Transcendence of digital expansions and continued fractions generated by a cyclic permutation and \( k \)-adic expansion

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Abstract
In this article, first we generalize the Thue-Morse sequence \((a(n))_{n=0}^{\infty}\) (the generalized Thue-Morse sequences) by a cyclic permutation and \( k \)-adic expansion of natural numbers, and consider the necessary-sufficient condition that it is non-periodic. Moreover we will show that, if the generalized Thue-Morse sequence is not periodic, then all equally spaced subsequences \((a(N + nl))_{n=0}^{\infty}\) (where \( N \geq 0 \) and \( l > 0 \)) of the generalized Thue-Morse sequences are not periodic. Finally we apply the criterion of [ABL], [Bu1] on transcendental numbers, to find that, for a non periodic generalized Thue-Morse sequences taking the values on \( \{0, 1, \ldots, \beta - 1\} \) (where \( \beta \) is an integer greater than 1), the series \( \sum_{n=0}^{\infty} a(N + nl)\beta^{-n-1} \) gives a transcendental number, and further that for non periodic generalized Thue-Morse sequences taking the values on positive integers, the continued fraction \([0 : a(N), a(N + l), \ldots, a(N + nl), \ldots]\) gives a transcendental number, too.

1 Introduction

In this paper we find many transcendental real numbers defined by using digit counting: Let \( k \) be an integer greater than 1. We define the \( k \)-adic expansion of natural number \( n \) as follows,

\[
n = \sum_{q=1}^{finite} s_{n,q}k^{w_n(q)},
\]  

(1.1)
where \(1 \leq s_{n,q} \leq k - 1, w_n(q + 1) > w_n(q) \geq 0\). Let \(e_s(n)\) denote the number of counting of \(s\) (where \(s \in \{1, \ldots, k - 1\}\)) in the base \(k\) representation of \(n\). For an integer \(L\) greater than 1, we define the sequence \((e^L_s(n))_{n=0}^{\infty}\) by
\[
e^L_s(n) \equiv e_s(n) \pmod{L},
\]
where \(0 \leq e^L_s(n) \leq L - 1, e_s(0) = 0\). Specifically \((e^2_1(n))_{n=0}^{\infty}\) (where \(k = 2\)) is known as the Thue-Morse sequence.

Now we introduce a new sequence as follows. Let \(K\) be a map,
\[
K : \{1, \ldots, k - 1\} \rightarrow \{0, 1, \ldots, L - 1\}.
\]
We put \((a(n))_{n=0}^{\infty}\) by
\[
a(n) \equiv \sum_{s=1}^{k-1} K(s) e^L_s(n) \pmod{L},
\]
where \(0 \leq a(n) \leq L - 1\). Then the authors [MM], [AB1] proved the following result.

**Theorem 1.1 (MM-AB1)** Let \(\beta \geq L\) be an integer. Then \(\sum_{n=0}^{\infty} a(n) \beta^n + 1\) is a transcendental number unless
\[
sK(1) \equiv K(s) \pmod{L} \text{ for all } 1 \leq s \leq k - 1 \text{ and } K(k - 1) \equiv 0 \pmod{L}.
\]

Now we define the generalized Thue-Morse sequences as follows: Let \(d(n; s^y)\) (where \(s \in \{1, \ldots, k - 1\}\) and \(y\) is a non negative integer) be 1 or 0, \(d(n; s^0) = 1\) if and only if there exists an integer \(q\) such that \(s_{n,q}k^{w_n(q)} = s^y\). Let \(\kappa\) be a map,
\[
\kappa : \{1, \ldots, k - 1\} \times \mathbb{N} \rightarrow \{0, 1, \ldots, L - 1\},
\]
where \(\mathbb{N}\) denotes the set of non-negative integers (or, natural numbers). We put \((a(n))_{n=0}^{\infty}\) by
\[
a(n) \equiv \sum_{y=0}^{\infty} \sum_{s=1}^{k-1} \kappa(s,y)d(n; s^y) \pmod{L},
\]
where \(0 \leq a(n) \leq L - 1, a(0) = 0\). In this article we will call \((a(n))_{n=0}^{\infty}\) the generalized Thue-Morse sequences of type \((L, k, \kappa)\). In this article, we generalize Theorem 1.1 as follows.

**Theorem 1.2** Let \(\beta \geq L\) be an integer. Then \(\sum_{n=0}^{\infty} a(N+n) \beta^n + 1\) (where \(N \geq 0\) and \(l > 0\)) is a transcendental number unless there exists an integer \(A\) such that
\[
\kappa(s, A+y) \equiv \kappa(1, A)s^y \pmod{L},
\]
where \(0 \leq s \leq k - 1\) and for all \(y \in \mathbb{N}\).
The proof of Theorem 1.2 rests on the results in section 3 (especially Lemma 3.1) and the transcendence criterion of [ABL].

This paper is organized as follows: In section 2 we review the basic concepts about the periodicity of sequences, and give the formally definition of the generalized Thue-Morse sequences. For a sequence \((a(n))_{n=0}^\infty\), we set its generating function \(g(z) \in \mathbb{C}[[z]]\) by

\[
g(z) := \sum_{n=0}^{\infty} a(n)z^n.
\]

For a generalized Thue-Morse sequence, one can prove that the generating function is convergent, and that it has an infinite product expansion. In section 3, first we prove the key lemma on the \(k\)-adic expansion of natural numbers. Then we will use this lemma and the infinite product expansion of the generating function of a generalized Thue-Morse sequence to prove a necessary-sufficient condition for the non-periodicity of the sequence. Moreover we prove that if the sequence is not periodic, then all equally spaced subsequences \((a(N + nl))_{n=0}^\infty\) (where \(N \geq 0\) and \(l > 0\)) of the sequence are not periodic. In section 4, first we introduce the concept of stammering sequence ([AB1], [Bu1]) and the combinatorial transcendence criterion of [ABL], [Bu1]. Then applying combinatorial transcendence criterion of [ABL], [Bu1] to the generalized non-periodic Thue-Morse sequence \((a(n))_{n=0}^\infty\) which take the values on \(\{0, 1, \ldots, \beta - 1\}\) where \(\beta\) is an integer greater than 1 (resp. which take the values on bounded positive integers), we show that \(\sum_{n=0}^{\infty} a(N + nl)\beta^{-n-1}\) are transcendental numbers. (resp. the continued fraction \([0 : a(N), a(N + l)\ldots, a(N + nl)\ldots]\) are transcendental numbers.) These results include Theorem 1.2. In section 5, first we consider the necessary-sufficient condition that a generalized Thue-Morse sequence of is \(k\)-automatic sequence. Then we can find many transcendental numbers whose irrationality exponent is finite in all equally spaced subsequences of the corresponding generalized Thue-Morse sequence by result of [AC]. Moreover we consider transcendency of the value at algebraic point of the generating function \(\sum_{n=0}^{\infty} a(N + nl)z^{-n-1}\) by [Bec].

