On the effective potential, Horava-Lifshitz-like theories and finite temperature

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We calculate the one-loop effective potential at finite temperature for the Horava-Lifshitz-like QED and Yukawa-like theories for arbitrary values of the critical exponent and the space-time dimension. Additional remarks on the zero temperature situation are also presented.

I. INTRODUCTION

The Horava-Lifshitz (HL) methodology, based on an asymmetry between time and space coordinates [1], has gained much attention within the context of the search for a perturbatively consistent gravity theory. The main advantage of that approach comes from the fact that, from one side, it improves the renormalizability of field theory models, and, from another side, it avoids the appearance of ghosts whose presence is characteristic of theories with higher time derivatives [2]. Therefore, this concept (or, more generally, the concept of time-space asymmetry) began to be applied not only within studies of gravity but also for other (f.e. scalar and vector) field theory models.

One line of studies of theories with time-space asymmetry is devoted to the investigation
of their renormalization. Within this context, the HL versions of the gauge field theories [3], scalar field theories [4], four-fermion theory [5] and $CP^{N-1}$ model [6] were considered. Another important result in this context is the generalization of the Ward identities for the HL-like theories [7].

Another line of investigations on HL-like theories concerns the study of the effective potential. In the works [8–11] the one-loop effective potential for scalar field theories with different forms of self-couplings and arbitrary values of the critical exponent $z$, for scalar QED and for the Yukawa models with $z = 2$ and $z = 3$ have been obtained. However, a remaining problem was the calculation of the (one-loop) effective potential for the same models with an arbitrary value of the critical exponent. The analysis of these models at zero temperature has been carried out in [12], and its extension for the non-zero temperature case is considered in this paper.

We begin our study of scalar quantum electrodynamics at finite temperature for generic $z$ and $d$ space dimensions by considering in the section 2 that the effective space-time dimension $d + z$ is odd. As we shall demonstrate, at these values of $d$ and $z$ no self-interaction of the scalar field is necessary to achieve the consistency of the model. However, as discussed in the section 3, for $d + z$ even, a divergence occurs demanding the inclusion of a self-interaction of the scalar field. In the section 4 we analyze an HL version of the Yukawa model and show that for odd $z$ the model is in general nonrenormalizable with only one exception happening if the needed counterterm is proportional to $\phi^4$. A summary and further comments of our results are presented in section 5.

II. SCALAR QUANTUM ELECTRODYNAMICS WITH $d + z$ ODD

The Lagrangian of the scalar QED with an arbitrary $z$ looks like

$$L = \frac{1}{2}F_{0i}F_{0i} - \frac{1}{4}F_{ij}(-\Delta)^{z-1}F_{ij} + D_0\phi(D_0\phi)^* - D_iD_i\ldots D_i\phi(D_iD_i\ldots D_i\phi)^*.$$  \hfill (1)

where $D_0 = \partial_0 - ieA_0$, $D_i = \partial_i - ieA_i$ is a gauge covariant derivative and we assume $\phi$ to be massless, for simplicity. By the same reason, we choose the critical exponents for the scalar and vector fields to be the same.

Adding the gauge fixing term [12]

$$L_{gf} = -\frac{1}{2} \left[(-\Delta)^{-(z-1)/2}\partial_0A_0 - (-\Delta)^{(z-1)/2}\partial_iA_i \right]^2,$$  \hfill (2)
the propagators acquire the simple forms
\[ <A_0A_0> = -\frac{i k^{2z-2}}{k_0^2 + k^{2z}}, \]
\[ <A_iA_j> = \frac{i \delta_{ij}}{k_0^2 + k^{2z}}. \] (3)

For the one-loop calculation of the effective potential, the only relevant vertices are
\[ e^2 A_0^2 \Phi \Phi^*, \]
\[ -ie(\Phi^* \phi - \Phi \phi^*) \partial_0 A_0, \]
\[ -ie(\Phi \phi^* - \Phi^* \phi) \partial_j (\Delta)^{z-1} A_j, \]
\[ -e^2 A_j (\Delta)^{z-1} A_j \Phi \Phi^* \] (4)

where \( \Phi \) is a constant background scalar field generated by the shift \( \phi \to \phi + \Phi \).

