FREEZING AND DECORATED POISSON POINT PROCESSES

ELIRAN SUBAG AND OFER ZEITOUNI

ABSTRACT. The limiting extremal processes of the branching Brownian motion (BBM), the two-speed BBM, and the branching random walk are known to be randomly shifted decorated Poisson point processes (SDPPP). In the proofs of those results, the Laplace functional of the limiting extremal process is shown to satisfy $L[\theta_y f] = g(y - \tau f)$ for any nonzero, nonnegative, compactly supported, continuous function $f$, where $\theta_y$ is the shift operator, $\tau_f$ is a real number that depends on $f$, and $g$ is a real function that is independent of $f$. We show that, under some assumptions, this property characterizes the structure of SDPPP. Moreover, when it holds, we show that $g$ has to be a convolution of the Gumbel distribution with some measure.

The above property of the Laplace functional is closely related to a 'freezing phenomenon' that is expected by physicists to occur in a wide class of log-correlated fields, and which has played an important role in the analysis of various models. Our results shed light on this intriguing phenomenon and provide a natural tool for proving an SDPPP structure in these and other models.

1. Introduction

The branching Brownian motion (BBM) is a continuous-time branching process described as follows. At time $t = 0$ a single particle starts a standard Brownian motion $x$ from the origin, continuing for a randomly distributed exponential time $T$ independent of $x$. At this moment, the particle splits into two particles. Each, in turn, performs a Brownian motion starting from $x(T)$ and is subject to the same splitting rule. Thus, at time $t$ there is a random number of particles $N(t)$ and we denote their positions by $X_1(t), \ldots, X_{N(t)}(t)$.

The BBM has been extensively studied over the last decades. The seminal works of McKean [43], Bramson [10, 11], and Lalley and Sellke [38] were mainly concerned with the maximum (or rightmost particle) $M_t = \max_{i \leq N(t)} X_i(t)$ and its relations to the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation [37]. In particular, it was shown in [38] that with appropriate recentering term $m_t$,

$$\lim_{t \to \infty} \mathbb{P}\{M_t - m_t \leq x\} = \mathbb{E}\exp\left\{-e^{-\sqrt{2}(\varepsilon \log(CZ)/\sqrt{2})}\right\},$$

where $Z$ is the limit of the so-called derivative martingale and $C$ is a constant.

Recently, BBM became the object of renewed interest with the main focus being the behavior of extreme values of the process [14, 34, 41, 4, 11, 13]. Perhaps the most important result in this direction is a remarkable description of the limiting extremal process, i.e. the limit in distribution

$$\xi = \lim_{t \to \infty} \xi_t = \lim_{t \to \infty} \sum_{i \leq N(t)} \delta_{X_i(t) - m_t},$$

Research partially supported by a grant of the Israel Science Foundation.
given independently by Arguin, Bovier and Kistler \[6\] and Aïdékon, Berestycki, Brunet and Shi \[1\].

In the sequel, we denote by \( \overset{d}{=} \) equality in distribution. For a point process \( D = \sum_{i \geq 1} \delta_{d_i} \), we denote by \( \theta_x D \) the shift of \( D \) by \( x \), i.e. \( \theta_x D = \sum_{i \geq 1} \delta_{d_i+x} \).

The following notions describe the structure of limits alluded to above.

**Definition 1.** (1) A point process \( \psi \) is a decorated Poisson point process (DPPP) of intensity \( \nu \) and decoration \( D \) (denoted \( \psi \sim \text{DPPP}(\nu, D) \)), if \( \psi \overset{d}{=} \sum_{i \geq 1} \theta_{\zeta_i} D_i \) where \( \zeta = \sum_{i \geq 1} \delta_{\zeta_i} \) is a Poisson process with intensity \( \nu \), \( D \) is some point process, and \( D_i, i \geq 1 \), are copies of \( D \), independent of each other and of \( \zeta \).

(2) A point process \( \varphi \) is a randomly shifted decorated Poisson point process (SDPPP) of intensity \( \nu \), decoration \( D \) and shift \( S \) (denoted \( \varphi \sim \text{SDPPP}(\nu, D, S) \)) if for \( \psi \sim \text{DPPP}(\nu, D) \) and some independent (of \( \psi \)) random variable \( S \) it holds that \( \varphi \overset{d}{=} \theta_S \psi \).

In this notation \[1, 6\] showed that, with some point process \( D \), the limiting extremal process \( \xi \) in \[1, 2\] satisfies
\[
\xi \sim \text{SDPPP} \left( e^{-\sqrt{2x}} dx, D, \log \left( \sqrt{2CZ} \right) / \sqrt{2} \right),
\]
where \( C, Z \) are as in \[14\].

Our work centers around the relations between two properties of the limiting extremal process, one being the specific structure we have just described. The other is related to an intriguing ‘freezing’ phenomenon observed first by Spohn and Derrida in \[23\], exploiting Bramson’s results on the F-KPP equation \[11\]. They define the function \( G_{t, \beta} (y) \triangleq \mathbb{E} \exp \left\{ -e^{-\beta y} \sum_{i=1}^{N(t)} e^{\beta X_i(t)} \right\} \) and conclude it exhibits the shape of a ‘traveling wave’ as \( t \to \infty \). That is, for some \( m_{t, \beta} \) increasing in \( t \),
\[
\lim_{t \to \infty} G_{t, \beta} (y + m_{t, \beta}) = g_{\beta} (y).
\]
Moreover, they show that the profile of the wave and the velocity ‘freeze’ at a certain transition temperature \( \beta = \beta_c \) (\( \beta \) is the inverse temperature):
\[
\text{for any } \beta > \beta_c : \quad g_{\beta} (x) = g_{\beta_c} (x) \quad \text{and} \quad m_{t, \beta} + c_{\beta} = m_t,
\]
with some constants \( c_{\beta} \) depending on \( \beta \).

We introduce the shift-Laplace functional of a point process \( \xi \):
\[
L_\xi [f \mid y] \triangleq \mathbb{E} \left\{ \exp \left( -\int \theta_y f d\xi \right) \right\},
\]
where \( f : \mathbb{R} \to \mathbb{R} \) is measurable, nonnegative function and where we abuse notation by writing \( \theta_y f(x) = f(x-y) \). Denote \( f \approx g \) whenever two functions are equal up to translation and let \([g]\) denote the equivalence class of \( g \) under this relation. Since \( \xi_t \to \xi \) in distribution, it is easily seen (under some boundedness condition, cf. Lemma \[38\]) that freezing, put in other words, means that
\[
\text{for any } f \in \{ e^{\beta y} : \beta > \beta_c \} : \quad L_\xi [f \mid \cdot] \approx g_{\beta_c} (\cdot).
\]

Inspired by this, we introduce the following notion, using the notation \( C^+_c (\mathbb{R}) \) for the class of nonnegative, compactly supported, continuous, real functions that are not identically equal to 0.

**Definition 2.** A shift-Laplace functional is uniquely supported on \([g]\) if \( L_\xi [f \mid \cdot] \approx g (\cdot) \) for any \( f \in C^+_c (\mathbb{R}) \).
The following is a direct corollary of our main result Theorem 9. As we shall see in Lemma 12, when the shift-Laplace functional is uniquely supported the corresponding function $g$ is monotone. Thus in order to simplify the notation we restrict below to the case where $g$ is increasing. The point process $D(\xi)$ appearing in the statement is defined as the limit, as $y \to \infty$, of $\theta_{-y} \xi$ conditioned on $\xi((y, \infty)) > 0$, shifted so its maximum is at 0 (see Section 6). Denote by $\text{Gum}(y) \triangleq \exp\{-e^{-y}\}$ the standard Gumbel distribution function.

**Corollary 3.** Let $\xi$ be a point process such that $P\{\xi(\mathbb{R}) > 0\} = 1$ and let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function. Consider the following statements:

- **(US)** $L_\xi [f \, | \, \cdot] \text{ is uniquely supported on } [g]$.
- **(SUS)** $L_\xi [f \, | \, \cdot] \text{ is uniquely supported on } [g]$ and for some random variable $Z$,

$$
(1.6) \quad g(y) = \int \text{Gum}(c(y-z)) \, d\mu_Z(z) = \mathbb{E}\{\text{Gum}(c(y-Z))\},
$$

where $\mu_Z$ is the law of $Z$ and $c > 0$.
- **(SDP)** $\xi \sim \text{SDPPP}(e^{-cx} dx, D, Z)$ for some point process $D$, random variable $Z$, and constant $c > 0$.

If $g$ satisfies (2.1) below, then (SUS) and (SDP) are equivalent and (SDP) holds with $D$ equal to $D(\xi)$ up to translation. If, in addition, $D(\xi)$ satisfies a certain boundedness condition (Assumption 8), or if $L_\xi [e^{\beta x} \, | \, \cdot] \approx g(\cdot)$ for all $\beta > c$, then all three conditions are equivalent.

By Corollary 14 below, in the presence of (US), the condition $L_\xi [e^{\beta x} \, | \, \cdot] \approx g(\cdot)$ is equivalent to the condition $P\{\int e^{\beta x} d\xi(x) < \infty\} = 1$. We conjecture (see Conjecture 10 below), that the assumptions in the last sentence in Corollary 3 are not needed for the conclusion.

In the case of BBM, where all three conditions are already known to occur, the implications of the result are rather conceptual than practical. However, in other models it allows one to prove that the limiting extremal process admits the representation of (SDP) by studying its Laplace functional. In fact, as we discuss in Section 3 a study of the Laplace functional is the main step in the approach taken for all the models for which the limiting extremal process is known to satisfy (SDP): the BBM [6], the two-speed BBM [9], and the branching random walk (BRW) [39]. In particular, Proposition 3.2 of [6] gives exactly Condition (SUS) above.

We mention the class of logarithmically-correlated (log-correlated, for short) Gaussian fields [7, 16, 25, 29, 40], which are of great importance in a variety of fields (see Section 3). Log-correlated Gaussian fields are known to share many properties with BBM. For example, the distribution of the maximum [8, 12, 40], the overlap of extremal points [3, 7], and their limiting Gibbs measures [46, 49] behave similarly. It is therefore widely believed that their limiting extremal processes exist and should also exhibit the structure of Condition (SDP). Interestingly, Carpentier and Le Doussal [16] postulate that the freezing phenomenon described above is also satisfied by a wide class of log-correlated Gaussian fields. For the sub-class of star scale invariant fields freezing was proved in [41]. Other works in the physics literature that are related to freezing include [28, 29, 31, 32]. From those studies it is evident that the freezing phenomenon plays an important role in the analysis of log-correlated fields. We hope the link we have made between freezing and the structure of the extremal process will improve the understanding of both phenomena. In light of the above we also believe that it provides a natural
tool for attacking the problem of proving (SDP) for the limiting extremal process of log-correlated fields.

Lastly, let us briefly mention the related problem of the characterization of decorated Poisson point processes (with no shift, $Z = 0$). A point process is said to be exponentially-$c$-stable, $c > 0$, if for three independent copies of it $\xi$, $\xi_1$ and $\xi_2$ and any two numbers $a, b$ such that $e^a + e^b = c$, $\xi \doteq \theta_a \xi_1 + \theta_b \xi_2$. Brunet and Derrida \[14\] (p. 18) conjectured that this property is equivalent to (SDP) with $Z = 0$. This can be proved using a representation of the Laplace functional for infinitely divisible processes (see Maillard \[42\]; implicitly, this also appears in \[19\]). It is fairly simple to see that exponential-$c$-stability is equivalent to uniqueness of the support of $L_{\xi}[f \mid x]$ up to translation with $g(x) = \text{Gum}(cx)$ (cf. Corollary \[11\]). Hence, our main result can be seen as a generalization of this characterization. Let us stress, however, that the assumption of exponential-stability constrains $g$ to be of the form just specified and so in this case the relation to Condition (US) is irrelevant. Apart from the equivalence of exponential-stability and (SDP) with $Z = 0$, a corollary of our main result yields a description of the decoration process $D$ in terms of the original process $\xi$.

In the next section we give some further definitions and state our main results. In Section 2 we discuss in more detail the relations to the works mentioned above from the mathematics and physics literature. We prove one (easy) direction of our main theorem in Section 1. Sections 3 and 4 are devoted to results we shall need in the proof of the other direction, given in Section 5. In particular, in Section 6 we shall describe a construction of a point process which will be used as the decoration of a specific SDPPP process we use in our proofs. In Section 5 we discuss the relations between freezing for a limiting process (1.5) and the definition in terms of the sequence of processes (1.3), (1.4). Finally, a short appendix is devoted to the proof of a relation between Gumbel distribution functions of different scales.

2. Main Results

Some of the results rely on the following assumption. As Proposition 16 of Section 5 states, equation (2.1) holds whenever the intensity measure of the process is finite on some nonempty, open set.

Assumption 4. The shift-Laplace functional $L_{\xi}[f \mid \cdot]$ is uniquely supported on $[g]$ where $g$ is increasing and $g$ satisfies, with some constant $c = c_\xi > 0$,

$$\lim_{x \to \infty} \frac{1 - g(x + y)}{1 - g(x)} = \lim_{x \to \infty} \frac{\log g(x + y)}{\log g(x)} = e^{-cy}. \quad (2.1)$$

In Section 6 we shall construct, under a weaker assumption than the above, the point process $D(\xi)$ (see Definition PP-D). For some of our results we will require it to have one of the following properties.

Definition 5. (1) The point process $D(\xi)$ is said to have exponential moments if there exist some $t, \epsilon > 0$ such that $\mathbb{E}[\exp \{tD(\xi)(-\epsilon, 0]\}] < \infty$.
(2) The point process $D(\xi)$ is said to satisfy a law of large numbers (LLN) with rate $c$ if for some function $\alpha : \mathbb{R} \to \mathbb{R}$ and some constant $u > 0$, the point process $\psi_c \sim \text{DP}(\mathbb{R} \to \mathbb{R})$ satisfies

$$\frac{\psi_c((-y, \infty))}{\alpha(y)} \xrightarrow{\text{prob.}} u, \quad \text{as } y \to \infty. \quad (2.2)$$
Theorem 9. Let $\inf \{ \xi (\infty) \}$ be the infimum to be particle) as $M$ be an increasing function. Then the following hold:

$$q \psi_c ((-y, \infty)) \overset{d}{=} \sum_{i=1}^{N_y} \theta_{X_i} (\langle 0, \infty \rangle),$$

where $N_y \sim \text{Pois}(e^{\alpha y}/c)$, $X_i \overset{d}{=} X \sim \exp(e)$, and $D_i \overset{d}{=} D(\xi)$ are all independent. Thus, if $\mathbb{E} \theta_X(\mathbb{D}(\xi)((0, \infty))) < \infty$, it follows by the law of large numbers that $D(\xi)$ satisfies an LLN with rate $c$. That is,

$$\frac{\psi_c ((-y, \infty))}{e^{\alpha y}/c} \xrightarrow{\text{prob.}} \mathbb{E} \theta_X(\mathbb{D}(\xi)((0, \infty))), \quad y \to \infty.$$

**Remark 6.** By similar arguments to those in Section 4 it can be shown that

$$\psi_c ((-y, \infty)) = \sum_{i=1}^{N_y} \theta_{X_i} (\langle 0, \infty \rangle),$$

where $N_y \sim \text{Pois}(e^{\alpha y}/c)$, $X_i \overset{d}{=} X \sim \exp(e)$, and $D_i \overset{d}{=} D(\xi)$ are all independent. Thus, if $\mathbb{E} \theta_X(\mathbb{D}(\xi)((0, \infty))) < \infty$, it follows by the law of large numbers that $D(\xi)$ satisfies an LLN with rate $c$. That is,

$$\frac{\psi_c ((-y, \infty))}{e^{\alpha y}/c} \xrightarrow{\text{prob.}} \mathbb{E} \theta_X(\mathbb{D}(\xi)((0, \infty))), \quad y \to \infty.$$

**Remark 7.** From (2.3) it follows that for any natural $k$,

$$\psi_c ((-y + \log k/c, \infty)) \overset{d}{=} \sum_{j=1}^{k} \psi^{(j)}_c ((-y, \infty)),$$

where $\psi^{(j)}_c$ are i.i.d copies of $\psi_c$. Assuming (2.2), it is therefore straightforward to verify that for any rational $q$,

$$\lim_{y \to \infty} \frac{\psi_c ((-y + \log q/c, \infty))}{\alpha(y + \log q/c)} = q \lim_{y \to \infty} \frac{\alpha(y)}{\alpha(y + \log q/c)} \frac{\psi_c ((-y, \infty))}{\alpha(y)},$$

and thus, since $\psi_c ((-y, \infty))$ is increasing in $y$, for any real $r$,

$$\lim_{y \to \infty} \frac{\alpha(y + t)}{\alpha(y)} = e^{ct}.$$

**Assumption 8.** Let $\xi$ be a point process that satisfies Assumption 4 with constant $c_\xi$. Assume that the decoration process $D(\xi)$ has exponential moments, or that $D(\xi)$ satisfies an LLN with rate $c_\xi$.

