Proper holomorphic Legendrian curves in $SL_2(\mathbb{C})$

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Abstract In this paper we prove that every open Riemann surface properly embeds in the Special Linear group $SL_2(\mathbb{C})$ as a holomorphic Legendrian curve, where $SL_2(\mathbb{C})$ is endowed with its standard contact structure. As a consequence, we derive the existence of proper, weakly complete, flat fronts in the real hyperbolic space $\mathbb{H}^3$ with arbitrary complex structure.

Keywords Complex contact manifolds, Riemann surfaces, holomorphic Legendrian curves, flat fronts.

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1. Introduction and main results

Let $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ be a positive integer. A complex contact manifold is a complex manifold $W$ of odd dimension $2n + 1 \geq 3$ endowed with a holomorphic contact structure $\mathcal{L}$. The latter is a holomorphic vector subbundle $\mathcal{L} \subset T W$ of complex codimension one in the tangent bundle $TW$, satisfying that every point $p \in W$ admits an open neighborhood $U \subset W$ such that

$$\mathcal{L}|_U = \ker \eta$$

for a holomorphic 1-form $\eta$ on $U$ which satisfies

$$\eta \wedge (d\eta)^n = \eta \wedge d\eta \wedge \cdots \wedge d\eta \neq 0 \quad \text{everywhere on } U.$$

We shall write $(W, \eta)$ instead of $(W, \mathcal{L})$ when the defining holomorphic contact 1-form $\eta$ is globally defined on $W$; in such case the complex contact manifold is said to be strict.

The model example of a complex contact manifold is the complex Euclidean space $\mathbb{C}^{2n+1}$ endowed with its standard holomorphic contact form

$$(1.1) \quad \eta_0 = dz + \sum_{j=1}^{n} x_j dy_j,$$

where $(x_1, y_1, \ldots, x_n, y_n, z)$ denote the complex coordinates on $\mathbb{C}^{2n+1}$. Another example of a complex contact manifold which is the focus of interest is the Special Linear group

$$SL_2(\mathbb{C}) = \left\{ z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} : \det z = z_{11}z_{22} - z_{21}z_{12} = 1 \right\}$$

endowed with its standard holomorphic contact form

$$\eta_{SL} = z_{11}dz_{22} - z_{21}dz_{12}.$$

From now on in this paper we will just write $\mathbb{C}^{2n+1}$ and $SL_2(\mathbb{C})$ for the complex contact manifolds $(\mathbb{C}^{2n+1}, \eta_0)$ and $(SL_2(\mathbb{C}), \eta_{SL})$, respectively.
Let \((W, \mathcal{L})\) be a complex contact manifold and \(M\) be a complex manifold. A holomorphic map \(F: M \to W\) is said to be Legendrian if it is everywhere tangent to the contact structure:
\[
dF_p(T_p M) \subset \mathcal{L}_{F(p)}\quad \text{for all} \quad p \in M.
\]
If \(\mathcal{L} = \ker \eta\) for a holomorphic contact 1-form \(\eta\), then \(F\) is Legendrian if, and only if,
\[
F^*\eta = 0.
\]
When \(M\) is an open Riemann surface, a holomorphic Legendrian map \(F: M \to W\) is called a Legendrian curve; the same definition applies when \(M\) is a compact bordered Riemann surface (i.e. a compact Riemann surface with nonempty boundary consisting of finitely many pairwise disjoint smooth Jordan curves; its interior \(\bar{M} = M \setminus bM\) is called a bordered Riemann surface) and \(F\) is a Legendrian map of class \(\mathcal{A}^1(M)\) (i.e. of class \(\mathcal{C}^1(M)\) and holomorphic in the interior \(\bar{M} = M \setminus bM\)).

A major problem in complex contact geometry is to determine whether a given complex contact manifold admits properly embedded holomorphic Legendrian curves with arbitrary complex structure. Thus, in the compact case, a celebrated result by Bryant from 1982 (see [6, Theorem G]) ensures that every compact Riemann surface embeds as a holomorphic contact manifold admits properly embedded holomorphic Legendrian curves with arbitrary contact manifold: the only holomorphic contact structure in \(CP^3\) endowed with the holomorphic contact form obtained by projectivizing the standard symplectic form of \(\mathbb{C}^4\) (this is in fact, up to contactomorphisms, the only holomorphic contact structure in \(CP^3\); see LeBrun [15]). In the open case, it has been a long-standing open problem, positively settled only very recently by Forstnerič, López, and the author, whether every open Riemann surface \(M\) admits a proper holomorphic Legendrian embedding \(M \hookrightarrow \mathbb{C}^3\) (see [4, Theorem 1.1]). In the opposite direction, Forstnerič [8] has recently proved that, for every \(n \in \mathbb{N}\), there exists a holomorphic contact form \(\eta_F\) on \(\mathbb{C}^{2n+1}\) such that the complex contact manifold \((\mathbb{C}^{2n+1}, \eta_F)\) is Kobayashi hyperbolic; in particular, any holomorphic \(\eta_F\)-Legendrian curve \(C \to \mathbb{C}^{2n+1}\) is constant. The aim of this paper is to settle the embedding problem for holomorphic Legendrian curves in \(SL_2(\mathbb{C})\).

**Theorem 1.1.** Every open Riemann surface \(M\) admits a proper holomorphic Legendrian embedding \(M \hookrightarrow SL_2(\mathbb{C})\).

The holomorphic map \(\mathcal{Y}: \mathbb{C}^3 \to SL_2(\mathbb{C})\) given by
\[
\mathcal{Y}(x, y, z) = \begin{pmatrix} e^{-z} & x e^z \\ y e^{-z} & (1 + xy) e^z \end{pmatrix}, \quad (x, y, z) \in \mathbb{C}^3,
\]
maps holomorphic Legendrian immersions \(M \to \mathbb{C}^3\) into holomorphic Legendrian immersions \(M \to SL_2(\mathbb{C})\) (see Martín, Umehara, and Yamada [16, p. 210]). However, there are two main problems with using the map \(\mathcal{Y}: \mathbb{C}^3 \to SL_2(\mathbb{C})\) in order to obtain properly embedded Legendrian curves in \(SL_2(\mathbb{C})\); namely, it is neither injective nor proper (see Remark 4.1 for a careful discussion of this assertion). The proof of Theorem 1.1 consists of, given an open Riemann surface \(M\), constructing a holomorphic Legendrian embedding \(F = (X, Y, Z): M \to \mathbb{C}^3\) such that
\[
(X, Y, e^Z): M \to \mathbb{C}^3\quad \text{is one-to-one}
\]
and
\[
\max \{ |X|, -\Re Z \}: M \to \mathbb{R}_+ = [0, +\infty)\quad \text{is a proper map},
\]
where \(\Re\) denotes real part; see Theorem 4.2 for a more precise statement. It easily follows from these two conditions that \(\mathcal{Y} \circ F: M \hookrightarrow SL_2(\mathbb{C})\), where \(\mathcal{Y}\) is the map (1.2), is a
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proper holomorphic Legendrian embedding. Forstnerič, López, and the author made in [4] a systematic investigation of holomorphic Legendrian curves in the complex Euclidean spaces; in particular, they proved several approximation results which will be exploited in the present paper.