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2 Definition of the generalized Thue-Morse sequences and their generating functions

Let \((a(n))_{n=0}^{\infty}\) be a sequence with values in \(\mathbb{C}\). We say that \((a(n))_{n=0}^{\infty}\) is ultimately periodic if there exist non-negative integers \(N\) and \(l(0 < l)\) such that
\[
a(n) = a(n + l) \quad (\forall n \geq N). \tag{2.1}
\]

An equally spaced subsequence of \((a(n))_{n=0}^{\infty}\) is defined to be a subsequence such as \((a(N + tl))_{t=0}^{\infty}\), where \(N \geq 0\) and \(l > 0\).

**Definition 2.1** Let \((a(n))_{n=0}^{\infty}\) be a sequence with values in \(\mathbb{C}\). The sequence \((a(n))_{n=0}^{\infty}\) is called almost everywhere non periodic if all equally spaced subsequences of \((a(n))_{n=0}^{\infty}\) do not take on only one value.

Now we show some lemmas about the almost everywhere non periodic sequences.

**Lemma 2.1** Let \((a(n))_{n=0}^{\infty}\) be almost everywhere non periodic. Then \((a(n))_{n=0}^{\infty}\) is not ultimately periodic.

**Proof.** We show in contraposition. Assume that \((a(n))_{n=0}^{\infty}\) is ultimately periodic. From the definition of almost everywhere non periodic, we see that there exist non-negative integers \(N\) and \(l(0 < l)\) such that
\[
a(n) = a(n + l) \quad (\forall n \geq N). \tag{2.2}
\]
From this, it follows that the equally spaced subsequence \((a(N + tl))_{t=0}^{\infty}\) takes on only one value. This completes the proof. \(\square\)

**Lemma 2.2** Let \((a(n))_{n=0}^{\infty}\) be almost everywhere non periodic. Then all equally spaced subsequences of this sequence are almost everywhere non periodic.

**Proof.** We show in contraposition. If \((a(N + tl))_{t=0}^{\infty}\) is not almost everywhere non periodic, then there exist non-negative integers \(k\) and \(J(0 < J)\) such that \((a(N + k + ml))_{m=0}^{\infty}\) takes on only one value. \((a(N + k + ml))_{m=0}^{\infty}\) is an equally spaced subsequence of \((a(n))_{n=0}^{\infty}\), too. Then \((a(n))_{n=0}^{\infty}\) is not almost everywhere non periodic. This completes the proof. \(\square\)

**Corollary 2.1** \((a(n))_{n=0}^{\infty}\) is almost everywhere non periodic if and only if all equally spaced subsequences of this sequence are not ultimately periodic.

**Proof.** We assume that \((a(n))_{n=0}^{\infty}\) is almost everywhere non periodic. By Lemma 2.1, 2.2, we see that all equally spaced subsequences of this sequence are not periodic.

We show the converse of collorally in contraposition. Assume \((a(n))_{n=0}^{\infty}\) is not almost everywhere non periodic. Then there exist non-negative integers \(N\) and \(l(0 < l)\) such that \((a(N + tl))_{t=0}^{\infty}\) takes on only one value. This sequence is ultimately periodic. This completes the proof. \(\square\)
Now we give the formally definition of the generalized Thue-Morse sequences.

**Definition 2.2** Let $L$ be an integer greater than 1 and $a_0, a_1, \cdots, a_{L-1}$ different $L$ complex numbers. Define a morphism $f$ from $\{a_0, a_1, \cdots, a_{L-1}\}$ to $\{a_0, a_1 \cdots a_{L-1}\}$ as follows,

$$f(a_i) = a_{i+1},$$

where the indice $i$ is defined mod $L$. Let $f^j$ be the $j$ times composed mapping of $f$ and $f^0$ an identity mapping. Let $A$ and $B$ be finite words on $\{a_0, a_1, \cdots, a_{L-1}\}$, and we denote by $AB$ concatenation of $A$ and $B$.

Let $A_0 = a_0$. Let $k$ be an integer greater than 1, $N$ the set of non negative integers. Let $\kappa$ be a map $\kappa: \{1, \cdots, p - 1\} \times N \to \{0, \cdots, L - 1\}$. We define $W_m$, a space of words, by

$$W_m := \{a_{i_1}a_{i_2} \cdots a_{i_m} | a_{i_1}, a_{i_2}, \cdots, a_{i_m} \in \{a_0, a_1 \cdots a_{L-1}\}\}.$$  

(2.4)

Define $A_{n+1} \in W_{k^{n+1}}$ recursively by

$$A_{n+1} := A_n f^{\kappa(1,n)}(A_n) \cdots f^{\kappa(k-1,n)}(A_n).$$

(2.5)

and denote the limit of $A_n$ by

$$A_\infty := \lim_{n \to \infty} A_n.$$  

(2.6)

We call $A_\infty$ the generalized Thue-Morse sequence of type $(L, k, \kappa)$. We call it $(L, k, \kappa)$-TM sequence for short.

**Example 2.1** Let $L = 2$, $a_0 = 0$, $a_1 = 1$, $\kappa(1,n) = 1$ for all $n$. Then the $(2, 2, 1)$-TM sequence is as follows,

$$A_0 = 0, \quad A_1 = 01, \quad A_2 = 0110, \quad A_3 = 01101001,$$

$$A_\infty = 0110100110010110100101100110100110010110011010010110100110 \cdots.$$

This is the Thue-Morse sequence.

