The Ruijsenaars-Schneider Model in the Context of Seiberg-Witten Theory

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The compactification of five dimensional \( N = 2 \) SUSY Yang-Mills (YM) theory onto a circle provides a four dimensional YM model with \( N = 4 \) SUSY. This supersymmetry can be broken down to \( N = 2 \) if non-trivial boundary conditions in the compact dimension, \( \phi(x_5 + R) = e^{2\pi i \epsilon} \phi(x_5) \), are imposed on half of the fields. This two-parameter \((R, \epsilon)\) family of compactifications includes as particular limits most of the previously studied four dimensional \( N = 2 \) SUSY YM models with supermultiplets in the adjoint representation of the gauge group. The finite-dimensional integrable system associated to these theories via the Seiberg-Witten construction is the generic elliptic Ruijsenaars-Schneider model. In particular the perturbative (weak coupling) limit is described by the trigonometric Ruijsenaars-Schneider model.

1 Introduction

Seiberg-Witten theory [1] provides a description of the effective low-energy actions of four dimensional \( N = 2 \) SUSY Yang-Mills theories in terms of finite-dimensional integrable systems [2, 3]. Such a description has been extended to five dimensional \( N = 2 \) SUSY theories [4, 5] with one dimension compactified on a circle of radius \( R \) [4]. By starting with such a five dimensional model one may obtain four dimensional \( N = 2 \) SUSY models (with fields only in the adjoint representation of the gauge group) by imposing non-trivial boundary conditions on half of the fields:

\[
\phi(x_5 + R) = e^{2\pi i \epsilon} \phi(x_5) .
\]

If \( \epsilon = 0 \) one obtains \( N = 4 \) SUSY in four dimensions, but when \( \epsilon \neq 2\pi n \) this is explicitly broken to \( N = 2 \). The low-energy mass spectrum of the four dimensional theory contains two towers of Kaluza-Klein modes:

\[
M = \frac{\pi n}{R} \quad \text{and} \quad M = \frac{\epsilon + \pi n}{R}, \quad n \in \mathbb{Z} .
\]
In accordance with \cite{4,6} the prepotential $F(a_i)$ for the group $SU(N)$ \((i = 1 \ldots N)\) should be

\[
F(a) = -\tau \sum_{i}^{N} a_i^2 + \frac{1}{i\pi} \left( \frac{1}{R} \right)^2 \sum_{i,j}^{'} \sum_{n=-\infty}^{\infty} \left\{ (Ra_{ij} + \pi n)^2 \log (Ra_{ij} + \pi n) - (Ra_{ij} + \pi n - \epsilon)^2 \log (Ra_{ij} + \pi n - \epsilon) + \text{regulator} \right\}
\]

+ non-perturbative (instanton-induced) corrections

\[
= \sum_{i,j}^{'} f(a_{ij}) + \text{corrections.}
\]

Here

\[
f(a) = -\tau N a^2 + \frac{1}{2i\pi R^2} \left( \text{Li}_3 \left( e^{-2iRa} \right) - \text{Li}_3 \left( e^{-2i(Ra+\epsilon)} \right) \right)
\]

and

\[
\frac{\partial^2 f}{\partial a^2} = -\frac{2\tau}{N} + \frac{1}{i\pi} \sum_{n=-\infty}^{\infty} \left( \log (Ra + \pi n) - \log (Ra + \epsilon + \pi n) \right) + \text{corrections}
\]

\[
= -\frac{2\tau}{N} + \frac{1}{i\pi} \log \frac{\sin aR}{\sin(aR + \epsilon)} + \text{corrections.}
\]

In these formulas $\tau = \frac{4\pi g^2}{\sigma_2} + \frac{\theta}{2\pi}$ is the bare coupling constant and $a_{ij} \equiv a_i - a_j$, where $a_i$ \((\sum_{i}^{N} a_i = 0)\) characterize the (eigenvalues of the) vacuum expectation values of the scalar fields which spontaneously break the $SU(N)$ gauge symmetry down to the abelian one, $U(1)^{N-1}$. Note that $T_{ij}(a) \equiv \frac{\partial^2 F}{\partial a_i \partial a_j}$ are couplings of the effective low-energy abelian theory and they depend on the choice of vacuum. Such an explicit occurrence of the bare coupling $\tau$ is typical of UV-finite YM theories which possess the highest possible supersymmetry, perhaps softly broken.

Now Seiberg-Witten theory (implicitly) provides exact expressions for the full non-perturbative prepotential $F(a)$ (i.e. with all the instantonic corrections included) via

\[
a_i = \oint_{A_i} dS,
\]

\[
\frac{\partial F}{\partial a_i} = \oint_{B_i} dS.
\]

Here $dS$ is the generating 1-form on the spectral curve $C$ of the associated one dimensional integrable model, and $A_i, B_i$ are the conjugate 1-cycles on $C$, $A_i \circ B_j = \delta_{ij}$. The parameters $a_i$ may be considered as (some) moduli of the complex structures on $C$.

According to the proposal of \cite{4}, the prepotential (5) arising from this five dimensional theory may be associated with the elliptic Ruijsenaars-Schneider integrable model \cite{7}. In various double-scaling limits it reduces to well-studied systems:

(a) If $R \rightarrow 0$ (with finite $\epsilon$) the (finite) mass spectrum (2) reduces to a single point $M = 0$. This is the standard four dimensional $N = 2$ SUSY YM model – the original and simplest example of Seiberg-Witten theory \cite{1} – associated with the periodic $A_{N-1}$ Toda chain for $SU(N)$ \cite{2} and with the appropriate periodic Toda chain for other gauge groups \cite{8}. In this situation $N = 2$ SUSY in four dimensions is insufficient to ensure UV-finiteness, thus $\tau \rightarrow i\infty$, but the

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\textsuperscript{1} Our conventions are to include a factor of $1/(2\pi i)$ in the definition of $\oint$. Thus $\oint dz/z = 1$.  

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The phenomenon of dimensional transmutation occurs whereupon one substitutes the dimensionless \( \tau \) by the new dimensionful parameter \( \Lambda^N = e^{2\pi i r} (\epsilon/R)^N \).

(b) If \( R \to 0 \) and \( \epsilon \sim mR \) for finite \( m \), then UV finiteness is preserved. The mass spectrum \((2)\) reduces to the two points \( M = 0 \) and \( M = m \). This is the four dimensional YM model with \( N = 4 \) SUSY softly broken to \( N = 2 \). The associated finite-dimensional integrable system \([9, 10]\) is the elliptic Calogero model \([11, 12]\). Case (a) is then obtained from (b) by Inosemtsev’s \([13]\) double scaling limit when \( m \to 0 \), \( \tau \to i\infty \) and \( \Lambda^N = m^N e^{2\pi i r} \) is fixed.

