Anholonomic Triads and New Classes of (2+1)–Dimensional Black Hole Solutions

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Abstract

We apply the method of moving anholonomic frames in order to construct new classes of solutions of the Einstein equations on (2+1)–dimensional pseudo–Riemannian spaces. The anholonomy associated to a class of off–diagonal metrics results in alternative classes of black hole solutions which are constructed following distinguished (by nonlinear connection structure) linear connections and metrics. There are investigated black holes with deformed horizons and renormalized locally anisotropic constants. We speculate on properties of such anisotropic black holes with characteristics defined by anholonomic frames and anisotropic interactions of matter and gravity. The thermodynamics of locally anisotropic black holes is discussed in connection with a possible statistical mechanics background based on locally anisotropic variants of Chern–Simons theories.

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1 Introduction

In recent years there has occurred a substantial interest to the (2+1)–dimensional gravity and black holes and possible connections of such objects with string/M–theory. Since the first works of Deser, Jackiv and ’t Hooft [10] and Witten [28] on three dimensional gravity and the seminal solution for (2+1)–black holes constructed by Bañados, Teitelboim, and Zanelli (BTZ) [3] the gravitational models in three dimensions have become a very powerful tool for exploring the foundations of classical and quantum gravity, black hole physics, as well the geometrical properties of the spaces on which the low–dimensional physics takes place [5].

On the other hand, the low–dimensional geometries could be considered as an arena for elaboration of new methods of solution of gravitational field equations. One of peculiar features of general relativity in 2+1 dimensions is that the bulk of physical solutions of Einstein equations are constructed for a negative cosmological constant and on a space of constant curvature. There are not such limitations if anholonomic frames modeling locally anisotropic (la) interactions of gravity and matter are considered.

In our recent works [25] we emphasized the importance of definition of frames of reference in general relativity in connection with new methods of construction of solutions of the Einstein equations. The former priority given to holonomic frames holds good for the ‘simplest’ spherical symmetries and is less suitable for construction of solutions with ‘deformed’ symmetries, for instance, of static black holes with elliptic (or ellipsoidal and/or torus) configurations of horizons. Such type of ‘deformed’, locally anisotropic, solutions of the Einstein equations are easily to be derived from the ansatz of metrics diagonalized with respect to some classes of anholonomic frames induced by locally anisotropic ‘elongations’ of partial derivatives. After the task has been solved in anholonomic variables it can be removed with respect to usual coordinate bases when the metric becomes off–diagonal and the (for instance, elliptic) symmetry is hiden in some nonlinear dependencies of the metric components.

The specific goal of the present work is to formulate the (2+1)–dimensional gravity theory with respect to anholonomic frames with associated nonlinear connection (N–connection) structure and to construct and investigate some new classes of solutions of Einstein equations on locally anisotropic spacetimes (modelled as usual pseudo–Riemannian spaces provided with an anholonomic frame structure). A material of interest are the properties of the locally anisotropic elastic media and rotating null fluid and anisotropic collapse described by gravitational field equations with locally anisotropic matter. We investigate black hole solutions that arise from coupling in a self–consistent manner the three dimensional (3D) pseudo–Riemannian geometry to the physics of locally anisotropic fluids formulated with respect to anholonomic frames of reference. For certain special cases the locally anisotropic matter gives the BTZ black holes with/or not rotation and electrical charge and variants of their anisotropic generalizations. For other cases, the resulting solutions are generic black holes with “locally anisotropic hair”.

We emphasize that the anisotropic gravitational field has very unusual properties. For instance, the vacuum solutions of Einstein anisotropic gravitational field equations could describe anisotropic black holes with elliptic symmetry. Some subclasses of such locally anisotropic spaces are teleparallel (with non–zero induced torsion but with vanishing
curvature tensor) another are characterized by nontrivial, induced from general relativity on anholonomic frame bundle, N–connection and Riemannian curvature and anholonomy induced torsion. In a more general approach the N–connection and torsion are induced also from the condition that metric and nonlinear connection must solve the Einstein equations.

The paper is organized as follows: In the next section we briefly review the locally anisotropic gravity in (2+1)–dimensions. Conformal transforms with anisotropic factors and corresponding classes of solutions of Einstein equations with dynamical equations for N–connection coefficients are examined in Sec. 3. In Sec. 4 we derive the energy–momentum tensors for locally anisotropic elastic media and rotating null fluids. Sec. 5 is devoted to the local anisotropy of (2+1)–dimensional solutions of Einstein equations with anisotropic matter. The nonlinear self–polarization of anisotropic vacuum gravitational fields and matter induced polarizations and related topics on anisotropic black hole solutions are considered in Sec. 6. We derive some basic formulas for thermodynamics of anisotropic black holes in Sec. 7. The next Sec. 8 provides a statistical mechanics background for locally anisotropic thermodynamics starting from the locally anisotropic variants of Chern–Simons and Wess–Zumino-Witten models of locally anisotropic gravity. Finally, in Sec. 9 we conclude and discuss the obtained results.

2 Anholonomic Frames and 3D Gravity

In this Section we wish to briefly review and reformulate the Cartan’s method of moving frames [8] for investigation of gravitational and matter field interactions with mixed subsets of holonomic (unconstrained) and anholonomic (constrained, equivalently, locally anisotropic, in brief, la) variables [25]. Usual tetradic (frame, or vielbein) approaches to general relativity, see, for instance, [20, 12], consider ‘non–mixed’ cases when all basic vectors are anholonomic or transformed into coordinate (holonomic) ones. We note that a more general geometric background for locally anisotropic interactions and locally anisotropic spacetimes, with applications in physics, was elaborated by Miron and Anastasiei [19] in their generalized Finsler and Lagrange geometry; further developments for locally anisotropic spinor bundles and locally anisotropic superspaces are contained in Refs [23, 24]. Here we restrict our constructions only to three dimensional (3D) pseudo–Riemannian spacetimes provided with a global splitting characterized by two holonomic and one anholonomic coordinates.

2.1 Anholonomic frames and nonlinear connections

We model the low dimensional spacetimes as a smooth (i. e. class $C^\infty$) 3D (pseudo) Riemannian manifolds $V^{(3)}$ being Hausdorff, paracompact and connected and enabled with the fundamental structures of symmetric metric $g_{\alpha\beta}$, with signature $(-, +, +)$ and of linear, in general nonsymmetric (if we consider anholonomic frames), metric connection $\Gamma^\alpha_{\beta\gamma}$ defining the covariant derivation $D_\alpha$. The indices of geometrical objects on $V^{(3)}$ are stated with respect to a frame vector field (triad, or dreibien) $e_\alpha$ and its dual $e^\alpha$. A
holonomic frame structure on 3D spacetime could be given by a local coordinate base

\[ \partial_\alpha = \partial / \partial u^\alpha, \quad (1) \]

consisting from usual partial derivatives on local coordinates \( u = \{ u^\alpha \} \) and the dual basis

\[ d^\alpha = du^\alpha, \quad (2) \]

consisting from usual coordinate differentials \( du^\alpha \).

An arbitrary holonomic frame \( e_\alpha \) could be related to a coordinate one by a local linear transform

\[ e_\alpha = A_\beta^\alpha (u) \partial_\beta, \quad (1) \]

for which the matrix \( A_\beta^\alpha \) is nondegenerate and there are satisfied the holonomy conditions

\[ e_\alpha e_\beta - e_\beta e_\alpha = 0. \]

Let us consider a 3D metric parametrized into (2 + 1) components

\[ g_{\alpha\beta} = \begin{pmatrix} g_{ij} + N_j^* N_i^* h^{\bullet \bullet} & N_i^* h^{\bullet \bullet} \\ N_i^* h^{\bullet \bullet} & h^{\bullet \bullet} \end{pmatrix}, \quad (3) \]

given with respect to a local coordinate basis (2), \( du^\alpha = (dx^i, dy) \), where the Greek indices run values 1, 2, 3, the Latin indices \( i, j, k, ... \) from the middle of the alphabet run values for \( n = 1, 2, ... \) and the Latin indices from the beginning of the alphabet, \( a, b, c, ... \), run values for \( m = 3, 4, ... \) if we want to consider imbeddings of 3D spaces into higher dimension ones. The coordinates \( x^i \) are treated as isotropic ones and the coordinate \( y^\bullet = y \) is considered anholonomic (anisotropic). For 3D we denote that \( a, b, c, ... = \bullet, \quad y^\bullet \rightarrow y, \quad h_{ab} \rightarrow h^{\bullet \bullet} = h \) and \( N_i^a \rightarrow N_i^\bullet = w_i \). The coefficients \( g_{ij} = g_{ij}(u), h^{\bullet \bullet} = h(u) \) and \( N_i^\bullet = N_i(u) \) are supposed to solve the 3D Einstein gravitational field equations. The metric (3) can be rewritten in a block \( (2 \times 2) \oplus 1 \) form

\[ g_{\alpha\beta} = \begin{pmatrix} g_{ij}(u) & 0 \\ 0 & h(u) \end{pmatrix}, \quad (4) \]

with respect to the anholonomic basis (frame, anisotropic basis)

\[ \delta_\alpha = (\delta_i, \partial_\bullet) = \frac{\delta}{\partial u^\alpha} = \left( \delta_i = \frac{\delta}{\partial x^i} = \frac{\partial}{\partial x^i} - N_i^\bullet (u) \frac{\partial}{\partial y}, \partial_\bullet = \frac{\partial}{\partial y} \right), \quad (5) \]

and its dual anholonomic frame

\[ \delta^\beta = (d^i, \delta^\bullet) = \delta u^\beta = \left( d^i = dx^i, \delta^\bullet = \delta y = dy + N_k^\bullet (u) dx^k \right). \quad (6) \]

where the coefficients \( N_j^\bullet (u) \) from (5) and (6) could be treated as the components of an associated nonlinear connection (N–connection) structure \([2, 19, 23, 24]\) which was considered in Finsler and generalized Lagrange geometries and applied in general relativity and Kaluza–Klein gravity for construction of new classes of solutions of Einstein equations by using the method of moving anholonomic frames \([24]\). On 3D (pseudo)–Riemannian spaces the coefficients \( N_j^\bullet \) define a triad of basis vectors (dreibein) with respect to which
the geometrical objects (tensors, connections and spinors) are decomposed into holonomic (with indices $i, j, ...$) and anholonomic (provided with $\bullet$–index) components.

