Topological phases, topological phase transition, and bulk-edge correspondence of magnetized cold plasmas

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Plasmas have been recently studied as topological materials. We attempt at a comprehensive picture of the topological phases, topological phase transitions, and the bulk-edge correspondence of magnetized cold plasmas. We find that there are 10 topological phases in the parameter space of density $n$, magnetic field $B$, and parallel wavenumber $k_{\parallel}$. They are separated by the surfaces of Langmuir wave-L wave resonance, Langmuir wave-cyclotron wave resonance, and zero magnetic field. For fixed $B$ and $k_{\parallel}$, only the phase transition at the Langmuir wave-cyclotron wave resonance corresponds to edge modes. A sufficient and necessary condition for the existence of this new type of edge modes is given and verified by numerical solutions. The edge modes fall into four categories characterized by different behaviors of the Fermi arcs or Fermi-arcs-like curves. We demonstrate that edge modes exist not only on a plasma-vacuum interface but also on more general plasma-plasma interfaces. This finding broadens the possible applications of these exotic excitations in space and laboratory plasmas.

Introduction.— Recently, the relation between the topological properties of the bulk modes and the chiral (unidirectional) edge modes has attracted growing interest in classical fluid [1–5] and plasma physics [6–10]. Originating in condensed matter physics, the bulk-edge correspondence [11–13] predicts that at the interfaces between two topologically different materials, there exist gapless edge modes across the common band gap. For condensed matter systems, the gap Chern number is a topological invariant for bulk modes to give a matter systems, the gap Chern number is a topological edge modes across the common band gap. For condensed two topologically different materials, there exist gapless spondence [11–13] predicts that at the interfaces between nating in condensed matter physics, the bulk-edge corre- in classical fluid [1–5] and plasma physics [6–10]. Origi- spondence between the gap of X waves can be physi- nological properties of the bulk modes and the chiral (uni- calized [20]. Another type of edge mode has been derived using a simplified analytical model [6] and linked to the Weyl degeneracies [7]. This edge mode was also numerically demonstrated [10] for a plasma-vacuum inter- face with continuous density falloff. However, the corre- sponding bulk topological phases and phase transition have not yet been identified.

In the present study, we attempt at a comprehensive picture of the topological phases, topological phase tran- sitions, and the bulk-edge correspondence of magnetized cold plasmas in the absence of a solid boundary. We find that in the parameter space of magnetic field $B$, density $n$, and parallel wavenumber $k_{\parallel}$, there are 10 topological phases, separated by the Langmuir wave-L wave (LL) resonance, the Langmuir wave-Cyclotron wave (LC) resonance, and the $B = 0$ surface. Their topological properties are classified by the integer Chern numbers of the spectrum. For fixed non-vanishing $B$ and $k_{\parallel}$, there are two possible topological phase transitions due to the two resonances, while only the transition at the LC resonance produces edge modes. There exists a critical density $n_c$ such that plasmas below and above $n_c$ are in different topological phases across the LC resonance. We find that edge modes exist at more general plasma-plasma interfaces when a necessary and sufficient condition, Eq. (5), is satisfied, and the edge modes can be categorized by different behaviors of the Fermi arcs or Fermi-arcs-like curves. This finding broadens the possible applications of these exotic edge modes in space and laboratory plasmas.

Bulk dispersion relation and eigenmodes.— Following Ref. [10], we use the linearized fluid equations for a magnetized cold plasma with stationary ion and uniform density $n_c$ to study the eigenmodes of the system. The constant background magnetic field is in the $z$-direction, i.e., $B_0 = B_0 \hat{z}$, and there is no equilibrium flow. After spacetime Fourier transform, $\partial_t \rightarrow -i\omega$, $\nabla \rightarrow ik$, and the governing equations can be written as $H(k)|\psi\rangle = \omega|\psi\rangle$, where $H(k)$ is a $9 \times 9$ Hermitian matrix, and $|\psi\rangle = (\nu, E, B)^T$ is a nine-dimensional vector consisting of the perturbed velocity and electromagnetic fields.

