Feed-forward chains of recurrent attractor neural networks with finite dilution near saturation

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Abstract

A stationary state replica analysis for a dual neural network model that interpolates between a fully recurrent symmetric attractor network and a strictly feed-forward layered network, studied by Coolen and Viana, is extended in this work to account for finite dilution of the recurrent Hebbian interactions between binary Ising units within each layer. Gradual dilution is found to suppress part of the phase transitions that arise from the competition between recurrent and feed-forward operation modes of the network. Despite that, a long chain of layers still exhibits a relatively good performance under finite dilution for a balanced ratio between inter-layer and intra-layer interactions.

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1 Introduction

It is now well established that biological and artificial neural networks propagate information by means of synaptic interactions which may be symmetric or non-symmetric. The latter appear in biological networks, while either of both have been used in artificial networks.

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There are two main classes of neural network models for associative information processing. One is that of recurrent networks with feed-back of either symmetric or non-symmetric synapses \([1, 2]\) and the other one is that of layered feed-forward networks, with no feed-back loops, in which the synapses are intrinsically non-symmetric \([3-7]\). Models of the first class can be solved by means of equilibrium statistical mechanics in the case of symmetric synapses \([8, 9]\), due to the presence of detailed balance, whereas one has to resort to a dynamical procedure in the search for stationary states in the case of non-symmetric synapses \([10]\). On the other hand, feed-forward networks are already dynamical systems which may reach stationary states in place of true equilibrium states.

A novel dual model of binary neurons which combines recurrent and feed-forward processing, with symmetric synapses in the recurrent part, has been proposed some time ago by Coolen and Viana \([11]\), and the model has been generalized recently \([12-14]\). The network model consists of layers of fully connected neurons in a recurrent architecture that receive inputs from the previous layer (except the first one) and pass the processed information by means of feed-forward synaptic interactions to the next layer. The latter processes further the information and feeds, in turn, the next layer. Every unit in each layer is assumed to be connected to every other unit in that layer as well as to every unit in the next and in the previous layer but the information goes only from one layer to the next one (the feed-forward nature of the model). It is important to note that each unit in every layer participates of both the recurrent and the feed-forward process. Thus, a single updating procedure (usually, either sequential or parallel) has to be applied to every unit in each layer.

The dual model becomes solvable by means of equilibrium statistical mechanics when all layers reach a stationary state and that may eventually be the case in the zero-temperature limit. In the case of finite temperature, thermal fluctuations destroy the Boltzmann form of the stationary states in the layers and one has to go over to a dynamical approach, which has not been done apparently so far. Actually, the model has been solved working out the free energy and the equations for the order parameters for finite temperature, assuming a Boltzmann form based on an underlying sequential updating dynamics, and letting the temperature go to zero after the thermodynamic limit. The close agreement between theory and numerical simulations justified the procedure and the order of these limits despite the absence of detailed balance \([11]\).

A feed-forward chain of recurrent networks is interesting as a model of dual information processing in which the feed-forward mode transmits the
outcome for the stationary overlaps within a layer to the next one. So far, only the case of a fully connected network has been considered, and one may ask how does the performance of the model change if the constraint of full connectivity between the units within each layer is relaxed by means of synaptic dilution keeping the full connectivity between units in adjacent layers. Synaptic dilution reduces the outcome from a layer and it could be that the cumulative effect in a long chain of layers would be a vanishingly small storage capacity even for a reduced amount of dilution.

This issue is of interest in statistical mechanics and also for the information processing in biological networks, in particular for the understanding of the role played by the CA3 region of the hippocampus in the primate brain where the mean connection between units (our parameter $c$ below) in the latter is of the order of $0.1 - 0.01$ \cite{15,16}, which may be considered as a finite dilution. Finite dilution may also be of interest if one thinks of defects in the connectivity within layers in artificial neural networks. Detailed studies of the dynamics and the equilibrium states that describe the performance have been done in recent works that deal with finite dilution in various neural network models on different architectures \cite{17} -\cite{21} following earlier works \cite{22} -\cite{24}.

The main purpose of the present paper is to study the stationary states and obtain the phase diagrams that describe the performance of the dual model with finite dilution of the synaptic interactions within layers. In consistency with the procedure, we first take into account the disorder due to dilution for non-zero temperature, solve the model in the thermodynamic limit and then let the temperature go to zero. A second, minor purpose, is to indicate the necessary steps that yield the recursion relation for the parameter that accounts for the input from the non-condensed overlaps which follows an unconventional derivation. It will be shown that, in addition to the well-known effect of finite dilution in replacing the intra-layer synaptic connection by an effective interaction that is the superposition of a Hebbian term for a fully connected network and a Gaussian noise, the only further contribution of the dilution on that parameter is an overall factor which can be absorbed by a simple rescaling in the case of dense networks. A further interest in the procedure to deal with the effects of synaptic dilution is that it may be extended to study the influence of general non-linear synapses \cite{22}.

The outline of the paper is the following. In Sec. 2 we review the model and introduce the alterations due to finite dilution. In Sec. 3 we solve the model to obtain the replica symmetric free energy near saturation and derive the recursion relations that describe the evolution of the parameters of the
network from one layer to the next. The phase transitions and some features of the performance are obtained and discussed in Sec. 4 and we end with concluding remarks in Sec. 5.

