COMPOSITE SPINOR BUNDLES
IN GRAVITATION THEORY.

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Abstract

In gravitation theory, the realistic fermion matter is described by spinor bundles associated with the cotangent bundle of a world manifold $X^4$. In this case, the Dirac operator can be introduced. There is the 1:1 correspondence between these spinor bundles and the tetrad gravitational fields represented by sections of the quotient $\Sigma$ of the linear frame bundle over $X^4$ by the Lorentz group. The key point lies in the fact that different tetrad fields imply nonequivalent representations of cotangent vectors to $X^4$ by the Dirac's matrices. It follows that a fermion field must be regarded only in a pair with a certain tetrad field. These pairs can be represented by sections of the composite spinor bundle $S \rightarrow \Sigma \rightarrow X^4$ where values of tetrad fields play the role of parameter coordinates, besides the familiar world coordinates. The goal is modification of the familiar gravitational equations.

1 Introduction

We follow the generally accepted geometric description of classical fields as sections of a fibred manifold $\pi : Y \rightarrow X$.

Remark. A fibred manifold $Y \rightarrow X$ is provided with fibred coordinates $(x^\lambda, y^i)$ where $x^\lambda$ are coordinates of the base $X$. A locally trivial fibred manifold is termed the bundle. We denote by $V Y$ and $V^* Y$ the vertical tangent bundle and the vertical cotangent bundle of $Y$ respectively. For the sake of simplicity, the pullbacks $Y \times TX$ and $Y \times T^* X$ are denoted by $TX$ and $T^* X$ respectively. □

In gauge theory, $Y \rightarrow X$ is a bundle with a structure group. In gravitation theory, all bundles must be associated with the tangent bundle $TX$ of a world manifold $X^4$. Its structure group is

$$GL_4 = GL^+(4, \mathbb{R}).$$

At the same time, the structure group of spinor bundles describing the Dirac fermion matter is the Lorentz group. In other words, gravitation theory is theory of spontaneous symmetry breaking which moreover displays on the classical level.
Let $LX$ be the principal bundle of linear frames in tangent spaces to $X^4$. In gravitation theory, its structure group $GL_4$ is reduced to the connected Lorentz group

$$L = SO(3, 1).$$

It means that there is a reduced subbundle $L^hX$ of $LX$ whose structure group is $L$. In accordance with the well-known theorem, there is the $1:1$ correspondence between the reduced $L$ subbundles $L^hX$ of $LX$ and the tetrad gravitational fields $h$ represented by global sections of the quotient bundle

$$\Sigma := LX/L \to X^4. \quad (1)$$

Several spinor models have been suggested in order to describe fermion matter, but all fermions observed are the Dirac fermions. We describe them as follows.

Let us consider a bundle of complex Clifford algebras $C_{3,1}$ over $X^4$, its spinor subbundle $S_M \to X^4$ and the subbundle $Y_M \to X^4$ of Minkowski spaces of generating elements of $C_{3,1}$. There exists the bundle morphism

$$\gamma : Y_M \otimes S_M \to S_M$$

which is representation of elements of $Y_M$ by the Dirac’s $\gamma$-matrices on elements of the spinor bundle $S_M$.

To describe Dirac fermion fields on a world manifold, one must require that the bundle $Y_M$ is isomorphic to the cotangent bundle $T^*X$ of $X^4$. It takes place if $Y_M$ is associated with some reduced $L$ subbundle $L^hX$ of the linear frame bundle $LX$. Then, there exists the representation

$$\gamma_h : T^*X \otimes S_h \to S_h$$

of cotangent vectors to a world manifold $X^4$ by Dirac’s $\gamma$-matrices on elements of the spinor bundle $S_h$. Sections of $S_h$ describe Dirac fermion fields in the presence of the tetrad gravitational field $h$.

The crucial point consists in the fact that, for different gravitational fields $h$ and $h'$, the representations $\gamma_h$ and $\gamma_{h'}$ fail to be equivalent [1, 2]. It follows that every Dirac fermion field must be regarded only in a pair with a certain tetrad gravitational field $h$. Therefore, gravitational fields and fermion fields can not be represented by sections of the familiar product $S \times \Sigma$ of the bundle $\Sigma$ and some standard spinor bundle $S \to X^4$. Their pairs constitute the so-called fermion-gravitation complex [3]. There is the $1:1$ correspondence between these pairs and the sections of the composite bundle

$$S \to \Sigma \to X^4 \quad (2)$$

where $S \to \Sigma$ is a spinor bundle associated with the $SL(2, C)$ lift of the $L$ principal bundle $LX \to \Sigma$. At the same time, every spinor bundle $S_h \to X^4$ is isomorphic to restriction of $S \to \Sigma$ to $h(X^4) \subset \Sigma$. 

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By a composite fibred manifold is meant the composition

\[ Y \to \Sigma \to X \]  \hspace{1cm} (3)

where \( Y \to \Sigma \) is a bundle denoted by \( Y_\Sigma \) and \( \Sigma \to X \) is a fibred manifold. In gauge theory, composite manifolds

\[ P \to P/K \to X \]

where \( P \) is a principal bundle whose structure group is reducible to its closed subgroup \( K \) describe spontaneous symmetry breaking [4]. Global sections of \( P/K \to X \) are treated the Higgs fields.

Application of composite manifolds to field theory is founded on the following. Given a global section \( h \) of \( \Sigma \), the restriction \( Y_h \) of \( Y_\Sigma \) to \( h(X) \) is a fibred submanifold of \( Y \to X \).

There is the 1:1 correspondence between the global sections \( s_h \) of \( Y_h \) and the global sections of the composite manifold \( (3) \) which cover \( h \). Therefore, one can think of sections \( s_h \) of \( Y_h \) as describing fields in the presence of a background parameter field \( h \), whereas sections of the composite manifold \( Y \) describe all the pairs \((s_h, h)\). It is important when the bundles \( Y_h \) and \( Y_{h\neq h'} \) fail to be equivalent in a sense. The configuration space of these pairs is the first order jet manifold \( J^1Y \) of the composite manifold \( Y \) and their phase space is the multisymplectic Legendre bundle \( \Pi \) over \( Y \).

Dynamics of fields represented by sections of a fibred manifold \( Y \to X \) is phrased in terms of jet manifolds \([3, 5, 6, 7, 8, 9, 10]\). The \( k \)-order jet manifold \( J^kY \) of a fibred manifold \( Y \) comprises the equivalence classes \( j^k_x s, x \in X \), of sections \( s \) of \( Y \) identified by the \((k+1)\) terms of their Taylor series at \( x \). Recall that a \( k \)-order differential operator on sections of a fibred manifold \( Y \), by definition, is a morphism of \( J^kY \) to a vector bundle over \( X \).

In field theory, we can restrict our consideration to the first order Lagrangian formalism where the jet manifold \( J^1Y \) plays the role of a finite-dimensional configuration space of fields.

**Remark.** The first order jet manifold \( J^1Y \) of \( Y \) is both the fibred manifold \( J^1Y \to X \) and the affine bundle \( J^1Y \to Y \) modelled on the vector bundle \( T^*X \otimes VY \). It is endowed with the adapted coordinates \((x^\lambda, y^i, y^i_\lambda)\) where

\[ y^i_\lambda = \left( \frac{\partial y^i}{\partial y^j} y^j_\mu + \frac{\partial y^i}{\partial x^\mu} \right) \frac{\partial x^\mu}{\partial x^j_\lambda}. \]

One identifies usually \( J^1Y \) to its image under the canonical bundle monomorphism

\[ \lambda : J^1Y \to T^*X \otimes TY, \]

\[ \lambda = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i). \]  \hspace{1cm} (4)

Every fibred morphism of \( \Phi : Y \to Y' \) over a diffeomorphism of \( X \) has the jet prolongation

\[ J^1\Phi : J^1Y \to J^1Y'. \]
\[ y_i^\nu \circ J^1 \Phi = (\partial_\lambda \Phi^i + \partial_j \Phi^i y_j^\lambda) \frac{\partial x^\lambda}{\partial y^\mu}. \]

A section \( s \) of \( Y \) gives rise to the section
\[ \sigma = J^1 s, \quad \sigma^i_\mu = \partial_\mu s^i, \]
of the fibred jet manifold \( J^1 Y \to X \). There is the 1:1 correspondence between the connections on \( Y \to X \) and global sections
\[ \Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i) \]
of the affine jet bundle \( J^1 Y \to Y \). These global sections form the affine space modelled on the linear space of soldering forms
\[ Y \to T^* Y \otimes VY \]
on \( Y \). Every connection \( \Gamma \) on \( Y \to X \) yields the first order differential operator
\[ D_\Gamma : J^1 Y \to T^* Y \otimes VY; \]
\[ D_\Gamma = (y^i_\lambda - \Gamma^i_\lambda) dx^\lambda \otimes \partial_i, \]
on \( Y \) which is called the covariant differential relative to the connection \( \Gamma \). \( \square \)

A Lagrangian density on the configuration space \( J^1 Y \) is defined to be a morphism
\[ L : J^1 Y \to \bigwedge^n T^* X, \quad n = \dim X, \]
\[ L = L \omega, \quad \omega = dx^1 \wedge ... \wedge dx^n. \]
Note that, since the jet bundle \( J^1 Y \to Y \) is affine, every polynomial Lagrangian density of field theory factors
\[ L : J^1 Y \to T^* Y \otimes VY \to \bigwedge^n T^* X. \] (6)

The feature of the dynamics of field systems on composite manifolds consists in the following.