The dispersion relation is given by the vanishing determinant of $H(k) - \omega I_9$, which can be simplified as

$$\det (NN - N^2 I_3 + \epsilon) = 0,$$

where $N = ck/\omega$, and $\epsilon$ is the $3 \times 3$ cold plasma dielectric
tensor. In terms of the standard notations in Ref. [21],
\[
\epsilon = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}, \quad P = 1 - \frac{\omega_p^2}{\omega^2},
\]
\[
S = 1 - \frac{\omega_p^2}{\omega^2 - \Omega^2}, \quad D = -\Omega \frac{\omega_p^2}{\omega^2 - \Omega^2},
\]
where \(\omega_p = \sqrt{n_e e^2/m_e c_0}\) is the plasma frequency, and \(\Omega = \text{sign}(B_0) eB_0/m_e\) is the signed electron cyclotron frequency. The plasma is called underdense if \(\omega_p < |\Omega|\), and overdense if \(\omega_p > |\Omega|\). It can be shown that the dispersion relation surfaces in the \((\omega, k)\) space are symmetric with respect to all four coordinate hyperplanes. For each given \(k_{\parallel}\), the system has 9 eigenvalues \(\omega_n\) and eigenvectors \(|\psi_n\rangle\) as functions of \(k_{\perp}\), where \(\omega_n = -\omega_p\) and \(n = -4, -3, ..., 3, 4\) is the index for the eigenmodes. Note that \(\omega_0 = 0\) is the zero frequency eigenmode of the system. The dispersion surfaces \(\omega(k_{\parallel}, k_{\perp})\) of the four positive-frequency branches for both overdense and underdense plasmas are shown in Figs. 1(a) and 1(c). To better illustrate the crossing between branches, the dispersion curves \(\omega(k_{\parallel})\) at different values of \(k_{\perp}\) are shown in Figs. 1(b) and 1(d). We can observe that branch crossing occurs only when \(k_{\perp} = 0\), represented by the coldest blue lines. In this case, the horizontal lines \(\omega = \omega_p\) are the Langmuir waves given by \(P = 0\). The other three branches are the R wave and the L wave given by \(N^2 = R\) and \(N^2 = L\), respectively, where \(R = S + D\) and \(L = S - D\). For convenience, we define functions
\[
k_{\pm} := \frac{\omega_p/c}{\sqrt{1 \pm \omega_p/\Omega}}.
\]
When \(k_{\parallel} > 0\), the Langmuir wave resonates with the L wave at \(k_\perp = k^+\). In an underdense plasma, the Langmuir wave also resonates with the lower branch of the R wave, a.k.a electron cyclotron wave, at \(k_{\perp} = k^-\). Notice that \(k^+ \rightarrow \infty\) when \(\omega_p \rightarrow \Omega\). These four resonant points at \(k_{\perp} = \pm k^\pm\), previously recognized as the Weyl points [7], play important roles in determining the topological properties of magnetized plasmas.

**Topological phase diagram.**—When fixing \(k_{\parallel}\) as a parameter, we can calculate the Chern number for each band, i.e., branch of the dispersion relation, in the \(k_{\perp} = (k_x, k_y)\) space. If the \(k_{\perp}\) space can be properly compactified [10, 14], Chern numbers in the system should be integers, which are invariant under continuous transforms. It means that each band’s Chern number is a topological invariant that can change only when different bands cross. As shown in Fig. 1, the band crossing in a cold plasma is only possible at the points \((k_x, k_y, k_{\parallel}) = (0, 0, \pm k^\pm)\) when \(B \neq 0\) and \(k_{\parallel} \neq 0\). Therefore, the locations of \(k_{\parallel} = \pm k^\pm\) defines the boundaries between topologically different regions if separated regions have different Chern numbers. In Fig. 2(a), the surfaces of \(k_{\parallel} = k^\pm\) and \(\Omega = 0\) in the \((\omega_p, \Omega, k_{\parallel})\) space are shown, which separate the parameter space into 10 different regions, each represents a different phase characterized by a set of Chern numbers. Although the first band touches the zero-frequency mode at \(k_{\parallel} = 0\), we will show later that this band crossing does not affect the topology. The cross sections of the 3D surfaces at \((\omega_p = 1, \Omega > 0, k_{\parallel} > 0)\) and \((\omega_p > 0, \Omega = 1, k_{\parallel} > 0)\) are shown in Figs. 2(b) and 2(c), each of which is separated into three phases. Notice that phase I in these cross sections only exists in underdense plasmas, i.e., when \(\omega_p < |\Omega|\).
that the Weyl point at density $n$ indicates the LC resonance surface, defines a critical behavior is observed between bands 2 and 3 when $k^\parallel$ changes from 0 to 1 and 2 crossing at $k^\perp = 0$, this is a new finding that was not reported previously. When the parameters cross the boundary of $k^\parallel = k^\parallel$ and change from phase I to phase II, bands 1 and 2 cross at $k^\perp = 0$ and change their Chern numbers from (0, 0) to (−1, 1). This change agrees with the fact that the Weyl point at $k^\parallel = k^\parallel$ has Chern number 1 [7]. Similar behavior is observed between bands 2 and 3 when parameters cross the boundary of $k^\parallel = k^\parallel$ and change from phase II to phase III.

