THE NOTION OF CATEGORY OVER AN ALGEBRAIC STACK

DENNIS GAITSGORY

Abstract. The goal of this note is to spell out the (apparently well-known and intuitively clear) notion of an abelian category over a stack. In the future we will discuss the (much less evident) notion, when instead of an abelian category one considers a triangulated one.

1. Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category. We will assume that $\mathcal{C}$ is closed under inductive limits, i.e., that the tautological embedding $\mathcal{C} \to \text{Ind}(\mathcal{C})$ admits a right adjoint $\text{limInd} : \text{Ind}(\mathcal{C}) \to \mathcal{C}$, and that the latter functor is exact. In particular, it makes sense to tensor objects of $\mathcal{C}$ by vector spaces.

2. The affine case. Let $A$ be a commutative algebra. We say that $\mathcal{C}$ is $A$-linear if we are given a map $A \to Z(\mathcal{C})$, i.e., if $A$ acts functorially on every Hom$(X,Y)$ for $X,Y \in \mathcal{C}$. We shall also say that in this case $\mathcal{C}$ "lives over $S = \text{Spec}(A)$".

We claim that we have a well-defined functor of tensor product

$$M,X \mapsto M \otimes_A X : A\text{-mod} \times \mathcal{C} \to \mathcal{C} :$$

If $M = A^I$ for some index set $I$, then $M \otimes_A X := X^I$, and if $M = \text{coker}(A^I \to A^J)$, then

$$M \otimes_A X := \text{coker}(X^I \to X^J),$$

where the $(i,j)$-entry of the matrix $X^I \to X^J$ is given by the action of the $(i,j)$-entry of the matrix $A^I \to A^J$.

Lemma 3. The above definition is independent of the presentation of $M$ as a quotient.

By construction, the functor of tensor product commutes with inductive limits with respect to both $M$ and $X$, and is right-exact. In addition, we have:

Lemma 4.

(a) If $M$ is flat, then the functor $X \mapsto M \otimes_A X$ is exact.

(b) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of $A$-modules with $M_3$ flat, then the sequence

$$0 \to M_1 \otimes_A X \to M_2 \otimes_A X \to M_3 \otimes_A X \to 0$$

is also short exact.

(c) If $M$ is projective and finitely generated, and $M^\vee$ is the dual module, then the above functor admits left and right adjoints, both given by $X \mapsto M^\vee \otimes_A X$.

\footnote{In what follows by an inductive limit we will mean a limit taken over a small filtering category}
Proof. First, let us note that if $M$ is a projective module given by an idempotent of $A$ for some set $I$, then $M \otimes_A X$ is given by the corresponding idempotent of $M^I$. This implies that the functor of tensor product with a projective $A$-module is exact. This implies point (a), since every flat $A$-module can be represented as an inductive limit of projective ones.

Similarly, for point (b) we can assume that $M_3$ is projective, in which case the short exact sequence splits and the assertion is obvious.

Point (c) is immediate, since we have the adjunctions maps

$$X \to M^Y \otimes_A (M \otimes_A X) \simeq M \otimes_A (M^Y \otimes X)$$

and

$$M^Y \otimes_A (M \otimes_A X) \simeq M \otimes_A (M^Y \otimes X) \to X$$

that satisfy the necessary conditions.

Finally, we have:

**Proposition 5.** Assume that $A'$ is a faithfully-flat algebra over $A$. Then $A' \otimes_A X \neq 0$ if $X \neq 0$.

**Proof.** (Drinfeld)

**Lemma 6.** If $A'$ is a faithfully flat algebra over $A$, then the quotient $A'/A$ is $A$-flat.

Clearly, the lemma implies Proposition 5, by Lemma 4(b).

**Proof.** (of the Lemma)

It is enough to show that $A'/A \otimes_A A'$ is $A'$-flat. But

$$A'/A \otimes_A A' \simeq \text{coker}(A' \to A' \otimes_A A'),$$

and the latter is a split injection.

We shall say that $X \in \mathcal{C}$ is flat over $A$ (or $S$) if the functor $M \mapsto M \otimes_A X : A\text{-mod} \to \mathcal{C}$ is exact.

