Cosmological Mass of the Photon and Dark Energy as its Bose-Einstein Condensate in de Sitter Space

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ABSTRACT — I develop a physical picture of dark energy (DE) based on fundamental principles and constants of quantum mechanics (QM) and general relativity (GR) theories. It derives from a conjecture of non-zero masses for nearly standard-model photons or gluons, based on QM localization at a cosmological scale. Dark energy is associated with de Sitter space and that has a fundamentally invariant event horizon, which provides the basis for my DE model. I conceive of DE as a Bose-Einstein condensate (BEC) of cosmologically massive photons and I estimate fundamentally the binding energy per particle originating from an effectively attractive QM potential in that BEC. Since massive photons may stand at rest in a de Sitter universe with flat spatial geometry, I solve the time-independent Schrödinger equation for a non-relativistic attractive spherical-well potential self-confining at the de Sitter horizon. The minimal critical potential depth that binds a particle state at the top of that well, combined with the prototypical condition of dark energy-pressure relation in the standard flat Λ-CDM model, provides an estimate of the photon mass, $m_g$. That is supported by an independent calculation of the vacuum energy of the BEC in a de Sitter static metric with coordinate-time slicing. I also consider classical gravitational collapse for a uniform dark energy density, approaching Schwarzschild condition. These QM and GR estimates provide compatible accounts of dark energy condensation, bridging a chasm between nuclear and cosmological scales. I then investigate statistical properties of equilibrium between the $g$-BEC phase and the ordinary ‘vapor’ phase of $m_g$ photons. Resulting corrections to the Planck spectrum of the CMB are too small to be detectable, at least currently. Most notably, I consider a system of cosmological units, or ‘$g$-units,’ that complements the fundamental system of Planck units in various ways. The geometric mean of Planck and $g$-mass turns out to be remarkably close to current estimates of neutrino masses, suggesting that even masses of the lightest known fermions may be deeply related to both GR and QM fundamental constants $\Lambda$, $G$, $c$ and $h$.

Keywords: photon mass; neutrino mass; cosmological constant; cosmological units; vacuum energy; dark energy; dark matter; Bose-Einstein condensate; general theory of relativity. — EMAIL: resca@cua.edu

1. INTRODUCTION

In a previous paper I proposed non-zero bare masses for photons and gluons, derived from quantum-mechanical localization at a cosmological scale.[1] My basic assumption is that

$$\lambda_g = \frac{\hbar}{m_g c} = C_g^{-1} L_g = C_g^{-1} \sqrt{\frac{\Lambda}{\Lambda}} \simeq C_g^{-1} \times 10^{10}\text{ly}. \tag{1}$$

In Eq. (1), $\Lambda$ is Einstein’s cosmological constant, while $\lambda_g$ is the Compton wavelength of originally massless gauge bosons in the standard model (SM) of elementary particles. Those become endowed with a minimal but finite mass

$$m_g c^2 = \frac{\hbar c}{\lambda_g} = C_g M_g c^2 = C_g \hbar c \sqrt{\Lambda} \simeq C_g (1.4 \times 10^{-41}) m_p c^2, \tag{2}$$

with a value given here in units of the proton rest-energy mass, $m_p c^2 \simeq 938$MeV.

Beyond my original conjecture, I introduce in Eq. (1) a numerical coefficient, $C_g$, which I estimate or constrain for photons by two alternative approaches. One approach involves a non-relativistic estimate of the effective binding energy for the Bose-Einstein condensate (BEC) of $m_g$-photons confined within de Sitter horizon. An alternative
approach is based on the vacuum or zero-point energy of the $g$-BEC in de Sitter static metric. Remarkably, that vacuum energy can match dark energy (DE) within a single order of magnitude.

My purpose is to formulate a basic physical picture of dark energy and pressure that may be reasonably consistent with the standard flat Λ-CDM (cold dark matter) model of hot big bang cosmology and most fundamental principles of quantum mechanics (QM) and general relativity (GR) theories.

A most notable outcome of my formulation is the introduction of cosmological units $L_g$, $M_g$, and $T_g$, derived from the universal constants $\Lambda$, $c$, and $h$. These ‘$g$-units’ complement Planck units, $lp$, $m_P$, and $t_P$, derived from $G$, $c$, and $h$, in various ways and corresponding uncertainty relations. The fundamental ratio between corresponding Planck and $g$-units (of mass or inverse length) is approximately $2 \times 10^3$, reflecting the vastness between microscopic and cosmological scales of the ‘observable’ universe. Yet the geometric mean of the ‘maximal’ Planck mass and the ‘minimal’ $g$-mass, which falls in the logarithmic middle of that vast range, stands within a single order of magnitude of current estimates of neutrino masses. Chances that this is yet another mere coincidence seem remote. I thus suppose that even masses of the lightest known fermions are related to basic combinations of QM and GR fundamental constants.

Namely, I discuss in this paper that neutrino masses may be related to dark-energy density as

$$m_\nu \simeq \sqrt{M_g m_P} = \left[ \frac{h^3 \Lambda}{c G} \right]^{1/4} \simeq c^{-2} \left[ 8\pi (hc)^3 \rho_\Lambda \right]^{1/4} \simeq 2 \times 10^{-2} eV/c^2. \quad (3)$$

Major developments of observational and theoretical dark energy (DE) and dark matter (DM) research continue to challenge the boundaries of the standard Λ-CDM model and other fundamental concepts of modern cosmology. For the basic perspective of this paper, I may assume that the standard Λ-CDM model holds with sufficient accuracy. Thus I maintain that, beside Einstein’s constant, $8\pi G/c^4$, only the cosmological term can be minimally introduced in Einstein field equations without upsetting either their stress-energy tensor conservation laws nor their general coordinate invariance, requiring a second universal and fundamental constant, $\Lambda$, with dimensions of an inverse square length.

The extent to which my basic description of DE and my estimates of photon and neutrino masses may be confirmed and contribute to far reaching endeavors remains to be seen. In this paper I may only provide estimates that follow from my basic assumptions and evaluate whether corresponding corrections may be testable at present. Nevertheless, if correct, developments of my conception may be bound to have a major impact on fundamental physical theories and observations.

Currently, there are advanced theories that develop techniques potentially applicable to my basic cosmological description of dark energy and dark matter as well. Those include studies of ultra-light bosonic scalar and vector field dark matter, and also theories of gravitational-vacuum and dark-energy stars.

This paper is developed at a basic level, designed to be fully accessible to general readers. Technical issues are kept to a minimum or mainly referenced to standard literature.

Thus my paper is structured as follows.

In Sec. 2, Sec. 3, and Sec. 4 I review the essential elements of the standard framework of hot big bang cosmology within which I am bound to frame my basic formulation of dark energy structure.

That is developed in Sec. 5, Sec. 6, and Sec. 7. Most remarkably in the latter, I can identify the dark energy density, $\rho_\Lambda$, with my zero-point energy density of the BEC, $\rho'_0$, within a single order of magnitude. By comparison, standard quantum field theory (QFT) calculations overestimate vacuum energy by as many as 120 orders of magnitude. In fact, the ratio of the Časimir electromagnetic vacuum energy density, $\rho_{CEM}$, to my $\rho'_0$ is of the order of $8\pi^2 (L_g/l_P)^2 \simeq 4.55 \times 10^{122}$.

Comparisons between my estimates of dark energy condensation and a classical GR perspective on self-gravitation and collapse are drawn in Sec. 8.

In Sec. 9 I investigate statistical properties of equilibrium between the $g$-BEC phase and the ordinary vapor phase of cosmologically massive photons. Relations and comparisons with the Planck spectrum and measurements of the cosmic microwave background (CMB) are then evaluated and discussed.

In Sec. 10 I refer to advanced theories of Gross-Pitaevskii-Poisson systems and their relativistic generalizations of hydrodynamic equations that may be applicable to my formulation of cosmologically massive $g$-photons as well.

In Sec. 11 I describe possible relations between QM ‘universal time’ and GR ‘cosmological time.’

In Sec. 12 I introduce cosmological units that complement Planck units in opposite limits. A remarkable outcome of that formulation, derived in Sec. 13 is that the geometric mean between the ‘minimal’ and the ‘maximal’ mass turns out to be tantalizingly close to observational values of neutrino masses: Eq. (3).

Some more technical and speculative considerations about gluons and gravitons, but relevant nonetheless, are summarized in Sec. 14.
In Sec. 15, I draw the main conclusions of my work. Last but not least, in the Appendix of Sec. 16, I refer to alternative work that may lead to remarkable developments and new perspectives.

2. EINSTEIN FIELD EQUATIONS AND FLRW GEOMETRY

My conjecture is framed in the context of the cosmological principle and observation that our universe is homogeneous and isotropic on a very large scale,[40] which lead to a relatively simple Robertson-Walker (RW) metric. That is derived, for example, on p. 343 of Ref. 41 as

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(c dt)^2 + R^2(t) \left[ \frac{(dr)^2}{1 - \kappa r^2} + r^2 \left( (d\theta)^2 + \sin^2 \theta (d\phi)^2 \right) \right]. \]  

(4)

Flat, closed, open universes correspond to ‘spatial curvatures’ \( \kappa = 0, +1, -1 \), respectively.[42] Einstein’s field equations may be expressed as

\[ G^{\mu\nu} = -\Lambda g^{\mu\nu} + \frac{8\pi G}{c^4} T^{\mu\nu}. \]  

(5)

One may assume that the stress-energy tensor originates from a material ‘perfect fluid’ with pressure, \( p \), and energy density, \( \rho \), given by

\[ T^{\mu\nu} = p g^{\mu\nu} + (\rho + p)U^\mu U^\nu / c^2, \]  

(6)

where \( U^\mu = dx^\mu / dt \) denotes the four-velocity of the fluid element.

