On the Cauchy problem for a weakly dissipative Camassa-Holm equation in critical Besov spaces

Zhiying Meng\textsuperscript{a} and Zhaoyang Yin\textsuperscript{a, b}

\textsuperscript{a}Department of Mathematics, Sun Yat-sen University, Guangzhou, People’s Republic of China; \textsuperscript{b}Faculty of Information Technology, Macau University of Science and Technology, Macau, People’s Republic of China

ABSTRACT
In this paper, we mainly consider the Cauchy problem of a weakly dissipative Camassa-Holm equation. We first establish the local well-posedness of equation in Besov spaces $B^{s,r}_p$ with $s > 1 + \frac{1}{p}$ and $s = 1 + \frac{1}{p}, r = 1, p \in [1, \infty)$. Then, we prove the global existence for small data, and present two blow-up criteria. Finally, we get two blow-up results, which can be used in the proof of the ill-posedness in critical Besov spaces.

ARTICLE HISTORY
Received 15 June 2022
Accepted 23 August 2022

COMMUNICATED BY
A. Constantin

KEYWORDS
A weakly dissipative Camassa-Holm equation; local well-posedness; global existence; blow up; ill-posedness

MATHEMATICS SUBJECT CLASSIFICATIONS
35Q53; 35B10; 35B65; 35C05

1. Introduction
In this paper, we consider the Cauchy problem for the following weakly dissipative Camassa-Holm equation

\begin{equation}
\begin{cases}
    u_t - u_{txxx} + 3uu_x + \lambda(u - u_{xx}) = 2uu_x + uu_{xxx} + \alpha u + \beta u^2 u_x + \gamma u^3 u_x + \Gamma u_{xxx}, & t > 0,
    
    u(0, x) = u_0, & x \in \mathbb{R},
\end{cases}
\end{equation}

where $\alpha, \beta, \gamma, \Gamma$ are arbitrary real constants and $\lambda > 0$. This equation was proposed by Freire in [1]. Let $h(u) = (\alpha + \Gamma)u + \beta \frac{1}{3} u^3 + \gamma \frac{1}{4} u^4, \Lambda^2 = (1 - \partial_{xx})^{-1}$, the Equation (1) can be rewritten as

\begin{equation}
\begin{cases}
    u_t + (u + \Gamma)u_x + \lambda u = Q, & t > 0,
    
    u(0, x) = u_0, & x \in \mathbb{R},
\end{cases}
\end{equation}

where $Q = \Lambda^{-2} \partial_x (h(u) - u^2 - \frac{1}{2} u_x^2)$.

If $\lambda = \beta = \Gamma = 0$ and $\alpha \neq 0$, it reduces to the Dullin-Gottwald-Holm equation [2, 3]. If $\beta = \gamma = 0$, $\lambda > 0$ and $\alpha \Gamma \neq 0$, the Equation (2) becomes the weakly dissipative DGH equation [4–6]. If $\alpha = \beta = \gamma = \Gamma = 0$ and $\lambda > 0$, the Equation (2) becomes weakly dissipative CH equation [7, 8]. If $\lambda = \alpha = \beta = \gamma = \Gamma = 0$, we deduce that

\begin{equation}
(1 - \partial_x^2)u_t = 3uu_x - 2u_x u_{xx} - uu_{xxx},
\end{equation}

which is the famous Camassa-Holm (CH) equation. There are many properties of the CH equation. For example, it is completely integrable [9–11] and has a Hamiltonian structure [12]. Its solitary waves
solutions and peakon solutions of the form $ce^{ix-ct}$ were studied in [13–16]. The local well-posedness and ill-posedness global strong solutions, blow-up strong solutions of the CH equation were investigated in [16–31]. The global weak solutions, global conservative solutions and dissipative solutions of the CH equation were also studied in [32–37].

Recently, Freire established the local well-posedness and blow-up phenomena for the Equation (2) in Sobolev spaces [1]. In [38], the authors studied the integrability and global existence of solution for the Equation (2) under some conditions. Therefore, it is worth noting that the local well-posedness and ill-posedness, global strong solutions, blow-up phenomena of (2) in Besov spaces have not yet been studied. Hence, in this paper, we will study these problems. It is worthy to note that the dissipative term $\lambda(u-u_{xx})$ in the Equation (2) do have impacts on the global existence, which is proved below in Theorem 4.1.

This article is organized as follows. In Section 2, we introduce some basic properties in Besov spaces and some prior estimates about the transport equation. In Section 3, we establish the local well-posedness of the Cauchy problem of (2). Section 4 is devoted to discussing the the global strong solution of (2) with small initial values, and give two blow-up criteria of the Equation (2). Then, we gain two blow-up results. Finally, we prove that the Cauchy problem of (2) is ill-posed in $B^s_{2, r}$ with $1 < r \leq \infty$.

2. Preliminaries

In this section, we will present some propositions about the Littlewood-Paley decomposition and Besov spaces. Now we state some useful results in the transport equation theory, which are important to the proof of our main theorem later. We first give the following equation

$$\begin{cases}
f_t + v \cdot \nabla f = g, & t > 0, \\
f(0, x) = f_0(x), & x \in \mathbb{R}.
\end{cases}$$

Lemma 2.1 ([39]): Let $s, t > 0$, $p, r \in [1, \infty]$. Define $R_j = \{v \cdot \nabla, \Delta_j\}f$. There exists a constant $C$ such that

$$\| (2^j \| R_j \|_{L^p})_j \|_{L^r(\mathbb{R})} \leq C \left( \| \nabla v \|_{L^\infty} \| f \|_{B^s_{p, r}} + \| \nabla v \|_{B^{s-1+\varepsilon}_{p, r}} \| \nabla f \|_{B^{-\varepsilon}_{\infty, \infty}} \right).$$

Then, if $f$ solves the Equation (4), we have

$$\| f(t) \|_{B^s_{p, r}} \leq \| f_0 \|_{B^s_{p, r}} + C \int_0^t \left( \| \nabla v \|_{L^\infty} \| f \|_{B^s_{p, r}} + \| \nabla v \|_{B^{s-1+\varepsilon}_{p, r}} \| \nabla f \|_{B^{-\varepsilon}_{\infty, \infty}} + \| g \|_{B^s_{p, r}} \right) dt'.
$$

Lemma 2.2 ([39]): Let $1 \leq p \leq p_1 \leq \infty$, $1 \leq r \leq \infty$, $s > -d \min\left(\frac{1}{p_1}, \frac{1}{p}\right)$. Let $f_0 \in B^s_{p, r}$, $g \in L^1([0, T]; B^d_{p_1, \infty})$, and let $v$ be a time-dependent vector field such that $v \in L^\rho([0, T]; B^{-M}_{\infty, \infty})$ for some $\rho > 1$ and $M > 0$, and

$$\nabla v \in L^1([0, T]; B^d_{p_1, \infty}), \quad \text{if } s < 1 + \frac{d}{p_1}$$

$$\nabla v \in L^1([0, T]; B^{s-1}_{p, r}), \quad \text{if } s > 1 + \frac{d}{p_1} \text{ or } \left( s = 1 + \frac{d}{p_1} \text{ and } r = 1 \right).$$

Then the Equation (4) has a unique solution $f$ in

- the space $C([0, T]; B^s_{p, r})$, if $r < \infty$;
- the space $\bigcap_{s' \leq s} C([0, T]; B^{s'}_{p, \infty}) \cap C_w([0, T]; B^s_{p, \infty})$, if $r = \infty$. 


Lemma 2.3 ([28,39]): Let \( s \leq 1 \), \( s \leq \infty \). There exists a constant \( C \) such that for all solutions \( f \in L^\infty([0,T];B^{s}_{p,r}) \) of (4) in one dimension with initial data \( f_{0} \in B^{s}_{p,r} \), and \( g \in L^{1}([0,T];B^{s}_{p,r}) \), we have for a.e. \( t \in [0,T] \),

\[
\|f(t)\|_{B^{s}_{p,r}} \leq \|f_{0}\|_{B^{s}_{p,r}} + \int_{0}^{t} \|g(t')\|_{B^{s}_{p,r}} \, dt' + \int_{0}^{t} V'(t') \|f(t)\|_{B^{s}_{p,r}} \, dt'
\]

or

\[
\|f(t)\|_{B^{s}_{p,r}} \leq e^{CV(t)} \left( \|f_{0}\|_{B^{s}_{p,r}} + \int_{0}^{t} e^{-CV(t')} \|g(t')\|_{B^{s}_{p,r}} \, dt' \right)
\]

with

\[
V'(t) = \begin{cases} 
\|\nabla v\|_{B^{s+1}_{p,r}}, & \text{if } s > \max \left(-\frac{1}{2}, \frac{1}{p} - 1\right), \\
\|\nabla v\|_{B^{s}_{p,r}}, & \text{if } s > \frac{1}{p} \text{ or } \left(s = \frac{1}{p}, p < \infty, r = 1\right),
\end{cases}
\]

and when \( s = \frac{1}{p} - 1, 1 \leq p \leq 2, r = \infty \), and \( V'(t) = \|\nabla v\|_{B^{s}_{p,1}} \).

If \( f = v \), for all \( s > 0 \), \( V'(t) = \|\nabla v(t)\|_{L^\infty} \).

