MULTIPLICATION OPERATORS ON $S^2(\mathbb{D})$

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ABSTRACT. In this paper, we study the multiplication operators on $S^2$, the space of analytic functions on the open unit disk $\mathbb{D}$ whose first derivative is in $H^2$. Specifically, we characterize the bounded and the compact multiplication operators, establish estimates on the operator norm, and determine the spectrum. Finally, we prove that the isometric multiplication operators are precisely those induced by a constant function of modulus one.

1. INTRODUCTION

Let $X$ be a Banach space of analytic functions on the open unit disk $\mathbb{D}$ in $\mathbb{C}$. Let $\psi$ be analytic on $\mathbb{D}$ and $\varphi$ an analytic self-map of $\mathbb{D}$. We define, for $f \in X$,

\[ C_\varphi(f) = f \circ \varphi, \]
\[ M_\psi(f) = \psi f, \]
\[ W_{\psi,\varphi}(f) = \psi(f \circ \varphi), \]

the composition, multiplication, and weighted composition operators, respectively, on $X$. The composition operator has a long history, for which the reader is referred to [18] and [9].

The multiplication operator is not as well studied of an operator. This is surprising since the study of multipliers on spaces can aid in the understanding of spaces of analytic functions, and can be a key tool in the study of weighted composition operators. There are many classical spaces on which to study such functions. These spaces include the weighted Hardy (see [19]), Bergman (see [3],[13]), Dirichlet (see [20]), and Bloch (see [2], [5], [1]) spaces. In addition to the classical spaces, the study of multiplication operators extends between various spaces of analytic functions (see [4], [12] and [15]). The main goal in the study of multiplication operators on these spaces is to link the properties of the operator $M_\psi$ with the properties of the symbol $\psi$.

In this paper, we study the multiplication operator on the space $S^2$, defined as the set of functions analytic on $\mathbb{D}$ whose derivative is a function in the Hardy Hilbert space $H^2$. This is a specific instance in the family of spaces $S^p$ for $1 \leq p < \infty$. The study of composition operators on $S^p$ began with Roan [17]. Subsequently, MacCluer characterized boundedness and compactness in terms of Carleson measures [16]. In her Ph.D. dissertation [14], the second author studied composition operators on $S^2$ in terms of the symbol.

Contreras and Hernández-Díaz in [6] studied the weighted composition operators between $S^p$ and $S^q$. The boundedness and compactness of these operators were characterized in terms of other weighted composition operators between Hardy spaces. In this paper, we wish to characterize boundedness and compactness of the multiplication operator in terms of its symbol. In addition, we also study other properties of the operator, including the spectrum, commutant, and isometries.

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1.1. **Organization of the paper.** In Section 2, we collect useful properties of functions in $S^2$ and $H^2$. In Section 3, we characterize the boundedness of a class of weighted composition operators on $S^2$. In addition, we establish bounds on the operator norm. From this, we obtain a characterization of the bounded multiplication operators on $S^2$ in terms of its symbol.

In Section 4, we determine the spectrum of the bounded multiplication operators on $S^2$. With this, in Section 5 we show the only compact multiplication operator on $S^2$ is the operator induced by the zero function.

Finally, in Section 6, we characterize the isometric multiplication operators on $S^2$ as those induced by constant functions of modulus one. We use this characterization to study isometric zero-divisors on $S^2$.

### 2. Preliminaries

Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$, and $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$. In the field of operator theory, the classical spaces on $\mathbb{D}$ include the Hardy space, standard weighted Bergman spaces, and the Dirichlet space. The Hardy space is defined by

$$H^2(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_{H^2}^2 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \right\},$$

where $d\theta$ is Lebesgue arc-length measure on the unit circle. For $\beta > -1$, the standard weighted Bergman space is defined by

$$A^{\beta}_2(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_{A^{\beta}_2}^2 = \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\beta \, dA < \infty \right\},$$

where $dA$ is Lebesgue area measure normalized so $A(\mathbb{D}) = 1$. The Dirichlet space is defined by

$$D(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_D^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA < \infty \right\}.$$

A reproducing kernel Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ has the property that for each $w \in \mathbb{D}$, there exists a unique function $K_w \in \mathcal{H}$, called the point-evaluation kernel, such that

$$f(w) = \langle f, K_w \rangle_{\mathcal{H}}.$$ 

The Hardy space, standard weighted Bergman spaces, and the Dirichlet space are all reproducing kernel Hilbert spaces with kernels:

$$H^2 : K_w(z) = \frac{1}{1 - \overline{w}z},$$

$$A^{\beta}_2 : K_w(z) = \frac{1}{(1 - \overline{w}z)^{\beta+2}},$$

$$D : K_w(z) = 1 + \log \frac{1}{1 - \overline{w}z},$$

where $\log z$ denotes the principal branch of the logarithm.