Let $(a(n))_{n=0}^\infty$ be a sequence with values in $\mathbb{C}$. The generating function of $(a(n))_{n=0}^\infty$ is a formal power series $g(z) \in \mathbb{C}[[z]]$ defined by

$$g(z) := \sum_{n=0}^\infty a(n)z^n.$$  

The following lemma clarifies the meaning of $(L, k, \kappa)$-TM sequence and will be used in the next section.

**Lemma 2.3** Let $A_\infty = ((b(n))_{n=0}^\infty$ be a $(L, k, \kappa)$-TM sequence with $a_j = \exp \frac{2\pi i j}{L}$ (where $0 \leq j \leq L - 1$). Let $G_{A_\infty}(z)$ be the generating function of $((b(n))_{n=0}^\infty,$

$$G_{A_\infty}(z) := \sum_{n=0}^\infty a(n)z^n.$$  

5
Then $G_{\infty}(z)$ has the infinite product on $|z| < 1$ as follows,

$$G_{\infty}(z) = \prod_{y=0}^{\infty} \left( 1 + \sum_{s=1}^{k-1} \exp \left( \frac{2\pi \sqrt{-1} \kappa(s, y)}{L} z^{sk^y} \right) \right).$$  

(2.7)

Proof. From the assumption that $a_j = \exp \frac{2\pi \sqrt{-1}}{L} j$, we have

$$f(a_j) = \exp \frac{2\pi \sqrt{-1}}{L} a_j.$$  

(2.8)

By the fact $(L, k, \kappa)$-TM sequence takes only finite values and the Cauchy-Hadamard theorem, we see that $G_{\infty}(z)$ and $\prod_{y=0}^{\infty} \left( 1 + \sum_{s=1}^{k-1} \exp \left( \frac{2\pi \sqrt{-1} \kappa(s, y)}{L} z^{sk^y} \right) \right)$ converge absolutely on the unit disk. Let $G_{\infty}(z)$ be the generating function of $A_n$ (We identifies $A_n0\cdots0\cdots$ with $A_n$).

First, we will show (2.9) by induction on $n$,

$$G_n(z) = \prod_{y=0}^{n-1} \left( 1 + \sum_{s=1}^{k-1} \exp \left( \frac{2\pi \sqrt{-1} \kappa(s, y)}{L} z^{sk^y} \right) \right).$$  

(2.9)

First we check the case $n=1$. By definition of $A_1$, we have

$$G_1(z) = 1 + \sum_{s=1}^{k-1} \exp \left( \frac{2\pi \sqrt{-1} \kappa(s, 0)}{L} z^s \right).$$  

(2.10)

Thus the case $n=1$ is true. By the induction hypothesis we may assume that,

$$G_j(z) = \prod_{y=0}^{j-1} \left( 1 + \sum_{s=1}^{k-1} \exp \left( \frac{2\pi \sqrt{-1} \kappa(s, y)}{L} z^{sk^y} \right) \right).$$  

(2.11)

Then we have,

$$G_{j+1}(z) = G_j(z) + \sum_{s=1}^{k-1} g_{f_{\kappa(s,j)}(A_j)}(z) z^{sk^j}. $$  

(2.12)

On the other hand we have,

$$G_{f_{\kappa(s,j)}(A_j)}(z) = \exp \left( \frac{2\pi \sqrt{-1} \kappa(s, j)}{L} \right) G_j(z).$$  

(2.13)

From (2.11)-(2.13), we get

$$G_{j+1}(z) = G_j(z) \left( 1 + \sum_{s=1}^{k-1} \exp \left( \frac{2\pi \sqrt{-1} \kappa(s, y)}{L} z^{sk^y} \right) \right) = \prod_{y=0}^{j} \left( 1 + \sum_{s=1}^{k-1} \exp \left( \frac{2\pi \sqrt{-1} \kappa(s, y)}{L} z^{sk^y} \right) \right).$$  

(2.14)
This completes the proof of (2.9). Next we will compare the coefficients of both sides $z^j$ of (2.7). The coefficient of $z^j$ in right-hand side of (2.7) is determined by $G_{A_N}(z)$ for large enough $N$. By definition of $A_{\infty}$, we see that the prefix $p^N$ words of $A_{\infty}$ is $A_N$. By the fact mentioned above and (2.9), the coefficients of both sides $z^j$ of (2.7) coincide. This completes the proof. □

Proposition 2.1 Let $A_{\infty} = ((b(n))_{n=0}^{\infty}$ be a $(L, k, \kappa)$-TM sequence with $a_j = \exp \frac{2\pi \sqrt{-1}}{L}$ (where $0 \leq j \leq L - 1$). Let $(a(n))_{n=0}^{\infty}$ be a sequence defined by (1.4). Then

$$\frac{L}{2\pi \sqrt{-1}} \log b(n) \equiv a(n) \pmod{L}. \quad (2.15)$$

Proof. Let the $k$-adic expansion of $n$ be as follows,

$$n = \sum_{q=1}^{n(k)} s_{n,q} k^{w_n(q)}, \quad (2.16)$$

where $1 \leq s_{n,q} \leq k - 1$, $0 \leq w_n(q) < w_n(q + 1)$. By the fact uniqueness of the $k$-adic expansion and Lemma 2.3, we have

$$b(n) = \prod_{q=1}^{n(k)} \exp \frac{2\pi \sqrt{-1}(\kappa(s_{n,q}, w_n(q)))}{L} = \exp \frac{2\pi \sqrt{-1}(\sum_{q=1}^{n(k)} \kappa(s_{n,q}, w_n(q)))}{L} \pmod{L}. \quad (2.17)$$

By (2.16), (2.17) and definition of $a(n)$, we get

$$\frac{L}{2\pi \sqrt{-1}} \log b(n) \equiv a(n) \pmod{L}. \quad (2.18)$$

This completes the proof.

