BALANCED MODULAR PARAMETERIZATIONS

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ABSTRACT. For prime levels $5 \leq p \leq 19$, sets of $\Gamma_0(p)$-permuted theta quotients are constructed that generate the graded rings of modular forms of positive integer weight for $\Gamma_1(p)$. An explicit formulation of the permutation representation and several applications are given, including a new representation for the number of $t$-core partitions. The $\Gamma_0(p)$-action induces coefficient symmetries within representations for modular forms and invariance subgroups for coupled systems of differential equations. The symmetry for levels $p = 5, 7, 11$ is linked to the Kleinian automorphism groups.

1. Introduction

The graded ring of modular forms for a finite index subgroup of the full modular group is isomorphic to a polynomial ring in two or more generators modulo a finite set of relations \([11], [38]\) p. 249. Certain classical polynomial representations for modular forms exhibit coefficient symmetry. For example, the Klein polynomials whose roots encode distinguished points of the stereographically projected circumsphere for a regular icosahedron, are symmetric in absolute value about the middle coefficients

\[
K_e(\Lambda) = (1 - 11\Lambda - \Lambda^2)^5, \quad K_f(\Lambda) = 1 + 228\Lambda + 494\Lambda^2 - 228\Lambda^3 + \Lambda^4, \quad (1.1)
\]

\[
K_f(\Lambda) = 1 - 522\Lambda - 10005\Lambda^2 - 10005\Lambda^4 + 522\Lambda^5 + \Lambda^6. \quad (1.2)
\]

The polynomials $K_e(\Lambda)$ and $K_f(\Lambda)$, encoding the edge and face points, correspond to representations for Eisenstein series in terms of two modular parameters of level five

\[
B^{20}K_e(\Lambda) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3q^n}{1 - q^n}, \quad B^{30}K_f(\Lambda) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5q^n}{1 - q^n}; \quad |q| < 1, \quad (1.3)
\]

\[
A^5(q) = q \frac{(q; q^2)^2}{(q^2; q^3; q^5)^5}, \quad B^5(q) = \left(\frac{q; q^2}{q^2; q^4; q^5}\right)_\infty, \quad \Lambda = A^5/B^5, \quad (1.4)
\]

where $(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$ and $(a_1, a_2, \ldots, a_r; q)_n = \prod_{j=1}^{r} (a_j; q)_n$ for $n \in \mathbb{N} \cup \infty$. These are special cases of more general symmetric parameterizations in $A^5, B^5$ from \([8]\). At level seven, certain modular forms are symmetric functions of the parameters

\[
x = q \frac{(q^2, q^5, q^7; q^7)_\infty}{(q, q^4, q^7; q^7)_\infty^2}, \quad y = -q \frac{(q, q^6, q^7, q^7; q^7)_\infty}{(q^2, q^5; q^7)_\infty^2}, \quad z = \frac{(q^3, q^4, q^7, q^7; q^7)_\infty}{(q, q^6, q^7; q^7)_\infty^2}. \quad (1.5)
\]

For example, the Hecke Eisenstein series twisted, respectively by the Jacobi symbol and trivial character $\chi_{1,7}$ modulo seven have the symmetric representations \([10], [21]\)

\[
x + y + z = 1 + 2 \sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{q^n}{1 - q^n}, \quad x^2 + y^2 + z^2 = 1 + 4 \sum_{n=1}^{\infty} \frac{\chi_{1,7}(n)nq^n}{1 - q^n}. \quad (1.6)
\]

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and are connected to one other by the curious identity \[ (x + y + z)^2 = x^2 + y^2 + z^2. \] (1.7)

This identity results from the Klein quartic equation, with symmetric form \[ xy + xz + yz = 0. \] (1.8)

Many formulations for modular forms of prime level, not necessarily in symmetric form, appear in the work of Klein and Ramanujan. Because the symmetry may appear incidental, no unified study of symmetric modular parameterizations or extensions to other settings have been undertaken. In the present work, such symmetric constructions are shown to be special cases of more general balanced parameterizations for modular forms of prime level \( p \) with \( 5 \leq p \leq 19 \). The coefficient symmetry is a hallmark of a special collection of theta quotients generating the graded ring of modular forms on

\[
\Gamma_1(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z}) \mid c \equiv 0, \ a \equiv d \equiv 1 \pmod{p} \right\}.
\]

(1.9)

In each case, the polynomial generators are permuted up to a change of sign by

\[
\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z}) \mid c \equiv 0 \pmod{p} \right\}.
\]

(1.10)

We show that balanced polynomial representations for modular forms on \( \Gamma_0(p) \) result from a nontrivial permutative action on generators for the graded rings of modular forms for \( \Gamma_1(p) \) induced by modular transformation formulas. In addition to the list of permuted generators of level five and seven already presented, a corresponding set of \( \Gamma_0(11) \)-permuted generators for the graded ring of forms on \( \Gamma_1(11) \) will be given by

\[
\frac{(q^4, q^7, q^{11}; q^{11})_\infty}{(q, q^{10}, q^2, q^9; q^{11})_\infty}, \frac{(q^5, q^6, q^{11}, q^{11}; q^{11})_\infty}{(q^3, q^8, q^4, q^7; q^{11})_\infty}, \frac{(q^2, q^{10}, q^{11}, q^{11}; q^{11})_\infty}{(q^6, q^6, q^{10}, q^{11})_\infty}, \frac{(q^3, q^9, q^{11}, q^{11}; q^{11})_\infty}{(q^9, q^{11}, q^{11}, q^{11})_\infty}.
\]

(1.11)

A similarly permuted set of generators for the graded ring of forms for \( \Gamma_1(13) \) is

\[
\frac{(q^6, q^7, q^{13}, q^{13}; q^{13})_\infty}{(q, q^{12}, q^2, q^{10}, q^{13})_\infty}, \frac{(q^5, q^8, q^{13}, q^{13}; q^{13})_\infty}{(q^3, q^{10}, q^4, q^9; q^{13})_\infty}, \frac{(q^2, q^{12}, q^4, q^9; q^{13})_\infty}{(q^6, q^6, q^7, q^{13})_\infty}.
\]

(1.13)

A set of \( \Gamma_0(17) \)-permuted generators for the graded ring of forms on \( \Gamma_1(17) \) is

\[
\frac{(q^8, q^9, q^{17}, q^{17}; q^{17})_\infty}{(q^2, q^{15}, q^3, q^{14}, q^{17})_\infty}, \frac{(q^5, q^{12}, q^{17}, q^{17}; q^{17})_\infty}{(q^3, q^{14}, q^4, q^{13}, q^{17})_\infty}, \frac{(q^3, q^{16}, q^{17}, q^{17}; q^{17})_\infty}{(q^4, q^{13}, q^6, q^{11}, q^{17})_\infty}, \frac{(q^4, q^{13}, q^{17}; q^{17})_\infty}{(q^2, q^{15}, q^3, q^{12}, q^{17})_\infty}.
\]

(1.15)

(1.16)
Finally, a set of permuted generators for the graded ring of forms for $\Gamma_1(19)$ is
\[
\begin{align*}
(q^8, q, q^9, q^{10}, q^{19}, q^{19}; q^{19})_\infty, & \quad (q^2, q^{17}, q^7, q^{12}, q^7, q^{12}; q^{19})_\infty, \\
(q^3, q^{16}, q^4, q^{15}, q^5, q^{14}; q^{19})_\infty, & \quad (q^2, q^{18}, q^4, q^{15}, q^6, q^{13}; q^{19})_\infty, \\
(q, q^{18}, q^4, q^{15}, q^{14}, q^{13}; q^{19})_\infty, & \quad (q^4, q^{15}, q^7, q^{12}, q^{19}, q^{19}; q^{19})_\infty,
\end{align*}
\]
\[
\begin{align*}
q^5 (q, q^{18}, q^5, q^{19}, q^{16}, q^{19}; q^{19})_\infty, & \quad (q^2, q^{17}, q^5, q^{14}, q^6, q^{13}; q^{19})_\infty, \\
(q^6, q^{13}, q^8, q^{11}; q^{10}, q^{19})_\infty, & \quad (q^2, q^{15}, q^7, q^{12}, q^8, q^{11}; q^{19})_\infty, \\
q^2 (q, q^{18}, q^6, q^{13}, q^{19}, q^{19}; q^{19})_\infty, & \quad (q^4, q^{15}, q^{14}, q^{11}, q^{19}; q^{19})_\infty, \\
q^2 (q^2, q^{17}, q^3, q^{16}, q^9, q^{10}; q^{19})_\infty, & \quad (q^5, q^{14}, q^8, q^{11}, q^{19}; q^{19})_\infty,
\end{align*}
\]
\[
q^2 (q^2, q^{17}, q^6, q^{13}, q^{19}; q^{19})_\infty.
\]

The distinguishing feature of each claimed set of polynomial generators appearing here is that any modular form of positive integer weight on a subgroup containing $\Gamma_0(p)$ enjoys a polynomial representation exhibiting a certain coefficient symmetry.

**Theorem 1.1.** For each prime $p$ with $5 \leq p \leq 19$, let $\left\{x_k\right\}_{k=1}^{p-1}$ denote the given generators for the graded ring of modular forms for $\Gamma_1(p)$. Any modular form of weight $k$ for a subgroup of $PSL(2, \mathbb{Z})$ containing $\Gamma_0(p)$ is representable in the form
\[
\sum_{k_1 + \cdots + k_{(p-1)/2} = k} a_{k_1, \ldots, k_{(p-1)/2}} \sum_{\sigma \in S_{(p-1)/2}} \epsilon_{\sigma(k_1), \ldots, \sigma(k_{(p-1)/2})} \frac{x_1^{\sigma(k_1)} \cdots x_{(p-1)/2}^{\sigma(k_{(p-1)/2})}}{2},
\]
over integers $k_i \geq 0$, $a_{k_1, \ldots, k_{(p-1)/2}} \in \mathbb{C}$, $S_n$ the symmetric group on $n$ elements, and
\[
\epsilon_{\sigma(k_1), \ldots, \sigma(k_{(p-1)/2})} \in \{\pm 1\}.
\]

In particular, the coefficient of each monomial in (1.23) agrees in absolute value with the coefficient of any other monomial obtained through a permutation of the exponents.

The coefficient symmetry is induced by transformation formulas satisfied by the generators. To describe the symmetry exhibited by forms of level $p$ in terms of the generators, let $\gamma \in PSL(2, \mathbb{Z})$ act on the upper half plane by Möbius transformation, and define the diamond operator on a modular form $f$ of weight $k$ on $\Gamma_1(p)$ by
\[
\langle \gamma \rangle (f) = (\gamma_{21} \tau + \gamma_{22})^{-k} f(\gamma \tau), \quad \gamma = (\gamma_{11}, \gamma_{12}; \gamma_{21}, \gamma_{22}) \in \Gamma_0(p).
\]

**Theorem 1.2.** The generators for $\Gamma_1(p)$ from (1.24) are permuted up to change of sign by $\Gamma_0(p)$ under action by $\langle \cdot \rangle$, with permutation representation $(\mathbb{Z}/p\mathbb{Z})^\ast/\{\pm 1\}$.

