Long large character sums

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Abstract
In this paper, we prove a lower bound for \( \max_{\chi \neq \chi_0} | \sum_{n \leq x} \chi(n) | \), when \( x = q/(\log q)^B \). This improves on a result of Granville and Soundararajan for large character sums when the range of summation is wide. When \( B \) goes to zero, our lower bound recovers the expected maximal value of character sums for most characters.

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1 | INTRODUCTION

Since their introduction in 1837, Dirichlet characters have played an important role in understanding questions about primes and integers. As is often the case with multiplicative functions, we would like to understand their mean value, in this case, the growth of character sums of the form

\[ \sum_{n \leq x} \chi(n), \]  

(1.1)

where \( x \) is a positive real number and \( \chi \) is a character modulo an integer \( q \). The best unconditional upper bound for (1.1) is given by the Pólya-Vinogradov inequality (1918):

\[ \sum_{n \leq x} \chi(n) \ll \sqrt{q} \log q. \]

Montgomery and Vaughan improved this, under the assumption of the generalized Riemann hypothesis, to

\[ \sum_{n \leq x} \chi(n) \ll \sqrt{q} \log \log q. \]  

(1.2)
This is best possible (up to the value of the constant), and indeed Granville and Soundararajan [5] proved that for any large prime \( q \) and \( \theta \in (-\pi, \pi] \) there are \( > q^{1 - \frac{c}{(\log \log q)^2}} \) odd characters \( \chi \) (mod \( q \)) for which

\[
\sum_{n \leq x} \chi(n) = e^{i \theta} \frac{\Theta'}{\pi} \sqrt{q \log \log q} + O\left( (\log \log q)^{\frac{1}{2}} \right)
\]

for almost all \( x \leq q \), where \( \gamma \) is the Euler–Mascheroni constant. In particular, this implies that there are a lot of characters for which

\[
\max_{x \leq q} \left| \sum_{n \leq x} \chi(n) \right| \geq \left( \frac{\Theta'}{\pi} + o(1) \right) \sqrt{q \log \log q}.
\]

(1.3)

It is also of interest to understand the behaviour of

\[
\max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right|,
\]

(1.4)

for different values of \( x \). Granville and Soundararajan [4] established lower bounds on (1.4) for \( x \) covering all ranges up to \( q \). For example, for small \( x \), they proved that for any fixed \( B > 0 \) and \( x = (\log q)^B \)

\[
\max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right| \gg x \rho(B),
\]

where \( \rho(B) \) is the Dickman–De Bruijn function. This is defined by \( \rho(u) = 1 \) for \( 0 \leq u \leq 1 \), and \( \rho(u) = \frac{1}{u} \int_0^u \rho(t)dt \) for all \( u > 1 \).

On the other hand, in the dual range, Granville and Soundararajan proved that

\[
\max_{\chi \neq \chi_0} \left| \sum_{n \leq \frac{q}{(\log q)^B}} \chi(n) \right| \gg \sqrt{q} \frac{\rho(B)}{\log \log q^{\frac{1}{2}} + o(1)},
\]

in which taking the maximum over \( x \leq \frac{q}{(\log q)^B} \), they get (see [4, Theorem 8])

\[
\max_{x < \frac{q}{(\log q)^B}} \max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right| \gg \sqrt{q} \rho(B).
\]

(1.5)

Using the resonance method, Hough [8] (see Theorem 1.2) improved on their result by removing the need for the second maximum over \( x \), thus obtaining a similar bound when \( x = \frac{q}{(\log q)^B} \).

Here we improve these results to the following:

**Theorem 1.** Let \( Q \) be a large integer, for all but at most \( Q^{\frac{1}{m}} \) primes \( q \leq Q \), if \( 0 \leq B < \frac{\log \log q}{\log \log \log \log q} \), then

\[
\max_{\chi \neq \chi_0} \left| \sum_{n \leq \frac{q}{(\log q)^B}} \chi(n) \right| \geq \frac{1}{\pi} \int_B^\infty \rho(u)du \cdot \sqrt{q \log \log q} + O(\sqrt{q \log \log q}).
\]
Since $\int_0^\infty \rho(u)du = e^\gamma$ (see [2, Lemma 3.3]), letting $B$ go to zero recovers (1.3). We also believe that Theorem 1 should hold for all prime moduli $q$, but we were unable to prove this due to a the limitation in our Fourier analysis argument. It has been pointed to us that a careful use of the resonance method could lead to a simpler proof and most importantly perhaps to a strengthening of Theorem 1 removing the exceptional set of moduli $q$ and allowing a better uniformity in $B$. However, our approach has the advantage that it helps identify for which characters the character sums get large and therefore, we have not pursued this different approach in the present paper. As it would still be interesting to develop this idea in order to improve on the uniformity of Theorem 1, we include a sketch of the alternative proof as the Appendix.

Our proof shows that large character sums in Theorem 1 all arise from odd characters and in particular it shows that 1—pretentious odd characters have large character sums. For even characters, we can obtain the following weaker bound, for all prime moduli $q$.

**Theorem 2.** Let $q$ be a large prime and let $1 \leq B < \frac{\log \log \log q}{\log \log \log \log q}$, then

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq \frac{q}{(\log q)^B}} \chi(n) \right| \geq \frac{\rho(B)}{2} \sqrt{q} + O\left(\frac{\sqrt{q} \log \log q}{\sqrt{\log \log q}}\right).$$

The case $B < 1$ was excluded from Theorem 2 as the focus of this work is Theorem 1, but this could be worked out with extra technicalities.

Observe that the Dickman–De Bruijn $\rho$-function appears in both Theorem 2 and (1.5) since it counts smooth numbers, and in both of these ranges, most of the contribution to the character sums comes from the smooth summands.

A similar observation leads us to conjecture that our result is best possible. Indeed, based on what is known on large character sums, namely by Granville and Soundararajan’s work and the present paper, we believe that character sums can only get large because of values of characters on small primes, so that the main contribution to character sums has to come from smooth numbers. Together with the fact that we do not expect exponentials to interfere with characters in an extraordinary fashion, this brings us to the following conjecture on a key component of our proof;

**Conjecture 1.** Let $q$ be large and let $\chi$ be a non-principal character modulo $q$. If $\log q \leq y \leq x \leq q^{1/2-\varepsilon}$

$$\max_{\alpha \in [0,1]} \left| \sum_{n \leq x \atop P(n) > y} \frac{\chi(n)}{n} e(an) \right| \ll y^{1/2},$$

where $P(n)$ denotes the largest prime factor dividing $n$.

Assuming the conjecture, the inequalities in Theorems 1 and 2 become equalities and therefore, we expect our results to be “best possible.”

Although our proof of Theorem 1 restricts the range to $B \leq \frac{\log \log \log q}{\log \log \log \log q}$, we believe that the lower bound in Theorem 1 should extend to a much wider range:

**Conjecture 2.** Let $q$ be a large prime and write $x = \frac{q}{(\log q)^B}$. If $0 < B \leq \sqrt{\log q}$, then

$$\max_{\chi \neq \chi_0} \left| \sum_{n \leq x} \chi(n) \right| \gtrsim \frac{1}{\pi} \int_B^\infty \rho(u)du \cdot \sqrt{q \log \log q}.$$
A key aspect of our investigation concerns the following question about lattices. Given an integer $M$ and a lattice vector $\mathbf{u} = \frac{1}{M}(u_1, \ldots, u_k) \in (\mathbb{R}/\mathbb{Z})^k$, is there an integer $1 \leq \ell \leq M - 1$ such that all components of $(\mathbf{u} \mod 1)$ are small? The pigeonhole principle allows us to find many such vectors $\ell \mathbf{u}$, but all of the $\ell$ that we find this way might be even, which for us would mean not being able to work with odd characters. What we need is to find many such odd $\ell$, and so we start by making the following more general definitions:

$$C_{n^+}(\eta, k) = \{0 \leq \ell \leq M-1, \ell \equiv 0 \pmod{n} : \|((\ell \mathbf{u})_j)\| \leq \eta \text{ for } 1 \leq j \leq k\},$$  
(1.6)

and

$$C_{n^-}(\eta, k) = \{0 \leq \ell \leq M-1, \ell \not\equiv 0 \pmod{n} : \|((\ell \mathbf{u})_j)\| \leq \eta \text{ for } 1 \leq j \leq k\},$$  
(1.7)

where $||x||$ is the distance from $x$ to the nearest integer. These two sets contain all the vector multiples $\ell \mathbf{u}$ with all small components. The pigeonhole principle gives the following:

**Proposition 1.** Fix a positive real number $N$. Let $\mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$, with $k$ large, be such that $M \mathbf{u} \in \mathbb{Z}^k$ where $M \in \mathbb{Z}$ is the smallest integer such that this holds. Then for any fixed integer $n < \frac{M}{N^k}$,

$$\#C_{n^+}\left(\frac{1}{N}, k\right) \geq \frac{M}{nN^k}.$$

Restricting our search to multiples $\ell \mathbf{u}$ with $\ell \not\equiv 0 \pmod{n}$, we obtain the following:

**Theorem 3.** Fix a positive real number $N$. Let $\mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$, with $k$ large, be such that $M \mathbf{u} \in \mathbb{Z}^k$ where $M \in \mathbb{Z}$ is the smallest integer such that this holds, and let $n$ be a divisor of $M$. Then either:

(i) there exists a non-zero vector $\mathbf{r} \in (\mathbb{R}/\mathbb{Z})^k$ such that $|r_j| \leq k^4N \log^2(N)$ for $j \leq k$ and $n(\mathbf{r} \cdot \mathbf{u}) \equiv 0 \pmod{1}$; or

(ii)

$$\#C_{n^-}\left(\frac{2}{N}, k\right) \geq \frac{M}{nN^k}.$$

Note that in Proposition 1 we can choose $n$ to be any integer, while $n$ is a divisor of $M$ in Theorem 3.

Although we cannot quite show the converse, in the opposite direction, we have:

**Theorem 4.** Let $\mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$ with $k$ large be such that $M \mathbf{u} \in \mathbb{Z}^k$ with $M \in \mathbb{Z}$, and let $n$ be a divisor of $M$. Suppose that there exists $\mathbf{r} \in \mathbb{Z}^k$ such that $\mathbf{r} \cdot \mathbf{u} \equiv \frac{t}{n} \pmod{1}$, where $(t, n) = 1$. Then for any integer $1 \leq \ell \leq M - 1$ such that $\ell \not\equiv 0 \pmod{n}$, the vector $\mathbf{x} = \ell \mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k$ satisfies

$$|\mathbf{r} \cdot \mathbf{x} \pmod{1}| \geq \frac{1}{n}.$$  
(1.8)

In particular, if $||\mathbf{r}||_2 \leq L$, then

$$||\mathbf{x}||_2 \geq \frac{1}{nL},$$

where $|| \cdot ||_2$ denotes the Euclidean norm.
2 | SMOOTH NUMBERS

In this section, we collect several important results about smooth numbers. Let \( \Psi(x, y) = \#\{n \leq x : p|n \Rightarrow p \leq y\} \) be the counting function for the \( y \)-smooth integers. Hildebrand [10, p. 369] proved the following key estimate; for \( \exp((\log \log x)^{5/3+\varepsilon}) \leq y = x^{1/u} \), we have

\[
\Psi(x, y) = x \rho(u) \left(1 + O\left(\frac{\log(u + 1)}{\log y}\right)\right).
\]

The following estimate for the size of \( \rho(u) \) follows from [2, Lemma 3.1]:

\[
\rho(u) \ll u^{-u} \quad \text{for all } u \geq 1. \tag{2.1}
\]

Corollary 8.3 in [10] states that for any integer \( k \geq 0 \) and real number \( u_1 > 1 \), if \( u \geq u_1 \), then we have

\[
\rho^{(k)}(u) = (-1)^k \xi(u)^k \rho(u) \left(1 + O\left(\frac{1}{u}\right)\right), \tag{2.2}
\]

where \( \xi(u) \) is the unique real non-zero root of the equation \( e^\xi = 1 + u\xi \). Lemma 8.1 in [10] states that if \( u \geq 3 \), then

\[
\xi(u) = \log(u \log u) + O\left(\frac{\log \log u}{\log u}\right). \tag{2.3}
\]

We immediately deduce that if \( \exp((\log \log x)^{5/3+\varepsilon}) \leq y \), then

\[
\Psi(x, y) = x \rho(u) + O\left(\frac{x |\rho'(u)|}{\log y}\right). \tag{2.4}
\]

We also deduce that perturbing \( u \) by a small amount does not affect too much the value of the \( \rho \)-function.

