Skew critical problems

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Abstract

Skew critical problems occur in continuous and discrete nonholonomic Lagrangian systems. They are analogues of constrained optimization problems, where the objective is differentiated in directions given by an apriori distribution, instead of tangent directions to the constraint. We show semiglobal existence and uniqueness for nondegenerate skew critical problems, and show that the solutions of two skew critical problems have the same contact as the problems themselves. Also, we develop some infrastructure that is necessary to compute with contact order geometrically, directly on manifolds.

1 Introduction

Let $M$ and $N$ be manifolds, suppose $f: M \to \mathbb{R}$ is $C^1$, and let $g: M \to N$ be a $C^1$ submersion. Given this data, $m_c \in M$ is a critical point at $n \in N$ if

$$
\begin{align*}
\{ & df(m_c)(v) = 0 \text{ for all } v \text{ such that } T_{m_c}g(v) = 0, \\
g(m_c) = n. 
\}
\end{align*}
$$

(1.1)

This is the standard constrained optimization problem that seeks critical points of the objective $f$ subject to the constraint $g$.

Appearing in (1.1) are the derivative of the objective $df$, the constraint function $g$, and $\ker Tg$, which is a distribution on $M$. Generalizing, we consider the data $(\alpha, D, g)$, where $\alpha$ is a one-form on $M$, $D$ is a distribution on $M$, and $g: M \to N$ is as above. We replace the first condition of (1.1) with the condition that $\alpha$ annihilates $D$, and we call the result a skew critical problem. Skew critical problems occur when an objective function is not differentiated in tangent directions to a constraint, but rather is differentiated in the directions specified by an apriori given distribution. We are interested in skew critical

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problems because, for nonholonomic mechanics, the relevant variational principle is skew [5], and this is also true of the variational discrete analogues of nonholonomic systems.

For mechanics we are interested in existence and uniqueness of skew critical problems, by direct perturbation from the point of zero-time change. We have a global solution of the (trivial) zero-time problem, and we are interested in semiglobal results, which means global along the unperturbed problem, but local transverse to that. For discrete nonholonomic systems, we are also interested to know that the solutions of two skew critical problems have the same contact as the data of the two problems. The skew critical problems of mechanics require desingularization at zero-time, essentially by dividing by time. This degrades the order matching, which is again recovered by a zero-time symmetry of the desingularized problem, and so we must consider the presence of symmetry. We are interested in applications to both the continuous and discrete mechanics, so we work in an appropriate context of infinite dimensional manifolds.

In this work, we collect some technical results related to skew critical problems. For such a problem \((\alpha, D, g)\), little can be inferred just from the equations \(\alpha(m)\big|_D = 0,\ g(m) = n\), without some control imposed on \(\alpha, D, \) and \(g\), so we begin in Section 2 with the definition of a nondegenerate skew critical point. This corresponds to infinitesimal conditions that, using the implicit function theorem, imply there is locally a unique skew critical point for every nearby constraint value (Lemma 2.5). If \(N\) is paracompact, then a manifold of nondegenerate skew critical points along a submanifold \(N_0 \subseteq N\) can be extended along the whole of \(N_0\). We call this result semiglobal because it establishes an extension over the whole of \(N_0\), rather that just at one point of \(N\).

Contact of solutions of skew critical problems is important for discretizations of constrained Lagrangian systems, because contact with the exact system determines the order of the corresponding numerical methods. Section 3 establishes the basic definitions and results about contact. Generally, it often happens that cancellations result in one higher contact that would normally be expected from data or computation. For example, any Taylor expansion to odd order of an even function, is actually the expansion to the next higher order; a less trivial example is the fact that any odd order self-adjoint one step numerical method is one higher (even) order [2]. It is best to understand the cancellations geometrically. This kind of “passage to the next order” occurs when a geometric object that we call the residual vanishes. In Section 3 we find that it is useful to consider the vector bundle analogue of blowing up near the zero of a function of a single variable i.e. the function \(f(t)/t\) where \(f(0) = 0\). The completion of the function is made with the help of the vertical bundle at the zero section, and the contact drops by one. We provide, for computing on manifolds, Equation (3.8), which computes the residuals of the composition of two maps in terms of the residuals of the maps themselves. For skew critical problems, it is necessary to consider the contact order of distributions, which are subsets rather that maps. This is naturally done using Grassmann manifolds: a distribution can be regarded as an assignment of subspaces to base points.

Finally, in Section 4 we consider contact for inverse functions and the prob-
lems of construction maps from graphs. For graphs, an exchange symmetry of the residuals implies that the contact increases by one. In Section 5 we consider contact for skew critical problems. In the presence of the action of a Lie group, we obtain equivariance of the residuals of the skew critical points given equivariance of the residuals of the skew critical problems.

The notations in this work follow those of [1]. We assume without mention that the manifolds and submanifolds we use are sufficiently differentiable to support whatever operations are involved.

2 Regular skew critical problems

Let $M$ and $N$ be Banach manifolds, $\alpha$ be a $C^k$ one-form on $M$, $\mathcal{D}$ be a $C^k$ distribution on $M$, and let $g: M \to N$ be a $C^k$ submersion i.e. $Tg$ is surjective with split kernel. We call $(\alpha, \mathcal{D}, g)$ a $C^k$ skew critical problem. 

**Definition 2.1.** A point $m_c \in M$ is a skew critical point of $(\alpha, \mathcal{D}, g)$ at $n \in N$ if

\[
\begin{align*}
\{ &\alpha(m_c)(v) = 0 \text{ for all } v \in \mathcal{D}_{m_c}, \\
&g(m_c) = n. \}
\end{align*}
\] (2.1)

A critical point $m_c$ of a constrained optimization problem with $n \equiv g(m_c)$ is called nondegenerate if the Hessian of $f|_{g^{-1}(n)}$ is nonsingular. The corresponding notion for skew critical problems is given below in Definitions 2.2 and 2.4.

**Definition 2.2.** Let $m_c$ be a skew critical point of $(\alpha, \mathcal{D}, g)$. Define the bilinear form $d_{\mathcal{D}\alpha}(m_c): T_{m_c}M \times \mathcal{D}_{m_c} \to \mathbb{R}$ by

\[
d_{\mathcal{D}\alpha}(m_c)(u, v) \equiv \langle d(i_V\alpha)(m_c), u \rangle,
\]

where $V$ is a (local) vector field with values in $\mathcal{D}$ such that $V(m_c) = v$. The skew Hessian of $\alpha$ with respect to $g$ and $\mathcal{D}$ is the bilinear form

\[
d_{\mathcal{D}, g}\alpha(m_c): \ker T_{m_c}g \times \mathcal{D}_{m_c} \to \mathbb{R}
\]

obtained by restriction of $d_{\mathcal{D}\alpha}(m_c)$. Define $d_{\mathcal{D}, g}\alpha(m_c)^\flat: \ker T_{m_c}g \to \mathcal{D}_{m_c}^*$ by

\[
d_{\mathcal{D}, g}\alpha(m_c)^\flat(u) \equiv d_{\mathcal{D}, g}\alpha(m_c)(u, \cdot).
\]

