Hidden Orders and RVB formation of the Four-Leg Heisenberg Ladder Model

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Abstract

The ground state of the four-chain Heisenberg ladder model is numerically investigated. Hidden-order correlations suitable for the system are introduced and calculated with an emphasis on the spatially isotropic point, where a corresponding material exists. The existence of a long-range hidden correlation indicates formation of a short-range RVB state in the case of the antiferromagnetic inter-chain coupling. A transition between the phase of the ferromagnetic inter-chain coupling and that of the antiferromagnetic one is discussed.

KEYWORDS: hidden correlation, ladder model, resonating valence bond, Haldane phase

1 Introduction

In recent years, considerable attention has been devoted to ground-state properties of low-dimensional quantum systems. The interest was greatly stimulated by the discovery of the high-temperature superconductivity [1] and by subsequent studies which revealed the importance of the CuO$_2$ plane of high-$T_c$ materials. However, complete understanding of the high-$T_c$ mechanism still appears to be beyond our reach. A possible new approach to the high-$T_c$ mechanism was provided by a recent experiment [2] on the novel material Sr$_{2n_{\text{leg}}-2}$Cu$_{2n_{\text{leg}}}$O$_{4n_{\text{leg}}-2}$. This material has periodic line defects in the CuO$_2$ plane, and thus is composed of ladders with $n_{\text{leg}}$ legs interacting weakly with each other. The limit $n_{\text{leg}} \to \infty$ yields the high-$T_c$ superconductors.

At half-filling this material may be described well by the antiferromagnetic Heisenberg model defined on a ladder:

$$\mathcal{H} = J \sum_{l=1}^{n_{\text{leg}}} \sum_{i=1}^{L} \mathbf{S}_{l,i} \cdot \mathbf{S}_{l,i+1} + J' \sum_{l=1}^{n_{\text{leg}}-1} \sum_{i=1}^{L} \mathbf{S}_{l,i} \cdot \mathbf{S}_{l+1,i}. \quad (1)$$

(Here $\mathbf{S}_{l,i}$ denotes the $S = 1/2$ spin at the $i$th site of the $l$th chain. We put $J = 1$ hereafter.) It was reported [3] indeed, that the double chain ($n_{\text{leg}} = 2$) with $J \sim J'$ develops the paring correlation upon doping. This report was followed by the suggestion [4, 5, 6] that a set of ladders interacting weakly with frustration might show the superconductivity. Thus, characterization of the ground state of the ladder models can be an essential step to understanding the high-temperature superconductivity.

It has been conjectured [4, 5, 6] for $J = J'$ that the ground state of the model (1) with even $n_{\text{leg}}$ is spin liquid with the energy gap, while that with odd $n_{\text{leg}}$ is critical, or gapless. This conjecture is supported by a theorem [7], numerical calculations up to $n_{\text{leg}} = 4$ [3, 8, 9] and a scaling theory [10].
In order to explain this remarkable conjecture intuitively, White et al.\cite{1} proposed a resonating-valence-bond (RVB) picture of the ground state of the antiferromagnetic ladder models. This was followed by a proposal\cite{1, 12, 13} of a hidden-order correlation for the RVB ground state of the double chain:

\[
\mathcal{O}^*_{\text{RVB}}(|i-j|) = \left\langle (S_{1,i}^z + S_{2,i+1}^z)e^{i\pi \sum_{k=1}^{j-1}(S_{1,k}^z + S_{2,k+1}^z)}(S_{1,j}^z + S_{2,j+1}^z) \right\rangle .
\]  

(2)

The existence of the long-range RVB correlation supported\cite{1, 12, 13} the RVB picture for \( n_{\text{leg}} = 2 \).

The present RVB correlation (2) was also shown\cite{11} to be very useful in discussing another issue, namely the criticality of the model (I). Suppose that we change the value of the coupling across the ladder, \( J' \) (the inter-chain coupling). Then the above conjecture arises the following question. The model is known to be critical, or gapless\cite{4} for \( J' = 0 \), whereas the model has the gap for \( J' = J \) according to the above conjecture. How does the energy gap emerge as \( J' \) changes? In the previous paper\cite{11} we presented numerical calculations of the RVB correlations as well as another hidden-order correlation, namely the string correlation:\cite{15}

\[
\mathcal{O}^*_{\text{string}}(|i-j|) = \left\langle (S_{1,i}^z + S_{2,i+1}^z)e^{i\pi \sum_{k=1}^{j-1}(S_{1,k}^z + S_{2,k+1}^z)}(S_{1,j}^z + S_{2,j+1}^z) \right\rangle .
\]  