Theorem 1.1 is in connection with a result by Forstnerič and the author asserting that every bordered Riemann surface properly embeds in $SL_2(\mathbb{C})$ as a holomorphic null curve (see [3, Corollary 1.5]). The latter is a holomorphic immersion $F: M \to SL_2(\mathbb{C})$ of an open Riemann surface $M$ into $SL_2(\mathbb{C})$ satisfying the nullity condition $\det dF = 0$ everywhere on $M$. It still remains open the question whether every open Riemann surface properly embeds (or at least immerses) in $SL_2(\mathbb{C})$ as a holomorphic null curve (cf. [3, Problem 1, p. 919]). The main difference between Legendrian and null curves is that the holomorphic distribution controlling Legendrian curves does depend on the base point, and hence the constructions in the Legendrian case become more delicate and involved.

Theorem 1.1 is also related to the embedding problem for open Riemann surfaces in $\mathbb{C}^2$, asking whether every such properly embeds in $\mathbb{C}^2$ as a complex curve (see Forstnerič and Wold [9, 10] and the references therein for a discussion of the state of the art of this long-standing, likely very difficult, open problem).

A flat front in the hyperbolic space $\mathbb{H}^3$ is a flat surface in $\mathbb{H}^3$ with admissible singularities. Here by a flat surface we mean a surface in $\mathbb{H}^3$ whose Gauss curvature vanishes at the regular points; a singular point of such surface is said an admissible singularity if the corresponding points on nearby parallel surfaces are regularly immersed (see e.g. Kokubu, Umehara, and Yamada [14, §2] for more details). It is classical that the only complete smooth flat surfaces in $\mathbb{H}^3$ are the horospheres and the hyperbolic cylinders (see Sasaki [21]), and this is why the study of flat surfaces with singularities in $\mathbb{H}^3$, and, in particular, of flat fronts, has been the focus of interest in this subject during the last decades (see e.g. [11, 14, 12, 13, 16, 17, 18] and the references therein). Flat fronts in $\mathbb{H}^3$ enjoy a nice theory; it is for instance well known that these objects admit a Weierstrass type representation formula in terms of holomorphic data on a Riemann surface (see Gálvez, Martínez, and Milán [11] and Gálvez and Mira [12]). It is also well known that, given an open Riemann surface $M$ and a holomorphic Legendrian immersion $F: M \to SL_2(\mathbb{C})$, the map

$$FF^t: M \to \mathbb{H}^3 = SL_2(\mathbb{C})/SU(2) = \{a\bar{a}^t : a \in SL_2(\mathbb{C})\},$$

where $\bar{\cdot}$ and $^t$ denote complex conjugation and transpose matrix, respectively, determines a flat front in $\mathbb{H}^3$ which is conformal with respect to the metric induced on $M$ by the second fundamental form of $FF^t$, and that this fact locally characterizes flat fronts in $\mathbb{H}^3$ (see [14, 12]). Moreover, the flat front $FF^t$ is said to be weakly complete (according to Kokubu, Rossman, Umehara, and Yamada [13, §3]) if its holomorphic lift $F$ is complete in the sense that the Riemannian metric induced on $M$ by the one in $SL_2(\mathbb{C})$ via $F$ is complete. Properness and weakly completeness are the most natural global assumptions in the theory of flat fronts in $\mathbb{H}^3$.

Since the map

$$SL_2(\mathbb{C}) \ni a \longmapsto a\bar{a}^t \in \mathbb{H}^3$$

is proper, Theorem 1.1 implies the following

**Corollary 1.2.** Every open Riemann surface $M$ is the complex structure associated to the second fundamental form of a proper, weakly complete, flat front in $\mathbb{H}^3$. 


If $M$ is an open Riemann surface and there is a complete flat front $M \to \mathbb{H}^3$ (in the sense of \cite{14,13}) being conformal with respect to the second fundamental form, then there are a compact Riemann surface $M'$ and a finite subset $\{p_1, \ldots, p_m\} \subset M'$ such that $M$ is biholomorphic to $M' \setminus \{p_1, \ldots, p_m\}$ (see again \cite{14,13}). Thus, completeness imposes strong restrictions on the complex structure, and even on the topology, of flat fronts in $\mathbb{H}^3$; in particular the examples given in Corollary \cite{1,2} are not complete in general.

**Organization of the paper.** In Section \ref{section2} we state the notation and preliminaries that will be needed throughout the paper. Section \ref{section3} is devoted to the proof of a general position result for holomorphic Legendrian curves in $\mathbb{C}^3$ which will be the key to ensure the embeddedness of the examples in Theorem \ref{thm1.1}. Finally, we prove Theorem \ref{thm1.1} in Section \ref{section4}.

### 2. Preliminaries

We denote by $\mathbb{C} = \mathbb{C} \setminus \{0\}$, $i = \sqrt{-1}$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, and $\mathbb{R}_+ = [0, +\infty)$. Given $N \in \mathbb{N}$, we denote by $| \cdot |$ the Euclidean norm in $\mathbb{C}^N$. Let $K$ be a compact topological space and $f : K \to \mathbb{C}^N$ be a continuous map, we denote by

$$\|f\|_{0,K} := \max\{|f(p)| : p \in K\}$$

the maximum norm of $f$ on $K$. Likewise, if $K$ is a subset of a Riemann surface $M$, then for any $r \in \mathbb{Z}_+$ we denote by

$$\|f\|_{r,K}$$

the standard $C^r$-norm of a function $f : K \to \mathbb{C}^n$ of class $C^r(K)$, where, if $r > 0$, the derivatives are measured with respect to any fixed Riemannian metric on $M$ (its precise choice will not be important in the paper).