The following is our main result which, under the assumptions above, shows that uniqueness of the support of the shift-Laplace functional - a property of the Laplace functional of the process - characterizes the structure of randomly shifted decorated Poisson point process. The conditions (US), (SUS), and (SDP) considered here are the ones stated in Corollary 3. Whenever $L_\xi [ f \cdot ]$ is uniquely supported on $[g]$, for any $f \in C^+_\mathbb{R}$, we define $\tau_f = \tau^Z_f (\xi)$ to be the (unique, whenever $P \{ \xi (\mathbb{R}) > 0 \} > 0$, as follows from Lemma 12) translation such that $L_\xi [ f \cdot y ] = g(y - \tau_f)$. For a point process $\xi$ define the maximum (or rightmost particle) as $\mathcal{M}(\xi) \overset{\text{d}}{=} \inf \{ y \in \mathbb{R} : \xi (\langle y, \infty \rangle) = 0 \}$, where if the set is empty we take the infimum to be $\infty$.

**Theorem 9.** Let $\xi$ be a point process such that $P \{ \xi (\mathbb{R}) > 0 \} = 1$ and let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function. Then the following hold:

**Converse part:** If (SDP) holds, then (SUS) holds with the same $Z$ and $c$ and, in particular, (US) holds. Moreover, in this case the corresponding shifts are given, for any $f \in C^+_\mathbb{R}$, by

$$\tau^Z_f (\xi) = c^{-1} \log \left( -\int_{-\infty}^{\infty} e^{-ct} (L_D [ f \cdot ] - t) dt \right).$$

**Direct part:** Let Assumption 4 hold with $c_\xi$. 
(a) If (SUS) holds with some random variable $Z$ and with $c = c_\xi$, then (SDP) holds with the same $Z$ and $c$ and with $D = \hat{D}(\xi) \triangleq \theta - \tau_{M}(\xi) - c^{-1} \log c D(\xi)$, where $\tau_{M}(\xi)$ is defined in Lemma 15 and $D(\xi)$ is the point process satisfying $M(D(\xi)) = 0$ a.s., given in Definition PP-D of Section 6.

(b) If (US) holds, then there exists a random variable $Z$ such that $\xi \overset{d}{=} \theta Z \psi$, where $\psi \sim DPPP\left(e^{-c_\xi x}dx, \hat{D}(\xi)\right)$ possibly depends on $Z$.

(c) In addition to Assumption 4, suppose that $L_{\xi}\left[e^{\beta x} \mid \cdot \right] \approx g(\cdot)$ for all $\beta > c = c_\xi$. If (US) holds then there exists a random variable $Z$ such that (SUS) holds with $c = c_\xi$.

(d) In addition to Assumption 4, suppose that Assumption 8 holds. If (US) holds then there exists a random variable $Z$ such that (SUS) holds with $c = c_\xi$.

We note again that, by Corollary 14, under the assumption (US) the condition in part (c) is equivalent to the condition $P\left\{ \int e^{\beta x}d\xi(x) < \infty \right\} = 1$ for all $\beta > c$.

We believe, but have been unable to prove, the following.

Conjecture 10. The assumptions in parts (c) and (d) of Theorem 9 are not needed; that is, under Assumption 4 (US), (SUS) and (SDP) are equivalent.

We remark that Lemma 32 of Section 7 below implies that freezing in the sense of (1.5), even without assuming (US), implies that $g$ is of the form (1.6).

The following corollary follows from the theorem. (However, we shall prove it without relying on Theorem 9, using a simpler result in Section 6.)

Corollary 11. Let $\xi$ be a point process such that $P\{\xi(\mathbb{R}) > 0\} > 0$ and let $c > 0$. The following are equivalent:

1. $\xi$ is a DPPP with density $e^{-cx}dx$.
2. $\xi$ is $c$-exponentially-stable.
3. $L_{\xi}\left[f \mid \cdot \right]$ is uniquely supported on the class $[\text{Gum}(cy)]$.

When any of the conditions hold, we can take the decoration to be $D = \hat{D}(\xi)$, i.e., $\xi \sim DPPP\left(e^{-cx}dx, \hat{D}(\xi)\right)$, where $\hat{D}(\xi)$ is defined in part (a) of Theorem 9.

3. Relations to other works

3.1. SDPPP in BBM, two-speed BBM and BRW. In this section we discuss three processes for which the extremal process, i.e. $\sum_{i \leq N(t)} \delta_{X_i(t) - m_t}$ with appropriate translations $m_t$, is known to converge to an SDPPP of exponential density. The case of BBM was proved independently by Arguin, Bovier and Kistler [6] and Aïdékon, Berestycki, Brunet and Shi [1], with each giving a different description for the decoration process. The approach of Aïdékon et al. relies on the so-called spinal decomposition - a tilted measure which distinguishes the path of a single particle that typically attains extreme values (i.e., the spine). They express the decoration as the limit, first letting $t \to \infty$ and then letting $\zeta \to \infty$, of the point measure of particles at time $t$ which have branched off the particle at $M_t$ after time $t - \zeta$, including the particle at $M_t$, all shifted by the position of $M_t$.

The proof of Arguin et al. starts with a computation of the Laplace functional of the limiting extremal process, based on its relation to the F-KPP equation. In our notation, they prove Condition (SUS) and express the corresponding shifts.
\(\tau_f\) in terms of a solution of the F-KPP equation with initial condition \(v(0, y) = \exp\{-f(y)\}\). They then show that the Laplace functional of the limiting extremal process is equal to that of an auxiliary process they construct - a limit of SDPPP processes of density \(-\sqrt{2/\pi}xe^{-x^2}dx\) (each has a different decoration processes). This shows that the two limiting processes are equal in distribution and allows them to study the latter in order to prove the required structure.

With \(\eta_t \triangleq \sum_{i \leq N(t)} \delta_{X_i(t) - \sqrt{2t}}\), Arguin et al. express the decoration of the extremal process as the limit of \(\theta_{-\mathcal{M}(\eta_t)}\eta_t\) conditioned on \(\mathcal{M}(\eta_t) > 0\) as \(t \to \infty\) (Chauvin and Rouault [17] studied the same process). Our description of the decoration process appearing in Theorem 9 also involves the behavior of the process \((\chi)\) around high levels conditioned on the maximum being sufficiently high. However, our results apply to the limiting process \(\xi\) directly. We study the process \(\theta_{-y} \xi \triangleq \lim_{t \to \infty} \sum_{i \leq N(t)} \delta_{X_i(t) - m_i - y}\) conditioned on \(\mathcal{M}(\xi) > y\), as \(y \to \infty\). For the purpose of comparison, in the case of BBM we can relate our results to studying the limiting behavior as \(t \to \infty\) and then \(y \to \infty\), and the approach of Arguin et al. can be seen as taking the limits simultaneously by defining \(y(t) = \sqrt{2t} - m_t = (3/2\sqrt{2}) \log t - c + o(1)\) and letting \(t \to \infty\).

Madaule [39] proved that the extremal process of the BRW is an SDPPP of exponential density. Theorem 2.3 of [39], which as the author notes is the key step to the main result, expresses the Laplace functional of the extremal process shifted by the derivative martingale, the definition of which is similar to that in the case of BBM, for functions of the form \(f(x) = \sum_{i \leq k} \theta_i \exp(\beta_i x)\) with \(\beta_i\) larger then a critical value. It also gives the independence of the derivative martingale and the limiting shifted process. Combined with Remark 3.2, Theorem 2.3 of [39] implies that for those functions \(L_\xi[f(\cdot)] \approx \text{Gum}\). Essentially, using an approximation argument the equivalence is extended to \(f \in \mathcal{C}_c^+(\mathbb{R})\) in order to show that this process is exponentially-stable (to be accurate, the approximation is done in terms of the characteristic function and not the Laplace transform). By the result of [12], this yields the required structure, without, however, saying anything about the decoration process. Applying Corollary 11 yields a description in terms of the limiting process.

Very recently the two-speed BBM was considered by Bovier and Hartung in [9]. The two-speed BBM is a variant of the BBM where instead of constant variance (per time unit), the Brownian motions describing the evolution of the particles have a certain variance \(\sigma_1^2\) up to a fraction \(bt\), \(b \in [0, 1]\), of the total time and some other variance \(\sigma_2^2\) for the rest of the evolution. As was shown in [9], the structure of the extremal process depends on the relation between \(\sigma_1\) and \(\sigma_2\), and in both cases is an SDPPP of exponential density. Their method of proof essentially follows that of [9].

Lastly, we mention that partial results on the structure of the extremal process of the 2-dimensional discrete Gaussian free field are proved by Biskup and Louidor [8].

3.2. Freezing and Log-correlated fields. Log-correlated random fields are fields whose covariance function decays logarithmically with the distance. They have been analyzed by Carpentier and Le Doussal (C&LD) [16] in a general setting in their study of random energy landscapes. Various specific physical models of log-correlated fields have been considered in [28, 29, 30, 31, 32]. Log-correlated fields are also of great significance in the area of Gaussian multiplicative chaos,
introduced by Kahane [34], which has recently became the object of renewed interest [2, 44, 45, 46, 47]. We also mention the 2-dimensional Gaussian free field which plays an important role in statistical physics, the theory of random surfaces, and quantum field theory [15, 24, 27, 48].

One of the main motivations of the current work is the conjectured, and in some cases proven, freezing phenomenon in log-correlated Gaussian fields. The analysis of C&LD [16], albeit non-rigorous, suggests that the freezing phenomenon occurs in a wide class log-correlated Gaussian fields. For the sub-class of star scale invariant fields freezing was proved by Madaule, Rhodes and Vargas in [41]. Freezing is also proved in the case of the Gaussian BRW in the work of Webb [49].

Discussions on the implications of freezing in different models can be found in [28, 29, 30, 31, 32]. Of particular importance to us is the work of Fyodorov and Bouchaud (F&B) [29]. Assuming freezing as their starting point, F&B analyze the distribution of the maximum of a specific log-correlated Gaussian field (see also [30] where the connection of this model to characteristic polynomials of the CUE matrix is discussed). This allows them to conjecture the limiting distribution of the recentered maximum to be

$$\lim_{t \to \infty} \mathbb{P} \left\{ M_t - m_t \leq x \right\} = g(x) = 2e^{-\beta_c x/2}K_1 \left( 2e^{-\beta_c x/2} \right),$$

where $g$ is the corresponding function from freezing, $K_1$ is the modified Bessel function, and $\beta_c$ is the inverse of the critical temperature.

In view of Theorem 9 and Lemma 32, one must wonder if the function $g$ above is of the structure of (1.6). Indeed, there is a result of Gumbel himself [33] by which, reassuringly, (3.1)

$$2e^{-\beta_c x/2}K_1 \left( 2e^{-\beta_c x/2} \right) = \text{Gum} \ast \text{Gum}' \left( \beta_c x \right),$$

where Gum’ is the derivative of the standard Gumbel distribution. Curiously, this implies that

$$M_t - m_t \overset{d}{\to} X' + X'', \text{ as } t \to \infty,$$

where $X'$ and $X''$ are i.i.d variables with distribution function Gum ($\beta_c x$). We do not have a good direct explanation or proof as to why the shift in the F&B model is itself Gumbel-distributed.

The last example naturally leads us to discuss the subject of ‘universality classes’. In the physics literature the term universality class refers to a class of models that share a certain property. In the context of log-correlated fields, C&LD and F&B were interested in the universality class of fields such that the limiting distribution function of the recentered maximum has certain properties.

The first random energy model considered by physicists is a collection of uncorrelated Gaussian random variables introduced by Derrida in [22]. In this case, by classical results from extreme value theory [20] the limiting distribution of the recentered maximum is the Gumbel distribution. C&LD and F&B emphasized the fact that for the models they considered the limiting distribution is different from this one, and thus the models are not in the same universality class. C&LD come to this conclusion by observing that the tails of the distributions are different.

However, as mentioned they do expect freezing, which also occurs in the case of uncorrelated variables, to occur in those models. In fact, freezing would follow, for example under the conditions of Lemma [38] if the limiting extremal process is an SDPPP. Moreover, on the heuristic level Theorem 9 says that freezing ‘almost’
implies such structure. The SDPPP structure would also allow us to interpret the difference in the limiting distribution simply as the difference in the corresponding random shift (in particular, in agreement with (5.1)).

4. Proof of Theorem 9: the converse part

While proving the direct part of Theorem 9 requires the development of new tools, the proof of the converse direction is immediate. Moreover, the two are essentially independent, and so, we shall deal with the latter now.

Note first that the case where \( \xi \sim SDPPP(e^{-ct}dx,D,Z) \) with general random shift \( Z \) easily follows, by conditioning, from the case where

\[
\xi \sim SDPPP(e^{-ct}dx,D,0) = DPPP(e^{-ct}dx,D).
\]

For simplicity we also assume that \( c = 1, i.e., \xi \sim DPPP(e^{-t}dx,D) \). The case with general \( c \neq 0 \) follows by scaling. The proof for the case \( \xi \sim DPPP(e^{-t}dx,D) \) in fact follows from Theorem 3.1 of [42]. We repeat the proof (though with a slightly more direct approach) for the sake of completeness.

Fix some \( f \in C_c^+(\mathbb{R}) \) throughout the proof. Let \( \zeta_i \) be the atoms of the Poisson process with intensity \( e^{-t}dx \) corresponding to the DPPP and define, for any \( T > 0 \), the random variable \( I(T) = \{ i : \zeta_i \geq -T \} \) and the point process \( \xi_T = \sum_{i \in I(T)} \theta_{\zeta_i} D_i \). By the monotone convergence theorem,

\[
L_{\xi} [f | y] = \lim_{T \to \infty} \mathbb{E} \exp \{-\langle \theta_y f, \xi_T \rangle \}.
\]

By definition, \( I(T) \) is a Poisson random variable of parameter \( e^T = \int_{-T}^{\infty} e^{-t}dx \) independent of \( D_i \), \( i \geq 1 \). Conditioned on the event \( \{ I(T) = k \} \), \( \xi_T \) is the sum of \( k \) i.i.d point processes, each has the same law as \( \theta_{X-T} D \), where \( X \sim \exp(1) \) (i.e., \( X-T \) has density \( e^{-t} \cdot 1_{(-T,\infty)}(t) dt/e^T \), where \( 1_A \) denotes the indicator function of the set \( A \)). Thus

\[
L_{\xi} [f | y] = \lim_{T \to \infty} \mathbb{E} [\mathbb{E} \{ \exp \{-\langle \theta_y f, \xi_T \rangle \} | I(T) \}]
\]

\[
= \lim_{T \to \infty} \mathbb{E} \left\{ (\mathbb{E} \{ L_D [f | y+T-X] \})^{I(T)} \right\}.
\]

Recall that if \( N \sim \text{Pois}(\lambda) \), then

\[
\mathbb{E} t^N = \sum_{k=0}^{\infty} t^k \frac{\lambda^k e^{-\lambda}}{k!} = e^{\lambda(t-1)} \text{ for any } t \in \mathbb{R}.
\]

Hence, by the monotone convergence theorem,

\[
L_{\xi} [f | y] = \lim_{T \to \infty} \mathbb{E} \{ \mathbb{E} \{ L_D [f | y+T-X] \} - 1 \}
\]

\[
= \lim_{T \to \infty} \exp \left\{ \int_{-T}^{\infty} e^{-t} (L_D [f | y-t] - 1) \, dt \right\}
\]

\[
= \exp \left\{ \int_{-\infty}^{\infty} e^{-t} (L_D [f | y-t] - 1) \, dt \right\}
\]

\[
= \exp \left\{ e^{-y} \int_{-\infty}^{\infty} e^{-t} (L_D [f | t] - 1) \, dt \right\},
\]

and therefore, with \( \tau_f^y = \log \left( -\int_{-\infty}^{\infty} e^{-t} (L_D [f | t] - 1) \, dt \right) \),

\[
L_{\xi} [f | y] = \exp \left\{ -e^{-(y-\tau_f^y)} \right\} = \text{Gum} \left( y - \tau_f^y \right).
\]

Since \( f \in C_c^+(\mathbb{R}) \) was arbitrary, \( L_{\xi} [f | \cdot] \) is uniquely supported on \( \text{Gum} \{ y \} \). \( \square \)
5. Basic properties under (US)

In this section we investigate basic properties of the function $g$ and the point process $\xi$, assuming $L_\xi [f \mid \cdot]$ is uniquely supported on $[g]$. In particular, we shall consider the implications of assuming the intensity measure of $\xi$, i.e., the Borel measure $\nu_\xi (B) = E \{ \xi (B) \}$, is boundedly finite.

