A HEREDITARILY INDECOMPOSABLE $\mathcal{L}_\infty$-SPACE THAT SOLVES THE SCALAR–PLUS–COMPACT PROBLEM

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Abstract. We construct a hereditarily indecomposable Banach space with dual space isomorphic to $\ell_1$. Every bounded linear operator on this space is expressible as $\lambda I + K$ with $\lambda$ a scalar and $K$ compact.
1. Introduction

The question of whether there exists a Banach space $X$ on which every bounded linear operator is a compact perturbation of a scalar multiple of the identity has become known as the “Scalar–plus–Compact Problem”. It is mentioned by Lindenstrauss as Question 1 in his 1976 list of open problems in Banach space theory [25]. Lindenstrauss remarks that, by the main theorem of [10] or [26], every operator on a space of this type has a proper non-trivial invariant subspace. Related questions go further back: for instance, Thorp [34] asks whether the space of compact operators $\mathcal{K}(X;Y)$ can ever be a proper complemented subspace of $\mathcal{L}(X;Y)$. On the Gowers–Maurey space $\mathcal{X}_{gm}$ [21], every operator is a strictly singular perturbation of a scalar, and other hereditarily indecomposable (HI) spaces also have this property. Indeed it seemed for a time that $\mathcal{X}_{gm}$ might already solve the scalar–plus–compact problem. However, after Gowers [20] had shown that there is a strictly singular, non-compact operator from a subspace of $\mathcal{X}_{gm}$ to $\mathcal{X}_{gm}$, Androulakis and Schlumprecht [4] showed that such an operator can be defined on the whole of $\mathcal{X}_{gm}$. Gasparis [18] has done the same for the Argyros–Deliyannis space $\mathcal{X}_{ad}$ of [5].

In the present paper, we solve the scalar–plus–compact problem by combining techniques that are familiar from other HI constructions with an additional ingredient, the Bourgain–Delbaen method for constructing special $\mathcal{L}_\infty$-spaces [12]. The initial motivation for combining these two constructions was to exhibit a hereditarily indecomposable predual of $\ell_1$; such a space is, in some sense, the extreme example of a known phenomenon—that the HI property does not pass from a space to its dual [17, 9, 6]. Serendipitously, it turned out that the additional structure was just what we needed to show that strictly singular operators are compact. It is interesting, perhaps, to note that the Schur property of $\ell_1$ does not play a role in our proof and, indeed, we have no general result to say that an HI predual of $\ell_1$ necessarily has the scalar–plus–compact property. We use in an essential way the specific structure of the BD construction, which embeds into our space some very explicit finite-dimensional $\ell_\infty$-spaces. As well as the (now) classical machinery of HI constructions—a space of Schlumprecht type, Maurey–Rosenthal coding and rapidly increasing sequences based on $\ell_1$-averages—we add the possibility of splitting an arbitrary vector into pieces of comparable norm, while staying in one of these $\ell_\infty$’s. This allows us to introduce two additional classes of rapidly increasing sequences, and these in turn lead to the stronger result about operators.

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2. Background

2.1. Notations. We use standard notations: if \( A \) is any set, \( \ell_\infty(A) \) is the space of all bounded (real-valued) functions on \( A \), equipped with the supremum norm \( \| \cdot \|_\infty \) and \( \ell_1(A) \) is the space of all absolutely summable functions on \( A \), equipped with the norm \( \| x \|_1 = \sum_{a \in A} |x(a)| \). The support of a function \( x \) is the set of all \( a \) such that \( x(a) \neq 0 \); \( c_{00}(A) \) is the space of functions of finite support. We shall write \( \ell_p \) for the space \( \ell_p(\mathbb{N}) \), where \( \mathbb{N} \) is the set \( \{1, 2, 3, \ldots\} \) of positive integers, and \( \ell_n^p \) for \( \ell_p(\{1, 2, \ldots, n\}) \). Even when we are dealing with these sequence spaces we shall use function notation \( x(m) \), rather than subscript notation, for the \( m^{th} \) coordinate of the vector \( x \).

When \( x \) and \( y \) are in \( c_{00}(A) \) (and more generally) we shall write \( \langle y, x \rangle \) for \( \sum_{a \in A} x(a)y(a) \). If we are thinking of \( y \) as a functional acting on \( x \) (rather than vice versa) we shall usually choose a notation involving a star, denoting \( y \) by \( f^* \), or something of this kind. In particular, \( e_a \) and \( e^*_a \) are two notations for the same unit vector in \( c_{00}(A) \) (given by \( e_a(a') = \delta_{a,a'} \)), to be employed depending on whether we are thinking of it as a unit vector or as the evaluation functional \( x \mapsto \langle e^*_a, x \rangle = x(a) \). We apologize to those readers who may find this kind of notation somewhat babyish.

We say that (finitely or infinitely many) vectors \( z_1, z_2, \ldots \) in \( c_{00} \) are successive, or that \( (z_i) \) is a block-sequence, if \( \max \text{supp } z_i < \min \text{supp } x_{i+1} \) for all \( i \). In a Banach space \( X \) we say that vectors \( y_j \) are successive linear combinations, or that \( (y_j) \) is a block sequence of a basic sequence \( (x_i) \) if there exist \( 0 = q_1 < q_2 < \cdots \) such that, for all \( j \geq 1 \), \( y_j \) is in the linear span \( [x_i : q_{j-1} < i \leq q_j] \). If we may arrange that \( y_j \in [x_i : q_{j-1} < i < q_j] \) we say that \( (y_j) \) is a skipped block sequence. More generally, if \( X \) has a Schauder decomposition \( X = \bigoplus_{n \in \mathbb{N}} F_n \) we say that \( (y_j) \) is a block sequence (resp. a skipped block sequence) with respect to \( (F_n) \) if there exist \( 0 = q_0 < q_1 < \cdots \) such that \( y_j \) is in \( \bigoplus_{q_{j-1} < n \leq q_j} F_n \) (resp. \( \bigoplus_{q_{j-1} < n < q_j} F_n \)). A block subspace is the closed subspace generated by a block sequence.

2.2. Hereditary indecomposability. A Banach space \( X \) is indecomposable if there do not exist infinite-dimensional closed subspaces \( Y \) and \( Z \) of \( X \) with \( X = Y \oplus Z \), and is hereditarily indecomposable (HI) if every closed subspace is indecomposable. The following useful criterion, like so much else in this in this area, goes back to the original paper of Gowers and Maurey [21].

Proposition 2.1. Let \( X \) be a an infinite dimensional Banach space. Then \( X \) is HI if and only if, for every pair \( Y, Z \) of infinite-dimensional subspaces, and every \( \epsilon > 0 \), there exist \( y \in Y \) and \( z \in Z \) with \( \|y + z\| > 1 \) and \( \|y - z\| < \epsilon \). If \( X \) has a finite-dimensional decomposition \( (F_n)_{n \in \mathbb{N}} \) it is enough that the above should hold for block subspaces.

We shall make use of the following well-known blocking lemma, the first part of which can be found as Lemma 1 of [27]. The proof of the second part is very similar, and, as Maurey remarks, both can be traced back to R.C. James [23].

Lemma 2.2. Let \( n \geq 2 \) be an integer, let \( \epsilon \in (0, 1) \) be a real number and let \( N \) be an integer that can be written as \( N = n^k \) for some \( k \geq 1 \). Let \( (x_i)_{i=1}^N \) be a sequence of vectors in the unit sphere of a Banach space \( X \).

(i) If \( \| \sum_{i=1}^N \pm x_i \| \geq (n - \epsilon)^k \) for all choices of signs \( \pm 1 \), then there is a block sequence \( y_1, y_2, \ldots, y_n \in [x_i : 1 \leq i \leq N] \) which is \( (1 - \epsilon)^{-1} \)-equivalent to the unit-vector basis of \( \ell_1^n \).

(ii) If \( \| \sum_{i=1}^N \pm x_i \| \leq (1 + \epsilon)^k \) for all choices of signs \( \pm 1 \), then there is a block sequence \( y_1, y_2, \ldots, y_n \in [x_i : 1 \leq i \leq N] \) which is \( (1 + \epsilon)^{-1} \)-equivalent to the unit-vector basis of \( \ell_\infty^n \).

2.3. \( L_\infty \)-spaces. A separable Banach space \( X \) is an \( L_\infty, \lambda \)-space if there is an increasing sequence \( (F_n)_{n \in \mathbb{N}} \) of finite dimensional subspaces of \( X \) such that the union \( \bigcup_{n \in \mathbb{N}} F_n \) is dense in \( X \) and, for each \( n \), \( F_n \) is \( \lambda \)-isomorphic to \( \ell_\text{dim}^\infty F_n \). It is known [24] that if a separable \( L_\infty \) space
X has no subspace isomorphic to ℓ₁, then the dual space X* is necessarily isomorphic to ℓ₁. Of course, this implies that the dual of a separable, hereditarily indecomposable L∞-space is isomorphic to ℓ₁.

The Bourgain–Delbaen spaces X_{a,b}, which inspired the construction given in this paper, were the first examples of L∞ spaces not containing c₀.

2.4. Mixed Tsirelson spaces. All existing HI constructions have, somewhere at the heart of them, a space of Schlumprecht type; rather than working with the original space of [33], we find it convenient to look at a different mixed Tsirelson space. We recall some notation and terminology from [8]. Let (l_j) be a sequence of positive integers and let (θ_j) be a sequence of real numbers with 0 < θ_j < 1. We define W[(\mathcal{A}_j, θ_j)_j] to be the smallest subset W of c₀₀ with the following properties

1. ±e_k^* ∈ W for all k ∈ N;
2. whenever f_1^*, f_2^*, ..., f_m^* ∈ W are successive vectors, \( θ_j \sum_{i=1}^{m} f_i^* ∈ W \), provided m ≤ l_j.

We say that an element f* of W is of Type 0 if f* = ±e_k^* for some k and of Type I otherwise; an element of type I is said to have weight θ_j if f* = θ_j \( \sum_{i=1}^{m} f_i^* \) for a suitable sequence (f_i) of successive elements of W.

The mixed Tsirelson space T[(\mathcal{A}_j, θ_j)_j] is defined to be the completion of c₀₀ with respect to the norm

\[ \|x\| = \sup\{⟨f^*, x⟩ : f^* ∈ W[\mathcal{A}_j, θ_j)_j] \}. \]

We may also characterize the norm of this space implicitly as being the smallest function x ↦ ∥x∥ satisfying

\[ \|x\| = \max \left\{ \|x\|_∞, \sup \theta_j \sum_{i=1}^{l_j} \|x_{E_i}\| \right\}, \]

where the supremum is taken over all j and all sequences of finite subsets E_1 < E_2 < ... < E_l_j. Schlumprecht’s original space is the result of taking l_j = j and θ_j = (log_2(j + 1))⁻¹.

In the rest of the paper we shall choose to work with two sequences of natural numbers (m_j) and (n_j). We require m_j to grow quite fast, and n_j to grow even faster. The precise requirements are as follows.

Assumption 2.3. We assume that (m_j, n_j)_{j ∈ N} satisfy the following:

1. m_1 ≥ 4;
2. m_{j+1} ≥ m_j^2;
3. n_{j+1} ≥ m_{j+1}(4n_j)^{2j+1}.

A straightforward way to achieve this is to assume that (m_j, n_j) is some subsequence of the sequence (2^{2j}, 2^{2j+1})_{j ∈ N}. From now on, whenever m_j and n_j appear, we shall assume we are dealing with sequences satisfying 2.3.

The following lemma can be found as II.9 of [8]. The proof is not affected by the small change we have made in the definition of the sequences (n_j)_j and (m_j)_j.

Lemma 2.4. If j_0 ∈ N and f ∈ W[(\mathcal{A}_{4n_j}, m_j^{-1})_j] is an element of weight m_h, then

\[ \left| ⟨f^*, n_{j_0}^{-1} \sum_{i=1}^{n_{j_0}} e_l⟩ \right| ≤ \begin{cases} 2m_h^{-1}m_{j_0}^{-1} & \text{if } i < j_0, \\ m_h^{-1} & \text{if } i ≥ j_0. \end{cases} \]

In particular, the norm of n_{j_0}^{-1} \sum_{i=1}^{n_{j_0}} e_l in T[(\mathcal{A}_{4n_j}, m_j^{-1})_j] is exactly m_{j_0}^{-1}. 
If we restrict attention to \( f \in W[(\mathcal{A}_{4n_j}, m_j^{-1})_{j \neq j_0}] \) then

\[
\left| \langle f^*, n^{-1}_{j_0} \sum_{l=1}^{n_{j_0}} e_l \rangle \right| \leq \begin{cases} 
2m_h^{-1}m_j^{-2} & \text{if } i < j_0 \\
m_h^{-1} & \text{if } i > j_0.
\end{cases}
\]

In particular, the norm of \( n^{-1}_{j_0} \sum_{l=1}^{n_{j_0}} e_l \) in \( T[(\mathcal{A}_{4n_j}, m_j^{-1})_{j \neq j_0}] \) is at most \( m_{j_0}^{-2} \).
3. The general Bourgain–Delbaen construction

In this section we shall present a generalization of the Bourgain–Delbaen construction of separable $\mathcal{L}_\infty$-spaces. Our approach is slightly different from that of [11] and [12], but the mathematical essentials are the same. We choose to set things out in some detail partly because we believe our approach yield new insights into the original BD construction, and partly because the calculations presented here are a good introduction to the notations and methods we use later. It is perhaps worth emphasizing here that BD constructions are very different from the majority of constructions that occur in Banach space theory. Normally we start with the unit vectors in the space $c_{00}$ and complete with respect to some (possibly exotic) norm. The only norms that occur in a BD construction are the usual norms of $\ell_\infty$ and $\ell_1$. What we construct here are exotic vectors in $\ell_\infty$ whose closed linear span is the space we want.

The idea will be to introduce a particular kind of (conditional) basis for the space $\ell_1$ and to study the subspace $X$ of $\ell_\infty$ spanned by the biorthogonal elements. Since $\ell_1$ is then in a natural way a subspace of (and in some cases the whole of) $X^*$, we shall be thinking of elements of $\ell_1$ as functionals and, in accordance with the convention explained earlier, denote them $b^*$, $c^*$ and so on. In our initial discussion we shall consider the space $\ell_1(\mathbb{N})$ (which we shall later replace with $\ell_1(\Gamma)$ with $\Gamma$ a certain countable set better adapted to our needs).

**Definition 3.1.** We shall say that a basic sequence $(d^*_n)_{n \in \mathbb{N}}$ in $\ell_1(\mathbb{N})$ is a triangular basis if $\text{supp} d^*_n \subseteq \{1, 2, \ldots, n\}$, for all $n$. We thus have

$$d^*_n = \sum_{m=1}^{n} a_{n,m} e^*_m,$$

where, by linear independence, we necessarily have $a_{n,n} \neq 0$. Notice that the linear span $[d^*_1, d^*_2, \ldots, d^*_n]$ is the same as $[e^*_1, e^*_2, \ldots, e^*_n]$, that is to say, the space $\ell_1(n)$, regarded as a subspace of $\ell_1(\mathbb{N})$ in the usual way. So, in particular, the basic sequence $(d^*_n)$ is indeed a basis for the whole of $\ell_1$. The biorthogonal sequence in $\ell_\infty$ will be denoted $(d^*_n)$; it is a weak* basis for $\ell_\infty$ and a basis for its closed linear span, which will be our space $X$.

**Proposition 3.2.** If $(d^*_n)$ is a triangular basis for $\ell_1(\mathbb{N})$, with basis constant $M$, then the closed linear span $X = [d_n : n \in \mathbb{N}]$ is a $\mathcal{L}_\infty$-$M$-space. If $(d^*_n)$ is boundedly complete, or equivalently $(d_n)$ is shrinking, then $X^*$ is naturally isomorphic to $\ell_1(\mathbb{N})$ with $\|g^*\|_{X^*} \leq \|g^*\|_1 \leq M \|g^*\|_{X^*}$.

**Proof.** In accordance with our “$\pi$-star” notation, let us write $P^*_n$ for the basis projection $\ell_1 \rightarrow \ell_1$ associated with the basis $(d^*_n)$. Thus $P^*_n(d^*_n)$ equals $d^*_m$ if $m \leq n$ and 0 otherwise; because $e^*_m \in \ell_1(n) = [d^*_1, \ldots, d^*_n]$, we also have $P^*_n e_m = e_m$ when $m \leq n$. If we modify $P^*_n$ by taking the codomain to be the image $\text{im} P^*_n = \ell_1(n)$, rather than the whole of $\ell_1$, what we have is a quotient operator, which we shall denote $q_n$, of norm at most $M$. The dual of this quotient operator is an isomorphic embedding $i_n : \ell_\infty(n) \rightarrow \ell_\infty(\mathbb{N})$, also of norm at most $M$. If $m \leq n$ and $u \in \ell_\infty(n)$ we have

$$(i_n u)(m) = \langle e^*_m, i_n u \rangle = \langle q_n e^*_m, u \rangle = \langle e^*_m, u \rangle = u(m).$$

So $i_n$ is an extension operator $\ell_\infty^n \rightarrow \ell_\infty(\mathbb{N})$ and we have

$$\|u\|_\infty \leq \|i_n u\|_\infty \leq M \|u\|_\infty$$

for all $u \in \ell_\infty^n$. In particular, the image of $i_n$, which is exactly $[d_1, \ldots, d_n]$ is $M$-isomorphic to $\ell_\infty^n$, which implies that $X$ is a $\mathcal{L}_\infty$-$M$-space.

In the case where $(d^*_n)$ is a boundedly complete basis of $\ell_1$ then $X^*$ may be identified with $\ell_1$ by standard result about bases. Moreover, for $g^* \in \ell_1$, we have

$$\|g^*\|_{X^*} = \sup \{ \langle g^*, x \rangle : x \in X \text{ and } \|x\|_\infty \leq 1 \} \leq \|g^*\|_1.$$
Theorem 3.4

On the other hand, if \( g^* \) has finite support, say \( \text{supp}g^* \subseteq \{1, 2, \ldots, n\} \), we can choose \( u \in \ell_\infty^n \) with \( \|u\| = 1 \) and \( \langle g^*, u \rangle = \|g^*\|_1 \). The extension \( x = i_n(u) \) is now in \( X \) and satisfies
\[
\|x\| \leq M, \quad \langle g^*, x \rangle = \|g^*\|.
\]

We shall say that \((d_n^*)\) is a \textit{unit-triangular basis} of \( \ell_1(\mathbb{N}) \) if it is a triangular basis and the non-zero scalars \( a_{n,n} \) are all equal to 1. We can thus write
\[
d_n^* = e_n^* - c_n^*.
\]
where \( c_n^* = 0 \) and \( \text{supp}c_n^* \subseteq \{1, 2, \ldots, n-1\} \) for \( n \geq 2 \). The clever part of the Bourgain–Delbaen construction is to find a method of choosing the \( c_n^* \) in such a way that \((d_n^*)\) is indeed a basic sequence. The idea is to proceed recursively assuming that, for some \( n \geq 1 \), we already have a unit-triangular basis \((d_m^*)_{m \leq n}\) of \( \ell_1^n \). The value of \( P_r^*b^* \) is thus already determined when \( 1 \leq r \leq n \) and \( b^* \in \ell_1^n \).