Let $M$ be an open Riemann surface. Given a subset $A \subset M$ we denote by $\mathcal{O}(A)$ the space of functions $A \to \mathbb{C}$ which are holomorphic on an unspecified open neighborhood (depending on the map) of $A$ in $M$. If $A \subset M$ is a smoothly bounded compact domain and $r \in \mathbb{Z}_+$, we denote by $\mathcal{O}^r(A)$ the space of $C^r$ functions $A \to \mathbb{C}$ which are holomorphic on the interior $\tilde{A} = A \setminus bA$; we just write $\mathcal{O}^r(A)$ for $(\mathcal{O}^r(A))^N = \mathcal{O}^r(A) \times \cdots \times \mathcal{O}^r(A)$ when there is no place for ambiguity. Thus, by a Legendrian curve $A \to \mathbb{C}^{2n+1}$ ($n \in \mathbb{N}$) of class $\mathcal{O}^r(A)$, $r \geq 1$, we simply mean a map of class $\mathcal{O}^r(A)$ whose restriction to $\tilde{A}$ is a holomorphic Legendrian curve.

A **compact bordered Riemann surface** is a compact Riemann surface $\overline{M}$ with nonempty boundary $bM \subset \partial M$ consisting of finitely many pairwise disjoint smooth Jordan curves. The interior $M = \overline{M} \setminus bM$ of $\overline{M}$ is called a **bordered Riemann surface**. It is classical that every compact bordered Riemann surface $\overline{M}$ is diffeomorphic to a smoothly bounded compact domain in an open Riemann surface $M'$. The space $\mathcal{O}^r(\overline{M})$ is defined as above.

#### 2.1. A Mergelyan theorem for Legendrian curves.

A compact subset $K$ of an open Riemann surface $M$ is said to be **Runge**, or **holomorphically convex**, if $M \setminus K$ has no relatively compact connected components in $M$. By Mergelyan’s theorem, $K \subset M$ is Runge if, and only if, every continuous function $K \to \mathbb{C}$, holomorphic in $K$, may be approximated uniformly on $K$ by holomorphic functions $M \to \mathbb{C}$ (see \cite{20,19}).

**Definition 2.1** ([4, Def. 4.2]). A compact subset $S$ of an open Riemann surface $M$ is called **admissible** if $S = K \cup \Gamma$, where $K = \bigcup_j \overline{D}_j$ is a union of finitely many pairwise disjoint, smoothly bounded, compact domains $\overline{D}_j$ in $M$ and $\Gamma = \bigcup_i \Gamma_i$ is a union of finitely many
The following Mergelyan type approximation result for generalized Legendrian curves in $\mathbb{C}^{2n+1}$ is a particular instance of [4, Theorem 5.1].

**Theorem 2.3.** Let $S$ be a Runge admissible subset of an open Riemann surface $M$. Every generalized Legendrian curve $S \to \mathbb{C}^{2n+1}$ ($n \in \mathbb{N}$) may be approximated in the $\mathcal{C}^1(S)$-topology by holomorphic Legendrian embeddings $M \hookrightarrow \mathbb{C}^{2n+1}$ having no constant component function.

The condition that the approximating embeddings in the above theorem can be chosen to do not have any constant component function is not explicitly mentioned in the statement of [4, Theorem 5.1] but it easily follows from an inspection of its proof (see in particular [4, Lemma 4.4 and proof of Lemma 5.2]).

3. A general position result for Legendrian curves

In this section we prove a desingularizing result for Legendrian curves in $\mathbb{C}^{2n+1}$ which will be the key to ensure the one-to-oneness of the examples in Theorem 1.1.

**Lemma 3.1.** Let $\overline{M} = M \cup bM$ be a compact bordered Riemann surface. Every Legendrian curve $F: \overline{M} \to \mathbb{C}^{2n+1}$ ($n \in \mathbb{N}$) of class $\mathcal{C}^1(\overline{M})$ may be approximated in the $\mathcal{C}^1(\overline{M})$-topology by Legendrian embeddings $\tilde{F} = (\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_{2n+1}): \overline{M} \hookrightarrow \mathbb{C}^{2n+1}$ of class $\mathcal{C}^1(\overline{M})$ having no constant component function and such that the map

$$(\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_{2n+1}): \overline{M} \to \mathbb{C}^{2n} \times \mathbb{C}_* \subset \mathbb{C}^{2n+1}$$
is one-to-one.

Lemma 3.1 is a subtle extension to [4, Lemma 4.4]; it will enable us to construct embedded Legendrian curves in $\mathbb{C}^3$ which, by composing with the non-injective map $\Psi: \mathbb{C}^3 \to SL_2(\mathbb{C})$ given in (1.2), provide embedded Legendrian curves in $\mathbb{C}^3$.

**Proof.** For simplicity of exposition we shall assume that $n = 1$; the same proof applies in general. Moreover, by Theorem 2.3 we may assume that the initial Legendrian curve $F$ is an embedding of class $A_1(M)$ having no constant component function.

Let us write $F = (X, Y, Z): M \to \mathbb{C}^3$. Consider the closed discrete subset

$$\Lambda := \{(0, 0, 2m\pi) \in \mathbb{C}^3 : m \in \mathbb{Z}\} \subset \mathbb{C}^3.$$

Since $F$ is an embedding, the difference map $\delta F: M \times M \to \mathbb{C}^3$ defined by

$$\delta F(p, q) = F(q) - F(p), \quad p, q \in \overline{M},$$

satisfies

$$\delta F^{-1}(0) = D_M := \{(p, p) : p \in \overline{M}\}.$$

Thus, since $\delta F$ is continuous, $\Lambda$ is closed and discrete, and $M$ is compact, there is an open neighborhood $U \subset M \times M$ of the diagonal $D_M$ such that

$$\delta F(U \setminus D_M) \cap \Lambda = \emptyset.$$

On the other hand, by [4, Proof of Lemma 4.4], there exists a holomorphic map $H: \overline{M} \times \mathbb{C}^N \to \mathbb{C}^3$ for some big $N \in \mathbb{N}$ such that the following conditions are satisfied for some $r > 0$:

i) $H(\cdot, 0) = F$.

ii) $H(\cdot, \zeta): \overline{M} \to \mathbb{C}^3$ is a Legendrian immersion of class $\mathcal{A}^1(M)$ for all $\zeta \in r\mathbb{B}$, where $\mathbb{B}$ denotes the unit ball in $\mathbb{C}^N$.

iii) The difference map $\delta H: \overline{M} \times \overline{M} \times r\mathbb{B} \to \mathbb{C}^3$, defined by

$$\delta H(p, q, \zeta) = H(q, \zeta) - H(p, \zeta), \quad p, q \in \overline{M}, \quad \zeta \in r\mathbb{B},$$

is a submersive family of maps on $\overline{M} \times \overline{M} \setminus U$, meaning that the derivative

$$\partial_{\zeta}|_{\zeta=0}\delta H(p, q, \zeta): \mathbb{C}^N \to \mathbb{C}^3$$

is surjective for all $(p, q) \in \overline{M} \times \overline{M} \setminus U$.