From now on, except where explicitly indicated, the propagators (as well as all momenta) will be taken in the Euclidean space, and everywhere \( k^2 \equiv \vec{k}^2 = k_i k_i \), with \( i \) running from 1 to \( d \). As we will show shortly, similarly to what happens in the relativistic QED, the perturbative consistency of the model may require the addition of a self-interaction term for the scalar field. For the time being, we discard such possibility so that, as it has been shown in [12] and will be discussed in more details in the next section, the effective potential turns out to be determined by a single integral
\[ U^{(1)} = U_a + U_b + U_c = \frac{d}{2} \int \frac{d^d k dk_0}{(2\pi)^{d+1}} \ln[k_0^2 + k^{2z} + M^2 k^{2z-2}], \] (5)

where \( M^2 = 2e^2 \Phi \Phi^* \) and \( U_a, U_b \) and \( U_c \) are the contributions coming from loops containing only \( <A_0A_0> \) or \( <A_iA_j> \) propagators and graphs with both the gauge and scalar field propagators, respectively.

In the finite temperature case, following the Matsubara prescription [13], observing that all propagators are bosonic ones, we must change \( k_0 \to 2\pi n T \), where \( T \) is the temperature, and \( n \) is an integer number. The integral over \( k_0 \) is then replaced by the sum:
\[ U^{(1)} = \frac{d}{2} T \sum_{n=-\infty}^{\infty} \int \frac{d^d k}{(2\pi)^d} \ln[4\pi^2 n^2 T^2 + k^{2z} + M^2 k^{2z-2}]. \] (6)

Using the known summation formula [14]:
\[ \sum_{n=-\infty}^{\infty} \ln(4\pi^2 n^2 T^2 + E^2) = \frac{E}{T} + 2 \ln(1 - e^{-E/T}) + \text{const}, \] (7)
where the additive constant does not depend on $E$ and will be omitted from now on, we have

$$U^{(1)} = \frac{d}{2} \int \frac{d^d k}{(2\pi)^d} \left\{ \left( k^{2z} + M^2 k^{2z-2} \right)^{1/2} + 2T \ln \left( 1 - \exp \left( - \frac{(k^{2z} + M^2 k^{2z-2})^{1/2}}{T} \right) \right) \right\}. \quad (8)$$

The first term identically reproduces the zero temperature result from [12]. Therefore (throughout this paper we adopt dimensional regularization with minimal subtraction),

$$U^{(1)} = -\frac{d\pi}{4(2\pi)^d} \frac{d^{d+z}}{\Gamma \left( \frac{d+z}{2} \right)} \Gamma \left( \frac{d+z-1}{2} \right) + U_T, \quad (9)$$

$$U_T = T d \int \frac{d^d k}{(2\pi)^d} \ln(1 - \exp \left( - \frac{(k^{2z} + M^2 k^{2z-2})^{1/2}}{T} \right)).$$

Notice that $U_0$, the first term in the above expression, is finite for $d + z$ odd while, for $d + z$ even, it diverges and requires a subtraction which may be carried out by adding a corresponding counterterm. Therefore, in principle, for the case $d + z = 2n$, one should introduce into the theory an additional vertex $\lambda (\Phi \Phi^*)^n$, to achieve multiplicative renormalizability; the presence of this new self-interaction vertex generates new Feynman diagrams making the evaluation of the one-loop effective potential much more complicated. We will defer the discussion of this situation to the next section and here we will restrict ourselves to the analysis of the case with $d + z = 2n + 1$. By making the change of variables $k^2 = \bar{k}$ (with $\bar{k}$ dimensionless), we obtain

$$U_T = \frac{d}{(4\pi)^{d/2}\Gamma(d/2)} T^{1+\frac{z}{2}} \int_0^\infty d\bar{k} \bar{k}^{d/z-1} \ln \left[ 1 - \exp \left( - (\bar{k}^2 + M^2 \bar{k}^{2(1-z)})^{1/2} \right) \right] \quad (10)$$

Thus, for large $T$, the leading contributions are

$$U_T = \frac{d}{(4\pi)^{d/2}\Gamma(d/2)} T^{1+\frac{z}{2}} \left[ A + B \frac{M^2}{T^{2z}} \right] + \ldots, \quad (11)$$

where

$$A = \int_0^\infty d\bar{k} \bar{k}^{d/z-1} \ln(1 - e^{-\bar{k}}) = \text{Li}_{d+1}(1) \Gamma(d/z), \quad (12)$$

and

$$B = \frac{1}{2} \int_0^\infty d\bar{k} \frac{\bar{k}^{d-2}}{e^{\bar{k}} - 1} = \frac{1}{2} \text{Li}_{d+2}(1) \Gamma \left( \frac{d-2}{z} + 1 \right), \quad (13)$$

where $\text{Li}_\nu(x)$ denotes the polylogarithm function of order $\nu$. We see that for $z = 1$ this expression reproduces the temperature dependence found in [14], but, for generic $z > 1$, the effective potential grows more slowly with the temperature.
III. SCALAR QUANTUM ELECTRODYNAMICS WITH $d + z$ EVEN

