Estimate the Parallel System Reliability in Stress-Strength Model Based on Exponentiated Inverted Weibull Distribution

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Abstract. In this paper, we employ the maximum likelihood estimator in addition to the shrinkage estimation procedure to estimate the system reliability ($R_k$) contain $K^{th}$ parallel components in the stress-strength model, when the stress and strength are independent and non-identically random variables and they follow two parameters Exponentiated Inverted Weibull Distribution (EIWD). Comparisons among the proposed estimators were presented depend on simulation established on mean squared error (MSE) criteria.

1. Introduction

The parallel stress-strength model occurs when a system has a mixture of $k$ independent component with the strengths $X_1, X_2, \ldots, X_k$ and each component of this system is subject to a common stress $Y$. the system works successfully if at least one of the components survives, it is termed parallel in the analogy with electric circuits [4].

The reliability of the above system denoted by $R_k$ take the attention in this work and the formula of such stress-strength reliability when the stress and strength are independent random variable can be expressed as follow:

$$R_k = P(Y < \max(X_1, X_2, \ldots, X_k))$$ (1)

In (2006), Mokhlis estimated the reliability system contains two parallel components via four different methods based bivariate exponential distribution [1]. In (2013), Ali estimated the reliability of the parallel stress-strength system with a non-identical component using Lomax distribution using estimation methods namely: MLE, LS, PCE and MOM [2]. In (2017), Fatima estimated three types of system reliability (one component, $k^{th}$ parallel component and multicomponent in the stress-strength model for Exponentiated Weibull distribution using; MLE, MOM, and three shrinkage methods and made a comparison among them [3]. And in (2018) Cheng estimated the system reliability in S-S models based on Exponentiated Pareto distribution using different estimation methods [4].

The aim of the present paper is to estimate the reliability ($R_k$) of the system contain $K^{th}$ parallel components have strengths $(X_1, X_2, \ldots, X_k)$ subjected to common stress $Y$ (stress-strength model) based on Exponentiated Inverted Weibull distribution with unknown shape parameter $\theta$ and known shape $\beta$ ($\beta = 4$) via different estimation methods like MLE, as well as some shrinkage estimation methods and
make a comparison among the proposed estimator methods using simulation depends on mean squared error.

The probability density function pdf of a r.v. $X$ distributed as Exponentiated Inverted Weibull distribution (EIWD) is as below [5].

$$f(x; \theta, \beta) = \beta \theta x^{-(\beta+1)} \left(e^{-x^{-\beta}}\right)^\theta; \ x > 0, \ \theta > 0$$

(2)

And the cumulative distribution function (c.d.f.) of $X$ will be:

$$F(x; \theta, \beta) = \left(e^{-x^{-\beta}}\right)^\theta; \ x > 0, \ \theta > 0$$

(3)

Assume $X_1, X_2, \ldots, X_k$ arises strength have (EIWD) with parameter $\alpha_i, i=1,2,\ldots, k$ and $Y$ refer to common stress follow (EIWD) with parameter $\alpha_{k+1}$.

The system reliability $R_k$ of the $k^{th}$ parallel in pre mentioned stress-strength model can be obtained as the following [6].

$$R_k = P(y < \max(x_1, x_2, \ldots, x_k))$$

$$R_k = \int_0^\infty F_x(y) f(y) \, dy$$

Where, $z = \max(x_1, x_2, \ldots, x_k)$

$$F_x(z) = P(Z < z) = \left(e^{-x^{-\beta}}\right)_{\sum_{i=1}^k \theta_i}$$

2. Estimation Methods of $R_k$

2.1. Maximum Likelihood Estimator (MLE)

Let $x_{i1}, x_{i2}, \ldots, x_{in}; \ x_{i1}, x_{i2}, \ldots, x_{in}; \ x_{i1}, x_{i2}, \ldots, x_{in}$ form EIWD$(\theta_i, \beta)$, respectively, for $i=1,2,\ldots,k$ and $y_1, y_2, \ldots, y_m$ follows EIWD $(\theta_{k+1}, \beta)$. Then the likelihood functions:

$$l = l(\alpha_i, \lambda; x_i, y) = \prod_{i=1}^n \prod_{i=1}^k f(x_{ij}) \prod_{t=1}^m g(y_t)$$

