Stable knots in the trapped Bose-Einstein condensates

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Abstract – The knot of the spin-texture is studied within the two-component Bose-Einstein condensates which are described by the nonlinear Gross-Pitaevskii equations. We start from the noninteracting equations including an axisymmetric harmonic trap to obtain an exact solution, which exhibits a nontrivial topological structure. The spin-texture is a knot with an integral Hopf invariant. The stability of the knot is verified by numerically evolving the nonlinear Gross-Pitaevskii equations along imaginary time.

Topological objects are often interesting topics in various fields of physics ranging from the condensed-matter physics [1,2] to the particle physics and the modern universe [3,4]. Although many works have been done, much less attention is paid to their mathematical existence [5–12]. In most works, the existence of the knot is discussed only qualitatively, using plausibility arguments, but the problem of constructing the corresponding solutions is rarely addressed. Among those, ultracold atoms provide an ideal pilot to study the complex topological excitations [13]. In particular, two-component Bose-Einstein condensates (BECs), in which the interactions between the atoms can be precisely tuned by the magnetic-field Feshbach resonance, have been widely used to create topological defects [14,15].

The properties of the BECs are described by the order parameters within the mean-field theory. By using the normalized spinor \( \xi(r) \) with \( \xi^\dagger \xi = 1 \), the wave function is represented as \( \psi(r) = \sqrt{n(r)} \xi(r) \), where \( n(r) \) is the density of the condensate [16–18]. When the symmetry group \( G \) of a system reduces to its subgroup \( H \) through spontaneous symmetry breaking, the topological excitations in the spinor condensates are characterized by homotopy classes of the order parameter (OP) space \( M \) identified by the quotient space \( G/H \) [16,17]. This examination can be carried out with the help of the homotopy groups of the OP space \( \pi_n(M) \). The homotopy groups not only determine the topological invariants, but also stipulates the rules of coalescence and disintegration of the topological excitations [19].

The 3D topological structures are classified by the third homotopy group \( \pi_3(M) \) [18]. Knots differ from other topological excitations such as vortices and skyrmions in that they are classified by a linking number while others are classified by winding numbers [20]. Knots are characterized by mapping from a three-dimensional sphere \( S^3 \) to \( S^2 \). The homotopy groups \( \pi_3(S^2) = \mathbb{Z} \).

In this paper, we construct an ansatz wave function which is the stationary solution in vanishing interactions limit. The state exhibits the topological structure with a knotted spin-texture. As the nonlinear interactions are switched on, we prove by numerical simulations that this topological structure is still energetically stable, providing the conservation of the particle number in each species atoms.

The paper is organized as follows. In the following section we present an exact solution in the limit of vanishing nonlinear interactions. In the second section we reveal the topological structure of the stationary state. In the third section we numerically verify the stability of the knot by taking into account the nonlinear coupling in the GPEs. A brief summary is included in the last section.

Stationary solution for noninteracting condensates. – We consider the two-component BECs that are confined in a 3D trap. The dynamics is described by the
In order to reveal the topological structure more clearly, we parameterized the order parameters of the condensates as [21]

$$
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
= \sqrt{n(r)} \begin{pmatrix}
\cos(\theta(r)/2)e^{i\varphi_1(r)} \\
\sin(\theta(r)/2)e^{i\varphi_2(r)}
\end{pmatrix},
$$

where $\theta \in [0, \pi]$, $\varphi_1 \in [0, 2\pi]$ and $\varphi_2 \in [0, 2\pi]$. The local pseudo-spin $S$ is defined by $S = \xi^i\sigma_\xi^j$, where $\sigma$ is the Pauli matrix.

The order parameter allows a topological classification $\pi_3(S^2)$ of a map from the real space to the spin vector which is described by

$$Q_{\text{H}} = \frac{1}{4\pi^2} \int \varepsilon_{ijk} F_{ij} A_k d^3x,$$

where the field strength $F_{ij} = \partial_i A_j - \partial_j A_i = S \cdot (\partial_i S \times \partial_j S)$ [2]. The integration is over the whole physical space where the density is assumed to be nonvanishing. By substituting the spin of eq. (3) into the formula (5) of the Hopf charge, we can directly calculate $Q_{\text{H}} = 1$.

We analyze the topological structure by using the cylindrically symmetric ansatz (4) in comparison with the exact solution (3). In the physical space, $\xi(r)$ is given by the continuous deformation of the mapping $\theta(r) = f_1(\varphi, z)$, $\varphi_1(r) = f_2(\varphi, z)$, $\varphi_2(r) = m \varphi$. For $f_1 \in [0, \pi], f_2 \in [0, 2\pi]$, there is a winding number $n$ in the $r$-$z$ plane which is defined as $q = \frac{1}{2\pi} \int_{\varphi} \varepsilon_{ijk} \theta \nabla_i \varphi_1 \nabla_j \varphi_2 d^3r$. Thus, we can classify the topological excitations in terms of the pair of integers $(m, n)$. Figure 2 display the distributions of the spin fields at various radii $r = 1, 2, 4, 12$, respectively. The color of the arrows indicate the values of $S_x (\times \pi)$. As the radius tends to infinity, all the spins gradually point to the north direction, namely, $S(r) \rightarrow (0, 0, 1)$, implying that the spin manifold is compactified to a point at spatial infinity.

