Exponentiated Weibull-Poisson distribution: model, properties and applications

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Abstract

In this paper we propose a new four-parameters distribution with increasing, decreasing, bathtub-shaped and unimodal failure rate, called as the exponentiated Weibull-Poisson (EWP) distribution. The new distribution arises on a latent complementary risk problem base and is obtained by compounding exponentiated Weibull (EW) and Poisson distributions. This distribution contains several lifetime sub-models such as: generalized exponential-Poisson (GEP), complementary Weibull-Poisson (CWP), complementary exponential-Poisson (CEP), exponentiated Rayleigh-Poisson (ERP) and Rayleigh-Poisson (RP) distributions.

We obtain several properties of the new distribution such as its probability density function, its reliability and failure rate functions, quantiles and moments. The maximum likelihood estimation procedure via a EM-algorithm is presented in this paper. Sub-models of the EWP distribution are studied in details. In the end, Applications to two real data sets are given to show the flexibility and potentiality of the new distribution.

Key words: EM-algorithm, Exponentiated Weibull distribution, Maximum likelihood estimation, Probability weighted moments, Residual life function.

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1. Introduction

The Weibull and EW distributions provide a simple and close form solution to many problems in lifetime and reliability. However, they do not provide a reasonable parametric fit for some practical applications, for example the Weibull distribution is not useful for modeling phenomenon with non-monotone failure rates. The bathtub-shaped and the unimodal failure

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rates which are common in reliability and biological studies cannot be modeled by the Weibull distribution.

Recently, attempts have been made to define new families of probability distributions that extend well-known families of distributions and at the same time provide great flexibility in modeling data in practice. The exponential-geometric (EG), exponential-Poisson (EP), exponential-logarithmic (EL), exponential-power series (EPS), Weibull-geometric (WG) and Weibull-power series (WPS) distributions were introduced and studied by Adamidis and Loukas [2], Kus [23], Tahmasbi and Rezaei [38], Chahkandi and Ganjali [10], Barreto-Souza et al. [7] and Morais and Barreto-Souza et al. [31], respectively.

Barreto-Souza and Cribari-Neto [5] and Louzada et al. [24] introduced the exponentiated exponential-Poisson (EEP) and the complementary exponential-geometric (CEG) distributions where the EEP is the generalization of the EP distribution and the CEG is complementary to the EG model proposed by Adamidis and Loukas [2]. Recently, Cancho et al. [9] introduced the two-parameter Poisson-exponential (PE) lifetime distribution with increasing failure rate. Mahmoudi and Jafari [25] introduced the generalized exponential-power series (GEPS) distribution by compounding the generalized exponential (GE) distribution with the power series distribution. Also exponentiated Weibull-logarithmic (EWL), exponentiated Weibull-geometric (EWG) and exponentiated Weibull-power series (EWP) distributions has been introduced and analyzed by Mahmoudi and Sepahdar [26] and Mahmoudi and Shiran [27, 28].

In this paper, we propose a new four-parameters distribution, referred to as the EWP distribution, which contains as special sub-models the GEP, CWP, CEP, ERP and RP distributions, among others. The main reasons for introducing the EWP distribution are: (i) This distribution due to its flexibility in accommodating different forms of the risk function is an important model that can be used in a variety of problems in modeling lifetime data. (ii) This distribution is a suitable model in a complementary risk problem base (Basu and Klein, [8]) in presence of latent risks, in the sense that there is no information about which factor was responsible for the component failure and only the maximum lifetime value among all risks is observed. (iii) It provides a reasonable parametric fit to skewed data that cannot be properly fitted by other distributions and is a suitable model in several areas such as public health, actuarial science, biomedical studies, demography and industrial reliability.

The paper is organized as follows. In Section 2, we review the EW distribution and its
properties. In Section 3, we define the EWP distribution. The density, survival and hazard rate functions and some of their properties are given in this section. We derive quantiles and moments of the EWP distribution in Section 4. Rényi and Shannon entropies of the EWP distribution are given in Section 5. Section 6 provides the moments of order statistics of the EWP distribution. Residual and reverse residual life functions of the EWP distribution are discussed in Section 7. In Section 8 we explain probability weighted moments. Mean deviations from the mean and median are obtained in Section 9. Section 10 is devoted to the Bonferroni and Lorenz curves of the EWP distribution. Estimation of the parameters by maximum likelihood via a EM-algorithm and inference for large sample are presented in Section 11. In Section 12, we studied some special sub-models of the EWP distribution. Applications to real data sets are given in Section 13 and conclusions are provided in Section 14.

2. EW distribution: A brief review

The EW distribution, introduced by Mudholkar and Srivastava [32] as extension of the Weibull family, contains distributions with bathtub-shaped and unimodal failure rates besides a broader class of monotone failure rates. The applications of the EW distribution in reliability and survival studies were illustrated by Mudholkar et al. [33], Mudholkar and Huston [34], Gupta and Kundu [19], Nassar and Eissa [36] and Choudhury [12].

The random variable $X$ has an EW distribution if its cumulative distribution function (cdf) takes the form

$$F_X(x) = \left(1 - e^{-\beta x^\gamma}\right)^\alpha, \quad x > 0,$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. The corresponding probability density function (pdf) is

$$f_X(x) = \alpha \gamma \beta x^{\gamma-1} e^{-(\beta x)^\gamma} \left(1 - e^{-(\beta x)^\gamma}\right)^{\alpha-1}.$$

The survival and hazard rate functions of the EW distribution are

$$S(x) = 1 - \left(1 - e^{-(\beta x)^\gamma}\right)^\alpha,$$

and

$$h(x) = \alpha \gamma \beta x^{\gamma-1} e^{-(\beta x)^\gamma} \left(1 - e^{-(\beta x)^\gamma}\right)^{\alpha-1} \left\{ \left[1 - \left(1 - e^{-(\beta x)^\gamma}\right)^\alpha\right] \right\}^{-1},$$

respectively. The $k$th moment about zero of the EW distribution is given by

$$E(X^k) = \alpha \beta^{-k} \Gamma\left(\frac{k}{\gamma} + 1\right) \sum_{j=0}^{\infty} (-1)^j \binom{\alpha - 1}{j} (j + 1)^{-\left(\frac{k}{\gamma} + 1\right)}.$$
Note that for positive integer values of $\alpha$, the index $j$ in previous sum stops at $\alpha - 1$ and the above expression takes the closed form

$$E(X^k) = \alpha \beta^{-k} \Gamma \left( \frac{k}{\gamma} + 1 \right) A_k(\gamma),$$

when

$$A_k(\gamma) = 1 + \sum_{j=1}^{\alpha-1} (-1)^j \binom{\alpha-1}{j} (j+1)^{-(\frac{k}{\gamma}+1)}, \quad k = 1, 2, 3, \cdots,$$

where $\Gamma(\frac{k}{\gamma} + 1)$ denotes the gamma function (see, Nassar and Eissa [36] for more details).

3. The EWP distribution

Suppose that the random variable $X$ has the EW distribution where its cdf and pdf are given in (1) and (2). Given $N$, let $X_1, \cdots, X_N$ be independent and identically distributed random variables from EW distribution. Let $N$ is distributed according to zero truncated Poisson distribution with pdf

$$P(N = n) = \frac{e^{-\theta} \theta^n}{n!(1 - e^{-\theta})}, \quad n = 1, 2, \cdots, \theta > 0.$$

Let $Y = \max(X_1, \cdots, X_N)$, then the cdf of $Y|N = n$ is given by

$$F_{Y|N=n}(y) = [1 - e^{-(\beta y)^\gamma}]^n,\quad n = 1, 2, \cdots,$$

which is EW distribution with parameters $n\alpha$, $\beta$ and $\gamma$. The EWP distribution, denote by EWP$(\alpha, \beta, \gamma, \theta)$, is defined by the marginal cdf of $Y$, i.e.

$$F(y) = e^{\theta(1-e^{-(\beta y)^\gamma})^\alpha - 1}. \quad (3)$$

This new distribution includes some sub-models such as CEP, GEP, GWP, ERP and RP distributions.

The pdf of EWP distribution is given by

$$f(y) = \frac{\alpha \gamma \beta^\gamma y^{\gamma-1}}{e^\theta - 1} e^{-(\beta y)^\gamma} (1 - e^{-(\beta y)^\gamma})^{\alpha-1} e^{\theta(1-e^{-(\beta y)^\gamma})^\alpha}, \quad (4)$$

where $\alpha, \beta, \gamma, \theta > 0$.

The survival and hazard rate functions of EWP distribution are given, respectively, by

$$S(y) = 1 - \frac{e^{\theta(1-e^{-(\beta y)^\gamma})^\alpha - 1}}{e^\theta - 1}, \quad (5)$$

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and
\[ h(y) = \frac{\alpha \gamma \beta^\gamma y^{\gamma-1} e^{-\beta y} (1 - e^{-\gamma y\beta})^{\alpha-1} e^\theta (1 - e^{-\gamma y\beta})^\alpha}{e^\theta - e^\theta (1 - e^{-\gamma y\beta})^\alpha}. \] (6)

The plots of density and hazard rate functions of EWP distribution for $\beta = 1$ and different values $(\alpha, \gamma, \theta)$ are given in Figs. 1 and 2, respectively.

**Proposition 1.** The limiting distribution of EWP $(\alpha, \beta, \gamma, \theta)$ where $\theta \to 0^+$ is
\[ \lim_{\theta \to 0^+} F(y) = (1 - e^{-\gamma y^\beta})^\alpha, \]
which is the cdf of EW distribution.

