Symmetric 2-Step 4-Point Hybrid Method for the Solution of General Third Order Differential Equations

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Abstract

This research considers a symmetric hybrid continuous linear multistep method for the solution of general third order ordinary differential equations. The method is generated by interpolation and collocation approach using a combination of power series and exponential function as basis function. The approximate basis function is interpolated at both grid and off-grid points but the collocation of the differential function is only at the grid points. The derived method was found to be symmetric, consistent, zero stable and of order six with low error constant. Accuracy of the method was confirmed by implementing the method on linear and non-linear test problems. The results show better performance over known existing methods solved with the same third order problems. AMS 2010 Subject Classification: 65D05; 65L05; 65L06.

Keywords: Symmetric; Hybrid method; Power series and exponential function; Continuous predictor-corrector method

Introduction

In this paper, we considered the solution of initial value problems for general third order ordinary differential equations of the form

\[ y''' = f(t, y, y', y''), \quad y(t_0) = y_0, y'(t_0) = y_1, y''(t_0) = y_2. \tag{1} \]

Where \( t, y \in \mathbb{R} \).

The numerical and theoretical studies of eqn. (1) have appeared in literature severally. The direct approach for solving this type of ordinary differential equations have been studied and appeared in different literatures [1-7]. This direct approach has demonstrated advantages over the popular approach (reduction to system of first order approach) in terms of speed and accuracy [8,9]. Many authors have focused on direct solution of general second order ivps of odes of the form

\[ y'' = f(t, y, y') \tag{2} \]

Majid et al. [10] proposed two point four step direct implicit block method for the solution of second order system of ordinary differential equations (ODEs), using variable step size. The method estimated the solutions of initial value problems at two points simultaneously by using four backward steps but with lower order of accuracies. Akinfenwa [11] presented ninth order hybrid block integrator for solving second order ordinary differential equations. In the paper, the proposed block integrator discretizes the problem using the main and the additional methods to generate system of equations. The resulting system was solved simultaneously in a block-by-block fashion but the order of accuracies is low compare to the order of the method. The authors came up with direct implementation of predictor-corrector methods [3,4,7]. The authors emphasized the need to develop the same order of accuracy of the main predictors and that of the correctors to ensure good accuracy of the method. The order of accuracies in these works improved significantly compare to the existing methods with lower order of their main predictors.

Attempts have also been made by these scholars [12-14,6,7,15]. Olabode [13] proposed a 5-step block scheme for the solution of special type of eqn. (1). The order of accuracy in Olabode et al. [13] improves more than that of Olabode et al [13]. Awoyemi et al. [6], developed a four-point implicit method for the numerical integration of third order ODEs using power series polynomial function [16]. Kuboye and Omar [7] proposed numerical solution of third order ordinary differential equations using a seven-step block method to improve on Awoyemi et al. [6] and Olabode [13] which are of lower order of accuracy. Furthermore, a symmetric hybrid linear multistep method of order six having two off-step points for the solution of eqn. (1) directly was presented by Obarhua and Kayode [15].

To improve on the study of Obarhua and Kayode [15] a symmetric of two-step four-point hybrid method for the solution of third order initial value problems of ordinary differential equations directly is therefore proposed using the combination of power series and exponential function as the approximate basis function [17].

Derivation of the Method

This research work considers the derivation of 2-step 4-point hybrid method for the solution of general third order initial value problems of ordinary differential equations. The approach is to solve eqn. (1) directly without reducing it to a system of first order differential equation. A combination of power series and exponential function is used as the basis function for eqn. (1). The approximate solution eqn. (1) and the resulting differential systems are respectively given as

\[ y(t) = \sum_{j=0}^{r+s-1} \lambda_j f^j + \lambda_{r+s} \sum_{j=0}^{r-1} \alpha_j^r f^j. \tag{3} \]

Where \( r \) and \( s \) are the number of interpolation and collocation points respectively.