Of particular interest in the present study is the transition between phases I and II. The boundary between them, i.e., the LC resonance surface, defines a critical density $n_c$, which expressed in terms of the corresponding plasma frequency $\omega_{p,c} = \sqrt{n_c e^2 / m_e \epsilon_0}$, is

$$\omega_{p,c} = \frac{|\Omega|}{2} \left[ \left( \frac{ck^\parallel}{\Omega} \right)^4 + 4 \left( \frac{ck^\parallel}{\Omega} \right)^2 - \left( \frac{ck^\perp}{\Omega} \right)^2 \right]. \quad (3)$$

For fixed $k^\parallel$ and $\Omega$, transition between phases I and II occurs at $n = n_c$. Notice that when $k^\parallel \to \infty$, $\omega_{p,c} \to |\Omega|$, and $n_c$ becomes $n_c \approx \Omega^2 m_e \epsilon_0 / e^2$.

The surface of $\Omega = 0$ is also a boundary between different topological phases because $C_n(-\Omega) = -C_n(\Omega)$. For the special case of $k^\parallel = 0$, the boundary between phases I and II and the boundary between phases II and III collapse to the lines defined by $\Omega = 0$ and $\omega_\up = 0$. In this case, the only possible nontrivial phase transition happens at the $\Omega = 0$ surface.

Band gaps.– Bulk-edge correspondence suggests that edge modes exist in the common band gaps at the interface between two topological materials with different gap Chern numbers. We now identify possible band gaps in a magnetized cold plasma. At fixed $k^\parallel$, when $\omega_\down \to \infty$, $(\omega_1, \omega_2, \omega_3, \omega_4) \to (0, \omega_{ub}, ck^\parallel, ck^\parallel)$, where $\omega_{ub}^2 = \omega_p^2 + \Omega^2$ is the upper hybrid frequency. Thus, when $|k^\parallel| \neq k^\parallel$, it is possible to have gaps between bands 1 and 2 and between bands 2 and 3. However, the gap between bands 2 and 3 does not always exist. When $k^\parallel = 0$, $\omega_\up(k^\parallel)$ is given by $N^2 = L$. The non-overlapping of bands 2 and 3 requires $\omega_\up(k^\parallel) > \omega_{ub}$, which leads to

$$|k^\parallel| > k^* := \left\lfloor \frac{|\Omega|}{c} \left( 1 + \frac{\omega_p^2}{\Omega^2} \right)^{1/4} \right\rfloor. \quad (4)$$

The locations of $|k^\parallel| = k^*$ in the parameter space are shown in Figs. 2(b) and 2(c). It is clear that the gap between bands 2 and 3 does not exist in phase III because condition (4) is not satisfied there.

The topology of a band gap is characterized by it gap Chern number, which is defined as $C_i, i+1 = \sum_{n=-4}^{i} C_n$ for the gap between the $i$-th and $(i+1)$-th bands. In phase I, both $C_{1,2}$ and $C_{2,3}$ are trivially zero, as in the phase of vacuum [11] that phase I neighbors at the boundary of $\omega_\up = 0$. Thus, as far as the gap topology is concerned, phase I plasmas are identical to the vacuum, which is interesting if not surprising. In phase II, the gap Chern numbers ($C_{1,2}, C_{2,3}$) become (−1, 0), indicating a topological phase transition at the boundary between phases I and II due to the crossing of the gap between bands 1 and 2. In phase III, the gap Chern numbers ($C_{1,2}, C_{2,3}$) are (−1, 1). Although $C_{2,3}$ is different between phase II and III, there is no band gap between bands 2 and 3 in phase III as proved above. When $|k^\parallel| \neq 0$, there is another band gap between bands 0 and 1. However, the gap Chern number $C_{0,1}$ for this gap is zero for all three phases, and it is a trivial band gap. Therefore, only the band gap between bands 1 and 2 shared by phases I and II is interesting in the context of bulk-edge correspondence.