**Proposition 2.** The limiting behavior of hazard function of EWP distribution in (6) is
(i) for $0 < \gamma < 1$, $\lim_{y \to 0} h(y) = \begin{cases} \infty & 0 < \alpha \leq 1 \\ 0 & \alpha > 1, \end{cases}$ and $\lim_{y \to \infty} h(y) = 0$.
(ii) for $\gamma = 1$, $\lim_{y \to 0} h(y) = \begin{cases} \frac{\theta \beta}{e^{\theta-1}} & \alpha = 1 \\ 0 & \alpha > 1, \end{cases}$ and $\lim_{y \to \infty} h(y) = \beta$.
(iii) for $\gamma > 1$, $\lim_{y \to 0} h(y) = 0$, for each value $\alpha > 0$ and $\lim_{y \to \infty} h(y) = \infty.$
Hazard $\alpha = 0.5, \gamma = 0.5, \theta = 0.5$

Hazard $\alpha = 3, \gamma = 0.5, \theta = 0.5$

Hazard $\alpha = 0.5, \gamma = 1, \theta = 0.5$

Hazard $\alpha = 3, \gamma = 1, \theta = 0.5$

Hazard $\alpha = 0.5, \gamma = 3, \theta = 0.5$

Hazard $\alpha = 3, \gamma = 3, \theta = 0.5$

Figure 2: Plots of hazard function of EWP distribution for $\beta = 1$ and different values $\alpha, \theta$ and $\gamma$.

**Proof.** The proof is a forward calculation and is omitted.

4. Quantiles and moments of the EWP distribution

Some of the most important features and characteristics of a distribution can be studied through its moments and quantiles such as tending, dispersion, skewness and kurtosis. Also, the quantiles of a distribution can be used in data generation from a distribution.

The $p$th quantile of the EWP distribution is given by

$$x_p = \frac{1}{\beta} \left[ - \log \left( 1 - \left( \frac{1}{\theta} \log \left( p \left( e^{\theta} - 1 \right) + 1 \right)^{1/\alpha} \right) \right) \right]^{1/\gamma},$$

which is used for data generation from the EWP distribution.

Now we obtain the moment generating function of the EWP distribution. Suppose that $Y \sim EWP(\alpha, \beta, \gamma, \theta)$ and $X(n) = \max(X_1, \cdots, X_n)$, where $X_i \sim EW(\alpha, \beta, \gamma)$ for $i = 1, 2, \cdots, n$,
\[ M_Y(t) = \sum_{n=1}^{\infty} P(N = n)M_{X_{(n)}}(t) \]
\[ = \sum_{n=1}^{\infty} P(N = n) \sum_{k=0}^{\infty} \frac{t^k}{k!} n\alpha \beta^{-k} \Gamma(1 + \frac{k}{\gamma}) \sum_{j=0}^{\infty} (-1)^j \left( \frac{n\alpha - 1}{j} \right)^{(n\alpha - 1)}(j + 1)^{-\left(1 + \frac{k}{\gamma}\right)} \]
\[ = \frac{\alpha \theta}{\beta^e (e^\theta - 1)} \sum_{n=1}^{\infty} \frac{\theta^{n-1}}{(n-1)!} \alpha \beta^{-k} \sum_{j=0}^{\infty} (-1)^j \left( \frac{n\alpha - 1}{j} \right)^{(n\alpha - 1)}(j + 1)^{-\left(1 + \frac{k}{\gamma}\right)} \Gamma(1 + \frac{k}{\gamma}) \]
\[ \overset{(8)}{=} \frac{\alpha \theta^\gamma (1 + \frac{k}{\gamma})}{\beta^e (e^\theta - 1)} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\alpha(n - 1)}{j} \right)^{(\alpha(n - 1))}(j + 1)^{-\left(1 + \frac{k}{\gamma}\right)} \]

One can use \( M_Y(t) \) to obtain the \( k \)th moment about zero of the EWP distribution. We have
\[ E(Y^k) = \sum_{n=1}^{\infty} P(N = n)E(X_{(n)}^k) \]
\[ = \frac{\alpha \theta^\gamma (1 + \frac{k}{\gamma})}{\beta^e (e^\theta - 1)} \sum_{n=1}^{\infty} \frac{\theta^{n-1}}{(n-1)!} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\alpha(n - 1)}{j} \right)^{(\alpha(n - 1))}(j + 1)^{-\left(1 + \frac{k}{\gamma}\right)} \]
\[ \overset{(9)}{=} \frac{\alpha \theta^\gamma (1 + \frac{k}{\gamma})}{\beta^e (e^\theta - 1)} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\alpha(n - 1)}{j} \right)^{(\alpha(n - 1))}(j + 1)^{-\left(1 + \frac{k}{\gamma}\right)} \]
\[ = \frac{\alpha \theta^\gamma (1 + \frac{k}{\gamma})}{\beta^e (e^\theta - 1)} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\alpha(n - 1)}{j} \right)^{(\alpha(n - 1))}(j + 1)^{-\left(1 + \frac{k}{\gamma}\right)} \]

Using the change of variable \( i = n - 1 \), another equivalent formula for \( E(Y^k) \) is given by
\[ E(Y^k) = \frac{\alpha \theta^\gamma (1 + \frac{k}{\gamma})}{\beta^e (e^\theta - 1)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\alpha(i + 1)}{j} \right)^{(\alpha(i + 1))}(j + 1)^{-\left(1 + \frac{k}{\gamma}\right)} \]
\[ \overset{(10)}{=} \frac{\alpha \theta^\gamma (1 + \frac{k}{\gamma})}{\beta^e (e^\theta - 1)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\alpha(i + 1)}{j} \right)^{(\alpha(i + 1))}(j + 1)^{-\left(1 + \frac{k}{\gamma}\right)} \]

The mean and variance of the EWP distribution are given, respectively, by
\[ E(Y) = \frac{\alpha \theta^\gamma (1 + \frac{1}{\gamma})}{\beta^e (e^\theta - 1)} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\alpha(n - 1)}{j} \right)^{(\alpha(n - 1))}(j + 1)^{-\left(1 + \frac{1}{\gamma}\right)} \]
\[ \overset{(11)}{=} \frac{\alpha \theta^\gamma (1 + \frac{1}{\gamma})}{\beta^e (e^\theta - 1)} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\alpha(n - 1)}{j} \right)^{(\alpha(n - 1))}(j + 1)^{-\left(1 + \frac{1}{\gamma}\right)} \]
\[ Var(Y) = \frac{\alpha \theta^\gamma (1 + \frac{2}{\gamma})}{\beta^e (e^\theta - 1)} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\alpha(n - 1)}{j} \right)^{(\alpha(n - 1))}(j + 1)^{-\left(1 + \frac{2}{\gamma}\right)} - E^2(Y), \]

where \( E(Y) \) is given in Eq. \( (11) \). Note that for positive integer values of \( \alpha \), the index \( j \) in Eqs. \( (8)-(12) \) stops at \( n\alpha - 1 \).

5. Rényi and Shannon entropies

If \( X \) is a random variable having an absolutely continuous cumulative distribution function \( F(x) \) and probability distribution function \( f(x) \), then the basic uncertainty measure for distribution \( F \) (called the entropy of \( F \) is defined as \( H(x) = E[-\log(f(X))] \)). Statistical entropy is a probabilistic measure of uncertainty or ignorance about the outcome of a random experiment, and is a measure of a reduction in that uncertainty. Numerous entropy and information indices, among them the Rényi entropy, have been developed and used in various disciplines and contexts. Information theoretic principles and methods have become integral parts of probability and statistics and have been applied in various branches of statistics and related fields.
Entropy has been used in various situations in science and engineering. The entropy of a random variable $Y$ is a measure of variation of the uncertainty. For a random variable with the pdf $f$, the Rényi entropy is defined by $I_R(r) = \frac{1}{1-r} \log \{ \int_R f^r(y)dy \}$, for $r > 0$ and $r \neq 1$. Using the power series expansion $(1 - z)^α = \sum_{j=0}^{∞} (-1)^j \binom{α}{j} z^j$ and change of variable $(βy)^{γ} = u$ gives

$$f_0^∞ f^r(y)dy = \left( \frac{αβγ}{e^α-1} \right)^r \sum_{j=0}^{∞} \frac{(rθ)^j}{j!} \sum_{k=0}^{∞} (-1)^k \binom{α(r+j)-r}{k} \int_0^∞ y^r(γ-1) e^{-(r+k)(θγ)γ} dy.$$ Using formula $\int_0^∞ y^r(γ-1) e^{-(r+k)(θγ)γ} dy = \Gamma \left( \frac{r(γ-1)+1}{γ} \right) \frac{γβγ(rγ)(γ-1)}{1}(r+k)^{-\left( \frac{r(γ-1)+1}{γ} \right)}$, we have

$$f_0^∞ f^r(y)dy = \frac{(αθ)^r(γβ)^{γ-1}}{(e^α-1)^r} \Gamma \left( \frac{r(γ-1)+1}{γ} \right) \sum_{j=0}^{∞} \sum_{k=0}^{∞} (-1)^k \frac{(rθ)^j}{j!} \frac{(α(r+j)-r)}{k} (r+k)^{-\left( \frac{r(γ-1)+1}{γ} \right)}.$$ Thus, according to the definition of Rényi entropy we have

$$I_R(r) = \frac{1}{1-r} \log \left[ \frac{(αθ)^r(γβ)^{γ-1}}{(e^α-1)^r} \Gamma \left( \frac{r(γ-1)+1}{γ} \right) \sum_{j=0}^{∞} \sum_{k=0}^{∞} (-1)^k \frac{(rθ)^j}{j!} \frac{(α(r+j)-r)}{k} (r+k)^{-\left( \frac{r(γ-1)+1}{γ} \right)} \right].$$ The Shannon entropy is defined by $E[-\log[f(Y)]]$. This is a special case derived from $\lim_{r→1} I_R(r)$.

6. Moments of order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Moments of order statistics play an important role in quality control testing and reliability, where a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictors are often based on moments of order statistics. We now derive an explicit expression for the density function and cumulative distribution function of the $r$th order statistic $Y_{r:n}$, in a random sample of size $n$ from the EWP distribution.

