Quantum aspects of antisymmetric tensor field with spontaneous Lorentz violation

Sandeep Aashish and Sukanta Panda

Department of Physics, Indian Institute of Science Education and Research, Bhopal 462066, India

(Dated: March 28, 2019)

Abstract

We study the quantization of a simple model of antisymmetric tensor field with spontaneous Lorentz violation in curved spacetime. We evaluate the 1-loop corrections at first order of metric perturbation, using a general covariant effective action approach. We revisit the issue of quantum equivalence, and find that it holds for non-Lorentz-violating modes but breaks down for Lorentz violating modes.
I. INTRODUCTION

The quest for quantizing gravity is ultimately related to understanding physics at the planck scale, candidates for which include string theory and loop quantum gravity. A difficulty that the development of such theories faces, is our inability to probe high energy scales, owing to the limitations of current particle physics experiments [1]. This has led to significant efforts towards finding low energy signatures using effective field theory tools that could be relevant in current and near future experiments in both particle physics and early universe cosmology. Phenomenologically, this amounts to detecting planck supressed variations to standard model and general relativity while maintaining observer independence, termed as standard model extension (SME) [2–5].

There is substantial evidence of SME effects from string theory and quantum gravity, according to which certain mechanisms could lead to violation of Lorentz symmetry [6–11], which is a fundamental symmetry in general relativity that relates all physical local Lorentz frames. In principle, Lorentz violation can be introduced in a theory either explicitly, in which case the Lagrange density is not Lorentz invariant, or spontaneously, so that the Lagrange density is Lorentz invariant but the physics can still display Lorentz violation [6]. However, theories with explicit Lorentz violation have been found to be problematic due to their incompatibility with Bianchi identities in Riemann geometry [4], and are therefore not favourable for studies involving gravity.

Another consequence of string theory, at low energies, is the appearance of antisymmetric tensor field along with a symmetric tensor (metric) and a dilaton (scalar field) as a result of compactification of higher dimensions [12, 13]. Until recently, antisymmetric tensor had not received serious consideration in studies of early universe cosmology, in particular inflation, due to some generic instability issues [14–16], but some recent studies have shown that presence of antisymmetric tensor field is likely to play a role during inflation era [17, 18]. Hence, as a natural extension, an interesting exercise is to consider Lorentz violation in conjunction with antisymmetric tensor.

Altschul et al. in Ref. [19] explored in detail spontaneous Lorentz violation with antisymmetric tensor fields, and found the presence of distinctive physical features with phenomenological implications for tests of Lorentz violation, even with relatively simple antisymmetric field models with a gauge invariant kinetic term.
Our interest in the present work is to take first steps to extend the classical analysis in Ref. [19] to quantum regime. We focus on the formal aspects of quantization of antisymmetric tensor field with spontaneous Lorentz violation, and primarily restrict ourselves to dealing with two issues. First, we set up the framework to evaluate the one-loop effective action using covariant effective action approach [20–24]. For simplicity, we consider an action with only quadratic order terms, but in a nearly flat spacetime (Minkowski metric $\eta_{\mu\nu}$ plus a classical perturbation $\kappa h_{\mu\nu}$). This yields one-loop corrections at $\mathcal{O}(\kappa h)$. Second, we check the quantum equivalence of the quadratic action considered in the first part with a classically equivalent vector theory, at 1-loop level. The issue of quantum equivalence in curved spacetime is interesting because a free massive antisymmetric tensor theory (no Lorentz violation) is known to be equivalent to a massive vector theory at classical and quantum level due to topological properties of zeta functions [25] but, such properties do not hold when Lorentz symmetry is spontaneously broken [26]. The method presented here is quite general in terms of its applicability to models with higher order terms in fields.

In Sec. II, we briefly review spontaneous Lorentz violation in antisymmetric tensor and introduce the classical action considered in this work, borrowing the notations of Ref. [19]. We discuss the covariant effective action technique and its application to derive 1-loop corrections in Sec. III. We also calculate the various propagators required to solve the 1-loop integrals. In Sec. IV we consider the classically equivalent vector theory and calculate 1-loop corrections to compare with the results of Sec. III to check the quantum equivalence.

II. SPONTANEOUS LORENTZ VIOLATION AND CLASSICAL ACTION

Spontaneous symmetry breaking occurs when the equations of motion obey a symmetry but the solutions do not, and is effected via fixing a preferred value of vacuum (ground state) solutions. In general relativity, physically equivalent coordinate (or observer) frames are related via general coordinate transformations and local Lorentz transformations. In a given observer frame, fixing the vacuum expectation value (vev) of a tensor or vector field leads to spontaneous breaking of Lorentz symmetry, since all couplings with vev have preferred directions in spacetime [5, 27].

Spontaneous Lorentz violation in tensor field Lagrangians can be introduced through a potential term that drives a nonzero vacuum value of tensor field. For an antisymmetric
2-tensor field $B_{\mu\nu}$, we assume,

$$\langle B_{\mu\nu} \rangle = b_{\mu\nu}. \quad (1)$$

The structure of $b_{\mu\nu}$ in any local Lorentz frame is given by \[19\],

$$b_{\mu\nu} = \begin{pmatrix}
0 & -a & 0 & 0 \\
 a & 0 & 0 & 0 \\
0 & 0 & 0 & b \\
0 & 0 & -b & 0
\end{pmatrix}, \quad (2)$$

provided at least one of the quantities $x_1 = -2(a^2 - b^2)$ and $x_2 = 4ab$ is nonzero. For later convenience, we choose $b_{\mu\nu}b^{\mu\nu} = 1$ and $a = 0$.