**Lemma 2.1.** For \( |v| \leq \frac{1}{\log u} \), we have

\[
\rho(u + v) = \rho(u)(1 + O(|v| \log(u + 1))).
\]

**Proof.** By the mean value theorem, there exists \( u_0 \in [u, u + v) \) such that

\[
\rho(u + v) = \rho(u) + v \rho'(u_0).
\]

By (2.2) and (2.3), we have

\[
\rho'(u_0) \ll \xi(u_0) \rho(u_0) \ll \log(u + v) \rho(u).
\]

and the result follows. \( \square \)
With the same proof, one can prove that for \( u \geq 2 \) and \( 0 < v \leq \frac{u}{\log u} \), we have \( \rho(u+v) = \rho(u)u^{-1+o(1)}v \).

The next lemma approximates the sum of reciprocals of \( y \)-smooth integers using the Dickman–De Bruijn’s function. It follows directly from the strong version of [2, Lemma 3.3]. (See Remark 3.1.)

**Lemma 2.2.** Let \( y \geq 2 \) and \( 0 < s \leq r \), then

\[
\sum_{y^s \leq n \leq y^r \atop P(n) \leq y} \frac{1}{n} = \log y \int_s^r \rho(t)dt + O(\rho(s)).
\]

**Proof.** First, suppose \( s > 1 \). Then using partial summation and (2.4), we have that

\[
\sum_{y^s \leq n \leq y^r \atop P(n) \leq y} \frac{1}{n} = \frac{1}{y^r} \Psi(y^r, y) - \frac{1}{y^s} \Psi(y^s, y) + \int_{y^s}^{y^r} \frac{\Psi(t, y)}{t^2} dt
\]

\[
= O(\rho(s)) + \int_{y^s}^{y^r} \frac{t \rho(u) + O\left(\frac{|\rho'(u)|}{\log y}\right)}{t^2} dt
\]

\[
= \int_{y^s}^{y^r} \frac{\rho(u) + O\left(\frac{|\rho'(u)|}{\log y}\right)}{t} dt + O(\rho(s)) = \log y \int_s^r \rho(u)du + O(\rho(s)),
\]

changing variable \( t = y^u \). If \( s \leq 1 \), then for \( s \leq u \leq 1 \) we have that \( \rho(u) = 1 \) and therefore,

\[
\sum_{y^s \leq n \leq y^r \atop P(n) \leq y} \frac{1}{n} = \sum_{y^s \leq n \leq y} \frac{1}{n} + \sum_{y^s \leq n \leq y^r \atop P(n) \leq y} \frac{1}{n}
\]

\[
= (1 - s) \log y + O(1) + \log y \int_1^r \rho(u)du + O(1)
\]

and the result follows. \( \square \)

### 3 | A FIRST RESULT ABOUT LATTICES

One of the main challenges in the proof of Theorem 1 arises from finding an odd character which takes values close to one on all primes up to some point \( T \). In order to handle this obstacle, we prove a corresponding result about lattices.

We say that \( u = \left(\frac{1}{M}(u_1, u_2, \ldots, u_k) \in (\mathbb{R}/\mathbb{Z})^k \right) \) has order \( M \), if \( M \) is the smallest positive integer for which \( Mu \in \mathbb{Z}^k \).

#### 3.1 | The easier case: vector multipliers \( \ell' \equiv 0 \pmod{n} \)

As an immediate corollary to Proposition 1, we have:
Corollary 3.1. Let \( \mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k \) be a lattice vector of order \( M \), then
\[
#C_2^+(rac{1}{N}, k) \geq \frac{M}{2Nk}.
\]

Proof of Proposition 1. Let \( x_\ell \equiv \ell \mathbf{u} \pmod{1} \), where \( n < \frac{M}{N^k} \) is fixed and the multipliers \( 0 \leq \ell \leq M - 1 \) satisfies \( \ell \equiv 0 \pmod{n} \). We split \((\mathbb{R}/\mathbb{Z})^k\) into \( N^k \) equal hypercubes, each side of which has length \( 1/N \). Note that for each integer \( 0 \leq \ell \leq M - 1 \), with \( \ell \equiv 0 \pmod{n} \), the vector \( x_\ell \) must belong to one of the cubes, and therefore, by the pigeonhole principle, we must have an hypercube \( C \) which contains at least \( \frac{M}{nN^k} \) vectors.

Now, fix \( x_r \in C \) where \( r > s \) for all other vectors \( x_s \in C \). By the construction of the cubes, for any other vector \( x_s \) in \( C \), we must have \( |x_{r,j} - x_{s,j}| \leq \frac{1}{N} \), for \( j \leq k \). Let \( \ell = r - s \) and observe that \( r - s \equiv 0 \pmod{n} \) and thus the vector \( x_\ell = x_r - x_s \equiv (r - s)\mathbf{u} \pmod{1} \) has multiplier \( \ell \equiv 0 \pmod{n} \), with each component of size at most \( 1/N \). As there are \( \frac{M}{nN^k} \) such vectors \( x_s \in C \), including \( x_r \), we deduce that there are at least \( \frac{M}{nN^k} \) integers \( 0 \leq \ell \leq M - 1 \), with \( \ell \equiv 0 \pmod{n} \), such that \( x_\ell \) has components \( |x_{\ell,j}| \leq \frac{1}{N} \) for all \( j \leq k \) and the result follows.

Corollary 3.1 is important for us, as it will allow us to show the existence of many even characters with small argument. However, we need to show that there are a lot of odd characters with small arguments. The next lemma shows that if we can find just one vector with multiplier \( \ell \not\equiv 0 \pmod{n} \) that is small, then we can find many of them.

Lemma 3.1. Given a lattice vector \( \mathbf{u} \in (\mathbb{R}/\mathbb{Z})^k \) of order \( M \), let \( x_\ell \equiv \ell \mathbf{u} \pmod{1} \).

Suppose that \( C_{n-}^-(\nu, k) \neq \emptyset \), then
\[
#C_{n-}^-(\nu + \eta, k) \geq #C_{n+}^+(\eta, k),
\]
where the sets are defined as in (1.6) and (1.7).

Proof. Suppose that there exists an integer \( 0 \leq r \leq M - 1 \), with \( r \not\equiv 0 \pmod{n} \), such that each component of \( x_r \) satisfies \( |x_{r,j}| \leq \nu \) for \( 1 \leq j \leq k \). For any integer \( s \equiv 0 \pmod{n} \) in the same range and such that the vector \( x_s \in C_+^+(\eta, k) \), then \( \ell \equiv r - s \pmod{M} \) satisfies \( \ell \not\equiv 0 \pmod{n} \) and the size of the components of the vector \( x_\ell \) is bounded by
\[
|x_{\ell,j}| = |x_{r,j} - x_{s,j}| \leq |x_{r,j}| + |x_{s,j}| \leq \nu + \eta.
\]
Hence \( x_\ell \in C_-(\eta + \nu, k) \). Therefore, distinct vectors in \( C_+^+(\eta, k) \) will give rise to distinct vectors in \( C_-(\eta + \nu, k) \), and therefore it follows that \( #C_-(\eta + \nu, k) \geq #C_+^+(\eta, k) \).

3.2 The harder case: vector multipliers \( \ell \not\equiv 0 \pmod{n} \)

Finding multipliers of the form \( \ell \not\equiv 0 \pmod{n} \) for our lattice vector \( \mathbf{u} \) is more subtle; indeed such vectors do not always occur as we see in Theorem 3. Theorem 3 follows directly from Proposition 1, Lemma 3.1 and the following key proposition.
**Proposition 3.1.** Let $N > 0$, $k$ be a large integer and let $u \in \mathbb{R}/\mathbb{Z}^k$ be a lattice vector of order $M$. Given a divisor $n$ of $M$, then either:

(i) there exists a non-zero vector $r \in \mathbb{R}/\mathbb{Z}^k$ such that $|r_j| \leq k^4N \log^2(N)$ for $j \leq k$ and $n(r \cdot u) \equiv 0 \pmod{1}$; or

(ii) $C_{n^{-1}} \left( \frac{1}{N}, k \right) \neq \emptyset$.

**Proof.** We will use Fourier analysis to construct a counting function detecting vectors with small components, and apply it to vectors of the form $x_\ell \equiv \ell u \pmod{1}$.

For now, suppose that there is a positive real number $L$ for which there is no vector $r \in \mathbb{Z}^k$, with $|r_j| < L$ for $j \leq k$ such that $n(r \cdot u) \equiv 0 \pmod{1}$.

Let

$$\phi(x) = \begin{cases} c_0 e^{1-(2x)^2} & \text{if } x \in \left( -\frac{1}{2}, \frac{1}{2} \right) \\ 0 & \text{otherwise,} \end{cases}$$

which is a positive valued Schwartz function, where $c_0$ is a normalizing constant so that $\int_{-\infty}^{\infty} \phi(x) dx = 1$.

The $k$-dimensional bump function

$$\Phi_N(x) = \prod_{j \leq k} \phi_N(x_j),$$

where $\phi_N(x) := N\phi(Nx)$, is non-negative and has support in $(-\frac{1}{2N}, \frac{1}{2N})^k$. Now let

$$F_N(x) = \sum_{v \in \mathbb{Z}^k} \Phi_N(v + x),$$

and we define our counting function by

$$S(N) = \sum_{x \in V} F_N(x)$$

for a generic set of vectors $V$. For $V = \{x \equiv \ell u \pmod{1} : 1 \leq \ell \leq M - 1, \ell \not\equiv 0 \pmod{n}\}$,

$$S(N) = \sum_{a=1}^{n-1} \sum_{\ell \equiv a \pmod{n} \atop 1 \leq \ell \leq M-1} F_N(\ell u).$$

If we can show that $S(N) > 0$, then $F_N(\ell u)$ is non-zero for some integer $\ell \not\equiv 0 \pmod{n}$, thus proving the existence of a vector $x \equiv \ell u \pmod{1}$ with components in $\left( -\frac{1}{N}, \frac{1}{N} \right)$. 

Letting $\hat{\Phi}_N(r) = \int_{\mathbb{R}^n} \Phi_N(x)e(x \cdot r)dx$ be the Fourier transform of $\Phi_N(x)$, by the Poisson summation formula, we have that

$$F_N(x) = \sum_{r \in \mathbb{Z}^k} e(x \cdot r)\hat{\Phi}_N(r),$$

and so

$$S(N) = \sum_{a=1}^{n-1} \sum_{r \in \mathbb{Z}^k} \sum_{\ell \equiv a \pmod{n}} e(\ell \cdot \ell u)\hat{\Phi}_N(r),$$

$$= \sum_{r \in \mathbb{Z}^k} \hat{\Phi}_N(r) \sum_{a=1}^{n-1} \sum_{0 \leq s \leq m-1} e((sn + a)u \cdot r),$$

where $mn = M$ with $\ell = sn + a$. The inner sum is

$$e(\ell u \cdot r) \sum_{0 \leq s \leq m-1} e(sn u \cdot r).$$

The components of $u$ are all of the form $\frac{u_j}{M}$ for some $1 \leq u_j \leq M - 1$, so that we have a complete exponential sum and thus

$$\sum_{0 \leq s \leq m-1} e(snu \cdot r) = \begin{cases} m & \text{if } n(u \cdot r) \equiv 0 \pmod{1}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we have that

$$S(N) = m \sum_{r \in \mathbb{Z}^k} \hat{\Phi}_N(r) \sum_{a=1}^{n-1} e(ar \cdot u).$$

Observe that

$$\sum_{a=1}^{n-1} e(ar \cdot u) = \begin{cases} n - 1 & \text{if } r \cdot u \equiv 0 \pmod{1}, \\ -1 & \text{otherwise.} \end{cases}$$