A common thread linking the symmetric constructions is the Hecke Eisenstein series for $\Gamma_0(p)$ twisted by Dirichlet character $\chi$ modulo $p$, defined for weight $k \in \mathbb{N}$ by
\[
E_{k, \chi}(\tau) = 1 + \frac{2}{L(1-k, \chi)} \sum_{n=1}^{\infty} \chi(n) \frac{n^{k-1} q^n}{1-q^n}, \quad q = e^{2\pi i \tau},
\]
where $L(1-k, \chi)$ is the analytic continuation of the associated Dirichlet $L$-series and $\chi(-1) = (-1)^k$. We prove the above claims in the form of Theorems 3.1, 3.3 by systematically expressing generating theta quotients in terms of Eisenstein series.
In Section 4 notable representations are highlighted for modular forms in terms of the generating parameters. In particular, the product of elements in each generating set is closely related to the generating functions for $p$-cores. A new convolution representation for $p$-cores is given in Corollary 4.2 for \(5 \leq p \leq 19\). We also derive coupled systems of differential equations for the generators of level \(5 \leq p \leq 19\). Each system is invariant under action by \(\Gamma_0(p)\). The differential systems of level five and seven appeared recently in \([20, 21]\). Their coefficient invariance is a common feature of the coupled differential systems derived for each set of generators of level \(5 \leq p \leq 19\).

Theorem 1.3. Let \(E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{na^n}{1-q^n}\) and \(\mathcal{P} = E_2(q^5)\). Then

\[
q \frac{d}{dq} A = \frac{1}{60} A \left(-5A^{10} - 66A^5 B^5 + 7B^{10} + 5\mathcal{P}\right),
\]

(1.26)

\[
q \frac{d}{dq} B = \frac{1}{60} B \left(-5B^{10} + 66A^5 B^5 + 7A^{10} + 5\mathcal{P}\right),
\]

(1.27)

\[
q \frac{d}{dq} \mathcal{P} = \frac{5}{12} \left(\mathcal{P}^2 - B^{20} + 12B^{15} A^5 - 14B^{10} A^{10} - 12B^5 A^{15} - A^{20}\right),
\]

(1.28)

Theorem 1.4. Let \(\mathcal{P}(q) = E_2(q^7)\). Then

\[
q \frac{d}{dq} x = \frac{x}{12} \left(5y^2 + 5z^2 - 7x^2 + 20yz + 52xy + 7\mathcal{P}\right),
\]

(1.29)

\[
q \frac{d}{dq} y = \frac{y}{12} \left(5z^2 + 5x^2 - 7y^2 + 20xz + 52yz + 7\mathcal{P}\right),
\]

(1.30)

\[
q \frac{d}{dq} z = \frac{z}{12} \left(5x^2 + 5y^2 - 7z^2 + 20xy + 52xz + 7\mathcal{P}\right),
\]

(1.31)

\[
q \frac{d}{dq} \mathcal{P}(q) = \frac{7}{12} \left(\mathcal{P}^2 - x^4 - 4x^3 y - 12x^3 z - 4y^3 z - 4x^3 y^3 - 12y^3 z^3 - z^4\right).
\]

Theorem 4.5 encodes in concise form the differential equations from Theorems 1.3 and 1.4 and provides coupled systems for the generators corresponding to higher prime levels. These systems are analogous to those for modular parameters of lower levels \([17, 18, 19, 30]\) and to Ramanujan’s differential system for Eisenstein series \([32]\)

\[
q \frac{d}{dq} E_2 = \frac{E_2^2 - E_4}{12}, \quad q \frac{d}{dq} E_4 = \frac{E_2 E_4 - E_6}{3}, \quad q \frac{d}{dq} E_6 = \frac{E_2 E_6 - E_4^2}{2},
\]

(1.32)

where the normalized Eisenstein series \(E_k = E_k(q)\) for \(\text{PSL}(2, \mathbb{Z})\) are defined by

\[
E_{2k}(q) = 1 + \frac{2}{\zeta(1 - 2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1}q^n}{1-q^n},
\]

(1.33)

and where \(\zeta\) is the analytic continuation of the Riemann \(\zeta\)-function. In fact, the last equations of Theorem 1.3 and 1.4 are equivalent to the first equation of (1.32). In view of Theorem 3.2, the coupled systems corresponding to each level may be viewed as systems of equations for certain linear combinations of twisted Eisenstein series.

The final part of the paper, Section 5, connects the permuted generators to results in classical representation theory. In particular, we derive parameterizations for the Klein quartic and analogous identities for the generating parameters at higher levels.
2. Elliptic modular preliminaries

Before embarking on proofs of the claims in the last section, we introduce some fundamental notions from the theory of elliptic modular forms. Let \( \mathcal{M}_k(\Gamma) \) denote the vector space of weight \( k \) modular forms for \( \Gamma \subseteq PSL(2, \mathbb{Z}) \). For a given prime \( p \), the \((p-1)/2\) linearly independent Eisenstein series of weight one and primitive character \( \chi \) are known to generate a subspace, called the Eisenstein subspace, of modular forms of weight one for \( \Gamma \subseteq PSL(2, \mathbb{Z}) \). For each prime \( p \) with \( 5 \leq p \leq 19 \), \[ \dim(\mathcal{M}_1(\Gamma_1(p))) = \frac{p-1}{2}. \] (2.1)

Hence, the set of weight one Eisenstein series of odd primitive character modulo \( p \) form a basis for \( \mathcal{M}_1(\Gamma_1(p)) \) over \( \mathbb{C} \). The claimed symmetric generators of this paper originate from theta function expansions for certain linear combinations of Eisenstein series of weight one. To formulate the change of bases from Eisenstein series to the permuted bases of products from (1.4)–(1.22), we first derive, in Theorem 3.1, representations for sums of Eisenstein series in terms of the Dedekind eta function, \( \eta(\tau) = q^{1/24}(q; q)_\infty \), a weight 1/2 modular form for \( SL(2, \mathbb{Z}) \) with multiplier given explicitly by [28, p. 51]. The relevant Eisenstein series representations will also involve the Jacobi theta function

\[
\theta_1(z \mid q) = -iq^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} e^{(2n+1)iz},
\] (2.2)

an odd function of \( z \) with a simple zero at the origin such that [36, p. 489]

\[
\frac{\theta_1'}{\theta_1}(z \mid q) = \cot z + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} \sin 2nz
\]

\[
= i - 2i \sum_{n=1}^{\infty} \frac{q^n e^{2iz}}{1 - q^n e^{2iz}} + 2i \sum_{n=0}^{\infty} \frac{q^n e^{-2iz}}{1 - q^n e^{-2iz}}.
\] (2.3)

Our subsequent calculations require the following easily verified functional equations

\[
\theta_1(z + n\pi) = (-1)^n \theta_1(z \mid q), \quad \theta_1(z + n\pi \tau \mid q) = (-1)^n q^{-n^2/2} e^{-2inz} \theta_1(z \mid q). \] (2.5)

Product representations result from the Jacobi Triple Product expansion given by [36]

\[
\theta_1(z \mid q) = -iq^{1/8} e^{iz} (q; q)_\infty (q e^{2iz}; q)_\infty (e^{-2iz}; q)_\infty.
\] (2.6)

In particular, we will make use of the special case

\[
\theta_1(r\pi \tau \mid q^p) = iq^{-r} q^{p/8} q^{r/2} (q^r, q^{p-r}, q^p, q^p)_\infty.
\] (2.7)

By differentiating (2.6) at the origin, we obtain

\[
\theta_1'(q) := \lim_{z \to 0} \frac{\theta_1(z \mid q)}{z} = 2q^{1/8} (q; q)_\infty^3.
\] (2.8)

To extend the bases of weight one forms on \( \Gamma_1(p) \) to homogeneous representations for any positive integer weight, it suffices to generate modular forms up to weight three. For prime levels \( N \geq 5 \), this is proven in [4], and for general \( N \) in [34]. The situation is better for principal congruence subgroups of level \( N \), where weight 1 suffices [22].
Lemma 2.1. Denote by $\mathcal{M}_k(\Gamma)$ the $\mathbb{C}$-vector space of weight $k$ modular forms for the congruence subgroup $\Gamma$, and let $\mathcal{M}(\Gamma) = \bigoplus_{k=1}^{\infty} \mathcal{M}_k(\Gamma)$ be the corresponding graded ring.

1. For $N \geq 5$, any algebra containing $\mathcal{M}_k(\Gamma_1(N))$, for $k \leq 3$, contains $\mathcal{M}(\Gamma_1(N))$.

2. For $N \geq 3$, any algebra containing $\mathcal{M}_1(\Gamma(N))$ contains $\mathcal{M}(\Gamma(N))$.

To show that $\Gamma_0(p)$ acts as indicated on the generating quotients, it will be convenient to apply transformation formulas for special values of the Jacobi theta function in the form of those for theta constants of odd order $k$ and index $\ell$, defined by [14]

$$\varphi_{k,\ell}(\tau) = \theta[\chi_{\ell,k}](0, k\tau), \quad \chi_{\ell,k} = \left[ \frac{2\ell-1}{k} \right], \quad 1 \leq \ell \leq \frac{k-1}{2}, \quad (2.9)$$

in turn, constructed from theta constants of characteristic $[\epsilon, \epsilon'] \in \mathbb{R}^2$

$$\theta \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (z, \tau) = \sum_{n \in \mathbb{Z}} \exp 2\pi i \left\{ \frac{1}{2} \left( n + \frac{\epsilon}{2} \right)^2 \tau + \left( n + \frac{\epsilon}{2} \right) \left( z + \frac{\epsilon'}{2} \right) \right\}. \quad (2.10)$$

Theorem 2.2. [14] pp. 215-219] For odd positive integers $k \geq 3$, let

$$\mathcal{V}_k(\tau) = \left[ \theta \left[ \begin{array}{c} (k-2)/k \\ 1 \end{array} \right] (k\tau), \theta \left[ \begin{array}{c} (k-4)/k \\ 1 \end{array} \right] (k\tau), \ldots, \theta \left[ \begin{array}{c} 1/k \\ 1 \end{array} \right] (k\tau) \right]^T. \quad (2.11)$$

