Research Article

A Leap-Frog Finite Difference Method for Strongly Coupled System from Sweat Transport in Porous Textile Media

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Received 25 March 2019; Accepted 15 May 2019; Published 12 June 2019

Academic Editor: Tepper L. Gill

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In this paper, we present an uncoupled leap-frog finite difference method for the system of equations arising from sweat transport through porous textile media. Based on physical mechanisms, the sweat transport can be viewed as the multicomponent flow that coupled the heat and moisture transfer, such that the system is nonlinear and strongly coupled. The leap-frog method is proposed to solve this system, with the second order accuracy in both spatial and temporal directions. We prove the existence and uniqueness of the solution to the system with optimal error estimates in the discrete $L^2$ norm. Numerical simulations are presented and analyzed, respectively.

1. Introduction

Single/multicomponent flow in porous textile media attracted considerable attention in the last several decades. See [1–4] for the single-component models and [5–9] for the multicomponent models. In this paper, we study the multicomponent sweat transport coupled with vapor and heat in porous textile media. In [10], Ye et al. proposed a quasi-steady-state single-component model which consists of a steady-state air equation and dynamic state equations for other components. Under certain conditions, the multicomponent model reduces to a new single-component model, and the physical process can be viewed as sweat transport (vapor and heat flow) governed by the conservation of mass and energy:

\[
\frac{\partial}{\partial t} (\varepsilon C) + \frac{\partial}{\partial x} (u_g \varepsilon C) = -\Gamma_{ce}
\]

\[
\frac{\partial}{\partial t} (C_{st} T) + \frac{\partial}{\partial x} (u_g C_{mg} C_T) - \frac{\partial}{\partial x} \left( \kappa \frac{\partial T}{\partial x} \right)
= \lambda M_w \Gamma_{ce},
\]

where $\varepsilon$ is the porosity of the media, $C$ is the vapor concentration, $T$ is the temperature, $\kappa$ is the thermal conductivity, $\lambda$ is the latent heat of evaporation/condensation, and $M_w$ is the molecular weight of water. The effective volumetric heat capacity $C_{st}$ is defined by

\[
C_{st} = \varepsilon C_{mg} C + (1 - \varepsilon) C_{vf},
\]

where $C_{mg}$ is the molar heat capacity and $C_{vf}$ is the volumetric heat capacity of fiber.

By Darcy’s law, the gas velocity $u_g$ is defined as

\[
u_g = -\frac{k \partial P}{\mu \partial x},
\]

where $k$ is the permeability and $\mu$ is the dynamic viscosity, which usually is density-dependent for the compressible flow. Here we choose a linear form of $\mu = \gamma C$, where $\gamma$ is a certain constant.

By the Hertz-Knudsen equation [11], the phase change rate $\Gamma_{ce}$ is defined as

\[
\Gamma_{ce} = \beta_t \left( \frac{P - P_{sat}(T)}{\sqrt{T}} \right),
\]
where \( \beta_T \) is a positive constant, the saturation pressure \( P_{sat} \) is determined from experimental measurements [12], and the pressure \( P \) is given by \( P = RCT \), where \( R \) is the universal gas constant.

With nondimensionalization, the sweat transport process (1)-(2) can be described by the following system:

\[
c_t - ((c\theta)_x)_x = -\Gamma(c, \theta),
\]

\[
0 < x < L, \ t > 0,
\]

\[
(c\theta)_t + (\sigma \theta)_x - ((c\theta)_x \theta)_x - \kappa \theta_{xx} = \lambda \Gamma(c, \theta),
\]

\[
0 < x < L, \ t > 0,
\]

where \((\cdot)_x = \partial/\partial x\), \((\cdot)_t = \partial/\partial t\), \( \Gamma(c, \theta) = c\sqrt{\theta} - p_i(\theta) \), and \( p_i(\theta) \sim P_{sat}(\theta)/\sqrt{\theta} \) is a smooth and increasing function satisfying \( p_i(0) = 0 \).

Since the right boundary is exposed to environment and the left boundary is connected to the body, we consider commonly used Robin type boundary conditions

\[
\begin{align*}
(c\theta)_x &= \alpha_1 \left( c - \mu_1 \right), \quad x = 0, \ 0 < t \leq T, \\
(c\theta)_x &= \alpha_2 \left( \mu_2 - c \right), \quad x = L, \ 0 < t \leq T,
\end{align*}
\]

where \( \alpha_1, \alpha_2 \) are nonnegative constants [1, 2, 6].

Due to the strong nonlinearity and the coupling of the equations, the method presented in [23] does not apply to the leap-frog scheme directly. One of the difficulties is to show convergence of the numerical solution without restriction on the grid ratio. In this paper, we assume that the solution \((c(x,t), \theta(x,t))\) to the system (6)-(12) satisfies that

\[
\begin{align*}
\epsilon_{\text{min}} &\leq c(x,t) \leq \epsilon_{\text{max}}, \\
\theta_{\text{min}} &\leq \theta(x,t) \leq \theta_{\text{max}},
\end{align*}
\]

for some positive constants \( \epsilon_{\text{min}}, \epsilon_{\text{max}}, \theta_{\text{max}}, \) and \( \theta_{\text{min}} \).

The manuscript is organized as follows: in Section 2, we present an uncoupled leap-frog finite difference method for the nonlinear sweat transport system. In Section 3, we prove the existence and uniqueness of the solution to the sweat transport system with the optimal error estimate in the discrete \( L^2 \) norm. Numerical results will be presented in Section 4 to support our theoretical results.

### 2. The Leap-Frog Finite Difference Scheme

For convenience of calculations, we add the equation (6) times \( \theta \) into the equation (7); thus the governing system (6)-(7) can be modified as

\[
\begin{align*}
c_t - ((c\theta)_x)_x &= -\Gamma(c, \theta), \\
0 &< x < L, \ t > 0,
\end{align*}
\]

\[
\begin{align*}
(c + \sigma) \theta_t - (c\theta)_x - \kappa \theta_{xx} &= (\lambda + \theta) \Gamma(c, \theta), \\
0 &< x < L, \ t > 0.
\end{align*}
\]

Due to the practical interest in a long time period, say 8–24 hours, we present an uncoupled leap-frog finite difference scheme in the temporal direction and the central finite difference (volume) scheme in the spatial direction for the above system with the initial/boundary conditions (8)-(12).

Let \( \mathcal{T} \) be a positive number, let \( \Omega_h = \{ x_i \mid x_i = ih, 0 \leq i \leq M \} \) be a uniform partition in \([0, L]\), and let \( \Omega_T = \{ t_n \mid t_n = rt, 0 \leq n \leq N \} \) be a uniform partition in \([0, \mathcal{T}]\), where \( h = L/M \) and \( r = \mathcal{T}/N \) are the step size in the spatial and temporal directions, respectively. Denote \( x_{i+1/2} = (x_i + x_{i+1})/2 \) in the spatial cell and let \( \{ v^i_j \mid 0 \leq i \leq M, 0 \leq n \leq N \} \) be a mesh function defined on \( \Omega_{hr} \times \Omega_T \). Some notations are introduced below:

\[
\begin{align*}
\bar{v}_i^n &= \frac{1}{2r} \left( v_{i+1}^{n+1} - v_{i-1}^{n-1} \right), \\
\bar{\bar{v}}_i^n &= \frac{1}{2} \left( v_{i+1}^{n+1} + v_{i-1}^{n-1} \right), \\
\tilde{v}_{i+1/2}^n &= \frac{1}{2} \left( v_{i+1}^n + v_{i+1}^{n+1} \right), \\
\tilde{\bar{v}}_{i+1/2}^n &= \frac{1}{2} \left( v_i^n + v_{i+1}^n \right),
\end{align*}
\]

where \( v_i^n \) is the numerical solution at point \( x_i \) and time \( t_n \).
\[ \begin{align*}
\delta_x v_i^{n+1/2} &= \frac{1}{h} (v_{i+1}^n - v_i^n), \\
\delta_x v_i^n &= \frac{1}{h} (v_{i+1/2}^n - v_{i-1/2}^n), \\
\delta_x v_i^{n+1/2} &= \frac{1}{2} (\delta_x v_{i+1/2}^n + \delta_x v_{i-1/2}^n), \\
\delta_x^2 v_i^{n+1/2} &= \frac{1}{h} (\delta_x v_i^{n+1/2} - \delta_x v_i^{n-1/2}), \\
\end{align*} \]

from which
\[ \left( \delta_x v_{i+1/2}^n \right) v_i^{n+1/2} + v_i^{n+1/2} \delta_x v_{i+1/2}^n = \delta_x \left( v_{i+1/2}^n v_{i+1/2}^n \right). \]

