QCD at Finite Extension

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Abstract

A study of QCD at finite extension is presented and the relation to QCD at finite temperature and in the infinite momentum frame is discussed. The dynamics of Polyakov loops is investigated and shown to be described by functional integrals with finite range of integration. Consequences of this non-Gaussian form of the generating functional concerning the QCD vacuum, gluonic excitations and confinement are discussed.

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The study of QCD at finite extension, i.e., in a geometry where the system is of finite extent \((L_3)\) in one direction \((x_3)\), is of interest for various reasons. First, with \(L_3\) a parameter is introduced which helps to control infrared ambiguities. This is of particular importance when using axial like gauges which are appropriate for such a geometry (for an early discussion of ambiguities in the axial gauge, see Ref. [1]).

Second, by covariance, QCD at finite extension is equivalent to finite temperature QCD. This equivalence of finite extension and finite temperature of relativistic field theories has been noted in Ref. [2] and used e.g. in a discussion of the finite temperature quark propagators [3]. By rotational invariance in the Euclidean, the value of the partition function of a system with finite extension \(L_3\) in 3 direction and \(\beta\) in 0 direction is invariant under the exchange of these two extensions,

\[
Z(\beta, L_3) = Z(L_3, \beta),
\]

provided bosonic (fermionic) fields satisfy periodic (antiperiodic) boundary conditions in both time and 3 coordinate. As a consequence of (1), energy density and pressure are related by

\[
\epsilon(\beta, L_3) = -p(L_3, \beta).
\]

For a system of non-interacting particles this relation connects energy density or pressure of the Stefan Boltzmann law with the corresponding quantities measured in the Casimir effect.

In QCD, covariance also implies by Eq. (2) that at zero temperature a confinement-deconfinement transition occurs when compressing the QCD vacuum (i.e. decreasing \(L_3\)). From lattice gauge calculations [4] we infer that this transition occurs at a critical extension \(L_3^c \approx 0.8\) fm in the absence of quarks and at \(L_3^c \approx 1.3\) fm when quarks are included. For extensions smaller than \(L_3^c\), the energy density and pressure reach values which are typically 80\% of the corresponding “Casimir” energy and pressure. The order parameter which characterizes the phases of QCD and in particular the realization of the center symmetry (in the absence of quarks) is the vacuum expectation value of the Polyakov loop operator at finite temperature [5] and correspondingly of the operator

\[
W(x_\perp) = N_c^{-1}\text{tr} P \exp \left\{ i g \int_0^{L_3} dx_3 A_3(x) \right\}
\]

at finite extension \((x_\perp = (x_0, x_1, x_2))\).
When compressing the system beyond the typical length scales of strong interaction physics (e.g. beyond a typical hadron radius $R$), two further limits are of interest. Again as supported by results from lattice calculations $[6]$, correlation functions at transverse momenta or energies $|p| \ll 1/L_3$ are dominated by the zero “Matsubara wave-numbers” in 3-direction and can therefore be expected to be given by the dimensionally reduced QCD$_{2+1}$. On the other hand we may consider excitations with large momenta $p$ in the 3 direction such that the Lorentz-contracted extension of a hadron (of mass $m$) is small in comparison to $L_3$,

$$mR/p \ll L_3 \ll R.$$  

For sufficiently high momenta, spectrum and structure of hadrons should not be affected by the finite extension, although the vacuum on which these excitations are built is not the confining ground state with broken chiral symmetry, but that of the quark gluon plasma with the chiral symmetry restored. This situation of large momentum excitations at small extension is closely related to the light cone or infinite momentum frame limit in which, because of the off-diagonal metric, a finite extension along the light-cone space axis $x^-$ actually describes an interval of vanishing invariant length $[7]$.