One may further suppose that the cosmological term also originates from a peculiar perfect fluid defined as having

\[ T_A^{\mu\nu} = \frac{c^4}{8\pi G} (-\Lambda g^{\mu\nu}) = p_A g^{\mu\nu} + (\rho_A + p_A)U^\mu U^\nu / c^2. \]  

(7)

This definition is based on the assumption that

\[ \rho_A = -p_A = \frac{c^4 \Lambda}{8\pi G} \simeq 3.6m_p c^2 / m^3 \simeq 5.3 \times 10^{-15} \text{atm}. \]  

(8)

Strict equality between a negative pressure or ‘tension’ and energy density provides a basis for the standard Λ-CDM model. Notwithstanding its wide range of successes,[41,42] some discrepancies by more than 5%, or above three standard deviations, mainly between supernova measurements and Planck spacecraft CMB data, keep challenging the applicability of the Λ-CDM model.[2–9] Thus it remains to be seen to what extent the ratio of \( p_\Lambda / \rho_\Lambda \) may or may not equal \(-1\) in particular.[12]

Resolving these issues is undoubtedly bound have most consequential theoretical and observational implications.[5–20]. Currently, however, that lies beyond the scope of estimates that I provide in this paper. Thus, let me assume that Eq. (8) holds more or less accurately for my most immediate concerns.[8]

Mathematically, the RW metric and its simplicity derive from application of Weyl’s postulate and the cosmological principle to a maximally symmetric space.[42–45] That applies to theories even outside GR. Within GR, application of Einstein field equations to the RW metric leads to two Friedmann-Lemaître (FL) equations that determine the dynamical evolution of the universe ‘scale factor,’ \( R(t) \), as a function of ‘cosmic time,’ \( t \). I refer to that metric evolution as FLRW geometry or geometrodynamics.[40,46] Derivations of FL equations are generally provided: see, for instance, p. 355 of Ref. [41] or p. 379 of Ref. [42].

The first FL equation derives from the \( tt \)-component of Eq. (5) and may be cast as

\[ (\dot{R})^2 = -c^2 \kappa - W(R) = -c^2 \kappa - \left[ -\frac{8\pi G}{3c^2} (\rho R^2) - \frac{c^2}{3} (\Lambda R^2) \right]. \]  

(9)

Differentiation with respect to time is generally denoted by a ‘dot’ over the variable, e.g., \( \dot{R} \equiv dR/dt \).
Formally, Eq. (9) may be interpreted as conserving the ‘total energy,’ $-c^2\kappa$, of a ‘kinetic’ term, $(\dot{R})^2$, plus an ‘effective potential’ term, $W(R)$. That $W(R)$ is generally ‘hill-shaped’ and negative for $\Lambda > 0$. At relatively small $R$, the first term, originating from the matter and radiation energy density, $\rho$, dominates $W(R)$, attractively. At relatively large $R$, the second term, originating from dark energy density, $\rho_\Lambda$, dominates $W(R)$, repulsively. At the maximum of $W(R)$, or least negative, these two terms equilibrate. Should that maximum exactly coincide with $-c^2\kappa$ for $\kappa = +1$, a static universe would result, as originally conceived by Einstein (1917). Finally it was noted by Eddington (1930) that such a static universe is hopelessly unstable: see p. 759 of Ref. 40 and p. 408 of Ref. 42. There is of course a lot more to that story.[14, 15] Yet, reports of Einstein’s blunders have been exaggerated.

Compared to the enormous complexity of Einstein’s Eq. (5), from which it was derived, the FL first-order ordinary differential Eq. (9) is remarkably simple, ultimately as a consequence of the maximal symmetry assumed in the RW metric, Eq. (4). Yet Eq. (9) fundamentally rules the geometrodynamics of all the observable universe on its largest scale!

A second and independent FL equation derives from a combination of $rr$- and $tt$-components of Eq. (5) and may be cast as

$$\dot{R} = \frac{4\pi G}{3c^2} (\rho + 3p)R + \frac{c^2}{3} \Lambda R.$$  

(10)

Both FL equations can be uniquely solved when an equation of state between pressure and energy density, $p = p(\rho)$, is specifically provided.

After a ‘hot big bang,’ energy densities of radiation and matter dominated a decelerating expansion of $R(t)$ in the early universe. Dark energy did not affect that until much later, when it turned that expansion into acceleration around $7\times10^9$ years. Indeed, energy densities of radiation and matter decrease as $R^{-4}$ and $R^{-3}$, respectively, whereas the dark energy density remains constant and independent of $R$.

In terms of cosmic time, $t$, the universe scale factor becomes asymptotically

$$R(t) = \dot{R}e^{\sqrt{\Lambda/3} t}$$  

(11)

on account of overwhelming dark energy. That expansion also creates an ‘event horizon’ (EH) that ultimately reduces to a constant

$$a_\Lambda = \sqrt{3/\Lambda}$$  

(12)

for all future times.[42, 44] The Hubble parameter, $H(t) \equiv \dot{R}/R$, also approaches asymptotically a constant $H_\Lambda = c/a_\Lambda$. In emptied space-time, with $T^{\mu\nu} = 0$, these results are exact for $\kappa = 0$, while they become asymptotically correct for $t >> a_\Lambda/c$ if $\kappa = \pm 1$.

Density parameters, $\Omega_i(t)$, are defined by dividing all terms in FL Eq. (9) by $(\dot{R})^2$. Following the $R(t)$ expansion, dark energy density currently amounts already to about $\Omega_{\Lambda,0} = c^2\Lambda[H(t_0)]^{-2}/3 \simeq 0.7$, and it is bound to further overwhelm exponentially all other density parameters, as $H(t)$ will further reduce to $H_\Lambda$ in the future.

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That certainly includes the spatial curvature parameter, $\Omega_\kappa(t) = -c^2\kappa/\dot{R}^2$, which raises nonetheless a fundamental question regarding the so called flatness problem. First of all, if $\kappa = \pm 1$, FL Eq. (9) is not manifestly independent of an arbitrary factor, $\dot{R}$, that should generally pertain to the universe scale factor, $R(t)$. More importantly, current cosmological observations indicate that, if $\kappa = \pm 1$, the magnitude of $\Omega_\kappa(t)$ had to be finely tuned to a very small value at early epochs, grow exponentially to still a rather small value at the deceleration/acceleration change-over around $7\times10^9$ years, and then ‘refocus’ back to a small value at the present time.

Rather, data from supernovae, cosmic microwave background (CMB), and studies of evolutions of galaxy clusters suggest that $\kappa = 0$. The power spectrum of matter-density perturbations in the CMB and the angular position and scale of the first acoustic peak provide further support for a flat $\Lambda$-CDM model.[41, 42] That mainly underlies my model of the dark energy $g$-BEC in Sec. 5.

3. DE SITTER SPACE

In $n = 4$ space-time dimensions, de Sitter space, $dS_4$, is the maximally symmetric Lorentzian manifold with constant positive curvature that can be embedded in a 5-dimensional Minkowski space-time, $M(1, 4)$. It is a solution of Einstein’s Eq. (5) with $T^{\mu\nu} = 0$. Thus, $dS_4$ is a 4-dimensional ‘Einstein manifold,’ with Ricci tensor $R^{\mu\nu} = \Lambda g^{\mu\nu}$.
and a positive constant Ricci scalar curvature, $g_{\mu\nu}R^{\mu\nu} = 4A > 0$, throughout the manifold. That corresponds to a subset of FLRW geometry.

Introducing particular definitions of a time-like coordinate, $x^0$, and a space-like coordinate, $x^4$, in the $M(1,4)$ ambient space, metrics and coordinates of $dS_4$ can be related to those of FLRW geometry. As shown by Lemaître in 1925, flat slicing of $dS_4$ corresponds to setting $\kappa = 0$ in Eq. (4), with Eq. (11) satisfied exactly for any arbitrary constant, $\dot{R}$. Closed or open slicing of $dS_4$ correspond to setting in Eq. (11) $\kappa = +1$ and $R(t) = a_\Lambda\cosh(\text{ct}/a_\Lambda)$, or $\kappa = -1$ and $R(t) = a_\Lambda\sinh(\text{ct}/a_\Lambda)$, respectively.

Other choices of $dS_4$ slicing and coordinates are possible. Most notably, static coordinates yield the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{r^2}{a_\Lambda^2}\right)(ct')^2 + \left(1 - \frac{r^2}{a_\Lambda^2}\right)^{-1}(dr')^2 + r'^2\left((d\theta)^2 + \sin^2(\theta)(d\phi)^2\right).$$

I will refer to Eq. (13) as de Sitter’s static metric, although de Sitter (1917) originally introduced other coordinates. De Sitter was influenced by Einstein’s cosmological idea of a static closed universe. In fact, de Sitter’s metric in Eq. (13) has closed spatial sections, corresponding to $\kappa = +1$ in FLRW geometry, and

De Sitter’s metric is ‘static’ in the sense that its $g_{\mu\nu}$ metric tensor components are time-independent and the line element is invariant under time-reversal. It looks similar to Schwarzschild metric, in the sense that we can cast both metrics in the form

$$ds^2 = -\left(1 + \frac{2}{c^2}V(r')\right)(ct')^2 + \left(1 + \frac{2}{c^2}V(r')\right)^{-1}(dr')^2 + r'^2\left((d\theta)^2 + \sin^2(\theta)(d\phi)^2\right).$$

For Schwarzschild metric, $V_S(r') = -GM/r'$ corresponds to the familiar Newtonian attractive gravitational potential. For de Sitter metric, $V_{dS}(r') = -(c^2/2)(r'^2/a_\Lambda^2)$ corresponds to a quadratic repulsive potential with a constant anti-gravitational acceleration, consistent with our understanding of dark energy.

