EXOTIC SMOOTH STRUCTURES AND SYMPLECTIC FORMS ON CLOSED MANIFOLDS

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Abstract. In this paper we discuss relations between symplectic forms and smooth structures on closed manifolds. Our main motivation is the problem if there exist symplectic structures on exotic tori. This is a symplectic generalization of a problem posed by Benson and Gordon. We give a short proof of the (known) positive answer to the original question of Benson and Gordon that there are no Kähler structures on exotic tori. We survey also other related results which give an evidence for the conjecture that there are no symplectic structures on exotic tori.

1. Introduction

One of the most fundamental problems in symplectic topology is the problem of existence of symplectic structures on closed manifolds. This problem can be understood as follows. Consider an almost complex 2n-manifold $(M, J)$ with the "cohomological candidate for the symplectic form", i.e. a class $a \in H^2(M; \mathbb{R})$ such that $a^n \neq 0$. Then we can ask whether $M$ carries a symplectic structure $\omega$ such that $[\omega] = a$ and $\omega$ is compatible with an almost complex structure homotopic to $J$? This fundamental question seems to be very difficult to answer in full generality. Moreover, the work of many authors [CFG, Cv, FeGG, FeS, FeM, IRTU, McD, RT, TO] gives a strong evidence that there are no homotopic properties specific for symplectic manifolds, except the obvious one given above. Various observations of particular classes of symplectic manifolds are manifested in the Thurston conjecture: for any graded commutative finite-dimensional algebra $H = \bigoplus_{i=0}^{2n} H^i$ satisfying the Poincaré duality and possessing an element $a \in H^2$ such that $a^n \neq 0$, there exists a closed symplectic manifold $(M, \omega)$ such that $H^*(M; \mathbb{R}) \cong H$ (see [Th]). In the sequel we will call such $H$ a cohomologically symplectic or $c$-symplectic algebra. If $M$ is a closed (not necessarily symplectic) 2n-manifold with $a \in H^2(M; \mathbb{R})$
such that $a^n \neq 0$, it is called a $c$-symplectic manifold. It is clear that to gain some understanding of the Thurston conjecture one needs at least examples of closed $c$-symplectic but non-symplectic manifolds.

In the four-dimensional case examples of almost complex cohomologically symplectic but non-symplectic manifolds are given by the Seiberg-Witten theory [T, FS, WG]. For example, it has become a classical result now that manifolds $k\mathbb{C}P^2 \# l\overline{\mathbb{C}P^2}$, $k > 1$ carry no symplectic structures compatible with the orientation given by the complex structure, although some of them are almost complex and they have the cohomology type of a closed symplectic manifolds. This follows, since their Seiberg-Witten invariant (which is a smooth invariant) vanishes, which contradicts the symplecticness by the Taubes theorem. There are several other results in this direction obtained by methods of the Seiberg-Witten theory (see, for example, [FS, WG]). On the other hand, there are no known results of this type in higher dimensions.

In view of [Cv, FeM, IRTU, FeS, TO] on homotopy of symplectic manifolds and of the crucial developments on smooth invariants in the four-dimensional case one can ask how the existence of symplectic structure does depend on smooth structure. Hence, the following problem seems to be very interesting.

**Problem 1.1.** It is known that for any $m \geq 5$, there are smooth manifolds $\mathbb{T}^m$ which are homeomorphic but not diffeomorphic to the standard torus $\mathbb{T}^m$. Does an exotic torus $\mathbb{T}^{2n}$, $n > 2$, carry symplectic structures?

This is a generalization of a similar problem, posed by Benson and Gordon in [BG] for Kähler manifolds. In the present paper we will give a short proof of the fact that there are no Kähler structures on exotic tori. When this paper was written we have learned that this fact was known to algebraic geometers [BC, CI].

In fact, Baues, Cortez and Catanese in these papers developed a theory of aspherical Kähler manifolds with solvable fundamental group. This theory has an application to another problem of Benson and Gordon [BG1]. A solvmanifold is a homogeneous space $G/H$ with solvable Lie group $G$. In the sequel we always assume that $H$ is a discrete co-compact subgroup (denoted by $\Gamma$). We use the notation $L(G)$ for the Lie algebra of $G$. A solvmanifold $G/\Gamma$ is called completely solvable, if for any $x \in L(G)$ the linear operator $\text{ad} \; x : L(G) \rightarrow L(G)$ has only real eigenvalues.

