THE MOTION OF A CHARGED PARTICLE ON A RIEHMANNIAN SURFACE UNDER A NON-ZERO MAGNETIC FIELD

CÉSAR CASTILHO

ABSTRACT. In this paper we study the motion of a charged particle on a Riemannian surface under the influence of a positive magnetic field $B$. Using Moser’s Twist Theorem and ideas from classical perturbation theory we find sufficient conditions to perpetually trap the motion of a particle with a sufficient large charge in a neighborhood of a level set of the magnetic field. The conditions on the level set of the magnetic field that guarantee the trapping are local and hold near all non-degenerate critical local minima or maxima of $B$. Using symplectic reduction we apply the results of our work to certain $S^1$-invariant magnetic fields on $\mathbb{R}^3$.

1. Introduction

The motion of a charge $e$ on a Riemannian surface $M$ can be described as a Hamiltonian dynamical system on $T^*M$ with metric Hamiltonian and twisted symplectic 2-form $\omega_e = d\lambda + e\pi^*\beta$. Here $\lambda$ is the standard Liouville 1-form, $\pi : T^*M \to M$ is the canonical projection, $e$ is the particle’s charge and $\beta$ is a 2-form on $M$ representing the magnetic field. A method for finding periodic orbits of this motion on a fixed energy level was introduced by V.I. Arnold ([Ar2, Ar3]) and developed by V. Ginzburg ([Gi1, Gi2, Gi3]). The method relies on symplectic topology techniques (Conley-Zehnder theorem [CZ]) or symplectic capacities) to prove the existence of some minimum number of periodic orbits. In this work we will combine ideas from classical perturbation theory and appropriately chosen changes of variables (cf. [Ar1, Ar2] and [Gi2]) to study motion when the charge is large. The approximation involved with large charge is known as the guiding center approximation. The motion of the particle is described as a fast rotation around a point that drifts slowly along a level line of the magnetic field. The point of rotation is known as the guiding center. Its behaviour is important since one is usually interested in confining charged particles to a bounded region of the phase space. We will find local conditions on a level set of a magnetic field on a surface with general Riemannian metric to perpetually trap the motion of the particle near this level set. By considering the twisted symplectic 2-form as the main element of our treatment we will write the system in a form suitable for applying Moser’s Twist Map Theorem after applying successive diffeomorphisms. The motion will be trapped between the invariant circles predicted by the theorem. The non-degeneracy condition of the theorem will be studied and neighborhoods where the condition of the

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theorem holds will be characterized. Throughout the work the energy level will be fixed at $E = \frac{1}{2}$. We state our main results:

**Theorem 1.1.** Let $M$ be a Riemannian surface. Let $\Omega$ be the area form on $M$. Let $w = d\lambda + e\pi^*(B\Omega)$, where $\lambda$ is the canonical 1-form on $T^*M$, $\pi^*: T^*M \to M$ is the natural projection, $B$ is a positive $C^2$ function on $M$ and $e > 0$ is a parameter. Denote by $L_c = \{B = c\}$ a non-critical level set of the function $B$ with the topology of an imbedded circle. Set $\beta \equiv B\Omega = dA$. Let $F = \nabla B / |\nabla B|^2$ where $\nabla B$ denotes the gradient of $B$. If the level set $L_c$ is such that

$$\int_{L_c} i_F d\{i_F dA + \frac{2}{e} A\} \neq 0$$

then for any neighborhood $N$ of $L_c$, there exists a number $e_\ast > 0$ such that for any $e > e_\ast$ the trajectories of the magnetic field problem with energy $E = \frac{1}{2}$ and charge $e$ will be contained on $N$ for all times.

**Corollary 1.2.** Let $p \in M$ be a non-degenerate maximum or minimum point of the function $B$. Then there exists arbitrarily small neighborhoods $N$ of $p$, and $e_\ast > 0$, such that for all $e > e_\ast$ the integral curves of the magnetic problem with charge $e$, energy $E = \frac{1}{2}$ and initial conditions in $N$ will remain in $N$ for all times.

**Corollary 1.3.** Let $s \in M$ be a maximum or minimum submanifold of the function $B$ with the topology of an imbedded circle. Then there exists arbitrarily small neighborhoods $N$ of $s$, and $e_\ast > 0$, such that for all $e > e_\ast$ the integral curves of the magnetic problem with charge $e$, energy $E = \frac{1}{2}$ and initial conditions in $N$ will remain in $N$ for all times.

The proof of the theorem will be based on a change of coordinates that will put the twisted symplectic 2-form $ed\lambda + B\Omega$ in the form $\gamma - e^2 dH_\epsilon \wedge d\theta$, where $H_\epsilon$ is an analytic function depending on $\epsilon$ such that its first zero order term is given by $B^{-1}$ i.e, we will construct a diffeomorphism $\Lambda$ such that $\Lambda^*(ed\lambda + B\Omega) = \gamma - e^2 dH_\epsilon \wedge d\theta$.

The characteristic line field of this form is spanned by $e^2 X_{H_\epsilon} + \frac{\partial}{\partial \theta}$ where $X_{H_\epsilon}$ is the hamiltonian vector field of $H_\epsilon$ with respect to the area form $\gamma$. Thinking of the variable $\theta$ as time allows us to introduce action-angle variables $I$, $\psi$ on a neighborhood of the level set $L_c$ of $B$, and to reduce the system to a time $2\pi$ map $\theta \to \theta + 2\pi$ from this neighborhood to itself. The particular form of $H_\epsilon$ and $\gamma$ will allow us to invoke the twist theorem provided condition \[ holds for $B$.

The paper is organized as follows. In section 2 we introduce the tools we need and describe the limit of large charge as a pertubative limit. In section 3 we construct the diffeomorphism $\Lambda$ (theorem 3.1). In section 4 we apply Moser’s twist theorem to our system and characterize the non-degeneracy condition. In section 5 we prove theorem 1.1 and its corollaries. In section 6 we apply our main theorem to a family of symmetric magnetic fields and finally in section 7 we prove some technical lemmas.

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2. The Magnetic Problem for Large Charge

Let $M$ be a 2-dimensional oriented Riemannian manifold. Let $β = BΩ$ be the magnetic field 2-form, where $Ω$ is the area form on $M$ and $B$ is a non-vanishing function on $M$. The motion of a charge on the Riemannian surface $M$ can be identified with a Hamiltonian flow (see [31]). Identifying the tangent and cotangent bundle by means of the Riemannian metric we denote by $H : TM → R$ the energy function $H(X) = \frac{1}{2}X^2$, where $X ∈ TM$. Consider the Hamiltonian flow of $H$ on $T^*M$ with respect to the twisted symplectic structure $w_ε = d\lambda + επ^*β$. Here $λ = p_1dq_1 + p_2dq_2$ is the canonical 1-form on $T^*M$, and thus $dλ$ is the standard symplectic 2-form, $π^* : TM ≃ T^*M → M$ is the natural projection and $ε$ is the charge of the particle. Henceforth $β$ and $π^*β$ will be identified.