Lemma 2.4 ([28,39]): Let \( y_{0} \in B^{\frac{1}{p}}_{p,1} \) with \( 1 \leq p < \infty \), and \( f \in L^{1}([0,T];B^{\frac{1}{p}}_{p,1}) \). Define \( \tilde{N} = \mathbb{N} \cup \{\infty\} \), for \( n \in \tilde{N} \), denote by \( y_{n} \in C([0,T];B^{\frac{1}{p}}_{p,1}) \) the solution of

\[
\begin{cases} 
\partial_{t}y_{n} + (A_{n}(u) + \Gamma)\partial_{x}y_{n} = f, & x \in \mathbb{R}, \\
y_{n}(t,x)|_{t=0} = y_{0}(x),
\end{cases}
\]

where \( \Gamma \) is a real number. Assume for some \( \alpha(t) \in L^{1}(0,T) \), \( \sup_{n \in \tilde{N}} \|A_{n}(u)\|_{B^{\frac{1}{p}}_{p,1}} + \Gamma \leq \alpha(t) \). If \( A_{n}(u) \) converges in \( A_{\infty}(u) \) in \( L^{1}([0,T];B^{\frac{1}{p,1}}) \), then the sequence \( (y_{n})_{n \in \tilde{N}} \) converges in \( C([0,T];B^{\frac{1}{p,1}}) \).

3. Local well-posedness

In this section, we consider the local well-posedness of (2) in Besov spaces, and divided it into two cases: (i) \( s > 1 + \frac{1}{p}, p, r \in [1, \infty] \); (ii) \( s = 1 + \frac{1}{p}, p \in [1, \infty] \), \( r = 1 \). Our main results can be stated as follows.

Theorem 3.1 ([39]): Let \( u_{0} \in B^{s}_{p,r} \) with \( 1 \leq p, r \leq \infty \), \( s > 1 + \frac{1}{p} \). Then, there exists a time \( T > 0 \) such that the equation (2) has a unique solution in

\[
E^{s}_{p,r}(T) \triangleq \begin{cases} 
C([0,T];B^{s}_{p,r}) \cap C^{1}([0,T];B^{s-1}_{p,r}), & \text{if } r < \infty, \\
C_{w}([0,T];B^{s}_{p,\infty}) \cap C^{0,1}([0,T];B^{s-1}_{p,\infty}), & \text{if } r = \infty.
\end{cases}
\]

Moreover, the solution depends continuously on the initial data.

Theorem 3.2: Let \( u_{0} \in B^{1 + \frac{1}{p}}_{p,1} \) with \( p \in [1, \infty] \). Then, there exists a time \( T > 0 \) such that (2) has a unique solution \( u \in E^{p}_{T} \triangleq C([0,T];B^{1 + \frac{1}{p}}_{p,1}) \cap C^{1}([0,T];B^{1}_{p,1}) \). Moreover, the solution depends continuously on the initial data.
**Proof:** First Step: Existence of a local solution.

We now construct approximate solutions \((u^n)_{n \in \mathbb{N}}\) which are smooth solutions of the following linear transport equation. Letting \(u^0 = 0\), we obtain

\[
\begin{aligned}
\begin{cases}
u^{n+1}_t + (u^n + \Gamma)\nu^{n+1} = f^n, \\
u^{n+1}(t, x)|_{t=0} = S^{n+1}u_0,
\end{cases}
\end{aligned}
\tag{6}
\]
with \( f^n = \Lambda^{-2} \partial_x(h(u^n) - (u^n)^2 - \frac{1}{2}(u^n)^2) - \lambda u^n \) and \( h(u^n) = (\alpha + \Gamma)u^n + \frac{\beta}{3}(u^n)^3 + \frac{\gamma}{4}(u^n)^4 \).

Applying \(\Delta_j\) to (6), we have

\[
\partial_t u_{j+}^{n+1} + u^n \cdot \partial_x u_{j+}^{n+1} + \Gamma \partial_x u_{j+}^{n+1} = f_{j+}^{n+1} + [u^n, \nabla, \Delta_j]u_{j+}^{n+1},
\tag{7}
\]
where \(\Delta_j u_{j+}^{n+1} = u_{j+}^{n+1} \), \(\Delta_j f^n = f_j\). Multiplying both sides of the equation (7) by \(\text{sgn}(u_{j+}^{n+1})|u_{j+}^{n+1}|^{p-1}\) and integrating over \(\mathbb{R}\). Using the fact that \(\int_{\mathbb{R}} \partial_t u_{j+}^{n+1} \cdot \text{sgn}(u_{j+}^{n+1})|u_{j+}^{n+1}|^{p-1} dx = 0\), it follows that

\[
\partial_t \|u_{j+}^{n+1}\|_{L^p}^p - \int_{\mathbb{R}} \text{div} u^n |u_{j+}^{n+1}|^p dx = p \int_{\mathbb{R}} f_j \cdot \text{sgn}(u_{j+}^{n+1})|u_{j+}^{n+1}|^{p-1} dx \\
+ p \int_{\mathbb{R}} [u^n \cdot \nabla, \Delta_j]u_{j+}^{n+1} \cdot \text{sgn}(u_{j+}^{n+1})|u_{j+}^{n+1}|^{p-1} dx.
\]

According to Lemma 2.3, we infer that

\[
1 + \|u_{j+}^{n+1}\|_{B_{p,1}^{1+\frac{1}{p}}} \leq C \exp \left( C \int_0^t \|u^n(t')\|_{B_{p,1}^{1+\frac{1}{p}}} dt' \right) \left[ 1 + \|u_0\|_{B_{p,1}^{1+\frac{1}{p}}} \right] \\
+ \int_0^t \exp \left( -C \int_0^{t'} \|u^n(t'')\|_{B_{p,1}^{1+\frac{1}{p}}} dt'' \right) \left( 1 + \|u^n\|_{B_{p,1}^{1+\frac{1}{p}}} \right)^4 dt'.
\tag{8}
\]

Suppose that \(u^n \in L^\infty([0, T]; B_{p,1}^{1+\frac{1}{p}})\). Then, we get

\[
\|f^n\|_{B_{p,1}^{1+\frac{1}{p}}} \leq C \left( 1 + \|u^n\|_{B_{p,1}^{1+\frac{1}{p}}} \right)^4,
\tag{9}
\]
which leads to \(f^n \in L^\infty([0, T]; B_{p,1}^{1+\frac{1}{p}})\). Using Lemmas 2.2-2.3, we conclude that there exists a global solution \(u^{n+1} \in C([0, T]; B_{p,1}^{1+\frac{1}{p}}) \cap C^1([0, T]; B_{p,1}^{\frac{1}{2}})\) for all \(T > 0\). Fixed a \(T > 0\) such that \(\frac{2}{3} \Lambda T(1 + \|u_0\|_{B_{p,1}^{1+\frac{1}{p}}})^\frac{1}{2} < 1\) and

\[
1 + \|u^n\|_{B_{p,1}^{1+\frac{1}{p}}} \leq \frac{(1 + \|u_0\|_{B_{p,1}^{1+\frac{1}{p}}})^\frac{1}{2}}{1 - \frac{2}{3} \Lambda T (1 + \|u_0\|_{B_{p,1}^{1+\frac{1}{p}}})^\frac{1}{2}},
\tag{10}
\]
Plugging (10) into (8), we have
\[ 1 + \|u^{n+1}\|_{B^m_{p,\tau}} \leq \frac{(1 + \|u_0\|_{B^m_{p,\tau}})^{\frac{1}{2}}}{1 - \frac{2}{3} C t (1 + \|u_0\|_{B^m_{p,\tau}})^{\frac{1}{2}}}, \quad \forall \, t \in [0, \tau]. \]

Hence, by introduction, we see that \((u^n)_{n \in \mathbb{N}}\) is bounded in \(L^\infty([0, \tau]; B^\frac{1}{p}_p)\).

It follows that the compactness method for the approximating sequence \((u^n)_{n \in \mathbb{N}}\) that we get a solution \(u\) of (2). Owing to the uniformly boundedness of \(u^n\) in \(L^\infty([0, \tau]; B^\frac{1}{p}_p)\), one can get \(\partial_t u^n\) is uniformly bounded in \(L^\infty([0, \tau]; B^\frac{1}{p}_p)\). Hence, we deduce that \(u^n\) is uniformly bounded in \(C([0, \tau]; B^\frac{1}{p}_p) \cap C^2([0, \tau]; B^\frac{1}{p}_p)\). On the other hand, suppose that \((\phi^j)_{j \in \mathbb{N}}\) be a sequence of smooth functions with value in \([0,1]\) supported in \([0,1]\) supported in \(B(0,j+1)\) and value equal to 1 on \(B(0,j)\). According to Theorem 2.94 in [39], we get the map \(u^n \mapsto \phi^j u^n\) is compact from \(B^\frac{1}{p}_p\) to \(B^\frac{1}{p}_p\). Ascoli’s theorem entails that there has some function \(u^j\) such that, up to extraction, \((\phi^j u^n)_{n \in \mathbb{N}}\) converges to \(u^j\). And using the Cantor’s diagonal process, we deduce that there exists a subsequence of \((u^j)_{j \in \mathbb{N}}\) (still denoted by \((u^j)_{j \in \mathbb{N}}\) ) such that for all \(j \in \mathbb{N}\), \(\phi^j u^n\) converges to \(u^j\) in \(C([0, \tau]; B^\frac{1}{p}_p)\). Since the uniform boundedness of \(u^n\) and Fatou property guarantee that \(u \in L^\infty([0, \tau]; B^\frac{1}{p}_p)\). Making use of interpolation, we get \(\phi u^n\) tends to \(\phi u\) in \(C([0, \tau]; B^\frac{1}{p}_p + \frac{1}{p} - \epsilon)\) for any \(\epsilon > 0\).