(c) If \( R \neq 0 \) but \( \epsilon \to i\infty \) the mass spectrum \((2)\) reduces to a single Kaluza-Klein tower, \( M = \pi n/R \), \( n \in \mathbb{Z} \). This compactification of the five dimensional model has \( N = 1 \) SUSY and is not UV-finite. Here \( \tau \to i\infty \) and \( \epsilon \to i\infty \), such that \( 2\pi \tau - N\epsilon \) remains finite. The corresponding integrable system \([4]\) is the relativistic Toda chain \([14]\).

(d) Finally, when \( R \neq 0 \) and \( \epsilon \) and \( \tau \) are both finite one distinguished case still remains: \( \epsilon = \pi/2 \). Here only periodic and antiperiodic boundary conditions occur in the compact dimension. This is the case analysed in \([15]\). It is clearly special from both the point of view of Yang-Mills theory and integrable systems.

The purpose of the present paper is to provide further details about the general Ruijsenaars-Schneider prepotential \((\ref{eq:prepotential})\) with arbitrary \( R \) and \( \epsilon \), as well as to show what is special about the point \( \epsilon = \pi/2 \). We shall mostly concentrate on the perturbative (weak coupling) limit \( \tau \to i\infty \) (with \( R \) and \( \epsilon \) fixed) when the instantonic corrections in \((\ref{eq:prepotential})\) can be neglected. In contrast with the well-studied cases (a) and (c) the perturbative limit for Calogero and Ruijsenaars-Schneider systems at first sight appears non-trivial: here elliptic systems are reduced to their trigonometric counterparts and the spectral curve appears complicated. As we shall show, however, the spectral curve may be recast in rational form\(^3\). In contrast to earlier works, this characterisation of the curve enables us to give the first calculation of the perturbative prepotential. In the perturbative limit of case (d) we reobtain the curve considered in \([15]\).

An outline of the paper is as follows. First we will briefly review the basics of Seiberg-Witten theory and the Ruijsenaars-Schneider model. These sections provide enough information to make the paper self-contained, and we reference works that examine these topics more extensively. Our strategy is to calculate the third derivatives of the prepotential via information about the spectral curves of the models. These curves are complicated, and section 4 of the paper treats the case of \( SU(2) \) as an illustrative example. After this we treat the case of general \( SU(N) \). One of our important results is that the spectral curves admit a nice separation of variables in the perturbative limit, which allows for calculation.

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\(^2\)The case \( \epsilon = 0 \) of fully unbroken five dimensional \( N = 2 \) supersymmetry is of course also distinguished, but trivial: there is no evolution of effective couplings (renormalisation group flows) and the integrable system is just that of \( N \) non-interacting (free) particles.

\(^3\) In \([8, 10]\) the perturbative period matrix

\[
\frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} = \log \frac{a_{ij}}{a_{ij} + m}
\]

was deduced from the elliptic case by ingenious yet tedious calculations, which did not make explicit use of the fact that elliptic Calogero model can be substituted by its trigonometric limit (the Sutherland model).
2 Basics of SW theory

A key feature relating SW theory and the theory of integrable systems—and one still lacking a complete explanation—is the following: the families of spectral curves \( \mathcal{C} \) arising from SW theory are parameterized by the Hamiltonians of an associated integrable system. For a given group and model the genus of \( \mathcal{C} \) is fixed and our family of curves is a certain subspace (normally of high codimension) in the moduli space of complex structures. An explicit parameterisation of curves is provided with the help of Lax operator of integrable system. For the group \( SU(N) \) this has the form of an \( N \times N \) matrix \( L \) given on “the bare spectral curve” \( \mathcal{E} \), and

\[
\mathcal{C} : \quad \det_{N \times N} (\lambda I - L(\xi)) = 0.
\]

(7)

The curve is an \( N \)-fold covering over \( \mathcal{E} \) parameterized by the spectral parameter \( \xi \).

For YM theories which are UV-finite the bare curve is elliptic (a torus) with its own modulus \( \tau \) and we choose a “bare co-ordinate” \( \xi \) so that \( d\xi \) is a holomorphic differential on \( \mathcal{E} \). In the double-scaling limits which lead to the UV-infinite theories described above this modulus disappears and is replaced by the dimensionful \( \Lambda \). In this situation the bare spectral curve degenerates into the doubly punctured Riemann sphere with spectral parameter

\[
w \sim e^{2i\xi},
\]

(8)

and \( d\xi = \frac{1}{2i} \frac{dw}{w} \). The Lax operators and integrable systems we encounter reflect the dimensionality of the underlying YM theories. For five dimensional YM theories the Lax operators are “group-like” objects (with integrable systems the Ruijsenaars-Schneider and relativistic Toda models) while for four dimensional theories they are “algebra-like” objects (with integrable systems the Calogero and ordinary Toda chain). The generating 1-forms are then found to be

\[
\begin{align*}
dS &= R^{-1} \log \lambda d\xi \quad \text{for 5d models}, \\
dS &= \lambda d\xi \quad \text{for 4d models}.
\end{align*}
\]

(9)

Suppose we parameterize a bare curve that is elliptic by the algebraic equation,

\[
\hat{y}^2 = \hat{x}^3 - \alpha \hat{x}^2 - \beta \hat{x} - \gamma.
\]

(10)

Then with \( \hat{x} = \varphi(\xi) + \alpha/3 \) and \( \hat{y} = -\frac{1}{2} \varphi'(\xi) \) we have that \( d\xi = -2 \frac{dx}{y} \). Here \( \varphi(\xi) \) is the doubly periodic Weierstrass \( \varphi \)-function with periods \( 2\omega \) and \( 2\omega' \), and \( \tau = \frac{\omega'}{\omega} \) [18, 19]. This satisfies the canonical equation

\[
\varphi'(\xi)^2 = 4(\varphi(\xi) - e_1)(\varphi(\xi) - e_2)(\varphi(\xi) - e_3), \quad e_1 + e_2 + e_3 = 0.
\]

We remark that although the Weierstrass function depends on two periods, the homogeneity relation

\[
\varphi(tz|t\omega,t\omega') = t^{-2}\varphi(z|\omega,\omega')
\]

Such alternative representations are crucially important for the description of matter in the fundamental representation of the gauge group [17]. The exact (duality) relation between these \( N \times N \) and \( 2 \times 2 \) representations still remains obscure and is beyond the scope of the present paper.
enables us to arbitrarily rescale one of these. Our final results are independent of such scaling and this allows us to choose the real period to be \( \pi \) (that is \( \omega = \pi / 2 \)). The perturbative limit is then given by the imaginary period \( 2\omega' \) becoming infinite, \( \tau \to i\infty \). In this weak coupling limit (without any double-scaling) the bare curve (10) becomes

\[
y^2 = x^2(x - 1).
\]

(11)

Then \( \hat{x} \to x - 1/3 \) with \( x = \frac{1}{\sin^2 \xi} \), \( \hat{y} \to y = -\frac{\cos \xi}{\sin \xi} \), \( \alpha = 1 \), \( \beta = \gamma = 0 \) and \( d\xi \to \frac{dx}{\sqrt{x(x - 1)}} \).