A functional Hilbert space is a Hilbert space $\mathcal{H}$ whose elements are complex-valued functions on a set $\Omega$, with the usual addition of functions and multiplication by scalars, and such that evaluation at each point of $\Omega$ is a bounded linear functional on $\mathcal{H}$ and there is no point in $\Omega$ at which all functions of $\mathcal{H}$ vanish. The Hardy space, standard weighted Bergman spaces, and the Dirichlet space are all
functional Banach spaces with:

\[
\|K_w\|_{H^2}^2 = \frac{1}{1 - |w|^2};
\]

\[
\|K_w\|_{A^2_{\beta}}^2 = \frac{1}{(1 - |w|^2)^{\beta + 2}};
\]

\[
\|K_w\|_{D}^2 = 1 + \log \frac{1}{1 - |w|^2}.
\]

To see the natural relationship between these spaces, it is often convenient to define the spaces and norms in terms of series representations. We have the alternative definitions of the three spaces:

\[
H^2(\mathbb{D}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ in } H(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\},
\]

\[
A^2_{\beta}(\mathbb{D}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ in } H(\mathbb{D}) : |a_0|^2 + \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{\beta + 1}} < \infty \right\},
\]

\[
D(\mathbb{D}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ in } H(\mathbb{D}) : |a_0|^2 + \sum_{n=1}^{\infty} n|a_n|^2 < \infty \right\}.
\]

From this, we see that \( D \subset H^2 \subset A^2 \). However, there seem to be many more spaces that can extend this containment. One such space, which has received a great deal of attention of late, is the space \( S^2 \), defined by

\[
S^2(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_{S^2}^2 = |f(0)|^2 + \|f'\|_{H^2}^2 < \infty \right\}
\]

\[
= \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ in } H(\mathbb{D}) : \|f\|_{S^2}^2 = |a_0|^2 + \sum_{n=1}^{\infty} n^2|a_n|^2 < \infty \right\}.
\]

Note that the integral and series norms are actually equal, not just equivalent. We see that \( S^2 \) fits with the classical spaces since \( S^2 \subset D \subset H^2 \subset A^2 \). Unlike the other spaces, there is no “nice” closed form for the reproducing kernels in \( S^2 \).

The following results are collected here for use in later sections of the paper.

**Theorem 2.1.** For the space \( S^2 \):

(a) every function in \( S^2 \) extends continuously to the boundary of \( \mathbb{D} \).

(b) every function in \( S^2 \) is bounded on \( \mathbb{D} \).

(c) \( S^2 \) contains the polynomials.

(d) evaluation at each point in \( \mathbb{D} \) is a bounded linear functional.

(e) \( S^2 \) is a functional Hilbert space.

**Proof.** Properties (b), (c), and (d) can be found in Chapter 4 of [9]. The fact that \( S^2 \) is a functional Hilbert space follows immediately from (c) and (d). We will provide a proof of property (a), which shows \( S^2 \) to be a boundary-regular space. Consider \( f \in S^2 \) with power series representation \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and let \( f_N \) denote the \( N^{th} \)-partial sum of this series. Assuming \( M \geq N \), it follows that \( f \) is uniformly Cauchy on \( \bar{\mathbb{D}} \) from Hölder’s inequality and the estimate

\[
|f_N(z) - f_M(z)| \leq \sum_{n=N+1}^{M} |a_n| \leq \left( \sum_{n=N+1}^{M} |a_n|^2 n^2 \right)^{1/2} \left( \sum_{n=N+1}^{M} \frac{1}{n^2} \right)^{1/2}.
\]

It follows immediately that \( f \) extends continuously to \( \partial \mathbb{D} \). \( \square \)
Proposition 2.1.1. Let \( f \in S^2 \). Then \( \|f\|_\infty \leq \frac{\pi \sqrt{3}}{3} \|f\|_{S^2} \).

