Proper quantization of the Gubser-Rocha Einstein-Maxwell-Dilaton model reveals an exactly marginal operator

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ABSTRACT: We show that the strongly coupled field theory holographically dual to the Gubser-Rocha anti-de-Sitter Einstein-Maxwell-Dilaton theory describes not a single non-trivial AdS$_2$ IR fixed point, but a one-parameter family. It is dual to a local quantum critical phase instead of a quantum critical point. This result follows from a detailed analysis of the appropriate quantization of the gravitational theory that is consistent with the thermodynamics of the analytical Gubser-Rocha black hole solution. Other quantizations correspond to different holographic theories where the analytic Gubser-Rocha black hole does not describe a finite temperature conformal fluid.
1 Introduction

One of the main insights holography has provided into the physics of strongly correlated systems is the existence of previously unknown (large $N$) non-trivial IR fixed points. These fixed points are characterized by an emergent scaling symmetry of the Lifshitz form categorized by a dynamical critical exponent $z$, a hyperscaling exponent $\theta$, and a charge anomalous dimension $\zeta$.

\[ x \to \lambda^{1/z} x, \quad t \to \lambda t, \quad F \to \lambda^{d-\theta/z} F, \quad \rho \to \lambda^{d-\theta+\zeta/z} \rho. \]  

(1.1)

Here $F$ is the free energy density and $\rho$ the charge density [1–4]. Within these Lifshitz fixed points those with $z = \infty$ are special. Such theories have energy/temperature scaling with no corresponding spatial rescaling. These are therefore systems with exact local quantum criticality. Phenomenologically this energy/temperature scaling without a corresponding spatial part is observed in high $T_c$ cuprates, heavy fermions and other strange metals, where this nomenclature originates (see e.g. [5]). In holography $z = \infty$ IR fixed points correspond to an emergent AdS$_2$ symmetry near the horizon of the extremal black hole. The two most well-known such solutions are the plain extremal Reissner-Nordström (RN) black hole and the extremal Gubser-Rocha (GR) black hole [6]. The Reissner-Nordström solution of AdS-Einstein-Maxwell theory has been studied extensively primarily because it is the simplest such model. Its simplicity also means it is too constrained to be realistic as a model of observed locally quantum critical metals. Notably the Reissner-Nordström has a non-vanishing ground-state entropy and emerges from a $d > 2$-dimensional conformal field theory. The more realistic Gubser-Rocha model arises from a non-conformal strongly correlated theory, where one isolates the leading irrelevant deformation from the IR fixed
point. This “universal” subsector gives it a chance to be applicable to observed local quantum critical systems. Moreover the groundstate now has vanishing entropy (to leading order). In the gravitational description this leading (scalar) (IR)-irrelevant operator is encoded in a dilaton field that couples non-minimally to both the Einstein-Hilbert action and the Maxwell action. Even with its more realistic appeal, the more complex nature of the Gubser-Rocha dynamics means it has been studied less; some examples are [7–11].

In the course of these studies of non-minimally coupled Einstein-Maxwell-Dilaton theories, it was noted in particular that the proper holographic interpretation of the analytic Gubser-Rocha black hole solution depends sensitively on the particular quantization [10, 11]. Within holography, relevant and marginally relevant scalars allow for different quantization schemes. A relevant operator of dimension $\frac{d}{2} < \Delta < d$ always has a conjugate operator of dimension $0 < \Delta_{\text{conj}} = d - \Delta < \frac{d}{2}$, and one can choose whether one considers the original operator as the dynamical variable (standard quantization) or the conjugate operator (alternate quantization) or any intermediate linear combination through a double-trace deformation [12, 13].

An additional complication results from the fact that the analytical (static and isotropic) Gubser-Rocha solution is a two-parameter solution depending on $T$ and $\mu$, whereas one expects a third independent parameter encoding the asymptotic source value of the dilaton field. A low-energy scalar can have a sourced (or unsourced) vacuum-expectation value; this changes the energy of the ground-state and hence should contribute to the thermodynamics. For minimally coupled scalars this was recently elucidated in [14].

In this paper we will show that the correct way to interpret the analytical Gubser-Rocha solution is as a two-parameter subset of solutions within the three-parameter thermodynamic phase diagram. For essentially all quantization schemes this constrains the source of the dilaton field in terms of the temperature and chemical potential of the solution. Crucially this implies that derivatives of thermodynamic potentials mix the canonical contribution with an additional contribution from the scalar response. We will show this explicitly in Section 3.1. A proper understanding of the solution requires one to carefully separate out this contribution.

It also turns out, however, that there is a specific quantization scheme where the dilaton corresponds to an exactly marginal operator in the theory. This was previously noted for another set of the Einstein-Maxwell-Dilaton actions [11]. In this special quantization choice the analytical Gubser-Rocha solution corresponds to a solution with no explicit source for the dilaton field. Within this special quantization scheme one can deform the analytical solution to a nearby solution with a finite scalar source. We do so in Section 4. We conclude with a brief discussion on the meaning of this newly discovered exactly marginal deformation.

