ON THE $q$-EXTENSIONS OF THE BERNOULLI AND EULER NUMBERS, RELATED IDENTITIES AND LERCH ZETA FUNCTION

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Abstract Recently, $\lambda$-Bernoulli and $\lambda$-Euler numbers are studied in [5, 10]. The purpose of this paper is to present a systematic study of some families of the $q$-extensions of the $\lambda$-Bernoulli and the $\lambda$-Euler numbers by using the bosonic $p$-adic $q$-integral and the fermionic $p$-adic $q$-integral. The investigation of these $\lambda$-$q$-Bernoulli and $\lambda$-$q$-Euler numbers leads to interesting identities related to these objects. The results of the present paper cover earlier results concerning $q$-Bernoulli and $q$-Euler numbers. By using derivative operator to the generating functions of $\lambda$-$q$-Bernoulli and $\lambda$-$q$-Euler numbers, we give the $q$-extensions of Lerch zeta function.

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1. Introduction, Definitions and Notations

Throughout this paper, the symbols $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers.

The symbol $q$ can be treated as a complex number, $q \in \mathbb{C}$, or as a $p$-adic number, $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we always assume that $|q| < 1$. If $q \in \mathbb{C}_p$, then we usually assume that $|1 - q|_p < 1$. Here $| \cdot |_p$ stands for the $p$-adic absolute value in $\mathbb{C}_p$ with $|p|_p = \frac{1}{p}$. The $q$-basic natural numbers are defined by $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ ($n \in \mathbb{N}$) and $[n]_{-q} = \frac{1 - (-q)^n}{1 + q}$. In this paper, we use the notation

$$[x]_q = \frac{1 - q^x}{1 - q} \quad \text{and} \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q},$$

see [1-19].

Hence $\lim_{q \to 1} [x]_q = x$ for any $x$ with $|x|_p \leq 1$ in the present $p$-adic case.

For $x \in \mathbb{Z}_p$, we say that $g$ is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $g \in UD(\mathbb{Z}_p)$, the set of uniformly differentiable function, if the difference quotients

$$F_g(x, y) = \frac{g(y) - g(x)}{y - x}$$

have a limit $l = g'(a)$ as $(x, y) \to (a, a)$. For $f \in UD(\mathbb{Z}_p)$, the $q$-deformed bosonic $p$-adic integral is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \frac{q^x}{[p^N]_q}, \text{ see [1-19]},$$

(1)
and the $q$-deformed fermionic $p$-adic integral is defined by

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \frac{(-q)^x}{[p^N]_q},$$

(see [1-19]).

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. Then

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l).$$

(2)

The classical Bernoulli polynomials $B_n(x)$ and the Euler polynomials $E_n(x)$ are defined as

$$t - 1 = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{and} \quad 2e^t - 1 = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$ 

(3)

The Bernoulli numbers $B_n$ and the Euler numbers $E_n$ are defined as $B_n = B_n(0)$ and $E_n = E_n(0)$, (see [1-19]).

From (1), we note that

$$q I_{-q}(f_1) = I_{-q}(f) + (q-1)f(0) + \frac{q-1}{\log q} f'(0),$$

(4)

for $f_1(x) = f(x+1)$. By (2), we see that $I_{1}(f_1) = I_{1}(f) + f'(0)$, (see [7]).

Let $u$ be algebraic in $\mathbb{C}_p$ (or $\mathbb{C}$). Then the Frobenius-Euler polynomials are defined as

$$\frac{1-u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(u, x) \frac{t^n}{n!},$$

(5)

(see [5]).

In case $x = 0$, $H_n(u, 0) = H_n(u)$, which are called the Frobenius Euler numbers.

Let $C_{p^n}$ be the cyclic group consisting of all $p^n$-th roots of unity in $\mathbb{C}_p$ for any $n \geq 0$ and $T_p$ be the direct limit of $C_{p^n}$ with respect to the natural morphisms, hence $T_p$ is the union of all $C_{p^n}$ with discrete topology.

For $\lambda \in T_p$ with $\lambda \neq 1$, if we use (4), then we have

$$\int_{\mathbb{Z}_p} e^{tx} \lambda^x d\mu_1(x) = \frac{t}{\lambda e^t - 1}.$$ 

(6)

From (5), the $\lambda$–Bernoulli numbers are defined as

$$\frac{t}{\lambda e^t - 1} = e^{B(\lambda)t} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!},$$

(7)

with the usual convention of replacing $B'(\lambda)$ by $B_1(\lambda)$. Thus, $B_k(\lambda)$ can be determined inductively by

$$\lambda(B(\lambda) + 1)^k - B_k(\lambda) = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

(8)

(see [5]).