The twisted symplectic form $w_ε = dλ + εβ$ is not defined in the limit in which the charge is infinite. We will rescale the twisted 2-form so as to have an analytic limit. Let $ε = \frac{1}{λ}$. Set

$$w_ε ≡ \frac{1}{ε}w_e = \frac{1}{ε}(dλ + εβ) = εdλ + β.$$  

Denoting by $X_H$ the hamiltonian vector field of the magnetic problem we have

$$dH = w_e(X_H, \cdot) = ew_e(X_H, \cdot) = w_e(εX_H, \cdot).$$

It follows that under the scaling $X_H$ gets mapped to $\frac{1}{ε}X_H$. This can be interpreted as a reparametrization of time. Thus if $q(t)$ is a solution for $X_H$, then $q(\frac{t}{ε})$ is a solution curve for $\frac{1}{ε}X_H$. Therefore the scaling changes periodic orbits of period $T$ for $X_H$ to periodic orbits of period $εT$ for $\frac{1}{ε}X_H$. We thus have two associated problems.

Given $H$ we can consider the Hamiltonian problem given by the symplectic 2-form $w_e$ whose vector field we denoted by $X_H$ or we can consider the Hamiltonian problem given by the symplectic 2-form $w_e$ whose vector field we denote by $X_e$. From now on we will work only with $w_e$. Write $X_e = \frac{1}{ε}X_H$. For future reference we collect these facts on the table 1.1.

| $ε = 1/ε$ | Expression | Vector Field | Time |
|-----------|------------|--------------|------|
| $w_e$     | $dλ + εβ$  | $X_H$        | $T$  |
| $w_e$     | $εdλ + β$  | $X_e = X_H/ε$| $εT$ |

**TABLE 1.1** Relations between the scaled and non-scaled problem.

The limit in which $ε$ goes to infinity can be understood as a perturbative limit. A closer look at Hamilton’s equations will clarify this point. First we introduce some concepts necessary for our study.

Denote by $S^1M ⊂ TM$ the circle bundle over $M$ with respect to the given Riemannian metric. Then $S^1M = \{H = \frac{1}{ε}\}$. Let $v$ be the geodesic vector field of $M$ and $v_⊥$ be the vector field on $S^1M$ perpendicular to $v$ (with respect to the orientation of the manifold). Denote by $φ$ the unit vector field in the direction of the fibers of the bundle $π : S^1M → M$ with direction defined by the orientation of $M$. Thus, $v$, $v_1$, and $φ$ form a global moving frame for $S^1M$. Denote by $v^*$, $v_1^*$ and $φ^*$ the dual frame. Since energy is preserved the magnetic Hamiltonian vector field $X_H$ at energy $E = \frac{1}{2}$ is a vector field over $S^1M$. The following lemma gives the structure equations for the circle bundle.
Lemma 2.1. The Lie-Brackets of $v, v_\perp$ and $\phi$ are given by

\[ [v, v_\perp] = K\phi, \quad [\phi, v] = v_\perp, \quad [\phi, v_\perp] = -v; \]

where $K$ is the Gaussian curvature of $M$.

Proof. See [KN].

Dual to this lemma is

Lemma 2.2.

\[ d\phi^* = Kv_\perp^* \wedge v^*, \quad dv^*_\perp = v^* \wedge \phi^*, \quad dv^* = \phi^* \wedge v_\perp^*. \]

Remark: $\phi^*$ is the connection 1-form for the Levi-Civita connection.

Lemma 2.3. Let $\lambda$ denote the canonical 1-form on $T^*M$. Then $\lambda = v^*$.

Proof. In Darboux coordinates

\[ \lambda = \sum_i p_i dq^i = \sum_{ij} g_{ij} v^j dq^i = (v, \cdot) = v^*. \]

Corollary 2.4. Let $\Omega = v^* \wedge v_\perp^*$ be the area form on $M$ pulled back to $S^1 M$, $\lambda$ the canonical 1-form and $v^*, v_\perp^*$ and $\phi^*$ as before. Then

\[ d(i_{v_\perp} \Omega) = \phi^* \wedge v_\perp^* = d\lambda. \]

Consider the Hamiltonian equation for $w_e$,

\[ w_e(X_H, \cdot) = -dH. \tag{2} \]

We write the Hamiltonian vector field as

\[ X_H = v + ef\phi, \tag{3} \]

i.e., as a geodesic component plus a fiber component. The function $f$ is calculated by inserting this decomposition in equation \[3\],

\[ (d\lambda + eB\Omega)(v + ef\phi) = dH. \]

Since $\Omega\phi = 0$, $d\lambda(v) = dH = 0$ (on $S^1 M$), $\Omega(v) = v_\perp^*$ and $d\lambda(\phi) = v_\perp^*$. We find that $f = -B$. Thus

\[ X_H = v - eB\phi. \]

in $S^1 M$. Rescaling yields

\[ \frac{X_H}{e} = ev - B\phi. \tag{4} \]

Set $\tilde{X} = X/eB$ and $w = \frac{\tilde{X}}{B}$. Dividing by $B$ we have that

\[ \tilde{X} = ew - \phi. \]

The integral curves of $X_e$ and $X$ are the same up to reparametrization. Thus the limit $0 < \epsilon < 1$ can be understood as the limit where the vertical vector field $-\phi$ is perturbed by a ‘small’ vector field $\epsilon w$.

This interpretation allow us to apply standard perturbation theory to eliminate, in the first approximation, the perturbation containing $w$ by means of an appropriate change of coordinates. We indicate how to make this change of coordinates. Let

\[ x^t = X(x) = \epsilon w(x) - \phi(x), \]
where $t$ denotes derivation with respect to the new time. The basic idea is to pull back the vector field $\mathbf{a}t$ by the $\epsilon$-time flow $\Phi_\epsilon$ of some vector field $Y$. For a small $\epsilon$ we have (formally):

$$\Phi_\epsilon^*(\mathbf{a}t) = \sum_{s=0}^{\infty} \frac{\epsilon^s}{s!} \left. \frac{d^s}{d\tau^s} \right|_{\tau=0} \Phi_\epsilon(\mathbf{a}t),$$

where $\tau = \frac{t}{\epsilon}$. But

$$\left. \frac{d}{d\tau} \right|_{\tau=0} \Phi_\epsilon^*(\mathbf{a}t) = [Y, \mathbf{a}];$$

$$\left. \frac{d^2}{d\tau^2} \right|_{\tau=0} \Phi_\epsilon^*(\mathbf{a}t) = [Y, [Y, \mathbf{a}]];$$

$$\left. \frac{d^3}{d\tau^3} \right|_{\tau=0} \Phi_\epsilon^*(\mathbf{a}t) = [Y, [Y, [Y, \mathbf{a}]]],$$

and so on. Since $\mathbf{a}t = \epsilon w - \phi$ the first order term of $\Phi_\epsilon^*(\mathbf{a}t)$ is given by (recall that $w = \frac{\mathbf{a}}{B}$):

$$\frac{\mathbf{a}}{B} - [Y, \phi].$$

The goal is to choose $Y$ such that the first order term of the expansion is zero, i.e.

$$\frac{\mathbf{a}}{B} - [Y, \phi] = 0.$$ (5)

In classical perturbation theory, equation (5) is called the eikonal equation (see [Ar]).

