On the Entanglement of Multiple CFTs via Rotating Black Hole Interior

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We study the minimal surfaces between two of the multiple boundaries of 3d maximally extended rotating black hole. Via AdS/CFT, this corresponds to investigating the behavior of entanglements of the boundary CFT with multiple sectors. Non-trivial time evolutions of such entanglements detect the geometry inside the horizon, and behave differently depending on the choice of the two boundaries.

I. INTRODUCTION

The gauge/gravity correspondence is a fascinating correspondence. In short, it gives the non-perturbative definition of quantum gravity in terms of the corresponding gauge theories. However, how and why the bulk spacetime and gravity appears out of the boundary theory is still in obscurity, and indeed it is one of the most fundamental questions in this correspondence.

Van Raamsdonk [1,2] pointed out that quantum entanglement between separated regions of the boundary theory is a key to the emergence of smoothly connectedness of the spacetime in the bulk. This idea has been extended and materialized in subsequent works, for example [3,4]. Especially in Hartman-Maldacena [3], they investigated the time evolution of entanglement between the two copies of boundary CFT, in eternal AdS black hole without angular momentum. In this letter, we extend the analysis of [3] to the case of rotating BTZ black hole. Unlike the non-rotating case, the spacetime boundary has 8 disconnected regions. Naively they seem to correspond to 8 decoupled sectors of the boundary CFT, which are (maximally) entangled to one another. However, as we will see, actually the story is not so simple.

This short letter is organized as follows. We first review the known global structure of the rotating BTZ black hole very briefly in section II (a bit more details are given in Appendices A and B). Then, we derive the lengths of the geodesics connecting the different boundaries in section III and by using it, calculate the entanglement between different boundaries in section IV using the holographic entanglement formula [5,6]. We will discuss and interpret the results in section V.

II. ROTATING BTZ AND ANALYTIC CONTINUATION

The rotating BTZ black hole geometry is expressed as

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\phi - N(r)dt)^2, \]

\[ f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}, \quad N(r) = \frac{r_+r_-}{r^2}, \]

where \( \phi \simeq \phi + 2\pi \). We set AdS scale to be unit in this letter. The outer/inner horizon radii \( r_+ \) and \( r_- \) are related to the “chiral temperatures” \( T_+ \) and \( T_- (T_+ \leq T_-) \) as \( r_\pm = \pi(T_\pm \pm T_+) \).

This geometry is obtained by an orbifold on the global AdS3, and the outer region of the horizon (\( r > r_+ \)) can be embedded in \( \mathbb{R}^{2,2} \), where \( ds^2 = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2 \), as

\[ x_1 = \eta_1 \left( \frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \cosh(\pi(T_+u^+ + T_-u^-)), \]

\[ x_2 = \eta_1 \left( \frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \sinh(\pi(T_+u^+ + T_-u^-)), \]

\[ x_3 = \eta_2 \left( \frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \cosh(\pi(T_+u^+ - T_-u^-)), \]

\[ x_0 = \eta_2 \left( \frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \sinh(\pi(T_+u^+ - T_-u^-)), \]

where \( u^\pm = \phi \pm t \) and \( \eta_i = \pm 1 \). The four combinations of the \( (\eta_1, \eta_2) \) represents distinct regions outside the black hole, which we call \( 1_{++}, 1_{+-}, 1_- \) and \( 1_{--} \). One can go from one to another only through the interior region. We explain the spacetime structure of this geometry slightly more in Appendix A. The orbifold to produce the periodicity for \( \phi \) is given in (A5). For more details, see [7].

Furthermore, these different regions can be connected to one another by analytic continuations of \( (t, \phi) \) or \( u^\pm \) coordinates to complex-valued regions, as TABLE 1.

