On the WDVV Equation and \( M \)-Theory

J. M. Isidro

Dipartimento di Fisica “G. Galilei”, Via F. Marzolo 8, 35131 Padova, Italy.
isidro@pd.infn.it

Abstract

A wide class of Seiberg–Witten models constructed by \( M \)-theory techniques and described by non-hyperelliptic Riemann surfaces are shown to possess an associative algebra of holomorphic differentials. This is a first step towards proving that also these models satisfy the Witten–Dijkgraaf–Verlinde–Verlinde equation. In this way, similar results known for simpler Seiberg–Witten models (described by hyperelliptic Riemann surfaces and constructed without recourse to \( M \)-theory) are extended to certain non-hyperelliptic cases constructed in \( M \)-theory. Our analysis reveals a connection between the algebra of holomorphic differentials on the Riemann surface and the configuration of \( M \)-theory branes of the corresponding Seiberg–Witten model.
1 Introduction

Seiberg–Witten (SW) models in 4 and more dimensions have received renewed attention in the context of $M$-theory and geometric engineering. The elegant techniques developed allow the construction of a much larger class of SW models than had been known previously. The basic elements used in are certain configurations of Dirichlet 4-branes and solitonic 5-branes of type IIA string theory, lifted to 11-dimensional $M$-theory. The generalisations of SW models so obtained lie along different directions. First, the gauge group $G$ need no longer be simple, and thus it can now be taken to be a product of several simple factors, $G = G_1 \times \cdots \times G_n$. Second, a large family of SW models with vanishing beta function can be generated by the inclusion of Dirichlet 6-branes. Finally, upon compactification of one spatial dimension, a class of SW models can now be constructed whose Coulomb branch is described by coverings of a torus (elliptic models). In the seminal work of, the gauge group was a product of $SU(N)$ factors; see also. Orthogonal and symplectic gauge groups, and products thereof, were studied by including 4- and 6-orientifolds. The inclusion of 6-orientifolds further allows one to consider matter hypermultiplets in representations other than the fundamental.

Along different lines, a generalisation of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation of topological field theory has been shown to hold in the (apparently) unrelated context of SW models. There is in fact a deep link between topological field theories, integrability and Whitham hierarchies, on the one hand, and SW theories (in various dimensions), on the other. This link has been explored more recently in, also in connection with the matrix model of $M$-theory; some reviews are. Other related issues that have been studied are the structure of the exact Wilsonian effective action beyond the prepotential $F$, instanton expansions, and properties of the beta function and the renormalisation group equation of these models, as well as weak and strong coupling expansions of the prepotential $F$. Related interest in the properties of SW models derived from $M$-theory has been recently expressed in.

For the simple case of a SW model with an $SU(3)$ gauge group, the WDVV equation satisfied by the prepotential $F$ was first established in. This was done from a study of the Picard–Fuchs (PF) equations governing the electric and magnetic periods $a$ and...
the latter are related to the prepotential through the equation $a_D = \partial F / \partial a$. This same approach has been undertaken more recently in [28], in order to extend it to an arbitrary simple gauge group. An alternative line was developed in [29], where the WDVV equation satisfied by a wide class of SW models was established from an analysis of the algebra of differential forms on the Riemann surface describing the corresponding SW model; see [30] for connected topics. The WDVV equation has also been extended in order to include the quantum scale $\Lambda$ of the effective gauge theory [31]; related issues have been addressed in [32]. It also appears to have an application in the theory of Donaldson–Witten invariants of 4-manifolds [33].

A common feature to all the approaches mentioned in the preceding paragraph is that they rely on a technical assumption concerning the Riemann surface $\Sigma_g$ that governs the SW model in question. Namely, the surface $\Sigma_g$ must be hyperelliptic, i.e., it must be a 2-fold branched covering of the Riemann sphere $\mathbb{CP}^1$ [34, 35]. From a physical point of view, this corresponds to the case of a simple (classical) gauge group $G$, possibly including matter hypermultiplets, but always in the fundamental representation. Such was the case of the “old” SW models, as described in [1] and [5]. The advent of M-theory and geometric engineering has made it possible to lift these hypotheses, as explained above, in order to consider products of gauge groups, or matter hypermultiplets in non-fundamental representations. However, even in those cases where the Coulomb branch of moduli space continues to be described by a family of Riemann surfaces $\Sigma_g$, the latter are typically non-hyperelliptic, i.e., they are $n$-fold coverings of the Riemann sphere $\mathbb{CP}^1$ with $n > 2$ [34, 35]. It thus seems natural to ask if the prepotentials $F$ governing these more general SW models constructed in M-theory continue to satisfy the WDVV equation.

It is the purpose of this paper to answer the above question in the affirmative, at least for a large family of generalised 4-dimensional SW models to be made precise below. The requirement of hyperellipticity of $\Sigma_g$ can be lifted under certain assumptions that appear very naturally in an M-theory context. In the absence of explicit expressions for the PF equations of these generalised SW models, our analysis is based on a study of the algebra of holomorphic 1-forms on $\Sigma_g$, along the lines of [29]. Once the algebra has been established, the “residue formula” [29] provides the passage to the WDVV equation; we reserve a proof of such a formula for an upcoming publication [36].
This paper is organised as follows. In section 2 we briefly review the techniques of [29] to be applied in later sections. Section 3 is devoted to an analysis of the simplest non-hyperelliptic SW models: those constructed solely with $M$-theory 4- and 5-branes [4] and, possibly, 4-orientifolds as well [7, 9]. In all these cases we give an explicit construction of an associative algebra of holomorphic differentials. In section 4 we present a simple “dictionary” that allows one to read off a number of properties of the Riemann surface $\Sigma_g$ from a knowledge of the $M$-theory brane configuration giving rise to the SW model in question. In retrospective, this allows one to explain why the algebra of differentials holds for the models of section 3. Applying the same techniques we examine in section 5 two new families of SW models whose construction requires 6-branes [4] and/or 6-orientifolds [8]. In neither case is it possible to define an associative algebra of differentials on the surface $\Sigma_g$ following the pattern of previous sections. Finally, in section 6 we summarise our work and present some concluding remarks.

2 Formulation of the problem

To begin with, let us briefly review the derivation of the WDVV equation following the approach of [29].

2.1 Non-hyperelliptic Riemann surfaces

Consider a connected, compact Riemann surface $\Sigma_g$ of genus $g$. As such, it will be an $n$-fold covering of the Riemann sphere $\mathbb{C}P^1$, for a certain $n \geq 2$. We call $v$ a local coordinate on $\mathbb{C}P^1$, while $t$ will denote a local coordinate on $\Sigma_g$. The latter can be understood as the vanishing locus in $\mathbb{C}P^2$ of an irreducible polynomial $F(t, v)$,

$$F(t, v) = \sum_{j=0}^{n} p_j(v) t^{n-j} = 0,$$

where the $p_j(v)$ are certain polynomials in $v$. Branching points are the simultaneous solutions of the algebraic equations

$$F(t, v) = 0, \quad F_t(t, v) = 0,$$

where $F_t$ denotes $\partial F/\partial t$. 

3
The space $\Omega^{(1,0)}(\Sigma_g)$ of holomorphic differentials on $\Sigma_g$ is $g$-dimensional; let $\{\omega_j, j = 1, 2, \ldots, g\}$, denote a basis. Any $\Sigma_g$ with $g \leq 2$ is necessarily hyperelliptic, and thus falls into the special case of section 2.2. For $g \geq 3$, pick three independent 1-forms $\phi_k \in \Omega^{(1,0)}(\Sigma_g)$, $k = 1, 2, 3$, and consider the algebra
\[
\omega_i \omega_j = C^k_{ij} \omega_k \phi_1 + D^k_{ij} \omega_k \phi_2 + E^k_{ij} \omega_k \phi_3. \tag{2.3}
\]
The products $\omega_i \omega_j$ are not linearly independent, as they belong to the space $\Omega^{(2,0)}(\Sigma_g)$ of quadratic holomorphic differentials on $\Sigma_g$. The latter is $(3g - 3)$-dimensional. Equation (2.3) expresses the decomposition $\Omega^{(2,0)}(\Sigma_g) \simeq \Omega^{(1,0)}(\Sigma_g) \cdot (\phi_1 + \phi_2 + \phi_3)$ in a particular basis. For $i$ and $j$ given, there are $3g$ parameters $C^k_{ij}$, $D^k_{ij}$ and $E^k_{ij}$ to adjust in equation (2.3), minus 3 zero modes, which matches the value of $\dim \Omega^{(2,0)}(\Sigma_g)$. This exact match proves the existence and uniqueness of the algebra (2.3) of holomorphic 1-forms on $\Sigma_g$.

In what follows we will mod out in the equation above by the last two terms, $D^k_{ij} \omega_k \phi_2 + E^k_{ij} \omega_k \phi_3$. This factor algebra be denoted symbolically by
\[
\omega_i \omega_j = C^k_{ij} \omega_k \phi_1 \mod (\omega_k \phi_2, \omega_k \phi_3). \tag{2.4}
\]
Now this factor algebra need not be associative. The condition of associativity,
\[
0 = (\omega_i \omega_j) \omega_k - \omega_i (\omega_j \omega_k) = (C^l_{ij} C^m_{lk} - C^m_{il} C^l_{jk}) \omega_m (\phi_1)^2 \mod (\omega_k \phi_2, \omega_k \phi_3), \tag{2.5}
\]
is equivalent to the statement that the matrices $C_i$ whose $(j, k)$ entries are $C^k_{ij}$ commute:
\[
[C_i, C_j] = 0. \tag{2.6}
\]
In the same vein as above one can perform a counting of the free parameters in equation (2.5) and compare it with $\dim \Omega^{(3,0)}(\Sigma_g) = 5g - 5$, the dimension of the space of cubic holomorphic differentials on $\Sigma_g$. It turns out that the number of free parameters to be adjusted is $6g - 8$, which does not match $\dim \Omega^{(3,0)}(\Sigma_g)$. So, in general, associativity breaks down, unless there is some special reason for it to survive.