Definition 3.3. In the set-up described above, we shall say that an element \( c^* \) of \( \ell_1(n) \) is a BD-functional (with respect to the triangular basis \((d_n^*)_{n=1}^\infty\)) if there are exist real numbers \( \alpha \geq 0 \) and \( \beta \in [0, \frac{\theta}{4}] \) such that we can express \( c^* \) in one of the following forms:

\begin{enumerate}
\item \( \alpha e_j^* \) with \( 1 \leq j \leq n \),
\item \( \beta(I - P_k^*)b^* \) with \( 0 \leq k < n \) and \( b^* \in \text{ball\ } \ell_1(k+1, \ldots, n) \),
\item \( \alpha e_j^* + \beta(I - P_k^*)b^* \) with \( 1 \leq j \leq k \) and \( b^* \in \text{ball\ } \ell_1(k+1, \ldots, n) \).
\end{enumerate}

The non-negative constant \( \beta \) will be called the \textit{weight} of the functional \( c^* \) (“weight 0” in case (0)). Note that (0) and (1) are “almost” special cases of (2), with \( \beta \) (resp. \( \alpha \)) equal to 0. In the construction presented in this paper, we do not use functionals of type (0) and the constant \( \alpha \) in case (2) is always equal to 1. However, it may be worth stating the following theorem in full generality.

Theorem 3.4 \([11, 12]\). Let \( \theta \) be a real number with \( 0 < \theta < \frac{1}{2} \) and let \( d_n^* = e_n^* - c_n^* \) in \( \ell_1 \) be such that, for each \( n \), \( c_{n+1}^* \in \ell_1^n \) is a BD-functional of weight at most \( \theta \) with respect to \((d_m^*)_{m=1}^n\). Then \((d_n^*)_{n \in \mathbb{N}} \) is a triangular basis of \( \ell_1 \), with basis constant at most \( M = 1/(1-2\theta) \). The subspace \( X = [d_n : n \in \mathbb{N}] \) of \( \ell_\infty \) is thus a \( \mathcal{Z}_{1,M} \)-space.

Proof. Despite the disguise, this is essentially the same argument as in the original papers of Bourgain and Delbaen. What we need to show is that \( P_m^* \) is a bounded operator, with \( \|P_m^*\| \leq M \) for all \( m \). Because we are working on the space \( \ell_1 \) it is enough to show that \( \|P_m^*e_n^*\| \leq M \) for every \( m \) and \( n \).

First, if \( n \leq m \), \( P_m^* e_n^* = e_n^* \), so there is nothing to prove. Now let us assume that \( \|P_k e_j^*\| \leq M \) for all \( k \leq m \) and all \( j \leq n \); we then consider \( P_m^* e_{n+1}^* \). We use the fact that
\[
e_{n+1}^* = d_{n+1}^* + c_{n+1}^*,
\]
with \( e_{n+1}^* \in \ell_1^n \) a BD-functional. We shall consider a functional of type (2), which presents the most difficulty. We thus have
\[
c_{n+1}^* = \alpha e_j^* + \beta(I - P_k^*)b^*,
\]
where \( 1 \leq j \leq k < n \) and \( \alpha, \beta, b^* \) are as in Definition 3.3 and \( \beta \leq \theta \) by our hypothesis. Now, because \( n+1 > m \) we have \( P_m^* d_{n+1}^* = 0 \) so
\[
P_m^* e_{n+1}^* = \alpha P_m^* e_j^* + \beta(P_m^* - P_{m\wedge k}^*)b^*.
\]

If \( k \geq m \) the second term vanishes so that
\[
\|P_m^* e_{n+1}^*\| = \alpha \|P_m^* e_j^*\| \leq \|P_m^* e_j^*\|,
\]
which is at most \( M \) by our inductive hypothesis.
If, on the other hand, \( k < m \), we certainly have \( j < m \) so that \( P_m^* e_j^* = e_j^* \), leading to the estimate
\[
\|P_m^* e_{n+1}^*\| \leq \alpha \|e_j^*\| + \beta \|P_m^* b^*\| + \beta \|P_k^* b^*\|.
\]
Now \( b^* \) is a convex combination of functionals \( \pm e_l^* \) with \( l \leq n \), and our inductive hypothesis is applicable to all of these. We thus obtain
\[
\|P_m^* e_{n+1}^*\| \leq \alpha + M\beta \leq 1 + 2M\beta = M,
\]
by the definition of \( M = 1/(1 - 2\theta) \) and the assumption that \( 0 \leq \beta \leq \theta \).

The \( L_\infty \) spaces of Bourgain and Delbaen, and those we construct in the present paper are of the above type. However, the “cuts” \( k \) that occur in the definition of BD-functionals are restricted to lie in a certain subset of \( \mathbb{N} \), thus naturally dividing the coordinate set \( \mathbb{N} \) into successive intervals. As in [22], it will be convenient to replace the set \( \mathbb{N} \) with a different countable set \( \Gamma \) having a structure that reflects this decomposition. This will also enable us later to use a notation in which an element \( \gamma \in \Gamma \) automatically codes the BD-functional associated with it.

**Theorem 3.5.** Let \( (\Delta_q)_{q \in \mathbb{N}} \) be a sequence of non-empty finite sets, with \( \#\Delta_1 = 1 \); write \( \Gamma_q = \bigcup_{1 \leq p \leq q} \Delta_p, \Gamma = \bigcup_{p \in \mathbb{N}} \Delta_p \). Assume that there exists \( \theta < \frac{1}{2} \) and a mapping \( \tau \) defined on \( \Gamma \setminus \Delta_1 \), assigning to each \( \gamma \in \Delta_{q+1} \) a tuple of one of the forms:
\[
(0) \ (\alpha, \xi) \text{ with } 0 < \alpha < 1 \text{ and } \xi \in \Gamma_q;
(1) \ (p, \beta, b^*) \text{ with } 0 \leq p < q, 0 < \beta \leq \theta \text{ and } b^* \in \text{ball} \ell_1(\Gamma_q \setminus \Gamma_p);
(2) \ (\alpha, \xi, p, \beta, b^*) \text{ with } 0 < \alpha \leq 1, 1 \leq p < q, \xi \in \Gamma_p, 0 < \beta \leq \theta \text{ and } b^* \in \text{ball} \ell_1(\Gamma_q \setminus \Gamma_p).
\]
Then there exist \( d^*_\gamma = e^*_\gamma - c^*_\gamma \in \ell_1(\Gamma) \) and projections \( P^*_{\{0,q\}} \) on \( \ell_1(\Gamma) \) uniquely determined by the following properties:
\[
(1) \ P^*_{\{0,q\}} d^*_\gamma = \begin{cases} d^*_\gamma & \text{if } \gamma \in \Gamma_q \\ 0 & \text{if } \gamma \in \Gamma \setminus \Gamma_q \end{cases}
\]
\[
(2) \ c^*_\gamma = \begin{cases} 0 & \text{if } \gamma \in \Delta_1 \\ \alpha e^*_\xi & \text{if } \tau(\gamma) = (\alpha, \xi) \\ \beta(I - P^*_{\{0,p\}}) b^* & \text{if } \tau(\gamma) = (p, \beta, b^*) \\ \alpha e^*_\xi + \beta(I - P^*_{\{0,p\}}) b^* & \text{if } \tau(\gamma) = (\alpha, \xi, p, \beta, b^*) \end{cases}
\]
The family \( (d^*_\gamma)_{\gamma \in \Gamma} \) is a basis for \( \ell_1(\Gamma) \) with basis constant at most \( M = (1 - 2\theta)^{-1} \). The norm of each projection \( P^*_{\{0,q\}} \) is at most \( M \). The biorthogonal elements \( d^*_\gamma \) generate a \( L_{\infty,M} \)-subspace \( X(\Gamma, \tau) \) of \( \ell_\infty(\Gamma) \). For each \( q \) and each \( u \in \ell_\infty(\Gamma_q) \), there is a unique \( i_q(u) \in [d^*_\gamma : \gamma \in \Gamma_q] \) whose restriction to \( \Gamma_q \) is \( u \); the extension operator \( i_q : \ell_\infty(\Gamma_q) \rightarrow X(\Gamma, \tau) \) has norm at most \( M \). The subspaces \( M_n = [d^*_\gamma : \gamma \in \Delta_n] = i_q(\ell_\infty(\Delta_q)) \) form a finite-dimensional decomposition (FDD) for \( X \); if this FDD is shrinking then \( X^* \) is naturally isomorphic to \( \ell_1(\Gamma) \).

**Proof.** We shall show that, with a suitable identification of \( \Gamma \) with \( \mathbb{N} \), this theorem is just a special case of Theorem 3.1. Let \( k_p = \#\Gamma_p \) and let \( n \mapsto \gamma(n) : \mathbb{N} \rightarrow \Gamma \) be a bijection with the property that \( \Delta_1 = \{\gamma(1)\} \), while, for each \( q \geq 2 \), \( \Delta_q = \{\gamma(n) : k_{q-1} < n \leq k_q\} \). There is a natural isometry: \( J : \ell_1(\mathbb{N}) \rightarrow \ell_1(\Gamma) \) satisfying \( J(e^*_n) = e^*_\gamma(n) \). It is straightforward to check that if \( d^*_n = J^{-1}(d^*_\gamma(n)) = e^*_n - c^*_n \), then the hypotheses of Theorem 3.1 are satisfied. (The cuts \( k \) that occur in the BD-functionals \( c^*_n \) are all of the form \( k = k_p \).) All the assertions in the present theorem are now immediate consequences. The projections \( P^*_{\{0,q\}} \) whose existence is claimed here are given by \( P^*_{\{0,q\}} = JP^*_k J^{-1} \), where \( P^*_n \) is the basis projection of Theorem 3.1. When ordered as \( (d^*_\gamma(n))_{n \in \mathbb{N}} \) the vectors \( d_n \) form a basis of their closed linear span, which is a \( L_{\infty,M} \)-space. The extension operator that (by abuse of notation) we here denote by \( i_q \) is just
The assertions about the subspaces $M_q = \{ d_{\gamma(n)} : k_{q-1} < n \leq k_q \}$ follow from the fact that $(d_{\gamma(n)})$ is a basis.

We now make a few observations about the space $X = (\Gamma, \tau)$ and the functions $d_\gamma$, taking the opportunity to introduce notation that will be used in the rest of the paper. We have seen that for each $\gamma \in \Delta_{n+1}$ the functional $d_\gamma^*$ has support contained in $\Gamma_n \cup \{ \gamma \}$. Using biorthogonality, we see that $d_\gamma$ is supported by $\{ \gamma \} \cup \Gamma \setminus \Gamma_{n+1}$. It should be noted that we should not expect the support of $d_\gamma$ to be finite; in fact, in all interesting cases, we have $X \cap c_0(\Gamma) = \{0\}$.

As noted above the subspaces $M_n = \{ d_{\gamma} : \gamma \in \Delta_n \}$ form a finite-dimensional decomposition for $X$. For each interval $I \subseteq \mathbb{N}$ we define the projection $P_I : X \to \bigoplus_{n \in I} M_n$ in the natural way; this is consistent with our use of $P^*_n(0, n]$ in Theorem 3.5. Most of our arguments will involve sequences of vectors that are block sequences with respect to this FDD. Since we are using the word “support” to refer to the set of $\gamma$ where a given function is non-zero, we need other terminology for the set of $n$ such that $x$ has a non-zero component in $M_n$. We define the range of $x$, denoted ran $x$, to be the smallest interval $I \subseteq \mathbb{N}$ such that $x \in \bigoplus_{n \in I} M_n$. It is worth noting that if ran $x = (p, q]$ then we can write $x = i_q(u)$ where $u = x \upharpoonright \Gamma_q \in \ell_\infty(\Gamma_q)$ satisfies $\Gamma_p \cap \text{supp } u = \emptyset$. 

\[ J_{ik_q} J^{-1} \]
4. Construction of $\mathcal{B}_mT$ and $\mathcal{X}_K$

We now set about constructing specific BD spaces which will be modelled on mixed Tsirelson spaces, in rather the same way that the original spaces of Bourgain and Delbaen have been found to be modelled on $\ell_p$. We shall adopt a notation in which elements $\gamma$ of $\Delta_{n+1}$ automatically code the corresponding BD-functionals. This will allow us to write $X(\Gamma)$ rather than $X(\Gamma, \tau)$ for the resulting $\mathcal{L}_\infty$-space. To be more precise, an element $\gamma$ of $\Delta_{n+1}$ will be a tuple of one of the forms:

(1) $\gamma = (n + 1, \beta, b^*)$, in which case $\tau(\gamma) = (0, \beta, b^*)$;
(2) $\gamma = (n + 1, \xi, \beta, b^*)$ in which case $\tau(\gamma) = (1, \xi, \text{rank } \xi, \beta, b^*)$.

In each case, the first co-ordinate of $\gamma$ tells us what the rank of $\gamma$ is, that is to say to which set $\Delta_{n+1}$ it belongs, while the remaining co-ordinates specify the corresponding BD-functional.

It will be observed that BD-functionals of Type 0 do not arise in this construction and that the $p$ in the definition of a Type 1 functional is always 0. In the definition of a Type 2 functional that the scalar $\alpha$ that occurs is always 1 and $p$ equals $\text{rank } \xi$. We shall make the further restriction the weight $\beta$ must be of the form $m_j^{-1}$, where the sequences $(m_j)$ and $(n_j)$ satisfy Assumption 2.3. We shall say that the element $\gamma$ has weight $m_j^{-1}$ (sometimes dropping the $^{-1}$ and referring to “weight $m_j$”). In the case of a Type 2 element $\gamma = (n + 1, \xi, m_j^{-1}, b^*)$ we shall insist that $\xi$ be of the same weight $m_j^{-1}$ as $\gamma$.

To ensure that our sets $\Delta_{n+1}$ are finite we shall admit into $\Delta_{n+1}$ only elements of weight $m_j$ with $j \leq n + 1$. A further restriction involves a recursively defined function which we call “age”. For a Type 1 element $\gamma = (n + 1, \beta, b^*)$ we define age $\gamma = 1$. For a Type 2 element $\gamma = (n + 1, \xi, m_j^{-1}, b^*)$, we define age $\gamma = 1 + \text{age } \xi$, and further restrict the elements of $\Delta_{n+1}$ by insisting that the age of an element of weight $m_j$ may not exceed $n_j$. Finally, we shall restrict the functionals $b^*$ that occur in an element of $\Delta_{n+1}$ by requiring them to lie in some finite subset $B_n$ of $\ell_1(\Gamma_n)$. It is convenient to fix an increasing sequence of natural numbers $(N_n)$ and take $B_n$ to be the set of all linear combinations $b^* = \sum_{\eta \in \Gamma_n \setminus \Gamma_p} a_\eta e^*_\eta$, where $\sum_{\eta} |a_\eta| \leq 1$ and each $a_\eta$ is a rational number with denominator dividing $N_n$.

Assumption 4.1.

$$\Delta_{n+1} \subseteq \bigcup_{j=1}^n \left\{ (n + 1, m_j^{-1}, b^*) : b^* \in B_{n,0} \right\}$$

$$\cup \bigcup_{0<p<n} \left\{ (n + 1, \xi, m_j^{-1}, b^*) : \xi \in \Delta_p, \text{weight } \xi = m_j^{-1}, \text{age } \xi < n_j, b^* \in B_{n,p} \right\}$$

We shall also assume that $\Delta_{n+1}$ contains a rich supply of elements of “even weight”, more exactly of weight $m_j$ with $j$ even.

Assumption 4.2.

$$\Delta_{n+1} \supseteq \bigcup_{j=1}^{\lfloor(n+1)/2\rfloor} \left\{ (n + 1, m_{2j}^{-1}, b^*) : b^* \in B_{n,0} \right\}$$

$$\cup \bigcup_{1 \leq p < n} \left\{ (n + 1, \xi, m_{2j}^{-1}, b^*) : \xi \in \Delta_p, \text{weight } \xi = m_{2j}^{-1}, \text{age } \xi < n_{2j}, b^* \in B_{n,p} \right\}$$
For our main HI construction, there are additional restrictions on the elements with “odd weight” \(m_{2j-1}\). However, there is some interest already in the space we obtain without making such restrictions. We denote this space \(\mathfrak{B}_{mT}\); it is an isomorphic predual of \(\ell_1\) that is unconditionally saturated but contains no copy of \(c_0\) or \(\ell_p\). An analogous space \(\mathfrak{B}_T\), modelled on the standard Tsirelson space, rather than a mixed Tsirelson space, was constructed a few years ago by the second-named author.

**Definition 4.3.** We define \(\mathfrak{B}_{mT} = \mathfrak{B}_{mT}[(m_j, n_j)_{j \in \mathbb{N}}]\) to be the space \(X(\Gamma)\) where \(\Gamma = \Gamma^{max}\) is defined by the recursion \(\Delta_1 = \{1\},\)

\[
\Delta_{n+1} = \bigcup_{j=1}^{n+1} \left\{ (n + 1, m_j^{-1}, b^*) : b^* \in B_{n,0} \right\}
\]

\[
\cup \bigcup_{j=1}^{n} \bigcup_{\xi < n_j} \left\{ (n + 1, \xi, m_j^{-1}, b^*) : \xi \in \Delta_p, \text{weight } \xi = m_j^{-1}, \text{ age } \xi < n_j, \ b^* \in B_{n,p} \right\}
\]

The extra constraints that we place on “odd-weight” elements in order to obtain hereditary indecomposability will involve a coding function that will produce the analogues of the “special functionals” that occur in [21] and other HI constructions. In our case, all we need is an injective indecomposability will involve a coding function that will produce the analogues of the “special functionals” that occur in [21] and other HI constructions. In our case, all we need is an injective

\[
\text{with weight } \eta = m_{4i-2} > n_{2j-1}^2, \text{ while a Type 2 element must be}
\]

\[
(n + 1, \xi, m_{2j-1}^{-1}, e_\eta^*)
\]

\[
\text{with weight } \eta = m_{4\sigma(\xi)}.
\]

**Definition 4.4.** We define \(\mathfrak{X}_K[(m_j, n_j)_{j \in \mathbb{N}}]\) to be the space \(X(\Gamma)\) where \(\Gamma = \Gamma^K\) is defined by the recursion \(\Delta_1 = \{1\},\)

\[
\Delta_{n+1} = \bigcup_{j=1}^{(n+1)/2} \left\{ (n + 1, m_{2j-1}^{-1}, b^*) : b^* \in B_{n,0} \right\}
\]

\[
\cup \bigcup_{p=1}^{n} \bigcup_{\eta \leq p} \left\{ (n + 1, \xi, m_{2j-1}^{-1}, b^*) : \xi \in \Delta_p, \text{weight } \xi = m_{2j-1}^{-1}, \text{ age } \xi < n_{2j}, \ b^* \in B_{n,p} \right\}
\]

\[
\cup \bigcup_{j=1}^{(n+1)/2} \left\{ (n + 1, \xi, m_{2j-1}^{-1}, e_\eta^*) : \xi \in \Delta_p, \text{weight } \xi = m_{2j-1}^{-1}, \text{ age } \xi < n_{2j-1}, \ b^* \in B_{n,p} \right\}
\]

With the definition readily at hand, this is a convenient moment to record an important “tree-like” property of odd-weight elements of \(\Gamma^K\), even though we shall not be exploiting these special elements until later on.

