Determinant representation for a quantum correlation function of the lattice sine–Gordon model

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ABSTRACT

We consider a completely integrable lattice regularization of the sine–Gordon model with discrete space and continuous time. We derive a determinant representation for a correlation function which in the continuum limit turns into the correlation function of local fields. The determinant is then embedded into a system of integrable integro–differential equations. The leading asymptotic behaviour of the correlation function is described in terms of the solution of a Riemann Hilbert Problem (RHP) related to the system of integro–differential equations. The leading term in the asymptotical decomposition of the solution of the RHP is obtained.

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1 Introduction

The sine–Gordon model is completely integrable (exactly solvable) both on the classical and on the quantum level [1]–[7]. We shall write the sine–Gordon equation in the following form:

\[ \frac{\partial^2}{\partial t^2} u(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) + \frac{m^2}{\beta} \sin \beta u(x, t) = 0. \]  

(1.1)

Here \( m \) is a mass, \( \beta \) is the coupling constant. For later convenience we also introduce

\[ \gamma = \frac{\beta^2}{8}. \]

In the classical case \( u(x, t) \) is a function of two variables, \( x \) and \( t \) are space and time coordinates. In the quantum case \( u(x, t) \) is a local quantum field of the sine–Gordon model. The Hamiltonian reads

\[ H = \int dx \left( \frac{1}{2} p^2 + \frac{1}{2} (\partial_x u)^2 + \frac{m^2}{\beta^2} (1 - \cos \beta u) \right). \]

(1.2)

Momentum and topological charge are given by

\[ P = -\int dx \ p \ \partial_x u, \quad Q = \frac{\beta}{2\pi} \int dx \ \partial_x u. \]

(1.3)

Here \( p(x, t) = \partial_t u(x, t) \) and \( u(x, t) \) satisfy Poisson brackets \( \{ p(x), u(y) \} = \delta(x - y) \). Equation (1.1) has a Lax representation and a classical \( r \)-matrix [1]–[6]. After quantization, the fields \( u \) and \( p \) satisfy canonical commutation relations \( [u(x), p(y)] = i \delta(x - y) \). The physical ground state \( |\Omega\rangle \) of the quantum system can be obtained by filling the Dirac sea of negative energy pseudoparticles [7].

Let us now consider the quantum operator

\[ \exp \left( \alpha Q(x) \right) = \exp \left\{ \frac{\alpha \beta}{2\pi} (u(x) - u(0)) \right\}, \quad Q(x) = \frac{\beta}{2\pi} \int_0^x dz \partial_z u(z), \]

(1.4)

where \( Q(x) \) measures the topological charge on the interval \([0, x]\). In this paper we show how to represent the correlation function

\[ \langle \Omega | \exp (\alpha Q(x)) |\Omega\rangle \]

(1.5)

as the determinant of an integral operator (in fact we shall see below, that the coefficient \( \alpha \) in (1.5) needs to be renormalized). Note that via differentiation with respect to \( \alpha \) we can obtain correlation functions of local quantum fields from (1.5). We shall consider the quantum version of (1.2) in the region \( \frac{\pi}{2} < \gamma < \frac{2\pi}{3} \) (many of our intermediate results hold in larger regions of coupling constant). Note that \( \gamma \to 0 \) is the quasiclassical region of the sine–Gordon model and at \( \gamma = \pi/2 \) the spectrum of the Hamiltonian is equivalent to free fermions. To deal with the ultraviolet divergences of the continuum model we shall employ a suitably chosen lattice regularization.

The determinant representation then permits to describe the correlation functions in terms of a system of integrable integro–differential equations. These equations can be solved by means
of a Riemann-Hilbert problem which in turn enables one to obtain elementary formulas for the asymptotics of the correlation functions. This program has first been applied to the nonlinear Schrödinger equation in [8] and is described in detail in the book [9] (see also [10]).

There has been previous work on determining correlation functions in the sine–Gordon model. Form factors were determined by Smirnov in [11, 12]. At the free fermionic point $\gamma = \pi/2$ a determinant representation of the correlation function (1.5) has been constructed using the coordinate Bethe Ansatz in [13]. A description of a different correlator at the free fermionic point through a Fredholm determinant (derived from a form factor sum) which in turn satisfies an integrable differential (sinh-Gordon) equation has been obtained in [14]. In this paper we start the investigation of correlation functions in the sine–Gordon model for general $\gamma$, in particular away from the free fermionic point in the framework of its solution [15, 16] by means of the Quantum Inverse Scattering Method.

The plan of this paper is as follows: in section 2 we review the integrable lattice regularization of the sine–Gordon model introduced in [16]. The Algebraic Bethe Ansatz is formulated and the construction of the ground state [17] is discussed. In section 3 we derive the determinant representation of the correlator (1.5) for the range of coupling constants stated above. As this part of the analysis is very similar to the analogous problem for the spin-$\frac{1}{2}$ Heisenberg XXZ model (which was treated in full detail in [22]) we omit many details and only give an account of the main steps without providing proofs (which can be found in [22]). In sections 4 and 5 we embed the determinant representation into a system of integrable integro-differential equations and in section 6 the related Riemann-Hilbert problem is formulated and the leading asymptotic behaviour of the correlation function is extracted.

2 Lattice Sine–Gordon

2.1 $\mathcal{L}$-Operator

We shall consider a lattice version of the sine–Gordon model which is also completely integrable. It will have exactly the same $r$-matrix (both in the classical and quantum case) as the continuous model. The elementary $\mathcal{L}$-operator of the LSG model is [13, 16]

$$
\mathcal{L}(n|\lambda) = \begin{pmatrix}
    e^{-i\beta p_n/8} \rho_n e^{-i\beta p_n/8} & \frac{1}{2} m \Delta \sinh(\lambda - i\beta u_n/2) \\
    -\frac{1}{2} m \Delta \sinh(\lambda + i\beta u_n/2) & e^{i\beta p_n/8} \rho_n e^{i\beta p_n/8}
\end{pmatrix}
$$

(2.1)

Here $\Delta$ is the lattice constant and $p_n, u_n$ are the dynamical variables on site $n$ of the lattice. In the quantum model they obey canonical commutation relations $[u_n, p_m] = i\delta_{nm}$. Furthermore, we have introduced

$$
\rho_n = (1 + 2S \cos \beta u_n)^{\frac{1}{2}} , \quad S = \left( \frac{1}{4} m \Delta \right)^2.
$$

(2.2)

The symmetries of the $\mathcal{L}$-operator of the LSG model are expressed by the identities (the asterisk means Hermitian conjugation of the quantum operators)

$$
\sigma^y \mathcal{L}^*(n|\lambda) \sigma^y = \mathcal{L}(n|\lambda) , \quad \sigma^z \mathcal{L}(n|\lambda) \sigma^z = \mathcal{L}(n|\lambda + i\pi) .
$$

(2.3)

Its quantum determinant [13, 16] is

$$
\det_q \mathcal{L}(n|\lambda) \equiv 1 + 2S \cosh 2\lambda .
$$

(2.4)
The $\mathcal{L}$-operator (2.1) satisfies the Yang-Baxter equation
\[ R(\lambda, \mu) \left( \mathcal{L}(n|\lambda) \otimes \mathcal{L}(n|\mu) \right) = \left( \mathcal{L}(n|\mu) \otimes \mathcal{L}(n|\lambda) \right) R(\lambda, \mu) \, . \] (2.5)

$R(\lambda, \mu)$ in Eq. (2.5) is the standard sine–Gordon $R$-matrix given by the following expression:
\[
R(\lambda, \mu) = \begin{pmatrix}
  f(\mu, \lambda) & 0 & 0 & 0 \\
  0 & g(\mu, \lambda) & 1 & 0 \\
  0 & 1 & g(\mu, \lambda) & 0 \\
  0 & 0 & 0 & f(\mu, \lambda)
\end{pmatrix} \, . \] (2.6)

Here
\[
f(\mu, \lambda) = \frac{\sinh(\mu - \lambda - i\gamma)}{\sinh(\mu - \lambda)} \, , \quad g(\mu, \lambda) = -i \frac{\sin \gamma}{\sinh(\mu - \lambda)} \, . \] (2.7)

In different sites of the lattice the matrix elements of $\mathcal{L}$ commute. As usual in the Quantum Inverse Scattering method (QISM) we define the monodromy matrix by taking products of the $\mathcal{L}$-operators in matrix space:
\[
\mathcal{T}(\lambda) = \mathcal{L}(L|\lambda) \mathcal{L}(L-1|\lambda) \ldots \mathcal{L}(1|\lambda)
\] (2.8)

where $L$ is the number of sites in the lattice which we take to be even. By construction this operator also satisfies a Yang Baxter equation
\[ R(\lambda, \mu) \left( \mathcal{T}(\lambda) \otimes \mathcal{T}(\mu) \right) = \left( \mathcal{T}(\mu) \otimes \mathcal{T}(\lambda) \right) R(\lambda, \mu) \, . \] (2.9)

It might be interesting to point out that the entries for the $\mathcal{L}$-operator (2.1) form a representation of a quantum group: The operators (we suppress the site index $n$)
\[
S^+ = \frac{2}{i \sin \gamma m \Delta} e^{i\beta p/8} \rho e^{i\beta p/8} \, , \\
S^- = \frac{-2}{i \sin \gamma m \Delta} e^{-i\beta p/8} \rho e^{-i\beta p/8} \, , \\
S^0 = e^{-i\beta u/2} \, , \quad S^1 = e^{i\beta u/2}
\]
satisfy the commutation relations of the quadratic (Sklyanin) algebra
\[
[S^+, S^-] = \frac{1}{q - q^{-1}} \left( (S^0)^2 - (S^1)^2 \right) \, , \quad [S^0, S^1] = 0 \, , \\
S^\pm S^0 = q^{\mp 1} S^0 S^\pm \, , \quad S^\pm S^1 = q^{\mp 1} S^1 S^\pm
\]
with $q = \exp(i\gamma)$. For $q$ being a root of unity this algebra has finite dimensional cyclic representations: for rational values of the parameter $\gamma/\pi = Q/P$ the quantum operators entering the $\mathcal{L}$-operator can be written as $2P \times 2P$ matrices with elements
\[
\chi = e^{i\beta u/2} \rightarrow \delta_{ab} e^{i\pi(a-1)/P} \, , \quad \pi = e^{i\beta p/4} \rightarrow \delta_{a+Q,b} \, , \quad a, b = 1, \ldots, 2P \, , \, a + 2P \equiv a \, .
\]

The definition of the $\mathcal{L}$ operator alone does not determine a definite lattice model: In addition the Hamiltonian of the lattice sine–Gordon model needs to be specified. For this
choice there exist several different possibilities (see [13, 16, 18]). All of them are completely integrable and can in fact be diagonalized simultaneously. Furthermore all of them have the same continuum limit [1.2]. They differ from one another by higher orders in the lattice spacing $\Delta$. While all of them can be considered equivalently as a lattice regularization of the continuum model we shall show below, how a unique lattice Hamiltonian can be chosen by requiring that it has the “same” ground state wave function as the continuum model. This choice of the Hamiltonian will bring the dynamics of the lattice model as close as possible to that of the continuum model.