(3)

We showed\cite{11} quite clearly for \( n_{\text{leg}} = 2 \) that the energy gap appears as \( \Delta E \sim J'^\nu \) with \( \nu = 1 \). This conclusion is consistent with other studies.\cite{10, 13, 17, 18}

Hidden correlations were first introduced to investigate the ground state of the integer-\( S \) anti-ferromagnetic Heisenberg chains. The hidden correlations relevant to the model are the following string correlations,\cite{13, 20, 21, 22, 23, 24, 25}:

\[
\langle S_i^\alpha e^{i\pi \sum_{k=1}^{j-1} S_k^\beta} S_j^\gamma \rangle \quad (\alpha = x, z),
\]  

(4)

where \( \{ S_i \} \) denote the \( S = 2 \) spins. The correlations successfully detected the valence-bond-solid (VBS) structure\cite{26, 27, 28} of the ground state of the integer-spin chains as well as their criticality.\cite{21, 22}

Now that we know the usefulness of the RVB correlation in the case \( n_{\text{leg}} = 2 \), it is a challenging problem to generalize the definition (2) to higher \( n_{\text{leg}} \), and to explore the ground state and the criticality of general \( n_{\text{leg}} \) ladder models. Here we consider the ground state of the ladder model with \emph{four} legs. The Hamiltonian is given by (1) with \( n_{\text{leg}} = 4 \). We generalize the definitions (2), and (3) for the four-leg ladder, and show the RVB structure of the ground state in terms of the correlations. We also confirm the prediction of the scaling theory\cite{10} that the ground-state phase diagram consists of two disordered phases covering \( J' > 0 \) and \( J' < 0 \), respectively, and the critical point between them, \( J'_c = 0 \). This phase diagram is the same as in the two-leg case.\cite{16, 17, 18}

The ground-state properties are known in some cases. In the limit \( J' \to \infty \), the model is decoupled to independent four-spin rungs. The ground state is given by the product of singlets formed on the rungs.\cite{29} Therefore, the energy gap is given by \( \Delta E(J') \sim J' \) in this strong-coupling region. At the isotropic point \( J' = 1 \), which is of experimental interest,\cite{4} the energy gap \( \Delta E \) and the correlation length \( \xi \) have been numerically estimated as \( \Delta E = 0.190 \) and \( \xi = 5 \sim 6 \). At \( J' = 0 \) the model is reduced to four \( S = \frac{1}{2} \) Heisenberg antiferromagnetic spin chains. Hence, it is critical as noted above. In the limit \( J' \to -\infty \), the model converges to the \( S = 2 \) Heisenberg antiferromagnetic chain, which is massive according to the Haldane conjecture.\cite{30, 31} The Haldane conjecture for \( S = 2 \) has been confirmed numerically.\cite{11, 25, 32, 33, 34}

The present paper is organized as follows. We introduce hidden correlations in the next section, and present a naive discussion on the existence of the hidden correlations in terms of the RVB argument. In section 3, we show numerical results of the hidden correlations with an emphasis on the case \( J' = 1 \), where the corresponding material is available. We also present an argument on the existence of the RVB structure by considering two double-chain ladders coupled weakly.
This argument intuitively shows the nature of the RVB ground state. In section 4, we elaborate on the criticality of the model in terms of the hidden correlations. In the last section, we give a summary of the present paper.

2 RVB state and hidden correlations

In this section, we introduce the hidden correlations for the present four-leg-ladder model (4). They are defined so that they can detect expected RVB patterns.

We present schematic drawings of the expected RVB patterns in Fig. 1 (a) for \( J' > 0 \) and (b) for \( J' < 0 \), respectively. Each RVB pattern is arranged as follows. First for \( J' > 0 \), White et al.\(^4\) proposed that the ground state is dominated by the RVB pattern with vertical singlets and horizontal singlets. The state is stabilized by the resonance between the configuration of two adjacent vertical singlets and the configuration of two horizontal singlets; see Fig. 1 (a). This RVB picture was reported\(^9\) to be very useful for understanding several features of the ladder models. Second, the expected RVB pattern for \( J' < 0 \) is given as follows. In the limit \( J' \to -\infty \), the system (4) with \( n_{\text{leg}} = 4 \) converges to the \( S = 2 \) Heisenberg antiferromagnetic chain. The ground state of the chain was described\(^{22, 23, 24}\) well in terms of the Heisenberg antiferromagnetic chain. The RVB state was described in terms of the hidden correlations. In the last section, we give a summary of the present paper.