We consider a simple model of a rank-2 antisymmetric tensor field, $B_{\mu\nu}$, with a spontaneous Lorentz violation inducing potential \[19\],

$$V(B) = -\frac{1}{2} \lambda \left( B_{\mu\nu}B^{\mu\nu} - b_{\mu\nu}b^{\mu\nu} \right)^2. \quad (3)$$

Again, for the purpose of present analysis, we would like to consider only quadratic order terms in $B_{\mu\nu}$. To this end, we consider only the fluctuations of $B_{\mu\nu}$ about a background value $b_{\mu\nu}$,

$$B_{\mu\nu} = b_{\mu\nu} + \tilde{B}_{\mu\nu}. \quad (4)$$

and neglect quartic and cubic terms in fluctuations $\tilde{B}_{\mu\nu}$. The resulting potential is,

$$V(B) = -\frac{1}{2} \lambda \left( b_{\mu\nu}\tilde{B}^{\mu\nu} \right)^2. \quad (5)$$

Although it may seem at this point that a quadratic Lagrangian might not lead to any significant physical result upon quantization, and it is actually true in case of a flat spacetime, nontrivial physical contributions appear in the 1-loop effective action in curved spacetime as demonstrated in the next section. For notational convenience, we do not explicitly write the tilde symbol for field fluctuations, and assume its use throughout. We thus work with the Lagrangian,

$$\mathcal{L} = -\frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{4} \alpha^2 \left( b_{\mu\nu}\tilde{B}^{\mu\nu} \right)^2. \quad (6)$$

where we have defined $\alpha \equiv 8\lambda$. The first term in Eq. \[6\] is the gauge invariant kinetic term,

$$H_{\mu\nu\lambda} \equiv \nabla_\mu B_{\nu\lambda} + \nabla_\lambda B_{\mu\nu} + \nabla_\nu B_{\lambda\mu}, \quad (7)$$
obeying the symmetry: \( B^{\mu\nu} \rightarrow B^{\mu\nu} + \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu \) for a gauge parameter \( \xi_\mu \). The gauge invariance of kinetic term in an otherwise non-gauge invariant Lagrangian (6) gives rise to redundancy problems in the energy spectrum [28], and cannot be removed via usual quantization method. A consistent method to treat this redundancy is given by the St"uckelberg procedure [29]. According to this procedure, a strongly coupled field called the St"uckelberg field is introduced in the symmetry breaking potential term such that the gauge symmetry is restored in a given Lagrangian. The original theory is still recovered in a special gauge (where St"uckelberg field is put to zero), however, the advantage is that the redundant degrees of freedom are now encompassed in the St"uckelberg field, and can be dealt with using well known quantization frameworks like the Faddeev-Popov method.

The above procedure is applied to (6) via the introduction of a strongly coupled vector field \( C_\mu \):

\[
\mathcal{L} = -\frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{4} \alpha^2 \left[ b_{\mu\nu} \left( B^{\mu\nu} + \frac{1}{\alpha} F^{\mu\nu} [C] \right) \right]^2,
\]

so that the Lagrangian (8) becomes gauge invariant (here, \( F_{\mu\nu} \equiv \partial_\mu C_\nu - \partial_\nu C_\mu \)), and reduces to original Lagrangian (6) in the gauge \( C_\mu = 0 \). The new Lagrangian is invariant under the symmetries,

\[
B^{\mu\nu} \rightarrow B^{\mu\nu} + \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu, \\
C_\mu \rightarrow C_\mu + \alpha \xi_\mu,
\]

and,

\[
C_\mu \rightarrow C_\mu + \nabla_\mu \Lambda, \\
B^{\mu\nu} \rightarrow B^{\mu\nu}.
\]

In addition to the above symmetries of fields, there exists a set of transformation of gauge parameters \( \Lambda \) and \( \xi_\mu \) that leaves the fields \( B_{\mu\nu} \) and \( C_\mu \) invariant,

\[
\xi_\mu \rightarrow \xi_\mu + \nabla_\mu \psi, \\
\Lambda \rightarrow \Lambda + \alpha \psi.
\]

The gauge fixing procedure for this Lagrangian is covered explicitly in Ref. [26], and to save space we quote directly the final result for the total gauge fixed Lagrangian,

\[
\mathcal{L}_{GF} = -\frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{4} \alpha^2 \left( b_{\mu\nu} B^{\mu\nu} \right)^2 - \frac{1}{4} \left( b_{\mu\nu} F^{\mu\nu} \right)^2 - \frac{1}{4} \left( b_{\mu\nu} b_{\rho\sigma} \nabla^\mu B^{\rho\sigma} \right)^2 - \frac{1}{2} \alpha^2 C_\nu C^\nu - \frac{1}{2} \left( \nabla_\mu \Phi \right)^2 - \frac{1}{2} \left( \nabla^\mu C_\mu \right)^2 - \frac{1}{2} \alpha^2 \Phi^2.
\]
The presence of a new scalar field $\Phi$ is a direct consequence of gauge-fixing of Stückelberg field, and explicitly displays a scalar degree of freedom that remains hidden in the original Lagrangian (6) with broken gauge symmetry.