\footnote{We thank Blaise Goutéraux for bringing this paper to our attention.}
The Gubser-Rocha black hole is a solution to the Einstein-Maxwell-Dilaton action

\[ S_{\text{bulk}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R - \frac{Z(\phi)}{4} F^2 - \frac{1}{2} \partial^2 - V(\phi) \right], \quad (2.1) \]

where the potentials are given by \( Z(\phi) = e^{\phi/\sqrt{3}} \) and \( V(\phi) = -6 \cosh(\phi/\sqrt{3}) \). This action is a consistent truncation of \( d = 11 \) supergravity compactified on \( AdS_4 \times S_7 \) [6]. The equations of motion for this system are

\[
R_{\mu\nu} = \frac{Z(\phi)}{2} \left[ F_{\mu\rho} F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F^2 \right] + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\mu\nu} V(\phi),
\]

\[
\nabla_\mu [Z(\phi) F^{\mu\nu}] = 0,
\]

\[
\Box \phi = V'(\phi) + \frac{Z'(\phi)}{4} F^2,
\]

where we used that, on-shell, \( R = 2V(\phi) + \frac{1}{2} \partial^2 \). The static and isotropic metric ansatz that is asymptotically AdS is

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{1}{z^2} \left[ -f(z) dt^2 + g(z) (dx^2 + dy^2) + \frac{dz^2}{f(z)} \right], \quad (2.3)
\]

where the coordinate \( z \) is the radial direction with \( z = 0 \) the AdS boundary (UV). The Gubser-Rocha solution [6] is then given by

\[
g(z) = (1 + Qz)^{3/2},
\]

\[
f(z) = \frac{1 - z/z_h}{g(z)} \left[ 1 + (1 + 3Qz_h) \frac{z}{z_h} + (1 + 3Qz_h + 3Q^2 z_h^2) \left( \frac{z}{z_h} \right)^2 \right],
\]

\[
A_t(z) = \mu j(z) = \frac{\sqrt{3} Qz_h (1 + Qz_h)}{z_h} \frac{1 - z/z_h}{1 + Qz},
\]

\[
\phi(z) = \frac{\sqrt{3}}{2} \log [1 + Qz],
\]

where \( z_h \) is the horizon of this non-extremal black hole. From hereon we choose units where \( 2\kappa^2 = 16\pi G = 1 \), such that the temperature, chemical potential and entropy-density of the GR-black hole are

\[
T = -\frac{f'(z)}{4\pi} \bigg|_{z = z_h} = 3\sqrt{1 + Qz_h^2}, \quad s = 4\pi a_h = 4\pi z_h^3 \left( 1 + Qz_h^2 \right)^{3/2}, \quad \mu = A_t(z = 0) = \sqrt{3} Qz_h (1 + Qz_h)/z_h, \quad (2.5)
\]

where \( a_h = \sqrt{g_{xx}(z_h) g_{yy}(z_h)} \) is the area density of the horizon. Expressed in terms of the temperature, it is easy to see that the entropy vanishes linearly \( s = 16\pi^2/3\sqrt{3} \mu T + \ldots \) at low temperatures with no remnant ground state entropy. Important in the remainder is (1) to

\[ \text{Note that the dilaton has dimension zero.} \]
recall that both the temperature and the entropy can be read off from the near-horizon behavior of the metric alone. As local properties of the black hole they do not depend on the boundary conditions. (2) The analytic solution depends on two parameters $Q$ and $z_h$. And (3) note that the solution is not of the Fefferman-Graham type in that the change in metric functions starts at order $z$ and not $z^3$.

3 Regularization, boundary terms and choice of quantization

We must add to the gravitational action (2.1) a boundary action. This is to regularize its on-shell value as well as to make the variational principle well-defined. In the case of the scalar it also prescribes the quantization of the scalar field. The most general form of the boundary action is [10]:

$$S_{\text{bdy}} = -\int_{z = \epsilon} d^3x \sqrt{-\gamma} \left[ 2K + 4 + (3)R_\gamma + \frac{1}{2} \Lambda_{\phi} \phi^2 + c_{\phi} \phi N^\mu \partial_\mu \phi \right],$$  (3.1)

Here $N^\mu = -\sqrt{g^{zz}(0,0,0,1)}$ is an outward pointing spacelike unit normal vector defining the hypersurface $z = \epsilon \ll z_h$ and $\gamma_{\mu\nu} = g_{\mu\nu} - N_\mu N_\nu$ is the induced metric on the surface. Furthermore $K \equiv \gamma^{ij} K_{ij}$ is the trace of the extrinsic curvature $K_{ij} \equiv -\gamma^\mu_i \gamma^\nu_j \nabla_\mu N_\nu$ and $(3)R_\gamma$ the Ricci scalar curvature of the hypersurface (Latin symbols correspond to coordinates on the hypersurface while the greek symbols are those of the original manifold). The first three terms correspond to the usual Gibbons-Hawking-York counterterms necessary to make the variational principle for the metric well-defined and also to regularize the Einstein-Hilbert-Cosmological Constant part of the action on shell. In our coordinatization Eq.(2.3) the induced metric is flat on-shell. The term with $\Lambda_{\phi}$ is there to regularize the dilatonic part of the bulk action and the fourth term will be used to choose the quantization of the scalar field, as we shall now show.

