Path integral polymer propagator of relativistic and non-relativistic particles

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A recent proposal to connect the loop quantization with the spin foam model for cosmology via the path integral is hereby adapted to the case of mechanical systems within the framework of the so called polymer quantum mechanics. The mechanical models we consider are deparametrized and thus the group averaging technique is used to deal with the corresponding constraints. The transition amplitudes are written in a vertex expansion form used in the spin foam models, where here a vertex is actually a jump in position. Polymer propagators previously obtained by spectral methods for a nonrelativistic polymer particle, both free and in a box, are regained with this method and as a new result we obtain the polymer propagator of the relativistic particle. All of them reduce to their standard form in the continuum limit for which the length scale parameter of the polymer quantization is taken to be small. Our results are robust thanks to their analytic and exact character which in turn come from the fact that presented models are solvable. They lend support to the vertex expansion scheme of the polymer path integral explored before in a formal way for cosmological models. Some possible future developments are commented upon in the discussion.

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I. INTRODUCTION

The space-time singularities of General Relativity [1] and the ultraviolet divergences in Quantum Field Theory (QFT) [2] are open physical problems deeply connected to the underlying structure of space-time which is classically assumed to be a smooth manifold with a metric obeying equations of general relativity. There have been numerous attempts to understand and deal with these issues in approaches like string theory [3-5] and loop quantum Gravity (LQG) [6-8].

The need to cope with these divergences has led to investigation proposals in quantum gravity that may alleviate such difficulties, possibly by modifying the underlying space-time structure at the Planck scale. More concretely, a common understanding coming out of many of these attempts is that the high energy structure of the spacetime is fundamentally discrete. As a salient candidate, LQG has been successful to resolve the classical gravitational singularities of various cosmological and black hole models (there is a huge body of research done in both of these topics; for a sample of historical and recent works see [9-13] and the references within). This theory derives the discreteness of the spacetime via the quantization of spacetimes degrees of freedom, the so called holonomies and their conjugate fluxes. LQG comes in two versions, a Hamiltonian one and a path integral version or the spin foam model [14] which is based on the transition amplitudes a la Feynman and thus a manifestly covariant theory.

The representation used in LQG is a departure from the regular representation of the Weyl relations. Such a representation is called polymer quantization. As we will see, this type of quantization leads to a representation of the classical algebra that is not unitarily equivalent to the Schrödinger representation. This is a consequence of the fact that the weak continuity assumption of the Stone-von Neumann theorem does not hold for such polymer representation. The lack of continuity, in turn, is the result of a specific choice of topology (i.e. a discrete one). Polymer representation exhibits two standard polarization in which either configuration variables or the momenta are inherently discrete, and the conjugate ones take values on a compact space (S^1 for example).

Applying this type of quantization to mechanical system, i.e. systems with finite number of degrees of freedom is known as polymer quantum mechanics [15], while combining it with Feynman path integral formulation leads to the polymer path integral formulation. Also, if it is applied to each of the infinite modes of the fields [16] it would yield a polymer field model.

Historically, combining path integration with polymer quantization was considered originally in [17] and more recently has been considered along different lines. For instance Bianchi I cosmological model [18] and its effective dynamics have been studied including the cosmological constant [19]. All the isotropic models both in the deparametrized and timeless frameworks were worked out in [20]. Even alternative dynamics has been used which however yields the same effective dynamics [21]. Similar techniques have been applied to consistent histories approach in cosmology [22-24]. Even a coherent state functional has been studied for some models [25, 26]. Interestingly, a parallelism between the two point functions for cosmology and the relativistic particle has shed some light on the timeless and deparametrized frameworks [27]. For other systems some mechanical models have been considered that resemble the problematics in analyzing the semiclassical approximation of certain black holes [28]. Also polymer field theories have been dealt with by adopting this approach [29, 30].

Recently, a detailed connection between the Hamiltonain LQG and the spin foam model for the case of homogeneous isotropic cosmic models has been developed [31, 32] showing that both Hamiltonian transition amplitudes, time deparametrized and time reparametrization invariant, can be written in a vertex expansion form, typical of the spin foam models, and they actually coincide. Based on this, it is interesting to apply this method also to mechanical systems with finite degrees of freedom (i.e. vertex expansion together with polymer quantum mechanics) to see how are the results different from normal Schrödinger representation. It is particularly useful since in these cases, the propagator can be obtained analytically, exactly and explicitly due to solvable nature of many of these systems. Furthermore, it provides a direct and exact consistency check for the above method and also hopefully sheds more light on the physics behind
the idea. In this spirit, we try to derive the polymer path integral of three mechanical systems, the free nonrelativistic particle, the same particle in a box, and the relativistic particle.

This work is organized as follows. In Sec. II we present an introduction to polymer representation and its kinematics and dynamics, for a reader that is not quite familiar with the subject. In Sec. III the tools that are used in the rest of the work are introduced. These include, the nonrelativistic and relativistic deparametrization framework of constrained systems, group averaging technique to deal with solving quantum constrains and defining inner product on their representation space, and the generic form of the polymer path integral of a deparametrized system. Sec. IV is devoted to present the method introduced in [31, 32] to cast loop quantum cosmology in spin foam framework. Here we develop it for a single particle to be used in the following sections. In Sec. V we use all the material introduced in previous sections to derive the polymer path integral of a nonrelativistic particle, both free and in a box. The continuum limit of such a path integral is also derived. In Sec. VI, the polymer path integral of a relativistic particle and its continuum limit are derived. This is a new result as far as the authors understand. Finally in Sec. VII we summarize our findings and make several remarks about the work.

II. POLYMER QUANTUM MECHANICS

A. Weyl relation and its Schrödinger representation

An important part of any (canonical) quantization procedure is how to represent the classical phase space variables and their algebra, and consequently the functions of phase space, as linear operators on a suitable Hilbert space. Usually the representations that are used are unitarily equivalent to the Schrödinger representation as we will see. However not all of them are so, and the polymer representation [15] is an example of such inequivalent representation. In order to understand the basic foundations of the polymer representation, we will briefly look at the Weyl group and its representations and then describe the basics of polymer representation of this group.

The classical Poisson algebra of a simple quantum mechanical system between a pair of canonical variables can be written as \( \{ q, p \} = 1 \). If we represent this on a Hilbert space such that

\[
\hat{q} \hat{p} - \hat{p} \hat{q} = i \hat{1},
\]

it can be shown that \( \hat{q} \) and \( \hat{p} \) cannot both be bounded operators [33]. This is called the Wintner theorem [34]. Obviously, the right hand side of this relation is well-defined over all of the Hilbert space. However, since in the left hand side, at least one of the operators is not bounded, this side is not well-defined unless we introduce further conditions on the operators to make it so. One way is to specify the domain of the operators in Hilbert space on which they are bounded, but this can get very complicated. A simpler way around this problem was proposed by Weyl [35] in which one works with the “exponential version” of these operators. Since these can be unitary operators and these types of operators are bounded, then we need not to worry about the issue of boundedness. Let us see this in more detail.

We first probe the classical theory and then move to the quantum regime. Following the Weyl proposal, we can write classically

\[
U = e^{i\alpha q}, \quad V = e^{i\beta p},
\]

which are classical unitary objects if \( q \) and \( p \) are real. Using the Baker-Campbell-Hausdorff theorem, for a pair \( (q, p) \) that obey the canonical commutation relations \( \{ q, p \} = 1 \), we can write

\[
U(\alpha)V(\beta) = e^{-\alpha\beta\{q,p\}}V(\beta)U(\alpha).
\]

1 A linear operator (transformation) \( L : V_1 \to V_2 \) from a normed vector space \( V_1 \) to another one \( V_2 \) is a map for which the ratio of the norm of \( L(v) \) to that of \( v \), for all \( v \in V_1 \) and \( v \neq 0 \) is bounded: \( \frac{\|L(v)\|}{\|v\|} < \infty, \forall v \in V_1 \land v \neq 0. \)
Note that this is strictly valid only for $U$ and $V$ that are generated by classical canonical variables, but not for any general $U$ and $V$ that are unitary.

Moving to the quantum version (i.e. the representation of the operators on the Hilbert space), we can see that if both of the canonical pair of operators $\hat{q}$ and $\hat{p}$ were well-defined on the Hilbert space, we could have written their quantum version of exponentiation

$$\hat{U} = e^{i\alpha \hat{q}}, \quad \hat{V} = e^{i\beta \hat{p}}$$

and (2.3) would have become (using the Dirac prescription $[\hat{q}, \hat{p}] = i\hbar \{q, p\}$)

$$\hat{U}(\alpha) \hat{V}(\beta) = e^{i\alpha \beta} \hat{V}(\beta) \hat{U}(\alpha).$$

This is called the Weyl relation. Now the Weyl prescription is that, since not always both of generators $\hat{q}, \hat{p}$ can be represented as well-defined operators on the Hilbert space due to boundedness issues (and also continuity, see below) and thus we can not always literally exponentiate them to get their associated unitary operators, we can just forget about (2.1) and instead take the Weyl relations (2.5) as fundamental relations, and also take $\hat{U}(\alpha)$ and $\hat{V}(\beta)$ as the basic operators of the theory on their own and not as exponentiation of a generator. Then, if under some conditions, it is possible to write $\hat{U}$ and $\hat{V}$ as exponentials of some generators as in (2.4), we can easily regain (2.1) by using Baker-Campbell-Hausdorff theorem. These conditions, which we will discuss briefly in the following, are the ones that distinguish between the Schrödinger and polymer representations.

