ON TWO THEOREMS ABOUT SYMPLECTIC REFLECTION ALGEBRAS

GEORGES PINCZON

ABSTRACT. We give a new proof and an improvement of two Theorems of J. Alev, M.A. Farinati, T. Lambre and A.L. Solotar [1]: the first one about Hochschild cohomology spaces of some twisted bimodules of the Weyl Algebra $W$, and the second one about Hochschild cohomology spaces of the smash product $G \ast W$ ($G$ a finite subgroup of $SP(2n)$) and, as a consequence, we then give a new proof of a Theorem of P. Etingof and V. Ginzburg [12], which shows that the Symplectic Reflection Algebras are deformations of $G \ast W$ (and, in fact, all possible ones).

INTRODUCTION

This paper belongs to a very fascinating context of non-commutative geometry, for which we refer to [12] and [11]: in these papers, this context is perfectly described and developed, with many deep results, applications and examples (and all references). The problem is the study of deformations of $G \ast W$, where $W$ is a Weyl Algebra, and $G$ a finite subgroup of $SP(2n)$. Following Gerstenhaber, one has to find $H^2(G \ast W)$ (first order deformations), then $H^3(G \ast W)$ (obstructions), and then universal models of deformations of $G \ast W$. It turns out that these models are the Symplectic Reflection Algebras of [12]. Let us quickly explain how this program was worked out.

In [1], J. Alev, M.A. Farinati, T. Lambre and A.L. Solotar have found:

1. (A.F.L.S. Theorem 1) the dimension of the Hochschild cohomology spaces $H^k(W_\sigma)$ of the twisted $W$-bimodule $W_\sigma$ ($\sigma$ a diagonalizable element of $SP(2n)$).

2. (A.F.L.S. Theorem 2) the dimension of the Hochschild cohomology spaces $H^k(G \ast W)$.

Only particular cases of the A.F.L.S. Theorems were known before: when $\sigma = Id$, i.e the Hochschild cohomology of the Weyl Algebra ([20] [10]), or when $\sigma$ is the parity of $W$ ([17]). Part 2 gives the cohomology of $W^G$, since $W^G$ and $G \ast W$ are...
Morita-equivalent ([15],[1]). Some more information about $H^\bullet(G*W)$, including a description of the cup-product, was obtained in [3]. For applications of the A.F.L.S. Theorem see. [1] and [3]. Later, a new proof of the AFLS Theorems was given by P. Etingof [11].

In [12] (see also [11]), among many other results, P. Etingof and V. Ginzburg have completely solved the problem of deformations of $G*W$, showing that their Symplectic Reflections Algebras are non trivial (algebraic) deformations of $G*W$, and describe (up to equivalence and change of parameter) all possible deformations (the E.G. Theorem).

The E.G. Theorem belongs to non-commutative geometry (see [12]), and also to deformation quantization theory [5], since the Symplectic Reflection Algebras are natural generalizations of algebras used to quantize Calogero-Moser systems (see [11], [12]).

The goal of the present paper is to give new proofs of the A.F.L.S. Theorems, and also of the E.G. Theorem. We do not pretend that our proofs are simpler than the original ones, they are different, and we believe that new different proofs of deep results may be of interest. Moreover, we prove a significant amelioration of the A.F.L.S. Theorems, let us call it the C.A. Theorems, which can be used to simplify the original proof of the E.G. Theorem, and is an essential argument to build the new proof of the E.G. Theorem.

Let us now describe the sections of the paper, and the results.

(1) In section 1, we revisit the Koszul complex of the Weyl Algebra, as defined in [17] [1]. Let us quickly explain why. When reading papers [17], [1], [3], [12], [11], one has the feeling that the A.F.L.S. Theorems are not achieved: they give the dimensions of cohomology spaces, but no information about cocycles themselves. For instance, in [12], P. Etingof and V. Ginzburg have to prove that their deformation is not trivial (Lemma (2.17)), and the proof is not at all trivial. To understand what is missing, let us give another example. Let $S$ be a polynomial algebra, the Hochschild cohomology of $S$ reduces to the cohomology of the Koszul complex, which is infinite dimensional at all degrees. But one can obtain much more information if one remarks that the Koszul complex is a subcomplex of the Bar resolution: it easily follows that the cohomology is the space of skewsymmetric multivectors, and many other useful consequences. Now, we come back to the Weyl Algebra $W$. In that case, the Koszul complex is not a subcomplex of the Bar resolution, nevertheless the following holds:

**Lemma**

The Koszul complex of $W$ is a subcomplex of the normalized Bar resolution.
To our knowledge, this result has never been stated up to now. As a consequence, we prove:

**THEOREM 1:**

Given a W-bimodule $M$, one has

1. The restriction map, from the Hochschild complex of $M$ to the Koszul complex of $M$, induces an isomorphism in cohomology.

2. Let $V = \text{span}(p_i,q_i, i = 1...n)$, and $\Lambda$ the exterior algebra of $V$. Any Koszul $k$-cocycle is the restriction to $\Lambda^k$ of a Hochschild $k$-cocycle.

In section 2, we prove Part 1 of the C.A. Theorem. Let $\sigma$ be a diagonalizable element of $\text{SP}(2n)$, $V_\sigma := \text{Range}(\sigma - \text{Id})$, $2k_\sigma := \dim V_\sigma$, $\omega_\sigma$ the form defined by $\omega_\sigma(v_1,...,v_{2k_\sigma}) := \frac{1}{k_\sigma!} \omega^{k_\sigma}(p_\sigma(v_1),...,p_\sigma(v_{2k_\sigma}))$, where $\omega$ is the canonical two-form and $p_\sigma$ the projection on $V_\sigma$ coming from $V = V_\sigma \oplus \text{Im}(\sigma - \text{Id})$, $W_\sigma$ the twisted $W$-bimodule associated to $\sigma$.

**C.A. THEOREM 1**

1. (A.F.L.S. Theorem 1) $\dim H^{2k_\sigma}(W_\sigma) = 1$, and $H^k(W_\sigma) = \{0\}$, if $k \neq 2k_\sigma$.

2. There exists a Hochschild cocycle $\Omega_\sigma \in Z^{2k_\sigma}(W_\sigma)$ such that: $\Omega_\sigma |_{\Lambda^{2k_\sigma}} = \omega_\sigma$, and $H^{2k_\sigma}(W_\sigma) = \mathbb{C} \Omega_\sigma$.

3. Let $S$ be a finite subgroup of $\text{SP}(2n)$. We assume that $S$ commutes with $\sigma$, and denote by $H^*_S(W_\sigma)$ the $S$-invariant Hochschild cohomology. Then $H^*_S(W_\sigma) = H^*(W_\sigma)$, and the cocycle $\Omega_\sigma$ in (2) can be chosen to be $S$-invariant.

Let us say of few words about the proof. As in [17], [1], or [11], we introduce the Koszul complex, but then we follow arguments of [17]: we replace the differential by an equivalent one, split the new complex, and introduce explicit homotopies to deduce (1), then (2) and (3) follow from Theorem 1. Let us remark that the equivalence we use is natural in this context: it is built using operators which appear when showing that standard ordering and Weyl ordering define equivalent star-products.

In section 3, we prove Part 2 of the C.A. Theorem. This is an easy consequence of Part 1: $\mathbb{C}[G]$ is separable, so $H^*_C[G](G \ast W) = H^*(G \ast W)$ [13], and using a convenient description of $\mathbb{C}[G]$-relative cocycles, the result follows.

Let $\Gamma$ be the set of conjugacy classes of $G$ and $\Gamma_{2k} := \{ \gamma \in \Gamma/k_\sigma = k, \forall \sigma \in \Gamma \}$.

**C.A. THEOREM 2**
(1) (A.F.L.S. Theorem 2) \(H^k(G \ast W) = \{0\}\), if \(k\) is odd, and \(\dim H^{2k}(G \ast W) = \text{card } \Gamma_{2k}\).

(2) Assuming that \(\Gamma_{2k} \neq \emptyset\), let \(\lambda \in \mathbb{C}^{\Gamma_{2k}}\). There exists a cocycle \(C_{\lambda} \in Z^{2k}_{\mathbb{C}[G]}(G \ast W)\) such that
\[
C_{\lambda}(X_1 \wedge \ldots \wedge X_{2k}) = \sum_{\gamma \in \Gamma_{2k}} \lambda(\gamma) \sum g \in \gamma \omega_{\lambda}(X_1, \ldots, X_{2k}) \otimes g, \forall X_i \in V.
\]
\(C_{\lambda}\) is a coboundary if and only if \(\lambda = 0\), and the map \(\lambda \rightarrow C_{\lambda}\) induces an isomorphism from \(\mathbb{C}^{\Gamma_{2k}}\) onto \(H^{2k}_{\mathbb{C}[G]}(G \ast W)\).

(3) If a cocycle \(C \in Z^{2k}_{\mathbb{C}[G]}(G \ast W)\) vanishes on \(\Lambda^{2k}\), then \(C\) is a coboundary.

(2) and (3) are of interest since they describe the ”emerged part” of cocycles. For instance Lemma (2.17) of [12] is a consequence of (2) and (3).

