Canonical binary matrices related to bipartite graphs

Krasimir Yordzhev

Faculty of Mathematics and Natural Sciences
South-West University ”Neofit Rilski”
2700 Blagoevgrad, Bulgaria
E-mail: yordzhev@swu.bg

Abstract
The current paper is dedicated to the problem of finding the number of mutually non isomorphic bipartite graphs of the type \(g = \langle R_g, C_g, E_g \rangle\) at given \(n = |R_g|\) and \(m = |C_g|\), where \(R_g\) and \(C_g\) are the two disjoint parts of the vertices of the graphs \(g\), and \(E_g\) is the set of edges, \(E_g \subseteq R_g \times C_g\). For this purpose, the concept of canonical binary matrix is introduced. The different canonical matrices unambiguously describe the different with exactness up to isomorphism bipartite graphs. We have found a necessary and sufficient condition an arbitrary matrix to be canonical. This condition could be the base for realizing recursive algorithm for finding all \(n \times m\) canonical binary matrices and consequently for finding all with exactness up to isomorphism binary matrices with cardinality of each part equal to \(n\) and \(m\).

Keyword: Bipartite graph; Canonical binary matrix; Semi-canonical binary matrix

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1 Introduction and notation

Let \(k\) and \(n\) be positive integers, \(k \leq n\). By \([n]\) we denote the set \([n] = \{1, 2, \ldots, n\}\) and by \([k, n]\) the set \([k, n] = \{k, k+1, \ldots, n\}\).

Bipartite graph is the ordered triplet
\[ g = \langle R_g, C_g, E_g \rangle, \]
where \(R_g\) and \(C_g\) are non-empty sets such that \(R_g \cap C_g = \emptyset\), the elements of which will be called vertices. \(E_g \subseteq R_g \times C_g = \{(r, c) \mid r \in R_g, c \in C_g\}\) - the set of edges. Multiple edges are not allowed in our considerations.

By \(S_n\) we denote the symmetric group of order \(n\), i.e. the group of all one-to-one mappings of the set \([n] = \{1, 2, \ldots, n\}\) in itself. If \(x \in [n]\), \(\rho \in S_n\), then the image of the element \(x\) in the mapping \(\rho\) we denote by \(\rho(x)\).

Definition 1 Let \(g' = \langle R_{g'}, C_{g'}, E_{g'} \rangle\) and \(g'' = \langle R_{g''}, C_{g''}, E_{g''} \rangle\) are two bipartite graphs. We will say that the graphs \(g'\) and \(g''\) are isomorphic and we will write
\[ g' \cong g'', \]
If \(|R_g'| = |R_g''| = n, |C_g'| = |C_g''| = m\) and there exist \(\rho \in S_n\) and \(\sigma \in S_m\) such that \((r, c) \in E_g' \iff (\rho(r), \sigma(c)) \in E_g''\).

In this paper we consider only bipartite graphs up to isomorphism.

For more details on graph theory and its applications see\[1\,2\].

The connection between the bipartite graphs and the popular puzzle Sudoku is described in details in\[3\,4\]. There it is shown that if we want to find the number of all \(n^2 \times n^2\) Sudoku grids, it is necessary to obtain all bipartite graphs of the type \(g = \langle R_g, C_g, E_g \rangle\) with exactness up to isomorphism and also some of their numerical characteristics, where \(|R_g| = |C_g| = n\).

We dedicate this paper on the problem to obtain all bipartite graphs of the type \(g = \langle R_g, C_g, E_g \rangle\) with exactness up to isomorphism at given \(n\) and \(m\), where \(n = |R_g|,\ m = |C_g|\). The set of all these graphs we will denote with \(G_{n \times m}\). For this purpose we will represent the set \(G_{n \times m}\) with the help of \(n \times m\) binary matrices.

Let us recall that binary (or boolean, or \((0,1)\)-matrix) is called a matrix whose elements belong to the set \(\mathbb{B} = \{0, 1\}\). With \(\mathbb{B}_{n \times m}\) we will denote the set of all \(n \times m\) binary matrices.

A square binary matrix is called a permutation matrix, if there is just one 1 in every row and every column. Let us denote by \(P_n\) the group of all \(n \times n\) permutation matrices. In effect is the isomorphism \(P_n \cong S_n\).

As it is well known (see\[3\,4\]) that the multiplication of an arbitrary real or complex matrix \(A\) from the left with a permutation matrix (if the multiplication is possible) leads to dislocation of the rows of the matrix \(A\), while the multiplication of \(A\) from the right with a permutation matrix leads to the dislocation of the columns of \(A\).