**Remark 2.3.** The definition of $d_{\mathcal{D}\alpha}(m_c)$ does not depend on the extension $V$: in a vector bundle chart of $\mathcal{D}$, the local setup has

\[
TM = U \times (\mathbb{D} \oplus \mathbb{F}), \quad \mathcal{D} = U \times (\mathbb{D} \oplus \{0\}), \quad \alpha = \alpha_D \oplus \alpha_F,
\]

where $U \subseteq \mathbb{D} \oplus \mathbb{F}$ is open, $\alpha_D: U \to \mathbb{D}^* \cong \text{ann } \mathbb{F}$, and $\alpha_F: U \to \mathbb{F}^* \cong \text{ann } \mathbb{D}$. Supposing that $x_c \in U$ is a skew critical point, two extensions $V_i: U \to \mathbb{D}$, $i = 1, 2$, with $V_1(x_c) = v = V_2(x_c)$ result in $i_{V_1 - V_2}\alpha = \langle \alpha_D, V_1 - V_2 \rangle$. By
the product rule, the derivative of this at $x_c$ is zero since both $\alpha_D$ and $V_1 - V_2$ vanish at $x_c$, so $d(i_{V_1} \alpha)(x_c) = d(i_{V_2} \alpha)(x_c)$. In contrast to the constrained critical problems, skew Hessians are not symmetric since their arguments assume values in different vector subspaces.

**Definition 2.4.** A skew critical point $m_c$ of $(\alpha, D, g)$ is called nondegenerate if $d_{D, g} \alpha(m_c)$ is a linear isomorphism.

In finite dimensions, the standard constrained optimization problem (1.1) has as many equations for $m$ as there are unknowns, because $g$ simultaneously constrains both $v$ and $m_c$. For the skew problem (2.1), the number of equations need not equal the number of unknowns, since $g$ and $D$ may be unrelated. Definition 2.4 controls this, because if $m_c$ is nondegenerate then the fiber dimensions of $\ker T_m g$ and $D_{m_c}$ are equal since $\ker T_m g$ and $D_{m_c}$ are isomorphic.

**Lemma 2.5.** Let $m_c$ be a nondegenerate skew critical point of a $C_k$ skew critical problem $(\alpha, D, g)$, $k \geq 1$, and let $n_c \equiv g(m_c)$. Then there are neighborhoods $U \ni m_c$ and $V \ni n_c$ such that, for every $n \in V$ there is a unique skew critical point $m \in U$ of $(\alpha, D, g)$ such that $g(m) = n$. Moreover, the map $\gamma : V \rightarrow U$ so defined is $C_k$.

**Proof.** Using vector bundle charts as in Remark 2.3, the skew critical points $x$ such that $g(x) = y$ are obtained by solving $F(x) = (0, y)$, where $F(x) \equiv (\alpha_D(x), g(x))$. The derivative of $F$ at a particular $x_c$ is

$$DF(x_c)u = (D\alpha_D(x_c)u, Dg(x_c)u). \quad (2.2)$$

The first component is a linear isomorphism on $\ker Dg(x_c)$ since $x_c$ is nondegenerate. Since $Dg(x_c)$ is onto with a split kernel, there is a closed subspace $\mathbb{K}$ such that $D \oplus \mathbb{K} = \ker Dg(x_c) \oplus \mathbb{K}$, and $Dg(x_c)|\mathbb{K}$ is a linear isomorphism. From (2.2), $DF(x_c)u = (w_1, w_2)$ is continuously inverted by

$$\hat{u} = (Dg(x_c)|\mathbb{K})^{-1}w_2,
$$

$$u = \hat{u} + (D\alpha_D(x_c)|\ker Dg(x_c))^{-1}(w_1 - D\alpha_D(x_c)\hat{u}),$$

and the result follows from the inverse function theorem. \qed

The following semiglobal inverse function theorem is found on page 97 of [4]. The semiglobal result for skew critical points which follows that, the proof of which is included for completeness, pre-supposes nondegeneracy along a given smooth map of skew critical points.

**Theorem 2.6.** Let $M$ and $N$ be manifolds and $f : M \rightarrow N$ be $C_k$, $k \geq 1$. Suppose that

1. $M_0$ is a closed submanifold of $M$, $N_0$ is a closed submanifold of $N$, and $f|M_0 : M_0 \rightarrow N_0$ is a diffeomorphism; and

2. $f$ is a local diffeomorphism at every $m \in M_0$. 
Then \( f \) is a \( C^k \) diffeomorphism from some open neighborhood \( U \supseteq M_0 \) to some open neighborhood \( V \supseteq N_0 \).

**Theorem 2.7.** Let \((\alpha, D, g)\) be a \( C^k \) skew critical problem, \( k \geq 1 \), where \( g : M \rightarrow N \). Suppose that \( N \) is paracompact, and that

a. \( M_0 \) is a closed submanifold of \( M \), \( N_0 \) is a closed manifold of \( N \) and \( \gamma_0 : N_0 \rightarrow M_0 \) is a \( C^k \) diffeomorphism; and

b. for all \( n \in N_0 \), \( \gamma_0(n) \) is a nondegenerate skew critical point of \((\alpha, D, g)\) at \( n \).

Then there are open neighborhoods \( U \supseteq M_0 \) and \( V \supseteq N_0 \) and a \( C^k \) extension \( \gamma : V \rightarrow U \) such that

1. for all \( n \in V \), \( \gamma(n) \) is a skew critical point of \((\alpha, D, g)\) at \( n \); and

2. \( \gamma(n) \) is the unique skew critical point of \((\alpha, D, g)\) in \( U \).

**Proof.** Applying Lemma 2.5 at all \( \gamma_0(n_0) \) as \( n_0 \) ranges through \( N_0 \), there are open covers \( U_i \) of \( M_0 \) and \( V_i \) of \( N_0 \), and \( C^k \) maps \( \gamma_i : V_i \rightarrow U_i \) such that, for all \( n \in V_i \), \( \gamma_i(n) \) is the unique skew critical point of \((\alpha, D, g)\) in \( U_i \). By shrinking \( V_i \) one can arrange \( \gamma_i(\operatorname{cl} V_i) \subseteq U_i \) where \( \gamma_i \) is defined on an open superset of \( V_i \). Because \( N \) is paracompact, its open cover \( \{ N \setminus N_0, V_i \} \) admits a locally finite refinement, so the collection \( \{ V_i \} \) can be assumed locally finite.

By Lemma 20.4 of [6], the collection \( \{ \operatorname{cl} V_i \} \) is also locally finite, so each \( n \in \bigcup_i V_i \) admits a neighborhood \( V_n \) that meets only finitely many \( \operatorname{cl} V_i \). For each \( n \in \bigcup_i V_i \), the set of indices

\[
\operatorname{St}(n) \equiv \{ i : n \in \operatorname{cl} V_i \}
\]

is finite. No \( \operatorname{St}(n) \) is empty because every \( n \in \bigcup_i V_i \) is contained in some \( V_i \) and hence is in some \( \operatorname{cl} V_i \). The set

\[
V_n \setminus \bigcup \{ \operatorname{cl} V_i : \operatorname{cl} V_i \text{ meets } V_n \text{ and } i \notin \operatorname{St}(n) \}
\]

an open neighborhood of \( n \) because it subtracts from \( V_n \) only finite many closed sets, and it has the property that if any of its members is in any \( \operatorname{cl} V_i \) then \( i \in \operatorname{St}(n) \). Replacing each \( V_n \) with \( \bigcup_i V_i \), it can be assumed that \( \operatorname{St}(n') \subseteq \operatorname{St}(n) \) for all \( n' \in V_n \).