(4) In section (4), we give a new proof of the E.G. Theorem. Assuming \(\Gamma_2 \neq \emptyset\), let 
\(H_{\bar{h}}\) be the Symplectic Reflection Algebra, with \(\lambda \in \mathbb{C}^{\Gamma_2}, \lambda \neq 0\). This algebra is defined by generators and relations (SRA-relations), see (4.2). We prove:

E.G. THEOREM

There exists a non trivial polynomial \(\mathbb{C}[G]\)-relative deformation of \(G \ast W\) where the SRA-relations hold. The subalgebra \((G \ast W)[h]\) of this deformation is isomorphic to \(H_{\bar{h}}\). We have
\[
<X_1, \ldots, X_k> := \frac{1}{k!} \sum_{\sigma \in S_k} X_{\sigma(1)} h \ast \ldots \ast h X_{\sigma(k)} = X_1 \ldots X_k, \forall X_i \in V
\]
(the last product is computed using the abelian product on \(W\)).

The last properties give the P.B.W. property: given a basis \(\{e_1, \ldots, e_{2n}\}\) of \(V\), \(\{e_1^i \ldots e_{2n}^j \otimes g, i_j \in \mathbb{N}, g \in G\}\) is, resp.: when \(h\) is formal, a \(\mathbb{C}[h]\)-basis of \(H_{\bar{h}} = (G \ast W)[h]\), resp.: when \(h \in \mathbb{C}\), a basis of \(H_{\bar{h}}\).

As stated, the theorem is completely equivalent to the E.G.-Theorem, only the order of the claims and the proof differ. Let us give some details:

The original proof in [12] has two steps:

- First step (main one): \(H_{\bar{h}}\), is a deformation due to the Koszul Deformation Principle of Beilinson, Ginzburg and Soergel (see [11], [12]).
- Second step: the deformation is not trivial. As mentioned before, this step can be simplified using the C.A. Theorems.

Our proof goes exactly in the opposite direction, giving another insight of the result:

- First step: using the C.A. Theorems, we prove that there exists a \(\mathbb{C}[G]\)-relative deformation where the SRA relations hold.
Second step: we normalize the deformation using an adapted equivalence defined by a symmetrization map. This is a classical argument (e.g. [8]). We then prove that the obtained deformation is polynomial, using a powerful formula of F.A. Berezin [6] [9].

Third step: we prove that the subalgebra \((G \ast W) [h]\) is isomorphic to \(H_\hbar\), which is therefore a deformation, and the PBW-property.

Obviously, such a proof can be tried because we know from the beginning what we want to find (i.e: \(H_\hbar\)), thanks to P. Etingof and V. Ginzburg! On the other hand, the formula of Berezin is very explicit, and could be used to give some more light on the structure of \(H_\hbar\), but this is to be done.

Acknowledgments

I am grateful to J. Alev for highlighting lectures on the A.F.L.S. Theorems. I thank G. Dito who gave me an essential argument, coming from his paper [8]. I also thank D. Arnal and R. Ushirobira for many discussions on the subject. Finally, I am indebted to Moshe Flato for many explanations about deformation quantization, when the theory was starting.

1) THE KOSZUL COMPLEX OF THE WEYL ALGEBRA

1.1) Let \(W = \mathbb{C} [p_1, q_1, \ldots, p_n, q_n]\); there are two algebra structures on \(W\): the first one is the usual commutative product and the second one is the Moyal \(*\)-product (see eg. [19] for a short introduction); \(W\), with the Moyal \(*\)-product, is the Weyl Algebra. Let \(V = \text{span} (p_1, q_1, \ldots, p_n, q_n)\), and \(\sigma \in SP(2n)\); then \(\sigma\) extends to an automorphism of both algebra structures of \(W\). A Darboux basis of \(V\) will be any basis of type \(\{\sigma(p_1), \sigma(q_1), \ldots, \sigma(p_n), \sigma(q_n)\}\), for some \(\sigma \in SP(2n)\).

1.2) We define operators \(\sim Z_i\) of \(W \otimes W\), \(i = 1, \ldots, 2n\), as follows: we set \(Z_{2i-1} = p_i, Z_{2i} = q_i, i = 1 \ldots n\), and

\[
\sim Z_i (a \otimes b) = a * Z_i \otimes b - a \otimes Z_i * b, a, b \in W.
\]

We denote by \(\Lambda = \bigoplus_{k \geq 0} \Lambda^k\) the exterior algebra of \(V\), and by \(i_x, x \in V^*\), the corresponding derivation of \(\Lambda\). The Koszul complex \(K = (K^k, d^K_k, k \geq -1)\) is defined by:

\[
0 \leftarrow W \xleftarrow{m} W \otimes W \xleftarrow{d^K_1} \cdots \xleftarrow{d^K_k} K^k = W \otimes \Lambda^k \otimes W \leftarrow \ldots
\]
where $m_*$ is the Moyal product and $d^K_k := \sum_{i=1}^{2n} Z_i \otimes iZ_i$, $k > 0$. It is known that $K$ is a free resolution of the bimodule $W$ (e.g.: [17]). Therefore, given a $W$-bimodule $M$, applying $\text{Hom}_{\text{bimod}}(\bullet, M)$ on $K$, the Hochschild cohomology $H^*(M)$ is isomorphic to the cohomology of the complex $K(M) = (K^k(M), \Delta_k, k \geq 0)$:

\[(1.2.3) \ M \xrightarrow{\Delta_k} M \otimes \Lambda^1 \xrightarrow{\Delta_1} \ldots \xrightarrow{\Delta_k} K^k(M) = \sum_{i=1}^{2n} Z_i \otimes \mu_{Z_i} \otimes \ldots \otimes Z_i \otimes b.
\]

The Koszul complex $K$ has a remarkable property with respect to the normalized Bar-resolution:

\[(1.4.1) \ a, b \in W, a \otimes (Z_{i_1}, \ldots, Z_{i_k}) \otimes b = \sum_{\sigma \in \Sigma_k} e(\sigma) a \otimes Z_{a(i_1)} \otimes \ldots \otimes Z_{a(i_k)} \otimes b.
\]

The Koszul complex $K$ is a subcomplex of the normalized Bar-resolution (i.e: the inclusion map (1.3.1) is a chain map).

**Proof:** One checks that $d^B|_K = d^K$ by a straightforward direct computation, with main argument that $[a, b] \in \mathbb{C}$ if $a, b \in V$.

\[(1.3.3) \text{Remark: a similar result holds in the case, e.g., of a polynomial algebra, and has useful consequences. For the Weyl Algebra, it also has useful consequences, as we shall show.}
\]

\[(1.4) \text{Let } \mathcal{C}(M) = (\mathcal{C}^k(M), k \geq 0, d) \text{ be the normalized Hochschild complex of a bimodule } M. \text{ Since } \mathcal{C}(M), \text{ and } K(M) \text{ are obtained when applying the same operation (namely } \text{Hom}_{\text{bimod}}(\bullet, M)) \text{ on } \mathcal{B} \text{ and } K, \text{ by (1.3.2), one gets a restriction map:}
\]

\[(1.4.1) R : \mathcal{C}(M) \xrightarrow{R} K(M), \text{ defined by } R(C) = C|_{\Lambda}, \text{ if } C \in \mathcal{C}(M), \text{ and satisfying } \Delta \circ R = R \circ d. \text{ By standard arguments ([7]), one has:}
\]

\[(1.4.2) \text{PROPOSITION:}
\]
The restriction map induces an isomorphism in a cohomology.

(1.4.2) is a useful improvement of the usual isomorphism $H^\bullet(M) \simeq H^\bullet(d) \simeq H^\bullet(\Delta)$, since the isomorphism is explicit: it comes from the restriction map. For instance, one has the following immediate consequence:

(1.4.3) COROLLARY
Let $c \in L(\Lambda^k, M)$ be a Koszul cocycle of $M$, then there exists a Hochschild cocycle $C \in C^k(W)$ such that $C|_{\Lambda^k} = c$. If a second Hochschild cocycle $C'$ has the same property, then $C - C'$ is a Hochschild coboundary.

(1.5) Remark: we have defined the Koszul resolution using the canonical Darboux basis $p_1, q_1, \ldots, p_n, q_n$, let us show that the Koszul resolution has an intrinsic nature, so that the formulas are valid in any basis of $V$. To do that, we define, for any $X \in V$, an operator $\tilde{X}$ of $W \otimes W$ by $\tilde{X}(a \otimes b) = a \ast X \otimes b - a \otimes X \ast b$, $\forall a, b \in W$. Then we define $\rho_k : L(V) \mapsto C^k(W)$ by $\rho_k(X \otimes \varphi) = \tilde{X} \otimes \varphi$, $X \in V$, $\varphi \in V^\ast$. Since $d^k_k = \rho_k(Id_V)$, the result follows. We shall use this remark in section 2.

(2) HOCHSCHILD COHOMOLOGY OF THE TWISTED BIMODULE $W_\sigma$

(2.1) Given any automorphism $\sigma$ of the Weyl Algebra, we denote by $W_\sigma$ the $W$-bimodule with underlying space $W$ and action:

(2.1.1) $a, b \in W, a \cdot b = a \ast b$, $b \cdot a = b \ast \sigma(a)$.

