On singularity properties of convolutions of algebraic morphisms

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Abstract
Let $K$ be a field of characteristic zero, $X$ and $Y$ be smooth $K$-varieties, and let $V$ be a finite dimensional $K$-vector space. For two algebraic morphisms $\varphi : X \to V$ and $\psi : Y \to V$ we define a convolution operation, $\varphi \ast \psi : X \times Y \to V$, by $\varphi \ast \psi(x, y) = \varphi(x) + \psi(y)$. We then study the singularity properties of the resulting morphism, and show that as in the case of convolution in analysis, it has improved smoothness properties. Explicitly, we show that for any morphism $\varphi : X \to V$ which is dominant when restricted to each irreducible component of $X$, there exists $N \in \mathbb{N}$ such that for any $n > N$ the $n$th convolution power $\varphi^n := \varphi \ast \cdots \ast \varphi$ is a flat morphism with reduced geometric fibers of rational singularities (this property is abbreviated (FRS)).

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1.1 Motivation

Given functions $f, g \in L^1(\mathbb{R}^n)$, their convolution is defined by $(f * g)(x) = \int_{\mathbb{R}^n} f(x - t)g(t)dt$. It is a basic fact that $f * g$ has better smoothness properties than both $f$ and $g$; indeed, if in addition we assume that $f \in C^k(\mathbb{R}^n)$ and $g \in C^l(\mathbb{R}^n)$, then $(f * g)' = f' * g = f * g'$ and therefore $f * g \in C^{k+l}(\mathbb{R}^n)$. In particular, if either $f$ or $g$ is smooth then $f * g$ is smooth. An interesting question is whether this phenomenon has an analogue in the setting of algebraic geometry. In this paper we give a partial answer to this question by exploring the following operation, as proposed by Aizenbud and Avni:

**Definition 1.1** Let $X$ and $Y$ be algebraic varieties, $G$ an algebraic group and let $\varphi : X \to G$ and $\psi : Y \to G$ be algebraic morphisms. We define their convolution $\varphi \ast \psi : X \times Y \to G$ by $\varphi \ast \psi(x, y) = \varphi(x) \cdot \psi(y)$. In particular, the $n$th convolution power of $\varphi$ is $\varphi^n(x_1, \ldots, x_n) = \varphi(x_1) \cdots \varphi(x_n)$.

**Remark 1.2** This operation is related to the classical notion of convolution in the following way. Given a finite ring $A$, the above morphisms $\psi$ and $\varphi$ induce maps $\varphi_A : X(A) \to G(A)$ and $\psi_A : Y(A) \to G(A)$. By defining $F_{\psi_A}(t) := |\varphi_A^{-1}(t)|$, we have,
\[
F_{\psi_A * \psi_A}(t) = \sum_{s \in G(A)} F_{\psi_A}(s) \cdot F_{\psi_A}(s^{-1} t) = F_{\psi_A} * F_{\psi_A}(t).
\]

Recall that a morphism \( \varphi : X \to Y \) between \( K \)-varieties is smooth if it is flat and the fiber \( \varphi^{-1} \circ \varphi(x) \) is smooth for all \( x \in X(\overline{K}) \). Analogously to the analytic situation, convolving a smooth morphism with any morphism whose domain is a smooth variety yields a smooth morphism. More generally, the convolution operation preserves properties of morphisms which are stable under base change and compositions.

**Proposition 1.3** (see Proposition 3.1 for a proof) Let \( X \) and \( Y \) be varieties over a field \( K \), let \( G \) be an algebraic group over \( K \) and let \( S \) be a property of morphisms that is preserved under base change and compositions. If \( \varphi : X \to G \) is a morphism that satisfies the property \( S \), the natural map \( i_K : Y \to \text{Spec}(K) \) has property \( S \) and \( \psi : Y \to G \) is arbitrary, then \( \varphi * \psi \) has property \( S \).

A natural question is then whether any morphism \( \varphi : X \to G \) becomes smooth after sufficiently many convolution powers. Clearly, we must first ensure that \( \varphi \) becomes dominant when raised to high enough convolution power, but even if this is indeed the case, we might have that \( \varphi^n \) is not smooth for all \( n \in \mathbb{N} \), as shown in the next example.

**Example 1.4** Consider the map \( \varphi(x) = x^2 \). Then \( d\varphi^n_{(0, \ldots, 0)} \) is not surjective for every \( n \in \mathbb{N} \), and thus \( \varphi^n \) is not smooth for all \( n \).

Although in Example 1.4 above \( \varphi^n \) is not smooth at \( (0, \ldots, 0) \) for all \( n \in \mathbb{N} \), it has better singularity properties as \( n \) grows. While the singular locus of \( \varphi \) is of codimension 1 and its fiber over 0 is non-reduced, \( \varphi^2 \) has reduced fibers and its singular locus is of codimension 2. For \( n = 3 \) we get that the singular locus of \( \varphi^3 \) is of codimension 3, and that the non-smooth fiber \( (\varphi^3)^{-1}(0) \) is reduced and has rational singularities [for rational singularities see Definition 2.12(2)]. In the case where \( G = (\mathbb{A}^1, +) \) as above, this is analogous to the following Thom–Sebastiani type result: if \( f_1, f_2 \in \mathbb{C}[x_1, \ldots, x_t] \), then \( \text{lct}(f_1 * f_2) = \min\{1, \text{lct}(f_1) + \text{lct}(f_2)\} \), where \( \text{lct}(h) \) denotes the log-canonical threshold of the hypersurface defined by \( h \in \mathbb{C}[x_1, \ldots, x_t] \) in \( \mathbb{C}^t \) (e.g. [22, Corollary 1], see also [24]).

This motivates us to introduce the (FRS) property of morphisms as defined in [1] (see Sect. 1.3 for further discussion of the (FRS) property). From now on, we assume that \( K \) is a field of characteristic zero.

**Definition 1.5** Let \( X \) and \( Y \) be smooth \( K \)-varieties. We say that a morphism \( \varphi : X \to Y \) is (FRS) if it is flat and if every geometric fiber of \( \varphi \) is reduced and has rational singularities.

This property plays a key role in this paper, and as seen in the above example, although we cannot hope to obtain smooth morphisms after sufficiently many convolution powers, we might be able to get morphisms with the (FRS) property. It is conjectured by Aizenbud and Avni that such a phenomenon occurs, under adequate assumptions, for a general algebraic group \( G \):

**Conjecture 1.6** Let \( X \) be a smooth, absolutely irreducible \( K \)-variety, \( G \) be an algebraic \( K \)-group and let \( \varphi : X \to G \) be a morphism such that \( \varphi(X) \notin g H \) for any translation
of an algebraic subgroup $H \leq G$ by an element $g \in G(\overline{K})$. Then there exists $N \in \mathbb{N}$ such that for any $n > N$, the $n$th convolution power $\varphi^n$ is (FRS).

1.2 Main results

In this paper we verify Conjecture 1.6 for the case of a vector group. Namely, we prove the following theorem:

**Theorem 1.7** Let $K$ be a field of characteristic zero. Let $X$ be a smooth $K$-variety with absolutely irreducible components $X_1, \ldots, X_l$, and let $V$ be a $K$-vector space of finite dimension. Then for every morphism $\varphi : X \to V$ such that $\varphi(X_i)$ is not contained in any proper affine subspace of $V$ for all $i$, there exists $N \in \mathbb{N}$ such that for any $n > N$ the $n$th convolution power $\varphi^n$ is (FRS).

In fact, we are able to prove a uniform analogue of Theorem 1.7 for families of algebraic morphisms.

**Definition 1.8** We call a morphism $\varphi : X \to Y$ strongly dominant if $\varphi$ is dominant when restricted to each absolutely irreducible component of $X$.

**Theorem 1.9** Let $K$ and $V$ be as in Theorem 1.7, let $Y$ be a $K$-variety, let $\tilde{X}$ be a family of varieties over $Y$, and let $\tilde{\varphi} : \tilde{X} \to V \times Y$ be a $Y$-morphism. Denote by $\tilde{\varphi}_y : \tilde{X}_y \to V$ the fiber of $\tilde{\varphi}$ at $y \in Y$. Then,

1. The set $Y' := \{y \in Y : \tilde{X}_y$ is smooth and $\tilde{\varphi}_y : \tilde{X}_y \to V$ is strongly dominant\} is constructible.

2. There exists $N \in \mathbb{N}$ such that for any $n > N$, and any $n$ points $y_1, \ldots, y_n \in Y'(K)$, the morphism $\tilde{\varphi}_{y_1} \ast \cdots \ast \tilde{\varphi}_{y_n} : \tilde{X}_{y_1} \times \cdots \times \tilde{X}_{y_n} \to V$ is (FRS).

**Definition 1.10** An affine variety $X$ has complexity at most $D \in \mathbb{N}$, if $X$ can be written as $K[X] = K[x_1, \ldots, x_m]/(f_1, \ldots, f_k)$, where $m$, $k$ and the maximal degree of all polynomials, max$_i \{f_i\}$, is at most $D$. The notion of complexity can be similarly defined for non-affine varieties and for morphisms (see Sect. 7).

As a corollary of Theorem 1.9, one can then show that given a complexity class $D \in \mathbb{N}$, there exists $N(D) \in \mathbb{N}$ such that the convolution of any $n > N(D)$ morphisms of complexity at most $D$ is (FRS).

**Corollary 1.11** Let $V$ be a $K$-vector space of dimension not greater than $D \in \mathbb{N}$. Then there exists an integer $N(D) \in \mathbb{N}$ such that for every $n > N(D)$ the morphism $\varphi_1 \ast \cdots \ast \varphi_n$ is (FRS) for any $n$ strongly dominant morphisms $\varphi_i : X_i \to V$ of complexity at most $D$, from smooth $K$-varieties $X_i$ to $V$.

1.3 A brief discussion of the (FRS) property

The notion of rational singularities can be regarded as a certain approximation of smoothness of varieties. Thus, we can view the (FRS) property, which is its relative analogue, as an approximation to smoothness of morphisms. In this sense, Theorem 1.7
supports the claim that the convolution operation improves the singularity properties of morphisms. To explain our particular interest in the (FRS) property, we present it from several different points of view:

1. **The number-theoretic point of view:** Let $X$ be a finite type $\mathbb{Z}$-scheme such that $X_{Q} = X \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q})$ is a local complete intersection. By [2, Theorem 3.0.3] and [15, Theorem 1.3], which are based on works of Denef [13], Igusa [20] and Mustata [23], it turns out that $X_{Q}$ has rational singularities if and only if there exists a positive constant $C \in \mathbb{R}$ such that for any prime $p$ and any $k \in \mathbb{N}$ we have

$$\frac{|X(\mathbb{Z}/p^k\mathbb{Z})|}{p^k \dim(X_{Q})} < C. \tag{1.1}$$

Now, given a $\mathbb{Z}$-morphism $\varphi : X \to Y$, between finite type, reduced $\mathbb{Z}$-schemes such that $\varphi_{Q} := \varphi \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q})$ is (FRS), then for any element $y \in Y(\mathbb{Z})$, the fiber $X_{y}$ satisfies that $(X_{y})_{Q}$ is a reduced, local complete intersection variety with rational singularities. By the above arguments we see that the size of the fiber $X_{y}(\mathbb{Z}/p^k\mathbb{Z})$, where $y \in Y(\mathbb{Z})$, behaves asymptotically as if $\varphi$ were smooth. Combining the above with Remark 1.2 allows one to interpret the (FRS) property from a probabilistic point of view.

2. **The probabilistic point of view:** Let $X$ be a smooth finite type $\mathbb{Z}$-scheme, let $G$ be an algebraic group over $\mathbb{Z}$, let $\varphi : X \to G$ be a morphism, and let $A$ be a finite ring. Recall we have defined $F_{\varphi_{A}}(t) = |\varphi_{A}^{-1}(t)|$. We then saw that taking the size of the fiber commutes with convolution, that is $F_{\varphi_{A}}^n = F_{\varphi_{A}} \ast \cdots \ast F_{\varphi_{A}}$. Now, taking the uniform probability measure $1_{X(A)}$ on $X(A)$ gives rise to a random walk on $G(A)$ with probability distribution $\varphi_{*}(1_{X(A)})$, whose $m$th step has the following probability distribution:

$$\varphi_{*}(1_{X(A)})^m = \varphi_{*}^m (1_{X(A)} \times \cdots \times 1_{X(A)}) = \frac{F_{\varphi_{A}}^m}{|X(A)|^m}.$$ 

Thus, for $A = \mathbb{Z}/p^k\mathbb{Z}$, existence of $m$ large enough such that $\varphi_{Q}^m$ is (FRS) would imply by Formula 1.1 that after $m$ steps the probability distribution of the random walk is not too far from being uniform on $G(A)$. Consequentially, verifying Conjecture 1.6 leads to interesting uniform results on random walks on families of finite groups and compact $p$-adic groups. Further work in this direction is in progress, and will appear in upcoming papers.

3. **The analytic point of view:** There is a useful analytic characterization of the (FRS) property of a morphism $\varphi : X \to Y$ defined over a non-Archimedean local field $F$ of characteristic 0. Explicitly, $\varphi$ is (FRS) if and only if for every locally constant and compactly supported measure $\mu_{X}$ on $X(F)$, the pushforward $\varphi_{*}(\mu_{X})$ has continuous density with respect to any smooth non-vanishing measure $\mu_{Y}$ on $Y(F)$ (see Theorem 2.17 or [1, Theorem 3.4]). Such a characterization exists also for smooth and strongly dominant morphisms:
(a) If \( \varphi \) is strongly dominant, then \( \varphi_*(\mu_X) \) is absolutely continuous with respect to any smooth non-vanishing measure \( \mu_Y \) on \( Y(F) \), and hence it has an \( L^1 \)-density ([1, Corollary 3.6]).

(b) If \( \varphi \) is a smooth morphism, then \( \varphi_*(\mu_X) \) has a smooth density with respect to any smooth non-vanishing measure \( \mu_Y \) on \( Y(F) \) (see [1, Proposition 3.5]).

The above approaches to the (FRS) property have some very interesting applications. In [1], Aizenbud and Avni show that for any semisimple algebraic group \( G \) over a field \( K \) of characteristic 0, the commutator map \( \varphi : G \times G \to G \) via \( \varphi(g, h) = ghg^{-1}h^{-1} \) becomes (FRS) after finitely many self-convolutions (see [1, Theorem VIII]). As an application, they give a polynomial bound (in \( N \)) on the number of irreducible \( N \)-dimensional representations of open compact subgroups of \( G(F) \), for any non-Archimedean local field \( F \) of characteristic 0 (see [1, Theorem A]), and the same for arithmetic groups of higher rank (see [2, Theorem B]).

These applications and different characterizations supply us with enough evidence that the (FRS) property encodes valuable information, and that Conjecture 1.6 is of interest. We believe that the case of a vector group \( G = V \) (Theorems 1.7 and 1.11) is an important step towards proving Conjecture 1.6.

1.4 Sketch of the proof of the main result and structure of the paper

In Sect. 2 we recall relevant background material: in Sects. 2.1 and 2.2 we give necessary preliminaries from model theory, and in Sect. 2.3 we mainly review required notions from algebraic geometry and analysis on manifolds. The definition of the (FRS) property and the statement of the equivalent analytic criterion for the (FRS) property also appear in Sect. 2.3.

The scheme of the proof of the main result, Theorem 1.7, is as follows. Our goal is to show that when raised to high enough convolution power, \( \varphi : X \to A^m_K \) satisfies the analytic criterion for the (FRS) property as given in Theorem 2.17. We first prove Proposition 3.5 to reduce our problem to the case of a strongly dominant morphism. We then prove the theorem in the case where \( K = \mathbb{Q} \) in the following way. We construct a family of non-negative Schwartz measures \( \{ \mu_p \}_{p \text{ prime}} \) such that \( \mu_p \) is a measure on \( \mathbb{Q}_p \) and \( \text{supp}(\mu_p) = X(\mathbb{Z}_p) \) for each prime \( p \) (this is Proposition 3.14). By [1, Corollary 3.6], for any \( p \), we deduce that \( \varphi_*(\mu_p) \) is a compactly supported measure which is absolutely continuous with respect to the normalized Haar measure \( \lambda \) on \( \mathbb{Q}_p^m \) and consequently its density lies in \( L^1(\mathbb{Q}_p^m) \).

We now want to show there exists \( n \in \mathbb{N} \), such that for any prime \( p \), the pushforward under the \( n \)th convolution power \( \varphi_n^*(\mu_p \times \cdots \times \mu_p) = \varphi_*^*(\mu_p) \ast \cdots \ast \varphi_*^*(\mu_p) \) has continuous density with respect to \( \lambda \). This implies Theorem 1.7 for \( K = \mathbb{Q} \), since for any point \( x \in (X \times \cdots \times X)(\mathbb{Q}) \) there exists a prime \( p \) such that \( x \in \text{supp}(\mu_p \times \cdots \times \mu_p) \), and we can then apply the analytic criterion for the (FRS) property as given in Theorem 2.17 (see Proposition 3.16 for the precise statement). Two main difficulties now arise:

1. It is not true in general that every compactly supported, \( L^1 \)-measure on \( \mathbb{Q}_p^m \) results in a measure with continuous density after finitely many self convolutions. This
means we need to choose a measure $\mu_p$ that is well behaved with respect to pushing forward by $\varphi$.

2. The required number of self convolutions needed for $\varphi_*(\mu_p)$ to become (FRS) might depend on the prime $p$, while we want this number to be independent of $p$. Thus, we need to construct a collection of measures $\{\mu_p\}_{p \text{ prime}}$ that behave well in a uniform manner.

We deal with these two difficulties using methods from the theory of motivic integration; in Sect. 3.3, we define a notion of a motivic measure on a $\mathbb{Q}$-variety, based on the notion of motivic functions in the Denef–Pas language (see [7,9,11,12], or Sect. 2.1). The Denef–Pas language allows us to obtain results over $\mathbb{Q}_p$, which are uniform in $p$ for $p$ large enough, using specialization arguments (see e.g [7, Section 4] and Lemma 2.4), thus dealing with the second difficulty mentioned above.

To overcome the first difficulty, we show in Theorem 4.2 that the class of motivic measures behaves well under pushforward by algebraic morphisms. We then show that any motivic measure $\sigma$ on $\mathbb{Q}^m_p$ whose density lies in $L^1(\mathbb{Q}^m_p)$ and is compactly supported has continuous density after sufficiently many self convolutions. We prove this by studying the decay properties of the Fourier transform $F(\sigma)$ of such motivic measures (this is Theorem 5.2). An application of the results to $\sigma = \varphi_*(\mu_p)$ finishes the proof of Theorem 1.7 for the case $K = \mathbb{Q}$. An alternative proof of Theorem 5.2, as suggested by the anonymous referee, is given in Sect. 5.2 using results of [10] on motivic exponential functions.

Finally, we show in Sect. 6, that it is indeed enough to prove Theorem 1.7 for $K = \mathbb{Q}$, thus finishing the proof of the theorem. In Sect. 7, we prove the relative version of Theorem 1.7, i.e Theorem 1.9 and as a consequence, obtain Corollary 1.11.

1.5 Conventions

Throughout the paper we use the following conventions:

- Unless explicitly stated otherwise, $K$ is a field of characteristic 0 and $F$ is a non-Archimedean local field of characteristic 0 whose ring of integers is $\mathcal{O}_F$.
- For an integral subscheme $A \subseteq X$ we denote by $K(A)$ its function field.
- For a morphism $\varphi : X \to Y$ of algebraic varieties, the scheme theoretic fiber at $y \in Y$ is denoted by either $\varphi^{-1}(y)$ or $\mathrm{Spec}(K((y))) \times_Y X$.
- For a field extension $K'/K$ and a $K$-variety $X$ (resp. $K$-morphism $\varphi : X \to Y$), we denote the base change of $X$ (resp. $\varphi$) by $X_{K'} := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K')$ (resp. $\varphi_{K'} : X_{K'} \to Y_{K'}$).
- If $X$ is a $K$-variety, we set the $N$-fold product of $X$ by $X^N := X \times \cdots \times X$.
- If $\varphi : X \to G$ is a morphism to an algebraic group $G$ we denote its $n$th convolution power by $\varphi^n := \varphi \ast \cdots \ast \varphi$. 
2 Preliminaries

In this section we review definitions and results on which we rely in this work. For this, we mostly follow the definitions and notations of [4, Section 3], [7, Section 4], [9] and [25] in Sections 2.2 and 2.3 and [1, Sections 3.1–3.3] and [2] in Section 2.4.