Definition 2.3 Let $(a(n))_{n=0}^{\infty}$ be a sequence with values in $\mathbb{C}$. Let $g(z)$ be the generating function of $(a(n))_{n=0}^{\infty}$. We say $(a(n))_{n=0}^{\infty}$ is a $k$-adic expansion sequence if the $g(z)$ has the following infinite product expansion for an integer $k$ greater than 1 and $t_{s,j} \neq 0$ for all $1 \leq s \leq k - 1$ and for all $j \in \mathbb{N}$,

$$g(z) = \prod_{y=0}^{\infty} (1 + \sum_{s=1}^{k-1} t_{s,y} z^{sk^y}). \quad (2.19)$$

3 The necessary-sufficient condition for the non-periodicity of a generalized Thue-Morse sequence
First, we show the following key lemma about the $k$-adic expansion of natural numbers.

**Lemma 3.1** Let $k > 1$ and $l > 0$ be integers. Let $t$ be a non negative integer. Then there exists an integer $x$ such that,

$$xl = \sum_{q=1}^{\text{finite}} s_{xl,q} k^{w_{xl}(q)},$$  \hspace{1cm} (3.1)

where $s_{xl,1} = 1$, $w_{xl}(2) - w_{xl}(1) > t$, $w_{xl}(q + 1) > w_{xl}(q) \geq 0$.

Moreover let $t'$ be other non negative integer. Then there exists an integer $X$ such that,

$$Xl = \sum_{q=1}^{\text{finite}} s_{Xl,q} k^{w_{Xl}(q)},$$  \hspace{1cm} (3.2)

where $s_{Xl,1} = 1$, $w_{Xl}(2) - w_{Xl}(1) > t'$, $w_{Xl}(q + 1) > w_{Xl}(q) \geq 0$, $w_{xl}(1) = w_{Xl}(1)$.

**Proof.** Assume the factorization into prime factors of $k$ as follows,

$$k = \prod_{t=1}^{k} p_t^{y_t},$$  \hspace{1cm} (3.3)

where $p_t$ (1 ≤ $t$ ≤ $k$) are $k$ prime numbers and $y_t$ (1 ≤ $t$ ≤ $k$) are $k$ positive integers. Let $l$ be represented as follows,

$$l = G \prod_{u=1}^{n} p_{u}^{x_u},$$  \hspace{1cm} (3.4)

where $G$ and $k$ are coprime, $p_u \in \{p_t | 1 \leq t \leq k\}$ and $x_u$ are $n$ positive integers. By the fact $G$ and $k$ are coprime, then there exist integers $D$ and $E$ such that

$$DG = 1 - k^{t+1}E.$$  \hspace{1cm} (3.5)

Let

$$F := \max\{A|y_u = y_{u}A + H, \ 0 \leq H < y_u, \ 1 \leq u \leq n\}.$$  

From the definition of $F$, we see that $k^{F+1} \prod_{u=1}^{n} p_{u}^{-x_u}$ is a natural number. Then we have

$$lD^2 G k^{F+1} \prod_{u=1}^{n} p_{u}^{-x_u} = k^{F+1} D^2 G^2.$$  \hspace{1cm} (3.6)

On the other hands, by (3.5), we have

$$D^2 G^2 = 1 + k^{t+1}E(k^{t+1}E - 2).$$  \hspace{1cm} (3.7)
Then we see that $E(k^{t+1}E - 2)$ is a natural number. Since the $k$-adic expansion of $E(k^{t+1}E - 2)$, if $E(k^{t+1}E - 2) > 0$, then $k^{F+1}D^2G^2$ satisfies lemma. If $E(k^{t+1}E - 2) = 0$, then $G = 1$. $k^{F+1}(1 + k^{t+1})$ satisfies lemma.

Since $F + 1$ independent of $t$, then the second claim is trivial. This completes the proof.

We will show almost everywhere nonperiodic result for $k$-adic expansion sequences by previous lemma.

**Proposition 3.1** Let $A_{\infty} = (a(n))_{n=0}^{\infty}$ be a $k$-adic expansion sequence and $G_{A_{\infty}}(z)$ the generating function of this sequence. If there exists an equally spaced subsequence of $(a(n))_{n=0}^{\infty}$ which is periodic, then there exist a non-negative integer $A$ and a constant $h$ which satisfy

$$G_{A_{\infty}}(z) = \left( \sum_{n=0}^{k^A-1} a(n)z^n \right) \prod_{y=0}^{\infty} (1 + \sum_{s=1}^{k-1} h^{sk^y} z^{sk^A+y}).$$

(3.8)

**Proof.** Let $n$ and $m$ be two natural numbers and their $k$-adic expansions are as follows,

$$n = \sum_{q}^{finite} s_{n,q}k^{w_n(q)}, m = \sum_{p}^{finite} s_{m,p}k^{w_m(p)}$$

(3.9)

where $1 \leq s_{n,q}, s_{m,p} \leq k-1, w_n(q+1) > w_n(q) \geq 0,$ and $w_m(p+1) > w_m(p) \geq 0.$ From the definition of $k$-adic expansion sequence and the uniqueness of $k$-adic expansion of natural numbers, we see that if $w_n(q) \neq w_n(p)$ for all pairs $(q,p)$, then

$$a(n+m) = a(n)a(m).$$

(3.10)

Assume there exists an equally spaced subsequence of $(a(n))_{n=0}^{\infty}$ which is periodic. By Corollary 2.1, $(a(n))_{n=0}^{\infty}$ is not almost everywhere nonperiodic. Then there exist non-negative integers $N$ and $l > 0$ such that

$$a(N) = a(N + tl) \quad (\forall t \in \mathbb{N}).$$

(3.11)

Let the $k$-adic expansion of $N$ be as follows,

$$N = \sum_{q=1}^{N(k)} s_{N,q}k^{w_N(q)}$$

(3.12)

where $1 \leq s_{N,q} \leq k-1, 0 \leq w_N(q) < w_N(q+1).$

By the definition of $k$-adic expansion sequence and (3.10), we have

$$a(N) = a(N + k^r tl) = a(N)a(k^r tl) \quad (\forall r > w_N(N(k))).$$

(3.13)

$$a(N) \neq 0.$$
From (3.13) and (3.14), we get
\[ a(k'^r l) = 1 \quad (\forall r > w_N(N(k))). \] (3.15)

By Lemma 3.1, we see that there exists an integer \( x \) greater than zero such that
\[ xl = \sum_{q=1}^{x(l(k))} s_{xl,q} k^{w_{xl}(q)}. \] (3.16)

where \( s_{xl,1} = 1 \) and \( w_{xl}(2) - w_{xl}(1) > 1 \).
Moreover, by Lemma 3.1, we see that there exists an integer \( X \) greater than zero such that
\[ Xl = \sum_{q=1}^{X(l(k))} s_{Xl,q} k^{w_{Xl}(q)}. \] (3.17)

where \( s_{Xl,1} = 1 \), \( w_{Xl}(2) - w_{Xl}(1) > w_{xl}(1) \) and \( w_{Xl}(1) = w_{xl}(1) \).