Consider a one-parameter group of linear operators $\{L_t\}$ defined on a Hilbert space $\mathcal{H}$,

$$L_t : \mathbb{R} \to \text{End}(\mathcal{H}) \quad \text{and} \quad L_t L_s = L_{t+s} \quad \forall t, s \in \mathbb{R}, \quad L_0 = 1.$$  

This one-parameter group is called weakly continuous [35] if

$$\lim_{t \to 0} \langle \psi | L_t | \phi \rangle = \langle \psi | L_0 | \phi \rangle, \quad \forall |\psi\rangle, |\phi\rangle \in \mathcal{H}, \quad \forall t_0 \in \mathbb{R}.$$  

It is called strongly continuous if

$$\lim_{t \to 0} L_t |\phi\rangle = L_0 |\phi\rangle, \quad \forall |\phi\rangle \in \mathcal{H}, \quad \forall t_0 \in \mathbb{R}.$$  

In the case where $\{L_t\}$ is strongly continuous, there are states for which the limit

$$\lim_{t \to 0} \frac{L_t |\psi\rangle - L_0 |\psi\rangle}{t} = ig |\psi\rangle, \quad |\psi\rangle \in \mathcal{H}$$

exists, and so within this domain of states, one can define the infinitesimal generators $g$ of the one-parameter group $\{L_t\}$ such that

$$\hat{L}_t = e^{i\hat{g}}$$

with the $g$’s being hermitian [33]. It turns out that if $\{L_t\}$ is a unitary group, weak and strong continuities are equivalent [36], and thus if the one-parameter group of unitary operators $\{L_t\}$ is weakly continuous, one can write its elements as exponentiation of some hermitian operators (generators $g$ of $\{L_t\}$) on $\mathcal{H}$ as in (2.10), or similarly as in (2.4). Thus, while the Weyl relation (2.5) is always valid, only for one-parameter group of unitary operators (whose generators are well-defined on $\mathcal{H}$) one can recover the algebra of the generators similar to (2.1). Also note that, since $g$’s are the generators of the infinitesimal transformations (while the group members $L_t$ generate finite transformations), the existence (or lack) of generators on $\mathcal{H}$ means the existence (or lack) of infinitesimal transformations. As a sidenote we mention that a one-parameter unitary group is bounded if it is weakly continuous, hence the connection between boundedness and continuity.

The Schrödinger representation is a representation of the Weyl relation (2.5) which is weakly continuous. The celebrated Stone-von Neumann theorem [37–39] states that every irreducible representation of the Weyl relations in which the operators are unitary and weakly continuous, are unitarily equivalent to each other and to the Schrödinger representation.

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2 Or equivalently, cannot always expand $\hat{U}(\alpha)$ or $\hat{V}(\beta)$ as Taylor series in $\hat{q}$ or $\hat{p}$ respectively.
B. Polymer representation: kinematics

In cases where the above condition of weak continuity is not valid anymore, the resulting representations of the Weyl relation is not unitarily equivalent to the Schrödinger representation \[39\]. The polymer representation is such a representation for which (some of) the generators $g$ are not well-defined on $\mathcal{H}$. This means that the theory based on this representation does not admit these infinitesimal transformations and only contains the finite ones. The full Hilbert space, $\mathcal{H}_{\text{poly}}$, then, possesses an uncountable orthonormal basis such that

$$\langle \alpha | \beta \rangle = \delta_{\alpha \beta}, \quad \alpha, \beta \in \mathbb{R},$$

(2.11)

where $\delta_{\alpha \beta}$ is the Kronecker delta.

As we mentioned before, while one can classically write $U = e^{i\mu q}$ and $V = e^{i\lambda p}$, this is not always allowed quantum mechanically for both of these operators. Thus, in polymer representation, one usually chooses one of the two “polarizations”: the $q$-polarization in which $\hat{q}$ (but not $\hat{p}$) is well-defined and we have

$$\hat{U}_\mu |q\rangle = e^{i\mu q} |q\rangle,$$

(2.12)

$$\hat{V}_\lambda |q\rangle = |q - \lambda\rangle,$$

(2.13)

and the $p$-polarization in which $\hat{p}$ (but not $\hat{q}$) is well-defined where

$$\hat{U}_\mu |p\rangle = |p - \mu\rangle,$$

(2.14)

$$\hat{V}_\lambda |p\rangle = e^{i\lambda p} |p\rangle.$$

(2.15)

One can see that in the $q$-polarization, $V_\lambda$ is not weakly continuous, $\lim_{\lambda \to 0} \langle q | \hat{V}_\lambda | q \rangle \neq \langle q | \hat{V}_{\lambda=0} | q \rangle$, since

$$\lim_{\lambda \to 0} \langle q | \hat{V}_\lambda | q \rangle = \lim_{\lambda \to 0} \langle q | q - \lambda \rangle = 0,$$

(2.16)

while

$$\langle \mu | \hat{V}_{\lambda=0} | \mu \rangle = \| \hat{V}_0 \| = 1,$$

(2.17)

where the inner products have been taken using (2.11). This proves our claim that while the infinitesimal generator $\hat{q}$ exists and thus quantum mechanically we can write $\hat{U}_\mu = e^{i\mu \hat{q}}$, the infinitesimal generator $\hat{p}$ does not exist and thus we only have finite translations (with steps $\lambda$) in $q$ space generated by $V_\lambda$ as can be seen from (2.13). More precisely, once one fixes a certain $\lambda$ as a free parameter of the theory, and starts from an initial state $|q_0\rangle$, it is seen from (2.13) (and also (2.12)) that the wave functions $\langle q | \Psi \rangle$ are restricted to the lattice points $\{q_n | q_n = q_0 + n\lambda, \ n \in \mathbb{Z}\}$ and the eigenvalues of the operator $\hat{q}$ are discrete. On the contrary, values of $p$ corresponding to the basis $|p\rangle$ are not discrete but take values on a circle, i.e., $-\frac{\pi \hbar}{\lambda} \leq p < \frac{\pi \hbar}{\lambda}$ (see for example appendix A of \[28\]). The discreteness in $q$ is thus inherent in polymer type of theories. As a result of the above discussion, in this polarization it is usual to write the $|q\rangle$ basis as $|q_n\rangle$ which are the members of a countable basis of the corresponding Hilbert space $\mathcal{H}_{q_0}$. Note, however, that the full polymer Hilbert space is

$$\mathcal{H}_{\text{poly}} = \bigoplus_{0 \leq q_0 < \lambda} \mathcal{H}_{q_0},$$

(2.18)

Thus $\mathcal{H}_{q_0}$ is a separable super-selected sector of $\mathcal{H}_{\text{poly}}$ while, as mentioned above, the full non-separable polymer Hilbert space, possesses an inner product as in (2.11).
For the $p$-polarization, (2.14) and (2.15), things are reversed but the ideas are essentially the same. There, $\hat{U}_\mu$ is not weakly continuous and thus its generator $\hat{q}$ is not well-defined on the Hilbert space. For a fixed $\mu$ we have a lattice in $p$ space such that starting from an initial state $|p_0\rangle$, the wave functions $\langle p|\Psi\rangle$ are restricted to the lattice points $\{p_m|p_m = p_0 + m\mu, m \in \mathbb{Z}\}$. Also $q$ takes continuous values on a circle, $-\frac{\pi\mu}{\hbar} \leq q < \frac{\pi\mu}{\hbar}$, and the corresponding Hilbert space $\mathcal{H}_{p_0}$ is a super-selected sector of $\mathcal{H}_{\text{poly}}$, such that $\mathcal{H}_{\text{poly}} = \bigoplus_{0 \leq p_0 < \mu} \mathcal{H}_{p_0}$.

C. Polymer representation: dynamics

Let us consider the $q$-polarization [28, 40] where $\hat{U}_\mu = e^{i\mu\hat{q}}$ and

$$\hat{q}|q_n\rangle = q_n|q_n\rangle,$$

$$\hat{V}_\lambda|q_n\rangle = |q_n - \lambda\rangle,$$

$$\hat{q}|p\rangle = \frac{\hbar}{i}\partial_p|p\rangle,$$

$$\hat{V}_\lambda|p\rangle = e^{i\lambda p}|p\rangle.$$  \hspace{1cm} (2.19) \hspace{1cm} (2.20) \hspace{1cm} (2.21) \hspace{1cm} (2.22)

Since in this case the generator $\hat{p}$ does not exist, we need to construct its analog to be able to represent e.g. the kinetic term $p^2/2m$ in the Hamiltonian. It turns out that it is better to start from an analog of $\hat{p}^2$. Classically, we have the following approximation

$$e^{i\frac{\lambda p}{\hbar}} + e^{-i\frac{\lambda p}{\hbar}} \approx 2 - \frac{\lambda^2 p^2}{\hbar^2}, \quad p \ll \frac{\hbar}{\lambda},$$

which together with (2.22) can be used to give the analog of $\hat{p}^2$ as

$$\hat{p}_\lambda^2 \approx \frac{\hbar^2}{\lambda^2} (2 - \hat{V}_\lambda - \hat{V}_{-\lambda}).$$

Its action on $|p\rangle$ basis can then be computed to yield

$$\hat{p}_\lambda^2|p\rangle = \frac{4\hbar^2}{\lambda^2} \sin^2 \left( \frac{\lambda p}{2\hbar} \right) |p\rangle.$$  \hspace{1cm} (2.23)

Using this as a guide, we can define the analog of $\hat{p}$ such that

$$\hat{p}_\lambda|p\rangle = \frac{\hbar}{\lambda} \left( \sin \left( \frac{\lambda p}{\hbar} \right) \right) |p\rangle.$$  \hspace{1cm} (2.24)

This can be achieved by defining the analog of $\hat{p}$ as

$$\hat{p}_\lambda = \frac{\hbar}{2i\lambda} \left( \hat{V}_\lambda - \hat{V}_{-\lambda} \right).$$  \hspace{1cm} (2.25)

The same construction can be used to represent the analog of $\hat{q}$ and $\hat{q}^2$ in the $p$-polarization where $\hat{q}$ is not well-defined on the Hilbert space.

As a result of these definitions, the Hamiltonian operator of a free particle can be represented as

$$\hat{H} = \frac{1}{2m} \frac{\hbar^2}{\lambda^2} (2 - \hat{V}_\lambda - \hat{V}_{-\lambda})$$

whose action on the $|p\rangle$ basis yields

$$\hat{H}|p\rangle = \frac{2\hbar^2}{m\lambda^2} \left( \sin^2 \left( \frac{\lambda p}{2\hbar} \right) \right) |p\rangle.$$  \hspace{1cm} (2.26)
III. POLYMER DEPARAMETRIZED PROPAGATOR

A. Deparametrization

There are certain physical systems whose action is invariant under reparametrization, $\tau \rightarrow f(\tau)$, of the time variable $\tau$, and thus the physics of the model will not change. So the mentioned transformation is a gauge transformation and the parameter $\tau$ is a gauge parameter which does not represent the true physical time. Thus the evolution with respect to $\tau$ is just unfolding gauge transformations. Expectedly, these systems are totally constrained which means that the Hamiltonian can be written as

$$H = \int d^3x N^i \mathcal{C}_i,$$  \hspace{1cm} (3.1)

where $N^i$ are Lagrange multipliers and $\mathcal{C}_i$ are (first class) constraints of the system. In order to get a true physical evolution (i.e. relational evolution), one should fully fix the gauge in these systems. Then it turns out that at least one of the introduced gauge fixing conditions $\chi_j = 0$, should explicitly depend on $\tau$ and hence $\tau$ can be expressed as a combination of canonical variables $\tau = t(Q,P)$, where $Q,P$ symbolically stands for a full set of phase space variables which has a dimension equal to or greater than four. Such a time variable $t$ is called an internal time.