**Lemma 12.** Let $\xi$ be a point process that is not identically 0, i.e. $P \{ \xi (R) > 0 \} > 0$. If the shift-Laplace functional $L_\xi [f \mid \cdot]$ is uniquely supported on $[g]$, then $g$ is a continuous, monotone function such that for any $x \in R$,

$$(5.1) \quad P \{ \xi (R) = 0 \} = \inf_{y \in R} g(y) < g(x) < \sup_{y \in R} g(y) = 1.$$ 

**Proof.** Since $g(y - \tau_f) = P \{ \xi (R) = 0 \}$

$$+ \ E \{ \xi (R) > 0 \} \ E \{ \exp \{ - (\theta_y f, \xi) \} \mid \xi (R) > 0 \},$$

it is enough to prove the lemma under the assumption that $\xi (R) > 0$ almost surely. Assume henceforth it is so.

Fix some $f \in C^+_R$ throughout the proof. By the bounded convergence theorem, $\lim_{y \to y_0} (\theta_y f, \xi) = \langle \theta_y f, \xi \rangle$ almost surely and $\lim_{y \to y_0} L_\xi [f \mid y] = L_\xi [f \mid y_0]$, i.e. $g$ is continuous.

Applying, again, the bounded convergence theorem to the sequence $\{ n^{-1} f \}$, we obtain from uniqueness of the support,

$$1 \geq \sup_{y \in R} g(y) \geq \lim_{n \to \infty} E \{ \exp \{ - (n^{-1} f, \xi) \} \} = 1.$$ 

Similarly, by considering a sequence $f_n \in C^+_R$ such that $f_n \geq n \cdot 1_{(-n,n)}$ pointwise and noting that $(f_n, \xi) \to \infty \cdot 1_{\xi(R) > 0}$ almost surely, we conclude that $\inf_{y \in R} g(y) = P \{ \xi (R) = 0 \} = 0$.

If there were a $y_0$ such that $g(y_0) = \inf_{y \in R} g(y)$, then for some $y_0$, $L_\xi [f \mid y_0] = 0$, which would imply that $\langle \theta_y f, \xi \rangle$ is almost surely. Since $f$ has compact support, this is a contradiction and thus the lower bound in (5.1) strict.

In order to show that $g$ is monotone, we prove by contradiction that for any $\alpha > 0$, the super-level set $\Psi_\alpha \triangleq \{ y : g(y) > \alpha \}$ is either empty or it is an infinite interval. Let $\alpha > 0$ and assume that $\Psi_\alpha$ is not such. Then, from the continuity of $g$, $\Psi_\alpha$ is open and thus it either has a bounded connected component, or it is the union of two disjoint infinite intervals. If we assume the latter, then $\{ y : g(y) \leq \alpha \}$ is compact and the minimum of $g$ is attained in it which contradicts the strict lower bound of (5.1) which was already proved.

Now assume that $\Psi_\alpha$ has a bounded connected component $(a, b)$. Let $f' = \theta_y f$, with $y$ chosen so that $L_\xi [f' \mid \cdot] = g(\cdot)$. Setting $x = (a + b)/2$, from the bounded convergence theorem, we have that $L_\xi [t \cdot f' \mid x]$ is continuous in $t$ and therefore $L_\xi [t, f' \mid x] > \alpha$ for any $t \in [1, \delta]$ for some $\delta$. For each $t \in [1, \delta]$ define $(a_t, b_t)$ to be the bounded connected component of $\{ y : L_\xi [t \cdot f' \mid y] > \alpha \}$ that contains $x$ and note that, since $\alpha > 0$, $a_t$ (respectively, $b_t$) is strictly increasing (decreasing) in $t$. Hence, each of the intervals $(a_t, b_t)$ has different length. However, each of them is, up to translation, also a connected component of $\Psi_\alpha$, which, as an open subset of $R$, can only have countably many connected components. This contradicts our assumption and implies that $g$ is monotone.
It remains to prove that the upper bound of (5.1) is strict which we shall prove by contradiction. Assume that the maximum of $g$ is attained. WLOG assume that $g$ is increasing and set $\eta = \min \{ y : L_\xi [ f \mid y ] = 1 \}$, where the existence of the minimum is assured by the fact that $g$ is continuous and $\mathbb{P} \{ \xi (\mathbb{R}) = 0 \} = 0$. Note that

$$
\min \{ y : L_\xi [ 2f \mid y ] = 1 \} = \min \{ y : L_\xi [ f \mid y ] = 1 \} = \eta.
$$

Since $L_\xi [ f \mid \cdot ]$ is uniquely supported,

$$
g (y) = L_\xi [ f \mid y - \tau_1 ] = L_\xi [ 2f \mid y - \tau_2 ].
$$

However, from (5.2), $\tau_1 = \tau_2$ and $L_\xi [ f \mid y ] = L_\xi [ 2f \mid y ]$ in contradiction to the fact that $\mathbb{P} \{ \xi (\mathbb{R}) = 0 \} = 0$. This completes the proof.

The following corollary easily follows from the lemma. In the sequel we write $\xi \gg 0$ whenever $\mathbb{P} \{ \xi (\mathbb{R}) > 0 \} = 1$.

**Corollary 13.** Let $\xi \gg 0$ be a point process such that $L_\xi [ f \mid \cdot ]$ is uniquely supported on $[g]$. Let $\tau_n \in \mathbb{R}$, $n = 1, 2, \ldots$ be a sequence such that $g (\cdot - \tau_n) \to h (\cdot)$ pointwise. Then either $\tau_n \to \pm \infty$ and $h$ is a constant function whose value belongs to $\{0, 1\}$, or $\tau_n \to \tau$ for some real $\tau$ and $h (\cdot) = g (\cdot - \tau)$.

Using Corollary 13, uniqueness of the support can be easily extended to functions not in $C^+_\mathbb{R} (\mathbb{R})$.

**Corollary 14.** Let $\xi \gg 0$ be a point process such that $L_\xi [ f \mid \cdot ]$ is uniquely supported on $[g]$. Let $f \geq 0$ be a measurable function on $\mathbb{R}$ and suppose that there exists a sequence of functions $f_n \in C^+_\mathbb{R} (\mathbb{R})$ that converges pointwise and monotonically to $f$. If $L_\xi [ f \mid y ] \in (0, 1)$ for some $y \in \mathbb{R}$, then $L_\xi [ f \mid \cdot ] \approx g (\cdot)$.

**Proof.** The monotone convergence theorem implies that

$$
L_\xi [ f \mid y ] = \lim_{n \to \infty} L_\xi [ f_n \mid y ] = g (y - \tau (f_n)).
$$

Therefore the corollary follows from Corollary 13.

**Lemma 15.** Let $\xi \gg 0$ be a point process such that $L_\xi [ f \mid \cdot ]$ is uniquely supported on $[g]$. If Assumption 4 holds, then $\mathcal{M} (\xi)$ is almost surely bounded and there exists $\tau_\mathcal{M} = \tau_\mathcal{M} (\xi) \in \mathbb{R}$ such that

$$
L_\xi [ \infty \cdot 1_{(0, \infty)} \mid y ] = \mathbb{P} \{ \mathcal{M} (\xi) \leq y \} = g (y - \tau_\mathcal{M}).
$$

**Proof.** From (5.1) it easily follows that $\mathbb{P} \{ \xi (\{0, 1\}) = 0 \} \in (0, 1)$. Thus, using the previous corollary, there exists a real $\tau$ such that for any $n \in \mathbb{N}$,

$$
\mathbb{P} \{ \mathcal{M} (\xi) \geq n \} \leq \sum_{i=n}^{\infty} \mathbb{P} \{ \xi (\{i, i+1\}) > 0 \}
= \sum_{i=n}^{\infty} (1 - L_\xi [ \infty \cdot 1_{(0,1)} \mid i ]) = \sum_{i=n}^{\infty} (1 - g (y - \tau)).
$$

From Assumption 4 it easily follows that as $n \to \infty$, the probability above converges to 0 and therefore $\mathcal{M} (\xi)$ is almost surely bounded. Thus, for the function $h = \infty \cdot 1_{(0,\infty)}$ with $y$ sufficiently high, $\mathbb{P} \{ \langle \theta_y h, \xi \rangle < \infty \} > 0$. From the uniqueness of the support of $L_\xi [ f \mid \cdot ]$ it easily follows that $\mathbb{P} \{ \langle \theta_y h, \xi \rangle > 0 \} = 0$. Hence, applying Corollary 13 to $f$ completes the proof.
In the remainder of the section we consider the case where the intensity measure of the point process is finite.

**Proposition 16.** Let $\xi \gg 0$ be a point process such that $L_\xi [f \mid \cdot] \text{ is uniquely supported on } [g]$. Assume for concreteness that $g$ is increasing. If there exists an open interval $I$ such that $\nu_\xi (I) < \infty$, then there exist some constants $C, c > 0$ such that:

1. For any finite, open interval $I$, $\nu_\xi (I) < \infty$.
2. The measure $\nu_\xi$ is absolutely continuous relative to Lebesgue measure and its density is given by $Ce^{-cx}$.
3. The maximum of the point process $\mathcal{M} = \mathcal{M} (\xi)$ is almost surely finite and $\mathbb{P} \{ \mathcal{M} \geq x \} \leq \frac{C}{x} e^{-cx}$, for any $x > 0$.
4. The function $g$ satisfies (2.1), i.e.

$$
\lim_{x \to \infty} \frac{1 - g (x + y)}{1 - g (x)} = \lim_{x \to \infty} \frac{\log g (x + y)}{\log g (x)} = e^{-cy}.
$$

The proposition, of course, also holds for the case where $g$ is decreasing with the obvious adjustments to the above statements.

**Remark 17.** Note, in particular, that Proposition 16 implies that Assumption 3 holds in the case that the intensity measure is boundedly finite.

**Proof.** Suppose that $\nu_\xi ((a, b)) < \infty$ for some $a < b \in \mathbb{R}$. Set $\eta = (a + b) / 2$ and $I = (a + \eta / 2, b - \eta / 2)$. For any $t > 0$ and $y \in \mathbb{R}$, by Corollary 14 $L_\xi [t \cdot 1_I \mid y] = g \left( y - \tau^I_t \right)$ for some $\tau^I_t \in \mathbb{R}$.

Recall that for any nonnegative random variable $X$, the expectation $\mathbb{E}X$ is finite if and only if the (one-sided) derivative $\lim_{t \to 0} \frac{1 - \mathbb{E}e^{-tX}}{t}$ exists and is finite (see, for instance, [20], p.435).

Thus, since for any $y \in [-\eta / 2, \eta / 2]$, we have $\nu_\xi (I + y) < \infty$, it follows that

$$
\nu_\xi (I + y) = \lim_{t \to 0} \left[ t^{-1} (1 - \mathbb{E} \{ \exp (-t \xi (I + y)) \}) \right]
$$

(5.3)

Therefore, for any $y_1, y_2 \in [-\eta / 2, \eta / 2],

$$
\lim_{t \to 0} \frac{1 - g (y_1 - \tau^I_t)}{1 - g (y_2 - \tau^I_t)} = \frac{\nu_\xi (I + y_1)}{\nu_\xi (I + y_2)},
$$

(5.4)

where $\nu_\xi (I + y) > 0$ from the strict upper bound of (5.1) of Lemma 14 considering some function $f \in C_+^+ (\mathbb{R})$ supported on $I + y$.

Note that for any $y \in \mathbb{R}$, $\mathbb{E} \{ \exp (-t \xi (I + y)) \}$ increases to 1 as $t \to 0$. Thus, since $g$ is increasing, by Lemma 14 $\tau^I_t \to -\infty$. Hence, for any $y_1, y_2 \in [-\eta / 2, \eta / 2]$, the limit of (5.4) depends, in fact, only on the difference $y_2 - y_1$,

$$
\lim_{t \to 0} \frac{1 - g (y_1 - \tau^I_t)}{1 - g (y_2 - \tau^I_t)} = \lim_{x \to \infty} \frac{1 - g (x + y_1)}{1 - g (x + y_2)} = \lim_{x \to \infty} \frac{1 - g (x + y_1 - y_2)}{1 - g (x)}.
$$

(5.5)

It follows, in particular, that for any $y \in [-\eta, \eta]$, $\lim_{x \to \infty} \frac{1 - g (y + y_0)}{1 - g (x)}$ exists.

Now, for any $y \in [-\eta, \eta]$, the limit

$$
\lim_{t \to 0} \frac{1 - g (y - \tau^I_t)}{t} = \lim_{t \to 0} \frac{1 - g (y - \tau^I_t)}{1 - g (y / 2 - \tau^I_t)} \cdot \lim_{t \to 0} \frac{1 - g (y / 2 - \tau^I_t)}{t}
$$
exists and thus \( \nu_\xi(Id + y) \) is finite, for any \( y \in [-\eta, \eta] \). By reiterating this argument we obtain that, actually, \( \nu_\xi(Id + y) \) is finite for any \( y \in \mathbb{R} \). This completes the proof of \( \Box \).

Since (11) holds, by the same arguments as in the previous part of the proof, (5.3), (5.4) and (5.5) hold for any finite, open interval \( I \) and any \( y, y_1, y_2 \in \mathbb{R} \), with some \( \tau_1^I \in \mathbb{R} \) that depends on \( I \).

Define \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \) by the relation

\[
e^{\alpha(y)} \triangleq \lim_{x \rightarrow \infty} \frac{1 - g(x + y)}{1 - g(x)} > 0
\]

(where the inequality follows from (5.4) and the remark on \( \nu_\xi(Id + y) > 0 \) right after it).

Note that \( \alpha \) is a solution to Cauchy’s functional equation,

\[
\exp \{ \alpha(y + z) \} = \lim_{x \rightarrow \infty} \frac{1 - g(x + y + z)}{1 - g(x + z)} \lim_{x \rightarrow \infty} \frac{1 - g(x + z)}{1 - g(x)}
\]

\[
= \lim_{x \rightarrow \infty} \frac{1 - g(x + y)}{1 - g(x)} \lim_{x \rightarrow \infty} \frac{1 - g(x + z)}{1 - g(x)} = \exp \{ \alpha(y) + \alpha(z) \}
\]

Since \( g \) is increasing, \( \alpha \) is decreasing, and therefore \( \alpha(y) = -cy \) for some \( c \geq 0 \).