Throughout we will denote by “hatted” quantities those expressions depending on an elliptic bare curve (such as \( \hat{x} \)) while “unhatted” quantities will denote their degenerations in the \( \tau \to i\infty \) limit (such as \( x \)).

The variations of \( dS \) with respect to moduli are, by definition, holomorphic 1-forms on the spectral curve. Particular choices of coordinates for the moduli may have natural properties. For example, if \( a_i \) defined by (6) are chosen as coordinates on the moduli space, then

\[
\frac{\partial dS}{\partial a_i} = d\Omega_i
\]

are canonical 1-differentials, \( \oint A_i d\Omega_j = \delta_{ij} \). When the prepotential satisfies the WDVV equations this choice is equivalent to specifying a flat structure [20, 21]. Further the second derivatives \( T_{ij} \) of the prepotential (5) with respect to these coordinates then form the period matrix of \( C \). An even simpler expression – the “residue formula” – exists for the variation of the period matrix, that is for the third derivatives of \( F(a) \) (see, e.g. [6]):

\[
\frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k} = \frac{\partial T_{ij}}{\partial a_k} = \frac{1}{2\pi i} \text{res}_{d\xi=0} \frac{d\Omega_i d\Omega_j d\Omega_k}{\delta dS},
\]

(13)

where \( \delta dS \equiv d \left( \frac{dS}{d\xi} \right) d\xi \), or, explicitly

\[
\delta dS = R^{-1} d \log \lambda d\xi \quad \text{for 5d models},
\]

\[
\delta dS = d\lambda d\xi \quad \text{for 4d models}.
\]

(14)

We remark that although \( d\xi \) does not have zeroes on the bare spectral curve when it is a torus or doubly punctured sphere, it does in general however possess them on the covering \( C \). Examples of this will be given later in the paper.

## 3 Basics of the Ruijsenaars-Schneider model

The Ruijsenaars-Schneider model is a remarkable completely integrable system whose various limits include the (finite dimensional) Toda and Calogero-Moser models. Here we shall review only a few of its salient features needed for our calculations. More comprehensive accounts of its structure and applications may be found in [22]. The \( GL(N) \) model has an explicit Lax representation with Lax operator [4, 23, 24]

\[
\mathcal{L}_{ij} = c(\xi|\epsilon) e^{\rho_i} \frac{F(q_{ij}\xi)}{F(q_{ij}\epsilon)}.
\]

(15)

Here

\[
e^{\rho_i} = e^{\rho_i} \prod_{l \neq i} \sqrt{\varphi(q_{il}) - \varphi(\epsilon)}.
\]

(16)
where the \( p_i \) and \( q_k \) are canonically conjugate momenta and coordinates, \( \{p_i, q_j\} = \delta_{ij} \). The commuting Hamiltonians may be variously written

\[
H_k = \sum_{j \in \{1, \ldots, n\}, j \neq k} e^{\sum_{j \in J} P_j} \prod_{j \in J, k \in \{1, \ldots, n\}, j \neq k} \frac{F(q_j | \xi)}{F(q_j | \epsilon)} = \sum_{1 \leq i_1 < \ldots < i_k \leq N} e^{P_{i_1} + \ldots + P_{i_k}} \prod_{\alpha < \beta} \frac{1}{\wp(q_{i_\alpha i_\beta}) - \wp(\epsilon)}. \tag{17}
\]

(The final product is taken to be unity in the case \( k = 1 \).) These Hamiltonians arise in the description of the spectral curve \([7]\). The special functions that appear above are defined to be \([12]\)

\[
F(q | \xi) = \frac{\sigma(q - \xi)}{\sigma(q) \sigma(\xi)}, \tag{18}
\]

where \( \sigma(\xi) \) denotes the Weierstrass \( \sigma \)-function \([18, 19]\). One identity used throughout is

\[
\frac{\sigma(u - v) \sigma(u + v)}{\sigma^2(u) \sigma^2(v)} = \wp(v) - \wp(u). \tag{19}
\]

The integrability of the model depends on the functional equations satisfied by \( F \) \([17]\); the connection with functional equations and integrable systems is part of a larger story \([25]\).

Some comment on the parameters \( \xi \) and \( \epsilon \) in these formulae is called for. Here \( \xi \) is precisely the spectral parameter we have encountered in \([6]\). Further, the additional parameter \( \epsilon \) in \([13, 15]\) above is the same parameter we introduced in \([1]\) characterising the boundary conditions. Actually the integrability of \([17]\) does not require \( \epsilon \) to be real, but such a choice guarantees the reality of the Hamiltonians. The identification of these two parameters has been simplified by our choice of the real period of \( \wp \) being \( \pi \) so that both \([1]\) and \([13, 15]\) are manifestly \( \pi \) periodic. The “non-relativistic limit” \( \epsilon \to 0 \) which leads to the Calogero-Moser system means we can identify the mass \( m = \epsilon / R \) of the gauge multiplet with the Calogero-Moser coupling constant. The special point \( \epsilon = \pi / 2 \) (\( \equiv \omega \)) singled out earlier is now a half-period of the \( \wp \)-function, and at this point \( \wp'(\frac{\pi}{2}) = 0 \).

From the point of view of classical integrability the overall normalisation factor \( c(\xi | \epsilon) \) of the Lax operator does not change the integrals of motion apart from scaling. This normalisation factor does however lead to the rescaling of \( \lambda \) and this effects the explicit form of the 1-form \( ds \) \([6]\). We shall see below that significant simplifications occur with the choice

\[
c(\xi | \epsilon) = c_0(\xi | \epsilon) = \frac{1}{\sqrt{\wp(\xi) - \wp(\epsilon)}} = \frac{\sigma(\xi) \sigma(\epsilon)}{\sqrt{\sigma(\epsilon - \xi) \sigma(\epsilon + \xi)}} = \sqrt{c_+(\xi | \epsilon) c_- (\xi | \epsilon)} \tag{20}
\]

where

\[
c_\pm(\xi | \epsilon) \equiv \frac{\sigma(\xi) \sigma(\epsilon)}{\sigma(\epsilon \pm \xi)} = \mp \frac{1}{F(\epsilon \mp \xi)}, \tag{21}
\]

Similar issues of normalisation enter into discussion of separation of variables \([20]\).

