On the Cosmic Ray Driven Firehose Instability

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Abstract. The role of the non-resonant firehose instability in conditions relevant to the precursors of supernova remnant shocks is considered. Using a second order tensor expansion of the Vlasov-Fokker-Planck equation we illustrate the necessary conditions for the firehose to operate. It is found that for very fast shocks, the diffusion approximation predicts that the linear firehose growth rate is marginally faster than it’s resonant counterpart. Preliminary hybrid MHD-Vlasov-Fokker-Planck simulation results using young supernova relevant parameters are presented.

INTRODUCTION

Supernova remnants offer the most likely candidate for production of the majority of Galactic cosmic-rays, with the diffusive acceleration of particles at the fast outer shocks being the most promising accelerating mechanism [1]. Despite considerable advances in recent years, regarding direct observations that imply significant magnetic field amplification in the vicinity of the outer shocks of several nearby supernova remnants, the interplay between accelerated particles and these amplified fields remains an area of ongoing investigation. The key issue that remains [2], based on our current understanding of magnetic field amplification, is that the maximum attainable cosmic-ray energy falls short of the knee feature on the cosmic ray spectrum at a few PeV assuming conditions relevant to the known young Galactic SNRs [3]. It has thus been suggested that younger, faster shocks may be the primary source of Galactic cosmic rays at and above the knee.

Motivated by this, we re-examine the growth of linear fluctuations in the precursor of a fast parallel shock which is efficiently accelerating cosmic rays. We focus, in these proceedings on the non-resonant firehose instability [4, 5], driven by cosmic-rays in the extended precursor of an efficiently accelerating shock.

VLASOV-FOKKER-PLANCK EQUATION

We assume that the background plasma satisfies the ideal MHD Ohm’s law $E = -(1/c)u \times B$, such that the electric field vanishes in the local frame. It is thus convenient to work in a mixed coordinate frame in which particle momentum is measured in the local fluid frame, while all other quantities are measured in a fixed inertial frame. To order $u/c$, the VFP equation thus reads

$$\frac{\partial f}{\partial t} + (u + v) \cdot \nabla f - \left[(p \cdot \nabla)u\right] \cdot \frac{\partial f}{\partial p} - eB \cdot \left(v \times \frac{\partial f}{\partial p}\right) = \left(\frac{\delta f}{\delta t}\right)_c,$$

where, for small angle scatterings, we take the following form for the collision operator:

$$\left(\frac{\delta f}{\delta t}\right)_c = \nu \left\{ \frac{1}{2} \left(1 - \mu^2\right) \frac{\partial f}{\partial \mu} + \frac{1}{1 - \mu^2} \frac{\partial^2 f}{\partial \phi^2} \right\},$$

with $\nu(p, B)$ the collision rate.
Since we seek to explore the role of cosmic-ray pressure anisotropy, we must consider a tensor expansion of the distribution function to at least second order:
\[ f(x, p, t) = f_0(p) + \frac{P}{p} \cdot f_1(p) + \frac{5 PP}{2 p^2} : S(p). \]  

(3)

By considering the various moments of the distribution, the physical significance of each component in the previously stated expansion is immediately apparent:
\[ n_{cr} = \int d^3 pf = 4\pi \int p^2 f_0 dp \quad \text{(density),} \]  
\[ j_{cr} = e \int d^3 pvf = \frac{4\pi e}{3} \int p^2 v f_1 dp \quad \text{(current),} \]  
\[ P_{cr} = \int d^3 pvpf = \frac{4\pi}{3} \int p^3 (f_0 I + S) dp \quad \text{(pressure),} \]  