The third derivative of eqn. (3) as compared with eqn. (1) gives

\[ f(t, y, y', y'') = \sum_{j=0}^{r+s-1} \left( \sum_{j=0}^{r-1} (j(j-1)/(j-2))\lambda_j f^j \right) + \alpha_{r+s} \sum_{j=0}^{r-1} \alpha_j^r f^j. \tag{4} \]

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Collocating eqn. (4) at only the grid points, \( t_{mj} \), \( j=0(1)2 \), and interpolating (3) at both grid and off-grid points, \( t_{mj} = 0 \left( \frac{1}{3} \right) \), leads to the following system of equations.

\[
A = \begin{bmatrix}
1 & t_n & t_n^2 & \cdots & t_n^5 & \delta_1 \\
1 & t_n & t_n^2 & \cdots & t_n^5 & \delta_2 \\
1 & t_n & t_n^2 & \cdots & t_n^5 & \delta_3 \\
1 & t_n & t_n^2 & \cdots & t_n^5 & \delta_4 \\
1 & t_n & t_n^2 & \cdots & t_n^5 & \delta_5 \\
1 & t_n & t_n^2 & \cdots & t_n^5 & \delta_6
\end{bmatrix}
\]

At=b

where \( t=\lambda \), \( \alpha \), \( \beta \) form the following:

\[
\delta_1 = \left[ a + t_n^r + \frac{a}{2!} + \frac{a}{3!} + \frac{a}{4!} + \frac{a}{5!} \right]
\]

\[
\delta_2 = \left[ a + t_n^r + \frac{a}{2!} + \frac{a}{3!} + \frac{a}{4!} + \frac{a}{5!} \right]
\]

\[
\delta_3 = \left[ a + t_n^r + \frac{a}{2!} + \frac{a}{3!} + \frac{a}{4!} + \frac{a}{5!} \right]
\]

\[
\delta_4 = \left[ a + t_n^r + \frac{a}{2!} + \frac{a}{3!} + \frac{a}{4!} + \frac{a}{5!} \right]
\]

\[
\delta_5 = \left[ a + t_n^r + \frac{a}{2!} + \frac{a}{3!} + \frac{a}{4!} + \frac{a}{5!} \right]
\]

\[
\delta_6 = \left[ a + t_n^r + \frac{a}{2!} + \frac{a}{3!} + \frac{a}{4!} + \frac{a}{5!} \right]
\]

Taking \( k=2 \), the coefficients \( \alpha(t) \) and \( \beta(t) \) are expressed as function of \( v=\frac{T-t_{m+1}}{h} \) as follows:

\[
\alpha_1(v) = \frac{913}{5530} \cdot v^2 - \frac{1215}{632} \cdot v^2 + \frac{12879}{3160} \cdot v^2 - \frac{729}{553} \cdot v^2
\]

\[
\alpha_2(v) = 1 - \frac{28213}{2212} \cdot v + \frac{49101}{316} \cdot v^2 - \frac{11421}{316} \cdot v^2 + \frac{22577}{2212} \cdot v^2
\]

The first and second derivatives of eqn. (8) are:

\[
\beta_1(v) = \frac{4 \cdot 4139}{331695} \cdot \frac{77}{1080} - \frac{1 \cdot 1327}{123240} \cdot \frac{1}{364} + \frac{0.575}{115024}
\]

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\]

Evaluating eqn. (7) at \( v=1 \) gives the discrete method:

\[
\psi_j = \frac{256}{99} \cdot y'(t) + \frac{395}{97} \cdot y''(t) + \frac{395}{97} \cdot y'''(t) + \frac{256}{99} \cdot y^{(4)}(t) + \frac{1}{97} \cdot y^{(5)}(t)
\]

Where \( y(t) \) is assumed to be continuously differentiable of high order. Therefore, expanding eqn. (11) in Taylor’s series and comparing the coefficient of \( h^2 \) to give the expression

\[
L \left[ y(x) \right] = \sum_{j=1}^{\infty} \left( a_j y(t) + b_j y'(t) \right)
\]

where \( j=1 \) (1-2) as \( j=5,6 \).

Solving eqn. (5) for \( \lambda \)’s and substituting back into eqn. (3), with some manipulation yields, a linear multistep method with continuous coefficients in the form:

\[
y(t) = \sum_{i=0}^{\infty} a_i(t) y(t) + \sum_{i=0}^{\infty} c_i(t) y' + \sum_{i=0}^{\infty} r_i(t) y'' + \sum_{i=0}^{\infty} s_i(t) y''' + \sum_{i=0}^{\infty} u_i(t) y^{(4)} + \sum_{i=0}^{\infty} v_i(t) y^{(5)}
\]

Equations (8-10) are of order six, symmetric, consistent, low error constants and capable of handling oscillatory problems.
Implementation of the Method