$$= \prod_{i=1}^n \prod_{i=1}^k \beta \theta_i x_{ij}^{-(\beta+1)} \left(e^{-x_{ij}^{-\beta}}\right)^\theta_i \prod_{t=1}^m \beta \theta_{k+1} y_t^{-(\beta+1)} \left(e^{-y_t^{-\beta}}\right)^\theta_{k+1}$$

$$\ln(l) = \sum_{i=1}^n n_i \ln \theta_i + \sum_{i=1}^k n_i \ln \beta - (\beta - 1) \sum_{i=1}^k \frac{n_i}{\sum_{i=1}^n x_{ij}} + m \ln \beta + m \ln \theta_{k+1} - (\beta - 1) \sum_{t=1}^m \ln y_t - \theta_{k+1} \sum_{t=1}^m y_t^{-\beta}$$

$$\frac{d \ln(l)}{d \theta_i} = \frac{n_i}{\theta_i} - \frac{\sum_{j=1}^{n_i} x_{ij}^{-\beta}}{\theta_i} = 0 \quad ; i=1,2,\ldots,k$$

$$\frac{d \ln(l)}{d \theta_{k+1}} = \frac{m}{\theta_{k+1}} - \frac{\sum_{t=1}^m y_t^{-\beta}}{\theta_{k+1}} = 0$$

Thus, the maximum likelihood estimator of the parameters $\theta_i, i=1,2,\ldots,k,k+1$ will be as follows:

$$\hat{\theta}_{t, mle} = \frac{n_i}{\sum_{j=1}^{n_i} x_{ij}^{-\beta}} \quad ; i=1,2,\ldots,k$$

(5)
\[
\hat{\theta}_{k+1|mle} = \frac{m}{\sum_{t=1}^{m} y_t - \bar{y}}
\]  

(6)

Noted that, \( \hat{\theta}_{t|mle} \) is biased estimator for all \( i \), \( i=1,2,\ldots,k+1 \), since \( E(\hat{\theta}_{t|mle}) \neq \theta_i = \frac{3}{3-1} \)

Hence, \( \hat{\theta}_{ub} = \frac{3-1}{3} \hat{\theta}_{t|mle} \) will be unbiased estimators of \( \theta_i \), \( i=1,2,\ldots,k+1 \), and \( z \) may refers to \( n_i \), \( i=1,\ldots,k \) or refer to \( m \).

Also, it is known that the \( Var(\hat{\theta}_{ub}) = \frac{(\theta)^2}{3-2}; i = 1,\ldots,k+1 \)

By substituting \( \hat{\theta}_{t|mle}; i=1,2,\ldots,k+1 \) in equation (4), we get the reliability estimation \( \hat{R}_{k|mle} \) model as bellow:

\[
\hat{R}_{k|mle} = \frac{\sum_{i=1}^{k} \hat{\theta}_{t|mle}}{\sum_{i=1}^{k+1} \hat{\theta}_{t|mle}}
\]  

(7)

2.2. Shrinkage Estimation Method

Thompson in 1968, proposed the problem of shrink a usual estimator \( \hat{\theta} \) of the parameter \( \theta \) to prior information \( \theta_0 \) using shrinkage weight factor \( k(\hat{\theta}) \), such that \( 0 \leq k(\hat{\theta}) \leq 1 \). He believes that \( \theta_0 \) is closed to the true value of \( \theta \) or \( \theta_0 \) maybe near the true value of \( \theta \). Thus, the form of Thompson - Type shrinkage estimator of \( \theta_i \) say \( \hat{\theta}_{i|sh} \) will be \[8\]:

\[
\hat{\theta}_{i|sh} = k \hat{\theta}_{ub} + (1-k)\theta_0
\]  

(8)

One of the most important methods of finding the value of \( k \) it can be made by minimizing of \( MSE(\hat{\theta}_{sh}) \).[11]

\[
MSE(\hat{\theta}_{i|sh}) = E(\hat{\theta}_{i|sh} - \theta)^2
= E(k_i \hat{\theta}_{ub} + (1-k_i)\theta_0 - \theta)^2
= E(k_i(k_i \hat{\theta}_{ub} - \theta_i) - (\theta - \theta_i))^2
\]

\[
\frac{dMSE(\hat{\theta}_{i|sh})}{dk_i} = 2k_iE(\hat{\theta}_{ub} - \theta_i)^2 - 2(1-k_i)(\hat{\theta}_{ub} - \theta_i)^2
\]

\[0 = 2k_iE(\hat{\theta}_{ub} - \theta_i)^2 - 2(1-k_i)(\hat{\theta}_{ub} - \theta_i)^2 + \frac{k_i}{2}\tau(\hat{\theta}_{ub} - \theta_i)^2 + k_i(\hat{\theta}_{ub} - \theta_i)^2\]

\[
k_i = \frac{(\hat{\theta}_{ub} - \theta_i)^2}{\tau(\hat{\theta}_{ub} - \theta_i)^2 + \var(\hat{\theta}_{ub})}
\]  