For future reference it is convenient to record here the value of these expressions in the perturbative (weak coupling) limit \( \tau \to i \infty \). We have already seen that \( \wp(\xi) \to \frac{1}{\sin^2 \xi - \frac{1}{3}} \). Further,

\[
c_0(\xi | \epsilon) \to \frac{1}{\sqrt{\sin^{-2} \xi - \sin^{-2} \epsilon}} = \frac{\sin \xi \sin \epsilon}{\sqrt{\sin(\epsilon - \xi) \sin(\epsilon + \xi)}}, \quad c_\pm(\xi | \epsilon) \to \frac{\sin \xi \sin \epsilon}{\sin(\epsilon \pm \xi)} e^{\mp \xi / 3} \equiv \bar{c}_\pm e^{\mp \xi / 3}. \tag{22}
\]

\[5\] Note that although the choice of normalisation \( c(\xi | \epsilon) \) (or, equivalently, rescaling \( \lambda \)) can change the manifest expression for \( ds \), it affects neither the symplectic 2-form \( \frac{d\lambda}{\lambda} \wedge \frac{d\omega}{\omega} \), nor the period matrix, which is the second derivative of the prepotential.
The corresponding integrable system is then nothing but the trigonometric Ruijsenaars-Schneider model. Finally the special point $\epsilon = \omega = \frac{\pi}{2}$ sees

$$\varphi(\omega) = e_1 \rightarrow \frac{2}{3}$$

and the particular combination (20) becomes

$$c_0(\xi|\epsilon = \pi/2) \rightarrow \tan \xi.$$  

(24)

4 The case of SU(2)

Before generalising to the higher rank situation it is instructive to first consider the case of gauge group SU(2). In the formulae of the previous section this corresponds to setting $N = 2$ and working in the center-of-mass frame $p_1 + p_2 = 0$. We define $p \equiv p_1$ and $q \equiv q_1 - q_2$. Having constructed the spectral curve we shall proceed to calculate the third derivative of $F(a)$, first directly, then via the residue formula. These will be seen to agree with those coming from (5).

4.1 Spectral curve

Using the explicit form of the Lax operator the spectral curve (7) is found to be

$$\lambda^2 - cu\lambda + c^2(\varphi(\xi) - \varphi(\epsilon)) = \lambda^2 - cu\lambda + \frac{c^2}{c_0^2} = 0,$$

(25)

where

$$u \equiv H_1(p, q) = 2 \cosh p \sqrt{\varphi(q) - \varphi(\epsilon)}$$

(26)

is the Hamiltonian (17). For the choice $c(\xi|\mu) = c_0(\xi|\mu)$ this simplifies to yield

$$\lambda^2 - u c_0 \lambda + 1 = 0,$$

(27)

or simply

$$c_0^{-1}(\xi) = \frac{u \lambda}{\lambda^2 + 1} = \frac{u}{\lambda + \lambda^{-1}}.$$  

(28)

Observe that with our choice (20) this equation, which describes the Seiberg-Witten spectral curve, may be expressed in the form

$$u = H(\log \lambda, \xi).$$

(29)

Comparison with (5) shows that the generating differential $RdS = \log \lambda d\xi$ takes the form $pdq$ here. This appears to be the case in all known examples [2]: the Seiberg-Witten generating differential takes the form $pdq$ for the corresponding integrable system. A general proof of this rather natural correspondence is still lacking.

Upon using (24) we see that (28) reduces to

$$\cot \xi = \frac{u \lambda}{\lambda^2 + 1}$$

(30)
at the special point \( \epsilon = \pi/2 \) and in the perturbative limit. The rational spectral curve for this situation was given in [31] as

\[
w = \frac{(\lambda - \lambda_1)(\lambda - \lambda_1^{-1})}{(\lambda + \lambda_1)(\lambda + \lambda_1^{-1})}.
\]

(31)

These two expressions are seen to agree upon setting \( w = -e^{2i\xi} \) and \( iu = (\lambda_1 + \lambda_1^{-1}) \).

4.2 The period matrix in the perturbative limit

As we described in section 2, the derivatives of the 1-form \( dS \) with respect to the moduli are holomorphic differentials. A particular choice of coordinates for the moduli will lead to the canonical holomorphic 1-differentials. For the example at hand there is only one modulus and

\[
dv = \frac{\partial dS}{\partial u} = R^{-1} \frac{1}{\lambda} \frac{\partial \lambda}{\partial u} d\xi = R^{-1} \frac{c_0 d\xi}{2 \lambda - c_0 u} = \frac{1}{R} \frac{d\xi}{\sqrt{u^2 - 4/c_0^2}} = \frac{1}{2R} \frac{d\xi}{\sqrt{\frac{u^2}{4} + \phi(\epsilon) - \phi(\xi)}}.
\]

(32)

If we order the roots \( e_3 \leq e_2 \leq e_1 \) we may take for the \( A \) integral

\[
\oint_A dv = \frac{1}{i\pi} \int_{e_3}^{e_1} dv = \frac{1}{i\pi R \sqrt{\left(\frac{u^2}{4} + \phi(\epsilon) - e_2\right)(e_1 - e_3)}} K \left( \frac{\left(\frac{u^2}{4} + \phi(\epsilon) - e_1\right)(e_2 - e_3)}{\left(\frac{u^2}{4} + \phi(\epsilon) - e_2\right)(e_1 - e_3)} \right).
\]

(33)

Here \( K(q) \) is the complete elliptic integral of the first kind. Dividing \( dv \) by the right hand side of this expression would then give us the canonical holomorphic 1-differential \( d\Omega \). Similarly for the \( B \) integral we have

\[
\oint_B dv = \frac{1}{i\pi} \int_{e_2}^{e_1} dv = \frac{1}{\pi R \sqrt{\left(\frac{u^2}{4} + \phi(\epsilon) - e_2\right)(e_1 - e_3)}} K \left( \frac{\left(\frac{u^2}{4} + \phi(\epsilon) - e_1\right)(e_2 - e_3)}{\left(\frac{u^2}{4} + \phi(\epsilon) - e_2\right)(e_1 - e_3)} \right),
\]

(34)

and so the period matrix is

\[
T = \oint_B d\Omega = -i K \left( \frac{\left(\frac{u^2}{4} + \phi(\epsilon) - e_1\right)(e_2 - e_3)}{\left(\frac{u^2}{4} + \phi(\epsilon) - e_2\right)(e_1 - e_3)} \right) / K \left( \frac{\left(\frac{u^2}{4} + \phi(\epsilon) - e_1\right)(e_2 - e_3)}{\left(\frac{u^2}{4} + \phi(\epsilon) - e_2\right)(e_1 - e_3)} \right).
\]

(35)

One can take the limit \( \epsilon \to 0 \) in this expression to obtain \( T = -\tau \). This agrees perfectly with formula (3) for \( SU(2) \) and justifies our identification of the gauge theory coupling constant and the modulus of the bare spectral curve.