**7. Change of rings.** Let $f : \text{Spec}(A') = S' \to S = \text{Spec}(A)$ is a morphism of affine schemes, corresponding to a homomorphism of algebras $A \to A'$. There exists a universal $A'$-linear category $\mathcal{C}'$, which admits an $A$-linear functor $\mathcal{C} \to \mathcal{C}'$. We will denote this category by $\mathcal{C} \times_S S'$, and it is constructed as follows:

Objects of $\mathcal{C}'$ are objects $X \in \mathcal{C}$, endowed with an additional action of $A'$, such that the two actions of $A$ (one coming from $A \to A'$, and another from $A \to \text{End}(M)$, coincide. Morphisms in $\mathcal{C}'$ are arrows $X_1 \to X_2$ in $\mathcal{C}$ that commute with the the $A'$-action.

The functor $\mathcal{C} \to \mathcal{C}'$ is given by $X \mapsto A' \otimes_A X$, and it will be denoted by $f^*$. This functor is the left adjoint to the forgetful functor $f_* : \mathcal{C}' \to \mathcal{C}$.

Set $S'' = S' \times_S S'$, and let $\mathcal{C}''$ denote the corresponding base-changed category over $S''$. One naturally defines the category of descent data on $\mathcal{C}'$ with respect to $S''$. We will denote it by $\text{Desc}_{S''}(\mathcal{C}')$, and we have a natural functor $\mathcal{C} \to \text{Desc}_{S''}(\mathcal{C}')$.

**Proposition 8.** Suppose that $S'$ is faithfully-flat over $S$. Then $\mathcal{C} \to \text{Desc}_{S''}(\mathcal{C}')$ is an equivalence.
Proof. This is proved by the usual argument, using Lemma 5.

9. Stacks: approach I. Let \( Y \) be a stack (algebraic in the faithfully flat sense), for which the diagonal morphism \( Y \to Y \times Y \) is affine. This is equivalent to demanding that any morphism \( S \to Y \), with \( S \) an affine scheme, is affine. We are going to introduce the notion of sheaf of abelian categories over \( Y \). In particular, we will obtain a notion of category over a separated scheme.

Let \( \text{Sch}^{aff}_Y \) be the category of affine schemes over \( Y \), endowed with the faithfully flat topology. A sheaf of categories \( \mathcal{C}^{sh} \) over \( Y \) is the following data:

- For each \( S = \text{Spec}(A) \in \text{Sch}^{aff}_Y \), a category \( \mathcal{C}_S \) over \( S \).
- For \( f : S_2 \to S_1 \in \text{Sch}^{aff}_Y \), an \( S_1 \)-linear functor \( f^* : \mathcal{C}_{S_1} \to \mathcal{C}_{S_2} \), which induces an equivalence \( \mathcal{C}_{S_1} \times S_2 \to \mathcal{C}_{S_2} \).  
- For two morphisms \( S_3 \to S_2 \to S_1 \in \text{Sch}^{aff}_Y \) an isomorphism of functors \( g^* \circ f^* \sim (f \circ g)^* \), such that the natural compatibility axiom for 3-fold compositions holds.

Given a sheaf of categories \( \mathcal{C}^{sh} \) over \( Y \) one can form a single category, denoted \( \Gamma(Y, \mathcal{C}^{sh}) \) or \( \mathcal{C}_Y \) (or simply \( \mathcal{C} \), where no confusion is likely to occur) as follows:

Let \( S \to Y \) be a faithfully flat cover. We define the category \( \mathcal{C}_Y \) to be the category of descent data of \( \mathcal{C}_S \) with respect to the two maps \( S \times S \to S \). Proposition 8 insures that \( \mathcal{C}_Y \) is well-defined, i.e., is canonically independent of the choice of the cover \( S \).

Again, by Proposition 8, we have the natural functor \( X \mapsto X_S : \mathcal{C}_Y \to \mathcal{C}_S \) for any \( S \in \text{Sch}^{aff}_Y \), and for \( f : S_2 \to S_1 \) a functorial isomorphism \( f^*(X_{S_1}) \simeq X_{S_2} \).

When \( Y \) is itself an affine scheme \( S = \text{Spec}(A) \), a data of a sheaf of categories \( \mathcal{C}^{sh} \) over \( S \) is equivalent to a single category over \( S \), which is reconstructed as \( \mathcal{C}_S \). In this case we will often abuse the notation and not distinguish between \( \mathcal{C}^{sh} \) and \( \mathcal{C}_S \).