For the theoretical treatment of QCD at finite extension an axial type gauge is particularly appropriate. Within the canonical formalism, the Gauss law can be resolved explicitly $[8]$. The result for the SU(2) Yang Mills theory is summarized in the generating functional

$$Z[J, j_3] = \int D[A_\mu] d[a_3] \exp \left\{ iS_{YM} [A_\mu, a_3] + iS_{gf} [A_3^\alpha] + i \int d^4x \left( A_\mu^a J_\mu^a + a_3 j_3 \right) \right\},$$  

where in the standard Yang Mills action the 3 component of the gluon field $A_3(x)$ is replaced by the neutral (i.e. diagonal) 2-dimensional field $a_3(x_\perp)$ (cf. also Ref. $[9]$). At finite extension $L_3$, this two dimensional field can be eliminated neither in QED nor in QCD. In QED it describes transverse photons polarized in 3 direction and propagating in the 1-2 plane, while in SU(2) QCD, $a_3(x_\perp)$ are the eigenvalues of the (untraced) “Polyakov loop” variable $W(x_\perp)$ of Eq. (3). After an appropriate choice of coordinates in color space, $W(x_\perp)$ can be written as

$$W(x_\perp) = \cos (gL_3 a_3 (x_\perp)/2).$$  

The presence of the two dimensional field $a_3(x_\perp)$ makes a further gauge fixing of the
transverse \((\mu = 0, 1, 2)\), neutral, \(x_3\) independent gauge fields necessary, which is achieved by the additional contribution to the action in Eq. (5). The following choice of the gauge fixing term is particularly convenient for perturbative calculations,

\[
S_{gf} \left[ A^3_\mu \right] = -\frac{1}{2L^2_3} \left( \int_0^{L_3} dx_3 \partial^\mu A^3_\mu (x) \right)^2. \tag{7}
\]

The integration measure for the \(a_3\) functional integral is given by

\[
d [a_3] = \prod_{y_\perp} \sin^2 (gL_3a_3(y_\perp)/2) \Theta (a_3(y_\perp)) \Theta (2\pi/gL_3 - a_3(y_\perp)) \, da_3 (y_\perp) \tag{8}
\]

and accounts for the fact that \(gL_3a_3/2\) appears in the parametrization of the group manifold \(S_3\) as the first polar angle. As a consequence the corresponding part of the Haar measure enters with its finite range of integration. In the canonical formulation, this Faddeev Popov determinant arises as a Jacobian modifying the kinetic energy of the Polyakov loop variables \(a_3(x_\perp)\) \[8\]. In QED the same procedure yields the standard flat measure for \(a_3(x_\perp)\).

In most of the approaches to finite temperature QCD in the temporal gauge, one ignores the finite range of integration and accounts for the factor \(\sin^2 (gL_3a_3/2)\) in the volume element by introducing ghost fields \(c_a (x)\),

\[
\prod_{y_\perp} \sin^2 (gL_3a_3(y_\perp)/2) = \int d[c]d[c^\dagger] \exp \left\{ i \int d^4x c^\dagger_a (x) \left( -i\partial_3\delta^{ab} + ig\epsilon^{abc}a^c_3 (x_\perp) \right) c_b (x) \right\}. \tag{9}
\]

In perturbative treatments, such ghosts have no effect, as is most easily seen in dimensional regularization, and therefore the QCD measure is effectively replaced by the flat measure appropriate for QED. Debye screening is obtained as a result of this procedure but also, as in QED, a shift symmetry \((Z)\) of the effective potential for the Polyakov loop variables \(a_3\) not present in the original theory \[10\].

We have not been able to justify the above procedure. We find that the functional integral over \(a_3\) cannot be approximated by a Gaussian integral with corresponding perturbative corrections. Rather, the relevant limit is that of a vanishing instead of an infinite range of integration, as is easily seen when considering the propagator for non-interacting Polyakov loop variables \(W(x_\perp)\) in the discretized Euclidean formulation (with lattice spacing \(\ell\) and unit vectors \(e\)),

\[
\langle \Omega | T [W(x_\perp)W(0)] | \Omega \rangle = \frac{1}{4} \delta_{x,0} + \mathcal{O} \left( \ell \right) \left( \delta_{x,0} + \sum_{\pm e} \delta_{x,e} \right)
\]