The discrete system is defined by
\[ \begin{align*}
\nabla_i c_i^n - \frac{1}{h} \left[ \left( c_i^{n+1/2} \delta_x \theta_i^{n+1/2} + \theta_i^{n+1/2} \delta_x c_i^n \right) \right] \\
- \left( c_i^{n-1/2} \delta_x \theta_i^{n-1/2} + \theta_i^{n-1/2} \delta_x c_i^n \right) &= -\Gamma \left(c_i^n, \theta_i^n\right), \\
1 \leq i \leq M - 1, \\
\nabla_i c_0^n - \frac{2}{h} \left[ \left( c_0^{n+1/2} \delta_x \theta_0^{n+1/2} + \theta_0^{n+1/2} \delta_x c_0^n \right) \right] \\
- \left( c_0^{n-1/2} \delta_x \theta_0^{n-1/2} + \theta_0^{n-1/2} \delta_x c_0^n \right) &= -\Gamma \left(c_0^n, \theta_0^n\right), \\
\nabla_i c_M^n - \frac{2}{h} \left[ \left( c_M^{n+1/2} \delta_x \theta_M^{n+1/2} + \theta_M^{n+1/2} \delta_x c_M^n \right) \right] \\
- \left( c_M^{n-1/2} \delta_x \theta_M^{n-1/2} + \theta_M^{n-1/2} \delta_x c_M^n \right) &= -\Gamma \left(c_M^n, \theta_M^n\right), \\
\end{align*} \]

where
\[ u_{i+1/2}^n = c_{i+1/2}^n \delta_x \theta_{i+1/2}^n + \theta_{i+1/2}^n \delta_x c_{i+1/2}^n. \]

The computational procedure of the uncoupled leap-frog scheme at each time step is listed below:

**Step 1.** The vapor concentration \( c_i^{n+1} \) can be calculated by solving the tridiagonal linear systems defined in (18)-(20).

**Step 2.** With the updated vapor concentration \( c_i^{n+1} \), we can get \( c_0^n \) and \( u_{i+1/2}^n \) correspondingly.

**Step 3.** Finally, the temperature \( \theta_i^{n+1} \) can be obtained by solving the tridiagonal linear system (21)-(23).

### 3. The Leap-Frog Scheme and the Optimal Error Estimate

In this section, we will show the existence and uniqueness of the solution to the system (18)-(26) with optimal error estimates in the discrete \( L^2 \) norm. Let \( v = \{v_j\}_{j=0}^M \) and \( z = \{z_j\}_{j=0}^M \) be two mesh functions on \( \Omega_h \). We define the inner product and norms by

\[ (v, z) = \left[ \frac{1}{2} V_0 z_0 + \sum_{i=1}^{M-1} v_i z_i + \frac{1}{2} v_M z_M \right], \]

\[ \|v\| = \sqrt{\left[ \frac{1}{2} (v_0)^2 + \sum_{i=1}^{M-1} (v_i)^2 + \frac{1}{2} (v_M)^2 \right]}, \]

\[ \|v\|_\infty \leq \frac{1}{\|v\|}, \]

\[ \|\delta_x v\|^2 = \frac{h}{2} \sum_{i=0}^{M-1} (\delta_x^2 v_i)^2. \]

Let \((C, \Theta)\) be the solution of the system (6)-(12) and \( C_i^n = c(x_i, t_n) \), \( \Theta_i^n = \theta(x_i, t_n) \). The error functions are defined by

\[ \begin{align*}
\varepsilon_i^n &= C_i^n - c_i^n, \\
\tilde{\varepsilon}_i^n &= \Theta_i^n - \theta_i^n, \\
0 \leq i \leq M, & \quad 0 \leq n \leq N - 1, \\
\end{align*} \]

We state our main result in the theorem below:

**Theorem 1.** Suppose that the solution \((c, v)\) of the system (6)-(12) is in \( C^{1,1}([0, L] \times [0, T]) \), satisfying (13). Then there exist positive constants \( h_0 \) and \( E_0 \), independent of \( h \) and \( \tau \), such that,
when \( t \leq E, h \leq h_0 \), the finite difference scheme (18)-(26) is uniquely solvable and

\[
\|\tilde{c}^{n+1}\| + \|\tilde{\theta}^{n+1}\|^2 + \tau \sum_{m=1}^{n} \left( \|\tilde{\delta}_x \tilde{c}^m\|^2 + \|\tilde{\delta}_x \tilde{\theta}^m\|^2 \right) \leq E_0 \left( t^2 + h^2 \right),
\]

(29)

holds for \( n \leq k - 1 \). We prove the assumption and the theorem by induction method. By the initial condition (26), this is true for \( n = 0 \). In the next subsection we will show that this is also true for \( n = 1 \). In this part we let \( E \) be a generic positive constant, which is associated with the physical parameters \( \sigma, \lambda, \kappa, \alpha, \beta, \lambda, \kappa, \mu \), the parameters involved in initial and boundary conditions and the solution of the system (6)-(12). \( E_0 \) is independent of time step \( n \), mesh size \( h \), \( \tau \), and constant \( E_0 \).

3.1. The Leap-Frog Scheme and Preliminaries. For convenience of calculations, we further introduce some notations. Let \( u = (c\vartheta)_x \), \( \vartheta = \vartheta_x \); thus the sweat transport system (6)-(7) can be reduced to

\[
c - u_x = -\Gamma(c, \theta), \quad 0 \leq x \leq L, \quad 0 < t \leq \tau,
\]

(31)

\[
\vartheta = (c\vartheta_x + \beta\vartheta), \quad 0 \leq x \leq L, \quad 0 < t \leq \tau,
\]

(32)

\[
(c + \sigma)\vartheta_t - u\vartheta_x - \mu\vartheta = (\lambda + \theta) \Gamma(c, \theta), \quad 0 \leq x \leq L, \quad 0 < t \leq \tau,
\]

(33)

\[
w = \vartheta_x, \quad 0 \leq x \leq L, \quad 0 < t \leq \tau,
\]

(34)

with the initial and boundary conditions

\[
u = \alpha_1 (c - \mu_1), \quad x = 0, \quad 0 < t \leq \tau,
\]

\[
u = \alpha_2 (\mu_2 - c), \quad x = L, \quad 0 < t \leq \tau,
\]

\[
w = \beta_1 (\theta - \nu_1), \quad x = 0, \quad 0 < t \leq \tau,
\]

\[
w = \beta_2 (\nu_2 - \theta), \quad x = L, \quad 0 < t \leq \tau,
\]

(35)

\[
c(x, 0) = c_0(x), \quad 0 \leq x \leq L.
\]

\[
\theta(x, 0) = \theta_0 (x), \quad 0 \leq x \leq L.
\]
The initial conditions are

\[
W_{\nu+1/2} = \theta_0(x_\nu), \quad \Theta_{\nu+1/2} = \theta_0(x_\nu + \tau \xi_{\nu+1/2}), \quad C_{\nu+1/2} = C_0(x_\nu + \tau \xi_{\nu+1/2}),
\]

where

\[
\begin{align*}
|R_{\nu,j}^\alpha|, |R_{\nu,j}^\beta| &\leq E_1 (\tau + h^2), \quad 0 \leq j \leq M - 1, \\
|R_{\nu,j}^\gamma|, |R_{\nu,j}^\delta| &\leq E_2 (\tau^2 + h^2), \quad 0 \leq j \leq M - 1, \\
|R_{\nu,j}^\epsilon|, |R_{\nu,j}^\zeta| &\leq E_3 (\tau^2 + h^2), \quad 0 \leq j \leq M - 1, \\
|R_{\nu,j}^\eta|, |R_{\nu,j}^\theta| &\leq E_4 (\tau^2 + h^2), \quad 1 \leq j \leq N - 1.
\end{align*}
\]