A local frame (local basis) structure $\delta_\alpha$ on $V^{(3)} \rightarrow V^{(2+1)}$ (by $(2+1)$ we denote the N–connection splitting into 2 holonomic and 1 anholonomic variables in explicit form; this decomposition differs from the usual two space and one time–like parametrizations) is characterized by its anholonomy coefficients $w^\alpha_{\beta \gamma}$ defined from relations

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = w^\gamma_{\alpha \beta} \delta_\gamma. \quad (7)$$

The rigorous mathematical definition of N–connection is based on the formalism of horizontal and vertical subbundles and on exact sequences in vector bundles \([2, 19]\). In this work we introduce a N–connection as a distribution which for every point $u = (x, y) \in V^{(2+1)}$ defines a local decomposition of the tangent space

$$T_u V^{(2+1)} = H_u V^{(2)} \oplus V_u V^{(1)}.$$

into horizontal subspace, $H_u V^{(2)}$, and vertical (anisotropy) subspace, $V_u V^{(1)}$, which is given by a set of coefficients $N^*_j (u^\alpha)$ . A N–connection is characterized by its curvature

$$\Omega^*_ij = \frac{\partial N^*_j}{\partial x^i} - \frac{\partial N^*_i}{\partial x^j} + N^*_i \frac{\partial N^*_j}{\partial y} - N^*_j \frac{\partial N^*_i}{\partial y}. \quad (8)$$

The class of usual linear connections can be considered as a particular case of N–connections when

$$N^*_j(x, y) = \Gamma_{ij}^k(x) y^k.$$

The elongation (by N–connection) of partial derivatives and differentials in the adapted to the N–connection operators \((5)\) and \((6)\) reflects the fact that on the (pseudo) Riemannian spacetime $V^{(2+1)}$ it is modeled a generic local anisotropy characterized by anholonomy relations \((7)\) when the anholonomy coefficients are computed as follows

\[
\begin{align*}
    w_{ij}^k &= 0, w_{ij}^k \cdot = 0, w_{ij}^k \cdot \cdot = 0, w_{ij} \cdot \cdot \cdot = 0, \\
    w_{ij}^\cdot &= -\Omega_{ij}^\cdot, w_{ij}^\cdot \cdot = -\partial N_i^\cdot, w_{ij}^\cdot \cdot \cdot = \partial N_i^\cdot.
\end{align*}
\]

\[w_{ij}^\cdot \cdot \cdot = 0, w_{ij}^\cdot \cdot \cdot \cdot = 0,
\]

The frames \((5)\) and \((6)\) are locally adapted to the N–connection structure, define a local anisotropy and, in brief, are called anholonomic bases. A N–connection structure distinguishes (d) the geometrical objects into horizontal and vertical components, i.e. transform them into d–objects which are briefly called d–tensors, d–metrics and d–connections. Their components are defined with respect to an anholonomic basis of type \((5)\), its dual \((6)\), or their tensor products (d–linear or d–affine transforms of such frames could also be considered). For instance, a covariant and contravariant d–tensor $Q$, is expressed

$$Q = Q^\alpha_{\beta \delta} \delta_\alpha \otimes \delta^\beta = Q^i_j \delta_i \otimes d^j + Q^i_j \cdot \delta_i \otimes \delta^\cdot + Q^i_j \cdot \cdot \delta_i \otimes d^j + Q^i_j \cdot \cdot \cdot \delta_i \otimes \delta^\cdot. \quad (10)$$

Similar decompositions on holonomic–anholonomic, conventionally on horizontal (h) and vertical (v) components, hold for connection, torsion and curvature components adapted to the N–connection structure.
2.2 Compatible N- and d–connections and metrics

A linear d–connection $D$ on a locally anisotropic spacetime $V^{(2+1)}$, $D_{\beta j}\delta^\beta = \Gamma^\alpha_{\beta\gamma}(x,y) \delta^\alpha$, is given by its h–v–components,

$$\Gamma^\alpha_{\beta\gamma} = \left(L^i_{jk}, L^i_{k}, C^i_{j}, C^i_{\cdot}\right) \quad (11)$$

where

$$D_{\beta j}\delta^\alpha = L^i_{jk}\delta^\alpha, \quad D_{\beta k}\partial^\cdot = L^i_{k}\partial^\cdot, \quad D_{\beta \cdot j} = C^i_{j}\delta^\alpha, \quad D_{\beta \cdot k} = C^i_{\cdot}\partial^\cdot \quad (12)$$

A metric on $V^{(2+1)}$ with its coefficients parametrized as (3) can be written in distinguished form (4), as a metric d–tensor (in brief, d–metric), with respect to an anholonomic base (6), i.e.

$$\delta s^2 = g_{\alpha\beta}(u) \delta^\alpha \otimes \delta^\beta = g_{ij}(x,y)dx^i dx^j + h(x,y)(\delta y)^2. \quad (13)$$

Some N–connection, d–connection and d–metric structures are compatible if there are satisfied the conditions

$$D_{\alpha}g_{\beta\gamma} = 0.$$

For instance, a canonical compatible d–connection

$$^c\Gamma^\alpha_{\beta\gamma} = \left(^cL^i_{jk}, ^cL^i_{k}, ^cC^i_{j}, ^cC^i_{\cdot}\right)$$

is defined by the coefficients of d–metric (13), $g_{ij}(x,y)$ and $h(x,y)$, and by the coefficients of N–connection,

$$^cL^i_{jk} = \frac{1}{2}g^{in}\left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right),$$

$$^cL^i_{k} = \partial^\cdot N^i_{k} + \frac{1}{2}h^{-1}\left(\delta_k h - 2h\partial^\cdot N^i_{k}\right),$$

$$^cC^i_{j} = \frac{1}{2}g^{jk}\partial^\cdot g_{jk},$$

$$^cC^i_{\cdot} = \frac{1}{2}h^{-1}(\partial^\cdot h). \quad (14)$$

The coefficients of the canonical d–connection generalize for locally anisotropic spacetimes the well known Christoffel symbols. By a local linear non–degenerate transform to a coordinate frame we obtain the coefficients of the usual (pseudo) Riemannian metric connection. For a canonical d–connection \([14]\), hereafter we shall omit the left–up index "c", the components of canonical torsion,

$$T^\cdot(\delta^\gamma, \delta^\beta) = T^\alpha_{\beta\gamma}\delta^\alpha,$$

$$T^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\cdot\gamma} + \omega^\alpha_{\beta\gamma}$$

are expressed via d–torsions

$$T^i_{jk} = T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{k} = C^i_{j}, \quad T^i_{\cdot} = -C^i_{\cdot},$$

$$T^\cdot_{bc} = S^\cdot_{bc} = C^\cdot_{bc} - C^\cdot_{cb} \rightarrow S^\cdot_{\cdot} \equiv 0,$$

$$T^\cdot_{ij} = -\Omega^\cdot_{ij}, \quad T^\cdot_{i} = \partial^\cdot N^i_{\cdot} - L^i_{\cdot}, \quad T^\cdot_{\cdot} = -T^\cdot_{i}. \quad (15)$$
which reflects the anholonomy of the corresponding locally anisotropic frame of reference on $V^{(2+1)}$; they are induced effectively. With respect to holonomic frames the d–torsions vanishes. Putting the non–vanishing coefficients (14) into the formula for curvature

$$R (\delta_\tau, \delta_\gamma) \delta_\beta = R^\beta_\alpha \gamma_\tau \delta_\alpha,$$

$$R^a_\beta \gamma_\tau = \delta_\tau \Gamma^a_\beta_\gamma - \delta_\gamma \Gamma^a_\beta_\tau + \Gamma^\gamma_\beta_\alpha \Gamma^a_\alpha_\tau - \Gamma^\gamma_\beta_\tau \Gamma^a_\alpha_\alpha + \Gamma^a_\beta_\phi w^\phi_\gamma_\tau$$

we compute the components of canonical d–curvatures

$$R^i_{hjk} = \delta_k L^i_{hj} - \delta_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - C^i_{h\bullet j} \Omega^\bullet_{jk},$$

$$P^i_{\bullet jk} = \delta_k L^i_{\bullet j} + C^i_{\bullet j} T^\bullet_{\bullet k} - (\delta_k C^i_{\bullet j} + L^l_{\bullet jk} C^i_{\bullet \bullet} - L^l_{\bullet jk} C^i_{\bullet \bullet} - L^l_{\bullet jk} C^i_{\bullet \bullet}),$$

$$P_{\bullet \bullet jk} = \delta_k L_{\bullet \bullet j} + C_{\bullet \bullet j} T_{\bullet \bullet k} - (\delta_k C_{\bullet \bullet j} - L_{\bullet \bullet j} C_{\bullet \bullet \bullet} - L_{\bullet \bullet j} C_{\bullet \bullet \bullet} - L_{\bullet \bullet j} C_{\bullet \bullet \bullet}),$$

$$S_{jbc} = \partial_j C^{i}_{bc} - \partial_b C^{i}_{jc} - C^{i}_{jc} C_{bc} - C^{i}_{jc} C_{bc} \to S_{jbc} \equiv 0,$$

$$S_{\alpha ed} = \partial_\alpha C^{i}_{bc} - \partial_b C^{i}_{\alpha c} + C^{i}_{bc} C^{c}_{\alpha d} - C^{i}_{bc} C^{c}_{\alpha d} \to S_{\alpha ed} \equiv 0.$$  

The h–v–decompositions for the torsion, (15), and curvature, (16), are invariant under local coordinate transforms adapted to a prescribed N–connection structure.

### 2.3 Anholonomic constraints and Einstein equations

The Ricci d–tensor $R_{\beta \gamma} = R^\alpha_\beta \gamma_\alpha$ has the components

$$R_{ij} = R^k_{i,jk}, \quad R_{\bullet} = -P_{i,\bullet} = -P^k_{i,k\bullet},$$

$$R_{\bullet i} = \frac{1}{2} P_{i,\bullet}, \quad R_{ab} = S^{c}_{a,bc} \to S_{ab} \equiv 0$$

and, in general, this d–tensor is non symmetric. We can compute the scalar curvature $\overleftarrow{R} = g^{\beta \gamma} R_{\beta \gamma}$ of a d-connection $D$,

$$\overleftarrow{R} = \widehat{R} + S,$$

where $\widehat{R} = g^{ij} R_{ij}$ and $S = h^{ab} S_{ab} \equiv 0$ for one dimensional anisotropies. By introducing the values (17) and (18) into the usual Einstein equations

$$G_{\alpha \beta} + \Lambda g_{\alpha \beta} = k \Upsilon_{\alpha \beta},$$

where

$$G_{\alpha \beta} = R_{\beta \gamma} - \frac{1}{2} g_{\beta \gamma} R$$

is the Einstein tensor, written with respect to an anholonomic frame of reference, we obtain the system of field equations for locally anisotropic gravity with N–connection structure [19]:

$$R_{ij} - \frac{1}{2} (\widehat{R} - 2 \Lambda) g_{ij} = k \Upsilon_{ij},$$

$$-\frac{1}{2} (\widehat{R} - 2 \Lambda) h_{\bullet \bullet} = k \Upsilon_{\bullet \bullet},$$

$$^1 P_{\bullet i} = k \Upsilon_{\bullet i},$$

$$^2 P_{\bullet i} = -k \Upsilon_{\bullet i}.$$  

7
where $\Upsilon_{ij}$, $\Upsilon_{\bullet\bullet}$, $\Upsilon_{\bullet i}$ and $\Upsilon_{i\bullet}$ are the components of the energy–momentum d–tensor field $\Upsilon_{\beta\gamma}$ which includes the cosmological constant terms and possible contributions of d–torsions and matter, and $k$ is the coupling constant.

The bulk of nontrivial locally isotropic solutions in 3D gravity were constructed by considering a cosmological constant $\Lambda = -1/l^2$, with and equivalent vacuum energy–momentum $\Upsilon_{(\Lambda)}^{\beta\gamma} = -\Lambda g^{\beta\gamma}$.

