \( \text{return } c_{\text{list}0}, \ldots, c_{\text{list}m}, c_{\text{alpha}}, s_{\text{list}0}, \ldots, s_{\text{list}m} \)
In addition, \( c_\alpha \geq 0 \) for all \( \alpha \in \mathbb{N}_n \). When the sdp procedure at line succeeds, the matrix \( \hat{G}_j \) is an approximate Gram matrix of the polynomial \( \hat{c}_j \) with \( \hat{G}_j \geq 2^{-\delta}I + \sqrt{\text{Tr}(\hat{G}_j^2)} \leq R \), we obtain a positive rational approximation \( \hat{\lambda}_j \geq 2^{-\delta} \) of the smallest eigenvalue of \( \hat{G}_j \), \( \hat{c}_\alpha \) is a rational approximation of \( c_\alpha \) with \( \hat{c}_\alpha \geq 2^{-\delta} \), and \( \hat{c}_\alpha \leq R \), for all \( j = 0, \ldots, m \) and \( \alpha \in \mathbb{N}_n \). This happens when \( 2^{-\delta} \leq \varepsilon \) and \( 2^{-\delta} \leq \varepsilon - \nu \), thus for \( \delta = O(2^c \log m) \).

Next, we will give an upper bound on the bit size of \( R \). Note that the size \( r_0 \) of the matrix \( \hat{G}_0 \) satisfies \( r_0 \geq r_j \) for all \( j = 1, \ldots, m \). In addition, \( \deg g_j \leq d \) implies

\[
\sum_{\bar{g}} |\bar{g}| \leq \left( \frac{n + \deg g}{n} \right)^{2^r} \leq \left( \frac{n + d}{n} \right)^{2^r} \leq d^{r+1}.
\]

Thus, the bit size of the entries of each matrix \( C_{j,j} \) is upper bounded by \([n \log d] + r + 1 = O(D)\). As in the proof of Theorem 11 let \( n_{\text{exp}} \) be the sum of the sizes of the matrices involved in SDP 5 and \( m_{\text{exp}} \) the number of equality constraints. Note that

\[
n_{\text{exp}} \leq (m + 1)r_0 \leq (m + 1)\left( \frac{n + D}{n} \right), \quad m_{\text{exp}} \geq \left( \frac{n + D}{n} \right).
\]

Therefore, as in the proof of Proposition 9 we obtain an upper bound of \((m + 1)D(\delta(D^r)) = ((m + 1)D)^{\nu(D^r)}\) for the bit size of \( R \). As for the second loop of Algorithm intos, the third loop of Putinaros terminates when the remainder polynomial

\[
u = f_x - \sum_{j=0}^m \bar{c}_j g_j - \sum_{|\alpha| \leq k} \hat{c}_\alpha (1 - X^{2\alpha})
\]

satisfies \(|u| \leq \frac{\varepsilon}{r_0} \), where \( r_0 = \binom{n+n}{n} \) is the size of \( Q = \mathbb{N}_n \). As in the proof of Proposition 10 one can show that this happens when \( \delta \) and \( \hat{\delta}_c \) are large enough. To bound the precision \( \hat{\delta}_c \) required for Cholesky’s decomposition, we do as in the proof of Proposition 10. The difference now is that there are \( m + \binom{n+n}{k} = m + r_0 \) additional terms in each equality constraint of SDP 5, by comparison with SDP 1. Thus, we need to bound for all \( j = 1, \ldots, m, \alpha \in \mathbb{N}_n \) and \( c \in \text{supp}(u) \) each term \(|\text{Tr}(\hat{G}_j C_{j,j}) - (g_j \hat{c}_j)_\gamma|\) related to the constraint \( g_j \geq 0 \) as well as each term (omitted for conciseness) involving \( \hat{c}_\alpha \) related to the constraint \( 1 - X^{2\alpha} \geq 0 \).