In some of these models, one can explicitly find (a combination of) canonical variable(s) as above, which has a monotonic gauge-dependent relationship with $\tau$. This (combination of) canonical variable(s) can then be taken as the internal time$^3$ and using a canonical transformation $(Q,P) \rightarrow (t,\pi,q,p)$, such that $t(Q,P) \rightarrow t$ and $\pi$ is conjugate to $t$, the system can be “deparametrized” and written in the following form$^4$

$$\mathcal{C} = G(\pi) - h(q,p).$$  \hspace{1cm} (3.2)

This deparametrization can be done in two ways, either $G(\pi) = \pi$ or $G(\pi) = \pi^2$ (or some linear, or quadratic polynomials of $\pi$ respectively). The former case is called non-relativistic while the latter one is called relativistic deparametrization. Note that these terms actually refer to the type of deparametrization not the physical model itself. They are called like this due to the similarity of their form to relativistic and non-relativistic dispersion relation (considering $\pi$ as the “energy” which is the momentum of “time”).

Now one has two choices: either use the non-fixed theory and find the gauge time evolution with respect to $\tau$ using constraint $\mathcal{C}$ for a function $f(q,p,t,\pi)$

$$\frac{\partial f(q,p,t,\pi)}{\partial \tau} = f(q,p,t,\pi) = \{f(q,p,t,\pi),\mathcal{C}\}$$

$$= \{f(q,p,t,\pi),G(\pi)\} - \{f(q,p,t,\pi),h(q,p)\}$$

$$= \frac{\partial f}{\partial \pi} \frac{\partial G}{\partial \pi} - \{f,h\},$$  \hspace{1cm} (3.3)

or go to the reduced phase space where $\frac{\partial f(q,p,t,\pi)}{\partial \pi} = 0$ and use the full gauge fixing condition, i.e. solving $\mathcal{C} = 0$ and get (using above and the form of $\frac{\partial f}{\partial \pi}(q,p)$ from solving $\mathcal{C} = 0$)

$$\frac{\partial f}{\partial t} = \frac{1}{\mathcal{C}} \{f,h\} = \left\{ \begin{array}{ll} \{f,h\} & \text{non-relativistic case} \\ \frac{1}{2\hbar} \{f,h\} = \{f,\sqrt{h}\} & \text{relativistic case} \end{array} \right.$$  \hspace{1cm} (3.4)

We see that in the gauge fixed method, in both case above, the evolution of $f$ with respect to $t$ is given by $\pi$ (after solving it in terms of $h$ from $\mathcal{C} = 0$). Hence $\pi(q,p) = h$ or $\pi(q,p) = \sqrt{h}$ provide us with the true evolution. We thus conclude that, from (3.2), for a non-relativistic deparametrized constraint we have

$$\mathcal{C} = \pi - h(q,p),$$  \hspace{1cm} (3.5)

$^3$ In principle one can have several internal times $T^j$. Here we assume there is only one.

$^4$ There are cases in which $h$ depends also on the physical time $T$, but models studied here do not have this property.
with $\pi = h(q, p)$ being the true Hamiltonian, while for a relativistic one we get

$$\mathcal{C} = \pi^2 - h(q, p),$$

where the true Hamiltonian is $\pi = \sqrt{h(q, p)}$. Later we will see that these actually correspond to the form of deparametrized constraints of non-relativistic and relativistic particles.

On the other hand, if one has a system with a true Hamiltonian $\pi(q, p)$ that generates true evolution with respect to (an internal) time $t$, under certain conditions it is possible to extend the phase space to also include $t$ and its conjugate $\pi$, and then turn the system into a gauge system by deriving a constraint in deparametrized from which gives the evolution of the system with respect to a non-physical time parameter $\tau$ which is of course a gauge transformation.

### B. Non-relativistic and relativistic particles as constrained systems

Following what we mentioned in the last subsection, we can write both non-relativistic and relativistic particles as constrained system. First consider the non-relativistic case. The action is simply

$$S = \int dt \frac{1}{2} m \left( \frac{dx}{dt} \right)^2,$$

This is a system with true Hamiltonian

$$H = \frac{p_x}{2m},$$

which generates true physical evolution with respect to $t$. In order to enlarge the phase space by including $t$ in it, we introduce the non-physical variable $\tau$ such that $\tau = f(t)$, to get

$$S = \int d\tau \frac{1}{2} m \left( \frac{dx}{d\tau} \right)^2 = \int d\tau \frac{1}{2} m \ddot{x},$$

where now $t$ is a canonical variable and (gauge) evolution is with respect to $\tau$. Canonical analysis of the theory shows that we have

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{m \ddot{x}}{t},$$

$$p_t = \frac{\partial L}{\partial \dot{t}} = -\frac{1}{2} \frac{m \ddot{x}^2}{t^2},$$

and the Hamiltonian is

$$H = p_x \ddot{x} + p_t \dot{t} - L = \frac{m \ddot{x}^2}{i} - \frac{1}{2} \frac{m \ddot{x}^2}{i} - \frac{1}{2} \frac{m \ddot{x}^2}{t} = 0,$$

which shows that we have a totally constrained system. The system exhibits a constraint that from definitions of $p_x$ and $p_t$ above reads

$$\mathcal{C} = p_t + \frac{p_x^2}{2m} = 0.$$

\[^{5}\text{In these cases the notion of relativistic and non-relativistic deparametrizations coincide with the properties of the physical systems themselves.}\]
Now we can see that this has the non-relativistic form of a deparametrized constraint (3.5) with $\pi = p_t$ and thus the true Hamiltonian after gauge fixing is $\pi = p_t = -\frac{p_x^2}{2m}$ which generates true evolution (with a minus sign) with respect to $t$ as expected.

In the case of the relativistic particle, we have

$$S = \int dt \, m \sqrt{-\eta_{ab} \frac{dx^a}{dt} \frac{dx^b}{dt}}.$$ (3.14)

Here the system is already constrained and is reparametrization-invariant as can be seen by the fact that introducing $t = f(s)$ yields the same action

$$S = \int ds \, m \sqrt{-\eta_{ab} \frac{dx^a}{ds} \frac{dx^b}{ds}}.$$ (3.15)

Counting $t$ as a canonical variables and writing $\dot{x}^a = \frac{dx^a}{ds}$, one can compute the momenta

$$p_a = \frac{\partial L}{\partial \dot{x}^a} = -m \frac{\eta_{ab} \dot{x}^b}{\sqrt{-\eta_{ab} \frac{dx^a}{ds} \frac{dx^b}{ds}}}$$ (3.16)

and it is easy to see that

$$H = p_a \dot{x}^a - L = 0,$$ (3.17)

hence a constraint system. The constraint is

$$\mathcal{C} = \eta^{ab} p_a p_b + m^2 = 0,$$ (3.18)

which is the famous relativistic dispersion relation. This can be written as

$$\mathcal{C} = p_t^2 - p_x^2 - m^2 = 0$$ (3.19)

which not surprisingly takes the form (3.6) with $G(\pi) = \pi^2 = p_t^2$ and the true Hamiltonian in this case (after complete gauge fixing) is $\pi = p_t = \sqrt{p_t^2 + m^2}$. Later we will use (3.13) and (3.19) to study each case.

### C. Quantization and group averaging

To quantize a classical theory a la Dirac, one represents the classical canonical variables and their algebra on a vector space such that there are no anomalies, i.e. the quantum algebra $\{\hat{q}, \hat{p}\}$ mimics the classical algebra $\{q, p\}$, and then equip this vector space with an inner product and Cauchy-complete it to get a Hilbert space. Given that the classical phase space functions can now be represented as operators on this Hilbert space, one can proceed to compute the desired quantities using these states, operators and the inner product.

If the classical theory has first class constraints $\mathcal{C}_I$, i.e. it is a gauge theory, then the resulting Hilbert space is actually not the Hilbert space $\mathcal{H}_{\text{phys}}$ of physical states $|\psi_{\text{phys}}\rangle$, but in general it is a bigger space called the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ with states $|\psi_{\text{kin}}\rangle$, not all of them corresponding to the physical solutions. To be able to derive physical results, one has to find $\mathcal{H}_{\text{phys}}$. This means finding a subset of states of $\mathcal{H}_{\text{kin}}$ that are physical, and then define an inner product between them (physical inner product $\langle \cdot | \cdot \rangle_{\text{phys}}$) so that this subspace of states becomes a Hilbert space.

---

6 In general, the set $\{|\psi_{\text{phys}}\rangle\}$ are not needed and can not be in $\mathcal{H}_{\text{kin}}$, hence the word “subset” here may be misleading. This is especially the case when the gauge group is non-compact. However, in that case, $\{|\psi_{\text{phys}}\rangle\}$ can exist in a certain algebraic dual of a dense subset of $\mathcal{H}_{\text{kin}}$. See for example [7] for details.
The set \( \{ |\psi_{\text{phys}} \rangle \} \) are the ones that are annihilated under the action of the quantum version of the constraints \( \hat{C}_j \). The reason for this criterion is that since the constraints generate infinitesimal gauge (i.e. non-physical) transformations, physical states should remain invariant under these transformations [41]. More precisely, since classical first class constraints commute with each other weakly (i.e. on the constraint surface)

\[
\{ C_I, C_J \} \approx 0 \Rightarrow \{ C_I, C_J \} = \xi^{K}_{IJ} C_K , \quad (3.20)
\]

with \( \xi^{K}_{IJ} \) being structure constants, and since the algebra is represented in an anomaly-free manner, then one has

\[
[\hat{C}_I, \hat{C}_J] |\psi_{\text{phys}} \rangle = 0 \Rightarrow \hat{C}_K |\psi_{\text{phys}} \rangle = 0 . \quad (3.21)
\]

This way one can also see that the physical states \( |\psi_{\text{phys}} \rangle \) are invariant under the action of the associated gauge group elements \( \hat{U} = e^{-i\alpha^I \hat{C}_I} \)

\[
\hat{U} |\psi_{\text{phys}} \rangle = |\psi_{\text{phys}} \rangle \quad (3.22)
\]
as desired, with \( \alpha^I \) the Lagrange multipliers that are the parameters of the group. If these states are found, then one should proceed to define an inner product between them to construct \( \mathcal{H}_{\text{phys}} \).