Varying the total action

$$S = S_{\text{bulk}} + S_{\text{bdy}},$$  (3.2)

to first order, a proper holographic interpretation demands that one obtains a variation of the form [15]

$$\delta S = \int_{z = \epsilon} d^3x \sqrt{-\gamma} \left[ \frac{1}{2} T_{\mu\nu} \delta \gamma^{\mu\nu} + J_\mu \delta A_\mu + O_\phi \delta \phi \right],$$  (3.3)

where the terms multiplying the EMD fields are interpreted as the operators in the CFT where $T_{\mu\nu}$ is the boundary stress tensor, $J_\mu$ the boundary current associated with the U(1) charge and $O_\phi$ the operator dual to a scalar which may be a non-linear function of the dilaton field. The important point is that the action evaluated on the black hole solution is equated with (minus) its Gibbs free energy. The variation of the action (restricted to preserve isotropy) thus includes thermodynamic variations. The expression above makes clear that in addition to the temperature and the chemical potential there ought to be a dependence of the Gibbs free energy on an external (source) variation of (the boundary value of) the scalar field [14].

Performing this variation on Eqs (2.1) plus (3.1), we can write it as a bulk integral of an integrand proportional to the equations of motion (2.2), that vanishes on-shell, and a
remaining boundary part. In the boundary part the normal derivatives of $\delta\gamma_{\mu\nu}$ cancel due to the Gibbons-Hawking-York term; there are no normal derivatives in $A_{\mu}$. Restricting to boundary indices we have\(^3\)

\[
T_{ij} = 2K_{ij} - 2\left(\delta R_{\gamma_{ij}}\right) - 2(K + 2)\gamma_{ij} + \gamma_{ij} \left[ c_\phi \phi N^z \partial_z \phi + \Lambda_\phi \phi^2 / 2 \right],
\]

\[
J_i = -Z(\phi)N^z F_{zi}.
\]

The expression for $O_\phi$ requires more detailed discussion. Focusing on the variation in the dilaton $\phi$ when varying (3.3), we have

\[
\delta S_\phi = \int_{z=\epsilon} d^3x \sqrt{-\gamma} \left[ \Lambda_\phi \phi \delta \phi + c_\phi \phi N^z \partial_z \phi + c_\phi \delta \phi N^z \partial_z \phi - \delta \phi N^z \partial_z \phi \right].
\]

From its linearized equation of motion the dilaton has the following expansion in the near-boundary region

\[
\phi(z) = \alpha z^{\lambda_-} + \beta z^{\lambda_+} + O(z^3),
\]

where $\lambda_\pm = \frac{3}{2} \pm \frac{1}{2} \sqrt{9 + 4m^2}$ and $m$ is the effective mass. In the Gubser-Rocha model the effective mass equals

\[
m^2 = \frac{\partial}{\partial \phi^2} \left[ V(\phi) + \frac{Z(\phi)}{4} F^2 \right] \bigg|_{\phi=0,z\to0} = -2.
\]

This value of the mass $-\frac{9}{4} < m^2 < 1 - \frac{9}{4} = -\frac{5}{4}$ is in the regime where two different quantizations are allowed, i.e. both $\lambda_\pm > 0$ and either $\alpha$ (standard) or $\beta$ (alternate) can be chosen as the source for the dual CFT operator with the other the response. One can also choose a mixture of the two, corresponding to a multi trace deformation \cite{12, 13}. The latter is what we will do, as was already studied in \cite{10, 11}. For the remainder of this section, we will choose a rather generic ansatz for the metric

\[
ds^2 = \frac{1}{z^2} \left[ -H_{tt}(z)dt^2 + H_{xz}(z)dx^2 + H_{yy}(z)dz^2 + H_{zz}(z)dz^2 \right],
\]

where we require Anti-deSitter (AdS) asymptotics $H_{\mu\nu}(z=0) = 1$ and use the equations of motion (2.2) to constrain the near-boundary expansion of $H_{\mu\nu}$ in terms of a small subset of degrees of freedom. Using that $N^z(z) = -z/\sqrt{H_{z\bar{z}}(z)}$, and substituting (3.6) into (3.5), we can expand the variation w.r.t. the dilaton as

\[
\delta S_\phi = \int_{z=\epsilon} d^3x \left[ \Lambda_\phi - \frac{(2c_\phi - 1)}{2} \alpha \delta \alpha + \left( \Lambda_\phi - 3c_\phi + 1 \right) \alpha \delta \beta 
\right.
\]

\[
+ \left( \beta(2 - 3c_\phi + \Lambda_\phi) + \alpha(4 - 8c_\phi + 3\Lambda_\phi) H'_{tt}(0) \right) \delta \alpha + O(\epsilon) \right].
\]

We must remove the leading divergence by imposing $\Lambda_\phi = 2c_\phi - 1$, leaving a finite contribution

\[
\delta S_\phi = \int_{z=\epsilon} d^3x \left[ \alpha \delta \alpha \frac{H'_{tt}(0)}{2}(1 - 2c_\phi) + \beta \delta \alpha (1 - c_\phi) - c_\phi \alpha \delta \beta + O(\epsilon) \right],
\]

\[
= \int_{z=\epsilon} d^3x \left[ \gamma (1 - c_\phi) \delta \alpha - c_\phi \alpha \delta \gamma + O(\epsilon) \right],
\]

\(^3\)The radial components of $T_{\mu\nu}$ and $J_\mu$ vanish due to the projection on the hypersurface.
where we have defined $\gamma \equiv \beta + \alpha H_0'(0)/2$. Our result differs from eq. (3.5) of [10] by the term proportional to $H_0'(0) = -H_0''_{zz}(0)$ which finds its origin in the first order correction to $N^2$ near the boundary. Its contribution was accounted for in [11]. This correction is a consequence of the non-Fefferman-Graham nature of the solution. In the RN solution or most other types of black holes the first non-trivial correction to $N^2$ would be quadratic in $z$ while here it is linear. This linear contribution has the effect of shifting the effective falloffs in standard, alternate and mixed quantizations.