**Lemma 2.5.** The vector field $\mathbf{v}_\perp/B$ solves the eikonal equation.

**Proof.** This follows directly from lemma 2.1:

$$\frac{\mathbf{a}}{B} - [\mathbf{v}_\perp/B, \phi] = \frac{\mathbf{a}}{B} + \frac{1}{B}[\phi, \mathbf{v}_\perp] = \frac{\mathbf{a}}{B} - \frac{\mathbf{a}}{B} = 0.$$

\[ \square \]

### 3. The Perturbative Approach

The flow of the vector field $\mathbf{v}_\perp/B$ gives the diffeomorphism that eliminates the first order term in the $\epsilon$ expansion. In this section we prove the following theorem:

**Theorem 3.1.** Assume $L_c = \{B = c\}$ is a simple closed curve. For $\epsilon$ sufficiently small ($\epsilon > 0$), there exists a neighborhood $N_c$ of $\pi^{-1}(L_c) \subset S^1M$ (depending on $\epsilon$) and a diffeomorphism $\Lambda : N_c \rightarrow N_c$ such that

$$\Lambda^* w_\epsilon = \gamma - (\epsilon^2 dB^{-1} + \epsilon^3 dF) \wedge d\theta,$$

where $\gamma$ is the area form on $M$, $F$ is a function on $S^1M$ and $\theta$ is a circular fiber coordinate on $S^1M$.

The change of coordinates $\Lambda$ is involved. The basic idea is to construct it first on a section $\theta = constant$ of the circle bundle and then extend it to the full circle bundle. The diffeomorphism on this section is built by pulling back the twisted 2-form by the flow of the vector field $\mathbf{v}_\perp/B$ introduced in section 2. In the flat case the triviality of the bundle implies that this is already sufficient to allow the application of the twist theorem. In the non-flat case we will need one diffeomorphism for each such section of the principal bundle. This apparent difficulty will be overcome by ‘gluing’ together all these diffeomorphims. The main tool used to make this gluing will be a version of Moser’s homotopy argument which says that two symplectic
2-forms that agree on a compact submanifold must agree on a neighborhood of the submanifold. The resulting diffeomorphism, now defined on the full bundle, will put the twisted symplectic 2-form in the final desired form.

Proof. We start to build the diffeomorphism. We compute the pull-back of the twisted symplectic 2-form by the flow of $v_\perp/B$. The choice of this vector field is dictated by its role in the perturbation problem as shown on the previous section.

Lemma 3.2. The pull-back of $w_\epsilon$ by $\varphi_\epsilon$, the $\epsilon$-time flow of $v_\perp/B$ is given by

$$\varphi_\epsilon^* w_\epsilon = B\Omega + \frac{\epsilon^2 K}{2} \frac{\Omega}{B} - \frac{\epsilon^2}{2} dB^{-1} \wedge (d\theta + f) + O(\epsilon^3),$$

where $\theta$ is the fiber coordinate function and $f$ is a fiber-independent 1-form.

Proof. This is a straightforward computation using Cartan’s Formula and the structure equations. The identity $\phi^* = d\theta + f$ follows from the fact that $\phi^*$ is the Levi-Civita connection restricted to $S_1 M$.

Corollary 3.3. The Flat Case: Suppose $M$ is flat. Then $K = 0$, the bundle is trivial and we can take $\phi^* = d\theta$ i.e. $f = 0$. The pull-back computations reduce to

$$L_{v_\perp/B} w_\epsilon = -d\lambda - \epsilon dB^{-1} \wedge d\theta.$$

$$L^2_{v_\perp/B} w_\epsilon = (dB^{-1} - \epsilon L_{v_\perp/B} dB^{-1}) \wedge d\theta.$$

$$L^3_{v_\perp/B} w_\epsilon = \left( L_{v_\perp/B} dB^{-1} - \epsilon L^2_{v_\perp/B} dB^{-1} \right) \wedge d\theta.$$

We obtain

$$\varphi_\epsilon^* w_\epsilon = \epsilon d\lambda + B\Omega + \epsilon \left( -d\lambda - \epsilon dB^{-1} \wedge d\theta \right) + \frac{\epsilon^2}{2} \left( dB^{-1} - \epsilon L_{v_\perp/B} dB^{-1} \right) \wedge d\theta + 

\frac{\epsilon^3}{3} \left( L_{v_\perp/B} dB^{-1} - \epsilon L^2_{v_\perp/B} dB^{-1} \right) \wedge d\theta + \ldots$$

which simplifies to

$$\varphi_\epsilon^* w_\epsilon = B\Omega - \frac{\epsilon^2}{2} dB^{-1} \wedge d\theta - \frac{\epsilon^3}{6} L_{v_\perp/B} dB^{-1} \wedge d\theta - \frac{\epsilon^4}{12} L^2_{v_\perp/B} dB^{-1} \wedge d\theta + \ldots$$

giving

$$\varphi_\epsilon^* w_\epsilon = B\Omega - \epsilon^2 dH_\epsilon \wedge d\theta,$$

where

$$H_\epsilon = \sum_{i=1}^{\infty} \frac{\epsilon^{i-1}}{i(i+1)} L^{(i-1)}_{v_\perp/B} B^{-1}.$$

So, for the planar case, the first diffeomorphism is already sufficient to bring the twisted symplectic 2-form to the desired form. One should compared this expression with the one found by Littlejohn [Li].

A second diffeomorphism will be necessary if $K \neq 0$. Denote $\tilde{w}_\epsilon = \varphi_\epsilon^* w_\epsilon$. First we will restrict $\tilde{w}_\epsilon$ to the level set $\theta = constant$. In this case we have that $d\theta = 0$ and

$$\tilde{w}_\epsilon = B\Omega + \frac{\epsilon^2 K}{2} \frac{\Omega}{B} - \frac{\epsilon^2}{2} dB^{-1} \wedge f + \ldots$$
Lemma 3.4. For a constant $\theta$ and a sufficiently small $\epsilon$ ($\epsilon > 0$) there exist neighborhoods $N_0$ and $N_1$ (depending on $\epsilon$) of $\pi^{-1}(L_c) \subset S^1 M$, and a diffeomorphism $\Phi_{\epsilon, \theta} : N_0 \to N_1$ such that $\Phi_{\epsilon, \theta}^* \bar{w}_\epsilon = B \Omega$. Moreover

$$\Phi_{\epsilon, \theta} = I d + \epsilon^2 C(\epsilon)$$

with $C(\epsilon)$ uniformly bounded in the $C^0$ norm.

**Proof.** $\pi^{-1}(L_c) \cap \{ \theta = \text{constant} \}$ has dimension 1, consequently the forms $\bar{w}_\epsilon$ and $B \Omega$ degenerate. Since $L_c$ is compact it follows from theorem of Weinstein (a version of Moser’s homotopy argument) that there exists a diffeomorphism $\Phi_{\epsilon, \theta}$ such that $\Phi_{\epsilon, \theta}^* \bar{w}_\epsilon = B \Omega$. The order of the diffeomorphism is given by tracing back Moser’s homotopy argument. This is done in section 7.

Composing the two diffeomorphisms $\Phi_{\epsilon, \theta}$ and $\varphi_{\epsilon}$ we obtain on the section $\theta = \text{constant}$ of the principal bundle that

$$(\varphi_{\epsilon} \circ \Phi_{\epsilon, \theta})^* w_\epsilon = \Phi_{\epsilon, \theta}^* \varphi_{\epsilon}^* (w_\epsilon) = \Phi_{\epsilon, \theta}^* \bar{w}_\epsilon = B \Omega.$$ 

For a fixed $\epsilon$, $\Phi_{\epsilon, \theta}$ is a family of diffeomorphisms parametrized by $\theta$, each defined only on its own constant section of the principal bundle. We will paste all those diffeomorphisms together. Let $p \in S^1 M$. With respect to our local trivialization we write $p = (m, \theta)$ for an arbitrary point of $S^1 M$ in the given neighborhood. Define $\Xi_{\epsilon} : S^1 M \to S^1 M$ by

$$\Xi_{\epsilon}(m, \theta) = (\Phi_{\epsilon, \theta}(m), \theta).$$

**Lemma 3.5.** Assume $L_c = \{ B = c \}$ is a simple closed curve. For $\epsilon$ sufficiently small ($\epsilon > 0$), there exists a neighborhood $N_c$ of $L_c$ (depending on $\epsilon$) and a diffeomorphism $\Xi : N_c \to N_c$ such that

$$\Xi^* w_\epsilon = B \Omega - (\epsilon^2 dB^{-1} + \epsilon^3 R) \wedge d\theta,$$

for some 1-form $R$.