2.4 Some ansatz for d–metrics

2.4.1 Diagonal d–metrics

Let us introduce on 3D locally anisotropic spacetime $V^{(2+1)}$ the local coordinates $(x^1, x^2, y)$, where $y$ is considered as the anisotropy coordinate, and parametrize the d–metric (13) in the form

$$\delta s^2 = a(x^i) \left(dx^1\right)^2 + b(x^i) \left(dx^2\right)^2 + h(x^i, y) \left(\delta y\right)^2,$$

where

$$\delta y = dy + w_1(x^i, y) dx^1 + w_2(x^i, y) dx^2,$$

i. e. $N^*_i = w_i(x^i, y)$.

With respect to the coordinate base (1) the d–metric (13) transforms into the ansatz

$$g_{\alpha\beta} = \begin{bmatrix} a + w_1^2 h & w_1 w_2 h & w_1 h \\ w_1 w_2 h & b + w_2^2 h & w_2 h \\ w_1 h & w_2 h & h \end{bmatrix}.$$  

(26)

The nontrivial components of the Ricci d–tensor (17) are computed

$$2abR^1_1 = 2abR^2_2 - \ddot{b} + \frac{1}{2b} \dot{b}^2 + \frac{1}{2a} \dot{a} \dot{b} + \frac{1}{2b} \dot{a} \dot{b} - \frac{1}{2a} \dot{a}^2$$

where the partial derivatives are denoted, for instance, $\dot{h} = \partial h/\partial x^1$, $h' = \partial h/\partial x^2$ and $h^* = \partial h/\partial y$. The scalar curvature is $R = 2R^1_1$.

The Einstein d–tensor has a nontrivial component

$$G^3_3 = -hR^1_1.$$  

In the vacuum case with $\Lambda = 0$, the Einstein equations (21)–(24) are satisfied by arbitrary functions $a(x^i), b(x^i)$ solving the equation

$$- \ddot{b} + \frac{1}{2b} \dot{b}^2 + \frac{1}{2a} \dot{a} \dot{b} + \frac{1}{2b} \dot{a} \dot{b} - \frac{1}{2a} \dot{a}^2 = 0$$

(27)

and arbitrary function $h(x^i, y)$. Such functions should be defined following some boundary conditions in a manner as to have compatibility with the locally isotropic limit.
2.4.2 Off–diagonal d–metrics

For our further investigations it is convenient to consider d–metrics of type

$$\delta s^2 = g \left( x^i \right) \left( dx^i \right)^2 + 2 dx^1 dx^2 + h \left( x^i, y \right) \left( \delta y \right)^2.$$  \hspace{1cm} (28)

The nontrivial components of the Ricci d–tensor are

$$R_{11} = \frac{1}{2} g \frac{\partial^2 g}{\partial (x^2)^2}, \quad R_{12} = R_{21} = \frac{1}{2} \frac{\partial^2 g}{\partial (x^2)^2},$$ \hspace{1cm} (29)

when the scalar curvature is \( R = 2 R_{12} \) and the nontrivial component of the Einstein d–tensor is

$$G_{33} = - \frac{h}{2} \frac{\partial^2 g}{\partial (x^2)^2}.$$  \hspace{1cm}

We note that for the both d–metric ansatz (25) and (28) and corresponding coefficients of Ricci d–tensor, (27) and (29), the \( h \)-components of the Einstein d–tensor vanishes for arbitrary values of metric coefficients, i. e. \( G_{ij} = 0 \). In absence of matter such ansatz admit arbitrary nontrivial anholonomy (N–connection and N–curvature) coefficients (4) because the values \( w_i \) are not contained in the 3D vacuum Einstein equations. The \( h \)-component of the d–metric, \( h(x^k, y) \), and the coefficients of d–connection, \( w_i(x^k, y) \), are to be defined by some boundary conditions (for instance, by a compatibility with the locally isotropic limit) and compatibility conditions between nontrivial values of the cosmological constant and energy–momentum d–tensor.

3 Conformal Transforms with Anisotropic Factors

One of peculiar proprieties of the d–metric ansatz (25) and (28) is that there is only one non–trivial component of the Einstein d–tensor, \( G_{33} \). Becouse the values \( P_{3i} \) and \( P_{33} \) for the equations (22) and (23) vanish identically the coefficients of N–connection, \( w_i \), are not contained in the Einstein equations and could take arbitrary values. For static anisotropic configurations the solutions constructed in Sections IV and V cand be considered as 3D black hole like objects embedded in a locally anisotropic background with prescribed anholonomic frame (N–connection) structure.

In this Section we shall proof that there are d–metrics for which the Einstein equations reduce to some dynamical equations for the N–connection coefficients.

3.1 Conformal transforms of d–metrics

A conformal transform of a d–metric

\[(g_{ij}, h_{ab}) \longrightarrow (\bar{g}_{ij} = \Omega^2 \left( x^i, y \right) g_{ij}, \bar{h}_{ab} = \Omega^2 \left( x^i, y \right) h_{ab}) \] \hspace{1cm} (30)

with fixed N–connection structure, \( \bar{N}_i^a = N_i^a \), deforms the coefficients of canonical d–connection,

$$\bar{\Gamma}_\beta^\alpha_\gamma = \Gamma_\beta^\alpha_\gamma + \tilde{\Gamma}_\beta^\alpha_\gamma,$$
where the coefficients of deformation d–tensor $\Gamma^a_{\beta\gamma} = \{\hat{L}_{jk}, \hat{L}_{bk}, \hat{C}_{jc}, \hat{C}_{bc}\}$ are computed by introducing the values (30) into (44),
\begin{align}
\hat{L}_{jk}^i &= \delta_j^i \psi_k + \delta_k^i \psi_j - g_{jk}g^{in}\psi_n, \quad \hat{L}_{bk}^a = \delta_a^b \psi_k, \\
\hat{C}_{jc}^i &= \delta_j^i \psi_c, \quad \hat{C}_{bc}^i = \delta_b^a \psi_c + \delta_c^a \psi_b - h_{bc}h^{ac}\psi_c
\end{align}
with $\delta_j^i$ and $\delta_a^b$ being corresponding Kronecker symbols in h– and v–subspaces and $\psi_i = \delta_i \ln \Omega$ and $\psi_a = \partial_a \ln \Omega$.

In this subsection we present the general formulas for a n–dimensional h–subspace, with indices $i, j, k... = 1, 2, ...n$, and m–dimensional v–subspace, with indices $a, b, c... = 1, 2, ...m$.

The d–connection deformations (31) induce conformal deformations of the Ricci d–tensor (17),
\begin{align}
\hat{R}_{hj} &= R_{hj} + \hat{R}_{[1]hj}, \quad \hat{R}_{ja} = R_{ja} + \hat{R}_{ja}, \\
\hat{R}_{bk} &= R_{bk} + \hat{R}_{bk}, \quad \hat{S}_{bc} = S_{bc} + \hat{S}_{bc},
\end{align}
where the deformation Ricci d–tensors are
\begin{align}
\hat{R}_{[1]hj} &= \partial_i \hat{L}_{hj} - \partial_j \hat{L}_h + \hat{L}_{mj}^i \hat{L}_m - \hat{L}_{mj}^i \hat{L}_m - \hat{L}_{mj}^i \hat{L}_m - \hat{L}_{mj}^i \hat{L}_m, \\
\hat{R}_{[2]hj} &= N_j^a \partial_a \hat{L}_{hj} - N_j^a \partial_a \hat{L}_h + \hat{C}_{ha} R_j^i; \\
\hat{R}_{ja} &= -\partial_a \hat{L}_j + \delta_i \hat{C}_{ja} + L_{kij} \hat{C}_{ja} - L_{ijk} \hat{C}_{ja} - L_{ija} \hat{C}_{ja} - \hat{C}_{jb} P_{ja} - C^{ib}_{ja} P_{ia} - \hat{C}^{ij}_{jb} P_{ia}; \\
\hat{R}_{bk} &= \partial_i \hat{L}_{bk} - \partial_k \hat{C}_b + \hat{C}_{bd} P_{ka} - \hat{C}_{bd} P_{ka}, \\
\hat{S}_{bc} &= \partial_a \hat{C}_{bc} - \partial_b \hat{C}_c + \hat{C}_{be} C_e - \hat{C}_{be} C_e.
\end{align}
when $\hat{L}_h = \hat{L}_{hi}$ and $\hat{C}_b = \hat{C}_{be}$.

### 3.2 An ansatz with adapted conformal factor and N–connection

We consider a 3D metric
\begin{align}
g_{\alpha\beta} &= \begin{bmatrix}
\Omega^2(a - w_1^2h) & -w_1w_2h\Omega^2 & -w_1h\Omega^2 \\
-w_1w_2h\Omega^2 & \Omega^2(b - w_2^2h) & -w_2h\Omega^2 \\
-w_1h\Omega^2 & -w_2h\Omega^2 & -h\Omega^2
\end{bmatrix}
\end{align}
where $a = a(x^i), b = b(x^i), w_i = w_i(x^k, y), \Omega = \Omega \left(x^k, y\right) \geq 0$ and $h = h \left(x^k, y\right)$ when the conditions $\psi_i = \delta_i \ln \Omega = \partial_{x^i} \ln \Omega - w_i \ln \Omega = 0$ are satisfied. With respect to anholonomic bases (6) the (33) transforms into the d–metric
\begin{align}
\delta s^2 &= \Omega^2(x^k, y)[a(x^k)(dx^1)^2 + b(x^k)(dx^1)^2 + h \left(x^k, y\right)(\delta y)^2].
\end{align}

By straightforward calculus, by applying consequently the formulas (16)–(24) we find that there is a non–trivial coefficient of the Ricci d–tensor (17), of the deformation d–tensor (22),
\begin{align}
\hat{R}_{j3} &= \psi_3 \cdot \delta_j \ln |h|,
\end{align}
which results in non–trivial components of the Einstein d-tensor (20),

$$G^3_3 = -hR^1_1$$ and $$P_{\mathbf{s}} = -\psi_3 \cdot \delta_j \ln \sqrt{|h|},$$

where $$R^1_1$$ is given by the formula (27).

We can select a class of solutions of 3D Einstein equations with $$P_{\mathbf{j}} = 0$$ but with the horizontal components of metric depending on anisotropic coordinate $$y$$, via conformal factor $$\Omega(x^k, y)$$, and dynamical components of the N–connection, $$w_i$$, if we choose

$$h(x^k, y) = \pm \Omega^2(x^k, y)$$

and state

$$w_i(x^k, y) = \partial_i \ln |\ln \Omega|. \tag{35}$$

Finally, we not that for the ansatz (33) (equivalently (34)) the coefficients of N–connection have to be found as dynamical values by solving the Einstein equations.