With \(T_n \subset P_n\) we denote the set of all transpositions in \(P_n\), i.e. the set of all \(n \times n\) permutation matrices, which multiplying from the left an arbitrary \(n \times m\) matrix swaps the places of exactly two rows, while multiplying from the right an arbitrary \(k \times n\) matrix swaps the places of exactly two columns.

**Definition 2** Let \(A, B \in \mathbb{B}_{n \times m}\). We will say that the matrices \(A\) and \(B\) are equivalent and we will write
\[A \sim B,\]
if there exist permutation matrices \(X \in P_n\) and \(Y \in P_m\), such that
\[A = XBY.\]

In other words \(A \sim B\) if \(A\) is received from \(B\) after dislocation of some of the rows and the columns of \(B\). Obviously, the introduced relation is an equivalence relation.

Let \(g = \langle R_g, C_g, E_g \rangle\) be a bipartite graph, where \(R_g = \{r_1, r_2, \ldots, r_n\}\) and \(C_g = \{c_1, c_2, \ldots, c_m\}\). Then we build the matrix \(A = [a_{ij}] \in \mathbb{B}_{n \times m}\), such that \(a_{ij} = 1\) if and only if \((r_i, c_j) \in E_g\). Inversely, let \(A = [a_{ij}] \in \mathbb{B}_{n \times m}\). We denote the \(i\)-th row of \(A\) with \(r_i\), while the \(j\)-th column of \(A\) with \(c_j\). Then we build the bipartite graph \(g = \langle R_g, C_g, E_g \rangle\), where \(R_g = \{r_1, r_2, \ldots, r_n\}\), \(C_g = \{c_1, c_2, \ldots, c_m\}\) and there exists an edge from the vertex \(r_i\) to the vertex \(c_j\) if and only if \(a_{ij} = 1\). It is easy to see that if \(g\) and \(h\) are two isomorphic graphs and \(A\) and \(B\) are the corresponding matrices, then \(A\) is obtained from \(B\) by a permutation of columns and/or rows. Thus we showed the following obvious relation between the bipartite graphs and the binary matrices:
Proposition 1  There exists one-to-one mapping  
\[ \varphi : \mathcal{G}_{n \times m} \to \mathcal{B}_{n \times m} \]

between the elements of \( \mathcal{G}_{n \times m} \) and \( \mathcal{B}_{n \times m} \), such that if  \( g, h \in \mathcal{G}_n \), then
\[ g \cong h \iff \varphi(g) \sim \varphi(h). \]

Thus, the combinatorial problem to obtain and enumerate all of \( n \times m \) binary matrices up to permutation of columns or rows naturally arises.

2 Semi-canonical and canonical binary matrices

Let \( A \in \mathcal{B}_{n \times m} \). With \( r(A) \) we will denote the ordered \( n \)-tuple
\[ r(A) = \langle x_1, x_2, \ldots, x_n \rangle, \]
where \( 0 \leq x_i \leq 2^m - 1, \; i = 1, 2, \ldots n \) and \( x_i \) is a natural number written in binary notation with the help of the \( i \)-th row of \( A \).

Similarly with \( c(A) \) we will denote the ordered \( m \)-tuple
\[ c(A) = \langle y_1, y_2, \ldots, y_m \rangle, \]
where \( 0 \leq y_j \leq 2^n - 1, \; j = 1, 2, \ldots m \) and \( y_j \) is a natural number written in binary notation with the help of the \( j \)-th column of \( A \).

We consider the sets:
\[ \mathcal{R}_{n \times m} = \{ \langle x_1, x_2, \ldots, x_n \rangle \mid 0 \leq x_i \leq 2^m - 1, \; i = 1, 2, \ldots n \} = \{ r(A) \mid A \in \mathcal{B}_{n \times m} \} \]

and
\[ \mathcal{C}_{n \times m} = \{ \langle y_1, y_2, \ldots, y_m \rangle \mid 0 \leq y_j \leq 2^n - 1, \; j = 1, 2, \ldots m \} = \{ c(A) \mid A \in \mathcal{B}_{n \times m} \} \]

Thus we define the following two mappings:
\[ r : \mathcal{B}_{n \times m} \to \mathcal{R}_{n \times m} \]
and
\[ c : \mathcal{B}_{n \times m} \to \mathcal{C}_{n \times m}, \]
which are bijective and therefore
\[ \mathcal{R}_{n \times m} \cong \mathcal{B}_{n \times m} \cong \mathcal{C}_{n \times m}. \]