2.2 Algebraic Bethe Ansatz for the lattice sine–Gordon model

We shall consider the monodromy matrix (2.8)

$$\mathcal{T}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \quad (2.10)$$

As a direct consequence of the Yang-Baxter equation (2.5) for $\mathcal{T}(\lambda)$ the trace of the monodromy matrix, the so-called transfer matrix

$$\tau(\lambda) = \text{trace } \mathcal{T}(\lambda) = A(\lambda) + D(\lambda) \quad (2.11)$$

commutes for different values of the spectral parameter $\lambda$, i.e. $[\tau(\lambda), \tau(\mu)] = 0$. Hence, it is the generator of commuting integrals for the system which are diagonalized by the algebraic Bethe Ansatz. Starting point is the “pseudo vacuum” (or reference state): To construct this simple eigenstate of $\tau(\lambda)$ we combine the $\mathcal{L}$-operators in pairs:

$$\hat{\mathcal{L}}(n|\lambda) = \mathcal{L}(2n|\lambda) \mathcal{L}(2n - 1|\lambda) \equiv \begin{pmatrix} \alpha_n(\lambda) & \beta_n(\lambda) \\ \gamma_n(\lambda) & \delta_n(\lambda) \end{pmatrix}. \quad (2.12)$$

Choosing

$$\langle u|0\rangle_n = \left\{ 1 - 2S \cos \frac{\beta}{2} (u_{2n} + u_{2n-1}) \right\}^{-\frac{1}{2}} \delta \left( u_{2n} - u_{2n-1} - \frac{\beta}{4} + \frac{2\pi}{\beta} \right) \quad (2.13)$$

(for rational $\gamma/\pi = Q/P$ the $\delta$-function can be replaced by a Kronecker $\delta$-symbol and $|0\rangle_n$ will become normalizable) we find from (2.12)

$$\gamma_n(\lambda) |0\rangle_n = 0, \quad \alpha_n(\lambda) |0\rangle_n = \{ 1 + 2S \cosh(2\lambda - i\gamma) \} |0\rangle_n \quad (2.14)$$

$$\delta_n(\lambda) |0\rangle_n = \{ 1 + 2S \cosh(2\lambda + i\gamma) \} |0\rangle_n$$

Now we can follow the standard steps of the algebraic Bethe Ansatz. As a consequence of (2.14) the “global pseudo vacuum”

$$|0\rangle = \prod_{n=1}^{L/2} |0\rangle_n \quad (2.15)$$

is an eigenstate of the operators $A(\lambda)$ and $D(\lambda)$ (and hence the transfer matrix (2.11)) with eigenvalues $a(\lambda)$ and $d(\lambda)$, respectively:

$$a(\lambda) = \{ 1 + 2S \cosh(2\lambda - i\gamma) \}^{\frac{1}{2}}, \quad d(\lambda) = \{ 1 + 2S \cosh(2\lambda + i\gamma) \}^{\frac{1}{2}}. \quad (2.16)$$
More eigenfunctions of the transfer matrix are found by acting with the operator \( B(\lambda) \) on the pseudo vacuum
\[
\prod_{j=1}^{N} B(\lambda_j) |0\rangle
\]
provided that the \( \{\lambda_j\} \) satisfy the Bethe Ansatz equations
\[
\left( \frac{1 + 2S \cosh(2\lambda_j - i\gamma)}{1 + 2S \cosh(2\lambda_j + i\gamma)} \right)^{\frac{1}{2}} = -\prod_{k=1}^{N} \frac{\sinh(\lambda_j - \lambda_k + i\gamma)}{\sinh(\lambda_j - \lambda_k - i\gamma)}
\]
The corresponding eigenvalue of the transfer matrix (2.11) is
\[
\Lambda(\lambda|\lambda_j) = a(\lambda) \prod_{j=1}^{N} f(\lambda, \lambda_j) + d(\lambda) \prod_{j=1}^{N} f(\lambda_j, \lambda)
\]
where \( f(\lambda, \mu) \) has been defined in (2.7).

The number \( N \) of the Bethe Ansatz roots \( \lambda_j \) can be identified with the topological charge (1.3). The correct lattice version in the quantum case is
\[
Q = \frac{4}{\beta} \sum_{n=1}^{L/2} (u_{2n} - u_{2n-1}) + L \frac{\pi - \gamma}{2\gamma}
\]
The difference in the coefficient compared to (1.3) is related to the fractional charge of the excitations. In (1.4) it was shown that the fractional charge appears due to the repulsion beyond the cutoff in the process of ultraviolet renormalization. (2.20) is the number operator for particles
\[
Q \prod_{j=1}^{N} B(\lambda_j)|0\rangle = N \prod_{j=1}^{N} B(\lambda_j)|0\rangle.
\]
One can prove that (here \( \sigma^z \) is the Pauli matrix in the matrix space)
\[
[Q, T(\lambda)] = \frac{1}{2} [\sigma^z, T(\lambda)].
\]

Now we can discuss our choice of the lattice Hamiltonian for the lattice sine–Gordon model. As mentioned above we want to construct a lattice version resembling the dynamics of the continuum model as closely as possible. According to the standard quantization of the sine–Gordon model the ground state of the continuum model contains no bound states (strings).

For possible lattice models we shall concentrate on the two integrable models introduced in Refs. (13, 16) and (18). The latter has been constructed by Tarasov, Takhtajan and Faddeev (TTF) such that it contains interactions of nearest neighbours on the lattice only. The ground state for this Hamiltonian was found in Ref. (20) in addition to a Dirac sea of elementary particles it contains bound states. In the continuum limit the density of the bound states vanishes, thus reproducing the known results for the continuum model. Apart from the Hamiltonian the QISM yields higher integrals of motion. These describe interactions over larger distances. Adding these interaction terms to the TTF Hamiltonian with coefficients vanishing in the continuum limit \( \Delta \to 0 \) produces different lattice Hamiltonians with the same continuum
limit while preserving integrability. This is the origin of the freedom in choice of the lattice hamiltonian.

Another Hamiltonian for the lattice sine–Gordon model has been introduced in [15, 16]: The corresponding ground state for this Hamiltonian has been constructed by Bogoliubov [17]:

He was able to prove that in the interval $\pi/3 \leq \gamma \leq 2\pi/3$ the ground state is built from elementary particles only—just as in the continuum model. Furthermore, he found that the set of observable excitations coincides with the continuum model. Hence, unlike the situation in the TTF model no phase transition is met in performing the continuum limit. For the reasons stated above we choose this Hamiltonian for our studies of correlation functions.

It is given in terms of trace identities. Expressing the zeroes $d(\kappa_{\pm}) = 0$ and $a(\nu_{\pm}) = 0$ of (2.16) as

$$e^{2\kappa_{\pm}} = -b^{\pm 1} e^{-i\gamma}, \quad e^{2\nu_{\pm}} = -b^{\pm 1} e^{i\gamma},$$

where $b = \frac{2S}{1 + \sqrt{1 - 4S^2}}$ (2.21) $(\lambda_{\pm} = \frac{1}{2}(i\pi \pm \ln b)$ are the zeroes of the quantum determinant (2.4) of $L$) the Hamiltonian of the lattice sine–Gordon model considered here is given by

$$H_{LSG} = -\frac{m^2 \Delta}{32b \sin \gamma} \left\{ e^{i\gamma} \left( \frac{\partial}{\partial \lambda} \ln \frac{\tau(\lambda)}{\tau(\lambda)} \bigg|_{\lambda=\kappa_+} \right) - e^{-i\gamma} \left( \frac{\partial}{\partial \lambda} \ln \frac{\tau(\lambda)}{\tau(\lambda)} \bigg|_{\lambda=\kappa_-} \right) + e^{-i\gamma} \left( \frac{\partial}{\partial \lambda} \ln \frac{\tau(\lambda)}{\tau(\lambda)} \bigg|_{\lambda=\nu_+} \right) - e^{i\gamma} \left( \frac{\partial}{\partial \lambda} \ln \frac{\tau(\lambda)}{\tau(\lambda)} \bigg|_{\lambda=\nu_-} \right) \right\}$$

(2.22)

This is the model studied in [15, 16]. From (2.19) one finds that (2.17) are eigenfunctions of this Hamiltonian with energy eigenvalues given by

$$H_{LSG} \prod_{j=1}^N B(\lambda_j)|0\rangle = \left( \sum_{k=1}^N h(\lambda_k) \right) \prod_{j=1}^N B(\lambda_j)|0\rangle$$

(2.23)

with the single particle energies

$$h(\lambda) = \frac{m^2 \Delta}{32bi} \left\{ \frac{e^{i\gamma}}{\sinh(\kappa_+ - \lambda) \sinh(\kappa_+ - \lambda - i\gamma)} - \frac{e^{-i\gamma}}{\sinh(\kappa_- - \lambda) \sinh(\kappa_- - \lambda - i\gamma)} \right.$$

$$\left. - \frac{e^{-i\gamma}}{\sinh(\nu_+ - \lambda) \sinh(\nu_+ - \lambda + i\gamma)} + \frac{e^{i\gamma}}{\sinh(\nu_- - \lambda) \sinh(\nu_- - \lambda + i\gamma)} \right\}.$$ 

(2.24)

In the continuum limit $\Delta \to 0$ (which is reached by letting $b \to 0$ here) one immediately reproduces the result

$$h(\lambda)|_{\Delta \to 0} = \frac{1}{2} m^2 \Delta \sin \gamma \cosh 2\lambda$$

for the single particle dispersion of the continuum model.

To find the solution of (2.18) corresponding to the ground state of the model it is necessary to classify the possible configurations of $\lambda_j$ in the complex plane according to the so called string hypothesis [21]. The details of this are not important in the present context. It was found by Bogoliubov [17] that the ground state of (2.22) is obtained by filling all permitted
states of pseudoparticles with rapidities $\lambda_j$ on the line $\text{Im}\lambda = \pi/2$. Taking the logarithm of Eq. (2.18) in such a state one obtains

$$\frac{L}{2} p(\lambda + i\pi/2) = 2\pi Q_j - i \sum_k \ln \left( \frac{\sinh(\lambda_j - \lambda_k + i\gamma)}{\sinh(\lambda_j - \lambda_k - i\gamma)} \right). \tag{2.25}$$

Here the $Q_j$ are distinct integers characterizing the state uniquely and

$$p(\lambda + i\pi/2) = -i \ln \left( \frac{1 - 2S \cosh(2\lambda - i\gamma)}{1 - 2S \cosh(2\lambda + i\gamma)} \right).$$

In the thermodynamic limit the density $\rho(\lambda_j) = \frac{1}{L} \partial Q_j / \partial \lambda_j$ is then given in terms of the integral equation

$$\frac{1}{2} p'(\lambda + i\pi/2) = 2\pi \rho(\lambda) + \int_{-\infty}^{+\infty} d\mu K(\lambda - \mu) \rho(\mu) \tag{2.26}$$

where

$$K(\lambda) = \frac{-\sin 2\gamma}{\sinh(\lambda + i\gamma) \sinh(\lambda - i\gamma)} = \frac{-2 \sin 2\gamma}{\cosh 2\lambda - \cos 2\gamma}. \tag{2.27}$$

This integral equation can be solved by Fourier transform resulting in ($\Lambda = -\ln b$)

$$\rho(\lambda) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} dk \ e^{-ik\lambda} \frac{\sin \frac{1}{2} k(\pi - \gamma)}{\sinh \frac{1}{2} k\gamma \cosh \frac{1}{2} k(\pi - \gamma)} \cos \frac{1}{2} k\Lambda. \tag{2.28}$$

Similarly one can compute the excitation energies. This is useful to find the correct mass renormalization formula: To perform the continuum limit of the sine–Gordon model one should let $\Delta \to 0$ and simultaneously $m \to \infty$ as

$$m = \text{const.} \Delta^{-\gamma/\pi}. \tag{2.29}$$

3 Algebraic formulation of correlation functions

For the evaluation of the correlation function (1.5) we shall make extensive use of the similarity (in the framework of the QISM) of the LSG model with the spin-$\frac{1}{2}$ XXZ Heisenberg chain which is derived from a monodromy matrix satisfying a Yang-Baxter equation with the same $R$-matrix (2.6) as the present model. The correlation functions corresponding to (1.5) in the XXZ model have have recently been studied in [22, 23].

First we note, that the symmetry of the $L$ operator (2.3) implies for the ground state configuration consisting of rapidities $\{\tilde{\lambda}_j = \lambda_j + i\pi/2\}$ with real $\lambda_j$

$$\left( B(\tilde{\lambda}_j) \right)^\dagger = C(\tilde{\lambda}_j).$$

In order to express the correlation function (1.5) in the algebraic framework outlined above we first need to define the lattice analog of the operator $Q(x)$ in (1.4). The correct expression is found to be

$$Q_1(n) = \frac{4}{\beta} \sum_{k=1}^{n/2} (u_{2k} - u_{2k-1}) + n \left( \frac{\pi - \gamma}{2\gamma} \right) \tag{3.1}$$
which counts the number of particles in the interval \([1, n]\) \((n\ \text{even})\). In the continuum limit this expression becomes

\[
Q_1(n) \to \frac{2}{\beta} (u(x) - u(0)) + \frac{x}{\Delta} \left( \frac{\pi - \gamma}{2\gamma} \right), \quad x = n\Delta
\]  
(3.2)

Hence the lattice analog of the correlation function \(\langle \Omega \rangle\) can be written as

\[
\langle \Omega | \exp(\alpha Q_1(n)) | \Omega \rangle = \frac{\langle 0 | \prod_{j=1}^{N} C(\tilde{\lambda}_j) \exp(\alpha Q_1(n)) \prod_{k=1}^{N} B(\tilde{\lambda}_k) | 0 \rangle}{\langle 0 | \prod_{j=1}^{N} C(\lambda_j) \prod_{k=1}^{N} B(\lambda_k) | 0 \rangle}
\]  
(3.3)

where \(\tilde{\lambda}_j\) are solutions of the Bethe Ansatz equations \((2.18)\) for the ground state configuration.