### 2.2 Hyperelliptic Riemann surfaces

There are special cases when one can still define an associative factor algebra of holomorphic differentials. The resulting algebra will be similar, but not exactly equal, to that in equation...
(2.4). One such case is that in which $\Sigma_g$ is hyperelliptic, i.e., when the number of sheets in the covering is 2. For these surfaces we have [34, 35], after a suitable change of variables in equation (2.1),

$$t^2 = p(v) = \sum_{i=0}^{2g+2} u_i v^{2g+2-i}. \quad (2.7)$$

An explicit basis of $\Omega^{(1,0)}(\Sigma_g)$ is given by the holomorphic 1-forms

$$\omega_j = \frac{v^j}{t} dv, \quad j = 0, 1, \ldots, g - 1. \quad (2.8)$$

From equation (2.7) it is obvious that $\sigma : (t, v) \rightarrow (-t, v)$ is an involution of $\Sigma_g$. We use the subindices $+$ and $-$ to denote the even and odd subspaces of $\Omega^{(n,0)}(\Sigma_g)$, for $n = 1, 2$ and 3. Equation (2.8) implies that $\Omega^{(1,0)}(\Sigma_g) = \Omega^{(1,0)}_-(\Sigma_g)$, i.e., all holomorphic 1-forms are odd under $\sigma$. We now set $E_{ij}^k = 0$ in equation (2.3), and define the algebra of differentials through

$$\omega_i \omega_j = C^k_{ij} \omega_k \phi_1 \mod (\omega_k \phi_2), \quad (2.9)$$

where $\phi_1, \phi_2 \in \Omega^{(1,0)}_-(\Sigma_g)$. Now the multiplication operation takes $\Omega^{(1,0)}_-(\Sigma_g)$ into $\Omega^{(2,0)}_+(\Sigma_g)$, whose dimension is $2g - 1$. Further multiplication by $\Omega^{(1,0)}_+(\Sigma_g)$ takes us into $\Omega^{(3,0)}_-(\Sigma_g)$, whose dimension is $3g - 2$. One can check [29] that these dimensions exactly match the number of free parameters required to define an associative algebra in equation (2.9). Thus the hyperelliptic involution $\sigma$ guarantees the existence and associativity of the algebra of differentials. Finally, one can reexpress the associativity condition given in equation (2.6) as the WDVV equation [29]:

$$F_i F_k^{-1} F_j = F_j F_k^{-1} F_i. \quad (2.10)$$

This proves that the WDVV equation holds in the “old” SW models of [1, 5], as they were all described by hyperelliptic surfaces $\Sigma_g$ when $G$ was a classical, simple gauge group.

### 3 SW models with 4- and 5-branes

#### 3.1 Unitary gauge groups

As a first example of a non-hyperelliptic SW model, let us consider the product gauge group $\prod_{\alpha=1}^n SU(k_\alpha)$, with matter hypermultiplets transforming in the representation $\sum_{\alpha=1}^{\alpha=n-1}(k_\alpha, \bar{k}_{\alpha+1})$. 
As shown in [4], the configuration of M-theory branes that produces this model is a chain of \( n + 1 \) parallel 5-branes labelled from 0 to \( n \), with \( k_\alpha \) 4-branes connecting the \( \alpha - 1 \) and \( \alpha \)-th 5-branes, for \( \alpha = 1, \ldots, n \). No semi-infinite 4-branes are assumed at either end of the chain of 5-branes. The family of surfaces \( \Sigma_g \) describing the Coulomb branch of the moduli space of this theory is [4]

\[
F(t, v) = \sum_{\alpha=0}^{n+1} p_{k_\alpha}(v) t^{n+1-\alpha} = 0, \tag{3.1}
\]

where the polynomials \( p_{k_\alpha}(v) \) are given by

\[
p_{k_\alpha}(v) = \sum_{j=0}^{k_\alpha} u^{(\alpha)}_{j} v^{k_\alpha-j}, \quad \alpha = 0, 1, \ldots, n + 1, \tag{3.2}
\]

and the genus is

\[
g = \sum_{\alpha=1}^{n} (k_\alpha - 1). \tag{3.3}
\]

The degrees \( k_\alpha \) satisfy the condition \( 1 < k_1 \leq k_2 \leq k_3 \leq \ldots \leq k_n \); this ensures that the coefficient \( b_{0,\alpha} \) of the 1-loop beta function of \( SU(k_\alpha) \), \( b_{0,\alpha} = -2k_\alpha + k_{\alpha+1} + k_{\alpha-1} \), is negative or zero for all \( \alpha \). The inexistence of semi-infinite 4-branes at either end of the chain of 5-branes implies that \( k_0 = 0 = k_{n+1} \). For every value of \( \alpha \), the leading coefficient \( u_0^{(\alpha)} \) of \( p_{k_\alpha}(v) \) is identified with the gauge coupling constant of the factor \( SU(k_\alpha) \), while \( u_1^{(\alpha)} \) determines the hypermultiplet bare mass. The \( u_j^{(\alpha)} \) for \( j = 2, 3, \ldots, k_\alpha \) are a set of moduli on the Coulomb branch of the \( SU(k_\alpha) \) factor of the gauge group.

The SW differential \( \lambda_{SW} \) is given by [37, 38]

\[
\lambda_{SW} = v \frac{dt}{t}. \tag{3.4}
\]

Its derivatives with respect to the moduli \( u_j^{(\alpha)}, j = 2, 3, \ldots, k_\alpha \) are holomorphic on \( \Sigma_g \) [1, 5]. A straightforward computation shows that

\[
\omega_j^{(\alpha)} = \frac{\partial \lambda_{SW}}{\partial u_j^{(\alpha)}} = -\frac{1}{F_t} t^{n-\alpha} v^{k_\alpha-j} dv + d(*), \quad j = 2, 3, \ldots, k_\alpha, \quad \alpha = 1, 2, \ldots, n, \tag{3.5}
\]

and that there are \( g = \sum_{\alpha=1}^{n} (k_\alpha - 1) \) of them.

Following [29], we now turn to an analysis of the algebra of the holomorphic differentials defined in equation (3.3). Let \( \omega_i^{(\alpha)} \) and \( \omega_j^{(\beta)} \) be given. Tentatively we set the product \( \omega_i^{(\alpha)} \omega_j^{(\beta)} \) equal to

\[
\omega_i^{(\alpha)} \omega_j^{(\beta)} = C_{i(\alpha),j(\beta)}^{(\beta)} \omega_i^{(\alpha)} \omega_j^{(\beta)} \mod \frac{t^{2n-\alpha-\beta} p'_{k_\beta}(v)}{F_t^2} (dv)^2. \tag{3.6}
\]
Some comments are in order. Comparing the above with equation (2.4), we are taking \( \phi_2 \) in such a way that \( D^k_{ij} \omega_k^{(\alpha)} \phi_2 = t^{2n-\alpha-\beta} p'_{k\beta} (v) (dv)^2 / (F_t)^2 \). Also, the differential \( \phi_1 \) of equation (2.4) is now chosen to be any \( \omega^{(\alpha)} \) whose numerator, as a polynomial in \( v \), is coprime with \( p'_{k\beta} (v) \); any such \( \omega^{(\alpha)} \) will serve our purposes [29]. A summation over \( l \) is implied in the above equation, but there is no summation over \( \alpha \) or \( \beta \). The product \( \omega_i^{(\alpha)} \omega_j^{(\beta)} \) therefore carries an overall factor of \( t^{2n-\alpha-\beta} \), which is also present on the right-hand side. We can thus clear this common factor and understand the remaining equation as a polynomial in \( v \).

Let us prove that the structure constants \( C_{i(\alpha), j(\beta)}^{l(\beta)} \) are uniquely determined by equation (3.6). The left-hand side is a polynomial of degree \( k_\alpha + k_\beta - 4 \) in \( v \), while \( p'_{k\beta} (v) \) has degree \( k_\beta - 1 \). Hence, for equation (3.6) to hold, the piece containing \( p'_{k\beta} (v) \) on the right-hand side must appear multiplied by a polynomial \( q_{i(\alpha), j(\beta)}(v) \) of degree \( k_\alpha - 3 \) in \( v \). Altogether, on the right-hand side of equation (3.6), the number of coefficients to be determined is \( k_\beta - 1 \) (from the structure constants \( C_{i(\alpha), j(\beta)}^{l(\beta)} \)), plus \( k_\alpha - 2 \) (from the polynomial \( q_{i(\alpha), j(\beta)}(v) \)), which add to a total of \( k_\alpha + k_\beta - 3 \). On the other hand, identifying polynomials of degrees \( k_\alpha + k_\beta - 4 \) on both sides we have \( k_\alpha + k_\beta - 3 \) independent equations at our disposal. As the number of available equations exactly matches the number of unknown coefficients, the structure constants \( C_{i(\alpha), j(\beta)}^{l(\beta)} \) defined in (3.6) exist and are unique.

Equation (3.6) above defines a set of \( k_\alpha - 1 \) matrices \( C_{i(\alpha)} \) with dimensions \( (k_\beta - 1) \times (k_\beta - 1) \), the \( (j(\beta), l(\beta)) \) entry of \( C_{i(\alpha)} \) being equal to the structure constant \( C_{i(\alpha), j(\beta)}^{l(\beta)} \). On the other hand, the algebra defined by equation (3.6) is associative, since it is a polynomial algebra in the variable \( v \). Therefore all these \( k_\alpha - 1 \) matrices commute among themselves, as in equation (2.6):

\[
[C_{i(\alpha)}, C_{r(\alpha)}] = 0. \tag{3.7}
\]

Next we observe that the right-hand side of equation (3.6) contains a sum over the differentials \( \omega_l^{(\beta)} \), for a fixed value of \( \beta \). Hence it is not symmetric under the exchange of \( \alpha \) and \( \beta \). This problem does not occur when the covering is hyperelliptic [29], since surfaces with just two sheets correspond to a simple gauge group. We could just as well have chosen to expand the left-hand side of equation (3.6) in terms of the differentials \( \omega_l^{(\alpha)} \), for fixed \( \alpha \). Repeating the above steps we can define a new set of structure constants, \( \tilde{C} \), through

\[
\omega_j^{(\beta)} \omega_i^{(\alpha)} = \tilde{C}_{j\beta, i(\alpha)} \omega_{l(\alpha)} \omega^{(\beta)} \mod \frac{t^{2n-\alpha-\beta} p'_{k\alpha} (v)}{F_t^2} (dv)^2. \tag{3.8}
\]
For the same reasons as above, equation (3.8) defines a set of $k\beta - 1$ matrices $\tilde{C}_{j(\beta)}$ with dimensions $(k\alpha - 1) \times (k\alpha - 1)$, all of which commute among themselves, as in equation (3.7):

$$[C_{j(\beta)}, C_{s(\beta)}] = 0.$$  \hspace{1cm} (3.9)

We finally define the algebra of holomorphic differentials on the non-hyperelliptic surface $\Sigma_g$ through

$$\omega_i^{(\alpha)} \omega_j^{(\beta)} = \frac{1}{2} \left[ C_{i(\alpha), j(\beta)} \omega_i^{(\beta)} \omega_i^{(\alpha)} + \tilde{C}_{j(\beta), i(\alpha)} \omega_i^{(\alpha)} \omega_i^{(\beta)} \right],$$  \hspace{1cm} (3.10)

with the $C$’s and the $\tilde{C}$’s defined by equations (3.6) and (3.8) above. That the algebra so defined is associative follows from equations (3.7) and (3.9). This definition trivially reduces to the one in [29] when the covering is hyperelliptic, and correctly generalises the concept of an associative algebra of holomorphic differentials to the non-hyperelliptic coverings considered in this section.

