Wave-corpuscle mechanics for elementary charges

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Abstract

It is well known that the concept of a point charge interacting with the electromagnetic (EM) field has a problem. To address that problem we introduce the concept of wave-corpuscle to describe spinless elementary charges interacting with the classical EM field. Every charge interacts only with the EM field and is described by a complex valued wave function over the 4-dimensional space time continuum. A system of many charges interacting with the EM field is defined by a local, gauge and Lorentz invariant Lagrangian with a key ingredient - a nonlinear self-interaction term providing for a cohesive force assigned to every charge. An ideal wave-corpuscle is an exact solution to the Euler-Lagrange equations describing both free and accelerated motions. It carries explicitly features of a point charge and the de Broglie wave. A system of well separated charges moving with nonrelativistic velocities are represented accurately as wave-corpuscles governed by the Newton equations of motion for point charges interacting with the Lorentz forces. In this regime the nonlinearities are "stealthy" and don’t show explicitly anywhere, but they provide for the binding forces that keep localized every individual charge.

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1 Introduction

We all know from textbooks that if there is a point charge $q$ of a mass $m$ in an external electromagnetic (EM) field its non-relativistic dynamics is governed by the equation

$$\frac{d}{dt}[mv(t)] = q \left[ E(r(t), t) + \frac{1}{c} v(t) \times B(r(t), t) \right]$$  \hspace{1cm} (1.0.1)

where $r$ and $v = \dot{r} = \frac{dr}{dt}$ are respectively the charge’s time-dependent position and velocity, $E(t, r)$ and $B(t, r)$ are the electric field and the magnetic induction, and the right-hand side of the equation (1.0.1) is the Lorentz force. We also know that if the charge’s time-dependent position and velocity are $r$ and $v$ then there is associated with them an EM field described by the equations

$$\frac{1}{c} \frac{\partial B}{\partial t} + \nabla \times E = 0, \quad \nabla \cdot B = 0,$$  \hspace{1cm} (1.0.2)

$$\frac{1}{c} \frac{\partial E}{\partial t} - \nabla \times B = -\frac{4\pi}{c} q \delta(x-r(t)) v(t), \quad \nabla \cdot E = 4\pi q \delta(x-r(t)), \quad v(t) = \dot{r}(t),$$  \hspace{1cm} (1.0.3)

where $\delta$ is the Dirac delta-function. But if naturally we would like to consider the equation (1.0.1) and (1.0.2)-(1.0.3) as a closed system “charge-EM field” there is a problem. The origin of the problems is in the divergence of the EM field exactly at the position of the point charge, as, for instance, for the electrostatic field $E$ with the Coulomb’s potential $\frac{q}{|x-r|}$ with a singularity at $x = r$. If (1.0.1) is replaced by a relativistic equation

$$\frac{d}{dt}[\gamma mv(t)] = q \left[ E(r(t), t) + \frac{1}{c} v(t) \times B(r(t), t) \right],$$  \hspace{1cm} (1.0.4)

where $\gamma = 1/\sqrt{1 - \frac{v^2(t)}{c^2}}$ is the Lorentz factor, the system constituted by (1.0.4) and (1.0.2)-(1.0.3) becomes Lorentz invariant and has a Lagrangian that yields it via the variational principle [Barut (4.21)], [Spohn (2.36)], but the problem still persists. Some studies indicate that, [Yaghjian1], ”a fully consistent classical equation of motion for a point charge, unlike that of an extended charge, does not exist”. If one wants to stay within the classical theory of electromagnetism a possible remedy is the introduction of an extended charge which, though very small, is not a point. There are two most well known models for such an extended charge: the Abraham rigid charge model and the Lorentz relativistically covariant model. These models are considered, studied and advanced in [Jackson Sections 16], [Pearle1], [Rohrlich Sections 2, 6], [Schwinger], [Spohn], [Yaghjian]. Importantly for what we do here, Poincaré suggested in 1905-1906, [Poincare] (see also [Jackson Sections 16.4-16.6], [Rohrlich Sections 2.3, 6.1- 6.3], [Pauli RT Section 63], [Schwinger], [Yaghjian Section 4.2] and references there in), to add to the Lorentz-Abraham model non-electromagnetic cohesive forces which balance the charge internal repulsive electromagnetic forces and remarkably restore also the covariance of the entire model. W. Pauli argues very convincingly based on the relativity principle in [Pauli RT Section 63] the necessity to introduce for the electron an energy of non-electromagnetic origin.

An alternative approach to deal with the above-mentioned divergences goes back to G. Mie who proposed to modify the Maxwell equations making them nonlinear, [Pauli RT]
Section 64], [Weyl STM Section 26], and a particular example of the Mie approach is the Born-Infeld theory, [Born Infeld 1]. M. Kiessling showed that, [Kiessling 1], ”a relativistic Hamilton–Jacobi type law of point charge motion can be consistently coupled with the nonlinear Maxwell–Born–Infeld field equations to obtain a well-defined relativistic classical electrodynamics with point charges”.

A substantially different approach to elementary charges was pursued by E. Schrödinger who tried to develop a concept of wavepacket as a model for spatially localized charge. The Schrödinger wave theory, [Schrodinger ColPap], was inspired by de Broglie ideas, [de Broglie 2], [Barut, Section II.1]. The theory was very successful in describing quantum phenomena in the hydrogen atom, but it had great difficulties in treating the elementary charge as the material wave as it moves and interacts with other elementary charges. M. Born commented on this, [Born1, Chapter IV.7]: ”To begin with, Schrödinger attempted to interpret corpuscles and particularly electrons, as wave packets. Although his formulae are entirely correct, his interpretation cannot be maintained, since on the one hand, as we have already explained above, the wave packets must in course of time become dissipated, and on the other hand the description of the interaction of two electrons as a collision of two wave packets in ordinary three-dimensional space lands us in grave difficulties.”

We develop here a concept of wave-corpuscle, which is understood as a spatially localized excitation in a dispersive medium, and which is to substitute for the point charge concept. Our approach to a spatially distributed but localized elementary charge has some features in common with the above discussed concepts of extended charge, but it differs from any of them substantially. In particular, our approach provides for an electromagnetic theory in which (i) a ”bare” elementary charge and the EM field described by the Maxwell equations form an inseparable entity; (ii) every elementary ”bare” charge interacts directly only with the EM field; (iii) the EM field is a single entity providing for the interaction between ”bare” elementary charges insuring the maximum speed of interaction not to ever exceed the speed of light. To emphasize the inseparability of the ”bare” elementary charge from the EM field we refer to their entity as to dressed charge.

The best way to describe our concept of a spatially distributed but localized dressed charge in one word is by the name wave-corpuscle, since it is a stable localized excitation of a dispersive medium propagating in the three-dimensional space. An instructive example of a wave-corpuscle is furnished by our nonrelativistic charge model. In that model, in the simplest case, an ideal wave-corpuscle is described by a complex-valued wave function ψ of the form

$$\psi = \psi(t, x) = \exp \left\{ \frac{i}{\hbar} \left[ \mathbf{p}(t) \cdot \mathbf{x} - \int_0^t \frac{\mathbf{p}(t')^2}{2m} dt' \right] \right\} \tilde{\psi}(|\mathbf{x} - \mathbf{r}(t)|),$$  \hspace{1cm} (1.0.5)

where $\tilde{\psi}(s), s \geq 0,$ is a non negative, monotonically decaying function which vanishes at infinity at a sufficiently fast rate. Importantly, for the above wave function $\psi$ to be an exact solution of corresponding field equations, the parameters $\mathbf{r}(t)$ and $\mathbf{p}(t)$ satisfy the Newton’s equations which in this simplest case have the form

$$m \frac{d^2 \mathbf{r}(t)}{dt^2} = q\mathbf{E}_{\text{ex}}, \quad \mathbf{p}(t) = m \frac{d\mathbf{r}(t)}{dt},$$  \hspace{1cm} (1.0.6)

where $m$ and $q$ are respectively its mass and the charge and $\mathbf{E}_{\text{ex}}(t)$ is an external homogeneous electric field. We would like to emphasize that the Newton’s equations are not postulated as in (1.0.1) or (1.0.4) but rather are derived from the field equations. The ideal wave-corpuscle wave function $\psi(t, \mathbf{x})$ defined by (1.0.5), (1.0.6) together with the corresponding
EM field forms an exact solution to the relevant Euler-Lagrange field equations describing an accelerating dressed charge. The point charge momentum \( p(t) \) turns out to be exactly equal to the total momentum of the charge as a wave-corpuscle and its electromagnetic field. Remarkably the point charge features appear in the phase and amplitude of the ideal wave-corpuscle in a transparent and direct way without any limit process. The wave-corpuscle is a material wave, the quantity \( q|\psi(t,x)|^2 \) corresponds to the charge density and the density \( |\psi(t,x)|^2 \) is not given a probabilistic interpretation. The wave-corpuscle provides we believe an alternative resolution to the wave-particle duality problem. In one word our theory can be characterized as neo-classical as suggested by Michael Kiessling.

2 Sketch of the wave-corpuscle mechanics

We describe a bare single elementary charge by a complex-valued scalar field \( \psi = \psi(x) = \psi(t,x) \), where \( x = (t, x) \in \mathbb{R}^4 \) is the space-time variable. The charge is coupled at all times with the classical EM field as described by its potentials \( A^\mu = (\varphi, A) \) related to the EM field by the standard formulas

\[
E = -\nabla \varphi - \frac{1}{c} \partial_\tau A, \quad B = \nabla \times A, \tag{2.0.7}
\]

where \( c \) is the speed of light. The dynamics of the system of a single charge and the EM field is described via its Lagrangian

\[
L_0 (\psi, A^\mu) = \frac{\chi^2}{2m} \left\{ \frac{1}{c^2} |\tilde{\partial}_t \psi|^2 - |\tilde{\nabla} \psi|^2 - \kappa_0^2 |\psi|^2 - G (\psi^* \psi) \right\} + \frac{1}{8\pi} \left[ \left( \nabla \varphi + \frac{1}{c} \partial_\tau A \right)^2 - (\nabla \times A)^2 \right], \tag{2.0.8}
\]

where \( \tilde{\partial}_t \) and \( \tilde{\nabla} \) are the covariant differentiation operators defined by

\[
\tilde{\partial}_t = \partial_t + \frac{iq\varphi}{\chi}, \quad \tilde{\nabla} = \nabla - \frac{iqA}{\chi c}, \quad \tilde{\partial}_t^* = \partial_t - \frac{iq\varphi}{\chi}, \quad \tilde{\nabla}^* = \nabla + \frac{iqA}{\chi c}, \tag{2.0.9}
\]

\( m > 0 \) is the charge mass, \( \chi > 0 \) is a constant similar to the Planck constant \( \hbar = \frac{h}{2\pi} \) and it might be dependent on the charge; \( q \) is the total charge of the particle.

Let us take a closer look at the components of the Lagrangian (2.0.8). It involves constants \( \kappa_0, c, \chi \) and \( m \) and, acting similarly to the case of the Klein-Gordon equation for a relativistic particle (see [Pauli PWM] Sections 1, 18, 19) and Section [11.10], we introduce a fundamental frequency \( \omega_0 \) relating it to the above constants by the following formulas

\[
\omega_0 = \frac{mc^2}{\chi}, \quad \kappa_0 = \frac{\omega_0}{c} = \frac{mc}{\chi}. \tag{2.0.10}
\]

A key component of the Lagrangian in (2.0.8) is a real-valued nonlinear function \( G(s), s \geq 0 \), providing for the charge cohesive self-interaction. The second part of the expression (2.0.8) is the standard Lagrangian of the EM field coupled to the charge via the covariant derivatives. Observe that the Lagrangian \( L_0 \) defined by (2.0.8)-(2.0.9) is manifestly (i) local; (ii) Lorentz and gauge invariant, and (iii) it has a local nonlinear term providing for a cohesive self-force similar to the Poincaré force for the Lorentz-Poincaré model of an extended charge.
Since a single charge is coupled at all times to the EM field we always deal with the system "charge-EM field", \( \{ \psi, \psi^*, A^\mu \} \), and call it for short dressed charge. The dressed charge motion is governed by the relevant Euler-Lagrange field equations (see (3.0.6), (3.0.7)), and when the charge is at rest in the origin \( x = 0 \) it is described by the fields

\[
\psi(t, x) = e^{-\omega_0 t} \psi^* (|x|), \quad \varphi(t, x) = \varphi^* (|x|), \quad A(t, x) = 0,
\]

where \( \psi^* \) is the complex conjugate to \( \psi \) and the real valued functions \( \dot{\psi} \) and \( \dot{\varphi} \) satisfy the following system of equations

\[
-\Delta \dot{\varphi} = 4\pi \rho, \quad \dot{\rho} = q \left( 1 - \frac{q \dot{\varphi}}{mc^2} \right) \psi^2,
\]

\[
-\Delta \dot{\psi} + \frac{m \dot{\varphi}}{\chi^2 q} \left( 2 - \frac{q \dot{\varphi} \psi}{mc^2} \right) \dot{\psi} + G' \left( |\psi|^2 \right) \dot{\psi} = 0.
\]

where \( \Delta = \nabla^2 \) is Laplace operator. We refer to the state of the dressed charge of the form (2.0.11) as \( \omega_0 \)-static. The functions \( \dot{\psi} \) and \( \dot{\varphi} \) in the above formulas are instrumental for our constructions and we refer to them respectively as the charge form factor and form factor potential. Using Green’s function to solve equation (2.0.12) we see that the charge form factor \( \dot{\psi} \) determines the Coulomb-like potential \( \dot{\varphi} = \dot{\varphi} \dot{\psi} \) by the formula

\[
\dot{\varphi} = \dot{\varphi} \dot{\psi} = 4\pi q \left( -\Delta + \frac{4\pi q^2 \dot{\psi}}{mc^2} \right)^{-1} \dot{\psi}^2.
\]

Consequently, plugging the above expression into the equation (2.0.13) we get the following nonlinear equation

\[
-\Delta \dot{\psi} + \frac{m \dot{\varphi} \dot{\psi}}{\chi^2 q} \left( 2 - \frac{q \dot{\varphi} \psi}{mc^2} \right) \dot{\psi} + G' \left( |\psi|^2 \right) \dot{\psi} = 0.
\]

The above equation (2.0.15) signifies a complete balance of the three forces acting upon the resting charge: (i) internal elastic deformation force associated with the term \(-\Delta \dot{\psi}\); (ii) charge’s electromagnetic self-interaction force associated with the term \( \frac{m \dot{\varphi} \dot{\psi}}{\chi^2 q} \left( 2 - \frac{q \dot{\varphi} \psi}{mc^2} \right) \dot{\psi} \); (iii) internal nonlinear self-interaction of the charge associated with the term \( G' \left( |\psi|^2 \right) \dot{\psi} \). In what follows we refer to the equation (2.0.15) as charge equilibrium equation. Importantly, the static charge equilibrium equation (2.0.15) establishes an explicit relation between the form factor \( \dot{\psi} \) and the self-interaction nonlinearity \( G \). Hence being given the form factor \( \dot{\psi} \) we can find from the equilibrium equation (2.0.15) the self-interaction nonlinearity \( G \) which exactly produces this factor under an assumption that \( \dot{\psi} (r), r \geq 0 \), is a nonnegative, monotonically decaying and sufficiently smooth function. The later is a key feature of our approach: it allows to choose the form factor \( \dot{\psi} \) and then to determine matching self-interaction nonlinearity \( G \) rather than to deal with solving a nontrivial nonlinear partial differential equation.

Thus, to summarize an important point of our method: we pick the form factor \( \dot{\psi} \) and then the nonlinear self interaction function \( G \) is determined based on a physically sound ground: the charge equilibrium equation (2.0.15). Needless to say that under this approach the nonlinearity \( G \) is not expected to be a simple polynomial function but rather a function with properties that ought to be established. Then having fixed the nonlinear self-interaction \( G \)
based on the charge equilibrium equation (2.0.15) the challenge is to figure out the dynamics of the charge as it interacts with other charges or is acted upon by an external EM field and hence accelerates. The nonlinear self-interaction $G$ evidently brings into the charge model non-electromagnetic forces, the necessity of which for a consistent relativistic electromagnetic theory was argued convincingly by W. Pauli in [Pauli RT, Section 63]. It is worth to point out that the nonlinearity $G$ introduced via the charge equilibrium equation (2.0.15) differs significantly from nonlinearities considered in similar problems in literature including attempts to introduce nonlinearity in the quantum mechanics, [Bialynicki], [Holland], [Weinberg]. Important features of our nonlinearity include: (i) the boundedness of its derivative $G'(s)$ for $s \geq 0$ with consequent boundedness from below of the wave energy; (ii) non analytic behavior for small $s$ that is for small wave amplitudes.

We would like to mention that an idea to use concept of a solitary wave in nonlinear dispersive media for modelling wave-particles was quite popular. Luis de Broglie tried to use it in his pursuit of the material wave mechanics. G. Lochak wrote in his preface to the de Broglie’s monograph, [de Broglie 2, page XXXIX]: ”...The first idea concerns the solitons, which we would call *ondes à bosses* (humped waves) at the *Institut Henri Poincaré*. This idea of de Broglie’s used to be considered as obsolete and too classical, but it is now quite well known, as I mentioned above, and is likely to be developed in the future, but only provided we realize what the obstacle is and has been for twenty-five years: It resides in the lack of a general principle in the name of which we would be able to choose one nonlinear wave equation from among the infinity of possible equations. If we succeed one day in finding such an equation, a new microphysics will arise.” G. Lochak raised an interesting point of the necessity of a general principle that would allow to choose one nonlinearity among infinitely many. We agree to G. Lochak to the extend that there has to be an important physical principle that would allow to choose the nonlinearity but whether it has to be unique is different matter. In our approach such a principle is the exact balance of all forces for the resting dressed charge via the static charge equilibrium equation (2.0.15). As to a possibility of spatially localized excitations such as wave-packets to maintain their basic properties when propagate in a dispersive medium with a nonlinearity we refer to our work [Babin Figotin 1]-[Babin Figotin 3].

The gauge invariance of the Lagrangian $L_0$ allows us to introduce in a standard fashion the microcharge density $\rho$ and the microcurrent density $J$ by

$$\rho = -\frac{\chi q}{2mc} i \left( \bar{\partial}_t \psi^* \psi - \psi^* \partial_t \psi \right), \quad J = \frac{\chi q}{2m} i \left( \bar{\nabla}^\ast \psi^* \psi - \psi^* \nabla \psi \right).$$

They satisfy the conservation (continuity) equation

$$\partial_t \rho + \nabla \cdot J = 0,$$  

and, consequently, the total charge is conserved:

$$\int_{\mathbb{R}^3} \rho(t, \mathbf{x}) \, d\mathbf{x} = \text{const.}$$

For the fundamental pair $\{\psi, \varphi\}$ the corresponding microcharge density defined by (2.0.16) turns into

$$\rho = \rho(|\mathbf{x}|) = q \left( \frac{e^\varphi(|\mathbf{x}|)}{mc^2} \right) \psi^2(|\mathbf{x}|).$$
Note that equation (2.0.12) turns into the classical equation for the Coulomb's potential if \( \rho \) is replaced by \( q \delta (\mathbf{x}) \) where delta function has standard property \( \int \delta (\mathbf{x}) \, d\mathbf{x} = 1 \). Since we want \( \hat{\varphi} \) to behave as Coulomb’s electrostatic potential at large distances and \( q \) to be the charge, we introduce the following charge normalization condition imposed on the form factor \( \hat{\psi} \):

\[
\int_{\mathbb{R}^3} \left( 1 - \frac{q \hat{\varphi} (|\mathbf{x}|)}{mc^2} \right) \hat{\psi}^2 (|\mathbf{x}|) \, d\mathbf{x} = 1. \tag{2.0.20}
\]

Notice that we introduced the above terms microcharge and microcurrent densities to emphasize their relation to the internal structure of elementary charges and the difference from commonly used charge and the current densities as macroscopic quantities. It is worth noticing though that if it comes to the interaction with the electromagnetic field the "micro" charges and microcurrents densities behave exactly the same way as the macroscopic charges and densities, but microcharges are also subjects to the internal elastic and nonlinear self-interaction forces of non-electromagnetic nature.

## 2.1 Energy considerations

Let us denote by \( \mathcal{E}_0 (\hat{\psi}, A^\mu) \) the energy of the dressed charge \( \{ \hat{\psi}, A^\mu \} \) derived from the Lagrangian \( L_0 \). We found that for the fundamental pair \( \{ \hat{\psi}, \hat{\varphi} \} \) the energy \( \mathcal{E}_0 \) can be written in the following form

\[
\mathcal{E}_0 (\hat{\psi}, \hat{\varphi}) = \mathcal{E}_0 (\hat{\psi}) = mc^2 + \mathcal{E}'_0 (\hat{\psi}), \tag{2.1.1}
\]

where we use the relation \( \hat{\varphi} = \hat{\varphi}_\hat{\psi} \) from (2.0.14) to emphasize an important fact that the above energy \( \mathcal{E}_0 \) is a functional of \( \hat{\psi} \) and the model constants only. We refer to the energy \( \mathcal{E}'_0 (\hat{\psi}) \) defined in (2.1.1) as the relative energy. The significance of the representation (2.1.1) for the energy \( \mathcal{E}_0 \) is in the fact that it does not explicitly involve the nonlinear self-interaction \( G \).

Applying to the energy \( \mathcal{E}_0 \) the Einstein principle of equivalence of mass and energy, namely \( E = mc^2 \), [Pauli RT, Section 41], we define the dressed charge mass \( \tilde{m} = \tilde{m} (\hat{\psi}) \) by the equality

\[
\mathcal{E}_0 (\hat{\psi}) = \tilde{m} c^2 = \tilde{m} (\hat{\psi}) c^2. \tag{2.1.2}
\]

Combining the relation (2.1.2) with (2.1.1) we readily obtain

\[
(\tilde{m} - m) c^2 = \mathcal{E}'_0 (\hat{\psi}) = \frac{2}{3} \int_{\mathbb{R}^3} \left[ \frac{\chi^2}{2m} \nabla \hat{\psi}^* \cdot \nabla \hat{\psi} - \frac{(\nabla \hat{\varphi})^2}{8\pi} \right] \, dx. \tag{2.1.3}
\]

We also want the fundamental frequency \( \omega_0 \) to satisfy the Einstein relation \( \mathcal{E}_0 = \hbar \omega_0 \), which
would determine \( \omega_0 \) as a function of \( \dot{\psi} \), constants \( c, m, q \) and, importantly, \( \chi \), namely

\[
\mathcal{E}_0 \left( \dot{\psi} \right) = \hbar \omega_0 \left( \dot{\psi}, \chi \right), \quad \text{or}
\]

\[
\omega_0 = \omega_0 \left( \dot{\psi}, \chi \right) = \frac{1}{\hbar} \left\{ mc^2 + \frac{2}{3} \int_{\mathbb{R}^3} \left[ \frac{\chi^2}{2m} \nabla \dot{\psi}^* \cdot \nabla \dot{\psi} - \frac{(\nabla \dot{\varphi})^2}{8\pi} \right] \, dx \right\}.
\]

Then to be consistent with the earlier relation (2.0.10) we have to set the value of \( \chi \) so that

\[
\chi \omega_0 \left( \dot{\psi}, \chi \right) = mc^2,
\]

which, in view of the representation (2.1.4) for \( \omega_0 \left( \dot{\psi}, \chi \right) \), is equivalent to the requirement for \( \chi = \chi \left( \dot{\psi} \right) \) to be a positive solution to the following cubic equation

\[
\chi \left\{ c_2 \chi^2 + c_1 \right\} = \hbar, \quad \text{where}
\]

\[
c_2 = \frac{1}{3m^2c^2} \int_{\mathbb{R}^3} \nabla \dot{\psi}^* \cdot \nabla \dot{\psi} \, dx, \quad c_1 = 1 - \frac{2}{3mc^2} \int_{\mathbb{R}^3} \frac{(\nabla \dot{\varphi})^2}{8\pi} \, dx,
\]

where \( \dot{\varphi} \), in view of defining it equation (2.0.12), depends only on \( \dot{\psi} \) and constants \( c, m, q \). Note also that we can choose \( \dot{\psi} \) independently of \( \chi \), and then determine \( G \). Notice that cubic equation (2.1.6) always has a positive solution, and if, in addition to that, we know that \( c_1 \geq 0 \), then the left-hand side of the equation (2.1.6) is a monotonically increasing function for \( \chi \geq 0 \) implying that the solution is unique.

In the case of a generic form factor \( \dot{\psi} \) the relative energy \( \mathcal{E}_0' \left( \dot{\psi} \right) \) does not necessarily have to vanish and, in view of the formula (2.1.3), the mass \( \tilde{m} \) may be different from \( m \). Then, as it follows from the relation (2.1.2)-(2.1.6) \( \chi \neq \hbar \). The very same relations also readily imply that

\[
\chi = \hbar \text{ if } \tilde{m} = m.
\]

For a number of reasons, mainly for the perfect agreement between the relativistic energy-momentum and its nonrelativistic approximation constructed below, it is attractive to have \( \tilde{m} = m \) implying also, in view of (2.1.7), \( \chi = \hbar \). The question though is if that is possible. The answer is affirmative and the equality \( \tilde{m} = m \) of the two masses can be achieved as follows. We introduce for the form factor its size representation involving size parameter \( a \) and normalization constant \( \tilde{C} \):

\[
\dot{\psi} \left( s \right) = \dot{\psi}_a \left( s \right) = \frac{\tilde{C}}{a^{3/2}} \dot{\psi}_1 \left( \frac{s}{a} \right), \quad a > 0, \quad s \geq 0,
\]

where the function \( \dot{\psi}_1 \left( |x| \right) \) satisfies the normalization condition

\[
\int_{\mathbb{R}^3} \dot{\psi}_1^2 \left( |x| \right) \, dx = 1.
\]

We consider then values of \( \tilde{C} \) and \( a \) in the representation (2.1.8), (2.1.9) that satisfy two conditions: (i) the charge normalization condition (2.0.20), namely

\[
\tilde{C}^2 \int_{\mathbb{R}^3} \left( 1 - \frac{q \dot{\varphi}_a \left( |x| \right)}{mc^2} \right) \dot{\psi}_a^2 \left( |x| \right) \, dx = 1,
\]
and the energy normalization condition
\[ E'_0 (\psi_{a_0}) = 0. \] (2.1.11)

We provide arguments in Subsection 7.2 based on the smallness of the Sommerfeld fine structure constant showing that there exist such \( \hat{C}, a = a_0 \) for which both the normalization conditions (2.1.10) and (2.1.11) can hold. In view of (2.1.1) and (2.1.7) the above equality implies
\[ E_0 (\hat{c} \psi_{a_0}) = m c^2 \text{ and } \tilde{m} = m, \ \chi = \hbar. \] (2.1.12)

In other words, the requirement \( \tilde{m} = m \) fixes the charge size \( a = a_0 \) as well as the constant \( \chi = \hbar \), the magnitude of \( a_0 \) is of the same order as Bohr’s radius. One might ask if it is necessary to require the exact equality \( \tilde{m} = m \)? For a good agreement between the relativistic energy-momentum and its nonrelativistic approximation constructed below it would be sufficient for \( \tilde{m} - m \) to be small enough rather than zero. For this reason and because of the scope of this paper we decided not to impose here the exact mass equality \( \tilde{m} = m \) leaving this decision for future work. So from now on we assume that the value of the constant \( \chi \) to be set by equations (2.1.5), (2.1.6).

### 2.2 Moving free charge

Using the Lorentz invariance of the system we can obtain, as it is often done, a representation for the dressed charge moving with a constant velocity \( v \) simply by applying to the rest solution (2.0.11) the Lorentz transformation from the original "rest frame" to the frame in which the "rest frame" moves with the constant velocity \( v \) as described by the formulas (11.1.6), (11.4.12) (so \( x' \) and \( x \) correspond, respectively, to the "rest" and "moving" frames).

Namely, introducing
\[ \beta = \frac{v}{c}, \quad \beta = |\beta|, \quad \gamma = \left(1 - \left(\frac{v}{c}\right)^2\right)^{-1/2}, \] (2.2.1)
we obtain the following representation for the dressed charge moving with velocity \( v \)
\[ \psi (t, x) = e^{-i(\omega t - k \cdot x)} \psi (x'), \quad \varphi (t, x) = \gamma \beta \varphi (|x'|), \quad A (t, x) = \gamma \beta \varphi (|x'|), \] (2.2.2)
\[ x' = x + \frac{\gamma - 1}{\beta^2} (\beta \cdot x) \beta - \gamma vt, \quad \text{or} \quad x'_\parallel = \gamma (x_\parallel - vt), \quad x'_\perp = x_\perp, \] (2.2.3)
\[ \omega = \gamma \omega_0, \quad k = \gamma \beta \frac{\omega_0}{c}, \] (2.2.4)

where \( x_\parallel \) and \( x_\perp \) refer, respectively, to the components of \( x \) parallel and perpendicular to the velocity \( v \), with the fields given by
\[ E (t, x) = -\gamma \nabla \varphi (|x'|) + \frac{\gamma^2}{\gamma + 1} (\beta \cdot \nabla \varphi (|x'|)) \beta, \quad B (t, x) = \gamma \beta \times \nabla \varphi (|x'|). \] (2.2.5)

The above formulas (which provide a solution to the field equations (3.0.6), (3.0.7) indicate that the fields of the dressed charge contract by the factor \( \gamma \) as it moves with the velocity \( v \) compared to their rest state. The first oscillatory exponential factor in (2.2.1) is the de Broglie plane wave of frequency \( \omega = \omega (k) \) and a wave-vector \( k \), satisfying
\[ \omega^2 - c^2 k^2 = \omega_0^2, \quad \chi \omega_0 = mc^2. \] (2.2.6)
Based on the Lagrangian $L_0$ we found the symmetric energy-momentum tensor, which shows that the dressed charge moving with a constant velocity $v$ and described by (2.2.1)-(2.2.3) has energy $E$ and momentum $P$, which satisfy the Einstein-de Broglie relations

$$E = \hbar \omega, \quad P = \hbar k.$$  

In addition to that, the charge velocity $v$ and its de Broglie wave vector $k$ satisfy the following relation

$$v = \nabla_k \omega (k),$$

signifying that the velocity $v$ of the dressed charge is the group velocity of the linear medium with the dispersion relation (2.2.6).

The second factor in the formula (2.2.2) for $\psi$ involves the form factor $\hat{\psi}(r), r \geq 0$, which is a monotonically decreasing function of $r$ decaying at infinity. For such a form factor the form factor potential $\hat{\phi}(r)$ decays at infinity as the Coulomb’s potential as it follows from (2.0.14), i.e. $\hat{\phi}(r) \sim q r^{-1}$ for large $r$. Consequently, the dressed charge moving with constant velocity $v$ as described by equations (2.2.2)-(2.2.3) remains well localized and does not disperse in the space at all times justifying its characterization as a wave-corpuscle.

### 2.3 Nonrelativistic approximation for the charge in an external EM field

Our nonrelativistic Lagrangian for a single charge in external EM field with potentials $\varphi_{ex}, A_{ex}$ has the form

$$\hat{L}_0(\psi, \psi^*, \varphi) = \frac{\chi}{2} \left \{ \psi^* \tilde{\partial}_t \psi - \psi \tilde{\partial}_t^* \psi^* \right \} - \frac{\chi^2}{2m} \left \{ \nabla \psi \nabla^* \psi^* + G(\psi^* \psi) \right \} + \frac{|\nabla \varphi|^2}{8 \pi},$$

where

$$\tilde{\partial}_t = \partial_t + \frac{iq \varphi}{\chi}, \quad \nabla = \nabla - \frac{iq A_{ex}}{\chi c}, \quad \tilde{\partial}_t^* = \partial_t - \frac{iq \varphi}{\chi}, \quad \nabla^* = \nabla + \frac{iq A_{ex}}{\chi c}, \quad \varphi = \varphi + \varphi_{ex}.$$  

Derivation of (2.3.1) from relativistic Lagrangian (2.0.8) is discussed in Section 7. For simplicity, we consider at first the case where an external magnetic field vanishes, $A_{ex} = 0$. The field equations for this case take the form

$$\chi i \tilde{\partial}_t \psi - q (\varphi + \varphi_{ex}) \psi = -\frac{\chi^2}{2m} [\Delta \psi - G'(\psi^* \psi)],$$

$$-\Delta \varphi = 4 \pi q \psi \psi^*, \quad \text{where} \quad G'(s) = \partial_s G,$$

and we refer to the pair $\{\psi, \varphi\}$ as dressed charge. Recall that $\psi^*$ is complex conjugate to $\psi$.

Let us consider now the case of resting charge with no external EM field described by a static time independent solution to the equations (2.3.3)-(2.3.4). These equations under the assumption $E_{ex}(t) = 0$ turn into the following rest charge equations for a static state, as described by time independent real-valued radial functions $\hat{\psi} = \psi(|x|)$ and $\hat{\varphi} = \varphi(|x|)$:

$$-\Delta \hat{\varphi} = 4 \pi q \left | \hat{\psi} \right |^2,$$

$$-\Delta \hat{\psi} + \frac{2m}{\chi^2} q \hat{\varphi} \hat{\psi} + G'(\left | \hat{\psi} \right |^{2}) \hat{\psi} = 0.$$
The quantities \( \hat{\psi} \) and \( \hat{\phi} \) are fundamental for our theory and we refer to them, respectively, as *form factor* and *form factor potential*. In view of the equation (2.3.5) the charge form factor \( \hat{\psi} \) determines the form factor potential \( \hat{\phi} \) by the formula

\[
\hat{\phi} (|x|) = \hat{\phi}_\psi (|x|) = 4\pi q \int_{\mathbb{R}^3} \frac{\psi^2(|y|)}{|y - x|} \, dy,
\]

and if we plug in the above expression into equation (2.3.6), we get the following nonlinear equation

\[
-\Delta \hat{\psi} + \frac{2mq}{\chi^2} \hat{\phi}_\psi \hat{\psi} + G'(\hat{\psi})^2 \hat{\psi} = 0.
\]

The equation (2.3.8) signifies a complete balance of the three forces acting upon the resting charge: (i) internal elastic deformation force associated with the term \( -\Delta \hat{\psi} \); (ii) the charge’s electromagnetic self-interaction force associated with the term \( \frac{2mq}{\chi^2} \hat{\phi}_\psi \hat{\psi} \); (iii) internal nonlinear self-interaction of the charge associated with the term \( G'(\hat{\psi})^2 \hat{\psi} \). We refer to the equation (2.3.8), which establishes an explicit relation between the form factor \( \hat{\psi} \) and the self-interaction nonlinearity \( G \), as the *charge equilibrium equation*. Hence, being given the form factor \( \hat{\psi} \) we can find from the equilibrium equation (2.3.8) the self-interaction nonlinearity \( G \) which exactly produces this factor under the assumption that \( \hat{\psi}(r), r \geq 0 \) is a nonnegative, monotonically decaying and sufficiently smooth function. Thus, we pick the form factor \( \hat{\psi} \), and then the nonlinear self-interaction function \( G \) is determined based on the charge equilibrium equation (2.3.8). One of the advantages of determining \( G \) in terms of \( \hat{\psi} \) is that we more often use properties of \( \hat{\psi} \) in our analysis rather than properties of \( G \).

Note that after the nonlinearity \( G \) is determined, it is fixed forever, and while solutions of equations (2.3.3)-(2.3.4) may evolve in time, they do not need to coincide with \( \{\hat{\psi}, \hat{\phi}\} \).

Details and examples of the construction of the nonlinear self-interaction function \( G \) based on the form factor are provided in the following sections.

The 4-microcurrent density \( J^\mu \) related to the Lagrangian \( \hat{L}_0 \) is

\[
J^\mu = (\rho, J), \quad \rho = q\psi^\ast \psi, \quad J = \frac{iq}{2m} [\psi \nabla^\ast \psi^\ast - \psi^\ast \nabla \psi] = \frac{\chi q}{m} \text{Im} \frac{\nabla \psi}{\psi} |\psi|^2,
\]

and it satisfies the conservation/continuity equations

\[
\partial_\nu J^\nu = 0, \quad \partial_t \rho + \nabla \cdot J = 0.
\]

For the fundamental pair \( \{\hat{\psi}, \hat{\phi}\} \) the corresponding microcharge density defined by (2.3.9) turns into

\[
\rho = \rho(|x|) = q \left| \psi \right|^2 (|x|).
\]

The *charge normalization condition* (2.0.20) which, in particular, ensures that \( \hat{\phi}(|x|) \) is close to the Coulomb’s potential with the charge \( q \) for large \( |x| \) now takes the simpler form

\[
\int_{\mathbb{R}^3} \psi^2(|x|) \, dx = 1.
\]

Interestingly the momentum and the current densities of the dressed charge derived from \( \hat{L}_0 \) are identical up to a factor \( \frac{m}{q} \), namely

\[
P = \frac{m}{q} J = \frac{iq}{2} [\psi \nabla^\ast \psi^\ast - \psi^\ast \nabla \psi] = \chi \text{Im} \frac{\nabla \psi}{\psi} |\psi|^2.
\]
It turns out that the field equations \((2.3.3)-(2.3.4)\) have a closed form solution in terms of (i) the static dressed charge state \(\{\hat{\psi}, \varphi\}\) satisfying \((2.3.5)-(2.3.6)\) and (ii) the basic quantities related to the point charge of mass \(m\) moving in external homogeneous electric field \(E_{ex}(t)\) with the electric potential \(\varphi_{ex}(t, x) = \varphi_{ex}^0(t) - E_{ex}(t) \cdot x\). Indeed, let \(r, v, p, L_p(v, r), s_p\) and \(H_p(p, r, t)\) be, respectively, the point charge position, velocity, momentum, Lagrangian, action and Hamiltonian satisfying the following familiar relations:

\[
L_p(v, r, t) = \frac{mv^2}{2} - q\varphi_{ex}(t, x),
\]
\[
\frac{ds_p}{dt} = L_p(v, r, t) = \frac{d}{dt}(p \cdot r) - \frac{p^2}{2m} - \varphi_{ex}^0(t),
\]
\[
H_p(p, r, t) = \frac{p^2}{2m} + q\varphi_{ex}(t, r), \quad p = mv,
\]

with point charge equations of motion

\[
v = \frac{dr}{dt}, \quad \frac{dp}{dt} = qE_{ex}(t) \quad \text{or equivalently} \quad m\frac{d^2r}{dt^2} = qE_{ex}.
\]

One can recognize in the expression \(qE_{ex}(t)\) in \((2.3.15)\) the Lorentz force due the external electric field \(E_{ex}(t)\). We refer to \(L_p(v, r)\) and the equations \((2.3.15)\) respectively as the complementary point charge Lagrangian and the equations of motion.

Then the field equations \((2.3.3)-(2.3.4)\) have the following closed form solution

\[
\psi = \psi(t, x) = e^{iS/\chi \psi'(|x - r|)}, \quad S = p \cdot (x - r) + s_p,
\]
\[
\varphi(t, x) = \varphi_0(|x - r|),
\]

where in view of relations \((2.3.14)\) \(\psi\) can be also represented as

\[
\psi = e^{iS/\chi \psi'(|x - r|)}, \quad S = p \cdot x - \int_0^t \frac{P^2}{2m} dt' - q \int_0^t \varphi_{ex}^0(t') dt'.
\]

A similar exact solution is given for the class of EM fields with non-zero, spatially constant magnetic field \(B_{ex}(t)\). Of course, in latter case the Lorentz force involves \(B_{ex}\) as in \((1.0.1)\) (see Section 5.5 for details). Looking at the exact solution \((2.3.16), (2.3.17)\) to the field equations that describes the accelerating charge we would like to acknowledge the truly remarkable simplicity and transparency of the relations between the two concepts of the charge: charge as a field \(\{\psi, \varphi\}\) in \((2.3.16), (2.3.17)\) and charge as a point described by equations \((2.3.14)\) and \((2.3.15)\). Indeed, the wave amplitude \(\hat{\psi}(|x - r(t)|)\) in \((2.3.16)\) is a soliton-like field moving exactly as the point charge described by its position \(r(t)\). The exponential factor \(e^{iS/\chi \psi'}\) is just a plane wave with the phase \(S\) that depends only on the point charge position \(r\) and momentum \(p\) and a time dependent gauge term, and it does not depend on the nonlinear self-interaction. The phase \(S\) has a term in which we readily recognize the de Broglie wave-vector \(k(t)\) described exactly in terms of the point charge quantities, namely

\[
\chi k(t) = p(t) = mv(t).
\]

Notice that the dispersion relation \(\omega = \omega(k)\) of the linear kinetic part of the field equations \((2.3.3)\) for \(\psi\) is

\[
\omega(k) = \frac{\chi k^2}{2m}, \quad \text{implying that the group velocity} \quad \nabla_k \omega (k) = \frac{\chi k}{m}.
\]
Combining the expression (2.3.19) for the group velocity $\nabla_k \omega (k)$ with the expression (2.3.18) for wave vector $k (t)$ we establish another exact relation, namely

$$v (t) = \nabla_k \omega (k(t)), \quad (2.3.20)$$

signifying the equality between the point charge velocity $v (t)$ and the group velocity $\nabla_k \omega (k(t))$ at the de Broglie wave vector $k (t)$. Using the relations (2.3.9) and (2.3.13) we readily obtain the following representations for the micro-charge, the micro-current and momentum densities

$$\rho (t, x) = q \psi^2 (|x - r (t)|), \quad J (t, x) = q v (t) \psi^2 (|x - r (t)|), \quad (2.3.21)$$

$$P (t, x) = \frac{m}{q} J (t, x) = p (t) \psi^2 (|x - r (t)|). \quad (2.3.22)$$

We also found that the total dressed charge field momentum $P (t)$ and the total current $J (t)$ for the solution (2.3.16) are expressed exactly in terms of point charge quantities, namely

$$P (t) = \frac{m}{q} J (t) = \int_{\mathbb{R}^3} \frac{\chi q}{m} \text{Im} \frac{\nabla \psi}{\psi} |\psi|^2 \, dx = p (t) = mv (t). \quad (2.3.23)$$

To summarize the above analysis we may state that even when the charge accelerates it perfectly combines the properties of a wave and a corpuscle, justifying the name wave-corpuscle mechanics. Its wave nature shows, in particular, in the de Broglie exponential factor and the equality (2.3.20), indicating the wave origin the charge motion. The corpuscle properties are manifested in the factor $\psi^2 (|x - r (t)|)$ and soliton like propagation with $r (t)$ satisfying the classical point charge evolution equation (2.3.15). Importantly, the introduced nonlinearities are stealthy in the sense that they don’t show in the dynamics and kinematics of what appears to be soliton-like waves propagating like classical test point charges.

### 2.4 Many interacting charges

A qualitatively new physical phenomenon in the theory of two or more charges compared with the theory of a single charge is obviously the interaction between them. In our approach any individual "bare" charge interacts directly only with the EM field and consequently different charges interact with each other only indirectly through the EM field. We develop the theory for many interacting charges for both the relativistic and nonrelativistic cases, and in this section we provide key elements of the nonrelativistic theory. The primary focus of our studies on many charges is the correspondence between our wave theory and the point charge mechanics in the regime of remote interaction when the charges are separated by large distances compared to their sizes.

Studies of charge interactions at short distances are not within the scope of this paper, though our approach allows to study short distance interactions and we provide as an example the hydrogen atom model in Section 8. The main purpose of that exercise is to show a similarity between our and Schrödinger’s hydrogen atom models and to contrast it to any kind of Kepler’s model. Another point we can make based on our hydrogen atom model is that our theory does provide a basis for a regime of close interaction between two charges which differs significantly from the regime of remote interaction, which is the primary focus of this paper.
Let us consider a system of $N$ charges described by complex-valued fields $\psi^\ell$, $\ell = 1, \ldots, N$. The system’s nonrelativistic Lagrangian $\hat{\mathcal{L}}$ is constructed based on individual charge Lagrangians $\hat{\mathcal{L}}^\ell$ of the form (2.3.1)-(2.3.2) and an assumption that every charge interacts directly only with the EM field, including the external one with potentials $\varphi_{\text{ex}}(t, \mathbf{x})$, $\mathbf{A}_{\text{ex}}(t, \mathbf{x})$, namely

$$\hat{\mathcal{L}} = \sum_\ell \mathcal{L}^\ell + \frac{|\nabla \varphi|^2}{8\pi}, \quad \text{where}$$

$$\mathcal{L}^\ell = \frac{\chi}{2} \left[ \psi^{*\ell} \bar{\partial}_t \psi^\ell - \psi^\ell \bar{\partial}_t \psi^{*\ell} \right] - \frac{\chi^2}{2m} \left\{ \nabla \psi^\ell \cdot \nabla \psi^{*\ell} + G^\ell (\psi^{*\ell} \psi^\ell) \right\},$$

$$\bar{\partial}_t^\ell = \partial_t + \frac{iq^\ell (\varphi + \varphi_{\text{ex}})}{\chi}, \quad \bar{\nabla} = \nabla - \frac{iq\mathbf{A}_{\text{ex}}}{\chi c}.$$

The nonlinear self-interaction terms $G^\ell$ in (2.4.1) are determined through the charge equilibrium equation (2.3.8), and they can be the same for different $\ell$, or there may be several types of charges, for instance, protons and electrons. The field equations for this Lagrangian are

$$i\chi \bar{\partial}_t \psi^\ell = -\frac{\chi^2 \nabla^2 \psi^\ell}{2m^\ell} - \frac{\chi q^\ell \mathbf{A}_{\text{ex}} \cdot \nabla \psi^\ell}{m^\ell c i} +$$

$$+ q^\ell (\varphi + \varphi_{\text{ex}}) \psi^\ell + \frac{q^2 A_{\text{ex}}^2 \psi^\ell}{2m^\ell c^2} + \left[ G^\ell \right]' \left( |\psi^\ell|^2 \right) \psi^\ell,$$

$$- \nabla^2 \varphi = 4\pi \sum_{\ell=1}^N q^\ell |\psi^\ell|^2, \quad \ell = 1, \ldots, N.$$

and $\psi^{*\ell}$ is complex conjugate of $\psi^\ell$. Obviously, the equations with different $\ell$ are coupled only through the potential $\varphi$ which is responsible for the charge interaction. The Lagrangian $\hat{\mathcal{L}}$ is gauge invariant with respect to the first and the second gauge transformations (11.7.6), (11.7.7), and consequently every $\ell$-th charge has a conserved current

$$J^\mu = (c \rho^\ell, \mathbf{J}^\ell), \quad \rho^\ell = q |\psi^\ell|^2, \quad \mathbf{J}^\ell = \left( \frac{\chi q^\ell}{m^\ell} \Im \nabla \psi^\ell \overline{\psi^{*\ell}} - \frac{q^2 A_{\text{ex}}^2}{m^\ell c} \right) |\psi^\ell|^2,$$

satisfying the conservation/continuity equations

$$\partial_t \rho^\ell + \nabla \cdot \mathbf{J}^\ell = 0 \quad \text{or} \quad m^\ell \partial_t |\psi^\ell|^2 + \nabla \cdot \left( \chi \Im \frac{\nabla \psi^\ell}{\psi^\ell} |\psi^\ell|^2 - \frac{q^\ell}{c} A_{\text{ex}} |\psi^\ell|^2 \right) = 0.$$  

The differential form (2.4.4) of the charge conservation implies the conservation of every total $\ell$-th charge, and we require every total $\ell$-th conserved charge to be exactly $q^\ell$, rather than just any constant, implying the following charge normalization conditions

$$\int_{\mathbb{R}^3} |\psi^\ell|^2 \, d\mathbf{x} = 1, \quad t \geq 0, \quad \ell = 1, \ldots, N.$$  

We attribute to every $\ell$-th charge its potential $\varphi^\ell$ by the formula

$$\varphi^\ell (t, \mathbf{x}) = \frac{q^\ell}{c} \int_{\mathbb{R}^3} \frac{|\psi^\ell|^2 (t, \mathbf{y})}{|\mathbf{y} - \mathbf{x}|} \, d\mathbf{y}.$$  

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Hence we can write the equation for $\varphi$ in (2.4.2) in the form

$$\nabla^2 \varphi^\ell = -4\pi q^\ell |\psi^\ell|^2, \quad \varphi = \sum_{\ell=1}^{N} \varphi^\ell.$$  

(2.4.7)

An analysis of the system of the charges energy-momentum and its partition between individual charges shows another important property of the Lagrangian: the charge gauge invariant momentum density $P^\ell$ equals exactly the microcurrent density $J^\ell$ multiplied by the constant $m^\ell/q^\ell$, namely:

$$P^\ell = \frac{m^\ell}{q^\ell} J^\ell = \frac{i\chi}{2} \left[ \psi^\ell \nabla^\ast \psi^\ell - \psi^\ell \nabla^\ast \psi^\ell \right] = \left( \chi \operatorname{Im} \frac{\nabla \psi^\ell}{\psi^\ell} - \frac{q^\ell A_{\text{ex}}}{c} \right) |\psi^\ell|^2, \quad (2.4.8)$$

that can be viewed as the momentum density kinematic representation:

$$P^\ell(t, x) = m v^\ell(t, x), \quad \text{where} \quad v^\ell(t, x) = J^\ell(t, x)/q^\ell \quad \text{is the velocity density.} \quad (2.4.9)$$

To quantify the conditions of the remote interaction we make use of the explicit dependence on the size parameter $a$ of the nonlinearity $G^\ell = G_a^\ell$ as in (4.1.13) and take the size parameter as a spatial scale characterizing sizes of the fields $\psi_a^\ell$ and $\varphi_a^\ell$. The charges separation is measured roughly by a minimal distance $R_{\text{min}}$ between any two charges. Another relevant spatial scale $R_{\text{ex}}$ measures a typical scale of spatial inhomogeneity of external fields near charges. Consequently, conditions of remote interaction are measured by the dimensionless ratio $a/R$ where the characteristic spatial scale $R = \min(R_{\text{min}}, R_{\text{ex}})$ with the condition $a/R \ll 1$ to define the regime of remote interaction.

Next we would like to take a look on how the point charge mechanics is manifested in our wave mechanics governed by the Lagrangian $\hat{\mathcal{L}}$. There are two distinct ways to correspond our field theory to the classical point charge mechanics in the case when all charges are well separated satisfying the condition $a/R \ll 1$. The first way is via averaged quantities in the spirit of the well known in quantum mechanics Ehrenfest Theorem, [Schiff, Sections 7, 23], and the second one via a construction of approximate solutions to the field equations (2.4.2) when every charge is represented as wave-corpuscle similar to one from (2.3.16).

We construct the correspondence to the point charges mechanics via averaged quantities by defining first the $\ell$-th charge location $r_a^\ell(t)$ and its potential $\varphi_a^\ell$ by the relations

$$r^\ell(t) = r_a^\ell(t) = \int_{\mathbb{R}^3} x |\psi_a^\ell(t, x)|^2 \, dx, \quad \varphi_a^\ell(t, x) = q^\ell \int_{\mathbb{R}^3} \frac{|\psi_a^\ell|^2(t, y)}{|y - x|} \, dy. \quad (2.4.10)$$

Then, for the start, we consider a simpler case when the external magnetic field vanishes, i.e. $A_{\text{ex}} = 0$, and introduce a potential $\varphi_{\text{ex}, a}^\ell$ describing the interaction of $\ell$-th charge with all other charges:

$$\varphi_{\text{ex}, a}^\ell = \varphi + \varphi_{\text{ex}} - \varphi_a^\ell = \varphi_{\text{ex}} + \sum_{\ell' \neq \ell} \varphi_a^\ell. \quad (2.4.11)$$

In this case, based on the momentum balance equation for every $\ell$-th charge combined with the momentum density kinematic representation (2.4.7), (2.4.8) and the representation (2.4.11), we obtain the following exact equations of motion

$$m^\ell \frac{d^2 r_a^\ell}{dt^2} = - \int_{\mathbb{R}^3} q^\ell |\psi_a^\ell|^2 \nabla \varphi_{\text{ex}, a}^\ell \, dx, \quad \ell = 1, ..., N, \quad (2.4.12)$$

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where we recognize in the integrand the Lorentz force density exerted on the \( \ell \)-th charge by other charges and the external field. Suppose now that fields \( \psi_a^\ell (t, x) \) are localized about point \( r^\ell_0 (t) \) with the localization radius of the order \( a \) and observe that the normalization (2.4.5) combined with the localization implies that

\[
|\psi_a^\ell (t, x)|^2 \to \delta (x - r^\ell_0 (t)), \quad a \to 0, \quad \text{implying } r^\ell_0 (t) = \lim_{a \to 0} r^\ell_a (t). \quad (2.4.13)
\]

The relations (2.4.10) and (2.4.13) imply consequently

\[
\varphi_a^\ell (t, x) \to \varphi^\ell_0 = \frac{q^\ell}{|x - r^\ell_0|} \quad \text{as } a \to 0, \quad \text{and}
\]

\[
\varphi_{ex,0}^\ell (t, x) = \varphi_{ex} (t, x) + \sum_{\ell' \neq \ell} \frac{q^{\ell'}}{|x - r^{\ell'}_0 (t)|}.
\]

Then combining relations (2.4.10), (2.4.13) we can pass in (2.4.12) to the limit \( a \to 0 \), obtaining Newton’s equations of motion for the system of \( N \) point charges

\[
m^\ell \frac{d^2 r^\ell_0}{dt^2} (t) = -q^\ell \nabla \varphi_{ex,a}^\ell (r^\ell_0 (t), t), \quad \ell = 1, ..., N,
\]

where in the right-hand side of equation (2.4.15) we see the Lorentz force acting on the \( \ell \)-th point charge interacting with other charges via the point charges Coulomb’s potentials (2.4.14).

In the case when the external magnetic field potential \( A_{ex} (t, x) \) does not vanish, based on the exact equations of motion similar to (2.4.12) we establish that in the limit as \( a \to 0 \) the following point charges equation of motion hold:

\[
m^\ell \frac{d^2 r^\ell_0}{dt^2} = f^\ell, \quad f^\ell = q^\ell E^\ell + \frac{q^\ell}{c} \frac{d r^\ell_0}{dt} \times B^\ell, \quad \ell = 1, ..., N,
\]

with the Lorentz force \( f^\ell \) involving electric and magnetic fields defined by the classical formulas:

\[
E^\ell = - \left[ \nabla \varphi_{ex,0}^\ell (r^\ell_0) + \frac{1}{c} \partial_t A_{ex} (r^\ell_0) \right], \quad B^\ell = \nabla \times A_{ex} (t, r^\ell_0).
\]

Obviously, the above force \( f^\ell \) coincides with the Lorentz force acting on a point charge as in (1.0.1). To summarize our first way of correspondence, we observe that the exact equations of motion (2.4.12) form a basis for relating the field and point mechanics under an assumption that charge fields remain localized during time interval of interest. Notice the equations of motion (2.4.12) do not involve the nonlinear interaction terms \( G^{\ell} \) in an explicit way justifying their characterization as "stealthy" in the regime of remote interactions. As to the assumption that the charge fields remain localized, it has to be verified based on the field equations (2.4.2) where the nonlinear interaction terms \( G^{\ell} \) provide for cohesive forces for individual charges. The fact that they can do just that is demonstrated for a single charge represented as wave-corpuscle (2.3.16)-(2.3.17) as it accelerates in an external EM field.

The second way of correspondence between the charges as fields and points is based on established by us fact that all charge fields \( \psi_a^\ell \) can be represented as wave-corpuscles (2.3.16)-(2.3.17) which, though do not satisfy the field equations (2.4.2) exactly, but they satisfy them with small discrepancies in the regime of remote interaction when \( a/R \ll 1 \). More detailed
presentation of that idea is as follows. Consider for simplicity a simpler case when \( A_{\text{ex}} = 0 \) and introduce the following wave-corpuscle representation similar to (2.3.16)-(2.3.17):

\[
\psi^\ell_a(t, x) = e^{iS^\ell_a(x - r_0^\ell)} \varphi^\ell_a(t, x) = \varphi^\ell(x - r_0^\ell),
\]

where

\[
S^\ell_a(t, x) = \frac{p_0^\ell}{2m^\ell} \int_0^t dt' - q^\ell_0 \int_0^t \varphi^0_{\text{ex}}(t', r_0^\ell) \, dt', \quad p_0^\ell = m^\ell \frac{dr_0^\ell}{dt},
\]

and the position functions \( r_0^\ell(t) \) satisfy the Newton’s equations of motion (2.4.16). It turns out that wave-corpuscles \( \{\psi^\ell_a, \varphi^\ell_a\} \) defined by (2.4.18), (2.4.19) and the complementary point charge Newton’s equations of motion (2.4.16) solve the field equations (2.4.2) with a small discrepancy which approaches zero as \( a/R \) approaches zero. The point charge mechanics features are transparently integrated into the fields \( \{\psi^\ell_a, \varphi^\ell_a\} \) in (2.4.18), (2.4.19) via the de Broglie factor phases \( S^\ell_a \) and spatial shifts \( r_0^\ell \). Comparing with the motion of a single charge in an external field we observe that now the acceleration of the wave-corpuscle center \( r_0^\ell(t) \) is determined not only by the Lorentz force due to the external field but also by electric interactions with the remaining charges \( \varphi^\ell', \ell' \neq \ell \) according to the Coulomb’s law (2.4.14).