**Problem 1.2.** Is every completely solvable solvmanifold $G/\Gamma$ which carries a Kähler structure diffeomorphic to a torus?
As a result of efforts of \( \mathbf{A, AN, Ha, Ha1, TK} \) together with developments from algebraic geometry (see \( \mathbf{BC, CI} \)) the final (affirmative) solution of this problem was recently achieved (See Section 3). To the authors knowledge, the final clean proof was given in \( \mathbf{Ha, Ha1} \). In \( \mathbf{C, C1, ABCKT} \) some properties of the Albanese map of Kähler \( K(\pi, 1) \)-manifolds with solvable \( \pi \) were established. These properties allow one to show that no Kähler structures can exist on exotic tori, hence to answer the Benson-Gordon question in the negative.

Since we are interested in the symplectic version of the problem, our approach goes in other direction, where explicit constructions of exotic differential structures play a role. Thus this paper can be considered as complementary to \( \mathbf{BC, CI} \), with the intersection exactly in the original Benson-Gordon problem.

In \( \mathbf{HT} \) we obtained a result which gives a partial negative answer to the symplectic question. Sections 5 and 6 contain an exposition of it together with other related problems. This provides some evidence for the following conjecture.

**Conjecture 1.3.** There are no symplectic structures on exotic tori.

The purpose of this article is to survey results around this conjecture as well as to discuss some relations between smooth structures and symplectic forms on closed manifolds.

### 2. Cohomologically symplectic manifolds

A lot of work has been done by many mathematicians with the aim of better understanding the homotopic properties of closed symplectic manifolds. These results were initiated by Thurston’s discovery that non-Kählerness of symplectic structures can be detected by homotopic invariants \( \mathbf{BT, CFG, FeGG, FeS, G1, McD, RT, Th, TO} \). It is understood now that all known homotopic properties of Kähler manifolds may be violated by closed symplectic manifolds. For example, it is known that Kähler manifolds

1. have even odd-dimensional Betti numbers;
2. satisfy the hard Lefschetz property;
3. have vanishing Massey products;
4. are formal.

For a thorough discussion of the homotopic properties (1)-(4) and their role in Kähler and symplectic theory we refer to the monograph \( \mathbf{TO} \). Examples of symplectic manifolds violating (1)-(4) are constructed, for instance in \( \mathbf{Th, G1, McD, FeM, FeS, BT, RT} \). Moreover, it was asked what are the relations between (1)-(3) in the following sense:
Problem 2.1. Can any combination of properties (1)-(3) or their negatives be realized by a closed symplectic manifold?

A rather detailed answer is contained in [IRTU, FeM, Cv]. Results from these papers are summarized in the following statement.

Theorem 2.2. For any combination of properties (1)-(3) with the following exceptions:

1. Massey products are trivial, the hard Lefschetz property holds and some odd-dimensional Betti number is odd;
2. Massey products are non-trivial, the hard Lefschetz property holds and some odd-dimensional Betti number is odd,

there exists a closed symplectic manifold \((M, \omega)\) possessing this combination of properties.

Note that (1) and (2) are mentioned only for completness, since the hard Lefschetz property implies the evenness of the odd-dimensional Betti numbers.

The described results provide an evidence for the Thurston conjecture. On the other hand there do exist relations between symplectic structure and smooth structure on a closed manifold. This relation is indicated by results of Taubes [T].

Theorem 2.3. Let \(X\) be a compact, oriented, 4-dimensional manifold with \(b^+_2 \geq 2\). Let \(\omega\) be a symplectic form on \(X\) with \(\omega \wedge \omega\) giving the orientation. Then the first Chern class of the associated almost complex structure on \(X\) has the Seiberg-Witten invariant equal to \(\pm 1\).

