THE GRÖBNER STRATIFICATION OF A TROPICAL VARIETY

DUSTIN CARTWRIGHT

Abstract. Each Gröbner stratum of a tropical variety is a connected set of points, all of which induce the same initial subscheme. The Gröbner stratification is a coarsening of the decomposition into Gröbner polyhedra, and has the advantage that it does not depend on a choice of compactification. We give an example of a curve over a field with non-trivial valuation whose Gröbner stratification is strictly finer than the coarsest polyhedral decomposition of the tropical variety. We also show that the Gröbner stratification of a locally matroidal tropical variety is completely determined by the underlying tropical variety.

Let \( K \) be a field with valuation \( \text{val} : K^\times \to \mathbb{R} \) and let \( \Gamma \subset \mathbb{R} \) be the image of its valuation. We assume and fix a splitting \( r : \Gamma \to K^\times \) of the valuation, which is guaranteed to exist if \( K \) is algebraically closed or if \( \Gamma \) is discrete. For any closed subscheme \( V \subset \mathbb{G}^N_m \) and any point \( w \) of \( \mathbb{R}^N \), it is possible to take its weighted initial scheme in \( w(V) \) (see Definition [ ]). The tropicalization \( \text{Trop}(V) \) is the set of all weights \( w \in \mathbb{R}^N \) such that \( w(V) \) is non-empty. There exists a polyhedral complex whose support is \( \text{Trop}(V) \), but the polyhedral structure is not unique, and there may not be a coarsest polyhedral structure \( [ST08, Example 5.2] \). In particular, the construction of the polyhedral complex depends on the choice of a compactification \( \mathbb{G}^N_m \subset \mathbb{P}^N \) in which we take \( V \) to be the closure of \( V \). Then each polyhedron of the Gröbner complex is the closure of the set of \( w \in \mathbb{R}^N \) such that \( w(V) \) is some fixed scheme.

Without choosing a compactification, we can stratify the tropical variety based on the initial ideal in \( w(V) \), taken within the torus \( \mathbb{G}^N_m \). More precisely, a stratum of the Gröbner stratification is a component of the set of points \( w \in \mathbb{R}^N \) such that \( w(V) \) is some fixed subscheme. Unlike the Gröbner complex, each stratum need not be the relative interior of a convex polyhedron, nor even contractible (for example, take two tropical planes meeting in a single point), but it is at least an open subset of an affine linear subspace of \( \mathbb{R}^N \).

The purpose of this paper is to contrast the Gröbner stratification with topological stratification, which depends solely on \( \text{Trop}(V) \) as a subset of \( \mathbb{R}^N \), and is defined to be the finest stratification which is compatible with local products in the tropical variety. More precisely, suppose that we have a linear isomorphism \( \mathbb{R}^N \cong \mathbb{R}^{N-m} \times \mathbb{R}^m \) and an open set \( U \subset \mathbb{R}^N \) such that \( U \cap \text{Trop}(V) \) factors as a product of a subset of \( \mathbb{R}^{N-m} \) with an open subset...
The topological stratification is defined to be the finest stratification such that for any local factoring of an open set \( U \), each stratum also factors as the product of a subset of \( \mathbb{R}^{N-m} \) with \( W \). Unlike the Gröbner stratification, the topological stratification depends only on \( \text{Trop}(V) \subset \mathbb{R}^N \) and not on \( V \) itself. Without using this terminology, Sturmfels and Tevelev gave a recipe for constructing a tropical variety whose Gröbner stratification is strictly finer than its topological stratification [ST08, Example 3.10]. We give an explicit such example as a curve over a field with non-trivial valuation in Section 1, and we examine the preimages of its Gröbner strata in the Berkovich skeleton.

The Berkovich analytification \( V^{\text{an}} \) maps surjectively onto \( \text{Trop}(V) \subset \mathbb{R}^N \) by taking the valuations of the coordinate functions. When \( V \) is a curve, \( V^{\text{an}} \) contains a distinguished graph, the skeleton, coming from a minimal semistable reduction of \( V \), and \( \text{Trop}(V) \) is the image of the skeleton under a map which is linear on each edge. Thus, there are finitely many points in \( \text{Trop}(V) \) which are the images of the vertices of the skeleton. Any point \( w \) whose neighborhood in \( \text{Trop}(V) \) is not a union of open segments must be a zero-dimensional Gröbner stratum and also the image of a vertex of the skeleton. Nonetheless, in general, the set of images of vertices of the skeleton neither contains nor is contained in the set of zero-dimensional Gröbner strata. Example 2.6 of [BPR11] is a curve whose skeleton has a vertex which maps to a point in the interior of a one-dimensional Gröbner stratum. On the other hand, our example in Section 1 has a zero-dimensional Gröbner stratum contained in an open half-ray, whose preimage in the skeleton is two disjoint edges.