We would like to point out that when analyzing the system of charges in the regime of remote interaction we do not use any specific form of the nonlinearities. Note also that solutions to the field equations (2.4.2) depend on the size parameter \( a \), which is proportional to the radius of the wave-corpuscle, through the nonlinearity \( G^\ell_a \), but the integral equations (2.4.12) do not involve explicit dependence on \( a \). Equation (2.4.18) which describes the structure of the wave-corpuscle involves \( a \) only through radial shape factors \( \varphi^\ell_a = \varphi^\ell_{r_0^a} \) and through the electric potential \( \varphi^\ell = \varphi^\ell_{\rho_0^a} \). The dependence of \( \varphi^\ell_a \) on \( a \) is explicitly singular at zero, as that is expected since in the singular limit \( a \to 0 \) the wave-corpuscle should turn into the point charge with the square of amplitude described by a delta function as in (1.0.3). Nevertheless, for arbitrary small \( a > 0 \) the wave-corpuscle structure of every charge is preserved including its principal wave-vector. The dependence of \( \varphi^\ell_a \) on small \( a \) can be described as a regular convergence to the classical singular Coulomb’s potential, see (4.6.9) for details. That allows to apply representation (2.4.18) to arbitrary small charges with radius proportional to \( a \) without compromising the accuracy of the description and, in fact, increasing the accuracy as \( a \to 0 \).

2.5 Comparative summary with the Schrödinger wave mechanics

The nonrelativistic version of our wave mechanics has many features in common with the Schrödinger wave mechanics. In particular, the charges wave functions are complex valued, they satisfy equations resembling the Schrödinger equation, the charge normalization condition is the same as in the Schrödinger wave mechanics. Our theory provides for a hydrogen atom model which has a lot in common with that of the Schrödinger one, but its detailed study is outside of the scope of this article. There are features of our wave theory though that distinguish it significantly from the Schrödinger wave mechanics, and they are listed below.

- Charges are always coupled with and inseparable from the EM field.
- Every charge has a nonlinear self-interaction term in its Lagrangian providing for a cohesive force holding it together as it moves freely or accelerates.
A single charge either free or in external EM field is described by a soliton-like wave function parametrized by the position and the momentum related to the corresponding point mechanics. It propagates in the space without dispersion even when it accelerates, and this addresses one of the above mentioned "grave difficulties" with the Schrödinger's interpretation of the wave function expressed by M. Born.

When dressed charges are separated by distances considerably larger than their sizes their wave functions and the corresponding EM fields maintain soliton-like representation.

The correspondence between the wave mechanics and a point mechanics comes through the closed form soliton-like representation of wave functions in which point mechanics positions and momenta enter as parameters. In particular, the wave function representation includes the de Broglie wave vector as an exact parameter, it equals (up to the Planck constant) the point mechanics momentum. In addition to that, the corresponding group velocity matches exactly the velocity of a soliton-like solution and the point mechanics velocity.

In the case of many interacting charges every charge is described by its own wave function over the same three dimensional space in contrast to the Schrödinger wave mechanics for many charges requiring multidimensional configuration space.

Our theory has a relativistic version based on a local, gauge and Lorentz invariant Lagrangian with most of the above listed features.

3 Single free relativistic charge

A single free charge is described by a complex scalar field $\psi = \psi(t, x)$ and it is coupled to the EM field described by its 4-potential $A^\mu = (\varphi, A)$. To emphasize the fact that our charge is always coupled with the EM field we name the pair $\{\psi, A^\mu\}$ dressed charge. So whenever we use the term dressed charge we mean the charge and the EM field as an inseparable entity. The dressed charge is called free if there are no external forces acting upon it. The free charge Lagrangian is defined by the following formula

$$L_0 (\psi, A^\mu) = \frac{\chi^2}{2m} \{ \psi^* \psi^\mu - \kappa_0^2 \psi^* \psi - G (\psi^* \psi) \} - \frac{F_{\mu\nu} F^{\mu\nu}}{16\pi}, \quad (3.0.1)$$

where

$$\kappa_0 = \frac{\omega_0}{c}, \quad \omega_0 = \frac{mc^2}{\chi}, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (3.0.2)$$

$$\psi^\mu = \bar{\partial}^\mu \psi, \quad \psi^{*\mu} = \bar{\partial}^{*\mu} \psi^*, \quad \bar{\partial}^\mu = \partial^\mu - \frac{i}{\chi c} A^\mu, \quad \bar{\partial}^{*\mu} = \partial^{*\mu} - \frac{i}{\chi c} A^{\mu} \quad (3.0.3)$$

In the above relations $m > 0$ is the mass of the charge, $q$ is the charge value, and $\chi > 0$ is a parameter similar to the Planck constant $\hbar$ the value of which will be set later to satisfy the Einstein relation $E = \hbar \omega_0$. The term $G (\psi^* \psi)$ corresponds to the nonlinear self-interaction and is to be determined later, $\psi^*$ is complex conjugate to $\psi$. The above Lagrangian involves the so-called covariant differentiation operators $\bar{\partial}^\mu$ and $\bar{\partial}^{*\mu}$ with abbreviated notations $\psi^\mu$.
and $\psi^{\mu*}$ for the corresponding covariant derivatives. In what follows we use also the following abbreviations

$$\partial^\mu \psi = \psi^\mu, \ \partial^\mu \psi^* = \psi^{\mu*}. \tag{3.0.4}$$

We remind the reader also that

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{1}{c} \partial_t, \nabla \right), \ \partial^\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{1}{c} \partial_t, -\nabla \right), \tag{3.0.5}$$

$$A^\mu = (\varphi, A), \ A_\mu = (\varphi, -A), \ E = -\nabla \varphi - \frac{1}{c} \partial_t A, \ B = \nabla \times A.$$ 

Evidently the Lagrangian $L_0$ defined by the formulas (3.0.1)-(3.0.3) is obtained from the Klein-Gordon Lagrangian, [Griffiths, Section 7.1, 11.2], [Barut, III.3], by adding to it the nonlinear term $G(\psi^*\psi)$. The Lagrangian expression indicates that the charge is coupled to the EM field through the covariant derivatives, and such a coupling is well known and called minimal. The Klein-Gordon Lagrangian is a commonly used model for a relativistic spinless charge, and the introduced nonlinearity $G(\psi^*\psi)$ can provide for a binding self-force. Nonlinear alterations of the Klein-Gordon Lagrangian were considered in the literature, see, for instance, [Griffiths, Section 11.7, 11.8] and [Benci Fortunato], for rigorous mathematical studies. Our way to choose of the nonlinearity $G(\psi^*\psi)$ differs from those.

Observe that the Lagrangian $L_0$ defined by (3.0.1)-(3.0.3) is manifestly Lorentz and gauge invariant, and it is a special case of a general one charge Lagrangian studied in Section 11.6. This allows us to apply to the Lagrangian $L_0$ formulas from there to get the field equations, the 4-microcurrent and the energy-momenta tensors. Consequently, the Euler-Lagrange field equations (11.6.3) take here the form

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \tag{3.0.6}$$

$$\left[ \partial_\mu \tilde{\partial}^\mu + \kappa_0^2 + G'(\psi^*\psi) \right] \psi = 0. \tag{3.0.7}$$

The formula (11.6.5) for the 4-microcurrent density $J^\mu$ turns into

$$J^\nu = -\frac{\chi q}{2m} i \left( \psi \tilde{\partial}^\mu \psi^* - \psi^* \tilde{\partial}^\mu \psi \right) = -\left( \frac{\chi q}{m} \Im \frac{\partial^\nu \psi}{\psi} + \frac{q^2}{mc^2} A^\nu \right) |\psi|^2, \tag{3.0.8}$$

or, in the time-space variables,

$$\rho = -\frac{\chi q}{2mc} i \left( \psi \tilde{\partial}_t \psi^* - \psi^* \tilde{\partial}_t \psi \right) = -\left( \frac{\chi q}{mc^2} \Im \frac{\partial_t \psi}{\psi} + \frac{q^2}{mc^2} \varphi \right) |\psi|^2, \tag{3.0.9}$$

$$J = \frac{\chi q}{2m} i \left( \psi \tilde{\nabla}^\nu \psi^* - \psi^* \tilde{\nabla}^\nu \psi \right) = \left( \frac{\chi q}{m} \Im \frac{\nabla \psi}{\psi} - \frac{q^2 A}{mc^2} \right) |\psi|^2. \tag{3.0.10}$$

The above formulas for the 4-microcurrent density $J^\mu = (\rho, J)$ are well known in the literature, see for instance, [Wentzel (11.3)], [Morse Feshbach II, Section 3.3, (3.3.27), (3.3.34), (3.3.35)]. It satisfies the conservation/continuity equations

$$\partial_\nu J^\nu = 0, \ \partial_\nu \rho + \nabla \cdot J = 0, \ J^\nu = (\rho c, J). \tag{3.0.10}$$

Consequently, the total charge $\int \rho(x) \, dx$ of the elementary charge remains constant in the course of evolution, and we impose a charge normalization condition which extends (2.0.20), namely

$$\int_{\mathbb{R}^3} \rho(t,x) \, dx = -\int_{\mathbb{R}^3} \left( \frac{\chi q}{mc^2} \Im \frac{\partial_t \psi}{\psi} + \frac{q \varphi}{mc^2} \right) |\psi|^2 \, dx = 1. \tag{3.0.11}$$

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We would like to stress that the equation (3.0.11) is perfectly consistent with the field equations and the conservation laws (3.0.10), and it constitutes an independent and physically significant constraint for the total charge to be exactly $q$ as in the Coulomb’s potential, rather than an arbitrary constant.

Applying the general formulas (11.6.8)-(11.6.9) to the Lagrangian $L_0$ defined by (3.0.1)-(3.0.3) we obtain the following representations for the symmetric and gauge invariant energy-momenta tensors $T^{\mu\nu}$ and $\Theta^{\mu\nu}$ for, respectively, the charge and the EM field:

$$T^{\mu\nu} = \frac{\chi^2}{2m} \left\{ [\psi^{*\mu}\psi^{,\mu} + \psi^{*\mu}\psi^{\mu}] - [\psi_{*\mu}\psi^{*\mu} - \kappa_0^2\psi^*\psi - G(\psi^*\psi)] g^{\mu\nu} \right\}, \quad (3.0.12)$$

$$\Theta^{\mu\nu} = \frac{1}{4\pi} \left( g^{\mu\gamma}F_{\gamma\xi}F^{\xi\nu} + \frac{1}{4} g^{\mu\nu}F_{\gamma\xi}F^{\gamma\xi} \right). \quad (3.0.13)$$

Energy conservation equations which we derive later in (11.5.29)-(11.5.30) turn here into

$$\partial_\mu T^{\mu\nu} = f^\nu, \quad \partial_\mu \Theta^{\mu\nu} = -f^\nu, \quad (3.0.14)$$

where

$$f^\nu = \frac{1}{c} J_\mu F^{\mu\nu} = \left( \frac{1}{c} J \cdot E, \frac{1}{c} E + \frac{1}{c} J \times B \right) \quad (3.0.15)$$

is the Lorentz force density.

### 3.1 Charge at rest

We say the dressed charge to be at rest at the origin $x = 0$ if it is a radial solution to the field equations (3.0.6)-(3.0.7) of the following special form

$$\psi(t, x) = e^{-\omega_0 t} \hat{\psi}(|x|), \quad \varphi(t, x) = \hat{\varphi}(|x|), \quad A(t, x) = 0, \quad \omega_0 = \frac{mc^2}{\chi}, \quad (3.1.1)$$

and we refer to such a solution as $\omega_0$-static. Observe that as it follows from (3.0.8), (3.0.9), the micro-density $\rho$ and micro-current $J$ for the 4-microcurrent $J^\nu = (\rho c, J)$ for the $\omega_0$-static solution (3.1.1) are

$$\rho = q \left( 1 - \frac{q\hat{\varphi}}{mc^2} \right) \hat{\psi}^2, \quad J = 0. \quad (3.1.2)$$

The charge normalization condition (3.0.11) then turns into (2.0.20).

For the charge at rest as described by relations (3.1.1) the field equations (3.0.6)-(3.0.7) turn into the following system of two equations for the real-valued functions $\hat{\psi}$ and $\hat{\varphi}$ which we call rest charge equations:

$$-\Delta \hat{\varphi} = 4\pi \hat{\rho}, \quad \hat{\rho} = q \left( 1 - \frac{q\hat{\varphi}}{mc^2} \right) \hat{\psi}^2, \quad (3.1.3)$$

$$-\Delta \hat{\psi} + \frac{mq\hat{\varphi}}{\chi^2} \left( 2 - \frac{q\hat{\varphi}}{mc^2} \right) \hat{\psi} + G' \left( \frac{\hat{\psi}^2}{2} \right) \hat{\psi} = 0. \quad (3.1.4)$$

The radial functions $\hat{\psi}$ and $\hat{\varphi}$ play instrumental role in our constructions, and we name them respectively charge form factor and form factor potential. As it follows from the equation (3.1.3), the charge form factor $\hat{\psi} = \hat{\varphi}_\psi$ determines the form factor potential $\hat{\varphi}$.
by the formula (2.0.14). Consequently, plugging in the above expression into the equation (2.0.13) we get the nonlinear equation (2.0.15) as follows:

\[-\Delta \dot{\psi} + \frac{m\phi\dot{\psi}}{\chi^2} q \left(2 - \frac{q\phi\dot{\psi}}{mc^2}\right) \dot{\psi} + G' \left(\left|\dot{\psi}\right|^2\right) \dot{\psi} = 0.\]

(3.1.5)

As it is shown in the next section, the equation (3.1.5) signifies a complete balance (equilibrium) of the three forces acting upon the resting charge: (i) internal elastic deformation force associated with the term $-\Delta \dot{\psi}$; (ii) charge’s electromagnetic self-interaction force associated with the term $\frac{m\phi\dot{\psi}}{\chi^2} \left(2q - \frac{q^2\phi\dot{\psi}}{mc^2}\right) \dot{\psi}$; (iii) internal nonlinear self-interaction of the charge associated with the term $G' \left(\left|\dot{\psi}\right|^2\right) \dot{\psi}$. We refer to the equation (3.1.5) as charge equilibrium equation or just the equilibrium equation. Importantly, the charge equilibrium equation (3.1.5) establishes an explicit relation between the form factor $\dot{\psi}$ and the self-interaction nonlinearity $G$.

Now we come to a key point of our construction: determination of the nonlinearity $G$ from the equilibrium equation (3.1.5). First we pick and fix a form factor $\dot{\psi}(r)$, $r \geq 0$, which is assumed to be a nonnegative, monotonically decaying and sufficiently smooth function. Then we determine consequently $G'$ and $G$ from the equilibrium equation (3.1.5). This gives us at once the desired state of the resting charge $\{\dot{\psi}, \dot{\phi}\}$ without solving any nontrivial nonlinear partial differential equation, which is a stumbling block in most theories involving nonlinearities. Of course such a benefit of our approach comes at a cost of dealing with a nontrivial nonlinearity $G$ at all further steps, but it turns out that the definition of the nonlinearity via the equilibrium equation (3.1.5) is constructive enough for representing many important physical quantities in terms of $\dot{\psi}$, $\dot{\phi}$ and $G$ without explicit formulas for them. Curiously, for certain choices of $\dot{\psi}$ one can find explicit formulas for $\dot{\phi}$, $G$ and other important physical quantities, as we show in Section 4.5.

### 3.2 Energy-momentum tensor, forces and equilibrium

In any classical field theory over the four-dimensional continuum of space-time the energy-momentum tensor is of a fundamental importance. It provides the density of the energy, the momentum and the surface forces as well as the conservation laws that govern the energy and momentum transport. It is worth to point out that it is the differential form of the energy-momentum conservation which involve the densities of energy, momentum and forces rather than the original field equations are more directly related to corpuscular properties of the fields. In particular, for the charge model we study here the Lorentz force density arises in the differential form of the energy-momentum conservation equations and not in the original field equations. For detailed considerations of the structure and properties of the energy-momentum tensor including its symmetry, gauge invariance and conservation laws we refer the reader to Section 11. Here, using the results of that section, we compute and analyze the energy-momentum tensor for the Lagrangian $L_0$ defined by the formulas (3.0.1)-(3.0.3) and for the $\omega$-static state defined by (3.1.1).

Using the interpretation form (11.2.19)-(11.2.20) of the energy-momentum $T^{\mu\nu}$ and for-
mulas (3.0.12)-(3.0.13) we find that the energy-momentum tensor takes the following form

\[ T_{\mu\nu} = \begin{bmatrix} u & c_{p1} & c_{p2} & c_{p3} \\ c^{-1}s_1 & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ c^{-1}s_2 & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ c^{-1}s_3 & -\sigma_{31} & -\sigma_{32} & -\sigma_{33} \end{bmatrix}, \]  

(3.2.1)

where the energy density \( u \), the momentum and the energy flux components \( p_j \) and \( s_j \) are as follows:

\[ u = \frac{\chi^2}{2m} \left[ (\nabla \dot{\psi})^2 + G \left( \dot{\psi}^2 \right) \right] + \left( mc^2 - q\dot{\varphi} + \frac{q\phi^2}{2mc^2} \right) \dot{\psi}^2, \]  

(3.2.2)

\[ p^j = 0, \ s^j = 0, \ j = 1, 2, 3, \]  

(3.2.3)

and the stress tensor components \( \sigma_{ij} \) are represented by the formulas

\[ \sigma_{ij} = -\frac{\chi^2}{m} \left[ \partial_i \dot{\psi} \partial_j \dot{\psi} - \frac{1}{2} \left( \nabla \dot{\psi} \right)^2 \delta_{ij} \right] + \left[ q \left( \dot{\varphi} - \frac{q}{2mc^2} \dot{\psi}^2 \right) \ddot{\psi}^2 + \frac{\chi^2}{2m} G \left( \dot{\psi}^2 \right) \right] \delta_{ij}, \ i,j = 1, 2, 3. \]  

(3.2.4)

Notice that the vanishing of the momentum \( p \) and the energy flux \( s \) in (3.2.3) is yet another justification for the name \( \omega_0 \)-static solution. Observe also that for the \( \omega_0 \)-static state defined by (3.1.1) the EM field is

\[ \mathbf{E} = -\nabla \dot{\varphi}, \ \mathbf{B} = 0. \]  

(3.2.5)

Using the representation (11.4.21)-(11.4.22) for the EM energy-momentum \( \Theta_{\mu\nu} \) combined with the formulas (3.2.5) for the EM field we obtain the following representation of \( \Theta_{\mu\nu} \) for the \( \omega_0 \)-static solution (3.1.1):

\[ \Theta_{\mu\nu} = \begin{bmatrix} (\nabla \dot{\varphi})^2 / (8\pi) & 0 \\ 0 & -\tau_{ij} \end{bmatrix}, \]  

(3.2.6)

where

\[ -\tau_{ij} = \Theta^{ij} = -\frac{1}{4\pi} \left[ \partial_i \dot{\varphi} \partial_j \dot{\varphi} - \frac{1}{2} \left( \nabla \dot{\varphi} \right)^2 \delta_{ij} \right], \ i,j = 1, 2, 3. \]  

(3.2.7)

Combining the conservation law (3.0.14) with the general representation (3.2.1) of the charge energy-momentum tensor \( T_{\mu\nu} \) we obtain

\[ \partial_t p_i = \sum_{j=1,2,3} \partial_j \sigma_{ji} + \left( \rho E_i + \frac{1}{c} (\mathbf{J} \times \mathbf{B})_i \right) = 0, \ i = 1, 2, 3. \]  

(3.2.8)

Notice that for the \( \omega_0 \)-static solution (3.1.1), in view of (3.2.5), (3.2.6), (3.2.7), the equation (3.2.8) turns into the equilibrium equations

\[ \sum_{j=1,2,3} \partial_j \sigma_{ji} - \rho \partial_i \dot{\varphi} = 0, \ i = 1, 2, 3. \]  

(3.2.9)

Observe now that the stress tensor (str.t.) \( \sigma_{ij} \) defined in (3.2.4) can be naturally decomposed into the three components which we name as follows

\[ \sigma_{ij} = \sigma_{ij}^{el} + \sigma_{ij}^{em} + \sigma_{ij}^{nl}, \ i,j = 1, 2, 3, \]  

(3.2.10)
where
\[ \sigma_{ij}^{el} = -\frac{\chi^2}{m} \left[ \partial_i \dot{\psi} \partial_j \dot{\psi} - \frac{1}{2} \left( \nabla \dot{\psi} \right)^2 \delta_{ij} \right] \] (3.2.11)
is the elastic deformation stress tensor,

\[ \sigma_{ij}^{em} = -p^{em} \delta_{ij}, \quad p^{em} = -q \left( \hat{\varphi} - \frac{q \varphi^2}{2mc^2} \right) \varphi^2 \] (3.2.12)
is the EM interaction stress tensor,

\[ \sigma_{ij}^{nl} = -p^{nl} \delta_{ij}, \quad p^{nl} = -\frac{\chi^2 G \left( \varphi^2 \right)}{2m} \] (3.2.13)
is the nonlinear self-interaction stress tensor, and consequently the respective volume force densities are

\[ \sum_{j=1,2,3} \partial_j \sigma_{ij}^{el} = f_{i}^{el} = -\frac{\chi^2}{m} \Delta \dot{\psi} \partial_i \dot{\psi}, \quad i = 1, 2, 3 \] (3.2.14)

\[ \sum_{j=1,2,3} \partial_j \sigma_{ij}^{em} = f_{i}^{em} + \rho \partial_i \hat{\varphi}, \] (3.2.15)

\[ f_{i}^{em} = q \left( 2 \hat{\varphi} - \frac{q}{mc^2} \varphi^2 \right) \varphi \partial_i \dot{\psi}, \] (3.2.16)

\[ \sum_{j=1,2,3} \partial_j \sigma_{ij}^{nl} = f_{i}^{nl} = \frac{\chi^2}{m} G' \left( \varphi^2 \right) \varphi \partial_i \dot{\psi}. \] (3.2.17)

Notice that the volume force density for the electromagnetic interaction stress in (3.2.15) has two parts: \( f_{i}^{em} \), which we call *internal electromagnetic force*, and \( \rho \partial_i \hat{\varphi} \), which is the negative of the Lorentz force. Observe that the stress tensor \( \sigma_{ij}^{el} \) has a structure similar to the one for compressional waves, see Section 11.9 and (11.9.6), whereas the both stress tensors \( \sigma_{ij}^{em} \) and \( \sigma_{ij}^{nl} \) have the structure typical for perfect fluids, [Moller, Section 6.6], with respective hydrostatic pressures \( p^{em} \) and \( p^{nl} \) defined by the relations (3.2.12)-(3.2.13).

Notice that the formula (3.2.13) provides an interpretation of the nonlinearity \( G \left( \varphi^2 \right) \): \( p^{nl} = -\frac{\chi^2 G \left( \varphi^2 \right)}{2m} \) is the hydrostatic pressure when the charge is at rest.

Based on the equalities (3.2.14)- (3.2.17) we can recast the equilibrium equation (3.2.9) as

\[ f_{i}^{el} + f_{i}^{em} + f_{i}^{nl} = 0, \quad i = 1, 2, 3, \] (3.2.18)
or

\[ \left[ -\frac{\chi^2}{m} \Delta \dot{\psi} + q \left( 2 \varphi - \frac{q}{mc^2} \varphi^2 \right) \varphi \dot{\psi} + \frac{\chi^2}{m} G' \left( \varphi^2 \right) \varphi \right] \partial_i \dot{\psi} = 0. \]

The equation (3.2.18) signifies the ultimate equilibrium for the \( \omega_0 \)-static charge. It is evident from equation (3.2.18) that the scalar expression in the brackets before \( \nabla \dot{\psi} \) up to the factor \( \frac{\chi^2}{m} \) is exactly the left-hand side of the equilibrium equation (3.1.5). In fact if \( \nabla \dot{\psi} \neq 0 \) then the equilibrium equation (3.2.18) is equivalent to the scalar equilibrium equation (3.1.5).

Notice that since the \( \varphi \left( |x| \right) \) and \( \hat{\varphi} \left( |x| \right) \) are radial and monotonically decaying functions of \( |x| \) we readily have

\[ \nabla \dot{\psi} \left( |x| \right) = -\left| \nabla \dot{\psi} \right| \hat{x}, \quad \nabla \hat{\varphi} \left( |x| \right) = -\left| \nabla \hat{\varphi} \right| \hat{x}, \quad \hat{x} = \frac{x}{|x|} = (\hat{x}_1, \hat{x}_2, \hat{x}_3). \] (3.2.19)
The relations (3.2.19) combined with (3.2.14)-(3.2.17) imply that for the resting charge all the forces $f_{\text{el}}, f_{\text{em}}$ and $f_{\text{nl}}$ and radial, i.e. they are functions of $|x|$ and point toward or outward the origin.

Notice that it follows from the relations (3.2.3) and (3.2.6) that the total momentum $P$ and the energy flux $S$ of the resting dressed charge, i.e. the charge and the EM field together, vanish, and hence we have $P^\nu = (u, P)$ with

$$P = 0, \quad S = 0,$$

(3.2.20)

where the energy density $u$ is represented by the formula (3.2.2). We would like to point out that the vanishing for the resting charge of the micro-current $J$ in (3.1.2) as well as the momentum $P$ and the energy flux $S$ in (3.2.20) justifies the name $\omega_0$-static solution.

Using the general formulas (11.2.16) for the angular momentum density $M_{\mu\nu\gamma}$ and combining them with the relations (3.2.1)-(3.2.3) for the energy-momentum tensor $T_{\mu\nu}$ we readily obtain that the total angular momentum $J^{\nu\gamma}$ vanishes, namely

$$J^{\nu\gamma} = \int_{\mathbb{R}^3} M_{\mu\nu\gamma}(x) \, dx = 0.$$

(3.2.21)

### 3.3 Frequency shifted Lagrangian and the reduced energy

The time harmonic factor $e^{-i\omega_0 t}$ which appears in $\omega_0$-static states as in (3.1.1) plays a very important role in this theory, including the nonrelativistic case. To reflect that we introduce a change of variables

$$\psi(t, x) \rightarrow e^{-i\omega_0 t}\psi(t, x)$$

(3.3.1)

and substitute it in the Lagrangian $L_0$ defined by (3.0.1) to obtain the Lagrangian $L_{\omega_0}$, which we call frequency shifted, namely

$$L_{\omega_0}(\psi, A^\mu) = \frac{\chi i}{2} \left( \psi^* \tilde{\partial}_t \psi - \psi \tilde{\partial}_t^* \psi^* \right) + \frac{\chi^2}{2m} \left\{ \frac{1}{c^2} \tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* - \tilde{\nabla} \psi \tilde{\nabla}^* \psi^* - G(\psi^* \psi) \right\} - \frac{F_{\mu\nu} F^{\mu\nu}}{16\pi},$$

(3.3.2)

where we use notation

$$\tilde{\partial}_t = \partial_t + \frac{iq\varphi}{\chi}, \quad \tilde{\nabla} = \nabla - \frac{iqA}{\chi c}, \quad \tilde{\partial}_t^* = \partial_t - \frac{iq\varphi}{\chi}, \quad \tilde{\nabla}^* = \nabla + \frac{iqA}{\chi c}.$$

If we use the relation (11.4.16) we can rewrite it in the form

$$L_{\omega_0}(\psi, \psi^*, A^\mu) = \frac{\chi i}{2} \left( \psi^* \tilde{\partial}_t \psi - \psi \tilde{\partial}_t^* \psi^* \right) + \frac{\chi^2}{2m} \left\{ \frac{1}{c^2} \tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* - \tilde{\nabla} \psi \tilde{\nabla}^* \psi^* - G(\psi^* \psi) \right\} + \frac{1}{8\pi} \left[ \left( \nabla \varphi + \frac{1}{c} \partial_t A \right)^2 - (\nabla \times A)^2 \right].$$

The Lagrangian $L_{\omega_0}$ defined by the formula (3.3.2) is manifestly gauge and space-time translation invariant, it also invariant with respect to space rotations but it is not invariant with...
respect to the entire group of Lorentz transformations. Notice also that \( \omega_0 \)-static states for the original Lagrangian defined by (3.0.1) turn into regular static states for the Lagrangian \( L_{\omega_0} \), and that was one of the reasons to introduce it.

For a \( \omega_0 \)-static state \( \left\{ e^{-i\omega_0 t} \psi, \varphi \right\} \) satisfying the field equations (3.1.3)-(3.1.4) its canonical density of energy \( \hat{u}_{L_{\omega_0}} (\psi, \varphi) \) as defined by (11.2.34) can be simply related to the canonical energy \( \hat{u}_{L_{\omega_0}} (\psi, \dot{\varphi}) \) of the frequency shifted Lagrangian \( L_{\omega_0} \). Indeed applying the arguments provided in Section 11.8.1 particularly relations (11.8.37)-(11.8.40), and combined with the representation (3.3.3) we find that

\[
\hat{u}_{L_{\omega_0}} (\psi, \varphi) = \frac{mc^2}{q} \rho - L_{\omega_0} (\psi, \dot{\psi}, \dot{\varphi}) = \frac{mc^2}{q} \rho + \frac{\chi^2}{2m} \left[ \left( \nabla \psi \right)^2 + \left( \frac{\omega_0}{c} - \frac{q}{\chi c} \dot{\varphi} \right)^2 \dot{\psi}^2 + G \left( \dot{\psi}^2 \right) \right] - \frac{(\nabla \varphi)^2}{8\pi},
\]

and that the total energy in this state can be represented in the form (2.1.1) using results of Section 11.8. The energy representation (2.1.1) is important to us since it does not involve explicitly the nonlinear self-interaction \( G \).

### 3.4 Moving charge

As it is often done in the literature we use the Lorentz invariance of the system to obtain the state of the dressed charge moving with a constant velocity \( \mathbf{v} \). Namely, we apply to the rest solution described by (3.1.1)-(3.1.4) the Lorentz transformation from the original "rest frame" to the frame in which the "rest frame" moves with the constant velocity \( \mathbf{v} \) as described by the formulas (11.1.6), (11.4.12) (so \( \mathbf{x}' \) and \( \mathbf{x} \) correspond respectively to the "rest" and "moving" frames) yielding

\[
\psi (t, \mathbf{x}) = e^{-i (\omega t - \mathbf{k} \cdot \mathbf{x})} \tilde{\psi} (\mathbf{x}'), \quad \varphi (t, \mathbf{x}) = \gamma \tilde{\varphi} (|\mathbf{x}'|), \quad \mathbf{A} (t, \mathbf{x}) = \gamma \mathbf{\beta} \tilde{\varphi} (|\mathbf{x}'|), \quad \mathbf{B} (t, \mathbf{x}) = \gamma \mathbf{\beta} \times \nabla \tilde{\varphi} (|\mathbf{x}'|),
\]

where

\[
\omega = \gamma \omega_0, \quad \mathbf{k} = \gamma \mathbf{\beta} \frac{\omega_0}{c}, \quad \mathbf{\beta} = \frac{\mathbf{v}}{c}, \quad \beta = |\mathbf{\beta}|, \quad \gamma = \left( 1 - \left( \frac{\mathbf{v}}{c} \right)^2 \right)^{-1/2}, \quad \mathbf{x}' = \mathbf{x} + \frac{\gamma - 1}{\mathbf{\beta}^2} (\mathbf{\beta} \cdot \mathbf{x}) \mathbf{\beta} - \gamma \mathbf{v} t, \quad \mathbf{x}_\parallel = \gamma (\mathbf{x}_\parallel - \mathbf{v} t), \quad \mathbf{x}_\perp = \mathbf{x}_\perp,
\]

where \( \mathbf{x}_\parallel \) and \( \mathbf{x}_\perp \) refer respectively to the components of \( \mathbf{x} \) parallel and perpendicular to the velocity \( \mathbf{v} \). The above formulas provide a solution to field equations (3.0.6), (3.0.7) and indicate that the fields of the dressed charge contract by the factor \( \gamma \) as it moves with the velocity \( \mathbf{v} \) compared to their rest state. The first oscillatory exponential factor in (3.4.1) is the de Broglie plane wave of the frequency \( \omega \) and the de Broglie wave-vector \( \mathbf{k} \). Notice that the equalities (3.4.3) readily imply the following relations between \( \omega \), \( \mathbf{k} \) and \( \mathbf{v} \)

\[
\omega = \omega (\mathbf{k}) = \sqrt{\omega_0^2 + c^2 \mathbf{k}^2}, \quad \mathbf{v} = \nabla_k \omega (\mathbf{k}),
\]
where
\[ \omega_0 = \frac{\mathcal{E}_0 (\psi)}{\hbar} = \frac{mc^2}{\chi}, \]
and we refer to Section 2 formulas (2.1.4)-(2.1.6), for the values of the frequency \( \omega_0 \) and the constant \( \chi \).

Notice that the above relations show, in particular, that for the freely moving dressed charge defined by equalities (3.4.1)-(3.4.4) its velocity \( v \) equals exactly the group velocity \( \nabla_k \omega (k) \) computed for the de Broglie wave vector \( k \). This fact clearly points to the wave origin of the charge kinematics as it moves in the three dimensional space continuum with the dispersion relation \( \omega = \sqrt{\omega_0^2 + c^2 k^2} \). Notice that this dispersion relation is identical to the dispersion relation of the Klein-Gordon equation as a model for a free charge, [Pauli PWM, Section 18].

Now we consider the total 4-momentum \( P \) obtained from its density by integration over the space \( \mathbb{R}^3 \). Since the dressed charge is a closed system, its total 4-momentum \( P^\nu = (E, cP) \) is 4-vector, see the end of Section 11.2. Using this vector property and the value \( P^\nu = \left( \mathcal{E}_0 (\psi), 0 \right) \) for the resting dressed charge we find, by applying the relevant Lorentz transformation, that the dressed charge 4-momentum \( P^\nu \) satisfies
\[ P^\nu = (E, cP), \quad E = h\omega, \quad P = h\mathbf{k}, \quad (3.4.6) \]
showing that the Einstein-de Broglie relations hold for the moving charge. We would like to point out that, though the above argument used to obtain the relations (3.4.6) is rather standard, in our case relations (3.4.6) are deduced rather than rationally imposed.

Observe that our relations (3.4.5) under the assumption that \( \chi = \hbar \) are identical to those of a free charge as described by the Klein-Gordon equation, [Pauli PWM, Sections 1, 18], (see also Section 11.10) but there are several significant differences between the two models which are as follows. First of all, our charge is a dressed charge described by the pair \( \{ \psi, A^\mu \} \). From the very outset it includes the EM field as its inseparable part whereas the Klein-Gordon model describes a free charge by a complex-valued wave function \( \psi \) which is not coupled to its own EM field (not to be confused with an external EM field). Second, our free dressed charge when it moves, evidently preserves its shape up to the natural Lorentz construction whereas any wavepacket satisfying Klein-Gordon equation spreads out in the course of time.

### 3.5 Correspondence with the point charge mechanics

The free dressed charge as described by equalities (3.4.1)-(3.4.4) allows for a certain reduction to the model of point charge (mass). Notice that combining the relations (3.4.6) with (3.4.3) we obtain the well known point mass kinematic representations (11.1.13) for the total energy \( E \) and the momentum \( P \) of the dressed charge, namely
\[ P = h\mathbf{k} = \gamma \beta \frac{\hbar \omega_0}{c} = \gamma \tilde{m} v, \quad \gamma = \left( 1 - \left( \frac{v}{c} \right)^2 \right)^{-1/2}, \quad (3.5.1) \]
\[ E = h\omega = h\gamma \omega_0 = \gamma \tilde{m} c^2 = c\sqrt{P^2 + \tilde{m}^2 c^2}, \quad (3.5.2) \]
where \( \tilde{m} \) is the dressed charge mass defined by (2.1.2). We can also reasonably assign to the dressed charge described by equalities (3.4.1)-(3.4.4) a location \( \mathbf{r} (t) \) at any instant \( t \) of time.
which is obtained from the (3.4.4) by setting there $x'$ and solving it for $x$, $r(t) = x(x', t)$.
Not surprisingly, its solution is
$$r(t) = vt.$$ (3.5.3)
An elementary examination confirms that $(ct, vt)$ transforms as a 4-vector implying that the definition (3.5.3) is both natural and relativistically consistent. From (3.5.3) we readily obtain another fundamental relation for the point charge
$$v = \frac{dr(t)}{dt}.$$ (3.5.4)

4 Single nonrelativistic free and resting charge
The nonrelativistic case, i.e. the case when a charge moves with a velocity much smaller than the velocity of light, is important for our studies for at least two reasons. First of all, we need it to relate the wave-corpuscle mechanics to the Newtonian mechanics for point charges in EM fields. Second of all, in the nonrelativistic case we can carry out rather detailed analytical studies of many physical quantities in a closed form. With that in mind, we would like to treat the nonrelativistic case not just as an approximation to the relativistic theory but rather as a case on its own, and we do it by constructing a certain nonrelativistic Lagrangian $\hat{L}_0$ intimately related to the relativistic Lagrangian defined in (3.0.1)-(3.0.3). This nonrelativistic Lagrangian constitutes a fundamental basis for our nonrelativistic studies including the construction of the nonlinear self-interaction. The relation between the relativistic and nonrelativistic Lagrangians is considered in Section 7.

The nonrelativistic Lagrangian $\hat{L}_0$ is constructed as a certain nonrelativistic modification of the frequency shifted Lagrangian $L_{\omega_0}(\psi, A^\mu)$ introduced in Section 3.3. The first step in this modification is the change of variables (3.3.1), namely
$$\psi(t, x) \rightarrow e^{-i\omega_0 t} \psi(t, x)$$ (4.0.5)
which was the initial step in the construction of the frequency-shifted Lagrangian $L_{\omega_0}$ defined by (3.3.2)-(3.3.3). Then a gauge invariant and nonrelativistic Lagrangian $\hat{L}_0$ is obtained from the Lagrangian $L_{\omega_0}$ by omitting in (3.3.3) the term $\frac{x^2}{2mc^2} \tilde{\partial}_i \psi \tilde{\partial}_i \psi^*$ and setting $A = 0$, namely
$$\hat{L}_0(\psi, \psi^*, \varphi) = \frac{\lambda^4}{2} \left[ \psi^* \tilde{\partial}_t \psi - \psi \tilde{\partial}_t^* \psi^* \right] - \frac{x^2}{2m} \left[ \nabla \psi \nabla \psi^* + G(\psi^* \psi) \right] + \frac{\left| \nabla \varphi \right|^2}{8\pi}, (4.0.6)$$
$$\tilde{\partial}_i = \partial_i + \frac{iq\varphi}{\lambda}, \quad \tilde{\partial}_i^* = \partial_i - \frac{iq\varphi}{\lambda}$$
where, we recall, the term $G(\psi^* \psi)$ corresponds to the charge nonlinear self-interaction. Observe that the assumption $A = 0$ in view of (11.4.6) readily implies
$$E = -\nabla \varphi, \quad B = 0.$$ (4.0.7)
Hence, the EM field tensor $F^{\mu\nu}$ defined by (11.4.5) takes here a simpler form
$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ -E_1 & 0 & 0 & 0 \\ -E_2 & 0 & 0 & 0 \\ -E_3 & 0 & 0 & 0 \end{bmatrix}, (4.0.8)$$
The mentioned gauge invariance is understood with respect first to the gauge transformation of the first kind (global) as in (11.5.7) and of the second (local) types as in (11.5.7)-(11.5.8), namely

\[ \psi \rightarrow e^{i\gamma} \psi, \quad \psi^* \rightarrow e^{-i\gamma} \psi^*, \]

where \( \gamma \) is any real constant, and with respect to a reduced version of the second type gauge transformation

\[ \psi \rightarrow e^{-i\frac{iq\lambda(t)}{\chi}} \psi, \quad \psi^* \rightarrow e^{i\frac{iq\lambda(t)}{\chi}} \psi^*, \quad \varphi \rightarrow \varphi + \partial_t \lambda(t), \]

which is similar to (11.5.8) but the function \( \lambda(t) \) may depend only on time.

Evidently the EM field of the charge is represented in the above Lagrangian \( \hat{L}_0 \) only by its scalar potential \( \varphi \) and the corresponding electric field \( E = -\nabla \varphi \), since \( A = 0 \). The charge’s magnetic field is identically zero in view of the equalities (4.0.7), and, consequently, any radiation phenomena are excluded in this model. The Lagrangian \( \hat{L}_0 \) can be viewed as a field version of the point charges model (6.1.30) that neglects all retardation effects in the static limit (zeroth order in \( \frac{v}{c} \)) with the ”instantaneous” interaction Lagrangian \( -\frac{q_1 q_2}{r_{12}} \) between two charges, [Jackson, Section 12.6]. More detailed discussions on the relations between relativistic and nonrelativistic Lagrangians and the corresponding Euler-Lagrange equations are provided in Section 7.

The Euler-Lagrange field equations for this Lagrangian are

\[ \chi i \partial_t \psi = \frac{\chi^2}{2m} \left[ -\Delta \psi + G'(\psi^* \psi) \psi \right], \quad (4.0.11) \]
\[ -\Delta \varphi = 4\pi q \psi \psi^*, \quad (4.0.12) \]

where \( G'(s) = \partial_s G(s) \), and we refer to the pair \( \{ \psi, \varphi \} \) as dressed charge. Taking into account the form of the covariant time derivative from (4.0.6) we can recast the field equations (4.0.11)-(4.0.12) for the dressed charge as

\[ \chi i \partial_t \psi = \frac{\chi^2}{2m} \left[ -\Delta + \frac{2mq}{\chi^2} \varphi + G'(|\psi|^2) \right] \psi, \quad -\Delta \varphi = 4\pi q |\psi|^2. \quad (4.0.13) \]

which imply (2.3.3), (2.3.4).

Applying the general formulas (11.7.12)-(11.7.15) for the charge and current densities to the Lagrangian \( \hat{L}_0 \) we obtain expressions (2.3.9) for the densities and since external fields are absent, the current \( J^\mu \) satisfies the conservation/continuity equations (2.3.10). Consequently, the total charge remains constant in the course of evolution, and as always we set this constant charge to be exactly \( q \), namely we impose charge normalization condition (2.3.12)

\[ \int_{\mathbb{R}^3} \rho(x) \, dx = q \int_{\mathbb{R}^3} |\psi|^2 \, dx = q \text{ or } \int_{\mathbb{R}^3} |\psi|^2 \, dx = 1. \quad (4.0.14) \]

As in the relativistic case the equation (2.3.10) follows from the field equations, therefore (4.0.14) is preserved for all times.

### 4.1 Symmetries and conservation laws

To carry out a systematic analysis of conservation laws associated with the Lagrangian \( \hat{L}_0 \) defined by (4.0.6) via Noether theorem, see Section 11.3, we need to find a Lie group of transformations which preserve it. The Lagrangian \( \hat{L}_0 \) is not invariant with respect to either the
Lorentz or the Galilean groups of transformations. But a straightforward examination shows that $\hat{L}_0$ is invariant with respect to the following Galilean-gauge group of transformations

$$t' = t, \ x' = x - vt,$$

or

$$x'^{0} = x^{0}, \ x' = x - \frac{v}{c}x^{0},$$

$$\psi (t, x) = e^{i\frac{m}{\hbar}x^{2}} \psi (t', x'),$$

or

$$\psi' (t', x') = e^{i\frac{m}{\hbar}(x'^{2} - 2v \cdot x)} \psi (t, x),$$

with $\varphi (t, x) = \varphi' (t', x')$. One can also verify that the above transformations form an Abelian group of transformations parametrized by the velocity parameter $v$. It is curious to observe that according to the Galilean-gauge transformations (4.1.1), (4.1.2) the charge wave function does not transform as a scalar as in the relativistic case. These transformations are known, [Gottfried, Section 7.3], and were used, in particular, in studies of nonlinear Schrödinger equations, [Sulem, Section 2.3].

The above-defined Galilean-gauge group is naturally extended to the general inhomogeneous Galilean-gauge group by adding to it the group of spatial rotations $O$ and space-time translations $a^{\mu}$, namely

$$t' = t + \tau, \ x' = Ox - vt + a,$$

$$\psi' (t', x') = e^{i\frac{m}{\hbar}[x^{2}(t+\tau) - 2v \cdot (Ox+a)]} \psi (t, x), \ \varphi' (t', x') = \varphi (t, x).$$

The infinitesimal form of the above group of transformations is as follows

$$t' = t + \tau, \ x' = x + \xi \times x - vt + a, \ a^{\mu} = (c\tau, a), \ a^{0} = c\tau, \ x^{0} = ct,$$

or, equivalently,

$$x^{\mu} = x^{\mu} + \xi^{\mu\nu} x_{\nu} + a^{\mu}, \ \xi^{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\frac{-v^{1}}{c} & 0 & -\xi^{2} & \xi^{3} \\
\frac{-v^{2}}{c} & \xi^{2} & 0 & -\xi^{1} \\
\frac{-v^{3}}{c} & -\xi^{3} & \xi^{1} & 0
\end{pmatrix},$$

where the real number $\tau$ and the coordinates of the three three-dimensional vectors $v, \xi, a$ provide the total of ten real parameters as in the case of the infinitesimal inhomogeneous Lorentz group defined by (11.1.10). The infinitesimal form of the transformations (4.1.4) is

$$\delta \psi = -i\frac{m}{\hbar} \left( \sum_{j=1,2,3} x^{j} \cdot \delta v^{j} \right) \psi, \ \delta \varphi = 0.$$

Coming back to the analysis of basic features of our model we acknowledge the use in this section of the relativistic conventions for upper and lower indices and the summation as in Section 11.1 including

$$x^{\mu} = (x^{0}, x), \ x_{\mu} = (x^{0}, -x), \ x^{0} = ct.$$
Carrying out the Noether currents analysis as in Section 11.3 for the Lagrangian $\hat{L}_0$ we obtain 10 conservation laws which, as it turns out, can be formulated in terms of the canonical energy-momentum tensor $\hat{T}^{\mu\nu}$, which in turn is obtained from the general formula (11.2.5):

$$\hat{T}^{\mu\nu} = \frac{\partial \hat{L}_0}{\partial \dot{\psi}_{\mu}} \dot{\psi}_\nu + \frac{\partial \hat{L}_0}{\partial \dot{\psi}^*_{\mu}} \dot{\psi}^*_\nu + \frac{\partial \hat{L}_0}{\partial \dot{\varphi}_{\mu}} \varphi_\nu - \hat{L}_0 g^{\mu\nu}. \quad (4.1.9)$$

Namely, we get the total of ten conservation laws:

$$\partial_\mu \hat{T}^{\mu\nu} = 0 \text{ - energy-momentum conserv.,} \quad (4.1.10)$$

$$\hat{T}^{ij} = \hat{T}^{ji}, \ i, j = 1, 2, 3 \text{ - space angular momentum conserv.,} \quad (4.1.11)$$

$$P^i = \hat{T}^{0i} = \frac{m}{q} j^i, \ i = 1, 2, 3 \text{ - time-space angular momentum conserv.} \quad (4.1.12)$$

The first four standard conservation laws (4.1.10) are associated with the Noether’s currents with respect to space-time translations $a^\mu$. The second three conservation laws in (4.1.11) are associated with space rotation parameters $\xi$, and they turn into the symmetry of the energy-momentum tensor $\hat{T}^{\mu\nu}$ for the spatial indices similarly to relations (11.2.16)-(11.2.17). The form of the last three conservation laws (4.1.12) is special to the nonrelativistic Lagrangian $\hat{L}_0$, and it is due to the Galilean-gauge invariance (4.1.2), (4.1.4). These relations indicate that the total momentum density $P^i$ is identically equal up to the factor $\frac{m}{q}$ to the microcurrent density $J^i$ defined by (2.3.10). This important identity is analogous to the kinematic representation $p^\mu = \frac{m}{q} v^\mu$ of the momentum $p$ of a point charge. It is related to the velocity components $v^\mu$ in the Galilean-gauge transformations (4.1.2), (4.1.4), and can be traced to the infinitesimal transformation (4.1.7) and the phases in (4.1.4). The proportionality of the momentum and the current is known to occur for systems governed by the nonlinear Schrödinger equations, [Sulem, Section 2.3].

The issue of fundamental importance of studies of the energy-momentum tensor has been already addressed in the beginning of Section 3.2. To find the energy-momentum tensor for the Lagrangian $\hat{L}_0$ we apply to it the general formulas from Section 11.6. The canonical energy-momentum $\Theta^{\mu\nu}$ for the EM field is obtained by applying the general formula (11.2.5) to the Lagrangian $\hat{L}_0$ yielding

$$\Theta^{\mu\nu} = \begin{bmatrix} 
\hat{w} & 0 & 0 & 0 \\
c^{-1}\hat{s}_1 & -\hat{\tau}_{11} & -\hat{\tau}_{12} & -\hat{\tau}_{13} \\
c^{-1}\hat{s}_2 & -\hat{\tau}_{21} & -\hat{\tau}_{22} & -\hat{\tau}_{23} \\
c^{-1}\hat{s}_3 & -\hat{\tau}_{31} & -\hat{\tau}_{32} & -\hat{\tau}_{33} 
\end{bmatrix}, \quad (4.1.13)$$

$$\hat{w} = -\frac{|\nabla \varphi|^2}{8\pi}, \ \hat{s}_j = 0, \ \hat{\tau}_{ij} = c\frac{\partial_j \varphi \partial_i \varphi}{4\pi}. \quad (4.1.14)$$

The gauge invariant energy-momentum of the EM field takes the form

$$\Theta^{\mu\nu} = \begin{bmatrix} 
\hat{w} & 0 & 0 & 0 \\
0 & -\tau_{11} & -\tau_{12} & -\tau_{13} \\
0 & -\tau_{21} & -\tau_{22} & -\tau_{23} \\
0 & -\tau_{31} & -\tau_{32} & -\tau_{33} 
\end{bmatrix}, \quad (4.1.15)$$

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with matrix entries
\[ \partial_0 w = \frac{\mathbf{J} \cdot \nabla \varphi}{c} = -\frac{\mathbf{J} \cdot \mathbf{E}}{c}, \quad g_j = 0, \quad s_j = 0, \]
\[ \tau_{jj} = \frac{\partial^2 \varphi}{8\pi}, \quad \tau_{ij} = \frac{\partial_i \varphi \partial_j \varphi}{4\pi}. \]
\[ (4.1.16) \]

As we can see the relation (4.1.16) involves the time derivative \( \partial_0 w \) for the energy density \( w \) rather than the density itself, and it follows from it that
\[ w = w(t, \mathbf{x}) = w_0(\mathbf{x}) + \int_{-\infty}^{t} \frac{\mathbf{J}(t', \mathbf{x}) \cdot \nabla \varphi(t', \mathbf{x})}{c} \, dt', \]
\[ (4.1.17) \]
where \( w_0(\mathbf{x}) \) is a time independent energy density. Notice also that combining the relations (2.3.9), (2.3.10), (4.0.12) and (4.1.17) we obtain the following identity
\[ \partial_0 \left( \int_{\mathbb{R}^3} w \, d\mathbf{x} \right) = \int_{\mathbb{R}^3} \frac{\mathbf{J} \cdot \nabla \varphi}{c} \, d\mathbf{x} = -\int_{\mathbb{R}^3} \frac{\varphi(\nabla \cdot \mathbf{J})}{c} \, d\mathbf{x} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \varphi \partial_t \rho \, d\mathbf{x} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla \varphi \cdot \partial_t \nabla \varphi \, d\mathbf{x} = \partial_0 \left( \int_{\mathbb{R}^3} \frac{(\nabla \varphi)^2}{8\pi} \, d\mathbf{x} \right). \]
\[ (4.1.18) \]

A choice for \( w_0(\mathbf{x}) \) in (4.1.17), consistent with the canonical energy-momentum, is
\[ w_0(\mathbf{x}) = \frac{(\nabla \varphi)^2}{8\pi}. \]

The canonical energy-momentum tensor \( \tilde{T}^{\mu\nu} \) is not gauge invariant, but the following decomposition holds for it
\[ \tilde{T}^{\mu\nu} = \bar{T}^{\mu\nu} + \frac{1}{c} \mathcal{J}^\mu A^\nu, \quad A^\nu = (\varphi, 0) \]
\[ (4.1.19) \]
where \( \bar{T}^{\mu\nu} \) is a gauge invariant energy-momentum obtained by applying formula (11.6.8) to the Lagrangian \( \hat{L}_0 \), namely
\[ \bar{T}^{\mu\nu} = \begin{bmatrix} \tilde{u} & \tilde{c} \tilde{p}_1 & \tilde{c} \tilde{p}_2 & \tilde{c} \tilde{p}_3 \\ c^{-1} \tilde{s}_1 & -\tilde{\sigma}_{11} & -\tilde{\sigma}_{12} & -\tilde{\sigma}_{13} \\ c^{-1} \tilde{s}_2 & -\tilde{\sigma}_{21} & -\tilde{\sigma}_{22} & -\tilde{\sigma}_{23} \\ c^{-1} \tilde{s}_3 & -\tilde{\sigma}_{31} & -\tilde{\sigma}_{32} & -\tilde{\sigma}_{33} \end{bmatrix}, \]
\[ (4.1.20) \]
where
\[ \tilde{u} = \frac{\chi^2}{2m} \left[ |\nabla \psi|^2 + G \left( |\psi|^2 \right) \right], \]
\[ (4.1.21) \]
\[ \tilde{p}_j = \frac{\chi^2 i}{2m} \left( \psi \partial_j \psi^* - \psi^* \partial_j \psi \right), \quad \tilde{s}_j = -\frac{\chi^2 i}{2m} \left( \partial_i \psi \partial_j \psi^* + \partial_j \psi^* \partial_i \psi \right), \quad j = 1, 2, 3, \]
\[ (4.1.22) \]
and the stress tensor components \( \sigma_{ij} \) are represented by the formulas
\[ \tilde{\sigma}_{ii} = \tilde{u} - \frac{\chi^2}{m} \partial_i \psi \partial_i \psi^* + \frac{\chi^2 i}{2m} \left( \psi \partial_i \psi^* - \psi^* \partial_i \psi \right), \]
\[ (4.1.23) \]
\[ \tilde{\sigma}_{ij} = \tilde{\sigma}_{ji} = -\frac{\chi^2}{2m} \left( \partial_i \psi \partial_j \psi^* + \partial_j \psi \partial_i \psi^* \right) \text{ for } i \neq j, \quad i, j = 1, 2, 3. \]
One can verify using the field equations (4.0.11), (4.0.12) and the current conservation law (2.3.10) that the canonical and gauge invariant energy-momentum tensors satisfy the following relations

\[ \mathcal{\Theta}^{\mu j} = \Theta^{\mu j}, \ T^{\mu j} = \mathcal{T}^{\mu j}, \ j = 1, 2, 3 \]  

and

\[ \partial_\mu \left[ (\Theta^{\mu 0} + \mathcal{T}^{\mu 0}) - (\mathcal{\Theta}^{\mu 0} + \mathcal{T}^{\mu 0}) \right] = 0, \]

and that the conservation laws in view of the representation (4.0.8) take the following form

\[ \partial_\mu \mathcal{T}^{\mu \nu} = f^\nu, \ \partial_\mu \Theta^{\mu \nu} = -f^\nu, \]  

where

\[ f^\nu = \frac{1}{c} J_\mu F^{\nu \mu} = \left( \frac{1}{c} \mathbf{J} \cdot \mathbf{E}, \rho \mathbf{E} \right), \]

and we recognize in \( f^\nu \) the Lorentz force density.