where \( I \) is the unit tensor. Using the relevant orthogonality relations, it is straightforward to show that the VFP equation leads to the following system of coupled equations
\[
\frac{\partial f_0}{\partial t} + \nabla \cdot (u f_0) + \frac{v}{3} \nabla \cdot f_1 = \frac{1}{p^3} \left\{ p^3 \left( \nabla \cdot u \right) f_0 + S_{ab} \frac{\partial f_0}{\partial x_b} \right\}, 
\]  
\[
\frac{\partial f_1}{\partial t} + u^b \frac{\partial f_1}{\partial x^b} + \epsilon_{abc} \Omega c f_1 + \frac{v}{3} \frac{\partial f_0}{\partial x^b} (f_0 \sigma^{ab}) + S_{a} = \frac{\partial u^b}{\partial x^b} f_1 + \frac{1}{3} \frac{\partial u^c}{\partial x_c} p \frac{\partial}{\partial p} \left( f_1 \right) + \frac{1}{5} p^2 \frac{\partial}{\partial p} \left( f_1 \right) \sigma^{ab}, \]  
\[
\frac{\partial S_{ab}}{\partial t} + \frac{\partial}{\partial x^c} \left( u^c S_{ab} \right) + v \Lambda^{ab} - \frac{p}{5} \frac{\partial f_0}{\partial p} \sigma^{ab} + \left( S_{ab} \frac{\partial u^a}{\partial x^a} + S_{ab} \frac{\partial u^b}{\partial x^b} - \frac{2}{3} S_{cd} \frac{\partial u^d}{\partial x^d} \right) \delta^{ab} - \left( \epsilon_{acd} S_{bc} + \epsilon_{bcd} S_{ac} \right) \Omega^d 
= \frac{5}{7} \frac{\partial}{\partial p^3} \left\{ p^3 \left( \frac{\partial u^a}{\partial x^a} + \frac{\partial u^b}{\partial x^b} + \frac{\partial u^c}{\partial x^c} \right) + S_{cd} \left( \frac{\partial u^a}{\partial x^a} + \frac{\partial u^b}{\partial x^b} + \frac{\partial u^c}{\partial x^c} \right) - \frac{2}{3} S_{cd} \left( \frac{\partial u^a}{\partial x^a} + \frac{\partial u^b}{\partial x^b} + \frac{\partial u^c}{\partial x^c} \right) \right\}, \]  

where \( \Omega = eB/\gamma mc \) is the directional relativistic gyrofrequency, \( \epsilon_{abc} \) the Levi-Civita symbol, and summation over repeated indices is implied. We also introduce the trace-free tensors
\[
\sigma^{ab} = \frac{\partial u^b}{\partial x^a} + \frac{\partial u^a}{\partial x^b} - \frac{2}{3} \frac{\partial u^c}{\partial x^c} \delta^{ab}, \]  
\[
\Lambda^{ab} = \frac{\partial f_1}{\partial x^b} + \frac{\partial f_1}{\partial x^a} - \frac{2}{3} \frac{\partial f_1}{\partial x^c} \delta^{ab}, \]  

which correspond to the rate-of-strain tensors for the background fluid and cosmic-rays respectively. We note that these equations, using slightly different notation, have previously been derived in [6], although an additional adiabatic term is included in Equation (9) previously omitted, that ensures conservation of the trace-free nature of \( S_{ab} \).

The above equations must be solved self-consistently with the equations governing the background fluid, namely mass conservation and the magnetic induction equation (again assuming ideal MHD), together with the cosmic-ray modified MHD momentum conservation equation
\[
\rho \frac{du}{dt} = -\nabla P_{bg} + \frac{1}{c} j_{bg} \times B + \eta j_{cr} 
= -\nabla P_{bg} - \frac{c}{4\pi} B \times (\nabla \times B) - \frac{1}{c} j_{cr} \times B + \eta j_{cr}, \]  

(10)

where we have made use of Ampère’s law in the last equality. The final term on the right hand side represents the collisional momentum transfer from cosmic-rays to the background, with \( \eta \) as an yet to be determined collisional transfer rate. Note that the distribution is already calculated in the local fluid frame, so it is not necessary to consider the relative drift.

Using the above definitions for cosmic-ray number density, current density and pressure, one can derive evolutionary equations for the relevant macroscopic cosmic-ray fluid quantities. This will ultimately require us to consider some simplifying closure relations, which we discuss in the next section.
COSMIC-RAY PRESSURE AND THE FIREHOSE INSTABILITY

