Existence of traveling waves for a nonlocal monostable equation: an abstract approach

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Abstract We consider the nonlocal analogue of the Fisher-KPP equation

$$u_t = \mu * u - u + f(u),$$

where $\mu$ is a Borel-measure on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$ and $f$ satisfies $f(0) = f(1) = 0$ and $f > 0$ in $(0,1)$. We do not assume that $\mu$ is absolutely continuous. The equation may have a standing wave solution (a traveling wave solution with speed 0) whose profile is a monotone but discontinuous function. We show that there is a constant $c_*$ such that it has a traveling wave solution with monotone profile and speed $c$ when $c \geq c_*$ while no periodic traveling wave solution with average speed $c$ when $c < c_*$. In order to prove it, we modify a recursive method for abstract monotone discrete dynamical systems by Weinberger. We note that the monotone semiflow generated by the equation does not have compactness with respect to the compact-open topology.

Keywords: discontinuous profile, convolution model, integro-differential equation, discrete monostable equation, nonlocal evolution equation, Fisher-Kolmogorov equation.

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1 Introduction

In 1930, Fisher [9] introduced the reaction-diffusion equation $u_t = u_{xx} + u(1-u)$ as a model for the spatial spread of an advantageous form of a single gene in a population. He [10] found that there is a constant $c_*$ such that the equation has a traveling wave solution with speed $c$ when $c \geq c_*$ while it has no such solution when $c < c_*$. Kolmogorov, Petrovsky and Piskunov [17] obtained the same conclusion for a monostable equation $u_t = u_{xx} + f(u)$ with a more general nonlinearity $f$, and investigated long-time behavior in
the model. Since the pioneering works, there have been extensive studies on traveling waves and long-time behavior for monostable evolution systems.

In this paper, we consider the following nonlocal analogue of the Fisher-KPP equation:

\[ u_t = \mu * u - u + f(u). \]

Here, \( \mu \) is a Borel-measure on \( \mathbb{R} \) with \( \mu(\mathbb{R}) = 1 \) and the convolution is defined by

\[ (\mu * u)(x) = \int_{y \in \mathbb{R}} u(x - y)d\mu(y) \]

for a bounded and Borel-measurable function \( u \) on \( \mathbb{R} \). The nonlinearity \( f \) is a Lipschitz continuous function with \( f(0) = f(1) = 0 \) and \( f > 0 \) in \( (0, 1) \). Then, we would show that there is a constant \( c_* \) such that the nonlocal monostable equation has a traveling wave solution with monotone profile and speed \( c \) when \( c \geq c_* \) while it has no periodic traveling wave solution with average speed \( c \) when \( c < c_* \), if there is a positive constant \( \lambda \) satisfying

\[ \int_{y \in \mathbb{R}} e^{\lambda|y|}d\mu(y) < +\infty. \]

Here, we say that the solution \( u(t, x) \) is a periodic traveling wave solution with average speed \( c \), if \( u(t + \tau, \cdot) \equiv u(t, \cdot + c\tau) \) holds for some positive constant \( \tau \) with \( 0 \leq u(t, \cdot) \leq 1 \), \( u(t, +\infty) = 1 \) and \( u(t, \cdot) \neq 1 \) for all \( t \in \mathbb{R} \). In order to prove this result, we employ the recursive method for monotone dynamical systems introduced by Weinberger [26] and Li, Weinberger and Lewis [18]. We note that the semiflow generated by the nonlocal monostable equation does not have compactness with respect to the compact-open topology. Further, we would also show that there is a smooth and monostable nonlinearity \( f \) such that the equation has a standing wave solution (i.e., a traveling wave solution with speed 0) whose profile is a monotone but discontinuous function, if \( \mu \) satisfies the extra condition \( \int_{y \in \mathbb{R}} yd\mu(y) > 0 \). In these results, we do not assume that \( \mu \) is absolutely continuous with respect to the Lebesgue measure. For example, not only the integro-differential equation

\[ \frac{\partial u}{\partial t}(t, x) = \int_{0}^{1} u(t, x - y)dy - u(t, x) + f(u(t, x)) \]

but also the discrete equation

\[ \frac{\partial u}{\partial t}(t, x) = u(t, x - 1) - u(t, x) + f(u(t, x)) \]

satisfies all the assumptions for the measure \( \mu \).
For the nonlocal monostable equation, Atkinson and Reuter [1] first studied existence of traveling wave solutions. Schumacher [22, 23] proved that there is the minimal speed $c_*$ and the equation has a traveling wave solution with speed $c$ when $c \geq c_*$, if the nonlinearity $f$ satisfies the extra condition

$$f(u) \leq f'(0)u.$$ 

Recently, Coville, Dávila and Martínez [6] showed that if the monostable nonlinearity $f \in C^1(\mathbb{R})$ satisfies $f'(1) < 0$ and the Borel-measure $\mu$ has a density function $J \in C(\mathbb{R})$ with

$$\int_{y \in \mathbb{R}} (|y| + e^{-\lambda y})J(y)dy < +\infty$$

for some positive constant $\lambda$, then there is a constant $c_*$ such that the nonlocal monostable equation has a traveling wave solution with monotone profile and speed $c$ when $c \geq c_*$ while it has no such solution when $c < c_*$. The approach employed in [6] is not of dynamical systems, but they directly solved the stationary problem

$$J \ast u - u - cu_x + f(u) = 0, \quad u(-\infty) = 0, \quad u(+\infty) = 1.$$ 

The proof in this paper is self-contained, and we would believe that it might be rather simple than in [6]. When h1. Introduction and main resultsh in [6] was read, it might be misunderstood that Schumacher [22] and Weinberger [26] assumed the isotropy of dynamical systems. The nonlocal equation is isotropic if and only if $\mu$ is symmetric with respect to the origin. Here, to make sure, we note that the isotropy is not assumed in the results by [22] and [26]. Further, the result by [26] is not limited at a linear determinate. If $f(u) \leq f'(0)u$ holds, then it is a linear determinate. Schumacher [22, 23], Carr and Chmaj [3] and Coville, Dávila and Martínez [6] also studied uniqueness of traveling wave solutions. In [6], we could see an interesting example of nonuniqueness, where the nonlocal monostable equation admits infinitely many monotone profiles for standing wave solutions but it admits no continuous one. See, e.g., [5, 7, 8, 11, 12, 13, 14, 15, 16, 19, 20, 24, 25, 27, 28] on traveling waves and long-time behavior in various monostable evolution systems, [2, 4] nonlocal bistable equations and [21] Euler equation.

In Section 2, we give abstract conditions such that a semiflow satisfying the conditions has a traveling wave solution with speed $c$ when $c \geq c_*$ while
it has no periodic traveling wave solution with average speed $c$ when $c < c^*$. In Section 3, we prove abstract theorems mentioned in Section 2. In Section 4, we precisely state our main results for the nonlocal monostable equation, which are Theorems 14 and 15. In Section 5, we show that the semiflow generated by the nonlocal monostable equation satisfies the conditions given in Section 2 to prove the main results. The proof given in this paper is self-contained.

## 2 Abstract theorems for monotone semiflows

In the abstract, we would treat a monostable evolution system. Put a set of functions on $\mathbb{R}$:

$$\mathcal{M} := \{ u \mid u \text{ is a monotone nondecreasing and left continuous function on } \mathbb{R} \text{ with } 0 \leq u \leq 1 \}.$$  

The followings are our basic conditions for discrete dynamical systems:

**Hypotheses 1** Let $Q_0$ be a map from $\mathcal{M}$ into $\mathcal{M}$.

(i) $Q_0$ is continuous in the following sense: If a sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{M}$ converges to $u \in \mathcal{M}$ uniformly on every bounded interval, then the sequence $\{Q_0[u_k]\}_{k \in \mathbb{N}}$ converges to $Q_0[u]$ almost everywhere.