III. 1-LOOP EFFECTIVE ACTION

The quantization of theories such as (8) is tricky, because of the symmetries in gauge parameters, as in Eq. (11). Such symmetries lead to a degeneracy in the ghost determinant appearing in the Faddeev-Popov procedure [30], and require special treatment for quantization [30–32]. We follow a general quantization procedure developed in Ref. [32] based on DeWitt-Vilkovisky’s approach [21, 22] that yields covariant and background independent results, to deal with the additional symmetries of gauge parameters and derive the 1-loop effective action.

For a quadratic action not involving quantization of metric the expression for 1-loop effective action in the DeWitt-Vilkovisky’s field space notation, about a set of background fields $\bar{\varphi}^i$, is given by [32],

$$\Gamma_1[\bar{\varphi}] = -\ln \det Q_{\alpha\beta}[\bar{\varphi}] + \ln \det \hat{Q}_{ab} + \frac{1}{2} \ln \det (S_{ij}^{GF}[\bar{\varphi}]) .$$  

(13)

where $S_{ij}^{GF}$ is the gauge-fixed action. Let us briefly explain the various (field-space) notations in Eq. (13) (see [24] for a detailed introduction). The index $i$ in field space corresponds to all the tensor indices and spacetime dependence of fields in the coordinate space. For example, fields $(B_{\mu\nu}(x), C_{\mu}(x), \Phi(x))$ are denoted by components of $\varphi^i$ ($i = 1, 2, 3$) in field space. The rest of the constructions in field space (tensors, scalar products, connections, field space metric, etc.) are similar to that in a coordinate space. The object $S_{ij}$ represents a derivative in field space, define by,

$$S_{ij}[\bar{\varphi}] = \frac{\delta^2}{\delta \varphi^j \delta \varphi^i} S[\bar{\varphi}] .$$  

(14)

Let, $\varepsilon^a$ parametrize the symmetry of gauge parameters, as in Eq. (11), and $\bar{\chi}^a$ be the corresponding fixing condition for gauge parameters $\epsilon^a$ (can be read off of Eqs. (9) and (10)), then [32]

$$Q^a_{\beta} = \left( \frac{\delta}{\delta \epsilon^\beta} \chi^a[\varphi, \epsilon, \bar{\chi}] \right)_{\epsilon = 0} ,$$  

(15)

where $\chi^a$ is the gauge fixing condition for fields $\varphi^i$. $\det Q^a_{\beta}$ is the ghost determinant factor.
In the present case, corresponding to the symmetries (9), (10) and (11), there are three
gauge conditions $\chi_{\xi\nu}, \chi_\Lambda$ and $\tilde{\chi}_\psi$ that lead to three operators $Q_{\xi\nu}^\chi, Q_\Lambda^\chi$ and $\tilde{Q}_\psi^\chi$ respectively. The results are displayed in TABLE I.

| \(\chi_{\xi\nu} = b_{\mu\nu} b_{\rho\sigma} \nabla^\mu B^\rho_{\sigma} + \alpha C_{\nu}\) | \(Q_{\xi\nu}^\chi = 2b_{\alpha\mu} b_{3\nu} \nabla^\alpha \nabla^\beta + \nabla^\mu \nabla^\nu - \alpha^2 \delta_{\mu\nu}\) |
| --- | --- |
| \(\chi_\Lambda = \nabla^\mu C_{\mu} + \alpha \Phi\) | \(Q_\Lambda^\chi = \Box - \alpha^2\) |
| \(\tilde{\chi}_\psi = \nabla^\mu \xi_{\mu} - \alpha \Lambda\) | \(\tilde{Q}_\psi^\chi = \Box - \alpha^2\) |

TABLE I. Results for $Q$ operators corresponding to choices of gauge condition $\chi$.

Using these results in Eq. (13), we get

\[
\Gamma_1 = - \ln \det Q_{\xi\nu}^\chi + \frac{1}{2} \ln \det \left( S_{ij}^{GF} [\varphi] \right). \tag{16}
\]

$S^{GF}$ is of course quadratic in fields, and the value of $\Gamma_1$ in operator form turns out to be,

\[
\Gamma_1 = \frac{i\hbar}{2} \left[ \ln \det(\Box - \alpha^2 b^{\mu\nu} b_{\rho\sigma}) - \ln \det(\Box - \alpha^2) + \ln \det(\Box - \alpha^2) \right], \tag{17}
\]

where,

\[
\dot{D}_2 B^{\mu\nu} \equiv D_2 B^{\mu\nu} + 2b^{\mu\nu} b_{\rho\sigma} b^{\alpha\gamma} \nabla^\rho \nabla^\beta B^{\alpha\gamma},
\]

\[
\dot{D}_1 C^{\mu} \equiv 2b^{\mu\nu} b_{\rho\sigma} \nabla_\nu \nabla^\rho C^\sigma + \nabla^\mu \nabla_\nu C^{\nu}. \tag{18}
\]

In flat spacetime, no physically interesting inferences can be extracted from the above expression. However, in curved spacetime, the operators in Eq. (17) are coupled to the metric $g_{\mu\nu}$. So, addressing certain issues, like that of quantum equivalence, then becomes nontrivial. Unfortunately, effective action cannot be calculated exactly in such cases \[26\], and the best way forward is to perform a perturbative study. Therefore, we will consider a nearly flat spacetime instead of a general curved one, so that,

\[
g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x). \tag{19}
\]

$\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is a perturbation, while $\kappa = 1/M_p (M_p$ is Planck mass) parametrizes the scale of perturbation.