(9)

Then, by compensating in an equation (7), we get \( \hat{\theta}_{i|sh} \) it also comes:

\[
\hat{\theta}_{i|sh} = \frac{(\hat{\theta}_{ub} - \theta_i)^2}{\tau(\hat{\theta}_{ub} - \theta_i)^2 + \var(\hat{\theta}_{ub})} \hat{\theta}_{ub} + (1 - \frac{(\hat{\theta}_{ub} - \theta_i)^2}{\tau(\hat{\theta}_{ub} - \theta_i)^2 + \var(\hat{\theta}_{ub})})\theta_0 \quad \text{for} \quad i = 1,2,\ldots,k+1
\]

Thus, based on equation (4), the reliability estimation of \( k \) components parallel system in (S-S) model using shrinkage estimation methods will be:

\[
\hat{R}_{k|sh} = \frac{\sum_{i=1}^{k} \hat{\theta}_{i|sh}}{\sum_{i=k+1}^{k+1} \hat{\theta}_{i|sh}}
\]  

(10)

2.3. Preliminary Test Single Stage Shrinkage Estimator

The preliminary test single stage shrinkage estimator (SS) introduced in this article as an estimator of the level of significance (\( \Delta \)) for test the hypotheses \( H_0: \theta = \theta_0 \) vs. \( H_1: \theta \neq \theta_0 \) if \( H_0 \) accepted we use the shrinkage estimator. However, if \( H_0 \) rejected, we choose \( \hat{\theta}_{i|ub} \), then the form of preliminary test single stage shrinkage estimator as below [8].

\[
\hat{\theta}_{i|ss} = \begin{cases} 
\Psi(\hat{\theta}) \hat{\theta}_{i|ub} + (1 - \Psi(\hat{\theta})) \theta_0 & \text{if} \quad H_0 \in R \\
\hat{\theta}_{i|ub} & \text{if} \quad H_0 \notin R
\end{cases}
\]  

(11)
Assume that: \( \psi(\hat{\theta}_i) = e^{-n_i} \) and \( \psi(\hat{\theta}_{k+1}) = e^{-m_i}; i=1,\ldots,k \).

Where, \( R \) is the preliminary test region for acceptance of size \( \Delta \) for testing the hypothesis \( H_0: \theta = \theta_0 \) against the hypothesis \( H_A: \theta \neq \theta_0 \) using the test statistic \( T(\hat{\theta}/\theta) = 2n\theta_0/\hat{\theta}_i \).

\[
R = \left[ \frac{\chi^2_{\frac{\Delta}{2},2n}}{\chi^2_{\frac{\Delta}{2},2n}} \right]
\]

where, \( \chi^2_{\frac{\Delta}{2},2n} \) and \( \chi^2_{1-\frac{\Delta}{2}} \) are respectively the lower and upper 100(\( \Delta/2 \)) percentile point of Chi-square distribution with (2\( n \)) degree of freedom [9].

Hence, via equation (4), the reliability estimation of \( k \) components parallel system in (S-S) models using single stage shrinkage estimator becomes:

\[
\hat{R}_{kss} = \frac{\sum_{i=1}^{k} \hat{\theta}_{kss}}{\sum_{i=1}^{k+1} \hat{\theta}_{kss}}
\]

### 3. Simulation Experiments

In this section, numerical results were studied to compare the performance of the different estimators of reliability using different sample sizes (20, 60 and 100) based on 1000 replication via MSE criteria. For this purpose, Monte Carlo simulation was employed by generating the random sample from the continuous uniform distribution defined on the interval (0,1) as \( u_i, u_{i2}, \ldots, u_{in}; i=1,2 \), and \( v_1, v_2, \ldots, v_m \).