We now check that \(u\) is the solution of the Equation (2). For fixed \(\psi \in B^\frac{1}{p}_p\), we have
\[ \{ (\phi u^n)_t - (\phi u)_t, \psi \} + \{ (\phi u^n + \Gamma)((\phi u^n)_x - ((\phi u) + \Gamma)(\phi u)_x), \psi \} - \{ f(\phi u^n) - f(\phi u), \psi \} \overset{n \to \infty}{\longrightarrow} 0. \tag{11} \]

The main difficulty is to prove that \(\Lambda^{-2} \partial_x((\phi u^n)_x^2 - (\phi u)_x^2), \psi \) \(\overset{n \to \infty}{\longrightarrow} 0\) in (11). For simplicity, we only handle the \(\Lambda^{-2} \partial_x((\phi u^n)^2 - (\phi u)^2), \psi \) \(\overset{n \to \infty}{\longrightarrow} 0\), the others are similar. Hence,
\[
\begin{align*}
|\Lambda^{-2} \partial_x((\phi u^n)_x^2 - (\phi u)_x^2), \psi | & = |\Lambda^{-2} \partial_x((\phi u^n)_x - (\phi u)_x)(\phi u^n)_x + (\phi u)_x), \psi | \\
& \leq \| \Lambda^{-2} \partial_x((\phi u^n)_x - (\phi u)_x)(\phi u^n)_x + (\phi u)_x) \|_{B^\frac{1}{p}_p} \| \psi \|_{B^\frac{1}{p}_p} \\
& \leq \| \partial_x((\phi u^n)_x - (\phi u)_x)(\phi u^n)_x + (\phi u)_x) \|_{B^\frac{1}{p}_p} \| \psi \|_{B^\frac{1}{p}_p} \\
& \leq C \| \partial_x((\phi u^n)_x - (\phi u)_x) \|_{L^\infty} \| ((\phi u^n)_x - (\phi u)_x) \|_{B^\frac{1}{p}_p} \| \psi \|_{B^\frac{1}{p}_p} \\
& \leq C \| \phi u^n + \phi u \|_{B^\frac{1}{p}_p} \| \phi u^n - \phi u \|_{B^\frac{1}{p}_p} \| \psi \|_{B^\frac{1}{p}_p}. \tag{12}
\end{align*}
\]

By using the fact that \(\phi u^n \overset{n \to \infty}{\longrightarrow} \phi u\) in \(C([0, \tau]; B^\frac{1}{p}_p + \frac{1}{p} - \epsilon)\), and that \(\phi u^n\) is bounded in \(L^\infty([0, \tau]; B^\frac{1}{p}_p + \frac{1}{p} - \epsilon)\), we get (12) tends to 0 uniformly on \([0, \tau]\) as \(n \to \infty\) and \(u \in B^\frac{1}{p}_p\).
Second Step: Uniqueness.

In order to prove the uniqueness of solution, we reformulate the equation using a transformation that corresponds to the transformation between Eulerian and Lagrangian coordinates. Let \( u = u(t, x) \) denote the solution, and \( t \mapsto y(t, \xi) \) be the characteristic as follows

\[
\begin{align*}
\begin{cases}
y_t(t, \xi) &= u(t, y(t, \xi)) + \Gamma, & x \in \mathbb{R}, \\
y(t, \xi)|_{t=0} &= \tilde{y}(\xi).
\end{cases}
\end{align*}
\]

(13)

Our new variables are \( y(t, \xi), U(t, \xi) = u(t, y(t, \xi)) \), then the Equation (2) can be rewritten as

\[
U_t(t, \xi) = u_t(t, y(t, \xi)) + y_t(t, \xi) u_x(t, y(t, \xi)) = \tilde{Q}(t, \xi) - \lambda U(t, \xi),
\]

(14)

where

\[
\tilde{Q}(t, \xi) \triangleq P_x \circ y(t, \xi) = \frac{1}{2} \int_{\mathbb{R}} \sgn(y(t, \xi) - x) e^{-|y(t, \xi) - x|} \left(-h(u) + u^2 + \frac{1}{2} u_x^2\right) (t, x) \, dx.
\]

(15)

After the change of variables \( x = y(t, \eta) \), we have

\[
\tilde{Q}(t, \xi) = \frac{1}{2} \int_{\mathbb{R}} \sgn(y(t, \xi) - y(t, \eta)) e^{-|y(t, \xi) - y(t, \eta)|} \left(-h(U) y_\xi + U^2 y_\eta + \frac{U_\xi^2}{2y_\xi}\right) (t, \eta) \, d\eta.
\]

(16)

For fixed \( t \), \( y(t, \cdot) \) is an increasing function, which implies that \( \sgn(y(t, \xi) - y(t, \eta)) = \sgn(\xi - \eta) \). That is,

\[
\tilde{Q}(t, \xi) = \frac{1}{2} \int_{\mathbb{R}} \sgn(\xi - \eta) e^{-|y(t, \xi) - y(t, \eta)|} \left(-h(U) y_\xi + U^2 y_\eta + \frac{U_\xi^2}{2y_\xi}\right) (t, \eta) \, d\eta
\]

\[
= \frac{1}{2} \left( \int_{-\infty}^{\xi} - \int_{\xi}^{+\infty} \right) e^{-|y(t, \xi) - y(t, \eta)|} \left(-h(U) y_\xi + U^2 y_\eta + \frac{U_\xi^2}{2y_\xi}\right) (t, \eta) \, d\eta.
\]

(17)

We now give another new variable \( \zeta \) defined as \( \zeta(t, \xi) = y(t, \xi) - \xi - \Gamma t \), and \( \zeta \) satisfies

\[
\zeta_t(t, \xi) = U(t, \xi).
\]

(18)

Therefore, the derivatives of \( \tilde{Q} \) is given by

\[
\tilde{Q}_\xi = (-h(U) + U^2 - P)(\zeta_\xi + 1) + \frac{U_\xi^2}{2y_\xi},
\]

(19)

with

\[
P(t, \xi) = -\frac{1}{2} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \eta)|} \left(-h(U) y_\xi + U^2 y_\eta + \frac{U_\xi^2}{2y_\xi}\right) (t, \eta) \, d\eta.
\]

From the above equation, we can deduce that

\[
U_\xi(t, \xi) = \tilde{Q}_\xi(t, \xi) - \lambda U_\xi.
\]

(20)

Combining (13) and (22), we infer that \( \zeta(t, \xi) \) satisfies the following integral form

\[
\zeta(t, \xi) = \int_0^t U(\tau, \xi) \, d\tau,
\]

(21)
Note that $\zeta^2 + 1 = y^2_\xi$. Using the uniform boundedness of $u$ in $C([0, T]; B_{p, 1}^{1 + \frac{1}{p}})$ and the embedding with $B_{p, 1}^{1 + \frac{1}{p}} \hookrightarrow W^{1, p} \cap W^{1, \infty}$, we get $u \in C([0, T]; W^{1, p} \cap W^{1, \infty})$. Thereby, we obtain that $\zeta^2$ is uniformly bounded in $L^\infty([0, T]; L^\infty)$, which implies $y_\xi \in L^\infty([0, T]; L^\infty)$. More importantly, we have $\frac{1}{2} \leq \zeta^2, y_\xi \leq C_{u_0}$ for $T > 0$ small enough. Without loss of generality, suppose that $t$ is sufficiently small, otherwise we use the continuity method. Now, we prove that $U(t, \xi)$ is bounded in $L^\infty([0, T]; W^{1, p})$ as follows.

\[
\|U_t\|_{L^p}^p = \int \|U(t, \xi)\|^p \, d\xi = \int |u(t, y(t, \xi))|^p \frac{1}{y^2_\xi} \, dy \leq \|u\|_{L^p}^p \frac{1}{y^2_\xi} \|\zeta\|_{L^\infty} \leq 2\|u\|_{L^p}^p \leq C;
\]

\[
\|U\|_{L^p}^p = \int \|U_\xi\|^p \|y_\xi\|^p \, d\xi = \int \|u_x(t, y(t, \xi))\|^p |y_\xi|^p \|\zeta\|_{L^\infty} \|y_\xi\|_{L^\infty} \leq C_{u_0} \|u_x\|_{L^p}^p \leq C.
\]

Applying the above inequalities, we get $U(t, \xi) \in L^\infty([0, T]; W^{1, p} \cap W^{1, \infty})$, $\zeta \in L^\infty([0, T]; W^{1, p} \cap W^{1, \infty})$ and $\frac{1}{2} \leq \zeta^2, y_\xi \leq C_{u_0}$ for any $t \in [0, T]$.