Let the random variable $Y_{r:n}$ be the $r$th order statistic $(X_{1:n} ≤ Y_{2:n} ≤ \cdots ≤ Y_{n:n})$ in a sample of size $n$ from a distribution with pdf $f(.)$ and cdf $F(.)$, respectively. The pdf of $Y_{r:n}$ for $r = 1, \cdots, n$ is given by

$$f_{r:n}(y) = \frac{1}{B(r, n-r+1)} f(y)[F(y)]^{r-1}[1-F(y)]^{n-r}, \quad y > 0.$$ Substituting from (3) and (4) into (13) and using the binomial expansion gives

$$f_{r:n}(y) = \frac{1}{B(r, n-r+1)} \frac{αθβγ}{(e^α-1)^r} y^{γ-1} e^{-(θγ)γ} (1-e^{-(θγ)γ})^{α-1} \times \sum_{i=0}^{r-1} \sum_{j=0}^{n-r} (-1)^{r-j-1} \binom{r-1}{i} \binom{n-r}{j} e^{(n-r-j)θ} e^{i+j}(1-e^{-(θγ)γ})^α.$$
Also the cdf of \( Y_{r:n} \) is given by

\[
F_{r:n}(y) = \sum_{k=r}^{n} \binom{n}{k} [F(y)]^k [1 - F(y)]^{n-k} = \frac{1}{(e^\theta - 1)^n} \sum_{k=r}^{n} \binom{n-k}{i} \sum_{j=0}^{k} \binom{n-k}{j} e^{(n-k-j)\theta} e^{(i+j)\theta} (1-e^{-(\beta y)^\gamma})^\alpha. \tag{15}
\]

Using the binomial expansion, after some calculations the \( k \)th moment of the \( r \)th order statistic \( Y_{r:n} \) is given by

\[
E(Y_{r:n}^k) = \frac{\alpha^k}{B(r,n-r+1)\beta^y} \Gamma \left( \frac{k}{\gamma} + 1 \right) \sum_{i=0}^{n-r} \frac{(-1)^i}{(e^\theta - 1)^{r+i}} \sum_{j=0}^{r+i-1} (-1)^{r+i-j-1} \binom{n+r+i-1}{r+i-j} \prod_{l=0}^{\infty} \frac{\theta^{j+l+1}}{t^l} \sum_{j=0}^{\infty} (-1)^t \binom{t}{r+i-1} (t+1)^{-(1+\frac{1}{\gamma})}. \tag{16}
\]

The pdf and cdf of the smallest and biggest order statistics \( Y_{1:n} \) and \( Y_{n:n} \) can be obtained using Eqs. (14) and (15) for special cases \( r = 1 \) and \( n \).

For the smallest order statistics \( Y_{1:n} \), we have

\[
f_{1:n}(y) = \frac{n\alpha\gamma\beta\gamma y^{\gamma-1} e^{-(\beta y)^\gamma}}{(e^\theta - 1)^n} (1 - e^{-(\beta y)^\gamma})^{\alpha-1} e^{\theta(1-e^{-(\beta y)^\gamma})^\alpha} \left(e^\theta - e^{\theta(1-e^{-(\beta y)^\gamma})^\alpha}\right)^{n-1},
\]

\[
F_{1:n}(y) = 1 - \frac{1}{(e^\theta - 1)^n} \left(e^\theta - e^{\theta(1-e^{-(\beta y)^\gamma})^\alpha}\right)^n,
\]

and for the biggest order statistics \( Y_{n:n} \), we have

\[
f_{n:n}(y) = \frac{n\alpha\gamma\beta\gamma y^{\gamma-1} e^{-(\beta y)^\gamma}}{(e^\theta - 1)^n} (1 - e^{-(\beta y)^\gamma})^{\alpha-1} e^{\theta(1-e^{-(\beta y)^\gamma})^\alpha} \left(e^{\theta(1-e^{-(\beta y)^\gamma})^\alpha} - 1\right)^{n-1},
\]

\[
F_{n:n}(y) = \frac{1}{(e^\theta - 1)^n} \left(e^{\theta(1-e^{-(\beta y)^\gamma})^\alpha} - 1\right)^n,
\]

The expectation of the smallest and biggest order statistics \( Y_{1:n} \) and \( Y_{n:n} \) can be obtained by setting \( r = 1, n \) and \( k = 1 \) in (16).

7. Residual life function of the EWP distribution

Given that a component survives up to time \( t \geq 0 \), the residual life is the period beyond \( t \) until the time of failure and defined by the conditional random variable \( X - t | X > t \). The mean residual life (MRL) function is an important function in survival analysis, actuarial science, economics and other social sciences and reliability for characterizing lifetime. Although the shape of the failure rate function plays an important role in repair and replacement strategies, the MRL function is more relevant as the latter summarizes the entire residual life function, whereas the former considers only the risk of instantaneous failure. In reliability, it is well
known that the MRL function and ratio of two consecutive moments of residual life determine
the distribution uniquely (Gupta and Gupta, \[18\]).

MRL function as well as failure rate function is very important, since each of them can be
used to determine a unique corresponding lifetime distribution. Lifetimes can exhibit IMRL
(increasing MRL) or DMRL (decreasing MRL). MRL functions that first decreases (increases)
and then increases (decreases) are usually called bathtub-shaped (upside-down bathtub), BMRL
(UUMRL). The relationship between the behaviors of the two functions of a distribution was
studied by many authors such as Ghitany \[14\], Mi \[30\], Park \[37\] and Tang et al. \[39\].

The \(r\)th order moment of the residual life of the EWP distribution is given by the general formula

\[
m_r(t) = E[(Y - t)^r|Y > t] = \frac{1}{S(t)} \int_t^\infty (y - t)^r f(y)dy,
\]

where \(S(t) = 1 - F(t)\), is the survival function.

In what seen this onwards, we use the expressions

\[
\int_t^\infty x^{\gamma + s - 1} e^{-(k+1)\beta x^\gamma} dx = \frac{1}{\gamma \beta^{\gamma + s}} (k + 1)^{-1+\frac{s}{\gamma}} \Gamma^{1+\frac{s}{\gamma}}((k + 1)(\beta t)^\gamma),
\]

and

\[
\int_0^t x^{\gamma + s - 1} e^{-(k+1)\beta x^\gamma} dx = \frac{1}{\gamma \beta^{\gamma + s}} (k + 1)^{-1+\frac{s}{\gamma}} \Gamma^{1+\frac{s}{\gamma}}((k + 1)(\beta t)^\gamma),
\]

where \(\Gamma^s(t) = \int_t^\infty x^{s-1}e^{-x}dx\) is the upper incomplete gamma function and \(\Gamma_s(t) = \int_0^t x^{s-1}e^{-x}dx\)
is the lower incomplete gamma function.

The \(r\)th order moment of the residual life of the EWP distribution is given by

\[
m_r(t) = \frac{\alpha \theta}{S(t)(e^\theta - 1)} \sum_{i=0}^r (-1)^r \binom{r}{i} \beta^{-i} t^{-i} \sum_{j=0}^\infty \sum_{k=0}^\infty (-1)^k \frac{\theta^j}{j!(k+1)^{1+\frac{s}{\gamma}}} \\
\times \left(\alpha \binom{j+1}{k} - 1 \right) \Gamma^{1+\frac{s}{\gamma}}((k + 1)(\beta t)^\gamma), \quad (17)
\]

where \(S(t)\) (the survival function of \(Y\)) is given in \[5\].

For the EWP distribution the MRL function which is obtained by setting \(r = 1\) in \[17\], is given
in the following theorem.

**Theorem 1.** The MRL function of the EWP distribution with cdf \[3\] is given by

\[
m_1(t) = \frac{1}{S(t)} \frac{\alpha \theta}{\beta(e^\theta - 1)} \sum_{j=0}^\infty \sum_{k=0}^\infty (-1)^k \frac{\theta^j}{j!(k+1)^{1+\frac{s}{\gamma}}} \left(\alpha \binom{j+1}{k} - 1 \right) \Gamma^{1+\frac{s}{\gamma}}((k + 1)(\beta t)^\gamma) - t.
\]
The second moment of the residual life function of the EWP distribution is
\[
m_2(t) = \frac{1}{S(t)} \left\{ \alpha \theta \beta \sum_{k=0}^{\infty} \left( -1 \right)^{k} \frac{\theta^j}{\beta^j (k+1)^{\gamma+\frac{1}{\gamma}} \Gamma \left( 1 + \frac{1}{\gamma} \right) \Gamma \left( 1 + \frac{1}{\gamma} \right) \left( (k+1) (\beta t)^\gamma \right) } \right\} + t^2.
\]

The variance of the residual life function of the EWP distribution can be obtained using \( m_1(t) \) and \( m_2(t) \).

On the other hand, we analogously discuss the reversed residual life and some of its properties.

The reversed residual life can be defined as the conditional random variable \( t - X \mid X \leq t \) which denotes the time elapsed from the failure of a component given that its life is less than or equal to \( t \). This random variable may also be called the inactivity time (or time since failure); for more details one may see Kundu and Nanda \[22\] and Nanda et al. \[35\]. Also, in reliability, the mean reversed residual life (MRRL) and ratio of two consecutive moments of reversed residual life characterize the distribution uniquely. Using (4) and (5), the reversed failure (or reversed hazard) rate function of the EWP is given by
\[
r(y) = \frac{f(y)}{F(y)} = \frac{\alpha \gamma \beta \theta y^{\gamma-1} e^{-(\beta y)^\gamma} (1 - e^{-(\beta y)^\gamma})^{\alpha-1} e^{\theta (1 - e^{-(\beta y)^\gamma})^\alpha}}{e^{\theta (1 - e^{-(\beta y)^\gamma})^\alpha} - 1}.
\]

The \( r \)th moment of the reversed residual life function can be obtained by the formula
\[
\mu_r(t) = E \left[ (t - Y)^r \mid Y \leq t \right] = \frac{1}{F(t)} \int_0^t (t - y)^r f(y) dy.
\]

Hence,
\[
\mu_r(t) = \frac{1}{F(t)} \frac{\alpha \theta}{\beta^r (e^\theta - 1)} \sum_{i=0}^{r} \left( -1 \right)^i \binom{r}{i} t^{r-i} \beta^{r-i} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left( -1 \right)^{k} \frac{\theta^j}{\beta^j (k+1)^{\gamma+\frac{1}{\gamma}}} \times \left( \alpha \beta \right)^{j+1} \left( 1 + \frac{1}{\gamma} \right) \left( (k+1) (\beta t)^\gamma \right).
\]

The mean and second moment of the reversed residual life of the EWP distribution can be obtained by setting \( r = 1 \) and \( 2 \) in (18). Also, using \( \mu_1(t) \) and \( \mu_2(t) \) one can obtain the variance of the reversed residual life function of the EWP distribution.