From this point forward, we assume all cosmic-rays are ultra-relativistic \((p = \gamma mc, \text{etc.})\) and consist exclusively of protons. We note first that it is possible to modify Equation (10) further, by writing it in a more familiar form. Introducing the energy flux/momentum density

\[
W = \int p f d^3 p = \frac{4\pi}{3} \int f_1 p^3 dp
\]

and assuming our scattering rate \(\nu = \Omega/h, \text{with } h \text{ typically } \gg 1\) a momentum independent constant, it follows from Equation (8)

\[
\frac{dW}{dt} + (\nabla \cdot u)W + (W \cdot \nabla)u = -\nabla P_{\text{cr}} + \frac{1}{c} j \times B - \eta j,
\]

where \(\eta = |B|/ch\). Thus, restricting our attention to low frequency \((\tau \ll \Omega^{-1}, \nu^{-1})\) behaviour, we can neglect the terms on the left hand side, and one recovers the familiar momentum conservation equation including cosmic-ray pressure

\[
\rho \frac{du}{dt} + \nabla \left( P_{\text{bg}} I + P_{\text{cr}} \right) + \frac{c}{4\pi} B \times (\nabla \times B) = 0.
\]

We can now make use of Equations (7) and (9) to determine the form of the anisotropic pressure tensor. Defining the rank 3 tensor

\[
Q^{abc} = \frac{1}{m} \int p^a p^b p^c f \frac{d^3 p}{p^0} = \left\{ \begin{array}{ll} \frac{4}{3m} \int dp p^4 \left( f_0 \delta^{bc} + S^{bc} \right) & a = 0, \ b, c > 0 \\ \frac{4}{3m} \int dp p^4 \left[ f_1^0 \delta^{bc} + f_1^0 \delta^{ac} + f_1^0 \delta^{cb} \right] dp & a, b, c > 0 \end{array} \right.
\]

it follows that in the ultra-relativistic limit

\[
\frac{dQ^{ab}}{dt} + c \frac{\partial Q^{abc}}{\partial x^c} + Q^{abc} \frac{\partial u^a}{\partial x^c} + Q^{abc} \frac{\partial u^b}{\partial x^c} - \frac{eB^d}{mc} (\epsilon_{abcd} P^{bc} + \epsilon_{bcd} P^{ac}) = -3\nu \Pi^{ab},
\]

where \(\Pi^{ab}\) is the trace free part of the cosmic-ray pressure tensor, and we have taken advantage of the fact that \(\hat{\nu} = eB/hmc\). We note that, for a \(f \propto p^{-4}\) spectrum with range \(p_1 < p < p_2\) it follows that \(Q^{ab}/P^{ab} \sim (p_2/mc) \log(p_2/p_1)\), while \(Q^{abc}/P^{abc}\) is typically smaller by a fraction \(u_{ab}/c\). Hence, in the same limit as before \((\tau \ll \Omega^{-1}, \nu^{-1}\) at the upper energy range), the leading terms in this equation are

\[
\left( \epsilon_{abcd} P^{bc} + \epsilon_{bcd} P^{ac} \right) B^d = 0.
\]

This equation is satisfied by any tensor of the form

\[
P_0 = p_{||} \mathbf{b} b + p_{\perp} (\mathbf{I} - \mathbf{b} \mathbf{b}),
\]

where \(\mathbf{b} = B/B\) is the unit vector along the field, and \(p_{||}, p_{\perp}\) as yet undetermined constants. In order to determine \(p_{||}, p_{\perp}\), we make the following approximations. We neglect collisions and the heat flux term (these can be checked \(\text{a posteriori}\)), and define a slowly varying weighted Lorentz factor

\[
\langle \gamma \rangle = \frac{\int \gamma v^a p^b f d^3 p}{\int v^a p^b f d^3 p} = \frac{Q^{ab}}{P^{ab}},
\]

allowing us to write, to next leading order (see also [7])

\[
\frac{dP^{ab}}{dt} + P^{ab} \frac{\partial u^c}{\partial x^c} + P^{ac} \frac{\partial u^b}{\partial x^c} + P^{ac} \frac{\partial u^b}{\partial x^c} = \left( \epsilon_{abcd} P^{bc} + \epsilon_{bcd} P^{ac} \right) \hat{\nabla}^d,
\]

\(^1\)Formally speaking the ratio of the second to third term is, in the diffusion approximation \(\sim (\nu ab/c^2)/(v/k\delta u)\). Assuming \(\delta u \sim \nu a\), it follows that the heat flow is negligible on scales \(k^{-1} \ll (c/\nu ab) M_A (\lambda_{adp})\), i.e. the shock crossing time is less than the time taken for an Alfvén wave to transmit information across the precursor scaleheight.
Again, using Equations (7)-(9), the steady state solution in the upstream plasma has
dissimilar to the firehose condition. However, while the growth rates are comparable, the firehose instability is purely
polarisations, or indeed in the presence of coherent curved magnetic field structure.