2 The model

The network model consists of $L$ layers with $N$ binary Ising units (neurons) on each layer $l$ in a microscopic state $\sigma^l = \{\sigma^l_1, \ldots, \sigma^l_N\}$, in which each $\sigma^l_i = \pm 1$. The state $+1$ represents a firing neuron and the state $-1$ a neuron at rest. The microscopic dynamics of the network is assumed to be a Glauber sequential stochastic alignment of each neuron $\sigma^l_i$ to a local field $h^l_i$ with probability

$$\text{Prob}(\sigma^l_i \rightarrow -\sigma^l_i) = \frac{1}{2} \left[ 1 - \tanh(\beta \sigma^l_i h^l_i) \right], \quad (1)$$

where

$$h^l_i(\sigma^l, \sigma^{l-1}) = \sum_{j=1}^N J^l_{ij} \sigma^l_j + \sum_{j=1}^N K^l_{ij} \sigma^{l-1}_j, \quad (2)$$

is due to the states of other (see below) neurons $\sigma^l$ on the same layer and of neurons $\sigma^{l-1}$ on the previous layer, as shown schematically in Fig.1. At each time step the neuron to be updated is taken at random from the set $\{\sigma^l\}$. The parameter $\beta = T^{-1}$ controls the synaptic noise such that the dynamics of the network becomes deterministic when $T \rightarrow 0$.

Figure 1: The model. Two consecutive layers in a chain of length $L$ with neuron states $\sigma^l$ and symmetrically diluted recurrent interactions $\{J^l_{ij}\}$ within layer $l$ and feed-forward interactions $\{K^l_{ij}\}$ from the previous layer.
A macroscopic set of $p = \alpha N$ independent and identically distributed random patterns $\xi_{\mu,l} = \{\xi_{1\mu,l}^\mu, \ldots, \xi_{N\mu,l}^\mu\}; \mu = 1, \ldots, p$, are stored on the sites of each layer in the learning stage following a Hebbian rule

$$J_{ij}^l = \frac{c_{ij}J_0}{cN}(1 - \delta_{ij}) \sum_{\mu=1}^p \xi_{i\mu,l}^\mu \xi_{j\mu,l}^\mu ; \quad K_{ij}^l = \frac{J}{N} \sum_{\mu=1}^p \xi_{i\mu,l}^\mu \xi_{j\mu,l}^{\mu-1}$$

(3)

where each $\xi_{i\mu,l}^\mu = \pm 1$, with probability $\frac{1}{2}$, is the component of the pattern $\mu$ on neuron $i$ of layer $l$. Thus, $\alpha = p/N$ is the storage ratio of patterns per site of the Hopfield model [1], whether the sites are connected or not within a layer, since they are always connected to the sites of the adjacent layers. The parameters $(J_0, J)$ control the relative strength of the recurrent and feed-forward interactions, respectively, and they are the same for all layers, except on the first layer where $K_{ij}^1 = 0$, for all $(i, j)$, and where one may distinguish between a free relaxation and a clamped operation defined below [11]. The recurrent part of the synaptic interaction is symmetrically diluted by means of the set of identically distributed random variables $\{c_{ij}\}$, with $c_{ij} = c_{ji}$, such that $c_{ij} = 1$ with probability $c$ and zero with probability $1 - c$. The average number of connected neurons in each layer, the so-called connectivity, is $cN$ and we focus our attention in this work on dense networks in which $c = O(1)$ implying that $cN \gg 1$ within each layer [22]. When $c = 1$ the network is fully connected and letting $c$ go to zero, after the thermodynamic limit $N \rightarrow \infty$, one has the network with strongly (symmetrical) diluted layers. One should expect that the synaptic dilution within the layers will have effects on the feed-forward information processing of the network, as long as $J_0 \neq 0$, and it is this that we want to study here.

When $J = 0$ the model reduces to a set of $L$ decoupled symmetrically diluted attractor Hopfield-like networks which can be solved by means of equilibrium statistical mechanics using the replica method [22, 23, 24]. In that case, the intra-layer synaptic interaction becomes an effective interaction which is the sum of a Hebbian term and a Gaussian noise that will be determined below. In the case of full connectivity ($c = 1$) the noise term vanishes and the critical storage ratio is that of the usual Hopfield model, $\alpha_c \approx 0.138$.

In contrast, when $J_0 = 0$, we have a purely feed-forward network with full inter-layer synaptic interactions between neurons on consecutive layers which can be solved exactly for all $\alpha$, including the saturation of stored patterns, in the large $N$ limit by means of a signal-to-noise analysis which is a truncated version of a neurodynamic approach developed some time ago [25, 26]. In that case, the local fields follow a Gaussian distribution allowing
for the derivation of exact recursion relations for the order parameters in the limit of $N \to \infty$ with a critical storage capacity $\alpha_c \approx 0.269$. We expect this capacity, for $J_0 = 0$, to remain the same under exclusive synaptic dilution within the layers but there should be a gradual change of $\alpha_c$ with increasing $J_0$ and an eventual maximum $\alpha_c$ for an intermediate situation between feed-forward and recurrent operation, as in the case of the fully connected network ($c = 1$) where $\alpha_c \approx 0.317$. [11]

Coming back to the input at the first layer, one may have either a free relaxation from a specific initialization as a recall cue or a so-called "clamped" operation where the recall cue is given by an externally specified but stationary random state vector on that layer. We will discuss the effects of dilution on both modes.