Subtracting the system (36) from the system (38), we get the error equations

\[
\begin{align*}
\nabla_i \psi_i^\nu - \delta_{\nu} u_i^{\nu} = &\ - [\Gamma(C_0^\alpha, \Theta_0^\alpha) - \Gamma(c_0^\nu, \Theta_0^\nu)] + R_{\nu,j}^\alpha, \\
\nabla_0 \psi_0^\nu - \frac{2}{h} [\nabla_{i-1/2} - \alpha_{i} z_i] = &\ - [\Gamma(C_0^\alpha, \Theta_0^\alpha) - \Gamma(c_0^\nu, \Theta_0^\nu)] + R_{\nu,j}^\alpha, \\
\nabla_{i-1/2} \psi_0^\nu - \frac{2}{h} [\nabla_{i-1/2} - \alpha_{i} z_i] = &\ - [\Gamma(C_{i+1/2}^\alpha, \Theta_{i+1/2}^\alpha) - \Gamma(c_{i+1/2}^\nu, \Theta_{i+1/2}^\nu)] + R_{\nu,j}^\alpha.
\end{align*}
\]

and by (40), we can directly derive the inequality

\[
\|\bar{e}_i\|^2 + \|\bar{\theta}_i\|^2 + \|\delta_{\nu} \bar{u}_i\|^2 \leq E_0 (\tau^2 + h^2)^2. \tag{50}
\]

To prove our main theorem, the following formula will be often used:

\[
\begin{align*}
\psi_{i+1/2}^\nu - \frac{[\nabla_i u_{i+1/2} - \alpha_i z_i]}{h} = &\ - [\Gamma(C_{i+1/2}^\alpha, \Theta_{i+1/2}^\alpha) - \Gamma(c_{i+1/2}^\nu, \Theta_{i+1/2}^\nu)] + R_{\nu,j}^\alpha = R_{\nu,j}^\alpha,
\end{align*}
\]

In the following lemma, we present discrete Sobolev interpolation formulas and the proof can be found in [24].
Lemma 2. Let \( v \) and \( z \) be two mesh functions. Then for any positive constant \( \epsilon \),

\[
\|v\|_{\infty}^2 \leq \epsilon \|\delta_x v\|_{\infty}^2 + \left( \frac{1}{\epsilon} + \frac{1}{L} \right) \|v\|_{\infty}^2, \\
\|\delta_x v\|_{\infty} \leq \epsilon \|\delta_x^2 v\|_{\infty} + E_{\epsilon} \|\delta_x v\|_{\infty}^2.
\]

(52) (53)

Lemma 3.

\[
\|c^n\|_{\infty} \leq 3E_{0}^{3/2} (\tau^{3/2} + h^{3/2}), \quad 1 \leq n \leq k,
\]

(54)

\[
\|\delta_x c^n\|_{\infty} \leq E_{0}^{3/2} (\tau^{3/2} + h), \quad 1 \leq n \leq k - 1.
\]

(55)

Proof. From (30) for \( 0 \leq n \leq k - 1 \), we have

\[
\|c^{n+1}\|_{\infty}^2 + \tau \|\delta_x c^{n+1}\|_{\infty}^2 \leq E_0 (\tau^2 + h^2)^2, \\
\|c^n\|_{\infty}^2 + \tau \|\delta_x c^n\|_{\infty}^2 \leq E_0 (\tau^2 + h^2)^2.
\]

(56)

When \( \tau \leq h \), with the inverse inequality we have

\[
\|c^{n+1}_i\|_{\infty}^2 \leq \|c^n_{i+1}\|_{\infty}^2 \leq 2h^{-1} \|c^{n+1}\|_{\infty}^2 \leq 2h^{-1} E_0 (\tau^2 + h^2)^2 \\
\leq 8E_0 h^3.
\]

(57)

When \( h \leq \tau \), by taking \( \epsilon = \tau^{3/2} \) in Lemma 2,

\[
\|c^{n+1}_i\|_{\infty}^2 \leq \|c^n_{i+1}\|_{\infty}^2 \\
\leq \tau^{3/2} \|\delta_x c^{n+1}\|_{\infty}^2 (\tau^{-1/2} + L^{-1}) \|c^{n+1}\|_{\infty}^2 \\
\leq (\tau^{-1/2} + \tau^{1/2} L^{-1}) E_0 (\tau^2 + h^2)^2 \leq 8E_0 \tau^{7/2}, \\
1 \leq i \leq M - 1, \quad 0 \leq n \leq k - 1.
\]

(58)

The first part of (54) is obtained and the second part and the inequality (55) can be proved similarly. \( \square \)

In addition, by Lemma 3, there exist constants \( E_3 > 0 \) and \( s_0 > 0 \) such that, when \( h, \tau \leq s_0 \),

\[
\frac{c_{\min}}{2} \leq c_i^0 \leq 2c_{\max}, \\
\frac{\theta_{\min}}{2} \leq \theta_i^0 \leq 2\theta_{\max},
\]

(59)

and

\[
\left[ |V_i \Omega_i^0|, |V_i C_i^0| \right] \leq E_3, \quad 0 \leq i \leq M - 1, \quad 1 \leq n \leq k, \\
\left| \delta_x C_{i+1/2}^n \right|, \left| \delta_x \Theta_{i+1/2}^n \right|, \left| \delta_x C_{i+1/2}^n \right|, \left| \delta_x \Theta_{i+1/2}^n \right| \leq E_3, \\
0 \leq i \leq M - 1, \quad 1 \leq n \leq k, \\
\left| c_{i+1/2}^n \right|, \left| \theta_{i+1/2}^n \right| \leq E_3, \quad 0 \leq i \leq M - 1, \quad 1 \leq n \leq k,
\]

\[
\left| c_{i+1/2}^n \right|, \left| \theta_{i+1/2}^n \right| \leq E_3, \quad 0 \leq i \leq M - 1, \quad 1 \leq n \leq k,
\]

\[
\left[ \|c^{n+1}\|_{\infty}, \|c^n\|_{\infty} \right] \leq E_3, \quad 0 \leq i \leq M - 1, \quad 1 \leq n \leq k,
\]

\[
\left| c_{i+1/2}^n \right|, \left| \theta_{i+1/2}^n \right| \leq E_3, \quad 0 \leq i \leq M - 1, \quad 1 \leq n \leq k,
\]

\[
\left| c_{i+1/2}^n \right|, \left| \theta_{i+1/2}^n \right| \leq E_3, \quad 0 \leq i \leq M - 1, \quad 1 \leq n \leq k.
\]

(60)

3.2. The Existence and Uniqueness. Since the coefficient matrix in the system (18)-(20) is strictly diagonally dominant, thus the system (18)-(20) has a unique solution \( c_i^{k+1} \). Here we will discuss the boundedness of \( c_i^{k+1} \).

