SOME \( \Pi_q \)-IDENTITIES OF GOSPER

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Abstract. In 2001 W. Gosper introduced the \( q \)-trigonometric functions and the constant \( \Pi_q \) and conjectured many intriguing identities on these \( q \)-trigonometric functions and \( \Pi_q \). In this paper we employ some knowledge of modular equations to confirm two groups of Gosper’s \( \Pi_q \)-identities. Two strange \( q \)-identities involving \( \Pi_q \) and a Lambert series conjectured by W. Gosper are also confirmed. As a consequence, a \( q \)-identity involving \( \Pi_q \) and two Lambert series, which was also conjectured by Gosper, is proved. As an application, we confirm an interesting \( q \)-trigonometric identity of Gosper, which is a theta function analogue for a well-known trigonometric identity.

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1. Introduction

Throughout this paper we assume that \( |q| < 1 \). W. Gosper \cite{Gosper} p. 85] first introduced the \( q \)-constant \( \Pi_q \):

\[ \Pi_q = q^{1/4} \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}, \]

where \((a; q)_{\infty}\) is defined by

\[ (a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n), \]

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and then stated without proofs many identities involving $\Pi_q$ \[5, pp. 102–104\] by employing empirical evidence based on a computer program called MACSYMA. In particular, he \[5, pp. 103–104\] conjectured the following interesting $\Pi_q$-identities:

\[(1.2) \quad \frac{\Pi_q^2}{\Pi_q^2 \Pi_q^4} - \frac{\Pi_q^2}{\Pi_q^2} = 4,\]

\[(1.3) \quad \Pi_q^2 + 3\Pi_q\Pi_q^6 = \sqrt{\Pi_q\Pi_q^6(\Pi_q + 3\Pi_q^6)},\]

\[(1.4) \quad \frac{\Pi_q^2 \Pi_q^2}{\Pi_q^2 \Pi_q^6} = \frac{\Pi_q^2 - \Pi_q^6}{\Pi_q^2 + 3\Pi_q^6},\]

\[(1.5) \quad \Pi_q^2 \Pi_q^6 = \Pi_q^6(\Pi_q^2 - \Pi_q^6)^3(\Pi_q^2 + 3\Pi_q^6),\]

\[(1.6) \quad \Pi_q^6 \Pi_q^4 = \Pi_q^2(\Pi_q^2 - \Pi_q^6)(\Pi_q^2 + 3\Pi_q^6)^3.\]

The formula (1.2) was deduced by Gosper \[5, p. 93\]. The $\Pi_q$-identity (1.3) was confirmed by the author and H.-C. Zhai \[6\] by establishing an identity involving the $q$-trigonometric functions and the constant $\Pi_q$, which is equivalent to a theta function identity that can be proved by using an addition formula for Jacobi's theta functions of Liu \[8, Theorem 1\] (See Section 6 for the definitions of the $q$-trigonometric functions and see \[7\] for applications of Liu's addition formula and the definitions of Jacobi’s theta functions). M. El Bachraoui \[3, Theorem 2.2\] just gave a partial proof of the identity (1.4). Namely, he proved that

\[
\left(\frac{\Pi_q^2 \Pi_q^2}{\Pi_q^2 \Pi_q^6}\right)^2 = \left(\frac{\Pi_q^2 - \Pi_q^6}{\Pi_q^2 + 3\Pi_q^6}\right)^2.
\]

In \[3, Theorem 2.3\] El Bachraoui only showed that (1.5) is equivalent to (1.6). See \[3\] for many other $\Pi_q$-identities not mentioned by Gosper \[5\].

In this paper we will consider the $\Pi_q$-identities (1.4), (1.5), (1.6) and many other $\Pi_q$-identities of Gosper and adopt the notations of \[1, Chapters 5 and 6\]. The definition of modular equations \[1, (6.3.2)\] is very important.

Let $0 < k, l < 1$ and let $n$ be a positive integer. A relation between $k$ and $l$ induced by the formula

\[
n \frac{\sum_{k=1}^{n} \frac{2F_1(1/2, 1/2; 1; 1 - k^2)}{2F_1(1/2, 1/2; 1; k^2)}} = \frac{\sum_{k=1}^{n} \frac{2F_1(1/2, 1/2; 1; 1 - l^2)}{2F_1(1/2, 1/2; 1; l^2)}}
\]

is called a modular equation of degree $n$. Take $\alpha = k^2$, $\beta = l^2$, we say that $\beta$ has degree $n$ over $\alpha$. The multiplier $m$ is given by

\[
m = \frac{z_1}{z_n},
\]

where

\[z_n = \varphi^2(q^n)\]

and

\[
\varphi(q) = \sum_{k=-\infty}^{\infty} q^{k^2}.
\]

In the next two sections we will consider many $\Pi_q$-identities conjectured by W. Gosper. These identities can be divided into two groups, one group of identities involving the $q$-constants $\Pi_q, \Pi_q^2, \Pi_q^3$ or $\Pi_q^6$ and the other group of identities concerning $\Pi_q, \Pi_q^2, \Pi_q^6$ or $\Pi_q^{10}$. They are confirmed by establishing several modular
equations of degrees 3 or 5. In Section 2 we confirm two strange \( q \)-identities involving \( \Pi_q \) and a Lambert series conjectured by W. Gosper:

\[
\sum_{n \geq 1} \frac{q^n}{(1-q^n)^2} - 5 \sum_{n \geq 1} \frac{q^{3n}}{(1-q^{3n})^2} = \frac{\Pi_q^3}{\Pi_q^5} - 2 \frac{\Pi_q^2}{\Pi_q^5} + 5 \frac{\Pi_q}{\Pi_q^5} = \frac{\Pi_q^2}{\Pi_q^{10}} + 16 \frac{\Pi_q^2}{\Pi_q^{10}} - 4 - \frac{\Pi_q}{\Pi_q^{10}}.
\]

As a result, we in Section 5 confirm a \( q \)-identity involving \( \Pi_q \) and two Lambert series, which was also conjectured by Gosper:

\[
6 \left( \sum_{n \geq 1} \frac{q^n}{(1-q^n)^2} - 5 \sum_{n \geq 1} \frac{q^{3n}}{(1-q^{3n})^2} \right) + 1
= \left( \frac{\Pi_q}{\Pi_q^5} + 2 + 5 \frac{\Pi_q^3}{\Pi_q^5} \right) \left( \sum_{n \geq 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1-q^{10n-5})^2} \right).
\]

As an application, we employ Theorem 4.1 to confirm an interesting \( q \)-trigonometric identity of Gosper:

\[
\sin_q 5z = \frac{\Pi_q}{\Pi_q^5} (\cos_q z)^4 \sin_q z - \sqrt{\frac{\Pi_q^3}{\Pi_q^5} - 2 \frac{\Pi_q^2}{\Pi_q^5} + 5 \frac{\Pi_q}{\Pi_q^5}} (\cos_q z)^2 (\sin_q z)^3 + (\sin_q z)^5
\]
in the last section.

2. Identities involving \( \Pi_q, \Pi_q^2, \Pi_q^3 \) or \( \Pi_q^6 \)

2.1. Statement of results. In \cite{5} p. 103 W. Gosper conjectured the following interesting identities:

\[
(2.1) \quad \sqrt{\Pi_q^2 \Pi_q^6} (\Pi_q^2 - 3 \Pi_q^4) = \sqrt{\Pi_q \Pi_q^3} (\Pi_q^2 + 3 \Pi_q^4),
\]

\[
(2.2) \quad \Pi_q^2 (\Pi_q^4 + 18 \Pi_q^2 \Pi_q^3 - 27 \Pi_q^4) = \Pi_q \Pi_q^3 (\Pi_q^4 + 16 \Pi_q^4),
\]

\[
(2.3) \quad \Pi_q^2 (\Pi_q^4 - 6 \Pi_q^2 \Pi_q^3 - 3 \Pi_q^4) = \Pi_q \Pi_q^3 (\Pi_q^4 + 16 \Pi_q^4),
\]

\[
(2.4) \quad \Pi_q \Pi_q^3 (\Pi_q^2 \pm 4 \Pi_q^2) = \Pi_q^2 (\Pi_q \mp \Pi_q^3) (\Pi_q \pm 3 \Pi_q^3)^3,
\]

\[
(2.5) \quad \Pi_q \Pi_q^3 (\Pi_q^2 \pm 4 \Pi_q^2) = \Pi_q^2 (\Pi_q \mp \Pi_q^3)^3 (\Pi_q \pm 3 \Pi_q^3).
\]

In this section we will confirm these identities by establishing the following results:

**Theorem 2.1.** The identities \((2.1) - (2.6)\) and \((2.1) - (2.5)\) are true.

Both of the identities \((2.1)\) and \((2.2)\) involve all of the four constants \( \Pi_q, \Pi_q^2, \Pi_q^3 \) and \( \Pi_q^6 \), however, each of the other identities in Theorem 2.1 contains only three of these four constants. We will show these \( \Pi_q \)-identities by establishing several modular equations of degree 3.
2.2. Auxiliary results. Some auxiliary results are required to prove Theorem 2.1.