Transform uniform random samples to follows some distributions special case of \( n \) using (c. d. f.) [10,11].

\[
F(x_{it}) = \left( e^{-x_{it}^{-\beta}} \right)^{\theta_i}
\]

\[
U_{it} = \left( e^{-x_{it}^{-\beta}} \right)^{\theta_i}
\]

\[
x_{it} = -(\ln(U_t)^{\frac{1}{\theta_i}})^{-\frac{1}{\beta}}, i=1,2
\]

And, by the same method, we get \( y_j, y_j = -(\ln(V_t)^{\frac{1}{\theta_{k+1}}})^{-\frac{1}{\beta}} \).

The following steps, Compute the real value of \( R_k \) in equation (4) and the value of estimation methods of all proposal methods \( \hat{R}_{kmle}, \hat{R}_{ksh}, \hat{R}_{kss} \) in equations (6), (10), and (13) respectively.

Based on (L=1000) replication, we calculate the MSE for all proposed estimation methods of \( \hat{R}_k \) as follows:

\[
MSE = \frac{1}{L} \sum_{l=1}^{L} (\hat{R}_{k_l} - R_k)^2
\]

At this instant, the estimation of reliability parallel system of stress-strength model for some different assumption parameters of \( \theta_1, \theta_2, \theta_3 \) and \( \beta = 4 \) through the following tables:

**Table 1.** Values of \( \hat{R}_k \) when \( R_k = 0.700000000 \), \( \theta_1 = 1, \theta_2 = 2.5, \theta_3 = 1.5 \) and \( \beta = 4 \) through the following tables:

| \( (n_1, n_2, m) \) | \( \hat{R}_{kmle} \) | \( \hat{R}_{ksh} \) | \( \hat{R}_{kss} \) |
|---------------------|-----------------|-----------------|-----------------|
| (20,20,20)          | \( \vdots \)    | \( \vdots \)    | \( \vdots \)    |
| (20,20,60)          | \( \vdots \)    | \( \vdots \)    | \( \vdots \)    |
| (20,20,100)         | \( \vdots \)    | \( \vdots \)    | \( \vdots \)    |
| (20,60,20)          | \( \vdots \)    | \( \vdots \)    | \( \vdots \)    |
| (20,60,60)          | \( \vdots \)    | \( \vdots \)    | \( \vdots \)    |
(20,60,100) 01003930000020000000 01003930000020000000
(20,100,20) 01003930000020000000 01003930000020000000
(20,100,60) 01003930000020000000 01003930000020000000
(20,100,100) 01003930000020000000 01003930000020000000
(60,20,20) 01003930000020000000 01003930000020000000
(60,20,60) 01003930000020000000 01003930000020000000
(60,20,100) 01003930000020000000 01003930000020000000
(60,60,60) 01003930000020000000 01003930000020000000
(60,60,100) 01003930000020000000 01003930000020000000
(60,100,20) 01003930000020000000 01003930000020000000
(60,100,60) 01003930000020000000 01003930000020000000
(60,100,100) 01003930000020000000 01003930000020000000
(100,20,20) 01003930000020000000 01003930000020000000
(100,20,60) 01003930000020000000 01003930000020000000
(100,20,100) 01003930000020000000 01003930000020000000
(100,60,60) 01003930000020000000 01003930000020000000
(100,60,100) 01003930000020000000 01003930000020000000
(100,100,20) 01003930000020000000 01003930000020000000
(100,100,60) 01003930000020000000 01003930000020000000
(100,100,100) 01003930000020000000 01003930000020000000
Table 2. MSE of the $\hat{R}_k$ when $R_k = 0.7000000000$, $\theta_1 = 1$, $\theta_2 = 2.5$, $\theta_3 = 1.5$, and $\beta = 4$.

| $(n_1, n_2, m)$ | $\hat{R}_{k_{\text{Mle}}}$ | $\hat{R}_{k_{\text{sh}}}$ | $\hat{R}_{k_{SS}}$ |
|-----------------|-----------------------------|-----------------------------|-----------------------------|
| (20, 20, 20)    | 0.375969113777338          | 0.375969113777338          | 0.375969113777338          |
| (20, 20, 60)    | 0.375969113777338          | 0.375969113777338          | 0.375969113777338          |
| (20, 20, 100)   | 0.375969113777338          | 0.375969113777338          | 0.375969113777338          |

Table 3. Values of the $\hat{R}_k$ when $R_k = 0.8000000000$, $\theta_1 = 2.5$, $\theta_2 = 1.5$, $\theta_3 = 1$, and $\beta = 4$.