Let $C^\bullet(W_\sigma)$ be the normalized Hochschild complex of $W_\sigma$; then $C^k(W_\sigma) = C^k(W) = L(V[K^k, K^{k-1}])$ by $\rho_k(X \otimes \varphi) = \tilde{X} \otimes \varphi$, $X \in V$, $\varphi \in V^\ast$. Since $d^k_k = \rho_k(Id_V)$, the result follows. We shall use this remark in section 2.

(2.1.1) When $\sigma = Id_W$, one obtains the usual Hochschild cohomology of $W$, which is well known ([20], [10]): $H^0(W) = \mathbb{C}$, and $H^k(W) = 0$, if $k > 0$. As a consequence, $W$ is rigid in Gerstenhaber deformation theory.

(2.1.2) When $\sigma$ is the parity of $W$, $H^\bullet(W_\sigma)$ was computed in [17]; one has $dim H^{2n}(W_\sigma) = 1$, and $H^k(W_\sigma) = \{0\}$, if $k \neq 2n$. As a consequence, if $n > 1$, $W$ is rigid in super-commutative deformation theory (see [16] [17]); when $n = 1$ the enveloping algebra $\mathcal{W}(osp(1, 2))$ provides a universal super-commutative deformation, with deformation parameter the ghost of $\mathcal{W}(osp(1, 2))$ ([17], [18], [4]).
(2.2) Let us assume that $\sigma$ is the automorphism of $W$ extending a diagonalizable element $\sigma$ of $SP(2n)$. We assume moreover that $\sigma \neq Id$, and introduce $V_\sigma = \text{range } (\sigma - Id)\vert_V$. Then $\dim V_\sigma$ is even, say $2k_\sigma$. (see [2])

Let $x_1, \ldots, x_{2k_\sigma}$ be a Darboux basis of $V_\sigma$, and $\omega_\sigma = x_1^* \wedge \ldots \wedge x_{2k_\sigma}^*$ (any choice of the Darboux basis will lead to the same $\omega_\sigma$) ; if $\omega$ is the canonical two form, one has:

$$\omega_\sigma(v_1, \ldots, v_{2k_\sigma}) = \frac{1}{k_\sigma!} \omega^k \sigma(P_\sigma(v_1), \ldots, P_\sigma(v_{2k_\sigma})), \text{ where } P_\sigma \text{ is the projection on } V_\sigma \text{ associated to } V = V_\sigma \oplus \text{Ker}(\sigma - Id) \text{ (see [2] [3] [11] for details). One has:}

(2.2.1) $\omega_\sigma(v_1, \ldots, v_{2k_\sigma}) = \frac{1}{k_\sigma} \omega^k \sigma(P_\sigma(v_1), \ldots, P_\sigma(v_{2k_\sigma}))$, where $P_\sigma$ is the projection on $V_\sigma$ associated to $V = V_\sigma \oplus \text{Ker}(\sigma - Id)$ (see [2] [3] [11] for details). One has:

(2.2.2) C.A. THEOREM 1

(1) $[1]$ $\dim H^{2k_\sigma}(W_\sigma) = 1$, and $H^k(W_\sigma) = \{0\}$, if $k \neq 2k_\sigma$.

(2) There exists a Hochschild cocycle $\Omega_\sigma \in Z^{2k_\sigma}(W_\sigma)$ such that: $\Omega_\sigma|_{\Lambda^{2k_\sigma}} = \omega_\sigma$, and $H^{2k_\sigma}(W_\sigma) = C.\Omega_\sigma$.

(3) Let $S$ be a finite subgroup of $SP(2n)$, commuting with $\sigma$. Denoting by $H^*_S(W_\sigma)$ the $S$-invariant Hochschild cohomology (i.e. computed from $S$-invariant cochains), one has $H^*_S(W_\sigma) = H^*(W_\sigma)$, and the cocycle of (2) can be chosen $S$-invariant.

Proof:

By [2], there exists a Darboux basis $P_1, Q_1, \ldots, P_n, Q_n$ of $V$ such that:

$$\sigma(P_i) = \alpha_i P_i, \quad \sigma(Q_i) = \alpha_i^{-1} Q_i, \text{ with } \alpha_i \neq 1, \text{ if } i \leq k_\sigma, \text{ and } \alpha_i = 1, \text{ if } i > k_\sigma.$$

We compute $H^*(W_\sigma)$ using the Koszul complex (see remark (1.5)). Using (1.2.3), and Moyal product, the differential is given by:

(2.2.3) $\Delta_\sigma = \sum_{i=1}^{2n} T_i \otimes \mu_{E_i}$, where $T_{2i-1} = (1 - \alpha_i)m_{P_i} + \frac{1}{2}(1 + \alpha_i) \frac{\partial}{\partial Q_i}, T_{2i} = (1 - \alpha_i^{-1})m_{Q_i} - \frac{1}{2}(1 + \alpha_i^{-1}) \frac{\partial}{\partial P_i}, m_{P_i}(a) = P_i a, m_{Q_i}(a) = Q_i a, a \in W$.

We define operators $\theta$ and $A$ of $W$ by:

(2.2.4) $\begin{cases} A(P_i) = (1 - \alpha_i)^{-1} \cdot P_i, & \text{if } i \leq k_\sigma, \\ A(Q_i) = (1 - \alpha_i^{-1})^{-1} \cdot Q_i, & \text{if } i > k_\sigma, \end{cases}$ and $\begin{cases} A(P_i) = Q_i, & \text{if } i \leq k_\sigma, \\ A(Q_i) = -P_i & \text{if } i > k_\sigma, \end{cases}$

(2.2.5) $\theta = \exp\left[-\frac{1}{2} \sum_{i \leq k_\sigma} \beta_i \frac{\partial^2}{\partial P_i \partial Q_i}\right]$, where $\beta_i = \frac{1 + \alpha_i}{1 - \alpha_i}$. 

Let now $\xi = A \circ \theta$, and $\Delta' = (\xi \otimes \text{Id}_{\mathcal{A}}) \circ \Delta_{\sigma} \circ (\xi \otimes \text{Id}_{\mathcal{A}})^{-1}$. It is easy to check that:

$$\tag{2.2.6} \Delta' = \sum_{i=1}^{2k_{\sigma}} m_{Z_i} \otimes \mu_{Z_i} + \sum_{i=2k_{\sigma}}^{2n} \frac{\partial}{\partial Z_i} \otimes \mu_{Z_i}.$$ 

Since $\Delta'^2 = 0$, we get a new complex, with cohomology $H^\ast(\Delta')$, and from the definition of $\Delta'$, $\xi^{-1} \otimes \text{Id} : Z(\Delta') \rightarrow Z(\Delta_{\sigma})$ induces an isomorphism $H(\Delta') \cong H(\Delta_{\sigma})$. So, we get a new equivalent differential $\Delta' = \Delta'_1 + \Delta'_2$, where $\Delta'_1$ corresponds to the case $\alpha_i = -1$, $\forall i$, and $\Delta'_2$ is a de-Rham type differential.

$$\tag{2.2.7} \text{It is obvious that } \omega_{\sigma} \in Z^{2k_{\sigma}}(\Delta') ; \text{ given } C \in B^{2k_{\sigma}}(\Delta') , \text{ one has } C(Z_1,...,Z_{2k_{\sigma}})(0) = 0, \text{ and since } \omega_{\sigma}(Z_1,...,Z_{2k_{\sigma}}) = 1, \omega_{\sigma} \notin B^{2k_{\sigma}}(\Delta'). \text{ Moreover } (\xi^{-1} \otimes \text{Id})(\omega_{\sigma}) = \omega_{\sigma}, \text{ so } \omega_{\sigma} \text{ is also a non trivial cocycle in } Z^{2k_{\sigma}}(\Delta_{\sigma}). \text{ By (1.4.3), there exists a non trivial Hochschild cocycle } \Omega_{\sigma} \in Z^{2k_{\sigma}}(W_\sigma) \text{ such that } \Omega_{\sigma} \wedge 2k_{\sigma} = \omega_{\sigma}. $$

$$\tag{2.2.8} \text{We need some more notations. Let } V_1 = V_{\sigma}, V_2 = \text{Ker}(\sigma - \text{Id}), \Lambda_1 \text{ and } \Lambda_2 \text{ the exterior algebras of } V_1 \text{ and } V_2, W_1 \text{ and } W_2 \text{ the symmetric algebras of } V_1 \text{ and } V_2. \text{ One has } \Lambda = \Lambda_1 \otimes \Lambda_2, \text{ and } W = W_1 \otimes W_2. \text{ We introduce the following operators:}$$

$$h_1 = \sum_{i=1}^{2k_{\sigma}} \frac{\partial}{\partial Z_i} \otimes i_{Z_i}, \quad h_2 = \sum_{i=2k_{\sigma}}^{2n} m_{Z_i} \otimes i_{Z_i}, \quad R_1 = \sum_{i=2k_{\sigma}}^{2n} Z_i \frac{\partial}{\partial Z_i}, \quad R_2 = \sum_{i=2k_{\sigma}}^{2n} Z_i \frac{\partial}{\partial Z_i},$$

$$\mathcal{R}_1 = \sum_{i=2k_{\sigma}}^{2n} Z_i^* \wedge i_{Z_i}, \quad \mathcal{R}_2 = \sum_{i=2k_{\sigma}}^{2n} Z_i^* \wedge i_{Z_i}.$$ 

