On the existence of optimal stationary policies for average Markov decision processes with countable states

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Abstract

For a Markov decision process with countably infinite states, the optimal value may not be achievable in the set of stationary policies. In this paper, we study the existence conditions of an optimal stationary policy in a countable-state Markov decision process under the long-run average criterion. With a properly defined metric on the policy space of ergodic MDPs, the existence of an optimal stationary policy can be guaranteed by the compactness of the space and the continuity of the long-run average cost with respect to the metric. We further extend this condition by some assumptions which can be easily verified in control problems of specific systems, such as queueing systems. Our results make a complementary contribution to the literature in the sense that our method is capable to handle the cost function unbounded from both below and above, only at the condition of continuity and ergodicity. Several examples are provided to illustrate the application of our main results.

Keywords: Markov decision process, countable states, optimal stationary policy, metric space

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1 Introduction

For finite Markov decision processes (MDPs), the optimality of various types of policies are well studied. For example, it is well known that the optimal value of finite MDPs with discounted or average criteria can be achieved by Markovian and deterministic policies, thus history-dependent and randomized policies are not needed to consider. More details can be referred to books on MDPs (Bertsekas, 2012; Puterman, 1994).

Countable-state MDPs are a type of widely existing models and are particularly useful for many problems, such as queueing systems, inventory management, etc. When the state space of MDPs is changed from finite to infinite (countable), the relevant analysis becomes more complicated and the algorithms need sophisticated discussion (Golubin, 2003; Meyn, 1997). Compared with the complete theoretical results for finite MDPs, there is no comprehensive theory for infinite MDPs with countable states and the long-run average criterion. The existence of an optimal stationary policy for countable-state MDPs needs specific discussion, and attracts research attention in recent decades. Although we can restrict our attention to stationary policies in finite MDPs, this is no longer true when the state space is countable. In general, the optimal value of a countable-state MDP may not be achievable by stationary policies, even not by history-dependent policies. Interesting counterexamples can be found in the excellent books on MDPs (see Examples 5.6.1&5.6.5&5.6.6 of Bertsekas (2012), Examples 8.10.1&8.10.2 of Puterman (1994), and Subsection 7.1 of Sennott (1999)).

Since a stationary policy is not necessarily optimal for countably infinite MDPs, there are literature works on the specific existence conditions of optimal stationary policies. Sennott studies the existence conditions for average cost optimality of stationary policies for discrete-time MDPs when state space is countable and action space is finite (Sennott, 1986, 1989). In Sennott’s studies, a distinguished state is introduced and the vanishing discount optimality approach is adopted to study the optimality inequality. Borkar (1989) also studies the condition of optimal stationary policies for discrete-time average cost MDPs with countable states, but from the characterization through the dynamic programming equations. For constrained
MDPs with countable states and long-run average cost, Borkar (1994) further establishes the existence of stationary randomized policies for the general case of nonnegative cost functions (or unbounded from below), which uses the method of occupation measures. Lasserre (1988) studies the stationary policies of denumerable state MDPs for not only the average cost optimality, but also the Blackwell optimality. Meyn (1999) studies the similar problem based on the stabilization of controlled Markov chains with algorithmic analysis. Cao and Xie (2015) study the existence condition of optimal stationary policies for a class of queueing systems, also from the analysis of system stability. Cavazos-Cadena (1991); Cavazos-Cadena and Sennott (1992) give a fairly complete summary and comparison of different results on existence conditions for discrete-time average cost MDPs with countable state space and finite action sets.

For more general cases rather than countable state space, Hernández-Lerma (1991) studies the existence condition on average cost optimal stationary policies in a class of discrete-time Markov control processes with Borel spaces and unbounded costs, where the action space is assumed setwise continuity instead of a compact set. Feinberg and Lewis (2007) present sufficient conditions for the existence of an optimal stationary policy of MDPs with the average cost optimality inequalities, where the state and action space are Borel subsets of Polish spaces. The derived result is also applied to a cash balance problem with an inventory model. For continuous-time MDPs with infinite state in Polish spaces, Guo and Rieder (2006) study the existence of optimal deterministic stationary policies by using the Dynkin formula and two optimality inequalities for the average cost criterion. Some other systematic discussion on this issue can also be found in the excellent books on MDPs, see Bertsekas (2012); Hernández-Lerma and Lasserre (1996); Puterman (1994); Sennott (1999) for discrete-time MDPs and Bertsekas (2012); Guo and Hernández-Lerma (2009) for continuous-time MDPs.

In summary, most of the existing results are about the sufficient conditions, which usually require constructing a set of functions satisfying several sophisticated assumptions. Although these conditions are quite general, they may be not easy to verify and may encounter difficulty of function construction during the application to practical problems. In this paper, we study the optimality condition of stationary policies for average cost MDPs with countable states.
and finite actions available at each state. By defining a proper metric in the policy space, we study the continuity of the system’s average cost and the compactness of the policy space, and we show that such continuity and compactness can induce the existence of an optimal stationary policy. We further extend the continuity requirement by assuming some reasonable conditions on transition rates and uniform convergence of un-normalized probabilities in MDPs. Compared with the existing literature work, our result holds at a weak condition of requiring continuity and ergodicity, and it can handle the cost function unbounded from both below and above. While some general results in the literature require the cost function unbounded only from below (e.g., see (Borkar, 1994)) or ω-geometric ergodicity (e.g., see (Hernández-Lerma and Lasserre, 1999)), which partly demonstrates the advantages of our method. Moreover, our result may be easier to verify for some MDPs, especially for queueing systems. The main results of the paper are illustrated by several examples, for one of which the cost function is unbounded from above and from below, as discussed in Remark 2 at the end of Section 3.

The remainder of the paper is organized as follows. In Section 2, we derive the existence condition by studying the continuity of the average cost in a defined compact metric space of policies. In Section 3, an example of scheduling problem in queueing systems is provided to demonstrate the validation process of our existence condition of an optimal stationary policy. In Section 4, we further extend the existence condition to several reasonable assumptions which may be easy to satisfy in practical problems. Finally, we conclude the paper in Section 5.