Defining

\[
U \equiv \bigcup_{n \in \bigcup_i V_i} \left( g^{-1}(n) \cap \bigcap_{i \in \operatorname{St}(n)} U_i \right), \quad V \equiv \bigcup_i \gamma_i^{-1}(U),
\]

we can show the following facts.

1. \( M_0 \subseteq U \): if \( m \in M_0 \) and \( n \equiv g(m) \) then \( m \in g^{-1}(n) \) and \( n \in \operatorname{cl} V_i \) for all \( i \in \operatorname{St}(n) \) so \( \gamma_i(n) \in \gamma_i(\operatorname{cl} V_i) \subseteq U_i \) for all \( i \in \operatorname{St}(n) \), hence \( m \in U \).
2. $U$ is an open neighborhood of $M_0$: if $m \in U$ and $n \equiv g(m)$ then

$$m \in g^{-1}(n) \cap \bigcap_{i \in \text{St}(n)} U_i \subseteq g^{-1}(V_n) \cap \bigcap_{i \in \text{St}(n)} U_i.$$

The last set is open because it is the intersection of finitely many open sets. Also,

$$g^{-1}(V_n) \cap \bigcap_{i \in \text{St}(n)} U_i = \bigcup_{n' \in V_n} \left( g^{-1}(n') \cap \bigcap_{i \in \text{St}(n')} U_i \right) \subseteq U.$$

Thus there is an open neighborhood of $m$ that is contained in $U$.

3. $U$ has the property that, for all $m_1, m_2 \in U$, $g(m_1) = g(m_2)$ implies that there is an $i$ such that $m_1$ and $m_2$ are both in $U_i$. Indeed, any such $m_1$ and $m_2$ are members of

$$g^{-1}(n) \cap \bigcap_{i \in \text{St}(n)} U_i,$$

where $n = g(m_1) = g(m_2)$, and so both $m_1$ and $m_2$ are members of any $U_i$ for any $i \in \text{St}(n)$.

Let $n \in V$. Then $n \in \gamma_i^{-1}(U)$ for some $i$ and $m = \gamma_i(n)$ is a skew critical point of $(\alpha, D, g)$ in $U$. If $m' \in U$ is another such skew critical point then $g(m) = g(m')$, and $m$ and $m'$ both lie in a single $U_j$. By definition of the $U_j$, there is only one skew critical point of $(\alpha, D, g)$ in $U_j$, so $m = m'$. Thus for all $n \in V$ there is a unique skew critical point of $(\alpha, D, g)$ in $U$. Define $\gamma : V \to U$ by this correspondence. By the uniqueness used to define $\gamma$, the restriction of $\gamma$ to any $\gamma_i^{-1}(U)$ is $\gamma_i$, which is $C^k$, and the $\gamma_i^{-1}(U)$ cover $V$, so $\gamma$ is $C^k$. ✷

3 Order Notation and Residuals

Given two functions $f_i(x)$, $i = 1, 2$, of a single variable $x \in \mathbb{R}$, the standard definition of $f_1(x) = f_2(x) + O(x^r)$ is that there are numbers $\delta > 0$ and $C > 0$ such that $|f_1(x) - f_2(x)| \leq C|x|^r$ for $|x| < \delta$. If the functions $f_i$ are $C^r$, $r \geq 1$, then $f_1(x) = f_2(x) + O(x^r)$ if and only if there is a continuous function, say $\delta f(x)$, such that $f_1(x) = f_2(x) + x^r \delta f(x)$. The following definitions export the
second formulation to the context of manifolds.

**Definition 3.1.**

1. Let $M$ be a manifold and $h_M: M \to \mathbb{R}$ be a $C^\infty$ function which has 0 as a regular value. The pair $(M, h_M)$ will be called a manifold.

2. Let $(M, h_M)$ and $(N, h_N)$ be manifolds. A $C^k$ mapping $f: (M, h_M) \to (N, h_N)$ is a $C^k$ mapping $f: M \to N$ such that $h_N \circ f = h_M$.

3. A $C^k$ mapping $f: (M, h_M) \to N$ or $f: M \to (N, h_N)$ is a mapping $f: M \to N$ without any conditions involving $h_M$ or $h_N$.

**Definition 3.2.** Let $(M, h_M)$ and $N$ be manifolds and $f_i: (M, h_M) \to N$, $i = 1, 2$, be such that $f_1 = f_2$ on $h_M^{-1}(0)$. Define $f_2 = f_1 + O(h_M^r)$, $r \geq 1$ if, for all $m_0 \in h_M^{-1}(0)$, there is a chart $\nu$ at $n_0 \equiv f_i(m_0) \in N$, and there is a function $(\delta f)_\nu$ defined near $m_0$, and continuous at $m_0$, such that

$$\nu(f_2(m)) - \nu(f_1(m)) = h_M(m)^r(\delta f)_\nu(m),$$

for all $m$ in some neighborhood of $m_0$.

The definition of $f_2 = f_1 + O(h_M^r)$ does not depend on the coordinate chart: if $\nu$ and $\nu$ are two coordinate charts at $n_0$, as in Definition 3.2 and for $m$ near to $m_0$,

$$\tilde{\nu}(f_2(m)) - \tilde{\nu}(f_1(m))$$

$$= (\tilde{\nu} \circ \nu^{-1})(\nu(f_2(m))) - (\tilde{\nu} \circ \nu^{-1})(\nu(f_1(m)))$$

$$= (\tilde{\nu} \circ \nu^{-1})(\nu(f_1(m)) + h_M(m)^r(\delta f)_\nu(m)) - (\tilde{\nu} \circ \nu^{-1})(\nu(f_1(m)))$$

$$= \int_0^1 \frac{d}{dt}(\tilde{\nu} \circ \nu^{-1})(\nu(f_1(m)) + t h_M(m)^r(\delta f)_\nu(m)) dt$$

$$= h_M(m)^r \left[ \int_0^1 D(\tilde{\nu} \circ \nu^{-1})(\nu(f_1(m)) + t h_M(m)^r(\delta f)_\nu(m)) dt \right](\delta f)_\nu(m),$$

as required.

The quantities $(\delta f)_\nu(m_0)$ and $(\delta f)_{\tilde{\nu}}(m_0)$ transform as tangent vectors. Indeed,

$$h_M(m)^r(\delta f)_{\tilde{\nu}}(m) = \tilde{\nu}(f_2(m)) - \tilde{\nu}(f_1(m)),$$

so

$$(\delta f)_{\tilde{\nu}}(m) = \left[ \int_0^1 D(\tilde{\nu} \circ \nu^{-1})(\nu(f_1(m)) + t h_M(m)^r(\delta f)_\nu(m)) dt \right](\delta f)_\nu(m).$$

At $m_0 \in h_M^{-1}(0)$, and setting $n_0 \equiv f_i(m_0)$,

$$(\delta f)_{\tilde{\nu}}(m_0) = D(\tilde{\nu} \circ \nu^{-1})(\nu(n_0))(\delta f)_\nu(m_0),$$

as required.
Definition 3.3. Let \((M, h_M)\) be a manifold, \(f_2 = f_1 + O(h_M^1)\), and \(m \in h_M^{-1}(0)\). The vector \(\text{res}^r(f_2, f_1)(m) \in T_n N\) with representation \((\delta f)_\nu(m)\) for any chart \(\nu\) is called the \(r\)-residual of \(f_2\) with respect to \(f_1\).