We can rewrite $\Gamma_1$ in integral form by introducing ghost fields $c_\mu$ and $\bar{c}_\mu$,

\[
\Gamma_1 = - \ln \int [d\eta][dc_\mu][d\bar{c}_\mu] e^{-S_{GH}}, \tag{20}
\]
where,

\[ S_{GH} = \eta^i S_{GH}^{GF} \eta^j + \bar{c}^\mu Q_{\xi^\nu}^{\mu,\rho}, \quad (21) \]

and \( \eta^i \) are the quantum fluctuations \( (\delta B_{\mu\nu}(x), \delta C_\mu(x), \delta \Phi(x)) \). Now, we use Eq. (19) in Eq. (21) and rearrange terms in orders of \( h_{\mu\nu} \):

\[ S_{GH} = S_0 + S_1 + O(h^2), \quad (22) \]

where the subscripts denote the power of \( h_{\mu\nu} \). Substituting Eq. (22) in Eq. (20), and treating \( S_1 \) as a perturbation, the integrand can be Taylor expanded to write,

\[ \Gamma_1 = -\ln \left( 1 + \langle S_1 \rangle + O(h^2) \right), \quad (23) \]

where we have used the normalization for path integral of \( S_0 \). The logarithm can be further expanded to yield, up to first order in \( h_{\mu\nu} \),

\[ \Gamma_1 = -\langle S_1 \rangle. \quad (24) \]

The calculation of \( \Gamma_1 \) thus amounts to evaluating \( \langle S_1 \rangle \), which is a collection of two-point correlation functions of fields. These correlations are just the flat spacetime propagators of fields and can be derived from \( S_0 \) using projection operator method. We obtained the expansions of \( S_{GH} \) using xAct packages [33, 34] for Mathematica, results of which are presented below:

\[ S_0 = \int d^nx \left( -\frac{1}{2} \alpha^2 \delta C_\mu \delta C_\nu - \frac{1}{2} \alpha^2 \delta \phi^2 - \frac{1}{4} \alpha^2 (\delta B^{\mu\nu} b_{\mu\nu})^2 
- \frac{1}{2} (b_\alpha^\nu b_\beta^\gamma b_{\mu\nu} b_{\rho\sigma} \delta B^{\beta\gamma,\alpha} \delta B^{\rho\sigma,\mu}) - \frac{1}{2} (\delta C^{\mu,\nu})^2 
- \frac{1}{4} (b_{\nu\rho} (\delta C^{\nu,\mu} - \delta C^{\mu,\nu}))^2 - \frac{1}{2} \delta \phi_{\mu} \delta \phi_{\mu} 
+ \delta \bar{c}^\mu (\alpha^2 \delta \bar{c}_{\mu} + \delta \bar{c}_{\nu,\mu} + 2b_{\nu\mu} b_{\rho\sigma} \delta \bar{c}_{\sigma,\rho,\nu}) 
+ \frac{1}{12} (-\delta B_{\mu,\nu} - \delta B_{\rho,\mu} - \delta B_{\mu,\nu})(\delta B^{\rho,\mu} + \delta B^{\mu,\nu} + \delta B^{\mu,\nu}) \right), \quad (25) \]
\[ S_1 = \int d^n x \left( \frac{1}{2} \alpha^2 \delta \mu \delta \nu \epsilon_{\mu \nu} - \alpha^2 \delta \mu \delta \nu \epsilon_{\mu \nu} - \frac{1}{2} \alpha^2 \delta^2 \delta \mu \epsilon_{\mu} \right) \]

\[-\frac{1}{2} \alpha^2 \delta \mu \delta \nu \epsilon_{\mu \nu} - \frac{1}{2} \alpha^2 \delta \mu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \alpha^2 \delta \mu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \alpha^2 \delta \mu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \alpha^2 \delta \mu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \alpha^2 \delta \mu \delta \nu \epsilon_{\mu \nu} \]

\[+ \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} \]

\[-2 \delta \mu \delta \nu \mu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} \]

\[+ \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} + \frac{1}{2} \delta \mu \delta \mu \delta \nu \delta \nu \epsilon_{\mu \nu} \]

\[\left( 26 \right)\]
A. Propagators

We use the projection operator method [35] to invert the operators in $S_0$ and derive the Green’s functions or propagators. In the operator form, $S_0$ can be recast as

$$S_0 = \int d^n x \left( \frac{1}{4} B^{\mu \nu} O^B_{\mu \nu, \alpha \beta} B_{\alpha \beta} + \frac{1}{2} C^{\mu} O^C_{\mu \nu} C^\nu + \frac{1}{2} \Phi O^\Phi \Phi \right)$$  \hspace{1cm} (27)

where,

$$O^B_{\mu \nu, \alpha \beta} = \frac{\Box}{2} (\eta_{\mu \alpha} \eta_{\nu \beta} - \eta_{\mu \beta} \eta_{\nu \alpha}) + \frac{1}{2} (\partial_\mu \partial_\beta \eta_{\nu \alpha} - \partial_\nu \partial_\beta \eta_{\mu \alpha} - \partial_\mu \partial_\alpha \eta_{\nu \beta} + \partial_\nu \partial_\alpha \eta_{\mu \beta})$$