**Proof.** Since $\Xi_{\epsilon}$ restricted to $\theta = \text{constant}$ is equal to $\Phi_{\epsilon, \theta}$ if follows that $\Xi_{\epsilon}^* \bar{w}_\epsilon$ differs from $\Phi_{\epsilon, \theta}^* \bar{w}_\epsilon$ by a factor of the form $\alpha \wedge d\theta$, for some 1-form $\alpha$, i.e., we can write

$$\Xi_{\epsilon}^* \bar{w}_\epsilon = B \Omega + \alpha \wedge d\theta.$$  

The one-form $\alpha$ is defined mod $d\theta$ and can be taken to be

$$\alpha = -i_{\frac{\partial}{\partial \theta}} \Xi_{\epsilon}^* w_\epsilon.$$ 

For any diffeomorphism $\delta$ we have that $\delta_*(i_X \alpha) = i_{\delta_* X} (\delta_\ast \alpha)$ (see e.g. [AM]). Therefore

$$i_{\frac{\partial}{\partial \theta}} \Xi_{\epsilon}^* \bar{w}_\epsilon = \Xi_{\epsilon}^* (i_{\Xi_{\epsilon} \ast \frac{\partial}{\partial \theta}} \bar{w}_\epsilon).$$

So we must calculate $\Xi_{\epsilon} \ast \frac{\partial}{\partial \theta}$. We do this on the following lemma.

**Lemma 3.6.**

$$\Xi_{\epsilon} \ast \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} + \epsilon^3 z$$

where $z$ is a horizontal vector field relative to our trivialization.

**Proof.** The proof of this lemma is given in section 7. \qed
Using this lemma we compute \((\mathbf{10})\),
\[
(11) \quad i_{\partial_{\varphi}} \Xi^* \bar{\varphi}_\varepsilon = \Xi^* (i_{\Xi^*} \partial_{\varphi} \bar{\varphi}_\varepsilon) = \Xi^* (i_{\Xi^*} \bar{\varphi}_\varepsilon) = \varepsilon^3 \Xi - \varepsilon^3 (i_{\Xi^*} \bar{\varphi}_\varepsilon).
\]
We recall the expression for \(\bar{\varphi}_\varepsilon\) (formula \((\mathbf{9})\))
\[
\bar{\varphi}_\varepsilon \equiv \varphi^*_\varepsilon \varphi_\varepsilon = B\varphi + \varepsilon^2 \frac{K}{B}\Omega - \frac{\varepsilon^2}{2} dB^{-1} \wedge (d\varphi + f) + O(\varepsilon^3).
\]
It follows that
\[
i_{\partial_{\varphi}} \bar{\varphi}_\varepsilon = -\frac{\varepsilon^2}{2} dB^{-1} - \varepsilon^3 G,
\]
for some 1-form \(G\). We have that \((\mathbf{11})\) can be written as
\[
i_{\partial_{\varphi}} \Xi^* \bar{\varphi}_\varepsilon = \Xi^* \left( -\frac{\varepsilon^2}{2} dB^{-1} - \varepsilon^3 G \right) + \varepsilon^3 \Xi^* (i_{\Xi^*} \bar{\varphi}_\varepsilon).
\]
Using lemma \(\text{3.4}\) we have
\[
i_{\partial_{\varphi}} \bar{\varphi}_\varepsilon = (1 + \varepsilon^2 C^*(\varepsilon, \theta)) \left( -\frac{\varepsilon^2}{2} dB^{-1} + \varepsilon^3 G \right) + \varepsilon^3 \Xi^* (i_{\Xi^*} \bar{\varphi}_\varepsilon),
\]
giving
\[
i_{\partial_{\varphi}} \bar{\varphi}_\varepsilon = -\frac{\varepsilon^2}{2} dB^{-1} - \varepsilon^3 R,
\]
for some 1-form \(R\). Finally this implies
\[
(12) \quad \Xi^* \bar{\varphi}_\varepsilon = B\Omega + \alpha \wedge d\theta = B\Omega - (\varepsilon^2 dB^{-1} + \varepsilon^3 R) \wedge d\theta
\]
which proves lemma \(\text{3.5}\). \(\square\)

The 2-form \((\mathbf{12})\) is almost in the final form. The 1-form \(R\) may not be necessarily exact. A final change of coordinates is needed. So far we have been working on a neighborhood of a constant level curve \(\{B = c\}\) of the magnetic field. By hypothesis this has the topology of an annulus. We introduce Fermi coordinates \((x, y)\) on this annulus (see e.g. [M.]). The Fermi coordinates parametrize the \(L_c\) neighborhood \(N_0\) on the following way: Given a point \(m \in N_0\), consider the segment of geodesic \(S\) that connects \(m\) to \(L_c\) and has minimal length. Let \(y\) denote its oriented length. Denote by \(x\) the oriented length from some arbitrary point \(x_0\) in \(L_c\) to the intersection of \(S\) with \(L_c\). Using these coordinates we define
\[
a(\theta) = \frac{1}{l(c)} \int_{B=c} Rdx,
\]
where \(l(c)\) denotes the length of the level set \(L_c\).

**Lemma 3.7.** With \(a(\theta)\) as above there exist functions \(F\) and \(g\) such that
\[
R = dF + a(\theta)dx + gd\theta.
\]

**Proof.** Since \(B\) is \(\theta\) independent and \(\Xi^* \bar{\varphi}_\varepsilon\) is closed \((\mathbf{12})\) implies that
\[
dR \wedge d\theta = 0.
\]
Let
\[
R = R_x dx + R_y dy + R_\theta d\theta,
\]
and denote by \(R^*\) the restriction of \(R\) to a constant section of \(S^1 M\), i.e.
\[
R^* = R|_{\theta=c} = R_x(x, y, c) dx + R_y(x, y, c) dy.
\]
Thus \((13)\) implies that
\[
dR \wedge d\theta = \left( \frac{\partial R_y}{\partial x} - \frac{\partial R_x}{\partial x} \right) dx \wedge dy \wedge d\theta = 0,
\]
from which we see that \(R^*\) is closed. Now define a 1-form \(\xi\) on the section \(\theta = c\) by
\[
(14) \quad \xi = R^* - a(c)dx.
\]
Thus
\[
(15) \quad d\xi = 0 \quad \text{and} \quad \int_{L_c} \xi = 0,
\]
giving that
\[
a(c) = \int_{L'} R^*
\]
for any \(L'\) homotopic to \(L_c\) (since \(R^*\) is closed). Since the section \(\{\theta = \text{constant}\}\) has the homotopy type of an annulus we have that equations \((13)\) together imply that
\[
\xi = dF
\]
for some function \(F\). Thus we can write \(14\) as
\[
R^* = dF + a(c)dx.
\]
But \(R^*\) differs from \(R\) by a factor of the form \(gd\theta\) for some function \(g\) and we can write that
\[
R = dF + a(\theta)dx + gd\theta,
\]
proving the lemma. \(\square\)