Let us now calculate the same quantities in the perturbative limit. Now \( d\xi \to \frac{dx}{x \sqrt{x - 1}} \), \( \varphi(\xi) \to \sin^{-2} \xi - 1/3 \) and upon substituting \( 1/c_0^2 = x - \sin^{-2} \epsilon \) from (21) we obtain

\[
dv \to \frac{1}{2R} \frac{dx}{x \sqrt{(x - 1)(U^2 - x)}},
\]

(36)

with \( U^2 = \frac{1}{\sin^2 \epsilon} + \frac{u^2}{4} \). We have \( e_2 = e_3 = -1/3 \) and \( e_1 = 2/3 \) in this limit and so the \( A \)-period in this case shrinks to a contour around \( x = 0 \). Now \( \oint_A dv = 1/(2iRU) \) and we may identify the canonical differential \( d\Omega = 2iRUdv \). This result also follows from (33) upon using

\[
K(q) = \frac{\pi}{2} \left( 1 + \frac{q^2}{4} + \ldots \right).
\]

This paper in fact used the slightly different normalisation \( w = -2e^{2i\xi} \) from that being used here and we have compensated for this in the expressions being presented.
The $B$-period of $d\Omega$ again gives the period matrix. The corresponding integral now goes (twice) between $x = 0$ and $x = 1$, and the integral $\oint_B dv$ diverges logarithmically in the vicinity of $x = 0$. This divergence was to be expected because the period matrix (3) contains a term $\tau$ on the right hand side and the perturbative limit is given by $\tau \to i\infty$. Upon making the rational substitution $x = \frac{v}{1+v}$ we obtain

$$T = \oint_B d\Omega = \frac{U}{\pi} \int_0^1 \frac{dx}{x\sqrt{(x-1)(U^2-x)}} = \lim_{\varepsilon \to 0} \frac{U}{i\pi \sqrt{U^2-1}} \int_0^\infty \frac{dv}{v\sqrt{v+\frac{U^2}{U^2-1}}}$$

(37)

where $\varepsilon$ is a small-$x$ cut-off. Thus, the $U$ dependent part of this integral is finite and can be considered as the “true” perturbative correction, while the divergent part just renormalises the bare “classical” coupling constant $\tau$. Again the same result follows from (35) upon using

$$\lim_{q \to 1} \left( K(q) - \frac{1}{2} \ln \left( \frac{16}{1-q^2} \right) \right) = 0.$$

The final ingredient we wish are the $a$-variables, i.e. the $A$-period of $dS$ itself. This will correspond to the integral of (33) with respect to $u$, which is a rather complicated integral. For our purposes the perturbative limit will suffice when there are several simplifications. From the definition of (6) and that of $dS$ we find

$$a = \frac{1}{R} \oint \frac{\log \lambda dx}{x\sqrt{x-1}} = \frac{\log \lambda|_{x=0}}{iR}.$$

Now at $x = 0$ we have that $c_0 = i\sin\epsilon$ while at the same time $\lambda + \lambda^{-1} = c_0 u$. Together these yield

$$2\cos aR = iu \sin \epsilon$$

and

$$U^2 = \frac{u^2}{4} + \frac{1}{\sin^2 \epsilon} = \frac{1}{\sin^2 \epsilon} (1 - \cos^2 aR) = \frac{\sin^2 aR}{\sin^2 \epsilon}.$$

(38)

Substituting this expression for $U^2$ into (37) we obtain for the finite part of the period matrix

$$T_{\text{finite}} = \frac{1}{i\pi} \log \frac{\sin^2 aR}{\sin^2 \epsilon - \sin^2 aR} = \frac{1}{i\pi} \log \frac{\sin^2 aR}{\sin(a + aR) \sin(a - aR)}.$$

(39)

Upon using $\sin^2 a - \sin^2 b = \sin(a + b) \sin(a - b)$ this yields precise agreement with (3) where in the case of $SU(2)$ we note that only two terms contribute to the sum: one with $a_{12} = a$ and one with $a_{21} = -a$.

### 4.3 The perturbative residue formula

We remark that the problems we encountered in the previous section of divergent integrals do not arise when working with the derivatives of the period matrix with respect to its moduli (i.e. with the third derivative of the prepotential). Indeed the most effective way to deal with perturbative prepotentials is to calculate them via residue formulae, which are finite. Here we will illustrate the residue formula (13) for the system at hand, reproducing the result (33) of the previous subsection.
We wish to calculate
\[ \frac{\partial^3 F}{\partial a^3} = \frac{\partial T}{\partial a} = \frac{1}{2\pi i} \text{res}_{\xi=0} \frac{d\Omega d\Omega d\Omega}{\delta dS}. \] (40)

From the previous section we have the canonical differential
\[ d\Omega = iU \frac{dx}{x \sqrt{(x-1)(U^2-x)}}. \] (41)

Also from
\[ x = 1/c_0^2 + \sin^{-2} \epsilon = \frac{\lambda^2 u^2}{(\lambda^2 + 1)^2} + \sin^{-2} \epsilon = U^2 - \frac{u^2}{4} \left(1 - \frac{\lambda^2}{\lambda^2 + 1}\right)^2 \]
we obtain that
\[ \frac{dx}{d\lambda} = 2\lambda u \frac{1 - \lambda^2}{(1 + \lambda^2)^3} \]
and consequently that
\[ d\xi = \frac{dx}{x \sqrt{x-1}} = \frac{d\lambda}{\sqrt{\frac{\lambda^2 u^2}{(\lambda^2 + 1)^2} + \sin^{-2} \epsilon - 1} \lambda^2 u^2 + \sin^{-2} \epsilon (\lambda^2 + 1)^2} \frac{1 - \lambda^2}{1 + \lambda^2}. \] (42)

Observe that although the expression for \(d\xi\) is nonvanishing as a function of the bare spectral parameter \(x\) (for finite \(x\)), it does however vanish as a function of the proper local parameter \(\lambda^2\) at \(\lambda^2 = 1\) (when \(x = U^2\)). That is \(d\xi\), while nonvanishing as a function of the bare spectral curve, does vanish on its cover \(C\).

Putting these various expressions together yields
\[ \frac{d\Omega d\Omega d\Omega}{\delta dS} = iR u U^3 \frac{16\lambda^2}{x^2(x-1)(1 + \lambda^2)^3} \frac{d\lambda^2}{1 - \lambda^2}. \] (43)

Now we can apply the residue formula (13), which gives
\[ \frac{\partial^3 F}{\partial a^3} = \frac{1}{\pi} \frac{u R}{(1 - U^2) U}. \] (44)

This should be compared with the derivative of (39) with respect to \(a\), that gives exactly the same result.