Let us first study the norm appearing in the denominator of this expression. To evaluate this expression one should commute the \(C(\lambda_j)\) to the right of the product where they annihilate the pseudovacuum \(|0\rangle\). Since the commutation relations between the elements of the monodromy matrix \((2.10)\) are completely determined by the \(R\)-matrix we can use the result of \([24,22]\) (see also \([25,26]\)) for the norm of Bethe Ansatz states (after identifying \(\gamma\) with \(2(\pi - \eta)\) in paper \([22]\))

\[
\langle 0 | \prod_{j=1}^{N} C(\lambda_j) \prod_{j=1}^{N} B(\lambda_k) | 0 \rangle = (-\sin \gamma)^N \left\{ \prod_{j \neq k} f(\lambda_j, \lambda_k) \right\} \left\{ \prod_{j=1}^{N} a(\lambda_j)d(\lambda_j) \right\} \det N
\]  
(3.4)

where the \(N \times N\) matrix \(N\) is given by

\[
N_{jk} = \delta_{jk} \left\{ i \frac{\partial}{\partial \lambda_j} \ln \left( \frac{a(\tilde{\lambda}_j)}{d(\lambda_j)} \right) + \sum_{n=1}^{N} K(\tilde{\lambda}_j - \tilde{\lambda}_n) \right\} - K(\tilde{\lambda}_j - \tilde{\lambda}_k).
\]

The functions \(K(\lambda)\) and \(a(\lambda), d(\lambda)\) have been introduced in the previous section. In the thermodynamic limit this expression can be further simplified: We rewrite \(N = \mathcal{I} \cdot \mathcal{J}\) where

\[
\mathcal{I}_{jk} = \delta_{jk} - \frac{K(\lambda_j - \lambda_k)}{\theta_k}, \quad \mathcal{J}_{jk} = \delta_{jk} \theta_j
\]

\[
\theta_j = i \frac{\partial}{\partial \lambda_j} \ln \left( \frac{a(\tilde{\lambda}_j)}{d(\lambda_j)} \right) + \sum_{n=1}^{N} K(\tilde{\lambda}_j - \tilde{\lambda}_n).
\]

Comparing the last expression with Eqs. \((2.25)\) and \((2.26)\) for the ground state density of particles one obtains \(\theta_j = -2\pi L\rho(\lambda_j)\). Performing the thermodynamic limit on the matrix \(\mathcal{I}\) one finds that it turns into a Fredholm integral operator \(\hat{\mathcal{I}} = 1 + \frac{1}{2\pi} \hat{K}\) acting as

\[
\hat{\mathcal{I}} \ast f |\lambda = f(\lambda) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\mu K(\lambda - \mu) f(\mu)
\]  
(3.5)

Here \(K(\lambda)\) is kernel given in \((2.27)\).

Putting everything together we find

\[
\langle 0 | \prod_{j=1}^{N} C(\tilde{\lambda}_j) \prod_{j=1}^{N} B(\tilde{\lambda}_k) | 0 \rangle = \left(2\pi L \sin \gamma\right)^N \left\{ \prod_{j \neq k} f(\lambda_j, \lambda_k) \right\}
\]

\[
\times \left\{ \prod_{j=1}^{N} a(\tilde{\lambda}_j)d(\tilde{\lambda}_j) \rho(\lambda_j) \right\} \det \left(1 + \frac{1}{2\pi} \hat{K}\right).
\]  
(3.6)
We now turn to the numerator of (3.3): to reduce the evaluation of the expectation value of \(\exp(\alpha Q_1(n))\) in a Bethe state (2.17) to the computation of scalar products we divide the lattice of length \(L\) into two sub-chains of length \(n\) and \(L - n\) and associate a monodromy matrix with each of them, namely

\[
\mathcal{T}(\lambda) = \mathcal{T}(2, \lambda)\mathcal{T}(1, \lambda), \quad \mathcal{T}(i, \lambda) = \begin{pmatrix} A_i(\lambda) & B(\lambda_i) \\ C_i(\lambda) & D(\lambda_i) \end{pmatrix}, \quad i = 1, 2. \tag{3.7}
\]

In terms of \(\mathcal{L}\)-operators they are given by

\[
\mathcal{T}(2, \lambda) = \mathcal{L}(L, \lambda) \mathcal{L}(L - 1, \lambda) \ldots \mathcal{L}(n + 1, \lambda) \\
\mathcal{T}(1, \lambda) = \mathcal{L}(n, \lambda) \mathcal{L}(n - 1, \lambda) \ldots \mathcal{L}(1, \lambda).
\]

By construction these monodromy matrices satisfy the same Yang-Baxter equation (2.9) as \(\mathcal{T}(\lambda)\). Similarly, the global reference state (2.17) can be decomposed into a direct product of pseudo vacua for the subchains \(|0\rangle_2 \otimes |0\rangle_1\) (remember that we have chosen \(n\) to be even) which are eigenstates of \(A_i(\lambda)\) and \(D_i(\lambda)\)

\[
A_i(\lambda)|0\rangle_i = a_i(\lambda)|0\rangle_i, \quad D_i(\lambda)|0\rangle_i = d_i(\lambda)|0\rangle_i, \tag{3.8}
\]

where \(a_i(\lambda)\) and \(d_i(\lambda)\) are given by (2.16) with \(L\) replaced by \(n\) and \(L - n\) for \(i = 1, 2\), respectively. The creation and annihilation operators \(B_i(\lambda)\) and \(C_i(\lambda)\) act according to

\[
C_i(\lambda)|0\rangle_i = 0, \quad \langle 0|B_i(\lambda) = 0. \tag{3.9}
\]

In this decomposed quantum space the numerator of (3.3) can be rewritten as (see e.g. [9, 22])

\[
\sum_1 \langle 0| \prod_{I_C} C_1 (\tilde{\lambda}^C_{I_C}) \prod_{I_B} B_1 (\tilde{\lambda}^B_{I_B}) |0\rangle_1 \times \langle 0| \prod_{I_C} C_2 (\tilde{\lambda}^C_{I_C}) \prod_{I_B} B_2 (\tilde{\lambda}^B_{I_B}) |0\rangle_2 \\
\times e^{\alpha m_1} \left\{ \prod_{I_B,I_C} a_2 (\tilde{\lambda}^B_{I_B}) d_2 (\tilde{\lambda}^C_{I_C}) \right\} \left\{ \prod_{I_B,I_C} a_1 (\tilde{\lambda}^C_{I_C}) d_1 (\tilde{\lambda}^B_{I_B}) \right\} \\
\times \left\{ \prod_{I_B,I_C} f (\tilde{\lambda}^B_{I_B}, \tilde{\lambda}^B_{I_B}) \right\} \left\{ \prod_{I_C,I_C} f (\tilde{\lambda}^C_{I_C}, \tilde{\lambda}^C_{I_C}) \right\} \tag{3.10}
\]

where the sum is over all partitions

\[
\{ \tilde{\lambda}^B_{I_B} \} \cup \{ \tilde{\lambda}^B_{I_B} \} = \{ \tilde{\lambda} \}, \quad \{ \tilde{\lambda}^B_{I_B} \} \cap \{ \tilde{\lambda}^B_{I_B} \} = \emptyset, \quad \{ \tilde{\lambda}^C_{I_C} \} \cup \{ \tilde{\lambda}^C_{I_C} \} = \{ \tilde{\lambda} \}, \quad \{ \tilde{\lambda}^C_{I_C} \} \cap \{ \tilde{\lambda}^C_{I_C} \} = \emptyset
\]

of the set \(\{ \tilde{\lambda} \}\) with \(\text{card}(\tilde{\lambda}^B_{I_B}) = \text{card}(\tilde{\lambda}^C_{I_C}) = n_1, \text{card}(\tilde{\lambda}^B_{I_B}) = \text{card}(\tilde{\lambda}^C_{I_C}) = N - n_1\). Due to (1.9) we only need to consider partitions such that the sizes of \(I_B\) and \(I_C\) (and \(II_B\) and \(II_C\)) are the same. We next turn to an investigation of the scalar products occurring in (3.10). Owing to (3.8) and (3.9) and the fact that the monodromy matrices \(\mathcal{T}(i, \lambda)\) fulfill the same Yang-Baxter equation (2.9) as \(\mathcal{T}(\lambda)\) it is sufficient to consider scalar products on the entire lattice

\[
S_N = \langle 0 | \prod_{j=1}^N C(\lambda^C_j) \prod_{k=1}^N B(\lambda^B_k) |0 \rangle.
\]
Here we do not assume that the sets of spectral parameters \( \{ \lambda^B \} \) and \( \{ \lambda^C \} \) are the same, and we also do not impose the Bethe equations (2.18). From (2.7) and the action on the reference state \( A(\lambda)|0\rangle = a(\lambda)|0\rangle \), \( D(\lambda)|0\rangle = d(\lambda)|0\rangle \) it follows that scalar products can be represented as

\[
S_N = \sum_{A,D} \prod_{j=1}^N a(\lambda^A_j) \prod_{k=1}^N d(\lambda^D_k) K_N \left( \begin{array}{c} \{ \lambda^C \} \\ \{ \lambda^A \} \end{array} \right) , \tag{3.11}
\]

where the sum is over all partitions of \( \{ \lambda^C \} \cup \{ \lambda^B \} \) into two sets \( \{ \lambda^A \} \) and \( \{ \lambda^D \} \). The coefficients \( K_N \) are functions of the \( \lambda_j \) and are completely determined by the intertwining relation (2.3). The \( R \)-matrix (2.6) is however identical to the one for the spin-\( \frac{1}{2} \) Heisenberg XXZ model (after appropriate identifications of the coupling constants). This implies that the coefficients \( K_N \) for the sine–Gordon model and the XXZ chain are identical, so that we can take over the result for the XXZ case (see e.g. [22]). The main point is that the \( K_N \)’s can be represented as determinants. This is done in two steps: first the so-called highest coefficients, which are obtained for the partition \( \{ \lambda^A \} = \{ \lambda^C \} \), \( \{ \lambda^B \} = \{ \lambda^B \} \), are represented as determinants

\[
K_N \left( \begin{array}{c} \{ \lambda^C \} \\ \{ \lambda^A \} \end{array} \right) = \left( \prod_{j<k} g(\lambda^B_j, \lambda^B_k) g(\lambda^C_j, \lambda^C_k) \right) \prod_{j,k} h(\lambda^C_j, \lambda^B_k) \det(M^B_C) ,
\]

\[
h(\mu, \nu) = \frac{f(\mu, \nu)}{g(\mu, \nu)} , \quad (M^B_C)_{jk} = \frac{g(\lambda^C_j, \lambda^B_k)}{h(\lambda^C_j, \lambda^B_k)} = t(\lambda^C_j, \lambda^B_k) . \tag{3.12}
\]

where from (2.7)

\[
h(\lambda, \mu) = \frac{\sinh(\lambda - \mu - i\gamma)}{-i \sin \gamma} , \quad t(\lambda, \mu) = \frac{-\sin^2 \gamma}{\sinh(\lambda - \mu - i\gamma) \sinh(\lambda - \mu)} .
\]

In the second step arbitrary coefficients \( K_N \) are then expressed in terms of highest coefficients as follows

\[
K_N \left( \begin{array}{c} \{ \lambda^C \} \\ \{ \lambda^A \} \end{array} \right) = \left( \prod_{j \in AC} \prod_{k \in DC} f(\lambda^AC_j, \lambda^DC_k) \right) \left( \prod_{l \in AB} \prod_{m \in DB} f(\lambda^AB_l, \lambda^DB_m) \right) \times K_n \left( \begin{array}{c} \{ \lambda^AB \} \nonumber \\ \{ \lambda^AC \} \nonumber \end{array} \right) \quad K_{n-n} \left( \begin{array}{c} \{ \lambda^AC \} \nonumber \\ \{ \lambda^DB \} \nonumber \end{array} \right) . \tag{3.13}
\]