Due to the feed-forward nature of the interactions between consecutive layers the network will attain a stationary state at $T = 0$. This will be the case for any layer $l$ when the input from layer $l - 1$ becomes stationary. The equilibrium states of the network may then be associated with the stationary states of the dynamics with a probability distribution given by a Boltzmann form $p(\sigma^l) \sim e^{-\beta H(\sigma^l)}$, for any layer, with the Hamiltonian

$$H(\sigma^l) = -\frac{1}{2} \sum_{ij} J_{ij}^l \sigma_i^l \sigma_j^l - \sum_{ij} K_{ij}^l \sigma_i^l \sigma_j^{l-1},$$

(4)

which enables one to solve the model first for finite $T$ and then taking the $T \to 0$ limit, as in previous work [11].

To study the performance of the network we introduce the macroscopic overlaps between the state and the stored patterns on that layer,

$$m_{\mu,l}(\sigma^l) = \frac{1}{N} \sum_i \xi_{\mu,l}^i \sigma_i^l$$

(5)

and assume, as usual, that in equilibrium only a finite number $k$ of the patterns is condensed with overlap of $O(1)$, for $\mu = 1, \ldots, k$, and the remaining $p - k$ overlaps are assumed to be macroscopically small of $O(1/\sqrt{N})$. The local field $h_i^l(\sigma^l, \sigma^{l-1})$ will not only have contributions from these overlaps but also from the stationary overlaps of the previous layer $l - 1$, due to the second term in Eq.(2),

$$\tilde{m}_\mu = \frac{1}{N} \sum_i \xi_{\mu,l-1}^i \sigma_i^{l-1},$$

(6)

with similarly defined condensed and non-condensed components.
3 Free energy and stationary states

We consider the partition function

$$Z = \sum_{\sigma} e^{-\beta H(\sigma)},$$

(7)

with $H(\sigma)$ given by Eq.(4), and calculate the free energy treating, as usual, the non-condensed patterns $\mu > k$ as quenched disorder over which we average the logarithm of the partition function by means of the replica method to obtain the free energy $f$ per site as

$$-\beta f = \lim_{n \to 0} \lim_{N \to \infty} \frac{1}{N n} \{ \langle [Z^n]_c \rangle_{\text{patt}} - 1 \}.$$  

(8)

Here, $\langle [Z^n]_c \rangle_{\text{patt}}$ is the average over the quenched disorder, due to the patterns, of the connected replicated partition function $[Z^n]_c$ obtained by taking the average over the random symmetrical dilution.

With the dilution only of synapses within the layers, we may write

$$[Z^n]_c = \sum_{\{\sigma^1\} \ldots \{\sigma^n\}} e^{\beta \sum_{\alpha} \sum_{i \neq j} K_{ij}^l \sigma_i^l \sigma_j^l - 1} \prod_{i < j} \left[ e^{\beta J_{ij}^l} \sum_{\alpha} \sigma_i^l \sigma_j^l \right]_c,$$

(9)

where $\alpha$ (and $\beta$, below) denotes the replica index, whenever it appears as a subindex, and $\{\sigma^n\}$ is the set of state vectors in the replica $n$. Using the now standard procedure to perform the configurational average over the symmetric dilution in the case of weak synapses $J_{ij}^l = O(\frac{J_0}{\sqrt{N}})$ in the large $N$ limit one finds

$$\prod_{i < j} \left[ \exp(\beta J_{ij}^l \sum_{\alpha} \sigma_i^l \sigma_j^l) \right]_c = \exp \left\{ \frac{\beta^2 J_0^2 \alpha (1 - c)}{4 c N} \sum_{\alpha \beta} \left( \sum_i \sigma_i^l \sigma_i^{l^\prime} \right)^2 \right\} \times \exp \left\{ \frac{\beta N J_0}{2} \sum_{\mu=1}^p \sum_{\alpha} m_{\mu,i}^2(\sigma_{\alpha}) \right\}. \tag{10}$$

Thus, the effect of the dilution amounts to the replacement of the intralayer synapses $J_{ij}^l$ by the effective interaction

$$J_{ij}^l \text{eff} = \frac{J_0}{N} \sum_{\mu=1}^p \xi_i^{\mu,l} \xi_j^{\mu,l} + \delta_{ij}^l,$$

(11)

for $i \neq j$ which is the sum of a Hebbian term for a fully connected recurrent network and a Gaussian noise $\delta_{ij}^l$ of mean zero and variance

$$\Delta^2 = \langle [\delta_{ij}^l]^2 \rangle_c = J_0^2 \alpha (1 - c)/c.$$

(12)
Coming back to the averaged partition function we obtain

\[
\langle [Z^n]_{c,\text{patt}} \rangle = e^{-\frac{1}{2} J_0 \alpha \beta n N} \times \left\langle \sum_{\langle \sigma^1 \rangle \ldots \langle \sigma^n \rangle} \exp \left\{ \frac{\beta^2 J_0^2 \alpha (1 - c)}{4 c N} \sum_{\alpha, \beta} \left( \sum_i \sigma_{i\alpha}^l \sigma_{i\beta}^l \right)^2 \right\} \times \exp \left\{ \beta N \sum_\mu \sum_\alpha \left[ \frac{J_0}{2} m_{\mu,l}^2 (\sigma_{\alpha}^l) + J m_{\mu,l} (\sigma_{\alpha}^l) \tilde{m}_\mu \right] \right\} \right\rangle \xi (13)
\]

where the average is over the patterns \( \xi \) in the overlaps \( m_{\mu,l} (\sigma_{\alpha}^l) \), and \( \tilde{m}_\mu \) are the already stationary overlaps in the previous layer. The sum over \( \mu \) in the exponent is over both, \( \mu \leq k \) (condensed) and \( \mu > k \) (non-condensed) patterns.