$$\tag{2.2.9} \text{The complex } (K(W_{\sigma}), \Delta') \text{ splits into three sub-complexes:}$$

$$K(W_{\sigma}) = \Lambda_1^{k_{\sigma}^+} \otimes H_1 \oplus H_2, \quad H_1 = W_1 \oplus (\bigoplus_{i=2k_{\sigma}}^{2n} \Lambda_1^i), \quad H_2 = W_1 \otimes \Lambda_1^+ \Lambda_2^+ + W_1 \otimes W_2^+ \otimes \Lambda_1^+ \Lambda_2^*, \text{ where the subscript } + \text{ stands for the kernel of the corresponding evaluation; (it is easy to see that } H_1 \text{ and } H_2 \text{ are } \Delta' \text{-stable, and } \Lambda \wedge 2k_{\sigma}^+ \text{ is a subcomplex by (2.2.7).}$$

$$\tag{2.2.10} \text{By a simple computation, one has } h_2 \circ \Delta' + \Delta' \circ h_2 = R_2 + \mathcal{R}_2 = T_2. \text{ Since } \tau_2 = T_2|_{H^2} \text{ is invertible, since moreover } \Delta' \circ \tau_2 = \Delta' \circ h_2 \circ \Delta' = \tau_2 \circ \Delta', \text{ we obtain that } \text{Id}_{H_2} = (h_2 \circ \tau_2^{-1}) \circ \Delta' + \Delta' \circ (h_2 \circ \tau_2^{-1}), \text{ so the complex } (H_2, \Delta') \text{ has trivial cohomology.}$$

$$\tag{2.2.11} \text{Let } \Delta'_1 = \sum_{i=2k_{\sigma}}^{2n} m_{Z_i} \otimes \mu_{Z_i} = \Delta' |_{H_1}. \text{ A computation gives } h_1 \circ \Delta'_1 + \Delta'_1 \circ h_1 = R_1 + 2k_{\sigma} \text{Id} - \mathcal{R}_1 = T_1. \text{ Let } \tau_1 = T_1 |_{H_1}, \text{ } \tau_1 \text{ commutes with } \Delta'_1, \text{ so } (h \circ \tau_1^{-1}) \circ \Delta'_1 = T_1.$$
\[ \Delta_1 + \Delta'_1 \circ (h_1 \circ \tau^{-1}_1) = Id_{H_1} \] it results that the complex \((H_1, \Delta')\) has trivial cohomology.

(2.2.10) Finally, we stay with the complex \((\Lambda^{2k\sigma}, \Delta' = 0)\) treated in (2.2.7): \(\omega_\sigma\) is a non trivial cocycle; so finally \(H^*(\Delta') = \mathbb{C}.\omega_\sigma\). Applying (2.2.6) and (2.2.7), it follows that \(H^*(\Delta_\sigma) = \mathbb{C}.\omega_\sigma\).

(2.2.11) By (1.4.2) and (1.4.3), there exists a Hochschild cocycle \(\Omega_\sigma \in Z^{2k\sigma}(W_\sigma)\) such that the Hochschild cohomology \(H^*(W_\sigma) = \mathbb{C}.\Omega_\sigma\), and \(\Omega_\sigma|_{\Lambda^{3\sigma}} = \omega_\sigma\).

(2.2.11) Let \(S\) be a finite subgroup of \(SP(2n)\) such that \(s \circ \sigma = \sigma \circ s\), \(\forall s \in S\). \(S\) acts on Hochschild cochains \(C \in \mathcal{C}^k(W)\) by:

\[ \pi_\sigma(C)(a_1, ..., a_k) = s(C(s^{-1}(a_1), ..., s^{-1}(a_k)), a_i \in W, and this action commutes with the Hochschild differential. So we can consider the \(S\)-invariant Hochschild cohomology \(H^*_S(W_\sigma)\), using the complex \(\mathcal{C}^*_S(W_\sigma)\), of \(S\)-invariant cochains. By standard arguments, since \(S\) is finite, one has an inclusion \(H^*_S(W_\sigma) \subset H^*(W_\sigma)\). We now show that this is an equality:

(2.2.12) Let \(P = \frac{1}{|S|} \sum s \), then \(\pi_\sigma\) is a projector from \(\mathcal{C}^k(W)\) onto its trivial isotopic component \(\mathcal{C}^*_S(W)\). So \(\pi_\sigma(\Omega_\sigma)\) is an \(S\)-invariant Hochschild cocycle, and one has \(\pi_\sigma(\Omega_\sigma)|_{\Lambda^{3\sigma}} = \pi_\sigma(\omega_\sigma) = \omega_\sigma\). (because \(\pi_\sigma(\omega_\sigma) = \omega_\sigma, \forall s \in S\), since \(s \in SP(V_\sigma)\)). Using (1.4.2), \(H^{2k\sigma}(W_\sigma) = \mathbb{C}.\pi_\sigma(\Omega_\sigma) = H^{2k\sigma}_S(W_\sigma)\), and \(H^*_S(W_\sigma) = H^k(W_\sigma) = \{0\} \) if \(k \neq 2k_\sigma\). Therefore \(H^*_S(W_\sigma) = H^*_S(W_\sigma) = \mathbb{C}.\pi_\sigma(\Omega_\sigma)\). Q.E.D.

(2.3) Remark : Let \(\sigma, V_\sigma, k_\sigma, \omega_\sigma, \Omega_\sigma\) be as in (2.2), and \(\tau = x \sigma x^{-1}, x \in SP(2n)\); we introduce corresponding \(V_\tau = \pi_\tau(V_\sigma), k_\tau = k_\sigma, \omega_\tau = \pi_\tau(\omega_\sigma)\). Let \(d_\sigma\) and \(d_\tau\) be the respective Hochschild differentials of the bimodules \(W_\sigma\) and \(W_\tau\), since \(d_\tau = \pi_\tau \circ d_\sigma \circ \pi^{-1}_\tau\), one has, for Hochschild cocycles, \(Z(W_\tau) = \pi_\tau(Z(W_\sigma))\), the same for coboundaries, and \(\pi_\tau\) induces an isomorphism from \(H^*(W_\sigma)\) onto \(H^*(W_\tau)\). The cocycle \(\pi_\tau(\Omega_\sigma)\) is a non trivial element in \(Z(W_\tau)\), which satisfies \(\pi_\tau(\Omega_\sigma)|_{\Lambda^{3\sigma}} = \omega_\tau\), so, applying (2.2.2) (2) to \(\tau\), we can choose \(\Omega_\tau = \pi_\tau(\Omega_\sigma)\). Let \(S\) be a finite subgroup of \(SP(2n)\), if \(S\) commutes with \(\sigma\), then \(xSx^{-1} = S'\) commutes with \(\tau\), and a cochain \(C\) is \(S\)-invariant if and only if \(\pi_\tau(C)\) is \(S'\)-invariant. It results that \(H^*_S(W_\tau) = \pi_\tau(H^*_S(W_\sigma)) = \pi_\tau(H^*(W_\sigma)) = H^*(W_\tau)\); moreover, using (2.2.2) (3), we can start with an \(S\)-invariant \(\Omega_\sigma\), then \(\Omega_\tau = \pi_\tau(\Omega_\sigma)\) is \(S'\)-invariant, and \(H^*(W_\tau) = \mathbb{C}.\Omega_\tau\).
(2.4) Remark: We develop (2.3) in a context which will be used in the next section: Let $G$ be a finite subgroup of $SP(2n)$. Any $g \in G$ satisfies $g^{\text{card}} G = 1$, so $g$ is diagonalizable, and we can apply all the results of (2.2) and (2.3). Given a conjugacy class $\gamma$ of $G$, we fix $\sigma \in \gamma$, and use the notations of (2.2) and (2.3). If $\tau \in \gamma$, one has $k_\tau = k_\sigma$ (denoted by $k_\gamma$) and $\omega_\tau = \pi_\sigma(\omega_\sigma)$ for any $x$ such that $\tau = x \sigma x^{-1}$ (2.3). Denote by $S_\tau$ the centralizer of $\tau \in \gamma$ in $G$. One has $S_\tau = x S_\sigma x^{-1}$, if $\tau = x \sigma x^{-1}$, and by (2.2.2) $H^*_S(W_\tau) = H^*(W_\tau)$, the cohomology being one-dimensional, concentrated in degree 2 $k_\gamma$. Starting with an $S_\sigma$-invariant $\Omega_\sigma$, given by (2.2.2) (3), we define $\Omega_\tau = \pi_\sigma(\Omega_\sigma)$, if $\tau = x \sigma x^{-1} \in \gamma$; first, we remark that if $\tau = x' \sigma x'^{-1}$, then $\pi_{x'}(\Omega_\sigma) = \pi_\sigma(\Omega_\sigma)$, secondly, by (2.3), $\Omega_\tau$ is $S_\tau$-invariant, $H^{2k_\tau}(W_\tau) = \mathbb{C} \Omega_\tau$, and one has $\Omega_\tau |_{\lambda^{2k_\tau}} = \omega_\tau$.