Let \( Y \) be the composite manifold \([\Sigma]\) coordinatized by \((x^\lambda, \sigma^m, y^i)\) where \((x^\lambda, \sigma^m)\) are fibred coordinates of \( \Sigma \). Every connection
\[ A_\Sigma = dx^\lambda \otimes (\partial_\lambda + \tilde{A}^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i) \] (7)
on \( Y \to \Sigma \) yields the first order differential operator
\[ \tilde{D} : J^1 Y \to T^* Y \otimes VY_\Sigma; \]
\[ \tilde{D} = dx^\lambda \otimes (y^i_\lambda - \tilde{A}^i_\lambda - A^i_m \sigma^m_\lambda) \partial_i, \] (8)
on $Y$. Let $h$ be a global section of $\Sigma$ and $Y_h$ the restriction of the bundle $Y_\Sigma$ to $h(X)$. The restriction of $\tilde{D}$ to $J^1Y_h \subset J^1Y$ comes to the familiar covariant differential relative to a certain connection $A_h$ on $Y_h$. Thus, it is $\tilde{D}$ that we may utilize in order to construct a Lagrangian density

$$L : J^1Y \xrightarrow{\tilde{D}} T^*X \otimes VY_\Sigma \to \wedge T^*X$$

for sections of a composite manifold. It should be noted that such a Lagrangian density is never regular because of the constraint conditions

$$A_m^i \partial_m^\mu \mathcal{L} = \partial_m^\mu \mathcal{L}.$$  

If a Lagrangian density is degenerate, the corresponding Euler-Lagrange equations are underdetermined. To describe constraint field systems, one can utilize the multimomentum Hamiltonian formalism where canonical momenta correspond to derivatives of fields with respect to all world coordinates, not only the temporal one [11, 12, 13, 9, 10].

In the framework of multimomentum Hamiltonian formalism, the phase space of fields is the Legendre bundle

$$\Pi = \wedge T^*X \otimes T X \otimes V^*Y$$

over $Y$ coordinatized by $(x^\lambda, y^i, p_\lambda^i)$. Note that every Lagrangian density $L$ on $J^1Y$ determines the Legendre morphism

$$\hat{L} : J^1Y \to \Pi, \quad (x^\mu, y^i, p_\mu^i) \circ \hat{L} = (x^\mu, y^i, \partial_\mu^i \mathcal{L}).$$

Its image plays the role of the Lagrangian constraint space.

The Legendre bundle $\Pi$ carries the multisymplectic form

$$\Omega = dp_\lambda^i \wedge dy^i \wedge \omega \otimes \partial_\lambda.$$  

We say that a connection $\gamma$ on the fibred Legendre manifold $\Pi \to X$ is a Hamiltonian connection if the form $\gamma|\Omega$ is closed. Then, a Hamiltonian form $H$ on $\Pi$ is defined to be an exterior form such that

$$dH = \gamma|\Omega$$

for some Hamiltonian connection $\gamma$. The key point consists in the fact that every Hamiltonian form admits splitting

$$H = p_\lambda^i dy^i \wedge \omega_\lambda - p_\lambda^i \Gamma_\lambda^i \omega - \mathcal{H} \gamma \omega = p_\lambda^i dy^i \wedge \omega_\lambda - \mathcal{H} \omega, \quad \omega_\lambda = \partial_\lambda \gamma \omega,$$

where $\Gamma$ is a connection on $Y \to X$ and $\mathcal{H} \gamma \omega$ is a horizontal density on $\Pi \to X$. Given the Hamiltonian form $\Pi$, the equality (13) comes to the first order Hamilton equations

$$\partial_\lambda r^i = \partial_\lambda^i \mathcal{H},$$

$$\partial_\lambda r^\lambda_i = -\partial_i^\lambda \mathcal{H}.$$  

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for sections $r$ of the fibred Legendre manifold $\Pi \rightarrow X$.

If a Lagrangian density $L$ is regular, there exists the unique Hamiltonian form $H$ such that the first order Euler-Lagrange equations and the Hamilton equations are equivalent, otherwise in general case. One must consider a family of different Hamiltonian forms $H$ associated with the same degenerate Lagrangian density $L$ in order to exhaust solutions of the Euler-Lagrange equations. Lagrangian densities of field models are almost always quadratic and affine in derivative coordinates $y_i^\mu$. In this case, given an associated Hamiltonian form $H$, every solution of the corresponding Hamilton equations which lives on the Lagrangian constraint space $\hat{L}(J^1Y) \subset \Pi$ yields a solution of the Euler-Lagrange equations. Conversely, for any solution of the Euler-Lagrange equations, there exists the corresponding solution of the Hamilton equations for some associated Hamiltonian form.

The feature of Hamiltonian systems on composite manifolds (i) every connection $A_\Sigma$ on $Y \rightarrow \Sigma$ yields splitting

$$\omega \otimes \partial \lambda \otimes [p^\lambda_i(dy^i - A^i_m d\sigma^m) + (p^\lambda_m + A^i_m p^\lambda_i) d\sigma^m]$$

of the Legendre bundle $\Pi$ over a composite manifold $Y$ and (ii) the Lagrangian constraint space is

$$p^\lambda_m + A^i_m p^\lambda_i = 0.$$  \hfill (16)

Moreover, if $h$ is a global section of $\Sigma \rightarrow X$, the submanifold $\Pi_h$ of $\Pi$ given by the coordinate relations

$$\sigma^m = h^m(x), \quad p^\lambda_m + A^i_m p^\lambda_i = 0$$

is isomorphic to the Legendre bundle over the restriction $Y_h$ of $Y_\Sigma$ to $h(X)$. The Legendre bundle $\Pi_h$ is the phase space of fields in the presence of the background parameter field $h$.

In the Hamiltonian gravitation theory, the constraint condition (16) takes the form

$$p^c_\mu + \frac{1}{8} \eta^{cb} \sigma^a(y^B [\gamma_a, \gamma_b]^A B p^A_\lambda + p^A_{+}[\gamma_a, \gamma_b]^{+B} A y^+_B) = 0$$  \hfill (17)

where $(\sigma^\mu, y^A)$ are respectively tetrad and spinor coordinates of the composite spinor bundle (2), $p^\lambda_\mu$ and $p^\lambda_A$ are the corresponding momenta and $\eta$ denotes the Minkowski metric. The condition (17) replaces the standard gravitational constraints

$$p^\lambda_\mu = 0.$$  \hfill (18)

The crucial point is that, when restricted to the constraint space (18), the Hamilton equations of the gravitation theory come to the familiar gravitational equations, otherwise on the constraint space (17).
2 Dirac fermion fields

By \(X^4\) is further meant an oriented world manifold which satisfies the well-known global topological conditions in order that gravitational fields, space-time structure and spinor structure can exist. To summarize these conditions, we assume that \(X^4\) is not compact and that the tangent bundle of \(X^4\) is trivial.

Given a Minkowski space \(M\), let \(C_{1,3}\) be the complex Clifford algebra generated by elements of \(M\). A spinor space \(V\) is defined to be a minimal left ideal of \(C_{1,3}\) on which this algebra acts on the left. We have the representation

\[
\gamma : M \otimes V \to V \tag{19}
\]

of elements of the Minkowski space \(M \subset C_{1,3}\) by Dirac’s matrices \(\gamma\) on \(V\).

Let us consider the transformations preserving the representation (19). These are pairs \((l, l_s)\) of Lorentz transformations \(l\) of the Minkowski space \(M\) and invertible elements \(l_s\) of \(C_{1,3}\) such that

\[
\gamma(lM \otimes l_sV) = l_s\gamma(M \otimes V).
\]

Elements \(l_s\) form the Clifford group whose action on \(M\) however is not effective. We restrict ourselves to its spinor subgroup \(L_s = SL(2, \mathbb{C})\). Its generators act on \(V\) by the representation

\[
I_{ab} = \frac{1}{4} [\gamma_a, \gamma_b].
\]

Let us consider a bundle of complex Clifford algebras \(C_{3,1}\) over \(X^4\). Its subbundles are both a spinor bundle \(S_M \to X^4\) and the bundle \(Y_M \to X^4\) of Minkowski spaces of generating elements of \(C_{3,1}\). To describe Dirac fermion fields on a world manifold \(X^4\), one must require of \(Y_M\) to be isomorphic to the cotangent bundle \(T^*X\) of \(X^4\). It takes place if the structure group of \(LX\) is reducible to the Lorentz group \(L\) and \(LX\) contains a reduced \(L\) subbundle \(L^hX\) such that

\[
Y_M = (L^hX \times M)/L.
\]

In this case, the spinor bundle

\[
S_M = S_h = (P_h \times V)/L_s \tag{20}
\]

is associated with the \(L_s\)-lift \(P_h\) of \(L^hX\). There is the above-mentioned 1:1 correspondence between the reduced subbubdles \(L^hX\) of \(LX\) and the global sections of the quotient bundle \(\Sigma\) [4].