Proof. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) for \( z \in \mathbb{D} \). For \( x, y \geq 0 \), recall the following inequality

\[
(x + y)^2 \leq 2(x^2 + y^2). \tag{2.1}
\]

Then from Hölder's inequality, we have

\[
|f(z)|^2 \leq \left[ a_0 + \sum_{n=1}^{\infty} |a_n| \right]^2 \\
\leq \left[ a_0 + \left( \sum_{n=1}^{\infty} n^2 |a_n|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \right]^2 \\
\leq \left[ a_0 \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} + \left( \sum_{n=1}^{\infty} n^2 |a_n|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \right]^2 \\
= \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right) \left[ (a_0^2)^{1/2} + \left( \sum_{n=1}^{\infty} n^2 |a_n|^2 \right)^{1/2} \right]^2 \\
\leq 2 \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right) \left( |a_0|^2 + \sum_{n=1}^{\infty} n^2 |a_n|^2 \right) \\
= \frac{\pi^2}{3} \|f\|^2_{S^2}.
\]

Thus \( \|f\|_\infty \leq \frac{\pi \sqrt{3}}{3} \|f\|_{S^2} \). \( \square \)

This result not only shows that \( S^2 \) is contained in the disk algebra \( A(\mathbb{D}) \), but also the inclusion map is continuous. Although this is a known result (see [10] or [6]), we establish an actual bound which will be used in developing the estimates on the norm of weighted composition operators on \( S^2 \).

Lemma 2.2. Let \( \varphi \) be an automorphism of \( \mathbb{D} \) and \( f, g \in S^2 \). Then

(a) \( |f(\varphi(0))|^2 \leq \|K_{\varphi(0)}\|_{S^2}^2 \|f\|^2_{S^2} \leq \frac{\|f\|^2_{S^2}}{1 - |\varphi(0)|^2} \)

(b) \( \|(f \circ \varphi)'\|^2_{H^2} \leq \left( \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right) \|f\|^2_{S^2} \)

(c) \( fg \in S^2 \) and \( \|(fg)'\|^2_{H^2} \leq \frac{2\pi \sqrt{3}}{3} \|f\|_{S^2} \|g\|_{S^2} \)

Proof. The proof of parts (a) and (b) can be found as part of the proof of Theorem 3.2 in [14]. We will provide a proof of part (c). Since both \( f \) and \( g \) are assumed to be in \( S^2 \), they are necessarily
bounded on $D$ by Theorem 2.1(b). By the triangle inequality and Proposition 2.1.1, we have

\[
\| (fg)' \|_{H^2} = \| f' g + fg' \|_{H^2} \\
\leq \| f' g \|_{H^2} + \| fg' \|_{H^2} \\
\leq \| f' \|_{H^2} \| g \|_{\infty} + \| f \|_{\infty} \| g' \|_{H^2} \\
\leq \| f \|_{S^2} \| g \|_{\infty} + \| f \|_{\infty} \| g \|_{S^2} \\
\leq \| f \|_{S^2} \left( \frac{\pi \sqrt{3}}{3} \| g \|_{S^2} \right) + \left( \frac{\pi \sqrt{3}}{3} \| f \|_{S^2} \right) \| g \|_{S^2} \\
= \frac{2\pi \sqrt{3}}{3} \| f \|_{S^2} \| g \|_{S^2},
\]

as desired. \hfill \Box

**Proposition 2.2.1.** If $f \in S^2$ and $w \in D$ is a zero of $f$ of order $n \in \mathbb{N}$, then there exists a function $g \in S^2$ such that $g(w) \neq 0$ and $f(z) = (z - w)^n g(z)$ for all $z \in D$.