2 The Basic Idea

In an MDP, the state space is denoted as \( S \), which is assumed to be countably infinite. Without loss of generality, we denote it as \( S = \{0, 1, \ldots\} \). Associated with every state \( i \in S \), there is a finite action set \( A(i) \). At state \( i \in S \), if action \( a \in A(i) \) is adopted, an instant cost \( f(i, a) \) will incur. Meanwhile, the system will transit to state \( j \in S \) with transition probability \( p^a(i, j) \) for discrete-time MDPs and with transition rate \( q^a(i, j) \) for continuous-time MDPs, respectively.
Let $u$ denote a (deterministic) stationary policy which is a mapping on $\mathcal{S}$ such that $u(i) \in \mathcal{A}(i)$ for all $i \in \mathcal{S}$. Let $\mathcal{U}$ denote the stationary policy space and $\mathcal{U} := \times_{i \in \mathcal{S}} \mathcal{A}(i) := \mathcal{A}(0) \times \mathcal{A}(1) \times \ldots$, with “$\times$” being the Cartesian product. Let $X(t)$ be the system state at time $t$. Under suitable conditions, the long-run average performance measure for MDPs, which does not depend on any initial state $x \in \mathcal{S}$, but depends on $u \in \mathcal{U}$, is defined as $\eta(u)$:

$$
\eta(u) := \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left\{ \sum_{t=0}^{T-1} f(X(t), u(X(t))) \middle| X(0) = x \right\},
$$

or

$$
\eta(u) := \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left\{ \int_{t=0}^{T} f(X(t), u(X(t))) dt \middle| X(0) = x \right\},
$$

for discrete-time and continuous-time ergodic MDPs, respectively, where the expectation operator $\mathbb{E}$ depends on $u \in \mathcal{U}$. However, such dependence is omitted below for notation simplicity.

The goal of optimization is to find a policy $u^*$ such that

$$
\eta(u^*) = \inf_{u \in \mathcal{U}} [\eta(u)], \quad \text{or} \quad \eta(u^*) = \sup_{u \in \mathcal{U}} [\eta(u)].
$$

Assume that $\eta(u)$ is bounded in $u \in \mathcal{U}$, so $\inf_{u \in \mathcal{U}} [\eta(u)]$ is finite. We aim to find conditions under which such an optimal stationary policy $u^*$ exists.

**Theorem 1.** Suppose $\mathcal{U}$ is a compact metric space and the function $\eta(u)$ is continuous in $\mathcal{U}$ with the metric, then an optimal policy $u^*$ exists.

**Proof:** Let $\eta^* := \inf_{u \in \mathcal{U}} [\eta(u)]$. By definition, there exists a sequence of policies, denoted as $u_0, u_1, \ldots$, such that

$$
\lim_{n \to \infty} \eta(u_n) = \eta^*.
$$

Because $\mathcal{U}$ is compact, there is a subsequence of $\{u_n, n = 0, 1, \ldots\}$ that converges to a limit (accumulation) point. Denote this subsequence as $\{u_{n_k}, k = 0, 1, \ldots\}$ and the limit point as $u^* \in \mathcal{U}$. Then

$$
\lim_{k \to \infty} u_{n_k} = u^* \in \mathcal{U}.
$$

By continuity of $\eta(u)$, we have

$$
\lim_{k \to \infty} \eta(u_{n_k}) = \eta(u^*).
$$
By (4), we obtain
\[ \eta(u^*) = \eta^* = \inf_{u \in U} [\eta(u)]; \]
i.e., \( u^* \in U \) is an optimal policy.

Theorem 1 requires a compact metric space defined for \( U \). Below, we introduce such a metric in the policy space. Note that a policy can be denoted as
\[ u = (u(0), u(1), \ldots). \]

Choosing a real number \( 0 < r < 0.5 \), (e.g., \( r = 0.1 \)), we define the distance between two policies \( u_1 = (u_1(0), u_1(1), \ldots) \) and \( u_2 = (u_2(0), u_2(1), \ldots) \) as
\[
d(u_1, u_2) := \sum_{i=0}^{\infty} ||u_1(i) - u_2(i)|| r^i,
\]
in which
\[
||u_1(i) - u_2(i)|| := \begin{cases} 1 & \text{if } u_1(i) \neq u_2(i), \\ 0 & \text{if } u_1(i) = u_2(i). \end{cases}
\]
It is easy to verify that
\[
d(u, u) = 0, \quad d(u_1, u_2) = d(u_2, u_1),
\]
and for any three policies \( u_1, u_2, \) and \( u_3 \), the following triangle inequality holds
\[
d(u_1, u_3) \leq d(u_1, u_2) + d(u_2, u_3).
\]
Thus, \( d(u_1, u_2), u_1, u_2 \in U \), indeed defines a metric on \( U \).

Suppose for two policies \( u_1 \) and \( u_2, u_1(i) = u_2(i) \) for all \( i = 0, 1, \ldots, k \). Then
\[
d(u_1, u_2) = \sum_{i=k+1}^{\infty} ||u_1(i) - u_2(i)|| r^i \leq \sum_{i=k+1}^{\infty} r^i = r^{k+1} \sum_{i=0}^{\infty} r^i = \frac{r^{k+1}}{1-r} < r^k,
\]
where the last inequality holds because we choose \( r < 0.5 \), so \( \frac{r}{1-r} < 1 \). By (6), we have

**Lemma 1.** \( d(u_1, u_2) < r^k \) if and only if \( u_1(i) = u_2(i) \) for all \( i \leq k \).
Proof: The “If” part follows directly from (6). Now we prove the “Only if” part using contradiction. Assume that there is an integer $n$ such that $u_1(n) \neq u_2(n)$ and $n \leq k$. By (5), we have $d(u_1, u_2) \geq r^n > r^k$, which is in contradiction with the condition $d(u_1, u_2) < r^k$. Thus, the assumption is not true and the “Only if” part is proved. \[\square\]

The metric defined by the distance function $d(u_1, u_2)$ induces a topology on $U$. First, we define an open ball around a point $u \in U$ as

$$O_\epsilon(u) := \{v \in U : d(u, v) < \epsilon\}, \quad \epsilon > 0.$$ (7)

We have $u \in O_\epsilon(u)$ for any $\epsilon > 0$. A set $N(u)$ is called a neighborhood of a point $u \in U$, if there is an open ball $O_\epsilon(u)$ for some $\epsilon > 0$ such that $O_\epsilon(u) \subseteq N(u)$.

By Lemma 1 we have the following fact: $u'(i) = u(i)$ for all $i \leq k$ if and only if $u' \in O_{r^k}(u)$.

