ON SOLUTIONS OF FRACTAL FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. New class of differential and integral operators with fractional order and fractal dimension have been introduced very recently and gave birth to new class of differential and integral equations. In this paper, we derive exact solution of some important ordinary differential equations where the differential operators are the fractal-fractional. We presented a new numerical scheme to obtain solution in the nonlinear case. We presented the numerical simulation for different values of fractional orders and fractal dimension.

1. Introduction. Nature within humankind leaves can only be understood, analyse and predicted using the concept of modeling. This concept of modeling usually in applied mathematics and other related science is achieved using some mathematical tools called differential and integral operators. As they are able to depict the variations in time and space of an object. Within the available literature, especially in calculus, there exist different classes of differential and integral operators. Classical integral and differential operators; differential and integral operator based on the fading memory, differential and integral operators based on power law process and finally differential and integral operators based on generalized Mittag-Leffler function. Many great innovative ideas were introduced, new theories and even applications around these three classes of differential and integral operators [1, 2]. Nevertheless, researchers working in calculus, especially within the framework of fractional calculus have realized that nature due to its complexities cannot be understood with only these three classes of differential operators therefore new concept were introduced [13, 3]. These new differential operators can be viewed as fractal derivative of convolution of a continuous function and power law, exponential decay law or the generalized Mittag-Leffler function. These operators can be considered as upper classes of differential and integral operators as they can be converted to classical, fractal and fractional differential and integral operators in the limit cases [14, 12, 22, 5, 17, 11]. For more details see [6, 7, 15, 16, 19, 4, 8, 10, 18, 9, 20, 21].

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The concept of fractal-fractional differential and integral operators have been introduced very recently and appear to be superior to existing fractional operators with constant orders. When the fractional order tends to zero we recover the fractal differential and integral operators. When the fractal dimension tends to zero then we recover all the well-known fractional derivative including Riemann-Liouville derivative, Caputo-Fabrizio derivative and the Atangana-Baleanu fractional derivative, this is also true for existing fractional integrals. Therefore, to answer the question, why these new operators are better than, existing one, the answer lies on the fact that they can capture better complexities of nature than existing operators as they have both memory effect and self-similar properties.

In this paper, we consider the differential equations with the fractal fractional derivative. We obtain the solutions of the problems and demonstrate them by figures.

We organize the paper as: We give the main definitions of fractal fractional derivatives and Laplace transform in Section 2. We obtain the solutions of fractal differential equations in Section 3. We demonstrate our results by some figures in this section. We present the conclusion in the last section.

2. Fractal fractional derivatives and Laplace transform.

**Definition 2.1.** We assume that \( u(t) \) is continuous in opened interval \( (a, b) \), if \( u \) is fractal differentiable on \( (a, b) \) with order \( \alpha \) then, the Fractal-Fractional derivative of \( u \) of order \( \alpha \) in Riemann-Liouville sense with power law is presented as [3]:

\[
\frac{\text{FFR}}{t}^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t u(y)(t-y)^{-\alpha} dy, \quad 0 < \alpha, \beta \leq 1,\tag{2.1}
\]

where

\[
\frac{du(y)}{dy^\alpha} = \lim_{t \to y} \frac{u(t) - u(y)}{t^\beta - y^\beta}.\tag{2.2}
\]

**Definition 2.2.** We assume that \( u(t) \) is continuous in opened interval \( (a, b) \), if \( u \) is fractal differentiable on \( (a, b) \) with order \( \beta \) then, the Fractal-Fractional derivative of \( u \) of order \( \alpha \) in Riemann-Liouville sense with the exponential decay kernel is presented as [3]:

\[
\frac{\text{FFE}}{t}^\alpha \beta u(t) = \frac{M(\alpha)}{1-\alpha} \frac{d}{dt^\beta} \int_a^t u(y) \exp\left(\frac{-\alpha}{1-\alpha}(t-y)\right) dy, \quad 0 < \alpha, \beta \leq 1.\tag{2.3}
\]

**Definition 2.3.** We assume that \( u(t) \) is continuous in opened interval \( (a, b) \), if \( u \) is fractal differentiable on \( (a, b) \) with order \( \beta \) then, the Fractal-Fractional derivative of \( u \) of order \( \alpha \) in Riemann-Liouville sense with the generalized Mittag-Leffler kernel is presented as [3]:

\[
\frac{\text{FFM}}{t}^\alpha \beta u(t) = \frac{AB(\alpha)}{1-\alpha} \frac{d}{dt^\beta} \int_a^t u(y) E_\alpha\left(\frac{-\alpha}{1-\alpha} (t-y)^\alpha\right) dy, \quad 0 < \alpha, \beta \leq 1,\tag{2.4}
\]

where \( AB(\alpha) = 1 - \alpha + \frac{\alpha}{1-\alpha} \).

There are many kinds of transforms out there in the world. Laplace transform is the most important kind of these transforms.

