HOMOLOGICAL RESONANCES FOR HAMILTONIAN
Diffeomorphisms AND REEB FLOWS

VIKTOR L. GINZBURG AND ELY KERMAN

Abstract. We show that whenever a Hamiltonian diffeomorphism or a Reeb
flow has a finite number of periodic orbits, the mean indices of these orbits
must satisfy a resonance relation, provided that the ambient manifold meets
some natural requirements. In the case of Reeb flows, this leads to simple
expressions (purely in terms of the mean indices) for the mean Euler charac-
teristics. These are invariants of the underlying contact structure which are
capable of distinguishing some contact structures that are homotopic but not
diffeomorphic.

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1. Introduction and main results

1.1. Introduction. In this paper, we establish new restrictions on Hamiltonian
diffeomorphisms and Reeb flows which have only finitely many periodic orbits.
While these dynamical systems are rare, there are many natural examples, such
as irrational rotations of the two-dimensional sphere and Reeb flows on irrational
ellipsoids. Moreover, these systems serve as important counterpoints to cases where
one can prove the existence of infinitely many periodic orbits, see for example
[Ek, EH, FH, Hi, Gi, GG2, GG3, SZ, Vi]. Our main theorems establish resonance
relations for the mean indices of the periodic orbits of these systems when the
ambient manifolds meet some additional requirements.

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For a Hamiltonian diffeomorphism $\varphi$, the additional requirement on the ambient symplectic manifold is that $N \geq n + 1$, where $N$ is the minimal Chern number and $n$ is half the dimension. A resonance relation in this case is simply a linear relation, with integer coefficients, between the mean indices $\Delta_i$, viewed as elements of $\mathbb{R}/2N\mathbb{Z}$. The existence of these relations can be established essentially by carrying out an argument from [SZ] modulo $2N$, cf. [BCE]. The specific form of the resonance relations established depends on $\varphi$, although conjecturally the relation $\sum \Delta_i = 0 \mod 2N$ is always satisfied.

For Reeb flows, we show that certain sums of the reciprocal mean indices are equal to the (positive/negative) mean Euler characteristic, an invariant of the contact structure defined via cylindrical contact homology, when it itself is defined. (This relation generalizes the one discovered by Ekeland and Hofer [Ek, EH] and Viterbo [Vi] for convex and star-shaped hypersurfaces in $\mathbb{R}^{2n}$, which served as the main motivation for the present work.) One can view these resonance relations as new expressions for the mean Euler characteristics written purely in the terms of the mean indices. As is shown below in Example 1.9, these invariants can be used to distinguish some contact structures which are homotopic but not diffeomorphic, such as those distinguished by Ustilovsky in [U] using cylindrical contact homology.

One forthcoming application of our results is the $C^\infty$-generic existence of infinitely many periodic orbits for Hamiltonian diffeomorphisms or Reeb flows established under the same hypotheses as the resonance relations; see [GG4]. For the Reeb flows, these results generalize the $C^\infty$-generic existence of infinitely many closed characteristics on convex hypersurfaces in $\mathbb{R}^{2n}$ (see [Ek]) and the $C^\infty$-generic existence of infinitely many closed geodesics (see [Ra1, Ra2]).

1.2. Resonances for Hamiltonian diffeomorphisms. Let $(M^{2n}, \omega)$ be a closed symplectic manifold, which throughout this paper is assumed to be weakly monotone; see, e.g., [HS] or [MS2] for the definition. Denote by $N$ the minimal Chern number of $(M, \omega)$, i.e., $N$ is the positive generator of the subgroup $\langle c_1(TM), \pi_2(M) \rangle \subset \mathbb{Z}$. (When $c_1(TM) | \pi_2(M) = 0$, we set $N = \infty$.) Recall that $(M, \omega)$ is said to be rational if the group $\langle \omega, \pi_2(M) \rangle \subset \mathbb{R}$ is discrete.

A Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ which is one-periodic in time, determines a vector field $X_H$ on $M$ via Hamilton’s equation $i_{X_H} \omega = -dH$. Let $\varphi = \varphi_H$ be the Hamiltonian diffeomorphism of $M$ given by the time-1 flow of $X_H$. Recall that there is a one-to-one correspondence between $k$-periodic points of $\varphi$ and $k$-periodic orbits of $H$. In this paper, we restrict our attention exclusively to periodic points of $\varphi$ such that the corresponding periodic orbits of $H$ are contractible. (One can show that contractibility of the orbit $\varphi^t(x)$ of $H$ through a periodic point $x \in M$ is completely determined by $x$ and $\varphi$ and is independent of the choice of generating Hamiltonian $H$.) To such a periodic point $x$, we associate the mean index $\Delta(x)$, which is viewed here as a point in $\mathbb{R}/2N\mathbb{Z}$, and hence is independent of the choice of capping of the orbit. The mean index measures the sum of rotations of the eigenvalues on the unit circle of the linearized flow $d\varphi_H^t$ along $x$. The reader is referred to [SZ] for the definition of the mean index $\Delta$; see also, e.g., [GG1] for a detailed discussion. We only mention here that

$$|\Delta(x) - \hat{\mu}_{CZ}(x)| \leq n,$$
where \( \hat{\mu}_{CZ}(x) \) is the Conley–Zehnder index of \( x \), viewed as a point in \( \mathbb{R}/2N\mathbb{Z} \) and given a nonstandard normalization such that for a critical point \( x \) of a \( C^2 \)-small Morse function one has \( \hat{\mu}_{CZ}(x) = \mu_{Morse}(x) - n \mod 2N \).

