A CACTUS THEOREM FOR END CUTS

ANASTASIA EVANGELIDOU AND PANOS PAPASOGLU

Abstract. Dinits-Karzanov-Lomonosov showed that it is possible to encode all minimal edge cuts of a graph by a tree-like structure called a cactus. We show here that minimal edge cuts separating ends of the graph rather than vertices can be 'encoded' also by a cactus. We apply our methods to finite graphs as well and we show that several types of cuts can be encoded by cacti.

1. Introduction and Preliminaries

Vertex and edge cuts of graphs have been studied extensively in several different contexts: graph theory, geometric group theory, topology and networks. They have played an important role in applications, notably in clustering algorithms, combinatorial optimization and network design.

E.A. Dinits, A.V. Karzanov, M.V. Lomonosov [7] (see also [10], sections 7.4,7.5) gave an elegant way to encode all minimal edge cuts of a graph by a cactus, a tree-like structure. For a recent short proof of their theorem see [9]. This structure theorem has found many important applications ([13], [12]). The crucial observation in [7] is that minimal edge cuts which 'cross' have a circular structure.

Tutte has studied vertex cuts and has shown that minimal vertex cuts of cardinality 2 can be encoded by a tree like structure ([15] Ch. IV, [16] ch. 11, [17]). In fact one can see Tutte's theorem as a cactus theorem for vertex cuts, but his theorem applies only to cuts of cardinality 2. In [6] Tutte’s theorem was extended to infinite, locally finite graphs.

There is a similar theory of cuts of connected metric spaces ([18], [2]) dealing with cut points and cut pairs. In particular in the case of cut pairs Bowditch shows that crossing cut pairs have a circular structure.

Stallings [14] (in the locally finite case) and Dunwoody [1] (in general) have shown that if \( \Gamma \) is a graph with more than one end then there is a set of minimal end cuts of \( \Gamma \) which is invariant under \( Aut(\Gamma) \) and which can be encoded by a tree. The main motivation of Stallings and Dunwoody was to classify groups with many ends.

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In this paper we show that one can encode the set of all minimal end cuts of a graph by a cactus. We note that Stallings and Dunwoody proceeded by finding a subset of minimal end cuts which can be encoded by a tree (and is invariant under the automorphism group) while our work reveals that the set of all minimal end cuts has the finer structure of a cactus. In particular we show that crossing end cuts have a circular structure. It follows from our result too that groups with many ends split over finite groups.

Let $\Gamma = (V,E)$ be a connected graph. A subset $K \subset E$ is called an edge cut if $\Gamma - K = (V,E - K)$ has at least two connected components. A subset $K \subset V$ is a vertex cut of $\Gamma$ if $\Gamma - K$ is not connected. If $A,B \subset \Gamma$ we say that $K$ separates $A$ from $B$ if any path joining a vertex of $A$ to a vertex of $B$ intersects $K$.

A ray of $\Gamma$ is an infinite sequence of distinct consecutive vertices $v_0, v_1, v_2, ...$ of $\Gamma$. We say that two rays $r_1, r_2$ are equivalent if for any finite edge cut $K$ all vertices of $r_1 \cup r_2$ except finitely many are contained in the same connected component of $\Gamma - K$. The ends of $\Gamma$ are the equivalence classes of rays. If $A \subset \Gamma$ and $e$ is an end we say that $A$ contains $e$ is almost all vertices of some (all) ray $r$ representing $e$ are contained in $A$. Let $K$ be a finite edge cut of $\Gamma$. We say that $K$ is an end cut of $\Gamma$ if there are at least two connected components of $\Gamma - K$ which contain rays. We say that an end cut is a minimal cut of $\Gamma$ if its cardinality is minimal among all end cuts of $\Gamma$. We remark that if $K$ is a minimal cut then $\Gamma - K$ has exactly two components. We say that a minimal cut $K$ separates two ends $e_1, e_2$ if there are two rays $r_1, r_2$ representing respectively $e_1, e_2$ such that $r_1, r_2$ are contained in distinct connected components of $\Gamma - K$. We say that two minimal cuts $K, L$ are equivalent if any two ends $e_1, e_2$ are separated by $K$ if and only if they are separated by $L$. We denote the equivalence class of $K$ by $[K]$. If $M$ is a subset of a graph we denote by $V(M)$ the set of vertices of $M$. We say that a set of consecutive edges of a graph $e_1 = [a_1, a_2], ..., e_k = [a_k, a_{k+1}]$ is a cycle if $a_1 = a_{k+1}$ and for $i \neq j$, $a_i \neq a_j$ if $\{i, j\} \neq \{1, k + 1\}$. A graph $\mathcal{C} = (V,E)$ is called a cactus if any two cycles in $\mathcal{C}$ have at most one vertex in common. We say that a vertex $v$ of $\mathcal{C}$ is an end-vertex if $\mathcal{C} - v$ is connected. We remark that the degree of an end vertex is 1 or 2. We state now the main result of this paper:

**Theorem 3.11** Let $\Gamma$ be a graph. Then there is a cactus $\mathcal{C}$, an onto map $f$ from the ends of $\Gamma$ to the union of the ends of $\mathcal{C}$ with the end vertices of $\mathcal{C}$ and a 1-1 and onto map $g$ from equivalence classes of minimal cuts of $\Gamma$ to the minimal edge cuts of $\mathcal{C}$ so that two ends
e_1, e_2$ of $\Gamma$ are separated by a minimal cut $K$ if and only if $f(e_1), f(e_2)$ are separated by $g([K])$. Moreover any automorphism of $\Gamma$ induces an automorphism of $\mathcal{C}$.

It turns out that one can show similar results for the structure of ‘small’ edge cuts of a finite graph if the finite graph contains ‘big’ subgraphs that can not be cut by ‘small’ cuts. We discuss this briefly in section 4.

Martin Dunwoody brought to our attention his work with Krön [5] which contains some arguments similar to the ones used here. He also told us about Tutte’s work and this led us to consider edge cuts of finite graphs as well. We would like to thank Aggelos Georgakopoulos for pointing out a mistake in an earlier version of this paper.

2. Pretrees

We will use the notion of pretrees ([2],[3]) to show that the set of minimal cuts can be represented by a cactus.

Informally a pretree can be thought of as a subset of a tree. Given a pretree one constructs a tree by ‘joining the dots’ of the pretree. In a subset of a tree there is a natural ternary relation, if $a, b, c$ are 3 points at most one is between the 2 others. We use this betweeness relation to give a formal definition of pretrees.