Therefore, it follows that

$$S(N) = m \begin{pmatrix} (n - 1) \sum_{r \cdot u \equiv 0 \pmod{1}} \hat{\Phi}_N(r) - \sum_{r \in \mathbb{Z}^k \setminus \{r \cdot u \equiv 0 \pmod{1}\}} \hat{\Phi}_N(r) \end{pmatrix}.$$
Only the small values of \( \mathbf{r} \) make a significant contribution, so that we can truncate the sums without too much loss. Indeed,

\[
\sum_{\mathbf{r} \in \mathbb{Z}^k \atop \max |r_j| > L} \Phi_N(\mathbf{r}) - \sum_{\mathbf{r} \in \mathbb{Z}^k \atop \max |r_j| > L} \Phi_N(\mathbf{r}) \leq \sum_{j=1}^{k} \sum_{\mathbf{r} \in \mathbb{Z}^k \atop |r_j| > L} |\Phi_N(\mathbf{r})| \leq k \sum_{\mathbf{r} \in \mathbb{Z} \atop |\mathbf{r}| > L} |\hat{\phi}(\mathbf{r})| \left( \sum_{\mathbf{r} \in \mathbb{Z}} |\hat{\phi}_N(\mathbf{r})| \right)^{k-1} \leq k \int_{|t| > L} \left| \hat{\phi}\left( \frac{t}{N} \right) \right| dt \left( \int_{-\infty}^{\infty} \left| \hat{\phi}\left( \frac{t}{N} \right) \right| dt \right)^{k-1},
\]

since \( \phi(x) \), and therefore \( \hat{\phi}(x) \), is a Schwartz function, hence in \( L^1 \), allowing us to trivially bound the sums by the integrals. With the appropriate change of variable, we get that

\[
\int_{-\infty}^{\infty} \left| \hat{\phi}\left( \frac{t}{N} \right) \right| dt = cN,
\]

for some absolute constant \( c \). Moreover, knowing that \( |\hat{\phi}(y)| \ll \frac{e^{-\sqrt{y}}}{y^{3/4}} \) (see [9, p. 3] and [11, p. 2]) for large enough positive \( y \in \mathbb{R} \), we have

\[
\int_{|t| > L} \left| \hat{\phi}\left( \frac{t}{N} \right) \right| dt \ll \int_{|t| > L} \frac{e^{-\sqrt{|t|}}}{|t|^{3/4}} dt = 2N \int_{u > L/N} \frac{e^{-\sqrt{u}}}{u^{3/4}} du \ll \left( \frac{N^7}{L^3} \right)^{1/4} e^{-\frac{1}{2} \sqrt{\frac{L}{N}}}.
\]

Hence, putting this together, we have

\[
k \int_{|t| > L} \left| \hat{\phi}\left( \frac{t}{N} \right) \right| \left( \int_{-\infty}^{\infty} \left| \hat{\phi}\left( \frac{t}{N} \right) \right| dt \right)^{k-1} \ll k(Nc)^{k-1} \left( \frac{N^7}{L^3} \right)^{1/4} e^{-\frac{1}{2} \sqrt{\frac{L}{N}}},
\]

and choosing \( L = k^4 N \log^2(N) \), we get that

\[
\left| (n-1) \sum_{\mathbf{r} \in \mathbb{Z}^k \atop \max |r_j| > L} \Phi_N(\mathbf{r}) - \sum_{\mathbf{r} \in \mathbb{Z}^k \atop \max |r_j| > L} \Phi_N(\mathbf{r}) \right| \ll n \frac{c^k}{N^{k^2-k}}.
\]
Therefore, we can truncate the sum to get

\[
S(N) = m \left( (n - 1) \sum_{\mathbf{r} \in \mathbb{Z}^k : |r_j| < L} \hat{\Phi}_N(\mathbf{r}) - \sum_{\mathbf{r} \in \mathbb{Z}^k : n(r \cdot \mathbf{u}) \equiv 0 \pmod{1}} \hat{\Phi}_N(\mathbf{r}) + o(n) \right).
\]

Now, by hypothesis, there are no non-zero vectors \(|r_j| \leq L\) satisfying \(n(r \cdot \mathbf{u}) \equiv 0 \pmod{1}\), and so

\[
S(N) = m((n - 1)\hat{\Phi}_N(0) + o(1)) = m(n - 1 + o(n)) > 0
\]
as \(\hat{\Phi}_N(0) = \left( \int_{-\infty}^{\infty} \phi_N(t) \, dt \right)^k = 1\). This means that there is an integer \(\ell \not\equiv 0 \pmod{n}\) with \(1 \leq \ell \leq M - 1\), for which \(F_N(\ell \mathbf{u}) > 0\); in other words, if \(\mathbf{x} \equiv \ell \mathbf{u} \pmod{1}\), then \(|x_j| < \frac{1}{N}\) for all \(1 \leq j \leq k\). □

We highlight the case \(n = 2\) as it will play a role in the proof of Theorem 1.

**Corollary 1.** Let \(\mathbf{u} \in (\mathbb{R} / \mathbb{Z})^k\) be a vector of order \(2m\), and suppose that there is no vector \(\mathbf{r} \in \mathbb{R}^k\) with \(|r_j| < k^4N(\log N)^2\) for all \(j \leq k\), such that \(2(\mathbf{r} \cdot \mathbf{u}) \equiv 0 \pmod{1}\), then

\[
C_2 - \left( \frac{2}{N}, k \right) \geq \frac{M}{2N^k}.
\]

Next we establish the complementary Theorem 4:

**Proof of Theorem 4.** Suppose that \(\mathbf{r} \cdot \mathbf{u} \equiv \frac{t}{n} \pmod{1}\). If \(\mathbf{x} = \ell \mathbf{u}\), then

\[
\mathbf{r} \cdot \mathbf{x} = \ell(\mathbf{r} \cdot \mathbf{u}) \equiv \frac{\ell t}{n} \pmod{1}.
\]

Since \((t, n) = 1\) and \(\ell \not\equiv 0 \pmod{n}\), we deduce that \(\ell t \not\equiv 0 \pmod{n}\), and therefore

\[
|r \cdot x \pmod{1}| \geq \frac{1}{n},
\]
and this proves the first part of the theorem.

The second part follows directly from Cauchy–Schwartz so that

\[
||\mathbf{r}||_2 ||\mathbf{x}||_2 \geq |\mathbf{r} \cdot \mathbf{u}| \geq \frac{1}{n},
\]
and thus

\[
||\mathbf{x}||_2 \geq \frac{1}{n ||\mathbf{r}||_2}.
\]

□
In the next section, we apply these results to the character setting to obtain important information on character sums that will be necessary in the proof of Theorem 1.

4 | WHEN $\chi$ PRETENDS TO BE 1

In order to derive a lower bound for our character sum, we would like to find a character that pretends to be 1, that is to say a character taking values close to 1 on all the small primes. It is believed that there are characters taking value 1 for all the primes $p \ll (\log q)^{1-\varepsilon}$, but showing this is out of reach, so we resort to a softer condition. Instead, we will consider the sets

$$A_{\pm}(T, N) = \left\{ \chi \pmod{q} : \chi(-1) = \pm 1, \max_{p \leq T} |\chi(p) - 1| \ll \frac{1}{N} \right\},$$  \hspace{1cm} (4.1)

where $T \geq 2$ and $N = N(T) \to \infty$ as $T \to \infty$.

In this section, we investigate properties of characters that belong to $A_{\pm}(T, N)$, and then proceed to confirm that the sets $A_{\pm}(T, N)$ do indeed contain many characters.

4.1 | What if $\chi$ pretends to be 1?

Proposition 4.1. Let $A_{\pm}(T, N)$ be as in (4.1), and suppose that $\chi \in A_{\pm}(T, N)$. Let also $\log \log y = (1 + O(\frac{1}{N})) \log \log T$, then

$$\sum_{p \leq y} \frac{\chi(p) - 1}{p} \ll h(T) \text{ where } h(T) : = \frac{\log \log T}{N}.$$

Proof. Using the bound from (4.1), $\max_{p \leq T} |\chi(p) - 1| \ll \frac{1}{N}$, we have

$$\sum_{p \leq y} \frac{\chi(p) - 1}{p} = \sum_{p \leq T} \frac{\chi(p) - 1}{p} + O\left( \log \left( \frac{\log y}{\log T} \right) \right) \leq \max_{p \leq T} |\chi(p) - 1| \sum_{p \leq T} \frac{1}{p} + O\left( \frac{\log \log T}{N} \right) \ll \frac{\log \log T}{N}. \hspace{1cm} \square$$

Now Proposition 4.1 allows us to show that we can indeed approximate $\chi$ by 1 when performing logarithmic sums.

Proposition 4.2. Let $A_{\pm}(T, N)$ be as in (4.1), suppose that $\chi \in A_{\pm}(T, N)$ and let $f(n)$ be any bounded function. Let $y > T$ be such that $\log \log y = (1 + O(\frac{1}{N})) \log \log T$ and let $0 \leq u \leq u' < \exp((\log y)^{3/5-\varepsilon})$. Then writing $w = \max\{0, u - 1\}$ and $w' = \max\{u' - u, u' - 1\}$, we have

$$\left| \sum_{\substack{y^u \leq n \leq y^{u'} \atop P(n) \leq y}} \frac{\chi(n)}{n} f(n) - \sum_{\substack{y^u \leq n \leq y^{u'} \atop P(n) \leq y}} \frac{f(n)}{n} \right| \ll h(T) \log y \int_{w}^{u'} \rho(t)dt + \rho(u - 1) + \frac{\log^2 y}{\sqrt{y}}.$$

We will need the following lemmas.
Lemma 4.1. Let $|\alpha| \leq 1$, then

$$|\alpha \beta - 1| \leq |\beta - 1| + |\alpha - 1|.$$ 

Proof. Observe that

$$|\alpha \beta - 1| \leq |\alpha \beta - \alpha| + |\alpha - 1| \leq |\beta - 1| + |\alpha - 1|.$$ 

As an immediate corollary, by complete multiplicativity of characters, we obtain

Corollary 4.1.

$$|\chi(n) - 1| \leq \sum_{p^k | n} k|\chi(p) - 1|,$$

where $p^k | n$ means that $p^k$ divides exactly $n$ and that $p^{k+1}$ does not divide $n$.

Lemma 4.2. Let $y > T$ be such that $\log \log y = (1 + O(\frac{1}{N})) \log \log T$ and assume that $\chi \in A_{\pm}(T, N)$, where $A_{\pm}(T, N)$ is as in (4.1). Then

$$\sum_{p \leq y \atop k \geq 1} \frac{|\chi(p) - 1|}{p^k} \ll h(T).$$

Proof. Since the sum over $k$ is a geometric series, we have

$$\sum_{p \leq y \atop k \geq 1} \frac{|\chi(p) - 1|}{p^k} = \sum_{p \leq y} \frac{|\chi(p) - 1|}{p - 1} \leq 2 \sum_{p \leq y} \frac{|\chi(p) - 1|}{p} \ll h(T)$$

by Proposition 4.1.

Proof of Proposition 4.2. Start with

$$\left| \sum_{y^u \leq n \leq y^{u'}} \frac{\chi(n)}{n} f(n) - \sum_{y^u \leq n \leq y^{u'}} \frac{f(n)}{n} \right| \ll \sum_{y^u \leq n \leq y^{u'}} \frac{|\chi(n) - 1|}{n},$$

then using Corollary 4.1, we have

$$\sum_{y^u \leq n \leq y^{u'} \atop P(n) \leq y} \frac{|\chi(n) - 1|}{n} \leq \sum_{y^u \leq n \leq y^{u'} \atop P(n) \leq y} \frac{1}{n} \sum_{p \leq y \atop k \geq 1} \frac{|\chi(p) - 1|}{p^k} \sum_{\frac{y^u}{p^k} \leq n \leq \frac{y^{u'}}{p^k} \atop P(n) \leq y} \frac{1}{m}$$

$$= \sum_{p \leq y \atop k \geq 1} \frac{|\chi(p) - 1|}{p^k} \left( \log y \int_{\max (0, u - v_p)}^{u' - v_p} \rho(t)dt + O(\rho(\max (0, u - v_p))) \right).$$
by Lemma 2.2, with $v_p = k \frac{\log p}{\log y}$. Now if $p^k \leq y$, then $v_p \leq \min\{u, 1\}$ and if $p^k > y$, then as $p \leq y$, we must have $k \geq 2$ and therefore $y^{1/k} < p \leq y$. So next we split the sum to cover these two cases.

$$
\sum_{y^u \leq n \leq y^{u'}} \frac{|\chi(n) - 1|}{n} \leq \log y \left[ \sum_{p^k \leq y} \frac{|\chi(p) - 1|}{p^k} \left( \int_{\max\{0, u - 1\}}^{\max\{u' - u, u' - 1\}} \rho(t) dt + O\left( \frac{\rho(u - 1)}{\log y} \right) \right) 
\right.