Using (3.12) and (3.13) in (3.11) we obtain the following expression for general scalar products in the lattice sine–Gordon model

\[
S_N = \prod_{j<k} g(\lambda^C_j, \lambda^C_k) g(\lambda^B_j, \lambda^B_k) \sum \text{sgn}(P_C) \text{sgn}(P_B) \prod_{j,k} h(\lambda^AB_j, \lambda^AC_k) \prod_{l,m} h(\lambda^AC_l, \lambda^DB_m) \times \prod_{l,k} h(\lambda^AC_l, \lambda^DC_k) \prod_{j,m} h(\lambda^AB_j, \lambda^DB_m) \det(M^AB_{DC}) \det(M^DC_{AB}) , \tag{3.14}
\]

where \( P_C \) is the permutation \( \{ \lambda^AC_1, \ldots, \lambda^AC_n, \lambda^DC_1, \ldots, \lambda^DC_n \} \) of \( \{ \lambda^C_1, \ldots, \lambda^C_n \} \), \( P_B \) is the permutation \( \{ \lambda^DB_1, \ldots, \lambda^DB_n, \lambda^AB_1, \ldots, \lambda^AB_n \} \) of \( \{ \lambda^B_1, \ldots, \lambda^B_n \} \), \( \text{sgn}(P) \) is the sign of the permutation \( P \), and

\[
(M^AB_{DC})_{jk} = t(\lambda^AB_j, \lambda^DC_k) a(\lambda^DC_k) d(\lambda^DC_k) a(\lambda^AB_j) . \tag{3.15}
\]
Following the steps first carried out in [27] it is now possible to represent $S_N$ as a single determinant. The discussion for sine–Gordon is identical to the only for the XXZ chain [22] so that we only present a brief discussion of the necessary steps and give the final result. We first note that the sum on the r.h.s. in (3.14) looks very similar to a Laplace decomposition of the determinant of the sum of two matrices $(S_1)_{jk} = t(\lambda^C_j, \lambda^B_k)a(\lambda^C_j)d(\lambda^B_k)$ and $(S_2)_{jk} = t(\lambda^B_j, \lambda^C_k)d(\lambda^C_j)a(\lambda^B_k)$ (see e.g. [3] p. 221). However this does not reproduce the $h(\lambda, \mu)$-factors. This leads to the introduction of a dual quantum field $\varphi(\lambda)$ acting in a bosonic Fock space with vacua $|0\rangle$ and $|\bar{0}\rangle$ according to

$$
\varphi(\lambda) = p(\lambda) + q(\lambda), \ [\varphi(\lambda), \varphi(\mu)] = 0, \ (\bar{0}|q(\lambda) = 0 = p(\lambda)|0\rangle, \ [p(\lambda), q(\mu)] = 0 = [q(\lambda), q(\mu)].
$$

We emphasize that the field $\varphi$ commutes for different values of spectral parameters. Using the dual field it is now possible to recast (3.14) as a single determinant of the sum of two matrices

$$
S_N = \prod_{j>k}g(\lambda^C_j, \lambda^C_k)g(\lambda^B_j, \lambda^B_k)\prod_{j=1}^N a(\lambda^C_j)d(\lambda^B_j)\prod_{j,k} h(\lambda^C_j, \lambda^B_k)(|0\rangle \det S|0\rangle),
$$

$$
S_{jk} = t(\lambda^C_j, \lambda^B_k) + t(\lambda^B_k, \lambda^C_j)\frac{r(\lambda^B_k)}{r(\lambda^C_j)}\exp \left(\varphi(\lambda^B_k) - \varphi(\lambda^C_j)\right)
$$

$$
\times \prod_{m=1}^N \frac{h(\lambda^B_k, \lambda^B_m)h(\lambda^C_j, \lambda^C_m)}{h(\lambda^C_j, \lambda^B_k)h(\lambda^B_j, \lambda^B_m)}
$$

(3.17)

where $r(\lambda) = \frac{a(\lambda)}{d(\lambda)}$. The price we pay for representing $S_N$ as a single determinant is the occurrence of the expectation value in the dual space.

Using (3.17) in (3.10) and then applying the dual field trick several times it is possible to represent (3.10) as a single determinant of the sum of four matrices. This analysis is completely analogous to the XXZ case treated in [22] so that we only state the result:

$$
\langle 0| \prod_{j=1}^N C(\tilde{\lambda}_j)\exp(aQ_1(n)) \prod_{k=1}^N B(\tilde{\lambda}_k)|0\rangle = \left\{ \prod_{j\neq k} f(\lambda_j, \lambda_k) \right\} \left\{ \prod_{j=1}^N a(\tilde{\lambda}_j)d(\tilde{\lambda}_j) \right\} \langle 0| \det G|0\rangle
$$

$$
G_{jk} = t(\tilde{\lambda}_j, \tilde{\lambda}_k) + t(\tilde{\lambda}_k, \tilde{\lambda}_j)\frac{r_1(\tilde{\lambda}_j)}{r_1(\tilde{\lambda}_k)}\exp \left(\varphi_2(\tilde{\lambda}_k) - \varphi_2(\tilde{\lambda}_j)\right)
$$

$$
+ \exp \left(\alpha + \varphi_4(\tilde{\lambda}_k) - \varphi_3(\tilde{\lambda}_j)\right) \left[t(\tilde{\lambda}_k, \tilde{\lambda}_j) + t(\tilde{\lambda}_j, \tilde{\lambda}_k)\frac{r_1(\tilde{\lambda}_j)}{r_1(\tilde{\lambda}_k)}\exp \left(\varphi_1(\tilde{\lambda}_j) - \varphi_1(\tilde{\lambda}_k)\right)\right]
$$

$$
-i \delta_{jk} \sin \gamma \frac{\partial}{\partial \lambda_j} \left(\ln(r(\tilde{\lambda}_j)) + \sum_{n=1}^N \ln \left[\frac{h(\tilde{\lambda}_j, \lambda_n)}{h(\lambda_n, \tilde{\lambda}_j)}\right]\right),
$$

(3.18)

where $r_1(\lambda) = a_1(\lambda)/d_1(\lambda) = \left(\frac{1+2S \cosh(2\lambda - i\gamma)}{1+2S \cosh(2\lambda + i\gamma)}\right)^\frac{1}{2}$ and the commuting dual fields $\varphi_a$ are defined according to

$$
\varphi_a(\lambda) = p_a(\lambda) + q_a(\lambda), \ (\bar{0}|q_a(\lambda) = 0 = p_a(\lambda)|0\rangle, \ (\bar{0}|0\rangle = 1, \ a = 1 \ldots 4, \)
$$

1We use the same notation as in [22].
where \( a, b = 1 \ldots 4 \). Here all terms not proportional to \( \delta_{jk} \) in \( G_{jk} \) are understood in the sense of l'Hôpital for the diagonal elements. In the thermodynamic limit further simplifications take place. Following the analysis for the norms above we express \( G \) as the product of two matrices \( J \) and \( W \)

\[
G = - (\sin \gamma) W J \quad , \quad J_{jk} = \delta_{jk} \theta_k \quad , \quad W_{jk} = \delta_{jk} - \frac{1}{\theta_k} \mathcal{V}(\bar{\lambda}_j, \bar{\lambda}_k) \ ,
\]

where \( \theta_j = -2\pi L \rho(\lambda_j) \) and

\[
(sin \gamma) \mathcal{V}(\lambda, \mu) = t(\lambda, \mu) + t(\mu, \lambda) \frac{r_1(\lambda)}{r_1(\mu)} \exp (\varphi_2(\mu) - \varphi_2(\lambda)) + \exp (\alpha + \varphi_4(\mu) - \varphi_3(\lambda)) \left[ t(\mu, \lambda) + t(\lambda, \mu) \frac{r_1(\lambda)}{r_1(\mu)} \exp (\varphi_1(\lambda) - \varphi_1(\mu)) \right].
\]

In the thermodynamic limit \( W \) turns into an integral operator \( \hat{W} = 1 + \frac{1}{2\pi} \hat{\nabla} \) acting as

\[
(1 + \frac{1}{2\pi} \hat{\nabla}) \ast f |_\lambda = f(\lambda) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\mu V(\lambda, \mu) f(\mu) \ ,
\]

where the integral kernel is obtained from (3.21) as (the arguments of the dual fields are shifted by \( i\pi/2 \) which does not alter the defining commutation relations (3.19))

\[
V(\lambda, \mu) = \frac{- \sin \gamma}{\sinh(\lambda - \mu)} \left\{ \frac{1}{\sinh(\lambda - \mu - i\gamma)} + \frac{e_2^{-1}(\lambda)e_2(\mu)}{\sinh(\lambda - \mu + i\gamma)} \right. \\
+ \exp(\alpha + \varphi_4(\mu) - \varphi_3(\lambda)) \left( \frac{1}{\sinh(\lambda - \mu + i\gamma)} + \frac{e_1^{-1}(\mu)e_1(\lambda)}{\sinh(\lambda - \mu - i\gamma)} \right) \right\} \ ,
\]

with

\[
e_2(\lambda) = \left( \frac{1 - 2S \cosh(2\lambda + i\gamma)}{1 - 2S \cosh(2\lambda - i\gamma)} \right)^{\frac{1}{2}} e^{\varphi_2(\lambda)} \quad , \quad e_1(\lambda) = \left( \frac{1 - 2S \cosh(2\lambda - i\gamma)}{1 - 2S \cosh(2\lambda + i\gamma)} \right)^{\frac{1}{2}} e^{\varphi_1(\lambda)} .
\]

Putting everything together we thus find

\[
\langle \Omega | \exp(\alpha Q_1(n)) | \Omega \rangle \equiv \frac{\langle 0 | \det \left( 1 + \frac{1}{2\pi} \hat{\nabla} \right) | 0 \rangle}{\det \left( 1 + \frac{1}{2\pi} \hat{K} \right)} \ ,
\]

where \( 1 + \frac{1}{2\pi} \hat{\nabla} \) and \( 1 + \frac{1}{2\pi} \hat{K} \) are integral operators acting according to (3.22) and (3.5) with kernels defined in (2.27) and (3.23).
4 Continuum limit

As mentioned in the introduction the purpose of the present work is to determine correlators for the SG Quantum Field Theory, and the lattice model studied above is used merely as a regulator for the UV divergences. We are therefore interested in the \textit{continuum limit} of the determinant representation \textit{(3.24)}. As mentioned above the SG Quantum Field Theory is recovered from the lattice regularization by taking the lattice spacing to zero $\Delta \to 0$ and simultaneously the bare mass $m$ to infinity keeping $m\Delta^2$ fixed \cite{17}. In order to take the continuum limit we now employ the following regularisation for the integral operator \textit{(3.24)}: we restrict the integration for the integral operator $1 + \frac{1}{2\pi} \hat{V}$ to the interval $[-\Lambda, \Lambda]$, and then take $\Delta \to 0$ in such a way that $S \cosh(2\lambda) \ll 1$ if $\forall \lambda \in [-\Lambda, \Lambda]$ (recall \textit{(2.2)} for the relation of $S$ and $\Delta$). Using this regularisation the $e_j(\lambda)$’s simplify to

\begin{equation}
    e_2(\lambda) = \exp(-ip\sinh(2\lambda) + \varphi_2(\lambda)), \quad e_1(\lambda) = \exp(ip\sinh(2\lambda) + \varphi_1(\lambda)),
\end{equation}

where

\begin{equation}
    p = c^2 \frac{\Delta}{8} \frac{\sinh(\pi n\Delta)}{\sin(n\Delta)}.
\end{equation}

Here we have used \textit{(2.23)} and $n\Delta = x$ should be identified with the continuum distance. The constant $c$ is given in terms of the physical soliton mass.