As it was pointed out before, for the consistency of the model when $d + z = 2n$, it is necessary the inclusion of a self-interaction term for the scalar field, so that the Lagrangian then becomes

$$L = \frac{1}{2} F_0 F_{0i} + (-1)^z \frac{1}{4} F_{ij} \Delta^{z-1} F_{ij} + D_0 \phi (D_0 \phi)^*$$

$$- D_{i_1} D_{i_2} \ldots D_{i_z} \phi (D_{i_1} D_{i_2} \ldots D_{i_z} \phi)^* - \lambda (\phi \phi^*)^n.$$  \hfill (14)

The gauge fixing term and propagators for the gauge field are the same as in the previous section. The propagator for the scalar field will be fixed shortly.

First, we can find the contribution to the effective potential coming from the gauge propagators only. The corresponding Feynman diagrams are depicted in Fig. 1.

![FIG. 1: Contributions involving gauge propagators only.](image)

There are two types of such contributions – the first of them, $U_a$, is given by the sum of loops of $< A_0 A_0 >$ propagators, and – the second one, $U_b$, of loops of $< A_i A_j >$ propagators. They are completely analogous, up to an overall factor ($U_b$ carries the factor $d$), and they together contribute as (cf. [11, 12])

$$U_a + U_b = \frac{1}{2} (d + 1) \int \frac{dk_0}{(2\pi)^{d+1}} \ln(1 + \frac{2e^2 \Phi \Phi^* k^{2z-2}}{k_0^2 + k^{2z}}).$$  \hfill (15)

Repeating the calculations of the Section II, one can show that the results for $U_a + U_b$ at zero and finite temperature, reproduce the expressions (9,10,11) with the only difference that the overall factor $d$ is replaced by $d + 1$.

Now, let us obtain the background-dependent effective propagators of the scalar fields. After the background-quantum splitting $\phi \rightarrow \Phi + \phi$, $\phi^* \rightarrow \Phi^* + \phi^*$, the part of the Lagrangian quadratic in the quantum field $\phi$ turns out to be nontrivial, being of the form

$$L_{2\phi} = -\phi [\partial_0^2 + (-\Delta)^z] \phi^* - \lambda \left\{ \frac{n(n-1)}{2} (\Phi \Phi^*)^{n-2} (\Phi^*)^2 \phi^2 + \Phi^2 (\phi^*)^2 \right\} + n^2 \Phi^{n-1} (\Phi^*)^{n-1} \phi \phi^*,$$  \hfill (16)
which generates the propagators for $\phi$:

$$
\begin{pmatrix}
<\phi\phi> & <\phi\phi^*> \\
<\phi^*\phi> & <\phi^*\phi^*> 
\end{pmatrix}
\begin{pmatrix}
\mathcal{M} & \partial_0^2 + (-\Delta)^z + \mu \\
\partial_0^2 + (-\Delta)^z + \mu & \mathcal{M}
\end{pmatrix}^{-1} =
$$

$$
= \frac{i}{(\partial_0^2 + (-\Delta)^z + \mu)^2 - \mathcal{M}\mathcal{M}}
\begin{pmatrix}
\mathcal{M} & -(\partial_0^2 + (-\Delta)^z + \mu) \\
-(\partial_0^2 + (-\Delta)^z + \mu) & \mathcal{M}
\end{pmatrix},
$$

where $\mathcal{M} = \lambda n (n - 1) (\Phi\Phi^*)^{n-2} (\Phi^*)^2$, $\bar{\mathcal{M}} = \lambda n (n - 1) (\Phi\Phi^*)^{n-2} \Phi^2$, and $\mu = \lambda n^2 (\Phi\Phi^*)^{n-1}$.

These propagators will be represented by bold straight lines.

Besides, we also must use the background-dependent propagators of the gauge field, $<A_0 A_0>$ and $<A_i A_j>$, which are introduced as a result of the following summation over the quartic vertices represented in Fig. 2:

$$
\begin{align*}
\text{FIG. 2: Background-dependent gauge propagator.}
\end{align*}
$$

Actually, we will use not these propagators themselves but the objects derived from them:

$$
\begin{align*}
G_1 &= <\partial_0 A_0 \partial_0 A_0>; \\
G_2 &= <\partial_i \Delta^{z-1} A_i \partial_j \Delta^{z-1} A_j>,
\end{align*}
$$

whose Fourier transforms in the Euclidean space are

$$
\begin{align*}
G_1(k) &= \frac{k_0^2 k^{2z-2}}{k_0^2 + \bar{k}^{2z} + 2e^2 k^{2z-2} \Phi\Phi^*}; \\
G_2(k) &= \frac{k_0^2 k^{4z-2}}{k_0^2 + \bar{k}^{2z} + 2e^2 k^{2z-2} \Phi\Phi^*}.
\end{align*}
$$