For both Schwarzschild and de Sitter metrics, ‘time slicing’ generates an ‘atemporal space.’ Notice, however, that the usual interpretation of time-like and space-like coordinates holds in the interior of the $a_\Lambda$ horizon in Eq. (13), but in the exterior of the Schwarzschild radius, $R'_s = 2MG/c^2$. Furthermore, neither metric is fundamentally static, as more deeply understood in terms of other coordinates.

Laboriously, one can find the transformation between $dS_4$ static coordinates and those corresponding to the FLRW geometry with $\kappa = +1$. In my formalism, that transformation is

$$r' = R(t)r = a_\Lambda\cosh(\text{ct}/a_\Lambda)r,$$

and

$$\tanh(\text{ct}/a_\Lambda) = (1 - r^2)^{-1/2}\tanh(\text{ct}/a_\Lambda).$$

In this context, I have not reported explicit expressions of event horizons. Suffice it to say that for $dS_4$ the EH is an invariant constant, $a_\Lambda$, for any choice of coordinates and any time-slicing. In FLRW coordinates, that can be expressed as $a_\Lambda = R(t)r^{(EH)}_t$, where $t$ is any current cosmic time. Thus, Eq. (15) applies to the curvature coordinate, $r$, of FLRW geometry with $\kappa = +1$ as long as $r < r^{(EH)}_t = 1/\cosh(\text{ct}/a_\Lambda)$ and $r' < a_\Lambda$ remain consistently within the EH.

It is my understanding that there is a bijection only between $dS_4$ static coordinates and FLRW coordinates with $\kappa = +1$. If there were other bijections between $dS_4$ static coordinates and FLRW coordinates with either $\kappa = 0$ or $\kappa = -1$, then there would also be bijections between FLRW coordinates with any $\kappa$, which would violate a basic tenet of FLRW geometry. On the other hand, Sec. II of Ref. may suggest that de Sitter’s original static coordinates are compatible with FLRW coordinates with $\kappa = 0$. As a matter of fact, that would strengthen the consistency of later assumptions of mine, particularly when comparing the results of Sec. and Sec. Technically, however, application of de Sitter’s original static coordinates to my formulation is more cumbersome and will not be pursued in this paper.

In any case, the assumption that dark energy is tied to $dS_4$ and the fact that $dS_4$ intrinsically has a robust event horizon for any coordinate system and whatever value of $\kappa$ is what matters most to support my following conjecture and model of dark energy and pressure.

Using techniques and notations previously developed, it is instructive to determine geodesic equations and
orbits in $dS_4$, using static coordinates in particular. For a particle with $m > 0$, I obtain
\[
\left( \frac{dr'}{d\tau} \right)^2 = c^2(\tilde{E}^2 - 1) + c^2 \frac{r'^2}{a_\Lambda^2} - \left( 1 - \frac{r'^2}{a_\Lambda^2} \right) \frac{L^2}{r'^2}.
\]  
(17)

Here we recognize the anti-gravitational potential, $V_{dS}(r') = -(c^2/2)(r'^2/a_\Lambda^2)$, introduced in Eq. (14). For radial time-like geodesics, having $L = 0$, that produces an outward acceleration, $d^2r'/d\tau^2 = c^2(r'/a_\Lambda^2) > 0$. No massive particle can remain at rest for any $r' > 0$. Generally, it will be pushed beyond the $a_\Lambda$ horizon in a finite interval of ‘proper time,’ $\Delta \tau$. This agrees with our construct of dark energy. Furthermore, an $m$-particle may stand at rest indefinitely in an unstable equilibrium position at $r' = 0$ only if it also has minimal $\tilde{E}^2 - 1 = \tilde{L}^2 = 0$. Thus, in order to keep $m_\gamma$-photons at rest in a $g$-BEC, a non-gravitational (QM) inward pressure or ‘tension’ must be exerted uniformly throughout their density.

For a particle with $m = 0$, I obtain
\[
\left( \frac{dr'}{d\lambda} \right)^2 = \frac{E^2}{c^2} - \left( 1 - \frac{r'^2}{a_\Lambda^2} \right) \frac{L^2}{r'^2}.
\]  
(18)

For radial null geodesics, having $L = 0$, $m = 0$ particles move freely, without any acceleration, $d^2r'/d\lambda^2 = 0$, in terms of any affine parameter, $\lambda$. In terms of ‘static coordinate time,’ $t'$, however, we have
\[
\frac{dr'}{dt'} = \pm c \left( 1 - \frac{r'^2}{a_\Lambda^2} \right).
\]  
(19)

A $m = 0$ particle can then stop or stand at rest at the $a_\Lambda$ horizon with $L = 0$, though it takes an infinite amount of coordinate time, $t'$, to reach (or escape from) that horizon.

All these results correspond to those of the Schwarzschild metric,[41–43, 56, 57] on account of Eq. (14). In particular, while approaching the horizon, material sources emit increasingly red-shifted light backward to the observer at $r' = 0$. That feature of Eq. (14) was soon realized and related to Slipher’s 1912 discovery of galactic red-shifts.[51] That also provides an alternative interpretation of the gravitational red-shift in $dS_4$. Rather than viewing it as a result of the $R(t)$ expansion, $V_{dS}(r')$ generates ‘an inverse gravitational red-shift,’ wherein incoming light in Eq. (19) must climb, in static coordinate time, $t'$, a potential hill in order to reach the observer.[52]

4. PERFECT Fluids

Exploiting twice-contracted Bianchi identities, Einstein constructed his $G^{\mu\nu}$ tensor in Eq. (5) as divergence free, thus allowing four continuity equations that express conservation of energy and momentum for stress-energy tensor components,
\[
(T^{\mu\nu})_{\cdot\nu} = 0,
\]  
(20)

where a semi-colon conventionally denotes covariant differentiation.

In FLRW geometry, contraction of Eq. (20) with $U_\mu$ results in the elimination of $\ddot{R}$ in the two Friedmann-Lemaître equations.[11, 12] That yields for cosmology an energy conservation equation reminiscent of the first law of thermodynamics for a perfect fluid.[11, 12, 47] having no viscosity nor heat conduction:
\[
\frac{d}{dt}(\rho R^3) + \frac{d}{dt}(\rho R^3) = 0.
\]  
(21)

In Eq. (21) the energy density, $\rho = T_{\mu\nu}U^\mu U^\nu/c^2$, and pressure, $p = \frac{1}{3}T_{\mu\nu}(g^{\mu\nu} + U^\mu U^\nu/c^2)$, are defined as scalar functions under general coordinate transformations.

It is further possible to show that[11, 42]
\[
(U^\mu)_{\cdot\mu} = \partial_\mu U^\mu + \Gamma^\mu_{\alpha\mu} U^\alpha = 3 \frac{\ddot{R}}{R}
\]  
(22)
in FLRW geometry. Then

\[(N^\mu)_{,\mu} \equiv (nU^\mu)_{,\mu} = \dot{n} + 3n \frac{\dot{R}}{R}, \quad (23)\]

where \(N^\mu\) denotes the number-flux four-vector. When set equal to zero, the continuity of the four-divergence Eq. (23) provides local conservation of particle number for a perfect fluid element in FLRW geometry.\[41, 42\] Einstein’s field Eq. (5) imply various energy conditions,\[42, 48, 49\] but they do not generally require conservation of particle number.

Let us then consider a matter-energy density, \(\rho_m\), with an equation of state

\[\rho_m < \rho_m = \frac{A}{[R(t)]^3}, \quad (24)\]

which obeys Eq. (21) approximately.\[60\] That occurs when variously called ‘random’ or ‘thermal’ or ‘peculiar’ relative velocities generate energies much smaller than the rest-energy mass, \(mc^2 > 0\), of the particles.\[11, 12, 17\] It is exactly so for a pressure-free perfect fluid called ‘dust,’ in which all particles are perfectly comoving with the FLRW geometry. That suggests consideration of a particle density

\[n_m = \frac{\rho_m}{mc^2}. \quad (25)\]

Contracting Eq. (20) with \(U_\mu\), it is easy to show that Eq. (23) vanishes and provides conservation of particle number for \(n_m\) just as approximately as Eq. (21) is obeyed for \(\rho_m\).

The situation is different for radiation energy density, which obeys an equation of state\[60\]

\[3p_r = \rho_r = \frac{B}{[R(t)]^4}, \quad (26)\]

In that case, Eq. (21) is exactly satisfied, even though neither Eq. (3) nor Eq. (23) apply, as a particle number density cannot be determined for radiation.\[50\] That is related to the fact that ordinary radiation is associated with transverse massless photons. It takes no energy to create any number of zero energy massless photons, which must then coexist with energetic photons in a two-phase equilibrium between a BEC and a ‘vapor’ phase with \(\mu = 0\) at all temperatures.

I view that as a mathematical idealization or a limiting case at best.\[1\]

For the dark energy density, as given in Eq. (8), the situation is profoundly different. In fact, \(\rho_\Lambda\) remains a uniform universal constant relative to the fixed \(a_\Lambda\) horizon in both FLRW geometry and \(dS_4\) static coordinates. Rather than comoving, I thus refer to the dark energy density as at rest in FLRW geometry. Correspondingly, a negative pressure or tension is needed to keep the dark energy density constant in FLRW geometry, while its scale factor, \(R(t)\), expands. That is completely different from the case of matter density or ‘dust’ in Eq. (24), which instead comoves and ‘stretches’ with the \(R(t)\) expansion in FLRW geometry.

These basic considerations thus provide the foundation for my model of the dark energy \(g\)-BEC that I develop in the next Sec. 5.