III. GEODESICS BETWEEN BOUNDARIES

Our purpose in this paper is to investigate the way how the degrees of freedom on different boundaries are
TABLE I: Analytic continuations from 1_{++} to 1_{n1n2}, up to the periodicity $(u^+, u^-) \simeq (u^+ + i/T_+, u^- \pm i/T_-)$. Their complex conjugates also work.
\[
\begin{array}{c|ccc}
    & u^+ & u^- & r \\
 1_{++} & \sinh(\pi T_-(\delta u - 2\pi n)) & \sinh(\pi T_+(\delta u + 2\pi n)) \\
1_{--} & -\sinh(\pi T_-(\delta u - 2\pi n)) & \sinh(\pi T_+(\delta u + 2\pi n)) \\
1_{++} & \cosh(\pi T_-(\delta u - 2\pi n)) & \cosh(\pi T_+(\delta u + 2\pi n)) \\
1_{--} & -\cosh(\pi T_-(\delta u - 2\pi n)) & \cosh(\pi T_+(\delta u + 2\pi n)) \\
\end{array}
\]

TABLE II: Lengths $L^{(n)}(P_1, P_2)$ of geodesics connecting $P_1$ on $1_{++}$ boundary and $P_2$ on each boundary, in terms of $X_n$ where $L^{(n)}(P_1, P_2) = \log X_n - \log (\pi^2 T_+ T_-) + \log r_\infty^2$ and $\delta u^\pm = u^\pm_2 - u^\pm_1$.

entangled. In 2d CFT, various entanglement entropies are expressed as combinations of the lengths of geodesics connecting points on the boundaries of the 3d spacetime, according to the Ryu-Takayanagi holographic entanglement entropy formula [5, 6].

First, we consider two points $P_1 = (t_1, \phi_1, r_\infty)$ and $P_2 = (t_2, \phi_2, r_\infty)$ on the boundary of the same region, say 1_{++}. The geodesic length connecting $P_1$ and $P_2$ is easily computed by the coordinate mapping to Poincare AdS$_3$ [8], giving

\[
L^{(n)}_{1_{++}}(P_1, P_2) = \log X_n - \log (\pi^2 T_+ T_-) + \log r_\infty^2,
\]

\[
X_n = \sinh(\pi T_-(\delta u^- + 2n\pi)) \sinh(\pi T_+(\delta u^+ + 2n\pi)),
\]

\[
\delta u^\pm = u^\pm_2 - u^\pm_1,
\]

where $n \in \mathbb{Z}$ is the “winding number” around the $\phi$-circle [2]. The minimum $X = \min_{n \in \mathbb{Z}} \{X_n\}$ is positive, if and only if $P_1$ and $P_2$ are spacelike separated on the cylindrical boundary of 1_{++}.

By applying the analytic continuation TABLE I for the point $P_2$ in [4], we obtain the geodesic length between 1_{++} and another boundary, as TABLE II.

From this TABLE II, we notice that $X_n$ are always positive for 1_{--} and negative for 1_{++}. It implies that whole of the 1_{++} boundary is spacelike separated to 1_{++} boundary, while the 1_{--} boundary is timelike. The most complicated is the case of 1_{--}. By taking very large winding number $n$, we can make $X_n$ arbitrarily negative, that is, make the geodesic more timelike.

IV. ENTANGLEMENT BETWEEN DIFFERENT BOUNDARIES

When we take a proper time-slice in this spacetime which connects the boundaries of 1_{++} and 1_{--}, the boundary dual is discussed in [9] — it is a maximally entangled pair of two CFT sectors with chiral temperatures $T_+ \neq T_-$ (in other words, with chemical potential for momentum). We expect that it would also be true when we take different time-slices connecting other boundary pairs.

In order to investigate the structures of such inter-boundary entanglement, we consider the entanglement entropy for a subsystem $A$ which is the union of two intervals $A_1$ and $A_2$ on different boundaries (FIG. 1). We fix $A_1$ on the boundary of 1_{++}, and $A_2$ is on another one (in FIG. 1, 1_{--}).

We set the endpoints of $A_1$ and $A_2$ as $P_1 = (t_1, \phi_1, r_\infty)$, $Q_1 = (t_1, \phi_1 + \ell_1, r_\infty)$ and $P_2 = (t_2, \phi_2, r_\infty)$, $Q_2 = (t_2, \phi_2 + \ell_2, r_\infty)$, respectively, where $0 < \ell_1, \ell_2 < 2\pi$.