Given \(h\), let \(\Psi^h\) be an atlas of \(LX\) such that the corresponding local sections \(z^h_\xi\) of \(LX\) take their values into \(L^hX\). With respect to \(\Psi^h\) and a holonomic atlas \(\Psi^T = \{\psi^T_\xi\}\) of \(LX\), a tetrad field \(h\) can be represented by a family of \(GL_4\)-valued tetrad functions

\[
h_\xi = \psi^T_\xi \circ z^h_\xi, \quad x^\lambda = h^\lambda_a(x) h^a. \tag{21}
\]
Given a tetrad field $h$, one can define the representation

$$\gamma_h : T^*X \otimes S_h = (P_h \times (M \otimes V))/L_s \rightarrow (P_h \times \gamma(M \otimes V))/L_s = S_h$$

of cotangent vectors to a world manifold $X^4$ by Dirac’s $\gamma$-matrices on elements of the spinor bundle $S_h$. With respect to an atlas $\{z_\xi\}$ of $P_h$ and the associated atlas $\{z_\xi^h\}$ of $LX$, the morphism (22) reads

$$\gamma_h(h^a \otimes y^A v_A(x)) = \gamma^{aA}_B y^B v_A(x)$$

where $\{v_A(x)\}$ are the associated fibre bases for $S_h$. As a shorthand, one can write

$$\tilde{dx}^\lambda = \gamma_h(dx^\lambda) = h^\lambda_a(x)\gamma^a.$$

We shall say that, given the representation (22), sections of the spinor bundle $S_h$ describe Dirac fermion fields in the presence of the gravitational field $h$. Indeed, let $A_h$ be a principal connection on $S_h$ and

$$D : J^1 S_h \rightarrow T^*X \otimes VS_h,$$

$$D = (y^A_\lambda - A^{ab}_\lambda(x)I_{ab}^A y^B)dx^\lambda \otimes \partial_A,$$

the corresponding covariant differential. Given the representation (22), one can construct the Dirac operator

$$D_h = \gamma_h \circ D : J^1 S_h \rightarrow T^*X \otimes VS_h \rightarrow VS_h,$$

$$\dot{y}^A \circ D_h = h^\lambda_a(\gamma^{aA}_B(y^B_\lambda - A^{ab}_\lambda I_{ab}^A y^B)).$$

We here use the fact that the vertical tangent bundle $VS_h$ admits the canonical splitting

$$VS_h = S_h \times S_h,$$

and $\gamma_h$ in the expression (22) is the pullback

$$\gamma_h : T^*X \otimes VS_h \rightarrow VS_h,$$

$$\gamma_h(h^a \otimes y^A \partial_A) = \gamma^{aA}_B y^B \partial_B,$$

over $S_h$ of the bundle morphism (22).

For different tetrad fields $h$ and $h'$, the representations $\gamma_h$ and $\gamma_{h'}$ (22) are not equivalent [1, 2]. It follows that a Dirac fermion field must be regarded only in a pair with a certain tetrad gravitational field. There is the 1:1 correspondence between these pairs and sections of the composite spinor bundle [2].
3 Composite manifolds

A composite manifold is defined to be composition of surjective submersions

$$\pi_{\Sigma X} \circ \pi_{Y \Sigma} : Y \to \Sigma \to X.$$ (24)

It is provided with the particular class of coordinate atlases \((x^\lambda, \sigma^m, y^i)\) where \((x^\mu, \sigma^m)\) are fibred coordinates of \(\Sigma\) and \(y^i\) are bundle coordinates of \(Y_\Sigma\). We further propose that \(\Sigma\) has a global section.

Recollect the following assertions [4, 8, 14].

(i) Let \(Y\) be the composite manifold (24). Given a section \(h\) of \(\Sigma\) and a section \(s_\Sigma\) of \(Y_\Sigma\), their composition \(s_\Sigma \circ h\) is a section of the composite manifold \(Y\). Conversely, if the bundle \(Y_\Sigma\) has a global section, every global section \(s\) of the fibred manifold \(Y \to X\) is represented by some composition \(s_\Sigma \circ h\) where \(h = \pi_{Y \Sigma} \circ s\) and \(s_\Sigma\) is an extension of the local section \(h(X) \to s(X)\) of the bundle \(Y_\Sigma\) over the closed imbedded submanifold \(h(X) \subset \Sigma\).

(ii) Given a global section \(h\) of \(\Sigma\), the restriction \(Y_h = h^*Y_\Sigma\) of the bundle \(Y_\Sigma\) to \(h(X)\) is a fibred imbedded submanifold of \(Y\).

(iii) There is the 1:1 correspondence between the sections \(s_h\) of \(Y_h\) and the sections \(s\) of the composite manifold \(Y\) which cover \(h\).

(iv) Given fibred coordinates \((x^\lambda, \sigma^m, y^i)\) of the composite manifold \(Y\), the jet manifolds \(J^1\Sigma, J^1Y_\Sigma\) and \(J^1Y\) are coordinatized respectively by

\[
(x^\lambda, \sigma^m, y^i, y^i_\lambda), \quad (x^\lambda, \sigma^m, y^i, \tilde{y}^i_\lambda, y^i_\lambda), \quad (x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, y^i_\lambda).
\]

There exists the canonical surjection

\[
\rho : J^1\Sigma \times \Sigma \to J^1Y_\Sigma \to J^1Y,
\]

\[
y^i_\lambda \circ \rho = y^i_m \sigma^m_\lambda + \tilde{y}^i_\lambda,
\]

where \(s_\Sigma\) and \(h\) are sections of \(Y_\Sigma\) and \(\Sigma\) respectively.

The following assertions are concerned with connections on composite manifolds.

Let \(A_\Sigma\) be the connection (6) on the bundle \(Y_\Sigma\) and \(\Gamma\) the connection (5) on the fibred manifold \(\Sigma\). Building on the morphism (25), one can construct the composite connection

\[
A = dx^\lambda \otimes [\partial_\lambda + \Gamma^m_\lambda \partial_m + (A^i_m \Gamma^m_\lambda + \tilde{A}^i_\lambda) \partial_i]\] (26)

on the composite manifold \(Y\).

Let a global section \(h\) of \(\Sigma\) be an integral section of the connection \(\Gamma\) on \(\Sigma\), that is, \(\Gamma \circ h = J^1h\). Then, the composite connection (26) on \(Y\) is reducible to the connection

\[
A_h = dx^\lambda \otimes [\partial_\lambda + (A^i_m \partial_m h^m + \tilde{A}^i_\lambda) \partial_i]\] (27)
on the fibred submanifold $Y_h$ of $Y \to X$. In particular, every connection $A_\Sigma$ \((7)\) on $Y_\Sigma$, whenever $h$, is reducible to the connection \((27)\) on $Y_h$.

Every connection \((7)\) on the bundle $Y_\Sigma$ yields the horizontal splitting

$$VY = VY_\Sigma \oplus (Y \times V_\Sigma),$$  \hspace{1cm} \text{(28)}$$

$$\dot{y}^i \partial_i + \dot{\sigma}^m \partial_m = (\dot{y}^i - A^i_m \dot{\sigma}^m) \partial_i + \dot{\sigma}^m (\partial_m + A^i_m \partial_i),$$

and the dual horizontal splitting

$$V^*Y = V^*Y_\Sigma \oplus (Y \times V^*_\Sigma),$$  \hspace{1cm} \text{(29)}$$

$$\dot{y}_idy^i + \dot{\sigma}_md\sigma^m = \dot{y}_i(dy^i - A^i_m d\sigma^m) + \dot{\sigma}_m (\partial_m + A^i_m \dot{y}_i)d\sigma^m.$$ 

It is readily observed that the splittings \((28)\) and \((29)\) are uniquely characterized by the form

$$\omega \wedge A_\Sigma = \omega \wedge d\sigma^m \otimes (\partial_m + A^i_m \partial_i),$$  \hspace{1cm} \text{(30)}$$

and different connections $A_\Sigma$ can define the same splittings \((28)\) and \((29)\).

Building on the horizontal splitting \((28)\), one can construct the first order differential operator \((8)\) on the composite manifold $Y$. This possesses the following property.

Given a global section $h$ of $\Sigma$, let $\Gamma$ be a connection on $\Sigma$ whose integral section is $h$, that is, $\Gamma \circ h = J^1h$. It is readily observed that the differential \((8)\) restricted to $J^1Y_h \subset J^1Y$ comes to the familiar covariant differential relative to the connection $A_h$ \((27)\) on $Y_h$. Thus, it is the differential \((8)\) that one may utilize in order to construct a Lagrangian density \((9)\) for sections of a composite manifold.

Let $\pi_P : P \to X$ be a principal bundle with a structure Lie group $G$ and $K$ its closed subgroup. We have the composite manifold

$$\pi_{\Sigma X} \circ \pi_{P_\Sigma} : P \to P/K \to X$$  \hspace{1cm} \text{(31)}$$

where

$$P_\Sigma := P \to P/K$$

is a principal bundle with the structure group $K$ and

$$\Sigma = P/K = (P \times G/K)/G$$

is the $P$-associated bundle.

Let the structure group $G$ be reducible to its closed subgroup $K$. By the well-known theorem, there is the $1:1$ correspondence

$$\pi_{P_\Sigma}(P_h) = (h \circ \pi_P)(P_h)$$
between global sections $h$ of the bundle $P/K \to X$ and the reduced $K$-principal subbundles $P_h$ of $P$ which consist with restrictions of the principal bundle $P_\Sigma$ to $h(X)$.

Given the composite manifold (31), the canonical morphism (25) results in the surjection

$$J^1P_\Sigma/K \times J^1\Sigma \to J^1P/K$$

over $J^1\Sigma$. Let $A_\Sigma$ be a principal connection on $P_\Sigma$ and $\Gamma$ a connection on $\Sigma$. The corresponding composite connection (26) on the composite manifold (31) is equivariant under the canonical action of $K$ on $P$. If the connection $\Gamma$ has an integral global section $h$ of $P/K \to X$, the composite connection (26) is reducible to the connection (27) on $P_h$ which consists with the principal connection on $P_h$ induced by $A_\Sigma$.

Let us consider the composite manifold

$$Y = (P \times V)/K \to P/K \to X$$

(32)

where the bundle

$$Y_\Sigma := (P \times V)/K \to P/K$$

is associated with the $K$-principal bundle $P_\Sigma$. Given a reduced subbundle $P_h$ of $P$, the associated bundle

$$Y_h = (P_h \times V)/K$$

is isomorphic to the restriction of $Y_\Sigma$ to $h(X)$.