We follow the extension of the standard procedure [11] to calculate \( \langle [Z^n]_{c,\text{patt}} \rangle \) and obtain the free energy per site in the large-\( N \) limit by means of the method of steepest descent. We find the replica-symmetric form

\[
f = \frac{1}{2} \alpha [J_0 + \beta r (1 - q)] + \frac{1}{2} J_0 \sum_{\mu \leq k} m_{\mu}^2 + \frac{\alpha}{2 \beta} \left\{ \log [1 - \beta J_0 (1 - q)] - \frac{J_0 \beta q}{1 - \beta J_0 (1 - q)} \right\} - \frac{\beta J^2 (1 - q)}{2 [1 - \beta J_0 (1 - q)]} \sum_{\mu > k} \tilde{m}_\mu^2 - \frac{1}{4} \beta \Delta^2 (1 - q^2) - \frac{1}{\beta} \left\langle \int D z \log \left\{ 2 \cosh \beta \left( \sum_{\mu \leq k} \xi_\mu (J_0 m_{\mu} + J \tilde{m}_\mu) + z \sqrt{\alpha r} \right) \right\} \right\rangle \left\{ \xi_{\mu} \right\} (14)
\]

where \( \Delta^2 \) is given by Eq. (12), \( Dz = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2} z^2} dz \) and all quantities refer to any given layer, except for \( \tilde{m}_\mu \) which refers to the previous layer. Here, \( m_{\mu}, q \) and \( r \) are the replica-symmetric parameters that will be determined as solutions of the saddle-point equations \( \partial f / \partial m_{\mu} = \partial f / \partial q = \partial f / \partial r = 0 \). The condensed overlap \( \tilde{m}_\mu \) for \( \mu \leq k \) will be taken as an input from the previous layer and the sum over non-condensed patterns, \( \sum_{\mu > k} \tilde{m}_\mu^2 \), provides a further link with that layer which will be determined below.

The parameters are interpreted as follows: \( m_{\mu} = \langle m_{\mu,l} (\sigma_{\alpha}^l) \rangle \) is the replica symmetric average overlap with pattern \( \xi_{\mu,l}^\alpha \) for all replicas \( \alpha \) given by Eq. (5), while \( q = \langle \sigma_{\alpha\alpha}^l \sigma_{\beta\beta}^l \rangle \) is the replica symmetric average of the spin-glass order parameter \( q_{\alpha\beta} = \frac{1}{N} \sum_i \sigma_{\alpha i}^l \sigma_{\beta i}^l \), for \( \alpha \neq \beta \), where the brackets stand for a thermal and configurational average over the patterns. As usual, \( r \) accounts for the introduction of the spin-glass order parameter into the
replicated partition function and it is given here by

\[ \alpha \beta^2 r \alpha x = \beta J^2 x_\alpha x_\beta \sum_{\mu > k} \tilde{m}^2_{\mu, l} + \beta^2 \Delta^2 q \]

\[ + \alpha \beta J_0 \int Dz \frac{z_\alpha z_\beta e^{-\frac{1}{2} \beta J_0 z_\alpha z_\beta}}{\frac{1}{2} \beta J_0 z_\alpha z_\beta}, \]  \quad (15)

for \( \alpha \neq \beta \), where \( x_\alpha = \sum_\gamma [I - \beta J_0 q_\alpha \gamma^{-1} \beta] \) in which \( I \) is the identity matrix and \( q = \{ q_\alpha \beta \} \), while \( z \in \mathbb{R}^n \) and \( Dz = (2\pi)^{-n/2} e^{-\frac{1}{2} z^2} d\mathbf{z} \).

In order to determine \( \sum_{\mu > k} \tilde{m}^2_{\mu} \), consider the formal relationship for the derivative of the free energy with respect to the control parameter \( J_0 \), which can be derived directly from Eqs. (8), (12) and (13)

\[ \sum_{\mu > k} \langle m^2_{\mu, l} (\sigma^l) \rangle = \alpha - \alpha \beta J_0 \left[ \frac{1-\beta J_0 (1-q^2)}{e} \right] - \sum_{\mu \leq k} \langle m^2_{\mu, l} (\sigma^l) \rangle - 2 \frac{\partial f}{\partial J_0}, \] \quad (16)

in replica symmetric form. Note that the second term on the right-hand side contains the noise in the form \(-\beta (1-q^2) \Delta^2 / J_0 \) that will be cancelled by a similar term from the last part of the equation. Using the explicit expression for the free energy in Eq. (14), one finds

\[ \sum_{\mu > k} \langle m^2_{\mu, l} (\sigma^l) \rangle = \frac{1}{[1-\beta J_0 (1-q)]^2} \left[ \alpha [1-\beta J_0 (1-q^2)] \right] 
\]

\[ + \beta^2 J^2 (1-q)^2 \sum_{\mu > k} \langle m^2_{\mu, l-1} (\sigma^{l-1}) \rangle \], \quad (17)

relating the non-condensed overlaps in two consecutive layers, where \( q \) is the spin-glass order parameter in layer \( l \). A similar relationship can be derived for layer \( l-1 \) where \( \langle m^2_{\mu, l-2} \rangle \) becomes the stationary input \( \tilde{m}^2_{\mu} \) for that layer. Using this relationship with the saddle-point equation \( \partial f / \partial q = 0 \) for layer \( l-1 \), in order to eliminate \( \sum_{\mu > k} \langle m^2_{\mu, l-2} \rangle \), and applying again the saddle-point equation to layer \( l \) we obtain the third recursion relation below between the macroscopic order parameters in two consecutive layers \( (\mathbf{m}, q, r) \rightarrow (\mathbf{m}', q', r') \).

