The violation of a uniqueness theorem and an invariant in the application of Poincaré–Perron theorem to Heun’s equation

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Abstract

The domain of convergence of a Heun function obtained through the Poincaré–Perron (P–P) theorem is not absolute convergence but conditional one [2]. We show that a uniqueness theorem is not available if we apply the P–P theorem into the Heun’s equation. We verify that the uniqueness theorem is only applicable when a local Heun function is absolutely convergent.

Keywords: Uniqueness theorems; Poincaré–Perron theorem; Heun’s equation; Three term recurrence relation

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1. Introduction

Heun’s equation appears in mathematical physics problems, in addition to that, it starts to arise in economic and financial problems (SABR model) recently [6, 10]. For instance, the Heun functions come out in atomic and nuclear physics (the hydrogen-molecule ion) [27], in the Schrödinger equation with doubly anharmonic potential [17] (its solution is the confluent forms of Heun functions), in the Stark effect [4], in gravitational waves and black holes such as perturbations of the Kerr metric [12, 26], in solid state physics (crystalline materials), in Collegero–Moser–Sutherland systems, water molecule, graphene electrons, quantum Rabi model [29], biophysics [21], etc [22, 23].

In the classical viewpoint and specialized flavour, power series solutions have been obtained by putting infinite series with unknown coefficients into linear ODEs. The recurrence relation of polynomial coefficients starts to arise, and there can be between 2-term and infinity-term in the recursive relation.

Even though the Heun equation is represented in many scientific areas, its closed forms are unknown until now because the recurrence relation in its series solution consists of the 3-term. It seems that its general solution cannot be reduced to a 2-term recurrence relation by changing an independent variable and coefficients. Until recently, we have constructed many physics solutions by utilizing hypergeometric-type functions having the 2-term recursive relation between consecutive coefficients in their series solutions traditionally. But since the Heun’s equation starts to appear in tremendous physical phenomenons, we cannot use hypergeometric...
functions any more. Today, it seems we require at least a three or four term recurrence relation in power series solutions.

It has been known that we obtain the radius of convergence for a local Heun function by applying the Poincaré–Perron (P–P) theorem [14, 15]. However, the author has shown that the domain of convergence of the Heun function obtained by applying the P–P theorem is not absolute convergence but conditionally one [2]. And the absolute convergence test for the Heun function has been well-established and it has been well explained why the P–P theorem is so problematic when applied to the Heun equation [2]. In this paper, we show the reason why a uniqueness theorem and an invariant are violated if the P–P theorem is applied into a local Heun function. It also shows that the domain of convergence of the local Heun function must be absolutely convergent in order for the uniqueness theorem to be preserved. Indeed, this idea is applicable to any linear ODEs having multi-term recurrence relation in their power series solutions.

The canonical form of the general Heun’s differential equation is taken as [9]

\[
\frac{d^2 y}{dx^2} + \left( \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a} \right) \frac{dy}{dx} + \frac{\alpha \beta x - q}{x(x-1)(x-a)} y = 0 \tag{1.1}
\]

This equation is of Fuchsian type with regular singularities at \(x = 0, 1, a, \infty\) with the condition \(\epsilon = \alpha + \beta - \gamma - \delta + 1\) to ensure regularity of the point at \(\infty\). The Mathieu, Lame, spheroidal wave, hypergeometric-type equations and etc are just particular cases of the Heun’s equation, referred as the 21st century successor of hypergeometric equation [7, 20].

Since the solution of (1.1) is analytic, \(y(x)\) can be represented by a power series of the form

\[
y(x) = \sum_{n=0}^{\infty} d_n x^{\lambda_n} \tag{1.2}
\]

where \(\lambda\) is an indicial root. Plug (1.2) into (1.1):

\[
d_{n+1} = A_n d_n + B_n d_{n-1} \quad ; \quad n \geq 1 \tag{1.3}
\]

with \(A_n = A \overline{A}_n\) and \(B_n = A B_n\)

\[
A = \frac{1+x}{a} \quad B = -\frac{1}{a} \quad A_0 = \frac{n^2 + (\alpha + \beta - \delta + 2 \lambda + 1)(\alpha + \beta + \gamma - 1 + 2 \lambda)}{1 + a} \quad B_0 = \frac{n^2 + (1 + \gamma + 2 \lambda) a + (1 + \lambda)(\beta - 1 + \lambda)}{n^2 + (\gamma + 1 + 2 \lambda) a + (1 + \lambda)(\beta - 1 + \lambda)}
\]

We have two indicial roots which are \(\lambda = 0\) and \(1 - \gamma\).