In order to detect the above RVB patterns, we define the following hidden correlations:

\[ O_{\text{RVB}}^z(\theta, |i-j|) = \left\langle (S_{1,i}^z + S_{2,i+1}^z + S_{3,i}^z + S_{4,i+1}^z)e^{i\theta \sum_{k=1}^{j-1}(S_{1,k}^z + S_{2,k+1}^z + S_{3,k}^z + S_{4,k+1}^z)}(S_{1,j}^z + S_{2,j+1}^z + S_{3,j}^z + S_{4,j+1}^z) \right\rangle \]  \hspace{1cm} (5)

and

\[ O_{\text{string}}^z(\theta, |i-j|) = \left\langle (S_{1,i}^z + S_{2,i}^z + S_{3,i}^z + S_{4,i}^z)e^{i\theta \sum_{k=1}^{j-1}(S_{1,k}^z + S_{2,k}^z + S_{3,k}^z + S_{4,k}^z)}(S_{1,j}^z + S_{2,j}^z + S_{3,j}^z + S_{4,j}^z) \right\rangle. \]  \hspace{1cm} (6)

We here generalized the hidden correlations (4) and (3), which have been applied to the double-chain ladder model.\(^{11, 12, 13}\) The angle \( \pi \) appearing in (4) and (3) is replaced by \( \theta \) in the present definitions (5) and (6).

We show in the present paper that the choice \( \theta = \pi/2 \) is the most relevant to the four-chain ladder model. We also show that the correlation (3) develops in the phase \( J' > 0 \) while the correlation (4) develops in the phase \( J' < 0 \). Let us explain the reasons of these facts briefly, before going into details in \( \S 3 \) and 4.

First, in the limit \( J' \to -\infty \), the model converges to the \( S = 2 \) Heisenberg antiferromagnetic chain, as is explained in the previous section. The correlation (3) then is reduced to

\[ \langle S_i^z e^{i\theta \sum_{k=1}^{j-1} S_k^z S_j^z} \rangle, \]  \hspace{1cm} (7)

where \( \{S_i\} \) denote the \( S = 2 \) spins. This correlation with \( \theta = \pi/2 \) is reported to remain finite in the limit \( |i-j| \to \infty \).\(^{24, 23}\) as was mentioned in \( \S 1 \). The correlation (3), on the other hand, remains finite in the limit \( J' \to \infty \); see the next section.
Second, as was done for the double-chain ladder model, we can show the existence of hidden orders schematically as follows. Consider any spin configurations of the four-leg ladder, satisfying the RVB pattern of Fig. 1 (a). We present an example in Fig. 2 (a). We notice that the set of the numbers \( \{ \tilde{S}_z \equiv S^z_{1,i} + S^z_{2,i} + S^z_{3,i} + S^z_{4,i} \} \) in Fig. 2 (a) satisfies the condition

\[
\left| \sum_{k=i}^{j} \tilde{S}_z^k \right| \leq 2.
\]

for \( \forall i, \forall j \). This is due to the formation of two singlets over two neighboring composite spins \( \{ \tilde{S}_i \} \).

It is easy to see that the quantity

\[
\tilde{S}_i e^{i \theta \sum_{k=1}^{i-1} \tilde{S}_k} \tilde{S}_j
\]

remains finite for any \( i \) and \( j \), when it is averaged over all configurations that satisfy the condition \( \tilde{S}_i \). This explains that the RVB correlation (3) remains finite if the RVB configurations as Fig. 1 (a) are dominant in the ground state. The above argument applies to the RVB patterns as Fig. 1 (b) if we define the composite spin as \( \{ \tilde{S}_i \equiv S^z_{1,i} + S^z_{2,i+1} + S^z_{3,i} + S^z_{4,i+1} \} \); see Fig. 2 (b). We can expect that local destruction of the RVB patterns does not result in vanishing of the long-range correlation.

3 RVB formation and the presence of the hidden order for \( J' > 0 \)

In this section, we investigate the ground state of the four-leg ladder by means of the hidden correlations (5) and (6) numerically. The RVB picture explained in the above section is confirmed. We show that the choice \( \theta = \frac{\pi}{2} \) is the most relevant to the present system.