The presence of the new propagators will lead to a contribution to the effective action emerged from the "crossed" sector (those graphs involving both gauge and matter propagators shown at Fig. 3) given by

$$
\begin{align*}
U_c &= -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \int \frac{d^d k d k_0}{(2\pi)^{d+1}} \left[ -e^2 (2\Phi\Phi^* <\phi\phi^*> + \Phi\Phi <\phi^*\phi^*> + \Phi^*\Phi^* <\phi\phi>) \times \\
&\quad \times (G_1 + G_2) \right]^n,
\end{align*}
$$

(20)
where

\[
2\Phi\Phi^* <\phi\phi^*> + \Phi <\phi^*\phi^*> + \Phi^*\Phi <\phi^*> = - \frac{2\Phi\Phi^*(k_0^2 + k^2z + \mu) + \mathcal{M}\Phi + \overline{\mathcal{M}}\Phi^*\Phi^*}{(k_0^2 + k^2z + \mu)^2 - \mathcal{M}\overline{\mathcal{M}}}.
\]

So, we can write

\[
U_c = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \int \frac{d^dk_0}{(2\pi)^{d+1}} \left( 1 - e^2 \frac{2\Phi\Phi^*(k_0^2 + k^2z + \mu) + \mathcal{M}\Phi + \overline{\mathcal{M}}\Phi^*\Phi^*}{(k_0^2 + k^2z + \mu)^2 - \mathcal{M}\overline{\mathcal{M}}} \times \frac{k_0^2k_{2z-2} + k_{4z-2}}{k_0^2 + k^2z + 2e^2k_{2z-2}\Phi^*} \right)^n =
\]

\[
= \frac{1}{2} \int \frac{d^dk_0}{(2\pi)^{d+1}} \ln \left( 1 - e^2 \frac{2\Phi\Phi^*(k_0^2 + k^2z + \mu) + \mathcal{M}\Phi + \overline{\mathcal{M}}\Phi^*\Phi^*}{(k_0^2 + k^2z + \mu - \sqrt{\mathcal{M}\overline{\mathcal{M}}})(k_0^2 + k^2z + \mu + \sqrt{\mathcal{M}\overline{\mathcal{M}}})} \times \frac{k_{2z-2}(k_0^2 + k^2z)}{k_0^2 + k^2z + M^2k_{2z-2}} \right).
\]

The integral over momenta, as well as the discretization of the zero component of the momentum in order to implement finite temperature, is straightforward but the result is highly cumbersome. Nevertheless it can be performed in the following way.

To simplify this expression, we introduce \( \mu_\pm = \mu \pm \sqrt{\mathcal{M}\overline{\mathcal{M}}} \), so that \( \mu_+ = \lambda n(2n - 1)(\Phi\Phi^*)^{n-1} \), and \( \mu_- = \lambda n(\Phi\Phi^*)^{n-1} \) and define \( q^2 = k_0^2 + k^2z \). Using that \( \mathcal{M}\Phi + \overline{\mathcal{M}}\Phi^*\Phi^* = 2\lambda n(n - 1)(\Phi\Phi^*)^n \) we get

\[
U_c = \frac{1}{2} \int \frac{d^dk_0}{(2\pi)^{d+1}} \ln \left( 1 - e^2 \frac{2\Phi\Phi^*(q^2 + \mu)}{(q^2 + \mu_+)(q^2 + \mu_-)} \frac{k_{2z-2}q^2}{q^2 + M^2k_{2z-2}} \right). \quad (23)
\]

After some algebraic transformations, we can rewrite this expression as

\[
U_c = \frac{1}{2} \int \frac{d^dk_0}{(2\pi)^{d+1}} \left[ \ln \left( (q^2 + \mu_+)(q^2 + \mu_-)(q^2 + k_{2z-2}^2M^2) - q^2k_{2z-2}^2M^2(q^2 + \mu_+) \right) - \ln(q^2 + \mu_+) - \ln(q^2 + \mu_-) - \ln(q^2 + k_{2z-2}^2M^2) \right]. \quad (24)
\]
By cancelling the term with $\ln(q^2 + \mu_+)$, we arrive at

$$U_c = \frac{1}{2} \int \frac{d^d k dk_0}{(2\pi)^{d+1}} \left[ \ln \left( (q^2 + \mu_-)(q^2 + k^{2z-2}M^2) - q^2 k^{2z-2}M^2 \right) - \ln(q^2 + \mu_-) - \ln(q^2 + k^{2z-2}M^2) \right].$$