5. DARK ENERGY AS A BEC OF COSMOLOGICALLY MASSIVE PHOTONS

For kinetically energetic photons in the ordinary vapor phase, FL equations, Eq. (9) and Eq. (10), must include a radiation contribution provided by Eq. (26). At the present cosmic time, \(t_0\), the \(R(t_0)\) expansion has already reduced that ordinary electromagnetic radiation contribution to only a small \(\Omega_r,0 \simeq 5x10^{-5}\) density fraction.\[42\]

Aside from that, dark energy may be modeled as follows. In a previous paper,\[1\] I conjectured that cosmologically minimally massive gauge bosons such as photons - let me leave aside gluons till Sec. 14 - may form a BEC at rest in FLRW geometry. According to Eq. (2) and Eq. (8), that BEC has a number density

\[n_g = \frac{\rho_\Lambda}{m_g c^2} = \frac{c^3 \sqrt{X/C_g}}{8\pi hG} = \frac{\sqrt{X}/C_g}{8\pi l_P} \simeq 2.6\times10^{41} m^{-3}/C_g, \quad (27)\]

where \(l_P = \sqrt{hG/c^3} = 4.051285\times10^{-35} m\) denotes the Planck length. Thus \(m_g\)-photons belonging to the BEC phase provide a stress-energy tensor contribution \(T^\mu_\nu = p_\Lambda g^\mu_\nu\) to Einstein’s Eq. (6) with \(T^\mu_\nu = 0\), as prescribed in Eq. (7) with \(\Lambda > 0\).
Thus there is a fundamental difference between a perfect fluid that involves an energy density of matter or dust with conserved particle number, as derived from Eq. (24) and Eq. (25), and the special perfect fluid that I associate with dark energy density, having a particle number density given in Eq. (27). In fact, my \( n_g \) does not yield particle number conservation in Eq. (23), even though its equation of state in Eq. (8) exactly obeys all energy-momentum conservation laws, Eq. (20) and Eq. (21). Rather, dark energy \( m_g \)-particles in the BEC are generated at a steady rate of

\[
(N^\mu)_g = 3n_g \frac{\dot{R}}{R} \simeq 3n_g \frac{c}{a\Lambda} = \frac{\sqrt{3}c}{8\pi C_g (lp L_g)^2},
\]

as \( R(t) \) of the FLRW geometry expands and the Hubble parameter approaches \( H\Lambda = c/a\Lambda \).

Equivalently, the relative rate of change of particle number is

\[
\frac{(N^\mu)_g}{n_g} = \frac{3}{R} \simeq \frac{\sqrt{3}}{T_g},
\]

where \( T_g = L_g/c \simeq 10^{10} \) years. On that cosmic time scale, each \( m_g \)-particle must then approximately double in order to keep my \( n_g \) particle density constant overall.

In order to appreciate these results more intuitively, define a space-volume \( V = R^3 \), which also expands at a rate \( \dot{V} = 3VR/R \). Now the number-flux four-vector per unit space-volume, \( F^\mu_g = N^\mu_g / V \), is conserved, i.e., \( (F^\mu_g)_\mu = 0 \). This means that \( m_g \)-particles must be created at exactly the same rate as space-time itself is created in FLRW geometry, in order to keep universally constant the \( n_g \) particle density, exactly as we expect from dark energy.

Now compute that

\[
U_\mu(T^\mu)^{\nu}_g = -(\rho\Lambda + p\Lambda)(U^\nu)_g.
\]

This means that we have conservation of dark energy and Eq. (21) only if the equation of state \( p\Lambda = -\rho\Lambda \) is precisely satisfied. Namely, creation of \( m_g \)-particles at the rate of Eq. (20) does not result in increase of dark energy only if a negative pressure balances exactly each \( m_g c^2 \) creation of particle energy.

How could that be fully justified in my QM-GR model? Perhaps non-linearities in a QM attractive photon-photon interaction field in the BEC extract each \( m_g c^2 \) quantum precisely from a corresponding space-time volume expansion. Although that can hardly be ‘proved’ at present, I will at least refer to some prospects in Sec. 10.

6. FLAT-SPACE ESTIMATE OF PHOTON MASS AND BEC INTER-PARTICLE DISTANCE

In this Sec. 6 I shall assume a Euclidean spatial geometry, which applies to FLRW geometry in a flat universe with \( \kappa = 0 \) in Eq. (4). These assumptions thus amount to the so called de Sitter universe, which satisfies the ‘perfect cosmological principle,’ assuming isotropy and homogeneity equally throughout space and time. \[61\] Correspondingly, both FL, Eq. (9) and Eq. (10) coincide for \( T^{\mu\nu} = 0 \) and \( \kappa = 0 \).

Now, this spatial flatness assumption may not be entirely consistent with the \( dS_4 \) static metric of Eq. (13), which has closed spatial sections. \[51\] Asymptotically, however, if not altogether, the \( \Omega_\kappa(t) \) density parameter hardly matters. Thus, at least approximately, I will refer alternatively to desirable ‘near-flatness’ properties of both the expanding flat slicing (Lemaitre) and the static slicing (de Sitter) of \( dS_4 \) that I later consider in Sec. 7.

Here let me then most simply assume that each particle, being almost free and virtually at rest in the BEC, is subject to a QM effective non-relativistic attractive spherical-well potential,

\[
V(r) = \begin{cases} 0, & r < a\Lambda, \\
V_{0c} > 0, & r > a\Lambda, \end{cases}
\]

extending the Euclidean geometry to all space at any cosmic time.

In Eq. (31) I then assume that \( V_{0c} \) provides the minimal QM critical depth that binds just one particle state at the top of the cosmologically constant Euclidean well. By solving the time-independent Schrödinger equation with \( m_g \) given in Eq. (28), I immediately obtain that

\[
V_{0c} = \frac{\hbar^2}{2m_g} \left( \frac{\pi}{2a\Lambda} \right)^2 = \frac{1}{96C_g^2} m_g c^2.
\]
Since there are no ‘walls’ at $a_{\Lambda}$, that horizon must raise steeply a QM square-well potential barrier therein, generating a corresponding gradient, hence, an inward force times a $dr$-displacement on each particle at the horizon, amounting to $V_{0c}$. I may then combine $n_g$ in Eq. (27) with Eq. (32) to conclude, by definition of pressure, that
\[-p_\Lambda = n_g V_{0c} = \frac{1}{96C_g^2} \rho_\Lambda \rightarrow \rho_\Lambda. \tag{33}\]

In the last step of Eq. (33) I recalled that the invariant continuity Eq. (20), which allowed the introduction of the cosmological term in Einstein’s field Eq. (5) in the first place, requires that Eq. (8) holds for a perfect fluid of dark energy. Thus I set
\[C_g = 1/\sqrt{96} \simeq 0.1. \tag{34}\]

Albeit qualitatively, this model illustrates how it is possible to constrain conjectures like those of Eq. (1), Eq. (2) and Eq. (27) with an independent requirement, such as that of Eq. (8). In this case, photons acquire a rest-energy mass
\[m_g c^2 = C_g h c / \sqrt{\Lambda} \simeq 1.3 \times 10^{-33} \text{eV}. \tag{35}\]

Following Eq. (27), I thus estimate that the average distance between photons in their BEC is
\[d_g \simeq n_g^{-1/3} = 2(\pi C_g)^{1/3} \left[ \frac{\rho}{\sqrt{\Lambda}} \right]^{1/3} \simeq 7.5 \text{fm}. \tag{36}\]

Because of dimensional constraints, it may appear that Eq. (33) contains some degree of circularity. Conceptually, however, that is not the case. The first step in Eq. (32) derives from the fundamentally non-relativistic version of the uncertainty principle of Heisenberg et al. Initially, $V_{0c}$ does not contain explicitly any $c$ factor, while $m_g$ stands in the denominator. On the other hand, $m_g$ derives from a fundamentally independent relativistic version of the uncertainty principle in Eq. (1), containing a $c$ factor. As a result of that, a factor of $c^2$ appears in the second step of Eq. (32), while $m_g$ switches to the numerator. Thus, ultimately, Eq. (32) connects vastly different non-relativistic and relativistic quantum uncertainties and bridges corresponding microscopic ($h$) and cosmological ($\Lambda$) scales. The other factor, $n_g$, that enters in Eq. (33) ultimately simplifies two $m_g c^2$ factors away, yielding a numerical proportionality with $\rho_\Lambda$, independently of $h$. Thus, fixing the numerical proportionality constant, i.e., $C_g$ in Eq. (34), according to Eq. (8), may derive from an extremely simplified model and estimate of dark energy, but conceptually that is neither trivial nor circular.

7. ZERO-POINT ENERGY OF THE BEC OF COSMOLOGICALLY MASSIVE PHOTONS

The estimate of $C_g$ in Eq. (34) derives from QM relations expressed in Eq. (11) and Eq. (22) in particular. The symmetry group that underlies SM-QFT is that of flat Minkowski space-time, $M(1,3)$. However, de Sitter space, $dS_4$, is an Einstein manifold with constant invariant Ricci scalar curvature, $g_{\mu\nu}R^{\mu\nu} = 4\Lambda > 0$, even for $\kappa = 0$. That space-time curvature may be negligible even up to galactic scales, but it clearly becomes sizeable at the event horizon scale of $dS_4$, Eq. (12). That is where the treatment of Sec. 6 and the result of Eq. (32) in particular can hardly be held as precise.

As an alternative and independent approach, let me then consider the zero-point energy of the BEC of cosmologically massive photons in de Sitter static metric, Eq. (13), with a corresponding coordinate-time $t'$-slicing. Referring to Eq. (11) and Eq. (17), I have already discussed the quadratic repulsive de Sitter potential, $V_{dS}(r')$, induced by the $dS_4$ anti-gravitational curvature. I then assume that in such static metric and $t'$-slicing the BEC of cosmologically massive photons is kept at rest in equilibrium by an opposite attractive QM harmonic oscillator (HO) potential per $m_g$-mass,
\[V_{0d}(r') = -V_{dS}(r') = \frac{c^2}{2a_{\Lambda}^2} r'^2 = \frac{1}{2} \frac{\omega_{dS}^2}{2} r'^2, \tag{37}\]
where $\omega_{dS} = c/a_{\Lambda}$ is the HO angular frequency.
That yields the zero-point energy density for the BEC as
\[ \rho_{0g} = \left( \frac{1}{2} \hbar \omega_{dS} \right) n'_g = \frac{1}{4 \pi \sqrt{3} C'_g} \rho_\Lambda \rightarrow \rho_\Lambda. \] (38)

Here I define again \( m'_g \) and \( n'_g \) as in Eq. (2) and Eq. (27), but the numerical coefficient, \( C'_g \), now differs from the previous \( C_g \) that is determined in Eq. (34). Two \( h \) factors cancel in Eq. (38) and \( \rho_{0g} \) is remarkably independent of \( h \). Thus, if I set
\[ C'_g = \frac{1}{4 \pi \sqrt{3}} \simeq 0.045944, \] (39)

I can identify the dark energy density, \( \rho_\Lambda \), with the zero-point energy density of the BEC, \( \rho_{0g} \).