Hence, from (5.4) and (5.5), for any \( y_1, y_2 \in \mathbb{R} \),

\[
\frac{\nu_\xi(Id + y_1)}{\nu_\xi(Id + y_2)} = e^{-c(y_1 - y_2)}.
\]

Now, to rule out the case \( c = 0 \), take some \( f \in C^+_c(\mathbb{R}) \) such that \( f(y) \leq 1_{(0,1)} \) and note that since \( L_\xi[f \cdot \cdot] \) is uniquely supported and by Lemma (12) for any \( \epsilon, t > 0 \) there exists \( y \in \mathbb{R} \) such that

\[
\epsilon > \mathbb{E} \{ \exp (- (\theta_y(t \cdot f), \xi)) \}
\]

thus for any \( \delta > 0 \)

\[
\epsilon > e^{-\delta} \mathbb{P} \{ (\theta_y(t \cdot f), \xi) \leq \delta \}
\]

and

\[
e^{\epsilon \delta} > \mathbb{P} \{ (\theta_yf, \xi) \leq \delta/t \}.
\]

Since \( \epsilon, t, \delta \) are arbitrary, it follows that \( \sup_{y \in \mathbb{R}} \nu_\xi((0,1) + y) = \infty \), which in light of (5.7) implies that \( c \neq 0 \).

Substituting \( \alpha(y) = -cy \) in (5.6) yields

\[
\lim_{x \rightarrow \infty} \frac{1 - g(x + y)}{1 - g(x)} = \lim_{x \rightarrow \infty} \frac{\log g(x + y)}{\log g(x)} = e^{-cy},
\]

where the equality of the two limits follows from the fact that \( g(x + y) \rightarrow 1 \) as \( x \rightarrow \infty \) for any \( y \in \mathbb{R} \), and that \( (1 - e^{-w})/w \rightarrow 1 \) as \( w \rightarrow 0 \). This is exactly part of (4) of the proposition.

To get rid of the nuisance of (5.7) being true only for open intervals \( I \), note that for any \( x > 0 \)

\[
\nu_\xi((-x,x)) \geq \nu_\xi((-x,-x/3)) + \nu_\xi((-x/3,x/3)) + \nu_\xi((x/3,x))
\]

which, by (5.7), implies that

\[
2^{-1} \nu_\xi((-x,x)) \geq \nu_\xi((-x/3,x/3)).
\]

Hence \( \nu_\xi(\{0\}) = 0 \). By a similar argument for any \( x \in \mathbb{R}, \nu_\xi(\{x\}) = 0 \) and thus (5.7) hold for any finite, nontrivial interval \( I \), not necessarily open.
Using this, observe that for any \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \),
\[
\nu_\xi ([0, nx)) = \nu_\xi ([0, x)) + \nu_\xi ([x, 2x)) + \ldots + \nu_\xi (((n-1)x, nx))
\]
\[
= \nu_\xi ([0, x)) \left( 1 + e^{-cx} + e^{-2cx} + \ldots + e^{-(n-1)cx} \right)
\]
\[
= \nu_\xi ([0, x)) \frac{1 - e^{-cnx}}{1 - e^{-cx}}.
\]
Setting \( C = c \nu_\xi ([0, 1)) / (1 - e^{-c}) \) and restricting to the case \( x = 1 \) we obtain
\[
\nu_\xi ([0, n)) = \frac{C}{e} (1 - e^{-cn}).
\]
Then, for any rational \( p/q \), \( p, q \in \mathbb{N} \), by setting \( n = q \) and \( x = p/q \),
\[
\nu_\xi ([0, p)) = \nu_\xi ([0, p/q)) \frac{1 - e^{-cp}}{1 - e^{-cp/q}}
\]
and
\[
\nu_\xi ([0, p/q)) = \frac{C}{e} \left( 1 - e^{-cp/q} \right).
\]
Since \( x \mapsto \nu_\xi ([0, x)) \), \( x \geq 0 \) is monotone, we have for all \( x \geq 0 \)
\[
\nu_\xi ([0, x)) = \frac{C}{e} \left( 1 - e^{-cx} \right).
\]
Since \( \nu_\xi ([1, 0)) = e^c \nu_\xi ([0, 1)) = \frac{C}{e} (e^c - 1) \), by similar arguments it is seen that
\[
\nu_\xi ([x, 0)) = \frac{C}{e} (e^{cx} - 1).
\]
Hence, \( \nu_\xi \) is absolutely continuous relative to Lebesgue measure and has density
\( Ce^{-cx} \), and part (2) is completed.

Lastly, note that part (3) simply follows from Markov’s inequality where \( \nu_\xi ([x, \infty)) \) is easily computed from the density just derived. This completes the proof. \( \square \)

6. (Re)construction of the decoration

Under Assumption 20 below, we show in this section that the following limiting point process exists.

**Definition 18.** Suppose that the point process \( \xi \) satisfies Assumption 20. Define \( \tilde{\xi} \) to be the limit in distribution as \( y \to \infty \) of \( \theta_{-y} \xi \) conditioned on \( \{ \xi((y, \infty)) > 0 \} \).

With \( \tilde{\xi} \) defined, we can define the following.

**Definition (PP-D).** Suppose that the point process \( \xi \) satisfies Assumption 20. Define the point process \( \tilde{D}(\xi) \) to be the point process \( \tilde{\xi} \) translated so its maximum is at 0, that is, \( D(\xi) \triangleq \theta_{-M(\xi)} \tilde{\xi} \).

Before we proceed, to see why this process is of interest, we investigate it in the case that \( \xi \) is a DPPP.

**Example 19.** Suppose \( \xi = \sum_{i \geq 1} \zeta_i D_i \sim DPPP(e^{-cx}) \) for some point process \( D \) and some constant \( c > 0 \), such that the maximum of the decoration, \( M(D) \), is 0 almost surely.

Now consider the process \( \xi \) conditioned on \( \{ \xi((y, \infty)) > 0 \} \). The number of decorations that intersect \( (y, \infty) \), i.e. the number of shifted copies \( \zeta_i D_i \) of \( D \) that attribute a positive measure to \( (y, \infty) \), is a Poisson random variable of parameter \( c^{-1}e^{-cy} \), conditioned on being positive. As \( y \to \infty \), this random variable converges to 1. Similarly, the number of decorations that intersect \( (y-t, y] \) converges to 0, for
any fixed $t > 0$. Moreover, the maximum of each of the decorations that intersect $(y, \infty)$ is distributed like $y + X$ where $X \sim \exp(c)$. Hence, one can extract the law of the process $\theta_y D$. $X \sim \exp(c)$, by investigating $\theta_y \xi$ under the conditioning above and letting $y \to \infty$. Once we have the law of this process, we can also recover the law of $D$

Of course, if $\psi \sim SDPPP (e^{-cx}, D, Z)$ and the right tail of the distribution of $Z$ decays fast enough, then we should also be able to study the law of $D$ from $\psi$. \hfill \Box

In order to prove that $\xi$ exists, we construct some other point process and show that it is equal in distribution to $\xi$. The construction is done in a few stages, in each of which we define (and, when needed, prove the existence of) a point process based on the previous.

The following assumption does not imply that $L_\xi [f | \cdot]$ is uniquely supported. However, as we shall see, when we do assume uniqueness, using $D(\xi)$ as the decoration to define a DPPP we recover the original process $\xi$ up to a random shift (under Assumption 4). This will be the key to the proof of Theorem 9.

**Assumption 20.** For some constant $c > 0$ and some real increasing function $g$, there exist real numbers $\bar{\tau}_f = \bar{\tau}_f(\xi)$ such that

\begin{align}
\lim_{x \to \infty} \frac{1 - L_\xi [f | x]}{1 - g(x - \bar{\tau}_f)} &= 1, \quad \forall f \in C_+^c(\mathbb{R}) \cup \{ \infty \cdot 1_{(0, \infty)} \} \\
\lim_{x \to \infty} \frac{1 - g(x + y)}{1 - g(x)} &= e^{-cg}, \quad \text{and} \\
\lim_{t \to 0} \bar{\tau}_f &= \infty, \quad \forall f \in C_+^c(\mathbb{R}).
\end{align}

Since we require (6.1) to hold with $f = \infty \cdot 1_{(0, \infty)}$, we remark that

$L_\xi [\infty \cdot 1_{(0, \infty)} | x] = \mathbb{E} \{ \exp \left( - \langle \cdot, 1_{(x, \infty)} \rangle \right) \} = \mathbb{P} \{ M(\xi) \leq x \}$.

We write $\bar{\tau}_M$ for $\bar{\tau}_\infty 1_{(0, \infty)}$.

**Remark 21.** By Lemma 15, Assumption 4 implies Assumption 20 with $\bar{\tau}_f(\xi) = \tau_f(\xi)$, the shifts corresponding to the uniqueness of the support. Equation (6.1) follows by definition and (6.3) follows by the monotone convergence theorem.

We now define the first of the point processes.

**Definition (PP1).** Given a point process $\xi$, let $\xi(y)$ denote the point process $\theta_y \left( \xi(y, \infty) \right)$ conditioned on $\{\xi((y, \infty)) > 0\}$.

**Definition (PP2).** Suppose Assumption 20 holds. Define the point process $\xi^{\infty}$ as the limit in distribution of $\xi(y)$ as $y \to \infty$.

We show that the definition makes sense in the following.

**Lemma 22.** Under Assumption 20, $\xi^{\infty}$ as in Definition PP2 exists. Further, for any $f \in C_+^c(\mathbb{R})$ whose support is contained in $(0, \infty)$ and any $y \geq 0$,

$L_{\xi^{\infty}} [f | y] = 1 - e^{-c(y + \bar{\tau}_M - \bar{\tau}_f)}$. 

Proof. Let $f \in C^+_c(\mathbb{R})$ be a function whose support is contained in $(0, \infty)$ and let $\tilde{\tau}_f \in \mathbb{R}$ be the shift from Assumption 20. Then

\[
\lim_{y \to \infty} L_{\xi^{(y)}}[f] = \lim_{y \to \infty} \mathbb{E} \exp \left\{ - \left\langle f, \xi^{(y)} \right\rangle \right\} \\
= \lim_{y \to \infty} \mathbb{E} \left[ \exp \left\{ - \left\langle \theta_y f, \xi \right\rangle ; \mathcal{M}(\xi) > y \right\} \right] \\
= \lim_{y \to \infty} \mathbb{E} \left[ \exp \left\{ - \left\langle \theta_y f, \xi \right\rangle \right\} \right] - \mathbb{P} \left[ \mathcal{M}(\xi) \leq y \right] \\
= \lim_{y \to \infty} \frac{L_\xi [f | y]}{1 - \mathbb{P} [\mathcal{M}(\xi) \leq y]} \\
= \lim_{y \to \infty} \frac{L_\xi [f | y]}{1 - \frac{L_\xi [\infty \cdot 1_{(0,\infty)} | y]}{\mathbb{E} [\exp \{ - \langle \theta_y f, \xi \rangle \} | y]}} \\
= 1 - \frac{1 - g(y - \tilde{\tau}_f)}{1 - g(y - \tilde{\tau}_M)} = 1 - e^{-c(\tilde{\tau}_M - \tilde{\tau}_f)}.
\]

(6.4)

Since the point process $\xi^{(y)}$ is supported on $(0, \infty)$, this shows that the Laplace functional $L_{\xi^{(y)}}[f]$ converges for any $f \in C^+_c(\mathbb{R})$.

It is easy to verify that $\tilde{\tau}_{0,f} = \tilde{\tau}_f - x$ for any $f \in C^+_c(\mathbb{R})$. Therefore, for any $f \in C^+_c(\mathbb{R})$ whose support is contained in $(0, \infty)$ and for any $x \geq 0$,

\[
\lim_{y \to \infty} \mathbb{E} \exp \left\{ - \left\langle \theta_x f, \xi^{(y)} \right\rangle \right\} = 1 - \lim_{y \to \infty} \frac{1 - g(y - \tilde{\tau}_f)}{1 - g(y - \tilde{\tau}_M)} = 1 - e^{-c(x + \tilde{\tau}_M - \tilde{\tau}_M)}.
\]

Thus all that remains is to prove that for any sequence $y_n \to \infty$, for any $f \in C^+_c(\mathbb{R})$, such that

\[
\lim_{n \to \infty} y_n \to \infty, \quad \lim_{n \to \infty} \mathbb{P} \left[ \mathcal{M}(\xi) > y_n \right] = 1 - e^{-c(y_n - \tilde{\tau}_M)}
\]

which will imply that $\xi^{(y_n)}$ converges in distribution as $y_n \to \infty$.

Note that, by definition, for any $y > 0$, $T > 1$,

\[
\mathbb{P} \left\{ \xi^{(y)} ((0, T - 1/T)) > t \right\} = \frac{\mathbb{P} \left\{ \xi^{(y-1/T)} ((1/T, T)) > t \right\}}{\mathbb{P} \left\{ \xi^{(y-1/T)} ((1/T, \infty)) > 0 \right\}}.
\]

and, using Assumption 20

\[
\mathbb{P} \left\{ \xi^{(y-1/T)} ((1/T, \infty)) > 0 \right\} = \frac{\mathbb{P} \left\{ \mathcal{M}(\xi) > y \right\}}{\mathbb{P} \left\{ \mathcal{M}(\xi) > y - 1/T \right\}} \cdot \frac{1 - L_\xi [\infty \cdot 1_{(0,\infty)} | y]}{1 - L_\xi [\infty \cdot 1_{(0,\infty)} | y - 1/T]} \lim_{y \to \infty} \frac{1 - g(y - \tilde{\tau}_M)}{1 - g(y - 1/T - \tilde{\tau}_M)} = e^{-c/T}.
\]

Therefore, for any sequence $\xi^{(y_n)}$, $n \geq 1$, as above,

\[
\lim_{n \to \infty} \sup_{t > 0} \mathbb{P} \left\{ \xi^{(y_n)} ((0, T - 1/T)) > t \right\} = e^{c/T} \lim_{n \to \infty} \sup_{t > 0} \mathbb{P} \left\{ \xi^{(y_n-1/T)} ((1/T, T)) > t \right\}.
\]

Hence, since $\xi^{(y_n)}$ is supported on $(0, \infty)$, it will be sufficient to show that (10.1) holds for any sequence $\xi^{(y_n)}$, $n \geq 1$, as above, with $B = (1/T, T)$ for any $T > 0$.

Fix a sequence $\xi^{(y_n)}$ as above, fix some $T > 1$, and fix some function $f_0 \in C^+_c(\mathbb{R})$ with support in $(0, \infty)$ such that $f_0(x) \geq 1_{(1/T, T)}(x)$, for any $x \in \mathbb{R}$. For any $m > 0$, abbreviate $\tilde{\tau}_{m,f_0}$ of Assumption 20 to $\tilde{\tau}_m$.

Note that, denoting $q_t(y) := \mathbb{P} \left\{ \xi^{(y)} ((1/T, T)) > t \right\}$, for any $y$, $m$, $t > 0$,

\[
\mathbb{E} \left[ \exp \left\{ - \left\langle m \cdot 1_{(1/T, T)}, \xi^{(y)} \right\rangle \right\} \right] \leq q_t(y) e^{-tm} + (1 - q_t(y))
\]
and therefore
\[ q_t(y) \left(1 - e^{-tm}\right) \leq 1 - E \left[ \exp \left\{ - \langle m \cdot 1_{(1/T,T)}, \xi(y) \rangle \right\} \right] \]
\[ \leq 1 - E \left[ \exp \left\{ - \langle mf_0, \xi(y) \rangle \right\} \right] . \]

Thus, for any \( m > 0 \),
\[ \lim_{t \to \infty} \limsup_{n \to \infty} q_t(y_n) = \lim_{t \to \infty} \limsup_{n \to \infty} q_t(y_n) \left(1 - e^{-tm}\right) \leq e^{-c(\bar{r}_M - \bar{r}_n)}, \]
and therefore, by (6.3),
\[ \lim_{t \to \infty} \limsup_{n \to \infty} q_t(y_n) = 0, \]
which proves tightness. \( \square \)

**Corollary 23.** Under Assumption 20 the maximum of \( \xi^- \), \( \mathcal{M}(\xi^-) \), is an exponential random variable with parameter \( c \).