To implement the implicit linear 2-step 4-point discrete scheme eqn. (8) and its first and second derivatives eqns.(9) and (10), respectively. The following symmetric explicit schemes and their derivatives are also developed by the same procedure for the evaluation of $y_{n+3}, y_{n+2}$ and $y'_{n+2}$.

$$y_{n+3} = \frac{10392}{2629} x^n + \frac{15753}{2629} x^{n+1} + \frac{11312}{2629} x^{n+2} - \frac{3003}{2629} x^n$$

$$p=6, c_{p+3}=3.107634 \times 10^{-7}, c_{p+3}=9.0 \times 10^{-6}, c_{p+3}=1.35 \times 10^{-4}$$

The methods eqns. (13), (14) and (15) are of order $p=6, c_{p+3}=9.0 \times 10^{-6}$ and $c_{p+3}=1.35 \times 10^{-4}$ respectively.

Other explicit schemes were also generated to evaluate other $y$-exact, the $y$-computed, the errors of the new method and the time of iteration for Problem 2 are shown.

### Table 2: Numerical solution and errors for problem 2.

| $x$   | $y_{exact}$     | $y_{computed}$     | Error in Olabode, (2009), p=7, k=5 | Error in new scheme, p=6, k=2 |
|-------|-----------------|---------------------|------------------------------------|------------------------------|
| 0.1   | 0.9148290819243523 | 0.9148290819245347 | 7.56477e-11 | 1.82410e-13 |
| 0.2   | 0.855972418398302  | 0.855972418415010 | 2.60170e-10 | 1.48598e-11 |
| 0.3   | 0.8301411924299970 | 0.8301411924299984 | 5.76003e-10 | 6.00142e-12 |
| 0.4   | 0.82187502358729 | 0.821875023735889 | 8.41270e-10 | 1.67078e-12 |
| 0.5   | 0.8512787292998718 | 0.8512787293299923 | 1.0013e-09 | 3.01205e-11 |
| 0.6   | 0.89781199604913 | 0.89781199663331 | 1.09051e-09 | 5.38411e-11 |
| 0.7   | 0.9662472925295238 | 0.9662472926178395 | 1.07048e-09 | 8.83157e-11 |
| 0.8   | 1.0544590715479328 | 1.0544590716435931 | 1.49247e-09 | 1.36060e-10 |
| 0.9   | 1.160398888430511 | 1.160398889429206 | 3.15695e-09 | 9.9870e-10 |
| 1.0   | 1.2817181715409554 | 1.2817181718237693 | 4.45956e-09 | 2.82814e-10 |

### Table 3: Numerical solution and errors for problem 3.

| $x$   | $y_{exact}$     | $y_{computed}$     | Error in Olabode, (2009), p=7, k=5 | Error in new scheme, p=6, k=2 |
|-------|-----------------|---------------------|------------------------------------|------------------------------|
| 0.1   | 1.0500417792784914 | 1.0500418242095606 | 9.496e-08 | 0.0027 |
| 0.2   | 1.1003353477310759 | 1.1003366664736403 | 1.326e-06 | 0.0024 |
| 0.3   | 1.1511404359634668 | 1.1511466804205729 | 5.656e-06 | 0.0026 |
| 0.4   | 1.2027325540540821 | 1.202743597246535 | 1.586e-05 | 0.0026 |
| 0.5   | 1.255412188289952 | 1.255448279429176 | 3.555e-05 | 0.0026 |
| 0.6   | 1.3095196042031119 | 1.3095893044089619 | 6.979e-05 | 0.0027 |
| 0.7   | 1.3654437542713962 | 1.365569060595540 | 1.25e-04 | 0.0027 |
| 0.8   | 1.4236489301936017 | 1.4238603658481614 | 2.11e-04 | 0.0026 |
| 0.9   | 1.4847002785940517 | 1.485041349663610 | 3.41e-04 | 0.0028 |
| 1.0   | 1.5493061443340548 | 1.5498382025358242 | 5.32e-04 | 0.0029 |
Table 3: Numerical solution and errors for problem 3.