**Lemma 4.5.** Let \(\gamma, \gamma'\) be two elements of \(\Gamma^K\) both of weight \(m_{2j-1}\) and of ages \(a \geq a'\), respectively. Let \((p_i, e_\eta^*, \xi_i)_{1 \leq i \leq a}\), resp. \((p'_i, e_\eta'^*, \xi'_i)_{1 \leq i \leq a'}\), be the analysis of \(\gamma\), resp. \(\gamma'\). There
exists \( l \) with \( 1 \leq l \leq a' \) such that \( \xi'_i = \xi_i \) when \( i < l \), while weight \( \eta_j \neq \text{weight } \eta'_j \) for all \( j \) when \( l < i \leq a' \).

**Proof.** If weight \( \eta'_i \neq \text{weight } \eta_j \) for all \( i \geq 2 \) and all \( j \) there is nothing to prove (we may take \( l = 1 \)). Otherwise, let \( 2 \leq l \leq a \) be maximal subject to the existence of \( j \) such that weight \( \eta_j = \text{weight } \eta'_j \). Now this weight is exactly \( m_{4r}(\xi'_i, \xi'_j) \), which means that \( j \) cannot be 1 (because the weight of \( \eta_1 \) has the form \( m_{4k-2} \)). Thus \( \sigma(\xi'_{i-1}) = \sigma(\xi_{j-1}) \), which implies that \( \xi'_{i-1} = \xi_{j-1} \). Since \( l - 1 = \text{age } \xi'_{i-1} \) and \( j - 1 = \text{age } \xi_{j-1} \), we deduce that \( j = l \). Moreover, since the elements \( \xi_i \) with \( i < l - 1 \) are determined by \( \xi_{l-1} \), we have \( \xi_i = \xi'_i \) for \( i < l \). \( \Box \)

Although the structure of the space \( X(\Gamma) \) is most easily understood in terms of the basis \( (d_{\gamma}) \) and the biorthogonal functionals \( d^*_\gamma \), it is with the evaluation functionals \( e^*_\gamma \) that we have to deal in order to estimate norms. The recursive definition of the functionals \( d^*_\gamma \) can be unpicked to yield the following proposition.

**Proposition 4.6.** Assume that the set \( \Gamma \) satisfies Assumption \[\text{a]} \] Let \( n \) be a positive integer and let \( \gamma \) be an element of \( \Delta_{n+1} \) of weight \( m_j \) and age \( a \leq n_j \). Then there exist natural numbers \( 0 = p_0 < p_1 < \cdots < p_a = n + 1 \), elements \( \xi_1, \ldots, \xi_a = \gamma \) of weight \( m_j \) with \( \xi_r \in \Delta_{p_r} \) and functionals \( b^*_r \in \text{ball } \ell_1 \left( \Gamma_{p_r - 1} \setminus \Gamma_{p_r - 1} \right) \) such that

\[
e^*_\gamma = \sum_{r=1}^{a} d^*_{\xi_r} + m_j^{-1} \sum_{r=1}^{a} P^*_r (p_{r-1}, \infty) b^*_r
\]

where \( c^*_1 \) is the Type 1 BD-functional

\[
c^*_1 = m_j^{-1} P^*_{(0, \infty)} b^*,
\]

with \( b^* \in B(n, 0) \subset \text{ball } \ell_1 \left( \Gamma_n \right) \). Since \( b^* \) is in the image of the projection \( P^*_{(0, n]} \) we have \( P^*_{(0, n]} b^* = b^* \) and so

\[
c^*_1 = d^*_{\xi_1} + m_j^{-1} P^*_{(p_0, \infty)} b^*_1 = d^*_{\xi_1} + m_j^{-1} P^*_{(p_0, p_1)} b^*_1,
\]

with \( p_0 = 0, p_1 = n + 1, b^*_1 = b^* \) and \( \xi_1 = \gamma \).

If \( a > 1 \) then \( \gamma \) has the form \( (n + 1, \xi, m_j^{-1}, b^*) \) and \( c^*_1 \) is the Type 2 BD-functional

\[
c^*_1 = e^*_r + m_j^{-1} P^*_{(p, \infty)} b^*.
\]

If we apply our inductive hypothesis to the element \( \xi \) of weight \( m_j \), rank \( p \) and age \( a - 1 \), we obtain the desired expression for \( e^*_r \).

We shall refer to the identity presented in the above proposition as the evaluation analysis of \( \gamma \) and shall use it repeatedly in norm estimations. The form of the second term in the evaluation analysis, involving a sum weighted by \( m_j^{-1} \), indicates that there is going to be a connection with mixed Tsirelson spaces; the first term, involving functionals \( d^*_\xi \), with no weight, can cause inconvenience in some of our calculations, but is an inevitable feature of the BD construction.

The data \( (p_r, b^*_r, \xi_r)_{1 \leq r \leq a} \) will be called the analysis of \( \gamma \). We note that if \( 1 \leq s \leq a \) the analysis of \( \xi_s \) is just \( (p_r, b^*_r, \xi_r)_{1 \leq r \leq s} \).

In the remainder of this section, and in the next, we shall be dealing with a space \( X = X(\Gamma) \) and shall be making the assumptions \[\text{a]} \) and \[\text{b]} \). Our results thus apply both to \( \mathfrak{B}_mT \) and \( \mathfrak{X}_k \).
We note that, since the weights $m_j^{-1}$ are all at most $\frac{1}{2}$, the constant $M$ in Theorem 3.3 may be taken to be 2. This leads to the following norm estimates for the extension operators $i_n$ and for the projections $P_l$ associated with the FDD $(M_n)$:

$$
\|i_n\| = \|P_{(0, n]}\| \leq 2, \quad \|P_{(n, \infty)}\| \leq 3, \quad \|P_{(m, n]}\| \leq 4, \quad \|d^*_\xi\| = \|P^*_{\text{rank}\xi, \infty}e^*_\xi\| \leq 3.
$$

The assumption 4.2 enables to write down a kind of converse to Proposition 4.6 which will lead to our first norm estimate.

**Proposition 4.7.** Let $j, a$ be positive integers with $a \leq n_{2j}$, let $0 = p_0 < p_1 < p_2 < \cdots < p_a$ be natural numbers with $p_1 \geq 2j$ and let $b^*_r$ be functionals in $B(p_r - 1, p_{r-1})$ for $1 \leq r \leq a$. Then there are elements $\xi_r \in \Gamma_{p_r}$ such that the analysis of $\gamma = \xi_a$ is $(p_r, b^*_r, \xi_r)_{1 \leq r \leq a}$.

**Proof.** This is another easy induction on $a$. For $a = 1$, the assumption 4.2 and the hypothesis that $p_1 \geq 2j$ guarantee that the tuple $\xi_1 = (p_1, m_{2j}^{-1}, b^*_1)$ is in $\Gamma_{p_1}$. We continue recursively, setting $\xi_{r+1} = (p_{r+1}, \xi_r, m_{2j}^{-1}, b^*_{r+1})$.

**Proposition 4.8.** Let $(x_r)_{r=1}^a$ be a skipped block sequence (with respect to the FDD $(M_n)$) in $X$. If $j$ is a positive integer such that $a \leq n_{2j}$ and $2j < \min\{\text{ran} x_2\}$, then there exists an element $\gamma$ of weight $m_{2j}$ satisfying

$$
\sum_{r=1}^a x_r(\gamma) \geq \frac{1}{2} m_{2j}^{-1} \sum_{r=1}^a \|x_r\|.
$$

Hence

$$
\| \sum_{r=1}^a x_r \| \geq \frac{1}{2} m_{2j}^{-1} \sum_{r=1}^a \|x_r\|.
$$

**Proof.** Let $p_0 = 0$, and choose $p_1, p_2, \ldots, p_a$ such that ran $x_r \subseteq (p_r - 1, p_r)$. Thus $x_r = i_{p_r-1}(u_r)$ where the element $u_r = x_r \upharpoonright \Gamma_{p_{r-1}}$ has support disjoint from $\Gamma_{p_{r-1}}$. Since $\|i_n\| \leq 2$ for all $n$ we have $\|u_r\| \geq \frac{1}{2} \|x_r\|$ and so there exist $\eta_r \in \Gamma_{p_r-1} \setminus \Gamma_{p_{r-1}}$ with

$$
|u_r(\eta_r)| \geq \frac{1}{2} \|x_r\|.
$$

The functional $b^*_r = \pm e^*_\eta_r$ is certainly in $B(p_{r-1}, p_{r-1})$ and with a suitable choice of sign we may arrange that

$$
\langle b^*_r, x_r \rangle = |u_r(\eta_r)| \geq \frac{1}{2} \|x_r\|.
$$

By Proposition 4.7 there is an element $\gamma$ of $\Delta_p$ whose analysis is $(p_r, b^*_r, \xi_r)_{1 \leq r \leq a}$. We shall use the evaluation analysis to calculate

$$
\sum_{s=1}^a x_s(\gamma) = \langle e^*_\gamma, \sum_{s=1}^a x_s \rangle.
$$

For any $r$ and $s$, $x_s \in [d^*_\xi : p_{s-1} < \text{rank} \xi < p_s]$, while $\text{rank} \xi_r = p_r$, whence

$$
\langle d^*_\xi, x_s \rangle = 0 \quad \text{for all } r, s,
$$

while

$$
\langle P^*_{(p_{r-1}, p_r)} b^*_r, x_s \rangle = \langle b^*_r, P_{(p_{r-1}, p_r)} x_s \rangle = 0,
$$

for all $r \neq s$. In the case $r = s$ we have

$$
\langle P^*_{(p_{r-1}, p_r)} b^*_r, x_r \rangle = \langle b^*_r, P_{(p_{r-1}, p_r)} x_r \rangle = \langle b^*_r, x_r \rangle.
$$
The evaluation analysis thus simplifies to yield
\[
\sum_{r=1}^{a} x_r(\gamma) = m_{2j}^{-1} \sum_{r=1}^{a} \langle b_r^*, x^* r \rangle \geq \frac{1}{2}m_{2j}^{-1} \sum_{r=1}^{a} \|x_r\|. 
\]
\def\qed{}

The lower estimate we have just obtained indicates that there is a close connection between our space $X$ and mixed Tsirelson spaces of the kind considered in Subsection 2.4. With a bit more work one can show that a normalized skipped-block sequence in $X$ dominates the unit vector basis of $T[(\mathcal{A}_n, m_{2j}^{-1})_{j \in \mathbb{N}}]$. We shall not need this more precise result in the present work.
5. Rapidly increasing sequences

We continue to work with the space \( X = X(\Gamma) \), where \( \Gamma \) satisfies the assumptions \[ \text{4.1} \] and \[ \text{4.2} \]. We saw in the last section that skipped block sequences admit useful Mixed Tsirelson lower estimates. We now pass to a class of block sequences that admit upper estimates of a similar kind. The following definition is a variant of something that is familiar from other III constructions.

**Definition.** Let \( I \) be an interval in \( \mathbb{N} \) and let \( (x_k)_{k \in I} \) be a block sequence (with respect to the FDD \( (M_n) \)). We say that \( (x_k) \) is a rapidly increasing sequence, or RIS, if there exists a constant \( C \) such that the following hold:

1. \( \|x_k\| \leq C \) for all \( k \in \mathbb{N} \),
2. \( j_{k+1} > \max \operatorname{ran} x_k \)
3. \( |x_k(\gamma)| \leq Cm_i^{-1} \) whenever weight \( \gamma = m_i \) and \( i < j_k \)

If we need to be specific about the constant, we shall refer to a sequence satisfying the above conditions as a \( C \)-RIS.

**Lemma 5.1.** Let \( (x_k) \) be a \( C \)-RIS and let \( (j_k) \) be an increasing sequence of natural numbers as in the definition. If \( \gamma \in \Gamma \) and weight \( \gamma = m_i \) then, for any natural number \( s \)

\[
\|\langle e_\gamma^*, P_{(s,\infty)}x_k \rangle \| \leq \begin{cases} 5Cm_i^{-1} & \text{if } i < j_k \\ 3Cm_i^{-1} & \text{if } i \geq j_k+1 \end{cases}
\]

**Proof.** We first consider the case where \( i \geq j_{k+1} \), noting that this implies that \( i > \max \operatorname{ran} x_k \) by RIS condition (2). As in Proposition \[ \text{4.6} \] we may write down the evaluation analysis of \( \gamma \) as

\[
e_\gamma^* = \sum_r d_{\xi_r}^* + m_i^{-1} \sum_r b_r^* \circ P_{(p_r-1, \infty)},
\]

where \( 0 = p_0 < p_1 < q_1 < p_2 < \cdots \), and \( b_r^* \) is a norm-1 element of \( \ell_1(\Gamma) \), supported by \( \Gamma_{p_i-1} \setminus \Gamma_{p_r-1} \), while \( \xi_r \) is of rank \( p_r \) and weight \( m_i \). Since \( \Delta_q \) contains no elements of weight \( m_i \) unless \( q \geq i \), it must be that \( p_1 \geq i \). Thus \( p_1 > \max \operatorname{ran} x_k \), from which it follows that \( P_{(p_r-\infty)} \circ P_{(s,\infty)}x_k = P_{(s\vee p_r-\infty)}x_k = 0 \) for all \( r \geq 1 \). For the same reason, we also have

\[
\langle d_{\xi_r}^*, P_{(s,\infty)}x_k \rangle = \langle e_\gamma^*, P_{(s\vee q_r-\infty)}P_{(p_r-\infty)}x_k \rangle = 0
\]

for all \( r \). We are left with

\[
\|\langle e_\gamma^*, P_{(s,\infty)}x_k \rangle \| = m_i^{-1} \|b_r^* \circ P_{(p_r-\infty)}x_k \| \leq m_i^{-1} \|P_{(s,\infty)}\| \|x_k \| \leq 3Cm_i^{-1}
\]

In the case where \( i < j_k \), we again use the evaluation analysis, but need to be more careful about the value of \( s \). Since we shall need this argument again, we state it as a separate lemma. Clearly the second part of the present lemma is an immediate consequence. \( \square \)

**Lemma 5.2.** Let \( i \) be a positive integer and suppose that \( x \in X \) has the property that \( \|x\| \leq C \) and \( |x(\xi)| \leq \delta \) whenever weight \( \xi = m_i \). Then for any \( s \) and any \( \gamma \) of weight \( m_i \) we have

\[
\|\langle e_\gamma^*, P_{(s,\infty)}x \rangle \| \leq 2\delta + 3Cm_i^{-1}
\]

**Proof.** As before we consider the evaluation analysis

\[
e_\gamma^* = \sum_{r=1}^a d_{\xi_r}^* + m_i^{-1} \sum_{r=1}^a b_r^* \circ P_{(p_r-\infty, \infty)}.
\]

If \( s \geq p_a \) then \( P_{(s,\infty)}e_\gamma^* = 0 \). If \( 0 < s < p_1 \), by applying \( P_{(s,\infty)} \) to each of the terms in the evaluation analysis, we see that

\[
P_{(s,\infty)}e_\gamma^* = e_\gamma^* - m_i^{-1}P_{(0,s]}b_1^*,
\]
which leads to
\[|\langle e_\gamma^*, P_{(s,\infty)}x_k \rangle| \leq \delta + m_{i}^{-1}||b_{l}^*|| ||P_{(p_{l+1},s)}|| ||x_k|| \leq \delta + 3Cm_{i}^{-1},\]
by our assumptions.

In the remaining case, there is some \(t\) with \(1 \leq t < a\) such that \(p_{t} \leq s\) while \(p_{t+1} > s\). We may rewrite the evaluation analysis of \(\gamma\) as
\[e_\gamma^* = e_{\xi t}^* + \sum_{r=t+1}^{a} d_{r}^* + m_{i}^{-1} \sum_{r=t+1}^{a} b_{r}^* \circ P_{(p_{r-1},\infty)},\]
which gives us
\[P_{(s,\infty)}e_\gamma^* = e_{\xi t}^* - m_{i}^{-1}P_{(p_{t},s)}b_{t+1}^* - \sum_{r=t+1}^{a} d_{r}^* + m_{i}^{-1} \sum_{r=t+1}^{a} b_{r}^* \circ P_{(p_{r-1},\infty)}\]
When we recall that weight \(\xi_{t} = \text{weight } \gamma\) this yields
\[|\langle e_\gamma^*, P_{(s,\infty)}x_k \rangle| \leq 2\delta + 3Cm_{i}^{-1},\]
as above.

**Proposition 5.3** (Basic Inequality). Let \((x_k)_{k \in I}\) be a \(C\)-RIS, let \(\lambda_k\) be real numbers, let \(s\) be a natural number and let \(\gamma\) be an element of \(\Gamma\). There exist \(k_0 \in I\) and and a functional \(g^* \in W((\varphi_{3n_j},m_{j})_{j \in \mathbb{N}})\) such that:

1. either \(g^* = 0\) or weight \((g^*) = \text{weight } (\gamma)\) and \(\text{supp } g^* \subseteq \{ k \in I : k > k_0 \} \);
2. \(|\langle e_\gamma^*, P_{(s,\infty)} \sum_{k \in I} \lambda_k x_k \rangle| \leq 5C|\lambda_{k_0}| + 5C(g^*, \sum_{k} |\lambda_k| e_k)|.

Moreover, if \(j_0\) is such that
\[|\langle e_\gamma^*, \sum_{k \in J} \lambda_k x_k \rangle| \leq 2C \max_{k \in J} |\lambda_k|,\]
for all subintervals \(J\) of \(I\) and all \(\xi \in \Gamma\) of weight \(m_{j_0}\), then we may choose \(g^*\) to be in \(W((\varphi_{3n_j},m_{j})_{j \neq j_0})\).

**Proof.** We proceed by induction of the rank of \(\gamma\), noting that if \(\gamma\) is of rank 1 we have \(P_{(s,\infty)}e_\gamma^* = 0\) whenever \(s \geq 1\), so that
\[\langle e_\gamma^*, P_{(s,\infty)} \sum_{k \in I} \lambda_k x_k \rangle = \begin{cases} 0 & \text{if } r \geq 1 \\ \lambda_{1} x_1(\gamma) & \text{if } r = 0. \end{cases}\]
Thus \(k_0 = 1\) and \(g^* = 0\) have the desired property.

Now consider an element \(\gamma\) of rank greater than 1, of age \(a\) and of weight \(m_{h}\). Taking \(j_k\) to be a sequence as in the definition of a RIS, we shall suppose that there is some \(l \in I\) such that \(j_l \leq h < j_{l+1}\). (The cases where \(h < j_k\) for all \(k \in I\) and where \(h \geq j_{k+1}\) for all \(k \in I\) are simpler.)