Let \( x \) be any integer greater than \( w_N(N(k)) + w_{xl}(1) \) and \( s \) any integer in \( \{1, \ldots, k - 1\} \).
By the definition of \( Xl \) and (3.10), we have
\[ a(k'^sXl) = a(sk^r)a(k'^sXl - sk^r). \] (3.18)

From (3.10) and (3.15), we get
\[ 1 = a(k'^x l), \]
\[ 1 = a(k'^sXl), \]
\[ 1 = a(k'^x l + k'^sXl). \] (3.21)

By (3.10), (3.18)-(3.21), the definitions of \( xl \) and \( Xl \), we have
\[ a(k'^r)a(k'^x l - k'^r) = 1, \] (3.22)
\[ a((sk)^r)a(sXlk^r - sk^r) = 1, \] (3.23)
\[ a(k'^r(s + 1))a(xlk^r - k'^r)a(sXlk^r - sk^r) = 1. \] (3.24)

From (3.22)-(3.24), we get
\[ a(k'^r(s + 1)) = a(k'^r)s. \] (3.25)

Let \( h := a(k^{w(N(k)) + w_{xl}(1) + 1}) \) and using the same notation of Definition 2.3.
By (3.10), we have
\[ a(sk^y) = t_{s,y}. \] (3.26)

for all \( 1 \leq s \leq k - 1 \) and for all \( y \in \mathbb{N} \).
By (3.14), (3.25), (3.26) and inductively computation, we get the following relations,
\[ t_{s,w(N(k)) + w_{xl}(1) + 1 + y} = h^{sk^y}, \] (3.27)

for all \( 1 \leq s \leq k - 1 \) and for all \( y \in \mathbb{N} \). Since the definition of \( k \)-adic expansion sequence, this completes the proof. \[ \square \]
Theorem 3.1 Let \( A_\infty = (a(n))_{n=0}^\infty \) be a \((L, k, \kappa)\)-TM sequence. Then \( A_\infty = (a(n))_{n=0}^\infty \) is ultimately periodic if and only if there exists an integer \( A \) such that

\[
\kappa(s, A + y) \equiv \kappa(1, A)s^p \pmod{L},
\]

for all \( 1 \leq s \leq k - 1 \) and for all \( y \in \mathbb{N} \).

Moreover if \((L, k, \kappa)\)-TM sequence is not ultimately periodic, then all equally spaced subsequences of \((L, k, \kappa)\)-TM sequence are not ultimately periodic.

Proof. Assume without loss of generality that \( A_\infty = (a(n))_{n=0}^\infty \) is a \((L, k, \kappa)\)-TM sequence with \( a_j = \exp^{\frac{2j\sqrt{-1}}{L}} \) (where \( 0 \leq j \leq L - 1 \)). From this assumption and Lemma 2.3, \((a(n))_{n=0}^\infty \) is the \( k \)-adic expansion sequence. By the fact mentioned above and previous Proposition, we see that (3.28) is the necessary condition.

We will show converse. Let \( G_{A_\infty}(z) \) be the generating function of \((a(n))_{n=0}^\infty \). We use the same notation of Definition 2.3. Assume \((a(n))_{n=0}^\infty \) satisfies (3.28), then there exists a non negative integer \( A \) such that

\[
t_{s, A+y} = h^{sk^y} \quad (\forall y \in \mathbb{N}).
\]

Then \( G_{A_\infty}(z) \) has the infinite product expansion as follows,

\[
G_{A_\infty}(z) = (\sum_{n=0}^{k^A-1} b(n)z^n) \prod_{y=0}^{\infty} \left( 1 + \sum_{s=1}^{k-1} (h z^{sk^y}) \right). \tag{3.30}
\]

Let \( Z = h z^{k^A} \). By the fact \( h \) is \( L \)-th root of 1 and Lemma 2.3 when \( \kappa \) is zero map, we have

\[
\prod_{y=0}^{\infty} (1 + \sum_{s=1}^{k-1} Z^{sk^y}) = \prod_{n=0}^{\infty} Z^n \quad \text{on} \ |Z| < 1. \tag{3.31}
\]

Put \( G(z) = \sum_{n=0}^{k^A-1} a(n)z^n \). From (3.30) and (3.31), we get

\[
G_{A_\infty}(z) = G(z) \left( \sum_{n=0}^{\infty} (h z^{k^A})^n \right). \tag{3.32}
\]

Since \( h \) is \( L \)-th root of 1 and (3.32), we have

\[
G_{A_\infty}(z) = (G(z) \sum_{n=0}^{L-1} (h z^{k^A})^n) \left( 1 + \sum_{s=1}^{\infty} z^{s Lk^A} \right) = \frac{G(z) \sum_{n=0}^{L-1} (h z^{k^A})^n}{1 - z^{Lk^A}}. \tag{3.33}
\]

By the fact the degree of \( G(z) \) is \( k^A - 1 \) and (3.33), then we see that the sequence \((a(n))_{n=0}^\infty \) which satisfies (3.28) has period \( Lk^A \).

By the fact mentioned above and Proposition 3.1, we see that if \((L, k, \kappa)\)-TM sequence is not ultimately periodic, then all equally spaced sequences of \((L, k, \kappa)\)-TM sequence are not ultimately periodic. This completes the proof. \( \Box \)
If \((L, k, \kappa)\)-TM sequence independent of \(n\), then \((\kappa(1), \kappa(2), \ldots, \kappa(k-1))\)-L denote \((L, k, \kappa)\)-TM sequence. The weak version of the following corollary can be found as Theorem 2 in [MM] (see also [AS2], [Fr]).

