Conceptual understanding about piecewise functions based on graphical representation

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Abstract. Understanding the concept of function is very important in learning mathematics. Students' understanding of functions can be seen from how students represent these functions with a graph. The present research aims to describe second-year mathematics students' conceptual understanding about piecewise functions seen from the graphic images they made. The subjects were 5 second-year students of mathematics study program in Universitas Negeri Malang, Indonesia. Data collected by conducting written tests and interviews with students. Students are given one problem about drawing graphs of polynomial functions that are defined piecewise. The collected data were analysed using descriptive qualitative method. After that, interviews were conducted related to graphic images made by students. From the findings, it was found that only one student understood the concept of the piecewise function correctly. He understood that each of the three polynomials defined in the piecewise function had different domains and could graph them accurately. The remaining students defined the piecewise function by using a single algebraic formula by summing each algebraic formula in the given piecewise function. Misunderstanding of the concept of piecewise functions is due to their lack of understanding of the domain and the continuity of a function at certain intervals.

1. Introduction
The concept of function has a very important position in learning mathematics, starting from high school level [1-3] to the university level [4-9]. Mathematics learning at the university level emphasizes more on understanding the concepts rather than focus on basic mathematical skills about functions [4]. A strong understanding of the concept of function will help in studying calculus because the concept of function underlies all material in calculus [2,6,7]. Another opinion says that the concept of function is the center of all material in mathematics [1] and its application [10], provides opportunities for students to develop algebraic thinking skills and to generalize findings on calculations [11]. Functions are concepts that connect other concepts in mathematics, so the role of functions is to unite all concepts in mathematics [7]. The concept of function is not only related to mathematics but also in other domains of sciences [12], such as astronomy [13,14], engineering [15-18], and physics [19].
Functions is a mathematical concept which can be represented in various ways, such as arrow diagrams, tables, graphs, formulas or algebraic representations, and phrases [20]. Most students learn functions using graphical and algebraic representations [21,22]. Communicating mathematical ideas, like the concept of function, requires external representations, such as graphic or image [23]. So, one way that can be used to see students’ understanding of the concept of functions is to look at the graphical representation of the function they made [10]. External representation, such as graphic, can explains [24] and demonstrates [8] students’ understanding about the concept of functions. It can simplify the analysis process and provide an explanation of understanding the concept of function [25].

There are several approaches taken by students when representing functions in graphical form. Moore et al [26] suggested static shape and emergent shape thinking to characterize the focus on shapes and points. Öçal [24] found students using rough sketches to draw graphs of functions. Roughly sketches are sketching graphs of functions using rote learning. Even [27] found two approaches that students do when representing functions with graphs, namely the pointwise and global thinking. The pointwise thinking is generating input-output pairs of a function, representing these as points in a coordinate system, and then connecting the points with a curve. The global thinking is using basic function graphs as a reference for drawing more complex function graphs. For example, students draw the graphic function of \( f(x) = (x + 5)^2 \) by drawing the basic graphic function of \( f(x) = x^2 \) first. After that, students shifted the graphic function of \( f(x) = x^2 \) five units to the left so that the graphic function of \( f(x) = (x + 5)^2 \) is obtained.

Students in junior high school are usually introduced to linear and quadratic functions through plotting of points represented by coordinates extracted from tables. The continuation of functions depends on some discreet points chosen in the table [11]. At the high school level, in addition to the concept of algebraic functions, students also learn about the concept of trigonometric functions. Students at high school level are required to be able to represented functions in the algebraic forms and graphs [11]. At the university level, students learn more complex functions, one of which is a piecewise function. The piecewise function is a function that is defined using different output formulas for different parts of the domain [28]. According to Bayazit [29], a piecewise function is a function defined by more than one rule on the sub-domains.

Understanding about piecewise functions is very important in learning mathematics. Charles [22] stated that there are two things that underlie the importance of learning piecewise function. First, the piecewise function acts as a means of learning the concepts of discontinuous functions and one and two-sided limits in calculus. Second, if students have a strong understanding of the concept of piecewise functions, the students’ understanding of the domain function will be stronger. This happened because in the piecewise function, students think about the domain of one piece, but in the context of multiple domains [22]. The concept of piecewise function is also very important to be understood by mathematic students who will take real analysis course and learn about pointwise convergence of function sequences [30]. In addition, the concept of piecewise functions is also needed as a basis for learning computer mathematical programming [28,31], applied mathematics [32,33], and other subject that is more complex in mathematics courses [22,30].

There are several studies on piecewise functions in mathematics education that have been conducted. Sokolowski [34] conducted research toward calculus students about the model of functions. This model is done by formulating position functions for objects moving along a horizontal path with multiple rules so that they form the piecewise function. Aliustaoglu [35] examining the pedagogical content knowledge of prospective mathematics teachers on the subject of limits by using piecewise function. Jones [36] conducted research about graphical representations of the calculus concept of the definite integral. One form of graphical representation is the piecewise function. Zazkis [37] conducted a research about generating function from scripting tasks. One of the functions generated by prospective secondary school
teachers is the piecewise function. Hohensee [22] conducted research about senior high school student difficulties with graphing and understanding piecewise functions.