This implies that connected sums of 4-manifolds with non negative-definite intersection forms do not admit symplectic forms which are compatible with the given orientation. The latter fact follows, since Taubes also proves that if \(X\) has \(b^+_2 \geq 2\) and can be split by an embedded 3-sphere into \(X_1 \# X_2\) where neither \(X_1\) nor \(X_2\) have negative definite intersection forms, then the Seiberg-Witten invariants of \(X\) vanish. In particular, we have the following corollary.

Corollary 2.4. Connected sums \(k\mathbb{C}P^2 \# l\overline{\mathbb{C}P^2}\) when \(k > 1\) do not admit symplectic forms which define the orientation given by the almost complex structure.

The Ehresmann-Wu theorem which characterizes 4-manifolds admitting almost complex structures [Be] easily implies that \(k\mathbb{C}P^2 \# l\overline{\mathbb{C}P^2}\) is almost complex if and only if \(k\) is odd.

Another kind of examples can be obtained by the Seiberg-Witten theory as follows.
**Theorem 2.5.** ([WG]) Let $\tilde{X}$ be a Kähler 4-manifold with positive basic class $K_{\tilde{X}} > 0$ and $b_2^+(\tilde{X}) > 3$. Suppose that $\sigma : \tilde{X} \to \tilde{X}$ is an antiholomorphic involution without fixed points. Then for $X = \tilde{X}/\sigma$, $SW(X) = 0$, and, hence, $X$ cannot carry symplectic structures.

Here is an explicit example.

**Example 2.6.** Let $\tilde{X}$ denote the hypersurface

$$\tilde{X} = \{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 \mid \sum_{i=0}^{3} z_i^{2d} = 0, d > 2 \} \subset \mathbb{C}P^3.$$  

Let $\mathbb{Z}^2$ act on $\tilde{X}$ by complex conjugation. By Theorem 2.3, $X = \tilde{X}/\mathbb{Z}^2$ has no symplectic structure.

Seiberg-Witten invariants yield also more sophisticated examples, for instance, in the simply-connected case. This is the knot construction due to Fintushel and Stern (see [FS] and [CS]). The relation between symplectic structures and Seiberg-Witten invariants is delicate, since there exist closed almost complex manifolds with Seiberg-Witten invariants satisfying the Taubes condition and having no symplectic structures [KMT].

In contrast to dimension 4, the following question is open.

**Problem 2.7.** Do there exist closed c-symplectic non-symplectic (almost complex) manifolds in dimensions $\geq 6$?

In fact, up to now, no general tools have been worked out. A rich algebraic theory being a source of potential examples was created by Lupton and Oprea in [LO]. However, although the examples available from [LO] are c-symplectic, there is no way in sight neither to prove the existence of symplectic structures nor to disprove this. To give the reader some taste of what is known in higher dimensions we recall here an example from [LO].

**Example 2.8.** Let $K = \mathbb{C}P^2 \times V$, where $V$ is a hypersurface in $\mathbb{C}P^4$ defined by a single equation of degree 3. The rational cohomology algebra of $K$ can be easily calculated, since the cohomology of hypersurfaces in $\mathbb{C}P^n$ is known. In the case considered here we have

$$H^*(K; \mathbb{Q}) \cong \mathbb{Q}[\omega]/(\omega^3) \otimes \mathbb{Q}[x, a_1, ..., a_5, a_1^*, ..., a_5^*]/\mathcal{R},$$

with the ideal $\mathcal{R}$ generated by

$$\{ xa_j, xa_j^* \}_{j=1,...,5}, \{ a_ja_k, a_j^*a_k^* \}_{1 \leq j < k \leq 5}, \{ a_ja_k^* \}_{j \neq k}$$
and

\[ x^3 - \left( \sum_{j=1}^{5} a_j a_j^* \right). \]

It is shown that there exists a non-formal space \( X \) such that \( H^*(X) \cong H^*(K) \). Note that \( K \) is Kähler, and it is well known that Kähler manifolds are formal. Using rational surgery \[\text{Ba}\], Lupton and Oprea show that there exists a simply connected 10-dimensional manifold with the same minimal model, and cohomology as \( X \) (and \( K \)) (see \[\text{TO}\] for the terminology). Finally, we have arrived at the conclusion: there exists a c-symplectic simply connected 10-dimensional smooth manifold which has the cohomology algebra of a Kähler manifold and homotopy type different from that of Kähler manifolds.