It is worth mentioning that when $k^\parallel = 0$, band 3 becomes the O wave, and bands 2 and 4 are the X wave. If one chooses to ignore the O wave [13–15, 17], then a band gap shows up between bands 2 and 4, as long as $\Omega \neq 0$. The physical properties of this gap has been extensively studied, including the violation of bulk-edge correspondence under certain conditions [20]. However, as an important eigenmode in magnetized cold plasmas,
band 3 always exists. Especially when $|k_y| > k^+$, band 3 has a non-zero Chern number and should not be ignored. In the present study, we include all 9 bands in magnetized cold plasmas.

**Bulk-edge correspondence.** Having established the topological phase diagram and possible band gaps in a magnetized cold plasma, we now investigate the edge modes at the interface between two different magnetized cold plasmas. As discussed above, at a fixed $B$, the only possible nontrivial gap that admits two different gap Chern numbers is the gap between bands 1 and 2 shared by phases I and II. The bulk-edge correspondence then predicts that edge modes exist in the common band gap at the interface between a phase I plasma and a phase II plasma. We now solve for these edge modes numerically in an 1D inhomogeneous plasma. The background magnetic field is constant, i.e., $B_0 = B_0 z$, and the plasma density is nonuniform only in the $z$-direction, as shown in Fig. 4(a). The density profile is given by $n(x) = \frac{1}{2}(n_1 - n_2)\left(\tanh(-x/l) + \tanh[(x + l)/\delta]\right) + n_2$, where $n_1$ and $n_2$ are the densities of the inner and outer plasmas, and $l$ and $\delta$ are the location and width of the interface. For realistic plasmas, the width of the interface is finite, i.e., $\delta > 0$.

Because the system is uniform in the $z$-direction, the parallel wavenumber $k_\parallel$ enters as a parameter. The inner and outer plasmas can be represented by two points, $(\omega_{p,1}, \Omega, k_\parallel)$ and $(\omega_{p,2}, \Omega, k_\parallel)$, in the phase diagram shown in Fig. 2. Here, $\omega_{p,1}$ and $\omega_{p,2}$ are the inner and outer plasma frequencies. As discussed above, when there is a common band gap shared by the inner and outer plasmas, chiral edge modes exist in the gap if and only if the inner plasma is in phase I and outer plasma is in phase II such that they have different gap Chern numbers. Furthermore, the number of chiral edge modes at the interface should be equal to the difference of the gap Chern numbers, which is 1 in the present case. Since $n_c$ is the critical density at the boundary between phases I and II, for given $k_\parallel$ and $\Omega$, the chiral edge mode exists if and only if

$$n_1 > n_c > n_2,$$  \hspace{1cm} (5)

which, expressed in terms of plasma frequencies, is

$$\frac{\omega_{p,1}}{\Omega} + \frac{\omega_{p,2}^2}{c^2 k_\parallel^2} > 1 > \frac{\omega_{p,2}^2}{\Omega} + \frac{\omega_{p,2}^2}{c^2 k_\parallel^2}. \hspace{1cm} (6)$$

Here, we observe that underdense and overdense plasmas behave differently. When both the inner and outer plasmas are overdense, edge mode cannot exist regardless of $k_\parallel$. If the inner plasma is overdense while the outer plasma is underdense, edge modes can be found when $k_\parallel > k^-(\omega_{p,2})$. If both the inner and outer plasma are underdense, the edge mode can only be found when $k^-(\omega_{p,1}) > |k_\parallel| > k^-(\omega_{p,2})$. Notice that when the outer side is a vacuum, the criteria given by Eq. (6) can be satisfied in two scenarios, either $k_\parallel^2$ is small enough or the inner plasma is overdense. Incidentally, all the parameters chosen in Ref. [10] belong to the first scenario.