**Definition 2.1.** Let $\mathcal{P}$ be a set and let $R \subseteq \mathcal{P} \times \mathcal{P} \times \mathcal{P}$. We say then that $R$ is a betweeness relation. If $(x, y, z) \in R$ then we write $xyz$ and we say that $y$ is between $x, z$. $\mathcal{P}$ equipped with this betweeness relation is called a pretree if the following hold:

1. there is no $y$ such that $xyx$ for any $x \in \mathcal{P}$.
2. $xzy \iff yzx$
3. For all $x, y, z$ if $y$ is between $x, z$ then $z$ is not between $x, y$.
4. If $xzy$ and $z \neq w$ then either $xzw$ or $yzw$.

**Example 2.1.** The obvious example of a pretree is the vertex set of a tree. Note also that any subset of a pretree is a pretree. Another example of a pretree is the edge set of a tree. Not all pretrees are subsets of trees. Indeed any linearly ordered set $(\mathcal{P}, <)$ can be seen as a pretree.

**Definition 2.2.** We say that a pretree $\mathcal{P}$ is discrete if for any $x, y \in \mathcal{P}$ there are finitely many $z \in \mathcal{P}$ such that $xzy$.

If there is no $z$ between $x, y \in \mathcal{P}$ we say that $x, y$ are adjacent.

Let $\mathcal{P}$ be a countable discrete pretree. We recall briefly how can one pass from $\mathcal{P}$ to a tree (see [3] for a more general construction).
We call a subset \( H \subset \mathcal{P} \) a star if any \( a, b \in H \) are adjacent. We define now a tree \( T \) as follows:

\[
V(T) = \mathcal{P} \cup \{ \text{maximal stars of } \mathcal{P} \}
\]

\[
E(T) = \{(v, H) : v \in \mathcal{P}, v \in H, H \text{ maximal star}\}
\]

We show that \( T \) is indeed a tree. Since \( \mathcal{P} \) is discrete \( T \) is connected.

If \( T \) contains a circuit then there are \( x_1, \ldots, x_n \) (\( n > 2 \)) in \( \mathcal{P} \) such that \( x_i \) is adjacent to \( x_{i+1} \) and \( x_i \) is not adjacent to \( x_{i+2} \) for all \( i \in \mathbb{Z}_n \).

Since \( x_i, x_{i+2} \) are not adjacent there is \( y \) such that \( x_i y x_{i+2} \). If \( y \neq x_{i+1} \) then either \( x_i y x_{i+1} \) or \( x_{i+2} y x_{i+1} \) but both these are impossible. Hence \( x_i x_{i+1} x_{i+2} \) holds. We claim that \( x_1 x_i x_{i+1} x_{i+2} \) holds for all \( i \leq n \). We argue by induction. Since \( x_{i+1} \neq x_{i-1} \), by 4, we have that \( x_1 x_{i-1} x_i x_{i+1} \) holds. Since \( x_{i-1} x_i x_{i+1} \) holds for \( n > i \geq 2 \) by 4 either \( x_1 x_i x_{i+1} \) or \( x_{i-1} x_i x_1 \) holds. However by induction \( x_1 x_{i-1} x_i \) holds, so by 3, necessarily \( x_1 x_i x_{i+1} \). So \( x_1 x_{n-1} x_n \) holds, contradicting our assumption that \( x_1, x_n \) are adjacent.

**Example 2.2.** Note that ‘adding’ the stars is necessary in order to get a tree from a pretree. Consider for example the case of a pretree \( \mathcal{P} \) consisting of three mutually adjacent elements \( x, y, z \). Then to get a tree one adds a new ‘star’ vertex \( w \) and joins it by edges to \( x, y, z \).

3. Cut sets

As our objective in this section is to study the end structure of graphs we will restrict, without loss of generality, to graphs that do not contain loops. Indeed if \( \Gamma \) is a graph and \( \Gamma' \) is the subgraph of \( \Gamma \) obtained from \( \Gamma \) by erasing all loops of \( \Gamma \) then there is an obvious 1-1 and onto map from the ends of \( \Gamma' \) to the ends of \( \Gamma \).

It will be convenient to replace edge cuts by cuts consisting of midpoints of edges. We set up some notation: If \( \Gamma = (V, E) \) is a graph then we have the incidence map \( \psi \) from the set \( E \) of edges to the set of unordered pairs of vertices. So if \( e \) is an edge \( \psi(e) = \{v, u\} \) where \( v, u \) are the endpoints of \( e \). In general \( v = u \) is possible (when \( e \) is a loop), however here since we assume that \( \Gamma \) has no loops, \( v, u \) are distinct. If \( K \subset V \) we say that \( K \) is a vertex cut if the graph we obtain from \( \Gamma \) by erasing all edges incident to \( K \) and with vertex set \( V - K \) has more than one connected component. Abusing notation slightly we denote this new graph by \( \Gamma - K \). If \( C \) is a component of \( \Gamma - K \) we denote by \( \partial C \) the set of vertices of \( \Gamma \) which do not lie in \( C \) and are incident to edges that intersect \( C \) (note that we may see \( C \) as a subgraph of \( \Gamma \)).
Finally we denote by $\bar{C}$ the graph obtained by adding to $C$ all edges that intersect $C$. So the vertex set of $\bar{C}$ is $V(\bar{C}) = V(C) \cup \partial C$.

To replace edge cuts by vertex cuts we use the barycentric subdivision of a graph:

**Definition 3.1.** If $\Gamma$ is a graph the barycentric subdivision $\Gamma^b$ of $\Gamma$ is the graph we obtain by subdividing each edge of $\Gamma$ into two edges.

More formally if $\Gamma = (V, E)$ then $\Gamma^b = (V^b, E^b)$ where $V^b = V \cup E$ and $E^b = \{ (e, v) : e \in E, v \in \psi(e) \}$ where $\psi$ is the incidence function of the graph $\Gamma$.

If $K$ is a minimal cut of $\Gamma$ then $K' = K \cap V^b$ is a vertex cut of $\Gamma^b$. We remark that if $C$ is a component of $\Gamma^b - K'$ then $K' = \partial C$.

If $K$ is a vertex cut we say that an end $e$ is contained in a component $C$ of $\Gamma - K$ if for any ray $r$ representing $e$ almost all vertices of $r$ are contained in $C$.

3.1. Equivalent Cuts.

**Definition 3.2.** Let $\Gamma$ be a graph with more than one end. Let $K_1, K_2$ be minimal cuts of $\Gamma$. We say that $K_1, K_2$ are equivalent if any two ends $e_1, e_2$ of $\Gamma$ are separated by $K_1$ if and only if they are separated by $K_2$. We write then $K_1 \sim K_2$ and we denote the equivalence class of $K_1$ by $[K_1]$.