Solving quantum constraints however, is not always an easy task, and when one can not solve (3.21) for the states \( |\psi_{\text{phys}} \rangle \), one obviously cannot define an inner product between them. So other methods are to be pursued. One way to deal with this issue is the so-called group averaging technique [7, 42, 43]. Not only this method solves for physical states in some sense, it also provides us with a physical inner product given that the kinematical space already has one, i.e. given that we already have a \( \mathcal{H}_{\text{kin}} \) with \( \langle \cdot | \cdot \rangle_{\text{kin}} \). We do not go into the details but just mention a few points that are important to us in this paper.

Given that \( \hat{U}(\alpha) = e^{-i\hat{c}^I \alpha^I} \) are the members of the gauge group, one writes physical states as

\[
|\psi_{\text{phys}} \rangle = \int d\alpha \langle \psi_{\text{kin}} | \hat{U}^\dagger(\alpha) \rangle \quad (3.23)
\]

and then, using the kinematical inner product, the physical inner product is defined as

\[
\langle \psi_{\text{phys}} | \phi_{\text{phys}} \rangle_{\text{phys}} := \int d\alpha \langle \psi_{\text{kin}} | \hat{U}^\dagger(\alpha) |\phi_{\text{kin}} \rangle_{\text{kin}} . \quad (3.24)
\]

This inner product has certain properties which amounts to solving for the states for which \( \hat{U}(\alpha) |\psi \rangle = |\psi \rangle \).

Finally note that in the non-relativistic deparametrization case, (3.5) or (3.13), if we represent the “time momentum” as a derivative \( \hat{\pi} = i\hbar \frac{\partial}{\partial t} \) and use the condition of physical states (3.21), we will get

\[
\hat{C} |\psi_{\text{phys}} \rangle = 0 \Rightarrow i \frac{\partial}{\partial t} |\psi_{\text{phys}} \rangle = \hat{h} |\psi_{\text{phys}} \rangle \quad (3.25)
\]

which is just the Schrödinger’s equation given that \( \hat{h} \) is the true Hamiltonian operator generating physical evolution with respect to the physical time \( t \). In the case of relativistic deparametrization (3.6) or (3.19), by the same argument and representation we get

\[
\hat{C} |\psi_{\text{phys}} \rangle = 0 \Rightarrow -\hbar^2 \frac{\partial^2}{\partial t^2} |\psi_{\text{phys}} \rangle = \hat{h} |\psi_{\text{phys}} \rangle , \quad (3.26)
\]

which for the case of a particle gives the equivalent of the Klein-Gordon equation. So in both cases, one can regain a physical evolution in terms of a true Hamiltonian operator that generates time evolution with respect to an internal physical time \( t \).
D. Polymer path integral

Most of the interesting properties of quantum systems are expressed in terms of transition amplitudes

$$\langle \psi_f | \hat{U}(t_f,t_i) | \psi_i \rangle, \quad t_i < t_f,$$

(3.27)

between initial and final states $| \psi_i \rangle$ and $| \psi_f \rangle$ respectively, with an evolution operator $\hat{U}(t_f,t_i)$. Using the completeness relations of some basis, say $| q \rangle$, one can write

$$\int dq_f \int dq_i \langle \psi_f | q_f \rangle \langle q_f | \hat{U}(t_f,t_i) | q_i \rangle \langle q_i | \psi_i \rangle = \int dq_f \int dq_i \psi_f^* (q_f) \psi_i(q_i) K(q_f,t_f;q_i,t_i),$$

(3.28)

where we have defined the kernel $K(q_f,t_f;q_i,t_i)$ as the matrix elements of the evolution operator

$$K(q_f,t_f;q_i,t_i) = \langle q_f | \hat{U}(t_f,t_i) | q_i \rangle.$$

(3.29)

The kernel itself may be written as a path integral

$$K(q_f,t_f;q_i,t_i) = \int \mathcal{D}' q \mathcal{D} p e^{i \mathcal{S}' p}$$

(3.30)

as we will see in details in what follows for different cases. Note that In this case, $\mathcal{S}'$ and $\mathcal{D}' q \mathcal{D} p$ are not the classical action and the usual measure that appears in the Schrödinger quantization, hence the primes used for them.

Given that the Hamiltonian is time independent, the kernel in the form (3.29) can be interpreted as the transition amplitude

$$K(q_f,t_f;q_i,t_i) = \langle q_f,t_f | q_i,t_i \rangle$$

(3.31)

between the kets $| q_i,t_i \rangle = e^{\hat{H}t_i} | q_i \rangle$ and $| q_f,t_f \rangle = e^{\hat{H}t_f} | q_f \rangle$. Note that these are not the Schrödinger-picture time evolution of states $| q_i \rangle$ and $| q_f \rangle$, since they have the wrong sign in the exponent in front of them, but they are actually eigenstates of the $\hat{q}$ operator in the Heisenberg picture at corresponding times $t_i$ and $t_f$.

If the states $| q_i, t_i \rangle$ and $| q_f, t_f \rangle$ belong to the $\mathcal{H}_{\text{phys}}$ of a single particle system, a gauge system with only a single first class constraint $\mathcal{C}$, the kernel due to (3.24) and (3.31) can be written as

$$K(q_f,t_f;q_i,t_i)_{\text{phys}} = \langle q_f,t_f | q_i,t_i \rangle_{\text{phys}} = \int d\alpha \langle q_f,t_f | \hat{U}(\alpha) | q_i,t_i \rangle_{\text{kin}} = \int d\alpha \langle q_f,t_f | e^{i\alpha \mathcal{C}} | q_i,t_i \rangle_{\text{kin}}.$$

(3.32)

From now on we drop the “phys” and “kin” subscripts and they are to be understood in the context. Using (3.5) and (3.6), the kernel for non-relativistic and relativistic deparametrized cases can then be written as

$$K(q_f,t_f;q_i,t_i) = \begin{cases} \int_\mathbb{R} d\alpha \langle q_f,t_f | e^{i\alpha(\hat{\pi} - \hat{H})} | q_i,t_i \rangle, & \text{non-relativistic,} \\ \int_\mathbb{R} d\alpha \langle q_f,t_f | e^{i\alpha(\hat{\pi}^2 - \hat{H})} | q_i,t_i \rangle, & \text{relativistic,} \end{cases}$$

(3.33)

since in general $\hat{\pi}$ and $\hat{H}$ commute. If we assume that the states $| q,t \rangle$ can be written as a tensor product $| q \rangle \otimes | t \rangle$, which can be taken as a consequence of $\hat{\pi}$ and $\hat{H}$ commuting, and choose a polarization of the polymer representation such that the form of $\hat{H}$ (rather than $\hat{\pi}$) is changed from Schrödinger to polymer form (or equivalently only polymerize the pair $\langle q,p \rangle$ and quantize the pair $\langle \pi,t \rangle$ using Schrödinger representation), then we can write

$$K_{\text{poly}}(q_f,t_f;q_i,t_i) = \begin{cases} \int_\mathbb{R} d\alpha \langle t_f | e^{|\alpha|\pi} | t_i \rangle \langle q_f | e^{-i\alpha h} | q_i \rangle, & \text{non-relativistic,} \\ \int_\mathbb{R} d\alpha A_\pi(t_f,t_i;\alpha) A_{\text{poly}}(q_f,q_i;\alpha), & \text{relativistic,} \end{cases}$$

(3.34)
where \( \langle f | \exp (\cdots) | i \rangle \) can be taken as the probability amplitudes of transiting from \( |i\rangle \) to \( |f\rangle \) under the unitary transformation \( \exp (\cdots) \). The amplitudes have been factorized into two terms due to \([\hat{h}, \hat{\pi}] = 0\) and the resulting factorization \(|q,t\rangle = |q\rangle \otimes |t\rangle\). We have defined the polymer amplitude in each case as

\[
A_{\text{poly}}(q_f, q_i; \alpha) := \langle q_f | e^{-i\alpha \hat{h}} | q_i \rangle. \tag{3.35}
\]

The other amplitude

\[
A_{\pi}(t_f, t_i; \alpha) = \begin{cases} 
\langle t_f | e^{i\alpha \hat{x}} | t_i \rangle, & \text{non-relativistic} \\
\langle t_f | e^{i\alpha \hat{x}} | t_i \rangle, & \text{relativistic}
\end{cases} \tag{3.36}
\]

corresponds to the momentum of the physical time. In the rest of the work, we will derive these amplitudes and subsequently the whole kernel for three cases of a non-relativistic free particle, non-relativistic particle in a box, and the relativistic particle.

## IV. GENERIC POLYMER AMPLITUDE

If \( \hat{h} \) in \( A_{\text{poly}}(q_f, q_i; \alpha) \) in (3.35) commutes with itself, which is the case for both free relativistic and non-relativistic particles (see (3.19) and (3.13)), one can write the polymer amplitude as

\[
A_{\text{poly}}(q_f, q_i; \alpha) = \left\langle q_f \right| \left( e^{-i\alpha \hat{h}} \right)^N | q_i \rangle = \left\langle q_f \right| e^{-i\alpha \hat{h}} \cdots e^{-i\alpha \hat{h}} \right| q_i \rangle \tag{4.1}
\]

with \( \varepsilon = \frac{1}{N} \). This is done so that we can proceed with the Feynman procedure of writing the amplitude as a sum over histories. Consequently this decomposition means that we are taking \( \hat{h} \) as an evolution operator and are considering the whole (gauge) temporal interval as \( \Delta \tau = N\varepsilon = 1 \). Note that, we are looking at these systems as constrained ones, since the evolution of the system is generated by the constraint \( \hat{\mathcal{C}} \), hence the whole evolution is a gauge transformation. So \( t \) is one of the canonical variables and the evolution steps are counted in \( \tau \) rather than internal time \( t \).