(ii) $Q_0$ is order preserving; i.e.,

$$u_1 \leq u_2 \implies Q_0[u_1] \leq Q_0[u_2]$$

for all $u_1$ and $u_2 \in \mathcal{M}$. Here, $u \leq v$ means that $u(x) \leq v(x)$ holds for all $x \in \mathbb{R}$.

(iii) $Q_0$ is translation invariant; i.e.,

$$T_{x_0}Q_0 = Q_0T_{x_0}$$

for all $x_0 \in \mathbb{R}$. Here, $T_{x_0}$ is the translation operator defined by $(T_{x_0}[u])(\cdot) := u(\cdot - x_0)$.

(iv) $Q_0$ is monostable; i.e.,

$$0 < \alpha < 1 \implies \alpha < Q_0[\alpha]$$

for all constant functions $\alpha$. 

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Remark 1°. If $Q_0$ satisfies Hypothesis 1 (iii), then $Q_0$ maps constant functions to constant functions. 2°. The semiflow generated by a map $Q_0$ satisfying Hypotheses 1 may not be compact with respect to the compact-open topology.

The following states that existence of suitable super-solutions of the form $\{v_n(x + cn)\}_{n=0}^{\infty}$ implies existence of traveling wave solutions with speed $c$ in the discrete dynamical systems on $\mathcal{M}$:

**Proposition 2** Let a map $Q_0 : \mathcal{M} \to \mathcal{M}$ satisfy Hypotheses 1, and $c \in \mathbb{R}$. Suppose there exists a sequence $\{v_n\}_{n=0}^{\infty} \subset \mathcal{M}$ with $(Q_0[v_n])(x-c) \leq v_{n+1}(x)$, $\inf_{n=0,1,2,\ldots} v_n(x) \neq 0$ and $\lim \inf_{n \to -\infty} v_n(x) \neq 1$. Then, there exists $\psi \in \mathcal{M}$ with $(Q_0[\psi])(x-c) \equiv \psi(x)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$.

In the discrete dynamical system on $\mathcal{M}$ generated by a map $Q_0$ satisfying Hypotheses 1, if there is a periodic traveling wave super-solution with average speed $c$, then there is a traveling wave solution with speed $c$:

**Theorem 3** Let a map $Q_0 : \mathcal{M} \to \mathcal{M}$ satisfy Hypotheses 1, and $c \in \mathbb{R}$. Suppose there exist $\tau \in \mathbb{N}$ and $\phi \in \mathcal{M}$ with $(Q_0^\tau[\phi])(x-c\tau) \leq \phi(x)$, $\phi \neq 0$ and $\phi \neq 1$. Then, there exists $\psi \in \mathcal{M}$ with $(Q_0[\psi])(x-c) \equiv \psi(x)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$.

The infimum $c_*$ of the speeds of traveling wave solutions is not $-\infty$, and there is a traveling wave solution with speed $c$ when $c \geq c_*$:

**Theorem 4** Suppose a map $Q_0 : \mathcal{M} \to \mathcal{M}$ satisfies Hypotheses 1. Then, there exists $c_* \in (-\infty, +\infty]$ such that the following holds:

Let $c \in \mathbb{R}$. Then, there exists $\psi \in \mathcal{M}$ with $(Q_0[\psi])(x-c\tau) \equiv \psi(x)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ if and only if $c \geq c_*$.

We add the following conditions to Hypotheses 1 for continuous dynamical systems on $\mathcal{M}$:

**Hypotheses 5** Let $Q^t$ be a map from $\mathcal{M}$ to $\mathcal{M}$ for $t \in [0, +\infty)$.

(i) $Q$ is a semigroup; i.e., $Q^t \circ Q^s = Q^{t+s}$ for all $t$ and $s \in [0, +\infty)$.

(ii) $Q$ is continuous in the following sense: Suppose a sequence $\{t_k\}_{k \in \mathbb{N}} \subset [0, +\infty)$ converges to 0, and $u \in \mathcal{M}$. Then, the sequence $\{Q^{t_k}[u]\}_{k \in \mathbb{N}}$ converges to $u$ almost everywhere.
As we would have Theorems 3 and 4 for the discrete dynamical systems, we would have the following two for the continuous dynamical systems:

**Theorem 6** Let \( Q^t \) be a map from \( \mathcal{M} \) to \( \mathcal{M} \) for \( t \in [0, +\infty) \). Suppose \( Q^t \) satisfies Hypotheses 1 for all \( t \in (0, +\infty) \), and \( Q \) Hypotheses 5. Then, the following holds:

Let \( c \in \mathbb{R} \). Suppose there exist \( \tau \in (0, +\infty) \) and \( \phi \in \mathcal{M} \) with \( (Q^\tau[\phi])(x - ct) \leq \phi(x) \), \( \phi \neq 0 \) and \( \phi \neq 1 \). Then, there exists \( \psi \in \mathcal{M} \) with \( \psi(-\infty) = 0 \) and \( \psi(+\infty) = 1 \) such that \( (Q^t[\psi])(x - ct) \equiv \psi(x) \) holds for all \( t \in [0, +\infty) \).

**Theorem 7** Let \( Q^t \) be a map from \( \mathcal{M} \) to \( \mathcal{M} \) for \( t \in [0, +\infty) \). Suppose \( Q^t \) satisfies Hypotheses 1 for all \( t \in (0, +\infty) \), and \( Q \) Hypotheses 5. Then, there exists \( c_* \in (-\infty, +\infty) \) such that the following holds:

Let \( c \in \mathbb{R} \). Then, there exists \( \psi \in \mathcal{M} \) with \( \psi(-\infty) = 0 \) and \( \psi(+\infty) = 1 \) such that \( (Q^t[\psi])(x - ct) \equiv \psi(x) \) holds for all \( t \in [0, +\infty) \) if and only if \( c \geq c_* \).

### 3 Proof of the abstract theorems

In this section, we would modify the recursive method introduced by Weinberger [26] and Li, Weinberger and Lewis [18] to prove the theorems stated in Section 2.

**Lemma 8** Let a sequence \( \{u_k\}_{k \in \mathbb{N}} \) of monotone nondecreasing functions on \( \mathbb{R} \) converge to a continuous function \( u \) on \( \mathbb{R} \) almost everywhere. Then, \( \{u_k\}_{k \in \mathbb{N}} \) converges to \( u \) uniformly on every bounded interval.

**Proof.** Let \( C \in (0, +\infty) \) and \( \varepsilon \in (0, +\infty) \). Then, there exists \( \delta \in (0, +\infty) \) such that, for any \( y_1 \) and \( y_2 \in [-C - 1, +C + 1] \), \( |y_2 - y_1| < \delta \) implies \( |u(y_2) - u(y_1)| < \varepsilon/4 \). So, we take \( N \in \mathbb{N} \) and a sequence \( \{x_n\}_{n=1}^N \) such that \( \lim_{k \to \infty} u_k(x_n) = u(x_n), -C - 1 \leq x_1 \leq -C, x_n < x_{n+1} < x_n + \delta \) and \( +C \leq x_N \leq +C + 1 \) hold.