Using lemma 3.7 we can rewrite \((12)\) obtaining that
\[
(16) \quad \Xi^* \tilde{w}_c = B\Omega - (\epsilon^2 dB^{-1} + \epsilon^3 dF + \epsilon^3 a(\theta)dx) \wedge d\theta.
\]
Define a new variable
\[
\bar{y} = \int_0^y Bdy.
\]
and the diffeomorphism
\[
(17) \quad \begin{cases} 
X = x, \\
Y = \bar{y} - \epsilon^3 \int_0^\theta a(\theta)d\theta, \\
\Theta = \theta.
\end{cases}
\]
Substituting back in \((16)\) and denoting by \(w_c\) the pull-back of \(\tilde{w}_c\) by \(\Xi^*_c\) we obtain that
\[
w_c = dX \wedge dY + \epsilon^3 a(\Theta)dX \wedge d\Theta - (\epsilon^2 dB^{-1} + \epsilon^3 dF + \epsilon^3 a(\Theta)dX) \wedge d\Theta,
\]
i.e.,
\[
w_c = dX \wedge dY - (\epsilon^2 dB^{-1} + \epsilon^3 dF) \wedge d\Theta,
\]
that we write as
\[
(18) \quad w_c = dX \wedge dY + \epsilon^2 dH_c \wedge d\Theta,
\]
where \(H_c = B^{-1} + \epsilon F\) which proves theorem 3.1 \(\square\)
4. Moser’s Twist Theorem

The characteristic line field of (18) can then be thought as being generated by the Hamiltonian system given by the Hamiltonian $H_\varepsilon$ and symplectic 2-form $dX \wedge dY$. Using the Fermi coordinates (see section 3) we have that the level set is given by $L_c = \{ y = 0, \theta = 0 \}$ and is contained in an open set $N = \{ -\delta \leq y \leq \delta, \theta = 0 \}$ for some $\delta \geq 0$ and $N \subset N_0$. Denote by $N$ the image of $N$ under the diffeomorphism.

Thus we have that

$$N = \{ -\delta B \leq Y \leq \delta B, \Theta = 0 \}.\quad (19)$$

Define $I(X,Y) = \int_{A(X,Y)} dX \wedge d\bar{Y}.$

We consider the Hamiltonian equations with $H_\varepsilon$ on $A(X,Y)$. By integrating and taking the time ($\theta = 2\pi$) flow we get

$$\begin{cases} I_1 = I + O(\varepsilon^3), \\ \psi_1 = \psi + \pi \varepsilon^2 \frac{\partial}{\partial I} B^{-1}(I) + O(\varepsilon^3). \end{cases}\quad (20)$$

We consider the Hamiltonian equations with $H_\varepsilon$ on $A(X,Y)$. By integrating and taking the time ($\theta = 2\pi$) flow we get

At this point we recall the statement of Moser’s Twist map theorem:

**Theorem 4.1.** (Moser, 1962 [Mos]) Let $\Phi$ be a measure preserving map $\Phi(R, \theta) = (R_1, \theta_1)$ given by

$$\begin{cases} R_1 = R + \varepsilon^k f(R, \theta, \varepsilon), \\ \theta_1 = \theta + \alpha + \varepsilon^k \gamma(R) + \varepsilon^l g(R, \theta, \varepsilon), \end{cases}\quad (21)$$

where $\gamma(R) \neq 0$, $k \leq l$, $f$ and $g$ bounded. Then for $\varepsilon$ sufficiently small there exists an invariant curve $\Gamma$ surrounding $R = 1$. More precisely there exists a differentiable closed curve

$$\begin{cases} R = F(\phi, \varepsilon), \\ \theta = \phi + G(\phi, \varepsilon) \end{cases}\quad (22)$$

with $F, G$ of period $2\pi$ in $\phi$ which is invariant under the map $\Phi$.

To apply the twist map theorem [11 to 20] it is sufficient that $\frac{\partial^2}{\partial I^2} B^{-1}(I) \neq 0$ at $L_c$. This is the so called non-degeneracy condition and plays an essential role in our problem. In what follows we will characterize the non-degeneracy condition.

4.1. The non-degeneracy condition.

**Definition 4.2.** A level set $L_c = \{ B = c \}$ is called non-degenerate if

$$\frac{\partial^2 B^{-1}(I)}{\partial I^2} \neq 0,\quad (23)$$

along $L_c$. 

Using the chain rule and assuming $L_c$ is a non-critical level set of $B$ we can write that (23) is equivalent to

\[ \frac{d^2 I}{dB^2} = -\frac{2}{B} \frac{dI}{dB} . \]

Form (24) is particularly useful since one can explicitly calculate the derivative of $I$ with respect to $B$. To do so we first note that $I$, the action variable, can be easily expressed in terms of the Fermi coordinates $x$ and $y$. Since

\[ I(X,Y) = \int_{A(X,Y)} dX \wedge dY , \]

we have that

\[ I(x,y) = \int_{\bar{A}(x,y)} Bdx \wedge dy . \]

where $\bar{A}(x,y)$ is the preimage of $A(X,Y)$ under the diffeomorphism $\bar{F}$.

Lemma 4.3.

\[ \frac{\partial I}{\partial B} = \int_{L_c} i_F dA = \int_{L_c} \left( \frac{B}{|\nabla B|} \right) ds , \]

where $ds$ is the arc length of $L_c$.

Proof. Choose a vector potential $A = A_x dx + A_y dy$ for the 2-form $Bdx \wedge dy$. Thus we have by Stokes theorem that

\[ I(B) = \int_{L_c} A . \]

Now consider the vector field $F = \frac{\nabla B}{|\nabla B|^2}$. Let $\phi_t$ be its time $t$ flow. Thus

\[ \frac{d}{dt}(B(\phi_t(x,y))) = \langle \nabla B, F \rangle = 1 , \]

implying that $\phi_t^* B = B + \epsilon$. We can write that

\[ I(B + \epsilon) = \int_{\phi_t(L_c)} A = \int_{L_c} \phi_t^* A . \]

Differentiating with respect to $\epsilon$ and using Cartan’s formula we obtain that

\[ \frac{d}{d\epsilon} I(B + \epsilon) = \int_{L_c} (di_F A + i_F dA) . \]

The first term on the integration vanishes since $di_F A$ integrates to zero along the closed curve $L_c$. To get the other term, observe that for any smooth simple closed curve with smooth normal $n$ we have $i_n dx \wedge dy = ds$ the arc length. Since $F = \frac{\nabla B}{|\nabla B|} n$ and $dA = Bdx \wedge dy$ we find that $i_F dA = \frac{B}{|\nabla B|} ds$ as desired.

Corollary 4.4.

\[ \frac{\partial^2 I}{\partial B^2} = \int_{L_c} i_F d(i_F dA) . \]

Proof. Since the second derivative is expressed as the integral of a 1-form, the same reasoning applied to compute the first derivative can be applied to compute the second.
The non-degeneracy condition can be written in a more explicit way by noting that $ds$ can be written as $f(x,y)dx$ for some function $f > 0$. A long and straightforward computation gives us that the level $L_c$ is nondegenerate if

$$\int_{L_c} \left\{ f \left( \frac{3}{|\nabla B|} - \frac{B}{|\nabla B|^3} < \nabla|\nabla B|, \nabla B > \right) + \frac{B}{|\nabla B|^3} < \nabla f, \nabla B > \right\} dx \neq 0,$$

This expression for the non-degeneracy condition will be used to prove corollary 1.2.