**Corollary 3.1** \((\kappa(1), \kappa(2), \ldots, \kappa(k-1))\)-L is periodic if and only if \(\kappa(s)\) (for all \(1 \leq s \leq k - 1\)) which satisfies

\[
s\kappa(1) \equiv \kappa(s), \kappa(k-1) \equiv 0 \pmod{L}.
\]

Moreover if \((\kappa(1), \kappa(2), \ldots, \kappa(k-1))\)-L is not periodic, then all equally spaced subsequences of \((\kappa(1), \kappa(2), \ldots, \kappa(k-1))\)-L are not periodic.

**Proof.** By Theorem 3.1, then we see that necessary-sufficient condition for the periodicity of \((\kappa(1), \kappa(2), \ldots, \kappa(k-1))\)-L is the following relations,

\[
\kappa(1, A+1) \equiv \kappa(1, A)k \pmod{L}, \kappa(k-1) \equiv (k-1)\kappa(1) \equiv 0 \pmod{L}.
\]

This completes the proof. \(\square\)

### 4 Transcendence results of the generalized Thue-Morse sequences

The authors [ABL] introduced the new class of sequences as follows: For any positive number \(y\), we define that \(\lfloor y \rfloor\) and \(\lceil y \rceil\) denote the floor and ceiling functions. Let \(W\) be finite word on \(\{a_0, a_1 \ldots a_{L-1}\}\). Let \(|W|\) be length of \(W\). For any positive number \(x\), we denote by \(W^x\) the word \(W[\lfloor x \rfloor]W\), where \(W^x\) is prefix of \(W\) of length \(\lfloor (x - \lfloor x \rfloor)|W|\rfloor\).

**Definition 4.1** \((a(n))_{n=0}^\infty\) is said to be a stammering sequence if \((a(n))_{n=0}^\infty\) satisfies the following conditions,

1. \((a(n))_{n=0}^\infty\) is a non periodic sequence.
2. There exist two sequences of finite words \((U_m)_{m \geq 1}\), \((V_m)_{m \geq 1}\) such that,
   - \((A)\) There exist a real number \(w > 1\) independent of \(n\) such that, the word \(U_mV_m^w\) is a prefix of the word \((a(n))_{n=0}^\infty\).
   - \((B)\) \(\lim_{m \to \infty} |U_m|/|V_m| < +\infty\),
   - \((C)\) \(\lim_{m \to \infty} |V_m| = +\infty\).

Let \((a(n))_{n=0}^\infty\) be a sequence of positive integers. We put the continued fractions as follows,

\[
[0 : a(0), a(1), \ldots, a(n) \ldots] := 0 + \frac{1}{\frac{1}{a(0)} + \frac{1}{\frac{1}{a(1)} + \frac{1}{\cdots + \frac{1}{a(n) + \cdots}}}}.
\]

The authors [ABL], [Bu1] proved the following amazing result by using Schmidt Subspace Theorem.
Theorem 4.1 (ABL-Bu1) Let \( \beta \) be an integer greater than 1. Let \((a(n))_{n=0}^{\infty}\) be a stammering sequence on \(\{0, 1, \ldots, \beta - 1\}\). Then \(\sum_{n=0}^{\infty} \frac{a(n)}{\beta^n}\) is a transcendental number. Moreover if \((a(n))_{n=0}^{\infty}\) be a stammering sequence on bounded positive integers, then the continued fraction \([0 : a(0), a(1) \cdots, a(n) \cdots]\) is a transcendental number, too.

By Theorem 3.1 and 4.1, we will show the next theorem that includes Theorem 1.2.

Theorem 4.2 Let \(A_{\infty} = (a(n))_{n=0}^{\infty}\) be a \((L, k, \kappa)\)-TM sequence. Let \(\beta\) be an integer greater than 1. If \((a(n))_{n=0}^{\infty}\) takes the values on \(\{0, 1, \ldots, \beta - 1\}\), then \(\sum_{n=0}^{\infty} \frac{a(N+nl)}{\beta^{n+1}}\) (where \(N \geq 0\) and \(l > 0\)) is a transcendental numbers unless there exists an integer \(A\) such that

\[
\kappa(s, A + y) \equiv \kappa(1, A)sk^y \pmod{L},
\]

for all \(1 \leq s \leq k - 1\) and for all \(y \in \mathbb{N}\).

Moreover if \((a(n))_{n=0}^{\infty}\) takes the values on positive integers, then \([0 : a(N), a(N + s) \cdots, a(N + nl) \cdots]\) (where \(N \geq 0\) and \(l > 0\)) is a transcendental numbers unless there exists an integer \(A\) such that

\[
\kappa(s, A + y) \equiv \kappa(1, A)sp^y \pmod{L},
\]

for all \(1 \leq s \leq k - 1\) and for all \(y \in \mathbb{N}\).

Proof. Let \(N\) and \(l > 0\) be positive integers. By Theorem 3.1, \((a(N + nl))_{n=0}^{\infty}\) is non periodic. Then we will prove only that \((L, k, \kappa)\)-TM satisfies the condition (2) of Definition 4.1.

There exists an integer \(M\) such that \(k^M > 2(N + l)\). Assume \(m > M\). Since \(f\) is a cyclic permutation of order \(L\) and the Definition 2.2, we see that prefix \((Ll + 1)k^m\) word of \((a(n))_{n=0}^{\infty}\) is as follows,

\[
A_{\infty} = (a(n))_{n=0}^{\infty} = A_m f^{i_1}(A_m) \cdots f^{i_L}(A_m) \cdots,
\]

where \(A_m\) is prefix \(k^m\) word of \((a(n))_{n=0}^{\infty}\), \(i_j(1 \leq j \leq L) \in \{0, \ldots, L - 1\}\).

By (4.4), we have

\[
f^{i_t}(a(n)) = a(n + k^m t),
\]

for all \(0 \leq n \leq k^m - 1\) and for all \(1 \leq t \leq L\).