We will now define a functor

\[
\mathcal{F}, X \mapsto \mathcal{F} \star X : \text{QCoh}_Y \times \mathcal{C}_Y \to \mathcal{C}_Y.
\]

Let \( \mathcal{F} \) be a quasi-coherent sheaf of \( Y \); for \( S = \text{Spec}(A) \in \text{Sch}^{aff}_Y \) we will denote by \( \mathcal{F}_S \) the corresponding quasi-coherent sheaf of \( S \). For \( X \in \mathcal{C}_Y \) we define

\[
(\mathcal{F} \star X)_S := \mathcal{F}_S \otimes_A X_S,
\]

which by descent gives rise to an object of \( \mathcal{C}_Y \).

The above functor has the following properties:

- (i) \( \text{QCoh}_Y \times \mathcal{C}_Y \to \mathcal{C}_Y \) is right exact and commutes with inductive limits.
- (ii) We have a functorial isomorphism \( \mathcal{O}_Y \star X \simeq X \).
- (iii) We have functorial isomorphisms \( \mathcal{F}_1 \star (\mathcal{F}_2 \star X) \simeq (\mathcal{F}_1 \otimes A_Y \mathcal{F}_2) \star X \), compatible with triple tensor products and the isomorphism of (ii).

By construction, the assertions of Lemma 4 hold in the present context, when we replace \( M \otimes X \) by \( \mathcal{F} \mapsto \mathcal{F} \star X \).

10. Descent of categories. Let \( f : Y' \to Y \) be a map of stacks, and \( \mathcal{C}^{sh} \) a sheaf of categories over \( Y \). It is clear that it gives rise to a sheaf of categories \( \mathcal{C}^{sh} := \mathcal{C}^{sh} \times Y' \) over \( Y' \), such that for \( S \in \text{Sch}^{aff}_Y \) the category \( \mathcal{C}'_S \) is by definition \( \mathcal{C}_S \), where \( S \) is regarded as an object of \( \text{Sch}^{aff}_Y \).
If $g : Y'' \to Y'$, it is clear that we have an equivalence of sheaves of categories

$$\mathcal{E}^\text{sh} \times_{Y} Y'' \simeq (\mathcal{E}^\text{sh} \times Y') \times_{Y'} Y''.$$

Suppose now $\mathcal{E}^\text{sh}$ is a sheaf of categories over $Y'$. Let $p_j$ be the projection on the $j$-th factor from the $i$-fold Cartesian product $Y(i)$ of $Y$ over $Y$. Let $\mathcal{E}^{(i)\text{sh}}$ denote the corresponding base-changed sheaf of categories categories over $Y(i)$.

Suppose we are given an equivalence of sheaves of categories over $Y$ of $\mathcal{E}^\text{sh}$ with an equivalence

$$\mathcal{E}^\text{sh} \times_{Y} Y' \simeq \mathcal{E}^\text{sh} \times_{Y} Y'$$

and which gives rise to the above functors and natural transformations.

**Proposition 11.** Suppose that $Y'$ is faithfully flat over $Y$. Then there exists a well-defined sheaf of categories $\mathcal{E}^\text{sh}$ over $Y$ with an equivalence $\mathcal{E}^\text{sh} \simeq \mathcal{E}^\text{sh} \times_{Y} Y'$, and which gives rise to the above functors and natural transformations.

**Proof.** The assertion readily reduces to the case when both $Y$ and $Y'$ are affine schemes, $\text{Spec}(A)$ and $\text{Spec}(A')$, respectively. Let $\Phi$ denote the functor $\mathcal{E}^{(2)} \to \mathcal{E}^{(2)}$, and $T$ the natural transformation between the functors $\Phi^{1,3}$ and $\Phi^{2,3} \circ \Phi^{1,2}$ between $\mathcal{E}^{(3)}$ and $\mathcal{E}^{(3)}$.