2.1 The Denef–Pas language, definable sets and functions, motivic functions and integration

In this section, we recall the definitions of the Denef–Pas language, definable sets, definable functions and motivic functions, and review an integration result concerning motivic functions.

2.1.1 The Denef–Pas language

The Denef–Pas language, denoted $L_{DP}$, is a first order language with three sorts of variables: the valued field sort $VF$, the residue field sort $RF$, and the value group sort $VG$. The $L_{DP}$ language consists of the following:

- The language of rings $L_{Val} = (+, -, \cdot, 0, 1)$ for the valued field sort $VF$.
- The language of rings $L_{Res} = (+, -, \cdot, 0, 1)$ for the residue field sort $RF$.
- The language $L_{\infty \text{Pres}} = L_{\text{Pres}} \cup \{ \infty \}$ for the value group sort $VG$, where $\infty$ is a constant, and $L_{\text{Pres}} = (+, -, \leq, \{ \equiv_{\text{mod } n} \}_{n>0}, 0, 1)$ is the Presburger language consisting of the language of ordered abelian groups along with constants 0, 1 and a family of 2-relations $\{ \equiv_{\text{mod } n} \}_{n>0}$ of congruence modulo $n$.
- A function $\text{val} : VF \rightarrow VG$ for a valuation map.
- A function $\text{ac} : VF \rightarrow RF$ for an angular component map.

Altogether, we write

$$L_{DP} = \{ L_{Val}, L_{Res}, L_{\infty \text{Pres}}, \text{val}, \text{ac} \}.$$

2.1.2 The angular component map

Let $K$ be a complete, discretely valued field, with valuation map $\text{val} : K \rightarrow \Gamma \cup \{ \infty \}$, where $\Gamma$ is an ordered abelian group which we usually take to be $\mathbb{Z}$. Let $O_K$ be the ring of integers of $K$ with maximal ideal $m_K = \{ x \in K : \text{val}(x) > 0 \}$, and let $k_K = O_K / m_K$ denote its residue field and $\text{Res} : O_K \rightarrow k_K$ be the canonical quotient map. An angular component map $\text{ac} : K \rightarrow k_K$ is a map satisfying the following:

1. $\text{ac}(0) = 0$.
2. $\text{ac}|_{O_K^\times} = \text{Res}|_{O_K^\times}$.
3. $\text{ac}|_{K^\times}$ is a multiplicative morphism from $K^\times$ to $k_K^\times$.

It is unique up to choosing a uniformizer $\pi \in O_K$ and setting $\text{ac}(\pi) = 1$. 
2.1.3 Definable sets, definable functions and motivic functions

Any pair \((F, \pi)\) of a non-Archimedean local field \(F\) and a uniformizer \(\pi\) of \(\mathcal{O}_F\) has a natural \(\mathcal{L}_{\text{DP}}\)-structure. Let \(\text{Loc}\) be the set of all such pairs, and \(\text{Loc}_M\) be the set of \((F, \pi) \in \text{Loc}\) such that \(F\) has residue characteristic larger than \(M\). Usually, we just write \(F \in \text{Loc}\), omitting the second term, as our results are independent of the choice of a uniformizer.

Given a formula \(\phi\) in \(\mathcal{L}_{\text{DP}}\), with \(n_1\) free valued field variables, \(n_2\) free residue field variables and \(n_3\) free value group variables we can naturally interpret it in \(F \in \text{Loc}\), yielding a subset \(F(\phi) \subseteq F^{n_1} \times k_F^{n_2} \times \mathbb{Z}^{n_3}\). We would like to study families of subsets of \(F^{n_1} \times k_F^{n_2} \times \mathbb{Z}^{n_3}\) which arise from a fixed \(\mathcal{L}_{\text{DP}}\)-formula \(\phi\). To do so, we introduce the following definitions:

**Definition 2.1** (See [9, Definitions 2.3–2.6]) Let \(n_1, n_2, n_3\) and \(M\) be natural numbers.

1. A collection \(X = (X_F)_{F \in \text{Loc}_{M}}\) of subsets \(X_F \subseteq F^{n_1} \times k_F^{n_2} \times \mathbb{Z}^{n_3}\) is called a **definable set** if there is an \(\mathcal{L}_{\text{DP}}\)-formula \(\phi\) and \(M' \in \mathbb{N}\) such that \(X_F = \phi(F)\) for every \(F \in \text{Loc}_{M'}\).
2. We denote by \(\text{VF}\) and \(\text{RF}\) the definable sets \((F)_{F \in \text{Loc}}\) and \((k_F)_{F \in \text{Loc}}\) respectively.
3. Let \(X\) and \(Y\) be definable sets. A **definable function** is a collection \(f = (f_F)_{F \in \text{Loc}_{M}}\) of functions \(f_F : X_F \to Y_F\), such that the collection of their graphs \(\{\Gamma_f\}_{F \in \text{Loc}_{M}}\) is a definable set.
4. Let \(X\) be a definable set. A collection \(h = (h_F)_{F \in \text{Loc}_{M}}\) of functions \(h_F : X_F \to \mathbb{R}\) is called a **motivic** (or **constructible**) function on \(X\), if there exists \(M' \in \mathbb{N}\) such that for all \(F \in \text{Loc}_{M'}\) it can be written in the following way (for every \(x \in X_F\)):

\[
h_F(x) = \sum_{i=1}^{N} |Y_{i,F,x}| q_F^{a_{i,F}(x)} \left( \prod_{j=1}^{N'} \beta_{i,j,F}(x) \right) \left( \prod_{l=1}^{N''} \frac{1}{1 - q_F^{a_{i,l}}} \right),
\]

where,

- \(N, N'\) and \(N''\) are integers and \(a_{i,l}\) are non-zero integers.
- \(\alpha_i : X \to \mathbb{Z}\) and \(\beta_{i,j} : X \to \mathbb{Z}\) are definable functions.
- \(Y_{i,F,x} = \{y \in k_F^{r_i} : (x, y) \in Y_{i,F}\}\) is the fiber over \(x\) where \(Y_i \subseteq X \times \text{RF}^{r_i}\) are definable sets and \(r_i \in \mathbb{N}\).
- The integer \(q_F\) is the size of the residue field \(k_F\).

The set of motivic functions on a definable set \(X\) forms a ring, which we denote by \(\mathcal{C}(X)\).

2.1.4 Integration of motivic functions

We want to show that the ring of motivic functions is preserved under integration. In order to do so, we first need to explain what does it mean to integrate a motivic function.

Take \(\{\mu_{1,F}\}_{F \in \text{Loc}}\) to be the family of normalized Haar measures on \(\text{VF}\), that is \((\mu_{1,F}(\mathcal{O}_F)) = 1\) for every \(F \in \text{Loc}\), and let \(\{\mu_{2,F}\}_{F \in \text{Loc}}\) and \(\{\mu_{3,F}\}_{F \in \text{Loc}}\) be
the families of counting measures on RF and Z respectively. Given a definable set \( X \subseteq \mathbb{V}F^n_1 \times \mathbb{R}F^n_2 \times \mathbb{Z}^n_3 \), we can consider the family of measures \((\mu_F)_{F \in \mathcal{L}_{\text{Loc}}} \) induced from the collection \((\mu^{n_1}_{1,F} \times \mu^{n_2}_{2,F} \times \mu^{n_3}_{3,F})_{F}\) for each \( F \in \mathcal{L}_{\text{Loc}} \). We call measures such as \((\mu_F)_{F \in \mathcal{L}_{\text{Loc}}} \) motivic measures. For other ways to obtain motivic measures, see [11, Section 8], [9, Section 2.3] or [8, Section 2.5]. Using \((\mu_F)_{F \in \mathcal{L}_{\text{Loc}}} \), we can integrate motivic functions. The following theorem shows that the ring of motivic functions is preserved under integration with respect to any motivic measure:

**Theorem 2.2** (See [7, Theorem 4.3.1]) Let \( X \) and \( Y \) be \( \mathcal{L}_{\text{DP}} \)-definable sets, \( f \) be in \( \mathcal{C}(X \times Y) \) and let \( \mu \) be a motivic measure on \( Y \). Then there exist a function \( g \in \mathcal{C}(X) \) and an integer \( M > 0 \) such that for every \( F \in \mathcal{L}_{\text{Loc}} \) and \( x_0 \in X_F \), if \( f_F(x_0, y) \in L^1(Y_F) \) then

\[
g_F(x) = \int_{Y_F} f_F(x, y) d\mu_F.
\]

**Remark 2.3** In Sect. 3.3 we extend the definition of motivic functions and motivic measures to smooth algebraic \( \mathbb{Q} \)-varieties.

### 2.2 Elimination of quantifiers and uniform cell decomposition

In this subsection we state a quantifier elimination result and a uniform cell decomposition theorem for the \( \mathcal{L}_{\text{DP}} \)-theory \( T_{H,\text{ac},0} \) which is defined below, and a Presburger cell decomposition theorem.

#### 2.2.1 Uniform cell decomposition in \( T_{H,\text{ac},0} \)

Denote by \( T_{H,\text{ac},0} \) the \( \mathcal{L}_{\text{DP}} \)-theory of Henselian valued fields \( K \) of residue characteristic zero such that there is an angular component map \( \text{ac} : K \to k_K \). The theory \( T_{H,\text{ac},0} \) has quantifier elimination (in the valued field sort), and there is a uniform cell decomposition theorem. This allows us, using specialization arguments, to obtain uniform results about non-Archimedean local fields with residue characteristic large enough:

**Lemma 2.4** (See e.g. [4, Lemma 3.5]) Let \( \phi \) be a sentence in \( \mathcal{L}_{\text{DP}} \). Assume that \( \phi \) holds in all models of \( T_{H,\text{ac},0} \), then there exists \( M = M(\phi) \in \mathbb{N} \) such that \( \phi \) holds in all \( F \in \mathcal{L}_{\text{Loc}}_M \).

We now wish to state the uniform cell decomposition theorem. Let \( f_i : \mathbb{V}F^m \times VF \to VF \) be functions who are polynomial in their second variable, and whose coefficients are definable functions in the first variable. The uniform cell decomposition theorem allows us to decompose an \( \mathcal{L}_{\text{DP}} \)-definable set in \( \mathbb{V}F^m \times VF \) into a disjoint union of smaller \( \mathcal{L}_{\text{DP}} \)-definable sets called cells, such that on each cell \( \text{val}(f_i) \) and \( \text{ac}(f_i) \) have simpler description, which depends on one less valued field variable.

For the sake of consistency and in order to avoid confusion, we follow the definitions and notations of [4, Section 3] and [25] with only minor changes. We start with the definition of a cell:
Definition 2.5 (Cells, see [4, Definition 3.1] or [25, Definition 2.9])

1. Let $y \in VF$ and $x = (x_1, \ldots, x_m) \in VF^m$ be valued field sort variables, $\xi = (\xi_1, \ldots, \xi_n) \in RF^n$ be residue field sort variables and let $\lambda \in \mathbb{Z}_{>0}$. Furthermore, take a definable subset $C \subseteq VF^m \times RF^n$, and definable functions $b_1, b_2, c : C \to VF$. We also denote by $\square_1, \square_2$ the relations $<, \leq$ or no condition. For each $\xi \in RF^n$, let $A(\xi)$ denote the definable set

$$\{ (x, y) \in VF^m \times VF : (x, \xi) \in C \land \mathrm{val}(b_1(x, \xi)) \square_1 \lambda \cdot \mathrm{val}(y - c(x, \xi)) \square_2 \mathrm{val}(b_2(x, \xi)) \land \mathrm{ac}(y - c(x, \xi)) = \xi_1 \}.$$ 

2. Suppose that the definable sets $A(\xi)$ are distinct; then $A = \bigcup_{\xi \in RF^n} A(\xi)$ is called a cell in $VF^m \times VF$ with parameters $\xi$ and center $c$ and we call $A(\xi)$ a fiber of the cell $A$. Furthermore, we denote by $\Theta(A) = (C, b_1(x, \xi), b_2(x, \xi), c(x, \xi), \lambda)$ the datum of the cell $A$.

Definition 2.6 (Uniform cell decomposition, see [4, Definition 3.2] or [25, Theorem 3.2]) Let $y \in VF$ and $x = (x_1, \ldots, x_m) \in VF^m$, and take $f_1(x, y), \ldots, f_r(x, y)$ to be polynomials in $y$ whose coefficients are definable functions in $x$, i.e. expressions of the form $\sum_k a_k(x)y^k$ where each $a_k$ is a definable function. We say that a collection of valued fields $\mathcal{F}$ have a uniform cell decomposition of dimension $m$ with respect to the functions $\{f_i\}_{i=1}^r$ if, for some positive integer $N$, there exist the following:

- A non-negative integer $n$ and definable sets $C_i \subseteq VF^m \times RF^n$,
- Definable functions $b_{i,1}(x, \xi), b_{i,2}(x, \xi), c_i(x, \xi)$ and $h_{ij}(x, \xi)$ from $VF^m \times RF^n$ into $VF$,
- Positive integers $\lambda_i$,
- Non-negative integers $w_{ij}$,
- Functions $\mu_i : \{1, \ldots, r\} \to \{1, \ldots, n\}$, where $1 \leq i \leq N$ and $1 \leq j \leq r$,

such that,

- For any $F \in \mathcal{F}$, we have $F^m \times F = \bigcup_{i=1}^N A_{i,F}$, where the cells $A_i$ are defined by the cell data $\Theta(A_i) = (C_i, b_{i,1}, b_{i,2}, c_i, \lambda_i)$ for $1 \leq i \leq N$.
- For every $1 \leq i \leq N$ and $1 \leq j \leq r$, and all $(x, y) \in A_i(\xi)$, we have

$$\mathrm{val}(f_j(x, y)) = \mathrm{val}(h_{ij}(x, \xi)(y - c(x, \xi))^{w_{ij}}) \land \mathrm{ac}(f_j(x, y)) = \xi_{\mu_i(j)},$$

where the integers $w_{ij}$ and the maps $\mu_i$ do not depend on $x, \xi$ and $y$.

The following uniform cell decomposition theorem was proved by Pas for the $\mathcal{L}_{\text{DP}}$ language:

Theorem 2.7 (Uniform cell decomposition theorem, see [4, Theorems 3.3, 3.6] or [25, Theorems 3.1–3.2, Remark 3.3]) Let $y$ and $x = (x_1, \ldots, x_m)$ be valued field sort variables, and $f_1(x, y), \ldots, f_r(x, y)$ be polynomials in $y$ whose coefficients are definable functions in $x$. Then,

1. The class of all models of $T_{\text{H}, \text{ac}, 0}$ has uniform cell decomposition of dimension $m$ with respect to the functions $\{f_i\}_{i=1}^r$.
2. There is a constant $M = M(m; f_1, \ldots, f_r)$ such that $\text{Loc}_M$ has uniform cell decomposition of dimension $m$ with respect to the functions $\{f_i\}_{i=1}^r$. 

2.2.2 Elimination of quantifiers in $T_{\text{H,ac,0}}$

The following theorem is due to Denef and Pas:

**Theorem 2.8** (Quantifier elimination theorem, see [4, Theorem 3.4], [11, Theorem 2.1.1] or [25, Theorem 4.1]) The theory $T_{\text{H,ac,0}}$ admits elimination of quantifiers in the valued field sort. More explicitly, every $L_{\text{DP}}$-formula $\psi(x, \xi, k) \subseteq VF^{n_1} \times RF^{n_2} \times VG^{n_3}$ is $T_{\text{H,ac,0}}$-equivalent to a finite disjunction of formulas of the form

$$\chi(\text{ac}(g_1(x)), \ldots, \text{ac}(g_s(x)), \xi) \land \theta(\text{val}(g_1(x)), \ldots, \text{val}(g_s(x)), k)$$

where $\chi$ is an $L_{\text{Res}}$ formula, $\theta$ is an $L_{\text{Pres}}$ formula and $g_i \in \mathbb{Z}[x_1, \ldots, x_{n_1}]$.

**Notation 2.9** Given an $L_{\text{DP}}$-formula $\psi$, we write $/X_{i=1}^r(\psi) = (\{g_j\}_{j=1}^s, \{\chi_i\}_{i=1}^r, \{\theta_i\}_{i=1}^r)$ for its data, where $\psi$ is equivalent to a formula

$$\bigvee_{i=1}^r \chi_i(\text{ac}(g_1(x)), \ldots, \text{ac}(g_s(x)), \xi) \land \theta_i(\text{val}(g_1(x)), \ldots, \text{val}(g_s(x)), k),$$

$x, \xi, k, \{g_j\}_{j=1}^s$ are as in Theorem 2.8, $\{\theta_i\}_{i=1}^r$ are $L_{\text{Pres}}^\infty$-formulas and $\{\chi_i\}_{i=1}^r$ are $L_{\text{Res}}$-formulas.

2.2.3 Presburger cell decomposition

We conclude our discussion of cell decomposition results with a result in the Presburger language. We begin with a definition.

**Definition 2.10** (See [5, Definition 1]) Let $X \subseteq VG^m$ be an $L_{\text{Pres}}$-definable set.

1. We call a definable function $f : X \to VG$ linear if there is a constant $\gamma \in VG$ and integers $a_i$ and $0 \leq c_i < n_i$ for $i = 1, \ldots, m$ such that $x_i - c_i \equiv 0 \mod n_i$ and $f(x) = \sum_{i=1}^m a_i \left(\frac{x_i - c_i}{n_i}\right) + \gamma$ for every $x \in X$.
2. We call a definable function $f : X \to VG$ piecewise linear if there exists a finite partition of $X = \bigcup_{i=1}^N A_i$, such that $f|_{A_i}$ is linear for all $i$.

The following cell decomposition result will be used in the proof of Theorem 5.2:

**Theorem 2.11** (Presburger cell decomposition, see [5, Theorem 1]) Let $X \subseteq VG^m$ and $f : X \to VG$ be $L_{\text{Pres}}$-definable. Then there exists a finite partition $P$ of $X$ into cells (see [5, Definition 2]), such that the restriction $f|_A : A \to VG$ is linear for each cell $A \in P$.

2.3 Resolution of singularities, rational singularities and the (FRS) property

In this section we recall necessary notions from algebraic geometry, as well as define the (FRS) property and state an analytic criterion for the (FRS) property.
2.3.1 Resolution of singularities, rational singularities and definition of the (FRS) property

Definition 2.12 Let $X$ be an algebraic variety over a field $K$.

1. A resolution of singularities of $X$ is a proper map $p : \tilde{X} \to X$ such that $\tilde{X}$ is smooth and $p$ is a birational equivalence. A strong resolution of singularities of $X$ is a resolution of singularities $p : \tilde{X} \to X$ which is an isomorphism over the smooth locus of $X$, denoted $X^{\text{sm}}$. It is a theorem of Hironaka [18,19], that any $X$ over a field $K$ of characteristic zero admits a strong resolution of singularities.

2. (See [21, I.3 pages 50–51] or [1, Definition 6.1]) We say that $X$ has rational singularities if for any (or equivalently, for some) resolution of singularities $p : \tilde{X} \to X$, the natural morphism $O_X \to Rp_* (O_{\tilde{X}})$ is a quasi-isomorphism, where $Rp_* (\cdot)$ is the higher direct image functor.