3
\[
\ell^3 \approx \frac{\ell^3}{4} g^{(3)} (x_\perp). \tag{10}
\]

Hopping terms are suppressed by powers of \( \ell / g^2 L_3 \), and the propagator becomes local in the continuum limit. We note that this property is due to the “macroscopic” nature of the variable \( a_3 (x_\perp) \). When describing a single link variable on the lattice, the factor \( \ell / L_3 \) does not appear and the discretized form of the free propagator is obtained. The winding of \( a_3 \) around the circle in 3 direction apparently provides an infinite inertia. Propagation of excitations induced by \( a_3 (x_\perp) \) can consequently only arise by coupling to the other microscopic degrees of freedom. Formally this suggests the Polyakov loop variables \( a_3 \) to be integrated out by disregarding the contribution of the free \( a_3 \) action, but keeping the coupling to the other degrees of freedom. Also disregarding possible effects from singular field configurations, the following effective action is obtained

\[
S_{\text{eff}} [A_\mu] = S_{\text{YM}} [A_\mu, A_3 = 0] + S_{\text{gf}} [A_3^a] + M^2 / 2 \sum_{a=1,2} \int d^4 x A_\mu^a (x) A^{a,\mu} (x). \tag{11}
\]

In this effective action, the Polyakov loop variables have left their signature in the geometrical mass term of the charged gluons

\[
M^2 = \left( \frac{\pi^2}{3} - 2 \right) / L_3^2 \tag{12}
\]

and in the change to antiperiodic boundary conditions

\[
A^{1,2}_\mu (x_\perp, L_3) = -A^{1,2}_\mu (x_\perp, 0), \tag{13}
\]

while the neutral gluons remain massless and periodic. The antiperiodic boundary conditions reflect the mean value of the Polyakov loop variables, the geometrical mass their fluctuations; notice that in both of these corrections, the coupling constant has dropped out. The role of the order parameter is taken over by the neutral color current in 3-direction,

\[
u (x_\perp) = \int_0^{L_3} dx_3 \epsilon^{abc} A^a_\mu (x) \partial_3 A^{b,\mu} (x), \tag{14}
\]

which determines the vacuum expectation value of the Polyakov loops

\[
\langle \Omega | W (x_\perp) | \Omega \rangle \propto \langle \Omega | u (x_\perp) | \Omega \rangle \tag{15}
\]

and the corresponding correlation function

\[
\langle \Omega | T \left[ W (x_\perp) W (0) \right] | \Omega \rangle \propto \langle \Omega | T \left[ u (x_\perp) u (0) \right] | \Omega \rangle. \tag{16}
\]
Formulated in these variables, the center symmetry acts as charge conjugation

\[ C : \quad A^3_\mu \to -A^3_\mu \quad , \quad A^1_\mu + iA^2_\mu \to A^1_\mu - iA^2_\mu . \quad (17) \]

The treatment of the Polyakov loop variables required by the finite range of integration leads to a picture of QCD which is quite different from that based on the treatment of the $a_3$ functional integral as an approximate Gaussian one. Among the most important differences, we note that the perturbative vacuum corresponding to the effective action (11) is an eigenstate of the center symmetry. In the perturbative ground state, no color currents are present, hence

\[ \langle \Omega_{pt} | W (x_\perp) | \Omega_{pt} \rangle = 0 . \quad (18) \]

Loosely speaking we can take this as an indication for an infinite free energy of a static quark. A more precise characterization of the realization of confinement can be obtained from properties of the associated correlation function (16) which, in the Euclidean, determines the static quark-antiquark interaction energy [F]. Due to the locality property (10), this static quark-antiquark potential is given directly by the $a = b = 3, \mu = \nu = 3$ component of the vacuum polarization tensor $\Pi^{ab}_{\mu\nu}$ and not by the zero mass propagator of the Gaussian approximation to the Polyakov loop action. Up to an irrelevant factor we have in the Euclidean

\[ \exp \left\{ -L_3 V (r) \right\} = \langle \Omega | T \left[ u (x_\perp^E) u (0) \right] | \Omega \rangle = x_\perp^E \quad (19) \]