We define $\mathcal{C}$ to have as objects $X' \in \mathcal{C}'$ endowed with an isomorphism

$$\alpha_{X'} : \Phi((p_{1}^2)^*(X')) \to (p_{2}^2)^*(X'),$$

such that the diagram

$$\begin{array}{ccc}
\Phi^{1,3}((p_{1}^2)^*(X')) & \xrightarrow{T} & \Phi^{2,3} \circ \Phi^{1,2}((p_{1}^2)^*(X')) \\
\downarrow & & \downarrow \\
(p_{1}^2)^*(X') & \leftarrow & \Phi^{2,3}((p_{2}^2)^*(X'))
\end{array}$$

commutes. Morphisms in this category are $\mathcal{C}'$-morphisms, commuting with the data of $\alpha_{X'}$. Evidently, this is an $A$-linear category.

By construction, we have a functor $\mathcal{C} \to \mathcal{C}'$, which gives rise to a functor

$$\mathcal{C} \underset{\text{Spec}(A)}{\times} \to \mathcal{C}'.$$

The fact that the latter is an equivalence is shown by the base-change technique as in the context of quasi-coherent sheaves.

---

12. **Example: categories with a group-action.** Let us consider an example of the above situation, when $Y' = \text{pt}$, $Y = \text{pt} / G$, where $G$ is an affine algebraic group. Let $\mathcal{E}^\text{sh}$ be a sheaf of categories over $Y'$, i.e., a plain category. Then the data of an equivalence $\mathcal{E}^{(2)} \to \mathcal{E}^{(2)}$ together with a natural transformation as above is what can be reasonably called an action of the group $G$ on $\mathcal{C}'$.

Let us spell this notion out in more detail. We claim that an action of $G$ on a category category $\mathcal{C}'$ is equivalent to a data of a functor

$$\text{act}^* : \mathcal{C}' \to \mathcal{O}_G\text{-mod} \otimes \mathcal{C},$$

(here $\mathcal{O}_G\text{-mod} \otimes \mathcal{C}'$ denotes the same thing as $\mathcal{C}' \times \text{Spec}(A)$), and two functorial isomorphisms related to this functor. This first isomorphism is between the identity functor on $\mathcal{C}'$ and the
composition $\mathcal{C}^\prime \overset{\text{act}^\ast}{\longrightarrow} \mathcal{O}_G\text{-mod} \otimes \mathcal{C}^\prime \rightarrow \mathcal{C}^\prime$, where the second arrow corresponds to the restriction to $1 \in G$.

To formulate the second isomorphism, note that from the existing data we obtain a natural functor

$$\text{act}^\ast : A\text{-mod} \otimes \mathcal{C}^\prime \rightarrow \mathcal{O}_G\text{-mod} \otimes \mathcal{C}^\prime \simeq (\mathcal{O}_G \otimes A)\text{-mod} \otimes \mathcal{C}^\prime$$

for any algebra $A$.

The second isomorphism is between the two functors $\mathcal{C} \rightarrow \mathcal{O}_{G \times G}\text{-mod} \otimes \mathcal{C}$ that correspond to the two circuits of the diagram

$$\begin{array}{ccc}
\mathcal{C} & \overset{\text{act}^\ast}{\longrightarrow} & \mathcal{O}_G\text{-mod} \otimes \mathcal{C} \\
\text{act}^\ast & \downarrow & \text{act}^\ast_G \\
\mathcal{O}_G\text{-mod} \otimes \mathcal{C} & \overset{\text{mult}^\ast}{\longrightarrow} & \mathcal{O}_{G \times G}\text{-mod} \otimes \mathcal{C},
\end{array}$$

where mult denoted the multiplication map $G \times G \rightarrow G$. These functors must satisfy the usual compatibility conditions.

From Proposition 11, it follows that an action of $G$ on a category $\mathcal{C}^\prime$ is equivalent to the data of a sheaf of categories $\mathcal{C}^\text{sh}$ over $\text{pt}/G$. (As we shall see later, the latter can be also reformulated as a category with an action of the tensor category $\text{Rep}(G)$.)

By definition, $\mathcal{C} := \mathcal{C}_{\text{pt}/G}$ can be reconstructed as the category of $G$-equivariant objects of $\mathcal{C}^\prime$. By definition, the latter consists of $X^\prime \in \mathcal{C}^\prime$, endowed with an isomorphism $\alpha_X : \text{act}^\ast(X^\prime) \simeq \mathcal{O}_G \otimes X^\prime$, which is compatible with unit and associativity constraints. Morphisms in the category are $\mathcal{C}^\text{p}$-morphisms, compatible with the data of $\alpha$.