Remark 1. Lemma 1 reveals the advantage of the metric (5): It shows that all the policies in a small neighborhood $O_{r^k}(u)$ of policy $u$ take the same actions in the first $k$ states. This property is very useful in proving the continuity of $\eta(u)$ in many optimization problems, in which the steady-state probability of state $i$, $\pi(i)$, goes to zero when $i$ goes to infinity; in other words, states $i > k$ are less important. \[\square\]

In a metric space $U$, a limit point can be defined by the metric, i.e., $\lim_{n \to \infty} u_n = u$ for some sequence $\{u_n\} \subseteq U$, if and only if $\lim_{n \to \infty} d(u_n, u) = 0$. In this sense, a continuous function is defined in the same way as a continuous function defined in a real space.

Since $A(i)$ is finite and $S$ is countable, it is well known that with the metric (5) the policy space $U = \times_{i \in S} A(i)$ is compact. In fact, every point $u \in U$ is an accumulation (limit) point, and every policy is in $U$. In order to apply Theorem 1 we have to prove the continuity of $\eta(u)$ in $U$ for the specific problems. Below, we use some examples to illustrate the applicability of Theorem 1 in MDPs.

Example 1. (A modification of Example 8.10.2 in Puterman’s book [Puterman, 1994]) Consider an MDP with $S = \{1, 2, \ldots\}$. At each state $i \in S$, there are two actions 1 and 0. If action 1 is taken, then the state transits from $i$ to $i + 1$ with probability 1 and the cost is $f(i, 1) = 0$; if action 0 is taken, then the state stays at $i$ with probability 1 and the cost is
f(i, 0) = \frac{1}{i}. The Markov chain (under any given policy) is denoted as X(t), t = 0, 1, \ldots. A stationary policy is denoted as a mapping u : S \to \{0, 1\}.

The performance measure for policy u = (u(1), u(2), \ldots) with initial state i is the long-run average

$$\eta(u, i) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left\{ \sum_{t=0}^{T-1} f(X(t), u(X(t))) \mid X(0) = i \right\}. \quad (8)$$

Note that the performance may depend on the initial state, i.e., it is a function of both the initial states and policies. To prove the existence of an optimal policy, we need to fix the initial state. In (8), we choose X(0) = 1. We wish to find a policy u* such that

$$\eta(u^*, 1) = \inf_{\omega \in \mathcal{U}} \{\eta(\omega, 1)\}.$$

We need to prove that such an optimal stationary policy exists.

Now, we prove that \(\eta(u, 1)\) is continuous in \(u \in \mathcal{U}\) with metric (8). Given a policy \(u_0\), for any small positive \(\epsilon\), we find the maximum \(k\) satisfying \(r^k > \epsilon\). By Lemma (a) if we choose a policy \(u\) satisfying \(d(u, u_0) < \epsilon\), then all the actions of such policies \(u\) and \(u_0\) at states \(i \leq k\) are the same. By the structure of \(\eta(u, i)\) defined in (8), we can conclude that

$$|\eta(u, 1) - \eta(u_0, 1)| < \frac{1}{k}.$$

More precisely, since \(u(i) = u_0(i)\) for all \(i \leq k\), we discuss it with two cases. Case 1: If \(u(i) = u_0(i) = 1\) for all \(i \leq k\), we have \(0 < \eta(u, 1), \eta(u', 1) < \frac{1}{k}\), thus \(|\eta(u, 1) - \eta(u_0, 1)| < \frac{1}{k}\).

Case 2: If there exists some state \(i \leq k\) such that \(u(i) = u_0(i) = 0\), we denote the smallest such state as \(i^*\) and we have \(\eta(u, 1) = \eta(u_0, 1) = \frac{1}{i^*}\), thus \(|\eta(u, 1) - \eta(u_0, 1)| = 0\). In summary, for any \(\epsilon > 0\), take \(k > 1\) such that \(\frac{1}{k} < \epsilon\), thus \(|\eta(u, 1) - \eta(u_0, 1)| < \epsilon\) for all \(u \in O_{\epsilon k}(u_0)\). Therefore, \(\eta(u, 1)\) is continuous at \(u_0\).

Finally, by Theorem (a) the optimal stationary policy exists. Actually, it is easy to verify that the optimal policy is \(u^* = (1, 1, \ldots, 1, \ldots)\) and the corresponding optimal cost is \(\eta^* = 0\).

\(\square\)

**Example 2.** (Example 8.10.2 in Puterman’s book (Puterman, 1994)) Consider an MDP with \(S = \{1, 2, \ldots\}\). At state \(i \in S\), there are two actions 1 and 0. If action 1 is taken, then the
state transits from \( i \) to \( i + 1 \) with probability \( 1 \) and the reward is \( f(i, 1) = 0 \); if action 0 is taken, then the state stays at \( i \) with probability \( 1 \) and the reward is \( f(i, 0) = 1 - \frac{1}{i} \). The Markov chain (under any policy) is denoted as \( X(t), t = 0, 1, \ldots \). A stationary policy is denoted as a mapping \( u : S \rightarrow \{0, 1\} \).

The performance measure for policy \( u = (u(1), u(2), \ldots) \) with initial state \( i \) is the long-run average reward as follows.

\[
\eta(u, i) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left\{ \sum_{t=0}^{T-1} f(X(t), u(X(t))) \bigg| X(0) = i \right\}.
\] (9)

We set the initial state always as \( X(0) = 1 \) and we wish to find a policy \( u^* \) such that

\[
\eta(u^*, 1) = \sup_{u \in \mathcal{U}} \{ \eta(u, 1) \}.
\]

The discussion is the same as Example 1 except that \( \eta(u, 1) \) is NOT continuous at \( u_0 = (1, 1, \ldots, 1, \ldots) \) with \( \eta(u_0, 1) = 0 \), while \( \eta(u, 1) \geq 1 - \frac{1}{k} \) for any neighboring policy \( u \) with \( d(u, u_0) < r^k \). Therefore, an optimal stationary policy may not exist for this example. Actually, it is easy to verify that the optimal reward of this problem is \( \eta^* = 1 \). A history-dependent policy \( u^* \) which uses action 0 \( i \) times in state \( i \), and then uses action 1 once, will yield a reward stream of \( (0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{2}{3}, \frac{2}{3}, 0, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \ldots) \). Thus, the history-dependent policy \( u^* \) can reach the optimal reward \( \eta^* = 1 \). However, any stationary deterministic policy yields possible rewards as either 0 or \( 1 - \frac{1}{i} \), which cannot reach the optimal reward \( \eta^* = 1 \). □

3 The \( c/\mu \)-Rule in Queueing Systems

In this section, we show that, with the metric space defined by (5), the basic idea presented in Section 2 can be applied to a class of optimal scheduling problems in queueing systems, called the \( c/\mu \)-rule problem, to establish the existence of an optimal stationary policy.