Then $\mathcal{V}_k$ is a vector-valued form of weight $1/2$ on $\text{PSL}(2, \mathbb{Z})$ inducing a representation

$$\pi_k : \text{PSL}(2, \mathbb{Z}) \to \text{PGL}(\text{dim}(k-1)/2, \mathbb{C}) \quad \text{via} \quad \mathcal{V}_k(\gamma \tau) = (\gamma_{21} \tau + \gamma_{22})^{1/2} \pi_k(\gamma) \mathcal{V}_k(\tau)$$

determined by the images of generators for $\text{SL}(2, \mathbb{Z})$, $S = (0, -1; 1, 0)$, $T = (1, 1; 0, 1)$,

$$\mathcal{V}_N(T\tau) = \mathcal{V}_N(\tau + 1) = \pi_N(T)\mathcal{V}_N(\tau), \quad \mathcal{V}_N(S\tau) = \mathcal{V}_N(-1/\tau) = \tau^{1/2} \pi_N(S)\mathcal{V}_N(\tau),$$

where the matrices $\pi_N(S)$ and $\pi_N(T)$ have $(\ell, j)$th entry, for $1 \leq \ell, j \leq (N-1)/2$,

$$\{\pi_N(T)\}_{(\ell, j)} = \begin{cases} \exp \left( \frac{(j-N\ell)(j-N\ell)}{4N} \right), & \ell = j, \\ 0, & \text{else}, \end{cases} \quad (2.12)$$

$$\{\pi_N(S)\}_{(\ell, j)} = \frac{\left( 1 + \frac{\ell \cdot (N-2\ell)}{N} \frac{\pi i}{2N} \right) \exp \left( \frac{(j-N\ell)(j-N\ell)}{4N} \pi i \right)}{\sqrt{2N}}. \quad (2.13)$$

Equivalent transformation formulas figure prominently in Klein’s representation of the automorphism group of the icosahedron [13, 25, 26]; in the septic extension to the Klein quartic [27]; as well as in Klein’s level 11 analysis of the symmetries of the Klein cubic [23]. Our formulation of the quartic and higher level extensions are equivalent to quadratic relations derivable from a classical theta function identity [36, p. 518]

$$\left( \frac{\theta_1' \theta_1}{\theta_1} \right) (x \mid q) - \left( \frac{\theta_1' \theta_1}{\theta_1} \right) (y \mid q) = \frac{\theta_1' (0 \mid q)^2 \theta_1 (x-y \mid q) \theta_1 (x+y \mid q)}{\theta_1^2 (x \mid q) \theta_1^2 (y \mid q)}. \quad (2.14)$$
3. Eisenstein expansions for permuted bases

The goal of this section is to formulate and prove claims made in the Introduction for generators of the graded rings of modular forms on \( \Gamma_1(p) \). We construct the generators and prove the claimed permutative action of \( \Gamma_0(p) \). In Theorem 3.1, product expansions are derived for the normalized sums of weight one Eisenstein series twisted by the odd primitive Dirichlet characters modulo \( p \). By considering the by \( \Gamma_0(p) \)-orbit of these series under modular transformation, we derive in Theorems 3.2 and 3.3, bases for the weight one forms on \( \Gamma_1(p) \) and explicitly characterize the permutative action by \( \Gamma_0(p) \). Theorem 3.4 demonstrates that the Eisenstein bases from Theorem 3.2 are expressible as quotients of theta functions. Finally, Theorem 3.5 proves the corresponding theta quotients generate the graded ring of modular forms for \( \Gamma_1(p) \) of positive integer weight.

**Theorem 3.1.** Define \( E_{\chi,k}(\tau) \) as in (1.25). For each prime \( 5 \leq p \leq 19 \), let

\[
E_p(\tau) = \frac{2}{p-1} \sum_{\chi(-1)=-1} E_{\chi,1}(\tau),
\]

where the sum is over the odd primitive Dirichlet characters modulo \( p \). Then

\[
E_5(\tau) = \frac{(q, q^2)}{(q, q^4; q^5)^5}, \quad E_7(\tau) = \frac{(q^3, q^4, q^7; q^7)}{(q, q^6; q^7)^2},
\]

\[
E_{11}(\tau) = \frac{(q^4, q^7, q^{11}; q^{11})}{(q, q^{10}; q^{11})}, \quad E_{13}(\tau) = \frac{(q^6, q^7, q^{13}; q^{13})}{(q, q^{12}; q^{13})},
\]

\[
E_{17}(\tau) = \frac{(q^8, q^9, q^{17}; q^{17})}{(q, q^{15}; q^{17})}, \quad E_{19}(\tau) = \frac{(q^8, q^{11}, q^{19}; q^{19})}{(q, q^{16}; q^{19})}.
\]

**Proof.** To prove each identity, we use the fact that the sum of the residues of an elliptic function on its period parallelogram is zero [36]. The challenge lies in writing down the relevant elliptic functions. We begin by proving the leftmost equation of (3.2). Let

\[
f_5(z) = \frac{e^{-2iz\theta^3_1(z - \pi \tau \mid q^5)}}{\theta_1^2(z \mid q^5)\theta_1(z + 2\pi \tau \mid q^5)}.
\]

Apply (2.5) to verify that \( f_5(z) \) is an elliptic function with periods \( \pi \) and \( 5\pi \tau \). From corresponding properties of the Jacobi theta function, observe that \( f_5(z) \) has a simple pole at \( z = -2\pi \tau \) and a double pole at \( z = 0 \). The residue of \( f_5(z) \) at \( z = -2\pi \tau \) is

\[
\lim_{z \to -2\pi \tau} \frac{(z + 2\pi \tau)}{\theta_1(z \mid q^5)} = \frac{q^2}{\theta_1^2(q^5)} \cdot \frac{\theta_1^3(-3\pi \tau \mid q^5)}{\theta_1^2(-2\pi \tau \mid q^5)}.
\]

The residue of \( f_5(z) \) at \( z = 0 \) is

\[
\lim_{z \to 0} (z^2 f_5(z))' = \lim_{z \to 0} \left( z^2 f'(z) \right) \left( 2 + \frac{f_5'(z)}{f_5(z)} \right)
\]

\[
= \left( \lim_{z \to 0} \frac{z^2}{\theta_1^2(z \mid q^5)} \right) \left( \lim_{z \to 0} \frac{e^{-2iz\theta^3_1(z - \pi \tau \mid q^5)}}{(z + 2\pi \tau \mid q^5)} \right) \left( \lim_{z \to 0} \frac{2}{z} + \frac{f_5'(z)}{f_5(z)} \right)
\]

\[
= -\frac{1}{\theta_1^2(q^5)} \left( \frac{\theta_1^3(\pi \tau \mid q^5)}{\theta_1(2\pi \tau \mid q^5)} \right) \left( \lim_{z \to 0} \frac{2}{z} + \frac{f_5'(z)}{f_5(z)} \right).
\]
Since the sum of the residues of \(f_5(z)\) is zero, we obtain from (2.7)
\[
-2i \left( \frac{q^2, q^3, q^5}{q, q^3, q^5} \right)^2 (q^5, q^5)^3 \frac{f(z)}{f_5(z)} = \lim_{z \to 0} \frac{2}{z} + \frac{f_5(z)}{f_5(z)},
\]
(3.10)
By applying identities (2.3)-(2.4), and the Laurent expansion for cot \(z\), we derive
\[
\lim_{z \to 0} \frac{2}{z} + \frac{f_5(z)}{f_5(z)} = \lim_{z \to 0} \left( \frac{2}{z} - 2 \frac{\theta_1}{\theta_1}(z \mid q^5) \right) - 2i - 3 \frac{\theta_1}{\theta_1}(\pi \tau \mid q^5) + \frac{\theta_1}{\theta_1}(2\pi \tau \mid q^5)
\]
(3.11)
\[
= -2i - 3 \frac{\theta_1}{\theta_1}(\pi \tau \mid q^5) - \frac{\theta_1}{\theta_1}(2\pi \tau \mid q^5) = -2i - \sum_{n=1}^{\infty} c_n q^n
\]
(3.12)
where, from (2.4), \(\{c_n\}_{n=1}^{\infty}\) is a periodic sequence modulo five such that
\[
c_1 = -6i, \quad c_2 = -2i, \quad c_3 = 2i, \quad c_4 = 6i, \quad c_5 = 0.
\]
(3.13)
If we denote the two odd primitive Dirichlet characters modulo five by \(\langle \chi_{2,5}(n) \rangle_{n=0}^{4} = \langle 0, 1, i, -i, -1 \rangle\), \(\langle \chi_{4,5}(n) \rangle_{n=0}^{4} = \langle 0, 1, -i, i, -1 \rangle\),
then, since, for \(\chi\) non-principal modulo \(p\), we may write [12, pp. 136-137],
\[
L(\chi, 0) = \sum_{n=0}^{p-1} \chi(n) \left( \frac{1}{2} - \frac{n}{p} \right),
\]
(3.15)
it follows from (3.14) and (3.15) that
\[
c_n = \frac{2i \chi_{2,5}(n)}{L(\chi_{2,5}, 0)} + \frac{2i \chi_{4,5}(n)}{L(\chi_{4,5}, 0)}
\]
(3.16)
Therefore, from (3.1) and identities (3.12), (3.13), and (1.25),
\[
\lim_{z \to 0} \frac{2}{z} + \frac{f_5(z)}{f_5(z)} = -2i \mathcal{E}_5(q).
\]
(3.17)
This completes the proof of the leftmost equation of (3.2). A formulation of rightmost equation of (3.2) from (3.18) is given in [29 Eq. (3.23)]. For prime levels \(7 \leq p \leq 19\), the claimed product expansions for the Eisenstein sums \(\mathcal{E}_p(\tau)\) may obtained by applying the residue theorem with the elliptic functions \(f_p(z)\) of period \(\pi, p\pi \tau\), defined by
\[
f_7(z) = e^{2iz} \frac{\theta_3^2(z + \pi \tau \mid q^7)}{\theta_3^2(z \mid q^7) \theta_3(z - 3 \pi \tau \mid q^7)},
\]
(3.18)
\[
f_{11}(z) = e^{-2iz} \frac{\theta_3(z - 2 \pi \tau \mid q^{11}) \theta_3(z - 3 \pi \tau \mid q^{11}) \theta_3(z - 5 \pi \tau \mid q^{11})}{\theta_3^2(z \mid q^{11}) \theta_3(z + \pi \tau \mid q^{11})},
\]
(3.19)
\[
f_{13}(z) = e^{-2iz} \frac{\theta_3(z - 3 \pi \tau \mid q^{13}) \theta_3(z - 4 \pi \tau \mid q^{13}) \theta_3(z - 5 \pi \tau \mid q^{13})}{\theta_3^2(z \mid q^{13}) \theta_3(z + \pi \tau \mid q^{13})},
\]
(3.20)
\[
f_{17}(z) = e^{-2iz} \frac{\theta_3(z - 3 \pi \tau \mid q^{17}) \theta_3(z - 5 \pi \tau \mid q^{17}) \theta_3(z - 7 \pi \tau \mid q^{17})}{\theta_3^2(z \mid q^{17}) \theta_3(z + 2 \pi \tau \mid q^{17})},
\]
(3.21)
\[
f_{19}(z) = e^{-2iz} \frac{\theta_3(z - 4 \pi \tau \mid q^{19}) \theta_3(z - 5 \pi \tau \mid q^{19}) \theta_3(z - 7 \pi \tau \mid q^{19})}{\theta_3^2(z \mid q^{19}) \theta_3(z + 3 \pi \tau \mid q^{19})},
\]
(3.22)
each constructed by writing \(\mathcal{E}_p(\tau)\) in terms of \((\theta_3/\theta_1)(k \pi \tau \mid q^p), 1 \leq k \leq (p-1)/2\). \(\square\)
We now construct alternative bases to the Eisenstein bases for $\mathcal{M}_1(\Gamma_1(p))$ by letting $\Gamma_0(p)$ act on the series $\mathcal{E}_p(\tau)$ and requiring that the first nonzero coefficient in the $q$-expansion of the image of $\mathcal{E}_p(\tau)$ under $\langle \cdot \rangle$ be 1. Since this action of $\langle \cdot \rangle$ depends only on the lower right entry of $\gamma \in \Gamma_0(p)$, we list only this element in subsequent results.