**Proof.** Since $f$ is analytic on $D$, there exists a function $g$ analytic on $D$ with $g(w) \neq 0$ and $f(z) = (z - w)^n g(z)$. It suffices to show that $g(z) = f(z)/(z - w)^n$ is in $S^2$. Let $f$ have a power series representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in D$. Then

\[
g(0) = \begin{cases} 
  a_{n+1}, & w = 0 \\
  \frac{f(0)}{(-w)^n}, & w \neq 0.
\end{cases}
\]

We will now show $\| g' \|_{H^2} < \infty$. Let $\alpha = \frac{1}{2}(1 + |w|)$. Then for $\alpha < r < 1$ and $|z| = r$, we have

\[
\frac{1}{|z - w|} \leq \frac{2}{1 - |w|}.
\]

By the previous estimate, the triangle inequality, Theorem 1.5 of [10] and inequality (2.1) we obtain

\[
\| g' \|_{H^2}^2 = \sup_{\alpha < r < 1} \int_0^{2\pi} |g'(re^{i\theta})|^2 \frac{d\theta}{2\pi} \\
\leq \frac{2\alpha^{n+3}}{(1 - |w|)^{2n+2}} \sup_{\alpha < r < 1} \int_0^{2\pi} \left( 4|f'(re^{i\theta})|^2 + n^2 |f(re^{i\theta})|^2 \right) \frac{d\theta}{2\pi},
\]

which is finite since $f \in S^2 \subset H^2$. Thus $g \in S^2$ as desired. \hfill \Box

For $w \in D$, we define $ev_w$ to be the linear functional on $S^2$ such that $ev_w(f) = f(w)$ for $f \in S^2$. Since $S^2$ is a functional Hilbert space, $ev_w$ is a bounded linear functional for every $w \in D$.

**Proposition 2.2.2.** Let $w \in D$. Then $\ker ev_w = \text{ran}(M_z - w)$.

**Proof.** Let $g \in \text{ran}(M_z - w)$. Then there exists an $f \in S^2$ such that $(z - w)f(z) = g(z)$ for $z \in D$. Then $0 = g(w) = ev_w(g)$, and so $\text{ran}(M_z - w) \subseteq \ker ev_w$.

Now suppose $f \in \ker ev_w$ with $f$ not identically zero. By Proposition 2.2.1, there exists $g \in S^2$ with $g(w) \neq 0$ and $f(z) = (z - w)^n g(z)$. It follows that $G(z) = (z - w)^{n-1} g(z)$ is in $S^2$ by Lemma 2.2(c). Thus $f(z) = (z - w)G(z)$, and hence $\ker ev_w \subseteq \text{ran}(M_z - w)$. \hfill \Box
3. BOUNDEDNESS AND NORM ESTIMATES

In this section we establish a characterization of the bounded multiplication operators on $S^2$, as well as estimates on the operator norm. Our aim is to characterize the boundedness of $M_\psi$ in terms of the properties of $\psi$. To accomplish this, we first investigate the weighted composition operators on $S^2$ induced by automorphisms of $\mathbb{D}$.

Contreras and Hernández-Díaz in [6] characterized bounded weighted composition operators on $S^2$, among other things, in terms of the boundedness of a different weighted composition operator on $H^2$. In this paper, we establish the boundedness of the weighted composition operator on $S^2$ in terms of the symbols. Specifically, we study weighted composition operators $W_{\psi,\varphi}$ induced by $\psi \in S^2$ and $\varphi$ an automorphism of $\mathbb{D}$. With this, we can establish the boundedness of $M_\psi$ on $S^2$, as well as operator norm estimates on $W_{\psi,\varphi}$ and $M_\psi$.

**Theorem 3.1.** Let $\psi$ be analytic on $\mathbb{D}$ and $\varphi$ an automorphism of $\mathbb{D}$. Then $W_{\psi,\varphi}$ is bounded on $S^2$ if and only if $\psi \in S^2$. Moreover, for $W_{\psi,\varphi}$ bounded on $S^2$,

$$
\|\psi\|_{S^2} \leq \|W_{\psi,\varphi}\| \leq \left(1 + \frac{8\pi^2}{3}\right) \left(1 + \frac{|\varphi(0)|}{1 - |\varphi(0)|}\right)^{1/2} \|\psi\|_{S^2}.
$$

**Proof.** Suppose $W_{\psi,\varphi}$ is bounded on $S^2$. Taking the test function $f$ identically 1, we have $\|\psi\|_{S^2} = \|W_{\psi,\varphi}f\|_{S^2} < \infty$. Thus $\psi \in S^2$.