The problem is about the on/off scheduling control of parallel servers in a group-server queue. More details of the problem setting can be referred to (Xia et al., 2018) and we give a brief introduction as follows. Consider a group-server queue with a single infinite-size buffer.
and $K$ groups of parallel servers, as illustrated by Fig. 1. Customers are homogeneous and customer arrival is assumed as a Poisson process with rate $\lambda$. Arriving customers will go to the idle servers at status ‘on’. If all the servers at status ‘on’ are busy, the arriving customer will wait in the buffer. Servers are providing service in parallel and categorized into $K$ groups. Servers in the same group are homogeneous in service rates and cost rates, while those in different groups are heterogeneous. Group $k$ has $M_k$ servers with service rate $\mu_k$ and cost rate $c_k$ per unit of time, $k = 1, 2, \ldots, K$. Without loss of generality, we assume $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_K$. The system cost includes two parts, the operating cost of servers and the holding cost of customers. The system state $n$ is the number of customers in the system. The state space is denoted as $S = \{0, 1, 2, \ldots\}$, which is countably infinite. We can turn on or off servers dynamically to reduce the system average cost. The action is the number of working servers at each group, which is denoted as $a = (a_1, a_2, \ldots, a_K)$, where $a_k$ is the number of working servers in group $k$ and $a_k \in \{0, 1, \ldots, M_k\}$. For any state $n \geq 1$, action space $A(n)$ is a subset of $\{1, \ldots, M_1\} \times \{0, \ldots, M_2\} \times \cdots \times \{0, \ldots, M_K\}$, where $a_1 \geq 1$ is reasonable to guarantee the system ergodic. Define a stationary policy as $u := (u(0), u(1), u(2), \ldots)$, where $u(n) := (u(n, 1), u(n, 2), \ldots, u(n, K)) \in A(n)$ is the action at state $n$ and $u(n, k)$ is the number
of working servers in group $k$ at state $n$. The cost function at state $n$ under policy $u$ is

$$f(n, u) := h(n) + \sum_{k=1}^{K} c_k u(n, k), \quad (10)$$

where $h(n)$ is the holding cost rate at state $n$. The system long-run average cost under policy $u$ is defined as

$$\eta(u) = \lim_{T \to \infty} \frac{1}{T} E \left\{ \int_{t=0}^{T} f(n(t), u) dt \right\}, \quad (11)$$

where $n(t)$ is the system state at time $t$. The optimal average cost is $\eta^* = \inf_u \eta(u)$. We aim at finding the optimal stationary policy $u^*$ which achieves the optimal average cost, i.e., $\eta(u^*) = \eta^*$, where $u^* \in \mathcal{U}$ and $\mathcal{U}$ is the stationary policy space. In (Xia et al., 2018), it is shown that the optimal policies (if one exists) follow the so called $c/\mu$-rule: Servers in the group with smaller values of $c/\mu$ should be turned on with higher priority. Here, we want to verify that an optimal stationary policy does exist for this problem with countable states.

It is natural to assume that the holding cost $h(n)$ is increasing in $n$; and thus, under optimal policies the queue should be ergodic. So we assume that $\{n(t)\}$ is ergodic (under each policy in $\mathcal{U}$) with a unique steady-state distribution $\pi(n, u)$, $n = 0, 1, \ldots$, $u \in \mathcal{U}$, and the long-run average (11) does not depend on the initial state.

Since our queue is a birth-death process, we can derive the steady-state distribution as below.

$$\pi(n, u) = \frac{1}{1 + G(u)} \prod_{l=1}^{n} \frac{\lambda}{u(l)\mu}, \quad n \geq 1, \quad (12)$$

where $\mu = (\mu_1, \ldots, \mu_K)^T$, and

$$u(l)\mu := \sum_{k=1}^{K} u(l, k)\mu_k, \quad (13)$$

and

$$G(u) := \sum_{n=1}^{\infty} \prod_{l=1}^{n} \frac{\lambda}{u(l)\mu} \quad (14)$$

The queue is stable if and only if $G(u) < \infty$ which also indicates

$$\lim_{n \to \infty} \sum_{m=n}^{\infty} \prod_{l=1}^{m} \frac{\lambda}{u(l)\mu} = 0. \quad (15)$$
The ergodicity of the system under a policy \( u \) can indicate a necessary condition: \( u(n)\mu \neq 0 \) for all \( n \geq 1 \). The stability of the system can be guaranteed by a sufficient condition: there exists an \( \bar{n} \) such that \( u(n)\mu > \lambda \) for all \( n > \bar{n} \).

For an ergodic policy \( u \in \mathcal{U} \), under suitable condition, the long-run average (11) equals

\[
\eta(u) = \sum_{n=0}^{\infty} \pi(n, u)f(n, u). \tag{16}
\]

For the analysis here, we need to make the following assumption:

**Assumption 1.** The normalizing factor \( G(u) \) in (14) (equivalently, the limit in (15)) and the performance limit (16) converge uniformly in \( \mathcal{U} \).

We use the metric definition (3) to quantify the distance between any two policies \( u_1 \) and \( u_2 \). In what follows, we will prove that when the two policies \( u \) and \( u' \) are infinitely close, their performance measures \( \eta(u) \) and \( \eta(u') \) are also infinitely close to each other. Denote the two policies by \( u = (u(0), u(1), \cdots, u(n), \cdots) \) and \( u' = (u'(0), u'(1), \cdots, u'(n), \cdots) \). By Lemma 1, we assume that

\[
u(l) = u'(l), \quad \text{for } l = 0, 1, \cdots, n. \tag{17}\]

which means that \( u' \in \mathcal{O}_{r*}(u) \).