Theorem 2.2. Let \( \beta \) have degree 3 over \( \alpha \). Then

\[
\begin{align*}
(2.6) & \quad m - \sqrt[4]{\frac{\beta}{\alpha}} = 1 + \frac{3}{m} \sqrt[4]{\frac{\beta}{\alpha}}, \\
(2.7) & \quad m^2 \alpha + 3\beta = 2(\alpha \beta)^{1/8}(m^2 \alpha^{1/2} - 3\beta^{1/2}), \\
(2.8) & \quad \frac{16}{m} \sqrt[3]{\frac{\beta}{\alpha}} = \alpha \left( 1 - \frac{1}{m} \sqrt[4]{\frac{\beta}{\alpha}} \right) \left( 1 + \frac{3}{m} \sqrt[4]{\frac{\beta}{\alpha}} \right)^3, \\
(2.9) & \quad \frac{1}{m} \sqrt[4]{\frac{\beta}{\alpha}} \left( 1 + \alpha \right) = \frac{\alpha^{1/2}}{4} \left( 1 + \frac{18}{m^2} \sqrt[4]{\frac{\beta}{\alpha}} - \frac{27 \beta}{m^2 \alpha} \right), \\
(2.10) & \quad \frac{m^4}{\beta} \left( 1 + \beta \right) = \frac{\beta^{1/2}}{4} \left( \frac{m^4 \alpha}{\beta} - 6m^2 \sqrt[4]{\frac{\alpha}{\beta}} - 3 \right), \\
(2.11) & \quad \frac{1}{m} \sqrt[4]{\frac{\beta}{\alpha}} \left( 1 \pm \alpha^{1/2} \right)^2 = \frac{\alpha^{1/2}}{4} \left( 1 \pm \frac{1}{m} \sqrt[4]{\frac{\beta}{\alpha}} \right) \left( 1 \pm \frac{3}{m} \sqrt[4]{\frac{\beta}{\alpha}} \right)^3, \\
(2.12) & \quad \frac{m}{\beta} \left( 1 \pm \beta^{1/2} \right)^2 = \frac{\beta^{1/2}}{4} \left( \frac{m}{\beta} \sqrt[4]{\beta} \mp 1 \right)^3 \left( \frac{m}{\beta} \sqrt[4]{\beta} \pm 3 \right). 
\end{align*}
\]

Proof. The identity (2.6) follows easily from [1, (6.3.23)].

We now show (2.7). It follows from [1, (6.3.19) and (6.3.20)] that

\[
(\alpha \beta)^{1/8}(m^2 \alpha^{1/2} - 3\beta^{1/2}) = \frac{(m-1)(3+m)}{8m} \left( (3+m)m - 3(m-1) \right) = \frac{(m-1)(3+m)(m^2 + 3)}{8m}
\]

and

\[
m^2 \alpha + 3\beta = \frac{(m-1)(3+m)}{16m} \left( (3+m)^2 + 3(m-1)^2 \right) = \frac{(m-1)(3+m)(m^2 + 3)}{4m}.
\]

From these identities the formula (2.7) follows readily.

The identity (2.8) can be obtained by combining [1, (6.3.19) and (6.3.23)].

We now prove (2.9). It can be deduced from [1, (6.3.19) and (6.3.23)] that

\[
\frac{1}{m} \sqrt[4]{\frac{\beta}{\alpha}} \left( 1 + \alpha \right) = \frac{1}{m} \sqrt{\frac{m(m-1)}{3+m} \left( 1 + \frac{(m-1)(3+m)^3}{16m^3} \right)} = \frac{m^4 + 24m^3 + 18m^2 - 27}{16m^3} \sqrt{\frac{m-1}{m(3+m)}}
\]
and

\[
\frac{\alpha^{1/2}}{4} \left( 1 + \frac{18}{m^2} \sqrt{\frac{\beta}{\alpha}} - \frac{27}{m^4} \right) = 3 + m \left( m - 1 \right) \left( 1 + \frac{18(m - 1)}{m(3 + m)} - \frac{27}{m^2(3 + m)} \right) \\
= \frac{m^4 + 24m^3 + 18m^2 - 27}{16m^3} \sqrt{\frac{m - 1}{m(3 + m)}}.
\]

From these two identities (2.9) follows quickly.

We then deduce (2.10). It follows from [1, (6.3.20) and (6.3.23)] that

\[
m^4 \sqrt{\frac{\beta}{\alpha}} \left( 1 + \beta \right) = \left( \frac{m(3 + m)}{m - 1} \right)^{1/2} \left( 1 + \frac{(m - 1)^3(3 + m)}{16m} \right) \\
= \frac{m^4 - 6m^2 + 24m - 3}{16} \left( \frac{3 + m}{m(m - 1)} \right)^{1/2}
\]

and

\[
\frac{\beta^{1/2}}{4} \left( \frac{m^4 \alpha}{\beta} - 6m^2 \sqrt{\frac{\alpha}{\beta}} - 3 \right) = \frac{m - 1}{16} \left( \frac{(m - 1)(3 + m)}{m} \right)^{1/2} \left( \frac{m^2(3 + m)^2}{(m - 1)^2} - \frac{6m(3 + m)}{m - 1} - 3 \right) \\
= \frac{m^4 - 6m^2 + 24m - 3}{16} \left( \frac{3 + m}{m(m - 1)} \right)^{1/2},
\]

from which (2.10) is obtained.

We now derive (2.11). Using [1 (6.3.19) and (6.3.23)] we get

\[
\frac{1}{m} \sqrt{\frac{\beta}{\alpha}} \left( 1 \pm \alpha^{1/2} \right)^2 \\
= \sqrt{\frac{m - 1}{m(3 + m)}} \left( 1 \pm \frac{3 + m}{4m} \right)^2 \sqrt{\frac{(m - 1)(3 + m)}{m}} \\
= m^4 + 24m^3 + 18m^2 - 27 \sqrt{\frac{m - 1}{m(3 + m)}} \pm \frac{m^2 + 2m - 3}{2m^2}
\]

and

\[
\frac{\alpha^{1/2}}{4} \left( 1 + \frac{1}{m} \sqrt[4]{\frac{\beta}{\alpha}} \right) \left( 1 \pm \frac{3}{m} \sqrt[4]{\frac{\beta}{\alpha}} \right)^3 \\
= \frac{3 + m}{16m} \sqrt{\frac{(m - 1)(3 + m)}{m}} \left( 1 \pm \sqrt{\frac{m - 1}{m(3 + m)}} \right)^3 \left( 1 \pm \frac{m - 1}{m(3 + m)} \right)^3 \\
= \frac{m^4 + 24m^3 + 18m^2 - 27}{16m^3} \sqrt{\frac{m - 1}{m(3 + m)}} \pm \frac{m^2 + 2m - 3}{2m^2}.
\]

From these two identities we obtain (2.11).
Finally, we show (2.12). We deduce from [1, (6.3.20) and (6.3.23)] that

\[
m \sqrt[4]{\frac{\alpha}{\beta}} (1 \pm \beta^{1/2})^2 = \sqrt{\frac{m(3 + m)}{m - 1}} \left( 1 \pm \frac{m - 1}{4} \sqrt{\frac{(m - 1)(3 + m)}{m}} \right)^2
\]

\[
= \frac{m^4 - 6m^2 + 24m - 3}{16m} \sqrt{\frac{m(3 + m)}{m - 1}} \pm \frac{m^2 + 2m - 3}{2}
\]

and

\[
\frac{\beta^{1/2}}{4} \left( m \sqrt[4]{\frac{\alpha}{\beta}} + 1 \right)^3 \left( m \sqrt[4]{\frac{\alpha}{\beta}} - 3 \right)
\]

\[
= \frac{m - 1}{16} \sqrt{\frac{(m - 1)(3 + m)}{m}} \left( \sqrt{\frac{m(3 + m)}{m - 1}} \mp 1 \right)^3 \left( \sqrt{\frac{m(3 + m)}{m - 1}} \mp 3 \right)
\]

\[
= \frac{m^4 - 6m^2 + 24m - 3}{16m} \sqrt{\frac{m(3 + m)}{m - 1}} \pm \frac{m^2 + 2m - 3}{2}.
\]

From these we arrive at (2.12). This completes the proof of Theorem 2.2. □

2.3. Proof of Theorem 2.1. In this subsection, we only consider modular equations of degree 3 and then we always assumed that \( n = 3 \) and

\[
m = \frac{z_1}{z_3}.
\]

We are now ready to show Theorem 2.1.