The residual \(\text{res}^r(f_2, f_1)\) is defined only on \(h_M^{-1}(0) \subset M\) and takes values in \(TN\). The condition \(f_2 = f_1 + O(h_M^r)\), can be localized to a point of \(M\) or a subset of \(M\) in the obvious way, and the residual will be correspondingly localized. In general, jets of mappings between manifolds carry an affine action by a geometrically based vector space, amounting basically to the first nonzero term of the Taylor series of the difference between two mappings. Also, the notion of contact below is the same as the contact equivalence in the definition of jets [3].

If \((M, h_M)\) is a manifold, then, since 0 is a regular value of \(h_M\), there are \(h_M\)-adapted charts at each \(m_0 \in h_M^{-1}(0)\) i.e. charts such that the local representative of \(h_M\) is the projection \((x, t) \mapsto t\). We can prove an equality or formula concerning residuals in any chart since residuals are geometric, and in particular, we can always use an \(h_M\)-adapted chart.

Suppose that \(f_i : (U, h_U) \rightarrow V \subseteq \mathbb{F}, \ i = 1, 2\), are \(C^r\), where \(U\) is an open subset of \(\mathbb{E} \times \mathbb{R}\), where \(\mathbb{E}\) and \(\mathbb{F}\) are Banach spaces, and where \(h_U(x, t) = t\). For fixed \(x\), the Taylor expansions in \(t\) about \(t = 0\) of the \(f_i\) are

\[
f_i(x, t) = f_i(x, 0) + t \frac{\partial f_i}{\partial t}(x, 0) + \cdots + \frac{t^r}{r!} \frac{\partial^r f_i}{\partial t^r}(x, 0) + R_{r,i}(x, t) t^r,
\]

where

\[
R_{r,i}(x, t) = \int_0^1 (1 - s)^{r-1} \left( \frac{\partial^r f_i}{\partial t^r}(x, st) - \frac{\partial^r f_i}{\partial t^r}(x, 0) \right) ds.
\]

The condition that \(f_2 = f_1 + O(h_U^r)\) at \((x, 0)\) is thus equivalent to the condition that these Taylor expansions match at \((x, 0)\) up to and including the degree \(r - 1\) term. So, given this,

\[
f_2(x, t) - f_1(x, t) = \frac{t^r}{r!} \left( \frac{\partial^r f_2}{\partial t^r}(x, 0) - \frac{\partial^r f_1}{\partial t^r}(x, 0) \right) + R_{r,2}(x, t) t^r - R_{r,1}(x, t) t^r,
\]

which identifies \((\delta f)_\nu(x, t)\) in these coordinates as

\[
(\delta f)_\nu(x, t) = \frac{1}{r!} \left( \frac{\partial^r f_2}{\partial t^r}(x, 0) - \frac{\partial^r f_1}{\partial t^r}(x, 0) \right) + R_{r,2}(x, t) - R_{r,1}(x, t).
\]

Setting \(t = 0\), the residual is

\[
(\delta f)_\nu(x, 0) = \left. \frac{1}{r!} \frac{\partial^r}{\partial t^r} \right|_{t=0} (f_2(x, t) - f_1(x, t)).
\]

If \(\text{res}^r(f_2, f_1) = 0\), then the Taylor series of \(f_1\) and \(f_2\) agree up to and including terms of degree \(r\), one more than the degree \(r - 1\) agreement implied by \(f_2 =
isomorphism into horizontal and vertical parts. This splitting can be defined by the linear
natural horizontal subspace, so any vector of vertical subbundle. Recall that the tangent space of the zero section defines a

**Definition 3.4.** If \( M, h_M, \) and \( f_i \) are as in Definition 3.2, then \( f_1 \) and \( f_2 \) have order \( h_M^{-1} \) contact, or just have contact \( r - 1 \), if \( f_2 = f_1 + O(h_M) \).

If \( \pi: E \to M \) is a vector bundle then ker \( T \pi \) is a subbundle of \( TE \), called the *vertical subbundle*. Recall that the tangent space of the zero section defines a natural horizontal subspace, so any vector of \( TE \) at the zero section can be split into horizontal and vertical parts. This splitting can be defined by the linear isomorphism

\[
TM \oplus E \to TE : \quad (v_m, w_m) \mapsto \frac{d}{dt} \bigg|_{t=0} 0_{m(t)} + \frac{d}{dt} \bigg|_{t=0} tw_m, \quad (3.1)
\]

where \( m(t) \) is a curve in \( M \) such that \( m'(0) = v_m \). If \( z \in T_{0_m} E \) then denote the horizontal and vertical parts of \( z \) by hor \( z \in T_{m} M \) and vert \( z \in E_{m} \), respectively i.e. the inverse of (3.1) is \( z \mapsto (\text{hor } z, \text{vert } z) \).

If \( f \) is a \( C^1 \) function such that \( f(0) = 0 \), then it is elementary that

\[
\hat{f}(t) \equiv \begin{cases} 
\frac{f(t)}{t}, & t \neq 0, \\
 f'(0), & t = 0,
\end{cases}
\]

is continuous. The purpose of Lemma 3.5 is to show that a mapping on a manifold can be smoothly divided by a real function that takes values in a vector bundle and is in the zero section \( 0(E) \) if the function vanishes.

**Proposition 3.5.** Let \( (M, h_M) \) and \( N \) be a manifolds, and let \( \pi: E \to N \) be a vector bundle. Suppose that \( f: U \to E \) is \( C^k \), \( k \geq 1 \), and that \( f(m) \in 0(E) \) whenever \( h_M(m) = 0 \). Then for all \( m \) such that \( h_M(m) = 0 \), there is a unique \( \epsilon(m) \in E_{\pi(f(m))} \) such that

\[
\text{vert } T_m f(v_m) = (dh_M(m)v_m)\epsilon(m), \quad v_m \in T_m M. \quad (3.2)
\]

Moreover, the function \( \hat{f}: M \to E \) defined by

\[
\hat{f}(m) \equiv \begin{cases} 
\frac{f(m)}{h_M(m)}, & h_M(m) \neq 0, \\
 \epsilon(m), & h_M(m) = 0,
\end{cases}
\]

is \( C^{k-1} \).

**Proof.** If \( h_M(m) = 0 \) and \( v_m \in \ker dh_M(m) \) then \( v_m = c'(0) \) for some curve \( c(t) \in h_M^{-1}(0) \). Since \( f \) is in the zero section whenever \( h_M \) is zero, it follows that \( f \circ c(t) \) takes values in the zero section, so \( (f \circ h_M)'/0 \) is horizontal. Thus \( \text{vert } T_m f(v_m) = 0 \) for all \( v_m \in \ker dh_M(m) \), so there is a unique \( \epsilon(m) \in E_{\pi(f(m))} \) satisfying (3.2).

