THE STUDY ON THE KERNEL OF SERIES

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Abstract. Taylor series is a very important tool to study functions, but the application of Taylor series is limited. Because some functions cannot be expanded into Taylor Series, but it can be expanded into the Fourier Series. However, in order to verify the convergence of partial sum of Fourier Series, we transform it into the convolution of the original function and the Dirichlet kernel, and we introduced the concept of Good Kernel. According to the Good kernel, we know the convergence of some function series. However, the Dirichlet kernel is NOT a Good Kernel. Thus, we introduce the notion about the Cesàro means and modify it into a Fejer kernel which is a Good Kernel.

1. Literature Review
Using a series of function to represent a function is very impressive and the study of the Fourier Series is extremely important in dealing with various problem in many fields. For example, [3] for transient heat transfer problem with uniform initial temperature, various methods can be applied and one of the most powerful methods is the Laplace transformation method. The given governing partial differential equation should be transformed into an algebraic equation (subsidary equation). Once the subsidiary equation can be transformed inversely, the required solution of the original differential equation will be obtained. The inverse transform is not easy, especially in some complicated cases. But the Fourier series technique is used to obtain the inversion of the subsidiary equation.

In this article, it mainly discusses about the superiority of the Fourier Series, the condition of Good kernel and why the Dirichlet kernel is NOT a Good kernel? And how to find the Fejer kernel as a substitute of Dirichlet kernel as a Good kernel.

2. Methodology
In the process of proving the Dirichlet kernel is not a Good kernel, we convert the summation form of the Dirichlet kernel into the formula with cos function and sin function using the identical formula. Because we can easily calculate the integration of the trigonometric function. After a series of transformation, we get the integration is approximate to clogN, which is not convergent. In this way we know the Dirichlet kernel does not satisfy the third condition of Good kernel.

In the process of proving Fejer kernel is a Good kernel, we firstly utilize the definition of the Cesàro summation to represent Fejer kernel using Dirichlet kernel, then we use the summation form to represent Dirichlet kernel, at the same time, we set \( \omega = e^{ix} \) to make the formula looks more concise, which is helpful in the following deducing and unite the like terms. Additionally, in processing the consecutive summation of sin function we put the additional sin \( \left[ \frac{x}{2} \right] \) into the bracket and use the double angel formula to divide the product of two sin function into the subtraction of the cos function, which can be eliminated by the following terms. After this process, it just leaves the first and the last terms not be eliminated. The result is got. About the proof of Fejer kernel satisfy the first
condition, which is the integration of the Fejer kernel is 1. Similarly, we convert the summation form into the cos function and sin function form using the Euler formula. Easily noticing the integration of cos function and sin function in the region \([-\pi, \pi]\) is 0 due to the fact that the period of cos function and sin function is the multiple of 2 \(\pi\). Thus, we get the integration of Fejer kernel is 1, which satisfy the first condition of the Good kernel.

3. Introduction
In 1715, in order to restore a function using very little information, the Brook Taylor proposed that Taylor Series which represent a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. This measure is excellent due to that it can depict a function grounded on a little information about the function---its derivative[5]. However, there is an obvious flaw in this method, that is the requirement for this representation is harsh and non-guaranteed, because it require the function is higher order differentiable. Thus, in 1807, Jean-Baptiste Joseph Fourier (1768–1830) introduced that Fourier Series which uses a series of trigonometric series to represent a function, which is also widely used in solving some mathematical problems like the differential equation.

4. Discussion
From 16 century Taylor Series is a very important conception, and plays an important role in solving some problem. Sometimes there are some functions cannot be expanded using Taylor Series, because they are not higher order differentiable. Even they are smooth, their Taylor Series’s convergence region may be proper set of entire domain or the summation of the function may not be the original function.

Example:
Case1: (The Function is not higher differentiable)
Such as \(f(x)=|x|\) is not differentiable in the point when \(x=0\).