The above described bijections \( r \) and \( c \) leads to the following statement, which is an analog of Proposition 1.
Proposition 2  There exist one-to-one mappings between the elements of $G_{n \times m}$ and the sets

\[ R_{n \times m} = \{ (x_1, x_2, \ldots, x_n) \mid 0 \leq x_i \leq 2^m - 1, \ i = 1, 2, \ldots n \} \]

and

\[ C_{n \times m} = \{ (y_1, y_2, \ldots, y_m) \mid 0 \leq y_j \leq 2^n - 1, \ j = 1, 2, \ldots m \} \]

Example 1  The shown in Figure 1 graph is unambiguously coded with the help of the matrix

\[
A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

and the help of the ordered set of nonnegative integers.

\[ r(A) = (12, 14, 1) \]

and

\[ c(A) = (6, 6, 2, 1). \]

The lexicographic orders in $R_{n \times m}$ and in $C_{n \times m}$ we will denote with $\prec$.

Definition 3  Let $A \in B_{n \times m}$, $r(A) = (x_1, x_2, \ldots, x_n)$ and $c(A) = (y_1, y_2, \ldots, y_m)$. We will call the matrix $A$ semi-canonical, if

\[
x_1 \leq x_2 \leq \cdots \leq x_n
\]

and

\[
y_1 \leq y_2 \leq \cdots \leq y_m.
\]

Proposition 3  Let $A = [a_{ij}] \in B_{n \times m}$ be a semi-canonical matrix. Then there exist integers $i, j$, such that $1 \leq i \leq n, 1 \leq j \leq m$ and

\[
a_{11} = a_{12} = \cdots = a_{1j} = 0, \quad a_{1j+1} = a_{1j+2} = \cdots = a_{1m} = 1, \quad (1)
\]

\[
a_{11} = a_{21} = \cdots = a_{i1} = 0, \quad a_{i+1 1} = a_{i+2 1} = \cdots = a_{n1} = 1. \quad (2)
\]
Proof. Let $r(A) = \langle x_1, x_2, \ldots, x_n \rangle$ and $c(A) = \langle y_1, y_2, \ldots, y_m \rangle$. We assume that there exist integers $p$ and $q$, such that $1 \leq p < q \leq m$, $a_{1p} = 1$ and $a_{1q} = 0$. In this case $y_p > y_q$, which contradicts the condition for semi-canonicity of the matrix $A$. We have proven (1). Similarly, we prove (2) as well. □

**Corollary 1** Let $A = [a_{ij}] \in \mathbb{B}_{n \times m}$ be a semi-canonical matrix. Then there exist integers $s, t$, such that $0 \leq s \leq m$, $0 \leq t \leq n$, $x_1 = 2^s - 1$ and $y_1 = 2^t - 1$. □

**Definition 4** We will call the matrix $A \in \mathbb{B}_{n \times m}$ canonical matrix, if $r(A)$ is a minimal element about the lexicographic order in the set $A = \{r(B) \mid B \sim A\}$.

If the matrix $A \in \mathbb{B}_{n \times m}$ is canonical and $r(A) = \langle x_1, x_2, \ldots, x_n \rangle$, then obviously

$$x_1 \leq x_2 \leq \cdots \leq x_n.$$  \hspace{1cm} (3)

From Definition 4 immediately follows that in every equivalence class about the relation $\sim$ (see Definition 2) there exists only one canonical matrix. Therefore, to find all bipartite graphs of type $g = \langle R_g, C_g, E_g \rangle$, $|R_g| = n$, $|C_g| = m$ up to isomorphism, it suffices to find all canonical matrices from the set $\mathbb{B}_{n \times m}$.

**Theorem 1** Let $A$ be an arbitrary matrix from $\mathbb{B}_{n \times m}$. Then:

a) If $X_1, X_2, \ldots, X_s \in \mathcal{T}_n$ are such that

$$r(X_1X_2\ldots X_sA) < r(X_2X_3\ldots X_sA) < \cdots < r(X_sA) < r(A),$$
then

$$c(X_1X_2\ldots X_sA) < c(A).$$

b) If $Y_1, Y_2, \ldots, Y_t \in \mathcal{T}_m$ are such that

$$c(AY_1Y_2\ldots Y_t) < c(AY_2Y_3\ldots Y_t) < \cdots < c(AX_t) < r(A),$$
then

$$r(AY_1Y_2\ldots Y_t) < r(A).$$

Proof. a) Induction by $s$.