If the \(|q\rangle\) basis is discrete, e.g. when we are in \( q \)-polarization of the polymer representation, we have the identity \( \hat{\pi}_q = \sum_n |q_n\rangle \langle q_n| \), and inserting \( N - 1 \) of them in between the exponentials in above yields

\[
A_{\text{poly}}(q_f, q_i; \alpha) = A_{\text{poly}}(\bar{q}_N, \bar{q}_0; \alpha) = \sum_{\bar{q}_1, \ldots, \bar{q}_{N-1}} \left\langle \bar{q}_{N} | e^{-i\alpha \hat{h}} \bar{q}_{N-1} \right| \left\langle \bar{q}_{N-1} | e^{-i\alpha \hat{h}} \bar{q}_{N-2} \right| \cdots \left\langle \bar{q}_1 | e^{-i\alpha \hat{h}} \bar{q}_0 \right|,
\]

where \( q_f = \bar{q}_N \) and \( q_i = \bar{q}_0 \) and we maintain that it is possible to have \( \bar{q}_k = \bar{q}_{k+1} \) for some or all of \( k \). If \( \bar{q}_{k-1} \neq \bar{q}_k \) we say that a position transition has happened. This way we can express \( A_{\text{poly}}(\bar{q}_N, \bar{q}_0; \alpha) \) in terms of smaller amplitudes \( U_{k,l} = \left\langle \bar{q}_k | e^{-i\alpha \hat{h}} \bar{q}_l \right| \)

\[
A_{\text{poly}}(\bar{q}_N, \bar{q}_0; \alpha) = \sum_{\bar{q}_1, \ldots, \bar{q}_{N-1}} \prod_{k=1}^N U_{k,k-1}. \tag{4.3}
\]

The picture here is one of a particle going from an initial \( \bar{q}_0 \) to a final \( \bar{q}_N \) in \( N \) steps, each step e.g. from \( \bar{q}_{k-1} \) to \( \bar{q}_k \) taken under a finite evolution \( U_{k,k-1} \). This way we have introduced a discrete path or history \( \sigma_N = (\bar{q}_0, \ldots, \bar{q}_N) \) for the particle. Typically at this point, one derives the path integral by taking the limit \( N \to \infty \). However this is not rigorous in our case since we are in \( q \)-polarization of the polymer representation and this means that the discreteness of \( q \) in e.g. (4.2) is intrinsic and not just due to arbitrary non-physical discretization of the interval \([q_i, q_f]\) often used in deriving path integral in ordinary quantum field theory.
Thus we need to find a rigorous manner to do this for our case. To do this, we reformulate the model in terms of position transitions instead of time steps as follows.

To implement the position transition framework, we first assume that in these \(N\) steps, \(M\) position transitions \(\bar{q}_{k-1} \rightarrow \bar{q}_k\) may happen and we call these distinct positions \(q_k\) without a bar. Thus in these \(N\) steps, we may have \(M+1\) positions \(q_k\) corresponding to \(M\) transitions. However, we only put a restriction on the set \(\{q_k\}\) such that although no two consecutive \(q_k\)'s can be the same, two non-consecutive ones, i.e. with \(q_j\) and \(q_l\) when \(j \neq l \pm 1\), can. So we have provided a discrete history \(\sigma_N^M\) for the particle’s position in the form of an ordered sequence

\[
\sigma_N^M = \left( \bar{q}_{N_{M-1}+1}, \ldots, \bar{q}_{N_N}, \bar{q}_{N_{M-1}}, \ldots, \bar{q}_{N_1}, \ldots, \bar{q}_{N_{M-1}+1}, \ldots, \bar{q}_{N_1} \right)
\]

of a particle that starts from \(\bar{q}_0 = q_0\), and in \(N_1 - 1\) steps (for the period \(\tau = (N_1-1)\varepsilon\) still remains at \(\bar{q}_{N_1-1} = q_0\) without any position transition. Then, at this time, it makes a transition from there to \(q_{N_1} = q_1 \neq q_0\), and so on. At the \(r\)'th transition, the particle goes from \(\bar{q}_{N_{r-1}} = q_{r-1}\) to \(\bar{q}_{N_r} = q_r\). Note that the total number of transitions \(M\) is smaller or equal to the total number of steps \(N\), and all the \(\bar{q}\) between (and equal to) \(\bar{q}_{N_1}\) to \(\bar{q}_{N_{r+1}-1}\) are equal to \(q_{r+1}\). Also note that \(\bar{q}_k = k\varepsilon\) is the position after \(k\) steps, while \(q_r = N_r\varepsilon\) is the position after the \(r\)'th transition. This history can be expressed more concisely in terms of two ordered sequences

\[
\sigma_N^M = \{ (q_N, \ldots, q_0), (N_M, \ldots, N_1) \}, \quad q_{k-1} \neq q_k, N_K > N_{K-1}.
\]

This way, the probability amplitude of a certain history \(\sigma_N^M\) is written as

\[
A_{\text{poly}}(\sigma_N^M) = U_{N,M}^{-N_{M-1}} U_{M,M-1} \cdots U_{1,1}^{N_2-N_1-1} U_{1,0}^{N_1-1},
\]

where now

\[
U_{k,l} = \langle q_k | e^{-i\alpha \hat{b}} | q_l \rangle.
\]

Next, we need to implement the sum over all the amplitudes corresponding to all possible histories. This is done in three steps:

**Step 1:** We consider all the possible transitions at any possible time step, e.g. the first transition may happen at any time between \(\tau = 0\) (particle at \(q_0\) only at \(\tau = 0\) and then goes from \(q_0\) to \(q_1\)) to \(\tau = (N_2-1)\varepsilon\) (particle staying at \(q_0\) for up to the end of \(\tau = (N_2-2)\varepsilon\), at that time goes from \(q_0\) to \(q_1\) and remains there until \(\tau = (N_2-1)\varepsilon\), and then goes from \(q_1\) to \(q_2\) such that at \(\tau = N_2\varepsilon\) it is at \(q_2\), and so forth. Summing over all these types of amplitudes for all possible minima to maxima of \(N_i\)'s, we get

\[
A_{\text{poly}}^N(q_M, \ldots, q_0; \alpha) = \sum_{N_M=M}^{N-1} \sum_{N_{M-1}=M-1}^{N_N} \cdots \sum_{N_1=1}^{N_{M-1}} A_{\text{poly}}(\sigma_N^M), \quad N = N_{M+1}.
\]
Step 2: Here we sum over all the paths with exactly $M$ transitions but with all possible intermediate values of $q_k$’s, given fixed initial and final positions $q_0$ and $q_M$, and such that $q_k-1 \neq q_k$ as mentioned before,

$$A_{\text{poly}}^N (M; \alpha) = \sum_{q_{M-1}, \ldots, q_1 \atop q_{k-1} \neq q_k} A_{\text{poly}}^N (q_M, \ldots, q_0; \alpha). \tag{4.9}$$

Step 3: Finally we sum over all possible number of transitions $0 \leq M \leq N$, and since this amplitude does not depend on $N$, we can take the limit $N \to \infty$ to get the full polymer amplitude

$$A_{\text{poly}} (q_f, q_i; \alpha) = \lim_{N \to \infty} \sum_{M=0}^N A_{\text{poly}}^N (M; \alpha). \tag{4.10}$$

Free particle

Since we are going to work with free particles, further simplifications of the above amplitude are possible. In that case, since the operator $\hat{h}$ in $U_{k,l} = \langle q_k \mid e^{-i\alpha \hat{h}} \mid q_l \rangle$ only contains the polymer kinetic term with no potential term present to introduce $q$ dependence, we will have

$$U_{0,0} = U_{1,1} = \cdots = U_{M,M}. \tag{4.11}$$

Using this, (4.6) will become

$$A_{\text{poly}} (\sigma_N^M) = U_{M,M-1} \cdots U_{1,0} \left[ U_{M,M}^{N-N_M-1} \cdots U_{M,M}^{N_{2}-N_1-1} U_{M,M}^{N_1-1} \right], \quad N = N_{M+1}$$

$$= \prod_{k=1}^{M} U_{k,k-1}^{N_{M+1}-1}. \tag{4.12}$$

On the other hand $U_{M,M}^{N-M-1}$ for large $N$ and finite $M$ is

$$U_{M,M}^{N-M-1} = e^{-i\alpha h_{MM}} \left[ 1 + O \left( N^{-1} \right) \right], \tag{4.13}$$

where $h_{MM} = \langle q_M \mid \hat{h} \mid q_M \rangle$ and $\epsilon = N^{-1}$. One also gets

$$U_{k,k-1} = \langle q_k \mid e^{-i\alpha \hat{h}} \mid q_i \rangle = \langle q_k \mid 1 - i\alpha \hat{h} + O \left( \epsilon^2 \right) \mid q_i \rangle = -i\alpha \epsilon h_{k,k-1} \left[ 1 + O \left( N^{-2} \right) \right]. \tag{4.14}$$

Substituting these into (4.12) yields

$$A_{\text{poly}} (\sigma_N^M) = \prod_{k=1}^{M} -i\alpha \epsilon h_{k,k-1} \left[ 1 + O \left( N^{-2} \right) \right] e^{-i\alpha h_{MM}} \left[ 1 + O \left( N^{-1} \right) \right]. \tag{4.15}$$

Now using this in the first step (4.8), one arrives at

$$A_{\text{poly}}^N (q_M, \ldots, q_0; \alpha) = (-i\alpha \epsilon)^M \left[ \prod_{k=1}^{M} h_{k,k-1} \right] e^{-i\alpha h_{MM}} \sum_{N_M=M}^{N-1} \sum_{N_{M-1}=M-1}^{N_M-1} \cdots \sum_{N_2=1}^{N_{M-1}} \left[ 1 + O \left( N^{-1} \right) \right]. \tag{4.16}$$

By changing $N_j \to N_j + j$, $\forall j$, the sums in above can be cast into the following more convenient form,

$$A_{\text{poly}}^N (q_M, \ldots, q_0; \alpha) = (-i\alpha \epsilon)^M \left[ \prod_{k=1}^{M} h_{k,k-1} \right] e^{-i\alpha h_{MM}} \sum_{N_M=0}^{N-M} \sum_{N_{M-1}=0}^{N_M} \cdots \sum_{N_1=0}^{N_2} \left[ 1 + O \left( N^{-1} \right) \right]. \tag{4.17}$$
In the following we first take the limit \( N \to \infty \) (partially performing step 3 above) and then proceed to sum over all possible \( q \) and then over all possible \( M \). Each time interval without transition \( \tau_j \) has a time length of \( \tau_j = N_j \varepsilon \). Using this, taking the limit \( N_j \to \infty \), and noting \( \varepsilon \sum_{N_j} \to \int d \tau_j \), we can convert the sums above into integrals

\[
A_{\text{poly}}(q_M, \ldots q_0; \alpha) = (-i\alpha)^M \left[ \prod_{k=1}^{M} h_{k,k-1} \right] e^{-i \alpha h_{MM}} \int_{0}^{1} d \tau_M \cdots \int_{0}^{\tau_2} d \tau_1 = \frac{1}{M!} (-i\alpha)^M \left[ \prod_{k=1}^{M} h_{k,k-1} \right] e^{-i \alpha h_{MM}},
\]

(4.18)

where in the limit of the last integral \( \varepsilon \sum_{N_j=M}^{N-1} \to \int_{0}^{1} d \tau_M \) we have used the fact that the total interval is of unit length, \( \tau = N \varepsilon = 1 \), and \( M \) is finite and we can neglect its contribution:

\[
(N-M) \varepsilon \approx N \varepsilon = 1.
\]

(4.19)

Now we perform step 2 mentioned above and sum over all possible intermediate \( q_k \)'s for a fixed \( M \). This leads to

\[
A_{\text{poly}}(q_f, q_i; M; \alpha) = \frac{1}{M!} (-i\alpha)^M e^{-i \alpha h_{MM}} \sum_{q_{M-1} \ldots q_1} \left[ \prod_{k=1}^{M} h_{k,k-1} \right].
\]

(4.20)

Finally step 3 is completed by summing over all possible number of transitions \( M \) which yields

\[
A_{\text{poly}}(q_f, q_i; \alpha) = \sum_{M=0}^{\infty} \frac{1}{M!} (-i\alpha)^M e^{-i \alpha h_{MM}} \sum_{q_{M-1} \ldots q_1} \left[ \prod_{k=1}^{M} h_{k,k-1} \right].
\]

(4.21)

Now we are ready to use this machinery to compute the full propagator of three types of system in the following sections.