4 Matter Energy Momentum D–Tensors

4.1 Variational definition of energy-momentum d–tensors

For locally isotropic spacetimes the symmetric energy momentum tensor is to be computed by varying on the metric (see, for instance, Refs. [12, 20]) the matter action

$$S = \frac{1}{c} \int {L}\sqrt{|g|}dV,$$

where $$L$$ is the Lagrangian of matter fields, $$c$$ is the light velocity and $$dV$$ is the infinitesimal volume, with respect to the inverse metric $$g^{\alpha\beta}$$. By definition one states that the value

$$\frac{1}{2} \sqrt{|g|}T_{\alpha\beta} = \frac{\partial (\sqrt{|g|}L)}{\partial g^{\alpha\beta}} - \frac{\partial }{\partial u^\tau} \frac{\partial (\sqrt{|g|}L)}{\partial g^{\alpha\beta}/\partial u^\tau} \tag{36}$$

is the symmetric energy–momentum tensor of matter fields. With respect to anholonomic frames (5) and (6) there are imposed constraints of type

$$g_{\mathbf{b}} - N_i^* h = 0$$

in order to obtain the block representation for d–metric (4). Such constraints, as well the substitution of partial derivatives into N–elongated, could result in nonsymmetric energy–momentum d–tensors $$\Upsilon_{\alpha\beta}$$ which is compatible with the fact that on a locally anisotropic spacetime the Ricci d–tensor could be nonsymmetric.

The gravitational–matter field interactions on locally anisotropic spacetimes are described by dynamical models with imposed constraints (a generalization of anholonomic analytic mechanics for gravitational field theory). The physics of systems with mixed holonomic and anholonomic variables states additional tasks connected with the definition of conservation laws, interpretation of non–symmetric energy–momentum tensors $$\Upsilon_{\alpha\beta}$$ on locally anisotropic spacetimes and relation of such values with, for instance, the
non–symmetric Ricci d–tensor. In this work we adopt the convention that for locally anisotropic gravitational matter field interactions the non–symmetric Ricci d–tensor induces a non–symmetric Einstein d–tensor which has as a source a corresponding non–symmetric matter energy–momentum tensor. The values $\Upsilon_{\alpha\beta}$ should be computed by a variational calculus on locally anisotropic spacetime as well by imposing some constraints following the symmetry of anisotropic interactions and boundary conditions.

In the next subsection we shall investigate in explicit form some cases of definition of energy momentum tensor for locally anisotropic matter on locally anisotropic spacetime.

### 4.2 Energy–Momentum D–Tensors for Anisotropic Media

Following DeWitt approach [27] and recent results on dynamical collapse and hair of black holes of Husain and Brown [13], we set up a formalism for deriving energy–momentum d–tensors for locally anisotropic matter.

Our basic idea for introducing a local anisotropy of matter is to rewrite the energy–momentum tensors with respect to locally adapted frames and to change the usual partial derivations and differentials into corresponding operators (5) and (6), ”elongated” by N–connection. The energy conditions (weak, dominant, or strong) in a locally anisotropic background have to be analyzed with respect to a locally anisotropic basis.

We start with DeWitt’s action written in locally anisotropic spacetime,

$$S [g_{\alpha\beta}, z^i] = - \int_V \delta^3 u \sqrt{-g} \rho \left( z^i, q_{jk} \right),$$

as a functional on region $V$, of the locally anisotropic metric $g_{\alpha\beta}$ and the Lagrangian coordinates $z^i = z^i(u^\alpha)$ (we use underlined indices $\underline{i}, \underline{j}, ... = 1, 2$ in order to point out that the 2–dimensional matter space could be different from the locally anisotropic spacetime). The functions $\zeta^i = z^i(u^\alpha)$ are two scalar locally anisotropic fields whose locally anisotropic gradients (with partial derivations substituted by operators (1)) are orthogonal to the matter world lines and label which particle passes through the point $u^\alpha$. The action $S [g_{\alpha\beta}, z^i]$ is the proper volume integral of the proper energy density $\rho$ in the rest anholonomic frame of matter. The locally anisotropic density $\rho \left( z^i, q_{jk} \right)$ depends explicitly on $z^i$ and on matter space d–metric $q^{ij} = (\delta_\alpha z^i) g^{\alpha\beta} (\delta_\beta z^j)$, which is interpreted as the inverse d–metric in the rest anholonomic frame of the matter.

Using the d–metric $q^{ij}$ and locally anisotropic fluid velocity $V^\alpha$, defined as the future pointing unit d–vector orthogonal to d–gradients $\delta_\alpha z^i$, the locally anisotropic spacetime d–metric [13] of signature (–,+,+) may be written in the form

$$g_{\alpha\beta} = -V_\alpha V_\beta + q_{jk} \delta_\alpha z^j \delta_\beta z^k,$$

which allow us to define the energy–momentum d–tensor for elastic locally anisotropic medium as

$$\Upsilon_{\beta\gamma} \equiv - \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\beta\gamma}} \rho V_\beta V_\gamma + t_{jk} b j_\beta z^j \delta_\gamma z^k,$$

(37)
where the locally anisotropic matter stress d–tensor \( t_{\mu\nu} \) is expressed as

\[
t_{\mu\nu} = 2 \frac{\delta \rho}{\partial q_{\mu\nu}} - \rho q_{\mu\nu} = \frac{2}{\sqrt{q}} \frac{\delta}{\partial q_{\mu\nu}} \left( \sqrt{q} \rho \right).
\]

Here one should be noted that on locally anisotropic spaces

\[
D_\alpha Y^{\alpha\beta} = D_\alpha \left( R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) = J^\beta \neq 0
\]

and this expression must be treated as a generalized type of conservation law with a geometric source \( J^\beta \) for the divergence of locally anisotropic matter d–tensor \([19]\).

The stress–energy–momentum d–tensor for locally anisotropic elastic medium is defined by applying N–elongated operators \( \delta_\alpha \) of partial derivatives \([1]\),

\[
T_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\alpha\beta}} = -\rho g_{\alpha\beta} + 2 \frac{\partial \rho}{\partial q^{ij}} \delta_\alpha \hat{z}^i \delta_\beta \hat{z}^j = -V_\alpha V_\beta + \tau_{\alpha\beta} \delta_\alpha \hat{z}^i \delta_\beta \hat{z}^j,
\]

where we introduce the matter stress d–tensor

\[
\tau_{\alpha\beta} = 2 \frac{\partial \rho}{\partial q_{\alpha\beta}} - \rho q_{\alpha\beta} = \frac{2}{\sqrt{q}} \frac{\partial}{\partial q_{\alpha\beta}} \left( \sqrt{q} \rho \right).
\]

The obtained formulas generalize for spaces with nontrivial N–connection structures the results on isotropic and anisotropic media on locally isotropic spacetimes.

### 4.3 Isotropic and anisotropic media

The isotropic elastic, but in general locally anisotropic medium is introduced as one having equal all principal pressures with stress d–tensor being for a perfect fluid and the density \( \rho = \rho(n) \), where the proper density (the number of particles per unit proper volume in the material rest anholonomic frame) is \( n = \bar{n}(\hat{z}) / \sqrt{q} \); the value \( \bar{n}(\hat{z}) \) is the number of particles per unit coordinate cell \( \delta^3 z \). With respect to a locally anisotropic frame, using the identity

\[
\frac{\partial \rho(n)}{\partial q_{\mu\nu}} = \frac{n}{2} \frac{\partial \rho}{\partial n q_{\mu\nu}}
\]

in \([38]\), the energy–momentum d–tensor \([37]\) for a isotropic elastic locally anisotropic medium becomes

\[
\Upsilon_{\beta\gamma} = \rho V_\beta V_\gamma + \left( \frac{n}{\partial \rho}{\partial n} - \rho \right) (g_{\beta\gamma} + V_\beta V_\gamma).
\]

This medium looks like isotropic with respect to anholonomic frames but, in general, it is locally anisotropic.

The anisotropic elastic and locally anisotropic medium has not equal principal pressures. In this case we have to introduce \((1+1)\) decompositions of locally anisotropic matter d–tensor \( q_{\mu\nu} \)

\[
q_{\mu\nu} = \left( \begin{array}{cc} \alpha^2 + \beta^2 & \beta \\ \beta & \sigma \end{array} \right),
\]
and consider densities $\rho \left( n_1, n_2 \right)$, where $n_1$ and $n_2$ are respectively the particle numbers per unit length in the directions given by bi–vectors $v^1_2$ and $v^2_2$. Substituting

$$\frac{\partial \rho \left( n_1, n_2 \right)}{\partial h^k_j} = \frac{n_1}{2} \frac{\partial \rho}{\partial n_1} v^1_1 v^1_k + \frac{n_2}{2} \frac{\partial \rho}{\partial n_2} v^2_2 v^2_k$$

into (38), which gives

$$t_{jk} = \left( n_1 \frac{\partial \rho}{\partial n_1} - \rho \right) v^1_1 v^1_k + \left( n_2 \frac{\partial \rho}{\partial n_2} - \rho \right) v^2_2 v^2_k,$$

we obtain from (37) the energy–momentum d–tensor for the anisotropic locally anisotropic matter

$$\Upsilon_{\beta\gamma} = \rho V_{\beta} V_{\gamma} + \left( n_1 \frac{\partial \rho}{\partial n_1} - \rho \right) v^1_1 v^1_k + \left( n_2 \frac{\partial \rho}{\partial n_2} - \rho \right) v^2_2 v^2_k.$$  (40)

So, the pressure $P_1 = \left( n_1 \frac{\partial \rho}{\partial n_1} - \rho \right) v^1_1$ differs from the pressure $P_2 = \left( n_2 \frac{\partial \rho}{\partial n_2} - \rho \right) v^2_2$. For instance, if for the (2+1)–dimensional locally anisotropic spacetime we impose the conditions $\Upsilon_1 = \Upsilon_2 = \Upsilon_3$, when

$$\rho = \rho \left( n_1 \right), \; z^1 (u^\alpha) = r, \; z^2 (u^\alpha) = \theta,$$

$r$ and $\theta$ are correspondingly radial and angle coordinates on locally anisotropic spacetime, we have

$$\Upsilon_1 = \Upsilon_2 = \rho, \; \Upsilon_3 = \left( n_1 \frac{\partial \rho}{\partial n_1} - \rho \right).$$  (41)

We shall also consider the variant when the coordinated $\theta$ is anisotropic ($t$ and $r$ being isotropic). In this case we shall impose the conditions $\Upsilon_1 \neq \Upsilon_2 = \Upsilon_3$ for

$$\rho = \rho \left( n_1 \right), \; z^1 (u^\alpha) = t, \; z^2 (u^\alpha) = r,$$

and

$$\Upsilon_1 = \left( n_1 \frac{\partial \rho}{\partial n_1} - \rho \right), \; \Upsilon_2 = \Upsilon_3 = \rho.$$  (42)

The anisotropic elastic locally anisotropic medium described here satisfies respectively weak, dominant, or strong energy conditions only if the corresponding restrictions are placed on the equation of state considered with respect to an anholonomic frame (see Ref. [13] for similar details in locally isotropic cases). For example, the weak energy condition is characterized by the inequalities $\rho \geq 0$ and $\partial \rho / \partial n_1 \geq 0$.