3. Albanese map and a solution of the Benson-Gordon problem

The theorem below yields a solution to the original problem \[\text{BG}\]. Although it is an evidence to our conjecture, since Kähler forms are symplectic, the proof is "non-symplectic" and probably cannot be generalized.

**Theorem 3.1.** There are no Kähler structures on exotic tori.

The proof of this result uses some properties of the Albanese map, which we recall now. Let \( X \) be a compact Kähler manifold. By definition (see \[\text{BPV}\]), the Albanese variety of \( X \) is the complex torus

\[ \text{Alb}(X) = \frac{H^0(X, \Omega^1_X)}{\text{im}(j(H_1(X, \mathbb{Z})))}, \]

where \( j \) is the homomorphism

\[ j : H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1_X)^*, \]

\[ j([\gamma]) = (\omega \to \int_\gamma \omega). \]

Here \( \Omega^1_X \) denotes the sheaf of germs of holomorphic 1-forms on \( X \). It is known (see \[\text{ABCKT}, \text{BPV}\]), that \( j(H_1(X, \mathbb{Z})) = H_1(X, \mathbb{Z})/\text{Torsion} \) is a lattice of rank \( b_1(X) = 2h^0(X, \Omega^1_X) \), and thus \( \text{Alb}(X) \) is a complex torus. Fixing a basepoint \( x_0 \in X \), one defines the Albanese map by

\[ \alpha_X : X \to \text{Alb}(X), \quad \alpha_X(x) = (\omega \to \int_{x_0}^x \omega). \]

The following properties of the Albanese map are known (\[\text{ABCKT}, \text{BPV}\]):
\alpha_X is a holomorphic map from X to the complex torus \( T^{b_1(X)} = Alb(X) \), of complex dimension \( \frac{1}{2}b_1(X) \);

\alpha_X induces a surjection \((\alpha_X)_* : \pi_1(X) \to \pi_1(Alb(X))\);

For any topological space X of finite type define \( a(X) \) to be the maximal integer for which the image of \( \Lambda^m H^1(X, \mathbb{R}) \) in \( H^m(X, \mathbb{R}) \) is non-trivial. The following result can be found in [ABCKT, C].

**Theorem 3.2.** Let X be a compact Kähler manifold. Then \( a(X) \) is the real dimension of its Albanese image:

\[ a(X) = \dim \alpha_X(X). \]

Having the above facts in mind we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Let X denote a smooth manifold homeomorphic to \( \mathbb{T}^{2n} \), the torus of dimension 2n. Clearly, \( a(X) = 2n \), since \( \Lambda^{2n} H^1(X, \mathbb{R}) \cong H^{2n}(X, \mathbb{R}) \) is spanned by \( x_1 \wedge \ldots \wedge x_{2n} \neq 0 \), where \( x_i \) denote the 1-dimensional generators of \( H^1(X, \mathbb{R}) \).

Assume that X carries a Kähler structure. Then we have the Albanese map \( \alpha_X : X \to \mathbb{T}^{2n} \).

Consider a regular point \( p \) in the image of \( \alpha_X(X) \). Since holomorphic map preserves orientations, its degree is equal to the cardinality of \( \alpha_X^{-1}(p) \). As \( \dim \alpha_X(X) = 2n = \dim Alb(X) \), it implies that \( \deg \alpha_X > 0 \) and \( \alpha_X \) is onto. Moreover, \( \alpha_X \) induces a surjection of free abelian groups of rank 2n,

\[ (\alpha_X)_* : \pi_1(X) \to \pi_1(\mathbb{T}^{2n}), \]

hence an isomorphism. This gives also that \( \alpha_X \) induces an isomorphism \( H^1(\mathbb{T}^{2n}; \mathbb{Z}) \to H^1(X; \mathbb{Z}) \) and consequently an isomorphism \( H^{2n}(\mathbb{T}^{2n}; \mathbb{Z}) \to H^{2n}(X; \mathbb{Z}) \).

Thus \( \alpha_X \) is a map of degree one.