We split the summation over \(k\) into three parts as follows:
\[\langle e_\gamma^*, P_{(s,\infty)} \sum_{k \in I} \lambda_k x_k \rangle = \sum_{I \ni k < l} \lambda_k \langle e_\gamma^*, P_{(s,\infty)} x_k \rangle + \langle e_\gamma^*, P_{(s,\infty)} \lambda x_l \rangle + \langle e_\gamma^*, P_{(s,\infty)} \sum_{I \ni k > l} \lambda_k x_k \rangle\]
and estimate the three terms separately.

When \(k < l\) we have \(h \geq j_l \geq j_{k+1}\) so that
\[|\langle e_\gamma^*, P_{(s,\infty)} \lambda_k x_k \rangle| \leq 3Cm_h^{-1}|\lambda_k| \leq 3Cm_{j_{k+1}}^{-1}|\lambda_k|,\]
by Lemma 5.1. Thus
\[|\sum_{I \ni k < l} \lambda_k \langle e_\gamma^*, P_{(s,\infty)} x_k \rangle| \leq 3C \sum_{k<l} m_{j_k}^{-1}|\lambda_k| \leq 3C \sum_{j=1}^{\infty} \sum_{k<l} m_{j_k}^{-1} \max_{k<l}|\lambda_k| \leq C \max_{k<l}|\lambda_k|.\]
For the second term, we have the immediate estimate
\[ |\langle e_\gamma^*, P_{(s,\infty)} \lambda_l x_l \rangle| \leq \| P_{(s,\infty)} \| \| \lambda_l \| \| x_l \| \leq 3C|\lambda_l|. \]
Thus putting the first two terms together we have
\[ |\langle e_\gamma^*, P_{(s,\infty)} \sum_{k \leq l} \lambda_k x_k \rangle| \leq C \max_{k \leq l} |\lambda_k| + 3C|\lambda_l| \leq 4C|\lambda_0|, \tag{5.1} \]
for a suitably chosen \( k_0 \leq l \).

We now have to estimate the last term
\[ |\langle e_\gamma^*, \sum_{k \in I'} \lambda_k x_k' \rangle|, \]
where \( I' = \{ k \in I : k > l \} \) and \( x_k' = P_{(s,\infty)} x_k \). We shall use the evaluation analysis of \( \gamma \)
\[ e_\gamma^* = \sum_{r=1}^{a} d_{\xi_r}^* + m_h^{-1} \sum_{r=1}^{a} b_r^* \circ P_{(p_r-1,\infty)}. \]

Let \( I_0' = \{ k \in I' : \text{ran } x_k' \text{ contains rank } \xi_r \text{ for some } r \} \) noting first that \( \# I_0' \leq a \) and secondly that for \( k \in I' \setminus I_0' \) the interval \( \text{ran } x_k' \) meets \( (p_r-1, p_r) \) for at most one value of \( r \). If we set \( I_r' = \{ k \in I' : \text{ran } x_k \text{ meets } (p_r-1, p_r) \} \text{ but no other } (p_r-1, p_r) \} \) then each \( I_r' \) is a subinterval of \( I' \) and we have
\[ \langle e_\gamma^*, x_k' \rangle = m_h^{-1} \langle b_r^*, P_{(p_r-1,\infty)} x_k' \rangle = m_h^{-1} \langle b_r^*, P_{(s \vee p_r-1,\infty)} x_k \rangle \]
if \( k \in I_r' \), while
\[ \langle e_\gamma^*, x_k' \rangle = 0 \quad \text{if } k \in I' \setminus \bigcup_r I_r' \]
Thus \( \langle e_\gamma^*, \sum_{k \in I'} \lambda_k x_k' \rangle = \langle e_\gamma^*, \sum_{k \in I_0'} \lambda_k x_k' \rangle + m_h^{-1} \sum_{r=1}^{a} \langle b_r^*, \sum_{k \in I_r'} \lambda_k x_k' \rangle. \)

Applying Lemma \( 5.1 \) we see that
\[ |\langle e_\gamma^*, \sum_{k \in I'} \lambda_k x_k' \rangle| \leq 5C m_h^{-1} \sum_{k \in I_0'} |\lambda_k| + m_h^{-1} \left| \sum_{r=1}^{a} \langle b_r^*, \sum_{k \in I_r'} \lambda_k x_k' \rangle \right|. \tag{5.2} \]
Now, for each \( r \), the functional \( b_r^* \) is a convex combination of functionals \( \pm e_\eta^* \) with \( p_r-1 < \text{rank } \eta < p_r \), so we may choose \( \eta_r \) to be such an \( \eta \) with
\[ \left| \langle b_r^*, \sum_{k \in I_r'} \lambda_k x_k' \rangle \right| \leq \left| \langle e_\eta^*, \sum_{k \in I_r'} \lambda_k x_k' \rangle \right|. \]
For each \( r \), we may apply our inductive hypothesis to the element \( \eta_r \in \Gamma \) and the RIS \( (x_k)_{k \in I_r'} \), obtaining \( k_r \in I_r' \) and \( g_r^* \in W[\langle \alpha_{3n_j}, m_j^{-1} \rangle_{j \in \mathbb{N}}] \) supported on \( \{ k \in I_r' : k > k_r \} \) satisfying
\[ |\langle e_\eta^*, P_{(s \vee p_r-\infty)} \sum_{k \in I_r'} \lambda_k x_k \rangle| \leq 5C|\lambda_{k_r}| + 5C \langle g_r^*, \sum_{k \in I_r'} \lambda_k e_k \rangle. \tag{5.3} \]
We now define \( g^* \) by setting
\[ g^* = m_h^{-1} \left( \sum_{k \in I_0'} e_k^* + \sum_{r=1}^{a} (e_k^* + g_r^*) \right). \]
This is a sum, weighted by \(m_h\), of at most \(3n_h\) functionals in \(W[\langle \mathcal{A}_{3n_j}, m_j^{-1}\rangle]_{j \in \mathbb{N}}\), supported by disjoint intervals, and is hence itself in \(W[\langle \mathcal{A}_{3n_j}, m_j^{-1}\rangle]_{j \in \mathbb{N}}\). Putting together 5.1, 5.2 and 5.3 we finally obtain

\[
|\langle e_\gamma^*, P_{(s, \infty)} \sum_{k \in I} \lambda_k x_k \rangle| \leq 4C|\lambda_{k_0}| + 5Cm_h^{-1} \sum_{k \in I_0^c} |\lambda_k| + m_h^{-1} \sum_{r=1}^a \langle b_r^*, \sum_{k \in I_r^c} \lambda_k x_k' \rangle |
\]

\[
\leq 4C|\lambda_{k_0}| + 5Cm_h^{-1} \sum_{k \in I_0^c} |\lambda_k| + m_h^{-1} \sum_{r=1}^a \langle e_\gamma^*, P_{(s, \infty)} \sum_{k \in I_r^c} \lambda_k x_k \rangle |
\]

\[
\leq 4C|\lambda_{k_0}| + 5Cm_h^{-1} \left( \sum_{k \in I_0^c} |\lambda_k| + \sum_{r=1}^a \langle |\lambda_{k_r}| + \langle g^*, \sum_{k \in I_r^c} |\lambda_k| e_k \rangle \rangle \right)
\]

\[
\leq 5C|\lambda_{k_0}| + 5C\langle g^*, \sum_{k \in I} |\lambda_k| e_k \rangle.
\]

If \(j_0\) satisfies the additional condition set out in the statement of the theorem, we proceed by the same induction. The base case certainly presents no problem and if weight \(\gamma = m_h\) with \(h = j_0\) we have a simple way to estimate

\[
\langle e_\gamma^*, P_{(s, \infty)} \sum_{k \in I} \lambda_k x_k \rangle.
\]

Indeed there is at most one value of \(k, l\), say, for which \(s\) is in ran \(x_k\) and \(P_{(s, \infty)} x_k = 0\) for \(k < l\). If we set \(J = \{k \in I : k > l\}\) we then have

\[
|\langle e_\gamma^*, P_{(s, \infty)} \sum_{k \in I_l} \lambda_k x_k \rangle| \leq |\lambda_l||P_{(s, \infty)}||x_l|| + |\lambda_l(\sum_{k \in J} \lambda_k x_k)|.
\]

By our usual estimate \(\|P_{(s, \infty)}\| \leq 3\) and the assumed additional condition, this is at most \(5C|\lambda_{k_0}|\) for some \(l \leq k_0 \in I\). We can then take \(g^* = 0\).

**Corollary 5.4.** Any RIS is dominated by the unit vector basis of \(T[\langle \mathcal{A}_{3n_j}, m_j^{-1}\rangle]_{j \in \mathbb{N}}\). More precisely, if \((x_k)\) is a C-RIS then, for any real \(\lambda_k\), we have

\[
\|\sum_k \lambda_k x_k\| \leq 10C\|\sum_k \lambda_k e_k\|
\]

where the norm on the right hand side is taken in \(T[\langle \mathcal{A}_{3n_j}, m_j^{-1}\rangle]_{j \in \mathbb{N}}\).

As well as this domination result, we shall need the following more precise lemma.

**Proposition 5.5.** Let \((x_k)_{k=1}^{n_{j_0}}\) be a C-RIS. Then

1. For every \(\gamma \in \Gamma\) with weight \(\gamma = m_h\) we have

\[
|n_{j_0}^{-1} \sum_{k=1}^{j_0} x_k(\gamma)| \leq \begin{cases} 11Cm_{j_0}^{-1}m_h^{-1} & \text{if } h < j_0 \\ 5Cn_{j_0}^{-1} + 5Cm_h^{-1} & \text{if } h \geq j_0 \end{cases}
\]

In particular,

\[
|n_{j_0}^{-1} \sum_{k=1}^{j_0} x_k(\gamma)| \leq 6Cm_{j_0}^{-2},
\]

if \(h > j_0\) and

\[
\|n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} x_k\| \leq 6Cm_{j_0}^{-1}.
\]
(2) If $\lambda_k$ $(1 \leq k \leq n_{j_0})$ are scalars with $|\lambda_k| \leq 1$ and having the property that
\[ |\sum_{k \in J} \lambda_k x_k(\gamma)| \leq 2C \max_{k \in J} |\lambda_k|, \]
for every $\gamma$ of weight $m_{j_0}$ and every interval $J \subseteq \{1, 2, \ldots, n_{j_0}\}$, then
\[ \|n_{j_0}^{-1} \sum_{k=1}^{j_0} \lambda_k x_k\| \leq 6Cm_{j_0}^{-2}. \]

Proof. This is a direct application of the Basic Inequality, with all the coefficients $\lambda_k$ equal to $n_{j_0}^{-1}$. Indeed, for $(1)$ there exists $g^* \in W[(\omega_{3n_j}, m_j^{-1})_{k \in \mathbb{N}}]$ (either zero or of weight $m_h$) such that
\[ |n_{j_0}^{-1} \sum_{k=1}^{j_0} x_k(\gamma)| \leq 5Cn_{j_0}^{-1} + 5Cg^*(n_{j_0}^{-1} \sum_{k=1}^{j_0} e_k)). \]

Using Lemma $2.4$ to estimate the term involving $g^*$, we obtain
\[ |n_{j_0}^{-1} \sum_{k=1}^{j_0} x_k(\gamma)| \leq \begin{cases} 5Cn_{j_0}^{-1} + 10Cm_{j_0}^{-1}m_h^{-1} & \text{if } h < j_0 \\ 5Cn_{j_0}^{-1} + 5Cm_h^{-1} & \text{if } h \geq j_0. \end{cases} \]

The formulae given in $(1)$ follow easily when we note that $n_{j_0}$ is (much) larger than $5m_{j_0}^2$ when $j_0 \geq 2$.

If the scalars $\lambda_k$ satisfy the additional condition, then the $g^*$ whose existence is guaranteed by the Basic Inequality may be taken to be in $W[(\omega_{3n_j}, m_j^{-1})_{j \neq j_0}]$ so that the second part of Lemma $2.4$ may be applied, yielding
\[ |n_{j_0}^{-1} \sum_{k=1}^{j_0} x_k(\gamma)| \leq \begin{cases} 5Cn_{j_0}^{-1} + 10Cm_{j_0}^{-2}m_h^{-1} & \text{if } h < j_0 \\ 5Cn_{j_0}^{-1} + 5Cm_h^{-1} & \text{if } h > j_0. \end{cases} \]

This leads easily to the claimed estimate for $\|n_{j_0}^{-1} \sum_{k=1}^{j_0} \lambda_k x_k\|$. □

It turns out that in our space there are three useful types of RIS. One of these is based on an idea that will be familiar from other constructions, that of introducing long $\ell_1$-averages. We defer our discussion of this construction until the next section. We shall deal first with the other two types of RIS, which involve the $L_\infty$ structure of our space, and provide the extra tool that we eventually use to solve the scalar-plus-compact problem.

We have already remarked that the support of an element of $X$ is not of great interest — indeed the support of any nonzero element of $X$ is an infinite set, and contains elements $\gamma$ of $\Gamma$ of all possible weights. There is, however, a related notion which is of much use. Recall that an element $x$ whose range is contained in the interval $[p, q]$ can be expressed as $i_q(u)$ where $u \in \ell_\infty(\Gamma_q)$ and $\text{supp}(u) \subseteq \Gamma_q \setminus \Gamma_p$. It turns out that the support of $u$ contains a lot of information about $x$. We shall refer to $\text{supp}(u)$ as the local support. A formal (and unambiguous) definition may be formulated as follows.

**Definition 5.6.** Let $x$ be an element of $\bigoplus_n M_n$ and let $q = \max \text{ran} \ x$; thus $x$ may be expressed as $i_q(u)$ with $u = x \mid \Gamma_q$. The subset $\text{supp} u = \{ \gamma \in \Gamma_q : x(\gamma) \neq 0 \}$ is defined to be the local support of $x$.

The following easy lemma uses an idea that has already occurred in Lemma $5.1$.

**Lemma 5.7.** Let $\gamma \in \Gamma$ be of weight $m_h$ and assume that weight $(\xi) \neq m_h$ for all $\xi$ in the local support of $x$. Then $|x(\gamma)| \leq 3m_h^{-1} \|x\|$. 
Proof. Let $q = \max \text{ran} \ x$ so that $x = i_q (x \mid \Gamma_q)$ and, by hypothesis, weight $\xi \neq m_h$ whenever $\xi \in \Gamma_q$ and $x(\xi) \neq 0$. If rank $\gamma \leq q$ we thus have $x(\gamma) = 0$ and there is nothing to prove. Otherwise we consider the evaluation analysis of $\gamma$

$$e^* \gamma = \sum_{r=1}^{a} d^* + m^{-1} \sum_{r=1}^{a} b^* \circ P_{(p_r, \infty)}$$

and let $s$ be chosen maximal subject to $p_s = \text{rank} \xi_s \leq q$. (Since $\gamma = \xi_s$ such an $s$ certainly exists.) For $r \geq s$ we have $r > \max \text{ran} \ x$, whence $d^* (x) = 0$ and $P_{(p_r, \infty)} x = 0$. Thus

$$x(\gamma) = (e^* \gamma, x) = \begin{cases} m^{-1} (b^*_s, P_{(p_s, \infty)} x) + (e^* \gamma, x) & \text{if } s > 1 \\ m^{-1} (b^*_1, x) = m^{-1} (b^*_1, P_{(p_0, \infty)} x) & \text{if } s = 1. \end{cases}$$

Since, in the first of the above cases, we have rank $\xi_{s-1} < q$ and weight $\xi_{s-1} = m_h$, which imply $e^* \gamma (x) = 0$, we deduce that in both cases

$$|x(\gamma)| = m^{-1} |(b^*_s, P_{(p_s, \infty)} x)| \leq 3m^{-1} \|x\|.$$

We can now introduce two classes of block sequence, characterized by the weights of the elements of the local support.

Definition 5.8. We shall say that a block sequence $(x_k)_{k \in \mathbb{N}}$ in $X$ has bounded local weight if there exists some $j_1$ such that weight $\gamma \leq m_{j_1}$ for all $\gamma$ in the local support of $x_k$, and all values of $k$. We shall say that $(x_k)_{k \in \mathbb{N}}$ has rapidly increasing local weight if, for each $k$ and each $\gamma$ in the local support of $x_{k+1}$, we have weight $\gamma > m_k$ where $i_k = \max \text{ran} x_k$.

Proposition 5.9. Let $(x_k)_{k \in \mathbb{N}}$ be a bounded block sequence. If either $(x_k)$ has bounded local weight, or $(x_k)$ has rapidly increasing local weight, the sequence $(x_k)$ is a RIS.

Proof. We start with the case of rapidly increasing local weight and let $m_{j_k}$ be the minimum weight of an element $\gamma$ in the local support of $x_k$. By hypothesis, $j_k + 1 > \max \text{supp} x_k$ so that RIS condition (2) is satisfied. Also, if $h < j_k$ and $\gamma$ is of weight $m_h$ then $|x_k(\gamma)| \leq 3m^{-1} \|x_k\|$ by Lemma 5.7. So $(x_k)$ is a C-RIS with $C = 3 \sup \|x_k\|$.

Now let us suppose that weight $\gamma \leq m_{j_1}$ for all $\gamma$ in the local support of $x_k$ and all $k$. For $k \geq 2$ define $j_k = 1 + \max \text{supp} x_{k-1}$, thus ensuring that RIS condition (2) is satisfied. If weight $\gamma = m_h$ where $h < j_k$ there are two possibilities: if $i > j_1$ then $|x_k(\gamma)| \leq 3m^{-1} \|x_k\|$ by Lemma 5.7 if $i \leq j_1$ then $|x_k(\gamma)| \leq \|x_k\| \leq m_i^{-1} m_{j_1} \|x_k\|$. Thus $(x_k)$ is a C-RIS, where $C$ is the (possibly quite large) constant $m_{j_1} \sup \|x_k\|$.

Proposition 5.10. Let $Y$ be any Banach space and $T : X(\Gamma) \to Y$ be a bounded linear operator. If $\|T(x_k)\| \to 0$ for every RIS $(x_k)_{k \in \mathbb{N}}$ in $X(\Gamma)$ then $\|T(x_k)\| \to 0$ for every bounded block sequence sequence in $X(\Gamma)$.

Proof. It is enough to consider a bounded block sequence $(x_k)$ and show that there is a subsequence $(x_k')$ such that $\|T(x_k')\| \to 0$. We may write $x_k = i_{q_k}(u_k)$ with $u_k = x_k \mid \Gamma_{q_k}$ supported by $\Gamma_{q_k} \setminus \Gamma_{q_k-1}$. For each $k$ and each $N \in \mathbb{N}$, we split $u_k$ as $v_k^N + w_k^N$, where, for $\gamma \in \Gamma_{q_k}$,

$$v_k^N(\gamma) = \begin{cases} u_k(\gamma) & \text{if weight } \gamma \leq m_N \\ 0 & \text{otherwise} \end{cases}$$

$$w_k^N(\gamma) = \begin{cases} u_k(\gamma) & \text{if weight } \gamma > m_N \\ 0 & \text{otherwise} \end{cases}$$
and set
\[ y_k^N = i_{q_k}(v_k^N), \quad z_k^N = i_{q_k}(v_k^N). \]

We notice that \( \| y_k^N \| \leq \| v_k^N \| \leq \| x_k \| \), with a similar estimate for \( \| z_k^N \| \), so that the sequences \((y_k^N)_k\) and \((z_k^N)_k\) are bounded. We note also that weight \( \gamma \leq N \) for all \( \gamma \) in the local support of \( y_k^N \) and weight \( \gamma > N \) for all \( \gamma \) in the local support of \( z_k^N \).