2. Poincaré–Perron theorem and its applications to Heun’s equation

According to the Poincaré–Perron (P–P) theory [11, 15, 13, 14], the characteristic polynomial equation of a recurrence (1.3) is given by

\[
t^2 - At - B = 0 \tag{2.1}
\]
An asymptotic recurrence relation of (2.1) is
\[ d_{n+1} = A d_n + B d_{n-1} ; \quad n \geq 1 \] (2.2)
and \( d_1 = A d_0 \) where \( d_0 = 1 \) for simplicity.

The spectral numbers \( \lambda_1 \) and \( \lambda_2 \) in (2.1) have two different moduli such as
\[ \lambda_1 = \frac{1 + a - |1 - a|}{2a} \quad \lambda_2 = \frac{1 + a + |1 - a|}{2a} \] (2.3)

More explanations are explained in Appendix B of part A [17], Wimp (1984) [28], Kristensson (2010) [11] or Erdélyi (1955) [5].

We know \( \lim_{n \to \infty} |d_{n+1}/d_n| = \lim_{n \to \infty} |\overline{d}_{n+1}/\overline{d}_n| \). And the Poincaré–Perron theory for the 3-term recurrence relation states that,

(i) if \( |\lambda_i| < |\lambda_j| \), then \( \lim_{n \to \infty} |\overline{d}_{n+1}/\overline{d}_n| = |\lambda_j| \), so that the radius of convergence for a 3-term recursion relation (2.1) is \( |\lambda_j|^{-1} \) where \( i, j \in \{1, 2\} \).

(ii) if \( |\lambda_i| = |\lambda_j| \) and \( \lambda_i \neq \lambda_j \), then \( \lim_{n \to \infty} |\overline{d}_{n+1}/\overline{d}_n| \) does not exist; if \( \lambda_i = \lambda_j \), \( \lim_{n \to \infty} |\overline{d}_{n+1}/\overline{d}_n| \) is convergent.

The proper domain of convergence for \( a, x \in \mathbb{R} \), is shown shaded in Fig. 1; it does not include dotted lines, there is no such solution at the origin, and maximum modulus of \( x \) is the unity.

![Figure 1: Domain of convergence of the series by applying Poincaré-Perron theorem](image)

Actually, even though there is no such solution at \( a = -1 \) in Fig. 1 because of the statement (ii), we are able to have suitable domain by leaving a solution as the recurrence relation. To verify this phenomenon, put \( a = -1 \) into (2.2)
\[ \overline{d}_{n+1} = A \overline{d}_n + B \overline{d}_{n-1} ; \quad n \geq 1 \] (2.4)
where \( \overline{d}_1 = 0 \). Put (2.4) into a series \( \sum_{n=0}^{\infty} \overline{d}_n x^n \) and its series solution is
\[ \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1 - x^2} \] (2.5)
Therefore, we have the proper domain \( |x^2| < 1 \) as \( a = -1 \) for a local solution. Or we can have its domain by putting \( a = -1 \) in (1.2), directly.

Unlike a series solution (3.1), taken by rearranging coefficients \( A_n \) and \( B_n \) in each sequence \( d_n \) in (1.3), a power series solutions on the domain of convergence which is obtained by Poincaré–Perron theory, is just left as a general solution of Heun equation as a solution of recurrences.

\[
y^{\alpha}(x) = \sum_{n=0}^{\infty} d_n x^{n+\alpha}
\]

\[
y^{\alpha}(x) = x^1 \left( 1 + A_0 x + (B_1 + A_{0,1}) x^2 + (A_0 B_2 + A_1 B_1 + A_{0,1,2}) x^3 + \right.
\]

\[
+ (B_{1,3} + A_0 B_{3,1} + A_{0,3} B_2 + A_{2,3} B_1 + A_{0,1,2,3}) x^4 + (A_0 B_{2,4} + A_2 B_{1,4} + A_4 B_{1,3} + A_{0,1,2} B_4 + A_{0,1,4} B_3 + A_{0,3,4} B_2 + A_{2,3,4} B_1 + A_{0,1,2,3,4}) x^5 + \cdots
\]

(2.6)

In (2.6), the definition of \( B_{i,j,k,l} \) refer to \( B_i B_j B_k B_l \). Also, \( A_{i,j,k,l} \) refer to \( A_i A_j A_k A_l \).