A Hamiltonian diffeomorphism \( \varphi \) is said to be perfect if it has finitely many (contractible) periodic points, and every periodic point of \( \varphi \) is a fixed point. Let \( \Delta_1, \ldots, \Delta_m \) be the collection of the mean indices of the fixed points of a perfect Hamiltonian diffeomorphism \( \varphi \) with exactly \( m \) fixed points. A resonance or resonance relation is a vector \( \vec{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m \) such that

\[
a_1 \Delta_1 + \ldots + a_m \Delta_m = 0 \mod 2N.
\]

It is clear that resonances form a free abelian group \( R = R(\varphi) \subset \mathbb{Z}^m \).

**Theorem 1.1.** Assume that \( n+1 \leq N < \infty \).

(i) Then \( R \neq 0 \), i.e., the mean indices \( \Delta_i \) satisfy at least one non-trivial resonance relation.

(ii) Assume in addition that there is only one resonance, i.e., \( \text{rk} R = 1 \), and let \( \vec{a} = (a_1, \ldots, a_m) \) be a generator of \( R \) with at least one positive component. Then all components of \( \vec{a} \) are non-negative, i.e., \( a_i \geq 0 \) for all \( i \), and

\[
\sum a_i \leq \frac{N}{N-n}.
\]

(iii) Furthermore, assume that \((M, \omega)\) is rational. Then assertion (i) holds when only irrational mean indices are considered (i.e., the irrational mean indices satisfy a non-trivial resonance relation) and assertion (ii) holds when only non-zero mean indices are considered.

We require \((M, \omega)\) to be weakly monotone here only for the sake of simplicity: this condition can be eliminated by utilizing the machinery of virtual cycles. Likewise, the hypothesis that \((M, \omega)\) is rational in (iii) is purely technical and probably unnecessary. However, the proof of (iii) relies on a result from [GG3] which has so far been established only for rational manifolds although one can expect it to hold without this requirement; see [GG3, Remark 1.19]. When \( N = \infty \), i.e., \( c_1(TM) \mid_{\pi_2(M)} = 0 \), perfect Hamiltonian diffeomorphisms probably do not exist and the assertion of the theorem is void. For instance, if \((M, \omega)\) is rational and \( N = \infty \), every Hamiltonian diffeomorphism has infinitely many periodic points; see [GG3].

We note that every \( \Delta_i \in \mathbb{Q} \) (e.g., \( \Delta_i = 0 \)) automatically gives rise to an infinite cyclic subgroup of resonances. Thus, assertion (iii) is much more precise than (i) or (ii) in the presence of rational or zero mean indices.

Finally we observe that the condition that \( \varphi \) is perfect can be relaxed and replaced by the assumption that \( \varphi \) has finitely many periodic points. Indeed, in this case, suitable iterations \( \varphi^k \) are perfect. Applying Theorem 1.1 to such a \( \varphi^k \), we then obtain resonance relations involving (appropriately normalized) mean indices of all periodic points of \( \varphi \).

**Example 1.2.** Let \( \varphi \) be the Hamiltonian diffeomorphism of \( \mathbb{CP}^n \) generated by a quadratic Hamiltonian \( H(z) = \pi(\lambda_0|z_0|^2 + \ldots + \lambda_n|z_n|^2) \), where the coefficients \( \lambda_0, \ldots, \lambda_n \) are all distinct. (Here, we have identified \( \mathbb{CP}^n \) with the quotient of the unit sphere in \( \mathbb{C}^{n+1} \). Recall also that \( N = n+1 \) for \( \mathbb{CP}^n \).) Then, the Hamiltonian diffeomorphism \( \varphi = \varphi_H \) is perfect and has exactly \( n+1 \) fixed points (the coordinate
The mean indices are
\[ \Delta_i = \sum_j \lambda_j - (n + 1)\lambda_i, \]
where now \( i = 0, \ldots, n \). Thus, \( \sum \Delta_i = 0 \) and this is the only resonance relation for a generic choice of the coefficients: the image of the map \((\lambda_0, \ldots, \lambda_n) \mapsto (\Delta_0, \ldots, \Delta_n)\) is the hyperplane \( \sum \Delta_i = 0 \).

More generally, we have

**Example 1.3.** Suppose that \((M, \omega)\) is equipped with a Hamiltonian torus action with isolated fixed points; see, e.g., [GGK, MS1] for the definition and further details. A generic element of the torus gives rise to a perfect Hamiltonian diffeomorphism \(\varphi\) of \((M, \omega)\) whose fixed points are exactly the fixed points of the torus action. One can show that in this case the mean indices again satisfy the resonance relation \( \sum \Delta_i = 0 \). (The authors are grateful to Yael Karshon for a proof of this fact; [Ka].)

Examples of symplectic manifolds which admit such torus actions include the majority of coadjoint orbits of compact Lie groups. One can also construct new examples from a given one by equivariantly blowing-up the symplectic manifold at its fixed points. The resulting symplectic manifold always inherits a Hamiltonian torus action and, in many instances, this action also has isolated fixed points.

These examples suggest that, in the setting of Theorem 1.1, the mean indices always satisfy the resonance relation \( \sum \Delta_i = 0 \). The next result can be viewed as a preliminary step towards proving this conjecture.

**Corollary 1.4.** Let \(\varphi\) be a perfect Hamiltonian diffeomorphism of \(\mathbb{C}P^n\) such that there is only one resonance, i.e., \(\text{rk } \mathcal{R} = 1\). Denote by \(\vec{a}\) a generator of \(\mathcal{R}\) as described in statement (ii) of Theorem 1.1, and assume that \(a_i \neq 0\) for all \(i\). Then \(\varphi\) has exactly \(n + 1\) fixed points and \(\vec{a} = (1, \ldots, 1)\), i.e., the mean indices satisfy the resonance relation \( \sum \Delta_i = 0 \).