We would like to associate to a graph $\Gamma$, in a canonical way, a cactus $\mathcal{C}$ which encodes the minimal cuts of the graph $\Gamma$. To be more precise we will encode minimal cuts up to equivalence. We will proceed by defining a pretree. There is a natural way to define betweenness of equivalence classes of minimal cuts, which we describe now. Let $\mathcal{E}$ be the set of ends of $\Gamma$. If $K$ is a minimal cut then $K$ partitions $\mathcal{E}$, so we may write $\mathcal{E} = K^{(1)} \cup K^{(2)}$ where 2 ends are separated by $K$ if and only if they lie in different sets of this partition. Clearly if $L$ is equivalent to $K$ then (after relabelling) $K^{(1)} = L^{(1)}$, $K^{(2)} = L^{(2)}$.

**Definition 3.3.** Let $\Gamma$ be a graph and let $K, L, M$ be inequivalent minimal cuts of $\Gamma$. We say that $[L]$ is between $[K], [M]$ if, possibly after relabelling,

$$K^{(1)} \subset L^{(1)} \subset M^{(1)}.$$ 

Clearly if $[L]$ is between $[K], [M]$ then we have also $K^{(2)} \supset L^{(2)} \supset M^{(2)}$.

It is easy to see that the axioms 1,2,3 of the pretree definition are satisfied. However axiom 4 is not satisfied because of ‘crossing’ cuts. We define formally crossing cuts in the next section and we show that
such cuts have some surprisingly simple structure. This will allow us
to remedy the problem of crossing cuts and define a pretree.

We give in the next lemmas an equivalent way to define betweeness.

**Lemma 3.1.** Let $\Gamma$ be a graph and let $K, L$ be inequivalent minimal
cuts of $\Gamma$ such that $K^{(1)} \subset L^{(1)}$. Then for any $L_1 \in [L]$ there is some
$K_1 \in [K]$ such that $K_1$ intersects a single connected component of
$\Gamma - L_1$.

**Proof.** It will be convenient to replace $\Gamma$ by its barycentric subdivision
and view edge cuts as vertex cuts of the barycentric subdivision. To
keep notation simple we keep denoting the barycentric subdivision by $\Gamma$.

Let $L_1 \in [L]$. Let $C_1, C_2$ be the two connected components of $\Gamma - L_1$,
where the ends of $L^{(1)}$ are contained in $C_1$. We set:

$$ c = |K \cap L_1|, \quad n = |K| = |L_1| $$

Consider $\Gamma - (L_1 \cup K)$. Clearly there are components $U_1, U_2$ of $\Gamma -
(L_1 \cup K)$ such that $U_1$ contains all ends in $K^{(1)}$ and $U_2$ contains all
ends in $L^{(2)}$. We have then

$$ \partial U_1 \subset K \cup L_1, \quad \partial U_2 \subset K \cup L_1, \quad \partial U_1 \cap \partial U_2 \subset K \cap L_1 $$

So

$$ 2n - c \geq |\partial U_1 \cup \partial U_2| = |\partial U_1| + |\partial U_2| - |\partial U_1 \cap \partial U_2| \geq 2n - |\partial U_1 \cap \partial U_2| $$

It follows that $|\partial U_1 \cap \partial U_2| = c$ and $\partial U_1 \cap \partial U_2 = K \cap L_1$. Since
all inequalities are equalities $|\partial U_1| = |\partial U_2| = n$ and $\partial U_1 \in [K]$. So
$K_1 = \partial U_1$ satisfies the requirements of the lemma.

□

**Lemma 3.2.** Let $\Gamma$ be a graph and let $K, L, M$ be inequivalent minimal
cuts of $\Gamma$. Then $[L]$ is between $[K], [M]$ if and only if for any $L_1 \in [L]$ there
are $K_1 \in [K], M_1 \in [M]$ such that $K_1$ intersects a single connected
component $C_1$ of $\Gamma - L_1$, $M_1$ intersects a single connected component
$C_2$ of $\Gamma - L_1$ and $C_1 \neq C_2$.

**Proof.** As before we replace $\Gamma$ by its barycentric subdivision and view
edge cuts as vertex cuts of the barycentric subdivision.

Suppose that for some $L_1 \in [L]$ there are $K_1 \in [K], M_1 \in [M]$ such
that $K_1$ intersects a single connected component $C_1$ of $\Gamma - L_1$, $M_1$
intersects a single connected component $C_2$ of $\Gamma - L_1$ and $C_1 \neq C_2$.
Set $L^{(1)}$ to be all ends of $\Gamma$ contained in $C_1$. Let $C_1'$ be the connected
component of $\Gamma - K_1$ contained in $C_1$ and let $C_2'$ be the connected
component of $\Gamma - M_1$ contained in $C_2$. Set $K^{(1)}$ to be all ends of $\Gamma$
contained in \( C_1 \) and \( M^{(2)} \) to be all ends of \( \Gamma \) contained in \( C_2 \). Then clearly \( K^{(1)} \subset L^{(1)} \subset M^{(1)} \), so \([L]\) is between \([K],[M]\). Conversely now, assume that \([L]\) is between \([K],[M]\). We have then \( K^{(1)} \subset L^{(1)} \subset M^{(1)} \). Let \( L_1 \in [L]\) and let \( C_1, C_2 \) be the connected components of \( \Gamma - L_1 \), where \( C_1 \) contains all ends in \( K^{(1)} \). By lemma 3.1 there is \( K_1 \in [K]\) such that \( K_1 \subset C_1 \). Since \( M^{(2)} \subset L^{(2)} \) by lemma 3.1 again there is \( M_1 \in [M]\) such that \( M_1 \subset C_2 \).

\[ \square \]

3.2. Crossing cuts.

**Definition 3.4.** Let \( \Gamma \) be a graph and let \( K, L \) be minimal cuts of \( \Gamma \). We say that \([K]\) crosses \([L]\) if \( K^{(i)} \cap L^{(j)} \neq \emptyset \) for all \( i,j = 1,2 \).

**Remark 1.** Clearly if \( K, L \) are inequivalent minimal cuts either \([K],[L]\) cross or, after relabelling, \( K^{(1)} \subset L^{(1)} \). From lemma 3.1 we have that the following are equivalent:

a. \([K]\) crosses \([L]\).

b. For some \( L_1 \in [L]\) any \( K_1 \in [K]\) intersects both connected components of \( \Gamma - L_1 \).

c. For any \( L_1 \in [L]\) any \( K_1 \in [K]\) intersects both connected components of \( \Gamma - L_1 \).