So for each \( N \), the sequence \((y_k^N)_k\) has bounded local weight and is thus a RIS, by Proposition 5.9. By hypothesis, \( \| T(y_k^N) \| \to 0 \) for each \( N \). Hence we can choose a sequence \((k_n)\) tending to \( \infty \) such that \( \| T(y_{k_n}^N) \| \to 0 \). If we put \( n_1 = 1 \) and then, recursively, set \( n_{j+1} = q_{k_{n_j}} \), it is easy to see that the sequence \((z_{k_{n_j}}^{n_j})\) has rapidly increasing local weight. Thus this sequence is a RIS and we hence have \( \| T(z_{k_{n_j}}^{n_j}) \| \to 0 \). Since \( x_{k_{n_j}} = y_{k_{n_j}}^{n_j} + z_{k_{n_j}}^{n_j} \), we have found a subsequence \((x_j') = (x_{k_{n_j}})\) of \((x_k)\) with \( \| T(x_j') \| \to 0 \). \( \square \)

The above proposition will play an important role in proving compactness of operators, but in the mean time we shall use it to give our promised proof that the dual of \( X \) is \( \ell_1 \). There is an alternative approach using \( \ell_1 \)-averages.

**Proposition 5.11.** The dual of \( X(\Gamma) \) is \( \ell_1(\Gamma) \).

**Proof.** As we have already noted in Theorem 5.5 it is enough to show that the FDD \((M_n)\) is shrinking, that is to say, that every bounded block sequence in \( X \) is weakly null. So let \( \phi \) be an element of \( X^* \). By the upper estimate of Proposition 5.5 we see that \( \phi(x_k) \to 0 \) for every RIS \((x_k)_{k \in \mathbb{N}}\). Now Proposition 5.10, applied with \( T = \phi \), shows that \( \phi(x_k) \to 0 \) for every bounded block sequence \((x_k)\). \( \square \)

**Remark.** We can see that \( X(\Gamma) \) has many reflexive subspaces. Indeed, suppose that \((q_n)\) is an increasing sequence of natural numbers, and that, for each \( n \), \( F_n \) is a finite dimensional subspace of \( \bigoplus_{q_n < k < q_{n+1}} M_k \). Then \((F_n)\) is an FDD for the subspace \( W = \overline{\bigoplus_{n \in \mathbb{N}} F_n} \), and \((F_n)\) is shrinking because \((M_n)\) is. But \((F_n)\) is also boundedly complete, by the lower estimate of Proposition 4.8. Thus \( W \) is reflexive. We shall see later in section 8 that \( W^* \) is hereditarily indecomposable whenever \( W \) is a subspace of this type in the space \( X_K \).
6. $\ell_1$-averages, Exact Pairs and the HI property

In the first part of this section, we shall still only be using the assumptions \[4.1\] and \[4.2\] so that our results will apply when $X$ is either of the spaces $\mathcal{B}_{mT}$ and $X_K$. The special properties of the second of these spaces will come into play only from Definition \[6.10\] onwards.

**Definition 6.1.** An element $x$ of $X$ will be called a $C$-$\ell_1^n$ average if there exists a block sequence $(x_i)_{i=1}^n$ in $X$ such that $x = n^{-1} \sum_{k=1}^n x_k$ and $\|x_k\| \leq C$ for all $k$. We say that $x$ is a normalized $C$-$\ell_1^n$ average if, in addition, $\|x\| = 1$.

A standard argument (c.f. II.22 of \[8\]) using the lower estimate of Lemma \[4.8\] and Lemma \[2.2\] leads to the following.

**Lemma 6.2.** Let $Z$ be any block subspace of $X$. For any $n$ and and $C > 1$, $Z$ contains a normalized $C$-$\ell_1^n$ average.

**Proof.** Write $C = (1 - \epsilon)^{-1}$ and choose an integer $l$ with $n(1 - \epsilon/n)^l < 1$; next choose $j$ sufficiently large as to ensure that $n_{2j} > (2m_{2j})^l$; finally let $k$ be minimal subject to

$$m_{2j} < (1 - \epsilon/n)^{-k}$$

Since $\frac{1}{2}(1 - \epsilon/n)^{-k} \leq (1 - \epsilon/n)^{-k+1} \leq m_{2j}$ we have

$$n_{2j} > (2m_{2j})^l \geq (1 - \epsilon/n)^{-kl} > n^k.$$  

If $(x_i)$ is any normalized skipped-block sequence in $Z$, we can apply Lemma \[4.8\] to see that

$$\| \sum_{i=1}^n x_i \| \geq m_{2j}^{-1} n^k > (n - \epsilon)^k.$$  

It now follows from Lemma \[2.2\] that there are normalized successive linear combinations $y_1, \ldots, y_n$ of $(x_i)$ such that

$$\| \sum_{i=1}^n a_i y_i \| \geq (1 - \epsilon) \sum_{i=1}^n |a_i|,$$

for all real $a_i$. In particular, there is a normalized $C$-$\ell_1^n$ average. \[\square\]

**Lemma 6.3.** Let $x$ be a $C$-$\ell_1^n_j$ average. For all $\gamma \in \Gamma$ we have $|\langle d_\gamma, x \rangle| \leq 3Cn_j^{-1}$. If $\gamma$ is of weight $m_i$ with $i < j$ and $p \in \mathbb{N}$ then $|x(\gamma)| \leq 2Cm_i^{-1}$.

**Proof.** Let $x = n_j^{-1} \sum_{k=1}^{n_j} x_k$, as in the definition of a $C$-$\ell_1^n$ average. For any $\gamma$ there is some $k$ such that $\langle d_\gamma, x \rangle = n_j^{-1} \langle d_\gamma^*, x_k \rangle$. Thus

$$|\langle d_\gamma, x \rangle| \leq n_j^{-1} \|d_\gamma^*\| \|x_k\| \leq 3Cn_j^{-1}.$$  

Let us now consider the case where weight $\gamma = m_i$, with $i < j$. From the evaluation analysis

$$e_\gamma = \sum_{r=1}^a d_\gamma^* r + m_i^{-1} \sum_{r=1}^a b_\gamma^* \circ P_{(p_{r-1}, \infty)},$$

it follows that

$$|x(\gamma)| \leq \sum_{r=1}^a |\langle d_\gamma^*, x \rangle| + m_i^{-1} \sum_{r=1}^a \|P_{(p_{r-1}, p_r)} x\|.$$  

By what we have already observed, we have

$$\sum_{r=1}^a |\langle d_\gamma^*, x \rangle| \leq 3Cn_j^{-1}.$$
To estimate the second term in (6.1) we follow the argument of page 33 of [8], letting \( I_r \) (resp. \( J_r \)) be the set of \( k \) such that \( \text{ran} x_k \) is contained in (resp. meets) the interval \((p_{r-1}, p_r)\). We have \( \# J_r \leq \# I_r + 2 \) and \( \sum_r \# I_r \leq n_j \). Moreover, for each \( r \), we have \( P_{(p_{r-1}, p_r)} x_k = x_k \) if \( k \in I_r \), while \( P_{(p_{r-1}, p_r)} x_k = 0 \) if \( k \notin J_r \) and
\[
\|P_{(p_{r-1}, p_r)} x_k\| \leq 4 \|x_k\| \leq 4C \quad \text{if} \quad k \in J_r \setminus I_r.
\]
It follows that
\[
\|P_{(p_{r-1}, p_r)} x\| \leq n^{-1}_j (C \# I_r + 8C) \leq C n^{-1}_j (\# I_r + 8).
\]
Summing over \( r \) leads us to
\[
(6.3) \quad \sum_{r \leq a} \|P_{(p_{r-1}, p_r)} x\| \leq C n^{-1}_j (n_j + 8a).
\]
Combining our inequalities, and using the fact that \( a \leq n_i \) we obtain
\[
|x(\gamma)| \leq 3C n^{-1}_j + m^{-1}_i n^{-1}_j (C n_j + 8C a) \leq C m^{-1}_i + 5C n_j m^{-1}_j < 2C m^{-1}_i.
\]

\[\square\]

**Lemma 6.4.** Let \( I \) be an interval in \( \mathbb{N} \), let \((x_k)_{k \in I}\) be a block sequence in \( X \) and let \((j_k)_{k \geq 1}\) be an increasing sequence of natural numbers. Suppose that, for each \( k \), \( x_k \) is a \( C \ell^1_{l_1} \)-average and that \( j_{k+1} > \max \text{ran} x_k \). Then \((x_k)\) is a \( 2C \)-RIS.

**Proof.** We just have to prove RIS condition (3) and this is an immediate consequence of Lemma 6.3. \[\square\]

**Corollary 6.5.** Let \( Z \) be a block subspace of \( X \), and let \( C > 2 \) be a real number. Then \( X \) contains a normalized \( C \)-RIS.

**Proof.** This is immediate from Lemmas 6.2 and 6.4. \[\square\]

**Definition 6.6.** Let \( C > 0 \) and let \( \varepsilon \in \{0, 1\} \). A pair \((x, \gamma) \in X \times \Gamma\) is said to be a \((C, j, \varepsilon)\)-exact pair if
\[(1) \quad |(d^x_\xi, x)| \leq C m_j^{-1} \text{ for all } \xi \in \Gamma;
(2) \text{ weight } \gamma = m_j, \quad \|x\| \leq C, \quad x(\gamma) = \varepsilon;
(3) \text{ for every element } \gamma' \text{ of } \Gamma \text{ with weight } \gamma' = m_i \neq m_j, \text{ we have}
\[
|x(\gamma')| \leq \begin{cases} C m_i^{-1} & \text{if } i < j \\ C m_j^{-1} & \text{if } i > j. \end{cases}
\]

**Remark.** It is an immediate consequence of Lemma 5.2 that a \((C, j, \varepsilon)\) exact pair also satisfies the estimates
\[
|(e^x_\xi, P_{(s, \infty)} x)| \leq \begin{cases} 5C m_i^{-1} & \text{if } i < j \\ 5C m_j^{-1} & \text{if } i > j \end{cases}
\]
for elements \( \gamma \) of \( \Gamma \) with weight \( \gamma = m_i \neq m_j \).

It will be seen that these estimates, as well as those in the definition, have much in common with those of Lemma 5.1. Our first task is to show how we can construct \((C, 2j, 1)\)-exact pairs, starting from a RIS.

**Lemma 6.7.** Let \( j \) be a positive integer and let \((x_k)_{k=1}^{n_2} \) be a skipped-block \( C \)-RIS, such that \( \min \text{ran} x_2 \geq 2j \) and \( \|x_k\| \geq 1 \) for all \( k \). Then there exists \( \theta \in \mathbb{R} \), with \( |\theta| \leq 2 \), and there exists \( \gamma \in \Gamma \), such that \((x, \gamma)\) is a \((22C, 2j, 1)\)-exact pair, where \( x \) is the weighted sum
\[
x = \theta m_{2j} n_{2j}^{-1} \sum_{k=1}^{n_{2j}} x_k.
\]
Proof. We may apply the construction of Lemma 6.8 to obtain an element $\gamma$ of $\Gamma$ of weight $m_{2j}$ such that

$$n_{2j}^{-1} \sum_{k=1}^{n_{2j}} x_k(\gamma) \geq \frac{1}{2} m_{2j}^{-1}.$$ 

For a suitably chosen $\theta \in \mathbb{R}$ with $0 < \theta \leq 2$ we have $x(\gamma) = 1$, where $x = \theta m_{2j} n_{2j}^{-1} \sum_{k=1}^{n_{2j}} x_k$. We thus have condition (2) in the definition of an exact pair.

There is no problem establishing condition (1) since, for any $\xi$, there is some $k$ satisfying $\langle d_\xi^*, x \rangle = \theta m_{2j} n_{2j}^{-1} \langle d_\xi, x \rangle$. By RIS condition (1), $\|x_k\| \leq C$ and we know that $\|d_\xi\| \leq 3$. Hence $|\langle d_\xi^*, x_k \rangle| \leq 6C m_{2j} n_{2j}^{-1} < C m_{2j}^{-1}$.

To establish condition (3), we shall use the fact that $(x_k)$ is a $C$-RIS and apply Proposition 5.5 with $j_0 = 2j$. If weight $\gamma' = m_i$ with $i \neq 2j$, we thus have

$$|x(\gamma)| = |\theta| m_{2j} n_{2j}^{-1} \sum_{k=0}^{n_{2j}} x_k(\gamma') \leq \begin{cases} 22C m_i^{-1} & \text{if } i < 2j \\ 10C m_{2j} n_{2j}^{-1} + 10C m_{2j} m_i^{-1} < 11C m_{2j}^{-1} & \text{if } i > 2j. \end{cases}$$

Using Lemma 6.5, we now immediately obtain the following.

Lemma 6.8. If $Z$ is a block subspace of $X$ then for every $j \in \mathbb{N}$ there exists a $(45, 2j, 1)$-exact pair $(x, \eta)$ with $x \in Z$.

The proof of the following lemma, which we shall need in Section 7, is very similar.

Lemma 6.9. Let $(x_k)_{k=1}^{n_{2j}}$ be a skipped-block $C$-RIS, and let $q_0 < q_1 < q_2 < \cdots < q_{n_{2j}}$ be natural numbers such that ran $x_k \subseteq (q_{k-1}, q_k)$ for all $k$. Let $z$ denote the weighted sum $x = m_{2j} n_{2j}^{-1} \sum_{k=1}^{n_{2j}} x_k$. For each $k$ let $b_k^*$ be an element of $B_{q_{k-1},q_k}$ with $b_k(x_k) = 0$. Then there exist $\zeta_i \in \Delta_{q_i} (1 \leq i \leq n_{2j})$ such that the element $\eta = \zeta_{n_{2j}}$ has analysis $(q_i, b_i^*, \zeta_i)_{1 \leq i \leq n_{2j}}$, and $(z, \eta)$ $(12C, n_{2j}, 0)$-exact pair.

We are finally ready to make use of the special conditions governing “odd-weight” elements of $\Gamma$. We need to consider a special type of rapidly increasing sequence whose members belong to exact pairs.

Definition 6.10. Consider the space $X_K = X(\Gamma)$ where $\Gamma = \Gamma_K$ as defined in 4.4. We shall say that a sequence $(x_i)_{i \leq n_{2j_{0}-1}}$ is a $(C, 2j_0 - 1, \varepsilon)$-dependent sequence if there exist $0 = p_0 < p_1 < p_2 < \cdots < p_{n_{2j}-1}$, together with $\eta_i \in \Gamma_{p_i-1} \setminus \Gamma_{p_{i-1}}$ and $\xi_i \in \Delta_{p_i} (1 \leq i \leq n_{2j_{0}-1})$ such that

1. (for each $k$) ran $x_k \subseteq (p_{k-1}, p_k)$;
2. (the element $\xi = \xi_{2j_0-1}$ of $\Delta_{2j_0-1}$) has weight $m_{2j_0-1}$ and analysis $(p_i, e_{q_i}^*, \zeta_i)_{1 \leq i \leq 2n_{2j_{0}-1} - 1}$;
3. $(x_1, \eta_1)$ is a $(C, 4j_1 - 2, \varepsilon)$-exact pair;
4. (for each $2 \leq i \leq n_{2j_{0}-1}$) $(x_i, \eta_i)$ is a $(C, 4j_i, \varepsilon)$-exact pair, with ran $x_i \subseteq (p_{i-1}, p_i)$.

We notice that, because of the special odd-weight conditions in 4.4, we necessarily have $m_{4j_{i}-2} = \text{weight } \eta_{i} > n_{2j_{0}-1}^2$ and $m_{4j_{i}+1} = \text{weight } \eta_{i+1} = m_{4j_{i}+1}$, where $j_{i+1} = \sigma(\xi_i)$ for $1 \leq i < n_{2j_{0}-1}$.

Lemma 6.11. A $(C, 2j_0 - 1, \varepsilon)$-dependent sequence in $X_K$ is a $C$-RIS.

Proof. For each $i \geq 1$ we have max ran $x_i < p_i$ and $j_{i+1} = \sigma(\xi_i) > \text{rank } \xi_i = p_i$. This establishes Condition (2) in the definition of a RIS. Condition (3) follows from the definition of a $C$-exact pair.

Lemma 6.12. Let $(x_i)_{i \leq n_{2j_{0}-1}}$ be a $(C, 2j_0 - 1, 1)$-dependent sequence in $X_K$ and let $J$ be a sub-interval of $[1, n_{2j_{0}-1}]$. For any $\gamma' \in \Gamma$ of weight $m_{2j_0-1}$ we have

$$|\sum_{i \in J} (-1)^i x_i(\gamma')| \leq 4C.$$
Proof. Let $\xi_i, \eta_i, p_i, j_i$ be as in the definition of a dependent sequence and let $\gamma$ denote $\xi_{2j_0-1}$, an element of weight $m_{4j_0-1}$. Let $(p'_i, e^*_\eta_i, \xi'_i)_{1 \leq i \leq a'}$ be the analysis of $\gamma'$ and let the weight of $\xi'_i$ be $m_{4j'_i-2}$ when $i = 1$, $m_{4j'_i}$ when $1 < i \leq a'$. We note that $a' \leq n_{2j_0-1}$ because $\gamma'$ is of weight $m_{2j_0-1}$. We may thus apply the tree-like property of Lemma 4.5 deducing that there exists $1 \leq l \leq a'$ such that $(p'_i, \eta'_i, \xi'_i) = (p_i, \eta_i, \xi_i)$ for $i < l$ while $j_k \neq j'_k$ for all $l < i \leq a'$ and all $1 \leq k \leq n_{2j_0-1}$. Since

$$e^*_\gamma \circ P_{(0,p_{n-1})} = e^*_\xi_{l-1} = e^*_\gamma \circ P_{(0,p_{l})},$$

we have

$$x_k(\gamma') = x_k(\gamma) = m_{2j_0-1}^{-1} e^*_\eta_k \circ P_{p_{l-1},\infty} x_k = m_{2j_0-1}^{-1} x_k(\eta_k) = m_{2j_0-1}^{-1},$$

for $1 \leq k < l$.