**Proof.** By the Arnold conjecture for \(\mathbb{C}P^n\), we have \(m \geq n + 1\), see [Fo, FW] and also [Fl, Sc]. By (ii), \(a_i \neq 0\) means that \(a_i \geq 1\). Hence, by (ii) again, \(m = n + 1\) and \(a_i = 1\) for all \(i\) since \(N = n + 1\) and \(\sum a_i \leq N/(N - n) = n + 1\). \(\square\)

Conjecturally, any Hamiltonian diffeomorphism of \(\mathbb{C}P^n\) with more than \(n + 1\) fixed points has infinitely many periodic points. (For \(n = 1\) this fact is established in [FH].) Corollary 1.4 implies that this is indeed the case, provided that the mean indices satisfy exactly one resonance relation (i.e., \(\text{rk } \mathcal{R} = 1\)) and all components of the resonance relation are non-zero. (We emphasize that by Theorem 1.1, \(\text{rk } \mathcal{R} \geq 1\).)

**Remark 1.5.** The resonances considered here are not the only numerical constraints on the fixed points of a perfect Hamiltonian diffeomorphism \(\varphi\): \(M \to M\). Relations of a different type, involving both the mean indices and action values, are established in [GG3] when \((M, \omega)\) is either monotone or negative monotone. For instance, it is proved there that the so-called augmented action takes the same value on all periodic points of \(\varphi\): \(\mathbb{C}P^n \to \mathbb{C}P^n\) whenever \(\varphi\) has exactly \(n + 1\) periodic points.

Note also that a perfect Hamiltonian diffeomorphism need not be associated with a Hamiltonian torus action as are the Hamiltonian diffeomorphisms in Examples 1.2 and 1.3. For instance, there exists a Hamiltonian perturbation \(\varphi\) of an irrational
rotation of $S^2$ with exactly three ergodic invariant measures: the Lebesgue measure and the two measures corresponding to the fixed points of $\varphi$; [AK, FK]. Clearly, $\varphi$ is perfect and not conjugate to a rotation.

Remark 1.6. As is immediately clear from the proof, one can replace in Theorem 1.1 the collection $\Delta_1, \ldots, \Delta_m$ of the mean indices of the fixed points of $\varphi$ by the set of all distinct mean indices. This is a refinement of the theorem, for an equality of two mean indices is trivially a resonance relation. Note also that, as a consequence of this refinement, all mean indices are distinct whenever $\text{rk} R = 1$.

1.3. Resonances for Reeb flows. Let $(W^{2n-1}, \xi)$ be a closed contact manifold such that the cylindrical contact homology $HC_\ast(W, \alpha)$ is defined. More specifically, we require $(W, \xi)$ to admit a contact form $\alpha$ such that

(CF1) all periodic orbits of the Reeb flow of $\alpha$ are non-degenerate, and
(CF2) the Reeb flow of $\alpha$ has no contractible periodic orbits $x$ with $|x| = \pm 1$ or $0$.

Here, $|x| = \mu_{cz}(x) + n - 3$, where $\mu_{cz}(x)$ stands for the Conley–Zehnder index of $x$ (with its standard normalization). For the sake of simplicity, we also assume that $c_1(\xi) = 0$. Then $HC_\ast(W, \xi)$ is the homology of a complex $CC_\ast(W, \alpha)$ which is generated (over a fixed ground field, say, $\mathbb{Z}_2$) by certain periodic orbits of the Reeb flow, and is graded via $| \cdot |$. To be more precise, the generators of $CC_\ast(W, \alpha)$ are all iterations of good Reeb orbits and odd iterations of bad Reeb orbits (See the definitions below.) The homology $HC_\ast(W, \xi)$ is independent of $\alpha$ as long as $\alpha$ meets requirements (CF1) and (CF2). The exact nature of the differential on $CC_\ast(W, \alpha)$ is inessential for our considerations. We refer the reader to, for instance, [Bo, BO, El] and the references therein for a more detailed discussion of contact homology.

Furthermore, assume that

(CH) there are two integers $l_+$ and $l_-$, such that the space $HC_l(W, \xi)$ is finite-dimensional for $l \geq l_+$ and $l \leq l_-$.

In the examples considered here, the contact homology is finite dimensional in all degrees and this condition is automatically met. By analogy with the constructions from [EH, Vi], we set

$$\chi^{\pm}(W, \xi) = \lim_{N \to \infty} \frac{1}{N} \sum_{l = l_{\pm}}^{N} (-1)^l \dim HC_{\pm l}(W, \xi),$$

provided that the limits exist. Clearly, when $HC_l(W, \xi)$ is finite–dimensional for all $l$, we have

$$\frac{\chi^+(W, \xi) + \chi^-(W, \xi)}{2} = \chi(W, \xi) := \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{l = -N}^{N} (-1)^l \dim HC_l(W, \xi).$$

We call $\chi^{\pm}(W, \xi)$ the positive/negative mean Euler characteristic of $\xi$. Likewise, we call $\chi(W, \xi)$ the mean Euler characteristic of $\xi$. (This invariant is also considered in [VK, Section 11.1.3].)

In what follows, we denote by $x^k$ the $k$th iteration of a periodic orbit $x$ of the Reeb flow of $\alpha$ on $W$. Recall that a simple periodic orbit $x$ is called bad if the linearized Poincaré return map along $x$ has an odd number of real eigenvalues strictly smaller than $-1$. Otherwise, the orbit is said to be good. (This terminology differs slightly from the standard usage, cf. [Bo, BO].) When the orbit $x$ is good,
the parity of the Conley–Zehnder indices \( \mu_{\text{CZ}}(x^k) \) is independent of \( k \); if \( x \) is bad, then the parity of \( \mu_{\text{CZ}}(x^k) \) depends on the parity of \( k \).

To proceed, we now assume that

(CF3) the Reeb flow of \( \alpha \) has finitely many simple periodic orbits.