$$- \left( \alpha^2 + 2 (b_{\rho \sigma} \partial^\rho)^2 \right) b_{\mu \nu} b_{\alpha \beta},$$  \hspace{1cm} (28)

$$O^C_{\mu \nu} = \frac{1}{2} (b_{\sigma \mu} b_{\rho \nu} \partial^\sigma \partial^\rho + b_{\sigma \nu} b_{\rho \mu} \partial^\sigma \partial^\rho) + \frac{1}{2} \partial_\mu \partial_\nu - \frac{1}{2} \alpha^2 \eta_{\mu \nu},$$  \hspace{1cm} (29)

$$O^\Phi = \Box - \alpha^2.$$  \hspace{1cm} (30)

At this point, we would like to point out that a calculation for green’s function for $B_{\mu \nu}$ was performed recently in Ref. [36] using projector method. However, their calculation did not account for the Stückelberg field and as a result our operator $(O^B)_{\mu \nu, \alpha \beta}$ is different from the one in Ref. [36], which misses the contribution from gauge-fixing term $2 (b_{\rho \sigma} \partial^\rho)^2 b_{\mu \nu} b_{\alpha \beta}$. Fortunately, this term is merely an addition to mass, $\alpha^2$ and ends up not contributing to the propagator, $(O^B)^{-1}_{\mu \nu, \alpha \beta}$. So, we end up getting an identical result for the propagator, barring complex infinity terms that can be ignored [36] (see appendix for details of projection operators $P^{(1)}, ..., P^{(6)}$).

$$ (O^B)^{-1}_{\mu \nu, \alpha \beta}(x, x') = \int \frac{d^n p}{(2 \pi)^n} e^{-ip \cdot (x-x')} \left( \frac{1}{p^2} P^{(1)}_{\mu \nu, \alpha \beta} + \frac{b^2}{(b_{\rho \sigma} p^\rho)^2} (P^{(4)}_{\mu \nu, \alpha \beta} + P^{(5)}_{\mu \nu, \alpha \beta}) \right),$$  \hspace{1cm} (31)

There are no massive propagating modes in Eq. (31) and only one a massive mode propagates, as concluded in Ref. [19]. The second pole describes a massive pole propagating in an anisotropic medium, which for our choice of $b_{\mu \nu}$ gives,

$$b^2 ((p^2)^2 + (p^3)^2) = 0.$$  \hspace{1cm} (32)

Contrary to the claim in Ref. [36] where these modes were described as non-physical due to a negative sign appearing in energy-momentum relations as a result of a different choice of $b_{\mu \nu}$, we note that for a different but equally legal choice of $b_{\mu \nu}$ energy terms $(p^0)$ disappear altogether, and hence it is not straightforward to judge the physical/nonphysical nature of these modes.
For the Stückelberg field $C_\mu$, spontaneous Lorentz violating term appears in the kinetic part (first term in Eq. (29)), which makes inverting $O^C_{\mu\nu}$, a little tricky. New projector operators have to be defined apart from the longitudinal and transverse momentum operators, that also have a closed algebra, so that any operator $D_{\mu\nu}$ can be then expanded in terms of these projectors. We define,

$$
P^{(1)}_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}; \quad P^{(2)}_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}; \quad P^{(3)}_{\mu\nu} = \frac{1}{2p^2} \left( b_{\sigma\mu} b_{\nu\rho} p^\sigma p^\rho + b_{\sigma\nu} b_{\mu\rho} p^\sigma p^\rho \right).$$

(33)

These operators satisfy a closed algebra, as shown in TABLE II. Using these operators, $O^C_{\mu\nu}$ in momentum space can be written as,

$$
O^C_{\mu\nu} = -2p^2 P^{(3)}_{\mu\nu} - \left( p^2 + \alpha^2 \right) P^{(1)}_{\mu\nu} - \alpha^2 P^{(2)}_{\mu\nu}.
$$

(34)

Assuming that $(O^C)^{-1}_{\mu\nu}$ in momentum space has the form,

$$
(O^C)^{-1}_{\mu\nu} = x_1 P^{(1)}_{\mu\nu} + x_2 P^{(2)}_{\mu\nu} + x_3 P^{(3)}_{\mu\nu},
$$

(35)

we use the identity $OO^{-1} = I$ to obtain,

$$
(O^C)^{-1}_{\mu\nu}(x, x') = \int \frac{d^np}{(2\pi)^n} e^{-ip(x-x')} \left( -\frac{1}{p^2 + \alpha^2} P^{(1)}_{\mu\nu} - \frac{1}{\alpha^2} P^{(2)}_{\mu\nu} + \frac{1}{\alpha^2} \left( b_{\sigma\mu} p^\sigma \right)^2 + \frac{1}{\alpha^2} P^{(3)}_{\mu\nu} \right).
$$

(36)

Here, a massive scalar mode with pole at $\alpha$ propagates while another anisotropic mode propagates with mass $\alpha/\sqrt{2}$. Terms in $P^{(2)}$ contain a massless pole and an additive poleless term which does not contribute to correlations and can be ignored. For $\Phi$, the scalar propagator is given by,

$$
(O^\Phi)^{-1}_{\mu\nu}(x, x') = \int \frac{d^np}{(2\pi)^n} e^{-ip(x-x')} \frac{1}{p^2 + \alpha^2}.
$$