Since $\overline{M} \times \overline{M} \setminus U$ is compact, iii) guarantees that the partial differential $\partial_{\zeta}(\delta H)$ is surjective on $(\overline{M} \times \overline{M} \setminus U) \times r'\mathbb{B}$ for some number $0 < r' < r$. It follows that the map $\delta H: (\overline{M} \times \overline{M} \setminus U) \times r'\mathbb{B} \to \mathbb{C}^3$ is transverse to any submanifold of $\mathbb{C}^3$, in particular, to the the closed discrete subset $\Lambda \subset \mathbb{C}^3$. By Abraham’s reduction to Sard’s theorem (see [1]; see also [7, §7.8] for the holomorphic case) we have that for a generic choice of $\zeta \in r'\mathbb{B}$ the difference map $\delta H(\cdot, \cdot, \zeta)$ is transverse to $\Lambda$ on $\overline{M} \times \overline{M} \setminus U$, and hence, by dimension reasons,

$$\delta H(\overline{M} \times \overline{M} \setminus U, \zeta) \cap \Lambda = \emptyset.$$

Choosing $\zeta$ sufficiently close to $0 \in \mathbb{C}^N$ we get a Legendrian immersion

$$\tilde{F} = (\tilde{X}, \tilde{Y}, \tilde{Z}) := H(\cdot, \zeta): \overline{M} \to \mathbb{C}^3.$$
Theorem 4.2. proper (in a strong sense) Legendrian embeddings in compact domain, and so the sequence 
\((\tilde{F}, \tilde{V}, e^{\tilde{Z}}): \overline{M} \to \mathbb{C}^2 \times \mathbb{C}_* \subset \mathbb{C}^3\) is one-to-one, and hence the same happens to 
\(\tilde{F}: \overline{M} \to \mathbb{C}^3\). This implies that \(\tilde{F}\) is an embedding, which concludes the proof. \(\square\)

4. Proof of Theorem 1.1

The aim of this section is to prove Theorem 1.1 in the introduction. Before proceeding with that, let us point out the following

Remark 4.1. Forstnerič, López, and the author proved in [4, Theorem 1.1] that every open Riemann surface, \(M\), carries a proper holomorphic Legendrian embedding \(F = (F_1, F_2, F_3): M \hookrightarrow \mathbb{C}^3\), but also that, given \(\{i, j\} \in \{1, 2, 3\}, i \neq j\), such an embedding \(F\) can be found so that \((F_i, F_j): M \to \mathbb{C}^2\) is a proper map. However, this fact does not imply Theorem 1.1 (even allowing self-intersections) by making use of the map \(\mathcal{Q}: \mathbb{C}^3 \to SL_2(\mathbb{C})\) given in (12). Indeed, consider the sequence \(\{p_j = (x_j, y_j, z_j)\}_{j \in \mathbb{N}}\), where \(p_j = (e^{-j}, -e^j, j) \in \mathbb{C}^3\) for all \(j \in \mathbb{N}\). Observe that the sequences \(\{(x_j, y_j)\}_{j \in \mathbb{N}}, \{(x_j, z_j)\}_{j \in \mathbb{N}}\) are all divergent in \(\mathbb{C}^2\), whereas

\[\mathcal{Q}(p_j) = \begin{pmatrix} e^{-j} & 1 \\ -1 & 0 \end{pmatrix}, \quad j \in \mathbb{N},\]

and so the sequence \(\{\mathcal{Q}(p_j)\}_{j \in \mathbb{N}}\) is convergent (and hence bounded) in \(SL_2(\mathbb{C})\).

We will obtain Theorem 1.1 as a consequence of the following approximation result by proper (in a strong sense) Legendrian embeddings in \(\mathbb{C}^{2n+1}\).

Theorem 4.2. Let \(M\) be an open Riemann surface, \(K \subset M\) be a smoothly bounded Runge compact domain, and \(F = (F_1, F_2, \ldots, F_{2n+1}): K \to \mathbb{C}^{2n+1}\) be a Legendrian curve of class \(\mathcal{A}^1(K)\). Then \(F\) may be approximated in the \(\mathcal{C}^1(K)\)-topology by holomorphic Legendrian embeddings \(\tilde{F} = (\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_{2n+1}): M \hookrightarrow \mathbb{C}^{2n+1}\) satisfying the following properties:

(i) The function

\[
\max\{|\tilde{F}_1|, -\Re\tilde{F}_{2n+1}\}: M \to \mathbb{R}_+ = [0, +\infty)
\]

is proper, where \(\Re\) denotes the real part.

(ii) The holomorphic map

\[
(\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_{2n+1}): M \to \mathbb{C}^{2n} \times \mathbb{C}_* \subset \mathbb{C}^{2n+1}
\]

is one-to-one.