**Lemma 3.3.** Let \( K, L \) be minimal cuts of a graph \( \Gamma \). If \([K]\) crosses \([L]\) then for any \( L_1 \in [L]\) and any \( K_1 \in [K]\), the following hold:

a. \( |K_1| = 2k \) for some \( k \in \mathbb{N} \).

b. \( K_1 \cap L_1 = \emptyset \), and the intersections of \( K_1 \) with both components of \( \Gamma - L_1 \) contain \( k \) elements.

c. \( \Gamma - (K_1 \cup L_1) \) has exactly 4 connected components, and each of these components contains at least one end of \( \Gamma \).

**Proof.** It will be convenient to replace \( \Gamma \) by its barycentric subdivision and view edge cuts as vertex cuts of the barycentric subdivision. To keep notation simple we keep denoting the barycentric subdivision by \( \Gamma \).

We denote the connected components of \( \Gamma - L_1 \) by \( C_1, C_2 \). Let’s say that ends in \( L^{(1)} \) are contained in \( C_1 \) and ends in \( L^{(2)} \) are contained in \( C_2 \). Since \([K],[L]\) cross there are ends \( e_{ij} \in K^{(i)} \cap L^{(j)} \) where \( i,j = 1,2 \).

We denote the connected components of \( \Gamma - K_1 \) by \( D_1, D_2 \) where ends in \( K^{(1)} \) are contained in \( D_1 \) and ends in \( K^{(2)} \) are contained in \( D_2 \).

So \( e_{ij} \in D_i \cap C_j \).

We set:

\[ k_1 = |K_1 \cap C_1|, \quad k_2 = |K_1 \cap C_2|, \quad l_1 = |L_1 \cap D_1|, \quad l_2 = |L_1 \cap D_2| \]
We denote \( n = |L_1| \) and \( m = |L_1 \cap K_1| \). Obviously
\[
k_1 + k_2 + m = n, \quad l_1 + l_2 + m = n
\]
Let’s pose further
\[
m_1 = |\partial(C_1 \cap D_1) \cap K_1 \cap L_1|
m_2 = |\partial(C_1 \cap D_2) \cap K_1 \cap L_1|
\]
We remark that
\[
\partial(C_2 \cap D_2) \cap K_1 \cap L_1 = \partial(C_1 \cap D_1) \cap K_1 \cap L_1
\]
\[
\partial(C_1 \cap D_2) \cap K_1 \cap L_1 = \partial(C_2 \cap D_1) \cap K_1 \cap L_1
\]
and
\[
m_1 + m_2 = m
\]
We have also
\[
|K_1| = k_1 + k_2 + m_1 + m_2 = n = |L_1| = l_1 + l_2 + m_1 + m_2 \quad (*)
\]
We remark that
\[
|\partial(C_1 \cap D_1)| = k_1 + l_1 + m_1
\]
Since \( e_{11} \in C_1 \cap D_1 \) and \( \partial(C_1 \cap D_1) \) separates \( e_{11} \) from \( e_{12} \) we have that
\[
k_1 + l_1 + m_1 \geq n \quad (1)
\]
Considering similarly \( C_2 \cap D_1, C_1 \cap D_2 \) and \( C_1 \cap D_2 \) we obtain the inequalities:
\[
k_2 + l_1 + m_2 \geq n \quad (2)
k_1 + l_2 + m_2 \geq n \quad (3)
k_2 + l_2 + m_1 \geq n \quad (4)
\]
Adding up (1),(2),(3),(4) and using (*) we obtain
\[
4n - 2(m_1 + m_2) \geq 4n
\]
It follows that \( m_1 = m_2 = 0 \).
Further we have that necessarily (1), (2), (3), (4) are equalities. From (1), (2) it follows that \( k_1 = k_2 \). Similarly from (1), (3) we get that \( l_1 = l_2 \). This proves assertions a and b of the lemma.
Part c also follows since \( |\partial(C_i \cap D_j)| = n \) for all \( i, j = 1, 2 \), \( C_i \cap D_j \) is necessarily connected, otherwise by considering its connected component containing \( e_{ij} \) we would obtain an end cut with less than \( n \) edges, a contradiction.
\( \square \)
Corollary 3.4. Let $\Gamma$ be a graph. If the cardinality of minimal cuts of $\Gamma$ is an odd number then the set of minimal cuts with the betweenness relation defined earlier is a discrete pretree. So the set of minimal cuts in this case can be represented by a tree.

3.3. Cyclic sets.

Definition 3.5. Assume that the cardinality of a minimal cut of a graph $\Gamma$ is $2k$. A subgraph $S$ of $\Gamma^b$ is called $k$-cyclic if it is a union of $m \geq 4$ finite subgraphs, $S = S_1 \cup S_2 \ldots \cup S_m$ and there are connected subgraphs $M_1, \ldots, M_m$ of $\Gamma^b$ so that for each $i \in \mathbb{Z}_m$:

1. $S_i \cap M_i = \{s_{i1}, \ldots, s_{ik}\}, S_i \cap M_{i+1} = \{s_{i1}', \ldots, s_{ik}'\}$ with $s_{ij}, s_{ij}'$ in $V(\Gamma^b) - V(\Gamma)$.
2. $M_i \cup M_{i+1}$ separates $S_i$ from $\bigcup_{j \neq i} S_j$ and $S_i \cup S_{i-1}$ separates $M_i$ from $\bigcup_{j \neq i} M_j$
3. $\bigcup (M_i \cup S_i) = \Gamma^b$ and for each $i$, $M_i$ contains at least one end of $\Gamma$.

We will often simply say that $S$ is cyclic rather than $k$-cyclic. We will say that the $M_i$’s are the beads and the $S_i$’s are the elements of the cyclic set $S$.

Lemma 3.5. Let $K_1, K_2$ be minimal cuts such that $[K_1], [K_2]$ cross each other. Then both $K_1, K_2$, are contained in a cyclic set $S$.

Proof. As usual we see $K_1, K_2$ as vertex cuts of $\Gamma^b$. Let $C_1, C_2$ be the connected components of $\Gamma^b - K_1$ and let $D_1, D_2$ be the connected components of $\Gamma^b - K_2$. Recall that if $C$ is a connected component of $\Gamma^b - K$ we denote by $\bar{C}$ the graph which is the union of $C$ with all edges that intersect $C$.

Then we can take $S = K_1 \cup K_2$ and

$M_1 = \bar{C}_1 \cap \bar{D}_1$
$M_2 = \bar{C}_1 \cap \bar{D}_2$
$M_3 = \bar{C}_2 \cap \bar{D}_2$
$M_4 = \bar{C}_1 \cap \bar{D}_1$

By lemma 3.3 $S, M_1, M_2, M_3, M_4$ satisfy the definition of a cyclic set.

We recall the edge version of Menger’s theorem (see eg. [1], thm. 7.17, p.170):

Menger’s Theorem. Let $\Gamma$ be a graph and let $a, b$ be vertices of $\Gamma$. Then the maximum number of edge disjoint paths joining $a, b$ is equal to the minimum number of edges in an edge cut separating $a, b$. 