+ \sum_{k \geq 2} \sum_{y^{1/k} < p \leq y} \frac{|\chi(p) - 1|}{p^k} \left( \int_{\max\{0, u - 1\}}^{\max\{u' - u, u' - 1\}} \rho(t) dt + O\left( \frac{\rho(u - 1)}{\log y} \right) \right) \left. \right].
$$

Bounding the second sum trivially, we have

$$
\sum_{k \geq 2} \sum_{y^{1/k} < p \leq y} \frac{|\chi(p) - 1|}{p^k} \left( \int_{\max\{0, u - 1\}}^{\max\{u' - u, u' - 1\}} \rho(t) dt + O\left( \frac{\rho(u - 1)}{\log y} \right) \right) \ll \sum_{2 \leq k \leq \log y} \int_{y^{1/k}}^y \frac{1}{t^k} dt
\ll \log y
\frac{1}{\sqrt{y}}.
$$

and using Lemma 4.2 to bound the first sum, we get

$$
\sum_{y^u \leq n \leq y^{u'}} \frac{|\chi(n) - 1|}{n} \ll h(T) \log y \int_0^{w'} \rho(t) dt + \log y \frac{\rho(u - 1) + \log^2 y}{\sqrt{y}},
$$

where $w = \max\{0, u - 1\}$ and $w' = \max\{u' - u, u' - 1\}$. 

The bound on the characters in $A_{\pm}(T, N)$ also allows us to evaluate logarithmic character sums over $y$-smooth numbers. So next we show

**Proposition 4.3.** Let $A_{\pm}(T, N)$ be as in (4.1) and assume that $\chi \in A_{\pm}(T, N)$, then for $y \geq T$ with $\log \log y = (1 + O(\frac{1}{N})) \log \log T$ and $B < \exp((\log y)^{3/5-\epsilon})$, we have

$$
\sum_{n>y^B \atop P(n) \leq y} \frac{\chi(n)}{n} = \log y \int_B^{\infty} \rho(u) du + O(1 + h(T) \log y).
$$

We start by writing the sum as

$$
\sum_{n>y^B \atop P(n) \leq y} \frac{\chi(n)}{n} = \sum_{n \geq 1 \atop P(n) \leq y} \frac{\chi(n)}{n} - \sum_{n \leq y^B \atop P(n) \leq y} \frac{\chi(n)}{n},
$$

and we first use Proposition 4.1 to evaluate the first sum on the right-hand side of (4.2).
Lemma 4.3. Let $A_{±}(T, N)$ be as in (4.1) and assume that $\chi \in A_{±}(T, N)$, then for $y \geq T$ with $\log \log y = (1 + O(\frac{1}{N})) \log \log T$,

$$\sum_{\substack{n \geq 1 \\ P(n) \leq y}} \frac{\chi(n)}{n} = e^{\gamma} \log y + O(h(T) \log y).$$

Proof. Taking the Euler product, we have

$$\sum_{\substack{n \geq 1 \\ P(n) \leq y}} \frac{\chi(n)}{n} = \prod_{p \leq y} \left(1 - \frac{\chi(p)}{p}\right)^{-1}. \tag{4.2}$$

Now taking absolute values, we have

$$0 \leq \left| \log \left( \prod_{p \leq y} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \right) \right| = \left| \sum_{p \leq y} \sum_{k \geq 1} \frac{\chi(p)^k - 1}{kp^k} \right| \leq \sum_{p \leq y} \sum_{k \geq 1} \left| \chi(p) - 1 \right| \frac{1}{kp^k}.$$ 

Applying Lemma 4.1 and computing the geometric series, we get, using the fact that $\max_{p \leq T} |\chi(p) - 1| \ll \frac{1}{N}$, that this is

$$\leq \sum_{p \leq y} \sum_{k \geq 1} \left| \chi(p) - 1 \right| \frac{1}{p^k} \leq \sum_{p \leq y} \frac{\chi(p) - 1}{p - 1} \ll h(T).$$

Using Mertens estimate, we deduce the result. \(\square\)

Proof of Proposition 4.3. Starting with (4.2) and using Lemmas 4.3 and Proposition 4.2 with $f \equiv 1$, we get

$$\sum_{\substack{n > y^B \\ P(n) \leq y}} \frac{\chi(n)}{n} = \sum_{\substack{n \geq 1 \\ P(n) \leq y}} \frac{\chi(n)}{n} - \sum_{\substack{n \leq y^B \\ P(n) \leq y}} \frac{\chi(n)}{n}$$

$$= e^{\gamma} \log y + O(h(T) \log y) - \sum_{n \leq y^B} \frac{1}{n} + O\left( h(T) \log y \int_{0}^{B} \rho(u)du \right).$$

Now using Lemma 2.2, we have

$$\sum_{\substack{n > y^B \\ P(n) \leq y}} \frac{\chi(n)}{n} = e^{\gamma} \log y - \log y \int_{0}^{B} \rho(u)du + O(1) + O(h(T) \log y)$$

$$= \log y \int_{B}^{\infty} \rho(u)du + O(1 + h(T) \log y),$$

as $\int_{0}^{\infty} \rho(u)du = e^{\gamma}$. \(\square\)
4.2 Finding 1-pretentious characters: incursion in the world of lattices

It remains to show that we can find characters that belong to $A_\pm(T, N)$. In order to do so, we turn to our theorems on lattices from Section 3. We start with the set containing even characters and we show the following bound which holds for all prime moduli $q$.

**Proposition 4.4.** Let $q$ be a prime, $N \geq 1$ and $T \geq 3$, then

$$\# A_+(T, N) \gg \frac{q}{N^{2T/\log T}}. \quad (4.3)$$

**Proof.** Let $\theta_p = \theta_p(\chi) = \frac{\arg(\chi(p))}{2\pi}$, and observe that

$$|\chi(p) - 1| = 2\pi |\theta_p| + O(\theta_p^2),$$

so that the bound in (4.1) is equivalent to showing that $\max_{p \leq T} |\theta_p| \ll \frac{1}{N}$. Thus, we are looking for a lower bound on the size of

$$C_+ \left( \frac{1}{N}, T \right) = \left\{ \chi \pmod{q} : \chi(-1) = 1, |\theta_p| \leq \frac{1}{N} \forall p \leq T \right\}.$$

So we let $k = \pi(T)$, we choose a generator $\chi$ for the group of characters and we consider the $k$-dimensional argument vector

$$V_\chi = (\theta_2, \theta_3, ..., \theta_{\pi(T)}) \in (\mathbb{R}/\mathbb{Z})^k.$$

As the subgroup of even characters arises from taking $\chi^{\ell}$ for even integers $1 \leq \ell \leq \phi(q)$, then each even character has an argument vector given by $\ell V_\chi \pmod{1}$ for some even $1 \leq \ell \leq \phi(q)$. Now, as $\chi$ has order $\phi(q)$ in the group of characters, then the lattice vector $V_\chi$ must have order $d$, where $d | \phi(q)$. However, since $\chi^{\ell}$ produces distinct characters for each $1 \leq \ell \leq \phi(q)$, then for every integer $1 \leq \ell \leq d$, there must be $\frac{\phi(q)}{d}$ characters $\Psi = \chi^{\ell}$ for which $r V_\chi \equiv \ell V_\chi \pmod{1}$, and choosing to view each of these as distinct vectors and we may consider the vector $V_\chi$ to have order $M = \phi(q)$. That is, taking $u = V_\chi$, by Corollary 3.1 from Section 3, we get that

$$\# C_+ \left( \frac{1}{N}, T \right) = \# C_2^+ \left( \frac{1}{N}, k \right) \geq \frac{\phi(q)}{2N^k}.$$

It follows that

$$\max_{p \leq T} |\chi(p) - 1| \ll \frac{1}{N}$$

for at least $\frac{\phi(q)}{2N^k}$ even characters $\pmod{q}$, which proves the proposition. \qed

**Remark.** For the clarity of the argument, we have chosen $q$ to be prime, but this is not necessary since the pigeonhole principle still holds even if $q$ is not prime which implies that there is an
hypercube containing vectors that belong to at least \( \frac{\phi(q)}{2N^k} \) even characters and therefore the rest of the argument can be worked out in the same fashion. In that case, the bound in Proposition 4.4 remains true with \( \phi(q) \) in place of \( q \).

For the set containing the odd characters, we obtain a slightly weaker result which holds for most of the prime moduli \( q \) except for a small exceptional set. This limitation comes from our inability to fully exploit the Fourier analysis argument in Theorem 3 and improving this argument by removing or improving the dependence on \( k \) in the upper bound for \( |r_j| \) would lead to a result holding for all prime moduli \( q \).

**Proposition 4.5.** Let \( Q \) be a large integer and let \( T \leq \frac{\log Q}{100} \) and \( N \leq \frac{T}{2(\log T)^3} \). For all but at most \( Q^{1/10} \) primes \( q \leq Q \),

\[
\#A_-(T, N) \gg \frac{q}{N^{2T/\log T}}.
\]

(4.4)

As in Proposition 4.4, the strategy to prove Proposition 4.5 will be to use our theorems on lattices from Section 3. In particular, the proposition will follow from Corollary 1 and in order to get the desired bound, we will be required to show that for most primes \( q \leq Q \), there are no small vector \( r \in \mathbb{Z}^k \) such that \( 2(r \cdot V_\chi) \equiv 0 \pmod{1} \). This is the purpose of the following Lemma.

So again, let

\[
V_\chi(k) = (\theta_2, \theta_3, \ldots, \theta_{p_k}) \quad \text{where each } \theta_{p_j} = \frac{\arg(\chi(p_j))}{2\pi}.
\]

**Lemma 4.4.** Let \( Q \) be a large integer and \( k \leq \frac{1}{60} \log \log \log Q \). Let \( \chi \pmod{q} \) be a character of order \( q - 1 \) and let \( u_q = V_\chi(k) \). For all but at most \( Q^{1/100} \) primes \( q \leq Q \), if \( n(r \cdot u_q) \equiv 0 \pmod{1} \), for some integer \( n \leq Q^{1/160} \), then there exists \( j \leq k \) such that \( |r_j| > k^5 \).

**Proof.** For given prime \( q \) and \( \chi \pmod{q} \) generating the group of character, let \( u_q = V_\chi(k) \) be the argument vector. Define

\[
S(Q) = \left\{ \frac{Q}{2} < q \leq Q : \exists r \in \mathbb{Z}^k \text{ with } |r_j| \leq k^5 \forall j \leq k \text{ and } n(r \cdot u_q) \equiv 0 \pmod{1}, n \leq Q^{1/160} \right\}.
\]

We will now show that \( \#S(Q) \leq Q^{1/23} \) which implies that for most primes \( q \), the condition \( n(r \cdot u_q) \equiv 0 \pmod{1} \) implies that at least one of the components of \( r \) is greater than \( k^5 \).