We may now estimate as follows

$$|\sum_{k \leq l} (-1)^k x_k(\gamma')| \leq |\sum_{k \leq l} m_{2j_0-1}^{-1} (-1)^k x_k(\gamma')| + |x_l(\gamma')| + \sum_{k \leq l, k > l} |x_l(\gamma')|$$

$$\leq m_{2j_0-1}^{-1} |\sum_{k \leq l} (-1)^k| + \|x_l\| + \sum_{k \leq l, k > l} |d^*_\eta x_k + m_{2j_0-1}^{-1} e^*_\eta \circ P_{p_{l-1},\infty} x_k|$$

$$\leq 1 + C + n_{2j_0-1}^{-2} \max_{l \leq k \leq l, k \leq a} |d^*_\eta x_k + m_{2j_0-1}^{-1} e^*_\eta \circ P_{p_{l-1},\infty} x_k|.$$

Now we know that, provided $k > l$, weight $\eta'_k \neq \eta_k$ for all $i$, so by the definition of an exact pair, we have

$$|d^*_\eta (x_i) + m_{2j_0-1}^{-1} P_{p_{l-1},\infty} x_i(\eta_k)| \leq C(\text{weight } \eta_k)^{-1} + 5 C m_{2j_0-1}^{-1} \max \{\text{weight } \eta'_k, \text{weight } \eta_k\}^{-1}$$

$$\leq 2 C \max \{\text{weight } \eta_k^{-1}, \text{weight } \eta'_k^{-1}\}$$

$$\leq 2 C \max \{m_{4j_1-2}, m_{4j'_1-2}\} \leq 2 C n_{2j_0-1}^{-2},$$

using the fact that $m_{4j_1-2}$ and $m_{4j'_1-2}$ are both at least $n_{2j_0-1}^2$. We now deduce the inequality

$$|\sum_{i \in J} (-1)^i x_i(\gamma')| \leq 4 C$$

as required. \qed

Lemma 6.13. Let $(x_i)_{i \leq n_{2j_0-1}}$ be a $(C, 2j_0 - 1, 1)$-dependent sequence in $X_K$. Then

$$\|n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} x_i\| \geq m_{2j_0-1}^{-1} \quad \text{but} \quad \|n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} (-1)^i x_i\| \leq 12 C m_{2j_0-1}^{-2}.$$

Proof. Using the notation of Definition 6.10 is easy to show by induction on $a$, as in Lemma 4.8 that

$$\sum_{i=1}^a x_i(\xi_a) = m_{2j_0-1}^{-1} a,$$

whence we immediately obtain

$$\|n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} x_i\| \geq \sum_{i=1}^a x_i(\xi_{2j_0-1}) \geq m_{2j_0-1}^{-1}.$$

To estimate $\|n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} (-1)^i x_i\|$ we consider any $\gamma \in \Gamma$ and apply the second part of Lemma 5.5 with $\lambda_i = (-1)^i n_{2j_0-1}^{-1}$ and with $2j_0 - 1$ playing the role of $j_0$. Lemma 6.12 shows that the extra hypothesis of the second part of Lemma 5.5 is indeed satisfied, provided we replace $C$ by $2 C$. We deduce that $\|n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} (-1)^i x_i\| \leq 12 C m_{2j_0-1}^{-2}$, as claimed. \qed

A very similar proof yields the following estimate, which we shall use in the next section.
Lemma 6.14. Let \((x_i)_{i \leq n^{2j_0-1}}\) be a \((C, 2j_0 - 1, 0)\)-dependent sequence in \(X_K\). Then

\[
\|n^{2j_0-1} \sum_{i=1}^{n^{2j_0-1}} x_i\| \leq 4Cm^{2j_0-2}.
\]

In the mean time, we may finish the proof of one of our main theorems.

Lemma 6.15. Let \(Y\) and \(Z\) be block subspaces of \(X_K\). Then, for each \(\epsilon > 0\), there exist \(y \in Y\) and \(z \in Z\) with \(\|y - z\| < \epsilon \|y + z\|\).

Proof. We start by choosing \(j_0, j_1\) with \(m^{2j_0-1} > 540\epsilon^{-1}\) and \(m^{4j_1-2} > n^{2j_0-1}\). Next we use Lemma 6.8 to choose a \((45, m^{4j_1-2}, 1)\)-exact pair \((x_1, \eta_1)\) with \(x_1 \in Y\). Now, for some \(p_1 > \text{rank} \eta_1 \vee \text{max ran } x_1\), we define \(\xi_1 \in \Delta_{p_1}\) to be \((p_1, m^{2j_0-1}, e^{\ast}_{\eta_1})\).

We now set \(j_2 = \sigma(\xi_1)\) and choose a \((45, m^{4j_2}, 1)\)-exact pair \((x_2, \eta_2)\) with \(x_2 \in Z\) and \(\text{min ran } x_2 > p_1\). We pick \(p_2 > \text{rank} \eta_2 \vee \text{max ran } x_2\) and take \(\xi_2\) to be the element \((p_2, \xi_1, m^{2j_0-1}e^{\ast}_{\eta_2})\) of \(\Delta_{p_2}\). Notice that this tuple is indeed in \(\Delta_{q_{1+1}}\) because we have ensured that weight \(\eta_2 = m^{4\sigma(\xi_1)}\).

Continuing in this way, we obtain a \((45, 2j_0 - 1)\)-dependent sequence \((x_i)\) such that \(x_i \in Y\) when \(i\) is odd and \(x_i \in Z\) when \(i\) is even. We define \(y = \sum_{i \text{ odd}} x_i\) and \(z = \sum_{i \text{ even}} x_i\), and observe that, by Lemma 6.10

\[
\|y + z\| = \|\sum_{i=1}^{n^{2j_0-1}} x_i\| \geq n^{2j_0-1}m^{2j_0-1}, \quad \text{while}\n\]

\[
\|y - z\| = \|\sum_{i=1}^{n^{2j_0-1}} (-1)^i x_i\| \leq 12 \times 45n^{2j_0-1}m^{2j_0-2}.
\]

Proposition 2.1 now yields the theorem.

Theorem 6.16. The space \(X_K\) is hereditarily indecomposable.
7. Bounded linear operators on $X_K$

For technical reasons it will be convenient in the first few results of this section to work with elements of $X_K$ all of whose coordinates are rational, that is to say with elements of $X_K \cap \mathbb{Q}^\Gamma$. Since (as may be readily checked) each $d_k$ is in $X_K \cap \mathbb{Q}^\Gamma$, as are all rational linear combinations of these, we see that $X_K \cap \mathbb{Q}^\Gamma$ is dense in $X_K$.

**Lemma 7.1.** Let $m < n$ be natural numbers and let $x \in X_K \cap \mathbb{Q}^\Gamma$, $y \in X_K$ be such that $\text{ran } x, \text{ran } y$ are both contained in the interval $(m, n)$. Suppose that $\text{dist}(y, \mathbb{R}x) > \delta$. Then there exists $b^* \in \text{ball } \ell_1(\Gamma_n \setminus \Gamma_m)$, with rational coordinates, such that $b^*(x) = 0$ and $\|b^*(y)\| > \frac{1}{2}\delta$.

**Proof.** Let $u, v \in \ell_\infty(\Gamma_n \setminus \Gamma_m)$ be the restrictions of $x, y$ respectively. Then $x = i_n u$, $y = i_n v$ and so, for any scalar $\lambda$, $\|y - \lambda x\| \leq \|i_n\| \|v - \lambda u\|$. Hence $\text{dist}(v, \mathbb{R}u) > \frac{1}{2}\delta$ and so, by the Hahn–Banach Theorem in the finite dimensional space $\ell_\infty(\Gamma_n \setminus \Gamma_m)$, there exists $a^* \in \ell_1(\Gamma_n \setminus \Gamma_m)$ with $a^*(u) = 0$ and $a^*(v) > \frac{1}{2}\delta$. Since $x$ has rational coordinates our vector $u$ is in $\mathbb{Q}^{\Gamma_n \setminus \Gamma_m}$. It follows that we can approximate $a^*$ arbitrarily well with $b^* \in \mathbb{Q}^{\Gamma_n \setminus \Gamma_m}$ retaining the condition $b^*(u) = 0$. \hfill $\square$

**Lemma 7.2.** Let $T$ be a bounded linear operator on $X_K$, let $(x_i)$ be a $C$-RIS in $X_K \cap \mathbb{Q}^\Gamma$ and assume that $\text{dist}(Tx_i, \mathbb{R}x_i) > \delta$ for all $i$. Then, for all $j, p \in \mathbb{N}$, there exist $z \in [x_i : i \in \mathbb{N}]$, $q > p$ and $\eta \in \Delta_q$ such that

1. $(z, \eta)$ is a $(12C, 2j, 0)$-exact pair;
2. $(Tz)(\eta) > \frac{7}{\eta^n} \delta$;
3. $\|(I - P_{(p,q)})Tz\| < m_2^{-1} \delta$;
4. $\langle P_{(p,q)}^* e_{\eta, j}^*, Tz \rangle > \frac{2560}{1279} \delta$.

**Proof.** Since the sequence $(Tx_i)$ is weakly null, we may, by taking a subsequence if necessary, assume that there exist $p < q_0 < q_1 < q_2 < \ldots$ such that, for all $i \geq 1$, $\text{ran } x_i \subseteq (q_{i-1}, q_i)$ and $\|(I - P_{(q_{i-1}, q_i)})Tx_i\| < \frac{1}{2} m_2^{-2} \delta \leq \frac{1}{80} m_2^{-1} \leq \frac{1}{1279} \delta$. It certainly follows from this that $\text{dist}(P_{(q_{i-1}, q_i)}Tx_i, \mathbb{R}x_i) > \frac{1279}{2560} \delta$. We may apply Lemma 7.1 to obtain $b_i^* \in \text{ball } \ell_1(\Gamma_{q_i - 1} \setminus \Gamma_{q_i - 1})$, with rational coordinates, satisfying

$$\langle b_i^*, x_i \rangle = 0, \quad \langle b_i^*, P_{(q_i - 1, q_i)}Tx_i \rangle > \frac{1279}{2560} \delta.$$ 

Taking a further subsequence if necessary, we may assume that the coordinates of $b_i^*$ have denominators dividing $N_{q_{i-1}}!$, so that $b_i^* \in B_{q_{i-1}, q_i}$, and we may also assume that $q_1 \geq 2j$. We are thus in a position to apply Lemma 6.4 getting elements $\xi_i$ of weight $m_2$ in $\Delta_{q_i}$ such that the element $\eta = \xi_{n_{2j}}$ of $\Delta_{n_{2j}}$ has evaluation analysis

$$e_{\eta, j}^* = \sum_{i=1}^{n_{2j}} d_{\xi_i}^* + m_2^{-1} \sum_{i=1}^{n_{2j}} P_{(q_{i-1}, q_i)} b_i^*.$$

and such that $(x, \eta)$ is a $(12C, 2j, 0)$-exact pair, where $z$ denotes the weighted average

$$x = m_2^{-1} \sum_{i=1}^{n_{2j}} x_i.$$

We next need to estimate $(Tz)(\eta)$. For each $k$, we have $\|(I - P_{(q_k, q_k)})Tx_k\| < \frac{1}{80} m_2^{-1} \delta$ so that

$$\langle Tz \rangle(\eta) \geq \langle e_{\eta, j}^*, P_{(q_k, q_k)}Tx_k \rangle - \frac{1}{80} m_2^{-1} \delta = m_2^{-1} \langle b_k^*, P_{(q_k, q_k)}Tx_k \rangle - \frac{1}{80} m_2^{-1} \delta > \frac{1247}{2560} m_2^{-1} \delta.$$
Proof. It will be enough to prove the result for a RIS in \(\mathcal{X}_K\). Let \(\xi = (x_i)\) be a (12, \(p_i, q_i\))-exact pair with \(n_{2j_0} \geq 1\) such that \(m_{j_0, 1} \geq 256C\|T\|\delta^{-1}\) and \(j_1 \geq m_{j_0, 1}^2\). Taking \(p = p_0 = 0\) and \(j = j_1 - 1\) in Lemma [7,2] we can find \(q_1\) and a (12, 4, \(j_1 - 2, 0\))-exact pair \((z_1, \eta_1)\) with rank \(\eta_1 = q_1\), \((Tz_1)(\eta_1) > \frac{7}{16}\delta\) and \(\|(I - P_{(p_0, q_1)})(Tz_1)\| < m_{4j_1, 1}^{-1}\delta\). Let \(p_1 = q_1 + 1\) and let \(\xi_1\) be the special Type 1 element of \(\Delta_{p_1}\) given by \(\xi_1 = (p_1, m_{j_0, 1} - 1, e_{\eta_1}^*).\)

Now, recursively for \(2 \leq i \leq n_{2j_0} - 1\), define \(j_i = \sigma(\xi_{i-1})\), and use the lemma again to choose \(q_i\) and a (12, 4, \(j_i, 0\))-exact pair \((z_i, \eta_i)\) with rank \(\eta_i = q_i\), \((Tz_i)(\eta_i) > \frac{7}{16}\delta\) and \(\|(I - P_{(p_i, q_i)})(Tz_i)\| < m_{4j_i, 1}^{-1}\delta\). We now define \(p_i = q_i + 1\) and let \(\xi_i\) to be the Type 2 element \((p_i, \xi_{i-1}, m_{2j_0, 1} - 1, e_{\eta_i}^*)\) of \(\Delta_{p_i}\).

It is clear that we have constructed a (12, 2, \(j_0 - 1, 0\))-dependent sequence \((z_i)_{1 \leq i \leq n_{2j_0} - 1}\).

By the estimate of Lemma [6,13] we have \[\|z\| \leq 48Cm_{2j_0, 1}^{-2}\]

for the average \[z = n_{2j_0, 1}^{-1} \sum_{i=1}^{n_{2j_0} - 1} z_i.\]

However, let us consider the element \(\gamma = \xi_{n_{2j_0} - 1}\) of \(\Delta_{m_{2j_0, 1}^{-1}}\), which has evaluation analysis \[e_{\gamma}^* = \sum_{i=1}^{n_{2j_0, 1}^{-1}} d_{\xi_i}^* + m_{2j_0, 1}^{-1} \sum_{i=1}^{n_{2j_0, 1}^{-1}} P_{(p_i, q_i)} e_{\eta_i}^*\].

Proposition 7.3. Let \(T\) be a bounded linear operator on \(\mathcal{X}_K\) and let \((x_i)_{i \in \mathbb{N}}\) be a RIS in \(\mathcal{X}_K\). Then \(\text{dist}(Tx_i, \mathbb{R}x_i) \to 0\) as \(i \to \infty\).

Proof. It will be enough to prove the result for a RIS in \(\mathcal{X}_K \cap \mathbb{Q}^\Gamma\). Suppose, if possible, that \(\text{dist}(Tx_i, \mathbb{R}x_i) > \delta > 0\) for all \(i\). The idea is to obtain a dependent sequence in rather the same way as we did in Lemma [6,15] except that this time it will be a 0-dependent sequence, rather than a 1-dependent sequence.

We start by choosing \(j_0\) such that \(m_{2j_0, 1} > 256C\|T\|\delta^{-1}\) and \(j_1\) such that \(m_{4j_1, 2} > n_{2j_0, 1}^2\). Taking \(p = p_0 = 0\) and \(j = j_1 - 1\) in Lemma [7,2] we can find \(q_1\) and a (12, 4, \(j_1 - 2, 0\))-exact pair \((z_1, \eta_1)\) with rank \(\eta_1 = q_1\), \((Tz_1)(\eta_1) > \frac{7}{16}\delta\) and \(\|(I - P_{(0, q_1)})(Tz_1)\| < m_{4j_1, 2}^{-1}\delta\). Let \(p_1 = q_1 + 1\) and let \(\xi_1\) be the special Type 1 element of \(\Delta_{p_1}\) given by \(\xi_1 = (p_1, m_{j_0, 1} - 1, e_{\eta_1}^*).\)

Now, recursively for \(2 \leq i \leq n_{2j_0} - 1\), define \(j_i = \sigma(\xi_{i-1})\), and use the lemma again to choose \(q_i\) and a (12, 4, \(j_i, 0\))-exact pair \((z_i, \eta_i)\) with rank \(\eta_i = q_i\), \((Tz_i)(\eta_i) > \frac{7}{16}\delta\) and \(\|(I - P_{(p_i, q_i)})(Tz_i)\| < m_{4j_i, 1}^{-1}\delta\). We now define \(p_i = q_i + 1\) and let \(\xi_i\) to be the Type 2 element \((p_i, \xi_{i-1}, m_{2j_0, 1}^{-1}, e_{\eta_i}^*)\) of \(\Delta_{p_i}\).

It is clear that we have constructed a (12, 2, \(j_0, 1, 0\))-dependent sequence \((z_i)_{1 \leq i \leq n_{2j_0} - 1}\).

By the estimate of Lemma [6,13] we have \[\|z\| \leq 48Cm_{2j_0, 1}^{-2}\]

for the average \[z = n_{2j_0, 1}^{-1} \sum_{i=1}^{n_{2j_0} - 1} z_i.\]
Theorem 7.4. Let $p_k = q_k + 1$ for $k \geq 1$, and that $m_{4j_i} > m_{4j_i - 2} > n_{2j_0 - 1}^2$, we may estimate $(Tz)(\gamma)$ as follows
\[
(Tz)(\gamma) = n_{2j_0 - 1}^{-1} \sum_{k=1}^{n_{2j_0 - 1}} (Tz_k)(\gamma)
\]
\[
\geq n_{2j_0 - 1}^{-1} \sum_{k=1}^{n_{2j_0 - 1}} \left( (P_{(p_{k-1}, p_k)}^* e^*_{\gamma}, Tz_k) - \| (I - P_{(p_{k-1}, q_k)}) (Tx_k) \| \right)
\]
\[
\geq n_{2j_0 - 1}^{-1} \sum_{k=1}^{n_{2j_0 - 1}} \left( m_{2j_0 - 1}^{-1} (P_{(p_{k-1}, p_k)}^* e^*_{\gamma}, Tx_k) - m_{4j_i - 2}^{-1} \delta \right)
\]
\[
\geq \delta n_{2j_0 - 1}^{-1} \sum_{k=1}^{n_{2j_0 - 1}} \left( \frac{3}{2} m_{2j_0 - 1}^{-1} - 5n_{2j_0 - 1}^2 \right) > \frac{1}{4} m_{2j_0 - 1}^{-1} \delta.
\]
So
\[
\| Tz \| \geq \frac{1}{14} m_{2j_0 - 1}^{-1} > \frac{1}{14} C^{-1} \delta m_{2j_0 - 1} \| z \|,
\]
which is a contradiction because $\frac{1}{14} C^{-1} \delta m_{2j_0 - 1} > \| T \|$ by our original choice of $j_0$. □

**Theorem 7.4.** Let $T$ be a bounded linear operator on $X$. Then there exists a scalar $\lambda$ such that $T - \lambda I$ is compact.

**Proof.** We start by considering a normalized RIS $(x_i)$ in $X$. By Proposition 7.3 there exist scalars $\lambda_i$ such that $\| Tx_i - \lambda_i x_i \| \to 0$. We claim that $\lambda_i$ necessarily tends to some limit $\lambda$. Indeed, if not, by passing to a subsequence, we may suppose that $|\lambda_{i+1} - \lambda_i| > \delta$ for all $i$. Now the sequence $(y_i)$ where $y_i = x_{2i-1} + x_{2i}$ is again a RIS, so that there exist $\mu_i$ with $\| Ty_i - \mu_i y_i \| \to 0$ by Proposition 7.3 again. We thus have
\[
\| (\lambda_{2i} - \mu_i) x_{2i} + (\lambda_{2i-1} - \mu_i) x_{2i-1} \| \leq \| Tx_{2i} - \lambda_{2i} x_{2i} \| + \| Tx_{2i-1} - \lambda_{2i-1} x_{2i-1} \| + \| Ty_i - \mu_i y_i \| \to 0.
\]
Since the RIS $(x_i)$ is a block sequence, there exist $l_i$ such that $P_{(0, l_i)} y_i = x_{2i-1}$ and $P_{(l_i, \infty)} y_i = x_{2i}$. Using the assumption that the sequence $(x_i)$ is normalized we now have
\[
|\lambda_{2i-1} - \mu_i| = \| (\lambda_{2i-1} - \mu_i) x_{2i-1} \| \leq \| P_{(0, l_i)} \| \| (\lambda_{2i} - \mu_i) x_{2i} + (\lambda_{2i-1} - \mu_i) x_{2i-1} \|,
\]
with a similar estimate for $|\lambda_{2i} - \mu_i|$. Each of these sequences thus tends to 0, so that $\lambda_{2i} - \lambda_{2i-1}$ also tends to 0, contrary to our assumption.

