On Cohomology Rings of Non-Commutative Hilbert Schemes and CoHa-Modules

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Abstract

We prove that Chow groups of certain non-commutative Hilbert schemes have a basis consisting of monomials in Chern classes of the universal bundle. Furthermore, we realize the cohomology of non-commutative Hilbert schemes as a module over the Cohomological Hall algebra.

Introduction

The cohomology of the Hilbert scheme of $d$ points in an $m$-dimensional affine space has been studied intensively by various authors (e.g. [6], [13] and [11]). The objective of this paper is to investigate cohomological properties of a certain non-commutative generalization of these Hilbert schemes. Observing that the Hilbert scheme of $d$ points in $\mathbb{k}^m$ parametrizes ideals of codimension $d$ of the polynomial algebra in $m$ variables, we might ask for the moduli space of left-ideals of codimension $d$ in the free non-commutative algebra in $m$ letters. This is the most prominent example of a non-commutative Hilbert scheme.

So far, we know that non-commutative Hilbert schemes possess a cell decomposition. This was shown by Reineke [15]. The cells are parametrized by $m$-ary trees with $d$ nodes. The existence of a cell decomposition implies that the Chow group (and also the singular cohomology) is a free group with a basis given by the closures of the cells. As a consequence, it is possible to give an explicit formula for the Poincaré polynomial and for the Euler characteristic.

But as a non-commutative Hilbert scheme is also a non-singular variety, we know that its Chow group possesses a ring structure. It turns out that calculating the intersection product of the cell closures is a difficult task and therefore, this basis is not so well-adapted to the multiplication. We provide another basis of the Chow ring that allows us insights into the multiplicative structure.

Again, let us turn our attention to classical Hilbert schemes for a moment. A result of Lehn and Sorger (cf. [11]) shows that the cohomology ring of the Hilbert scheme of $d$ points in the affine plane is isomorphic to the ring of class functions of the symmetric group $S_d$. This is done by using results of Grojnowski (cf. [9]) and Nakajima (cf. [13]) which show that the direct sum (over all $d$) of all

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these cohomology groups (tensored to the rationals) has the structure of a vertex algebra isomorphic to the bosonic Fock space.

It turns out that an appropriate analog in the non-commutative case would be to give the cohomology of non-commutative Hilbert schemes a module structure over Kontsevich’s and Soibelman’s Cohomological Hall algebra (cf. [10]).

The two major results of this paper are the following: As a non-commutative Hilbert scheme arises as a fine moduli space, it is equipped with a universal bundle. We will exhibit a basis of the Chow group consisting of monomials in the Chern classes of the universal bundle in Corollary 2.5. In particular, this gives a description of the Chow ring as a quotient of a polynomial ring. We thus circumvent the problem that a result of King and Walter [9] does not apply here. Their theorem states that fine quiver moduli are generated by Chern classes of tautological bundles if the quiver is acyclic.

Moreover, we realize the cohomology of non-commutative Hilbert schemes (which equals their Chow ring after extending scalars to the rationals) as a quotient of the Cohomological Hall algebra and describe the kernel of the quotient map explicitly. This is Theorem 3.6.

The paper is organized as follows: In the first section, we recollect some facts on non-commutative Hilbert schemes that are essential for our purposes. In particular, we present Reineke’s cell decomposition (cf. Theorem 1.4).

The first main result of this paper, the existence of a basis of the Chow ring of the non-commutative Hilbert scheme consisting of monomials in Chern classes of the universal bundle, is Corollary 2.5. Not unsurprisingly, it is contained in section 2. It is a direct consequence of Theorem 2.4 which states that certain monomials in Chern classes, parametrized by \(m\)-ary trees (or \(m\)-ary forests for a slightly more general type of non-commutative Hilbert schemes) can be expressed as integer linear combinations of cell closures. These linear combinations provide an upper unitriangular base change matrix. This theorem is proved by expressing the filtration steps of Reineke’s cells decomposition as intersections of degeneracy loci and proving that every irreducible component of this intersection has the “correct” dimension.

The third section is devoted to the description of the module structure over the Cohomological Hall algebra (CoHa, for short). The major result of this section is Theorem 3.6 which states that the kernel of the quotient map from the CoHa to the Chow rings of non-commutative Hilbert schemes can be described using the CoHa-multiplication. The proof of this theorem relies on the Harder-Narasimhan stratification, when interpreting a non-commutative Hilbert scheme as a framed quiver moduli space. It is natural to ask if a similar statement holds true for the non-commutative Hilbert scheme and the CoHa of an arbitrary quiver. In fact, there is a generalization of Theorem 3.6. This will be discussed in future work.

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1 Terminology and Facts

Fix an algebraically closed field \( k \).

1.1 Notation

We basically keep the notation of [13]. Fix positive integers \( d, m \) and \( n \) and vector spaces \( V \) of dimension \( n \) and \( W \) of dimension \( d \). Define \( \hat{R} \) to be the vector space \( \text{Hom}(V, W) \oplus \text{End}(W)^{m} \) and let \( G \) be the algebraic group \( \text{Gl}(W) \) which acts on \( \hat{R} \) via

\[
g \cdot (f, \varphi_1, \ldots, \varphi_m) = (gf, g\varphi_1g^{-1}, \ldots, g\varphi_mg^{-1}).
\]

An element \((f, \varphi) = (f, \varphi_1, \ldots, \varphi_m)\) is called stable if \( k(\varphi_1, \ldots, \varphi_m)f(V) = W \), i.e. the image of \( f \) generates \( W \) regarded as a representation of the free non-commutative algebra \( A = k(x_1, \ldots, x_m) \) in \( m \) variables. On the set \( \hat{R}_{\text{st}} \) of stable points of \( \hat{R} \), a geometric \( G \)-quotient

\[
\pi : \hat{R}_{\text{st}} \to \text{Hilb}^{(m)}_{d,n}
\]

exists. It is even a principal \( G \)-bundle. The variety \( \text{Hilb}^{(m)}_{d,n} \) is called a non-commutative Hilbert scheme. As \( m \) is fixed throughout this text, we sometimes suppress the dependency on \( m \) and write \( \text{Hilb}_{d,n} \), for convenience. It is a smooth and irreducible variety of dimension \( N := (m-1)d^2 + nd \). Its points parametrize \( A \)-subrepresentations of codimension \( d \) of the free representation \( A^n \). Denote the image \( \pi(f, \varphi) \) by \([f, \varphi] \).

On \( \hat{R}_{\text{st}} \), we consider the \( G \)-equivariant vector bundle \( \hat{R}_{\text{st}} \times W \to \hat{R}_{\text{st}} \), the trivial bundle, equipped with the \( G \)-action \( g \cdot ((f, \varphi), v) = ((gf, g\varphi g^{-1}), gv) \). This descends to a vector bundle \( \mathcal{U} \) on \( \text{Hilb}_{d,n} \) such that \( \pi^{-1}\mathcal{U} \) with its canonical \( G \)-action equals the \( G \)-bundle we have just described. The \( G \)-linear endomorphisms \( \hat{R}_{\text{st}} \times W \to \hat{R}_{\text{st}} \times W \) mapping a point \(((f, \varphi), v)\) to \(((f, \varphi), \varphi_i v)\) descend to endomorphisms \( \Phi_i : \mathcal{U} \to \mathcal{U} \). Choosing a basis \( e_1, \ldots, e_n \) of \( V \) gives rise to \( G \)-linear sections \( \hat{R}_{\text{st}} \to \hat{R}_{\text{st}} \times W \) defined by sending \((f, \varphi)\) to \(((f, \varphi), fe_i)\). These sections, in turn, induce sections \( s_1, \ldots, s_n \) of \( \mathcal{U} \).

The variety \( \text{Hilb}^{(m)}_{d,n} \) is a so-called framed quiver moduli space (cf. [12], [2], or [16]). Consider the \( m \)-loop quiver \( Q \) consisting of a single vertex \( i \) and \( m \) loops. A dimension vector for this quiver is just a natural number, say \( d \). A representation of \( Q \) of dimension vector \( d \) is a \( d \)-dimensional vector space \( W \) together with \( m \) endomorphisms \( \varphi_1, \ldots, \varphi_m \). In the sense of [8], every representation of \( Q \) is stable, when choosing the linear form \( \theta : \mathbb{Z} \to \mathbb{Z} \) to be zero.

Define a quiver \( \hat{Q}_n \) with two vertices \( \infty \) and \( i \) and with \( n \) arrows pointing from \( \infty \) to \( i \) and \( m \) loops at \( i \). In a picture

\[
\infty \to \{n\} \to i \to \mathcal{P}(m).
\]

For this quiver, we consider the dimension vector \((1, d)\). A representation of \( \hat{Q}_n \) consists of a representation of \( Q \) of dimension vector \( d \) and additionally, a linear map \( f \) from an \( n \)-dimensional space \( V \) to \( W \). When choosing the extended stability condition \( \tilde{\theta} \) as in [16] according to \( \theta = 0 \), we obtain that such a representation \((f, \varphi)\) is stable if and only if it is \( \tilde{\theta} \)-stable.
1.2 Words and Forests

Let $\Omega := \Omega^{(m)}$ be the set of words on the alphabet $\{1, \ldots, m\}$. The empty word will be denoted by $\varepsilon$.

**Definition.** A finite subset $S$ of $\Omega$ is called a ($m$-ary) tree if it is closed under taking left subwords, that means, $w \in S$ provided $ww' \in S$ for some $w' \in \Omega$.

A ($m$-ary) forest with $n$ roots is an $n$-tuple $S_s = (S_1, \ldots, S_n)$ of ($m$-ary) trees.

Let $\mathcal{F}_{d,n} := \mathcal{F}^{(m)}_{d,n}$ be the set of $m$-ary forests with $n$ roots and $d$ nodes. Here, a forest $S_s = (S_1, \ldots, S_n)$ is said to have $d$ nodes if $\#S_1 + \cdots + \#S_n = d$. For a word $w$ with $w \in S_k$, we write $(k, w) \in S_s$.

A pair $(k', w')$ consisting of an index $1 \leq k' \leq n$ and a word $w' \in \Omega$ is called critical for a forest $S_s$ if either $w' = \varepsilon$ and $S_{k'} = \emptyset$ or if $w' \notin S_{k'}$ but there exists a word $w \in S_{k'}$ and a letter $i \in \{1, \ldots, m\}$ with $w' = wi$. We define $C(S_s)$ to be the set of critical pairs of $S_s$. Its cardinality $c(S_s)$ equals $(m-1)\#S_s + n$.