**Definition 2.4.** Let \( f(t) \) be defined for \( t \geq 0 \). The Laplace transform of \( f(t) \) defined by \( F(s) \) or \( L\{f(t)\} \), is an integral transform given by the Laplace integral:

\[
L\{f(t)\} = F(s) = \int_0^\infty \exp(-st)f(t)dt.\tag{2.5}
\]
Provided that this (improper) integral exists, i.e. that the integral is convergent. The Laplace transform is an operation that transforms a function of \( t \) (i.e., a function of time domain), defined on \([0, \infty)\), to a function of \( s \) (i.e., of frequency domain). \( F(s) \) is the Laplace transform, or simply transform, of \( f(t) \). Together the two functions \( f(t) \) and \( F(s) \) are called a Laplace transform pair.

We give some examples of the Laplace transform as:
\[
L\{t^{-\alpha}\} = s^{\alpha-1} \Gamma(1 - \alpha). \tag{2.6}
\]
\[
L\{t^\beta\} = s^{-\beta-1} \Gamma(1 + \beta). \tag{2.7}
\]

We have the following relations for the Caputo, Caputo-Fabrizio and Atangana-Baleanu derivatives,
\[
C_0^\alpha D_t^\alpha u(t) = \frac{du(t)}{dt} * \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}. \tag{2.8}
\]
\[
CF_0^\alpha D_t^\alpha u(t) = \frac{du(t)}{dt} * \frac{M(\alpha)}{1 - \alpha} \exp\left(-\frac{\alpha}{1 - \alpha} t\right). \tag{2.9}
\]
\[
ABC_0^\alpha D_t^\alpha u(t) = \frac{du(t)}{dt} * \frac{AB(\alpha)}{1 - \alpha} E_\alpha \left[\frac{-\alpha}{1 - \alpha} t^\alpha\right]. \tag{2.10}
\]

We have the Laplace transforms of these derivatives as:
\[
L\{C_0^\alpha D_t^\alpha u(t)\} = \left(sL\{u(t)\} - u(0)\right) s^{\alpha-1}. \tag{2.11}
\]
\[
L\{CF_0^\alpha D_t^\alpha u(t)\} = -\left(sL\{u(t)\} - u(0)\right) \frac{M(\alpha)}{s\alpha - s - \alpha}. \tag{2.12}
\]
\[
L\{ABC_0^\alpha D_t^\alpha u(t)\} = \left(sL\{u(t)\} - u(0)\right) \frac{AB(\alpha)s^{\alpha-1}}{s^\alpha(1 - \alpha) + \alpha}. \tag{2.13}
\]

3. Solutions of fractal fractional differential equations. In this section we investigate four problems with Caputo, Caputo-Fabrizio and Atangana-Baleanu derivatives.

3.1. Problem 1. We consider the following fractal fractional differential equation with the power law kernel as:
\[
_{0}^{FFP}D_t^{\alpha,\beta} u(t) = t. \tag{3.1}
\]
We have
\[
\frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t u(y)(t - y)^{-\alpha} dy = t. \tag{3.2}
\]
Then, we obtain
\[
\frac{1}{\Gamma(1 - \alpha)} \frac{1}{\beta t^{\beta-1}} \frac{d}{dt} \int_0^t u(y)(t - y)^{-\alpha} dy = t. \tag{3.3}
\]
Thus, we get
\[
_{0}^{RL}D_t^{\alpha} u(t) = \beta t^\beta. \tag{3.4}
\]
After using the relation between Riemann-Liouville and Caputo derivatives, we acquire
\[
C_0^\alpha D_t^\alpha u(t) = \beta t^\beta - \frac{u(0)}{\Gamma(1 - \alpha)} t^{-\alpha}. \tag{3.5}
\]
Let’s take Laplace transform of both sides of Eq. (3.5):
\[
L\{C_0^\alpha D_t^\alpha u(t)\} = L\{\beta t^\beta\} - L\left(\frac{u(0)}{\Gamma(1 - \alpha)} t^{-\alpha}\right). \tag{3.6}
\]
Then, we have
\[ s^\alpha L(u) - u(0)s^{\alpha-1} = \beta \Gamma(1 + \beta)s^{-\beta-1} - \frac{u(0)}{\Gamma(1 - \alpha)}s^{\alpha-1}\Gamma(1 - \alpha). \tag{3.7} \]

After simplifying this relation, we get
\[ L(u) = \beta \Gamma(1 - \beta)s^{\alpha-\beta-1}. \tag{3.8} \]

Then, we obtain
\[ u(t) = \frac{\beta \Gamma(1 + \beta)}{\Gamma(1 + \alpha + \beta)}t^{\alpha + \beta} . \tag{3.9} \]

We consider the following fractal fractional differential equation with the exponential decay kernel as:
\[ _0^{\text{FFE}} D_t^{\alpha, \beta} u(t) = t. \tag{3.10} \]

We have
\[ \frac{M(\alpha)}{1 - \alpha} \frac{d}{dt} \int_0^t u(y) \exp(-\alpha (t - y)) \, dy = t. \tag{3.11} \]

Then, we obtain
\[ \frac{M(\alpha)}{1 - \alpha} \frac{1}{\beta t^{\beta-1}} \frac{d}{dt} \int_0^t u(y) \exp(-\alpha (t - y)) \, dy = t. \tag{3.12} \]