This regularisation allows to embed the determinant \textit{(3.24)} into a system of integrable integro-differential equations which we shall need later to determine the subleading terms in the asymptotic expansion of the correlation functions: with \textit{(4.1)} the kernel \textit{(3.23)} can be brought into standard form \cite{8}: We perform a change of variables $z = \exp(2\lambda)$, and replace the factors $(\sinh(\lambda - \mu \pm i\gamma))^{-1}$ in \textit{(3.23)} by an integration over an exponential. Then the transpose of the kernel \textit{(3.23)} reads (up to a similarity transform which leaves the determinant unchanged)

\begin{equation}
    \frac{1}{2\pi} V^T(z_1, z_2) = i \sum_{j=1}^{4} E_j(z_2|s) e_j(z_1|s), \quad (4.3)
\end{equation}

where

\begin{align}
    e_1(z|s) &= \frac{\kappa}{\sqrt{2\pi}} \exp(\varphi_4(z)) |2, z, s|, \quad E_1(z|s) = -\frac{\kappa}{\sqrt{2\pi}} \exp(-\varphi_3(z)) |2, z, s| \\
    e_2(z|s) &= \frac{1}{\sqrt{2\pi}} |1, z, s|, \quad E_2(z|s) = \frac{1}{\sqrt{2\pi}} |1, z, s| \\
    e_3(z|s) &= \frac{1}{\sqrt{2\pi}} \exp(-ipk(z) + \varphi_2(z)) |2, z, s|, \quad E_3(z|s) = -\frac{1}{\sqrt{2\pi}} \exp(ipk(z) - \varphi_2(z)) |2, z, s| \\
    e_4(z|s) &= \frac{\kappa}{\sqrt{2\pi}} \exp(-ipk(z) - \varphi_1(z) + \varphi_4(z)) |1, z, s| \\
    E_4(z|s) &= -\frac{\kappa}{\sqrt{2\pi}} \exp(ipk(z) + \varphi_1(z) - \varphi_3(z)) |1, z, s|.
\end{align}

(4.4)

Here we use the notation $k(z) = \frac{1}{2}(z - z^{-1})$, $w = \exp(i\gamma)$, $\kappa = \exp(\frac{x}{w})$, and

\begin{align}
    |1, z, s| &= \sqrt{2z \sin(\gamma)} \exp(izws) = \langle 2, z, s |, \quad |2, z, s| = \sqrt{2z \sin(\gamma)} \exp(-izws) = \langle 1, z, s | \\
    (4.5)
\end{align}
are normalized in such a way that \( \langle 1|1 \rangle = \int_0^\infty ds \, \langle 1,z,s|1,z,s \rangle = 1 \), and similarly \( \langle 2|2 \rangle = 1 \).

The inverse of the integral operator \( 1 + \frac{1}{2\pi} \hat{V}^T \) is defined by
\[
(1 - \hat{R}) \ast (1 + \frac{1}{2\pi} \hat{V}^T) = 1 = (1 + \frac{1}{2\pi} \hat{V}^T) \ast (1 - \hat{R}) ,
\]
\[
\hat{R} = (1 + \frac{1}{2\pi} \hat{V}^T)^{-1} \ast \frac{1}{2\pi} \hat{V}^T . \tag{4.6}
\]

In terms of the functions \( f_j(z|s), F_j(z|s) \)
\[
(1 - \hat{R}) \ast e_j \bigg|_{z,s} = f_j(z|s) , \quad E_j \ast (1 - \hat{R}) \bigg|_{z,s} = F_j(z|s) . \tag{4.7}
\]
the kernel of \( \hat{R} \) can be written in a form similar to (4.3)
\[
R(z_1, z_2) = i \sum_{j=1}^{4} \int_0^\infty ds \, f_j(z_1|s)F_j(z_2|s) . \tag{4.8}
\]
as can be seen by acting with \( 1 + \frac{1}{2\pi} \hat{V}^T \) on (4.8).

## 5 Integro-Differential equations

Let us now derive integro-differential equations determining the functions \( f_j(z|s) \) and \( F_j(z|s) \). The analog of these equations in the case of impenetrable bosons proved very useful for the analysis of the corresponding Riemann-Hilbert problem and we expect the equations below to play a similar role for the problem at hand. To this end we consider derivatives with respect to \( p \) and the integration boundary \( \Lambda \). For the \( \Lambda \)-derivatives we find
\[
\partial_\Lambda f_j(z|s) + \sum_{l=1}^{4} \int_0^\infty dt U_{jl}(z|s,t) f_l(z|t) = 0 ,
\]
\[
\partial_\Lambda F_j(z|s) - \sum_{l=1}^{4} \int_0^\infty dt F_l(z|t) U_{lj}(z|t, s) = 0 , \tag{5.1}
\]
where
\[
U_{jk}(z|s,t) = \frac{2ie^{2\Lambda}}{z - e^{2\Lambda}} f_j(e^{2\Lambda}|s) F_k(e^{2\Lambda}|t) + \frac{2ie^{-2\Lambda}}{z - e^{-2\Lambda}} f_j(e^{-2\Lambda}|s) F_k(e^{-2\Lambda}|t) . \tag{5.2}
\]
The \( p \)-derivatives of the functions \( f_j(z|s) \) obey the integro-differential equations
\[
\partial_p f_j(z|s) = \left( -ik(z) f_j(z|s) + \frac{1}{2} \sum_{l=1}^{4} \left[ B_{jl}^{(0)} + \frac{1}{z} B_{jl}^{(1)} \right] * f_l \bigg|_{z,s} \right) (\delta_{j,3} + \delta_{j,4})
\]
\[
- \frac{1}{2z} \sum_{k=3}^{4} C_{jk}^{(1)} * \sum_{l=1}^{4} \left[ I - iB_{kl}^{(1)} \right] * f_l \bigg|_{z,s} - \frac{1}{2} \sum_{k=3}^{4} C_{jk}^{(0)} * f_k \bigg|_{z,s} , \tag{5.3}
\]
These identities can be easily proved by using (4.7). From now on we will replace expressions by
\[ B^{(n)}_{jk}(s, t) = \int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} \frac{dz}{zn} C_j(z|s) F_k(z|t) , \]
\[ C^{(n)}_{jk}(s, t) = \int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} \frac{dz}{zn} f_j(z|s) E_k(z|t) . \] (5.4)

We note the following relations between the integral operators \( B^{(n)}_{jk} \) and \( C^{(n)}_{jk} \)
\[ B^{(0)}_{jk}(s, t) = C^{(0)}_{jk}(s, t) , \quad [I - iB^{(1)}]_{jk} * [I + iC^{(1)}]_{kl} \bigg|_{s,t} = \delta_{jl} \delta(s - t) . \] (5.5)

These identities can be easily proved by using (4.7). From now on we will replace \( B^{(0)}_{jk} \) in all expressions by \( C^{(0)}_{jk} \). The IDE for \( F_j(z|s) \) are found to be
\[ \partial_p F_j(z|s) = \left( ik(z) F_j(z|s) + \frac{1}{2} \sum_{l=1}^{4} F_l * \left[ C^{(0)}_{lj} + \frac{1}{z} C^{(1)}_{lj} \right] \bigg|_{z,s} \right) \left( \delta_{j,3} + \delta_{j,4} \right) \]
\[ - \frac{1}{2z} \sum_{k=3}^{4} \sum_{l=1}^{4} \frac{1}{2} \sum_{m=3}^{4} F_l * \left[ I + iC^{(1)} \right]_{lk} * B^{(1)}_{kj} \bigg|_{z,s} - \frac{1}{2} \sum_{k=3}^{4} \frac{1}{2} \sum_{m=3}^{4} F_k * C^{(0)}_{kj} \bigg|_{z,s} , \] (5.6)

The “potentials” \( B^{(n)} \) and \( C^{(n)} \) obey the equations
\[ \partial_p C^{(n)}_{jk}(s, t) = -\frac{1}{2} \sum_{m=3}^{4} C^{(0)}_{jm} * C^{(n)}_{mk} \bigg|_{s,t} \]
\[ - \frac{1}{2} \sum_{m=3}^{4} \sum_{l=1}^{4} C^{(1)}_{jm} * C^{(n)}_{ml} \bigg|_{s,t} \]
\[ - i \left( \delta_{j,3} + \delta_{j,4} \right) \left[ C^{(n-1)}_{jk}(s, t) - C^{(n+1)}_{jk}(s, t) + i \sum_{l=1}^{4} C^{(0)}_{jk} * C^{(n)}_{sk} \bigg|_{s,t} + B^{(1)}_{jk} * C^{(n+1)}_{sk} \bigg|_{s,t} \right] \]
\[ + i \left( \delta_{k,3} + \delta_{k,4} \right) \left[ C^{(n-1)}_{jk}(s, t) - C^{(n+1)}_{jk}(s, t) \right] , \] (5.7)

\[ \partial_p B^{(n)}_{jk}(s, t) = -\frac{1}{2} \sum_{m=3}^{4} B^{(n)}_{jm} * C^{(0)}_{mk} \bigg|_{s,t} \]
\[ - \frac{1}{2} \sum_{m=3}^{4} \sum_{l=1}^{4} B^{(n+1)}_{jl} * C^{(0)}_{mk} \bigg|_{s,t} \]
\[ - i \left( \delta_{k,3} + \delta_{k,4} \right) \left[ B^{(n-1)}_{jk}(s, t) - B^{(n+1)}_{jk}(s, t) - i \sum_{l=1}^{4} B^{(n)}_{jl} * C^{(0)}_{sk} \bigg|_{s,t} + B^{(n+1)}_{jk} * C^{(1)}_{sk} \bigg|_{s,t} \right] \]
\[ - i \left( \delta_{j,3} + \delta_{j,4} \right) \left[ B^{(n-1)}_{jk}(s, t) - B^{(n+1)}_{jk}(s, t) \right] . \] (5.8)

The derivatives with respect to \( \Lambda \) are given by
\[ \partial_{\Lambda} C^{(0)}_{jk}(s, t) = 2 \Delta F_j(s) F_k(e^{2\Lambda}|t) + 2 e^{-2\Lambda} f_j(e^{-2\Lambda}|s) F_k(e^{-2\Lambda}|t) , \]
\[ \partial_{\Lambda} C^{(1)}_{jk}(s, t) = 2 F_j(e^{2\Lambda}|s) \left( F_k(e^{2\Lambda}|t) + i F_t * C^{(1)}_{tk} \bigg|_{e^{2\Lambda},t} \right) + \Lambda \rightarrow -\Lambda \]
\[ \partial_{\Lambda} C^{(2)}_{jk}(s, t) = 2 e^{-2\Lambda} f_j(e^{2\Lambda}|s) \left( F_k(e^{2\Lambda}|t) + i F_t * C^{(1)}_{tk} \bigg|_{e^{2\Lambda},t} \right) + i e^{2\Lambda} F_t * C^{(1)}_{tk} \bigg|_{e^{2\Lambda},t} \right) + \Lambda \rightarrow -\Lambda \]
\[ \partial_{\Lambda} B^{(1)}_{jk}(s, t) = 2 F_k(e^{2\Lambda}|t) \left( f_j(e^{2\Lambda}|s) - i B^{(1)}_{jt} \bigg|_{e^{2\Lambda},s} \right) + \Lambda \rightarrow -\Lambda . \] (5.9)
Eqs. (5.1), (5.3) and (5.6) form a Lax pair. Their consistency is implied by the following relation for the cross-derivatives

\[ \partial_p \partial_A f_j(z|s) = \partial_A \partial_p f_j(z|s) . \]  

In order to simplify the computations we first introduce some notation. We rewrite (5.3) as

\[ \partial_p f_j(z|s) = -\frac{i}{2} z f_j(z|s) (\delta_{j,3} + \delta_{j,4}) + \sum_{l=1}^{4} a_{jl} * f_l + \sum_{l=1}^{4} \frac{z}{z} b_{jl} * f_l, \]

where

\[ a_{jl}(s, t) = \frac{1}{2} C^{(0)}_{jl} (\delta_{j,3} + \delta_{j,4} - \delta_{l,3} - \delta_{l,4}) , \]

\[ b_{jl}(s, t) = \frac{i}{2} \delta_{j,l} (s - t) + \frac{1}{2} P^{(1)}_{jl} (\delta_{j,3} + \delta_{j,4}) - \frac{1}{2} \sum_{k=3}^{4} C_{jk}^{(1)} [I - i B_{kl}^{(1)}] . \]

In the same notation (5.6) can be written as

\[ \partial_p F_j(z|s) = \frac{i}{2} z F_j(z|s) (\delta_{j,3} + \delta_{j,4}) - \sum_{l=1}^{4} F_l * a_{lj} + \frac{1}{z} \sum_{l=1}^{4} F_l * b_{lj} . \]