### 3.2 Orthogonal and symplectic gauge groups

In this subsection we will analyse some SW models whose gauge group is given by a product of orthogonal and symplectic factors [7]. In order to be specific we will consider a gauge group of the type $Sp(2k_1) \times SO(2k_2) \times \ldots \times Sp(2k_{n-1}) \times SO(2k_n)$, with $n$ even. Somewhat different (though closely related) product gauge groups can be described similarly; see [7] for details. The matter content of this theory will be $n - 1$ half hypermultiplets transforming as $\sum_{\alpha=0}^{n}(2k_\alpha, 2k_{\alpha+1})$, where $2k_\alpha$ denotes the fundamental representation of the corresponding orthogonal or symplectic group $G_\alpha$. With respect to each $G_\alpha$ there is always an even number of half hypermultiplets.

The brane configuration describing this model is a chain of $n + 1$ parallel 5-branes, with a set of $2k_\alpha$ 4-branes stretching between the 5-branes at sites $\alpha - 1$ and $\alpha$. An orientifold 4-plane is placed parallel to the 4-branes, in such a way that every object not lying on top of it must have a mirror image. The 4-orientifold traverses the whole configuration at $v = 0$. In particular, between 5-branes $\alpha - 1$ and $\alpha$ there will be $k_\alpha$ 4-branes “above” the 4-orientifold, and another $k_\alpha$ “below” it. No semi-infinite 4-branes are assumed at the ends of the configuration.

The family of surfaces $\Sigma_g$ describing the Coulomb branch of the moduli space of this
theory is

\[ F(t, v) = \sum_{\alpha=0}^{n/2} p_{k_{2\alpha}}(v^2) t^{n+1-2\alpha} + \sum_{\alpha=0}^{n/2} \left[ v^2 p_{k_{2\alpha+1}}(v^2) + c_{2\alpha+1} \right] t^{n-2\alpha}, \]  

(3.11)

where the polynomials \( p_{k_{\alpha}}(v^2) \) are given by

\[ p_{k_{\alpha}}(v^2) = \sum_{j=0}^{k_{\alpha}} u_{2j}^{(\alpha)} v^{2k_{\alpha} - 2j}, \quad \alpha = 0, 1, \ldots, n + 1, \]  

(3.12)

and the \( c_{2\alpha+1} \) are certain numerical constants irrelevant for our purposes. The inexistence of semi-infinite 4-branes at either end of the brane configuration implies that \( k_0 = 0 = k_{n+1} \), while the \( k_{\alpha} \) when \( \alpha = 1, \ldots, n \) satisfy a constraint imposed by the requirement of asymptotic freedom. Namely, let \( q_{\alpha} = (-1)^{\alpha+1} \) denote the charge of the 4-orientifold “to the left” of the 5-brane at site \( \alpha \). Then the coefficient \( b_{0,\alpha} \) of the 1-loop beta function of the group factor \( G_{\alpha} \) is proportional to \( a_{\alpha} - a_{\alpha-1} \), where \( a_{\alpha} = 2k_{\alpha+1} - 2k_{\alpha} - 2q_{\alpha} \). Asymptotic freedom therefore requires that \( a_0 \geq a_1 \geq \ldots \geq a_{n-1} \geq a_n \). Furthermore, for every value of \( \alpha \), the leading coefficient \( u_{0}^{(\alpha)} \) of the polynomial \( p_{k_{\alpha}}(v^2) \) is interpreted as the gauge coupling constant of the group factor \( G_{\alpha} \), while the \( u_{2j}^{(\alpha)} \) for \( j = 1, 2, \ldots, k_{\alpha} \) are a set of moduli on the Coulomb branch of moduli space. Contrary to the case of unitary gauge groups, all hypermultiplet bare masses are zero.

The genus \( g \) of the family of surfaces \( \Sigma_g \) defined in equations (3.11) and (3.12) can be easily computed with the aid of the Riemann–Hurwitz formula [34, 35]. One finds

\[ g = \sum_{\alpha=1}^{n} (2k_{\alpha} - 1). \]  

(3.13)

This value of \( g \) is greater than \( \sum_{\alpha=1}^{n} k_{\alpha} \), which is the number of independent moduli (i.e., the dimension of the Coulomb branch or, equivalently, the rank of the product gauge group). As explained in [4], one must restrict to a subvariety of the full Jacobian [34, 35] of \( \Sigma_g \) in order to obtain physically meaningful values for the electric and magnetic periods \( a \) and \( a_D \) entering the BPS mass formula. This subvariety is the so-called Prym variety [16] of \( \Sigma_g \), whose dimension is \( 2 \sum_{\alpha=1}^{n} k_{\alpha} \), i.e., twice that of the Coulomb branch of moduli space.

In fact, the surface \( \Sigma_g \) defined by equations (3.11) and (3.12) possesses an involution \( \sigma: (v, t) \rightarrow (-v, t) \). The SW differential given in equation (3.4) is odd under this involution, i.e., \( \sigma(\lambda_{SW}) = -\lambda_{SW} \). Hence the holomorphic differentials generated by modular differenti-
ation of $\lambda_{SW}$ will also be odd under $\sigma$. One finds using equations (3.11) and (3.12)
\[
\omega_j^{(2\alpha+1)} = \frac{\partial \lambda_{SW}}{\partial u_j^{(2\alpha+1)}} = -\frac{1}{F_t} t^{n-1-2\alpha} v^{2k_{2\alpha+1}+2-2j} \, dv + d(*), \quad j = 1, 2, \ldots, k_{2\alpha+1}
\]
\[
\omega_j^{(2\alpha)} = \frac{\partial \lambda_{SW}}{\partial u_j^{(2\alpha)}} = -\frac{1}{F_t} t^{n-2\alpha} v^{2k_{2\alpha}-2j} \, dv + d(*), \quad j = 1, 2, \ldots, k_{2\alpha}.
\]

The above differentials span a basis of the subspace of $\sigma$-odd holomorphic 1-forms. The
dimension of the latter is $\sum_{\alpha=1}^{n} k_\alpha$. These are the differentials that are to be integrated in
order to construct the Prym variety.

Again following [29], we now turn to an analysis of the algebra of the holomorphic differen-
tials in equation (3.14). We have to define the products $\omega_i^{(2\alpha)} \omega_j^{(2\beta)}$, $\omega_i^{(2\alpha)} \omega_j^{(2\beta+1)}$ and
$\omega_i^{(2\alpha+1)} \omega_j^{(2\beta+1)}$. These three cases must be studied separately, although the conclusions turn
out to be the same, so we will just present the details pertaining to the case of $\omega_i^{(2\alpha)} \omega_j^{(2\beta)}$.

We tentatively define it through
\[
\omega_i^{(2\alpha)} \omega_j^{(2\beta)} = C_{i(2\alpha),j(2\beta)}^t \omega_i^{(2\alpha)} \omega_j^{(2\beta)} \mod \frac{t^{2n-2\alpha-2\beta} p_{k_{2\beta}}'(v^2)}{F_t^2} (dv)^2,
\]

where, as in the previous subsection, a summation is implied over $l$, but not over $\alpha$ nor $\beta$. Also, $\omega^{(2\alpha)}$ can be taken to be any differential whose numerator, as a polynomial in $v$, is co-
prime with $p_{k_{2\beta}}'(v^2)$. The product $\omega_i^{(2\alpha)} \omega_j^{(2\beta)}$ carries an overall factor of $t^{2n-2\alpha-2\beta} (dv)^2/(F_t)^2$,
which is also present on the right-hand side. We can clear this common factor and understand
the remaining equation as a polynomial in $v$.

We first observe that the left-hand side is a polynomial of degree $2k_{2\alpha} + 2k_{2\beta} - 4$ in $v$, while $p_{k_{2\beta}}'(v^2)$ has degree $2k_{2\beta} - 1$. Hence, for equation (3.15) to hold, the piece containing
$p_{k_{2\beta}}'(v^2)$ on the right-hand side must appear multiplied by a polynomial $q_{i(2\alpha),j(2\beta)}(v)$ of
degree $2k_{2\alpha} - 3$ in $v$. Altogether, on the right-hand side of equation (3.15), the number of
coefficients to be determined add to a total of $2k_{2\alpha} + 2k_{2\beta} - 3$. There is a contribution of
$2k_{2\alpha} - 2$ to this quantity from the polynomial $q_{i(2\alpha),j(2\beta)}(v)$, while the structure constants
$C_{i(2\alpha),j(2\beta)}^t$ contribute $2k_{2\beta} - 1$. This latter number comes from the fact that, although there are only $k_{2\beta}$ $\sigma$-odd differentials $\omega_i^{(2\beta)}$, one must also impose the condition that all $\sigma$-even
terms vanish. On the other hand, identifying polynomials of degrees $2k_{2\alpha} + 2k_{2\beta} - 4$ on
both sides we have $2k_{2\alpha} + 2k_{2\beta} - 3$ independent equations at our disposal. As the number
of available equations exactly matches the number of unknown coefficients, the structure
constants $C^{(2\beta)}_{i(2\alpha), j(2\beta)}$ exist and are unique. They define a set of $2k_{2\alpha} - 1$ commuting matrices $C_{i(2\alpha)}$ of dimensions $(2k_{2\beta} - 1) \times (2k_{2\beta} - 1)$, as corresponds to an associative algebra.

Next we define a new set of structure constants $\tilde{C}^{(2\alpha)}_{j(2\beta), i(2\alpha)}$ by exchanging the indices $\alpha$ and $\beta$ above:

$$\omega^{(2\beta)}_j \omega^{(2\alpha)}_i = \tilde{C}^{(2\alpha)}_{j(2\beta), i(2\alpha)} \omega^{(2\alpha)}_i \omega^{(2\beta)}_j \mod \frac{t^{2n-2\alpha-2\beta} p'_{k_{2\alpha}}(v^2)}{F^2_t} (dv)^2. \quad (3.16)$$

Again this defines a set of $2k_{2\beta} - 1$ commuting matrices $\tilde{C}^{(2\alpha)}_{j(2\beta), i(2\alpha)}$ of dimensions $(2k_{2\alpha} - 1) \times (2k_{2\alpha} - 1)$. Finally, the complete product $\omega^{(2\alpha)}_i \omega^{(2\beta)}_j$ is defined as the half-sum of the right-hand sides of equations (3.15) and (3.16). As in the previous subsection, the algebra so defined is associative.

### 4 Brane configuration and structure of $\Sigma_g$

We have found in section 3 that an associative factor algebra of holomorphic differentials can be defined on the non-hyperelliptic Riemann surfaces describing certain families of generalised SW models. The algebra satisfied conforms to the pattern of equation (2.9), which was seen in section 2 to be the algebra of hyperelliptic surfaces. There might seem to be an inconsistency between the conclusions of sections 2 and 3. We devote this section to a resolution of this apparent puzzle. As it turns out, there is an intimate link between the brane configuration that gives rise to the SW model in question, the structure of its corresponding Riemann surface $\Sigma_g$, and the possibility of defining an associative factor algebra of holomorphic differentials following equation (2.9). For the sake of simplicity, we will concentrate for the rest of this section on the case of unitary gauge groups dealt with in section 3.1. This dispenses with the need to project onto a certain subspace of differentials or, equivalently, onto a certain Prym subvariety. However, it will become clear that our conclusions can be easily generalised to all the models dealt with in the previous section.