## 4 Composite spinor bundles

In gravitation theory, we have the composite manifold

$$\pi_{\Sigma X} \circ \pi_{P_\Sigma} : LX \to \Sigma \to X^4$$

(33)

where $\Sigma$ is the quotient bundle (11) and

$$LX_\Sigma := LX \to \Sigma$$

is the $L$-principal bundle.

Building on the double universal covering of the group $GL_4$, one can perform the $L_s$-principal lift $P_\Sigma$ of $LX_\Sigma$ such that

$$P_\Sigma/L_s = \Sigma, \quad LX_\Sigma = r(P_\Sigma).$$

In particular, there is imbedding of the $L_s$-lift $P_h$ of $L^hX$ onto the restriction of $P_\Sigma$ to $h(X^4)$.
Let us consider the composite spinor bundle (2) where

\[ S_\Sigma = (P_\Sigma \times V)/L_s \]

is associated with the \( L_s \)-principal bundle \( P_\Sigma \). It is readily observed that, given a global section \( h \) of \( \Sigma \), the restriction \( S_\Sigma \) to \( h(X^4) \) is the spinor bundle \( S_h \) whose sections describe Dirac fermion fields in the presence of the tetrad field \( h \).

Let us provide the principal bundle \( LX \) with a holonomic atlas \( \{ \psi^T_\xi, U_\xi \} \) and the principal bundles \( P_\Sigma \) and \( LX_\Sigma \) with associated atlases \( \{ z_\xi^s, U_\xi \} \) and \( \{ z_\xi = r \circ z_\xi^s \} \). With respect to these atlases, the composite spinor bundle is endowed with the fibred coordinates \((x_\lambda, \sigma^\mu_a, y^A, \tilde{y}^A_\lambda)\) where \((x_\lambda, \sigma^\mu_a)\) are fibred coordinates of the bundle \( \Sigma \) such that \( \sigma^\mu_a \) are the matrix components of the group element

\[ GL_4 \ni (\psi^T_\xi \circ z_\xi^s)(\sigma) : \mathbb{R}^4 \to \mathbb{R}^4, \quad \sigma \in U_\xi, \quad \pi_{\Sigma X}(\sigma) \in U_\xi. \]

Given a section \( h \) of \( \Sigma \), we have

\[ z_\xi^h(x) = (z_\xi \circ h)(x), \quad h(x) \in U_\xi, \quad x \in U_\xi, \]

\[ (\sigma^\mu_a \circ h)(x) = h^\mu_a(x), \]

where \( h^\mu_a(x) \) are tetrad functions (21).

The jet manifolds \( J^1 \Sigma \), \( J^1 S_\Sigma \) and \( J^1 S \) are coordinatized respectively by

\[ (x_\lambda, \sigma^\mu_a, \sigma^\mu_{a\lambda}), \quad (x_\lambda, \sigma^\mu_a, y^A, \tilde{y}^A_\lambda, y_{A\mu}), \quad (x_\lambda, \sigma^\mu_a, y^A, \sigma^\mu_{a\lambda}, y_{A\lambda}). \]

Note that, whenever \( h \), the jet manifold \( J^1 S_h \) is a fibred submanifold of \( J^1 S \to X^4 \) given by the coordinate relations

\[ \sigma^\mu_a = h^\mu_a(x), \quad \sigma^\mu_{a\lambda} = \partial_\lambda h^\mu_a(x). \]

Let us consider the bundle of Minkowski spaces

\[ (LX \times M)/L \to \Sigma \]

associated with the \( L \)-principal bundle \( LX_\Sigma \). Since \( LX_\Sigma \) is trivial, it is isomorphic to the pullback \( \Sigma \times T^*X \) which we denote by the same symbol \( T^*X \). Building on the morphism (19), one can define the bundle morphism

\[ \gamma_\Sigma : T^*X \otimes S_\Sigma = (P_\Sigma \times (M \otimes V))/L_s \to (P_\Sigma \times \gamma(M \otimes V))/L_s = S_\Sigma, \quad (34) \]

\[ \hat{dx}_\lambda = \gamma_\Sigma(dx^\lambda) = \sigma^\lambda_a \gamma_a, \]
over $\Sigma$. When restricted to $h(X^4) \subset \Sigma$, the morphism (34) comes to the morphism $\gamma_h$ (22). Because of the canonical vertical splitting

$$VS_\Sigma = S_\Sigma \times S_\Gamma,$$

the morphism (34) yields the corresponding morphism

$$\gamma_\Sigma : T^* X \otimes VS_\Sigma \to VS_\Sigma.$$ (35)

We use this morphism in order to construct the total Dirac operator on sections of the composite spinor bundle $S$ (2).

Let

$$\tilde{\Gamma} = dx^\lambda \otimes (\partial_\lambda + \tilde{A}_\lambda^B \partial_B) + d\sigma_\mu^a \otimes (\partial_\mu + A_\mu^a \partial_B)$$ (36)

be a connection on the bundle $S_\Sigma$. It determines the horizontal splitting (28) of the vertical tangent bundle $VS$ and the differential (8). The composition of the morphisms (35) and (8) is the first order differential operator

$$D = \gamma_\Sigma \circ \tilde{D} : J^1 S \to T^* X \otimes VS_\Sigma \to VS_\Sigma,$$

$$\dot{y}^A \circ D = \sigma_\mu^a \gamma_\Sigma^a \sigma_\mu^b (\dot{y}_\lambda^B - \tilde{A}_\lambda^B - A_\mu^a \sigma_\mu^c),$$

on $S$. One can think of it as being the total Dirac operator since, whenever a tetrad field $h$, the restriction of $D$ to $J^1 S_h \subset J^1 S$ comes to the Dirac operator $D_h$ (23) relative to the connection

$$A_h = dx^\lambda \otimes (\partial_\lambda + (\tilde{A}_\lambda^B + A_\mu^a \partial_\lambda h_\mu^a) \partial_B)$$

on $S_h$.

To construct the connection (36) in explicit form, let us set up the principal connection on the bundle $LX_\Sigma$ which is given by the local connection form

$$A_\Sigma = (\tilde{A}_\mu^a dx_\lambda^\mu + A_\mu^a d\sigma_\mu^c) \otimes I_{ab},$$ (37)

$$\tilde{A}_\mu^a = \frac{1}{2} K^\nu_{\lambda \mu \sigma_c} (\eta^{cb} \sigma_c^{\nu a} - \eta^{ca} \sigma_c^{b \nu}),$$

$$A_{\lambda \mu \sigma_c} = \frac{1}{2} (\eta^{cb} \sigma_c^a - \eta^{ca} \sigma_c^b),$$ (38)

where $K$ is some symmetric connection on $TX$ and (38) corresponds to the canonical left-invariant free-curvature connection on the bundle

$$GL_4 \to GL_4/L.$$ 

Given a tetrad field $h$, the connection (37) is reduced to the Levi-Civita connection

$$A_h = \frac{1}{2} [K^\nu_{\lambda \mu} \sigma_c^{\lambda a} (\eta^{cb} \sigma_c^{\nu a} - \eta^{ca} \sigma_c^{b \nu}) + \partial_\mu h_\nu^c (\eta^{cb} \sigma_c^a - \eta^{ca} \sigma_c^b)]$$

13
on $L^h X$.

The connection (36) on the spinor bundle $S_\Sigma$ which is associated with $A_\Sigma$ (37) reads

$$\tilde{A}_\Sigma = dx^\lambda \otimes (\partial_\lambda + \frac{1}{2}\tilde{A}^a_{\lambda}I_{ab}y^A\partial_B) + d\sigma^\mu_\epsilon \otimes (\partial^\epsilon_\mu + \frac{1}{2}A^{abc}_{\mu}I_{ab}y^A\partial_B).$$

(39)

It determines the canonical horizontal splitting (28) of the vertical tangent bundle $VS$ given by the canonical form (30)

$$\omega \wedge d\sigma^\mu_\epsilon \otimes [\partial^\epsilon_\mu + \frac{1}{8}\eta^{ab}\sigma^\mu_\epsilon[\gamma_a,\eta_b]B_Ay^A\partial_B].$$

### 5 Multimomentum Hamiltonian formalism

Let $\Pi$ be the Legendre bundle (10) over a fibred manifold $Y \rightarrow X$. This is the composite manifold

$$\pi_{\Pi X} = \pi \circ \pi_{\Pi Y} : \Pi \rightarrow Y \rightarrow X$$

provided with fibred coordinates $(x^\lambda, y^i, p^\lambda_i)$ where

$$p^\lambda_i = \det(\frac{\partial x^\lambda}{\partial x^{\mu}}, \frac{\partial y^j}{\partial x^{\mu}}, \frac{\partial x^\lambda}{\partial x^{\mu}}p^\mu_j).$$

By $J^1\Pi$ is meant the first order jet manifold of $\Pi \rightarrow X$. It is coordinatized by

$$(x^\lambda, y^i, p^\lambda_i, y^i_{(\mu)}, p^\lambda_{i\mu}).$$

**Remark.** We call by a momentum morphism any bundle morphism $\Phi : \Pi \rightarrow J^1Y$ over $Y$. Given a momentum morphism $\Phi$, its composition with the monomorphism (4) is represented by the horizontal pullback-valued 1-form

$$\Phi = dx^\lambda \otimes (\partial_\lambda + \Phi^\lambda_i \partial_i)$$

(40)

on $\Pi \rightarrow Y$. For instance, let $\Gamma$ be a connection on $Y$. Then, the composition $\tilde{\Gamma} = \Gamma \circ \pi_{\Pi Y}$ is a momentum morphism. The corresponding form (10) on $\Pi$ is the pullback $\tilde{\Gamma}$ of the form $\Gamma$ (3) on $Y$. Conversely, every momentum morphism $\Phi$ defines the associated connection $\Gamma_\Phi = \Phi \circ \tilde{\Phi}$ on $Y \rightarrow X$ where $\tilde{\Phi}$ is the global zero section of $\Pi \rightarrow Y$. Every connection $\Gamma$ on $Y$ gives rise to the connection

$$\tilde{\Gamma} = dx^\lambda \otimes [\partial_\lambda + \Gamma_i^\lambda(y)\partial_i + (-\partial_j\Gamma_i^\lambda(y)p^\mu_i - K^{\mu}_{\nu\lambda}(x)p^\nu_j + K^{\nu\alpha\lambda}(x)p^\mu_i)p^\mu_j]$$

(41)

on $\Pi \rightarrow X$ where $K$ is a linear symmetric connection on $T^*X$. $\Box$

The Legendre manifold $\Pi$ carries the multimomentum Liouville form

$$\theta = -p^\lambda_i dy^i \wedge \omega \otimes \partial_\lambda$$

(42)
and the multisymplectic form \( \Omega \).