First, we compare the difference of the normalization factors \( 1 + G(u) \) and \( 1 + G(u') \) of these two policies. We have

\[
\frac{1 + G(u)}{1 + G(u')} = \frac{1 + \sum_{m=1}^{\infty} \prod_{l=1}^{m} \frac{\lambda}{u(l)\mu}}{1 + \sum_{m=1}^{\infty} \prod_{l=1}^{m} \frac{\lambda}{u'(l)\mu}}
\]

\[
= \left( \frac{1 + \sum_{m=1}^{n} \prod_{l=1}^{m} \frac{\lambda}{u(l)\mu}}{1 + \sum_{m=1}^{n} \prod_{l=1}^{m} \frac{\lambda}{u'(l)\mu}} \right) + \sum_{m=n+1}^{\infty} \frac{\prod_{l=1}^{m} \frac{\lambda}{u(l)\mu}}{1 + \sum_{m=1}^{\infty} \prod_{l=1}^{m} \frac{\lambda}{u'(l)\mu}}
\]

\[
= \frac{1 + \sum_{m=n+1}^{\infty} \prod_{l=1}^{m} \frac{\lambda}{u(l)\mu}}{1 + \sum_{m=1}^{n} \prod_{l=1}^{m} \frac{\lambda}{u'(l)\mu}} < 1 + \frac{\sum_{m=n+1}^{\infty} \prod_{l=1}^{m} \frac{\lambda}{u(l)\mu}}{1 + \sum_{m=1}^{n} \prod_{l=1}^{m} \frac{\lambda}{u'(l)\mu}} < 1 + \delta(n, u), \tag{18}\]
where
\[ \delta(n, u) := \sum_{m=n+1}^{\infty} \prod_{l=1}^{m} \frac{\lambda}{u(l)\mu}. \]  

(19)

Similarly, we can also have

\[
\frac{1 + G(u)}{1 + G(u')} > \frac{1 + \sum_{m=n+1}^{\infty} \prod_{l=1}^{m} \frac{\lambda}{u(l)\mu}}{1 + \sum_{m=n+1}^{\infty} \prod_{l=1}^{m} \frac{\lambda}{u'(l)\mu}} > 1 - \sum_{m=n+1}^{\infty} \prod_{l=1}^{m} \frac{\lambda}{u'(l)\mu} = 1 - \delta(n, u'), 
\]

(20)

where

\[ \delta(n, u') := \sum_{m=n+1}^{\infty} \prod_{l=1}^{m} \frac{\lambda}{u'(l)\mu}. \]  

(21)

Therefore, we have

\[ 1 - \delta(n, u') < \frac{1 + G(u)}{1 + G(u')} < 1 + \delta(n, u). \]  

(22)

Let \( \sigma(n, u, u') \) be determined by

\[ \frac{1 + G(u)}{1 + G(u')} = 1 + \sigma(n, u, u'). \]  

(23)

Then,

\[ -\delta(n, u') < \sigma(n, u, u') < \delta(n, u). \]  

(24)

With (12), (17), and (23), the steady-state distributions under these two policies \( u \) and \( u' \) have the following relation.

\[ \pi(m, u') = (1 + \sigma(n, u, u'))\pi(m, u), \quad m = 0, 1, \ldots, n. \]  

(25)

Next, we study the difference between the associated long-run average costs \( \eta \) under policies \( u \) and \( u' \). The cost functions are denoted by \( f(m, u) \) and \( f(m, u') \), respectively. By (10) and
\[ \eta(u') - \eta(u) = \sum_{m=0}^{\infty} [\pi(m, u') f(m, u') - \pi(m, u) f(m, u)] + \sum_{m=n+1}^{\infty} [\pi(m, u') f(m, u') - \pi(m, u) f(m, u)]. \]

Applying (25), we have
\[ \eta(u') - \eta(u) = \sigma(n, u, u') \sum_{m=0}^{n} \pi(m, u) f(m, u) + \sum_{m=n+1}^{\infty} [\pi(m, u') f(m, u') - \pi(m, u) f(m, u)]. \] (26)

Now we are ready to prove the continuity of \( \eta(u) \) in the metric space \( \mathcal{U} \) with metric (5).

With (15), we have
\[ \lim_{n \to \infty} \delta(n, u) = 0, \quad \lim_{n \to \infty} \delta(n, u') = 0. \]

Let \( \epsilon > 0 \) be any small number. Under Assumption 1 by the uniformity of \( G(u) \) in (14) and (15), there exists a large integer \( N_1 \) such that if \( n > N_1 \), we have \( \delta(n, u) < \epsilon \) for any \( u \in \mathcal{U} \).

By (24), we have
\[ |\sigma(n, u, u')| < \epsilon, \quad \forall u, u' \in \mathcal{U}. \]

Next, because (16) converges, there is a large integer \( N_2 \) such that
\[ \left| \sum_{m=0}^{n} \pi(m, u) f(m, u) \right| < |\eta(u)| + 1, \quad \forall n > N_2. \]

Furthermore, under Assumption 1 by the uniformity of the convergence of (16), there is a large integer \( N_3 \) such that
\[ \left| \sum_{m=n+1}^{\infty} [\pi(m, u') f(m, u') - \pi(m, u) f(m, u)] \right| < 2\epsilon, \quad \forall n > N_3 \text{ and } u, u' \in \mathcal{U}. \]

Finally, let \( N^* := \max\{N_1, N_2, N_3\} \). Then, by (26) and Lemma 1, we have
\[ |\eta(u) - \eta(u')| \leq |\sigma(n, u, u')| \left| \sum_{m=0}^{n} \pi(m, u) f(m, u) \right| + \left| \sum_{m=n+1}^{\infty} [\pi(m, u') f(m, u') - \pi(m, u) f(m, u)] \right| \]
\[ \leq (|\eta(u)| + 3)\epsilon, \quad \text{for all } u' \in O_{\epsilon, N^*}(u). \] (27)
Since $\eta(u)$ is bounded, we conclude that $\eta(u)$ is continuous at $u$ in the metric space. Therefore, the existence of optimal stationary policy $u^*$ for this $c/\mu$-rule problem directly follows by Theorem $\blacksquare$.