3.1 Hidden correlations in the limit \( J' \to \infty \)

In this subsection, we concentrate on the four-leg ladder in the limit \( J' \to \infty \). The limiting point is expected to be the fixed point of the phase \( J' > 0 \). Hence, characteristics at the point may be relevant to those of the whole region \( J' > 0 \).

In this limit \( J' \to \infty \), the Hamiltonian is reduced to a set of independent rungs. The ground state is given by the direct product of singlets that are formed on the rungs. We can calculate the long-range limit of the hidden correlations (3) and (4) exactly. Though the ground state is spin liquid, the hidden correlation (3) is long-ranged. The correlation \( O_{\text{RVB}}^z(\theta, |i-j| \to \infty) \) is plotted against \( \theta \) in Fig. 3. The correlation has the maximum around \( \theta = \pi/2 \), while the correlations at \( \theta = 0 \) and \( \pi \) are both vanishing. These behaviors are intrinsic to the whole region of the phase \( J' > 0 \) as we see below. It is apparent, on the other hand, that the correlation (4) is of short range, \( O_{\text{string}}^z(\theta, |i-j| \to \infty) = 0 \).

3.2 The hidden orders at \( J' = 1 \)

In this subsection, we concentrate on the system at \( J' = 1 \), where the interaction is spatially isotropic. This isotropic system is of experimental interest. A strong-inter-chain-coupling expansion starts to fail at this point. We employed the density-matrix renormalization-group method in order to treat large systems approximately. We conclude that ground state properties are indeed consistent with the RVB picture given in the previous section and the recent proposal that the point \( J' \to \infty \) is the fixed point of the phase \( J' > 0 \).
approximate level $m$. The parameter $m$ is the number of states kept; in this method, we treat only $m$ states in the course of the renormalization. It should be noted that the precision of the correlations is worse than that of the ground-state energy. The reason may be as follows. Because the Hilbert space is restricted to the $m$ states, this renormalization-group method may be regarded as a kind of variational method. In many cases, the variational calculation yields worse estimation for the correlation functions.

The correlations $O_{\text{RVB}}^z(\theta, 21)$ and $O_{\text{string}}^z(\theta, 21)$ are plotted in Fig. 5 (a) and (b), respectively. The maxima of the correlations are located around $\theta = \frac{\pi}{2}$. This is also reported in the previous subsection for the system with $J' \to \infty$. We hence observe that the most relevant angle for the correlations is given by $\theta = \frac{\pi}{2}$ as is expected from the discussions in §2. In order to estimate the infinite distance limit of the correlations, we plotted $O_{\text{RVB}}^z \left( \frac{\pi}{2}, |i-j| \right)$ and $O_{\text{string}}^z \left( \frac{\pi}{2}, |i-j| \right)$ against $1/|i-j|$ in Fig. 6 (a) and (b), respectively. The correlation (5) remains finite; $O_{\text{RVB}}^z \left( \frac{\pi}{2}, |i-j| \to \infty \right) \approx 0.65$. (10)

(The correlation would have developed fully as $O^z(\frac{\pi}{2}) = 1$, if the ground state satisfied the condition (3) completely.) On the other hand, it is seen that the correlation (6) is very small. Considering the precision of the numerical calculators, we conclude that the correlation (6) is of short range.

Finally, we show the results of the correlation (3) with the angle $\theta = 0$. Note that this is nothing but the Néel correlation. We present a semi-logarithmic plot of the correlation against $|i-j|$; see Fig. 7. It is seen that it does decrease exponentially. The correlation length is somewhat consistent with an estimate in Ref. [9].

### 3.3 Mechanism of the RVB order with $\theta = \frac{\pi}{2}$

In this subsection, we clarify the essential mechanism of the development of the correlation (3) with $\theta = \frac{\pi}{2}$ in the region $J' > 0$.

For this purpose, let us decouple the four-leg ladder system into two ladder systems. The Hamiltonian is given by

$$
\mathcal{H} = \sum_{l=1}^{4} \sum_{i=1}^{L} S_{l,i} \cdot S_{l,i+1}^+ + J' \sum_{i=1}^{L} S_{1,i} \cdot S_{2,i}^+ + J'' \sum_{i=1}^{L} S_{2,i} \cdot S_{3,i}^+ + J' \sum_{i=1}^{L} S_{3,i} \cdot S_{4,i}^+.
$$

The effect of the inter-ladder coupling $J''$ is analysed with the aid of numerical simulations below. Reigrotzki et al. [29] suggested that the decoupled system ($J'' = 0$) is a convenient starting point for investigating the four-leg ladder.