Multiplying (41)-(43) by \( h c_i^n, h c_0^n/2 \), and \( h c_M^n/2 \), respectively, we get

\[
\left( V_i c_i^n, c_i^n \right) + \alpha_1 |c_0^n| + \alpha_2 |c_M^n| = \left[ \begin{array}{c} u_{1/2}^1 \cr \end{array} \right] \cdot \left( \begin{array}{c} \frac{h}{2} R_{c_0^n} + \frac{h}{2} R_{c_M^n} \cr \end{array} \right)
\]

(61)

\[
- \frac{1}{2} \left\{ \Gamma (C_0^n, \Theta_0^n) - \Gamma (c_0^n, \theta_0^n) \right\} c_0^n \\
+ \sum_{i=1}^{M-1} \left\{ \Gamma (C_i^n, \Theta_i^n) - \Gamma (c_i^n, \theta_i^n) \right\} c_i^n \\
+ \frac{1}{2} \left\{ \Gamma (C_M^n, \Theta_M^n) - \Gamma (c_M^n, \theta_M^n) \right\} c_M^n \right\} = J_1 + J_2
\]

(62)

With (44), (51), (59), (60), and (40), we see that

\[
J_1 = -u_{1/2}^1 \cdot h \sum_{i=1}^{M-1} \delta_x u_i^n \delta_x c_i^n + u_{M-1/2}^n \cdot \delta_x c_{M-1/2}^n
\]

(63)

\[
+ R_{c_0^n} \delta_x c_0^n + \delta_x c_0^n \theta_0^n + R_{c_M^n} \delta_x c_M^n + \delta_x c_M^n \theta_M^n
\]

(64)

\[
|\delta_x c_0^n| - E_3 \left( \|c_0^n\|_{\infty} + \|\theta_0^n\|_{\infty} \right) \|\delta_x c_0^n\|_{\infty}
\]

\[
- h \left( \frac{\alpha_1}{4} \left| \delta_x c_0^n \right|^2 + 2E_3 \|\delta_x c_0^n\|_{\infty} \right) \leq \left( \frac{\alpha_2}{2} + \frac{\alpha_1}{4} \right) \left( \|c_0^n\|_{\infty} + \|\theta_0^n\|_{\infty} \right)
\]

(65)

\[
- \frac{\alpha_1}{4} \left| \delta_x c_0^n \right|^2 + 2E_3 \|\delta_x c_0^n\|_{\infty} \right) \leq \left( \frac{\alpha_2}{2} + \frac{\alpha_1}{4} \right) \left( \|c_0^n\|_{\infty} + \|\theta_0^n\|_{\infty} \right)
\]

(66)

\[
- E_3 \left( \|\delta_x \theta_0^n\| + \|\delta_x c_0^n\|_{\infty} \right) - \frac{\alpha_1}{4} \left| \delta_x c_0^n \right|^2
\]

(67)
and by using (60) again, we have
\[ |J_3| \leq E_3 \left( \|c^2\| + \|\bar{\theta}^2\| \right) \|c^2\|, \tag{63} \]

and with (52),
\[ |J_2| \leq \frac{1}{2} h \sum_{i=1}^{M-1} \left( \left( R_{c,i}^2 \right)^2 + \left( R_{c,i+1}^2 \right)^2 \right) + \frac{h^2}{2} \left| R_{c,0} \right|^2 + \frac{1}{2} \left| c_0^2 \right|^2 \]
\[ + \frac{h^2}{2} \left| R_{c,M} \right|^2 + \frac{1}{2} \left| c_M^2 \right|^2 \]
\[ \leq \|c^2\|_{\infty}^2 + E_c \left( \|c^2\|^2 + \|\bar{\theta}^2\| + \|c^2\|^2 \right) \]
\[ + E_c \left( \tau^2 + h^2 \right)^2, \tag{64} \]

Substituting the last three equations into (61) results in
\[ \frac{1}{2} V \|c^2\|^2 + \frac{\theta_{\min}}{8} \left( \|\delta_x c^2\|^2 + \alpha_1 \|c_0\|^2 + \alpha_2 \|c_M\|^2 \right) \]
\[ \leq E_c \left( \|\delta_x \bar{\theta}^2\|^2 + \|c^2\|^2 + \|\bar{\theta}^2\|^2 + \|c^2\|^2 \right) \]
\[ + E_c \left( \tau^2 + h^2 \right)^2, \tag{65} \]

where we have noted \((V \|c^2\|, \|c^2\|) = (1/4\tau)(\|c^{n+1}\| - \|c^{n-1}\|)^2 = (1/2) V \|c^2\|^2\). Moreover, by the assumption of the induction,
\[ \|c^{k+1}\|^2 + 4 \tau \left( \frac{\theta_{\min}}{8} \|\delta_x c^2\|^2 + \alpha_1 \|c_0\|^2 + \alpha_2 \|c_M\|^2 \right) \]
\[ \leq \|c^{k-1}\|^2 \]
\[ + 4 E_c \tau \left( \|\delta_x \bar{\theta}^2\|^2 + \|c^2\|^2 + \|\bar{\theta}^2\|^2 + \|c^2\|^2 \right) \]
\[ + 4 E_c \left( \tau^2 + h^2 \right)^2. \tag{66} \]

Since we have the fact that \(\|c^2\|^2 \leq (1/2) \|c^{k+1}\|^2 + \|c^{k-1}\|^2\), thus
\[ (1 - 2 \tau E_c) \|c^{k+1}\|^2 + \frac{\tau \theta_{\min}}{2} \|\delta_x c^2\|^2 \]
\[ \leq (E_0 + 4 \tau E_c + 4 E_c E_0 + \bar{E} R E_0) \left( \tau^2 + h^2 \right)^2. \tag{67} \]

When \(\tau E_c < 1/4\), we can get the inequality as
\[ \|c^{k+1}\|^2 + \tau \theta_{\min} \|\delta_x c^2\|^2 \leq E_4 \left( \tau^2 + h^2 \right)^2. \tag{68} \]

Since \(E_4\) are independent of \(k\), by (13) when \(h\) and \(\tau\) are small enough,
\[ c_i^{k+1} \geq 0, \quad 0 \leq i \leq M. \tag{69} \]

Now we try to prove our main theorem. By noting (44), (60), (40), and Lemma 3,
\[ \|\tilde{u}\|^2 \leq \sum_{i=0}^{M-1} \left( \left( \sum_{i=1}^{M-1} c_i^{k+1} \bar{\theta}_i^2 \right)^2 + \left( \sum_{i=1}^{M-1} c_i^{k+1} \bar{\theta}_i^2 \right)^2 \right) \]
\[ + \bar{c}_i^{k+1} \left( \sum_{i=1}^{M-1} \left( \sum_{i=1}^{M-1} c_i^{k+1} \bar{\theta}_i^2 \right)^2 + \left( \sum_{i=1}^{M-1} c_i^{k+1} \bar{\theta}_i^2 \right)^2 \right) \]
\[ \leq \sum_{i=0}^{M-1} \left( \left( \sum_{i=1}^{M-1} c_i^{k+1} \bar{\theta}_i^2 \right)^2 + \left( \sum_{i=1}^{M-1} c_i^{k+1} \bar{\theta}_i^2 \right)^2 \right) \]
\[ + \bar{c}_i^{k+1} \left( \sum_{i=1}^{M-1} \left( \sum_{i=1}^{M-1} c_i^{k+1} \bar{\theta}_i^2 \right)^2 + \left( \sum_{i=1}^{M-1} c_i^{k+1} \bar{\theta}_i^2 \right)^2 \right) \]
\[ \leq 20 \theta_{\min} \left( \|\delta_x \bar{\theta}^2\|^2 + 20 \theta_{\min} \|\delta_x c^2\|^2 + 20 \|\bar{\theta}^2\|^2 + 5 E_3 \left( \|c^2\|^2 \right) + 5 E_2 L \left( \tau^2 + h^2 \right)^2, \quad 1 \leq n \leq k \]

We can see that, when \(\tau \leq h\), the assumption of induction and (68) show that
\[ \|\tilde{u}\|^2 \leq 40 \theta_{\min} \frac{\|c^{k+1}\|^2}{h^2} + E_c h^4 \tag{71} \]
\[ \leq 160 \theta_{\min}^2 E_c h^2 + E_c h^4, \]

and when \(h \leq \tau\), by (68),
\[ \|\tilde{u}\|^2 \leq 20 \theta_{\min} \frac{E_c \left( \tau^2 + h^2 \right)^2}{\tau \theta_{\min}} + E_c \tau^4 \tag{72} \]

which means there exists an \(E_5\) independent of \(k\), such that
\[ \|\tilde{u}\| \leq E_5 \left( \tau^{3/2} + h \right). \tag{73} \]