Similarly we introduce the notation

\[ U_{jl}(z|s, t) = A_{jl}(\Lambda|s, t)\frac{z - e^{2\Lambda}}{z - e^{-2\Lambda}} + A_{jl}(-\Lambda|s, t)\frac{z - e^{-2\Lambda}}{z - e^{2\Lambda}} , \]

where \( A_{jk}(\Lambda|s, t) = 2i e^{2\Lambda} f_j(e^{2\Lambda}|s) F_k(e^{2\Lambda}|t). \) In what follows we will denote by \( \tilde{A}_{jk}(\Lambda) \) the integral operator in the \( s \)-variable with kernel \( A(\Lambda|s, t). \) After some calculations we arrive at the following equations

\[ \partial_A \partial_p f_j(z|s) = \left( \sum_{m=1}^{4} \tilde{A}(\Lambda)_{jm} * f_m \right) \left( \frac{z - e^{2\Lambda}}{z - e^{-2\Lambda}} z,s + \Lambda \to -\Lambda \right) \]

\[ + \sum_{m=1}^{4} \partial_A a_{jm} * f_m |_{z,s} - \sum_{l=1}^{4} a_{jl} * \left( \sum_{m=1}^{4} \tilde{A}_{lm}(\Lambda) * f_m \right) \frac{z - e^{2\Lambda}}{z - e^{-2\Lambda}} + \Lambda \to -\Lambda |_{z,s} \]

\[ + \frac{1}{z} \sum_{m=1}^{4} \partial_A b_{jm} * f_m |_{z,s} - \frac{1}{z} \sum_{l=1}^{4} b_{jl} * \left( \sum_{m=1}^{4} \tilde{A}_{lm}(\Lambda) * f_m \right) \frac{z - e^{2\Lambda}}{z - e^{-2\Lambda}} + \Lambda \to -\Lambda |_{z,s} . \] (5.15)

\[ \partial_p \partial_A f_j(z|s) = -\sum_{m=1}^{4} \left( \frac{\partial_p \tilde{A}(\Lambda)_{jm} * f_m}{z - e^{2\Lambda}} \right) |_{z,s} + \Lambda \to -\Lambda \]

\[ + \frac{iz}{2} \sum_{m=1}^{4} \left( \frac{\tilde{A}(\Lambda)_{jm} * f_m}{z - e^{2\Lambda}} \right) |_{z,s} + \Lambda \to -\Lambda \left( \delta_{m,3} + \delta_{m,4} \right) \]

\[ - \sum_{l=1}^{4} \sum_{m=1}^{4} \left( \frac{\tilde{A}(\Lambda)_{jl}}{z - e^{2\Lambda}} \right) |_{z,s} + \Lambda \to -\Lambda \left( a_{lm} * f_m + \frac{1}{z} b_{lm} * f_m \right) |_{z,s} . \] (5.16)
In order to equate (5.15) and (5.16) we first rewrite both equations in the form $O_{jm} \ast f_m$, where $O$ are complicated integral operators, and then “truncate” the $f_m$’s from the resulting expressions, which amounts to supposing that they form an independent set of functions in the space the integral operators act in. In the next step we then compare the resulting expressions (which are both meromorphic functions of $z$) at the singular points $z = \infty, 0, e^{\pm 2\Lambda}$. If they (their residues) are equal at these points the expressions coincide for all values of $z$. For $z \to \infty$ we get the condition

$$\partial_{\Lambda} a_{jm}(s, t) = \frac{i}{2}(\delta_{m,3} + \delta_{m,4} - \delta_{j,3} - \delta_{j,4})(A_{jm}(\Lambda|s, t) + A_{jm}(-\Lambda|s, t)), \quad (5.17)$$

At $z = 0$ we obtain

$$\partial_{\Lambda} b_{jm}(s, t) = \sum_{l=1}^{4} \left( e^{-2\Lambda} \left[ \tilde{A}_{jl}(\Lambda) \ast b_{lm} - b_{jl} \ast \tilde{A}_{lm}(\Lambda) \right] + \Lambda \to -\Lambda \right) \bigg|_{s,t}. \quad (5.18)$$

At $z = e^{2\Lambda}$ we get

$$\partial_{p} A_{jm}(\Lambda|s, t) = \frac{i}{2} e^{2\Lambda}(\delta_{m,3} + \delta_{m,4} - \delta_{j,3} - \delta_{j,4})A_{jm}(\Lambda|s, t)$$

$$+ \sum_{l=1}^{4} \left( [a_{jl} + e^{-2\Lambda} b_{jl} \ast \tilde{A}_{lm}(\Lambda) - \tilde{A}_{jl}(\Lambda) \ast [a_{lm} + e^{-2\Lambda} b_{lm}]] \right) \bigg|_{s,t}, \quad (5.19)$$

whereas the condition from $z = e^{-2\Lambda}$ is obtained by taking $\Lambda \to -\Lambda$ in (3.19). It is straightforward to show that these equations hold by inserting the expressions for $a$, $b$ and $A$ and using the identities for the $p$- and $\Lambda$-derivatives of $C_{jk}^{(n)}$ and $B_{jk}^{(n)}$ written above.

Finally, to relate the functional determinant in (3.24) to the quantities introduced above we turn to the logarithmic derivatives of $\det(1 + \frac{1}{2\pi} \hat{V}^T)$.

The derivative with respect to $p$ is given by

$$\partial_{p} \ln \left( \det(1 + \frac{1}{2\pi} \hat{V}^T) \right) = \text{tr} \left( (1 - \hat{R}) \ast \frac{1}{2\pi} \partial_{p} \hat{V}^T \right). \quad (5.20)$$

Using (1.4) we find that

$$\frac{1}{2\pi} \partial_{p} V^T(z_1, z_2) = \frac{k(z_1) - k(z_2)}{z_1 - z_2} \int_{0}^{\infty} ds \sum_{j=3}^{4} e_j(z_1|s) E_j(z_2|s)$$

$$= \frac{1}{2} \left( 1 + \frac{1}{z_1 z_2} \right) \int_{0}^{\infty} ds \sum_{j=3}^{4} e_j(z_1|s) E_j(z_2|s). \quad (5.21)$$

This implies that

$$(1 - \hat{R}) \ast \left. \frac{1}{2\pi} \partial_{p} \hat{V}^T \right|_{z_1, z_2} = \frac{1}{2} \int_{0}^{\infty} ds \sum_{j=3}^{4} \int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} dz \left[ \delta(z_1 - z) - R(z_1, z) \right] \frac{e_j(z|s) E_j(z|s)}{z_2}$$

$$+ \frac{1}{2} \int_{0}^{\infty} ds \sum_{j=3}^{4} f_j(z_1|s) E_j(z_2|s). \quad (5.22)$$
Using the representation (4.8) of \( R(z_1, z_2) \) we rewrite the r.h.s. as
\[
\text{r.h.s.} = \frac{1}{2} \int_0^\infty ds \sum_{k=3}^4 \int_{-\infty}^{\exp(2\Lambda)} dz \left[ \delta(z_1 - z) - R(z_1, z) \right] \frac{e_k(z|s)E_k(z_2|s)}{z_1z_2} \\
\quad - \frac{i}{2} \int_0^\infty ds \int_0^\infty dt \sum_{k=3}^4 \sum_{l=1}^4 \int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} dz \frac{f_l(z_1|t)F_l(z|t)}{z_1z} \frac{e_k(z|s)E_k(z_2|s)}{z_2}
\]
\[
+ \frac{1}{2} \int_0^\infty ds \sum_{j=3}^4 f_j(z_1|s)E_j(z_2|s) .
\]

Using this with (5.4) in (5.20) we finally arrive at
\[
\partial_p \ln \left( \det(1 + \frac{1}{2\pi i} \hat{V}^T) \right) = \frac{1}{2} \sum_{k=3}^4 \int_0^\infty ds \left[ C_k^{(0)}(s, s) + C_k^{(2)}(s, s) - \sum_{l=1}^4 B_{kl}^{(1)} * C_l^{(2)} \right] .
\]

The logarithmic derivative of the determinant with respect to \( \Lambda \) is
\[
\partial_\Lambda \ln \left( \det(1 + \frac{1}{2\pi i} \hat{V}^T) \right) = 2e^{2\Lambda} R(e^{2\Lambda}, e^{2\Lambda}) + 2e^{-2\Lambda} R(e^{-2\Lambda}, e^{-2\Lambda}) .
\]

After some manipulations similar to the case of the Bose gas (see [1]) this can be rewritten as
\[
R(e^{2\Lambda}, e^{2\Lambda}) = \frac{i e^{-2\Lambda}}{2} \sum_{j=1}^4 \int_0^\infty ds \ F_j(e^{2\Lambda}|s) \frac{d}{d\Lambda} f_j(e^{2\Lambda}|s) \\
- \frac{e^{-4\Lambda}}{2 \sinh(2\Lambda)} \left[ \sum_{j=1}^4 \int_0^\infty ds \ f_j(e^{-2\Lambda}|s)F_j(e^{2\Lambda}|s) \right] \left[ \sum_{l=1}^4 \int_0^\infty dt \ f_l(e^{2\Lambda}|t)F_l(e^{-2\Lambda}|t) \right]
\]
\[
R(e^{-2\Lambda}, e^{-2\Lambda}) = -\frac{i e^{2\Lambda}}{2} \sum_{j=1}^4 \int_0^\infty ds \ F_j(e^{-2\Lambda}|s) \frac{d}{d\Lambda} f_j(e^{-2\Lambda}|s) \\
- \frac{e^{4\Lambda}}{2 \sinh(2\Lambda)} \left[ \sum_{j=1}^4 \int_0^\infty ds \ f_j(e^{2\Lambda}|s)F_j(e^{-2\Lambda}|s) \right] \left[ \sum_{l=1}^4 \int_0^\infty dt \ f_l(e^{-2\Lambda}|t)F_l(e^{2\Lambda}|t) \right]
\]

This embeds the determinant into the system of integrable integro-differential equations derived above.

6 The Riemann-Hilbert Problem

In this section we show that the results of the previous section can be reformulated in terms of an infinite-dimensional Riemann-Hilbert problem (RHP) for an integral operator valued function \( Y(z) \). This connection will enable us to determine the asymptotic behaviour of the correlation function. We introduce the conjugation matrix \( G(z) \) of this Riemann-Hilbert problem as
\[
[G(z|s, t)]_{ij} = \delta_{ij} \delta(s - t) + 2\pi \epsilon_i(z|s)E_j(z|t) .
\]

It’s elements can be expressed in terms of the projectors (4.3), e.g.
\[
[G(z|s, t)]_{11} = \delta(s - t) - \kappa^2 \exp(\varphi_4(z) - \varphi_3(z))|2, z, s\rangle\langle 2, z, t| \\
[G(z|s, t)]_{12} = \kappa \exp(\varphi_4(z))|2, z, s\rangle\langle 1, z, t|
\]
\[
\ldots
\]
Consider now an integral-operator valued function $Y(z)$ with kernel $Y_{jk}(z|s,t)$, $j, k = 1, \ldots, 4$, $s, t \in [0, \infty)$ acting on a vector $f$ of functions of $z$ and $s$ according to

$$[Y(z) * f(z)]_j = \int_0^\infty dt \sum_{k=1}^4 Y_{jk}(z|s,t)f_k(z|t) \ .$$  \hspace{1cm} (6.2)

$Y(z)$ is solution to the following RHP

- $Y(z) = I + \sum_{k=1}^\infty \frac{M_k}{z^k}$ for $z \to \infty$.

- $Y(z)$ is analytic throughout the complex plane with the exception of the contour $C$, which is the interval $[\exp(-2\Lambda), \exp(2\Lambda)]$ on the real axis (see Fig. 1).

- $Y^-(z) = Y^+(z)G(z)$ on $C$ where $Y^\pm(z)$ are the boundary values as indicated in Fig. 1 and $G(z)$ is the conjugation matrix (6.1).