In contrast with (CF1) and (CH) and even (CF2), this is a very strong restriction on \( \alpha \). Denote the good simple periodic orbits of the Reeb flow by \( x_i \) and the bad simple periodic orbits by \( y_i \). Then, \( \text{CC}_n(W, \alpha) \) is generated by the \( x_i^k \), for all \( k \), together with the \( y_i^k \) for \( k \) odd. Whenever (CF3) holds, condition (CH) is automatically satisfied with \( l_- = -2 \) and \( l_+ = 2n - 4 \). Moreover, in this case the spaces \( \text{CC}_l(W, \alpha) \) are finite–dimensional. (This fact can, for instance, be extracted from the proof of Theorem 1.7; see (3.1) and (3.2).) Likewise, all spaces \( \text{CC}_l(W, \alpha) \) (and hence \( \text{HC}_l(W, \xi) \)) are finite–dimensional, provided that all of the orbits \( x_i \) and \( y_i \) have non-zero mean indices. We denote the mean index of an orbit \( x \) by \( \Delta(x) \) and set \( \sigma(x) = (-1)^{|x|} = -(-1)^n(-1)^{\mu_{\text{CZ}}(x)} \). In other words, \( \sigma(x) \) is, up to the factor \(-(-1)^n\), the topological index of the orbit \( x \) or, more precisely, of the Poincaré return map of \( x \).

**Theorem 1.7.** Assume that \( \alpha \) satisfies conditions (CF1)–(CF3). Then the limits in (1.1) exist and

\[
\sum_{x}^\pm \frac{\sigma(x)}{\Delta(x)} + \frac{1}{2} \sum_{y}^\pm \frac{\sigma(y)}{\Delta(y)} = \chi^\pm(W, \xi),
\]

where \( \sum^+ \) (respectively, \( \sum^- \)) stands for the sum over all orbits with positive (respectively, negative) mean index.

This theorem will be proved in Section 3. Here we only mention that the specific nature of the differential on the complex \( \text{CC}_*(W, \xi) \) plays no role in the argument. Also note that a similar result holds when the homotopy classes of orbits are restricted to any set of free homotopy classes closed under iterations, provided that (CF1)–(CF3) hold for such orbits. For instance, (1.2) holds when only contractible periodic orbits are taken into account in the calculation of the left-hand side and the definition of \( \chi^\pm \). Also it is worth pointing out that for non-contractible orbits the definitions of the Conley–Zehnder and mean indices involve some additional choices (see, e.g., [Bo]) which effect both the right- and the left-hand side of (1.2).

**Example 1.8.** Let \( \xi_0 \) be the standard contact structure on \( S^{2n-1} \). Then, as is easy to see, \( \chi^-(S^{2n-1}, \xi_0) = 0 \) and \( \chi^+(S^{2n-1}, \xi_0) = 1/2 \). In this case, the resonance relations (1.2) were proved in [Vi]. (The case of a convex hypersurface in \( \mathbb{R}^{2n} \) was originally considered in [Ek, EH].)

By definition (1.1), the mean Euler characteristics \( \chi^\pm(W, \xi) \) are invariants of the contact structure \( \xi \). (Strictly speaking this is true only when \( (W, \xi) \) is equipped with some extra data or \( W \) is simply connected.) Theorem 1.7 implies that, whenever there is a contact form \( \alpha \) for \( \xi \) which satisfies conditions (CF1)–(CF3), these invariants can, in principle, be calculated by purely elementary means (without first calculating the contact homology) via the mean indices of closed Reeb orbits. The following example shows that the mean Euler characteristics can distinguish some non-diffeomorphic contact structures within the same homotopy class.

**Example 1.9.** In [U], Ustilovsky considers a family of contact structures \( \xi_p \) on \( S^{2n-1} \) for odd \( n \) and positive \( p \equiv \pm 1 \mod 8 \). For a fixed \( n \), the contact structures \( \xi_p \) fall
within a finite number of homotopy classes, including the class of the standard structure $\xi_0$. By computing $\text{HC}_*(S^{2n-1}, \xi_p)$, Ustilovsky proves that the structures $\xi_p$ are mutually non-diffeomorphic, and that none of them are diffeomorphic to $\xi_0$.

It follows from [U], that $\xi_p$ can be given by a contact form $\alpha_p$ satisfying conditions (CF1)–(CF3). Furthermore, it is not hard to show (see below or [VK]) that

$$
\chi^+(S^{2n-1}, \xi_p) = \frac{1}{2} \left( \frac{p(n-1) + 1}{p(n-2) + 2} \right)
$$

(1.3)

and $\chi^-(S^{2n-1}, \xi_p) = 0$. The right-hand side of (1.3) is a strictly increasing function of $p > 0$. Hence, the positive mean Euler characteristic distinguishes the structures $\xi_p$ with $p > 0$. Note also that $\chi^+(\xi_p) > \chi^+(S^{2n-1}, \xi_0) = 1/2$ when $p > 1$ and $\chi^+(\xi_1) = 1/2$. In particular, $\chi^+$ distinguishes $\xi_p$ with $p > 1$ from the standard structure $\xi_0$.

Formula (1.3) can be established in two ways, both relying on [U]. The first way is to use the contact form $\alpha_p$, constructed in [U]. The indices of periodic orbits of $\alpha_p$ are determined in [U] and the mean indices can be found in a similar fashion or obtained using the asymptotic formula $\Delta(x) = \lim_{k \to \infty} \mu_{cz}(x^k)/k$; see [SZ]. Then (1.2) is applied to calculate $\chi^+(S^{2n-1}, \xi_p)$. (Note that this calculation becomes even simpler when the Morse–Bott version of (1.2) is used, reducing the left-hand side of (1.2) to just one term for a suitable choice of contact form, see [Es].) Alternatively one can use the definition (1.1) of $\chi^+$ and the calculation of $\dim \text{HC}_*(S^{2n-1}, \xi_p)$ from [U]; see [VK, Section 11.1.3].