By using the fact that \( \text{Tr}(\hat{G}_j C_{j,j}) = \sum_{\delta} \delta \sum_{\alpha + \beta = \gamma} \hat{G}_{j,\delta} \beta \), we obtain

\[
|\text{Tr}(\hat{G}_j C_{j,j}) - (g_j \hat{c}_j)_\gamma| \leq \sum_{\delta} \delta |\bar{g}| \frac{\sqrt{\alpha} (r_j + 1) 2^{-\hat{\delta}_r} R}{1 - (r_j + 1) 2^{-\hat{\delta}_r}},
\]

where \( r_j \) is the size of \( \hat{G}_j \).

This yields an upper bound of \( d^r 2^{r+1} \frac{\sqrt{\alpha} (r_j + 1) 2^{-\hat{\delta}_r} R}{1 - (r_j + 1) 2^{-\hat{\delta}_r}} \). We obtain a similar bound (omitted for conciseness) for each term involving \( \hat{c}_\alpha \). Then, we take the smallest \( \delta \) such that \( 2^{-\delta} \leq \frac{\varepsilon}{r_0} \) and the smallest \( \hat{\delta}_c \) such that

\[
d^r 2^{r+1} \frac{\sqrt{\alpha} (r_0 + 1) 2^{-\hat{\delta}_c} R}{1 - (r_0 + 1) 2^{-\hat{\delta}_c}} \leq \frac{\varepsilon}{2 r_0 (m + 1) + r_0}.
\]

Thus, one can choose \( \delta \) and \( \hat{\delta}_c \) of bit size upper bounded by \((m + 1)D)^{\nu(D^r)}\) in order to ensure that Putinaros terminates. As in the proof of Proposition 10, one shows that the output
To bound the boolean run time, we consider the cost of solving SDP (5), which is performed in 
\(O(n_{sdp}^2 \log_2 (2^n n_{sdp} R 2^n))\) iterations of the ellipsoid method, where each iteration requires \(O(n_{sdp}^2 (m_{sdp} + n_{sdp}))\) arithmetic operations over \(\log_2 (2^n n_{sdp} R 2^n)\)-bit numbers. Since \(m_{sdp}\) is bounded by \(\binom{n+D}{D} \leq 2D^n\), we obtain the desired result.

The complexity is singly exponential in the degree \(D\) of the representation. On all the examples we tackled, it was close to the degrees of the involved polynomials, as shown in Section 6.

5.4. Comparison with the rounding-projection algorithm of Peyrl and Parrilo

We now state a constrained version of the rounding-projection algorithm from Peyrl and Parrilo (2008).

Algorithm 7 RoundProjectPutinar

Input: lists \(f, g_1, \ldots, g_m \in \mathbb{Z}[X]\) of degrees less than \(d \in \mathbb{N}\), \(S := \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}\), rounding precision \(\delta_e \in \mathbb{N}\), precision parameters \(\delta, R \in \mathbb{N}\) for the SDP solver, precision \(\delta, R \in \mathbb{N}\) for the Cholesky’s decomposition

Output: lists \(c_{\text{list}_0}, \ldots, c_{\text{list}_m}\) of numbers in \(\mathbb{Q}\) and lists \(s_{\text{list}_0}, \ldots, s_{\text{list}_m}\) of polynomials in \(\mathbb{Q}[X]\)