Let \( k \in \mathbb{N} \) be sufficiently large. Then, \( \max\{|u_k(x_n) - u(x_n)|\}_{n=1}^N < \varepsilon/4 \) holds. Let \( x \in [-C, +C] \). There exists \( n \) such that \( x_n \leq x \leq x_{n+1} \) holds. So, we get \( |u_k(x) - u(x)| \leq |u_k(x_n) - u(x)| + |u_k(x_{n+1}) - u(x)| \leq |u_k(x_n) - u(x_n)| + |u(x_n) - u(x)| + |u_k(x_{n+1}) - u(x_{n+1})| + |u(x_{n+1}) - u(x)| < \varepsilon \). ■

The set of discontinuous points of a monotone function on \( \mathbb{R} \) is at most countable. So, if a sequence \( \{u_k\}_{k \in \mathbb{N}} \) of monotone functions on \( \mathbb{R} \) converges to
a monotone function $u$ on $\mathbb{R}$ at every continuous point of $u$, then it converges to $u$ almost everywhere. The converse also holds:

**Lemma 9** Let a sequence $\{u_k\}_{k \in \mathbb{N}}$ of monotone nondecreasing functions on $\mathbb{R}$ converge to a monotone nondecreasing function $u$ on $\mathbb{R}$ almost everywhere. Then, $\lim_{k \to \infty} u_k(x) = u(x)$ holds for all continuous points $x \in \mathbb{R}$ of $u$.

**Proof.** We take $x_n \in (x - 2^{-n}, x]$ and $\overline{x}_n \in [x, x + 2^{-n})$ satisfying $\lim_{k \to \infty} u_k(x_n) = u(x_n)$ and $\lim_{k \to \infty} u_k(\overline{x}_n) = u(\overline{x}_n)$ for $n \in \mathbb{N}$. Then, $u(x_n) \leq \liminf_{k \to \infty} u_k(x) \leq \limsup_{k \to \infty} u_k(x) \leq u(\overline{x}_n)$ holds. Hence, we have $\lim_{k \to \infty} u_k(x) = u(x)$ as $x$ is a continuous point of $u$. ■

Hypotheses 1 imply more strong continuity than Hypothesis 1 (i):

**Proposition 10** Let a map $Q_0 : \mathcal{M} \to \mathcal{M}$ satisfy Hypotheses 1 (i), (ii) and (iii). Suppose a sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{M}$ converges to $u \in \mathcal{M}$ almost everywhere. Then, $\lim_{k \to \infty} (Q_0[u_k])(x) = (Q_0[u])(x)$ holds for all continuous points $x \in \mathbb{R}$ of $Q_0[u]$.

**Proof.** We take a cutoff function $\rho \in C^\infty(\mathbb{R})$ with

$|x| \geq 1/2 \implies \rho(x) = 0,$

$|x| < 1/2 \implies \rho(x) > 0$

and

$\int_{x \in \mathbb{R}} \rho(x) dx = 1.$

We put smooth functions

$\rho_n(\cdot) := 2^n \rho(2^n \cdot),$

$\underline{u}^n(\cdot) := (\rho_n \ast u)(\cdot - 2^{-(n+1)})$

and

$\overline{u}^n(\cdot) := (\rho_n \ast u)(\cdot + 2^{-(n+1)})$

for $n \in \mathbb{N}$. Then, we obtain

$u(\cdot - 2^{-n}) \leq \underline{u}^n(\cdot) \leq u(\cdot) \leq \overline{u}^n(\cdot) \leq u(\cdot + 2^{-n}).$

The sequence $\{\min\{u_k, \underline{u}^n\}\}_{k \in \mathbb{N}}$ converges to $\underline{u}^n$ almost everywhere, and $\{\max\{u_k, \overline{u}^n\}\}_{k \in \mathbb{N}}$ also $\overline{u}^n$. Hence, by Lemma 8, the sequence $\{\min\{u_k, \underline{u}^n\}\}_{k \in \mathbb{N}}$
converges to \( u^n \) uniformly on every bounded interval, and \( \{ \max\{u_k, \overline{u}^n\}\}_{k \in \mathbb{N}} \) also \( \overline{u}^n \). Then, by Hypothesis 1 (i), the sequence \( \{Q_0[\min\{u_k, u^n\}]\}_{k \in \mathbb{N}} \) converges to \( Q_0[u^n] \) almost everywhere, and \( \{Q_0[\max\{u_k, \overline{u}^n\}]\}_{k \in \mathbb{N}} \) also \( Q_0[\overline{u}^n] \). From Hypothesis 1 (ii), \( Q_0[\min\{u_k, u^n\}] \leq Q_0[u_k] \leq Q_0[\max\{u_k, \overline{u}^n\}] \) holds. Therefore, \( Q_0[u^n] \leq \liminf_{k \to \infty} Q_0[u_k] \leq \limsup_{k \to \infty} Q_0[u_k] \leq Q_0[\overline{u}^n] \) holds almost everywhere. So, by Hypotheses 1 (ii) and (iii), \( Q_0[u](\cdot - 2^{-n}) \leq \liminf_{k \to \infty} Q_0[u_k](\cdot) \leq \limsup_{k \to \infty} Q_0[u_k](\cdot) \leq Q_0[u](\cdot + 2^{-n}) \) holds almost everywhere. Hence, \( \lim_{k \to \infty} Q_0[u_k](\cdot) = Q_0[u](\cdot) \) holds almost everywhere, because \( \lim_{n \to \infty} Q_0[u](\cdot - 2^{-n}) = \lim_{n \to \infty} Q_0[u](\cdot + 2^{-n}) = Q_0[u](\cdot) \) holds almost everywhere. So, from Lemma 9, \( \lim_{k \to \infty} (Q_0[u_k])(x) = (Q_0[u])(x) \) holds for all continuous points \( x \in \mathbb{R} \) of \( Q_0[u] \).

Combining Proposition 10 with Helly’s theorem, we can make the argument in Weinberger [26] and Li, Weinberger and Lewis [18] work to prove Proposition 2.

**Proof of Proposition 2.**

We put \( w(\cdot) := \lim_{h \downarrow 0} \inf_{n=0,1,2, \ldots} v_n(\cdot - h) \), and \( u^k_0 := 2^{-k}w \in \mathcal{M} \) for \( k \in \mathbb{N} \). We also take functions \( u^n_k \in \mathcal{M} \) such that

\[
u^n_k(\cdot) = \max\{Q_0[u^n_{k-1}](\cdot - c), 2^{-k}w(\cdot)\}
\]

holds for \( k \) and \( n \in \mathbb{N} \).

We show \( u^n_k \leq u^n_{k+1} \). We have \( u^n_0 \leq u^n_1 \). As \( u^n_{k-1} \leq u^n_k \) holds, we get \( Q_0[u^n_{k-1}] \leq Q_0[u^n_k] \) and \( u^n_k \leq u^n_{k+1} \). So, we have

\[
u^n_k \leq u^n_{k+1}.
\]

In virtue of (3.2), we put \( u^k := \lim_{n \to \infty} u^n_k \in \mathcal{M} \). Then, by (3.1) and Proposition 10,

\[
u^k(\cdot) = \max\{Q_0[u^k](\cdot - c), 2^{-k}w(\cdot)\}
\]

holds. Because \( \lim_{m \to \infty} Q_0[u^k(\cdot + m)] = Q_0[u^k(+\infty)] \) holds from Proposition 10, we have

\[
u^k(+) = \lim_{m \to \infty} \max\{Q_0[u^k](m - c), 2^{-k}w(m)\}
\]

\[
= \lim_{m \to \infty} \max\{Q_0[u^k(\cdot + m)](-c), 2^{-k}w(m)\}
\]

\[
= \max\{Q_0[u^k(+\infty)], 2^{-k}w(+\infty)\}.
\]
Hence, $u^k(+\infty) \geq Q_0[u^k(+\infty)]$ and $u^k(+\infty) \geq 2^{-k}w(+\infty) > 0$ hold. So, from Hypothesis 1 (iv), we obtain

$$u^k(+\infty) = 1. \quad (3.4)$$

We show $u^k_n \leq v_n$. We get $u^k_0 \leq w \leq v_0$. As $u^k_{n-1} \leq v_{n-1}$ holds, we have

$$Q_0[u^k_{n-1}](\cdot - c) \leq Q_0[v_{n-1}](\cdot - c) \leq v_n(\cdot)$$

and $u^k_n \leq v_n$ because of $2^{-k}w \leq w \leq v_n$. So, we have

$$u^k_n \leq v_n. \quad (3.5)$$

From (3.5),

$$u^k(-\infty) \leq \lim_{m \to -\infty} \liminf_{n \to \infty} v_n(-m) < 1 \quad (3.6)$$

holds. Also, $\lim_{m \to -\infty} Q_0[u^k(\cdot - m)] = Q_0[u^k(-\infty)]$ holds from Proposition 10. Hence, by (3.3), we have

$$u^k(-\infty) = \lim_{m \to -\infty} \max\{Q_0[u^k(\cdot - m - c), 2^{-k}w(-m)] \geq Q_0[u^k(-\infty)].$$