By the definition of the Frobenius-Euler numbers, we see that

$$\frac{t}{\lambda e^t - 1} = \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \frac{(m+1)H_m(\lambda^{-1})}{\lambda - 1} t^{m+1},$$

(9)

(see [7]).
For $m \geq 1$ and $\lambda \neq 1$, we have

\begin{equation}
B_m(\lambda) = \int_{\mathbb{Z}_p} x^m \lambda^x d\mu_1(x) = \frac{m}{\lambda - 1} H_{m-1}(\lambda^{-1}), \quad \text{(see [5]).}
\end{equation}

We can also easily see that $\int_{\mathbb{Z}_p} \lambda^x d\mu_1(x) = 0$ and

$$e^x x = \lim_{m \to \infty} \sum_{\lambda \in \mathbb{C}_p} \frac{t^m \lambda^x}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \lim_{m \to \infty} \sum_{\lambda \in \mathbb{C}_p} \int_{\mathbb{Z}_p} x^m \lambda^x d\mu_1(x) \lambda^x.$$

Consequently, we have

\begin{align*}
x^n &= B_n(1) + \sum_{\substack{\lambda \in \mathbb{Z}_p \setminus \{0\} \setminus \{1\} \setminus \{\lambda\}}} \frac{1}{\lambda - 1} H_{n-1}(\lambda^{-1}) \lambda^n \\
&= B_n(1) + \sum_{\lambda \in \mathbb{C}_p, \lambda \neq 1} \frac{B_n(\lambda)}{\lambda}.
\end{align*}

From (6) and (8), we note that $B_0(\lambda) = 0$, $B_1(\lambda) = \frac{1}{\lambda - 1}$, $B_2(\lambda) = -\frac{2\lambda}{(\lambda - 1)^2}$, \ldots.

The Genocchi numbers are defined by the generating function

\begin{equation}
\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}.
\end{equation}

These numbers satisfy the relation $G_0 = 0$, $G_1 = 1$, $G_3 = 0 = G_5 = \cdots = G_{2k+1} = 0$, and the even coefficients are $G_n = 2(1 - 2^n)B_n$.

For $\lambda \in \mathbb{C}_p$ with $|\lambda| < 1$, by (2), we have

\begin{equation}
\int_{\mathbb{Z}_p} \lambda^x e^t d\mu_{-1}(x) = \frac{2}{\lambda e^t + 1}.
\end{equation}

By (11), we define the $\lambda$-Euler numbers as follows:

\begin{equation}
\frac{2}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \frac{E_n(\lambda)}{n!} t^n, \quad \text{(see [7, 9, 10]).}
\end{equation}

Note that $E_n(\lambda) = \frac{2}{\lambda + 1} H_n(-\lambda^{-1})$.

From (12), we can easily derive

\begin{equation}
\int_{\mathbb{Z}_p} x^n \lambda^x d\mu_{-1}(x) = E_n(\lambda) = \frac{2}{\lambda + 1} H_n(-\lambda^{-1}).
\end{equation}

The $\lambda$-Genocchi numbers are also defined as

\begin{equation}
t \int_{\mathbb{Z}_p} x^n \lambda^x d\mu_{-1}(x) = \frac{2t}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \frac{G_n(x) t^n}{n!}.
\end{equation}

Thus, we have $G_0(\lambda) = 0$, $G_1(\lambda) = \frac{2}{\lambda + 1}$, \ldots, $E_n(\lambda) = \frac{G_{n+1}(\lambda)}{n+1}$.

In this paper, we study the $q$-extension of $\lambda$-Bernoulli number and $\lambda$-Euler numbers related to Lerch zeta function. The purpose of this paper is to present a systematic study of some families of the $q$-extension of the $\lambda$-Bernoulli and $\lambda$-Euler numbers by using the bosonic $p$-adic $q$-integral and the fermionic $p$-adic $q$-integral.
The investigation of these $\lambda$-$q$-Bernoulli and $\lambda$-$q$-Euler numbers leads to interesting identities related to these objects. The results of the present paper cover earlier results concerning $q$-Bernoulli and $q$-Euler numbers. By using derivative operator to the generating functions of $\lambda$-$q$-Bernoulli and $\lambda$-$q$-Euler numbers, we can give the $q$-extension of Lerch zeta function.