In order to see the topology more clearly, the spin-textures in different sections are illustrated in fig. 3. Figure 3(a) shows the distributions of $S(r)$ in the horizontal section of $z = 0$. The spin rotates $2\pi$ around the azimuth direction ($\phi$). Figure 3(b) shows the rotation of
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Fig. 3: (Color online) The spin-textures for (a) the horizontal section in the \( z = 0 \) plane and (b) along the radial direction. Each figure reveals a wind of the spin. (c) The spin-texture for the vertical section of \( \phi = 0 \) (a half-plane). It is a 2D skyrmion. The color of the arrows represent the value \( S_r \).

\( \mathbf{S}(r) \) along the \( z \)-direction. The spin twists \( 2\pi \) from \( S_z = 1 \) in the center to \( S_z = 1 \) at infinity, completing a whole winding in this direction. Figure 3(c) shows the distributions of \( \mathbf{S}(r) \) in the vertical section of \( \phi = 0 \). This section reveals a 2D skyrmion with winding number 1. The knot can be viewed as the 2D skyrmion (\( S^2 \)) rotating a circle (\( S^1 \)) around the \( z \)-axis. We can see the map as \( S^3 \rightarrow S^2 \times S^1 \rightarrow S^2 \), which can be thought as concentric spheres. As discussed above, we have a pair of integers (1,1) which indicates the topological invariant of the knot \( Q_H = 1 \times 1 \). It can be verified by a deformation and recombination of eq. (5) when substituting the solution (3) into the formula of the Hopf charge. A similar analysis and deformation can be found in ref. [22].

Apart from the direct calculation, the Hopf charge can also be viewed by the linking number from the image. As we know, \( \mathbf{S} \) defines a map from the 3D physical domain (\( x \)) into a 2D sphere. Consequently, the preimage of a point on the target \( S^2 \) corresponds to a closed loop in the compactified \( S^3 \). The Hopf charge \( Q_H \) characterizes that the two loops corresponding to the preimages of any two distinct points on the target \( S^2 \) should be linked \( Q_H \) times [18]. Figure 4 illustrates the knotted features of the spin fields. The torus in fig. 4(a) is the isosurface of \( S_z = 0 \). The red and green linked loops are the preimages of two points on the \( S^2 \) sphere. They are essentially two twisted tubes since we plot the isosurfaces of \( S_x = 0.98 \) and \( S_y = 0.98 \), respectively. In fig. 4(b), the torus is the isosurface of \( S_z = 0.98 \), where the variations of color specify the angle of \((S_x, S_y)\) which depicts the twist and chirality of the knot [23]. The ring in the inner of the torus is the preimage of \( S_z = 0.98 \). From the figures, we can see that the linking number for this spin-texture is 1, which indicates that the Hopf charge is one.

**Stability in the presence of nonlinear interactions.** – In the previous works, a “knot” or helical baby skyrmion whose physical space is \( R^2 \times S^1 \) (periodic in \( z \)-direction) is widely studied [22,24]. But exactly, it is not a strict knot, as the linking number or Hopf invariant is not a homotopy invariant where the tube is not closed [3,25,26]. Knot in BEC is rarely addressed, in the paper [2], a knot is created in the spin-1 BEC by manipulating an external magnetic field, however the lifetime for this knot is short.

Next, we numerically calculate the stability or dynamics of our knot by solving the time-dependent Gross-Pitaevskii equation. From the homotopy theory, the continuous deformation of eq. (3), with the satisfied boundary condition, does not change the topological charge. In our case, by taking the immiscible regimes of interactions \( a_1a_{22} < a_{12}^2 \), and the number of atoms in each species separately conserved, we find the knot can be stabilized. The density ring in the \( \psi_1 \) component is fixed by the interaction of the two species, which prevent the knot from shrinkage.