Case2: (The Function is not differentiable in its entire domain)
Such as
\[
 f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots
\]
its convergence region[4] \(r = \frac{a_{n+1}}{a_n} = 1\)
it means that in the entire domain \((-1,1)\),
\[
 f(x) = 1 + x + x^2 + \cdots + x^n + \cdots
\]
if \(x \geq 1\) or \(x \leq -1\),
\[
 f(x) = 1 + x + x^2 + \cdots + x^n + \cdots
\]

Case3: (the summation of the function is not the original function [2])
Such as
\[
 f(x) = \begin{cases} 
 e^{-\frac{1}{x^2}}, & x \neq 0 \\
 0, & x = 0 
\end{cases}
\]
\[
 f'(x) = e^{-\frac{1}{x^2}} \cdot \left(-\frac{2!}{x^3}\right)
\]
\[
 f''(x) = e^{-\frac{1}{x^2}} \cdot \left(-\frac{2!}{x^3} + \frac{3!}{x^4}\right)
\]
\[
 f^{(n)}(x) = e^{-\frac{1}{x^2}} \cdot \left(-\frac{2!}{x^3} + \frac{3!}{x^4} + \cdots + \frac{n!}{x^{n+1}}\right)
\]
we have \(f^{(n)}(0) = 0\) \(n \in N\)

Because \(f(x)=e^{-\frac{1}{x^2}}\) is continuous and higher differentiable in its entire domain
According to Taylor series
\[ f(x) = f(0) + f'(x)x + f''(x)x^2 + \cdots + f^{(n)}(x)x^n + \cdots \]
That is to say
\[ f(x) = e^{-\frac{1}{x^2}} = f(0) + f'(x)x + f''(x)x^2 + \cdots + f^{(n)}(x)x^n + \cdots = 0 \]
But \( f(x) = e^{-\frac{1}{x^2}} \neq 0 \) in its entire domain. So, the Taylor series is inapplicable.

Compared with the Taylor Series, Fourier Series is better due to that nearly every continuous function has its Fourier Series. \( S_n(f,x) = \sum_{n=1}^{N} A_n \cos nx + \sum_{n=0}^{N} B_n \sin nx = \sum_{n=-N}^{N}\hat{f}(n)e^{inx} = (f*\hat{D}) (x) \) Moreover, its Fourier Series always convergent to itself. And for any continuous function, its product with sin function, cos function is still continuous, so we can compute integrals and get Fourier coefficients. Starting from the Fourier coefficient, Fourier series can be defined.

The convergence of the series of functions is essentially the convergence of its partial sum sequence It is known from the calculation of its partial summation sequence which is the convolution of the original function and a particular function, that is to say, \( S_n(f,x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)D_n(t)dt \), and the particular function is defined as the Dirichlet kernel \( D_n(x) = \sum_{n=-N}^{N} e^{inx} \).

If the partial summation, the convolution of Dirichlet kernel and original function, is convergent to original function, and then we can expand this function into the Fourier Series. So, we just need to verify the convergence of the convolution of the original function and the Dirichlet kernel.

For this reason, the following sufficient conditions are introduced, the definition of good kernel is given, and the properties of good kernel are proved below.

A family of kernels \( \{K_n\}_1^\infty \) on the circle is said to be a family of good kernels if it satisfies the following properties[1]:

a) For all \( n \geq 1 \),
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x)dx = 1 \]

b) There exists \( M \geq 0 \) such that for all \( n \geq 1 \),
\[ \int_{-\pi}^{\pi} |K_n(x)|dx \leq M \]

c) For every \( \delta > 0 \),
\[ \int_{\delta \leq |x| \leq \pi} |K_n(x)|dx \to 0, \quad \text{as } n \to \infty \]

The proof given below explains why the good kernel can solve the problem and is a sufficient condition to the convergence of the partial summation.

(proof[1]: ) Let \( \{K_n\}_1^\infty \) be a family of kernels, and \( f \) an integrable function on the circle. If \( \epsilon > 0 \) and \( f \) is continuous at \( x \), choose \( \delta \) so that \( |y| < \delta \) implies \( |f(x-y) - f(x)| < \epsilon \). Then, by the first property of good kernels we can we write
\[ (f * K_n)(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)f(x-y)dy - f(x) \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)[f(x-y)dy - f(y)]dy \]

Hence,
\[ |(f * K_n)(x) - f(x)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)[f(x-y)dy - f(y)]dy \right| \]
\[ \leq \frac{1}{2\pi} \int_{|y|<\delta} |K_n(y)||f(x-y) - f(x)|dy + \frac{1}{2\pi} \int_{\delta \leq |y| \leq \pi} |K_n(y)||f(x-y) - f(x)|dy \]

\[ \leq \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} |K_n(x)|dx + \frac{2B}{2\pi} \int_{\delta \leq |y| \leq \pi} |K_n(y)|dy \quad \# \]

where \( B \) is a bound for \( f \). The first term is bounded by \( \epsilon M/2 \) because of the second property of good kernels. By the third property we say that for all large \( n \), the second term will be less than \( \epsilon \). Therefore, for some constant \( C > 0 \) and all large \( n \) we have \( |(f * K_n)(x) - f(x)| \leq C\epsilon \), thereby proving the assertion. If \( f \) is continuous everywhere, then it is uniformly continuous, and \( \delta \) can be chosen independent of \( x \). This provides the desired conclusion that \( f * K_n \to f \) uniformly.