3. Heun function and its domain of convergence

By rearranging coefficients \( A_n \) and \( B_n \) in each sequence \( d_n \) in (1.3) where \( d_0 = 1 \) chosen for simplicity from now on, a local Heun series solution is given by

\[
y^{\alpha}(x) = x^1 \left( y^{\alpha}_0(x) + y^{\alpha}_1(x) \eta + \sum_{n=2}^{\infty} \eta^{\alpha}_n(x) \eta^n \right)
\]

(3.1)

where

\[
y^{\alpha}_0(x) = \sum_{i_0=0}^{\infty} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \eta^{i_0}
\]

\[
y^{\alpha}_1(x) = \sum_{i_0=0}^{\infty} \prod_{i_1=0}^{i_0-1} B_{2i_1+1} \sum_{i_2=0}^{i_1-1} \prod_{i_3=0}^{i_2-1} B_{2i_3+1} \eta^{2i_0}
\]

\[
y^{\alpha}_n(x) = \sum_{t=2}^{\infty} \left( \sum_{i_0=0}^{\infty} \prod_{i_1=0}^{i_0-1} A_{2i_1} \prod_{i_2=0}^{i_1-1} B_{2i_2+1} \eta^{i_0} \prod_{k=1}^{t-1} \prod_{i_2=i_0+2t-1}^{i_1} \prod_{i_3=i_2+2t-1}^{i_3-1} \prod_{i_4=i_3+2t-1}^{i_4-1} B_{2i_4+1} \sum_{i_5=0}^{i_4-1} \prod_{i_6=0}^{i_5-1} \prod_{i_7=0}^{i_6-1} B_{2i_7+1} \right)
\]

and

\[
\eta = \frac{(1+a)x}{a}, \\
z = -1 \frac{1}{a} x^2
\]

The sequence \( d_n \) combines into combinations of \( A_n \) and \( B_n \) terms in (1.3): (3.1) is done by letting \( A_n \) in the sequence \( d_n \) is the leading term in a series (1.2); we observe the term of sequence \( d_n \) which includes zero term of \( A'_n \)'s for a sub-power series \( y^{\alpha}_{0}(x) \), one term of \( A'_n \)'s for the sub-power series \( y^{\alpha}_{1}(x) \), two terms of \( A'_n \)'s for a \( y^{\alpha}_{2}(x) \), three terms of \( A'_n \)'s for a \( y^{\alpha}_{3}(x) \), etc.

The radius of convergence for (3.1) is taken as (2)

\[
\left| \frac{1 + a}{a} \right| + \left| -\frac{1}{4} \frac{a}{a} \right| < 1
\]

(3.2)
For $a, x \in \mathbb{R}$, Fig. 2 represents a graph for the radius of convergence of (3.2) in the $a$-$x$ plane; the shaded area represents the domain of convergence of the series for a Heun’s equation around $x = 0$; it does not include dotted lines, there is no such solution at the origin (the black colored point), and maximum modulus of $x$ is the unity. For the simple numeric calculation, we treat $A_n$ and $B_n$ as the unity in (1.4). A generating function of (2.2) is

$$
\sum_{n=0}^{\infty} d_n x^n = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(n + m)!}{n! m!} z^n \eta^m = \frac{1}{1 - (z + \eta)}
$$

(3.3) generally speaking, where $|z| + |\eta| < 1$.

Figs. 1 and 2 tell us that both intervals of convergence of Heun function at $x = 0$ are not equal to each other analytically. Because Fig. 2 is build by rearranging coefficients $A$ and $B$ in each sequence $d_n$ in (2.2). In contrast, Fig. 1 is taken by observing the ratio of sequence $d_{n+1}$ to $d_n$ at the limit $n \to \infty$ in (2.2).