To numerically verify the criteria in Eqs. (5) or (6), we Fourier-transform in $y$, $z$ and $t$, then spatially discretize the Hamiltonian $H$ in the $x$-direction using a finite difference method. A periodic boundary condition at $x = \pm 2l$ is adopted, as in Ref. [1]. More details of the numerical methods are provided in the Supplementary Material. The structures of the positive-frequency bands for different $(n_1, n_2)$ at $ck_\parallel/\Omega = 0.7$ are shown in Fig. 4(d)-4(f). For the cases of $n_c > n_1 > n_2 = 0$ and $n_1 > n_2 > n_c$, there is no edge mode in the gap between the first and second bands. In particular, Fig. 4(f) shows that the edge modes can be absent at a plasma-vacuum interface. In Fig. 4(e), $n_1 > n_c > n_2$ and there are two gapless edge modes in the band gap. Due to the symmetry of $\omega(k_y) = \omega(-k_y)$, the two edge modes cross at $k_y = 0$. The electric field structure of the two gapless modes localized at different edges is shown in Figs. 4(b) and 4(c). The number of edge modes at each edge is the same as the difference of the gap Chern number, consistent with the prediction of bulk-edge correspondence.

Figure 4. The band structure of nonuniform plasma for $ck_\parallel/\Omega = 0.7$. The critical plasma frequency given by Eq. (3) is $\omega_{p,pc}/\Omega \approx 0.5$. (a) Schematic diagram of the density profile in $x$ direction, where $c\Omega/\Omega = 40$ and $c\delta/\Omega = 4$. (b) and (c) The non-zero components of electric fields of edge modes in (e) at $ck_\parallel/\Omega = 0.05$. (d),(e), and (f) The band structure $\omega(k_y)$ with various inner and outer densities. The plasma frequencies $(\omega_{p,1}, \omega_{p,2})/\Omega$ are (0.8, 0.6), (0.6, 0.4), and (0.4, 0) in (d), (e), and (f), respectively.

To further understand the edge modes, band structures at $k_y = 0$ with various densities are plotted in Fig. 5, which shows that edge modes can be classified...
by behaviors of the Fermi arc or Fermi-arc-like curves. The inner plasma is underdense in Figs. 5(a) and 5(c) and overdense in Figs. 5(b) and 5(d). The outer is underdense in Figs. 5(c) and 5(d) and vacuum in Figs. 5(a) and 5(b). The bulk dispersions are shown by orange and blue lines for inner and outer plasmas in each case. The topological edge modes are shown in red lines representing the crossing points between the left and right edge modes at each given $k_\parallel$. We can see that the range where edge modes exist coincides with Eq. (6) exactly. Noticeably, in Fig. 5(a), edge modes exist when $|k_\parallel| < k^- (\omega_{p,1})$ between a underdense plasma and a vacuum. The dispersion surface $\omega = \omega(k_\parallel, k_y)$ of the edge modes connects the two Weyl points of the inner plasma at $k_\parallel = \pm k^- (\omega_{p,1})$. The intersection of this dispersion surface and $\omega = \omega_{p,1}$ is known as the Fermi arc connecting two Weyl points [22, 23]. When the inner plasma is overdense in Fig. 5(b), the Weyl points $\pm k^- (\omega_{p,1})$ disappear, but the edge modes still exist and the dispersion surface connects to $k_\parallel = \pm \infty$. When the inner and outer plasmas are all underdense in Fig. 5(c), edge modes are prohibited if $-k^- (\omega_{p,2}) < k_\parallel < k^- (\omega_{p,2})$, then the dispersion surface no longer connects the Weyl points of the inner plasma at positive and negative $k_\parallel$. Instead, a Fermi-arc-like curve connects $k_\parallel = k^- (\omega_{p,1})$ and $k_\parallel = \omega_{p,2}$, the Weyl points of the inner and outer plasmas. In Fig. 5(d), the inner is overdense and the outer underdense, a Fermi-arc-like curve connects the Weyl point of the outer plasma to infinity. These numerical results also confirm the condition given by Eqs. (5) or (6) for the existence of the edge modes.

As a side note, the gap between bands 1 and 2 of the inner plasma may overlap with the gap between bands 2 and 3 of the outer plasma. It is reasonable to suggest that if $C_{\text{in,1.2}} \neq C_{\text{out,2.3}}$, edge modes might exist within this gap. However, this common gap is filled with other eigenmodes in reality. The upper hybrid frequency $\omega_{uh}$, which depends on plasma density, sets the upper range for band 2. Since the density profile is continuous, the local upper hybrid frequency $\omega_{uh}(x)$ will always fill in the gap between the second band of the inner and outer plasmas. An example is illustrated in Fig. 6, where the inner and outer plasmas belong to phases II and I, respectively.