Having accounted for this, one sees in the expansion (3.10) that the simple choice $c_\phi = 0$ means $\delta S_\phi = \int \gamma \delta \alpha$. This is standard quantization where the leading falloff coefficient $\alpha$ is the source and $\gamma$ is the response/VEV of the field. On the other hand, choosing $c_\phi = 1$ means that $\delta S_\phi = -\int \alpha \delta \gamma$ so the source is now $\gamma$ and the response/VEV is $-\alpha$. This is alternate quantization modified by For general $c_\phi$, i.e. a multi-trace deformed dual theory, the sources and responses are functions of the leading and subleading falloffs. Assuming these functions are invertible, we can express the falloffs as $\alpha(\phi_S, \phi_V), \beta(\phi_S, \phi_V)$ where $\phi_S$ is the general source and $\phi_V$ the general VEV. We can substitute this ansatz into (3.10) and require that the variation takes the form $\delta S_\phi = \int \phi_V \delta \phi_S$. This leads to two differential equations which can be further simplified by decomposing $\gamma(\phi_S, \phi_V) = u(\phi_S, \phi_V)\alpha(\phi_S, \phi_V)^{1/c_\phi}/c_\phi$. The equations are then

$$\partial_{\phi_S} u = \frac{\phi_V}{c_\phi} \alpha^{-1/c_\phi} \tag{3.11}$$

$$\partial_{\phi_V} u = 0 \tag{3.12}$$

The second equation means that $u(\phi_S, \phi_V)$ is independent of $\phi_V$. Therefore, this means that

$$\partial_{\phi_V} \left( \frac{\phi_V}{c_\phi} \alpha^{-1/c_\phi} \right) = 0 \text{ i.e., } \alpha(\phi_S, \phi_V) = \phi_V^{c_\phi} K(\phi_S) \tag{3.12}$$

A priori, there is a family of solutions depending on the function $K(\phi_S)$. We can then make the choice $K(\phi_S) = 1$ such that

$$u(\phi_S) = C_0 - \phi_S/c_\phi \tag{3.13}$$

leading to $\gamma = \phi_V^{1-c_\phi}(C_0 - \phi_S/c_\phi)$. We can invert the expressions for $\gamma, \alpha$ in terms of $\phi_V = O_\varphi, \phi_S$ as

$$O_\varphi = \alpha^{1/c_\phi} \quad \phi_S = c_\phi \left( C_0 - \gamma \alpha^{1-1/c_\phi} \right) \tag{3.14}$$

We see therefore that the VEV of the operator has dimension $1/c_\phi$. It is naturally connected to the alternate quantization solution with $c_\phi = 1$. This motivates our choice $K(\phi_S) = 1$ after the fact. The previous derivation is valid for any $1 > c_\phi > 0$ while the solution for $c_\phi = 0$ was already mentioned previously with $\phi_S = \alpha$ and $O_\varphi = \gamma$. One can now recognize that this corresponds to a multi-trace deformation (based on alternate quantization and ignoring that the metric sources the scalar as well) of the form $W[\alpha] = \alpha^{1/c_\phi}(\sqrt{3}c_\phi - 6\phi_S)/6$ [12, 13].

Using our scalar expansion, we can now also compute the contribution to the trace of the stress tensor

$$T_i^i = (1 - 3c_\phi) \alpha \gamma \tag{3.15}$$
This points to the quantization with $c_\phi = 1/3$ as a special value where the stress tensor remains traceless even in the presence of a source for the dilaton field. Indeed for $c_\phi = 1/3$ the VEV has dimension 3 equal to that of a marginal operator in the theory. This result does not a priori depend on the exact form of the potentials $V$ and $Z$, but only requires that the scalar mass remains in the mixed quantization window and that a proper Einstein-Maxwell-Dilaton black hole solution exists, as was previously shown in [11].

### 3.1 Choice of quantization and thermodynamics

In this subsection, we will derive the thermodynamics of a black hole solution in a general quantization choice. This goes beyond the analyses in [10, 11] where only the thermodynamics of a marginal scalar were considered. In view of extending the choice of possible theories to non-marginal ones, we will show that the thermodynamics space is extended from a 2-parameter to a 3-parameter space, as also emphasized for Einstein-Scalar theory in [14].