\textbf{Remark 2.} The condition of uniform convergence in Assumption $\blacksquare$ is easy to validate in queueing systems. For example, we can set the condition for the control of our group-server queues as follows: 1 there exists a constant $\tilde{n}$ such that for any $n > \tilde{n}$, every feasible action $u(n) \in A(n)$ always satisfies $u(n)\mu > \lambda$. Therefore, we define $\rho_0 := \max_{u(n) \in A(n), n > \tilde{n}} \left\{ \frac{\lambda}{u(n)\mu} \right\} < 1$. We directly have $G(u) \leq \sum_{n=0}^{\tilde{n}} \prod_{i=1}^{n} \frac{\lambda}{u(i)\mu} + \sum_{n=\tilde{n}+1}^{\infty} \rho_0^n < \infty$, which indicates that the queueing system is stable and the normalizing factor $G(u)$ in (14) converges uniformly in $u \in U$. Compared with (12), we further define a pseudo probability $\tilde{\pi}(n, u) := \frac{1}{1 + G(u)} \rho_0^n$. Obviously, we always have $\tilde{\pi}(n, u) \geq \pi(n, u)$ for any policy $u$ and $n > \tilde{n}$. Thus, for the performance limit (16), we have $|\eta(u)| \leq \sum_{n=0}^{\tilde{n}} \pi(n, u)|f(n, u)| + \sum_{n=\tilde{n}+1}^{\infty} \tilde{\pi}(n, u)|f(n, u)| = \sum_{n=0}^{\tilde{n}} \pi(n, u)|f(n, u)| + \frac{1}{1 + G(u)} \sum_{n=\tilde{n}+1}^{\infty} \rho_0^n |f(n, u)|$, where the first part is always finite and we only need to guarantee the second part bounded. Thus, any cost function $|f(n, u)|$ polynomially increasing to infinity along with $n$ will be controlled by the exponential factor $\rho_0^n$. Therefore, with 1 and 2, we can easily validate Assumption $\blacksquare$ that $G(u)$ and $\eta(u)$ converge uniformly, and thus an optimal stationary policy exists. More specifically, for the cost function (10), we have $f(n, u) = h(n) + \sum_{k=1}^{K} c_k u(n, k)$, where the operating cost $\sum_{k=1}^{K} c_k u(n, k)$ is obviously bounded and the holding cost $h(n)$ can be unbounded. From the above analysis, we can see that $f(n, u)$ can be unbounded from both below and above sides. For example, we can set $h(n) = (-1)^n \cdot n$, which is unbounded both below and above while satisfies our condition 2. However, this kind of cost function may not be handled by other methods in the literature (Borkar, 1994) because the cost function thereof is required to be unbounded from below. This is also one of the advantages of our method in this paper.

We have demonstrated the applicability of Theorem $\blacksquare$ for proving the existence of optimal stationary policies in a scheduling problem of queueing systems. In the next section, we further show that this approach also applies to more general cases.
4 More General Cases

In general, we consider a continuous-time MDP with a countable state space denoted as $\mathcal{S} = \{0, 1, \ldots \}$. Let $\pi(i, u)$ be the steady-state probability of state $i \in \mathcal{S}$ under given policy $u \in \mathcal{U}$, and $q^a(i, j)$ be the transition rate from state $i$ to $j$ under action $a \in \mathcal{A}(i)$, $i, j \in \mathcal{S}$. Obviously, we have $q^a(i, j) \geq 0$ for $i \neq j$ and $q^a(i, i) = -\sum_{j \in \mathcal{S}; j \neq i} q^a(i, j) \leq 0$, where $|q^a(i, i)|$ can be understood as the total rates transiting out from state $i$ if action $a$ is adopted. Then we know that the steady-state probabilities $\pi(i, u)$’s must satisfy the following equations.

$$\sum_{j=0}^{\infty} \pi(j, u)q^{u(j)}(j, i) = 0, \quad i \in \mathcal{S},$$  \hspace{1cm} (28)

$$\sum_{i=0}^{\infty} \pi(i, u) = 1,$$  \hspace{1cm} (29)

where (29) is called a normalization equation. Given a policy $u \in \mathcal{U}$, any sequence $\nu(i, u) \geq 0$ (depending on $u$), $i \in \mathcal{S}$, that satisfies

$$\sum_{j=0}^{\infty} \nu(j, u)q^{u(j)}(j, i) = 0, \quad \forall i \in \mathcal{S} \quad \text{and} \quad \sum_{i=0}^{\infty} \nu(i, u) < \infty,$$  \hspace{1cm} (30)

is called an un-normalized steady-state vector. From (30), we have

$$\pi(i, u) = \frac{\nu(i, u)}{\sum_{i=0}^{\infty} \nu(i, u)}, \quad i \in \mathcal{S},$$

is the steady-state probability.

In the rest of the paper, it is more convenient to deal with the un-normalized vector because it does not contain the denominator. Moreover, it is convenient to set $\nu(0, u) = 1$ to obtain an un-normalized probability.

First, we make the following assumptions to simplify the problem setting.

Assumption 2. (a) $q^a(i, j)$ is bounded, i.e., $|q^a(i, j)| < \Lambda$, for all $i, j \in \mathcal{S}, a \in \mathcal{A}(i)$.

(b) There is an integer $M > 0$ such that $q^a(j, i) = 0$, for all $j > i + M$, $i \in \mathcal{S}$ and $a \in \mathcal{A}(i)$.

Assumption 2(a) indicates that the transition rate from any state $i$ has an upper bound $\Lambda$, which is reasonable for most cases in practice. Assumption 2(b) means that the transition
rate from state $j$ back to $i$ is 0 if state $j$ is far away from state $i$. This assumption is also reasonable in many practical systems, especially it is usually true for queueing systems since state $j$ always transits back only to state $j - 1$ caused by a service completion event.

Given any $u \in \mathcal{U}$, at a state $i \in \mathcal{S}$, we may take an action denoted by $u(i)$, which determines the value of $q^{u(i)}(i, j)$, $j \in \mathcal{S}$. Then $u := (u(0), u(1), \ldots)$ denotes a policy. Let $\mathcal{U}$ be the space of all policies. The steady-state probability at state $i$ is denoted by $\pi(i, u)$, which depends on policy $u$. The reward or cost function at state $i$ with action $u(i)$ is denoted by $f(i, u(i))$. We assume that the Markov processes under all policies in $\mathcal{U}$ are ergodic and the long-run average performance under policy $u$ is

$$\eta(u) := \sum_{i=0}^{\infty} \pi(i, u)f(i, u(i)).$$

(31)

Denoting $\nu(i, u)$ as the un-normalized steady-state vector satisfying (30) under policy $u$, we give one more assumption as follows (cf. Assumption 1).

**Assumption 3.** $\sum_{i=0}^{N} \nu(i, u)$, with $\nu(0, u) = 1$, converges uniformly in $\mathcal{U}$ as $N \to \infty$, and $\sum_{i=0}^{N} \pi(i, u)f(i, u(i))$ converges uniformly in $\mathcal{U}$, as $N \to \infty$.

Assumption 3 holds for many Markov systems, especially when the system is stable under the neighborhood of policies. In fact, it holds if there is a sequence, denoted as $\overline{\nu}(i)$, $i = 0, 1, \ldots$, such that $\nu(i, u) \leq \overline{\nu}(i)$ and $\sum_{i=0}^{\infty} \overline{\nu}(i) < \infty$.