We introduce an ordering on $\Omega$, the so-called lexicographic ordering. For two words $w = i_1 \ldots i_s$ and $w' = i'_1 \ldots i'_t$, let $p$ be the largest index such that $i_p = i'_p$. Formally, we define $p = 0$ if such an index does not exist. Define $w \leq w'$ if either $p = s$ (note that $s$ is the length of $w$) or $i_{p+1} < i'_{p+1}$.

This ordering can be extended to an ordering on the set of trees. Let $S$ and $S'$ be two distinct trees. Define $S < S'$ if either $\#S > \#S'$ or $\#S = \#S'$ and, writing $S = \{w_1 < \ldots < w_s\}$ and $S' = \{w'_1 < \ldots < w'_t\}$, we obtain $w_{p+1} < w'_{p+1}$ for the maximal index $p$ such that $w_p = w'_p$.

Let us enlarge this to an ordering on the set of forests. For two distinct forests $S_s, S'_s \in \mathcal{F}_{d,n}$, let $p$ be the largest index with $S_p = S'_p$ (again, $p = 0$ if $S_1 \neq S'_1$). Define $S_s < S'_s$ if $S_{p+1} < S'_{p+1}$.

Let $S_s$ be a forest. Define $D(S_s)$ to be the set of all quadruples $(k, w, k', w')$ consisting of indexes $1 \leq k, k' \leq n$ and words $w, w' \in \Omega$ satisfying

- $(k, w) \in S_s$,
- $(k', w') \in C(S_s)$, and
- $(k, w) < (k', w')$, that means either $k < k'$ or $k = k'$ and $w < w'$.

The cardinality of $D(S_s)$ will be denoted $d(S_s)$.

**Example.** Let us describe all $S_s \in \mathcal{F}_{d,n}^{(m)}$ for $m = 2$, $d = 3$ and $n = 1$. This example will accompany us throughout the text. When displaying $\Omega$ as follows

```
   ε
  /\  \
 1  2
/ \ / \ /
11 12 21 22
```

The forest $S_s = (S_1, S_2)$ is a valid representation of this set. When displaying $\Omega$ further, we omit some of the critical pairs.
then the 2-ary - let us call them binary - trees with 3 nodes are exactly those:

Compare this to Stanley’s list of descriptions of the Catalan numbers (cf. [18, Ex. 6.19]). The above is part (c) of the list. When considering the sets $S_s \sqcup C(S_s)$, we also get a tree (or a forest, in general), more precisely a plane binary tree with $(m-1)d + n = 7$ vertices. This is part (d) of Stanley’s list.

In the sketch below, the vertices belonging to $S_s$ are displayed in gray:

Now, let us determine the sets $D(S_s)$. They are given by

$$
\begin{array}{cccccccc}
\varepsilon & 111 & 112 & 12 & 2 & \varepsilon & 11 & 121 & 22 \\
1 & \bullet & \bullet & \bullet & \bullet & 1 & \bullet & \bullet & \bullet \\
111 & \bullet & \bullet & \bullet & \bullet & 2 & \bullet & \bullet & \bullet
\end{array}
$$

and when viewing the missing entries as Young diagrams (after turning them upside down) fitting in some triangular shape, we obtain (v) of [18, Ex. 6.19].

1.3 A Cell Decomposition

For a word $w \in \Omega$, say $w = i_1 \ldots i_s$, and a point $(f, \varphi) \in \hat{R}$, define the endomorphism $\varphi_w$ of $W$ to be the composition $\varphi_{i_s} \ldots \varphi_{i_1}$. In the same vein, define $\Phi_w := \Phi_{i_s} \ldots \Phi_{i_1}$ to obtain an endomorphism of the bundle $\mathcal{W}$. Finally, define the section $s_{(k, w)}$ of $\mathcal{W}$ to be $\Phi_w s_k$.

**Definition.** Let $S_s \in \mathcal{F}_{d,n}^{(m)}$ be a forest. Define $U_{S_s}$ to be the subset of all $[f, \varphi] \in \text{Hilb}_{d,n}^{(m)}$ such that the vectors $\varphi_w f e_k$ with $(k, w) \in S_s$ form a basis of $W$.

Reineke shows that for every point $[f, \varphi]$ of $\text{Hilb}_{d,n}$ and every forest $S'_s$ for which the tuple of vectors $(\varphi_w f e_k \mid (k, w) \in S'_s)$ is linearly independent, there exists a forest $S_s \in \mathcal{F}_{d,n}$ containing $S'_s$ such that $[f, \varphi]$ in $U_{S_s}$. Furthermore, by expressing $[f, \varphi]$ in terms of the basis $\varphi_w f e_k$ with $(k, w) \in S_s$, he shows that $U_{S_s}$ is isomorphic to an affine space. This implies:

**Lemma 1.1.** The variety $\text{Hilb}_{d,n}^{(m)}$ is covered by the open subsets $U_{S_s}$ with $S_s \in \mathcal{F}_{d,n}^{(m)}$, each of which is isomorphic to an affine space of dimension $N = (m-1)d^2 + nd$.

Next, we define certain closed subsets of the $U_{S_s}$. These subsets will be the cells of the cell decomposition we are about to describe.
**Definition.** Let \( S_* \in \mathcal{S}^{(m)}_{d,n} \) be a forest. Define \( Z_{S_*} \) to be the set of all \([f, \varphi] \in U_{S_*}\) such that for all critical pairs \((k', w') \in C(S_*)\), the vector \( \varphi_w f e_k \) is contained in the span of all \( \varphi_w f e_k \) with \((k, w) \in S\) and \((k, w) < (k', w')\).

In [13], Theorem 3.6 gives a description of \( Z_{S_*} \) as a set in terms of the \( U_{S_*}' \) for \( S_*' < S_* \). It reads as follows:

**Theorem 1.2.** For all forests \( S_* \in \mathcal{S}^{(m)}_{d,n} \), we obtain \( Z_{S_*} = U_{S_*} \setminus \bigcup_{S_*' < S_*} U_{S_*'} \).

Let us equip \( Z_{S_*} \) with the reduced closed subscheme structure of \( U_{S_*} \). Displaying \([f, \varphi]\) in terms of the basis \( \varphi_w f e_k \) with \((k, w) \in S_*\) shows that \( Z_{S_*} \) is also an affine space. Precisely:

**Lemma 1.3.** The closed subset \( Z_{S_*} \), viewed as a reduced closed subscheme of \( U_{S_*} \), is isomorphic to an affine space of dimension \( d(S_*) \).

These results culminate in a main result of [13], the existence of a cell decomposition of \( \text{Hilb}_{d,n} \). By definition, a cell decomposition of a variety is a descending sequence of closed subsets such that the successive complements are isomorphic to affine spaces. Define \( A_{S_*} := \text{Hilb}_{d,n} - \bigcup_{S_*' < S_*} U_{S_*'} \). Enumerating the forests of \( \mathcal{S}^{(m)}_{d,n} \) lexicographically, say \( S_*^1 < \ldots < S_*^m \), and abbreviating \( A_i := A_{S_*^i} \), we obtain a filtration

\[
\text{Hilb}_{d,n} = A_1 \supseteq A_2 \supseteq \ldots \supseteq A_{u} \supseteq A_{u+1} := \emptyset
\]

by closed subsets satisfying \( A_i - A_{i+1} = Z_{S_*^i} \). Cutting a long story short:

**Theorem 1.4.** The variety \( \text{Hilb}^{(m)}_{d,n} \) possesses a cell decomposition parametrized by forests \( S_* \in \mathcal{S}^{(m)}_{d,n} \), whose cells \( Z_{S_*} \) are of dimensions \( d(S_*) \).

An immediate application yields a basis of the Chow group of \( \text{Hilb}_{d,n} \). Denote by \( \mathcal{Z}_{S_*} \) the closure of \( Z_{S_*} \) in \( \text{Hilb}_{d,n} \) equipped with the reduced closed subscheme structure. As \( \mathcal{Z}_{S_*} \) is irreducible, it becomes a closed subvariety of \( \text{Hilb}_{d,n} \).

**Corollary 1.5.** The Chow group \( A_*(\text{Hilb}^{(m)}_{d,n}) \) is the free abelian group with basis \( [\mathcal{Z}_{S_*}] \) for \( S_* \in \mathcal{S}^{(m)}_{d,n} \).

**Example (continued).** Again, let \( m = 2, d = 3 \) and \( n = 1 \). We describe the cells \( Z_{S_*} \) belonging to the binary trees \( S_* \) with 3 nodes. A point of \( R \) may be viewed as a triple \((v, A, B)\), where \( v \in k^3 \) and \( A \) and \( B \) are \((3 \times 3)\)-matrices. Write \([v, A, B]\) for its image in the non-commutative Hilbert scheme. The cells are

\[
\begin{align*}
Z_{\cdot}^r &= \{ [v, A, B] | v, Av, A^2v \text{ basis of } k^3 \}, \\
Z_{\cdot}^l &= \{ [v, A, B] | v, Av, BAv \text{ basis of } k^3 \text{ and } A^2v \in \langle v, Av \rangle \}, \\
Z_{\cdot}^\ast &= \{ [v, A, B] | v, Av, Bv \text{ basis of } k^3 \text{ and } A^2v, BAv \in \langle v, Av \rangle \}, \\
Z_{\cdot}^\ast & = \{ [v, A, B] | v, Bv, ABv \text{ basis of } k^3 \text{ and } Av \in \langle v \rangle \}, \text{ and} \\
Z_{\cdot}^\ast & = \{ [v, A, B] | v, Bv, B^2v \text{ basis of } k^3, \text{ and } Av \in \langle v \rangle, \text{ and } ABv \in \langle v, Bv \rangle \}.
\end{align*}
\]

Their dimensions allow us to determine the Poincaré polynomial \( \sum_i \dim_q(A_i(\text{Hilb}^{(m)}_{d,n})/q)t^i \) at once. It reads \( t^{12} + t^{11} + 2t^{10} + t^9 \).
2 Another Basis of the Chow Group

We are interested in the ring structure on the Chow group of the non-commutative Hilbert scheme $\text{Hilb}_{d,n}^{(m)}$. It turns out that computing the intersection product of two cell closures is rather difficult. We would therefore like to find another basis that is better adapted to the multiplication. This basis will be provided by Chern classes of the universal bundle $\mathcal{W}$ which we have already introduced in the previous section.