Let $s = 1$ and let $X \in \mathcal{T}_n$ be a transposition which multiplying an arbitrary matrix $A = [a_{ij}] \in \mathbb{B}_{n \times m}$ from the left swaps the places of the rows of $A$ with numbers $u$ and $v$ ($1 \leq u < v \leq n$), while the remaining rows stay in their places. In other words if

$$A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1m} \\
    a_{21} & a_{22} & \cdots & a_{2r} & \cdots & a_{2m} \\
    \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
    a_{u1} & a_{u2} & \cdots & a_{ur} & \cdots & a_{um} \\
    \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
    a_{v1} & a_{v2} & \cdots & a_{vr} & \cdots & a_{vm} \\
    \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nr} & \cdots & a_{nm}
\end{bmatrix}$$
then

\[
X_1 A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1m} \\
  a_{21} & a_{22} & \cdots & a_{2r} & \cdots & a_{2m} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_{v1} & a_{v2} & \cdots & a_{vr} & \cdots & a_{vm} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_{u1} & a_{u2} & \cdots & a_{ur} & \cdots & a_{um} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nr} & \cdots & a_{nm}
\end{bmatrix},
\]

where \(a_{ij} \in \{0,1\}, 1 \leq i \leq n, 1 \leq j \leq m\).

Let

\[r(A) = \langle x_1, x_2, \ldots, x_u, \ldots, x_v, \ldots, x_n \rangle.\]

Then

\[r(XA) = \langle x_1, x_2, \ldots, x_v, \ldots, x_u, \ldots, x_n \rangle.\]

Since \(r(XA) < r(A)\), then according to the properties of the lexicographic order \(x_u < x_v\). Let the representation of \(x_u\) and \(x_v\) in binary notation with an eventual addition if necessary with unessential zeros in the beginning be respectively as follows:

\[x_u = a_{u1}a_{u2}\cdots a_{um},\]

\[x_v = a_{v1}a_{v2}\cdots a_{vm}.\]

Since \(x_v < x_u\), then there exists an integer \(r \in \{1,2,\ldots,m\}\), such that \(a_{uj} = a_{vj}\) when \(j < r\), \(a_{ur} = 1\) and \(a_{vr} = 0\). Hence if \(c(A) = \langle y_1, y_2, \ldots, y_m \rangle\), \(c(XA) = \langle z_1, z_2, \ldots, z_m \rangle\), then \(y_j = z_j\) when \(j < r\), while the representation of \(y_r\) and \(z_r\) in binary notation with an eventual addition if necessary with unessential zeros in the beginning is respectively as follows:

\[y_r = a_{1r}a_{2r}\cdots a_{u-1, r}a_{ur}\cdots a_{vr}\cdots a_{nr},\]

\[z_r = a_{1r}a_{2r}\cdots a_{u-1, r}a_{vr}\cdots a_{ur}\cdots a_{nr}.\]

Since \(a_{ur} = 1\), \(a_{vr} = 0\), then \(z_r < y_r\), whence it follows that \(c(XA) < c(A)\).

We assume that for every \(s\)-tuple of transpositions \(X_1, X_2, \ldots, X_s \in \mathcal{T}_n\) and for every matrix \(A \in \mathcal{B}_{n \times m}\) from

\[r(X_1X_2\ldots X_s A) < r(X_2 \cdots X_s A) < \cdots < r(X_s A) < r(A)\]

it follows that

\[c(X_1X_2\ldots X_s A) < c(A)\]

and let \(X_{s+1} \in \mathcal{T}_n\) be such that

\[r(X_1X_2\ldots X_sX_{s+1} A) < r(X_2 \cdots X_{s+1} A) < \cdots < r(X_{s+1} A) < r(A).\]

According to the induction assumption \(c(X_{s+1} A) < c(A)\).

We put

\[A_1 = X_{s+1} A.\]
According to the induction assumption from
\[ r(X_1X_2\cdots X_sA_1) < r(X_2\cdots X_sA_1) < \cdots < r(X_sA_1) < r(A_1) \]
it follows that
\[ c(X_1X_2\cdots X_sX_{s+1}A) = c(X_1X_2\cdots X_sA_1) < c(A_1) = c(X_{s+1}A) < c(A), \]
with which we have proven a).

b) is proven similarly to a). \(\square\)

Obviously in effect is also the dual to Theorem 1 statement, in which everywhere instead of the sign \(<\) we put the sign \(>\).