If $n > 1$, the surface $\Sigma_g$ as defined by equations (3.1) and (3.2) is non-hyperelliptic. According to [29], the algebra of differentials (2.4) always exists, but it is not guaranteed to be associative. However, the structure of $\Sigma_g$ is such that it allows one to establish an associative factor algebra of holomorphic 1-forms, in a way that closely resembles the hyperelliptic case of equation (2.9). In fact we have already exhibited the algebra; it remains to explain why
it can be established. We will do so using two alternative, though substantially equivalent arguments. The first one provides a dictionary that allows one to read off a number of properties of the Riemann surface from the underlying brane configuration. The second argument, more concise, relies on a counting of moduli.

Let us review some properties of $\Sigma_g$ from the construction of this model in [4]. There are as many sheets in the covering as there are 5-branes, so $\Sigma_g$ is an $(n+1)$-fold covering of the base $\mathbb{C}P^1$. Every sheet of $\Sigma_g$ is a copy of the complex $v$-plane $\mathbb{C}$, later compactified to $\mathbb{C}P^1$. Assume factorising $F(t,v)$ as $\prod_{\alpha=0}^n (t - t_\alpha(v))$. Then $t_\alpha(v)$ is a local coordinate on the $\alpha$-th sheet. There is a branching between adjacent sheets at sites $\alpha - 1$ and $\alpha$ whenever the coordinate $v$ on the base $\mathbb{C}P^1$ is such that $t_{\alpha-1}(v) = t_\alpha(v)$ for that particular value of $v$. This indicates the presence of a 4-brane with coordinate $v$; there are $k_\alpha$ such values of $v$, all different, each corresponding to one of the $k_\alpha$ 4-branes that stretch between sheets $\alpha - 1$ and $\alpha$.

No single 4-brane can connect non-adjacent sheets, i.e., sheets $\alpha - 1$ and $\alpha - 1 + s$ for $s > 1$. However, a branching between sheets $\alpha - 1$ and $\alpha - 1 + s$ for $s > 1$ can occur if $s$ 4-branes are positioned as follows. For a fixed $v_0$ on the base $\mathbb{C}P^1$, it must hold that $t_{\alpha-1}(v_0) = t_\alpha(v_0) = \ldots = t_{\alpha-1+s}(v_0)$. In this case, for every $r = 0, 1, \ldots, s - 1$, one 4-brane out of the $k_{\alpha+r}$ between sheets $\alpha - 1 + r$ and $\alpha + r$ has a projection $v_0$ on the base $\mathbb{C}P^1$. All these $s$ 4-branes lie “one after another”, thus producing a branching between $s + 1$ sheets of $\Sigma_g$, with a branching index $B = s + 1$.

It is clear that if non-adjacent sheets $\alpha - 1$ and $\alpha - 1 + s$ for some $s > 1$ are branched together, then it must be in the manner just described. In particular, all intermediate sheets $\alpha, \alpha + 1, \ldots, \alpha - 2 + s$ will be involved in the branching; none of them are bypassed. It also holds that $v = \infty$ is not a branching point; this follows from the compactification of each sheet of $\Sigma_g$ [4]. We will also assume that there is no branching at $v = 0$.

For later purposes it will be instructive to compute the genus $g$. This we do with the aid of the Riemann–Hurwitz formula [34, 35]. For an $(n + 1)$-fold covering of $\mathbb{C}P^1$, it holds that

$$
\sum_{p \in \Sigma_g} (B(p) - 1) = 2g + 2n,
$$

(4.1)

where $B(p)$ denotes the branching index at point $p \in \Sigma_g$. The summand $(B(p) - 1)$ vanishes except at a finite number of points (branching points) [34, 35]. Let us first consider a situation
in which \( B(p) = 2 \) at all branching points. This corresponds to a brane configuration in which no 4-branes lie one after another in the manner described above. Hence in this case the sum \( \sum_{p \in \Sigma_g}(B(p) - 1) \) equals the total number of branching points. On the \( \alpha \)-th sheet there are \( k_\alpha \) 4-branes “coming in” and \( k_{\alpha+1} \) 4-branes “going out”, so the total number of branching points on \( \Sigma_g \) is \( \sum_{\alpha=0}^n(k_\alpha + k_{\alpha+1}) = 2 \sum_{\alpha=1}^n k_\alpha \). From here we conclude \( g = \sum_{\alpha=1}^n (k_\alpha - 1) \) as in [4]. This value of the genus stays the same if the requirement that the branching index be \( B = 2 \) is lifted. There is then a decrease in the number of branching points, but it is compensated by an equal increase in the branching index \( B \).

Now let \( \Sigma_g \) be non-singular, i.e., assume that the derivatives \( F_t \) and \( F_v \) never vanish simultaneously on \( F = 0 \) \([34, 35]\), and consider the 1-forms given by

\[
\phi_j^{(n)} = \frac{v^{k_\alpha-j}}{\partial^{n-\alpha+1} F/\partial t^{n-\alpha+1}} dv, \quad j = 2, 3, \ldots, k_\alpha, \quad \alpha = 1, 2, \ldots, n. \tag{4.2}
\]

For \( \alpha = n \), the holomorphicity of \( \phi_j^{(n)} \) on \( \Sigma_g \) when \( j = 2, 3, \ldots, k_n \) follows simply from the fact that \( \phi_j^{(n)} = -\omega_j^{(n)} \), as per equations (3.3) and (4.2). However, let us provide an alternative argument that will be useful in what follows.

The simultaneous equations \( F_t = 0 \) and \( F = 0 \) hold at the branching points of \( \Sigma_g \). Now, from \( dF = F_v dv + F_t dt = 0 \) and the assumption of non-singularity, whenever \( F_t = 0 \) on \( F = 0 \) we can write \( dv/F_t = -dt/F_v \), with \( F_v \neq 0 \). This alternative expression for \( \phi_j^{(n)} \) proves that it has no poles at finite points \( v \neq 0 \). If it has any poles at all, then they will be at \( v = 0 \) or \( v = \infty \). In fact one can prove that the divisor \([\phi_j^{(n)}]\) is given by

\[
[\phi_j^{(n)}] = (k_n - j)(0_{\alpha-1} + 0_n) + \left( \sum_{\alpha=1}^{n-1} k_\alpha + j - (n + 1) \right)(\infty_{n-1} + \infty_n), \tag{4.3}
\]

where \( 0_\alpha \) (respectively, \( \infty_\alpha \)) denotes the point on the \( \alpha \)-th sheet of the covering \( F = 0 \) lying above \( v = 0 \) (respectively, \( v = \infty \)) on the base \( \mathbb{CP}^1 \). In our conventions, zeroes (respectively, poles) carry positive (respectively, negative) coefficients in the divisor. A proof of equation (4.3) is given in the appendix.

Now for any meromorphic 1-form \( \varphi \) on \( \Sigma_g \) it holds that \([14, 33]\)

\[
\sum_{p \in \Sigma_g} \text{ord}_p (\varphi) = 2g - 2, \tag{4.4}
\]

i.e., the zeroes minus the poles of \( \varphi \) must equal \( 2g - 2 \). The divisor \([\phi_j^{(n)}]\) satisfies this
requirement since, by equation (4.3),
\[ 2(k_n - j) + 2\left(\sum_{\alpha=1}^{n-1} k_{\alpha} + j - (n + 1)\right) = 2g - 2. \] (4.5)

Hence $\phi_j^{(n)}$ is holomorphic on the surface $F(t, v) = 0$ precisely when $j = 2, 3, \ldots, k_n$. However, when $\alpha < n$, the $\phi_j^{(\alpha)}$ are not a priori assured to be holomorphic on $\Sigma_g$.

Let us observe that, if the surface $F(t, v) = 0$ corresponds to the configuration of $n$ 5-branes, then the operation of taking the derivative $\partial / \partial t$ corresponds to the removal of the 5-brane at site $\alpha = n + 1$ (the one “farthest to the right” in the conventions of [3]). So the surface $\partial F / \partial t = 0$ describes a configuration of $n$ 5-branes, with $k_n$ semi-infinite 4-branes to the right of the $n$-th 5-brane. Applying the Riemann–Hurwitz formula of equation (4.1), its genus turns out to be $g_{n-1} = \sum_{\alpha=1}^{n-1} (k_{\alpha} - 1)$. Similarly, the second derivative $\partial^2 F / \partial t^2 = 0$ describes a configuration of $n - 1$ 5-branes, with $k_{n-1}$ semi-infinite 4-branes to the right of the $(n - 1)$-th 5-brane, and genus $g_{n-2} = \sum_{\alpha=1}^{n-2} (k_{\alpha} - 1)$. In general, the surface $\partial^l F / \partial t^l = 0$ corresponds to a configuration of $(n - l + 1)$ 5-branes, with $k_{n-l+1}$ semi-infinite 4-branes to the right of the 5-brane at site $(n - l + 1)$, and genus $g_{n-l} = \sum_{\alpha=1}^{n-l} (k_{\alpha} - 1)$. After $l$ derivatives have been taken, the gauge group is $\prod_{\alpha=1}^{n-l} SU(k_{\alpha})$. For $l = n - 1$ we are left with a configuration of just two 5-branes, with $k_1$ 4-branes stretched across them, and $k_2$ semi-infinite 4-branes to the right: this is an $SU(k_1)$ gauge theory with $N_f = k_2$ fundamental flavours. As such it is already hyperelliptic.

Consider now $\alpha = n - 1$. The above arguments establish that $\phi_j^{(n-1)}$, for $j = 2, 3, \ldots, k_{n-1}$, is holomorphic on the surface $\partial F / \partial t = 0$, provided the latter is non-singular. In general, assume that the surfaces $\partial^{n-\alpha} F / \partial t^{n-\alpha} = 0$ are non-singular for all values of $\alpha = n - 1, n - 2, \ldots, 1$. Then, for every fixed value of $\alpha = n - 1, n - 2, \ldots, 1$, the 1-forms $\phi_j^{(\alpha)}$ for $j = 2, 3, \ldots, k_{\alpha}$ are holomorphic on the surface $\partial^{n-\alpha} F / \partial t^{n-\alpha} = 0$. An expression for the divisor $[\phi_j^{(\alpha)}]$ on the surface $\partial^{n-\alpha} F / \partial t^{n-\alpha} = 0$ can be easily given, provided that $v = 0$ is not a branching point on the surface $\partial^{n-\alpha} F / \partial t^{n-\alpha} = 0$. As proved in the appendix, it is given by
\[ [\phi_j^{(\alpha)}] = (k_{\alpha} - j) (0_{\alpha-1} + 0_{\alpha}) + \left(\sum_{l=1}^{\alpha-1} k_l + j - (\alpha + 1)\right) (\infty_{\alpha-1} + \infty_{\alpha}). \] (4.6)

It also satisfies the requirement of equation (4.4), for a value of the genus $g_{\alpha} = \sum_{l=1}^{\alpha} (k_l - 1)$.