2. $q$-extension of $\lambda$-Bernoulli numbers and polynomials

For $\lambda \in T_p$, let us consider the $q$-extension of $\lambda$-Bernoulli numbers as follows.

\[ \beta_{k,q}(\lambda) = \int_{Z_p} \lambda^x [x]_q^k d\mu_q(x). \]  

From (14), we note that

\[
\beta_{k,q}(\lambda) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} \lambda^x [x]_q^k q^x
\]

\[
= \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} (\lambda q)^x \left( \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^l q^lx}{1 - q^l+1} \right) \frac{1}{(1 - q)^k}
\]

\[
= \frac{1 - q}{(1 - q)^k} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^l \frac{1 - (\lambda q^{l+1})^N}{1 - \lambda q^{l+1}}}{l + 1}
\]

Therefore, we obtain the following theorem.

**Theorem 1.** For $k \in \mathbb{N} \cup \{0\}$ and $\lambda \in T_p$, we have

\[
\beta_{k,q}(\lambda) = \frac{1}{(1 - q)^{k-1}} \sum_{l=0}^{k} \binom{k}{l} (-1)^l \frac{l + 1}{1 - \lambda q^{l+1}}.
\]

Let $F(t, \lambda : q)$ be the generating functions of $\beta_{n,q}(\lambda)$ with

\[
F(t, \lambda : q) = \sum_{n=0}^{\infty} \beta_{n,q}(\lambda) \frac{t^n}{n!}.
\]
Then we have

\[
F(t, \lambda : q) = \sum_{n=0}^{\infty} \beta_{n,q}(\lambda) \frac{t^n}{n!} = \int_{\mathbb{R}} \lambda^x e^{tx} d\mu_q(x)
\]

(15)

\[
= \sum_{n=0}^{\infty} \int_{\mathbb{R}} \lambda^x [x]_q^n d\mu_q(x) \frac{t^n}{n!}
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{(1-q)^{k-1}} \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} (-1)^l (l+1) \sum_{m=0}^{\infty} \lambda^m q^{(l+1)m} \frac{t^k}{k!}
\]

(16)

\[
= \sum_{k=0}^{\infty} \frac{1}{(1-q)^{k-1}} \sum_{m=0}^{\infty} q^m \lambda^m \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} (-1)^l q^m \frac{t^k}{k!}
\]

(17)

Since \( l \binom{k}{l} = k \binom{k-1}{l-1} \), the first term of the last equation in (15) equals

\[
\sum_{m=0}^{\infty} q^m \lambda^m \sum_{k=0}^{\infty} \frac{1}{(1-q)^{k-1}} \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} (-1)^l q^m \frac{t^k}{k!}
\]

(16)

\[
= - \sum_{m=0}^{\infty} q^m \lambda^m \sum_{k=0}^{\infty} \frac{1}{(1-q)^{k-1}} \sum_{l=0}^{k-1} \frac{k!}{l!(k-l)!} (-1)^l q^m \frac{t^k}{k!}
\]

The second term of the last equation in (15) equals

\[
\sum_{k=0}^{\infty} \frac{1}{(1-q)^{k-1}} \sum_{m=0}^{\infty} q^m \lambda^m (1-q)^m \frac{t^k}{k!}
\]

(17)

\[
= (1-q) \sum_{m=0}^{\infty} q^m \lambda^m \sum_{k=0}^{\infty} \frac{1}{q^k} \frac{t^k}{k!} = (1-q) \sum_{m=0}^{\infty} q^m \lambda^m [m]_q t.
\]

From (15), (16) and (17), we obtain the following proposition.

**Proposition 2.** Let \( F(t, \lambda : q) = \sum_{n=0}^{\infty} \beta_{n,q}(\lambda) \frac{t^n}{n!} \). Then we have

\[
F(t, \lambda : q) = -t \sum_{m=0}^{\infty} q^m \lambda^m [m]_q t + (1-q) \sum_{m=0}^{\infty} q^m \lambda^m [m]_q t.
\]
Since $q^{2m} = q^{m}\{m\}_q(q - 1) + 1$, it follows that

$$\beta_{k,q}(\lambda) = \frac{d^k F_q(t, \lambda : q)}{(dt)^k}_{t=0}$$

$$= -k \sum_{m=0}^{\infty} q^{2m} \lambda^m [m]_q^{k-1} + (1 - q) \sum_{m=0}^{\infty} q^{m} \lambda^m [m]_q^{k}$$

$$= -k(q - 1) \sum_{m=0}^{\infty} q^{m} \lambda^m [m]_q^{k} - k \sum_{m=0}^{\infty} q^{m} \lambda^m [m]_q^{k-1} + (1 - q) \sum_{m=0}^{\infty} q^{m} \lambda^m [m]_q^{k}$$

$$= (1-q)(k+1) \sum_{m=0}^{\infty} q^{m} \lambda^m [m]_q^{k} - k \sum_{m=0}^{\infty} q^{m} \lambda^m [m]_q^{k-1}.$$  

Therefore, we obtain the following theorem.