5 The case of general \(SU(N)\)

We now consider the general \(SU(N)\) model. The first step is to evaluate the spectral curve (7) for the Lax matrix (15). Expanding the determinant about the diagonal yields
\[ \sum_{k=0}^{N} (-\lambda)^{N-k} e^k \left\{ \sum_{1 \leq i_1 < \ldots < i_k \leq \lambda} e^{p_{i_1} + \ldots + p_k} \text{det}_{(ab)} F(q_{iai_k}; \xi) \right\} = 0, \] (45)

and Ruijsenaars [4] expressed the determinants appearing here in terms of the Hamiltonians (17) by means of a generalised Cauchy formula. Here we shall reobtain this expansion using a simple fact that generalised Cauchy formulae can be derived in terms of free-fermion correlators via Wick’s theorem for fermions. The Ruijsenaars-Schneider model (and its spin generalisations) may be understood [24, 27] in terms of a reduction of the Toda lattice hierarchy. The Hirota
bilinear identities of that hierarchy may be expressed in terms of free-fermion correlators \[25\], and are the origin of those here. We believe that these free field expansions will ultimately lead to a better field theoretic understanding of the appearance of these integrable systems. The machinery of free-fermion correlators has already found use, for example, in calculating within the context of the Whitham hierarchy for these integrable systems \[29\]. Having obtained the spectral curve we will then show that remarkable simplifications occur for the parameterisation of the curve, both in the perturbative limit and for the nonperturbative special point \(\epsilon = \omega = \pi/2\). It is the existence of these simpler rational forms of the curves that enables us to calculate the prepotential.

### 5.1 Handling determinants

The determinants in \((13)\) can be evaluated by making use of Wick’s theorem for fermionic correlators on a Riemann surface. (Appendix A of \[29\] provides both a summary and references for this machinery.) First observe that

\[
\frac{F(q_{ab},\xi)}{F(q_{ab},\epsilon)} = \frac{\sigma(q_a - q_b - \xi)\sigma(\epsilon)}{\sigma(q_a - q_b - \epsilon)\sigma(\xi)} \approx \frac{\theta_e(q_a - q_b - \epsilon)}{\theta_e(q_a - q_b - \epsilon)\theta_e(0)} \cdot \frac{\theta_*(\epsilon)}{\theta_*(\xi)}. \tag{46}
\]

Here \(*\) is the odd theta-characteristic (such that \(\theta_*(-z) = -\theta_*(z)\), traditionally labelled \(\theta_1\) for genus one), \(\epsilon = \epsilon - \xi\) and \(\theta_*(z) \equiv \theta_*(z + \epsilon)\). The symbol \(\equiv\) denotes here equality modulo standard factors like quadratic and linear exponents and Dedekind functions, which will cancel in the final expressions. They may be simply restored upon using

\[
\sigma(z) = \frac{2\omega}{\pi} e^{\pi \frac{q^2}{\pi^2}} \frac{\theta_1(\frac{\pi \tau}{2})}{\theta_1(0)}.
\]

Now the fermionic correlator on a torus is simply

\[
\psi_e(\zeta|\eta) = \langle \psi(\zeta)\tilde{\psi}(\eta) \rangle = \frac{1}{E(\zeta,\eta)} \frac{\theta_e(\zeta - \eta)}{\theta_e(0)} = \frac{\theta'_1(0)}{\theta_1(\zeta - \eta)} \frac{\theta_e(\zeta - \eta)}{\theta_e(0)} \sqrt{d\zeta \sqrt{d\eta}},
\]

where \(E(\zeta,\eta)\) is the prime form. With \(\zeta = q_a\) and \(\eta = q_b + \epsilon\) we see that the first factor on the right hand side of \((13)\) is simply the fermionic correlator \(\psi_e(q_a|q_b + \epsilon)\) on the torus. Further, the multi-fermionic correlator also has a simple expression:

\[
\psi_e(\zeta_1,\ldots,\zeta_k|\eta_1,\ldots,\eta_k) = \prod_{a=1}^k \psi_e(\zeta_a|\eta_a) = \frac{\prod_{a,b} E(\zeta_a,\zeta_b) E(\eta_a,\eta_b)}{\prod_{a,b} E(\zeta_a,\eta_b)} \cdot \frac{\theta_e(\sum \zeta_a - \sum \eta_a)}{\theta_e(0)}.
\tag{47}
\]

Wick’s theorem \[30\] expresses the fact that such a correlator has determinant form:

\[
det_{(ab)} \psi_e(\zeta_a|\eta_b) = (-)^{\frac{k(k-1)}{2}} \psi_e(\zeta_1,\ldots,\zeta_k|\eta_1,\ldots,\eta_k). \tag{48}
\]

(In fact \((17)\) \((18)\) hold true for any genus, but here we only need them on the torus.)

We see then that (up to factors) the determinants \(\det_{(ab)} F(q_{ia,ib},\xi)\) are the determinants of free fermion correlators, which may be evaluated using \((17)\) and \((18)\). Upon substituting \(\zeta_a = q_{ia}\), \(\eta_a = q_{ia} + \epsilon\) and \(\epsilon = \epsilon - \xi\), and collecting these factors we obtain the result of Ruijsenaars \[7\]:

\[
\hat{D}_k \equiv \det_{(ab)} F(q_{ia,ib},\xi) = (-)^{\frac{k(k-1)}{2}} \frac{\sigma^{k-1}(\xi - \epsilon)\sigma(\xi + (k-1)\epsilon)\prod_{a < b} \sigma(\epsilon + q_{ia,ib})\sigma(\epsilon - q_{ia,ib})}{\sigma^k(\xi)\sigma^{k(k-1)}(\epsilon)} \tag{49}
\]
where \( I_k = \{ i_1, \ldots, i_k \} \) denotes the set of \( k \) indices and \( a, b = 1, \ldots, k \). The right hand side of this expression is in fact a doubly periodic function of all the arguments \( (q_i, \xi, \epsilon) \) and so may rewritten in terms of Weierstrass functions. In particular, upon using (19), we find that

\[
\prod_{a < b} \frac{\sigma(q_{ia} \epsilon)^2}{\sigma(q_{ia} \epsilon) \sigma(q_{ia} \epsilon)} = \prod_{a < b} \frac{1}{\varphi(q_{ia} \epsilon) - \varphi(\epsilon)},
\]

thus recovering the Hamiltonians (17) noted earlier.

### 5.2 Spectral curve

Upon substitution of (49) into (45) the Ruijsenaars-Schneider spectral curve takes the form

\[
\sum_{k=0}^{N} (-\lambda)^{N-k} c^k \left( \sum_{k} e^{P_{11}+\cdots+P_{kk}} D_{1k} \right) = \sum_{k=0}^{N} (-\lambda)^{N-k} c^k D_k(\xi|\epsilon) H_k = 0.
\] (51)

Here we have collected the \( q \)-independent factors in (43) into \( D_k \) where

\[
D_k(\xi|\epsilon) = (-\frac{k(k-1)}{2}) \frac{\sigma^{k-1}(\xi - \epsilon) \sigma(\xi + (k-1) \epsilon)}{\sigma^k(\xi) \sigma^{k-1}(\epsilon)} = \frac{1}{c^k} \frac{\sigma(\xi + (k-1) \epsilon)}{\sigma(\xi - \epsilon)} \frac{(-1)^{(k-1)}}{\sigma^{k-2}(\epsilon)}.
\] (52)

Again this is doubly periodic function in both \( \xi \) and \( \epsilon \) and so is expressible in terms of the Weierstrass function and its derivative, though the explicit formulae are rather complicated. Although it may appear from the last expression in (52) that (51) simplifies when \( c = c_- \), such a choice would break the double periodicity of our spectral curve and so is inappropriate for the fully elliptic model. However, in the perturbative limit when one of the periods becomes infinite this choice is then available.