First, suppose that \( q \in S(Q) \) and consider

\[
\chi\left( \prod_{j \leq k} p_j^{r_jn} \right) = \prod_{j \leq k} \chi(p_j)^{r_jn} = e^{2\pi in(u_q \cdot r)} = e^0 = 1.
\]

As \( \chi \) is a generator for the group of characters, we deduce that \( \prod_{j \leq k} p_j^{r_jn} \equiv 1 \pmod{q} \), which means that

\[
\prod_{r_j > 0} p_j^{r_jn} \equiv \prod_{r_j < 0} p_j^{|r_j|n} \pmod{q},
\]
from which we deduce that
\[ q \text{ divides } \prod_{r_j > 0} p_j^{r_j n} - \prod_{r_i < 0} p_i^{\lvert r_i \rvert n}. \]  
(4.5)

Now fixing \( n \) and \( r \), we wish to count the number of primes for which (4.5) can hold. So let
\[ s(r, n) = \# \left\{ \frac{Q}{2} < q < Q : q \text{ divides } \prod_{r_j > 0} p_j^{r_j n} - \prod_{r_i < 0} p_i^{\lvert r_i \rvert n} \right\}, \]
and observe that
\[ \prod_{q \in s(r, n)} q \text{ divides } \prod_{r_j > 0} p_j^{r_j n} - \prod_{r_i < 0} p_i^{\lvert r_i \rvert n}, \]
so that
\[ \prod_{q \in s(r, n)} q \leq \left| \prod_{r_j > 0} p_j^{r_j n} - \prod_{r_i < 0} p_i^{\lvert r_i \rvert n} \right|. \]

Using the lower bound on \( q \), we have that
\[ \left( \frac{Q}{2} \right)^{\#s(r, n)} \leq \left| \prod_{r_j > 0} p_j^{r_j n} - \prod_{r_i < 0} p_i^{\lvert r_i \rvert n} \right| \leq \prod_{j \leq k} p_j^{\frac{1}{160}} \]
\[ \leq \left( \prod_{j \leq k} p_j \right)^{k^5 Q^{\frac{1}{160}}} \leq e^{k \log k (1 + o(1)) k^5 Q^{\frac{1}{160}}} \leq e^{2k^6 \log k Q^{\frac{1}{160}}}. \]

It follows that
\[ \#s(r, n) \leq \frac{2k^6 \log k Q^{\frac{1}{160}}}{\log \left( \frac{Q}{2} \right)}. \]

Now summing over all values of \( n \) and possible \( r \), we get that
\[ \#S(Q) = \sum_{n, r} s(r, n) \leq \sum_{\lvert r_j \rvert \leq k^5} \sum_{j \leq k \text{ and } n \leq Q^{\frac{1}{160}}} k^6 \log k Q^{\frac{1}{160}} \]
\[ \leq (2k^5 + 1) k^6 \log k Q^{\frac{1}{80}} \leq (3k)^{sk} Q^{\frac{1}{80}} \leq Q^{\frac{1}{12}} Q^{\frac{1}{80}} = Q^{\frac{23}{240}} \]
since
\[ (3k)^{sk} \leq \left( \frac{3 \log Q}{60 \log \log Q} \right)^{\frac{1}{12}} \log Q^{\frac{1}{12}} \log Q^{\frac{1}{12}} \leq \exp \left( \log \log Q \frac{\log Q}{12 \log \log Q} \right) = Q^{\frac{1}{12}}. \]
as \( k \leq \frac{\log Q}{60 \log \log Q} \).
Finally, the number of exceptional primes \( q \leq Q \) is
\[
\sum_{l=0}^{\infty} S \left( \frac{Q}{2^l} \right) \leq \sum_{l=0}^{\infty} \left( \frac{Q}{2^l} \right)^{23} \sum_{l=0}^{\infty} \left( \frac{1}{2^{23l}} \right)^{l} \leq Q^{\frac{1}{10}}.
\]

With this restriction on the vector \( \mathbf{r} \) at our disposal, we now prove Proposition 4.5.

**Proof of Proposition 4.5.** Let \( q \) be a prime and let \( \mathbf{u}_q = \mathbf{V}_q(k) = (\theta_{p_1}, \ldots, \theta_{p_k}) \) be the argument vector for \( \Psi \), where \( \Psi \) is chosen to be a generator for the group of characters \((\mod q)\). Because \( \Psi \) has order \( \phi(q) = q - 1 \) in the group of characters, we view \( \mathbf{u}_q \) as a vector of order \( q - 1 = 2m \).

Now, as in the even case, Proposition 4.5 is equivalent to finding a lower bound for
\[
C_-(\nu, T) = \{ \chi \pod{\mod q} : \chi(-1) = -1, |\theta_p| \leq \nu, \forall p \leq T \},
\]
for \( \nu \ll \frac{1}{N} \).

Letting \( k = \pi(T) \) be the number of primes up to \( T \), we observe that taking \( d = 2 \) as the divisor of the order \( q - 1 = 2m \), we have
\[
C_-(\nu, T) = C_{2-}(\nu, k).
\]

That is, by Corollary 1, we have that
\[
\#C_-(\frac{2}{N}, T) \geq \frac{2m}{2N^k}
\]
provided that there are no vector \( \mathbf{r} \in \mathbb{Z}^k \), with \( |r_j| \leq k^4 N \log^2 N \) for all \( j \leq k \), such that \( 2(r \cdot \mathbf{u}_q) \equiv 0 \pod{1} \). But Lemma 4.4 states that for all but at most \( Q^{\frac{1}{10}} \) primes \( q \leq Q \), the condition \( 2(r \cdot \mathbf{u}_q) \equiv 0 \pod{1} \), implies that there is a \( j \leq k \) for which \( |r_j| > k^5 \). As we chose \( N \leq \frac{T}{2(\log T)^3} \), we have that
\[
N \log^2 N < \frac{T}{2(\log T)^3} \log^2 T = \frac{T}{2 \log T} \leq \pi(T) = k.
\]

It follows that \( k^4 N \log^2 N < k^5 \) and therefore, the conditions for Corollary 1 to hold are satisfied, and we conclude that for all of these primes \( q \), we must indeed have that \( \gg \frac{q - 1}{2N^k} \) odd characters such that
\[
|\chi(p) - 1| \ll |\theta_p| \ll \frac{1}{N}.
\]

This proves the proposition. \( \square \)

Finding these 1-pretentious characters plays a key role in the proof of Theorem 1, as such characters will provide us with large character sums.


5 | PRELIMINARY ESTIMATES

Before diving into the proof of Theorem 1, we gather in this section some estimates on exponential sums and smooth numbers that will be of use in section 6.

5.1 | Some estimates on exponential sums

What stands out when investigating logarithmic exponential sums of the form

\[ \sum_{n \in I} \frac{e(\pm \alpha n)}{n} \]  

is that all the action occurs when \( n \) is around \( \frac{1}{\alpha} \). As we will see, this will have a direct impact on the logarithmic character sums that we evaluate in Theorem 1.

We start with a technical lemma that will allow us to handle the error terms in Lemmas 5.2 and 5.4.

Lemma 5.1. Let \( \alpha \in (0, 1) \) and let \( Y \geq 1 \), then

\[ \int_Y^\infty \frac{[t]e(\pm \alpha t)}{t} \, dt \ll 1 + \frac{1}{\alpha Y}. \]

Proof.

\[
\int_Y^\infty \frac{[t]e(\pm \alpha t)}{t} \, dt = \sum_{n \geq [Y]} \int_0^1 \frac{te(\pm \alpha(t+n))}{t+n} \, dt + O\left(\frac{1}{Y}\right) \\
= \int_0^1 te(\pm \alpha t) \left( \sum_{n \geq [Y]} \frac{e(\pm \alpha n)}{t+n} \right) \, dt + O\left(\frac{1}{Y}\right).
\]

Observe that

\[
\left| \frac{e(\pm \alpha n)}{t+n} - \frac{e(\pm \alpha n)}{n} \right| \leq \left| \frac{1}{n+1} - \frac{1}{n} \right| \leq \frac{1}{n^2},
\]

and therefore

\[
\int_Y^\infty \frac{[t]e(\pm \alpha t)}{t} \, dt = \int_0^1 te(\pm \alpha t) \left( \sum_{n \geq [Y]} \frac{e(\pm \alpha n)}{n} + O\left(\frac{1}{n^2}\right) \right) \, dt.
\]

Now it is not hard to see that the integral on the right-hand side is bounded by 1 and by partial summation, we have that

\[
\sum_{n \geq [Y]} \frac{e(\pm \alpha n)}{n} = \int_Y^\infty \sum_{n \in I} e(\pm \alpha n) \frac{dt}{t^2} + O(1) \\
= \int_Y^\infty \frac{e(\pm \alpha(\lfloor t \rfloor + 1)) - 1}{e(\pm \alpha) - 1} \frac{dt}{t^2} + O(1) \ll \frac{1}{\alpha Y} \int_Y^\infty \frac{1}{t^2} \, dt + 1 \ll \frac{1}{\alpha Y} + 1.
\]
Putting this together, it follows that
\[
\int_Y^\infty \frac{\{t\} e(\pm \alpha t)}{t} \, dt \ll 1 + \frac{1}{\alpha Y}.
\]

The next lemma emphasizes that most contributions to (5.1) happen around \(\frac{1}{\alpha}\) by showing that the tail of the sum is negligible.

**Lemma 5.2.** Let \(\alpha \in (0, \frac{1}{e}]\), then
\[
\sum_{n \geq \frac{1}{\alpha}} \frac{e(\pm \alpha n)}{n} = \sum_{\frac{1}{\alpha} < n \leq \frac{\log \alpha}{\alpha}} \frac{e(\pm \alpha n)}{n} + O\left(\frac{1}{|\log \alpha|^c}\right).
\]

**Proof.**
\[
\sum_{n \geq \frac{1}{\alpha}} \frac{e(\pm \alpha n)}{n} - \sum_{\frac{1}{\alpha} \leq n \leq \frac{\log \alpha}{\alpha}} \frac{e(\pm \alpha n)}{n} = \sum_{n > \frac{\log \alpha}{\alpha}} \frac{e(\pm \alpha n)}{n}
\]
\[
= \int_{\frac{\log \alpha}{\alpha}}^\infty \frac{e(\pm \alpha t)}{t} \, d(t - \{t\})
\]
\[
= \int_{\frac{\log \alpha}{\alpha}}^\infty \frac{e(\pm \alpha t)}{t} \, dt - \int_{\frac{\log \alpha}{\alpha}}^\infty \frac{e(\pm \alpha t)}{t} \, d\{t\}
\]
\[
= \int_{\frac{\log \alpha}{\alpha}}^\infty \frac{e(\pm \alpha t)}{t} \, dt \mp 2\pi i \alpha \int_{\frac{\log \alpha}{\alpha}}^\infty \frac{t e(\pm \alpha t)}{t} \, dt + O\left(\frac{\alpha}{|\log \alpha|^c}\right).
\]

Now the second term is \(O(\alpha)\) by Lemma 5.1 and noticing that
\[
\int_n^{n+1} \frac{e(\pm w)}{w} \, dw = \int_n^{n+1} \frac{e(\pm w)}{n} \, dw + \int_n^{n+1} e(\pm w) \left(\frac{1}{w} - \frac{1}{n}\right) \, dw
\]
\[
= \frac{1}{n} \int_0^1 e(\pm w) \, dw + \int_n^{n+1} e(\pm w) \left(\frac{n-w}{nw}\right) \, dw \ll \frac{1}{n^2}
\]
allows us to deduce that
\[
\int_{\frac{\log \alpha}{\alpha}}^\infty \frac{e(\pm \alpha t)}{t} \, dt = \int_{\frac{\log \alpha}{\alpha}}^\infty \frac{e(\pm w)}{w} \, dw \ll \sum_{n = \lfloor \log \alpha c\rfloor - 1}^{\infty} \frac{1}{n^2} \ll \frac{1}{|\log \alpha|^c}.
\]

Analogously, it is easy to see that the beginning of the following sum does not contribute too much.

**Lemma 5.3.** Let \(\alpha \in (0, \frac{1}{e}]\), then
\[
\sum_{n \leq \frac{1}{\alpha}} \frac{1 - e(\pm \alpha n)}{n} = \sum_{\frac{1}{\alpha |\log \alpha|} < n \leq \frac{1}{\alpha}} \frac{1 - e(\pm \alpha n)}{n} + O\left(\frac{1}{|\log \alpha|}\right).
\]
Proof.

\[ \sum_{n \leq \frac{1}{\alpha} | \log \alpha |} \frac{1 - e(\pm \alpha n)}{n} \ll \sum_{n \leq \frac{1}{\alpha} | \log \alpha |} \frac{\alpha n}{n} \ll \frac{\alpha}{\alpha | \log \alpha |} \ll \frac{1}{| \log \alpha |}. \]

Interestingly, putting the sums in Lemmas 5.3 and 5.2 together gives rise to a constant. This will play an important role for the proof of Theorem 2.

**Lemma 5.4.** Let \( \alpha \in (0, 1) \), then

\[ \sum_{n \leq \frac{1}{\alpha}} \frac{1 - e(\pm \alpha n)}{n} - \sum_{n > \frac{1}{\alpha}} \frac{e(\pm \alpha n)}{n} = \log(2\pi) + \gamma \pm \frac{i\pi}{2} + O(\alpha | \log \alpha |), \]

where \( \gamma \) is the Euler–Macheronin constant.