(25)

It follows from (15) that $U_a = \frac{1}{2} \int \frac{d^d k dk_0}{(2\pi)^{d+1}} \ln(q^2 + M^2 k^{2z-2})$. Thus, we can cancel some additional terms and get

$$U_a + U_c = \frac{1}{2} \int \frac{d^d k dk_0}{(2\pi)^{d+1}} \left[ \ln \left( (q^2 + \mu_-)(q^2 + k^{2z-2}M^2) - q^2 k^{2z-2}M^2 \right) - \ln(q^2 + \mu_-) \right];$$

$$U_b = \frac{d}{2} \int \frac{d^d k dk_0}{(2\pi)^{d+1}} \ln(q^2 + M^2 k^{2z-2}).$$

(26)

To close the calculations, let us obtain the contribution to the effective potential generated by the self-coupling of the scalar fields. It follows from (17) that in this case we have a new contribution to the effective action

$$U^{(1)}_d = -\frac{i}{2} \ln \det \left( \begin{array}{cc} \mathcal{M} & \partial^2_0 + (-\Delta)^z + \mu \\ \partial^2_0 + (-\Delta)^z + \mu & \bar{\mathcal{M}} \end{array} \right).$$

(27)

To calculate this determinant it is convenient to perform the Fourier transform, which after a Wick rotation to the Euclidean space yields

$$U^{(1)}_d = \frac{1}{2} \int \frac{d^d k dk_0}{(2\pi)^{d+1}} \ln \det \left( \begin{array}{cc} \mathcal{M} & k^2_0 + k^{2z} + \mu \\ k^2_0 + k^{2z} + \mu & \bar{\mathcal{M}} \end{array} \right).$$

(28)

Up to an irrelevant additive constant, the evaluation of this expression gives

$$U^{(1)}_d = \frac{1}{2} \int \frac{d^d k dk_0}{(2\pi)^{d+1}} \ln \left( (k^2_0 + k^{2z} + \mu)^2 - \mathcal{M} \bar{\mathcal{M}} \right) =$$

$$= \frac{1}{2} \int \frac{d^d k dk_0}{(2\pi)^{d+1}} \left( \ln[q^2 + \mu_-] + \ln[q^2 + \mu_+] \right).$$

(29)

Using (26), we can write the complete one-loop effective potential:

$$U_a + U_b + U_c + U_d = \frac{1}{2} \int \frac{d^d k dk_0}{(2\pi)^{d+1}} \left[ \ln \left( (q^2 + \mu_-)(q^2 + k^{2z-2}M^2) - q^2 k^{2z-2}M^2 \right) + d \ln(q^2 + M^2 k^{2z-2}) + \ln(q^2 + \mu_+) \right].$$

(30)

The term proportional to $d$ (that is, $U_b$) is given by the expressions (9)–(11). The last term, that is, those originated from $U_d$ involving $\ln(q^2 + \mu_+)$, yields the zero temperature result

$$U^{(1)}_{d+} = -\frac{1}{2\sqrt{\pi}} \frac{1}{(2\pi)^d} \frac{1}{\Gamma(d/2)} \Gamma(\frac{d}{2z}) \Gamma(-\frac{1}{2} - \frac{d}{2z}) \mu_+^{1/2 + d/(2z)}.$$

(31)
plus the finite temperature contribution

\[ U_{d^+}^{(1)}(T) = 2T \int \frac{d^d k}{(2\pi)^d} \ln \left\{ 1 - \exp \left[ -\left( \frac{\sqrt{k^2 z + \mu^+}}{T} \right) \right] \right\}. \]  

(32)

At high temperatures we obtain

\[ U_{d^+}^{(1)}(T) = \frac{2}{(4\pi)^{d/2} \Gamma(d/2)} T^{1+d/2} (A + \mu_{+} B_0 \frac{T}{T_2}) + \ldots, \]  

(33)

where \( A \) is given by (12), and

\[ B_0 = \int_0^\infty \frac{k^{d/z - 2}}{e^k - 1} = \text{Li}_{d/z - 1}(1) \Gamma(d/z - 1). \]  

(34)

It remains to analyse the first term from (30) which looks like

\[ I = \frac{1}{2} \int \frac{d^d k d k_0}{(2\pi)^{d+1}} \ln \left( (q^2 + \mu_-)(q^2 + k^{2z - 2} M^2) - q^2 k^{2z - 2} M^2 \right). \]  

(35)

Unfortunately, this integral cannot be done in a closed form. We present the results only for two particular cases.