However, the Dirichlet kernel is not a good kernel.

(Proof:)

we define \( L_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(x)|dx \)

and Dirichlet kernel have a closed form formula:

\[ D_N(x) = \frac{\sin((N + \frac{1}{2})x)}{\sin(x/2)} \]

using this formula, we derive that

\[ L_n(x) \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((N + \frac{1}{2})x)}{\sin x} \right| dx \]

let \( t = (N + \frac{1}{2})x \)

\[ \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin t}{t} \right| dt \]

\[ \geq c \int_{\pi}^{\pi N} \left| \frac{\sin t}{t} \right| dt + O(1) \quad (c > 0) \]

\[ = \sum_{k=1}^{N-1} \int_{\pi k}^{\pi (k+1)} \left| \frac{\sin t}{t} \right| dt + O(1) = c \log N + O(1) \quad \# \]

Thus, we conclude that

\[ \int_{-\pi}^{\pi} |D_N(x)|dx \geq c \log N , \text{ as } N \to \infty. \]

This violates the third condition of the good kernels, so the Dirichlet kernel is not a good kernel.
For this reason, the concept of Cesaro summation and Fejer kernel must be introduced to verify some of its properties, and it is used to prove that the Fourier series can converge back to itself.

Proposition:
The average of the first N partial sum denoted by \( \sigma_N = \frac{S_0 + s_1 + \cdots + s_{N-1}}{N} \), is called the \( N^{th} \) Cesaro mean of the sequence \( \{s_k\} \). To see this, we form the \( N^{th} \) Cesaro mean of the Fourier series, which by definition is

\[
\sigma_N(f)(x) = \frac{S_0(f)(x) + S_1(f)(x) + \cdots + S_{N-1}(f)(x)}{N}
\]

Since \( S_N(f) = f \ast D_N \), we find that \( \sigma_N(f)(x) = (f \ast F_N) \), where \( F_N(x) \) is the \( N^{th} \) Fejer kernel given by

\[
F_N = \frac{D_0(x) + D_1(x) + \cdots + D_{N-1}(x)}{N}
\]

and the Fejer kernel is a good kernel.

(Proof:)
Note that

\[
NF_N(x) = D_0(x) + D_1(x) + \cdots + D_{N-1}(x), \quad \text{where } D_n(x) \text{ is Dirichlet kernel.}
\]

\[
NF_N(x) = \sum_{n=-N}^{0} e^{inx} + \sum_{n=-1}^{1} e^{inx} + \cdots + \sum_{n=-(N-1)}^{N-1} e^{inx}
\]

Set \( \omega = e^{ix} \)

\[
= 1 + (\omega^{-1} + 1 + \omega) + \cdots + (\omega^{-(N-1)} + \omega^{-(N-1)} + \cdots + \omega^{N-1})
\]

\[
= 1 + \frac{\omega^{-1}(1 - \omega^2)}{1 - \omega} + \cdots + \frac{\omega^{-(N-1)}(1 - \omega^{2N-1})}{1 - \omega}
\]

\[
= 1 + \frac{\omega^{-1} - \omega^2}{1 - \omega} + \cdots + \frac{\omega^{-(N-1)} - \omega^N}{1 - \omega}
\]

\[
= \sum_{n=0}^{N-1} \frac{\omega^{-n} - \omega^{n+1}}{1 - \omega}
\]

Because

\[
D_N(x) = \sum_{n=-N}^{N} e^{inx} \quad \text{set } \omega = e^{ix}
\]