Let $u_1, u_2$ be two solutions of (2), and $U_i(t, \xi) = u(t, y_i(t, \xi))$ ($i = 1, 2$) satisfy

\[
U_i(t, \xi) = u_h(t, y_i) + U_\xi(t, \xi) u_{tx}(t, y_i(t, \xi)) = \tilde{Q}_i(t, \xi) - \lambda U_i(t, \xi).
\]

Likewise, we have $U_i(t, y_i(t, \xi)) \in L^\infty([0, T]; W^{1, p} \cap W^{1, \infty})$, $\zeta_i(t, \xi) \in L^\infty([0, T]; W^{1, p} \cap W^{1, \infty})$ and $\frac{1}{2} \leq \zeta^2, y_\xi \leq C_{u_0}$ for sufficiently small $T > 0$.

We now demonstrate the following estimate.

\[
\|\tilde{Q}_1(t, \xi) - \tilde{Q}_2(t, \xi)\|_{W^{1, p} \cap W^{1, \infty}} \leq C(\|U_1 - U_2\|_{W^{1, p} \cap W^{1, \infty}} + \|y_1 - y_2\|_{W^{1, p} \cap W^{1, \infty}}).
\]

According to (17), we can deduce that

\[
\tilde{Q}_1(t, \xi) - \tilde{Q}_2(t, \xi) = \frac{1}{4} \left( \int_{-\infty}^\xi - \int_{+\infty}^\xi \right) e^{-|2(\xi - y_1(\eta))|} \left( \frac{U_1^2}{y_1^\eta} - \frac{U_2^2}{y_2^\eta} \right) (\eta) \, d\eta
\]

\[
+ \frac{1}{4} \left( \int_{-\infty}^\xi - \int_{+\infty}^\xi \right) \left( e^{-|y_1(\xi) - y_1(\eta)|} - e^{-|2(\xi - y_2(\eta))|} \right) \frac{U_1^2}{y_1^\eta} (\eta) \, d\eta
\]

\[
+ \frac{1}{2} \left( \int_{-\infty}^\xi - \int_{+\infty}^\xi \right) \left( e^{-|y_1(\xi) - y_1(\eta)|} - e^{-|y_2(\xi) - y_2(\eta)|} \right) (-h(U_1)y_1^\eta + U_1^2y_1^\eta) (\eta) \, d\eta
\]

\[
+ \frac{1}{2} \left( \int_{-\infty}^\xi - \int_{+\infty}^\xi \right) e^{-|y_2(\xi) - y_2(\eta)|} (h(U_2)y_2^\eta - h(U_1)y_1^\eta + U_2^2y_1^\eta - U_2^2y_2^\eta) (\eta) \, d\eta
\]

\[
= \sum_{i=1}^4 T_i.
\]

We first give an estimate of $T_1$ as follows.

\[
T_1 = \frac{1}{4} \int_{-\infty}^\xi e^{-|2(\xi - y_2(\eta))|} \left( \frac{U_1^2}{y_1^\eta} - \frac{U_2^2}{y_2^\eta} \right) (\eta) \, d\eta - \frac{1}{4} \int_{-\infty}^\xi e^{-|2(\xi - y_2(\eta))|} \left( \frac{U_1^2}{y_1^\eta} - \frac{U_2^2}{y_2^\eta} \right) (\eta) \, d\eta
\]
From the above analysis, we get
\[
\| \tilde{C} \|_{\tilde{F}_1} \leq \max \{ \| e^{-|x|} \|, \| e^{-|y|} \| \} \cdot |x - y|, \text{ and } \frac{1}{2} \leq x, y \leq C_{u_0} \text{ for } T > 0 \text{ small enough.}
\]

According to \( U_i(t, \xi) \) and \( \gamma_i(t, \xi) \) are bounded in \( L^\infty([0, T]; W^{1,\infty}) \), we get
\[
\mathcal{I}_i \leq C \left( \left\| \frac{U_{1i} + U_{2i}}{y_{1i}} \right\|_{L^\infty} + \left\| \frac{U_{2i}}{y_{1i}y_{2i}} \right\|_{L^\infty} \right) \int_{-\infty}^{+\infty} e^{-\eta} \left( \| U_{1i} - U_{2i} \| + |y_{1i} - y_{2i}| \right) (\eta) \, d\eta
\]
\[
+ C \left( \left\| \frac{U_{1i} + U_{2i}}{y_{1i}} \right\|_{L^\infty} + \left\| \frac{U_{2i}}{y_{1i}y_{2i}} \right\|_{L^\infty} \right) \int_{-\infty}^{\xi} e^{-\eta} \left( \| U_{1i} - U_{2i} \| + |y_{1i} - y_{2i}| \right) (\eta) \, d\eta
\]
\[
\leq C \left[ 1_{\geq 0}(x) e^{-|x|} \left( \| U_{1i} - U_{2i} \| + |y_{1i} - y_{2i}| \right) + 1_{\leq 0}(x) e^{-|x|} \left( \| U_{1i} - U_{2i} \| + |y_{1i} - y_{2i}| \right) \right].
\]

Since the other terms can be proved similarly, then we conclude that
\[
\tilde{Q}_1(t, \xi) - \tilde{Q}_2(t, \xi) \leq C \left[ 1_{\geq 0}(x) e^{-|x|} \left( \| U_{1i} - U_{2i} \| + |y_{1i} - y_{2i}| \right) \right]
\]
\[
+ C \left[ 1_{\leq 0}(x) e^{-|x|} \left( \| U_{1i} - U_{2i} \| + |y_{1i} - y_{2i}| \right) \right]
\]
\[
+ C \left| U_{1i} - U_{2i} \right|_{L^\infty} \left[ 1_{\geq 0}(x) e^{-|x|} \left( h(U_{1i}) y_{1i} + \frac{U_{2i}^2}{y_{1i}} + U_{1i}^2 y_{1i} \right) \right]
\]
\[
+ 1_{\leq 0}(x) e^{-|x|} \left( h(U_{1i}) y_{1i} + \frac{U_{2i}^2}{y_{1i}} + U_{1i}^2 y_{1i} \right).\]

That is
\[
\| \tilde{F}_1 - \tilde{F}_2 \|_{L^\infty \cap L^P} \leq C \left( \| U_{1i} - U_{2i} \|_{L^\infty \cap L^P} + \| U_{1i} - U_{2i} \|_{L^\infty \cap L^P} + \| y_{1i} - y_{2i} \|_{L^\infty \cap L^P} \right). \tag{26}
\]

It is now clear that the same estimate holds for \( \| \tilde{F}_1 - \tilde{F}_2 \|_{L^\infty \cap L^P} \) as follows
\[
\| \tilde{F}_1 - \tilde{F}_2 \|_{L^\infty \cap L^P} \leq C \left( \| U_{1i} - U_{2i} \|_{L^\infty \cap L^P} + \| U_{1i} - U_{2i} \|_{L^\infty \cap L^P} + \| y_{1i} - y_{2i} \|_{L^\infty \cap L^P} \right). \tag{27}
\]

Making use of (26) and (27) yields that
\[
\| \tilde{F}_1(t, \xi) - \tilde{F}_2(t, \xi) \|_{W^{1,P} \cap W^{1,\infty}} \leq C \left( \| U_{1i} - U_{2i} \|_{W^{1,P} \cap W^{1,\infty}} + \| y_{1i} - y_{2i} \|_{W^{1,P} \cap W^{1,\infty}} \right). \tag{28}
\]

From the above analysis, we get
\[
\| U_{1i} - U_{2i} \|_{W^{1,P} \cap W^{1,\infty}} + \| y_{1i} - y_{2i} \|_{W^{1,P} \cap W^{1,\infty}}
\]
\[
\leq C \left( \| U_{1i}(0) - U_{2i}(0) \|_{W^{1,P} \cap W^{1,\infty}} + \| y_{1i}(0) - y_{2i}(0) \|_{W^{1,P} \cap W^{1,\infty}} \right)
\]
\[
+ C \int_0^T \left( \| U_{1i} - U_{2i} \|_{W^{1,P} \cap W^{1,\infty}} + \| y_{1i} - y_{2i} \|_{W^{1,P} \cap W^{1,\infty}} \right)(t) \, dt. \tag{29}
\]

Plugging (28) into (29) and using the fact that \( y_{1i}(0) = y_{2i}(0) = \xi \), we infer that
\[
\| U_{1i} - U_{2i} \|_{W^{1,P} \cap W^{1,\infty}} + \| y_{1i} - y_{2i} \|_{W^{1,P} \cap W^{1,\infty}}
\]
\[ \leq C e^{CT} \left( \| U_1(0) - U_2(0) \|_{W^{1,p} \cap W^{1,\infty}} + \| y_1(0) - y_2(0) \|_{W^{1,p} \cap W^{1,\infty}} \right) \]
\[ \leq C e^{CT} \| u_1(0) - u_2(0) \|_{B_{p,1}^{1+\frac{1}{p}}} \]
from which it follows,
\[ \| u_1 - u_2 \|_{L^p} \leq C \| u_1 \circ y_1 - u_2 \circ y_2 \|_{L^p} \]
\[ \leq C \| u_1 \circ y_1 - u_2 \circ y_1 + u_2 \circ y_1 - u_2 \circ y_2 \|_{L^p} \]
\[ \leq C \| U_1 - U_2 \|_{L^p} + C \| u_{2x} \|_{L^\infty} \| y_1 - y_2 \|_{L^p} \]
\[ \leq C \| u_1(0) - u_2(0) \|_{B_{p,1}^{1+\frac{1}{p}}} \]
The embedding \( L^p \hookrightarrow B_{p,\infty}^0 \) guarantees that
\[ \| u_1 - u_2 \|_{B_{p,\infty}^0} \leq C \| u_1 - u_2 \|_{L^p} \leq C \| u_1(0) - u_2(0) \|_{B_{p,1}^{1+\frac{1}{p}}} \]
This completes the proof of the uniqueness of the solution of (2) if \( u_1(0) = u_2(0) \).