Substituting the homogeneous solution (2.4) into the action, the free energy of the black hole solution is given by

$$
\Omega = -S_{\text{on-shell}}^{\text{reg}} \quad \text{so} \quad \Omega = - \left( \frac{1}{z_h} + Q \right)^3 + \frac{3c_\phi - 1}{16} Q^3. \quad (3.16)
$$

Furthermore, the holographic dictionary tells us that the chemical potential and the temperature of the boundary theory are given by (2.5). One might be inclined to use this to deduce a variation of $\Omega$ in the 2-parameter grand canonical ensemble $d\Omega = -s_1dT - \rho_1d\mu$ and derive from it the thermodynamic entropy and charge density of the theory

$$
s_1 = -\left( \frac{\partial \Omega}{\partial T} \right)_\mu, \quad \rho_1 = -\left( \frac{\partial \Omega}{\partial \mu} \right)_T. \quad (3.17)
$$

However, we have seen from Eq. (3.3) that the free energy variation should be corrected by a scalar contribution of the form (see also [14])

$$
d\Omega = -s_2dT - \rho_2d\mu - \phi_Vd\phi_S. \quad (3.18)
$$

This is the full 3-parameter thermodynamics of the system. The fact the free energy (3.16) of the analytical GR solution only depends on $T$ and $\mu$, and not on the value of the scalar source means that the Gubser-Rocha solution should be seen as a 2-parameter constrained solution within this 3-parameter space. This family of solutions is only a subset of all the possible ones for any given quantization scheme. A direct corollary is that to explore only this analytical set of solutions, variations of $\phi_S, T, \mu$ are not independent. Denoting $\phi_S$ as the dependent variable, i.e. it is not independent but is a function of both $T$ and $\mu$, then the grand canonical potential varies as

$$
d\Omega = - \left( s_2 + \phi_V \frac{\partial \phi_S(T, \mu)}{\partial T} \right) dT - \left( \rho_2 + \phi_V \frac{\partial \phi_S(T, \mu)}{\partial \mu} \right) d\mu \quad (3.19)
$$

if one constrains one’s considerations to Gubser-Rocha solutions only.
The precise relation of $\phi_V$ and $\phi_S$ to the fall-off of the dilaton depends on the quantization scheme as we have just shown. A choice of quantization is not a canonical transformation, as shown by [14]. Therefore the value of the free energy will depend on this choice. This is evident in the dependence on $c_\phi$ in Eq.(3.16). In the full 3-parameter space of solutions this quantization choice dependence would only appear in the dilaton contribution part. In the constrained 2-parameter space of solutions, it would appear to imply that now also the thermodynamic entropy $s_1$ and charge density $\rho_1$ deduced from Eq.(3.17) depend on the quantization, as

\[
s_1 = 4\pi \left( \frac{1 + Qz_h}{z_h^2} \right)^{3/2} \left[ 1 + (3c_\phi - 1) \frac{Q^3z_h^3}{16(1 + Qz_h)^3} \right],
\]

\[
\rho_1 = \mu \frac{1 + Qz_h}{z_h} \left[ 1 - (3c_\phi - 1) \frac{Q^2z_h^2(2 + Qz_h)}{16(1 + Qz_h)^3} \right],
\]

(3.20)

This is strange, as the entropy and the charge density are properties of the black hole and do not depend on the boundary action which sets the quantization. Indeed they can be read off directly from the geometry.

\[
s_2 = 4\pi \sqrt{g_{xx}(z_h)g_{yy}(z_h)} = \frac{4\pi(1 + Qz_h)^{3/2}}{z_h^2} \quad \text{the area of the horizon of the black hole},
\]

\[
\rho_2 = -\partial_z A_t(z \to 0) = \mu \frac{1 + Qz_h}{z_h} \quad \text{the global U(1) charge}.
\]

(3.21)

The solution is of course that in the constrained system $s_1$ and $\rho_1$ are not the true entropy and charge density, as they include the contribution from varying $\phi_S(T, \mu)$. In a given quantization scheme one can in principle carefully disentangle the contributions to the entropy and the charge density. However, one can also choose the quantization $c_\phi = 1/3$ where the geometric black hole expressions directly agree with the thermodynamic black hole expressions. For this value of $c_\phi$, we have that the source corresponds to

\[
\phi_S = \left( C_0 - \gamma/\alpha^2 \right) / 3,
\]

(3.22)

In the analytic Gubser-Rocha black hole one has $\alpha = \sqrt{3}Q/2$ and $\gamma = \sqrt{3}Q^2/8$ and hence $\phi_S^{(GR)} = (C_0 - \frac{1}{2\sqrt{3}})/3$. To determine $C_0$ we note that when we set $Q = 0$, we recover the AdS-Schwarzschild black hole solution – characterized by $f(z) = 1 - (z/z_h)^3$, $g(z) = 1$ and $A_t(z) = \phi(z) = 0$ – which therefore also belongs to this family. For this solution $\phi$ is identically zero and the correct normalization choice is $\phi_S = 0$ i.e., $C_0 = \frac{1}{2\sqrt{3}}$. This will set the normalization for the entire GR family.

In the dual holographic theory the operator $\mathcal{O}_\phi$ thus has no explicit source. One might be tempted to call this a spontaneously ordered solution, however that is not the case. The crucial observation is that for this quantization $c_\phi = 1/3$ the trace of the stress tensor vanishes. The theory remains that of a conformal fluid with exact equation of state $\epsilon = 2P$ at all scales. Note that the vanishing trace of the stress tensor occurs regardless of the value of the source $\phi_S$ we impose on the scalar field. This can only happen if $\phi$ is dual to an exactly marginal operator. This is consistent with our observation that for $c_\phi = 1/3$ $\mathcal{O}_\phi$
has dimension $1/c_\phi = 3$. The Gubser-Rocha solution (2.4) is therefore a point on a line of fixed points characterized by its value of $\phi_S$.

In the next section we shall construct examples of this one-parameter family of marginally deformed solutions and prove the continued existence of conformal symmetry through the vanishing of the trace of the stress tensor.