**Theorem 3.** For $k \in \mathbb{N} \cup \{0\}$ and $\lambda \in T_p$, we have

$$\beta_{k,q}(\lambda) = (1-q)(k+1) \sum_{m=0}^{\infty} q^{m} \lambda^m [m]_q^{k} - k \sum_{m=0}^{\infty} q^{m} \lambda^m [m]_q^{k-1}.$$  

Now we consider another $q$-extension of $\lambda$-Bernoulli numbers as follows.

\begin{equation}
B_{n,q}(\lambda) = \int_{\mathbb{Z}_p} q^{-x} \lambda^x [x]_q^n \, d\mu_q(x).
\end{equation}

From (18), we can derive

$$B_{n,q}(\lambda) = \int_{\mathbb{Z}_p} q^{-x} \lambda^x [x]_q^n \, d\mu_q(x)$$

$$= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) \int_{\mathbb{Z}_p} q^{-x} (-1)^l \lambda^l q^{lx} \, d\mu_q(x)$$

$$= \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) (-1)^l \frac{l}{1-\lambda q}.$$  

Thus, we obtain the following theorem.

**Theorem 4.** For $n \in \mathbb{N} \cup \{0\}$ and $\lambda \in T_p$, we have

$$B_{n,q}(\lambda) = \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) (-1)^l \frac{l}{1-\lambda q}.$$

Let $F^*(t, \lambda : q)$ be the generating functions of $B_{n,q}(\lambda)$ with

$$F^*(t, \lambda : q) = \sum_{n=0}^{\infty} B_{n,q}(\lambda) \frac{t^n}{n!}.$$
Then we have
\[
F^*(t, \lambda : q) = \sum_{n=0}^{\infty} B_{n,q}(\lambda) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} q^{-x} \lambda^x e^{x|x|^{-1}} d\mu_q(x)
\]
\[
= \sum_{n=0}^{\infty} \{ \int_{\mathbb{Z}_p} q^{-x} \lambda^x [x]^n d\mu_q(x) \} \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \{ \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l}{1-\lambda q^{-1}} \} \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \{ \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{m=0}^{\infty} \lambda^m q^m \} \frac{t^n}{n!}
\]
\[
= \sum_{m=0}^{\infty} \lambda^m \left( \sum_{n=1}^{\infty} \frac{n}{(1-q)^{n-1}} \sum_{l=1}^{n} \binom{n-1}{l-1} (-1)^l q^m \right) \frac{t^n}{n!}
\]
\[
= -\sum_{m=0}^{\infty} \lambda^m q^m \sum_{n=1}^{\infty} \frac{n}{(1-q)^{n-1}} (1-q^m)^{n-1} \frac{t^n}{n!}
\]
\[
= -\sum_{m=0}^{\infty} \lambda^m q^m \sum_{n=0}^{\infty} \frac{(1-q^m)^n t^{n+1}}{(1-q)^n} \frac{t^n}{n!}
\]
\[
= -t \sum_{m=0}^{\infty} \lambda^m q^m e^{[m]_q t}.
\]

Therefore we obtain the following lemma.

**Lemma 5.** Let \( F^*(t, \lambda : q) = \sum_{n=0}^{\infty} B_{n,q}(\lambda) \frac{t^n}{n!} \). Then we have
\[
F^*(t, \lambda : q) = -t \sum_{m=0}^{\infty} \lambda^m q^m e^{[m]_q t}.
\]

We also have
\[
B_{k,q}(\lambda) = \frac{d^k F_q(t, \lambda : q)}{(dt)^k} \bigg|_{t=0} = -k \sum_{m=0}^{\infty} q^m \lambda^m [m]_q^{k-1}.
\]

Therefore we obtain the following theorem.

**Theorem 6.** For \( k \in \mathbb{N} \cup \{0\} \) and \( \lambda \in T_p \), we have
\[
B_{k,q}(\lambda) = -k \sum_{m=0}^{\infty} q^m \lambda^m [m]_q^{k-1}.
\]

### 3. q-extension of \( \lambda \)-Euler numbers and polynomials

In this section, we assume that \( p \) is an odd prime number and \( \lambda \in \mathbb{C}_p \) with \( |1-\lambda|_p < 1 \). By using the fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \), we consider the \( q \)-extensions of \( \lambda \)-Euler numbers as follows.