2.1 A Connection Between Cell Closures and Chern Classes of the Universal Bundle

Let $S_\ast \in \mathcal{F}_{d,n}$ be a forest. Order the words of the trees lexicographically, i.e. $S_k = \{w_{k,1} < \ldots < w_{k,d_k}\}$. Consider all pairs $(k,w) \in S_\ast$ and order them lexicographically, too. This gives

$$(1, w_{1,1}) < \ldots < (1, w_{1,d_1}) < \ldots \ldots < (n, w_{n,1}) < \ldots < (n, w_{n,d_n})$$

and we denote these pairs as $x_1 < \ldots < x_d$. This means $(k, w_{k,\nu}) = x_{d_1 + \ldots + d_{\nu - 1} + \nu}$. For a critical pair $x' = (k', w')$ of $S_\ast$, let $j = j_{S_\ast}(x')$ be the maximal index such that $x_j < x'$. Formally, let $j = 0$ if such an index does not exists.

We express $j$ in a slightly different way. If $w' = \varepsilon$, then $j = d_1 + \ldots + d_{k' - 1}$. Otherwise, it is $j = d_1 + \ldots + d_{k' - 1} + \nu$, where $\nu$ is the maximal index such that $w_\nu < w'$ (and $\nu$ is not zero in this case, but possibly $d_{k'}$).

As $D(S_\ast)$ is clearly in bijection to the disjoint union of the sets $\{(x_1, x'), \ldots, (x_{j(x')}, x')\}$, with $x'$ ranging over all critical pairs of $S_\ast$, we see that $\sum_{x' \in C(S_\ast)} j(x') = d(S_\ast)$. Define $i(x') := i_{S_\ast}(x')$ to be $d - j(x')$. We will show the following:

**Theorem 2.1.** For all forests $S_\ast \in \mathcal{F}_{d,n}^{(m)}$, we have

$$\prod_{x' \in C(S_\ast)} c_{i(x')}(\mathcal{W}) \cap [\text{Hilb}_{d,n}^{(m)}]_{[\mathcal{Z}^S_{S_\ast}]} = [\mathcal{Z}^S_{S_\ast}] + \sum_{\mathcal{Z}^S_{S'_{\ast}} \supseteq S_\ast, d(S'_{\ast}) \neq d(S_\ast)} n_{S_\ast, S'_{\ast}} [\mathcal{Z}^S_{S'_{\ast}}]$$

for some positive integers $n_{S_\ast, S'_{\ast}}$.

Recall the section $s_x = s_{(k,w)}$ associated to any pair $x = (k,w)$ consisting of an index $1 \leq k \leq n$ and a word $w$. For a forest $S_\ast$, define $D_{S_\ast}$ as the intersection of the degeneracy loci

$$D_{S_\ast} = \bigcap_{x' \in C(S_\ast)} D_{S_\ast}(x'),$$

where $D_{S_\ast}(x') := D(s_{x_1}, \ldots, s_{x_{j(x')}}, s_{x'})$ is the degeneracy locus as defined in [5, Chap. 14]. As all degeneracy loci possess a natural structure of a closed subscheme of $\text{Hilb}_{d,n}$, the subset $D_{S_\ast}$ does, too.

**Lemma 2.2.** The underlying closed subset of $D_{S_\ast}$ is $A_{S_\ast} = \text{Hilb}_{d,n}^{(m)} \cap \bigcup_{S'_{\ast} < S_\ast} U_{S'_{\ast}}$. 

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2.1 A Connection Between Cell Closures and Chern Classes of the Universal Bundle

**Proof.** Order the pairs of $S_*$, i.e. $x_1 = (k_1, w_1) < \ldots < x_d = (k_d, w_d)$. A point $[f, \varphi]$ lies in $D_{S_*}$ if and only if the vectors 

$$\varphi_{w_1} fe_{k_1}, \ldots, \varphi_{w_d} fe_{k_d}$$

are linearly dependent for all critical pairs $x' = (k', w')$ of $S_*$. Let $S'_* \in \mathcal{F}$ with $S'_* < S_*$, say $S' = \{x'_1 < \ldots < x'_d\}$. Define $p$ to be the maximal index such that $x_p = x'_p$. We have $x'_{p+1} < x_{p+1}$ and therefore $x'_{p+1} \notin S_*$. We write $x'_{p+1}$ as $(k', w')$, this means $w' \notin S'$. If $w' = \varepsilon$, then $S_{k'} = \emptyset$ and if $w' \neq \varepsilon$, we write $w' = w$ for some $w \in S'_k$. As $(k', w) < x'_{p+1}$, we obtain $w \in S'_{k'}$. This means $x'_{p+1}$ is a critical pair for $S_*$. Moreover, we get $j(x'_{p+1}) = p$. Therefore, the vectors

$$\varphi_{w_1} fe_{k_1}, \ldots, \varphi_{w_p} fe_{p}, \varphi_{w'} fe_{k'}$$

are linearly dependent and this implies that $[f, \varphi]$ is not contained in $U_{S'_*}$.

Conversely, assume that $[f, \varphi]$ does not belong to the union $\bigcup_{S'_* < S_*} U_{S'_*}$. Let $x' = (k', w')$ be a critical pair for $S_*$. Let $j := j(x')$ and write $x_j = (k'_j, w'_j)$. Consider the forest $S'_*$ consisting of $S'_k := S_k$ for all $k < k'$, of

$$S'_k = \{w_{k'_j} < \ldots < w_{k', w'} < w'\}$$

and of $S'_k = \emptyset$ for $k > k'$. Assume that the vectors $\varphi_{w_1} f e_{k_1}, \ldots, \varphi_{w_p} f e_{p}, \varphi_{w'} f e_{k'}$ were linearly independent. By [15 Lemma 3.2], there exists a forest $S''_*$ containing $S'_*$ such that $[f, \varphi]$ belongs to $U_{S''_*}$. But this forest fulfills $S''_k < S_*$. A contradiction. □

**Example (continued).** Let us determine the underlying closed subsets $A_{S_*}$ of the $D_{S_*}$ in the - by now well known - case $m = 2$, $d = 3$ and $n = 1$. We have

\[
\begin{align*}
A_{S_*} &= \text{Hilb}_{3,1}^{(2)}, \\
A_{S_*}^\prime &= \{[v, A, B] \mid v, Av, A^2v \text{ linearly dependent}\}, \\
A_{S_*}^\prime &\prime = \{[v, A, B] \mid v, Av, A^2v \text{ and } v, Av, BAv \text{ both linearly dependent}\}, \\
A_{S_*}^\prime \prime &= \{[v, A, B] \mid v, Av \text{ linearly dependent}\}, \text{ and} \\
A_{S_*}^\prime \prime \prime &= \{[v, A, B] \mid v, Av \text{ and } Bv, ABv \text{ both linearly dependent}\}.
\end{align*}
\]

We can easily see that the successive complements are, indeed, the cells $Z_{S_*}$.

For general reasons (cf. [15 Thm. 14.4]), we know that every irreducible component of $D_{S_*}(x')$ has dimension at least $N - i(x')$. We will show that, in fact, equality holds.

**Lemma 2.3.** Let $T_* = \{x_1 < \ldots < x_j < x'\}$ be a forest. Then, the closed subset $D_{T_*} = D(s_{x_1}, \ldots, s_{x_j}, s_{x'})$ has pure dimension $N - (d - j)$ (or is empty).

**Proof.** The proof proceeds by induction on $j$. In the case $j = 0$, the forest $T_*$ equals $\{(k', \varepsilon)\}$ for an index $1 \leq k' \leq n$. Choose a forest $S_* \in \mathcal{F}_{d,n}$ such that $(k', \varepsilon) \in C(S_*)$. If $n = 1$, such a forest does not exist and $D_{T_*}$ is empty. Otherwise, $D_{T_*} \cap U_{S_*} \neq \emptyset$. We apply the isomorphism $U_{S_*} \cong \mathbb{A}^N$ from [15], provided by the functions $\lambda_{x,x'}$ on $U_{S_*}$ for every $x' = (k', w') \in C(S_*)$ and every $x \in S_*$. By definition, $\lambda_{x,x'}[f, \varphi]$ is the coefficient occurring in the linear combination

$$\varphi_{w'} fe_{k'} = \sum_{x=(k,w)\in S_*} \lambda_{x,x'}[f, \varphi] \cdot \varphi_{w} fe_{k}$$

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for \( x' = (k', w') \in C(S_*) \). The closed subscheme \( D_{T_s} \cap U_{S_*} \) is defined by annihilation of all functions \( \lambda_{x, (k', x)} \). This describes an affine space of dimension \( N - d \).

Assume that \( j > 0 \). Let \( T'_s := \{ x_1 < \ldots < x_j \} \). This is also a forest. By induction hypothesis, the closed subset \( D_{T'_s} \), which is contained in \( D_{T_s} \), has pure dimension \( N - (d - j) - 1 \) (or is empty). But as we know that every irreducible component of \( D_{T_s} \) has dimension at least \( N - (d - j) \), it suffices to show that \( D_{T'_s} \cap U_{S_*} \) has pure dimension \( N - (d - j) \) for every forest \( S_* \in \mathcal{F}_{d,n} \) containing \( T'_s \). For \( D_{T_s} \cap U_{S_*} \) not being empty, we require \( x' \notin S_* \). This means that \( x' \) is a critical pair for \( S_* \). Via the isomorphism \( U_{S_*} \cong \mathbb{A}^N \) described above, \( D_{T_s} \cap U_{S_*} \) is given by the ideal generated by all functions \( \lambda_{x, x'} \) with \( x \in S_* \) and \( x > x' \). This describes an affine space of dimension \( N - (d - j) \).

The above lemma implies, using [3, Ex. 14.4.2], that the cycle \([D_{S_*}(x')]\) associated to the closed subscheme \( D_{S'_*}(x') \), regarded as an element of \( A_{N-i(x')}(\text{Hilb}_{d,n}) \), equals \( c_i(x')(\mathcal{U}) \cap [\text{Hilb}_{d,n}] \). We use this observation to prove Theorem 2.1.