1: \(k := \lceil d/2 \rceil, D := 2k, g_0 := 1\)
2: while \(f \notin Q_D(S)\) do \(k := k + 1, D := D + 2\)
3: done
4: ok := false
5: while not ok do
6: \([G_0, \ldots, G_m, \lambda_0, \ldots, \lambda_m], := \text{sdp}(f, \delta, R, S)\)
7: \(G' := \text{round}(G_0, \delta)\)
8: for \(j \in \{1, \ldots, m\}\) do
9: \((s_{ij}, \ldots, s_{rj}) := \text{cholesky}(G_j, \lambda_j, \delta), \sigma_j := \sum_{i=1}^j s_{ij}\)
10: \(c_{\text{list}_j} := [1, \ldots, 1], s_{\text{list}_j} := [s_{ij}, \ldots, s_{rj}]\)
11: done
12: \(u := f = \sum_{i=1}^m \sigma_j\)
13: \(Q := \mathbb{N}_{k}^m\)
14: for \(\alpha, \beta \in Q\) do \(\eta(\alpha + \beta) := \#(\alpha', \beta' \in Q^2 \mid \alpha' + \beta' = \alpha + \beta)\)
15: \(G(\alpha, \beta) := G'(\alpha, \beta) - \frac{1}{\eta(\alpha + \beta)}\left(\sum_{\alpha' + \beta' = \alpha + \beta} G'_{\alpha', \beta'} - u_{\alpha' + \beta'}\right)\)
16: done
17: \((c_{10}, \ldots, c_{00}, s_{10}, \ldots, s_{r0}) := 1d1(G)\)
18: if \(c_{10}, \ldots, c_{m0} \in \mathbb{Q}_{\geq 0}, s_{00}, \ldots, s_{m0} \in \mathbb{Q}[X]\) then \(ok := \text{true}\)
19: else \(\delta_j := 2\delta, \delta := 2\delta, R := 2R, \delta_i := 2\delta_i\)
20: end
21: done
22: \(c_{\text{list}_0} := [c_{10}, \ldots, c_{00}], s_{\text{list}_0} := [s_{10}, \ldots, s_{r0}]\)
23: return \(c_{\text{list}_0}, \ldots, c_{\text{list}_m}, s_{\text{list}_0}, \ldots, s_{\text{list}_m}\)

For \(f \in \mathbb{Z}[X]\) positive over a basic compact semi-algebraic set \(S\) satisfying Assumption [21], Algorithm RoundProjectPutinar starts as in Algorithm Putinaros (see Section 5.2), it
outputs the smallest $D$ such that $f \in Q_D(S)$, solves SDP \((5)\) in Line\textsuperscript{6} and performs Cholesky’s factorization in Line\textsuperscript{8} to obtain an approximate Putinar’s representation of $f$. Note that the approximate Cholesky’s factorization is obtained to perform weighted SOS decompositions associated to the constraints $g_1, \ldots, g_m$ (i.e. $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_m$, respectively).

Next, the algorithm applies in Line\textsuperscript{14} the same projection procedure of Algorithm RoundProject (see Section\textsuperscript{3.4}) on the polynomial $u := f - \sum_{j=1}^m \tilde{\sigma}_j g_j$. Note that when there are no constraints, one retrieves exactly the projection procedure from Algorithm RoundProject. Exact $\text{LDL}^T$ is then performed on the matrix $G$ corresponding to $u$.

If all input precision parameters are large enough, $G$ is a Gram matrix associated to $u$ and $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_m$ are weighted SOS polynomials, yielding the exact Putinar’s representation $f = u + \sum_{j=1}^m \tilde{\sigma}_j g_j$. As for Theorem\textsuperscript{12} and Theorem\textsuperscript{27} Algorithm RoundProjectPutinar has a similar bit complexity than Putinarsos.

6. Practical experiments

We provide experimental results for Algorithms intsos, Reznicksos and Putinarsos. These are implemented in a procedure, called multivsos, and integrated in the RealCertify library by [Magron and Safey El Din \(2018\)](http://www.home.math.uwo.ca/faculty/franz/convex), written in Maple. More details about installation and benchmark execution are given on the dedicated webpage\textsuperscript{3}. All results were obtained on an Intel Core i7-5600U CPU \((2.60 \text{ GHz})\) with \(16\text{Gb}\) of RAM. We use the Maple Convex package\textsuperscript{4} to compute Newton polytopes. Our subroutine \texttt{sdp} relies on the arbitrary-precision solver SDPA-GMP by [Nakata \(2010\)](https://gricad-gitlab.univ-grenoble-alpes.fr/magronv/RealCertify) and the cholesky procedure is implemented with \texttt{LDL} available within Maple. Most of the time is spent in the \texttt{sdp} procedure for all benchmarks. To decide non-negativity of polynomials, we use either RAGLib or the \texttt{sdp} procedure as oracles. Recall that RAGLib relies on critical point methods whose runtime strongly depends on the number of (complex) solutions to polynomial systems encoding critical points. While these methods are more versatile, this number is generically exponential in $n$. Hence, we prefer to rely at first on a heuristic strategy based on using \texttt{sdp} first (recall that it does not provide an exact answer).