For \( n \in \mathbb{N} \cup \{0\} \), we define the \( q \)-extension of \( \lambda \)-Euler numbers as
\[
E_{n,q}(\lambda) = \int_{\mathbb{Z}_p} q^{-x} \lambda^x [x]^n d\mu_{-q}(x).
\]
From (19), we note that
\[
E_{n,q}(\lambda) = \int_{\mathbb{Z}} q^{-x} \lambda^x [x]^n_q d\mu_{-q}(x)
\]
\[
= \lim_{N \to \infty} \frac{1 + q}{1 + q^N} \sum_{x=0}^{2^{N-1}} (-1)^x [x]^n_q \lambda^x
\]
\[
= \frac{2}{2} \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} \frac{n}{l} (-1)^l \lim_{N \to \infty} \frac{1 + q^N \lambda^N}{1 + q^N}
\]
\[
= \frac{2}{2} \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} \frac{n}{l} (-1)^l \frac{2}{1 + q^N}
\]
\[
= \frac{2}{2} \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} \frac{n}{l} (-1)^l \frac{1}{1 + q^N}
\]

Therefore we obtain the following theorem.

**Theorem 7.** For \( n \in \mathbb{N} \cup \{0\} \), we have
\[
E_{n,q}(\lambda) = \frac{2}{2} \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} \frac{n}{l} (-1)^l \frac{1}{1 + q^N}
\]

Let \( g(t, \lambda : q) \) be the generating function of \( E_{n,q}(\lambda) \) with
\[
g(t, \lambda : q) = \sum_{n=0}^{\infty} E_{n,q}(\lambda) \frac{t^n}{n!}
\]

Then we have
\[
g(t, \lambda : q) = \sum_{n=0}^{\infty} E_{n,q}(\lambda) \frac{t^n}{n!} = \int_{\mathbb{Z}} q^{-x} \lambda^x e^{[x]_q t} d\mu_{-q}(x)
\]
\[
= \sum_{n=0}^{\infty} \{ \int_{\mathbb{Z}} q^{-x} \lambda^x [x]^n_q d\mu_{-q}(x) \} \frac{t^n}{n!}
\]
\[
= [2q] \sum_{n=0}^{\infty} \frac{1}{(1 - q)^n} \sum_{l=0}^{n} \frac{n}{l} (-1)^l \left( \frac{1}{1 + \lambda q^l} \right) \frac{t^n}{n!}
\]
\[
= [2q] \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^m}{(1 - q)^n} \sum_{l=0}^{n} \frac{n}{l} (-1)^l \left( \sum_{m=0}^{\infty} (-1)^m \lambda^m q^{lm} \right) \frac{t^n}{n!}
\]
\[
= [2q] \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^m}{(1 - q)^n} \sum_{l=0}^{n} \frac{n}{l} (-1)^l \left( \sum_{m=0}^{\infty} (-1)^m \lambda^m q^{lm} \right) \frac{t^n}{n!}
\]
\[
= [2q] \sum_{m=0}^{\infty} \frac{(-1)^m \lambda^m}{(1 - q)^n} \sum_{n=0}^{\infty} \frac{n}{l} \frac{t^n}{n!}
\]
\[
= [2q] \sum_{m=0}^{\infty} (-1)^m \lambda^m e^{[m]_q t}
\]

Thus, we have the following lemma.
Lemma 8. Let \( g(t, \lambda : q) = \sum_{n=0}^{\infty} E_{n,q}(\lambda) \frac{t^n}{n!} \). Then we have

\begin{equation}
 g(t, \lambda : q) = [2]_q \sum_{m=0}^{\infty} (-1)^m \lambda^m e^{[m]_q t}.
\end{equation}

By (20), we can also consider the \( \lambda \)-\( q \)-Genocchi numbers as follows.

\begin{equation}
 t \int_{\mathbb{Z}_p} q^{-x} x^n e^{[x]_q t} d\mu_q(x) = [2]_q t \sum_{m=0}^{\infty} (-1)^m \lambda^m e^{[m]_q t} = \sum_{n=0}^{\infty} G_{n,q}(\lambda) \frac{t^n}{n!}.
\end{equation}

From (21), we note that \( G_{0,q}(\lambda) = 0 \) and

\begin{equation}
 \int_{\mathbb{Z}_p} q^{-x} \lambda^x [x]_q^m d\mu_q(x) = \frac{G_{n+1,q}(\lambda)}{n+1}.
\end{equation}

Thus, we see that

\begin{equation}
 E_{n,q}(\lambda) = \int_{\mathbb{Z}_p} q^{-x} \lambda^x [x]_q^n d\mu_q(x) = \frac{G_{n+1,q}(\lambda)}{n+1}.
\end{equation}