We say that a connection \( \gamma \) on the fibred Legendre manifold \( \Pi \to X \) is a Hamiltonian connection if the exterior form \( \gamma \} \Omega \) is closed. An exterior \( n \)-form \( H \) on the Legendre manifold \( \Pi \) is called a Hamiltonian form if there exists a Hamiltonian connection satisfying the equation (13).

Let \( H \) be a Hamiltonian form. For any exterior horizontal density \( \tilde{H} = \tilde{H} \omega \) on \( \Pi \to X \), the form \( H - \tilde{H} \) is a Hamiltonian form. Conversely, if \( H \) and \( H' \) are Hamiltonian forms, their difference \( H - H' \) is an exterior horizontal density on \( \Pi \to X \). Thus, Hamiltonian forms constitute an affine space modelled on a linear space of the exterior horizontal densities on \( \Pi \to X \).

In particular, let \( \Gamma \) be a connection on \( Y \to X \) and \( \tilde{\Gamma} \) its lift (41) onto \( \Pi \to X \). We have the equality

\[
\tilde{\Gamma} \} \Omega = \partial_i \lambda H
\]

A glance at this equality shows that \( \tilde{\Gamma} \) is a Hamiltonian connection and

\[
H_H = \tilde{\Gamma} \} \theta = p_i^\lambda dy^i \wedge \omega_\lambda - p_i^\lambda \Gamma_i^\lambda \omega
\]

is a Hamiltonian form. It follows that every Hamiltonian form on \( \Pi \) can be given by the expression (14) where \( \Gamma \) is some connection on \( Y \to X \). Moreover, a Hamiltonian form has the canonical splitting (14) as follows. Given a Hamiltonian form \( H \), the vertical tangent morphism \( VH \) yields the momentum morphism

\[
\tilde{H} : \Pi \to J^1 Y, \quad y_i^\lambda \circ \tilde{H} = \partial_i^\lambda H,
\]

and the associated connection \( \Gamma_H = \tilde{H} \circ 0 \) on \( Y \). As a consequence, we have the canonical splitting

\[
H = H_{\Gamma_H} - \tilde{H}.
\]

The Hamilton operator \( \mathcal{E}_H \) for a Hamiltonian form \( H \) is defined to be the first order differential operator

\[
\mathcal{E}_H = \partial H - \hat{\Omega} = [(y_i^\lambda - \partial_i^\lambda H)dp_i^\lambda - (p_i^\lambda + \partial_i H)dy_i^\lambda] \wedge \omega
\]

where \( \hat{\Omega} \) is the pullback of the multisymplectic form \( \Omega \) onto \( J^1 \Pi \).

For any connection \( \gamma \) on \( \Pi \to X \), we have

\[
\mathcal{E}_H \circ \gamma = \partial H - \gamma \} \Omega.
\]

It follows that \( \gamma \) is a Hamiltonian connection for a Hamiltonian form \( H \) iff it takes its values into \( \text{Ker} \mathcal{E}_H \), that is, satisfies the algebraic Hamilton equations

\[
\gamma_i^\lambda = \partial_i^\lambda H, \quad \gamma^\lambda_{i\lambda} = -\partial_i H.
\]
Let a Hamiltonian connection has an integral section \( r \) of \( \Pi \to X \). Then, the Hamilton equations \((45)\) are brought into the first order differential Hamilton equations \((15a)\) and \((15b)\).

Now we consider relations between Lagrangian and Hamiltonian formalisms in fibred manifolds in case of degenerate Lagrangian densities.

**Remark.** The repeated jet manifold \( J^1J^1Y \), by definition, is the first order jet manifold of \( J^1Y \to X \). It is coordinatized by \((x^\lambda, y^i, y^i_\lambda, y^i_\mu)\). Its subbundle \( \hat{J}^2Y \) with \( y^i_\lambda = y^i_\lambda \) is called the sesquiholonomic jet manifold. The second order jet manifold \( J^2Y \) of \( Y \) is the subbundle of \( \hat{J}^2Y \) with \( y^i_\mu = y^i_\mu \). □

Let \( Y \to X \) be a fibred manifold and \( L = \mathcal{L}\omega \) a Lagrangian density on \( J^1Y \). One can construct the exterior form

\[
\Lambda_L = [y^i_\lambda - y^i_\lambda]d\pi^\lambda_i + (\partial_i - \hat{\partial}_\lambda\partial^\lambda_i)\mathcal{L}dy^i \wedge \omega,
\]

\[(46)\]

on the repeated jet manifold \( J^1J^1Y \). Its restriction to the second order jet manifold \( J^2Y \) of \( Y \) reproduces the familiar variational Euler-Lagrange operator

\[
\mathcal{E}_L = [\partial_i - (\partial_\lambda + y^i_\lambda\partial_i + y^i_\mu\partial^\mu_i)\partial^\lambda_i]\mathcal{L}dy^i \wedge \omega.
\]

\[(47)\]

The restriction of the form \((46)\) to the sesquiholonomic jet manifold \( \hat{J}^2Y \) defines the sesquiholonomic extension \( \mathcal{E}'_L \) of the Euler-Lagrange operator \((47)\). It is given by the expression \((47)\), but with nonsymmetric coordinates \( y^i_\mu \).

Let \( \bar{s} \) be a section of the fibred jet manifold \( J^1Y \to X \) such that its first order jet prolongation \( J^1\bar{s} \) takes its values into \( \text{Ker} \mathcal{E}'_L \). Then, \( \bar{s} \) satisfies the first order differential Euler-Lagrange equations

\[
\partial_\lambda \bar{s}^i = \bar{s}^i_\lambda,
\]

\[
\partial_i \mathcal{L} - (\partial_\lambda + \bar{s}^i_\lambda\partial_j + \partial_\lambda \bar{s}^i_\mu \partial^\mu_j)\partial^\lambda_i \mathcal{L} = 0.
\]

\[(48)\]

They are equivalent to the second order Euler-Lagrange equations

\[
\partial_i \mathcal{L} - (\partial_\lambda + \partial_\lambda s^i j \partial_j + \partial_\lambda \partial_\mu s^i j \partial^\mu_j)\partial^\lambda_i \mathcal{L} = 0.
\]

\[(49)\]

for sections \( s \) of \( Y \) where \( \bar{s} = J^1s \).

Given a Lagrangian density \( L \), the vertical tangent morphism of \( L \) yields the Legendre morphism \((11)\). We say that a Hamiltonian form \( H \) is associated with a Lagrangian density \( L \) if \( H \) satisfies the relations

\[
\hat{L} \circ \hat{H} |_Q = \text{Id}_Q, \quad Q = \hat{L}(J^1Y),
\]

\[
H = H \hat{H} + L \circ \hat{H},
\]

\[(50a)\] \(\)(50b)
or in the coordinate form

\[
\partial^\mu \mathcal{L}(x^\lambda, y^j, \partial^j \mathcal{H}) = p^\mu_i, \quad p^\mu_i \in Q,
\]

\[
\mathcal{L}(x^\lambda, y^j, \partial^j \mathcal{H}) = p^\mu_i \partial^i \mathcal{H} - \mathcal{H}.
\]

Note that different Hamiltonian forms can be associated with the same Lagrangian density.

Let us restrict our consideration to the semiregular Lagrangian densities \( \mathcal{L} \) when the preimage \( \hat{L}^{-1}(q) \) of each point of \( q \in Q \) is the connected submanifold of \( J^1Y \).

All Hamiltonian forms associated with a semiregular Lagrangian density \( \mathcal{L} \) consist with each other on the Lagrangian constraint space \( Q \), and the Hamilton operator \( \mathcal{E}_H \) satisfies the relation

\[
\Lambda_L = \mathcal{E}_H \circ J^1 \hat{L}.
\]

Let a section \( r \) of \( \Pi \rightarrow X \) be a solution of the Hamilton equations (15a) and (15b) for a Hamiltonian form \( H \) associated with a semiregular Lagrangian density \( L \). If \( r \) lives on the constraint space \( Q \), the section \( \mathfrak{s} = \widetilde{H} \circ r \) of \( J^1Y \rightarrow X \) satisfies the first order Euler-Lagrange equations (18). Conversely, given a semiregular Lagrangian density \( L \), let \( \mathfrak{s} \) be a solution of the first order Euler-Lagrange equations (18). Let \( H \) be a Hamiltonian form associated with \( L \) so that

\[
\widetilde{H} \circ \hat{L} \circ \mathfrak{s} = \mathfrak{s}.
\]  

(51)

Then, the section \( r = \hat{L} \circ \mathfrak{s} \) of \( \Pi \rightarrow X \) is a solution of the Hamilton equations (15a) and (15b) for \( H \). For sections \( \mathfrak{s} \) and \( r \), we have the relations

\[
\mathfrak{s} = J^1 s, \quad s = \pi_{HY} \circ r
\]

where \( s \) is a solution of the second order Euler-Lagrange equations (19).