Proof of Theorem 2.1. According to [3] Theorem 2.3, we know that (1.5) is equivalent to (1.0), so we only need to prove one of them. In this section we shall show the identities (1.4), (1.6), (2.1)–(2.5). If the identities in Theorem 2.1 hold for \( 0 < q < 1 \), then, by analytic continuation, these identities are also true for \( |q| < 1 \). So we can assume that \( 0 < q < 1 \).

Let

\[
\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.
\]

It follows from (1.1) and [11] (1.3.14)] that

\[
(2.14) \quad \Pi_q = q^{1/4} \psi^2(q).
\]
Then the identities (2.14), (2.15), (2.16)–(2.18) are respectively equivalent to

\[
\frac{\psi^2(q^2) \psi^4(q^3)}{\psi^2(q^6) \psi^4(q)} = \psi^2(q^2) - q\psi^2(q^6),
\]

(2.16)

\[
\frac{\psi^2(q^6) \psi^8(q)}{\psi^2(q^2) \psi^8(q^2)} = \left(1 - q\frac{\psi^2(q^6)}{\psi^2(q^2)}\right) \left(1 + 3q\frac{\psi^2(q^6)}{\psi^2(q^2)}\right)^3,
\]

(2.17)

\[
\psi(q^2) \psi(q^4)(\psi^4(q) - 3q\psi^4(q^3)) = \psi(q) \psi(q^3)(\psi^4(q^2) + 3q^2\psi^4(q^6)),
\]

(2.18)

\[
\frac{\psi^2(q^3)}{\psi^2(q)} \left(1 + 16q\frac{\psi^3(q^2)}{\psi^3(q)}\right) = \psi^4(q) \left(1 + 18q\frac{\psi^3(q^3)}{\psi^3(q)} - 27q^2\frac{\psi^3(q^3)}{\psi^3(q)}\right),
\]

(2.19)

\[
\frac{\psi^2(q^4)}{\psi^2(q^2)} \left(1 + 16q^3\frac{\psi^4(q^3)}{\psi^4(q)}\right) = \psi^4(q^6) \left(\psi^8(q) \psi^4(q^3) - 6q\frac{\psi^4(q)}{\psi^4(q^3)} - 3q^2\right),
\]

(2.20)

\[
\frac{\psi^2(q^6)}{\psi^2(q^4)} \left(1 + 4q^{1.5}\frac{\psi^4(q^4)}{\psi^4(q)}\right)^2 = \psi^4(q^2) \left(1 + q^{1.5}\frac{\psi^2(q^3)}{\psi^2(q)}\right) \left(1 + 3q^{1/2}\frac{\psi^2(q^3)}{\psi^2(q)}\right)^3,
\]

(2.21)

\[
\frac{\psi^2(q^3)}{\psi^2(q^5)} \left(1 + 4q^{3/2}\frac{\psi^4(q^6)}{\psi^4(q^3)}\right)^2 = \psi^4(q^3) \left(\psi^2(q) \psi^4(q^3) + q^{1/2}\right) \left(\psi^2(q) \psi^2(q^3) \pm 3q^{1/2}\right).
\]

We first prove (2.14). Let \( \beta \) have degree 3 over \( \alpha \). Then, by [1, Theorem 5.4.2, (i) and (iii)],

(2.22)

\[
\psi(q) = \sqrt[2]{\frac{\alpha}{2}} (\alpha/q)^{1/8},
\]

(2.23)

\[
\psi(q^2) = \frac{1}{2} \sqrt{z_1} (\alpha/q)^{1/4},
\]

(2.24)

\[
\psi(q^3) = \sqrt[2]{\frac{\alpha}{2}} (\beta/q^3)^{1/8},
\]

(2.25)

\[
\psi(q^6) = \frac{1}{2} \sqrt{z_3} (\beta/q^3)^{1/4},
\]

and so

(2.26)

\[
\frac{\psi(q^2)}{\psi(q)} = q^{-1/4} \sqrt[2]{z_2} (\beta/\alpha)^{1/8},
\]

(2.27)

\[
\frac{\psi(q^6)}{\psi(q^2)} = q^{-1/2} \sqrt[2]{z_3} (\beta/\alpha)^{1/4}.
\]

Hence, the formula (2.15) follows by dividing both sides of (2.6) by \( m(1 + \frac{1}{m} \sqrt[2]{\alpha}) \) and then using (2.26), (2.27) and (2.13) in the resulting identity.

We now prove (2.16). It follows from (2.22) and (2.23) that

(2.28)

\[
\frac{\psi(q)}{\psi(q^2)} = \frac{\sqrt[2]{\alpha}}{\alpha/q^{1/8}}.
\]

Then (2.16) can be obtained by dividing both sides of (2.28) by \( \alpha \) and then employing (2.27), (2.28) and (2.13) in the resulting identity.
The identity (2.17) follows easily by multiplying both sides of (2.7) by \( \frac{(\alpha \beta)^{1/8}}{\sqrt{z_1 z_2 / q^3}} \) and then using (2.13) in the resulting equation.

The formula (2.18) can be deduced by dividing both sides of (2.9) by \( \sqrt{q} \) and then using (2.20) and (2.13).

We then show (2.19). It follows from (2.28) that
\[
\psi(q^3) \psi(q^6) = \sqrt{2} \left( \frac{\beta}{q^3} \right)^{1/8}
\]
and so
\[
(2.29)
\psi(q^6) \psi(q^3) = \left( \frac{\beta}{q^3} \right)^{1/8} \sqrt{2}.
\]
We multiply both sides of (2.10) by \( q^{1/2} \) and then apply (2.26), (2.29) and (2.13) in the resulting identity to obtain (2.19).

The identity (2.20) can be derived by dividing both sides of (2.11) by \( q^{1/2} \) and then using (2.26), (2.29) and (2.13) in the resulting formula.

The identity (2.21) follows readily by multiplying both sides of (2.12) by \( q^{1/2} \) and then employing (2.26), (2.29) and (2.13) in the resulting identity. This finishes the proof of Theorem 2.1.

\[ \square \]

3. Identities involving \( \Pi_q, \Pi_{q^2}, \Pi_{q^5} \) or \( \Pi_{q^{10}} \)

3.1. Statement of results. Gosper [5] pp. 103–104 conjectured the following \( \Pi_q \)-identities:

\[
(3.1) \quad \Pi_q^2 \Pi_{q^5}^5(16 \Pi_{q^{10}}^4 - \Pi_{q^5}^5) = \Pi_{q^5}^5(5 \Pi_{q^{10}} - \Pi_q^2)(\Pi_{q^2} - \Pi_{q^{10}})^5,
\]
\[
(3.2) \quad \Pi_{q^{10}}^4 \Pi_q^4(16 \Pi_{q^2}^4 - \Pi_q^5) = \Pi_q^5(5 \Pi_{q^{10}} - \Pi_{q^2})^5(\Pi_{q^2} - \Pi_{q^{10}}),
\]
\[
(3.3) \quad \Pi_q \Pi_{q^5}^4(16 \Pi_{q^2}^4 - \Pi_q^5)^2 = \Pi_{q^2}^2(5 \Pi_{q^5} - \Pi_q)^5(\Pi_{q^5} - \Pi_q),
\]
\[
(3.4) \quad \Pi_q \Pi_{q^5}^4(16 \Pi_{q^{10}}^4 - \Pi_q^5)^2 = \Pi_{q^{10}}^4(5 \Pi_{q^5} - \Pi_q)(\Pi_{q^5} - \Pi_q)^5,
\]
\[
(3.5) \quad (\Pi_q \Pi_{q^{10}} - \Pi_{q^2} \Pi_{q^5})^2 = \Pi_{q^2} \Pi_{q^{10}}(5 \Pi_{q^5} - \Pi_q)(5 \Pi_{q^5} - \Pi_q).
\]

In this section we will confirm these results.

Theorem 3.1. The identities (3.1)–(3.5) are true.

The identities (3.1)–(3.4) only contain three of the constants \( \Pi_q, \Pi_{q^2}, \Pi_{q^5} \) and \( \Pi_{q^{10}} \), but the formula (3.5) includes all of these four constants. These five identities have similar styles so that our proofs share the same pattern. We will show these identities by setting up some modular equations of degree 5.