**Proof.** Approximating \( \infty \cdot 1_{(0,\infty)} \) by an increasing sequence of functions \( f_n \in C_c^+ (\mathbb{R}) \) whose support is contained in \( (0, \infty) \) easily yields
\[ \mathbb{P} \left[ \mathcal{M}(\xi^-) \leq y \right] = L_{\xi^-} \left[ \infty \cdot 1_{(0,\infty)} \big| y \right] = 1 - e^{-c(y+\bar{r}_M-\bar{r}_M)} = 1 - e^{-cy}. \]

**Definition** (PP3). Suppose Assumption 20 holds. For any \( y > 0 \) define \( \xi(y) \) to be the point process \( \theta_y \xi^- \) conditioned on \( \{\xi^-((y, \infty)) > 0\} \) (\( \xi(y) \) is supported on \((-y, \infty))\).

**Lemma 24.** Under Assumption 20 for any \( 0 \leq y, t \) and any \( f \in C_c^+ (\mathbb{R}) \) whose support is contained in \( (0, \infty) \),
\[ L_{\xi(y)}[f] = 1 - e^{-c(\bar{r}_M-\bar{r}_f)} \quad \text{and} \quad L_{\xi(y+t)}[\theta_y f] \text{ is independent of } t. \]

**Proof.** By Lemma 22 and Corollary 23 for any \( 0 \leq y \) and any \( f \in C_c^+ (\mathbb{R}) \) whose support is contained in \( (0, \infty) \),
\[ L_{\xi(y)}[f] = E \left\{ \exp \left\{ - \langle \theta_y f, \xi^- \rangle \right\} \mid \mathcal{M}(\xi^-) > y \right\} \]
\[ = \frac{E \left\{ \exp \left\{ - \langle \theta_y f, \xi^- \rangle \right\} \mid \mathcal{M}(\xi^-) > y \right\}}{\mathbb{P} \left[ \mathcal{M}(\xi^-) > y \right]} \]
\[ = \frac{E \left\{ \exp \left\{ - \langle \theta_y f, \xi^- \rangle \right\} \right\} - E \left\{ \mathcal{M}(\xi^-) \leq y \right\}}{1 - e^{-c(y+\bar{r}_M-\bar{r}_f)}} \]
\[ = \frac{1 - e^{-c(y+\bar{r}_M-\bar{r}_f)}}{1 - e^{-cy}} \]
and therefore, for any \( 0 \leq y, t \),
\[ \xi(y) \big|_{(0, \infty)} \overset{d}{=} \xi(y+t) \big|_{(0, \infty)}. \]
Now, for any $0 \leq y, t$ and any $f \in C_c^+ (\mathbb{R})$ whose support is contained in $(0, \infty)$,

$$
L_{\xi(t+t)} \{ \theta - t \} = \mathbb{E} \left[ \exp \left\{ - \langle \theta y f, \xi \rangle \right\} \right| \mathcal{C} \left( \xi^{-} \right) > y + t
$$

$$
= \frac{\mathbb{P} \left[ \mathcal{C} \left( \xi^{-} \right) > y + t \right]}{\mathbb{P} \left[ \mathcal{C} \left( \xi^{-} \right) > y + t \right]} \mathbb{E} \left[ \exp \left\{ - \langle \theta y f, \xi \rangle \right\} \right| \mathcal{C} \left( \xi^{-} \right) > y
$$

$$
= e^{ct} \mathbb{E} \left[ \exp \left\{ - \langle f, \theta - y \xi \rangle \right\} \right| \mathcal{C} \left( \xi^{-} \right) > y] .
$$

According to (6.6), the last expression is independent of $y$, which completes the proof.

Equation (6.5) allows us to define the following.

**Definition (PP4).** Suppose Assumption 20 holds. Define $\xi^{*+}$ as the limit in distribution of $\xi(y)$ as $y \to \infty$.

We are now ready to prove that the limiting point process $\bar{\xi}$ of definition 18 exists.

**Lemma 25.** Under Assumption 20, the point process $\bar{\xi}$ exists and is equal in distribution to $\xi^{*+}$.

**Proof.** Let $n \geq 0$, let $A_i \subset \mathbb{R}$, $i \leq n$, be Borel sets, and let $k_i \in \mathbb{R}$, $i \leq n$. We need to show that

$$
\lim_{y \to \infty} \frac{\mathbb{P} \left\{ \xi(A_i + y) \geq k_i, i \leq n, \xi((y, \infty)) \right\}}{\mathbb{P} \left\{ \xi((y, \infty)) \right\}} = \mathbb{P} \left\{ \xi^{*+}(A_i) \geq k_i, i \leq n \right\} .
$$

From Definition PP3 and Definition PP4,

$$
\mathbb{P} \left\{ \xi^{*+}(A_i) \geq k_i, i \leq n \right\} = \lim_{y \to \infty} \frac{\mathbb{P} \left\{ \xi^{*+}(A_i + y) \geq k_i, i \leq n, \xi^{*+}((y, \infty)) \right\}}{\mathbb{P} \left\{ \xi^{*+}((y, \infty)) \right\}} .
$$

For large enough $y$, from Definition PP1 and Definition PP2,

$$
\lim_{t \to \infty} \frac{\mathbb{P} \left\{ \xi^{*+}(A_i + y) \geq k_i, i \leq n, \xi^{*+}((y, \infty)) \right\}}{\mathbb{P} \left\{ \xi^{*+}((y, \infty)) \right\}} = \lim_{t \to \infty} \mathbb{P} \left\{ \xi(A_i + y + r) \geq k_i, i \leq n, \xi((y + r, y + r + t)) \right\} .
$$

Note that, using Assumption 20

$$
\lim_{t \to \infty} \frac{\mathbb{P} \left\{ \xi(A_i + y + r) \geq k_i, i \leq n, \xi((y + r, \infty)) \right\}}{\mathbb{P} \left\{ \xi((y, \infty)) \right\}} - \lim_{t \to \infty} \mathbb{P} \left\{ \xi(A_i + y + r) \geq k_i, i \leq n, \xi((y + r, y + r + t)) \right\}
$$

$$
\leq \lim_{t \to \infty} \frac{\mathbb{P} \left\{ \xi((y + r + t, \infty)) \right\}}{\mathbb{P} \left\{ \xi((r, \infty)) \right\}} = \lim_{t \to \infty} \frac{1 - g(y + r + t - \tilde{\tau}_M)}{1 - g(r - \tilde{\tau}_M)}
$$

$$
= \lim_{t \to \infty} e^{-c(y + t)} = 0 ,
$$

which completes the proof.
and therefore
\[
\mathbb{P}\{\xi^{-}(A_{i} + y) \geq k_{i}, \ i \leq n, \xi^{-}((y, \infty))\} = \lim_{r \to \infty} \frac{\mathbb{P}\{\xi(A_{i} + y + r) \geq k_{i}, \ i \leq n, \xi((y + r, \infty))\}}{\mathbb{P}\{\xi((r, \infty))\}}.
\]
Setting \(k_{i} = 0\), we similarly obtain
\[
\mathbb{P}\{\xi^{-}((y, \infty))\} = \lim_{r \to \infty} \frac{\mathbb{P}\{\xi((y + r, \infty))\}}{\mathbb{P}\{\xi((r, \infty))\}} > 0.
\]
Substituting in (6.8) yields
\[
\mathbb{P}\{\xi^{+}(A_{i}) \geq k_{i}, \ i \leq n\} = \lim_{y \to \infty} \lim_{r \to \infty} \frac{\mathbb{P}\{\xi(A_{i} + y + r) \geq k_{i}, \ i \leq n, \xi((y + r, \infty))\}}{\mathbb{P}\{\xi((y + r, \infty))\}}.
\]
which implies (6.7) and completes the proof. \(\square\)

From (6.5), we have that \(\xi^{+}|_{(-\infty, y)} \overset{d}{=} \xi(y)\) for any \(0 \leq y\). The following then easily follows.

**Corollary 26.** Suppose Assumption 20 holds. Then for any \(f \in C^{+}((\mathbb{R})\),
\[
(6.9) \quad \mathbb{E}\left[\exp\left\{-\langle f, \theta_{y}\xi^{-}\rangle\right\}\mid \mathcal{M}(\xi^{+}) > y\right] \quad \text{is independent of } y > 0,
\]
and \(\mathcal{M}(\xi^{+})\) is an exponential random variable with parameter \(c\), and therefore the point process defined as \(\theta_{y}\xi^{+}\) conditioned on the event \(\{\mathcal{M}(\xi^{+}) > y\}\) is independent of \(y > 0\).

**Proof.** Let \(f \in C^{+}((\mathbb{R})\) and \(y \geq 0\). Then for \(x\) large enough such that the support \(\theta_{y}f\) is contained in \((-x, \infty)\),
\[
\mathbb{E}\left[\exp\left\{-\langle f, \theta_{y}\xi^{+}\rangle\right\}\mid \mathcal{M}(\xi^{+}) > y\right] = \mathbb{E}\left[\exp\left\{-\langle f, \theta_{y}\xi(x)\rangle\right\}\mid \mathcal{M}(\xi(x)) > y\right]
\]
\[
= \mathbb{E}\left[\exp\left\{-\langle f, \theta_{-y}\xi^{+}\rangle\right\}\cdot 1_{\{\mathcal{M}(\xi(x)) > y\}}\right],
\]
\[
= \mathbb{E}\left[\exp\left\{-\langle f, \theta_{-y}\theta_{-x}\xi^{-}\rangle\right\}\cdot 1_{\{\mathcal{M}(\theta_{-x}\xi^{-}) > y\}}\mid \mathcal{M}(\xi^{-}) > x\right]
\]
\[
= \mathbb{E}\left[\exp\left\{-\langle f, \theta_{-(x+y)}\xi^{-}\rangle\right\}\cdot 1_{\{\mathcal{M}(\xi^{-}) > x+y\}}\right],
\]
\[
= \mathbb{E}\left[\exp\left\{-\langle f, \theta_{-(x+y)}\xi^{-}\rangle\right\}\mid \mathcal{M}(\xi^{-}) > x+y\right].
\]
Thus, for two different \(y_{1}, y_{2} > 0\), choosing \(x_{1}\) and \(x_{2}\) large enough and such that \(x_{1} + y_{1} = x_{2} + y_{2}\), we have
\[
\mathbb{E}\left[\exp\left\{-\langle f, \theta_{y_{1}}\xi^{+}\rangle\right\}\mid \mathcal{M}(\xi^{+}) > y_{1}\right] = \mathbb{E}\left[\exp\left\{-\langle f, \theta_{-(x_{1}+y_{1})}\xi^{+}\rangle\right\}\mid \mathcal{M}(\xi^{+}) > x_{1} + y_{1}\right]
\]
\[
= \mathbb{E}\left[\exp\left\{-\langle f, \theta_{y_{2}}\xi^{+}\rangle\right\}\mid \mathcal{M}(\xi^{+}) > y_{2}\right].
\]
Relying on the fact that \(\xi^{+}|_{(-\infty, y)} = \xi_{y}\) with \(y = 0\) we have \(\xi^{+}|_{(0, \infty)} = \xi^{-}\) and thus, from Corollary 20, \(\mathcal{M}(\xi^{+})\) is exponentially distributed with parameter \(c\). \(\square\)
Note that by Lemma 25 and Definition PP-D, \( D(\xi) \triangleq \theta_{-M(\xi^{*+})}\xi^{*+} \). Corollary 26 immediately gives the following.

**Corollary 27.** The point process \( D(\xi) \) is independent of the maximum \( M(\xi^{*+}) \).

**Proof.** It is sufficient to show that \( \mathbb{E}[\exp \{-f, D(\xi)\} \mid M(\xi^{*+}) > y] \) is independent of \( y \geq 0 \) for any \( f \in C^+(\mathbb{R}) \). This follows from Corollary 26 since

\[
\mathbb{E}[\exp \{-f, D(\xi)\} \mid M(\xi^{*+}) > y]
= \mathbb{E}[\exp \{-\langle \theta_{M(\xi^{*+})}f, \xi^{*+}\rangle\} \mid M(\xi^{*+}) > y]
= \mathbb{E}[\exp \{-\langle \theta_{M(\xi^{*+})}f, \xi^{*+}\rangle\} \mid M(\xi^{*+}) > y]
= \mathbb{E}[\exp \{-\langle \theta_{M(\xi^{*+})}f, \xi^{*+}\rangle\} \mid M(\xi^{*+}) > y].
\]

\[\square\]

The two last corollaries directly give the following.

**Corollary 28.** If Assumption 20 holds, then \( \xi^{*+} \overset{d}{=} \theta_X D(\xi) \), where \( X \sim \exp(c) \) is independent of \( D(\xi) \).

The main result of this section can now be proven.

**Lemma 29.** Suppose Assumption 27 holds. If

\[ \psi(\xi) \sim DPPP \left( e^{-cx}dx, \theta_{-\tau_{M(\xi)}} - c^{-1}\log c D(\xi) \right), \]

then \( L_{\psi(\xi)}(f \mid \cdot) \) is uniquely supported on the class \([\text{Gum}(cy)]\) with shifts \( \tau_{Gump(cy)}(\psi(\xi)) = \tilde{\tau}_{f} \xi^{*+} = \tilde{\tau}_{M(\xi)}(\xi) - c^{-1}\log c, \) for any \( f \in C^+(\mathbb{R}) \).

**Proof.** Note that the lemma follows if we show that for \( \psi' = \psi(\xi) \sim DPPP \left( e^{-cx}, D(\xi) \right), \)

\[ L_{\psi}(f \mid \cdot) \] is uniquely supported on the class \([\text{Gum}(cy)]\) with shifts \( \tau_{f}(\psi') = \tilde{\tau}_{f} \xi^{*+} = \tilde{\tau}_{M(\xi)}(\xi) - c^{-1}\log c, \) for any \( f \in C^+(\mathbb{R}) \).

From the calculations of Section 4, \( L_{\psi}(f \mid \cdot) \) is uniquely supported on the class \([\text{Gum}(cy)]\) and the shifts are given, for any \( f \in C^+(\mathbb{R}) \), by

\[ \tau_{f}(\psi') = \tau_{f,c} = c^{-1}\log \left( -\int_{-\infty}^{\infty} e^{-ct} \left( L_{D(\xi)}[f] - t \right) - 1 dt \right). \]

(6.10)

Note that it is sufficient to verify that (6.10) holds for any \( f \in C^+(\mathbb{R}) \) such that \( \inf \{ x \in \mathbb{R} : f(x) > 0 \} = 0 \). If \( f \) is such a function, then, since \( M(D(\xi)) = 0 \) a.s.,

\[ \int_{-\infty}^{\infty} e^{-ct} \left( L_{D(\xi)}[f] - t \right) - 1 dt = \int_{0}^{\infty} e^{-ct} \left( L_{D(\xi)}[\theta_{-t}f] - 1 \right) dt \]

\[ = -c^{-1}e^{-c(\tilde{\tau}_{M(\xi)}(\xi) - \tilde{\tau}_{f}(\xi))}, \]

where we used Corollary 28 and Lemma 22. Substituting this in (6.11) yields (6.10).

Using Lemma 29, Corollary 11 can be now easily proved.
Proof. First, recall that, as stated in Remark 21, Assumption 20 holds with shifts 
Part Corollary 30. We begin with the following corollary of Lemma 29. Observe that the Laplace functionals of 
dependent. We begin with the following corollary of Lemma 29. Observe that the Laplace functionals of 
\[ \mathcal{L}_\varphi \left( f \right) = \mathcal{L}_\varphi \left( f \right) \cdot \mathcal{L}_\varphi \left( f \right) \cdot \mathcal{L}_\varphi \left( f \right) \], 
Thus, if there exists some \( t \) such that, for any \( f \in C_+^e (\mathbb{R}) \), \( \varphi (x) = \varphi (1) x \). Hence 
\[ \mathcal{L}_\varphi \left( f \right) = \exp \left\{ -e^{-(y - \tau_f)} \right\} \approx \text{Gum} (y), \]
which proves (3).