We shall say that a family of Hamiltonian forms \( H \) associated with a semiregular Lagrangian density \( L \) is complete if, for each solution \( \mathfrak{s} \) of the first order Euler-Lagrange equations (18), there exists a solution \( r \) of the Hamilton equations (15a) and (15b) for some Hamiltonian form \( H \) from this family so that

\[
r = \hat{L} \circ \mathfrak{s}, \quad \mathfrak{s} = \widetilde{H} \circ r, \quad \mathfrak{s} = J^1(\pi_{HY} \circ r).
\]

Such a complete family exists iff, for each solution \( \mathfrak{s} \) of the Euler-Lagrange equations for \( L \), there exists a Hamiltonian form \( H \) from this family so that the condition (51) holds.

The most of field models possesses affine and quadratic Lagrangian densities. Complete families of Hamiltonian forms associated with such Lagrangian densities always exist [8, 9].

As a test case, let us examine the gauge theory of principal connections.

In the rest of this Section, the manifold \( X \) is assumed to be oriented world manifold provided with a nondegenerate fibre metric \( g_{\mu\nu} \) in the tangent bundle of \( X \). We denote \( g = \text{det}(g_{\mu\nu}) \).
Let $P \to X$ be a principal bundle with a structure Lie group $G$ which acts on $P$ on the right. There is the 1:1 correspondence between the principal connections $A$ on $P$ and the global sections of the bundle $C = J^1 P/G$. It is the affine bundle modelled on the vector bundle

$$\mathcal{C} = T^*X \otimes V^G P, \quad V^G P = VP/G.$$ 

Given a bundle atlas $\Psi^P$ of $P$, the bundle $C$ is provided with the fibred coordinates $(x^\mu, k^m_\mu)$ so that

$$(k^m_\mu \circ A)(x) = A^m_\mu(x)$$

are coefficients of the local connection 1-form of a principal connection $A$ with respect to the atlas $\Psi^P$. The first order jet manifold $J^1 C$ of the bundle $C$ is coordinatized by $(x^\mu, k^m_\mu, k^{m}_{\mu\lambda})$.

There exists the canonical splitting

$$J^1 C = C_+ \oplus C_- = (J^2 P/G) \oplus (\wedge^2 T^*X \otimes V^G P),$$

over $C$ with the corresponding surjections

$$S : J^1 C \to C_+, \quad S^m_{\lambda \mu} = k^m_{\mu \lambda} + k^m_{\lambda \mu} + c^m_{nl} k^l_n k^l_\mu,$$

$$F : J^1 C \to C_-, \quad F^m_{\lambda \mu} = k^m_{\mu \lambda} - k^m_{\lambda \mu} - c^m_{nl} k^l_n k^l_\mu.$$ 

The Legendre bundle over the bundle $C$ is

$$\Pi = \wedge^n T^*X \otimes TX \otimes [C \times \overline{\mathcal{C}}]^*.$$ 

It is coordinatized by $(x^\mu, k^m_\mu, p^m_{\mu \lambda})$.

On the configuration space (72), the conventional Yang-Mills Lagrangian density $L_{YM}$ is given by the expression

$$L_{YM} = \frac{1}{4e^2} a^{G}_{mn} g^{\lambda \mu} g^{\beta \tau} F^m_{\lambda \beta} F^n_{\mu \tau} \sqrt{|g|} \omega$$

where $a^G$ is a nondegenerate $G$-invariant metric in the Lie algebra of $G$. The corresponding Legendre morphism takes the form

$$p^{(\mu \lambda)}_{m} \circ \hat{L}_{YM} = 0,$$

$$p^{[\mu \lambda]}_{m} \circ \hat{L}_{YM} = e^{-2} a^{G}_{mn} g^{\lambda \alpha} g^{\mu \beta} F^n_{\alpha \beta} \sqrt{|g|}.$$ 

Let us consider connections on the bundle $C$ which take their values into $\text{Ker} \hat{L}_{YM}$:

$$S : C \to C_+, \quad S^m_{\lambda \mu} - S^m_{\mu \lambda} - c^m_{nl} k^l_n k^l_\mu = 0.$$
For all these connections, the Hamiltonian forms

\[ H = p^\mu k^m_\mu \omega - p^\mu S_{B^m_\mu \lambda} \omega - \widetilde{\mathcal{H}}_{YM} \omega, \]  

\[ \widetilde{\mathcal{H}}_{YM} = \frac{\varepsilon^2}{4} a_{G}^m g_{\mu \nu} g_{\lambda \beta} f[m^\lambda \mu_{\nu \beta}] - g^{-1/2}, \]  

are associated with the Lagrangian density \( L_{YM} \) and constitute the complete family. Moreover, we can minimize this complete family if we restrict ourselves to connections (55) of the following type. Given a symmetric linear connection \( K \) on the cotangent bundle \( T^*X \) of \( X \), every principal connection \( B \) on \( P \) gives rise to the connection \( S_B \) (55) such that

\[ S_B \circ B = S \circ J^1 B, \]

\[ S_B^m_{\mu \lambda} = \frac{1}{2} [c^m_{lm} k^l_{\mu \lambda} + \partial_\mu B^m_{\lambda} + \partial_\lambda B^m_{\mu} - \varepsilon^m_{nm} (k^\mu_{\nu} B^\lambda_{\nu} + k^\nu_{\lambda} B^\mu_{\nu})] - K^\beta_{\mu \lambda}(B^m_{\beta} - k^m_{\beta}). \]

The corresponding Hamilton equations for sections \( r \) of \( \Pi \to X \) read

\[ \partial_\lambda P^\mu_{m} = -c^m_{lm} k^l_{\mu \nu} P^\nu_{n} + c^m_{nm} B^{l}_{\nu} P^\mu_{l} - K^\mu_{\lambda \nu} P^\nu_{m}, \]  

\[ \partial_\lambda k^m_{\mu} + \partial_\mu k^m_{\lambda} = 2S_{B^m_{(\mu \lambda)}} \]

plus the equation (54b). When restricted to the constraint space (54a), the equations (54b) and (57) are the familiar Yang-Mills equations for \( A = \pi_{\Pi C} \circ r \). Different Hamiltonian forms (56) lead to different equations (58) which play the role of the gauge-type conditions.

### 6 Hamiltonian systems on composite manifolds

The major feature of Hamiltonian systems on a composite manifold \( Y \) (24) lies in the following. The horizontal splitting (29) yields immediately the corresponding splitting of the Legendre bundle \( \Pi \) over the composite manifold \( Y \). As a consequence, the Hamilton equations (15a) for sections \( h \) of the fibred manifold \( \Sigma \) reduce to the gauge-type conditions independent of momenta. Thereby, these sections play the role of parameter fields. Their momenta meet the constraint conditions (16).

Let \( Y \) be a composite manifold (24). The Legendre bundle \( \Pi \) over \( Y \) is coordinatized by

\[ (x^\lambda, \sigma^m, y^i, p^\lambda_m, p^\mu_i). \]

Let \( A_\Sigma \) be a connection (7) on the bundle \( Y_\Sigma \). With a connection \( A_\Sigma \), the splitting

\[ \Pi = \wedge^n T^*Y \otimes T Y \otimes [V^*Y_\Sigma \oplus (Y_\Sigma \times V^*Y)] \]  

(59)
of the Legendre bundle $\Pi$ is performed as an immediate consequence of the splitting (29). Given the splitting (59), the Legendre bundle $\Pi$ can be provided with the corresponding coordinates
\[
\bar{p}_i^\lambda = p_i^\lambda, \quad \bar{p}_m^\lambda = p_m^\lambda + A_i^m p_i^\lambda. \tag{60}
\]
Let $h$ be a global section of the fibred manifold $\Sigma$. It is readily observed that, given the splitting (59), the submanifold
\[
\{ \sigma = h(x), \bar{p}_m^\lambda = 0 \} \tag{61}
\]
of the Legendre bundle $\Pi$ over $Y$ is isomorphic to the Legendre bundle $\Pi_h$ over the restriction $Y_h$ of $Y_\Sigma$ to $h(X)$.

Let the composite manifold $Y$ be provided with the composite connection (26) determined by connections $A_\Sigma$ on $Y_\Sigma$ and $\Gamma$ on $\Sigma$. Relative to the coordinates (60), every Hamiltonian form on the Legendre bundle $\Pi$ over $Y$ can be given by the expression
\[
H = (p_i^\lambda dy^i + p_m^\lambda d\sigma^m) \wedge \omega_\lambda - [\bar{p}_i^\lambda \bar{A}_i^\lambda + \bar{p}_m^\lambda \Gamma_m^\lambda + \bar{H}(x^\mu, \sigma^m, y^i, p_i^\mu, \bar{p}_m^\mu)] \omega. \tag{62}
\]
The corresponding Hamilton equations are written
\[
\begin{align*}
\partial_\lambda p_i^\lambda &= -p_i^\lambda [\partial_i (\bar{A}_i^\lambda + \partial_i A_i^\lambda (\Gamma_m^\lambda + \partial_m \bar{H}))] - \partial_i \bar{H}, \tag{63a} \\
\partial_\lambda y^i &= \bar{A}_i^\lambda + A_i^\lambda (\Gamma_m^\lambda + \partial_m \bar{H}) + \partial^\lambda \bar{H}, \tag{63b} \\
\partial_\lambda p_m^\lambda &= -p_i^\lambda \partial_m (\bar{A}_i^\lambda + \partial_m A_i^\lambda (\Gamma_m^\lambda + \partial_m \bar{H})) - \bar{p}_m^\lambda \partial_m \Gamma_m^\lambda - \partial_m \bar{H}, \tag{63c} \\
\partial_\lambda \sigma^m &= \Gamma_m^\lambda \tag{63d}
\end{align*}
\]
plus constraint conditions.