According to the minimal area prescription [5, 6], the corresponding entanglement entropy is given by

\[
S_A = \min \left\{ S_A^{(c)}(, S_A^{(d)}) \right\},
\]

\[
S_A^{(d)} = L^{(0)}(P_1, Q_1) + L^{(0)}(P_2, Q_2),
\]

\[
S_A^{(c)} = \min \left\{ L^{(n)}(P_1, P_2) + L^{(n)}(Q_1, Q_2) \right\},
\]

where $n = 1$ for simplicity. These $S_A^{(d)}$ and $S_A^{(c)}$ correspond to different topologies of the minimal surface drawn in FIG. 1 by red (“disconnected”) and blue (“connected”) lines. Physical quantity

\[
I(A_1, A_2) = S_{A_1} + S_{A_2} - S_A,
\]

plays the role of the order parameter which distinguishes these two phases. That is, the red disconnected surface

\footnote{Actually $S_A^{(d)}$ has another candidate which corresponds to the surface going around the other side of the $\phi$-circle (with winding number $n = -1$). Hereafter we assume that the red line in FIG. 1 is always smaller than it. This is possible without loss of generality because we can redefine $A_1 \rightarrow A_1^c$ and $\ell_i \rightarrow 2\pi - \ell_i$ ($i = 1, 2$ at the same time) without changing $S_A^{(c)}$ [7].}

\footnote{In the connected phase, the two geodesics in FIG. 1 must have same winding numbers, in order that the union of the two geodesics should be homotopic to $A$.}
corresponds to $I(A_1, A_2) = 0$ phase while the blue connected one is $I(A_1, A_2) > 0$ phase, and it is a sharp phase transition only in the classical approximation, i.e., large $N$ on the CFT side.

The entanglement entropy of the disconnected phase $S_A^{(d)}$ can be written as

$$S_A^{(d)} = \log[\sinh(\pi T_- \ell_1) \sinh(\pi T_+ \ell_2)] - 2 \log(\pi^2 T_+ T_-) + 2 \log r_\infty^2 \,.$$  \hspace{1cm} (9)

regardless of which boundary $A_2$ lives on. In particular, when the black hole is nearly extremal, we have $T_+ \gg \ell_1$ and then

$$S_A^{(d)} \sim \log[\sinh(\pi T_- \ell_1) \sinh(\pi T_+ \ell_2)] + \log(\ell_1 \ell_2) - 2 \log (\pi T_+) + 2 \log r_\infty^2 \,.$$  \hspace{1cm} (10)

**A. (1++, 1-)**

First let us put $A_2$ in Region 1++.. This is what corresponds to the setup investigated in [3]. Let us take

$$\phi_2 - \phi_1 = \delta \phi, \quad \ell_2 - \ell_1 = \delta \ell \,. \quad t_2 - t_1 = \delta t \,.$$  \hspace{1cm} (11)

Since the time coordinate $t$ flows to opposite directions between 1++ and 1+- regions, we regard this $\delta t$ as the time flow of the total system. From TABLE [11] we obtain

$$S_A^{(c)} = \log X_n^P + \log X_n^Q - 2 \log(\pi^2 T_+ T_-) + 2 \log r_\infty^2 \,,$$

$$X_n^P = \cosh(\pi T_-(\delta \phi - \delta t + 2\pi n))$$

$$\times \cosh(\pi T_+(\delta \phi + \delta t + 2\pi n))$$

$$X_n^Q = \cosh(\pi T_-(\delta \phi - \delta t + \delta \ell + 2\pi n))$$

$$\times \cosh(\pi T_+(\delta \phi + \delta t + \delta \ell + 2\pi n)) \,. \hspace{1cm} (12)$$

Of course, when $T_- = T_+$, $\delta \phi = \delta \ell = 0$ and $n = 0$, this reproduces the corresponding result in [3] (eq.(3.32)).