We now define the notion of an (FRS) map, which is the relative version (i.e. for morphisms) of having rational singularities:

Definition 2.13 ([1, Section 1.2.1, Definition II]) Let $\varphi : X \to Y$ be a morphism between smooth varieties $X$ and $Y$.

1. We say that $\varphi : X \to Y$ is (FRS) at $x \in X(K)$ if it is flat at $x$, and there exists an open $x \in U \subseteq X$ such that $U \times_Y \{\varphi(x)\}$ is reduced and has rational singularities.

2. We say that $\varphi : X \to Y$ is (FRS) if it is flat and it is (FRS) at $x$ for all $x \in X(\bar{K})$.

A useful theorem is given in [14], and implies in particular that the (FRS)-locus of a morphism is open.

Theorem 2.14 (See [1, Theorem 6.3] or [14, Theorem 4.5]) Let $\varphi : X \to S$ be a flat morphism of finite type $K$-schemes and let $x \in X$ be such that $\varphi(x)$ is a rational singularity in $S$. Assume that $x$ is a rational singularity of its fiber $\varphi^{-1}(\varphi(x))$, then we have the following:

1. $x$ is a rational singularity in $X$.

2. The set $\{x \in X(K) : x$ is a rational singularity of $\varphi^{-1}(\varphi(x))\}$ is open in $X(K)$.

2.3.2 Measures on $p$-adic analytic varieties and an analytic criterion for the (FRS) property

Let $X$ be a $d$-dimensional smooth algebraic variety over $K$. We denote by $\Omega^r_X$ the sheaf of differential $r$-forms on $X$ and by $\Omega^r_X[X]$ (resp. $\Omega^r_X(X)$) the regular $r$-forms (resp. rational $r$-forms). Given a non-Archimedean local field $F \supseteq K$, then $X(F)$ has a structure of an $F$-analytic manifold.

For $\omega \in \Omega^{\text{top}}_X(X)$, we can define a measure $|\omega|_F$ on $X(F)$ as follows. Let $U \subseteq X(F)$ be a compact open set and let $\phi$ be an $F$-analytic diffeomorphism from an open subset $W \subseteq F^d$ to $U$. We can write $\phi^* \omega = g dx_1 \wedge \cdots \wedge dx_d$, for some $g : W \to F$, and define

$$|\omega|_F (U) = \int_W |g|_F d\lambda,$$
where $|·|_F$ is the normalized absolute value on $F$ and $\lambda$ is the normalized Haar measure on $F^d$ (i.e. $\lambda(O_F^d) = 1$). Note that this definition is independent of the diffeomorphism $\phi$, and that the measure $|\omega|_F$ obtained in this way is unique after fixing $\omega$.

**Definition 2.15** Let $X$ be as above.

1. A measure $\mu$ on $X(F)$ is called smooth if every point $x \in X(F)$ has an analytic neighborhood $U$ and an $(F$-analytic) diffeomorphism $\phi : U \to O_F^d$ such that $\phi^\ast \mu|_U$ is a Haar measure on $O_F^d$.
2. A measure on $X(F)$ is called Schwartz if it is smooth and compactly supported.
3. A measure $\mu$ on $X(F)$ has continuous density, if there is a smooth measure $\tilde{\mu}$ and a continuous function $f : X(F) \to \mathbb{C}$ such that $\mu = f \cdot \tilde{\mu}$.

Schwartz measures and measures with continuous density can be characterized in the following way:

**Proposition 2.16** ([1, Proposition 3.3]) Let $X$ be a smooth variety over a non-Archimedean local field $F$.

1. A measure $\mu$ on $X(F)$ is Schwartz if and only if it is a linear combination of measures of the form $f|\omega|_F$, where $f$ is a locally constant and compactly supported function on $X(F)$, and $\omega \in \Omega_1^{\text{top}}(X)$ has no zeroes or poles in the support of $f$.
2. A measure $\mu$ on $X(F)$ has continuous density if and only if for every point $x \in X(F)$ there is a neighborhood $U$ of $x$, a continuous function $f : U \to \mathbb{C}$, and $\omega \in \Omega_1^{\text{top}}(X)$ with no poles in $U$ such that $\mu = f|\omega|_F$ on $U$.

We can now state an analytic criterion which is equivalent to the (FRS) property. It is often much easier to use this criterion (specifically the third condition) than to use Definition 2.13 directly.

**Theorem 2.17** ([1, Theorem 3.4]) Let $\phi : X \to Y$ be a map between smooth algebraic varieties defined over a finitely generated field $K$ of characteristic 0, and let $x \in X(K)$. Then the following conditions are equivalent:

1. $\phi$ is (FRS) at $x$.
2. There exists a Zariski open neighborhood $x \in U \subseteq X$ such that, for any non-Archimedean local field $F \supseteq K$ and any Schwartz measure $\mu$ on $U(F)$, the measure $(\phi|_{U(F)})_\ast(\mu)$ has continuous density.
3. For any finite extension $K'/K$, there exists a non-Archimedean local field $F \supseteq K'$ and a non-negative Schwartz measure $\mu$ on $X(F)$ that does not vanish at $x$ such that $(\phi|_{X(F)})_\ast(\mu)$ has continuous density.

3 Reduction of Theorem 1.7 to an analytic problem

3.1 Convolution of morphisms preserves smoothness properties

We would like to show that the convolution operation preserves certain properties of morphisms, and in particular that it preserves the (FRS) property (see Definition 1.5). We use the following proposition:
Proposition 3.1 Let $X$ and $Y$ be varieties over a field $K$, let $G$ an algebraic $K$-group and let $S$ be a property of morphisms that is preserved under base change and compositions. If $\psi : Y \to G$ is arbitrary, $\varphi : X \to G$ is a morphism that satisfies property $S$, and the natural map $i_K : Y \to \text{Spec}(K)$ has property $S$, then $\varphi \ast \psi$ has property $S$.

Proof Since $i_K$ satisfies $S$ and $S$ is preserved under base change, the projection to the first coordinate $\pi_G : G \times Y \to G$ satisfies $S$. Now consider the following fibered diagram:

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\pi_X} & X \\
\downarrow \alpha & & \downarrow \varphi, \\
G \times Y & \xrightarrow{\beta} & G
\end{array}
$$

where $\alpha(x, y) = (\varphi \ast \psi(x, y), y)$ and $\beta(g, y) = g \cdot \psi(y)^{-1}$. This implies that $\alpha$ satisfies $S$, and since $\varphi \ast \psi = \pi_G \circ \alpha$ we are done. \qed

Proposition 3.2 (The (FRS) property is preserved under compositions and base change) Let $X$, $Y$ and $Z$ be smooth $K$-varieties, and let $\varphi : X \to Y$ be an (FRS) morphism.

1. If $\psi : Y \to Z$ is (FRS), then $\psi \circ \varphi$ is (FRS).
2. Consider the following base change diagram,

$$
\begin{array}{ccc}
X \times Y \times Z & \xrightarrow{\tilde{\psi}} & Z \\
\downarrow & & \downarrow \psi \\
X & \xrightarrow{\varphi} & Y
\end{array}
$$

where $\psi : Z \to Y$ is arbitrary. Then $\tilde{\psi}$ is (FRS).

Proof

1. Since flatness is preserved by compositions, we have that $\psi \circ \varphi$ is flat. As a consequence, for any $z \in Z$ the fiber $X_z := (\psi \circ \varphi)^{-1}(z)$ is a local complete intersection scheme, and in particular Cohen–Macaulay. By the $(S_1 + R_0)$-criterion (see e.g. [26, Lemma 10.151.3]), in order to show that $X_z$ is reduced it is enough to show that $X_z$ is generically reduced, or equivalently, that its non-smooth locus is of codimension $\geq 1$. By [17, III.10.2], since $\psi \circ \varphi$ is flat, the smooth locus of $X_z$ is equal to the set $X_{z,\text{sm},\psi \circ \varphi} := \{x \in X_z : \psi \circ \varphi \text{ is smooth at } x\}$. Thus, we would like to show that $X_{z,\text{sm},\psi \circ \varphi}$ is dense in $X_z$. As above, define $X_y, Y_z$ and $X_{y,\text{sm},\varphi}, Y_{z,\text{sm},\psi}$ for any $y \in Y$ and $z \in Z$. Since $\psi$ and $\varphi$ are (FRS), we have that $Y_{z,\text{sm},\psi}$ is dense in $Y_z$ and $X_{y,\text{sm},\varphi}$ is dense in $X_y$ for any $y \in Y$. Since smoothness of morphisms is preserved under composition, we have $X_{z,\text{sm},\psi \circ \varphi} \supseteq \bigcup_{y \in Y_{z,\text{sm},\psi}} X_{y,\text{sm},\varphi}$. Now let $U$ be an open set in $X_z$. By flatness, $\varphi^{-1}(Y_{z,\text{sm},\psi})$ is open and dense in $X_z$, thus there exists $y \in Y_{z,\text{sm},\psi}$ such that $U \cap X_y$ is a non-empty open subset of $X_y$, and thus
$U \cap X_{Y}^{\text{sm}, \varphi}$ is non empty. This shows that $X_{z}^{\text{sm}, \psi \circ \varphi}$ is dense in $X_{z}$, and hence $X_{z}$ is reduced. We therefore showed that $\psi \circ \varphi$ is flat, with reduced fibers.

Now for $z \in Z$, let $x \in X_{z}$ and consider the map $\varphi|_{X_{z}} : X_{z} \to Y_{z}$. By our assumption, $y := \varphi(x)$ is a rational singularity of $Y_{z}$ and $x$ is a rational singularity of $X_{y}$. Since $\varphi$ is flat and $\varphi|_{X_{z}}$ is a base change of $\varphi$, it follows that $\varphi|_{X_{z}}$ is flat as well. By Theorem 2.14, $x$ is a rational singularity of $X_{z}$. Hence, the fibers of $\psi \circ \varphi$ have rational singularities and we are done.

2. First, notice that the fibers of $\tilde{\varphi}$ are the base change of the fibers of $\varphi$. Indeed, for every $y \in Y$ and $z \in Z$ such that $\psi(z) = y$, we have:

$$\text{Spec}(K(\{z\})) \times_{Z} (Z \times_{Y} X) \simeq \text{Spec}(K(\{z\})) \times_{Y} X$$

$$\simeq \text{Spec}(K(\{z\})) \times_{\text{Spec}(K(\{y\}))} (\text{Spec}(K(\{y\})) \times_{Y} X).$$

Since reduceness, having rational singularities and flatness are preserved under base change (recall that $\text{char}(K) = 0$), we deduce that the fibers of $\tilde{\varphi}$ are reduced and have rational singularities and that $\tilde{\varphi}$ is flat. Therefore that $\tilde{\varphi}$ is (FRS).

By Propositions 3.1 and 3.2 above, it is immediate that the convolution operation preserves the (FRS) property. The same holds for dominance, flatness and smoothness.

**Corollary 3.3** Let $G$ be an algebraic $K$-group, and suppose that $\varphi : X \to G$ is (FRS) (resp. dominant/flat/smooth) and let $\psi : Y \to G$ be any morphism. Then the morphisms $\varphi \ast \psi : X \times Y \to G$ and $\psi \ast \varphi : X \times Y \to G$ are (FRS) (resp. dominant/flat/smooth).

### 3.2 Reduction of Theorem 1.7 to the case of strongly dominant morphisms

We want to show that under reasonable assumptions, high enough convolution power of a given morphism yields a morphism whose restriction to every absolutely irreducible component is dominant. This will imply it is enough to prove Theorem 1.7 for such morphisms.

**Definition 3.4** Let $G$ be an algebraic group, let $\varphi : X \to G$ be a morphism of $K$-varieties and let $\{X_{i}\}_{i=1}^{l}$ be the absolutely irreducible components of $X$.

1. We say that $\varphi$ is generating if $\varphi(X) \not\subseteq gH$ for any algebraic subgroup $H \leq G$ and $g \in G(\overline{K})$.
2. We say that $\varphi$ is strongly generating if $\varphi|_{X_{i}}$ is generating for all $1 \leq i \leq l$.
3. We say that $\varphi$ is strongly dominant if $\varphi|_{X_{i}}$ is dominant for all $1 \leq i \leq l$.

**Proposition 3.5** Let $X$ be a smooth $K$-variety, $G$ be a commutative algebraic $K$-group and $\varphi : X \to G$ be a strongly generating morphism. Then there exists $n \in \mathbb{N}$ such that $\varphi^{n}$ is strongly dominant.

**Proof** Assume that $X$ is absolutely irreducible. By exchanging $\varphi$ with its translation by $g \in G(\overline{K})$, we may assume that $e \in \varphi(X)$. Set $U_{n} := \text{Im}(\varphi^{n})$ and note that
$U_n \subseteq U_{n+1}$ for all $n \in \mathbb{N}$. By dimension considerations, there exists $n_0 \in \mathbb{N}$ such that $\overline{U_n} = \overline{U_m}$ for all $m, n > n_0$, and in particular $\overline{U_n^2} = \overline{U_{2n}} = \overline{U_n} \subseteq \overline{U_n^2}$.

Now, since the multiplication map $m : G \times G \to G$ is an open map, by Chevalley’s theorem, $\overline{U_n^2}$ contains an open set in $\overline{U_n^2}$, and thus by the irreducibility of $\overline{U_n^2} = m(\text{Im} \varphi^n \times \text{Im} \varphi^n)$ we get that $\overline{U_n^2} = \overline{U_n^2}$. Setting $H := \overline{U_n}$ for $n$ large enough, we get that $H \cdot H = H$, so $H$ is a closed algebraic semigroup.

Given $h \in H(\overline{K})$, since $L_h : H \to H$ (left translation by $h$) is an injective map, by the Ax–Grothendieck theorem [3,16] we deduce that it is also surjective (over $\overline{K}$). As $e \in \text{Im} L_h$ for all $h \in H(\overline{K})$, we get that $H$ is an algebraic group. Our assumption implies that $H = G$, and hence $\varphi^n$ is dominant.

We now move to prove the general case. Let $\{X_i\}_{i=1}^l$ be the absolutely irreducible components of $X$. By the above argument, since $\varphi$ is strongly generating there exist $n_i \in \mathbb{N}$ such that $\varphi|_{X_i \times \cdots \times X_i}$ is dominant for all $i$. Set $n = \max_i\{n_i\}$, we claim that $\varphi^n$ is strongly dominant. Indeed, all the absolutely irreducible components of $X^n$ are of the form $X_{i_1} \times \cdots \times X_{i_l}$, where $1 \leq i_k \leq l$, and therefore there exists some $1 \leq j \leq l$ such that $X_j$ appears at least $n$ times in $X_{i_1} \times \cdots \times X_{i_l}$. Since dominance is preserved by convolution, and $G$ is commutative, we are done. \(\square\)

As a corollary, we get the desired reduction:

**Corollary 3.6** It is enough to prove Theorem 1.7 for $\varphi : X \to V$ strongly dominant.

### 3.3 Motivic measures on algebraic $\mathbb{Q}$-varieties

The goal of this subsection is to define the notion of a motivic measure on a smooth algebraic $\mathbb{Q}$-variety $X$. This will allow us to construct a collection of measures $\{\mu_p\}_{p \text{ prime}}$ on $\{X(\mathbb{Z}_p)\}_p$ which behave well with respect to pushforward under a strongly dominant map $\varphi : X \to V$. For such a collection $\{\mu_p\}_{p \text{ prime}}$, we will be able to show (in Sect. 5) that after sufficiently many self convolutions of $\varphi_*(\mu_p)$, we get a measure with continuous density, and that the number of convolutions required does not depend on $p$. Using Proposition 3.16, we will then deduce our main theorem (Theorem 1.7) for the case $K = \mathbb{Q}$.

Let $X$ be a reduced, finite type affine $\mathbb{Z}$-scheme. An embedding $\psi : X \hookrightarrow \mathbb{A}^N_\mathbb{Z}$ naturally gives rise to an $\mathcal{L}_{\text{DP}}$-definable subset $\{\psi(X)(F)\}_{F \in \text{Loc}}$ of $\text{VF}_N$. This allows us to define definable subsets and motivic functions on $\mathbb{Q}$-varieties.

**Definition 3.7**

1. Let $X$ be a finite type, affine $\mathbb{Z}$-scheme.

   (a) A collection $\{Y_F\}_{F \in \text{Loc}}$ of subsets $Y_F \subseteq X(F)$ is called a **definable subset** of $X$ if there exists an embedding $\psi : X \hookrightarrow \mathbb{A}^N_\mathbb{Z}$ such that $\{\psi(Y_F)\}_{F \in \text{Loc}}$ is a definable subset of $\{\psi(X)(F)\}_{F \in \text{Loc}}$.

   (b) A collection $h = (h_F)_F$ of functions $h_F : X(F) \to \mathbb{R}$ is called a **motivic function** on $X$, and denoted $h \in \mathcal{C}(X)$, if there exists an embedding $\psi : X \hookrightarrow \mathbb{A}^N_\mathbb{Z}$ and $f \in \mathcal{C}(\psi(X))$ such that $h = \psi^*(f)$. 
2. Let $X$ be a finite type $\mathbb{Z}$-scheme.

(a) A collection $\{Y_F\}_{F \in \text{Loc}}$ of subsets $Y_F \subseteq X(F)$ is called a \textit{definable subset} of $X$ if there exists an affine cover $X = \bigcup_{i=1}^l U_i$, with embeddings $\psi_i : U_i \leftrightarrow \mathbb{A}^N_{\mathbb{Z}}$, such that $\{\psi_i(U_i(F) \cap Y_F)\}_{F \in \text{Loc}}$ is a definable subset of $\psi_i(U_i)$ for all $1 \leq i \leq l$.

(b) A collection $h = (h_F)_F$ of functions $h_F : X(F) \rightarrow \mathbb{R}$ is called a \textit{motivic function} on $X$ if there exists an affine cover $X = \bigcup_{i=1}^l U_i$, with embeddings $\psi_i : U_i \leftrightarrow \mathbb{A}^N_{\mathbb{Z}}$, and a collection $f_1, \ldots, f_l$ where $f_i \in C(\psi_i(U_i))$ and $\psi_i^*(f_i) = h|_{U_i}$.

3. Let $X$ be an algebraic $\mathbb{Q}$-variety.

(a) A collection $\{Y_F\}_{F \in \text{Loc}}$ of subsets $Y_F \subseteq X(F)$ is called a \textit{definable subset} of $X$ if there exists a $\mathbb{Z}$-model $\widetilde{X}$ of $X$, such that $\{Y_F\}_{F \in \text{Loc}}$ is a definable subset of $\widetilde{X}$. We denote the set of definable subsets of $X$ by $\mathcal{D}(X)$.

(b) A collection $h = (h_F)_F$ of functions $h_F : X(F) \rightarrow \mathbb{R}$ is called a \textit{motivic function} on $X$, if there exists a $\mathbb{Z}$-model $\widetilde{X}$ of $X$ such that $h \in C(\widetilde{X})$.

Remark 3.8 Note that the notions above are independent of the embedding $\psi$ into affine space. Given two embeddings $\psi : X \leftrightarrow \mathbb{A}^N_{\mathbb{Z}}$ and $\psi' : X \leftrightarrow \mathbb{A}^N_{\mathbb{Z}}$, we have an algebraic $\mathbb{Z}$-isomorphism between $\psi(X)$ and $\psi'(X)$, which induces a definable isomorphism $\{\psi(X)_F\}_{F \in \text{Loc}} \sim \{\psi'(X)_F\}_{F \in \text{Loc}}$.

Lemma 3.9 Let $X$ be an algebraic $\mathbb{Q}$-variety, let $Y = \{Y_F\}_{F \in \text{Loc}}$ be a collection of subsets $Y_F \subseteq X(F)$ and let $h = (h_F)_F$ be a collection of functions $h_F : X(F) \rightarrow \mathbb{R}$.