Proof of Theorem 1.1 assuming Theorem 4.2 Let \(M\) be an open Riemann surface. By Theorem 4.2 there is a holomorphic Legendrian embedding \(F = (X, Y, Z): M \hookrightarrow \mathbb{C}^3\) such that

i) \(\max\{|X|, -\Re Z\}: M \to \mathbb{R}_+\) is a proper map and

ii) \((X, Y, e^Z): M \to \mathbb{C}^3\) is one-to-one.
Consider the holomorphic Legendrian immersion
\[ \mathcal{Y} \circ F = \left( e^{-Z} Y e^{-Z}, X e^{-Z}, (1 + XY) e^{-Z} \right) : M \to SL_2(\mathbb{C}), \]
where \( \mathcal{Y} : \mathbb{C}^3 \to SL_2(\mathbb{C}) \) is the map (1.2). It trivially follows from property (ii) that \( \mathcal{Y} \circ F \) is one-to-one, and hence, since proper injective immersions \( M \hookrightarrow SL_2(\mathbb{C}) \) are embeddings, to finish the proof it suffices to show that \( \mathcal{Y} \circ F : M \to SL_2(\mathbb{C}) \) is a proper map. Indeed, pick a divergent sequence \( \{p_j\}_{j \in \mathbb{N}} \) in \( M \) and let us check that \( \{\mathcal{Y}(F(p_j))\}_{j \in \mathbb{N}} \) diverges in \( SL_2(\mathbb{C}) \subset \mathbb{C}^4 \); equivalently,
\[ \lim_{j \to \infty} |\mathcal{Y}(F(p_j))|_1 = +\infty, \]
where
\[ |\mathcal{Y}(F(p))|_1 = e^{-\Re Z(p)}(1 + |Y(p)|) + e^{\Re Z(p)}(|X(p)| + |1 + X(p)Y(p)|), \quad p \in M. \]
In view of property (i) we may assume that either \( \lim_{j \to \infty} |X(p_j)| = +\infty \) or \( \lim_{j \to \infty} \Re Z(p_j) = -\infty \); let us distinguish cases.

**Case 1.** Assume that \( \lim_{j \to \infty} \Re Z(p_j) = -\infty \). It follows that
\[ +\infty = \lim_{j \to \infty} e^{-\Re Z(p_j)} \leq \lim_{j \to \infty} |\mathcal{Y}(F(p_j))|_1, \]
which proves (4.1).

**Case 2.** Assume that \( \lim_{j \to \infty} |X(p_j)| = +\infty \). We reason by contradiction and, up to passing to a subsequence, assume that \( \{|\mathcal{Y}(F(p_j))|_1\}_{j \in \mathbb{N}} \) is a bounded sequence. It turns out that \( \{e^{\Re Z(p_j)}|X(p_j)|\}_{j \in \mathbb{N}} \) is also bounded, and hence, since we are assuming that \( \lim_{j \to \infty} |X(p_j)| = +\infty \), we infer that \( \lim_{j \to \infty} e^{\Re Z(p_j)} = 0 \); equivalently, \( \lim_{j \to \infty} \Re Z(p_j) = -\infty \). This reduces the proof to Case 1, and hence concludes the proof of the theorem.

Theorem (4.2) will follow from a standard recursive application of Lemma 3.1 and the following approximation result, which is the kernel of this section.

**Lemma 4.3.** Let \( M \) be an open Riemann surface, \( \emptyset \neq K \subset K' \subset M \) be smoothly bounded, Runge compact domains such that the Euler characteristic \( \chi(K' \setminus K) \in \{-1, 0\} \), \( F = (F_1, F_2, \ldots, F_{2n+1}) : K \to \mathbb{C}^{2n+1} \) be a Legendrian curve of class \( \mathcal{C}^1(K), \rho > 0 \) be a positive number, and assume that
\[ (4.2) \quad \max\{|F_1|, -\Re F_{2n+1}\} > \rho \quad \text{everywhere on } bK. \]
Then \( F \) may be approximated in the \( \mathcal{C}^1(K) \)-topology by Legendrian curves \( \tilde{F} = (\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_{2n+1}) : K' \to \mathbb{C}^{2n+1} \) of class \( \mathcal{C}^1(K') \) satisfying the following conditions:

(i) \( \max\{|\tilde{F}_1|, -\Re \tilde{F}_{2n+1}\} > \rho \quad \text{everywhere on } K' \setminus \hat{K}. \)

(ii) \( \max\{|\tilde{F}_1|, -\Re \tilde{F}_{2n+1}\} > \rho + 1 \quad \text{everywhere on } bK'. \)

**Proof.** For simplicity of exposition we assume that \( n = 1 \); the same proof applies in general. Write \( F = (X, Y, Z) : K \to \mathbb{C}^3 \). We distinguish cases.

**Case 1:** Assume that \( \chi(K' \setminus \hat{K}) = 0 \). In this case \( K' \setminus \hat{K} \) consists of finitely many pairwise disjoint compact annuli. Again for simplicity of exposition we assume that \( K \) (and so \( K' \)) has a single boundary component, and hence \( K' \setminus \hat{K} \) is connected (an annulus); otherwise we would reason analogously on each connected component of \( K' \setminus \hat{K} \).
By inequality (4.2) there are an integer \( m \geq 3 \) and arcs \( \alpha_j \subset bK, j \in \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z} = \{0, \ldots, m-1\} \), meeting the following requirements:

(a1) \( \bigcup_{j \in \mathbb{Z}_m} \alpha_j = bK. \)

(a2) \( \alpha_j \cap \alpha_{j+1} \) consists of a single point \( p_j \) and \( \alpha_j \cap \alpha_k = \emptyset \) for all \( k \in \mathbb{Z}_m \setminus \{j-1, j, j+1\} \).

(a3) There are disjoint subsets \( I_X \) and \( I_Z \) of \( \mathbb{Z}_m \) such that \( I_X \cup I_Z = \mathbb{Z}_m, \ |X| > \rho \) everywhere on \( \alpha_j \) for all \( j \in I_X \), and \( -\Re Z > \rho \) everywhere on \( \alpha_j \) for all \( j \in I_Z \).

Let us now take a family of pairwise disjoint smooth Jordan arcs \( \gamma_j \subset K' \setminus K, j \in \mathbb{Z}_m \), having an endpoint \( p_j \in bK \) and the other endpoint \( q_j \in bK' \) and being otherwise disjoint from \( bK \cup bK' \). We choose such arcs so that the compact set

\[
S := K \cup \left( \bigcup_{j \in \mathbb{Z}_m} \gamma_j \right) \subset K'
\]

is admissible in \( M \) in the sense of Def. 2.1. It follows that \( K' \setminus S \) consists of \( m \) pairwise disjoint disks; we denote by \( \Omega_j \) the one whose closure contains \( \alpha_j \), and by \( \beta_j \) the arc \( bK' \cap \bar{\Omega}_j \). (See Figure 4.1.)