3.2. One lemma. The value of the multiplier \( m \) depends on \( n \), but throughout this subsection and the next subsection we only consider modular equations of degree 5, then it is always assumed that \( n = 5 \) and
\[
(3.6) \quad m = \frac{z_1}{z_5}.
\]
In order to prove Theorem 3.1 we need several auxiliary results.
Theorem 3.2. If $\beta$ has degree 5 over $\alpha$, then

\begin{equation}
256\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/2}\frac{1}{\beta}\left(1 - \frac{1}{\beta}\right) = \left(5 - \frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/2}\right)\left(\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/2} - 1\right)^5,
\end{equation}

\begin{equation}
256\frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/2}\frac{1}{\alpha}\left(1 - \frac{1}{\alpha}\right) = \left(5\frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/2} - 1\right)\left(1 - \frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/2}\right),
\end{equation}

\begin{equation}
\frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/4}\left(\frac{\alpha}{\beta} - 1\right)^2 = \frac{\alpha}{16}\left(5\frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/4} - 1\right)\left(\frac{z_5}{z_1}\left(\frac{\beta}{\alpha}\right)^{1/4} - 1\right),
\end{equation}

\begin{equation}
\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/4}\left(\frac{\beta}{\alpha} - 1\right)^2 = \frac{\beta}{16}\left(5 - \frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/4}\right)\left(1 - \frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/4}\right)^5,
\end{equation}

\begin{equation}
\left(\frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/4} - \frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/2}\right)^2 = \frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/2}\left(1 - \frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/4}\right)\left(5 - \frac{z_1}{z_5}\left(\frac{\alpha}{\beta}\right)^{1/4}\right).
\end{equation}

Proof. We first prove (3.7) and (3.10). According to [2] Chapter 19, (13.12)–(13.15)] we have

\begin{equation}
\left(\frac{\alpha}{\beta}\right)^{1/4} = \frac{2m + \rho}{m(m - 1)},
\end{equation}

\begin{equation}
\left(\frac{\beta}{\alpha}\right)^{1/4} = \frac{2m - \rho}{5 - m},
\end{equation}

\begin{equation}
\left(\frac{1 - \beta}{1 - \alpha}\right)^{1/4} = \frac{2m + \rho}{5 - m},
\end{equation}

\begin{equation}
(\alpha\beta)^{1/2} = \frac{4m^3 - 16m^2 - 20m + \rho(m^2 - 5)}{16m^2},
\end{equation}

\begin{equation}
\{(1 - \alpha)(1 - \beta)\}^{1/2} = \frac{4m^3 - 16m^2 + 20m - \rho(m^2 - 5)}{16m^2},
\end{equation}

where

\[ \rho = (m^3 - 2m^2 + 5m)^{1/2}. \]

Then

\begin{equation}
\beta = \left(\frac{2m - \rho}{5 - m}\right)^2 \frac{4m^3 - 16m^2 + 20m + \rho(m^2 - 5)}{16m^2},
\end{equation}

\begin{equation}
1 - \beta = \left(\frac{2m + \rho}{5 - m}\right)^2 \frac{4m^3 - 16m^2 + 20m - \rho(m^2 - 5)}{16m^2}.
\end{equation}

Substituting (3.8), (3.12), (3.16) and (3.17) into both sides of each of the identities (3.7) and (3.10), noticing that $\rho = (m^3 - 2m^2 + 5m)^{1/2}$ and then simplifying we find that both sides of each of the identities (3.7) and (3.10) are respectively equal to

\[ \left(\frac{2}{m - 1}\right)^2 A(m) \]

and

\[ \frac{m - 5}{256m(m - 1)} B(m), \]
where
\[ A(m) = 4m^9 - 24m^8 + m^8 \rho + 64m^7 - 2m^7 \rho - 200m^6 + 6m^6 \rho - 40m^5 - 98m^5 \rho \\
- 40m^4 + 80m^4 \rho - 1280m^3 - 470m^3 \rho - 504m^2 - 470m^2 \rho - 28m - 70m \rho - \rho \]

and
\[ B(m) = 2m^6 - 10m^5 + m^5 \rho - 5m^4 \rho + 4m^4 + 10m^3 \rho \\
- 4m^3 - 102m^2 - 42m^2 \rho - 18m - 27m \rho - \rho. \]

These prove (3.7) and (3.10).

We now show (3.8) and (3.9). According to [2, Chapter 19, (13.12)] we get
\[ (1 - \alpha)^{1/4} = \frac{2m - \rho}{m(m - 1)}. \]

It follows from (3.12), (3.14), (3.15) and (3.18) that
\[ \alpha = \left( \frac{2m + \rho}{m(m - 1)} \right)^2 \frac{4m^3 - 16m^2 + 20m + \rho(m^2 - 5)}{16m^2}, \]
\[ 1 - \alpha = \left( \frac{2m - \rho}{m(m - 1)} \right)^2 \frac{4m^3 - 16m^2 + 20m - \rho(m^2 - 5)}{16m^2}. \]

We substitute (3.10), (3.13), (3.19) and (3.20) into both sides of each of the identities (3.8) and (3.9), note that \( \rho = (m^3 - 2m^2 + 5m)^{1/2} \) and then simplify to deduce that both sides of each of the identities (3.8) and (3.9) equal
\[ -\frac{2^{12}m^2}{(m - 5)^{12}} C(m) \]
and
\[ \frac{1 - m}{256m^6(m - 5)} D(m) \]
respectively, where
\[ C(m) = 28m^9 + 2520m^8 - m^8 \rho + 32000m^7 - 350m^7 \rho + 5000m^6 - 11750m^6 \rho \\
+ 25000m^5 - 58750m^4 \rho + 62500m^4 + 50000m^4 \rho - 1000000m^3 - 306250m^3 \rho \\
+ 187500m^2 + 9375m^2 \rho - 1562500m - 156250 \rho + 390625, \]
and
\[ D(m) = 18m^6 + 510m^5 - m^5 \rho + 100m^4 - 135m^4 \rho - 1050m^3 \rho \\
- 500m^3 + 1250m^2 \rho + 6250m^2 - 6250m - 3125m \rho + 3125 \rho, \]
which prove (3.8) and (3.9).

We finally prove (3.11). We substitute (3.10) and (3.12) into both sides of (3.11) and then simplify using the identity \( \rho = (m^3 - 2m^2 + 5m)^{1/2} \) to derive that both sides of (3.11) are equal to
\[ \frac{(m - 5)^2(m^3 - m^2 + 7m + 2m \rho + 1 + 2 \rho)}{(m - 1)^4}, \]
from which (3.11) follows readily. This concludes the proof of Theorem 3.2.
3.3. Proof of Theorem 3.1. In this subsection we will prove Theorem 3.1.

Proof of Theorem 3.1. Using the equation (2.14) we see that the identities (3.1)–(3.5) are respectively equivalent to

\[ \frac{\psi^2(q^2)}{\psi^2(q^{10})} \frac{\psi^8(q^5)}{\psi^8(q^{10})} \left( 16q^5 - \frac{\psi^8(q^5)}{\psi^8(q^{10})} \right) = \left( 5q^2 - \frac{\psi^2(q^2)}{\psi^2(q^{10})} \right) \left( \frac{\psi^2(q^2)}{\psi^2(q^{10})} - q^2 \right)^5, \]  

(3.21)

\[ \frac{\psi^2(q^{10})}{\psi^2(q^2)} \frac{\psi^8(q)}{\psi^8(q^2)} \left( 16q - \frac{\psi^8(q)}{\psi^8(q^2)} \right) = \left( 5q^2 \frac{\psi^2(q^{10})}{\psi^2(q^2)} - 1 \right) \left( 1 - q^2 \frac{\psi^2(q^{10})}{\psi^2(q^2)} \right), \]  

(3.22)

\[ \frac{\psi^2(q^5)}{\psi^2(q)} \left( 16q - \frac{\psi^8(q^2)}{\psi^8(q)} \right) = \left( \psi^2(q^2) \right)^2 \left( \psi^8(q) \right) \left( 5q\psi^2(q^5) - 1 \right) \left( \frac{\psi^2(q^2)}{\psi^2(q^5)} - 1 \right), \]  

(3.23)

\[ \frac{\psi^2(q)}{\psi^2(q^5)} \left( 16q \frac{\psi^8(q^{10})}{\psi^8(q)} - 1 \right) \left( \frac{\psi^2(q^{10})}{\psi^2(q^5)} \right) = \left( \psi^2(q) \right)^2 \left( \psi^8(q^2) \right) \left( 5q - \psi^2(q) \right) \left( q - \frac{\psi^2(q)}{\psi^2(q^5)} \right)^5, \]  

(3.24)

\[ \left( q \frac{\psi^2(q^2)}{\psi^2(q^{10})} - \frac{\psi^2(q^2)}{\psi^2(q^{10})} \right)^2 = \left( \frac{\psi^2(q^2)}{\psi^2(q^{10})} \right) \left( q - \frac{\psi^2(q)}{\psi^2(q^5)} \right) \left( 5q - \psi^2(q) \right). \]  

(3.25)

We temporarily assume that \( 0 < q < 1 \). Let \( \beta \) have 5 degree over \( \alpha \). According to [1, Theorem 5.4.2 (i) and (iii)] we have

\[ \psi(q) = \sqrt{\frac{z_1}{2}} (\alpha/q)^{1/8}, \]  

(3.26)

\[ \psi(q^2) = \frac{1}{2} \sqrt{z_1} (\alpha/q)^{1/4}, \]  

(3.27)

\[ \psi(q^5) = \sqrt{\frac{z_5}{2}} (\beta/q^5)^{1/8}, \]  

(3.28)

\[ \psi(q^{10}) = \frac{1}{2} \sqrt{z_5} (\beta/q^5)^{1/4}. \]  

(3.29)

It follows from (3.27), (3.28) and (3.29) that

\[ \frac{\psi(q^2)}{\psi(q^{10})} = \sqrt{\frac{z_1}{z_5}} \left( \frac{\alpha}{\beta} \right)^{1/4} q, \]  

(3.30)

\[ \frac{\psi(q^5)}{\psi(q^{10})} = \frac{\sqrt{2}}{(\beta/q^5)^{1/8}}. \]  

(3.31)

Multiplying both sides of (3.7) by \( q^{12} \) and then using (3.30) and (3.31) in the resulting equation we can easily obtain the identity (3.21).