After using the relation between Riemann-Liouville and Caputo derivatives with non-singular kernels, we acquire
\[ _0^{\text{CF}} D_t^{\alpha} u(t) = \beta t^\beta. \tag{3.13} \]

After taking the inverse Laplace transform of the above equation, we get
\[ u(t) = \beta \Gamma(1 + \beta)(1 + \beta)^{-1} \Gamma(1 - \alpha) \frac{1}{1 - \alpha + s}. \tag{3.16} \]

After simplifying this relation, we get
\[ L(u) = \frac{\beta \Gamma(1 + \beta)}{M(\alpha)} \left[ \alpha s^{-\beta-2} + s^{\beta-1}(1 - \alpha) \right]. \tag{3.17} \]

We consider the following fractal fractional differential equation with the generalized Mittag-Leffler kernel as:
\[ _0^{\text{FM}} D_t^{\alpha, \beta} u(t) = t. \tag{3.19} \]

We have
\[ \frac{AB(\alpha)}{1 - \alpha} \frac{d}{dt} \int_0^t u(y) E_\alpha \left( -\frac{\alpha}{1 - \alpha} (t - y)^\alpha \right) \, dy = t. \tag{3.20} \]

Then, we obtain
\[ \frac{AB(\alpha)}{1 - \alpha} \frac{1}{\beta t^{\beta-1}} \frac{d}{dt} \int_0^t u(y) E_\alpha \left( -\frac{\alpha}{1 - \alpha} (t - y)^\alpha \right) \, dy = t. \tag{3.21} \]
Then, we have
\[ A^{0}_{BR} D_t^\alpha u(t) = \beta t^\beta. \] (3.22)

After using the relation between Riemann-Liouville and Caputo derivatives with the generalized Mittag-Leffler kernel, we obtain
\[ A^{0}_{BC} D_t^\alpha u(t) = \beta t^\beta - \frac{AB(\alpha)}{1 - \alpha} u(0) E_\alpha \left( \frac{-\alpha}{1 - \alpha} t^\alpha \right). \] (3.23)

Let's take Laplace transform of both sides of the above equation:
\[ L \left[ A^{0}_{BC} D_t^\alpha u(t) \right] = L(\beta t^\beta) - L \left( \frac{AB(\alpha)}{1 - \alpha} u(0) E_\alpha \left( \frac{-\alpha}{1 - \alpha} t^\alpha \right) \right). \] (3.24)

Then, we obtain
\[ (sL(u(t)) - u(0)) \frac{AB(\alpha)s^{\alpha-1}}{s^\alpha(1 - \alpha) + \alpha} = \beta \Gamma(1 + \beta) s^{-1-\beta} - \frac{AB(\alpha)}{1 - \alpha} u(0) \frac{s^{\alpha-1}}{s^\alpha + \frac{\alpha}{1 - \alpha}}. \] (3.25)

Thus, we get
\[ u(t) = \frac{\beta \Gamma(1 + \beta)}{AB(\alpha)} \left[ 1 - \alpha + \alpha \frac{t^\alpha}{\Gamma(1 + \beta) + \Gamma(1 + \alpha + \beta)} \right]. \] (3.27)

3.2. Problem 2. We consider the following fractal fractional differential equation with the power law kernel as:
\[ ^{FFP}_0 D_t^{\alpha,\beta} u(t) = \exp(-tu). \] (3.28)

We have
\[ \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t u(y)(t - y)^{-\alpha} = \exp(-tu). \] (3.29)

Then, we obtain
\[ \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t u(y)(t - y)^{-\alpha} dy = \exp(-tu). \] (3.30)

Thus, we get
\[ ^{RL}_0 D_t^\alpha u(t) = \beta t^{\beta-1} \exp(-tu). \] (3.31)

After using the relation between Riemann-Liouville and Caputo derivatives, we acquire
\[ ^{C}_0 D_t^\alpha u(t) = \beta t^{\beta-1} \exp(-tu) - \frac{u(0)}{\Gamma(1 - \alpha)} t^{-\alpha}. \] (3.32)

Let's take Laplace transform of both sides of Eq. (3.32):
\[ L \left[ ^{C}_0 D_t^\alpha u(t) \right] = L(\beta t^{\beta-1} \exp(-tu)) - L \left( \frac{u(0)}{\Gamma(1 - \alpha)} t^{-\alpha} \right). \] (3.33)

Then, we have
\[ s^\alpha L(u) - u(0)s^{\alpha-1} = L \left( \beta t^{\beta-1} \exp(-tu) \right) - \frac{u(0)}{\Gamma(1 - \alpha)} s^{\alpha-1} \Gamma(1 - \alpha). \] (3.34)

\[ s^\alpha L(u) = L \left( \beta t^{\beta-1} \exp(-tu) \right). \] (3.35)

\[ L(u) = s^{-\alpha} L \left( \beta t^{\beta-1} \exp(-tu) \right). \] (3.36)
We use the inverse Laplace transform and the convolution for the above equation. Then, we get