Part of the difficulty in dealing with the elliptic Ruijsenaars-Schneider and Calogero-Moser models is the complicated nature of these spectral curves. For example, in the case of \( SU(3) \) the spectral curve (1) is

\[
\lambda^3 - cu\lambda^2 + c^2 v(\varphi(\xi) - \varphi(\epsilon)) + c^3 \left( \frac{1}{2} \varphi'(\xi) \varphi'(\epsilon) - 3\varphi(\xi) \varphi'(\epsilon) + \varphi(\epsilon) + 2a\varphi(\xi) \varphi(\epsilon) + 2b(\varphi(\xi) + \varphi(\epsilon)) + 2\gamma \right) = 0
\] (53)

\[ u \equiv H_1 = H_+, v \equiv H_2 = H_-
\]

where

\[
H_\pm = e^{\pm p_1} \sqrt{\varphi(q_{12}) - \varphi(\epsilon)} \sqrt{\varphi(q_{13}) - \varphi(\epsilon)} + e^{\pm p_2} \sqrt{\varphi(q_{12}) - \varphi(\epsilon)} \sqrt{\varphi(q_{23}) - \varphi(\epsilon)} + e^{\pm p_3} \sqrt{\varphi(q_{13}) - \varphi(\epsilon)} \sqrt{\varphi(q_{23}) - \varphi(\epsilon)}, \quad p_1 + p_2 + p_3 = 0.
\] (54)

Again the issue is whether some choice of the normalisation \( c \) might simplify matters. With our choice of (21) for example, the third term in (53) turns into just \( v\lambda \), but no drastic simplification occurs in the constant term unless at the special point \( \epsilon = \frac{\pi}{2} \) (where \( \varphi'(\epsilon = \pi/2) = 0 \)) when – modulo \( c \) – the whole equation becomes linear in \( x = \varphi(\xi) \).

Simplifications do however take place in the perturbative limit which for our purposes suffices to establish the prepotential. In this limit we shall establish a convenient factorisation of our curves. We note similar simplifications were found in (16, 31) in the study of the trigonometric
limit of the elliptic Calogero-Moser models. Indeed, in the perturbative limit and at the special point $\epsilon = \pi/2$, which was the context of [15], quite dramatic simplifications occur. We shall study this case first, before turning to the case of general $\epsilon$. Before performing this analysis for general $N$ however, let us consider the simplifications for the example of (53) given above.

In perturbative limit and at the special point $\epsilon = \pi/2$ (53) turns into

$$
\lambda^3 + v \lambda = (u \lambda^2 + 1) \tan \xi
$$

(55)

for the choice $c = c_0$. Equivalently,

$$
\tan \xi = \frac{\lambda(\lambda^2 + v)}{u \lambda^2 + 1}.
$$

(56)

This may be readily compared with the rational spectral curve from [15]. To make this comparison more immediate, let us slightly change the choice of $c$: $c = -ic_0$. With this choice we get

$$
i \tan \xi = \frac{\lambda(\lambda^2 + v)}{u \lambda^2 + 1}.
$$

(57)

and should compare it with the curve from [15]:

$$
w = -\frac{(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)}{(\lambda + \lambda_1)(\lambda + \lambda_2)(\lambda + \lambda_3)}.
$$

(58)

Again, (57) is equivalent to (58) after the identification (8), $w = -e^{2\xi}$ and with $u = \lambda_1 + \lambda_2 + \lambda_3$ and $v = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = \lambda_1 \lambda_2 \lambda_3 \left(\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1}\right)$, $\lambda_1 \lambda_2 \lambda_3 = 1$.

### 5.3 The special point $\epsilon = \omega = \pi/2$

Here we separately consider the special point where $\epsilon$ equals the real half-period $\omega = \pi/2$. We shall see that simplifications occur even before taking the perturbative limit. Using the fact that $\sigma(\xi + 2\omega) = -\sigma(\xi)e^{2\eta(\xi + \omega)}$ with $\eta = \xi(\omega)$ [18, 19], we find that

$$
\frac{\sigma(\xi)\sigma(\omega)}{\sigma(\xi + \omega)} \equiv c_+ (\xi | \omega) = \alpha c_0(\xi | \omega) = \alpha^2 c_-(\xi | \omega)
$$

(59)

with $\alpha = e^{-\xi n}$. Then

$$
D_k(\xi | \epsilon) = \begin{cases} 
-\frac{k}{2} H_1^{-k(k-2)/2} & k \text{ even}, \\
-\frac{k}{2} H_1^{-k(k-2)/2} c_0 H_1^{-1/2} & k \text{ odd}.
\end{cases}
$$

(60)

Here $H_1 = 2e_1^2 + e_2 e_3$ is independent of $\xi$ and we note that in the trigonometric limit $H_1 = 1$. The choice $c \sim c_0$ now essentially removes all $\xi$-dependence in the spectral curve apart from a term linear in $c_0$ that multiplies the odd Hamiltonians. Similar to the $SU(3)$ case, we choose $c = -ic_0$. By absorbing the (inessential) constant factors into the Hamiltonians, $h_k = i^{k} H_1^{-k(k-2)/2} H_k$ for even $k$ and $h_k = i^{k-1} H_1^{-1/2} H_1^{-k(k-2)/2} H_k$ for odd $k$, one obtains the non-perturbative spectral curve (51) at the special point $\epsilon = \omega$:

$$
-ic_0^{-1}(\xi | \omega) = \frac{1}{\sqrt{\varphi(\xi) - \varphi(\omega)}} = -\frac{\sum_{k, odd}^N h_k(-\lambda)^{N-k}}{\sum_{k, even}^N h_k(-\lambda)^{N-k}} = \frac{P(\lambda) - (-)^N P(-\lambda)}{P(\lambda) + (-)^N P(-\lambda)}.
$$

(61)
where

\[ P(\lambda) = \sum_k h_k(-\lambda)^{N-k} = \lambda^N + \ldots + 1 \]  

(62)

These formulae may also be easily expressed in terms of the Jacobi elliptic functions. For example, \( \sigma(\xi) - \omega(\omega) = (e_1 - e_3)c_{n(u,k)}sn(u,k) \) and, therefore, \( c_0(\xi|\omega) = \frac{1}{\sqrt{e_1-e_3}cn(u,k)} \); here \( sn(u,k) \) and \( cn(u,k) \) are the Jacobi functions \([18, 19]\), \( u = (e_1 - e_3)^{1/2} \xi \), \( k \) is elliptic modulus.