13. Example: categories acted on by a groupoid. Generalizing the above set-up, let $S$ be a base scheme, and $\mathcal{G} \overset{p_2}{\twoheadrightarrow} S$ be an affine groupoid, such that the maps $p_1, p_2$ (or, equivalently, one of them) are flat. Let $\mathcal{C}^\text{sh}$ be a sheaf of categories over the quotient stack $\mathcal{Y} = S/\mathcal{G}$. This data can be rewritten as a sheaf of categories $\mathcal{C}^\text{sh}$ over $S$, acted on by $\mathcal{G}$, which means the following:

We must be given a functor $\text{act}^\ast : \mathcal{C}^\prime \rightarrow \mathcal{C}^\prime \times \mathcal{G}$, which is $\mathcal{O}_S$-linear if we regard $\mathcal{C}^\prime \times \mathcal{G}$ as a category over $S$ via $\mathcal{G} \overset{p_2}{\twoheadrightarrow} S$, and two functorial isomorphisms related to it. The first isomorphism is a unit constraint, i.e., an isomorphism between the functor

$$\mathcal{C}^\prime \overset{\text{act}^\ast}{\longrightarrow} \mathcal{C}^\prime \times \mathcal{G} \overset{1_S \otimes \text{id}_{\mathcal{G}}}{\longrightarrow} \mathcal{C}^\prime \times \mathcal{G} \times S \simeq \mathcal{C}^\prime,$$

where $1_S : S \rightarrow \mathcal{G}$ is the unit map.

Formulate the second isomorphism note that for any scheme $S'$, mapping to $S$, we obtain a functor

$$\text{act}^\ast \times \text{id}_{S'} : \mathcal{C}^\prime \times S' \rightarrow \mathcal{C}^\prime \times (\mathcal{G} \times S').$$

---

2To simplify the notation, we will assume here that $S$ is affine as well.
The second isomorphism is an associativity constraint, i.e., an isomorphism between the two functors in the diagram

\[
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{\text{act}^*} & \mathcal{C}' \times S_{p_1} \\
\downarrow \text{act}^* & & \downarrow \text{act}^* \times \text{id}_S \\
\mathcal{C}' \times S_{p_1} & \xrightarrow{\text{mult}^*} & \mathcal{C}' \times (S_{p_1} \times S_{p_1})
\end{array}
\]

such that the natural compatibility conditions hold.

**Lemma 14.**

(a) Let \(0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0\) be a short exact sequence of quasi-coherent sheaves on \(\mathcal{S}\) with \(\mathcal{F}_3\) being \(O_{S}\)-flat with respect to \(p_2\). Then for \(X \in \mathcal{C}'\), the sequence

\[0 \to \mathcal{F}_1 \circ \text{act}^*(X) \to \mathcal{F}_2 \circ \text{act}^*(X) \to \mathcal{F}_3 \circ \text{act}^*(X) \to 0\]

is also short exact.

(b) If \(X \in \mathcal{C}'\) is \(O_{S}\)-flat, then \(\text{act}^*(X)\) is \(O_{\mathcal{S}}\)-flat.

**Proof.** Let \(S'\) be a scheme with a map \(\phi : S' \to \mathcal{S}\); let \(\psi_i = p_i \circ \phi, i = 1, 2\). We claim that there exists a natural \(O_{S'}\)-linear equivalence

\[\text{act}^*_\phi : \mathcal{C}' \times S' \to \mathcal{C}' \times S',\]

defined by

\[X \mapsto (\phi \times \text{id}_{S'})^* \circ (\text{act}^* \times \text{id}_{S'})(X),\]

where \(\phi \times \text{id}_{S'} : S' \to \mathcal{S} \times S'\). Its quasi-inverse is defined using the map \(\gamma \circ \phi : S' \to \mathcal{S}\), where \(\gamma\) is the inversion on \(\mathcal{S}\).