1. $\{Y_F\}_{F \in \text{Loc}}$ is a definable subset of $X$ if and only if for any $\mathbb{Z}$-model $\widetilde{X}$ of $X$, we have that $\{Y_F\}_{F \in \text{Loc}}$ is a definable subset of $\widetilde{X}$.

2. $h = (h_F)_F$ is a motivic function on $X$ if and only if for any $\mathbb{Z}$-model $\widetilde{X}$ of $X$, we have that $h \in C(\widetilde{X})$.

Proof We prove (1), the proof for (2) is similar. For any two $\mathbb{Z}$-models $\widetilde{X}_1$ and $\widetilde{X}_2$ there exists a $\mathbb{Q}$-isomorphism $\varphi : \widetilde{X}_1 \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q}) \rightarrow \widetilde{X}_2 \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q})$. Let $Y = \{Y_F\}_{F \in \text{Loc}}$ be a definable subset of $\widetilde{X}_2$ and consider the collection $\{\varphi^{-1}(Y_F)\}_{F \in \text{Loc}}$, where $\varphi^{-1}(Y_F) \subseteq X(F)$. We want to show that $\{\varphi^{-1}(Y_F)\}_{F \in \text{Loc}}$ is a definable subset of $\widetilde{X}_1$. It is enough to prove for the case where $\widetilde{X}_1$ and $\widetilde{X}_2$ are affine with $\widetilde{X}_1 \leftrightarrow \mathbb{A}^N_{\mathbb{Z}}$ and $\widetilde{X}_2 \leftrightarrow \mathbb{A}^{N_2}_{\mathbb{Z}}$. By Theorem 2.8, $Y$ is defined by an $\mathcal{L}_{\text{DP}}$-formula $\widetilde{\phi}$ which is a disjunction of formulas of the form

$$
\chi_i(\text{ac}(g_1(x)), \ldots, \text{ac}(g_s(x))) \wedge \theta_i(\text{val}(g_1(x)), \ldots, \text{val}(g_s(x))),
$$

where $\chi_i$ is an $\mathcal{L}_{\text{Res}}$-formula, $\theta_i$ is an $\mathcal{L}_{\text{pres}}$-formula and $g_j \in \mathbb{Z}[x_1, \ldots, x_{N_2}]$. Notice that if the formula $\phi$ with data $\Sigma(\phi) = (\{g_j\}_{j=1}^s, \{\chi_i\}_{i=1}^r, \{\theta_i\}_{i=1}^s)$ (recall Notation 2.9) defines a set $Z \subseteq \widetilde{X}_2(F)$, then the formula $\phi^*\phi$ with data $\Sigma(\phi^*\phi) = (\{g_j \circ \varphi\}_{j=1}^s, \{\chi_i\}_{i=1}^r, \{\theta_i\}_{i=1}^s)$ defines the set $\varphi^{-1}(Z) \subseteq \widetilde{X}_1(F)$. Thus, our main candidate for a formula for $\{\varphi^{-1}(Y_F)\}_{F \in \text{Loc}}$ is $\phi^*\phi$.

1 Recall that a $\mathbb{Z}$-model of $X$ is a $\mathbb{Z}$-scheme $\widetilde{X}$ such that $\widetilde{X} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q}) \simeq X$. 

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The problem which arises is that \( \{g_j \circ \varphi\}_{j=1}^s \subset \mathbb{Q}[y_1, \ldots, y_{N_j}] \) do not necessarily have integral coefficients. In order to solve this, define \( \Xi(\phi') := \{(N \cdot g_j \circ \varphi)^{s}_{j=1}, \{\chi_i\}_{i=1}^s, \{\theta_i\}_{i=1}^s\} \) for \( N \in \mathbb{N} \) large enough such that \( N \cdot g_j \circ \varphi \in \mathbb{Z}[y_1, \ldots, y_{N_j}] \) for any \( j \). Notice that for \( M \in \mathbb{N} \) large enough, we have \( ac\left(N \cdot g_j \circ \varphi(x)\right) = N \cdot ac\left(g_j \circ \varphi(x)\right) \) and \( val\left(N \cdot g_j \circ \varphi(x)\right) = val\left(g_j \circ \varphi(x)\right) \) for any \( F \in \text{Loc}_M \), so we only need to take care of \( \{\chi_i\}_{i=1}^s \).

It is left to show that for any \( \mathcal{L}_{\text{Res}} \)-formula \( \chi(t_1, \ldots, t_s) \) there exists an \( \mathcal{L}_{\text{Res}} \)-formula \( \chi' \) such that \( \chi'(t_1, \ldots, t_s) = \chi(N \cdot t_1, \ldots, N \cdot t_s) \), and then we are done, by setting \( \Xi(\phi'') := \{(N \cdot g_j \circ \varphi)^{s}_{j=1}, \{\chi_i\}_{i=1}^s, \{\theta_i\}_{i=1}^s\} \), and observing that \( \phi'' \) defines \( \{\varphi^{-1}(Y_F)\}_{F \in \text{Loc}} \).

Firstly, every \( \mathcal{L}_{\text{Res}} \)-formula \( \chi(t_1, \ldots, t_s) \) is defined by zeros of polynomials \( P_j(t_1, \ldots, t_s) \), and possibly has quantifiers. Let \( D_j \) be the maximal degree in each \( P_j \) and consider the polynomials \( \tilde{P}_j \) which are obtained by replacing each monomial \( M_{ij} \) in \( P_j \) by \( N^{D_j} \cdot M_{ij} \cdot M_{ij} \). Now, notice that \( \tilde{P}_j(N \cdot t_1, \ldots, N \cdot t_s) = N^{D_j} \cdot P_j(t_1, \ldots, t_s) \), and define \( \chi' \) by replacing each \( P_j \) by \( \tilde{P}_j \) in \( \chi \). It is easy to see that \( \chi' \) satisfies \( \chi'(t_1, \ldots, t_s) = \chi(N \cdot t_1, \ldots, N \cdot t_s) \) and we are done. \( \square \)

As a conclusion, we can pull back definable sets and motivic functions with respect to \( \mathbb{Q} \)-morphisms.

**Lemma 3.10** Let \( X \) and \( Y \) be two algebraic \( \mathbb{Q} \)-varieties and let \( \varphi : X \to Y \) be a \( \mathbb{Q} \)-morphism. Then for any definable subset \( \{Z_F\}_{F \in \text{Loc}} \) of \( Y \) we have that \( \{\varphi^{-1}(Z_F)\}_{F \in \text{Loc}} \) is a definable subset of \( X \) and for any \( f \in \mathcal{C}(Y) \) we have \( f \circ \varphi \in \mathcal{C}(X) \).

**Proof** We may assume that \( \varphi : X \to Y \) is defined over \( S^{-1}\mathbb{Z} \), the localization of \( \mathbb{Z} \) by a finite set of primes \( S \). Since \( S^{-1}\mathbb{Z} \) is of finite type over \( \mathbb{Z} \) we may choose a \( \mathbb{Z} \)-model \( \tilde{\varphi} : \tilde{X} \to \tilde{Y} \) of the morphism \( \varphi \) (which includes \( \mathbb{Z} \)-models of \( X \) and \( Y \)). Since in the setting of the \( \mathcal{L}_{\text{DP}} \) language pullbacks are well defined, by reducing to the affine case, we have well defined pullbacks \( \tilde{\varphi}^* : \mathcal{C}(\tilde{Y}) \to \mathcal{C}(\tilde{X}) \) and \( \tilde{\varphi}^* : \mathcal{D}(\tilde{Y}) \to \mathcal{D}(\tilde{X}) \).

Since \( \tilde{X} \times_{\text{Spec}(\mathbb{Q})} \text{Spec}(\mathbb{Q}) \cong X \) and \( \tilde{Y} \times_{\text{Spec}(\mathbb{Q})} \text{Spec}(\mathbb{Q}) \cong Y \), and since \( \mathbb{Q} \subset F \) for any \( F \in \text{Loc} \), there are identifications \( \tilde{X}(F) \cong X(F) \) and \( \tilde{Y}(F) \cong Y(F) \), under which \( \varphi \) and \( \tilde{\varphi} \) induce the same map \( X(F) \to Y(F) \). This implies the lemma. \( \square \)

The next lemma follows easily by reducing to the affine case and choosing a \( \mathbb{Z} \)-model.

**Lemma 3.11** Let \( X \) be an algebraic \( \mathbb{Q} \)-variety.

1. Any \( \mathbb{Q} \)-subvariety \( Y \subseteq X \) is definable.
2. \( \mathcal{D}(X) \) is closed under intersections, unions and complements.

**Definition 3.12** Let \( X \) be a smooth algebraic \( \mathbb{Q} \)-variety. We say that a collection of measures \( \mu = \{\mu_F\}_{F \in \text{Loc}} \) on \( \{X(F)\}_{F \in \text{Loc}} \) is a *motivic* measure on \( X \) if there exists an open affine cover \( X = \bigcup_{j=1}^l U_j \), such that \( \mu_F|_{U_j(F)} = (f_j)_F \left|_{\omega_j} \right|_F \), where \( f_j \in \mathcal{C}(U_j) \) and \( \omega_j \) is a non-vanishing top form on \( U_j \).

**Lemma 3.13** Let \( X \) be a smooth algebraic \( \mathbb{Q} \)-variety and let \( \mu = \{\mu_F\}_{F \in \text{Loc}} \) be a collection of measures on \( \{X(F)\}_{F \in \text{Loc}} \). Then \( \mu \) is motivic if and only if there exists an open affine cover \( X = \bigcup_{j=1}^l U_j \), such that \( \mu \) can be written as \( \mu_F := \sum_{j=1}^l f_{j,F} \cdot \left|\omega_j\right|_F \) for some \( f_j \in \mathcal{C}(U_j) \) and \( \omega_j \) non-vanishing top forms on \( U_j \) respectively.
Let $X$ be a smooth algebraic $\mathbb{Q}$-variety, then there exists a motivic measure $\mu = \{\mu_F\}_{F \in \text{Loc}}$ on $X$, such that for every $F \in \text{Loc}$, $\mu_F$ is a non-negative Schwartz measure and $\text{supp}(\mu_F) = X(O_F)$.

Choose an affine open cover $X = \bigcup_{i=1}^l U_i$, with embeddings $\psi_i : U_i \hookrightarrow \mathbb{A}_{\mathbb{Q}}^{N_i}$ and non-vanishing top forms $\omega_j$ on $U_i$ (it is possible since $X$ is smooth). We can construct a disjoint open cover $\{V_i\}_{i=1}^{l-1}$ of $X(O_F)$ by setting $V_1 = U_1(O_F)$ and $V_i = U_i(O_F) \setminus \bigcup_{j=1}^{i-1} U_j(O_F)$.

Define the measure $\mu_F := \sum_{i=1}^l 1_{V_i} \cdot |\omega_i|_F$. Then by Proposition 2.16, $\mu_F$ is a Schwartz measure and it is clearly non-negative and supported on $X(O_F)$ for any $F \in \text{Loc}$. It is left to show that $\{\mu_F\}_{F \in \text{Loc}}$ is motivic. By Lemmas 3.11 and 3.13, it is enough to show that $U_i(O_F)$ is a definable subset of $U_i$ for each $i$.

Choose $\mathbb{Z}$-models $\tilde{U}_i$ of $U_i$ and $\tilde{\psi}_i : \tilde{U}_i \hookrightarrow \mathbb{A}_{\mathbb{Z}}^{N_i}$ of $\psi_i$, and let

$$B_1^{N_i}(0) = \{\tilde{y} \in VF_{N_i} : \text{val}(y_j) \geq 0 \text{ for } 0 \leq j \leq N_i\}$$

be the $L_{\text{DP}}$-definable set whose points over $F \in \text{Loc}$ are the unit disc in $F_{N_i}$. Clearly, the set

$$\left\{\tilde{\psi}_i(\tilde{U}_i)(F) \cap B_1^{N_i}(0)(F)\right\}_{F \in \text{Loc}}$$

is $L_{\text{DP}}$-definable for any $i$. This proves the proposition. \hfill \Box

### 3.4 Reduction to an analytic problem

Our next goal is to use the equivalent characterization of the (FRS) property given in Theorem 2.17 in order to reduce our main problem, as stated in Theorem 1.7 (with $K = \mathbb{Q}$), to an analytic question (see Proposition 3.16). To do so, we need the following lemma:

**Lemma 3.15** Let $X$ be a smooth algebraic $\mathbb{Q}$-variety and let $x_1, \ldots, x_k \in X(\mathbb{Q})$. Then for any finite extension $K / \mathbb{Q}$ such that $x_1, \ldots, x_k \in X(K)$ there exist infinitely many prime numbers $p$ with $i_p : K \hookrightarrow \mathbb{Q}_p$ such that $i_p(x_1), \ldots, i_p(x_k) \in X(\mathbb{Z}_p)$.

**Proof** Let $K$ be such that $x_1, \ldots, x_k \in X(K)$ and let $K'$ be the Galois closure of $K$. We can write $K' = \mathbb{Q}(\alpha)$ where $\alpha$ is a root of a monic minimal polynomial
\(x^r + \sum_{i=0}^{r-1} a_i x^i = q(x) \in \mathbb{Z}[x]\). Denote \(\overline{q}(x)\) for the reduction modulo \(p\) of \(q(x)\), then by a conclusion of Chebotarev’s density theorem, \(\overline{q}(x)\) splits in \(\mathbb{F}_p\) for infinitely many primes \(p\). Set

\[S_N = \{p \text{ prime} : \overline{q}(x) \text{ splits in } \mathbb{F}_p[x] \text{ and } p > N\}\]

where \(N = r \max_i |a_i|\), then \(\overline{q}(x)\) is separable in \(\mathbb{F}_p[x]\) for every \(p \in S_N\).

Now, use Hensel’s lemma for each \(p \in S_N\) to lift a root of \(\overline{q}(x) \in \mathbb{F}_p[x]\) to a root \(\alpha'\) of \(q(x)\) in \(\mathbb{Z}_p\). This gives rise to an embedding \(i_p : K' \hookrightarrow \mathbb{Q}_p\) by \(\alpha \mapsto \alpha'\), where \(\alpha\) maps to \(\mathbb{Z}_p\) as \(\alpha'\) lies in \(\mathbb{Z}_p\), which in turn gives rise to a map \(i_{p*} : X(K') \rightarrow X(\mathbb{Q}_p)\).

Finally, for each \(x_i \in X(K) \subset X(K')\) take an open affine neighborhood \(U_i\) of \(x_i\) and let \(\psi : U_i \hookrightarrow \mathbb{A}^{N_i}_{K'}\) be a closed embedding, for some \(N_i \in \mathbb{N}\). We may write each \(x_i \in U_i(K') \subset (K')^{N_i}\) in the form \(x_i = (x_{i,1}(\alpha'), \ldots, x_{i,N_i}(\alpha'))\) where \(x_{i,j}(\alpha') = \sum_{t=0}^{r-1} b_{i,j,t}(\alpha')^t\) with \(b_{i,j,t}, c_{i,j,t} \in \mathbb{Z}\). Taking \(N' = \max\{\{|c_{i,j,t}||_{i,j,t}, N\}\}\), we get that \(i_{p*}(x_1), \ldots, i_{p*}(x_k) \in X(\mathbb{Z}_p)\) for all \(p \in S_{N'}\). \(\square\)

Using Theorem 2.17 and Lemma 3.15 we can now reduce our main problem to the following.

**Proposition 3.16** Let \(X\) be a smooth algebraic \(\mathbb{Q}\)-variety, \(\mu\) be a motivic measure on \(X\) as in Proposition 3.14, and let \(\varphi : X \rightarrow \mathbb{A}^m_{\mathbb{Q}}\) be a strongly dominant morphism. Assume that there exists \(n \in \mathbb{N}\), such that for every large enough prime \(p\) the measure \(\varphi^n(\mu_{\mathbb{Q}_p} \times \cdots \times \mu_{\mathbb{Q}_p})\) has continuous density with respect to the normalized Haar measure on \((\mathbb{Q}_p)^m\). Then the map \(\varphi^n : X \times \cdots \times X \rightarrow \mathbb{A}^m_{\mathbb{Q}}\) is (FRS).

**Proof** Let \(x = (x_1, \ldots, x_n) \in (X \times \cdots \times X)(\overline{\mathbb{Q}})\). There exists a finite extension \(K/\mathbb{Q}\) such that \(x_1, \ldots, x_n \in X(K)\). By [1, Theorem 3.4], in order to show that \(\varphi^n\) is (FRS) at \(x\) it is enough to show that for any finite extension \(K'/K\), there exists a non-Archimedean local field \(F\) containing \(K'\) and a non-negative Schwartz measure \(\mu\) on \((X \times \cdots \times X)(F)\) that does not vanish at \(x\), such that \(\varphi^n(\mu)\) has continuous density. Given such \(K'/K\), by Lemma 3.15, we can choose \(F = \mathbb{Q}_p\) for large enough \(p\) such that \(K' \hookrightarrow \mathbb{Q}_p\) and \(i_{p*}(x_1), \ldots, i_{p*}(x_n) \in X(\mathbb{Z}_p)\). By our assumption, \(\varphi^n(\mu_{\mathbb{Q}_p} \times \cdots \times \mu_{\mathbb{Q}_p})\) has continuous density with respect to the normalized Haar measure on \((\mathbb{Q}_p)^m\), and \(\mu_{\mathbb{Q}_p} \times \cdots \times \mu_{\mathbb{Q}_p}\) does not vanish at \(x\), so we are done. \(\square\)

### 4 Pushforward of a motivic measure under a strongly dominant map

In the last section we have reduced Theorem 1.7, in the case \(K = \mathbb{Q}\), to an analytic question on the pushforward of motivic Schwartz measures under a strongly dominant morphism \(\varphi : X \rightarrow \mathbb{A}^m_{\mathbb{Q}}\) (Proposition 3.16), where \(X\) is a smooth \(\mathbb{Q}\)-variety. In this section, we show that the pushforward under \(\varphi\) of a motivic Schwartz measure \(\mu = \{\mu_F\}_{F \in \text{Loc}}\) yields a motivic measure \(\{\varphi_*(\mu_F)\}_{F \in \text{Loc}}\) with \(L^1\)-density with respect to the normalized Haar measure on \(F^m\) (see Corollary 4.3). This will be a conclusion of the more general Theorem 4.2, whose proof relies on the following consequence of [11, Theorem 10.1.1]:

\[x^r + \sum_{i=0}^{r-1} a_i x^i = q(x) \in \mathbb{Z}[x].\]
**Lemma 4.1** Suppose that $X$ and $Y$ are smooth algebraic $\mathbb{Q}$-varieties, $\varphi : X \to Y$ is a morphism with finite fibers and let $f \in \mathcal{C}(X)$. Then the function $I_{f,F}(y) = \sum_{x \in \varphi^{-1}(y)} f_F(x)$ is in $\mathcal{C}(Y)$.