### 4.2 Resting charge

For a resting charge the representations (4.1.15)-(4.1.17) and (4.1.20)-(4.1.23) for the energy-momentum tensors \( \Theta^{\mu \nu} \) and \( \mathcal{T}^{\mu \nu} \) turn into

\[
\Theta^{\mu \nu} = \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & -\tau_{11} & -\tau_{12} & -\tau_{13} \\ 0 & -\tau_{21} & -\tau_{22} & -\tau_{23} \\ 0 & -\tau_{31} & -\tau_{32} & -\tau_{33} \end{bmatrix}, \quad \mathcal{T}^{\mu \nu} = \begin{bmatrix} \tilde{u} & 0 & 0 & 0 \\ 0 & -\tilde{\sigma}_{11} & -\tilde{\sigma}_{12} & -\tilde{\sigma}_{13} \\ 0 & -\tilde{\sigma}_{21} & -\tilde{\sigma}_{22} & -\tilde{\sigma}_{23} \\ 0 & -\tilde{\sigma}_{31} & -\tilde{\sigma}_{32} & -\tilde{\sigma}_{33} \end{bmatrix},
\]

showing, in particular, that the momentum and flux densities for the charge and for the EM field are all identically zero. Consequently, the total momentum \( P \) and the energy flux \( S \) of the resting dressed charge vanish and we have \( P^\nu = (\tilde{u}, P) \) with

\[ P = 0, \ S = 0 \]  

and

\[ \tilde{u} = \frac{\chi^2}{2m} \left[ (\nabla \dot{\psi})^2 + G \left( \dot{\psi}^2 \right) \right]. \]  

Observe now that the stress tensor (str. t.) \( \sigma_{ij} \) defined by relations (4.1.23) in the case of resting charge can be naturally decomposed into three components which we name as follows

\[ \sigma_{ij} = \sigma_{ij}^{el} + \sigma_{ij}^{em} + \sigma_{ij}^{nl}, \ i, j = 1, 2, 3, \]  

where

\[ \sigma_{ij}^{el} = -\frac{\chi^2}{m} \left[ \mathcal{\Theta}^{\dot{\psi} \dot{\psi}} - \frac{1}{2} \left( \nabla \dot{\psi} \right)^2 \delta_{ij} \right], \]  

is elastic deformation stress tensor,

\[ \sigma_{ij}^{em} = -p_{ij}^{em} \delta_{ij}, \ p_{ij}^{em} = -q \dot{\varphi} \dot{\psi}^2 \]  

is EM interaction stress tensor,

\[ \sigma_{ij}^{nl} = -p_{ij}^{nl} \delta_{ij}, \ p_{ij}^{nl} = -\frac{\chi^2 G \left( \dot{\psi}^2 \right)}{2m}, \]  

and

We have

\[ \text{33} \]
is nonlinear self-interaction stress tensor. Consequently, the respective volume force densities are

\[ \sum_{j=1,2,3} \partial_j \sigma_{ij}^{el} = f_i^{el} = -\frac{\chi^2}{m} \Delta \tilde{\psi} \partial_i \tilde{\psi}, \ i = 1, 2, 3, \]

(4.2.7)

\[ \sum_{j=1,2,3} \partial_j \sigma_{ij}^{em} = f_i^{em} = \rho \partial_i \tilde{\varphi}, \ f_i^{em} = 2q \tilde{\varphi} \partial_i \tilde{\psi}, \]

(4.2.8)

\[ \sum_{j=1,2,3} \partial_j \sigma_{ij}^{nl} = f_i^{nl} = \frac{\chi^2}{m} G' \left( \tilde{\psi}^2 \right) \tilde{\psi} \partial_i \tilde{\psi}. \]

(4.2.9)

Notice that the volume force density for the electromagnetic interaction stress in (4.2.8) has two parts: \( f_i^{em} \), which we call internal electromagnetic force, and \( \rho \partial_i \tilde{\varphi} \) which is the negative of the Lorentz force. Observe that the stress tensor \( \sigma_{ij}^{el} \) has the structure similar to the one for compressional waves, see Section 11.9 and (11.9.6), whereas the both stress tensors \( \sigma_{ij}^{em} \) and \( \sigma_{ij}^{nl} \) have the structure typical for perfect fluids, [Moller, Section 6.6], with respective hydrostatic pressures \( p^{em} \) and \( p^{nl} \) defined by the relations (4.2.5)-(4.2.6).

Based on the equalities (4.2.7)-(4.2.9) we can recast the equilibrium equation (3.2.9) as

\[ f_i^{el} + f_i^{em} + f_i^{nl} = 0, \ i = 1, 2, 3, \]

or

(4.2.10)

The equation (4.2.10) signifies the ultimate equilibrium for the static charge. It is evident from equation (4.2.10) that the scalar expression in the brackets before \( \nabla \tilde{\psi} \) up to the factor \( \frac{m}{\chi^2} \) is exactly the left-hand side of the equilibrium equation (2.3.6). In fact if \( \nabla \tilde{\psi} \neq 0 \) then the equilibrium equation (4.2.10) is equivalent to the scalar equilibrium equation (2.3.6).

Notice that since the \( \tilde{\psi}(|x|) \) and \( \tilde{\varphi}(|x|) \) are radial and monotonically decaying functions of \(|x|\) we readily have

\[ \nabla \tilde{\psi} (|x|) = \hat{x} - \left| \nabla \tilde{\psi} \right| \hat{r}, \ \nabla \tilde{\varphi} (|r|) = - \left| \nabla \tilde{\varphi} \right| \hat{x}, \ \hat{x} = \frac{x}{|x|} = (\hat{x}_1, \hat{x}_2, \hat{x}_3). \]

(4.2.11)

The relations (4.2.11) combined with (4.2.7)-(4.2.9) imply that for the resting charge all the forces \( f_i^{el}, f_i^{em} \) and \( f_i^{nl} \) radial, i.e. they are functions of \(|x|\) and point toward or outward the origin.

The total energy of the resting dressed charge \( \mathcal{E} (\tilde{\psi}) \) can be estimated based on either canonical energy-momentum or the gauge invariant one. If we use the canonical energy-momentum tensors defined by (4.1.13), (4.1.14) and (4.1.19)-(4.1.21) we find the following expressions for the respectively the charge energy density \( \tilde{u} \), the EM field energy density \( \tilde{w} \) and the total energy density of the dressed charge \( \tilde{u} + \tilde{w} \):

\[ \tilde{u} + \tilde{w} = (\tilde{u} + \tilde{\psi}^2 \tilde{\varphi}) \]

\[ + \tilde{w} = \frac{\chi^2}{2m} \left[ (\nabla \tilde{\psi})^2 + G (\tilde{\psi}^2) \right] + \tilde{\psi}^2 \tilde{\varphi} - \frac{(\nabla \tilde{\varphi})^2}{8\pi}. \]

(4.2.12)

Using now the results of Section 11.8 including the relation (11.8.15) we obtain the following representation for the total energy of the resting dressed charge

\[ \mathcal{E} (\tilde{\psi}) = \int_{\mathbb{R}^3} (\tilde{u} + \tilde{w}) \, dx = \frac{2}{3} \int_{\mathbb{R}^3} \left[ \frac{\chi^2 (\nabla \tilde{\psi})^2}{2m} - \frac{(\nabla \tilde{\varphi})^2}{8\pi} \right] dx. \]

(4.2.13)
If we wanted to use the gauge invariant energy-momentum tensors (4.1.15)-(4.1.23) for the same evaluation, a choice for \( w_0(x) \) in (4.1.17) consistent with the canonical energy-momentum would be \( w_0(x) = \dot{w} = -\frac{(\nabla \phi)^2}{8\pi} \).

### 4.3 Freely moving charge

We can use the invariance of the Lagrangian \( \hat{L}_0 \) with respect to Galilean-gauge transformations (4.1.1)-(4.1.2) to obtain a freely moving charge solution to the field equations (4.0.11)-(4.0.12) based on the resting charge solution (2.3.5)-(2.3.6) similarly to what is done in the relativistic case where we obtain a freely moving charge solution applying Lorentz transformation to the resting one. Namely, the field equations (2.3.3)-(2.3.4) have the following closed form solution

\[
\psi = \psi(t, x) = e^{iS/\chi} \hat{\psi}(|x - vt|),
\]

\[
S = \frac{m}{2} \left[ v^2 t + 2v \cdot (x - vt) \right], \quad \varphi(t, x) = \dot{\varphi}_0 (|x - vt|),
\]

where \( \psi \) in view of relations (2.3.14) can be also represented as

\[
\psi = \psi(t, x) = e^{iS/\chi} \hat{\psi}(|x - vt|), \quad S = p \cdot x - \frac{p^2 t}{2m}, \quad p = mv.
\]

Solutions of a similar form propagating with a constant speed are well-known in the theory of Nonlinear Schrödinger equations, see [Sulem] and references therein. In what follows we refer to a wave function represented by the formulas (4.3.1), (4.3.2) as a wave-corpuscle. Looking at the exact solution (4.3.1), (4.3.2) to the field equations describing the freely moving charge we observe that it harmoniously integrates the features of the point charge. Indeed, the wave amplitude \( \hat{\psi}(|x - vt|) \) in (4.3.1) is a soliton-like field moving exactly as a free point charge described by its position \( r = vt \). The exponential factor \( e^{iS/\chi} \) is a plane wave with the phase \( S \) that depends only on the point charge position \( vt \) and momentum \( p = mv \), and it does not depend on the nonlinear self-interaction. The phase \( S \) has a term in which we readily recognize the de Broglie wave-vector \( k \) described exactly in terms of the point charge quantities, namely

\[
k = \frac{p}{\chi} = \frac{m}{\chi} v.
\]

Notice that the dispersion relation \( \omega = \omega(k) \) of the linear part of the field equations (4.0.11) for \( \psi \) is

\[
\omega(k) = \frac{\chi k^2}{2m}, \quad \text{implying that the group velocity } \nabla_k \omega(k) = \frac{\chi k}{m}. \quad \text{(4.3.4)}
\]

Combining the expression (4.3.4) for the group velocity \( \nabla_k \omega(k) \) with the expression (4.3.3) for wave vector \( k \) we establish another exact relation

\[
v = \nabla_k \omega(k),
\]

signifying the equality between the point charge velocity \( v \) and the group velocity \( \nabla_k \omega(k) \) at the de Broglie wave vector \( k \). Using the relations (2.3.9) and (4.1.12) we readily obtain the following representations for the micro-charge, the micro-current and momentum densities

\[
\rho(t, x) = q \psi^2(|x - vt|), \quad J(t, x) = qv \psi^2(|x - vt|),
\]

\[
P(t, x) = \frac{m}{q} J(t, x) = p \psi^2(|x - vt|).
\]

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The above expressions and the charge normalization condition (4.0.14) readily imply the following representations for the total dressed charge field momentum \( P \) and the total current \( J \) for the solution (2.3.16) in terms of point charge quantities, namely

\[
P = \frac{m}{q} J = \int_{\mathbb{R}^3} \frac{\chi q}{m} \text{Im} \frac{\nabla \psi}{\psi} \hat{\psi}^2 \, dx = p = mv. \tag{4.3.8}
\]

### 4.4 Nonlinear self-interaction and its basic properties

As we have already explained in the beginning of Section 4, the nonlinear self interaction function \( G \) is determined from the charge equilibrium equation (2.3.8) based on the form factor \( \hat{\psi} \) and the form factor potential \( \hat{\varphi} \). It is worth to point out that such a nonlinearity differs significantly from nonlinearities considered in similar problems in literature. Important features of our nonlinearity include: (i) the boundedness of its derivative \( G'(s) \) for \( s \geq 0 \) with consequent boundedness from below of the wave energy; (ii) non analytic behavior for small \( s \) that is for small wave amplitudes.

In this section we consider the construction of the function \( G \), study its properties and provide examples for which the construction of \( G \) is carried out explicitly. Throughout this section we have \( \psi, \hat{\psi} \geq 0 \) and hence \( |\psi| = \psi \).

We introduce explicitly the size parameter \( a > 0 \) through the following representation of the fundamental functions \( \hat{\psi}_a \) and \( \hat{\varphi}_a \)

\[
\hat{\psi}(r) = \hat{\psi}_a(r) = a^{-3/2} \hat{\psi}_1 \left( a^{-1} r \right),
\]

\[
\hat{\varphi}(r) = \hat{\varphi}_a(r) = a^{-1} \hat{\varphi}_1 \left( a^{-1} r \right),
\]

where \( \hat{\psi}_1(r) \) and \( \hat{\varphi}_1(r) \) are functions of the dimensionless parameter \( r \), and, as a consequence of (4.0.14), the function \( \hat{\psi}_a(r) \) satisfies the charge normalization condition

\[
\int_{\mathbb{R}^3} \hat{\psi}_a^2(|x|) \, dx = 1 \text{ for all } a > 0. \tag{4.4.2}
\]

The size parameter \( a \) naturally has the dimension of length, but we do not yet identify it with the size. Indeed, any properly defined spatial size of \( \hat{\psi}_a \), based, for instance, on the variance or on an energy-based scale as in (4.7.3), is proportional to \( a \) with a coefficient depending on \( \hat{\psi}_1 \). The charge equilibrium equation (2.3.8) can be written in the following form

\[
-\frac{\chi^2}{2m} \nabla^2 \hat{\psi}_a + q \hat{a} \hat{\psi}_a^2 + \frac{\chi^2}{2m} G'(\hat{\psi}_a^2) \hat{\psi}_a = 0, \tag{4.4.3}
\]

\[
-\nabla^2 \hat{\varphi}_a = 4\pi q \left| \hat{\psi}_a \right|^2. \tag{4.4.4}
\]

The function \( \hat{\psi}_a(r) \) is assumed to be a positive, monotonically decreasing function of \( r \geq 0 \), and to satisfy the charge normalization condition (4.4.2). Recall that \( \hat{\psi}_a(|x|) \) and \( \hat{\varphi}_a(|x|) \) are radial functions and consequently, when solving the equation (4.4.4) for \( \hat{\varphi}_a \), we obtain (see Section 4.6 and (4.6.7) for details) the formula

\[
\hat{\varphi}_a(r) = \frac{q}{r} \left[ 1 - \frac{4\pi}{a} \int_{r/a}^\infty \left( r_1 - \frac{r}{a} \right) r_1 \hat{\psi}_1^2(r_1) \, dr_1 \right]. \tag{4.4.5}
\]
Obviously, if $\psi_1^2(r)$ decays sufficiently fast as $r \to \infty$ and $a$ is sufficiently small then the potential $\dot{\varphi}_a(r)$ is very close to the Coulomb’s potential $q/r$, as we show in Section 4.6.

Let us look first at the case $a = 1$, $\psi_a = \psi_1$, $\varphi_a = \dot{\varphi}_1$, for which the equation (4.4.3) yields the following representation for $G' (\psi_1^2)$ from (4.4.3)

$$G' (\psi_1^2 (r)) = \frac{(\nabla^2 \dot{\psi}_1) (r)}{\dot{\psi}_1 (r)} - \frac{2m}{\chi^2} q \dot{\varphi}_1 (r). \quad (4.4.6)$$

Since $\psi_1^2(r)$ is a monotonic function, we can find its inverse $r = r (\psi^2)$, yielding

$$G' (s) = \left[ \frac{\nabla^2 \dot{\psi}_1}{\dot{\psi}_1} - \frac{2m}{\chi^2} q \dot{\varphi}_1 \right] (r (s)), \quad 0 = \psi_1^2 (\infty) \leq s \leq \psi_1^2 (0). \quad (4.4.7)$$

We extend $G' (s)$ for $s \geq \psi_1^2 (0)$ to be a constant, namely

$$G' (s) = G' (\psi_1^2 (\infty)) \text{ if } s \geq \psi_1^2 (\infty). \quad (4.4.8)$$

Observe that the positivity and the monotonicity of the form factor $\dot{\psi}_1$ was instrumental for recovering the function $G' (s)$ from the charge balance equation (4.4.3).

Using the representation (4.4.7) for the function $G' (s)$ we decompose it naturally into two components:

$$G' (s) = G'_\nabla (s) - \frac{2}{a_\chi} G'_{\varphi} (s), \quad (4.4.9)$$

where

$$a_\chi = \frac{\chi^2}{mq^2} \quad (4.4.10)$$

and for all $r \geq 0$

$$G'_{\nabla} (\psi_1^2) (r) = \frac{(\nabla^2 \dot{\psi}_1)}{\dot{\psi}_1} (r), \quad G'_{\varphi} (\psi_1^2) (r) = \frac{\dot{\varphi}_1 (r)}{q} = \dot{\varphi}_1 (r). \quad (4.4.11)$$

We refer to $G'_{\nabla} (s)$ and $G'_{\varphi} (s)$ as elastic and EM components respectively. In the case of arbitrary size parameter $a$ we find first that

$$G'_{\nabla,a} (s) = a^{-2} G'_{\nabla,1} (a^3 s), \quad G'_{\varphi,a} (s) = a^{-1} G'_{\varphi,1} (a^3 s), \quad a > 0, \quad (4.4.12)$$

and then combining (4.4.12) with (4.4.9) and (4.4.11) we obtain the following representation for the function $G'_a (s)$

$$G'_a (s) = \frac{G'_{\nabla,1} (a^3 s)}{a^2} - \frac{2G'_{\varphi,1} (a^3 s)}{aa_\chi}. \quad (4.4.13)$$

Let us take a look at general properties of $G' (s)$ and its components $G'_{\nabla} (s)$ and $G'_{\varphi} (s)$ as they follow from defining them relations (4.4.7)-(4.4.13). Starting with the EM component $G'_{\varphi} (s)$ we notice that $\dot{\varphi}_1 (|x|)$ is a radial solution to the equation (4.4.4). Combining that with $\psi^2 \geq 0$ and using the Maximum principle we conclude that $\dot{\varphi}_1 (|x|)/q$ is a positive function without local minima, implying that it is a monotonically decreasing function of $|x|$. 37
Consequently, $G'_{\varphi}(s)$ defined by (4.4.11) is a monotonically increasing function of $s$, and hence

$$G'_{\varphi}(s) > 0 \text{ for all } s > 0 \text{ and } G'_{\varphi}(0) = 0. \tag{4.4.14}$$

Note that $G'_{\varphi}(s)$ is not differentiable at zero, which can be seen by comparing the behavior of $\varphi_1(r)$ and $\dot{\varphi}_1(r)$ at infinity. Indeed, $\varphi_1(r) / q \sim r^{-1}$ as $r \to \infty$ and since $\dot{\varphi}_1^2(|x|)$ is integrable, it has to decay faster than $|x|^{-3}$ as $|x| \to \infty$. Consequently, $|G'_{\varphi}(s)|$ for small $s$ has to be greater than $s^{1/3}$ which prohibits its the differentiability at zero. One has to notice though that the nonlinearity $G'(|\psi|^2)\psi$ as it enters the field equation (4.0.13) is differentiable for all $\psi$ including zero, hence it satisfies a Lipschitz condition required for uniqueness of solutions of initial value problem for (4.0.13).

Let us look at the elastic component $G'_{\nabla}(s)$ defined by the relations (4.4.11). Since $\dot{\psi}(|x|) > 0$ the sign of $G'_{\nabla}(|\psi|^2)$ coincides with the sign of $\nabla^2 \dot{\psi}_1(|x|)$. At the origin $x = 0$ the function $\dot{\psi}_1(|x|)$ has its maximum and, consequently, $G'_{\nabla}(s) \leq 0$ for all $s$ close to $s = \psi_1^2(\infty)$, implying

$$G'_{\nabla}(s) \leq 0 \text{ for } s \gg 1. \tag{4.4.15}$$

The Laplacian applied to radial functions $\dot{\psi}_1$ takes the form $\frac{1}{r} \frac{\partial^2}{\partial r^2} \left(r \dot{\psi}_1 |x| \right)$. Consequently, if $r \dot{\psi}_1(r)$ is convex at $r = |x|$ we have $\nabla^2 \dot{\psi}_1(|x|) \geq 0$. Since $r^2 \dot{\psi}(r)$ is integrable we can naturally assume that $|x| \dot{\psi}_1(|x|) \to 0$ as $|x| \to \infty$. Then if the second derivative of $r \dot{\psi}_1(r)$ has a constant sign near infinity, it must be non-negative. For an exponentially decaying $\dot{\psi}_1(r)$ the second derivative of $r \dot{\psi}_a(r)$ is positive implying

$$G'_{\nabla}(s) > 0 \text{ for } s \ll 1. \tag{4.4.16}$$

Combining this with the equality $G'_{\varphi}(0) = 0$ from (4.4.14) we readily obtain

$$G'(s) > 0 \text{ for } s \ll 1. \tag{4.4.17}$$

From the relations (4.4.9), (4.4.14), (4.4.15) we also obtain

$$G'(s) < 0 \text{ if } s \gg 1. \tag{4.4.18}$$

We remind the reader that the sign of the $G'(s)$ according to the representation (4.2.9) for nonlinear self-interaction force density $f_{nl}^a$ controls its direction.

### 4.5 Examples of nonlinearities

In this section we provide two examples of the form factor $\dot{\psi}$ for which the form factor potential $\dot{\varphi}_0$ and the corresponding nonlinear self-interaction function $G$ can be constructed explicitly. The first example is for the form factor $\dot{\psi}(r)$ decaying as a power law as $r \to \infty$. In this case both $\dot{\varphi}_0$ and $G$ are represented by rather simple, explicit formulas, but some properties of these functions are not as appealing. Namely, the variance of the function $\psi$ is infinite and the rate of approximation of the exact Coulomb’s potential by $\dot{\varphi}_a(x)$ for small $a$ is not as fast. The second example is for the form factor $\dot{\psi}(r)$ decaying exponentially as $r \to \infty$. In this case the representations for $\dot{\varphi}_0$ and $G$ are more involved compared with the power law form factor but all the properties of $\dot{\psi}$ and $\dot{\varphi}$ are satisfactory in any regard.
4.5.1 Nonlinearity for the form factor decaying as a power law

We introduce here a form factor \( \psi_1(r) \) decaying as a power law of the form

\[
\psi_1(r) = \frac{c_{pw}}{(1 + r^2)^{5/4}}, \tag{4.5.1}
\]

where \( c_{pw} \) is the normalization factor,

\[
c_{pw} = \frac{3^{1/2}}{(4\pi)^{1/2}}.
\]

This function evidently is positive and monotonically decreasing as required. Let us find now \( G'_V(s) \) and \( G'_\varphi \) based on the relations (4.4.11). An elementary computation shows that

\[
\nabla^2 \psi_1 = \frac{15}{4c_{pw}^{4/5}} \left( 1 - \frac{3}{c_{pw}} \frac{\psi_{1/5}}{\psi_1} \right) \psi_1^{1+4/5},
\]

implying

\[
G'_V(s) = \frac{15s^{2/5}}{4c_{pw}^{4/5}} - \frac{45s^{4/5}}{4c_{pw}^{8/5}}, \quad G_V(s) = \frac{75s^{7/5}}{28c_{pw}^{4/5}} - \frac{25s^{9/5}}{4c_{pw}^{8/5}}, \quad \text{for } 0 \leq s \leq c_{pw}^2. \tag{4.5.2}
\]

To determine \( G'_\varphi \) we find by a straightforward examination that function

\[
\varphi_1 = \frac{q}{c_{pw}^{2/5}} \psi_1^{2/5}, \tag{4.5.3}
\]

solves equation

\[
\nabla^2 \varphi_1 = -4\pi q \psi_1^2,
\]

and that together with (4.4.11) yields

\[
G'_\varphi(s) = \frac{s^{1/5}}{c_{pw}^{2/5}}, \quad G_\varphi(s) = \frac{5s^{6/5}}{6c_{pw}^{2/5}}, \quad \text{for } 0 \leq s \leq c_{pw}^2. \tag{4.5.4}
\]

Observe that the both components \( G'_V(s) \) and \( G'_\varphi(s) \) in (4.5.2), (4.5.4) of the total nonlinearity \( G'_s(s) \) defined by (4.4.9) are not differentiable at \( s = 0 \).

If we explicitly introduce size parameter \( a \) into the form factor, namely

\[
\psi_a(r) = \frac{ac_{pw}}{(a^2 + r^2)^{5/4}}, \tag{4.5.5}
\]

then combining (4.5.2), (4.5.4) with (4.4.12) we obtain the following representation for the nonlinearity components

\[
G'_V(a) = \frac{15a^{2/5}}{4a^{4/5}c_{pw}^{4/5}} - \frac{45a^{2/5}a^{4/5}}{4c_{pw}^{8/5}}, \tag{4.5.6}
\]

\[
G_\varphi(s) = \frac{75a^{7/5}}{28a^{4/5}c_{pw}^{4/5}} - \frac{25a^{2/5}a^{9/5}}{4c_{pw}^{8/5}}, \quad \text{for } 0 \leq s \leq c_{pw}^2 a^{-3}.
\]
\[ G'_{\varphi, a}(s) = \frac{s^{1/5}}{a^{2/5} c_{pw}^{2/5}}, \quad G_{\varphi}(s) = \frac{5s^{6/5}}{6a^{2/5} c_{pw}^{2/5}} \text{ for } 0 \leq s \leq c_{pw}^2 a^{-3}. \quad (4.5.7) \]

Notice that the variance of the form factor \( \psi_1^2 (|x|) \) decaying as a power law (4.5.1) is infinite, i.e.
\[
\int_{\mathbb{R}^3} |x|^2 \psi_1^2 (|x|) \, dx = 4\pi \int_0^\infty \frac{c_{pw}^2}{(1 + r^2)^{5/2}} r^4 \, dr = 3 \int_0^\infty \frac{1}{(1 + r^2)^{5/2}} r^4 \, dr = \infty. \quad (4.5.8)
\]

### 4.5.2 Nonlinearity for the form factor decaying exponentially

We introduce here an exponentially decaying form factor \( \psi_1 \) of the form
\[
\psi_1 (r) = c e^{-\left(\frac{r^2}{2} + 1\right)^{1/2}} \quad (4.5.9)
\]
where \( c \) is the normalization factor,
\[
c = \left(4\pi \int_0^\infty r^2 e^{-2\left(\frac{r^2}{2} + 1\right)^{1/2}} \, dr\right)^{-1/2} \approx 0.79195.
\]
Evidently \( \psi_1 (r) \) is positive and monotonically decreasing as required. The dependence \( r (s) \) defined by the relation (4.5.9) is as follows:
\[
r = \left[\ln^2 \left(\frac{c e}{\sqrt{s}}\right) - 1\right]^{1/2}, \text{ if } \sqrt{s} \leq \psi_1 (0) = c e^{-1}. \quad (4.5.10)
\]
An elementary computation shows that \( \nabla^2 \psi_1 = -W \psi_1 \) where
\[
W = \frac{2}{(r^2 + 1)^{1/2}} + \frac{1}{(r^2 + 1)} + \frac{1}{(r^2 + 1)^{3/2}} - 1, \quad (4.5.11)
\]
implying
\[
G'_{\nabla} \left( \psi_1^2 (r) \right) = -W (r) = 1 - \frac{2}{(r^2 + 1)^{1/2}} - \frac{1}{(r^2 + 1)} - \frac{1}{(r^2 + 1)^{3/2}}. \quad (4.5.12)
\]
Combining (4.5.10) with (4.5.12) we readily obtain the following function for \( \sqrt{s} \leq c e^{-1} \approx 0.29134 \)
\[
G'_{\nabla, 1} (s) = \left[1 - \frac{4}{\ln (c e^2 / s)} - \frac{4}{\ln^2 (c e^2 / s)} - \frac{8}{\ln^3 (c e^2 / s)}\right], \quad (4.5.13)
\]
which is evidently monotonically decreasing. We extend it for larger \( s \) as follows:
\[
G'_{\nabla, 1} (s) = G'_{\nabla, 1} \left( c e^2 e^{-2} \right) = -3 \text{ if } \sqrt{s} \geq c e^{-1}. \quad (4.5.14)
\]
The relations (4.5.13) and (4.5.14) imply \( G'_{\nabla, 1} (s) \) takes values in the interval \([1, -3]\). It also follows from (4.5.13) that
\[
G'_{\nabla, 1} (s) \approx 1 - \frac{4}{\ln 1/s} \text{ as } s \to 0, \quad (4.5.15)
\]
implying that the function \( G'_{\nabla, 1} (s) \) is not differentiable at \( s = 0 \) and consequently is not analytic about 0.
To determine the second component \( G'_\varphi \) we need to solve (4.4.11). Using the fact that \( \varphi_1, \psi_1 \) are radial functions we obtain the following equation for \( \varphi_1(r) \)

\[
-\frac{1}{4\pi r} \partial_r^2 (r\varphi_1) = qe^2 e^{-2\sqrt{r^2+1}}.
\] (4.5.16)

We seek a solution of equation (4.5.16) that is regular at zero and behaves like the Coulomb’s potential \( \frac{\varphi}{r} \) for large \( r \). Taking that into account we obtain after the first integration of (4.5.16)

\[
\partial_r (r\varphi_1) = \pi q e^2 [1 + 2 (r^2 + 1)^{1/2}] e^{-2(r^2+1)^{1/2}},
\] (4.5.17)

and integrating (4.5.17) yields the ultimate formula for the form factor potential:

\[
\varphi_1(r) = \frac{q}{r} - \frac{\pi q e^2}{r} \int_0^r \left[ 1 + 2 (r^2 + 1)^{1/2} \right] e^{-2(r^2+1)^{1/2}} \, dr_1
\] (4.5.18)

The above formula shows that the form factor potential \( \varphi_1(r) \) is exponentially close to the Coulomb’s potential \( q/r \) for large \( r \). But if we use the substitution \( (r^2+1)^{1/2} = u \) in the second integral in (4.5.18) we can recast \( \varphi_1(r) \) in even more convenient form for estimations of its proximity to the Coulomb’s potential \( q/r \), namely

\[
\varphi_1(r) = \frac{q}{r} - \frac{\pi q e^2}{r} \int_{(r^2+1)^{1/2}}^\infty \left[ \frac{(2u + 1)}{(1 - u^2)^{1/2}} - 2u \right] e^{-2u} \, du.
\] (4.5.19)

Then, based on the relation (4.4.11) and (4.5.18), we find consequently

\[
G'_{\varphi,1}(\psi^2) = \frac{1}{q} \varphi_1(r(\psi)) = \frac{\pi q e^2}{r(\psi)} \int_0^{r(\psi)} \left[ 1 + 2 (r^2 + 1)^{1/2} \right] e^{-2(r^2+1)^{1/2}} \, dr_1,
\] (4.5.20)

where

\[
r(\psi) = \left[ \ln^2(c_e/\psi) - 1 \right]^{1/2} \text{ for } \psi \leq c_e e^{-1}.
\]

We extend \( G'_{\varphi,1}(\psi^2) \) for larger values of \( \psi \) as a constant:

\[
G'_{\varphi,1}(\psi^2) = \lim_{r \to 0} \frac{\varphi(r)}{q} = 3\pi c_e^2 e^{-2} \simeq 0.79998, \text{ for } \psi \geq c_e e^{-1}.
\] (4.5.21)

Using the representation (4.5.19) we obtain the following formula for \( G'_{\varphi,1}(\psi^2) \)

\[
G'_{\varphi,1}(\psi^2) = \frac{1}{\left[ \ln^2(c_e/\psi) - 1 \right]^{1/2}} - \frac{\pi \left[ 1 + 2 \ln(c_e/\psi) \right] \psi^2}{\left[ 2 \ln^2(c_e/\psi) - 1 \right]^{1/2}}
\]

\[-\frac{\pi c_e^2}{\left[ \ln^2(c_e/\psi) - 1 \right]^{1/2}} \int_0^{\infty} \left[ \frac{(2u + 1)}{(1 - u^2)^{1/2}} - 2u \right] e^{-2u} \, du.
\] (4.5.22)
To find $G'_a (s)$ for arbitrary $a$ we use its representation (4.4.13), i.e.

$$G'_a (s) = \frac{G'_{\nabla,1} (a^3 s)}{a^2} - \frac{2G'_{\varphi,1} (a^3 s)}{aa\chi}$$

(4.5.23)

and combine with the formulas (4.5.13) and (4.5.22). We don’t write the final formula since it is quite long but it is clear from formulas (4.5.13) and (4.5.22) that $G'_a (s)$ does not depend analytically on $s$ at $s = 0$, and that the following asymptotic formula holds

$$G'_a (s) = \frac{1}{a^2} - \left( \frac{1}{a^2} + \frac{1}{aa\chi} \right) \frac{4}{\ln (c_2^2 / (a^3 s))}$$

for $s \to 0$. (4.5.24)

The variance of the exponential form factor $\tilde{\psi}_1 (r)$ is

$$\int_{\mathbb{R}^3} |x|^2 \tilde{\psi}_1^2 (|x|) \ dx = 4\pi \int_0^\infty r^4 e^{-2(r^2+1)^{1/2}} \ dr \simeq 3.8268.$$ (4.5.25)

4.6 Form factor potential proximity to the Coulomb’s potential

In this subsection we study the proximity of the potential form factor $\tilde{\varphi}_a (|x|)$ to the Coulomb’s potential $q / |x|$ for small $a$. This is an important issue since it is a well known experimental fact that the Coulomb’s potential $q / |x|$ represents the electrostatic field of the charge very accurately even for very small values of $|x|$.

According to the rest charge equation (2.3.8) and the equation (2.3.7) the potential $\tilde{\varphi}_a (|x|) = \tilde{\varphi}_a (|x|) / q$ satisfies

$$\nabla^2 \tilde{\varphi}_a (|x|) = -4\pi \tilde{\psi}_a^2 (|x|),$$

hence

$$\tilde{\varphi}_a (|x|) = \int_{\mathbb{R}^3} \frac{\tilde{\psi}_a^2 (|y|)}{|x - y|} \ dy > 0.$$ (4.6.1)

In view of the relations (4.4.1) the dependence of the potential $\tilde{\varphi}_a (r)$ on the size parameter $a$ is of the form

$$\tilde{\varphi}_a (r) = a^{-1} \phi_1 (a^{-1} r),$$ (4.6.2)

and consequently its behavior for small $a$ is determined by the behavior of $\phi_1 (r)$ for large $r$.

To find the latter, consider the radial solution $\zeta (r)$ to the Poisson equation

$$\frac{1}{r} \left( \frac{d}{dr} \right)^2 \zeta (r) = -4\pi \tilde{\psi}_1^2 (r), \ \zeta (r) = r\phi_1 (r), \ r \geq 0.$$ (4.6.3)

We seek a solution $\zeta (r)$ to the above equation that is close to the Coulomb’s potential $1/r$ and hence satisfies the following condition

$$\zeta (r) = r\phi_1 (r) \to 1 \text{ as } r \to \infty.$$ (4.6.4)

Taking into account (4.6.4) when integrating equation (4.6.3) twice yields

$$\zeta (r) = 1 - 4\pi \int_r^\infty \int_{r_2}^\infty r_1 \tilde{\psi}_1^2 (r_1) \ dr_1 dr_2 = 1 - 4\pi \int_r^\infty (r_1 - r) r_1 \tilde{\psi}_1^2 (r_1) \ dr_1,$$ (4.6.5)
where the second equality in (4.6.5) is obtained by rewriting the preceding repeated integral as as a double integral and changing the order of integration, namely
\[
\int_r^\infty \int_{r_2}^\infty r_1 \psi_1^2 (r_1) \, dr_2 \, dr_1 = \int_r^\infty \int_r^r r_1 \psi_1^2 (r_1) \, dr_2 \, dr_1 = \int_r^\infty (r_1 - r) r_1 \psi_1^2 (r_1) \, dr_1.
\]

In view of the charge normalization condition (4.4.2) we readily obtain from (4.6.5)
\[
\zeta (0) = 1 - 4\pi \int_{r}^{\infty} r_1^2 \psi_1^2 (r_1) \, dr_1 = 1 - \int_{\mathbb{R}^3} \psi_1^2 (|x|) \, dx = 0. \tag{4.6.6}
\]

The representation (4.6.5) for \( \zeta (r) = r \phi_1 (r) \) readily implies the following representation for the potential \( \phi_1 (r) \):
\[
\phi_1 (r) = \frac{1}{r} \left[ 1 - 4\pi \int_r^\infty (r_1 - r) r_1 \psi_1^2 (r_1) \, dr_1 \right]. \tag{4.6.7}
\]

Combining (4.6.7) with (4.6.6) we conclude that \( \phi_1 (r) \) is regular for small \( r \geq 0 \). Using (4.6.7) once more we obtain the following expression for the difference \( D_C \) between \( \phi_1 (r) \) and \( 1/r \)
\[
D_C (\phi_1) = \phi_1 (r) - \frac{1}{r} = \frac{4\pi}{r} \int_r^\infty (r_1 - r) r_1 \psi_1^2 (r_1) \, dr_1. \tag{4.6.8}
\]

The relation (4.6.8) together with (4.6.2) implies
\[
D_C (\phi_a) = \phi_a (r) - \frac{1}{r} = \frac{4\pi}{r} \int_{a^{-1} r}^\infty (r_1 - a^{-1} r) r_1 \psi_1^2 (r_1) \, dr_1, \tag{4.6.9}
\]

showing in particular that the difference \( D_C \) becomes small for small \( a \). More precisely, if \( \psi_1^2 \) decays exponentially as in (4.5.9) then
\[
|D_C (\phi_a) (r)| = \left| \phi_a (r) - \frac{1}{r} \right| \leq \frac{4\pi}{r} \int_{a^{-1} r}^\infty (r_1 - a^{-1} r) r_1 c_e^2 e^{-2(r_1^2 + 1)^{1/2}} \, dr_1 \leq \frac{4\pi c_e^2}{r} \int_0^\infty (r_1 + a^{-1} r) r_1 e^{-2r_1} \, dr_1 = \frac{\pi c_e^2 (a^{-1} r + 1)}{r} e^{-2a^{-1} r}. \tag{4.6.10}
\]

For instance, for \( r \geq 10a \) the difference \( D_C \) between the potential \( \phi_a (r) \) and the Coulomb’s potential \( 1/r \) is extremely small:
\[
|D_C (\phi_a) (r)| = \pi c_e^2 (a^{-1} r + 1) e^{-2a^{-1} r} \lesssim 4.4674 \times 10^{-8}. \tag{4.6.11}
\]

Similar estimates for the power law decaying \( \psi_1^2 \) as in (4.5.1) yields
\[
|D_C (\phi_a) (r)| = \left| \phi_a (r) - \frac{1}{r} \right| \leq \frac{1}{r} \int_{a^{-1} r}^\infty \frac{3 (r_1 - a^{-1} r) r_1}{r_1^3} \, dr_1 = \frac{1}{r} \left( \frac{3a^2}{2r^2} - \frac{3a^3}{5r^3} \right), \tag{4.6.12}
\]

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implying

\[ |D_C (\phi_a)| \leq \frac{0.009}{r}, \text{ for } r \geq 10a. \]

Notice that if we would take \( \hat{\psi}_1 (r) = 0 \) for all \( r \geq r_0 \), as it is the case in the Abraham-Lorentz model, the formula (4.6.7) would imply that \( \hat{\varphi}_a (r) \) would be exactly the Coulomb’s potential for \( r \geq a r_0 \). But for such a \( \hat{\psi}_1 (r) \) we would not be able to construct the nonlinear self-interaction component \( G_\varphi' \) which would satisfy (4.4.11) since it requires \( \hat{\psi}_1^2 (r) \) to be strictly positive for all \( r \geq 0 \).

### 4.7 Energy related spacial scale

An attractive choice for the spacial scale can be obtained based on the requirement that the total energy \( \mathcal{E} (\hat{\psi}) \) of the resting dressed charge defined by the expression (4.2.13) be exactly 0, which readily reduces to the requirement

\[ \mathcal{E}_1 (\hat{\psi}) = \mathcal{E}_2 (\hat{\varphi}), \quad (4.7.1) \]

where

\[ \mathcal{E}_1 (\hat{\psi}) = \frac{\lambda^2}{2m} \int_{\mathbb{R}^3} \left| \nabla \hat{\psi} \right|^2 \, dx, \quad \mathcal{E}_2 (\hat{\varphi}) = \frac{1}{8\pi} \int_{\mathbb{R}^3} (\nabla \hat{\varphi})^2 \, dx \]

(this condition is similar to (2.1.11)). Plugging \( \hat{\psi} = \hat{\psi}_a \) and \( \hat{\varphi} = \hat{\varphi}_a \) defined by (4.4.1) into the equalities (4.7.1) we obtain

\[ \mathcal{E}_1 (\hat{\psi}_a) = a^{-2} \mathcal{E}_1 (\hat{\psi}_1), \quad \mathcal{E}_2 (\hat{\varphi}_a) = a^{-1} \mathcal{E}_2 (\hat{\varphi}_1) = a^{-1} q^2 \mathcal{E}_2 (\hat{\varphi}_1), \quad \hat{\varphi}_1 = q^{-1} \hat{\varphi}_1. \quad (4.7.2) \]

Hence, the requirement \( \mathcal{E}_1 (\hat{\psi}) = \mathcal{E}_2 (\hat{\varphi}) \) in view of the relations (4.7.2) is equivalent to the following choice \( a = a_\psi \) of size parameter \( a \), with

\[
\begin{align*}
    a_\psi &= \frac{\mathcal{E}_1 (\hat{\psi}_1)}{\mathcal{E}_2 (\hat{\varphi}_1)} = \frac{4\pi\lambda^2}{m} \int_{\mathbb{R}^3} \left| \nabla \hat{\psi}_1 \right|^2 \, dx = \alpha \chi \theta \psi, \\
    \theta \psi &= \frac{4\pi \int_{\mathbb{R}^3} \left| \nabla \hat{\psi}_1 \right|^2 \, dx}{\int_{\mathbb{R}^3} \left( \nabla \hat{\varphi}_1 \right)^2 \, dx}, \quad \alpha = \frac{\lambda^2}{mq^2}. 
\end{align*}
\]

Since the functions \( \hat{\psi}_1, \hat{\varphi}_1 \) in the above relations are radial, the Dirichlet integrals in (4.7.3) can be recast as

\[ \int_{\mathbb{R}^3} \left| \nabla \hat{\psi}_1 \right|^2 \, dx = 4\pi \int_0^\infty \left( \partial_r \left( r \hat{\psi}_1 (r) \right) \right)^2 \, dr \quad (4.7.4) \]

with the similar formula for \( \hat{\varphi}_1 \). We refer the space scale \( a_\psi \) in (4.7.3) obtained based on the equality \( \mathcal{E}_1 (\hat{\psi}) = \mathcal{E}_2 (\hat{\varphi}) \) as energy-based spacial scale.

The energy-based spatial scale \( a_\psi \) defined by (4.7.3) for the power law form factor (4.5.1) \( a_\psi = \theta \psi a_\chi \), with

\[
\theta \psi = \frac{4\pi \int_{\mathbb{R}^3} \left( \nabla \hat{\psi}_1 \right)^2 \, dx}{\int_{\mathbb{R}^3} \left( \nabla \hat{\varphi}_1 \right)^2 \, dx} = \frac{40}{7\pi} \approx 1.8189. \quad (4.7.5)
\]
For the exponentially decaying form factor we get $a_\psi = \theta_\psi a_\chi$ with 

$$\theta_\psi \approx 1.2473$$

Note that the energy based spatial scales for power law form factor and exponential form factor are of the same order, though their variances are absolutely different (infinite variance for the power law as in (4.5.1)).

5 Accelerated motion of a single nonrelativistic charge in an external EM field

The key objective of this section is an extension of the wave-corpuscle representation defined by formulas (4.3.1), (4.3.2) to the case of a single nonrelativistic charge accelerating in an external EM field. Here we discuss in detail results sketched in Sections 2.3, 2.4. Recall that as in (4.0.6) we neglect the charge’s own magnetic field and set $\mathbf{A} = 0$ taking into account only external magnetic field. In the case of a general external EM field no exact closed form solution to the field equations seems to be available, but there is an approximate wave-corpuscle solution and its accuracy is a subject of our studies in this case; this solution is exact for special external fields.

The total EM fields are described by their potentials $\varphi, \mathbf{A}$ which involve potentials of the external field and particle’s own field, namely

$$\varphi = \varphi_{\text{ex}} + \varphi, \quad \mathbf{A} = \mathbf{A}_{\text{ex}}. \quad (5.0.6)$$

The nonrelativistic Lagrangian $\hat{L}_0$ for the charge in external field is obtained from the one for the free charge in (4.0.6) by modifying there the covariant derivative to include the external potential, namely

$$\hat{L}_0(\psi, \psi^*, \varphi) = \frac{\chi}{2} \left[ \psi^* \tilde{\partial}_t \psi - \psi \tilde{\partial}_t^* \psi^* \right] - \frac{\chi^2}{2m} \left\{ \tilde{\nabla} \psi \tilde{\nabla}^* \psi^* + G(\psi^* \psi) \right\} - \frac{|\nabla \varphi|^2}{8\pi}, \quad (5.0.7)$$

where

$$\tilde{\partial}_t = \partial_t + \frac{i q \varphi}{\chi}, \quad \tilde{\nabla} = \nabla - \frac{i q \mathbf{A}_{\text{ex}}}{\chi c}, \quad \tilde{\partial}_t^* = \partial_t - \frac{i q \varphi}{\chi}, \quad \tilde{\nabla}^* = \nabla + \frac{i q \mathbf{A}_{\text{ex}}}{\chi c}.$$ 

This modified Lagrangian remains to be gauge invariant with respect to the transformations (4.0.9) and the general formulas (11.7.12)-(11.7.15) for the charge and current densities applied to the Lagrangian $\hat{L}_0$ yield $J^\mu = (\rho, \mathbf{J})$ with

$$\rho = q \psi \psi^*, \quad \mathbf{J} = \frac{\chi q}{2m} \left[ \psi \tilde{\nabla}^* \psi^* - \psi^* \tilde{\nabla} \psi \right] = \frac{\chi q}{2m} \left( \frac{\chi}{m} \frac{\nabla \psi}{\psi} - \frac{q^2 \mathbf{A}_{\text{ex}}}{mc} \right) |\psi|^2. \quad (5.0.8)$$

This current satisfies the conservation/continuity equations $\partial_\nu J^\nu = 0$ which take the form

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0. \quad (5.0.9)$$
The Euler-Lagrange field equations in this case are

\[ \chi \partial_t \psi = \frac{\chi^2}{2m} \left[ -\nabla^2 \Delta \psi + G' (\psi^* \psi) \psi \right], \quad (5.0.10) \]

\[ -\Delta \varphi = 4\pi q \psi \psi^*, \quad (5.0.11) \]

where \( G' (s) = \partial_s G \) and, as always, \( \psi^* \) is complex conjugate to \( \psi \). Then the field equations (5.0.10)-(5.0.11) can be recast as the following field equations

\[ i \chi \partial_t \psi = -\frac{\chi^2 \nabla^2 \psi}{2m} - \frac{\chi q A_{ex} \cdot \nabla \psi}{mc} + q \left( \varphi + \varphi_{ex} + \frac{q A_{ex}^2}{2mc^2} \right) \psi + \frac{\chi^2 G' \psi}{2m}, \quad (5.0.12) \]

\[ \nabla^2 \varphi = -4\pi q |\psi|^2. \quad (5.0.13) \]

As in the case of a free charge we set the total conserved charge to be exactly \( q \) and, similarly to (4.0.14), we have the following charge normalization condition

\[ \int_{\mathbb{R}^3} |\psi|^2 \, dx = 1. \quad (5.0.14) \]

The presence of the external EM field turns the dressed charge into an open system with consequent subtleties in the treatment of the energy-momentum. All elements of the proper treatment of the energy and momentum densities in such a situation are provided in Section 11.7 and we apply them to the Lagrangian \( \hat{L}_0 \) defined by (5.0.7). An instrumental element in the analysis of the energy-momentum tensor is its partition between the charge and the EM field. In carrying out such a partition we are guided by two principles: (i) both the energy-momenta tensors and the forces have to be gauge invariant; (ii) the forces must be of the Lorentz form. The second principle is evidently special to the EM system consisting of the charge and the EM field.

### 5.1 Wave-corpuscle concept for an accelerating charge

In Section 4.3 we introduced a wave-corpuscle by the relations (4.3.1), (4.3.2) for a free moving dressed charge. In this section we study wave-corpuscles in an external EM field. Recall that the wave-corpuscle (4.3.1), (4.3.2) for a free moving dressed charge is an exact solution to the fields equations (2.3.3), (2.3.4), and when constructing the wave-corpuscle for a dressed charge in external EM field we also want it to be an exact solutions to the field equation (5.0.12)-(5.0.13). It turns out that it is possible if the external EM field is a homogeneous electric field, but no closed form solution seems to be available for a general external EM field as defined by its potentials \( \varphi_{ex}, A_{ex} \). To describe explicitly the class of external EM fields for which wave-corpuscles are exact solutions, we introduce auxiliary field equations. If the coefficients of the auxiliary field equations are linear functions of spatial variables, we write explicit solutions. The coefficients of the auxiliary field equations are in a simple, explicit correspondence with the coefficients of the original field equations, therefore we obtain wave-corpuscles which exactly solve original ones (5.0.12)-(5.0.13) for certain classes of external EM fields. If external EM fields are general, we construct a wave-corpuscle so that it exactly solves properly defined auxiliary field equations which differ from the original ones (5.0.12)-(5.0.13) by an explicitly defined discrepancy \( D \), based on which we judge the accuracy of
the approximation. Executing this plan we introduce the following system of auxiliary field equations

\[
\begin{align*}
\imath \chi \partial_t \psi &= -\frac{\chi^2 \nabla^2 \psi}{2m} - \frac{\chi q \tilde{A}_{ex} \cdot \nabla \psi}{mc} + q \left( \phi + \tilde{\varphi}_{ex} \right) \psi + \frac{\chi^2 G' \psi}{2m}, \\
\nabla^2 \phi &= -4\pi q |\psi|^2,
\end{align*}
\]

where the auxiliary potentials \( \tilde{\varphi}_{ex} (t, x) \), \( \tilde{A}_{ex} (t, x) \) are linear in \( x \) and generally may differ from the original potentials \( \varphi_{ex} \), \( A_{ex} \). Evidently, in addition to the alteration of the potentials \( \varphi_{ex}, A_{ex} \), the auxiliary field equations differ from the original ones (5.0.12), (5.0.13) only by a single term \( \chi q^2 A_{ex}^2 \psi^2 / (2mc^2) \) (which, in fact, is absorbed in \( \tilde{\varphi}_{ex} \)).

We define the wave-corpuscle \( \psi, \varphi \) by the formula similar to (4.3.1), (4.3.2), namely

\[
\psi (t, x) = e^{iS/x} \hat{\psi}, \quad S = m \mathbf{v} (t) \cdot [x - r (t)] + s_p (t) , \quad \hat{\psi} = \psi ([x - r (t)]), \quad \varphi = \tilde{\varphi} ([x - r (t)]).
\]

In the above formula \( \hat{\psi} \) and \( \tilde{\varphi} \) are, respectively, the form factor and the form factor potential satisfying (2.3.5), (2.3.6); the function \( r (t) \) is determined by the following complementary point charge equations

\[
m \frac{d^2 \mathbf{r}}{dt^2} = q \mathbf{E}_{ex} (t, \mathbf{r}) + \frac{q}{c} \frac{dr}{dt} \times \mathbf{B}_{ex} (t, \mathbf{r}), \tag{5.1.3}
\]

\[
r (0) = r_0, \quad \frac{dr}{dt} (0) = \dot{r}_0
\]

where

\[
\mathbf{E}_{ex} = -\nabla \varphi_{ex} - \frac{\partial \mathbf{A}_{ex}}{c}, \quad \mathbf{B}_{ex} = \nabla \times \mathbf{A}_{ex},
\]

are based on the EM potentials of the original equations (5.0.12)-(5.0.13) and \( r_0, \dot{r}_0 \) are initial data. The functions \( \mathbf{v} (t), s_p (t) \) in (5.1.2) are determined based on the solution \( r (t) \) by the formulas

\[
\mathbf{v} (t) = \frac{d\mathbf{r}}{dt} (t) + \frac{q}{mc} \mathbf{A}_{ex} (r (t)), \quad \frac{ds_p}{dt} = \frac{mv^2 (t)}{2} - q \varphi_{ex} (\mathbf{r} (t)). \tag{5.1.4}
\]

We readily recognize in the equation (5.1.3) the point charge motion in the external EM field equation. Notice that the function \( \mathbf{v} (t) \) defined by the first equation in (5.1.4) if \( \mathbf{A}_{ex} \neq 0 \) is not the charge velocity \( \dot{\mathbf{r}} (t) \), but it is simply related to canonical momentum \( \mathbf{p} \) (see (11.1.18)-(11.1.22)) by the formula

\[
\mathbf{v} (t) = \frac{\dot{\mathbf{p}} (t)}{m}, \quad \mathbf{p} = \mathbf{p} + \frac{q}{c} \mathbf{A}_{ex}, \quad \text{where} \quad \mathbf{p} = m \frac{d\mathbf{r}}{dt} \text{ is the kinetic momentum.} \tag{5.1.5}
\]

We refer to the function \( r (t) \) as wave-corpuscle center or wave-corpuscle position. Note that since \( \hat{\psi} \) is center-symmetric, this definition agrees with (6.1.22).

Now we define the auxiliary linear in \( x \) potentials \( \tilde{\varphi}_{ex} (t, x), \tilde{A}_{ex} (t, x) \) by the following formulas

\[
\tilde{\varphi}_{ex} = \varphi_{0,ex} (t) + \varphi'_{0,ex} (t) \cdot (x - r (t)), \tag{5.1.6}
\]

where

\[
\varphi_{0,ex} (t) = \varphi_{ex} (t, r (t)), \quad \varphi'_{0,ex} (t) = \nabla \varphi_{ex} (t, r (t)),
\]

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and
\[ \tilde{A}_{\text{ex}} = A_{\text{ex},0}(t) + \frac{1}{2}B_0(t) \times [x - r(t)], \]
where
\[ A_{\text{ex},0}(t) = A_{\text{ex}}(t, r(t)), \quad B_0(t) = [\nabla \times A_{\text{ex}}](t, r(t)), \]

The verification of the fact that the wave-corpuscle defined by the relations (5.1.3)-(5.1.5) is either exact or an approximate solution to the field equations (5.0.12)-(5.0.13) with estimated accuracy is carried out in the following sections.