Varying density is a convenient but not the only way to create interfaces between topologically different plasmas. Since plasma topology varies with the strength and direction of the magnetic field, one can create topologically nontrivial interfaces by assembling two plasmas with different background magnetic field. In particular, when $k_\parallel = 0$, the only possible topological phase transition in a cold plasma occurs at the $\Omega = 0$ surface, as discussed above. In this case, the edge modes at the interface between two plasmas with opposite magnetic fields has been studied for the X wave [13, 18]. However, as a whole system, such a setup is inhomogeneous, and the O wave cannot be decoupled. More thorough analysis is need.

Discussion and conclusion.— In summary, we found that a magnetized cold plasma has 10 different topological phases in the $(\omega, \Omega, k_\parallel)$ space. The different phases are separated by the surfaces of Langmuir wave-L wave resonance, Langmuir wave-cyclotron wave resonance, and $\Omega = 0$. We found that at fixed $\Omega$ and $k_\parallel$, only the band gap between bands 1 and 2 shared by phases I and II is interesting in the context of bulk-edge correspondence.
and that the necessary and sufficient condition for the existence of edge modes is $n_1 > n_c > n_2$. These findings were verified by numerical studies of the corresponding chiral edge modes in 1D inhomogeneous plasmas. The edge modes exist not only on the plasma-vacuum interface, but also on more general plasma-plasma interfaces. No violation of bulk-edge correspondence is observed in this system.

Our study of edge modes focused on the general gaseous plasma-plasma interfaces. We didn’t consider gaseous plasma-solid interfaces, which, unlike the interfaces between solid state materials [23], involve more complex physical processes, such as the plasma sheath and plasma-wall interactions [24]. It is not appropriate to model gaseous plasma-solid interfaces only as simple interfaces between two different topological materials.

The present investigation on the topology phases and edge modes is carried out for the model of magnetized cold plasmas. When other physical effects, such finite temperature, plasma collisions, and kinetic interactions, are important, band structures of the system can change significantly [21]. The bulk topology and the validity of the general bulk-edge correspondence in these regimes need to be further investigated. In particular, linear dynamics in these plasmas is expected to be non-Hermitian [25, 26], permitting unstable and damped eigenmodes. Applying the methods of topological phases for non-Hermitian systems will bring new insights and discoveries in the study of plasma instabilities for laboratory and astrophysical plasmas.

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Supplementary Material

I. REVIEW OF WAVES IN COLD PLASMA

A. Basic equations

In this section, we outline the band structure of cold plasma waves. Assume the background plasma is uniform and stationary, the ions are motionless and the magnetic field is constant, i.e., $B_0 = B_0 \hat{z}$. The linearized fluid equations are [21]

$$
\partial_t v = -\frac{e}{m_e} (E + v \times B_0),
$$

$$
\partial_t E = c^2 \nabla \times B + \frac{en_c}{\epsilon_0} v,
$$

$$
\partial_t B = -\nabla \times E,
$$

(S1)

where $v, B, E$ are perturbed velocity, magnetic field and electric field, $e > 0$ is the elementary charge, $m_e$ and $n_e$ are electron mass and density, and $c$ is light speed. Define plasma frequency $\omega_p^2 = ne^2/\epsilon_0 m_e$ and signed cyclotron frequency $\Omega = \text{sign}(B_0) |eB_0/m_e|$. For convenience, let $c = 1$ and define new variables, $\tilde{v} = \omega_p v$, $\tilde{E} = c E/m_e$, $\tilde{B} = e B/m_e$. The equation system becomes

$$
\partial_t \tilde{v} = -\omega_p \tilde{E} - \Omega \tilde{v} \times \hat{z},
$$

$$
\partial_t \tilde{E} = \nabla \times \tilde{B} + \omega_p \tilde{v},
$$

$$
\partial_t \tilde{B} = -\nabla \times \tilde{E}.
$$

(S2)

Notice that when the density is not uniform, it suffices to change $\omega_p$ to $\omega_p(r)$ in Eq. S2. From now on, we omit the tilde for convenience. After space Fourier transform, $\partial_t \rightarrow -i \omega$, $\nabla \rightarrow ik$, and the governing equations can be written as $H|\psi\rangle = \omega|\psi\rangle$, where $|\psi\rangle = (v, E, B)^\dagger$ and

$$
H(\omega_p, \Omega, k) = \begin{pmatrix} i\Omega \hat{z} \times & -i \omega_p & 0 \\ i\omega_p & 0 & -k \times \\ 0 & k \times & 0 \end{pmatrix}.
$$

(S3)

Here, $H$ is a $9 \times 9$ Hermitian matrix. The dispersion relation is given by $\text{det}(H - \omega I_9) = 0$, which simplifies to Eq. (1).