Thus, altogether, the saddle-point equations for the dense network in which \( c = O(1) \) yield the recursion relations

\[ m' = \langle \xi \int Dz \tanh \beta \left\{ \xi_\alpha (J_0 m' + J m) + z \sqrt{\alpha [\tilde{r}' + J_0^2 (1-c) q'/c]} \right\} \rangle \xi \] \quad (18)

\[ q' = \langle \int Dz \tanh^2 \beta \left\{ \xi_\alpha (J_0 m' + J m) + z \sqrt{\alpha [\tilde{r}' + J_0^2 (1-c) q'/c]} \right\} \rangle \xi \] \quad (19)
\[
\tilde{r}'[1 - \beta J_0(1 - q')]^2 - J_0^2 q' = \beta^2 J^2 (1 - q)^2 \tilde{r} - J^2 q + \frac{J^2 (1 + q)}{1 - \beta J_0 (1 - q)} ,
\]
where the last equation is formally the same as that for the fully connected recurrent layers in terms of the rescaled variable \( \tilde{r} = r/c \). Here, \( \mathbf{m} = (m_1, \ldots, m_k) \) is the condensed overlap vector in one layer and \( \mathbf{m}' = (m_1', \ldots, m_k') \) in the next layer. Given a state of the first layer, Eqs.(18)-(20) describe the evolution of the states of the network. We recover the fixed-point equations for a diluted purely recurrent network when \( J_0 = 1 \) \[24\] and the recursion relations for the layered feed-forward network, with full connection of units between layers, when \( J_0 = 0 \) for \( J = 1 \) \[7\]. For \( c = 1 \) we recover the equations of Coolen and Viana \[11\].

Note that the synaptic dilution in the case of the dense network enters only through the variance of the noise \( \Delta^2 \) in the effective intra-layer interactions. In contrast, in the extremely dilute limit \( c \to 0 \), it is more appropriate to rescale \[23,24\] \( \alpha \to \alpha' = \alpha/c = p/cN \), the ratio of stored patterns per mean number of connected sites, in place of rescaling \( r \). Eq.(20) becomes then a recursion relation that depends explicitly on \( c \).

Following earlier work, we consider either a clamped input or a free relaxation in the first layer \[11\]. In the first case, the state \( \mathbf{m}^1 \) in that layer may be any randomly chosen configuration with a given condensed overlap \( \mathbf{m} \). The first two recursion relations remain the same and the last one is replaced by

\[
\tilde{r}'[1 - \beta J_0(1 - q')]^2 - J_0^2 q' = J^2 .
\]

On the other hand, in the case of a free relaxation the macroscopic state \( (\mathbf{m}, q, r) \) of the first layer that follows from Eqs.(18)-(20) when \( J = 0 \) is determined by the equations

\[
\mathbf{m} = \langle \xi \int Dz \tanh \beta \left\{ \xi \cdot J_0 \mathbf{m} + z \sqrt{\alpha \tilde{r} + J_0^2 (1 - c) q/c} \right\} \rangle_\xi \tag{22}
\]

\[
q = \langle \int Dz \tanh^2 \beta \left\{ \xi \cdot J_0 \mathbf{m} + z \sqrt{\alpha \tilde{r} + J_0^2 (1 - c) q/c} \right\} \rangle_\xi \tag{23}
\]

\[
\tilde{r} = \frac{J_0^2 q}{[1 - \beta J_0 (1 - q)]^2} . \tag{24}
\]

We recover the known equations for the diluted recurrent network when \( J_0 = 1 \) \[24\]. Thus, Eqs.(18)-(24) extend the model of Coolen and Viana for the case of symmetric synaptic dilution in the recurrent layers. We take next the limit \( T \to 0 \) and eliminate the parameter redundancy writing \( J_0 = \frac{1}{2}(1 + \omega) \) and \( J = \frac{1}{2}(1 - \omega) \) \[11\].
4 Phase transitions

We turn next to the effects of dilution on the phase transitions in the model and consider three situations as in earlier work [11]. First, we look for the storage capacity of infinitely long chains in which a stationary state is reached. Then we focus attention on the first and the second layer in a search for multiple transitions between replica-symmetric states. The third transition we discuss is the replica-symmetry breaking transition at the de Almeida-Thouless (AT) line.