### 4 A one-parameter family of deformed Gubser-Rocha black holes

We will now construct a one-parameter family of fixed $T$, fixed $\mu$ Gubser-Rocha black holes that differ in the “source” for the scalar operator. This corresponds to a different boundary condition for the dilaton field. However, for each such new solution, its interpretation depends on the quantization one considers, i.e. what the on-shell value of the action including boundary terms reads.

We will solve the Gubser-Rocha model equations of motion (2.2) numerically implemented using the following parametrization

$$\phi = \sqrt{3} \frac{2}{z} \psi(z), \quad A_t(z) = \mu j(z)a_t(z),$$

(4.1)

and with metric ansatz

$$ds^2 = \frac{1}{z^2} \left[ -f(z)G_{tt}(z)dt^2 + \frac{dz^2}{f(z)}G_{zz}(z) + g(z)G(z) \left( dx^2 + dy^2 \right) \right],$$

(4.2)

where $f(z), g(z), j(z)$ are held fixed to their expressions in the analytical Gubser-Rocha solution (2.4) and $\psi(z), a_t(z), G_{tt}(z), G_{zz}(z), G(z)$ are the dynamical fields. The radial coordinate $z$ spans the range from the boundary at $z = 0$ to the outer horizon at $z = z_h$. The IR boundary conditions are chosen to have a single zero horizon corresponding to a non-extremal black hole and to impose regularity at the horizon for other fields (see e.g., [16]).

The UV boundary conditions are chosen to impose AdS asymptotics for the metric components and $A_t(0) = \mu$. Parametrizing $\mu = \sqrt{3Qz_h(1 + Qz_h)} / z_h$ as in the Gubser-Rocha solution, the one-parameter family of solution we consider is given by the scalar UV-boundary conditions

$$\psi'(0) = \frac{1 - 6\sqrt{3} \xi}{4} \psi(0)^2 - \frac{3Q}{4} \psi(0)$$

(4.3)

in terms of the dimensionless parameter $\xi$. It is readily verified that $\xi = 0$ corresponds to the analytic Gubser-Rocha solution Eq.(2.4) for which $\gamma = \frac{1}{2\sqrt{3}} \alpha^2$. The parametrization in terms of $\xi$ is naturally inspired by the choice $c_\phi = 1/3$ in which quantization $\xi = \phi_S$ is in fact the source of dual holographic operator. For some other quantization choices, i.e. other choices of $c_\phi$, the source/VEV in the dual theory are given in the Table 1.

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4The boundary conditions from regularity imply in particular that $G_{tt}(z_h) = G_{zz}(z_h)$. This conveniently allows us to set the temperature with the parameters $Q$ and $z_h$ just like in the analytical GR model in Eq. (2.5), as the temperature of this generalised model is given by $T = T_{GR} \sqrt{G_{tt}(z_h)/G_{zz}(z_h)} = 3\sqrt{1 + Qz_h/4\pi z_h}$. All solutions are generated by setting $\mu = 1$ and varying $T$. 

-- 9 --
The boundary condition we impose on the scalar is simply a way to parametrize how we choose a bulk solution constrained to have a black hole in the interior. It would be equally valid to impose Dirichlet boundary conditions on the scalar, i.e. fixing the value of \( \psi(0) = \frac{2}{\sqrt{3}} \alpha \), although this obscures the connection between the solution chosen and the source in the dual holographic theory in a generic quantization scheme. At the same time, in the regime of solutions where \( \xi \) is positive in (4.3), we have found it is numerically more stable to choose Dirichlet boundary conditions at the price of having to carefully reconstruct the source. This is because the parameter \( \xi \) is non-linearly dependent on the parameter \( \alpha \) and in that regime becomes a non-unique parametrization of the solution. Of course a Dirichlet boundary condition is also the appropriate choice from a standard quantization point of view as it directly connects to the source.

Let us briefly describe the effect of the deformation \( \xi \). By looking at the gravitational action (2.1), we see that changing the boundary conditions of the dilaton should tune the effective screening of the charge in the EMD solution through the coupling \( Z(\phi) \). Increasing the value of the dilaton, which through Eq.(4.3) can be seen to correspond to decreasing \( \xi \), suppresses the effective electric charge. Since the theory only depends on the ratio \( T/\mu \), this is roughly equivalent to raising the temperature without changing the source. Vice versa, decreasing the value of the dilaton through sourcing should roughly correspond the lowering the temperature without sourcing.