**Corollary 2**  *If the matrix* \(A \in \mathcal{B}_{n \times m}\) *is a canonical matrix, then* \(A\) *is a semi-canonical matrix.*

**Proof.** Let \(A \in \mathcal{B}_{n \times m}\) be a canonical matrix and \(r(A) = \langle x_1, x_2, \ldots, x_n \rangle\). Then from (4) it follows that \(x_1 \leq x_2 \leq \cdots \leq x_n\). Let \(c(A) = \langle y_1, y_2, \ldots, y_m \rangle\). We assume that there are \(s\) and \(t\) such that \(s \leq t\) and \(y_s > y_t\). Then we swap the columns of numbers \(s\) and \(t\). Thus we obtain the matrix \(A' \in \mathcal{B}_{n \times m}\), \(A' \neq A\). Obviously \(c(A') < c(A)\). From Theorem 1 it follows that \(r(A') < r(A)\), which contradicts the minimality of \(r(A)\). \(\square\)

In the next example, we will see that the opposite statement of Corollary 2 is not always true.

**Example 2** We consider the matrices:

\[
A = \begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

and

\[
B = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix}.
\]

After immediate verification, we find that \(A \sim B\). Furthermore \(r(A) = \langle 3, 3, 4, 8 \rangle\), \(c(A) = \langle 1, 2, 12, 12 \rangle\), \(r(B) = \langle 1, 6, 6, 8 \rangle\), \(c(B) = \langle 1, 6, 6, 8 \rangle\). So \(A\) and \(B\) are two equivalent to each other semi-canonical matrices, but they are not canonical. Canonical matrix in this equivalence class is the matrix

\[
C = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix},
\]

where \(r(C) = \langle 1, 2, 12, 12 \rangle\), \(c(C) = \langle 3, 3, 4, 8 \rangle\). \(\square\)
From example 2 immediately follows that in a given equivalence class it is possible to exist more than one semi-canonical element.

In [1] we described and we implemented with help of C++ programming language an algorithm for finding all \( n \times n \) semi-canonical binary matrices taking into account the number of 1 in each of them. In the described algorithm, the bitwise operations are substantially used.

Let us denote with \( \beta(n, k) \) the number of all \( n \times n \) semi-canonical binary matrices with exactly \( k \) 1’s, where \( 0 \leq k \leq n^2 \). In [2], we received the following integer sequences:

\[
\{\beta(2, k)\}_{k=0}^{4} = \{1, 1, 3, 1, 1\}
\]

\[
\{\beta(3, k)\}_{k=0}^{9} = \{1, 1, 3, 8, 10, 9, 8, 3, 1, 1\}
\]

\[
\{\beta(4, k)\}_{k=0}^{16} = \{1, 1, 3, 8, 25, 49, 84, 107, 121, 101, 72, 41, 24, 8, 3, 1, 1\}
\]

\[
\{\beta(5, k)\}_{k=0}^{25} = \{1, 1, 3, 8, 25, 80, 220, 524, 1057, 1806, 2671, 3365, 3680, 3468, 2865, 2072, 1314, 723, 362, 166, 72, 24, 8, 3, 1, 1\}
\]

\[
\{\beta(6, k)\}_{k=0}^{36} = \{1, 1, 3, 8, 25, 80, 283, 925, 2839, 7721, 18590, 39522, 74677, 125449, 188290, 252954, 305561, 332402, 326650, 290171, 233656, 170704, 113448, 68677, 37996, 19188, 8910, 3847, 1588, 613, 299, 72, 24, 8, 3, 1, 1\}
\]

3 A necessary and sufficient condition for a binary matrix to be canonical

Let \( A = [a_{ij}] \in \mathcal{B}_{n \times m}, \ r(A) = \langle x_1, x_2, \ldots, x_n \rangle \). We denote the following notations:

\[
\varepsilon_i(A) = \varepsilon(x_i) = \sum_{j=1}^{m} a_{ij} - \text{the number of 1 in the } i \text{-th row of } A, \ i = 1, 2, \ldots, n.
\]

\[
Z_i(A) = Z(x_i) = \{x_k \in r(A) | x_k = x_i\} - \text{the set of all rows, equal to } i \text{-th row of } A. \text{ By definition } x_i \in Z(x_i), \ i = 1, 2, \ldots, n.
\]

\[
\zeta_i(A) = \zeta(x_i) = |Z_i(A)|, \ i = 1, 2, \ldots, n.
\]

The next four statements are obvious and their proof is trivial.