Multiplying the error equation (41) by \(\delta_x \tilde{u}_i^k\) leads to
\[ \|\delta_x \tilde{u}_i^k\|^2 \leq \sum_{i=1}^{M-1} \left( \|\delta_x \tilde{u}_i^k\|^2 + \|\bar{\theta}^2\|^2 + \|c^2\|^2 \right) \]
\[ \leq \sum_{i=1}^{M-1} \|\delta_x \tilde{u}_i^k\|^2 + \bar{c}_i^{k+1} \left( \|\delta_x \tilde{u}_i^k\|^2 + \|\bar{\theta}^2\|^2 + \|c^2\|^2 \right) \]
\[ \leq 4 \left( \|\delta_x \tilde{u}_i^k\|^2 + \|\bar{\theta}^2\|^2 + \|c^2\|^2 \right) \]
\[ + \frac{1}{4} \|\delta_x \tilde{u}_i^k\|^2 + 4 E_2 L \left( \tau^2 + h^2 \right)^2, \tag{74} \]

that is,
\[ \|\delta_x \tilde{u}_i^k\|^2 \leq 16 \|\delta_x \tilde{u}_i^k\|^2 + 32 E_3 \left( \|\bar{\theta}^2\|^2 + \|c^2\|^2 \right) \]
\[ + 16 E_2 L \left( \tau^2 + h^2 \right)^2. \tag{75} \]

We can see that, when \(\tau \leq h\),
\[ \|\tilde{u}\| \leq h^{-1/2} \|\tilde{u}\| \leq E_5 \left( h + h^{1/2} \right). \tag{76} \]
and when $h \leq \tau$, with Lemma 2,
\[
\left\| \hat{u}^k \right\|_\infty^2 \leq \delta^2 \left\| \hat{w}^k \right\|_\infty^2 + (1 + L^{-1}) \left\| \hat{w}^k \right\|_\infty^2 \leq E_\delta \tau^2, \tag{77}
\]
where $E_\delta$ is independent of $k$. Then there exists $s_0 > 0$, when $h, \tau \leq s_0$,
\[
\left\| \hat{u}^k \right\|_\infty = \max_{1 \leq i \leq M} \left| (\hat{u}^k)_i \right| \leq 2E_3. \tag{78}
\]
With a time step condition $\tau \leq E_h h$, we can see that the coefficient matrix of the system (21)-(23) is strictly diagonally dominant. Thus this system has a unique solution $\hat{\theta}^{k+1}$.

3.3. The Optimal Error Estimate. We have proved the existence and uniqueness of the solution to the system and have derived the estimate (65) for $\hat{w}^{k+1}$. In this part, we try to derive an estimate for $\theta^{k+1}$.

Multiplying (45)-(47) by $h\hat{\theta}_i^0, h\hat{\theta}_M^0/2$, and $h\hat{\theta}_M/2$, respectively, we try to estimate each term below:
\[
h \left[ \frac{1}{2} (\tilde{c}_i + \sigma) \left( V_i \hat{\theta}_0^i \right) \hat{\theta}_0^i + \frac{1}{2} (\tilde{c}_i + \sigma) \left( V_i \hat{\theta}_M^i \right) \hat{\theta}_M^i \right]
+ \frac{1}{2} (\tilde{c}_i + \sigma) \left( V_i \hat{\theta}_M/2 \right) \hat{\theta}_M/2
+ \frac{1}{2} \left( \tilde{c}_i + \sigma \right) \left( V_i \hat{\theta}_i \right)^2
+ \tilde{c}_i \left( V_i \hat{\theta}_i \right)^2
+ \frac{1}{2} \left( \tilde{c}_i + \sigma \right) \left( V_i \hat{\theta}_i \right)^2
+ \frac{1}{2} \left( \tilde{c}_i + \sigma \right) \left( V_i \hat{\theta}_i \right)^2.
\]
According to Lemma 2, (40), (51), and (60), three terms on the left can be bounded by
\[
h \left[ \frac{1}{2} (\tilde{c}_i + \sigma) \left( V_i \hat{\theta}_0^i \right) \hat{\theta}_0^i + \frac{1}{2} (\tilde{c}_i + \sigma) \left( V_i \hat{\theta}_M^i \right) \hat{\theta}_M^i \right]
+ \frac{1}{2} \left( \tilde{c}_i + \sigma \right) \left( V_i \hat{\theta}_M/2 \right) \hat{\theta}_M/2
+ \frac{1}{2} \left( \tilde{c}_i + \sigma \right) \left( V_i \hat{\theta}_i \right)^2
+ \frac{1}{2} \left( \tilde{c}_i + \sigma \right) \left( V_i \hat{\theta}_i \right)^2.
\]
and
\[
\left| \left[ \frac{1}{2} (\tilde{c}_i + \sigma) \left( V_i \hat{\theta}_0^i \right) \hat{\theta}_0^i + \frac{1}{2} (\tilde{c}_i + \sigma) \left( V_i \hat{\theta}_M^i \right) \hat{\theta}_M^i \right]
+ \frac{1}{2} \left( \tilde{c}_i + \sigma \right) \left( V_i \hat{\theta}_M/2 \right) \hat{\theta}_M/2
+ \frac{1}{2} \left( \tilde{c}_i + \sigma \right) \left( V_i \hat{\theta}_i \right)^2
+ \frac{1}{2} \left( \tilde{c}_i + \sigma \right) \left( V_i \hat{\theta}_i \right)^2 \right|
\]
By (70), for those terms in the right hand side, we obtain
\[
|J_1| \leq E_3 \left( \left\| \hat{u}^k \right\|_\infty + \left\| \hat{u}^{k+1} \right\|_\infty \right),
\]
\[
|J_4| \leq hE_3 \left( \left\| \hat{u}^{k+1} \right\|_\infty + \sum_{i=1}^{M-1} \left| (\hat{u}^i - 1/2 + \hat{u}^{i+1/2}) \right| \left\| \theta^i \right\|_\infty \right)
\]
and
\[
|J_6| \leq hE_3 \left( \left\| \hat{u}^{k+1} \right\|_\infty + \sum_{i=1}^{M-1} \left| (\hat{u}^i - 1/2 + \hat{u}^{i+1/2}) \right| \left\| \theta^i \right\|_\infty \right).
\]
\[+ |\tilde{\theta}_{M-1/2}^n - \tilde{\theta}_M^n| \leq 2E_3 \left( \|\tilde{\theta}\|^2 \right)
+ \sum_{i=0}^{M-1} h^2 u_{i+1/2} u_{i+1}^2 \left( \|\tilde{\theta}_M^n\|^2 \right)
+ E_c \left( \|\tilde{\theta}_M^n\|^2 + \|\tilde{\theta}_M^n\|^2 + \|\tilde{\theta}_M^n\|^2 + E_2 \left( \tau^2 + h^2 \right)^2 \right) ,
\]

\[|J_a| \leq E_3 \left( \|\tilde{\theta}_M^n\|^2 + \|\tilde{\theta}_M^n\|^2 \right).\]

Taking the last six equations into (79), we obtain
\[
\begin{align*}
\frac{h}{2} \left( \frac{1}{2} \left( c_i^\alpha + \sigma \right) \nabla \left[ \left( \tilde{\theta}_i^\alpha \right)^2 \right] + \sum_{i=1}^{M-1} \left( c_i^\alpha + \sigma \right) \nabla \left[ \left( \tilde{\theta}_i^\alpha \right)^2 \right] \right) \\
+ \frac{1}{2} \left( c_i^\alpha + \sigma \right) \nabla \left[ \left( \tilde{\theta}_i^\alpha \right)^2 \right] + \frac{\kappa}{4} \left\| \tilde{\theta}_i^\alpha \right\|^2 \\
+ \kappa \beta_1 \left\| \tilde{\theta}_i^\alpha \right\|^2 + \kappa \beta_2 \left\| \tilde{\theta}_i^\alpha \right\|^2 \leq 40 E_2 \left( \left\| \tilde{\theta}_i^\alpha \right\|^2 + \left\| \tilde{\theta}_i^\alpha \right\|^2 + \left\| \tilde{\theta}_i^\alpha \right\|^2 \right) \\
+ E_c \left( \|\tilde{\theta}_i^\alpha \|^2 + \|\tilde{\theta}_i^\alpha \|^2 + \|\tilde{\theta}_i^\alpha \|^2 + \|\tilde{\theta}_i^\alpha \|^2 \right) \\
+ E_c \left( \tau^2 + h^2 \right)^2, \quad 1 \leq n \leq k.
\end{align*}
\]