### 5.2 Energy-momentum tensor

The canonical energy-momentum \( \Theta^{\mu\nu} \) for the EM field is obtained by applying the general formula (11.25) to the Lagrangian \( \hat{L}_0 \), yielding
\[ \hat{\Theta}^{\mu\nu} = \begin{bmatrix} \dot{w} & c\dot{g}_1 & c\dot{g}_2 & c\dot{g}_3 \\ c^{-1}s_1 & -\dot{\tau}_{11} & -\dot{\tau}_{12} & -\dot{\sigma}_{13} \\ c^{-1}s_2 & -\dot{\tau}_{21} & -\dot{\tau}_{22} & -\dot{\sigma}_{23} \\ c^{-1}s_3 & -\dot{\tau}_{31} & -\dot{\tau}_{32} & -\dot{\tau}_{33} \end{bmatrix} = \begin{bmatrix} \dot{w} & 0 & 0 & 0 \\ c^{-1}s_1 & -\dot{\tau}_{11} & -\dot{\tau}_{12} & -\dot{\tau}_{13} \\ c^{-1}s_2 & -\dot{\tau}_{21} & -\dot{\tau}_{22} & -\dot{\tau}_{23} \\ c^{-1}s_3 & -\dot{\tau}_{31} & -\dot{\tau}_{32} & -\dot{\tau}_{33} \end{bmatrix}, \] (5.2.1)

with
\[ \dot{w} = -\frac{1}{8\pi} \left| \nabla \varphi \right|^2, \quad \dot{g}_j = 0, \quad \dot{s}_j = c \frac{\partial \varphi \partial \varphi}{4\pi}, \quad \dot{\tau}_{ij} = \frac{\partial^2 \varphi}{4\pi} - \frac{1}{8\pi} \left| \nabla \varphi \right|^2, \quad \dot{\tau}_{ij} = \frac{\partial \varphi \partial \varphi}{4\pi}. \] (5.2.2)

The gauge invariant energy-momentum of the EM field takes the form
\[ \Theta^{\mu\nu} = \begin{bmatrix} w & cg_1 & cg_2 & cg_3 \\ c^{-1}s_1 & -\tau_{11} & -\tau_{12} & -\tau_{13} \\ c^{-1}s_2 & -\tau_{21} & -\tau_{22} & -\tau_{23} \\ c^{-1}s_3 & -\tau_{31} & -\tau_{32} & -\tau_{33} \end{bmatrix} = \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & -\tau_{11} & -\tau_{12} & -\tau_{13} \\ 0 & -\tau_{21} & -\tau_{22} & -\tau_{23} \\ 0 & -\tau_{31} & -\tau_{32} & -\tau_{33} \end{bmatrix}, \] (5.2.3)

\[ w(t, x) = w_0(x) + \int_{-\infty}^{t} \frac{\mathbf{J}(t', x) \cdot \nabla \varphi(t', x)}{c} \, dt', \quad g_j = 0, \quad s_j = 0, \] (5.2.4)

\[ \tau_{ij} = \frac{\partial \varphi \partial \varphi}{4\pi}, \quad \tau_{jj} = \frac{\partial^2 \varphi}{4\pi} - \frac{1}{8\pi} \left| \nabla \varphi \right|^2. \] (5.2.5)

The canonical energy-momentum tensor \( \tilde{T}^{\mu\nu} \) is not gauge invariant, but the following decomposition holds for it
\[ \tilde{T}^{\mu\nu} = \tilde{T}^{\mu\nu} + \frac{1}{c} J^{\mu} \tilde{A}^{\nu}, \quad \tilde{A}^{\nu} = (\varphi_{\text{ex}} + \varphi, A_{\text{ex}}), \] (5.2.6)

where \( \tilde{T}^{\mu\nu} \) is a gauge invariant energy-momentum obtained from formula (11.7.20) applied to the Lagrangian \( \tilde{L}_0 \) yielding
\[ \tilde{T}^{\mu\nu} = \begin{bmatrix} \tilde{u} & c\tilde{p}_1 & c\tilde{p}_2 & c\tilde{p}_3 \\ c^{-1}\tilde{s}_1 & -\tilde{\sigma}_{11} & -\tilde{\sigma}_{12} & -\tilde{\sigma}_{13} \\ c^{-1}\tilde{s}_2 & -\tilde{\sigma}_{21} & -\tilde{\sigma}_{22} & -\tilde{\sigma}_{23} \\ c^{-1}\tilde{s}_3 & -\tilde{\sigma}_{31} & -\tilde{\sigma}_{32} & -\tilde{\sigma}_{33} \end{bmatrix}, \] where
(5.2.7)
\[ \tilde{u} = \frac{\chi^2}{2m} \left[ \tilde{\nabla} \psi \cdot \tilde{\nabla}^* \psi^* + G(\psi^* \psi) \right], \quad (5.2.8) \]

\[ \tilde{p}_j = \frac{\chi i}{2} \left( \psi \tilde{\partial}_j \psi^* - \psi^* \tilde{\partial}_j \psi \right), \quad \tilde{s}_j = -\frac{\chi^2 i}{2m} \left( \tilde{\partial}_t \psi \tilde{\partial}_j^* \psi^* + \tilde{\partial}_j^* \psi^* \tilde{\partial}_t \psi \right), \quad j = 1, 2, 3, \quad (5.2.9) \]

and the stress tensor components \( \sigma_{ij} \) are represented by the formulas

\[ \tilde{\sigma}_{ii} = \tilde{u} - \frac{\chi^2}{m} \tilde{\partial}_i \psi \tilde{\partial}_i^* \psi^* + \frac{\chi i}{m} \left( \psi \tilde{\partial}_i \psi^* - \psi^* \tilde{\partial}_i \psi \right), \quad (5.2.10) \]

\[ \tilde{\sigma}_{ij} = -\frac{\chi^2 i}{2m} \left( \tilde{\partial}_i \psi \tilde{\partial}_j^* \psi^* + \tilde{\partial}_j \psi \tilde{\partial}_i^* \psi^* \right) \quad \text{for} \quad i \neq j, \quad i, j = 1, 2, 3. \]

It follows from (5.0.8) and (5.2.9) that the charge gauge invariant momentum \( P \) equals exactly the microcurrent density \( J \) multiplied by the constant \( m/q \), namely the following identity holds

\[ P = \frac{m}{q} J = \frac{i \chi}{2} \left[ \psi \tilde{\nabla}^* \psi^* - \psi^* \tilde{\nabla} \psi \right] = \left( \frac{\chi}{\psi} \text{Im} \frac{\nabla \psi}{\psi} - \frac{q \bar{A}}{c} \right) |\psi|^2, \quad (5.2.11) \]

which can be viewed as the kinematic representation of the momentum density

\[ P = mv, \quad v = J/q. \quad (5.2.12) \]

So we refer to the identities (5.2.11)-(5.2.12) as \textit{momentum density kinematic representation}.

Using the field equations we can also verify that the following conservations laws for the charge and its EM field hold:

\[ \partial_{\mu} \tilde{T}^{\mu\nu} = f^{\nu}, \quad \partial_{\mu} \Theta^{\mu\nu} = -f^{\nu}, \quad \partial_{\mu} T^{\mu\nu} = \partial_{\mu} \left( \tilde{T}^{\mu\nu} + \Theta^{\mu\nu} \right) = f^{\nu}_{\text{ex}}, \quad (5.2.13) \]

where

\[ f^{\nu} = \frac{1}{c} J_{\mu} F^{\nu\mu} = \left( \frac{1}{c} J \cdot E, \rho E \right), \quad (5.2.14) \]

\[ f^{\nu}_{\text{ex}} = \frac{1}{c} J_{\mu} F^{\nu\mu}_{\text{ex}} = \left( \frac{1}{c} J \cdot E_{\text{ex}}, \rho E_{\text{ex}} + \frac{1}{c} J \times B_{\text{ex}} \right), \]

We readily recognize in \( f^{\nu} \) and \( f^{\nu}_{\text{ex}} \) in equations (5.2.13) respectively the Lorentz force densities for the charge in its own and the external EM fields. We also see to our satisfaction from the first two equations in (5.2.13) that the Newton’s principle “action equals reaction” does manifestly hold for all involved densities at every point of the space-time.

\section*{5.3 Point charge mechanics via averaged quantities}

Combining now the conservations laws (5.2.13) with energy-momentum tensors representations (5.2.3)-(5.2.5) and (5.2.7)-(6.2.29) we obtain the following equations for the total dressed charge momentum density \( P = (P^1, P^2, P^3) \) and the energy density \( U \)

\[ \partial_t P^i = \partial_j \left( \tilde{p}^j + g^j \right) = \sum_{j=1,2,3} \partial_j \left( \tilde{\sigma}^{ji} + \tau_{ij} \right) + \left( \rho E_{\text{ex}} + \frac{1}{c} J \times B_{\text{ex}} \right)^i, \quad i = 1, 2, 3, \quad (5.3.1) \]
\[ \partial_t U = \partial_t (\tilde{u} + w) = - \sum_{j=1,2,3} \partial_j (\bar{s}_1 + s_1) + \mathbf{J} \cdot \mathbf{E}_{\text{ex}}. \] (5.3.2)

Integrating the above conservation laws over the entire space \( \mathbb{R}^3 \) we obtain the following equations for the total momentum \( \mathbf{P} \) and the total energy \( \mathbf{E} \)

\[
\frac{d\mathbf{P}}{dt} = \int_{\mathbb{R}^3} \left[ \rho \mathbf{E}_{\text{ex}} + \frac{1}{c} \mathbf{J} \times \mathbf{B}_{\text{ex}} \right] (t, \mathbf{x}) \, d\mathbf{x}, \quad (5.3.3)
\]
\[
\frac{d\mathbf{E}}{dt} = \int_{\mathbb{R}^3} \mathbf{J} \cdot \mathbf{E}_{\text{ex}} (t, \mathbf{x}) \, d\mathbf{x}.
\]

Let us introduce a charge average position \( \mathbf{r}(t) \) and average velocity \( \mathbf{v}(t) \) by the following relations

\[
\mathbf{r}(t) = \int_{\mathbb{R}^3} \mathbf{x} |\psi(t, \mathbf{x})|^2 \, d\mathbf{x}, \quad (5.3.4)
\]
\[
\mathbf{v}(t) = \frac{1}{q} \int_{\mathbb{R}^3} \mathbf{J}(t, \mathbf{x}) \, d\mathbf{x}. \quad (5.3.5)
\]

Then using the charge conservation law \((5.0.9)\) we find

\[
\frac{d\mathbf{r}(t)}{dt} = \int_{\mathbb{R}^3} \mathbf{x} \partial_t |\psi|^2 \, d\mathbf{x} = -\frac{1}{q} \int_{\mathbb{R}^3} \mathbf{x} \nabla \cdot \mathbf{J} \, d\mathbf{x} = \frac{1}{q} \int_{\mathbb{R}^3} \mathbf{J} \, d\mathbf{x} = \mathbf{v}(t). \quad (5.3.6)
\]

Utilizing the momentum density kinematic representation \((5.2.11)-(5.2.12)\) and the fact the momentum density of the charge’s EM field is identically zero according to \((5.2.4)\) we obtain the following kinematic representation for charge and hence the dressed charge total momentum:

\[
\mathbf{P}(t) = \frac{m}{q} \int_{\mathbb{R}^3} \mathbf{J}(t, \mathbf{x}) \, d\mathbf{x} = m \mathbf{v}(t). \quad (5.3.7)
\]

Notice now that for the spatially homogeneous EM fields \( \mathbf{E}_{\text{ex}}(t) \) and \( \mathbf{B}_{\text{ex}}(t) \) the equations \((5.3.3)\) take a simpler form

\[
\frac{d\mathbf{P}}{dt} = q \mathbf{E}_{\text{ex}}(t) + \frac{q \mathbf{v}(t)}{c} \times \mathbf{B}_{\text{ex}}(t), \quad (5.3.8)
\]
\[
\frac{d\mathbf{E}}{dt} = q \mathbf{v}(t) \cdot \mathbf{E}_{\text{ex}}(t).
\]

In addition to that, in this case combining the first equality in \((5.3.8)\) with the momentum kinematic representation \((5.3.7)\) we get

\[
\frac{dm \mathbf{v} \cdot \mathbf{v}}{2dt} = \mathbf{v} \cdot \frac{dm \mathbf{v}}{dt} = q \mathbf{v} \cdot \mathbf{E}_{\text{ex}}(t), \quad (5.3.9)
\]

and this combined with the second equality in \((5.3.8)\) implies the following energy kinematic representation:

\[
\mathbf{E} = \frac{m \mathbf{v} \cdot \mathbf{v}}{2} + \text{constant}. \quad (5.3.10)
\]

Combining the relations \((5.3.6)-(5.3.8)\) we also obtain

\[
\frac{m \mathbf{d}^2 \mathbf{r}(t)}{dt^2} = q \mathbf{E}_{\text{ex}}(t) + \frac{q \mathbf{d} \mathbf{r}(t)}{c dt} \times \mathbf{B}_{\text{ex}}(t), \quad (5.3.11)
\]

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in which we recognize the point charge in a homogeneous EM field dynamic equation with the familiar expression for the Lorentz force. Notice the above found correspondence between field quantities and point mechanics quantities via the charge position and velocity defined as average values \((5.3.4)\) is similar to the well known Ehrenfest Theorem in quantum mechanics, \cite{Schiff} Sections 7, 23]. This is, of course, not accidental as one can see from the Lagrangian representation of the Schrödinger wave mechanics briefly discussed in Section 11.11. The key argument for the Ehrenfest theorem as in our case is the momentum density kinematic representation \((5.2.11)-(5.2.12)\).

5.4 Accelerated motion in an external electric field

In this subsection we consider a purely electric external EM field, i.e. when \(A_{\text{ex}} = 0\), for which the field equations \((5.0.12), (5.0.13)\) take the form

\[
\begin{align*}
i\chi \partial_t \psi &= -\frac{\lambda^2 \nabla^2 \psi}{2m} + q(\varphi + \varphi_{\text{ex}}) \psi + \frac{\lambda^2}{2m} \chi'(|\psi|^2) \psi, \\
\n\n\n\n\end{align*}
\]

\((5.4.1)\)

In this case the wave-corpuscle is defined by the formula \((5.1.2)'\) with the complementary point charge equations \((5.1.3)\) taking the form

\[
m \frac{d^2 r(t)}{dt^2} = qE_{\text{ex}}(t, r),
\]

\((5.4.2)\)

\[
r(0) = r_0, \quad \frac{dr(t)}{dt}(0) = \dot{r}_0,
\]

where \(E_{\text{ex}}(t, x) = -\nabla \varphi_{\text{ex}}(t, x)\), \(r_0\) and \(\dot{r}_0\) are the initial data and in \((5.1.3)\)

\[
\begin{align*}
v(t) &= \frac{dr(t)}{dt}, \quad \frac{ds_p(t)}{dt} = \frac{mv^2(t)}{2} - q\varphi_{\text{ex}}(t, r(t)).
\end{align*}
\]

\((5.4.3)\)

In the case when the external electric field is homogeneous we show that the wave-corpuscle is an exact solution to the field equations \((5.4.1)\), and if the external electric field is inhomogeneous we show that the wave-corpuscle is an accurate approximation. Since the electric field homogeneity plays a role in the wave-corpuscle representation, it is convenient to extract from the external electric field potential \(\varphi_{\text{ex}}(t, x)\) its linear in \(x\) part \(\tilde{\varphi}_{\text{ex}}(t, x)\) about the trajectory \(r(t)\), namely we represent \(\varphi_{\text{ex}}(t, x)\) in the form

\[
\begin{align*}
\varphi_{\text{ex}}(t, x) &= \tilde{\varphi}_{\text{ex}}(t, x) + \varphi_{\text{ex}}^{(1)}(t, x),
\end{align*}
\]

\((5.4.4)\)

\[
\begin{align*}
\tilde{\varphi}_{\text{ex}}(t, x) &= \varphi_{0, \text{ex}}(t) + \varphi'_{0, \text{ex}}(t) \cdot (x - r(t)) , \quad \text{and}
\end{align*}
\]

\[
\begin{align*}
\varphi_{0, \text{ex}}(t) &= \varphi_{\text{ex}}(t, x)|_{x=r(t)} , \quad \varphi'_{0, \text{ex}}(t) = \nabla_x \varphi_{\text{ex}}(t, x)|_{x=r(t)} .
\end{align*}
\]

The remainder \(\varphi_{\text{ex}}^{(1)}(t, x)\) in \((5.4.4)\) is defined by

\[
\begin{align*}
\varphi_{\text{ex}}^{(1)}(t, x) &= \varphi_{\text{ex}}(t, x) - \varphi_{\text{ex}}(t, r) - \nabla \varphi_{\text{ex}}(t, r) (x - r), \quad \text{where} \ r = r(t)
\end{align*}
\]

\((5.4.5)\)

and, consequently, it satisfies

\[
\begin{align*}
\varphi_{\text{ex}}^{(1)}(t, x)|_{x=r(t)} = 0 , \quad \nabla \varphi_{\text{ex}}^{(1)}(t, x)|_{x=r(t)} = 0 .
\end{align*}
\]

Notice that in Section 2.3 we used a slightly different from \((5.4.4)\) form for the linear external potential \(\varphi_{\text{ex}}(t, x)\), namely \(\varphi_{\text{ex}}(t, x) = \varphi_{\text{ex}}^0(t) - E_{\text{ex}}(t) \cdot x\), where

\[
\varphi_{\text{ex}}^0(t) = \varphi_{0, \text{ex}}(t) + r(t) \cdot \nabla \varphi_{\text{ex}}(t, r(t)).
\]
5.4.1 Accelerated motion in an external homogeneous electric field

If the external field is a purely electric and homogeneous field \( E_{\text{ex}}(t) \) then its potential \( \varphi_{\text{ex}}(t, x) \) is linear in \( x \) and the representation (5.4.4) turns into

\[
\varphi_{\text{ex}}(t, x) = \tilde{\varphi}_{\text{ex}}(t, x) = \varphi_{0,\text{ex}}(t) + \varphi'_{0,\text{ex}}(t) \cdot (x - r(t)),
\]

(5.4.6)

where

\[
\varphi'_{0,\text{ex}}(t) = \nabla_x \varphi_{\text{ex}}(r(t), t) = -E_{\text{ex}}(t).
\]

The main result of this section is that the wave-corpuscle as defined by formula (5.1.2) is an exact solution to the field equation (5.4.1) that can be verified by a straightforward examination. One can alternatively establish that result by considering the expression for \( \psi \) in (5.1.2) and assuming that the real valued functions \( r(t), v(t) \) and \( s_p(t) \) are unknown and to be found, if possible, from the field equations (5.4.1). Indeed, observe that the representation (5.1.2) implies

\[
\partial_t \psi = \exp \left( \frac{iS}{\hbar} \right) \left\{ \left[ \frac{i m}{\hbar} (\hat{\mathbf{v}} \cdot (\mathbf{x} - \mathbf{r}) - \mathbf{v} \cdot \hat{\mathbf{r}}) + \frac{i \dot{s}_p}{\hbar} \right] \hat{\psi} - \hat{\mathbf{r}} \cdot \nabla \hat{\psi} \right\},
\]

(5.4.7)

and by the Leibnitz formula we have

\[
\nabla^2 \hat{\psi} = \hat{\psi} (\mathbf{x} - \mathbf{r}) \frac{\mathbf{x} - \mathbf{r}}{|\mathbf{x} - \mathbf{r}|},
\]

(5.4.8)

To find if the expression (5.1.2) for \( \psi \) can solve the field equations (5.4.1) we substitute the expression into the field equations (5.4.1) obtaining the following equation for functions \( v, r, s_p \):

\[
\left[-m \hat{\mathbf{v}} \cdot (\mathbf{x} - \mathbf{r}) - \mathbf{v} \cdot \hat{\mathbf{r}} - \dot{s}_p\right] \hat{\psi} - i\chi \hat{\mathbf{r}} \cdot \nabla \hat{\psi} = 0,
\]

(5.4.9)

\[
- \frac{m}{2} \mathbf{v}^2 \hat{\psi} + i\chi \mathbf{v} \cdot \nabla \hat{\psi} + \frac{\hbar^2}{2m} \nabla^2 \hat{\psi} - q (\tilde{\varphi}_{\text{ex}} + \varphi) \hat{\psi} - \frac{\hbar^2}{2m} G' \hat{\psi} = 0.
\]

Then using the charge equilibrium equation (2.3.8) we eliminate the nonlinearity \( G \) in the above equation (5.4.9) and obtain the following equation equivalent to it:

\[
- \left\{ m \left[ \hat{\mathbf{v}} \cdot (\mathbf{x} - \mathbf{r}) - \mathbf{v} \cdot \hat{\mathbf{r}} \right] + \frac{m}{2} \mathbf{v}^2 + \dot{s}_p + q \tilde{\varphi}_{\text{ex}} \right\} \hat{\psi} - i\chi (\hat{\mathbf{r}} - \mathbf{v}) \nabla \hat{\psi} = 0.
\]

(5.4.10)

Now to determine if there is a triple of functions \( \{r(t), v(t), s_p(t)\} \) for which the equation (5.4.10) holds we equate to zero the coefficients before \( \nabla \hat{\psi} \) and \( \hat{\psi} \) in that equation, resulting in two equations:

\[
\mathbf{v} = \hat{\mathbf{r}}, \quad m \left[ \hat{\mathbf{v}} \cdot (\mathbf{x} - \mathbf{r}) - \mathbf{v} \cdot \hat{\mathbf{r}} \right] + \frac{m}{2} \mathbf{v}^2 + \dot{s}_p + q \tilde{\varphi}_{\text{ex}} = 0,
\]

(5.4.11)

where, in view of the representation (5.4.6), the second equation in (5.4.11) can be recast as

\[
m \left[ \hat{\mathbf{v}} \cdot (\mathbf{x} - \mathbf{r}) - \mathbf{v} \cdot \hat{\mathbf{r}} \right] + \dot{s}_p + \frac{m\mathbf{v}^2}{2} + q \left[ \varphi_{0,\text{ex}} + \varphi'_{0,\text{ex}} \cdot (\mathbf{x} - \mathbf{r}) \right] = 0.
\]

(5.4.12)
To find out if there is a triple of functions \( \{ r(t), v(t), s_p(t) \} \) solving the equation (5.4.12) we equate to zero the coefficient before \((x - r)\) and the remaining coefficient and obtain the following pair of equations

\[
mv = -q\varphi'_{0,ex}(t), \quad \dot{s}_p - mv \cdot \dot{r} + \frac{mv^2}{2} + q\varphi_{0,ex}(t) = 0. \tag{5.4.13}
\]

Thus, based on the first equation (5.4.11) and the equations (5.4.13) we conclude that the wave-corpuscle defined by the formula (5.1.3) with the complementary point charge equations (5.4.2), (5.4.3) is indeed an exact solution to the field equations (5.4.1).

It is instructive to compare now construction of the exact solutions (5.1.2) with the quasi-classical approach based on the WKB theory. The trajectories of the charges centers as described by our model coincide with ones obtained based on the well-known quasiclassical asymptotics if one neglects the nonlinearity. Note though that there are two important effects of the nonlinearity that are not presented in the formal quasiclassical approach. First of all, due to the nonlinearity the charge preserves its shape in the course of evolution whereas in the linear model any wavepacket disperses over time. Second of all, the quasiclassical asymptotic expansions produce infinite asymptotic series which provide for a formal solution, whereas the properly introduced nonlinearity as in (2.3.5), (2.3.6) allows one to obtain an exact solution. For a treatment of a nonlinear wave mechanics based on the WKB asymptotic expansions we refer the reader to [Komech05] and references therein.

### 5.4.2 Accelerated motion in an external inhomogeneous electric field

In this section we consider a general external electric field \( E_{ex}(t, r) \) which can be inhomogeneous with the corresponding electric potential \( \varphi_{ex}(t, x) \) as described by relations (5.4.4), (5.4.5) with nonzero remainder \( \varphi^{(1)}_{ex}(t, x) \). For an inhomogeneous external electric field \( E_{ex}(t, r) \) no closed form solution to the field equations (5.4.1) seems to be available but the wave-corpuscle defined by the relations (5.1.2) with complementary point charge equations (5.4.2), (5.4.3) turns out to be a good approximation with the accuracy dependent on (i) the size parameter \( a \) defined by relations (4.4.1) and (ii) the degree of spatial inhomogeneity of the electric field measured by the electric field inhomogeneity length \( R_{ex} \) introduced below.

The parameter \( R_{ex} \) is similar to the radius of curvature of the graph of \( \varphi_{ex}(t, x) \), and large or small values \( R_{ex} \) correspond, respectively, to almost homogeneous or highly inhomogeneous electric fields. It turns out that the wave-corpuscle solves the field equations (5.4.1) with the discrepancy \( D = O\left((a/R_{ex})^2\right) \) for \( a \ll R_{ex} \) as we show below. We fix now for the rest of this section the initial data \( r_0 \) and \( v_0 \) in (5.4.2) and consequently the function \( r(t) \). We assume here that the factor \(|\psi_1(|x|)|^2 \) decays exponentially as \(|x| \to \infty\), and, in particular, for some constant \( C_0 \)

\[
\int_{\mathbb{R}^3} |\psi_1(|x|)|^2 |x|^2 \, dx \leq C_0, \quad \int_{\mathbb{R}^3} |\psi_1(|x|)|^2 |x| \, dx \leq C_0. \tag{5.4.14}
\]

To assess the accuracy of the wave-corpuscle solution defined by relations (5.1.2), (5.4.2), (5.4.3) we follow an approach discussed in Section 5.1. Namely, we introduce write auxiliary field equation (5.1.1) with \( \tilde{A}_{ex} = 0 \) in the form

\[
i\chi \partial_t \psi - \frac{\chi^2\nabla^2 \psi}{2m} + q (\varphi + \tilde{\varphi}_{ex}(t, x)) \psi - \frac{\chi^2 G' \psi}{2m} = 0,
\]

\[
\nabla^2 \varphi = -4\pi q |\psi|^2,
\]

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where $\tilde{\varphi}_{\text{ex}}(t, \mathbf{x})$ is defined by (5.4.4). In view of the relations (5.4.6) the corresponding external field $E_{\text{ex}}(t) = -\nabla_x \tilde{\varphi}_{\text{ex}}(r(t), t)$ is homogeneous. A straightforward examination shows that the results of Section 5.4.1 apply and the wave-corpuscle defined by (5.4.12) with complementary point charge equations (5.4.2), (5.4.3) is an exact solution to the auxiliary field equations (5.4.15). Now notice that the auxiliary field equations differ from the original ones (5.4.1) only by the replacement of $\varphi_{\text{ex}}(t, \mathbf{x})$ with $\tilde{\varphi}_{\text{ex}}(t, \mathbf{x})$ with the consequent \textit{discrepancy}

\[ D_0(t, \mathbf{x}) = q \left[ \tilde{\varphi}_{\text{ex}}(t, \mathbf{x}) - \varphi_{\text{ex}}(t, \mathbf{x}) \right] \psi(t, \mathbf{x}) = -q\varphi^{(1)}(t, \mathbf{x}) \psi(t, \mathbf{x}). \tag{5.4.16} \]

Based on the above discrepancy and taking into account the dependence on size parameter $a$ for $\psi = \psi_a$ as in (4.4.1) we introduce the integral \textit{discrepancy}

\[ \tilde{D}_0 = \int_{\mathbb{R}^3} D_0(t, \mathbf{x}) \psi^* d\mathbf{x} = \int_{\mathbb{R}^3} -q\varphi^{(1)}(t, \mathbf{x}) \psi \psi^* d\mathbf{x} \tag{5.4.17} \]

\[ = \int_{\mathbb{R}^3} -qa^{-3} \left| \dot{\psi}_1 \left( a^{-1} |\mathbf{x} - \mathbf{r}| \right) \right|^2 \varphi^{(1)}(t, \mathbf{x}) d\mathbf{x}. \]

Notice that the relations (5.4.4) for $\tilde{\varphi}_{\text{ex}}(t, \mathbf{x})$ and the charge normalization condition (5.0.14) imply that a similar integral involving $\tilde{\varphi}_{\text{ex}}$ equals

\[ \int_{\mathbb{R}^3} q\tilde{\varphi}_{\text{ex}}(t, \mathbf{x}) |\psi|^2 d\mathbf{x} = q\varphi_{\text{ex}}(t, \mathbf{r}(t)), \tag{5.4.18} \]

which coincides with the potential energy of a point charge $q$ in the potential $\varphi_{\text{ex}}(t, \mathbf{x})$, therefore it is natural to compare $\tilde{D}_0$ with variation of this energy in the dynamics. To assess typical scales of inhomogeneity of the external field we introduce the potential variation quantity

\[ \varphi_{0,T} = \max_{0 \leq t \leq T} |\varphi_{\text{ex}}(t, \mathbf{r}(t)) - \varphi_{\text{ex}}(0, \mathbf{r}(0))|. \tag{5.4.19} \]

Note that $|q| \varphi_{0,T}$ equals the point charge potential energy variation on time interval $[0, T]$.

We also introduce a parameter $\sigma_{|\psi|$ which plays a role similar to $3\sigma$ for Gaussian probability but with respect to the function $\psi_1(|z|)^2$ in (4.3.1), namely

\[ \psi_1(r) \simeq 0 \text{ if } r \geq \sigma_{|\psi|}. \tag{5.4.20} \]

The above approximation means that the discrepancy created by replacing $\tilde{\psi}_1(r)$ by zero for large $r$ is smaller than other discrepancies that appear below.

Let us introduce the following characteristic lengths $R_{\varphi}(t, \mathbf{r})$ and $R_{\varphi}$ similar to the radius of the curvature:

\[ \frac{1}{R_{\varphi}^2(t, \mathbf{r})} = \sup_{0 < |z| \leq a\sigma_{|\psi|}} \frac{\varphi_{\text{ex}}^{(1)}(t, \mathbf{r} + \mathbf{z})}{|\mathbf{z}|^2 |\tilde{\varphi}|}, \quad \frac{1}{R_{\varphi}^2} = \max_{0 \leq t \leq T} \frac{1}{R_{\varphi}^2(t, \mathbf{r}(t))}, \tag{5.4.21} \]

where

\[ |\tilde{\varphi}| = \max_{0 \leq t \leq T} \max_{0 < |z| \leq a\sigma_{|\psi|}} |\varphi(t, \mathbf{r}(t) + \mathbf{z}) - \varphi(0, \mathbf{r}(0))|. \tag{5.4.22} \]

The quantity $R_{\varphi}$ represents the typical spatial scale at which the spatially curvilinear component $\varphi_{\text{ex}}^{(1)}(t, \mathbf{x})$ of the external field $\varphi_{\text{ex}}(t, \mathbf{x})$ changes significantly in a vicinity of $\mathbf{r}(t)$. For small $a$ the quantity $R_{\varphi}$ is essentially determined by the maximal eigenvalue $|\lambda_{\max}|$ of
the matrix of the second spatial derivatives of $\varphi_{ex}(t, x)$ at $x = r(t)$. In addition to that, $R_\varphi(t, r) \to 1/|\lambda_{max}|$ as $a \to 0$ where $1/|\lambda_{max}|$ is the minimal curvature radius of the graph of the normalized potential $\varphi^{(1)}_{ex}(t, x) / |\varphi|$ at the point $r(t)$. It follows from (5.4.3) and (5.4.21) that $1/R_\varphi^2$ is bounded as long as the curve $r(t)$ is not close to singularities of the external field $\varphi_{ex}(t, x)$ if any. Then we estimate the integral discrepancy as follows:

$$
|\tilde{D}_0| = |q| \left| \int_{\mathbb{R}^3} a^{-3} \left| \varphi_1 (a^{-1} |x - r|) \right|^2 \varphi^{(1)}_{ex}(t, r + x - r) \, dx \right|
$$

$$
\lesssim \frac{|q| |\varphi|}{R_\varphi^2} \int_{\mathbb{R}^3} a^{-3} \left| \varphi_1 (a^{-1} |x - r|) \right|^2 (x - r)^2 \, dx \leq C_0 \frac{a^2 |q| |\varphi|}{R_\varphi^2}.
$$

Combining inequality (5.4.23) with relations (5.4.18), (5.4.19), (5.4.22) we can judge the quality of approximation by requiring the relative dimensionless discrepancy $|\tilde{D}_0| / (|q| \varphi_{0,T})$ to be small, namely

$$
\frac{|\tilde{D}_0|}{|q| \varphi_{0,T}} \lesssim \frac{a^2 |\varphi|}{R_\varphi^2 \varphi_{0,T}} \ll 1 \text{ is a requirement for an accurate approximation.}
$$

For further applications we briefly consider an effect on the discrepancy of a perturbation of the external potential $\varphi_{ex}(t, x)$ when it is substituted with a slightly different potential $\hat{\varphi}_{ex}(t, x, \epsilon)$ with $\epsilon$ being a small perturbation parameter and the approximate solution is determined based on $\varphi_{ex}(t, x) = \hat{\varphi}_{ex}(t, x, 0)$. Supposing the initial data $r_0, \dot{r}_0$ and hence the position function (trajectory) $r(t)$ solving the equation of motion (5.4.2) being fixed we assume that there exists fixed positive constants $C, C_1, T$ and $\epsilon_1$ such that for any small $\epsilon$ we have

$$
|\varphi_{ex}(t, x) - \hat{\varphi}_{ex}(t, x, \epsilon)| \leq C_1 \epsilon, \quad |\nabla \varphi_{ex}(t, x) - \nabla \hat{\varphi}_{ex}(t, x, \epsilon)| \leq C_1 \epsilon, \quad (5.4.25)
$$

for any $t$ and $x$ such that

$$
|x - r(t)| \leq \epsilon_1, \quad 0 \leq t \leq T.
$$

The above condition simply requires the external perturbed field potential to be close to the original one in a small vicinity of the trajectory $r(t)$. Substitution of the original wave-corpse solution $\{\psi, \varphi\}$ defined by (5.1.2) (corresponding to $\epsilon = 0$) and the original complementary point charge equations of motion (5.4.2) for $r(t)$ into the equation (5.4.1) with the external potential $\hat{\varphi}_{ex}(t, x, \epsilon)$ produces the total discrepancy

$$
\hat{D}_0(t, x) = \hat{D}_0(t, x) + \hat{D}_1(t, x), \quad \hat{D}_1(t, x) = -[\varphi_{ex}(t, x) - \hat{\varphi}_{ex}(t, x, \epsilon)] \psi. \quad (5.4.26)
$$

Note that if $a$ is small $a \sigma_\psi \leq \epsilon_1$ and using (5.4.26) we get

$$
\left| \int_{\mathbb{R}^3} \hat{D}_1(t, x) \psi^* \, dx \right| \lesssim \int_{|r - r(t)| \leq \epsilon_1} |\varphi_{ex}(t, x) - \hat{\varphi}_{ex}(t, x, \epsilon)| |\psi|^2 \, dx \lesssim \sup_{|r - r(t)| \leq \epsilon_1} |\varphi_{ex}(t, r) - \hat{\varphi}_{ex}(t, r, \epsilon)| \leq C_1 \epsilon, \quad (5.4.27)
$$

where $C_1 / |\varphi|$ is a dimensionless constant. Combining (5.4.23), (5.4.26), (5.4.27) we get the following rough estimate

$$
\frac{|\hat{D}_0(t, x)| + |\hat{D}_1(t, x)|}{|\varphi|} \lesssim \left( C_0 \frac{a^2}{R_\varphi^2} + \frac{C_1 \epsilon}{|\varphi|} \right) |q|.
$$

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It is instructive to look at the trajectory $r(t)$ determined from (5.4.2) for previously constructed exact and approximate wave-corpuscle solutions from another point of view. Namely, let us introduce a moving frame $y = x - r(t)$, where $r = r(t)$, $s_p = s_p(t)$, $v = v(t)$ solve (5.4.2), (5.4.3). Notice that the origin of the new frame is at the center $r(t)$ of the wave-corpuscle. Let us change variables in equations (5.4.1)

$$
\psi(t, x) = \exp \left\{ \frac{imv \cdot y}{\chi} + \frac{is_p}{\chi} \right\} \hat{\psi}(t, y), \quad \varphi(x) = \hat{\varphi}(y),
$$

where $\hat{\psi}(t, y)$ is a new unknown function. Then repeating the above calculations (without using (2.3.8)) we obtain an equivalent equation of the same form as (5.4.1), namely

$$
i\chi \partial_t \hat{\psi} = -\frac{\chi^2}{2m} \nabla_y^2 \hat{\psi} + q (\hat{\varphi} + \hat{\varphi}_{ex}) \hat{\psi} + \frac{\chi^2}{2m} G' \left( |\hat{\psi}|^2 \right) \hat{\psi},
$$

with an external potential $\hat{\varphi}_{ex}(t, y)$ which satisfies an additional condition $\hat{\varphi}_{ex}(0) = 0$, $\nabla \hat{\varphi}_{ex}(0) = 0$. If the original potential $\varphi_{ex}(t, x)$ is linear in $x$, the external potential $\hat{\varphi}_{ex}(t, y)$ in the moving frame vanishes, i.e. $\hat{\varphi}_{ex}(t, y) = 0$ for all $t, y$. In this case (5.4.30) coincides with the equilibrium condition (2.3.8), hence $\hat{\psi} = \hat{\psi}$ and the wave-corpuscle rests at the origin of the moving frame.

### 5.5 Accelerated motion in a general external EM field

In this subsection we consider a single charge in a general external EM field which can have nonzero magnetic component. The primary goal of this section is to show that the wave-corpuscle defined by relations (5.1.2) and the complementary point charge equations (5.1.3), (5.1.4) is an approximate solution to the field equations (5.0.12), (5.0.13). We accomplish this by following the method of Section 5.1 and showing first that the wave-corpuscle is an exact solution to the auxiliary field equations (5.1.1) and then provide estimations of the discrepancy between the auxiliary and the original field equations (5.0.12), (5.0.13).

#### 5.5.1 Wave-corpuscle as an exact solution

In this section we show that the wave-corpuscle defined by relations (5.1.2) and the complementary point charge equations (5.1.3), (5.1.4) is an exact solution to the auxiliary field equations (5.1.1). One way to do that is to plug in $\psi, \varphi$ defined by the formulas (5.1.2) into the auxiliary field equations (5.1.1) and, using the complementary point charge equations (5.1.3), (5.1.4), verify that the equality does hold. An alternative and more instructive, we believe, way to accomplish the same goal is (i) to assume for the sake of the argument that the point charge functions $r(t)$, $v(t)$, $s_p(t)$ are unknown; (ii) to find out if there is a way to choose those functions so that the wave-corpuscle fields $\psi, \varphi$ solves exactly the auxiliary field equations (5.1.1); (iii) verify that the chosen $r(t)$, $v(t)$, $s_p(t)$ satisfy the complementary point charge equations. Following this way we substitute (5.1.2) into (5.1.1) and obtain the
following equation for \( r, v, s_p \):

\[
i\{im\left[ \hat{v} \cdot (x - r) - v \cdot \hat{r} \right] + is_p \} \hat{\psi} - i\chi \hat{r} \cdot \hat{\psi}' \frac{x - r}{|x - r|} - m\frac{v^2\hat{\psi}}{2} + i\chi \hat{\psi}' \frac{x - r}{|x - r|} + \frac{\chi^2 v^2 \hat{\psi}}{2m} + \left[ \frac{qv \hat{\psi}}{c} + \frac{\chi q \hat{\psi}'}{mc} \right] \left[ \frac{A_{ex,0} + \frac{B_0 \times (x - r)}{2}}{|x - r|} \right] \psi - q \left( \varphi_{ex} + \varphi' \right) \hat{\psi} - \frac{\chi^2 G' \hat{\psi}}{2m} = 0.
\] (5.5.1)

Taking into account the obvious identity \((x - r) \cdot [B_0 \times (x - r)] = 0\) we recast the above equation as follows:

\[
- m (\hat{v} \cdot (x - r) - v \cdot \hat{r}) - s_p \right] \hat{\psi} - i\chi \hat{r} \cdot \hat{\psi}' \frac{x - r}{|x - r|} - m\frac{v^2\hat{\psi}}{2} + i\chi \hat{\psi}' \frac{x - r}{|x - r|} + \frac{\chi^2 v^2 \hat{\psi}}{2m} + \left[ \frac{qv \hat{\psi}}{c} + \frac{\chi q \hat{\psi}'}{mc} \right] \left[ \frac{A_{ex,0} + \frac{B_0 \times (x - r)}{2}}{|x - r|} \right] \hat{\psi} - q \left( \varphi_{ex} + \varphi' \right) \hat{\psi} - \frac{\chi^2 G' \hat{\psi}}{2m} = 0.
\] (5.5.2)

Using the charge equilibrium equation (2.3.8) we can eliminate \( G' \) in equation (5.5.2), obtaining the following equation equivalent to it:

\[
- [m (\hat{v} \cdot (x - r) + v \cdot \hat{r}) + s_p] \hat{\psi} - m\frac{v^2\hat{\psi}}{2} + i\chi \hat{\psi}' \frac{x - r}{|x - r|} + \frac{\chi^2 v^2 \hat{\psi}}{2m} + \left[ \frac{qv \hat{\psi}}{c} + \frac{\chi q \hat{\psi}'}{mc} \right] \left[ \frac{A_{ex,0} + \frac{B_0 \times (x - r)}{2}}{|x - r|} \right] \hat{\psi} + q \left( \varphi_{ex} + \varphi' \right) \hat{\psi} = 0.
\] (5.5.3)

For equation (5.5.3) to hold we may require the coefficient before \( \hat{\psi} \) and \( \hat{\psi}' \) in it to be zero. Executing that by collecting terms with \( \hat{\psi} \) and \( \hat{\psi}' \) and using \( \varphi_{ex} = \varphi_{0,ex} + \varphi'_{0,ex} \cdot (x - r) \) we obtain the following equations:

\[
\hat{r} = v - q \frac{A_{ex,0}}{mc},
\] (5.5.4)

\[
-m [\hat{v} \cdot (x - r) - v \cdot \hat{r}] - s_p - m\frac{v^2}{2} + \left[ \frac{qv}{c} + \frac{B_0 \times (x - r)}{2} \right] - q (\varphi_{0,ex} + \varphi'_{0,ex} \cdot (x - r)) = 0.
\] (5.5.5)

To solve (5.5.5) we require the coefficient of \((x - r)\) and the remaining one to be zero, and this with the help of an elementary identity \( v \cdot (B_0 \times (x - r)) = (x - r) \cdot (v \times B_0) \) yields the
being given \( r(0) \) we obtain the following equations for \( s(0) \) and the complementary point charge equations (5.1.3), (5.1.4) if the following (5.0.13) are compatible with the auxiliary system (5.1.1) for the wave-corpuscle defined by following pair of equations:

\[
m\ddot{v} = -q \left[ \frac{1}{2c} B_0(t) \times v + \varphi_{0,ex}'(t) \right], \tag{5.5.6}
\]

\[
mv \cdot \dot{r} - \dot{s}_p + \frac{q}{c} v \cdot A_{ex,0} - \frac{mv^2}{2} - q\varphi_{0,ex} = 0. \tag{5.5.7}
\]

Now being given \( v(0) \) we readily find \( v(t) \) from the linear equation (5.5.6). Then using \( v(t) \) and being given \( r(0) \) we immediately find \( r(t) \) from equation (5.5.4). Combining equations (5.5.4) and (5.5.7) we obtain the following equations for \( s_p(t) \)

\[
\dot{s}_p = mv \cdot \dot{r} + \frac{q}{c} v \cdot A_{ex,0} - \frac{mv^2}{2} - q\varphi_{0,ex} = \frac{m v^2}{2} - q\varphi_{0,ex}. \tag{5.5.8}
\]

It remains to verify that the triple \( \{ r(t), v(t), s_p(t) \} \) satisfies the complementary point charge equations (5.1.3), (5.1.4). Indeed, combining the relations (5.5.6) and (5.5.4) we obtain

\[
m\ddot{r} = -q \left[ \frac{1}{2c} B_0 \times \left( -\dot{r} - \frac{q}{mc} A_{ex,0} \right) + \varphi_{0,ex}' + \frac{1}{c} \partial_t A_{ex,0} \right]. \tag{5.5.9}
\]

A straightforward comparison taking into account (5.1.6) shows that the above equation (5.5.9) coincides with the point charge equation of motion (5.1.3), and the equations (5.5.4) and (5.5.8) provide the point charge equations (5.1.4). With that we completed the desired verification of the fact the wave-corpuscle does solve exactly the auxiliary field equations (5.1.1).

As to the exact solvability issue let us compare the coefficients of auxiliary equations (5.1.1) and the original field equations (5.0.12), (5.0.13). Taking into account the relation (5.1.6)-(5.1.7) we find that the EM potentials \( \varphi_{ex}(t,x) \), \( A_{ex}(t,x) \) in the field equations (5.0.12), (5.0.13) are compatible with the auxiliary system (5.1.1) for the wave-corpuscle defined by (5.1.2) and the complementary point charge equations (5.1.3), (5.1.4) if the following exact solvability condition holds

\[
\varphi_{ex}(t,x) = \varphi_{0,ex} + \varphi_{0,ex}' \cdot (x - r) - \frac{q}{2mc^2} \left( A_{ex,0} + \frac{1}{2} B_0(t) \times (x - r(t)) \right)^2,
\]

\[
A_{ex}(t,x) = \tilde{A}_{ex}(t,x) = A_{ex,0} + \frac{1}{2} B_0(t) \times (x - r(t)).
\]

Note that if \( B_0 \neq 0 \) the electrical field potential \( \varphi_{ex} \) involves a quadratic term which vanishes at the center of the wave-corpuscle. One can naturally ask how broad is the class of external EM fields as in (5.5.10) for which there are exact solutions as the wave-corpuscles? The class of such EM field is sufficiently broad in the sense that for any accelerated motion of a point charge in an arbitrary EM field there is wave-corpuscle as an exact solution to the field equations with an external field from the class. To see that let us consider a point charge in an arbitrary external EM field and find its trajectory \( r(t) \). Then we introduce a special EM field defined by (5.1.6), (5.1.7) with \( B_0 = B(t) \) and \( \varphi_{0,ex}(t) \) defined by (5.5.11) and for this external field the wave-corpuscle (5.1.2) is an exact solution to the field equations (5.0.12), (5.0.13): its center moves exactly according to the trajectory \( r(t) \). More than that, an arbitrary vector-function \( r(t) \) can be obtained as a solution of (5.1.3) with appropriate choice.
of \( \mathbf{E}(t, r) \), \( \mathbf{B}(t, r) \). Indeed, let \( \mathbf{E}(t, r) = m \mathbf{\dot{r}}(t) / q \), and such \( \mathbf{r}(t) \) is a solution of (5.1.3). Note that for the given \( \mathbf{E}(t) \) along the trajectory we can take \( \mathbf{A}_{\text{ex},0}(t) \) to be arbitrary and determine \( \varphi'_{0,\text{ex}}(t) \) by the following formula

\[
\varphi'_{0,\text{ex}}(t) = - \left\{ \frac{\partial_t \mathbf{A}_{\text{ex},0}(t)}{c} - \frac{\mathbf{B}_0(t) \times \mathbf{\dot{r}}(t)}{2c} + \mathbf{E}(t) + \frac{q}{mc^2} \left[ \frac{1}{2} \mathbf{B}_0(t) \times \mathbf{A}_{\text{ex},0}(t) \right] \right\}. \tag{5.5.11}
\]

Thus, we can conclude that the wave-corpuscle (5.1.2) as an exact solution to the field equations (5.0.12), (5.0.13) with an appropriate choice of the external EM field can model any motion of a point charge.

### 5.5.2 de Broglie factor for accelerating charge

In this subsection we would like to take a look at the de Broglie exponential factor in the wave-corpuscle defined by (5.1.2) and the complementary point charge equations (5.1.3), (5.1.4). Let \( \tilde{\psi}(k) = \left[ \mathcal{F}\psi \right](k) \) be the Fourier transform of the wave function \( \psi(x) \)

\[
\tilde{\psi}(k) = \left[ \mathcal{F}\psi \right](k) = \int_{\mathbb{R}^3} e^{-ik \cdot x} \psi(x) \, dx. \tag{5.5.12}
\]

Then in view of the charge normalization condition (5.0.14) and by the Parseval theorem \( \tilde{\psi}(k) \) satisfies similar condition, namely

\[
(2\pi)^{-3} \int_{\mathbb{R}^3} \left| \tilde{\psi}(k) \right|^2 \, dk = 1, \tag{5.5.13}
\]

and we can introduce the center \( k_*(\psi) \) for \( \tilde{\psi}(k) \) as follows:

\[
k_*(\psi) = k_* = (2\pi)^{-3} \int_{\mathbb{R}^3} k \left| \tilde{\psi}(k) \right|^2 \, dk. \tag{5.5.14}
\]

Note that the following identity holds

\[
k \left| \tilde{\psi}(k) \right|^2 = k \tilde{\psi}(k) \tilde{\psi}^*(k) = \frac{1}{2i} \left\{ i k \tilde{\psi}(k) \tilde{\psi}^*(k) - \tilde{\psi}(k) \left[ i k \tilde{\psi}(k) \right]^* \right\},
\]

implying together with (5.5.14) the following representation

\[
k_* = \int_{\mathbb{R}^3} \text{Im} \frac{\nabla \tilde{\psi}(x)}{\tilde{\psi}(x)} \left| \psi(x) \right|^2 \, dx. \tag{5.5.15}
\]

Observe that the Fourier transform (5.5.12) of the wave-corpuscle defined by (5.1.2) is

\[
\tilde{\psi}(t, k) = \exp \left\{ i \mathbf{r}(t) \cdot k - \frac{i s_p(t)}{\chi} \right\} \left[ \mathcal{F}\left[ \psi \right] \right](k) \left( k - m \mathbf{v}(t) \right) \chi, \tag{5.5.16}
\]

and since \( \tilde{\psi}(\left|k\right|) \) is a radial function, its Fourier transform \( \mathcal{F}\left[ \tilde{\psi} \right](k) \) is a radial function as well. Let us consider \( k_* \) defined by the formula (5.5.14) that corresponds to the wave-corpuscle (5.5.16). Using the fact that \( \mathcal{F}\left[ \tilde{\psi} \right](k) \) is a radial function and the relations (5.1.4), (5.1.3) we readily find that

\[
k_* = \frac{m \mathbf{v}(t)}{\chi}, \quad \mathbf{v}(t) = \mathbf{\dot{r}}(t) + \frac{q}{mc} \mathbf{A}_{\text{ex},0} = \mathbf{\dot{P}} / m. \tag{5.5.17}
\]
The dispersion relation \( \omega (\mathbf{k}) \) for the linear part of equation (5.1.1) and the corresponding group velocity \( \nabla_k \omega (\mathbf{k}) \) are, respectively,
\[
\omega (\mathbf{k}) = \frac{\chi}{2m} k^2 - \frac{q}{mc} A_{ex,0} \cdot \mathbf{k},
\]
\[
\nabla_k \omega (\mathbf{k}) = \frac{\chi}{m} k - \frac{q}{mc} A_{ex,0}.
\]
Combining relations (5.5.18) and (5.5.17)
\[
\nabla_k \omega (\mathbf{k}_*) = \frac{\chi}{m} k_* - \frac{q}{mc} A_{ex,0} = \mathbf{v} (t) - \frac{q}{mc} A_{ex,0} = \dot{\mathbf{r}} (t),
\]
we find that the charge velocity \( \mathbf{v} (t) \) is identical with the group velocity \( \nabla_k \omega (\mathbf{k}_*) \) indicating the wave origin of the charge motion.

5.5.3 General external EM field

The subject of this section and the treatment are similar to the ones in Section 5.4.2, but estimates in the presence of external magnetic field are more involved. Below we provide the most essential estimates related to this case omitting tedious details. The main result of this section is that the wave-corpuscle defined by (5.1.2) and the complementary point charge equations (5.1.3), (5.1.4) is an approximate solution to the field equations (5.0.12), (5.0.13) with a discrepancy of the order \( O \left( \frac{a}{R_{ex}} \right)^2 \) for \( a \ll R_{ex} \), where \( a \) is the size parameter defined by relations (4.4.1) and \( R_{ex} \) is a typical length for inhomogeneity of the external field. We assume here that the function \( |\varphi_1 (s)|^2 \) decays exponentially as \( s \to \infty \), and (5.4.14) holds.

We define the coefficients (5.1.6)- (5.1.7) of the auxiliary field equations in (5.1.1) as follows
\[
\varphi_{0,ex} (t) = \varphi_{ex} (\mathbf{r} (t), t) + \frac{q}{2mc^2} A_{ex}^2 (\mathbf{r} (t), t),
\]
\[
\varphi'_{0,ex} (t) = \nabla_x \varphi_{ex} (\mathbf{r} (t), t) + \frac{q}{2mc^2} [A_{ex,0} \times B_0 (t)],
\]
implying
\[
\bar{\varphi}_{ex} (t, \mathbf{x}) = \varphi_{0,ex} (t) + \varphi'_{0,ex} (t) \cdot (\mathbf{x} - \mathbf{r} (t)),
\]
\[
B_0 (t) = B (\mathbf{r} (t), t), \quad A_{ex,0} (t) = A_{ex} (\mathbf{r} (t), t).
\]
Note that the wave-corpuscle defined by (5.1.2) and the complementary point charge equations (5.1.3), (5.1.4) generally speaking is not an exact solution of (5.0.12) since the potentials of (5.0.12) satisfy more general relations than (5.5.10):
\[
\varphi_{ex} (t, \mathbf{x}) = \varphi_{0,ex} (t) + \varphi'_{0,ex} (t) \cdot (\mathbf{x} - \mathbf{r} (t)) - \frac{q}{2mc^2} \left( A_{ex,0} + \frac{1}{2} B_0 (t) \times (\mathbf{x} - \mathbf{r} (t)) \right)^2 + \varphi^{(l)}_{ex} (t, \mathbf{x}),
\]
\[
B (t, \mathbf{x}) = B_0 (t) + B_1 (t, \mathbf{x}),
\]
\[
A_{ex} (t, \mathbf{x}) = A_{ex,0} (t) + \frac{1}{2} B_0 (t) \times (\mathbf{x} - \mathbf{r} (t)) + A_{ex,1} (t, \mathbf{x}),
\]

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with non-zero terms \( \varphi_{\text{ex}}^{(1)} (t, x) \), \( B_1 (t, x) \), \( A_{\text{ex},1} (t, x) \). These extra terms are small in the vicinity of \( r (t) \) where \( \psi \) is localized since

\[
\varphi_{\text{ex}}^{(1)} (x, r (t)) = 0, \quad \nabla \varphi_{\text{ex}}^{(1)} (x, r (t)) = 0, \quad B_1 (r (t), t) = 0, \quad A_{\text{ex},1} (r (t), t) = 0.
\] (5.5.23)

To estimate of the discrepancy resulting from magnetic field we introduce quantities

\[
|\ddot{B}| = \max_{0 \leq t \leq T} B(r (t), t),
\]

\[
|\ddot{A}| = \max_{0 \leq t \leq T} \max_{0 < |z| \leq a \sigma_{\psi}} |A(r (t) + z, t) - A(r (0), 0)|.
\] (5.5.24)

\[
\frac{1}{R_B (t, r)} = \sup_{0 < |z| \leq a \sigma_{\psi}} \frac{|A(r (t) + z, t)|}{|z| |\ddot{B}|},
\]

\[
\frac{1}{R_A (t, r)} = \sup_{0 < |z| \leq a \sigma_{\psi}} \frac{|A_{\text{ex},1} (r (t) + z, t)|}{|z| |\ddot{A}|},
\]

\[
\frac{1}{R_M} = \max_{0 \leq t \leq T} \left( \frac{1}{R_B (t, r)} \frac{1}{R_A (t, r)} \right).
\] (5.5.25)

The quantity \( R_A (t, r) \) is a spatial distance at which the local variation of \( A \) is of the same order as the global variation \( |\ddot{A}| \) of \( A \) itself, and consequently it represents a spatial scale at which the local variation of \( A \) is not negligible. By (5.5.23) the quantity \( 1/R_M \) is bounded. We substitute (5.1.2) in (5.0.12) and observe that according to Subsection 5.5, \( \psi, \varphi \) exactly satisfy (5.0.12) if \( \varphi_{\text{ex}}^{(1)} (t, x), B_1 (t, x), A_{\text{ex},1} (t, x) \) are identically zero. If they are not zero, we obtain a discrepancy

\[
D_1 = \frac{\chi q}{mc} |A_{\text{ex},1} + B_1 (t, x) \times (x - r (t))| \cdot \nabla \psi.
\] (5.5.26)

Similarly to (5.4.17) we introduce the integral discrepancy \( \tilde{D}_1 \):

\[
\tilde{D}_1 = \text{Re} \int_{\mathbb{R}^3} \frac{\chi q}{mc} \left( A_{\text{ex},1} + B_1 (t, x) \times (x - r (t)) \right) \cdot \nabla \psi \psi^* \, dx.
\] (5.5.27)

\[
= \int_{\mathbb{R}^3} \frac{\chi q}{mc} \left[ A_{\text{ex},1} + B_1 (t, x) \times (x - r (t)) \right] \cdot \nabla \left( \frac{\nabla \psi}{\psi} \right) |\psi|^2 \, dx.
\]

Note that for solutions of the form (5.1.2) we have

\[
\text{Im} \left( \frac{\nabla \psi}{\psi} \right) = \text{Im} \nabla \left( \frac{im}{\lambda} v \cdot (x - r) + \frac{i}{\lambda} s_p \right) = \frac{m}{\lambda} v,
\]

implying when combined with (5.5.27)

\[
\tilde{D}_1 = \frac{q}{c} \int_{\mathbb{R}^3} \left[ A_{\text{ex},1} + B_1 (t, x) \times (x - r (t)) \right] \cdot v |\psi|^2 \, dx,
\]

which, in turn, yields the following estimate

\[
|\tilde{D}_1| \lesssim \frac{|v|}{c} \int_{\mathbb{R}^3} \left| \dddot{A} \right| |x - r (t)| |\psi_{\text{a}}|^2 \frac{dx}{R_M (t, r)} + |\dddot{B}| \int_{\mathbb{R}^3} \frac{1}{R_B (t, r)} |x - r (t)|^2 |\psi_{\text{a}}|^2 \, dx \leq C_0 \frac{|v|}{c} \left( |q| |\dddot{A}| + q a |\dddot{B}| \right), \quad 0 \leq t \leq T.
\] (5.5.28)
Combining relation (5.4.23) and (5.5.28) we get

\[ |\bar{D}_0 + \bar{D}_1| \lesssim C_0 \frac{a^2 |q| |\bar{\varphi}|}{R^2_\varphi} + \left| \frac{\mathbf{v}}{c} \right| \frac{a}{R_M} C_0 \left( |q| |\bar{A}| + qa |\bar{B}| \right). \tag{5.5.29} \]

yielding the following conditions for the discrepancy to be relatively small

\[ \left| \frac{\mathbf{v}}{c} \right| \frac{a}{R_M} \ll 1, \quad \frac{a^2}{R^2_\varphi} \ll 1. \tag{5.5.30} \]

5.6 Stability issues

A comprehensive analysis of the stability is complex, involved and beyond the scope of this paper. Nevertheless, we would like to give a concise consideration to three aspects of stability for well separated charges in the nonrelativistic regime: (i) no "blow-up" or "collapse"; (ii) preservation with high accuracy of the form of a wave-corpuscle solution for a limited time; (iii) preservation of spatial localization for certain solutions on long time intervals.