Third Step: The continuous dependence.
Assume that we are given \( u^n \) and \( u^\infty \), two solutions of (2) with the initial data \( u^n_0 \) and \( u^\infty_0 \) satisfying \( u^n_0 \) tends to \( u^\infty_0 \) in \( B_{p,1}^{1+\frac{1}{p}} \). The First Step and Second Step guarantee that \( u^n \) and \( u^\infty \) are uniformly bounded in \( L^\infty([0, T]; B_{p,1}^{1+\frac{1}{p}}) \), and
\[ \| (u^n - u^\infty)(t) \|_{B_{p,\infty}^0} \leq C \| u^n_0 - u^\infty_0 \|_{B_{p,1}^{1+\frac{1}{p}}} \]
This implies \( u^n \) converges to \( u^\infty \) in \( C([0, T]; B_{p,\infty}^{1+\frac{1}{p}-\varepsilon}) \) for any \( \varepsilon > 0 \). If \( \varepsilon = 1 \), we have
\[ u^n \rightarrow u^\infty \quad \text{in} \quad C([0, T]; B_{p,1}^{\frac{1}{p}}). \] (30)
In order to get \( u^n \) tends to \( u^\infty \) in \( C([0, T]; B_{p,1}^{1+\frac{1}{p}}) \), we now prove that \( u^n_0 \) tends to \( u^\infty_0 \) in \( C([0, T]; B_{p,1}^{\frac{1}{p}}) \). Letting \( v^n = u^n \), we split \( v^n = z^n + w^n \) with \((z^n, w^n)\) satisfying
\[ \begin{align*}
  &w^n + u^n w_x + \Gamma w^n_x = f^n, \\
  &w^n(t, x)|_{t=0} = u^n_0, \\
  &z^n + u^n z^n_x + \Gamma z^n_x = f^n - f^\infty,
\end{align*} \]
and
\[ \begin{align*}
  &z^n(t, x)|_{t=0} = u^n_0 - u^\infty_0, \\
  &z^n(t, x) \quad \text{are uniformly}
\end{align*} \]
with
\[ f^n = -\lambda v^n - h(u^n) + (u^n)^2 - \frac{1}{2} (v^n)^2 + \Lambda^{-2} \left( h(u^n) - (u^n)^2 - \frac{1}{2} (v^n)^2 \right), \quad \forall n \in \mathbb{N}. \]
Combining Lemma 2.1 with the boundeness of \( u^n \) and \( u^\infty \) in \( C([0, T]; B_{p,1}^{1+\frac{1}{p}}) \), we deduce that
\[ \| u^n \|_{B_{p,1}^{1+\frac{1}{p}}} \leq C, \]
and
\[ \| f^n - f^\infty \|_{B_{p,1}^{\frac{1}{p}}} \leq C \| u^n - u^\infty \|_{B_{p,1}^{1+\frac{1}{p}}} \left( 1 + \| u^n \|_{B_{p,1}^{1+\frac{1}{p}}} + \| u^\infty \|_{B_{p,1}^{1+\frac{1}{p}}} \right)^3 \]
\[
\leq C \left( \|u^n - u^\infty\|_{B^p_{p,1}} + \|u_x^n - u_x^\infty\|_{B^{p,1}_{p,1}} \right)
\]
\[
\leq C \left( \|u^n - u^\infty\|_{B^p_{p,1}} + \|z^n\|_{B^{p,1}_{p,1}} + \|w^n - w^\infty\|_{B^{p,1}_{p,1}} \right).
\] (31)

Using Lemma 2.3 and (31), for any \( n \in \mathbb{N}, t \in [0, T] \), we obtain
\[
\|z_n(t)\|_{B^{p,1}_{p,1}} \leq C \left( \|u_0^n - u_0^\infty\|_{B^{p,1}_{p,1}} + \int_0^t \|f^n - f^\infty\|_{B^{p,1}_{p,1}} \, dt' \right)
\]
\[
\leq C \left( \|u_0^n - u_0^\infty\|_{B^{p,1}_{p,1}} + \int_0^t \left( \|u^n - u^\infty\|_{B^{p,1}_{p,1}} + \|z^n\|_{B^{p,1}_{p,1}} + \|w^n - w^\infty\|_{B^{p,1}_{p,1}} \right)(t') \, dt' \right),
\] (32)

where we used the fact that \( \int_{\mathbb{R}} \Delta_j z \cdot \text{sgn}(\Delta_j z) |\Delta_j z|^{p-1} \, dx = 0 \) when the linear term \( \Gamma z \) is estimated in \( B^\frac{3}{2}_{p,1} \). Hence, we have
\[
\|z^n\|_{B^{p,1}_{p,1}} \leq e^{Ct} \left( \|v_0^n - v_0^\infty\|_{B^{p,1}_{p,1}} + \int_0^t \left( \|w^n - w^\infty\|_{B^{p,1}_{p,1}} + \|z^n\|_{B^1_{p,1}} \right) (t') \, dt' \right).
\]

Notice that
\[
u^n \to u^\infty \quad \text{in } C([0, T]; B^\frac{1}{2}_{p,1});
\]
\[v_0^n \to v_0^\infty \quad \text{in } C([0, T]; B^\frac{1}{2}_{p,1});
\]
\[w_0^n \to w_0^\infty \quad \text{in } C([0, T]; B^\frac{1}{2}_{p,1}).
\]

Hence, we have \( z^n \xrightarrow{n \to \infty} 0 \) in \( C([0, T]; B^\frac{1}{2}_{p,1}) \). Lemmas 2.2-2.3 and \( z^\infty = 0 \) entail that \( z^n \) converges to \( z^\infty \) in \( C([0, T]; B^\frac{1}{2}_{p,1}) \). Moreover, we thus conclude that \( u^n \xrightarrow{n \to \infty} u^\infty \) in \( C([0, T]; B^{\frac{1}{2} + \frac{1}{p}}_{p,1}) \).

Thus, we complete the proof of Theorem 3.2.

### 4. Global existence and blow-up

In this section, we consider the global existence and blow-up of solutions for (2).

#### 4.1. Global existence for small data

We first prove the global existence for small data.

**Theorem 4.1:** Let \( u_0 \in B^s_{p,r} \) with \((s, p, r)\) being as in Theorems 3.1-3.2. There is a constant \( \epsilon \) such that if
\[
H_0 \equiv |\alpha| + |\Gamma| + \|u_0\|_{B^p_{p,r}} + \frac{|\beta|}{3} \|u_0\|_{B^p_{p,r}}^2 + \frac{|\gamma|}{4} \|u_0\|_{B^p_{p,r}}^3 \leq \lambda \epsilon,
\] (33)
for all real numbers \( \alpha, \beta, \gamma, \Gamma \) and \( \lambda > 0 \), then, there exists a unique global solution \( u \) of the Equation (2), and for any \( t \in [0, \infty) \) we have
\[
H(t) \equiv |\alpha| + |\Gamma| + \|u(t)\|_{B^p_{p,r}} + \frac{|\beta|}{3} \|u(t)\|_{B^p_{p,r}}^2 + \frac{|\gamma|}{4} \|u(t)\|_{B^p_{p,r}}^3 \leq H_0.
\] (34)
Proof: According to (2) and the proofs of Theorems 3.1-3.2, for any $t \in [0, T]$, we get
\[
\|u(t)\|_{B^p_{r,t}} + \lambda \|u(t)\|_{L^{p}_{t}(B^p_{r,t})} \leq \|u_0\|_{B^p_{r,t}} + C \int_0^t \|u(t')\|_{B^p_{r,t}} \left( |\alpha| + |\Gamma| + \|u(t')\|_{B^p_{r,t}} \right) \frac{\beta}{3} \|u(t')\|_{B^p_{r,t}}^2 + \frac{|\gamma|}{4} \|u(t')\|_{B^p_{r,t}}^3 \, dt',
\]
where we used Lemmas 2.2-2.3. For any $\epsilon > 0$ sufficiently small, suppose that for all $t \in [0, T]$, we have
\[
|\alpha| + |\Gamma| + \|u(t)\|_{L^{p}_{t}(B^p_{r,t})} + \frac{\beta}{3} \|u(t)\|_{B^p_{r,t}}^2 + \frac{|\gamma|}{4} \|u(t)\|_{B^p_{r,t}}^3 \leq 2\lambda \epsilon.
\]
This means
\[
\int_0^T \|u\|_{B^p_{r,t}} \left( |\alpha| + |\Gamma| + \|u(t)\|_{B^p_{r,t}} + \frac{\beta}{3} \|u(t)\|_{B^p_{r,t}}^2 + \frac{|\gamma|}{4} \|u(t)\|_{B^p_{r,t}}^3 \right) \, dt \leq 2\lambda \epsilon \int_0^T \|u(t)\|_{B^p_{r,t}} \, dt,
\]
for some sufficiently small $\epsilon$.