First, we consider the completely decoupled case $J'' = 0$. The hidden correlation (3) is reduced to

$$
O_{\text{RVB}}^z(\theta, |i-j|) = 2 \langle \sigma_i^z e^{i\theta \sum_{k=1}^{L} \sigma_k^z} \sigma_j^z \rangle \langle \bar{e}^{i\theta \sum_{k=1}^{L} \tau_k^z} \rangle + 2 \langle e^{i\theta \sum_{k=i+1}^{L} \sigma_k^z} \sigma_j^z \rangle \langle e^{i\theta \sum_{k=1}^{L} \tau_k^z} \rangle,
$$

where $\sigma_i^z = S_{1,i}^z + S_{2,i+1}^z$ and $\tau_i^z = S_{3,i}^z + S_{4,i+1}^z$. The brackets here denote the ground-state expectation value of a single two-leg ladder system.
In Fig. 8, we show an example of spin configurations \( \{\sigma_i^z\} \) and \( \{\tau_i^z\} \) for each single-ladder system. The configurations are expressed in terms of the height of steps. The upward (downward) step at the position \( i \) of the upper drawing stands for \( \sigma_i^z = 1 \) (\( \sigma_i^z = -1 \)). The same rule applies to the lower drawing. The example here satisfies the conditions

\[
\left| \sum_{k=i}^{j} \sigma_k^z \right| \leq 1 \quad \text{and} \quad \left| \sum_{k=i}^{j} \tau_k^z \right| \leq 1 \tag{13}
\]

for \( \forall i, \forall j \); therefore the sites with the magnetization 1 and \(-1\) appear alternately with the sites with 0 being inserted between them. Indeed, such configurations are expected to dominate the ground state of the two-leg ladder system, according to the RVB theory for the two-leg ladder RVB pattern as in Fig. 9 satisfies the condition (13). We calculated the two terms in eq. (12) for the exemplified configuration in Fig. 8 fixing the angle \( \theta = \frac{\pi}{2} \). The expectation values indicated in Fig. 8 are the average over the depicted configuration and the reflected \( (\sigma_i^z \rightarrow -\sigma_i^z, \tau_i^z \rightarrow -\tau_i^z) \) one. We can see that the correlation (12) does not vanish only at the sites where the steps of the upper and lower ladders synchronize coincidently.

We can also show after a similar analysis that the hidden correlation (3) does not develop for \( \theta = 0 \) and \( \theta = \pi \), if we restrict the configurations with the condition (13). This is the same as what we observe in the previous subsection. Thus, the analysis based on the decoupled system appears to be fairly relevant.

Now we investigate the effect of the inter-ladder coupling in terms of the above RVB picture numerically. As a consequence, we present more detailed information on the RVB pattern. In Fig. 10, we show the hidden correlation (3) for the system with \( L = 6, J' = 1 \) and \( J'' \) varied. We observe that the inter-ladder coupling \( J'' \) stabilizes the hidden correlation. Recall that the essential mechanism of the development of the RVB correlation is the synchronization of the configurations of the upper-half ladder and the lower-half ladder; see Fig. 8. The RVB correlation \( \mathcal{O}_{\text{RVB}}^z \) for \( J'' = 0 \) expresses the contribution of the coincidental synchronization. We can observe from Fig. 10 that the inter-ladder coupling enhances the synchronization.

We can explain the reason of the enhancement as follows. According to the RVB picture, the ground state is stabilized by the resonance between the configuration of two vertical singlets and that of two horizontal singlets. \( 3, 11 \) When the inter-ladder coupling \( J'' \) is turned on, the resonance shown in Fig. 11 becomes possible. To take advantage of the resonance, singlets along the upper-half ladder and those along the lower-half ladder may tend to appear at the same position \( i \). This effect can enhance the synchronization of the configurations.

## 4 Criticality and the phase diagram

In this section, we investigate the development of the hidden correlations (3) and (4) and their criticality when we change the parameter \( J' \). We show the results of the exact-diagonalization method under the periodic-boundary condition. The result is consistent with that of the scaling theory (11).