**Proof of Theorem 2.1** By the cell decomposition, we obtain that \( A_{S_*} = Z_{S_*} \cup \bigcup_{S'_* > S_*} Z_{S'_*} = \mathcal{Z}_{S_*} \cup \bigcup_{S'_* > S_*} \mathcal{Z}_{S'_*} \). The proper components of the intersection of the \( D_{S_*}(x') \) with \( x' \in C(S_*) \) are among those \( \mathcal{Z}_{S'_*} \) with \( S'_* \geq S_* \) and \( d(S_*) = d(S'_*) \). Hence, using [3, Ex. 8.2.1], there are positive integers \( n_{S_*, S_*} \) and \( n_{S_*, S'_*} \) such that

\[
\prod_{x' \in C(S_*)} c_i(x')(\mathcal{U}) \cap [\text{Hilb}_{d,n}] = \prod_{x' \in C(S_*)} [D_{S_*}(x')] = n_{S_*, S_*}[\mathcal{Z}_{S_*}] + \sum_{S'_* > S_*} n_{S_*, S'_*}[\mathcal{Z}_{S'_*}].
\]

It remains to prove that the coefficient \( n_{S_*, S_*} \) is 1. In order to do so, it suffices to prove that \( D_{S_*} \cap U_{S_*} = Z_{S_*} \) as schemes. As mentioned above, Reineke shows that an isomorphism \( U_{S_*} \rightarrow \mathbb{A}^N \) (with \( N := (m-1)d^2 + nd \)) is given by the functions \( \lambda_{x, x'} \) with \( x \in S_* \) and \( x' \in C(S_*) \) assigning to every point \([f, \varphi]\) of \( U_{S_*} \) the coefficient \( \lambda_{x, x'}[f, \varphi] \) that occurs displaying \( \varphi_{w'}e_k \) as a linear combination

\[
\varphi_{w'}e_k = \sum_{x = (k, w) \in S_*} \lambda_{x, x'}[f, \varphi] \cdot \varphi_{w'}e_k
\]

where \( x' = (k', w') \). Over \( U_{S_*} \), the bundle \( \mathcal{U} \) trivializes. Moreover, for every pair \( x_0 = (k_0, w_0) \), the sections \( s_{x_0} \) of \( \mathcal{U} \) correspond to the sections of the trivial rank \( d \)-bundle on \( U_{S_*} \) assigning to \([f, \varphi]\) the matrix \((a_{x, x_0} | x \in S_*)\) of coefficients of the linear combination \( \varphi_{w_0} e_k = \sum_{x = (k, w) \in S_*} a_{x, x_0} \varphi_{w} e_k \).

Therefore, enumerating \( S_* = \{ x_1 < \ldots < x_d \} \), the section \( s_{x_0} \) restricted to \( \mathbb{A}^N \) maps a matrix \( \lambda \) to the \( i \)-th coordinate vector and \( s_{x'_*} \) maps \( \lambda \) to the vector \((\lambda_{x_1, x'}, \ldots, \lambda_{x_d, x'}) \). Under the isomorphism \( U_{S_*} \rightarrow \mathbb{A}^N \), the degeneracy locus is therefore given by the vanishing of all \( j(x') \)-minors of the matrices

\[
\begin{pmatrix}
  1 & \lambda_{x_1, x'} \\
  \vdots & \ddots & \ddots \\
  \lambda_{x_1, x'} & \cdots & 1 \\
 \end{pmatrix}.
\]
Thus, locally on $U_S \cong \mathbb{A}^N$, the degeneracy locus $D_S$ is given by the ideal generated by the coordinate functions $\lambda_{x_j(x')_i+1,x'}, \ldots, \lambda_{x_d,x'}$ with $x'$ ranging over all critical pairs of $S$. It is therefore an affine space of dimension $d(S)$ and thus isomorphic to $Z_S$.

**Remark 2.4.** Fixing the notation as in Theorem 2.1 we are able to determine the numbers $n_{S,S'}$ - at least in principle. They are given as intersection multiplicities

$$n_{S,S'} = i\left(\mathcal{Z}_{S'}, D_S(x_1) \ldots D_S(x_r); \text{Hilb}_{d,n}^{(m)}\right)$$

as defined in [5, Ex. 8.2.1]. Here, $\{x_1', \ldots, x_r'\} = C(S')$. As the non-commutative Hilbert scheme is non-singular, it is also Cohen-Macaulay. Applying Lemma 2.3 and [5, Ex. 14.4.2], we obtain that every $D_S(x')$ is Cohen-Macaulay, too. Thus, [5] Prop. 8.2 implies that

$$n_{S,S'} = l\left(\mathcal{O}_{D_S \cap \mathcal{Z}_{S'}}, \mathcal{O}_{\mathcal{Z}_{S'}}\right),$$

$D_S$ being equipped with its natural scheme structure.

### 2.2 A Basis Consisting of Monomials in Chern Classes

We reformulate Theorem 2.1 slightly. Let $S_r \in S_{d,n}$ be a forest. Enumerate $S_r = \{x_1 < \ldots < x_d\}$ and $C(S) = \{x_1', \ldots, x_{(m-1)d+n}'\}$. Define $j_\nu$ to be the index $j_S(x_\nu')$. One can show that this induces a bijection

$$\mathcal{F}_{d,n}^{(m)} \to \mathcal{F}_{d,n}^{(m)},$$

where the right hand side is the set of tuples $(j_1, \ldots, j_{(m-1)d+n})$ of integers $0 \leq j_1 \leq \ldots \leq j_{(m-1)d+n} \leq d$ such that $j_\nu \geq l$ for every $1 \leq l \leq d$ and every $(m-1)(l-1) + n \leq \nu \leq (m-1)l + n - 1$.

For a tuple $(j_1, \ldots, j_{(m-1)d+n})$, define values $b_0, \ldots, b_{d-1}$ by letting $b_i$ be the number of $\nu$ with $j_\nu = i$. This, in turn induces a bijection

$$\mathcal{J}_{d,n}^{(m)} \to \mathcal{B}_{d,n}^{(m)},$$

the latter being the set of all tuples $(b_0, \ldots, b_{d-1})$ of non-negative integers such that $\sum_{r=0}^{i} b_r < (m-1)i + n$ for all $0 \leq i \leq d - 1$. This proves the following:

**Corollary 2.5.** The underlying additive group of the Chow ring $A^*(\text{Hilb}_{d,n}^{(m)})$ is a free abelian group with basis

$$c_1(\mathcal{U})^{b_{d-1}} \ldots c_d(\mathcal{U})^{b_0},$$

where $(b_0, \ldots, b_{d-1})$ ranges over all tuples of non-negative integers satisfying $b_0 + \ldots + b_i < (m-1)i + n$ for every $0 \leq i \leq d - 1$.

**Example (continued).** Let us illustrate Corollary 2.5 in our favorite example $m = 2, d = 3$ and $n = 1$. The bijections $\mathcal{F} \to \mathcal{J} \to \mathcal{B}$ yield the following result: The trees
again displayed together with the sets $C(S_\ast)$, give rise to the following sequences of numbers in $\mathcal{I}_{3,1}^{(2)}$

\[
3333 \quad 2333 \quad 2233 \quad 1333 \quad 1233
\]

which, in turn, correspond to the following sequences of integers in $\mathcal{B}_{3,1}^{(2)}$

\[
000 \quad 001 \quad 002 \quad 010 \quad 011
\]

By forming the sequences of partial sums and then increasing every entry by one, we end up at Stanley’s (s) (cf. [13, Ex. 6.19]). We have thus obtained a basis for the underlying (free) abelian group of the Chow ring of the non-commutative Hilbert scheme. It reads

\[
A^*(\text{Hilb}_{3,1}^{(2)}) = \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot c_1(\mathcal{V}) \oplus \mathbb{Z} \cdot c_1(\mathcal{V})^2 \oplus \mathbb{Z} \cdot c_2(\mathcal{V}) \oplus \mathbb{Z} \cdot c_1(\mathcal{V})c_2(\mathcal{V}).
\]

Applying Theorem 2.1 also gives us a relation between these (monomials in) Chern classes and the cycles associated to the cell closures. The theorem tells us that

\[
1 \cap [\text{Hilb}_{3,1}^{(2)}] = [\mathcal{Z}_\ast],
\]

\[
c_1(\mathcal{V}) \cap [\text{Hilb}_{3,1}^{(2)}] = [\mathcal{Z}_\ast],
\]

\[
c_1(\mathcal{V})^2 \cap [\text{Hilb}_{3,1}^{(2)}] = [\mathcal{Z}_\ast] + n[\mathcal{Z}_\ast],
\]

\[
c_2(\mathcal{V}) \cap [\text{Hilb}_{3,1}^{(2)}] = [\mathcal{Z}_\ast],
\]

\[
c_1(\mathcal{V})c_2(\mathcal{V}) \cap [\text{Hilb}_{3,1}^{(2)}] = [\mathcal{Z}_\ast]
\]

for some positive integer $n$.

As we have remarked (cf. Remark 2.4), the integer $n$ is precisely the intersection multiplicity

\[
i \left( \mathcal{Z}_\ast, D_\ast \cap \text{Hilb}_{3,1}^{(2)} \right)
\]

which equals the length of the local ring $\mathcal{O}_{D_\ast \cap U_\ast \mathcal{Z}_\ast}$. Employing the isomorphism $U_\ast \cong \mathbb{A}^{12}$ from [13], we regard an element $[v, A, B] \in U_\ast$ as a tuple

\[
1 \quad x_{v,1} \quad 0 \quad x_{v,211} \quad 0 \quad x_{v,22} \quad x_{v,212} \quad 1 \quad x_1 \quad 0 \quad x_2 \quad 0 \quad x_5 \quad x_8
\]

\[
0 \quad x_{2,1} \quad 1 \quad x_{2,211} \quad 0 \quad x_{2,22} \quad x_{2,212} =: 0 \quad y \quad 1 \quad x_3 \quad 0 \quad x_6 \quad x_9
\]

\[
0 \quad x_{21,1} \quad 0 \quad x_{21,211} \quad 1 \quad x_{21,22} \quad x_{21,212} \quad 0 \quad z \quad 0 \quad x_4 \quad 1 \quad x_7 \quad x_{10}
\]

which comes from displaying $[v, A, B]$ in terms of the basis $v, Bv, ABv$. The closed subscheme $D_\ast \cap U_\ast$ is defined by the vanishing of the determinants

\[
\det(v \mid Av \mid A^2v) = \begin{vmatrix} 1 \quad x_1 \quad x_1^2 + x_2z \\ 0 \quad y \quad x_1y + x_3z \\ 0 \quad z \quad x_1z + y + x_4z \end{vmatrix} = y^2 + x_4yz - x_3z^2 \quad \text{and}
\]

\[
\det(v \mid Av \mid BAv) = \begin{vmatrix} 1 \quad x_1 \quad x_5y + x_8z \\ 0 \quad y \quad x_1 + x_6y + x_9z \\ 0 \quad z \quad x_7y + x_{10}z \end{vmatrix} = x_7y^2 + (x_{10} - x_6)yz - x_1z - x_9z^2.
\]
On the other hand, the closed subvariety $Z_\varphi$ is given by the vanishing of $y$ and $z$. Therefore, the local ring of $D_{\varphi} \cap U_\varphi$ along the closed subvariety $Z_\varphi$ is
\[ k(x_1, \ldots, x_{10})[y, z]/(y^2 + x_4yz - x_3z^2, x_7y^2 + (x_{10} - x_6)yz - x_1z - x_9z^2). \]
A lengthy computation shows that the length of this (artinian) ring - which equals its dimension over $k(x_1, \ldots, x_{10})$ - is 4. The author has determined this using SINGULAR.