Our detailed examination of this example occupies the entirety of Section 1. In Section 2, we compare the two strata on general tropical varieties. Proposition 6 implies that in any case where the Gröbner refines the tropical stratification, the corresponding initial ideals will have the same support, but different non-reduced structures. We prove that the two strata agree for locally matroidal tropical varieties and for curves whose multiplicities are all one (Theorem 8 and Corollary 9). Throughout the paper, \( R \) will denote the valuation ring of \( K \) with maximal ideal \( m \subset R \) and residue field \( k = R/m \).

Acknowledgments. I have been supported by the National Science Foundation under award DMS-1103856. I would like to thank Sam Payne for many helpful suggestions, including the question which prompted this paper. Walter Gubler pointed me towards Example 3.10 in [ST08]. I also thank the Institut Mittag-Leffler and the Max Planck Institute in Bonn for their hospitality during my work on this paper.

1. A non-trivial Gröbner stratification

In this section, we give an example of a curve whose tropicalization has an open ray, which contains 3 distinct Gröbner strata. First, it will be useful to define the initial degeneration \( \text{in}_w(V) \) in greater generality than
has appeared in the literature. In particular, the following does not require
the entries of \( w \) to be in the value group.

**Definition 1.** For any \( w \in \mathbb{R}^N \), we let \( R[x_1^\pm, \ldots, x_N^\pm]^w \) denote the \( R \)-subalgebra of \( K[x_1^\pm, \ldots, x_N^\pm] \) generated by all terms \( ax^u \) such that \( \text{val}(a) + u \cdot w \geq 0 \), where \( u \in \mathbb{Z}^N \) is the vector of exponents. We define \( V^w \) to be the closure of \( V \) in \( \text{Spec} \, R[x_1^\pm, \ldots, x_N^\pm]^w \). We then define a ring homomorphism \( \phi: R[x_1^\pm, \ldots, x_N^\pm] \to k[x_1^\pm, \ldots, x_N^\pm] \) by sending the monomial \( ax^u \) to

\[
\phi(ax^u) = \begin{cases} ar(u \cdot w)x^u & \text{if } \text{val}(a) + r \cdot u = 0 \\ 0 & \text{if } \text{val}(a) + r \cdot u > 0, \end{cases}
\]

where \( ar(u \cdot w) \) denotes the image of \( ar(u \cdot w) \) in the residue field \( k \). The initial degeneration \( \text{in}_w(V) \) is defined to be \( (\phi^*)^{-1}(V^w) \), where \( \phi^* \) is the map of schemes induced by \( \phi \).

If the entries of \( w \) lie in the value group, then \( \phi^* \) induces an isomorphism from \( \text{in}_w(V) \) to the special fiber of \( V^w \), and if not, \( \text{in}_w(V) \) can be thought of as the special fiber of \( V^w \) after extending \( K \) to a field whose valuation group contains the coordinates of \( w \).

For our example, we let \( \pi \in K \) be an element with non-trivial valuation, and for convenience, we rescale the valuation such that \( \text{val}(\pi) = 1 \). Throughout this section, \( I \subseteq K[x^\pm, y^\pm, z^\pm] \) will denote the ideal generated by the \( 2 \times 2 \) minors of the matrix

\[
\begin{bmatrix}
 x - 1 & \pi(y - 1) & \pi(y - 1 - \pi z) \\
\pi(y - 1) & \pi(y - 1 - \pi z) & x - 1 - \pi
\end{bmatrix},
\]

and \( V \) will be the subvariety of \( \mathbb{G}_m^3 \) defined by \( I \). Abstractly, \( V \) is isomorphic to \( \mathbb{P}^1 \) with a finite number of points removed, and it can be given explicitly as the image of the rational map

\[
(x, y, z) = \left( \frac{\pi}{1 - u^3} + 1, \frac{u}{1 - u^3} + 1, \frac{u}{\pi(1 + u + u^2)} \right).
\]

We will use \( X, Y, \) and \( Z \) as coordinates for \( \mathbb{R}^3 \). The intersection of the tropical variety of \( V \) with the half space defined by \( Z > -1 \) is the single half ray with \( X = Y = 0 \). The initial stratification divides this half-ray into three strata, based on the ideal defining \( \text{in}_w(V) \):

\[
I(\text{in}_{(0,0,Z)}(V)) = \begin{cases} 
((x - 1), (y - 1)^2) & \text{if } -1 < Z < 0 \\
((x - 1)^2, (x - 1)z - (y - 1)) & \text{if } Z = 0 \\
((x - 1)^2, (y - 1)) & \text{if } 0 < Z.
\end{cases}
\]

Note that all three of these initial schemes have the same support, namely the subtorus defined by \( x = y = 1 \), but different non-reduced structures.