So, from Hypothesis 1 (iv) and (3.6), we obtain

$$u^k(-\infty) = 0. \quad (3.7)$$

In virtue of (3.4) and (3.7), there exists $x_k$ such that $u^k(-x_k) \leq 1/2 \leq \lim_{h \to 1+0} u^k(-x_k + h)$ for $k \in \mathbb{N}$. We put $\psi^k(\cdot) := u^k(\cdot - x_k) \in \mathcal{M}$. Then, we have

$$\psi^k(0) \leq 1/2 \leq \lim_{h \to 1+0} \psi^k(h) \quad (3.8)$$

and

$$\psi^k(\cdot) = \max\{Q_0[\psi^k(\cdot - c), 2^{-k}w(\cdot - x_k)] \} \quad (3.9)$$

from (3.3). By Helly’s theorem, there exist a subsequence $\{k(n)\}_{n \in \mathbb{N}}$ and $\psi \in \mathcal{M}$ such that $\lim_{n \to \infty} \psi^{k(n)}(x) = \psi(x)$ holds for all continuous points $x \in \mathbb{R}$ of $\psi$. So, from (3.8), (3.9) and Proposition 10,

$$\psi(0) \leq 1/2 \leq \lim_{h \to 1+0} \psi(h) \quad (3.10)$$

and

$$\psi(\cdot) = Q_0[\psi](\cdot - c) \quad (3.11)$$
holds. Because $\psi(-\infty) = Q_0[\psi(-\infty)]$ and $\psi(+\infty) = Q_0[\psi(+\infty)]$ also hold by (3.11) and Proposition 10, from Hypothesis 1 (iv) and (3.10), we have $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$. ■

Proof of Theorem 3.

We take functions $v_n \in M$ for $n = 0, 1, 2, \cdots$ such that

$$v_{n+mr} = (Q_0^n[\phi])(\cdot - cn)$$

holds for all $n = 0, 1, 2, \cdots, \tau - 1$ and $m = 0, 1, 2, \cdots$. Then, we see

$$v_{n+1}(\cdot) \geq Q_0[v_n](\cdot - c)$$

and

$$\liminf_{n \to \infty} v_n = \inf_{n=0,1,2,\cdots} v_n = \min_{n=0,1,2,\cdots,\tau-1} v_n.$$  (3.12)  

We show $v_n(+\infty) > 0$. We have $v_0(+\infty) > 0$. As $v_{n-1}(+\infty) > 0$ holds, we get $v_n(+\infty) \geq Q_0[v_{n-1}(+\infty)] > 0$ by (3.12), Proposition 10, Hypotheses 1 (ii) and (iv). So, we have $v_n(+\infty) > 0$. Hence, because $\lim_{n \to \infty} \min_{n=0,1,2,\cdots,\tau-1} v_n (m) > 0$ holds, from (3.13), we see $\inf_{n=0,1,2,\cdots} v_n \neq 0$. Because $\min_{n=0,1,2,\cdots,\tau-1} v_n \leq \phi$ holds, by (3.13) and $\phi(-\infty) < 1$, we have $\liminf_{n \to \infty} v_n \neq 1$. Therefore, by Proposition 2, there exists $\psi \in M$ with $Q_0[\psi](\cdot - c) = \psi(\cdot)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$. ■

Lemma 11 Let a sequence $\{u_k\}_{k \in \mathbb{N}}$ of monotone nondecreasing functions on $\mathbb{R}$ converge to a monotone nondecreasing function $u$ on $\mathbb{R}$ almost everywhere. Then, $\lim_{k \to \infty} u_k(x - x_k) = u(x)$ holds for all sequences $\{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ with $\lim_{k \to \infty} x_k = 0$ and continuous points $x \in \mathbb{R}$ of $u$.

Proof. We put $y_n := \sup_{k=n,n+1,n+2,\cdots} |x_k|$ for $n \in \mathbb{N}$. Then, $u_k(\cdot - y_n) \leq u_k(\cdot - x_k) \leq u_k(\cdot + y_n)$ holds when $k \geq n$. Hence, $u(\cdot - y_n) \leq \liminf_{k \to \infty} u_k(\cdot - x_k) \leq \limsup_{k \to \infty} u_k(\cdot - x_k) \leq u(\cdot + y_n)$ holds almost everywhere. So, $\lim_{k \to \infty} u_k(\cdot - x_k) = u(\cdot)$ holds almost everywhere, because $\lim_{n \to \infty} u(\cdot - y_n) = \lim_{n \to \infty} u(\cdot + y_n) = u(\cdot)$ holds almost everywhere. Hence, from Lemma 9, $\lim_{k \to \infty} u_k(x - x_k) = u(x)$ holds for all continuous points $x \in \mathbb{R}$ of $u$. ■

Proof of Theorem 4.

[Step 1] Let $c_* \in [-\infty, +\infty]$ be the infimum of $c \in \mathbb{R}$ such that there exists $\psi \in M$ with $Q_0[\psi](\cdot - c) = \psi(\cdot)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$. 10
Then, we have the following: Let $c \in \mathbb{R}$. Then, there exists $\psi \in \mathcal{M}$ with $Q_0[\psi](\cdot - c) = \psi(\cdot)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ only if $c \geq c_*$.

[Step 2] In this step, we show the following: Let $c \in (c_*, +\infty)$. Then, there exists $\psi \in \mathcal{M}$ with $Q_0[\psi](\cdot - c) = \psi(\cdot)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$.

There exist $c_0 \in (-\infty, c)$ and $\phi \in \mathcal{M}$ with $Q_0[\phi](\cdot - c_0) = \phi(\cdot)$, $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Then, because we have $Q_0[\phi](\cdot - c) \leq \phi(\cdot)$, by Theorem 3, there exists $\psi \in \mathcal{M}$ with $Q_0[\psi](\cdot - c) = \psi(\cdot)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$.

[Step 3] In this step, we show the following: Let $c_* \in \mathbb{R}$. Then, there exists $\psi \in \mathcal{M}$ with $Q_0[\psi](\cdot - c_*) = \psi(\cdot)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$.