Besides, we introduce a notation as \( \tilde{\theta}^* = (1/2)\left[ (\tilde{\theta}_i^a)^2 + (\tilde{\theta}_i^r)^2 \right] \), and adding the first three equations into (36) by \( h\tilde{\theta}_i^a/2, h\tilde{\theta}_i^r/4, \) and \( h\tilde{\theta}_M^a/4 \), respectively, we have
\[
\begin{align*}
\frac{h}{2} \left[ \frac{1}{2} \left( \nabla \left( \tilde{\theta}_i^a \right)^2 \right) \right] + \frac{\alpha}{2} \left( \tilde{\theta}_i^a - \mu_1 \right) \tilde{\theta}_i^a + \frac{\alpha}{2} \left( \tilde{\theta}_M^a - \mu_2 \right) \tilde{\theta}_M^a \\
\left. - \frac{1}{2} \left[ u_{i+1/2} u_{i+1} \tilde{\theta}_{i+1} - u_{i-1/2} \tilde{\theta}_i \right] \right|_{M-1/2} \\
+ \frac{1}{2} \left( \nabla \tilde{\theta}_i^r \right) \tilde{\theta}_i^r + \frac{M-1}{2} \tilde{\theta}_M^r \\
+ \frac{1}{2} \tilde{\theta}_M^r \frac{\partial^2}{M} \left( \tilde{\theta}_M^r \right), \quad 1 \leq n \leq k.
\end{align*}
\]

We now estimate the terms in (84). By (51), we denote
\[
\begin{align*}
J_a &= \frac{1}{2} \left[ u_{i+1/2} u_{i+1} \tilde{\theta}_{i+1} - u_{i-1/2} \tilde{\theta}_i \right] \\
&\leq \frac{1}{2} \sum_{i=0}^{M-1} u_{i+1/2} \left( \tilde{\theta}_{i+1} - \tilde{\theta}_i \right) \\
&\leq E_3 h \sum_{i=0}^{M-1} \left( \tilde{\theta}_{i+1/2} - \tilde{\theta}_{i+1/2} \right) + \tilde{\theta}_{i+1/2} \tilde{\theta}_{i+1/2} \\
&\leq E_3 \left( \left\| \tilde{\theta}_i^a \right\|^2 + \left\| \tilde{\theta}_i^r \right\|^2 + \left\| \tilde{\theta}_i^a \right\|^2 + \left\| \tilde{\theta}_i^r \right\|^2 \right).
\end{align*}
\]

Using (60) again, we get
\[
\begin{align*}
\frac{h}{2} \left[ \frac{1}{2} \tilde{\theta}_0^a \Gamma (c_0^a, \theta_0^a) + \sum_{i=1}^{M-1} \tilde{\theta}_i^a \Gamma (c_i^a, \theta_i^a) \right] \\
+ \frac{1}{2} \tilde{\theta}_M^a \Gamma (c_M^a, \theta_M^a) \right|_{M-1} \leq \frac{E_4}{4} \left( \|\tilde{\theta}_i^a \|^2 + \|\tilde{\theta}_i^r \|^2 \right).
\end{align*}
\]

and with Lemma 2, we have
\[
\begin{align*}
\frac{h}{2} \left[ \frac{1}{2} \left( \nabla \left( \tilde{\theta}_i^a \right)^2 \right) \right] + \frac{\alpha}{2} \left( \tilde{\theta}_i^a - \mu_1 \right) \tilde{\theta}_i^a + \frac{\alpha}{2} \left( \tilde{\theta}_M^a - \mu_2 \right) \tilde{\theta}_M^a \\
+ \frac{\kappa}{4} \left\| \tilde{\theta}_i^a \right\|^2 \\
+ \kappa \beta_1 \left\| \tilde{\theta}_i^a \right\|^2 + \kappa \beta_2 \left\| \tilde{\theta}_i^a \right\|^2 \leq 40 E_2 \left( \left\| \tilde{\theta}_i^a \right\|^2 + \left\| \tilde{\theta}_i^a \right\|^2 + \left\| \tilde{\theta}_i^a \right\|^2 \right) \\
+ E_c \left( \|\tilde{\theta}_i^a \|^2 + \|\tilde{\theta}_i^a \|^2 + \|\tilde{\theta}_i^a \|^2 + \|\tilde{\theta}_i^a \|^2 \right) \\
+ E_c \left( \tau^2 + h^2 \right)^2, \quad 1 \leq n \leq k.
\end{align*}
\]

Moreover, by noting the fact that
\[
\left[ (c_i^a + \sigma) \right] \nabla \left[ (\tilde{\theta}_i^a)^2 \right] + \tilde{\theta}_i^a \nabla c_i^a = \nabla \left[ (c_i^a + \sigma) (\tilde{\theta}_i^a)^2 \right], \quad \text{for} \quad 1 \leq n \leq k.
\]
Multiplying the last equation with \( s_1 = \theta_{\min}/320E_2\bar{\rho}^2_{\max} \) and adding it into equation (65), we get

\[
\begin{align*}
\nabla_i \|c^i\|^2 + \frac{\theta_{\min}}{8} \|\partial_x \tilde{c}^i\|^2 + \left[ 2\alpha_1 \|c_0^i\| + 2\alpha_2 \|c_{M}^i\| \right] \\
+ s_1 \kappa_1 \|\tilde{b}_1\| + s_1 \kappa_2 \|\tilde{b}_M\| \\
+ s_1 \frac{h}{2} \left[ \frac{1}{2} \nabla_i \left( (c_0^i + \sigma) \|\tilde{b}_1\|^2 \right) \\
+ \sum_{i=1}^{M-1} \nabla_i \left( (c_0^i + \sigma) \|\tilde{b}_M\|^2 \right) \right] \\
+ \frac{M-1}{2} \nabla_i \left[ (c_0^i + \sigma) \|\tilde{b}_1\|^2 \right] + \frac{1}{2} \nabla_i \left[ (c_0^i + \sigma) \|\tilde{b}_M\|^2 \right] \right] \\
+ \frac{\kappa s_1 \|\partial_x \tilde{b}\|^2}{4} \leq E_c \left( \|\partial_x \tilde{b}\|^2 + \|\tilde{b}^{i+1}\|^2 + \|c^{i+1}\|^2 \right) + E_c \left( r^2 + h^2 \right), \quad 1 \leq n \leq k.
\end{align*}
\]

Finally, we estimate \( \|\partial_x \tilde{b}\| \). Multiplying the error equation (45) by \(-h\partial_x \tilde{b}/(c_0^i + \sigma)\) and summing up the resulting equations for \( i = 1, 2, \ldots, M - 1 \), we have

\[
\begin{align*}
- h \sum_{i=1}^{M-1} \left( \nabla_i \tilde{b}_i \right) \partial_x^2 \tilde{b}_i + \frac{\kappa}{\bar{c}^i + \sigma} \sum_{i=1}^{M-1} \|\partial_x \tilde{b}_i\|^2 + \frac{\kappa}{\bar{c}^i + \sigma} \sum_{i=1}^{M-1} \|\tilde{b}^{i+1}\|^2 \\
- \frac{\kappa}{\bar{c}^i + \sigma} \sum_{i=1}^{M-1} \|\tilde{b}^{i+1}\|^2 + \frac{\kappa}{\bar{c}^i + \sigma} \sum_{i=1}^{M-1} \|\tilde{b}^i\|^2 \right] = \frac{h}{\bar{c}^i + \sigma}
\end{align*}
\]