Here is an argument for why there can not be a "blow-up" in finite time. A "blow-up" is an issue since the nonlinearity \( G'(s) \) provides focusing properties with consequent soliton-like solutions \( \psi^\ell, \phi^\ell \). In our model the possibility of "blow up" is excluded when we define \( G^\ell_a \) to be a constant for large amplitudes of the fields, namely for \( s \geq \left( \psi^\ell(0) \right)^2 \) as in (4.4.8). This factor combined with the charge normalization condition (5.0.14) implies that the energy is bounded from below. Indeed, according to (6.1.8), (6.1.10) and (4.1.18) the energy of a free charge can be written in the form

\[ \mathcal{E}(\psi, \varphi) = \int_{\mathbb{R}^3} (w + \tilde{u}) \, \mathrm{d}x = \int_{\mathbb{R}^3} \frac{\nabla \varphi^2}{8\pi} + \frac{\chi^2}{2m} [\nabla |\psi|^2 + G(|\psi|^2)] \, \mathrm{d}x, \tag{5.6.1} \]

where \( \varphi = \varphi_\psi \) is determined from (2.4.6). In view of relations (4.4.8) the nonlinearity derivative \( G'(s) \) is bounded, implying \( G(|\psi|^2) \geq -C |\psi|^2 \) for a constant \( C \). That combined with the charge normalization condition (5.0.14) implies boundedness of the energy from below, namely

\[ \mathcal{E}(\psi, \varphi) \geq -C \text{ for all } \psi, \varphi, \quad \|\psi\|^2 = \int_{\mathbb{R}^3} |\psi|^2 \, \mathrm{d}x = 1. \tag{5.6.2} \]

A similar argument in the case of many interacting charges also shows that the energy is bounded from below. Since energy is a conserved quantity, using the boundedness of the energy from below one can prove along lines of [Kato89] the global existence of a unique solution \( \psi^\ell(t, x), \phi^\ell(t, x) \) to (2.4.2), (2.4.7) for all times \( 0 \leq t < \infty \) for given initial data \( \psi^\ell(0, x) \).

The second aspect of stability is about a preservation of the wave-corpuscle shape with high accuracy for limited times. A basis for it is provided in Section 5.4.2. Since discrepancies in the equations (5.4.15) are of the order \( |q| |\bar{\varphi}| a^2/R^2 \) for the charge in an external EM field the fields \( \psi, \varphi \) have to be close to the wave-corpuscle of the form (5.1.2) on time intervals of order \( |q| |\bar{\varphi}| a^2/ (\chi R^2) \) where \( R \) is a spatial scale of inhomogeneity of the external field, and \( |q| |\bar{\varphi}| \) is a global variation of the external field potential energy near the trajectory of the wave-corpuscle.

The third aspect is a stability on very long time intervals which is understood in a broader sense, namely when a charge maintains its spatial localization without necessarily preserving
the exact form of a wave-corpuscle It is shown in Section 6.1.1 that such a broad localization assumption is sufficient to identify the corresponding point charge trajectory. Now let us consider the following argument for the charge stability based on properties of the energy. For simplicity let us consider a single free charge with energy (5.6.1). The energy conservation law implies

$$E(\psi(t), \varphi(t)) = E(\psi(0), \varphi(0)), \text{ for all } 0 \leq t < \infty. \quad (5.6.3)$$

Note that the rest solution \( \check{\psi} \) as in (2.3.5), (2.3.6) is a critical point of \( E \) defined by (5.6.1). Let us assume that the rest solution \( \check{\psi} \) is the global minimum under the charge normalization constraint, namely

$$E\left(\check{\psi}, \varphi \check{\psi}\right) = \min_{\|\psi\| = 1} E(\psi, \varphi) = E_0. \quad (5.6.4)$$

Consider then the initial data \( \psi_0 \) for (4.0.11)-(4.0.12) at \( t = 0 \) that (i) satisfies the charge normalization condition (5.0.14); (ii) is close to \( \check{\psi} \) and has almost the same energy, i.e. \( |E(\psi(0), \varphi(0)) - E_0| \ll 1 \). Note that since every spatial shift \( \psi(x - r), \varphi_\psi(x - r) \) of \( \psi(x), \varphi_\psi(x) \) produces fields satisfying the charge normalization condition (5.0.14) and of of the the same energy, the minimum in (5.6.1) has to be degenerate. But if we assume that all the degeneracy is due to spatial translations, rotations and the multiplication by \( e^{i\sigma} \), then the condition \( |E(\psi(t), \varphi(t)) - E_0| \ll 1 \) though allowing for spatial translation of \( \psi(x), \varphi_\psi(x) \) to large distances and times, still implies that form of \( |\psi(t, x - r(t))|, \varphi(t, x - r(t)) \) has to be almost the same as the form of \( \check{\psi}(x), \check{\varphi}(x) \). The same argument works for a local minimum that is non-degenerate modulo spatial translations, rotations and the multiplication by \( e^{i\sigma} \).

6 Many interacting charges

A qualitatively new physical component in the theory of two or more charges compared to the theory of a single charge is obviously the interaction between them. In our approach any individual "bare" charge interacts directly only with the EM field, and consequently different charges interact with each other only indirectly through the EM field. In this section we develop the Lagrangian theory for many interacting charges for the both relativistic and nonrelativistic cases based on Lagrangians for single charges studies in Sections 3, 4, 5. The primary focus of our studies on many charges is ways of integration into our wave theory the point charge mechanics in the regime of remote interaction when the charges are separated by large distances compare to their sizes. A system of many charges can have charges of different type, for instance electrons and protons. In that case we naturally assume that individual Lagrangians for charges of the same type to have identical Lagrangians with the same mass \( m \), charge \( q \), form factor \( \check{\psi} \) and consequently the same nonlinear self-interaction \( G \). We use here general results of the Lagrangian field theory for many interacting charges including symmetries, conservation laws and the construction of gauge-invariant and symmetric energy-momentum tensors described in Section 11.5.

Let us introduce a system of \( N \) charges interacting directly only with the EM field described by its 4-vector potential \( A^\mu = (\varphi, A) \). The charges are described by their wave functions \( \psi^\ell \) with the superscript index \( \ell = 1, \ldots, N \) labeling them. In this section we study the dynamics of the system of charges in the regime of remote interaction, that is when any two different charges of the system are well separated so that the distance between them is much larger compare to their typical sizes. We show here that under the assumption of remote interaction the charges interact which is other by Lorentz forces and that in non-relativistic
case their dynamics is perfect correspondence with the dynamics of the corresponding system of point charges. In the relativistic case the correspondence with the point mechanics is more subtle because of fundamental limitations. In turns out that in the regime of remote interaction the nonlinear self-interaction terms associated with charges do not manifest themselves in any way but making charges to behave as wave-corpuscles similar to ones studied in Sections 3, 4, 5.

In non-relativistic case we provide a big picture of interaction and dynamics via a single charge in an external field and charges interacting instantaneously via their individual electric fields.

6.1 Non-relativistic theory of interacting charges

The purpose of this section is to develop a non-relativistic theory of many interacting charges that would be sufficient for establishing its intimate relation to the point charges mechanics. Developed here non-relativistic theory for many interacting charges naturally integrates the theory of single non-relativistic charge developed in Sections 4 and 5, including the set up for the interaction between the bare charge and the EM field as described by its electric potential $\varphi$. Our nonrelativistic Lagrangian $\hat{\mathcal{L}}$ for many charges is constructed based on (i) individual charges nonrelativistic Lagrangians $\hat{\mathcal{L}}_{\ell}$ of the form (4.0.6) and (ii) the assumption that every charge interacts directly only with the EM field as defined by its electric potential $\varphi$, namely

$$\hat{\mathcal{L}} = \hat{\mathcal{L}} (\psi^\ell, \psi_{\mu}^\ell, \psi^{\ast \ell}, \nabla \varphi, \varphi, x^\mu) = \frac{|\nabla \varphi|^2}{8\pi} + \sum_{\ell} \hat{\mathcal{L}}_{\ell} (\psi^\ell, \psi^{\ast \ell}, \varphi), \quad (6.1.1)$$

where

$$\hat{\mathcal{L}}_{\ell} = \frac{\chi}{2} \left[ \psi^{\ast \ell} \tilde{\partial}^\ell \psi^\ell - \psi^\ell \tilde{\partial}^{\ast \ell} \psi^{\ast \ell} \right] - \frac{\chi^2}{2m^\ell} \left\{ \tilde{\nabla} \psi^\ell \tilde{\nabla}^{\ast \ell} \psi^{\ast \ell} + G^\ell (\psi^{\ast \ell} \psi^\ell) \right\},$$

$$\tilde{\partial}^\ell = \partial_t + \frac{iq^\ell (\varphi + \varphi_{\mathrm{ex}})}{\chi}, \quad \tilde{\nabla}^\ell = \nabla - \frac{iq^\ell A_{\mathrm{ex}}}{\chi c},$$

where $(\varphi_{\mathrm{ex}}, A_{\mathrm{ex}})$ is the potential of the external EM field. Evidently, according to this Lagrangian, every charge is coupled to the EM field exactly as if it were a single charge, but since there is just one EM field all charges are coupled. The Euler-Lagrange field equations for this Lagrangian are

$$\chi i \tilde{\partial}^\ell \psi^\ell = \frac{\chi^2}{2m^\ell} \left[ -\tilde{\nabla}^{\ell 2} \psi^\ell + \left[ G^\ell \right]' \left( |\psi^\ell|^2 \right) \psi^\ell \right], \quad (6.1.2)$$

$$-\Delta \varphi = 4\pi \sum_{\ell} q^\ell |\psi^\ell|^2, \quad (6.1.3)$$

where $[G^\ell]' (s) = \partial_s G^\ell (s)$, and as in the case of a single charge $\psi^{\ast \ell}$ is the complex conjugate to $\psi^\ell$ for all $\ell$. The nonlinear self-interaction terms $G^\ell$ in (6.1.1) are determined based on the corresponding form factors $\hat{\psi}^\ell$ from $\ell$-th charge equilibrium equation (2.3.8).

The Lagrangian $\hat{\mathcal{L}}$ defined by (6.1.1) is gauge invariant with respect to the first and the second gauge transformations (11.7.6), (11.7.7) and consequently every $\ell$-th charge has a conserved current $J_{\mu}^\ell = (c\rho^\ell, J^\ell)$ which can be found from relations (11.7.12), (11.7.13).
yielding the following formulas similar to (6.1.8)

\[ \rho^\ell = q |\psi^\ell|^2, \]  
\[ J^\ell = \frac{i\chi q^\ell}{2m^\ell} \left[ \psi^\ell \nabla^\ell \psi^\ell - \psi^\ell \nabla^\ell \psi^\ell \right] = \left( \frac{\chi q^\ell}{m^\ell} \Im \frac{\nabla\psi^\ell}{\psi^\ell} - \frac{q^\ell}{mc} A_{ex} \right) |\psi^\ell|^2, \]  
\[ \frac{\partial}{\partial t} |\psi^\ell|^2 + \nabla \cdot \left( \frac{m^\ell}{\chi} \Im \frac{\nabla\psi^\ell}{\psi^\ell} |\psi^\ell|^2 - \frac{q^\ell}{c} A_{ex} |\psi^\ell|^2 \right) = 0. \]  

Every current \( J^\ell_{\mu} \) satisfies the conservation/continuity equations \( \partial_{\mu} \rho^\ell + \nabla \cdot J^\ell = 0 \) or explicitly

\[ m^\ell \frac{\partial}{\partial t} |\psi^\ell|^2 + \nabla \cdot \left( \frac{\chi}{m^\ell} \Im \frac{\nabla\psi^\ell}{\psi^\ell} |\psi^\ell|^2 - \frac{q^\ell}{c} A_{ex} |\psi^\ell|^2 \right) = 0. \]  

The equations (6.1.5) imply the conservation of the total \( \ell \)-th conserved charge. Similarly to the case of a single charge we require every total \( \ell \)-th conserved charge to be exactly \( q^\ell \) and, hence, to satisfy the following charge normalization condition of the form (2.4.5)

\[ \int_{\mathbb{R}^3} |\psi^\ell|^2 \, dx = 1 \text{ for all } t. \]  

Next based on the equation (6.1.3) it is natural to introduce for every \( \ell \)-th charge the corresponding potential \( \varphi^\ell \) using the Green function (12.0.11), namely

\[ \varphi^\ell (t, x) = q^\ell \int_{\mathbb{R}^3} \frac{|\psi^\ell|^2 (t, y)}{|y - x|} \, dy, \varphi = \sum_{\ell} \varphi^\ell. \]  

Taking into account the expression for the covariant derivatives from (6.1.1) we can recast the field equations (6.1.2), (6.1.3) as (2.4.2). The charge conservation equations (6.1.5) can be alternatively derived directly from the field equations (2.4.2) by multiplying them and their complex conjugate, respectively, by \( \psi^\ell \), \( \psi^\ell \) and subtracting from one another.

One can see in the integral expression (6.1.7) instantaneous action-at-a-distance, a feature which occurs in the nonrelativistic point Lagrangian mechanics.

Many technical aspects needed for the treatment of many charges in the regime of remote interaction are already developed in our studies of a single non-relativistic charge in an external EM field in Section 5, and to avoid repetition whenever the case we use relevant results from there. In fact, an accurate guiding principle for the treatment of distant interaction of non-relativistic charges is to view every charge as essentially a single one in the slowly varying in the space and time external EM field created by other charges.

To study the motion of the energy and momentum for the involved charges and the EM field we introduce for every \( \ell \)-th charge its gauge invariant energy-momentum tensor \( \tilde{T}^{\mu\nu}_\ell \) based on the formulas (5.2.7)-(5.2.10) substituting there \( \psi^\ell, \psi^{\ell*} \) in place of \( \psi, \psi^* \) and the covariant derivatives with the index \( \ell \) from (6.1.1) in place of the covariant derivatives for \( \psi, \psi^* \). That yields the following formulas for energy and momentum densities for individual charges

\[ \tilde{u}^\ell = \frac{\chi^2}{2m} \left[ \tilde{\nabla}^\ell \psi^\ell \cdot \nabla^{\ell*} \psi^{\ell*} + G^\ell (\psi^{\ell*} \psi^\ell) \right], \]  
\[ \tilde{p}^\ell_j = \frac{\chi}{2} \left( \psi^\ell \tilde{\partial}^\ell_j \psi^{\ell*} - \psi^{\ell*} \tilde{\partial}^\ell_j \psi^\ell \right), \quad j = 1, 2, 3. \]  

The gauge invariant energy-momentum tensor \( \Theta^{\mu\nu} \) for the EM field is defined by (5.2.3)-(5.2.5) and, in particular, its energy, momentum and energy flux densities are

\[ \partial_0 w = \frac{\mathbf{J} \cdot \nabla \varphi}{c} = - \frac{\mathbf{J} \cdot \mathbf{E}}{c}, \quad g_j = 0, \quad s_j = 0. \]
Using the field equations (6.1.2), (6.1.3) and the representation (4.0.8) for $F^{\nu\mu}$ we can also verify that following conservations laws for the individual charges and the EM field:

$$\partial_{\mu} \tilde{T}^{\ell\mu\nu} = f_{\nu}^{\ell} + f_{\text{ex}}^{\ell\nu}, \quad \partial_{\mu} \Theta^{\mu\nu} = -\sum_{\ell} f_{\nu}^{\ell}, \tag{6.1.11}$$

where

$$f_{\nu}^{\ell} = \frac{1}{c} J_{\mu}^{\ell} F_{\nu\mu}^{\ell} = \left( -\frac{1}{c} J_{\mu}^{\ell} \cdot E_{\nu}^{\text{ex}}, \varphi_{\ell}^{\text{ex}} \right),$$

$$f_{\text{ex}}^{\ell\nu} = \frac{1}{c} J_{\mu}^{\ell} F_{\nu\mu}^{\text{ex}} = \left( -\frac{1}{c} J_{\mu}^{\ell} \cdot E_{\nu}^{\text{ex}}, \rho_{\text{ex}}^{\ell} E_{\text{ex}}^{\ell} + \frac{1}{c} J_{\mu}^{\ell} \times B_{\text{ex}}^{\ell} \right).$$

The energy-momentum conservation equations (6.1.11) can be viewed as equations of motion in an elastic continuum, [Moller, Section 6.4, (6.56), (6.57)], similar to the case of kinetic energy-momentum tensor for a single relativistic particle, [Pauli RT, Section 37, (3.24)]. It is important to remember though that in contrast to the point mechanics the equations of motion (6.1.11) must always be complemented with the field equations (6.1.2), (6.1.3) or (2.4.2) without which they do not have to hold and are not alone sufficient to determine the motion. We also recognize in $f_{\nu}^{\ell}$ and $f_{\text{ex}}^{\ell\nu}$ in the equations of motion (6.1.11) respectively the Lorentz force densities for the charge in the EM field (of charges) and the same for the external EM field. Observe that equations (6.1.11) satisfy manifestly the Newton’s principle “action equals reaction” for all involved densities at every point of the space-time.

In the regime of remote interactions it makes sense to introduce dressed charges and attribute to every charge its EM field via the potential $\varphi_{\ell}$ as defined by relations (2.4.2) and (6.1.7). Based on the potentials $\varphi_{\ell}$ we define the corresponding energy-momentum tensor $\Theta^{\ell\mu\nu}$ by formulas (5.2.3)-(5.2.5) where we substitute $\varphi_{\ell}$ and $J_{\mu}^{\ell}$ defined by equalities (6.1.7), (6.1.4) in place of $\varphi$ and $J$. One can verify then that the conservation law (5.2.13) for $\Theta^{\ell\mu\nu}$ takes here the form

$$\partial_{\mu} \Theta^{\ell\mu\nu} = \frac{1}{c} J_{\mu}^{\ell} F_{\nu\mu}^{\ell}. \tag{6.1.12}$$

Now for every $\ell$-th dressed charge we define its energy-momentum tensor $T^{\ell\mu\nu}$ by the formula

$$T^{\ell\mu\nu} = \tilde{T}^{\ell\mu\nu} + \Theta^{\ell\mu\nu}. \tag{6.1.13}$$

It is also natural and useful to introduce for every $\ell$-th charge the EM field $E_{\text{ex}}^{\ell}$ and $F_{\text{ex}}^{\ell\nu\mu}$ of all other charges $\ell' \neq \ell$ by

$$E_{\text{ex}}^{\ell} = \sum_{\ell' \neq \ell} E_{\ell'}^{\ell}, \quad F_{\text{ex}}^{\ell\nu\mu} = \sum_{\ell' \neq \ell} F_{\ell'}^{\ell\nu\mu}, \quad E_{\ell}^{\ell} = -\nabla \varphi_{\ell}. \tag{6.1.14}$$

Then combining relations (6.1.11), (6.1.12) with (4.0.8) we obtain the following equations of motion for dressed charges

$$\partial_{\mu} T^{\ell\mu\nu} = f_{\nu}^{\ell} + f_{\text{ex}}^{\ell\nu}, \tag{6.1.15}$$

$$f_{\nu}^{\ell} = \frac{1}{c} J_{\mu}^{\ell} \sum_{\ell' \neq \ell} F_{\ell'}^{\ell\nu\mu} = \left( -\frac{1}{c} J_{\mu}^{\ell} \cdot E_{\text{ex}}^{\ell}, \rho_{\text{ex}}^{\ell} E_{\text{ex}}^{\ell} \right).$$
describing the motion of energies and momenta of the dressed charges in the space-time continuum. Importantly, the Lorentz force \( f^{\nu\ell} \) in the right-hand of (6.1.15) excludes manifestly the self-interaction in contrast to the Lorentz force acting upon bare charge as in (6.1.11) which explicitly includes the self-interaction term \( \frac{1}{c} J^\mu_\ell F^\mu_\nu \). Thus, we can conclude that when the charge and its EM field are treated as a single entity, namely dressed charge, there is no self-interaction as signified by the exact equations (6.2.26).

It follows from (6.1.4) and (6.1.9) that the charge gauge invariant momentum density \( P^\ell \) equals exactly the microcurrent density \( J^\ell \) multiplied by the constant \( m^\ell / q^\ell \), namely the following identity holds

\[
P^\ell = \frac{m^\ell}{q^\ell} J^\ell = \frac{i \chi}{2} \left[ \psi^\ell \bar{\nabla}^\ell \psi^\ell - \psi^\ell \bar{\nabla}^\ell \bar{\psi}^\ell \right] = \left( \chi \text{Im} \frac{\bar{\nabla}^\ell \psi^\ell}{\bar{\psi}^\ell} - \frac{q^\ell \bar{A}}{c} \right) |\psi^\ell|^2,
\]

that can be viewed as the momentum density kinematic representation. We can also recast the above equality as

\[
P^\ell (t, x) = m v^\ell (t, x), \quad \text{where} \quad v^\ell (t, x) = J^\ell (t, x) / q^\ell \quad \text{is the velocity density.}
\]

Up to this point we introduced the basic elements of theory of interacting charges described as fields via the Lagrangian (6.1.1). A natural question then is in what ways the point charge mechanics is integrated into this Lagrangian classical field theory? There are two distinct ways to correspond our field theory to the point charge mechanics: (i) via averaged quantities in spirit of the well known in quantum mechanics Ehrenfest Theorem, [Schiff, Sections 7, 23]; (ii) via a construction of approximate solutions to the field equations (6.1.2), (6.1.3), (2.4.2) in terms of radial wave-corpuscles similar to (5.1.2). We consider these two ways in the next subsections.

6.1.1 Point mechanics via averaged quantities

Introducing the total individual momenta \( P^\ell \) and energies \( E^\ell \) for \( \ell \)-th dressed charge

\[
P^\ell = \int_{\mathbb{R}^3} p^\ell \, dx, \quad E^\ell = \int_{\mathbb{R}^3} \bar{u}^\ell \, dx,
\]

and using arguments similar to (5.3.1)-(5.3.3) combined with relations (6.1.14), (6.1.15) we obtain the following equations

\[
\frac{dP^\ell}{dt} = q^\ell \int_{\mathbb{R}^3} \left[ \left( \sum_{\ell' \neq \ell} E^{\ell'} + E_{\text{ex}} \right) |\psi^{\ell'}|^2 + \frac{1}{c} v^{\ell} \times B_{\text{ex}} \right] \, dx, \quad \text{where (6.1.19)}
\]

\[
\int_{\mathbb{R}^3} \sum_{\ell' \neq \ell} E^{\ell'} |\psi^{\ell'}|^2 \, dx = \int_{\mathbb{R}^3 \times \mathbb{R}^3} (y - x) \left| \psi^{\ell'} (t, y) \right|^2 \left| \psi^{\ell} (t, x) \right|^2 \frac{dy}{|y - x|^3} \, dx,
\]

\[
\frac{dE^\ell}{dt} = \int_{\mathbb{R}^3} J^\ell \cdot E_{\text{ex}}^\ell (t, x) \, dx = q^\ell \int_{\mathbb{R}^3} v^\ell \cdot \left( \sum_{\ell' \neq \ell} E^{\ell'} + E_{\text{ex}} \right) \, dx.
\]

(6.1.21)
Let us introduce the $\ell$-th charge position $\mathbf{r}^\ell(t)$ and velocity $\mathbf{v}^\ell(t)$ as the following averages

$$
\mathbf{r}^\ell(t) = \int_{\mathbb{R}^3} \mathbf{x} |\psi^\ell(t, \mathbf{x})|^2 \, d\mathbf{x}, \quad \mathbf{v}^\ell(t) = \frac{1}{q} \int_{\mathbb{R}^3} \mathbf{J}^\ell(t, \mathbf{x}) \, d\mathbf{x}.
$$

(6.1.22)

Then using the charge conservation law (6.1.5) we find the following identities

$$
\frac{d\mathbf{r}^\ell(t)}{dt} = \int_{\mathbb{R}^3} \mathbf{x} \partial_t |\psi^\ell|^2 \, d\mathbf{x} = -\frac{1}{q} \int_{\mathbb{R}^3} \mathbf{x} \nabla \cdot \mathbf{J}^\ell \, d\mathbf{x} = \frac{1}{q} \int_{\mathbb{R}^3} \mathbf{J}^\ell \, d\mathbf{x} = \mathbf{v}^\ell(t),
$$

(6.1.23)

showing the positions and velocities defined by formulas (6.1.22) are related exactly as in the point charge mechanics. Then utilizing the momentum density kinematic representation (6.1.16)-(6.1.17) and the fact that the momentum density of the $\ell$-th charge EM field is identically zero according to (5.2.4) we obtain the following kinematic representation for charge and, hence, the dressed charge total momentum

$$
P^\ell(t) = \frac{m^\ell}{q^\ell} \int_{\mathbb{R}^3} \mathbf{J}^\ell(t, \mathbf{x}) \, d\mathbf{x} = m^\ell \mathbf{v}^\ell(t),
$$

(6.1.24)

which also is exactly the same as in point charges mechanics. Combining relations (6.1.19), (6.1.23) and (6.1.24) we obtain the following system of equations of motion for $N$ charges:

$$
m^\ell \frac{d^2 \mathbf{r}^\ell(t)}{dt^2} = \frac{d\mathbf{P}^\ell}{dt} = q^\ell \int_{\mathbb{R}^3} \left[ \sum_{\ell' \neq \ell} \mathbf{E}^\ell + \mathbf{E}_{ex} \right] |\psi^\ell|^2 + \frac{1}{c} \mathbf{v}^\ell \times \mathbf{B}_{ex} \right] \, d\mathbf{x},
$$

(6.1.25)

where

$$
\mathbf{E}^\ell(t, \mathbf{x}) = -\nabla \varphi^\ell(t, \mathbf{x}), \quad \mathbf{E}_{ex} = -\nabla \varphi_{ex}(t, \mathbf{x}).
$$

The above system is analogous to the well known in quantum mechanics Ehrenfest Theorem, [Schiff, Sections 7, 23]. Notice that the system of the equations of motion (6.1.25) departs from the corresponding system for point charges by having the averaged Lorentz density force instead of the Lorentz force at exact position $\mathbf{r}^\ell(t)$. Observe, also that the system of equations of motion (6.1.25) is consistent with Newton’s third law of motion ”action equals reaction” as it follows from the relations (6.1.20).

Let us suppose now that charges and current densities $|\psi^\ell|^2$ and $q^\ell \mathbf{v}^\ell$ are localized near a point $\mathbf{r}^\ell(t)$, and have localization scales $a^\ell$ which are small compared with the typical variation scale $R_{EM}$ of the EM field. Then for a bounded $R_{EM}$ and $a^\ell$ converging to 0 we have as $a^\ell \to 0$,

$$
|\psi^\ell|^2(t, \mathbf{x}) \to \delta \left( \mathbf{x} - \mathbf{r}^\ell(t) \right), \quad \mathbf{v}^\ell(t, \mathbf{x}) \to \mathbf{v}^\ell(t) \delta \left( \mathbf{x} - \mathbf{r}^\ell(t) \right)
$$

(6.1.26)

where the coefficients at the delta-functions are determined from the charge normalization conditions (6.1.6) and relations (6.1.22). Using potential representations (6.1.7) we infer from (6.1.26) the convergence of the potentials $\varphi^\ell$ to the corresponding Coulomb’s potentials, namely

$$
\varphi^\ell(t, \mathbf{x}) \to \varphi_0^\ell(t, \mathbf{x}) = \frac{q^\ell}{|\mathbf{x} - \mathbf{r}^\ell(t)|} \quad \text{as} \quad a^\ell \to 0.
$$

(6.1.27)
Hence, we can recast the equations of motion (6.1.25) as the system

\[
m^\ell \frac{d^2 \mathbf{r}^\ell}{dt^2} = \frac{dP^\ell}{dt} = f^\ell + \epsilon_{p^\ell},
\]

(6.1.28)

where

\[
f^\ell = \sum_{\ell' \neq \ell} q^\ell E_{0}^{\ell'} + q^\ell E_{ex} (\mathbf{r}^\ell) + \frac{1}{c} \mathbf{v}^\ell \times \mathbf{B}_{ex} (\mathbf{r}^\ell), \ \ell = 1, \ldots, N,
\]

with small discrepancies \(\epsilon_{p^\ell} \rightarrow 0\) as \(a^\ell / R_{EM} \rightarrow 0\). Notice that terms \(q^\ell E_{0}^{\ell'}\) in equations (6.1.28) coincide with the Lorentz forces for the Coulomb’s potentials

\[
q^\ell E_{0}^{\ell'} = -q^\ell \nabla \varphi_{0}^{\ell'} (t, \mathbf{x}) = -q^\ell q^{\ell'} (\mathbf{r}^{\ell'} - \mathbf{r}^\ell) / |\mathbf{r}^{\ell'} - \mathbf{r}^\ell|^3.
\]

(6.1.29)

In the case when there is no external EM field the point charges equations of motion (6.1.28) in the limit \(a^\ell \rightarrow 0\) are associated with the following Lagrangian (“static limit, zeroth order in \((v/c)\)”, [Jackson, Section 12.6])

\[
\mathcal{L}_{p} = \sum_{\ell} m^\ell (\partial_{t} \mathbf{r}^\ell)^2 - \frac{1}{2} \sum_{\ell' \neq \ell} q^\ell q^{\ell'} / |\mathbf{r}^{\ell'} - \mathbf{r}^\ell|,
\]

(6.1.30)

and the equations of motion (6.1.28) take the form

\[
m^\ell \frac{d^2 \mathbf{r}^\ell}{dt^2} = -\sum_{\ell' \neq \ell} q^\ell q^{\ell'} (\mathbf{r}^{\ell'} - \mathbf{r}^\ell) / |\mathbf{r}^{\ell'} - \mathbf{r}^\ell|^3, \ \ell = 1, \ldots, N.
\]

(6.1.31)

Using similar arguments we obtain from (6.1.21)

\[
\frac{d\mathbf{E}^\ell}{dt} = \mathbf{v}^\ell \cdot \mathbf{f}^\ell + \epsilon_{E^\ell},
\]

(6.1.32)

with small discrepancies \(\epsilon_{E^\ell} \rightarrow 0\) as \(a^\ell / R_{EM} \rightarrow 0\). Combining equalities (6.1.24) (6.1.28) (6.1.32) we get

\[
\frac{dm^\ell v^\ell \cdot v^\ell}{2 dt} = v^\ell \cdot \frac{dm^\ell v^\ell}{dt} = \mathbf{v}^\ell \cdot \mathbf{f}^\ell + \mathbf{v}^\ell \cdot \epsilon_{p^\ell} = \frac{d\mathbf{E}^\ell}{dt} + \mathbf{v}^\ell \cdot \epsilon_{p^\ell} - \epsilon_{E^\ell},
\]

(6.1.33)

implying

\[
\frac{d}{dt} \left( \mathbf{E}^\ell - \frac{m^\ell \mathbf{v}^\ell \cdot \mathbf{v}^\ell}{2} \right) = \epsilon_{E^\ell} - \mathbf{v}^\ell \cdot \epsilon_{p^\ell},
\]

which, up to small errors, are kinematic representations for the energies of individual charges well known from the point charge mechanics. Note that to obtain point charges equations of motion (6.1.28) it is sufficient to assume localization only for \(\psi^\ell\).
6.1.2 Point mechanics via wave-corpuscles

In this section we construct approximate solution of the field equations (6.1.2), (6.1.3) (or, equivalently, (2.4.2)) for \(N\) interacting charges in which every charge is a wave-corpuscle defined by (5.1.2) with properly chosen complementary point charges equations of motion. The construction proposed here is valid for any external EM field, but to avoid involved expressions we consider the case when the external EM field is purely electric with \(\mathbf{A}_{\text{ex}} = 0\). The general case when \(\mathbf{A}_{\text{ex}} \neq 0\) is treated similarly based on the results of Section 5.5. We assume here that the shape factor \(\left| \hat{\psi}_1 (|x|) \right|^2\) decays exponentially as \(|x| \to \infty\) for every \(\ell\)-th charge and (5.4.14) holds. The wave-corpuscle approximation (2.4.18) is based on trajectories \(\mathbf{r}_0^{\ell}\) for the wave-corpuscle centers determined from equations (2.4.15) which involve the exact Coulomb’s potentials \(\hat{\varphi}_0^{\ell}\) corresponding to the size parameter \(a = 0\). To show that the approximation is accurate for small \(a > 0\) we use the results obtained for a single charge motion in an external field. As the first step for an estimate we introduce an auxiliary system of equations to determine all center trajectories. This system has the following property. If \(\ell\)-th charge is singled out and the potentials \(\hat{\varphi}_0^{\ell} (x - \mathbf{r}_0^{\ell} (t))\) of remaining charges are replaced by the linear approximation of \(\hat{\varphi}_0^{\ell} (x - \mathbf{r}_0^{\ell} (t))\) based on the position of the \(\ell\)-th charge, then the exact wave-corpuscle solution for the \(\ell\)-th charge is available in so modified field. In addition to that, the motion of the center \(\mathbf{r}_0^{\ell}\) of the exact solution to the auxiliary equation has the same trajectories \(\mathbf{r}_0^{\ell}\). Replacing \(\hat{\varphi}_0^{\ell}\) by the exact Coulomb’s potential \(\hat{\varphi}_0^{\ell}\) produces a contribution to the discrepancy, and the second source of the discrepancy is the field linearization at \(\mathbf{r}_0^{\ell}\). To estimate these discrepancies we use the results of Section 5.4.2. First, we find trajectories \(\mathbf{r}_0^{\ell} (t)\) from the auxiliary equations

\[
m^\ell \frac{d^2 \mathbf{r}_0^{\ell}}{dt^2} = -q^\ell \nabla \varphi_{\text{ex},0} (\mathbf{r}_0^{\ell}), \tag{6.1.34}
\]

with initial data

\[
\mathbf{r}_0^{\ell} (0) = \mathbf{r}_0^{\ell}, \quad \frac{d \mathbf{r}_0^{\ell}}{dt} (0) = \mathbf{v}_0^{\ell}, \quad \ell = 1, ..., N.
\]

The electrostatic potential \(\varphi_{\text{ex},0}\) in (6.1.34) is the Coulomb’s potential as in (2.4.14), and for \(a > 0\) we introduce an intermediate external potential for the \(\ell\)-th charge as follows:

\[
\varphi_{\text{ex},\ell} (t, x) = \varphi_{\text{ex}} (t, x) + \sum_{\ell' \neq \ell} \hat{\varphi}_0^{\ell'} (x - \mathbf{r}_0^{\ell'}). \tag{6.1.35}
\]

We define then an approximate solution \(\psi_{\text{ap}}^{\ell}\) to be of the form of wave-corpuscles (2.4.18), (2.4.19), namely:

\[
\psi_{\text{ap}}^{\ell} (t, x) = e^{iS/\hat{\psi}_0^{\ell}} \left(|x - \mathbf{r}_0^{\ell}|\right), \quad \varphi_{\text{ap}}^{\ell} (t, x) = q^\ell \hat{\varphi}_a^{\ell} (x - \mathbf{r}_0^{\ell}), \tag{6.1.36}
\]

where

\[
\mathbf{p}_0^{\ell} = m^\ell \frac{d \mathbf{r}_0^{\ell}}{dt},
\]

\[
S^{\ell} (t, x) = \mathbf{p}_0^{\ell} \cdot x - \int_0^t \frac{\mathbf{p}_0^{\ell 2}}{2m^\ell} dt' - q \int_0^t \varphi_{\text{ex},0} (t', \mathbf{r}_0^{\ell}) dt',
\]

and \(\mathbf{r}_0^{\ell}\) is solution of (6.1.34). Recall that the dependence on the size parameter \(a\) of the form factor \(\hat{\psi}_0^{\ell} = \hat{\psi}_a^{\ell}\) and corresponding potential \(\varphi_0^{\ell} = \varphi_a^{\ell}\) is given by (4.4.1). Notice that
according to relations (4.6.7) the interaction force term in (6.1.34) approaches the Lorentz forces based on the Coulomb’s potential, namely

$$\mathcal{F}_\ell (x) = q \int_{\mathbb{R}^3} \frac{\psi_\ell (t, y)}{|y - x|} \, dy \to \mathcal{F}_0 (x) = \frac{q^2}{|x|} \quad \text{as} \ a \to 0. \quad (6.1.37)$$

Let us introduce an auxiliary spatially linear potential $\tilde{\varphi}_{\ell, 0} (t, x)$ for the $\ell$-th charge by formula (5.4.4) with $\varphi_{ex}$ replaced by $\varphi_{\ell, 0}$. Observe that for every $\ell$ the pair $\{ \psi_\ell, \varphi_\ell \}$ is an exact solution to the auxiliary equation (5.4.15) with the external potential $\tilde{\varphi}_{\ell, 0} (t, x)$ obtained by the linearization of $\varphi_{\ell, 0} (t, x)$ at $r_0 (t)$ according to (5.4.4). It remains to examine if the so defined $\{ \psi_\ell, \varphi_\ell \}$ yield an approximation to the field equations (2.4.2). Indeed, the $\ell$-th wave-corpuscles $\{ \psi_\ell, \varphi_\ell \}$ is an exact solution to equations (5.4.15). To obtain from (5.4.15) the $\ell$-th equation (2.4.2) we have to replace $\tilde{\varphi}_{\ell, 0} (t, x)$ with $\tilde{\psi}_\ell (x)$ resulting in a discrepancy

$$\tilde{\varphi}_{\ell, 0} (t, x) = \tilde{\varphi}_{\ell, a} (x) = \left[ \tilde{\varphi}_{\ell, 0} (t, x) - \tilde{\varphi}_{\ell, a} (t, x) \right] \tilde{\varphi}_{a} (x) = \left[ \tilde{\varphi}_{\ell, 0} (t, x) - \tilde{\varphi}_{\ell, 0} (t, x) \right] \tilde{\varphi}_{a} (x) + \left[ \tilde{\varphi}_{\ell, 0} (t, x) - \tilde{\varphi}_{\ell, 0} (t, x) \right] \tilde{\varphi}_{a} (x). \quad (6.1.38)$$

The first term of the above discrepancy is the same as in (5.4.16), and the corresponding integral discrepancy is estimated as in (5.4.23). The second term has the form

$$\tilde{\varphi}_{\ell, 0} (t, x) = \sum_{\ell' \neq \ell} q_{\ell'} \left[ \tilde{\varphi}_{\ell'} (x - r_{\ell'}^0) - \tilde{\varphi}_{0} (x - r_{\ell'}^0) \right] e^{i S / \chi} \tilde{\varphi}_{\ell'} (x). \quad (6.1.39)$$

Taking into account that $\tilde{\psi}_{\ell} (|x|)$ decays exponentially as $|x| \to \infty$ we find that every term in (6.1.39) is small if $r_0^\ell$ is separated from $r_0^\ell$ for $\ell' \neq \ell$. To take into account point charges separation we introduce a quantity

$$R_{min} = \min_{\ell \neq \ell', 0 \leq t \leq T} \left| r_0^\ell (t) - r_0^\ell (t) \right|, \quad (6.1.40)$$

and assume it to be positive, i.e. $R_{min} > 0$. Under this condition using (4.4.5) we obtain

$$\mathcal{F}_\ell (x) = \mathcal{F}_0 (x) = \frac{q^2}{|x|} \quad \text{as} \ a \to 0, \quad (6.1.41)$$

where constant $C$, $\sigma \psi = \sigma_\psi$ are the same as in (5.4.20) (note that $\varepsilon (a/R_{min}) \to 0$ exponentially fast if $\tilde{\psi}_{\ell} (r)$ decays exponentially). Hence, we can take $\tilde{\varphi}_{\ell, a} (t, x)$ as $\varphi_{ex} (t, r, \varepsilon)$ and $\tilde{\varphi}_{\ell, 0} (t, x)$ as $\varphi_{ex} (t, x)$ in (5.4.25) with $\varepsilon = \varepsilon (a/R_{min})$. Then we use the integral discrepancy estimate (5.4.28) which implies

$$\left| \tilde{D}_0 (x, t) + \tilde{D}_1 (x, t) \right| \leq C_0 \frac{a^2}{R_\varphi} + \frac{C_{4}}{\varphi} \left( \frac{a}{R_{min}} \right) \max_{\ell} \left| q^\ell \right| \left| \varphi^\ell \right|, \quad (6.1.42)$$

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where the potential curvature radius $R_\varphi^2$ is based on $\varphi^{\ell}_{ex,0}(x,t)$ as in (6.1.35), $\varphi^\ell$ is based on $\varphi^{\ell}_{ex,0}$ and $r^\ell_0$. The factors $|q^\ell|/|\varphi^\ell|$ are bounded uniformly in $a$. Consequently, we conclude that the integral discrepancy resulting from the substitution of $\psi^{\ell}_{ap}(t,x)$ into (2.4.2) and given by (6.1.41) tends to zero as $a/R \to 0$ where $R = \min(R_{\max}, R_\varphi)$. Note that for exponentially decaying $\psi^\ell$ the function $\varepsilon(a/R_{\min})$ decays exponentially as $a/R_{\min} \to 0$ and hence

$$\left| \hat{D}_0(x,t) + \hat{D}_1(x,t) \right| \simeq O\left( \frac{a^2}{R_\varphi^2} \right) U, \quad U = \max_{\ell} |q^\ell|/|\varphi^\ell|.$$

Interestingly, an additional analysis of the exact equations of motion (2.4.12) shows that though the integral discrepancy decays as $a^2/R_\varphi^2$, the positions $r^\ell(t)$ given by (6.1.22) are approximated by $r^\ell_0(t)$ with accuracy of the order $a^3/R_\varphi^3$.

### 6.2 Relativistic theory of interacting charges

Relativistic theory of many interacting point particles is known to have fundamental difficulties. "The invariant formulation of the motion of two or more interacting particles is complicated by the fact that each particle will have a different proper time. ... No exact general theory seems to be available", [Barut, Section II.1, System of colliding particles]. Some of these difficulties are analyzed by H. Goldstein in his classical monograph, [Goldstein, Section 7.10]: "The great stumbling block however is the treatment of the type of interaction that is so natural and common in nonrelativistic mechanics - direct interaction between particles. ... To say that the force on a particle depends upon the positions or velocities of other particles at the same time implies propagation of effects with infinite velocity from one particle to another - "action at a distance." In special relativity, where signals cannot travel faster than the speed of light, action-at-a-distance seems outlawed. And in a certain sense this seems to be the correct picture. It has been proven that if we require certain properties of the system to behave in the normal way (such as conservation of total linear momentum), then there can be no covariant direct interaction between particles except through contact forces." Another argument, due to von Laue, on the incompatibility of the relativity with any finite dimensional mechanical system was articulated by W. Pauli, [Pauli RT, Section 45]: "...This in itself raised strong doubts as to the possibility of introducing the concept of a rigid body into relativistic mechanics. The final clarification was brought about in a paper by Laue, who showed by quite elementary arguments that the number of kinematic degrees of freedom of a body cannot be limited, according to the theory of relativity. For, since no action can be propagated with a velocity greater than that of light, an impulse which is given to the body simultaneously at $n$ different places, will, to start of with, produce a motion to which at least $n$ degrees of freedom must be ascribed.”

Now we ask ourselves what features of point charges mechanics can be integrated into a relativistic mechanics of fields? It seems that the above arguments by Goldstein, von Laue and Pauli completely rule out any Lagrangian mechanics with finitely many degrees of freedom even as an approximation because of its incompatibility with a basic relativity requirement for the signal speed not to exceed the speed of light. On the constructive side, these arguments suggest that (i) the EM field has to be an integral part of charges mechanics, (ii) every charge of the system has to be some kind of elastic continuum coupled to the EM field. We anticipate though that point mechanics features that can be integrated into a relativistic field mechanics are limited and have subtler manifestation compared to the nonrelativistic theory. We expect
point mechanics features to manifest themselves in (i) identification of the energy-momentum
tensor for every individual bare charge; (ii) certain partition of the EM field into a sum of
EM fields attributed to individual charges with consequent formation of dressed charges,
that is bare charges with attached to them EM fields. That energy-momentum partition
between individual charges must comply with the Newton’s “action equals to reaction” law,
the interaction forces densities have to be of the Lorentzian form and every dressed charge
has not to interact with itself.

In the theory proposed here we address the above challenges by (i) the principal depar-
ture from the concept of point charge, which is substituted by a concept of wave-corpuscle
described by a complex valued function in the space-time; (ii) requirement for every charge
to interact directly to only the EM field implying that different charges interact only via the
EM field. With all that in mind we introduce the system Lagrangian \( L \) to be of the general
form as in Section 11.5

\[
L \left( \{ \psi^\ell, \psi^\ell_\mu \}, \{ \psi^{\ell*}, \psi^{\ell*}_\mu \}, A^\mu \right) = \sum \ell L^\ell \left( \psi^\ell, \psi^\ell_\mu, \psi^{\ell*}, \psi^{\ell*}_\mu \right) - \frac{F^{\mu\nu}F_{\mu\nu}}{16\pi},
\]

(6.2.1)

with every \( \ell \)-th charge Lagrangian \( L^\ell \) to be of the form of single relativistic charge (3.0.1)-(3.0.2)

\[
L^\ell \left( \psi^\ell, \psi^\ell_\mu, \psi^{\ell*}, \psi^{\ell*}_\mu \right) = \frac{\chi^2}{2m^\ell} \left\{ \psi^{\ell*}_\mu \psi^\ell_\mu - \kappa^2 \psi^{\ell*} \psi^\ell - G^\ell \left( \psi^{\ell*} \psi^\ell \right) \right\},
\]

(6.2.2)

where \( \kappa^\ell = \frac{\omega^\ell}{c} = \frac{m^\ell c}{\chi}, \quad \omega^\ell = \frac{m^\ell c^2}{\chi} \),
and \( \psi^\ell_\mu \) and \( \psi^{\ell*}_\mu \) are the covariant derivatives defined by the following formulas

\[
\psi^\ell_\mu = \tilde{\partial}^\ell_\mu \psi^\ell, \quad \psi^{\ell*}_\mu = \tilde{\partial}^{\ell*}_\mu \psi^\ell, \quad \tilde{\partial}^\ell_\mu = \partial^\mu + \frac{i q^\ell A^\mu}{\chi c}, \quad \tilde{\partial}^{\ell*}_\mu = \partial^\mu - \frac{i q^\ell A^\mu}{\chi c},
\]

(6.2.3)

where \( \tilde{\partial}^\ell_\mu \) and \( \tilde{\partial}^{\ell*}_\mu \) are called the covariant differentiation operators. We also assume that for
every \( \ell \): (i) \( m^\ell > 0 \) is the charge mass; (ii) \( q^\ell \) is a real valued (positive or negative) charge; (iii)
\( \kappa^\ell > 0 \); (iv) \( G^\ell \) is a nonlinear self-interaction function. Notice that charges interaction with
the EM field enters the Lagrangian \( L \) via the covariant derivatives (6.2.3). The Lagrangian \( L \) defined by (6.2.1)-(6.2.4) is manifestly local, Lorentz invariant, and gauge invariant with
respect to the second-kind (local) gauge transformation

\[
\psi^\ell \rightarrow e^{-i q^\ell \chi^\ell (x)} \psi^\ell, \quad A^\mu \rightarrow A^\mu + \partial^\mu \chi^\ell (x),
\]

(6.2.4)
as well as with respect to the group of global (the first-kind) gauge transformations

\[
\psi^\ell \rightarrow e^{-i q^\ell \chi^\ell} \psi^\ell, \quad A^\mu \rightarrow A^\mu
\]

(6.2.5)
for any real numbers \( \chi^\ell \). The Euler-Lagrange field equations (11.5.12)-(11.5.13) for the
Lagrangian \( L \) defined by (6.2.1)-(6.2.4) take the form:

\[
\left\{ \tilde{\partial}^\ell_\mu \tilde{\partial}^{\ell*}_\mu + \kappa^2 + G^{\ell*} \left( |\psi^\ell|^2 \right) \right\} \psi^\ell = 0, \quad \tilde{\partial}^\ell_\mu \psi^\ell = \partial^\mu + \frac{i q^\ell A^\mu}{\chi c},
\]

(6.2.6)
together with the conjugate equation for $\psi^*\ell$

$$\left[\bar{\partial}_\mu^\ell \bar{\partial}^{*\mu} + \kappa^2 + G^\ell \left(|\psi^\ell|^2\right)\right] \psi^*\ell = 0, \; \bar{\partial}^{*\mu} = \partial^\mu - \frac{iq^\ell A^\mu}{\chi c}, \tag{6.2.7}$$

and equations for the EM field 4-potentials

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \; J^\nu = \sum_\ell J^\ell_{\nu}, \; F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{6.2.8}$$

where the $\ell$-th charge 4-vector EM micro-current $J^\ell_{\nu}$ defined by (11.5.14) takes here the form

$$J^\ell_{\mu\nu} = -\frac{q^\ell}{2m^\ell} \left( \Im \frac{\partial^\mu}{\psi^\ell} + \frac{q^\ell A^\mu}{c} \right). \tag{6.2.9}$$

Observe that the equations (6.2.8) are the Maxwell equations (11.4.7) with currents $J^\ell_{\nu}$. As in the case of the more general Lagrangian (11.5.5) the gauge invariance (6.2.5) implies that every individual $\ell$-th charge 4-vector micro-current $J^\ell_{\nu}$ satisfies the continuity equation

$$\partial_\nu J^\ell_{\nu} = 0, \; \partial_t \rho^\ell + \nabla \cdot J^\ell = 0, \; J^\ell_{\nu} = \left(\rho^\ell c, J^\ell\right), \tag{6.2.10}$$

under the assumption that the fields $\{\psi^\ell, F^{\mu\nu}\}$ satisfy the Euler-Lagrange field equations (6.2.6)-(6.2.8). Notice that in view of (6.2.9)

$$\rho^\ell = -\frac{q^\ell}{m^\ell c^2} \left( \chi \Im \frac{\partial_\mu}{\psi^\ell} + q^\ell \varphi \right), \tag{6.2.11}$$

$$J^\ell = \frac{q^\ell}{m^\ell} \left( \chi \Im \frac{\nabla}{\psi^\ell} - \frac{q^\ell A}{c} \right).$$

As a consequence of the continuity equations (6.2.10) the space integral of every $\rho^\ell (x)$ is a conserved quantity which we assign to be exactly $q^\ell$, i.e. we assume the following charge normalization

$$\int_{\mathbb{R}^3} \rho^\ell (x) \, dx = \frac{1}{m^\ell c^2} \int_{\mathbb{R}^3} \left( \chi \Im \frac{\partial_\mu}{\psi^\ell} + q^\ell \varphi \right) |\psi^\ell|^2 \, dx = 1, \; \ell = 1, \ldots N. \tag{6.2.12}$$

To summarize, the equations (6.2.6)-(6.2.8) constitute a complete set of equations describing the state of the all fields $\{\psi^\ell, F^{\mu\nu}\}$ in the space-time. Notice that (6.2.6) and the Maxwell equations can be recast as

$$\left[ c^{-2} \tilde{\partial}_t^2 - \tilde{\nabla}^2 + \kappa^2 + G^\ell \left(|\psi^\ell|^2\right)\right] \psi^\ell = 0, \tag{6.2.13}$$

where

$$\tilde{\partial}_t^\ell = \partial_t + \frac{iq^\ell \varphi}{\chi}, \; \tilde{\nabla}^\ell = \nabla - \frac{iq^\ell A}{\chi c},$$

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\[ \nabla \cdot (\partial_t \mathbf{A} + \nabla \varphi) = 4\pi \sum_{\ell} \frac{q_{\ell}}{m_{\ell} c^2} \left( \chi \text{Im} \frac{\partial \psi_{\ell}^*}{\psi_{\ell}} + q_{\ell} \varphi \right), \quad (6.2.14) \]

\[ \nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c} \partial_t (\partial_t \mathbf{A} + \nabla \varphi) = \]

\[ = \frac{4\pi}{c} \sum_{\ell} \left( \chi \frac{q_{\ell}}{m_{\ell}} \text{Im} \left( \frac{\nabla \psi_{\ell}^*}{\psi_{\ell}^*} - \frac{q_{\ell}^2}{m_{\ell} c^2} \mathbf{A} \right) \right) |\psi_{\ell}|^2. \quad (6.2.15) \]

In order to see point charge features in the charges described as fields over the continuum of the space-time we have to identify the energy-momentum tensor \( T^{\ell\mu\nu} \) for every \( \ell \)-th charge and the EM field energy-momentum \( \Theta^{\mu\nu} \). Notice that the system Lagrangian \( \mathcal{L} \) defined by (6.2.1)-(6.2.4) satisfies the symmetry condition (11.5.9) and consequently the general construction of the symmetric energy-momenta from Section 11.5 applies here. Namely, using the formulas (11.5.24)-(11.5.25) we get the following representation for the energy-momenta

\[ T^{\ell\mu\nu} = \frac{\chi^2}{2m_{\ell}} \left\{ \left( \psi_{\ell}^* \psi_{\ell} + \psi_{\ell} \psi_{\ell}^* \right) - \left[ \psi_{\ell}^* \psi_{\ell} + \kappa^2 \psi_{\ell} \psi_{\ell}^* - \mathcal{G}^\ell \left( \psi_{\ell}^* \psi_{\ell} \right) \right] \delta^{\mu\nu} \right\}, \quad (6.2.16) \]

\[ \Theta^{\mu\nu} = \frac{1}{4\pi} \left( g^{\mu\gamma} F_{\gamma\xi} F^{\xi\nu} + \frac{1}{4} g^{\mu\nu} F_{\gamma\xi} F^{\gamma\xi} \right). \quad (6.2.17) \]

The above defined energy-momentum tensors satisfy the equations (11.5.29)-(11.5.30), namely

\[ \partial_\mu T^{\ell\mu\nu} = f^{\ell\nu}, \quad (6.2.18) \]

\[ \partial_\mu \Theta^{\mu\nu} = -f^{\nu}, \quad (6.2.19) \]

where

\[ f^{\ell\nu} = \frac{1}{c} J^\ell F^{\mu\nu}, \quad f^{\nu} = \sum_\ell f^{\ell\nu} = \frac{1}{c} J^\mu F^{\mu\nu}, \quad J^\mu = \sum_\ell J^\ell. \]

The energy-momentum conservation equations (6.2.18)-(6.2.19) can be viewed as equations of motion in elastic continuum, [Moller Section 6.4, (6.56), (6.57)], similar to the case of kinetic energy-momentum tensor for a single relativistic particle, [Pauli RT, Section 37, (3.24)]. We recognize in the 4-vectors \( f^{\ell\nu} \) in right-hand side of conservation equations (6.2.18) the density of the Lorentz force acting upon \( \ell \)-th charge, and we also see the density of the Lorentz force with the minus sign in the right-hand side of EM energy-momentum conservation equation (6.2.18). We remind to the reader that equations (6.2.18)-(6.2.19) hold only under the assumption that the involved fields satisfy the field equations (6.2.6)-(6.2.9). Consequently, in contrast to the case of the point mechanics the conservation/equations of motion (6.2.18)-(6.2.19) in the elastic continuum alone cannot substitute for the field equations and determine the motion.