V. NON-RELATIVISTIC PARTICLE

A. Free particle

This system, written in deparametrized form, has two configuration variables, \( t \) and \( q \). We will use the Schrödinger representation for \( t \) and its canonical conjugate \( p_t \), while \( q \) and its counterpart \( \hat{V}_\lambda \) are quantized using polymer representation. Their algebras read

\[
[h, p_t] = i\hbar, \quad [\hat{q}, \hat{V}_\lambda] = -\frac{\lambda}{\hbar} \hat{V}_\lambda, \quad [\hat{r}, \hat{q}] = 0.
\]

(5.1)

We derive the full polymer propagator of this model using (3.13) and (3.34)-(3.36). First notice that as mentioned before and as can be seen from (3.13) and (3.5), the amplitude \( A_\pi \) for this system, using the identity \( \hat{p}_t = \int_{\mathbb{R}} dp_t |p_t\rangle \langle p_t| \), is

\[
\langle t_f | e^{i \alpha \hat{p}_t} | t_i \rangle = \int_{\mathbb{R}} dp_t e^{i \alpha p_t e^{\pm \hbar \hat{V}_\lambda (t_f - t_i)}} = \delta \left( \frac{t_f - t_i}{\hbar} + \alpha \right)
\]

(5.2)

Now we set to compute the polymer amplitude \( A_{\text{poly}}(q_f, q_i; \alpha) \) using (4.21). For this, we need to compute \( h_{MM} \) and \( h_{k,k-1} \). Looking at (4.13) one can see that in this case, \( \hbar = -\frac{p_t^2}{2m} \). Since we are in \( q \)-polarization, the generator corresponding to \( p_\lambda \) does not exist on the Hilbert space and we should use (2.24) instead. This then leads to

\[
\hbar = -\frac{1}{2m} \frac{\hbar^2}{\lambda^2} (2 - \hat{V}_\lambda - \hat{V}_{-\lambda}).
\]

(5.3)
Using this and the form of the action of $\hat{V}_\lambda$, \[(2.20)\], we can write

$$h_{jk} = -\frac{1}{2m \lambda^2} \langle q_j | (2 - \hat{V}_\lambda - \hat{V}_{-\lambda}) | q_k \rangle = -\frac{1}{2m \lambda^2} \left(2\delta_{q_j,q_k} - \delta_{q_j,q_k-\lambda} - \delta_{q_j,q_k+\lambda}\right),$$

and hence

$$h_{jk} = \begin{cases} \frac{-\hbar^2}{m \lambda^2}, & j = k \\ \frac{1}{2m \lambda^2} \left(\delta_{q_j,q_k-\lambda} + \delta_{q_j,q_k+\lambda}\right), & j \neq k. \end{cases} (5.5)$$

Substituting these into \[(4.20)\] yields

$$A_{\text{poly}}(q_f, q_i; M; \alpha) = \frac{1}{M!} e^{i\alpha \frac{\hbar^2}{m \lambda^2}} \left(-\frac{i\alpha}{2m \lambda^2}\right)^M S(q_f, q_i; M),$$

with

$$S(q_f, q_i; M) = \sum_{q_{M-1}, \ldots, q_1} \prod_{k=1}^M \left(\delta_{q_k,q_{k-1}-\lambda} + \delta_{q_k,q_{k-1}+\lambda}\right) = \sum_{q_{M-1}, \ldots, q_1} \prod_{k=0}^{M-1} \left(\delta_{q_{M-k},q_{M-k-1}+\lambda} + \delta_{q_{M-k},q_{M-k-1}-\lambda}\right).$$

It can be shown by induction that this can be rewritten in a more useful way

$$S(q_f, q_i; M) = \sum_{j=0}^M \binom{M}{j} \delta_{q_{M+j}, q_{j+2j-M}},$$

where $q_i$ is the initial position of the particle. This function has several important properties which will be used in the following computations. The first property is invariance under translations

$$S(q_2, q_1; M) = S(q_2 + r\lambda, q_1 + r\lambda; M), \quad r \in \mathbb{Z}. (5.9)$$

Another one is the symmetry under change of the signs

$$S(q_2, q_1; M) = S(-q_2, -q_1; M). (5.10)$$

The third property is what we call the “Pascal triangle” property which is related to the points in which $S$ has nontrivial values. More precisely, if the whole spatial interval is $q_f - q_i = \lambda L$, then it is easily seen that $S$ in \[(5.8)\] (and consequently $A_{\text{poly}}(q_f, q_i; M; \alpha)$ in \[(5.6)\]) has only one nonvanishing term for which $j = \lfloor (L + M)/2 \rfloor$. Since $0 \leq j \leq M$, this means that $|L| \leq M$. The graph of spatial position of these terms for different $M$’s becomes like a triangle, on each point of which, $S$ has a value corresponding to the binomial coefficients, hence the Pascal triangle (Fig. \[(1)\]).

We now use these properties to simplify the polymer amplitude. First we assume that $L$ is even. Then $A_{\text{poly}}(q_f, q_i; M; \alpha)$ will only have nontrivial values for an even $M = 2k$, $k \in \mathbb{Z}$, and greater than $|L|$. Then the final polymer amplitude, using step 3 mentioned before (for \[(5.6)\] and \[(5.8)\]), is

$$A_{\text{poly}}(q_f, q_i; \alpha) = \sum_{M=|L|}^\infty A_{\text{poly}}(q_f, q_i; M; \alpha) = \sum_{k=|\frac{L}{2}|}^\infty \frac{1}{(2k)!} \left(-\frac{i\alpha}{2m \lambda^2}\right)^{2k} e^{i\alpha \frac{\hbar^2}{m \lambda^2}} \left(\frac{2k}{k + \frac{L}{2}}\right).$$

\[(5.11)\]
Using a change of index, \( k \to k + \frac{|L|}{2} \), in the above yields

\[
A_{\text{poly}}(q_f, q_i; \alpha) = (-1)^L i^L e^{i \frac{L \alpha^2}{m \hbar^2} t} J_L \left( \frac{\alpha \hbar^2}{m \lambda^2} \right),
\]

with \( J_L \) being the Bessel function of the first kind and we have used the property \( J_n(-x) = (-1)^n J_n(x) \). If we would have assumed that \( L \) is odd, the odd values of \( M \) for which \( M = 2k + 1 \) and \( M \geq |L| \) would have given rise to nonvanishing terms, but then again, we would have gotten the same result. Now the full propagator using the above, combined with (3.34) and (5.2), becomes

\[
K_{\text{poly}}(q_f, t_f; q_i, t_i) = i^L e^{-i[L - t_i] \frac{\hbar}{m \alpha^2} J_L \left( \frac{|t_f - t_i| \hbar}{m \lambda^2} \right)},
\]

This result coincides with earlier results based on Hamiltonian approach [44]. We will now discuss the continuum limit of this propagator.

1. The continuum limit

One can get the continuum limit of a polymer quantized theory by letting \( \lambda \to 0 \), namely taking the fundamental discrete scale to be zero. In our case, we assume that while \( \lambda \to 0 \) and \( L \to \infty \), the spatial interval remains finite, \((q_f - q_i) = \lambda L < \infty\). To find the continuum limit, we first rewrite (5.13) in the form

\[
K_{\text{poly}}(q_f, t_f; q_i, t_i) = i^L e^{-iL \lambda^2 \Delta} J_L \left( \frac{h}{\hbar} \Delta \right), \quad \Delta = \frac{h(t_f - t_i)}{m(q_f - q_i)^2}.
\]

Using an identity of the Bessel functions,

\[
J_L(\lambda \sec(\beta)) \approx \exp[i L (\tan(\beta) - \beta)] \quad \text{for} \quad \sec(\beta) = L \Delta,
\]

in the above \( K_{\text{poly}}(q_f, t_f; q_i, t_i) \) and then Taylor expanding each term separately around \( z := \sec(\beta) = L \Delta \to 0 \) (i.e. taking \( \lambda \to 0 \)) and keeping only the first terms yields

\[
K_{\text{poly}}(q_f, t_f; q_i, t_i) \approx \lambda K_0(q_f, t_f; q_i, t_i),
\]

Figure 1. a) The set of points for which the function \( S(q_f, q_i; M) \) is nontrivial. Only the coordinates of boundary points are written. b) Nontrivial values of the function \( S(q_f, q_i; M) \) that form the values of a Pascal triangle.
where

\[ K_0(q_f, t_f; q_i, t_i) = \sqrt{\frac{m}{2\pi\hbar(t_f - t_i)}} \exp \left[ \frac{im(q_f - q_i)^2}{2\hbar(t_f - t_i)} \right], \tag{5.17} \]

is the standard propagator of the free particle. Note that the polymer propagator corresponds to a countable measure due to the presence of \( \lambda \) in (5.16). This permits us to go from discrete sum over positions to a corresponding integral. More precisely, the presence of \( \lambda \) in the right side of (5.16) is necessary for following reason. To compute the evolution of an initial wave equation, \( \psi_0(q) \), we make use of the propagator and the measure of the Hilbert space in the following way,

\[ \psi(q, t) = \sum_{q' \in \text{Lattice}} K_{\text{Poly}}(q, t; q', 0) \psi_0(q) = \sum_{q' \in \text{Lattice}} \lambda K_0(q, t; q', 0) \psi_0(q) + O(\lambda^2). \tag{5.18} \]

If we consider the limit \( \lambda \to 0 \) in this equation, we get

\[ \sum_{q' \in \text{Lattice}} K_{\text{Poly}}(q, t; q', 0) \psi_0(q) \to \int dq' K_0(q, t; q', 0) \psi_0(q), \tag{5.19} \]

which shows the presence of \( \lambda \) allows us to go from the discrete sum to the corresponding integral, \( \sum_0 \lambda \to \int dq \), cf. [45].