We plotted the values of \( \mathcal{O}_{\text{RVB}}^z(\frac{\pi}{2}, \frac{\pi}{2}) \) and \( \mathcal{O}_{\text{string}}^z(\frac{\pi}{2}, \frac{\pi}{2}) \) for the system of the size \( L = 4 \) and 6 in Fig. 12. As is expected, we observe that the correlation \( \mathcal{O}_{\text{RVB}}^z(\frac{\pi}{2}) \) develops in the region \( J' > 0 \), while the correlation \( \mathcal{O}_{\text{string}}^z(\frac{\pi}{2}) \) develops in the region \( J' < 0 \). We readily know the values of the correlations in two limiting cases. In the limit \( J' \rightarrow \infty \), we showed \( \mathcal{O}_{\text{RVB}}^z(\frac{\pi}{2}) \approx 0.386895 \). In the limit \( J' \rightarrow -\infty \), the correlation (3) corresponds to the string correlation of the \( S = 2 \) Heisenberg chain. It is reported \( 24, 25 \) that we have \( \mathcal{O}_{\text{string}}^z(\frac{\pi}{2}) \approx 0.65 \).

In order to see the development of the correlations and the criticality, we calculated the squared order parameters,

\[
\langle O_{\text{RVB}}^{\dagger}(\frac{\pi}{2}) O_{\text{RVB}}(\frac{\pi}{2}) \rangle \tag{14}
\]
and
\[ \langle O_{\text{string}}^\dagger \left( \frac{\pi}{2} \right) O_{\text{string}} \left( \frac{\pi}{2} \right) \rangle, \] (15)
where
\[ O_{\text{RVB}}(\theta) = \frac{1}{L} \sum_{i=1}^{L} e^{i\theta \sum_{k=1}^{L-1} (S_{1,k}^{z} S_{2,k+1}^{z} + S_{3,k}^{z} S_{4,k+1}^{z}) (S_{1,i}^{z} + S_{2,i+1}^{z} + S_{3,i}^{z} + S_{4,i+1}^{z})} \] (16)
and
\[ O_{\text{string}}(\theta) = \frac{1}{L} \sum_{i=1}^{L} e^{i\theta \sum_{k=1}^{L-1} (S_{1,k}^{z} S_{2,k+1}^{z} + S_{3,k}^{z} S_{4,k}^{z}) (S_{1,i}^{z} + S_{2,i}^{z} + S_{3,i}^{z} + S_{4,i}^{z})}. \] (17)

We plotted the results in Fig. 13 (a) and (b), scaling the data by the factor \( L^\frac{4}{5} \). This is because of the following reason. At the critical point \( J_c = 0 \), the correlations are reduced to
\[ O_\alpha^z(\theta, r) = (O_{S=1/2}^z(\theta, r))^4 \quad (\alpha = \text{string, RVB}) \] (18)
\[ \sim r^{-\eta_\alpha(\theta)}, \]
where \( O_{S=1/2}^z \) denotes the string correlation for the \( S = 1/2 \) Heisenberg chain:
\[ O_{S=1/2}^z(\theta, |i-j|) = \left\langle S_i^z e^{i\theta \sum_{k=1}^{L-1} S_k^z S_j^z} \right\rangle. \] (19)

If we know the correlation exponent of (19) defined in
\[ O_{S=1/2}^z(\theta, r) \sim r^{-\eta_{S=1/2}(\theta)}, \] (20)
the relation (18) is immediately followed by
\[ \eta_\alpha(\theta) = 4\eta_{S=1/2}(\theta) \] (21)
both for \( \alpha = \text{string} \) and \( \alpha = \text{RVB} \). Generalizing Hida’s analysis[37] we obtain the formula \( \eta_{S=1/2}(\theta) = \frac{1}{4} \left( \frac{\theta}{2} \right)^2 \). Therefore, the exponent is given by \( \eta_\alpha(\frac{\pi}{4}) = \frac{1}{4} \).

The scaled order parameters are invariant at the critical point for various system sizes. In Fig. 13 (a) and (b), we observe the crossing points close to the expected critical point \( J_c' = 0 \).[10] (Another crossing point which appears in Fig. 13 (b) around \( J' \approx 1.5 \) may have arose owing to the limited system size.) The hidden correlations appear to obey the behavior explained in §2.

### 5 Summary

The ground state of the four-chain ladder model with both ferromagnetic and antiferromagnetic inter-chain coupling \( J' \) has been analysed by means of the hidden correlations (3) and (4). The hidden correlations can indicate the development of the corresponding RVB states.