5. Proof of Theorem 1

Proof. We look to the magnetic problem given by the 2-form $w_\epsilon = \epsilon d\lambda + B\Omega$ for $H = \frac{1}{2}$ where $\epsilon = \frac{1}{2}$. According to table 1.1 the Hamiltonian vector field for $w_\epsilon$ is given by a scaling of the Hamiltonian vector field for $w$. Thus their characteristics differ only by a time reparametrization. By theorem 1.1 there is a neighborhood $N$ of $L_c$ and a diffeomorphism $\Xi_\epsilon : N \to N$ such that

$$\Xi_\epsilon^* w_\epsilon = \gamma - (\epsilon^2 B^{-1} + \epsilon^3 F) \wedge d\theta.$$ 

Thus the characteristic line bundle of $\Xi_\epsilon^* w_\epsilon$ is spanned by the Hamiltonian vector field given by the Hamiltonian system with $H_\epsilon = \epsilon^2 B^{-1} + \epsilon^3 F$ and symplectic 2-form $\Omega$ (where by abuse of notation we denote $\Xi_\epsilon^* \Omega$ by $\Omega$). In section 4 we introduced action-angle coordinates on the neighborhood of $\Xi_\epsilon^* N$ and reduced the dynamics of the Hamiltonian system $(H_\epsilon, \Omega)$ on this neighborhood to the dynamics of a twist map. The nondegeneracy condition of Moser’s twist theorem is calculated to be equation 1 according to (24) and lemma 4.3. So Moser’s twist theorem applies and invariant circles exist. By dimensionality, they trap the charge for all times $t > 0$.

Proof. (of Corollary 1.3) Observe that each term in (27) depends on different powers of $|\nabla B|$. A careful analysis of this fact (done in section 7) implies that (27) cannot be satisfied near a critical point of $B$. So for a sufficiently large $\epsilon$ we can find a small neighborhood of $p$ where condition (23) is satisfied everywhere.

Proof. (of Corollary 1.4) The proof is by contradiction. Suppose that given a neighborhood $N$ of $L_c$ we have that all the level sets contained on $N$ are degenerate. This implies that exists a constant $K$ such that

$$\frac{\partial^2 B^{-1}}{\partial^2 I} = K,$$

on $N$. This is equivalent to

$$-\frac{1}{B^2} \frac{\partial B}{\partial I} = -\frac{1}{B^2} \frac{\partial B}{\partial B} = K.$$

What implies that

$$\frac{-1}{KB^2} = \frac{\partial I}{\partial B} = \int_{L_c} \frac{B}{|\nabla B|} ds.$$

Since the right hand side is positive, the constant $K$ must be negative and writing $|K| = -K$ it follows that

$$\frac{|K|}{B^3} = \int_{L_c} \frac{1}{|\nabla B|} ds \geq \frac{l(c)}{\max(|\nabla B|)}.$$
where \(l(c)\) denotes the length of the level \(L_c\). Thus we have that

\[
(28) \quad l(c) \leq \frac{|K| \text{max} (|\nabla B|)}{m^3}.
\]

where \(m\) is the minimum of \(B\) on \(\overline{N}\), the closure of \(N\). (28) implies that \(l(c)\) is dominated by \(\text{max} (|\nabla B(c)|)\). Since \(N\) was arbitrary this implies that \(l(c)\) the length of the critical level set must be zero, but this contradicts the hypothesis.

6. The Symmetric Case

In this section we deal with symmetric magnetic fields on a three dimensional manifold, namely \(\mathbb{R}^3\) with the z-axis removed. Using symplectic reduction the dimension of the system will be reduced allowing us to apply the main theorem of this work. The non-degeneracy condition of one of the components of the magnetic field will be sufficient to apply Moser’s Twist theorem.

Let \(M\) be the standard euclidean three dimensional space with the z-axis removed. Let \(S^1\) act on \(\mathbb{R}^3\) by rotation around the z-axis. Let an \(S^1\) invariant 1-form \(A\) on \(M\) be given. Using cylindrical coordinates we write

\[
A = A_r dr + A_\theta d\theta + A_z dz.
\]

The magnetic system given by \(\beta = dA\) and the purely metric hamiltonian is equivalent to the Hamiltonian system given by

\[
H = \frac{(p_\theta - eA_\theta)^2}{2r^2} + \frac{(p_r - eA_r)^2}{2} + \frac{(p_z - eA_z)^2}{2}
\]

and symplectic 2-form

\[
w = dp_\theta \wedge d\theta + dp_r \wedge dr + dp_z \wedge dz.
\]

The momentum map for rotation around the z-axis is \(p_\theta\). Fixing \(p_\theta = M\) it follows by a theorem of M. Kummer [Ku] that the reduced Hamiltonian is

\[
\hat{H} = \frac{p_r^2}{2} + \frac{p_z^2}{2} + V_{eff}
\]

where

\[
V_{eff} = \frac{(M - eA_\theta)^2}{2r^2},
\]

the reduced symplectic 2-form is given by

\[
\Omega = dp_r \wedge dr + dp_z \wedge dz + e \left( \frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z} \right) dr \wedge dz
\]

and the reduced manifold \(N\) is given by \(N = \mathbb{R} \times \mathbb{R}^+\) with coordinates \((z, r)\) which can be visualized as a half-plane with boundary the z-axis. Writing \(\beta\) as

\[
\beta = B_\theta dr \wedge dz + B_r d\theta \wedge dz + B_z dr \wedge d\theta
\]

we see that

\[
\Omega = dp_r \wedge dr + dp_z \wedge dz + eB_\theta dr \wedge dz = d\lambda + eB_\theta dr \wedge dz
\]

here \(d\lambda\) is the canonical 1-form on \(T^*N\). Assuming that

\[
E > V_{eff}
\]

we have that the hypersurfaces

\[
M_E = \{ \hat{H} = E \}\]
and
\[ M_1 = \left\{ (E - V_{\text{eff}})^{-1} \left( \frac{p_r^2}{2} + \frac{p_z^2}{2} \right) = 1 \right\} \]
are equal. It follows that the closed characteristics of \( \Omega|_{M_E} \) are equivalent to the closed characteristics of \( \Omega|_{M_1} \). Thus we have that the reduced dynamics is (up to reparametrization) given by the purely metric Hamiltonian
\[ \bar{H} = \frac{\|p\|^2}{2g_E} \]
(where \( p = (p_z, p_r) \) and \( g_E \) is the metric \( g_E = (E - V_{\text{eff}})^{-1}g \) where \( g \) denotes the Euclidean metric on \( T^*N \)) and symplectic 2-form \( \Omega \). Assume that a level set of \( B = (B_\theta, B_z, B_r) \) restricted to \( N \) is a simple closed curve, call it \( L_c \), and that for a neighborhood of \( L_c \), the non-degeneracy condition holds, that is to say that \( B_\theta \) satisfies (27) on a neighborhood of \( L_c \). Then we can apply Moser’s twist theorem as in the proof of our main theorem.