It is easily deduced from (3.26) and (3.27) that

\[ \frac{\psi(q)}{\psi(q^2)} = \sqrt{\frac{\sqrt{2}}{(\alpha/q)^{1/8}}}. \]  

(3.32)

Then (3.22) follows by substituting (3.30) and (3.32) into (3.8).

It is easily seen from (3.26) and (3.28) that

\[ \frac{\psi(q^5)}{\psi(q)} = \sqrt{\frac{z_5}{z_1}} \left( \frac{\beta}{\alpha} \right)^{1/8}/q^{1/2}. \]  

(3.33)

Then (3.23) follows easily by dividing both sides of (3.9) by \( q \) and then using (3.32) and (3.33) in the resulting identity.
Multiplying both sides of (3.10) by $q$ and then employing (3.31) and (3.33) in the resulting equation we can attain (3.24).

The identity (3.25) follows readily by multiplying both sides of (3.11) by $q^4$ and then using (3.30) and (3.33) in the resulting identity.

From these we see that (3.21)–(3.25) holds for $0 < q < 1$.

By analytic continuation, these identities are also true for $|q| < 1$.

This completes the proof of Theorem 3.1.

□

4. Two strange $q$-identities involving $\Pi_q$ and a Lambert series

Two strange $q$-identities involving $\Pi_q$ and a Lambert series, conjectured by W. Gosper, is presented in [5, p. 102]:

\[
\sum_{n \geq 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1-q^{10n-5})^2} = \sqrt{\frac{\Pi_q^3}{\Pi_q^5}} - 2 \frac{\Pi_q^2}{\Pi_q^5} + 5 \frac{\Pi_q}{\Pi_q^5} = \frac{\Pi_q^5}{\Pi_q^3} - 4 - \frac{\Pi_q^5}{\Pi_q^2},
\]

In this section we will confirm these results in the following theorem.

**Theorem 4.1.** The equalities in (4.1) are true.

The first equality of (4.1) involves a Lambert series:

\[
\sum_{n \geq 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1-q^{10n-5})^2}
\]

and an irrational expression of a rational function in $\Pi_q$ and $\Pi_q^5$:

\[
\sqrt{\frac{\Pi_q^3}{\Pi_q^5}} - 2 \frac{\Pi_q^2}{\Pi_q^5} + 5 \frac{\Pi_q}{\Pi_q^5},
\]

which leads to its huge appearance and complicated proof. The key to our proof of the identity (4.1) is to deal with this Lambert series and these $q$-constants.

**Proof of Theorem 4.1.** We assume that $0 < q < 1$ temporarily. We first prove the first equality of (4.1). Let $P(q)$ be one of the Ramanujan Eisenstein series:

\[
P(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.
\]

Since

\[
\frac{x}{(1-x)^2} = \sum_{n \geq 1} nx^n, \ |x| < 1,
\]

we see that

\[
\sum_{m \geq 1} \frac{q^m}{(1-q^m)^2} = \sum_{m,n \geq 1} nq^{m+n} = \sum_{n \geq 1} \frac{nq^n}{1-q^n} = \frac{1}{24}(1 - P(q)).
\]
Then
\[
\sum_{n \geq 1} \frac{q^{2n-1}}{(1 - q^{2n-1})^2} = \sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2} - \sum_{n \geq 1} \frac{q^{2n}}{(1 - q^{2n})^2} = \frac{1}{24} (1 - P(q)) - \frac{1}{24} (1 - P(q^2)) = \frac{1}{24} (P(q^2) - P(q))
\]
and so
\[
\sum_{n \geq 1} \frac{q^{10n-5}}{(1 - q^{10n-5})^2} = \frac{1}{24} (P(q^{10}) - P(q^5)).
\]
Combining the above two identities we get
\[
\sum_{n \geq 1} \frac{q^{2n-1}}{(1 - q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1 - q^{10n-5})^2} = \frac{1}{24} [(P(q^2) - 5P(q^{10})) - (P(q) - 5P(q^5))].
\]
Let \( \beta \) have degree 5 over \( \alpha \). According to \cite{1}, Theorem 5.4.9] we have
\[
(4.4) \quad P(q) = (1 - 5\alpha)z_1^2 + 12\alpha(1 - \alpha)z_1 \frac{dz_1}{d\alpha},
\]
\[
(4.5) \quad P(q^2) = (1 - 2\alpha)z_1^2 + 6\alpha(1 - \alpha)z_1 \frac{dz_1}{d\alpha},
\]
\[
(4.6) \quad P(q^5) = (1 - 5\beta)z_5^2 + 12\beta(1 - \beta)z_5 \frac{dz_5}{d\beta},
\]
\[
(4.7) \quad P(q^{10}) = (1 - 2\beta)z_5^2 + 6\beta(1 - \beta)z_5 \frac{dz_5}{d\beta}.
\]
In view of \cite{2} Chapter 18, Entry 24(vi)] we conclude that
\[
(4.8) \quad \beta(1 - \beta)z_5 \frac{dz_5}{d\beta} = \frac{m\alpha(1 - \alpha)}{5} z_1 \frac{dz_1}{d\alpha}.
\]
Since \( z_1 = mz_5 \), we know that
\[
\frac{dz_1}{d\alpha} = m \frac{dz_5}{d\alpha} + z_5 \frac{dm}{d\alpha}.
\]
Substituting the identity
\[
\frac{m}{z_5} = \frac{dz_1}{d\alpha} - z_5 \frac{dm}{d\alpha}
\]
into \(4.8\) we get
\[
\beta(1 - \beta)z_5 \frac{dz_5}{d\beta} = \frac{\alpha(1 - \alpha)}{5} z_1 \frac{dz_1}{d\alpha} - \frac{\alpha(1 - \alpha)}{5} z_1 z_5 \frac{dm}{d\alpha}.
\]
Substituting this equation into \(4.9\) and \(4.10\) gives
\[
(4.9) \quad P(q^5) = (1 - 5\beta)z_5^2 + \frac{12}{5} \alpha(1 - \alpha)z_1 \frac{dz_1}{d\alpha} - \frac{12}{5} \alpha(1 - \alpha)z_1 z_5 \frac{dm}{d\alpha},
\]
\[
(4.10) \quad P(q^{10}) = (1 - 2\beta)z_5^2 + \frac{6}{5} \alpha(1 - \alpha)z_1 \frac{dz_1}{d\alpha} - \frac{6}{5} \alpha(1 - \alpha)z_1 z_5 \frac{dm}{d\alpha}.
\]
Differentiating the identity \([2\) Chapter 19, (14.2)]\) with respect to \(\alpha\) using the method of logarithmic differentiation and then simplifying yields

\[
\frac{dm}{d\alpha} = \frac{1 - 2\alpha}{\alpha(1 - \alpha)} \frac{m(m - 1)(5 - m)}{25 - 20m - m^2}.
\]

We substitute (4.4), (4.5), (4.9) and (4.10) into (4.3) and then employ (4.11) in the following identity to get

\[
\sum_{n \geq 1} \frac{q^{2n-1}}{(1 - q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1 - q^{10n-5})^2} = \frac{1}{8} \left( \alpha z_1^2 - 5\beta z_5^2 - 2\alpha(1 - \alpha)z_1z_5 \frac{dm}{d\alpha} \right)
\]

\[
= \frac{z_2^2}{8} \left( \alpha m^2 - 5\beta - 2\alpha(1 - \alpha)m \frac{dm}{d\alpha} \right)
\]

\[
= \frac{z_2^2}{8} \left( \alpha m^2 - 5\beta - 2(1 - 2\alpha) \frac{m^2(m - 1)(5 - m)}{25 - 20m - m^2} \right)
\]