In virtue of Step 2, there exists $\psi_k \in \mathcal{M}$ with $Q_0[\psi_k](\cdot - (c_* + 2^{-k})) = \psi_k(\cdot)$, $\psi_k(-\infty) = 0$ and $\psi_k(+\infty) = 1$ for $k \in \mathbb{N}$. We also take $x_k$ such that $\psi_k(-x_k) \leq 1/2 \leq \lim_{h \downarrow 0} \psi_k(-x_k + h)$, and put $\psi^k(\cdot) := \psi_k(\cdot - x_k) \in \mathcal{M}$. Then, we have

$$\psi^k(0) \leq 1/2 \leq \lim_{h \downarrow 0} \psi^k(h)$$

(3.14)

and

$$Q_0[\psi^k(\cdot - 2^{-k})](\cdot - c_*) = \psi^k(\cdot).$$

(3.15)

By Helly’s theorem, there exist a subsequence $\{k(n)\}_{n \in \mathbb{N}}$ and $\psi \in \mathcal{M}$ such that $\lim_{n \to \infty} \psi^{k(n)}(x) = \psi(x)$ holds for all continuous points $x \in \mathbb{R}$ of $\psi$. Also, by Lemma 11, $\lim_{n \to \infty} \psi^{k(n)}(x - 2^{-k(n)}) = \psi(x)$ holds for all continuous points $x \in \mathbb{R}$ of $\psi$. Therefore, from (3.14), (3.15) and Proposition 10,

$$\psi(0) \leq 1/2 \leq \lim_{h \downarrow 0} \psi(h)$$

(3.16)

and

$$Q_0[\psi](\cdot - c_*) = \psi(\cdot)$$

(3.17)

holds. Because $Q_0[\psi(-\infty)] = \psi(-\infty)$ and $Q_0[\psi(+\infty)] = \psi(+\infty)$ also hold by (3.17) and Proposition 10, from Hypothesis 1 (iv) and (3.16), we have

$$\psi(-\infty) = 0 \quad \text{and} \quad \psi(+\infty) = 1.$$

[Step 4] Finally, we show $c_* \in (-\infty, +\infty]$.

Suppose $c_* = -\infty$. Then, in virtue of Step 2, there exists $\phi_k \in \mathcal{M}$ with $Q_0[\phi_k](\cdot + 2^k) = \phi_k(\cdot)$, $\phi_k(-\infty) = 0$ and $\phi_k(+\infty) = 1$ for $k \in \mathbb{N}$. We also take $x_k$ such that $\phi_k(-x_k) \leq 1/2 \leq \lim_{h \downarrow 0} \phi_k(-x_k + h)$, and put $\phi^k(\cdot) := \phi_k(\cdot - x_k) \in \mathcal{M}$. Then, we have

$$\phi^k(0) \leq 1/2 \leq \lim_{h \downarrow 0} \phi^k(h)$$

(3.18)
and
\[ Q_0[\phi^k(\cdot + 2^k)](\cdot) = \phi^k(\cdot). \]  
(3.19)

Put \( \chi \in \mathcal{M} \) such that \( \chi(x) = 0 (x \leq 0) \) and \( \chi(x) = 1/2 (0 < x) \). Then, \( \chi \leq \phi^k \) holds from (3.18). Hence, by (3.18) and (3.19), we see \( Q_0[\chi(\cdot + 2^k)](0) \leq 1/2 \). So, from \( \lim_{k \to \infty} \chi(\cdot + 2^k) = 1/2 \) and Proposition 10, we obtain \( Q_0[1/2] \leq 1/2 \). This is a contradiction with Hypothesis 1 (iv).

\[ \blacksquare \]

**Lemma 12** Let \( Q^t \) be a map from \( \mathcal{M} \) to \( \mathcal{M} \) for \( t \in [0, +\infty) \). Suppose \( Q \) satisfies Hypothesis 5 (ii). Then, \( \lim_{t \to 0}(Q^t[u])(x - ct) = u(x) \) holds for all \( c \in \mathbb{R}, u \in \mathcal{M} \) and continuous points \( x \in \mathbb{R} \) of \( u \).

**Proof.** Let a sequence \( \{t_k\}_{k \in \mathbb{N}} \subset [0, +\infty) \) converge to 0. Then, by Hypothesis 5 (ii) and Lemma 11, \( \lim_{k \to \infty} Q^{t_k}[u](x - ct_k) = u(x) \) holds for all continuous points \( x \in \mathbb{R} \) of \( u \).

**Proof of Theorem 6.**

By Theorem 3, there exists \( \psi_k \in \mathcal{M} \) with \( Q^{\tau} \psi_k(\cdot - \frac{ct}{2^k}) = \psi_k(\cdot) \), \( \psi_k(-\infty) = 0 \) and \( \psi_k(\infty) = 1 \) for \( k \in \mathbb{N} \). We also take \( x_k \) such that \( \psi_k(-x_k) \leq 1/2 \leq \lim_{h \downarrow 0} \psi_k(-x_k + h) \), and put \( \psi^k(\cdot) := \psi_k(\cdot - x_k) \in \mathcal{M} \). Then, we have
\[ \psi^k(0) \leq 1/2 \leq \lim_{h \downarrow 0} \psi^k(h) \]  
(3.20)
and
\[ Q^{\tau} \psi^k(\cdot - \frac{ct}{2^k}) = \psi^k(\cdot). \]  
(3.21)

By Helly’s theorem, there exist a subsequence \( \{k(n)\}_{n \in \mathbb{N}} \) and \( \psi \in \mathcal{M} \) such that \( \lim_{n \to \infty} \psi^{k(n)}(x) = \psi(x) \) holds for all continuous points \( x \in \mathbb{R} \) of \( \psi \).

Let \( k_0 \in \mathbb{N} \) and \( m_0 \in \mathbb{N} \). As \( n \in \mathbb{N} \) is sufficiently large,
\[ Q^{\frac{m_0 \tau}{2^{k_0}} \psi^{k(n)}}(\cdot - \frac{m_0 \tau}{2^{k_0}}) \]
\[ = (Q^{\frac{m_0 \tau}{2^{k_0}}})^{m_0 2^{k(n)-k_0}} \psi^{k(n)}(\cdot - \frac{ct}{2^{k(n)}} m_0 2^{k(n)-k_0}) = \psi^{k(n)}(\cdot) \]
holds because of \( k(n) \geq k_0 \) and (3.21). Therefore, by Proposition 10, we obtain
\[ Q^{\frac{m_0 \tau}{2^{k_0}}} \psi(\cdot - \frac{m_0 \tau}{2^{k_0}}) = \psi(\cdot). \]
(3.22)

From (3.20), we also see
\[ \psi(0) \leq 1/2 \leq \lim_{h \downarrow 0} \psi(h). \]  
(3.23)
Let $t \in [0, +\infty)$. Then, by (3.22), there exists a sequence $\{t_k\}_{k \in \mathbb{N}} \subset [0, +\infty)$ with $\lim_{k \to \infty} t_k = 0$ such that $Q^{t+t_k}[\psi](\cdot - c(t + t_k)) = \psi(\cdot)$ holds for all $k \in \mathbb{N}$. So, by $Q^{t_k}[Q^{\cdot}[\psi](\cdot - ct)](\cdot - ct_k) = Q^{t+t_k}[\psi](\cdot - c(t + t_k))$ and Lemma 12, we obtain

$$Q^t[\psi](\cdot - ct) = \psi(\cdot).$$

Hence, because $Q^t[\psi(-\infty)] = \psi(-\infty)$ and $Q^t[\psi(+\infty)] = \psi(+\infty)$ hold by Proposition 10, from (3.23), we see $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$. $\blacksquare$

**Proof of Theorem 7.**

In virtue of Theorem 4, we take $c^* \in (-\infty, +\infty]$ such that the following holds: Let $c \in \mathbb{R}$. Then, there exists $\phi \in \mathcal{M}$ with $(Q^1[\phi])(\cdot - c) \equiv \phi(\cdot)$, $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$ if and only if $c \geq c^*$.