(i) When the contribution of the gauge coupling dominates, we can choose \( \lambda \simeq 0 \). Then one has \( \mu_- = 0 \) and the integral \( I \) is just an irrelevant constant, independent of the classical fields. The complete contribution to the effective potential in this case comes from the terms (9–11), while the term proportional to \( \lambda \) is essential only on the tree level.

(ii) When the contribution of the scalar self-coupling dominates, we can choose \( g \simeq 0 \) in this term. In this case, one has \( M = 0 \), so, \( I \simeq \frac{1}{2} \int \frac{d^d k d k_0}{(2\pi)^{d+1}} \ln(q^2 + \mu_-) \), which, similarly to expressions (31) and (33), yields

\[ U_{d^+}^{(1)} = -\frac{1}{2\sqrt{\pi}} \frac{1}{(2\pi)^d} \frac{\pi^{d/2}}{\Gamma(d/2)} \Gamma\left( \frac{d}{2z} \right) \Gamma\left( -\frac{1}{2} - \frac{d}{2z} \right) \mu_-^{1/2 + d/(2z)}, \]  

(36)

at zero temperature and

\[ U_{d^+}^{(1)}(T) = \frac{2}{(4\pi)^{d/2} \Gamma(d/2)} T^{1+d/2} (A + \mu_{+} B_0 \frac{T}{T_2}) + \ldots. \]  

(37)

at high temperature. Here the effective potential is reduced to the sum of the expressions (31) and (36) at zero temperature, and of (33) and (37) at the high temperature.

We close this section with a discussion of the renormalizability of the model. First, one reminds that we introduced a self-coupling of the scalar field since, at \( \lambda = 0 \), the one-loop
effective potential diverges if \( d + z = 2n \) with \( n \) integer (see (33)) so, the counterterms \((\Phi \Phi^*)^n\) is needed. At the same time, the new vertex \((\Phi \Phi^*)^n\) generates new contributions as in (31), which diverge if \( 1 + \frac{z}{2} = 2\tilde{n} \), with \( \tilde{n} \) an integer. These contributions are proportional to \((\Phi \Phi^*)^{\tilde{n}(n-1)}\), therefore, to achieve multiplicative renormalizability, one must, in principle, introduce a new vertex \((\Phi \Phi^*)^{\tilde{n}(n-1)}\), which, again modifies the classical action. The only exceptional situation, when this modification is not necessary, is the case \( \tilde{n} = \frac{n}{n-1} \). For \( n \) and \( \tilde{n} \) integer, the only solution is \( n = \tilde{n} = 2 \). Therefore, we conclude that only the vertex \((\Phi \Phi^*)\) corresponds to the renormalizable interaction, with \( d = 3 \) and \( z = 1 \), that is, just the usual scalar QED. We note, however, that in the cases, where either \( d + z \) is odd (that is, the case considered in the previous section), or \( \frac{d}{2} \) is not an odd number (i.e. either even, or fractionary one), this problem simply will not arise, since, for \( d + z \) odd, there is no divergent contributions to the one-loop effective potential. There are, of course, additional restrictions on \( d \) and \( z \) arising from the fact that, in the renormalizable theories, dimensions of couplings must be non-negative, i.e. \( z - d + 2 \geq 0 \) (for the coupling \( e \)) and \( d + z - n(d - z) \geq 0 \) (for the coupling \( \lambda \)). However, these restrictions play a role only at higher loop orders.

IV. YUKAWA THEORY

Let us now formulate the arbitrary \( z \) version of the Yukawa theory whose Lagrangian density is

\[
L = \bar{\psi}(i\gamma^0\partial_0 + (i\gamma^i\partial_i)^z + h\Phi)\psi. \tag{38}
\]

To study the one-loop effective potential, it is enough to treat the scalar field as purely external, and to consider the spinor field to be massless since a nontrivial mass implies only in a redefinition of the \( \Phi \) field. In this case, the loop expansion ends at the one-loop contribution. However, if we assume that \( \Phi \) is also dynamical (which, in particular, is necessary to proceed renormalization if the contribution to the effective potential diverges), with the same critical exponent \( z \) as the \( \psi \), its free Lagrangian is the same as in the theory (1). Notice that the mass dimension of \( h \) is \((3z - d)/2\), and the theory is (super)renormalizable for \( z \geq d/3 \) – in particular, it is renormalizable in the usual case \( (z = 1) \) Yukawa model in \((3 + 1)\)-dimensional space.
The one-loop effective potential corresponding to the Lagrangian \([38]\), looks like

\[
U^{(1)} = i \text{Tr} \ln(i\gamma^0 \partial_0 + (i\gamma^i \partial_i)z + h\Phi). \tag{39}
\]