Fig. 3 represents two different shaded areas of convergence in Figs. 1 and 2: (1) There are no analytic solutions at $a = 0$ for both domains of convergence, (3) in the bright shaded area where $a > 0$, the domain of convergence of the Heun series around $x = 0$, obtained by rearranging coefficients $A$ and $B$ terms in each sequence $d_n$, in (2.2), is not available; it only provides the domain of convergence obtained by the Poincaré–Perron theory, (4) in the dark shaded area at $a = 0$, two different domains of convergence are equivalent to each other.

It has been believed that the Poincaré–Perron theory provides us the domain of convergence for a power series solution of Heun equation. If this is true, a series solution converges absolutely whether we rearrange its terms for the series solution or not. However, at any points of $(a, x)$ in the bright shaded area where $a > 0$ on Fig. 3, a geometric series solution (3.3), constructed by rearranging of $A$ and $B$ terms in (2.2), is not convergent any more; and $y^A(x)$ is the divergent infinite series in (3.1).

By putting (2.2) into $\sum_{n=0}^{\infty} d_n x^n$ with $d_0 = 1$, the series solution of the 3-term recurrence
The absolutely convergent series solution is decided uniquely when
\[ \sum_{n=0}^{\infty} \left| d_n \right| x^n = 1 + Ax + \left( A^2 + B \right)x^2 + \left( A^3 + 2AB \right)x^3 + \left( A^4 + 3A^2B + B^2 \right)x^4 + \left( A^5 + 4A^3B + 3AB^2 \right)x^5 + \cdots \] (3.4)

The absolutely convergent series solution is decided uniquely when \( \sum_{n=0}^{\infty} \left| d_n \right| |x|^n \) is convergent. According to the Cauchy ratio test, if the condition \( \lim_{n \to \infty} \left| d_{n+1} \right|/\left| d_n \right| < 1 \) is satisfied, a series solution is absolute convergent. And the Poincaré–Perron theorem tell us that \( \left| d_{n+1} \right|/\left| d_n \right| \) as \( n \to \infty \) is equivalent to one of roots of the characteristic polynomial in (2.1). This approach gives us that a series solution is not absolutely convergent but conditionally one. Taking all absolute values inside parentheses of (3.4) to make an absolutely convergent series solution, we obtain a proper radius of convergence for a solution.

\[ \sum_{n=0}^{\infty} \left| d_n \right| |x|^n = 1 + |A||x| + \left( |A|^2 + |B| \right)|x|^2 + \left( |A|^3 + 2|A||B| \right)|x|^3 + \left( |A|^4 + 3|A|^2|B| + |B|^2 \right)|x|^4 + \left( |A|^5 + 4|A|^3|B| + 3|A||B|^2 \right)|x|^5 + \cdots \] (3.5)

If we take moduli of \( A \) and \( B \) in (2.1), its domain of convergence is equivalent to Fig. 2 except the case of \( a = -1 \). More explicit proof is explained [2].

Therefore, we conclude that we can not use Poincaré–Perron theorem to obtain the radius of convergence for a power series solution. And a series solution for an infinite series, obtained by applying Poincaré–Perron theorem, is not absolutely convergent but only conditionally one.

4. Uniqueness theorem and Poincaré–Perron theorem

According to Fig. 3, we have

\[ D_a = \text{Domain of absolute convergence} \subseteq D_c = \text{Domain of conditional convergence} \] (4.1)
As we all know, (4.1) is not only available for the 3-term recurrence relation of linear ODEs but also multi-term cases.

Uniqueness theorem for Poisson’s equation tells us that its equation has an unique solution for a given boundary conditions: If $\Omega$ is a boundary domain in $\mathbb{R}^n$, a function $u : \Omega \to \mathbb{R}$ such that $u \in C^2(\Omega) \cap C(\overline{\Omega})$, and $\nabla^2 u = \rho$ in $\Omega$ and either $u = f$ or $\partial u / \partial n = g$ on $\partial \Omega$, where $\rho$, $f$ and $g$ are given functions, then $u$ is unique (at least to within an additive constant) [16]. Also, its theorem is available in the Klein-Gordon equation; the solution for the scalar field $\Phi(\mathbf{r})$ in $\Omega$ bounded by $\partial \Omega$ is unique if either Dirichlet or Neumann boundary conditions are specified on $\partial \Omega$.