In particular, let the Hamiltonian form (62) be associated with a Lagrangian density (9) which contains the velocities $\sigma_i^m$ only in the vertical covariant differential (8). Then, the Hamiltonian density $\bar{H} \omega$ appears independent of the momenta $\bar{p}_m^\mu$ and the Lagrangian constraints read
\[
\bar{p}_m^\mu = 0. \tag{64}
\]
In this case, the Hamilton equation (63d) comes to the gauge-type condition
\[
\partial_\lambda \sigma^m = \Gamma_m^\lambda
\]
independent of momenta.

In particular, let us consider a Hamiltonian system in the presence of a background parameter field $h(x)$. After substituting the equation (63c) into the equations (63a) - (63d) and restricting them to the submanifold (61), we obtain the equations
\[
\begin{align*}
\partial_\lambda p_i^\lambda &= -p_i^\lambda \partial_i [(\bar{A} \circ h)_i^\lambda + A_i^\mu \partial_\lambda h^m] - \partial_i \bar{H}, \\
\partial_\lambda y^i &= (\bar{A} \circ h)_i^\lambda + A_i^\mu \partial_\lambda h^m + \partial^\lambda \bar{H} \tag{65}
\end{align*}
\]
for sections of the fibred Legendre manifold $\Pi_h \to X$ of the bundle $Y_h$ endowed with the connection (27). Equations (65) are the Hamilton equations corresponding to the Hamiltonian form

$$H_h = p_i^\lambda dy^i \land \omega_\lambda - [p_i^\lambda A_{h,i} + \widehat{\mathcal{H}}(x^\mu, h^m(x), y^j, p_i^\mu, \eta^m = 0)]\omega$$

on $\Pi_h$ which is induced by the Hamiltonian form (62) on $\Pi$.

7 Dynamics of the fermion-gravitation complex

At first, let us consider Dirac fermion fields in the presence of a background tetrad field $h$. Recall that they are represented by global sections of the spinor bundle $S_h$ (20). Their Lagrangian density is defined on the configuration space $J^1(S_h \oplus S_h^*)$ provided with the adapted coordinates $(x^\mu, y^A, y^+_A, y^A_\mu, y^+_{A\mu})$. It is the affine Lagrangian density

$$L_D = \left\{ \frac{i}{2} [y^+_A (\gamma^0 \gamma^\mu) A_B (y^B_\mu - A^B C_\mu y^C) - (y^+_A - A^+ C A_\mu y^C)(\gamma^0 \gamma^\mu) A_B y^B] \ight.$$

$$-my^+_A (\gamma^0) A_B y^B \right\} h^{-1} \omega, \quad \gamma^\mu = h^a(x) \gamma^a, \quad h = \text{det}(h^a_d),$$

(66)

where

$$A^A_{B\mu} = \frac{1}{2} A^{ab}_{\mu}(x) I_{ab}^A$$

is a principal connection on the principal spinor bundle $P_h$.

The Legendre bundle $\Pi_h$ over the spinor bundle $S_h \oplus S_h^*$ is coordinatized by

$$(x^\mu, y^A, y^+_A, p^\mu_A, p^\mu_{-A}).$$

Relative to these coordinates, the Legendre morphism associated with the Lagrangian density (66) is written

$$p^\mu_A = \pi^\mu_A = \frac{i}{2} y^+_B (\gamma^0 \gamma^\mu) A_B h^{-1},$$

$$p^\mu_{-A} = \pi^\mu_{-A} = -\frac{i}{2} (\gamma^0 \gamma^\mu) A_B y^B h^{-1}.\quad (67)$$

It defines the constraint subspace of the Legendre bundle $\Pi_h$. Given a soldering form

$$S = S^A_{B\mu}(x)y^B dx^\mu \otimes \partial_A$$

on the bundle $S_h$, let us consider the connection $A + S$ on $S_h$. The corresponding Hamiltonian form associated with the Lagrangian density (67) reads

$$H_S = \left( p^\mu_A dy^A + p^\mu_{-A} dy^+_A \right) \land \omega_\mu - \mathfrak{H}_S \omega,$$

$$\mathfrak{H}_S = p^\mu_A A^A_{B\mu} y^B + y^+_B A^+_{B\mu} p^\mu_{-A} + my^+_A (\gamma^0) A_B y^B h^{-1} +$$

$$\left( p^\mu_A - \pi^\mu_A \right) S^A_{B\mu} y^B + y^+_B S^+_{A\mu} (p^\mu_{-A} - \pi^\mu_{-A}).$$

(68)
The corresponding Hamilton equations for a section $r$ of the fibred Legendre manifold $\Pi_h \to X$ take the form

\[
\begin{align*}
\partial_\mu y^\lambda_A &= y^\mu_B \left( A^+B_{A\mu} + S^+B_{A\mu} \right), \\
\partial_\mu p^\gamma_A &= -p^\mu_B A^B_A - \left( p^\mu_B - \pi^\mu_B \right) S^B_A - m y^\gamma_B (\gamma^0)^B A + \frac{i}{2} y^\gamma_B S^+B_{C\mu} (\gamma^0 \gamma^\mu)^C A h^{-1}.
\end{align*}
\]

plus the equations for the components $y^A$ and $p^\mu_A$. The equation (69a) and the similar equation for $y^A$ imply that $y$ is an integral section of the connection $A + S$ on the spinor bundle $S_h$. It follows that the Hamiltonian forms (68) constitute the complete family. On the constraint space (67), the equation (69b) comes to

\[
\partial_\mu \pi^\mu_A = -\pi^\mu_B A^B_A - m y^\gamma_B (\gamma^0)^B A + \frac{i}{2} y^\gamma_B S^+B_{C\mu} (\gamma^0 \gamma^\mu)^C A h^{-1}.
\]

Substituting the equation (69a) into the equation (70), we obtain the familiar Dirac equation for fermion fields in the presence of a tetrad gravitational field $h$.

We now consider gravity without matter.

In the gauge gravitation theory, dynamic gravitational variables are pairs of tetrad gravitational fields $h$ and gauge gravitational potentials $A_h$ identified with principal connections on $P_h$. Following the general procedure, one can describe these pairs $(h, A_h)$ by sections of the composite bundle

\[
C_L := J^1LX/L \to J^1\Sigma \to \Sigma \to X^4.
\]

The corresponding configuration space is the jet manifold $J^1C_L$ of $C_L$. The Legendre bundle

\[
\Pi = \wedge^4 T^*X^4 \otimes T^*X^4 \otimes V^*C_L
\]

over $C_L$ plays the role of a phase space of the gauge gravitation theory.

The bundle $C_L$ is endowed with the local fibred coordinates

\[
(x^\mu, \sigma^\lambda_a, k^{ab}_\lambda = -k_{ba}^\mu, \sigma_{a\mu}^\lambda)
\]

where $(x^\mu, \sigma^\lambda_a, \sigma_{a\mu}^\lambda)$ are coordinates of the jet bundle $J^1\Sigma$. The jet manifold $J^1C_L$ of $C_L$ is provided with the corresponding adapted coordinates

\[
(x^\mu, \sigma^\mu_a, k^{ab}_\lambda = -k_{ba}^\mu, \sigma^\mu_{a\lambda} = \sigma^\mu_{a(\lambda)}, k^{ab}_{\mu\lambda}, \sigma^\mu_{a\lambda\nu}).
\]

The associated coordinates of the Legendre manifold (71) are

\[
(x^\mu, \sigma^\lambda, k^{ab}_\lambda, \sigma_{a\mu}^\gamma, P_{a\lambda}^\mu, P_{a\mu}^\gamma, P_{\lambda}^{a\mu\nu}).
\]

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where \((x^\mu, \sigma^\lambda_a, p^{a\mu}_\lambda)\) are coordinates of the Legendre manifold of the bundle \(\Sigma\).

For the sake of simplicity, we here consider the Hilbert-Einstein Lagrangian density of classical gravity

\[
L_{\text{HE}} = -\frac{1}{2\kappa} F^{ab}_{\mu\lambda} \sigma^\mu_a \sigma^\lambda_b \sigma^{-1} \omega, \quad (72)
\]

\[F^{ab}_{\mu\lambda} = k^{ab}_{\mu\lambda} - k^{ab}_{\mu\lambda} + k^{a}_{c\mu} k^{cb}_{\lambda} - k^{a}_{c\lambda} k^{cb}_{\mu}, \]

\[\sigma = \text{det}(\sigma^\mu_a).\]

The corresponding Legendre morphism \(\hat{L}_{\text{HE}}\) is given by the coordinate expressions

\[
p^{ab}_{\lambda\mu} = \pi^{ab}_{\lambda\mu} = -\frac{1}{\kappa \sigma} \sigma^{\mu}_{a} \sigma^{\lambda}_{b}. \quad (73a)
\]

\[
p^{ab}_{\lambda\mu} = 0, \quad p^{a\mu}_{\lambda} = 0, \quad p^{a\nu\mu}_{\lambda} = 0, \quad (73b)
\]

We construct the complete family of Hamiltonian forms associated with the affine Lagrangian density \((72)\). Let \(K\) be a world connection associated with a principal connection \(B\) on the linear frame bundle \(LX\). To minimize the complete family, we consider the following connections on the bundle \(C_K\):

\[
\Gamma^\lambda_{\alpha\mu} = B^b_{ab\alpha} \sigma^\lambda_b - K^\lambda_{\nu\mu} \sigma^\nu_a,
\]

\[
\Gamma^\lambda_{a\nu\mu} = \partial_{\mu} B^d_{ab\nu} \sigma^\lambda_d - \partial_{\mu} K^\lambda_{\beta\nu} \sigma^\beta_a + B^d_{ab\nu} (\sigma^\gamma_d - \Gamma^\gamma_{ab\nu}) + K^\gamma_{a\nu\mu} (\sigma^\lambda_d - \Gamma^\lambda_{a\nu\mu})
\]

\[
\Gamma_{\lambda\mu} = \frac{1}{2} [k^{a}_{c\lambda\mu} k^{cb}_{\lambda} - k^{a}_{c\mu} k^{cb}_{\lambda} + \partial_{\lambda} B^{ab}_{\mu} + \partial_{\mu} B^{ab}_{\lambda}]
\]

\[
- B^b_{c\mu} k^{ac}_{\lambda} - B^b_{c\lambda} k^{ac}_{\mu} - B^a_{c\mu} k^{cb}_{\lambda} - B^a_{c\lambda} k^{cb}_{\mu} + K^{\nu}_{\mu\lambda} k^{ab\nu} - K^{\nu}_{(\mu\lambda)} B^{ab\nu} - \frac{1}{2} R^{ab}_{\lambda\mu},
\]

where \(R\) is the curvature of the connection \(B\).