Now suppose $\psi \in S^2$. For $f \in S^2$, we have $f \circ \varphi \in S^2$. By Lemma 2.2, we obtain

$$
\|W_{\psi,\varphi}f\|_{S^2}^2 = |\psi(0)f(\varphi(0))|^2 + \|(\psi(f \circ \varphi))'\|_{H^2}^2
\leq \|\psi\|_{S^2}^2 |f(\varphi(0))|^2 + \frac{4\pi^2}{3} \|\psi\|_{S^2}^2 \|f \circ \varphi\|_{S^2}^2
\leq \left[1 + \frac{4\pi^2}{3}\right] |f(\varphi(0))|^2 + \frac{4\pi^2}{3} \|(f \circ \varphi)'\|_{H^2}^2 \|\psi\|_{S^2}^2
\leq \left[1 + \frac{4\pi^2}{3}\right] \frac{1}{1 - |\varphi(0)|^2} + \frac{4\pi^2}{3} \left(1 + |\varphi(0)| \right) \|\psi\|_{S^2}^2 \|f\|_{S^2}^2
\leq \left(1 + \frac{8\pi^2}{3}\right) \left(1 + \frac{|\varphi(0)|}{1 - |\varphi(0)|}\right) \|\psi\|_{S^2}^2 \|f\|_{S^2}^2.
$$

Thus $W_{\psi,\varphi}$ is bounded on $S^2$ and the upper estimate follows. To obtain the lower bound, we take the function $f$ identically 1 and observe $\|\psi\|_{S^2} = \|W_{\psi,\varphi}f\|_{S^2}$. $\square$

As a direct consequence we obtain a characterization of the multiplication operators on $S^2$ in terms of the symbol $\psi$ by taking $\varphi(z) = z$.

**Corollary 3.2.** The multiplication operator $M_\psi$ is bounded on $S^2$ if and only if $\psi \in S^2$.

We now establish bounds on the norm of $M_\psi$ on $S^2$. From Theorem 3.1, we can conclude that

$$
\|\psi\|_{S^2} \leq \|M_\psi\| \leq \left(1 + \frac{8\pi^2}{3}\right)^{1/2} \|\psi\|_{S^2}.
$$

However, many of the estimates used in the proof are not sharp. This is due, in part, to the composition operator. While this is not a concern when determining boundedness of an operator, it poses problems when determining norm estimates.

When considering the multiplication operator in isolation, we obtain sharper estimates on the norm.
Theorem 3.3. Let $\psi \in S^2$. Then

$$\max \{ \|\psi\|_{S^2}, \|\psi\|_{\infty} \} \leq \|M_\psi\| \leq \left( 1 + \frac{4\pi^2}{3} \right)^{1/2} \|\psi\|_{S^2}.$$  

Proof: From Lemma 11 of [11], we have $\|\psi\|_{\infty} \leq \|M_\psi\|$. Thus, the lower bound is established.

By Lemma 2.2, for $f \in S^2$

$$\|M_\psi f\|_{S^2}^2 = \|\psi(0)f(0)\|^2 + \|(\psi f)'\|_{H^2}^2$$

$$\leq \|\psi\|_{S^2}^2 \|f\|_{S^2}^2 + \frac{4\pi^2}{3} \|\psi\|_{S^2}^2 \|f\|_{S^2}^2$$

$$= \left( 1 + \frac{4\pi^2}{3} \right) \|\psi\|_{S^2}^2 \|f\|_{S^2}^2.$$  

Thus

$$\|M_\psi f\|_{S^2} \leq \left( 1 + \frac{4\pi^2}{3} \right)^{1/2} \|\psi\|_{S^2} \|f\|_{S^2},$$

and the upper estimate follows. \qed

4. Spectrum

We now turn our attention to the spectrum of the bounded multiplication operators on $S^2$. Recall that the resolvent set of a bounded linear operator $T$ on a complex Banach space $X$ is defined as

$$\rho(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is invertible} \},$$

where $I$ is the identity operator. The spectrum of $T$ is defined as $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