Then, from Theorem 6, we have the conclusion of this theorem. $\blacksquare$

**4 The main results for the nonlocal monostable equation**

Let a Lipschitz continuous function $f$ on $\mathbb{R}$ be a monostable nonlinearity; $f(0) = f(1) = 0$ and $f(u) > 0$ in $(0, 1)$. Let a Borel-measure $\mu$ on $\mathbb{R}$ satisfy $\mu(\mathbb{R}) = 1$. (We do not assume that $\mu$ is absolutely continuous with respect to the Lebesgue measure.) Then, we consider the following nonlocal monostable equation:

$$u_t = \mu * u - u + f(u), \quad (4.1)$$

where $(\mu * u)(x) := \int_{y \in \mathbb{R}} u(x - y) d\mu(y)$ for a bounded and Borel-measurable function $u$ on $\mathbb{R}$. Then, $G(u) := \mu * u - u + f(u)$ is a map from the Banach space $L^\infty(\mathbb{R})$ into $L^\infty(\mathbb{R})$ and it is Lipschitz continuous. (We note that $u(x - y)$ is a Borel-measurable function on $\mathbb{R}^2$, and $\|u\|_{L^\infty(\mathbb{R})} = 0$ implies $\|\mu * u\|_{L^1(\mathbb{R})} \leq \int_{y \in \mathbb{R}} (\int_{x \in \mathbb{R}} |u(x - y)| dx) d\mu(y) = 0$.) So, because the standard theory of ordinary differential equations works, we have well-posedness of (4.1) and the equation generates a flow in $L^\infty(\mathbb{R})$. The following gives two positively invariant sets:

**Proposition 13** If $u_0 \in L^\infty(\mathbb{R})$ satisfies $0 \leq u_0 \leq 1$, then there exists a solution $\{u(t)\}_{t \in [0, +\infty)} \subset L^\infty(\mathbb{R})$ to (4.1) with $u(0) = u_0$ and $0 \leq u(t) \leq 1$.

For any $u_0 \in \mathcal{M}$, then there exists a solution $\{u(t)\}_{t \in [0, +\infty)} \subset \mathcal{M}$ to (4.1) with $u(0) = u_0$. 

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Here, we recall that $M$ has been defined at the beginning of Section 2.

If the semiflow generated by (4.1) has a periodic traveling wave solution with average speed $c$ (even if the profile is not a monotone function), then it has a traveling wave solution with monotone profile and speed $c$.

**Theorem 14** Let a Borel-measure $\mu$ have $\lambda \in (0, +\infty)$ satisfying

$$
\int_{y \in \mathbb{R}} e^{\lambda|y|}d\mu(y) < +\infty,
$$

and $c \in \mathbb{R}$. Suppose there exist $\tau \in (0, +\infty)$ and a solution $\{u(t, x)\}_{t \in \mathbb{R}} \subset L^\infty(\mathbb{R})$ to (4.1) with $0 \leq u(t, x) \leq 1$, $\lim_{x \to +\infty} u(t, x) = 1$ and $\|u(t, x) - 1\|_{L^\infty(\mathbb{R})} \neq 0$ such that

$$
u(t + \tau, x) = u(t, x + c\tau)
$$

holds for all $t$ and $x \in \mathbb{R}$. Then, there exists $\psi \in M$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $\{\psi(x + ct)\}_{t \in \mathbb{R}}$ is a solution to (4.1).

**Remark** The condition (4.2) with some positive constant $\lambda$ ensures that the semiflow on $M$ generated by (4.1) satisfies Hypothesis 1 (i). See Proof of Proposition 19 in Section 5.

The infimum $c_*$ of the speeds of traveling wave solutions is not $\pm\infty$, and there is a traveling wave solution with speed $c$ when $c \geq c_*$:

**Theorem 15** Let a Borel-measure $\mu$ have $\lambda \in (0, +\infty)$ satisfying (4.2). Then, there exists $c_* \in \mathbb{R}$ such that the following holds:

Let $c \in \mathbb{R}$. Then, there exists $\psi \in M$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $\{\psi(x + ct)\}_{t \in \mathbb{R}}$ is a solution to (4.1) if and only if $c \geq c_*$.

**Remark** The condition

$$
\int_{y \in \mathbb{R}} e^{-\lambda y}d\mu(y) < +\infty
$$

with some positive constant $\lambda$ ensures $c_* \neq +\infty$. See Proof of Theorem 15 in Section 5.

Coville and Dupaigne [7] showed that the minimal speed $c_*$ is positive, if the Borel-measure $\mu$ has a density function $J \in C^1(\mathbb{R})$ with the extra condition

$$J(-y) \equiv J(y).$$
It implies \( \int_{y \in \mathbb{R}} y d\mu(y) = 0 \). On the other hand, if the Borel-measure \( \mu \) satisfies \( \int_{y \in \mathbb{R}} y d\mu(y) > 0 \), then the minimal speed \( c_\ast \) is negative and, so, the semiflow has a standing wave solution (a traveling wave solution with speed 0) for a sufficiently small nonlinearity \( f \):

**Proposition 16** Let a Borel-measure \( \mu \) have \( \lambda \in (0, +\infty) \) satisfying (4.2), and \( \int_{y \in \mathbb{R}} y d\mu(y) > 0 \). Then, there exists \( \gamma \in (0, +\infty) \) such that the minimal speed \( c_\ast \) is negative when \( f(u) \leq \gamma u \) \((0 \leq u \leq 1)\).

The solutions to (4.1) are continuous in \( L^\infty(\mathbb{R}) \). Hence, if the profile of a traveling wave solution with speed \( c \neq 0 \) is monotone, then it is a continuous function on \( \mathbb{R} \). However, for some nonlinearity \( f \), if the profile of a standing wave solution (a traveling wave solution with speed 0) is monotone, then it is a discontinuous one:

**Proposition 17** Let a nonlinearity \( f \in C^1(\mathbb{R}) \) satisfy \( f'(\alpha) > 1 \) for some constant \( \alpha \in [0, 1] \), and \( \psi \in \mathcal{M} \psi(-\infty) = 0 \) and \( \psi(+\infty) = 1 \). Suppose \( u(t, x) := \psi(x) \) is a solution to (4.1). Then, \( \psi \) is a discontinuous function.

### 5 Semiflows generated by nonlocal monostable equations

In this section, we show the results for the nonlocal monostable equation (4.1) stated in Section 4.

First, we have the comparison theorem on the phase space \( L^\infty(\mathbb{R}) \):

**Proposition 18** Let \( T \in (0, +\infty) \), and functions \( u^1 \) and \( u^2 \in C^1([0, T], L^\infty(\mathbb{R})) \). Suppose that for any \( t \in [0, T] \), the inequality

\[
 u^1_t - (\mu * u^1 - u^1 + f(u^1)) \leq u^2_t - (\mu * u^2 - u^2 + f(u^2))
\]

holds almost everywhere in \( x \). Then, the inequality \( u^1(T, x) \leq u^2(T, x) \) holds almost everywhere in \( x \) if the inequality \( u^1(0, x) \leq u^2(0, x) \) holds almost everywhere in \( x \).