Remark 1.10. A different version of contact homology, the linearized contact homology, is defined when $(W, \xi)$ is equipped with a symplectic filling $(M, \omega)$. The chain group for this homology is still described via the closed orbits of a Reeb flow for $\xi$. In particular, it is still generated by all iterations of good orbits and the odd iterations of bad ones. The differential is defined via the augmentation, associated with $(M, \omega)$, on the full contact homology differential algebra; see, e.g., [BO, El]. Thus, the linearized contact homology, in general, depends on $(M, \omega)$.

A key point in this construction is that one does not need to assume condition (CF2) in order to define the linearized contact homology. (When (CF2) is satisfied, the linearized contact homology coincides with the cylindrical contact homology.) Hence, for fillable contact manifolds, Theorem 1.7 holds without this assumption. This follows immediately from the fact that our proof of the theorem makes no use of the specific nature of the differential. One still requires the contact form $\alpha$ to satisfy (CF1) and (CF3), and the filling $(M, \omega)$ is assumed to be such that $\langle \omega, \pi_2(M) \rangle = 0$ and $c_1(TM) = 0$.

Remark 1.11. An argument similar to the proof of Theorem 1.7 also establishes the following “asymptotic Morse inequalities”

$$
\sum_{l=1}^{N} \sum_{i=1}^{\pm} \frac{1}{\Delta(x_i)} + \frac{1}{2} \sum_{l=1}^{N} \sum_{i=1}^{\pm} \frac{1}{\Delta(y_i)} \geq \lim_{N \to \infty} \sup \frac{1}{N} \sum_{l=1}^{N} \dim \text{HC}_n(W, \xi),
$$

provided that $\alpha$ satisfies (CF1)–(CF3) or in the setting of Remark 1.10. A similar inequality (as well as some other relations between mean indices and actions) is proved in [EH] for convex hypersurfaces in $\mathbb{R}^{2n}$.

Remark 1.12. Finally note that the quite restrictive requirement that the Reeb flow is non-degenerate and has finitely many periodic orbits can be relaxed in a variety
of ways. For instance, the Morse–Bott version of Theorem 1.7 is proved in [Es],
generalizing the resonance relations for geodesic flows established in [Ra1].

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2. RESONANCES IN THE HAMILTONIAN CASE

2.1. Resonances and subgroups of the torus. Consider a closed subgroup \( \Gamma \) of
\( T^m = \mathbb{R}^m/\mathbb{Z}^m \) which is topologically generated by an element \( \Delta = (\Delta_1, \ldots, \Delta_m) \in T^m \).
In other words, \( \Gamma \) is the closure of the set \( \{k\Delta \mid k \in \mathbb{N} \} \), which we will call
the orbit of \( \bar{\Delta} \). Note that \( \Gamma \) is a Lie group since it is a closed subgroup of a Lie
group. (We refer the reader to, e.g., [DK, K] for the results on Lie groups and
duality used in this section.) Moreover, the connected component of the identity
\( \Gamma_0 \) in \( \Gamma \) is a torus, for \( \Gamma_0 \) is compact, connected and abelian. Denote by \( \mathcal{R} \) the
group of characters \( T^m \rightarrow S^1 \) which vanish on \( \Gamma \) or equivalently on \( \bar{\Delta} \). Thus, \( \mathcal{R} \)
is a subgroup of the dual group \( \mathbb{Z}^m \) of \( T^m \). We can think of \( \mathcal{R} \) as the set of linear
equations determining \( \Gamma \). In other words, \( \bar{a} \) belongs to \( \mathcal{R} \) if and only if
\[
a_1 \bar{\Delta}_1 + \ldots + a_m \bar{\Delta}_m = 0 \mod 1.
\]
We will refer to \( \mathcal{R} \) as the group of resonances associated to \( \Gamma \). Clearly, \( \Gamma \) is com-
pletely determined by \( \mathcal{R} \). When the role of \( \bar{\Delta} \) or \( \Gamma \) needs to be emphasized, we will
use the notation \( \Gamma(\bar{\Delta}) \) and \( \mathcal{R}(\Gamma) \) or \( \mathcal{R}(\bar{\Delta}) \), etc. Furthermore, we denote by \( \mathcal{R}_0 \supset \mathcal{R} \)
the group of resonances associated to \( \Gamma_0 \).

We will need the following properties of \( \Gamma \) and \( \mathcal{R} \):

- \( \text{codim} \Gamma = \text{rk} \mathcal{R} \);
- \( \Gamma \subset \Gamma' \) iff \( \mathcal{R}(\Gamma) \supset \mathcal{R}(\Gamma') \);
- \( \Gamma/\Gamma_0 \) and \( \mathcal{R}_0/\mathcal{R} \) are finite cyclic groups dual to each other.

Here the second assertion is obvious. To prove the first and the last ones, first note
that \( \Gamma/\Gamma_0 \) is finite and cyclic since \( \Gamma \) is compact and has a dense cyclic subgroup.
Further note that \( \mathcal{R} \) can be identified with the dual group of \( T^m/\Gamma \) and that
the first assertion is clear when \( \Gamma = \Gamma_0 \), i.e., \( \Gamma \) is a torus. Dualizing the exact sequence
\[
0 \rightarrow \Gamma/\Gamma_0 \rightarrow T^m/\Gamma \rightarrow T^m/\mathcal{R} \rightarrow 0
\]
we obtain the exact sequence \( 0 \rightarrow \mathcal{R} \rightarrow \mathcal{R}_0 \rightarrow \mathcal{R}/\mathcal{R} \rightarrow 0 \) (see, e.g., [Ki, Chapter 12]) and the last assertion follows. (The
dual of a finite cyclic group, say \( \mathbb{Z}_k \), is isomorphic to \( \mathbb{Z}_k \).) It also follows that
\( \text{rk} \mathcal{R} = \text{rk} \mathcal{R}_0 = \text{codim} \Gamma_0 = \text{codim} \Gamma \).