\[
= \sum_{n=-N}^{N-1} \omega^{n} + \sum_{n=0}^{N} \omega^{n}
\]
\[
\begin{align*}
\omega^{-N} - \omega^{-1} + \frac{1 - \omega^{N+1}}{1 - \omega} &= 1 - \omega^{-1} - \omega^{N+1} \frac{1}{1 - \omega} \\
\omega^{-N} - \omega^{N+1} &= \omega^{-1/2} - \omega^{N+1/2} \\
\frac{\omega^{-N-1/2} - \omega^{N+1/2}}{\omega^{-1/2} - \omega^{1/2}} &= \frac{\sin((N + 1/2)x)}{\sin(x/2)} \\
\text{so } F_N(x) &= \frac{1}{N} \sum_{n=0}^{N-1} \frac{\omega^{-n} - \omega w^{n+1}}{1 - \omega} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \frac{\sin((N + 1/2)x)}{\sin(x/2)} \\
&= \frac{1}{N \sin^2(x/2)} \sin \left( \frac{x}{2} \right) \left( \sin \frac{x}{2} + \sin \frac{3x}{2} + \cdots + \sin \frac{(2N-1)x}{2} \right) \\
&= \frac{1}{N \sin^2(x/2)} \left( \sin \frac{x}{2} \sin \frac{x}{2} + \sin \frac{x}{2} \sin \frac{3x}{2} + \cdots + \sin \frac{x}{2} \sin \frac{(2N-1)x}{2} \right) \\
&= \frac{1}{2N \sin^2(x/2)} \left[ \cos \left( \frac{x}{2} - \frac{3x}{2} \right) - \cos \left( \frac{x}{2} + \frac{3x}{2} \right) \right] \\
&= \frac{1}{2N \sin^2(x/2)} \left[ \cos \left( \frac{x}{2} - \frac{(2N-1)x}{2} \right) - \cos \left( \frac{x}{2} + \frac{(2N-1)x}{2} \right) \right] \\
&= \frac{1}{2N \sin^2(x/2)} \left[ 1 - \cos(-x) + \cos x - \cos(-2x) + \cos 2x + \cdots + \cos(-N - 1)x - \cos Nx \right] \\
&= \frac{1}{2N \sin^2(x/2)} (1 - \cos Nx) \\
&= \frac{1}{2N \sin^2 \left( \frac{x}{2} \right)} \left( 1 - \left( 1 - 2 \sin^2 \left( \frac{N x}{2} \right) \right) \right)
\end{align*}
\]
Therefore, we have \( F_N(x) = \frac{1}{N} \sin^2 \left( \frac{Nx}{2} \right) \sin^2 \left( \frac{x}{2} \right) \)

Note that \( F_N \) is positive and \( \frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(x) \, dx = 1 \)

Because \( \frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(x) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{D_0(x) + D_1(x) + \cdots + D_{N-1}(x)}{N} \, dx \)

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sum_{k=0}^{N-1} \sum_{n=-k}^{k} e^{in(x)}}{N} \, dx \]

\[ = \frac{1}{2\pi} \sum_{k=0}^{N-1} \int_{-\pi}^{\pi} e^{in(x)} \, dx \]

\[ = \frac{1}{2\pi} \sum_{k=0}^{N-1} \int_{-\pi}^{\pi} (\cos nx + i \sin nx) \, dx \]

\[ = \frac{1}{2\pi} \sum_{k=0}^{N-1} \left( \int_{-\pi}^{\pi} \cos nx \, dx + i \int_{-\pi}^{\pi} \sin nx \, dx \right) \]

\[ = \frac{1}{2\pi} \sum_{k=0}^{N-1} \left( \int_{-\pi}^{\pi} (1 + \cos x + \cos 2x + \cdots + \cos kx) \, dx + 0 \right) \]

\[ = \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} (1 + 0) + 0 \right) \]

\[ = 1 \]

And we have \( F_N(x) = \frac{1}{N} \frac{\sin^2 \left( \frac{Nx}{2} \right)}{\sin^2 \left( \frac{x}{2} \right)} \)

\( \sin^2 (x/2) \geq c_\delta > 0, \text{if} \ \delta \leq |x| \leq \pi, \text{hence} \ F_N (x) \leq 1/(Nc_\delta), \text{from which it follows that} \)
\[ \int_{\delta \leq |x| \leq \pi} |F_N(x)| \, dx \to 0, \quad \text{as} \ N \to \infty \]

All these above confirm the condition of the good kernels, so, the Fejer kernel is a good kernel.

5. Conclusion
Even though the Dirichlet kernel is not a Good Kernel and cannot converge to the original function directly, we can use the Fejer kernel as a substitute and can obtain the following conclusion theorem. Theorem [1]:
If \( f \) is integrable on the circle, then the Fourier series of \( f \) is Ces\'aro summable to \( f \) at every point of continuity of \( f \).
Moreover, if \( f \) is continuous on the circle, then the Fourier series of \( f \) is uniformly Ces\'aro summable to \( f \).

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