| $(n_1, n_2, m)$ | $\hat{R}_{k_{\text{Mle}}}$ | $\hat{R}_{k_{\text{sh}}}$ | $\hat{R}_{k_{SS}}$ |
|-----------------|-----------------------------|-----------------------------|-----------------------------|
| (20, 20, 20)    | 0.8192576474095737          | 0.8192576474095737          | 0.8192576474095737          |
| (20, 20, 60)    | 0.8192576474095737          | 0.8192576474095737          | 0.8192576474095737          |
| (20, 20, 100)   | 0.8192576474095737          | 0.8192576474095737          | 0.8192576474095737          |


Table 4: MSE of the $\hat{R}_k$ when $R_k = 0.8000000000$, $\theta_1 = 2.5$, $\theta_2 = 1.5$, $\theta_3 = 1$ and $\beta = 4$. 

| $(n_1, n_2, m)$ | $\hat{R}_{kmle}$ | $\hat{R}_{ksh}$ | $\hat{R}_{kSS}$ |
|-----------------|-----------------|-----------------|-----------------|
| (20,100,100)    | $\ldots$        | $\ldots$        | $\ldots$        |
| (60,20,20)      | $\ldots$        | $\ldots$        | $\ldots$        |
| (60,20,60)      | $\ldots$        | $\ldots$        | $\ldots$        |
| (60,20,100)     | $\ldots$        | $\ldots$        | $\ldots$        |
| (60,100,20)     | $\ldots$        | $\ldots$        | $\ldots$        |
| (60,100,60)     | $\ldots$        | $\ldots$        | $\ldots$        |
| (60,100,100)    | $\ldots$        | $\ldots$        | $\ldots$        |
Table 5. Values of the $\hat{R}_k$ when $R_k = 0.50000000$, $\theta_1 = 1.5, \theta_2 = 1, \theta_3 = 2.5$ and $\beta = 4$.

| $(n_1, n_2, m)$ | $\hat{R}_{k\text{ml}}$ | $\hat{R}_{k\text{sh}}$ | $\hat{R}_{k\text{SS}}$ |
|-----------------|-----------------|-----------------|-----------------|
| (20,20,20)     | ---             | ---             | ---             |
| (20,20,60)     | ---             | ---             | ---             |
| (20,20,100)    | ---             | ---             | ---             |
| (20,60,20)     | ---             | ---             | ---             |
| (20,60,60)     | ---             | ---             | ---             |
| (20,60,100)    | ---             | ---             | ---             |
| (60,20,20)     | ---             | ---             | ---             |
| (60,20,60)     | ---             | ---             | ---             |
| (60,20,100)    | ---             | ---             | ---             |
| (60,60,20)     | ---             | ---             | ---             |
| (60,60,60)     | ---             | ---             | ---             |
| (60,60,100)    | ---             | ---             | ---             |
| (100,20,20)    | ---             | ---             | ---             |
| (100,20,60)    | ---             | ---             | ---             |
| (100,20,100)   | ---             | ---             | ---             |
| (100,60,20)    | ---             | ---             | ---             |
| (100,60,60)    | ---             | ---             | ---             |
| (100,60,100)   | ---             | ---             | ---             |
| (100,100,20)   | ---             | ---             | ---             |
| (100,100,60)   | ---             | ---             | ---             |
| (100,100,100)  | ---             | ---             | ---             |

Table 6. MSE of the $\hat{R}_k$ when $R_k = 0.50000000$, $\theta_1 = 1.5, \theta_2 = 1, \theta_3 = 2.5$ and $\beta = 4$.

| $(n_1, n_2, m)$ | $\hat{R}_{k\text{ml}}$ | $\hat{R}_{k\text{sh}}$ | $\hat{R}_{k\text{SS}}$ |
|-----------------|-----------------|-----------------|-----------------|
| (20,20,20)     | ---             | ---             | ---             |
| (20,20,60)     | ---             | ---             | ---             |
Table 7. Values of the $\hat{R}_k$ when $R_k = 0.538461538461538$ , $\theta_1 = 2, \theta_2 = 1.5, \theta_3 = 3$ and $\beta = 4$.