We now compute the skeleton of \( V^{\text{an}} \) and show that the map from this skeleton to \( \text{Trop}(V) \) does not distinguish \((0,0,0)\) from nearby points. We
Table 1. Boundary components of the compactification of $V$, over an algebraically closed field of characteristic not 3 or 23. The value of $\ast$ is $-1$ for the root of $u^3 - \pi - 1$ such that $\text{val}(u - 1) = 1$ and $-2$ for the other two roots, for which $\text{val}(u - 1) = 0$.

| Component $D$ | $(\text{val}_D(x), \text{val}_D(y), \text{val}_D(z))$ | $\text{Trop}(D)$ |
|---------------|--------------------------------------|------------------|
| $u^3 - \pi - 1$ | $(1, 0, 0)$ | $(\infty, -1, \ast)$ |
| $u^3 - u - 1$ | $(0, 1, 0)$ | $(0, 0, -1)$ |
| $u$ | $(0, 0, 1)$ | $(0, 0, \infty)$ |
| $u^2 + u + 1$ | $(1, 1, 1)$ | $(-\infty, -\infty, -\infty)$ |
| $u - 1$ | $(1, 1, 0)$ | $(-\infty, -\infty, -1)$ |
| $\infty$ | $(0, 0, 1)$ | $(0, 0, \infty)$ |

will assume that $K$ is algebraically closed and that $k$ does not have characteristic 3 or 23. The purpose of these two assumptions is so that the boundary $\mathbb{P}^1 \setminus V$ consists of exactly 11 distinct closed points which are the roots of the polynomials shown in the first column of Table 1. In characteristics 3 and 23, there are extra coincidences among the divisors in Table 1 which change the skeleton, but not the part mapped to the half-space $Z > -1$.

There is a unique $\mathbb{R}$-model of $V$ defined by taking $u$ to be a non-constant rational function on the special fiber. This model is not semistable because each root of $u^3 - \pi - 1$ intersects a corresponding root of $u^3 - 1$ in the special fiber. If we blow up the special fiber at each of the three roots of $u^3 - 1$, we obtain a semistable model whose special fiber has 4 components. The original component maps to $(0, 0, -1)$. The exceptional divisor of the blow-up at $u = 1$ maps to $(0, -1, -1)$ and the other two exceptional divisors map to $(0, -1, -2)$. The dual graph of the special fiber is a claw graph and the Berkovich skeleton consists of this graph together with 11 infinite edges.

The map of the Berkovich skeleton to $\mathbb{R}^3$ is shown in Figure 1.

The preimage of the half-plane $Z > -1$ in the skeleton consists of two disjoint rays. Since $V^{\text{an}}$ has a deformation retract onto its skeleton, this implies that $V^{\text{an}}$ itself decomposes into two connected components in the corresponding analytic domain $D \subset \mathbb{C}^3$ defined by $\text{val}(z) > -1$. We now compute this decomposition explicitly. Since $I$ is generated by all $2 \times 2$ minors of (1), it also contains the determinant

$$
\frac{1}{\pi^2} \begin{vmatrix} 1 + \pi z - y & 0 & y - 1 \\ x - 1 & \pi(y - 1) & \pi(y - 1 - \pi z) \\ \pi(y - 1) & \pi(y - 1 - \pi z) & x - 1 - \pi \end{vmatrix}
$$

$$
= (1 - 3\pi z)(y - 1)^2 + \pi z(3\pi z - 1)(y - 1) - \pi^3 z^3
$$

This is the equation of the projection of $V$ onto the $y$ and $z$ coordinates. We now assume that $K$ does not have characteristic 2 in order to use the quadratic formula to find analytic representations of the two branches of $y$.
Figure 1. Skeleton of the Berkovich analytification of $V$, shown using its projection onto the $Y$ and $Z$ coordinates. Pairs and triples of lines which map to the same place are drawn slightly separated from each other. The short dotted lines represent line segments pointed into the positive $X$ direction. The other lines lie in the $X = 0$ plane except for the two southwest-most rays, which point in the $(-1, -1, -1)$ direction.

in terms of $z$:

$$y = 1 + \frac{\pi z}{2} \left( 1 \pm \sqrt{\frac{1 - 3\pi z}{1 + \pi z}} \right),$$

which can be expanded as a power series in $z$, convergent on the analytic disk defined by $\text{val}(z) > -1$. Using the left-most minor of (1), we have

$$x - 1 = \frac{\pi (y - 1)^2}{y - 1 - \pi z},$$

and we can use the expansion of (3) as a power series to get analytic formulas:

$$x = 1 - \pi + 2\pi^2 z + \cdots$$

$$x = 1 - \pi^4 z^3 + \pi^6 z^5 + \cdots$$

for the positive and negative branch of $y$ respectively. Thus, (3) and (4) give equations for each branch of $V$ as the graph of an analytic function from the $z$ coordinate to the $x$ and $y$ coordinates. Although we only defined $\text{in}_w(V)$ for subschemes of the torus, we can also take initial ideals in the analytic domain $D$. For $X = Y = 0$ and $Z > -1$, the formulas (3) and (4) give $x - 1 = y - 1 = 0$. Thus, either branch on its own degenerates to the subtorus defined by $x = y = 1$, and it is only when the two branches degenerate together that the “twisted” non-reduced structure appears in (2).
An important use of the polyhedral complex decomposition of a tropical variety is the theory of tropical compactifications, as introduced by Tevelev [Tev07] and later extended to fields with non-trivial valuation by Qu [Qu09] (see also LQ11) and Gubler [Gub11, Sec. 11]. The topological stratification of Trop(V) has polyhedral strata, and we can compactify V by taking its closure inside the corresponding toric scheme. This compactification cannot be a tropical compactification since a tropical compactification must come from a polyhedral complex which refines the Gröbner stratification [Gub11, Cor. 12.9]. We examine this compactification explicitly.

A toric scheme over \( \mathbb{R} \) corresponds to a fan in \( \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \) [Gub11, Sec. 7], and we will look at one affine chart of the scheme coming from the cone over the topological stratification of Trop(V), placed at height 1. Our usual ray of Trop(V) produces the cone generated by the vectors \((0,0,1,0)\) and \((0,0,-1,1)\). The corresponding affine toric scheme is \( \text{Spec} \mathbb{R}[x^\pm, y^\pm, v] \), where the subtorus \( \text{Spec} \mathbb{K}[x^\pm, y^\pm, z^\pm] \) is defined by \( v = \pi z \). Near the point in the special fiber defined by \( x = y = 1 \) and \( v = 0 \), the closure \( \overline{V} \subset \text{Spec} \mathbb{R}[x^\pm, y^\pm, v] \) of \( V \) has two branches, given by substituting \( w = \pi z \) into the analytic equations (3) and (4).

\[
\begin{align*}
y &= 1 + \frac{w}{2} \left( 1 + \sqrt{\frac{1 - 3w}{1 + w}} \right) \\
x &= 1 - \pi + 2\pi w + \cdots
\end{align*}
\]

Thus, \( \overline{V} \) consists of two analytic branches, each isomorphic to the completion of the two-dimensional scheme \( \text{Spec} \mathbb{R}[w] \), meeting at the single closed point \((x, y, w) = (1, 1, 0)\). This is a textbook example of a non-Cohen-Macaulay singularity, which was the ingredient suggested in [ST08, Ex. 5.2] for building a tropical variety with a polyhedral structure which is too coarse to be tropical. As they argued, the non-Cohen-Macaulay point, together with Theorem 1.2 of [Tev07] give another proof that this (partial) compactification is not tropical.

2. Locally matroidal tropical varieties

In this section, we more systematically compare the Gröbner and topological stratifications. Proposition 6 relates the topological stratification to the structure of the initial ideals. The main result of this section is Theorem 8 which gives criterion for the two stratifications to agree.

We first need several lemmas, which extend standard results to initial ideals at points whose coordinates are not necessarily in the value group. In particular, Lemma 3 tells us that even when \( \text{in}_w(V) \) is not the special fiber of \( V^w \), it can be realized as the special fiber of a flat family over some extension of \( \mathbb{R} \).
Lemma 2. Let $\overline{K} \supset K$ be an extension of valued fields and $w$ any point in $\mathbb{R}^N$. Then $\text{in}_w(V \times \text{Spec} \overline{K}) = \text{in}_w(V) \times \text{Spec} \overline{k}$ as subschemes of the torus, where $\overline{k}$ is the residue field of $\overline{K}$.

Proof. We let $\overline{V}^w$ be the closure of the extension $V \times \text{Spec} \overline{K}$ in the tilted torus $R[x_1^\pm, \ldots, x_N^\pm]^w$, as in Definition 1. We have a commutative diagram:

$$
\begin{array}{ccc}
\overline{R}[x_1^\pm, \ldots, x_N^\pm]^w & \xrightarrow{\overline{\phi}} & \overline{k}[x_1^\pm, \ldots, x_N^\pm] \\
\uparrow & & \uparrow \\
R[x_1^\pm, \ldots, x_N^\pm]^w & \xrightarrow{\phi} & k[x_1^\pm, \ldots, x_N^\pm],
\end{array}
$$

where $\phi$ and $\overline{\phi}$ are as in Definition 1. By [OP10, Thm. A.3], the ideal of $\overline{V}^w$ is the image of the ideal of $V^w$ under the inclusion on the left (although [OP10] assumes that $K$ is algebraically closed, this isn’t used in the proof of Theorem A.3). Thus, the ideal of $\text{in}_w(V \times \text{Spec} \overline{K})$ is the image of the ideal of $V^w$ by the upper-left path around the square. The ideal of $\text{in}_w(V) \times \text{Spec} \overline{k}$ is the image by the lower right path, and so the two schemes are equal. \qed

Lemma 3. Let $w$ be a point in $\text{Trop}(V)$. Then there exists a valuation ring $\overline{R}$ dominating $R$ and a flat family over $\overline{R}$ whose special fiber is isomorphic to $\text{in}_w(V)$ and whose general fiber is $V \times \text{Spec} \overline{K}$, where $\overline{K}$ is the fraction field of $\overline{R}$.

