ON SUFFRIDGE POLYNOMIALS

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Abstract. We consider some known and some new properties of polynomials introduced by Ted Suffridge in 1969. We give a brief overview of their extremal properties in classic and more recent work. Finally, we give a compact form for Suffridge polynomials which surprisingly had not appeared earlier in the literature.

1. Introduction

Suffridge polynomials are a central object of study in the theory of univalent polynomials. For example, in the book by P. Duren [9], the “reference” for univalent functions, the section on univalent polynomials is devoted almost exclusively to Suffridge polynomials.

These polynomials were introduced by Ted Suffridge in 1969 [12] and are still essential to modern research because of their remarkable properties and because of the otherwise very limited number of interesting known examples of univalent polynomials. They are defined as

$$S_{N,j}(z) = \sum_{k=1}^{N} \frac{N + 1 - k \sin \frac{kj\pi}{N+1}}{N \sin \frac{j\pi}{N+1}} z^k, \quad j = 1, \ldots, N; \quad N = 1, 2, \ldots$$

and Suffridge (ibid.) showed that they are a good approximation for the Koebe function

$$K(z) = \frac{z}{(1 - z)^2}.$$ 

In this note, we present an overview of some of the main results regarding Suffridge polynomials (Section 2) and we also offer the first closed formula describing their general form (Section 3).

2. A Tale of Suffridge Polynomials

Some properties can be checked in a straightforward manner, such as the fact $S$ can be rewritten as

$$S_{N,j}(z) = z + \cdots + (-1)^{j-1}z^n/N.$$ 

Suffridge was the first to study the essential properties of these polynomials. In particular, he proved the univalency in $\mathbb{D}$, thus $S_{N,j}(z)$ are schlicht functions in $\mathbb{D}$, i.e. univalent with zero coefficient zero and the degree one coefficient equal to one.

Suffridge polynomials are also extremal: since the derivative of a function univalent in $\mathbb{D}$ never vanishes in $\mathbb{D}$ the leading coefficient of the univalent polynomial of the degree $N$ cannot exceed $1/N$ in absolute value.
In that sense Suffridge polynomials are on the verge of univalence - the roots of the derivative allowed on the boundary, which can be seen from the image of the unit circle - it has cusps. This fact is illuminated on the hand drawn pictures below (borrowed from Suffridge’s original paper [12]).

A careful observation of Figure 1 indicates that the value \(|S_{5,1}(-1)|\) might not be the minimal distance from the boundary of the region \(S_{5,1}(\mathbb{D})\) to the origin. More about this phenomena as well as some other interesting results can be found in [6].

Moreover, they are extremal as an existing substitute for Koebe functions in these settings. The Koebe function is extremal in two famous theorems of geometric complex analysis: the one quarter Koebe theorem and in the Bieberbach conjecture - now de Branges’ theorem. Let us to remind these results.

The Koebe theorem states that there exists a constant \(r\), such that \(f(\mathbb{D}) \supset D_r\) for any schlicht in \(\mathbb{D}\) function \(f\). Here \(D_r = \{|z| < r\}\). In 1916 Bieberbach proved that \(r = 1/4\) and that the extremal function is unique (up to rotation). The extremal function is called Koebe function

\[
K(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + ...
\]

It maps the unit disc \(\mathbb{D}\) in the slit region \(\mathbb{C}\backslash(-\infty, -1/4]\). In the same article Bieberbach proved that for any schlicht function \(f(z) = z + a_2z^2 + 3z^3 + \ldots\) the estimate

\[
|a_2| \leq 2
\]

and conjectured that \(|a_k| \leq k, k = 1, 2, \ldots\). For over 70 years this conjecture was a driving force behind the development of the geometric complex analysis. It was finally proved in
1984 by de Branges. Thus, it was established that the solution to both extremal problems function is the Koebe function \( K(z) \).

It is natural to ask what is the polynomial version of these results. Note that the Koebe function is also on the verge of univalence - any increase in any of its coefficients eliminates the function from the univalency class; it is thus natural to consider polynomials with the last coefficient \( 1/N \).

Denote the coefficients of the Suffridge polynomial \( S_{N,1}(z) \) by \( A_j \), \( j = 1, ..., N \). Let \( P_N(z) = z + a_2 z^2 + ... + a_N z^N \) be any schlicht polynomial with real coefficients \(^1\) and the normalization to be as described above. Then

\[
|a_j| \leq A_j, j = 1, ..., N.
\]

Further, Suffridge found an expression for the values of the polynomial on the unit circle (formula (5) in [12])

\[
S_{N,j}(e^{it}) = \frac{N + 1}{2N(cot - cos \alpha)} + \frac{1}{2N(cot - cos \alpha)} \sin t(1 - (-1)^j e^{i(N+1)t})
\]

where \( \alpha = j \pi/(N + 1) \). Surprisingly, the compact form of the Suffridge polynomials has not been found yet. Let us derive such a form below.