8. Probability weighted moments

Probability weighted moments (PWMs) are expectations of certain functions of a random variable defined when the ordinary moments of the random variable exist. The PWMs method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly.
Estimates based on PWMs are often considered to be superior to standard moment-based estimates. They are sometimes used when maximum likelihood estimates are unavailable or difficult to compute. They may also be used as starting values for maximum likelihood estimates.

The PWMs method, which has been investigated by many researchers, was originally proposed by Greenwood et al. \[17\]. Since then it has been used widely in practice and for research purposes. Hosking et al. \[20\] investigated the properties of parameters estimated by the PWMs method for the generalized extreme value (GEV) distribution using fairly long observed series, and they gave a good summary of the PWMs method. Hosking \[21\] showed that the PWMs method is superior to the Maximum Likelihood (ML) method in parameter estimations when a distribution can be obtained by setting $\tau = 1$ and $2$ in above equation.

In this paper we calculate the PWMs of the EWP distribution since they can be used for estimating the EWP parameters. For a random variable with the pdf $f(.)$ and cdf $F(.)$, the PWMs function are defined by

$$\tau_{s,r} = E [X^sF(X)^r] = \int_{0}^{\infty} x^s(F(x))^r f(x)dx.$$

For the EWP distribution the PWMs are given by

$$\tau_{s,r} = \frac{\alpha\gamma\beta}{(e^{\theta}-1)^{r+1}} \sum_{i=0}^{r}(\gamma+s-1) \int_{0}^{\infty} x^s e^{-\beta x}(1-e^{-\beta x})^{\gamma} \alpha-1 e^{\theta(i+1)}(1-e^{-\beta x})^\alpha dx$$

$$= \frac{\alpha\gamma\beta}{(e^{\theta}-1)^{r+1}} \sum_{i=0}^{r}(\gamma+s-1) \sum_{j=0}^{\infty} \frac{\theta(j+1)}{j!} \int_{0}^{\infty} x^s e^{-\beta x}(1-e^{-\beta x})^{\gamma} \alpha(j+1)-1 dx$$

$$= \frac{\alpha\gamma\beta}{(e^{\theta}-1)^{r+1}} \sum_{i=0}^{r}(\gamma+s-1) \sum_{j=0}^{\infty} \frac{\theta(j+1)}{j!} \sum_{k=0}^{\infty} (-1)^k \left( \frac{\alpha(j+1)-1}{k} \right) \int_{0}^{\infty} x^s e^{-\beta x} \Gamma(1+\frac{\alpha}{\gamma}) dx.$$ 

Using formula $\int_{0}^{\infty} x^s e^{-\beta x} \Gamma(1+\frac{\alpha}{\gamma}) dx = \frac{\Gamma(1+\frac{\alpha}{\gamma})}{\gamma e^{s}(\beta+\gamma)^{1+\frac{\alpha}{\gamma}}}$, we have

$$\tau_{s,r} = \frac{\alpha\theta}{\beta^s(e^{\theta}-1)^{r+1}} \sum_{i=0}^{r}(\gamma+s-1) \sum_{j=0}^{\infty} \frac{\theta(j+1)}{j!} \sum_{k=0}^{\infty} (-1)^k \left( \frac{\alpha(j+1)-1}{k} \right) \Gamma(1+\frac{\alpha}{\gamma}) \left( \frac{1}{(k+1)^{1+\frac{\alpha}{\gamma}}} \right).$$ \hspace{1cm} (19)

**Remark 1.** The $s$th moment of the EWP distribution can be obtained by putting $r = 0$ in \hspace{1cm} (19). Therefore

$$E(X^s) = \frac{\alpha\theta}{\beta^s(e^{\theta}-1)} \sum_{j=0}^{\infty} \frac{\theta(j+1)}{j!} \sum_{k=0}^{\infty} (-1)^k \left( \frac{\alpha(j+1)-1}{k} \right) \Gamma(1+\frac{\alpha}{\gamma}) \left( \frac{1}{(k+1)^{1+\frac{\alpha}{\gamma}}} \right),$$

which is equal to Eq. \hspace{1cm} (11), if $s$ is replaced by $k$. Also, the mean and variance of the EWP distribution can be obtained by setting $s = 1$ and $2$ in above equation.
9. Mean deviations

The amount of scatter in a population can be measured by the totality of deviations from the mean and median. The mean deviation from the mean is a robust statistic, being more resilient to outliers in a data set than the standard deviation. In the standard deviation, the distances from the mean are squared, so on average, large deviations are weighted more heavily, and thus outliers can heavily influence it. In the mean deviation from the mean, the magnitude of the distances of a small number of outliers is irrelevant. The mean deviation from the median is a measure of statistical dispersion. It is a more robust estimator of scale than the sample variance or standard deviation. It thus behaves better with distributions without a mean or variance, such as the Cauchy distribution.

For a random variable $X$ with pdf $f(x)$, cdf $F(x)$, mean $\mu$ and median $M$, the mean deviation from the mean and the mean deviation from the median are defined by

$$\delta_1(X) = \int_0^\infty |x - \mu| f(x) dx = 2\mu F(\mu) - 2I(\mu),$$

and

$$\delta_2(X) = \int_0^\infty |x - M| f(x) dx = 2MF(M) - M + \mu - 2I(M),$$

respectively, where $I(b) = \int_0^b x f(x) dx$.

**Theorem 2.** The Mean deviations function of the EWP distribution are

$$\delta_1(X) = \frac{2}{e^{\theta}-1}\mu \left( e^{(1-e^{-\theta})\gamma} - 1 \right) - \frac{\alpha \theta}{\beta} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{\theta^j}{j!(k+1)^{1+\gamma}} \left( \frac{\alpha(j+1)-1}{k} \right) \Gamma \left( \frac{1}{1+\gamma} \right) \Gamma \left( \frac{1}{1+\gamma} (k+1)(\beta \mu)^\gamma \right),$$

and

$$\delta_2(X) = \mu - 2\alpha \theta \beta (e^\theta - 1) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{\theta^j}{j!(k+1)^{1+\gamma}} \left( \frac{\alpha(j+1)-1}{k} \right) \Gamma \left( \frac{1}{1+\gamma} \right) \Gamma \left( \frac{1}{1+\gamma} (k+1)(\beta M)^\gamma \right),$$

where $\mu$ is given in (11) and $M$ can be obtained by setting $p = .5$ in (7).

10. Bonferroni and Lorenz curves

Study of income inequality has gained a lot of importance over the last many years. Lorenz curve and the associated Gini index are undoubtedly the most popular indices of income inequality. However, there are certain measures which despite possessing interesting characteristics have
not been used often for measuring inequality. Bonferroni curve and scaled total time on test transform are two such measures, which have the advantage of being represented graphically in the unit square and can also be related to the Lorenz curve and Gini ratio (Giorgi, [15]). These two measures have some applications in reliability and life testing as well (Giorgi and Crescenzi, [16]). The Bonferroni and Lorenz curves and Gini index have many applications not only in economics to study income and poverty, but also in other fields like reliability, medicine and insurance.

For a random variable $X$ with cdf $F(.)$, the Bonferroni curve is given by

$$B_F[F(x)] = \frac{1}{\mu F(x)} \int_0^x u f(u) du.$$  

From the relationship between the Bonferroni curve and the mean residual lifetime, the Bonferroni curve of the EWP distribution is given by

$$B_F[F(x)] = \frac{\alpha \theta}{\beta \mu (e^{\theta} - 1)} \left[ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{\theta^j}{j!(k+1)^{1+\gamma}} \frac{(\alpha(j+1)-1)}{k} \right] \times \Gamma_{(1+\frac{1}{\gamma})}((k+1)(\beta x)^\gamma).$$

The Lorenz curve of the EWP distribution can be obtained via the expression

$$L_F[F(x)] = B_F[F(x)] F(x) = \frac{\alpha \theta}{\beta \mu (e^{\theta} - 1)} \left[ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{\theta^j}{j!(k+1)^{1+\gamma}} \frac{(\alpha(j+1)-1)}{k} \right] \times \Gamma_{(1+\frac{1}{\gamma})}((k+1)(\beta x)^\gamma),$$

where $\mu$ is the mean of the EWP distribution.

The scaled total time on test transform of a distribution function $F$ is defined by

$$S_F[F(t)] = \frac{1}{\mu} \int_0^t S(u) du.$$  

If $F(t)$ denotes the cdf of the EWP distribution then

$$S_F[F(t)] = \frac{1}{\mu (e^{\theta} - 1)} \left[ t e^{\theta} - \frac{1}{\gamma} \beta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{\theta^j}{j!} \left( \frac{\alpha j}{k} \right) k^{-1/\gamma} \Gamma_{(\frac{1}{\gamma})}(k(\beta t)^\gamma) \right].$$

The cumulative total time can be obtained by using formula $C_F = \int_0^1 S_F[F(t)] f(t) dt$ and the Gini index can be derived from the relationship $G = 1 - C_F$.