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t F(z)(t-z)^{\alpha-1} \, dz,$$

where $F(z) = \beta z^{\alpha-1} \exp(-z u(z))$. Therefore, we obtain

$$u(t) = J_0^\alpha F(t).$$

We consider the following fractal fractional differential equation with the exponential decay kernel as:

$$_{0}^{FF}D_t^{\alpha,\beta} u(t) = \exp(-tu).$$

We have

$$\frac{M(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t u(y) \exp\left(-\frac{\alpha}{1-\alpha}(t-y)\right) \, dy = \exp(-tu).$$

Then, we obtain

$$\frac{M(\alpha)}{1-\alpha} \frac{1}{\beta t^{\beta-1}} \frac{d}{dt} \int_0^t u(y) \exp\left(-\frac{\alpha}{1-\alpha}(t-y)\right) \, dy = \exp(-tu).$$

$$L\left(_{0}^{FF}D_t^{\alpha} u(t)\right) = \left(\beta t^{\beta-1} \exp(-tu)\right) - L\left(\frac{M(\alpha)}{1-\alpha} u(0) \exp\left(-\frac{\alpha}{1-\alpha} t\right)\right).$$

Then, we have

$$-(sL(u) - u(0)) \frac{M(\alpha)}{s^{\alpha} - \alpha - s} = L\left(\beta t^{\beta-1} \exp(-tu)\right) - L\left(\frac{M(\alpha)}{1-\alpha} u(0) \frac{1}{1-\alpha} + s\right).$$

After simplifying this relation, we get

$$L(u) = \frac{s + \alpha - s\alpha}{sM(\alpha)} L\left(\beta t^{\beta-1} \exp(-tu)\right).$$

We use the inverse Laplace transform for the above equation. Then, we get

$$u(t) = \beta \left[-(-1 + \alpha) \exp(-tu(t))t^{\beta-1} + \alpha \int_0^t \exp(-su(s))s^{\alpha-1} \, ds\right].$$

We consider the following fractal fractional differential equation with the generalized Mittag-Leffler kernel as:

$$_{0}^{FM}D_t^{\alpha,\beta} u(t) = \exp(-tu).$$

We have

$$AB(\alpha) \frac{d}{dt} \int_0^t u(y) E_{\alpha} \left(-\frac{\alpha}{1-\alpha}(t-y)^{\alpha}\right) \, dy = \exp(-tu).$$

Then, we obtain

$$AB(\alpha) \frac{1}{\beta t^{\beta-1}} \frac{d}{dt} \int_0^t u(y) E_{\alpha} \left(-\frac{\alpha}{1-\alpha}(t-y)^{\alpha}\right) \, dy = \exp(-tu).$$

$$_{0}^{ABR}D_t^{\alpha} u(t) = \beta t^{\beta-1} \exp(-tu).$$
After using the relation between Riemann-Liouville and Caputo derivatives in the sense of Atangana-Baleanu derivatives, we get

$$0^ABC D_t^\alpha u(t) = \beta t^{\beta-1} \exp(-tu) - \frac{AB(\alpha)}{1-\alpha} u(0) E_\alpha \left( -\frac{\alpha}{1-\alpha} t^\alpha \right).$$

(3.52)

Let's take Laplace transform of both sides of the above equation:

$$L \left( 0^ABC D_t^\alpha u(t) \right) = L \left( \beta t^{\beta-1} \exp(-tu) \right) - L \left( \frac{AB(\alpha)}{1-\alpha} u(0) E_\alpha \left( -\frac{\alpha}{1-\alpha} t^\alpha \right) \right).$$

(3.53)

Then, we have

$$(sL(u(t)) - u(0)) \frac{AB(\alpha) s^{\alpha-1}}{s^\alpha(1-\alpha) + \alpha} = L \left( \beta t^{\beta-1} \exp(-tu) \right) - \frac{AB(\alpha)}{1-\alpha} u(0) \frac{s^{\alpha-1}}{s^\alpha + \frac{\alpha}{1-\alpha}}.$$  

(3.54)

If we simplify the above equation we will get

$$\frac{AB(\alpha)}{s^\alpha(1-\alpha) + \alpha} L(u(t)) = L \left( \beta t^{\beta-1} \exp(-tu) \right).$$

(3.55)

We use the inverse Laplace transform and the convolution for the above equation. Then, we get

$$u(t) = \frac{1}{AB(\alpha)} \int_0^t F(z) \left[ (1-\alpha) \delta(t-z) + \frac{\alpha}{\Gamma(\alpha)} (t-z)^{\alpha-1} \right] dz,$$

(3.56)

where $F(z) = \beta z^{\beta-1} \exp(-zu(z))$.