2.2. Proof of Theorem 1.1. Let \( \Delta = (\Delta_1, \ldots, \Delta_m) \) where the components \( \Delta_i \in \mathbb{R}/2\mathbb{N}Z \)
are the mean indices of the \( m \) periodic points of the perfect Hamiltonian
diffeomorphism \( \varphi \). Set \( \bar{\Delta} = \Delta/2N \). Then, \( \bar{\Delta} \) belongs to \( T^m \) and we have \( \mathcal{R}(\varphi) = \mathcal{R}(\bar{\Delta}) \). Recalling that \( n < N \), we define \( \Pi \) to be the cube \((n/N, 1)^m \) in \( T^m \).
In other words, \( \Pi \) consists of points \( \theta = (\theta_1, \ldots, \theta_m) \in T^m \) such that \( \theta_i \) is in the arc
\((n/N, 1) \) for all \( i \). We will refer to \( \Pi \) as the prohibited region of \( T^m \).

2.2.1. Proof of (i). By a standard argument, for every \( k \in \mathbb{N} \) at least one component
of \( k\bar{\Delta} \) is in the arc \([0, n/N) \). (See [SZ] or, e.g., [GG3]; we will also briefly recall
the argument in the proof of (iii) below.) In other words, none of the points of the
orbit \( \{k\bar{\Delta} \mid k \in \mathbb{N} \} \) lies in the prohibited region \( \Pi \). Since \( \Pi \) is open, we conclude
that \( \Gamma \cap \Pi = \emptyset \). Hence, \( \text{codim} \Gamma > 0 \) and \( \mathcal{R} \neq 0 \).
2.2.2. Proof of (ii). Here we assume that \( \text{rk} \mathcal{R} = 1 \). Let \( \vec{a} \) be a generator of \( \mathcal{R} \). Then, \( \Gamma \) is given by the equation
\[
\vec{a} \cdot \theta := \sum_i a_i \theta_i = 0 \text{ in } \mathbb{R}/\mathbb{Z},
\]
where \( \theta = (\theta_1, \ldots, \theta_m) \in \mathbb{T}^m \). Note that \( \Pi \) can also be viewed as the product of the arcs \( (-1 + n/N, 0) \) in \( \mathbb{T}^m \). Thus the intersection of \( \Pi \) with a neighborhood of \((0, \ldots, 0) \in \mathbb{T}^m \) fills in the (open) portion of the negative quadrant in that neighborhood. Since \( \Gamma \cap \Pi = \emptyset \), all non-zero components \( a_i \) must have the same sign. Hence, if \( \vec{a} \) has at least one positive component we have \( a_i \geq 0 \).

Let \( L = \{(t, \ldots, t) \mid t \in S^1\} \) be the “diagonal” one-parameter subgroup of \( \mathbb{T}^m \). The point of \( L \) with \( t = -1/\sum a_i \) lies in \( \Gamma \). Hence, this point must be outside \( \Pi \) and so \(|t| \geq 1 - n/N \). It follows that \( \sum a_i \leq N/(N - n) \).

2.2.3. Proof of (iii). The proofs of (i) and (ii) above are based on the observation that for every \( k \), there exists a capped \( k \)-periodic orbit \( \bar{y} \) such that the local Floer homology of \( \varphi^k \) at \( \bar{y} \) is non-zero in degree \( n \): \( \text{HF}_n(\varphi^k, \bar{y}) \neq 0 \). Then \( \Delta(\bar{y}) \in [0, 2n] \).
(See [Gi, GG2, GG3] for the proofs of these facts; however, the argument essentially goes back to [SZ]. Note also that here we use the grading of the Floer homology by \( \mu_{cz} \), i.e., the fundamental class has degree \( n \).) The orbit \( y \) is the \( k \)th iteration of some orbit \( x_i \), and hence \( \Delta(y) = k\Delta_i \) in \( \mathbb{R}/2N\mathbb{Z} \). We claim that necessarily \( \Delta_i \neq 0 \), provided that \( M \) is rational and, as before, \( N \geq n + 1 \). As a consequence, the orbits with \( \Delta_i = 0 \) can be discarded in the proofs of (i) and (ii).

To show that \( \Delta_i \neq 0 \), we argue by contradiction. Assume the contrary: \( \Delta_i = 0 \). Then \( \Delta(y) = 0 \mod 2N \) and, in fact, \( \Delta(y) = 0 \), since we also have \( \Delta(y) \in [0, 2n] \) and \( N \geq n + 1 \). The condition that \( \Delta(y) = 0 \) and \( \text{HF}_n(\varphi^k, \bar{y}) \neq 0 \) is equivalent to that \( \bar{y} \) is a symplectically degenerate maximum of \( \varphi^k \); [GG2, GG3]. By [GG3, Theorem 1.18], a Hamiltonian diffeomorphism with symplectically degenerate maximum necessarily has infinitely many periodic points whenever \( M \) is rational. This contradicts the assumption that \( \varphi \) is perfect.

Thus, we have proved that (i) and (ii) hold with only non-zero mean indices (in \( \mathbb{R}/2N\mathbb{Z} \)) taken into account. To finish the proof of (iii), it suffices to note that replacing \( \varphi \) by \( \varphi^k \), for a suitably chosen \( k \), we can make every rational mean index zero. Since every resonance relation for \( \varphi^k \) is also a resonance relation for \( \varphi \), we conclude that the irrational mean indices of \( \varphi \) satisfy a resonance relation.