For the first term, we have

\[
\begin{align*}
- h \sum_{i=1}^{M-1} \left( \nabla_i \tilde{b}^i \right) \partial_x^2 \tilde{b}^i \\
= \frac{h}{2\alpha_1} \sum_{i=1}^{M-1} \|\partial_x \tilde{b}^{i+1}\|^2 + \frac{h}{2\alpha_2} \sum_{i=1}^{M-1} \|\partial_x \tilde{b}^i\|^2 \\
= \frac{1}{2r} \sum_{i=1}^{M-1} \|\partial_x \tilde{b}^{i+1}\|^2 - \partial_x \tilde{b}^{i+1} \left( \partial_x \tilde{b}^{i+1} - \partial_x \tilde{b}^{i+1} \right) \\
+ \frac{1}{2r} \sum_{i=1}^{M-1} \|\partial_x \tilde{b}^i\|^2 - \partial_x \tilde{b}^i \left( \partial_x \tilde{b}^i - \partial_x \tilde{b}^i \right).
\end{align*}
\]
\[
\sum_{i=1}^{M-1} \left( u_i^{\hat{\tilde{\varphi}}}_{i-1/2} \delta_{x_i^{\hat{\tilde{\varphi}}}} + u_{i+1/2}^{\hat{\tilde{\varphi}}} \delta_{x_{i+1/2}} \right) \delta_i^{\hat{\tilde{\varphi}}}
+ E_c \left( r^2 + h^2 \right)^2 = J_6 + J_9 + J_{10} + J_{11}, \quad 1 \leq n \leq k.
\] (95)

Then we estimate the term \( \nabla_i \delta_{x_i^{\hat{\tilde{\varphi}}}} \delta_{x_{i-1/2}}^{\hat{\tilde{\varphi}}} - \nabla_i \delta_{x_i^{M-1/2}} \delta_{x_{i-1/2}}^{M-1/2} \) and \( J_6 \) to \( J_{11} \), respectively. From (46), we have
\[
\delta_{x_i^{\hat{\tilde{\varphi}}}} = \beta_i \delta_0^{\hat{\tilde{\varphi}}}
+ \frac{h}{2k} \left[ (c_i^{\hat{\tilde{\varphi}}} + \sigma) \nabla_i \delta_0^{\hat{\tilde{\varphi}}} - u_i^{\hat{\tilde{\varphi}}} \delta_{x_i^{\hat{\tilde{\varphi}}}} - \nabla_i \delta_{x_i^{\hat{\tilde{\varphi}}}} - R_{0,0}^{\hat{\tilde{\varphi}}} \right] \] (96)

A straightforward calculation with Lemma 2 leads to
\[
\nabla_i \delta_{x_i^{\hat{\tilde{\varphi}}}} \delta_{x_{i-1/2}}^{\hat{\tilde{\varphi}}} = \beta_i \nabla_i \delta_0^{\hat{\tilde{\varphi}}} + \frac{h}{2k} \left[ (c_i^{\hat{\tilde{\varphi}}} + \sigma) \nabla_i \delta_0^{\hat{\tilde{\varphi}}} - u_i^{\hat{\tilde{\varphi}}} \delta_{x_i^{\hat{\tilde{\varphi}}}} - \nabla_i \delta_{x_i^{\hat{\tilde{\varphi}}}} - R_{0,0}^{\hat{\tilde{\varphi}}} \right] \] (97)

such that
\[
\nabla_i \delta_{x_i^{\hat{\tilde{\varphi}}}} \delta_{x_{i-1/2}}^{\hat{\tilde{\varphi}}} - \nabla_i \delta_{x_i^{M-1/2}} \delta_{x_{i-1/2}}^{M-1/2} \geq \frac{\beta_i}{2} \] (98)

and
\[
J_6 \leq E_c \left| \hat{r} \right| \left( c_{\max} + \sigma \right) \left| d_{\hat{\tilde{\varphi}}} \right|^2 \] (100)

\[
J_9 \leq \frac{k}{16} \frac{\left( c_{\max} + \sigma \right)}{\left( c_{\min} + \sigma \right)} \int \left| \hat{d}_{\hat{\tilde{\varphi}}} \right|^2 \] (101)

\[
J_10 \leq \frac{k}{16} \frac{\left( c_{\max} + \sigma \right)}{\left( c_{\min} + \sigma \right)} \int \left| \hat{d}_{\hat{\tilde{\varphi}}} \right|^2 \] (102)

where we noted the fact that \( (R_{0,1/2}^{\hat{\tilde{\varphi}}}/2)^2 \) and \( (R_{0,1/2}^{\hat{\tilde{\varphi}}}/2)^2 \). Similarly we estimate
\[
- \nabla_i \delta_{x_i^{\hat{\tilde{\varphi}}}} \delta_{x_{i-1/2}}^{\hat{\tilde{\varphi}}} \] (103)

\[
\nabla_i \delta_{x_i^{\hat{\tilde{\varphi}}}} \delta_{x_{i-1/2}}^{\hat{\tilde{\varphi}}} \delta_{x_{i+1/2}}^{\hat{\tilde{\varphi}}} \] (104)

\[
+ \frac{k}{16} \frac{\left( c_{\max} + \sigma \right)}{\left( c_{\min} + \sigma \right)} \int \left| \hat{d}_{\hat{\tilde{\varphi}}} \right|^2 \] (105)

\[
+ 64E_c \left( c_{\max} + \sigma \right) \left| \hat{d}_{\hat{\tilde{\varphi}}} \right|^2 \] (106)
Plugging the last six equations into (95), we get
\[
\frac{1}{2} \sum V_i \| \delta_x \tilde{\vartheta}^i \|^2 + \frac{K}{4} \| \delta_x^2 \tilde{\vartheta}^i \|^2 + \frac{\beta}{2} V_i \left( | \tilde{\vartheta}_M^i |^2 \right) + \frac{\beta}{2} V_i \left( | \tilde{\vartheta}_N^i |^2 \right) + \frac{\beta}{2} V_i \left( | \tilde{\vartheta}_2^i + \tilde{\vartheta}_3^i |^2 \right) \leq E^i \| \delta_x \tilde{c}^i \|^2 + E_s \| \delta_x \tilde{\vartheta}^i \|^2 + E_c \left( \tau^2 + h^2 \right)^2 \quad (104)
\]
Multiplying the last equation with \(0 \leq s \leq E_q\) and adding it into (90), we have
\[
\sum_{i=1}^{M-1} V_i \left( (c_i^q + \sigma) | \tilde{\vartheta}_i^q |^2 \right) + \frac{1}{2} V_i \left( (c_i^q + \sigma) | \tilde{\vartheta}_i^q |^2 \right) \leq E_c \left( \| \delta_x \tilde{c}^i \|^2 + \| \tilde{c}^i \|^2 + \| \tilde{\vartheta}^i \|^2 + \| \tilde{\vartheta}^i + \tilde{\vartheta}^i \|^2 \right) + E_c \left( \tau^2 + h^2 \right)^2 \quad (105)
\]
Letting \(s = \min(s_1/2, s_2, s_3), \theta_{\min}/8, k_{s_1}/4, 1, s_1, s_2, s_3/2, 2, s_2, s_2/2, 2\), we get
\[
\sum_{i=1}^{M-1} V_i \left( (c_i^q + \sigma) | \tilde{\vartheta}_i^q |^2 \right) + \frac{1}{2} V_i \left( (c_i^q + \sigma) | \tilde{\vartheta}_i^q |^2 \right) \leq E_c \left( \| \delta_x \tilde{c}^i \|^2 + \| \tilde{c}^i \|^2 + \| \tilde{\vartheta}^i \|^2 + \| \tilde{\vartheta}^i + \tilde{\vartheta}^i \|^2 \right)
\]
Letting \(s = \min(s_1/2, s_2, s_3, k_{s_1}/4, 1, s_1, s_2, s_3/2, 2, s_2, s_2/2, 2\), we get
\[
\sum_{i=1}^{M-1} V_i \left( (c_i^q + \sigma) | \tilde{\vartheta}_i^q |^2 \right) + \frac{1}{2} V_i \left( (c_i^q + \sigma) | \tilde{\vartheta}_i^q |^2 \right) \leq E_c \left( \| \delta_x \tilde{c}^i \|^2 + \| \tilde{c}^i \|^2 + \| \tilde{\vartheta}^i \|^2 + \| \tilde{\vartheta}^i + \tilde{\vartheta}^i \|^2 \right)
\]
Letting \(s = \min(s_1/2, s_2, s_3, k_{s_1}/4, 1, s_1, s_2, s_3/2, 2, s_2, s_2/2, 2\), we get
\[
\sum_{i=1}^{M-1} V_i \left( (c_i^q + \sigma) | \tilde{\vartheta}_i^q |^2 \right) + \frac{1}{2} V_i \left( (c_i^q + \sigma) | \tilde{\vartheta}_i^q |^2 \right) \leq E_c \left( \| \delta_x \tilde{c}^i \|^2 + \| \tilde{c}^i \|^2 + \| \tilde{\vartheta}^i \|^2 + \| \tilde{\vartheta}^i + \tilde{\vartheta}^i \|^2 \right)
\]
Since $E_k$ is independent of $E_0$, with