Now we would like to identify the EM field attributed to every individual bare charge. That can be accomplished by partitioning the total EM \( F^{\mu\nu} \) defined as a causal solution to the linear Maxwell equation (6.2.8) (see Section 11.4.1) with a source \( \frac{4\pi}{c} J^\mu \) according to the partition of the current \( J^\mu = \sum_\ell J^\ell \). Namely we introduce the EM potentials \( A^{\ell\mu} \) and the corresponding EM field \( F^{\ell\mu\nu} \) for every individual \( \ell \)-th charge as the causal solution of the form (11.4.39) to the following Maxwell equation

\[ \partial_\mu F^{\ell\mu\nu} = \frac{4\pi}{c} J^{\ell\nu}. \quad (6.2.20) \]
In view of the linearity of the Maxwell equation (6.2.8) we evidently always have
\[ F^{\mu\nu} = \sum_{\ell} F_{\ell}^{\mu\nu}. \] (6.2.21)

Being given individual EM fields \( F_{\ell}^{\mu\nu} \) we introduce the corresponding EM energy-momentum tensor \( \Theta_{\ell}^{\mu\nu} \) via the general formula (11.4.20), namely
\[ \Theta_{\ell}^{\mu\nu} = \frac{1}{4\pi} \left( g_{\mu\gamma} F_{\ell}^{\gamma\xi} F_{\ell}^{\xi\nu} + \frac{1}{4} g_{\mu\nu} F_{\ell}^{\gamma\xi} F_{\ell}^{\gamma\xi} \right). \] (6.2.22)

Then combining (6.2.20), (6.2.22) with (11.4.27) we obtain
\[ \partial_{\mu} \Theta_{\ell}^{\mu\nu} = -\frac{1}{c} J_{\ell}^{\mu} F_{\ell}^{\nu\mu}. \] (6.2.23)

Notice also that from (6.2.21) and (6.2.19) we have
\[ \partial_{\mu} T_{\ell}^{\mu\nu} = \frac{1}{c} J_{\ell}^{\mu} F_{\ell}^{\nu\mu} + \frac{1}{c} J_{\ell}^{\mu} \sum_{\ell' \neq \ell} F_{\ell'}^{\nu\mu}. \] (6.2.24)

If we introduce now the energy-momentum \( T_{\ell}^{\mu\nu} \) of the dressed charge, i.e. the charge with its EM field, by the formula
\[ T_{\ell}^{\mu\nu} = T^{\mu\nu} + \Theta_{\ell}^{\mu\nu}, \] (6.2.25)
then the sum of two equalities (6.2.23)-(6.2.24) readily yields the following equations of motion for dressed charges
\[ \partial_{\mu} T_{\ell}^{\mu\nu} = \frac{1}{c} J_{\ell}^{\mu} \sum_{\ell' \neq \ell} F_{\ell'}^{\nu\mu}, \ \ell = 1, \ldots, N. \] (6.2.26)

describing the motion of energies and momenta of the dressed charges in the space-time continuum. Importantly, the Lorentz force in the right-hand of (6.2.26) excludes manifestly the self-interaction in contrast to the Lorentz force acting upon a bare charge as in (6.2.25) which explicitly includes the self-interaction term \( \frac{1}{c} J_{\ell}^{\mu} F_{\ell}^{\nu\mu} \). Thus, we can conclude that when the charge and its EM field are treated as a single entity, namely dressed charge, there is no self-interaction as signified by the exact equations (6.2.26).

We would like to point out certain subtleties related to the individual EM fields \( F_{\ell}^{\mu\nu} \). Namely, the individual currents \( J_{\ell}^{\mu} \) constructed via solutions to the Euler-Lagrange field equations (6.2.6)-(6.2.8) are implicitly related to each other. Those implicit relations manifest themselves in particular in the fact that the sum of the energy-momentum tensors of the individual EM fields does not exactly equal the energy-momentum tensor of the total EM field, i.e.
\[ \Theta^{\mu\nu} \neq \sum_{\ell} \Theta_{\ell}^{\mu\nu}, \] (6.2.27)

since \( \Theta^{\mu\nu} \) defined by (6.2.18) is a quadratic, and hence nonlinear, function of the EM \( F^{\mu\nu} \). Nevertheless, the approximate equality holds
\[ \Theta^{\mu\nu} \approx \sum_{\ell} \Theta_{\ell}^{\mu\nu}, \] (6.2.28)
when the supports of charges wave functions $\psi^\ell$ are well separated in the space for the time period of interest and, hence,

$$F^\ell_\gamma F^{\ell\nu} \approx 0 \text{ for } \ell' \neq \ell$$  \hfill (6.2.29)

implying (6.2.28). In other words, for well separated charges the above mentioned subtle correlations between individual currents become insignificant for their EM energy-momenta.

We can ask now how far one can go in extracting from the equations of motion (6.2.26) equations of point charges similarly to the non-relativistic theory in previous subsection. We can argue that in the relativistic theory in the regime of remote interaction we study, every charge can behave similarly to a wave-corpuscle but their motion can not be reduced to a system of differential equations obtained from a conventional finite-dimensional Lagrangian since it is prohibited by presented above arguments by Goldstein, von Laue and Pauli. The next option in simplicity can be the motion governed by a system of integro-differential equations which can account for retardation effects similar to the Sommerfeld integro-differential-difference equation of motion for the nonrelativistically rigid electron and its relativistic generalizations, [Pearle1 Sections 8-10], but we are not going to pursue this problem any further in this paper.

### 7 Equations in dimensionless form

We introduce here changes of variables allowing to recast the original field equations into a dimensionless form. These equations in dimensionless form allow to clarify three aspects of the theory for a single charge: (i) out of all the constants involved there is only one parameter of significance denoted by $\alpha$, and it coincides with the Sommerfeld fine structure constant $\alpha_S \simeq 1/137$ if $\chi = \hbar$ and $q, m$ are the electron charge and mass respectively; (ii) the non-relativistic Lagrangian (4.0.6) can be obtained from the relativistic one via the frequency-shifted Lagrangian (3.3.3) by setting there $\alpha = 0$; (iii) the fulfillment of charge and energy normalization conditions in relativistic case follows from smallness of $\alpha$.

Recall that the single charge nonrelativistic Lagrangian $\L_0$ defined by (4.0.6) is constructed in Section 4 based on the relativistic one via the the frequency shifted Lagrangian $\L_{0\omega}$ defined by (3.3.1)-(3.3.3) (see also Sections 3.3 1.8.1). The corresponding to $\L_{0\omega}$ Euler-Lagrange field equations are

$$\frac{\tilde{\partial}_t^2 \psi}{c^2} - \frac{im}{\chi} \left( 2\partial_t \psi + 2\frac{2q\varphi}{\chi} \frac{\psi}{\chi} \right) - \tilde{\nabla}^2 \psi + G' \left( |\psi|^2 \right) \psi = 0,$$  \hfill (7.0.30)

$$\frac{1}{4\pi} \nabla \cdot \left( \frac{\partial_t A}{c} + \nabla \varphi \right) = \left( \frac{\chi q}{mc^2} \frac{\psi}{\chi} + \frac{q^2 \varphi}{mc^2} \right) |\psi|^2 - q |\psi|^2,$$

$$- \left( \nabla \times (\nabla \times A) + \frac{\partial_t A}{c} \left( \frac{\partial_t A}{c} + \nabla \varphi \right) \right) = \frac{4\pi}{c} \left( -\frac{\chi q}{m} \frac{\nabla \varphi}{\chi} + \frac{q^2 \tilde{A}}{mc} \right) |\psi|^2,$$

where

$$\tilde{\partial}_t = \partial_t + \frac{ig\varphi}{\chi}, \quad \varphi = \varphi + \varphi_{ex}, \quad \tilde{A} = A + A_{ex}.$$  \hfill (7.0.31)

Let us introduce the following constants and new variables:

$$a_\chi = \frac{\chi^2}{mq^2}, \quad \alpha = \frac{q^2}{\chi c}, \quad \omega_0 = \frac{mc^2}{\chi} = \frac{c}{aa_\chi},$$  \hfill (7.0.32)
\[\alpha^2 \omega_0 t = \tau, \ x = a_x y, \quad (7.0.33)\]

\[\psi (x) = \frac{1}{a_x^{3/2}} \Psi \left(\frac{x}{a_x}\right), \ \varphi (x) = \frac{q}{a_x} \Phi \left(\frac{x}{a_x}\right), \ A (x) = \frac{q}{a_x} A \left(\frac{x}{a_x}\right).\]

In those new variables the field equations (7.0.30) turn into the following dimensionless form:

\[\alpha^2 (\partial_\tau + i \bar{\Phi})^2 \Psi - 2i (\partial_\tau + i \bar{\Phi}) \Psi - (\nabla_y - i \alpha \bar{A})^2 \Psi + G' \left(|\Psi|^2\right) \Psi = 0, \quad (7.0.34)\]

\[\frac{1}{4\pi} \nabla_y \cdot (\alpha \partial_\tau A + \nabla_y \Phi) = \left(\alpha^2 \text{Im} \frac{\partial_\tau \Psi}{\Psi} + \alpha^2 \Phi\right) |\Psi|^2 - |\Psi|^2, \quad \nabla_y \times (\nabla_y \times A) = -4\pi \alpha \left(\text{Im} \frac{\nabla_y \Psi}{\Psi} + \alpha \bar{A}\right) |\Psi|^2, \quad (7.0.36)\]

We would like to show that the dimensionless form of the non-relativistic equations field equations (5.0.12), (5.0.13) can be obtained from the field equations (7.0.34) in the limit \(\alpha \to 0\). To have a nonvanishing external magnetic field in the limit \(\alpha \to 0\) we set

\[A_{\text{ex}} = \alpha^{-1} A_{\text{ex}}^0 \quad (7.0.35)\]

Plugging in the expression (7.0.35) into the equations (7.0.34) we obtain in the limit \(\alpha \to 0\) the following dimensionless version of the field equations (5.0.12), (5.0.13):

\[i \partial_\tau \Psi = -\frac{1}{2} (\nabla_y - i A_{\text{ex}}^0)^2 \Psi + (\Phi + \Phi_{\text{ex}}) \Psi + \frac{1}{2} G' \left(|\Psi|^2\right) \Psi, \quad (7.0.36)\]

\[\nabla_y^2 \Phi = 4\pi \left|\Psi\right|^2, \ \nabla_y \times (\nabla_y \times A) = 0. \quad (7.0.36)\]

To get an insight in the nonrelativistic case as an approximation to the relativistic one we would like to make a few comments on the relative magnitude of terms that have to be omitted in equation (7.0.34) in order to obtain equation (7.0.36). The nonrelativistic case is defined as one when the charge velocity \(v\) is much smaller than the speed of light \(c\), and a careful look at those omitted terms in (7.0.34) that have factors \(\alpha\) and \(\alpha^2\) shows that they can be small not only because of \(\alpha\), but also because of the smallness of typical values of velocities compared to the speed of light. Indeed, every term that has factor \(\alpha\) involves time derivatives with only one exception: the term \(\alpha^2 (i \bar{\Phi})^2 \Psi\). An estimation of the magnitude of the omitted terms for solutions of the form of wave-corpuscles (5.1.2) indicated that every such a term is of order \(\alpha |v| / c\) where \(v\) is the wave-corpuscle velocity. The only omitted term in (7.0.34) which does not involve time derivatives is \(\alpha^2 \Phi^2 \Psi\) and, in fact, it can be preserved in the nonrelativistic system which would be similar to (7.0.36) with properties analogous to (5.0.12). Analysis of that system is more involved and the treatment is similar to the one for the rest solution of the relativistic equation involving that term considered in next Subsection 7.1.

### 7.1 Single relativistic charge at rest

In this section we consider a single relativistic charge. Using the new constants and variables (7.0.32), (7.0.33) we get the following dimensionless version of the resting charge equations (2.0.12), (2.0.13) and the charge normalization condition (2.0.20)

\[-\frac{1}{2} \nabla_y^2 \Psi + \left(\Phi - \frac{\alpha^2 \Phi^2}{2}\right) \Psi + \frac{\hat{G}' (\Psi^2)}{2} \Psi = 0, \quad (7.1.1)\]

\[-\nabla^2 \Phi = 4\pi \left(1 - \alpha^2 \Phi\right) \left|\Psi\right|^2 = 0, \quad (7.1.2)\]
\[
\int_{\mathbb{R}^3} \left[ (1 - \alpha^2 \Phi) |\Psi|^2 \right] \, dx = 1. \tag{7.1.3}
\]

Setting in the above equations \( \alpha = 0 \) we obtain the dimensionless form nonrelativistic equilibrium equations (2.3.5), (2.3.6) and the charge normalization condition (2.3.12), namely

\[
-\frac{1}{2} \nabla_y^2 \Psi + \Phi \Psi + \hat{G}_0 \left( |\Psi|^2 \right) \hat{\Psi} = 0, \quad -\nabla^2 \Phi = 4\pi |\Psi|^2, \tag{7.1.4}
\]

\[
\int_{\mathbb{R}^3} |\Psi|^2 \, dx = 1. \tag{7.1.5}
\]

Now using perturbations analysis we argue that for small \( \alpha \) the solution \( \Psi_\alpha, \Phi_\alpha \) to the equations (7.1.2) is close to the solution \( \Psi_0, \Phi_0 \) of the equations (7.1.4). Indeed, for zero approximation \( \Phi_0 (x) \)

\[
\Phi_0 (x) = \int_{\mathbb{R}^3} \frac{|\Psi_0 (y)|^2}{|x - y|} \, dy,
\]

and the first order approximation \( \Phi_1 \) is a solution to

\[
-\nabla^2 \Phi_1 = 4\pi \left( 1 - \alpha^2 \Phi_0 \right) |\Psi_0|^2.
\]

Using the Maximum principle we get

\[
0 < \Phi_1 (x) < \Phi (x) < \Phi_0 (x) = \hat{\Phi} (x) \quad \text{for all } x. \tag{7.1.6}
\]

Obviously,

\[
\Phi_1 (x) = \Phi_0 (x) + \alpha^2 \Phi_{01} (x), \quad \text{where } \nabla^2 \Phi_{01} = 4\pi \Phi_0 |\Psi_0|^2, \tag{7.1.7}
\]

and hence

\[
\Phi_{01} (x) = -\int_{\mathbb{R}^3} \frac{\Phi_0 (y) |\Psi_0 (y)|^2}{|x - y|} \, dy.
\]

Consequently, inequalities (7.1.6) imply explicit estimate

\[
\alpha^2 \Phi_{01} (x) < \Phi (x) - \Phi_0 (x) < 0 \quad \text{for all } x \in \mathbb{R}^d. \tag{7.1.8}
\]

### 7.2 Charge and energy simultaneous normalization

We consider here some technical details related to the size representation (2.1.8), (2.1.9) and the problem of simultaneous normalization of the charge and the energy by equations (2.1.10) and (2.1.11). Let function \( \Psi_\theta (y) \) be the dimensionless version of the function \( \psi_a \) in (2.1.8), (2.1.9), namely

\[
\psi_a (x) = \frac{\theta^{3/2}}{a^{3/2}} \left( \frac{\theta^{3/2} x}{a^{3/2}} \right) = \Psi_\theta (y) = C_\psi \theta^{3/2} \Psi_1 (\theta y), \tag{7.2.1}
\]

where \( \theta = \frac{a}{\chi} \) and \( \int_{\mathbb{R}^3} |\Psi_1|^2 \, dy = 1. \)

Then the charge and the energy normalization conditions (2.1.10) and (2.1.11) take the form

\[
\mathcal{N} = \int_{\mathbb{R}^3} \left( 1 - \alpha^2 \Phi_\theta \right) |\Psi_\theta|^2 \, dy = 1. \tag{7.2.2}
\]
Using the relation (11.8.36) and (11.8.40) we obtain the following representation for the energy $E(\psi_a, \varphi_a)$, which is a version of the Pohozhaev-Derrik formula (see Section 11.8.1),

$$E_0(\psi_a, \varphi_a) = mc^2 + \mathcal{E}_0'(\psi_a, \varphi_a),$$  

(7.2.3)

$$\mathcal{E}_0'(\psi_a, \varphi_a) = \frac{2}{3} \int_{\mathbb{R}^3} \left( \chi^2 \frac{|\nabla \psi_a|^2}{2m} - \frac{|\nabla \varphi_a|^2}{8\pi} \right) \, dx.$$

Then the energy normalization condition (2.1.11), namely $\mathcal{E}_0' = 0$, turns into the dimensionless variables into the condition

$$\mathcal{E}_0'(\Psi_\theta, \Phi_\theta) = \frac{q^2}{3a_\chi} \int_{\mathbb{R}^3} \left( |\nabla \Psi_\theta|^2 - \frac{|\nabla \Phi_\theta|^2}{4\pi} \right) \, dy = 0.$$  

(7.2.4)

First, let us consider a simple case $\alpha = 0$ using for it the notation $\tilde{\Phi}_\theta = \Phi_\theta|_{\alpha = 0}$, $\tilde{\Psi}_\theta = \Psi_\theta|_{\alpha = 0}$. In this case the charge normalization condition (7.2.2), in view of the normalization condition in (7.2.1), is satisfied for $C_\Psi = 1$ and

$$\tilde{\Psi}_\theta (y) = \theta^{3/2} \tilde{\Psi}_1 (\theta y), \quad \tilde{\Phi}_\theta (y) = \theta \tilde{\Phi}_1 (\theta y) \quad \text{for} \quad \alpha = 0.$$  

(7.2.5)

It follows then from (7.2.4) that

$$\mathcal{E}_0'(\tilde{\Psi}_\theta, \tilde{\Phi}_\theta) = \frac{q^2}{3a_\chi} \int_{\mathbb{R}^3} \left( \theta^2 |\nabla \tilde{\Psi}_1|^2 - \frac{|\nabla \tilde{\Phi}_1|^2}{4\pi} \right) \, dy.$$  

(7.2.6)

implying that $\mathcal{E}_0'(\tilde{\Psi}_\theta, \tilde{\Phi}_\theta)$ is a quadratic function of the parameter $\theta$. Since $\Psi$ is fixed we set in (7.2.1) $C_\Psi = 1$ and $\theta = \theta_0$, where $\theta_0$ is defined by

$$\theta_0 = b_\Psi, \quad b_\Phi = \frac{1}{4\pi} \int_{\mathbb{R}^3} |\nabla \tilde{\Phi}_1|^2 \, dy, \quad b_\Psi = \int_{\mathbb{R}^3} |\nabla \tilde{\Psi}_1|^2 \, dy,$$  

(7.2.7)

and obtain the desired energy normalization condition (7.2.4), namely

$$\mathcal{E}_0'(\tilde{\Psi}_\theta, \tilde{\Phi}_\theta) = 0, \quad \int_{\mathbb{R}^3} |\tilde{\Psi}_\theta|^2 \, dx = 1.$$  

(7.2.8)

Note that $\theta_0$ as in (7.2.7) coincides with $\theta_\psi$ as in (4.7.3).

Let us consider now the case $\alpha > 0$. We would like to show that for a given form factor $\Psi$ and small $\alpha > 0$ there exist constants $C_\Psi$ and $\theta$ such that the following two equation hold

$$\mathcal{E}_0'(\tilde{\Psi}_\theta, \tilde{\Phi}_\theta, \alpha) = 0, \quad \mathcal{N}(C_\Psi, \theta, \alpha) - 1 = 0,$$  

(7.2.9)

where $\mathcal{E}_0'$ and $\mathcal{N}_1$ are defined respectively by relations (7.2.3) and (7.2.22). In other words we need to find two parameters $C_\Psi$ and $\theta$ from a system of two nonlinear equations (7.2.9). We have already established that for $\alpha = 0$ : the solution is $C_\Psi = 1$, $\theta = \theta_0$ as in (7.2.7).

A geometrical argument shows that for sufficiently small $\alpha$ equations (7.2.9) have a solution $\{C, \theta\}$ which is close for small $\alpha$ to the solution $C_\Psi = 1$, $\theta = \theta_0$. The complete argument is based on the inequality (7.1.3) but its details are rather technical and we omit them here.
8 Hydrogen atom model

The purpose of this section is to introduce a hydrogen atom model within the framework of our theory for two interacting charges: an electron and a proton. We have no intend though to study this model in detail here. Our modest effort on the subject in this paper is, first, to indicate a similarity between our and Schrödinger’s hydrogen atom models and to contrast it to any kind of Kepler’s model. Another point we can make based on our hydrogen atom model is that our theory does provide a basis for a regime of close interaction between two charges which differs significantly from the regime of remote interaction which is the primary focus of this paper. Clearly the results on interaction of many charges as in Section 6.1.2 do not apply to the hydrogen atom model. Indeed in this case the two charges are in close proximity and the potentials can vary significantly over their locations, and hence, other methods have to be developed for the hydrogen atom model.

To model interaction of two charges at a short distance we must consider the original system (2.4.2) for two charges with \(-q_1 = q_2 = q > 0\), that is

\[
i\chi \partial_t \psi_1 = -\frac{\chi^2 \nabla^2 \psi_1}{2m_1} - q (\varphi_1 + \varphi_2) \psi_1 + \frac{\chi^2 G'_1 (|\psi_1|^2) \psi_1}{2m_1},
\]

\[
\nabla^2 \varphi_1 = -4\pi q |\psi_1|^2,
\]

\[
i\chi \partial_t \psi_2 = -\frac{\chi^2 \nabla^2 \psi_2}{2m_2} + q (\varphi_1 + \varphi_2) \psi_2 + \frac{\chi^2 G'_2 (|\psi_2|^2) \psi_2}{2m_2},
\]

\[
\nabla^2 \varphi_2 = -4\pi q |\psi_2|^2.
\]

Note that the model describes proton-electron interaction if \(q = e\) is the electron charge, \(\chi = \hbar\) is the Planck constant, \(m_1\) and \(m_2\) are the electron and the proton masses respectively. Let us look now at time-harmonic solutions to the system (8.0.10) in the form

\[
\psi_1 (t, \mathbf{x}) = e^{-i\omega_1 t} u_1 (\mathbf{x}), \quad \psi_2 = e^{-i\omega_2 t} u_2 (\mathbf{x}), \quad \Phi_1 = -\frac{\varphi_1}{q}, \quad \Phi_2 = \frac{\varphi_2}{q}.
\]

The system (8.0.10) for such solutions turns into the following nonlinear eigenvalue problem:

\[
-\frac{a_1}{2} \nabla^2 u_1 + (\Phi_1 - \Phi_2) u_1 + \frac{a_1}{2} G'_1 (|u_1|^2) u_1 = \frac{\chi}{q^2} \omega_1 u_1,
\]

where \(a_1 = \frac{\chi^2}{q^2 m_1}\),

\[
-\frac{a_2}{2} \nabla^2 u_2 + (\Phi_2 - \Phi_1) u_2 + \frac{a_2}{2} G'_2 (|u_2|^2) u_2 = \frac{\chi}{q^2} \omega_2 u_2,
\]

where \(a_2 = \frac{\chi^2}{q^2 m_2}\).

Here \(a_1\) coincides with Bohr radius. We seek solutions of (8.0.12) satisfying the charge normalization conditions

\[
\int_{\mathbb{R}^3} |u_1|^2 \, d\mathbf{x} = 1, \quad \int_{\mathbb{R}^3} |u_2|^2 \, d\mathbf{x} = 1.
\]

The potentials \(\Phi_i\) are presented using (8.0.10) as follows:

\[
\Phi_i = 4\pi (\nabla^2)^{-1} |u_i|^2 = 4\pi \int_{\mathbb{R}^3} \frac{|u_i|^2 (\mathbf{y})}{|\mathbf{y} - \mathbf{x}|} \, d\mathbf{y}, \quad i = 1, 2.
\]

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Let us introduce now the following energy functional

\[ E(u_1, u_2) = \]

\[ q^2 \int_{\mathbb{R}^3} \left\{ \frac{a_1}{2} \left[ |\nabla u_1|^2 + G_1 \left( |u_1|^2 \right) \right] + \frac{a_2}{2} \left[ |\nabla u_2|^2 + a_2 G_2 \left( |u_2|^2 \right) \right] \]

\[ - (\Phi_1 - \Phi_2) \left( |u_2|^2 - |u_1|^2 \right) - \frac{\nabla (\Phi_1 - \Phi_2)}{8\pi} \right\} \, dx, \]

where \( \Phi_1, \Phi_2 \) are determined in terms of \( u_1, u_2 \) by (8.0.14). Notice that the equations (8.0.12) describe stationary points of the functional \( E \) and can be obtained by its variation under constraints (8.0.13) with the frequencies \( \omega_1, \omega_2 \) being the Lagrange multipliers. Observe that the energy functional and the constraints are invariant with respect to multiplication by \(-1\), and that allows us to apply the Lusternik-Schnirelman theory which guarantees the existence of an infinite set of critical points under appropriate conditions. The critical points are the eigenfunctions of the corresponding Schrödinger operators (see, for example, [Heid], [Lions]). Our preliminary analysis shows that properties of solutions are similar to those in the spectral theory of the hydrogen atom described by the linear Schrödinger equation. The smallness of the ratio \( m_1/m_2 \approx 1/1836 \) of electron to proton masses plays an important role in the analysis. For the critical points with low energies the potential \( \Phi_2 \) of the proton is close to the Coulomb’s potential \( 1/|x| \) at spatial scales of the order \( a_1 \). For a properly chosen nonlinearity the quadratic part of the energy functional and the corresponding linear Schrödinger equation can be used to find discrete low levels of energy. Rough estimates of the energy levels of approximate solutions of the nonlinear problem based on the eigenfunctions of the linear Schrödinger operator show good agreement with well-known energy levels for the hydrogen atom if choose the size parameter \( a_1 \) 10-20 times greater then the Bohr radius. It seems reasonable to assume that the wave function of a free electron contracts when it becomes bound to a positively charged proton and apparently its size is reduced by an order of magnitude.

9 Comparison with the Schrödinger wave theory

We already made some points on common features and differences between our theory and the Schrödinger wave mechanics in Section 2.5 and here we discuss in more detail a few significant differences between the two theories. Recall that the Schrödinger wave mechanics is constructed based on the classical point particle Hamiltonian

\[ E = H(p, x) = \frac{p^2}{2m} + V(x) \]

by substituting there, [Pauli WM Section 2, 11], [Pauli PWM Sections 3, 4],

\[ p \rightarrow -i\hbar \nabla, \quad E \rightarrow i\hbar \frac{\partial}{\partial t}, \]

that yields the celebrated Schrödinger equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2 \nabla^2 \psi}{2m} + V(x) \psi. \]
The substitution (9.0.17) is essentially the quantization procedure allowing to correspond the classical point Hamiltonian (9.0.16) to the quantum mechanical wave equation (9.0.18). If we introduce the polar representation

\[ \psi = e^{i\frac{S}{\hbar}}R, \quad R \geq 0 \]  

(9.0.19)

then the Schrödinger equation (9.0.18) can be recast as the following system of two coupled equations for the real valued phase function \( S \) and the amplitude \( R \), [Holland, Section 3.2.1],

\[
\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2 \nabla^2 R}{2mR} = 0, \\
\frac{\partial R^2}{\partial t} + \nabla \cdot \left( \frac{R^2 \nabla S}{m} \right) = 0.
\]

(9.0.20) (9.0.21)

If we expand the phase \( S \) into power series with respect to \( \hbar \) i.e.

\[ S = S_0 + \hbar S_1 + \hbar^2 S_2 + \ldots, \]  

(9.0.22)

we obtain from the equation (9.0.20) the so called WKB approximation, [Pauli PWM, Section 12]. In particular, the function \( S_0 \) satisfies the Hamilton-Jacobi equation

\[
\frac{\partial S_0}{\partial t} + \frac{(\nabla S_0)^2}{2m} + V = 0,
\]

(9.0.23)

which shows, in particular, that the Schrödinger wave equation (9.0.18) does ”remember” how it was constructed by ”returning back” the original Hamiltonian \( H \) via the Hamilton-Jacobi equation (9.0.23) for \( S_0 \).

Our approach works the other way around. We introduce the Lagrangian (3.0.1)-(3.0.3) and the corresponding field equations as a fundamental basis and deduce from them the classical Newtonian mechanics as a certain approximation (see Sections 5, 6.1). To appreciate the difference let us look at a system of \( N \) charges. The here introduced wave-corpuscle mechanics would have \( N \) wave functions and the EM fields defined over the same 3 dimensional space, whereas the same system of \( N \) charges in the Schrödinger wave mechanics has a single wave function defined over a \( 3N \)-dimensional ”configuration space”.

It is quite instructive to compare the polar representation (2.3.16) for wave-corpuscle \( \psi \) with the same for the Schrödinger wave function \( \psi \) for the potential \( V(\mathbf{x}) = -q \mathbf{E}_{\text{ex}} \cdot \mathbf{x} \) corresponding to a homogeneous external electric field (the eigenfunctions of the corresponding Schrödinger equations can be expressed in terms of the Airy functions, [Pauli WM, Section 26]). The amplitude \( \psi( |\mathbf{x} - \mathbf{r}(t)| ) \) of the wave-corpuscle in the expression (2.3.16) for \( \psi \) is a soliton-like field moving exactly as the point charge described by its position \( \mathbf{r}(t) \) in contrast to the amplitude \( R \) of the Schrödinger wave function which describes the location of the charge rather implicitly via the coupled equations (9.0.21). The difference between the phases is equally significant. Indeed, the exponential factor \( e^{iS/\chi} \) for the accelerating wave-corpuscle is just a plane wave with the phase \( S \) that depends only on the point charge position \( r \) and momentum \( p \) whereas the same for the Schrödinger equations captures the features of the point charge only via WKB approximation and the Hamilton-Jacobi equation (9.0.23) which holds for the phase \( S \) only in the limit \( \hbar \to 0 \).

Let us now take a look at uncertainty relations which constitute a very important consequence of the Schrödinger wave mechanics as a linear wave theory. Detailed studies of
this subject is not in the scope of this paper but we can already see significant alterations of the uncertainty relations brought by the nonlinearity. W. Pauli writes in section "The uncertainty principle", [Pauli WM, Section 3]): "The kinematics of waves does not allow the simultaneous specification of the location and the exact wavelength of a wave. Indeed, one can only speak of the location of a wave in the case of a spatially localized wave packet. The number of different wavelengths contained in the Fourier spectrum increases as the wave packet becomes more localized. A relation of the form $\Delta k \Delta x \geq \text{constant}$ seems reasonable, and we now want to derive this relation quantitatively". Then in the same section he derives the well known Heisenberg uncertainty principle for a wavepacket in the form

$$\Delta k \Delta x \geq \frac{1}{2}, \quad \Delta p \Delta x \geq \frac{h}{2}, \quad (9.0.24)$$

where $\Delta x$, $\Delta k$ and $\Delta p = h\Delta k$ are respectively the spacial range, the wave number range and the momentum range for the wavepacket $\psi(x)$ defined by

$$\Delta x^2 = \int_{\mathbb{R}^3} (x - \bar{x})^2 |\psi(x)|^2 \, dx, \quad \bar{x} = \int_{\mathbb{R}^3} x |\psi(x)|^2 \, dx, \quad (9.0.25)$$

$$\Delta k^2 = \int_{\mathbb{R}^3} (k - \bar{k})^2 |\hat{\psi}(k)|^2 \, dk, \quad \text{where}$$

$$\bar{k} = \int_{\mathbb{R}^3} \psi^*(x) \left(-i \frac{\partial}{\partial x}\right) \psi(x) \, dx = \int_{\mathbb{R}^3} k |\hat{\psi}(k)|^2 \, dk,$$

$$\hat{\psi}(k) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ikx} \psi(x) \, dx.$$

Importantly, in the orthodox quantum wave mechanics, which is a linear theory, if $\psi(x)$ is the wave function $|\psi(x)|^2$ is interpreted as the probability density for a point particle to be at a location $x$. Hence, in this theory the uncertainty is already in the very interpretation of the wave function, and $\Delta x$ as in (9.0.25) is an uncertainty in the location of the point particle with a similar interpretation holding for $\Delta p$. Thus the exact form of the Heisenberg uncertainty relation (9.0.24) is a direct consequence of the fundamental definition (9.0.17) of the momentum and the definition of the uncertainty as in (9.0.25) based on the probabilistic interpretation of the wave function.

An important feature of the uncertainty relations in the linear theory is that any freely propagating wavepacket spreads out as a quadratic function of the time $t$ and such a spread out takes a particularly simple form for a Gaussian wavepacket, [Pauli WM, Section 3],

$$(\Delta x)^2 = \frac{1}{4} (\Delta k)^2 + \frac{\hbar^2 (\Delta k)^2}{m^2} t^2. \quad (9.0.26)$$

We would like to point out that the very concept of wavepacket is based on the medium linearity, and the same is true for of the uncertainty relations (9.0.24), (9.0.26) as general wave phenomena.

Let us consider now the here proposed wave-corpuscle mechanics from the uncertainty relations point of view. In wave-corpuscle mechanics we denote uncertainties in the position $x$ and the momentum $p$ by respectively by $\Delta x$ and $\Delta p$ using a different symbol $\tilde{\Delta}$ to emphasize the difference in their definition since the wave-function does not carry a probabilistic interpretation. We also limit our considerations of the uncertainties $\Delta x$ and $\Delta p$ to special
cases when a wave-corpuscle is an exact solution to either relativistic or nonrelativistic field equations, since in these cases we can argue more convincingly what constitutes uncertainty without giving its definition in a general case.

Notice that a common feature of the dressed charge (wave-corpuscle) and a wavepacket is the wave origin of their kinematics as manifested by the equality of the velocity to the group velocity of the underlying linear medium. But, in contrast to a wavepacket in a linear medium, the free relativistic dressed charge described by the relations (3.4.1)-(3.4.4) does not disperse as it moves and preserves its shape up the Lorentz contraction. The de Broglie vector $\mathbf{k}$ and the frequency $\omega$ can be determined from the factor $e^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})}$ in (3.4.1), and consequently the same applies to the total momentum $P = \hbar \mathbf{k}$. In the case of a nonrelativistic wave-corpuscle as defined by relations (4.3.1), (4.3.2) similarly its total momentum is $P = m \mathbf{v} = \hbar \mathbf{k}$. More than that as we show in Section 5 the wave-corpuscle as defined by relations (5.1.2) is an exact solution to the field equations and it propagates in space without dispersion even when accelerates.

To further clarify differences with the uncertainties in Schrödinger wave mechanics and the wave-corpuscle mechanics introduced here notice the following. The dressed charge described by the relations (3.4.1)-(3.4.4) is a material wave for which we can reasonably assign size $D$ in relativistic and nonrelativistic cases by formulas

$$D^2 (t) = \int_{\mathbb{R}^3} (\mathbf{x} - \bar{\mathbf{x}})^2 \frac{\rho (t, \mathbf{x})}{q} \, d\mathbf{x}, \quad (9.0.27)$$

where

$$\bar{\mathbf{x}} = \int_{\mathbb{R}^3} \mathbf{x} \frac{\rho (t, \mathbf{x})}{q} \, d\mathbf{x},$$

and, respectively,

$$\frac{\rho (t, \mathbf{x})}{q} = q \left( 1 - \frac{q \dot{\phi} (\mathbf{x} - \mathbf{v} t)}{mc^2} \right) \psi^2 (\mathbf{x} - \mathbf{v} t),$$

and

$$\frac{\rho (t, \mathbf{x})}{q} = \psi^2 (\mathbf{x} - \mathbf{v} t).$$

In the case of a relativistic or nonrelativistic wave-corpuscle we define "safely" the uncertainty $\tilde{\Delta} x = D (t)$ and notice that it follows from the definition (9.0.24) and charge normalization conditions (3.0.11), (4.0.14) that in fact $\Delta x = D (t) = D$ does not depend on $t$. Hence, the uncertainty $\Delta x = D$ unlike the uncertainty $\Delta x$ from (9.0.26) does not grow without bound for large times. As to the charges momentum $P = m \mathbf{v} = \hbar \mathbf{k}$ we can argue that $\Delta \mathbf{p} = 0$ for the exact wave-corpuscle solutions. Indeed, for a wave-corpuscle as defined by relations (3.4.1)-(3.4.4) or (4.3.1), (4.3.2) respectively in the relativistic and nonrelativistic cases the motion of dressed charge is obtained by application of space translations (or the Lorentz transformations) to a fixed rest charge, therefore "every point" of the dressed charge moves with exactly the same velocity $\mathbf{v}$ similarly to a rigid body, which allows naturally define its velocity and momentum without uncertainty. Summarizing, we can conclude that in the wave-corpuscle mechanics the Heisenberg uncertainty principle cannot be a universal principle.
10 Relation to the Quantum Mechanics and hidden variables theories

Since the wave-corpuscle mechanics (WCM) naturally covers all spatial scales one can wonder how it relates to the quantum mechanics (QM), including the probabilistic interpretation of the wave function, the hydrogen atom frequencies, the double-slit experiment, scattering of a charge by a potential created by another charge and more. In sections 8, 9 we already provided comparisons of some features of the WCM and the QM. But a systematic comparison of all fundamental features of the two mechanics requires more extensive studies of the WCM, and at this point we can only formulate hypotheses on some significant elements of the WCM and their relations to fundamentals of the QM.

A key element of the WCM that has not been studied yet is a regime of time limited close interaction. More precisely, it is a regime when initially free moving charge undergoes for a naturally limited time a close interaction with another charge or an external EM field after which the charge continues to move freely again but with altered location and velocity. A typical example of such an interaction is when one moving charge is scattered by another one, or when a charge passes through a bounded domain in the space with a strong external EM field. Let us try to imagine what can happen according to the WCM to a charge during a time limited close interaction. Recall that in the WCM when charges are far apart every charge is represented by a particle-like well localized wave function as in (2.3.16) as a result of a fine balance of forces including the nonlinear self-interaction. Importantly, the cohesive action of the nonlinear self-interaction is very subtle, and by no means it is a brute attractive force since there is no action at distance in the WCM. Now, when one charge comes close to another or if it enters a domain with an external EM field varying at a sufficiently small scale, a fine balance of forces holding the charge together is disrupted. We can already see consequences of such a disruption in the WCM hydrogen atom model in section 8 where the electron size reduces by a factor of order 10 compare to that of the free electron under attractive action of a single proton. A disruption of the subtle cohesive action of the nonlinear self-interaction can also cause the charge wave function to spread out substantially and become wave-like. We can imagine further that during the time of interaction the evolution of the extended wave function is determined by an interplay of two factors: (i) the linear Schrödinger component of the field equation; (ii) its nonlinear component due the nonlinear self-interaction. Shortly after the interaction ends the wave-function of the charge restores the particle-like form but its position and velocity after the contraction will depend sensitively on details of the interaction. So, effectively, a limited time interaction switches one particle-like state of the charge to another.

Based the above hypothetical features of a time limited close interaction one can explain how the entirely deterministic WCM can conceivably lead to some of probabilistic aspects of the QM. Suppose for the sake of argument that the time scale of the interaction process is smaller than an observer can resolve, and, consequently, he sees the interaction result as a transition from one particle-like state to another. The interaction process can alter the total momentum of the charge quite considerably. This momentum alteration combined with effects of the nonlinear self-interaction can cause an extreme sensitivity to the initial data and that in turn can make the transition look like it is random and, hence, a subject to a statistical theory. An interesting feature of the nonlinear self-interaction in the WCM that might be relevant to the extreme sensitivity is that it is not analytic and singular for small
wave-amplitudes (see examples of the WCM nonlinearity in [4.5]). Consequently, small wave amplitudes can play far more important role in the WCM than in the case of conventional nonlinearities which are analytic for small amplitudes. Going further we observe that the WCM field equations are similar to the Schrödinger equation. Hence, it is conceivable that the statistics of the transition will be determined with certain degree of accuracy by a wave function satisfying an effective linear Schrödinger equation. Some general ideas on the “determinism beneath quantum mechanics” at the Planck scale were put forward recently by ’t Hooft (see [t Hooft] and references therein).

Let us use the described above hypothetical scenario of interaction to explain the double-slit experiment. Suppose that a single electron is fired by a device and moves freely as a particle-like wave toward a double-slit apparatus. As the electron approaches and interacts with the double-slit apparatus its wave function spreads out quite substantially, its amplitude reduces and consequently the electron turns into a ”real wave”. This extended wave penetrates through the both slits and over a limited time leaves the apparatus. Being outside of the apparatus in a free space the electron wave function contracts back to its particle-like shape and proceeds toward a sensitive screen until it hits it at a well defined impact location. Assume for the sake of argument that the initial condition of the fired electron can not be controlled with a sufficient accuracy (which may be higher than in the existing experiments). Then, the impact location can appear to be random with a statistics consistent with well known interference pattern as described in the modern double-slit experiments, [Greenstein, Section 1.1].

Another qualitatively important regime is a regime of close interaction for an extended or even infinite period of time. This regime can occur, in particular, in complex systems involving many charges such as atoms, molecules or solids. As we have already indicated in Section 8 the WCM hydrogen atom has features which are very similar the Schrödinger one. In particular, the primary binding force in that model between the electron and the nucleus is the EM attraction. As to solids let us briefly recall basics of their treatment in the QM. As any theory of many particles, the fundamental QM theory of a solid is of enormous complexity, but the standard simplified QM treatment of charges in crystalline solids is based on a free-electron model with the following basics assumptions, [Ashcroft, Chapter 1]: (i) positively charged ionized atoms, consisting of nuclei and tightly bound to them ”core electrons”, form an immobile periodic lattice structure; (ii) ”valence electrons” called also conductance electrons are ”allowed to wander far away from their parent atoms”; (iii) the conductance are non-interacting and independent and the interaction between a conductance electron and the periodic lattice is modeled via a periodic potential. Such a simplified QM theory is effectively reduced to the one-electron theory for the Schrödinger operator with the periodic potential. Consequently, the eigenfunctions of such an operator are of the Bloch form and are extended over the entire crystal. The fundamental WCM theory of a solid is of an enormous complexity as well, but similarly to the QM theory we can introduce a simplified WCM model for a solid based on the same assumptions as in the QM theory. Hence, as in the QM model there is an immobile periodic lattice of ionized atoms described by a periodic potential corresponding to an external electric field. The one-electron WCM model is similar to the QM one, but it differs from it by the presence of nonlinear self-interaction. In this non-relativistic WCM model a mobile conductance/valence electron is subjected to (i) attractive forces of the periodic lattice; (ii) electric force of its own electric field; (iii) nonlinear-self interaction forces. Since in a solid the distance between atoms is pretty small, it is of order the atom Bohr radius, the cohesive action of the nonlinear self-
interaction can be disrupted and the wave function can spread out significantly and even it might resemble a Bloch eigenmode, in which case the electron would occupy the entire crystal sample.

The above considerations bring us naturally to issues of the charge size and the WCM theory locality. As the above considerations suggest in the WCM the electron size can vary significantly depending on whether it is free or if it is bound in a atom, or if it is a conductance electron in a crystalline solid. In particular, the size of the electron can increase dramatically when it undergoes a strong close interaction with an external EM field or a system of other charges. As to the locality of the WCM it is perfectly local in one sense but can be non-local in another sense. Namely, the WCM theory is perfectly local in the sense that there is no action of distance. But the charge evidently is not perfectly local since in the WCM it is not a point but at the best a localized wave which under indicated above conditions can spread out significantly. Consequently, it is conceivable within the WCM that an elementary charge being a spatially distributed quantity can be simultaneously at two distant spatial locations, and in this sense the WCM might deviate significantly from being a local theory. If one excepts that, then the Bell’s inequalities [Bell, Section 2], [Gottfried, Section 12.3] do not apply to the WCM. A question if a modification of the Bell’s approach is possible for the WCM would require more studies including, in particular, an introduction of the spin concept in the theory.

One can also wonder what is a relation between the WCM and hidden variables theories, see [Gottfried, 12.2], [Holland, Section 1.5, 3.7.2], and a review article [Genovese] with references therein. Particularly, it is interesting to look how does the WCM compare with the Bohmian Mechanics (BM), [Holland, Section 3.1, 3.2], a well known example of hidden variables theories. Even a brief look on the WCM and the BM shows their significant differences: (i) in the BM an elementary charge is a point whereas in the WCM it is a distributed quantity, a wave; (ii) the WCM theory is local in the sense that there is no action at distance, and it is no so for the BM; (iii) the WCM is a genuine Lagrangian Mechanics and, consequently, the Third Newton Law is always satisfied, and it is not so for the BM, [Holland, Section 3.3.2]. In addition to that, as we have already indicated above, the WCM might account for the QM statistics via the dynamic instability approximately, and the verification of that, including the accuracy of approximation, is a subject of future studies. But it is absolutely clear already that the statistical predictions of the WCM can not be precisely the same as those of the QM, since the WCM field equations might only approximately and under certain conditions produce the QM evolution equation. The later factor evidently differs the WCM from the BM in which the Schrödinger equation is an exact equation for the wave function as a part of the BM variables.

11 Classical field theory

In this section we discuss important elements of the classical field theory including the variational principles and the Lagrangian formalism, gauge invariance and conservation laws. There are many classical references on the classical field theory: [Barut], [Goldstein], [LandauLif F], [Morse Feshbach 1, Section 3.4], [Pauli RFTh] and more. So we picked and chose different parts of theory from different sources to emphasize concepts and constructions important for our own arguments. Often we gave multiple references to provide different and complementary points of view on the same subjects. We also extended some aspects of the theory.
as needed. In particularly, we did that for an important for us subject of many charges interacting with the EM.

11.1 Relativistic Kinematics

Here where provide very basic facts and notations related to the relativistic kinematics following to [Barut Section 1], [LandauLif F Sections 1.1-1.4, 2], [Jackson Section 11.3], [Goldstein Section 7].

The time-space four vector in its contravariant $x^\mu$ and covariant $x_\mu$ forms are represented as follows

$$ x = x^\mu = (x^0, x^1, x^2, x^3) = (ct, \mathbf{x}), \quad \mu = 0, 1, 2, 3; \quad (11.1.1) $$

$$ x_\mu = g_{\mu\nu} x^\nu = (x^0, -x^1, -x^2, -x^3), \quad (11.1.2) $$

with the common convention on the summation of the same indices, and where $g_{\mu\nu}$ or $g^{\mu\nu}$, called metric tensor, $4 \times 4$ matrix, are defined by

$$ \{ g_{\mu\nu} \} = \{ g^{\mu\nu} \} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (11.1.3) $$

Notice also that, [Jackson Section 11.6], [Schwabl Section 6.1],

$$ g^\mu_\nu = g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu_\nu \quad \text{where} \quad \delta^\mu_\nu \quad \text{is the Dirac symbol}, \quad \{ \delta^\mu_\nu \} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (11.1.4) $$

and

$$ \partial_\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{1}{c} \partial_t, \nabla \right), \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left( \frac{1}{c} \partial_t, -\nabla \right). \quad (11.1.5) $$

The elementary Lorentz transformation to a moving with a velocity $\mathbf{v}$ frame is

$$ x'^{0} = \gamma \left( x^{0} - \beta \cdot \mathbf{x} \right), \quad x'_t = x_t + \frac{\gamma - 1}{\beta^2} (\beta \cdot \mathbf{x}) \beta - \gamma \beta x^0, \quad (11.1.6) $$

$$ \beta = \frac{\mathbf{v}}{c}, \quad |\beta| = \beta, \quad \gamma = \frac{1}{\sqrt{1 - \left( \frac{\mathbf{v}}{c} \right)^2}}. $$

If for a space vector $\mathbf{x}$ we introduce $\mathbf{x}_||$ and $\mathbf{x}_\perp$ so that they are respectively its components parallel and perpendicular to the velocity $\mathbf{v}$, i.e. $\mathbf{x} = \mathbf{x}_|| + \mathbf{x}_\perp$, then (11.1.6) can be recast as

$$ x'^{0} = \gamma \left( x^{0} - \beta \cdot \mathbf{x}_|| \right), \quad x'_t = \gamma \left( x_t - \beta x^0 \right), \quad x'_1 = x_\perp, \quad (11.1.7) $$

which in the case when $\mathbf{v}$ is parallel to the axis $x^1$ turns into

$$ x'^{0} = \gamma \left( x^{0} - \beta x^1 \right), \quad x'^{1} = \gamma (x^{1} - \beta x^0), \quad x'^{2} = x^2, \quad x'^{3} = x^3. \quad (11.1.8) $$

The Lorentz invariance then of a 4-vector $x$ under the above transformation reduces to

$$ (x'^0)^2 - |x'|^2 = (x^0)^2 - |x|^2. \quad (11.1.9) $$
The general infinitesimal form of the inhomogeneous Lorentz transformation is, [Moller, Section 6.1],
\[ x'^\mu = x^\mu + \xi^{\mu\nu} x_\nu + a^\mu, \quad \xi^{\mu\nu} = -\xi^{\nu\mu}, \] (11.1.10)
where \( \xi^{\mu\nu} \) and \( a^\mu \) are its ten parameters.

The Lagrangian \( L_p \) of the moving point mass is
\[ L_p = -mc^2 \sqrt{1 - \left(\frac{v}{c}\right)^2}, \] (11.1.11)
implying the following nonrelativistic approximation
\[ L_p \approx -mc^2 + \frac{mv^2}{2}, \quad \text{for} \quad \left|\frac{v}{c}\right| \ll 1. \] (11.1.12)

The momentum (the ordinary kinetic momentum) and the energy of the point mass for the relativistic Lagrangian \( L_p \) defined by (11.1.11) are, [LandauLif F, Section 2.9],
\[ p = \frac{mv}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, \quad E = p^0 c = p \cdot v - L = \frac{mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = c \sqrt{p^2 + m^2c^2}. \] (11.1.13)

The relativistic Lagrangian \( L_p \) of a point charge \( q \) with a mass \( m \) in an external EM field as described by electric potential \( \varphi \) and vector potential \( A \) is defined by, [Jackson, Section 12.1]
\[ L_p = -mc^2 \sqrt{1 - \left(\frac{v}{c}\right)^2} - q\varphi + \frac{q}{c} v \cdot A. \] (11.1.14)
For this Lagrangian the ordinary kinetic momentum \( p \), the canonical (conjugate) momentum \( \hat{p} \), and the Hamiltonian \( H_p \) are defined by the following relations
\[ p = \frac{mv}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, \quad \hat{p} = p + \frac{q}{c} A, \] (11.1.15)
\[ H_p = \hat{p} \cdot v - L = \sqrt{(cp - qA)^2 + m^2c^4 + q\varphi}, \] (11.1.16)
and the Euler-Lagrange equations are
\[ \frac{dp}{dt} = F = qE + \frac{q}{c} v \times B, \quad \text{and} \quad \frac{dE}{dt} = qv \cdot E, \] where \( E = -\nabla \varphi - \frac{\partial A}{\partial t}, \quad B = \nabla \times A, \) (11.1.17)
where \( F \) is the Lorentz force, and \( E \) and \( B \) are respectively the electric field and the magnetic induction.

The nonrelativistic version of the above Lagrangian in view of (11.1.12) is, [Goldstein, Section 1.5]
\[ L_p = \frac{m\dot{r}^2}{2} - q\varphi (r) + \frac{q}{c} A (r) \cdot \dot{r}, \quad \dot{r} = \frac{dr}{dt}. \] (11.1.18)
The corresponding ordinary kinetic momentum \( p \), the canonical (conjugate) momentum \( \hat{p} \), and the Hamiltonian \( H_p \) are defined by the following relations
\[ p = m\dot{r}, \quad \hat{p} = p + \frac{q}{c} A, \quad H_p = \hat{p} \cdot \dot{r} - L = \frac{m\dot{r}^2}{2} + q\varphi. \] (11.1.19)
The canonical Euler-Lagrange equations for the nonrelativistic Lagrangian (11.1.18) take the form
\[ \frac{d\mathbf{p}}{dt} = -q\nabla\varphi + \frac{q}{c} [DA]\dot{\mathbf{r}}, \quad [DA]_j = \frac{\partial}{\partial x^j} A, \quad j = 1, 2, 3, \] (11.1.20)
or, if use \( \mathbf{p} = \mathbf{p} + \frac{q}{c} A \) and the identity
\[ \frac{dA}{dt} = \frac{\partial A}{\partial t} + [DA]\dot{\mathbf{r}}, \] (11.1.21)
we can recast (11.1.20) as
\[ \frac{d\mathbf{p}}{dt} = m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} = q\mathbf{E} + \frac{q}{c} \frac{d\mathbf{r}}{dt} \times \mathbf{B}, \] (11.1.22)
where the right-hand side of the equation (11.1.22) is the Lorentz force. Importantly for what we study in this paper the canonical Euler-Lagrange equation (11.1.20) involves the canonical momentum \( \mathbf{p} \) and the canonical force \(-q\nabla\varphi + \frac{q}{c} [DA]\dot{\mathbf{r}} \) which manifestly depend on the EM potentials \( \varphi \) and \( A \) rather than EM fields \( \mathbf{E} \) and \( \mathbf{B} \). Consequently, the equation (11.1.20) involves quantities which are not directly measurable in contrast to the equivalent to it equation (11.1.22) which is gauge invariant and involves measurable quantities, namely the kinematic momentum \( \mathbf{p} \) and the Lorentz force \( q\mathbf{E} + \frac{q}{c} \dot{\mathbf{r}} \times \mathbf{B} \).

### 11.2 Lagrangians, field equations and conserved quantities

In this section we collect basic well known facts on the Lagrangian formalism for classical fields following to [Barut, Section III.3], [Morse Feshbach 1, Section 3.4], [Pauli RFTh] and other classical sources. Let us assume that physical systems of interest are described by fields real-valued \( q^\ell (x), \ell = 1, \ldots, N \), with the Lagrangian density
\[ \mathcal{L}(\{ q^\ell (x) \}, \{ q^\ell,_{\mu} (x) \}, x), \quad q^\ell,_{\mu} (x) = \partial^\mu q^\ell (x), \quad \mu = 0, 1, 2, 3, \] (11.2.1)
According to the Lagrangian formalism the field equation are derived from the variational principle
\[ \delta \int_R \mathcal{L} \, dx = 0 \] (11.2.2)
where \( R \) is four-dimensional space-like region with a three-dimensional boundary \( \partial R \). Importantly, the variation \( \delta \) is such that \( \delta q^\ell \) vanish on the boundary \( \partial R \). Then the corresponding Euler-Lagrange field equations take the form
\[ \Lambda^\ell = \frac{\partial \mathcal{L}}{\partial q^\ell} - \partial^\mu \frac{\partial \mathcal{L}}{\partial (q^\ell,_{\mu})} = 0. \] (11.2.3)
The equation (11.2.3) can be recast in a Hamiltonian form as
\[ \frac{\partial \pi^\ell}{\partial x_0} = F^\ell, \quad \pi^\ell = \frac{\partial \mathcal{L}}{\partial \dot{q}^\ell}, \quad F^\ell = \frac{\partial \mathcal{L}}{\partial \dot{q}^\ell} - \sum_{\mu=1}^3 \partial^\mu \frac{\partial \mathcal{L}}{\partial (q^\ell,_{\mu})}, \quad x_0 = ct, \] (11.2.4)
where \( \pi^\ell \) is interpreted as the canonical momentum density of the field \( q^\ell \) and \( F^\ell \) is the canonical force density acting upon the field. The term \( \frac{\partial \mathcal{L}}{\partial \dot{q}^\ell} \) of the canonical force density \( F^\ell \)
has to do with external forces acting on the field \( q^\ell \). In the view of the last remark, if the Lagrangian \( L \) involves a nonlinear terms \( G_\ell (q^\ell) \) its derivative \( \frac{\partial G}{\partial q^\ell} \) can be interpreted as a self-force.

In our considerations the fields \( q^\ell (x) \) describe elementary charges (and in fact they are complex-valued) and the potential 4-vector field \( A_\mu \) describes the classical EM field. The extension of the Lagrangian formalism to complex-valued in considered in a following section.

The canonical energy-momentum tensor (also called stress-energy tensor or stress-tensor) is defined by the following formula, \[ \bar{T}^{\mu\nu} = \sum_\ell \frac{\partial L}{\partial q_{\mu}^\ell} q^{\ell\nu} - g^{\mu\nu} L, \quad (11.2.5) \]
with the energy conservation laws in the form, \[ \partial_\mu \bar{T}^{\mu\nu} = - \frac{\partial L}{\partial x_\nu}, \quad (11.2.6) \]
Notice that in \[ \text{Morse Feshbach 1, (3.4.2)}, \quad \text{Goldstein, Section 13.3} \] the canonical energy-momentum tensor is defined as matrix-transposed to \( \bar{T}^{\mu\nu} \) defined by \( (11.2.5) \), namely \( \bar{T}^{\mu\nu} \rightarrow \bar{T}^{\nu\mu} \). The conservation laws for the energy-momentum are examples of conservation laws obtained from the Noether’s theorem considered below in Section 11.3.

In the case of the Lagrangian \( L \) does not depend explicitly on \( x_\nu \), in other words invariant under translations \( x_\nu \rightarrow x_\nu + a_\nu, \ \partial L/\partial x_\nu = 0 \) and the conservation laws \( (11.2.6) \) turn into the following continuity equations,
\[ \partial_\mu T^{\mu\nu} = 0. \quad (11.2.7) \]
A typical situation when the general conservation laws \( (11.2.6) \) apply rather than \( (11.2.7) \) is the presence of external forces which can “drive” our field. For instance, an external EM field driving a charge or an external ”imposed” current which becomes a source for the EM field.