**Proposition 4** Let \( A = [a_{ij}] \in \mathcal{B}_{n \times m}, \ r(A) = \langle x_1, x_2, \ldots, x_n \rangle \) and let \( x_1 \leq x_2 \leq \cdots \leq x_n \). Let \( s \) and \( t \) are integers so that \( 0 \leq s < n, \ 0 \leq t < m \) and let the matrix \( B = \mathcal{B}_{n-s \times m-s} \) be obtained from \( A \) removing the first \( t \) rows and the last \( s \) columns and let \( r(B) = \langle x'_{t+1}, x'_{t+2}, \ldots, x'_n \rangle \). Then \( x'_{t+1} \leq x'_{t+2} \leq \cdots \leq x'_n \).

\[ \square \]

**Proposition 5** Let \( A = [a_{ij}] \in \mathcal{B}_{n \times m}, \ r(A) = \langle x_1, x_2, \ldots, x_n \rangle \) and let \( x_1 \leq x_2 \leq \cdots \leq x_n \). Then for each \( i = 2, 3, \ldots, n \), for which \( x_{i-1} < x_i \), or \( i = 1 \) the condition

\[
Z(x_i) = \{x_i, x_{i+1}, \ldots, x_{i+\zeta(x_i)}\}
\]

is fulfilled.

\[ \square \]
Proposition 6 Let $A = [a_{ij}] \in \mathfrak{B}_{n \times m}$, $r(A) = \langle x_1, x_2, \ldots, x_n \rangle$, $x_1 \leq x_2 \leq \cdots \leq x_n$ and let $\varepsilon_1(A) = m$, i.e. $x_1 = 2^m - 1$. Then $A$ is canonical. \hfill $\Box$

Proposition 7 Let $A = [a_{ij}] \in \mathfrak{B}_{n \times m}$, $r(A) = \langle x_1, x_2, \ldots, x_n \rangle$, $x_1 = 2^s - 1$ for some integer $s$, $0 \leq s \leq m$ and let $\zeta_1(A) = n$. Then $A$ is canonical. \hfill $\Box$

Theorem 2 Let $A = [a_{ij}] \in \mathfrak{B}_{n \times m}$,

$$r(A) = \langle x_1, x_2, \ldots, x_n \rangle,$$

$$c(A) = \langle y_1, y_2, \ldots, y_m \rangle.$$  

Then $A$ is canonical if and only if the next condition are true:

1. $x_1 \leq x_2 \leq \cdots \leq x_n$;
2. $x_1 = 2^s - 1$, where $s = \varepsilon_1(A)$;
3. For each $i = 2, 3, \ldots, n$ is fulfilled $\varepsilon_1(A) \leq \varepsilon_i(A)$;
4. If for some integer $i$ such that $\zeta_1(A) < i \leq n$ is fulfilled $\varepsilon_i(A) = \varepsilon_1(A)$, then $\zeta_1(A) \geq \zeta_i(A)$;
5. $y_{m-s+1} \leq y_{m-s+2} \leq \cdots \leq y_m$, where $s = \varepsilon_1(A)$;
6. Let for some integer $i$ such that $\zeta_1(A) < i \leq n$ are fulfilled $\varepsilon_1(A) = \varepsilon_i(A) = s$ and $\zeta_1(A) = \zeta_i(A) = t$. Let $\Upsilon(x_i) = \{y_i \in c(A) \mid a_{ij} = 1\} = \{y_{u_1}, y_{u_2}, \ldots, y_{u_s}\}$. Let $A'$ is the matrix which is obtained from $A$ replacing the places of the rows from the set $C(x_i)$ with the rows of the set $Z(x_i)$, the place of the column $y_{u_1}$ with the place of the column $y_{m-s}$, the place of the column $y_{u_2}$ with the place of the column $y_{m-s+1}$ and so on, the place of the column $y_{u_s}$ with the place of the column $y_m$. Then $r(A) \leq r(A')$;
7. If $s = \varepsilon_1(A) < m$ and $t = \zeta_1(A) < n$ then the matrix $B \in \mathfrak{B}_{(n-s) \times (m-t)}$, which is obtained from $A$ removing the first $s$ rows and the last $t$ columns is canonical.

Proof.
Necessity. Let $A = [a_{ij}] \in \mathfrak{B}_{n \times m}$ be a canonical matrix and let $r(A) = \langle x_1, x_2, \ldots, x_n \rangle$, $c(A) = \langle y_1, y_2, \ldots, y_m \rangle$.