We have two possibilities. In the first one, \(z\) is even, so, \((i\gamma^i \partial_i)z = (-\Delta)^{z/2}\), and we find that the effective potential in the Euclidean space is

\[
U^{(1)} = -\frac{\delta}{2} \int \frac{d^dkd\theta_0}{(2\pi)^{d+1}} \ln \left[ \frac{k_0^2 + (k^2)^{z/2} + h\Phi}{k_0^2} \right], \tag{40}
\]

where \(\delta\) is the dimension of the Dirac matrices. Taking into account the discretization of the zero component of the momentum, we get

\[
U^{(1)} = -\frac{T}{2} \sum_{n=-\infty}^{\infty} \int \frac{d^dk}{(2\pi)^d} \ln \left[ 4\pi^2 T^2 (n + \frac{1}{2})^2 + ((k^2)^{z/2} + h\Phi)^2 \right]. \tag{41}
\]

Using the expression for the sum

\[
\sum_{n=-\infty}^{\infty} \ln((\pi^2(2n + 1)^2 + E^2) = \frac{E}{T} + 2\ln(1 + e^{-E/T}). \tag{42}
\]

(cf. \([14]\); note that the presence of \(\ln(1 + \cdots)\) instead of \(\ln(1 - \cdots)\) comes from a difference between bosonic and fermionic cases), we arrive at

\[
U^{(1)} = -\frac{\delta}{2} \int \frac{d^dk}{(2\pi)^d} \left[ (k^2)^{z/2} + h\Phi + 2T \ln(1 + \exp[-\frac{(k^2)^{z/2} + h\Phi}{T}]) \right]. \tag{43}
\]

Integration of the first term gives zero result, and hence, no renormalization is needed. So, we get

\[
U^{(1)} = -T\delta \int \frac{d^dk}{(2\pi)^d} \ln(1 + \exp[-\frac{(k^2)^{z/2} + h\Phi}{T}]). \tag{44}
\]

This expression is non-trivial only for non-zero temperature. Proceeding just as in the previous sections, we find at large \(T\):

\[
U^{(1)} = -\frac{T^{1+d/z}\delta}{2^d\pi^{d/2}\Gamma(d/2)z} \left( A_1 - \frac{B_1 h\Phi}{T} \right), \tag{45}
\]

where

\[
A_1 = \int_0^\infty d\bar{k}\bar{k}^{d/z-1} \ln(1 + e^{-\bar{k}}) = (-1 + 2^{d/z})\Gamma(d/z)\zeta(d/z)2^{-d/z},
\]

\[
B_1 = \int_0^\infty \frac{d\bar{k}\bar{k}^{d/z-1}}{e^{\bar{k}} + 1} = (2^{d/z} - 2)\Gamma(d/z)\zeta(d/z)2^{-d/z}, \tag{46}
\]
where $\zeta(x)$ is the Riemann zeta function.

The second possibility when $z$ is odd, $z = 2l + 1$, so that, $(i\gamma^i\partial_i)^z = (-\Delta)^l i\gamma^i\partial_i$. In this case we have

$$U^{(1)} = -i\text{Tr} \ln(i\gamma^0\partial_0 + i(-\Delta)^l i\gamma^i\partial_i + h\Phi) = -\frac{1}{2} \delta \int \frac{dk_0 d^dk}{(2\pi)^{d+1}} \ln(k_0^2 + (k^2)^z + M^2),$$

(47)

where $M = h\Phi$. We replace the zero component of the momentum by the discrete one, $k_0 = (2n + 1)\pi T$, and have

$$U^{(1)} = -\frac{1}{2} \delta T \sum_{n=-\infty}^{\infty} \int \frac{d^dk}{(2\pi)^d} \ln(4\pi^2 T^2(n + \frac{1}{2})^2 + (k^2)^z + M^2)^{1/2}$$

(48)

After evaluating the sum we get

$$U^{(1)} = -\frac{1}{2} \delta \int \frac{d^dk}{(2\pi)^d} \ln(1 + \exp[-\frac{(k^2 + M^2)^{1/2}}{T}]).$$

(49)

Performing the integration, we find

$$U^{(1)} = U_0 + U_T =$$

$$= -\delta \frac{\pi^{d/2-1/2}}{(2\pi)^d} \frac{1}{\Gamma(d/2)} \Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{1}{2} - \frac{d}{2}\right) \Gamma\left(\frac{1}{2} \left[1 - \frac{d}{z}\right]\right) \left(h\Phi\right)^{d/2+1} -$$

$$- \frac{1}{2} T \delta \int \frac{d^dk}{(2\pi)^d} \ln(1 + \exp[-\frac{(k^2 + M^2)^{1/2}}{T}]).$$