### B. Particle in a box

This case is essentially the free particle with a specific boundary condition. We implement this boundary condition by the method of images used in optics as will be explained in detail in what follows.

To find the polymer amplitude, we start by deriving the form of the function \( \bar{S}(q_f, q_i; M) \) which is the equivalent of the function \( S(q_f, q_i; M) \) in the case of the free particle. To implement the boundary conditions, we maintain that

\[ \bar{S}(q_f, q_i; M) \bigg|_{q_f=x^\pm} = 0, \tag{5.20} \]

where \( x^- \) and \( x^+ \) are the positions of the left and right walls of the box respectively. Following the method of images, we propose that the first contribution to \( \bar{S}(q_f, q_i; M) \) would be \( \bar{S}(q_f, q_i; M) \) of the free particle. The other contributions come from the implementation of the boundary conditions through images.

Now consider the image of the point \( q_i \) with respect to the left wall at \( x^- \). We call the position of this image, \( x^- \) (see Fig. 2). On the other hand, one can see from the properties of \( S \) in (5.9) and (5.10) that \( S(q_i + a, q_i; M) = S(q_i - a, q_i; M) \). Using these, and noting that \( q_i - x^- = x^- - x^- \), the boundary condition at \( x^- \), which is \( \bar{S}(x^-, q_i; M) = 0 \), is satisfied if we assume

\[ \bar{S}(q_f, q_i; M) = S(q_f, q_i; M) - S(q_f, x^-; M). \tag{5.21} \]

Now we use the same argument to implement the boundary condition at \( x^+ \), \( \bar{S}(x^+, q_i; M) = 0 \). Consider the images \( x^+_1 \) and \( x^+_2 \) of the point \( q_i \) and \( x^- \), respectively, with respect to the right wall at \( x^+ \) (see Fig. 2). In this case we have \( x^+ - q_i = x^+_1 - x^- \) and \( x^+ - x^- = x^+_2 - x^- \). Then the boundary condition at \( x^+ \) is satisfied if we take

\[ \bar{S}(q_f, q_i; M) = S(q_f, q_i; M) - S(q_f, x^-; M) - S(q_f, x^+_1; M) + S(q_f, x^+_2; M). \tag{5.22} \]

Obviously these combination of functions does not satisfy the boundary condition at \( x^- \) and we have to again introduce images of \( x^+_1 \) and \( x^+_2 \) with respect to the left wall and repeat the same procedure again. This
process should be done indefinitely. To generalize this method, we note that the position of images can be written as
\[ x_n^\pm = 2x^\pm - x_{n-1}^\pm, \quad n \in \mathbb{N}, \quad x_0^\pm = q_i, \]  
(5.23)
and the contributions corresponding to even \( n \) are positive while the ones corresponding to the odd ones are negative. The above leads to
\[ x_n^\pm = \pm 2L + x_{n-2}^\pm, \]  
(5.24)
where \( L = x^+ - x^- \) is the length of the box. The solutions to these difference equations for even and odd \( n \) are
\[ x_{2k}^\pm = \pm 2kL + q_i, \quad n = 2k, \quad k \in \mathbb{N} \]  
(5.25)
\[ x_{2k-1}^\pm = \pm 2(k-1)L + 2x^\pm - q_i, \quad n = 2k-1, \quad k \in \mathbb{N}. \]  
(5.26)
From these one can see that
\[ x_{2k}^- \bigg|_{k \to k} = x_{2k}^+, \]  
(5.27)
\[ x_{2k-1}^- \bigg|_{k \to k+1} = x_{2k-1}^+. \]  
(5.28)
Now, the full contribution to \( \bar{S}(q_f, q_i; M) \) comes from the sum of three groups of functions: \( S(q_f, x_{2k}^\pm; M) \) with positive sign, \( S(q_f, x_{2k-1}^\pm; M) \) with negative sign, and \( S(q_f, q_i; M) \) also with positive sign,
\[ \bar{S}(q_f, q_i; M) = S(q_f, q_i; M) + \sum_{k=1}^{\infty} \left[ S(q_f, x_{2k}^+; M) + S(q_f, x_{2k}^-; M) \right] - \sum_{k=1}^{\infty} \left[ S(q_f, x_{2k-1}^+; M) + S(q_f, x_{2k-1}^-; M) \right]. \]  
(5.29)
Using \((5.27)\) and \((5.28)\), and letting \(k \in \mathbb{Z}\), the above function can be expressed more concisely as

\[
\bar{S}(q_f, q_i; M) = \sum_{k=-\infty}^{\infty} \left[ S(q_f, x_{2k}^+; M) - S(q_f, x_{2k-1}^+; M) \right]
\]

\[
= \sum_{k=-\infty}^{\infty} \left[ S(q_f, 2kL + q_i; M) - S(q_f, 2kL + 2x^- - q_i; M) \right]
\]

where in the second line we have used \((5.25)\) and \((5.26)\).

Finally using this function together with \((5.8)\) in \((5.6)\) and summing over \(M\) (or using \((5.11)\)), and then using the resulting expression together with \((5.2)\) in \((3.34)\), yields

\[
K^{\text{Box}}_{\text{poly}}(q_f, t_f; q_i, t_i) = \sum_{k=-\infty}^{\infty} \left[ K_{\text{poly}}(q_f, t_f; q_i + 2kL, t_i) - K_{\text{poly}}(q_f, t_f; -q_i + 2kL, t_i) \right],
\]

where \(K_{\text{poly}}(q_f, t_f; q_i, t_i)\) is the polymer propagator of the free particle derived in the previous section, and we have chosen \(x^- = 0\). This is precisely the result that has been obtained by spectral methods in [44].

1. The continuum limit

Due to the form of \(K^{\text{Box}}_{\text{poly}}(q_f, t_f; q_i, t_i)\), its continuum limit simply results from the continuum limit of the propagator of the free particle,

\[
K^{\text{Box}}_{\text{poly}} \approx \lambda \sum_{k=-\infty}^{\infty} \left[ K_0(q_f, t_f; q_i + 2kL, t_i) - K_0(q_f, t_f; -q_i + 2kL, t_i) \right]
\]

\[
\approx \lambda K^{\Box}_{\text{poly}}(q_f, t_f; q_i, t_i),
\]

with \(K^{\Box}_{\text{poly}}(q_f, t_f; q_i, t_i)\) being the usual propagator of the particle in a box. The presence of \(\lambda\) here was discussed earlier at the end of Sect. VI A 1.

VI. RELATIVISTIC PARTICLE

For this case we assume that the system is four dimensional. The deparametrized constraint is \(\tilde{\mathcal{C}} = \pi^2 - \hbar(q, p) = p_i^2 - \sum_{j=1}^{3} (p_j)^2 - m^2\) with \(\pi^2 = p_i^2\), which is the four dimensional generalization of \((3.6)\) and \((3.19)\). The representation of this constraint on the Hilbert space can be written as

\[
\hat{\mathcal{C}} = \hat{p}_i^2 - \sum_{j=1}^{3} (\hat{p}_j)^2_{\lambda} - m^2,
\]

where \(\hat{p}_i^2\) is the usual Schrödinger representation of \(p_i\) while \((\hat{p}_j)^2_{\lambda}\) is

\[
(\hat{p}_j)^2_{\lambda} = \frac{\hbar^2}{\lambda^2} \left( 2 - \hat{V}^j_{\lambda} - \hat{V}_{-\lambda}^j \right),
\]

which is the polymer representation of \((p_j)^2\) similar to \((2.24)\). The full polymer propagator from \((3.34)\) becomes

\[
K_{\text{poly}}(q_f, t_f; q_i, t_i) = \int_{\mathbb{R}} d\alpha \, e^{-i\alpha m^2} \left\langle t_f \right| e^{i\alpha \hat{p}_i^2} \left| t_i \right\rangle \prod_{j=1}^{3} \left\langle q_f^j \right| e^{-i\alpha (\hat{p}_j)^2_{\lambda}} \left| q_i^j \right\rangle,
\]

\[(6.3)\]
where the superscript $j$ on $q$, corresponds to the three spatial dimensions. The amplitude of the relativistic momentum of the physical time, (3.36), turns out to be

$$A_\pi(t_f, t; \alpha) = \langle t_f | e^{i\alpha x^2} | t_i \rangle = \int_{\mathbb{R}} d\rho_i e^{i\alpha \rho_i^2} e^{i\phi_i(t_f - t_i)} = \sqrt{\frac{i\pi}{\alpha}} e^{-\frac{i}{\alpha \eta} (t_f - t_i)^2}. \quad (6.4)$$

Due to the similarity of $\hat{\eta}$ in both relativistic and nonrelativistic particle cases (see (3.13), (3.19) and (3.34)), the relativistic polymer amplitudes can be computed, to a great extent, in the same manner as in section 3.1. The only difference is that we need to make a simple change, $\lambda^2 \to \lambda^2 / 2m$, in the result in (5.12), and keep in mind that now there are three lengths $L_j = \frac{q_j - q_f}{\lambda}$ instead of just one length $L$. Therefore we will get

$$A_{\text{poly}}(q_f, q_i; \alpha) = \langle q_f | e^{-i\alpha(p_j)^2} | q_i \rangle = (-1)^L \prod_{j=1}^L e^{-i\alpha \bar{\alpha}^2 \frac{2\lambda^2}{\lambda^2}} J_{L_j} \left( 2\alpha \frac{\hbar^2}{\lambda^2} \right), \quad j = 1, 2, 3. \quad (6.5)$$

Putting (6.3), (6.4) and (6.5) together yields

$$K_{\text{poly}}^{\text{Rel}} (q_f, t_f; q_i, t_i) = \left( \prod_{j=1}^3 J_{L_j} \right) \int_{\mathbb{R}} d\alpha \sqrt{\frac{i\pi}{\alpha}} e^{-i\alpha \eta^2} e^{-\frac{i}{\alpha \eta} (t_f - t_i)^2} e^{-i\alpha \bar{\alpha}^2} \prod_{j=1}^3 J_{L_j} \left( 2\alpha \frac{\hbar^2}{\lambda^2} \right), \quad (6.6)$$

which is the polymer propagator of the free relativistic particle.