**Proof** Let $\bigcup_j V_j$ be an open affine cover of $Y$, and for each $j$ take an open affine cover $\bigcup_{i=1}^r U_{ji}$ of $\varphi^{-1}(V_j)$. By the definition of a motivic function on a $\mathbb{Q}$-variety, it is enough to prove the lemma for $\varphi_j := \varphi|_{\varphi^{-1}(V_j)} : \varphi^{-1}(V_j) \to V_j$. By choosing a $\mathbb{Z}$-model for the diagram consisting of $X$, $Y$, $\{V_j\}_{j=1}^r$, $\{U_{ji}\}_{i=1}^r$, their intersections, and the morphisms between these objects, we can assume they are all defined over $\mathbb{Z}$. Construct a definable disjoint cover of $\varphi^{-1}(V_j)$ by setting $U'_{j1} = U_{j1}$, and $U'_{ji} = U_{ji} \setminus \bigcup_{k=1}^{i-1} U_{jk}$, and let $\varphi_{ji} := \varphi|_{U'_{ji}}$ and $f_{ji} := f|_{U'_{ji}}$. Then we have the following:

$$I_{f_j,F}(y) = \sum_{i=1}^r \sum_{x \in \varphi_{ji}^{-1}(y)} f_{ji,F}(x) = \sum_{i=1}^r \varphi_{ji*}(f_{ji}).$$

Since by [11, Theorem 10.1.1] every summand of the RHS of the above formula is a constructible function on $Y$, we are done. \qed

Using Lemma 4.1, we can now show the following variant of [1, Proposition 5.6 and Corollary 3.6]:

**Theorem 4.2** Let $X$ and $Y$ be smooth algebraic $\mathbb{Q}$-varieties, let $M \in \mathbb{N}$, and let $\varphi : X \to Y$ be a strongly dominant morphism. Let $\mu_X$ be a motivic measure on $X$ such that $\mu_{X,F}$ is a Schwartz measure for every $F \in \text{Loc}_M$, and let $\mu_Y$ be a motivic measure on $Y$ such that $\mu_{Y,F}$ is smooth and non-vanishing on $Y(F)$ for every $F \in \text{Loc}_M$.

1. $\varphi_*(\mu_X)$ is a motivic measure, and $\varphi_*(\mu_{X,F})$ is absolutely continuous with respect to $\mu_{Y,F}$ for any $F \in \text{Loc}_M$.
2. In particular, if $Y$ has a non-vanishing top form $\omega_Y$, then there exists $g \in \mathcal{C}(Y)$ such that $\varphi_*(\mu_{X,F})$ is absolutely continuous with respect to $|\omega_Y|_F$ with density $g_F$, for any $F \in \text{Loc}_M$.

**Proof** Since $Y$ is smooth, by Definition 3.12 we may assume that $\Omega^{\text{top}}_{Y/\mathbb{Q}}$ is free, and in particular that $Y$ has a non-vanishing top form $\omega_Y$. Hence, it is enough prove part (2) of the theorem. By Lemma 3.13 and the smoothness of $X$, we may assume that $\Omega^{\text{top}}_{X/\mathbb{Q}}$ is free and that there exists $f \in \mathcal{C}(X)$ such that $\mu_{X,F} = f_F |\omega_X|_F$ for some top form $\omega_X$.

Denote by $X^{\text{sm},\varphi}$ the smooth locus of $\varphi$ and by $i_{\text{sm}} : X^{\text{sm},\varphi} \hookrightarrow X$ the inclusion. Since $\varphi$ is strongly dominant, we have by [1, Corollary 3.6] that for any $F \in \text{Loc}_M$, the measure $\varphi_*(f_F |\omega_X|_F)$ is absolutely continuous with respect to $|\omega_Y|_F$ and has an $L^1$-density $g_F$ such that

$$g_F(y) = \int_{\varphi^{-1}(y) \cap i_{\text{sm}}(X^{\text{sm},\varphi})(F)} f_F(x) \left| \frac{\omega_X}{\varphi_* \omega_Y} |\varphi^{-1}(y) \cap i_{\text{sm}}(X^{\text{sm},\varphi})|_F \right|^r.$$
To finish the theorem, we need to show that \( g \in \mathcal{C}(Y) \). Let \( j : U \hookrightarrow X^{\text{sm}, \varphi} \) be an open dense affine subvariety of \( X^{\text{sm}, \varphi} \) and set \( \psi := i_{\text{sm}} \circ j : U \hookrightarrow X \) and \( \varphi_U := \varphi \circ \psi : U \to Y \). Since \( \varphi_U \) is a smooth map and \( \Omega^1_{X/Q} \) and \( \Omega^1_{Y/Q} \) are free, \( \varphi_U \) factors as:

\[
\varphi_U : U \xrightarrow{\tilde{\varphi}} \mathbb{A}^{\dim X - \dim Y}_Q \times Y \xrightarrow{\pi} Y,
\]

where \( \pi \) is the projection, and \( \tilde{\varphi} \) is an étale map. Since \( U \) is open and dense in \( X^{\text{sm}, \varphi} \), we have that \( \psi(U(F)) \) is dense in \( X^{\text{sm}, \varphi}(F) \) for any \( F \in \text{Loc} \). This implies

\[
\int_{\psi^{-1}(Y) \cap i_{\text{sm}}(X^{\text{sm}, \varphi}(F))} f_F \cdot \left| \frac{\omega_X}{\varphi^* \omega_Y} \right|_{\psi^{-1}(Y) \cap i_{\text{sm}}(X^{\text{sm}, \varphi})} = \int_{\psi^{-1}(Y) \cap (U(F))} f_F \cdot \left| \frac{\omega_X}{\varphi^* \omega_Y} \right|_{F} = \int_{\psi_U^{-1}(Y)(F)} (f \circ \psi)_F \left| \frac{\psi^*(\omega_X)}{(\varphi_U^*)^* \omega_Y} \right|_{F},
\]

and therefore

\[
\varphi_*(f_F \mid \omega_X|_F) = \varphi_U^*((f \circ \psi)_F \mid \psi^*(\omega_X)|_F) = \pi_*(\tilde{\varphi}_*((f \circ \psi)_F \mid \psi^*(\omega_X)|_F)).
\]

Now, by Lemma 3.10, we have that \( f \circ \psi \in \mathcal{C}(U) \). As \( \tilde{\varphi} \) is étale, it has finite fibers, so we can write:

\[
\tilde{\varphi}_*((f \circ \psi)_F \mid \psi^*(\omega_X)|_F) = h_F \mid \omega \times \omega_Y|_F,
\]

where \( \omega \) is a top form on \( \mathbb{A}^{\dim X - \dim Y}_Q \) which induces the normalized Haar measure \( \lambda = |\omega|_F \) on \( F^{\dim X - \dim Y} \), and

\[
h_F(t, y) = \sum_{x \in \tilde{\varphi}^{-1}(t, y)(F)} (f \circ \psi)_F(x) \frac{\psi^*(\omega_X)}{\tilde{\varphi}^*(\omega \times \omega_Y)}|_F(x).
\]

By Lemma 4.1, we have that \( h = \{h_F\}_{F \in \text{Loc}} \) is in \( \mathcal{C}(\mathbb{A}^{\dim X - \dim Y}_Q \times Y) \). Notice that for any \( F \in \text{Loc}_M \), we have \( (g_F \mid \omega_Y|_F) (y) = \pi_*(h_F \mid \omega \times \omega_Y|_F) (y), \) that is:

\[
g_F(y) = \int_{F^{\dim X - \dim Y} \times \{y\}} h_F(t, y) d\lambda.
\]

Using [7, Theorem 4.3.1], we have \( g = \{g_F\}_{F \in \text{Loc}_M} \in \mathcal{C}(Y) \) and we are done. \( \square \)

As a conclusion, we obtain the following result, which is the main goal of this section:

**Corollary 4.3** Let \( X \) be a smooth algebraic \( \mathbb{Q} \)-variety, \( \varphi : X \to \mathbb{A}^m_{\mathbb{Q}} \) be a strongly dominant morphism and \( \mu \) a motivic measure on \( X \) as in Proposition 3.14. Then there exist \( M \in \mathbb{N} \) and \( f \in \mathcal{C}(\mathbb{A}^m_{\mathbb{Q}}) \), such that for every \( p > M \) the measure \( \varphi_* (\mu_{|_{\mathbb{Q}_p}}) \) is absolutely continuous with respect to the normalized Haar measure on \( \mathbb{Q}_p^m \) with density \( f_{\mathbb{Q}_p} \in L^1 \).
5 Uniform bounds on decay rates of Fourier transform of motivic measures

In this section we finish the proof of Theorem 1.7 for the case \( k = \mathbb{Q} \):

**Theorem 5.1** Let \( X \) be a smooth \( \mathbb{Q} \)-variety and let \( V = \mathbb{A}_\mathbb{Q}^m \) be the \( m \)-dimensional affine space. Let \( \varphi : X \to V \) be a strongly generating morphism (Definition 3.4), then there exists \( N \in \mathbb{N} \) such that for any \( n > N \), the \( n \)-th convolution power \( \varphi^n \) is (FRS).

Let \( \mu \) be a motivic measure on \( X \) as in Proposition 3.14. We saw in Corollary 3.6 that \( \varphi \) can be taken to be a strongly dominant morphism. We have further showed, using Proposition 3.16, that it is enough to show that there exists \( n \in \mathbb{N} \), such that for large enough prime \( p \), the measure \( \varphi_n^*(\mu_{\mathbb{Q}_p} \times \cdots \times \mu_{\mathbb{Q}_p}) \) has continuous density with respect to the normalized Haar measure on \((\mathbb{Q}_p)^m\).

Notice that
\[
\varphi_n^*(\mu_{\mathbb{Q}_p} \times \cdots \times \mu_{\mathbb{Q}_p}) = \varphi^*(\mu_{\mathbb{Q}_p}) \ast \cdots \ast \varphi^*(\mu_{\mathbb{Q}_p}),
\]
and that by Corollary 4.3, the measure \( \{\varphi^*(\mu_F)\}_F \) is motivic, and for any \( F \in \text{Loc} \) we have that \( \varphi^*(\mu_F) \) is compactly supported and has \( L^1 \)-density. Hence, our next goal is to show that given a motivic measure \( \sigma = \{\sigma_F\}_{F \in \text{Loc}} \) on \( \text{VF}^m \) such that \( \sigma_F \) is compactly supported and has \( L^1 \)-density for all \( F \in \text{Loc} \), then there exists \( N \in \mathbb{N} \), such that the \( N \)-th convolution power \( \{\sigma_F^N\}_{F \in \text{Loc}} \) has continuous density for any \( F \in \text{Loc}_M \) and \( M \) large enough.

Recall that the Fourier transform \( \mathcal{F}(f) \) of an \( L^1 \)-function \( f : \mathbb{Q}_p^m \to \mathbb{C} \) is a continuous function, and that \( \mathcal{F} \circ \mathcal{F}(f)(x) = f(-x) \). Thus, in order to show that \( \sigma_F \ast \cdots \ast \sigma_F \) is continuous, it is enough to show that \( \mathcal{F}(\sigma_F \ast \cdots \ast \sigma_F) = \mathcal{F}(\sigma_F)^N \) is in \( L^1 \) for some \( N \) that does not depend on \( F \in \text{Loc} \), but firstly, we need to make sense of the Fourier transform of a motivic function (or measure).

In [12, Section 7], Cluckers and Loeser defined a class of motivic exponential functions and an analogue of Fourier transform for this class of functions (see also [7, Section 4] and [9, Section 2]). Specializing to a non-Archimedean local field \( F \), the Fourier transform operation can be established as follows:

Fix a collection \( \{\psi_F\}_{F \in \text{Loc}} \) of non-trivial additive characters by \( \psi_F(x) = \exp \frac{2\pi i}{F} \text{Tr}_{k_F/\mathbb{F}_p}(x) \) for any \( x \in \mathcal{O}_F \), where \( k_F = \mathcal{O}_F/m_F \) is the residue field (of characteristic \( p \)), \( \mathcal{O}_F \) is the reduction modulo \( m_F \) of \( x \) and \( \text{Tr}_{k_F/\mathbb{F}_p} \) is the trace. Let \( \langle , \rangle : F^m \times F^m \to F \) be the standard inner product on \( F^m \) through which we identify \( F^m \) with \( (F^m)^\vee \). Notice that \( \psi_F \) is trivial on \( m_F \). We denote by \( \lambda_F \) the normalized Haar measure on \( F^m \) (i.e. \( \lambda(\mathcal{O}_F^m) = 1 \)). For \( f \in \mathcal{C}(\text{VF}^m) \) with \( f_F \) absolutely integrable for any \( F \in \text{Loc} \), we define the Fourier transform by

\[
\mathcal{F}(f_F)(y) = \int_{F^m} f_F(x) \psi_F(\langle y, x \rangle) dx.
\]

We wish to prove the following uniform variant of [6, Theorem 4.1] for the \( \mathcal{L}_{\text{DP}} \)-language:
Theorem 5.2 Let $h \in C(VF^m)$. Assume that there exist a definable set $L \subseteq VF^m$ and a natural number $M'$ such that for any $F \in \text{Loc}_{M'}$ the set $L_F$ is compact, $\supp(h_F) \subseteq L_F$ and that $|h_F|$ is integrable. Then there exist a real constant $\alpha < 0$ and a natural number $M > M'$, such that for any $F \in \text{Loc}_M$

$$|F(h_F)(y)| < d(F) \cdot \min\{|y|^{-\alpha}, 1\},$$

for some constant $d(F)$ which depends on $F$.

Corollary 5.3 Theorem 5.2 implies Theorem 5.1.

Proof Let $\mu$ be a motivic measure on $X$ as in Proposition 3.14. By Theorem 5.2, there exists a real constant $\alpha < 0$ such that for any $F \in \text{Loc}_M$

$$|F(\varphi_*(\mu_F))(y)| < d(F) \cdot \min\{|y|^{-\alpha}, 1\}.$$

Set $N = \lceil -\frac{2}{\alpha} \rceil$, then we have

$$|F(\varphi_*^N(\mu_F \times \cdots \times \mu_F))(y)| = |F(\varphi_*(\mu_F))(y)|^N < d(F)^N \cdot \min\{|y|^{-2}, 1\}$$

and in particular it is $L^1$. Thus $\varphi_*^N(\mu_F \times \cdots \times \mu_F)$ has continuous density for any $F \in \text{Loc}_M$, for some $M \in \mathbb{N}$, and by Proposition 3.16 this implies Theorem 5.1. □

5.1 Proof of Theorem 5.2

We now prove Theorem 5.2. Firstly, it is clear that $F(h_F)$ is bounded since $|F(h_F)| \leq \int_{F^m} |h_F(x)| \, dx < \infty$. It is enough to show, for $1 \leq i \leq m$, that $|F(h_F)(y)| < C_i(F) \cdot |y|^{\alpha_i}$ for some $\alpha_i < 0$ and $C_i(F) > 0$. We prove this by reducing to a one dimensional analogue of the problem (see Lemma 5.4) which is easier to solve (Proposition 5.5).

Lemma 5.4 Let $M' \in \mathbb{N}$, and let the motivic function $h \in C(VF^m)$ and the definable set $L \subseteq VF^m$ be as in Theorem 5.2. Then there exist $M \in \mathbb{N}$ and a motivic function $g \in C(VF)$ such that for any $F \in \text{Loc}_M$ and $y_1, \ldots, y_m \in F$ we have

$$|F(h_F)(y_1, \ldots, y_m)| \leq g_F(y_m) and \lim_{|y_m| \to \infty} g_F(y_m) = 0.$$

Proof By Theorem 2.8, the function $h \in C(VF^m)$ is determined by finitely many polynomials $\{g_j\}_{j=1}^s$ in $\mathbb{Z}[x_1, \ldots, x_m]$. Write $x = (\hat{x}, x_m)$ with $\hat{x} = (x_1, \ldots, x_{m-1})$. By uniform cell decomposition applied to the functions $\{g_j\}_{j=1}^s$ (see Theorem 2.7), there exist cells $A_i$ with cell data $\Theta(A_i) = (C_i, b_{i,1}, b_{i,2}, c_i, \lambda_i)$ such that $F^m = \bigcup_{i=1}^N A_i,F$ and $A_i,F = \bigcup_{\xi \in k_F} A_i,F(\xi)$, where each fiber $A_i(\xi)$ is as in Definition 2.5. Furthermore, for any $1 \leq i \leq N$ and $1 \leq j \leq s$, and any $(\hat{x}, x_m) \in A_i(\xi)$, we have

$$\text{val}(g_{j,F}(\hat{x}, x_m)) = \text{val}(h_{ij,F}(\hat{x}, \xi)(x_m - c_{i,F}(\hat{x}, \xi))^{w_{ij}}) and \text{ac}(g_{j,F}(\hat{x}, x_m)) = \xi_{\mu_{i,j}},$$
where \( \mu_l, w_{ij}, h_{ij} \) are as in Definition 2.6. Hence, for \( y = (\hat{y}, y_m) \in F^{m-1} \times F \) we have,

\[
\mathcal{F}(h_F)(y) = \int_{F^m} h_F(\hat{x}, x_m)\psi_F((\hat{y}, x))dx_md\hat{x} \\
= \sum_{i=1}^{N} \int_{A_i} h_F(\hat{x}, x_m)\psi_F((\hat{y}, x))dx_md\hat{x}.
\]

Let \( A \in \{A_1, \ldots, A_N\} \) be one of the above cells. For simplicity, we assume it has datum \( \Theta(A) = (C, b_1, b_2, c, \lambda) \), and similarly we replace \( \mu_l, w_{ij} \) and \( h_{ij} \) with \( \mu, w_j \) and \( h_j \). Note that

\[
A_F = \bigcup_{\xi \in k'_{F}} A_F(\xi) = \bigcup_{\xi \in k'_{F}} \bigcup_{l \in \mathbb{Z}} \{(\hat{x}, x_m) \in A_F(\xi) : \text{val}(x_m - c(\hat{x}, \xi)) = l\}
\]

\[
= \bigcup_{\xi \in k'_{F}} \bigcup_{l \in \mathbb{Z}} \{(\hat{x}, x_m) \in F^m : \hat{x} \in W(l, \xi) \land x_m \in B(l, \xi, c, \hat{x})\},
\]

where

\[
B(l, \xi, c, \hat{x}) := \{x_m \in VF : \text{val}(x_m - c(\hat{x}, \xi)) = l, \ ac(x_m - c(\hat{x}, \xi)) = \xi_1\},
\]

\[
W(l, \xi) := \{\hat{x} \in VF^{m-1} : (\hat{x}, \xi) \in C, \text{val}(b_1(\hat{x}, \xi)) \sqcap_1 \lambda \land \sqcap_2 \text{val}(b_2(\hat{x}, \xi))\},
\]

and \( \sqcap_{1,2} \) are either \( \leq, < \) or no condition, as in the cell \( A_F \) (recall Definition 2.5). Calculating the integral over \( A_F \) yields:

\[
\int_{A_F} h_F(\hat{x}, x_m)\psi_F((\hat{y}, x))dx_md\hat{x} \\
= \sum_{\xi \in k'_{F}} \sum_{l \in \mathbb{Z}} \int_{W(l, \xi)} \int_{B(l, \xi, c, \hat{x})} h_F(\hat{x}, x_m)\psi_F((\hat{y}, \hat{x}) + y_mx_m)dx_md\hat{x} \\
= \sum_{\xi \in k'_{F}} \sum_{l \in \mathbb{Z}} \int_{W(l, \xi)} \psi_F((\hat{y}, \hat{x}) + c_F(\hat{x}, \xi) \cdot y_m) \\
\cdot \left( \int_{B(l, \xi, c, \hat{x})} h_F(\hat{x}, x_m) \cdot \psi_F((x_m - c_F(\hat{x}, \xi))y_m)dx_m \right) d\hat{x}. \tag{5.1}
\]

Note that \( h_F(\hat{x}, x_m) \) depends only on \( \hat{x} \) when \( \text{val}(x_m - c(\hat{x}, \xi)) = l \), since for any \( j \) we have \( \text{val}(g_{j,F}(\hat{x}, x_m)) = \text{val}(h_{j,F}(\hat{x}, \xi)) + w_j \cdot l \) and \( \text{ac}(g_{j,F}(\hat{x}, x_m)) \) does not depend on \( x_m \). Therefore after a linear change of variables \( u := x_m - c(\hat{x}, \xi) \) we can write (5.1) as:

\[
\sum_{\xi \in k'_{F}} \sum_{l \in \mathbb{Z}} \int_{W(l, \xi)} \psi_F((\hat{y}, \hat{x}) + c_F(\hat{x}, \xi) \cdot y_m) \cdot h_F(\hat{x}, x_m) \cdot \left( \int_{B(l, \xi, 0, \hat{x})} \psi_F(u \cdot y_m)du \right) d\hat{x}. \tag{5.2}
\]

\[
\int_{B(l, \xi, 0, \hat{x})} \psi_F(u \cdot y_m)du 
\]
Now, for every \( \hat{x} \in F^{m-1} \), we have \( B_F(l, \xi, 0, \hat{x}) = [\xi_1] \pi^l + \pi^{l+1} O_F \), where \([\xi_1] \in O_F^\times\) is the unique lift of \( \xi_1 \in k_F \) to \( O_F \). If we take \( y = (\hat{y}, y_m) \) with \( y_m \) such that \( \text{val}(y_m) \leq -\text{val}(u) - 2 = -l - 2 \), then for \( j = -(l + \text{val}(y_m) + 1) \geq 1 \) we have

\[
\{u \cdot y_m : u \in B_F(l, \xi, 0, \hat{x})\} = a \cdot \pi^{-1-j} + \pi^{-j} O_F,
\]

for some \( a \in O_F^\times \). Set \( u y_m = a \pi^{-1-j} + z \pi^{-j} \) where \( z \in O_F \) depends on \( u \), and recall that \( \psi_F(O_F) = 1 \). For any \( y_m \) with \( \text{val}(y_m) \leq -\text{val}(u) - 2 \), we obtain

\[
\int_{B_F(l, \xi, 0, \hat{x})} \psi_F(u \cdot y_m) \, du = 0,
\]

since this integral is essentially a sum of a non-trivial character over a finite group:

\[
\int_{B_F(l, \xi, 0, \hat{x})} \psi_F(u \cdot y_m) \, du = \psi_F(a \cdot \pi^{-1-j}) \cdot \int_{\pi^{l+1} O_F} \psi_F(z \cdot y_m) \, dz
\]

\[
= \psi_F(a \cdot \pi^{-1-j}) \cdot \left| \frac{1}{y_m} \right|_{F} \cdot \int_{\pi^{-j} O_F} \psi_F(z) \, dz
\]

\[
= \psi_F(a \cdot \pi^{-1-j}) \cdot \left| \frac{1}{y_m} \right|_{F} \cdot \sum_{z \in \pi^{-j} O_F / O_F} \tilde{\psi}_F(z) = 0,
\]

where \( \tilde{\psi}_F \) is the character induced on \( \pi^{-j} O_F / O_F \) from \( \psi_F \).