Furthermore, one can show that

$$S_A^{(c)} > 4\pi T_- |\delta t| - 4 \log 2 - 2 \log(\pi^2 T_+ T_-) + 2 \log r_\infty^2 \,.$$  \hspace{1cm} (13)

for arbitrary choice of $n$. Therefore in any cases, $S_A^{(c)}$ becomes very large in proportion to $|\delta t|$, therefore $S_A^{(d)} < S_A^{(c)}$ and $S_A = S_A^{(d)}$ in late time.

In particular, in near-extremal case, we find that the right-hand side also has a very large constant term $-2 \log T_+$. It corresponds to the divergence of the distance to the horizon in the extremal black hole, which can also be observed in the case of 5D non-rotating charged extremal black hole [11]. In terms of the boundary theory, it is closely related to the residual entropy, coming from IR degrees of freedom. As a result, the disconnected phase is always favored and we experience no transition in the near-extremal setup.

**B. (1++, 1-)**

When we put $A_2$ in Region 1++ in the same way as [11], we obtain from TABLE [11]

$$S_A^{(c)} = \log X_n^P + \log X_n^Q - 2 \log(\pi^2 T_+ T_-) + 2 \log r_\infty^2 \,,$$

$$X_n^P = - \sinh(\pi T_-(\delta \phi - \delta t + 2\pi n))$$

$$\times \sinh(\pi T_+(\delta \phi + \delta t + 2\pi n))$$

$$X_n^Q = - \sinh(\pi T_-(\delta \phi - \delta t + \delta \ell + 2\pi n))$$

$$\times \sinh(\pi T_+(\delta \phi + \delta t + \delta \ell + 2\pi n)) \,. \hspace{1cm} (14)$$

As we noted at the end of the previous section, these $X_n^P$ and $X_n^Q$ are not positive in general. They tend to be positive in late time for fixed values of $n$, but for any fixed time and other parameters, they become negative by taking sufficiently large $n$.

To avoid this strange property of the periodicity, let us consider a decompactifying limit and ignore the windings (i.e., set $n = 0$)\footnote{In [3], this limit is taken implicitly. This can be explicitly given as the scale transformation of AdS$_3$, as $\phi = \Lambda^{-1} \tilde{\phi}, \quad t = \Lambda^{-1} \tilde{t}, \quad r = \Lambda \tilde{r}$, \hspace{1cm} (15) where $\Lambda \to \infty$. Then in terms of $\tilde{\phi}$, the periodicity is $2\pi \Lambda \to \infty$. Accordingly, the parameters of our black hole and subsystem $A$ are also written as $r_\pm = \Lambda \tilde{r}_\pm, \quad T_{\pm} = \Lambda \tilde{T}_{\pm}, \quad r_\infty = \Lambda \tilde{r}_\infty \,,$

$\phi_i = \Lambda^{-1} \tilde{\phi}_i, \quad t_i = \Lambda^{-1} \tilde{t}_i, \quad \ell_i = \Lambda^{-1} \tilde{\ell}_i \,,$ \hspace{1cm} (16) and we regard the tilded quantities as $O(\Lambda^0)$. This means a huge black hole in the bulk and tiny intervals on the boundary. We omit tildes hereafter.}. After fixing $n = 0$, the lengths of the both geodesics are real when $\delta t > \delta t_0 \equiv \max\{|\delta \phi|, |\delta \phi + \delta \ell|\}$. We restrict the time in this regime and consider the time evolution of the entanglement entropy after $\delta t_0$.

At $\delta t \to \delta t_0$, $S_A^{(c)}$ is negatively divergent. From there it increases monotonically along with $\delta t$, and when $\delta t$ becomes large (i.e., $\delta t \gg T_+^{-1}, |\delta t_0|, |\delta \phi|$),

$$S_A^{(c)} \simeq 2 r_+ \delta t - r_- (2 \delta \phi + \delta \ell) - 2 \log(\pi^2 T_+ T_-) + 2 \log r_\infty^2 \,.$$  \hspace{1cm} (17)

Therefore in this setup, we always experience a transition from the connected phase to the disconnected one.

**C. (1++, 1-)**

As we noted in section [III] the boundary of 1-- is completely timelike to that of 1++, and so it is not reasonable to consider the entanglement between 1++ and 1--.