Lastly, assume that (3) holds. Note that \( \xi \) satisfies Assumption (1) and thus, as noted in Remark 21, it satisfies Assumption (20). Let \( \psi (\xi) \) be the corresponding DPPP from Lemma (29). Observe that the Laplace functionals of \( \psi (\xi) \) and \( \xi \) coincide on \( C_+^e (\mathbb{R}) \) and thus both processes have the same law. That is, (1) is satisfied with 
\[ \xi \sim \text{DPPP} \left( e^{-x} dx, \mathcal{D}_f (\xi) \right) \] and the proof is completed.

7. Proof of Theorem (2) the direct part

We shall prove each of the assertions of the direct part of Theorem (2) separately. The notation \( X \perp Y \) will be used whenever two random variables \( X \) and \( Y \) are independent. We begin with the following corollary of Lemma (29).

Corollary 30. Part (a) of the direct part of Theorem (2) holds.

Proof. First, recall that, as stated in Remark 21, Assumption (20) holds with shifts \( \tau_f^q (\xi) \) equal to the shifts of Assumption (3) \( \tau_f^q (\xi) \).

Part Corollary 30. We begin with the following corollary of Lemma 29. Observe that the Laplace functionals of \( \psi (\xi) \) and \( \xi \) coincide on \( C_+^e (\mathbb{R}) \) and thus both processes have the same law. That is, (1) is satisfied with 
\[ \xi \sim \text{DPPP} \left( e^{-x} dx, \mathcal{D}_f (\xi) \right) \] and the proof is completed.

Proof. First, recall that, as stated in Remark 21, Assumption (20) holds with shifts \( \tau_f^q (\xi) \).

Suppose (SUS) holds with some random variable \( Z \) and with \( c = c_\xi \). Let \( \psi = \psi (\xi) \) be the corresponding DPPP from Lemma (29) which by the lemma satisfies \( \tau_f^{\text{Gum}(c, y)} (\psi) = \tau_f^q (\xi) \). Define \( \psi \) and \( Z \) on the same probability space such that \( Z \perp \psi \) and consider the point process \( \theta_Z \psi \). Observe that, by conditioning on \( Z \), the Laplace functional of \( \theta_Z \psi \) is given, for any \( f \in C_+^e (\mathbb{R}) \), by 
\[ \mathcal{L}_{\theta_Z} \psi \left( f \right) = \int \mathcal{L}_{\theta_\psi} \left( f \right) \, d\mu_Z (z) = \int \mathcal{L}_\psi \left( f \right) \, d\mu_Z (z) 
= \int \text{Gum} (c \left( -z - \tau_f \right)) \, d\mu_Z (z) = g (-\tau_f) = \mathcal{L}_\xi \left( f \right), \]
that is, the Laplace functionals of \( \theta_Z \psi \) and \( \xi \) coincide on \( C_+^e (\mathbb{R}) \). This implies 
\[ \xi \sim \text{SDPPP} \left( e^{-c} dx, \mathcal{D}_f (\xi) \right) \] as required.
We now need to treat the case where the only assumption on $g$ is Assumption 4. In this case, in order to obtain a process that satisfies (SUS) we consider the process $\theta g\xi$, where the a random variable $S$ has itself a Gumbel distribution and is independent of $\xi$. Doing so easily yields the following.

**Lemma 31.** Part (b) of the direct part of Theorem 4 holds.

**Proof.** From Lemma 29 the DPPP $\psi = \psi(\xi)$ has shifts $\tau_f^{\text{Gum}(c)}(\psi) = \tau_f^{\text{Gum}(c)}(\xi)$, where $\text{Gum}(c)(y) \triangleq \text{Gum}(cy)$. Observe that, if $Z_G$ and $Z_g$ are random variables with cumulative distribution functions $\text{Gum}(c)$ and $g$ independent of $\xi$ and $\psi$, respectively, then for any $f \in C_+^c(\mathbb{R})$,

$$L_{\theta z_G} [f] = \text{Gum}(c) * g(-\tau_f) = g * \text{Gum}(c)(-\tau_f) = L_{\theta z_g} [f] ,$$

that is, the Laplace functionals of $\theta_{Z_G}\xi$ and $\theta_{Z_g}\psi$ coincide on $C_+^c(\mathbb{R})$ and therefore $\theta_{Z_G}\xi$ and $\theta_{Z_g}\psi$ have the same law.

Denote by $\mathcal{N}$ the space of positive, locally finite, counting measures on $\mathbb{R}$ (cf. Section 10). Defining the measurable function $h : \mathcal{N} \times \mathbb{R} \rightarrow \mathcal{N}$, $(\zeta, Z) \mapsto \theta_{Z}\zeta$, we have $\theta_{Z_g}\psi = h(\psi, Z_g)$ and $\theta_{Z_G}\xi \overset{d}{=} h(\psi, Z_g)$. Hence, according to the transfer principle [36, Corollary 6.11] (and due to the fact that $(\mathcal{N}, A)$ is a Borel space), we can in fact couple the random variables $Z_G$, $Z_g$, $\xi$ and $\psi$ such that $Z_G \perp \xi$, $Z_g \perp \psi$, and $\theta_{Z_G}\xi = \theta_{Z_g}\psi$ almost surely. On this space, we also have

$$\xi = \theta_{Z_g - Z_G}\psi \overset{d}{=} \theta_{Z}\psi, \quad \text{almost surely.}$$

This completes the proof. Note that $Z_G$, and hence $Z$, may depend on $\psi$. \qed

What remains is to show that under the assumptions of parts (c) and (d) of the direct part of Theorem 4 (SUS) holds. We shall begin with part (c), the proof of which will follow from the following result implying that the structure of the function $g$ specified in (SUS) is determined by the freezing phenomenon discussed in the introduction.

**Lemma 32.** Let $\xi \gg 0$ be a point process and let $c > 0$ such that

$$\text{for any } \beta > c : \ L_\xi \left[ e^{\beta x} \right] \approx g(\cdot),$$

where $g : \mathbb{R} \rightarrow (0, 1)$ is a function for which $\sup_{y \in \mathbb{R}} g(y) = 1$. Then there exists a random variable on $\mathbb{R}$, $Z$, with law $\mu_Z$, such that

$$g(y) = \int \text{Gum}(\beta(y - z)) \, d\mu_Z(z) = \mathbb{E}\{ \text{Gum}(\beta(y - Z)) \} .$$

**Proof.** Fix some $\beta' > \beta > c$. Let $\eta$ be the random measure given by

$$\eta = \sum_{i \geq 1} \theta_{\xi_i} P_i,$$

where $\xi = \sum_{i \geq 1} \delta_{\xi_i}$, and where $P_i$ are Poisson point processes of intensity $e^{-\beta x} dx$, independent of each other and $\xi$. Since

$$\mathbb{E} \exp \left\{ - \langle \theta_y e^{\beta x}, \xi \rangle \right\} \leq 1 - \mathbb{P} \left\{ \sum_{i \geq 1} e^{\beta \xi_i} = \infty \right\} ,$$
it follows from the assumptions that $\mathbb{P}\left\{ \sum_{i \geq 1} e^{\beta \xi_i} < \infty \right\} = 1$, and the same with $\beta'$.

Conditioned on $\xi$, the process $\eta$ is a sum of independent Poisson processes, and therefore $\eta$ is a Poisson point process of finite intensity

$$
\sum_{i \geq 1} e^{-\beta x + \beta \xi_i} \, dx = e^{-\beta x} \sum_{i \geq 1} e^{\beta \xi_i} \, dx.
$$

Equivalently, with $Z_\beta \triangleq \frac{1}{\beta} \log \left( \sum_{i \geq 1} e^{\beta \xi_i} \right)$,

$$
(7.1) \quad \eta \sim SDPPP \left( e^{-\beta x}, \delta_0, Z_\beta \right).
$$

Each $P_i$ satisfies (SDP) and thus (SUS) with the function $\text{Gum}(\beta y)$ (cf. the converse part of Theorem 9). From Corollary $14$.

From the assumptions that $\{ Z_\beta + \tau_2(\beta) \}_{\beta > c}$ is a tight family. Therefore there exists a sequence $\beta_i$ that decreases to $c$ as $i \to \infty$ such that $Z_{\beta_i} + \tau_2(\beta_i)$ converges in distribution to some limiting random variable $Z$. For this variable,

$$
g(y) = \lim_{i \to \infty} \mathbb{E}\left[ \text{Gum}(\beta_i (y - Z_{\beta_i} + \tau_2(\beta_i))) \right] = \mathbb{E}\left[ \text{Gum}(c (y - Z)) \right],
$$

which completes the proof.

We remark that when $\xi' = \theta S \xi$ with $S$ and $\xi$ independent, reassuringly,

$$
\frac{1}{\beta} \log \left( \sum_{i \geq 1} e^{\beta \xi_i} \right) = S + \frac{1}{\beta} \log \left( \sum_{i \geq 1} e^{\beta \xi_i} \right),
$$

and the limiting procedure above ‘extracts’ the shift.
Remark 33. In Proposition 39 of the appendix, we use the reasoning above to relate Gumbel distribution functions of different scales to each other by convolution with a Borel measure, a result of possible independent interest.

We now complete the proof of part (c).

**Corollary 34.** Part (c) of the direct part of Theorem 3 holds.

**Proof.** We wish to apply Lemma 32 with $c = c_\xi$. The first condition (equivalence to $g$) is assumed directly. What remains is to show that $\sup_{y \in \mathbb{R}} g(y) = 1$. This follows from (US) and Lemma 12. \qed

We continue with the proof of part (d).

**Lemma 35.** Part (d) of the direct part of Theorem 3 holds if $D(\xi)$ satisfies an LLN with rate $c\xi$.

**Proof.** Throughout the proof let $c = c_\xi$. Let $\psi = \psi(\xi) \sim DPPP(e^{-c\xi}dx, D(\xi))$. From Lemma 31 we know that $\xi \overset{d}{=} \theta_{Z,\psi} \overset{d}{=} \theta_{Z,\psi'}$ for some random variable $Z'$ and that

$$Z \perp (Z', \psi') \quad \text{and} \quad Z_g \perp \psi,$$

where $\psi \overset{d}{=} \psi$ and where $Z_g$ and $Z'$ are random variables with cumulative distribution functions $G_{\psi}^{(c)}$ and $g$, respectively.

From the same argument as in the proof of Lemma 31, appealing to Corollary 6.11 of [36], we can define all of the variables on the same probability space such that (7.2) is preserved and such that

$$\theta_{Z_g,\theta_{Z',\psi'}} = \theta_{Z,\psi}, \text{ almost surely}.$$  

Since $D(\xi)$ satisfies the LLN, there exists a sequence $y_n \in \mathbb{R}$ that increases to $\infty$, such that $\psi((-(y_n, \infty))/\alpha(y_n)) \to u$ almost surely, as $n \to \infty$, for some $u > 0$. Note that, almost surely,

$$\frac{\psi((-(y_n, \infty))/\alpha(y_n))}{\alpha(y_n)} = \frac{\theta_{Z_g,\psi}((-(y_n + Z, \infty))/\alpha(y_n))}{\alpha(y_n)} = \frac{\theta_{Z_g + Z',\psi'}((-(y_n + Z, \infty))/\alpha(y_n))}{\alpha(y_n)} = \frac{\psi'((-(y_n + Z - Z', \infty))/\alpha(y_n))}{\alpha(y_n)},$$

and therefore the same convergence holds for the rightmost term of the equation.

We shall prove by contradiction that $Z_g - Z_G - Z' \geq 0$ almost surely. Assume otherwise, i.e., there exists $\epsilon > 0$ such that

$$\mathbb{P}\{Z_g - Z_G - Z' > \epsilon\} > 0.$$  

Since $\psi((-(y, \infty))$ is increasing in $y$, by (2.4), almost surely on the event $\{Z_g - Z_G - Z' > \epsilon\}$,

$$\liminf_{n \to \infty} \frac{\psi((-(y_n - \epsilon, \infty))/\alpha(y_n - \epsilon))}{\alpha(y_n - \epsilon)} \geq \lim_{n \to \infty} \frac{\psi((-(y_n + Z_g - Z_G - Z', \infty))/\alpha(y_n - \epsilon))}{\alpha(y_n - \epsilon)} = u e^\epsilon,$$

and therefore, for large enough $N$,

$$\mathbb{P}\left\{\forall n \geq N : \frac{\psi((-(y_n - \epsilon, \infty))/\alpha(y_n - \epsilon)}{\alpha(y_n - \epsilon)} \geq u e^\epsilon / 2\right\} > 0,$$

in contradiction to the LLN. Hence $Z_g - Z_G - Z' \geq 0$ almost surely.
One obtains similarly that \( Z_g - Z_G - Z' \leq 0 \) and thus \( Z_g = Z_G + Z' \), almost surely. Since \( Z_G \perp Z' \), this implies that \( Z_g \) is distributed like a shifted Gumbel, i.e. Condition (SUS) holds. The proof is completed. \( \square \)

We finish the proof of the direct part of Theorem 9 with the following.

**Lemma 36.** Assuming \( D(\xi) \) has exponential moments, part (d) of the direct part of Theorem 9 holds.

**Proof.** As in the proof of Lemma 35, abbreviate \( c = c_x \) and couple the shifts \( Z_G \), \( Z_g \), \( Z' \) and point processes \( \psi \), \( \psi' \) defined there, so that (7.2) and (7.3) hold.

For any point process \( \eta \) such that \( \mathcal{M}(\eta) < \infty \) almost surely, we define \( \pi(\eta) \triangleq \theta_{-\mathcal{M}(\eta)} \). That is, \( \pi(\eta) \) is \( \eta \) shifted so its rightmost particle is exactly at zero almost surely. Clearly,

\[
\pi(\theta_{Z_G\theta_{Z},\psi'}) = \pi(\psi') \quad \text{and} \quad \pi(\theta_{Z_g\psi}) = \pi(\psi),
\]

and thus \( \pi(\psi') = \pi(\psi) \) almost surely.

Now, suppose \( A \subseteq \mathcal{N} \) is a measurable set such that \( \{\pi(\psi) \in A\} \) has positive probability. Note that, since \( \pi(\psi') = \pi(\psi) \) almost surely, the probability of the symmetric difference \( \{\pi(\psi) \in A\} \triangle \{\pi(\psi') \in A\} \) is 0. Therefore, conditioned either on \( \{\pi(\psi) \in A\} \) or on \( \{\pi(\psi') \in A\} \), the random vector \( (Z_G, Z', \mathcal{M}(\psi')) \) has the same distribution.

Define \( X_A \) and \( Y_A \) to be random variables distributed as \( Z' + \mathcal{M}(\psi') \) and \( \mathcal{M}(\psi) \) conditioned on \( \{\pi(\psi) \in A\} \), respectively. Then from (7.2), by considering the rightmost particle of the point processes in (7.3),

\[
Z_G + X_A \overset{d}{=} Z_g + Y_A, \quad \text{where} \quad Z_G \perp X_A \quad \text{and} \quad Z_g \perp Y_A.
\]

Let \( A_n \subseteq \mathcal{N}, \) \( n \geq 1, \) be a sequence of measurable sets such that \( \{\pi(\psi) \in A_n\} \) has positive probability and abbreviate \( Y_n = Y_{A_n}, X_n = X_{A_n} \). Suppose that \( Y_n - c_n \overset{d}{\to} 0 \) as \( n \to \infty \) for some deterministic sequence \( c_n \in \mathbb{R} \). Then \( Z_G + X_n - c_n \overset{d}{\to} Z_g \) as \( n \to \infty \) (with \( Z_G \perp X_n - c_n \) for any \( n \geq 1 \)), thus the characteristic function of \( Z_g \) satisfies

\[
\mathbb{E}e^{-itZ_g} = \lim_{n \to \infty} \mathbb{E}e^{-it(Z_G+X_n-c_n)} = \mathbb{E}e^{-itZ_G} \lim_{n \to \infty} \mathbb{E}e^{-it(X_n-c_n)}.
\]

It is easily seen from the convergence of \( Z_G + X_n - c_n \) and the independence of \( Z_G \) on \( X_n - c_n \) that \( X_n - c_n \), \( n \geq 1 \), is tight. By the continuity theorem (cf. Theorem 2, p. 481 of [26]), this implies that the limit of \( \mathbb{E}e^{-it(X_n-c_n)} \) is the characteristic function of some random variable \( Z \) and thus \( Z_g = Z_G + Z \) with \( Z_G \perp Z \).