The hidden correlation (3) develops in the phase \( J' > 0 \), while the hidden correlation (4) develops in the phase \( J' < 0 \). Though both the two phases are disordered, they are characterized in terms of the hidden long-range orders. It indicates that the RVB structure varies drastically at the critical point \( J_c' = 0 \). These results are consistent with a recent proposal[10] that the point \( J' \to \infty \) \((-\infty)\) is the fixed point of the region \( J' > 0 \) \(< 0 \). A detailed analysis of RVB pattern has been reported for the phase \( J' > 0 \). It implies that valence bonds formed along the ladder are not negligible as are in the case of the double-chain ladder model.[11] This tendency may be more apparent as the number of the legs is increased.

We are in position to conclude that, in the phase \( J' > 0 \), the ground state of the ladder models with two and four legs is the RVB state proposed by White et al.[3] The RVB pattern may be common to the arbitrary ladder models with even legs.
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Figure captions
Fig. 1 Schematic drawings of the expected RVB states. The RVB pattern of the type (a) ((b)) is dominant in the phase \( J' > 0 \) \( (J' < 0) \).
Fig. 2 Examples of spin configurations. The configurations (a) and (b) are arranged so as to satisfy the RVB patterns in Fig. 1 (a) and (b), respectively. The composite magnetization \( \tilde{S}_i \) is shown below each configurations. The full string order develops in both cases.
Fig. 3 The long-range hidden correlation \( \mathcal{O}_{\text{RVB}}^z(\theta, |i-j| \rightarrow \infty) \) against the angle \( \theta \) at \( J' \rightarrow \infty \).
Fig. 4 The relative error of the ground-state energy that is calculated by means of the density-matrix renormalization-group method for the system with the parameter \( J' = 1 \) and the system size \( L = 6 \) under the open-boundary condition.
Fig. 5 The hidden correlations (a) \( \mathcal{O}_{\text{RVB}}^z(\pi/2, |i-j|) \) and (b) \( \mathcal{O}_{\text{string}}^z(\pi/2, |i-j|) \) plotted against the inverse of the distance \( 1/|i-j| \).
Fig. 6 A semi-logarithmic plot of \( \mathcal{O}_{\text{RVB}}^z(0, |i-j|) \) against \(|i-j|\). For reference, we show the expected long-range behavior \( \propto e^{-|i-j|/5.5} \) as the broken line.
Fig. 7 As the inter-ladder coupling \( J'' \) is turned on, the resonance between the upper and lower ladders appears.
Fig. 8 The \( J' \) dependence of the hidden correlations \( \mathcal{O}_{\text{RVB}}^z(\pi/2, L/2) \) and \( \mathcal{O}_{\text{string}}^z(\pi/2, L/2) \).
Fig. 9 Example of the spin configuration for the upper double-chain ladder and the lower ladder. Two terms that appear in eq. \( (\ref{eq:12}) \) are calculated for the configurations, respectively. The expectation values are the average over the depicted configuration and the reflected one \( (\sigma_i^z \rightarrow -\sigma_i^z, \tau_i^z \rightarrow -\tau_i^z) \).
Fig. 10 The \( J'' \) dependence of the scaled hidden orders: (a) \( L^4 \mathcal{O}_{\text{RVB}}^z(\frac{\pi}{2}) \mathcal{O}_{\text{RVB}}^z(\frac{\pi}{2}) \) and (b) \( L^4 \mathcal{O}_{\text{string}}^z(\frac{\pi}{2}) \mathcal{O}_{\text{string}}^z(\frac{\pi}{2}) \). The intersection point may indicate the critical point.
(a)

(b)
configuration for the upper double chain ladder

\[ \langle \sigma^z_i e^{i\pi/2} \sum_{k=1}^{i-1} \sigma^+_k \sigma^-_i \rangle \times \begin{array}{ccccccccccc} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \end{array} \]

configuration for the lower double chain ladder

\[ \langle e^{i\pi/2} \sum_{k=1}^{i-1} \tau^+_k \rho \rangle \]

\[ \langle e^{i\pi/2} \sum_{k=2}^{i} \sigma^+_k \rho \rangle \times \begin{array}{ccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \end{array} \]

\[ \langle e^{i\pi/2} \sum_{k=2}^{i} \tau^+_k \rho \rangle \]

\[ O_{RVB}^z (\pi/2) \begin{array}{ccccccccccc} 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 \\ \end{array} \]