(37)
B. Quantum corrections

Since all terms in $\langle S_1 \rangle$ are local, they correspond to tadpole diagrams. We solve these integrals in two steps: first, the derivatives of field fluctuations are transformed to momentum space by substituting Eqs. (31), (36), and (37). We also perform by-parts integrals to get rid of derivatives of $h_{\mu\nu}$, so that in all expressions below, a coefficient $h_{\mu\nu}$ is understood to be present but not explicitly written. The Fourier transformed $\langle S_1 \rangle$ then has terms of the form,

$$\int d^n x A(x) \langle \partial^m \delta \partial^n \delta \rangle \rightarrow \int d^n x \frac{d^n p}{(2\pi^n)} A(x)(-ip)^m (ip)^n \langle \delta_p \delta_p \rangle,$$

(38)

where, tensor indices of $A(x)$ and $\delta$ have been omitted for convenience. $\delta$ is the quantum field fluctuation, and $\langle \delta_p \delta_p \rangle$ represents the propagator(s) in momentum space.

The second step is to replace $\langle \delta_p \delta_p \rangle$ with values of Green’s function and evaluate the integrals. We primarily use the results in Ref. [37] to evaluate the divergent terms of most of the integrals, except those involving anisotropic term $(b_{\rho\sigma} p^\rho)^2$. There are two types of poleless integrals coming from Eq. (31):

$$\int d^n x \frac{d^n p}{(2\pi^n)} A(x) \frac{p^\mu...p^\beta}{p^2}; \quad \int d^n x \frac{d^n p}{(2\pi^n)} A(x) \frac{p^\mu...p^\beta}{(b_{\rho\sigma} p^\rho)^2},$$

(39)

with up to four $p^\mu$’s in the numerator. The first integral vanishes due to the lack of a physical scale [37]. To solve the second integral, we use the approach developed in [38, 39], and find that it also does not have any physical contribution.

Next, there are broadly three types of integrals with non-zero poles arising from the rest of propagators:

$$\int d^n x \frac{d^n p}{(2\pi^n)} A(x) \frac{p^\mu...p^\beta}{p^2 + \alpha^2}; \quad \int d^n x \frac{d^n p}{(2\pi^n)} A(x) \frac{p^\mu...p^\beta}{p^2(p^2 + \alpha^2)^2};$$

$$\int d^n x \frac{d^n p}{(2\pi^n)} A(x) \frac{p^\mu...p^\beta}{(b_{\rho\sigma} p^\rho)^2 + \alpha^2/2};$$

(40)

Again, the solutions to first two types of integrals are available in Ref. [37]. We solve the third type of integral as follows. Following [38], we write

$$\int d^n p \frac{1}{(b_{\rho\sigma} p^\rho)^2 + \alpha^2/2} = \int d^n p \int_0^\infty d\theta \exp[-\theta((b_{\rho\sigma} p^\rho)^2 + \alpha^2/2)].$$

(41)

Integrating over $d^n p$, followed by writing the integral over $\theta$ in terms of $\Gamma$ function leads to familiar expressions encountered in dimensional regularization, which finally yields the
divergent part as \((\epsilon = n - 4)\),
\[
divp\left(\int d^n p \frac{1}{(b_{\mu \nu} p^2 + \alpha^2 / 2)}\right) = -\frac{\pi^2 m^2}{\sqrt{\det(b_{\mu \nu} b_{\mu \nu})}} \frac{2}{\epsilon},
\]
which is identical to that of a scalar propagator integral except for the \(\sqrt{\det(b_{\mu \nu} b_{\mu \nu})}\) in the denominator. For our choice of \(b_{\mu \nu}\), Eq. (2) with \(a = 0\) and \(b = 1/\sqrt{2}\), this term becomes a diagonal matrix,
\[
b_{\mu \nu} b^{\mu \nu} = \text{diag}(0 \ 0 \ 1/2 \ 1/2),
\]
implying that the determinant is zero. It turns out however, that this determinant appears as a factor in the denominator of the divergent part of effective action, and hence we use a regularization factor \(\epsilon'\) to write,
\[
b_{\mu \nu} b^{\mu \nu} = \lim_{\epsilon' \to 0} \text{diag}(\epsilon' \ \epsilon' \ 1/2 \ 1/2).
\]