11. Estimation and inference

In this section, we discuss the estimation of the parameters of the EWP distribution. Let $Y_1, Y_2, \cdots, Y_n$ be a random sample with observed values $y_1, y_2, \cdots, y_n$ from EWP distribution
with parameters $\alpha, \beta, \gamma$ and $\theta$. Let $\Theta = (\alpha, \beta, \gamma, \theta)^T$ be the parameter vector. The total log-likelihood function is given by

$$l_n \equiv l_n(y; \Theta) = -n \log [e^\theta - 1] + n [\log \theta + \log \alpha + \log \gamma + \gamma \log \beta]$$

$$+ (\gamma - 1) \sum_{i=1}^n \log y_i - \sum_{i=1}^n (\beta y_i)^\gamma + (\alpha - 1) \sum_{i=1}^n \log (1 - e^{-(\beta y_i)^\gamma})$$

$$+ \theta \sum_{i=1}^n (1 - e^{-(\beta y_i)^\gamma})^\alpha.$$

The associated score function is given by $U_n(\Theta) = (\partial l_n/\partial \alpha, \partial l_n/\partial \beta, \partial l_n/\partial \gamma, \partial l_n/\partial \theta)^T$, where

$$\frac{\partial l_n}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log (1 - e^{-(\beta y_i)^\gamma}) + \theta \sum_{i=1}^n \log (1 - e^{-(\beta y_i)^\gamma})(1 - e^{-(\beta y_i)^\gamma})^{\alpha},$$

$$\frac{\partial l_n}{\partial \beta} = \frac{n\gamma}{\beta} - \gamma \beta^{-1} \sum_{i=1}^n y_i^\gamma + (\alpha - 1) \gamma \beta^{-1} \sum_{i=1}^n \frac{y_i^\gamma e^{-(\beta y_i)^\gamma}}{1 - e^{-(\beta y_i)^\gamma}}$$

$$+ \theta \alpha \beta^{-1} \sum_{i=1}^n y_i^\gamma e^{-(\beta y_i)^\gamma}(1 - e^{-(\beta y_i)^\gamma})^{\alpha-1},$$

$$\frac{\partial l_n}{\partial \gamma} = \frac{n}{\gamma} + n \log \beta + \sum_{i=1}^n \log y_i - \sum_{i=1}^n (\beta y_i)^\gamma \log (\beta y_i)$$

$$+ (\alpha - 1) \sum_{i=1}^n \frac{(\beta y_i)^\gamma \log (\beta y_i) e^{-(\beta y_i)^\gamma}}{1 - e^{-(\beta y_i)^\gamma}} + \theta \alpha \sum_{i=1}^n (\beta y_i)^\gamma \log (\beta y_i) e^{-(\beta y_i)^\gamma}(1 - e^{-(\beta y_i)^\gamma})^{\alpha-1},$$

$$\frac{\partial l_n}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^n (1 - e^{-(\beta y_i)^\gamma})^\alpha - \frac{n}{1 - e^{-(\beta y_i)^\gamma}}.$$

The MLE of $\Theta$, say $\hat{\Theta}$, is obtained by solving the nonlinear system $U_n(\Theta) = 0$. The solution of this nonlinear system of equation has not a closed form. For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The $4 \times 4$ observed information matrix is

$$I_n(\Theta) = -\begin{bmatrix}
I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\gamma} & I_{\alpha\theta} \\
I_{\alpha\beta} & I_{\beta\beta} & I_{\beta\gamma} & I_{\beta\theta} \\
I_{\alpha\gamma} & I_{\beta\gamma} & I_{\gamma\gamma} & I_{\gamma\theta} \\
I_{\alpha\theta} & I_{\beta\theta} & I_{\gamma\theta} & I_{\theta\theta}
\end{bmatrix},$$

whose elements are given in Appendix.

Applying the usual large sample approximation, MLE of $\Theta$ i.e. $\hat{\Theta}$ can be treated as being approximately $N_4(\Theta, J_n(\Theta)^{-1})$, where $J_n(\Theta) = E[I_n(\Theta)]$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\Theta} - \Theta)$ is $N_4(0, J(\Theta)^{-1})$, where $J(\Theta) = \lim_{n \to \infty} n^{-1} I_n(\Theta)$ is the unit information matrix. This asymptotic behavior remains valid if $J(\Theta)$ is replaced by the average sample information matrix evaluated at $\hat{\Theta}$, say $n^{-1} I_n(\hat{\Theta})$. The estimated asymptotic multivariate normal $N_4(\Theta, I_n(\hat{\Theta})^{-1})$ distribution of $\hat{\Theta}$ can be used to construct approximate confidence intervals.
for the parameters and for the hazard rate and survival functions. A $100(1 - \gamma)$ asymptotic confidence interval for each parameter $\Theta_r$ is given by

$$ACI_r = (\hat{\Theta}_r - Z_{\frac{\gamma}{2}} \sqrt{\hat{I}_r}, \hat{\Theta}_r + Z_{\frac{\gamma}{2}} \sqrt{\hat{I}_r}),$$

where $\hat{I}_r$ is the $(r, r)$ diagonal element of $I_n(\hat{\Theta})^{-1}$ for $r = 1, 2, 3, 4$, and $Z_{\frac{\gamma}{2}}$ is the quantile $1 - \gamma/2$ of the standard normal distribution.

11.1. EM-algorithm

The MLEs of the parameters $\alpha, \beta, \gamma$ and $\theta$ in previous section must be derived numerically. Newton-Raphson algorithm is one of the standard methods to determine the MLEs of the parameters. To employ the algorithm, second derivatives of the log-likelihood are required for all iterations. The EM-algorithm is a very powerful tool in handling the incomplete data problem (Dempster et al., [13]; McLachlan and Krishnan, [29]). It is an iterative method by repeatedly replacing the missing data with estimated values and updating the parameters. It is especially useful if the complete data set is easy to analyze.

Let the complete-data be $Y_1, \ldots, Y_n$ with observed values $y_1, \ldots, y_n$ and the hypothetical random variable $Z_1, \ldots, Z_n$. The joint probability density function is such that the marginal density of $Y_1, \ldots, Y_n$ is the likelihood of interest. Then, we define a hypothetical complete-data distribution for each $(Y_i, Z_i)$ $i = 1, \ldots, n$, with a joint probability density function in the form

$$g(y, z; \Theta) = f(y|z)f(z) = \frac{\theta^z}{z!(e^{\theta} - 1)} z^{\alpha \gamma \beta^\gamma y^{\gamma - 1} e^{-(\beta y)\gamma} (1 - e^{-(\beta y)\gamma})^{z \alpha - 1}},$$

where $\Theta = (\alpha, \beta, \gamma, \theta)$, $y > 0$ and $z \in \mathbb{N}$.

Under the formulation, the E-step of an EM cycle requires the expectation of $(Z|Y; \Theta^{(r)})$ where $\Theta^{(r)} = (\alpha^{(r)}, \beta^{(r)}, \gamma^{(r)}, \theta^{(r)})$ is the current estimate of $\Theta$ (in the $r$th iteration).

The pdf of $Z$ given $Y$, say $g(z|y)$ is given by

$$g(z|y) = \frac{e^{-\theta(1-e^{-(\beta y)\gamma})^\alpha} [\theta(1-e^{-(\beta y)\gamma})^\alpha]^z}{(z-1)!},$$

with the expectation

$$E[Z|Y = y] = 1 + \theta(1-e^{-(\beta y)\gamma})^\alpha.$$

The EM cycle is completed with the M-step by using the maximum likelihood estimation over $\Theta$, with the missing $Z$’s replaced by their conditional expectations given above.
The log-likelihood for the complete-data is

\[ l_n(y_1, \cdots, y_n; z_1, \cdots, z; \Theta) \propto \sum_{i=1}^{n} z_i \log \theta - n \log(e^\theta - 1) + n \log \alpha + n \log \gamma + n \gamma \log \beta + (\gamma - 1) \sum_{i=1}^{n} \log y_i - \sum_{i=1}^{n} (\beta y_i)^\gamma + \sum_{i=1}^{n} (z_i \alpha - 1) \log(1 - e^{-(\beta y_i)^\gamma}). \]

The components of the score function \( U_n^*(\Theta) = (\frac{\partial l_n^*}{\partial \alpha}, \frac{\partial l_n^*}{\partial \beta}, \frac{\partial l_n^*}{\partial \gamma}, \frac{\partial l_n^*}{\partial \theta})^T \) are given by

\[
\frac{\partial l_n^*}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} z_i \log(1 - e^{-(\beta y_i)^\gamma}),
\frac{\partial l_n^*}{\partial \beta} = \frac{n \gamma}{\beta} - \gamma \beta^\gamma - 1 \sum_{i=1}^{n} y_i^\gamma + \gamma \beta^\gamma - 1 \sum_{i=1}^{n} (z_i \alpha - 1) \frac{y_i^\gamma e^{-(\beta y_i)^\gamma}}{1 - e^{-(\beta y_i)^\gamma}},
\frac{\partial l_n^*}{\partial \gamma} = \frac{n}{\gamma} + n \log \beta + \sum_{i=1}^{n} \log y_i - \sum_{i=1}^{n} (\beta y_i)^\gamma \log(\beta y_i) + \sum_{i=1}^{n} (z_i \alpha - 1) \frac{(\beta y_i)^\gamma \log(\beta y_i) e^{-(\beta y_i)^\gamma}}{1 - e^{-(\beta y_i)^\gamma}},
\frac{\partial l_n^*}{\partial \theta} = 1 - \frac{n}{1 - e^{-\beta}}.
\]