Conditions 1 and 2 are due to the fact that every canonical matrix is semi-canonical (Corollary 5), so $x_1 \leq x_2 \leq \cdots \leq x_n$ $y_1 \leq y_2 \leq \cdots \leq y_m$.

Condition 3 comes from Corollary 1.

We assume that an integer $i$, $2 \leq i \leq n$ exists, such that $\varepsilon_i(A) < \varepsilon_1(A) = s$ and let $\varepsilon_1(A) = u < s$. Then a matrix $A' = [a'_{ij}] \sim A$ exists such that $a_{i1} = a'_{i1} = \cdots = a'_{im-u} = 0$ and $a'_{im-u+1} = a'_{im-u+2} = \cdots = a_{im} = 1$. We move the $i$-th row of $A'$ at first place and we obtain a matrix $A''$. Obviously $A'' \sim A$. Let $r(A'') = \langle x'_1, x'_2, \ldots, x'_n \rangle$. Then $x'_1 = 2^u - 1 < 2^s - 1 = x_1$. Therefore $r(A'') < r(A)$, which is impossible, due to the fact that $A$ is canonical.

\footnote{If for some $j$ is satisfied $a_{ij} = m - s + j$, then $y_{u_j}$ remains at its place.}
Let $s = \varepsilon_1(A)$ and $t = \zeta_1(A)$. According to the proved above condition and Proposition 5 we have $x_1 = x_2 = \cdots = x_t = 2^s - 1 < x_{t+1}$. We assume that an integer $i$, $1 \leq i \leq n$, exists such that $\varepsilon_i(A) = \varepsilon_1(A) = s$ and $\zeta_i(A) > \zeta_1(A)$. Let $\zeta_1(A) = v, v > t$. Then a matrix $A' \sim A$ exist, such that $r(A') = (x'_1, x'_2, \ldots, x'_n)$, where $x'_1 = x'_2 = \cdots = x'_v = 2^s - 1 < x_{v+1}$. Due to the fact that $t+1 \leq v$, consequently $x'_{t+1} = x'_v = 2^s - 1 = x_t < x_{t+1}$. We obtained that $x_k = x'_k$ for $k = 1, 2, \ldots, t$ and $x'_{t+1} < x_{t+1}$. From here it follows that $r(A') < r(A)$, which is contrary to the canonicity of $A$.

Condition 6 comes directly from the fact that $A$ is canonical and $r(A) \leq r(A')$ for each matrix $A' \sim A$.

From the already proved condition 1 and Proposition 5 it follows that $A$ is presented in this type:

$$A = \begin{bmatrix} O & E \\ B & C \end{bmatrix}, \quad (4)$$

where $O$ is $t \times (m-s)$ matrix, all element of which are equal to $0$, $E$ is $t \times s$ matrix, all element of which are equal to $1$, $B \in \mathcal{B}_{(n-t) \times (m-s)}$, as the first rows of $B$ are not entirely null, $C \in \mathcal{B}_{(n-t) \times s}$, $s = \varepsilon_1(A)$ and $t = \zeta_1(A)$.

Let $B' \sim B$ and $B'$ is $(n-t) \times (m-s)$ canonical binary matrix. Then the following matrices $A' \in \mathcal{B}_{n \times n}$ and $C' \in \mathcal{B}_{(n-t) \times s}$ exist, such that $A' \sim A$, $C' \sim C$, $A' = \begin{bmatrix} O & E \\ B' & C' \end{bmatrix}$, and $C'$ is obtained from $C$ after eventual change some of the rows. Let $r(B) = \langle b_1, b_2, \ldots, b_{m-s} \rangle$, $r(B') = \langle b'_1, b'_2, \ldots, b'_{m-s} \rangle$, $r(C) = \langle c_{m-s+1}, c_{m-s+2}, \ldots, c_m \rangle$, $r(C') = \langle c'_{m-s+1}, c'_{m-s+2}, \ldots, c'_m \rangle$, $r(A') = \langle x'_1, x'_2, \ldots, x'_n \rangle$. Obviously $x'_i = x_i = 2^s - 1$ for each $i = 1, 2, \ldots, t$. Because $B'$ is canonical, and $0 \leq c_k, c'_k < 2^s$ for each $k \in [m-s+1, m]$ there exist $i \in \{ t+1, t+2, \ldots, n \}$ such that $b'_1 = b_1, b'_2 = b_2, \ldots, b'_{i-1} = b_{i-1}$ and $b'_i < b_i$. Then $x'_1 = x_1, x'_2 = x_2, \ldots, x'_{i-1} = x_{i-1}$ and $x'_i = b'_i 2^s + c'_i \leq b_i 2^s + c_i = x_i$. Consequently $r(A') \leq r(A)$. But $A$ is canonical, i.e. $r(A) \leq r(A')$. Therefore $A' = A$, from where $B' = B$ and $B$ is canonical.