Evidently, Eq. (38) and Eq. (39) imply that
\[ \epsilon'_g = m'_g c^2 = \frac{1}{2} \hbar \omega_{dS} = C'_g h c \sqrt{\Lambda} \simeq 0.59 \times 10^{-33} \text{eV}, \] (40)

and conversely. This confirms that the \( g' \)-photon rest-energy mass corresponds to the zero-point energy of the attractive QM-HO potential that balances \( V_{dS}(r') \).

From an order-of-magnitude perspective, switching between my numerical coefficients hardly matters, since \( C'_g \) has only about half the value of \( C_g \). By comparison, zero-point energies resulting from standard QFT calculations may overestimate vacuum fluctuations by as many as 120 orders of magnitude.\[10–16, 38, 39, 42, 49\] In fact, such calculations typically involve quantum field frequencies up to an ultra-violet cut-off at the Planck scale, whereas Eq. (38) requires only a single frequency, \( \omega_{dS} \), associated with an infra-red cosmological cut-off at the horizon scale. I will return to discuss that in Sec. 12.

8. CONSTANT DARK ENERGY DENSITY AND GRAVITATIONAL COLLAPSE

It is instructive to further consider a classical GR perspective of dark energy condensation in terms of impending gravitational collapse.

Thus consider a spherically symmetric and uniform matter density, \( \rho_\Lambda/c^2 \), extending initially in a spatial Euclidean geometry. Consider the radial function
\[ f(R') = 2 \frac{G}{c^4} \rho_\Lambda \frac{4\pi}{3} R'^3 - R'. \] (41)

Gravitational collapse is bound to occur when the radius, \( R' \), of \( \rho_\Lambda \) reaches a Schwarzschild radius, \( R'_S \), such that \( f(R'_S) = 0 \). This type of basic argument is often used to estimate gravitational collapse of black holes, galaxies and other astronomical objects.\[43\]

For dark energy, as given in Eq. (8), one immediately obtains that
\[ f(R'_S) = 0 \implies R'_S = a_\Lambda. \] (42)

This is a remarkably interesting result, independent of any further theory or supposition. It is related to the fact that the \( a_\Lambda \) horizon appears explicitly in the static metric of \( dS_4 \) and that static metric is formally similar to the Schwarzschild metric, as shown with Eq. (14).

Qualitatively, these results may indicate that relativistic QM, Eq. (1), non-relativistic QM, Eq. (32), and classical GR, Eq. (42), provide at least compatible accounts of dark energy condensation within de Sitter horizon, bridging a chasm between corresponding nuclear and cosmological scales, as defined by \( d_g \) and \( a_\Lambda \).

9. COSMIC MICROWAVE BACKGROUND

Description of dark energy as a BEC of cosmologically massive photons at rest in de Sitter space involves consideration of statistical properties of equilibrium between such BEC phase and a vapor phase of \( m_g \) photons. Analysis of cosmic microwave background (CMB) data may inform or constrain that description.

One should also be mindful of the so-called ‘strong energy condition,’ requiring \( \rho \geq -3p \) for the equation of
state. Most notably, that does not hold for dark energy in the standard Λ-CDM, while it should apply to a standard BEC in a Euclidean or Minkowskian space-time. On the other hand, a standard BEC is confined by an external volume, whereas the condensate that I envision, to which I refer more specifically as a $g$-BEC, must be self-confining within the de Sitter horizon.

Furthermore, $m_g$-photons in the ordinary vapor phase do not need to condense directly into the $g$-BEC. Rather, they may maintain that equilibrium via interactions with other particles, such as absorption, emission or scattering by electrons and protons, or even neutrinos. In fact, it is generally possible to investigate electro-weak processes including a massive photon propagator. Such calculations are quite involved, however, even to low perturbative orders.

Although I may not enter into any such technical issues at this time, let me at least note yet another ‘curious coincidence.’ The low-energy elastic scattering of photons by free electrons results in Thomson cross section, proportional to the square of the classical radius of the electron, $r_0 = e^2/m_e c^2 \approx 2.818 fm$. In value, that is remarkably close to the average distance between photons in the $g$-BEC, which is given in Eq. (35) as $d_g \approx 7.5 fm$. For all that is currently known (but see Postscript, Sec. 17) the electron charge, $e$, that enters $r_0$ seems unrelated to the fundamental constants $h,c,G,Λ$ that enter $d_g$, and yet $r_0 \sim d_g$. Somehow the $fm$-scale seems to be the microscopic length of convergence not only of classical and quantum electromagnetic and nuclear interactions, but also of $g$-BEC interactions derived from a cosmological horizon.

In any case, relative corrections derived from a photon mass, $m_g$, to QFT scattering cross sections in non-relativistic limits are at most of the order of $m_g/m_e \sim 3x10^{-39}$. Such relative corrections to atomic spectral lines up to the UV/X-ray range become at most of the order of $\alpha^{-2} m_g/m_e \sim 5x10^{-35}$. Any such corrections appear too small to be directly observable at present.

Let me then address at least some questions of compatibility between $g$-BEC equilibrium and CMB observations for ‘ordinary’ photons at a most basic level of approximation.

The $g$-BEC QM-average of momentum vector, $p$, of $m_g$-photons is $<p>_g = 0$. Consistently with Eq. (31) and Eq. (22), we may thus assume for the vapor phase a dispersion relation

$$\epsilon = \sqrt{(m_g c^2)^2 + c^2 p^2} - m_g c^2,$$

between kinetic energy, $\epsilon$, and average momentum magnitude, $p = |p|$. Thus $\epsilon$ excludes a zero-point energy, $V_0 = m_g c^2$, and varies in the range $(0, +\infty)$.

After bringing $m_g c^2$ to the left-hand side of Eq. (43) and squaring the result, the ratio between $dp$ and $d\epsilon$ can be obtained in terms of $\epsilon$. From that the exact expression for the density of states,

$$g(\epsilon)d\epsilon = g_S(V/h^3)4\pi p^2(d\epsilon/d\epsilon)d\epsilon = g_S V \frac{1}{(hc)^3}4\pi\sqrt{\epsilon + 2m_g c^2(\epsilon + m_g c^2)}d\epsilon,$$

can be easily derived. The spin degeneracy for $m_g$-photons must be $g_S = 3$, while it is $g_S = 2$ for massless transverse photons.

The average kinetic energy of $m_g$-photons in the vapor phase can thus be obtained as

$$U = \langle E \rangle = \int_0^\infty \frac{\epsilon g(\epsilon)d\epsilon}{e^{\beta \epsilon} - 1}.$$

Approximate integration in a small initial $(0, m_g c^2)$ interval contributes negligibly to Eq. (45). Thus an expansion for $m_g c^2 << \epsilon$ can be assumed to hold for the entire integral. The resulting correction to $U_p$ as provided by Planck’s law for massless photons then is

$$\Delta U \approx \frac{3}{2} \left( \frac{2\zeta(3)}{3\zeta(4)} \right) \left( \frac{m_g c^2}{k_B T} \right),$$

where $\frac{2\zeta(3)}{3\zeta(4)} \approx 0.74$. Independently of that, the first $(3/2)$ factor derives from the assumption of $g_S = 3$ for massive $(g_S = 2$ for massless $)$ photons in the numerator (denominator) of Eq. (40).

The current temperature of the CMB is $T_0 \approx 2.725 K$, corresponding to $k_B T_0 \approx 2.35x10^{-4} eV$. Comparison of that $k_B T_0$ value with the $m_g c^2$ value obtained in Eq. (35) yields a relative correction to Planck’s law for the current CMB of $\Delta U/U_p(T_0) \approx 6.2x10^{-30}$, which seems currently undetectable. At the recombination epoch of the optical horizon, $T_{rec} \approx 3000 K$ further reduced $\Delta U/U_p$ by another factor of $T_0/T_{rec} \approx 9x10^{-4}$, thus yielding $\Delta U/U_p(T_{rec}) \approx 5.6x10^{-33}$. So, there may not be any directly detectable deviation from current measurements of the CMB caused by any such tiny attribution of $m_g c^2 \approx 1.3x10^{-33} eV$ to the actual photon rest energy mass. However, the $n_g$ density of $m_g$-photons in the $g$-BEC is extremely high, as given in Eq. (27). Thus, measurable effects of the
$g$-BEC on other kinds of observations are entirely possible, if not likely, as I will mention in the next Sec. [10]

In any case, a spin degeneracy factor of $g_S = 2$ must be attributed to transverse massless photons for Planck’s law to describe the CMB with such a remarkable precision. However, massive photons must have $g_S = 3$, no matter how light they may be cosmologically. This apparent discrepancy must then be associated with a mathematical singularity in dealing with a $m_{\gamma} \to 0$ ‘sick’ limit vs. setting $m_{\gamma} = 0$ in the first place. This problem in fact recurs in QFT. [66–68] For the moment, I propose to regard the vapor phase as essentially or overwhelmingly composed of transverse massless photons, having $g_S = 2$. By contrast, the $g$-BEC phase, having no wavevector direction, since $<p>_\gamma = 0$, makes no distinction between transverse and longitudinal photons, corresponding to a ‘vacuum ground state’ to which $g_S = 3$ applies.