Thus,

\[
(4.3) \quad K' \setminus K = \bigcup_{j \in \mathbb{Z}_m} \bar{\Omega}_j, \quad bK' = \bigcup_{j \in \mathbb{Z}_m} \beta_j,
\]

and

\[
b\Omega_j = \bar{\Omega}_j \setminus \Omega_j = \gamma_{j-1} \cup \alpha_j \cup \gamma_j \cup \beta_j, \quad j \in \mathbb{Z}_m.
\]

**Figure 4.1.** \( K' \setminus K \).

Since every compact path \([0, 1] \to \mathbb{C}^3\) may be uniformly approximated by Legendrian paths (cf. [4, Theorem A.6]), property (a3) enables us to extend \( F = (X, Y, Z) \) with the same name, to a generalized Legendrian curve \( S \to \mathbb{C}^3 \) (see Def. 2.2) satisfying:

(b1) \( |X| > \rho \) everywhere on \( \gamma_{j-1} \cup \alpha_j \cup \gamma_j \) and \( |X| > \rho + 1 \) at \( q_{j-1} \) and \( q_j \) for all \( j \in I_X \).

(b2) \( -\Re Z > \rho \) everywhere on \( \gamma_{j-1} \cup \alpha_j \cup \gamma_j \) and \( -\Re Z > \rho + 1 \) at \( q_{j-1} \) and \( q_j \) for all \( j \in I_Z \).

Next, by Theorem 2.3 we may approximate \( F \) in the \( \mathcal{C}^1(S) \)-topology by Legendrian curves \( K' \to \mathbb{C}^3 \) of class \( \mathcal{C}^1(K') \) having no constant component function. We still denote by \( F = (X, Y, Z) \) to a such approximating curve and assume that the approximation is close enough so that properties (b1) and (b2) remain to hold. Thus, by continuity of \( F \), for each \( j \in \mathbb{Z}_m \) there is a smoothly bounded, closed disk

\[
T_j \subset \bar{\Omega}_j \setminus (\gamma_{j-1} \cup \alpha_j \cup \gamma_j) = \Omega_j \cup (\beta_j \setminus \{q_{j-1}, q_j\})
\]
such that:

1. \( \beta'_j := Y_j \cap \beta_j \neq \emptyset \) is an arc contained in the relative interior of \( \beta_j \).
2. \(|X| > \rho\) everywhere on \( \Omega_j \setminus Y_j \) and \(|X| > \rho + 1\) everywhere on \( \beta_j \setminus \beta'_j \) for all \( j \in I_X \).
3. \( -\Re Z > \rho \) everywhere on \( \Omega_j \setminus Y_j \) and \( -\Re Z > \rho + 1 \) everywhere on \( \beta_j \setminus \beta'_j \) for all \( j \in I_Z \).

See Figure 4.1.

Let us now assume that \( I_X \neq \emptyset \); otherwise \( I_Z = \mathbb{Z}_m \) and the following deformation procedure is not required. For each \( j \in I_X \) choose an arc \( \delta_j \subset \Omega_j \setminus Y_j \) having an endpoint in \( Y_j \setminus \beta_j \) and the other endpoint in the relative interior of \( \alpha_j \), and being otherwise disjoint from \( Y_j \cup b \Omega_j \). (See Figure 4.1.) Further, we may choose the arcs \( \delta_j, j \in \mathbb{Z}_m \), so that

[4, Lemma 4.3] guarantees that such a generalized Legendrian curve can be constructed as follows. Set \( X_1 := X |_{S_X} \), hence (d2) holds, and choose any map \( Z_1 : S_X \to \mathbb{C} \) of class \( \mathcal{A}^1(S_X) \) meeting the following requirements:

1. \( Z_1 = Z \) everywhere on \( S \cup \left( \bigcup_{j \in I_Z} \overline{\Omega}_j \right) \).
2. \( -\Re Z_1 > \rho + 1 \) everywhere on \( \bigcup_{j \in I_X} Y_j \).
3. \( dZ_1 \) (cf. (2.1)) has no zeros on \( \bigcup_{j \in I_X} \delta_j \) and its zeros on \( \bigcup_{j \in I_X} Y_j \) are those of \( X_1 \), with the same order.

Such a function trivially exists; to ensure (ii) recall that \( X_1 = X \) and \( Z \) have neither zeros nor critical points in \( \bigcup_{j \in I_X} \delta_j \). Now fix a point \( u_0 \in K \) and define \( Y_1 : S_X \to \mathbb{C} \) by

\[
S_X \ni u \mapsto Y_1(u) = \begin{cases} 
Y(u) & \text{for all } u \in S \cup \left( \bigcup_{j \in I_Z} \overline{\Omega}_j \right) \\
Y(u_0) - \int_{u_0}^{u} \frac{dZ_1}{X_1} & \text{for all } u \in \bigcup_{j \in I_X} \delta_j \cup Y_j.
\end{cases}
\]

By (i), (d2), (iii), and the facts that \( F = (X, Y, Z) : K' \to \mathbb{C}^3 \) is Legendrian, that \( S_X \) is a strong deformation retract of \( K' \), that \( X \) vanishes nowhere on \( \bigcup_{j \in I_X} \delta_j \), and that \( \delta_j \cup Y_j \) is simply-connected for all \( j \in I_X \), we infer that \( Y_1 : S_X \to \mathbb{C} \) is a well-defined function of class \( \mathcal{A}^1(S_X) \) and, taking also (ii) into account, the map \( G_1 := (X_1, Y_1, Z_1) : S_X \to \mathbb{C}^3 \) is a generalized Legendrian curve satisfying conditions (d1) and (d3).

In view of (d2) we have that \( X_1 = X \) is nonconstant and of class \( \mathcal{A}^1(K') \), and so [4, Lemma 4.3] guarantees that \( G_1 \) may be approximated in the \( \mathcal{C}^1(S_X) \)-topology by Legendrian curves \( \tilde{G}_1 = (\tilde{X}_1, \tilde{Y}_1, \tilde{Z}_1) : K' \to \mathbb{C}^3 \) of class \( \mathcal{A}^1(K') \) having no constant component function and with

\[
\tilde{X}_1 = X.
\]
Further, in view of (c3), (d1), and (d3), we may choose such an approximation \( \tilde{G}_1 \) of \( G_1 \) so that:

(e1) \(-\Re \tilde{Z}_1 > \rho \) everywhere on \( \bigcup_{j \in I_x} \overline{Y}_j \setminus Y_j \).

(e2) \(-\Re \tilde{Z}_1 > \rho + 1 \) everywhere on \( \bigcup_{j \in I_x} Y_j \) and on \( \bigcup_{j \in I_x} \overline{\beta_j \setminus \beta_j'} \) (for the latter take into account that \( \beta_j \subset \overline{Y}_j \) for all \( j \in \mathbb{Z}_m \supset I_Z \)).