**Theorem 3.2.** Define $\langle \cdot \rangle$ by (1.21) and $\mathcal{E}_p$ by (1.31). For prime $5 \leq p \leq 19$, and a set of distinct elements $\{a_{k,p}\}_{k=1}^{(p-1)/2} \subset (\mathbb{Z}/p\mathbb{Z})^* / \{\pm 1\}$, there exists a basis decomposition

$$\mathcal{M}_1(\Gamma_1(p)) = \bigoplus_{k=1}^{(p-1)/2} \mathbb{C}\langle a_{k,p}\rangle(\mathcal{E}_p).$$

(3.23)

Moreover, if the constants $a_{k,p}$ are as follows, the basis elements $\langle a_{k,p}\rangle(\mathcal{E}_p)$ of (3.23) are normalized so that the first nonzero coefficient in their $q$-expansion is 1:

\[
\begin{align*}
(a_{1,5})^2_{k=1} &= (1, 2), \quad (a_{1,7})^3_{k=1} = (1, 2, 3), \quad (a_{1,11})^5_{k=1} = (1, 2, 3, 5, 7), \\
(a_{1,13})^6_{k=1} &= (1, 2, 3, 4, 5, 7), \quad (a_{1,17})^8_{k=1} = (1, 2, 3, 5, 7, 8, 11, 13), \\
(a_{1,19})^9_{k=1} &= (1, 2, 3, 4, 5, 7, 9, 11, 13).
\end{align*}
\]

**Proof.** The orthogonality of the Dirichlet characters modulo $p$ may be used to derive

$$\sum_{\chi(-1)=-1} \chi(a)\overline{\chi}(b) = \begin{cases} \pm \varphi(p)/2, & a \equiv \pm b \pmod{p}, \\ 0, & a \not\equiv \pm b \pmod{p}, \end{cases}$$

(3.24)

Therefore, if $\{a_{k,p}\}_{k=1}^{(p-1)/2} = (\mathbb{Z}/p\mathbb{Z})^* / \{\pm 1\}$ and $\{\chi_{2s}\}_{s=1}^{(p-1)/2}$ are odd, the rows of

$$(B)_{k,s} = \chi_{2s}(a_{k,p}), \quad 1 \leq k, s \leq (p-1)/2,$$

(3.25)

are orthogonal with respect to the Hermitian inner product. Hence, the matrix $B$ is an invertible linear transformation corresponding to the change of basis for $\mathcal{M}_1(\Gamma_1(p))$

$$B \left( E_{1,\chi_2}(\tau), \ldots, E_{1,\chi_{2p}}(\tau) \right)^T = \left( \langle a_{1,p}\rangle(\mathcal{E}_p), \ldots, \langle a_{(p-1)/2,p}\rangle(\mathcal{E}_p) \right)^T.$$  

(3.26)

The normalization claims of Theorem 3.2 may be verified from $q$-expansions for the linear combination of Eisenstein series defining each basis element in the image. □

**Theorem 3.3.** The elements from (3.23) are permuted up to a change of sign by $\Gamma_0(p)$ under $\langle \cdot \rangle$, with permutation representation isomorphic to $(\mathbb{Z}/p\mathbb{Z})^* / \{\pm 1\}$.

**Proof.** If the set of odd primitive Dirichlet characters modulo $p$ is given by $\{\chi_{2s}\}_{s=1}^{(p-1)/2}$, the elements of $\mathcal{D} = \{\langle d \rangle(\mathcal{E}_p) \mid d \in (\mathbb{Z}/p\mathbb{Z})^* \}$ are in bijective correspondence with

$$\mathcal{E} = \{ \mathcal{A}(d) := diag(\chi_2(d), \chi_4(d), \ldots, \chi_{p-1}(d)) \in GL(\frac{p-1}{2}, \mathbb{C}) \mid d \in (\mathbb{Z}/p\mathbb{Z})^* \}.$$  

Note that $\mathcal{E}$ is a group under multiplication corresponding to action by $\Gamma_0(p)$ on $\mathcal{D}$, and $\mathcal{A}(-d) = -\mathcal{A}(d)$. Therefore, modulo a change of sign, the permutation representation for the action of $\Gamma_0(p)$ on $D$ is isomorphic to the homomorphic image of $\Gamma_0(p)$ in $PGL(\frac{p-1}{2}, \mathbb{C})$ under $\kappa : \gamma \mapsto diag(\chi_2(\gamma_{22}), \chi_4(\gamma_{22}), \ldots, \chi_{p-1}(\gamma_{22}))$, were $\gamma_{22}$ is the lower right-hand entry of $\gamma$. Since the map $\delta : \gamma \mapsto \gamma_{22} \pmod{p}$ is a surjection from $\Gamma_0(p)$ to $(\mathbb{Z}/p\mathbb{Z})^*$ with kernel $\Gamma_1(p)$, we conclude $\Gamma_0(p)/\Gamma_1(p) \cong (\mathbb{Z}/p\mathbb{Z})^*$. Therefore, the projection of the image of $\kappa$ in $PGL((p-1)/2, \mathbb{C})$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^* / \{\pm 1\}$. □
The bases for $M_{10}$ are as follows:

$$d(E_p) = \pm a_{k,p} \in \{a_{k,p}\}_{k=1}^{(p-1)/2}. \quad (3.27)$$

We now show that the normalized Eisenstein sums from Theorem 3.3 are synonymous with the products (1.4)–(1.22) from the Introduction. To prove this, we show that each basis element of level $p$ for $M_1(\Gamma_1(p))$ from Theorem 3.2 is representable as a quotient of modified theta constants with Jacobi triple product representation [14, p. 141]

$$\theta \left[ \frac{m/n \cdot \pi m}{2n} \right] (n \tau) = \exp \left( \frac{\pi i m}{2n} \right) q^{m^2/(8n)} (q^{(n-m)/2})^q (q^{(n+m)/2})^q. \quad (3.28)$$

We derive each theta quotient by writing the product formulations of Theorem 3.2 in terms of modified theta constants and applying transformations for the theta constants.

**Theorem 3.4.** Define $\varphi_{\ell,k}$ by (2.9), and, for $[b_1, \ldots, b_{(p-1)/2}] \in \mathbb{Z}^{(p-1)/2}$, denote

$$\mathfrak{T}_p[b_1, \ldots, b_{(p-1)/2}] (\tau) = \eta^3(p \tau) \prod_{k=1}^{(p-1)/2} \exp \left( -\frac{\pi i b_k (2k-1)}{2p} \right) \varphi_{b_k}^{p,k}(\tau). \quad (3.29)$$

The bases for $M_1(\Gamma_1(p))$ from Theorem 3.2 have the theta representations:

| Level, $p$ | Basis for $M_1(\Gamma_1(p))$ |
|-----------|-----------------------------|
| 5         | (1)($E_5$) = $\mathfrak{T}_5[2,-3]$, (2)($E_5$) = $\mathfrak{T}_5[-3,2]$ |
| 7         | (1)($E_7$) = $\mathfrak{T}_7[1,0,-2]$, (2)($E_7$) = $\mathfrak{T}_7[-2,1,0]$, (3)($E_7$) = $\mathfrak{T}_7[0,-2,1]$ |
| 11        | (1)($E_{11}$) = $\mathfrak{T}_{11}[0,1,0,-1,-1]$, (2)($E_{11}$) = $\mathfrak{T}_{11}[-1,0,0,1,1]$ |
| 13        | (1)($E_{13}$) = $\mathfrak{T}_{13}[1,0,0,-1,0,-1]$, (2)($E_{13}$) = $\mathfrak{T}_{13}[-1,-1,0,1,0,0]$, (3)($E_{13}$) = $\mathfrak{T}_{13}[-1,0,-1,1,0,0]$, (4)($E_{13}$) = $\mathfrak{T}_{13}[1,0,1,-1,1,0]$, (5)($E_{13}$) = $\mathfrak{T}_{13}[0,1,-1,0,1,0]$, (7)($E_{13}$) = $\mathfrak{T}_{13}[-1,0,0,0,-1,1]$ |
| 17        | (1)($E_{17}$) = $\mathfrak{T}_{17}[1,0,0,0,0,0,-1,-1,0]$, (2)($E_{17}$) = $\mathfrak{T}_{17}[0,-1,0,0,0,0,-1,0]$, (3)($E_{17}$) = $\mathfrak{T}_{17}[0,0,0,0,0,0,-1,0]$ |
| 19        | (1)($E_{19}$) = $\mathfrak{T}_{19}[1,1,0,0,-1,-1,1,0,0]$, (2)($E_{19}$) = $\mathfrak{T}_{19}[0,-1,-1,0,1,0,-1,0]$, (3)($E_{19}$) = $\mathfrak{T}_{19}[1,-1,0,0,-1,0,1,0]$ |
|           | (4)($E_{19}$) = $\mathfrak{T}_{19}[0,0,1,-1,0,1,1,-1]$, (5)($E_{19}$) = $\mathfrak{T}_{19}[0,0,-1,1,0,0,-1,1]$ |
|           | (7)($E_{19}$) = $\mathfrak{T}_{19}[0,0,1,-1,1,0,-1,0]$ |
|           | (9)($E_{19}$) = $\mathfrak{T}_{19}[-1,-1,0,-1,0,0,1,0]$ |
|           | (11)($E_{19}$) = $\mathfrak{T}_{19}[-1,0,0,1,0,0,-1,1]$ |
|           | (13)($E_{19}$) = $\mathfrak{T}_{19}[-1,1,-1,0,1,-1,0,0,0]$ |
Proof. The theta quotient representations for \( E_p(\tau) = \langle 1 \rangle (E_p)(\tau) \) may be deduced from the product representations proved in Theorem 3.2. Transformation formula for these theta quotient representations proved in Theorem 3.2. For each prime \( p \), we may deduce the product representations for each normalized Eisenstein sum \( \langle a_k p \rangle (E_p) \) from the modular transformation formulas for these building blocks. We illustrate the general procedure with \( p = 5 \). From Theorem 3.1 and (3.28),

\[
\langle 1 \rangle (E_5) = \frac{(q; q)_{\infty}^2}{(q; q^4; q^5)_{\infty}^2} = \eta^3(5\tau) e^{-2\pi i/10} \varphi_{5,1}^2(\tau) = \frac{\eta^3(5\tau) e^{-2\pi i/10} \varphi_{5,1}^2(\tau)}{\varphi_{5,2}^3(\tau)} = \mathfrak{T}_5[2, -3](\tau). \tag{3.30}
\]