9
We can set up an $h_M$-adapted chart $\{(x,t)\}$ on $M$ and a vector bundle chart on $N$, so that $E = \{(y,e)\}$, and $f(x,t) = (f_0(x,t), f_1(x,t))$. Then $f_1(x,0) = 0$ for all $x$, so
\[
\text{vert } T_m f(x,0)(\delta x, \delta t) = \frac{\partial f_1}{\partial x}(x,0)\delta x + \frac{\partial f_1}{\partial t}(x,0)\delta t = \frac{\partial f_1}{\partial t}(x,0)\delta t \quad (3.3)
\]
and
\[
dh_M(x,t)(\delta x, \delta t) = \delta t. \quad (3.4)
\]
By comparison of (3.2) with (3.3) and (3.4),
\[
e(x,0) = \frac{\partial f_1}{\partial t}(x,0),
\]
and it is required to show that $\hat{f}_1$ defined by
\[
\hat{f}_1(x,t) \equiv \begin{cases} f_1(x,t), & t \neq 0, \\ \frac{\partial f_1}{\partial t}(x,0), & t = 0, \end{cases}
\]
is $C^{k-1}$. At any $(x_0,0)$ the Taylor expansion of $f_1$ is
\[
f_1(x,t) = D f_1(x_0,0)(\delta x, t) + \cdots + D^k f_1(x_0,0)(\delta x, t)^k + R(x, t)(\delta x, t)^k \quad (3.5)
\]
where $\delta x = x - x_0$ and $R(x_0,0) = 0$. By differentiating $f_1(x,0) = 0$ in $x$, $D^i f_1(x_0,0)(\delta x, 0)^i = 0$ for $1 \leq i \leq k$, and substituting $t = 0$ into (3.5) gives $R(x, 0)(\delta x, 0)^k = 0$. Thus the left side of (3.3) has $t$ as a factor and
\[
\hat{f}_1(x,t) = \frac{\partial f_1}{\partial t}(x,0) + \frac{1}{t} D^2 f_1(x_0,0)(\delta x, t)^2 + \cdots + \frac{1}{t} D^k f_1(x_0,0)(t,h)^k + \frac{1}{t} R_k(x, t)(\delta x, t)^k. \quad (3.6)
\]
Each of the first $k$ terms of (3.6) are polynomial in $(\delta x, t)$ and the remainder is polynomial in $(\delta x, t)$ of degree $k - 1$ with coefficients functions of $(x, t)$ that vanish at $(x_0,0)$. Thus, by the converse of Taylor’s theorem [1], $f_1(x,t)$ is $C^{k-1}$ at any $(x_0,0)$.

**Proposition 3.6.** Let $(M,h_M)$ and $N$ be a manifolds, let $\pi: E \to N$ a vector bundle, and suppose $f_i$ and $\tilde{f}_i$ are as in Proposition 3.3 with $k \geq r$. Then $\tilde{f}_2 = \tilde{f}_1 + O(h_M^{r-1})$ if $f_2 = f_1 + O(h_M^r)$, $r \geq 2$. Moreover, $\text{res}^r(f_2, f_1)$ takes values in the vertical bundle of $E$ and $\text{res}^{r-1}(\tilde{f}_2, \tilde{f}_1) = \text{res}^r(f_2, f_1)$. \hfill $\blacksquare$
Proof. Assume the context and notations of the proof of Proposition 3.5 so that

\[ f_1(x, t) = (f_{1,0}(x, t), f_{1,1}(x, t)) \quad \text{and} \quad f_2(x, t) = (f_{2,0}(x, t), f_{2,1}(x, t)). \]

Since \( f_2(x, t) = f_{1,0}(x, t) + O(t) \), the \( r - 1 \) residual of the first components of \( f_2 \) and \( f_1 \) is zero, and it suffices show that

\[ \hat{f}_{2,1}(x, t) = \hat{f}_{1,1}(x, t) + t^{r-1} \delta f(x, t) \quad (3.7) \]

given \( f_2 = f_{1,1} + t^r \delta f \), where \( \delta f \) is continuous and

\[ \hat{f}_{i,1}(x, t) = \begin{cases} \frac{f_{i,1}(x, t)}{t}, & t \neq 0, \\ \frac{\partial f_{i,1}}{\partial t}(x, 0), & t = 0. \end{cases} \]

Equation (3.7) can be shown in the two cases \( t = 0 \) and \( t \neq 0 \): For \( t = 0 \),

\[ \frac{\partial f_{2,1}}{\partial t}(x, 0) - \frac{\partial f_{1,1}}{\partial t}(x, 0) = \lim_{t \to 0} \frac{t^r \delta f(x, t)}{t} = \lim_{t \to 0} t^{r-1} \delta f(x, t) = 0, \]

so even \( \hat{f}_{2,1}(x, t) = \hat{f}_{1,1}(x, t) \) in this case, whereas for \( t \neq 0 \),

\[ \hat{f}_{2,1}(x, t) = \frac{f_{2,1}(x, t)}{t} = \frac{f_{1,1}(x, t) + t^r \delta f(x, t)}{t} = \hat{f}_{1,1}(x, t) + t^{r-1} \delta f(x, t). \]

\[
\]

Proposition 3.7 is a key result because it can be used to compute residuals without the invocation of local charts. Note that if \( (M, h_M) \) and \( (N, h_N) \) are manifolds and \( f: M \to N \) is a \( C^1 \) function such that \( f(h_M^{-1}(0)) \subset h_N^{-1}(0) \), then for all \( m \in h_M^{-1}(0), \) \( d(h_N \circ f)(m)v_m = 0 \) for all \( v_m \) such that \( dh_M(m)v_m = 0 \). So one can define \( \hat{f}: h_M^{-1}(0) \to \mathbb{R} \) by

\[ d(h_N \circ f)(m) = \hat{f}(m)dh_M(m). \]

This is an instance of Proposition 3.5, and it follows that \( \hat{h}_{N,f} \) defined by extending \( d(h_N \circ f)/h_M \) to \( \hat{f} \) on \( h_M^{-1}(0) \) is continuous.

**Proposition 3.7.** Let \( (M, h_M) \), \( (N, h_N) \), and \( P \) be manifolds, and suppose \( f_i: M \to N \) and \( g_i: N \to P \), \( i = 1, 2 \) are \( C^1 \) and satisfy \( f_i(h_M^{-1}(0)) \subset h_N^{-1}(0), \)

\( f_2 = f_1 + O(h_M^r) \), and \( g_2 = g_1 + O(h_N^r) \). Then \( g_2 \circ f_2 = g_1 \circ f_1 + O(h_M^r) \). Moreover, if \( h_M(m) = 0 \) and \( n = f_i(m) \), then

\[ \text{res}^r(g_2 \circ f_2, g_1 \circ f_1)(m) = \hat{f}_2(m)^r \text{res}^r(g_2, g_1)(n) + T_n g_1 \text{res}^r(f_2, f_1)(m). \quad (3.8) \]
\textbf{Remark 3.10.} Subspace $\iota$ is defined the information on Grassmann manifolds in the Banach space context, see \cite{1}.