Proof. We build the extension $\overline{K} \supset K$, together with a splitting of its valuation homomorphism one coordinate of $w$ at a time. Let $K_0$ and $\Gamma_0$ equal $\overline{K}$ and $\Gamma$ respectively, and we inductively define $K_i$ as follows. If no multiple of $w_i$ lies in $\Gamma_{i-1}$, then we take $K_i$ to be the transcendental extension $K_{i-1}(t_i)$ with the unique extension of the valuation given by $\text{val}(t_i) = w_i$. Otherwise, if $m \text{val}(w_i)$ is in $\Gamma_{i-1}$, with $m$ minimal, then $K_i$ is constructed as the extension $K(t_i)$, where $t_i$ is a primitive $m$th root of $r(m \text{val}(w_i))$. In both cases, we also extend the splitting, which by abuse of notation, we still call $r$, by setting $r(w_i) = t_i$. In the end, we have produced a valued field $\overline{K} = K_N$ whose value group $\overline{\Gamma} = \Gamma_N$ includes the entries of $w$. We also note that at each stage of our construction of $\overline{K}$, the residue field of the valuation ring is unchanged.

The desired family will be $\overline{V}^w$, which is flat and finite presentation over $\text{Spec} \overline{R}$ [OP10, Prop. A.1]. By Lemma 2 we know that $\text{in}_w(V \times \text{Spec} \overline{K})$ is $\text{in}_w(V)$, and so it only remains to show that the map $\overline{\phi}$ as in Definition 1 induces an isomorphism between $\text{in}_w(V)$ and the special fiber of $V^w$. First, since $w_i \in \overline{\Gamma}$, each variable $x_i$ of $k[x_1^\pm, \ldots, x_N^\pm]$ is the image of $r(-w_i)x_i$ under $\overline{\phi}$, so $\overline{\phi}$ is surjective. Second, for any monomial $ax^w$ in the kernel of $\overline{\phi}$, i.e. with $\text{val}(a) + w \cdot u > 0$, then $w \cdot u$ is in $\overline{\Gamma}$, so there exists $b \in \overline{R}$, with $\text{val}(b) = -w \cdot u$. Thus, $bx^w \in R[x_1^\pm, \ldots, x_N^\pm]$, and $a/b \in \overline{R}$ with positive
valuation, so $ax^w$ vanishes on the special fiber. We’ve shown that the special fiber of $\tilde{V}^w$ is $\text{in}_w(\tilde{V}) = \text{in}_w(V)$, so $\tilde{V}^w$ forms the desired family. \hfill \Box

**Lemma 4.** Let $w$ be a point in $\text{Trop}(V)$. Then for sufficiently small vectors $\delta$, $\text{in}_{w+\delta}(V) = \text{in}_\delta(\text{in}_w(V))$, where the last initial ideal is taken with respect to the trivial valuation on $k$.

**Proof.** By choosing a compactification $\mathbb{G}_m^N$ into $\mathbb{P}^N$, we can partition $\mathbb{R}^N$ into a Gröbner complex as in [Gub11, Sec. 10]. Now we claim that the lemma is true if we take $\delta$ sufficiently small such that any polyhedron containing $w + \delta$ also contains $w$. If so, we can choose an extension $\tilde{K} \subset K$ whose valuation group contains the coordinates of $w$ and $w + \delta$ as in the proof of Lemma 3. Then Proposition 10.9 of [Gub11] tells us that $\text{in}_{w+\delta}(V_{\tilde{K}}) = \text{in}_\delta(\text{in}_w(V_{\tilde{K}}))$, and Lemma 2 tells us that the same equality holds over $K$. \hfill \Box

**Proposition 5.** A point $w \in \text{Trop}(V)$ is in a Gröbner stratum of dimension $m$ if and only if the maximal subtorus preserving $\text{in}_w(V)$ has dimension $m$.