We see then that substantial simplifications occur at the special point \( \epsilon = \omega \) before even considering the perturbative limit. Now in the perturbative limit \( \sigma(z) \rightarrow \sin(z)e^{\epsilon^2/6} \). Using this we easily obtain that

\[ i \cot \xi = \frac{P(\lambda) - (-)^N P(-\lambda)}{P(\lambda) + (-)^N P(-\lambda)}, \quad \text{and} \quad w \equiv -e^{2i\epsilon} = (-)^N \frac{P(\lambda)}{P(-\lambda)}. \]  

(63)

With \( dS \sim \log \frac{\lambda^w w}{\epsilon} \) this exactly reproduces the curves of ref.[15] where the prepotential (3) was calculated.

### 5.4 The perturbative result for generic \( \epsilon \)

We finally turn to the case of generic \( \epsilon \) in the perturbative limit. The system in this limit is described by the trigonometric Ruijsenaars-Schneider model. We have already seen that in this limit \( \sigma \)-functions are proportional to sines with exponential factors. However, the periodicity of our spectral curve means that these exponential factors must cancel amongst themselves. For example we find that in this limit

\[ D_k(\xi|\epsilon) = \frac{1}{\bar{c}_- k} \sin(\xi + (k-1)\epsilon) \frac{(-1)^{k(k-1)}}{\sin(k(k-2)\epsilon)}, \]  

(64)

where \( \bar{c}_- \) is given by (22). Now the ratio

\[ \frac{\sin(\xi + (k-1)\epsilon)}{\sin(\xi - \epsilon)} = \cos k\epsilon + \sin k\epsilon \cot(\xi - \epsilon) \]  

(65)

is expressible in terms of the single function \( \cot(\xi - \epsilon) \). This simple observation enables us to separate the variables \( \xi \) and \( \lambda \) in the equation for the spectral curve upon choosing\( c = -ie^{i\epsilon} \bar{c}_- \). With \( \bar{h}_k = H_k/\sin^{k(k-2)}(\epsilon) \) we may simplify the spectral curve to give

\[ i \cot(\xi - \epsilon) = \frac{\sum_{k=0}^N \bar{h}_k(-\lambda)^{N-k} (e^{2i{k\lambda}} + 1)}{\sum_{k=0}^N \bar{h}_k(-\lambda)^{N-k} (e^{2i{k\lambda}} - 1)} = \frac{P(\lambda) + e^{2iN\epsilon} P(\lambda e^{-2i\epsilon})}{P(\lambda) - e^{2iN\epsilon} P(\lambda e^{-2i\epsilon})} \]  

(66)

where \( P(\lambda) = \sum_{k=0}^N \bar{h}_k(-\lambda)^{N-k} \prod_{i}^N (\lambda - e^{2i{\epsilon}_i}) \) with some constants \( a_i, \sum_i^n a_i = 0 \). Introducing the variable \( w = e^{2i(\xi - \epsilon)} \), one finally arrives the spectral curve in the form

\[ w = e^{-2iN} \frac{P(\lambda)}{P(\lambda e^{-2i\epsilon})} \]  

(67)
Thus, we have shown that our system leads in the perturbative limit to the rational spectral curve (67) and the generating differential (68). One may now calculate the corresponding prepotential using the residue formula. This calculation has been done quite generally in [32], where the residue formula is applied to a general rational function of the form

\[ w \sim \frac{\prod_{i}^{N_c} (\lambda - \lambda_i)}{\sqrt{\prod_{\alpha}^{N_f} (\lambda - \lambda_{\alpha})}}, \]

with \( \lambda_i = e^{2a_i}, \sum_i a_i = 0, \lambda_{\alpha} = e^{2m_{\alpha}}. \) By choosing \( N_c = N \) and \( N_f = 2N, \) with the hypermultiplets masses pairwise coinciding and equal to \( a_i + \epsilon \) we obtain our curve (67). One then finds the prepotential from (3.37) of [32] gives the stated prepotential (3), and we are done. Further, upon setting \( \epsilon = \pi/2 \) we reproduce the results of the previous subsection.

6 Conclusion

In the context of Seiberg-Witten theory we have analysed some properties of the most general compactification from five dimensions to four dimensions with all of the fields belonging to adjoint representation of the gauge group \( SU(N). \) Here Seiberg-Witten theory describes the relevant low-energy effective action in terms of the finite-dimensional elliptic Ruijsenaars-Schneider integrable model, of which the elliptic Calogero, relativistic and ordinary Toda chains are particular limits. Our work unifies these previous treatments. Special attention was devoted to the perturbative (weak coupling) limit, when \( \tau \to i\infty \) and the elliptic models degenerate into trigonometric ones. Two topics are still beyond this general model: compactification from six dimensions and the inclusion of supermultiplets in the fundamental representation of the gauge group. For both these cases the elliptic Sklyanin (XYZ) spin-chain model [33] seems to be relevant (see arguments in [34, 32] and [17] respectively). In the present paper the Ruijsenaars-Schneider model has been discussed with the help of an \( N \times N \) Lax operator and we did not address the issue of a possible spin-chain-like description. This is an area warranting further investigation. The most immediate difficulty in extending this work to other groups is the lack of a suitable Lax formulation for general root systems, and we also highlight this as an interesting problem.

Another open question concerns the generalized WDVV equations [21]. At the special point \( \epsilon = \pi/2 \) these are known to hold [17], at least in the perturbative limit. Equally, they are known to fail [8, 6] for the perturbative limit of the elliptic Calogero model, that is the trigonometric Calogero-Sutherland system. This implies that there can be problems with the naive WDVV equations for the Ruijsenaars-Schneider system under consideration, unless the additional moduli \( \tau \) and \( \epsilon \) are cleverly taken into account. It may indeed be more illuminating to consider the more general Ruijsenaars-Schneider setting rather than that of the Calogero system, especially given the result of [15] at \( \epsilon = \pi/2. \) Perhaps the most intriguing property of these models is that perturbatively they are always described by the rational curves we have exhibited in this paper. Nevertheless, even at the perturbative level, the WDVV equations have only been established at the special point \( \epsilon = \pi/2. \) This issue deserves further analysis.
7 Acknowledgements

We acknowledge discussions with A. Gorsky, S. Kharchev, N. Nekrasov and A. Zabrodin.

A. Mar., A. Mir. and A. Mor. are grateful for the hospitality of University of Edinburgh and the support of EPSRC (grant GR/M08134). A. Mir. and H.W.B. also acknowledge the Royal Society for support under a joint project.

Our research is partly supported by the RFBR grants 98-01-00344 (A. Mar.), 98-01-00328 (A. Mir.), 98-02-16575 (A. Mor.), INTAS grants 96-482 (A. Mar.), 97-0103 (A. Mir.), the program for support of the scientific schools 96-15-96798 (A. Mir.) and the Russian President’s grant 96-15-96939 (A. Mor.).

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