\begin{proof}
It suffices to consider the local setup where $E$, $F$, and $G$ are Banach spaces, $U \subseteq E$ and $V \subset F$ are open, $f: U \to V$, $g: V \to W$, $h_U: U \to \mathbb{R}$, and $h_V: V \to \mathbb{R}$. Then

$$(g_2 \circ f_2)(x) = g_1(f_1(x) + h_U(x)\delta f(x)) + h_V(f_2(x)) \delta g(f_2(x))$$

$$= g_1(f_1(x)) + \int_0^1 \frac{d}{ds} g_1(f_1(x) + sh_U(x)\delta f(x)) \, ds$$

$$+ h_V(f_2(x)) \delta g(f_2(x))$$

$$= g_1(f_1(x)) + h_U(x)^\tau \left[ \int_0^1 Dg_1(f_1(x) + s h_U(x)\delta f(x)) \, ds \right] \delta f(x)$$

$$+ h_V(x)^\tau \hat{h}_V f_2(x) \delta g(f_2(x)).$$

Assuming $x$ satisfies $h_U(x) = 0$ and putting $y \equiv f_1(x)$, results in

$$\text{res}^r(g_2 \circ f_2, g_1 \circ f_1)(x) = Dg_1(y)\delta f(x) + \hat{h}_V f(x)^\tau \delta g(y),$$

which is the local form of \textbf{(3.8)}. \hfill \square

\textbf{Remark 3.8.} If $r \geq 2$, then $T_n g_1$ and $\hat{f}_2$ can be replaced by $T_n g_2$ and $\hat{f}_1$ respectively in Equation \textbf{(3.8)}. If $h_N \circ f_i = h_M$ then $\hat{f}_i = 1$ in any case. Also, if $g_1 = g_2$, one can dispense with $h_N$ and the assumption that $f_i(h_M^{-1}(0)) \subseteq h_N^{-1}(0)$, obtaining the formula

$$\text{res}^r(g \circ f_2, g \circ f_1)(m) = T_n g \text{ res}^r(f_2, f_1)(m).$$

\textbf{Remark 3.9.} If $E'$ is a $C^r$ subbundle of $E$, with typical fiber $E'$, then there is defined the $C^r$ map $\iota_{E'}: M \to G(E', E)$ that assigns to any $m \in M$ the subspace $\iota_{E'}(m) = E'_m$.

\textbf{Remark 3.10.} As is well known, the tangent space at $B \in G_{E_0}(E)$ is canonically hom$(B, E/B)$. Indeed, if $B(t)$ is a $C^1$ curve in $G_{E_0}(E)$ with $B(0) = B$, then choose a splitting $E = B \oplus F$ and define $\hat{B}: B \to F$ by

$$\hat{B} \equiv \left. \frac{d}{dt} \right|_{t=0} \pi_{E_0/B} \circ \left( \pi_{B \oplus F} \left| B(t) \right. \right)^{-1},$$

where $\pi_{E_0/B}$ denotes the projection of $E$ to the quotient $E/B$, and $\pi_{B \oplus F}$ denotes the projection to $B$ using the decomposition $E = B \oplus F$. One verifies that $\hat{B}$ is
Thus res obtains because Proof. Suppose then the common element $(D_1)\, h_M^1(0) = D_2\, h_M^1(0)$. Define $D_2 = D_1 + O(h_M^r)$ if $\nu_{D_2} = \nu_{D_1} + O(h_M^r)$, and define $\text{res}^r(D_2, D_1) = \text{res}^r(\nu_{D_2}, \nu_{D_1})$.

In the context of Definition 3.11 note that the assignment of the fibers of subbundles into the Grassmann bundle preserves fibers, so $\pi^G_{M}(TM) \circ t_{D_1} = \mathbf{1}_M$, and

$$\text{T}_{\pi^G_{M}(TM)} \text{res}(t_{D_2}, t_{D_1}) = \text{res}(\pi^G_{M}(TM) \circ t_{D_2}, \pi^G_{M}(TM) \circ t_{D_1}) = \text{res}(\mathbf{1}_M, \mathbf{1}_M) = 0,$$

which shows that, for all $m \in h_M^2(0)$, $\text{res}^r(D_2, D_1)(m)$ is a vertical vector in $T(G_D(TM))$. Such vertical vectors are derivatives of curves in the corresponding fiber i.e. derivatives of curves in the Grassmann manifold $G_D(T_m M)$. Thus, the residual of two vector bundles $D_i$ at $m$ is an element of the tangent space at the common element $(D_i)_m$ of the Grassmann manifold $G_D(TM)$, which, by Remark 3.10, can be regarded as an element of hom$(D_i)_m, T_m M/(D_i)_m$.

4 Equations

Computing with the order notation on manifolds might require the determining the contact or residual of the solutions of two implicit equations with a given contact or residual. A most basic result that enables this sort of argument is Proposition 4.1, which guarantees the contact of two inverse mappings, given the contact of two diffeomorphisms.

**Proposition 4.1.** Let $(M, h_M)$ and $(N, h_N)$ be manifolds, and let $f_i : M \to N$ be $C^k$ diffeomorphisms, $k \geq 1$, be such that $f_i$ maps $h_M^{-1}(0)$ into $h_N^{-1}(0)$, $i = 1, 2$. Then $f_2 = f_1 + O(h_M^r)$ implies $f_2^{-1} = f_1^{-1} + O(h_N^r)$.

**Proof.** Suppose $l$ is such that $g_2 \circ f_2 = g_1 \circ f_1 + O(h_M^l)$. This is true for $l = 1$, because $g_2 \circ f_2 = g_1 \circ f_1$ on $h_M^{-1}(0)$. Taking the residuals of $f_i \circ g_i = \mathbf{1}$, one obtains

$$0 = \text{res}^l(f_2 \circ g_2, f_1 \circ g_1)(n) = \hat{g}_2(n)^l \text{res}^l(f_2, f_1)(m) + T_n f_1 \text{ res}^l(g_2, g_1)(n).$$

Thus $\text{res}^l(g_2, g_1)(n) = 0$. If $\text{res}^l(f_2, f_1)(m) = 0$ i.e. $g_2 = g_1 + O(h_N^l)$ if $f_2 = f_1 + O(h_M^l)$, which inductively gives $g_2 = g_1 + O(h_N^l)$.

Another requirement is to semiglobally construct mappings from graphs. Proposition 4.2 uses the semiglobal inverse function theorem to provide such a result for a perturbation of an identity mapping.
Proposition 4.2. Let $M$ and $(N,h_N)$ be manifolds. Let $\gamma: U \subseteq N \to M \times M$ be $C^k$, $k \geq 1$. Suppose that $h^{-1}_N(0) \subseteq U$ and $\gamma| h^{-1}_N(0)$ is a diffeomorphism to $\Delta(M \times M)$. Then there are neighborhoods $\tilde{U} \subseteq U$ of $h^{-1}_N(0)$ and $V \subseteq M \times \mathbb{R}$ of $M \times \{0\}$ such that, for all $(m, h) \in V$, there is a unique $\tilde{m} \in M$ such that, for some $n \in \tilde{U}$, $\gamma(n) = (m, \tilde{m})$ and $h_N(n) = h$. The map $f_\gamma: V \to M$ defined by $f_\gamma(m, h) \equiv \tilde{m}$ is $C^k$.