**Proof.** For sufficiently small vectors $\delta$, the initial degeneration $\text{in}_{w+\delta}(V)$ equals the initial degeneration $\text{in}_\delta(\text{in}_w(V))$ by Lemma 4. Therefore, $w$ lies in a $m$-dimensional Gröbner stratum if and only if $\text{in}_\delta(\text{in}_w(V))$ is equal to $\text{in}_w(V)$ for all $\delta$ in a $m$-dimensional vector space. The scheme $\text{in}_w(V)$ is unchanged by taking initial with respect to some rational weight vector $\delta$ if and only if $\text{in}_w(V)$ is invariant under the action of the corresponding one-dimensional subtorus of $\mathbb{G}_m^N$. Thus, the vector space of all weights under which $\text{in}_w(V)$ is unchanged corresponds to the maximal subtorus under which $\text{in}_w(V)$ is invariant, and in particular, they have the same dimension. \hfill \Box

The proof of Proposition 5 already shows that the Gröbner stratification refines the topological one. However, it can also be seen through the following characterization of the topological stratification in terms of initial ideals.

**Proposition 6.** A point $w \in \text{Trop}(V)$ is in the topological stratum of dimension $m$ if and only if the maximal subtorus preserving the reduced subscheme of $\text{in}_w(V)$ has dimension $m$.

**Proof.** First, let $W$ denote the reduced induced subscheme of $\text{in}_w(V)$ and suppose that $W$ is preserved by some $m$-dimensional subtorus, and thus the tropical variety of $W$ is a product with the vector space corresponding to the subtorus. Since taking reduced subschemes does not change the underlying set of a tropical variety, the tropical variety of $\text{in}_w(V)$ now has the same property. Thus, $\text{Trop}(V)$ is also a product with a vector space in a neighborhood of $w$ by Lemma 4, so $w$ is in an $m$-dimensional topological stratum.

Conversely, suppose that $\text{Trop}(V)$ is a product with a $m$-dimensional linear space in a neighborhood of $w$. Then $\text{Trop}(\text{in}_w(V))$ is globally a product
with a vector space, and we can change coordinates to assume that the vector space generated by the first $m$ coordinate vectors. If we let $p$ be the projection of $\mathbb{G}_m^N$ onto the last $N - m$ coordinates, then $\text{Trop}(p(\text{in}_w(V)))$ is the projection of $\text{Trop}(\text{in}_w(V))$ onto the last $N - m$ coordinates. Therefore, the Bieri-Groves theorem tells us that $p(\text{in}_w(V))$ has dimension $d - m$, where $d$ is the dimension of $V$. For any irreducible component $V'$ of $\text{in}_w(V)$, the projection $p(V')$ must have dimension at most $d - m$, so $p^{-1}(p(V'))$ has dimension at most $d$. However, $V'$ is a $d$-dimensional variety and is contained in $p^{-1}(p(V'))$, so they must be equal. Therefore, $V'$ is invariant under the subtorus which acts on the first $m$ coordinates, as is the reduced subscheme of $\text{in}_w(V)$.

**Corollary 7.** If $\text{Trop}(V)$ is a linear space in a neighborhood of a point $w$, then the support of $\text{in}_w(V)$ is a finite union of torus orbits.

Recall that at a point $w$ in the interior of a maximal cone of the Gröbner complex, the multiplicity is the sum of the multiplicities of all primary components of $\text{in}_w(V)$. By the balancing condition, the multiplicities in any neighborhood of a locally linear point will all be equal. However, we can given an intrinsic definition of the multiplicity at any locally linear point $w$ as the sum

$$\sum_{W \in \text{Min}(\text{in}_w(V))} \text{length}(O_{V,W}) \left[ K(W)^T : k \right]$$

where $\text{Min}(\text{in}_w(V))$ is the set of irreducible components of $\text{in}_w(V)$, and $K(W)^T$ is the field of rational functions on $W$ which are invariant under the maximal subtorus of $\mathbb{G}_m^N$ which preserves $W$.

The tropicalizations of linear spaces have a purely combinatorial description in terms of the associated matroid [AK06], which we recall here. Let $M$ be a matroid with ground set $[N]$ and bases $B_1, \ldots, B_r$. For any basis $B = \{b_1, \ldots, b_m\}$ and $w \in \mathbb{R}^N$, let $w_B$ be the sum $w_{b_1} + w_{b_2} + \cdots + w_{b_m}$. Let $M_w$ be the set of bases $B$ for which $w_B$ is minimized, and it can be shown that $M_w$ is also the set of bases of a matroid. The Bergman fan of $M$ is the set of vectors $w$ for which $M_w$ has no loops. We say that a tropical variety is **locally matroidal** at a point $w \in \mathbb{R}^N$ if there exists a neighborhood of $w$ in which all multiplicities are 1 and which is equal to a neighborhood of the Bergman fan of some matroid $M$, possibly after a change of coordinates from an element of $\text{GL}_N(\mathbb{Z})$. Locally matroidal fans have been studied before as analogues of smooth varieties, for example in [Sha10] and [KS10, Sec. 7].

**Theorem 8.** Let $w$ be locally matroidal point of $\text{Trop}(V)$. Then, in appropriate coordinates, $\text{in}_w(V)$ is a linear subspace of $\mathbb{G}_m^N$. In particular, the Gröbner stratification and the topological stratification agree in a neighborhood of $w$.