### 3.3. Problem 3

We consider the following fractal fractional differential equation with the power law kernel as:

$$0^FFP D_t^{\alpha,\beta} u(t) = E_\alpha(-\lambda t^\alpha).$$

(3.57)

We have

$$\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t u(y)(t-y)^{-\alpha} dy = E_\alpha(-\lambda t^\alpha).$$

(3.58)

Then, we obtain

$$\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \frac{1}{\beta t^{\beta-1}} \int_0^t u(y)(t-y)^{-\alpha} dy = E_\alpha(-\lambda t^\alpha).$$

(3.59)

Thus, we get

$$0^RL D_t^\alpha u(t) = \beta t^{\beta-1} E_\alpha(-\lambda t^\alpha).$$

(3.60)

After using the relation between Riemann-Liouville and Caputo derivatives, we acquire

$$0^\alpha D_t^\alpha u(t) = \beta t^{\beta-1} E_\alpha(-\lambda t^\alpha) - \frac{u(0)}{\Gamma(1-\alpha)} t^{-\alpha}.$$  

(3.61)

Let's take Laplace transform of both sides of Eq. (3.61):

$$L \left( 0^\alpha D_t^\alpha u(t) \right) = L \left( \beta t^{\beta-1} E_\alpha(-\lambda t^\alpha) \right) - L \left( \frac{u(0)}{\Gamma(1-\alpha)} t^{-\alpha} \right).$$

(3.62)

Then, we have

$$s^\alpha L(u) - u(0)s^{\alpha-1} = L \left( \beta t^{\beta-1} E_\alpha(-\lambda t^\alpha) \right) - \frac{u(0)}{\Gamma(1-\alpha)} s^{\alpha-1} \Gamma(1-\alpha) s^{\alpha-1}.$$  

(3.63)

$$s^\alpha L(u) = L \left( \beta t^{\beta-1} E_\alpha(-\lambda t^\alpha) \right).$$

(3.64)
After taking the inverse Laplace transform of the above equation and using the convolution, we get

\[ u(t) = \int_0^t I_0^\alpha \beta t^{\alpha-1} E_\alpha(-\lambda t^\alpha) \, dt. \]  

(3.65)

We consider the following fractal fractional differential equation with the exponential decay kernel as:

\[ 0^\beta FFE D^\alpha_t u(t) = E_\alpha(-\lambda t^\alpha). \]  

(3.66)

We have

\[ M(\alpha) \frac{d}{dt} \int_0^t u(y) \exp\left(-\frac{\alpha}{1-\alpha}(t-y)\right) dy = E_\alpha(-\lambda t^\alpha). \]  

(3.67)

Then, we obtain

\[ M(\alpha) \frac{1}{1-\alpha} \frac{d}{dt} \int_0^t u(y) \exp\left(-\frac{\alpha}{1-\alpha}(t-y)\right) dy = E_\alpha(-\lambda t^\alpha). \]  

(3.68)

After taking the inverse Laplace transform of the above equation, we get

\[ 0^\beta CFR D^\alpha_t u(t) = \beta t^{\alpha-1} E_\alpha(-\lambda t^\alpha). \]  

(3.69)

After using the relation between Riemann-Liouville and Caputo derivatives with non-singular kernels, we acquire

\[ 0^\beta CFR D^\alpha_t u(t) = \beta t^{\alpha-1} E_\alpha(-\lambda t^\alpha) - \frac{M(\alpha)}{1-\alpha} u(0) \exp\left(-\frac{\alpha}{1-\alpha} t\right). \]  

(3.70)

Let’s take Laplace transform of both sides of the above equation:

\[ L\left(0^\beta CFR D^\alpha_t u(t)\right) = L\left(\beta t^{\alpha-1} E_\alpha(-\lambda t^\alpha)\right) - L\left(\frac{M(\alpha)}{1-\alpha} u(0) \exp\left(-\frac{\alpha}{1-\alpha} t\right)\right). \]  

(3.71)

Then, we have

\[ -(sL(u) - u(0)) = L\left(\beta t^{\alpha-1} E_\alpha(-\lambda t^\alpha)\right) - \frac{M(\alpha)}{1-\alpha} u(0) \frac{1}{\frac{1}{1-\alpha}}. \]  

(3.72)

After simplifying this relation, we get

\[ -\frac{M(\alpha)}{s\alpha - \alpha - s} L(u) = L\left(\beta t^{\alpha-1} E_\alpha(-\lambda t^\alpha)\right). \]  

(3.73)

After taking the inverse Laplace transform of the above equation, we get

\[ u(t) = \int_0^t \beta t^{\alpha-1} E_\alpha(-\lambda t^\alpha) (\alpha + \delta(t - \tau) + \alpha \delta(t - \tau)) d\tau. \]  

(3.74)

We consider the following fractal fractional differential equation with the generalized Mittag-Leffler kernel as:

\[ 0^\beta FFM D^\alpha_t u(t) = E_\alpha(-\lambda t^\alpha). \]  

(3.75)

We have

\[ AB(\alpha) \frac{d}{dt} \int_0^t u(y) E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-y)^\alpha\right) dy = E_\alpha(-\lambda t^\alpha). \]  

(3.76)

Then, we obtain

\[ AB(\alpha) \frac{1}{1-\alpha} \frac{d}{dt} \int_0^t u(y) E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-y)^\alpha\right) dy = E_\alpha(-\lambda t^\alpha). \]  

(3.77)

After using the relation between Riemann-Liouville and Caputo derivatives with the generalized Mittag-Leffler kernel, we obtain

\[ 0^\beta ABC D^\alpha_t u(t) = \beta t^{\alpha-1} E_\alpha(-\lambda t^\alpha) - \frac{AB(\alpha)}{1-\alpha} u(0) E_\alpha\left(-\frac{\alpha}{1-\alpha} t^\alpha\right). \]  