The numerical simulation is carried out with the imaginary time evolution scheme [27–30]. During each time step in the numerical simulation, we preserve the number of particles in the system while the chemical potential is adjusted correspondingly. This treatment has been widely used for studying various kinds of topological excitations in the BEC system [27]. We use the split-step method with a spatial grid of \( 151 \times 151 \times 151 \). The parameters of \( ^{87} \text{Rb} \), whose scattering length can be tuned by the magnetic-field Feshbach resonance, are employed in the numerical calculations. We have tested different particle numbers with different proportion of \( N_r = N_1/N_2 \). For a small number of atoms, the nonlinear repulsion between the two-species atoms is too weak to prevent the knot shrinking. So it is important to adjust the particle numbers and the interaction coefficients to keep the topological structure. Figure 5 shows the stable density distributions after a long time evolution for \( N_1 = N_2 = 4.5 \times 10^5 \). The trap parameters are \( \omega_z/\omega = 2 \) and \( \omega = 2\pi \times 7.8 \) Hz. We note that the stability is insensitive to the shape of the trap. This holds for an isotropic harmonic trap.

We have traced the topological invariants of \( Q_H \) during the imaginary time evolution and found that it is always
kept one for the stable knot. On the contrary, if we employ total particle number conservation, the knot becomes unstable and will disappear. Proposals for experimental realizations of the knots can be found in refs. [2,31].

In summary, we have presented the exact 3D solution which exhibits a knot for the linear GPEs. The energetic stability of the topology is numerically demonstrated as the nonlinear interaction is switched on, provided the particle conservation of each component is adopted. The Hopf charge keep invariant during the dynamical evolution.

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REFERENCES

[1] Babaev E., Faddeev L. D. and Niemi A. J., Phys. Rev. B, 65 (2002) 120512.
[2] Kawaguchi Y., Nitta M. and Ueda M., Phys. Rev. Lett., 100 (2008) 180403.
[3] Jäykkä J. and Hietarinta J., Phys. Rev. D, 79 (2009) 125027.
[4] Vilenkin A. and Shellard E. P. S., Cosmic Strings and Other Topological Defects (Cambridge University Press, Cambridge, England) 1994.
[5] Liang J., Liu X. and Duan Y., EPL, 86 (2009) 10008.
[6] Matthews R., Louis A. A. and Yeomans M., EPL, 89 (2010) 29001.
[7] Savage C. M. and Ruostekoski J., Phys. Rev. Lett., 91 (2003) 010403.
[8] Ruostekoski J. and Anglin J. R., Phys. Rev. Lett., 86 (2001) 3934.
[9] Battye R. A., Cooper N. R. and Sutcliffe P. M., Phys. Rev. Lett., 88 (2002) 080401.
[10] Wüster S., Argue T. E. and Savage C. M., Phys. Rev. A, 72 (2005) 043616.
[11] Kawakami T., Mizushima T., Nitta M. and Machida K., Phys. Rev. Lett., 109 (2012) 015301.
[12] Radu E. and Volkov M. S., Phys. Rep., 468 (2008) 101.
[13] Pethick C. J. and Smith H., Bose-EinsteinCondensation in Dilute Gases, 2nd edition (Cambridge University Press, Cambridge, UK) 2008.
[14] Papp S. B., Pino J. M. and Wieman C. E., Phys. Rev. Lett., 101 (2008) 040402.
[15] Tojo S., Taguchi Y., Masuyama Y., Hayashi T., Saito H. and Hirano T., Phys. Rev. A, 82 (2010) 033609.
[16] Kasamatsu K., Takeuchi H., Tsubota M. and Nitta M., Phys. Rev. A, 88 (2013) 013620.
[17] Mäkelä H., J. Phys. A: Math. Gen, 39 (2006) 7423.
[18] Kawaguchi Y. and Ueda M., Phys. Rep., 520 (2012) 253.
[19] Kobayashi S., Kobayashi M., Kawaguchi Y., Nitta M. and Ueda M., Nucl. Phys. B, 856 (2012) 577.
[20] Kawaguchi Y., Kobayashi M., Nitta M. and Ueda M., Prog. Theor. Phys. Suppl., 186 (2010) 455.
[21] Yang S. J., Wu Q. S., Zhang S. N. and Feng S., Phys. Rev. A, 77 (2008) 033621.
[22] Cho Y. M., Phys. Lett. B, 603 (2004) 88.
[23] Liu Y.-K. and Yang S.-J., Phys. Rev. A, 87 (2013) 063632.
[24] Liu Y.-K., Zhang C. and Yang S.-J., Phys. Lett. A, 377 (2013) 3300.
[25] Kobayashi M. and Nitta M., Nucl. Phys. B, 876 (2013) 605.
[26] Olsen K. W. and Bohr J., EPL, 103 (2013) 30002.
[27] Mizushima T., Machida K. and Kita T., Phys. Rev. A, 66 (2002) 053610.
[28] Bao W., Jaksch D. and Markovich P. A., J. Comput. Phys., 187 (2003) 318.
[29] Bao W. and Wang H., J. Comput. Phys., 217 (2006) 612.
[30] Bao W. and Cai Y., Kinet. Relat. Mod., 6 (2013) 1.
[31] Choi J.-Y., Kwon W. J. and Shin Y.-I., Phys. Rev. Lett., 108 (2012) 035301.