From a nonlinear system of equations \( U_n^*(\Theta) = 0 \), we obtain the iterative procedure of the EM-algorithm as

\[
\hat{\alpha}^{(t+1)} = \frac{\sum_{i=1}^{n} z_i^{(t)} \log(1 - e^{-(\hat{\beta}^{(t)} y_i)^{\hat{\gamma}^{(t)}}})}{-n \sum_{i=1}^{n} z_i^{(t)}},
\frac{n \gamma^{(t)}}{\hat{\beta}^{(t+1)}} - \hat{\gamma}^{(t)} (\hat{\beta}^{(t+1)})^{\hat{\gamma}^{(t)}} - 1 \sum_{i=1}^{n} y_i^{\hat{\gamma}^{(t)}} + \hat{\gamma}^{(t)} (\hat{\beta}^{(t+1)})^{\hat{\gamma}^{(t)}} - 1 \sum_{i=1}^{n} (z_i^{(t)} \hat{\alpha}^{(t)} - 1) \frac{y_i^{\hat{\gamma}^{(t)}} e^{-(\hat{\beta}^{(t+1)} y_i)^{\hat{\gamma}^{(t)}}}}{1 - e^{-(\hat{\beta}^{(t+1)} y_i)^{\hat{\gamma}^{(t)}}}} = 0,
\frac{n}{\hat{\gamma}^{(t+1)}} + n \log \hat{\beta}^{(t)} + \sum_{i=1}^{n} \log y_i - \sum_{i=1}^{n} \log(\hat{\beta}^{(t)} y_i)(\hat{\beta}^{(t)} y_i)^{\hat{\gamma}^{(t+1)}} + \sum_{i=1}^{n} (z_i^{(t)} \hat{\alpha}^{(t)} - 1) \frac{(\hat{\beta}^{(t)} y_i)^{\hat{\gamma}^{(t+1)}} \log(\hat{\beta}^{(t)} y_i) e^{-(\hat{\beta}^{(t+1)} y_i)^{\hat{\gamma}^{(t+1)}}}}{1 - e^{-(\hat{\beta}^{(t+1)} y_i)^{\hat{\gamma}^{(t+1)}}}} = 0,
\frac{n \gamma^{(t)}}{\hat{\theta}^{(t+1)}} - \frac{n}{1 - e^{-\hat{\beta}^{(t+1)}}} = 0,
\]

where \( \hat{\alpha}^{(t+1)}, \hat{\beta}^{(t+1)} \) and \( \hat{\gamma}^{(t+1)} \) are found numerically. Hence, for \( i = 1, \cdots, n \), we have that

\[ z_i^{(t)} = 1 + \hat{\gamma}^{(t)} (1 - e^{-(\hat{\beta}^{(t)} y_i)^{\hat{\gamma}^{(t)}}}) \hat{\alpha}^{(t)}. \]

12. Sub-models of the EWP distribution

The EWP distribution contains some sub-models for the special values of \( \alpha, \beta \) and \( \gamma \). Some of these distributions are discussed here in details.

12.1. The CWP distribution

The CWP distribution is a special case of EWP distribution for \( \alpha = 1 \). Our approach here is complementary to that of Morais and Barreto-Souza [31] in introducing the Weibull-Poisson
The pdf, cdf and hazard rate function of the CWP distribution are given, respectively by

\[ f(y) = \frac{\gamma \theta \beta y^{\gamma-1} e^{-(\beta y)^\gamma} \theta^{(1-e^{-(\beta y)^\gamma})}}{e^{\theta}-1}, \]

\[ F(y) = \frac{e^{\theta(1-e^{-(\beta y)^\gamma})} - 1}{e^{\theta}-1}, \]

and

\[ h(y) = \frac{\gamma \theta \beta y^{\gamma-1} e^{-(\beta y)^\gamma} \theta^{(1-e^{-(\beta y)^\gamma})}}{e^{\theta}-e^{\theta(1-e^{-(\beta y)^\gamma})}}. \]

According to (11) and (12), the mean and variance of the CWP distribution are given, respectively by

\[ E(Y) = \frac{\theta \Gamma(1 + \frac{\gamma}{\beta})}{\beta (e^{\theta} - 1)} \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} (-1)^j \frac{\theta^{n-1}}{(n-1)!} \left( \frac{n-1}{j} \right) (j+1)^{-\left(1+\frac{\gamma}{\beta}\right)}, \tag{20} \]

and

\[ Var(Y) = \frac{\theta \Gamma(1 + \frac{2}{\beta})}{\beta^2 (e^{\theta} - 1)} \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} (-1)^j \frac{\theta^{n-1}}{(n-1)!} \left( \frac{n-1}{j} \right) (j+1)^{-\left(1+\frac{2}{\beta}\right)} - E^2(Y), \]

where \( E(Y) \) is given by Eq. (20). One can obtain the Weibull distribution from the CWP distribution by taking \( \theta \) to be close to zero.

12.2. The GEP distribution

GEP distribution is a special case of the EWP distribution for \( \gamma=1 \). This distribution is introduced and analyzed in details by Mahmoudi and jafari [25]. The pdf, cdf and hazard rate function of the GEP distribution are given, by

\[ f(y) = \frac{\alpha \beta \theta e^{-\beta y}(1 - e^{-(\beta y)^\alpha} - 1) e^{\theta(1-e^{-\beta y})^\alpha}}{e^{\theta}-1}, \]

\[ F(y) = \frac{e^{\theta(1-e^{-\beta y})^\alpha} - 1}{e^{\theta}-1}, \]

and

\[ h(y) = \frac{\alpha \beta \theta e^{-\beta y}(1 - e^{-(\beta y)^\alpha} - 1) e^{\theta(1-e^{-\beta y})^\alpha}}{e^{\theta}-e^{\theta(1-e^{-\beta y})^\alpha}}, \]

respectively. The mean and variance of GEP distribution are

\[ E(Y) = \frac{\alpha \theta}{\beta (e^{\theta} - 1)} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j \frac{\theta^{n-1}}{(n-1)!} \left( \frac{n-1}{j+1} \right)^{\alpha}, \]

\[ = \frac{\theta}{\beta (e^{\theta} - 1)} \sum_{n=1}^{\infty} \frac{\theta^{n-1}}{n!} \psi(n\alpha + 1) - \frac{1}{\beta} \psi(1), \tag{21} \]

where \( \psi(x) \) is the digamma function.
and

\[ \text{Var}(Y) = \frac{2\alpha^2}{\beta^2(e^\theta - 1)} \sum_{n=1}^\infty \sum_{j=0}^{n-1} (-1)^j \frac{\alpha^n}{(n-1)!} \frac{(n-1)}{j+1}^2 - E^2(Y) \]
\[ = \frac{1}{\beta^2} \psi'(1) - \frac{\theta}{\beta^2(e^\theta - 1)} \sum_{n=1}^\infty \frac{\alpha^n}{n!} \left( \psi(n+1) + (\psi(n+1) - \psi(1))^2 \right) - E^2(Y), \]

where \( E(Y) \) is given in (21), \( \psi(.) \) is the digamma function and \( \psi'(.) \) is its derivative.

12.3. The CEP distribution

The CEP distribution is another special case of EWP distribution obtained by replacing \( \alpha = \gamma = 1 \), in EWP distribution. Cancho et al. \[9\] introduced this distribution as the name Poisson-exponential distribution. This distribution has an increasing failure rate and arises on a latent complementary risk problem base. The pdf, cdf and hazard rate function of the CEP distribution are given, respectively by

\[ f(y) = \frac{\theta \beta}{e^\theta - 1} e^{-\beta y} e^{\theta(1-e^{-\beta y})}, \]
\[ F(y) = \frac{e^{\theta(1-e^{-\beta y})} - 1}{e^\theta - 1}, \]

and

\[ h(y) = \frac{\theta \beta e^{-\beta y} e^{\theta(1-e^{-\beta y})}}{e^\theta - e^{\theta(1-e^{-\beta y})}}, \]

The mean and variance of the CEP distribution are given by

\[ E(Y) = \frac{\theta}{\beta(e^\theta - 1)} \sum_{n=1}^\infty \sum_{j=0}^{n-1} (-1)^j \frac{\alpha^n}{(n-1)!} \frac{(n-1)}{j+1}^2 \]
\[ = \frac{\theta}{\beta(e^\theta - 1)} \sum_{n=1}^\infty \frac{\alpha^n}{n!} \psi(n+1) - \frac{1}{\beta} \psi'(1), \]

and

\[ \text{Var}(Y) = \frac{2\theta}{\beta^2(e^\theta - 1)} \sum_{n=1}^\infty \sum_{j=0}^{n-1} (-1)^j \frac{\alpha^n}{(n-1)!} \frac{(n-1)}{j+1}^2 - E^2(Y) \]
\[ = \frac{1}{\beta^2} \psi'(1) - \frac{\theta}{\beta^2(e^\theta - 1)} \sum_{n=1}^\infty \frac{\alpha^n}{n!} \left( \psi(n+1) + (\psi(n+1) - \psi(1))^2 \right) - E^2(Y), \]

respectively, where \( E(Y) \) is given in Eq. (22).

13. Applications of the EWP distribution

To show the superiority of the EWP distribution, we compare the results of fitting this distribution to some of theirs sub-models such as EW, GE and Weibull distributions, using
two real data sets. The required numerical evaluations are implemented using the SAS and R softwares. The empirical scaled TTT transform (Aarset, [1]) and Kaplan-Meier curve can be used to identify the shape of the hazard function.

The first data consists of 63 record of the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Barreto-Souza et al. [6] used this data to fit the beta-generalized exponential (BGE) distribution. The TTT plot and Kaplan-Meier curve for the first data in Fig. 3 shows an increasing hazard rate function and, therefore, indicates that appropriateness of the EWP distribution to fit this data. Table 1 lists the MLEs with the standard deviations of the parameters, the values of K-S (Kolmogorov-Smirnov) statistic with its respective $p$-value, -2log(L), AIC (Akaike Information Criterion), AD (Anderson-Darling statistic) and CM (Cramér-von Mises statistic) for the first data. These values show that the EWP distribution provide a better fit than the EW, GE and Weibull for fitting the first data.