### 4.4 Spherical symmetry with respect to holonomic and anholonomic frames

In radial coordinates $(t, r, \theta)$ (with $-\infty \leq t < \infty$, $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$) for a spherically symmetric 3D metric (26)

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\theta^2,$$  (43)
with the energy–momentum tensor \((36)\) written
\[
T_{\alpha\beta} = \rho(r)(v_\alpha w_\beta + v_\beta w_\alpha) + P(r)(g_{\alpha\beta} + v_\alpha w_\beta + v_\beta w_\alpha),
\]
where the null d–vectors \(v_\alpha\) and \(w_\beta\) are defined by
\[
V_\alpha = \left(\sqrt{f}, -\frac{1}{\sqrt{f}}, 0\right) = \frac{1}{\sqrt{2}}(v_\alpha + w_\alpha),
\]
\[
q_\alpha = \left(0, \frac{1}{\sqrt{f}}, 0\right) = \frac{1}{\sqrt{2}}(v_\alpha - w_\alpha).
\]
In order to investigate the dynamical spherically symmetric \([5]\) collapse solutions it is more convenient to use the coordinates \((v, r, \theta)\), where the advanced time coordinate \(v\) is defined by \(dv = dt + (1/f) \, dr\). The metric \((43)\) may be written
\[
ds^2 = -e^{2\psi(v, r)} F(v, r) \, dv^2 + 2e^{\psi(v, r)} dv \, dr + r^2 \, d\theta^2,
\]
where the mass function \(m(v, r)\) is defined by \(F(v, r) = 1 - 2m(v, r)/r\). Usually, one considers the case \(\psi(v, r) = 0\) for the type II \([12]\) energy–momentum d–tensor
\[
T_{\alpha\beta} = \frac{1}{2\pi r^2} \frac{\delta m(v, r)}{\partial v} v_\alpha v_\beta + \rho(v, r)(v_\alpha w_\beta + v_\beta w_\alpha) + P(v, r)(g_{\alpha\beta} + v_\alpha w_\beta + v_\beta w_\alpha)
\]
with the eigen d-vectors \(v_\alpha = (1, 0, 0)\) and \(w_\alpha = (F/2, -1, 0)\) and the non–vanishing components
\[
T_{vv} = \rho(v, r) \left(1 - \frac{2m(v, r)}{r}\right) + \frac{1}{2\pi r^2} \frac{\delta m(v, r)}{\partial v},
\]
\[
T_{v\theta} = -\rho(v, r), \quad T_{\theta\theta} = P(v, r) g_{\theta\theta}.
\]
To describe a locally isotropic collapsing pulse of radiation one may use the metric
\[
ds^2 = \left[\Lambda r^2 + m(v)\right] \, dv^2 + 2dv \, dr - j(v) \, dv \, d\theta + r^2 \, d\theta^2,
\]
with the Einstein field equations \((19)\) reduced to
\[
\frac{\partial m(v)}{dv} = 2\pi \rho(v), \quad \frac{\partial j(v)}{dv} = 2\pi \omega(v)
\]
having non–vanishing components of the energy–momentum d–tensor (for a rotating null locally anisotropic fluid),
\[
T_{vv} = \frac{\rho(v)}{r} + \frac{j(v) \omega(v)}{2r^3}, \quad T_{v\theta} = -\frac{\omega(v)}{r},
\]
where \(\rho(v)\) and \(\omega(v)\) are arbitrary functions.
In a similar manner we can define energy–momentum d–tensors for various systems of locally anisotropic distributed matter fields; all values have to be re–defined with respect
to anholonomic bases of type (5) and (6). For instance, let us consider the angle $\theta$ as the anisotropic variable. In this case we have to 'elongate' the differentials,

$$d\theta \rightarrow \delta\theta = d\theta + w_i (v, r, \theta) \, dx^i,$$

for the metric (43) (or (48)), by transforming it into a d–metric, substitute all partial derivatives into N–elongated ones,

$$\partial_i \rightarrow \delta_i = \partial_i - w_i (v, r, \theta) \frac{\partial}{\partial \theta},$$

and 'N–extend' the operators defining the Riemannian, Ricci, Einstein and energy–momentum tensors $T_{\alpha\beta}$, transforming them into respective d–tensors. We compute the components of the energy–momentum d–tensor for elastic media as the coefficients of usual energy–momentum tensor redefined with respect to locally anisotropic frames,

$$\Upsilon_{11} = T_{11} + (w_1)^2 T_{33}, \quad \Upsilon_{33} = T_{33},$$

$$\Upsilon_{22} = T_{22} + (w_2)^2 T_{33}, \quad \Upsilon_{12} = \Upsilon_{21} = T_{21} + w_2 w_1 T_{33},$$

$$\Upsilon_{i3} = T_{i3} + w_i T_{33}, \quad \Upsilon_{3i} = T_{3i} + w_i T_{33},$$

where the $T_{\alpha\beta}$ are given by the coefficients (47) (or (49)). If the isotropic energy–momentum tensor does not contain partial derivatives, the corresponding d–tensor is also symmetric which is less correlated with the possible antisymmetry of the Ricci tensor (for such configurations we shall search solutions with vanishing antisymmetric components).

## 5 3D Solutions Induced by Anisotropic Matter

We investigate a new class of solutions of (2+1)–dimensional Einstein equations coupled with anisotropic matter \[5, 13, 3, 9, 22\] which describe locally anisotropic collapsing configurations.

Let us consider the locally isotropic metric

$$\hat{g}_{\alpha\beta} = \begin{bmatrix} g(v, r) & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & r^2 \end{bmatrix}$$

which solves the locally isotropic variant of Einstein equations (19) if

$$g(v, r) = -[1 - 2g(v) - 2h(v)r^{1-k} - \Lambda r^2],$$

where the functions $g(v)$ and $h(v)$ define the mass function

$$m(r, v) = g(v)r + h(v)r^{2-k} + \frac{\Lambda}{2}r^2$$

satisfying the dominant energy conditions

$$P \geq 0, \rho \geq P, T_{ab}w^aw^b > 0$$

if

$$\frac{dm}{dv} = \frac{dg}{dv}r + \frac{dh}{dv}r^{2-k} > 0.$$

Such solutions of the Vadya type with locally isotropic null fluids have been considered in Ref. [13].
5.1 Solutions with generic local anisotropy in spherical coordinates

By introducing a new time–like variable

$$t = v + \int \frac{dr}{g(v, r)}$$

the metric(51) can be transformed in diagonal form

$$ds^2 = -g(t, r) dt^2 + \frac{1}{g(t, r)} dr^2 + r^2 d\theta^2$$ (53)

which describe the locally isotropic collapse of null fluid matter.

A variant of locally anisotropic inhomogeneous collapse could be modeled, for instance, by the N–elongation of the variable \(\theta\) in (53) and considering solutions of vacuum Einstein equations for the d–metric (a particular case of (54))

$$ds^2 = -g(t, r) dt^2 + \frac{1}{g(t, r)} dr^2 + r^2 \delta\theta^2,$$ (54)

where

$$\delta\theta = d\theta + w_1 (t, r, \theta) dt + w_2 (t, r, \theta) dr.$$ The coefficients \(g(t, r), 1/g(t, r)\) and \(r^2\) of the d–metric were chosen with the aim that in the locally isotropic limit, when \(w_i \rightarrow 0\), we shall obtain the 3D metric (53). We note that the gravitational degrees of freedom are contained in nonvanishing values of the Ricci d–tensor (17),

$$R_{11} = R_{22} = \frac{1}{2g^3} [\left(\frac{\partial g}{\partial r}\right)^2 - g^3 \frac{\partial^2 g}{\partial t^2} - g \frac{\partial^2 g}{\partial r^2}]$$, (55)

of the N–curvature (8),

$$\Omega_{12}^3 = -\Omega_{21}^3 = -\frac{\partial w_1}{\partial r} + \frac{\partial w_2}{\partial t} - w_2 \frac{\partial w_1}{\partial \theta} + w_1 \frac{\partial w_2}{\partial \theta},$$

and d–torsion (15)

$$P_{13}^3 = \frac{1}{2} (1 + r^4) \frac{\partial w_1}{\partial \theta}, \quad P_{23}^3 = \frac{1}{2} (1 + r^4) \frac{\partial w_2}{\partial \theta} - r^3.$$ We can construct a solution of 3D Einstein equations with cosmological constant \(\Lambda\) (19) and energy momentum d–tensor \(\gamma_{\alpha\beta}\), when \(\gamma_{ij} = T_{ij} + w_i w_j T_{33}, \gamma_{3j} = T_{3j} + w_j T_{33}\) and \(\gamma_{33} = T_{33}\) when \(T_{\alpha\beta}\) is given by a d–tensor of type (42), \(T_{\alpha\beta} = \{n_1 \frac{\partial \rho}{\partial n_1}, P, 0\}\) with anisotropic matter pressure \(P\). A self–consistent solution is given by

$$\Lambda = \kappa n_1 \frac{\partial \rho}{\partial n_1} = \kappa P,$$ \(h = \frac{\kappa \rho}{R_1 + \Lambda}\) (56)

where \(R_1^i\) is computed by the formula (55) for arbitrary values \(g(t, r)\). For instance, we can take the \(g(\nu, r)\) from (52) with \(\nu \rightarrow t = \nu + \int g^{-1}(\nu, r) dr\).
For $h = r^2$, the relation (56) results in an equation for $g(t, r)$,

$$\left(\frac{\partial g}{\partial r}\right)^2 - g^3 \frac{\partial^2 g}{\partial t^2} - g \frac{\partial^2 g}{\partial r^2} = 2g^3 \left(\frac{\kappa \rho}{r^2} - \Lambda\right).$$

The static configurations are described by the equation

$$gg'' - (g')^2 + \varpi(r)g^3 = 0,$$

(57)

where

$$\varpi(r) = 2 \left(\frac{\kappa \rho(r)}{r^2} - \Lambda\right)$$

and the prime denote the partial derivative $\partial/\partial r$. There are four classes (see ([17])) of solutions of the equation (57), which depends on constants of the relation

$$(\ln |g|)' = \pm \sqrt{2|\varpi(r)|}(C_1 \mp g),$$

where the minus (plus) sign under square root is taken for $\varpi(r) > 0$ (\varpi(r) < 0) and the constant $C_1$ can be negative, $C_1 = -c^2$, or positive, $C_1 = c^2$. In explicit form the solutions are

$$g(r) = \begin{cases} 
  c^{-2} \cosh^{-2} \left[\frac{\sqrt{2}}{2} |\varpi(r)| (r - C_2) \right], & \text{for } \varpi(r) > 0, C_1 = c^2; \\
  c^{-2} \sinh^{-2} \left[\frac{\sqrt{2}}{2} |\varpi(r)| (r - C_2) \right], & \text{for } \varpi(r) < 0, C_1 = c^2; \\
  c^{-2} \sin^{-2} \left[\frac{\sqrt{2}}{2} |\varpi(r)| (r - C_2) \right] \neq 0, & \text{for } \varpi(r) < 0, C_1 = -c^2; \\
  -2\varpi(r)^{-1}(r - C_2)^{-2}, & \text{for } \varpi(r) < 0, C_1 = 0,
\end{cases}$$

(58)

where $C_2 = \text{const}$. The values of constants are to be found from boundary conditions. In dependence of prescribed type of matter density distribution and of values of cosmological constant one could fix one of the four classes of obtained solutions with generic local anisotropy of 3D Einstein equations.