By the fact \(f\) is a cyclic permutation of order \(L\), (4.4), (4.5) and Dirichlet Schubfachprinzip, then we have

\[
(a(N + nl))_{n=0}^{\infty} = W_{1,m}W_{2,m}W_{3,m}W_{2,m} \cdots,
\]

where \(W_{i,m} (i \in \{1, 2, 3\})\) are finite words and

\[
|W_{1,m}| \leq (Ll + 1)k^m - N) / l + 1,
\]

\[
|W_{2,m}| \geq (k^m - N) / l - 1,
\]

\[
W_{2,m} + |W_{3,m}| \leq (Ll + 1)k^m - N) / l + 1.
\]
We put $U_m := W_{1,m}$, $V_m := W_{2,m}W_{3,m}$ and $w := 1 + \frac{1}{2L + 3}$.

Since (4.7)-(4.9) and the assumption of $m$, then we get
\[
\left[ (w - 1) |V_m| \right] = \left[ \frac{1}{2L + 3} (|W_{2,m}| + |W_{3,m}|) \right] \leq \frac{1}{2L + 3}((Ll + 1)k^m - N + l)/l \leq \frac{k^m}{2L} < |W_{2,m}|.\
\] (4.10)

From (4.10), we see that $(a(n))_{n=0}^\infty$ satisfies the condition $(A)$. Furthermore we have,
\[
|U_m|/|V_m| = |W_{1,m}|/|W_{2,m}W_{3,m}| \leq ((Ll + 1)k^m - N + l)/l \times l/(k^m - N - l) \leq 2Ll + 3.\
\] (4.11)

From (4.11), we see that $(a(n))_{n=0}^\infty$ satisfies the condition $(B)$. Obviously, $(V_m)_{m \geq 1}$ satisfies the condition $(C)$. This completes the proof. \(\square\)

**Corollary 4.1** Let $(a(n))_{n=0}^\infty$ be a $(L, k, \kappa)$-TM sequence. Let $\beta$ be an integer greater than 1. If $(a(n))_{n=0}^\infty$ takes the values on $\{0, 1, \ldots, \beta - 1\}$, then the generating function $f(z) := \sum_{n=0}^\infty \frac{a(N+n)}{z^{N+n}}$ (where $N \geq 0$ and $l > 0$) are transcendental over $\mathbb{C}(z)$ unless there exists an integer $A$ such that
\[
\kappa(s, A + y) \equiv \kappa(1, A)s^ky \mod L,
\] (4.12)
for all $1 \leq s \leq k - 1$ and for all $y \in \mathbb{N}$.

**Proof.** We assume $f(z)$ is algebraic over $\mathbb{C}(z)$. By the fact $f(z)$ is algebraic over $\mathbb{Q}(z)$ if and only if $f(z)$ is algebraic over $\mathbb{C}(z)$ (see Remark of Theorem 1.2 in [N]), then there exist $c_i(z) \in \mathbb{Q}(z)$ ($0 \leq i \leq n$) such that $c_0(z)c_0(z) \neq 0$, $c_i(z)$ ($0 \leq i \leq n$) are coprime and satisfy following equation
\[
c_n(z)f^n(z) + c_{n-1}(z)f^{n-1}(z) + \cdots + c_0(z) = 0.\] (4.13)

From Theorem 4.2, we see that $f(\frac{1}{2})$ is a transcendental number. By the fact mentioned above and (4.13), we get $c_i(\frac{1}{2}) = 0$ (for all $0 \leq i \leq n$). This contradics the assumption that $c_i(z)$ ($0 \leq i \leq n$) are coprime. \(\square\)

5 The $k$-automatic generalized Thue-Morse sequences and some results

First we introduce some definitions.

**Definition 5.1** Let $\alpha$ be an irrational real number. The irrationality exponent $\mu(\alpha)$ of $\alpha$ is the supremum of the real numbers $\mu$ such that the inequality
\[
|\alpha - \frac{p}{q}| < \frac{1}{q^\mu},
\] (5.1)
has infinitely many solutions in non zero integers $p$ and $q$.
Definition 5.2 The $k$-kernel of $(a(n))_{n=0}^\infty$ is the set of all subsequences of the form $(a(k^n + j))_{n=0}^\infty$ where $e \geq 0$ and $0 \leq j \leq k^e - 1$.

Definition 5.3 We say $(a(n))_{n=0}^\infty$ is a $k$-automatic sequence if the $k$-kernel of $(a(n))_{n=0}^\infty$ is the finite set.

Definition 5.4 We say $\sum_{n=0}^\infty a(n)z^n \in \mathbb{C}[[z]]$ is a $k$-automatic power series if $(a(n))_{n=0}^\infty$ is a $k$-automatic sequence.

Definition 5.5 We say $(L, k, \kappa)$-TM sequence is $n$-period if there exist non-negative integers $N$ and $t (0 < t)$ such that

$$\kappa(s, n) = \kappa(s, n + t), \quad (5.2)$$

for all $1 \leq s \leq k - 1$ and for all $n \geq N$.

Now we introduce the following two results.

Theorem 5.1 (AC) Let $\beta$ be an integer greater than 1. Let $(a(n))_{n=0}^\infty$ is a non-periodic $k$-automatic sequence on $\{0, 1, \cdots, \beta-1\}$. Then $\mu(\sum_{n=0}^\infty a(n)\beta^n)$ is finite.

This theorem is obtained in [AC].

Theorem 5.2 (Be) Let $f(z) \in \mathbb{Q}[[z]]$ be a $k$-automatic power series. Let $0 < R < 1$. If $f(z)$ is transcendental over $\mathbb{Q}(z)$, then $f(\alpha)$ is transcendental for all but finitely many algebraic numbers $\alpha$ with $|\alpha| \leq R$.

This theorem is obtained in [Bec].

Now we consider $(L, k, \kappa)$-TM sequence of the necessary-sufficient condition that it is a $k$-automatic sequence.

Proposition 5.1 A $(L, k, \kappa)$-TM sequence is $n$-period if and only if it is a $k$-automatic sequence.