**Proof.** We have

\[
\sum_{n \leq \frac{1}{\alpha}} \frac{1 - e(\pm \alpha n)}{n} - \sum_{n > \frac{1}{\alpha}} \frac{e(\pm \alpha n)}{n} = \int_{\frac{1}{\alpha}}^{1} \frac{1 - e(\pm \alpha t)}{t} d\lfloor t \rfloor - \int_{\frac{1}{\alpha}}^{\infty} \frac{e(\pm \alpha t)}{t} d\lfloor t \rfloor
\]

\[
= \int_{1}^{\frac{1}{\alpha}} \frac{1 - e(\pm \alpha t)}{t} d(t - \{t\}) - \int_{\frac{1}{\alpha}}^{1} \frac{e(\pm \alpha t)}{t} d(t - \{t\})
\]

\[
= \int_{1}^{\frac{1}{\alpha}} \frac{1}{t} dt - \int_{1}^{\infty} \frac{e(\pm \alpha t)}{t} dt - \int_{1}^{\frac{1}{\alpha}} \frac{1 - e(\pm \alpha t)}{t} d\{t\} + \int_{\frac{1}{\alpha}}^{\infty} \frac{e(\pm \alpha t)}{t} d\{t\}. \tag{5.2}
\]

Now, by integrating by parts the third integral and noting that \( |1 - e(\alpha t)| \ll \alpha t \) for \( t < \frac{1}{\alpha} \), we have that

\[
\int_{1}^{\frac{1}{\alpha}} \frac{1 - e(\pm \alpha t)}{t} d\{t\} = \{t\} \frac{1 - e(\pm \alpha t)}{t} \bigg|_{1}^{\frac{1}{\alpha}} + \int_{1}^{\frac{1}{\alpha}} \{t\} \left( \frac{\pm 2\pi i\alpha e(\pm \alpha t)}{t} + \frac{1 - e(\pm \alpha t)}{t^2} \right) dt
\]

\[
\ll \alpha + \alpha \int_{1}^{\frac{1}{\alpha}} \frac{1}{t} dt + \int_{1}^{\frac{1}{\alpha}} \frac{\alpha t}{t^2} dt \ll \alpha | \log \alpha |.
\]

Similarly, integrating by parts the last integral in (5.2), we have

\[
\int_{\frac{1}{\alpha}}^{\infty} \frac{e(\pm \alpha t)}{t} d\{t\} = \mp 2\pi i\alpha \int_{\frac{1}{\alpha}}^{\infty} \{t\} e(\pm \alpha t) \frac{1}{t} dt + O(\alpha),
\]

and by Lemma 5.1 with \( Y = \frac{1}{\alpha} \), the integral is \( \ll 1 \), and we obtain

\[
\int_{\frac{1}{\alpha}}^{\infty} \frac{e(\pm \alpha t)}{t} d\{t\} \ll \alpha.
\]
Going back to (5.2), in which we rewrite the exponential integral as sine and cosine integrals, we obtain

\[
\sum_{n \leq 1/\alpha} \frac{1 - e(\pm \alpha n)}{n} - \sum_{n > 1/\alpha} \frac{e(\pm \alpha n)}{n} = \int_1^{1/\alpha} \frac{1}{t} \, dt - \int_1^{\infty} \frac{e(\pm \alpha t)}{t} \, dt + O(\alpha |\log \alpha|)
\]

\[
= \log \left( \frac{1}{\alpha} \right) - \left( \int_{2\pi \alpha}^{\infty} \frac{\cos t}{t} \, dt \pm i \int_{2\pi \alpha}^{\infty} \frac{\sin t}{t} \, dt \right) + O(\alpha |\log \alpha|).
\]

The cosine integrals can be estimated using the Taylor expansions and referring to [6, p. (106)], we know that

\[- \int_{2\pi \alpha}^{\infty} \frac{\cos t}{t} \, dt = \gamma + \log(2\pi \alpha) + O(\alpha^2)\]

hence we deduce that

\[- \int_{2\pi \alpha}^{\infty} \frac{\cos t}{t} \, dt = \gamma + \log(2\pi \alpha) + O(\alpha^2).\]

Now it is easily seen, using the Taylor series for sine, that

\[\int_{2\pi \alpha}^{\infty} \frac{\sin t}{t} \, dt = \int_0^{\infty} \frac{\sin t}{t} \, dt + O(\alpha),\]

and it is known (see, e.g., [1, p. 232]) that

\[\int_0^{\infty} \frac{\sin t}{t} \, dt = \frac{\pi}{2}.\]

Putting this together, we reach the conclusion that

\[
\sum_{n \leq 1/\alpha} \frac{1 - e(\pm \alpha n)}{n} - \sum_{n > 1/\alpha} \frac{e(\pm \alpha n)}{n} = \log \left( \frac{1}{\alpha} \right) + \log(2\pi \alpha) + \gamma \mp \frac{i\pi}{2} + O(\alpha |\log \alpha|)
\]

\[= \log(2\pi) + \gamma \mp \frac{i\pi}{2} + O(\alpha |\log \alpha|),\]

as desired. \(\square\)

### 5.2 Some estimates on smooth numbers

We start this section with an estimate showing that the tail of a logarithmic sum over \(y\)-smooth integers is small. This will help us bound the error term in the proof of Theorem 1. The argument follows the proof of [2, Lemma 3.2].
Lemma 5.5. Let $y \geq 100$, then

$$\sum_{n > \frac{\log \log y}{P(n) \leq y}} \frac{1}{n} \ll \frac{1}{(\log y)^{\log_3 y - \frac{3}{2}}}.$$  

Proof. We have

$$\sum_{n > \frac{\log \log y}{P(n) \leq y}} \frac{1}{n} \ll \sum_{y^{\frac{1}{2} \log \log y} < n \leq y^{\frac{1}{2} \log \log y}} \frac{1}{n} + \sum_{n > y \frac{1}{2} \log \log y} \frac{1}{n}.$$  

For the first sum of the right-hand side, we use Lemma 2.2 and (2.1) to get

$$\ll (\log y)^{3/2} \rho(\log \log y) \ll \frac{\log y \sqrt{\log y}}{(\log \log y)^{\log \log y}} \ll \frac{1}{(\log y)^{\log_3 y - \frac{3}{2}}}.$$  

For the second sum, given $\varepsilon = \frac{1}{\log y}$, we have

$$\sum_{n > y \frac{1}{2} \log \log y} \frac{1}{n} \ll \sum_{n > y \frac{1}{2} \log \log y} \frac{1}{n} \left(\frac{n}{y \sqrt{\log y}}\right)^{\varepsilon} \ll e^{-\sqrt{\log y}} \sum_{P(n) \leq y} \frac{1}{n^{1-\varepsilon}} \ll e^{-\sqrt{\log y}} \prod_{p \leq y} \left(1 - \frac{1}{p^{1-\varepsilon}}\right).$$  

As $p^{\varepsilon} = 1 + O\left(\frac{\log p}{\log y}\right)$ for $p \leq y$, we have

$$\sum_{p \leq y} \frac{1}{p^{1-\varepsilon}} - \sum_{p \leq y} \frac{1}{p} \ll \sum_{p \leq y} \frac{1}{p} \left(\frac{\log p}{\log y}\right) \ll 1$$  

and thus, for $y$ large enough, putting this together we deduce that

$$\sum_{y^{\frac{1}{2} \log \log y} < n \leq y} \frac{\chi(n)\varepsilon(n\alpha)}{n} \ll \frac{1}{(\log y)^{\log_4 y - \frac{3}{2}}}.$$  

\[\square\]

Even though smooth numbers are often major allies in evaluating sums over integers, they can also be an obstacle to our ability to evaluate sums. The following lemma shows that on small intervals, the smoothness condition can be removed.
Lemma 5.6. Let $y \geq 2$ and let $f(t)$ be a differentiable bounded function such that $|f'(t)| \leq \frac{1}{\log t}$ on any interval $I \subset \left[ \frac{y^B}{\log y}, y^B (\log y)^c \right]$, then for $B < \exp((\log y)^{3/5-\epsilon})$ and uniformly for $0 \leq c \leq B(\log \log y)$, we have

$$\sum_{n \in I \atop P(n) \leq y} \frac{f(n)}{n} = \rho(B) \sum_{n \in I} \frac{f(n)}{n} + O\left(\frac{\rho(B) \log(B + 1)(\log \log y)^2}{\log y}\right).$$

Proof. Let $I$ be any subinterval of $\left[ \frac{y^B}{\log y}, y^B (\log y)^c \right]$. By partial summation, we have

$$\sum_{n \in I \atop P(n) \leq y} \frac{f(n)}{n} = \int_I \frac{f(t)}{t} d(\Psi(t, y)) = \int_I \frac{f(t)}{t} d\left(\frac{t \rho(u)}{1 + O\left(\frac{\log u}{\log y}\right)}\right).$$

Now for $t$ in that range we have that $\log u = O(\log B)$ and by Lemma 2.1, $\rho(u) = \rho(B) + O\left(\frac{\rho(B) \log(B + 1) \log \log y}{\log y}\right)$, therefore

$$\sum_{n \in I \atop P(n) \leq y} \frac{f(n)}{n} = \left(\rho(B) + O\left(\frac{\rho(B) \log(B + 1) \log \log y}{\log y}\right)\right) \int_I \frac{f(t)}{t} dt.$$

On the other hand, using partial summation again, we have

$$\sum_{n \in I} \frac{f(n)}{n} = \frac{f(t)}{t} (t + O(1)) \bigg|_I - \int_I \left(\frac{f'(t)}{t} - \frac{f(t)}{t^2}\right)(t + O(1)) dt$$

$$= f(t) \bigg|_I + \int_I \frac{f(t)}{t} - f'(t) dt + O\left(\frac{(c + 1) \log \log y}{B \log y}\right)$$

$$= f(t) \bigg|_I - f(t) \bigg|_I + \int_I \frac{f(t)}{t} dt + O\left(\frac{(c + 1) \log \log y}{B \log y}\right)$$

$$= \int_I \frac{f(t)}{t} dt + O\left(\frac{(c + 1) \log \log y}{B \log y}\right).$$

Hence comparing both sides, we deduce that

$$\sum_{n \in I \atop P(n) \leq y} \frac{f(n)}{n} = \left(\rho(B) + O\left(\frac{\rho(B) \log(B + 1) \log \log y}{\log y}\right)\right) \left(\sum_{n \in I} \frac{f(n)}{n} + O\left(\frac{(c + 1) \log \log y}{B \log y}\right)\right)$$

$$= \rho(B) \sum_{n \in I} \frac{f(n)}{n} + O\left(\frac{\rho(B) \log(B + 1) \log \log y^2}{\log y}\right),$$

which ends the proof of the lemma. \(\square\)
As we undergo the proof of Theorem 2, we will have to face such a sum and Lemma 5.6 will come in handy. We are now ready for the proof of our main theorem.

6 PROOF OF THE MAIN THEOREM

In the following, we let \( y = \log q \), \( \alpha = \frac{1}{y^B} \) for some \( 0 \leq B \leq \frac{\log \log \log q}{\log \log \log \log q} \) and we let \( z = q^{11/21} \). Pólya’s Fourier expansion gives

\[
\sum_{n \leq \alpha y} \chi(n) = \frac{\tau(\chi)}{2\pi i} \sum_{1 \leq |n| \leq z} \overline{\chi}(n) \frac{1 - e(-\alpha n)}{n} + O\left(\frac{q \log q}{z}\right),
\]

(6.1)

where \(|\tau(\chi)| = \sqrt{q}\). For \( 1 \leq y \leq z \) and \( \delta \in \left[ \frac{1}{\log y}, 1 \right] \), we define

\[
A_\delta = \left\{ \chi \ (\text{mod} \ q) : \left| \sum_{1 \leq |n| \leq z \atop P(n) > y} \frac{\chi(n)}{n} (1 - e(-\alpha n)) \right| \leq e^{\gamma \delta} \right\}.
\]

(6.2)

We believe that the bound in (6.2) should hold for all characters modulo \( q \), for \( q \) large enough, as we saw in Conjecture 1. If Conjecture 1 holds, then the proof shows that Theorem 1 is best possible for most prime moduli \( q \), as the inequality sign then becomes an equality sign. For the purpose of our proof, [2, Theorem 4.2] states that

\[
\#\{\chi \ (\text{mod} \ q) : \chi \notin A_\delta \} \ll q^{1 - \frac{\delta^2}{\log \log q}} + q^{1 - \frac{1}{500 \log \log q}}.
\]

(6.3)

We only need \( \delta = 1 \) for the case of odd characters. However, the main term in Theorem 2 is much smaller so we have to be a little more delicate with the choice of \( \delta \), taking \( \delta \) to be of size \( \frac{\log \log y}{\sqrt{\log y}} \).