In general, local Heun functions require the uniqueness theorem in mathematical physics: For instance, the Heun equation is derived from the Klein-Gordon equation for $D = 4$ Kerr-de Sitter metric. [3, 23, 25]

Suppose that a series solution $y^x(x)$ of the Heun function at $x = 0$ is unique and its domain of convergence $D_c$ is obtained by the Poincaré–Perron theory. Then, a series $y^x(x)$ should be equivalent or proportional to a solution $y^p(x)$ with a constant value. However, at the region $D_c - D_a$ on Fig.3 (in the bright shaded area where $a > 0$), a series $y^x(x)$ is divergent; it means that $y^x(x)$ and $y^p(x)$ are independent to each other. Therefore, the uniqueness theorem is broken and is not available any more by applying the Poincaré–Perron theorem.

Now, let assume that $y^x(x)$ is unique in a boundary domain $D_a$, then a function $y^p(x)$ is equivalent (or proportional) to a series $y^x(x)$ because of $D_a \subseteq D_c$. It means the uniqueness theorem is available in this situation. So, the domain of a local Heun function must to be absolutely convergent. And absolutely convergent series behave “nicely” because all rearrangements of the series are convergent to the same sum. We give you more explanation in details.

Poisson’s equation (Gauss’s law in differential form) for an electrostatic problem is known as

$$\nabla \cdot (\epsilon_0 \nabla \phi) = -\rho_f$$

where $\phi$ is the electric potential and $E = -\nabla \phi$ is the electric field.

The uniqueness theorem of the electric field is proven for a large class of boundary conditions (outer boundary could be at infinity) in the following way.

Assume that there are two solutions $\phi_1$ and $\phi_2$, and define the difference of the two solutions as $\phi = \phi_2 - \phi_1$. We know that both $\phi_1$ and $\phi_2$ satisfy Poisson’s Equation, $\phi$ should obey Laplace’s equation such as

$$\nabla \cdot (\epsilon_0 \nabla \phi) = 0$$

Applying the following identity

$$\nabla \cdot (\phi \epsilon_0 \nabla \phi) = \epsilon_0 (\nabla \phi)^2 + \phi \nabla \cdot (\epsilon_0 \nabla \phi) = \epsilon_0 (\nabla \phi)^2$$

Taking the volume integral over all space specified by the boundary conditions gives

$$\int_V \nabla \cdot (\phi \epsilon_0 \nabla \phi) d^3 \mathbf{r} = \int_V \epsilon_0 (\nabla \phi)^2 d^3 \mathbf{r}$$

Applying the divergence theorem, (4.3) is rewritten as

$$\sum \int_{S_i} (\phi \epsilon_0 \nabla \phi) \cdot d\mathbf{S} = \sum \int_{S_i} \epsilon_0 (\nabla \phi)^2 d^3 \mathbf{r}$$

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Here $S_i$ cover all boundaries of the region.

We know $\varepsilon_0 > 0$ and $(\nabla \phi)^2 \geq 0$, then $\nabla \phi$ is also zero on everywhere (it means that $\nabla \varphi_1 = \nabla \varphi_2$) when the surface integral vanishes.

It means that the gradient of the solution is unique if only if

$$\sum_i \int_{S_i} (\phi \varepsilon_0 \nabla \phi) \cdot dS = 0 \quad (4.7)$$

As we see the above proof (see (4.2)–(4.7)), the uniqueness theorem is proved by applying the concept of an integral basically; the divergence theorem is also developed by utilizing an integral.

A surface (or volume) integral originally implies the concept of infinite sum.

Figure 4: Partial sum

Figure 5: Integral

Area $= \int_0^3 y(x)dx = \lim_{\Delta x \to 0} \sum_{n=1}^N y_n(x)\Delta x$

Figs. 4 and 5 shows us how we define an integral typically. On the left hand sided graph, there are finite partial sums. But there are infinite sum as $\Delta x \to 0$. Then, we must find the same area value no matter how we arrange each term of $y_n(x)\Delta x$’s. To put it more simply, an integral has an invariant. The invariant is the essence of nature; it is an indispensable condition, not an option.

No matter how a function $\phi$ is arranged or added in (4.2)–(4.7), we have to obtain the same value of an integral. Although not mentioned, an invariant is already included in the uniqueness theorem. Therefore, the P-P theorem cannot be applied into differential equation at least.