Now, for \( a \in VF \) and \( \xi \in RF^r \) define \( B'(a, \xi) = \{(\hat{x}, x_m) \in A(\xi) : \text{val}(a \cdot (x_m - c(\hat{x}, \xi))) \geq -1\} \) and \( B'(a) = \bigcup_{\xi \in RF^r} B'(a, \xi) \). By the above computation,

\[
\int_{A_F \setminus B'_F(y_m)} h_F(\hat{x}, x_m) \psi_F(\langle y, x \rangle) \, dx = 0.
\]

We get,

\[
\left| \int_{A_F} h_F(\hat{x}, x_m) \psi_F(\langle y, x \rangle) \, dx \right| = \int_{B'_F(y_m)} h_F(\hat{x}, x_m) \psi_F(\langle y, x \rangle) \, dx \leq \int_{B'_F(y_m)} \left| h_F(\hat{x}, x_m) \right| \, dx.
\]

Note that \( |h| \) is a motivic function, and that \( |h_F| \) is integrable on \( B'_F(y_m) \) for any \( F \in Loc_{M'} \) and any \( y_m \in F \). Recall we assumed there exists a definable set \( L \) such that \( \text{supp}(h_F) \subseteq L_F \) for all \( F \in Loc_{M'} \). By Theorem 2.2, there exist a motivic function \( g_A^F \in C(VF) \) and an integer \( M > M' \) such that for any \( F \in Loc_{M} \) we have,

\[
\int_{B'_F(y_m)} \left| h_F(\hat{x}, x_m) \right| \, dx = \int_{B'_F(y_m) \cap L_F} \left| h_F(\hat{x}, x_m) \right| \, dx = g_A^F(y_m).
\]

This implies,

\[
\left| \int_{A_F} h_F(\hat{x}, x_m) \psi_F(\langle y, x \rangle) \right| \leq g_A^F(y_m).
\]
Now, for any $F \in \text{Loc}_M$, the set $L_F$ is compact, and since the measure of $L_F \cap B^c_F(y_m)$ tends to zero as $|y_m|$ tends to infinity, we get that $\lim_{|y_m| \to \infty} g^A_F(y_m) = 0$. By repeating these arguments for the other cells, and possibly enlarging $M$, we obtain that for any $F \in \text{Loc}_M$,

$$|\mathcal{F}(h_F)(y)| \leq \sum_{i=1}^N \left| \int_{A_i,F} h_F(\hat{x}, x_m)\psi((x, y))dx \right| \leq \sum_{i=1}^N g^A_i(y_m),$$

where $\lim_{|y_m| \to \infty} \left( \sum_{i=1}^N g^A_i(y_m) \right) = 0$, as required. \hfill \Box

The following proposition, along with Lemma 5.4, finishes the proof of Theorem 5.2. Indeed, for $g \in \mathcal{C}(VF)$ as in Lemma 5.4, Proposition 5.5 yields constants $d$ and $\alpha$ such that,

$$|\mathcal{F}(h_F)(y_1, \ldots, y_m)| \leq g_F(y_m) \leq d(F) \cdot \min\{|y|^\alpha, 1\}.$$

**Proposition 5.5** Let $f \in \mathcal{C}(VF)$ and suppose that $\lim_{|y| \to \infty} f_F(y) = 0$ for any $F \in \text{Loc}_M$. Then there exists a real number $\alpha < 0$ such that for any $F \in \text{Loc}_M$,

$$|f_F(y)| < d(F) \cdot \min\{|y|^\alpha, 1\},$$

where $d(F) > 0$ is a constant that depends only on $F$.

Recall that $f_F$ is of the following form, where $\alpha_i, \beta_{ij}$ and $Y_i$ are as in Definition 2.1:

$$f_F(y) = \sum_{i=1}^N |Y_{i,F,y}| q^{\alpha_{i,F}(y)} \left( \prod_{j=1}^{N'} \beta_{ij,F}(y) \right) \left( \prod_{l=1}^{N'^n} \frac{1}{1 - q^{\tilde{\alpha}_l}} \right).$$

By quantifier elimination (Theorem 2.8), there exist polynomials $\{g_l\}_{l=1}^s \in \mathbb{Z}[y]$ such that (recall Notation 2.9):

1. $\alpha_i$ has datum $\Xi(\alpha_i) = (\{g_l\}_{l=1}^s, \{\chi_k^i(y)\}_{k=1}^{N_i}, \{\theta_k^i(y, t)\}_{k=1}^{N_i})$, with $t \in \mathbb{Z}$, $y \in VF$.
2. $\beta_{ij}$ has datum $\Xi(\beta_{ij}) = (\{g_l\}_{l=1}^s, \{\chi_k^{ij}(y)\}_{k=1}^{N_{ij}}, \{\theta_k^{ij}(y, t)\}_{k=1}^{N_{ij}})$, with $t \in \mathbb{Z}$, $y \in VF$.
3. $Y_i \subseteq VF \times RF^{r_i}$ has datum $\Xi(Y_i) = (\{g_l\}_{l=1}^s, \{\chi_k^i(y, \eta)\}_{k=1}^{M_i}, \{\tilde{\theta}_k^i(y)\}_{k=1}^{M_i})$, with $r_i \in \mathbb{N}$, $\eta \in RF^{r_i}$, $y \in VF$.

By the uniform cell decomposition theorem, we can decompose VF as a finite disjoint union of cells, where each cell is of the form $A = \bigcup_{\xi \in RF^r} A(\xi)$ with datum $\Theta(A) = (C, b_1(\xi), b_2(\xi), c(\xi), \lambda)$ for a definable set $C \subseteq RF^r$ and definable functions $b_1, b_2$ and $c$ from $C$ to $VF$. Moreover, we have $\text{val}(g_l(y)) = \text{val}(h_l(\xi)(y - c(\xi))^{w_l})$ and $ac(g_l(y)) = \xi_{\mu(l)}$ for all $1 \leq l \leq s$.

Since there are finitely many cells, it is enough to prove Proposition 5.5 for $f|_A$ and any cell $A$. We do the latter by proving several smaller lemmas.
Lemma 5.6 We may assume that \( \text{val}(g_l(y)) = w_l \cdot \text{val}(y) + \text{val}(h_l(\xi)) \) for all \( 1 \leq l \leq s \).

**Proof** For a given cell \( A \), consider the definable sets \( L(\xi) := \{ y \in A(\xi) : \text{val}(c(\xi)) > \text{val}(y) \} \) and \( L = \bigcup_{\xi \in \mathbb{R}^s} L(\xi) \). Since we are interested in asymptotic behavior (i.e. when \( \text{val}(y) \to -\infty \)), it is enough to prove the claim for \( f|_L \) and each such \( L \), but notice that for any \( y \in L \) we have

\[
\text{val}(g_l(y)) = \text{val}(h_l(\xi)(y - c(\xi)))^{w_l} = w_l \cdot \text{val}(y) + \text{val}(h_l(\xi)).
\]

\( \square \)

Lemma 5.7 Each cell \( A \) has a partition \( A = \bigcup_{b=1}^{N_A} A_b \) such that on each \( A_b \) we have the following:

1. The functions \( \beta_{ij} \) and \( \alpha_i \) can be written as compositions

\[
\alpha_i(y) = \tilde{\alpha}_i \circ (\text{val}(g_1(y)), \ldots, \text{val}(g_s(y))) \quad \text{and} \quad \beta_{ij}(y) = \tilde{\beta}_{ij} \circ (\text{val}(g_1(y)), \ldots, \text{val}(g_s(y))),
\]

where \( \tilde{\alpha}_i \) and \( \tilde{\beta}_{ij} \) are \( \mathcal{L}^\infty_{\text{Pres}} \)-definable functions from \( \mathcal{L}^\infty_{\text{Pres}} \)-definable subsets of \( \mathbb{Z}^s \) to \( \mathbb{Z} \).

2. \( |Y_{i,F,y}| \) depends only on \( \xi \) (and \( F \)). In particular, for any \( \xi \in k_F^* \), the value \( |Y_{i,F}(y)| \) is constant on \( A_b(\xi) \).

**Proof** We prove for \( \alpha_i \), the proof for \( \beta_{ij} \) is similar. We start with constructing a partition of \( A \) which satisfies (1). Recall that \( \alpha_i \) is defined by the formula

\[
\psi_{\alpha_i}(y, t) = \bigvee_{k=1}^{N_i} \chi_k(\text{ac}(g_1(y)), \ldots, \text{ac}(g_s(y))) \land \theta_k^l(\text{val}(g_1(y)), \ldots, \text{val}(g_s(y)), t).
\]

For simplicity we set \( \text{ac}(g(y)) := (\text{ac}(g_1(y)), \ldots, \text{ac}(g_s(y))) \), and use similar notation for \( \text{val}(g(y)) \). Then \( \alpha_i^{-1}(l') = \bigcup_{k=1}^{N_i} \{ y : \chi_k(\text{ac}(g(y))) \land \theta_k^l(\text{val}(g(y)), l') \} \) for any \( l' \in \mathbb{Z} \). Let \( I \in \{0, 1\}^{N_i} \) and consider the set

\[
A_I = \{ y \in A : \chi_k^I(\text{ac}(g(y))) = I(k), \; 1 \leq k \leq N_i \}.
\]

If we restrict to \( A_I \), we get that \( \alpha_i(y) = l' \) if and only if \( \sigma_i(y, l') := \bigvee_{1 \leq k \leq N_i, I(k)=1} \theta_k^l(\text{val}(g(y)), l') \) is true. Since \( \alpha_i \) is a definable function, for any \( y \in A_I \) there is at most one \( l' \) such that \( \sigma_i(y, l') \) holds, and thus \( \sigma_i(y, l') \) is a graph of a definable function. Now, note that \( \sigma_i(z) := \bigvee_{1 \leq k \leq N_i, I(k)=1} \theta_k^l(z) \) is an \( \mathcal{L}^\infty_{\text{Pres}} \)-formula in \( \mathbb{Z}^{s+1} \). Thus, it is a graph of a definable function into \( \mathbb{Z} \) when restricted to the \( \mathcal{L}^\infty_{\text{Pres}} \)-definable set \( \{ z \in \mathbb{Z}^s : \exists! z' \text{ s.t. } \sigma_i(z, z') \text{ holds} \} \). Since \( \sigma_i(y, l') = \sigma_i(\text{val}(g(y)), l') \), we get that \( \alpha_i|_{A_I} \) is of the required form.

To prove the second property, recall that \( Y_i \) is defined by the formula

\[
\psi_{Y_i} = \bigvee_{k=1}^{M_i} \tilde{\chi}_k^I(\text{ac}(g(y)), \eta) \land \tilde{\theta}_k^l(\text{val}(g(y))).
\]
By a process similar to before, we can find a partition $A = \bigcup_{b=1}^{\tilde{N}_A} \tilde{A}_b$ such that for every $b$ each $\tilde{\theta}_b^i(\text{val}(g(y)))$ is either identically true on $\tilde{A}_b$ or identically false. Moreover, on each $\tilde{A}_b \subseteq A$, since $\text{ac}(g_l(y)) = \xi_{\mu(l)}$ for any $l$, the formula $\tilde{\theta}_b^i(\text{ac}(g(y)), \eta)$ depends only on $\xi$ and $\eta$. This implies that the assignment $y \mapsto Y_{i,F,y} = \{ \eta \in k_F^r : (\eta,y) \in Y_{i,F} \}$ is constant on $\tilde{A}_b(\xi)$ for any $F \in \text{Loc}_M$, and thus $|Y_{i,F}(y)|$ is constant on $\tilde{A}_b(\xi)$ and depends only on $\xi$ in $\tilde{A}_b$, as required. To finish the proof, we take $\{A_b\}_{b=1}^{\tilde{N}_A}$ to be the joint refinement of $\{\tilde{A}_b\}_{b=1}^{\tilde{N}_A}$ and $\{A_I\}$ as constructed above. \qed

Recall we are trying to bound the decay rate of a motivic function $f \in \mathcal{C}(\mathcal{VF})$ which tends to zero at infinity. The following lemma allows us, after further refining our cover, to give a simple description of $f$ on each piece.

**Lemma 5.8** By further refining the cells $\{A_b\}_{b=1}^{\tilde{N}_A}$, we can assume the restriction of $f$ to each cell is of the form

$$f_F(y) = \sum_{i=1}^N a_i(\xi) \cdot q_F^{t_i(\xi)+t_{2i} \cdot \text{val}(y)} \cdot P_i,\xi(\text{val}(y)) \cdot Q_i(q_F),$$

where $a_i(\xi)$ are real numbers, $t_{1i}(\xi)$ and $t_{2i}$ are rational numbers, $P_i,\xi \in \mathbb{Q}[x]$ are polynomials with coefficients that may depend on $\xi$, and $Q_i(q_F)$ is an expression of the form $\prod_{l=1}^{N'} \frac{1}{1-q_F^{a_l}}$.

**Proof** Let $L(\xi) = \{ y \in A(\xi) : \text{val}(c(\xi)) > \text{val}(y) \}$ and $L = \bigcup_{\xi \in \mathbb{R}^r} L(\xi)$ as in the proof of Lemma 5.6, set $1 \leq b \leq N_A$ and consider $A_b, \tilde{\alpha}_i$ and $\tilde{\beta}_{ij}$, as defined in Lemma 5.7. By the Presburger cell decomposition (Theorem 2.11), $\tilde{\alpha}_i$ and $\tilde{\beta}_{ij}$ are piecewise linear in the sense of Definition 2.10. Thus, we have a finite partition $\{X_{b'}\}$ of $\{\text{val}(g_1(y)), \ldots, \text{val}(g_s(y)) : y \in A_b \}$ such that on each $X_{b'}$ the functions $\tilde{\alpha}_i$ and $\tilde{\beta}_{ij}$ are linear. Pulling $\{X_{b'}\}$ back to $A_b$ under $\text{val}(g(-))$, we then obtain a partition $A_b = \bigcup_{b'=1}^{N_{A_b}} A_{b'}$. By Lemmas 5.6 and 5.7, and the linearity of $\tilde{\alpha}_i$ and $\tilde{\beta}_{ij}$ on $A_{b'}$, the restriction $f_F|_{A_{b'},F \cap L_F}$ is a sum of terms of the following form:

$$|Y_{F,y}| \cdot q_F^{\rho} \cdot \left( \prod_{j=1}^N \rho_j \right) \cdot \left( \prod_{l=1}^{N'} \frac{1}{1-q_F^{a_l}} \right),$$

where

1. $\rho = \sum_{t=1}^s b_t \cdot \frac{w_{t,\text{val}(y)} + \text{val}(h_t(\xi)) - c_t}{n_t} + \gamma$.
2. $\rho_j = \sum_{i=1}^s b_{jt} \cdot \frac{w_{t,\text{val}(y)} + \text{val}(h_t(\xi)) - c_{jt}}{n_{jt}} + \gamma_j$.
3. The constants $b_t, n_t, c_t, b_{jt}, n_{jt}, c_{jt}$ and $\gamma_j$ are integers.
4. It holds that $0 \leq c_t < n_t$ and $0 \leq c_{jt} < n_{jt}$ for all $1 \leq t \leq s$ and $1 \leq j \leq N$ and furthermore,

$$w_{t,\text{val}(y)} + \text{val}(h_t(\xi)) \equiv c_t(\text{mod} n_t) \text{ and } w_{t,\text{val}(y)} + \text{val}(h_t(\xi)) \equiv c_{jt}(\text{mod} n_{jt}).$$
Now for a fixed \( F \in \text{Loc}_M \) and \( \xi \in k_F^t \), the functions \(|Y_{F,y}|\) and \( h_\xi(\xi) \) are constant on \( L_F(\xi) \cap A_{b_\xi}^\prime(\xi) \). This gives \( f_F \mid_{L_F \cap A_{b_\xi}^\prime} \) the required form. \( \square \)

We are ready to prove Proposition 5.5:

**Proof of Proposition 5.5** Fix \( F \in \text{Loc}_M \) and \( \xi \in k_F^t \). By Lemma 5.8, we have a partition of VF into cells, such that on each cell \( \tilde{A} \) the function \( f_F \) is of the form

\[
f_F(y) = \sum_{i=1}^{N} a_i \cdot q_i^{t_1 + t_2} \cdot \val(y) \cdot P_i(\val(y)) Q_i(q_F),
\]

for any \( y \in \tilde{A} \) with \( \val(c(\xi)) > \val(y) \). Denote the above summands of \( f_F \) by \( f_i \). If \( t = \max_i \{t_2i\} > 0 \), consider the sum \( \sum_{i:t_2i = t} f_i \) of all terms \( f_i \) such that \( t_2i = t \). We want to show this sum is zero. Since \( q_F^{\val(y)} \neq 0 \) it is enough to prove

\[
\sum_{i:t_2i = t} a_i \cdot q_i^{t_1i} \cdot P_i(\val(y)) Q_i(q_F) = 0.
\]

Note that the above sum is polynomial in \( \val(y) \), and denote it by \( P'_i(\val(y)) \). If \( P'_i \neq 0 \), then we have \( \lim_{\val(y) \to -\infty} \left| P'_i(\val(y)) \right| = \infty \). But then \( \lim_{|y| \to \infty} \frac{|f_F(y)|}{d'_{\val(y)}} = \infty \) and thus \( \lim_{|y| \to \infty} f_F(y) \neq 0 \), yielding a contradiction. Thus, we can write \( f_F \) in a reduced way, where \( \max_i \{t_2i\} < 0 \). Let \( \alpha_{\tilde{A}} := \frac{1}{2} \max_i \{t_2i\} \). The above argument implies that for any \( F \in \text{Loc}_M \), any \( \xi \in k_F^t \), and any \( y \in \tilde{A} \) we have:

\[
|f_F(y)| < d(F, \xi) \cdot \min\{|y|^\alpha_{\tilde{A}}, 1\},
\]

for some constant \( d(F, \xi) > 0 \) that depends on \( \xi \) and \( F \). Since there are finitely many cells \( \tilde{A} \) (and fibers \( \tilde{A}(\xi)) \), by taking the maximum over all \( \alpha_{\tilde{A}} \) above, there exists \( \alpha < 0 \) such that for any \( F \in \text{Loc}_M \) and \( y \in F \),

\[
|f_F(y)| < d(F) \cdot \min\{|y|^\alpha, 1\}
\]

for some constant \( d(F) > 0 \). \( \square \)

**5.2 Alternative proof of Theorem 5.2**

In this subsection we give another proof to Theorem 5.2 as suggested by the anonymous referee. The main idea is to study the Fourier transform \( \mathcal{F}(h) \) of an \( L^1 \)-motivic function \( h \) as a special case of a motivic exponential function (Definition 5.9) which decays to 0 at infinity, rather than to study the decay of \( \mathcal{F}(h) \) directly.