In Table\textsuperscript{1} we compare the performance of multivsos for nine univariate polynomials being positive over compact intervals. More details about these benchmarks are given in [Chevillard et al. \(2011\)](http://www.home.math.uwo.ca/faculty/franz/convex), Section 6) and [Magron et al. \(2018\)](http://www.home.math.uwo.ca/faculty/franz/convex), Section 5). In this case, we use Putinarsos. The main difference is that we use SDP in multivsos instead of complex root isolation in univsos2. The results emphasize that univsos2 is faster and provides more concise SOS certificates, especially for high degrees (see e.g. # 5). For # 3, we were not able to obtain a decomposition within a day of computation with multivsos, as meant by the symbol – in the corresponding column entries. Large values of $d$ and $r$ require more precision. The values of $\epsilon$, $\delta$ and $\delta_i$ are respectively between $2^{-80}$ and $2^{-240}$, 30 and 100, 200 and 2000.

Next, we compare the performance of multivsos with other tools in Table\textsuperscript{2}. The two first benchmarks are built from the polynomial $f = (X_1^2 + 1)^2 + (X_2^2 + 1)^2 + 2(X_1 + X_2)^2 - 268849736/10^5$ from [Lasserre \(2001\)](http://www.home.math.uwo.ca/faculty/franz/convex), Example 1), with $f_{12} := f^3$ and $f_{20} := f^5$. For these two benchmarks, we apply intsos. We use Reznicksos to handle $M_{20}$ (resp. $M_{100}$), obtained as in Example\textsuperscript{14} by adding $2^{-20}$ (resp. $2^{-100}$) to the positive coefficients of the Motzkin polynomial and $r$, which is a randomly generated positive definite quartic with $i$ variables. We implemented
Table 1: multivsos vs univsos2 [Magron et al. 2018] for benchmarks from Chevillard et al. [2011].

| #  | n   | d   | \(\tau_1\) (bits) | \(t_1\) (s) | \(\tau_2\) (bits) | \(t_2\) (s) |
|----|-----|-----|-------------------|-------------|-------------------|-------------|
| 1  | 13  | 22  | 22692             | 387179      | 51992            | 0.83        |
| 2  | 32  | 47  | 14701             | 1279065     | 106797           | 1.78        |
| 3  | 34  | 117 | 102789            | 69.3        | 265330           | 5.21        |
| 4  | 17  | 26  | 713865            | 1.15        | 59926            | 1.03        |
| 5  | 43  | 67  | 10360440          | 16.3        | 152277           | 11.2        |
| 6  | 22  | 78  | 1123152           | 1.95        | 63630            | 1.86        |
| 7  | 30  | 47  | 896342            | 1.54        | 68664            | 1.61        |
| 8  | 17  | 26  | 713865            | 1.15        | 59926            | 1.03        |
| 9  | 43  | 67  | 10360440          | 16.3        | 152277           | 11.2        |
| 10 | 22  | 78  | 1123152           | 1.95        | 63630            | 1.86        |

Table 2: multivsos vs RoundProject [Peyrl and Parrilo 2008] vs RAGLib vs CAD (Reznick).