Research on students' understanding of the piecewise function at the university level is still limited. Research conducted by Hohensee [22] is still limited to the high school level. Burns-Childers et al [2] in their research on university student understanding related to the concept of quadratic functions suggests that there is further research on understanding university student concepts in other functions. Because, in calculus, there are various types of functions that must be well understood by university students. Therefore, in this study, researchers wanted to know about second-year mathematics students' conceptual understanding about piecewise functions.

The aims of this study are to describe the second-year mathematics students' conceptual understanding about piecewise functions based on the graphical representations they made and the approach they use when making graphical representations of piecewise functions. By knowing students' understanding of the piecewise function, we as lecturers can find out the difficulties experienced by students in learning the piecewise function concept and help overcome the student's difficulties.

2. Method

This research is an explorative descriptive research through a qualitative approach. The subjects in this research were selected using purposeful sampling approach. Purposeful sampling is choosing a subject who can best help researchers understand their phenomenon or research objectives [38]. First, we conducted written test to 25 undergraduate students of mathematics study program in Universitas Negeri Malang, Indonesia. The test consisted of one question about graphing the piecewise function. The test used in this research was first validated by the expert. This validation is intended so that the test questions can be said to be feasible and precisely measure what should be measured or revealed in this research. The question can be seen in Figure 1 below.

![Figure 1. Question about Piecewise Function](image)

After giving the test, the results of student work are grouped based on piecewise function graph and the process obtained. There are 5 groups based on the answers generated. One student will be selected from each group. Students selected from each group will be subject to research. Then, the interviews were conducted on 5 selected subjects.

The data in this study were in the form of test results and interviews of the five selected subjects. The collected data were analyzed using transcription, segmentation, coding and categorizing techniques and drawing conclusions [38]. Conclusions drawn related to students' understanding of the concept of the piecewise function based on the graphical representations they made and the approach they use when making graphical representations of piecewise functions.

3. Result and Discussion

3.1 Conceptual Understanding about Piecewise Function
S1 is the only one that can make a graphical representation of the piecewise function correctly. From the results of the graph representation made by S1, it can be seen that S1 has understood the concept of the piecewise function correctly. S1 understood that for each of the three equations defined in the piecewise function, they have different domains and can graph them precisely. From the interviewed, S1 understood that $h(x)$ is a function defined by three different domains and equations. S1 does not defined $h(x)$ as a linear, constant, and cubic function that stands alone. The results of the S1 answer can be seen in Figure 2 below.

![Figure 2. S1 Work Result](image)

S1 drawn the graph for the first equation in the $h(x)$ function, that is $h(x) = -x + 3$, with the domain at interval $(-\infty, -2]$ correctly. At the coordinates of the point $(-2,5)$, S1 marked it with a full circle which means $x = -2$ included in the domain of $h(x)$ which is defined $h(x) = -x + 3$. The second equation in the $h(x)$ function, that is $h(x) = 4$, is drawn with the domain at interval $(-2,1)$ by S1. At the coordinates of the points $(-2,4)$ and points $(1,4)$, S1 marked it with an empty circle which means $x = -2$ and $x = 1$ are not included in the domain $h(x)$ which is defined $h(x) = 4$. The third equation in the $h(x)$ function, that is $h(x) = \frac{1}{2}x^3$, is drawn with the domain at interval $[1, \infty)$ by S1. At the coordinates of the points $(1, \frac{1}{2})$, S1 marked it with a full circle which means $x = 1$ included in the domain $h(x)$ which is defined $h(x) = \frac{1}{2}x^3$.

Test result and interview showed that S2 does not understand the concept of piecewise function. S2 assumed that the function $h(x)$ defined in the problem is the same as $h(x) = \frac{1}{2}x^3 - x + 3 + 4 = \frac{1}{2}x^3 - x + 7$ with the domain $x \in \mathbb{R}$. S2 added up each equation in the $h(x)$ function and combined the domains so that it turns into a real number. S2 does not understand that each domain that is defined in the $h(x)$ function has different rules and equations, so it should not be added to a single equation. The results of the S2 answer can be seen in Figure 3 below.

![Figure 3. S2 Work Result](image)
S2 has the understanding that each function must be defined using just one formula, including the given piecewise function, so S2 adds up every equation to the given piecewise function so that it only becomes one equation or formula. This is consistent with the result of Breidenbach et al [39], Bayazit [29], and Hohensee [22] research. They said that most students tend to believe that all functions must be defined by a single algebraic formula or rule over the whole domain. This focus often prevents flexible thinking about the function situation and can lead to wrong conclusions such as thinking that all functions must be "good" functions [39].