This result should be compared with the works of F. Truc [Tr] and M. Braun [Br]. They considered symmetric magnetic fields on \( \mathbb{R}^3 \) given by vector potentials of the type \( A_\theta dr \wedge dz \), which is to say \( B_\theta = 0 \). Those are extremely degenerate magnetic fields in the sense of (27) and so our work does not apply. Also in their case, the absence of the \( B_\theta \) component of their magnetic fields simplified the problem, after the reduction of the symmetry, to a problem of the type kinetic plus potential. Braun’s work is, to the author’s knowledge, the first to apply Moser’s twist theorem to a magnetic problem. Truc’s work is remarkable since she was able to prove the trapping of the particle in the case where the magnetic moment was not convex. To our knowledge we are the first to generalize the problem to a Riemannian surface and to deal with problems where the magnetic field could not be eliminated by the reduction of the symmetry. It is in this sense that our work complement theirs since we dealt with disjoint classes of magnetic field.

7. Technical Proofs

Proof. (of Corollary 3.4) We follow McDuff [McD]. The first part of 3.4 is an immediate consequence of Moser’s homotopy argument. The diffeomorphism \( \Phi_{\epsilon,\theta} \) is built by realizing it as the flow of a vector field \( X \) on \( M \). First we consider the family of 2-forms \( w_t = (1 - t)B\Omega + t\bar{w}_\epsilon \) for \( t \in [0, 1] \). We want a vector field \( X \) such that its flow \( \Phi_{\epsilon,\theta} \) is such that
\[ \Phi_{\epsilon,\theta}^* w_t = B\Omega. \]
(in what follows we will omit the \( \theta \) and \( \epsilon \) dependency from the notation). Differentiating this relation we obtain that
\[ \frac{d}{dt} \Phi_{t}^* w_t + \Phi_{t}^* \frac{dw_t}{dt} = 0, \]
i.e. that,
\[ \Phi^* (L_X w_t + (\bar{w}_\epsilon - B\Omega)) = 0. \]
Since \( \Phi \) is a diffeomorphism and \( w_t \) is closed it follows that this can be satisfied if
(29) \[ d(i_X w_t) + (\bar{w}_\epsilon - B\Omega) = 0. \]
We want to solve equation (29) for $X$. Since $w_t$ is non-degenerate, it suffices to find a family of 1-forms $\sigma_t$ such that

$$\sigma_t|_{TL_c} = 0, \quad d\sigma_1 = w_e - B\Omega. \tag{30}$$

In fact, conditions (30) imply

$$\sigma_t + iX_t w_t = 0. \tag{31}$$

To construct $\sigma_t$ we consider the restriction of the exponential map to the normal bundle $TL_c^\perp$ of the submanifold $L_c$ with respect to the Riemannian metric on $M$. We denote this restriction by

$$\exp : TL_c^\perp \to M. \tag{32}$$

Consider the neighborhood of the zero section

$$U_\epsilon = \{(m, v) \in TM \mid m \in M, v \in TL_c^\perp, |v| < \epsilon\}.$$

Then the restriction of the exponential map to $U_\epsilon$ is a diffeomorphism onto $N_0 = \exp(U_\epsilon)$ for $\epsilon > 0$ sufficiently small. Let $(p, v) \in TM$. Define $\pi_2$ as

$$\pi_2(p, v) = v.$$

Now define $\psi_t : N_0 \to N_0$ for $0 \leq t \leq 1$ by

$$\psi_t(q) = \exp(q, tv), \quad \text{where} \quad v = \pi_2(\exp^{-1}(q)).$$

Then $\psi_t$ is a diffeomorphism for $t > 0$ and we have $\psi_0(N_0) \subset L_c$, $\psi_1 = id$, and $\psi_1|L_c = id$. Calling $\tau = w_e - B\Omega$ we define

$$\sigma_t = \int_0^t \frac{d}{dt} \psi_t^* \tau. \tag{33}$$

It's a direct computation to check that the family of 1-forms $\sigma_t$ has the desired properties. Thus the family $(\sigma_t)$ satisfy conditions (31) and it follows that $\Phi_{\epsilon, \theta}$ the flow of $X$, the vector field that satisfies (31) is such that

$$\Phi_{\epsilon, \theta}^* \bar{w}_\epsilon = B\Omega.$$

It follows that

$$\sigma_t = \int_0^t \frac{d}{dt} \psi_t^* (w_e - B\Omega) = \int_0^t \frac{d}{dt} \psi_t^* \left\{ \epsilon^2 \left( \frac{K}{2B} \Omega - \frac{dB^{-1}}{2} \right) + \epsilon^3 R \right\}$$

where $R$ stands for the tail of the expansion of $\tau$. Denoting $F = (\frac{K}{2B} \Omega - \frac{dB^{-1}}{2})$ we have

$$\sigma_t = \epsilon^2 \int_0^t \frac{d}{dt} \psi_t^* F + \epsilon^3 \int_0^t \frac{d}{dt} \psi_t^* R,$$

that we write as,

$$\sigma_t = \epsilon^2 \sigma_0,t + \epsilon^3 \sigma_1,t,$$

where $\sigma_{0,t} = \int_0^t \frac{d}{dt} \psi_t^* F$ and $\sigma_{1,t} = \int_0^t \frac{d}{dt} \psi_t^* R$. To solve equation (29) we write

$$X = \epsilon^2 X_{0,t} + \epsilon^3 X_{1,t},$$

where $X_{i,t}$ is the solution of

$$\sigma_{i,t} + iX_{i,t} w = 0.$$
for $i = 0,1$. We write $X = \epsilon^2 Y$ where $Y = X_{0,t} + \epsilon X_{1,t}$. Since $X$ is a continuous vector field it follows that $|Y|$ is bounded on a neighborhood of $N_0$ by a constant $K$. The equation for the flow $\Phi$ is

$$\frac{d\Phi}{dt} = X(\Phi(t)).$$

Choosing $\Phi(0) = p_0 \in N_0$ and integrating it follows

$$\Phi(p_0,t) - \Phi(p_0,0) = \int_0^t X(\Phi(t))dt.$$

Writing $\Phi$ the flow of $X$ as $\Phi = \text{Id} + \epsilon^2 C(\epsilon)$ we obtain

$$p_0 + \epsilon^2 C(\epsilon) - p_0 = \epsilon^2 \int_0^1 Y(\Phi(t))dt,$$

giving

$$|\epsilon^2 C(\epsilon)| \leq \epsilon^2 \int_0^1 |Y|dt \leq \epsilon^2 K$$

and the result follows.

\[\square\]

**Remark 7.1.** An important consequence of this lemma is that $X_{0,t}$ is fiber independent. This follows directly from the definition.