Lately I became aware that BE condensation of photons in an optical microcavity has been achieved. [67] That experimental realization is formally equivalent to an ideal gas of massive bosons having an effective mass of about $4\text{eV}/c^2$, far greater than my estimate in Eq. (35). Though fundamentally important and suggestive, how such experimental findings may or may not be related to my considerations remains to be established.

10. GROSS-PITAЕVSKII-POISSON SYSTEMS AND RELATIVISTIC GENERALIZATIONS

At this point I must at least refer to great advances that have been made in the study and terrestrial realizations of BE condensates, such as those of superfluids and trapped-cooled dilute gases. [68, 69] A vast literature has further developed regarding astrophysical and cosmological applications. Excellent papers and reviews provide relatively concise and accessible entry points to that field. [21–25, 30–33] Some include most lucid derivations and complete accounts of the development of the field, starting from the original hydrodynamical formulations of quantum mechanical wave equations. [22, 23] They then focus on Gross-Pitaevskii-Poisson (GPP) systems and their relativistic generalizations. Their emphasis is on dark matter related to BEC of ultra-light particles with masses of the order of $m \sim 10^{-24}\text{eV}/c^2$, although a cosmological limit of $m \sim 10^{-35}\text{eV}/c^2$ may ultimately be attained.

From my perspective, a starting point may be provided by Eqs. [27–29] of Ref. 24. Although those Eqs. [27–29] apply to a pressureless gas in the non-relativistic limit, they can account for the cosmological expansion of $dS_4$ in particular. Exclusive consideration of a constant $\rho_\Lambda$ eliminates from Eq. [28] the effects of both the quantum potential arising from the Heisenberg uncertainty principle and the quantum scattering pressure that is typical of GPP systems with a corresponding scattering length. In fact, both such quantum effects are assumed to become negligible at the cosmological scale. [22] A term involving the gradient of a constant pressure, such as $p_\Lambda$ that I assume in my Eq. (35), would also vanish in the Euler-like Eq. [28] of Ref. 24. Based on such simplifying assumptions, my solution to the hydrodynamic Eq. [28] and its corresponding continuity Eq. [27] provides a gravitational potential that is half that of the $dS_4$ cosmologically expanding Poisson Eq. [29] of Ref. 24. This shows again that at least the orders of magnitude in my $dS_4$ $g$-BEC theory of dark energy are consistently viable.

A more accurate relativistic generalization of GPP systems should take into account at least some quantum potential and scattering pressure non-linear effects. [28, 29, 31] In my $g$-BEC that may take perturbatively advantage of a short scattering length, since scattering of light by light begins only at fourth-order for Feynman diagrams in standard QFT. [35, 62] Thus it may be possible to determine whether the fundamental equation of state of the standard $\Lambda$-CDM model, Eq. (35), can be satisfied for a $g$-BEC at least in Minkowskian space-time. On the other hand, full extension of $g$-BEC to the cosmological scale may eventually require full consideration of QFT in curved space-times. [70, 71] Such applications may involve profound conceptual and technical complexities, even for a ‘simple’ Einstein manifold such as $dS_4$.

Whenever the case, it is worth noting that correct evaluation of the low-energy cross section of photon-photon scattering yields an estimate of mean-free-path for visible light in the CMB frame of about $7\times 10^{52}$ly, at least $10^{42}$ times greater than the size of the ‘observable’ universe. [62] I estimated in Eqs. [6–8] of Ref. 1 the critical temperature and number density of the BEC of $g$-photons, implying that the latter is about $6\times 10^{32}$ times greater than the current CMB photon density. Thus my $g$-BEC should still remain largely transparent to visible light, by at least a $10^{9}$ scale factor. On the other hand, my $g$-photons must also gravitate and clump as dark matter around ordinary matter. Thus it may be possible to observe some dimming of most distant objects when their radiation grazes or lenses around compact objects in particular. Precise models and estimates of such effects lie beyond the scope of this paper, evidently, but at least in principle they are quite calculable, since most parameters relating to $g$-photons and the $g$-BEC medium are fairly narrowly specified. [1]
11. QUANTUM-MECHANICAL AND COSMOLOGICAL TIME

As much as anything else, conceptions of time differ profoundly in GR and QM. Precise definitions of the relativity of time are hallmarks of GR, whereas a conception of ‘universal time’ still underlies QM. On the other hand, ‘timelessness’ appears in the Wheeler-DeWitt equation, perhaps a first step toward a quantum theory of gravity.

Let us return, however, to Weyl’s postulate and the time-like world-lines of ‘fundamental observers’ comoving with the FLRW geometry. Ideal ‘atomic’ clocks carried by fundamental observers can thus be synchronized and all measure the same ‘proper’ or ‘synchronous’ or ‘cosmic’ time. Thus, at least within the idealization of the cosmological principle, one may identify the cosmic time of the entire universe with the ‘universal time’ that may apply to QM at a corresponding cosmological scale. That is indeed what Eq. (1) assumes, implying simultaneous delocalization and entanglement of photons uniformly within the cosmological horizon.

It is still necessary, however, to specify the causal sequence of events involved in the picture that I propose. My basic assumption is that the photon mass, \( m_g \), is created at the big bang. That creates the cosmological constant, \( \Lambda \), by seeding Eq. (1) and Eq. (2) at the big bang, regardless of the fact that de Sitter space, \( dS_4 \), with its ultimate event horizon, \( a_\Lambda \), is bound to dominate the universe evolution only at a much later epoch. In other words, it is \( m_g \) that ‘originates’ \( a_\Lambda \) in Eq. (2) and Eq. (12), rather than the other way around, as Eq. (1) may formally suggest, based on the uncertainty principle. Likewise, the \( g \)-BEC of photons also forms at a very early time of the order of \( 10^{-6} \) s, as estimated in Ref. 1, regardless of the fact that the CMB develops at a much later optical horizon, around 370 kyr after the big bang. Similarly, the non-relativistic QM localization reflected in Eq. (32) is also seeded by \( m_g \) at the big bang, regardless of the fact that the universe expansion is originally singular. It is only for descriptive purposes that I may envision the generation of \( m_g \) as arising from QM confinement within \( a_\Lambda \), simultaneously and independently of time.

12. COSMOLOGICAL VERSUS PLANCK UNITS AND UNCERTAINTY RELATIONS

Now let me introduce an alternative system of cosmological units, or ‘\( g \)-units,’ that complements the fundamental system of Planck units in various ways, including uncertainty relations to which either set of units inherently correspond.

Planck units fundamentally include Planck length, \( l_P \equiv (\Delta l)_P = \sqrt{\hbar G/c^3} = 4.051285 \times 10^{-35} \text{m} \), Planck mass, \( m_P = \hbar/(cl_P) = 3.060368 \times 10^{28} \text{eV}/c^2 \), and Planck time, \( t_P \equiv (\Delta t)_P = l_P/c = 1.351385 \times 10^{-43} \text{s} \).

Formally, one may combine these fundamental constants to form ‘relativistic uncertainty relations’

\[
(\Delta l)_P (m_P c) = (\Delta t)_P (m_P c^2) = \hbar,
\]

stemming from a Compton wavelength interpretation of the exact relation between \( l_P \) and \( m_P \).

Conversely, I define \( g \)-units of length, \( L_g \equiv (\Delta L)_g = 1/\sqrt{\Lambda} \approx 10^{10} \text{ly} \approx 0.946 \times 10^{20} \text{m} \), mass, \( M_g \equiv \hbar/(cL_g) \approx 1.3 \times 10^{-32} \text{eV}/c^2 \), and time, \( T_g \equiv (\Delta T)_g = L_g/c = 1/\sqrt{3\omega_{dS}} \approx 10^{10} \text{years} \approx 3.2 \times 10^{17} \text{s} \).

I may then combine these other fundamental constants to form relativistic uncertainty relations

\[
(\Delta L)_g (M_g c) = (\Delta T)_g (M_g c^2) = \hbar.
\]

Planck and \( g \)-units share the fundamental constants \( c \) of special relativity (SR) and \( h \) of QM. Planck units add to \( c \) and \( h \) the universal gravitational constant, \( G \). Instead of \( G \), \( g \)-units add to \( c \) and \( h \) the cosmological constant, \( \Lambda \). Both \( \Lambda \) and \( G \) rightfully belong to Einstein’s field Eq. (5). While \( \Lambda \) purely defines an inverse square length for any coordinate system, \( G \) requires the introduction of \( h \) to generate a fundamental length, i.e., \( l_P \). Thus, all three Planck units require the presence of \( h \), whereas only \( M_g \) requires that \( h \) in \( g \)-units, in order to form a Compton wavelength as \( L_P \). Thus, Planck units characterize the Planck era at high energies of the universe at its microscopic beginning, whereas \( g \)-units characterize the de Sitter era at low energies as the universe demises cosmologically.

Thus both \( G \) and \( \Lambda \) are very small on corresponding scales. That suggests taking their product to form a minimal fundamental a-dimensional ratio,

\[
\sqrt{\frac{\hbar G}{c^3}} \sqrt{\Lambda} = \frac{l_P}{L_g} = \frac{t_P}{T_g} = \frac{M_g}{m_P} \approx 4.2 \times 10^{-61}.
\]

(49)
There are some interesting alternative interpretations of Eq. (49). For example, the inverse ratio

$$\frac{L_g}{l_P} = 1 + z_\Lambda \simeq 2.4 \times 10^{60}$$

(50)

may be interpreted as the maximal red-shift from the $l_P$ to the $L_g$ scale, at the ‘look-back time’ from $t_P$ to $T_g$.