Assume for a moment that \( I_Z = \emptyset \) and let us show that \( \tilde{F} := \tilde{G}_1 \) solves the lemma. Indeed, in this case \( I_X = \mathbb{Z}_m \) and hence, in view of (4.3),

\[
\frac{e_2}{e_1}
\]

Thus, (4.5), (4.6), and (4.7) guarantee conditions (i) and (ii) in the statement of the lemma. Moreover, if the approximation of \( G_1 \) by \( \tilde{G}_1 \) is close enough in the \( C_1(S_X) \)-norm, property (d1) and the fact that \( K \subset S \subset S_X \) enable us to assume that \( \tilde{G}_1 \) is as close as desired to \( F \) in the \( C_1(K) \)-topology. This would conclude the proof of the lemma in case \( I_Z = \emptyset \).

Assume now that \( I_Z \neq \emptyset \). Analogously to what has been done in the previous deformation procedure, for each \( j \in I_Z \) we choose an arc \( \delta_j \subset \overline{Y}_j \setminus Y_j \) having an endpoint in \( Y_j \setminus \beta_j \) and the other endpoint in the relative interior of \( \alpha_j \), being otherwise disjoint from \( Y_j \cup bY_j \), and such that the set

\[
S_Z := S \cup \left( \bigcup_{j \in I_X} \overline{\Omega}_j \right) \cup \left( \bigcup_{j \in I_x} \delta_j \cup Y_j \right)
\]

is admissible in \( M \) and the functions \( \tilde{X}_1, \tilde{Y}_1, \) and \( \tilde{Z}_1 \) have neither zeros nor critical points in \( \bigcup_{j \in I_Z} \delta_j \); see Figure 4.1. For the latter, recall that the concerned functions are nonconstant and of class \( \mathcal{C}^3 \). Let \( G_2 = (X_2, Y_2, Z_2) : (S_Z) \to \mathbb{C}^3 \) be a generalized Legendrian curve satisfying the following properties:

(f1) \( G_2 = \tilde{G}_1 \) everywhere on \( S \cup \left( \bigcup_{j \in I_X} \overline{\Omega}_j \right) \).

(f2) \( Z_2 = \tilde{Z}_1 \) everywhere on \( S_Z \).

(f3) \( |X_2| > \rho + 1 \) everywhere on \( \bigcup_{j \in I_Z} Y_j \).

(f4) \( X_2 \) and \( Y_2 \) are nonconstant on \( Y_j \), \( X_2 \) has no zeros in \( \delta_j \), and \( Y_2 \) has no critical points in \( \delta_j \) for all \( j \in I_Z \).

Such may be constructed as follows. Set \( Z_2 := \tilde{Z}_1 |_{S_Z} \); this implies (f2). Choose any map \( X_2 : (S_Z) \to \mathbb{C} \) of class \( \mathcal{C}^3 \) satisfying the following conditions:

I) \( X_2 = \tilde{X}_1 \) everywhere on \( S \cup \left( \bigcup_{j \in I_X} \overline{\Omega}_j \right) \).

II) \( |X_2| > \rho + 1 \) everywhere on \( \bigcup_{j \in I_Z} Y_j \) and \( X_2 \) is nonconstant on \( Y_j \) for all \( j \in I_Z \).

III) \( X_2 \) vanishes nowhere on \( \bigcup_{j \in I_Z} \delta_j \).
Existence of such a function is clear; recall that $\tilde{X}_1$ does not vanish anywhere on $\bigcup_{j \in I_Z} \delta_j$.

Fix a point $u_0 \in \tilde{K}$ and define $Y_2: S_Z \to \mathbb{C}$ by

$$S_Z \ni u \mapsto Y_2(u) = \begin{cases} \tilde{Y}_1(u) & \text{for all } u \in S \cup \left( \bigcup_{j \in I_X} \overline{\Omega}_j \right) \\ \tilde{Y}_1(u_0) - \int_{u_0}^u \frac{dZ_2}{X_2} & \text{for all } u \in \bigcup_{j \in I_Z} \delta_j \cup \Upsilon_j. \end{cases}$$

In view of (f2), (I), (II), (III), the facts that $\tilde{G}_1$ is a Legendrian curve of class $\mathscr{A}^1(K')$, that $S_Z$ is a strong deformation retract of $K'$, and that $\delta_j \cup \Upsilon_j$ is simply-connected for all $j \in I_Z$, imply that $Y_2$ is a well-defined map of class $\mathscr{A}^1(S_Z)$ (observe that $X_2$ vanishes nowhere on $\bigcup_{j \in I_Z} \Upsilon_j$ since $\rho > 0$) and $G_2 := (X_2, Y_2, Z_2): S_Z \to \mathbb{C}^3$ is a generalized Legendrian curve satisfying properties (f1) and (f3). Finally, since $\tilde{Z}_1$ has no critical points in $\delta_j$ and is nonconstant on $\Upsilon_j$ for all $j \in I_Z$, properties (II), (III), and (f2) guarantee (f4).

Now we may apply [4, Lemma 4.3] to $G_2$ inferring that it may be approximated in the $\mathscr{C}^1(S_Z)$-topology by Legendrian curves $F = (\tilde{X}, \tilde{Y}, \tilde{Z}): K' \to \mathbb{C}^3$ of class $\mathscr{A}^1(K')$ having no constant component function and satisfying

$$\tilde{Z} = \tilde{Z}_1. \tag{4.8}$$

We claim that a close enough such approximation $\tilde{F}$ of $G_2$ satisfies the conclusion of the lemma. Indeed, since $K \subset S \subset S_X \cap S_Z$ then, by (d1), (f1), and choosing $\tilde{G}_1$ close enough to $G_1$ in the $\mathscr{C}^1(S_X)$-norm and $\tilde{F}$ close enough to $G_2$ in the $\mathscr{C}^1(S_Z)$-norm, we may choose $\tilde{F}$ to be as close as desired to $F$ in the $\mathscr{C}^1(K)$-norm. On the other hand, (e1), the second part of (e2), (f3), and (4.3) give that

$$\max\{|\tilde{X}|, -\Re\tilde{Z}\} > \rho \quad \text{everywhere on } \bigcup_{j \in I_Z} \Upsilon_j \cup \Omega_j \setminus \bar{\Upsilon}_j \tag{4.9}$$