Proof. Assume without loss of generality that $A_\infty = (a(n))_{n=0}^\infty$ is a $(L, k, \kappa)$-TM sequence with $a_j = \exp \frac{2\pi \sqrt{-1}}{L} (\text{where } 0 \leq j \leq L - 1)$. We assume $(a(n))_{n=0}^\infty$ is a $k$-automatic sequence. Since the $k$-kernel of $(a(n))_{n=0}^\infty$ is finite set, we see that there exist integers $e$ and $0 < t$ such that

$$a(k^en) = a(k^{e+t}n) \quad (\forall n \geq 0). \quad (5.3)$$

Let $s$ be any integer in $\{1, 2, \cdots, k-1\}$ and $y$ any integer in $\mathbb{N}$. By Lemma 2.3 with (3.10) and (5.3) with substituting $sk^y$ for $n$, we have

$$\exp \frac{2\pi \sqrt{-1}\kappa(s, e + y)}{L} = a(k^esy) = a(k^{e+t}sy) = \exp \frac{2\pi \sqrt{-1}\kappa(s, e + y + t)}{L}. \quad (5.4)$$
Since the definition of \((L, k, \kappa)\)-TM sequence and (5.4), then \((a(n))_{n=0}^{\infty}\) is \(n\)-period.

We will show converse. If \((L, k, \kappa)\)-TM sequence \(A_{\infty} = (a(n))_{n=0}^{\infty}\) is \(n\)-period, then there exist non-negative integers \(e\) and \(0 < t\) such that

\[
\kappa(s, e + n) = \kappa(s, e + n + t) \quad (\forall n \geq e \text{ and } 1 \leq \forall s \leq k - 1).
\]

Let \(t\) be any integer greater than \(t - 1\) and \((a(k^{e+l}n + j))_{n=0}^{\infty}\) any sequence in \(k\)-kernel of \((a(n))_{n=0}^{\infty}\).

From Lemma 2.3 with (3.10), we get

\[
a(k^{e+l}n + j) = a(k^{e+l}n)a(j).
\]

By the fact \((a(n))_{n=0}^{\infty}\) takes on only finitely many values, then \(a(j)\) takes on only finitely many values, too.

Let the \(k\)-adic expansion of \(n\) as follows,

\[
n = \sum_{q=1}^{N(n)} s_q k^{w(q)} \quad \text{where } 1 \leq s_q \leq k - 1, w(q + 1) > w(q) \geq 0.
\]

Let \(l(t) \equiv l \pmod{t}\) where \(0 \leq l(t) \leq t - 1\). By Lemma 2.3 with (3.10), we have

\[
a(k^{e+l}n) = a\left(\sum_{q=1}^{N(n)} s_q k^{w(q) + e + l}\right) = \prod_{q=1}^{N(n)} a(s_q k^{w(q) + e + l}).
\]

From (5.7), (5.8), and Lemma 2.3 with (3.10), we get

\[
a(k^{e+l}n) = \prod_{q=1}^{N(n)} a(s_q k^{w(q) + e + l}) = \prod_{q=1}^{N(n)} a(s_q k^{w(q) + e + l(t)})
\]

\[
= a\left(\sum_{q=1}^{N(n)} s_q k^{w(q) + e + l(t)}\right) = a(k^{e+l(t)}n).
\]

By the fact \(a(j)\) takes on only finitely many values, (5.6) and (5.9), then the \(k\)-kernel of \((a(n))_{n=0}^{\infty}\) is finite set. This completes the proof. \(\Box\)

**Theorem 5.3** Let \((a(n))_{n=0}^{\infty}\) be a \((L, k, \kappa)\)-TM. Let \(\beta\) be an integer greater than 1. If \((a(n))_{n=0}^{\infty}\) takes on \(\{0, 1, \cdots, \beta - 1\}\) and \(n\)-period, then \(\mu(\sum_{n=0}^{\infty} \frac{a(N+n)}{\beta^{n+1}})\) (where \(N \geq 0\) and \(l > 0\)) is finite unless there exists an integer \(A\) such that

\[
\kappa(s, A + y) \equiv \kappa(1, A)sk^{y} \pmod{L},
\]

for all \(1 \leq s \leq k - 1\) and for all \(y \in \mathbb{N} \).

**Proof.** By previous proposition, \((a(n))_{n=0}^{\infty}\) is a \(k\)-automatic sequence. By the fact if \((a(n))_{n=0}^{\infty}\) is \(k\)-automatic, then \((a(N+n))_{n=0}^{\infty}\) (where \(N \geq 0\) and \(l > 0\)) are \(k\)-automatic, too. (see Theorem 2.6 in [ASJ].) From Theorem 4.2 and 5.1, we see that \(\mu(\sum_{n=0}^{\infty} \frac{a(N+n)}{\beta^{n+1}})\) is finite. \(\Box\)

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Theorem 5.4 Let \((a(n))_{n=0}^{\infty}\) be a \((L, k, \kappa)\)-TM. Let \(\beta\) be an integer greater than 1. Let \(f(z) := \sum_{n=0}^{\infty} \frac{a(N+nl)}{z^{n+1}}\) where \(N \geq 0\) and \(l > 0\). Let \(0 < R < 1\). If \((a(n))_{n=0}^{\infty}\) takes on \(\{0, 1, \cdots, \beta - 1\}\) and \(n\)-period, then \(f(\alpha)\) is transcendental number for all but finitely many algebraic numbers \(\alpha\) with \(|\alpha| \leq R\) unless there exists an integer \(A\) such that

\[
\kappa(s, A + y) \equiv \kappa(1, A)sk^y \pmod{L}, \tag{5.11}
\]

for all \(1 \leq s \leq k - 1\) and for all \(y \in \mathbb{N}\).

Proof. By Corollary 4.1, \(f(z)\) is transcendental over \(\mathbb{Q}(z)\). From Proposition 5.1, we see that \((a(N + nl))_{n=0}^{\infty}\) (where \(N \geq 0\) and \(l > 0\)) are \(k\)-automatic sequences, too. Then \(f(z)\) is a \(k\)-automatic power series. Since Theorem 5.2, then \(f(\alpha)\) is transcendental for all but finitely many algebraic numbers \(\alpha\) with \(|\alpha| \leq R\). This completes the proof. \(\square\)

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