Substituting (3.16) and (3.19) into (4.12) and then simplifying using the identity \(\rho^2 = m^3 - 2m^2 + 5m\) we arrive at

\[
\sum_{n \geq 1} \frac{q^{2n-1}}{(1 - q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1 - q^{10n-5})^2} = \frac{z_2^2}{16} (m^2 - 5 + 2\rho).
\]

Taking squares on both sides of this identity and simplifying yields

\[
\left( \sum_{n \geq 1} \frac{q^{2n-1}}{(1 - q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1 - q^{10n-5})^2} \right)^2
\]

\[
= \frac{z_2^4}{256} (m^4 + 4m^3 - 18m^2 + 4m^2\rho + 20m + 25 - 20\rho).
\]

Using the identity \(2.14\) we know that

\[
\Pi_2^3 \Pi_4^3 - 2\Pi_2^2 \Pi_4^3 + 5\Pi_2 \Pi_4^3
\]

\[
= q^6 \psi^6(q) \psi^2(q^5) - 2q^3 \psi^4(q) \psi^4(q^5) + 5q^4 \psi^2(q) \psi^6(q^5).
\]

Substituting the equations (3.26) and (3.28) into (4.15) yields

\[
\Pi_2^3 \Pi_4^3 - 2\Pi_2^2 \Pi_4^3 + 5\Pi_2 \Pi_4^3
\]

\[
= \frac{z_1^2 z_5^2}{16} (\alpha^3 \beta)^{1/4} - 2z_1 z_5 (\alpha \beta)^{1/2} + 5z_5^2 (\alpha^3 \beta)^{1/4}
\]

\[
= \frac{m z_1^4}{16} (m^2 (\alpha^3 \beta)^{1/4} - 2m(\alpha \beta)^{1/2} + 5(\alpha^3 \beta)^{1/4}).
\]

According to \([2\) Chapter 19, (13.10) and (13.11)]\) we get

\[
(\alpha^3 \beta)^{1/8} = \frac{\rho + 3m - 5}{4m}, \quad (\alpha^3 \beta)^{1/8} = \frac{\rho + 3m - 5}{4m}.
\]

We substitute these two equations and (3.14) into (4.16), note that \(\rho^2 = m^3 - 2m^2 + 5m\) and then simplify to obtain

\[
\Pi_2^3 \Pi_4^3 - 2\Pi_2^2 \Pi_4^3 + 5\Pi_2 \Pi_4^3
\]

\[
= \frac{z_1^4}{256} (m^4 + 4m^3 - 18m^2 + 4m^2\rho + 20m + 25 - 20\rho).
\]
Combining (4.14) and (4.17) we are led to
\[
\left( \sum_{n \geq 1} \frac{q^{2n-1}}{(1 - q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1 - q^{10n-5})^2} \right)^2 = \Pi_q^3 \Pi_{q^5} - 2 \Pi_q^2 \Pi_{q^2} + 5 \Pi_q \Pi_{q^5}.
\]
Comparing the coefficient of \( q \) in \( \sum_{n \geq 1} \frac{q^{2n-1}}{(1 - q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1 - q^{10n-5})^2} \) and that of \( q^2 \) in \( \Pi_q^3 \Pi_{q^5} - 2 \Pi_q^2 \Pi_{q^2} + 5 \Pi_q \Pi_{q^5} \) we deduce that
\[
\sum_{n \geq 1} \frac{q^{2n-1}}{(1 - q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1 - q^{10n-5})^2} = \sqrt{\Pi_q^3 \Pi_{q^5} - 2 \Pi_q^2 \Pi_{q^2} + 5 \Pi_q \Pi_{q^5}}.
\]
Dividing both sides of the above identity by \( \Pi_{q^5}^2 \) we see that (4.1) holds for \( 0 < q < 1 \).

We now show the second equality. It follows from (2.14), (3.28) and (3.31) that
\[
(4.18) \quad \Pi_{q^5}^2 = q^{5/2} \psi(q^5) = \frac{\zeta_5^2}{\beta^{1/2}},
\]
\[
\Pi_{q^{10}} = \frac{\psi^2(q^5)}{q^{5/4} \psi^2(q^{10})} = \frac{2}{\beta^{1/4}}.
\]
Then
\[
\Pi_{q^5}^2 \cdot \left( \frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2} + 16 \frac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2} \right) = \frac{\rho m^2 + 24m - 4 \rho m - \rho}{16m} \zeta_5^2.
\]

It can be deduced from (2.14) and (3.33) that
\[
\Pi_q = \frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2} - 4 - \frac{\Pi_{q^{10}}}{\Pi_q} = \frac{\zeta_5}{\zeta_5^2} \left( \frac{\alpha}{\beta} \right)^{1/4} - 4 - \frac{\zeta_5^2}{10} \left( \frac{\beta}{\alpha} \right)^{1/4}.
\]

Applying (3.6), (3.12) and (3.13) in this identity and simplifying yields
\[
\Pi_q = \frac{2m^3 - 16m^2 - \rho m^2 + 22m + 6 \rho m - \rho}{(m-1)m(5-m)}.
\]
We multiply this equation by (4.13) and simplify using the identity \( \rho^2 = m^3 - 2m^2 + 5m \) to arrive at
\[
(4.20) \quad \left( \frac{\Pi_q}{\Pi_{q^5}} - 4 - \frac{\Pi_{q^5}}{\Pi_q} \right) \left( \sum_{n \geq 1} \frac{q^{2n-1}}{(1 - q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1 - q^{10n-5})^2} \right)
\]
\[
= \frac{\rho m^2 + 24m - 4 \rho m - \rho \zeta_5^2}{16m}.
\]
Combining (4.19) and (4.20) we deduce that
\[
\left( \frac{\Pi_q}{\Pi_{q^5}} - 4 - \frac{\Pi_{q^5}}{\Pi_q} \right) \left( \sum_{n \geq 1} \frac{q^{2n-1}}{(1 - q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1 - q^{10n-5})^2} \right)
\]
\[
= \Pi_{q^5} \cdot \left( \frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2} + 16 \frac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2} \right).
\]
Dividing both sides of this identity by
\[ \Pi_{q^5} \left( \frac{\Pi_q}{\Pi_{q^5}} - 4 - \frac{\Pi_{q^5}}{\Pi_q} \right) \]
gives that the second equality of (4.1) holds for \(0 < q < 1\).

From the above we see that these two equalities of (4.1) are true for \(0 < q < 1\). By analytic continuation, these two equalities hold for \(|q| < 1\). This finishes the proof of Theorem 4.1. □

5. A \(q\)-identity involving \(\Pi_q\) and two Lambert series

Gosper [5, p. 104] conjectured the following \(q\)-identity:

\[
6 \left( \sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2} - 5 \sum_{n \geq 1} \frac{q^{5n}}{(1 - q^{5n})^2} \right) + 1
\]

\[
= \left( \frac{\Pi_q}{\Pi_{q^5}} + 2 + 5 \frac{\Pi_{q^5}}{\Pi_q} \right) \left( \sum_{n \geq 1} \frac{q^{2n-1}}{(1 - q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1 - q^{10n-5})^2} \right),
\]

which includes two Lambert series.

In this section we will use (4.13) to confirm this identity. The key to our proof of (5.1) is to handle the first Lambert series

\[
\sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2} - 5 \sum_{n \geq 1} \frac{q^{5n}}{(1 - q^{5n})^2}.
\]

Theorem 5.1. The identity (5.1) is true.

From (5.1) and the second equality of (4.1) we deduce

\[
\sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2} - 5 \sum_{n \geq 1} \frac{q^{5n}}{(1 - q^{5n})^2} = \frac{\Pi_{q^5} \left( \frac{\Pi_q}{\Pi_{q^5}} + 2 + 5 \frac{\Pi_{q^5}}{\Pi_q} \right) \left( \frac{\Pi_{q^5}^2}{\Pi_{q^{10}}} + 16 \frac{\Pi_{q^5}^2}{\Pi_{q^5}} \right) - \left( \frac{\Pi_q}{\Pi_{q^5}} - 4 - \frac{\Pi_{q^5}}{\Pi_q} \right)}{6 \left( \frac{\Pi_{q^5}}{\Pi_{q^5}} - 4 - \frac{\Pi_{q^5}}{\Pi_q} \right)}.
\]

This indicates that the Lambert series

\[
\sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2} - 5 \sum_{n \geq 1} \frac{q^{5n}}{(1 - q^{5n})^2}
\]

can be represented as a rational function of \(\Pi_q\), \(\Pi_{q^5}\) and \(\Pi_{q^{10}}\).