| #  | n   | d   | multivsos \(\tau_1\) (bits) | multivsos \(t_1\) (s) | RoundProject \(\tau_2\) (bits) | RoundProject \(t_2\) (s) | RAGLib \(t_3\) (s) | CAD \(t_4\) (s) |
|----|-----|-----|-----------------------------|-----------------------|-------------------------------|----------------------|-----------------|----------------|
| f  | 12  | 2   | 316479                      | 3.99                  | 3274148                       | 3.87                 | 0.15            | 0.07           |
| f  | 20  | 2   | 754168                      | 113                   | 53661174                      | 137                  | 0.16            | 0.03           |
| M  | 3   | 8   | 4397                        | 0.14                  | 3996                          | 0.16                 | 0.13            | 0.05           |
| M  | 8   | 3   | 56261                       | 0.26                  | 12200                         | 0.20                 | 0.15            | 0.03           |
| r  | 2   | 4   | 1680                        | 0.11                  | 1031                          | 0.12                 | 0.09            | 0.01           |
| r  | 4   | 4   | 13351                       | 0.14                  | 47133                         | 0.15                 | 0.32            | -              |
| r  | 6   | 4   | 52446                       | 0.24                  | 475359                        | 0.37                 | 623             | -              |
| r  | 8   | 4   | 145933                      | 0.70                  | 2251511                       | 1.08                 | -               | -              |
| r  | 10  | 4   | 317906                      | 3.38                  | 8374082                       | 4.32                 | -               | -              |
| r  | 6   | 8   | 1180699                     | 13.4                  | 146103466                     | 112                  | 10.9            | -              |

In Maple the projection and rounding algorithm from Peyrl and Parrilo [2008] (stated in Section 3.4) also relying on SDP, denoted by RoundProject. For multivsos, the values of \(\varepsilon, \delta\) and \(\delta_c\) lie between \(2^{-100}\) and \(2^{-10}\), 60 and 200, 10 and 60.

In most cases, multivsos is more efficient than RoundProject and outputs more concise representations. The reason is that multivsos performs approximate Cholesky’s decompositions while RoundProject computes exact \(LDL^T\) decompositions of Gram matrices obtained after the two steps of rounding and projection. Note that we could not solve the examples of Table 2 with less precision.

We compare with RAGLib [Safey El Din 2007a] based on critical point methods (see e.g. Safey El Din and Schost [2003]; Hong and Safey El Din [2012]) and the SamplePoints procedure Lemaire et al. [2005] (abbreviated as CAD) based on CAD Collins [1975], both available in Maple. Observe that multivsos can tackle examples which have large degree but a rather small number of variables \((n \leq 3)\) and then return certificates of non-negativity. The runtimes are slower than what can be obtained with RAGLib and/or CAD (which in this setting have polynomial complexity when \(n \leq 3\) is fixed). Note that the bit size of the certificates which are obtained here is quite large which explains this phenomenon.

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When the number of variables increases, CAD cannot reach many of the problems we considered. Note that multivsos becomes not only faster but can solve problems which are not tractable by RAGLib.

Recall that multivsos relies first on solving numerically Linear Matrix Inequalities; this is done at finite precision in time polynomial in the size of the input matrix, which, here is bounded by \( (n+d)^d \). Hence, at fixed degree, that quantity evolves polynomially in \( n \). On the other hand, the quantity which governs the behaviour of fast implementations based on the critical point method is the degree of the critical locus of some map. On the examples considered, this degree matches the worst case bound which is the Bézout number \( d^n \). Besides, the doubly exponential theoretically proven complexity of CAD is also met on these examples.

These examples illustrate the potential of multivsos and more generally SDP-based methods: at fixed degree, one can hope to take advantage of fast numerical algorithms for SDP and tackle examples involving more variables than what could be achieved with more general tools.