We say that a cyclic set \( S = S_1 \cup ... \cup S_m \) is contained in a cyclic set \( S' = S'_1 \cup ... \cup S'_n \) if for each \( i = 1, ..., m \) there is some \( j = 1, ..., n \) such that \( S_i \subset S'_j \).

**Lemma 3.6.** Any \( k \)-cyclic set \( S \) is contained in a maximal \( k \)-cyclic set \( \Sigma \).

**Proof.** Let’s say that \( S = S_1 \cup ... \cup S_m \) is a cyclic set of a graph \( \Gamma \). Let \( M_2 \) be a bead of \( S \) and let \( S_1 \cap M_2 = \{ s_1, ..., s_k \}, S_2 \cap M_2 = \{ t_1, ..., t_k \} \) with \( s_i, t_i \) in \( V(\Gamma^b) - V(\Gamma) \). We identify all vertices \( s_1, ..., s_k \) to a single vertex \( a \) and all vertices \( t_1, ..., t_k \) to a single vertex \( b \) to obtain a new graph \( \Gamma' \). \( \Gamma' - \{a, b\} \) has two connected components \( C_1, C_2 \) (where, say, \( C_1 \) is obtained from \( M_2 \) by the vertex identifications indicated above).

Then no set of less than \( k \) edges separates \( a, b \) in \( C_1 \) or \( C_2 \). Therefore, by the edge version of Menger’s theorem, there are edge disjoint simple paths \( p_1, ..., p_k \) in \( C_1 \) and \( q_1, ..., q_k \) in \( C_2 \) such that for each one of these paths one endpoint lies in \( \{ s_1, ..., s_k \} \) and the other lies in \( \{ t_1, ..., t_k \} \). We lift the paths \( p_1, ..., p_k \) and \( q_1, ..., q_k \) to \( \Gamma \) and we keep denoting them the same way. We remark now that if \( M_i, (i \neq 1, i \in \mathbb{Z}_k) \) is another bead of \( S \) then at least one of \((M_i \cap S_i) \cup \{ s_1, ..., s_k \}, (M_i \cap S_i) \cup \{ t_1, ..., t_k \}\)
is a minimal cut. It follows that $M_i \cap S_i$ intersects each one of $q_1, \ldots, q_k$. Similarly $M_i \cap S_{i-1}$ intersects each one of $q_1, \ldots, q_k$.

Let $S' = S'_1 \cup \ldots \cup S'_n$ be a cyclic set containing $S$. Clearly $n \geq m$ since distinct elements of $S$ are contained in distinct elements of $S'$. This is because the elements of $S'$ do not contain any minimal cuts.

Let $S_i'$ be an element of $S'$. For any $j \in Z_n, j \neq i$,

$$K_{ij} = (S'_i \cap M'_i) \cup (S'_j \cap M'_j)$$

is a minimal cut (where $M'_i, M'_j$ are beads of $S'$). Choosing $j, p, r$ appropriately we may find a minimal cut:

$$L_{pr} = (S_p \cap M_p) \cup (S_r \cap M_r)$$

so that $K_{ij}, L_{pr}$ cross each other. Specifically if $S'_i$ contains come $S_t$ we may pick $S'_j$ so that it contains $S_{t+2}$ and take $p = t + 1, r = t + 3$. If $S'_i$ contains no $S_t$ then there are $i_1, i_2$ so that $i$ is between $i_1, i_2$, some $S_p$ is contained in $S'_i$ and $S_{p+1}$ is contained in $S'_i$. Take then $r = p + 1$ and $j$ so that $S'_j$ contains $S_{p+2}$.

It follows that $S'_i \cap M'_i$ intersects the union of the arcs $p_1 \cup q_1 \cup \ldots \cup p_k \cup q_k$ at $k$ points, so it is contained in this union. By the same reasoning, the same holds for $S'_i \cap M'_{i+1}$. Since $S'$ is determined by the sets $S'_i \cap M'_{i+1}, S'_i \cap M'_i$ we have that there are finitely many cyclic sets containing $S$. Therefore there is a maximal such set $\Sigma$ containing $S$.

**Lemma 3.7.** Let $\Sigma$ be a maximal cyclic set of $\Gamma$ and let $K$ be a minimal cut crossing some minimal cut contained in $\Sigma$. Then $K$ is equivalent to a minimal cut contained in $\Sigma$.

**Proof.** Let’s say that

$$\Sigma = S_1 \cup \ldots \cup S_m$$

Assume that $K$ is not equivalent to any minimal cut contained in $\Sigma$. Then there is some bead of $\Sigma$, say $M_i$ such that $K$ separates some ends of $M_i$. If

$$K' = (M_i \cap S_i) \cup (M_i \cap S_{i-1})$$

then $K$ crosses $K'$ so by lemma 3.3 $K = K_1 \cup K_2$, $K_1$ is contained in $M_i$, $K_2$ is contained in

$$\bigcup_{j \neq i} M_j$$

and $|K_1| = |K_2|$. Let

$$A = M_i \cap S_i, B = M_i \cap S_{i-1}$$

We claim that $K$ separates each vertex of $A$ from each vertex of $B$. We distinguish two cases. If $K$ separates some ends of some bead $M_j$,
\( j \neq i \) then \( K_2 \subset M_j \). It follows that \( K \) does not intersect at least one of the beads \( M_{i-1}, M_{i+1} \). Let’s say it does not intersect \( M_{i-1} \). If \( M_{i-1} \cap S_{i-1} = C \) then by Menger’s lemma each vertex of \( B \) is connected to some vertex of \( C \) by a path that does not intersect \( K \). It follows that all vertices of \( B \) are connected in the same component of \( \Gamma - K \).

By lemma 3.3 all vertices of \( A \) are contained in the same component of \( \Gamma - K \) too. If \( K \) does not intersect \( M_{i+1} \) we argue similarly using \( A \) rather than \( B \).

Assume now that \( K \) does not separate ends of any \( M_j \) with \( j \neq i \). Then for some \( j \neq i, i-1 \) \( K \) separates all ends in \( M_j \) from all ends in \( M_{j+1} \). It follows that \( K \) crosses the cut

\[
L = (M_{j+1} \cap S_{j+1}) \cup (M_j \cap S_{j-1})
\]

If \( L_1 = M_{j+1} \cap S_{j+1} \) then, by Menger’s lemma, each vertex in \( L_1 \) is connected by a path that does not intersect \( K \) to some vertex of \( B \). It follows that all vertices of \( B \) are contained in the same component of \( \Gamma - K \). By lemma 3.3 all vertices of \( A \) are contained in the same component of \( \Gamma - K \) too.