Substituting (37) into (35) yields
\[
\|u(t)\|_{B^p_{r,t}} \leq \|u_0\|_{B^p_{r,t}}, \quad \forall \, t \in [0, T].
\]
Analogously, we infer that
\[
|\alpha| + |\Gamma| + \|u(t)\|_{L^{p}_{t}(B^p_{r,t})} + \frac{\beta}{3} \|u(t)\|_{B^p_{r,t}}^2 + \frac{|\gamma|}{4} \|u(t)\|_{B^p_{r,t}}^3 \leq H_0 + C \int_0^T \left( |\alpha| + |\Gamma| + \|u(t')\|_{B^p_{r,t}} + \frac{\beta}{3} \|u(t')\|_{B^p_{r,t}}^2 + \frac{|\gamma|}{4} \|u(t')\|_{B^p_{r,t}}^3 \right) \, dt' \cdot \left( |\alpha| + |\Gamma| + \|u(t')\|_{B^p_{r,t}} + \frac{\beta}{3} \|u(t')\|_{B^p_{r,t}}^2 + \frac{|\gamma|}{4} \|u(t')\|_{B^p_{r,t}}^3 \right) \, dt'.
\]
Combining Theorems 3.1-3.2 with (36)–(39), we have the unique global solution on time initial $[0, T]$, and $\sup_{t \in [0, T]} H(t) \leq H_0$. Moreover, we have
\[
H(T) \leq H_0.
\]
Applying Theorems 3.1-3.2 again, we have the unique local solution and $H(t) \leq \lambda \epsilon$ on $t \in [T, 2T]$. From (36)–(38), we get $\sup_{t \in [T, 2T]} H(t) \leq H_0$. Hence, we deduce that
\[
\sup_{[0, 2T]} H(t) \leq H_0.
\]
Repeating the bootstrap argument, we prove the global existence of (2). \qed

4.2. Blow-up criteria

In this subsection, we present two blow-up criteria for the Equation (2).

Lemma 4.2 ([1]): Let $u_0 \in H^s$ with $s > \frac{3}{2}$. Assume that $T^*$ be the maximal existence time of the corresponding solution $u$ to (2). For any $t \in [0, T^*)$, we have
\[
\|u(t)\|_{H^1} = e^{-\lambda t} \|u_0\|_{H^1}.
\]
Lemma 4.3: Let \( u \in B^{s}_{p,r} \) with \((s, p, r)\) being as in Theorems 3.1-3.2. Assume that \( T^* \) be the maximal existence time of the corresponding solution \( u \) to (2). Then the solution of (2) blows up in finite time \( T^* < \infty \) if and only if

\[
\int_{0}^{T^*} \| u_x(t') \|_{L^\infty} \, dt' = \infty.
\]

Proof: Combining (2) with Lemma 2.1, it follows that

\[
\| u(t) \|_{B^{s}_{p,r}} + \lambda \| u(t) \|_{L^1_t([0,T];B^{s}_{p,r})} \leq \| u_0 \|_{B^{s}_{p,r}} + C \int_{0}^{t} \left( \| u_x \|_{L^\infty} \| u \|_{B^{s}_{p,r}} + \| u \|_{B^{s}_{p,r}} + \| u_x \|_{B^{s-1}_{p,r}} \| u_x \|_{B^{0}_{\infty,\infty}} + \| Q \|_{B^{s}_{p,r}} \right) (t') \, dt'.
\]

Moreover, we have

\[
G \leq C \| u(t) \|_{B^{s}_{p,r}} \left( 1 + \| u(t) \|_{L^\infty} \right)^3 + \| u_x(t) \|_{L^\infty}.
\]

Plugging (41) into (40), it follows that

\[
\| u(t) \|_{B^{s}_{p,r}} + \lambda \| u(t) \|_{L^1_t([0,T];B^{s}_{p,r})} \leq \| u_0 \|_{B^{s}_{p,r}} + C \int_{0}^{t} \| u \|_{B^{s}_{p,r}} \left( 1 + \| u \|_{L^\infty} \right)^3 + \| u_x \|_{L^\infty} (t') \, dt',
\]

which implies

\[
\| u(t) \|_{B^{s}_{p,r}} \leq \| u_0 \|_{B^{s}_{p,r}} e^{C \int_{0}^{t} \| u_x(t') \|_{L^\infty} \, dt'}. \tag{43}
\]

If \( T^* < \infty \), and \( \int_{0}^{T^*} \| u_x \|_{L^\infty} (t') \, dt' < \infty \), we know that \( u \in L^\infty((0, T^*); B^{s}_{p,r}) \), this contradicts the definition that \( T^* \).

According to Theorem 3.2 and the fact that \( B^{s}_{p,r} \hookrightarrow L^\infty \), if \( \int_{0}^{T^*} \| u_x(t') \|_{L^\infty} \, dt' = \infty \), then \( u \) must blow up in finite time.

Lemma 4.4 ([23]): Let \( u \in H^2 \) be a solution to (2). Then, we have

\[
\| u_x \|_{L^\infty} \leq C \left( \| u_x \|_{B^{0}_{\infty,\infty}} \cdot \ln (2 + \| u \|_{H^2}) + 1 \right).
\]

Next we give another blow-up criterion for (2) to claim norm inflation in critical Besov spaces.

Lemma 4.5: Let \( u_0 \in H^2 \). Assume that \( T^* \) be the maximal existence time of the corresponding solution \( u \) to (2). Then \( u \) blows up in finite time \( T^* < \infty \) if and only if

\[
\int_{0}^{T^*} \| u_x(t') \|_{B^{0}_{\infty,\infty}} \, dt' = \infty.
\]

Proof: Combining Lemma 2.1 with \( \| u(t) \|_{H^1} \leq \| u_0 \|_{H^1} \) for \( t \in [0, T^*) \), we get

\[
\| u(t) \|_{H^2} + \lambda \| u(t) \|_{L^1_t(H^2)}
\]
\[
\begin{align*}
\leq & \|u_0\|_{H^2} + C \int_0^t \left( \|u_x\|_{L^\infty} \|u\|_{H^1} + \|u_x\|_{H^1} \|u_x\|_{B_{\infty,\infty}^0} + \|u\|_{H^2} + \|Q\|_{H^2} \right) (t') \, dt' \\
\leq & \|u_0\|_{H^2} + C \int_0^t \|u\|_{H^2} (1 + \|u_x\|_{L^\infty}) (t') \, dt' .
\end{align*}
\]

Using Lemma 4.4, we have
\[
\|u\|_{H^2} \leq \|u_0\|_{H^2} + C \int_0^t \left( 1 + \|u_x\|_{B_{\infty,\infty}^0} \ln(e + \|u\|_{H^2}) \right) \|u\|_{H^2} \, dt' .
\]

The Gronwall inequality entails that
\[
\|u\|_{H^2} \leq \|u_0\|_{H^2} e^{C t + \int_0^t \|u_x\|_{B_{\infty,\infty}^0} \ln(e + \|u\|_{H^2}) \, dt'}.
\]

Hence, it follows that
\[
\ln(e + \|u\|_{H^2}) \leq \left( \ln(e + \|u_0\|_{H^2}) + C t \right) e^{C \int_0^t \|u_x\|_{B_{\infty,\infty}^0} \, dt'}.
\]

If \( T^* < \infty \), and \( \int_0^{T^*} \|u_x(t')\|_{B_{\infty,\infty}^0} \, dt' < \infty \), we deduce that \( u \in L^\infty ([0, T^*]; H^2) \), which contradicts the assumption that \( T^* \) is the maximal existence time. On the other hand, since the fact that \( L^\infty \hookrightarrow B_{\infty,\infty}^0 \), we get \( u \) must blow up in finite time whenever \( \int_0^{T^*} \|u_x(t')\|_{B_{\infty,\infty}^0} \, dt' = \infty \). \( \square \)

### 4.3. Blow-up

The following lemma shows that if the initial value satisfies certain conditions, the strong solution of (2) will blow up in finite time.

**Theorem 4.6 (1)**: Given the initial datum \( u_0 \in H^3(\mathbb{R}) \), let \( u = u(t, x) \) be the corresponding solution of (2), \( f(t) = \sup_{x \in \mathbb{R}} \kappa : = \max\{\|\alpha\|, |\beta|/3, |\gamma|/4, |\Gamma|\} \). There exists a \( \eta \in (0, \eta_0] \) and \( x_0 \in \mathbb{R} \) such that \( \eta u_{x0}(x_0) < \min\{-\|u_0\|_{H^1}^2, -\|u_0\|_{H^1}^2\} \) with \( \eta_0 = \sqrt{\frac{2}{1+2\kappa}} \).