B. Symmetries of the system

As a real system, the equations of motion for plasmas admit an unbreakable particle-hole symmetry [9, 27]. For our system, the symmetry states that $H(-k)^* = -H(k)$, where * denotes complex conjugate. It ensures that the dispersion relation has the symmetry of $\omega(-k) = -\omega(k)$. Notice that Eq. (1) remains invariant under $k \rightarrow -k$, so the dispersion relation satisfy $\omega(k) = \omega(-k)$ as well. In addition, since the system is isotropic in the direction perpendicular to the background magnetic field, $\omega$ only depends on $k_\parallel \equiv k_z$ and $k_\perp \equiv \sqrt{k_x^2 + k_y^2}$. Therefore, the dispersion relation surfaces in the $(\omega, k_\perp, k_y, k_z)$ space are symmetric with respect to the reflections of all four coordinate hyperplanes.

The symmetries of eigenvalues can also be obtained, which will be useful during the calculation of the Chern numbers. $H(\omega_p, \Omega, k)$ has 9 eigenvalues, one of which is identically zero. The eigenvalues and corresponding eigenvectors can be labeled as $\omega_n|\psi_n\rangle$, where $n = -4, -3, \cdots, 3, 4$, $\omega_i < \omega_j$ if $i < j$. Assume that at some $(\omega_p, \Omega, k)$, the $n$-th eigenvalue and eigenvector are $\omega_n = \omega$ and $|\psi_n\rangle = (v, E, B)^\dagger$. We can verify the following symmetries of eigenvalues and eigenvectors:

1. For the reflection of band number $n \rightarrow -n$, $\omega_{-n} = -\omega$ and $|\psi_{-n}\rangle = (v^*, E^*, -B^*)^\dagger$.
2. For the reflection of magnetic field $\Omega \rightarrow -\Omega$, $\omega_n = \omega$ and $|\psi_n\rangle = (-v^*, E^*, B^*)^\dagger$.
3. For the reflection of wavenumber $k \rightarrow -k$, $\omega_n = \omega$ and $|\psi_n\rangle = (v, E, -B)^\dagger$.

II. CALCULATION OF CHERN NUMBERS

In this section, we briefly describe the calculation and symmetries of Chern numbers. For any given parallel wavenumber $k_\parallel$, the Chern number can be calculated in the $k_\perp$ space for each band by [12] $C_n = (2\pi)^{-1} \int dS \cdot F_n(k)$, where $F_n = \nabla_k \times A_n$ is the Berry curvature, and $A_n = i(|\psi_n| \nabla_k |\psi_n\rangle)$ is the Berry connection. Let $|\psi_n(\omega_p, \Omega, k)\rangle = (v, E, B)^\dagger$. The Berry connection is $A_n(\omega_p, \Omega, k) = i(v \nabla_k v^\dagger + E \nabla_k E^\dagger + B \nabla_k B^\dagger)$, where $\dagger$ denotes conjugate transpose. Based on the symmetries of eigenvectors, we obtain the following symmetries of Chern numbers:

1. For the reflection of band number $n \rightarrow -n$,
$$
A_{-n}(\omega_p, \Omega, k) = i(|\psi_{-n}| \nabla_k \psi_{-n}\rangle
= i(v \nabla_k v^\dagger + E \nabla_k E^\dagger + B \nabla_k B^\dagger)
$$
(S4)
$$
= -A_n(\omega_p, \Omega, k).
$$

Thus, $C_{-n}(\omega_p, \Omega, k_\parallel) = -C_n(\omega_p, \Omega, k_\parallel)$.

2. For the reflection of magnetic field $\Omega \rightarrow -\Omega$, $A_n(\omega_p, -\Omega, k) = -A_n(\omega_p, \Omega, k)$. Thus, $C_n(\omega_p, -\Omega, k_\parallel) = -C_n(\omega_p, \Omega, k_\parallel)$.