We see then that in both cases \( K_1 \) separates \( A \) from \( B \) in \( M_i \).

It follows that we can enlarge \( \Sigma \) by adding \( K_1 \). We obtain a cyclic set

\[
\Sigma' = S_1 \cup ... \cup S_{i-1} \cup K_1 \cup S_i \cup ... \cup S_m
\]

This contradicts the maximality of \( \Sigma \). \( \square \)

**Lemma 3.8.** Let \( \Sigma \) be a maximal cyclic set of \( \Gamma \) containing a given \( k \)-cyclic set \( S \) and let \( K \) be a minimal cut which crosses a minimal cut contained in \( \Sigma \). Then \( K \) is contained in \( \Sigma \). In particular \( S \) is contained in a unique maximal cyclic set.

**Proof.** Let’s say that

\[
\Sigma = S_1 \cup ... \cup S_m
\]

By lemma 3.7 \( K \) is equivalent to \( K' \) where \( K' \) is contained in a union \( S_i \cup S_j \) for some \( i, j, j \neq i, i+1 \). We set

\[
S_{i-1} \cap M_i = \{a_1, ..., a_k\}, \quad S_{i+1} \cap M_{i+1} = \{b_1, ..., b_k\}
\]

\[
S_{j-1} \cap M_j = \{a'_1, ..., a'_k\}, \quad S_{j+1} \cap M_{j+1} = \{b'_1, ..., b'_k\}
\]

We remark that both minimal cuts

\[
L_1 = \{a_1, ..., a_k, b_1, ..., b_k\}, \quad L_2 = \{a'_1, ..., a'_k, b'_1, ..., b'_k\}
\]

cross \( K \). So by lemma 3.3 \( K \) can be written as disjoint union of two sets, \( K_1, K_2 \) where \( K_1 \) is contained in the connected component \( C \) of \( \Gamma - L_1 \) containing \( M_i \) and \( K_2 \) is contained in the connected component \( D \) of \( \Gamma - L_2 \) containing \( M_j \).
Applying Menger’s theorem as earlier we see that there are edge disjoint paths \( p_1, ..., p_k \) in \( C \) and \( q_1, ..., q_k \) in \( D \) such that \( p_i \) joins \( a_i \) to \( b_i \), and \( q_i \) joins \( a'_i \) to \( b'_i \) for all \( i = 1, ..., k \) (this of course up to relabeling of the \( a_i \)’s, \( b_i \)’s). Since \( K \) separates \( M_i \) from \( M_{i+1} \) and \( M_j \) from \( M_{j+1} \), \( K_1 \) intersects each \( p_i \) at one point and \( K_2 \) intersects each \( q_i \) at one point.

Let \( M'_i, M'_{i+1} \) be the infinite components of \( \Gamma - (S_{i-1} \cup S_i \cup K) \) contained in \( M_i, M_{i+1} \) respectively. Then
\[
\partial M'_i \subset K_1 \cup S_i \cup S_{i-1}
\]
Clearly
\[
S_{i-1} \cap M_i \subset \partial M'_i
\]
We remark now that if \( \partial M'_i \) intersects a path \( p_t \) in 3 points then the first point is \( a_t \) the second lies on \( K_1 \) and the third point say \( c \) lies on \( S_1 \). However in this case there is a path joining \( b_t \) to \( a_t \) which does not intersect \( K \). Indeed take \( p_t \) from \( b_t \) to \( c \) and then continue with a path in \( M'_t \) joining \( c \) to \( a_t \). This is clearly a contradiction since \( K \) separates \( a_t, b_t \). It follows that \( \partial M'_i \) intersects each \( p_t \) at at most 2 points, and since \( \partial M'_i \) has at least \( 2k \) points we conclude that \( \partial M'_i \) intersects each \( p_t \) at exactly 2 points. We argue similarly for \( M'_{i+1} \). We conclude that if \( S'_i \) is the union of the finite connected components of
\[
\Gamma - (M'_i \cup M'_{i+1})
\]
then \( S_i \subset S'_i \) and \( |S'_i \cap M'_i| = k \) and \( |S'_i \cap M'_{i+1}| = k \). So we may replace \( S_i \) by \( S'_i \) in the cyclic set \( \Sigma \). This contradicts the maximality of \( \Sigma \) unless \( S'_i = S_i \) and \( K_1 \subset S_i \). Arguing similarly for \( K_2 \) we have that \( K_2 \subset S_i \), so \( K \) is contained in \( \Sigma \).

We show now that \( \Sigma \) is unique. Let \( \Sigma_1, \Sigma_2 \) be two maximal cyclic sets containing \( S \). Then for any element \( S_i \) in \( \Sigma_1 \) there is some \( S_j \in \Sigma_1 \) such that the cuts
\[
K_1 = (S_i \cap M_i) \cup (S_i \cap M_i)
\]
and
\[
K_1 = (S_i \cap M_{i+1}) \cup (S_i \cap M_i)
\]
cross some minimal cut contained in \( \Sigma_2 \). By lemma 3.7 and the proof above it follows that \( K_1, K_2 \) are contained in \( \Sigma_2 \). This implies that for all \( i, S_i \) is contained in \( \Sigma_2 \) so \( \Sigma_1 \subset \Sigma_2 \). By symmetry \( \Sigma_2 \subset \Sigma_1 \) so \( \Sigma_1 = \Sigma_2 \).

\( \square \)

**Corollary 3.9.** If a minimal cut \( K \) crosses some other minimal cut \( L \) then every minimal cut \( K' \in [K] \) is contained in a cyclic set.
Proof. By lemmas 3.5 and 3.6, $K, L$ are contained in a maximal cyclic set $S$. $K'$ crosses $L$, so by lemmas 3.7, 3.8, $K'$ is contained in $S$ too. □

**Definition 3.6.** We say that a minimal cut $K$ is **isolated** if it does not cross any other minimal cut.

We can now define a pretree $\mathcal{P}$ ‘encoding’ all minimal cuts of $\Gamma$. The elements of $\mathcal{P}$ are the maximal cyclic sets of $\Gamma$ and the equivalence classes of the isolated minimal cuts of $\Gamma$. We make now some observations that will allow us to define betweeness in $\mathcal{P}$.