**V. DISCUSSIONS**

In this short letter, we discussed the entanglement in the pairs of (1++, 1+) or (1++, 1--) boundaries, by
computing the entanglement entropy of the union of two intervals $A_1$ and $A_2$. In $(1++, 1-_-)$ case, we have two candidates for the minimal surfaces — connected and disconnected ones —, and we can also have a freedom of the winding $n$ around the periodicity $[A9]$, for the connected surface. Phase transition between the two phases may or may not happen, depending on the parameters $T_+, \delta \phi$ and $\delta \ell$. In particular, in the near-extremal regime ($T_+ \to 0$), the disconnected phase is always favored and no transition takes place. In $(1++, 1-_-)$ case, the story is complicated because of the counterintuitive winding modes which contribute negatively to the spacelike distance. After removing them by decompactification, we find that the phase transition always occurs.

We can also write down the entanglement entropies for $(1++, 3_{\eta_1, \eta_2})$ pairs. However, the periodicity $[A9]$ makes problems again, because it is clearly a closed timelike curve and so it is doubtful whether such sectors have physically consistent description as a field theory. Furthermore, since the boundaries of region 3 are surrounded by the conical singularities (see FIG. 2 (a)), we are not sure that we can rely on the standard prescription of the minimal area surface. The naive computation itself is an easy problem by using TABLE III and we leave it to the reader.

In this letter, we analyzed the relation between entanglement and multi-boundary connected spacetime in the three dimensional bulk. It would be interesting to generalize this to higher dimensional spacetime. For deeper understanding of how generic multi-boundary spacetime are emerging related to the boundary entanglement like [4], we need to find a proper interpretation or counterparts of these results in the boundary CFT. Hopefully, we would return to these problems in near future.

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Appendix A: Spacetime Structure of Maximally Extended Rotating BTZ

In this appendix, we briefly review the spacetime structure of the rotating BTZ black hole. Large part of the contents here was examined in [7], and we use basically the same notation as theirs.

The AdS$_3$ spacetime is given as an $\mathbb{R}^{2,2}$-embedded hyperboloid, expressed by

\[
x_0^2 + x_1^2 + x_2^2 - x_3^2 = R^2, \\
ds^2 = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2. \tag{A1}
\]

It is obvious that this space is invariant under $SO(2, 2) \simeq SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, and the AdS boundary is given by

\[
x_0^2 + x_1^2 \to \infty, \quad x_2^2 + x_3^2 \to \infty. \tag{A2}
\]

We take the AdS radius $R = 1$ hereafter. By introducing $U$ and $V$ as

\[
U = x_1^2 - x_2^2, \quad V = x_0^2 - x_3^2, \tag{A3}
\]

the AdS hyperboloid $[A1]$ represents a straight line on the $(U, V)$-plane,

\[
U + V = 1. \tag{A4}
\]

At the same time, $[A3]$ can be regarded as hyperbolae on $(x_1, x_2)$- and $(x_0, x_3)$-planes for each fixed pair $(U, V)$. That is, each point $(U, V)$ on the line $(A4)$ represents the direct product of a pair of these hyperbolae. At $(U, V) = (1, 0)$ and $(0, 1)$, one of these two hyperbolae becomes a pair of straight lines crossing at the origin. Note that from $(A4)$, we can decompose $(U, V)$-plane into three regions, 1: $U \geq 0, V \leq 0$; 2: $U \geq 0, V \geq 0$, and 3: $U \leq 0, V \geq 0$. This decomposition will be used later.

In this context of $(A3)$, the AdS boundary $[A2]$ corresponds to going to infinity on either (or both) of $(x_1, x_2)$- and $(x_0, x_3)$-planes along with the hyperbolae. Therefore obviously, every point $(U, V)$ on $(A4)$ touches the AdS boundary.