The constructed in this section static solutions of 3D Einstein equations are locally anisotropic alternatives (with proper phases of anisotropic polarizations of gravitational field) to the well know BTZ solution. Such configurations are possible if anholonomic frames with associated N–connection structures are introduced into consideration.

### 5.2 An anisotropic solution in $(\nu, r, \theta)$–coordinates

For modeling a spherical collapse with generic local anisotropy we use the d–metric (28) by stating the coordinates $x^1 = v, x^2 = r$ and $y = \theta$. The equations (19) are solved if

$$\kappa \rho(v, r) = \Lambda \text{ and } \kappa P(v, r) = -\Lambda - \frac{1}{2} \frac{\partial^2 g}{\partial \nu^2},$$

for

$$g = \frac{\kappa}{\Lambda} \left[ \rho(v, r) \left(1 - \frac{2m(v, r)}{r}\right) + \frac{1}{2\pi r^2} \frac{\delta m(v, r)}{\delta v} \right].$$
5.3 A solution for rotating two locally anisotropic fluids

The anisotropic configuration from the previous subsection admits a generalization to a two fluid elastic media, one of the fluids being of locally anisotropic rotating configuration. For this model we consider an anisotropic extension of the metric (48) and of the sum of energy–momentum tensors (47) and (49). The coordinates are parametrized $x^1 = v, x^2 = r, y = \theta$ and the d–metric is given by the ansatz

$$g_{ij} = \begin{pmatrix} g(v, r) & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad h = h(v, r, \theta).$$

The nontrivial components of the Einstein d–tensor is

$$G_{33} = -\frac{1}{2} h \frac{\partial^2 g}{\partial r^2}.$$

We consider a non–rotating fluid component with nontrivial energy–momentum components

$$(1)T_{vv} = (1)\rho(v, r) \left(1 - \frac{2(1)m(v, r)}{r}\right) + \frac{1}{2\pi r^2} \frac{\delta(1)m(v, r)}{\partial v}, \quad (1)T_{vr} = -(1)\rho(v, r). \quad (59)$$

and a rotating null locally anisotropic fluid with energy–momentum components

$$(2)T_{vv} = \frac{(2)\rho(v)}{r} + \frac{(2)j(v)}{2r^3}, \quad (2)T_{v\theta} = \frac{(2)\omega(v)}{r}. \quad (60)$$

The nontrivial components of energy momentum d–tensor $\Upsilon_{\alpha\beta} = (1)\Upsilon_{\alpha\beta} + (2)\Upsilon_{\alpha\beta}$ (associated in the locally anisotropic limit to (49) and/or (47)) are computed by using the formulas (60), (59) and (50).

The Einstein equations are solved by the set of functions

$$g(v, r), (1)\rho(v, r), (1)m(v, r), (2)\rho(v), (2)j(v), (2)\omega(v)$$

satisfying the conditions

$$g(v, r) = \kappa \Lambda \left[ (1)T_{vv} + (2)T_{vv} \right], \quad \text{and} \quad \Lambda = \frac{1}{2} \frac{\partial^2 g}{\partial r^2} = \kappa (1)T_{vr},$$

where $h(v, r, \theta)$ is an arbitrary function which results in nontrivial solutions for the N–connection coefficients $w_i(v, r, \theta)$ if $\Lambda \neq 0$. In the locally isotropic limit, for $(1)\rho, (1)m = 0$, we could take $g(v, r) = g_1(v) + \Lambda r^2, w_1 = -j(v)/(2r^2)$ and $w_2 = 0$ which results in a solution of the Vadya type with locally isotropic null fluids [9].

The main conclusion of this subsection is that we can model the 3D collapse of inhomogeneous null fluid by using vacuum locally anisotropic configurations polarized by an anholonomic frame in a manner as to reproduce in the locally isotropic limit the usual BTZ geometry.

We end this section with the remark that the locally isotropic collapse of dust without pressure was analyzed in details in Ref. [22].
6 Gravitational Anisotropic Polarization and Black Holes

If we introduce in consideration anholonomic frames, locally anisotropic black hole configurations are possible even for vacuum locally anisotropic spacetimes without matter. Such solutions could have horizons with deformed circular symmetries (for instance, elliptic one) and a number of unusual properties comparing with locally isotropic black hole solutions. In this Section we shall analyze two classes of such solutions. Then we shall consider the possibility to introduces matter sources and analyze such configurations of matter energy density distribution when the gravitational locally anisotropic polarization results into constant renormalization of constants of BTZ solution.

6.1 Non–rotating black holes with ellipsoidal horizon

We consider a metric (33) for local coordinates \((x^1 = r, x^2 = \theta, y = t)\), where \(t\) is the time–like coordinate and the coefficients are parametrized

\[
a(x^i) = a(r), b(x^i) = b(r, \theta) \tag{61}
\]

and

\[
h(x^i, y) = h(r, \theta). \tag{62}
\]

The functions \(a(r)\) and \(b(r, \theta)\) and the coefficients of nonlinear connection \(w_i(r, \theta, t)\) will be found as to satisfy the vacuum Einstein equations (27) with arbitrary function \(h(x^i, y)\) (62) stated in the form in order to have compatibility with the BTZ solution in the locally isotropic limit.

We consider a particular case of d-metrics (34) with coefficients like (61) and (62) when

\[
h(r, \theta) = 4\Lambda^3(\theta) \left(1 - \frac{r^2(\theta)}{r^2}\right)^3 \tag{63}
\]

where, for instance,

\[
r^2_+(\theta) = \frac{p^2}{[1 + \varepsilon \cos \theta]^2} \tag{64}
\]

is taken as to construct a 3D solution of vacuum Einstein equations with generic local anisotropy having the horizon given by the parametric equation

\[
r^2 = r^2_+(\theta)
\]

describing a ellipse with parameter \(p\) and eccentricity \(\varepsilon\). We have to identify

\[
p^2 = r^2_+[0] = -M_0/\Lambda_0,
\]

where \(r_+[0], M_0\) and \(\Lambda_0\) are respectively the horizon radius, mass parameter and cosmological constant of the non–rotating BTZ solution [3] if we wont to have a connection with locally isotropic limit with \(\varepsilon \to 0\). We can consider that the elliptic horizon (64) is modeled by the anisotropic mass

\[
M(\theta) = M_0/[1 + \varepsilon \cos \theta]^2.
\]
For the coefficients (61) the equations (27) simplifies into
\[-\ddot{b} + \frac{1}{2b} \dot{b}^2 + \frac{1}{2a} \dot{a} \dot{b} = 0, \quad (65)\]
where (in this subsection) \( \dot{b} = \partial b / \partial r \). The general solution of (65), for a given function \( a(r) \) is defined by two arbitrary functions \( b_{[0]}(\theta) \) and \( b_{[1]}(\theta) \) (see [17]),
\[b(r, \theta) = \left[ b_{[0]}(\theta) + b_{[1]}(\theta) \int \sqrt{|a(r)|} \, dr \right]^2.\]
If we identify
\[b_{[0]}(\theta) = 2 \frac{\Lambda(\theta)}{\sqrt{|\Lambda_0|}} r_+^2(\theta) \quad \text{and} \quad b_{[1]}(\theta) = -2 \frac{\Lambda(\theta)}{\Lambda_0},\]
we construct a d–metric locally anisotropic solution of vacuum Einstein equations
\[\delta s^2 = \Omega^2(r, \theta) \left[ 4r^2|\Lambda_0| dr^2 + \frac{4}{|\Lambda_0|} \Lambda^2(\theta) \left[ r_+^2(\theta) - r^2 \right]^2 d\theta^2 - \frac{4}{|\Lambda_0|r^2} \Lambda^3(\theta) \left[ r_+^2(\theta) - r^2 \right]^3 \delta t^2 \right], \quad (66)\]
where
\[\delta t = dt + w_1(r, \theta) dr + w_2(r, \theta) d\theta\]
is to be associated to a N–connection structure
\[w_r = \partial_r \ln |\ln \Omega| \quad \text{and} \quad w_\theta = \partial_\theta \ln |\ln \Omega|\]
with \( \Omega^2 = \pm h(r, \theta) \), where \( h(r, \theta) \) is taken from (63). In the simplest case we can consider a constant effective cosmological constant \( \Lambda(\theta) \simeq \Lambda_0 \).

The matrix
\[g_{\alpha\beta} = \Omega^2 \begin{bmatrix} a - w_1^2 h & -w_1 w_2 h & -w_1 h \\ -w_1 w_2 h & b - w_2^2 h & -w_2 h \\ -w_1 h & -w_2 h & -h \end{bmatrix},\]
parametrizes a class of solutions of 3D vacuum Einstein equations with generic local anisotropy and nontrivial N–connection curvature (8), which describes black holes with variable mass parameter \( M(\theta) \) and elliptic horizon. As a matter of principle, by fixing necessary functions \( b_{[0]}(\theta) \) and \( b_{[1]}(\theta) \) we can construct solutions with effective (polarized by the vacuum anisotropic gravitational field) variable cosmological constant \( \Lambda(\theta) \).

We emphasize that this type of anisotropic black hole solutions have been constructed by solving the vacuum Einstein equations without cosmological constant. Such type of constants or varying on \( \theta \) parameters were introduced as some values characterizing anisotropic polarizations of vacuum gravitational field and this approach can be developed if we are considering anholonomic frames on (pseudo) Riemannian spaces. For the examined anisotropic model the cosmological constant is induced effectively in locally isotropic limit via specific gravitational field vacuum polarizations.
6.2 Rotating black holes with running in time constants

A new class of solutions of vacuum Einstein equations is generated by a d–metric written for local coordinates \( x^1 = r, x^2 = t, y = \theta \), where as the anisotropic coordinate is considered the angle variable \( \theta \) and the coefficients are parametrized

\[
a(x^i) = a(r), b(x^i) = b(r, t) \quad (67)
\]

and

\[
h(x^i, y) = h(r, t). \quad (68)
\]

Let us consider a 3D metric

\[
ds^2 = 4\frac{\psi^2}{r^2}dr^2 - \frac{N^4_{[s]}}{r^4}dt^2 + \frac{N^2_{(s)}}{r^4}[d\theta + N_{[\theta]}dt]^2 \quad (69)
\]

which is conformally equivalent (if multiplied to the conformal factor \( 4N^2_{(s)}/r^4 \)) to the rotating BTZ solution with

\[
N^2_{[s]}(r) = -\Lambda_0 \frac{r^2}{\psi^2} (r^2 - r^2_{+[0]}), N_{[\theta]}(r) = -\frac{J_0}{2\psi},
\]

\[
\psi^2(r) = r^2 - \frac{1}{2} \left( \frac{M_0}{\Lambda_0} + r^2_{+[0]} \right),
\]

\[
r^2_{+[0]} = -\frac{M_0}{\Lambda_0} \sqrt{1 + \Lambda_0 \left( \frac{J_0}{M_0} \right)^2},
\]

where \( J_0 \) is the rotation moment and \( \Lambda_0 \) and \( M_0 \) are respectively the cosmological and mass BTZ constants.