We now show that the scalar $\lambda$ is the same for all rapidly increasing sequences. Indeed, if $(x_i)$ and $(x'_i)$ are RIS with $\| Tx_i - \lambda x_i \| \to 0$ and $\| Tx'_i - \lambda' x'_i \| \to 0$, we may find $i_1 < i_2 < \cdots$ such that the sequence $(y_k)$ defined by
\[
y_k = \begin{cases} x_{ik} & \text{if } k \text{ is odd} \\ x'_{ik} & \text{if } k \text{ is even} \end{cases}
\]
is again a RIS. By the first part of the proof we must have $\lambda = \lambda'$.

We have now obtained $\lambda$ such that $\|(T - \lambda I) x_i \| \to 0$ for every RIS. By Proposition 5.10 we deduce that $\|(T - \lambda I) x_i \| \to 0$ for every bounded block sequence in $X$. This, of course, implies that $T - \lambda I$ is compact. □
8. Reflexive subspaces with HI duals

We devote this section to a proof that $X_K$ is saturated with reflexive HI subspaces having HI duals. The proof involves reworking much of the construction of Section 6 in the context of a subspace of $X_K$ and its dual. By standard blocking arguments, it is enough to prove the following theorem.

**Theorem 8.1.** Let $L = \{l_0, l_1, l_2, \ldots \}$ be a set of natural numbers satisfying $l_{n-1} + 1 < l_n$, and for each $n \geq 1$ let $F_n$ be a subspace of the finite-dimensional space $P_{(l_{n-1}, l_n)} X_K = \bigoplus_{l_{n-1} < k < l_n} M_k$. Then the subspace $W = \bigoplus_{n \in N} F_n$ of $X_K$ is reflexive and has HI dual.

We note in passing the following corollary, which gives an indication of the “very conditional” nature of the basis of $\ell_1$ that we have constructed. For the purposes of the statement we briefly abandon the “$\Gamma$ notation” and revert to the notation of Definition 3.1 and Theorem 3.4.

**Corollary 8.2.** There exist a basis $(d_n^*)_{n \in \mathbb{N}}$ of $\ell_1$ and natural numbers $k_1 < k_2 < \cdots$ with the property that the quotient $\ell_1 / [d_n^* : n \in M]$ is hereditarily indecomposable whenever the subset $M$ of $\mathbb{N}$ contains infinitely many of the intervals $(k_p, k_{p+1}]$.

The rest of this section will be devoted to the proof of Theorem 8.1. We have already remarked at the end of Section 5 that the subspace $W$ defined in the statement of the theorem is reflexive. The subspaces $F_n$ form a finite-dimensional decomposition of $W$, the corresponding FDD projections being $Q_{(m,n)} = P_{(l_m, l_n)} | W = P_{(l_m, l_n)} | W$, when $0 \leq m < n$. The dual space $W^*$ has a dual FDD $(F_n^*)$ and corresponding projections $Q_{(m,n)}^*$. We shall establish hereditary indecomposability of $W^*$ via the criterion Proposition 2.7. We write $R$ for the quotient mapping $X_K^* = \ell_1 \to W^*$ and observe that if $f_n^* \in F_n^*$ for $1 \leq n \leq N$ then the norm of $f^* = \sum_{n=1}^N f_n^*$ in $W^*$ is given by

$$\| \sum f_n^* \|_{W^*} = \inf \{ \|g^*\| : g^* \in X_K^* \text{ and } Rg^* = f^* \}.$$

**Lemma 8.3.** If $f^* \in \text{im} Q_{(m,N)}^* = \bigoplus_{M < n \leq N} F_n^* \subset W^*$ then there exists $h^* \in X_K^* = \ell_1(\Gamma)$ with supp $h^* \subseteq \Gamma_{N-1} \setminus \Gamma_M$ and $\|h^*\|_1 \leq 4\|f^*\|$ and $RP_{(m,N)}^* h^* = RP_{(m,\infty)}^* h^* = f^*$.

**Proof.** We extend $f^*$ by the Hahn–Banach theorem to obtain $g^* \in X_K^* = \ell_1(\Gamma)$ with $Rg^* = f^*$ and $\|g^*\|_{X_K^*} = \|f^*\|_{W^*}$. We set $h^* = P_{(0,N)} g^* \in \ell_1(\Gamma_{N-1})$ and $b^* = h^* |_{\Gamma_{N-1} \setminus \Gamma_{i+1}}$, noting that

$$\|b^*\|_1 \leq \|h^*\|_1 \leq 2\|g^*\|_1 \leq 4\|g^*\|_{X_K^*} = 4\|f^*\|.$$

To check that $RP_{(m,N)}^* h^* = RP_{(m,\infty)}^* b^* = f^*$, we first note that

$$P_{(m,\infty)}^* b^* = P_{(m,\infty)}^* h^*,$$

because $P_{(m,\infty)}^* b^* = 0$ whenever supp $k^* \subseteq \Gamma_M$. Since both $b^*$ and $h^*$ are supported by $\Gamma_{N-1}$ we have

$$P_{(m,N)}^* b^* = P_{(m,\infty)}^* P_{(0,N)} b^* = P_{(m,\infty)}^* b^* = P_{(m,\infty)}^* P_{(m,N)}^* h^* = P_{(m,\infty)}^* P_{(0,N)} h^* = P_{(m,N)}^* g^*.$$

It follows that

$$R^* P_{(m,N)}^* b^* = R^* P_{(m,N)}^* g^* = g^* \circ P_{(m,N)} | W = g^* \circ Q_{(m,N)} = f^*.$$

**Lemma 8.4.** Let $j \geq 1$, $1 \leq a \leq n_{2j}$ and $M_0 < M_1 < M_2 < \cdots < M_a$ be natural numbers, with $2j \leq M_1$. For each $i \leq a$, let $f_i^*$ be in ball $\bigoplus_{M_i < n \leq M_{i+1}} F_n^*$ and write $f^* = \sum_{i=1}^a f_i^*$. Then there exists $\gamma \in \Gamma$ with $P_{(0,M)}^* e^*_\gamma = 0$ and $\|4m_{2j} R(e^*_\gamma) - f^*\| \leq 2^{-l_M+3}$; in particular $\|f^*\|_{Y^*} \leq 5m_{2j}$.
Proof. By Lemma 8.3 there exist \( h_i^* \in \ell_1(\Gamma_{I_{M+1}} \setminus \Gamma_{I_{M-1}}) \) with \( \|h_i^*\|_1 \leq 4 \) and \( R(P_{(I_{M+1},I_{M+1})}^*) = f_i^* \). Since \( B_{(I_{M+1},I_{M+1})} \) is an \( \epsilon \)-net in \( \ell_1(\Gamma_{I_{M+1}} \setminus \Gamma_{I_{M-1}}) \), with \( \epsilon = 2^{-l_{M+1}} \leq 2^{-l_{M-2}i+1} \), we can choose \( b_i^* \in B_{(I_{M+1},I_{M+1})} \) such that \( \|h_i^* - 4b_i^*\|_1 \leq 2^{-l_{M-2}i+3} \).

Now write \( p_i = I_{M_i} \) for \( 1 \leq i \leq a \) and apply the construction of Proposition 4.7 to obtain \( \gamma \in \Delta_{p_i} \) with evaluation analysis

\[
e_i^\gamma = \sum_{i=1}^{a} d_{\xi_i} + m_{2j} \sum_{i=1}^{a} P_{(p_{k-1},\infty)}^* b_k^*.
\]

Since rank \( \xi_i = p_i \in L \) for all \( i \), we have \( Rd_{\xi_i}^* = 0 \) and so

\[
\|f^* - 2m_{2j} R(e_i^\gamma)\| = \left\| \sum_{i=1}^{a} (f_i^*-2RP_{(p_{k-1},\infty)}^* b_i^*) \right\|
\leq \sum_{i=1}^{n_{2j}} \|RP_{(p_{k-1},\infty)}^* h_i^* - 2RP_{(p_{k-1},\infty)}^* b_i^*\|
\leq 3 \sum_{i=1}^{a} \|h_i^* - 2b_i^*\| \leq 3 \sum_{i=1}^{\infty} 2^{-l_{M-2}i+2} = 2^{-l_{M+2}}.
\]

It follows that \( \|f^*\| \leq \|4m_{2j} R(e_i^\gamma)\| + 8 \leq 5m_{2j} \). \( \square \)

**Lemma 8.5.** Let \( Y \) be any block subspace of \( W^* \) and let \( n, M \) be positive integers. For every \( C > 1 \) there exists a \( 4C^{-l_0^\gamma} \)-average \( w \in W \), with \( Q_{(0,M)}^* w = 0 \), and a functional \( g^* \in \text{ball } Y \) with \( Q_{(0,M)}^* g^* = 0 \) and \( \langle g^*, w \rangle \geq 1 \).

**Proof.** The proof is a dualized version of Lemma 6.2. We suppose, without loss of generality, that \( C < 2 \) and choose \( l, j \) such that \( C^l > n \) and \( n_{2j} > (10n_{2j})^l \); we take \( k \) minimal subject to \( C^k > 5m_{2j} \) noting that

\[ n_{2j} > (10n_{2j})^l \geq (2C^{l-1})^l \geq C^{kl} > n^k. \]

Now take \( (f_i^*)^n_{i=1} \) to be a normalized block sequence in \( Y \cap Q_{(0,M)}^* \); we may apply Lemma 8.4 to obtain

\[
\left\| \sum_{i=1}^{n} \pm f_i^* \right\| \leq 5m_{2j} < C^k.
\]

So by part (ii) of Lemma 2.2 (with \( C = 1 + \epsilon \)) there are successive linear combinations \( g_1^*, \ldots, g_n^* \) such that \( \|g_i^*\| \geq C^{-1} \) for all \( i \), while

\[
\left\| \sum_{i=1}^{n} \pm g_i^* \right\| \leq 1,
\]

for all choices of sign. Since \( (g_i^*) \) is a block sequence in \( \text{ker } Q_{(0,M)}^* \) we can choose \( M \leq N_0 < N_1 < \ldots \) such that \( Q_{(N_{i-1},N_i]}^* g_i^* = g_i^* \). Now we choose, for each \( i \) an element \( w_i \) of \( W \) such that \( \|w_i\| \leq C \) and \( \langle g_i^*, w_i \rangle = 1 \). If we set \( w'_i = Q_{(N_{i-1},N_i]} w_i \) then we have \( \|w'_i\| \leq 4C \) and \( \langle g_i^*, w'_i \rangle = \langle g_i^*, w_i \rangle = 1 \), while \( \langle g_i^*, w'_h \rangle = 0 \) when \( h \neq i \). The element \( w = n^{-1} \sum_{i=1}^{n} w'_i \) is thus a \( 4C^{-l_0^\gamma} \)-average, with \( Q_{(0,M)}^* w = 0 \), and satisfies \( \langle g^*, w \rangle = 1 \), where \( g^* = \sum_{i=1}^{n} g_i^* \in \text{ball } Y \). \( \square \)

**Lemma 8.6.** Let \( Y \) be any block subspace of \( W^* \) and let \( n, j \) be positive integers. There exists a \((600,2j,1)\)-exact pair \((z, \gamma)\) with \( z \in W \), \( Q_{(0,n]}^* z = 0 \), \( P_{(0,\infty)}^* e_{\gamma} = 0 \) and \( \text{dist}(Re_{\gamma}^*, Y) < 2^{-l_{\gamma}} \).

**Proof.** By repeated applications of Lemma 8.5 we construct natural numbers \( N \leq M_0 < M_1 < M_2 < \ldots \) and \( j_1 < j_2 < \ldots \), elements \( w_i = Q_{(M_{i-1},M_i]} w_i \) of \( W \), and functionals \( g_i^* = Q_{(M_{i-1},M_i]}^* g_i^* \in \text{ball } Y \) such that
(1) $w_i$ is a $5\cdot 1^{n_i}$-average;
(2) $\langle g_i^*, w_i \rangle \geq 1$;
(3) $j_{i+1} > M_i$.

It follows from Lemma 6.4 that $(w_i)$ is a 10-RIS.

Writing $g^* = \sum_{i=1}^{n_2j} g_i^*$ and applying Lemma 6.4 we find $\gamma$ of weight $m_{2j}$ such that $\|4m_{2j}R(e_\gamma^*) - g^*\| \leq 2^{-N+3}$. We thus have

$$\text{dist}(Re_\gamma^*, Y) \leq \|Re_\gamma^* - \frac{1}{4}m_{2j}^{-1}g^*\| \leq 2^{-lN+1}m_{2j}^{-1} \leq 2^{-lN},$$

and

$$4m_{2j} \sum_{i=1}^{n_{2j}} w_i(\gamma) \leq \sum_{i=1}^{n_{2j}} \langle g_i^*, w_i \rangle - 2^{-lN+3} \geq n_{2j} - 16.$$ We now set $z = \theta m_{2j} n_{2j}^{-1} \sum_{i=1}^{n_{2j}} w_i$ where $\theta$ is chosen so that $z(\gamma) = 1$; by the above inequality $0 < \theta \leq 4 + 128n_{2j}^{-1} < 5$.

To estimate $\|z\|$ and $|z(\gamma')|$ when weight $\gamma' = m_h \neq m_{2j}$ we return to Lemma 5.5 deducing that

$$\|z\| \leq 60\theta \quad \text{and} \quad |z(\gamma')| \leq \begin{cases} 110\theta m_h^{-1} & \text{if } h < 2j \\ 60\theta m_{2j}^{-1} & \text{if } h > 2j. \end{cases}$$

So $(z, \gamma)$ is certainly a $(600, 2j, 1)$-exact pair.

\begin{lemma}
Let $Y_1$ and $Y_2$ be block subspaces of $W^*$ and let $j_0$ be a natural number. There exists a sequence $(x_i)_{i \leq n_{2j_0}-1}$ in $W$, together with natural numbers $0 = p_0 < p_1 < p_2 < \cdots < p_{n_{2j_0}-1}$, and elements $\eta_i \in \Gamma_{p_i-1} \setminus \Gamma_{p_i-1}$, $\xi_i \in \Delta_{p_i}$ (1 \leq i \leq n_{2j_0}-1), satisfying the conditions (1) to (4) of Definition 6.10 with $C = 600$, $\varepsilon = 1$, and such that, for all $i \geq 1$, the following additional properties hold

(5) rank $\xi_i = p_i \in L$;
(6) $P_{[p_i-1, p_i]}^* e_{\eta_i}^* = 0$, $P_{[p_i-1, p_i]}(x_i) = x_i$;
(7) dist$(Re_{\eta_i}^*, Y_k) < 2^{-p_i-1}$, where $k = 1$ for odd $i$ and $k = 2$ for even $i$.
\end{lemma}

\begin{proof}
We start by choosing $j_1$ such that $m_{4j_1-2} > n_{2j_0-1}^2$ and then applying Lemma 8.6 to obtain a $(600, 4j_1-2, 1)$-exact pair $(x_1, \eta_1)$ with $x_1 \in W$. Set $p_1 = l_{N_1}$, where $N_1$ is large enough to ensure that $P_{[0, p_1]}x_1 = Q_{[0, N_1]}x_1 = x_1$, rank $\eta_1 < p_1$ and $2^{p_1} > 2n_{2j_0-1}$. Let $\xi_1 = (p_1, m_{2j_0-1}^{-1}, \eta_1) \in \Delta_{p_1}$.

Continuing recursively, if for some $i < n_{2j_0-1}$, we have defined $\xi_i \in \Delta_{p_i}$, where $p_i = l_{N_i}$, we set $j_{i+1} = \sigma(\xi_i)$ and apply Lemma 8.6 to get a $(600, 4j_{i+1}, 1)$-exact pair $(x_i, \eta_{i+1})$ with $x_{i+1} \in W$, $Q_{[0, N_i]}x_{i+1} = P_{[0, p_i]}x_{i+1} = 0$, $P_{[0, p_i]}e_{\eta_{i+1}}^* = 0$ and dist$(Re_{\eta_{i+1}}^*, Y_k) < 2^{-p_i}$, where $k$ depends on the parity of $i+1$. We now take $N_{i+1}$ large enough, set $p_{i+1} = l_{N_{i+1}}$ and define $\xi_{i+1} = (p_{i+1}, \xi_i, m_{2j_{i+1}-1}^{-1}, \eta_{i+1}) \in \Delta_{p_{i+1}}$.

We are now ready to finish the proof of the theorem. We consider any two infinite-dimensional subspaces $Y_1$ and $Y_2$ of $W^*$ and apply Lemma 8.7 obtaining a dependent sequence satisfying (1) to (7). By property (7) we may choose, for each $i$, an element $y_i^*$ of $Y_k$ with

$$\|y_i^* - Re_{\eta_i}^*\| < 2^{-p_i}.$$ We set

$$y^* = m_{2j_0-1}^{-1} \sum_{i \text{ odd}} y_i^* \in Y_1, \quad z^* = m_{2j_0-1}^{-1} \sum_{i \text{ even}} y_i^* \in Y_2.$$
If \( \gamma \) is the element \( \xi_{n2j_0-1} \) then the evaluation analysis of \( \gamma \) is

\[
e^*_{\gamma} = \sum_{i=1}^{n2j_0-1} d^i_{\xi_i} + m_{2j_0-1}^{-1} \sum_{i=1}^{n2j_0-1} p^*_{(p_i-1, \infty)} e^*_{\eta_i}
\]

\[
= \sum_{i=1}^{n2j_0-1} d^i_{\xi_i} + m_{2j_0-1}^{-1} \sum_{i=1}^{n2j_0-1} e^*_{\eta_i},
\]

because \( P^*_{(0,p_i-1)} e^*_{\eta_i} = 0 \). Since \( \text{rank} \xi_i = p_i \in L \) for all \( i \) we have

\[
\Re e^*_{\gamma} = m_{2j_0-1}^{-1} \sum_{i=1}^{n2j_0-1} \Re e^*_{\eta_i},
\]

which leads to

\[
\|y^* + z^*\| \leq 1 + \|m_{2j_0-1}^{-1} \sum_{i=1}^{n2j_0-1} \Re e^*_{\eta_i}\| = 1 + \|\Re e^*_{\gamma}\| \leq 2.
\]

We shall prove that \( \|y^* - z^*\| \) is very large by estimating \( \langle y^* - z^*, x \rangle \), where \( x \) is the average

\[
x = m_{2j_0-1}^{-1} \sum_{k=1}^{n2j_0-1} (-1)^k x_k,
\]

about which we know from Lemma 6.13 that

\[
\|x\| \leq 7200 m_{2j_0-1}^{-2}.
\]

By (7) and the definition of a 1-exact pair, we have

\[
\langle e^*_{\eta_i}, x_k \rangle = \begin{cases} 
1 & \text{if } i = k \\
0 & \text{if } i \neq k,
\end{cases}
\]

so that

\[
\langle y^* - z^*, x \rangle = m_{2j_0-1}^{-1} m_{2j_0-1}^{-1} \sum_{i,k} i, k \langle y^*_i, x_k \rangle
\]

\[
\geq m_{2j_0-1}^{-1} m_{2j_0-1}^{-1} \sum_{i,k} (\langle e^*_{\eta_i}, x_k \rangle - 2^{-p_i})
\]

\[
\geq m_{2j_0-1}^{-1} (1 - m_{2j_0-1}^{-2}) \geq \frac{1}{2} m_{2j_0-1}^{-1},
\]

the last step following from our choice of \( p_1 \) with \( 2^{p_1} > 2m_{2j_0-1} \).