If instead of considering the 1-form $\phi_j^{(\alpha)}$ on the surface $\partial^{n-\alpha} F / \partial t^{n-\alpha} = 0$ we consider it
on \( F = 0 \), then its divisor is given by

\[
[\phi_j^{(\alpha)}] = (k_\alpha - j)(0_{\alpha-1} + 0_\alpha) + \left( \sum_{l \neq \alpha}^n k_l + j - (n + 1) \right)(\infty_{\alpha-1} + \infty_\alpha).
\]  

(4.7)

Of course, the actual value of the genus that will now satisfy the requirement of equation (4.4) is

\[ g = g_n = \sum_{l=1}^n (k_l - 1). \]

We observe in the above equation that the coefficients of the divisor \([\phi_j^{(\alpha)}]\) are positive precisely when \( j = 2, 3, \ldots, k_\alpha \) and \( \alpha = 1, 2, \ldots, n \); this ensures holomorphicity of the 1-forms \( \phi_j^{(\alpha)} \) on the surface \( F = 0 \).

This completes the proof that the 1-forms given in equation (4.2) constitute a basis of \( \Omega^{(1,0)}(\Sigma_g) \), under the assumption of simultaneous non-singularity of the \( \partial^{n-\alpha} F/\partial t^{n-\alpha} = 0 \) for all \( \alpha = n - 1, n - 2, \ldots, 1 \). Following [29], it is now immediate to establish an associative factor algebra for the holomorphic differentials defined in equation (4.2). Let \( \phi_i^{(\alpha)} \) and \( \phi_j^{(\beta)} \) be given. For any fixed values of \( \alpha \) and \( \beta \), we first define a set of structure constants \( C \) through

\[
\phi_i^{(\alpha)} \phi_j^{(\beta)} = C_i^{(\alpha)} j^{(\beta)} \phi_l^{(\gamma)} \phi^{(\delta)} \mod \frac{p'_{k_\alpha}(v)(dv)^2}{(\partial^{n-\alpha+1} F/\partial t^{n-\alpha+1})(\partial^{n-\beta+1} F/\partial t^{n-\beta+1})},
\]  

(4.8)

where, as usual, a summation is implied over \( l \), but not over \( \alpha \) nor \( \beta \), and we require that the numerator of \( \phi^{(\alpha)} \) be coprime with \( p'_{k_\alpha}(v) \). It suffices to repeat the argument provided at the end of subsection 3.1 in order to prove that the above equation uniquely defines a set of structure constants \( C \). The whole argument goes through, without the need to cancel any \( t \)-dependent factors from the differentials as done in that subsection. The structure constants \( C \) in the above equation are actually coincident with those of equation (3.6). Next one defines a new set of structure constants \( \tilde{C} \), by simply exchanging \( \alpha \) and \( \beta \) in equation (4.8). Finally, the complete product \( \phi_i^{(\alpha)} \phi_j^{(\beta)} \) is defined as the half-sum of the piece with the \( C \)'s and the piece with the \( \tilde{C} \)'s, as in equation (3.10). Associativity is a simple consequence of the fact that the algebra itself has been reduced to a polynomial algebra.

For future reference in section 5 we make the following observation. In the basis of equation (3.3), for any fixed value of \( \alpha \), we span a subspace of differentials by letting \( j \) run over the range \( j = 2, 3, \ldots, k_\alpha \). We can understand this subspace as contributing by an amount \( k_\alpha - 1 \) to the overall genus \( g \) given in equation (3.3). As \( \alpha \) runs over the range \( 1, 2, \ldots, n \), we can interpret multiplication by the prefactor \( t^{n-\alpha} \) for \( \alpha = 1, 2, \ldots, n \) as taking us from one pair of sheets \((\alpha - 1, \alpha)\) to the next. There is a well-defined explicit dependence
of the basis $\omega_j^{(a)}$ on the variable $t$, namely, a simple monomial $t^{n-\alpha}$. In passing from the $\omega_j^{(a)}$ to the $\phi_j^{(a)}$ as a basis, we are cancelling this explicit dependence in the numerator, at the cost of increasing the order of $t$-derivatives in the denominators of the differentials. It can be done without losing holomorphicity. This property of $\Sigma_g$ follows from the underlying brane configuration. As a consequence, establishing the algebra of differentials in the non-hyperelliptic models of section 3 has been reduced, basically, to that of $n(n-1)/2$ “equivalent hyperelliptic problems”. Every pair of adjacent sheets defines a “hyperelliptic building block”; the structure of $\Sigma_g$ can be understood, roughly speaking, as a superposition of $n$ such blocks.

This brings us to a counting of moduli, in order to clarify the structure of $\Sigma_g$ as a “superposition of hyperelliptics”. The moduli of the models just examined are the order parameters $u_j^{(a)}$ on the Coulomb branch. The latter are associated with the gauge group $G$. Let us start from the $SU(k_1)$ gauge theory described by a hyperelliptic surface with genus $g = k_1 - 1$ and $k_1 - 1$ independent moduli. The addition of the group factor $SU(k_2)$ to the gauge group corresponds to adding one more sheet to the covering, with an additional $k_2 - 1$ new moduli, and a contribution of $k_2 - 1$ to the genus. In general, with respect to the hyperelliptic case, when $G$ was a simple factor, the only new moduli that appear in these non-hyperelliptic models are those associated with a product gauge group $G = G_1 \times \cdots \times G_n$. This is an equivalent statement of the fact, already observed, that the counting of undetermined parameters in the algebra of differentials, versus that of available equations, closely resembles the hyperelliptic case of [29].

It remains to explain, in $M$-theory terms, why any hyperelliptic building block admits an associative algebra of holomorphic differentials. We saw in section 2.2 that this can be traced back to the existence of the hyperelliptic involution $\sigma$. The latter has a very natural interpretation in $M$-theory. Namely, let us recall from [4] that the $4$-brane and the $5$-brane lift to one and the same basic object in $M$-theory. The type IIA $5$-brane on $\mathbb{R}^{10}$ is simply an $M$-theory $5$-brane on $\mathbb{R}^{10} \times S^1$ whose worldvolume, roughly, is located at a point in $S^1$ and spans a 6-manifold in $\mathbb{R}^{10}$. A type IIA $4$-brane is an $M$-theory $5$-brane that is wrapped over the $S^1$. The type IIA configuration of parallel 5-branes joined by 4-branes can be reinterpreted in $M$-theory as a single 5-brane with a more complicated world history. It sweeps out arbitrary values of the first four coordinates $x^0, x^1, x^2, x^3$ of 11-dimensional space–time. It is located at $x^7 = x^8 = x^9 = 0$. In the remaining four coordinates $x^4, x^5, x^6$ and $x^{10}$, which parametrise a
4-manifold $Q \simeq \mathbb{R}^3 \times S^1$, the 5-brane worldvolume spans a two-dimensional surface $\Sigma_g$. Then our coordinates $v$ and $t$ are defined \([4]\) as $v = x^4 + i x^5$ and $t = \exp[-(x^6 + i x^{10})/R]$, where $R$ is the radius of $S^1$. So the hyperelliptic involution $\sigma: (t, v) \rightarrow (-t, v)$ is nothing but the statement that, in its propagation, the $M$-theory 5-brane crosses $x^{10}$ and its diametrically opposed point $x^{10} + \pi R$.

In section 3 we found it convenient to use the basis given by the $\omega_j^{(\alpha)}$. This was natural, as it was the basis obtained by straight modular differentiation of the SW differential $\lambda_{SW}$. The construction given in this section by means of the auxiliary surfaces $\partial^{n-\alpha} F/\partial t^{n-\alpha} = 0$ for $\alpha = 1, 2, \ldots, n-1$ has a simple $M$-theory origin that highlights the similarities between this non-hyperelliptic case and the hyperelliptic surfaces dealt with in \([27]\). In the following section we will exhibit some new SW models where these similarities cease to exist. We will again resort to their $M$-theory construction in order to reveal the effects caused by the loss of these similarities.

## 5 SW models with 6-orientifolds and 6-branes

### 5.1 Models with 6-orientifolds

Let us now study the effect of introducing one 6-orientifold into the brane configuration. Following \([8]\), there are basically two different choices to place it. In the first one, the 6-orientifold is located between the 4-branes and the 5-branes, in such a way that the resulting gauge group is of the type $\Pi_\alpha SU(k_\alpha) \times SO(k_\alpha)$ or $\Pi_\alpha SU(k_\alpha) \times Sp(k_\alpha)$, with a certain hypermultiplet content that typically transforms as a sum of bifundamental (and/or vector) representations. We will not be interested in these configurations. For brevity, we will be interested in placing the 6-orientifold on top of one 5-brane. This will bring us to interesting conclusions without substantial loss of generality.

Specifically, it is known \([8]\) that a SW model with an $SU(N)$ gauge group and one matter hypermultiplet can be generated by the following brane configuration: three parallel 5-branes (labelled $\alpha = 0, 1, 2$), the middle one ($\alpha = 1$) on top of an orientifold 6-plane, with $N$ 4-branes stretched across from $\alpha = 0$ through $\alpha = 2$. The 6-orientifold at $\alpha = 1$ enforces the condition that the configuration be left/right-symmetric with respect to the 5-brane at site $\alpha = 1$. This
implies that the branching index $B$ is 3 at all branching points. When the orientifold 6-plane carries RR charge +4, the matter hypermultiplet turns out to transform in the symmetric representation of $SU(N)$, while it transforms in the antisymmetric if it carries charge $-4$. We will analyse these two cases separately.

We first consider the symmetric representation. The surface is given by

$$F(t, v) = v^2 t^3 + f(v) t^2 + (-1)^N A^{N-2} g(v) t + A^{3N-6} v^2 = 0,$$

where

$$f(v) = \prod_{i=1}^{N} (v - a_i) = \sum_{j=0}^{N} (-1)^{j} u_j v^{N-j}$$

$$g(v) = \prod_{i=1}^{N} (v + a_i) = \sum_{j=0}^{N} u_j v^{N-j}.$$  \tag{5.2}

The $u_j$ for $j = 2, 3, \ldots, N$ are a set of moduli parametrising the Coulomb branch, while $u_1$ is proportional to the bare mass of the hypermultiplet, and $u_0 = 1$.

The surface $\Sigma_g$ defined by equations (5.1) and (5.2) has genus $g = 3N - 2$, as one finds by application of the Riemann–Hurwitz formula of equation (4.1). On the other hand, the dimension of the Coulomb branch is $N - 1$. The full Jacobian of $\Sigma_g$ contains a Prym subvariety that is invariant under the involution $\sigma: (v, t) \rightarrow (-v, \Lambda^{2N-4}/t)$ of the surface [8].