(3.79)
Let's take Laplace transform of both sides of the above equation:

\[
L\left(\text{ABC} D_t^\alpha u(t)\right) = L\left(\beta t^{\beta-1} E_\alpha(-\lambda t^\alpha)\right) - L\left(\frac{AB(\alpha)}{1-\alpha} u(0) E_\alpha\left(\frac{-\alpha}{1-\alpha}\right)\right).
\]  

(3.80)

Then, we obtain

\[
(sL(u(t)) - u(0)) \frac{AB(\alpha) s^{\alpha-1}}{s^\alpha(1-\alpha) + \alpha} = L\left(\beta t^{\beta-1} E_\alpha(-\lambda t^\alpha)\right) - \frac{AB(\alpha)}{1-\alpha} u(0) \frac{s^{\alpha-1}}{s^\alpha + \frac{\alpha}{1-\alpha}}.
\]

(3.81)

After taking the inverse Laplace transform of the above equation, we get

\[
u(t) = \frac{1}{AB(\alpha)} \int_0^t \beta \tau^{\beta-1} E_\alpha(-\lambda \tau^\alpha) \left(\delta(t-\tau) + \alpha \delta(t-\tau) + \frac{\alpha(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\right) d\tau.
\]

(3.83)

Then, we obtain

\[
u(t) = \frac{1}{AB(\alpha)} \int_0^t \beta \tau^{\beta-1} E_\alpha(-\lambda \tau^\alpha) (\delta(t-\tau) + \alpha \delta(t-\tau)) d\tau + \frac{1}{AB(\alpha)} \alpha \beta_0 I_t^\beta t^{\beta-1} E_\alpha(-\lambda t^\alpha).
\]

(3.84)

3.4. **Problem 4.** We consider the following fractal fractional differential equation with the power law kernel as:

\[
\text{FFP} D_t^{\alpha,\beta} u(t) = \sin(t).
\]

(3.85)

We have

\[
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t u(y)(t-y)^{-\alpha} dy = \sin(t).
\]

(3.86)

Then, we obtain

\[
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \frac{1}{\beta t^{\beta-1}} \int_0^t u(y)(t-y)^{-\alpha} dy = \sin(t).
\]

(3.87)

Thus, we get

\[
\text{RL} D_t^\alpha u(t) = \beta t^{\beta-1} \sin(t).
\]

(3.88)

After using the relation between Riemann-Liouville and Caputo derivatives, we acquire

\[
C_0 D_t^\alpha u(t) = \beta t^{\beta-1} \sin(t) - \frac{u(0)}{\Gamma(1-\alpha)} t^{-\alpha}.
\]

(3.89)

Let's take Laplace transform of both sides:

\[
L\left(C_0 D_t^\alpha u(t)\right) = L\left(\beta t^{\beta-1} \sin(t)\right) - L\left(\frac{u(0)}{\Gamma(1-\alpha)} t^{-\alpha}\right).
\]

(3.90)

Then, we have

\[
s^\alpha L(u) - u(0) s^{\alpha-1} = L\left(\beta t^{\beta-1} \sin(t)\right) - \frac{u(0)}{\Gamma(1-\alpha)} s^{\alpha-1} \Gamma(1-\alpha).
\]

(3.91)

\[
s^\alpha L(u) = L\left(\beta t^{\beta-1} \sin(t)\right).
\]

(3.92)

After taking the inverse Laplace transform of the above equation and using the convolution, we get

\[
u(t) = \beta \frac{1}{\Gamma(\alpha)} \int_0^t \tau^{\beta-1} \sin(\tau)(t-\tau)^{\alpha-1} d\tau.
\]

(3.93)
Thus, we obtain
\[ u(t) = \beta \left( a I_0^\alpha t^{\beta-1} \sin(t) \right). \tag{3.94} \]

We consider the following fractal fractional differential equation with the exponential decay kernel as:
\[ \frac{D_{\alpha}^\beta}{0^F} u(t) = \sin(t). \tag{3.95} \]
We have
\[ M(\alpha) \frac{d}{dt} \int_0^t u(y) \exp \left( \frac{-\alpha}{1-\alpha} (t-y) \right) dy = \sin(t). \tag{3.96} \]

Then, we obtain
\[ M(\alpha) \frac{d}{dt} \int_0^t u(y) \exp \left( \frac{-\alpha}{1-\alpha} (t-y) \right) dy = \sin(t). \tag{3.97} \]

After using the relation between Riemann-Liouville and Caputo derivatives with non-singular kernels, we acquire
\[ \frac{D_{\alpha}^\beta}{0^F} u(t) = \beta t^{\beta-1} \sin(t). \tag{3.98} \]

Let's take Laplace transform of both sides of the above equation:
\[ L \left( \frac{D_{\alpha}^\beta}{0^F} u(t) \right) = L \left( \beta t^{\beta-1} \sin(t) \right) \]

Then, we have
\[ - (s L(u) - u(0)) \frac{M(\alpha)}{s \alpha - \alpha - s} = L \left( \beta t^{\beta-1} \sin(t) \right) \]