We can now deduce that

\[
\|y^* - z^*\| \geq \frac{m_{2j_0-1}}{14400}.
\]

We have shown that the subspaces \( Y_1 \) and \( Y_2 \) of \( W^* \) contain elements \( y^* \), \( z^* \) with \( \|y^* + z^*\| \leq 2 \) and \( \|y^* - z^*\| \) arbitrarily large. By Proposition 2.1, we have established hereditary indecomposability of \( W^* \).
9. Concluding Remarks

9.1. Operators on subspaces of $\mathcal{X}_K$. If we are looking at a bounded linear operator $T : Y \to \mathcal{X}_K$ defined only on a subspace $Y$ of $\mathcal{X}_K$, rather than on the whole space, then, as in other HI constructions, the arguments of the preceding section can be used to show that $T$ can be expressed as $\lambda Y + S$ with $S$ strictly singular. However, as we shall now see, in this case the perturbation need not be compact.

Proposition 9.1. There exists a subspace $Y$ of $\mathcal{X}_K$ and a strictly singular, non-compact operator $T$ from $Y$ into $\mathcal{X}_K$. In fact, for a suitably chosen $Y$, we may choose $T$ mapping $Y$ into itself.

Proof. By a theorem of Androulakis, Odell, Schlumprecht and Tomczak-Jaegermann [3], in order to find $Y$ and a strictly singular, non-compact $T : Y \to \mathcal{X}_K$, it is enough to exhibit normalized sequences $(x_i)$ and $(y_i)$ in $\mathcal{X}_K$ such that $(y_i)$ has a spreading model equivalent to the usual $\ell_1$-basis, while $(x_i)$ has a spreading model that is not equivalent to that basis. For $(x_i)$ we may take any normalized RIS; indeed, by Proposition 5.3, the spreading model associated with any RIS is dominated by the unit vector basis of the Mixed Tsirelson space $\mathcal{T}(\omega\mathcal{S}_{m_j}, m_j^{-1})_{j \in \mathbb{N}}$, and so is not equivalent to the $\ell_1$-basis. For $(y_i)$ we may take a specific sequence, setting

$$y_n = \sum_{\xi \in \Delta_n} d_\xi.$$  

The result we need is a lemma about norms of linear combinations of these vectors.

Lemma 9.2. Let $F$ be a finite set of natural numbers with $\min F \geq j$ and $\# F < 2n_{2j}$. Then, for all real scalars $a_n$,

$$\| \sum_{n \in F} a_n y_n \| \geq \frac{1}{2} \sum_{n \in F} |a_n|.$$  

Proof. Without loss of generality, we may suppose that $\sum_{n \in F} a_n^+ \geq \frac{1}{2} \sum_{n \in F} |a_n|$ and we may choose $p_1, p_2, \ldots, p_r$ in $F$, with $p_{i+1} > p_i + 1$, $r \leq p_{2j}$, and

$$\sum_{i=1}^r a_i \geq \frac{1}{2} \sum_{n \in F} |a_n|.$$  

Since $p_1 \geq \min F \geq 2j$, $\Delta_n$ does contain Type 1 elements of the form $(p_1, m_{2j}^{-1}, \pm e_{\eta_1}^*)$, with $\eta_1 \in \Gamma_{n_{2j}}$. We take $\xi_1$ to be such an element, and continue recursively, for $1 \leq i < r$, taking $\eta_{i+1}$ to be any element of $\Delta_{p_{i+1}}$ and $\xi_{i+1}$ to be the Type 2 element $(p_{i+1}, \xi, m_{2j}^{-1}, \pm e_{\eta_{i+1}}^*)$ of $\Delta_{n_{2j}}$. If $\gamma = \xi_r$ then the evaluation analysis of $\Gamma$ is

$$e_\gamma^* = \sum_{i=1}^r d_{\xi_i}^* + m_{2j}^{-1} \sum_{i=1}^r \pm P^*_{(n_{i-1}, n_i)} e_{\eta_i}^*.$$  

If we write $y = \sum_{n \in F} a_n y_n$, we have $(d_{\xi_i}^*, y) = a_{n_i}$ for each $i$, so that

$$e_\gamma^*(y) = \sum_{i=1}^r a_{p_i} + m_{2j}^{-1} \sum_{i=1}^r \pm P^*_{(n_{i-1}, n_i)} e_{\eta_i}^*(y).$$  

We have not until now been explicit about how the signs $\pm$ were chosen, but it is now clear that this may be done in such a way that $e_\gamma^*(y) \geq \sum_{i=1}^r a_{p_i} \geq \frac{1}{2} \sum_{n \in F} |a_n|$.

It is now clear that the theorem of Androulakis et al may be applied. In order to get the refined version where $T$ takes $Y$ into itself, it is enough to look a little more closely at the proof given in [3]. It turns out that we may take $(y_i)$ as above and $Y$ to be the closed linear span $[y_i : i \in \mathbb{N}]$. It may be shown that, for any RIS $(x_i)$, the mapping $y_i \mapsto x_i$ extends to a bounded
linear operator from $Y$ to $X_K$. Since $Y$, like all other infinite dimensional subspaces, contains a RIS, we may choose the $x_i$ to lie in $Y$.

9.2. Very incomparable Banach spaces. The original spaces $X_{a,b}$ of Bourgain and Delbaen provided, for the first time, a continuum of non-isomorphic $\mathcal{L}_\infty$ spaces. It has also been noted \cite{1} that if we take $Y$ to be Hilbert space and $X$ to be $X_{a,b}$ with (for instance) $0 < b < \frac{1}{2} < a < 1$, $a^4 + b^4 = 1$, then all operators from $X$ to $Y$ and all operators from $Y$ to $X$ are compact. The constructions in the present paper allow us to exhibit a continuum of spaces $X_\alpha$ ($\alpha \in \mathcal{C}$) such that, for all $\alpha \neq \beta$, $\mathcal{L}(X_\alpha, X_\beta) = \mathcal{K}(X_\alpha, X_\beta)$.

We start by taking an almost-disjoint family $(L_\alpha)_{\alpha \in \mathcal{C}}$ of infinite subsets of $\mathbb{N}$. For each $\alpha$ we enumerate $L_\alpha$ in increasing order as $l^\alpha_j$ and define

$$m^\alpha_j = m^\alpha_{j+1}, \quad n^\alpha_j = n^\alpha_{j+1},$$

where $(m_j, n_j) = (2^{2^j}, 2^{2^j+1})$ is the sequence mentioned in Subsection 2.4.

Now we may take $X_\alpha$ to be either $\mathcal{B}(\mathbb{N})[([\mathcal{A}_{m^\alpha_j}, 1/m^\alpha_j])_{j \in \mathbb{N}}]$ or $X_K([([\mathcal{A}_{m^\alpha_j}, 1/m^\alpha_j])_{j \in \mathbb{N}}]$.

**Proposition 9.3.** Assume that $\alpha \neq \beta$ and let $T : X_\alpha \to X_\beta$ be a bounded linear operator. For any RIS $(x_i)_{i \in \mathbb{N}}$ in $X_\alpha$, we have $\|T(x_i)\| \to 0$ as $i \to \infty$.

**Proof.** Let $(x_i)$ be a C-RIS in $X_\alpha$ and suppose, if possible, that $\|T(x_i)\| > \delta > 0$ for all $i$. Since $(T(x_i))$ is weakly null we may, by taking a subsequence, assume that $(T(x_i))$ is a small perturbation of a skipped-block sequence in $X_\beta$. Thus, if $l = l^\beta_{2j} \in L_\beta$, we may apply Proposition 4.8 to conclude

$$\|n^\alpha_{l-1} \sum_{i=1}^{n_l} T x_r\|_{X_\beta} \geq \frac{1}{4} m^{-1}_{2j} n^\alpha_{l-1} \sum_{r=1}^{n_l} \|T x_r\| \geq \frac{1}{4} \delta m^{-1}_{2j}.$$

On the other hand, Corollary 5.4 tells us that

$$\|n^\alpha_{l-1} \sum_{i=1}^{n_l} x_r\|_{X_\alpha} \leq 10 C \|n^\alpha_{l-1} \sum_{i=1}^{n_l} e_i\|,$$

where the norm on the right-hand side is calculated in $T([([\mathcal{A}_{m^\alpha_j}, 1/m^\alpha_j])_{j \in L_\alpha}$]. If $l$ is not in $L_\alpha$ then this norm is at most $m^{-2}_l$ by Lemma 2.4, so that

$$\|n^\alpha_{l-1} \sum_{i=1}^{n_l} x_r\|_{X_\alpha} \leq 10 C m^{-2}_l.$$

By the assumed almost-disjointness of $L_\beta$ and $L_\alpha$ we can certainly choose $j$ such that $l^\beta_{2j} \notin L_\alpha$ and $m_j > 40 \|T\|^{-1}$, yielding a contradiction.

**Remark.** The topologies $\sigma(\ell_1, X_\alpha)$ provide a continuum of very incomparable weak* topologies on $\ell_1$; indeed, any linear mapping on $\ell_1$ which is continuous from $\sigma(\ell_1, X_\alpha)$ to $\sigma(\ell_1, X_\beta)$, with $\alpha \neq \beta$ is necessarily compact.

9.3. The space of operators $\mathcal{L}(X_K)$. Of course, the spaces $\mathcal{L}(X)$ and $\mathcal{K}(X)$ of bounded (respectively compact) linear operators on an infinite-dimensional Banach space $X$ are always decomposable. (Indeed, for finite dimensional subspaces $E \subset X$ and $F \subset X^*$, the subspaces $X^* \otimes E$ and $F \otimes X$ are complemented.) So we must not hope for too much exotic structure in these spaces of operators. In this section we shall look briefly at subspaces of $\mathcal{L}(X_K)$. Certainly, $\mathcal{L}(X_K) = \mathcal{K}(X_K) \oplus \mathbb{R}I$ has HI subspaces, such as those isomorphic to $X_K$ and subspaces isomorphic to $X_K = \ell_1$. It has no subspace isomorphic to $c_0$ by a result of Emmanuele. (The main result of \cite{14} shows that $c_0$ does not embed into $\mathcal{K}(X,a,b)$ and the same proof works for $X_K$.) We shall now see that $(X_K)$ does have other subspaces with unconditional basis. It is a general fact
that if \((x_n)\) is a basic sequence in a Banach space \(X\) then the injective tensor product \(\ell_1 \hat{\otimes} X\) contains a sequence equivalent to the “unconditionalization” of the basic sequence \((x_n)\). This follows immediately from the following exact formula for the norm of a finite sum of elementary tensors in \(\ell_1 \hat{\otimes} X\):

\[
\| \sum_{j=1}^{n} e_j^* \otimes x_j \| = \sup \| \sum_{j=1}^{n} \pm x_j \|,
\]

where the supremum is over all choices of signs.

In the case of \(X_K\) the space of compact operators \(\mathcal{K}(X_K)\) is isomorphic to \(\ell_1 \hat{\otimes} X_K\) and so contains the unconditionalization of any basic sequence in \(X_K\). An interesting special case is that of the basis \((d_n)\); we have chosen to prove the following proposition in a way that does not depend on the general theory of tensor products.

**Proposition 9.4.** The family \((e_\gamma^* \otimes d_\gamma)_{\gamma \in \Gamma}\) is an unconditional basis of a reflexive subspace of \(\mathcal{K}(X_K)\).

**Proof.** Let us write \(U_\gamma = e_\gamma^* \otimes d_\gamma\) considered as the rank–1 operator

\[U_\gamma : X_K \rightarrow X_K; x \mapsto x(\gamma)d_\gamma.\]

For a finite linear combination \(W = \sum_{\gamma \in \Gamma_n} w(\gamma)U_\gamma\) and any \(x \in \text{ball } X_K\) we have

\[\|W(x)\| = \| \sum_{\gamma \in \Gamma_n} (w\gamma)x(\gamma)d_\gamma \| \leq \max_{\pm} \| \sum_{\gamma \in \Gamma_n} \pm w(\gamma)d_\gamma \|.
\]

We shall write \(||W|||\) for the last expression on the line above. We have thus shown that \(||W||| \leq ||W||\).

On the other hand, if we choose \(u(\gamma) = \pm 1\) for \(\gamma \in \Gamma_n\) in such a way as to achieve the maximum in the definition of \(||W|||\) and then set \(y = i_n(u)\) we have

\[\|W(y)\| = \| \sum_{\gamma \in \Gamma_n} u(\gamma)d_\gamma \| = \| W(y) \| \leq ||W|| \| i_n \| \leq 2||W||.
\]

Thus the operator norm \(|\cdot||\) and the unconditionalized norm \(|||W|||\) are equivalent on \([U_\gamma : \gamma \in \Gamma]\).

It will be convenient to work with the latter norm.

Given a linear combination \(V = \sum_{\gamma} v(\gamma)U_\gamma\), any vector \(\sum_{\gamma} \pm v(\gamma)d_\gamma\) in \(X_K\), (whether or not the signs achieve the supremum in the definition of the unconditionalized norm), will be called a realization of \(W\).

If the subspace \([U_\gamma : \gamma \in \Gamma]\) is not reflexive then by unconditionality there is a skipped block sequence equivalent to the unit vector basis of either \(c_0\) or \(\ell_1\). We shall treat the case of \(\ell_1\), leaving the (very easy) other case to the reader.

We consider a normalized skipped block sequence with \(V_i = \sum_{\gamma \in \Gamma_{n_{i-1}} \setminus \Gamma_{n_{i-2}}} v(\gamma)U_\gamma\) and suppose, if possible, that \((V_i)\) is \(C\)-equivalent to the usual \(\ell_1\)-basis for the norm \(||\cdot|||\). More precisely, let us suppose that \(||V_i||| \leq C\) for all \(i\) and that

\[\| \sum_i a(i)V_i \| \geq \sum_i |a(i)|
\]

for all scalars \(a_i\). Let us note that if \(W\) is a linear combination of the form

\[W = n^{-1} \sum_{i=l+1}^{l+n} V_i,
\]

then any realization \(\hat{W}\) of \(W\) is a \(C-\ell_1\)-average as in Definition 6.1. Indeed \(\hat{W}\) is expressible as \(n^{-1} \sum_{i=l+1}^{l+n} \hat{V}_i\) where the \(\hat{V}_i\) are realizations of \(V_i\), and so satisfy \(||\hat{V}_i||| \leq |||V_i||| \leq C\) for all \(i\).
We now look at Lemma 6.4. It should be clear that, by choosing sequences \((j_k)_{j \in \mathbb{N}}\) and \((l_k)_{j \in \mathbb{N}}\) growing sufficiently fast, we may define
\[
W_k = n_{j_0}^{-1} \sum_{i=l_j+1}^{l_j+n_{j_k}} V_i,
\]
in such a way that any realizations \(\hat{W}_k\) form a 2C-RIS in \(X_K\). In particular
\[
\|n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} W_k\| = \|n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} \hat{W}_k\|
\]
for suitable realizations \(\hat{W}_k\), yielding
\[
\|n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} W_k\| \leq 12Cm_{j_0}^{-1},
\]
by Proposition 5.5. On the other hand,
\[
\|n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} W_k\| = \|n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} \sum_{i=l_k+n_{j_k}}^{l_k+n_{j_k}} V_i\|
\]
which is at least 1, by our assumption on \((V_i)\).

So we have a contradiction for suitably large values of \(j_0\). □

9.4. Open problems. Our constructions give no clue as to whether there exists a reflexive Banach space on which all operators are scalar–plus–compact. The construction of such a space, if one exists, will need new ideas. We thus have no example of a reflexive space on which all operators have non-trivial proper invariant subspaces. It is piquant to observe that, at the other end of the spectrum, the construction of a reflexive space on which some operator has no non-trivial proper invariant subspace has also proved to be very resistant to attack. We refer the reader to the papers of Enflo [15, 16] and Read [30, 31] for more about the Invariant Subspace Problem, noting the more recent paper [32] of Read, in which a strictly singular operator is constructed which has no non-trivial proper invariant subspace.

As we remarked in the introduction, we do not know whether an isomorphic predual of \(\ell_1\) which has the “few-operators” property in the scalar–plus–strictly-singular sense necessarily also has this property in the scalar–plus–compact sense. An answer to this would follow from an affirmative solution to the following more general problem.

Problem 9.5. Let \(X\) be a \(\mathcal{L}_\infty\)-space. Is every strictly singular operator on \(X\) weakly compact?

Working with Ch. Raikoftsalis, the present authors have recently constructed another counterexample to the scalar–plus–compact problem. Like the space presented here, it is a \(\mathcal{L}_\infty\)-space constructed by the Bourgain–Delbaen method. However, the new space has non-separable dual and has a subspace isomorphic to \(\ell_1\). We believe it to be the first example of an indecomposable space containing \(\ell_1\). The only obvious obstruction to embeddability of a given Banach space \(X\) into an indecomposable space is the existence in \(X\) of a subspace isomorphic to \(c_0\). We therefore are led to pose another problem.

Problem 9.6. Let \(X\) be a separable Banach space with no subspace isomorphic to \(c_0\). Does \(X\) necessarily embed in an indecomposable space? Does \(X\) necessarily embed in a \(\mathcal{L}_\infty\)-space with the scalar–plus–compact property?
It is tempting to believe that some combination of the techniques developed in this paper with the Bourgain–Pisier method [13] for embedding arbitrary Banach spaces into $L_\infty$-spaces might provide an answer.

We should like to draw the reader’s attention to the problems posed by Bourgain [11, page 46] about the spaces $X_{a,b}$ and $L_\infty$-spaces in general. Problems 1, 2 and 3 remain open. Hoping that his or her appetite has been whetted by the present paper, we leave it to the reader to find out what these problems are. Concerning Problem 4, we now know [19] that there is an infinite-dimensional Banach space with separable dual, no reflexive subspace and no subspace isomorphic to $c_0$. The present paper yields an example of an $L_\infty$ space with no unconditional basis sequence. But we still do not have an example of a space $X$ with $X^*$ isomorphic to $\ell_1$ and not containing $c_0$ or a reflexive subspace. Only Problem 5 has been completely settled: each $X_{a,b}$ is saturated with $\ell_p$ for some $p$ [22].
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