This is exactly what we need to show, hence all that remains is to construct the sets \( A_n \) as above. We shall show that for \( A_n^\epsilon \triangleq \{\eta \in \mathcal{N} : \mathcal{M}(\eta) = 0, \eta([\epsilon,0] \geq n)\} \), with \( \epsilon > 0 \) being a parameter to be determined below, the convergence \( Y_n - c_n \overset{d}{\to} 0 \) is achieved. Of course, \( \mathbb{P}(\pi(\psi) \in A_n^\epsilon) > 0 \) for any \( \epsilon > 0, n \geq 1 \), since with positive probability there are \( n \) atoms of the Poisson process corresponding to \( \psi \) in the interval \( (\mathcal{M}(\xi) - \epsilon, \mathcal{M}(\xi)) \).

For simplicity we shall assume \( c = 1 \), the general case follows by scaling. Recall that the decoration \( D(\xi) \) satisfies \( \mathcal{M}(D(\xi)) = 0 \), almost surely. Therefore, the density of \( \mathcal{M}(\psi) \) relative to Lebesgue measure is given by \( \exp\{-e^{-x} - x\} \).
By conditioning on $\mathcal{M}(\psi)$, for any Borel set $B \subset \mathbb{R}$,
\[
\mu_n(B) \triangleq \mathbb{P}\{\mathcal{M}(\psi) \in B \mid \pi(\psi) \in A_n^\prime\}
\]
(7.4)
\[
= \frac{\int_B \exp\{-e^{-x} - x\} \mathbb{P}\{(\eta_x + \theta_x D_0)([x - \epsilon, x]) \geq n\} \, dx}{\int_B \exp\{-e^{-x} - x\} \mathbb{P}\{(\eta_x + \theta_x D_0)([x - \epsilon, x]) \geq n\} \, dx},
\]
where $\eta_x \sim DPPP(e^{-cy} \cdot 1_{(-\infty, x)} dy, D(\xi))$ and $D_0$ is an independent copy of $D(\xi)$.

The number of the decorations composing $\eta_x$ that attribute a positive measure to the interval $[x - \epsilon, x]$ is a Poisson variable of parameter $e^{-x} (e^\epsilon - 1)$. Conditioned on this number, the decorations are independent, and each is equal in distribution to $\bar{S}(\epsilon)$ where the p.d.f. of the random variable $\bar{S}(\epsilon)$ is
\[
e^{-x} 1_{[x-\epsilon, x]}(s) \, ds/ (e^{-x} (e^\epsilon - 1)).
\]
Note also that $\bar{S}(\epsilon) \overset{d}{=} x + S(\epsilon)$ where the p.d.f. of $S(\epsilon)$ is $e^{-x} 1_{[-\epsilon, 0]}(s) \, ds/ ((e^\epsilon - 1))$.

Hence, if
\[
U_0^* \overset{d}{=} \theta_x D_0 ([x - \epsilon, x]) = D_0 ([\epsilon, 0]),
\]
\[
W_i^* \overset{d}{=} \bar{S}(\epsilon)(D(\xi)) ([x - \epsilon, x]) = \bar{S}(\epsilon)(D(\xi)) ([\epsilon, 0]), \ \forall i \geq 1,
\]
are independent random variables, with $\{W_i^*\}_{i \geq 1}$ an i.i.d sequence, then
\[
(\eta_x + \theta_x D_0)([x - \epsilon, x]) \overset{d}{=} U_0^* + \sum_{i=1}^{\tilde{N}_x} W_i^*,
\]
where $\tilde{N}_x \sim \text{Pois}(e^{-x} (e^\epsilon - 1))$ is a random variable independent of $U_0, W_i, i \geq 1$.

From this, by a change of variables $y = e^{-x}/n$, we obtain from (7.4), for any $y_n^* = \theta_x D_0([\epsilon, 0])/n$, we obtain from (7.3), for any $y_n^*$, $\delta > 0$,
\[
\mu_n([-\log(ny_n^*) - \delta, -\log(ny_n^*) + \delta]) = \frac{\int_{-\log(ny_n^*)}^{-\log(ny_n^*)} h(y, n) \, dy}{\int_{-\infty}^{\log(ny_n^*)} h(y, n) \, dy},
\]
(7.5)
where
\[
h(y, n) \triangleq e^{-ny} \mathbb{P}\left\{U_0^* + \sum_{i=1}^{N_{ny}} W_i^* \geq n\right\} = e^{-ny} \mathbb{P}\left\{\frac{1}{n} U_0^* + \frac{1}{n} \sum_{i=1}^{N_{ny}} W_i^* \geq 1\right\}
\]
and where $N_{ny} \sim \text{Pois}(ny (e^\epsilon - 1))$ is a random variable independent of $U_0^*, W_i^*, i \geq 1$.

In order to prove the convergence of $Y_n - c_n \overset{d}{\to} 0$ as $n \to \infty$, it suffices to show that there exist $y_n^* \in (0, \infty), n \geq 1$, such that the ratio of (7.3) converges to 1 as $n \to \infty$, for any $\delta > 0$; one then sets $c_n = -\log(ny_n^*)$. In our proof, we will choose $y_n = m^*$ where $m^*$ is an $n$-independent constant to be determined below.

We first choose the parameter $\epsilon > 0$. Recall that, by assumption, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in [0, \epsilon_0]$, there exists $t > 0$ for which $\mathbb{E}[\exp\{tD(\xi)(-\epsilon,0)\}] < \infty$. For each such $\epsilon$, we define
\[
\lambda^* (U_0^*) \triangleq \inf\{\lambda : \mathbb{E}\{\exp(\lambda U_0^*)\} = \infty\},
\]
with $\lambda^* (U_0^*) = \infty$ in case the set is empty. Define $\lambda^* (W_i^*)$ similarly and note that since $0 \leq W_i^* \leq U_0^* \ a.s.$, $\lambda^* (U_0^*) \leq \lambda^* (W_i^*)$. 
If \( \lambda^* (U_0^{(n)}) = \infty \), and therefore also \( \lambda^* (W_1^{(n)}) = \infty \), we set \( \epsilon = \epsilon_0 \). Otherwise, note that \( \lambda^* (U_0^{(n)}) \) is a bounded, decreasing function of \( \epsilon \) on \([0, \epsilon_0]\), thus there exists some \( \epsilon' > 0 \) at which it is continuous, and we set \( \epsilon = \epsilon' \). For any \( \lambda > \lambda^* (U_0^{(n)}) \),

\[
\mathbb{E} \{ \exp (\lambda W_1^{(n)}) \} \geq \mathbb{P} \{ S (\epsilon) \geq -\delta \} \mathbb{E} \{ \exp (\lambda \cdot \theta_{-\delta} D (\xi) ([\epsilon, 0])) \} = \mathbb{P} \{ S (\epsilon) \geq -\delta \} \mathbb{E} \{ \exp (\lambda \cdot U_0^{(n) - \delta}) \}.
\]

Choosing \( \delta > 0 \) small enough such that \( \lambda > \lambda^* (U_0^{(n) - \delta}) \), we conclude that \( \mathbb{E} \{ \exp (\lambda W_1^{(n)}) \} = \infty \). It follows that \( \lambda^* (W_1^{(n)}) = \lambda^* (U_0^{(n)}) \). Henceforth we fix \( \epsilon \) as above, abbreviate \( \lambda^* \equiv \lambda^* (U_0^{(n)}) \), and suppress \( \epsilon \) from the notation.

For each \( y > 0 \), define the logarithmic moment generating function

\[
(7.6) \quad \Lambda_y (\lambda) \equiv \log \mathbb{E} \left\{ \exp \left\{ \lambda \sum_{i=1}^{N_n} W_i \right\} \right\} = y (e^\epsilon - 1) \left( \mathbb{E} \{ e^{\lambda W_1} \} - 1 \right) , (\lambda \in \mathbb{R})
\]

and its Fenchel-Legendre transform

\[
\Lambda_y^* (x) \equiv \sup_{\lambda \in \mathbb{R}} \{ \lambda x - \Lambda_y (\lambda) \} , (x \in \mathbb{R}).
\]

We next claim that

\[
(7.7) \quad \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left\{ \frac{1}{n} U_0 + \frac{1}{n} \sum_{i=1}^{N_n} W_i \geq 1 \right\} = - \inf_{x \geq 1} \Lambda_y^* (x).
\]

Indeed, introduce the representation

\[
(7.8) \quad \sum_{i=1}^{N_n} W_i = \sum_{j=1}^{n} \sum_{i=1}^{N_n} W_i^{(j)}
\]

with the i.i.d. sequences of variables \((i, j) W_i^{(j)} \overset{d}{=} W_i \) and \( N_n^{(j)} \overset{d}{=} N_n \) independent of each other. The lower bound in \((7.7)\) follows from \( U_0 \geq 0 \) using Cramér’s theorem \[21\] Corollary 2.2.19:

\[
\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left\{ \frac{1}{n} U_0 + \frac{1}{n} \sum_{i=1}^{N_n} W_i \geq 1 \right\} \geq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left\{ \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{N_n^{(j)}} W_i^{(j)} \geq 1 \right\} = - \inf_{x \geq 1} \Lambda_y^* (x).
\]

To see the corresponding upper bound, introduce the logarithmic moment generating function

\[
\tilde{\Lambda}_n^{(y)} (\lambda) \equiv \log \mathbb{E} \left\{ \exp \left\{ \lambda \left( U_0/n + \sum_{i=1}^{N_n} W_i/n \right) \right\} \right\} , (\lambda \in \mathbb{R}),
\]

and its scaled limit and Fenchel-Legendre transform

\[
\tilde{\Lambda}_y (\lambda) \equiv \lim_{n \to \infty} \frac{1}{n} \tilde{\Lambda}_n^{(y)} (n \lambda) , \quad \tilde{\Lambda}_y^* (x) \equiv \sup_{\lambda \in \mathbb{R}} \{ \lambda x - \tilde{\Lambda}_y (\lambda) \} , (x \in \mathbb{R}).
\]

For \( \lambda < \lambda^* \) we have (using the representation \((7.8)\) that \( \Lambda_y (\lambda) = \tilde{\Lambda}_y (\lambda) \). Obviously, for \( \lambda > \lambda^* \) we have \( \Lambda_y (\lambda) = \tilde{\Lambda}_y (\lambda) = \infty \). Thus, \( \Lambda_y \) and \( \tilde{\Lambda}_y \) coincide in the interior.
of their (common) domain; since both are convex, they are necessarily continuous in the interior of their domain and
\[
\limsup_{\lambda \to \lambda^*} \Lambda_y(\lambda) \leq \Lambda_y(\lambda^*), \quad \limsup_{\lambda \to \lambda^*} \tilde{\Lambda}_y(\lambda) \leq \tilde{\Lambda}_y(\lambda^*).
\]
In particular, for any \( x \in \mathbb{R} \),
\[
\tilde{\Lambda}_y^*(x) = \sup_{\lambda \in \mathbb{R}} \{ \lambda x - \tilde{\Lambda}_y(\lambda) \} = \sup_{\lambda \leq \lambda^*} \{ \lambda x - \tilde{\Lambda}_y(\lambda) \}
\]
\[
= \sup_{\lambda < \lambda^*} \{ \lambda x - \Lambda_y(\lambda) \} = \sup_{\lambda \in \mathbb{R}} \{ \lambda x - \Lambda_y(\lambda) \} = \Lambda_y^*(x).
\]

Applying the upper bound of the Gärtner–Ellis theorem \cite[Theorem 2.3.6]{21}, we obtain
\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left\{ U_0/n + \sum_{i=1}^{N_{ny}} W_i/n \geq 1 \right\} \leq -\inf_{x \geq 1} \Lambda_y^*(x),
\]
which completes the proof of (7.7).

Note that with \( \bar{x} \triangleq (e^\varepsilon - 1) EW_1 \), we have by \cite[Lemma 2.2.5]{21} that \( \Lambda_1^*(\bar{x}) = 0 \) and that \( \Lambda_1^*(x) \) is nondecreasing for \( x \geq \bar{x} \). Thus, we can rewrite (7.7) as
\[
\hat{h}(y) \triangleq \lim_{n \to \infty} \frac{1}{n} \log \hat{h}(y, n) = -y - \inf_{x \geq 1} \Lambda_y^*(x) = -y - \inf_{x \geq 1/y} \Lambda_1^*(x)
\]
(7.9)
\[
= \begin{cases} 
-y - \Lambda_1^*(1/y), & \text{if } y \in (0, 1/\bar{x}) \\
-y, & \text{if } y \in [1/\bar{x}, \infty),
\end{cases}
\]
where the next-to-last equality follows from (7.6) and the definition of the Fenchel-Legendre transform.

Before returning to the evaluation of (7.5), we analyze the function \( \hat{h} \). As a preliminary, note that \( \Lambda_1^* \) is finite and continuous in a neighborhood of \( \bar{x} \) because 0 is in \( \mathcal{D}_{\Lambda_1} \), the interior of the domain of \( \Lambda_1 \), and \( \sum_{i=1}^{N_y} W_i \) is not deterministic, and thus \( \bar{x} \in \{ \Lambda_1^*(\lambda) : \lambda \in \mathcal{D}_{\Lambda_1} \} \), so that \cite[Lemma 2.2.5(c) and Exercise 2.2.24]{21} can be applied.

Let \( I_1 = (0, 1/\bar{x}) \) and \( I_2 = [1/\bar{x}, \infty) \). We have that \( \hat{h} \) is strictly decreasing on \( I_2 \). Further, since \( 1/y \) is strictly convex in \( y \), \( \hat{h}(y) \) is strictly concave in \( 1/y \) on \( I_1 \). Thus, since \( \lim_{y \to 0} \hat{h}(y) = -\infty \),
\[
\sup_{y \in \mathbb{R}} \hat{h}(y) = \sup_{z \in [\bar{x}, \infty)} \hat{h}(1/z) = \hat{h}(m^*)
\]
for a unique \( m^* \in (0, 1/\bar{x}] \), and, fixing an arbitrary \( \delta > 0 \), with \( J = (e^{-\delta} m^*, e^\delta m^*) \) we have
\[
\rho \triangleq \hat{h}(m^*) > \kappa \triangleq \sup_{y \in J^c} \hat{h}(y).
\]
Set \( \Delta \triangleq \rho - \kappa \).