Let me further consider the Casimir electromagnetic vacuum energy density with a cut-off at the Planck length, which is readily evaluated as $\rho_{CEM} = \frac{\pi hc}{l_P^4}$, following the derivation of Eq. [19.37] on p. 534 of Ref. 39, for example. The ratio of $\rho_{CEM}$ to my $\rho_{0g}$ from Eq. (38) is

$$\frac{\rho_{CEM}}{\rho_{0g}} = \frac{8\pi^2(L_g/l_P)^2}{\rho_{0g} \rightarrow \rho_\Lambda} \simeq 4.55 \times 10^{122}.$$ 

(51)

This ratio illustrates the discrepancy between vacuum energy densities according to standard QFT calculations[10–16, 38, 39, 42, 49] and my estimate of $\rho_{0g}$ with a cosmological cut-off at $L_g$, which is as large as anyone may get.

Correspondingly, let me consider

$$n_g' \left( \frac{4\pi}{3} a_\Lambda^3 \right) = \frac{\sqrt{3}}{2C_g' \left( \frac{L_g}{l_P} \right)^2} \simeq 1.1 \times 10^{122},$$

(52)

where $C_g'$ is given in Eq. (39). This may be interpreted as the total number of $g$-photons in the BEC that fill and still keep together the ‘observable’ universe up to its ultimate event horizon.

Thus, switching between relativistic uncertainty relations at the most microscopic and cosmological scales, as expressed in Eq. (47) and Eq. (48), respectively, may provide complementary views of the QM-GR ‘beginning’ and ‘demise’ of the universe, spanning all of the ‘observable’ space-time. Through it all, this causally connected universe has always been and will always remain an indivisible ‘atom,’ as originally conceived by Lemaître and always prescribed by QM and GR combined.

Planck and $g$-uncertainty relations may also imply that the heaviest ‘elementary particle,’ born at the beginning of the universe, has a mass of the order of $m_P$, while the lightest elementary particle, surviving at its end, has a mass of the order of $M_g$. That ‘ultimate survivor’ is the photon, I propose, which is just about as light, common and easy to produce as ultimately allowed.

13. COSMOLOGICAL MASS OF NEUTRINOS

Standing between so vastly different $M_g$ and $m_P$ extreme values, one may wish to find some other characteristic mass. Thus, rather than taking the product, one may wish to consider the ratio between those two fundamental constants of GR, $\Lambda$ and $G$, and then form another QM mass, namely, the geometric mean

$$\sqrt{M_g m_P} = \sqrt[1/4]{\left( \frac{\hbar^3 \Lambda}{c G} \right)^{1/4}} \simeq 2 \times 10^{-2} eV/c^2 \simeq m_\nu.$$ 

(53)

Thus, in units of $eV/c^2$, we have

$$M_g \simeq 1.3 \times 10^{-32} < m_\nu \simeq 2 \times 10^{-2} < m_P \simeq 3.06 \times 10^{28}.$$ 

(54)

While $M_g$ and $m_P$ masses span a range of about 60 orders of magnitudes, their $\sqrt{M_g m_P}$ geometric mean falls close to the logarithmic middle of that range, curiously matching current estimates of neutrino masses almost within a single order of magnitude! Chances that all this happens again entirely by coincidence seem remote. Beside $m_g$-photons, if those exist, neutrinos are the lightest known fermions. They are electrically neutral and only weakly interacting. It is thus at least plausible that neutrino masses are most directly related to both GR and QM theories, perhaps fundamentally deriving from Eq. (53). The central position of neutrinos in both GR and elementary particle physics has been long foreseen.

Unlike Eq. (1), which is grounded on a fundamental uncertainty principle of QM, extended to further involve only $\Lambda$ in GR, Eq. (53) is not as simply related to that. However, the ratio of $\Lambda$ and $G$ is fundamentally related to $\rho_\Lambda$ in Eq. (38). Namely,

$$c^2 \left[ \frac{\hbar^3 \Lambda}{c G} \right]^{1/4} = \left[ 8\pi (hc)^3 \rho_\Lambda \right]^{1/4} \simeq 2 \times 10^{-2} eV,$$ 

(55)
independently of any supposition. Though not all that shines is gold, Eq. [53] and Eq. [55] look at least as beautiful to my eyes. They may hint at a weakly-interacting g-photon-neutrino plasma that may hold the universe together or provide a component of dark matter, perhaps.

Also consider that

$$\sqrt{t_p T_\rho} = \frac{h}{c^2 \sqrt{M_g m_p}} = \left[ \frac{h G}{c^2 \Lambda} \right]^{1/4} \approx 2.1 \times 10^{-13} s$$

(56)

turns out to be closely related to the time of the weak-electro phase transition at about $10^{-12} s$ in the chronology of the universe. So the cosmological constant, $\Lambda$, originally introduced to account for the largest scale of cosmic geometrodynamics in GR, may also be involved, upon taking the geometric mean of the corresponding $T_\rho$ with the Planck time, $t_P$, with the quintessential Higgs mechanism of electro-weak gauge symmetry breaking in QFT. [17] [38]

Introduction of the $M_g$ unit further suggests to reconsider of a famous dictum not to ask ‘why is gravity so feeble,’ but rather ‘why is the proton’s mass so small?’ That is, compared to Planck mass. [73] [81] However, compared to $M_g$, the proton’s mass, $m_p \approx 938$ MeV/c$^2$, is very large! In fact, while $m_p$ still lies well within the vast range of $M_g$ and $m_P$ values, it already stands about 11 orders of magnitudes above the value of $\sqrt{M_g m_p}$ in Eq. (53) due to other quantum chromodynamics (QCD) elements and interactions. Thus, ‘scaling mount Planck’ may have to start at a ‘zero-point base camp,’ as from Eq. (13) at the ‘bottom’!

In fact, let us consider the next weighted geometric mean

$$[M_g m_p^2]^{1/3} \approx 230$ MeV/c$^2 \approx m_p/4.1$$

(57)

This is precisely what sets the inter-particle distance in the $g$-BEC almost at the fermi scale in Eq. (56), i.e., $d_g \approx 5.66 \lambda_p \approx 7.5 fm$, where $\lambda_p \approx 1.32 fm$ is the Compton wavelength of the proton. Namely, via Eq. (1), the fundamental constants of GR and QM combine in Eq. (27) to yield the volume $L_g^2 p^2$, setting apparently unrelated $d_g$ and $\lambda_p$ lengths at about the same $fm$-scale. However ‘coincident’ is this perhaps the most tantalizing or consequent ‘occurrence’ that I remarked both in this and in my previous paper. [1] In fact, the dark energy density in Eq. (8) can be expressed fundamentally as

$$\rho_\Lambda = \frac{h c}{8\pi L_g^2 p^2} \approx 3.38 \times 10^9 eV m^{-3}$$

(58)

14. MASSIVE GLUONS AND GRAVITONS

Gauge invariance does not preclude the photon from acquiring mass. Beside Higgs mechanisms, a Stückelberg mechanism applied to an Abelian hypercharge group, $U(1)_Y$, can provide photons (or other $U(1)_{B-L}$ gauge bosons) with mass. [53] [55] In fact, it should be harder to ‘prove’ that the photon is exactly massless than to admit that it may have a small mass, since only the latter can afford technical naturalness. [64] [65] The problem for the latter is rather to determine what value that ‘small mass’ may actually have!

To generalize Stückelberg mechanisms to non-Abelian gauge theories, while still preserving their unitarity and renormalizability, is far more complicated. However, there are proposals to do that and avoid the plague of almost intractable infrared divergences in QCD, for example. [63]

Furthermore, even if QCD requires massless gluons, that applies only to flat Minkowski space-time. My proposal of Eq. (11) refers instead to a cosmologically curved space-time. I maintain that Eq. (11) is consistent with fundamental principles of both QM and GR applied at a cosmological scale. Those are the uncertainty principle, fundamental to QM, and a finite maximum/supremum speed, $c$, hence causality-limited horizons, fundamental to SR and GR. Thus, a cosmologically minimal mass, such as $M_g$, should limit theoretically massless gluons in the SM as well. [1]

Now, in QED, photons are neutral, whereas in QCD gluons carry color and strongly interact with themselves. Thus, the far greater complexity of QCD interactions and vacuum fluctuations render inapplicable to gluons any elementary argument that I have put forward in this paper to further estimate $m_g$ values for the mass of the photon. Thus, gluon masses may well be higher than predicted by the cosmological $M_g$ minimum.

On the other hand, it remains tempting to maintain some arguments of Ref. [1] regarding the nearly asymptotic freedom ‘coincidence’ already noted after Eq. [5] therein, as well as the the critical temperature of BEC in Eqs. [6-7], reaching to the end of the quark epoch, at about $10^{-8}$s and $10^{12}$K. Far more advanced theories have also considered that $\Lambda$ may be fundamentally related to masses of elementary particles such as pions or Higgs bosons, [2] or that cold dark matter may derive from a gluonic BEC in anti-de Sitter space-time. [82]

Remarkably, recent measurements of the positive muon anomalous magnetic moment suggest new physics beyond
the SM. From my perspective, one could at least consider QCD calculations on a lattice that may determine parametric values of gluon masses starting from the minimal $M_g$ that I estimated.

Furthermore, some of the elementary argument that I have put forward in this paper may also apply to $g$-gravitons. Starting with the Pauli-Fierz approach (1939), theories of graviton mass have a long history, although that is still complicated and far from resolution.

Famously, Einstein repeatedly failed to correctly predict any property of elementary particles exclusively from GR theory, although his ‘mistakes’ notoriously led to most fruitful developments, such as that of the Einstein-Rosen ‘bridge,’ for example. In the starkest of contrasts, standard QFT has proved immensely successful in predicting all kinds of properties of elementary particles, including many mass relations, without any regard to GR. If or when QFT, and mechanisms of symmetry breaking in particular, may further account for space-time curvature of GR, then the universal and cosmological constants $G$ and $\Lambda$ may be finally proved to enter fundamental relations such as those of photon and neutrino masses that I have barely estimated.