We apply this to \(S' = \mathcal{S}\) and \(\phi = \gamma\). We obtain an equivalence

\[\text{act}^*_\gamma : \mathcal{C}' \times \mathcal{S} \to \mathcal{C}' \times \mathcal{S},\]

such that for \(X \in \mathcal{C}'\),

\[\text{act}^*_\gamma(\text{act}^*(X)) \simeq p_2^*(X).\]

This readily implies both points of the lemma. \(\square\)

We say that an object \(X \in \mathcal{C}'\) is \(\mathcal{S}\)-equivariant, if we are given an isomorphism

\[p_1^*(X) \simeq \text{act}^*(X) \in \mathcal{C}' \times \mathcal{S}_{p_1}\]

compatible with the unit and associativity constraints. Let us denote by \(\mathcal{C}'^\mathcal{S}\) the category of \(\mathcal{S}\)-equivariant objects in \(\mathcal{C}'\).

From the definitions we obtain:

**Lemma 15.**

(a) For any \(X \in \mathcal{C}'\), the object \((p_1)_*(\text{act}^*(X))\) is naturally \(\mathcal{S}\)-equivariant.

(b) The functor \(X \mapsto (p_1)_*(\text{act}^*(X))\) is the right adjoint to the forgetful functor \(\mathcal{C}'^\mathcal{S} \to \mathcal{C}'\).

In addition, we have:

**Lemma 16.** Assume that \(\mathcal{S}\) is flat over \(S \times S\). Then every \(\mathcal{S}\)-equivariant object of \(\mathcal{C}'\) is \(O_{S}\)-flat.
Proof. This follows from the fact that for $\mathcal{F} \in \text{QCoh}_S$ and $X \in \mathcal{C}'$, 
$$\text{act}^* (\mathcal{F} \ast X) \simeq p_2^*(\mathcal{F}) \ast \text{act}^*(X).$$

\[\square\]

17. Stacks: approach II. Let $\text{Vect}_Y$ denote the tensor category of locally free sheaves of finite rank on $Y$.

Assume now that we are given a category $\mathcal{C}_Y$ endowed with an action of the tensor category $\text{Vect}_Y$:
$$\ast : \text{Vect}_Y \times \mathcal{C}_Y \rightarrow \mathcal{C}_Y,$$
which is exact. I.e., for a fixed $\mathcal{P} \in \text{Vect}_Y$ the functor $X \mapsto \mathcal{P} \ast X$ is exact, and whenever $0 \rightarrow \mathcal{P}_1 \rightarrow \mathcal{P}_2 \rightarrow \mathcal{P}_3 \rightarrow 0$ is a short exact sequence of objects of $\text{Vect}_Y$, the corresponding sequence
$$0 \rightarrow \mathcal{P}_1 \ast X \rightarrow \mathcal{P}_2 \ast X \rightarrow \mathcal{P}_3 \ast X \rightarrow 0$$
is also exact. We shall call such a data "a category over $\mathcal{C}_Y$".

We will now make an additional assumption on the stack $Y$:

- The stack $Y$ is locally Noetherian and every quasi-coherent sheaf on it is an inductive limit of coherent ones.
- Every coherent sheaf on $Y$ can be covered by an object of $\text{Vect}_Y$.

As in the affine case, this implies that every flat quasi-coherent sheaf on $Y$ can be represented as an inductive limit of objects of $\text{Vect}_Y$.

**Theorem 18.** Under the above assumption on $Y$, a data of a category over $Y$ is equivalent to that of a sheaf of categories over $Y$.

The rest of this subsection and the next one are devoted to the proof of this theorem. One direction has been explained above: given a sheaf of categories $\mathcal{C}^{sh}$ over $Y$, we reconstruct $\mathcal{C}_Y$ as $\Gamma(Y, \mathcal{C}^{sh})$. To carry out the construction in the opposite direction we will use the above additional assumption on $Y$.

We claim that the above data extends to an action of the monoidal category $\text{QCoh}_Y$ on $\mathcal{C}_Y$, satisfying the conditions (i),(ii),(iii) of Sect. 9 and assertions (a), (b) and (c) of Lemma 4.