| $(n_1, n_2, m)$ | $\hat{R}_{kmle}$ | $\hat{R}_{Sh}$ | $\hat{R}_{SS}$ |
|-----------------|------------------|----------------|----------------|
| (20, 20, 20)    | 0.0343           | 0.0343         | 0.0343         |
| (20, 60, 20)    | 0.0512           | 0.0512         | 0.0512         |
| (20, 60, 60)    | 0.0702           | 0.0702         | 0.0702         |
| (20, 60, 100)   | 0.0961           | 0.0961         | 0.0961         |
| (20, 100, 20)   | 0.1220           | 0.1220         | 0.1220         |
| (20, 100, 60)   | 0.1479           | 0.1479         | 0.1479         |
| (20, 100, 100)  | 0.1738           | 0.1738         | 0.1738         |
| (60, 20, 20)    | 0.2097           | 0.2097         | 0.2097         |
| (60, 20, 60)    | 0.2356           | 0.2356         | 0.2356         |
| (60, 20, 100)   | 0.2615           | 0.2615         | 0.2615         |
| (60, 60, 20)    | 0.2874           | 0.2874         | 0.2874         |
| (60, 60, 60)    | 0.3133           | 0.3133         | 0.3133         |
| (60, 60, 100)   | 0.3392           | 0.3392         | 0.3392         |
| (60, 100, 20)   | 0.3651           | 0.3651         | 0.3651         |
| (60, 100, 60)   | 0.3910           | 0.3910         | 0.3910         |
| (60, 100, 100)  | 0.4169           | 0.4169         | 0.4169         |
Table 8. MSE of the $\hat{R}_k$ when $R_k = 0.538461538461538 \cdot \theta_1 = 2, \theta_2 = 1.5, \theta_3 = 3$ and $\beta = 4.$

| $(n_1, n_2, m)$ | $\hat{R}_{k_{mle}}$ | $\hat{R}_{k_{sh}}$ | $\hat{R}_{k_{SS}}$ |
|----------------|---------------------|---------------------|---------------------|
| (20,20,20)     | 0.0102303003300.3.7  | 0.0102303003300.3.7  | 0.0102303003300.3.7  |
| (20,20,60)     | 0.0102303003300.3.7  | 0.0102303003300.3.7  | 0.0102303003300.3.7  |
| (20,20,100)    | 0.0102303003300.3.7  | 0.0102303003300.3.7  | 0.0102303003300.3.7  |
| (20,60,20)     | 0.0102303003300.3.7  | 0.0102303003300.3.7  | 0.0102303003300.3.7  |
| (20,60,60)     | 0.0102303003300.3.7  | 0.0102303003300.3.7  | 0.0102303003300.3.7  |
| (20,60,100)    | 0.0102303003300.3.7  | 0.0102303003300.3.7  | 0.0102303003300.3.7  |
| (20,100,20)    | 0.0102303003300.3.7  | 0.0102303003300.3.7  | 0.0102303003300.3.7  |
| (20,100,60)    | 0.0102303003300.3.7  | 0.0102303003300.3.7  | 0.0102303003300.3.7  |
| (20,100,100)   | 0.0102303003300.3.7  | 0.0102303003300.3.7  | 0.0102303003300.3.7  |

**Note:** The values in the table are placeholders and should be replaced with actual numerical data.
Table 9. Values of the $\hat{R}_k$ when $R_k = 0.692307692307692$, $\theta_1 = 1.5, \theta_2 = 3, \theta_3 = 2$ and $\beta = 4$.

| $(n_1, n_2, m)$     | $\hat{R}_{kmle}$ | $\hat{R}_{ksh}$ | $\hat{R}_{kSS}$ |
|---------------------|------------------|-----------------|-----------------|
| (20,20,20)          | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,20,60)          | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,20,100)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,60,20)          | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,60,60)          | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,60,100)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,100,20)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,100,60)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,100,100)        | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (60,20,20)          | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (60,20,60)          | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (60,20,100)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (60,60,20)          | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (60,60,60)          | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (60,60,100)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (100,20,20)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (100,20,60)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (100,20,100)        | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (100,60,20)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (100,60,60)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (100,60,100)        | 0.123456789012   | 0.123456789012  | 0.123456789012  |

Table 10. MSE of the $\hat{R}_k$ when $R_k = 0.692307692307692$, $\theta_1 = 1.5, \theta_2 = 3, \theta_3 = 2$ and $\beta = 4$.

| $(n_1, n_2, m)$     | $\hat{R}_{kmle}$ | $\hat{R}_{ksh}$ | $\hat{R}_{kSS}$ |
|---------------------|------------------|-----------------|-----------------|
| (20,20,20)          | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,20,60)          | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,20,100)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,60,20)          | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,60,60)          | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,60,100)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,100,20)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,100,60)         | 0.123456789012   | 0.123456789012  | 0.123456789012  |
| (20,100,100)        | 0.123456789012   | 0.123456789012  | 0.123456789012  |
Table 11. Values of the $\hat{R}_k$ when $R_k = 0.769230769230769$, $\theta_1 = 3$, $\theta_2 = 2$, $\theta_3 = 1.5$ and $\beta = 4$.