(3) COHOMOLOGY OF $G \ast W$

(3.1) In this section, $G$ is a finite subgroup of $SP(2n)$, $G \ast W$ is the algebra with underlying space $W \otimes \mathbb{C}[G]$, and product such that:

(3.1.1) $\sigma \ast a = \sigma(a) \otimes \sigma$, $a \ast \sigma = a \otimes \sigma$, $\forall \sigma \in G$, $a \in W$, Moyal product on $W = W \otimes 1$, and convolution product on $\mathbb{C}[G] = 1 \otimes \mathbb{C}[G]$. The group $G$ acts on $G \ast W$ by conjugation:

(3.1.2) $Ad_\sigma(b) = b \ast \sigma \ast b^{-1}$, $\forall \sigma \in G$, $b \in G \ast W$, and the product is invariant:

(3.1.3) $Ad_\sigma(b_1 \ast b_2) = Ad_\sigma(b_1) \ast Ad_\sigma(b_2)$. On $W$, one has: $Ad_\sigma(a) = \sigma(a)$, and $Ad_\sigma(a_1 \ast a_2) = \sigma(a_1) \ast \sigma(a_2)$, $\forall \sigma \in G$, $a, a_1, a_2 \in W$.

(3.2) $\mathbb{C}[G]$ is a separable algebra, so $H^*(G \ast W) = H^*_{\mathbb{C}[G]}(G \ast W)$, $[13]$, hence Hochschild cohomology of $G \ast W$ can be computed using $\mathbb{C}[G]$-relative cochains, i.e cochains $C$ which satisfy:

(3.2.2) $C(b_1, ..., b_k) = 0$, if some $b_i \in \mathbb{C}[G]$, $\sigma \ast C(b_1, ..., b_k) = C(\sigma \ast b_1, b_2, ..., b_k)$, $C(b_1, ..., b_i \ast \sigma, b_{i+1}, ..., b_k) = C(b_1, ..., b_i, \sigma \ast b_{i+1}, ..., b_k)$, $\forall i$, and $C(b_1, ..., b_k \ast \sigma) = C(b_1, ..., b_k) \ast \sigma$, $\forall \sigma \in G$, $b_i \in G \ast W$.

It results that:

(3.2.3) $Ad_\sigma(C(b_1, ..., b_k)) = C(Ad_\sigma(b_1), ..., Ad_\sigma(b_k))$, $\forall \sigma \in G$, $b_i \in G \ast W$, and, as a particular case, that if $D = C \otimes_k W$, one has:

(3.2.4) $Ad_\sigma(D(a_1, ..., a_k)) = D(\sigma(a_1), ..., \sigma(a_k))$, $\forall \sigma \in G$, $a_i \in W$, i.e: $D$ is $G$-invariant.
Conversely, given \( D \in \mathcal{L}(\otimes_k W, G \ast W) \), \( G \)-invariant and normalized, one defines a \( \mathbb{C}[G] \)-relative cochain \( C \) of \( G \ast W \), such that \( C| \otimes_k W = D \), by:

\[
(3.2.5) \quad C(a_1 \otimes g_1, ..., a_k \otimes g_k) = D(a_1, g_1(a_2), ..., g_1 \cdots g_{k-1}(a_k)) * g_1 \cdots g_k.
\]

So we can replace \( \mathbb{C}[G] \)-relative \( k \)-cochains on \( G \ast W \) by their restrictions to \( \otimes_k W \), and this is what we shall now do. We denote by \( \mathcal{C}^k_{\mathbb{C}[G]}(G \ast W) \) the corresponding space, i.e \( C \in \mathcal{L}(\otimes_k W, G \ast W) \) such that:

\[
(3.2.6) \quad C \text{ is normalized: } C(a_1, ..., a_k) = 0, \text{ if some } a_i \in C, \text{ and } C \text{ is } G\text{-invariant.}
\]

Let us quickly recall what is a \( \mathbb{C}[G] \)-relative cochain \( C \) of \( G \ast W \). Observe that the product \( m_s \) is \( G\)-invariant by \((3.1.3)\). It results that \( m_h \) is \( \mathbb{C}[G] \)-relative if and only if

\[
(3.2.7) \quad Ad \sigma(m_h(a_1, a_2)) = m_h(\sigma(a_1), \sigma(a_2)), \quad \forall \sigma \in G, a_1, a_2 \in W ; \text{ in that case, one has:}
\]

\[
(3.2.8) \quad Ad \sigma(m_h(b_1, b_2)) = m_h(Ad \sigma(b_1), Ad \sigma(b_2)), \quad \forall \sigma \in G, b_1, b_2 \in G \ast W ; \text{ i.e } m_h \text{ is } G\text{-invariant.}
\]

(3.3) When \( C \in \mathcal{C}^k_{\mathbb{C}[G]}(G \ast W) \) we write \( C = \sum_{g \in G} C_g \otimes g \), with \( C_g \in \mathcal{C}^k(W) \), the space of normalized cochains on \( W \). We recall that \( G \) acts on \( \mathcal{C}^k(W) \) by:

\[
(3.3.1) \quad \pi_\sigma(C)(a_1, ..., a_k) = \sigma(C(\sigma^{-1}(a_1), ..., \sigma^{-1}(a_k))).
\]

The \( G \)-invariance condition becomes:

\[
(3.3.2) \quad C_{Ad \sigma(g)} = \pi_\sigma(C_g), \quad \forall \sigma, g \in G.
\]

Let \( S_\sigma \) be the centralizer of \( \sigma \) in \( G \), observe that \( \pi_\sigma(C_{\sigma}) = C_{\sigma}, \quad \forall s \in S_\sigma \), so that \( C_{\sigma} \) has to be \( S_\sigma\)-invariant. We shall use the notation:

\[
(3.3.3) \quad \mathcal{C}^k_{\sigma}(W) = \mathcal{C}^k(W)^{S_\sigma}.
\]

Let \( \Gamma \) be the set of conjugacy classes of \( G \); we fix a section \( \sigma_\gamma \in \gamma, \forall \gamma \in \Gamma \), and, with an abuse of notation, we write \( \gamma = \{ \tilde{x} \in G/S_\gamma \}, S_\gamma \) being the centralizer of \( \sigma_\gamma \) in \( G \). Given \( C \in \mathcal{C}^k_{\mathbb{C}[G]}(G \ast W) \), let \( \tilde{C}_\gamma := C_{\sigma_\gamma}, \quad \gamma \in \Gamma \), then, from \((3.3.2)\), \( \tilde{C}_\gamma \in \mathcal{C}^k_{\sigma_\gamma}(W) := \mathcal{C}^k_{\sigma_\gamma}(W) \), and, using \((3.3.2)\), one gets:

\[
(3.3.4) \quad C = \sum_{\gamma \in \Gamma} \sum_{s \in G/S_\gamma} \pi_s(\tilde{C}_\gamma) \otimes Ad x(\sigma_\gamma).
\]
On the other hand, given $\bar{C} = (\bar{C}_\gamma, \gamma \in \Gamma) \in \prod_{\gamma \in \Gamma} \mathcal{C}^{k}_{\gamma}(W)$, the map $B(x) := \pi_s(\bar{C}_\gamma), x \in G$, satisfies $B(xs) = B(x), \forall s \in S_\gamma$, so, if we define $C$ by formula (3.3.4), then $C \in \mathcal{C}^{k}_{C[G]}(G \ast W)$. So we have proved:

(3.3.5) LEMMA

*The map $T : C \rightarrow \bar{C}$ is an isomorphism from $\mathcal{C}^{k}_{C[G]}(G \ast W)$ onto $\prod_{\gamma \in \Gamma} \mathcal{C}^{k}_{\gamma}(W)$.*

(3.4) Let $d$ and $d_g, g \in G$, be the differentials of the Hochschild complex respectively of $G \ast W$ and $W_g$. One has:

(3.4.1) If $C = \sum_{g \in G} C_g \otimes g \in \mathcal{C}^{k}_{C[G]}(W \ast G)$, then $d(C) = \sum_{g \in G} d_g(C_g) \otimes g$.

(3.4.2) LEMMA

*With the notations of (3.4.1) and (3.3.4), $C \in B^{k}_{C[G]}(G \ast W)$ if and only if $\bar{C}_\gamma \in B^{k}_{\gamma}(W_\gamma)$, $\forall \gamma, \gamma \in \Gamma$.*

**Proof:** Let us assume that $C \in Z^{k}_{C[G]}(G \ast W)$, and that $\bar{C}_\gamma \in B^{k}_{\gamma}(W_\gamma)$, $\forall \gamma \in \Gamma$. One has $C = \sum_{\gamma \in \Gamma} \sum_{x \in G/S_\gamma} \pi_s(\bar{C}_\gamma) \otimes Ad x(\sigma_\gamma)$, $\bar{C}_\gamma$ is $S_\gamma$-invariant, therefore, by (2.2.2) there exists an $S_\gamma$-invariant cochain $\bar{B}_\gamma$ such that $\bar{C}_\gamma = d_{\sigma_\gamma} \bar{B}_\gamma$. Using (2.3), one has

$\pi_s(\bar{C}_\gamma) = d_{Ad x(\sigma_\gamma)}(\pi_s(\bar{B}_\gamma)), \forall x \in G$; defining $B := \sum_{\gamma \in \Gamma} \sum_{x \in G/S_\gamma} \pi_s(\bar{B}_\gamma) \otimes Ad x(\sigma_\gamma)$, $B$ is a $C[G]$-relative cochain by (3.3.5), and one has $C = dB$ by (3.4.1) Q.E.D.