Sufficiency. Let $A \in \mathcal{B}_{n \times n}$ satisfy the conditions 1 and 7 and let $A' \in \mathcal{B}_{n \times m}$ be a canonical matrix, $A' \sim A$. Since the conditions $1 \sim 7$ are necessary for the canonicity of a matrix, consequently $A'$ also satisfies these conditions.

For $A' \sim A$ and having in mind conditions 1 and 7 it is easy to see that

$$\varepsilon_1(A') = \varepsilon_1(A) = s \quad \text{and} \quad \zeta_1(A') = \zeta_1(A) = t. \quad (5)$$

If $s = m$, according to Proposition 6 the matrix $A$ is canonical. If $t = n$, according to Proposition 7 the matrix $A$ is canonical.

Let $1 \leq s < m$, $1 \leq t < n$. In this case conditions 1 and 7, Proposition 5 and equations 6 guarantee that $A$ and $A'$ are presented in the type

$$A = \begin{bmatrix} O & E \\ B & C \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} O & E \\ B' & C' \end{bmatrix}, \quad (6)$$

where $O$ is $t \times (m-s)$ matrix, all elements of which are equal to $0$, $E$ is $t \times s$ matrix, all elements of which are equal to $1$, $B, B' \in \mathcal{B}_{(n-t) \times (m-s)}$ as the first rows of $B$ and $B'$ are not entirely null and $C, C' \in \mathcal{B}_{(n-t) \times s}$.

According to conditions 5 and 7, and the equations 6, it is easy to see that if $c(A) = \langle y_1, y_2, \ldots, y_m \rangle$ and $c(A') = \langle y'_1, y'_2, \ldots, y'_m \rangle$, then $y_1 \leq y_2 \leq \cdots \leq y_m$ and $y'_1 \leq y'_2 \leq \cdots \leq y'_m$. Therfore the matrices $A$ and $A'$ are semi-canonical.
We assume that $A' \neq A$ and $A'$ we obtain that the change of $A$ of some of the columns. Let as change the places of column with numbers $k$ and $l$, $1 \leq k < l \leq m$. The inequality $m - s < k < l \leq m$, where $s = \varepsilon_1(A) = \varepsilon_1(A')$ is impossible due the condition [5]. The inequality $1 \leq k < l \leq m - s$ is impossible due the condition [7]. Consequently $1 \leq k \leq m - s < l \leq m$. But then having in mind [10], it is easy to see that in this case it is necessary also to change the places of some of the rows of $A$.

Let $A'$ be obtained after changing the places of some of the rows and afterwards possibly of some of of the columns of $A$. Let us change the places of the rows with the number $i$ and $j$ of the matrix $A$, where $1 \leq i < j \leq n$. If $1 \leq i < j \leq t = \zeta_1(A) = \zeta_1(A')$, the change of these rows does not lead to alteration of the matrix. If $t < i < j \leq n$, then the condition [7] will be broken. So $1 \leq i \leq t < j \leq n$. According to conditions [2] and [3] $\varepsilon_j(A) = \varepsilon_j(A') = s$ and $\zeta_j(A) = \zeta_j(A') = t$. Consequently we have changed the place of the first equal to each other $t = \zeta_1(A)$ rows with another equal to each rows of the set $\mathcal{Z}_j(A)$. After that in order to obtain a matrix of kind [4] it is necessary to change the places of some columns of the matrix $A$. But this contrary to condition [6] and to the assumption that $A' \neq A$ and $A'$ is canonical.

Therefore $A = A'$, i.e. $A$ is canonical.

\[\blacksquare\]

4 Conclusions and future work

The formulation of Theorem [2] is a good basis for the creation of an algorithm receiving all $n \times m$ canonical binary matrices, which on the other hand describe all bipartite graphs (Proposition [1]) of the type $g = \langle R_g, C_g, E_g \rangle$ up to isomorphism, where $|R_g| = n, C_g = m$. We will have to settle this problem in the near futures. This paper will be very useful for its solving.

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