Note that if $S = S_1 \cup \ldots \cup S_n$ is a maximal cyclic set with beads $M_1, \ldots, M_n$ then each end of $\Gamma$ is contained in exactly one of the $M_i$’s. So $S$ partitions the set of ends

$$\mathcal{E} = M^{(1)} \cup \ldots \cup M^{(n)}$$

where we denote by $M^{(i)}$ the set of ends contained in $M_i$. If $K$ is a minimal cut not contained in $S$ such that $K$ separates some ends $e_1, e_2$ that lie in some $M^{(i)}$ then $K$ many not separate any two ends $e'_1, e'_2$ that lie in some $M^{(j)}$ with $i \neq j$. Indeed in that case $K$ would cross the cut $(M_i \cup S_i) \cup (M_i \cup S_{i-1})$ so by lemma 3.7, $K$ would be contained in $S$. We remark further that if $K$ does not separate ends that lie in any $M^{(i)}$ then there is some $i$ such that $K$ separates all ends in $M^{(i)}$ from all ends in $M^{(j)}$ for all $j \neq i$. Indeed if not, as before, $K$ crosses some minimal cut contained in $S$, so $K$ lies in $S$. We conclude that in all cases the following holds: if $K$ is a minimal cut not contained in $S$ and if we denote by $K^{(1)} \cup K^{(2)}$ the partition of ends of $\Gamma$ induced by $K$ then $K^{(1)}$ or $K^{(2)}$ is contained in some $M^{(i)}$. By lemma 3.1, it follows further that if $K$ is a minimal cut that is not contained in $S$ then it is equivalent to a cut $K'$ such that $K' \subset M_i$ for some $i$.

Let $S' = S'_1 \cup \ldots \cup S'_n$ be another maximal cyclic set. Then for each minimal cut $K$ contained in $S'$ there is an $i$ such that $K$ is equivalent to a minimal cut contained in $M_i$. However if there are minimal cuts $K, L$ in $S'$ such that, say $K$ is equivalent to a minimal cut in $M_i$ and $L$ is equivalent to a minimal cut in $M_j$ with $j \neq i$ then there is a minimal cut in $S$ that crosses a minimal cut in $S'$. But this implies that $S = S'$. It follows that all minimal cuts in $S'$ are equivalent to cuts that are contained in a single bead $M_i$ of $S$.

If $S_1, S_2, S_3$ are distinct elements of $\mathcal{P}$ we define betweeness as follows: If $S_1$ is cyclic, then it is between $S_2, S_3$ if the minimal cuts in $S_2, S_3$ are equivalent to cuts which are contained in distinct beads of $S_1$. If $S_1 = [K_1]$ is not cyclic then $S_1$ is between $S_2, S_3$ if all minimal cuts $K_2$ in $S_2, K_3$ in $S_3$ are equivalent to cuts that lie in distinct components of $\Gamma - K_1$. 
Theorem 3.10. \( \mathcal{P} \) with the betweenness relation defined above is a pretree.

Proof. Axioms 1 and 2 of the pretree definition obviously hold. We show that axiom 3 holds. Assume that \( S_1 \) is between \( S_2, S_3 \). We will show that \( S_3 \) is not between \( S_1, S_2 \). Assume first that \( S_1 \) is cyclic. Since \( S_1 \) is between \( S_2, S_3 \) then the minimal cuts in \( S_2, S_3 \) are equivalent to minimal cuts which are contained in distinct beads, say \( M_i, M_j \) of \( S_1 \). This implies that for any minimal cut in \( S_3 \) the minimal cuts in \( S_1, S_2 \) are equivalent to minimal cuts that are contained in the same bead of \( S_3 \) if \( S_3 \) is cyclic or in the same component of \( \Gamma - K \) for \( K \in S_3 \) if \( S_3 \) is not cyclic. In both cases \( S_1S_3S_2 \) does not hold. Assume now that \( S_1 = [K] \) where \( K \) is an isolated cut. Then the minimal cuts in \( S_2, S_3 \) are equivalent to cuts that lie in distinct components of \( \Gamma - K \). Then, if \( S_3 \) is cyclic, the minimal cuts in \( S_1, S_2 \) are equivalent to minimal cuts that lie in the same bead of \( S_3 \) so \( S_1S_3S_2 \) does not hold. Similarly in \( S_3 = [K] \) all minimal cuts in \( S_1, S_2 \) are equivalent to minimal cuts that lie in the same component of \( \Gamma - K \). So \( S_1S_3S_2 \) does not hold in this case either.

We show finally axiom 4. Assume that \( S_2S_3 \) holds and that \( S_4 \neq S_1 \). If \( S_1 \) is cyclic then then the minimal cuts in \( S_2, S_3 \) are equivalent to cuts contained in distinct beads of \( S_1 \). If all minimal cuts in \( S_4 \) are equivalent to cuts contained in the same bead as \( S_2 \) then \( S_3S_1S_4 \) holds. Otherwise \( S_2S_1S_4 \) holds. If \( S_1 \) is an equivalence class of an isolated cut \( K \) then the minimal cuts in \( S_2, S_3 \) are equivalent to cuts contained in distinct components of \( \Gamma - K \). If the minimal cuts in \( S_4 \) are equivalent to minimal cuts contained in the same component of \( \Gamma - K \) as the minimal cuts of \( S_2 \) then \( S_3S_1S_4 \) holds. Otherwise \( S_2S_1S_4 \) holds. This shows that axiom 4 is satisfied.

Clearly the pretree \( \mathcal{P} \) is discrete, so it can be completed to a tree \( T \) encoding all minimal cuts of \( \Gamma \). We give now a detailed description of \( T \). The vertices of \( T \) are of three types: isolated cuts, cyclic sets and ‘star’ vertices (see section 2 for the ‘star’ vertices). We remark that if \( S \) is a cyclic set in \( \mathcal{P} \), \( S \) is adjacent to the isolated cuts that correspond to its beads. So if \( S = S_1 \cup \ldots \cup S_n \) and \( M_i \) is a bead of \( S \) then \( S \) is adjacent to the isolated cut \((M_i \cap S_{i-1}) \cup (M_i \cap S_i)\). It follows that all star vertices adjacent to \( S \) have degree 2. If \( K \) is an isolated cut of \( \mathcal{P} \) and \( H \) is a star of \( \mathcal{P} \) containing \( K \) then either \( H \) consists of a cyclic set \( S \) and \( K \) or \( H \) consists of at least 3 isolated cuts.

To retain the cyclic structure of the crossing cuts we replace \( T \) by a cactus \( \mathcal{C} \) as follows: A cyclic set \( S \) of \( \mathcal{P} \) gives a cycle \( C \) of \( \mathcal{C} \) with
vertices corresponding to the beads of the cyclic set $S$. Each vertex of $C$ is joined to the corresponding star vertex. In this way we obtain a cactus. We further simplify this cactus as follows: If $K$ is an isolated cut adjacent to a cyclic set $S$ in $P$ then $K$ is joined to $S$ by a path of two edges (from $K$ to the star vertex and then from the star vertex to $S$). We collapse all these 2-edge paths joining isolated cuts to cyclic sets.