To confirm our intuition, we can compare solutions at fixed \( T/\mu \) and varying \( \xi \) to (analytically known GR-)solutions with fixed \( \xi = 0 \) but different \( T/\mu \). We will choose to focus on the gauge field \( A_t(z) \) and more specifically the component \( a_t(z) \) defined in (4.1). Formally, \( a_t(z) = A_t(z)/(\mu j(z,T_0/\mu)) \) for a fixed \( T_0/\mu \). Since the analytically known GR solution at a different temperature \( T/\mu \) will have a gauge field \( A_t(z) = \mu j(z,T/\mu) \), the correct field to compare with will be \( a_t^{\xi=0}(z,T/\mu \neq T_0/\mu) = j(z,T/\mu)/j(z,T_0/\mu) \). We plot the profiles \( a_t^{\xi \neq 0,T=T_0}(z) \) in Figure 1 and compare these to \( a_t^{\xi=0,T>T_0}(z) \) (purple) and \( a_t^{\xi=0,T<T_0}(z) \) (red). We see that indeed, starting from \( \xi = 0 \), as we decrease \( \xi \), the solution becomes similar to a GR solution at higher \( T/\mu \) while when we increase \( \xi \), the solution becomes similar to a GR solution at lower \( T/\mu \). This qualitative connection between dilaton sourcing and temperature explains why in most of our numerical data, we were limited to a range of values for \( \xi \), as we encounter numerical accuracy issues at low temperatures.
Figure 1. Gauge field component $a_t(z)$ as defined in (4.1) for various values of $\xi$ and $T/\mu = 0.1$. We compare with the equivalent function $a_t^{\text{GR}}$ of the Gubser-Rocha solution at different temperatures $T/\mu = 0.11$ (purple) and $T/\mu = 0.098$ (red). This illustrates that qualitatively the effect of sourcing of the dilaton through its screening of the charge of the black hole has similarities to changing the ratio $T/\mu$.

4.1 The holographic dual of the one-parameter family of solutions in different quantization choices

Having numerically constructed instances of this one-parameter deformation of fixed $T$, fixed $\mu$ Gubser-Rocha black holes, each instance in turn has multiple holographic dual interpretations depending on the quantization. We will focus on two specific choices: the conformal symmetry preserving quantization $c_\phi = 1/3$ and the standard quantization $c_\phi = 0$. Using (3.4) we can compute the energy and the pressure of a solution in a specific quantization scheme and construct the trace of the stress tensor $T^{ii} = -\epsilon + 2P$ for each of these solutions as a function of $\phi_S$ as given in Table 1. As we can see in Figure 2, the stress tensor remains traceless for any sourcing of the scalar in the quantization $c_\phi = 1/3$, confirming the analytic result Eq.(3.15). This is what we expect from a marginal operator as we have previously argued. On the other hand, in standard quantization, we see that generically conformality is broken and the stress tensor acquires a non zero trace. In this quantization, this is also true for the Gubser-Rocha solution. There are two exceptions: the first one is when $\phi_S = 0$ (but $O_\phi \neq 0$) – which is reminiscent of a $\mathbb{Z}_2$ spontaneously symmetry breaking solution but here, the finite charge of the black hole actually always leads to an explicitly symmetry broken (ESB) solution $\phi(z) \neq 0$. The second solution would happen around $\mu \phi_S \approx 0.35$ such that $O_\phi = 0$. These are consistent with what we would have expected from (3.15).

Each one of these new black hole solutions has a different thermodynamics compared
Figure 2. Trace of the boundary stress tensor as a function of the scalar source for $c_\phi = 1/3$ and $c_\phi = 0$ and $T/\mu = 0.2$. GR denotes the analytically known Gubser-Rocha solution. (Left) We see that for $c_\phi = 1/3$, $T_{ij}$ remains traceless regardless of $\phi_S$ which is consistent with a marginal operator. (Right) For $c_\phi = 0$ the trace is generically not zero, but this can happen for specific boundary theories (marked ESB). In both figures, we have used the appropriate $\phi_S$ for the x-axis i.e., $\phi_S = \xi$ in the left-hand figure and $\phi_S = \alpha$ in units of $\mu = 1$ in the right-hand figure.

to the analytic Gubser-Rocha solution. A clean way to exhibit this is to show the entropy, which is geometrically determined and therefore manifestly independent of the quantization choice. In Figure 3, we plot the entropy as a function of temperature for various values of the dominant falloff $\alpha$. Numerically this is more convenient than parametrizing in terms of $\xi$ – but as we will see, the solutions do not depend on the quantization. It is clear from this figure that the entropy as a function of $T/\mu$ is dependent on the deformation of the boundary theory and the deformed solution describes a different state, even if the change is small. To emphasize that the solutions do not depend on the quantization, but that the interpretation does differ, we show in Figure 4 the entropy as a function of the dominant falloff $\alpha$ computed from the horizon area for several temperatures. Since both the entropy and the temperature are geometrically determined by the horizon, they do not depend on the quantization. The deformation needed to obtain a particular entropy-temperature curve does depend on the quantization, however. We can parametrize the solution the entropy for both type of quantizations, by alternatively choosing $c_\phi = 0$ and $c_\phi = 1/3$. This is seen here through the $\xi$-equipotential lines. Viewed from the point of $c_\phi = 1/3$ quantization, the necessary $\xi$-deformation needed to obtain the same entropy-temperature relation for a given $\alpha$ changes with temperature.

5 Conclusion

In this short paper, we have clarified how the Gubser-Rocha black hole thermodynamics works in the context of holography and the appropriate quantization thereof. The well-known analytical solution (2.4) of [6] covers only a 2-parameter subspace of the full 3-parameter thermodynamics of black hole solutions to the action (2.1). The 2-parameter analytical GR black hole solution has been used widely as a physically sound version of the
Figure 3. Black hole entropy as a function of the temperature $T/\mu$ when imposing Dirichlet boundary conditions, i.e. when fixing $\alpha$. As the entropy is a geometric quantity, it is independent of the quantization choice. At low temperature, we see that the entropy is linear in $T/\mu$ just like we have in the analytical Gubser-Rocha solution (2.4).