Recall however that multivsos computes rational certificates of non-negativity in some “easy” situations: roughly speaking, these are the situations where the input polynomial lies in the interior of the SOS cone and has coefficients of moderate bit size. This fact is illustrated by Table 3.

| Id | \( n \) | \( d \) | \( t_1 \) (bits) | \( t_1 \) (s) | \( t_2 \) (s) | \( t_3 \) (s) |
|----|-----|-----|----------------|---------|---------|---------|
| \( S_1 \) | 4 24 | - - | 1788 | - |
| \( S_2 \) | 4 24 | - - | 1840 | - |
| \( V_1 \) | 6 8 | - - | 5.00 | - |
| \( V_2 \) | 5 18 | - - | 1180 | - |
| \( M_1 \) | 8 8 | - - | 351 | - |
| \( M_2 \) | 8 8 | - - | 82.0 | - |
| \( M_3 \) | 8 8 | - - | 120 | - |
| \( M_4 \) | 8 8 | - - | 84.0 | - |

This table reports on problems appearing enumerative geometry (polynomials \( S_1 \) and \( S_2 \) communicated by Sottile and appearing in the proof of the Shapiro conjecture Sottile (2000)), computational geometry (polynomials \( V_1 \) and \( V_2 \) appear in Everett et al. (2009)) and in the proof of the monotone permanent conjecture in Haglund et al. (1999) (\( M_1 \) to \( M_4 \)).

We were not able to compute certificates of non-negativity for these problems which we presume do not lie in the interior of the SOS cone. This illustrates the current theoretical limitation of multivsos. These problems are too large for CAD but RAGLib can handle them. Note that some of these examples involve 8 variables; we observed that the Bézout number is far above the degree of the critical loci computed by the critical point algorithms in RAGLib. This explains the efficiency of such tools on these problems. Recall however that RAGLib did not provide a certificate of non-negativity.

This whole set of examples illustrates first the efficiency and usability of multivsos as well as its complementarity with other more general and versatile methods. This shows also the need of further research to handle in a systematic way more general non-negative polynomials than what it does currently. For instance, we emphasize that certificates of non-negativity were
computed for $M_i$ (1 $\leq$ $i$ $\leq$ 4) in \cite{kaltofen2009} (see also \cite{kaltofen2008}).

### Table 4: multivsos vs RoundProjectPutinar vs RAGLib vs CAD (Putinar).

| Id   | n   | d | multivsos | RoundProject | RAGLib | CAD |
|------|-----|---|-----------|--------------|--------|-----|
|      |     |   | $k$ | $\tau_1$ (bits) | $t_1$ (s) | $\tau_2$ (bits) | $t_2$ (s) | $t_3$ (s) | $t_4$ (s) |
| $p_{46}$ | 2   | 4 | 3   | 45 168 | 0.17 | 230 101 | 0.19 | 0.15 | 0.81 |
| $f_{360}$ | 6   | 3 | 2   | 251 411 | 2.35 | 5 070 043 | 3.60 | 0.12 | – |
| $f_{491}$ | 6   | 3 | 2   | 245 392 | 4.63 | 4 949 017 | 5.63 | 0.01 | 0.05 |
| $f_{752}$ | 6   | 2 | 2   | 23 311 | 0.16 | 74 536 | 0.15 | 0.07 | – |
| $f_{859}$ | 6   | 7 | 4   | 13 596 376 | 299 | 2 115 870 194 | 5339 | 5896 | – |
| $f_{863}$ | 4   | 2 | 1   | 12 753 | 0.13 | 30 470 | 0.13 | 0.01 | 0.01 |
| $f_{884}$ | 4   | 4 | 3   | 423 325 | 13.7 | 10 122 450 | 16.1 | 0.21 | – |
| $f_{890}$ | 4   | 4 | 2   | 80 587 | 0.48 | 775 547 | 0.56 | 0.08 | – |
| butcher | 6   | 3 | 2   | 538 184 | 1.36 | 8 963 044 | 3.35 | 47.2 | – |
| heart   | 8   | 4 | 2   | 1 316 128 | 3.65 | 35 919 125 | 14.1 | 0.54 | – |
| magnetism | 7   | 2 | 1   | 19 606 | 0.29 | 16 022 | 0.28 | 434 | – |

Finally, we compare the performance of multivsos (Putinar\textsubscript{os}) on positive polynomials over basic compact semi-algebraic sets in Table 4. The first benchmark is from \cite{lasserre2001} Problem 4.6. Each benchmark $f_i$ comes from an inequality of the Flyspeck project \cite{hales2013}. The three last benchmarks are from \cite{munoz2013}. The maximal degree of the polynomials involved in each system is denoted by $d$. We emphasize that the degree $D = 2k$ of each Putinar representation obtained in practice with Putinar\textsubscript{os} is very close to $d$, which is in contrast with the theoretical complexity estimates obtained in Section 5. The values of $\varepsilon$, $\delta$ and $\delta_c$ lie between $2^{-30}$ and $2^{-10}$, 60 and 200, 10 and 30.