We now restrict our attention to characters in \( A_\delta \) and split the remaining sum as

\[
\sum_{1 \leq n \leq z \atop P(n) \leq y} \frac{\chi(n)}{n} (1 - e(\pm \alpha n)) = S_1 + S_2^\pm + S_3^\pm,
\]

(6.4)

where the sum

\[
S_1 = \sum_{n > y^B \atop P(n) \leq y} \frac{\chi(n)}{n}
\]

will give the main contribution in the odd character case, the sum

\[
S_2^\pm = \sum_{y^B < n \leq y^B \atop P(n) \leq y} \frac{\chi(n)}{n} - \sum_{y^B < n \leq y^B (\log y)^5 \atop P(n) \leq y} \frac{\chi(n) e(\pm \alpha n)}{n}
\]
will give the main term in the even character case, and finally

$$S_3^\pm = \sum_{1 \leq n \leq \frac{y^B}{P(n) \leq y}} \frac{\chi(n) \left(1 - e^{\pm \alpha n} \right)}{n} - \sum_{y^{B \log y} < n \leq y^{B \log \log y}} \frac{\chi(n) e^{\pm \alpha n}}{n} - \sum_{y \log \log y < n \leq y} \frac{\chi(n)}{n}$$

(6.5)

will contribute the error term.

### 6.1 | $S_3$: ranges with small contribution

In this section, we dissect $S_3^\pm$ to show that it provides only a small contribution to 6.4.

**Proposition 6.1.** For $y$ large enough and $1 \leq B < \exp((\log y)^{3/5} - \epsilon)$, we have

$$S_3^\pm \ll \sqrt{B \log y}.$$ 

Further, if $\chi \in A_\pm(N, T)$, then for $0 < B < 1$

$$S_3^\pm \ll \log \log y + h(T) \log y.$$ 

We treat the sums in $S_3^\pm$ one at a time, Lemma 6.1 dealing with the first sum, Lemma 6.2 the second and the last two sums in Lemma 6.3. First we have

**Lemma 6.1.** Let $y \geq 2$ and $\frac{\log \log y}{\log y} < B < \exp((\log y)^{3/5} - \epsilon)$ and let $\alpha = \frac{1}{y^B}$, then

$$\sum_{1 \leq n \leq \frac{y^B}{P(n) \leq y}} \frac{\chi(n) \left(1 - e^{\pm \alpha n} \right)}{n} \ll \frac{\rho(B)}{\log y}.$$ 

**Proof.** We have $|\alpha n| < 1$ since $\alpha = \frac{1}{y^B}$, and thus $\frac{1 - e^{\pm \alpha n}}{n} \ll \frac{\alpha n}{n} = \alpha$. Therefore, as each $|\chi(n)| \leq 1$,

$$\sum_{1 \leq n \leq \frac{y^B}{P(n) \leq y}} \frac{\chi(n) \left(1 - e^{\pm \alpha n} \right)}{n} \ll \alpha \sum_{n \leq \frac{y^B}{P(n) \leq y}} \frac{\rho \left( B - \frac{\log \log y}{\log y} \right)}{\log y} \ll \frac{\rho(B)}{\log y}$$

by Lemma 2.1. □
The second sum in $S_{\pm}^2$ requires the use of a result from De la Bretèche for exponential sums with multiplicative coefficients over smooth numbers [3]. We obtain

**Lemma 6.2.** Let $y \geq 2$ and let $c \geq 5$. For $\alpha = \frac{1}{y^B}$, if $B \geq 1$, then uniformly for $c$

$$\sum_{y^B (\log y)^c < n \leq y \log \log y \atop P(n) \leq y} \frac{\chi(n)e(\pm \alpha n)}{n} \ll \frac{\sqrt{B}}{\log y}.$$  

If further $\chi \in A_{\pm}(N, T)$, then for $0 < B < 1$

$$\sum_{y^B (\log y)^c < n \leq y \log \log y \atop P(n) \leq y} \frac{\chi(n)e(\pm \alpha n)}{n} \ll \log y + h(T) \log y.$$  

In order to prove Lemma 6.2, we use the following theorem which appears as [3, Proposition 1].

**Theorem 6.1.** Let $f(n)$ be a multiplicative function with $\sum_{n \leq t} |f(n)|^2 \leq A^2 t$ for $A \in \mathbb{R}_{\geq 0}$, and suppose that there is $(a, m) = 1$ such that $|\alpha - \frac{a}{m}| \leq \frac{1}{m^2}$, then

$$\sum_{n \leq x \atop P(n) \leq y} f(n)e(\alpha n) \ll A^2 x \sqrt{\log x} \log y \left(\frac{\sqrt{y}}{\sqrt{x}} + \frac{\sqrt{m}}{\sqrt{x}} + \frac{1}{\sqrt{m}} + e^{-\sqrt{\log x}}\right).$$

A simple use of partial summation and the results just stated allow us to deduce Lemma 6.2.

**Proof of Lemma 6.2.** Let $\kappa = \max\{1, B\}$. Given $\alpha = \frac{1}{y^B}$, taking $m$ to be the closest integer to $y^B$, we can apply Theorem 6.1 with $A = 1$. That is, we have

$$\sum_{y^B (\log y)^c < n \leq y \log \log y \atop P(n) \leq y} \frac{\chi(n)e(\pm \alpha n)}{n} \ll \frac{1}{y^B \log y} \sum_{n \leq y^B (\log y)^c \atop P(n) \leq y} \chi(n)e(\pm \alpha n) - \frac{1}{y^B \log y} \sum_{n \leq y^B (\log y)^c \atop P(n) \leq y} \chi(n)e(\pm \alpha n)$$

$$+ \int_{y^B (\log y)^c}^{y \log \log y} \frac{\chi(n)e(\pm \alpha n)}{t^2} \, dt$$

$$\ll \frac{\sqrt{\log y \log y}}{y^{B/2}} \frac{3}{2} + \frac{\sqrt{\kappa}}{(\log y)^{\frac{c-1}{2}}}$$

$$+ y^{x/2} \log y \int_{x^B (\log y)^c}^{2B} \frac{\sqrt{\log t}}{t^{3/2}} \, dt + \log y \int_{x^B (\log y)^c}^{y^2 B} \frac{\sqrt{\log t}}{t} e^{-\sqrt{\log t}} \, dt$$

$$+ \frac{\log y}{y^{B/2}} \int_{2B}^{\log \log y} \frac{\sqrt{\log t}}{t} \, dt.$$
Computing the integrals gives

\[
\sum_{y^c (\log y)^c \leq n \leq y \log y} \frac{\chi(n)e(\pm an)}{n} \ll \frac{\sqrt{x}}{(\log y)^{\frac{c}{2} - \frac{3}{2}}} + Be^{-\frac{\sqrt{B \log y}}{2} \log^2 y} + \frac{\log y}{y^{B/2}} (\log y \log \log y)^{3/2}
\]

whenever \(c \geq 5\), proving the first part of the lemma when \(B \geq 1\).

Now if \(B < 1\), then \(\kappa = 1\) and we still have to estimate the sum over the range \([y^B (\log y)^c, y (\log y)^c]\). In that case, write

\[
\sum_{y^B (\log y)^c < n \leq y (\log y)^c} \frac{\chi(n)e(\pm an)}{n} = \sum_{y^B (\log y)^c < n \leq y} \frac{\chi(n)e(\pm an)}{n} + \sum_{y < n \leq (\log y)^c} \frac{\chi(n)e(\pm an)}{n}
\]

As the first sum is \(y\)-smooth, we can remove the smoothness condition, and using Proposition 4.2 and Lemma 5.2, we obtain

\[
\sum_{y^B (\log y)^c < n \leq y} \frac{\chi(n)e(\pm an)}{n} = \sum_{y^B (\log y)^c < n \leq y} \frac{e(\pm an)}{n} + O(h(T) \log y)
\]

\[
= O\left(\frac{1}{(\log y)^c} + h(T) \log y\right).
\]

Now for the second sum, bounding trivially the numerator gives

\[
\sum_{y < n \leq (\log y)^c} \frac{\chi(n)e(\pm an)}{n} = \sum_{y < n \leq (\log y)^c} \frac{1}{n} \ll \log \log y,
\]

therefore, putting this together, we get

\[
\sum_{y^B (\log y)^c < n \leq y (\log y)^c} \frac{\chi(n)e(\pm an)}{n} \ll \log y + h(T) \log y,
\]

which proves the second part of the Lemma. \(\square\)

The next lemma deals with the two last sums of (6.5) and follows directly from Lemma 5.5.
Lemma 6.3. Let \( \chi \) be a character modulo \( q \), let \( \alpha \) be any real number in \((0,1] \) and \( z \geq y^{\log \log y} \) for \( y \geq 100 \), then

\[
\sum_{\substack{y^{\log \log y} \leq n \leq z \\ P(n) \leq y}} \frac{\chi(n)e(\pm \alpha n)}{n} + \sum_{\substack{n > z \\ P(n) \leq y}} \frac{\chi(n)}{n} \ll \frac{1}{(\log y)^{\log_3 y - 3/2}}.
\]

Finally, putting Lemmas 6.1, 6.3, and 6.2 together gives Proposition 6.1.

6.2 \ | \ \( S_1 \) and \( S_2 \): The main contributions

Our strategy in order to evaluate \( S_1 \) and \( S_2 \) will be to use characters that pretend to be 1, so that \( \chi \in A_\pm(N, T) \). This supposes that our choice of character will satisfy

\[
\max_{p \leq T} |\chi(p) - 1| \ll \frac{1}{N},
\]

and using this hypothesis brings us back to the results we derived in Section 4. With this hypothesis in hand, Proposition 4.3 gives that

\[
S_1 = \log y \int_B^\infty \rho(u)du + O(1 + h(T) \log y)
\]

and this constitutes our main term in Theorem 1. It now remains to evaluate \( S_2 \).

6.2.1 \ | \ The constant arising from \( S_2 \)

We show that if \( \chi \) pretends to be 1, then \( S_2 \) gives rise to a constant.

Proposition 6.2. Let \( y \geq 2 \), let \( 0 \leq B < \exp((\log y)^{3/5 - \varepsilon} \) and let \( \chi \) be in \( A_\pm(N, T) \). Then for \( \kappa = \max\{1, B\} \)

\[
S_2^\pm = \rho(B)\left(\gamma + \log(2\pi) \mp \frac{i\pi}{2}\right) + O\left(h(T)\rho(\kappa - 1)\log \log y + \frac{\rho(B) \log(B + 1)(\log \log y)^2}{\log y}\right).
\]

Proof. We start by using Proposition 4.2 with \( f(n) = 1 - e(\pm \alpha n) \) for the first sum and \( f(n) = e(\pm \alpha n) \) for the second sum to approximate \( \chi \) by 1. We have

\[
\begin{align*}
&\sum_{\substack{y^{\log \log y} \leq n \leq y^B \\ P(n) \leq y}} \chi(n) \frac{1 - e(\pm \alpha n)}{n} - \sum_{\substack{y^B \leq n \leq y^B(\log y)^5 \\ P(n) \leq y}} \chi(n) \frac{e(\pm \alpha n)}{n} \\
&= \sum_{\substack{y^{\log \log y} \leq n \leq y^B \\ P(n) \leq y}} \frac{1 - e(\pm \alpha n)}{n} - \sum_{\substack{y^B \leq n \leq y^B(\log y)^5 \\ P(n) \leq y}} \frac{e(\pm \alpha n)}{n} + O\left(h(y) \log y \int_{\kappa-1+\frac{\log \log y}{\log y}}^{\kappa-1} \rho(u)du \right).
\end{align*}
\]
\[
\sum_{\frac{y^B}{\log y} \leq n \leq y^B} \frac{1 - e(\pm \alpha n)}{n} - \sum_{\frac{y^B}{\log y} \leq n \leq y^B (\log y)^5} \frac{e(\pm \alpha n)}{n} + O(h(T)\rho(\kappa - 1) \log \log y),
\]

where we bounded the integral with Lemma 2.1. Next, to evaluate the right-hand side, we start by removing the smoothness condition with Lemma 5.6 and then we throw back in the end ranges to the summations using Lemmas 5.3 and 5.2 with \( c = 5 \left(1 - \frac{\log B}{\log |\log \alpha|}\right) \). This gives us

\[
\sum_{\frac{y^B}{\log y} \leq n \leq y^B} \chi(n) \frac{1 - e(\pm \alpha n)}{n} - \sum_{\frac{y^B}{\log y} \leq n \leq y^B (\log y)^5} \chi(n) \frac{e(\pm \alpha n)}{n} = \rho(B) \left( \gamma + \log(2\pi) \mp \frac{i\pi}{2} \right)
\]

\[
+ O\left(h(T)\rho(\kappa - 1) \log \log y + \frac{\rho(B) \log(B + 1)(\log \log y)^2}{\log y}\right),
\]