and

$$\max\{|\tilde{X}|, -\Re\tilde{Z}\} > \rho + 1 \quad \text{everywhere on } \bigcup_{j \in I_Z} \Upsilon_j \cup \beta_j \setminus \bar{\beta}_j, \tag{4.10}$$

provided that $\tilde{X}$ is chosen close enough to $X_2$ uniformly on $S_Z \supset \bigcup_{j \in I_Z} \Upsilon_j$. Finally, since $Z_m = I_X \cup I_Z$,

$$K' \setminus \tilde{K} = \bigcup_{j \in Z_m} \Omega_j = \bigcup_{j \in Z_m} \Upsilon_j \cup \Omega_j \setminus \bar{\Upsilon}_j, \quad \text{and} \quad bK' = \bigcup_{j \in Z_m} \beta_j \subset \bigcup_{j \in Z_m} \Upsilon_j \cup \beta_j \setminus \bar{\beta}_j,$$

properties (4.6), (4.7), (f1), (4.8), (4.9), and (4.10) guarantee conditions (i) and (ii) in the statement of the lemma, whenever that the approximation of $X_2$ by $\tilde{X}$ on the set $S_Z \supset \bigcup_{j \in I_X} \Omega_j = \bigcup_{j \in I_X} \Upsilon_j \cup \Omega_j \setminus \bar{\Upsilon}_j$ is sufficiently close. This concludes the proof in the case when the Euler characteristic $\chi(K' \setminus \tilde{K}) = 0$.

Case 2: Assume that $\chi(K' \setminus \tilde{K}) = -1$. In this case there is a Jordan arc $\gamma \subset K' \setminus \tilde{K}$ such that the two endpoints of $\gamma$ lie in $bK'$ and $\gamma$ is otherwise disjoint from $K'$, and $S := K \cup \gamma \subset K'$ is Runge and admissible in $M$ (in the sense of Def. 2.1) and a strong deformation retract of $K'$. Since every compact path in $\mathbb{C}^3$ may be uniformly approximated by Legendrian paths (see [4, Theorem A.6]), inequality (4.22) enables us to extend $F$, with the same name, to a generalized Legendrian curve $S \to \mathbb{C}^3$ such that

$$\max\{|X|, -\Re Z\} > \rho \quad \text{everywhere on } bK \cup \gamma. \tag{4.11}$$
Now, by Theorem 2.3, we may approximate $F$ uniformly in the $C^1(S)$-topology by holomorphic Legendrian curves $F_1 = (X_1, Y_1, Z_1): M \rightarrow \mathbb{C}^3$. Since $S$ is a strong deformation retract of $K'$ then, if the approximation of $F$ by $F_1$ is close enough, ensures the existence of a smoothly bounded Runge compact domain $K''$ such that $S \subseteq K'' \subseteq K'$, the Euler characteristic $\chi(K' \setminus K'') = 0$, and $\max\{||X_1|, -\Re Z_1|| > \rho$ everywhere on $K'' \setminus K$. This reduces the proof to Case 1, and hence concludes the proof of the lemma. 

To finish the proof of Theorem 1.1 we now prove the main result of this section.

Proof of Theorem 1.1. For simplicity of exposition we assume that $n = 1$ and write $F = (X, Y, Z)$; the same proof applies in general. By approximation, we may also assume in view of Theorem 2.3 that $F$ extends, with the same name, to a holomorphic Legendrian embedding on an open neighborhood of $K$ having no constant component functions. Thus, up to slightly enlarging $K$ if necessary, we may assume that $X$ does not vanish anywhere on $bK$, and hence there is a number $\rho_0 > 0$ such that

$$(4.12) \quad \max\{||X|, -\Re Z|| \} > \rho_0 \quad \text{everywhere on } bK.$$ 

Let

$$K_0 := K \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq \bigcup_{j \in \mathbb{Z}_+} K_j = M$$

be an exhaustion of $M$ by smoothly bounded, Runge compact domains such that the Euler characteristic $\chi(K_j \setminus K_{j-1}) \in \{-1, 0\}$ for all $j \in \mathbb{N}$. The existence of such an exhaustion is well known; see for instance [5] Lemma 4.2 for a simple proof.

Set $F_0 := F$. For any sequence of positive numbers $\{\epsilon_j\}_{j \in \mathbb{N}} \downarrow 0$, a standard recursive application of Lemma 4.3 provides a sequence of Legendrian curves $\{F_j = (X_j, Y_j, Z_j): K_j \rightarrow \mathbb{C}^3\}_{j \in \mathbb{N}}$ of class $\mathscr{A}^1(K_j)$ such that the following conditions hold for all $j \in \mathbb{N}$:

(a) $||F_j - F_{j-1}||_{1,K_{j-1}} < \epsilon_j$.

(b) $\max\{||X_j|, -\Re Z_j|| \} > j - 1$ everywhere on $K_j \setminus \hat{K}_{j-1}$.

(c) $\max\{||X_j|, -\Re Z_j|| \} > j$ everywhere on $bK_j$.

Furthermore, by Lemma 5.1 we may also assume that

(d) $(X_j, Y_j, e^Z_j): K_j \rightarrow \mathbb{C}^3$ is one-to-one for all $j \in \mathbb{N}$.

Thus, choosing the number $\epsilon_j > 0$ small enough at each step in the recursive construction, (a) and (d) ensure that the sequence $\{F_j\}_{j \in \mathbb{N}}$ converges uniformly on compact subsets of $M$ to a holomorphic Legendrian immersion $\tilde{F} = (\tilde{X}, \tilde{Y}, \tilde{Z}): M \rightarrow \mathbb{C}^3$ which is as close as desired to $F_0 = F$ in the $C^1$-norm on $K_0 = K$ and such that the holomorphic map $(\tilde{X}, \tilde{Y}, e^{Z}): M \rightarrow \mathbb{C}^2 \times \mathbb{C}_+ \subset \mathbb{C}^3$ is one-to-one. It follows that $\tilde{F}: M \rightarrow \mathbb{C}^3$ is one-to-one as well. Moreover, if each $\epsilon_j > 0$ is chosen sufficiently small, condition (b) guarantees that $\max\{||\tilde{X}|, -\Re Z|| \}: M \rightarrow \mathbb{R}_+$ is a proper map, and hence $\tilde{F}$ is an embedding. This concludes the proof. 

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