We would like to remind the reader that the canonical tensor of energy-momentum defined by \( (11.2.5) \) is not the only one that satisfies the conservation laws \( (11.2.6) \) or \( (11.2.7) \). For instance, any tensor of the form, \[ \text{Barut (3.73)}, \quad \text{LandauLif F (32.7)}, \quad \text{Pauli RFTh (14)} \]
\[ T^{\mu\nu} = \bar{T}^{\mu\nu} - \partial_\gamma f^{\gamma\mu\nu}, \quad \text{where} \quad f^{\gamma\mu\nu} = - f^{\gamma\nu\mu}, \quad (11.2.8) \]
would satisfy \( (11.2.7) \) as long as \( \bar{T}^{\mu\nu} \) does. In view of the \( (11.2.8) \) the energy-momentum tensor \( T^{\mu\nu} \) satisfy the same conservation laws \( (11.2.6) \) or \( (11.2.7) \) as \( \bar{T}^{\mu\nu} \), namely
\[ \partial_\mu T^{\mu\nu} = - \frac{\partial L}{\partial x_\nu}, \quad (11.2.9) \]
or, if the Lagrangian \( L \) does not depend explicitly on \( x_\nu \) and, hence, invariant under time and space translations, the above conservation laws turn into
\[ \partial_\mu T^{\mu\nu} = 0. \quad (11.2.10) \]
In fact, this flexibility in choosing the energy-momentum can be used to define \( f^{\gamma\mu\nu} \) and construct a symmetric energy-momentum tensor \( T^{\mu\nu} \), i.e. \( T^{\mu\nu} = T^{\nu\mu} \), which is a necessary
and sufficient condition for the field angular momentum density to be represented by the usual formula in terms of the field momentum density, [LandauLifF, Section 32], [Barut, Section III.4]. The symmetry of the energy-momentum tensor for matter fields is also fundamentally important since it is a source for the gravitational field, [Nair, Section 3.8], [Misner, Section 5.7]. As to the uniqueness of the energy-momentum we can quote [Misner, Section 21.3]: "... the theory of gravity in the variational formulation gives a unique prescription for fixing the stress-energy tensor, a prescription that, besides being symmetric, also automatically satisfies the laws of conservation of momentum and energy". This unique form of the symmetric energy-momentum can be derived based on a variational principle involving charge of bound-
ary, with varied boundary, [Barut, Section III.3(B)], and under the following assumptions: (i) the Lagrangian does not depend explicitly on $x$; (ii) the fields $q^\ell (x)$ satisfy the fields equations (11.2.3); (iii) the fields vanish at the spacial infinity sufficiently fast. The result is a symmetric Belinfante-Rosenfeld energy-momentum tensor $T^{\mu \nu}$, [Barut, (3.73)-(3.75)], [Pauli RFTh, (13a), (13b), (13c), (14)], [Belinfante1], [Belinfante 2], [Rosenfeld], namely

$$
T^{\mu \nu} = \hat{T}^{\mu \nu} - \partial_\gamma f^{\mu \nu} = \sum_\ell \frac{\partial \mathcal{L}}{\partial q^\ell_{\mu}} q^\ell_{\nu} - g^{\mu \nu} \mathcal{L} - \partial_\gamma f^{\mu \nu},
$$

(11.2.11)

where $f^{\mu \nu}$ describes the infinitesimal transformation of the involved fields $q^\ell (x) \rightarrow q^\ell (x')$ along with the infinitesimal inhomogeneous Lorentz transformation of the space time vector $x \rightarrow x'$ as described by (11.2.10), namely,

$$
x'^\mu = x^\mu + \xi^{\mu \nu} x_\nu + a^\mu, \quad \xi^{\mu \nu} = -\xi^{\nu \mu},
$$

(11.2.12)

where $\xi^{\mu \nu}$ and $a^\mu$ are the ten parameters, and

$$
q^\ell (x') = q^\ell (x) + \xi^{\mu \nu} \sum_\ell \xi_{\ell}^{\mu \nu} q^{\ell} (x), \quad S^{\ell \mu \nu} = -S^{\ell \nu \mu}.
$$

(11.2.13)

In particular, [Barut III.4(A)],

$$
S^{\ell \mu \nu} = 0 \text{ if } q \text{ is a scalar field}, \quad S_{\beta}^{\alpha \mu \nu} = g^{\alpha \mu} g^{\nu \beta} - g^{\alpha \nu} g^{\mu \beta} \text{ if } q \text{ is a vector field}.
$$

(11.2.14)

(11.2.15)

The conserved quantities are, [Barut, (3.76)-(3.77)], [Pauli RFTh, (6)]

$$
P_\nu = \int_\sigma T^{\mu \nu} d\sigma_\mu, \quad J^{\nu \gamma} = \int_\sigma M^{\mu \nu \gamma} d\sigma_\mu, \quad M^{\mu \nu \gamma} = T^{\mu \nu} x^\gamma - T^{\mu \gamma} x^\nu,
$$

(11.2.16)

where $\sigma$ is any space-like surface, for instance $x_0 = \text{const}$. $P_\nu$ is four-vector of the total energy-momentum and $J^{\nu \gamma} = -J^{\gamma \nu}$ is the total angular momentum tensor. The differential form of the conservation laws is

$$
\partial_\mu T^{\mu \nu} = 0, \quad \partial_\mu M^{\mu \nu \gamma} = T^{\gamma \nu} - T^{\nu \gamma} = 0,
$$

(11.2.17)

Observe that the conservation of the angular momentum $M^{\mu \nu \gamma}$ in (11.2.17) implies the symmetry of the energy-momentum tensor and, in view of (11.2.11), the following identities, [Barut (3.81')],

$$
T^{\mu \nu} = T^{\nu \mu}, \quad \sum_\ell \frac{\partial \mathcal{L}}{\partial q^\ell_{\mu}} q^\ell_{\nu} - \partial_\gamma f^{\mu \nu} = \sum_\ell \frac{\partial \mathcal{L}}{\partial q^\ell_{\nu}} q^\ell_{\mu} - \partial_\gamma f^{\mu \nu}.
$$

(11.2.18)
For an alternative insightful derivation of the symmetric energy-momentum tensor based on kinosthenic (ignorable) variables and the Noether’s method as a way to generate such variables we refer to [Lanczos VP, Section 3.5, 3.6, 3.10]. Interestingly under this approach the conservations laws take the form of the Euler-Lagrange equations for those kinosthenic variables. We would like to point out that the symmetry of the energy-momentum tensor and the corresponding identities (11.2.18) are nontrivial relations which hold provided that the involved fields satisfy the field equations (11.2.3).

Since the symmetric energy-momentum tensor \( T^{\mu\nu} \) is the one used in most of the cases we often refer to it just as the energy-momentum tensor, while the tensor \( \tilde{T}^{\mu\nu} \) defined by (11.2.5) is referred to as the canonical energy-momentum tensor.

The interpretation of the symmetric energy-momentum tensor \( T^{\mu\nu} \) entries is as follows, \[ \text{LandauLif F, Section 32], [Goldstein, Sections 13.2, 13.3], [Morse Feshbach 1, Chapter 3.4] } \]

\[
T^{\mu\nu} = \begin{bmatrix}
  u & cp_1 & cp_2 & cp_3 \\
  c^{-1}s_1 & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\
  c^{-1}s_2 & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\
  c^{-1}s_3 & -\sigma_{31} & -\sigma_{32} & -\sigma_{33}
\end{bmatrix}, \tag{11.2.19}
\]

where

- \( u \) field energy density,
- \( p_j, j = 1, 2, 3, \) field momentum density,
- \( s_j = c^2 p_j, j = 1, 2, 3, \) field energy flux density,
- \( \sigma_{ij} = \sigma_{ji}, j = 1, 2, 3, \) field symmetric stress tensor.

If we introduce the 4-momentum and 4-flux densities by the formulas

\[
p^\nu = \frac{1}{c} T^{0\nu} = \left( \frac{1}{c} u, p \right), \quad s^\nu = c T^{\nu 0} = (cu, s).
\]

then the energy-momentum conservation law (11.2.9) can be recast as follows

\[
\partial_\nu p^\nu + \sum_{j=1,2,3} \partial_j T^{j\nu} = -\frac{\partial L}{\partial x^\nu}, \tag{11.2.21}
\]

or, in other words,

\[
\partial_t p_i = \sum_{j=1,2,3} \partial_j \sigma_{ji} - \frac{\partial L}{\partial x^i}, \quad \text{where} \quad p_i = \frac{1}{c} T^{0i}, \quad \sigma_{ji} = -T^{ji}, \quad i, j = 1, 2, 3, \tag{11.2.22}
\]

\[
\partial_t u + \sum_{j=1,2,3} \partial_j s_j = -\frac{\partial L}{\partial t}, \quad \text{where} \quad u = T^{00}, \quad s_i = c T^{i0} = c^2 p_i, \quad i = 1, 2, 3. \tag{11.2.23}
\]

If the Lagrangian \( L \) does not depend explicitly on \( x_\mu \) the above energy-momentum conservation laws turn into

\[
\partial_t p_i = \sum_{j=1,2,3} \partial_j \sigma_{ji}, \quad \text{where} \quad p_i = \frac{1}{c} T^{0i}, \quad \sigma_{ji} = -T^{ji}, \quad i, j = 1, 2, 3, \tag{11.2.24}
\]

\[
\partial_t u + \sum_{j=1,2,3} \partial_j s_j = 0, \quad \text{where} \quad u = T^{00}, \quad s_i = c T^{i0} = c^2 p_i, \quad i = 1, 2, 3. \tag{11.2.25}
\]
Consequently, the total conserved energy momentum 4-vector takes the form

\[ P_\nu = \frac{1}{c} \int_{\mathbb{R}^3} T^{0\nu} (x) \, dx, \]  

(11.2.26)

and its components, the total energy and momentum are respectively

\[ H = cP^0 = \int_{\mathbb{R}^3} T^{00} (x) \, dx, \quad P^j = \int_{\mathbb{R}^3} T^{0j} (x) \, dx, \quad j = 1, 2, 3. \]  

(11.2.27)

Evidently, the formulas (11.2.26), (11.2.27) are particular cases of the formulas (11.2.16) for the special important choice of \( \sigma = \{ x = (x_0, x) : x \in \mathbb{R}^3 \} \).

Importantly, for closed systems the conserved total energy-momentum \( P^\nu \) and \( J^{\nu\gamma} \) angular momentum as defined by formulas (11.2.16) and (11.2.20) transform respectively as 4-vector and 4-tensor under Lorentz transformation and that directly related to the conservation laws (11.2.17), [Moller, Section 6.2], [Jackson, Section 12.10 A]. But for open (not closed) systems generally the total energy-momentum \( P^\nu \) and \( J^{\nu\gamma} \) angular momentum do not transform as respectively 4-vector and 4-tensor, [Moller, Section 7.1, 7.2], [Jackson, Section 12.10 A, 16.4].

### 11.3 Noether’s theorem

In this section following to [Goldstein, Section 13.7] we provide basic information on the Noether’s theorem which relates symmetries to conservation laws based on the Lagrangian formalism. Suppose that the Lagrangian \( \mathcal{L} \) as defined (11.2.1) does not depend explicitly on the field variable \( q^\ell \). Then the Euler-Lagrange equations (11.2.3) for that variable turns into

\[ \partial_\mu J^\mu_\ell = 0, \quad J^\mu_\ell = \frac{\partial \mathcal{L}}{\partial q^\ell_{\mu}}, \]  

(11.3.1)

which is a conservation law (continuity equation) for the four-”current” \( J^\mu_\ell \). Noether theory interprets the above situation as certain invariance (symmetry) of the Lagrangian and provides its far reaching generalization allowing to obtain conservation laws based on the Lagrangian invariance (symmetry) with respect to a general Lie group of transformation. Symmetry under coordinate transformation refers to the effects of infinitesimal transformation of the form

\[ x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu, \]  

(11.3.2)

where the infinitesimal change \( \delta x^\mu \) may depend on other \( x^\nu \). The field transformations are assumed to be of the form

\[ q^\ell (x) \rightarrow q'^\ell (x') = q^\ell (x) + \delta q^\ell (x), \]  

(11.3.3)

where \( \delta q^\ell (x) \) measures to total change of \( q^\ell \) due to both \( x \) and \( q^\ell \) and it can depend on other field variables \( q^{\ell_1} \). We also consider a local change \( \delta q^\ell (x) \) of \( q^\ell (x) \) at a point \( x \)

\[ q'^\ell (x) = q^\ell (x) + \delta q^\ell (x), \]  

(11.3.4)

We assume the following three conditions to hold for the transformations (11.3.2)-(11.3.3): (i) the 4-space (time and the space) is flat; (ii) the Lagrangian density \( \mathcal{L} \) displays the same
functional form in terms of the transformed quantities as it does of the original quantities, that is,
\[ L\left(\{q^\ell(x)\}, \{q^\ell_\mu(x)\}, x\right) = L\left(\{q'^\ell(x')\}, \{q'^\ell_\mu(x')\}, x'\right); \quad (11.3.5) \]

(iii) The magnitude of the action integral is invariant under the transformation, that is
\[ \int_{\Omega} L\left(\{q^\ell(x)\}, \{q^\ell_\mu(x)\}, x\right) dx = \int_{\Omega'} L\left(\{q'^\ell(x')\}, \{q'^\ell_\mu(x')\}, x'\right) dx'; \quad (11.3.6) \]

where \( \Omega \) is 4-dimensional region bounded by two space-like 3-dimensional surfaces and \( dx = \sqrt{|\det g|} dx_0 dx_1 dx_2 dx_3 \).

If the three above conditions are satisfied the following conservation law holds
\[ \partial_\mu J^\mu = 0, \quad J^\mu = \sum_\ell \frac{\partial L}{\partial q^\ell_\mu} \delta q^\ell (x) + L \delta x^\mu. \quad (11.3.7) \]

The above conservation law can be made more specific if we introduce infinitesimal parameters \( \xi_r \) related to Lie group of transformations and represent \( \delta x^\mu \) and \( \delta q^\ell \) in their terms, namely
\[ \delta x^\mu = \sum_r X^\mu_\nu \xi_\nu, \quad \delta q^\ell = \sum_r Q^\ell_\nu \xi_\nu, \quad (11.3.8) \]

where the functions \( X^\mu_\nu \) and \( Q^\ell_\nu \) may depend upon the other coordinates and field variables, respectively. Then we for \( r \) the following conservation law holds for the respective Noether’s current \( J^\mu_r \):
\[ \partial_\mu J^\mu_r = 0, \quad J^\mu_r = \left[ \sum_\ell \frac{\partial L}{\partial q^\ell_\mu} \delta q^\ell_\nu - L \delta^\mu_\nu \right] X^\nu_r - \sum_\ell \frac{\partial L}{\partial q^\ell_\mu} Q^\ell_\nu. \quad (11.3.9) \]

For instance, in the case of the group of inhomogeneous Lorentz transformations defined by (11.1.10) there are ten parameters \( a^\mu \) and \( \xi^{\mu\nu} \) and consequently there are ten corresponding conserved quantities \( P^\mu \) and \( J^{\nu\mu} = -J^{\mu\nu} \) defined by (11.2.16). Another important example is the group of gauge transformation of the first kind defined by (11.5.7) below. For this group there is a conserved current \( J^{\nu} \) for every \( \ell \) defined by (11.5.14).

11.4 Electromagnetic fields and the Maxwell equations

We consider the Maxwell equations
\[ \nabla \cdot E = 4\pi \rho, \quad \nabla \cdot B = 0 \quad (11.4.1) \]
\[ \nabla \times E + \frac{1}{c} \partial_t B = 0, \quad \nabla \times B - \frac{1}{c} \partial_t E = \frac{4\pi}{c} j. \quad (11.4.2) \]
for the EM fields and their covariant form following to [Jackson, Section 11.9], [LandauLif EM, Sections 23, 30], [Griffiths, Sections 7.4, 11.2], in CGS units. To represent Maxwell equations in a manifestly Lorentz invariant form it is common to introduce a four-vector potential \( A^\mu \) and a four-vector current density \( J^\nu \)
\[ A^\mu = (\varphi, A^r), \quad J^\mu = (c\rho, J^r), \quad (11.4.3) \]
\[ \partial_\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{1}{c} \partial_t, \nabla \right), \quad \partial^\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{1}{c} \partial_t, -\nabla \right), \]
and, then, an antisymmetric second-rank tensor, the "field strength tensor,

\[ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \]  

so that

\[
F^{\mu\nu} = \begin{bmatrix}
0 & -E_1 & -E_2 & -E_3 \\
E_1 & 0 & -B_3 & B_2 \\
E_2 & B_3 & 0 & -B_1 \\
E_3 & -B_2 & B_1 & 0 \\
\end{bmatrix}, \quad F_{\mu\nu} = \begin{bmatrix}
0 & E_1 & E_2 & -E_3 \\
-E_1 & 0 & -B_3 & B_2 \\
-E_2 & B_3 & 0 & -B_1 \\
-E_3 & -B_2 & B_1 & 0 \\
\end{bmatrix},
\]

and

\[ E = -\nabla \phi - \frac{1}{c} \partial_t A, \quad B = \nabla \times A. \]  

Then the two inhomogeneous equations and the two homogeneous equations from the four Maxwell equations (11.4.1) take respectively the form

\[ \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \]  

\[ \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \quad \alpha, \beta, \gamma = 0, 1, 2, 3. \]  

It follows from the asymmetry of \( F^{\mu\nu} \), the Maxwell equation (11.4.7) and (11.4.3)-(11.4.4) that the four-vector current \( J^\mu \) must satisfy the continuity equation

\[ \partial_\mu J^\mu = 0 \text{ or } \partial_t \rho + \nabla \cdot A = 0. \]  

The Maxwell equations (11.4.7) turn into the following equations for the four-vector potential \( A^\mu \)

\[ \Box A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J^\nu, \]  

where

\[ \Box = \partial_\mu \partial^\mu = \frac{1}{c^2} \partial_t^2 - \nabla^2 \]  

d’Alembertian operator). (11.4.11)

According to [Jackson, Section 11.10] the electric and magnetic fields are transformed from one frame to another one moving relatively with the velocity \( \mathbf{v} \) by the following formulas

\[
E' = \gamma (E + \beta \times B) - \frac{\gamma^2}{\gamma + 1} (\beta \cdot E) \beta, \\
B' = \gamma (B - \beta \times E) - \frac{\gamma^2}{\gamma + 1} (\beta \cdot B) \beta, \\
\beta = \frac{\mathbf{v}}{c}, \quad \beta = |\beta|, \quad \gamma = \frac{1}{\sqrt{1 - (\frac{\mathbf{v}}{c})^2}},
\]

which also can be recast as, [Grainer EM, Section 22],

\[
E'_\perp = \gamma (E_\perp + \beta \times B), \quad E'_\parallel = E_\parallel, \\
B'_\perp = \gamma (B_\perp - \beta \times E), \quad B'_\parallel = B_\parallel.
\]
where subindices $\perp$ and $\parallel$ stand for vector components respectively parallel and perpendicular to $v$. Observe that for $\beta \ll 1$ the formulas (11.4.12), (11.4.13) yield the following approximations with an error proportional to $\beta^2$ where $J_\mu$ is an external four-vector current.

$$E'_\perp \approx E_\perp + \beta \times B, \quad E'_\parallel = E_\parallel, \quad B'_\perp \approx B_\perp - \beta \times E, \quad B'_\parallel = B_\parallel.$$  \hfill (11.4.14)

The EM field Maxwell Lagrangian is, [Jackson, Section 12.7], [Barut, Section IV.1]

$$L_{em}(A^\mu) = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu,$$  \hfill (11.4.15)

where $J_\mu$ is an external (impressed) current. Using (11.4.5), (11.4.6) and (11.4.3) we can recast (11.4.15) as

$$L_{em}(A^\mu) = \frac{1}{8\pi} \left[ \nabla \varphi + \frac{1}{c} \partial_0 A \right]^2 - \left( \nabla \times A \right)^2 - \rho \varphi + \frac{1}{c} J \cdot A. \hfill (11.4.16)$$

In particular, if there are no sources the above Lagrangians turn into

$$L_{em}(A^\mu) = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} = \frac{1}{8\pi} (E^2 - B^2) = \frac{1}{8\pi} \left[ \left( \nabla \varphi + \frac{1}{c} \partial_0 A \right)^2 - \left( \nabla \times A \right)^2 \right] - \rho \varphi + \frac{1}{c} J \cdot A. \hfill (11.4.17)$$

The canonical stress (power-momentum) tensor $\hat{\Theta}^{\mu\nu}$ for the EM field is as follows, [Jackson, (12.104)], [Barut, Section III.4.D]

$$\hat{\Theta}^{\mu\nu} = -\frac{F^{\mu\gamma} A_\gamma}{4\pi} + \frac{g^{\mu\nu} F^{\xi\gamma} F_{\xi\gamma}}{16\pi}, \hfill (11.4.18)$$

or, in particular, for $i, j = 1, 2, 3$

$$\hat{\Theta}^{00} = -\frac{E^2 - B^2}{8\pi} + \rho \varphi - \frac{1}{c} J \cdot A - \frac{\partial_0 A \cdot E}{4\pi}, \hfill (11.4.19)$$

$$\hat{\Theta}^{0i} = -\frac{\partial_i A \cdot E}{4\pi}, \quad \hat{\Theta}^{i0} = -\frac{E_i \partial_0 \varphi}{4\pi} + \frac{(B \times \partial_0 A)_i}{4\pi},$$

$$\hat{\Theta}^{ij} = -\frac{E_i \partial_j \varphi}{4\pi} + \frac{(B \times \partial_j A)_i}{4\pi} + \frac{E^2 - B^2}{8\pi} - \rho \varphi + \frac{1}{c} J \cdot A,$$

whereas the symmetric one $\Theta^{\alpha\beta}$ for the EM field is, [Jackson, Section 12.10, (12.113)], [Barut, Section III.3]

$$\Theta^{\alpha\beta} = \frac{1}{4\pi} \left( g^{\alpha\mu} F_{\mu\nu} F^{\nu\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right), \hfill (11.4.20)$$

implying the following formulas for the field energy density $w$, the momentum $c g_i$ and the Maxwell stress tensor $\tau_{ij}$:

$$w = \Theta^{00} = \frac{E^2 + B^2}{8\pi}, \quad c g_i = \Theta^{0i} = \Theta^{i0} = \frac{E \times B}{4\pi}, \hfill (11.4.21)$$
\[ \Theta^{ij} = -\frac{1}{4\pi} \left[ E_i E_j + B_i B_j - \frac{1}{2} \delta_{ij} (E^2 + B^2) \right], \quad (11.4.22) \]

\[ \Theta^{\alpha\beta} = \begin{bmatrix} w & cg \\ cg & -\tau_{ij} \end{bmatrix}, \quad \Theta_{\alpha\beta} = \begin{bmatrix} w & -cg \\ -cg & -\tau_{ij} \end{bmatrix}, \quad (11.4.23) \]

\[ \Theta^{\alpha}_\beta = \begin{bmatrix} w & -cg \\ cg & -\tau_{ij} \end{bmatrix}, \quad \Theta^\beta_\alpha = \begin{bmatrix} w & cg \\ -cg & -\tau_{ij} \end{bmatrix}. \]

Note that in the special case when the vector potential \( \mathbf{A} \) vanishes and the scalar potential \( \varphi \) does not depend on time using the expressions (11.4.6) we get the following representation for the canonical energy density defined by (11.4.19)

\[ \dot{\Theta}^{00} = -\frac{(\nabla \varphi)^2}{8\pi} + \rho \varphi \text{ for } \mathbf{A} = 0 \text{ and } \partial_0 \varphi = 0, \quad (11.4.24) \]

whereas \( \Theta^{00} = \frac{(\nabla \varphi)^2}{8\pi} \).

It is instructive to observe a substantial difference between the above expressions \( \dot{\Theta}^{00} \), which is the Hamiltonian density of the EM field, and the energy density \( \Theta^{00} \) defined by (11.4.21).

If there no external currents the with differential conservation law takes the form

\[ \partial_\alpha \Theta^{\alpha\beta} = 0, \quad (11.4.25) \]

and, in particular, the energy conservation law

\[ 0 = \partial_\alpha \Theta^{\alpha\beta} = \frac{1}{c} \left( \frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} \right), \quad \text{where } w \text{ is the energy density, and} \quad (11.4.26) \]

\[ \mathbf{S} = c^2 \mathbf{g} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \text{ is the Poynting vector}. \]

In the presence of external currents the conservation laws take the form, [Jackson, Section 12.10]

\[ \partial_\alpha \Theta^{\alpha\beta} = -f^\beta, \quad f^\beta = \frac{1}{c} F^{\beta\nu} J_\nu, \quad (11.4.27) \]

and the time and space components of the equations (11.4.27) are the conservation of energy \( w \) and momentum \( \mathbf{g} \) which can be recast as

\[ \frac{1}{c} \left( \frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} \right) = -\frac{1}{c} \mathbf{J} \cdot \mathbf{E}, \quad (11.4.28) \]

\[ \frac{\partial g_i}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x^j} \tau_{ij} = - \left[ \rho E_i + \frac{1}{c} (\mathbf{J} \times \mathbf{B})_i \right]. \quad (11.4.29) \]

The 4-vector \( f^\beta \) in the conservation law (11.4.27) is known as the Lorentz force density

\[ f^\beta = \frac{1}{c} F^{\beta\nu} J_\nu = \left( \frac{1}{c} \mathbf{J} \cdot \mathbf{E}, \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right). \quad (11.4.30) \]
11.4.1 Green functions for the Maxwell equations

This section in an excerpt from [Jackson, Section 12.11]. The EM fields $F^{\mu\nu}$ arising from an external source $J^\nu$ satisfy the inhomogeneous Maxwell equations

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$  \hspace{1cm} (11.4.31)

which take the following form for the potentials $A^\nu$

$$\Box A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J^\nu.$$  \hspace{1cm} (11.4.32)

If the potentials satisfy the Lorentz condition, $\partial_\mu A^\mu = 0$, they are then solutions of the four-dimensional wave equation,

$$\Box A^\nu = \frac{4\pi}{c} J^\nu.$$  \hspace{1cm} (11.4.33)

The solution of (11.4.33) is accomplished by finding a Green function $D(x, x')$ for the equation

$$\Box D(z) = \delta^{(4)}(z), \quad D(x, x') = D(x - x'),$$  \hspace{1cm} (11.4.34)

where $\delta^{(4)}(z) = \delta(z_0) \delta(z)$ is a four-dimensional delta function. One can introduce then the so-called retarded or causal Green function solving the above equation (11.4.34), namely

$$D_r(x - x') = \frac{\theta(x_0 - x'_0) \delta(x_0 - x'_0 - R)}{4\pi R}, \quad R = |x - x'|,  \hspace{1cm} (11.4.35)$$

where $\theta(x_0)$ is the Heaviside step function. The name causal or retarded is justified by the fact that the source-point time $x'_0$ is always earlier than the observation-point time $x_0$. Similarly one can introduce the advanced Green function

$$D_a(x - x') = \frac{\theta[-(x_0 - x'_0)] \delta(x_0 - x'_0 + R)}{4\pi R}, \quad R = |x - x'|.  \hspace{1cm} (11.4.36)$$

These Green functions can be written in the following covariant form

$$D_r(x - x') = \frac{1}{2\pi} \theta(x_0 - x'_0) \delta \left[ (x - x')^2 \right],  \hspace{1cm} (11.4.37)$$

$$D_a(x - x') = \frac{1}{2\pi} \theta(x'_0 - x_0) \delta \left[ (x - x')^2 \right],  \hspace{1cm} (11.4.38)$$

where $(x - x')^2 = (x_0 - x'_0)^2 - |x - x'|^2$ and

$$\delta \left[ (x - x')^2 \right] = \frac{1}{2R} [\delta(x_0 - x'_0 - R) + \delta(x_0 - x'_0 + R)].  \hspace{1cm} (11.4.39)$$

The solution to the wave equation (11.4.33) can be written in terms of the Green functions

$$A^\nu(x) = A^\nu_{in}(x) + \frac{4\pi}{c} \int D_r(x - x') J^\nu(x') \, dx$$  \hspace{1cm} (11.4.39)$$

or

$$A^\nu(x) = A^\nu_{out}(x) + \frac{4\pi}{c} \int D_a(x - x') J^\nu(x') \, dx$$  \hspace{1cm} (11.4.40)$$
where \( A^\nu_{\text{in}}(x) \) and \( A^\nu_{\text{out}}(x) \) are solutions to the homogeneous wave equation. In \((11.4.39)\) the retarded Green function is used. In the limit \( x_0 \rightarrow -\infty \), the integral over the sources vanishes, assuming the sources are localized in space and time, because of the retarded nature of the Green function, and \( A^\nu_{\text{in}}(x) \) can be interpreted as ”incident” or ” incoming” potential, specified at \( x_0 \rightarrow -\infty \). Similarly, in \((11.4.40)\) with the advanced Green function, the homogeneous solution \( A^\nu_{\text{out}}(x) \) is the asymptotic ”outgoing” potential, specified at \( x_0 \rightarrow +\infty \). The radiation fields are defined as the difference between the ” outgoing” and ” incoming” fields, and their 4-vector potential is

\[
A^\nu_{\text{rad}}(x) = A^\nu_{\text{out}}(x) - A^\nu_{\text{in}}(x) = \frac{4\pi}{c} \int D(x - x') J^\nu(x') \, dx, \quad \text{where} \quad (11.4.41)
\]

\[
D(x - x') = D_t(x - x') - D_a(x - x').
\]

11.5 Many charges interacting with the electromagnetic field

In this section we consider the Lagrange formalism for complex-valued fields \( \psi^\ell, \ell = 1, \ldots, N \) that describe charges following to \cite{Pauli RFTh, Part I} and \cite{Wentzel, Section I.3}. For every complex-valued \( \psi^\ell \) we always assume the presence of its conjugates \( \psi^{\ell*} \), so the product \( \psi^\ell \psi^{\ell*} \) is real. The Lagrangian is assumed to be real valued and its general form is

\[
\mathcal{L} = \mathcal{L} \left( \{ \psi^\ell, \psi^\ell*, \psi^{\ell*}, \psi^{\ell*}, \} , \{ V^g, V^g_{\mu} \} , x^\ell \right), \quad (11.5.1)
\]

where \( \{ V^g \} \) are real-valued quantities. In the Lagrangian \((11.5.1)\) the fields \( \psi^\ell \) and their conjugates \( \psi^{\ell*} \) are treated as independent and the corresponding Euler-Lagrange field equations are, \cite{Morse Feshbach 1, Section 3.3, Wentzel, Section II.3, (3.3)],

\[
\frac{\partial \mathcal{L}}{\partial \psi^{\ell*}} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \psi^\ell_{\mu*}} \right) = 0, \quad (11.5.2)
\]

\[
\frac{\partial \mathcal{L}}{\partial \psi^{\ell*}} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \psi^\ell^{\mu*}} \right) = 0, \quad \frac{\partial \mathcal{L}}{\partial V^g} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial V^g_{\mu}} \right) = 0.
\]

The canonical energy-momentum tensor for the Lagrangian \((11.5.1)\) is similar to the general formula \((11.2.5)\), namely

\[
\mathcal{T}^{\mu\nu} = \sum_\ell \frac{\partial \mathcal{L}}{\partial \psi^\ell_{\mu*}} \psi^{\ell*}_{\nu} + \frac{\partial \mathcal{L}}{\partial \psi^{\ell*}_{\mu}} \psi^\ell_{\nu} + \sum_g \frac{\partial \mathcal{L}}{\partial V^g_{\mu*}} V^g_{\nu} - g^{\mu\nu} \mathcal{L}. \quad (11.5.3)
\]

In the case when the Lagrangian \( \mathcal{L} \) depends on only complex-valued fields \( \psi^\ell \) and \( \psi^{\ell*} \) the canonical stress tensor is symmetric and is of the form, \cite{Morse Feshbach 1 (3.3.23), Wentzel (3.8)],

\[
\mathcal{T}^{\mu\nu} = \sum_\ell \frac{\partial \mathcal{L}}{\partial \psi^\ell_{\mu*}} \psi^{\ell*}_{\nu} + \frac{\partial \mathcal{L}}{\partial \psi^{\ell*}_{\mu}} \psi^\ell_{\nu} - g^{\mu\nu} \mathcal{L}. \quad (11.5.4)
\]

An important for us special case of the Lagrangian \((11.5.1)\) is when there are several charges described by complex valued fields \( \psi^\ell \) and \( \psi^{\ell*} \) interacting with the EM field described by the real-valued four-potential \( A^\mu \). For this case we introduce the Lagrangian of the form

\[
\mathcal{L} \left( \{ \psi^\ell, \psi^\ell_{\mu*}, \psi^{\ell*}, \psi^{\ell*}, \} , A^\mu \right) = \quad (11.5.5)
\]

\[
= \sum_\ell L^\ell \left( \psi^\ell, \psi^\ell_{\mu*}, \psi^{\ell*}, \psi^{\ell*} \right) - \frac{F^{\mu\nu} F_{\mu\nu}}{16\pi}, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.
\]
where we have introduced the so-called \textit{covariant derivatives} \( \psi_{\ell\mu}^{\mu} \) and \( \psi_{\ell\mu}^{*} \) by the following formulas

\[
\psi_{\ell\mu}^{\mu} = \bar{\psi}^{\mu} \psi_{\ell}^{\mu}, \quad \psi_{\ell\mu}^{*} = \bar{\psi}^{\mu*} \psi_{\ell}^{\mu}, \quad (11.5.6)
\]

where

\[
\bar{\psi}^{\mu} = \partial^{\mu} + \frac{iq_{\ell}}{\lambda c} A^{\mu}, \quad \bar{\psi}^{\mu*} = \partial^{\mu} - \frac{iq_{\ell}}{\lambda c} A^{\mu}.
\]

In the above formula for every \( \ell \) the real number \( q_{\ell} \) is the charge of the \( \ell \)-th elementary charge and \( \bar{\psi}^{\mu} \) and \( \bar{\psi}^{\mu*} \) are called the \textit{covariant differentiation operators}. The particular forms \((11.5.5)-(11.5.6)\) of the multiparticle Lagrangian \( L \) and its \( \ell \)-the charge components \( L^{\ell} \) originates from the condition of \textit{gauge invariance}. More precisely, one introduces the \textit{gauge transformation of the first or the second kind} (known also as respectively \textit{global and local gauge transformation}) for the fields \( \psi_{\ell}^{\mu} \) and \( \psi_{\ell}^{\mu*} \). These transformations are described respectively by the following formulas namely, \([\text{Pauli RFTh}, (17), (23a), (23b)], [\text{Wentzel}, \text{Section 11, (11.4)}]\),

\[
\psi_{\ell}^{\mu} \rightarrow e^{i\gamma_{\ell}^{\mu} \rho} \psi_{\ell}^{\mu}, \quad \psi_{\ell}^{\mu*} \rightarrow e^{i\gamma_{\ell}^{\mu*} \rho} \psi_{\ell}^{\mu*}, \quad \text{where} \quad \gamma_{\ell}^{\mu} = \text{any real constant}, \quad (11.5.7)
\]

\[
\psi_{\ell}^{\mu} \rightarrow e^{-i\frac{q_{\ell}}{\lambda c} x^{\mu}} \psi_{\ell}^{\mu}, \quad \psi_{\ell}^{\mu*} \rightarrow e^{i\frac{q_{\ell}}{\lambda c} x^{\mu}} \psi_{\ell}^{\mu*}, \quad A^{\mu} \rightarrow A^{\mu} + \partial^{\mu} \lambda, \quad (11.5.8)
\]

and the Lagrangian is assumed to be invariant with respect to the all gauge transformations \((11.5.7), (11.5.8)\). Notice that for the multi-charge Lagrangian \( L \) as defined by \((11.5.5)-(11.5.6)\) the following is true: (i) every charge interacts with the EM field described by the four-potential \( A^{\mu} \); (ii) different charges don’t interact directly with each other, but they interact only indirectly through the EM field.

We also introduce the following symmetry condition on charges Lagrangians \( L^{\ell} \):

\[
\frac{\partial L^{\ell}}{\partial \psi_{\ell\mu}^{\mu}} \psi_{\ell\nu}^{\mu} + \frac{\partial L^{\ell}}{\partial \psi_{\ell\mu}^{*}} \psi_{\ell\nu}^{*\mu} = \frac{\partial L^{\ell}}{\partial \psi_{\ell\mu}^{\mu}} \psi_{\ell\nu}^{\mu} + \frac{\partial L^{\ell}}{\partial \psi_{\ell\mu}^{*}} \psi_{\ell\nu}^{*\mu}. \quad (11.5.9)
\]

As we show below the symmetry condition \((11.5.9)\) implies that energy-momentum assigned to every individual charge is symmetric and gauge invariant energy-momentum. A simple sufficient condition for the symmetry condition \((11.5.9)\) is a requirement for the Lagrangians \( L^{\ell} \) to depend on the field covariant derivatives only through the combination \( \psi_{\ell\mu}^{\mu} \psi_{\ell\mu}^{*\mu} \), in other words if there exist such functions \( K^{\ell}(\psi_{\ell}, \psi_{\ell}^{*}, b) \) that

\[
L^{\ell}(\psi_{\ell}, \psi_{\ell\mu}^{\mu}, \psi_{\ell\mu}^{*\mu}, \psi_{\ell\mu}^{*\mu}) = K^{\ell}(\psi_{\ell}, \psi_{\ell}^{*}, \psi_{\ell\mu}^{\mu} \psi_{\ell\mu}^{*\mu}). \quad (11.5.10)
\]

Indeed, in this case

\[
\frac{\partial L^{\ell}}{\partial \psi_{\ell\mu}^{\mu}} \psi_{\ell\nu}^{\mu} + \frac{\partial L^{\ell}}{\partial \psi_{\ell\mu}^{*}} \psi_{\ell\nu}^{*\mu} = \frac{\partial K^{\ell}}{\partial b} \left( \psi_{\ell\mu}^{\mu} \psi_{\ell\mu}^{*\mu} + \psi_{\ell\mu}^{\mu} \psi_{\ell\mu}^{*\mu} \right), \quad (11.5.11)
\]

readily implying that the symmetry condition \((11.5.9)\) does hold.

The field equations for the Lagrangian \( L \) defined by \((11.5.5)-(11.5.6)\) are

\[
\frac{\partial L^{\ell}}{\partial \psi_{\ell}^{\mu}} - \bar{\psi}_{\mu}^{*} \left[ \frac{\partial L^{\ell}}{\partial \psi_{\ell\mu}^{\mu}} \right] = 0, \quad \frac{\partial L^{\ell}}{\partial \psi_{\ell}^{*\mu}} - \bar{\psi}_{\mu}^{*} \left[ \frac{\partial L^{\ell}}{\partial \psi_{\ell\mu}^{*\mu}} \right] = 0. \quad (11.5.12)
\]
\[ \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad J^\nu = \sum_\ell J^{\ell\nu}, \] (11.5.13)

where \( J^{\ell\nu} \) is the four-vector current related to the \( \ell \)-th charge is defined as follows. Under the gauge invariance conditions (11.5.7), (11.5.8) for the Lagrangian \( \mathcal{L} \) using the Noether’s theorem and the formula (11.3.9) one can introduce for every charge \( \psi^\ell \) the following 4-vector current, \[ J^{\ell\nu} = -i q^\ell \chi \left( \frac{\partial L^\ell}{\partial \psi^\ell_{\nu}} \psi^\ell - \frac{\partial L^\ell}{\partial \psi^\ell_{\nu}^*} \psi^\ell_* \right), \] (11.5.14)
or, since \( J^\nu = (c \rho, J) \),
\[ \rho^\ell = -i q^\ell \chi \left( \frac{\partial L^\ell}{\partial \psi^\ell_{\rho}} \psi^\ell - \frac{\partial L^\ell}{\partial \psi^\ell_{\rho}^*} \psi^\ell_* \right), \] (11.5.15)\[ J^j_j = -i q^\ell \chi \left( \frac{\partial L^\ell}{\partial \psi^\ell_{j}} \psi^\ell - \frac{\partial L^\ell}{\partial \psi^\ell_{j}^*} \psi^\ell_* \right), \quad j = 1, 2, 3, \]
which satisfy for every \( \ell \) the charge conservation/continuity equations
\[ \partial_\nu J^{\ell\nu} = 0, \quad \text{or} \quad \partial_\nu \rho^\ell + \nabla \cdot J^\ell = 0, \quad J^{\ell\nu} = (c \rho^\ell, J^\ell). \] (11.5.16)

Notice that in view of the relations (11.5.5)-(11.5.6) the following alternative representation holds for the four-current \( J^{\ell\nu} \) defined by (11.5.14)
\[ J^{\ell\nu} = -i q^\ell \chi \left( \frac{\partial L^\ell}{\partial \psi^\ell_{\nu}} \psi^\ell - \frac{\partial L^\ell}{\partial \psi^\ell_{\nu}^*} \psi^\ell_* \right) = -c \frac{\partial L^\ell}{\partial A^\nu}. \] (11.5.17)

We would like to emphasize here the physical significance of identity (11.5.17) equating two complementary views on the electric current: (i) as a source current in the Maxwell equations (11.5.13); (ii) as the gauge electric current (11.5.14) satisfying the continuity equation (11.5.16). Notice that the equality (11.5.17) originates from a particular form of the coupling between the EM field and charges in the Lagrangian (11.5.5), namely the coupling through the covariant derivatives (11.5.6). One may also view the electric currents identity (11.5.17) as a physical rational for introducing the coupling exactly as it is done in the expressions (11.5.5)-(11.5.6).

### 11.5.1 Gauge invariant and symmetric energy-momentum tensors

In this subsection we consider a Lagrangian defined by formulas (11.5.5)-(11.5.6) and assume it to be gauge invariant with respect to transformations if the first and the second type. To obtain an expression for the total symmetric energy-momentum tensor \( \mathcal{T}^{\mu\nu} \) for such a Lagrangian we use the theory described in Section 11.2 formulas (11.2.11) and (11.5.4), namely
\[ \mathcal{T}^{\mu\nu} = \mathcal{\dot{T}}^{\mu\nu} - \partial_\gamma f^{\mu\gamma\nu}, \quad \mathcal{\dot{T}}^{\mu\nu} = \mathcal{\dot{\Theta}}^{\mu\nu} + \sum_\ell \mathcal{\dot{T}}^{\ell\mu\nu}, \] (11.5.18)
where the canonical energy-momentum of EM field $\hat{T}^{\mu
u}$ and the energy-momentum tensor $\hat{T}^{\mu
u}_\ell$ of $\ell$-th charge are represented as follows (see [Jackson, (12.104)], [Barut, Section III.4.D] for $\hat{\Theta}^{\mu
u}$ and (11.5.4) for $\hat{T}^{\mu
u}_\ell$)

$$\hat{T}^{\mu
u}_\ell = \frac{\partial L_\ell}{\partial \psi^{\ell,\mu}} \psi^{\ell,\nu} + \frac{\partial L_\ell}{\partial \psi^{\ell,*\mu}} \psi^{\ell,*\nu} - g^{\mu\nu} L_\ell, \quad (11.5.19)$$

$$\hat{\Theta}^{\mu\nu} = -\frac{1}{4\pi} F^{\mu\gamma} \partial_{\nu} A_\gamma + g^{\mu\nu} \frac{F^{\xi\gamma} F_{\xi\gamma}}{16\pi}. \quad (11.5.20)$$

The above canonical energy-momenta tensors are neither gauge invariant nor symmetric. To find a representation for $f^{\mu\gamma\nu}$ in the formula (11.5.18) for $\hat{T}^{\mu\nu}$ we use the formulas (11.2.11) noticing that for the scalar fields $\psi^{\ell}$ and $\psi^{\ell,*}$ we apply the formula (11.2.14), whereas for the vector field $A^\mu$ we apply the formula (11.2.15). This yields

$$f^{\mu\gamma\nu} = -\frac{1}{4\pi} F^{\mu\gamma} A^\nu, \quad (11.5.21)$$

and, consequently

$$-\partial_{\gamma} \frac{1}{4\pi} f^{\mu\gamma\nu} = \frac{1}{4\pi} \partial_{\gamma} (F^{\mu\gamma}) A^\nu + \frac{1}{4\pi} F^{\mu\gamma} \partial_{\gamma} A^\nu = -\frac{1}{c} J^\mu A^\nu + \frac{1}{4\pi} F^{\mu\gamma} \partial_{\gamma} A^\nu, \quad (11.5.22)$$

where we used the Maxwell equations (11.5.13) producing a term with the current $J^\mu = \sum_\ell J^{\ell\mu}$. We introduce now the following energy-momentum tensors

$$T^{\mu\nu}_\ell = \hat{T}^{\mu\nu}_\ell - \frac{1}{c} J^{\ell\mu} A^\nu, \quad \Theta^{\mu\nu} = \hat{\Theta}^{\mu\nu} + \frac{1}{4\pi} F^{\mu\gamma} \partial_{\gamma} A^\nu, \quad (11.5.23)$$

and using relations (11.5.17) and (11.5.19) we find that they have the following representations

$$\Theta^{\mu\nu} = \frac{1}{4\pi} \left( g^{\mu\gamma} F_{\gamma\xi} F^{\xi\nu} + \frac{1}{4} g^{\mu\nu} F_{\gamma\xi} F^{\gamma\xi} \right), \quad (11.5.24)$$

$$T^{\mu\nu}_\ell = \frac{\partial L_\ell}{\partial \psi^{\ell,\mu}} \psi^{\ell,\nu} + \frac{\partial L_\ell}{\partial \psi^{\ell,*\mu}} \psi^{\ell,*\nu} - g^{\mu\nu} L_\ell. \quad (11.5.25)$$

The formula (11.5.24) is a well known representation (11.4.20) for the symmetric and gauge invariant energy-momentum tensor of the EM field (see [Jackson, Section 12.10], [Barut, III.3]). Notice also that each tensors $T^{\mu\nu}_\ell$ defined by (11.5.25) is manifestly gauge invariant. In the case when symmetry condition (11.5.9) is satisfied $T^{\mu\nu}_\ell$ is also symmetric.

Using (11.5.24) and (11.5.25) we define now the total energy-momentum tensor by

$$T^{\mu\nu} = \Theta^{\mu\nu} + \sum_\ell T^{\mu\nu}_\ell, \quad (11.5.26)$$

and that it is an admissible since it differs from the canonical one by the divergence $\partial_{\gamma} f^{\mu\gamma\nu}$.

In the case of the Lagrangian of the form (11.5.10) in view of (11.5.11) the energy-momentum expression (11.5.25) turns into

$$T^{\mu\nu}_\ell = \frac{\partial K_\ell}{\partial b} \left( \psi^{\ell,\mu} \psi^{\ell,*\nu} + \psi^{\ell,*\mu} \psi^{\ell,\nu} \right) - g^{\mu\nu} K_\ell. \quad (11.5.27)$$
Consequently, as expected the energy conservation law for the total system takes the familiar form

$$\partial_\mu T^{\mu\nu} = \sum_\ell \partial_\mu T^{\ell\mu\nu} + \partial_\mu \Theta^{\mu\nu} = 0.$$  \hfill (11.5.28)

### 11.5.2 Equations for the energy-momentum tensors

Notice that using the field equations (11.5.12)-(11.5.13) and the expression (11.5.24) for the energy-momentum $\Theta^{\mu\nu}$ of the EM field we get (see details of the derivation in Jackson, Section 12.10C) the following equation

$$\partial_\mu \Theta^{\mu\nu} = -\frac{1}{c} J_\mu F^{\nu\mu}, \text{ where } J_\mu = \sum_\ell J^{\ell}_\mu.$$  \hfill (11.5.29)

We show below that the above equation for the energy-momentum $\Theta^{\mu\nu}$ is complemented by the following equations for the energy-momenta $T^{\ell\mu\nu}$ defined by (11.5.25) of individual charges

$$\partial_\mu T^{\ell\mu\nu} = \frac{1}{c} J^{\ell}_\mu F^{\nu\mu}.$$  \hfill (11.5.30)

Observe now that in view of the representation (11.4.30) for the Lorentz force the following is true: (i) the right-hand side of the equation (11.5.30) is the Lorentz force exerted by the EM field on the $\ell$-th charge; (ii) the right-hand side of the equation (11.5.29) is the force exerted by all the charges on the EM field and, as one can expect based on the Third Newton’s Law, “every action has an equal and opposite reaction”, this force it is exactly the negative sum of all the Lorentz forces for involved charges. In fact, based on general consideration of the equations for energy-momenta as in relations (11.2.19)-(11.2.23) we can view the equations (11.5.29)-(11.5.30) with involved Lorentz forces as a continuum version of classical equations of motion. An important difference though of the equation (11.5.29)-(11.5.30) unlike the equations of motion for point particles do not by themselves determine the dynamics of all involved fields, and, in fact, they hold only under an assumption that the field equations (11.5.12)-(11.5.13) are satisfied.

To verify the identities (11.5.30) we, following to Pauli RFTh, Part I, Section 2, introduce a useful computational tool for dealing with the covariant differentiation operators $\tilde{\partial}_\mu$ and $\tilde{\partial}^{\mu*}$ as defined in (11.5.6). Namely, let us consider a function $f(\psi, \psi;_\mu, \psi^*;_\mu)$, where

$$\psi;_\mu = \tilde{\partial}_\mu \psi, \quad \psi^*;_\mu = \tilde{\partial}^{\mu*} \psi,$$

(11.5.31)

$$\tilde{\partial}_\mu = \partial_\mu + \frac{i\gamma}{\lambda c} A^\mu, \quad \tilde{\partial}^{\mu*} = \partial^{\mu*} - \frac{i\gamma}{\lambda c} A^\mu,$$

which is invariant with respect to the gauge transformations of the first kind (global) as in (11.5.7):

$$\psi \rightarrow e^{i\gamma} \psi, \quad \psi^* \rightarrow e^{-i\gamma} \psi^*, \text{ where } \gamma \text{ is any real constant.}$$  \hfill (11.5.32)

The invariance of $f$ readily implies the following identity

$$\frac{d}{d\gamma} \left( f(\psi, \psi;_\mu, \psi^*;_\mu) \right) \bigg|_{\gamma=0} = 0,$$

where $f(\psi, \psi;_\mu, \psi^*;_\mu)$ is a function of $\psi, \psi;_\mu, \psi^*;_\mu$. This identity is satisfied if

$$\frac{d}{d\gamma} f(\psi, \psi;_\mu, \psi^*;_\mu) = 0.$$  \hfill (11.5.33)
Observe also that from the definition (11.5.6) of the covariant differentiation operators $\bar{\partial}^{\mu}$ and $\bar{\partial}^{\mu\nu}$ we have

$$\bar{\partial}^{\mu} \bar{\partial}^{\nu} - \bar{\partial}^{\nu} \bar{\partial}^{\mu} = \frac{iq}{\lambda c} (\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}) = \frac{iq}{\lambda c} F^{\mu\nu}$$

(11.5.34)

$$\bar{\partial}^{\mu\nu} \bar{\partial}^{\nu\mu} - \bar{\partial}^{\nu\mu} \bar{\partial}^{\mu\nu} = - \frac{iq}{\lambda c} (\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}) = - \frac{iq}{\lambda c} F^{\mu\nu}$$

Now for a gauge invariant $f$ we have

$$\partial^{\nu} f = \frac{\partial f}{\partial \psi} \bar{\partial}^{\nu} \psi + \frac{\partial f}{\partial \psi^{*\mu}} \bar{\partial}^{\nu} \bar{\partial}^{\mu} \psi + \frac{\partial f}{\partial \psi^{*\nu}} \bar{\partial}^{\nu} \bar{\partial}^{\nu} \psi^{*} + \frac{\partial f}{\partial \psi^{*\nu}} \bar{\partial}^{\nu} \bar{\partial}^{\nu} \psi^{*} = \frac{iq}{\lambda c} A^{\nu}$$

(11.5.35)

$$\frac{\partial f}{\partial \psi} \bar{\partial}^{\nu} \psi + \frac{\partial f}{\partial \psi^{*\mu}} \bar{\partial}^{\nu} \bar{\partial}^{\mu} \psi + \frac{\partial f}{\partial \psi^{*\nu}} \bar{\partial}^{\nu} \bar{\partial}^{\nu} \psi^{*} + \frac{\partial f}{\partial \psi^{*\nu}} \bar{\partial}^{\nu} \bar{\partial}^{\nu} \psi^{*}$$

(11.5.35)

which together with (11.5.33) implies the following identity

$$\partial^{\nu} (fg^{*}) = \left( \partial^{\nu} f \right) g^{*} + f \left( \partial^{\nu} g^{*} \right).$$

(11.5.37)

With an argument similar to the above on can verify that if $f$ and $g^{*}$ are functions of $\psi, \psi^{*}, \bar{\partial}^{\mu} \psi, \bar{\partial}^{\mu\nu} \psi^{*}$ which transform under the gauge transformations (11.5.32) as $e^{i\gamma} f$ and $e^{-i\gamma} g^{*}$ then the following identity holds

$$\partial^{\nu} (fg^{*}) = \left( \partial^{\nu} f \right) g^{*} + f \left( \partial^{\nu} g^{*} \right).$$

(11.5.37)

One can verify that the function $T^{\mu\nu}$ defined by (11.5.25) is an expression for which the identities (11.5.36) and (11.5.37) can be applied. Now applying these identities to $\partial_{\mu} T^{\mu\nu}$ and using the field equations (11.5.12) together with identities (11.5.34) and the representation (11.5.17) for the current $J^{\nu}_{\mu}$ we obtain

$$\partial_{\mu} T^{\mu\nu} = \partial_{\mu} \left( \frac{\partial L}{\partial \psi^{*\mu}} \bar{\partial}^{\nu} \psi + \frac{\partial L}{\partial \psi^{*\nu}} \bar{\partial}^{\nu} \psi^{*} \right) - \partial^{\nu} L^{\mu} = \frac{iq}{\lambda c}$$

(11.5.38)

$$= \left[ \bar{\partial}^{\mu} \left( \frac{\partial L}{\partial \psi^{*\mu}} \right) \right] \bar{\partial}^{\nu} \psi^{*} + \frac{\partial L}{\partial \psi^{*\nu}} \bar{\partial}^{\nu} \psi^{*} + \frac{\partial L}{\partial \psi^{*\nu}} \bar{\partial}^{\nu} \psi^{*} -$$

$$- \left( \frac{\partial L}{\partial \psi^{*\mu}} \bar{\partial}^{\nu} \psi^{*} + \frac{\partial L}{\partial \psi^{*\nu}} \bar{\partial}^{\nu} \psi^{*} + \frac{\partial L}{\partial \psi^{*\nu}} \bar{\partial}^{\nu} \psi^{*} \right)$$

which is the desired equation (11.5.30).

Observe that the equation (11.5.29)-(11.5.30) for the energy-momentum are evidently consistent with the total energy conservation (11.5.28).
11.5.3 Gauge invariant and partially symmetric energy-momentum tensors

In our studies, in particular of non-relativistic approximations, we have Lagrangians which are gauge invariant and invariant with respect to space and time translations but they might not be invariant with respect to the entire Lorentz group of transformations. This subsection is devoted to this kind of Lagrangians with the main point that essentially all important results of the subsections 11.5.1 and 11.5.2 apply to them with the only difference that the energy-momentum tensor is not fully symmetric but commonly its space part, the stress tensor, is symmetric.

As in the previous subsection we assume the Lagrangian to be of the form described by formulas (11.5.5)-(11.5.6) and assume it to be gauge invariant with respect to transformations if the first and the second type and invariant with respect space and time translations. A careful analysis of the arguments in subsections 11.5.1 and 11.5.2 which produced the expressions (11.5.24) and (11.5.25) for respectively energy-momentum $\Theta^{\mu\nu}$ of the EM field and energy-momenta $T^{\ell\mu\nu}$ of charges show the same expressions hold for gauge and translation invariant Lagrangian even if it is not invariant with respect to the entire Lorentz group of transformation.