In particular, we see that the Chow ring is generated by the Chern classes $c_1(\mathcal{U}), \ldots, c_d(\mathcal{U})$. A result of King and Walter (cf. [9]) asserts that the Chow ring of a fine quiver moduli space is generated by the Chern classes of the universal bundle if the quiver has no oriented cycles. This theorem is not applicable here, yet the statement holds nonetheless.

3 A Module Structure over the Cohomological Hall Algebra

The Cohomological Hall algebra (CoHa), which we will call CoHa in the following, was invented by Kontsevich and Soibelman in [10]. We will consider the CoHa for the $m$-loop quiver and define a module structure on the Chow rings of non-commutative Hilbert schemes in the case $n = 1$.

3.1 Cohomological Hall Algebra of a Loop Quiver

We consider the $m$-loop quiver $Q$. A dimension vector for $Q$ is just a non-negative integer $d$. Denote by $R_d$ the vector space $M_{d \times d}^m$ of $m$-tuples of $(d \times d)$-matrices. On this space, we have an action of the reductive linear algebraic group $G_d := \text{GL}_d$ by conjugation. We define
\[ \mathcal{H}_d := A^*_G(A_d Q) \]
the $G_d$-equivariant Chow ring as defined by Edidin and Graham (cf. [3]). See also Brion’s article [1]. Note that, in case $k = \mathbb{C}$, the equivariant cycle map $A^*_G(A_d Q) \rightarrow H^*_G(A_d Q)$ is an isomorphism (of rings) that doubles the degrees. Therefore, we may work with equivariant Chow rings (with rational coefficients) rather than with equivariant cohology rings (like Kontsevich and Soibelman do).

We make the following convention: In this section, all (equivariant) Chow rings are extended to the rationals, i.e. write $A^*_G(X)$ instead of $A^*_G(A_d Q)$. On the direct sum $\mathcal{H} := \bigoplus_{d \geq 0} \mathcal{H}_d$, Kontsevich and Soibelman define a “convolution like” multiplication $\mathcal{H}_p \otimes \mathcal{H}_q \rightarrow \mathcal{H}_{d=p+q}$, assigning $f \otimes g \rightarrow f \ast g$, as the composition of the horizontal maps in
\[
\begin{align*}
A^i_G(p) \otimes A^j_G(q) & \xrightarrow{x} A^{n+i+j}_{G \times G}(R_p \times R_q) \\
& \cong A^i_{p,q}(R_p)(R_p) \xrightarrow{i_*} A^{n+pq}_{p,q}(R_d) \xrightarrow{\pi_*} A^{n+(m-1)pq}_{G_d}(R_d).
\end{align*}
\]
Here, $P_{p,q}$ denotes the upper parabolic of $G_d$ to the decomposition of $k^d$ into the coordinate space of the first $p$ and the last $q$ unit vectors. The subspace $R_{p,q}$ of $R_d$ is the space of all tuples $(\varphi_1, \ldots, \varphi_m)$.
of \((d \times d)\)-matrices, mapping the coordinate space \(k^p\) into itself. From now on, write \(L_{p,q} := G_p \times G_q\). It is the Levi factor of the parabolic \(P_{p,q}\). The above maps arise as follows: The map \(\times\) is the equivariant exterior product. For the other maps, choose a \(i\) the non-horizontal maps being affine space bundles, \(G\) morphism with fiber 
\[\dim V \leq \dim(V - U) > n + (m - 1)pq.\]
We have morphisms

\[
\begin{array}{ccc}
(R_p \times R_q) \times L_{p,q} U & R_{p,q} \times P_{p,q} U & R_d \times P_{p,q} U \\
\downarrow & \downarrow & \downarrow \\
R_{p,q} \times L_{p,q} U & R_d \times G_d U,
\end{array}
\]

the non-horizontal maps being affine space bundles, \(i\) is a closed embedding and \(\pi\) is a smooth morphism with fiber \(G_d/P_{p,q} = \text{Gr}_p(k^d)\). In particular, \(\pi\) is proper.

This multiplication makes \(\mathcal{H}\) an associative graded algebra with a unit \(1 \in \mathcal{H}_0\). Moreover, we can define a \((\mathbb{Z}_{\geq 0} \times \mathbb{Z})\)-bigrading on \(\mathcal{H}\) by putting

\[
\mathcal{H}_{d,k} := A^{(m-1)(\frac{d}{2})-k}_{Gr_d}(R_d).
\]

This bigrading is compatible with the multiplication. It is a little different from the one used in [10]. The bigrading we use here was suggested in [17].

In [10], an explicit formula for the multiplication is derived. Identifying \(\mathcal{H}_d = A^*_d(R_d)\) with \(A^*_d(\text{pt}) \cong A^*_{R_d}(\text{pt})^W = \mathbb{Q}[x_1, \ldots, x_d]^S_d\), we obtain

\[
(f \ast g)(x_1, \ldots, x_d) = \sum_{1 \leq i_1 < \ldots < i_p \leq d} f(x_{i_1}, \ldots, x_{i_p})g(x_{j_1}, \ldots, x_{j_q}) \prod_{\mu=1}^p \prod_{\nu=1}^q (x_{j_\nu} - x_{i_\mu})^{m-1},
\]

where two sequences \(1 \leq i_1 < \ldots < i_p \leq d\) and \(1 \leq j_1 < \ldots < j_q \leq d\) are called complementary if the union of these numbers is \(\{1, \ldots, d\}\). Using this formula, it is evident that the multiplication \(\ast\) is graded commutative if \(m\) is even, and commutative if \(m\) is odd.

### 3.2 A Computational Approach

Our goal is to realize the direct sum \(\bigoplus_d A^*(\text{Hilb}^{(m)}_{d,1})\) as an \(\mathcal{H}\)-module. In fact, it will turn out to be a quotient of \(\mathcal{H}\) by some ideal, thus it inherits the structure of an algebra itself. The idea stems from a purely algebraic observation.

A point \((v, \varphi) \in R_d\) is stable if and only if \(v\) generates \(k^d\) as a \(k(\varphi_1, \ldots, \varphi_m)\)-module. In other words, this means that a proper subspace \(U\) containing \(v\) cannot be invariant under all \(\varphi_i\). Consider the universal bundle \(\mathcal{V}\) on \(\text{Hilb}_d := \text{Hilb}^{(m)}_{d,1}\) of rank \(d\). Let \(\text{Fl} := \text{Fl}(\mathcal{V}) \to \text{Hilb}_d\) be the complete flag bundle. It possesses a universal flag

\[
\mathcal{V}^r : 0 = \mathcal{V}^0 \subseteq \mathcal{V}^1 \subseteq \ldots \subseteq \mathcal{V}^d = \mathcal{V}_{\text{Fl}}
\]

with \(\text{rk} \mathcal{V}^i = i\). A point \(y\) of \(\text{Fl}(U)\) is a pair consisting of a \([v, \varphi] \in \text{Hilb}_d\) and a complete flag \(W^r\) of \(\mathcal{V}_{[v, \varphi]} \cong k^d\). The universal flag is defined by the property that its fiber \(\mathcal{V}^y_{[v, \varphi]}\) in the point \(y\) is precisely
We might wonder if these tautological relations provide a presentation of the Chow ring of \( C^* U \) of vanishes at the Chern

In order to obtain the ideal of tautological relations, we may compute

For all \( \tau \) there exist a unique \( \Delta = \Delta \)

In this context, \( \phi \) defined by

if it contains \( v \). In terms of the above sections, this means that

cannot all be zero. In other words, the intersection \( Z(s^p) \cap Z(\Phi_1^p) \cap \ldots \cap Z(\Phi_m^p) \) of the zero loci of the sections is empty. In particular, this implies that

\( Z(s^p) \cdot Z(\Phi_1^p) \ldots Z(\Phi_m^p) = 0, \)

\( Z \) denoting the localized top Chern class as in \([5, 14.1]\). The image of the left-hand expression in the Chow ring \( A^*(\text{Fl}) \) equals

\[
c_{\text{top}} \left( \mathcal{W}_{\text{Fl}} / \mathcal{W}^P \right) \cdot c_{\text{top}} \left( \mathcal{W}_{\text{Fl}} / \mathcal{W}^P \otimes (\mathcal{W}^P)^\vee \right)^m = \xi_{p+1} \cdots \xi_d \prod_{\mu=1}^{p} \prod_{\nu=p+1}^{d} (\xi_{\nu} - \xi_{\mu})^m = f^{(p)}(\xi_1, \ldots, \xi_d)
\]

if we denote \( \xi_{\nu} := c_1(\mathcal{W}^\nu / \mathcal{W}^{\nu-1}) \). In terms of \([4]\), the expression \( f^{(p)} \) is called a forbidden polynomial.

By a result of Grothendieck (cf. \([7]\)), the Chow ring \( A^*(\text{Fl}) \) is isomorphic to \( A^*(\text{Hilb}_{d}) \otimes \mathcal{H}_d \), where \( \mathcal{H}_d = C_{d}^{S_d} = \mathbb{Q}[e_1, \ldots, e_d] \) is the ring of symmetric polynomials. It is a basic fact that \( \mathcal{H}_d \) is a free \( C_d \)-module. Pick a basis \( \mathcal{B} \). Displaying \( f^{(p)} \) in terms of \( \mathcal{B} \) we obtain coefficients \( \tau^{(p)}(y) \) for every \( y \in \mathcal{B} \). As \( f^{(p)} \) vanishes at the Chern roots of \( \mathcal{W} \), the coefficient \( \tau^{(p)}(y) \) (which is a polynomial in the elementary symmetric functions \( e_1, \ldots, e_d \)) vanishes at the Chern classes of \( \mathcal{W} \). Using the language of \([4]\), \( \tau^{(p)}(y) \) is called a tautological relation.

We might wonder if these tautological relations provide a presentation of the Chow ring of \( \text{Hilb}_{d} \).