We apply the AD and CM statistics, in order to verify which distribution fits better to this data. The AD and CM test statistics are described in details in Chen and Balakrishnan [11]. In general, the smaller the values of AD and CM, the better the fit to the data. According to these statistics in Table 1, the EWP distribution fit the first data set better than the others.

Using the likelihood ratio (LR) test, we test the null hypothesis H0: EW versus the alternative hypothesis H1: EWP, or equivalently, H0: $\theta = 0$ versus H1: $\theta \neq 0$. The value of the LR test statistic and the corresponding $p$-value are 3.4 and 0.0651, respectively. Therefore, the null hypothesis (EW model) is rejected in favor of the alternative hypothesis (EWP model).
Table 1: MLEs(stds.), K-S statistics, p-values, $-2 \log(L)$ and AIC for the strengths of 1.5 cm glass fibres.

| Dist. | MLEs(stds.) | K-S p-value | $-2 \log(L)$ | AIC | AD | CM |
|-------|-------------|-------------|--------------|-----|----|----|
| EWP   | $\hat{\alpha} = 0.578(0.3424), \hat{\beta} = 0.647(0.0508)$ | 0.1154 | 0.3731 | 26.0 | 34.0 | 0.682 | 0.195 |
|       | $\hat{\gamma} = 5.502(1.395), \hat{\theta} = 2.782(1.568)$ |              |              |     |     |     |     |
| EW    | $\hat{\alpha} = 0.671(0.2489), \hat{\beta} = 0.582(0.0292)$ | 0.1901 | 0.0211 | 29.4 | 35.4 | 1.086 | 0.279 |
|       | $\hat{\gamma} = 7.285(1.707)$ |              |              |     |     |     |     |
| GE    | $\hat{\alpha} = 31.349(9.519), \hat{\beta} = 2.612(0.2380)$ | 0.2291 | 0.0027 | 62.8 | 66.8 | 4.340 | 0.881 |
|       | $\hat{\gamma} = 5.781(0.5761)$ |              |              |     |     |     |     |
| Weibull | $\hat{\alpha} = 0.614(0.0140), \hat{\beta} = 5.781(0.5761)$ | 0.1589 | 0.0830 | 30.4 | 34.4 | 3.135 | 0.297 |

for a significance level $> 0.0651$. For test the null hypothesis $H_0$: GE versus the alternative hypothesis $H_1$: EWP, or equivalently, $H_0$: $(\gamma, \theta) = (1, 0)$ versus $H_1$: $(\gamma, \theta) \neq (1, 0)$, the value of the LR test statistic is 36.8 ($p$-value = 1e-08), which includes that the null hypothesis (GE model) is rejected in favor of the alternative hypothesis (EWP model) for any significance level.

We also test the null hypothesis $H_0$: Weibull versus the alternative hypothesis $H_1$: EWP, or equivalently, $H_0$: $(\alpha, \theta) = (1, 0)$ versus $H_1$: $(\alpha, \theta) \neq (1, 0)$. The value of the LR test statistic is 4.4 ($p$-value = 0.1108), which includes that the null hypothesis (Weibull model) is rejected in favor of the alternative hypothesis (EWP model) for a significance level $> 0.1108$, and for any significance level $< 0.1108$, the null hypothesis is not rejected but the values of AD and CM in Table 1 show that the EWP distribution gives the better fit to the first data set than the Weibull distribution.

As a second application, we consider the data show the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 90 stress level until all had failed. The failure times in hours are shown in Andrews and Herzberg [3] and Barlow et al. [4]. The TTT plot and Kaplan-Meier curve for this data in Fig. 3 shows bathtub-shaped hazard rate function and, therefore, indicates that appropriateness of the EWP distribution to fit this data. The MLEs with the standard deviations of the parameters, the values of K-S statistic, $p$-value, $-2\log(L)$, AIC, AD and CM are listed in Table 2. From these values, we note that the EWP model is better than the EW, GE and Weibull distributions in terms of fitting to this data.

Plots of the estimated pdf, cdf and survival function of the EWP, EW, GE and Weibull models fitted to the data sets corresponding to Tables 1 and 2, respectively, are given in Fig. 4. These plots suggest that the EWP distribution is superior to the EW, GE and Weibull
Table 2: MLEs(stds.), K-S statistics, p-values, $-2 \log(L)$ and AIC for the strengths of 1.5 cm glass fibres.

| Dist. | MLEs(stds.) | K-S | p-value | $-2 \log(L)$ | AIC | AD | CM |
|-------|-------------|-----|---------|--------------|-----|----|----|
| EWP   | $\hat{\alpha} = 0.859(0.3681)$, $\hat{\beta} = 1.303(0.7398)$ | 0.0725 | 0.6640 | 204.6 | 212.6 | 0.841 | 0.218 |
|       | $\hat{\gamma} = 0.8717(0.2409)$, $\hat{\theta} = 1.266(1.2007)$ |       |         |           |     |     |     |
| EW    | $\hat{\alpha} = 0.793(0.2870)$, $\hat{\beta} = 0.821(0.2651)$ | 0.0844 | 0.4677 | 205.6 | 211.6 | 0.955 | 0.247 |
|       | $\hat{\gamma} = 1.060(0.2399)$ |       |         |           |     |     |     |
| GE    | $\hat{\alpha} = 0.866(0.1098)$, $\hat{\beta} = 0.888(0.1201)$ | 0.0887 | 0.4041 | 205.6 | 209.6 | 1.021 | 0.263 |
| Weibull | $\hat{\beta} = 1.010(0.1141)$, $\hat{\gamma} = 0.926(0.0726)$ | 0.0907 | 0.3777 | 206.0 | 210.0 | 1.224 | 0.2789 |

distributions in fitting these two data sets.

14. Conclusion

A new four-parameter distribution called the EWP distribution is introduced. This distribution is a generalization of the EW distribution and contains several lifetime sub-models such as: GEP, CWP, CEP, ERP and RP. A characteristic of the EWP distribution is that its failure rate function can be decreasing, increasing, bathtub-shaped and unimodal depending on its parameter values. Several properties of the new distribution such as its probability density function, its reliability and failure rate functions, quantiles and moments is obtained. The maximum likelihood estimation procedure via a EM-algorithm is presented. Fitting the EWP model to two real data sets indicates the flexibility and capacity of the proposed distribution in data modeling. In view of the density and failure rate function shapes, it seems that the proposed model can be considered as a suitable candidate model in reliability analysis, biological systems, data modeling, and related fields.

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Figure 4: Fitted pdf and cdf of the EWP, EW, GE and Weibull distributions for the data sets corresponding to Tables 1 and 2, respectively.
Appendix

The elements of $4 \times 4$ observed information matrix $I_n(\Theta)$ are given by

\[
I_{\alpha\alpha} = -\frac{n}{\alpha} + \theta \sum_{i=1}^{n} \left( 1 - e^{-(\beta y_i)\gamma} \right)^{\alpha} \log \left( 1 - e^{-(\beta y_i)\gamma} \right)^2 ,
\]

\[
I_{\alpha\beta} = \gamma \sum_{i=1}^{n} y_i e^{-(\beta y_i)\gamma} \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1} + \theta \gamma \sum_{i=1}^{n} y_i e^{-(\beta y_i)\gamma} \left( 1 - e^{-(\beta y_i)\gamma} \right)^{\alpha-1} \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1}
+ \theta \alpha \gamma \sum_{i=1}^{n} y_i e^{-(\beta y_i)\gamma} \left( 1 - e^{-(\beta y_i)\gamma} \right)^{\alpha-1} \log \left( 1 - e^{-(\beta y_i)\gamma} \right) \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1} ,
\]

\[
I_{\alpha\theta} = \sum_{i=1}^{n} \left( 1 - e^{-(\beta y_i)\gamma} \right)^{\alpha} \log \left( 1 - e^{-(\beta y_i)\gamma} \right) ,
\]

\[
I_{\alpha\gamma} = \sum_{i=1}^{n} \frac{e^{-(\beta y_i)\gamma} \log(\beta y_i)(\beta y_i)^{\gamma}}{1-e^{-(\beta y_i)\gamma}} + \theta \sum_{i=1}^{n} e^{-(\beta y_i)\gamma} \left( 1 - e^{-(\beta y_i)\gamma} \right)^{\alpha-1} \log \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right) \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1}
+ \alpha \theta \sum_{i=1}^{n} e^{-(\beta y_i)\gamma} \left( 1 - e^{-(\beta y_i)\gamma} \right)^{\alpha-1} \log \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right) \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1} ,
\]

\[
I_{\beta\beta} = -\frac{n}{\beta^2} - \gamma (\gamma - 1) \sum_{i=1}^{n} y_i^2 \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-2} + (\gamma - 1) \sum_{i=1}^{n} \frac{y_i^2 e^{-(\beta y_i)\gamma}}{1-e^{-(\beta y_i)\gamma}}
- (\alpha - 1) \gamma^2 \sum_{i=1}^{n} y_i^2 e^{-(\beta y_i)\gamma} \left( 1 - e^{-(\beta y_i)\gamma} \right)^{\alpha-1} \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1}
- \alpha \theta \gamma (\gamma - 1) \sum_{i=1}^{n} y_i^2 e^{-(\beta y_i)\gamma} \left( 1 - e^{-(\beta y_i)\gamma} \right) \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1} \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1}
+ \alpha (\alpha - 1) \theta \gamma^2 \sum_{i=1}^{n} y_i^2 e^{-(\beta y_i)\gamma} \left( 1 - e^{-(\beta y_i)\gamma} \right)^{\alpha-2} \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1} ,
\]

\[
I_{\beta\theta} = \alpha \gamma \sum_{i=1}^{n} y_i e^{-(\beta y_i)\gamma} \left( 1 - e^{-(\beta y_i)\gamma} \right)^{\alpha-1} \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1} ,
\]