**Proof.** (of Corollary 3.6) Using the Fermi coordinates let $p = (x, y, \theta)$ be a point of $S^1 M$. Omitting the $\epsilon$ dependence of $\Xi$ for notational convenience we can write that

$$\Xi(p) = (\Xi_x(p), \Xi_y(p), \theta).$$

The derivative of $\Xi$ can be calculated as

$$\Xi_* = \begin{pmatrix} \frac{\partial \Xi_x}{\partial x} & \frac{\partial \Xi_x}{\partial y} & \frac{\partial \Xi_x}{\partial \theta} \\ \frac{\partial \Xi_y}{\partial x} & \frac{\partial \Xi_y}{\partial y} & \frac{\partial \Xi_y}{\partial \theta} \\ 0 & 0 & 1 \end{pmatrix},$$

which give us that

$$\Xi_* (p) \frac{\partial}{\partial \theta} = \Xi_* (p) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left( \frac{\partial \Xi_x}{\partial \theta}(pt), \frac{\partial \Xi_y}{\partial \theta}(pt), 1 \right),$$

where $pt = \Xi(p)$. Recalling the definitions and lemma 3.4 we can write that

$$\begin{cases} \frac{\partial \Xi_x}{\partial \theta} = \epsilon^2 \frac{\partial C_x}{\partial \theta}, \\ \frac{\partial \Xi_y}{\partial \theta} = \epsilon^2 \frac{\partial C_y}{\partial \theta}. \end{cases}$$

Since $\Phi_{t,\theta} = 1 + \epsilon^2 C(\epsilon, \theta)$ is the flow of $X = \epsilon^2 X_{0,t} + \epsilon^3 X_{1,t}$ we have

$$\frac{d\Phi_{t,\theta}}{dt} = X(\Phi_{t,\theta}).$$

Integrating this equation we have that

$$\epsilon^2 C(\epsilon, \theta) = \int_0^t \left( \epsilon^2 X_{0,t} + \epsilon^3 X_{1,t} \right) \circ (1 + \epsilon^2 C(\epsilon, \theta)) dt,$$

that simplifies to,

$$C(\epsilon, \theta) = \int_0^t \left\{ X_{0,t}(1 + \epsilon^2 C(\epsilon, \theta)) + \epsilon X_{1,t} + \epsilon^3 X_{1,t}(C(\epsilon, \theta)) \right\} dt.$$
Expanding $X_{0,t}(1 + \epsilon^2 C(\epsilon, \theta))$ in $\epsilon$, inserting the resulting expression in (35) and collecting terms we have

$$C(\epsilon, \theta) = \int_0^t X_{0,t} dt + \epsilon \int_0^t X_R dt,$$

for some $X_R$. Since $X_{0,t}$ is $\theta$ independent, we obtain

$$\frac{\partial C(\epsilon, \theta)}{\partial \theta} = \epsilon \frac{\partial}{\partial \theta} \int_0^t X_R dt.$$

This is to say that $\frac{\partial C(\epsilon, \theta)}{\partial \theta}$ is of order at least one in $\epsilon$ and we write that

$$\frac{\partial C(\epsilon, \theta)}{\partial \theta} = \epsilon \bar{C}(\epsilon, \theta),$$

where

$$\bar{C}(\epsilon, \theta) = \frac{\partial}{\partial \theta} \int_0^t X_R dt.$$

Equation (34) gives

$$\Xi^*(p) \frac{\partial}{\partial \theta} = \Xi^*(p) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^3 \bar{C}_x (p') \\ e^3 \bar{C}_y (p') \end{pmatrix},$$

and we write

$$\Xi^*(p) \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} + \epsilon^3 z,$$

where

$$z = \begin{pmatrix} \bar{C}_x \\ \bar{C}_y \\ 0 \end{pmatrix},$$

proving the lemma.

Proof. (of Corollary 7.3) We will study each term of (27) individually. Let $B(p) = a$. Since $p$ is a nondegenerate minimum (or maximum) point we have by Morse’s lemma that there is a neighborhood $V$ and a system of coordinates $(u, v)$ on $V$ such that $B(u, v) = a \pm \frac{1}{2} (u^2 + v^2)$ where the plus sign is used if the point is a minimum and the minus sign is used if the point is a maximum.

Lemma 7.2. Let $r^2_e = u^2 + v^2$ and let $r^2_g = |(u, v)|^2$. Then we have that

$$|\nabla B|^2 = \frac{\text{trace}[g_{ij}] r^2_e - r^2_g}{g}.$$  

The proof of this lemma is straightforward. Note that (36) implies that $\text{trace}[g_{ij}] r^2_e - r^2_g > 0$ on $V - \{p\}$. For a level set $c$ not equal to $a$ and such that $L_c \subset V - \{p\}$ we have that $r_e$ is constant (since $B$ is) and we can write that

$$g |\nabla B|^2 \geq \text{trace}[g_{ij}] r^2_e,$$

i.e.,

$$|\nabla B| \geq \left( \frac{\text{trace}[g_{ij}]}{g} \right)^{\frac{1}{2}} r_e.$$  

Now we can see that the first term of (27) is bounded, in fact we have that

$$\int_{L_c} \frac{3}{|\nabla B|} f dx < 3 \left( \frac{g}{\text{trace}[g_{ij}]} \right)^{\frac{1}{2}} \int_{L_c} f dx,$$
but on $V$ we have that $g$ and $f$ are bounded above and $\text{trace}[g_{ij}]$ is bounded below. So we can find constants $M$ and $K$ such that
\[
M \geq \left( \frac{g}{\text{trace}[g_{ij}]} \right)^{\frac{1}{2}}, \quad \text{and} \quad K \geq f
\]
on $V$, allowing us to write that
\[
\int_{L_c} \frac{3}{|\nabla B|} f dx < \frac{MK}{r_e} \int_{L_c} dx = \frac{MK}{r_e} 2\pi r_e = 2\pi MK.
\]
This gives us that the first term of (27) is bounded for any level set close enough to $p$ as claimed. Now we look to the second term
\[
\int_{L_c} \frac{B}{|\nabla B|^4} < \nabla |\nabla B|, \nabla B > f dx.
\]
For a constant metric one computes that
\[
< \nabla |\nabla B|, \nabla B > = |\nabla B|.
\]
(38) becomes
\[
\int_{L_c} \frac{B}{|\nabla B|^3} f dx,
\]
considering lemma (36), and proceeding as before we have that
\[
\int_{L_c} \frac{B}{|\nabla B|^3} f dx \geq \frac{c}{K} \int_{L_c} \frac{f}{r_e} dx \geq \frac{2\pi c K}{M r_e^2};
\]
The case of a nonconstant metric can be reduced to the case of a constant metric by choosing normal coordinates on a neighborhood of $p$.

The last term of (27) can be minorated by noticing that
\[
\left| \int_{L_c} \frac{B}{|\nabla B|^3} f dx \right| \leq \int_{L_c} \frac{B}{|\nabla B|^2} |\nabla f| \leq G \int_{L_c} \frac{1}{|\nabla B|^2},
\]
for some constant $G$ (since $|\nabla f|$ and $B$ are bounded on $V$). Now observing that $|\nabla B|^2 = r_e^2$, (36) implies
\[
|\nabla B|^2 = \left( \frac{\text{trace}[g_{ij}]}{g+1} \right) r_e^2.
\]
And we can write
\[
\frac{1}{|\nabla B|^2} \leq \frac{H}{r_e^2}
\]
for some constant $H$ such that
\[
\frac{g+1}{\text{trace}[g_{ij}]} \leq H
\]
on $V$. It follows that
\[
\int_{L_c} \frac{B}{|\nabla B|^3} < \nabla f, \nabla B > dx \leq 2\pi G \frac{H}{r_e^2}.
\]
Now considering (37), (38), (40) we see that (27) implies
\[
MK \geq \frac{c K}{M r_e^2} - G \frac{H}{r_e},
\]
which cannot be satisfied if $r_e$ is small enough. Thus for a sufficiently small neighborhood of $p$ all the level sets are non-degenerate.\[\square\]
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Departamento de Matemática, Universidade Federal de Pernambuco, Recife, PE, CEP 50740-540, Brazil.

E-mail address: castilho@dmat.ufpe.br