Proof. Let $\pi_1$ and $\pi_2$ be the projections on $M \times M$ i.e. $\pi_i(m_1, m_2) \equiv m_i$, $i = 1, 2$. Define $\psi: U \to M \times \mathbb{R}$ by $\psi(n) \equiv ((\pi_1 \circ \gamma)(n), h_N(n))$. The map $\psi$ is a diffeomorphism from $h^{-1}_N(0)$ to $M \times \{0\}$ and, by the inverse function theorem, is a local diffeomorphism at each point of $h^{-1}_N(0)$. By Lemma 2.6, $\psi$ is a diffeomorphism from a neighborhood $\tilde{U} \subseteq U$ of $h^{-1}_N(0)$ to a neighborhood $V$ of $M \times \{0\}$.

If $(m, h) \in V$, then let $n \in \tilde{U}$ be such that $\psi(n) = (m, h)$, and define $\tilde{m} \equiv \pi_2(\gamma(n))$, so that $f_\gamma(m, h) \equiv \tilde{m} = (\pi_2 \circ \gamma \circ \psi^{-1})(m, h)$. From $\psi(n) = (m, h)$ follows $(\pi_1(\gamma(n)), h_N(n)) = (m, \tilde{m})$ and $\gamma(n) = (m, \tilde{m})$ and $h_N(n) = h$, which are the required properties of $\tilde{m}$. If there is another such, say $\tilde{m}'$, then would have to be an $n' \in \tilde{U}$ such that $\gamma(n') = (m, \tilde{m}')$ and $h_N(n') = h$, so $\psi(n') = (m, h) = \psi(n)$ which, since $\psi$ is a diffeomorphism, implies $n = n'$. Hence $(m, \tilde{m}) = \gamma(n) = \gamma(n') = (m, \tilde{m}')$, so $\tilde{m} = \tilde{m}'$.

Proposition 4.2 establishes the contact of the mappings constructed from graphs is equal to the contact of the graphs. Further, the mappings have one higher contact if there is present a symmetry condition for the residuals of the graphs.

Proposition 4.3. Let $(M, h_M)$ and $(N, h_N)$ be manifolds and $\gamma_i$ and $f_i$ be as in Proposition 4.2. Then $f_{\gamma_2} = f_{\gamma_1} + O(h^r_M)$ if $\gamma_2 = \gamma_1 + O(h^r_N)$. If $\text{res}^r(\gamma_2, \gamma_1)$ is symmetric i.e. $\delta^r \gamma(n) = \delta^r \gamma(n')$ for all $n \in N$, where $\text{res}^r(\gamma_2, \gamma_1) = (\partial \gamma^r, \partial \gamma^r)$, then $f_{\gamma_2} = f_{\gamma_1} + O(h^{r+1})$.

Proof. Assume the context and notations of the proof of Proposition 4.2. Since $f_{\gamma_1} = \pi_3 \circ \gamma_i \circ \psi_i^{-1}$, where $\psi_i = (\gamma_i, h_N)$, Propositions 3.7 and 4.1 imply $f_{\gamma_2} = f_{\gamma_1} + O(h^r_M)$ if $\gamma_2 = \gamma_1 + O(h^r)$. Then
\[
\pi_2 \circ \gamma_i = f_{\gamma_1} \circ (\pi_1 \circ \gamma_i, h_N),
\]
so, taking the residuals of this equation at $n \in h^{-1}_N(0)$, and setting $m \equiv \pi_1(\gamma_i(n))$, gives
\[
T_{(m, m)}\pi_2 \text{res}^r(\gamma_2, \gamma_1)(n) = \text{res}^r(f_{\gamma_2} \circ (\pi_1 \circ \gamma_2, h_N), f_{\gamma_1} \circ (\pi_1 \circ \gamma_1) h_N)(n) = \text{res}^r(f_{\gamma_2}, f_{\gamma_1})(m, 0) + T_{(m, 0)}f_{\gamma_1} \text{res}^r((\pi_1 \circ \gamma_2, h_N), (\pi_1 \circ \gamma_1, h_N))(n) = \text{res}^r(f_{\gamma_2}, f_{\gamma_1})(m, 0) + T_{(m, 0)}f_{\gamma_1}(T_{\pi_1} \text{res}^r(\gamma_2, \gamma_1), 0)(n).
\]
Also, \( f_{\gamma_1}(\pi_1(m_1, m_2), 0) = m_1 \) for all \( m_1 \in M \), so the last term of the equation immediately above is \( T_{(m, m)}\pi_1(\text{res}^r(\gamma_2, \gamma_1)(n)) \), and hence

\[
\text{res}^r(f_{\gamma_2}, f_{\gamma_1})(m, 0) = T_{(m, m)}\pi_2 \text{res}^r(\gamma_2, \gamma_1)(n) - T_{(m, m)}\pi_1 \text{res}^r(\gamma_2, \gamma_1)(n),
\]

which is zero if \( \text{res}^r(\gamma_2, \gamma_1)(n) \) is symmetric.

\[\square\]

5 Skew critical problems

Theorem 5.1 is a main objective of this work. It uses the infrastructure we have developed to show that the contact of solutions of nondegenerate skew critical problems is the same as the contact of their data. Moreover, the residuals of the solutions are determined geometrically through the residuals of the data.

**Theorem 5.1.** Let \((M, h_M)\) and \((N, h_N)\) be manifolds and suppose \(a^i, g_i, \gamma_i\) and \(D_i, i = 1, 2\) are as in Theorem 2.1 and \(M_0 \subseteq h^{-1}_M(0)\), \(N_0 \subseteq h^{-1}_N(0)\). If \(a^2 = a^1 + O(h^*_M)\), \(g_2 = g_1 + O(h^*_M)\), and \(D_2 = D_1 + O(h^*_N)\), then \(\gamma_2 = \gamma_1 + O(h^*_N)\).

**Proof.** It suffices to consider the local setup at \(x = x_c\), where

1. \(x_c \in U \subseteq \mathbb{E}\) and \(V \subseteq \mathbb{F}\) are open in Banach spaces \(\mathbb{E}\) and \(\mathbb{F}\), respectively, and \(g(x_c) = y_c\);
2. the fiber of \(D_i\) at \(x\) is the graph \(\{e + \Delta_i(x)e : e \in \mathbb{D}\}\), where \(\mathbb{E} = \mathbb{D} \oplus \mathbb{D}^\perp\), \(\Delta_i(x) : \mathbb{D} \rightarrow \mathbb{D}^\perp\), and \(\Delta_i(0) = 0\);
3. \(a^i : U \rightarrow \mathbb{E}^*\).

In this setup, \(x = \gamma_i(y)\) are determined by the equations \(F_i(x) = (0, y)\) such that \(F_i : U \rightarrow \mathbb{D}^* \times V\) is defined by

\[
F_i(x) \equiv (a_{\Delta_i}(x), g_i(x)), \quad a_{\Delta_i} \equiv a^i(x) \circ (\mu_{\mathbb{D}^*} + \Delta_i(x)),
\]

where \(\mu_{\mathbb{D}^*}\) is the inclusion of \(\mathbb{D}\) into \(\mathbb{E}\). The domain of the \(F_i\) has the local representative \(h_U\) of \(h_M\), and the codomain of the \(F_i\) has the function \(h_{V^*} \times \mathbb{V}(\alpha, y) \equiv h_V(y)\) where \(h_V\) locally represents \(h_N\). \(F_2 = F_1 + O(h^*_U)\) since \(a^1 = a^2 + O(h^*_U)\) and \(\Delta_1 = \Delta_2 + O(h^*_U)\), and since composition of linear maps is continuous and bilinear. Also, \(x_c\) is a nondegenerate skew critical point for both problems corresponding to \(i = 1, 2\), so each \(F_i\) is a local diffeomorphism at \(x_c\). From Proposition 4.1 and near \((0, y_c), F_1^{-1} = F_2^{-1} + O(h^*_U), which from \(\gamma_i(y) = F_i^{-1}(0, y)\) implies \(\gamma_1 = \gamma_2 + O(h^*_U)\), as required.