$$E_0 = \frac{\delta E_i, \tau}{s},$$

we find that (30) holds for $k = n$. The induction and the proof of the theorem are completed.

### 4. Numerical Examples

We now numerically evaluate the performance of the proposed leap-frog scheme.

**Example 1.** First, we test the accuracy of our algorithm in an artificial example which is taken from [23]. The system is

$$C_t + (u_gC)_x = -\Gamma + f_C(x,t),$$

$$(C + \sigma)T_t - (\kappa T)_x + u_gC_xT_x = [\lambda + \sigma T] \Gamma + f_T(x,t),$$

with the boundary conditions (8)-(12), where $f_C, f_T, \mu$, and $v_i$ are coefficients decided by the exact solution

$$C(x,t) = e^{-0.72t/\epsilon} x^2 (1-x)^2 + a (1-x) + bx,$$

$$T(x,t) = 50e^{-0.72t/\epsilon} x^2 (1-x)^2 + c (1-x) + dx,$$

with $a, b, c, d$ being constants.

We apply the uncoupled leap-frog finite difference method to solve the artificial example. We choose $\tau = h$ and $L = 1$. Since the proposed scheme is of the second order in both spatial and temporal directions, we take $\tau = h$ such that the error bound is proportional to $h^2$. We present the $L^2$-norm errors and the order of convergence $h^r$ in Table 1 with $h = L/200, L/400, L/800$ at different time level. We can see clearly from Table 1 that the $L^2$-norm errors for both components are proportional to $h^r$, which confirms our theoretical analysis.

**Example 2.** In the second example, we discuss a typical clothing assembly in the textile industry [2, 4, 25]. The clothing assembly consists of three layers, in the middle is porous fibrous media, and the outside cover is exposed to a cold environment with fixed temperature and relative humidity while the inside cover is exposed to a mixture of air and vapor at higher temperature and relative humidity. In this paper, polyester porous media with laminated or nylon cover materials are tested. To compare with the experimental data in [12], a water equation is added to equations (1)-(2):

$$\frac{\partial}{\partial t}(\rho_w (1 - \epsilon')) W = M_w T_{ce}$$

where $W$ is water content, $\rho_w$ is the density of water, $\epsilon$ is the porosity with liquid water content, and $\epsilon'$ is the porosity without liquid water content. We have

$$\epsilon = \epsilon' - \frac{\rho_f}{\rho_w} W (1 - \epsilon')$$

and the effective heat conductivity is defined by

$$\kappa = \kappa_f + (1 - \epsilon) \kappa_\epsilon$$

where $\kappa_f$ is the thermal conductivity of gas and $\kappa_\epsilon$ is the thermal conductivity of the fiber-water mixture [2, 6, 7], given by

$$\kappa_\epsilon = \left( \frac{1}{\rho_f} + \frac{W}{\rho_w} \right)^{-1} \left( \frac{1}{\rho_f} + \frac{W}{\rho_w} \right) \kappa_f.$$

The values of these physical parameters for polyester media are presented in Table 2. Other parameters values can be found in [2, 6, 7].

### Table 1: Numerical results of Example 1 with $\tau = h$.

| $t$  | $h = L/200$ | $h = L/400$ | $h = L/800$ | Order $r$ |
|------|-------------|-------------|-------------|-----------|
| $t = 0.25$ | 2.941E-03   | 1.004E-03   | 0.254E-03   | 1.99      |
| $t = 0.50$ | 8.163E-03   | 2.178E-03   | 0.527E-03   | 2.03      |
| $t = 0.75$ | 1.397E-02   | 0.350E-02   | 0.084E-02   | 2.04      |
| $t = 1.00$ | 2.032E-02   | 0.496E-02   | 0.118E-02   | 2.05      |

### Table 2: Physical parameters for batting materials.

| Parameter | Polyester |
|-----------|-----------|
| $\rho_f$  | $1.39 \times 10^1$ kgm$^{-3}$ |
| $\rho_w$  | $1 \times 10^3$ kgm$^{-3}$   |
| $\epsilon$ | 0.993      |
| $\kappa_f$ | $1 \times 10^{-1}$ W m$^{-1}$K$^{-1}$ |
| $\kappa_w$ | $5.7 \times 10^{-1}$ W m$^{-1}$K$^{-1}$ |
| $C_{ef}$  | $1.17 \times 10^6$ J m$^{-1}$K$^{-1}$ |
| $L$       | $4.92 \times 10^{-2}$ m         |
The initial conditions for the vapor, temperature, and water content are given by
\[ C = 65\% \frac{P_s(T)}{RT}, \quad T = 25^\circ C, \quad W = 0, \quad \text{at } t = 0. \quad (119) \]

We apply the uncoupled leap-frog finite difference method for solving the sweat transport system defined in (1)-(2) coupled with the water equation (115). Since only the right side of the water equation includes \( c \) and \( \theta \), therefore, the water equation is calculated separately. Numerically, at each time step, we first find solution \(c^{n+1}_j, \theta^{n+1}_j\) by procedure (18)-(26), and then \( W^{n+1}_j \) can be solved by following nonnormalized discrete formate:
\[ \rho_j \left(1 - e^n_j\right) \frac{1}{\tau} \left(W^{n+1}_j - W^n_j\right) = M_w \Gamma^n_{cell}, \quad (120) \]

Then we evaluate the parameters explicitly in (18)-(26) based on \( W^{n+1}_j \). Here all numerical results are obtained by taking the time step size \( \tau = 20s \) and spatial mesh size \( h = L/100 \). We present numerical results of vapor, temperature, and water content at 8 hours and 24 hours, respectively, for

5. Conclusion
As a subsequent work of [23], we have presented an uncoupled leap-frog finite difference method for the sweat transport system in porous textile media, which is governed by a strongly coupled, nonlinear parabolic system. Optimal \( L^2 \) error estimates were presented, which imply that the numerical scheme is unconditionally stable. Both theoretical analysis and numerical example indicate that the current scheme is second order accurate in both the temporal and spatial directions. Since the scheme is decoupled for the system, the method can be applied efficiently for problems in higher-dimensional space. Under certain time-step restrictions, the analysis can also be extended to the multidimensional problems.
Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors wish to thank Professors Z. Sun and W. Sun for helpful discussions. This research was partially supported by National Natural Science Foundation of China (Nos. 11601346, 11501377), Shenzhen Key Laboratory of Advanced Machine Learning and Applications, Guangdong Key Laboratory of Intelligent Information Processing, and Interdisciplinary Innovation Team of Shenzhen University.

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