With these inputs in xAct\[\text{[34]}\], the final result for the divergent part of 1-loop effective after some further manipulations, is obtained as,
\[
divp(\Gamma_1) = \frac{1}{16\pi^2} \left(\alpha^4 \kappa h^a_{\cdot a} + \frac{1}{\sqrt{\det(b_{\mu \nu} b_{\mu \nu})}} \left( -\frac{1}{16} \alpha^4 \kappa h_{\mu \nu} b_a^\mu b_a^\nu - \frac{3}{32} \alpha^4 \kappa h^b_{\mu \nu} b_{a \mu} b_{a \nu} a \right. \right.
\]
\[
-\frac{1}{12} \alpha^4 \kappa h_{\mu \nu} b_a^\mu b_b^\nu b_c^\nu b_a^\mu - \frac{5}{192} \alpha^4 \kappa h_{\mu \nu} b_a^\mu b_b^\nu b_c^\nu b_b^\mu + \frac{1}{192} \alpha^4 \kappa h_{\mu \nu} b_b^c b_{\mu} b_{\nu} b_{\mu} b_{\nu} a
\]
\[
+ \frac{5}{192} \alpha^4 \kappa h_{\mu \nu} b_a^\nu b_b^\nu b_b^\mu b_a^\mu - \frac{1}{192} \alpha^4 \kappa h_{\mu \nu} b_a^\mu b_b^\nu b_b^\nu b_{a \mu} b_{a \nu} a + \frac{5}{384} \alpha^4 \kappa h_{\mu \nu} b_b^c b_{\mu} b_{\nu} b_{\mu} b_{\nu} a
\]
\[
+ \frac{5}{192} \alpha^4 \kappa h_{\mu \nu} b_a^\nu b_b^\nu b_b^\mu b_a^\mu + \frac{1}{192} \alpha^4 \kappa h_{\mu \nu} b_a^\mu b_b^\nu b_b^\nu b_{a \mu} b_{a \nu} a
\]
\[
- \frac{7}{192} \alpha^4 \kappa h_{\mu \nu} b_b^c b_{\mu} b_{a \mu} b_{a \mu} b_{a \mu} b_{a \nu} b_{a \nu} a - \frac{1}{96} \alpha^4 \kappa h_{\mu \nu} b_b^c b_{\mu} b_{a \mu} b_{a \mu} b_{a \nu} b_{a \nu} a
\]
\[
- \frac{1}{96} \alpha^4 \kappa h_{\mu \nu} b_b^c b_{\mu} b_{a \mu} b_{a \mu} b_{a \nu} b_{a \nu} a + \frac{1}{96} \alpha^4 \kappa h_{\mu \nu} b_b^c b_{\mu} b_{a \mu} b_{a \mu} b_{a \nu} b_{a \nu} a b_{\mu} a b_{\nu} a.\]
\]

**IV. QUANTUM EQUIVALENCE**

Checking classical equivalence of two theories is an interesting theoretical exercise, because it provides insight into the degrees of freedom and dynamical properties of theories that may be described by very different fields, like in 2-form, 1-form or a scalar field theories, and thus may lead to several simplifications in a given theory. This problem naturally extends to the quantum regime, and it is certainly not trivial to prove quantum equivalence of two classically equivalent theories especially in curved spacetime. For instance, it can be
shown that a massive 2-form field is quantum equivalent to a massive vector field because of some special topological properties of zeta functions \[25\]. However, it is extremely difficult to perform similar analyses when, for example, the Lorentz symmetry is broken \[26\]. In flat spacetime, establishing quantum equivalence is indeed trivial, because there is no field dependence in \(\Gamma_1\) (Eq. \(\text{(17)}\)) and hence effective actions of two theories do not possess any physical distinction.

On the contrary, in curved spacetime, the presence of metric makes things interesting. Only problem is, the effective action cannot be calculated exactly. So, our best bet, in this case, is to do a perturbative study like the one in the previous section.

Classical equivalence of Eq. \[6\] was explored in Ref. \[19\], it was found to be equivalent to,

\[
\mathcal{L} = \frac{1}{2} B_{\mu\nu} F^{\mu\nu} - \frac{1}{2} C^{\mu} C_{\mu} - \frac{1}{4} \alpha^2 (b_{\mu\nu} B^{\mu\nu})^2, \tag{46}
\]

where,

\[
F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}. \tag{47}
\]

\(C_\mu\) is a vector field and \(F_{\mu\nu}\) is as defined before. We choose to continue with the same symbol for vector and St"uckelberg field to avoid unnecessary complications. Eq. \[46\] can be written exclusively in terms of \(C_\mu\) through the use of projection operators,

\[
T_{||\mu\nu} = b_{\rho\sigma} T^{\rho\sigma} b_{\mu\nu},
\]

\[
T_{\perp\mu\nu} = T_{\mu\nu} - T_{||\mu\nu}, \tag{48}
\]

for any two-rank tensor \(T_{\mu\nu}\), and subsequently using the equations of motion for \(B_{||\mu\nu}\) and \(B_{\perp\mu\nu}\), to obtain,

\[
\alpha^2 \mathcal{L} = \frac{1}{4} \left( b_{\mu\nu} F^{\mu\nu} \right)^2 - \frac{1}{2} \alpha^2 C^{\mu} C_{\mu}, \tag{49}
\]

where we have defined \(\tilde{b}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} b^{\rho\sigma}\). Note that Lorentz violation enters Eq. \[49\] through the kinetic term, although it is still gauge-symmetric. A similar exercise of applying St"uckelberg procedure leads to the gauge fixed action in flat spacetime,

\[
\tilde{S}_0 = \int d^n x \left( \frac{1}{2} C^{\mu} C_{\mu} + \frac{1}{2} \Phi \mathcal{O} \Phi \right) \tag{50}
\]

where,

\[
\mathcal{O}^{\mu\nu} = -\frac{1}{2} \left( \tilde{b}_{\sigma\mu} \tilde{b}_{\rho\nu} \partial^\sigma \partial^\rho + \tilde{b}_{\sigma\nu} \tilde{b}_{\rho\mu} \partial^\sigma \partial^\rho \right) + \frac{1}{2} \partial_{\mu} \partial_{\nu} - \frac{1}{2} \alpha^2 \eta_{\mu\nu}. \tag{51}
\]
Upon comparing Eqs. (45) and (53), we can immediately notice that the first term is identical to contributions from propagator of Lorentz violating modes. Hence, the quantum equivalence from the propagator of non-Lorentz violating modes, while all the other terms correspond to contributions from propagator of Lorentz violating modes. In the present work, this problem is overcome by adopting a perturbative approach to evaluating effective action, that is also general enough to be applied to more complicated models including interaction terms.