In the context of Theorem 5.1, we will need to know that the residuals of the solutions \(\gamma_i\) depend only on the residuals of \(a^i, D_i, \) and \(g_i\). For this, it suffices to show that, given a skew critical point \(m_c \in M_c\) at \(n_c \in N_0\), \(u_c = \text{res}^r(\gamma_2, \gamma_1)(n_c)\) is the unique solution of \(F(u) = 0\) subject to the constraint \(G(u) = 0\), where \(F : T_m M \rightarrow D^*_{m_c}\) is defined by

\[
F(u) = \gamma_2(n_c) d_\theta a^1(m_c)(u) + \text{res}^r(a^2, a^1)(m_c) + \tilde{a}^1(m_c) \circ \text{res}^r(D_2, D_1)(m_c) \quad (5.1)
\]
and $G: T_{n_c} M \to T_n N$ is defined by

$$G(u) = Tg_1(m_c)u + \text{res}^r(g_2, g_1)(m_c).$$  \hfill (5.2) \ \ \ \ \text{(5.2)}$$

Here $\alpha^i(m_c)$ annihilates $\langle D_1 \rangle_{m_c}$ and so descends to $\bar{\alpha}^i(m_c)$ in the quotient $T_{m_c} M/\langle D_1 \rangle_{m_c}$. If $r \geq 2$ then the index 1 occurring asymmetrically in (5.1) and (5.2), such as in the fragment $d\alpha^1(m_c)$, can be replaced by the index 2 because of the skew critical problems are assumed to match to order.

To show (5.1) and (5.2), note that, in the local setup, a vector field in $\mathcal{D}_i$ extending any $\alpha$ of the unperturbed problem are equivariant. Then Proposition 5.2, such as in the fragment $\frac{d}{dt}$, can be replaced by the index 2 because of the skew critical problems are assumed to match to order.

To show (5.1) and (5.2), note that, in the local setup, a vector field in $\mathcal{D}_i$ extending any $\alpha$ of the unperturbed problem are equivariant. Then Proposition 5.2, such as in the fragment $\frac{d}{dt}$, can be replaced by the index 2 because of the skew critical problems are assumed to match to order.

$$d\mathcal{D}_i \alpha^i(x_c)(u, e) = \left. \frac{d}{dt} \right|_{t=0} \langle \alpha^i(x_c + ut), e + \nabla_i(x_c + ut)e \rangle = \langle D\alpha_{\nabla_i}(x_c)u, e \rangle.$$

Since $\alpha_{\nabla_i}(\gamma_i(y)) = 0$, the residuals of this for $i = 1, 2$ are zero, so

$$0 = \gamma_i(\delta_{\nabla_i}(\alpha_{\nabla_i}, \alpha_{\nabla_i})(x_c) + D\alpha_{\nabla_i}(x_c) \text{res}^r(\gamma_2, \gamma_1)(y_c)$$

$$= \gamma_i(\delta_{\nabla_i}(\alpha^2, \alpha^1)(x_c) + \alpha^1(x_c) \circ \text{res}^r(\gamma_2, \gamma_1)(x_c)$$

$$+ D\mathcal{D}_i \alpha^1(x_c) \circ \text{res}^r(\gamma_2, \gamma_1)(y_c),$$

which is the local version of (5.1). The constraint (5.2) follows from the equation $\text{res}^r(g_2 \circ \gamma_2, g_1 \circ \gamma_1)(n_c) = 0$, since $g_1 \circ \gamma_1(n) = g_2 \circ \gamma_2(n) = n$.

Suppose one has skew critical problems as in Theorem 5.1, where the unperturbed problem is equivariant under the action of a Lie group. For the application we have in mind, $\mathcal{G}$ is not a symmetry group of the full critical problem: only the residuals of the unperturbed problem are equivariant. Then Proposition 5.2 below shows that the residuals of the solutions are equivariant. Recall that, if a Lie group acts on a manifold $M$, then it acts by lifts on $TM$ and $T^*M$, and also in the obvious way on the vertical bundles of $TM$ and $T^*M$, and on any Grassmann bundle of $TM$.

**Proposition 5.2.** Let $(M, h_M)$ and $(N, h_N)$ be manifolds suppose $\alpha^i, g_i, \gamma_i$, and $\mathcal{D}_i$, $i = 1, 2$ are as in Theorem 5.1. Suppose that a Lie group $\mathcal{G}$ acts on $M$ and $N$, and

1. $g_i: (M, h_M) \to (N, h_N)$ i.e. $h_N \circ g_i = h_M$;
2. $\mathcal{D}_i|M$ are tangent to $h_M^{-1}(0)$ and are invariant, $\alpha^i|T(h_M^{-1}(0))$ are invariant, and $g_i|h_M^{-1}(0)$ are equivariant;
3. $\text{res}^r(\alpha^2, \alpha^1), \text{res}^r(g_2, g_1)$, and $\text{res}^r(\gamma_2, \gamma_1)$ are equivariant.

Then $h_M \circ \gamma_i = h_N$, and $\text{res}(\gamma_2, \gamma_1): h_N^{-1}(0) \to T(h_M^{-1}(0))$ is equivariant.

**Proof.** Since $g_i \circ \gamma_i(n) = g_i \circ \gamma_i(n) = h_M \circ \gamma_i(n) = h_N(n)$,
and \( h_M \circ \gamma_i = h_N \) follows. Fix \( \tau \in G \) and let \( \tilde{\alpha}^i = \tau^* \alpha, \tilde{D}_i = \tau D_i, \) and \( \tilde{g}_i = \tau^* g_i, \) where \( \tau^* \) denotes pull-back by \( m \mapsto \tau m. \) Note that the maps \( \tilde{\gamma}_i \equiv \tau \gamma_i \) give the skew critical points of \((\tilde{\alpha}^i, \tilde{D}_i, \tilde{g}_i).\) By (5.1) and (5.2), the residuals \( \text{res}^\tau(\gamma_2, \gamma_1) \) and \( \text{res}^\tau(\tilde{\gamma}_2, \tilde{\gamma}_1) \) are determined by the residuals of the data in the skew problems \((\alpha^i, D_i, g_i)\) and \((\tilde{\alpha}^i, \tilde{D}_i, \tilde{g}_i),\) respectively. So the assumed equivariance of the data residuals implies that the residuals \( \text{res}^\tau(\gamma_2, \gamma_1) \) and \( \text{res}^\tau(\tilde{\gamma}_2, \tilde{\gamma}_1) \) are equal, and

\[
\tau \text{res}^\tau(\gamma_2, \gamma_1)(\tau n) = \text{res}^\tau(\tau \gamma_2, \tau \gamma_1)(\tau n) = \text{res}^\tau(\tilde{\gamma}_2, \tilde{\gamma}_1)(\tau n) = \text{res}^\tau(\gamma_2, \gamma_1)(\tau n),
\]
as required.

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