Proof of Theorem 5.1. We first assume that \(0 < q < 1\). It follows from (4.2) that

\[
\sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2} - 5 \sum_{n \geq 1} \frac{q^{5n}}{(1 - q^{5n})^2} = \frac{1}{24} (1 - P(q)) - \frac{5}{24} (1 - P(q^5))
\]

\[
= \frac{1}{24} (5P(q^5) - P(q)) - \frac{1}{6}.
\]
Using (4.4), (4.9) and (4.11) in the above identity gives

\[
\sum_{n \geq 1} \frac{q^n}{(1-q^n)^2} - 5 \sum_{n \geq 1} \frac{q^{5n}}{(1-q^{5n})^2} = \frac{1}{24} \left( 5(1 - 5\beta)z_5^2 - (1 - 5\alpha)z_1^2 - 12\alpha(1 - \alpha)z_1z_5 \frac{dm}{d\alpha} \right) - \frac{1}{6} = \frac{z_5^2}{24} \left( 5(1 - 5\beta) - (1 - 5\alpha)m^2 - 12(1 - 2\alpha)m^2(5m - m) \right) - \frac{1}{6}.
\]

Substituting (3.16) and (3.19) into this identity and simplifying using the equality \(\rho^2 = m^3 - 2m^2 + 5m\) we get

\[
\sum_{n \geq 1} \frac{q^n}{(1-q^n)^2} - 5 \sum_{n \geq 1} \frac{q^{5n}}{(1-q^{5n})^2} = \frac{6m^3 + m^2\rho + 14m\rho - 30m + 5\rho}{96m} z_5^2 - \frac{1}{6}.
\]

Then

\[
6 \left( \sum_{n \geq 1} \frac{q^n}{(1-q^n)^2} - 5 \sum_{n \geq 1} \frac{q^{5n}}{(1-q^{5n})^2} \right) + 1 = \frac{6m^3 + m^2\rho + 14m\rho - 30m + 5\rho}{16m} z_5^2.
\]

It is deduced from (2.14) and (3.33) that

\[
\frac{\Pi_q}{\Pi_q^5} + 2 + 5 \frac{\Pi_{q^5}}{\Pi_q} = \frac{\psi^2(q)}{q^2 \psi^2(q^5)} + 2 + \frac{5q^2 \psi^2(q^5)}{\psi^2(q)} = m \left( \frac{\alpha}{\beta} \right)^{1/4} + 2 + \frac{5}{m} \left( \frac{\beta}{\alpha} \right)^{1/4}.
\]

Then, by (3.12) and (3.13),

\[
\frac{\Pi_q}{\Pi_q^5} + 2 + 5 \frac{\Pi_{q^5}}{\Pi_q} = \frac{2m + \rho}{m - 1} + 2 + \frac{5(2m - \rho)}{m(5 - m)}.
\]

Multiplying this equation by (4.13) and then simplifying by employing the identity \(\rho^2 = m^3 - 2m^2 + 5m\) we have

\[
\left( \frac{\Pi_q}{\Pi_q^5} + 2 + 5 \frac{\Pi_{q^5}}{\Pi_q} \right) \left( \sum_{n \geq 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5 \sum_{n \geq 1} \frac{q^{10n-5}}{(1-q^{10n-5})^2} \right) = \frac{6m^3 + m^2\rho + 14m\rho - 30m + 5\rho}{16m} z_5^2.
\]

Combining (5.2) and (5.3) produces that (5.1) holds for \(0 < q < 1\). By analytic continuation, we see that (5.1) holds for \(|q| < 1\). This ends the proof of Theorem 5.1. \(\square\)
6. An Application

The Jacobi theta functions \( \theta_j(z|\tau) \) for \( j = 1, 2 \) are defined by \[10\] [12, p. 464]:

\[
\theta_1(z|\tau) = -i q^{\frac{1}{2}} \sum_{k=-\infty}^{\infty} (-1)^k q^k e^{(2k+1)zi},
\]

\[
\theta_2(z|\tau) = q^{\frac{1}{2}} \sum_{k=-\infty}^{\infty} q^k e^{(2k+1)zi},
\]

where \( q = \exp(\pi i \tau) \) and \( \tau \) is a complex number with \( \text{Im} \tau > 0 \). The notations \( \vartheta'_1(\tau) = \vartheta'_1(0|\tau) \) and \( \vartheta_2(\tau) = \vartheta_2(0|\tau) \) will be used in this section. We have the following relations:

\[
\theta_1\left(z + \frac{\pi i}{2} \tau\right) = \theta_2(z|\tau), \quad \theta_2\left(z + \frac{\pi i}{2} \tau\right) = -\theta_1(z|\tau).
\]

In [5] Gosper introduced \( q \)-analogues of \( \sin z \) and \( \cos z \):

\[
\sin_q(z) := q^{\frac{1}{2}} \prod_{n=1}^{\infty} \frac{(1 - q^{2n-2z})(1 - q^{2n+2z-2})}{(1 - q^{2n-1})^2},
\]

\[
\cos_q(z) := q^{\frac{1}{2}} \prod_{n=1}^{\infty} \frac{(1 - q^{2n-2z-1})(1 - q^{2n+2z-1})}{(1 - q^{2n-1})^2}.
\]

Gosper also gave two identities between \( \sin_q z \), \( \cos_q z \) and the functions \( \theta_1 \) and \( \theta_2 \), which are equivalent to the following:

\[
\sin_q z = \frac{\theta_1(z|\tau')}{\vartheta_2(\tau')}, \quad \cos_q z = \frac{\theta_2(z|\tau')}{\vartheta_2(\tau')},
\]

where \( \tau' = -\frac{1}{2}. \) He conjectured various identities involving \( \sin_q z \) and \( \cos_q z \). In particular, he stated [5, pp. 99–100]

\[
\sin_q 2z = \frac{\Pi q}{\Pi q^2} \sin_q z \cos_q z
\]

\[
= \frac{1}{2} \frac{\Pi q}{\Pi q^2} \sqrt{(\sin_q z)^2 - (\sin_q z)^4}
\]

\[
\sin_q 3z = \frac{\Pi q^3}{\Pi q^6} (\cos_q z)^2 \sin_q z - (\sin_q z)^3
\]

\[
= \frac{1}{3} \frac{\Pi q}{\Pi q^6} \sin_q z - \left(1 + \frac{1}{3} \frac{\Pi q}{\Pi q^6}\right) (\sin_q z)^3
\]

and

\[
\sin_q 5z = \frac{\Pi q^2}{\Pi q^6} (\cos_q z)^4 \sin_q z - \sqrt{\frac{\Pi q^3}{\Pi q^6} - 2 \frac{\Pi q^2}{\Pi q^6} + \frac{3}{5} \frac{\Pi q}{\Pi q^6} (\cos_q z)^2 (\sin_q z)^3} + (\sin_q z)^5.
\]

The first equality in (6.3) was confirmed by Mező [11] by using the method of logarithmic derivatives. The identity (6.4) and the second equality in (6.3) were proved by M. El Bachraoui [11] by employing the theory of elliptic functions. Since

\[
\left\{ \lim_{q \to 1} \sin_q z, \lim_{q \to 1} \cos_q z, \lim_{q \to 1} \frac{\Pi q}{\Pi q^6} \right\} = \{ \sin z, \cos z, 5 \},
\]

\[
\text{ON } \Pi_q \text{-IDENTITIES OF GOSPER} \quad 18
\]
Theorem 6.1. \( \text{The } q\text{-trigonometric identity (6.5) holds for any complex number } z. \)

Our proof of the \( q\text{-trigonometric identity (6.5) is different from those of (6.3) and (6.4)} \) and more complicated. The key to proving the identity (6.5) is to determine the constant

\[
\frac{\Pi_2}{\Pi_4} - 2 \frac{\Pi_2}{\Pi_6} + 5 \frac{\Pi_2}{\Pi_8},
\]

in front of \((\cos_q^2 z)^2(\sin_q^4 z)^3\). This can be done by employing Theorem 4.1.

In order to show Theorem 6.1 we also need the following interesting result.

Theorem 6.2. \[9, \text{Theorem 2.2}\] Suppose that \( f_1(z), f_2(z), \ldots f_r(z) \) are \( r \) linearly independent nonzero entire functions of \( z \) and satisfy the functional equations:

\[
f(z) = (-1)^r f(z + \pi) = (-1)^r q^r e^{2riz} f(z + \pi \tau).
\]

Let \( f(z) \) be any nonzero entire function satisfying (6.6). Then \( f(z) \) is a linear combination of the functions \( f_1(z), f_2(z), \ldots f_r(z) \).

We are now in the position to prove Theorem 6.1.