### 1. The continuum limit

The Bessel function in this case becomes

$$J_{L_j} \left( 2\alpha \frac{\hbar^2}{\lambda^2} \right) = J_{L_j} \left( 2\alpha L_j^2 \frac{\hbar^2}{L_j^2 \lambda^2} \right) = J_{L_j} \left( L_j^2 \frac{2\alpha \hbar^2}{(q_f - q_i)^2} \right) = J_{L_j} \left( L_j^2 \Delta \right), \quad (6.7)$$

where we have used $\Delta = \frac{2\alpha \hbar^2}{(q_f - q_i)^2}$, and $L_j = \frac{q_j - q_f}{\lambda}$ is the length of the $j$’th dimension of the particle’s motion. By applying the same approximation method used in section 3.1 one gets (for the limit $\lambda \to 0$)

$$J_{L_j} \left( L_j^2 \Delta \right) \approx \frac{1}{\sqrt{2\pi L_j^2}} \frac{\sqrt{\Delta}}{\sqrt{i}} (-i)^{L_j} \exp \left[ iL_j z^{-1} + \frac{1}{2}iL_j \right], \quad \sec(\beta) = L_j \Delta = z^{-1}. \quad (6.8)$$

Substituting back for the values of $L_j$ and $z$, yields

$$J_{L_j} \left( 2\alpha \frac{\hbar^2}{\lambda^2} \right) = \frac{\lambda}{\sqrt{4\pi i \alpha \hbar^2}} (-i)^{L_j} \exp \left[ 2i\alpha \frac{\hbar^2}{\lambda^2} + i \frac{(q_f - q_i)^2}{4\alpha \hbar^2} \right]. \quad (6.9)$$

Finally using it in the propagator (6.6), one arrives at

$$K_{\text{poly}}^{\text{Rel}} (q_f, t_f; q_i, t_i) = \frac{\lambda^3}{\hbar^3} \frac{1}{8\pi} \int_{\mathbb{R}} d\alpha \frac{e^{-i\alpha \eta^2}}{\alpha^2} \exp \left\{ i \frac{(\Delta \Delta^2 - 4\alpha^2 \hbar^2 m^2)}{4\alpha \hbar^2} \right\} = \frac{\lambda^3}{\hbar^3} \frac{1}{8\pi} \int_{\mathbb{R}} d\alpha \frac{e^{-i\alpha \eta^2}}{\alpha^2} \left\{ \eta_{\mu \nu} \left( x_{\mu}^f - x_{\mu}^i \right) \left( x_{\mu}^f - x_{\mu}^i \right) \right\}, \quad (6.10)$$

where

$$(\Delta \Delta^2 - 4\alpha^2 \hbar^2 m^2) = -(t_f - t_i)^2 + \sum_j \left( q_j - q_f \right)^2 = \eta_{\mu \nu} \left( x_{\mu}^f - x_{\mu}^i \right) \left( x_{\nu}^f - x_{\nu}^i \right). \quad (6.11)$$
is the Lorentz invariant spacetime interval between initial and final points of the particle’s motion in flat spacetime. This can be written as

\[ K^\text{Rel}_\text{poly}(q_f, t_f; q_i, t_i) = \lambda^3 K^\text{Rel}, \]

(6.12)

where \( K^\text{Rel} \) is the well-known propagator of the nonpolymer relativistic particle. Again, the presence of \( \lambda^3 \) is due to the reasons discussed at the end of Sect. \[\text{V A 1}\].

One can make an interesting observation using (6.10): assuming that the particle is a massless one that moves on a null curve in spacetime, one has \((\Delta S)^2 = 0\) and the propagator becomes

\[ K^\text{Rel}_\text{poly}(q_f, t_f; q_i, t_i) = \frac{\lambda^3}{\hbar^3} \lim_{m \to 0} \int_{\mathbb{R}} d\alpha \frac{e^{-i\alpha m^2}}{\alpha^2}. \]

(6.13)

The integral can be computed by analytic continuation and by using the incomplete Gamma function \( \Gamma(a, z) \). This will yield

\[ K^\text{Rel}_\text{poly}(q_f, t_f; q_i, t_i) = \frac{\lambda^3}{\hbar^3} \lim_{m \to 0} \left[ -\frac{1}{\alpha} e^{-i\alpha m^2} + im^2 \Gamma(0, i\alpha m^2) \right] = -\frac{\lambda^3}{\hbar^3} \frac{1}{8\pi\alpha}. \]

(6.14)

We can recognize the usual Feynman propagator of a massless particle on the right hand side of the above equation. Once again the appearance of the factor \( \lambda^3 \) can be understood as in the previous cases.

VII. DISCUSSION

It is a general expectation that modifying the underlying structure of spacetime can alleviate the difficulties posed by the spacetime singularities of General Relativity and the ultraviolet divergences of field theory. In particular proposals like Loop Quantum Gravity which lead to a discrete geometry may play an important role in this regard. Indeed some cosmological models as well as black hole interiors have been shown to avoid the classical singularities when they are described using LQG methods. Moreover, along the same lines, field theories improve their high energy behavior. Now there are two approaches that have been developed to implement loop quantization, a Hamiltonian one and a path integral version, better known as spin foam models. A recent proposal was put forward to connect both approaches in the case of cosmological models that is based on a vertex expansion of a transition amplitude. However such an amplitude has not been calculated completely in an explicit way in spite of the fact that the models considered are soluble.

In this work we have applied a previous proposal to connect the Hamiltonian and path integral polymer quantizations to three mechanical systems in order to calculate their explicit analytic polymer propagators. These systems are the nonrelativistic particle, both free (5.13) and in a box (5.31), and the relativistic particle (6.6). Although the first two had been computed before through spectral methods [44], they are regained here using path integration. The third one is a completely new result. All three are shown here to reduce to the known forms, a la Schrödinger, when the polymer scale parameter is small.

After giving a brief introduction to polymer representation, constrained systems and their deparametrization, and group averaging technique, we started by considering the deparametrized form of the classical canonical description of a free particle where there is only one associated constraint present. Next we set up the polymer quantization of this system for the two independent degrees of freedom, time and position. The former is quantized a la Schrödinger while the latter is subject to the polymer quantization, thus differing in their Hilbert spaces, as well as the commutator algebra to be fulfilled, as can be seen from (5.1). This is similar to what is actually done in the cosmological model [31] where the clock variable is taken to be a scalar field quantized in the standard way, whereas the gravitational degrees of freedom are treated using polymer quantization.
Thanks to $t$ and $q$ commuting, the generic form of the polymer propagator can be expressed as the product of a polymer amplitude corresponding to $q$ and a Schrödinger one associated to $t$. As for the polymer contribution the splitting into small propagators which amounts to a sum over histories can be given the form of a sum over a number of transitions that involve a change in position of the particle. Such resummation is just the analogue of the vertex expansion of the spin foam models that was identified in [31] [32]. In contrast to the latter, however, for our mechanical models we have been able to compute the sums explicitly and hence derive an analytic propagator for each case.

The calculations are remarkably simplified in the case of a free non relativistic particle. For a particle in a box, the boundary walls are implemented through combinations of free particle contributions via the image method. Moreover, this deparametrized approach turns out to be very convenient for the relativistic particle due to the fact that the time and position variables decouple.

Our explicit results for the polymer propagators lend support to the proposal to connect the Hamiltonian and the path integral loop quantizations.

The present work can be extended in several directions. One possibility is to pursue the path integral formulation of the polymer harmonic oscillator which will have various applications in polymer field theory [16]. Our analysis may also suggest some approximations in the case of the cosmological models for which the explicit propagator has not been obtained, in particular in regard to the resummation of the amplitude in terms of a vertex expansion. Additionally, there are indications that polymer field theory could lead to violation of Lorentz invariance symmetry [16], although this has been shown to be phenomenologically constrained [46]. More work is needed to settle this issue, perhaps using the path integral method, since these types of effects could be an artifact of the truncation used to derive the dispersion relation of the model [47] (see also [30]).

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[1] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time* (Cambridge University Press, 1973).
[2] S. Weinberg, *The Quantum Theory of Fields*, Vol. I (Cambridge University Press, 1995).
[3] B. Zwiebach, *A First Course in String Theory* (Cambridge University Press, 2004).
[4] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, 1988).
[5] J. Polchinski, *String Theory: An Introduction to the Bosonic String*, Vol. I (Cambridge University Press, 1998).
[6] M. Bojowald, *Quantum Cosmology: A Fundamental Description of the Universe* (Springer-Verlag New York, 2011).
[7] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, 2007).
[8] C. Rovelli, *Quantum Gravity* (Cambridge University Press, 2004).
[9] A. Ashtekar, T. Pawlowski, and P. Singh, “Quantum nature of the big bang,” Phys. Rev. Lett. 96, 141301 (2006), arXiv:gr-qc/0602086.
[10] A. Ashtekar, “Loop quantum cosmology: An overview,” Gen.Rel.Grav. 41, 707–741 (2009), arXiv:0812.0177 [gr-qc].
[11] A. Ashtekar and M. Bojowald, “Quantum geometry and the schwarzschild singularity,” Class. Quantum Grav. 23, 391–411 (2006), gr-qc/0509075.
[12] R. Gambini, J. Olmedo, and J. Pullin, “Quantum black holes in loop quantum gravity,” Class. Quantum Grav. 31, 095009 (2014), arXiv:1310.5996 [gr-qc].
[13] A. Corichi, J. Olmedo, and S. Rastgoo, “Callan-giddings-harvey-strominger vacuum in loop quantum gravity and singularity resolution,” Phys. Rev. D94, 084050 (2016), 1608.06246 [gr-qc].
[43] D. Giulini and D. Marolf, “On the generality of refined algebraic quantization,” Class. Quantum Grav. 16, 2479–2488 (1999).

[44] J. D. Reyes E. Flores-González, H. A. Morales-Técotl, “Propagators in polymer quantum mechanics,” Annals Phys. 336, 394–412 (2013), arXiv:1302.1906 [math-ph]

[45] M. Creutz, Quarks, Gluons and Lattices (Cambridge University Press, 1985).

[46] V. Husain and J. Louko, “Low energy Lorentz violation from modified dispersion at high energies,” Phys. Rev. Lett. 116, 061301 (2016).

[47] M. Bojowald, H. A. Morales-Técotl, and H. Sahlmann, “Loop quantum gravity phenomenology and the issue of Lorentz invariance,” Phys. Rev. D71, 084012 (2005).