Hence

\begin{equation}
 G_{n,q}(\lambda) = [2]_q \frac{n}{1-q^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \frac{1}{1+q^l \lambda},
\end{equation}

where \( n = 1, 2, 3, \cdots \). Indeed,

\begin{align*}
 G_{1,q}(\lambda) &= \frac{[2]_q}{1+\lambda}, \\
 G_{2,q}(\lambda) &= \frac{2[2]_q}{1-q} \sum_{l=0}^{1} \binom{1}{l} (-1)^l \frac{1}{1+q^l \lambda} = \frac{2[2]_q}{1-q} \left( \frac{2}{1+\lambda} - \frac{2}{1+q\lambda} \right) \\
 &= -2[2]_q \frac{\lambda}{(1+\lambda)(1+q\lambda)}.
\end{align*}

Now, we consider the \( q \)-extension of \( \lambda \)-Euler polynomials as follows.

\begin{equation}
 E_{n,q}(\lambda, x) = \int_{\mathbb{Z}_p} q^{-y} x^n e^{[x+y]_q t} d\mu_q(y).
\end{equation}

From (22), we can easily derive

\begin{equation}
 E_{n,q}(\lambda, x) = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n \binom{n}{l} (-1)^l q^l x \frac{1}{1+q^l \lambda}.
\end{equation}

Let \( g(x, \lambda : q) = \sum_{n=0}^{\infty} E_{n,q}(\lambda, x) \frac{t^n}{n!} \). Then we have

\begin{align*}
 g(x, \lambda : q) &= \sum_{n=0}^{\infty} E_{n,q}(\lambda, x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} q^{-y} x^n e^{[x+y]_q t} d\mu_q(y) \\
 &= \sum_{n=0}^{\infty} \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n \binom{n}{l} (-1)^l q^l x \left( \sum_{m=0}^{\infty} (-1)^m q^m \lambda^m \right) \frac{t^n}{n!} \\
 &= [2]_q \sum_{m=0}^{\infty} (-1)^m \lambda^m e^{[m]_q t}.
\end{align*}
It follows that
\[ E_{n,q}(\lambda, x) = \frac{d^n(g(x, \lambda; g))}{(dt)^n} \bigg|_{t=0} = [2]_q \sum_{m=0}^{\infty} (-1)^m \lambda^m [m + x]_q^n. \]

Then we obtain the following theorem.

**Theorem 9.** For \( n \in \mathbb{N} \cup \{0\} \), we have
\[ E_{n,q}(\lambda, x) = [2]_q \sum_{m=0}^{\infty} (-1)^m \lambda^m [m + x]_q^n. \]

By the same method, we consider the \( \lambda \)-q-Genocchi polynomials as follows.
\[
(23) \quad t \int_{\mathbb{Z}_p} q^{-x} \lambda^x e^{[x+y]q^t} d\mu_q(x) = [2]_q t \sum_{m=0}^{\infty} (-1)^m \lambda^m [m + x]_q^n.
\]

By (23), we see
\[
E_{n,q}(\lambda, x) = \frac{1}{n!} \sum_{n=0}^{\infty} G_{n,q}(\lambda, x) \frac{t^n}{n!}.
\]

It is easy to see that
\[
qI_q(f_1) + I_q(f) = [2]_q f(0),
\]
where \( f_1(x) = f(x + 1) \). Thus, we have
\[
q \int_{\mathbb{Z}_p} q^{-y-1} \lambda^y [x+1+y]_q^n d\mu_q(y) + \int_{\mathbb{Z}_p} q^{-y} \lambda^y [x+y]_q^n d\mu_q(y) = [2]_q [x]_q^n.
\]

Therefore, we obtain the following theorem.

**Theorem 10.** For \( n \in \mathbb{N} \cup \{0\} \), we have
\[ \lambda E_{n,q}(\lambda, x + 1) + E_{n,q}(\lambda, x) = [2]_q [x]_q^n. \]

By Theorem 10 and (24), we have the following result.
Corollary 11. For \( n \in \mathbb{N} \cup \{0\} \), we have
\[
\lambda G_{n,q}(\lambda, x + 1) + G_{n,q}(\lambda, x) = [2]_q^n[x]_{q}^{n-1}.
\]

It is easy to see that
\[
\frac{\partial}{\partial x} \left[ x+y \right]_q^n = n[x+y]_{q}^{n-1} \frac{\log q}{q-1}x+y
\]
\[
= n \log q [x+y]_{q}^{n-1} + \frac{\log q}{q-1} n[x+y]_{q}^{n-1}.
\]

From (22), we note that
\[
\frac{\partial}{\partial x} E_{n,q}(\lambda, x) = \frac{\partial}{\partial x} \int_{\mathbb{Z}_p} q^{-y} \lambda^y [x+y]_{q}^n d\mu_{-q}(y).
\]

The right side of (26) equals
\[
n \log q \int_{\mathbb{Z}_p} q^{-y} \lambda^y [x+y]_{q}^n d\mu_{-q}(y) + \frac{\log q}{q-1} \int_{\mathbb{Z}_p} q^{-y} \lambda^y [x+y]_{q}^n d\mu_{-q}(y)
\]
\[
= n \log q E_{n,q}(\lambda, x) + \frac{\log q}{q-1} nE_{n-1,q}(\lambda, x).
\]

Therefore, we obtain the following lemma.