Remark 2.1. The requirement that \( M \) be rational enters the proof of (iii) only at the last point where [GG3, Theorem 1.18] is utilized. The role of this requirement in the proof of this theorem is purely technical and it is likely that the requirement can be eliminated. Note also that we do not assert that (ii) holds when only irrational mean indices are considered. However, an examination of the above argument shows that the following is true. Assume that the resonance group for the irrational mean indices has rank one for a perfect Hamiltonian diffeomorphism \( \varphi \). Then these mean indices satisfy a non-trivial resonance relation of the form \( r\hat{b} \), where \( r \) is a natural number, \( b_i \geq 0 \) for all \( i \), and \( \sum b_i \leq N/(N - n) \).

2.3. Perfect Hamiltonian flows on \( \mathbb{CP}^n \). In this section, we state (without proof) another result asserting, roughly speaking, that \( \sum \Delta_i = 0 \) for perfect flows on \( \mathbb{CP}^n \) satisfying some additional, apparently generic, requirements.

Consider an autonomous, Morse Hamiltonian \( H \) on \( \mathbb{CP}^n \) with exactly \( n + 1 \) critical points \( x_0, \ldots, x_n \). Let us call \( \tau \in \mathbb{R}, \tau > 0 \), a critical period if at least one
of the critical points of $H$ is degenerate when viewed as a $\tau$-periodic orbit of $H$ or equivalently as a fixed point of $\varphi_H^t$. We denote the collection of critical times by $C_{t} \subset \mathbb{R}$ and call $t \in (0, \infty) \setminus C_{H}$ regular. Assume furthermore that for every regular $t > 0$ the points $x_0, \ldots, x_n$ are the only fixed points of $\varphi_H^t$ and that for every critical time $\tau > 0$ at least one of the points $x_i$ is non-degenerate as a fixed point of $\varphi_H^\tau$. Then $\sum \Delta(x_i, t) = 0$ for any regular $t > 0$, where $\Delta(x_i, t)$ is the mean index of $\varphi_H^t$ at $x_i$ equipped with trivial capping.

In particular, $\sum \Delta_i = 0$ in the setting of Theorem 1.1 with $\varphi = \varphi_H^t$. (To ensure that $\varphi$ satisfies the hypotheses of the theorem it suffices to require that $kt \notin C_H$ for all $k \in \mathbb{N}$.) A quadratic Hamiltonian on $\mathbb{C}P^n$ with $n \geq 2$ from Example 1.2 meets the above conditions for generic eigenvalues $\lambda_i$ or, more precisely, if and only if $H$ generates a Hamiltonian action of a torus of dimension greater than one.

The proof of this result, to be detailed elsewhere, goes beyond the scope of the present paper. The argument is conceptually similar to the proof of [GG3, Theorem 1.12] but is technically more involved, for it relies on a more delicate version of Ljusternik–Schnirelman theory than the one considered in that paper.

3. Reeb flows: the proof of Theorem 1.7

Our goal in this section is to prove Theorem 1.7. We focus on establishing the result for $\chi^+(W, \xi)$. The case of $\chi^-(W, \xi)$ can be handled in a similar fashion.

First recall that for every periodic orbit $x$ of the Reeb flow, we have

$$|\mu_{cz}(x^k) - k\Delta(x)| < n - 1$$

(see, e.g., [SZ]), and hence

$$-2 < |x^k| - k\Delta(x) < 2n - 4.$$  \hfill (3.2)

In particular, it follows from (3.2) that condition (CF3) implies condition (CH) with $l_+ = 2n - 4$ and $l_- = -2$. Moreover, the dimension of $CC_l(W, \alpha)$ with $l \geq l_+$ or $l \leq l_-$ is bounded from above by a constant independent of $l$.

To simplify the notation, let us set $C_l := CC_l(W, \alpha)$. Denote by $C_s^{(N)}$ the complex $C_s$ truncated from below at $l_+$ and from above at $N > l_+$. In other words,

$$C_s^{(N)} = \begin{cases} C_l & \text{when } l_+ \leq l \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

The complex $C_s^{(N)}$ is generated by the iterations $x_i^k$ and $y_i^k$ (with odd $k$) such that $|x_i^k|$ and $|y_i^k|$ are in the range $[l_+, N]$. Note that by (3.2) this can only happen when $\Delta(x_i) > 0$ and $\Delta(y_i) > 0$. By (3.2) again, an orbit $x_i^k$ or $y_i^k$ with positive mean index $\Delta$ is in $C_s^{(N)}$ for $k$ ranging from some constant (depending on the orbit, but not on $N$) to roughly $N/\Delta$, up to a constant independent of $N$. Furthermore, the parity of $|x_i^k|$ and $|y_i^k|$ (odd $k$) is independent of $k$, i.e., $\sigma(x_i^k) = \sigma(x_i)$ and $\sigma(y_i^k) = \sigma(y_i)$. Thus, the contribution of the iterations of $x_i$ to the Euler characteristic

$$\chi(C_s^{(N)}):= \sum (-1)^i \dim C_i^{(N)} = \sum_{i=l_+}^{N} (-1)^i \dim C_l$$

is $\sigma(x_i)N/\Delta(x_i) + O(1)$ as $N \to \infty$. Likewise, the contribution of the iterations of $y_i$ is $\sigma(y_i)N/2\Delta(y_i) + O(1)$, since $k$ assumes only odd values in this case. Summing
up over all $x_i$ and $y_i$ with positive mean index, we have
\[ \chi(C^{(N)}_*) = N \left( \sum^+ \frac{\sigma(x_i)}{\Delta(x_i)} + \frac{1}{2} \sum^+ \frac{\sigma(y_i)}{\Delta(y_i)} \right) + O(1), \]
and hence
\[ \lim_{N \to \infty} \chi(C^{(N)}_*) / N = \sum^+ \frac{\sigma(x_i)}{\Delta(x_i)} + \frac{1}{2} \sum^+ \frac{\sigma(y_i)}{\Delta(y_i)}. \]

To finish the proof it remains to show that
\[ \chi^+(W, \xi) = \lim_{N \to \infty} \chi(C^{(N)}_*) / N, \tag{3.3} \]
which is nearly obvious. Indeed, by the very definition of $C^{(N)}_*$, we have $H_l(C^{(N)}_*) = HC_l(W, \xi)$ when $l_* < l < N$. Furthermore, $|H_l(C^{(N)}_*) - HC_l(W, \xi)| = O(1)$ since $\dim C_N = O(1)$. Hence,
\[ \chi(C^{(N)}_*) = \sum_l (-1)^l \dim H_l(C^{(N)}_*) = \sum_{l=l_*}^N (-1)^l \dim HC_l(W, \xi) + O(1) \]
and (3.3) follows. This completes the proof of the theorem.