15. CONCLUSIONS

I developed an essential physical picture of dark energy based on most fundamental principles of quantum mechanics (QM) and general relativity (GR) theories. That derives from an ultimate mechanism of mass generation for initially massless gauge bosons in the standard model (SM) of quantum field theory (QFT) associated with quantum-mechanical confinement within a cosmological horizon. That is expressed in Eq. (1), which is in fact the only premise that I assume: all the rest follows from logic, well-established theory and elementary calculations.

Dark energy is intrinsically tied to de Sitter space, $dS_4$, and that has in turn a fundamentally invariant event horizon. I discussed how these matters are essential to my conjecture and model of dark energy and pressure, based on the standard $\Lambda$-CDM model of hot big bang cosmology.

According to a possible solution of the flatness problem, I have considered a de Sitter universe with a Euclidean spatial geometry. That corresponds to flat slicing of $dS_4$ in FLRW geometry. I have also considered a de Sitter static metric, which has closed spatial sections. Based on further analysis, I have alternatively referred to properties of both flat and static slicings of $dS_4$.

I conceive of dark energy as a Bose-Einstein condensate (BEC) of cosmologically massive photons and I have estimated fundamentally the binding energy per particle originating from the QM effectively attractive inter-particle potential in that BEC. Since massive photons may stand at rest in a de Sitter universe with flat spatial geometry, I solved the time-independent Schrödinger equation for a non-relativistic attractive spherical-well potential self-confining at the de Sitter horizon. That provides the minimal critical potential depth that binds just one particle state at the top of that well. Combining that with the fundamental relativistic version of the uncertainty principle that cosmologically constrains the photon mass through its Compton wavelength at de Sitter horizon, Eq. (1), I obtained in Eq. (35) a specific estimate of that mass, $m_g$, consistent with the dark energy-pressure relation of the standard flat $\Lambda$-CDM model.

My non-relativistic or ‘flat’ estimate of $m_g$ is essentially confirmed by an independent relativistic or ‘curved’ estimate of the QM zero-point energy of the BEC in de Sitter static metric with coordinate-time slicing. Remarkably, I can identify the dark energy density, $\rho_\Lambda$, with my zero-point energy density of the BEC, $\rho_{0g}$, within a single order of magnitude. The Casimir electromagnetic energy density of the vacuum is of the order of $\rho_{CEM} \approx \pi c^3/h^4$ at a cut-off of the order of the Planck length. Thus the ratio of that $\rho_{CEM}$ to my $\rho_{0g}$, in Eq. (38), is of the order of $4.55 \times 10^{122}$, according to Eq. (39). On a logarithmic scale, that discrepancy is as large as anyone may get.

Qualitative estimates that I provided in Eq. (34) and Eq. (39) are only meant to illustrate that numerical coefficients $C_g$ or $C'_g$ are of the order of 1 or so. A far more precise evaluation of Eq. (1) may only derive from QFT calculations of the $p_A/\rho_{\Lambda}$ equation of state and $g$-BEC properties in curved or de Sitter space-time.

I further investigated properties of equilibrium between the $g$-BEC phase and the ordinary vapor phase of kinetically energetic $m_g$-photons. I made comparisons with the Planck spectrum of the cosmic microwave background (CMB) and I found that corrections introduced by the tiny photon mass may be currently undetectable.

Remarkably, I considered a system of cosmological units, or ‘$g$-units,’ that complements the fundamental system of Planck units in various ways, including uncertainty relations to which either set of units inherently correspond. Planck and $g$-units stand in a fundamental ratio given by Eq. (49), spanning about 60 orders of magnitude. In the logarithmic middle of that range, the geometric mean of Planck and $g$-mass, Eq. (53), or an equivalent but theory-independent Eq. (55), turn out to be tantalizingly close to current estimates of neutrino masses, almost within a single order of magnitude. I thus suggested that masses of the lightest known fermions may also be associated with both QM and GR fundamental constants, $\Lambda$, $G$, $c$ and $h$.

Although based on entirely different physical grounds, there are common elements between my cosmological picture of dark energy and theories of ultra-light bosonic scalar and vector field dark matter, as well as theories of
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After I completed my work and I submitted it for publication, its review by the Journal made me aware of some already published findings that are most relevant and possibly equivalent or deeply related to mine. [88–90]

Specifically, the fundamental length and mass scales that I introduced in my Eq. (1) and Eq. (2) correspond to those of Eq. [C.9] and Footnote [8] of Ref. 88. The corresponding particle, called cosmon, is associated with the current Hubble parameter, since in the logotropic model the cosmological constant is ultimately time-dependent. Effectively, however, the current Hubble radius concurs with a fundamental cosmological constant corresponding to that of Einstein. [88, 89] A cosmon mass scale has been interpreted as the smallest mass of bosonic particles of dark matter predicted by string theory. [91] or as the upper bound on the mass of gravitons. [88, 89, 92]

On the other hand, I argued that a mass of the order of $10^{-32}$ eV/c$^2$ must provide a lower bound to the mass of any existing particle in the observable universe. [1] That would rule out ‘cosminos’ with an energy of the order of $10^{-67}$ eV, which have been proposed as ‘quanta’ of dark energy or the cosmological constant. [90] That paper asserts ‘that cosminos did not condense gravitationally, and hence the particles associated with dark energy fail to represent dark matter, which is in complete agreement with the present standard model of cosmology.’ Whether conceptions of cosmons and cosminos developed in Ref. 90 may be supported or not, perspectives of that paper are inspiring.

In relation to models of dark matter halos, it has also been shown that the cosmon mass scale corresponds to the fundamental mass scale of BEC bosons: see Abstract, Eq. [158] and Eq. [I10] of Appendix I-2 in Ref. 89.

The neutrino mass scale that I introduce in my Eq. (3) and discuss in my Sec. I-3 was also introduced in Ref. 89. It is related to models of dark matter halos, consisting of fermions: see Abstract, Eq. [175] and Eq. [I4] of Appendix I-1 in Ref. 89.

Using various arguments and equations, Chavanis even relates in Eq. [37] and Eq. [38] of Ref. 88 the electron mass and the fine structure constant to the GR and QM fundamental constants that I also consider in relation to

16. APPENDIX: LOGOTROPIC MODEL AND COSMONS
electron classical radius and its fermi scale, whose ‘coincidence’ with $d_g$ I simply remark in the fourth paragraph of my Sec. 9.

So, the fundamental a-dimensional ratio that I consider in my Eq. (49) essentially coincides with Eq. [C.15] of Ref. 88. Likewise, my Eq. (51) essentially coincides with Eq. [C.11] of Ref. 88. An estimate of the number of bosons in the observable universe is given in my Eq. (52) and correspondingly in Footnote [12] of Ref. 88. Finally, the most tantalizing ‘coincidence’ of the proton mass with my Eq. (57) is also essentially found in Eq. [G.4] of Ref. 88. An equivalent relation is provided in Eq. [11] of Ref. 90. Further referring to the electron classical radius, Böhmer and Harko suggest a ‘small number hypothesis’ as an extension of the famous ‘large number hypothesis’ by Dirac. They propose that ‘the numerical equality between two very small quantities with a very similar physical meaning cannot be a simple coincidence.’ I shall leave it at that, but Chavanis further provides incisive historical accounts of ‘curious coincidences’ that intrigued famous scientists and surely many others time after time.

Evidently, the foundation, modelling and estimates of my work are altogether different and independent of those in Refs. 88–90 and possibly other works, quoted therein. What may then lie at the root of these remarkable ‘concurrences?’ Well, at least some basic lines of reasoning seem relatively common throughout all these papers. But could there also be a deeper relation, or perhaps the actual coincidence, between the relatively ‘ordinary’ $g$-photon that I propose at the root of it all and ‘cosmons’ that have been variously envisioned in other work? Only much deeper and encompassing theory, observation and experiment may answer that fundamental question conclusively. But there will be an answer.

17. POSTSCRIPT: FINE STRUCTURE CONSTANT

The fine structure constant, $\alpha = e^2/\hbar c \simeq 1/137$, is an a-dimensional parameter that fundamentally characterizes quantum electrodynamics (QED). Using Planck units, I define $\alpha_P = e^2/\hbar c \simeq 1/861$. Crude estimates of a numerical coefficient, $C_g$, that I provided in Eq. (34) and in Eq. (39) may reflect a far more fundamental relation, such as $C_g \sim \alpha_P$. If so, the photon mass in Eq. (2) becomes

$$m_g c^2 \sim \hbar \alpha_P \sqrt{\Lambda} = e^2/L_g \simeq 1.5 \times 10^{-41} \text{MeV}.$$  \hfill (59)

That of course does not mean that $L_g$ represents some sort of “classical radius” of the photon, which is neutral. Rather, Eq. (59) indicates that the Coulomb potential of the electron, $e^2/r$, which is carried by a massless photon to an infinite range for $r \to \infty$, reduces instead to a still vast, but cosmologically finite range, $L_g \equiv 1/\sqrt{\Lambda} \simeq 10^{40} \text{ly} \simeq 0.946 \times 10^{26} \text{m}$, when the electrostatic potential is carried by a massive $m_g$-photon. Thus, if confirmed, Eq. (59) should demonstrate how profoundly a cosmological horizon and constant can affect and constrain quantum properties, propagation and interactions of QED and QCD elementary particles, including their mass generation through Higgs and Stückelberg mechanisms.

In fact, the counterpart of Eq. (59) in the logotropic model is expressed in Eq. [32] and Eq. [37] of Ref. 88. With my units and definition of $\Lambda$, that Eq. [37] of Chavanis yields

$$m_e = 1.03 \alpha \left(\frac{\Lambda \hbar^4}{G^2}\right)^{(1/6)} \simeq 0.511 \text{MeV}/c^2$$ \hfill (60)

for the mass of the electron, $m_e$. 