A set of generators for \( \Gamma_0(5) \) is given by

\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}, \quad \beta = \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}. \tag{3.31}
\]

We now employ transformation formulas up to a constant multiple for the weight 1/2 vector of modified theta constants \( \varphi_{5,2}; \varphi_{5,1} \)

We begin with parameterizations for the generators of \( \Gamma_0(5) \) in terms of those for the full modular group

\[
\alpha = TST^2ST^3S, \quad \beta = TST^3ST^2S. \tag{3.32}
\]

Transformation matrices for the vectors of modified theta constants may be computed from their images in \( \text{PGL}((p - 1)/2, \mathbb{C}) \) via the representation \( \pi_p \) given in (2.12)–(2.13)

\[
\pi_5(T) = \begin{pmatrix} e^{\frac{9\pi i}{20}} & 0 \\ 0 & e^{\frac{9\pi i}{20}} \end{pmatrix}, \quad \pi_5(\alpha) = \begin{pmatrix} 0 & e^{\frac{9\pi i}{20}} \\ -e^{\frac{9\pi i}{20}} & 0 \end{pmatrix}, \quad \pi_5(\beta) = \begin{pmatrix} 0 & e^{\frac{9\pi i}{20}} \\ -e^{\frac{9\pi i}{20}} & 0 \end{pmatrix}. \tag{3.33}
\]

Hence, by (3.30), and the modular transformation formula for \( \eta(\tau) \), we deduce that up to a constant multiple, \( C \),

\[
\langle 2 \rangle (E_5) = (5\tau - 3)^{-1} \mathfrak{T}_5[2, -3](\beta\tau) = C\eta^3(5\tau) \frac{\varphi_{5,2}^2(\tau)}{\varphi_{5,1}^3(\tau)} = C\frac{e^{6\pi i/10}}{e^{3\pi i/10}} q + O(q^2). \tag{3.34}
\]

On the other hand, from transformation formulas satisfied by \( E_{\chi_{2,5,1}}(\tau) \) and \( E_{\chi_{4,5,1}}(\tau) \),

\[
\langle 2 \rangle (E_5) = E_5(\beta\tau) = \chi_{2,5,1}(2) E_{\chi_{2,5,1}}(\tau) + \chi_{4,5,1}(2) E_{\chi_{4,5,1}}(\tau) = q + O(q^2). \tag{3.35}
\]

Therefore, \( C = e^{-3\pi i/10} \), and so

\[
\langle 2 \rangle (E_5) = e^{-3\pi i/10} \eta^3(5\tau) \frac{\varphi_{5,2}^2(\tau)}{\varphi_{5,1}^3(\tau)} = \mathfrak{T}_5[-3, 2](\tau) = q \frac{(q; q)_{\infty}^2}{(q^2; q^4; q^5)_{\infty}}. \tag{3.36}
\]

For higher levels \( 7 \leq p \leq 19 \), we similarly use the fact that the image of \( \Gamma_0(p) \) under the presentation \( \pi_p \) defined by Theorem 2.2 is a matrix with a single nonzero entry in each row and column. We obtain the theta quotient representations of the bases for \( \mathcal{N}_1(\Gamma_1(p)) \) from those for \( E_p(\tau) \). In each case, we permute the theta quotients according to the image of \( \pi_N \) and apply the transformation formulas for Eisenstein series on \( \Gamma_0(p) \).

\[
(\gamma_{21}\tau + \gamma_{22})^{-k} E_{k,\chi}(\gamma\tau) = \chi(\gamma_{22}) E_{k,\chi}(\tau) \quad \text{for } \gamma \in \Gamma_0(p), \text{ to each component of } E_p(\tau). \]

We then compare the first nonzero entry in the resulting \( q \)-expansions. By repeating this process with an independent set of generators for \( \Gamma_0(p) \), we ultimately obtain the linearly independent sets of theta quotient representations claimed in Theorem 3.4. \( \square \)
We next extend the permuted bases of weight one forms for \( \Gamma_1(p) \) from Theorems 3.2 and 3.4 to generators for the graded algebra of positive integer weight modular forms for \( \Gamma_1(p) \). By Lemma 2.1, it suffices to prove that monomials of degree \( k = 2, 3 \) in the prospective weight one generators span the vector space \( M_k(\Gamma_1(p)) \). Lemma 3.5 demonstrates the existence of \( \dim M_k(\Gamma_1(p)) \) linearly independent monomials of degree \( k \) in the generators. The dimensions of \( M_k(\Gamma_1(p)) \) for \( k = 2, 3 \) are given by [12]:

| \( p \) | \( p = 5 \) | \( p = 7 \) | \( p = 11 \) | \( p = 13 \) | \( p = 17 \) | \( p = 19 \) |
|---|---|---|---|---|---|---|
| \( \dim M_2(\Gamma_1(p)) \) | 3 | 5 | 10 | 13 | 20 | 24 |
| \( \dim M_3(\Gamma_1(p)) \) | 4 | 7 | 15 | 20 | 32 | 39 |

**Theorem 3.5.** Let \( \langle a_{k,p} \rangle (E_p)(\tau) \) be defined as in Theorems 3.2 and 3.4. For each prime \( 5 \leq p \leq 19 \), the set \( \{ \langle a_{k,p} \rangle (E_p)(\tau) \}_{k-1}^{(p-1)/2} \) generates the graded ring \( M(\Gamma_1(p)) \).

**Proof.** Denote the image of the normalized Eisenstein sum under \( a \in (\mathbb{Z}/n\mathbb{Z})^* \) by
\[
E_{a,p}(\tau) := \langle a \rangle (E_p)(\tau).
\]
Since, by (3.30) and (3.36),
\[
E_{1,5}(\tau) = 1 + O(q), \quad E_{2,5} = q + O(q^2),
\]
any set of distinct monomials of degree \( k \) forms a linearly independent set of modular forms of weight \( k \) for \( \Gamma_1(5) \). Hence, bases for \( M_k(\Gamma_1(5)) \), \( k = 2, 3 \), respectively, are
\[
\{ E_{1,5}^2, E_{1,5}E_{2,5}, E_{2,5}^2 \}, \quad \{ E_{1,5}^3, E_{1,5}E_{2,5}, E_{1,5}E_{2,5}, E_{1,5}E_{2,5} \}.
\]
For the higher levels \( p = 7, 11, 13, 17, 19 \), we introduce the complete homogeneous symmetric polynomial in \( n \) variables of degree \( k \),
\[
h_k(\bar{x}) = h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1}x_{i_2}\ldots x_{i_k}.
\]
For each fixed \( p, k \), and \( 1 \leq i, j \leq \left( \frac{k+1}{2} \right) \), let \( H(i, j) \) denote the coefficient of \( q^2 \) in the \( q \)-expansion of the \( i \)th monomial of
\[
h_k \left( E_{a_1,p}, E_{a_2,p}, \ldots, E_{a_{(k-1)/2},p} \right)
\]
under lexicographic ordering of the monomials of \( h_k(\bar{x}) \). Define the matrix \( H_{p,k} = \{ H(i, j) \} \). For each prime \( 5 \leq p \leq 19 \), a computer algebra system may be used to show
\[
\text{Rank}(H_{p,k}) = \dim M_k(\Gamma_1(p)), \quad k = 1, 2, 3.
\]
This proves that a subset of the terms in the polynomial (3.41) span \( M_k(\Gamma_1(p)) \) for \( k = 2, 3 \). The proof of Theorem 3.5 may be completed by applying Lemma 2.1. \( \square \)

For large values of \( p \), the verification of (3.42) is nontrivial. In particular, the case \( p = 19, k = 3 \) requires knowledge of the first 165 terms in the \( q \)-expansions of the 165 monomials of degree 3 in the parameters from (1.18)–(1.22). This calculation and the rank of the corresponding matrix was accomplished in less than 24 hours on a CAS. For each \( p \), an explicit basis for \( M_k(\Gamma_1(p)) \) may be constructed by row reducing \( H_{k,p} \) and selecting the \( \dim M_k(\Gamma_1(p)) \) monomials corresponding to linear independent rows.
4. Symmetric representations for modular forms of level $p$

This section is devoted to applications of the symmetric representations for modular forms in terms of the permuted generators for $\mathcal{M}(\Gamma_1(p))$. Theorem 4.1 presents a uniform parameterization for an important class of combinatorial generating functions, that of $t$-cores. For a given partition $\lambda$, each square in the Young diagram representation for $\lambda$ defines a hook consisting of that square, all the squares to the right of that square, and all the squares below that square. The hook number of a given square is the number of squares in the that hook. A partition $\lambda$ is said to be a $t$-core if it has no hook numbers that are multiples of $t$. The generating function for the number of $t$-cores is

$$\sum_{n=0}^{\infty} c_t(n)q^n = \frac{(q^{tn}; q^{tn})_\infty}{(q; q)_\infty}.$$ (4.1)

Series from the first three cases of Theorem 4.1 play a fundamental role in the derivation of Ramanujan’s famous congruences for the partition function modulo 5, 7, 11 [5, 33].

**Theorem 4.1.** Let $a_{k,p}$ be defined by Theorem 3.2, $E_{a,p}$ by (3.37), and let $\delta_p = \frac{p^2 - 1}{24}$. Then, for each prime $p$ with $5 \leq p \leq 19$,

$$q^{\delta_p} \left( \frac{q^p; q^p)_\infty}{(q; q)_\infty} = \prod_{k=1}^{p-1} E_{a_k,p}(\tau).$$ (4.2)

**Proof.** The claim (4.2) for each level $p$ may be deduced by utilizing the product representations, given explicitly by (1.4)–(1.22), for the generators from Theorem 3.4. □

From Theorem 4.1, we deduce new divisor sum representations for $p$-cores, $p$ prime, $5 \leq p \leq 19$, in terms of $L$-function values for odd Dirichlet characters at the origin.