$z = \infty$ AdS$_2$ IR critical point that preserves the quantum critical properties but does so with a vanishing zero temperature entropy. While it was already pointed out [10, 11] that an unusual quantization choice could preserve conformal thermodynamics and hence stay within the analytical known 2-parameter family, we have shown that correctly accounting for the non-trivial $H'_{tt}(0)$ contribution confirms this, but also indicates the existence of a marginal operator in this specific quantization. More importantly, the analytic solution has a specific value for the source of this marginal operator. To prove this point we have numerically computed the solutions corresponding to a deformation by this marginal operator. This fills out the full 3-parameter thermodynamic phase space. The filled out phase-space also elucidates that other quantization choices are just as valid as the one we chose to focus on. This had to be so, but the tradeoff that one must make is to properly account for various scalar contributions to the general thermodynamics of the theory in line with the findings in [14].

Because the Gubser-Rocha action is a consistent truncation of $d = 11$ supergravity compactified on AdS$_4 \times S_7$ and has ABJM theory as its known holographically dual CFT, in principle one should be able to identify this marginal operator in the CFT. The fact that marginality is associated with a multitrace deformation makes this not as straightforward as may seem. In particular as it originates naturally in alternate quantization, it is likely that it is an operator which is only marginal in the large $N$ limit where the classical gravity description applies. We leave this for future research.

Our focus and interest is the use of the Gubser-Rocha and other EMD models as
Figure 4. Black hole entropy as a function of the dominant falloff $\alpha$ computed from the horizon area as a function of the temperature $T/\mu$. The lines correspond to Dirichlet boundary conditions naturally encoding $\phi_S$ in the quantization $c_\phi = 0$, while the dots correspond to the boundary condition (4.3) naturally encoding $\phi_S$ in the quantization $c_\phi = 1/3$. The fact that the two datasets give the same entropy-temperature relation is not a coincidence. As a geometrical quantity the entropy does not see the difference in quantization. The only difference between the two is that the interpretation of the source changes.

phenomenological descriptions of AdS$_2$ fixed points, especially due to its resemblance to the experimental phenomenology of strange metals. In this comparison thermodynamic susceptibilities and (hydrodynamic) transport play an important role. Our result here shows that in EMD models one must be precise in the choice of boundary conditions and scalar quantization as they will directly affect the long-wavelength regime of the dual boundary theory as well as correct the thermodynamics of any extension of the Gubser-Rocha model. A proper understanding of the boundary conditions is necessary both for the thermodynamics of the background and the hydrodynamic fluctuations on top of that background.

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References

[1] Thomas Faulkner, Hong Liu, John McGreevy, and David Vegh. Emergent quantum criticality, Fermi surfaces, and AdS(2). *Phys. Rev. D*, 83:125002, 2011.

[2] Christos Charmousis, Blaise Gouteraux, Bom Soo Kim, Elias Kiritsis, and Rene Meyer. Effective Holographic Theories for low-temperature condensed matter systems. *JHEP*, 11:151, 2010.

[3] B. Gouteraux and E. Kiritsis. Generalized Holographic Quantum Criticality at Finite Density. *JHEP*, 12:036, 2011.

[4] Liza Huijse, Subir Sachdev, and Brian Swingle. Hidden Fermi surfaces in compressible states of gauge-gravity duality. *Phys. Rev. B*, 85:035121, 2012.

[5] Qimiao Si, Silvio Rabello, Kevin Ingersent, and J. Lleweilun Smith. Locally critical quantum phase transitions in strongly correlated metals. *Nature*, 413(6858):804–808, October 2001.

[6] Steven S. Gubser and Fabio D. Rocha. Peculiar properties of a charged dilatonic black hole in AdS5. *Phys. Rev. D*, 81:046001, 2010.

[7] Kevin Goldstein, Shamit Kachru, Shiroman Prakash, and Sandip P. Trivedi. Holography of Charged Dilaton Black Holes. *JHEP*, 08:078, 2010.

[8] Yi Ling, Chao Niu, Jian-Pin Wu, and Zhuo-Yu Xian. Holographic Lattice in Einstein-Maxwell-Dilaton Gravity. *JHEP*, 11:006, 2013.

[9] Richard A. Davison, Koenraad Schalm, and Jan Zaanen. Holographic duality and the resistivity of strange metals. *Phys. Rev. B*, 89(24):245116, 2014.

[10] Bom Soo Kim. Holographic Renormalization of Einstein-Maxwell-Dilaton Theories. *JHEP*, 11:044, 2016.

[11] Marco M. Caldarelli, Ariana Christodoulou, Ioannis Papadimitriou, and Kostas Skenderis. Phases of planar AdS black holes with axionic charge. *JHEP*, 04:001, 2017.

[12] Edward Witten. Multitrace operators, boundary conditions, and AdS / CFT correspondence. 12 2001.

[13] Wolfgang Mueck. An Improved correspondence formula for AdS / CFT with multitrace operators. *Phys. Lett. B*, 531:301–304, 2002.

[14] Li Li. On Thermodynamics of AdS Black Holes with Scalar Hair. *Phys. Lett. B*, 815:136123, 2021.

[15] Vijay Balasubramanian and Per Kraus. A Stress tensor for Anti-de Sitter gravity. *Commun. Math. Phys.*, 208:413–428, 1999.

[16] Gary T. Horowitz, Jorge E. Santos, and David Tong. Optical Conductivity with Holographic Lattices. *JHEP*, 07:168, 2012.