Proof of Theorem 6.1. It is clear that all of the five entire functions

\[
\frac{\theta_1(5z|\tau)}{\theta_1(z|\tau)}, \theta_2^1(z|\tau), \theta_2^1(z|\tau)\theta_2^2(z|\tau), \theta_1^1(z|\tau), \theta_1(2z|\tau)
\]

satisfy the functional equations:

\[
f(z) = f(z + \pi) = q^r e^{rzi} f(z + \pi \tau).
\]

We now prove that the four functions

\[
\theta_2^1(z|\tau), \theta_2^1(z|\tau)\theta_2^2(z|\tau), \theta_1^1(z|\tau), \theta_1(2z|\tau)
\]

are linearly independent over \( \mathbb{C} \). Assume that

\[
C_1 \theta_2^1(z|\tau) + C_2 \theta_2^1(z|\tau)\theta_2^2(z|\tau) + C_3 \theta_1^1(z|\tau) + C_4 \theta_1(2z|\tau) = 0
\]

for some complex numbers \( C_1, C_2, C_3, C_4 \). Setting \( z = 0 \) in (6.7) gives \( C_1 = 0 \).

Replacing \( z \) by \(-z\) in (6.7) we have \( C_1 = 0 \). Substituting \( C_1 = C_4 = 0 \) into (6.7), dividing both sides of the resulting identity by \( \theta_2^2(z|\tau) \) and then setting \( z = 0 \) we obtain \( C_2 = 0 \) and so \( C_3 = 0 \). Hence, these four functions are linearly independent over \( \mathbb{C} \).

In view of Theorem 6.2 we get

\[
\frac{\theta_1(5z|\tau)}{\theta_1(z|\tau)} = D_1 \theta_2^1(z|\tau) + D_2 \theta_2^1(z|\tau)\theta_2^2(z|\tau) + D_3 \theta_1^1(z|\tau) + D_4 \theta_1(2z|\tau)
\]
for some complex numbers $D_1$, $D_2$, $D_3$, $D_4$. These four constants are independent of $z$ but depend on $\tau$, and we sometimes denote $D_i$ as $D_i(\tau)$ in the sequel. Putting $z = 0$ in (6.8) we are led to

\begin{equation}
(6.9) \quad D_1 = \frac{1}{\vartheta_2'(\tau)} \lim_{z \to 0} \frac{\theta_1(5z|5\tau)}{\theta_1(z|\tau)} = \frac{5\vartheta_1'(5\tau)}{\vartheta_2'(\tau)}.
\end{equation}

Replacing $z$ by $-z$ in (6.8) gives

\begin{equation}
(6.10) \quad D_4 = 0.
\end{equation}

We set $z = \frac{\pi}{2}$ in (6.8) to obtain

\begin{equation}
(6.11) \quad D_3 = \frac{\vartheta_2(5\tau)}{\vartheta_2'(5\tau)}.
\end{equation}

Multiplying both sides of (6.8) by $\theta_1(z|\tau)$, replacing $z$ by $z + \pi/2$ and substituting (6.9), (6.10) and (6.11) into the resulting identity we get

\begin{equation}
(6.12) \quad \theta_2(5z|5\tau) = \frac{5\vartheta_1'(5\tau)}{\vartheta_2'(\tau)}\theta_1(z|\tau)\theta_2(z|\tau) + D_2(\tau)\theta_1(z|\tau)\theta_2(z|\tau) + \frac{\vartheta_2(5\tau)}{\vartheta_2'(\tau)}\theta_2(z|\tau).
\end{equation}

It follows from (6.2) that

\begin{equation}
(6.13) \quad \sin_{q^5} z = \frac{\theta_1(z|\tau'/5)}{\vartheta_2(\tau'/5)}, \quad \cos_{q^5} z = \frac{\theta_2(z|\tau'/5)}{\vartheta_2(\tau'/5)}.
\end{equation}

According to [6, Lemma 3.1] we have

\begin{equation}
(6.14) \quad \frac{\Pi_q}{\Pi_{q^5}} = \frac{5\vartheta_1'(\tau')\vartheta_2(\tau'/5)}{\vartheta_1(\tau'/5)\vartheta_2(\tau')}.
\end{equation}

Dividing both sides of (6.12) by $\vartheta_2(5\tau)$, replacing $\tau$ by $\tau'/5$ and then applying (6.13) and (6.14) in the resulting identity we find that

\begin{equation}
(6.15) \quad \cos_{q^5} 5z = \frac{\Pi_q}{\Pi_{q^5}}(\sin_{q^5} z)^4 \cos_{q^5} z + E(q)(\sin_{q^5} z)^2(\cos_{q^5} z)^3 + (\cos_{q^5} z)^5,
\end{equation}

where

\[ E(q) = \frac{D_2(\tau'/5)\vartheta_2(\tau'/5)}{\vartheta_2(\tau')} = \frac{25\cos_{q^5}' 0 - 5\cos_{q^5}' 0}{2(\sin_{q^5}' 0)^2}.
\]

Noticing the difference between the identities (6.15) and (6.16), we only need to determine the constant $E(q)$ not $D_2$. We now calculate the constant $E(q)$.

Subtracting $(\cos_{q^5} z)^5$ from both sides of (6.15), dividing the resulting identity by $(\sin_{q^5} z)^2$, setting $z \to 0$ and then using L'Hôpital's rule two times we deduce that

\begin{equation}
(6.16) \quad E(q) = \frac{25\cos_{q^5}' 0 - 5\cos_{q^5}' 0}{2(\sin_{q^5}' 0)^2}.
\end{equation}

From the definition of $\Pi_q$ [5, p. 85] we see that

\[ \sin_{q^5}' 0 = -\frac{2\ln q}{\pi} \Pi_q,
\]

and so

\begin{equation}
(6.17) \quad \sin_{q^5}' 0 = -\frac{10\ln q}{\pi} \Pi_{q^5}.
\end{equation}
Differentiating both sides of (6.1) with respect to $z$ using the method of logarithmic differentiation and then setting $z = 0$ we have

(6.18) \[ \cos'' q_0 = \frac{2 \ln q}{\pi^2} \left( 1 - 4 \ln q \sum_{n \geq 1} \frac{q^{2n-1}}{(1 - q^{2n-1})^2} \right). \]

Then

(6.19) \[ \cos'' q_5 = \frac{10 \ln q}{\pi^2} \left( 1 - 20 \ln q \sum_{n \geq 1} \frac{q^{10n-5}}{(1 - q^{10n-5})^2} \right). \]

We substituting (6.17), (6.18) and (6.19) into (6.16) and then employ the first equality of (4.1) to get

\[ E(q) = -\sqrt{\frac{\Pi_3^2 q}{\Pi_3^2 q^5} - 2 \frac{\Pi_3^2 q^3}{\Pi_5 q^3} + 5 \frac{\Pi_3 q}{\Pi_5 q^3}}. \]

Then (6.5) follows readily by substituting this equation into (6.15) and then replacing $z$ by $z + \pi/2$ in the resulting identity. This concludes the proof of Theorem 6.1. \qed

Remark. Similar to the proof of the $q$-trigonometric identity (6.5) supplied above, we can also employ Theorem 6.2 to give new proofs of the $q$-trigonometric identities (6.3) and (6.4).

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References

[1] B.C. Berndt, Number Theory in the Spirit of Ramanujan, American Mathematical Society, Providence, RI, 2006.
[2] B.C. Berndt, Ramanujan’s Notebooks, Part III, Springer-Verlag, New York, 1991.
[3] M. El Bachraoui, On the Gosper’s $q$-constant $\Pi_q$, Acta Mathematica Sinica, English Series. 34(11)(2018), 1755–1764
[4] M. El Bachraoui, Proving some identities of Gosper on $q$-trigonometric functions. arXiv:1801.03654v1.
[5] R.W. Gosper, Experiments and discoveries in q-trigonometry, in: F.G. Garvan, M.E.H. Ismail (Eds.), Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics, Kluwer, Dordrecht, Netherlands, 2001, pp.79–105.
[6] B. He and H.-C. Zhai, Proofs for certain $q$-trigonometric identities of Gosper, Science China–Mathematics, 2019, http://engine.scichina.com/doi/10.1007/s11425-019-9555-1.
[7] Z.-G. Liu, Addition formulas for Jacobi theta functions, Dedekind’s eta functions, and Ramanujan’s congruences, Pacific J. Math. 240(1) (2009), 135–150.
[8] Z.-G. Liu, An addition formula for the Jacobian theta function and its applications, Adv. Math. 212(1) (2007), 389–406.
[9] Z.-G. Liu, Elliptic functions and the Appell theta functions, Int. Math. Res. Not., IMRN 11 (2010), 2064–2093.
[10] Z.-G. Liu, Residue theorem and theta function identities. Ramanujan J. 5(2) (2001), 129–151.
[11] I. Mezó, Duplication formulae involving Jacobi theta functions and Gosper’s $q$-trigonometric functions, Proc. Amer. Math. Soc. 141 (7) (2013) 2401–2410.
[12] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis. 4th ed., Cambridge University Press, Cambridge, 1990.
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