4.1 Saturation transition in an infinitely long diluted chain

We consider here the transition where the overlap disappears with an increasingly large storage of patterns. A stationary state is reached along the network when \((m', q', \tilde{r}') = (m, q, \tilde{r})\) and Eqs.(18)-(20) become

\[
m = \left\langle \xi \int Dz \tanh \beta \left\{ \xi \cdot m + z \sqrt{\alpha[\tilde{r} + J_0^2(1-c)/c]} \right\} \right\rangle \xi \tag{25}
\]

\[
q = \left\langle \int Dz \tanh^2 \beta \left\{ \xi \cdot m + z \sqrt{\alpha[\tilde{r} + J_0^2(1-c)/c]} \right\} \right\rangle \xi \tag{26}
\]

\[
\tilde{r} = \frac{(1-\omega)^2 + q(1+\omega)^2 - 2qC\omega(1+\omega)}{4[1 - \frac{\omega}{2}C(1+\omega)][1 - C(1+\omega) + \omega C^2]} \tag{27}
\]

where \(C = \beta (1-q)\). Taking the limit \(T \to 0\) and focusing attention on pure states, where \(m_\mu = m_\delta \mu \lambda, \) for some \(\lambda,\) we can do the integrations and the averages and follow the usual procedure [9] to reduce Eqs.(25)-(27) to a single equation, now in terms of the scaled overlap

\[
x = m/\sqrt{2\alpha[\tilde{r} + J_0^2(1-c)/c]} \tag{28}
\]

where \(\tilde{r}\) is given by Eq. (27) and \(q \to 1\) in the limit \(T \to 0\). We also have

\[
C \sim \left[ \frac{2}{\pi \alpha[\tilde{r} + J_0^2(1-c)/c]} \right]^{\frac{1}{2}} \exp(-x^2) \tag{29}
\]

and

\[
x \sqrt{\alpha} = \frac{erf(x)}{[1 + \omega^2]} [A(x, \omega)]^{\frac{1}{2}} \frac{1}{\sqrt{\pi}} \frac{(1-\omega)}{(1+\omega)^2} \left( \frac{1-c}{c} \right) A(x, \omega) \frac{1}{2} \tag{30}
\]
where

\[
A(x, \omega) = [\text{erf}(x) - \frac{2x}{\sqrt{\pi}} \exp(-x^2)][\text{erf}(x) - \frac{(1+\omega)x}{\sqrt{\pi}} \exp(-x^2)]
\times [\text{erf}(x) - \frac{2\omega x}{\sqrt{\pi}} \exp(-x^2)],
\]

(31)

\[
B(x, \omega) = \text{erf}(x) - \frac{(\omega + \omega^2)2x}{(1+\omega^2)} \frac{2}{\sqrt{\pi}} \exp(-x^2).
\]

(32)

When \(c = 1\) one recovers the relationship for the fully connected dual model [11], and when \(\omega = -1\) \((J_0 = 0)\) or \(\omega = 1\) \((J = 0)\) one gets back the result for the layered network [7] or the result for the diluted recurrent network [24], respectively.

The numerical solution of Eqs.(30)-(32) yields the critical storage capacity \(\alpha_c(\omega, c)\), for a given \(\omega\) and \(c\), as the value of \(\alpha\) where the solution with finite overlap (non-zero \(x\)) disappears. The result is shown in Fig. 2 for various values of \(c\). When \(\omega = 1\) (the purely recurrent network) one recovers the \(\alpha_c = 0.138\) for the fully connected network with \(c = 1\) [9]. On the other hand, when \(\omega = -1\) (the purely layered network) \(\alpha_c = 0.269\) for any value of \(c\), since the dilution is only in the recurrent part [7]. There is a maximum value of \(\alpha_c \sim 0.317\) for \(\omega \sim -0.12\), when \(c = 1\), in consistency with a previous result [11], and the maximum decreases and shifts towards the behavior of the layered network with increasing dilution. It has also been checked that although \(\alpha_c\) decreases with increasing dilution, the ratio \(\alpha_c/c = 0.629\), as one would expect in the extreme diluted limit \(c \to 0\) for \(\omega = 1\).

Although \(\alpha_c\) is a decreasing function of \(c\) for all values of \(\omega\), as one would expect, the network still has a relatively large storage capacity for a moderate amount of dilution and a balanced ratio of interaction strengths. For instance, for \(c = 0.1\) and \(\omega \approx -0.20\) \((J/J_0 \approx 1.5)\), \(\alpha_c\) is still the same as that for the fully connected network. Thus, the synaptic interactions between layers do not need to be much stronger than the interactions within the layers for a typical good performance with a moderately finite degree of dilution. As pointed out in the introduction, this may be of use for biological networks.

4.2 Multiple phase transitions in the first layers

We consider now the operation of the first two layers and we study the effects of dilution on multiple phase transitions that already appear at the
junction between the first and the second layer \[\Pi\]. Even limiting the study
to pure states as we do here, with \(m_\mu = m_\delta \mu \lambda\) for some \(\lambda\), the overlap is
layer dependent and we denote it by \(m\) and \(m'\) for the first and the second
layer, respectively.

We deal first with the clamped operation in the first layer, with a given
overlap vector \(m\), and take the \(T \to 0\) limit in Eqs.(18),(19) to obtain
\(m' = erf(y)\), where

\[
y = \frac{J_0 m' + J m}{\sqrt{2 \alpha \tilde{r}'^2 + J_0^2 (1 - c)/c}},
\]

(33)
is the appropriate scaled overlap and \(\tilde{r}'\) is given by Eq.(21). In that limit,
\(q' \to 1\) and \(C' = \beta (1 - q')\) becomes

\[
C' \sim \left[\frac{2}{\pi \alpha [\tilde{r}'^2 + J_0^2 (1 - c)/c]}\right]^{\frac{1}{2}} \exp(-y^2).
\]

(34)
When this is used we obtain the non-linear equation for \(y\),

\[
G(y, \alpha, m) \left[ F(y) + m \frac{1 - \omega}{1 + \omega} \right] = y \sqrt{2 \alpha} \left[ 1 + \rho \left( \frac{1 - \omega}{1 + \omega} \right)^2 \right]^\frac{1}{2},
\]

(35)
in which

\[
F(y) = erf(y) - \frac{2y}{\sqrt{\pi}} e^{-y^2},
\]

(36)

\[
G^2(y, \alpha, m) = 1 - \left( \frac{1 - c}{c} \right) \left[ \frac{y \sqrt{2 \alpha}}{erf(y) + m \frac{1 - \omega}{1 + \omega}} \right]^2
\]

(37)
and \(\rho = 1\). Solution of Eq.(35) with Eqs.(36) and (37) yields the overlap \(m'\)
for the second layer and the phase diagrams presented below.