(50)

As before, we have a sum of the zero temperature result with an additive term which is non-trivial only at the non-zero temperature. As for the zero temperature term, it is proportional to $\Gamma\left(\frac{1}{2} \left[1 - \frac{d}{z}\right]\right)$, thus, it diverges if $1 - \frac{d}{z} = -2n$, with $n$ integer. In this case the divergence will be proportional to $\Phi^{2n+2}$. Thus, similarly to the previous section, the $\Phi^{2n+2}$ term must be present in the Lagrangian from the very beginning. Its presence will give an additional contribution to the effective action. Such a contribution, at the one-loop level, is a sum of all one-loop scalar graphs. Therefore it enters the one-loop effective action only as an additive term. Indeed, if the action of the scalar field looks like

$$S = \int dtd^dx \left(\frac{1}{2} \phi^2 - \frac{1}{2} (-1)^z \phi\Delta^z \phi - V(\phi)\right),$$

(51)

the corresponding one-loop effective potential, in the case $1 - \frac{d}{z} = -2n$, is

$$U^{(1)} = -\frac{1}{2} \frac{\pi^{d/2}}{2\sqrt{\pi} (2\pi)^d} \frac{1}{\Gamma(d/2)} \Gamma(n + \frac{1}{2}) \Gamma(-1 - n)(V''(\Phi))^{n+1}.$$
with \( \Phi \) being a background field, and \( V(\Phi) = \frac{f}{(2n+2)(2n+1)} \Phi^{2n+2} \), so that, \( V''(\Phi) = f \Phi^{2n} \).

The corresponding quantum correction, for \( 1 - \frac{d}{z} = -2n \), with \( n \) integer, is divergent being proportional to \( \Gamma(-n-1) \Phi^{2n(n+1)} \), which, of course, needs a counterterm, and, hence, the presence of this vertex in the action from the very beginning, which, consequently, modifies \( V(\Phi) \) once more. The only special case is \( n = 1 \), where this modification does not happen, it corresponds to \( d = 3z \). This divergent term reproduces the structure of the potential, i.e. if the coupling looks like \( V(\Phi) = \frac{f}{2} \Phi^4 \), the divergences arising both from spinor and scalar sectors will be proportional to \( \Phi^4 \), so, no other coupling is needed in this case. Actually, we have shown that it is the only renormalizable case.

The temperature dependent term from (50), after the corresponding change of variables, is

\[
U_T = -\frac{T^{1+d/z} \delta}{2^{d+1} \pi^{d/2} \Gamma(d/2) \delta} \int_0^\infty d\bar{k} \frac{\bar{k}^{d/z-1}}{} \ln(1 + e^{-\sqrt{k^2 + M^2}/T^2}).
\] (53)

Again, we can obtain leading and subleading terms:

\[
U_T = -\frac{T^{1+d/z} \delta}{2^{d+1} \pi^{d/2} \Gamma(d/2) \delta} (A_1 - B_2 \frac{M^2}{T^2}),
\] (54)

where

\[
B_2 = \frac{1}{2} \int_0^\infty \frac{d\bar{k} \bar{k}^{d/z-2}}{} \frac{\bar{k} + 1}{e^\bar{k} + 1} = (2^{d/z} - 4) \Gamma\left(\frac{d}{z} - 1\right) \zeta\left(\frac{d}{z} - 1\right) 2^{-(d+z)/z},
\] (55)

and \( A_1 \) is just the same one defined earlier in (46). We note that this integral is well defined if \( d > z \). Thus, we obtained the high-temperature asymptotic expressions for the effective potential, both in bosonic and fermionic case.

**V. CONCLUSIONS**

In this work we studied the one-loop effective potential at finite temperature for the HL QED and Yukawa models. We made also important remarks on the zero temperature situation which extend an earlier study by some of us. Indeed, for \( d + z \) even in the case of QED and also for the Yukawa model with \( z \) odd there occurs divergences whose elimination require the addition of self-interactions of the scalar fields. These new terms produce new divergences in a way that invalidates the usual renormalization procedure unless for the usual case, \( z = 1 \) and \( d = 3 \). In the cases with \( d + z \) odd for the HL QED and \( z \) even...
for the HL Yukawa models there are no one-loop divergences. This, of course, does not preclude the existence of divergences in higher orders which must be removed by an adequate renormalization scheme.

The models have the same high temperature limit proportional to $T^{1+d/z}$ as they should but different next to leading behaviors unless $d = z$. Thus for $z > 1$ the effective potential decays with the temperature more slowly than in the usual case [14].

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