This RHP can be rewritten as the system of singular integral equations

$$Y^+(z) = I + \frac{1}{2\pi i} \int_{-\infty}^{\infty} dz' \frac{Y^+(z')[I - G(z')]}{z' - z - i0} \ .$$  \hspace{1cm} (6.3)

The solution of (6.3) can be expressed in terms of the functions $E$ and $f$ defined in Section 4 as

$$Y_{ij}^+(z|s,t) = \delta_{ij}\delta(s-t) + i \int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} dz' \frac{f_i(z'|s)E_j(z'|t)}{z' - z - i0} \ .$$  \hspace{1cm} (6.4)

which follows from the identity

$$f_j(z|s) = \int_0^\infty dt \ Y_{jk}(z|s,t) e_j(z|t) \ .$$  \hspace{1cm} (6.5)

The potentials $B^{(1)}$ and $C^{(n)}$ (5.4) can be related to the solution $Y(z)$ of the RHP through asymptotic expansions around 0 and $\infty$. We find

$$Y_{jk}(z) \to I + iC^{(1)} + izC^{(2)} + iz^2C^{(3)} + O(z^3) \quad \text{for } z \to 0 \ ,$$  \hspace{1cm} (6.6)

$$Y_{jk}(z) \to I - \frac{i}{z}C^{(0)} + O(z^{-2}) \quad \text{for } z \to \infty.$$  \hspace{1cm} (6.7)

From (5.6) and (5.5) we find

$$[I - iB^{(1)}] * C^{(2)} = -i(Y^{-1}(0) \frac{d}{dz}\big|_{z=0} Y(z)) \ .$$  \hspace{1cm} (6.8)

Together with (5.24) this expresses the correlation function (3.23) in terms of the solution $Y(z)$ of our RHP.
6.1 Analysis of the RHP

While the operator-valued RHP defined above determines the correlation functions completely, its solution appears to be a daunting task in general. In what follows we concentrate on the leading term in the asymptotical decomposition of the solution of the RHP in the region of its solution appears to be a daunting task in general. In what follows we concentrate on the leading term in the asymptotical decomposition of the solution of the RHP with respect to $\gamma$. The reason for this restriction is the following: the upper bound on $\gamma$ stems from the construction of the ground state of our lattice regularization. The lower bound ensures that the parameter $p$ defined in (4.2) will go to infinity in the continuum limit, which essentially simplifies the analysis of the RHP: it permits us to study the asymptotical decomposition of the solution of the RHP with respect to $p$ (recall that $p$ contains the continuum distance as well). Due to the fact that this parameter will be not only large but diverge the number of terms in the asymptotical decomposition will be very small – in fact we expect only three contributions (see also below). As we shall show in our analysis of the leading contribution, the special form of the conjugation matrix in addition to our interest in partial traces of $Y$ only allows to reduce the RHP to a tractable scalar one (still containing the auxiliary dual fields, of course). The analysis of the subleading terms is technically much more involved and is currently under investigation. We will report on this work elsewhere.

Let us now turn to the calculation of the leading term. First, we note that the conjugation matrix can be decomposed into the product of an upper and lower triangular matrix as follows

$$[G(z|s,t)]_{ab} = \sum_{c=1}^{4} \int_{0}^{\infty} ds' [T_{1}(z|s,s')]_{ac} [T_{2}(z|s',t)]_{cb} .$$  \hspace{1cm} (6.9)

Here

$$T_{1}(z|s,t) = \begin{pmatrix} 1 & \alpha_{1}(z|s,t) & \alpha_{2}(z|s,t) \exp(ipk(z)) & \alpha_{3}(z|s,t) \exp(ipk(z)) \\ 0 & 1 & \alpha_{4}(z|s,t) \exp(ipk(z)) & \alpha_{5}(z|s,t) \exp(ipk(z)) \\ 0 & 0 & 1 & \alpha_{6}(z|s,t) \\ 0 & 0 & 0 & 1 \end{pmatrix},$$  \hspace{1cm} (6.10)

with matrix elements

$$\begin{align*}
\alpha_{1}(z|s,t) &= \frac{\kappa \exp(\varphi_{4}(z))}{1 + \kappa^{2} \exp(\varphi_{4}(z) - \varphi_{3}(z))}\langle 2, z, s | 1, z, t \rangle , \\
\alpha_{2}(z|s,t) &= -\frac{1}{\kappa} \exp(-\varphi_{2}(z) + \varphi_{3}(z))\langle 2, z, s | 2, z, t \rangle , \\
\alpha_{3}(z|s,t) &= \frac{\kappa^{2} \exp(\varphi_{1}(z) - \varphi_{3}(z) + \varphi_{4}(z))}{1 + \kappa^{2} \exp(\varphi_{4}(z) - \varphi_{3}(z))}\langle 2, z, s | 1, z, t \rangle , \\
\alpha_{4}(z|s,t) &= -\frac{1}{\kappa} \exp(-\varphi_{2}(z) + \varphi_{3}(z) - \varphi_{4}(z))\langle 1, z, s | 2, z, t \rangle , \\
\alpha_{5}(z|s,t) &= \frac{\kappa \exp(\varphi_{1}(z) - \varphi_{3}(z))}{1 + \kappa^{2} \exp(\varphi_{4}(z) - \varphi_{3}(z))}\langle 1, z, s | 1, z, t \rangle , \\
\alpha_{6}(z|s,t) &= \frac{\kappa \exp(\varphi_{2}(z) + \varphi_{1}(z) - \varphi_{3}(z))}{1 + \kappa^{2} \exp(\varphi_{4}(z) - \varphi_{3}(z))}\langle 2, z, s | 1, z, t \rangle .
\end{align*}$$  \hspace{1cm} (6.11)
Similarly, we find

\[
T_2(z|s,t) = \begin{pmatrix}
c_1(z|s,t) & 0 & 0 & 0 \\
\beta_1(z|s,t) & c_2(z|s,t) & 0 & 0 \\
\beta_2(z|s,t) \exp(-ipk(z)) & \beta_4(z|s,t) \exp(-ipk(z)) & c_3(z|s,t) & 0 \\
\beta_3(z|s,t) \exp(-ipk(z)) & \beta_5(z|s,t) \exp(-ipk(z)) & \beta_6(z|s,t) & c_4(z|s,t)
\end{pmatrix}.
\] (6.12)

The matrix elements of \(T_2\) are given by

\[
\begin{align*}
\beta_1(z|s,t) &= -\frac{1}{\kappa} \exp(-\varphi_4(z))|1, z, s\rangle\langle 2, z, t|, \\
\beta_2(z|s,t) &= -\frac{\kappa \exp(\varphi_2(z) - \varphi_3(z))}{1 + \kappa^2 \exp(\varphi_4(z) - \varphi_3(z))}|2, z, s\rangle\langle 2, z, t|, \\
\beta_3(z|s,t) &= -\kappa^2 \exp(-\varphi_1(z) - \varphi_3(z) + \varphi_4(z))|1, z, s\rangle\langle 2, z, t|, \\
\beta_4(z|s,t) &= \frac{\exp(\varphi_2(z))}{1 + \kappa^2 \exp(\varphi_4(z) - \varphi_3(z))}|2, z, s\rangle\langle 1, z, t|, \\
\beta_5(z|s,t) &= \kappa \exp(-\varphi_1(z) + \varphi_4(z))|1, z, s\rangle\langle 1, z, t|, \\
\beta_6(z|s,t) &= -\kappa \exp(-\varphi_1(z) - \varphi_2(z) + \varphi_4(z))|1, z, s\rangle\langle 2, z, t|,
\end{align*}
\] (6.13)

and finally

\[
\begin{align*}
c_1(z|s,t) &= 1 - \frac{\kappa^2 \exp(\varphi_4(z) - \varphi_3(z))}{1 + \kappa^2 \exp(\varphi_4(z) - \varphi_3(z))}|2\rangle\langle 2|, \\
c_2(z|s,t) &= 1 + \frac{1}{\kappa^2} \exp(-\varphi_4(z) + \varphi_3(z))|1\rangle\langle 1|, \\
c_3(z|s,t) &= 1 - \frac{1}{1 + \kappa^2 \exp(\varphi_4(z) - \varphi_3(z))}|2\rangle\langle 2|, \\
c_4(z|s,t) &= 1 + \kappa^2 \exp(\varphi_4(z) - \varphi_3(z))|1\rangle\langle 1|.
\end{align*}
\] (6.14)

Let us now go through a “deformation” of the RHP like for the case of the Bose gas \[28\]. We define an integral-operator valued function \(\tilde{Y}(z)\) in the following way:

- \(\tilde{Y}(z) = Y(z)\) outside the “bubble” defined in Fig 2. In particular \(\tilde{Y}(z) = Y(z)\) for \(z \to 0, \infty\), which will be important later.

- \(\tilde{Y}(z) = Y(z)T_1(z)\) in the region enclosed by the real axis and the contour \(\Gamma_1\). Note that in this region \(\text{Im}k(z) \geq 0 \forall z\).

- \(\tilde{Y}(z) = Y(z)[T_2(z)]^{-1}\) in the region enclosed by the real axis and the contour \(\Gamma_2\). Note that in this region \(\text{Im}k(z) \leq 0 \forall z\).

It can be easily seen that the function \(Y(z)\) defined in the above way has the following properties: \(Y(z)\) is analytic in the whole complex plane with the exception of the contours \(\Gamma_1\) and \(\Gamma_2\). On the contours \(\Gamma_j\) \(\tilde{Y}\) satisfies the conjugation equations

\[
\begin{align*}
(\tilde{Y})^{-}(z) &= \tilde{Y}^{+}(z)T_1(z), \quad z \in \Gamma_1, \\
(\tilde{Y})^{-}(z) &= \tilde{Y}^{+}(z)T_2(z), \quad z \in \Gamma_2.
\end{align*}
\] (6.15)
Since we are only interested in the asymptotic behaviour of the determinant for $p \gg 1$ we can use the fact that in this limit $T_{1(2)}$ become blockdiagonal in the vicinity of the contour $\Gamma_{1(2)}$ from which we find that
\[ Y(z) \sim \begin{pmatrix} \Phi_1(z) & 0 \\ 0 & \Phi_2(z) \end{pmatrix}. \] (6.16)

Here $\Phi_j(z)$ are solutions to $2 \times 2$ operator-valued RHPs
\[ \Phi_j^-(z) = \Phi_j^+(z) * G_j(z), \quad j = 1, 2 \] (6.17)
with the same conjugation contour $C$ as the original RHP and conjugation matrices
\[
G_1(z) = \begin{pmatrix} 1 - |2\rangle\langle 2| & 0 \\ 0 & 1 - |1\rangle\langle 1| \end{pmatrix} + \begin{pmatrix} 0 & \frac{\exp(\varphi_1(z))}{\kappa} |1\rangle\langle 2| \\ \frac{\exp(\varphi_2(z))}{\kappa^2} |2\rangle\langle 1| & 1 + \frac{\exp(\varphi_3(z))}{\kappa^2} |1\rangle\langle 1| \end{pmatrix},
\]
(6.18)
\[
G_2(z) = \begin{pmatrix} 1 - |2\rangle\langle 2| & 0 \\ 0 & 1 - |1\rangle\langle 1| \end{pmatrix} + \begin{pmatrix} 0 & \kappa \exp(\varphi_1(z) + \varphi_2(z) - \varphi_3(z)) |2\rangle\langle 1| \\ -\kappa \exp(-\varphi_1(z) - \varphi_2(z) + \varphi_4(z)) |1\rangle\langle 2| & 1 + \kappa^2 \exp(\varphi_4(z) - \varphi_3(z)) |1\rangle\langle 1| \end{pmatrix}.
\]

Using the fact that $G_j(z)$ form representations of $GL(2|C)$ we can now calculate the determinants of $G_j(z)$ as is shown in the appendix
\[ \det(G_1(z)) = \exp(-\alpha + \varphi_3(z) - \varphi_4(z)), \quad \det(G_2(z)) = \exp(\alpha - \varphi_3(z) + \varphi_4(z)). \] (6.19)

The scalar RHPs for the determinants
\[ \det(\Phi_j^-) = \det(\Phi_j^+) \det(G_j), \quad j = 1, 2 \]
is now easily integrated to give
\[
\begin{align*}
\det(\Phi_1(z)) &= \exp\left(-\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz_1 \frac{-\alpha + \varphi_3(z_1) - \varphi_4(z_1)}{z_1 - z} \right), \\
\det(\Phi_2(z)) &= \exp\left(-\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz_1 \frac{\alpha - \varphi_3(z_1) + \varphi_4(z_1)}{z_1 - z} \right).
\end{align*}
\] (6.20)

7 Leading Term in the Asymptotics of the Correlator

Let us now relate the solution of the scalar RHPs to the logarithmic derivative of $\det(1 + \frac{1}{2\pi i} \hat{V}T)$. The contribution due to $C^{(0)}$ in (5.24) can be obtained from (5.4) and (5.20) as
\[
\frac{1}{2} \sum_{k=3}^{4} \int_0^{\infty} ds C_{kk}^{(0)}(s,s) = \lim_{z \to \infty} \frac{i}{2} \ln \left( \det(\Phi_2(z)) \right) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dz [\alpha - \varphi_3(z) + \varphi_4(z)].
\] (7.1)
Similarly, the second contribution in (5.24) is with (6.7)

\[ \frac{1}{2} \sum_{k=3}^{4} \int_0^\infty ds \left( [I - iB^{(1)}] * C^{(2)} \right)_{kk} (s, s) = -\frac{i}{2} \left. \frac{d}{dz} \ln[\det(\Phi_2(z))] \right|_{z=0} \]

\[ = \frac{1}{4\pi} \int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} dz \frac{\alpha - \varphi_3(z) + \varphi_4(z)}{z^2} . \tag{7.2} \]