First we define an action of the monoidal category $\text{Coh}_Y$ on $\mathcal{C}_Y$: By assumption, every $\mathcal{F} \in \text{Coh}_Y$ can be represented as $\text{coker}(\mathcal{P} \rightarrow \mathcal{Q})$ with $\mathcal{P}, \mathcal{Q} \in \text{Vect}_Y$. We set
$$\mathcal{F} \ast X := \text{coker}(\mathcal{P} \ast X \rightarrow \mathcal{Q} \ast X).$$

To show that this is well-defined, we must consider a commutative diagram of objects of $\text{Vect}_Y$

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
\mathcal{P} & \mathcal{Q} & \mathcal{F} \rightarrow 0 \\
\uparrow & \uparrow & \uparrow \\
\mathcal{P}' & \mathcal{Q}' & \mathcal{F} \rightarrow 0 \\
\uparrow & \uparrow & \uparrow \\
\mathcal{P}'' & \mathcal{Q}'' & 0
\end{array}
\]
with exact rows and columns, and show that the map
\[
coker(\phi \ast \text{id}_X) \to coker(\phi' \ast \text{id}_X)
\]
is an isomorphism. But this follows from the assumption.

It is clear that the resulting functor is right-exact and satisfies properties (ii) and (iii) of Sect. 9.

Next, we have to extend the above action of \( \text{Coh}_Y \) on \( \mathcal{C}_Y \) to that of \( \text{QCoh}_Y \) by setting for \( F \cong \lim_{\to} F_i \) with \( F_i \in \text{Coh}_Y \), \( F \in \text{QCoh}_Y \),
\[
F \ast X := \lim_{\to} F_i \ast X.
\]
The resulting action satisfies properties (i), (ii) and (iii) of Sect. 9. By assumption, the functor of tensor product with an object of \( \text{Vect}_Y \) is exact. This implies properties (a) and (b) Lemma 4, by repeating the proof of loc.cit. Property (c) stated in Lemma 4 is evident.

19. We shall now show how the data of an action of \( \text{QCoh}_Y \) on \( \mathcal{C}_Y \) with the above properties reconstructs the categories \( \mathcal{C}_S \) for \( S \in \text{Sch}_Y \).

Let \( S = \text{Spec}(A) \), let us denote by \( \mathcal{O}_S \) the direct image of the structure sheaf of \( S \) onto \( Y \), regarded as an algebra in \( \text{QCoh}_Y \). We introduce \( \mathcal{C}_S \) as the category, consisting of objects \( X \in \mathcal{C}_Y \) acted on by \( A \) in \( \text{QCoh}_Y \), and the morphisms being \( \mathcal{C}_Y \)-morphisms compatible with the action.

We have a map of algebras \( A \otimes \mathcal{O}_Y \rightarrow \mathcal{O}_S \) in \( \text{QCoh}_Y \); this makes \( \mathcal{O}_S \) into an \( A \)-linear category. We also have a functor \( \mathcal{C}_Y \rightarrow \mathcal{C}_S \) given by \( X \mapsto X_S := \mathcal{O}_S \ast X \).

Let \( f : S_2 = \text{Spec}(A_2) \rightarrow \text{Spec}(A_1) = S_1 \) be a morphism in \( \text{Sch}_Y^{\text{eff}} \). We define a functor \( f^* : \mathcal{C}_{S_1} \rightarrow \mathcal{C}_{S_2} \) by
\[
X \mapsto \mathcal{O}_{S_2} \otimes_{\mathcal{O}_{S_1}} X,
\]
where for an algebra \( A \) in \( \text{QCoh}_Y \), a sheaf \( \mathcal{F} \) of \( A \)-modules and an object \( X \in \mathcal{C}_Y \) acted on by \( A \), we set
\[
\mathcal{F} \ast X := \text{coker}((A \cdot A) \ast X \Rightarrow \mathcal{F} \ast X).
\]

We claim that the induced functor \( \mathcal{C}_{S_1} \times \mathcal{S}_2 \rightarrow \mathcal{C}_{S_2} \) is an equivalence. This follows from the fact that \( \mathcal{O}_{S_2} \simeq A_2 \otimes_{A_1} \mathcal{O}_{S_1} \).

Note in addition that for \( X \in \mathcal{C}_Y \),
\[
\mathcal{O}_{S_2} \ast X \simeq A_2 \otimes_{A_1} (\mathcal{O}_{S_1} \ast X).
\]

This implies that for \( X \in \mathcal{C}_Y \), we have a natural isomorphism \( f^*(X_{S_1}) \simeq X_{S_2} \).

Thus, we have constructed a sheaf of categories over \( Y \), and it remains to show that the initial category \( \mathcal{C}_Y \) can be reconstructed by the descent procedure. The usual proof for coherent sheaves works, once we establish the following:

Lemma 20. If \( A \in \text{QCoh}_Y \) be an algebra, faithfully flat over \( \mathcal{O}_Y \). Then the functor \( X \mapsto A \ast X : \mathcal{C}_Y \rightarrow \mathcal{C}_Y \) is exact and faithful.