For \( F \in \text{Loc} \), we denote by \( \mathcal{D}_F \) the collection of additive characters of \( F \) that are trivial on \( \mathfrak{m}_F \), and satisfy \( \psi_F(x) = \exp \frac{2\pi i}{p} \text{Tr}_{k_F/p}(x) \) for any \( x \in O_F \), where \( \text{char}(k_F) = p \).
Definition 5.9 (see [9, Definition 2.7]) Let $X$ be an $L_{DP}$-definable set. A collection of functions $f = \{f_F, \psi : X_F \to \mathbb{C}\}_{F \in \mathcal{Loc}_M, \psi \in \mathcal{D}_F}$ is called a motivic exponential function if there exist $N \in \mathbb{N}$ and non-negative integers $\{r_i\}_{i=1}^N$, such that for any $F \in \mathcal{Loc}_M$, any $\psi \in \mathcal{D}_F$, and any $x \in X_F$, the function $f$ can be written as,

$$f_{F, \psi}(x) = \sum_{i=1}^{N} f_{i, F}(x) \cdot \left( \sum_{y \in Y_{i,x,F}} \psi( g_{i,F}(x,y) + e_{i,F}(x,y) ) \right),$$

where $f_i \in C(X)$, $Y_i \subseteq X \times \mathbb{R}^{r_i}$ are definable sets, and $e_i : Y_i \to \mathbb{R}$ and $g_i : Y_i \to \mathbb{R}$ are definable functions for any $1 \leq i \leq N$ (we make sense of the expression inside $\psi$ by lifting the values of $e_i, f_i(x, y)$ to $O^\times_F$). The set of motivic exponential functions on a definable set $X$ forms a ring, which we denote by $C_{exp}(X)$.

The ring $C_{exp}(X)$ is closed under integration (see [9, Theorem 2.8]), and in particular, the Fourier transform $\mathcal{F}(f)$ of an $L^1$-function $f \in C(X)$ belongs to $C_{exp}(X)$. We now turn to the proof of Theorem 5.2.

Alternative proof of Theorem 5.2 Let $h \in C(VF^m)$, and assume that $h_{F}$ is absolutely integrable for any $F \in \mathcal{Loc}_M$. We may drop the assumption that $h_{F}$ is compactly supported. Set $\hat{h}(y) := \mathcal{F}(h)(y) \in C_{exp}(VF^m)$. For any $F \in \mathcal{Loc}_M$ and $\psi \in \mathcal{D}_F$ the function $|\hat{h}_{F,\psi}(y)|$ is bounded on $F^m$, and by the Riemann-Lebesgue lemma it decays to 0 as $|y| \to \infty$. We can divide $VF^m$ to $2^m$ definable subsets

$$A_I := \{y \in VF^m : 1 \leq i \leq m, \text{ val}(y_i) \geq 0 \text{ if } I(i) = 1 \text{ and } \text{val}(y_i) < 0 \text{ if } I(i) = 0\},$$

with $I \in \{0, 1\}^m$. It is enough to prove the claim for $\hat{h}|_{A_I}$ for each such $I$. Fix $I$, and set $n_I = \# \{i : I(i) = 1\}$ Without loss of generality, assume that $I(i) = 1$ for $i = 1, \ldots, n_I$ and $I(i) = 0$ otherwise. For $t \in \mathbb{Z}^m$, set $t_{\geq 0} := (t_1, \ldots, t_{n_I})$ and $t_{< 0} := (t_{n_I+1}, \ldots, t_m)$, and define $H \in C_{exp}(VF^m \times \mathbb{Z}^m)$ by $H(y, t_{\geq 0}, t_{< 0}) := \hat{h}(y) \cdot 1_{B(t_{\geq 0}, t_{< 0})}$, where

$$B(t_{\geq 0}, t_{< 0}) := \{y \in A_I : \text{val}(y_i) = t_i \text{ for } 1 \leq i \leq m\}.$$ 

Set $W = VF^m \times \mathbb{Z}^{n_I}$ and $X = \mathbb{Z}^{m-n_I}$. By [10, Theorem 2.1.3], we can find $G \in C_{exp}(\mathbb{Z}^{m-n_I})$ and $k \in \mathbb{Z}$ such that for each $F \in \mathcal{Loc}_{M'}$ with $0 \ll M' \in \mathbb{N}$, and each $\psi \in \mathcal{D}_F$ the following holds:

$$\sup_{(y, t_{\geq 0}) \in W_F} |H_{F,\psi}(y, t_{\geq 0}, t_{< 0})|^2 = \sup_{y \in B_{t_{< 0}}F} |\hat{h}_{F,\psi}(y)|^2 \leq G_{F,\psi}(t_{< 0})$$

$$\leq q_k^f \sup_{(y, t_{\geq 0}) \in W_F} |H_{F,\psi}(y, t_{\geq 0}, t_{< 0})|^2, \quad (5.3)$$

where $B_{t_{< 0}} := \bigcup_{t_{\geq 0} \in \mathbb{Z}^{n_I}} B(t_{\geq 0}, t_{< 0})$. Note that $G_{F,\psi}(t_{< 0})$ is bounded. Set $\|t_{< 0}\| := \sum_{i=n_I+1}^{m} \|t_i\|$, then it is clear from the definition of $H$ that $H_{F,\psi}(y, t_{\geq 0}, t_{< 0}) = 0$ whenever $t_i \geq 0$ for some $n_I + 1 \leq i \leq m$. It then follows from (5.3) that
\[
\lim_{\|t_{\prec 0}\| \to \infty} |G_{F,\psi}(t_{\prec 0})| = 0. \quad \text{By [10, Theorem 3.1.1(2)]}, \quad \text{we can find } r \in \mathbb{Q}_{< 0} \quad \text{and } a \in \mathbb{Z} \quad \text{such that for any } F \in \text{Loc}_{M'} \quad (\text{we may assume that } M' \text{ is large enough}) \quad \text{and each } \psi \in \mathcal{D}_F \quad \text{we have}
\]
\[
|G_{F,\psi}(t_{\prec 0})| \leq q_F^r \|t_{\prec 0}\|
\]
for any \( t_{\prec 0} \) with \( \|t_{\prec 0}\| > a \). We then deduce from (5.3)
\[
\left| \hat{h}_{F,\psi}(y) \right|^2 < q_F^{r \sum_{i=m+1}^\infty \text{val}(y_i)} = q_F^{-r \sum_{i=m+1}^\infty \text{val}(y_i)} \leq |y|^{r(m-nI)}
\]
for any \( y \in A_I \) such that \( |y| > q^d \). To deal with small \( y \), we use [10, Theorem 2.1.1]; there exist integers \( b \) and \( d' \) such that for any \( F \in \text{Loc}_{M'} \), any \( \psi \in \mathcal{D}_F \) and any \( t_{\prec 0} \) we have
\[
\sup_{y \in B_{t_{\prec 0}, F}} \left| \hat{h}_{F,\psi}(y) \right|^2 \leq |G_{F,\psi}(t_{\prec 0})| \leq q_F^{b \|t_{\prec 0}\| + d'}.
\]
Repeating the above argument for each \( A_I \), and setting \( a_0 \) (resp. \( b_0, d'_0, \alpha \)) to be maximal amongst the \( a \)'s (resp. \( b, d', \frac{r(m-nI)}{4} \)) above, over all possible \( I \), we get
\[
\left| \mathcal{F}(f)_{F,\psi}(y) \right| < q_F^{\frac{1}{2} (b_0 a_0 + d'_0)} \cdot \min\{|y|^{\alpha}, 1\} \quad (5.4)
\]
as required.

**Remark 5.10** Since the results in [10] used above work also for definable families of motivic exponential functions, one can provide a version of Theorem 5.2 for families. This can be used to prove part (2) of Theorem 1.9, for the case that \( K = \mathbb{Q} \). For a general field of characteristic zero, one still needs to use arguments similar to those of Sects. 6 and 7.

### 6 Reduction of Theorem 1.7 to the case of \( \mathbb{Q} \)-morphisms

In this section we complete the proof of Theorem 1.7, by reducing to the case \( K = \mathbb{Q} \) which we have proved in the last section (Theorem 5.1). Explicitly, we show the following:

**Proposition 6.1** It is enough to prove Theorem 1.7 for \( K = \mathbb{Q} \) and \( \varphi : X \to V \) strongly dominant.

We prove the claim by a series of reductions:

**Lemma 6.2** It is enough to prove Theorem 1.7 for a field \( K' \) which is finitely generated over \( \mathbb{Q} \), and \( \varphi : X \to V \) a strongly dominant \( K' \)-morphism.
Let $K$ be a field of characteristic 0. We have already seen in Corollary 3.6 that we may assume that $\varphi$ is strongly dominant. Notice that since $\mathbb{Q} \subseteq K$, we have that $\varphi$ is defined over a finitely generated field $K'/\mathbb{Q}$. Since the (FRS) property is preserved under base change (Proposition 3.2), we are done. □

**Proposition 6.3** Let $K'$ and $\varphi : X \to V$ be as in Lemma 6.2. Assume there exists $N \in \mathbb{N}$ such that the $N$th convolution power $\varphi^N$ is (FRS) at $(x_1, \ldots, x)$ for any $x \in X(K')$. Then $\varphi^{2N}$ is (FRS).

**Proof** We use the analytic criterion for the (FRS) property (see Theorem 2.17) to show that $\varphi^{2N}$ is (FRS) at any point $(x_1, \ldots, x_{2N}) \in X^{2N}(K')$. Let $(x_1, \ldots, x_{2N}) \in X^{2N}(K')$, then there exists a finite extension $K''/K'$ such that $(x_1, \ldots, x_{2N}) \in X^{2N}(K'')$. We need to show that for any finite extension $K'''$ of $K''$, there exist $K''' \subseteq F \in \text{Loc}$ and a non-negative Schwartz measure $\mu$ on $X^{2N}(F)$ that does not vanish at $(x_1, \ldots, x_{2N})$, such that $\varphi^{2N}_*(\mu)$ has continuous density.

Fix such $K'''$, and let $U_i \subseteq X$ be a Zariski open neighborhoods of $(x_i, \ldots, x_i)$ such that $\varphi^N$ is (FRS) at any $x \in U_i$ (it is possible by Theorem 2.14). Notice that for any $K''' \subseteq F \in \text{Loc}$ the set $U_i(F)$ contains a set of the form $V_{i,F} \times \cdots \times V_{i,F}$, where $V_{i,F} \subseteq X(F)$ is open and $x_i \in V_{i,F}$. For any $i \in \{1, \ldots, 2N\}$, let $\mu_i$ be a non-negative Schwartz measure on $X(F)$ supported on $V_{i,F}$, that does not vanish at $x_i$. Notice that $\text{supp}(\mu_i \times \cdots \times \mu_i) \subseteq U_i(F)$.

Since $\varphi^N$ is (FRS) at $(x_i, \ldots, x_i)$, the measure $\varphi^N_*(\mu_i \times \cdots \times \mu_i)$ has continuous density with respect to the normalized Haar measure on $V(F) = F^m$. Now, use the standard identification between measures and functions on a locally compact group, and recall that the Fourier transform $\mathcal{F}(f)$ of an $L^2$-function $f$ on $F^m$ is an $L^2$-function and that the Fourier transform $\mathcal{F}(f)$ of an $L^1$-function $f : F^m \to \mathbb{C}$ is a continuous function. Since $\varphi^N_*(\mu_i \times \cdots \times \mu_i)$ is a compactly supported measure with continuous density, its density is $L^2$, and hence

$$\mathcal{F} \left( \varphi^N_*(\mu_i \times \cdots \times \mu_i) \right) = \mathcal{F} \left( \varphi_*(\mu_i) \ast \cdots \ast \varphi_*(\mu_i) \right) = \mathcal{F}(\varphi_*(\mu_i))^N$$

has $L^2$-density. This implies $\mathcal{F}(\varphi_*(\mu_i))$ has $L^{2N}$-density. By a generalization of Hölder’s inequality, it follows that

$$\mathcal{F} \left( (\varphi_*(\mu_1) \ast \cdots \ast (\varphi_*(\mu_{2N})) \right) = \prod_{i=1}^{2N} \mathcal{F}(\varphi_*(\mu_i))$$

has $L^1$-density, and hence $(\varphi_*(\mu_1) \ast \cdots \ast (\varphi_*(\mu_{2N}))$ has continuous density. As a consequence, we get that $\varphi^{2N}$ is (FRS) at $(x_1, \ldots, x_{2N})$. □

Let $\varphi : X \to V$ be a strongly dominant $K'$-morphism. Any finitely generated field $K'/\mathbb{Q}$ is a finite extension of some $\mathbb{Q}(t_1, \ldots, t_n)$. Since $X$ and $\varphi$ are defined using finitely many polynomials with coefficients in $K'$, there exists $f \in \mathbb{Q}[t_1, \ldots, t_n]$, such that $X$ and $\varphi$ are defined over a ring $A$, which is finite over $\mathbb{Q}[t_1, \ldots, t_n, f^{-1}]$. Since $X$ is smooth over the generic point $\text{Spec}(K')$, by further localizing, we may assume
that $X$ is smooth over $A$. We denote the resulting $A$-model for the diagram $\varphi$ by $\varphi_A : X_A \to V_A$, i.e. $X$, $V$ and $\varphi$ are base changes of $X_A$, $V_A$ and $\varphi_A$ to $\Spec(K')$.

Since $A$ is finite type over $\mathbb{Q}$, the morphisms $X_A \to \Spec(A) \to \Spec(\mathbb{Q})$ and $V_A \to \Spec(A) \to \Spec(\mathbb{Q})$ endow $X_A$ and $V_A$ with a natural structure of $\mathbb{Q}$-varieties, denoted by $\tilde{X}_A$ and $\tilde{V}_A$ (and we similarly denote $\tilde{\varphi}_A$). Hence, $\tilde{X}_A$ (resp. $\tilde{V}_A$) is a family of smooth $\mathbb{Q}$-varieties (resp. $\mathbb{Q}$-vector spaces) over $\Spec(A)$. Notice that the convolution operation can be generalized to such $\Spec(A)$-families by $\varphi \ast \varphi := \mult_A \circ (\varphi, \varphi)$, where $\mult_A : G \times \Spec(A) \to G$ is the multiplication map over $\Spec(A)$, and $(\varphi, \varphi) : X \times \Spec(A) \to G \times \Spec(A) G$. With the above terminology, we can now prove the following:

**Lemma 6.4** Let $\tilde{X}_a$ and $\tilde{V}_a$ be the fibers of the varieties $\tilde{X}_A$ and $\tilde{V}_A$ over $a \in \Spec(A)$. It is enough to show that for every $a \in \Spec(A)$ there exists $n_a \in \mathbb{N}$ such that $\tilde{\varphi}^n_a : \tilde{X}_a \times \cdots \times \tilde{X}_a \to \tilde{V}_a$ is (FRS) at $y = (x, \ldots, x)$ for any $x \in \tilde{X}_a$.

**Proof** Notice that $\tilde{\varphi}^n_a$ is the fiber of $\tilde{\varphi}^n_A$ over $a \in \Spec(A)$. Let $\pi : \tilde{X}_A \to \Spec(A)$ be the $\mathbb{Q}$-structure map and consider $\pi^n : \tilde{X}_A^n \to \Spec(A)$, where $\tilde{X}_A := \tilde{X}_A \times \Spec(A) \times \cdots \times \Spec(A) \tilde{X}_A$. It follows by [1, Corollary 2.2], that the set

$$U_n := \{x \in \tilde{X}_A(\mathbb{Q}) : \tilde{\varphi}^n_{\pi^n(y)} : \tilde{X}^n_{\pi^n(y)} \to \tilde{V}_{\pi^n(y)} \text{ is (FRS) at } y = (x, \ldots, x)\}$$

is open for any $n \in \mathbb{N}$. By our assumption, $\tilde{X}_A(\mathbb{Q}) = \bigcup_{n=1}^{\infty} U_n$, and by the fact that the $\{U_n\}_{n=1}^{\infty}$ are increasing combined with quasi-compactness, there exists $N \in \mathbb{N}$ such that $U_N = \tilde{X}_A$, implying that $\tilde{\varphi}^N_A$ is (FRS) at $(x, \ldots, x)$ for any $x \in \tilde{X}_A(\mathbb{Q})$. By the last proposition, the morphism $\tilde{\varphi}^{2N}_A$ is (FRS). Since $\varphi^{2N} : X^{2N} \to V$ is the generic fiber of $\tilde{\varphi}^{2N}_A$, then $\varphi^{2N}$ is (FRS) as well. \qed

The next lemma is the final reduction before we can prove Proposition 6.1.

**Lemma 6.5** It is enough to prove Theorem 1.7 for a number field $K$ and $\varphi : X \to V$ a strongly dominant $K$-morphism.

**Proof** Let $a \in \Spec(A)(\mathbb{Q})$. Then there exists a finite extension $K/\mathbb{Q}$ such that $a \in \Spec(A)(K)$. By our assumption, there exists $n \in \mathbb{N}$ such that the morphism $\tilde{\varphi}^n_a : \tilde{X}_a \times \cdots \times \tilde{X}_a \to \tilde{V}_a$ is (FRS) and by Lemma 6.4 we are done. \qed

We can now finish the proof of Proposition 6.1.

**Proof of Proposition 6.1** Let $\varphi : X \to V$ be a strongly dominant $K$-morphism, where $K$ is a number field. We may assume that $K/\mathbb{Q}$ is Galois. By restriction of scalars we obtain a $\mathbb{Q}$-morphism $\Res^K_Q \varphi : \Res^K_Q(X) \to \Res^K_Q(V)$. By our assumption, $\Res^K_Q(\varphi^n) = \Res^K_Q(\varphi^n)$ is (FRS) for some $n \in \mathbb{N}$. Since the (FRS) property is preserved under base change to $K$, we obtain that

$$\left((\Res^K_Q(\varphi))^n\right)_K = \left((\Res^K_Q(\varphi))^n\right) : \left(\Res^K_Q(X)_K\right)^n \to \left(\Res^K_Q(V)_K\right)^n$$
is (FRS). Finally, notice that Res\(^K\)(X\(_K\)) (resp. Res\(^K\)(V\(_K\))) is a product of \(l := [K : \mathbb{Q}]\) K-varieties \(X_1, \ldots, X_l\) (resp. K-vector spaces \(V_1, \ldots, V_l\)), each of which is \(\mathbb{Q}\)-isomorphic to \(X\) (resp. \(V\)), and that Res\(^K\)\((\varphi^n)\) can be written as,

\[\varphi^n \times \cdots \times \varphi^n : X^n_1 \times \cdots X^n_l \rightarrow V_1 \times \cdots \times V_l,\]

where each \(\varphi_i\) is a twist of \(\varphi_K\) by some Galois element \(\sigma_i \in \text{Gal}(K/\mathbb{Q})\), with \(\varphi_1 := \varphi\). This allows us to deduce that for each \(i\), the morphism \(\varphi^n_i\) is (FRS), and in particular \(\varphi^n\) is (FRS) as well. \(\square\)

7 Convolution properties of algebraic families of morphisms—a uniform version of Theorem 1.7

Let \(K\) be a field of characteristic 0. Our goal in this section is to prove Theorem 1.9:

**Theorem 7.1** (Theorem 1.9) Let \(Y\) be a K-variety, set \(V = \mathbb{A}^n_K\), let \(\tilde{X}\) be a family of varieties over \(Y\) and let \(\tilde{\varphi} : \tilde{X} \rightarrow V \times Y\) be a \(Y\)-morphism. Then,

1. The set \(Y' := \{y \in Y : \tilde{X}_y\) is smooth and \(\tilde{\varphi}_y : \tilde{X}_y \rightarrow V\) is strongly dominant\} is constructible.
2. There exists \(N \in \mathbb{N}\) such that for any \(n > N\), and any \(n\) points \(y_1, \ldots, y_n \in Y'(K)\), the morphism \(\tilde{\varphi}_{y_1} \ast \cdots \ast \tilde{\varphi}_{y_n} : \tilde{X}_{y_1} \times \cdots \times \tilde{X}_{y_n} \rightarrow V\) is (FRS).