**Corollary 4.2.** Let $\delta_p$ be defined by Theorem 4.1. For primes $5 \leq p \leq 19$, and $n \geq \delta_p$,

$$c_p(n - \delta_p) = \left( \frac{2}{p-1} \right)^{\frac{n-3}{2}} \sum_{r_1 + \cdots + r_{(p-1)/2} = n} \left( \prod_{k=1}^{(p-1)/2} \ell_p(a_{k+1,p} \cdot d_k) \right)$$

$$+ \left( \frac{2}{p-1} \right)^{\frac{n-1}{2}} \sum_{r_1 + \cdots + r_{(p-1)/2} = n} \left( \prod_{k=1}^{(p-1)/2} \ell_{p,a_{k,p}}(a_{k,p} \cdot d_k) \right),$$ (4.3)

where, $r_i \geq 1$, $a_{k,p}$ is defined by Theorem 3.2 and for $d \in \mathbb{Z}$, we define

$$\ell_p(d) = 2 \sum_{\chi(-1) = -1 \text{ primitive mod } p} \frac{\chi(d)}{L(0, \chi)}.$$ (4.4)

**Proof.** Identity (4.3) may be derived from (4.1), (4.2) and since, for $1 \neq a \in (\mathbb{Z}/p\mathbb{Z})^*$,

$$\mathcal{E}_{a,p}(\tau) = \frac{2}{p-1} \sum_{\chi(-1) = -1} \chi(a)E_{1,\chi}(\tau) = \frac{2}{p-1} \sum_{n=1}^{\infty} \left( \sum_{d|n} \chi(-1) = -1 \sum_{\chi(\frac{ad}{n})} \frac{2\chi(ad)}{L(0, \chi)} \right) q^n.$$ (4.5)

The right side of (4.3) is the Cauchy product of coefficients of the series from (4.2). □
For the primes $p = 5, 7, 13$, the Eisenstein series of weight two and trivial character modulo $p$ have common representations in terms of the squares of the generators.

**Theorem 4.3.** If $\chi_{1,p}$ is the principal character modulo $p$, and $E_{a,p}$ is given by (3.37),

\[
1 + 6 \sum_{n=1}^{\infty} \frac{\chi_{1,5}(n)q^n}{1 - q^n} = E_{1,5}^2 + E_{2,5}^2, \quad 1 + 4 \sum_{n=1}^{\infty} \frac{\chi_{1,7}(n)q^n}{1 - q^n} = E_{1,7}^2 + E_{2,7}^2 + E_{3,7}^2, \quad (4.6)
\]

\[
1 + 2 \sum_{n=1}^{\infty} \frac{\chi_{1,13}(n)q^n}{1 - q^n} = E_{1,13}^2 + E_{2,13}^2 + E_{3,13}^2 + E_{4,13}^2 + E_{5,13}^2 + E_{7,13}^2. \quad (4.7)
\]

We next turn our attention to the common coefficients within the differental systems of Theorems 4.3 and 4.4. We will apply the following standard result with $h = pk/12$.

**Lemma 4.4.** [4] Lemma 2.1 Suppose that $p$ is a prime and that $f$ is a meromorphic modular form of weight $k$ on $\Gamma_0(p)$. If $h$ is any constant, then the function

\[
F_f := \frac{d}{dq}f + \left\{ \frac{\frac{k}{12} - h}{p - 1} \cdot pE_2(p\tau) + h - \frac{pk}{p - 1} \cdot E_2(\tau) \right\} \cdot f \in \mathcal{M}_{k+2}(\Gamma_0(p)). \quad (4.8)
\]

The proof of Lemma 4.4 shows that the Lemma remains true for the subgroup $\Gamma_1(p)$ of $\Gamma_0(p)$. Moreover, if $f, g \in \mathcal{M}_k(\Gamma_1(p))$, and $\gamma \in \Gamma_0(p)$ with $\langle \gamma \rangle(f) = g$, then $\langle \gamma \rangle(F_f) = F_g$. Thus, for $a \in \left(\mathbb{Z}/p\mathbb{Z}\right)^*$, $(a \frac{d}{dq}E_{a,p}(\tau))/E_{a,p}(\tau) \in \mathcal{M}_2(\Gamma_0(p))$.

Relevant basis expansions are encoded by Theorem 4.3 These subsume the systems of level 5 and 7 of Theorems 4.3 and 4.4 and new coupled systems of level 11 $\leq p \leq 19$. The differential equations are permuted by $\Gamma_0(p)$, with invariance subgroup $\left(\mathbb{Z}/p\mathbb{Z}\right)^*/\{\pm 1\}$.

**Theorem 4.5.** Let $E_{a,p}$ be given as in (3.37) and $a_{k,p}$ as in Theorem 4.2, and define

\[
F_5(x_1, x_2) = -5x_1^2 + 66x_1x_2 + 7x_2^2, \quad (4.9)
\]

\[
F_7(x_1, x_2, x_3) = -7x_1^2 + 5x_2^3 + 5x_3^2 - 20x_2x_3 + 52x_1x_2, \quad (4.10)
\]

\[
F_{11}(x_1, x_2, \ldots, x_5) = -11x_1^2 + x_2^2 + 13x_3^2 + x_4^2 + x_5^2 + 34x_3x_4 + 42x_2x_5 - 40x_2x_3 + 38x_3x_5 - 10x_4x_5, \quad (4.11)
\]

\[
F_{13}(x_1, x_2, \ldots, x_6) = -13x_1^2 + 11x_2^2 - x_3^2 - x_4^2 + 11x_5^2 + 16x_2x_1 + 38x_1x_5 + 2x_2x_3 - 20x_2x_4 + 40x_2x_5 - 8x_2x_6 + 14x_4x_6, \quad (4.12)
\]

\[
F_{17}(x_1, x_2, \ldots, x_8) = -17x_1^2 + 19x_2^2 + 7x_3^2 - 5x_4^2 - 5x_5^2 - 5x_6^2 + 5x_7^2 + 19x_8^2 + 12x_2x_3 + 54x_2x_5 + 12x_5x_6 + 30x_2x_7 - 42x_5x_7 - 60x_2x_8 + 12x_3x_8 - 54x_5x_8 - 12x_7x_8, \quad (4.13)
\]

\[
F_{19}(x_1, x_2, \ldots, x_9) = -19x_1^2 + 41x_2^2 + 5x_3^2 - 7x_4^2 - 7x_5^2 + 5x_6^2 + 5x_7^2 - 7x_8^2 - 7x_9^2 + 16x_2x_5 - 36x_3x_5 + 8x_4x_5 + 32x_5x_6 + 12x_4x_7 - 16x_5x_7 - 16x_5x_9 + 28x_5x_9 - 12x_6x_8 - 4x_2x_9 + 68x_3x_9 + 16x_5x_9 - 28x_8x_9. \quad (4.14)
\]

Then, the generators satisfy a system of $(p - 1)/2$ differential equations subsumed by

\[
\frac{12}{E_{a,p}} \frac{dq}{dq}E_{a,p} = F_f(E_{a_{1,p}}, E_{a_{2,p}}, \ldots, E_{a_{(p-1)/2,p}}) + pE_2(p\tau), \quad a \in \left(\mathbb{Z}/p\mathbb{Z}\right)^*. \quad (4.15)
\]
A differential system for a full set of generators for \( \mathcal{M}(\Gamma_1(p)) \) may be formulated from these relations (4.15), and the group action formulation (3.27). For \( p = 11 \),

\[
\begin{align*}
\frac{d}{dq} \mathcal{E}_{1,11} &= F_{11}(\mathcal{E}_{1,11}, \mathcal{E}_{2,11}, \mathcal{E}_{3,11}, \mathcal{E}_{5,11}, \mathcal{E}_{7,11}) + 11P(q^{11}), \\
\frac{d}{dq} \mathcal{E}_{2,11} &= F_{11}(\mathcal{E}_{2,11}, -\mathcal{E}_{7,11}, -\mathcal{E}_{5,11}, -\mathcal{E}_{1,11}, \mathcal{E}_{3,11}) + 11P(q^{11}), \\
\frac{d}{dq} \mathcal{E}_{3,11} &= F_{11}(\mathcal{E}_{3,11}, -\mathcal{E}_{5,11}, -\mathcal{E}_{2,11}, -\mathcal{E}_{7,11}, -\mathcal{E}_{1,11}) + 11P(q^{11}), \\
\frac{d}{dq} \mathcal{E}_{5,11} &= F_{11}(\mathcal{E}_{5,11}, -\mathcal{E}_{1,11}, -\mathcal{E}_{7,11}, \mathcal{E}_{3,11}, \mathcal{E}_{2,11}) + 11P(q^{11}), \\
\frac{d}{dq} \mathcal{E}_{7,11} &= F_{11}(\mathcal{E}_{7,11}, \mathcal{E}_{3,11}, -\mathcal{E}_{1,11}, \mathcal{E}_{2,11}, \mathcal{E}_{5,11}) + 11P(q^{11}).
\end{align*}
\]

(4.16) (4.17) (4.18) (4.19) (4.20)

5. QUADRATIC RELATIONS AND KLEIN’S AUTOMORPHISM GROUPS

The following results relate permuted generators for \( \mathcal{M}(\Gamma_1(p)) \) for primes \( 5 \leq p \leq 19 \) to Klein’s classical automorphism groups and extensions. We begin with symmetries of the icosahedron. Define the normalized theta constant of order \( p \) and index \( k \) by

\[
\phi_{p,k}(\tau) = \exp\left(-\frac{(2k-1)\pi i}{2p}\right) \varphi_{p,k}(\tau).
\]

(5.1)

To study the geometry of \( X(N) \), the extended upper half plane modulo the principal congruence subgroup \( \Gamma(N) \), Klein devised vector-valued modular forms and used them to formulate explicit maps into the relevant moduli spaces. At level \( p = 5 \), Klein considered \( \mathcal{V}_5(\tau) = [\phi_{5,2}(\tau), \phi_{5,1}^T(\tau)] \) and showed \( \mathcal{V}_5 \) satisfies transformation formulas under generators for the full modular group

\[
\mathcal{V}_5(S\tau) := V(-1/\tau) = \tau^{1/2} \rho_5(S) \mathcal{V}_5(\tau), \quad \mathcal{V}_5(T\tau) := \mathcal{V}_5(\tau + 1) = \rho_5(T) \mathcal{V}_5(\tau)
\]

(5.2)

Klein showed that \( \rho_5 : PSL(2, \mathbb{Z}) \to \text{PLG}(2, \mathbb{C}) \) is a representation for the automorphism group of the projected icosahedron. Namely, \( (\rho_5(S), \rho_5(T)) = PSL(2, \mathbb{F}_5) = A_5 \).

Our symmetric representations for modular forms in terms of \( \mathcal{E}_{1,5}, \mathcal{E}_{2,5} \) make use of the fact that, up to a constant multiple, \( \rho_5 |_{\Gamma_0(5)} \) induces a permutation of \( \phi_{5,1}, \phi_{5,2} \). A similar analysis at level \( p = 7 \) gives rise to the Klein quartic curve defined by the locus

\[
a^3b + b^3c + c^3a = 0, \quad [a : b : c] \in \mathbb{C}P^2.
\]

(5.3)

At levels \( p = 7, 11 \), certain quadratic relations between the \( \Gamma_0(p) \)-permuted generators for \( \mathcal{M}(\Gamma_1(p)) \) define the relevant curves in complex projective space isomorphic to those defining the modular curve \( X(p) \). At level seven, apply Theorem 3.1 to observe that

\[
\mathcal{E}_{1,7} = \frac{\phi_{7,1}}{\phi_{7,3}^2}, \quad \mathcal{E}_{2,7} = \frac{\phi_{7,2}}{\phi_{7,1}^2}, \quad \mathcal{E}_{3,7} = \frac{\phi_{7,3}}{\phi_{7,2}^2} = -\mathcal{E}_{4,7}.
\]

(5.4)

With \( a = \phi_{7,1}, b = \phi_{7,2}, c = -\phi_{7,3} \), Theorem 5.1 provides a parameterization for (5.3).