We consider next the case of free relaxation of the first layer, and restrict
ourselves again to pure states. First, Eqs.(22)-(24) have to be solved for the
scaled overlap in the limit \(T \to 0\) and when the solution is taken as an input
in Eqs.(18)-(20), in the same limit, one obtains a new overlap \(m'\) on the
second layer for this mode of operation. Thus, we find first \(m = erf(w)\), where

\[
w = J_0 m' / \sqrt{2 \alpha \tilde{r}'^2 + J_0^2 (1 - c)/c}
\]

(38)
and \(C = \beta (1 - q)\) is now given by Eq.(29) with \(w\) in place of \(x\) and \(F(w)\) is
given by the same expression as in Eq.(36). We find the equation

\[
F(w) = \frac{w \sqrt{2 \alpha}}{G(w, \alpha, 0)}
\]

(39)
that yields the scaled overlap $w$ for free relaxation of the first layer. Following
similar steps as for the clamped operation, with the full expression for $r'$ in
Eq. (20), in place of Eq. (21), we find formally the result given by Eq. (35)
but now with
\[ \rho = \left[ \frac{m}{F(w)} \right]^2. \]  \hfill (40)

When $c = 1$, $G(y, \alpha, m) = 1$ for any $m$, and we recover the results for the
fully connected network.

It can be seen from the above equations that, as long as the dilution re-
 mains finite, there is a clear distinction between the two modes of operation,
as in the fully connected network \cite{11}. Already when $m = 0$, in the case of a
clamped input with $\rho = 1$, Eqs. (35)-(37) yield a critical $\alpha_c$ for a bifurcation
of a solution with $y \neq 0$ (meaning $m' \neq 0$), for any finite $c$. In contrast, one
finds that $\rho = \infty$ for free relaxation of the first layer and in that case the
only solution is $y = 0$, that is, $m' = 0$.

To obtain all the solutions that appear for general values of $m$ (in the
case of clamped input) and $c$, we look for the bifurcations from Eq. (35) that
are given by this equation and its derivative with respect to $y$ which have
to be solved simultaneously. This can be done either for the clamped input,
with $\rho = 1$, or for the case of a free relaxation of the first layer, with $\rho$ given
by Eq. (40). The outcome for the case of a clamped input is shown in Fig.
3 for $m = 0$ and in Fig. 4 for $m = 1$, in both cases for various values of $c$,
together with the saturation transition for the infinite chain in each case, for
reference. This is not the saturation transition for the chain of two layers
we are discussing here, except for $\omega = 1$, as will be seen below.

The phase diagram for the two-layer chain when $m = 0$ has two regions.
In region I, everywhere above the solid phase boundary $m' = 0$ is the only
solution. In region II, below the solid line, there are three stable solutions:
$m'_1 \sim 1$, $m'_2 \sim -1$ and $m'_3 = 0$, such that $|m'_1| \neq |m'_2|$, except for $\omega = 1$
where $|m'_1| = |m'_2|$. The phase boundary of multiple solutions of region II
meets the saturation transition of the infinitely long chain at $\omega = 1$. This is
the place where the model becomes a set of purely recurrent networks, with
no interaction between layers and, hence, the stationary states of the second
layer are already those of the infinite chain. The solution with overlap zero
and $q' = 1$, above and below the phase boundary, describes a spin-glass
state.

In the case of an input $m = 1.0$, one obtains two further regions of
coeexisting stable states shown in Fig. 4. Now in region II one has two
solutions, $m'_1 \sim 1$ and $m'_2 \sim -1$, such that $|m'_1| \neq |m'_2|$. In region III
there are two stable retrieval solutions with $m'_1 \sim 1$, $m'_2$ smaller than one.
In region IV there are three stable retrieval solutions \( m'_1 \sim 1 \), \( m'_2 \sim -1 \) and \( m'_3 \) small and positive, such that \( |m'_1| \neq |m'_2| \), except for \( \omega = 1 \) where \( |m'_1| = |m'_2| \). The small solution \( m'_3 \) vanishes precisely at \( \omega = 1 \), leaving the known results for the purely recurrent network, regardless of dilution. The phase boundaries that go up to \( \omega = 1 \) again meet there the saturation transition of the infinitely long chain.

Again, all three regions of retrieval are reduced with increasing dilution, particularly regions III and IV, but they are still there for a finite dilution of \( c = 0.1 \). In the case of extreme dilution only region II is left over a tiny part of the phase diagram in terms of \( \alpha = p/N \). But in this limit it is more appropriate to consider \( \alpha' = \alpha/c \), which remains finite. Similar results are obtained for other finite values of the fixed input overlap \( m \). We also analyzed the effects of free relaxation of the first layer on the performance of the second layer and found qualitatively similar results to those described here for the case of clamped input.