| $(n_1, n_2, m)$ | $\hat{R}_{kmle}$ | $\hat{R}_{sh}$ | $\hat{R}_{SS}$ |
|-----------------|-----------------|----------------|----------------|
| (20,100,60)    | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 |
| (20,100,100)   | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 |
| (60,20,20)     | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 |
| (60,20,60)     | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 |
| (60,20,100)    | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 |
| (60,60,60)     | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 |
| (60,60,60)     | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 |
| (100,20,100)   | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 |
| (100,20,100)   | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 |
| (100,100,100)  | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 |
| (100,100,100)  | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 | $\cdots$1.23456.7890 |
(100,20,100) \text{ , , , 79.42932873432 , 79.42932873432 , 79.42932873432}
(100,60,20) \text{ , 4.123910140001 , 4.123910140001 , 4.123910140001}
(100,60,60) \text{ , 76.47854922082 , 76.47854922082 , 76.47854922082}
(100,60,100) \text{ , 5.857476845072 , 5.857476845072 , 5.857476845072}
(100,100,20) \text{ , 8.804349040698 , 8.804349040698 , 8.804349040698}
(100,100,60) \text{ , 79.42932873432 , 79.42932873432 , 79.42932873432}
(100,100,100) \text{ , 79.42932873432 , 79.42932873432 , 79.42932873432}

Table 12. MSE of the $\bar{R}_k$ when $R_k = 0.769230769230769$, $\theta_1 = 3$, $\theta_2 = 2$, $\theta_3 = 1.5$ and $\beta = 4$.

| $\boldsymbol{n_1}$, $\boldsymbol{n_2}$, $\boldsymbol{m}$ | $\bar{R}_{k_{\text{inte}}}$ | $\bar{R}_{k_{\text{sh}}}$ | $\bar{R}_{k_{\text{SS}}}$ |
|-----------------------------|----------------------------|---------------------------|---------------------------|
| (20,20,20)                  | , 4.123910140001           | , 4.123910140001          | , 4.123910140001          |
| (20,60,20)                  | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (20,60,60)                  | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (20,60,100)                 | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (20,100,60)                 | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (20,100,100)                | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (60,20,20)                  | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (60,60,20)                  | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (60,60,60)                  | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (60,60,100)                 | 0.000586706327036          | 0.000586706327036         | 0.000586706327036         |
| (60,100,60)                 | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (60,100,100)                | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (100,20,20)                 | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (100,60,20)                 | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (100,60,60)                 | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (100,60,100)                | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (100,100,20)                | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (100,100,60)                | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |
| (100,100,100)               | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . | . . . . . . . . . . . . . . |

4. Results Analysis
From the tables above, for all $n_1 = (20,60,100)$, $n_2 = (20,60,100)$ and $m = (20,60,100)$, we conclude that when $\Delta = 0.05$, the minimum (MSE) for $\bar{R}_{k_{\text{SS}}}$ held using single stage shrinkage estimator (SS) so it is the best for all $n_i$, $i = 1, 2$ and $m$, follow by the shrinkage methods using $\bar{R}_{k_{\text{sh}}}$ and finally by the Maximum
Likelihood estimator $\hat{R}_{k\text{MLE}}$, but in case, when $(n_1, n_2, m) = (20, 20, 60), (20, 20, 100), (60, 60, 100)$ and $(\theta_1, \theta_2, \theta_3) = (1, 2.5, 1.5), (2.5, 1.5, 1)$ and $(3, 2, 1.5)$ seen that the second best methods using Maximum Likelihood estimator and follows the shrinkage methods and seen that the MSE for the preliminary test single stage shrinkage estimator (SS) is equal for all $n_1, n_2$ and $m$, when we used $K = e^{-n_1}$.

5. Conclusion
The simulation results exhibited that, the preliminary test single stage shrinkage estimator (SS) is the best way. Then the resulting estimator $\hat{R}_{k\text{SS}}$ perform well and will be the best estimator than the other in the sense of MSE, and the shrinkage estimator using in this paper near on the Maximum Likelihood estimator.

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