**Theorem 5.1.** Define \( \mathcal{E}_{a,p} = \mathcal{E}_{a,p}(\tau) \) as in (3.37). Then

\[
\mathcal{E}_{2,7}\mathcal{E}_{4,7} + \mathcal{E}_{2,7}\mathcal{E}_{1,7} + \mathcal{E}_{4,7}\mathcal{E}_{1,7} = 0.
\]

(5.5)
Proof. Replace $q$ by $q^7$ in (2.14), and make the respective substitutions
\[(x, y) = (\pi \tau, 2\pi \tau), \ (\pi \tau, 3\pi \tau), \ (2\pi \tau, 3\pi \tau)\] (5.6)
in the resultant identities to derive, from (2.4), (2.6), and (2.8),
\[D_2(q) - D_1(q) = \mathcal{E}_{3,7}\mathcal{E}_{1,7}, \quad D_1(q) - D_3(q) = \mathcal{E}_{2,7}\mathcal{E}_{1,7},\] (5.7)
\[D_3(q) - D_2(q) = \mathcal{E}_{2,7}\mathcal{E}_{3,7},\] (5.8)
where $D_1(q), D_2(q),$ and $D_3(q)$ take the form
\[D_1(q) = \sum_{n=1}^{\infty} \frac{nq^{7n}}{1-q^{7n}}, \quad D_2(q) = \sum_{n=1}^{\infty} \frac{nq^{6n}}{1-q^{7n}} + \sum_{n=1}^{\infty} \frac{nq^{5n}}{1-q^{7n}},\] (5.9)
\[D_3(q) = \sum_{n=1}^{\infty} \frac{nq^{3n}}{1-q^{7n}} + \sum_{n=1}^{\infty} \frac{nq^{4n}}{1-q^{7n}}.\] (5.10)
Identity (5.5) follows immediately from (5.7)–(5.8). \(\square\)

The level 11 analogue of the Klein quartic is the Klein cubic [1, 23, 15, Band II]
\[v^2 w + w^2 x + x^2 y + y^2 z + z^2 v = 0, \quad [v, w, x, y, z] \in \mathbb{CP}^4.\] (5.11)
The curve $X(11)$ is isomorphic to the locus in $\mathbb{CP}^4$ consisting of $[v, w, x, y, z]$ with [3, 24]
\[\text{Rank } \begin{pmatrix} w & v & 0 & 0 & z \\ v & x & w & 0 & 0 \\ 0 & w & y & x & 0 \\ 0 & 0 & x & z & y \\ z & 0 & 0 & y & v \end{pmatrix} = 3.\] (5.12)

The following theorem features relations between the level eleven generators equivalent to theta function parameterizations for the second and third minors along the top row.

**Theorem 5.2.** Let $\mathcal{E}_{a,p} = \mathcal{E}_{a,p}(\tau)$ be defined as in (3.37). Then
\[\mathcal{E}_{3,11}\mathcal{E}_{7,11} + \mathcal{E}_{1,11}\mathcal{E}_{7,11} - \mathcal{E}_{3,11}\mathcal{E}_{5,11} - \mathcal{E}_{5,11}\mathcal{E}_{7,11} = 0,\] (5.13)
\[\mathcal{E}_{1,11}\mathcal{E}_{5,11} - \mathcal{E}_{1,11}\mathcal{E}_{5,11} + \mathcal{E}_{5,11}\mathcal{E}_{7,11} = 0.\] (5.14)

**Proof.** The claimed identities may be derived directly as special cases of (2.14) or from Theorem 3.2 and the $q$-expansions of the generators $\mathcal{E}_{i,11}$ for $\mathcal{M}(\Gamma_1(11))$. \(\square\)

From Theorem 3.2, (5.13), and (5.14), we deduce the quartic theta constant identities
\[\phi_{11,3}\phi_{11,2}^3 - \phi_{11,3}\phi_{11,5}\phi_{11,2} + \phi_{11,1}\phi_{11,4}\phi_{11,5}\phi_{11,2} - \phi_{11,1}\phi_{11,3}^2 = 0,\] (5.15)
\[\phi_{11,2}\phi_{11,4}\phi_{11,1}^2 - \phi_{11,1}\phi_{11,2}\phi_{11,3}^2 + \phi_{11,3}\phi_{11,4}\phi_{11,5}^2 = 0.\] (5.16)
These are theta function parameterizations for the zeros of the second and third minors along the top row of the matrix from (5.12)
\[v^2 x^2 - v^2 y z + v y^3 + w x y z, -v x^3 + v x y z - x y^3.\] (5.17)
Under action by $(\mathbb{Z}/11\mathbb{Z})^* \times \{\cdot\}$ and (3.27), identities (5.13)–(5.14) are equivalent to 10 theta function identities providing a parameterization for $X(11)$ (see [3, p. 10]).
At higher levels, the geometry of the modular curves $X(p)$ was not examined as closely by Klein, though these cases have been the subject of \cite{K, L}. The ensuing quadratic relations may be relevant in describing the modular curves at levels 13, 17, 19. We give only a subset of all quadratic relations and point out that many more may be derived by acting on each identity by $\Gamma_0(p)$. The identities may be proved from the Sturm bound for each subgroup $\Gamma_1(p)$ \cite{St} §3.3 and the $q$-expansions of the generators.

**Theorem 5.3.** If $E_{a,p}$ is defined by \eqref{3.3}, the following level 13 relations hold:

\begin{align}
E_{1,13}E_{3,13} + E_{2,13}E_{3,13} - E_{1,13}E_{5,13} - E_{1,13}E_{7,13} - E_{2,13}E_{7,13} - E_{5,13}E_{7,13} &= 0, \\
E_{1,13}E_{4,13} - E_{1,13}E_{5,13} + E_{3,13}E_{7,13} + E_{4,13}E_{7,13} - E_{5,13}E_{7,13} &= 0, \\
E_{3,13}E_{5,13} - E_{1,13}E_{7,13} - E_{3,13}E_{7,13} &= 0.
\end{align}

**Theorem 5.4.** With the above notation, the theta quotient identities of level 17 hold:

\begin{align}
-3E_{2,17}E_{3,17} + 2E_{7,17}E_{3,17} + 3E_{13,17}E_{3,17} + E_{2,17}E_{7,17} + E_{2,17}E_{8,17} + 3E_{2,17}E_{11,17} \\
-3E_{5,17}E_{11,17} + 3E_{7,17}E_{11,17} - E_{2,17}E_{13,17} - 3E_{7,17}E_{13,17} - 3E_{11,17}E_{13,17} &= 0,
\end{align}

\begin{align}
-9E_{2,17}E_{3,17} + 5E_{7,17}E_{3,17} + 9E_{13,17}E_{3,17} - 5E_{2,17}E_{7,17} - 5E_{2,17}E_{8,17} + 9E_{1,17}E_{11,17} \\
+ 9E_{2,17}E_{11,17} + 9E_{7,17}E_{11,17} + 5E_{2,17}E_{13,17} - 9E_{11,17}E_{13,17} &= 0,
\end{align}

\begin{align}
-3E_{1,17}E_{5,17} - 3E_{7,17}E_{5,17} + 3E_{1,17}E_{7,17} + E_{2,17}E_{7,17} + 2E_{3,17}E_{7,17} + E_{2,17}E_{8,17} \\
-3E_{2,17}E_{11,17} - 3E_{7,17}E_{11,17} + 2E_{2,17}E_{13,17} &= 0
\end{align}

\begin{align}
-9E_{2,17}E_{3,17} + 7E_{7,17}E_{3,17} + 9E_{8,17}E_{3,17} + 2E_{2,17}E_{7,17} + 2E_{2,17}E_{8,17} + 9E_{7,17}E_{11,17} \\
+ 7E_{2,17}E_{13,17} - 9E_{7,17}E_{13,17} &= 0,
\end{align}

\begin{align}
16E_{2,17}E_{7,17} - 16E_{3,17}E_{7,17} + 9E_{8,17}E_{7,17} + 9E_{1,17}E_{8,17} + 16E_{2,17}E_{8,17} \\
-9E_{2,17}E_{11,17} - 7E_{2,17}E_{13,17} &= 0,
\end{align}

\begin{align}
E_{2,17}E_{7,17} - 3E_{3,17}E_{5,17} + 2E_{3,17}E_{7,17} + E_{2,17}E_{8,17} - E_{2,17}E_{13,17} - 3E_{7,17}E_{13,17} &= 0.
\end{align}

**Theorem 5.5.** With the above notation, the theta quotient identities of level 19 hold:

\begin{align}
E_{2,19}E_{5,19} - E_{4,19}E_{5,19} + E_{7,19}E_{5,19} - 3E_{9,19}E_{5,19} + E_{11,19}E_{5,19} + E_{13,19}E_{5,19} \\
+ E_{7,19}E_{9,19} - E_{7,19}E_{11,19} - E_{2,19}E_{13,19} - 3E_{19}E_{13,19} + E_{9,19}E_{13,19} - E_{11,19}E_{13,19} &= 0,
\end{align}

\begin{align}
- E_{1,19}E_{5,19} + E_{2,19}E_{5,19} + E_{3,19}E_{5,19} - E_{4,19}E_{5,19} + E_{7,19}E_{5,19} - E_{11,19}E_{5,19} \\
- E_{13,19}E_{5,19} + E_{1,19}E_{7,19} - E_{2,19}E_{13,19} - E_{3,19}E_{13,19} + E_{11,19}E_{13,19} &= 0.
\end{align}
\[-\mathcal{E}_{2,19}\mathcal{E}_{5,19} - \mathcal{E}_{3,19}\mathcal{E}_{5,19} + \mathcal{E}_{7,19}\mathcal{E}_{5,19} - \mathcal{E}_{9,19}\mathcal{E}_{5,19} + \mathcal{E}_{11,19}\mathcal{E}_{5,19} + \mathcal{E}_{13,19}\mathcal{E}_{5,19} + \mathcal{E}_{2,19}\mathcal{E}_{7,19} - \mathcal{E}_{7,19}\mathcal{E}_{11,19} + \mathcal{E}_{2,19}\mathcal{E}_{13,19} - \mathcal{E}_{11,19}\mathcal{E}_{13,19} = 0,\]

\[\mathcal{E}_{2,19}\mathcal{E}_{5,19} + \mathcal{E}_{3,19}\mathcal{E}_{5,19} - \mathcal{E}_{7,19}\mathcal{E}_{5,19} - \mathcal{E}_{11,19}\mathcal{E}_{5,19} - \mathcal{E}_{13,19}\mathcal{E}_{5,19} + \mathcal{E}_{3,19}\mathcal{E}_{11,19} - \mathcal{E}_{2,19}\mathcal{E}_{13,19} - \mathcal{E}_{3,19}\mathcal{E}_{13,19} + \mathcal{E}_{11,19}\mathcal{E}_{13,19} = 0,\]