### 3. Closed Form

**Lemma 1 (Key ingredient).** Let \( S_N = \sum_{k=1}^{N} \sin(k\alpha) z^k \) and \( C_N = \sum_{k=1}^{N} \cos(k\alpha) z^k \) then

\[
S_N = z \sin(\alpha) - \sin((N+1)\alpha) z^N + \sin(N\alpha) z^{N+1} + \frac{z}{1 - 2 \cos(\alpha) z + z^2}
\]

and

\[
C_N = z \cos(\alpha) - z - \cos((N+1)\alpha) z^N + \cos(N\alpha) z^{N+1} + \frac{z}{1 - 2 \cos(\alpha) z + z^2}
\]

**Proof.** By using the exponential notation, we can express \( S_N \) and \( C_N \) simultaneously as a finite geometric sequence

\[
C_N + iS_N = \sum_{k=1}^{N} (e^{i\alpha} z)^k
\]

which can then be rewritten as

\[
\frac{1 - (e^{i\alpha} z)^N}{1 - e^{i\alpha} z} e^{i\alpha} z.
\]

The final result is then obtained by clearing the denominator of complex terms and using the real-imaginary decomposition. \( \square \)

From this result we derive (no pun intended):

**Lemma 2.** Let \( T_N = \sum_{k=1}^{N} k \sin(k\alpha) z^k \) then

\[
T_N = \frac{z}{(1 - 2 \cos(\alpha) z + z^2)^2} \left( \sin(\alpha) + \ldots - \sin(\alpha) z^2 + \ldots \right)
\]

\(^1\)Everywhere below we will assume all coefficients to be real
−(N + 1) \sin((N + 1)\alpha)z^N + \ldots
+((N + 2) \sin(N\alpha) + 2N \cos(\alpha) \sin((N + 1)\alpha))z^{N+1} + \ldots
−((N − 1) \sin((N + 1)\alpha) + 2(N + 1) \cos(\alpha) \sin(N\alpha))z^{N+2} + \ldots
+N \sin(\alpha N)z^{N+3}\}

\textbf{Proof.} This identity follows from Lemma 1 and from the observation that \( T_N = z \frac{d}{dz} S_N. \)

□

A first application of the above Lemmas is the derivation of a closed form for Suffridge polynomials. We do not know of this form in the literature though Suffridge have used a similar argument to look at the image of the circle \( \partial \mathbb{D}. \)

**Proposition 1** (Suffridge in Closed Form). The Suffridge polynomials can be written as

\[ S_{N,j}(z) = z \frac{N - 2(N + 1) \cos \left( \frac{j\pi}{N+1} \right) z + (N + 2)z^2 + (-1)^j z^{N+1} - (-1)^j z^{N+3}}{N \left( 1 - 2 \cos \left( \frac{j\pi}{N+1} \right) z + z^2 \right)^2} \]

\textbf{Proof.} Recall that the Suffridge polynomials are defined as

\[ S_{N,j}(z) = \sum_{k=1}^{N} A_k z^k \]

where

\[ A_k = A_k(N,j) = \frac{N - k + 1 \sin \left( \frac{k \pi}{N+1} \right) \sin \left( \frac{j \pi}{N+1} \right)}{N \sin \left( \frac{j \pi}{N+1} \right)}. \]

We can decompose \( A_k \) as

\[ \frac{N + 1}{N} \frac{\sin \left( \frac{k \pi}{N+1} \right) \sin \left( \frac{j \pi}{N+1} \right)}{\sin \left( \frac{j \pi}{N+1} \right)} - k \frac{\sin \left( \frac{k \pi}{N+1} \right) \sin \left( \frac{j \pi}{N+1} \right)}{\sin \left( \frac{j \pi}{N+1} \right)}. \]

Applying Lemmas 1 and 2 respectively to the first and second term gives the above result. □

\textbf{Example.} \( S_{5,2}(z) = \frac{z(5-6z+7z^2-z^3-z^4)}{5(1-z+z^2)^2} = -\frac{z^5}{5} - \frac{2z^4}{5} + \frac{4z^2}{5} + z \)

4. \textbf{Brandt Representation}

Note that Brandt [5] suggested a general form of typically real polynomials \( z + a_2 z^2 + \ldots + a_n z^n. \) The set of such polynomials was denoted by \( T_n. \) His idea in itself is quite remarkable, but the coefficients included in it are difficult to choose. Here is a theorem from [5]

**Theorem 1.** Let \( f(z) \) be a rational function normalized by

\[ f(0) = 0, \quad f'(0) = 1. \]

Then the following statements are equivalent:

(a) \( f(z) \) belongs to the class \( T_n \)
(b) There are \( n \) real numbers \( b_k \) and \( b'_k \) with

\[
f(z) = 4n \sum_{k=1}^{\lfloor n/2 \rfloor} \left( b_k^2 \cos^2 \frac{(2k-1)\pi}{2n} + b'_k^2 \sin^2 \frac{(2k-1)\pi}{2n} \right) \frac{z}{1 - 2z \cos \frac{(2k-1)\pi}{n} + z^2} \\
-(1 + z^n)(1 - z^2)(1 + z^2) \left( \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{b_k}{1 - 2z \cos \frac{(2k-1)\pi}{n} + z^2} \right)^2 \\
+(1 + z^n)(1 - z^2)(1 + z^2) \left( \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{b'_k}{1 - 2z \cos \frac{(2k-1)\pi}{n} + z^2} \right)^2 \\
+ \left( \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} b_k^2 \right)^2 - \left( \sum_{k=1}^{\lfloor n/2 \rfloor} b'_k^2 \right)^2.
\]