Let \( \mathcal{P} = \{ 0 = p_0 < p_1 < \cdots < p_k = T \} \) be a finite partition containing each of the ends of the interval \( J \), such that \( ||\mathcal{P}|| \triangleq \max_{1 \leq i \leq k} (p_i - p_{i-1}) < \Delta/6 \), and where \( T \) satisfies \( -T < \kappa + \Delta/3 \). Define for any point \( y \in [0, T] \),
\[
p^*(y) = \min \{ p_i : p_i \geq y \} \quad \text{and} \quad p^*_y(y) = \max \{ p_i : p_i \leq y \}.
\]
Set \( J^c = J^c \cap \{0, T\} \). Since \( Q_y \triangleq P\{ \sum_{i=1}^{N_y} W_i \geq 1 \} \) is nondecreasing in \( y \), we have the upper bound
\[
\int_{J^c} h(y, n) \, dy 
\leq \int_{J^c} e^{-np^*(y)} Q_{p^*(y)} dy 
\leq T e^n \max_{y \in (J^c) \cap \mathcal{P}} \{ e^{-ny} Q_y \} 
\leq T e^{n/3} \max_{y \in (J^c) \cap \mathcal{P}} h(y, n),
\]
and, for any \( \alpha < (e^{\delta} - 1)m^* \), the lower bound
\[
\int_{J} h(y, n) \, dy 
\geq \int_{J} h(y, n) \, dy 
\geq \alpha h(m^*, n) e^{-n\alpha}.
\]

Therefore, for \( n \) large enough, using the fact that \( \mathcal{P} \) is a finite set which does not depend on \( n \),
\[
\frac{1}{n} \log \left( \int_{J^c} h(y, n) \, dy \right) < \kappa + \Delta/3,
\]
\[
\frac{1}{n} \log \left( \int_{J} h(y, n) \, dy \right) > \rho - \Delta/3.
\]

For bounding the integral on \((T, \infty)\) we use the simple bound
\[
\frac{1}{n} \log \left( \int_{T} h(y, n) \, dy \right) \leq \frac{1}{n} \log \left( \int_{T} e^{-ny} dy \right) \leq -T < \kappa + \Delta/3.
\]

Combining the three bounds we obtain that
\[
\lim_{n \to \infty} \frac{\int_{J^c} h(y, n) \, dy}{\int_{J} h(y, n) \, dy} = \infty.
\]

Since \( \delta > 0 \) was arbitrary, with our choice \( y^*_n = m^* \), the ratio of (7.5) converges as \( n \to \infty \) to 1, which completes the proof. \( \square \)

8. FROM UNIQUENESS TO FREEZING

In our notation, the freezing phenomenon considered in the physics literature \cite{16, 23, 28, 29, 30, 31, 32} can be written as
\[
\forall \beta > \beta_c : \lim_{n \to \infty} L_{\xi_n} \left[ e^{\beta x} \mid \cdot \right] \approx g(\cdot), \text{ where } \xi_n \triangleq \sum_{i \leq N(n)} \delta_{X_i(n)-m_n}.
\]

Under the convergence \( \xi_n \overset{d}{\to} \xi \), as \( n \to \infty \), it is natural to examine the relations of the condition above with
\[
\forall \beta > \beta_c : L_{\xi} \left[ e^{\beta x} \mid \cdot \right] \approx g(\cdot).
\]

Essentially, if enough is known about a point process \( \xi \) for which \( L_{\xi} \left[ f \mid \cdot \right] \) is uniquely supported (i.e., the above holds for functions in \( C^+_c(\mathbb{R}) \)), one can extend the equivalence \( L_{\xi} \left[ f \mid \cdot \right] \approx g(\cdot) \) to functions with unbounded support, e.g. \( e^{\beta x} \), by appealing to Corollary \cite{14} and approximating functions by sequences in \( C^+_c(\mathbb{R}) \).

\}
However, since convergence of point processes in distribution (with respect to the vague topology) depends only on the convergence of \((f, \xi_n)\) for \(f\) with bounded support, some care is needed in concluding that
\[
\lim_{n \to \infty} L_{\xi_n} \left[ e^{\beta x} \mid \cdot \right] = L_\xi \left[ e^{\beta x} \mid \cdot \right].
\]
The purpose of this section is to give a sufficient condition for this in terms of the sequence \(\xi_n\).

We begin with a remark on the assumption of tightness. Assume that \(\xi_n, n \geq 1\), is a sequence of point processes such that \(\lim_{n \to \infty} L_{\xi_n} [f \mid y] \equal{} \varphi (f \mid y)\) exists for any \(f \in C_0^+ (\mathbb{R})\), \(\varphi (f \mid \cdot) \approx g (\cdot)\). Since the limit of the Laplace functionals exists, the sequence converges in distribution if and only if it is tight. One then may hope that since we are not only assuming convergence of the Laplace functionals, but also assume this limit to have a very specific form, tightness should follow. The next example shows it is not so.

**Example 37.** Let \(\xi_n = \sum_{x \in \mathbb{Z}} \delta_x\), where \(\mathbb{Z}^- \triangleq \mathbb{Z} \cap (-\infty, 0]\). Then for any \(f \in C_0^+ (\mathbb{R})\), \(\lim_{n \to \infty} L_{\xi_n} [f \mid y] = 1_{[0, \infty)} (y + \tau_f)\) with \(\tau_f \triangleq \inf \{x : f (x) > 0\}\).

Let \(g : \mathbb{R} \to \mathbb{R}\) be a distribution function and let \(Z\) be a random variable whose distribution function is \(g\), independent of \(\xi_n\). Defining \(\xi''_n = \theta_2 \xi_n\) yields
\[
\lim_{n \to \infty} L_{\xi''_n} [f \mid y] = \mathbb{E} \left\{ 1_{[0, \infty)} (y - Z + \tau_f) \right\} = \mathbb{P} \{ Z \leq y + \tau_f \} = g (y + \tau_f).
\]

Defining \(\xi''_n = A \xi''_n\) where \(A\) is a random variable independent of \(\xi''_n\) that is equal to 0 with probability \(p\) and is equal to 1 with probability \(1 - p\), we obtain a sequence such that \(\lim_{n \to \infty} L_{\xi''_n} [f \mid y] \approx g_p (y) = p + (1 - p) g (y)\). Similarly, by reflecting the point processes around zero we obtain a sequence of point processes such that \(\lim_{n \to \infty} L_{\xi''_n} [f \mid y] \approx -g_p (y)\).

In particular, for any function \(g\) that satisfy the conditions stated in Lemma 12, we can construct a sequence of point processes \(\xi''_n\) which is not tight and such that \(\lim_{n \to \infty} L_{\xi''_n} [f \mid y] \approx g (y)\) for any \(f \in C_0^+ (\mathbb{R})\).

In light of the above, we assume in the following convergence of the sequence of processes and not just the convergence of the Laplace functionals.

**Lemma 38.** Let \(\xi_n, n \geq 1\), be a sequence of point processes such that \(\xi_n \xrightarrow{d} \xi\) where \(L_\xi [f \mid \cdot] \) is uniquely supported on \([g]\). Then any nonnegative, continuous function \(f \neq 0\) such that
\[
(8.1) \quad \forall y \in \mathbb{R}, \quad \lim_{T \to \infty} \limsup_{n \to \infty} \mathbb{P} \left\{ \int_{-T}^T \theta_y f \cdot 1_{(-T, T]^c} d\xi_n > \epsilon \right\} = 0,
\]
satisfies
\[
\lim_{n \to \infty} L_{\xi_n} [f \mid \cdot] = L_\xi [f \mid \cdot] \approx g (\cdot)
\]

**Proof.** For any \(T > 0\) let \(h_T : \mathbb{R} \to [0, 1]\) be the continuous function that is equal to 1 on \([-T, T]\), is equal to 0 on \([- (T + 1), T + 1]^c\), and is linear in each of the two remaining intervals. Denote \(h_T^c \triangleq 1 - h_T\). Let \(f \neq 0\) be a nonnegative, continuous function such that (8.1) holds and let \(\epsilon > 0\).
First, note that for any $T > 0$ and $y \in \mathbb{R}$, abbreviating $f_y \triangleq \theta_y f$,

$$
\mathbb{P} \left\{ \int f_y \cdot h_T^c d\xi > \epsilon \right\} = \lim_{R \to \infty} \mathbb{P} \left\{ \int f_y \cdot (h_T^c - h_R^c) d\xi > \epsilon \right\} 
\leq \lim_{R \to \infty} \liminf_{n \to \infty} \mathbb{P} \left\{ \int f_y \cdot (h_T^c - h_R^c) d\xi_n > \epsilon \right\}
\leq \liminf_{n \to \infty} \mathbb{P} \left\{ \int f_y \cdot h_T^c d\xi_n > \epsilon \right\},
$$

(8.2)

where the first inequality follows from the convergence in distribution of $\int f_y \cdot (h_T^c - h_R^c) d\xi_n \to \int f_y \cdot (h_T^c - h_R^c) d\xi$ and the portmanteau theorem. Hence, since

$$
\mathbb{P} \left\{ \int f_y d\xi = \infty \right\} = \mathbb{P} \left\{ \int f_y \cdot h_T^c d\xi = \infty \right\},
$$

by (8.1), $\int f_y d\xi < \infty$ almost surely.

Let $T > 0$. Embed $\xi, \xi_n, n \geq 1$, in the same probability space such that $\int f_y \cdot h_T d\xi_n \to \int f_y \cdot h_T d\xi$ in probability (for example, by using Skorohod coupling, cf. Corollary 6.12 of [36]). Write

$$
\limsup_{n \to \infty} \mathbb{P} \left\{ \left| \int f_y d\xi_n - \int f_y d\xi \right| > 3\epsilon \right\} \leq \mathbb{P} \left\{ \int f_y \cdot h_T^c d\xi > \epsilon \right\}
+ \limsup_{n \to \infty} \mathbb{P} \left\{ \int f_y \cdot h_T^c d\xi_n > \epsilon \right\} + \limsup_{n \to \infty} \mathbb{P} \left\{ \left| \int f_y \cdot h_T d\xi_n - \int f_y \cdot h_T d\xi \right| > \epsilon \right\}
$$

and note that as $T \to \infty$, by (5.2) and (5.4), the first and second summands tend to 0, and, from convergence in probability, so does the third summand.

Therefore $\int f_y d\xi_n \to \int f_y d\xi$ in distribution and thus $\lim_{n \to \infty} L_n [ f | \cdot ] = L [ f | \cdot ]$.

Note that if $\xi = 0$ then, obviously, $L_\xi [ f | \cdot ] = g ( \cdot ) = 1$. Assume henceforth that $\xi \neq 0$. By the monotone convergence theorem, for any $y \in \mathbb{R}$,

$$
(8.3) \quad L_\xi [ f | y ] = \lim_{T \to \infty} \mathbb{E} \left\{ \exp \left( - \langle \theta_y ( f \cdot h_T ) , \xi \rangle \right) \right\} = \lim_{T \to \infty} g ( y - \tau_T ),
$$

where the second equality follows, with appropriate $\tau_T$, from the fact that $f \cdot h_T \in C^+_T ( \mathbb{R} )$ for large $T$ and since $L_\xi [ f | \cdot ]$ is uniquely supported.

By Lemma 12 and the fact that $\int f_y d\xi < \infty$ almost surely,

$$
1 > L_\xi [ f | h_T | y ] \geq L_\xi [ f | y ] = \mathbb{E} \left\{ \exp \left( - \langle \theta_y f , \xi \rangle \right) \right\}
> \mathbb{E} \left\{ \exp \left( - \infty \cdot 1_{\{ \xi (R) > 0 \}} \right) \right\} = \inf_{y \in \mathbb{R}} g ( y )
$$

and thus, by Corollary 13 $L_\xi [ f | y ] = g ( y - \tau )$, with $\lim_{T \to \infty} \tau_T = \tau \in \mathbb{R}$, for all $y \in \mathbb{R}$, which completes the proof. \hfill \Box

9. Appendix I: Gumbel distributions of different scales

This short appendix is devoted to a proposition which relates Gumbel distribution functions of different scales. We suspect that it must be known to experts but we have not been able to locate neither such a statement in the literature nor a direct, analytic proof. The proposition follows from the observation made in the proof of Lemma 32 by composing an SDPPP with a Poisson process of exponential density one obtains a process which has two substantially different representation as an SDPPP.
Proposition 39. For any $c_2 > c_1 > 0$,

\[ \text{Gum} (c_1 (y)) = \int \text{Gum} (c_2 (y - z)) \, d\mu (z), \]

where $\mu = \mu_{c_1, c_2}$ is the law of $S - \tau$ for some $\tau \in \mathbb{R}$, with $S \triangleq \frac{1}{c_2} \log (\langle e^{c_2 x}, \xi^c_1 \rangle)$ and $\xi^c_1 \sim \text{DPPP} (e^{-c_2 x} \, dx, \delta_0)$.

Proof. For any $c > 0$ let $\xi^c \sim \text{DPPP} (e^{-c_2 x} \, dx, \delta_0)$. Note that

\[ \mathbb{E} \left\{ \langle e^{c_2 x}, \xi^c_1 \rangle \big| (-\infty, 0) \right\} = \sum_{n=0}^{\infty} \mathbb{E} \left\{ \langle e^{c_2 x}, \xi^c_1 \rangle \big| -(n+1), -n \right\} \leq \frac{1}{c_1} \sum_{n=0}^{\infty} e^{-c_2 n} \left( e^{c_1 (n+1)} - e^{c_1 n} \right) < \infty. \]

Thus, since $\xi^c_1 ([0, \infty)) < \infty$ a.s., $\langle e^{c_2 x}, \xi^c_1 \rangle < \infty$ a.s.

Let

\[ \eta \sim \text{SDPPP} (e^{-c_1} \, dx, \xi^c_2, 0), \]

By the argument leading to (7.4) in the proof of Lemma 32, we also have

\[ \eta \sim \text{SDPPP} (e^{-c_2} \, dx, \delta_0, S), \]

where $S$ is defined in the statement of the proposition.

From the converse part of Theorem 9, $\eta$ satisfies (SUS) with both the functions (of $y$) appearing in the two sides of (9.1). The converse part also gives $\tau$ explicitly, using (2.5). This completes the proof. \qed

10. Appendix II: point processes

Denote by $\mathcal{N}$ the space of positive, locally finite, counting measures on $\mathbb{R}$. That is, Borel measures $\mu$ such that $\mu (B) \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ for any bounded Borel set $B$. Endow $\mathcal{N}$ with the vague topology (cf. Appendix 7 of [35]) and let $\mathcal{A}$ be the corresponding Borel $\sigma$-algebra on $\mathcal{N}$. Note that with this choice of topology $\mathcal{N}$ is Polish (cf. A 7.7 of [35]) and therefore $(\mathcal{N}, \mathcal{A})$ is a Borel space (a fact we shall need for some coupling arguments in Section 7). Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a point process (on $\mathbb{R}$) is a measurable mapping $(\Omega, \mathcal{F}) \to (\mathcal{N}, \mathcal{A})$. (Since all the point processes we consider are on $\mathbb{R}$, henceforth we shall not state the space on which they are defined)

The Laplace functional of a point process $\xi$ is the mapping $M^+ (\mathbb{R}) \to [0, 1]$ given by

\[ L_{\xi} [f] \triangleq \mathbb{E} \{ \exp (-\langle f, \xi \rangle) \}, \quad f \in M^+ (\mathbb{R}). \]

When discussing convergence of a sequence of point processes $\xi_n, n \geq 1$, we shall consider convergence in distribution with respect to the vague topology which we denote by $\Rightarrow$. As Theorem 4.2 of [35] states, convergence in distribution of the point processes $\xi_n \Rightarrow \xi$ is equivalent to convergence in distribution of the random variables $\langle f, \xi_n \rangle \Rightarrow \langle f, \xi \rangle$, for any $f \in C^+_c (\mathbb{R})$, and to convergence of the Laplace functionals $L_{\xi_n} [f] \Rightarrow L_{\xi} [f]$, for any $f \in C^+_c (\mathbb{R})$.

According to Lemma 4.5 of [35], relative compactness of a sequence $\xi_n, n \geq 1$, with respect to convergence in distribution in the vague topology is equivalent to tightness of the sequence which is equivalent to

\[ \lim_{t \to \infty} \limsup_{n \to \infty} \mathbb{P} \{ \xi_n (B) > t \} = 0, \quad \text{for any bounded Borel set } B. \]
In particular, it follows that if (10.1) holds and \( L_{\xi_n} [f] \to \varphi[f] \), for any \( f \in C^+_c(\mathbb{R}) \), for some function \( \varphi : C^+_c(\mathbb{R}) \to \mathbb{R} \), then there exists a point process \( \xi \) such that \( \varphi[f] = L_{\xi} [f] \) on \( C^+_c(\mathbb{R}) \) and \( \xi_n \xrightarrow{d} \xi \).

Acknowledgments. The authors would like to thank Pascal Maillard for many helpful discussions.

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