A d–metric defines a locally anisotropic extension of (69) if the solution of (65), in variables \( x^1 = r, x^2 = t \), with coefficients (67) and (68), is written

\[
b(r, t) = -\left[ b_{[0]}(t) + b_{[1]}(t) \int \sqrt{|a(r)|}dr \right]^2 = -\Lambda^2(t) \left[ r^2_{+}(t) - r^2 \right]^2, \quad (70)
\]

for

\[
a(r) = 4\Lambda_0 r^2, b_{[0]}(t) = \Lambda(t) r^2_{+}(t), b_{[1]}(t) = 2\Lambda(t) / \sqrt{\Lambda_0}
\]

with \( \Lambda(t) \sim \Lambda_0 \) and \( r_{+} \sim r_{+[0]} \) being some running in time values.

The functions \( a(r) \) and \( b(r, t) \) and the coefficients of nonlinear connection \( w_i(r, t, \theta) \) must solve the vacuum Einstein equations (27) with arbitrary function \( h(x^i, y) \) (62) stated in the form in order to have a relation with the BTZ solution for rotating black holes in the locally isotropic limit. This is possible if we choose

\[
w_1(r, t) = -\frac{J(t)}{2\psi(r, t)}, \quad h(r, t) = \frac{4N^2_{[s]}(r, t) \psi^6(r, t)}{r^4}, \quad (71)
\]

for an arbitrary function \( w_2(r, t, \theta) \) with \( N_{[\theta]}(r, t) \) and \( \psi(r, t) \) computed by the same formulas (70) with the constant substituted into running values,

\[
\Lambda_0 \rightarrow \Lambda(t), M_0 \rightarrow M(t), J_0 \rightarrow J(t).
\]
We can model a dissipation of 3D black holes, by anisotropic gravitational vacuum polarizations if for instance,
\[ r_+^2(t) \simeq r_{+,[0]}^2 \exp[-\lambda t] \]
for \( M(t) = M_0 \exp[-\lambda t] \) with \( M_0 \) and \( \lambda \) being some constants defined from some "experimental" data or a quantum model for 3D gravity. The gravitational vacuum admits also polarizations with exponential and/or oscillations in time for \( \Lambda(t) \) and/or of \( M(t) \).

### 6.3 Anisotropic Renormalization of Constants

The BTZ black hole [3] in “Schwarzschild” coordinates is described by the metric
\[ ds^2 = -(N^\perp)^2 dt^2 + f^{-2} dr^2 + r^2 \left( d\phi + N^\phi dt \right)^2 \]  
with lapse and shift functions and radial metric
\[ N^\perp = f = \left( -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} \right)^{1/2}, \]
\[ N^\phi = -J \frac{2r^2}{2r^2} \quad (|J| \leq M\ell). \]
which satisfies the ordinary vacuum field equations of (2+1)-dimensional general relativity [19] with a cosmological constant \( \Lambda = -1/\ell^2 \).

If we are considering anholonomic frames, the matter fields ”deform” such solutions not only by presence of a energy–momentum tensor in the right part of the Einstein equations but also via anisotropic polarizations of the frame fields. In this Section we shall construct a subclass of d–metrics (54) selecting by some particular distributions of matter energy density \( \rho(r) \) and pressure \( P(r) \) solutions of type (72) but with renormalized constants in (73),
\[ M \rightarrow \overline{M} = \alpha^{(M)} M, J \rightarrow \overline{J} = \alpha^{(J)} J, \Lambda \rightarrow \overline{\Lambda} = \alpha^{(\Lambda)} \Lambda, \]
where the receptivities \( \alpha^{(M)}, \alpha^{(J)} \) and \( \alpha^{(\Lambda)} \) are considered, for simplicity, to be constant (and defined ”experimentally” or computed from a more general model of quantum 3D gravity) and tending to a trivial unity value in the locally isotropic limit. The d–metric generalizing (72) is stated in the from
\[ \delta s^2 = -F(r)^{-1} dt^2 + F(r) dr^2 + r^2 \delta \theta^2 \]  
where
\[ F(r) = \left( -\overline{M} - \overline{\Lambda} r^2 + \frac{J^2}{4r^2} \right), \quad \delta \theta = d\theta + w_1 dt \text{ and } w_1 = -\overline{J} \frac{2r^2}{2r^2}. \]

The d–metric (75) is a static variant of d–metric (54) when the solution (58) is constructed for a particular function
\[ \overline{\omega}(r) = 2 \left( \frac{\kappa \overline{\rho}(r)}{r^2} - \overline{\Lambda} \right) \]
is defined by corresponding matter distribution $\rho (r)$ when the function $F(r)$ is the solution of equations $(57)$ with coefficient $\varpi(r)$ before $F^3$, i.e.

$$FF'' - (F')^2 + \varpi(r)F^3 = 0.$$  

The d–metric $(75)$ is singular when $r = r_\pm$, where

$$\bar{r}_\pm^2 = -\frac{M}{2\Lambda} \left\{ 1 \pm \left[ 1 + \Lambda \left( \frac{J}{M} \right)^2 \right]^{1/2} \right\}, \tag{76}$$

i.e.,

$$M = -\bar{\Lambda}(\bar{r}_+^2 + \bar{r}_-^2), \quad J = \frac{2\bar{r}_+ \bar{r}_-}{\ell}, \quad \bar{\Lambda} = -1/\ell^2.$$

In locally isotropic gravity the surface gravity was computed $[16]$

$$\sigma^2 = -\frac{1}{2} D^\alpha \chi^\beta D_\alpha \chi_\beta = \frac{r_+^2 - r_-^2}{\ell^2 r_+},$$

where the vector $\chi = \partial_\nu - N^\alpha (r_+) \partial_\theta$ is orthogonal to the Killing horizon defined by the surface equation $r = r_+$. For locally anisotropic renormalized (overlined) values we have

$$\bar{\chi} = \delta_\nu = \partial_\nu - w_1 (\bar{r}_+) \partial_\theta$$

and

$$\bar{\sigma}^2 = -\frac{1}{2} D^\alpha \bar{\chi}^\beta D_\alpha \bar{\chi}_\beta = \bar{\Lambda} \bar{r}_-^2 - \bar{r}_+^2.$$

The renormalized values allow us to define a corresponding thermodynamics of locally anisotropic black holes.

### 6.4 Ellipsoidal black holes with running in time constants

The anisotropic black hole solution of 3D vacuum Einstein equations $[66]$ with elliptic horizon can be generalized for a case with varying in time cosmological constant $\Lambda_0(t)$. For this class of solutions we choose the local coordinates $(x^1 = r, x^2 = \theta, y = t)$ and a d–metric of type $(34)$,

$$\delta s^2 = \Omega_{(el)}^2 (r, \theta, t) [a(r)(dr)^2 + b(r, \theta)(d\theta)^2 + h(r, \theta, t)(dt)^2], \tag{77}$$

where

$$h(r, \theta, t) = -\Omega_{(el)}^2 (r, \theta, t) = -\frac{4\Lambda^3 (\theta)}{|\Lambda_0(t)| r^2} \left[ r_+^2 (\theta, t) - r^2 \right]^3,$$

for

$$r_+^2 (\theta, t) = \frac{p(t)}{(1 + \varepsilon \cos \theta)^2}, \quad \text{and} \quad p(t) = r_+^2 (0)(\theta, t) = -M_0/\Lambda_0(t) \tag{78}$$

and it is considered that $\Lambda_0(t) \simeq \Lambda_0$ for static configurations.
The $d$–metric (77) is a solution of 3D vacuum Einstein equations if the 'elongated' differential

$$\delta t = dt + w_r(r, \theta, t)dr + w_\theta(r, \theta, t)d\theta$$

has the $N$–connection coefficients are computed following the condition (35),

$$w_r = \partial_r \ln |\ln \Omega(\ell t)| \quad \text{and} \quad w_\theta = \partial_\theta \ln |\ln \Omega(\ell t)|.$$ 

The functions $a(r)$ and $b(r, \theta)$ from (77) are arbitrary ones of type (61) satisfying the equations (65) which in the static limit could be fixed to transform into static locally anisotropic elliptic configurations. The time dependence of $\Lambda_0(t)$ has to be computed, for instance, from a higher dimension theory or from experimental data.

7 On the Thermodynamics of Anisotropic Black Holes

A general approach to the anisotropic black holes should be based on a kind of nonequilibrium thermodynamics of such objects imbedded into locally anisotropic gravitational (locally anisotropic ether) continuous, which is a matter of further investigations (see the first works on the theory of locally anisotropic kinetic processes and thermodynamics in curved spaces [26]).

In this Section, we explore the simplest type of locally anisotropic black holes with anisotropically renormalized constants being in thermodynamic equilibrium with the locally anisotropic spacetime "bath" for suitable choices of $N$–connection coefficients. We do not yet understand the detailed thermodynamic behavior of locally anisotropic black holes but believe one could define their thermodynamics in the neighborhoods of some equilibrium states when the horizons are locally anisotropically deformed and constant with respect to an anholonomic frame.

In particular, for a class of BTZ like locally anisotropic spacetimes with horizons radii (76) we can still use the first law of thermodynamics to determine an entropy with respect to some fixed anholonomic bases (6) and (5) (here we note that there are developed some approaches even to the thermodynamics of usual BTZ black holes and that uncertainty is to be transferred in our considerations, see discussions and references in [3]).

In the approximation that the locally anisotropic spacetime receptivities $\alpha^{(m)}, \alpha^{(J)}$ and $\alpha^{(A)}$ do not depend on coordinates we have similar formulas as in locally isotropic gravity for the locally anisotropic black hole temperature at the boundary of a cavity of radius $r_H$.

$$T = -\frac{\sigma}{2\pi \left(M + \Lambda r_H^2\right)^{1/2}},$$

and entropy

$$S = 4\pi\tau_+$$

in Plank units.
For a elliptically deformed locally anisotropic black hole with the outer horizon $r_+(\theta)$ given by the formula (64) the Bekenstein–Hawking entropy,

$$S^{(a)} = \frac{L_+}{4G^{(a)}_{(gr)}},$$

were

$$L_+ = 4 \int_0^{\pi/2} r_+(\theta) \, d\theta$$

is the length of ellipse’s perimeter and $G^{(a)}_{(gr)}$ is the three dimensional gravitational coupling constant in locally anisotropic media, has the value

$$S^{(a)} = \frac{2p}{G^{(a)}_{(gr)}\sqrt{1 - \varepsilon^2}} \arctg \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}}.$$

If the eccentricity vanishes, $\varepsilon = 0$, we obtain the locally isotropic formula with $p$ being the radius of the horizon circumference, but the constant $G^{(a)}_{(gr)}$ could be locally anisotropically renormalized.

In dependence of dispersive or amplification character of locally anisotropic ether with $\alpha^{(m)}$, $\alpha^{(J)}$ and $\alpha^{(A)}$ being less or greater than unity we can obtain temperatures of locally anisotropic black holes less or greater than that for the locally isotropic limit. For example, we get anisotropic temperatures $T^{(a)}(\theta)$ if locally anisotropic black holes with horizons of type (64) are considered.