The BTZ black hole $[1]$ is obtained as an orbifold,

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_0 \end{pmatrix} \sim \begin{pmatrix} \cosh \gamma_+ & \sinh \gamma_+ & 0 & 0 \\ \sinh \gamma_+ & \cosh \gamma_+ & 0 & 0 \\ 0 & 0 & \cosh \gamma_- & \sinh \gamma_- \\ 0 & 0 & \sinh \gamma_- & \cosh \gamma_- \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_0 \end{pmatrix}, \tag{A6a}
\]

\[
\gamma_\pm = \pm 2\pi r \pm \varepsilon, \tag{A5}
\]

of the global AdS$_3$ spacetime $[A1]$. This orbifolded spacetime can be covered by using 12 patches, each of which has the metric of the form of $[1]$. Those are:

**Region 1**: (outside the black hole, $r \geq r_+$)

\[
x_1 = \eta_1 \left( \frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right)^{1/2} \cosh(\pi (T_+ u^+ + T_- u^-)), \tag{A6a}
\]

\[
x_2 = \eta_1 \left( \frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right)^{1/2} \sinh(\pi (T_+ u^+ + T_- u^-)), \tag{A6b}
\]

\[
x_3 = \eta_2 \left( \frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right)^{1/2} \cosh(\pi (T_+ u^+ - T_- u^-)), \tag{A6c}
\]

\[
x_0 = \eta_2 \left( \frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right)^{1/2} \sinh(\pi (T_+ u^+ - T_- u^-)), \tag{A6d}
\]
Hereafter $u^\pm = \phi \pm t$, and the pair $(\eta_1, \eta_2)$ takes (+1,+1), (+1,-1), (-1,+1), (-1,-1). This region 1 covers all the sign of $x_1$ and $x_3$ in the $(U,V)$-plane with $U > 0$, $V \leq 0$.

Region 2: (between the outer and inner horizons, $r_- \leq r \leq r_+$.)

$$x_1 = \eta_1 \left( \frac{r^2 - r_-^2}{r^2 - r_+^2} \right)^{1/2} \cosh(\pi (T^+_u + T^-_u)), \quad (A7a)$$

$$x_2 = \eta_1 \left( \frac{r^2 - r_-^2}{r^2 - r_+^2} \right)^{1/2} \sinh(\pi (T^+_u + T^-_u)), \quad (A7b)$$

$$x_3 = \eta_2 \left( \frac{r^2 - r_-^2}{r^2 - r_+^2} \right)^{1/2} \sinh(\pi (T^+_u - T^-_u)), \quad (A7c)$$

$$x_0 = \eta_2 \left( \frac{r^2 - r_-^2}{r^2 - r_+^2} \right)^{1/2} \cosh(\pi (T^+_u - T^-_u)). \quad (A7d)$$

This region 2 covers $U \geq 0$, $V \geq 0$ in the $(U,V)$-plane. Note that from region 1 to region 2, the range of $r$ changes from $r \geq r_+$ to $r \leq r_+$, and the sign of $V = x_2^3 - x_3^3$ changes, while the sign of $U = x_1^2 - x_3^2$ unchanged. This explains the $r$-dependent factor changes and the “sinh”-“cosh” flip between $A6c$ and $A7c$, and between $A6d$ and $A7d$.

Region 3: (inside the inner horizon, $r \geq r_+$.)

$$x_1 = \eta_1 \left( \frac{r^2 - r_-^2}{r^2 - r_+^2} \right)^{1/2} \sinh(\pi (T^+_u + T^-_u)), \quad (A8a)$$

$$x_2 = \eta_1 \left( \frac{r^2 - r_-^2}{r^2 - r_+^2} \right)^{1/2} \cosh(\pi (T^+_u + T^-_u)), \quad (A8b)$$

$$x_3 = \eta_2 \left( \frac{r^2 - r_-^2}{r^2 - r_+^2} \right)^{1/2} \sinh(\pi (T^+_u + T^-_u)), \quad (A8c)$$

$$x_0 = \eta_2 \left( \frac{r^2 - r_-^2}{r^2 - r_+^2} \right)^{1/2} \cosh(\pi (T^+_u + T^-_u)). \quad (A8d)$$

This region 3 covers $U \leq 0$, $V > 0$ in the $(U,V)$-plane. Note that region 1 and region 3 are related by $U(x_1^2 - x_2^2)$ and $V(x_2^3 - x_3^3)$ exchange, therefore, region 1’s $(x_1,x_2,x_3,x_0)$ and region 3’s $(x_0,x_3,x_2,x_1)$ are exchanged.\(^5\)