growing, the real and imaginary parts of the frequency ion-cyclotron are comparable in the strongly modified case, 
and is dominated by the real part in the unmodified case. Additionally, the ion-cyclotron instability is only expected
to grow for waves with polarisation in the same sense as the cosmic-rays’ zeroth order helical motion in the mean 
field, while the firehose, being non-resonant, does not depend on the sense of rotation, and is thus unstable to both 
polarisations, or indeed in the presence of coherent curved magnetic field structure.

\[ \nabla \cdot \mathbf{u} = -\frac{1}{\rho} \frac{d \rho}{dt} \] 
and
\[ \frac{dB}{dt} = \mathbf{b} \cdot \frac{d \mathbf{B}}{dt} = \mathbf{b} \cdot [(\mathbf{B} \cdot \nabla)\mathbf{u} - \mathbf{B}(\nabla \cdot \mathbf{u})] \]

the standard double adiabatic equations follow:

\[ \frac{d}{dt} \left( \frac{p B^2}{\rho^3} \right) = \frac{d}{dt} \left( \frac{p_\perp}{\rho B} \right) = 0 . \]  

This closes our system of equations, which are now in exactly the form that reproduces the well-known result for parallel modes (see for example [8]):

\[ \omega^2 = \frac{k^2}{\rho} \left[ \frac{B^2}{4\pi} + p_\perp - p_\parallel \right] , \]  

which is purely growing in the limit

\[ p_\parallel - p_\perp > \frac{B^2}{4\pi} . \]  

**FIREHOSE IN SNR PRECURSORS**

While we have identified the necessary conditions for the onset of firehose instability, the discussion up to this point has not made any connection to actual supernovae, and specifically what physical values \( p_\parallel \) and \( p_\perp \) might take. For simplicity we consider a planar steady shock with velocity \( u_{sh} \), with magnetic field and shock normal along the \( x \)-axis. Again, using Equations (7)-(9), the steady state solution in the upstream plasma has

\[ f_1^i = \frac{5}{3} \frac{\mu_{sh}}{c} f_0 , \quad S^{11} = -2S^{22} = -2S^{33} = \frac{4}{5} \left( \frac{\mu_{sh}}{c} \right)^2 f_0 . \]  

Using these numbers, the firehose condition, Equation (18) can be expressed as

\[ 6 \left( \frac{\mu_{sh}}{c} \right)^2 M_A^2 \left( \frac{\rho_0}{\rho_{sh}} \right) > 1 , \]  

where \( P_0^0 \) is the isotropic cosmic-ray scalar pressure and \( M_A \) the Alfvén Mach number of the shock. We also recall, 
that in the previous section, we neglected the role of collisions and heat flux in the pressure tensor equation. The first of 
these approximations is clearly justified, provided \( v \ll \Omega \ (h >> 1) \), i.e. scattering is far from the Bohm limit. The 
latter approximation, of negligible heat flux is more controversial [4], although it generally valid provided the shock 
velocity remains non-relativistic.

Assuming the above conditions are satisfied, the corresponding growth rate is

\[ \Gamma_{FH} \sim k u_{sh} \left( \frac{\mu_{sh}}{c} \right)^2 \left( \frac{P_{cr}}{\rho_{sh} u_{sh}^2} \right)^{1/2} . \]  

It is interesting to note that the growth rate is very similar to that of the strongly modified resonant ion-cyclotron 
instability, [9]

\[ \Gamma_{IC} \approx k u_{sh} \left( \frac{1}{\ln(p_{max}/p_{min})} \left( \frac{\mu_{sh}}{c} \right) \left( \frac{P_{cr}}{\rho_{sh} u_{sh}^2} \right) \right)^{1/2} . \]  