If we adapt the Euclidean path integral formalism of Gibbon and Hawking [14] to locally anisotropic spacetimes, by performing calculations with respect to an anholonomic frame, we develop a general approach to the locally anisotropic black hole irreversible thermodynamics. For locally anisotropic backgrounds with constant receptivities we obtain similar to [4, 7, 5] but anisotropically renormalized formulas.

Let us consider the Euclidean variant of the d–metric (75)

$$\delta s^2_E = (F_E) \, dt^2 + (F_E)^{-1} \, dr^2 + r^2 \, d\theta^2$$

(81)

where $t = i\tau$ and the Euclidean lapse function is taken with locally anisotropically renormalized constants, as in (74) (for simplicity, there is analyzed a non–rotating locally anisotropic black hole), $F = (-\overline{M} - \overline{r}r^2)$, which leads to the root $\tau_+ = [-\overline{M}/\overline{r}]^{1/2}$. By applying the coordinate transforms

$$x = \left(1 - \left(\frac{\tau_+}{r}\right)^2\right)^{1/2} \cos(-\overline{r}\tau_+ \theta) \exp\left(\sqrt{|\overline{r}|}\tau_+ \theta\right),$$

$$y = \left(1 - \left(\frac{\tau_+}{r}\right)^2\right)^{1/2} \sin(-\overline{r}\tau_+ \theta) \exp\left(\sqrt{|\overline{r}|}\tau_+ \theta\right),$$

$$z = \left(\left(\frac{\tau_+}{r}\right)^2 - 1\right)^{1/2} \exp\left(\sqrt{|\overline{r}|}\tau_+ \theta\right),$$

26
the d–metric (81) is rewritten in a standard upper half–space ($z > 0$) representation of locally anisotropic hyperbolic 3–space,

$$\delta s^2_E = -\frac{1}{\Lambda} (z^2 dz^2 + dy^2 + \delta z^2).$$

The coordinate transform (82) is non–singular at the $z$–axis $r = \tau_+$ if we require the periodicity

$$(\theta, \tau) \sim (\theta, \tau + \overline{\beta}_0)$$

where

$$\overline{\beta}_0 = \frac{1}{T_0} = -\frac{2\pi}{\Lambda \tau_+}$$

is the inverse locally anisotropically renormalized temperature, see (79).

To get the locally anisotropically renormalized entropy from the Euclidean locally anisotropic path integral we must define a locally anisotropic extension of the grand canonical partition function

$$Z = \int [dg] e^{I_E[g]},$$

where $T_E$ is the Euclidean locally anisotropic action. We consider as for usual locally isotropic spaces the classical approximation $Z \sim \exp\{T_E[g]\}$, where as the extremal d–metric $g$ is taken (81). In (83) there are included boundary terms at $\tau_+$ and $\infty$ (see the basic conclusions and detailed discussions for the locally isotropic case [4, 7, 5] which are also true with respect to anholonomic bases).

For an inverse locally anisotropic temperature $\overline{\beta}_0$ the action from (83) is

$$T_E[g] = 4\pi \tau_+ - \overline{\beta}_0 M$$

which corresponds to the locally anisotropic entropy (80) being a locally anisotropic renormalization of the standard Bekenstein entropy.

8 Chern–Simons Theories and Locally Anisotropic Gravity

In order to compute the first quantum corrections to the locally anisotropic path integral (83), inverse locally anisotropic temperature (82) and locally anisotropic entropy (80) we take the advantage of the Chern–Simons formalism generalized for (2+1)–dimensional locally anisotropic spacetimes.

By using the locally anisotropically renormalized cosmological constant $\Lambda$ and adapting the Achucarro and Townsend [1] construction to anholonomic frames we can define two $SO(2,1)$ gauge locally anisotropic fields

$$A^\mu = \omega^\mu + \frac{1}{\sqrt{|\Lambda|}} e^\mu$$

and

$$\bar{A}^\mu = \omega^\mu - \frac{1}{\sqrt{|\Lambda|}} e^\mu.$$
where the index $\alpha$ enumerates an anholonomic triad $e^\alpha = e^\alpha_\mu \delta x^\mu$ and $\omega^\alpha = \frac{1}{2} \epsilon^{abc} \omega_{\mu bc} \delta x^\mu$ is a spin d–connection (d–spinor calculus is developed in [23]). The first–order action for locally anisotropic gravity is written
\[ I_{\text{grav}} = I_{\text{CS}}[A] - I_{\text{CS}}[\tilde{A}] \] (84)
with the Chern–Simons action for a (2+1)–dimensional vector bundle $\tilde{E}$ provided with N–connection structure,
\[ I_{\text{CS}}[A] = \frac{\overline{k}}{4\pi} \int_{\tilde{E}} Tr \left( A \wedge \delta A + \frac{2}{3} A \wedge A \wedge A \right) \] (85)
where the coupling constant $\overline{k} = \sqrt{|\Lambda|/(4\sqrt{2}G_{(gr)})}$ and $G_{(gr)}$ is the gravitational constant. The one d–form from (85) $A = A^\alpha_\mu \delta x^\mu$ is a gauge d–field for a Lie algebra with generators $\{T_a\}$. Following [6] we choose
\[ (T_a)_c^d = -\epsilon_{abc} \eta^{dc}, \quad \eta_{ab} = \text{diag}(-1, 1, 1), \quad \epsilon_{012} = 1 \]
and considering $Tr$ as the ordinary matrix trace we write
\[ [T_a, T_b] = g_{ab} T_c, \quad g_{ab} = 2 \eta_{ab} \epsilon^{cd} \eta^{ef} f_{ab} \epsilon_{def} = -2 \eta_{ab} \epsilon^{de} \eta^{ef} f_{ab} \epsilon_{def} \]

If the manifold $\tilde{E}$ is closed the action (84) is invariant under locally anisotropic gauge transforms
\[ \tilde{A} \rightarrow A = q^{-1} \tilde{A} q + q^{-1} \delta q. \]
This invariance is broken if $\tilde{E}$ has a boundary $\partial \tilde{E}$. In this case we must add to (85) a boundary term, written in $(v, \theta)$–coordinates as
\[ \mathcal{T}_{\text{CS}} = -\frac{\overline{k}}{4\pi} \int_{\partial \tilde{E}} Tr A_\theta A_v \] (86)
which results in a term proportional to the standard chiral Wess–Zumino–Witten (WZW) action [21, 11]:
\[ (\mathcal{T}_{\text{CS}} + \mathcal{T}_{\text{CS}}^+) [A] = (\mathcal{T}_{\text{CS}} + \mathcal{T}_{\text{CS}}^-) [\tilde{A}] - \overline{k} \mathcal{T}_{\text{WZW}}^+[q, \tilde{A}] \]
where
\[ \mathcal{T}_{\text{WZW}}^+[q, \tilde{A}] = \frac{1}{4\pi} \int_{\partial \tilde{E}} Tr \left( q^{-1} \delta q \right) \left( q^{-1} \delta_v q \right) \]
\[ + \frac{1}{2\pi} \int_{\partial \tilde{E}} Tr \left( q^{-1} \delta_v q \right) \left( q^{-1} \delta_{\theta} q \right) + \frac{1}{12\pi} \int_{\tilde{E}} Tr \left( q^{-1} \delta q \right)^3. \] (87)

With respect to a locally anisotropic base the gauge locally anisotropic field satisfies standard boundary conditions
\[ A_v^+ = A_v^- = \tilde{A}_v^+ = \tilde{A}_v^- = 0. \]
By applying the action\(^{(84)}\) with boundary terms\(^{(86)}\) and\(^{(87)}\) we can formulate a statistical mechanics approach to the \((2+1)\)-dimensional locally anisotropic black holes with locally anisotropically renormalized constants when the locally anisotropic entropy of the black hole can be computed as the logarithm of microscopic states at the anisotropically deformed horizon. In this case the Carlip’s results\(^[6, 15]\) could be generalized for locally anisotropic black holes. We present here the formulas for one-loop corrected locally anisotropic temperature\(^{(79)}\) and locally anisotropic entropy\(^{(80)}\)

\[
\beta_0 = -\frac{\pi}{4\Lambda \hbar G_{(gr)}} \tau_+ \left(1 + \frac{8\hbar G_{(gr)}}{\sqrt{|\Lambda|}}\right) \text{ and } S^{(a)} = \frac{\pi \tau_+}{2\hbar G_{(gr)}} \left(1 + \frac{8\hbar G_{(gr)}}{\sqrt{|\Lambda|}}\right).
\]

We do not yet have a general accepted approach even to the thermodynamics and its statistical mechanics foundation of locally isotropic black holes and this problem is not solved for locally anisotropic black holes for which one should be associated a model of nonequilibrium thermodynamics. Nevertheless, the formulas presented in this section allows us a calculation of basic locally anisotropic thermodynamical values for equilibrium locally anisotropic configurations by using locally anisotropically renormalized constants.

9 Conclusions and Discussion

In this paper we have aimed to justify the use of moving frame method in construction of metrics with generic local anisotropy, in general relativity and its modifications for higher and lower dimension models\(^[25, 26]\).

We argued that the 3D gravity reformulated with respect to anholonomic frames (with two holonomic and one anholonomic coordinates) admits new classes of solutions of Einstein equations, in general, with nonvanishing cosmological constants. Such black hole like and another type ones, with deformed horizons, variation of constants and locally anisotropic gravitational polarizations in the vacuum case induced by anholonomic moving triads with associated nonlinear connection structure, or (in the presence of 3D matter) by self-consistent distributions of matter energy density and pressure and dreibein (3D moving frame) fields.

The solutions considered in the present paper have the following properties: 1) they are exact solutions of 3D Einstein equations; 2) the integration constants are to be found from boundary conditions and compatibility with locally isotropic limits; 3) having been rewritten in ‘pure’ holonomic variables the 3D metrics are off–diagonal; 4) it is induced a nontrivial torsion structure which vanishes in holonomic coordinates; for vacuum solutions the 3D gravity is transformed into a teleparallel theory; 5) such solutions are characterized by nontrivial nonlinear connection curvature.

The arguments in this paper extend the results in the literature on the black hole thermodynamics by elucidating the fundamental questions of formulation of this theory for anholonomic gravitational systems with local frame anisotropy. We computed the entropy and temperature of black holes with elliptic horizons and/or with anisotropic variation and renormalizations of constants.

We also showed that how the 3D gravity models with anholonomic constraints can be transformed into effective Chern–Simons theories and following this priority we computed
the locally anisotropic quantum corrections for the entropy and temperature of black holes.

Our results indicate that there exists a kind of universality of inducing locally anisotropic interactions in physical theories formulated in mixed holonomic–anholonomic variables: the spacetime geometry and gravitational field are effectively polarized by imposed constraints which could result in effective renormalization and running of interaction constants.

Finally, we conclude that problem of definition of adequate systems of reference for a prescribed type of symmetries of interactions could be of nondynamical nature if we fix at the very beginning the class of admissible frames and symmetries of solutions, but could be transformed into a dynamical task if we deform symmetries (for instance, a circular horizon into an elliptic one) and try to find self-consistently a corresponding anholonomic frame for which the metric is diagonal but with generic anisotropic structure).

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