**Corollary 9.** If $V$ is a curve and all multiplicities of $\text{Trop}(V)$ are 1, then the Gröbner and topological stratifications on $\text{Trop}(V)$ coincide.
Proof. Since the Gröbner stratification refines the topological stratification, it suffices to prove that each one-dimensional topological stratum of $\text{Trop}(V)$ is also a Gröbner stratum. However, any one-dimensional topological stratum is locally linear by definition, and thus matroidal, so the claim follows from Theorem 8. □

Recall that a subvariety $V$ of a torus is called schön if the initial degeneration $\text{in}_w(V)$ is smooth for every $w \in \mathbb{R}^N$.

Corollary 10. If $\text{Trop}(V)$ is locally matroidal at all points $w \in \text{Trop}(V)$, then $V$ is schön.

One ingredient in the proof of Theorem 8 is the fact that linear spaces are characterized by their tropicalizations, which was proved by Katz and Payne [KPTI, Prop. 4.2]. Given this characterization, the remaining difficulty is to prove that the $\text{in}_w(V)$ is reduced, for which we use the following.

Theorem 11. If $\text{in}_w(V)$ is generically reduced and its reduced induced subscheme is geometrically integral and normal, then $\text{in}_w(V)$ is reduced.

Theorem 11 is a generalization of a result first obtained by Hironaka for degenerations over the localization of a ring of finite type over a field [Hir58, Lemma 4] (see also [Har77, Lemma III.9.12]). We use the Hironaka’s argument for the case when $R$ is an excellent DVR, and use Noetherian approximation to reduce to this case when $R$ is a general rank 1 valuation.

Proof of Theorem 11. Using Lemma 3, we let $S$ denote the coordinate ring of the flat family over the valuation ring, for which we drop the tilde and write $R$. We let $p$ denote the unique minimal prime of $S/mS$.

We first prove the theorem under the assumption that $R$ is an excellent discrete valuation ring and write $\pi$ to denote a uniformizer of $R$. We’ve assumed that $S/mS$ is generically reduced, so $\pi$ generates the maximal ideal in $S_p$, meaning that $S_p$ is a discrete valuation ring. Therefore, $\tilde{S}$, the normalization of $S$ is a subring of $S_p$, and so $\tilde{S}/p\tilde{S}$ is a subring $S_p/pS_p$. However, $S_p/pS_p$ is the field of fractions of $S/p$, which we’ve assumed to be normal, so $\tilde{S}/p\tilde{S}$ must equal $S/pS$.

Since $S/mS$ is generically reduced, it will be sufficient to show that it has no embedded primes. Suppose that $q$ is any prime containing $p$ and we will show that $q$ is not an embedded prime of $S/mS$. Since $R$ is excellent, then, in particular, the normalization $\tilde{S}$ is a finite $S$-module. We’ve shown that $\tilde{S}/q\tilde{S}$ equals $S/qS$, so Nakayama’s Lemma implies that $\tilde{S}_q$ equals $S_q$. In other words, $S_q$ is normal. Thus, the principal ideal in $S_q$ generated by $\pi$ cannot have any embedded primes, and thus $q$ cannot be an embedded prime of $S/mS$. This completes the proof when $R$ is an excellent DVR.

Now, we allow $R$ to be an arbitrary valuation ring and we reduce to the excellent case by Noetherian approximation. By [Gro66, Cor. 11.2.7], there exists a finite type $\mathbb{Z}$-algebra $R_0$, contained in $R$, and a finite type, flat $R_0$-algebra $S_0$ such that $R_0 \subset R$ and $S \cong R \otimes_{R_0} S_0$ as $R$-algebras. Set $R_1$ to
be the localization \((R_0)_{m \cap R_0}\) and let \(m_1\) be its maximal ideal. By [Gröbner, Prop. 7.1.7], there exists a discrete valuation ring \((R_2, m_2)\) dominating \(R_1\) with the same field of fractions. From the proof of that proposition, we see that \(R_2\) is the normalization of a localization of a ring of finite type over \(R_0\), so \(R_2\) is excellent [Gröbner, Schol. 7.8.3(iii)]. These various bases are summarized in the following commutative diagram:

\[
\begin{array}{c}
S_0 \otimes_{R_0} R_2 & \leftarrow & S_0 \otimes_{R_0} R_1 & \rightarrow & S = S_0 \otimes_{R_0} R \\
\uparrow & & \uparrow & & \uparrow \\
R_2 & \leftarrow & R_1 & \rightarrow & R
\end{array}
\]

We will write \(S_0 \otimes_{R_0} R_i\) as \(S_i\) for \(i = 1, 2\).