In order to obtain the ideal of tautological relations, we may compute \( \rho(bf^{(p)}) \), where \( b \) runs through the polynomial ring \( C_d := \mathbb{Q}[x_1, \ldots, x_d] \) and \( \rho = \rho_d : C_d \rightarrow \mathcal{H}_d = \mathbb{Q}[x_1, \ldots, x_d]^{S_d} \) is the \( A \)-linear map defined by

\[
\rho(f) = \Delta^{-1} \sum_{w \in S_d} \text{sign}(w)w.f.
\]

In this context, \( \Delta = \Delta_d \) is the discriminant \( \prod_{i<j}(x_j - x_i) \). We now use the fact that for every \( w \in S_d \), there exist a unique \( \tau \in S_p \times S_q \) and a unique \((p,q)\)-shuffle permutation \( \sigma \) with \( w = \sigma \tau \). Being a \((p,q)\)-shuffle permutation means that \( \sigma \) is of the form

\[
\sigma = \begin{pmatrix} 1 & \ldots & p & p+1 & \ldots & d \\ i_1 & \ldots & i_p & j_1 & \ldots & j_q \end{pmatrix}
\]
We insert the definition of $f$ as a rational vector space by all expressions \[ \rho(bf^{(p)}) = \Delta^{-1} \sum_{\sigma} \sum_{\tau \in S_p \times S_q} \text{sign}(\sigma \tau)(\sigma \tau).bf^{(p)} \]

As sequences $i_1 < \ldots < i_p$ for sequences $\Delta$ we obtain

\[ \rho(bf^{(p)}) = \Delta^{-1} \sum_{\sigma} \sum_{\tau \in S_p \times S_q} \text{sign}(\sigma \tau)(\sigma \tau).bf^{(p)} \]

\[ = \Delta^{-1} \sum_{\sigma} \sum_{\tau \in S_p \times S_q} \text{sign}(\sigma \tau)(\sigma \tau).b \]

\[ = \Delta^{-1} \sum_{\sigma} \sum_{\tau \in S_p \times S_q} \text{sign}(\sigma \tau)(\sigma \tau).b \]

As $\sum_{\tau} \text{sign}(\tau)b$ is alternating under the action of $S_p \times S_q$, it is divisible by $\Delta_{p \times q}$ which we define to be $\Delta_p(x_1, \ldots, x_p)\Delta_q(x_{p+1}, \ldots, x_{d})$. Putting $\delta := \prod_{i=1}^{d} \prod_{j=p+1}^{d} (x_j - x_i)$, we obtain $\Delta = \delta \Delta_{p \times q}$, and therefore,

\[ \text{sign}(\sigma)\Delta = \sigma.\Delta = (\sigma,\delta)(\sigma.\Delta_{p \times q}) \]

With $\rho_{p \times q}(b) = \Delta_{p \times q}^{-1} \sum_{\tau \in S_p \times S_q} \text{sign}(\tau)b$, we get

\[ \rho(bf^{(p)}) = \sum_{\sigma} (\sigma,\delta)^{-1} \cdot \sigma.\rho_{p \times q}(b) \]

We insert the definition of $f$. We write it as $f^{(p)} = x_{p+1} \ldots x_d \delta^m$. This yields

\[ \rho(bf^{(p)}) = \sum_{\text{complementary}} x_{j_1} \ldots x_{j_p} \prod_{\mu=1}^{p} \prod_{\nu=1}^{q} (x_{j_{\nu}} - x_{i_{\mu}})^{m-1} \cdot (\rho_{p \times q}(b))(x_{i_1}, \ldots, x_{i_p}, x_{j_1}, \ldots, x_{j_q}) \]

As $b$ runs through $C_d$, the image $\rho_{p \times q}(b)$ runs through $\mathbb{Q}[x_1, \ldots, x_p, x_{p+1}, \ldots, x_d]^{S_p \times S_q}$ which we may identify with the tensor product $\mathbb{Q}[x_1, \ldots, x_p]^{S_p} \otimes \mathbb{Q}[x_{p+1}, \ldots, x_d]^{S_q}$. Thus:

**Lemma 3.1.** The ideal $\rho(C_d \cdot f^{(p)})$ consists of expressions

\[ \sum_{\text{complementary}} f(x_{i_1}, \ldots, x_{i_p})g(x_{j_1}, \ldots, x_{j_q})x_{j_1} \ldots x_{j_q} \prod_{\mu=1}^{p} \prod_{\nu=1}^{q} (x_{j_{\nu}} - x_{i_{\mu}})^{m-1} \]

with $f$ ranging over all symmetric polynomials in $p$ variables and $g$ over those in $q$ variables.

The similarity of the above theorem with the multiplication in the CoHa is too obvious to be coincidental. By abuse of notation, we write $e_i \in \mathcal{H}_q$ for the $i$-th elementary symmetric function in $n$ variables whenever it is obvious where these elements live. We obtain that $\rho(C_d \cdot f^{(p)})$ is generated as a rational vector space by all expressions $f \ast (e_q \cup g)$ with $f \in \mathcal{H}_p$ and $g \in \mathcal{H}_q$, writing $\cup$ for the ordinary multiplication in the ring of symmetric functions (which coincides with the cup product) in order to distinguish it from the CoHa-multiplication. Summarizing, we have seen:

**Lemma 3.2.** The ideal of tautological relations $\sum_{p < d} \sum_{\gamma \in \mathcal{H}_d} \mathcal{H}_d \cdot \tau^{(p)}(\gamma) = \sum_{p < d} \rho(C_d \cdot f^{(p)})$ equals

\[ \sum_{p+q=d, \ q \neq 0} \mathcal{H}_p \ast (e_q \cup \mathcal{H}_q) \].
3.3 Construction of the CoHa-Module Structure

Put $\mathcal{A} := \bigoplus_d \mathcal{A}_d$, where $\mathcal{A}_d := A^*(\text{Hilb}_d)$ is the Chow ring of the non-commutative Hilbert scheme of codimension $d$ ideals. We are going to define an $\mathcal{A}$-module structure on $\mathcal{A}$ and show that we can realize it as a quotient of $\mathcal{A}$ itself and describe the kernel of the quotient map.

Note that we may construct a similar diagram as above for the “decorated” representation variety $\hat{R}_d^\text{st}$:

We have morphisms

\[
\begin{array}{cccc}
(R_p \times \hat{R}_q) \times L_{p,q} U & \rightarrow & \hat{R}_{p,q} \times L_{p,q} U & \rightarrow \hat{R}_d \times \text{G}_d U,
\end{array}
\]

where the arrows without names are again affine space bundles. This induces an $\mathcal{A}$-module structure on the direct sum $\bigoplus_d A^*_\text{G}_d(\hat{R}_d)$. But as $\hat{R}_d \rightarrow R_d$ is also a $\text{G}_d$-equivariant affine bundle, the direct sum of these Chow groups coincides with $\mathcal{A}$, as a vector space. It is not hard to see that the induced module structure on $\bigoplus_d A^*_\text{G}_d(\hat{R}_d)$ coincides with the natural $\mathcal{A}$-module structure on $\mathcal{A}$ itself.

As a next step, we pass to the stable locus of $\hat{R}_d$. It consists of the tuples $(v, \varphi_*)$ such that $k(\varphi_1, \ldots, \varphi_m)v = k^d$. Consider the open subset $\hat{R}^\text{st}_{p,q}$ of $\hat{R}_{p,q} = R_{p,q} \times k^d$ which is the intersection $\hat{R}_{p,q} \cap \hat{R}^\text{st}_{p,q}$. An element $(v, \varphi_*)$ of $\hat{R}^\text{st}_{p,q}$ is of the form

\[
\left( \begin{array}{c}
 v' \\
 v''
\end{array} \right) = \left( \begin{array}{c}
 \varphi'_1 \\
 \vdots \\
 \varphi'_m \\
 \varphi''_1 \\
 \vdots \\
 \varphi''_m
\end{array} \right) \cdot \left( \begin{array}{c}
 1 \\
 \vdots \\
 1 \\
 \varphi''_1 \\
 \vdots \\
 \varphi''_m
\end{array} \right) = \left( \begin{array}{c}
 P(v'_1, \ldots, v'_m) \\
 P(v''_1, \ldots, v''_m)
\end{array} \right).
\]

This implies that $(v'', \varphi''_*)$ is a stable point of $\hat{R}_q$. By restricting the projection $\hat{R}_{p,q} \rightarrow R_p \times \hat{R}_q$, we obtain a well defined morphism

\[
f : \hat{R}^\text{st}_{p,q} \rightarrow R_p \times \hat{R}^\text{st}_q.
\]

Being an affine space bundle, the projection $\hat{R}_{p,q} \rightarrow R_p \times \hat{R}_q$ is flat. This implies at once that $f$ is a flat morphism, too. We can draw the, by now, well known diagram

\[
\begin{array}{cccc}
(R_p \times \hat{R}^\text{st}_{q}) \times L_{p,q} U & \rightarrow & \hat{R}^\text{st}_{p,q} \times L_{p,q} U & \rightarrow \hat{R}^\text{st}_d \times \text{G}_d U,
\end{array}
\]

which gives us maps in equivariant Intersection Theory as follows:
3.3 Construction of the CoHa-Module Structure

\[ A^i_{G_p}(R_p) \otimes A^j_{G_q}(\hat{R}^\text{st}_q) \cong A^n_{G_p \times G_q}(R_p \times \hat{R}^\text{st}_q) \xrightarrow{i^*} A^n_{P_{p,q}}(\hat{R}^\text{st}_{p,q}) \xrightarrow{i_{p,q}} A^{n+mpq}_{P_{p,q}}(\hat{R}^\text{st}_d) \xrightarrow{\pi_*} A^{n+(m-1)pq}_{G_d}(\hat{R}^\text{st}_d). \]

Composing these maps, we get \( \mathcal{H}_p \otimes \mathcal{A}_q \to \mathcal{A}_d \). Let us write \( f \otimes g \) for the image of \( f \otimes g \) under this map. A similar argument to [10, 2.3] shows:

**Proposition 3.3.** The maps \( \mathcal{H}_p \otimes \mathcal{A}_q \to \mathcal{A}_{p+q} \) constructed above make \( \mathcal{A} \) into an \( \mathcal{H} \)-module.

If we define a bigrading on \( \mathcal{A} \) by letting \( \mathcal{A}_{d,k} := A^{(m-1)\left(\frac{d}{2}\right) - k}_{G_d}(\hat{R}^\text{st}_d) \), we obtain that \( \mathcal{A} \) also becomes a bigraded \( \mathcal{H} \)-module.