As for Table 2, RAGLib and multivsos can both solve large problems (involving e.g. 8 variables) but note that multivsos outputs certificates of emptiness which cannot be computed with implementations based on the critical point method. In terms of timings, multivsos is sometimes way faster (e.g. magnetism, $f_{859}$) but that it is hard here to draw some general rules. Again, it is important to keep in mind the parameters which influence the runtimes of both techniques. As before, for multivsos, the size of the SDP to be solved is clearly the key quantity. Also, it is important to write the systems in an appropriate way also to limit the size of those matrices (e.g. write $1 - x^2 \leq 0$ to model $-1 \leq x \leq 1$). For RAGLib, it is way better to write $-1 \leq x$ and $x \leq 1$ to better control the Bézout bounds governing the difficulty of solving systems with purely algebraic methods. Note also that the number of inequalities increase the combinatorial complexity of those techniques.

Finally, note that CAD can only solve 3 benchmarks out of 10 and all in all multivsos and RAGLib solve a similar amount of problems; the latter one however does not provide certificates of emptiness. As for Table 2, multivsos and RoundProjectPutinar yield similar performance, while the former provides more concise output than the latter.

### 7. Conclusion and perspectives

We designed and analyzed new algorithms to compute rational SOS decompositions for several sub-classes of non-negative multivariate polynomials, including positive definite forms and
polynomials positive over basic compact semi-algebraic sets. Our framework relies on SDP solvers implemented with interior-point methods. A drawback of such methods, in the context of unconstrained polynomial optimization, is that we are restricted to non-negative polynomials belonging to the interior of the SOS cone. We shall investigate the design of specific algorithms for the sub-class of polynomials lying in the border of the SOS cone. We also plan to adapt our framework, either for problems involving non-commutative polynomial data, or for alternative certification schemes, e.g. in the context of linear/geometric programming relaxations.

Appendix A. Appendix

Let $f \in \mathbb{Z}[X_1, \ldots, X_n]$ of degree $d$ and $\tau$ be the maximum bit size of the coefficients of $f$ in the standard monomial basis.

Let $V \subset \mathbb{C}^n$ be the algebraic set defined by

$$f = \frac{\partial f}{\partial X_1} = \cdots = \frac{\partial f}{\partial X_n} = 0 \quad (A.1)$$

By the algebraic version of Sard’s theorem (see e.g. Safey El Din and Schost, 2017, Appendix B), when $V$ is equidimensional and has at most finitely singular points, the projection of the set $V \cap \mathbb{R}^n$ on the $X_1$-axis is finite (and hence a real algebraic set of $\mathbb{R}^n$); we denote it by $Z_{\mathbb{R}}$. Hence, it is defined by the vanishing of some polynomial in $\mathbb{Z}[X_1]$.

The goal of this Appendix is to provide a proof of Proposition 16 which states that under the above notation and assumption, there exists a polynomial $w \in \mathbb{Z}[X_1]$ of degree $\leq d_n$ with coefficients of bit size $\leq \tau \cdot (4d + 2)^m$ such that its set of real roots contains $Z_{\mathbb{R}}$. To prove Proposition 16 our strategy is to rely on algorithms computing sample points in real algebraic sets: letting $C \subset V$ be a finite set of points which meet all connected components of $V \cap \mathbb{R}^n$, it is immediate that the projection of $C$ on the $X_1$-axis contains $Z_{\mathbb{R}}$.