| x    | y_{exact}         | y_{computed}       | Error in new scheme, p=6, k=2 | Time(s) |
|------|-------------------|--------------------|-------------------------------|---------|
| 1.05 | 1.0526315789473684| 1.052631579293796  | 1.017989e-09                  | 0.0291  |
| 1.10 | 1.111111111111112 | 1.111110853730750  | 2.573804e-08                  | 0.0313  |
| 1.15 | 1.1764705882352944 | 1.176470464899964  | 1.217663e-07                  | 0.0316  |
| 1.20 | 1.2500000000000002| 1.2499996375184190 | 3.624816e-07                  | 0.0320  |
| 1.25 | 1.3333333333333337| 1.3333246878923525 | 8.664510e-07                  | 0.0323  |
| 1.30 | 1.4265714265714290| 1.4265619003086576 | 1.819265e-06                  | 0.0325  |
| 1.35 | 1.5384615384615392| 1.538457871365800  | 3.551325e-06                  | 0.0328  |
| 1.40 | 1.6666666666666667| 1.666660337776776  | 6.632899e-06                  | 0.0332  |
| 1.45 | 1.8181818181818195| 1.818180033777676  | 1.211591e-05                  | 0.0335  |
| 1.50 | 2.0000000000000000| 1.9999779695058428 | 2.20E-05                      | 0.0338  |

oscillatory type of problems. The numerical tests results obtained were compared with block method of Olabode [13] which was found to perform favorably than the existing method.

References

1. Jator S (2001) Improvements in Adams-Moulton Methods for the First Order Initial Value Problems. Journal of the Tennessee Academy of Science 76: 57-60.
2. Jator SN, Li J (2009) A Self Starting Linear Multistep Method for the General Second Order Initial Value Problems. Int J Comput Math 86: 817-836.
3. Kayode SJ, Adeyeye O (2011) A 3-step hybrid method for direct solution of second order initial value problems. Aust J Basic Appl Sci 5: 2121-2126.
4. Kayode SJ, Obarhua FO (2013) Continuous y-function Hybrid Methods for Direct Solution of Differential Equations. International Journal of Differential Equations and Applications 6: 191-196.
5. Areo EA, Adeniyi RB (2013) A Self-Starting Linear Multistep Method for Direct Solution of Initial Value Problems of Second Order Ordinary Differential Equations. International Journal of Pure and Applied Mathematics 82: 345-354.
6. Awoyemi DO, Kayode SJ, Adoghe LO (2014) A four-point fully implicit method for numerical integration of third-order ordinary differential equations. Int J Phys Sci 9: 7-12.
7. Kuboye JO, Omar Z (2015) Numerical Solution of Third Order Ordinary Differential Equations Using a Seven-Step Block Method. International Journal of Mathematical Analysis 9: 743-754.
8. Yap LK, Ismail F, Senu N (2014) An Accurate Block Hybrid Collocation Method for Third Order Ordinary Differential equations. Journal of Applied Mathematics 2014: 1-9.
9. Kayode SJ, Obarhua FO (2015) 3-step y-function hybrid methods for direct numerical integration of second order IVPs in ODEs. Theoretical Mathematics & Applications 5: 39-51.
10. Majid ZA, Azimi NA, Suleiman M (2009) Solving second order ordinary differential equations using two point four step direct implicit block method. European Journal of Scientific Research 31: 29-36.
11. Akinfenwa OA (2013) Ninth Order Block Piecewise Continuous Hybrid Integrators for Solving Second Order Ordinary Differential Equations. International Journal of Differential Equations and Applications, 12: 49-67.
12. Awoyemi DO, Idowu OM (2005) A Class of Hybrid Collocation Methods for Third Order Ordinary Differential Equations. Int J Computer Math 82: 1287-1293.
13. Olabode BT (2009) An Accurate Scheme by Block Method for Third Order Ordinary Differential Equation. Pacific Journal of Science and Technology 10: 129-142.
14. Olabode BT (2013) Block Multistep Method for the Direct Solution of Third Order Ordinary Differential Equations. FUTA Journal of Research in Sciences 2: 194-200.
15. Obarhua FO, Kayode SJ (2016) Symmetric Hybrid Linear Multistep Method for General Third Order Differential Equations. Open Access Library Journal 3: 2583.
16. Okunuga SA, Ehijie J (2009) New Derivation of Continuous Multistep Methods using Power Series as Basis Function. J Modern Math Stat 3: 43-50.
17. Majid ZA, Azimi NA, Suleiman M, Ibrahim ZB (2012) Solving Directly General Third Order Ordinary Differential Equations Using Two-Point Four-Step Block Method. Sians Malaysiana 41: 623-632.