This is because such isolated cuts are already represented in the cycle (by the two edges adjacent to the bead). For symmetry’s sake finally we ‘double’ all separating edges of the cactus. Clearly now we have a 1-1 correspondence between the minimal cuts of $\Gamma$ and the minimal edge cuts of $C$. We state this formally:

**Theorem 3.11.** Let $\Gamma$ be a graph. Then there is a cactus $C$, an onto map $f$ from the ends of $\Gamma$ to the union of the ends of $C$ with the end vertices of $C$ and a 1-1 and onto map $g$ from equivalence classes of minimal cuts of $\Gamma$ to the minimal edge cuts of $C$ so that two ends $e_1, e_2$ of $\Gamma$ are separated by a minimal cut $K$ if and only if $f(e_1), f(e_2)$ are separated by $g([K])$. Moreover any automorphism of $\Gamma$ induces an automorphism of $C$.

**Corollary 3.12.** (Stallings end theorem) Let $G$ be a group acting transitively on a graph $\Gamma$ with more than 2 ends. Then $G$ splits as $G = A_1 * F$ or $G = A_2 * F$ where $F$ has a finite index subgroup which is a stabilizer of an edge of $\Gamma$.

**Proof.** We associate a tree $T$ to $P$. The action is non trivial since the action of $G$ on $\Gamma$ is transitive. It follows that $G$ splits over a stabilizer of an edge. Edges correspond to equivalence classes of minimal cuts. So edge stabilizers stabilize equivalence classes of minimal cuts. Since such equivalence classes contain finitely many edges the result follows. □

Stallings’ theorem covers the 2-ended case too. However our cactus is this case reduces to a single point. We remark that if $G$ is a finitely generated group and $\Gamma$ its Cayley graph the 2-ended case is simpler and it is easy to show that in this case $G$ has a finite index subgroup isomorphic to $\mathbb{Z}$. We note finally that Krön [11] has given recently a very elegant proof of Stallings theorem using the methods of [7].

4. **Generalizations**

One can show that the ‘cactus structure’ of cuts exists in other settings as well. We discuss here some such generalizations, which are interesting for finite graphs.
Definition 4.1. Let $\Gamma$ be a graph and $K$ a set of edges of $\Gamma$. We say that $K$ is an $n$-cut if $\Gamma - K$ has more than one component and $|K| = n$.

Definition 4.2. Let $\Gamma$ be a graph and let $S$ be a set of vertices of $\Gamma$. We call $S$ $n$-inseparable if for any $r$-cut $K$, with $r \leq n$, $S$ is contained in a single component of $\Gamma - K$.

We remark that Dunwoody and Krön [5] consider a similar notion of inseparable sets but their definition is slightly stronger, they require further that $|S| > n$.

Definition 4.3. Let $\Gamma$ be a graph and $k \in \mathbb{N}$. We define $N(k)$ to be the smallest $n$ such that there are at least two distinct maximal $n$-inseparable subsets of $\Gamma$ with at least $k$ vertices each. If there is no such $n$ we set $N(k) = \infty$. If for some $k$, $N(k) < \infty$ we say that $\Gamma$ is a $k$-thin graph. We call an $N(k)$-cut $K$ essential if both components of $\Gamma - K$ contain some $N(k)$-inseparable set.

We remark that if $k_1 > k_2$ then $N(k_1) \geq N(k_2)$. In particular if a graph is $k$-thin for some $k > 2$ then it is also 2-thin. Clearly every graph with at least two vertices is 1-thin. Assume that $\Gamma$ is $k$-thin, and set $n = N(k)$. If $S_1, S_2$ are $n$-inseparable subsets and $K$ is an $n$-cut which separates $S_1, S_2$ then $\Gamma - K$ has exactly 2 components.

If $K_1, K_2$ are $n$-cuts of $\Gamma$ we say that that $K_1, K_2$ are equivalent if for any two $n$-inseparable subsets of $\Gamma$, $S_1, S_2$ are separated by $K_1$ if and only if they are separated by $K_2$. We denote the equivalence class of $K_1$ by $[K_1]$.

Lemma 3.3 applies in this context as well and one can show exactly as in the case of minimal end cuts that all equivalence classes of $N(k)$-cuts of a $k$-thin graph are encoded by a cactus.

Clearly every graph with at least 2 vertices is 1-thin. In this case $N(1)$ is the cardinality of a minimal edge cut and every equivalence class has a single element. So this case amounts to the classical cactus theorem of Dinits-Karzanov-Lomonosov ([7]).

Definition 4.4. Let $\Gamma$ be a graph and $K$ a set of edges of $\Gamma$. We say that $K$ is an $(n, k)$-cut if $|K| = n$ and $\Gamma - K$ has at least 2 components which have each at least $k$ vertices. Let $S$ be a set of vertices of $\Gamma$ containing at least $k$ elements. We call $S$ $(n, k)$-inseparable if for any $(r, k)$-cut $K$, with $r \leq n$, $S$ is contained in a single component of $\Gamma - K$.

We note that $(n, k)$-cuts have been studied extensively in network theory (see e.g. [8], [19]).

Definition 4.5. Let $\Gamma$ be a graph. We say that $\Gamma$ is $(n, k)$-large if $\Gamma$ has at least 2 distinct maximal $(n, k)$-inseparable subsets. We set
$M(k)$ to be the least $n$ such that $\Gamma$ has at least two distinct maximal $(n, k)$-inseparable subsets. If $M(k) < \infty$ we say that $\Gamma$ is $k$-slim.

**Definition 4.6.** Let $\Gamma$ be a $k$-slim graph. Let $K_1, K_2$ be $(M(k), k)$-cuts of $\Gamma$. We say that $K_1, K_2$ are equivalent if for any two $(n, k)$-inseparable subsets, $S_1, S_2$ of $\Gamma$ we have that $S_1, S_2$ are contained in distinct components of $\Gamma - K_1$ if and only if they are contained in distinct components of $\Gamma - K_2$. We write then $K_1 \sim K_2$ and we denote the equivalence class of $K_1$ by $[K_1]$.

**Definition 4.7.** We call an $(M(k), k)$ cut essential if both components of $\Gamma - K$ contain some $(M(k), k)$-inseparable set.

Lemma 3.3 applies to equivalence classes of essential $(M(k), k)$-cuts, so such cuts are also encoded by a cactus.

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E-mail address, Anastasia Evangelidou: aneva@spidernet.com.cy
E-mail address, Panos Papasoglu: papazoglou@maths.ox.ac.uk

(Anastasia Evangelidou) Mathematics Department, University of Athens, Athens 157 84, Greece

(Panos Papasoglu) Mathematical Institute, University of Oxford, 24-29 St Giles’, Oxford, OX1 3LB, U.K.