Proof. The exactness part follows by property (a) of Lemma 4. The faithfulness part follows as in Proposition 5 using property (b) of Lemma 4. □
21. Example: de-equivariantization. Let \( Y \) be the stack \( \text{pt}/G \), where \( G \) is an affine algebraic group. Given a category \( \mathcal{C} \), a structure of category over \( \text{pt}/G \) on it is by definition the same as an action of the tensor category \( \text{Rep}(G) \) of finite-dimensional representations of \( G \) on it:

\[
V \in \text{Rep}(G), X \in \mathcal{C} \mapsto V \ast X \in \mathcal{C},
\]

which has the exactness property of Sect. 17.

By Theorem 18, such a data gives rise to a sheaf of categories \( \mathcal{C}^{\text{sh}} \) over \( \text{pt}/G \).

Let us show how to reconstruct the category \( \mathcal{C}' := \mathcal{C}^{\text{sh}} \times_{\text{pt}/G} \text{pt} \). By definition, this is the category, whose objects are \( X' \in \mathcal{C} \), endowed with an associative action \( \mathcal{O}_G \ast X' \to X' \), and morphisms are \( \mathcal{C} \)-morphisms, compatible with this action.

According to [AG], this data can be rewritten as follows. For every \( V \in \text{Rep}(G) \) we must be given a map \( \beta_V : V \ast X \to X \otimes V \), for every \( V \in \text{Rep}(G) \) (here \( V \) denoted the vector space underlying a representation), which satisfy the compatibility conditions of [AG], Sect. 2.2. One easily shows that the maps \( \beta_V \) are necessarily isomorphisms. Morphisms in this category are \( \mathcal{C} \)-morphisms, compatible with the data of \( \beta \).

Thus, \( \mathcal{C}' \) is the category of Hecke eigen-objects in \( \mathcal{C} \) with respect to the action of \( \text{Rep}(G) \). By construction, \( \mathcal{C}' \) carries a canonical action of \( G \). Explicitly, for \( X' \in \mathcal{C}' \), the \( \mathcal{O}_G \)-family \( \text{act}^\ast(X') \) is isomorphic to \( X' \otimes \mathcal{O}_G \) as an object of \( \mathcal{C} \). The isomorphisms \( \beta \) are given by

\[
V \ast (X' \otimes \mathcal{O}_G) \xrightarrow{\beta_V \otimes \text{id}_{\mathcal{O}_G}} X' \otimes V \otimes \mathcal{O}_G \to X' \otimes \mathcal{O}_G,
\]

where the second arrow is given by the co-action map \( V \to V \otimes \mathcal{O}_G \).

According to Sect. 6, the category \( \mathcal{C} \) is reconstructed from \( \mathcal{C}' \) as the category \( \mathcal{C}^G \) of \( G \)-equivariant objects. We will refer to \( \mathcal{C}' \) as the de-equivariantization of \( \mathcal{C} \).

22. Another example. Generalizing the previous example, let us take \( Y = S/G \), where \( S = \text{Spec}(A) \) is an affine scheme, and \( G \) an affine algebraic group acting on it. Let \( \mathcal{C} \) be an abelian category. A structure on \( \mathcal{C} \) of category over \( S/G \) is by definition expressed as follows:

An action of the tensor category \( \text{Rep}(G) \) on \( \mathcal{C} \): \( V, X \mapsto V \ast X \) as above, and a functorial map \( \alpha_X : A \ast X \to X \), where \( A \) is regarded as an algebra in \( \text{Rep}(G) \), such that for \( V \in \text{Rep}(G) \) the diagram

\[
\begin{array}{ccc}
V \ast (A \ast X) & \xrightarrow{\sim} & A \ast (V \ast X) \\
V \ast \alpha_X \downarrow & & \alpha_{V \ast X} \downarrow \\
V \ast X & \xrightarrow{\text{id}} & V \ast X
\end{array}
\]

commutes.

References

[AG] S. Arkhipov and D. Gaitsgory, Another realization of the category of modules over the small quantum group, math.QA/0010270, Adv. Math. 173 (2003), 114–143.