Lemma 12. For \( n \in \mathbb{N} \), we have
\[
\frac{\partial}{\partial x} E_{n,q}(\lambda, x) = n \log q E_{n,q}(\lambda, x) + \frac{\log q}{q-1} nE_{n-1,q}(\lambda, x).
\]

Remark 1. Note that
\[
\frac{\partial}{\partial x} G_{n,q}(\lambda, x) = nE_{n-1,q}(\lambda, x)
\]
\[
= \frac{n}{(1-q)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^l \frac{1}{1+q^l}.
\]

Remark 2. Note that
\[
E_{n,q}(\lambda, dx) = \int_{\mathbb{Z}_p} q^{-y} \lambda^y [dx+y]_{q}^n d\mu_{-q}(y)
\]
\[
= [d]_q^n \frac{[2]_q}{[2]_q^d} \sum_{a=0}^{n-1} (-1)^a \lambda^a \int_{\mathbb{Z}_p} [x+a+\frac{a}{d}] \lambda^d q^{-d} dy d\mu_{-q^a}(y)
\]
\[
= [d]_q^n \frac{[2]_q}{[2]_q^d} \sum_{a=0}^{n-1} (-1)^a \lambda^a E_{n,q^a}(\lambda^d, x+\frac{a}{d}),
\]

for \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \).

For \( n \in \mathbb{N} \), it is known that
\[
q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \quad \text{see [7]},
\]

where \( f_n(x) = f(x+n) \). By (27), we obtain the following lemma.
Lemma 13. For \( n \in \mathbb{N} \), we have
\[
q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]^q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l).
\]

For \( n \equiv 1 \pmod{2} \), we also have
\[
q^n I_{-q}(f_n) + I_{-q}(f) = [2]^q \sum_{l=0}^{n-1} (-1)^l q^l f(l).
\]

If we take \( f(x) = \lambda^x q^{-x} [x]^m \) with \( m \in \mathbb{N} \cup \{0\} \), then we see that
\[
q^n \int_{\mathbb{Z}_p} q^{-x-n} \lambda^x [x+n]_q^m d\mu_{-q}(x) + \int_{\mathbb{Z}_p} q^{-x} \lambda^x [x]_q^m d\mu_{-q}(x) = [2]^q \sum_{l=0}^{n-1} (-1)^l \lambda^l [l]_q^m.
\]

Thus we have
\[
E_{m,q}(\lambda, n) + E_{m,q}(\lambda) = [2]^q \sum_{l=0}^{n-1} (-1)^l \lambda^l [l]_q^m.
\]

For \( m \equiv 1 \pmod{2} \), we note that
\[
E_{n,q}(\lambda, dx) = \frac{[2]^q}{[2]^m} \sum_{a=0}^{m-1} (-1)^a \lambda^a E_{n,q}^{m}(\lambda^m, \frac{a}{m})
\]
\[
= \frac{[2]^q}{[2]^m} \sum_{a=0}^{m-1} (-1)^a \lambda^a E_{E_{1,q}^{m}}(\lambda^m) \sum_{l=0}^{n-1} (-1)^a \lambda^a q^a [a]_q^{n-l}.
\]

Remark 3. Note that
\[
\frac{G_{m+1,q}(\lambda, n)}{m+1} + \frac{G_{m+1,q}(\lambda)}{m+1} = [2]^q \sum_{l=0}^{n-1} (-1)^l \lambda^l [l]_q^m.
\]

Now we can also consider the following DC type \( \lambda-q \)-Euler numbers and polynomials. For \( \lambda \in \mathbb{C}_p \) with \( |1-\lambda|_p < 1 \), we define the DC type \( \lambda-q \)-Euler numbers as
\[
E_{n,q}^*(\lambda) = \int_{\mathbb{Z}_p^n} \lambda^x [x]_q^m d\mu_{-q}(x)
\]
\[
= \frac{[2]^q}{(1-q)^n} \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) (-1)^l \frac{1}{1 + q^{l+1} \lambda}
\]
\[
= [2]^q \sum_{m=0}^{\infty} (-1)^m \lambda^m q^m [m]_q^n.
\]