This identity implies a linearly rising potential at large distances, if the vacuum expectation value of the Polyakov loop operator vanishes and if the spectrum of states excited by $u$ exhibits a gap $\Delta E$,

\[ V (r) \to \sigma r = \Delta E r / L_3 . \quad (20) \]

Thus in this axial like gauge, confinement is connected to a shift in the spectrum of gluonic excitations to excitation energies

\[ E \geq \sigma L_3 . \quad (21) \]

After a gradual decrease of this threshold with decreasing $L_3$, the whole spectrum of excitations becomes suddenly available when at the deconfinement transition the string
tension vanishes. At the perturbative level already a linearly rising potential is obtained, however with a string tension decreasing with increasing extension \((\propto L_3^{-2})\). For small distances, Coulomb-like behaviour requires this particular component of the vacuum polarization tensor to possess an essential singularity at infinite momentum

\[
\int d^3xe^{ipx}\langle\Omega|T[u(x)u(0)]|\Omega\rangle \rightarrow e^{-\sqrt{g^2 L_3 p/\pi}}
\]  

which, in the Euclidean, implies a density of intermediate states contributing to the vacuum polarization which increases exponentially with the square root of the excitation energy. A perturbative evaluation of the vacuum polarization therefore has to yield increasingly high powers of \(L_3 p\) with increasing order \([12]\).

With respect to perturbative treatments, we remark that the difficulties encountered in finite temperature perturbation theory \([13]\) are not expected to arise in this perturbative, confining phase. With the charged gluons satisfying antiperiodic boundary conditions, the infrared properties of QCD at finite extension should resemble those of QED. Moreover, the finite extension gluon propagator is well defined and, unlike the continuum axial gauge propagator, does not need a prescription how to handle spurious double poles \([14]\).

A final remark concerns the perturbative phase with its signatures of confinement. This phase is most likely not relevant for QCD at extensions smaller than \(L_3^c\). Not only do we expect the center symmetry to be broken at small extensions but also dimensional reduction to QCD\(_{2+1}\) to happen. On the other hand, in the process of dimensional reduction, charged gluons decouple from the low-lying excitations due to their antiperiodic boundary conditions; the small extension or high temperature limit in this phase is QED\(_{2+1}\). Thus in the deconfinement phase transition arising when compressing the QCD vacuum, a change to periodic boundary conditions as well as the disappearance of the geometrical mass must occur. This change in boundary conditions results in a change in Casimir energy density and pressure which for non-interacting gluons (and neglecting the effects of the geometrical mass) is given by

\[
\Delta \epsilon = -\pi^2/12L_3^4 , \quad \Delta p = 3\Delta \epsilon .
\]  

(23)

This estimate is of the order of magnitude of the change in the energy density across the confinement-deconfinement transition when compressing the system,

\[
\Delta \epsilon = -0.45/L_3^4 ,
\]  

(24)
deduced from the finite temperature lattice calculation of Ref. [15]. If QCD in the high temperature or small extension quark gluon plasma phase is to be described perturbatively, one has to abandon the non-perturbative resolution which is implicit to the temporal or axial like gauge with its characteristic $N - 1$ Polyakov loops as zero modes. Starting point has to be QCD at $g = 0$, and for this $U(1)^{N^2-1}$ theory an axial gauge with $N^2 - 1$ photons as zero modes is e.g. appropriate. This procedure breaks the center symmetry and the high temperature limit results [12].

In summary we have investigated QCD at finite extension in an axial like gauge. Novel aspects in the theoretical description have emerged due to the peculiar dynamics of the Polyakov loops; these variables are described by functional integrals which do not have a natural Gaussian limit. By elimination of these variables an effective description has been derived. It yields a QCD ground state which already at the perturbative level exhibits certain characteristics of the confining vacuum and permits to connect confinement with properties of the spectrum of gluonic excitations.

Details of this investigation, the development of perturbation theory and its application to the static quark-antiquark potential as well as generalizations to SU(3) and full QCD with fermions will be presented elsewhere [12].

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