(3.4.3) PROPOSITION

(1) The map $T$ of (3.3.5) induces an isomorphism from $H^{k}_{C[G]}(G \ast W)$ onto $\prod_{\gamma \in \Gamma} H^{k}_{\gamma}(W_\gamma)$

(2) [1] $H^{k}(G \ast W) = \{0\}$, if $k$ is odd. Let $\Gamma_{2k} = \gamma \in \Gamma / k_\sigma = k, \forall \sigma \in \gamma$, then $\dim H^{2k}(G \ast W) = \text{card } \Gamma_{2k}$.

**Proof**

(1) We need only to prove that $T$ is onto, but this is an immediate consequence of formula: $d_{Ad x(\sigma_\gamma)}(\pi_s(\bar{C}_\gamma)) = \pi_s(d_{\sigma_\gamma}(\bar{C}_\gamma))$, which is proved in (2.3), so if $C$ is defined by (3.3.4), $dC = 0$. 


(2) We apply (1) and (2.2.2). Q.E.D.

(3.4.4) Remark: As proved in [1], by dimension argument, there is an isomorphism $H^{2k}(G \ast W) \simeq \mathbb{C}^{\Gamma_{2k}}$.

(3.5) Let us precise the isomorphism of (3.4.4). We assume that $\Gamma_{2k} \neq \emptyset$, and take $\gamma \in \Gamma_{2k}$. Using (2.2.2), there exists $\Omega_{\gamma} \in \mathbb{Z}_{2k}^{\Gamma_{2k}}(W_{\gamma})$ such that $H^{2k}(W_{\gamma}) = \mathbb{C} \cdot \Omega_{\gamma}$ and $\Omega_{\gamma} |_{\Lambda_{2k}} = \omega_{\sigma_{\gamma}}$. It results from (2.3) that for any $\tau = Adx(\sigma_{\gamma}) \in \gamma$, one has $\pi_{\gamma}(\Omega_{\gamma}) |_{\Lambda_{2k}} = \omega_{\tau}$, and $H^{2k}(W_{\tau}) = \mathbb{C} \cdot \pi_{\gamma}(\Omega_{\gamma})$. Let us define: $C_{\gamma} := \sum_{x \in G/S_{\gamma}} \pi_{\gamma}(\Omega_{\gamma}) \otimes Adx(\sigma_{\gamma})$, and decompose: $C_{\gamma} = \sum_{g \in G} C_{g}^{\gamma} \otimes g$, then one has $C_{g}^{\gamma} = 0$, if $g \notin \gamma$, $C_{g}^{\gamma} |_{\Lambda_{2k}} = \omega_{g}$, if $g \in \gamma$, and $C_{\gamma} := C_{g}^{\gamma} = \Omega_{\gamma}$.

(3.5.1) PROPOSITION

(1) $C_{\gamma}$, $\gamma \in \Gamma_{2k}$ is a basis of $H^{2k}_{\mathbb{C}[G]}(G \ast W)$.

(2) Given $\lambda \in \mathbb{C}^{\Gamma_{2k}}$, there exists a cocycle $C_{\lambda} \in \mathbb{Z}_{\mathbb{C}[G]}^{2k}(G \ast W)$ such that:

$$C_{\lambda}(X_{1} \land \ldots \land X_{2k}) = \sum_{\gamma \in \Gamma_{2k}} \lambda(\gamma) \sum_{g \in G} \omega_{g}(X_{1}, \ldots, X_{2k}) \otimes g, \forall X_{i} \in V, g \in G.$$

$C_{\lambda}$ is a coboundary if and only if $\lambda = 0$, and the map $\lambda \to C_{\lambda}$ induces an isomorphism from $\mathbb{C}^{\Gamma_{2k}}$ onto $H^{2k}_{\mathbb{C}[G]}(G \ast W)$.

(3) If a cocycle $C \in \mathbb{Z}_{\mathbb{C}[G]}^{2k}(G \ast W)$ vanishes on $\Lambda_{2k}$, then $C$ is a coboundary.

Proof: To obtain (1), we apply (3.4.3); then, we define $C_{\lambda} = \sum_{\gamma \in \Gamma_{2k}} \lambda(\gamma) C_{\gamma}$, and prove (2) using (3.5) and (3.5.1) (1). (3) is consequence of (3.4.2) and (1.4.3). Q.E.D.

(3.5.2) Remark: (3.4.3) and (3.5.1) give the C.A. Theorem 2.

(4) AN ALTERNATIVE PROOF OF A THEOREM OF P. ETINGOF AND V. GINZBURG ABOUT SYMPLECTIC REFLECTION ALGEBRAS

(4.1) With the notations of (2.2), a symplectic reflection (in the finite subgroup $G$ of $SP(2n)$), is an element $g$ such that $dim V_{g} = 2$. When there are no symplectic reflections in $G$, by (3.4.3), $G \ast W$ is rigid. So let us assume the contrary. Given any $\lambda \in \mathbb{C}^{\Gamma_{2k}}$, $\lambda \neq 0$, we construct a non-trivial 2-cocycle $C_{\lambda}$ by (3.5.1) (2); since $H^{3}_{\mathbb{C}[G]}(G \ast W) = \{0\}$, $C_{\lambda}$ is not obstructed, so there exists a $\mathbb{C}[G]$-relative non trivial deformation of $G \ast W$, with leading cocycle $C_{\lambda}$. Using once more $H^{3}_{\mathbb{C}[G]}(G \ast W) =$
{0}, and varying $\lambda$, the procedure will provide a universal deformation formula of $G*W$ (see [11]). Using (3.5.1), one has:

$$\forall X, Y \in V, [X, Y]_h^2 = \omega(X, Y) + h \sum_{\gamma \in \Gamma_2} \lambda(\gamma) \sum_{g \in \gamma} \omega_g(X, Y) \otimes g + h^2(...).$$

At first order, we find exactly the relations of the Symplectic Reflection Algebra $H_{\hbar}$ of [12]. We call these relations the SRA-relations.

(4.1.2) By construction, since $C_\lambda$ is non trivial, the corresponding deformation is non trivial.

(4.2) Let us quickly recall the definition of the symplectic reflection algebra $H_{\hbar}$, following [11] [12]. Let $T(V)$ be the tensor algebra of $V$, $G* T(V)$ the smash product of $T(V)$ with $\mathbb{C}[G]$, and, given $\lambda \in \mathbb{C}^{G_2}$, $I_\lambda$ the ideal in $[G* T(V)] [\hbar]$ generated by:

$$R_{\hbar}(X, Y) = X \otimes Y - Y \otimes X - \omega(X, Y) - h \sum_{\gamma \in \Gamma_2} \lambda(\gamma) \sum_{g \in \gamma} \omega_g(X, Y) \otimes g, \forall X, Y \in V.$$ 

Then $H_{\hbar} = (G* T(V)) [\hbar] / I_\lambda$.

By definition, the SRA-relations hold in $H_{\hbar}$.

Theorem (2.16) of [12] ((9.5) of [11]) proves that, when varying $\lambda$ in $\mathbb{C}^{G_2}$, $H_{\hbar}$ provides an algebraic deformation of $G*W$, non trivial, as a deformation, if $\lambda \neq 0$.

We recall that an algebraic deformation of an algebra $A$ is a $\mathbb{C}[h]$-algebra structure on $A[\hbar]$, and that a polynomial deformation is a formal deformation on $A[[\hbar]]$ such that $A[\hbar]$ is a subalgebra (and therefore an algebraic deformation), in other words: $\forall a, b \in A, a \ast \hbar b \in A[\hbar]$.

(4.3) We shall now give a completely different proof of the E.G. Theorem (2.16) in [12] ((9.5) in [11]), and PBW Theorem (1.3) in [12] (8.3) in [11]): we show that there exists a $\mathbb{C}[G]$-relative deformation of $G*W$, satisfying the SRA-relations, that, up to an equivalence, this deformation is polynomial, and that the polynomial part is isomorphic to $H_{\hbar}$; the PBW property is a consequence. The proof uses essentially the C.A. Theorems, classical deformation theory, and a formula of Berezin ([6], [9]).

(4.4) E.G. THEOREM

There exists a non trivial polynomial $\mathbb{C}[G]$-relative deformation of $G*W$, satisfying the SRA-relations. The subalgebra $(G*W) [\hbar]$ of this deformation is isomorphic to the Symplectic Reflection Algebra $H_{\hbar}$, and the PBW property holds for $H_{\hbar}$ ($\hbar$ formal, or $\hbar \in \mathbb{C}$).