Let \( g^*(t, \lambda : q) = \sum_{n=0}^{\infty} E_{n,q}^*(\lambda) \frac{t^n}{n!} \). Then we see that
\[
g^*(t, \lambda : q) = \int_{\mathbb{Z}_p^n} \lambda^x [x]_q^m d\mu_{-q}(x) = [2]^q \sum_{m=0}^{\infty} (-1)^m \lambda^m q^m e^{[m]_q}.
\]
The DC type $\lambda$-$q$-Euler polynomials are also defined as

$$E^*_n(\lambda, x) = \int_{\mathbb{Z}} \lambda^y[x + y]^n d\mu_q(y).$$

Thus we can give the generating function of the DC type $\lambda$-$q$-Euler polynomials as follows.

$$\sum_{n=0}^{\infty} E^*_n(\lambda, x) \frac{t^n}{n!} = \int_{\mathbb{Z}} \lambda^y e^{(x+y)t} d\mu_q(y) = \sum_{m=0}^{\infty} (-1)^m \lambda^m q^m [m + x]^n.$$

4. Further Remarks and Observation for the $q$-extension Lerch zeta function

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. It is well-known that Lerch type zeta function is defined as

$$\zeta(x, s, a) = \sum_{n=0}^{\infty} \frac{x^n}{(n + a)^s},$$

where $a \in \mathbb{C}$ with $a \not= 0, -1, -2, \ldots$, and $s \in \mathbb{C}$ when $|x| < 1$, $Re(S) > 1$ when $|x| = 1$, and Hurwitz zeta function is defined as

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s},$$

where $Re(S) > 1$ and $a \not= 0, -1, -2, \ldots$. The Lerch zeta function is known that

$$\zeta(s, \eta) = \sum_{n=0}^{\infty} \frac{e^{2\pi in}}{n^s} = e^{2\pi in} \zeta(e^{2\pi in}, 1),$$

where $\eta \in \mathbb{R}$ and $Re(S) > 1$.

Now we consider the first kind of the $q$-extension of Lerch type zeta function as follows.

$$\zeta_q(\lambda, s) = (1 - q)^{s - 1} \sum_{m=1}^{\infty} \frac{q^m \lambda^m}{[m]_q^s} + \sum_{m=1}^{\infty} \frac{q^m \lambda^m}{[m]_q^s},$$

where $q \in \mathbb{C}$ with $|q| < 1$, and $\lambda \in \mathbb{C}$ with $\lambda = e^{2\pi i f}$ ($f \in \mathbb{N}$).

By Theorem 3, we see that

$$\frac{-\beta_{k, q}(\lambda)}{k} = (q - 1) \frac{k + 1}{k} \sum_{m=1}^{\infty} q^m \lambda^m [n]_q^k + \sum_{m=1}^{\infty} q^m \lambda^m [m]_q^{k-1},$$

for $k \in \mathbb{N}$.

By (28) and (29), we obtain the following theorem.
**Theorem 14.** For $k \in \mathbb{N}$, we have
\[
\zeta_q(\lambda, 1-k) = -\frac{\beta_{k,q}(\lambda)}{k}.
\]

Now, we define the second of the $q$-extension of Lerch zeta function as follows. For $s \in \mathbb{C}$ and $\lambda = e^{2\pi i/f}$ ($f \in \mathbb{N}$), define
\[
\zeta^*_q(\lambda, s) = \sum_{m=1}^{\infty} q^{m} \lambda^m \left[\frac{m}{s}\right]_q.
\]

By Theorem 6, we easily see that
\[
-\frac{\beta_{k,q}(\lambda)}{k} = \sum_{m=1}^{\infty} q^{m} \lambda^m [m]_q^{k-1}.
\]

By (30) and (31), we obtain the following theorem.

**Theorem 15.** For $k \in \mathbb{N}$, we have
\[
\zeta^*_q(\lambda, 1-k) = -\frac{\beta_{k,q}(\lambda)}{k}.
\]

**Remark 4.** The extension of Hurwitz’s type $q$-Euler zeta function is defined as
\[
\zeta_{q,E}(\lambda, s) = [2]_q \sum_{m=0}^{\infty} \frac{(-1)^m \lambda^m}{[m+x]_q^s},
\]
where $s \in \mathbb{C}$, $\lambda \in \mathbb{C}$ with $\lambda = e^{2\pi i/f}$ ($f \in \mathbb{N}$). Then we have
\[
\zeta_q(\lambda, 1-k) = E_{k,q}(\lambda, x), \quad (k \in \mathbb{N}).
\]

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