Some scholars obtain a solution in the case of $\sum_{n=-\infty}^{\infty} a_n$ rather than $\sum_{n=0}^{\infty}$. Then, Laurent series is applied. In this case, the P-P theorem also cannot be utilized because it violates the uniqueness theorem.

Assume that $f(z)$ is a holomorphic function on the annulus $r < |z - z_0| < R$ which has two Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n \quad (4.8)$$

where the $a_n$ is defined by a line integral which is a generalization of Cauchy’s integral formula:

$$a_n = \frac{1}{2\pi i} \oint_{c} \frac{f(z)dz}{(z - c)^{n+1}}.$$ 

Multiply both sides with $(z - c)^{-m-1}$ in (4.8) where $m \in \mathbb{Z}$, and integrate on a
We know the series converges uniformly on $r < |z - z_0| < R$, so the integration and summation can be interchanged. Substituting the identity
\[ \oint_C (z - z_0)^{r-m-1} \, dz = \frac{2\pi i}{n!} \delta_{nm} \] (4.10)
into the summation, we conclude that $a_m = b_m$. Therefore the Laurent series is unique. As we observe Cauchy’s integral formula and (4.9), the concept of an integral is applied and we cannot use the P-P theorem by a similar reason.

Anyway, ignoring the above examples we have presented, using the P–P theorem causes another problem. When a function is given, we often try to find an integral of it. It is essential to assume that the following summation symbol and integral symbol can then be exchanged:
\[ \int_a^b \sum_n \rightarrow \sum_n \int_a^b \] (4.11)
For (4.11) to be true, $f = \sum_n f_n$ must to be absolute convergent rather than the P–P type conditional convergent. Otherwise, we are very likely to make a mistake in obtaining a certain function value.

In general, we are interested in converting one function into another. Then two different local functions should be absolute convergent. For instance, Robert S. Maier obtained 192 local solutions of the Heun equation in 2004. [19] They are analogous to Kummer’s 24 solutions of the Gauss hypergeometric equation. As we see (5.5)–(5.11) on his paper, he obtain several transformation of Heun functions. In (5.8), he say
\[ H_l(a, q; \alpha, \beta, \gamma, \delta; x) = \left(1 - \frac{x}{a}\right)^{-\alpha} H_l\left(\frac{1}{1-a}, \frac{q - y\alpha}{1-a}; -\beta + \gamma + \delta, \alpha, \gamma, \alpha - \beta + 1; \frac{x}{x-a}\right) \] (4.12)
And we require that $H_l(a, q; \alpha, \beta, \gamma, \delta; x)$ and $H_l\left(\frac{1}{1-a}, \frac{q - y\alpha}{1-a}; -\beta + \gamma + \delta, \alpha, \gamma, \alpha - \beta + 1; \frac{x}{x-a}\right)$ should be absolute convergent. Of course, we only take the radius of convergence of (4.12) as overlap of domains of $H_l(a, q; \alpha, \beta, \gamma, \delta; x)$ and $H_l\left(\frac{1}{1-a}, \frac{q - y\alpha}{1-a}; -\beta + \gamma + \delta, \alpha, \gamma, \alpha - \beta + 1; \frac{x}{x-a}\right)$.

Because of these reasons, we claim that we cannot use the P-P theorem into differential equations at least. This theorem violates the uniqueness theorem and an invariant. Actually, an invariant is not mentioned in uniqueness theorem directly. But it is already included as we observe the concept of an integral.

We always use arithmetic operations with discrete quantities to solve linear differential equations naturally when we compute any mathematical or physical quantity. Here, “discrete quantity” involves the meaning of “we are able to measure, observe and see” in our words. Then, we have numerous solutions about infinite series. Fortunately, however, there is only one correct solution due to the uniqueness theorem. Even if arithmetic operations are applied, there is only one invariant, which is to force the function values to have the same value even though the order of terms in a power series is rearranged. Because of uniqueness theorem, we use the power series.
In fact, in the case of linear differential equations, the radius of convergence is constructed by applying a power series.

Because of the above reasons, we obtain the following conclusion: For the d-term recurrence relation of a homogeneous linear ODE where \( d \geq 2 \), if a power series solution is conditionally convergent, then the uniqueness theorem is not available any more. Its theorem is only valid as a series is absolutely convergent.

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