From the computation of an exact representation of such a set $C$, one will be able to analyze the bit size of a polynomial whose set of roots contains $Z_{\mathbb{R}}$. We focus on algorithms based on the critical point method. Those yield the best complexity estimates which are known in theory and practical implementations reflecting these complexity gains have been obtained in Safey El Din (2007a) from e.g. Safey El Din and Schost (2003), Hong and Safey El Din (2012). Here, we focus on (Basu et al., 2006, Algorithm 13.3) since it is the more general one and it does not depend on probabilistic choices which make it easy to analyze from a bit complexity perspective.

It starts by computing the polynomial

$$g = f^2 + \left(\frac{\partial f}{\partial X_2}\right)^2 + \cdots + \left(\frac{\partial f}{\partial X_n}\right)^2.$$ 

Observe that the set of real solutions of $g = 0$ coincides with $V \cap \mathbb{R}^n$. Next, one introduces two infinitesimals $\epsilon$ and $\eta$ (see Basu et al., 2006, Chap. 2) for an introduction on Puiseux series and infinitesimals). Consider the polynomial:

$$g_1 = g + \left(\eta(X_1^2 + \cdots + X_{n+1}^2) - 1\right)^2.$$ 

Its vanishing set over $\mathbb{R}(\eta)^{n+1}$ corresponds to the intersection of the lifting of the vanishing set of $g$ in $\mathbb{R}^n$ with the hyperball of $\mathbb{R}(\eta)^{n+1}$ centered at the origin of radius $\frac{1}{\eta}$.
Let $d_i$ be the degree of $g_1$ in $X_i$. Without loss of generality, up to reordering the variables, we assume that $d_1 \geq d_2 \geq \cdots \geq d_n$; we assume that after this process $X_1$ has been sent to $X_k$. Now, we let

$$h = g_1(1-\varepsilon) + \varepsilon(X_1^{2d_1+1} + \cdots + X_n^{2d_n+1} + X_{n+1}^6 - (n + 1)\zeta^{d+1})$$

We finally focus on the polynomial system:

$$h = \frac{\partial h}{\partial X_2} = \cdots = \frac{\partial h}{\partial X_{n+1}} = 0$$

The rationale behind the last infinitesimal deformation is twofold (see Basu et al., 2006, Chap. 12 and Chap. 13):

- the algebraic set defined by the vanishing of $h$ is smooth;
- the above polynomial system is finite and forms a Gröbner basis $G$ for any degree lexicographical ordering with $X_1 \succ \cdots \succ X_{n+1}$.

Besides, Basu et al. (2006, Prop. 13.30) states that taking the limits (when infinitesimals tend to zero) of projections on the $(X_1, \ldots, X_n)$-space of a finite set of points meeting each connected component of the real algebraic set defined by $h = 0$ provides a finite set of points in the real algebraic set defined by $g = 0$.

In our situation, we do not need to go into such details. We only need to compute a non-zero polynomial $w \in \mathbb{Z}[X_k]$ whose set of real roots contains $\mathbb{Z}_R$. Using Stickelberger’s theorem (Basu et al., 2006, Theorem 4.98) and the process for computing limits in (Basu et al., 2006, Algorithm 12.14) and Rouillier et al. (2000), it suffices to compute the characteristic polynomial of the multiplication operator by $X_k$ in the ring of polynomials with coefficients in $\mathbb{Q}[\eta, \zeta]$ quotiented by the ideal $(G)$. This is done using (Basu et al., 2006, Algorithm 12.9).

In order to analyze the bit size of the coefficients of the output characteristic polynomial, we need to bound the bit size of the entries in the matrix output by (Basu et al., 2006, Algorithm 12.9). Following the discussion in the complexity analysis of (Basu et al., 2006, Algorithm 13.1), we deduce that the coefficients of these entries have bit size dominated by $\tau(2(2d + 1))^m$. Besides, this matrix has size bounded by $(2(2d + 1))^m$. We deduce that the coefficients of its characteristic polynomial have bit size bounded by $(2(2d + 1))^m$.

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