We notice first the field equations and expressions for conserved currents are provided by (11.5.12), (11.5.13), (11.5.14), (11.5.15), (11.5.17), namely

$$\frac{\partial L^\ell}{\partial \dot{\psi}^\ell} - \tilde{\partial}^\mu \left[ \frac{\partial L^\ell}{\partial \dot{\psi}^\ell_\mu} \right] = 0, \quad \frac{\partial L^\ell}{\partial \dot{\psi}^\ell_\mu} - \tilde{\partial}^\mu \left[ \frac{\partial L^\ell}{\partial \dot{\psi}^\ell_\mu} \right] = 0, \quad (11.5.39)$$

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad J^\nu = \sum_\ell J^{\ell\nu}, \quad (11.5.40)$$

where $J^{\ell\nu}$ is the four-vector current related to the $\ell$-th charge is defined by

$$J^{\ell\nu} = -i \frac{q^\ell}{\chi} \left( \frac{\partial L^\ell}{\partial \dot{\psi}_0^\ell} \psi^\ell - \frac{\partial L^\ell}{\partial \dot{\psi}^\ell_0} \dot{\psi}^\ell_0 \right) = -c \frac{\partial L^\ell}{\partial A^\nu}, \quad (11.5.41)$$

or, since $J^\nu = (c\rho, J)$,

$$\rho^\ell = -i \frac{q^\ell}{\chi} \left( \frac{\partial L^\ell}{\partial \dot{\psi}_0^\ell} \psi^\ell - \frac{\partial L^\ell}{\partial \dot{\psi}^\ell_0} \dot{\psi}^\ell_0 \right), \quad (11.5.42)$$

$$J^\ell_j = -i \frac{q^\ell}{\chi} \left( \frac{\partial L^\ell}{\partial \dot{\psi}_j^\ell} \psi^\ell - \frac{\partial L^\ell}{\partial \dot{\psi}^\ell_j} \dot{\psi}^\ell_j \right), \quad j = 1, 2, 3. \quad (11.5.43)$$

Then we assign to the energy-momenta of the EM field $\Theta^{\mu\nu}$ and the $\ell$-th charge $T^{\ell\mu\nu}$ respectively expressions (11.5.24) and (11.5.25), namely

$$\Theta^{\mu\nu} = \frac{1}{4\pi} \left( g^{\mu\gamma} F_{\gamma\xi} F_{\xi\nu} + \frac{1}{4} g^{\mu\nu} F_{\gamma\xi} F^{\gamma\xi} \right), \quad (11.5.43)$$

$$T^{\ell\mu\nu} = \frac{\partial L^\ell}{\partial \dot{\psi}_0^\ell} \psi^{\ell\mu\nu} + \frac{\partial L^\ell}{\partial \dot{\psi}^\ell_0} \dot{\psi}^{\ell\nu}{\psi}^{\ell\mu} - g^{\mu\nu} L^\ell, \quad (11.5.44)$$

The above expressions for the energy-momenta are manifestly gauge invariant.
Looking at the arguments in subsections 11.5.1 and 11.5.2 we compare the above expressions of the energy-momenta of the EM field $\Theta^{\mu\nu}$ and the $\ell$-th charge $T^{\ell\mu\nu}$ with their canonical expression and observe that

$$T^{\ell\mu\nu} = \hat{T}^{\ell\mu\nu} - \frac{1}{c} J^{\ell\mu} A^{\nu}, \quad \Theta^{\mu\nu} = \tilde{\Theta}^{\mu\nu} + \frac{1}{4\pi} F^{\mu\gamma} \partial_\gamma A^{\nu}. \quad (11.5.45)$$

It remains to verify that the difference between the total energy-momentum and its canonical value is a 4-divergence. Indeed it follows from \((11.5.45)\) that

$$\sum_{\ell} \left( T^{\ell\mu\nu} - \hat{T}^{\ell\mu\nu} \right) + \left( \Theta^{\mu\nu} - \tilde{\Theta}^{\mu\nu} \right) =$$

$$= \frac{1}{4\pi} F^{\mu\gamma} \partial_\gamma A^{\nu} - \sum_{\ell} \frac{1}{c} J^{\ell\mu} A^{\nu} =$$

$$= \frac{1}{4\pi} F^{\mu\gamma} \partial_\gamma A^{\nu} - \frac{1}{c} J^\mu A^\nu = -\partial_\gamma \frac{1}{4\pi} f^{\mu\gamma\nu},$$

where $f^{\mu\gamma\nu} = -\frac{1}{4\pi} F^{\mu\gamma} A^{\nu}$.

Using the arguments of the subsections 11.5.2 we also find that the relations \((11.5.29)\) and \((11.5.30)\) hold here, namely

$$\partial_\mu \Theta^{\mu\nu} = -\frac{1}{c} J_\mu F^{\nu\mu}, \text{ where } J_\mu = \sum_{\ell} J^{\ell\mu}, \quad (11.5.47)$$

$$\partial_\mu T^{\ell\mu\nu} = \frac{1}{c} J^{\ell\mu} F^{\nu\mu}, \quad (11.5.48)$$

where once again we recognize in the right-hand sides of equalities \((11.5.47)-(11.5.48)\) the relevant Lorentz force densities. Consequently, as expected the energy conservation law for the total system takes the familiar form

$$\partial_\mu T^{\mu\nu} = \sum_{\ell} \partial_\mu T^{\ell\mu\nu} + \partial_\mu \Theta^{\mu\nu} = 0. \quad (11.5.49)$$

The equations \((11.5.47)-(11.5.49)\) reconfirm that our assignment \((11.5.43)-(11.5.44)\) of energy-momenta to the EM field and individual charges is physically sound.

We would like to notice now that even if a Lagrangian of the form \((11.5.5)-(11.5.6)\) is not invariant with respect the entire Lorentz group of transformations it often satisfies a reduced version of the symmetry condition \((11.5.9)\) which holds for the space indices only, namely

$$\frac{\partial L^\ell}{\partial \psi_{i j}^\ell} \psi_{i j}^\ell + \frac{\partial L^\ell}{\partial \psi_{i j}^{\ell*}} \psi_{i j}^{\ell*} = \frac{\partial L^\ell}{\partial \psi_{i j}^{\ell*}} \psi_{i j}^{\ell*} + \frac{\partial L^\ell}{\partial \psi_{i j}^\ell} \psi_{i j}^\ell, \quad i, j = 1, 2, 3. \quad (11.5.50)$$

Under the reduced symmetry condition \((11.5.50)\) the space part of the energy-momenta $T^{\ell\mu\nu}$, known as the stress tensor, is symmetric, namely

$$T^{\ell ij} = T^{\ell ji}, \quad i, j = 1, 2, 3. \quad (11.5.51)$$

We remind that the symmetry of the stress tensor is a very important property equivalent to the space angular momentum conservation, see Section 11.2 and, for instance, [Moller, Section 6.1, 6.2].
11.6 A single free charge

A single free charge interacting with the EM field is evidently a particular case of considered above system of many charges in Section 11.5 and the Lagrangian (11.5.5) takes the form

\[ \mathcal{L}_0 = L_0 (\psi, \psi^*, \psi^*; \psi^*; \mu) - \frac{F_{\mu\nu} F^{\mu\nu}}{16\pi}, \quad F_{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \] (11.6.1)

where \( \psi^\mu \) and \( \psi^{*\mu} \) are the covariant derivatives with respect to the covariant differential operators \( \tilde{\partial}^\mu \) and \( \tilde{\partial}^{*\mu} \) defined by

\[ \psi^{\mu} = \tilde{\partial}^\mu \psi, \quad \psi^{*\mu} = \tilde{\partial}^{*\mu} \psi^*, \quad \tilde{\partial}^\mu = \partial^\mu + \frac{iq}{\chi c} A^\mu, \quad \tilde{\partial}^{*\mu} = \partial^{*\mu} - \frac{iq}{\chi c} A^\mu. \] (11.6.2)

The Lagrangian is assumed to be Lorentz and gauge invariant with respect to the gauge transformations of the first and the second type (11.5.7)-(11.5.8). The field equations are

\[ \frac{\partial L_0}{\partial \psi} - \tilde{\partial}^\nu \left[ \frac{\partial L_0}{\partial \psi^{*\nu}} \right] = 0, \quad \frac{\partial L_0}{\partial \psi^*} - \tilde{\partial}^\nu \left[ \frac{\partial L_0}{\partial \psi^{*\nu}} \right] = 0, \] (11.6.3)

\[ \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \] (11.6.4)

where \( J^\mu \) is the four-vector micro-current related to the charge is defined by

\[ J^\mu = -\frac{i}{\chi} \left( \frac{\partial L_0}{\partial \psi^\mu} \psi - \frac{\partial L_0}{\partial \psi^{*\mu}} \psi^* \right) = -c \frac{\partial L_0}{\partial A^\nu}. \] (11.6.5)

or, since \( J^\nu = (c \rho, J) \),

\[ \rho = -\frac{i}{\chi} \left( \frac{\partial L_0}{\partial \psi^\mu} \psi - \frac{\partial L_0}{\partial \psi^{*\mu}} \psi^* \right), \] (11.6.6)

\[ J_j = -\frac{i}{\chi} \left( \frac{\partial L_0}{\partial \psi^\mu} \psi - \frac{\partial L_0}{\partial \psi^{*\mu}} \psi^* \right), \quad j = 1, 2, 3, \]

which satisfy the charge conservation/continuity equations

\[ \partial_\nu J^\nu = 0, \quad \partial_t \rho + \nabla \cdot J = 0, \quad J^\nu = (c \rho, J). \] (11.6.7)

The energy-momentum of the charge and the EM field according to the formulas (11.5.24)-(11.5.25) are respectively as follows

\[ T^{\mu\nu} = \frac{\partial L_0}{\partial \psi^\mu} \psi^{*\nu} + \frac{\partial L_0}{\partial \psi^{*\mu}} \psi^{*\nu} - g^{\mu\nu} L_0, \] (11.6.8)

\[ \Theta^{\mu\nu} = \frac{1}{4\pi} \left( g^{\mu\gamma} F_{\gamma\xi} F^{\xi\nu} + \frac{1}{4} g^{\mu\nu} F_{\gamma\xi} F^{\gamma\xi} \right), \] (11.6.9)

and the energy conservation equations (11.5.29)-(11.5.30) turn here into

\[ \partial_\mu T^{\mu\nu} = \frac{1}{c} J^{\mu\nu}, \quad \partial_\mu \Theta^{\mu\nu} = -\frac{1}{c} J^{\mu\nu}. \] (11.6.10)
11.7 A single charge in an external electromagnetic field

Here we consider a single dressed charge in an external EM field. The very presence of external forces turns the dressed charge into an open system and that brings up subtleties in the set up of gauge invariant expressions for the energy-momentum tensor. One can find signs of those subtleties already in a simple case of a point charge in an external EM. Indeed for the point charge model the canonical momentum and force are not gauge invariant as discussed briefly in Section 11.1. The principle source of the problems lies in the openness of the system with consequent uncertainty of the energy and the momentum as system changes under action of external forces. That can be seen, in particular, based on the relativity grounds, [Moller, Section 7.1, 7.2], when seemingly well defined 4-momenta for a number of open systems do not transform as 4-vectors, that, in general, can be taken as a proof of openness of a system.

Coming back to our case we want, first of all, to define a Lagrangian for a dressed charge in EM field based on (i) our studies in Section 11.5 of a closed system of many dressed charges and (ii) the Lagrangian of a singe free charge considered in Section 11.6. We do that by altering the EM potential $A^{\mu}$ in the expressions (11.2.1)-(11.2.3) for the Lagrangian of the free single dressed charge with $\bar{A}^{\mu} = A^{\mu} + A_{\text{ex}}^{\mu}$, where $A_{\text{ex}}^{\mu}$ is the 4-potential of an the external EM field. Namely, we set

$$L_0 = L_0(\psi, \psi^*; \partial^\mu, \psi^*; \partial^\mu) - \frac{F_{\mu\nu} F^{\mu\nu}}{16\pi}, \quad F_{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu},$$  \hspace{1cm} (11.7.1)

where $\psi_{\mu}$ and $\psi^*_{\mu}$ are the covariant derivatives with respect to the covariant differential operators $\tilde{\partial}^{\mu}$ and $\tilde{\partial}^{\mu*}$ defined by

$$\psi_{\mu} = \tilde{\partial}^{\mu} \psi, \quad \psi^*_{\mu} = \tilde{\partial}^{\mu*} \psi^*,$$  \hspace{1cm} (11.7.2)

$$\tilde{\partial}^{\mu} = \partial^{\mu} + \frac{iq}{\lambda c} \bar{A}^{\mu}, \quad \tilde{\partial}^{\mu*} = \partial^{\mu} - \frac{iq}{\lambda c} \bar{A}^{\mu}, \quad \bar{A}^{\mu} = A^{\mu} + A_{\text{ex}}^{\mu}.$$  \hspace{1cm} (11.7.3)

To justify the expressions (11.7.1)-(11.7.2) for the Lagrangian let us look at a closed system of many charges studied in Section 11.5. We find there, in particular, that every individual charge with an index $\ell$ has a conserved current $J_{\ell\nu}^{\mu}$ and the total current is $J_{\mu} = \sum_{\ell} J_{\ell\nu}^{\mu}$. Hence, based on the linearity of Maxwell equation (11.5.13) we can introduce individual EM potentials $A_{\ell\mu}$ and the corresponding EM fields $F_{\ell\mu\nu}$ as the causal solutions to the following Maxwell equations

$$\partial_{\mu} F_{\ell\mu\nu} = \frac{4\pi}{c} J_{\ell\nu}^{\mu},$$  \hspace{1cm} (11.7.4)

implying

$$A^{\mu} = \sum_{\ell} A_{\ell\mu}, \quad F_{\mu\nu} = \sum_{\ell} F_{\ell\mu\nu}.$$  \hspace{1cm} (11.7.5)

Notice that every individual charge satisfies its field equation (11.5.12) with the EM field entering it via the potential $A^{\mu}$ in the covariant derivatives (11.5.31), and we can always represent it as

$$A^{\mu} = A_{\ell\mu} + A_{\text{ex}}^{\mu}, \quad A_{\text{ex}}^{\mu} = \sum_{\ell' \neq \ell} A_{\ell'\mu}.$$  \hspace{1cm} (11.7.6)

This representation indicates that we can account for the interaction of the $\ell$-th charge with remaining charges via an external field as in (11.7.5) justifying the expressions (11.7.1)-(11.7.2) for the Lagrangian $L_0$. 

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The Lagrangian \((11.7.1)-(11.7.2)\) is assumed to be invariant with respect to the first and
the second type gauge transformations \((11.5.7), (11.5.8)\), which in this case take the form
\[
\psi \rightarrow e^{i\gamma} \psi^\ell, \quad \psi^* \rightarrow e^{-i\gamma} \psi^*, \quad \text{where } \gamma \text{ is any real constant,}
\]
\[
\psi \rightarrow e^{-ie^{i\lambda(x)}} \psi, \quad A^\mu \rightarrow A^\mu + \partial^\mu \lambda.
\]
Similarly to the case of many charges we also assume the charge Lagrangian \(L_0\) to satisfy the
following symmetry condition
\[
\frac{\partial L_0}{\partial \psi^\nu} \psi^\prime_{\mu} + \frac{\partial L_0}{\partial \psi^\prime_{\mu}} \psi^\nu = \frac{\partial L_0}{\partial \psi^*_{\mu}} \psi^\nu + \frac{\partial L_0}{\partial \psi^\nu} \psi^*_{\mu}.
\]
(11.7.8)

As in already considered case of many charges there is a simple sufficient condition for the
symmetry condition \((11.7.8)\) to hold. It is when the Lagrangians \(L\) depends on the field
covariant derivatives only through the combination \(\psi^\ell_{\mu}, \psi^\prime_{\mu}\), in other words if there exist such
functions \(K(\psi, \psi^*, b)\) that
\[
L_0(\psi, \psi^\ell_{\mu}, \psi^*_{\mu}) = K(\psi, \psi^*, \psi^\mu_{\mu}).
\]
(11.7.9)

The field equations for the Lagrangian \(L_0\) defined by \((11.7.1)-(11.7.2)\) are
\[
\frac{\partial L_0}{\partial \psi} - \tilde{\partial}^\mu \left[ \frac{\partial L_0}{\partial \psi^*_{\mu}} \psi^\nu - \frac{\partial L_0}{\partial \psi^*_{\mu}} \psi^\nu \right] = 0,
\]
(11.7.10)

\[
\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,
\]
(11.7.11)

where \(J^\mu\) is the four-vector current related to the charge is defined by manifestly gauge
invariant expression
\[
J^\nu = -\frac{i}{\chi} \left( \frac{\partial L_0}{\partial \psi_{\nu}} \psi - \frac{\partial L_0}{\partial \psi^*_{\nu}} \psi^* \right),
\]
(11.7.12)
or, since \(J^\nu = (c\rho, J)\), the
\[
\rho = -\frac{i}{\chi} \left( \frac{\partial L_0}{\partial \psi_{0}} \psi - \frac{\partial L_0}{\partial \psi^*_{0}} \psi^* \right),
\]
(11.7.13)

\[
J_j = -\frac{i}{\chi} \left( \frac{\partial L_0}{\partial \psi_{j}} \psi - \frac{\partial L_0}{\partial \psi^*_{j}} \psi^* \right), \quad j = 1, 2, 3,
\]
which satisfy the charge conservation/continuity equations
\[
\partial_\nu J^\nu = 0, \quad \text{or } \partial_t \rho + \nabla \cdot J = 0, \quad J^\nu = (c\rho, J).
\]
(11.7.14)

Notice as in the case of many charges the relations \((11.7.1)-(11.7.2)\) imply the following
alternative representation for the four-current \(J^\nu\) defined by \((11.7.12)\)
\[
J^\nu = -\frac{i}{\chi} \left( \frac{\partial L_0}{\partial \psi_{\nu}} \psi - \frac{\partial L_0}{\partial \psi^*_{\nu}} \psi^* \right) = -c \frac{\partial L_0}{\partial A_\nu} = -c \frac{\partial L_0}{\partial A^\nu} = -c \frac{\partial L_0}{\partial A^{ex\nu}}.
\]
(11.7.15)
Now we would like to construct gauge invariant energy-momentum tensors for the single charge and its EM field. For that we start with their canonical expressions (11.5.4)-(11.4.18) obtained via Noether’s theorem

\[ \dot{T}^{\mu\nu} = \frac{\partial L_0}{\partial \psi_{\mu}} \psi^{;\nu} + \frac{\partial L_0}{\partial \psi^{*\mu}} \psi^{;\nu*} - g^{\mu\nu} L_0, \]  
(11.7.16)

\[ \dot{\Theta}^{\mu\nu} = -\frac{F_{\mu\gamma} \partial^\nu A_\gamma}{4\pi} + g^{\mu\nu} \frac{F_{\xi\gamma} F_{\xi\nu}}{16\pi}. \]  
(11.7.17)

The conservation law for the total energy-momentum tensor \( \dot{T}^{\mu\nu} + \dot{\Theta}^{\mu\nu} \) in view of the general conservation law (11.2.6) and the current representation (11.7.15) take the form

\[ \partial_\mu (\dot{T}^{\mu\nu} + \dot{\Theta}^{\mu\nu}) = -\frac{\partial L_0}{\partial x_\nu} = -\frac{\partial L_0}{\partial A_{\text{ex}\mu}} \partial^\nu A_{\text{ex}\mu} = \frac{1}{c} J^{\mu \text{ex}} \partial^\nu A_{\text{ex}\mu}. \]  
(11.7.18)

Observe now that the both canonical expressions (11.7.16), (11.7.17) as well as the density of the generalized force \( \frac{1}{c} J^\mu \partial^\nu A_{\text{ex}\mu} \) in (11.7.18) are evidently not gauge invariant, and this is very similar to what we already observed for the point charge model in Section 11.1. We recall that there is a general way to alter the canonical energy-momentum tensor as in the relation (11.2.8), namely

\[ T^{\mu\nu} + \Theta^{\mu\nu} = \dot{T}^{\mu\nu} + \dot{\Theta}^{\mu\nu} - \partial_\gamma f_{\mu\gamma\nu}, \quad f_{\mu\gamma\nu} = -f^{\gamma\mu\nu}. \]  
(11.7.19)

But any such alteration alone can not be satisfactory since by its very construction it would keep unchanged the not gauge invariant density of generalized force \( \frac{1}{c} J^\mu \partial^\nu A_{\text{ex}\mu} \) in the right-hand side of (11.7.18). Therefore, a more profound alteration of the energy-momenta is required that would change the expression for the force density in the right-hand side of (11.7.18) so that it becomes gauge invariant. More than that we expect it to produce exactly the density of the Lorentz force associated with the external EM potential \( A^\mu_{\text{ex}} \).

The results of Section 11.5.1 suggest a satisfactory choice for gauge invariant energy-momentum tensors and it is as in formulas (11.5.19)-(11.5.20), namely we set

\[ T^{\mu\nu} = \frac{\partial L_0}{\partial \psi_{\mu}} \psi^{;\nu} + \frac{\partial L_0}{\partial \psi^{*\mu}} \psi^{;\nu*} - g^{\mu\nu} L_0, \]  
(11.7.20)

\[ \Theta^{\mu\nu} = \frac{1}{4\pi} \left( g^{\mu\gamma} F_{\gamma\xi} F_{\xi\nu} + \frac{1}{4} g^{\mu\nu} F_{\gamma\xi} F^{\gamma\xi} \right). \]  
(11.7.21)

Observe that the EM energy-momentum \( \Theta^{\mu\nu} \) is also manifestly symmetric and the charge energy-momentum \( T^{\mu\nu} \) is symmetric if the the symmetry condition (11.7.8) is satisfied. The energy-momenta conservation laws here take the form

\[ \partial_\mu \Theta^{\mu\nu} = \partial_\mu \frac{1}{4\pi} \left( g^{\mu\gamma} F_{\gamma\xi} F_{\xi\nu} + \frac{1}{4} g^{\mu\nu} F_{\gamma\xi} F^{\gamma\xi} \right) = -\frac{1}{c} J_\mu F^{\mu\nu}, \]  
(11.7.22)

\[ \partial_\mu T^{\mu\nu} = \frac{1}{c} J_\mu \bar{F}^{\mu\nu}, \quad \bar{F}^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu. \]  
(11.7.23)

Indeed as in the case of derivation of the similar formula (11.5.30) we observe that the \( T^{\mu\nu} \) defined by (11.7.20) has an expression for which the identities (11.5.36) and (11.5.37) can be applied. Now we literally repeat the calculation (11.5.38). Namely applying the mentioned
identities to $\partial_\mu T^{\mu\nu}$ and using the field equations (11.7.10) together with identities (11.5.34) and the representation (11.5.17) for the current $J_\nu$ we obtain

$$
\partial_\mu T^{\mu\nu} = \partial_\mu \left( \frac{\partial L_0}{\partial \psi_{\mu}^*} \tilde{\psi}^* + \frac{\partial L_0}{\partial \psi_{\mu}^*} \tilde{\psi}^* \right) - \partial_\nu L_0 = \left( \frac{\partial L_0}{\partial \psi} \tilde{\psi}^* + \frac{\partial L_0}{\partial \psi} \tilde{\psi}^* \right)\partial_\nu L_0 - \frac{1}{c} J_\mu F^{\mu\nu} = \frac{1}{c} J_\mu \tilde{F}^{\mu\nu},
$$

(11.7.24)

implying the desired relation (11.7.23).

Now adding up the equalities (11.7.22) and (11.7.23) we get the conservation law for the total energy-momentum $T^{\mu\nu} = T^{\mu\nu} + \Theta^{\mu\nu}$, i.e.

$$
\partial_\mu T^{\mu\nu} = \partial_\mu \left( T_0^{\mu\nu} + \Theta^{\mu\nu} \right) = \frac{1}{c} J_\mu F_{\text{ex}}^{\mu\nu}, \quad F_{\text{ex}}^{\mu\nu} = \partial_\mu A_\nu^{\text{ex}} - \partial_\nu A_\mu^{\text{ex}}.
$$

(11.7.25)

Notice as we expected we have the density of the Lorentz force $\frac{1}{c} J_\mu F_{\text{ex}}^{\mu\nu}$ in the right-hand side of the conservation laws (11.7.25) and (11.7.23).

### 11.8 Energy partition for static and time harmonic fields

Let us consider the Lagrangian of the form

$$
\mathcal{L} = \mathcal{L} \left( \{ \psi^\ell, \psi_{\mu}^\ell, \psi_{\mu}^\ast, \psi_{\mu}^{\text{ex}}, V^g, V_{\mu}^g \} \right),
$$

(11.8.1)

where $\{ V^g \}$ are real-valued quantities. The corresponding Euler-Lagrange field equations are

$$
\begin{align*}
\frac{\partial \mathcal{L}}{\partial \psi^\ell} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \psi_{\mu}^\ell} \right) &= 0, \\
\frac{\partial \mathcal{L}}{\partial \psi_{\mu}^\ast} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \psi_{\mu}^{\text{ex}}} \right) &= 0, \\
\frac{\partial \mathcal{L}}{\partial V^g} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial V_{\mu}^g} \right) &= 0.
\end{align*}
$$

(11.8.2)

Static regime is characterized as one when the fields $\{ \psi^\ell \}$, $\{ \psi_{\mu}^\ast \}$ and $\{ V^g \}$ are time independent and hence depend only on the space variable, and we will use the following abbreviated notation for it

$$
\text{stat} \equiv \partial_t \psi^\ell = 0, \quad \partial_t \psi_{\mu}^\ast = 0, \quad \partial_t V^g = 0.
$$

(11.8.3)

Then using the canonical energy-momentum $\tilde{T}^{\mu\nu}$ as defined by (11.5.30) we readily obtain the following formulas for the total energy $\mathcal{E}_{\text{stat}}$ of static field

$$
\mathcal{E}_{\text{stat}} = \int_{\mathbb{R}^3} \mathcal{U} \, dx, \quad \text{where} \quad \mathcal{U} = \tilde{T}^{00} \left( \{ \psi^\ell, \psi_{\mu}^\ell, \psi_{\mu}^\ast, \psi_{\mu}^{\text{ex}}, V^g, V_{\mu}^g \} \right)_{\text{stat}}.
$$

(11.8.4)
The energy $\mathcal{E}_{\text{stat}}$ in a static regime can be identified with a potential energy. Notice that in view of the formula (11.5.3) for the canonical energy-momentum $\mathcal{T}^{\mu\nu}$ the corresponding energy density $\mathcal{T}^{00}_{\text{stat}}$ is represented by

$$\dot{\mathcal{U}} = \mathcal{T}^{00}_{\text{stat}} = -\mathcal{L} \left( \{\psi^\ell, \psi^{\ell*}, V^g\} \right)_{\text{stat}}.$$  (11.8.5)

Consequently, we have the following representation for the potential energy $\mathcal{E}_{\text{stat}}$

$$\mathcal{E}_{\text{stat}} = \int_{\mathbb{R}^3} -\mathcal{L} \left( \{\psi^\ell, \psi^{\ell*}, V^g\} \right)_{\text{stat}} \, dx.$$  (11.8.6)

which we use to establish the following variational principle. Based (11.8.5) we can conclude that a static solution $\{\psi^\ell, \psi^{\ell*}, V^g\}$ to the Euler-Lagrange field equations (11.8.2) evidently transforms into a solution to the equation

$$\sum_{j=1,2,3} \partial_j \left( \frac{\partial \mathcal{U}}{\partial \psi_j} \right) = 0, \quad \sum_{j=1,2,3} \partial_j \left( \frac{\partial \mathcal{U}}{\partial \psi_j^*} \right) = 0,$$

$$\sum_{j=1,2,3} \partial_j \left( \frac{\partial \mathcal{U}}{\partial V^g} \right) = 0,$$

and, hence, in view of the representation (11.8.6) it is a stationary point of the static energy functional $\mathcal{E}_{\text{stat}}$ in the complete agreement with the principle of virtual work for the state of equilibrium, [Lanczos VPM, Section III.1], [Sommerfeld M, Section II.8].

Let us expand now the potential energy density $\mathcal{U}$ defined by (11.8.5) into the series with respect to the derivatives $\nabla \psi^\ell, \nabla \psi^{\ell*}, \nabla V^g$, namely

$$\mathcal{U} = \sum_{n=0}^\infty \mathcal{U}^{(n)}, \quad \text{where}$$

$$\mathcal{U}^{(n)} = \sum_{\sum n_\ell + n_{\ell*} + n_g = n} \mathcal{U}_{\{n_\ell, n_{\ell*}, n_g\}} \left( \{\psi^\ell, \psi^{\ell*}, V^g\} \right) \prod_{\ell,g} \partial_{n_\ell} \psi^\ell \partial_{n_{\ell*}} \psi^{\ell*} \partial_{n_g} V^g.$$

This expansion via the representation (11.8.6) for the potential energy $\mathcal{E}_{\text{stat}}$ readily implies the the corresponding expansion for $\mathcal{E}_{\text{stat}}$:

$$\mathcal{E}_{\text{stat}} = \sum_{n=0}^\infty \mathcal{E}_{\text{stat}}^{(n)}, \quad \text{where} \quad \mathcal{E}_{\text{stat}}^{(n)} = \int_{\mathbb{R}^3} \mathcal{U}^{(n)} \, dx.$$  (11.8.9)

Now being given a static solution $\{\psi^\ell, \psi^{\ell*}, V^g\}$ we use its established above property to be a stationary point of the functional $\mathcal{E}_{\text{stat}}$ as defined by formula (11.8.6) and (11.8.9). Namely, we introduce the following family of fields

$$\psi^\ell_\xi(x) = \psi^\ell(\xi x), \quad \psi^{\ell*}_\xi(x) = \psi^{\ell*}(\xi x),$$

$$V^g_\xi(x) = V^g(\xi x) \quad \text{where} \quad \xi \text{ is real},$$  (11.8.10)
and observe that since \( \{ \psi^\ell, \psi^{\ell^*}, V^g \} \) is a stationary point of the functional \( E_{\text{stat}} \) we have

\[
\frac{d}{d\xi} E_{\text{stat}} \left( \{ \psi^\ell, \psi^{\ell^*}, V^g, \nabla\psi^\ell, \nabla\psi^{\ell^*}, \nabla V^g \} \right) \bigg|_{\xi=1} = \sum_{n=0}^{\infty} \xi_{n-3} E_{\text{stat}}^{(n)} \bigg|_{\xi=1} = \sum_{n=0}^{\infty} (n-3) E_{\text{stat}}^{(n)} = 0.
\]

(11.8.11)

In other words, for a static solution its energy components \( E_{\text{stat}}^{(n)} \) always satisfy the identity

\[
\sum_{n=0}^{\infty} (n-3) E_{\text{stat}}^{(n)} = 0.
\]

(11.8.12)

Very often the density \( \hat{U} \) of the potential energy depends on the field derivatives so that

\[
\hat{U} \left( \{ \psi^\ell, \psi^{\ell^*}, V^g, \nabla\psi^\ell, \nabla\psi^{\ell^*}, \nabla V^g \} \right) = U^{(2)} \left( \{ \psi^\ell, \psi^{\ell^*}, V^g, \nabla\psi^\ell, \nabla\psi^{\ell^*}, \nabla V^g \} \right) + U^{(0)} \left( \{ \psi^\ell, \psi^{\ell^*}, V^g \} \right).
\]

(11.8.13)

where \( U^{(2)} \) satisfies the following identity for any real \( \theta \)

\[
U^{(2)} \left( \{ \psi^\ell, \psi^{\ell^*}, V^g, \theta\nabla\psi^\ell, \theta\nabla\psi^{\ell^*}, \theta\nabla V^g \} \right) = \theta^2 U^{(2)} \left( \{ \psi^\ell, \psi^{\ell^*}, V^g, \nabla\psi^\ell, \nabla\psi^{\ell^*}, \nabla V^g \} \right).
\]

(11.8.14)

In this case the identity (11.8.12) turns into the following important identity for the two constituting components \( E_{\text{stat}}^{(2)} \) and \( E_{\text{stat}}^{(0)} \) of the total potential energy \( E_{\text{stat}} \):

\[
E_{\text{stat}} = E_{\text{stat}}^{(2)} + E_{\text{stat}}^{(0)}, \quad E_{\text{stat}}^{(0)} = -\frac{1}{3} E_{\text{stat}}^{(2)} \text{ implying } E_{\text{stat}} = \frac{2}{3} E_{\text{stat}}^{(2)}.
\]

(11.8.15)

The significance of the above identity for our goals is that in the cases of interest the energy component \( E_{\text{stat}}^{(0)} \) accounts for the energy of nonlinear self-interactions and the formula \( E_{\text{stat}} = \frac{2}{3} E_{\text{stat}}^{(2)} \) shows the total energy has a representation that does not depend explicitly on the nonlinear self-interactions. This is one among other properties allowing us to characterize the introduced nonlinear self-interactions as stealthy.

The identity (11.8.15) for a single field is known as the Pokhozhaev-Derrick identity, [Pokhozhaev, Derrick](see also [Kapitanski] and [Coleman Section 2.4]). It was often used to prove the nonexistence of nonzero solutions to the corresponding field equations in situations when a priori the both energies \( E_{\text{stat}}^{(2)} \) and \( E_{\text{stat}}^{(0)} \) are nonnegative and vanish for the zero field. Indeed if the nonnegativity of the energy components is combined with the identity (11.8.15) the both energies \( E_{\text{stat}}^{(2)} \) and \( E_{\text{stat}}^{(0)} \) must vanish implying that the field must vanish as well.

### 11.8.1 Time-harmonic fields

The above statements for static fields can be generalized for the case when complex valued fields \( \psi^\ell \) are time harmonic, namely when

\[
\psi^\ell = e^{-i\omega t} \tilde{\psi}^\ell, \quad \psi^{\ell^*} = e^{i\omega t} \tilde{\psi}^{\ell^*},
\]

(11.8.16)
where $\tilde{\psi}^\ell$ and $\tilde{\psi}^{\ell*}$ are static, i.e. time independent. We provide such a generalization for the Lagrangian of the form (11.5.5)-(11.5.6) describing many charges coupled with the EM field $F^{\mu\nu}$, i.e.

$$
L \left( \{ \psi^\ell, \psi^\ell_\mu, \psi^{\ell*}, \psi^{\ell*}_\mu \}, A^\mu \right) = \sum_\ell L^\ell \left( \psi^\ell, \psi^\ell_\mu, \psi^{\ell*}, \psi^{\ell*}_\mu \right) - \frac{F^{\mu\nu}F_{\mu\nu}}{16\pi},
$$

(11.8.17)

with an additional assumption on the charge Lagrangians $L^\ell$ to be of the form

$$
L^\ell \left( \psi^\ell, \psi^\ell_\mu, \psi^{\ell*}, \psi^{\ell*}_\mu \right) = K^\ell \left( \psi^{\ell*}, \psi^{\ell*}_\mu, \psi^\ell, \psi^\ell_\mu \right),
$$

(11.8.18)

where $K^\ell (a, b)$ is a function of real variables $a$ and $b$. In the case of interest represented by the Lagrangian (11.8.17)-(11.8.18) we add to the assumption (11.8.16) an assumption that the EM field is static namely

$$
\partial_t \varphi = 0, \ A = 0.
$$

(11.8.19)

Treating the equalities (11.8.16) as variables change let us recast the charges Lagrangians in the new variable $\tilde{\psi}^\ell$ and $\tilde{\psi}^{\ell*}$. Notice first that

$$
\psi^{\ell*}_0 = \tilde{\psi}^{\ell*}_0, \quad \psi^{\ell*}_j = \tilde{\psi}^{\ell*}_j.
$$

(11.8.20)

Then using (11.1.1), (11.2.2) and (11.4.3) we obtain

$$
\psi^{\ell*}_0 = \delta^{\ell*}_\mu \psi^\ell_\mu = \mathrm{e}^{i\omega^\ell t} \left( \frac{\partial_\ell}{c} - \frac{i\omega^\ell}{c} \frac{iq^\ell \varphi}{\chi c} \right) \tilde{\psi}^{\ell*}_0 = \mathrm{e}^{i\omega^\ell t} \left( \frac{\partial_\ell + i\omega^\ell}{c} - \frac{iq^\ell \varphi}{\chi c} \right) \tilde{\psi}^{\ell*}_0, \quad (11.8.21)
$$

$$
\psi^{\ell*}_j = \delta^{\ell*}_j \psi^\ell_j = \mathrm{e}^{i\omega^\ell t} \left( \partial_j - \frac{iq^\ell A_j}{\chi c} \right) \tilde{\psi}^{\ell*}_j, \quad \psi^{\ell*}_j = \tilde{\psi}^{\ell*}_j = \mathrm{e}^{i\omega^\ell t} \left( \partial_j + \frac{iq^\ell A_j}{\chi c} \right) \tilde{\psi}^\ell. \quad (11.8.22)
$$

Observe that in the case when there is just a single charge the expressions (11.8.21) show that the time derivatives $\psi^{\ell*}_0$ and $\psi^{\ell*}_\mu$ are modified so as the potential $\varphi$ is added a constant, namely

$$
\varphi \to \varphi - \frac{\chi \omega}{\rho}.
$$

(11.8.23)

Substituting (11.8.20) and (11.8.21) into the Lagrangian $L^\ell$ we get the Lagrangian which denote $L^{\omega^\ell}$ as a function of the variables $\tilde{\psi}^\ell$ and $\tilde{\psi}^{\ell*}$ and we obtain

$$
L^{\omega^\ell} = K^\ell \left( \tilde{\psi}^\ell, \tilde{\psi}^{\ell*}, \psi^\ell_\mu, \psi^{\ell*}_\mu \right), \quad \text{where} \quad \psi^{\ell*}_\mu = \psi^{\ell*}_0 = \delta^{\ell*}_\mu \psi^\ell_\mu = \left( \frac{\partial_\ell + i\omega^\ell}{c} - \frac{iq^\ell \varphi}{\chi c} \right) \tilde{\psi}^{\ell*}_0
$$

(11.8.24)

$$
= \left( \frac{\partial_\ell + i\omega^\ell}{c} - \frac{iq^\ell \varphi}{\chi c} \right) \tilde{\psi}^{\ell*}_0 - \left( \nabla + \frac{iq^\ell A^\ell}{\chi c} \right) \tilde{\psi}^{\ell*}_0 \cdot \left( \nabla - i\frac{q^\ell A^\ell}{\chi c} \right) \tilde{\psi}^\ell.
$$

(11.8.23)

We can apply now to the Lagrangian (11.8.17), (11.8.24) as a function of the fields $\tilde{\psi}^\ell$, $\tilde{\psi}^{\ell*}$ and $A^\mu$ the obtained above results for the static fields taking into account also the assumption (11.8.19) for the EM field to static. In this case the static regime is characterized by the time
independence of the charge fields $\tilde{\psi}^\ell, \tilde{\psi}^{\ell*}$ and the assumption (11.8.19) on the potential $A^\mu$ and these conditions which are abbreviated by the symbol \[ \text{stat} \equiv \partial_t \tilde{\psi}^\ell = 0, \partial_{\ell^\mu} \tilde{\psi}^{\ell*} = 0, \partial_{\ell^\mu} \varphi = 0, A = 0, A^\mu = (\varphi, A). \tag{11.8.25} \]

Hence the Lagrangian of interest now is

\[
\mathcal{L}_\omega \left( \{ \psi^\ell, \psi_{\mu}^\ell, \tilde{\psi}^{\ell*}, \tilde{\psi}_{\mu}^{\ell*} \} , A^\mu \right) = \sum_\ell \mathcal{L}_\ell^\omega \left( \psi^\ell, \psi_{\mu}^\ell, \tilde{\psi}^{\ell*}, \tilde{\psi}_{\mu}^{\ell*} \right) - \frac{F^{\mu\nu} F_{\mu\nu}}{16\pi}. \tag{11.8.26} \]

Now applying the formula (11.8.5) to the Lagrangian $\mathcal{L}_\omega$, as defined by (11.8.26), (11.8.18), (11.8.23)-(11.8.24), and the formula (11.4.17) for the Lagrangian of EM field we obtain the following expression for the energy density $\mathcal{U}_{\omega \text{stat}}$ of the system of charges and EM field:

\[
\mathcal{U}_{\omega \text{stat}} = \frac{\mathcal{T}_{\omega}^{00}}{\text{stat}} - \frac{1}{8\pi} \left( \nabla \varphi \right)^2 - \sum_\ell K^\ell \left( \psi^{\ell*} \tilde{\psi}^\ell, \left( \frac{\omega^\ell}{c} - \frac{q^\ell}{\chi c} \right)^2 \tilde{\psi}^{\ell*} \tilde{\psi}^\ell - \nabla \tilde{\psi}^{\ell*} \cdot \nabla \tilde{\psi}^\ell \right). \tag{11.8.27} \]

Let us take now the function $K^\ell (a, b)$ to be of a more special form

\[
K^\ell (a, b) = k_2^\ell (a) b + k_0^\ell (a), \text{ implying} \tag{11.8.28} \]

\[
L_\omega^\ell = K^\ell \left( \psi^{\ell*} \tilde{\psi}^\ell, \tilde{\psi}_{\mu}^{\ell*} \psi_{\mu}^\ell \right) = k_2^\ell \left( \psi^{\ell*} \tilde{\psi}^\ell \right) \tilde{\psi}_{\mu}^{\ell*} \psi_{\mu}^\ell + k_0^\ell \left( \psi^{\ell*} \tilde{\psi}^\ell \right). \tag{11.8.29} \]

Notice that the term $k_0^\ell (a)$ in the cases of interest contains the nonlinear self-interaction. In the special case (11.8.28) the expression (11.8.27) for the total energy density of the charges and the EM field takes the form

\[
\mathcal{U}_{\omega \text{stat}} = \mathcal{U}_{\omega \text{stat}}^{(2)} + \mathcal{U}_{\omega \text{stat}}^{(0)}, \text{ where} \tag{11.8.30} \]

\[
\mathcal{U}_{\omega \text{stat}}^{(2)} = -\frac{1}{8\pi} \left( \nabla \varphi \right)^2 + \sum_\ell k_2^\ell \left( \tilde{\psi}^{\ell*} \tilde{\psi}^\ell \right) \nabla \tilde{\psi}^{\ell*} \cdot \nabla \tilde{\psi}^\ell, \]

\[
\mathcal{U}_{\omega \text{stat}}^{(0)} = -\sum_\ell \left[ k_2^\ell \left( \tilde{\psi}^{\ell*} \tilde{\psi}^\ell \right) \left( \frac{\omega^\ell}{c} - \frac{q^\ell}{\chi c} \right)^2 \tilde{\psi}^{\ell*} \tilde{\psi}^\ell + k_0^\ell \left( \tilde{\psi}^{\ell*} \tilde{\psi}^\ell \right) \right]. \]

The corresponding expression for the total energy is

\[
\mathcal{E}_{\omega \text{stat}} = \mathcal{E}_{\omega \text{stat}}^{(2)} + \mathcal{E}_{\omega \text{stat}}^{(0)}, \text{ where} \tag{11.8.31} \]

\[
\mathcal{E}_{\omega \text{stat}}^{(2)} = \int_{\mathbb{R}^3} \left[ -\frac{1}{8\pi} \left( \nabla \varphi \right)^2 + \sum_\ell k_2^\ell \left( \tilde{\psi}^{\ell*} \tilde{\psi}^\ell \right) \nabla \tilde{\psi}^{\ell*} \cdot \nabla \tilde{\psi}^\ell \right] \, dx, \tag{11.8.32} \]

\[
\mathcal{E}_{\omega \text{stat}}^{(0)} = -\sum_\ell \int_{\mathbb{R}^3} \left[ k_2^\ell \left( \tilde{\psi}^{\ell*} \tilde{\psi}^\ell \right) \left( \frac{\omega^\ell}{c} - \frac{q^\ell}{\chi c} \right)^2 \tilde{\psi}^{\ell*} \tilde{\psi}^\ell + k_0^\ell \left( \tilde{\psi}^{\ell*} \tilde{\psi}^\ell \right) \right] \, dx. \tag{11.8.33} \]

Applying now the formula (11.8.15) we get

\[
\mathcal{E}_{\omega \text{stat}}^{(0)} = \frac{1}{3} \mathcal{E}_{\omega \text{stat}}^{(2)}. \tag{11.8.34} \]

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implying the following representation for the total system energy

\[ E_{\omega\text{ stat}} = \frac{2}{3} E^{(2)}_{\omega\text{ stat}} = \frac{2}{3} \int_{\mathbb{R}^3} \left[ -\frac{(\nabla \varphi)^2}{8\pi} + \sum\limits_{\ell} k^{(\ell)}_{2} (\psi^{\ell*} \psi^{\ell}) \nabla \tilde{\psi}^{\ell*} \cdot \nabla \tilde{\psi}^{\ell} \right] dx. \]  

(11.8.35)

In the case of a single charge the above formula turns into

\[ E_{\omega\text{ stat}} = \frac{2}{3} E^{(2)}_{\omega\text{ stat}} = \frac{2}{3} \int_{\mathbb{R}^3} \left[ -\frac{(\nabla \varphi)^2}{8\pi} + k^{(\ell)}_{2} (\psi^{\ell*} \psi^{\ell}) \nabla \tilde{\psi}^{\ell*} \cdot \nabla \tilde{\psi}^{\ell} \right] dx. \]  

(11.8.36)

We want to emphasize once more the importance of the representation (11.8.35) in comparison with the original formula (11.8.31)-(11.8.33), which shows that the total energy of the system of charges interacting with EM field does not explicitly depend on the terms \( k^{(\ell)}_{0} (\psi^{\ell*} \psi^{\ell}) \) which include the nonlinear self-interactions.

We would like to point out now that so far we carried computation for the energy computation for the Lagrangian \( L_{\omega} \), as defined by (11.8.26), (11.8.18), (11.8.23)-(11.8.24). But, in fact, what we really need is the energy for fields of the form (11.8.16) under static conditions (11.8.25) for the initial Lagrangian \( L \) as defined (11.8.17)-(11.8.18). In turns out, as one may expect, the difference between the two is just the sum of the rest energies. Indeed, using once more the formula (11.5.3) for the canonical energy \( \tilde{T}_{00} \) under static conditions (11.8.25), the formulas (11.8.5), (11.8.27) together with the formulas (11.5.14), (3.0.10) for microcharge density \( \rho^{\ell} \) we obtain

\[
\tilde{U} \left( e^{-i \omega^\ell t} \tilde{\psi}^\ell, e^{i \omega^\ell t} \tilde{\psi}^\ell, \varphi \right) = \sum\limits_{\ell} \frac{m^\ell c^2}{q} + \tilde{U}_{\omega\text{ stat}} = \sum\limits_{\ell} \frac{m^\ell c^2}{q} \rho^\ell + \tilde{U}_{\omega\text{ stat}}.
\]

(11.8.37)

Now integrating the above density over the entire space and using the micro-charge normalization condition (2.0.18) obtain

\[ E \left( e^{-i \omega^\ell t} \tilde{\psi}^\ell, e^{i \omega^\ell t} \tilde{\psi}^\ell, \varphi \right) = \sum\limits_{\ell} \frac{m^\ell c^2}{q} + E_{\omega\text{ stat}}. \]  

(11.8.38)

For the special case (11.8.28) combining the last formula with formulas (11.8.35), (11.8.36) we obtain the following important formulas for respectively many charges and a single charge

\[ E \left( e^{-i \omega^\ell t} \tilde{\psi}^\ell, e^{i \omega^\ell t} \tilde{\psi}^\ell, \varphi \right) = \sum\limits_{\ell} \frac{m^\ell c^2}{q} + E_{\omega\text{ stat}}. \]  

(11.8.39)
\[ E \left( e^{-i\omega t} \tilde{\psi}, e^{i\omega t} \tilde{\psi}^*, \varphi \right) = \]
\[ = mc^2 + \frac{2}{3} \int_{\mathbb{R}^3} \left[ -\frac{(\nabla \varphi)^2}{8\pi} + k_2 (\tilde{\psi}^* \psi) \nabla \tilde{\psi}^* \cdot \nabla \psi \right] \, dx. \]

The formulas (11.8.39) and (11.8.40) give important representation for the energy of time harmonic fields which does not explicitly involve the nonlinear self-interactions.

### 11.9 Compressional waves in nonviscous compressible fluid

In this section following to [Morse Feshbach, Section 3.3] and [Morse Ingard, Section 6.2] we consider here compressional waves in nonviscous and compressible fluid which are described by the pressure field \( p \) and velocity field \( \mathbf{v} \) and governed by the following system of equations

\[
\rho \partial_t \mathbf{v} = -\nabla p, \quad \kappa \partial_t p = -\nabla \cdot \mathbf{v}, \quad c^2 = \frac{1}{\rho \kappa} \]  

where \( \rho \) and \( \kappa \) are respectively uniform constant mass density and compressibility (adiabatic) of the fluid at equilibrium and \( c \) is the velocity of wave propagation. We also have

\[
\frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \text{ is the kinetic energy and } \frac{1}{2} \kappa p^2 \text{ is the potential energy} \]  

Then we introduce the velocity potential \( \psi \) so that

\[
p = \rho \partial_t \psi, \quad \mathbf{v} = -\nabla \psi, \]  

it immediately follows from (11.9.3) that \( \psi \) satisfies the classical wave equation

\[
\frac{1}{c^2} \partial_t^2 \psi - \nabla^2 \psi = 0. \]  

The compressional waves have the following Lagrangian density

\[
L = \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} - \frac{1}{2} \kappa p^2 = \frac{1}{2} \rho \left[ \frac{1}{c^2} (\partial_t \psi)^2 - (\nabla \psi)^2 \right] \]  

and the following canonical energy-momentum tensor

\[
T^{\mu\nu} = \begin{bmatrix} T^{00} & \rho \partial_0 \psi \partial_1 \psi & \rho \partial_0 \psi \partial_2 \psi & \rho \partial_0 \psi \partial_3 \psi \\ \rho \partial_1 \psi \partial_0 \psi & T^{11} & -\rho \partial_1 \psi \partial_2 \psi & -\rho \partial_1 \psi \partial_3 \psi \\ \rho \partial_2 \psi \partial_0 \psi & -\rho \partial_2 \psi \partial_1 \psi & T^{22} & -\rho \partial_2 \psi \partial_3 \psi \\ \rho \partial_3 \psi \partial_0 \psi & -\rho \partial_3 \psi \partial_1 \psi & -\rho \partial_3 \psi \partial_2 \psi & T^{33} \end{bmatrix}, \quad \partial_\theta = \frac{1}{c} \partial_t, \]  

\[
T^{00} = \frac{\rho}{2} \left[ (\partial_0 \psi)^2 + (\nabla \psi)^2 \right], \quad T^{ij} = \frac{\rho}{2} \left[ (\nabla \psi)^2 - 2 (\partial_i \psi)^2 - (\partial_0 \psi)^2 \right]. \]  

### 11.10 Klein-Gordon equation and Yukawa potential

Klein-Gordon equation is well known model for a free charge, [Pauli PWM, Section 18]. In particular, its certain modification describes a charge interacting with an external EM field, [Schwabl, Section 8.1]. Here we follow to [Martin, Section 1.5.2]. If the spin is neglected a
freely propagating particle \( X \) of the rest mass \( m_X \) is described by a complex-valued wave function \( \varphi(x) \) satisfying the \textit{Klein-Gordon equation}

\[
-\frac{1}{c^2} \partial_t^2 \varphi = \left\{ -\Delta + \left(\frac{m_X c}{\hbar}\right)^2 \right\} \varphi.
\]

This equation is obtained from the fundamental relativistic mass-energy relation

\[
\frac{E^2}{c^2} = p^2 + m_X c^2,
\]

where \( E \) is the particle energy, \( p \) is the three-dimensional space momentum, by the substitution \( E = \hbar \partial_t \) and \( p = -\hbar \nabla r \). A static solution \( V \) to the Klein-Gordon equation (11.10.1) with a \( \delta \)-function source, i.e.

\[
\left\{ -\Delta + \mu^2 \right\} V = -g^2 \delta(x),
\]

is called the \textit{Yukawa potential}

\[
V(|x|) = -\frac{g^2}{4\pi} \frac{e^{-\mu|x|}}{|x|} = - \left(\mu^2 - \Delta\right)^{-1} g^2 \delta(x), \quad \mu = \frac{m_X c}{\hbar},
\]

The quantity \( \mu^{-1} = \frac{\hbar}{m_c} \) is called the \textit{range of the potential} \( V \), is and it is also known as the \textit{Compton wavelength} of the relativistic particle of the mass \( m_X \). The constant \( g \) is a so-called coupling constant representing the basic strength of the interaction.

There is an interpretation of Klein-Gordon equation as a flexible string with additional stiffness forces provided by the medium surrounding it. Namely, if the string is embedded in a thin sheet of rubber or if it is along the axis of a cylinder of rubber whose outside surface kept fixed, [Morse Feshbach I, Section 2.1].

\section{11.11 Schrödinger Equation}

The Schrödinger equation with the potential \( V \) is

\[
\hbar i \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi.
\]

It is the Euler-Lagrange field equation (together with its conjugate) for the following Lagrangian, [Morse Feshbach I, (3.3.20)]

\[
L = i \hbar \left( \psi^* \partial_t \psi - \partial_t \psi^* \psi \right) - \frac{\hbar^2}{2m} \nabla \psi^* \cdot \nabla \psi - \psi^* V \psi.
\]

The stress-tensor here

\[
\tilde{T}^{\mu\nu} = \frac{\partial L}{\partial (\partial_{\psi^*}^{\nu})} \psi^{\nu} + \frac{\partial L}{\partial (\partial_{\psi^*}^{\nu})} \psi^* \psi - \delta^{\mu\nu} L,
\]

implying the following formula for the energy density

\[
H = \tilde{T}^{00} = -\frac{\hbar^2}{2m} \nabla \psi^* \cdot \nabla \psi + \psi^* V \psi.
\]
The energy flow vector $S$, the momentum density vector $P$ and the current density vector $J$ for the Schrödinger equation (11.11.1) are respectively, \[ S = -\frac{\hbar^2}{2m} [\partial_t \psi^* \cdot \nabla \psi + \partial_t \psi \cdot \nabla \psi^*] , \] (11.11.5)

\[ P = i \frac{\hbar}{2} [\psi^* \cdot \nabla \psi - \psi \cdot \nabla \psi^*] , \] (11.11.6)

\[ J = -\frac{q}{m} P , \] (11.11.7)

with the equation of continuity $\partial_t H + \nabla \cdot S = 0$.

Quantum mechanical charged particle in an external EM field with the 4-potential $A^\mu = (\varphi, A)$ is described by the following Schrödinger equation, \[ \hbar i \frac{\partial}{\partial t} \psi = \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - \frac{q}{c} A \right) \cdot \left( \frac{\hbar}{i} \nabla - \frac{q}{c} A \right) \psi + q \varphi \psi \] (11.11.6)

or

\[ \hbar i \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + i \frac{q \hbar}{2mc} \cdot \nabla \psi + \left[ \frac{q^2 |A|^2}{2mc^2} + q \varphi \right] \psi , \] (11.11.7)

with the charge density $\rho = q \psi \psi^*$ and the current $J$ as follows, \[ \rho = q \psi \psi^* , \] (11.11.8)

\[ J = i \frac{q \hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*) . \] (11.11.9)

The quantities $\rho$ and $J$ satisfy the continuity equation

\[ \partial_t \rho + \nabla \cdot J = 0 . \] (11.11.9)

12 Appendix: Fourier transforms and Green functions

The polar coordinates representation of the Laplace operator in $\mathbb{R}^3$, \[ \Delta = \Delta_r + \frac{1}{r^2} \Delta_s = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{1}{r^2} \Delta_s , \] \[ x \in \mathbb{R}^3 , \ r = |x| , \] (12.0.10)

where $\Delta_s$ is the Laplace operator on the unit sphere $S^2$. We also have, \[ (\kappa^2 - \Delta)^{-1} \delta (x) = \frac{e^{-\kappa |x|}}{4\pi |x|} , \ x \in \mathbb{R}^3 , \ \kappa \geq 0 . \] (12.0.11)

Notice that the action of the operator $\Delta$ on radial functions $g(r)$, i.e. functions depending on $r = |x|$, is reduced to the action of $\Delta_r$ only for smooth functions, i.e.

\[ \Delta g (r) = \Delta_r g (r) \text{ if } g (r) \text{ is continuos and smooth for } r \geq 0 . \] (12.0.12)

Indeed, in view of (12.0.11)

\[ \Delta_r \frac{1}{r} = 0 , \text{ whereas } \Delta_r \frac{1}{r} = -4\pi \delta (x) . \] (12.0.13)

Let us consider the Fourier transform of radial functions following to \[ \text{Taylor} 1 \text{ Section 3.6}: \]

\[ \hat{f} (k) = \int_{0}^{\infty} f (r) \psi_d (r |k|) r^{n-1} dr , \ \psi_d (|k|) = \int_{|x|=1} e^{-ik \cdot x} ds . \] (12.0.14)
Then the following identity holds

\[ \hat{f}(k) = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \frac{1}{|k|} \int_0^\infty f(r) \sin(r|k|) \, dr. \]  

(12.0.15)

Let \( w(x), x \in \mathbb{R}^3 \) be a real function satisfying

\[ 0 \leq w(x) \leq w_\infty < \infty. \]  

(12.0.16)

Then the Green function \( G(x, y) = (-\Delta + w)^{-1}(x, y) \) defined as a fundamental solution to the equation

\[ (-\Delta + w) G(x, y) = \delta(x - y), \]  

(12.0.17)

satisfies the following inequalities

\[ \frac{e^{-\sqrt{w_\infty}|x-y|}}{4\pi |x - y|} \leq (-\Delta + w_\infty)^{-1}(x, y) \leq (-\Delta + w)^{-1}(x, y) \leq \frac{1}{4\pi |x - y|}, \]  

(12.0.18)

which follow from the Feynman-Kac formula for the heat kernel, [Oksendal, Section 8.2], applied to the operator \(-\Delta + w\).

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