If
\[
\lambda \in \left( 0, -\frac{f(0) \eta^2 u_{x0}^2(x_0)}{4} - \frac{\eta^2 u_{x0}^2(x_0)}{\max\{-\|u_0\|_{H^1}^2, -\|u_0\|_{H^1}^2\}} \right),
\]
then the solution of (2) blows up in finite time.

Now we give another blow-up result.

**Theorem 4.7**: Let \( u_0 \in H^s(\mathbb{R}) \) with \( s > \frac{3}{2} \). Assume \( u(t, x) \) is the corresponding solution of (2) with the initial value \( u_0 \). For \( n \in \mathbb{N}/\{0\} \), if the slope of \( u_0 \) satisfies
\[
f(0) \triangleq \int_{\mathbb{R}} u_{x0}^{2n+1} \, dx < - \left( \frac{8cK^2\|u_0\|_{H^1}^{2n-1}}{2n-1} \right)^{\frac{2n-1}{2n}},
\]
with
\[
K^2 = \frac{\lambda(n+1) \left( 4c\lambda(n+1)\|u_0\|_{H^1}^{2n-1} \right)^{2n-1}}{2n2n} + \frac{(2n+1)(C_1+C_3)}{n+1} \left( \frac{8n(2n+1)C_2}{(2n-1)(n+1)} \right)^n,
\]

\[n\]
and $c$ is the Gagliardo-Nirenberg constant, then there exists a lifespan $T < \infty$ such that the solution $u$ blows up in finite time $T$. Moreover, the above bound of lifespan $T$ satisfies

$$T \leq \int_{-f(0)}^{\infty} \frac{dy}{\frac{2n-1}{4c} \|u_0\|_{H^1}^2 y^{\frac{2n}{n-1}} - K^2},$$

(46)

where $n \in \mathbb{N}/\{0\}$, $c$ is a universal Gagliardo-Nirenberg constant.

**Proof:** Letting the function $p = \frac{1}{2} e^{-|x|}$, then $p \ast f = \Lambda^{-2} f$ for any $f \in L^2(\mathbb{R})$. Differentiating (2) with respect to variable $x$, we have

$$u_{tx} + (u + \Gamma)u_{xx} + \lambda u_x = -h(u) + u^2 - \frac{1}{2} u_x^2 + P,$$

(47)

with $P = \Lambda^{-2}(-h(u) + u^2 + \frac{1}{2} u_x^2)$.

In view of (47), for any $n \in \mathbb{N}/\{0\}$, we infer that

$$\frac{d}{dt} \int_{\mathbb{R}} u_x^{2n+1} \, dx + \lambda (2n + 1) \int_{\mathbb{R}} u_x^{2n+1} \, dx = -\frac{2n-1}{2} \int_{\mathbb{R}} u_x^{2n+2} \, dx - (2n + 1) \int_{\mathbb{R}} h(u) u_x^{2n} \, dx$$

$$+ (2n + 1) \int_{\mathbb{R}} u_x^2 u_x^{2n} \, dx - (2n + 1) \int_{\mathbb{R}} P \cdot u_x^{2n} \, dx,$$

(48)

Using Lemma 4.2 and Hölder’s inequality yields that

$$\int_{\mathbb{R}} (p \ast u) u_x^{2n} \, dx \leq \left( \int_{\mathbb{R}} p_{1}^{n+1} \, dx \right)^{\frac{1}{n+1}} \left( \int_{\mathbb{R}} u_x^{n+2} \, dx \right)^{\frac{n}{n+1}}$$

$$\leq \frac{\|P_1\|_{L^{\infty}} \cdot \|P_1\|_{L^2}^2}{(n+1)\epsilon^{n+1}} + \frac{n \epsilon^{n+1}}{n+1} \int_{\mathbb{R}} u_x^{2n+2} \, dx$$

$$\leq \frac{\|u_0\|_{H^1}^{n+1}}{\sqrt{2}^{n-1} (n+1)\epsilon^{n+1}} + \frac{n \epsilon^{n+1}}{n+1} \int_{\mathbb{R}} u_x^{2n+2} \, dx,$$

(49)

where we use the fact that $\|P_1\|_{L^{\infty}} \leq \|u\|_{L^{\infty}} \leq \frac{\|u_0\|_{H^1}}{\sqrt{2}}$ and $\|P_1\|_{L^2} \leq \|u\|_{L^2} \leq \|u_0\|_{H^1}$. Note that

$$\int_{0}^{\infty} p(x) \ast (u^2 + \frac{1}{2} u_x^2) \cdot u_x^{2n} \, dx > 0.$$ Similar to the above inequality, we deduce that

$$I + II + III \leq \frac{2n+1}{(n+1)\epsilon^{n+1}} \left[ \frac{2(|\alpha| + |\Gamma|) \|u_0\|_{H^1}^{n+1}}{\sqrt{2}^{n-1}} + \frac{|\beta| (\|u_0\|_{H^1}^{3(n+1)} + \|u_0\|_{H^1}^{3n+5})}{3(\sqrt{2})^{3n+1}} \right],$$

(50)
Choosing $\epsilon = (\frac{2n-1}{2n+1})^{n+1}$, and defining $f(t) = \int_{\mathbb{R}} u_x^{2n+1} dx$, we obtain
\[
\frac{d}{dt} f(t) \leq -\frac{2n-1}{4} \int_{\mathbb{R}} u_x^{2n+2} dx + \lambda (2n+1) |f(t)| + K_1^2,
\]
with $K_1^2 = (\frac{2n+1}{2n-1})(\frac{n}{2n+1})^n$.

Combining the Gagliardo-Nirenberg inequality with the Young inequality, it follows that
\[
\left( \int_{\mathbb{R}} u_x^{2n+1} dx \right)^{\frac{2n}{2n+1}} \leq c \left( \int_{\mathbb{R}} u_x^2 dx \right)^{\frac{1}{2n+1}} \int_{\mathbb{R}} u_x^{2n+2} dx \leq c \| u_0 \|_{H^1}^{\frac{2n}{2n+1}} \int_{\mathbb{R}} u_x^{2n+2} dx,
\]
and
\[
\lambda (2n+1) |f(t)| \leq \frac{1}{2n} \left( \frac{\lambda (2n+1)}{\epsilon_0} \right)^{\frac{2n}{2n-1}} + \frac{2n-1}{2n} \left( |f(t)| \epsilon_0 \right)^{\frac{2n}{2n-1}}.
\]
Let $\epsilon_0 = (\frac{n}{4c \| u_0 \|_{H^1}^{2n+1}})^{\frac{2n-1}{2n}}$. After a few calculations, we have
\[
\frac{d}{dt} f(t) \leq -\frac{2n-1}{8c \| u_0 \|_{H^1}^{2n-1}} f^{\frac{2n}{2n-1}} (t) + K^2,
\]
where $K^2 = K_1^2 + \frac{\lambda (n+1)(4c \lambda (n+1) \| u_0 \|_{H^1}^{2n+1})^{n-1}}{2n}$. Using the fact that $f(0) < -\frac{8c K^2 \| u_0 \|_{H^1}^{2n-1}}{2n}$, we then get $f'(t) < 0$ and $f(t)$ is a decreasing function. Suppose that $u$ is a global solution, there exists a time $t_1 > 0$ such that
\[
f'(t) \leq -\frac{2n-1}{16c \| u_0 \|_{H^1}^{2n-1}} f^{\frac{2n}{2n-1}} (t), \quad t \geq t_1,
\]
which implies that
\[
\frac{1}{16c \| u_0 \|_{H^1}^{2n-1}} (t - t_1) + f^{\frac{1}{2n-1}} (t_1) \leq f^{\frac{1}{2n-1}} (t) \leq 0, \quad t \geq t_1.
\]
Thanks to $f(t_1) < 0$, if $t \geq t_1$ large enough, we deduce that the above inequality is not valid. Therefore, there exists a lifespan $T < \infty$ such that $\lim_{t \to T} f(t) = -\infty$. By solving (55), we end up with
\[
T \leq - \int_{0}^{T} \frac{f'(t) dt}{f^{\frac{2n}{2n-1} - \frac{2}{2n-1}} f^{\frac{2n}{2n-1}} (t) - K^2}.
\]
which means that
\[
T \leq \int_{f(0)}^{\infty} \frac{dy}{\frac{2n-1}{8e} \|u_0\|_{H^1} y^{\frac{n}{2n-1}} - K^2}.
\]
Then Lemma 4.6 guarantees that
\[
\lim \inf_{t \to T} \left( \inf_{x \in \mathbb{R}} u_x(t, x) \right) = -\infty.
\]

4.4. Ill-posedness

**Theorem 4.8:** Let \(1 \leq p \leq \infty\) and \(1 < r \leq \infty\). For any \(\epsilon > 0\), there exists \(u_0 \in H^\infty\) such that the following holds

1. \(\|u_0\|_{B^{1+\frac{1}{p}}_{p,r}} \leq \epsilon\);
2. There is a unique solution \(u \in C([0, T]; H^\infty)\) to the Equation (2) with a maximal lifespan \(T < \epsilon\);
3. \(\limsup_{t \to T^-} \|u\|_{B^{1+\frac{1}{p}}_{p,r}} = \lim_{t \to T^-} \|u\|_{B^{1+\frac{1}{p}}_{1,\infty}} = \infty\).