\[-\mathcal{E}_{2,19}\mathcal{E}_{5,19} - \mathcal{E}_{3,19}\mathcal{E}_{5,19} + \mathcal{E}_{7,19}\mathcal{E}_{5,19} + \mathcal{E}_{9,19}\mathcal{E}_{5,19}
    + \mathcal{E}_{13,19}\mathcal{E}_{5,19} + \mathcal{E}_{2,19}\mathcal{E}_{11,19} + \mathcal{E}_{3,19}\mathcal{E}_{13,19} - \mathcal{E}_{11,19}\mathcal{E}_{13,19} = 0,\]

\[-\mathcal{E}_{2,19}\mathcal{E}_{3,19} + \mathcal{E}_{5,19}\mathcal{E}_{3,19} + \mathcal{E}_{9,19}\mathcal{E}_{3,19} + \mathcal{E}_{2,19}\mathcal{E}_{5,19} - \mathcal{E}_{4,19}\mathcal{E}_{5,19} - \mathcal{E}_{2,19}\mathcal{E}_{13,19} = 0,\]

\[-\mathcal{E}_{3,19}\mathcal{E}_{5,19} - \mathcal{E}_{9,19}\mathcal{E}_{5,19} + \mathcal{E}_{13,19}\mathcal{E}_{5,19} + \mathcal{E}_{1,19}\mathcal{E}_{13,19} + \mathcal{E}_{3,19}\mathcal{E}_{13,19} = 0,\]

\[\mathcal{E}_{3,19}\mathcal{E}_{5,19} - \mathcal{E}_{4,19}\mathcal{E}_{5,19} + \mathcal{E}_{4,19}\mathcal{E}_{11,19} - \mathcal{E}_{3,19}\mathcal{E}_{13,19} = 0,\]

\[\mathcal{E}_{2,19}\mathcal{E}_{4,19} - \mathcal{E}_{5,19}\mathcal{E}_{9,19} - \mathcal{E}_{3,19}\mathcal{E}_{13,19} = 0.\]

For each prime level \( p \geq 23 \), a similar permuted basis construction exists for the Eisenstein subspace of weight one forms realized as the orbit of \( \mathcal{E}_p \) under \( \langle \cdot \rangle \), namely

\[\left\{ \langle a \rangle (\mathcal{E}_p) \mid a \in (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\} \right\}.
\]

These generators do not appear to be expressible as theta quotients, as is the case in the lower levels, and the generated graded algebra, in general, does not span \( \mathcal{M}(\Gamma_1(p)) \). The situation may arise because 23 is the first prime level admitting nonzero weight one cusp forms \([\text{C. Corollary 3}]\). Since the space of weight one cusp forms for \( \Gamma_1(23) \) is spanned by \( \eta(\tau)\eta(23\tau) \), a form for \( \Gamma_0(23) \) of multiplier \( \left( \frac{23}{23} \right) \), we may construct a corresponding \( \Gamma_0(23) \)-permuted basis of weight one forms for \( \mathcal{M}_1(\Gamma_1(23)) \)

\[\mathcal{M}_1(\Gamma_1(23)) = \mathbb{C}\eta(\tau)\eta(23\tau) \bigoplus_{a \in (\mathbb{Z}/23\mathbb{Z})^*/\{\pm 1\}} \mathbb{C}\langle a \rangle \mathcal{E}_{23}(\tau).\]

Although our calculations suggest that product representations do not exist for \( \mathcal{E}_{a,23} \), we nevertheless conjecture that the basis of weight one forms from \( \langle \cdot \rangle \) generate the graded algebra \( \mathcal{M}(\Gamma_1(23)) \). Imitating the construction at higher levels to obtain a similar \( \Gamma_0(p) \)-permuted basis of weight one forms for \( \mathcal{M}(\Gamma_1(p)) \) is made difficult by the lack of formulas for the dimensions of the spaces of cusp forms of weight one for \( p > 23 \).
REFERENCES

[1] A. Adler. On the automorphism group of a certain cubic threefold. *Amer. J. Math.*, 100(6):1275–1280, 1978.
[2] A. Adler. Cubic invariants for $\text{SL}_2(F_q)$. *J. Algebra*, 145(1):178–186, 1992.
[3] A. Adler. The Mathieu group $M_{11}$ and the modular curve $X(11)$. *Proc. London Math. Soc. (3)*, 74(1):1–28, 1997.
[4] S. Ahlgren. The theta-operator and the divisors of modular forms on genus zero subgroups. *Math. Res. Lett.*, 10(5-6):787–798, 2003.
[5] B. C. Berndt and K. Ono. Ramanujan’s unpublished manuscript on the partition and tau functions with proofs and commentary. *Sém. Lothar. Combin.*, 42:Art. B42c, 63 pp. (electronic), 1999. The Andrews Festschrift (Maratea, 1998).
[6] L. A. Borisov and P. E. Gunnells. Toric modular forms of higher weight. *J. Reine Angew. Math.*, 560:43–64, 2003.
[7] K. Buzzard. Computing weight one modular forms over $\mathbb{C}$ and $\overline{\mathbb{F}}_p$. *Preprint*, arXiv:1205.5077 [math.NT], 2012.
[8] R. Charles, T. Huber, and A. Mendoza. Parameterizations for quintic Eisenstein series. *J. Number Theory*, 133(1):195–214, 2013.
[9] Y.-H. Chen, Y. Yang, and N. Yui. On the $\text{PSL}_2(\mathbb{F}_{19})$-invariant cubic sevenfold. *Preprint*, arXiv:1301.1142v1 [math.AG], 2013.
[10] S. Cooper and P. C. Toh. Quintic and septic Eisenstein series. *Ramanujan J.*, 19(2):163–181, 2009.
[11] P. Deligne and M. Rapoport. Les schémas de modules de courbes elliptiques. In *Modular functions of one variable, II* (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pages 143–316. Lecture Notes in Math., Vol. 349. Springer, Berlin, 1973.
[12] F. Diamond and J. Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.
[13] W. Duke. Continued fractions and modular functions. *Bull. Amer. Math. Soc. (N.S.)*, 42(2):137–162 (electronic), 2005.
[14] H. M. Farkas and I. Kra. *Theta constants, Riemann surfaces and the modular group*, volume 37 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. An introduction with applications to uniformization theorems, partition identities and combinatorial number theory.
[15] R. Fricke and F. Klein. *Vorlesungen über die Theorie der automorphen Funktionen. Band 1: Die gruppentheoretischen Grundlagen. Band II: Die funktionentheoretischen Ausführungen und die Anwendungen*, volume 4 of *Bibliotheca Mathematica Teubneriana, Bände 3*. Johnson Reprint Corp., New York, 1965.
[16] F. Garvan, D. Kim, and D. Stanton. Cranks and $t$-cores. *Invent. Math.*, 101(1):1–17, 1990.
[17] H. Hahn. Eisenstein series associated with $\Gamma_0(2)$. *Ramanujan J.*, 15:235–257, 2008.
[18] T. Huber. Coupled systems of differential equations for modular forms of level $n$. In *Ramanujan Rediscovered: Proceedings of a Conference on Elliptic Functions, Partitions, and $q$-Series in memory of K. Venkatashaliengar*: Bangalore, 1 – 5 June, 2009, pages 139–146, Bangalore, 2009. The Ramanujan Mathematical Society.
[19] T. Huber. Differential equations for cubic theta functions. *Int. J. Num. Thy.*, 7(7):1945–1957, 2011.
[20] T. Huber. A theory of theta functions to the quintic base. *J. Number Theory*, 134:49–92, 2014.
[21] T. Huber and D. Lara. Differential equations for septic theta functions. *Ramanujan Journal*, To appear, arXiv:1304.0694 [math.NT].
[22] K. Khuri-Makdisi. Moduli interpretation of Eisenstein series. *Int. J. Number Theory*, 8(3):715–748, 2012.
[23] F. Klein. über die transformation elfter ordnung der elliptischen functionen. *Gesammelte Mathematische Abhandlungen III* (Julius Springer, Berlin, 1923), pages 140–165.
[24] F. Klein. Ueber gewisse Theilwerthe der Θ-Function. *Math. Ann.*, 17(4):565–574, 1880.

[25] F. Klein. *Lectures on the icosahedron and the solution of equations of the fifth degree*. Dover Publications, Inc., New York, N.Y., revised edition, 1956. Translated into English by George Gavin Morrice.

[26] F. Klein. Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade. Birkhäuser Verlag, Basel, 1993. Reprint of the 1884 original, Edited, with an introduction and commentary by Peter Slodowy.

[27] F. Klein. On the order-seven transformation of elliptic functions. In *The eightfold way*, volume 35 of *Math. Sci. Res. Inst. Publ.*, pages 287–331. Cambridge Univ. Press, Cambridge, 1999. Translated from the German and with an introduction by Silvio Levy.

[28] M. Knopp. *Modular functions in analytic number theory*. Markham Publishing Co., Chicago, Ill., 1970.

[29] Z-G. Liu. On certain identities of Ramanujan. *J. Number Theory*, 83(1):59–75, 2000.

[30] R. Maier. Nonlinear differential equations satisfied by certain classical modular forms. *manuscripta mathematica*, pages 1–42, 2010.

[31] S. Ramanujan. *Notebooks. Vols. 1, 2*. Tata Institute of Fundamental Research, Bombay, 1957.

[32] S. Ramanujan. On certain arithmetical functions [Trans. Cambridge Philos. Soc. 22 (1916), no. 9, 159–184]. In *Collected papers of Srinivasa Ramanujan*, pages 136–162. AMS Chelsea Publ., Providence, RI, 2000.

[33] S. Ramanujan. Some properties of p(n), the number of partitions of n [Proc. Cambridge Philos. Soc. 19 (1919), 207–210]. In *Collected papers of Srinivasa Ramanujan*, pages 210–213. AMS Chelsea Publ., Providence, RI, 2000.

[34] N. Rustom. Generators of graded rings of modular forms. (preprint), [arXiv:1209.3864v3 [math.NT]], 2012.

[35] W. Stein. *Modular forms, a computational approach*, volume 79 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007. With an appendix by Paul E. Gunnells.

[36] E. T. Whittaker and G. N. Watson. *A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions: with an account of the principal transcendental functions*. Fourth edition. Reprinted. Cambridge University Press, New York, 1962.

[37] L. Yang. Exotic arithmetic structure on the first Hurwitz triplet. [arXiv:1209.1783v5 [math.NT] (preprint)], 2013.

[38] D. Zagier. Introduction to modular forms. In *From number theory to physics (Les Houches, 1989)*, pages 238–291. Springer, Berlin, 1992.

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