**Example 3.** Consider a controlled $M/M/1$ queue with arrival rate $\lambda(i, u)$ and service rate $\mu(i, u)$ (under a given control policy $u$) when the number of customers is $i$, $i \in \mathcal{S} = \{0, 1, \ldots, \}$. Let $X(t) \in \mathcal{S}$ be the Markov process of the queue. The un-normalized steady-state vector is $\nu(i, u) = \prod_{l=0}^{i} \frac{\lambda(l, u)}{\mu(l, u)}$. The process is stable if

$$\sum_{i=0}^{\infty} \nu(i, u) = \sum_{i=0}^{\infty} \prod_{l=0}^{i} \frac{\lambda(l, u)}{\mu(l, u)} < \infty.$$

Therefore, Assumption 3 is the same as Assumption 1, and if there is a bound $\overline{\nu} < 1$ and state $i^*$ such that $\frac{\lambda(i, u)}{\mu(i, u)} < \overline{\nu}$ for all policies $u$ and states $i \geq i^*$, then Assumption 3 holds. $\square$
Now, let us understand the role of Assumptions 2 and 3. For any integer \( N > 0 \), we consider the first \( K \) equations in (30), where \( K > N \). Given any \( u \in \mathcal{U} \), by Assumption 2(b), the summation in (30) is over only finitely many states, resulting in

\[
\sum_{j=0}^{i+M} \nu(j, u) q_u^{u(j)}(j, i) = 0, \quad i = 0, 1, \ldots, K, \tag{32}
\]

which can be further rewritten as

\[
\sum_{j=0}^{K} \nu(j, u) q_u^{u(j)}(j, i) + \sum_{j=K+1}^{i+M} \nu(j, u) q_u^{u(j)}(j, i) = 0, \quad i = 0, 1, \ldots, K. \tag{33}
\]

For (33), the last summation is nonzero only if \( i + M > K \). Thus, only the last \( M \) equations in (33) contain nonzero terms of the last summation, whose values are small enough to be ignored, as shown by the following analysis.

For any \( \epsilon > 0 \) and \( N > 0 \), by Assumption 3, there is a large enough \( K \) such that

\[
\sum_{i=K+1}^{\infty} \nu(i, u) < \frac{\epsilon}{N}, \quad \text{for all } u \in \mathcal{U}. \tag{34}
\]

By Assumption 2, the last summation of (33) can be written as

\[
\sum_{j=K+1}^{i+M} \nu(j, u) q_u^{u(j)}(j, i) < \sum_{j=K+1}^{\infty} \nu(j, u) q_u^{u(j)}(j, i) < \frac{\epsilon}{N} \Lambda = O\left(\frac{\epsilon}{N}\right), \quad \text{for all } u \in \mathcal{U}. \tag{35}
\]

Substituting the above result into (33), we see that solving (33) becomes solving the following equations

\[
0 = \sum_{j=0}^{K} \nu(j, u) q_u^{u(j)}(j, i), \quad i = 0, 1, \ldots, K - M, \\
0 = \sum_{j=0}^{K} \nu(j, u) q_u^{u(j)}(j, i) + O\left(\frac{\epsilon}{N}\right), \quad i = K - M + 1, \ldots, K, \tag{36}
\]

where we have \( K + 1 \) variables and \( K + 1 \) linear equations. Thus, the variables \( \nu(i, u) \)'s can be solved and we state the results as (37) in the following lemma, where \( F_i(q_u^{u(j)}(j, k); j, k = 0, 1, \ldots, K) \) denotes a function \( F_i(\cdot) \) with variables \( q_u^{u(j)}(j, k), i = 0, 1, \ldots, K \).
Lemma 2. Under Assumptions 2 and 3 for any policy $u \in U$, integer $N > 0$, and small number $\epsilon > 0$, there exists an integer $K > 0$ such that

$$\nu(i, u) = F_i(q^{u(j)}(j, k); j, k = 0, 1, \ldots, K) + \kappa_i(N), \quad i = 0, 1, \ldots, N, \quad (37)$$

and $\kappa_i(N) < \frac{\epsilon}{N}$. In words, we say that roughly for any finite $N$, $\nu(0, u), \ldots, \nu(N, u)$ depend only on the transition rates among finitely many states. The functions $F_i$, $i = 0, 1, \ldots, N$, are the same for any policy $u' \in O_{\epsilon, K}(u)$.

Note that we can set $\nu(0, u) = 1$ for solving (36) since $c\nu$ is also a solution to (36) for any feasible solution $\nu$, where $c$ is a constant. Moreover, ignoring the term of $O(\frac{\epsilon}{N})$, (36) is a set of linear equations determined by the values of $\{q^{u(j)}(j, k); j, k = 0, 1, \ldots, K\}$. Therefore, for any two policies $u'$ and $u$ such that $u'(i) = u(i)$ for all $0 \leq i \leq K$, $F_i$’s take the same form for such policies, $i = 0, 1, \ldots, K$.

With Assumptions 2 and 3, we can further extend the existence condition of optimal stationary policies in Theorem 1 and derive the following theorem.

Theorem 2. Under Assumptions 2 and 3 there exists an optimal stationary policy for the average cost MDP with a countable state space.

Proof: Let $N > 0$ be any integer and $\epsilon > 0$ be any small number. Consider any two policies $u$ and $u'$, which determine the corresponding transition rates $q(i, j) := q^{u(i)}(i, j)$ and $q'(i, j) := q^{u'(i)}(i, j)$, as well as the steady-state vectors $\nu(i)$ and $\nu'(i)$, respectively. By Lemma 2 and Assumption 3 if $K$ is large enough, then we have

$$\nu'(i) = F_i(q'(j, k); j, k = 0, 1, \ldots, K) + \kappa_i'(N), \quad i = 0, 1, \ldots, N,$$

and

$$\nu(i) = F_i(q(j, k); j, k = 0, 1, \ldots, K) + \kappa_i(N), \quad i = 0, 1, \ldots, N,$$

where $\kappa_i'(N) < \frac{\epsilon}{N}$ and $\kappa_i(N) < \frac{\epsilon}{N}$.