We note that the condition for strong modification is \( M_A^2 \left( \mu_{sh}/c \right) \left( P_{cr}^{00}/\rho_{sh} u_{sh}^2 \right)/ \ln(p_{max}/p_{min}) > 1 \), which is also not 
dissimilar to the firehose condition. However, while the growth rates are comparable, the firehose instability is purely 
growing, the real and imaginary parts of the frequency ion-cyclotron are comparable in the strongly modified case, 
and is dominated by the real part in the unmodified case. Additionally, the ion-cyclotron instability is only expected 
to grow for waves with polarisation in the same sense as the cosmic-rays’ zeroth order helical motion in the mean 
field, while the firehose, being non-resonant, does not depend on the sense of rotation, and is thus unstable to both 
polarisations, or indeed in the presence of coherent curved magnetic field structure.
FIGURE 1. Growth of magnetic field fluctuations and driving current as a function of time from 3D simulations. Magnetic field is normalised to the initial mean field, while the cosmic ray current is normalised to its value in the diffusive approximation. Time is in units of $\Omega_g^{-1}$.

HYBRID MHD-VFP SIMULATIONS

We present here some preliminary simulations to explore the non-linear behaviour of cosmic-rays interacting with an MHD plasma on large scales. The simulations were performed in 3D, using Equations (7)-(10) together with the equations for mass and energy conservation, and the magnetic induction equation. We consider a periodic domain, with magnetic field along the $x$-axis, and background fluid initially at rest. To minimise numerical memory requirements, we employ the same technique used in [10], and solve the VFP equations for a single particle momentum $p_0$, replacing all momentum derivatives assuming a $p^{-4}$ power-law. This technique is valid provided the energy changes are small, but is essential to capture the important $E \times B$ drifts with respect to the background fluid.

Figure 1 shows the evolution of the cosmic-ray current and magnetic field fluctuations as a function of time for two different simulations, R1 & R2, both of which satisfy the firehose condition. Simulations R1 had a cosmic-ray pressure $P_{cr}/\rho u^2_{sh} = 0.02$ while this number is 0.01 for R2. The shock velocity and collision frequency in both simulations were $0.1c$ and $\Omega_g/100$ respectively. A grid resolution of $\Delta x = r_{g,0}/5$ was used, where $r_{g,0}$ is the gyroradius in the mean field. Both simulations included a uniform external driving term (see [10]), which proves to be essential, since any initial anisotropy would be damped on a timescale $\nu^{-1}$. To explore different possibilities, R1 starts from rest ($f(t = 0)$ is isotropic), while R2 is initialised with the diffusive solution given above.

Since both the firehose and ion-cyclotron instability have a growth rate proportional to $k$, the fastest growing mode appears to be occurring close to the grid scale. The growth rate of the field is consistent with either Equation (21) and (22) with wavelength $\lambda \sim 2r_g$. The polarisation appears to vary with position of the line-out in the $y-z$ plane making it ambiguous as to which instability is dominating. However, the modes appear to be purely growing which is more characteristic of the firehose instability. It has been previously suggested by [10] that sub-Larmor scattering of the cosmic-rays allows them to decouple from long wavelength fluctuations, and allow purely growing modes. However, this is unlikely to be the case here as there is insufficient structure below the Larmor scale.

Finally we note, in both cases, the cosmic-ray current is rapidly damped when $\delta B/B_0$ exceeds a few percent level. Given that $\nu = \Omega_g/100$ it is expected at about this level that gyration in the non-uniform fields dominates over the imposed small angle scattering. Future simulations can alter the driving to maintain a steady current.

CONCLUSIONS

We have identified the minimal conditions for the cosmic-ray driven firehose to occur in SNR precursors, in particular with regards the necessary approximations. As has been previously pointed out by [4], the biggest limitation concerning these approximations may well be the neglect of the heat flux, which for fast shocks ($> 0.1c$) can be comparable to other first order terms. We note that we have only considered the case of cosmic-ray current and pressure anisotropy driven by a large scale gradient, using the so-called diffusive approximation. If scattering is weak, and particles can escape the accelerator more freely, it is conceivable that the pressure anisotropy takes a different form. However, how
to implement such an effect without performing full shock simulations is not obvious.

Preliminary simulations indicate that, irrespective of the instability operating, a significant reduction in the mean free path, or equivalently, an enhancement of the CR confinement, can be achieved in a relatively modest number of Larmor periods ($\sim 10 - 50$ for our chosen simulation parameters). If for example, one considers a very young SNR shock in a highly magnetised but dense plasma (such that the Alfvén Mach number is still large), there is a clear prospect for self confinement on timescales that are consistent with the expansion time of the remnant, even at PeV energies.

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