After simplifying this relation, we get
\[ - \frac{M(\alpha)}{s \alpha - \alpha - s} s L(u) = L \left( \beta t^{\beta-1} \sin(t) \right) \tag{3.102} \]

After taking the inverse Laplace transform of the above equation, we get
\[ u(t) = \int_0^t \beta \tau^{\beta-1} \sin(\tau) \left( \alpha + \delta(t-\tau) - \alpha \delta(t-\tau) \right) d\tau. \tag{3.103} \]

We consider the following fractal fractional differential equation with the generalized Mittag-Leffler kernel as:
\[ \frac{D_{\alpha}^\beta}{0^F} u(t) = \sin(t). \tag{3.104} \]
We have
\[ \frac{AB(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t u(y) E_\alpha \left( \frac{-\alpha}{1-\alpha} (t-y)^\alpha \right) dy = \sin(t). \tag{3.105} \]

Then, we obtain
\[ \frac{AB(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t u(y) E_\alpha \left( \frac{-\alpha}{1-\alpha} (t-y)^\alpha \right) dy = \sin(t). \tag{3.106} \]

After using the relation between Riemann-Liouville and Caputo derivatives with the generalized Mittag-Leffler kernel, we obtain
\[ \frac{D_{\alpha}^\beta}{0^F} u(t) = \beta t^{\beta-1} \sin(t) - \frac{AB(\alpha)}{1-\alpha} u(0) E_\alpha \left( \frac{-\alpha}{1-\alpha} t^{\alpha} \right). \tag{3.108} \]
Then, we obtain
\[ L\left(\frac{ABC}{1-\alpha}D_t^\alpha u(t)\right) = L\left(\beta t^{\beta-1} \sin(t)\right) - L\left(\frac{AB(\alpha)}{1-\alpha} u(0)E_{\alpha}\left(\frac{-\alpha}{1-\alpha} t^\alpha\right)\right). \] (3.109)

Then, we obtain
\[ (sL(u(t)) - u(0)) \frac{AB(\alpha)s^{\alpha-1}}{s^{\alpha}(1-\alpha) + \alpha} = L\left(\beta t^{\beta-1} \sin(t)\right) - \frac{AB(\alpha)}{1-\alpha} u(0) \frac{s^{\alpha-1}}{s^{\alpha} + \frac{\alpha}{1-\alpha}}. \] (3.110)

\[ \frac{AB(\alpha)s^\alpha}{s^{\alpha}(1-\alpha) + \alpha} L(u(t)) = L\left(\beta t^{\beta-1} \sin(t)\right). \] (3.111)

After taking the inverse Laplace transform of the above equation, we get
\[ u(t) = \frac{1}{AB(\alpha)} \int_0^t \beta \tau^{\beta-1} \sin(t) \left(\delta(t-\tau) + \alpha \delta(t-\tau) + \frac{\alpha(t-\tau)^\alpha}{\Gamma(\alpha)}\right) d\tau. \] (3.112)

Then, we obtain
\[ u(t) = \frac{1}{AB(\alpha)} \int_0^t \beta \tau^{\beta-1} \sin(t) \left(\delta(t-\tau) + \alpha \delta(t-\tau)\right) d\tau + \frac{1}{AB(\alpha)} \alpha \beta_0 I_t^\alpha t^{\beta-1} \sin(t). \] (3.113)

4. Numerical simulations. We consider the general Cauchy problem where the time derivative is replaced by fractal fractional differential operator. With the power law kernel, we have
\[ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t u(y)(t-y)^{-\alpha} dy = f(t, u(t)). \] (4.1)

Thus, since the integral is differentiable, we have
\[ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t u(y)(t-y)^{-\alpha} dy = \beta t^{\beta-1} f(t, u(t)). \] (4.2)

The above equation can be transformed as:
\[ u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \beta y^{\beta-1} f(y, u(y))(t-y)^{\alpha-1} dy. \] (4.3)

Thus at \( t_n \), we have
\[ u(t_n) = \frac{1}{\Gamma(\alpha)} \int_0^{t_n} \beta y^{\beta-1} f(y, u(y))(t_n-y)^{\alpha-1} dy, \] (4.4)

following the derivation presented in [3], we obtain
\[ u^n = u^0 + \beta \alpha \left(\xi_n f(t_0, y_0) + \sum_{j=0}^{n} \gamma_{n-j} t_j^{\beta-1} f(t_j, u_j)\right). \] (4.5)

Now if we consider the Cauchy problem when the derivative is the fractal fractional with the exponential decay law, we have
\[ M(\alpha) \frac{d}{dt} \int_a^t u(y) \exp\left(\frac{-\alpha}{1-\alpha}(t-y)\right) dy = f(t, u(t)). \] (4.6)

As presented earlier, the above equation can be reformulated as:
\[ M(\alpha) \frac{d}{dt} \int_a^t u(y) \exp\left(\frac{-\alpha}{1-\alpha}(t-y)\right) dy = \beta t^{\beta-1} f(t, u(t)). \] (4.7)
Applying the integral, we get

$$u(t) = \frac{1 - \alpha}{M(\alpha)} \beta t^{\beta - 1} f(t, u(t)) + \frac{\alpha \beta}{M(\alpha)} \int_0^t f(y, u(y)) dy.$$