**Proof:** Fix \(1 \leq p \leq \infty\) and \(1 < r \leq \infty\), and \(\epsilon > 0\). Let
\[
g(x) = \sum_{j \geq 1} \frac{1}{2j^{\frac{1}{p}}} g_j(x),
\]
with \(\hat{g}_j(\xi) = i2^{-j} \hat{g}(2^{-j} \xi)\). Let \(\chi\) be a non-negative, non-zero \(C_0^\infty\) function satisfies \(\hat{\chi} \chi_0 = \hat{\chi}\). Hence, we obtain that \(\Delta_j g(x) = \frac{1}{2j^{\frac{1}{p}}} g_j(x)\), \(\|\Delta_j g(x)\|_{L^p} \sim \frac{2^j}{2j^{\frac{1}{p}}}\), and
\[
\|g\|_{B^{1+\frac{1}{p}}_{p,r}} \sim \left\|\frac{1}{2j^{\frac{1}{p}}}\right\|_p,
\]
from which it follows \(g \in B^{1+\frac{1}{p}}_{p,r} \setminus B^{1+\frac{1}{p}}_{1,\infty}\), and
\[
g'(0) = \int \hat{g}'(\xi) \, d\xi = \int 2\pi i\xi \hat{g}(\xi) \, d\xi = -\epsilon \sum_{j \geq 1} \frac{1}{j^{\frac{1}{p}}} = -\infty.
\]
For any \(\epsilon > 0\), suppose that \(u_{0,\epsilon} = \|g\|^{-1}_{B^{1+\frac{1}{p}}_{p,r}} \epsilon S_K(g)\) with \(K\) is a large enough such that \(\eta u_{0,\epsilon}(0) < \min\{-\|u_0\|_{H^1}^{\frac{1}{2}}, \|u_0\|_{H^1}^2\}\). Therefore, \(u_{0,\epsilon} \in H^\infty\), \(\|u_{0,\epsilon}\|_{B^{1+\frac{1}{p}}_{p,r}} \leq \epsilon\). Taking advantage of Theorem 4.6, we get there exists an unique associated solution \(u \in C([0, T]; H^\infty)\) with maximal lifespan \(T < \epsilon\). By Lemmas 4.4-4.5, we can prove that \(\limsup_{t \to T^-} \|u\|_{B^{1+\frac{1}{p}}_{1,\infty}} = \infty\).

**Acknowledgments**

The authors thank the referee for valuable comments and suggestions.
Disclosure statement
No potential conflict of interest was reported by the author(s).

Funding
This work was partially supported by the National Natural Science Foundation of China (Grant No. 12171493), the Macao Science and Technology Development Fund (Grant No. 0091/2018/A3), the Guangdong Special Support Program (Grant No. 8-2015).

References
[1] Freire IL. Wave breaking for shallow water models with time decaying solutions. J Differ Equations. 2020;269:3769–3793.
[2] da Silva PL. Classification of bounded travelling wave solutions for the Dullin-Gottwald-Holm equation. J Math Anal Appl. 2019;471(1-2):481–488.
[3] Dullin H, Gottwald G, Holm DD. An integrable shallow water equation with linear and nonlinear dispersion. Phys Rev Lett. 2001;87(19):194501.
[4] da Silva PL, Freire IL. An equation unifying both Camassa-Holm and Novikov equations. 10th AIMS Conference on Number Dynamical Systems, Differential Equations and Applications; 2015. 304–311.
[5] Novruzov E. Blow-up phenomena for the weakly dissipative Dullin-Gottwald-Holm equation. J Math Phys. 2013;54(9):092703, 8.
[6] Novruzov E. Blow-up of solutions for the dissipative Dullin-Gottwald-Holm equation with arbitrary coefficients. J Differ Equations. 2016;261(2):1115–1127.
[7] Wu S, Yin Z. Blow-up, blow-up rate and decay of the solution of the weakly dissipative Camassa-Holm equation. J Math Phys. 2006;47(1):013504, 12.
[8] Wu S, Yin Z. Global existence and blow-up phenomena for the weakly dissipative Camassa-Holm equation. J Differ Equations. 2009;246(11):4309–4321.
[9] Constantin A. On the scattering problem for the Camassa-Holm equation. R Soc Lond Proc Ser A Math Phys Eng Sci. 2001;457(2008):953–970.
[10] Constantin A, McKean HP. A shallow water equation on the circle. Comm Pure Appl Math. 1999;52(8):949–982.
[11] Constantin A, Gerdjikov VS, Ivanov RI. Inverse scattering transform for the Camassa-Holm equation. Inverse Probl. 2006;22(6):2197–2207.
[12] Fokas A, Fuchssteiner B. Symplectic structures, their bcklund transformations, hereditary symmetries. Phys D. 1981;4(1):47–66.
[13] Camassa R, Holm DD, Hyman JM. A new integrable shallow water equation. Adv Appl. 1994;31:1–33.
[14] Constantin A, Strauss W. Stability of peakons. Comm Pure Appl Math. 2000;53(5):603–610.
[15] Constantin A, Strauss W. Stability of a class of solitary waves in compressible elastic rods. Phys Lett A. 2000;270(3-4):140–148.
[16] Ye W, Yin Z, Guo Y. A new result for the local well-posedness of the Camassa-Holm type equations in critical Besov spaces $B_{p,1}^{1+\frac{1}{p}}$, $1 \leq p < +\infty$, 2021. arXiv: 2101.00803.
[17] Brandolese L, Cortez ME. Blowup issues for a class of nonlinear dispersive wave equations. J Differ Equations. 2014;256(12):3981–3998.
[18] Constantin A. Existence of permanent and breaking waves for a shallow water equation: a geometric approach. Ann Inst Fourier (Grenoble). 2000;50(2):321–362.
[19] Constantin A, Escher J. Global existence and blow-up for a shallow water equation. Ann Scuola Norm Sup Pisa Cl Sci (4). 1998;26(2):303–328.
[20] Constantin A, Escher J. Wave breaking for nonlinear nonlocal shallow water equations. Acta Math. 1998;181(2):229–243.
[21] Constantin A, Escher J. Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation. Comm Pure Appl Math. 1998;51(5):475–504.
[22] Danchin R. A few remarks on the Camassa-Holm equation. Differ Integral Equations. 2001;14(8):953–988.
[23] Guo Z, Liu X, Molinet L, et al. Ill-posedness of the Camassa-Holm and related equations in the critical space. J Differ Equations. 2019;266(2-3):1698–1707.
[24] Guo Y, Ye W, Yin Z. Ill-posedness for the Cauchy problem of the Camassa-Holm equation in $B_{1,\infty}^{1}(\mathbb{R})$. J Differ Equations. 2022;327:127–144.
[25] Li AY, Olver JP. Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation. J Differ Equations. 2000;162(1):27–63.
[26] Li J, Yu Y, Guo Y, et al. Ill-posedness for the Camassa-Holm equation in $B_{p,1}^{1} \cap C^{0,1}$, 2022. arXiv: 2201.02839.
[27] Li J, Yu Y, Zhu W. Ill-posedness for the Camassa-Holm and related equations in Besov spaces. J Differ Equations. 2022;306:403–417.

[28] Li J, Yin Z. Remarks on the well-posedness of Camassa-Holm type equations in Besov spaces. J Differ Equations. 2016;261(11):6125–6143.

[29] Liu Y, Yin Z. Global existence and blow-up phenomena for the Degasperis-Procesi equation. Comm Math Phys. 2006;267(3):801–820.

[30] Rodríguez-Blanco G. On the Cauchy problem for the Camassa-Holm equation. Nonlinear Anal Theory Methods Appl. 2001;46:309–327.

[31] Yin Z. Well-posedness, blowup, and global existence for an integrable shallow water equation. Discrete Contin Dyn Syst. 2004;11(2-3):393–411.

[32] Bressan A, Constantin A. Global dissipative solutions of the Camassa-Holm equation. Anal Appl. 2007;5(1):1–27.

[33] Bressan A, Constantin A. Global conservative solutions of the Camassa-Holm equation. Arch Ration Mech Anal. 2007;183(2):215–239.

[34] Constantin A, Molinet L. Global weak solutions for a shallow water equation. Comm Math Phys. 2000;211(1):45–61.

[35] Holden H, Raynaud X. Global conservative solutions of the Camassa-Holm equation – a lagrangian point of view. Comm Partial Differ Equations. 2007;32(10-12):1511–1549.

[36] Holden H, Raynaud X. Global conservative solutions of the generalized hyperelastic-rod wave equation. J Differ Equations. 2007;233(2):448–484.

[37] Xin Z, Zhang P. On the weak solutions to a shallow water equation. Comm Pure Appl Math. 2000;53(11):1411–1433.

[38] da Silva PL, Freire IL. Integrability, existence of global solutions, and wave breaking criteria for a generalization of the Camassa-Holm equation. Stud Appl Math. 2020;145(3):537–562.

[39] Bahouri H, Chemin JY, Danchin R. Fourier analysis and nonlinear partial differential equations. Vol. 343. Berlin, Heidelberg: Springer; 2011.