Depending on these signs, we refer each of the 12 regions as $1_{++}$, $2_{++}$, etc. It can be easily shown that each of the embeddings $(A6)(A7)(A8)$ leads to the same induced metric $\tilde{g}$, while the orbifold $(A5)$ becomes

$$\phi \simeq \phi + 2\pi \quad \text{(for region 1,2)}, \quad (A9a)$$

$$t \simeq t + 2\pi \quad \text{(for region 3)}. \quad (A9b)$$

Due to the $t \simeq t + 2\pi$ identification in region 3, there is a conical singularity at the radius $r = \sqrt{r_+^2 + r_-^2}$, where $g_{tt} = 0$ in region 3. The Penrose diagram for this spacetime can be drawn as FIG.2(a).\(^6\)

Since every point $(U,V)$ reaches to the AdS boundary, the AdS boundary is also divided into the 12 different regions $1_{\eta_1\eta_2}$, $2_{\eta_1\eta_2}$ and $3_{\eta_1\eta_2}$, although the region 2 becomes just straight-lines on the boundary. The arrangements of $1_{\eta_1\eta_2}$ and $3_{\eta_1\eta_2}$ on the AdS global coordinate boundary, can be seen, from \(\frac{\theta}{\pi} \) and \(\frac{\phi}{\pi} \) identification in region 3, there is a conical singularity at the radius $r = \sqrt{r_+^2 + r_-^2}$, where $g_{tt} = 0$ in region 3. The Penrose diagram for this spacetime can be drawn as FIG.2(b).\(^7\)

$\textbf{Appendix B: Analytic Continuations}$

The different patches $(A6)(A7)(A8)$ can be connected to one another, by various analytic continuations of $(t,\phi,r)$ or $(u^\pm, r)$ coordinates to complex-valued regions.

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\(^5\) This explains relations between $A6a$ and $A8d$, $A6b$ and $A8c$, $A6c$ and $A8b$, and $A6d$ and $A8a$.

\(^6\) Note that this diagram represents the null surface but the trajectory of the light is not necessary on this diagram, due to the constraint $d\phi = N(r)dt$. 

\(^7\)
The list of the ones from $1_{++}$ to $3_{1/2}$ is given in TABLE III. For completeness, we list the other formulas of analytic continuations in TABLE III.

| $\eta$ | $u^+$ | $u^-$ | $r$ |
|-------|------|------|-----|
| $1_{++}$ | $u^+ - \frac{i}{2T_+}$ | $u^- + \frac{i}{2T_-}$ | $r$ |
| $1_{--}$ | $u^+ + \frac{3i}{2T_+}$ | $u^- - \frac{3i}{2T_-}$ | $r$ |
| $2_{--}$ | $u^- - \frac{i}{2T_-}$ | $u^- + \frac{i}{2T_-}$ | $r$ |
| $2_{++}$ | $u^- - \frac{3i}{2T_+}$ | $u^- + \frac{3i}{2T_+}$ | $r$ |
| $3_{++}$ | $u^+ - \frac{i}{2T_+}$ | $u^- - \frac{i}{2T_-}$ | $i\sqrt{r^2 - (r_+^2 + r_0^2)}$ |
| $3_{--}$ | $u^+ + \frac{i}{2T_+}$ | $u^- + \frac{i}{2T_-}$ | $i\sqrt{r^2 - (r_+^2 + r_0^2)}$ |
| $3_{+-}$ | $u^- - \frac{i}{2T_-}$ | $u^- + \frac{i}{2T_-}$ | $i\sqrt{r^2 - (r_+^2 + r_0^2)}$ |

TABLE III: Analytic continuations from $1_{++}$ to $2_{1/2}$ and $3_{1/2}$, up to the periodicity $(u^+, u^-) \simeq (u^+ + i/T_+, u^- + i/T_-)$.

In region 2, we promise that $(r^2 - r_0^2)^{1/2} = i(r_+^2 - r_0^2)^{1/2}$.

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