1In fact, this equation holds for $i = 0, 1, \ldots, K$, $K > N$, but to prove Theorem 2 we only need it for the first $N \nu(i, u)$’s.
By Lemma [1] if \( u \) and \( u' \) are close enough such that \( d(u, u') < r^K \), then \( u(i) = u'(i) \) for all \( i < K \). This means \( q(i, j) = q'(i, j) \) for all \( i < K \) and \( j = 0, 1, \ldots \). Therefore, we have
\[
\nu'(i) = \nu(i) + \kappa_i'(N) - \kappa_i(N), \quad i = 0, 1, \ldots, N. \tag{38}
\]

The rest analysis is similar to (18)–(25). First, we have
\[
\pi(i) = \frac{\nu(i)}{\sum_{j=0}^{\infty} \nu(j)},
\]
and
\[
\pi'(i) = \frac{\nu'(i)}{\sum_{j=0}^{\infty} \nu'(j)} = \frac{\sum_{j=0}^{\infty} \nu(j)}{\sum_{j=0}^{\infty} \nu'(j)} \left\{ \frac{\nu(i) + \kappa_i(N) - \kappa_i'(N)}{\sum_{j=0}^{\infty} \nu(j)} \right\}, \quad i = 0, 1, \ldots, N. \tag{39}
\]

With (38), we have
\[
\frac{\sum_{j=0}^{\infty} \nu(j)}{\sum_{j=0}^{\infty} \nu'(j)} = \frac{\sum_{j=0}^{N} \nu(j) + \sum_{j=N+1}^{\infty} \nu(j)}{\sum_{j=0}^{N} \nu'(j) + \sum_{j=N+1}^{\infty} \nu'(j)} = \frac{\sum_{j=0}^{N} \nu(j) + \sum_{j=N+1}^{\infty} \nu(j)}{\sum_{j=0}^{N} \nu'(j) + \sum_{j=N+1}^{\infty} \nu'(j) + \sum_{j=0}^{N} [\kappa_i'(N) - \kappa_i(N)]}.
\]

If \( K \) is large enough (i.e., \( d(u, u') \) is small enough), it holds
\[
\left| \sum_{j=0}^{N} [\kappa_i'(N) - \kappa_i(N)] \right| < 2\epsilon.
\]

Therefore,
\[
\frac{\sum_{j=0}^{\infty} \nu(j)}{\sum_{j=0}^{\infty} \nu'(j)} = \frac{1 + \sum_{j=N+1}^{\infty} \nu(j)}{\sum_{j=0}^{N} \nu(j) + \sum_{j=N+1}^{\infty} \nu'(j)} + \frac{\sum_{j=0}^{N} \nu'(j)}{\sum_{j=0}^{N} \nu'(j)}\left\{ \frac{\nu(i) + \kappa_i(N) - \kappa_i'(N)}{\sum_{j=0}^{\infty} \nu(j)} \right\} = 1 + \frac{\sum_{j=N+1}^{\infty} \nu'(j)}{\sum_{j=0}^{N} \nu'(j)} + \epsilon(N, u, u'),
\]

with \( |\epsilon(N, u, u')| := \left| \sum_{j=0}^{N} [\kappa_i'(N) - \kappa_i(N)] \right| < \left| \sum_{j=0}^{N} [\kappa_i'(N) - \kappa_i(N)] / \sum_{j=0}^{N} \nu(j) \right| = 2\epsilon \), where we use the preset condition \( \nu(0) = 1 \).
The rest proof follows the same procedure as (20)–(25). First, as in (25), we can derive
\[ \pi'(i) = (1 + \sigma(N, u, u'))\pi(i), \quad i = 1, 2, \ldots, N, \tag{40} \]
where \(|\sigma(N, u, u')| < 3\epsilon\) (with a large \(N\) such that \(\sum_{i=N+1}^{\infty} \nu(i) < \epsilon\), when \(K\) is large enough.

Then, similar to (26), we have
\[
\begin{align*}
\eta(u') - \eta(u) & = \sigma(N, u, u') \sum_{m=0}^{N} \pi(i, u)f(i, u(i)) + \sum_{i=N+1}^{\infty} [\pi(i, u')f(m, u'(i)) - \pi(i, u)f(i, u(i))] \tag{41} \\
& \leq 3|\eta(u)| + 5\epsilon.
\end{align*}
\]

Now we are ready to prove the continuity of \(\eta(u)\) in the metric space \(U\) with metric \((5)\).

Let \(\epsilon > 0\) be any small number. First, as discussed above, under Assumptions 2 and 3, by the uniformity of \(\sum_{i=0}^{\infty} \nu(i)\), there is a large integer \(N_1\) such that if \(n > N_1\), we have \(|\sigma(N_1, u, u')| < 3\epsilon\) for any \(u\) and \(u'\). Next, because \(\sum_{i=0}^{\infty} \pi(i, u)f(i, u(i))\) converges, there is an \(N_2\) such that \(|\sum_{i=0}^{N} \pi(i, u)f(i, u(i))| < |\eta(u)| + 1\), for all \(N > N_2\). Furthermore, under Assumption 3 by the uniformity of the convergence of (31), there is a large \(N_3\) such that for all \(n > N_3\), it holds
\[
\left| \sum_{i=n+1}^{\infty} [\pi(i, u')f(i, u') - \pi(i, u)f(i, u)] \right| < 2\epsilon.
\]

Therefore, by (11) and Lemma 11 for \(\hat{N} := \max\{N_1, N_2, N_3\}\), we have
\[
|\eta(u) - \eta(u')| < [3|\eta(u)| + 5]\epsilon, \quad \forall \ u' \in O_{r, \delta}(u). \tag{42}
\]

Thus, \(\eta(u)\) is continuous at \(u\) in the metric space, and then the existence of optimal stationary policy \(u^*\) follows from Theorem 11.

In summary, we have extended the existence condition of optimal stationary policies for average MDPs with countable state space from Theorem 11 for the \(c/\mu\)-rule problem to Theorem 12 for the more general case. As stated by Assumptions 2 and 3 if the system has bounded and limited-distance backward transition rates, and with the uniformity of the convergence of the un-normalized probabilities and the performance sequences, the existence of optimal stationary policies can be guaranteed by Theorem 22. The theorem may be easily verified in practice, especially for queueing systems, as demonstrated in the aforementioned examples.
5 Conclusion

In this paper, we derive the existence conditions of optimal stationary policies for countable state MDPs with long-run average criterion. By defining a suitable metric on the policy space forming a compact metric space, the existence condition can be guaranteed by proving the continuity of the long-run average cost as a function in the policy space under the metric. With some assumptions on the transition rates and the uniformity of the convergence of the un-normalized probabilities of the processes, the existence of the optimal policies can be proved for the MDPs with countable states in a general form. Compared with other conditions studied in the literature, the condition in this paper may be easier to verify when applied to practical MDP problems, especially in queueing systems. Some examples are studied to illustrate the applicability of our results. Future research topics may include the extensions to MDPs with other criteria, such as the discounted ones.

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