Since antisymmetric tensor fields are likely to play a significant role in the early universe cosmology, studying their quantum aspect is a natural extension of classical analyses. In a past study [26], it was found that issues like quantum equivalence are difficult to address in a general curved spacetime. Hence, the quantum equivalence holds along non-Lorentz violating modes but not along Lorentz violating modes involving $b_{\mu\nu}$.

V. CONCLUSION

Study of spontaneous Lorentz violation with rank-2 antisymmetric tensor is interesting because of the possibility of rich phenomenological signals of SME in future experiments. Since antisymmetric tensor fields are likely to play a significant role in the early universe cosmology, studying their quantum aspect is a natural extension of classical analyses. In a past study [26], it was found that issues like quantum equivalence are difficult to address in a general curved spacetime. In the present work, this problem is overcome by adopting a perturbative approach to evaluating effective action, that is also general enough to be applied to more complicated models including interaction terms.

We quantized a simple action of an antisymmetric tensor field with a nonzero vev driving potential term that introduces spontaneous Lorentz violation, using a covariant effective
action approach at one-loop. The one-loop corrections were calculated in a nearly flat spacetime, at \( O(h) \). We revisited the issue of quantum equivalence, and found that for the non-Lorentz-violating modes (independent of vev \( b_{\mu\nu} \)), antisymmetric tensor field is quantum-equivalent to a vector field. However, contributions from the Lorentz violating part of the propagator leads to different terms in effective actions, and as a result, \( \Delta \Gamma = \Gamma_1 - \tilde{\Gamma}_1 \neq 0 \), i.e. the theories are not quantum equivalent.

**ACKNOWLEDGMENTS**

This work was partially funded by DST (Govt. of India), Grant No. SERB/PHY/2017041.

**Appendix: Projection operators for \( B_{\mu\nu} \)**

The basic projection operators for an antisymmetric tensor are defined as \[40\],

\[
P^{(1)}_{\mu\nu,\alpha\beta} = \frac{1}{2} (\theta_{\mu\alpha} \theta_{\nu\beta} - \theta_{\mu\beta} \theta_{\nu\alpha}),
\]

\[
P^{(2)}_{\mu\nu,\alpha\beta} = \frac{1}{4} (\theta_{\mu\alpha} \omega_{\nu\beta} - \theta_{\nu\alpha} \omega_{\mu\beta} - \theta_{\mu\beta} \omega_{\nu\alpha} + \theta_{\nu\beta} \omega_{\mu\alpha}),
\]

where,

\[
\theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}, \quad \omega_{\mu\nu} = \frac{\partial_{\mu} \partial_{\nu}}{\Box},
\]

are the longitudinal and transverse projection operators along the momentum. To account for the Lorentz violation induced by nonzero vev, four new operators need to be introduced as follows \[36\]:

\[
P^{(3)}_{\mu\nu,\alpha\beta} = P^{\perp}_{\mu\nu,\alpha\beta},
\]

\[
P^{(4)}_{\mu\nu,\alpha\beta} = \frac{1}{2} \left( \omega_{\mu\lambda} P^{\|}_{\nu\lambda,\alpha\beta} - \omega_{\nu\lambda} P^{\|}_{\mu\lambda,\alpha\beta} \right),
\]

\[
P^{(5)}_{\mu\nu,\alpha\beta} = \frac{1}{2} \left( \omega_{\alpha\lambda} P^{\|}_{\nu\mu,\beta\lambda} - \omega_{\beta\lambda} P^{\|}_{\mu\nu,\alpha\lambda} \right),
\]

\[
P^{(6)}_{\mu\nu,\alpha\beta} = \frac{1}{4} \left( \omega_{\mu\alpha} P^{\|}_{\nu\rho,\beta\sigma} \omega^{\rho\sigma} - \omega_{\nu\alpha} P^{\|}_{\mu\rho,\beta\sigma} \omega^{\rho\sigma} - \omega_{\mu\beta} P^{\|}_{\nu\rho,\alpha\sigma} \omega^{\rho\sigma} + \omega_{\nu\beta} P^{\|}_{\mu\rho,\alpha\sigma} \omega^{\rho\sigma} \right).
\]

The operators \( P^{(1)}_{\mu\nu,\alpha\beta}, \ldots, P^{(6)}_{\mu\nu,\alpha\beta} \) obey a closed algebra \[36\].

The identity element is given by,

\[
\mathcal{I}_{\mu\nu,\alpha\beta} = \frac{1}{2} (\eta_{\mu\alpha} \eta_{\nu\beta} - \eta_{\mu\beta} \eta_{\nu\alpha}) = \left[ P^{(1)} + P^{(2)} \right]_{\mu\nu,\alpha\beta}.
\]
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