Let us look at the map \( j^* : \mathcal{H} \to \mathcal{A} \) which is induced by the open embeddings \( \hat{R}^\text{st}_d \to \hat{R}_d \). It is clearly surjective. It is also \( \mathcal{H} \)-linear as the following commutative diagram asserts:

\[
\begin{array}{cccc}
(R_p \times \hat{R}^\text{st}_q) \times L_{p,q} \ U & \xrightarrow{f} & (R_p \times \hat{R}^\text{st}_d) \times L_{p,q} \ U \\
(R_p \times \hat{R}_q) \times L_{p,q} \ U & \xrightarrow{\hat{R}^\text{st}_{p,q}} \ U & \hat{R}^\text{st}_{p,q} \times L_{p,q} \ U & \xrightarrow{i} \hat{R}^\text{st}_{p,q} \times P_{p,q} \ U \\
\hat{R}_{p,q} \times L_{p,q} \ U & \xrightarrow{i} \hat{R}^\text{st}_{d} \times P_{p,q} \ U & \hat{R}^\text{st}_{d} \times G_d \ U & \xrightarrow{\pi} \hat{R}^\text{st}_{d} \times G_d \ U \\
\hat{R}^\text{st}_{d} \times G_d \ U & \xrightarrow{\pi} \end{array}
\]

In this diagram, all maps pointing from north-east to south-west are induced by the open embeddings. Note that every “square”, except for the uppermost, is cartesian. Hence, passing to intersection theory, the outer arrows of the diagram give two ways to go from \( A^*_{L_{p,q}}(R_p \times \hat{R}_q) \) to \( A^*_{G_d}(\hat{R}^\text{st}_d) \) which coincide. One way describes \( f \otimes j^* g \), whereas the other represents \( j^*(f \otimes g) \). In a picture:

\[
A^*_{L_{p,q}}(R_p \times \hat{R}_q) \xrightarrow{f \otimes j^* g} A^*_{G_d}(\hat{R}^\text{st}_d).
\]

Considering the above defined bigrading on \( \mathcal{A} \), the map \( j^* : \mathcal{H} \to \mathcal{A} \) is also homogeneous of bidegree \((0,0)\). Summarizing:
3.3 Construction of the CoHa-Module Structure

Proposition 3.4. The map \( j^*: \mathcal{H} \to \mathcal{A} \) induced by the open embeddings \( j: \hat{R}^* \to \hat{R}_d \) is \( \mathcal{H} \)-linear, surjective and homogeneous of bidegree \((0,0)\).

In other words, \( \mathcal{A} \) can be written as a quotient of \( \mathcal{H} \). Taking into account that \( \mathcal{H} \) is either commutative (if \( m \) is odd) or graded commutative (if \( m \) is even), we obtain:

Corollary 3.5. The vector space \( \mathcal{A} \) inherits the structure of a bigraded \( \mathcal{H} \)-algebra.

Motivated by the calculations we made using forbidden polynomials (cf. Lemma 3.2), we want to prove the following result about the kernel of the quotient map \( j^* \).

Theorem 3.6. The kernel of \( j^*: \mathcal{H} \to \mathcal{A} \) equals \( \bigoplus_{p \geq 0, q \geq 0} \mathcal{H}_p \ast (e_q \cup \mathcal{H}_q) \).

Proof. Let \( s_0: \mathcal{R} \to \hat{\mathcal{R}}_d \) be the zero section of \( \hat{\mathcal{R}}_d \) considered as a \( \mathcal{G}_d \)-linear bundle on \( \hat{\mathcal{R}}_d \). Under the identifications \( \mathcal{H}_d = A^*_{G_d}(\mathcal{R}_d) \cong A^*_{\hat{\mathcal{G}}_d}(\hat{\mathcal{R}}_d) \cong A^*_{G_d}(pt) \), the map

\[
A^*_{\hat{\mathcal{G}}_d}(pt) \cong A^*_{G_d}(\mathcal{R}_d) \xrightarrow{(s_0)_*} A^*_{\hat{\mathcal{G}}_d}(\hat{\mathcal{R}}_d) \cong A^*_{G_d}(pt),
\]

is the multiplication with the top \( G_d \)-equivariant Chern class of \( \hat{\mathcal{R}}_d \). Identifying \( A^*_{\hat{\mathcal{G}}_d}(pt) \) with the ring of symmetric functions in \( d \) variables, the top \( G_d \)-equivariant Chern class of \( \hat{\mathcal{R}}_d \) is the \( d \)-th elementary symmetric polynomial. Taking this into account, the statement to prove is equivalent to showing that the horizontal sequence in the diagram

\[
\bigoplus_{p+q=d, q \geq 0} \mathcal{H}_p \otimes \mathcal{H}_q \xrightarrow{j^*} \mathcal{A}_d \xrightarrow{\ast} 0
\]

is exact. For all \( p + q = d \), we have the Künneth isomorphism

\[
\bigoplus_{i+j=n} \mathcal{H}_i \otimes \mathcal{H}_j \xrightarrow{\cong} A_{G_p \times G_q}^{n-q}(R_{p,q}).
\]

Modulo these isomorphisms, the maps that we are interested in are \( A_{L_{p,q}}^{n-q}(R_{p,q}) \to A_{G_d}^{n+(m-1)pq}(\hat{\mathcal{R}}_d) \), where we write \( L_{p,q} = G_p \times G_q \) as we have already done before. These maps arise from the following morphisms

\[
\begin{align*}
R_{p,q} \times L_{p,q} \ U &\xrightarrow{s_0} (R_{p,q} \times k^q) \times L_{p,q} \ U \\
\hat{\mathcal{R}}_{p,q} \times L_{p,q} \ U &\xrightarrow{i} \hat{\mathcal{R}}_d \times L_{p,q} \ U \quad \xrightarrow{\pi} \hat{\mathcal{R}}_d \times G_d \ U,
\end{align*}
\]
3.3 Construction of the CoHa-Module Structure

the non-horizontal ones being affine space bundles. Considering the cartesian squares

$$
\begin{array}{ccc}
R_{p,q} \times L_{p,q} U & \xrightarrow{s_0} & (R_{p,q} \times k^q) \times L_{p,q} U \\
\downarrow & & \downarrow \\
(R_{p,q} \times k^q) \times L_{p,q} U & \xrightarrow{s_0} & \hat{R}_{p,q} \times L_{p,q} U \\
\downarrow & & \downarrow \\
(R_{p,q} \times k^q) \times P_{p,q} U & \xrightarrow{s_0} & \hat{R}_{p,q} \times P_{p,q} U
\end{array}
$$

and using the commutativity of flat pull-back and proper push-forward, we are bound to show the exactness of

$$
\bigoplus_{d=p+q, \ q>0} A^{k-q-(m-1)pq}_{P_{p,q}}(R_{p,q} \times k^p) \rightarrow A^k_{G_d} (\hat{R}_d) \rightarrow A^k_{G_d} (\hat{R}_{p,q}^* \rightarrow 0. \quad (*)
$$

Let \((v, \varphi)\) be an instable point of \(\hat{R}_d\). Then, the linear subspace \(L(v, \varphi) := k(\varphi_1, \ldots, \varphi_m) v\) is a proper subspace of \(k^d\). Let \(X_p\) be the closed \(G_d\)-subset of all \((v, \varphi)\) where the subspace \(L(v, \varphi)\) has dimension at most \(p\) (its natural scheme structure is the reduced one). This induces a filtration

$$
\hat{R}_{p,q}^{\text{inst}} = X_{d-1} \supseteq X_{d-2} \supseteq \ldots \supseteq X_0 = R_d \times \{0\}
$$

by closed subsets. This is the Harder-Narasimhan filtration, as defined in [14]. Denote \(W_p := R_{p,q} \times k^p\). Evidently, the \(G_d\)-saturation of \(W_p\) lies in \(X_p\). Therefore, we obtain a morphism \(\psi_p\) as the composition

$$
W_p \times P_{p,q} U \rightarrow X_p \times P_{p,q} U \rightarrow X_p \times G_d U.
$$

As the first map is a closed immersion and the latter is a \(G_d/P_{p,q}\)-bundle, \(\psi_p\) is proper. Consider the open subsets \(X_p^o := X_p \setminus X_{p-1}\) of \(X_p\) and \(W_p^o\) of \(W_p\) defined as the subset of all \((v, \varphi)\) such that \(L(v, \varphi) = k^p\). We claim that \(\psi_p\) induces an isomorphism

$$
W_p^o \times P_{p,q} U \xrightarrow{\cong} X_p^o \times G_d U.
$$

As \((W_p^o \times P_{p,q} G_d) \times G_d U\) is naturally isomorphic to \((W_p \times P_{p,q} G_d) \times G_d U\) as a \(G_d\)-variety and as \(X_p^o \times U \rightarrow X_p^o \times G_d U\) is a \(G_d\)-principal fiber bundle, it suffices by faithfully flat descent to show that

$$
W_p^o \times P_{p,q} G_d \rightarrow X_p^o
$$

is an isomorphism of \(G_d\)-varieties. This will be done in Lemma [14].

Denote by \(W_p^c\) the complement of \(W_p^o\) in \(W_p\). Applying [3 Ex. 1.8.1], the cartesian diagram

$$
\begin{array}{ccc}
W_p^c \times P_{p,q} U & \xrightarrow{\pi'} & W_p \times P_{p,q} U \\
\downarrow & & \downarrow \pi \\
X_{p-1} \times G_d U & \rightarrow & X_p \times G_d U
\end{array}
$$
induces an exact sequence
\[ A_n^{P,p,q}(W_p^c) \to A_n^{G_d}(X_{p-1}) \oplus A_n^{P,p,q}(W_p) \to A_n^{G_d}(X_p) \to 0, \]
where the first map sends \( \alpha \) to \( \pi'_s \alpha + (-\alpha) \) and the second maps \( \beta + \beta' \) to \( \beta + \pi_s \beta' \). By induction on \( p \), we obtain that the natural map
\[ A_n^{P,p,q}(W_p) \oplus \ldots \oplus A_n^{P,p,q}(W_p) \to A_n^{G_d}(X_p) \]
is onto. Inserting \( p = d - 1 \) finally yields the exactness of the sequence (*).

**Lemma 3.7.** With the notation as in the proof of Theorem 3.6, the natural map \( W_p^o \times P_{p,q} G_d \to X_p^o \) is an isomorphism.