One can check that the SW differential given in equation (3.4) is invariant under $\sigma$, i.e., $\sigma(\lambda_{SW}) = \lambda_{SW}$. Hence the holomorphic differentials obtained by modular differentiation of $\lambda_{SW}$ will also be invariant under $\sigma$. A basis of such $\sigma$-invariant differentials can be obtained with the help of equations (3.4), (5.1) and (5.2). One finds

$$\omega_j =: \frac{\partial \lambda_{SW}}{\partial u_j} = -\frac{1}{F_i} \left[ (-1)^j t + (-1)^N A^{N-2} \right] v^{N-j} \, dv + d(*), \quad j = 2, 3, \ldots, N. \tag{5.3}$$

As in previous sections, let us try to define an algebra of holomorphic differentials following equation (2.9). We first observe from the definition of $F(t, v) = 0$ that there are two apparently inequivalent choices for the term to be modded out, namely $f'(v)/(F_i)^2$ and $g'(v)/(F_i)^2$. In fact these two choices are related by a moduli redefinition, $u_j \rightarrow (-1)^j u_j$, so they are not independent. We tentatively set the product $\omega_i \omega_j$ equal to

$$\omega_i \omega_j = C_{ij} \omega_l \omega \mod \frac{f'(v)}{(F_i)^2} (dv)^2 \tag{5.4}$$

18
and examine whether or not the above equation can uniquely define a set of structure constants $C_{ij}^l$. The left-hand side of equation (5.4) carries a $t$-dependence given by $[(-1)^i t + (-1)^N \Lambda^{N-2}] \cdot [(-1)^j t + (-1)^N \Lambda^{N-2}]$. None of these terms can be cancelled against the prefactor $[(-1)^i t + (-1)^N \Lambda^{N-2}]$ of $\omega_l$, as there is a summation over $l$ on the right-hand side of equation (5.4). As a consequence, if the algebra is to hold, then equation (5.4) must be understood as a polynomial in the two variables $t$ and $v$, once the common factors $(dv)^2/(F_l)^2$ have been cleared. In particular, the left-hand side has degrees $\deg_t(\omega_i \omega_j) = 2$ and $\deg_v(\omega_i \omega_j) = 2N - 4$ in $t$ and $v$, respectively, while those of $f'(v)$ are $\deg_t(f'(v)) = 0$ and $\deg_v(f'(v)) = N - 1$. Let $q_{ij}(t,v)$ denote the polynomial multiplying term modded out on the right-hand side. Then we have $\deg_t(q_{ij}(t,v)) = 2$ and $\deg_v(q_{ij}(t,v)) = N - 3$. A straightforward computation gives $4N - 7$ as the total number of coefficients to be determined if the algebra is to hold. On the other hand, the number of available equations obtained by identification of two polynomials in $t$ and $v$ with respective degrees 2 and $2N - 4$ is $6N - 9$. We have an overdetermined system of equations. The algebra of holomorphic differentials does not exist as defined in equation (5.4).

It is in fact no surprise that we have not been able to define an algebra of holomorphic differentials following the hyperelliptic pattern of equation (2.9). Not only is the surface $\Sigma_g$ defined by equations (5.1) and (5.2) non-hyperelliptic; it also cannot be understood as a superposition of hyperelliptics. This conclusion can be arrived at by a counting of moduli, or by the following argument.

Given that the branching index $B$ is always 3, completing three loops around any one branching point $v_0$ on the base $\mathbb{CP}^1$ takes us from sheet $\alpha = 0$, through sheet $\alpha = 1$, to sheet $\alpha = 2$. This property is reflected in the presence of the prefactor $[(-1)^i t + (-1)^N \Lambda^{N-2}]$ in the differential $\omega_j$ of equation (5.3). This prefactor is no longer a monomial in $t$, as was the case for the models of sections 3.1 and 3.2. Rather, it is a sum of two monomials in $t$. The one of order $t^1$ can be understood as being associated with sheets $\alpha = 0$ and $\alpha = 1$, while that of order $t^0$ can be assigned to sheets $\alpha = 1$ and $\alpha = 2$.

Next let us apply $\partial/\partial t$ to this brane configuration. The resulting surface,

$$\frac{\partial F}{\partial t} = 3v^2 t^2 + 2 f(v) t + (-1)^N \Lambda^{N-2} g(v) = 0,$$

(5.5)

corresponds to an $SU(N - 1)$ gauge theory with $N$ fundamental flavours [39], as a counting
of powers of $v$ and $t$ reveals. Its genus is $g = N - 2$. We also observe that the power $N - 2$ to which the quantum scale $\Lambda$ is raised is indeed the correct one for an $SU(N - 1)$ gauge theory with $N$ matter hypermultiplets in the fundamental representation. It is known that a reduction in the rank of the gauge group can be achieved by taking the double scaling limit. However, we are not taking this limit here. Differentiation with respect to $t$ removes the 5-brane at site $\alpha = 2$, while $N$ semi-infinite 4-branes remain to its right. The latter account for the $N$ fundamental hypermultiplets, but the $N$ remaining 4-branes between the 5-branes at sites $\alpha = 0$ and $\alpha = 1$ are in excess for a gauge group $SU(N - 1)$.

This inconsistency between the brane configuration corresponding to the surface $\partial F/\partial t = 0$ and the SW model that it actually describes can be easily interpreted. It is a consequence of the fact that the initial surface $F = 0$ cannot be understood as a superposition of hyperelliptics. The basic building block of the original model consisted of three 5-branes, plus one 6-orientifold on top of the middle 5-brane to enforce a left/right symmetry with respect to $\alpha = 1$. The effect of this symmetry on the surface $\Sigma_g$ is to enforce a constant branching index $B = 3$, so this property is lost when one 5-brane is removed. However, we should emphasise that our arguments do not prevent the existence of a non-hyperelliptic associative algebra, according to the pattern of equation (2.4).

As a final example we will consider the antisymmetric representation of $SU(N)$. This case is very similar to the previous one, so we will briefly report the final results. The surface is described by

$$F(t, v) = t^3 + \left[v^2 f(v) + 3\Lambda^{N+2}\right] t^2 + \Lambda^{N+2} \left[(-1)^N v^2 g(v) + 3\Lambda^{N+2}\right] t + \Lambda^{3N+6} = 0, \quad (5.6)$$

with $f(v)$ and $g(v)$ given in equation (5.2). As in the symmetric representation, there is an involution $\sigma$ of the surface that is automatically taken account of when considering modular derivatives of the SW differential, since $\sigma(\lambda_{SW}) = \lambda_{SW}$. A basis of $\sigma$-invariant holomorphic differentials on the above surface is found to be

$$\omega_j =: \frac{\partial \lambda_{SW}}{\partial u_j} = -\frac{1}{F_t} \left[(-1)^j t + (-1)^N \Lambda^{N+2}\right] v^{N+2-j} dv + d(*), \quad j = 2, 3, \ldots, N. \quad (5.7)$$

In trying to define an algebra of holomorphic differentials as in equation (5.4), by modding out a term in $f'(v)(dv)^2/(F_t)^2$, one again finds that the prefactors $[(-1)^j t + (-1)^N \Lambda^{N+2}]$ in the forms $\omega_j$ cannot be cancelled. Therefore the algebra, if it exists, must be defined as a
polynomial equation in the two variables $t$ and $v$. The number of undetermined coefficients turns out to be $4N + 5$, while that of available equations is $6N + 3$. Again we have an overdetermined system of equations. Conclusions analogous to those that were found for the symmetric representation continue to hold for the antisymmetric representation as well.

5.2 Models with 6-branes

In this section we extend our analysis to the models of [4] that include 6-branes. The gauge group is $\prod_{\alpha=1}^{n} SU(k_{\alpha})$, with matter hypermultiplets transforming in the sum of bi-fundamental representations $\sum_{\alpha=1}^{n-1}(k_{\alpha}, \overline{k}_{\alpha+1})$. Starting from the same brane configuration as in section 3.1, we place $d_{\alpha}$ 6-branes between the 5-branes at sites $\alpha - 1$ and $\alpha$. This adds $d_{\alpha}$ hypermultiplets in the fundamental representation of $SU(k_{\alpha})$. According to [4], the family of surfaces $\Sigma_g$ describing the Coulomb branch of this theory is

$$F(t, v) = \sum_{\alpha=0}^{n+1} p_{k_{\alpha}}(v) \prod_{s=1}^{\alpha-1} J_{s}^{\alpha-s}(v) t^{n+1-\alpha} = 0,$$

(5.8)

where $p_{k_{\alpha}}(v)$ is given in equation (3.2). The polynomials $J_{s}(v)$ vanish (with multiplicity 1) at the projections $e_{a}$ on the base $\mathbb{CP}^{1}$ of the $d_{\alpha}$ 6-branes that are located between sites $\alpha - 1$ and $\alpha$, i.e.,

$$J_{\alpha}(v) = \prod_{a=1}^{d_{\alpha}} (v - e_{a}).$$

(5.9)

Now, on the above surface, the branchings between sheets are not only effected by the 4-branes, but also by the 6-branes [4]. A 6-brane placed between sheets $\alpha - 1$ and $\alpha$ effects a multiple branching of all sheets with $\beta \geq \alpha$; the number of the latter is therefore equal to the corresponding branching index. The $v$ coordinate of such a branching point on the base $\mathbb{CP}^{1}$ is given by the value $e_{a}$ of the corresponding 6-brane.

Let us compute the genus $g$ of the surface defined by equation (5.8). For the sake of simplicity we will assume that, on the base $\mathbb{CP}^{1}$, no 4-brane ever has the same $v$ coordinate as a 6-brane. This simplifying assumption allows us to write the Riemann–Hurwitz formula of equation (4.1) as

$$g = \frac{1}{2} \sum_{p \in \Sigma_g} (B^{(4)}(p) - 1) + \frac{1}{2} \sum_{p \in \Sigma_g} (B^{(6)}(p) - 1) - n,$$

(5.10)
where the superindices (4) and (6) indicate that the branching is effected by a 4- or a 6-brane, respectively. We immediately see that the genus \( g \) is greater than the one given in equation (3.3). When some of the \( d_\alpha \) are non-vanishing, there is a non-zero contribution \( g^{(6)} \),

\[
g^{(6)} = \frac{1}{2} \sum_{p \in \Sigma_g} (B^{(6)}(p) - 1), \tag{5.11}
\]

to add to the amount \( g^{(4)} \) contributed by the 4-branes,

\[
g^{(4)} = \frac{1}{2} \sum_{p \in \Sigma_g} (B^{(4)}(p) - 1) - n = \sum_{\alpha=1}^{n} (k_\alpha - 1), \tag{5.12}
\]

so that \( g = g^{(4)} + g^{(6)} \). In order to compute the contribution \( g^{(6)} \) we will make the assumption, to be justified presently, that no two 6-branes ever have the same \( v \) coordinate on the base \( \mathbb{CP}^1 \). Then it suffices to know how many 6-branes we have, plus how many 5-branes are placed to the right of any given 6-brane. The first number tells us how many branchings are effected by 6-branes; the second one tells us the corresponding branching index. The precise value of \( g^{(6)} \) so obtained is however immaterial to the discussion. It suffices to know that \( g^{(6)} > 0 \) except when all the \( d_\alpha \) vanish. The contribution \( g^{(6)} \) also vanishes in the limiting case that \( d_\alpha = 0 \) for all \( \alpha = 1, 2, \ldots, n-1 \) and \( d_n \neq 0 \). Then all the 6-branes are located between the last two sheets of \( \Sigma_g \), so they don’t effect any branchings at all. Physically, this is equivalent to having \( d_\alpha = 0 \) for all \( \alpha = 1, 2, \ldots, n \) and having all the 6-branes between the last two sheets replaced with the same number of semi-infinite 4-branes, but now placed to the right of the last sheet \([4]\).