We first reduce to a similar statement about self convolutions:

**Theorem 7.2** Let \(Y, V, \tilde{X}, \tilde{\varphi}\) and \(Y'\) be as in Theorem 7.1. Then \(Y'\) is constructible and there exists \(N \in \mathbb{N}\) such that for any \(n \geq N\) and any \(y \in Y'(K)\), the morphism \(\varphi^n : \tilde{X}_y \rightarrow V\) is (FRS).

**Lemma 7.3** Theorem 7.2 implies Theorem 7.1.

**Proof** Let \(\tilde{\varphi}, Y, V, Y'\) and \(N \in \mathbb{N}\) be as in Theorem 7.2, and let \(y_1, \ldots, y_{2N} \in Y'(K)\). Notice that for any \(1 \leq i \leq 2N\) we have that \(\tilde{\varphi}_{y_i}^N\) is (FRS). For each \(i\), let \(\mu_i\) be a non-negative Schwartz measure on \(\tilde{X}_{y_i}(F)\) for a non-Archimedean field \(K \subseteq F \in \text{Loc}\), supported on \(y_i\), such that \(N\)th convolution power \((\tilde{\varphi}_{y_i})_*(\mu_i)^N\) has continuous density. By an argument similar to the one in Proposition 6.3, we can deduce that \(\mathcal{F}(\tilde{\varphi}_{y_i}(*(\mu_i)))\) has an \(L^2\)-density and that \((\tilde{\varphi}_{y_1})_*(\mu_1) \ast \cdots \ast (\tilde{\varphi}_{y_{2N}})_*(\mu_{2N})\) has continuous density. As a consequence, we get that \((\tilde{\varphi}_{y_1} \ast \cdots \ast \tilde{\varphi}_{y_{2N}}\) is (FRS). \(\square\)

We now wish to prove Theorem 7.2. We start by proving Lemmas 7.4 and 7.5 in order to deduce that \(Y'\) is constructible.

**Lemma 7.4** Let \(\tilde{X}\) and \(Y\) be as in Theorem 7.1. Then the set \(\hat{Y} := \{y \in Y : \tilde{X}_y\) is smooth\} is constructible.

**Proof** Let \(\pi_{\tilde{X}} : \tilde{X} \rightarrow Y\) be the structure morphism. By Chevalley’s Theorem, it is enough to show that the set \(\pi_{\tilde{X}}^{-1}(\hat{Y}) = \{x \in \tilde{X} : \tilde{X}_{\pi_{\tilde{X}}(x)}\) is smooth\} is constructible. Consider the following functions from \(X\) to \(\mathbb{Z}\):
1. \( x \mapsto \dim_x \hat{X}_{\pi_X(x)} := \max_{X_i \in \text{Ir}(\hat{X}_{\pi_X(x)})} \{ \dim X_i : x \in X_i \} \).

2. \( x \mapsto \dim T_x \hat{X}_{\pi_X(x)} \), assigning to each \( x \in \hat{X} \) the dimension of the Zariski tangent space of \( \hat{X}_{\pi_X(x)} \) at \( x \).

Let \( \hat{X} := \{ x \in \hat{X} : \dim T_x \hat{X}_{\pi_X(x)} \neq \dim_x \hat{X}_{\pi_X(x)} \} \) and notice that \( \pi_{\hat{X}}(\hat{X}) = \{ y \in Y : \hat{X}_y \text{ is not smooth} \} \) and that \( \hat{Y} \) is the complement of \( \pi_{\hat{X}}(\hat{X}) \) in \( Y \). By a corollary of Chevalley’s theorem, the function \( \dim \hat{X}_{\pi_X(x)} \) is upper semi-continuous. Since \( \dim T_x \hat{X}_{\pi_X(x)} \) is also upper semi-continuous, it follows that \( \hat{X} \) is a constructible set. This implies that \( \hat{Y} \) is constructible.

**Lemma 7.5** Let \( S \) be a finite type \( K \)-scheme, \( Z \) be an absolutely irreducible finite type \( K \)-scheme and \( \phi : X \to Z \times S \) be an \( S \)-morphism. Then

\[ \hat{S} := \{ s \in S \text{ such that } \phi_s \text{ is strongly dominant} \} \]

is a constructible subset of \( S \).

**Proof** We may assume that \( S \) is irreducible. Let \( \pi_X : X \to S \) and \( \pi_{Z \times S} : Z \times S \to S \) be the structure morphisms, and \( \eta \) be the generic point of \( S \). In order to prove the lemma, it is enough to prove that \( \eta \not\in \hat{S} \) (resp. \( \eta \not\in \hat{S} \)) implies the existence of an open set \( U \), such that \( U \subseteq \hat{S} \) (resp. \( U \cap \hat{S} = \emptyset \)). By showing this we deduce the lemma by Noetherian induction.

By [26, Lemma 36.22.8], we may assume that we have an affine irreducible scheme \( S' \), a surjective finite étale morphism \( \psi : S' \to S \), and a fibered diagram

\[
\begin{array}{ccc}
X_{S'} & \longrightarrow & X \\
\downarrow \pi_{X_{S'}} & & \downarrow \pi_X \\
S' & \psi \longrightarrow & S
\end{array}
\]

such that all irreducible components of the generic fiber of \( \pi_{X_{S'}} \) are absolutely irreducible. Denote by \( \tilde{\psi} : X_{S'} \to Z \times S' \) the corresponding base change of \( \psi \) to \( S' \). Since \( \psi \) is finite, we have for any \( s' \in S' \) that \( \tilde{\psi}_{s'} \) is a base change of \( \phi \psi(s') \) by a finite field extension, and hence \( \tilde{\psi}_{s'} \) is strongly dominant if and only if \( \phi \psi(s') \) is strongly dominant. Since \( \psi \) is étale, it is an open map, so we may assume that \( S = S' \) and that all irreducible components of \( X_\eta := \pi_{X_{S'}}^{-1}(\eta) \) are absolutely irreducible.

By [16, Proposition 9.7.8, 9.7.12] (or [2, Theorem 3.43]), there exists an open set \( U \subseteq S \) such that for any \( s \in U \) the following holds:

1. There is a bijection \( \Phi_{\eta,s} \) between the number of irreducible components \( X_{\eta,1}, \ldots, X_{\eta,t} \) of \( X_\eta \) and the number of irreducible components of \( X_s \).
2. All irreducible components of \( X_s \) are absolutely irreducible.

The bijection \( \Phi_{\eta,s} \) is constructed as follows. For each irreducible component \( X_{\eta,j} \) of \( X_\eta \), we have that \( X_{\eta,j} \cap X_s \) is an irreducible component of \( X_s \). Under this bijection, we deduce that the set

\[ \tilde{X}_j := \{ x \in \pi_{X_{S'}}^{-1}(U) : x \in j \text{th irreducible component of } X_{\pi_X(x)} \} \]
is locally closed. For \( s \in U \), the condition that \( \varphi_s \) is strongly dominant is equivalent to \( (\varphi)_s := (\varphi|_{\widetilde{X}_j})_s : (\widetilde{X}_j)_s \to Z \times \text{Spec}(K) \text{Spec}(K(\{s\})) \) being dominant for any \( 1 \leq j \leq t \). Hence, it is enough to show that the set
\[
\widetilde{U}_j := \{ s \in U \text{ such that } (\varphi)_{s} \text{ is dominant} \},
\]
is constructible. Notice that \( (\varphi)_{\eta} \) is dominant if and only if \( \varphi_j(\widetilde{X}_j)_{\eta} = (\varphi_j)_{\eta}(\widetilde{X}_j)_{\eta} \) is dense in \( Z \times \text{Spec}(K) \text{Spec}(K(\{\eta\})) \). It now follows from [26, Lemma 36.22.3 and 36.22.4] that we may find an open set \( W \subseteq U \) such that,
1. If \( (\varphi)_{\eta} \) is dominant then \( (\varphi)_{s} \) is dominant for any \( s \in W \).
2. If \( (\varphi)_{\eta} \) is not dominant then \( (\varphi)_{s} \) is not dominant for any \( s \in W \).

By Noetherian induction we deduce that \( \widetilde{U}_j \) is constructible for any \( j \) and thus we are done.

Combining the above lemmas, we obtain that the set
\[
Y' := \{ y \in Y : \widetilde{X}_y \text{ is smooth, and } \widetilde{\varphi}_y : \widetilde{X}_y \to V \text{ is strongly dominant} \}
\]
is constructible. To complete the proof of Theorem 7.2 we need the following lemma:

**Lemma 7.6** Let \( \varphi : X \to Y \) be a morphism of irreducible \( K \)-varieties and let \( U \subseteq X \) be an open set that contains the generic fiber of \( \varphi \). Then there exists an open set \( W \subseteq Y \) such that \( \varphi^{-1}(W) \subseteq U \).

**Proof** We may assume that \( X \) and \( Y \) are affine. Let \( \eta_Y \) be the generic point of \( Y \) and \( \varphi^* : K[Y] \to K[X] \) be the ring map corresponding to \( \varphi \). The generic fiber \( \varphi^{-1}(\eta_Y) \) can be written as
\[
\varphi^{-1}(\eta_Y) = \{ [p] \in \text{Spec}(K[X]) : (\varphi^*)^{-1}(p) = (0) \}.
\]

Let \( U \subseteq X \) be an open set containing \( \varphi^{-1}(\eta_Y) \), then we can write
\[
U = D(I) := \{ [p] \in \text{Spec}(K[X]) : I \nsubseteq p \}.
\]

By Noetherianity, \( I \) is an intersection of finitely many prime ideals, so it is enough to prove the lemma for the case where \( I \) is prime. Notice that the condition \( D(I) \supseteq \varphi^{-1}(\eta_Y) \) implies that \( (\varphi^*)^{-1}(I) \neq \{0\} \). Hence, there exists some \( f \in K[Y] \) such that \( \varphi^*(f) = f \circ \varphi \in I \). But then, we have \( D(f \circ \varphi) \subseteq U \), and thus \( D(f \circ \varphi) = \varphi^{-1}(D(f)) \), so we are done.

We can now finish the proof of Theorem 7.2:

**Proof of Theorem 7.2** Since \( Y' \) is a constructible subset of \( Y \), we may write \( Y' = \bigcup_{i=1}^{j} Y_i \), where each \( Y_i \) is a locally closed subset of \( Y \). It is enough to prove the theorem
for each \( Y_i \subseteq Y' \). By decomposing \( Y_i \) into irreducible components \( Y_{i,1}, \ldots, Y_{i,d} \), it is enough to prove the theorem for the case where \( Y = Y' \) and \( Y \) is irreducible. Note that, for any \( y \in Y \), the fiber \( \tilde{X}_y \) is smooth and \( \tilde{\varphi}_y : \tilde{X}_y \to V \) is a strongly dominant morphism.

Denote by \( \eta_y \) the generic point of \( Y \). We first want to show the existence of an open set \( W \subseteq Y \) and \( N_W \in \mathbb{N} \) such that \( \tilde{\varphi}_y^{N_W} : \tilde{X}_y^{N_W} \to V \) is (FRS) for all \( y \in W \). By generic flatness ([16, Theorem 6.9.1]), we may assume that \( \tilde{X} \) is flat over \( Y \). Denote by \( \tilde{\pi}_y : \tilde{X} \to Y \) its structure map. By Theorem 1.7, there exists \( N(\eta) \in \mathbb{N} \) such that the \( N(\eta) \)-th convolution power \( \tilde{\varphi}_\eta^{N(\eta)} \) is (FRS). Denote \( \tilde{X}^{N(\eta)}, Y := \tilde{X} \times_Y \cdots \times_Y \tilde{X} \) and let \( \tilde{\pi}_\tilde{X}^{N(\eta)} : \tilde{X}^{N(\eta)}, Y \to Y \) be the corresponding (flat) structure map. By [1, Corollary 2.2], the set

\[
U := \{ x \in \tilde{X}^{N(\eta)}, Y (\tilde{K}) : \tilde{\varphi}_\tilde{X}^{N(\eta)}(x) : \tilde{X}^{N(\eta)}, Y \to V \text{ is (FRS) at } x \}
\]

is open. Note that we used here the flatness of \( \tilde{\pi}_\tilde{X}^{N(\eta)} \). Since \( U \) contains the generic fiber \( \left( \tilde{\pi}_\tilde{X}^{N(\eta)} \right)^{-1}(\eta) \), by Lemma 7.6 we have \( U \supseteq \left( \tilde{\pi}_\tilde{X}^{N(\eta)} \right)^{-1}(W) \) for some open \( W \subseteq Y \), and hence \( \tilde{\varphi}_\tilde{X}^{N(\eta)} : \tilde{X}_y^{N(\eta)}, Y \to V \) is (FRS) for any \( y \in W \). We can now repeat the process for the closed subvariety \( Y_1 := Y \setminus W \). By Noetherian induction, we can deduce the existence of \( N \in \mathbb{N} \) such that \( \tilde{\varphi}_y^N : \tilde{X}_y^N \to V \) is (FRS) for any \( y \in Y \) as required.

We are now ready to prove Corollary 1.11. We first introduce the notion of complexity.

**Definition 7.7** Denote by \( \mathcal{C}_D \) the class of all \( K \)-schemes and \( K \)-morphisms of complexity at most \( D \), which is defined in the following way:

1. An affine \( K \)-scheme \( X \) has complexity at most \( D \) (i.e \( X \in \mathcal{C}_D \)) if \( X \) has a closed embedding \( \psi_X : X \hookrightarrow \mathbb{A}_K^N \) with \( K[X] = K[x_1, \ldots, x_N]/(f_1, \ldots, f_k) \), where \( N, k, a \) and \( \max\{\deg(f_i)\} \) are at most \( D \).
2. A morphism \( \varphi : X \to Y \) of affine \( K \)-schemes is said to be of complexity at most \( D \) (i.e \( \varphi \in \mathcal{C}_D \)), if \( X, Y \in \mathcal{C}_D \) with embeddings \( \psi_X : X \hookrightarrow \mathbb{A}_K^{N_1} \) and \( \psi_Y : Y \hookrightarrow \mathbb{A}_K^{N_2} \) as above, such that the polynomial map \( \varphi : \mathbb{A}_K^{N_1} \to \mathbb{A}_K^{N_2} \) induced from \( \varphi \) is of the form \( \varphi = (\varphi_1, \ldots, \varphi_{N_2}) \), where the degrees of \( \{\varphi_i\}_{i=1}^N \) are at most \( D \).
3. A separated \( K \)-scheme \( X \) is in \( \mathcal{C}_D \) if it has an open affine cover \( X = \bigcup_{i=1}^N U_i \) where \( U_1, \ldots, U_N \in \mathcal{C}_D \), such that \( N \leq D \), and the transition maps of the (affine) intersections \( \{U_i \cap U_j\}_{i,j} \) are in \( \mathcal{C}_D \).
4. A morphism \( \varphi : X \to Y \) of separated \( K \)-schemes is in \( \mathcal{C}_D \), if there exist open affine covers \( Y = \bigcup_{j=1}^N V_j \), and \( X = \bigcup_{j=1}^N \bigcup_{i=1}^{N_j} U_{ij} \) with \( N, N_j \leq D \) for any \( j \), and such that the following holds for any \( 1 \leq j, j' \leq N \) and \( 1 \leq i, i' \leq N_j \):
   (a) \( \varphi(U_{ij}) \subseteq V_j \).
   (b) The morphisms \( \varphi|_{U_{ij}} : U_{ij} \to V_j \) are in \( \mathcal{C}_D \).
   (c) The transition maps of the intersections \( \{U_{i'j'} \cap U_{ij}\}_{i',j',j} \) are in \( \mathcal{C}_D \).
(d) The transition maps of the intersections \( \{ V_{j'} \cap V_j \}_{j, j'} \) are in \( C_D \).

5. A \( K \)-scheme \( X \) is in \( C_D \) if it has an open affine cover \( X = \bigcup_{i=1}^N U_i \) by \( U_1, \ldots, U_N \in C_D \), such that \( N \leq D \), and the transition maps of the intersections \( \{ U_i \cap U_j \} \) are in \( C_D \) (note that the schemes \( \{ U_i \cap U_j \} \) are not necessarily affine but are separated). We say that a morphism \( \varphi : X \to Y \) of \( K \)-schemes is in \( C_D \) if it satisfies the same demands as in (4).

Notice that the set \( Z_D \) of all affine \( K \)-schemes \( X \in C_D \) is an algebraic family. Indeed, let \( X \in Z_D \) be an affine \( K \)-scheme with coordinate ring \( K[X] = K[x_1, \ldots, x_m]/\langle f_1, \ldots, f_k \rangle \) consisting of \( m \) generators and \( k \) relations. Since the degrees and the number of variables of \( \{ f_i \}_{i=1}^k \) are bounded by \( D \), the set of all such schemes has a natural structure of an affine space \( Z_{m,k} \). We can write \( Z_D \) as a disjoint union of affine spaces \( Z_D = \bigsqcup_{m,k=1}^D Z_{m,k} \).

Let \( V = \mathbb{A}^m_K \) and \( D > m \). By a similar argument, the set of all morphisms \( \varphi : W \to \mathbb{A}^m_K \) in \( C_D \) forms an algebraic family, denoted \( Y_D \). We can now prove Corollary 1.11:

**Corollary 7.8** For any \( m < D \in \mathbb{N} \), there exists \( N(D) \in \mathbb{N} \) such that for any \( n > N(D) \) and \( n \) strongly dominant morphisms \( \{ \varphi_i : X_i \to \mathbb{A}^m_K \}_{i=1}^n \) of complexity at most \( D \), with \( \{ X_i \}_{i=1}^n \) smooth \( K \)-varieties, the morphism \( \varphi_1 \cdots \varphi_n \) is (FRS).

**Proof** Consider the variety \( Y_D \) as constructed above. Let \( \tilde{X} := \{ (w, (W, \varphi)) : w \in W, (W, \varphi) \in Y_D \} \) and define \( \tilde{\varphi} : \tilde{X} \to \mathbb{A}^m_K \times Y_D \) by

\[
\tilde{\varphi}(w, (W, \varphi)) = (\varphi(w), (W, \varphi)).
\]

Note that \( \tilde{X} \) is a \( Y_D \)-variety, \( \tilde{\varphi} \) is a \( Y_D \)-morphism and that for any \( (W, \varphi) \in Y_D \), the fiber \( \tilde{\varphi}^{-1}(W, \varphi) \) is just the map \( \varphi : W \to \mathbb{A}^m_K \). The corollary now follows from Theorem 7.1. \( \square \)

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