The complete family of Hamiltonian forms associated with the Lagrangian density \((72)\) consists of the forms given by the coordinate expressions

\[
H_{\text{HE}} = (p^{\lambda\mu}_{ab} d k^{ab}_{\lambda\mu} + p^{a\mu}_{\lambda} d \sigma^{\lambda}_{a} + p^{a\nu\mu}_{\lambda} d \sigma^{\lambda}_{a} \wedge \omega_{\mu} - \mathcal{H}_{\text{HE}} \omega,
\]

\[
\mathcal{H}_{\text{HE}} = (p^{\lambda\mu}_{ab} \Gamma^{\lambda\mu}_{ab} + p^{a\mu}_{\lambda} \Gamma^{\lambda}_{a\mu} + p^{a\nu\mu}_{\lambda} \Gamma^{\lambda}_{a\nu\mu}) + \frac{1}{2} R^{ab}_{\lambda\mu} (p^{\lambda\mu}_{ab} - \pi^{\lambda\mu}_{ab}).
\]

The Hamilton equations corresponding to such a Hamiltonian form read

\[
F^{ab}_{\mu\lambda} = R^{ab}_{\mu\lambda}, \quad (74a)
\]

\[
\partial_{\mu} k^{ab}_{\lambda} + \partial_{\lambda} k^{ab}_{\mu} = 2 \Gamma^{ab}_{(\mu\lambda)}, \quad (74b)
\]
\[ \partial_\mu \sigma^\lambda_a = \Gamma^\lambda_{a\mu}; \]  
\[ \partial_\mu \sigma^\lambda_{a\nu} = \Gamma^\lambda_{a\mu
u}; \]  
\[ \partial_\mu p^\lambda_{ac} = -\frac{\partial \mathcal{H}_{HE}}{\partial \sigma^\lambda_{ac}}; \]  
\[ \partial_\mu p^\mu_{a\nu} = -\frac{\partial \mathcal{H}_{HE}}{\partial \sigma^\lambda_{a\nu}}; \] 

plus the equations which are reduced to the trivial identities on the constraint space (73a). The equations (74a) - (74d) make the sense of gauge-type conditions. The equation (74d) has the solution 

\[ \sigma^\lambda_{a\mu} = \partial_\nu \sigma^\lambda_a. \]

The gauge-type condition (74b) has the solution 

\[ k = B(x). \]

On the constraint space, the equations (74e) and (74f) are brought into the form 

\[ \partial_\mu \pi^\lambda_{ac} = 2k^b_{c\mu} \pi^\lambda_{ab} + \pi^\beta \gamma \Gamma^\lambda_{\beta\gamma}, \]  
\[ R^{\beta\mu}_{\mu} \pi^\alpha_{ab} \gamma^\beta \lambda = 0. \] 

The equation (75a) shows that \( k(x) \) is the Levi-Civita connection for the tetrad field \( h(x) \). Substitution of the equations (74a) into the equations (75b) leads to the familiar Einstein equations.

Turn now to the fermion matter. Given the \( L_s \)-principal lift \( P_\Sigma \) of \( LX_\Sigma \), let us consider the composite spinor bundle \( S \) (2) where the bundle \( S_\Sigma \) is provided with the connection (39).

The total configuration space of the fermion-gravitation complex is the product

\[ J^1S \times J^1C_L. \] 

On this configuration space, the Lagrangian density \( L_{FG} \) of the fermion-gravitation complex is the sum of the Hilbert-Einstein Lagrangian density \( L_{HE} \) (2) and the modification \( L_{D} \) (1) of the Lagrangian density (66) of fermion fields:

\[ L_D = \{ \frac{i}{2} [\bar{y}_{A}^\lambda (\gamma^0 \gamma^\mu)^A_B (y_{B}^\mu - \frac{1}{2} (k^a_{\mu - A}^{abc} (\sigma_{c\mu} - \Gamma_{c\mu}) I_{ab}^{C} C_{\mu} y_{C}^\nu) - (y_{A\mu}^\lambda - \frac{1}{2} (k^{a\mu}_{\mu - A}^{abc} (\sigma_{c\mu} - \Gamma_{c\mu}) I_{ab}^{C} A_{\mu} y_{C}^\nu) (\gamma^0 \gamma^\mu)^A_B y_{B}^\nu - m y_{A}^\lambda (\gamma^0)^A_B y_{B}^\nu \} \} \}

where

\[ \Gamma^\nu_{c\mu} = \frac{1}{2} k^{mn}_{\mu} (\eta_{cn} \delta^d_m - \eta_{cm} \delta^d_n) - K^\nu_{c\mu} \sigma^\lambda_c, \]

\[ \gamma^\mu = \sigma^\mu_a \gamma^a, \quad \sigma = \det(\sigma^\mu_a). \]
The total phase space $\Pi$ of the fermion-gravitation complex is coordinatized by

$$(x^\lambda, \sigma^\mu_c, \sigma^{\mu c}_c, y^A, y^+_A, k^{ab}_{\mu}, p^{c\lambda}_{\mu}, p^{c\mu}_{\lambda}, p_A^\lambda, p_+^\mu, p_{ab}^{\mu \lambda})$$

and admits the corresponding splitting (59). The Legendre morphism associated with the Lagrangian density $L_{FG}$ defines the constraint subspace of $\Pi$ given by the relations (67), (73a), the conditions

$$p_{ab}^{(\lambda \mu)} = 0, \quad p_c^{\lambda \mu} = 0$$

and the constraint (64) which takes the form (17).

Hamiltonian forms associated with the Lagrangian density $L_{FG}$ are the sum of the Hamiltonian forms $H_{HE}$ and $H_S$ (68) where

$$A^A_{B\mu} = \frac{1}{2} k^{ab}_{\mu} I_{ab} A^B y^B. \quad (76)$$

The corresponding Hamilton equations for spinor fields consist with the equations (69a) and (69b) where $A$ is given by the expression (76). The Hamilton equations (74a) - (74d) remain true. The Hamilton equations (74e) and (74f) contain additional matter sources. On the constraint space

$$p_{\lambda}^{a \mu} = 0$$

the modified equations (74f) would come to the familiar Einstein equations

$$G_{\mu}^a + T_{\mu}^a = 0$$

where $T$ denotes the energy-momentum tensor of fermion fields, otherwise on the modified constraint space (17). In the latter case, we have

$$D_{A\mu} c^{\lambda} = G_{\mu}^a + T_{\mu}^a \quad (77)$$

where $D_{\lambda}$ denotes the covariant derivative with respect to the Levi-Civita connection which acts on the indices $\mu$. Substitution of (17) into (77) leads to the manifested modification of the Einstein equations for the total system of fermion fields and gravity:

$$-\frac{1}{2} J_{ab}^\lambda D_{\lambda} A^{abc}_{\mu} = G_{\mu}^c + T_{\mu}^c$$

where $J$ is the spin current of the fermion fields.
References

1. Sardanashvily G 1991 *Int. J. Theor. Phys.* **30** 721
2. Sardanashvily G and Zakharov O 1992 *Gauge Gravitation Theory* (Singapore: World Scientific)
3. Ne’eman Y and Šijački Dj 1979 *Ann. of Phys.* **120** 292
4. Sardanashvily G 1992 *J. Math. Phys.* **33** 1546
5. M. Gotay M 1991 in *Mechanics, Analysis and Geometry: 200 Years after Lagrange* (Amsterdam: North Holland)
6. Kolar I, Michor P W and Slovák J 1993 *Natural operations in differential geometry*, (Berlin: Springer-Verlag)
7. de Leon M, Marin-Solano J and Marrero J 1994 in *Proc. Colloquium on Diff. Geom.* (Debrecen, Hungary)
8. Sardanashvily G 1993 *Gauge Theory in Jet Manifolds* (Palm Harbor: Hadronic Press)
9. Sardanashvily G 1994 *J. Math. Phys.* **35** 6584
10. Sardanashvily G 1994 *Five lectures on the jet manifold methods in field theory* E-print: hep-th/9411089.
11. Cariñena J, Crampin M and Ibort L 1991 *Diff. Geom. Appl.* **1** 345
12. Günther C 1987 *J. Diff. Geom.* **25** 23
13. Sardanashvily G and Zakharov O 1993 *Diff. Geom. Appl.* **3** 245
14. Saunders D 1989 *The Geometry of Jet Bundles*, (Cambridge: Cambridge Univ. Press)