**Proof.** Consider the morphism \( L : X_p^o \to Gr_{p,d} := Gr_p(k^d) = G_d/P_{p,q} \) assigning to every point \((v, \varphi) \in X_p^o\) the subspace \( L(v, \varphi) \). We show that
\[ G_d \times W_p \xrightarrow{\text{act}} X_p^o \]
is a cartesian diagram of varieties. In fact, \( G_d \times_{Gr_{p,d}} X_p^o \) consists of those pairs \((g, (v, \varphi))\) such that \( L(v, \varphi) \) equals \( g \cdot k^p \). An isomorphism
\[ G_d \times W_p \to G_d \times_{Gr_{p,d}} X_p^o \]
is therefore given by mapping \((g, (v, \varphi))\) to \((g \cdot k^p, (v, \varphi))\). \(\Box\)

We deduce from Theorem 3.6 that the Chow ring of a non-commutative Hilbert scheme is tautologically presented. We make this statement a little more precise. For every \( 0 \leq p < d \), choose a basis \( P_{p,q} \) of \( \mathcal{H}_p \otimes \mathcal{H}_q \) as an \( \mathcal{H}_d \) module. It has cardinality \( \binom{d}{p} \). Without loss of generality, we may assume that every basis element is a tensor product \( f_{\lambda,P} \otimes g_{\lambda,q} \). Making this choice, we obtain:

**Corollary 3.8.** The kernel of the homomorphism \( j^* : \mathcal{H}_d = \mathbb{Q}[e_1, \ldots, e_d] \to A^*(\text{Hilb}_d) \), sending \( e_\nu \) to the \( \nu \)-th Chern class of the universal bundle of \( \text{Hilb}_d \), is the ideal of \( \mathcal{H}_d \) generated by the expressions
\[ f_{\lambda,p} \ast (e_q \cup g_{\lambda,q}) \]
with \( 0 \leq p < d, q := d - p \) and \( \lambda = 1, \ldots, \binom{d}{p} \).

**Example (continued).** Let us illustrate this result using once again our favorite non-commutative Hilbert scheme. Let \( m = 2 \) and \( d = 3 \) (and still \( n = 1 \)). We have \( \mathcal{H}_3 = \mathbb{Q}[e_1, e_2, e_3] = \mathbb{Q}[x, y, z]^S_3 \).

- Let \( p = 0 \). We obtain \( \mathcal{H}_p \otimes \mathcal{H}_q = \mathcal{H}_d \). Inserting \( g = 1 \) yields the relation \( e_3 = 0 \). 

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For $p = 1$, we get $\mathcal{H}_p \otimes \mathcal{H}_q = \mathbb{Q}[x][y, z]^{S_2}$. A basis as an $\mathcal{H}_d$-module is given by $1, x, x^2$. Putting $f(x) = 1$ yields

$$0 = yz(y - x)(z - x) + xz(x - y)(z - y) + xy(x - z)(y - z) \equiv (xy + xz + yz)^2 \equiv e^2_2$$

when employing the relation $xyz = 0$. The other basis vectors result in multiples of $xyz$.

Finally, let $p = 2$. Then, a basis of $\mathcal{H}_p \otimes \mathcal{H}_q = \mathbb{Q}[x, y]^{S_2}[z]$ over $\mathcal{H}_d$ is $1, z, z^2$. We consider $g(z) = 1$ first and obtain, using $xyz = 0$,

$$0 \equiv e^3_1 - 4e_1e_2.$$

After some lengthy computation, we see that for $g(z) = z$, we obtain the relation $e^4_1 = 0$. The basis element $z^2$ does not provide a new relation.

All in all, we have computed a presentation for the Chow ring of $\text{Hilb}^{(2)}_{1,2}$. It is isomorphic to

$$\mathbb{Q}[e_1, e_2]/(e^3_1 - 4e_1e_2, e^2_2, e^4_1).$$

### 3.4 Two Examples

For $m = 0$ and $m = 1$, there is an explicit description of the CoHa as an exterior algebra and as a symmetric algebra, respectively, both over a vector space of countably infinite dimension. We would like to describe the ideal $\ker j^*$ under these isomorphisms.

Let $m = 0$. In this case, the (non-commutative) Hilbert schemes $\text{Hilb}_0$ and $\text{Hilb}_1$ are singletons while $\text{Hilb}_d$ is empty for $d > 1$. The multiplication in $\mathcal{H}$ is given by the formula

$$(f * g)(x_1, \ldots, x_d) = \sum_{1 \leq i_1 < \ldots < i_p \leq d, \text{complementary}} f(x_{i_1}, \ldots, x_{i_p})g(x_{j_1}, \ldots, x_{j_q}) \prod_{\mu=1}^p \prod_{\nu=1}^q (x_{i_\mu} - x_{j_\nu})^{-1}.$$

It is easy to see that $(f * f)(x, y) = 0$ for all $f \in \mathcal{H}_1$. Therefore, the embedding $\mathcal{H}_1 \hookrightarrow \mathcal{H}$ induces a homomorphism of (graded) algebras $\varphi : \bigwedge^* (\mathcal{H}_1) \rightarrow \mathcal{H}$. We identify the ring $\mathcal{H}_1$ (equipped with the cup product $\cup$) with the polynomial ring $\mathbb{Q}[x]$. Let $\psi_0, \psi_1, \psi_2, \ldots$ be the basis of $\mathcal{H}_1$ that corresponds to $x^0, x^1, x^2, \ldots$ under this isomorphism. Note that $\psi_i$ lives in bidegree $(1, -i)$. A basis of $\bigwedge^d (\mathcal{H}_1)$ is given by expressions $\psi_{k_1} \wedge \ldots \wedge \psi_{k_d}$, where $k_1 < \ldots < k_d$ is an increasing sequence of $d$ non-negative integers. By induction on $d$, we can show that

$$(\psi_{k_1} \ast \ldots \ast \psi_{k_d})(x_1, \ldots, x_d) = s_\lambda(x_1, \ldots, x_d),$$

where $s_\lambda$ is the Schur function belonging to the partition $\lambda = (k_d - d + 1, \ldots, k_1)$. Hence, it follows that $\varphi$ is an isomorphism.

Let us determine $\mathcal{I} := \ker j^*$. Denoting $\mathcal{I}_d \subseteq \mathcal{H}_d$ the $d$-th homogeneous component, Theorem 3.13 implies that $\mathcal{I}_d = \sum_{q=1}^d \mathcal{H}_{d-q} \ast (e_q \cup \mathcal{H}_q)$. We obtain that $\mathcal{I}_0 = 0$, and $\mathcal{I}_1 \subseteq \mathcal{H}_1$ is $e_1 \cup \mathcal{H}_1$, which equals the ideal generated by $x$ under the isomorphism $(\mathcal{H}_1, \cup) \cong \mathbb{Q}[x]$. For $d \geq 2$, the element

$$(\psi_0 \ast \psi_2 \ast \ldots \ast \psi_{d-1}) \ast \psi_1$$
is contained in $\mathcal{I}_d$. But as $\mathcal{H}$ is graded commutative, this element is up to a sign just 
\[(\psi_0 * \psi_1 * \ldots * \psi_{d-1})(x_1, \ldots, x_d) = s_{(0, \ldots, 0)}(x_1, \ldots, x_d) = 1.\]

Therefore, $\mathcal{I}_d = \mathcal{H}_d$. This shows that $\varphi^{-1}(\mathcal{I}) \subseteq \Lambda^*(\mathcal{H})$ is the ideal generated by $\psi_1, \psi_2, \ldots$ (with respect to the multiplication $\wedge$). Consequently:

**Corollary 3.9.** In the case $m = 0$, the $\mathcal{H} = \Lambda^*(\psi_0, \psi_1, \ldots)$-algebra $\mathcal{A} = \bigoplus_d A^*(\text{Hilb}_{d,1}^{(0)})$ equals $\Lambda^*(\mathcal{H}) \cong \mathbb{Q}[\psi]/(\psi^2)$ with a generator of bidegree $(1,0)$.

Let us turn to the case $m = 1$. The (non-commutative) Hilbert scheme $\text{Hilb}_d$ is an affine space of dimension $d$. The CoHa-multiplication has the form

\[(f * g)(x_1, \ldots, x_d) = \sum_{1 \leq i_1 < \ldots < i_p \leq d} f(x_{i_1}, \ldots, x_{i_p})g(x_{j_1}, \ldots, x_{j_q}).\]

Again, the immersion $\mathcal{H}_0 \hookrightarrow \mathcal{H}$ yields a homomorphism $\varphi : \text{Sym}^d(\mathcal{H}) \to \mathcal{H}$ of algebras (which respects the grading). A basis element $\psi_{k_1} \ldots \psi_{k_d}$ of $\text{Sym}^d(\mathcal{H})$ with $k_1 \geq \ldots \geq k_d$ is mapped to 
\[(\psi_{k_1} * \ldots * \psi_{k_d})(x_1, \ldots, x_d) = c_\lambda m_\lambda(x_1, \ldots, x_d).\]

In the above equation, $m_\lambda$ denotes the monomial symmetric function of $\lambda = (k_1, \ldots, k_d)$ and $c_\lambda$ is some positive integer. This proves that $\varphi$ is an isomorphism.

We compute $\mathcal{I}$ and its inverse image under $\varphi$. We see that $\mathcal{I}_0 = 0$. For $d \geq 1$, the element

\[\left(1_{\mathcal{H}^{d-q}} * (e^{(q)} \cup 1_{\mathcal{H}^{q}})\right)(x_1, \ldots, x_d) = \sum_{1 \leq j_1 < \ldots < j_q \leq d} x_{j_1} \ldots x_{j_q} = e_q(x_1, \ldots, x_d)\]

is contained in $\mathcal{I}_d$ for all $q = 1, \ldots, d$. As the unit of $\mathcal{H}_d$ is not contained in $\mathcal{I}_d$, we obtain that $\mathcal{I}_d$ is generated by the elementary symmetric functions $e_1, \ldots, e_d$ in $d$ variables. It follows that $\varphi^{-1}(\mathcal{I}) \subseteq \text{Sym}^d(\mathcal{H})$ is the ideal generated by $\psi_1, \psi_2, \ldots$ and thus:

**Corollary 3.10.** For $m = 1$, the $\mathcal{H} = \text{Sym}^d(\psi_0, \psi_1, \ldots)$-algebra $\mathcal{A} = \bigoplus_d A^*(\text{Hilb}_{d,1}^{(1)})$ coincides with the polynomial ring $\text{Sym}^d(\mathbb{Q}[\psi]) \cong \mathbb{Q}[\psi]$ in one variable of bidegree $(1,0)$.

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