The spectrum is a non-empty compact subset of the closed disk centered at the origin of radius $\|T\|$. The set of eigenvalues of $T$, also known as the point spectrum, is a subset of the spectrum and defined as

$$\sigma_p(T) = \{ \lambda \in \mathbb{C} : \ker(T - \lambda I) \neq \{0\} \}.$$

Theorem 4.1. Let $\psi \in S^2$. Then $M_\psi$ has no eigenvalues unless $\psi$ is a constant function, that is,

$$\sigma_p(M_\psi) = \begin{cases} \{\lambda\}, & \text{if } \psi(z) = \lambda \text{ for all } z \in \mathbb{D}; \\ \emptyset, & \text{otherwise}. \end{cases}$$

Proof: Suppose $\psi(z) = \lambda$ for all $z \in \mathbb{D}$. Then for any non-zero function $f \in S^2$, it is true that $M_\psi f - \lambda f$ is identically zero on $\mathbb{D}$. Thus $\lambda$ is an eigenvalue of $M_\psi$. Let $\mu \in \mathbb{C}$ such that $\mu \neq \lambda$. If $f \in S^2$ such that $M_\psi f - \mu f$ is identically zero on $\mathbb{D}$, then $f$ is identically zero. Thus $\mu$ is not an eigenvalue of $M_\psi$. So $\sigma_p(M_\psi) = \{\lambda\}$.

Now suppose $\psi$ is not a constant function. Assume $\lambda \in \mathbb{C}$ is an eigenvalue of $M_\psi$ and $f \in S^2$ is a corresponding (non-zero) eigenfunction. Since $M_\psi f - \lambda f$ is identically zero on $\mathbb{D}$, it must be the case that $f$ is identically zero (see Chapter IV Theorem 3.7 of [7]), a contradiction. Thus $M_\psi$ has no eigenvalues. \qed

Remark 4.2. If the multiplication operator $M_\psi$ is bounded on some Banach space of analytic functions, then $\lambda \in \sigma(M_\psi)$ if and only if $M_\psi - \lambda I$ is not invertible. Since $\lambda I = M_\lambda$, and $M_\psi - M_\lambda = M_{\psi - \lambda}$, we see that $\lambda \in \sigma(M_\psi)$ if and only if $M_{\psi - \lambda}$ is not invertible. Since $M_{\psi - \lambda}^{-1} = M_{(\psi - \lambda)^{-1}}$, we finally arrive at $\lambda \in \sigma(M_\psi)$ if and only if $M_{(\psi - \lambda)^{-1}}$ is not well-defined.

Theorem 4.3. Let $\psi \in S^2$. Then $\sigma(M_\psi) = \psi(\overline{\mathbb{D}})$. 
Proof. Suppose \( \lambda \not\in \sigma(M_\psi) \). Then \( M_\psi - \lambda I \) is onto. So there exists a function \( f \in S^2 \) such that \((\psi(z) - \lambda)f(z) = 1 \) for all \( z \in \mathbb{D} \). It follows that \( \lambda \not\in \psi(\mathbb{D}) \). Thus \( \psi(\mathbb{D}) \subseteq \sigma(M_\psi) \).

Now suppose \( \lambda \not\in \psi(D) \). Then the function \( \psi - \lambda \) is bounded away from zero, that is there exists \( c > 0 \) such that \( |\psi(z) - \lambda| \geq c \) for all \( z \in \mathbb{D} \). Thus the function \( g = (\psi - \lambda)^{-1} \) is a bounded analytic function on \( \mathbb{D} \). We see that \( g \in S^2 \) since

\[
\|g\|_{S^2}^2 = |g(0)|^2 + \|g'\|_{H^2}^2 = \frac{1}{\|\psi(0) - \lambda\|^2} + \|\frac{\psi'}{(\psi - \lambda)^2}\|_{H^2}^2 \leq \frac{1}{c^2} + \frac{1}{c^4}\|\psi'\|_{H^2}^2 \leq \frac{1}{c^2} + \frac{1}{c^4}\|\psi\|_{S^2}^2 < \infty.
\]

Thus \( M_g \) is bounded on \( S^2 \) by Corollary 3.2, and so \( \lambda \not\in \sigma(M_\psi) \) (see Remark 4.2). Since the spectrum is closed, we have \( \psi(D) = \sigma(M_\psi) \).

5. Compactness

In this section, we characterize the compact multiplication operators on \( S^2 \). As with many spaces, the only compact multiplication operator is that induced by the zero function. To prove this, we utilize the spectral theorem for compact operators due to F. Riesz (see Chapter VII Theorem 7.1 of [8]), and the results from the previous section.