Proof:
(4.4.1) Let $C = C_\Lambda$. As seen in (4.1), there exists a non-trivial first order $\mathbb{C}[G]$-relative deformation: $a*b = a*b + \hbar C(a,b)$, $\forall a,b \in W$ (is the Moyal product), and we have to look for a second order still $\mathbb{C}[G]$-relative deformation $a*b = a*b + \hbar C(a,b) + \hbar^2 D(a,b)$, such that $D|_{\Lambda^2} = 0$, which is the SRA-condition. Using Gerstenhaber bracket, the associativity condition at order 2 is $d\Lambda = -\frac{1}{2} [C,C]$. Since $C \in Z^2_{\mathbb{C}[G]}(G \ast W)$, one has $[C,C] \in Z^3_{\mathbb{C}[G]}(G \ast W) = B^3_{\mathbb{C}[G]}(G \ast W)$, so $[C,C] = dB$, $B \in \mathcal{C}^2_{\mathbb{C}[G]}(G \ast W)$, moreover, since $C(\Lambda^2) \subset \mathbb{C}[G]$, one has $[C,C]|_{\Lambda^2} = 0$. With the notations of (3.3), and $d\gamma := d\sigma$, one has $[C,C]|_{\gamma} = d\gamma B^{\gamma}, B^{\gamma} \in \mathcal{C}^2_{\gamma}(W), \forall \gamma \in \Gamma$.

Let $b^{\gamma} := B^{\gamma}|_{\Lambda^2}$, and $\Delta^{\gamma}$ the Koszul differential, then $[C,C]|_{\Lambda^2} = 0$ implies that $d\gamma B^{\gamma}|_{\Lambda^2} = \Delta^{\gamma}b^{\gamma} = 0$, so $b^{\gamma}$ is a Koszul cocycle, and it is $S^{\gamma}$-invariant, therefore, as in (2.2.12), there exists an $S^{\gamma}$-invariant Hochschild cocycle $Z^{\gamma} \in Z^2_{\gamma}(W^{\gamma})$ such that $Z^{\gamma}|_{\Lambda^2} = b^{\gamma}$. By (3.4.3), there exists $Z \in Z^2_{\mathbb{C}[G]}(G \ast W)$ such that $Z^{\gamma} = Z^{\gamma}, \forall \gamma \in \Gamma$; let $D = \frac{1}{2} (Z-B)$, then $dD = -\frac{1}{2} [C,C]$, and $dD|_{\Lambda^2} = 0, \forall \gamma \in \Gamma$, moreover $D$ is $\mathbb{C}[G]$-relative, so $D|_{\Lambda^2} = 0$. Now, we have the wanted second order deformation. The same proof can be repeated for next orders: for instance, at third order, we have to find $E$ such that $dE = -[D,C]$, and $E|_{\Lambda^2} = 0$. Since $D$ is $\mathbb{C}[G]$-relative, and $D|_{\Lambda^2} = 0$, one has $[D,C]|_{\Lambda^2} = 0$, so the above arguments do apply.

(4.4.2) We have constructed a non trivial $\mathbb{C}[G]$-relative deformation satisfying the SRA-relations. We are going to renormalize, using an equivalence, to obtain a polynomial deformation still satisfying SRA-relations. We use arguments inspired of [8] and [14]. We recall that $\ast$ is the Moyal product, that $\ast$ is the abelian product, and that $W$ is linearly generated by elements of type $X^{*k} = X^k$, $X \in V$, $k \in \mathbb{N}$ (see e.g [19]). Let $\tilde{A} = G \ast W \ [[\hbar]]$, with product $\ast_{\hbar}$ defined in (4.4.1), there exists a $\mathbb{C}[[\hbar]]$-linear map $\rho : W[[\hbar]] \rightarrow \tilde{A}$ such that: $\rho(1) = 1$ and

$$\rho(X_1...X_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} X_{\sigma(1)} \ast_{\hbar} ... \ast_{\hbar} X_{\sigma(k)}, \forall X_i \in V,$$

and therefore $\rho(X^k) = X^{*k}$, $\forall X \in V$. Being $\mathbb{C}[G]$-relative, and a deformation of the Moyal product, $\ast_{\hbar}$ satisfies:

$$Ad_{g}(X_1 \ast_{\hbar} X_2 \ast_{\hbar} ... \ast_{\hbar} X_k) = g(X_1) \ast_{\hbar} g(X_2) \ast_{\hbar} ... \ast_{\hbar} g(X_k), \forall g \in G, X_i \in V.$$
(see (3.2.8)), so if we define a \(\mathbb{C}[[h]]\)-linear map \(\tilde{\rho} : \tilde{A} \rightarrow \tilde{A}\) by \(\tilde{\rho}(a \ast g) = \rho(a) \ast g, \ \forall a \in W, \ g \in G\), then \(\tilde{\rho}(g \ast a) = g \ast \tilde{\rho}(a)\), therefore \(\tilde{\rho} = Id + \sum_k h^k \cdot \rho_k\), with \(\rho_k \ \mathbb{C}[G]\)-relative. It results that we can define a new \(\mathbb{C}[G]\)-relative product on \(\tilde{A}\) by: \(\tilde{a} \ast' \tilde{b} = \tilde{\rho}^{-1} [\tilde{\rho}(\tilde{a}) \ast \tilde{\rho}(\tilde{b})]\), \(\tilde{a}, \tilde{b} \in \tilde{A}\). By definition, one has: \(X^{*k} = X^k = X^{*k}, \ \forall X \in V, \ k \in \mathbb{N}\), and therefore

\[
\frac{1}{k!} \sum_{\sigma \in \Sigma_k} X_{\sigma(1)} \ast' X_{\sigma(2)} \ast' \ldots \ast' X_{\sigma(k)} = X_1 \ldots X_k
\]

for all \(X_i \in V\) and \(k \in \mathbb{N}\). Moreover:

\[
[X, Y]_{\ast'} = \rho^{-1} (\omega(X, Y) + h C(X \wedge Y)) = \omega(X, Y) + h C(X \wedge Y),
\]

so the SRA-relations are still verified by \(\ast'\).

(4.4.3) Let us now show that \(\ast'\) is a polynomial deformation. Let \((W_k, k \geq 0)\) by the canonical filtration of \(W\). One has \(X_1 \ast' X_2 = \frac{1}{2}(X_1 \ast X_2 + X_2 \ast X_1) + \frac{1}{2} [X_1, X_2]_{\ast'} = X_1 X_2 + \frac{1}{2} [X_1, X_2]_{\ast'}, \) and \([X_1, X_2]_{\ast'} \in (\mathbb{C}[G]) [h]\).

(4.4.4) By induction, we assume: \(X_1 \ast' X_2 \ast' \ldots \ast' X_j = X_1 X_2 \ldots X_j + R, \ \forall X_i \in V, j \leq k,\) with \(R \in (W_{j-2} \otimes \mathbb{C}[G]) [h]\). We need the following Lemma (a direct consequence of a formula of F.A. Berezin [6], [9]):

(4.4.5) Lemma

Let \(A\) be an algebra, \(< a_1, \ldots, a_k > : = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} a_{\sigma(1)} \ldots a_{\sigma(k)}, \ a_i \in A,\) then:

\[
a \ast < a_1, \ldots, a_k > = < a, a_1, \ldots, a_k > + \sum_{1 \leq j \leq k} \frac{B_j}{j!} \sum_{i_1 < \ldots < i_j} < ada_{i_1}(a), a_{i_1}, \ldots, a_{i_j}, \ldots, a_k >
\]

where the \(B_j\) are the Bernoulli numbers.

Proof:

Consider \(A\), with its natural bracket, as a Lie Algebra, and let \(\mathcal{U}\) be its enveloping algebra. By [9], the formula is valid in \(\mathcal{U}\) (i.e: with product of \(\mathcal{U}\)). But there exists an algebra morphism \(\mu : \mathcal{U} \rightarrow A\) such that \(\mu|_A = Id_A\), and applying \(\mu\) on the formula written in \(\mathcal{U}\), one obtains the formula written in \(A\), as wanted. Q.E.D.

From the induction assumption (4.4.4), taking first \(X_j = X\), next ones = \(Y\), and using \(X^{\ast'} = X', Y^{\ast'} = Y\), it results that: \(X' \ast' Y^s = X', Y^s + R',\) with \(R' \in \)
(W_{r+s-2} \otimes \mathbb{C}[G]) [\hbar], if r + s \leq k, and then, since W_j is linearly generated by 
\{X^\ell, \ell \leq j, X \in V\}:

(4.4.6) a \ast' b = ab + R'' , with R'' \in (W_{r+s-2} \otimes \mathbb{C}[G]) [\hbar], if r + s \leq k, \forall a \in W_r, b \in W_s. Now, using Lemma (4.4.5):

\begin{align*}
X \ast' (X_1...X_k) &= X \ast' <X_1,...,X_k> \\
&= X.X_1...X_k + \\
&\sum_{j \geq k} B_{j} \sum_{i_1 < ... < i_j} <\text{ad}_X \tau_{i_1} \ldots \text{ad}_X \tau_{i_j}(X),X_1,\ldots,\hat{X}_{i_1},\ldots,\hat{X}_{i_j},\ldots,X_k>_{\ast'}
\end{align*}

Since \( \xi := \text{ad}_X \tau_{i_1} \ldots \text{ad}_X \tau_{i_j}(X) \) is a \( \mathbb{C}[\hbar]\)-linear combination of terms of type \( (V_1 \ast' \ldots \ast' V_{j-1}) \ast t, \) with \( V_i \in V, t \in \mathbb{C}[G], \) therefore an element of \( (W_{j-1} \otimes \mathbb{C}[G])[\hbar], \) and the \text{term ad}_X \tau_{i_1} \ldots \text{ad}_X \tau_{i_j}(X) \ast' X_1 \ast' \ldots \ast' \hat{X}_{i_1} \ast' \ldots \ast' \hat{X}_{i_j} \ast' \ldots \ast' X_k \in (W_{k-1} \otimes \mathbb{C}[G]) [\hbar]. \) Similar arguments show that all terms in the development of

\begin{align*}
<\text{ad}_X \tau_{i_1} \ldots \text{ad}_X \tau_{i_j}(X),X_1,\ldots,\hat{X}_{i_1},\ldots,\hat{X}_{i_j},\ldots,X_k>_{\ast'}
\end{align*}

are elements of \( (W_{k-1} \otimes \mathbb{C}[G]) [\hbar]. \) Therefore \( X \ast' (X_1...X_k) = X.X_1...X_k + S, S \in (W_{k-1} \otimes \mathbb{C}[G]) [\hbar]. \) Now \( X \ast' X_1 \ast' \ldots \ast' X_k = X \ast' (X_1...X_k) + X \ast' R = X.X_1...X_k + S + X \ast' R, \) and since \( S \) and \( X \ast' R \in (W_{k-1} \otimes \mathbb{C}[G]) [\hbar], \) our induction is complete.