(4.8)

Here we apply lagrange interpolation on the integral. To achieve this, we consider the above equation at $t_n$ and $t_{n+1}$:

$$u(t_{n+1}) = \frac{1 - \alpha}{M(\alpha)} \beta t_{n+1}^{\beta - 1} f(t_{n+1}, u(t_{n+1})) + \frac{\alpha \beta}{M(\alpha)} \int_{t_n}^{t_{n+1}} y^{\beta - 1} f(y, u(y)) dy. \quad (4.9)$$

$$u(t_n) = \frac{1 - \alpha}{M(\alpha)} \beta t_{n}^{\beta - 1} f(t_n, u(t_n)) + \frac{\alpha \beta}{M(\alpha)} \int_{t_n}^{t_{n+1}} y^{\beta - 1} f(y, u(y)) dy. \quad (4.10)$$

Then, we get

$$u^{n+1} - u^n = \frac{1 - \alpha}{M(\alpha)} \beta [t_{n+1}^{\beta - 1} f(t_{n+1}, u(t_{n+1})) - t_n^{\beta - 1} f(t_n, u(t_n))]$$

$$+ \frac{\alpha \beta}{M(\alpha)} \int_{t_n}^{t_{n+1}} y^{\beta - 1} f(y, u(y)) dy. \quad (4.11)$$

Then applying the lagrange polynomial on $y^{\beta - 1} f(y, u(y))$, we obtain

$$u^{n+1} = u^n + \frac{1 - \alpha}{M(\alpha)} \beta [t_{n+1}^{\beta - 1} f(t_{n+1}, u(t_{n+1})) - t_n^{\beta - 1} f(t_n, u(t_n))]$$

$$+ \frac{\alpha \beta}{M(\alpha)} \left[ \frac{1}{2} \Delta t_n^{\beta - 1} f(t_n, u^n) - \frac{1}{2} \Delta t_{n+1}^{\beta - 1} f(t_{n+1}, u^{n+1}) \right]. \quad (4.12)$$

Finally with the ABC case, we have

$$\frac{AB(\alpha)}{1 - \alpha} \frac{d}{dt} \int_a^t u(y) \left( \frac{\alpha}{1 - \alpha} (t - y)^\alpha \right) dy = f(t, u(t)). \quad (4.13)$$

Following the routines presented earlier, we have

$$u^n = u^0 + \frac{1 - \alpha}{AB(\alpha)} \beta t_n^{\beta - 1} f(t_n, u^n) + \frac{\beta \alpha}{AB(\alpha)} \left( \xi_n f(t_0, y_0) + \sum_{j=0}^n \gamma_{n-j} t_j^{\beta - 1} f(t_j, u^j) \right). \quad (4.14)$$

Where in both cases, for Caputo and ABC derivatives

$$\xi_n = \frac{(n-1)^{\alpha+1} - n^{\alpha}(n-\alpha+1)}{\Gamma(\alpha+2)}, \quad \gamma_n = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(\alpha+2)}, & \text{if } n = 0 \\
\frac{(n-1)^{\alpha+1} - 2n^{\alpha+1} + (n+1)^{\alpha+1}}{\Gamma(\alpha+2)}, & \text{if } n = 1, 2, \ldots
\end{array} \right. \quad (4.15)$$

5. Conclusion. We made use of the properties of the Laplace transform to derive exact solution of some new differential equations. These new class of differential operators with fractal-fractional operators have not yet attracted attention of many researchers as it was recently introduced in the literature. However their applicability to capture heterogeneity of nature has been approved and revealed in some few published papers. One of the great advantage of such differential operators is perhaps their abilities to recover the existing differential operators. For instance when the fractional order tends to one we recover the fractal derivative, when the fractal dimension tends to 1 we recover the fractional differential operator. The solutions obtained here can be reduced to those of associate fractional, fractal and even classical equations.
Figure 1. Solutions of the first problem for Caputo, Caputo-Fabrizio and Atangana Baleanu derivatives for $\alpha = 0.1 = \beta$.

Figure 2. Solutions of the first problem for Caputo, Caputo-Fabrizio and Atangana Baleanu derivatives for $\alpha = 0.5 = \beta$.

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Figure 3. Solutions of the first problem for Caputo, Caputo-Fabrizio and Atangana Baleanu derivatives for $\alpha = 1.0 = \beta$.

Figure 4. Solutions of the first problem for Caputo and Caputo-Fabrizio derivatives for $\alpha = 0.1 = \beta$.

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Figure 5. Solutions of the first problem for Caputo and Caputo-Fabrizio derivatives for $\alpha = 0.5 = \beta$.

Figure 6. Solutions of the first problem for Caputo and Caputo-Fabrizio derivatives for $\alpha = 1.0 = \beta$. 
Figure 7. Solutions of the first problem for Caputo-Fabrizio and Atangana-Baleanu derivatives for $\alpha = 0.1 = \beta$.

Figure 8. Solutions of the first problem for Caputo-Fabrizio and Atangana-Baleanu derivatives for $\alpha = 0.5 = \beta$.

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