References

[AK] D.V. Anosov, A.B. Katok, New examples in smooth ergodic theory. Ergodic diffeomorphisms, (in Russian), Trudy Moskov. Mat. Obˇsˇ c., 23 (1970), 3–36.
[Bo] F. Bourgeois, Introduction to contact homology. Lecture notes available at http://homepages.vub.ac.be/~fbourgeo/.
[BCE] F. Bourgeois, K. Cieliebak, T. Ekholm, A note on Reeb dynamics on the tight 3-sphere, J. Mod. Dyn., 1 (2007), 597–613.
[BO] F. Bourgeois, A. Oancea, An exact sequence for contact- and symplectic homology, Invent. Math., 175 (2009), 611–680.
[DK] J.J. Duistermaat, J.A.C. Kolk, Lie Groups, Springer-Verlag, Berlin, New York, 2000.
[Ek] I. Ekeland, Une th´eorie de Morse pour les syst`emes hamiltoniens convexes, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), 19–78.
[EH] I. Ekeland, H. Hofer, Convex Hamiltonian energy surfaces and their periodic trajectories, Comm. Math. Phys., 113 (1987), 419–469.
[El] Y. Eliashberg, Symplectic field theory and its applications, International Congress of Mathematicians, Vol. I, 217–246, Eur. Math. Soc., Zürich, 2007.
[Es] J. Espina, Work in progress.
[FK] B. Fayad, A. Katok, Constructions in elliptic dynamics, Ergodic Theory Dynam. Systems, 24 (2004), 1477–1520.
[Fl] A. Floer, Cuplength estimates on Lagrangian intersections, Comm. Pure Appl. Math., 42 (1989), 335–356.
[Fo] B. Fortune, A symplectic fixed point theorem for $\mathbb{C}P^n$, Invent. Math., 81 (1985), 29–46.
[FW] B. Fortune, A. Weinstein, A symplectic fixed point theorem for complex projective spaces, Bull. Amer. Math. Soc. (N.S.), 12 (1985), 128–130.
[FH] J. Franks, M. Handel, Periodic points of Hamiltonian surface diffeomorphisms, Geom. Topol., 7 (2003), 713–756.
[Gi] V.L. Ginzburg, The Conley conjecture, Preprint 2006, math.SG/0610956.
[GG1] V.L. Ginzburg, B.Z. Gürel, Periodic orbits of twisted geodesic flows and the Weinstein–Moser theorem, Preprint 2007, arXiv:0705.1818; to appear in Comment. Math. Helv.
[GG2] V.L. Ginzburg, B.Z. Gürel, Local Floer homology and the action gap, Preprint 2007, arXiv:0709.4077v2.
[GG3] V.L. Ginzburg, B.Z. Gürel, Action and index spectra and periodic orbits in Hamiltonian dynamics, Preprint 2008, arXiv:0810.5170; to appear in Geom. Topol.
V.L. Ginzburg, B.Z. Gürel, On the generic existence of periodic orbits in Hamiltonian dynamics, in preparation.

V. Guillemin, V. Ginzburg, Y. Karshon, *Moment Maps, Cobordisms, and Hamiltonian Group Actions*, Mathematical Surveys and Monographs, 98. American Mathematical Society, Providence, RI, 2002.

N. Hingston, Subharmonic solutions of Hamiltonian equations on tori, Preprint 2004; to appear in *Ann. of Math.*, available at http://comet.lehman.cuny.edu/sormani/others/hingston.html.

H. Hofer, D. Salamon, Floer homology and Novikov rings, in *The Floer memorial volume*, 483–524, Progr. Math., 133, Birkhäuser, Basel, 1995.

Y. Karshon, Private communication, September 2008.

A.A. Kirillov, *Elements of the Theory of Representations*, Springer-Verlag, Berlin, New York, 1976.

D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, The Clarendon Press, Oxford University Press, New York, 1995.

D. McDuff, D. Salamon, *J-holomorphic Curves and Symplectic Topology*, Colloquium publications, vol. 52, AMS, Providence, RI, 2004.

H.B. Rademacher, On the average indices of closed geodesics, *J. Differential Geom.*, 29 (1989), 65–83.

H.B. Rademacher, On a generic property of geodesic flows, *Math. Ann.*, 298 (1994), 101–116.

D. Salamon, E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, *Comm. Pure Appl. Math.*, 45 (1992), 1303–1360.

M. Schwarz, A quantum cup-length estimate for symplectic fixed points, *Invent. Math.*, 133 (1998), 353–397.

O. van Koert, Open books for contact five-manifolds and applications of contact homology, Inaugural-Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Universität zu Köln, 2005; available at http://www.math.sci.hokudai.ac.jp/~okoert/.

C. Viterbo, Equivariant Morse theory for starshaped Hamiltonian systems, *Trans. Amer. Math. Soc.*, 311 (1989), 621–655.

I. Ustilovsky, Infinitely many contact structures on $S^{4m+1}$, *Internat. Math. Res. Notices* 1999, no. 14, 781–791.

VG: Department of Mathematics, UC Santa Cruz, Santa Cruz, CA 95064, USA
E-mail address: ginzburg@math.ucsc.edu

EK: Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA
E-mail address: ekerman@math.uiuc.edu