Finally, appealing to Lemma 5.4, we obtain

\[
\sum_{\frac{y^B}{\log y} \leq n \leq y^B} \chi(n) \frac{1 - e(\pm \alpha n)}{n} - \sum_{\frac{y^B}{\log y} \leq n \leq y^B (\log y)^5} \chi(n) \frac{e(\pm \alpha n)}{n} = \rho(B) \left( y + \log(2\pi) \mp \frac{i\pi}{2} \right)
\]

\[
+ O\left(h(T)\rho(\kappa - 1) \log \log y + \frac{\rho(B) \log(B + 1)(\log \log y)^2}{\log y}\right),
\]

which proves the proposition.

\[\square\]

### 6.3 Smooth 1-pretentious characters

We already know from Propositions 4.4 and 4.5 that there are many characters pretending to be 1. From now on, assume that \( q \) is an admissible prime for the bounds to hold. Now, recall that we have restricted our characters to be in the set \( A_\delta \) defined as in (6.2), so we need to choose \( N \) and \( T \) to make sure that \( A_\delta \cap A_{\pm}(N, T) \neq \emptyset \).
Proposition 6.3. Let \( y = \log q \), and let \( A_\pm = A_\pm (T, N) \) for \( N = \log y \) and \( T = \frac{y}{4 \log y} \). Then

\[
\#A_\pm \gg q^{1 - \frac{\log \log q}{(\log \log q)^2}}.
\]

Proof. We know, as stated in Propositions 4.4 and 4.5, that

\[
\#A_\pm (N, T) \gg \frac{q}{N^{2T}}.
\]

Now, as \( y = \log q \), and given our choice \( N = \log y \) and \( T = \frac{y}{4 \log y} \), we have

\[
N^{2T} = \exp \left( \frac{y}{2 \log y} \frac{\log \log y}{\log T} \right) \leq \exp \left( \frac{y \log \log y}{\log^2 y} \right) = q^{\frac{\log y}{\log^2 y}},
\]

from which we deduce that

\[
\#A_\pm \gg q^{1 - \frac{\log \log q}{(\log \log q)^2}}. \tag*{\□}
\]

Corollary 6.1. Let \( A_\delta, A_\pm \) be the sets defined as above. If \( \delta > \left( \frac{\log \log \log q}{\log \log q} \right)^{\frac{1}{2}} \), then

\[
\#A_\delta \cap A_\pm \gg q^{1 - \frac{\log \log q}{(\log \log q)^2}}.
\]

Proof. Let \( \mathcal{A} = \{ \chi (\mod q) : \chi \not\in A_\delta \} \) be the exceptional set of \( A_\delta \) and suppose that \( \delta > \left( \frac{\log \log \log q}{\log \log q} \right)^{\frac{1}{2}} \), then by (6.3) we have that

\[
\#A \ll q^{1 - \left( \frac{\log \log q}{\log \log q} \right)^2}.
\]

That is, using Proposition 6.3, we get

\[
\#A_\delta \cap A_\pm = \#A_\pm - \#A_\pm \cap A \geq \#A_\pm - \#A \\
\gg q^{1 - \frac{\log \log q}{(\log \log q)^2}} - q^{1 - \left( \frac{\log \log q}{\log \log q} \right)^2} \gg q^{1 - \frac{\log \log q}{(\log \log q)^2}},
\]

as claimed. \tag*{\□}

Now that we have found at least a character to work with, we finally have the ingredients we need and are ready to go forward with the proof of Theorem 1.
6.4 | Proof of theorem 1

We are now ready to prove our main theorem, along with Theorem 2.

Proof of Theorem 1. Let $q$ be an admissible prime, let $y = \log q$ and let $\alpha = \frac{1}{y^{11}}$. Starting with Pólya’s Fourier expansion, we have

$$
\sum_{n \leq \alpha q} \chi(n) = \frac{\tau(\chi)}{2\pi i} \sum_{1 \leq |n| \leq z} \frac{\chi(n)}{n} (1 - e(-\alpha n)) + O\left(\frac{q \log q}{z}\right),
$$

where we let $z = q^{11/21}$.

Now we let $\delta = \frac{1}{\log \log y} (\log y)^{1/2}$ in (6.2), so that by Corollary 6.1 $|A_\delta \cap A_{\pm}| \neq \emptyset$, and we choose a character $\chi$ in the intersection. We have

$$
\sum_{1 \leq |n| \leq z} \frac{\chi(n)}{n} (1 - e(-\alpha n)) = \sum_{1 \leq n \leq z} \frac{\chi(n)}{n} (1 - e(-\alpha n)) - \chi(-1) \sum_{1 \leq n \leq z} \frac{\chi(n)}{n} (1 - e(\alpha n))
$$

$$
= (S_1 + S_2^- + S_3^-) - \chi(-1)(S_1 + S_2^+ + S_3^+) + O(\delta).
$$

At this point, we need to treat the odd and even character cases separately. If $\chi$ is an even character, then we get cancellation of $S_1$ and we are left with a contribution from $S_3^\pm$ and an error term from $S_3^\pm$. Because the main term from $S_3^\pm$ is a constant, we need to take $B \geq 1$ for the error from $S_3^\pm$ to be small enough. With this restriction, using Propositions 6.1 and 6.2, with $h(T) = h(y) = \frac{\log \log y}{\log y}$, we get

$$
\sum_{1 \leq |n| \leq z} \frac{\chi(n)}{n} (1 - e(-\alpha n)) = i\pi \rho(B) + O\left(\frac{(\rho(B) - 1)(\log \log y)^2}{\log y}\right) + O(\delta),
$$

and thus, going back to (6.1), we obtain

$$
\sum_{n \leq \alpha q} \overline{\chi}(n) = \frac{\tau(\chi)}{2\pi i} (i\pi \rho(B) + O(\delta)) + O(\rho(B) \log q)
$$

$$
= \frac{\tau(\chi)\rho(B)}{2} + O\left(\sqrt{q}\delta\right).
$$

Recalling that $y = \log q$ in $\delta = \frac{\log \log y}{(\log y)^{1/2}}$ and that $|\tau(\chi)| = \sqrt{q}$, we get

$$
\max_{\chi \neq \chi_0} \left| \sum_{n \leq q} \chi(n) \right| \geq \frac{\rho(B)}{2} \sqrt{q} + O\left(\frac{\sqrt{q} \log \log q}{\sqrt{\log \log q}}\right),
$$

as desired.
As for the odd character case, given \( \chi \in A_\delta \cap A_- \), we allow \( B > 0 \) and we use Propositions 6.1 for \( S_3^+ \) and \( S_3^- \), Proposition 6.2 for \( S_2^+ \) and \( S_2^- \) and Proposition 4.3 for \( S_1 \), to obtain

\[
\sum_{1 \leq |n| \leq z} \frac{\chi(n)}{n} (1 - e(-\alpha n)) = 2 \log y \int_B^\infty \rho(u)du + 2 \rho(B)(y \log(2\pi)) + O(\log \log y)
\]

\[
= 2 \log y \int_B^\infty \rho(u)du + O(\log \log y),
\]

where the error term is arising from Propositions 4.3 and 6.1. As a consequence, using (6.1), we deduce that

\[
\sum_{n \leq \alpha \sqrt{q}} \chi(n) = \frac{\tau(\chi)}{2\pi} \left( 2 \log y \int_B^\infty \rho(u)du + O(\log \log y) \right)
\]

\[
= \frac{\tau(\chi)}{\pi} \log y \int_B^\infty \rho(u)du + O(\sqrt{q} \log \log y),
\]

from which we conclude that

\[
\max_{\substack{\chi \neq \chi_0 \\chi \text{ odd}}} \left| \sum_{n \leq \alpha \sqrt{q}} \chi(n) \right| \geq \frac{\sqrt{q}}{\pi} \log \log q \int_B^\infty \rho(u)du + O(\sqrt{q} \log \log \log q),
\]

thus proving the theorem.

\[\square\]

**APPENDIX: ALTERNATIVE PROOF OF THEOREM 1**

We include here the sketch of an alternative proof that was generously provided to us by an anonymous referee who reviewed the present paper. We only include the even character case, the odd one being entirely similar.

**A.1  |  Sketch of the proof**

Let \( q \) be a large prime, and \( \chi \) be a non-principal character \( (\mod q) \). We are interested in finding large values of \( \sum_{n \leq \alpha q} \chi(n) \), especially in the case where \( \alpha = \frac{1}{(\log q)^{\beta}} \) for small values of \( B \). Using \( z = q^{\frac{3}{2}} \), the Fourier expansion of character sums (6.1) gives

\[
\left| \sum_{n \leq \alpha q} \chi(n) \right| = \frac{\sqrt{q}}{2\pi} \left| \sum_{1 \leq n \leq z} \frac{\chi(n)}{n} (1 - e(-n\alpha)) \right| + O(q^{\frac{1}{3}} \log q). \tag{A.1}
\]

Thus the problem is now to obtain lower bounds on the character sum in the right side of A.1. Letting \( S(x, y) \) denote the set of \( y \)-smooth integers up to \( x \) and \( \Psi(x, y) \) denote the cardinality of this set, we use the resonance method to show
Proposition A.1. Put $x = q^{\frac{1}{2}}$, and let $y$ be any parameter below $x$. Then

$$\max_{\chi \neq \chi_0 \atop \chi \text{ even}} \left| \sum_{n \leq \alpha q} \chi(n) \right| \geq \frac{\sqrt{q}}{\pi} \left| \sum_{n \in S(x,y)} \frac{1}{n} \sin(2\pi n\alpha) \frac{\Psi(x/n,y)}{\Psi(x,y)} \right| + O(q^{\frac{1}{3}} \log q),$$

while

$$\max_{\chi \neq \chi_0 \atop \chi \text{ odd}} \left| \sum_{n \leq \alpha q} \chi(n) \right| \geq \frac{\sqrt{q}}{\pi} \left| \sum_{n \in S(x,y)} \frac{1}{n} (1 - \cos(2\pi n\alpha)) \frac{\Psi(x/n,y)}{\Psi(x,y)} \right| + O(q^{\frac{1}{3}} \log q).$$

Proof. We deal with the case of even characters, the odd case being entirely similar. Define the resonator

$$R(\chi) = \sum_{n \in S(x,y)} \chi(n)$$

and consider

$$\sum_{\chi \neq \chi_0 \atop \chi \text{ even}} \sum_{n \leq z} \frac{\overline{\chi}(n)}{n} \sin(2\pi n\alpha) |R(\chi)|^2 = \sum_{\chi \text{ even}} \sum_{n \leq z} \frac{\overline{\chi}(n)}{n} \sin(2\pi n\alpha) |R(\chi)|^2 + O(\Psi(x,y)^2 \log z),$$

upon adding back the contribution of the principal character. Expanding out $|R(\chi)|^2$ and using orthogonality, the above equals

$$\frac{\phi(q)}{2} \sum_{n \leq z} \frac{\sin(2\pi n\alpha)}{n} \sum_{a,b \in S(x,y)} 1_{na \equiv \pm b \pmod{q}} + O(q^{\frac{1}{2}} \log q).$$

Since $xz \leq q^2$, we can only have $na \equiv b \pmod{q}$ if $na = b$ and this is only possible if $n$ is $y$-smooth. Therefore the above equals

$$\frac{\phi(q)}{2} \sum_{n \in S(x,y)} \frac{\sin(2\pi n\alpha)}{n} \Psi(x/n,y) + O(q^{\frac{1}{2}} \log q).$$

On the other hand,

$$\sum_{\chi \neq \chi_0 \atop \chi \text{ even}} |R(\chi)|^2 \leq \sum_{\chi \text{ even}} |R(\chi)|^2 = \frac{\phi(q)}{2} \Psi(x,y).$$

It follows that

$$\max_{\chi \neq \chi_0 \atop \chi \text{ even}} \left| \sum_{n \leq \alpha q} \chi(n) \right| \geq \frac{\sqrt{q}}{\pi} \left| \sum_{n \in S(x,y)} \frac{1}{n} \sin(2\pi n\alpha) \frac{\Psi(x/n,y)}{\Psi(x,y)} \right| + O(q^{\frac{1}{2}} \log q),$$

proving proposition. \qed
We can apply the proposition to produce large values of character sums by taking $y$ to be a bit smaller than $\log q$, so that whenever $n = y^{O(1)}$, we can evaluate $\frac{\Psi(x/n, y)}{\Psi(x, y)}$ using [7, Theorem 3], showing that it is roughly of size 1. The proposition together with partial summation and the estimates from Section 5.1 will lead to the desired lower bound for the character sums.

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