\[
I_{\beta\gamma} = -\sum_{i=1}^{n} y_i \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1} + \gamma \sum_{i=1}^{n} y_i \log(\beta y_i) \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1} + (\alpha - 1) \sum_{i=1}^{n} \frac{y_i e^{-(\beta y_i)\gamma}}{1-e^{-(\beta y_i)\gamma}}
+ (\alpha - 1) \gamma \sum_{i=1}^{n} \frac{e^{-(\beta y_i)\gamma} \log(\beta y_i)(\beta y_i)^{\gamma}}{1-e^{-(\beta y_i)\gamma}}
- \alpha \gamma \sum_{i=1}^{n} y_i e^{-(\beta y_i)\gamma} \left( 1 - e^{-(\beta y_i)\gamma} \right)^{\alpha-1} \log \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right) \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1}
+ \alpha \theta \gamma \sum_{i=1}^{n} y_i e^{-(\beta y_i)\gamma} \left( 1 - e^{-(\beta y_i)\gamma} \right) \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1} \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1}
- \alpha \theta \gamma \sum_{i=1}^{n} y_i e^{-(\beta y_i)\gamma} \left( 1 - e^{-(\beta y_i)\gamma} \right)^{\alpha-2} \log \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right) \left( \frac{\gamma}{1-e^{-(\beta y_i)\gamma}} \right)^{-1} ,
\]

\[24\]
\[ I_{\theta \theta} = \frac{n e^{-\theta}}{(e^{-\theta}-1)^2} - \frac{n}{\theta^2}, \]

\[ I_{\theta \gamma} = \alpha \sum_{i=1}^{n} e^{-(\beta y_i)\gamma} \left(1 - e^{-(\beta y_i)\gamma}\right)^{\alpha-1} \log (\beta y_i) (\beta y_i) \gamma, \]

\[ I_{\gamma \gamma} = -\frac{n}{\gamma^2} - \sum_{i=1}^{n} \log (\beta y_i)^2 (\beta y_i) \gamma + (\alpha - 1) \sum_{i=1}^{n} \frac{e^{-(\beta y_i)\gamma} \log(\beta y_i)^2(\beta y_i)^\gamma}{1-e^{-(\beta y_i)\gamma}} \]

\[ - (\alpha - 1) \sum_{i=1}^{n} \frac{e^{-2(\beta y_i)\gamma} \log(\beta y_i)^2(\beta y_i)^{2\gamma}}{1-e^{-(\beta y_i)\gamma}} - (\alpha - 1) \sum_{i=1}^{n} \frac{e^{-(\beta y_i)\gamma} \log(\beta y_i)^2(\beta y_i)^{2\gamma}}{1-e^{-(\beta y_i)\gamma}} \]

\[ + \alpha \theta \sum_{i=1}^{n} e^{-\beta y_i)\gamma} \left(1 - e^{-(\beta y_i)\gamma}\right)^{\alpha-1} \log (\beta y_i)^2 (\beta y_i) \gamma \]

\[ - \alpha \theta \sum_{i=1}^{n} e^{-\beta y_i)\gamma} \left(1 - e^{-(\beta y_i)\gamma}\right)^{\alpha-1} \log (\beta y_i)^2 (\beta y_i) \gamma \]

\[ + \alpha(\alpha - 1) \theta \sum_{i=1}^{n} e^{-2(\beta y_i)\gamma} \left(1 - e^{-(\beta y_i)\gamma}\right)^{\alpha-2} \log (\beta y_i)^2 (\beta y_i) \gamma. \]
References

[1] M.V. Aarset, How to identify bathtub hazard rate, IEEE Transactions Reliability 36 (1987) 106-108.

[2] K. Adamidis, S. Loukas, A lifetime distribution with decreasing failure rate, Statistics and Probability Letters 39 (1998) 35-42.

[3] D.F. Andrews, A.M. Herzberg, Data: A Collection of Problems from Many Fields for the Student and Research Worker, Springer Series in Statistics, New York, 1985.

[4] R.E. Barlow, R.H. Toland, T. Freeman, A Bayesian analysis of stress-rupture life of kevlar 49/epoxy spherical pressure vessels, In: Proceeding of Canadian Conference in Applied Statistics, Marcel Dekker, New York, 1984.

[5] W. Barreto-Souza, F. Cribari-Neto, A generalization of the exponential-Poisson distribution, Statistics and Probability Letters 79 (2009) 2493-2500.

[6] W. Barreto-Souza, A.H.S., Santos, G.M. Cordeiro, The beta generalized exponential distribution, Journal of Statistical Computation and Simulation 80 (2010) 159-172.

[7] W. Barreto-Souza, A.L. Morais, G.M. Cordeiro, The Weibull-geometric distribution, Journal of Statistical Computation and Simulation 81 (2011) 645-657.

[8] A. Basu, J. Klein, Some recent development in competing risks theory, In: J. Crowley, R.A. Johnson(Eds.), Survival Analysis, IMS, Hayward, 1982, pp. 216-229.

[9] V.G. Cancho, F. Louzada-Neto, G.D.C. Barriga, The Poisson-exponential lifetime distribution, Computational Statistics and Data Analysis 55 (2011) 677-686.

[10] M. Chahkandi, M. Ganjali, On some lifetime distributions with decreasing failure rate, Computational Statistics and Data Analysis 53 (2009) 4433-4440.

[11] G. Chen, N. Balakrishnan, A general purpose approximate goodness-of-fit test, Journal of Quality Technology 27 (1995) 154-161.

[12] A. Choudhury, A Simple derivation of moments of the exponentiated Weibull distribution, Metrika 62 (2005) 17-22.
[13] A.P. Dempster, N.M. Laird, D.B. Rubin, Maximum likelihood from incomplete data via the EM algorithm (with discussion), Journal of Royal Statistical Society Ser. B 39 (1977) 1-38.

[14] M.E. Ghitany, On a recent generalization of gamma distribution, Communications in Statistics Theory and Methods 27 (1998) 223-233.

[15] G.M. Giorgi, Concentration index, Bonferroni. Encyclopedia of Statistical Sciences, vol. 2, Wiley, New York, 1998, pp. 141-146.

[16] G.M. Giorgi, M. Crescenzi, A look at the Bonferroni inequality measure in a reliability framework. Statistica LXL 4 (2001) 571-583.

[17] J.A. Greenwood, J.M. Landwehr, N.C. Matalas, J.R. Wallis, Probability weighted moments: Definition and relation to parameters of several distribution expressible in inverse form, Water Resources Research 15 (1979) 1049-1054.

[18] P.L. Gupta, R.C. Gupta, On the moments of residual life in reliability and some characterization results, Communications in Statistics Theory and Methods 12 (1983) 449-461.

[19] R.D. Gupta, D. Kundu, Exponentiated exponential family: an alternative to gamma and Weibull distributions, Biometrika Journal 43 (2001) 117-130.

[20] J.R.M. Hosking, J.R. Wallis, E.F. Wood, Estimation of the generalized extreme value distribution by the method of probability weighted moments, Technometrics 27 (1985) 251-261.

[21] J.R.M. Hosking, The Theory of Probability Weighted Moments, IBM Research Report. RC 12210, Yorktown Heights, NY., 1986.

[22] C. Kundu, A.K. Nanda, Some reliability properties of the inactivity time, Communications in Statistics Theory and Methods 39 (2010) 899-911.

[23] C. Kus, A new lifetime distribution, Computational Statistics and Data Analysis 51 (2007) 4497-4509.

[24] F. Louzada, M. Roman, V.G. Cancho, The complementary exponential geometric distribution: model, properties, and comparison with its counter part, Computational Statistics and Data Analysis 55 (2011) 2516-2524.
[25] E. Mahmoudi, A.A. Jafari, Generalized exponential-power series distributions, Computational Statistics and Data Analysis 56 (2012) 4047-4066.

[26] E. Mahmoudi, A. Sepahdar, Exponentiated Weibull-logarithmic distribution, submitted for publication.

[27] E. Mahmoudi, M. Shiran, Exponentiated Weibull-geometric distribution and its applications, submitted for publication, [http://arxiv.org/abs/1206.4008v1](http://arxiv.org/abs/1206.4008v1)

[28] E. Mahmoudi, M. Shiran, Exponentiated Weibull-power series distribution, submitted for publication.

[29] G.J. McLachlan and T. Krishnan, The EM Algorithm and Extension, Wiley, NewYork, 1997.

[30] J. Mi, Bathtub failure rate and upside-down bathtub mean residual life, IEEE Transactions on Reliability 44 (1995) 388-391.

[31] A.L. Morais, W. Barreto-Souza, A compound class of Weibull and power series distributions, Computational Statistics and Data Analysis 55 (2011) 1410-1425.

[32] G.S. Mudholkar, D.K. Srivastava, Exponentiated Weibull family for analyzing bathtub failure-rate data, IEEE Transactions on Reliability 42 (1993) 299-302.

[33] G.S. Mudholkar, D.K. Srivastava, M. Freimer, The exponentiated Weibull family: a reanalysis of the bus-motor-failure data, Technometrics 37 (1995) 436-445.

[34] G.S. Mudholkar, A.D. Hustson, The exponentiated Weibull family: some properties and a flood data application, Communications in Statistics-Theory and Methods 25 (1996) 3059-3083.

[35] A.K. Nanda, H. Singh, N. Misra, P. Paul, Reliability properties of reversed residual lifetime, Communications in Statistics-Theory and Methods 32 (2003) 2031-2042.

[36] M.M. Nassar, F.H. Eissa, On the exponentiated Weibull distribution, Communications in Statistics-Theory and Methods 32 (2003) 1317-1336.

[37] K.S. Park, Effect of burn-in on mean residual life, IEEE Transactions on Reliability 34 (1985) 522-523.
[38] R. Tahmasbi S. Rezaei, A two-parameter lifetime distribution with decreasing failure rate, Computational Statistics and Data Analysis 52 (2008) 3889-3901.

[39] L.C. Tang, Y. Lu, E.P. Chew, Mean residual life distributions, IEEE Transactions on Reliability 48 (1999) 68-73.