(c) There are \( n \) real numbers \( c_k \) and \( c'_k \) with

\[
f(z) = 4n \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \left( c_k^2 \cos^2 \frac{k\pi}{n} + c'_k^2 \sin^2 \frac{k\pi}{n} \right) \frac{z}{1 - 2z \cos \frac{2k\pi}{n} + z^2} \\
-(1 + z^n)(1 - z^2)(1 + z^2) \left( \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{c_k}{1 - 2z \cos \frac{k\pi}{n} + z^2} \right)^2 \\
+(1 + z^n)(1 - z^2)(1 + z^2) \left( \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \frac{c'_k}{1 - 2z \cos \frac{2k\pi}{n} + z^2} \right)^2 \\
+ \left( \sum_{k=1}^{\lfloor n/2 \rfloor} c_k^2 \right)^2 - \left( \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} c'_k^2 \right)^2.
\]

Since Suffridge polynomials are typically real, they should have an appropriate Brandt representation. We managed to find the coefficients.

Namely, to represent Suffridge polynomials in the form of rational function we can use formula (3) of Theorem 1 for odd \( j \) and formula (4) if \( j \) is even. Let

\[ n = N + 1, \quad k = \left\lfloor \frac{j + 1}{2} \right\rfloor \quad \text{and} \quad b_k = b'_k = \frac{1}{2\sqrt{N}}, \quad c_k = c'_k = \frac{1}{2\sqrt{N}}. \]

Then it is easy to check the formulas (3) and (4) coincides with our Proposition 1.

5. Extremal properties of Suffridge polynomials

Formula (1) allows to compute the quantity

\[ S_{N,1}(-1) = -\frac{1}{4} \frac{N + 1}{N} \sec^2 \frac{\pi}{2(N+1)} \rightarrow -\frac{1}{4}, \quad N \rightarrow \infty. \]

Thus Suffridge polynomials can be used to prove that the \( 1/4 \) constant in Koebe’s theorem is sharp. Hence, in the above sense, Suffridge polynomials can be considered as a substitute for Koebe functions.
By the way, Brandt [1, p.79 (466)] solved the extreme problem of evaluating the modulus of schlicht polynomials of degree $N$, showing that the extreme polynomial is a Suffridge polynomial, and that the maximum value is

$$S_{N,1}(1) = \frac{1}{4} \frac{N+1}{N} \csc^2 \frac{\pi}{2(N+1)}.$$

Dmitrishin and Khamitova [7] announced a new non-obvious extreme properties of Suffridge polynomials. Namely, they are, after appropriate renormalization, the only optimal polynomials for the following extremal problem

$$\sup_{a_1 + \ldots + a_N = 1} \left( \min_t \{ \Re (F_N(e^{it})) : \Im (F_N(e^{it})) = 0 \} \right)$$

where $F_N(z)$ is any polynomial of degree $N$ with zero coefficient zero. The solution to this extreme problem as well very elegant and surprising applications to discrete dynamic systems are given in [8].

Another nice feature of Suffridge polynomials was observed by Genthner, Ruscheweyh and Salinas. In [11], they proposed an interesting characterization of the boundary of simply connected domains. More precisely, an oriented closed curve $\gamma : [0, 2\pi) \to \mathbb{C}$ is called quasi-simple if it represents the positively oriented boundary of a simply connected domain $D_\gamma \subset \mathbb{C}$. In this paper the authors give a nonstandard criterion for closed plane curves to be quasi-simple. Along the way, Genthner et al. define the concept of quasi-extremal polynomial.

**Definition 1 ([11], Definition 11).** Let $P$ be a complex polynomial of degree $n$. We call $P$ quasi-extremal (q-e) if there exists a simply connected domain $\Omega \subset \mathbb{C}$ such that

1. $P(\mathbb{D}) \subset \Omega$
2. $P'$ has $n - 1$ zeros on $\partial \mathbb{D}$, say in $e^{i\theta_k}$, where the angles $\theta_k$ are labeled such that $\theta_1 < \theta_2 < \ldots < \theta_n = \theta_1 + 2\pi$.
3. There exists $\tau_j \in [\theta_j, \theta_{j+1})$ with $P(e^{i\tau_j}) \in \partial \Omega, j = 1, \ldots, n - 1$.

The pertinence of q-e polynomials is a consequence of the following theorem.

**Theorem 2 ([10, 11]).** Every quasi-extremal polynomial is univalent in $\mathbb{D}$.

Quasi-extremal polynomials are interesting candidates as extremal polynomials for maximal range problems - see [2, 3, 4]. In particular, Suffridge polynomials are a particular example of q-e polynomials as one can observe from their boundary representation.

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