Theorem 5.1. Let \( \psi \in S^2 \). Then \( M_\psi \) is compact on \( S^2 \) if and only if \( \psi \) is identically zero.

Proof. Clearly if \( \psi \) is identically zero, then \( M_\psi \) is compact on \( S^2 \). Suppose \( M_\psi \) is compact on \( S^2 \). The spectrum of \( M_\psi \) is connected since it is precisely \( \psi(\mathbb{D}) \) by Theorem 4.3. Since \( \sigma(M_\psi) \) is countable and contains \( 0 \), by the Spectral Theorem for Compact Operators, it must be the case that \( \psi(\mathbb{D}) = \sigma(M_\psi) = \{0\} \). Thus \( \psi \) is identically zero.

This result is also a consequence of Theorem 2.1 of [6]. However, our proof is independent of the theory of compact multiplication operators on \( H^2 \).

6. Isometries

In this section, we characterize the isometric multiplication operators, as well as the isometric zero-divisors on \( S^2 \).

Theorem 6.1. Let \( \psi \in S^2 \). Then \( M_\psi \) is an isometry on \( S^2 \) if and only if \( \psi \) is a constant function of modulus one.

Proof. It is clear that a constant function of modulus one induces an isometric multiplication operator on \( S^2 \). Now suppose \( \psi \in S^2 \) such that \( M_\psi \) is an isometry on \( S^2 \). For \( f \) identically 1, we have

\[
\|\psi\|_{S^2} = \|M_\psi f\|_{S^2} = \|f\|_{S^2} = 1.
\]

Thus, if \( \psi \) has power series representation \( \psi(z) = \sum_{n=0}^{\infty} a_n z^n \), then

\[
1 = \|\psi\|_{S^2}^2 = |a_0|^2 + \sum_{n=1}^{\infty} n^2|a_n|^2, \tag{6.1}
\]
and
\[ z\psi'(z) + \psi(z) = a_0 + \sum_{n=1}^{\infty} (n+1)a_n z^n. \]

Let \( g(z) = z \). Since \( M_\psi \) is an isometry on \( S^2 \), we have
\[
1 = \|g\|_{S^2}^2 = \|M_\psi g\|_{S^2}^2 = \|z\psi' + \psi\|_{H^2}^2
\]
\[
= |a_0|^2 + \sum_{n=1}^{\infty} (n+1)^2 |a_n|^2
\]
\[
= |a_0|^2 + \sum_{n=1}^{\infty} n^2 |a_n|^2 + 2 \sum_{n=1}^{\infty} n |a_n|^2 + \sum_{n=1}^{\infty} |a_n|^2
\]
\[
= \|\psi\|_{S^2}^2 + 2 \sum_{n=1}^{\infty} n |a_n|^2 + \sum_{n=1}^{\infty} |a_n|^2.
\]

From (6.1), it follows immediately that
\[
\sum_{n=1}^{\infty} |a_n|^2 = 0,
\]
and thus \( a_n = 0 \) for all \( n \geq 1 \). So \( \psi \) is the constant function \( \psi(z) = a_0 \), and by (6.1) again, \( |\psi(z)| = |a_0| = 1 \), for all \( z \in \mathbb{D} \).

The following discussions are consequences of the characterization of the isometric multiplication operators on \( S^2 \).

6.1. Isometric Zero-Divisors. A nonempty sequence (finite or infinite) \( \{z_k\} \) of \( \mathbb{D} \) is called a zero-set of \( S^2 \) if there exists a function \( f \in S^2 \) which vanishes precisely on \( \{z_k\} \). For such a zero-set, a function \( g \in S^2 \) is called a zero-divisor if it vanishes precisely on \( \{z_k\} \) and \( f/g \in S^2 \) for all \( f \in S^2 \) which vanish on \( \{z_k\} \). If \( \|f/g\|_{S^2} = \|f\|_{S^2} \) for every such function \( f \), then \( g \) is said to be an isometric zero-divisor.

Aleman, et. al. proved the following necessary condition for the existence of isometric zero-divisors for a class of functional Banach spaces which includes \( S^2 \).