In the presence of more than one 6-brane with the same value of \( v \), the contribution \( g^{(6)} \) of the 6-branes to the overall genus \( g \) decreases. Let us for simplicity take two 6-branes, one placed between sites \( \alpha - 1 \) and \( \alpha \) in the chain of 5-branes, the other one between sites \( \beta - 1 \) and \( \beta \). Without loss of generality we can assume \( \alpha \leq \beta \leq n \). Denote their respective projections on the base \( \mathbb{CP}^1 \) by \( e_\alpha \) and \( e_\beta \). When \( e_\alpha \neq e_\beta \), we have one branching point at \( v = e_\alpha \) with branching index \( B_\alpha = n + 1 - \alpha \), and another one at \( v = e_\beta \) with branching index \( B_\beta = n + 1 - \beta \). The summand \( B(p) - 1 \) in \( g^{(6)} \) therefore receives a contribution \( 2n - (\alpha + \beta) \). On the other hand, when \( e_\alpha = e_\beta \), the two branching points have melted into one, with branching index \( B_\alpha = n + 1 - \alpha \), and a contribution of \( n - \alpha \) to the summand \( B(p) - 1 \) of \( g^{(6)} \) in equation (5.11). As \( n - \alpha \leq 2n - (\alpha + \beta) \), we see that the contribution
$g^{(6)}$ decreases. The equality holds if and only if $\beta = n$; this corresponds to the trivial case mentioned above, i.e., when one of the 6-branes is placed between the last two sheets of $\Sigma_g$.

The mechanism just described, whereby two 6-branes are made to have coincident $v$-projections on the base $\mathbb{CP}^1$, corresponds to a transition to a Higgs phase [4]. We observe that the models of section 3 did not exhibit this behaviour. As explained there, any one 4-brane between sites $\alpha - 1$ and $\alpha$ could be made to have the same $v$-projection as any other 4-brane between sites $\alpha$ and $\alpha + 1$, with the genus $g$ remaining constant. In section 3, the decrease in the number of branching points when two or more 4-branes had the same $v$-projection was compensated by an increase in the branching index. This was due to the structure of the surface $\Sigma_g$ as a superposition of hyperelliptics: no single 4-brane ever connected more than two adjacent sheets. We observe that the surfaces of this section no longer enjoy this property, due to the branchings effected by the 6-branes.

Hence, by differentiation of the SW differential $\lambda_{SW}$ with respect to the moduli $u_j^{(\alpha)}$ we do not obtain a complete basis of holomorphic differentials on this surface (unless all the $d_\alpha = 0$). All that one obtains is a basis for the $g^{(4)}$ holomorphic differentials that can be associated with the 4-branes. The property that the surface $\Sigma_g$ ceases to be a superposition of hyperelliptics is reflected in the appearance of new moduli in the theory, other than those associated with the order parameters $u_j^{(\alpha)}$ on the Coulomb branch.

We observe from [4] that the 4-manifold $Q$ in which the surface $\Sigma_g$ is immersed is no longer the space of section 3. On $\mathbb{R}^3 \times S^1$, the hyperelliptic involution $\sigma$ was a discrete transformation that squared to unity. In the presence of 6-branes, the 4-manifold $Q$ becomes multi–Taub–NUT space [40]. On the latter there is a continuous $\mathbb{C}^*$-action given by the complexification of a $U(1)$ rotation symmetry around the $S^1$ direction of $M$-theory [4]. The hyperelliptic involution on $\mathbb{R}^3 \times S^1$ is in fact a discrete remnant (in the limiting case when all the $d_\alpha$ vanish) of this $\mathbb{C}^*$-action on multi–Taub–NUT space.

Therefore, if $d_\alpha \neq 0$ for some $\alpha < n$, we are missing $g^{(6)}$ holomorphic differentials, so we cannot define an algebra. However, we can draw some conclusions. Given that the structure of $\Sigma_g$ isn’t a superposition of hyperelliptics, if any algebra of holomorphic differentials is to hold at all, we do not expect it to conform to the hyperelliptic pattern of equation (2.9). On physical grounds, the number $d_\alpha$ of 6-branes placed between sites $\alpha - 1$ and $\alpha$ is a free parameter that one can vary, in order to obtain a theory with a vanishing beta function [4].
A vanishing beta function indicates the existence of a new modulus in the theory, namely, the gauge coupling constant. From \[29\] we do not expect the WDVV equation to hold in this case.

6 Summary and conclusions

Using $M$-theory techniques, large classes of “new” SW models have been constructed recently whose moduli spaces (in their Coulomb branches) are described by non-hyperelliptic Riemann surfaces. “Old” SW models (i.e., prior to the advent of $M$-theory and geometric engineering) were typically described by hyperelliptic Riemann surfaces, with their corresponding prepotentials satisfying the WDVV equation. In this paper we have posed the question of whether or not the prepotentials associated with these new SW models continue to satisfy the WDVV equation, despite the loss of the property of hyperellipticity of the corresponding Riemann surfaces. The answer to this question comes in two steps. One first needs to define an associative algebra for the holomorphic 1-forms on the Riemann surface. Next one expresses the (third derivatives of the) prepotential in terms of those differentials (the so-called residue formula). Associativity of the algebra of 1-forms is then an equivalent statement of the validity of the WDVV equation.

We have taken the first of the two steps mentioned above, deferring a proof of the residue formula for an upcoming publication. We find two substantially different classes of non-hyperelliptic SW models. In the first one, it turns out to be possible to define an associative algebra of holomorphic differentials. Although non-hyperelliptic, these Riemann surfaces can be understood (roughly speaking) as a superposition of hyperelliptic building blocks. The construction of these surfaces is carried out in such a way that all properties of the hyperelliptic building block pertaining to the algebra of holomorphic 1-forms are maintained. Characteristically, the SW models so described correspond to product gauge groups, with matter hypermultiplets transforming in (sums of) bifundamental representations. The loss of hyperellipticity in these models is of no import, and therefore their prepotentials can be expected to satisfy the WDVV equation, much as their hyperelliptic ancestors did. From the viewpoint of their $M$-theory construction, these theories involve 4- and 5-branes only (plus, possibly, 4-orientifolds as well). We observe that, in these cases, $\Sigma_g$ is a surface in
the 4-manifold $Q = \mathbb{R}^3 \times S^1$. All the moduli in the theory are those associated with the physical order parameters on the Coulomb branch. The latter are determined solely by the gauge group; no new moduli appear in the passage from the “old” hyperelliptic SW models to these “new” non-hyperelliptic cases.

This allows us to formulate a sufficient condition for the algebra of holomorphic differentials to be associative. Namely, if the surface $\Sigma_g$ can be decomposed as a superposition of hyperelliptics (in the manner described in the body of the paper) then an associative algebra of holomorphic differentials will hold. Whether or not this condition is also necessary remains an open question.

A second class of non-hyperelliptic SW models is analysed, in which it turns out to be impossible to define an associative algebra of holomorphic differentials following the pattern of the hyperelliptic case. We would like to underline the fact that this does not rule out the possibility of defining an associative algebra. However, such an algebra (if it exists at all) will have to conform to the non-hyperelliptic pattern established in section 2.1. Geometrically, it is observed already at the level of the corresponding Riemann surfaces that hyperellipticity is lost in a more fundamental way, because it is no longer possible to “decompose” the surface as a superposition of hyperelliptic building blocks. From a physical viewpoint, their $M$-theory construction requires the inclusion of 6-branes and/or 6-orientifolds.

Specifically, in the presence of 6-orientifolds, the branching index is compelled to take fixed values greater than 2. The increase in the value of the genus (with respect to the hyperelliptic case with the same number of moduli) is compensated by a restriction to a certain Prym subvariety. The dimension of the latter is twice the number of independent moduli. However, even after this restriction, an associative algebra following the pattern of the hyperelliptic case is not possible. Typically, these SW models describe gauge theories with matter hypermultiplets in representations higher than the fundamental.

In the presence of 6-branes, the loss of hyperellipticity is more profound, because it can be ascribed to the appearance of new moduli. For example, the number of 6-branes included in the brane configuration can be fine-tuned in such a way that the beta function vanishes. The gauge coupling constant then becomes a modulus. Even before reaching that critical value in the number of 6-branes, when the beta function continues to be negative, the 6-branes cause an increase in the value of the genus with respect to the case with the lowest value of $g$ that
is compatible with the same number of moduli, i.e., the case with no 6-branes at all. This increase cannot be compensated by restricting to a certain Prym subvariety. In consequence, modular derivatives of the SW differential no longer provide us with a complete basis of holomorphic differentials. This can be rephrased by saying that we are missing moduli, so we cannot write down a complete basis of differentials. In these cases we observe that \( \Sigma_g \) is a surface in the 4-manifold \( Q \) given by a multi–Taub–NUT space. A natural question to ask is whether or not the 4-manifold \( Q \) can provide the missing moduli.

Our analysis reveals a connection between the algebra of holomorphic differential forms on the Riemann surface and the configuration of \( \mathcal{M} \)-theory branes used in the construction of the corresponding SW model. We hope these observations may provide some insight into a purely \( \mathcal{M} \)-theoretic derivation of the WDVV equation, i.e., one without recourse to an underlying algebra of differentials. Looking beyond, one could pose the question of whether or not there is some generalisation of the WDVV equation that would hold in the models examined in section 5.

We hope the observations made here may prove useful in clarifying these issues.

**Acknowledgements**

It is a great pleasure to thank M. Matone and M. Tonin for discussions and encouragement. This work has been supported by the Commission of the European Community under contract FMRX-CT96-0045.

**Appendix**

Below we present a proof of equations (4.3), (4.6) and (4.7).