Since the set of regular points of a holomorphic onto map is dense, we have that \( \alpha_X \) is bijective on a dense subset, thus everywhere. It is a standard fact that a holomorphic homeomorphism is biholomorphic (cf. [FG], Th. 8.5), thus \( \alpha \) is a diffeomorphism. The proof is complete.

**Remark 3.3.** As we have mentioned in the introduction, Baues and Cortéz [BC] gave an account on recent developments concerning the classification of compact aspherical Kähler manifolds whose fundamental groups contain a solvable subgroup of finite index. These developments lead to solutions of the both Benson-Gordon problems. In fact, more general results hold.
Theorem 3.4. Let $M$ be an infra-solvmanifold which admits a Kähler metric. Then $M$ is diffeomorphic to a flat Riemannian manifold.

Theorem 3.5. Let $X$ be a compact Kähler manifold of complex dimension $n$. Assume that $X$ satisfies $2 \dim X = \dim H^1(X; \mathbb{C})$. If $\dim H^2(X; \mathbb{C}) \leq \binom{n}{2}$, and $H^{2n}(X; \mathbb{Z})$ is generated by integral classes of degree 1, then the Albanese morphism $\alpha_X : X \to \text{Alb}(X)$ is a biholomorphic map.

The proof of the last theorem is the same as Theorem 3.1, which follows then as a corollary. The same result follows from [C1]. Proposition 4.8 in that paper implies Theorem 3.1, with the proof going along the same lines, although the solution of the Benson and Gordon problem is apparently a byproduct, and is not mentioned.

4. Examples of exotic structures on tori

Various examples of different smooth structures within the homotopy type of manifolds can be found in [HS], [HW], [W]. However, our aim is to present explicit constructions which enable to formulate our problems in the language of groups $\text{Diff}(M)$ and $\text{Symp}(M, \omega)$. The simplest examples of exotic tori are obtained as connected sums of the standard tori with homotopy spheres. Namely, consider a homotopy sphere $\Sigma$ of dimension $k$ and the manifold $T^k \# \Sigma = T^k \# \Sigma \times T^{2n-k}$.

The connected sum operation can be understood as follows. Let $\Sigma = D^k \cup_f D^k$, when $f \in \text{Diff}(S^{k-1}, S^{k-1})$, which means that we glue two copies of the disk using a diffeomorphism of the boundary sphere supported in the upper half-sphere. Then $T^k \# \Sigma$ is obtained by cutting $T^k$ along an embedded $(k-1)$-disk and gluing it again along the disk using $f$. Up to an orientation choice (which replaces $\Sigma$ by $-\Sigma$), this does not depend on the choice of the disk, thus we can choose it in a subtorus $T^{k-1} \subset T^k$. This is equivalent to cutting $T^k$ along the subtorus and gluing again with a diffeomorphism $f \in \text{Diff}(T^{k-1})$ extending $f$ by the identity. Thus what we get the mapping torus of $f$, i.e., the fibration over the circle with fiber $T^{k-1}$ and gluing diffeomorphism $f$, (cf. [H]).

For $T = (T^k \# \Sigma) \times T^{2n-k}$ we get a fibration over $T^{2n-k+1}$ and fiber $T^{k-1}$. If $k$ is odd, there exist symplectic structures on base and fiber, so we can ask if there exist such symplectic structures which induce a symplectic structure on $T$. Since the fibration is homotopically (and even topologically) trivial, a theorem of Thurston [MS] says that this is the case if $f$ is isotopic to a symplectomorphism (with respect to some
symplectic structure on the fiber). The main purpose of [HT] was to show that, in general, there is an obstruction to such isotopy.

In general, exotic differential structures on tori are obtained by iterating the operation described above, but then we have to apply the cutting and pasting along possibly exotic subtori. It follows from [W], “Fake tori” chapter, that the examples above give in fact all nonstandard differential structures on tori. However, it is not easy to distinguish diffeomorphism type of two manifolds obtained by the construction above. The classification resulting from surgery or from smoothing theory identifies smooth structures up to diffeomorphisms isotopic topologically to the identity. Here we need to know when two structures are diffeomorphic, thus we have to see what is the action of $\pi_0 \text{Diff}(\mathbb{T}^k)$ on the set of smoothings. Obviously, diffeomorphisms of the torus act nontrivially on homology, hence regluing along two subtori of the same dimension and using the same homotopy sphere yields diffeomorphic structures. Simply one can exchange the subtori by a diffeomorphism and it gives a diffeomorphism of resulting manifolds. However, $\pi_0 \text{Diff}(\mathbb{T}^k)$ is unknown for $k > 4$, which makes further analysis difficult.