Combining these to the leading asymptotical behaviour of \( \partial_p \ln(\det(1 + \frac{1}{2\pi} \hat{V}^T)) \) and using the fact that they are \( p \)-independent we obtain

\[ \det(1 + \frac{1}{2\pi} \hat{V}^T) = A \exp\left(\frac{\alpha p \sinh(2\Lambda)}{\pi}\right) \exp\left(\frac{p}{4\pi} \int_{\exp(-2\Lambda)}^{\exp(2\Lambda)} dz \left(1 + \frac{1}{z^2}\right) (\varphi_4(z) - \varphi_3(z))\right) \]

\[ = A \exp\left(\frac{\alpha p \sinh(2\Lambda)}{\pi}\right) \exp\left(\frac{p}{\pi} \int_{-\Lambda}^{\Lambda} d\lambda \cosh(2\lambda)(\varphi_4(\lambda) - \varphi_3(\lambda))\right) , \tag{7.3} \]

where \( A \) is a \( p \)-independent constant and where in the last step we have changed back to the original \( \lambda \)-variables. Decomposing the combination of dual fields into "momenta" and "coordinates" and using the commutation relations (3.19) we find

\[ \varphi_4(\lambda) - \varphi_3(\lambda) = P(\lambda) + Q(\lambda) , \quad [Q(\mu), P(\lambda)] = 0 . \tag{7.4} \]

This enables us to trivially evaluate the expectation value with respect to the dual fields in this approximation: the dual fields are found not to contribute at all leading to the following result for the leading asymptotical behaviour of the correlator

\[ \langle \Omega | \exp(\alpha Q_1(n)) | \Omega \rangle \sim \tilde{A} \exp\left(\frac{\alpha p \sinh(2\Lambda)}{\pi}\right) , \tag{7.5} \]

where \( \tilde{A} \) is a constant independent on \( p \).

We will now argue that the approximation (7.5) is too crude due to the fact that we have neglected the influence of the dual fields in the subleading factors in the solution of the RHP. We expect the final answer for the solution of the RHP to be of the form

\[ \det(1 + \frac{1}{2\pi} \hat{V}^T) = C(\{\varphi_j\}) \exp(\zeta(\{\varphi_j\}) \ln(p)) \]

\[ \times \exp\left(\frac{\alpha p \sinh(2\Lambda)}{\pi}\right) \exp\left(\frac{p}{\pi} \int_{-\Lambda}^{\Lambda} d\lambda \cosh(2\lambda)(\varphi_4(\lambda) - \varphi_3(\lambda))\right) , \tag{7.6} \]

where we keep in mind that \( p \to \infty \) as the lattice spacing \( \Delta \to 0 \). In (7.6) \( C \) is \( p \)-independent and we have conjectured that the subleading term in the solution of the RHP is a power-law in \( p \). Evaluating the expectation value of (7.6) in the dual bosonic Fock space the dual fields will contribute in the exponential term, i.e

\[ \langle \exp(\alpha Q_1(n)) \rangle \sim \langle 0 | \det(1 + \frac{1}{2\pi} \hat{V}^T) | 0 \rangle \]

\[ = \tilde{C} \tilde{p}^\tilde{\zeta} \exp(\tilde{m}p) \exp(\xi p \ln(p)) \exp\left(\frac{\sinh(2\Lambda)}{\pi} + \omega |\alpha p|\right) . \tag{7.7} \]

Here \( \tilde{\zeta}, \omega \) and \( \tilde{m} \) are functions of \( \gamma \), the soliton mass etc. For this answer to be of the correct qualitative form, the following conditions have to be satisfied:
• $\xi = 0$, as the leading asymptotic behaviour should be $\exp(\text{const } p)$.

• The last factor in (7.7) has to be cancelled by a suitable regularization procedure for the result to make sense. In the continuum limit we have (3.2) which implies that

$$\langle \exp(\alpha Q_1(n)) \rangle \rightarrow \langle \exp \left( \frac{2\alpha}{\beta} [u(x) - u(0)] \right) \rangle \times \exp \left( \frac{\alpha x}{\Delta} \frac{\pi - \gamma}{2\gamma} \right). \quad (7.8)$$

We see that this expression contains a divergent factor depending both on $\alpha$ and on the distance $x$. We now adjust our “cutoff” $\Lambda$ in such a way that the divergent factor in (7.7) precisely reproduces the divergent factor in (7.8), i.e.

$$\exp \left( \frac{\alpha x}{\Delta} \frac{\pi - \gamma}{2\gamma} \right) = \exp \left( \frac{\sinh(2\Lambda)}{\pi} + \omega \alpha p \right).$$

If $\omega = 0$ this leads to the following relation between the “cutoff” $\Lambda$ and the lattice spacing $\Delta$

$$\exp(2\Lambda) = \Delta^{-\frac{2\pi}{\pi} \frac{\pi - \gamma}{\pi} \left( \frac{8\pi(\pi - \gamma)}{c^2 \sin(\gamma) \gamma} \right)} \quad (7.9)$$

with a finite constant $c$. The procedure outlined above fixes the relation between $\Lambda$ and $\Delta$ and thus between the divergent part of $p$ and $\Lambda$. Note, however, that the result (7.9) for this relation is not consistent with the requirement $S \cosh(2\Lambda) \ll 1$, which we have used in order to simplify the kernel of $\hat{V}$ in section 4. Therefore the assumption $\omega = 0$ has to be wrong and we do need a $\Lambda$-dependence of $\omega$ instead which corrects (7.9).

8 Summary and Conclusion

In this paper we have applied the method of [9] to correlation functions in the sine–Gordon model. In order to deal with the UV divergences we used an integrable lattice regularization of the sine–Gordon model to derive a determinant representation for quantum correlation functions. We then took the continuum limit and obtained a determinant representation for the sine–Gordon QFT. Furthermore we embedded the determinant in a system of integrable integro-differential equations which we showed to be associated with an operator-valued Riemann-Hilbert problem. The quantum correlation function was expressed in terms of the solution of this RHP. We then presented a general approach to obtain the leading asymptotical behaviour of the solution of the RHP, which in turn yields the leading term in the asymptotics of the quantum correlation function. We showed that the subleading terms in the asymptotical decomposition are essential for obtaining explicit expressions for the asymptotics of the correlation function due to the presence of the dual quantum fields. For the case at hand there appear to be only two subleading terms in the asymptotical decomposition which is very encouraging! The analysis of the subleading terms is a difficult mathematical problem by itself and we will report on it in a separate publication.

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A \ Gl(2|C) \ Representation by Integral Operators

In this appendix we show how to essentially simplify the analysis of the operator-valued RHP through the use of \( GL(2|C) \) representation theory. We closely follow the discussion of \[28\].

Let us consider an integral-operator valued \( 2 \times 2 \) matrix with kernel

\[
\mathcal{O}(s, t) = \begin{pmatrix} \mathcal{O}_{11}(s, t) & \mathcal{O}_{12}(s, t) \\ \mathcal{O}_{21}(s, t) & \mathcal{O}_{22}(s, t) \end{pmatrix}, \quad s, t \in [0, \infty).
\]

(A.1)

Multiplication of integral-operator valued matrices \( \mathcal{O} \) and \( \mathcal{P} \) is defined in the usual way as

\[
[\mathcal{O}\mathcal{P}]_{ij}(s, t) = \sum_{k=1}^{2} \int_{0}^{\infty} dr \ \mathcal{O}_{ik}(s, r) \mathcal{P}_{kj}(r, t), \quad i, j = 1, 2.
\]

(A.2)

The left (right) action of the integral operators \( \mathcal{O}_{ij} \) on functions defined on the interval \([0, \infty)\) is given by

\[
\mathcal{O}_{ij} * f \bigg|_{a} = \int_{a}^{\infty} dt \ \mathcal{O}_{ij}(s, t) f(t), \quad g * \mathcal{O}_{ij} \bigg|_{b} = \int_{0}^{\infty} ds \ f(s) \ \mathcal{O}_{ij}(s, t).
\]

(A.3)

Let us now construct a special class of such operators \( \hat{\mathcal{O}} \) which form a representation of \( GL(2, C) \): we start with two pairs of functions \((\alpha(s), \beta(s))\) and \((A(s), B(s))\) on \([0, \infty)\) which we represent in Dirac notation as \( \alpha(s) \equiv |1\rangle \), \( \beta(s) \equiv |2\rangle \), \( A(s) \equiv \langle 1| \) and \( B(s) \equiv \langle 2| \). These functions are chosen in such a way that

\[
\langle 1|1\rangle \equiv \int_{0}^{\infty} ds A(s) \alpha(s) = 1 \equiv \langle 2|2\rangle \equiv \int_{0}^{\infty} ds B(s) \beta(s).
\]

(A.4)

In this notation we may write left multiplication by \( \hat{\mathcal{O}}_{ik} \) as

\[
\hat{\mathcal{O}}_{ik}|1\rangle = \int_{0}^{\infty} dt \mathcal{O}_{ik}(s, t) \alpha(t).
\]

(A.5)

Observe now that one may define a representation \( \hat{\mathcal{A}} \) of \( GL(2, C) \) in terms of integral operators via

\[
M \in Gl(2, C) \mapsto \hat{\mathcal{A}}(M) = \begin{pmatrix} I - |1\rangle \langle 1| & 0 \\ 0 & I - |2\rangle \langle 2| \end{pmatrix} + \begin{pmatrix} M_{11}|1\rangle \langle 1| & M_{12}|1\rangle \langle 2| \\ M_{21}|2\rangle \langle 1| & M_{22}|2\rangle \langle 2| \end{pmatrix}.
\]

(A.6)

Here \( M_{11}, M_{12}, M_{21} \) and \( M_{22} \) are complex numbers and \( I \) is the identity operator in the space of integral operators on \([0, \infty)\). Multiplication by the integral operators \(|1\rangle \langle 1|, |1\rangle \langle 2|, |2\rangle \langle 1| \) and \(|2\rangle \langle 2| \) is given by e.g

\[
|1\rangle \langle 2| f(s) = \left( \int_{0}^{\infty} ds B(s) f(s) \right) |1\rangle.
\]

(A.7)
Therefore $|i\rangle\langle j|$ act like projectors on the “states” $|i\rangle$ and $\langle j|$. In particular identities like $|I - 1\rangle\langle 1| |1\rangle\langle 1| = 0$ are seen to hold. Indeed for any $M, N \in \text{Gl}(2, \mathbb{C})$ the representation $\hat{A}$ has the following properties

\begin{align*}
\text{(P1)} & \quad \hat{A}(MN) = \hat{A}(M)\hat{A}(N) ; \quad \hat{A}(I) = I ; \quad \hat{A}(M^{-1}) = \hat{A}^{-1}(M) \\
\text{(P2)} & \quad \text{Tr} \left( \hat{A}(M) - \begin{pmatrix} I - |1\rangle\langle 1| & 0 \\ 0 & I - |2\rangle\langle 2| \end{pmatrix} \right) = \text{tr}M = M_{11} + M_{22} , \\
\text{(P3)} & \quad \text{Det}\hat{A}(M) = \det M = M_{11}M_{22} - M_{12}M_{21} .
\end{align*}

Properties (P1) and (P2) can be established by direct computation using the rules given above. Property (P3) shows that the determinant of the integral operator $\hat{A}$ is simply equal to the determinant of the $2 \times 2$ matrix $M$, which is quite remarkable. It is established by expressing the determinant as a trace via $\ln \text{Det}\hat{A} = \text{Tr} \ln \hat{A}$, then using (P1) in the expansion of the logarithm, using (P2) to express the operator trace in terms of the matrix trace, and finally expressing the sum over traces back as determinant of the matrix $M$.

It can be easily checked that the representation (6.18) of the conjugation matrices $G_j(z)$ is precisely of the above form (here $z$ plays the role of a parameter), which in turn allows us to evaluate the determinants of the conjugation matrices.
Figures

Figure 1: Conjugation Contour for the RHP

Figure 2: Deformation of the Conjugation Contour for the RHP
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