**Theorem 6.2** (Theorem 5 of [?]). Let \( X \) be a functional Banach space, and suppose \( X \) is invariant under multiplication by polynomials. Then every isometric zero-divisor in \( X \) induces an isometric multiplication operator on \( X \).

Due to the characterization of isometric multiplication operators on \( S^2 \), we obtain the following.

**Theorem 6.3.** There are no isometric zero-divisors in \( S^2 \).

**Proof.** It has already been shown that \( S^2 \) is a functional Hilbert space. Since the polynomials are contained in \( S^2 \), Lemma 2.2(c) implies that \( S^2 \) is invariant under multiplication by polynomials. Thus by Theorem 6.2, if \( g \) is an isometric zero-divisor of \( S^2 \), then \( g \) induces an isometric multiplication operator on \( S^2 \). However, by Theorem 6.1 it must be the case that \( g \) is a constant function of modulus one, which does not vanish anywhere. Thus \( S^2 \) contains no isometric zero-divisors.

**REFERENCES**

1. Robert F. Allen and Flavia Colonna, *Isometries and spectra of multiplication operators on the Bloch space*, Bull. Aust. Math. Soc. 79 (2009), no. 1, 147–160. MR 2486890
2. Jonathan Arazy, *Multipliers of bloch functions*, University of Haifa Mathematics Publication Series 54 (1982).
3. Sheldon Axler, *Multiplication operators on Bergman spaces*, J. Reine Angew. Math. 336 (1982), 26–44. MR 671320
4. J. Bonet, P. Domański, and M. Lindström, *Pointwise multiplication operators on weighted Banach spaces of analytic functions*, Studia Math. 137 (1999), no. 2, 177–194. MR 1734396
5. Leon Brown and A. L. Shields, *Multipliers and cyclic vectors in the Bloch space*, Michigan Math. J. 38 (1991), no. 1, 141–146. MR 1091517
6. Manuel D. Contreras and Alfredo G. Hernández-Díaz, *Weighted composition operators on spaces of functions with derivative in a Hardy space*, J. Operator Theory 52 (2004), no. 1, 173–184. MR 2091466
7. John B. Conway, *Functions of one complex variable*, second ed., Graduate Texts in Mathematics, vol. 11, Springer-Verlag, New York-Berlin, 1978. MR 503901
8. , *A course in functional analysis*, second ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990. MR 1070713
9. Carl C. Cowen and Barbara D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR 1397026
10. Peter L. Duren, *Theory of Hp spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York-London, 1970. MR 0268655
11. Peter L. Duren, Bernhard W. Romberg, and Allen L. Shields, *Linear functionals on Hp spaces with 0 < p < 1*, J. Reine Angew. Math. 238 (1969), 32–60. MR 259579
12. Nathan S. Feldman, *Pointwise multipliers from the Hardy space to the Bergman space*, Illinois J. Math. 43 (1999), no. 2, 211–221. MR 1703183
13. Kunyu Guo and Hansong Huang, *On multiplication operators on the Bergman space: similarity, unitary equivalence and reducing subspaces*, J. Operator Theory 65 (2011), no. 2, 355–378. MR 2785849
14. Katherine Cason Heller, *Composition Operators on Sp(D)*, ProQuest LLC, Ann Arbor, MI, 2010, Thesis (Ph.D.)–University of Virginia. MR 2844199
15. Yongmin Liu and Yanyan Yu, *The multiplication operator from F(p, q, s) spaces to nth weighted-type spaces on the unit disk*, J. Funct. Spaces Appl. (2012), Art. ID 343194, 21. MR 2923799
16. Barbara D. MacCluer, *Composition operators on Sp*, Houston J. Math. 13 (1987), no. 2, 245–254. MR 904956
17. Raymond C. Roan, *Composition operators on the space of functions with Hp-derivative*, Houston J. Math. 4 (1978), no. 3, 423–438. MR 512878
18. Joel H. Shapiro, *Composition operators and classical function theory*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993. MR 1237406
19. Sunil Kumar Sharma and B. S. Komal, *Multiplication operators on weighted Hardy spaces*, Lobachevskii J. Math. 33 (2012), no. 2, 165–169. MR 2943800
20. David A. Stegenga, *Multipliers of the Dirichlet space*, Illinois J. Math. 24 (1980), no. 1, 113–139. MR 550655