The simplest case, but still nontrivial, is that of the connected sum. It follows from [K] that $\mathbb{T} \# \Sigma_1 \cong \mathbb{T} \# \Sigma_2$ if and only if $\Sigma_1 \cong \Sigma_2$.

If we need only some examples of non diffeomorphic exotic tori, one can use the Atiyah - Milnor - Singer invariant $\hat{a}$ which distinguishes cobordism classes of framed manifolds. In fact we need only its $\mathbb{Z}_2$ part which is known as Hitchin invariant. The instructive example is $k = 2n - 1 = 8s + 1$. We have $\mathcal{T} = T^{8s+1} \# \Sigma \times \mathbb{T}^1$.

In dimension $8s + 1$ and $8s + 2$ half of homotopy spheres have nontrivial generalized $\hat{a}$-genus [H]. The $\hat{a}$-genus can be defined, for any closed spin manifold $M^m$, as the $KO$-theoretical index of the Dirac operator with values in $KO^{-m}(pt)$. It is known that the coefficient groups $KO^{-*}(pt)$ are the following

$$KO^{-m}(pt) = \begin{cases} \mathbb{Z} & \text{for } m \equiv 0 \text{ (mod 4)}; \\ \mathbb{Z}_2 & \text{if } m \equiv 1, 2 \text{ (mod 8)}; \\ 0 & \text{for any other } m. \end{cases}$$

Let $f : M^m \to \{pt\}$ denote the obvious collapsing map. For a spin structure on $M^m$ we have the Gysin map $f_! : KO^0(M^m) \to KO^{-m}(pt)$. By definition, the $\hat{a}$-genus of $M^m$ is an element of $KO^{-m}(pt)$ given by the formula

$$\hat{a}(M) = f_!(1).$$

The $\hat{a}$-genus has the following properties (see [LM]):
for any closed spin manifolds $X$ and $Y$

\[ \hat{a}(X \# Y) = \hat{a}(X) + \hat{a}(Y), \quad \text{(when } \dim X = \dim Y) \]

and

\[ \hat{a}(X \times Y) = \hat{a}(X)\hat{a}(Y), \]

(2) $\hat{a}$ is a spin cobordism invariant,

(3) for any $m > 2$ and any spin structure on the standard torus $\mathbb{T}^m$ we have $\hat{a}(\mathbb{T}^m) = 0$.

Note that (3) follows from (1) and (2) since any spin structure on $\mathbb{T}^m$ is given as a product of spin structures on circles. One of the two possible structures on $S^1$ has nonzero $\hat{a}$-genus, but the third power of the nonzero element of $KO^{-1}(pt)$ vanishes.

**Proposition 4.1.** The manifold $\mathcal{T} = (\mathbb{T}^{8s+1} \# \Sigma) \times S^1$ is homeomorphic, but not diffeomorphic to the standard torus $\mathbb{T}^{8s+2}$ if $\hat{a}(\Sigma) \neq 0$. Moreover, the $\hat{a}$-genus of $\mathcal{T}$ does depend on the choice of the spin structure.

By properties (1) and (2), if we choose the trivial spin structure on $\mathbb{T}^{8s+1}$ and nontrivial one on $S^1$, one has $\hat{a}(\mathcal{T}) \neq 0$, while it is always zero for the standard torus. This argument works also for $k = 8k+1$ or $8k+2$ and $2n - k > 1$. One may consider the map $A : H^1(\mathcal{T}; \mathbb{Z}^{2n-k}) \to \mathbb{Z}_2$ given as follows. Let $\phi : \mathcal{T} \to \mathbb{T}^{2n-k}$ correspond to $x \in H^1(\mathcal{T}; \mathbb{Z}^{2n-k})$ (note that $\mathbb{T}^r$ is the Eilenberg-MacLane space $K(\mathbb{Z}^r, 1)$). Let $A(x) = \hat{a}(\phi^{-1}(p))$, where $p$ is a regular value of $\phi$. In our examples there exists a spin structure such that $A$ is nontrivial, but it is trivial for the standard torus and $k > 2$.