As one can see, the presence of these regions and the coexistence of various retrieval states is a feature of the competition between the recurrent and the layer information processing, which is considerably diminished with increasing dilution. When \( c = 1 \) we recover the earlier results for the fully connected network [11].

4.3 The replica-symmetry breaking (RSB) transition

We look here for the effect of synaptic dilution on the de Almeida-Thouless (AT) transition [27] where the replica symmetric solution for the saddle point equations ceases to be valid at the AT line. There is RSB at sufficiently low \( T \) in the fully recurrent network, that is when \( \omega = 1 \), and there is no RSB in the purely layered network with \( \omega = -1 \). In both, the fully connected and the diluted dual model RSB appears as soon as \( \omega > -1 \), that is already for an arbitrarily small recurrent interaction [11]. Although the AT line is a boundary in the \((T, \alpha)\) plane and our work, as well as that in that ref. [11], are restricted to \( T = 0 \) one may get an idea of the size of the RSB region by looking at the deformation of that boundary with a decrease of \( \omega \) towards \( \omega = -1 \), and that is what we do next.

The AT line in the present model is obtained considering the fluctuations around the replica symmetric forms for the spin glass order parameter \( q_{\alpha\beta} = \langle \sigma_i \sigma_j \rangle \) and the auxiliary parameter \( \hat{q}_{\alpha\beta} \) given by the right-hand-side of Eq.(15) for \( \alpha \neq \beta \). Proceeding in the usual way on the free energy prior to the assumption of replica symmetry we obtain the AT line for the diluted
dual model,

\[
1 = \alpha \beta^2 \left\{ \frac{J_0^2}{[1 - \beta J_0(1 - q)]^2} + J_0^2 (1 - c)/c \right\} \langle \int Dz \right. \\
\times \left. \cosh^{-4} \beta \left\{ \xi (J_0 m + J_0 \tilde{m}) + z \sqrt{\alpha [r + J_0^2 (1 - c)q/c]} \right\} \right\} \xi . \tag{41}
\]

We recover the expression for the purely recurrent network when \( J_0 = 1 \) and \( J = 0 \) [24] and that for the fully connected dual model when \( c = 1 \) [11]. The replica symmetric solution Eqs.(18)-(20) is stable when the right-hand side of Eq.(41) is less than one. This corresponds to the regions above the AT lines shown in Fig. 5 for three different values of \( \omega \) when \( c = 1 \), \( c = 0.1 \) and \( c = 0.001 \). It can be seen that the domains of RSB below the AT lines continue to become smaller with a decrease of \( \omega \) (that is with reduced recurrent interactions) towards \( \omega = -1 \) where the AT line coincides with the \( T = 0 \) axis as in the fully connected network model [11]. The effect of (finite) dilution is to increase the region of RSB with a shift of the AT lines towards smaller values of \( \alpha = p/N \). Nevertheless, the effects of RSB are smaller than in the purely recurrent network justifying the analysis in this work based on the assumption of replica symmetry.

5 Conclusions

We studied in this work the effects of finite symmetric dilution on the performance of a dual model that combines information processing of recurrent and feed-forward networks. The model consists of a feed-forward chain of recurrent networks and the dilution is in the symmetric synaptic interactions of the recurrent layers. Our analysis extends the original work of Coolen and Viana with symmetric synaptic interactions between units within the layers and non-symmetric interactions between units in consecutive layers. The competition between these interactions is responsible for the behavior shown in the phase diagrams. On one hand, a small inter-layer feed-forward interaction between layers produces the multiple transitions that are found either in the case of clamped operation with finite \( m \) or in the case of free relaxation. In the other extreme, an infinitesimal symmetric recurrent interaction already yields a RSB transition.

A stationary state replica analysis was carried out in this work assuming a frozen-in dilution of synaptic interactions in order to study the saturation transition in infinitely long chains and the multiple transitions that already appear at the junction between the first and second layer. We found that
finite synaptic dilution produces a gradual change of the performance of the model reducing all the regions of stable retrieval states, in particular those where multiple solutions appear due to the competition between recurrent and feed-forward processing. One would expect that the study of further initial layers should yield a performance closer to that of long chains.

Although the performance of the dual model is reduced by synaptic dilution, we showed that there can still be a considerable output in a long chain of layers in the case of a finite amount of dilution of the synapses within the layers and pointed out that this feature could be relevant for dual models of biological networks.

There are several extensions that one may conceive of the work presented in this paper. It would be interesting to consider the effects of non-linear synapses of a general form [22] and one could also think of relaxing the symmetry of the synaptic interactions within the layers. A dynamical approach would be necessary in that case.

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**Figures**

![Figure 2: Critical storage capacity $\alpha_c$ for an infinitely long chain with various degrees of dilution, as indicated. The fully connected network, with $c = 1$, is also shown for reference. Increasing dilution is described by a decreasing value of $c$.](image-url)
Figure 3: Phase diagram for the two-layer network under clamped operation with overlap $m = 0$ in the first layer for various degrees of dilution, as indicated. The saturation transition (in dotted lines) is shown for reference and the phases I and II are described in the text.
Figure 4: Phase diagram for the two-layer network under clamped operation with overlap $m = 1$ in the first layer for various degrees of dilution, as indicated. The saturation transition (in dotted lines) is shown for reference and the phases I-IV are described in the text.
Figure 5: The replica-symmetry breaking transition (AT line) for \( \omega = 1.0, 0, -0.5 \) and various degrees of dilution. For \( \omega = -1 \) (purely layered network) the AT line coincides with the \( T = 0 \) line.