The rings in the bottom row of (5) are local rings and we now wish to the corresponding special fibers of the rings in the top row. The inclusions of \(R_1\) into \(R\) and \(R_2\) are local homomorphisms, so if we take the quotients of the rings in (5) by the maximal ideals in the bottom row, we get a commutative diagram whose squares are tensor products:

\[
\begin{array}{c}
S_2/m_2S_2 & \leftarrow & S_1/m_1S_1 & \rightarrow & S/mS \\
\uparrow & & \uparrow & & \uparrow \\
R_2/m_2 & \leftarrow & R_1/m_1 & \rightarrow & R/m
\end{array}
\]

By hypothesis, \(S/mS\) has a unique minimal prime. Thus, the only minimal prime of \(S_1/m_1S_1\) is the contraction \(p \cap (S_1/m_1S_1)\) [Gröbner, Prop. 4.2.7(i)], and we will call this prime \(p_1\). Moreover, localizing at this prime kills off any embedded components in \(S/mS\), so the localization of \(S_1/m_1S_1\) at \(p_1\) is a subring of an integral domain and thus \(S_1/p_1\) is generically reduced. Since \(S/p\) is geometrically integral, then \(S_2/p_1S_2\) is integral and \(p_1S_2\) is the unique minimal prime over \(m_2S_2\). Therefore, we can apply the proof of the excellent case to conclude that \(m_2S_2\) is radical. Since \(S_2/m_2S_2\) has no embedded primes, then neither do \(S_1/m_1S_1\) and \(S/mS\) [Gröbner, Prop. 4.2.7(i)], which is what we wanted to prove. \(\square\)

**Proof of Theorem 8.** The degeneration \(\text{in}_w(V)\) and its reduced subscheme have the same tropicalizations as sets. In general, the tropicalization of the reduced subscheme could have lower multiplicities, but the fact that all the multiplicities are 1 means that \(\text{in}_w(V)\) and its reduced subscheme have the same multiplicities as well. Therefore, the reduced subscheme of \(\text{in}_w(V)\) is isomorphic to the complement of a hyperplane arrangement in projective space [KP11, Prop. 4.2], and in particular it is normal. Thus, Theorem 11 implies that \(\text{in}_w(V)\) is reduced. The initial degenerations of a linear space are also linear spaces, so by Lemma 1, the initial subschemes are also reduced sufficiently close to \(w\). Therefore, the criteria in Propositions 5 and 6 agree, so the Gröbner and topological stratifications agree in this neighborhood. \(\square\)
References

[AK06] F. Ardila, C. Klivans: The Bergman complex of a matroid and phylogenetic trees, J. Combin. Theory Ser. B 96:1 (2006) 38–49.

[BPR11] M. Baker, S. Payne, J. Rabinoff: Nonarchimedean geometry, tropicalization, and metrics on curves, preprint, arXiv:1104.0320.

[Gro61] A. Grothendieck: Éléments de géométrie algébrique: II. Étude globale élémentaire de quelques classes de morphismes, Publ. Math. de l’IHÉS 8 (1961).

[Gro65] A. Grothendieck: Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Seconde partie, Publ. Math. de l’IHÉS 24 (1965).

[Gro66] A. Grothendieck: Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Troisième partie, Publ. Math. de l’IHÉS 28 (1966).

[Gub11] W. Gubler: A guide to tropicalizations, preprint, arXiv:1108.6126.

[Har77] R. Hartshorne: Algebraic Geometry, Graduate Texts in Mathematics 52, Springer, New York, 1977.

[Hir58] H. Hironaka: A note on algebraic geometry over ground rings: the invariance of Hilbert characteristic functions under the specialization process, Illinois J. Math. 2:3 (1958) 355-366.

[KP11] E. Katz, S. Payne: Realization spaces for tropical fans, Combinatorial aspects of commutative algebra and algebraic geometry, Abel Symp. 6 (2011) 73–88.

[KS10] E. Katz, A. Stapledon: Tropical geometry and the motivic nearby fiber, Compositio Math., to appear, arXiv:1007.0511

[LQ11] M. Luxton, Z. Qu: Some results on tropical compactifications, Trans. Amer. Math. Soc. 363:9 (2011) 4853–4876.

[OP10] B. Osserman, S. Payne: Lifting tropical intersections, preprint, arXiv:1007.1314.

[Qu09] Z. Qu: Toric schemes over a discrete valuation ring and tropical compactifications. PhD thesis, University of Texas at Austin (2009).

[ST08] B. Sturmfels, J. Tevelev: Elimination theory for tropical varieties, Mathematical Research Letters 15:3 (2008) 534–562.

[Sha10] K.M. Shaw: A tropical intersection product in matroidal fans, preprint, arXiv:1010.3967.

[Tev07] J. Tevelev: Compactifications of subvarieties of tori, Am. J. Math. 129: 4 (2007) 1087–1104.