Finally, it is known that all homotopy tori are parallelizable (see [W] or [HT]). Thus any such manifold admits an almost complex structure and it is cohomologically symplectic. Therefore, an exotic torus with no symplectic structure would be an example of a non-symplectic manifold satisfying the two ”obvious” conditions necessary for symplecticness.

It is remarkable that our assumption on dimension plays a role in the conjecture about the non-existence of symplectic structures. In dimension 4 the situation is different [P1, P2, P3]. Using the Gompf symplectic sum construction [G1, MW], it is possible to construct families of closed 4-manifolds which are homeomorphic to $(2m+1)\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ but not mutually diffeomorphic. It follows that $(2m+1)\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ admits infinitely many symplectic structures which are all ”exotic”, since this manifold does not admit symplectic structures coming from
the standard smooth structure, by the Taubes theorem. Let us give an exact formulation of the main result in [PT].

**Theorem 4.2.** For each positive integer $m$ and $n$ satisfying $2m + 8 \leq n \leq 10m + 9$ there exists a family of simply connected, closed, nonspin, irreducible symplectic 4-manifolds

$$\{ X_{2m+1,n}(p) \mid p \text{ is a positive integer} \}$$

which are all homomorphic to $(2m+1)CP^2 \# n\overline{CP}^2$, but not mutually diffeomorphic.

5. **Diffeomorphisms of tori and non-existence of fiberwise symplectic structures on exotic tori**

In this section we will describe the results of [HT]. We consider the following question:

**Problem 5.1.** Is there a symplectic structure on $T = (T^{8s+1} \# \Sigma) \times S^1$ compatible with the fiber bundle structure

$$T^{8s} \to T = (T^{8s+1} \# \Sigma) \times S^1 \to \mathbb{T}^2?$$

The necessary and sufficient conditions for a fibration over a symplectic manifold with a symplectic fiber to have a symplectic structure such that each fiber is symplectic was given by Thurston [MS]: one has to know that the fibration is symplectic and a cohomological condition should be satisfied. Clearly, the cohomology ring $H^*(T)$ is isomorphic to $H^*(\mathbb{T}^{2n}) \otimes H^*(\mathbb{T}^2)$. This easily implies that the cohomological condition of Thurston’s construction is satisfied. The symplecteness of the fibration is that the gluing diffeomorphism $f \in \text{Diff}(T^{8s})$ must be isotopic to a symplectomorphism.

Thus we come to the following question.

**Problem 5.2.** Given a diffeomorphism $f : T^{8s} \to T^{8s}$ supported in an embedded disc but non-isotopic to the identity, is there a symplectomorphism in the isotopy class of $f$?

We have seen above that the positive answer to Problem 5.2 would imply that $T$ admits a symplectic structure compatible with the fibration. Our goal is to give negative examples to this problem in the particular case of the standard symplectic structure on $\mathbb{T}^{2n}$. Let $\pi_0(\text{Diff}_+(M))$ denote the group of isotopy classes of orientation preserving diffeomorphisms of a smooth oriented manifold $M$. Assume now that $M$ is $2n$-dimensional and admits almost complex structures,
and let $JM$ denote the set of homotopy classes of such structures, compatible with the given orientation. Any diffeomorphism $f$ acts on the set of all almost complex structures by the rule
\[ f_*J = df J df^{-1}, \]
where $df : TM \to TM$ denotes the differential of $f$. This action clearly descends to the action of $\pi_0(\text{Diff}_+ (M))$ on $JM$.

Let now $\mathfrak{G}(M)$ denote the subgroup of $\pi_0(\text{Diff} (M))$ generated by diffeomorphisms with supports in discs.