S3 has the understanding that each equation in the piecewise function is a different and independent function. S3 thinks that there are three functions in the problem, namely \( h(x) = -x + 3 \), \( h(x) = 4 \), and \( h(x) = \frac{1}{2}x^3 \). S3 draws every equation in the \( h(x) \) function as a different function with each domain being a real number. S3 is too fixated on the concept of functions which must be special functions [22], "good" functions [39], or prototypes functions [40]. S3 does not considered that a piecewise-defined function such as \( h(x) \) is a function as well. The results of the S3 answer can be seen in Figure 4 below.

From the results of the interviews and work results, S4 failed to understand the domain of the piecewise function. S4 misunderstood the meaning of the domain in each equation of \( h(x) \) function. The second equation in the \( h(x) \) function, that is \( h(x) = 4 \), has domain at interval \((-2,1)\). S4 considered that because \( x = -2 \) and \( x = 1 \) are not included in the domain, the domain changed to \([-1,0]\). So S4 draw the second equation in the \( h(x) \) function, that is \( h(x) = 4 \), with domain \([-1,0]\). Hohensee [22] in his research found that students who did not understand the concept of the domain function would have difficulty to understand the concept of piecewise functions. That is what S4 experienced when understanding the piecewise function. Difficulties in understanding the domain of the piecewise function also occurred in the Cho et all [5] research. The results of the S4 answer can be seen in Figure 5 below.
S5 does not understand the domains defined in each equation for the $h(x)$ function. At $x = -2$, graph for the first and second equations are connected by a line, even though they should not. The same thing happened at $x = 1$ for the graphs in the second and third equations. S5 assumed that the graph of functions must not be cut off and must always be continuous. The same thing also appeared in Hohensee [22] research. The results of the S5 answer can be seen in Figure 6 below.

Figure 6. S5 Work Result

3.2 The Approach Used to Draw Graph Functions
S1 has no difficulty in drawing graph of functions defined by three different domains. S1 used the pointwise thinking approach in drawing a graph of the $h(x)$ function. This approach is also carried out by students in Hohensee [22] and Even [27] research. S1 used the boundary points at each domain interval as an assisting point in making graphs. S1 wrote how to get these assisting points in the table by entering the value of $x$ in the equation corresponding to the domain to get the value of $y$. The table created by S1 can
be seen in Figure 2. During the interview, S1 explained how to draw a graph of the function \( h(x) \) defined
\[
h(x) = \frac{1}{2}x^3.
\]
S1 remembered that the bottom of the graph of the function \( f(x) = x^3 \) is slightly curved. So based on these memories, S1 drawn a graph of the function \( h(x) \) defined \( h(x) = \frac{1}{2}x^3 \) with a slightly curved shape. The method used by S1 in drawing graphs of the function \( h(x) = \frac{1}{2}x^3 \) was also found in Öcal [24] research. Öcal [24] called this method a roughly sketch, which is sketching graphics based on memory.

S2 looks for the roots of the equation in the added function \( h(x) \). From the results of the interview, it is known that S2 will make the found roots become an assist points in drawing graphic function. However, because S2 is running out of time, S2 cannot continue counting these assist points. S2 finally thought that the \( h(x) \) function must be in the form of a curve because it was a polynomial function, so S2 made a rough sketch of the \( h(x) \) function which can be seen in Figure 3. In the beginning, the approach that S2 wanted to take in drawing \( h(x) \) function graph was the pointwise method. But at the last moment, S2 switched to using rough sketches. It can be seen that S2 has difficulty in drawing a graph of the function \( h(x) = \frac{1}{2}x^3 - x + 7 \) which is a cubic function. Jojo [11] also found that students have difficulty when drawing graphs of cubic functions. These difficulties occur because of the procedure that must be performed to draw a cubic function graph is too long [11]. A procedure that is too long caused S2 to run out of time and experienced confusion when graphs of cubic functions.

From the results of the graph representation and interviewed, it can be known that S3, S4, and S5 used the pointwise method approach when drawing graph functions. They do not understand the concept of domains in piecewise functions correctly, so their graphs are wrong. Similar to S2, S3 also has difficulty in drawing graphs of cubic functions. It can be seen in Figure 4 that \( h(x) = \frac{1}{2}x^3 \) is drawn in a straight line, even though it should not. S4 and S5 actually understand how to draw graphs of functions, but they are doesn’t understand the domains of functions defined piecewise, such as the \( h(x) \) function, correctly.

4. Conclusion
The results of this research indicate that there are still many second-year mathematics students who do not understand the concept of piecewise functions. The cause of the problem is because they do not understand the domain of function and the continuity of a function at certain intervals. In this study, it was also found that students still had difficulty in drawing graphs, mainly related to graphs of piecewise functions and cubic functions. The approaches used by students in drawing graphs of functions are mostly pointwise methods and rough sketches.

This research is limited to knowing students' understanding of the concept of piecewise functions and the approach they use when making graphical representations of piecewise functions. Further research is needed to find out what difficulties are experienced by students when understanding the concept of piecewise function. In addition, there is a need for research on learning that can improve student understanding of the concept of piecewise function and ability to draw a graph of functions.

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