3. For the reflection of parallel wavenumber $k_\parallel \rightarrow -k_\parallel$, since the system is isotropic in $k_\perp$ plane, it is equivalent to the reflection of wavenumber $k \rightarrow -k$. Due to the symmetries of the eigenvectors, we have $A_n(\omega_p, \Omega, -k) = -A_n(\omega_p, \Omega, k)$ and $F_n(\omega_p, \Omega, -k) = F_n(\omega_p, \Omega, k)$. Therefore, $C_n(\omega_p, \Omega, -k_\parallel) = C_n(\omega_p, \Omega, k_\parallel)$.
For numerical evaluation of the Chern numbers, the following alternative formula of Berry curvature is used,

$$ F_n = \sum_{m \neq n} \frac{\langle \psi_n | \nabla_k H | \psi_m \rangle \times \langle \psi_m | \nabla_k H | \psi_n \rangle}{(\omega_m - \omega_n)^2}. \quad (S5) $$

To integer Chern numbers, we adopt the same regularization strategy used in Ref. [10]. At large $k_\perp$, we regularize the plasma frequency in Eq. (S3) by replacing $\omega_p$ with $\omega_p/(1+k_{\perp}^2/k_c^2)$, where $k_c$ is a large-enough cutoff wavenumber.

### III. NUMERICAL CALCULATION OF EIGENMODES USING GEOMETRIC STRUCTURE-PRESERVING ALGORITHMS

Here we introduce the numerical methods for eigenmodes calculation. When density $n(x)$ is nonuniform in the $x$-direction, we Fourier-transform Eq. (S2) in $y, z, t$ and but not in $x$. The simulation region in the $x$-direction is $[-2l, 2l]$ and is discretized into $N$ grids. The interval of grids is $\Delta x = 4l/N$ and the grid points are $x_i = i\Delta x - 2l$, $i = 0, \ldots, N-1$. Similar to Ref. [1], a periodic boundary condition is applied at $x = \pm 2l$. Next, we adopt the strategy of structure-preserving geometric algorithms in plasma physics to discretize $v$ and $E$ on integer grid points. $v_i \equiv v(x_i)$, $E_i \equiv E(x_i)$, and $B_{i+1/2} \equiv B(x_{i+1/2})$. Such discretization ensures centered discretization of $x$-derivatives and preserves the geometric relations between different components of the field. Periodic boundary condition enforces that $v_N = v_0$, $E_N = E_0$, $B_{N+1/2} = B_{1/2}$. Define $\omega_{p,j} \equiv \omega_p(x_j) \sim \sqrt{n(x_j)}$. Then, Eq. (S2) is discretized as

$$
\begin{align*}
\omega v_{x,i} &= -i\Omega v_{y,i} - i\omega_p x E_{x,i}, \\
\omega v_{y,i} &= i\Omega v_{x,i} - i\omega_p y E_{y,i}, \\
\omega v_{z,i} &= -i\omega_p z E_{z,i}, \\
\omega E_{x,i} &= i\omega_p v_{x,i} + k_z \frac{B_{y,i+1/2} + B_{y,i-1/2}}{2} - k_y \frac{B_{x,i+1/2} + B_{x,i-1/2}}{2}, \\
\omega E_{y,i} &= i\omega_p v_{y,i} - k_x \frac{B_{z,i+1/2} + B_{x,i-1/2}}{2} - k_z \frac{B_{y,i+1/2} + B_{y,i-1/2}}{2}, \\
\omega E_{z,i} &= i\omega_p v_{z,i} + k_y \frac{B_{x,i+1/2} + B_{x,i-1/2}}{2} + k_x \frac{B_{y,i+1/2} - B_{y,i-1/2}}{2}, \\
\omega B_{x,i+1/2} &= -k_x \frac{E_{y,i+1} + E_{y,i}}{2} + k_y \frac{E_{x,i+1} + E_{x,i}}{2}, \\
\omega B_{y,i+1/2} &= k_x \frac{E_{y,i+1} + E_{y,i}}{2} - k_y \frac{E_{x,i+1} - E_{x,i}}{2}, \\
\omega B_{z,i+1/2} &= -k_y \frac{E_{x,i+1} + E_{x,i}}{2} - k_x \frac{E_{y,i+1} - E_{y,i}}{2}. \\
\end{align*}
$$

Equations (S6)-(S8) are now a standard matrix eigenvalue problem, which can be solved by standard matrix eigenvalue algorithms. We note that the method of structure-preserving geometric algorithms enables the reformulation of the eigenvalue problem of ODEs as an eigenvalue problem for Hermitian matrices.