**Proof.** Put \( K \in \mathbb{R} \) by

\[
 K := 1 - \inf_{h > 0, u \in \mathbb{R}} \frac{f(u + h) - f(u)}{h},
\]

(5.1)
and \( v \in C^1([0, T], L^\infty(\mathbb{R})) \) by
\[
v(t) := e^{Kt}(u^2 - u^1)(t).
\] (5.2)

Then, we have the ordinary differential equation
\[
\frac{dv}{dt} = F(t, v)
\] (5.3)
in \( L^\infty(\mathbb{R}) \) with \( v(0) = (u^2 - u^1)(0) \) as we define a map \( F : [0, T] \times L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R}) \) by
\[
F(t, w) := \mu * w + (K - 1)w + e^{Kt} (f(u^1(t) + e^{-Kt}w) - f(u^1(t))) + e^{Kt}a(t),
\]
where
\[
a := \left( \frac{du^2}{dt} - (\mu * u^2 - u^2 + f(u^2)) \right) - \left( \frac{du^1}{dt} - (\mu * u^1 - u^1 + f(u^1)) \right).
\]

For any \( t \in [0, T] \), we see the inequality
\[
a(t, x) \geq 0 \quad (5.4)
\]
audience everywhere in \( x \). Take the solution \( \tilde{v} \in C^1([0, T], L^\infty(\mathbb{R})) \) to
\[
\tilde{v}(t) = v(0) + \int_0^t \max\{F(s, \tilde{v}(s)), 0\} ds.
\] (5.5)

Then, for any \( t \in [0, T] \), we have
\[
\tilde{v}(t, x) \geq v(0, x) = (u^2 - u^1)(0, x) \geq 0 \quad (5.6)
\]
audience everywhere in \( x \). By using (5.1), (5.4) and (5.6), for any \( t \in [0, T] \), we also have the inequality \( F(t, \tilde{v}(t)) \geq 0 \) audience everywhere in \( x \). Hence, from (5.5), \( \tilde{v}(t) \) is the solution to the same ordinary differential equation (5.3) in \( L^\infty(\mathbb{R}) \) as \( v(t) \) with \( \tilde{v}(0) = v(0) \). So, in virtue of (5.2) and (5.6),
\[
(u^2 - u^1)(T, x) = e^{-KT}v(T, x) = e^{-KT}\tilde{v}(T, x) \geq 0
\]
holds almost everywhere in \( x \). \( \Box \)
Proof of Proposition 13.

The constants 0 and 1 are solutions to (4.1). So, by using Proposition 18, for any \( u_0 \in L^\infty(\mathbb{R}) \) with \( 0 \leq u_0 \leq 1 \), there exists a solution \( \{u(t)\}_{t \in [0, +\infty)} \) to (4.1) with \( u(0) = u_0 \) and \( 0 \leq u(t) \leq 1 \).

Let \( u_0 \in \mathcal{M} \). We take the solution \( \{u(t)\}_{t \in [0, +\infty)} \) to (4.1) with \( u(0) = u_0 \). Let \( t \in [0, +\infty) \) and \( h \in [0, +\infty) \). Then, by Proposition 18, we see \( u(t, x) \leq u(t, x + h) \) almost everywhere in \( x \). We take a cutoff function \( \rho \in C^\infty(\mathbb{R}) \) with

\[
|x| \geq 1/2 \implies \rho(x) = 0,
\]

\[
|x| < 1/2 \implies \rho(x) > 0
\]

and

\[
\int_{x \in \mathbb{R}} \rho(x) dx = 1.
\]

As we put

\[
v_n(x) := \int_{y \in \mathbb{R}} 2^n \rho(2^n(x - y)) u(t, y) dy \]

for \( n \in \mathbb{N} \), we see \( v_n(x) \leq v_n(x + h) \) for all \( x \in \mathbb{R} \). Therefore, \( v_n \) is a smooth and monotone nondecreasing function. By Helly's theorem, there exist a subsequence \( n_k \) and \( \psi \in \mathcal{M} \) such that \( \lim_{k \to \infty} v_{n_k}(x) = \psi(x) \) almost everywhere in \( x \). Then, \( \|u(t, x) - \psi(x)\|_{L^1([-C, +C])} \leq \lim_{k \to \infty} (\|u(t, x) - v_{n_k}(x)\|_{L^1([-C, +C])}) \leq \lim_{k \to \infty} (\|u(t, x) - v_{n_k}(x)\|_{L^1([-C, +C])} + \|v_{n_k}(x) - \psi(x)\|_{L^1([-C, +C])}) = 0 \) holds for all \( C \in (0, +\infty) \). Hence, we obtain \( \|u(t, x) - \psi(x)\|_{L^\infty(\mathbb{R})} = 0 \).

Proposition 19 Let a Borel-measure \( \mu \) have \( \lambda \in (0, +\infty) \) satisfying (4.2), and \( T \in (0, +\infty) \). Suppose a sequence \( \{u_n\}_{n=0}^\infty \subset C^1([0, T], L^\infty(\mathbb{R})) \) of solutions to (4.1) with \( \sup_{n \in \mathbb{N}, x \in \mathbb{R}} |u_n(0, x) - u_0(0, x)| \leq 1 \) satisfies

\[
\lim_{n \to \infty} \sup_{x \in [-I, +I]} |u_n(0, x) - u_0(0, x)| = 0
\]

for all \( I \in (0, +\infty) \). Then,

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \|u_n(t, x) - u_0(t, x)\|_{L^\infty([-J, +J])} = 0
\]

holds for all \( J \in (0, +\infty) \).
Proof. Let $J \in (0, +\infty)$ and $\varepsilon \in (0, +\infty)$. We take $K \in [0, +\infty)$ such that

$$K \geq \int_{y \in \mathbb{R}} e^{\lambda|y|} d\mu(y) - 1 + \sup_{h > 0, u \in \mathbb{R}} \frac{f(u + h) - f(u)}{h}.$$ 

Put positive constants $\delta := \min\{\varepsilon e^{-(K + \lambda J)}, 1\}$ and $I := \frac{1}{\lambda} \log(\frac{2}{\delta})$. Let $n \in \mathbb{N}$ be sufficiently large. Then, we have

$$\sup_{x \in [-I, +I]} |u_n(0, x) - u_0(0, x)| \leq \delta. \quad (5.7)$$

We consider the following two functions

$$\underline{u}(t, x) := u_0(t, x) - e^{Kt}w(x)$$

and

$$\overline{u}(t, x) := u_0(t, x) + e^{Kt}w(x),$$

where $w(x) := \min\{\delta e^{\lambda x} + e^{-\lambda x}, 1\}$. We see

$$(\mu * w)(x) \leq \min \left\{ \delta \frac{\left( \int_{y \in \mathbb{R}} e^{-\lambda y} d\mu(y) \right) e^{\lambda x} + \left( \int_{y \in \mathbb{R}} e^{\lambda y} d\mu(y) \right) e^{-\lambda x}}{2}, \mu(\mathbb{R}) \right\} \leq \left( \int_{y \in \mathbb{R}} e^{\lambda|y|} d\mu(y) \right) w(x).$$

So, $\overline{u}$ is a super-solution to (4.1), because of

$$\frac{d\overline{u}}{dt} - (\mu * \overline{u} - \overline{u} + f(\overline{u})) = (K + 1)e^{Kt}w - \left( e^{Kt}(\mu * w) + (f(u_0 + e^{Kt}w) - f(u_0)) \right) \geq 0$$

almost everywhere in $x$. We can also see that $\underline{u}$ is a sub-solution. Because of $w(0) = \delta$, $w(\pm I) = 1$ and (5.7), we get $\underline{u}(0, x) \leq u_n(0, x) \leq \overline{u}(0, x)$. Hence, by Proposition 18, $\underline{u}(t, x) \leq u_n(t, x) \leq \overline{u}(t, x)$ holds almost everywhere in $x$. So, we have $\|u_n(t, x) - u_0(t, x)\|_{L^\infty([-I, +I])} \leq e^{KT}w(\pm J) \leq \varepsilon$. 

In virtue of Propositions 13, 18 and 19, if $\mu$ has a constant $\lambda \in (0, +\infty)$ satisfying (4.2), then $Q^t$ ($t \in (0, +\infty)$) satisfies Hypotheses 1 and 5 for the semiflow $Q = \{Q^t\}_{t \in [0, +\infty)}$ on the set $\mathcal{M}$ generated by (4.1). So, Theorems 6 and 7 can work for this semiflow.
Proof of Theorem 14.