To conclude the proof, from \( X_1 \ast' \ldots \ast' X_k = X_1...X_k + R, \) with \( R \in (W_{k-2} \otimes \mathbb{C}[G]) [\hbar], \forall X_i \in V, k \in \mathbb{N}, \) we deduce, as was done for (4.4.6), \( a \ast' b = ab + T, \forall a \in W_r, b \in W_s, \) with \( T \in (W_{r+s-2} \otimes \mathbb{C}[G]) [\hbar]. \) This proves that \( \ast' \) is a polynomial deformation.

(4.4.7) We now prove that the subalgebra \( (W \otimes \mathbb{C}[G])[\hbar] \) of \( \tilde{A} \) is isomorphic to the Symplectic Reflexion Algebra \( H_{h\lambda} \) of (4.2), and the P.B.W. will follow.

We need some notations: \( \times_h \) will be the product of \( H_{h\lambda}, \) which is generated, as an algebra, by \( X, \mathfrak{g} \) and \( h, X \in V, g \in G. \) We denote by \( \ast_h \) the product on \( G \ast W[\hbar] \) constructed (and denoted by \( \ast' \)) in (4.4.2). We denote by \( \times \) the product on \( G \ast W \) coming from the abelian product of \( W. \) We also use the notation \( <a_1,...,a_k> \) of (4.4.5). We observe that \( R_{h\lambda}(g(X),g(Y)) = Ad g (R_{h\lambda} (X,Y)), \forall X,Y \in V, g \in G, \) so the natural action of \( G \) on \( V \) is preserved in the quotient \( H_{h\lambda} = G \ast T(V)[\hbar] / I_{\lambda} : g(\bar{X}) = g(\bar{X}) = Ad g(X) = Ad \mathfrak{g}(\bar{X}), \forall g \in G, X \in V. \)
(4.4.8) There exists a morphism $\pi : H_{h\lambda} \rightarrow G \ast W[h] \rightarrow G \ast W$ (with Moyal product), which is onto. We define a section $\sigma$ of $\pi$ by:

$$
\sigma(X_1 \ldots X_k \otimes g) = \langle \overline{X}_1, \ldots, \overline{X}_k \rangle_{\times h} \overline{g}.
$$

The map $\sigma$ is one to one, so we can identify the spaces $G \ast W$ and $\sigma(G \ast W)$. This being done, we have now $G \ast W \subset H_{h\lambda}$ and the product $\times$ of $G \ast W$ becomes:

(4.4.9) $X_1 \ldots X_k = \langle X_1, \ldots, X_k \rangle_{\times h} \forall X_i \in V, g \times (X_1 \ldots X_k) = (g(X_1) \ldots g(X_k))_{\times h}$

$g = g \times_h (X_1 \ldots X_k), \forall X_i \in V, g \in G$.

Denote by $G \ast W (h)$ the subspace of $H_{h\lambda}$ of elements which are polynomial of $h$, with coefficients in $G \ast W$, by $G \ast W_k (h)$ elements which are polynomial of $h$ with coefficients in $G \ast W_k$. Repeating identically the arguments of (4.6), we obtain:

(4.4.10) $X_1 \times_h X_2 \times_h \ldots \times_h X_k = X_1 \ldots X_k + R$, with $R \in G \ast W_{k-2}(h)$, $\forall X_i \in V$. It results that $H_{h\lambda} = G \ast W(h)$. Now, fix any basis $\{e_1, \ldots, e_n\}$ of $V$, the natural morphism from $H_{h\lambda}$ onto $G \ast W [h]$ maps the generator system $\{e_1^i \ldots e_n^j \times_h g, i_1 \ldots i_n \in \mathbb{N}, g \in G\}$ of the $\mathbb{C}[h]$-algebra $H_{h\lambda}$ onto a basis of $G \ast W[h]$, so we have the wanted isomorphism.

(4.4.11) From the isomorphism $H_{h\lambda} \simeq G \ast W[h]$, we deduce that $H_{h\lambda}$ is a deformation of $G \ast W$, and the P.B.W. property for $H_{h\lambda}$. Given $c \in \mathbb{C}$, $H_{\lambda}$ is defined as the quotient of $G \ast T(V)$ by relations $R_{\lambda} (X, Y)$, $X, Y \in V$ (see (4.2)). It is easy to check that algebras $H_{\lambda}$ and $H_{h\lambda}/H_{h\lambda}(h - c)$ are isomorphic, and the P.B.W. property for $H_{\lambda}$ follows.

Q.E.D.

References

[1] Alev J., Farinati M.A, Lambre T., Solotar A.L.: Homologie des invariants d’une algèbre de Weyl sous l’action d’un groupe fini, J. of Algebra, 232 (2000), 564-577.

[2] Alev J., Lambre T.: Homologie des invariants d’une algèbre de Weyl, K-Theory 18 (1999), 401-411.

[3] Alvarez M.S.: Algebra structure on the Hochschild cohomology of the ring of invariants of a Weyl algebra under a finite group, J. of Algebra 248 (2002), 291-306.

[4] Arnal D., Ben Amor H., Pinczon G.: The structure of $\mathfrak{sl}(2,1)$-supersymmetry, Pac. J. Math. 165 (1994), 17-49.

[5] Bayen F., Flato M., Fronsdal C., Lichnerowicz A. Sternheimer D.: Deformation theory and quantization, Ann. Phys. I, II, (1978), 61-110, 111-151.
[6] Berezin F.A.: Quelques remarques sur l’enveloppe associative d’une algèbre de Lie, Funct. Anal. i evo prilojenie, 1 (1967), 1-14.
[7] Cartan H., Eilenberg S.: Homological Algebra, Princeton Univ. Press, Princeton NJ, 1956.
[8] Dito G.: Kontsevich star product on the dual of a Lie algebra, Lett. Math. Phys., 48 (1999), 307-322.
[9] Dixmier J.: Algèbres Enveloppantes, Gauthier-Villars Paris, 1974.
[10] Du Cloux F.: Extensions entre représentations unitaires irréductibles des groupes de Lie nilpotents, Astérisque, 125 (1985), 129-211.
[11] Etingof P.: Lectures on Calogero-Moser systems, math.QA / 0606233.
[12] Etingof P., Ginzburg V.: Symplectic reflection algebras, Calogero-Moser space and deformed Harish-Chandra homomorphism. Invent. Math. 147 (2002), 243-348.
[13] Gestenhaber M., Schack S.D.: Algebraic cohomology and deformation Theory, in: Deformation theory of Algebras and Structures, NATO-ASI Series C.297, Kluwer Academic Publishers, Dordrecht, 1988.
[14] Gutt S.: An explicit \(^-*\)-product on the cotangent bundle of a Lie group, Lett. Math. Phys. 7 (1983), 249-258.
[15] Montgomery S.: Fixed Rings of Finite Automorphism Groups of Associative Rings, Lect. Notes in Math., vol. 818, Springer-Verlag, New-York / Berlin 1980.
[16] Nadaud F.: Generalized deformations, Koszul resolutions, Moyal products, Rev. Math. Phys. 10 (5) (1998), 685-704.
[17] Pinczon G.: Non commutative Deformation Theory, Lett. Math. Phys. 41 (1997), 101-117.
[18] Pinczon G.: The enveloping algebra of the Lie superalgebra \(osp(1,2)\), J. of Algebra 132 (1) (1990), 219-242.
[19] Pinczon G., Ushirobira R.: Supertrace and Superquadratic Lie structure on the Weyl Algebra, and Applications to Formal Inverse Weyl Transform, Lett. Math. Phys. 74 (2005), 263-291.
[20] Sridharan R.: Filtered algebras and representations of Lie algebras. Trans. Amer. Math. Soc., 100 (1961), 530-550.

INSTITUT DE MATHEMATIQUES DE BOURGOGNE, UNIVERSITE DE BOURGOGNE, B.P. 47870, F-21078 DIJON CEDEX, FRANCE
E-mail address: gpinczon@u-bourgogne.fr