In the sequel we will show that there exist diffeomorphisms $f : \mathbb{T}^{8k} \to \mathbb{T}^{8s}$ supported in a disc, whose isotopy classes $[f] \in \mathfrak{G}(M)$ do not preserve the homotopy class $[J_0] \in JM$ of the standard complex structure. Therefore, they cannot be isotopic to symplectomorphisms with respect to the standard symplectic structure $\omega_0$. Indeed, any symplectomorphism carries any almost complex structure compatible with a symplectic form to another almost complex structure compatible with the same symplectic form, but the space of all such almost complex structures is contractible.

Now, let us give a sketch of the main line of proof that such $f$ exist. We give first a necessary homotopic condition on a diffeomorphism to be isotopic to a symplectomorphism.

**Theorem 5.3.** Let $f \in \text{Diff} (\mathbb{T}^{4n})$ be supported in a disc $D^{4n} \subset \mathbb{T}^{4n}$. If $f$ is isotopic to a symplectomorphism with respect to the standard symplectic structure, then $df$ restricted to its support disc $D^{4n}$ gives in $\pi_{4n}SO(4n)$ the trivial homotopy class.

Since the $\hat{a}$-genus is additive with respect to the operation of connected sum and it is nontrivial for some homotopy spheres in dimension $8k + 1$, one easily concludes that there exist isotopy classes of diffeomorphisms $f$ with support in a disc such that $\hat{a}(\mathbb{T}^{8k+1}_f) \neq 0$. Here $\mathbb{T}^{8k+1}_f$ denotes the mapping torus of $f$ (see Section 4). On the other hand, we prove the following result.

**Theorem 5.4.** In the notation of Theorem 5.3, if $[df] = 0$, then $\hat{a}(\mathbb{T}_f) = 0$.

Comparing Theorem 5.3 and Theorem 5.4 we get the conclusion.

**Theorem 5.5.** For any $k > 0$ there exist diffeomorphisms $f : \mathbb{T}^{8k} \to \mathbb{T}^{8k}$ with support in a disc which are not isotopic to a symplectomorphism of $(\mathbb{T}^{8k}, \omega_0)$. 

6. FURTHER QUESTIONS AND PERSPECTIVES

We have shown in the preceding section that our question on existence of symplectic structure on exotic tori is related to the problem whether a diffeomorphism of an even dimensional torus, supported in a disk and non-isotopic to the identity, can be isotopic to a symplectomorphism.

Some more questions of that type were considered in symplectic topology and give additional evidence for Conjecture 1.2. A similar rigidity question was posed by McDuff and Salamon [MS]: Is it true that any symplectomorphism of a torus which induces identity on homology is isotopic to the identity?

In [CLO] the action of $\pi_4 SO(4)$ on almost complex structures on 4-manifolds was considered. The action is defined by changing an almost complex structure in a disk by a map $(D^4, \partial D^4) \to (SO(4), id)$ representing an element of $\pi_4 SO(4)$. The main result is that if an almost complex structure is compatible with a symplectic structure, then the almost complex structure twisted by a nontrivial element such that the corresponding $\text{spin}^c$–structure changes, has no compatible symplectic structure.

Another related question is the following. Consider the space of symplectomorphisms of $\mathbb{R}^{2n}$ with compact supports. It was proven by Gromov [G] that for $n = 2$ this space is contractible. The higher dimensional case is open. From what we have said before, a weaker question seems to be natural: is any symplectomorphism with compact support in $\mathbb{R}^{2n}$ isotopic (smoothly) to the identity? Or, more generally, is the image of the space of compactly supported symplectomorphisms in the group of all compactly supported diffeomorphism contractible?

The latter question (at least for $\pi_0$) can be attacked along the following lines motivated by [MS2]. Let $f$ be such a symplectomorphism and $\omega$ be the standard symplectic structure. Then $f^* \omega = \omega$, and for an almost complex structure $J$ compatible with $\omega$ we have $f^* J$ also compatible with $\omega$. Thus there exists a path $J_t$ of almost complex structures compatible with $\omega$ connecting $J$ with $f^* \omega$. Because of the compact support assumption we can consider all this in $T^{2n}$ or in $T^{2n-2} \times S^2$. Then the path of the evaluation maps for the space of $J_t$-holomorphic curves should give an isotopy between $f$ and the identity. We plan to address this problem in a forthcoming paper.

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