Put monotone nondecreasing functions \( \varphi(x) := \max\{\alpha \in \mathbb{R} | \alpha \leq u(0, y) \} \) holds almost everywhere in \( y \in (x, +\infty) \) and \( \phi(x) := \lim_{h \to 0^+} \varphi(x - h) \). Then, \( \phi \in \mathcal{M} \), \( \phi(-\infty) < 1 \) and \( \phi(+\infty) = 1 \) hold. We take a cutoff function \( \rho \in C^\infty(\mathbb{R}) \) with

\[
|x + 1/2| \geq 1/2 \implies \rho(x) = 0,
\]

\[
|x + 1/2| < 1/2 \implies \rho(x) > 0
\]

and

\[
\int_{x \in \mathbb{R}} \rho(x) dx = 1.
\]

As we put

\[
v_n(x) := \int_{y \in \mathbb{R}} 2^n \rho(2^n(x - y)) u(0, y) dy
\]

for \( n \in \mathbb{N} \), we see \( \phi \leq v_n \). Let \( N \in \mathbb{N} \). Because of \( \lim_{n \to \infty} \|v_n(x) - u(0, x)\|_{L^1([-N, +N])} = 0 \), there exists a subsequence \( n_k \) such that \( \lim_{k \to \infty} v_{n_k}(x) = u(0, x) \) almost everywhere in \( x \in [-N, +N] \). Therefore, we have \( \phi(x) \leq u(0, x) \) almost everywhere in \( x \in \mathbb{R} \). So, by Proposition 18, we obtain \( Q^t[\phi](x - ct) \leq u(\tau, x - ct) = u(0, x) \) almost everywhere in \( x \). Hence, because \( Q^t[\phi](x - ct) \leq \varphi(x) \) holds, we get \( Q^t[\phi](x - ct) \leq \phi(x) \). Therefore, by Theorem 6, there exists \( \psi \in \mathcal{M} \) with \( \psi(-\infty) = 0 \) and \( \psi(+\infty) = 1 \) such that \( Q^t[\psi](x - ct) \equiv \psi(x) \) holds for all \( t \in [0, +\infty) \).

Proof of Theorem 15.

By Theorem 7, there exists \( c_* \in (-\infty, +\infty] \) such that the following holds: Let \( c \in \mathbb{R} \). Then, there exists \( \psi \in \mathcal{M} \) with \( \psi(-\infty) = 0 \) and \( \psi(+\infty) = 1 \) such that \( \{\psi(x + ct)\}_{t \in \mathbb{R}} \) is a solution to (4.1) if and only if \( c \geq c_* \).

We show \( c_* \neq +\infty \). Take \( K \in [0, +\infty) \) such that

\[
K \geq \max \left\{ \int_{y \in \mathbb{R}} e^{-\lambda y} d\mu(y), \mu(\mathbb{R}) \right\} - 1 + \sup_{h > 0} \frac{f(h)}{h}.
\]

As we put \( \phi(x) := \min\{e^{\lambda x}, 1\} \in \mathcal{M} \), we see

\[
(\mu \ast \phi)(x) \leq \min \left\{ \left( \int_{y \in \mathbb{R}} e^{-\lambda y} d\mu(y) \right) e^{\lambda x}, \mu(\mathbb{R}) \right\}
\]

\[
\leq \max \left\{ \int_{y \in \mathbb{R}} e^{-\lambda y} d\mu(y), \mu(\mathbb{R}) \right\} \phi(x).
\]
So, $e^{Kt}\phi(x)$ is a super-solution to (4.1), because of

$$e^{Kt}(\mu \ast \phi) - e^{Kt}w + f(e^{Kt}\phi) \leq Ke^{Kt}\phi.$$  

Hence, by Proposition 18, we obtain $Q^1[\phi](x) \leq e^{K}\phi(x) \leq e^{\lambda(x+\frac{K}{\lambda})}$, and $Q^1[\phi](x - \frac{K}{\lambda}) \leq \phi(x)$. Therefore, from Theorem 6, there exists $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $Q^t[\psi](x - \frac{K}{\lambda}t) \equiv \psi(x)$ holds for all $t \in [0, +\infty)$. So, $c_* \leq \frac{K}{\lambda}$ holds. \hfill \blacksquare

**Proof of Proposition 16.**

We have $g(0) = 1$ and $g'(0) = -\int_{y \in \mathbb{R}} yd\mu(y) < 0$ for $g(\zeta) := \int_{y \in \mathbb{R}} e^{-\xi y}d\mu(y)$ ($\zeta \in [-\lambda, +\lambda]$). Hence, there exists $\xi \in (0, +\infty)$ with $\int_{y \in \mathbb{R}} e^{-\xi y}d\mu(y) < 1$. Then, we take $\gamma \in (0, 1 - \int_{y \in \mathbb{R}} e^{-\xi y}d\mu(y))$.

We consider the equation

$$u_t = \mu \ast u - u + \tilde{f}(u) \quad (5.8)$$  

instead of (4.1), where $\tilde{f}(u) := \gamma u$ in $(-\infty, 0)$, $f(u)$ in $[0, 1]$, $-2(u-1)$ in $(1, 2)$ and $-u$ in $[2, +\infty)$. Also, we put $K := \int_{y \in \mathbb{R}} e^{-\xi y}d\mu(y) - 1 + \gamma \in (-1, 0)$. Then, we show that the function $\psi(t, x) := 2e^{K(t-1)}\min\{e^{\xi x}, 1\}$ is a super-solution to (5.8) on $t \in [0, 1]$. For $x \in (-\infty, 0)$, we can see $\mu \ast \psi - \psi + \tilde{f}(\psi) \leq \frac{d\psi}{dt}$ from $(\mu \ast \psi)(t, x) \leq 2e^{K(t-1)}(\int_{y \in \mathbb{R}} e^{-\xi y}d\mu(y))e^{\xi x}$, $\tilde{f}(u) \leq \gamma u$ and $\frac{d\psi}{dt}(t, x) = 2Ke^{K(t-1)}e^{\xi x}$. For $x \in [0, +\infty)$, we can also see it from $(\mu \ast \psi)(t, x) \leq 2e^{K(t-1)}$, $f(\psi(t, x)) = -\tilde{f}(t, x)$ and $\frac{d\psi}{dt}(t, x) = 2Ke^{K(t-1)}$. Hence, by Proposition 18, we obtain $Q^1[\psi](x) \leq \psi(1, x) \leq 2e^{\xi x}$, as we put $\phi(x) := \min\{2e^{\xi x} - K, 1\} \in \mathcal{M}$. So, because $Q^1[\phi](x - \frac{K}{\lambda}) \leq \phi(x)$ holds, by Theorem 6, there exists $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $\{\psi(x + \frac{K}{\lambda}t)\}_{t \in \mathbb{R}}$ is a solution to (5.8). Because it is also one to (4.1), by Theorem 15, we have $c_* \leq \frac{K}{\lambda} < 0$. \hfill \blacksquare

**Proof of Proposition 17.**

Suppose $\psi$ is a continuous function. We take a interval $(a, b) \subset (0, 1)$ such that

$$\inf_{u \in (a, b)} f'(u) > 1. \quad (5.9)$$  

Let $x \in \psi^{-1}((a, \frac{a+b}{2}))$ and $y \in \psi^{-1}((\frac{a+b}{2}, b))$. Then, because of $x < y$ and (5.9), we have

$$(\mu \ast \psi)(x) - \psi(x) + f(\psi(x))$$  

$$\leq (\mu \ast \psi)(y) - \psi(x) + f(\psi(x))$$  

$$< (\mu \ast \psi)(y) - \psi(y) + f(\psi(y)).$$  

20
It is a contradiction, as $\psi^{-1}((a, \frac{a+b}{2}))$ and $\psi^{-1}((\frac{a+b}{2}, b))$ are open intervals.

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