Let us start with equation (4.3). We have observed that the 1-form \( \phi_j^{(n)} \) is holomorphic on the surface \( F = 0 \), because it coincides with \( \partial \lambda_{SW}/\partial u_j^{(n)} \). We also know that its zeroes will be at the points on the surface \( F = 0 \) lying above \( v = 0 \) and \( v = \infty \) on the base \( \mathbb{CP}^1 \). In fact, from the observation made after equation (4.8), the zeroes of \( \phi_j^{(n)} \) will lie on the sheets \( \alpha = n \) and \( \alpha = n + 1 \) of the surface \( F = 0 \). We have also made the assumption that neither \( v = 0 \) nor \( v = \infty \) are branching points of \( F = 0 \). From here we conclude a divisor \([\phi_j^{(n)}]\) of the general form

\[
[\phi_j^{(n)}] = c_1 (0_{n-1} + 0_n) + c_2 (\infty_{n-1} + \infty_n),
\]

where \( c_1 \) and \( c_2 \) are certain integers. Again, the fact that \( v = 0 \) is not a branching point of
\[ F = 0, \text{ together with equation (4.2), dictates that } c_1 = k_n - j. \] It now suffices to use equation (4.4) in order to conclude that the remaining coefficient \( c_2 \) is \( \sum_{l=1}^{n-1} k_l + j - (n + 1) \), as stated in equation (4.3).

Equation (4.6) is proved along the same lines as (4.3). We start from the observation that, for every fixed value of \( \alpha = n - 1, n - 2, \ldots, 1 \), the 1-form \( \phi_j^{(\alpha)} \) is holomorphic on the surface \( \partial^{n-\alpha} F / \partial t^{n-\alpha} = 0 \). From here conclude that the divisor \( [\phi_j^{(\alpha)}] \) on \( \partial^{n-\alpha} F / \partial t^{n-\alpha} = 0 \) is as stated in equation (4.6). The applicable value of the genus is now \( g_{\alpha} = \sum_{l=1}^{n} (k_l - 1) \).

Finally, in proving equation (4.7) we start from the observation that \( v = 0 \) has been assumed not to be a branching point of \( \partial^{n-\alpha} F / \partial t^{n-\alpha} = 0 \) for any \( \alpha = 1, 2, \ldots, n \) (the value \( \alpha = n \) corresponding by convention to the surface \( F = 0 \)). Hence the coefficient \( c_1 = k_{\alpha} - j \) multiplying \( (0_{\alpha-1} + 0_{\alpha}) \) in the divisor \( [\phi_j^{(\alpha)}] \) is correct not only on \( \partial^{n-\alpha} F / \partial t^{n-\alpha} = 0 \) for \( \alpha = n - 1, n - 2, \ldots, 1 \), as per equation (4.6), but also on \( F = 0 \). No new poles or zeroes appear when extending the 1-form \( \phi_j^{(\alpha)} \) from \( \partial^{n-\alpha} F / \partial t^{n-\alpha} = 0 \) to \( F = 0 \), except possibly at \( v = \infty \), i.e., at \( \infty_{\alpha-1} \) and \( \infty_{\alpha} \), for the same reasons as previously. Hence all that remains to determine is the coefficient \( c_2 \) in front of the term \( (\infty_{\alpha-1} + \infty_{\alpha}) \). This is again fixed by the requirement in equation (4.4), after observing that the applicable value of the genus is now \( g = g_n = \sum_{l=1}^{n} (k_l - 1) \). Hence the divisor in equation (4.7) is correct. Now, holomorphicity of \( \phi_j^{(\alpha)} \) on \( F = 0 \) is equivalent to the requirement that both coefficients in (4.7) be positive.

This happens if, and only if, for every \( \alpha \) in the range \( 1, 2, \ldots, n \), we have that \( j \) runs over the range \( 2, 3, \ldots, k_{\alpha} \). Indeed, from the coefficient \( c_1 \) of \( (0_{\alpha-1} + 0_{\alpha}) \) we obtain the condition that \( k_{\alpha} \geq j \). Next let \( j \geq 2 \). Then, from the coefficient \( c_2 \) of \( (\infty_{\alpha-1} + \infty_{\alpha}) \), we have that \( j + \sum_{l \neq \alpha} k_l \geq 2 + \sum_{l \neq \alpha} 1 = 2 + (n - 1) = n + 1 \), so also this \( c_2 \) is positive. This finally establishes the holomorphicity of \( \phi_j^{(\alpha)} \) on \( F = 0 \).

References

[1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19 (hep-th/9407087); Nucl. Phys. B431 (1994) 484 (hep-th/9408099).

[2] For reviews see, e.g., T. Banks (hep-th/9710234); D. Bigatti and L. Susskind (hep-th/9712072); A. Bilal (hep-th/9710136); A. Giveon and D. Kutasov
[3] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, Nucl. Phys. B459 (1996) 537 (hep-th/9508155); A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, Nucl. Phys. B477 (1996) 746 (hep-th/9604034); S. Katz, A. Klemm and C. Vafa, Nucl. Phys. B497 (1997) 173 (hep-th/9609235); S. Katz, D. Morrison and M. Plesser, Nucl. Phys. B477 (1996) 105 (hep-th/9601102); S. Katz, P. Mayr and C. Vafa, Adv. Theor. Math. Phys. 1 (1998) 53 (hep-th/9706110).

[4] E. Witten, Nucl. Phys. B500 (1997) 3 (hep-th/9703166).

[5] For reviews see, e.g., L. Alvarez-Gaumé and S. Hassan, Fortsch. Phys. 45 (1997) 159 (hep-th/9701069); L. Alvarez-Gaumé and F. Zamora (hep-th/9709180); A. Bilal (hep-th/9601007); P. Di Vecchia, Surv. High Energy Phys. 10 (1997) 119 (hep-th/9608090); S. Ketov, Fortsch. Phys. 45 (1997) 237 (hep-th/9611209, hep-th/9710085); A. Klemm (hep-th/9705131); W. Lerche, Nucl. Phys. Proc. Suppl. 55B (1997) 83 (hep-th/9611190).

[6] J. Erlich, A. Naqvi and L. Randall (hep-th/9801108).

[7] K. Landsteiner, E. López and D. Lowe, Nucl. Phys. B507 (1997) 197 (hep-th/9705199).

[8] K. Landsteiner and E. López (hep-th/9708118).

[9] A. Brandhuber, J. Sonnenschein, S. Theisen and S. Yankielowicz, Nucl. Phys. B504 (1997) 175 (hep-th/9705232); Nucl. Phys. B502 (1997) 125 (hep-th/9704044).

[10] E. Witten, Surv. Diff. Geom. 1 (1991) 243.

[11] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B352 (1991) 59.

[12] B. Dubrovin, Nucl. Phys. B379 (1992) 627; for a review see, e.g., B. Dubrovin (hep-th/9407018).

[13] R. Donagi and E. Witten, Nucl. Phys. B460 (1996) 299 (hep-th/9510101); for a review see, e.g., R. Donagi (alg-geom/9705010).

[14] I. Krichever, Comm. Pure Appl. Math., 47 (1994) 437.

[15] T. Nakatsu and K. Takasaki, Mod. Phys. Lett. A11 (1996) 157 (hep-th/9509162); K. Takasaki (hep-th/9803217).
[16] E. Martinec and N. Warner, Nucl. Phys. B459 97 (hep-th/9509161); E. Martinec, Phys. Lett. B367 (1996) 91 (hep-th/9510204).

[17] E. D’Hoker and D. Phong (hep-th/9804126, hep-th/9804125, hep-th/9804124); Nucl. Phys. B513 (1998) 405 (hep-th/9709053).

[18] I. Krichever and D. Phong, J. Diff. Geom. 45 (1997) 349 (hep-th/9604199).

[19] A. Gorski, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. B355 (1995) 466 (hep-th/9505035).

[20] A. Gorsky, S. Gukov and A. Mironov (hep-th/9707120, hep-th/9710239).

[21] S. Gukov (hep-th/9709138).

[22] R. Carroll (hep-th/9712110, hep-th/9802130, hep-th/9804086).

[23] M. Matone, Phys. Lett. B357 (1995) 342 (hep-th/9506102); Phys. Rev. D53 (1996) 7354 (hep-th/9506181); Phys. Rev. Lett. 78 (1997) 1412 (hep-th/9610204); G. Bonelli, M. Matone and M. Tonin, Phys. Rev. D55 (1997) 6466 (hep-th/9610026); G. Bonelli and M. Matone, Phys. Rev. Lett. 76 4107 (hep-th/9602174); D. Bellisai, F. Fucito, M. Matone and G. Travaglini, Phys. Rev. D56 (1997) 5218 (hep-th/9706093).

[24] E. D’Hoker and D. Phong, Phys. Lett. B397 (1997) 94 (hep-th/9701055); E. D’Hoker, I. Krichever and D. Phong, Nucl. Phys. B489 (1997) 211 (hep-th/9609145); Nucl. Phys. B489 (1997) 179 (hep-th/9609041).

[25] S. Naculich, H. Rhedin and H. J. Schnitzer (hep-th/9804105); I. Ennes, S. Naculich, H. Rhedin and H. J. Schnitzer (hep-th/9804151).

[26] G. Bonelli and M. Matone, Phys. Rev. Lett. 77 (1996) 4712 (hep-th/9605090).

[27] A. Klemm, W. Lerche and S. Theisen, Int. J. Mod. Phys. A11 (1996) 1929 (hep-th/9505150); J. M. Isidro, A. Mukherjee, J. P. Nunes and H. J. Schnitzer, Nucl. Phys. B492 (1997) 647 (hep-th/9609116); Int. J. Mod. Phys. A13 (1998) 233 (hep-th/9703176); Nucl. Phys. B502 (1997) 363 (hep-th/9704174); M. Alishahiha, (hep-th/9703186).

[28] K. Ito and S.-K. Yang (hep-th/9803126).
[29] A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. B389 (1996) 43 (hep-th/9607109); Mod. Phys. Lett. A 12 (1997) 773 (hep-th/9701014); (hep-th/9701123).

[30] A. Morozov (hep-th/9711194); A. Mironov and A. Morozov (hep-th/9712177).

[31] G. Bertoldi and M. Matone, Phys. Lett. B425 (1998) 104 (hep-th/9712039); Phys. Rev. D57 (1998) 6483 (hep-th/9712105).

[32] G. Bonelli and M. Matone (hep-th/9712025).

[33] G. Moore and E. Witten (hep-th/9709193); A. Losev, N. Nekrasov and S. Shatashvili (hep-th/971108); M. Mariño and G. Moore (hep-th/9802185).

[34] H. M. Farkas and I. Kra, Riemann Surfaces, 2nd edn., Graduate Texts in Mathematics vol. 71, Springer-Verlag, New York, 1991.

[35] W. Fulton, Algebraic Topology, Graduate Texts in Mathematics vol. 153, Springer-Verlag, New York, 1995.

[36] J. M. Isidro, in preparation.

[37] W. Lerche and N. P. Warner (hep-th/9608183).

[38] A. Fayyazuddin and M. Spaliński, Nucl. Phys. B508 (1997) 219 (hep-th/9706087).

[39] P. Argyres and A. Faraggi, Phys. Rev. Lett. 74 (1995) 3931 (hep-th/9411057).

[40] P. Townsend, Phys. Lett. B350 (1995) 184 (hep-th/9501068); G. Gibbons and P. Rychenkova (hep-th/9608085).