ON \textit{p}-ADIC PERIODS FOR MIXED TATE MOTIVES OVER A NUMBER FIELD

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Abstract. For a number field, we have a Tannaka category of mixed Tate motives at our disposal. We construct \textit{p}-adic points of the associated Tannaka group by using \textit{p}-adic Hodge theory. Extensions of two Tate objects yield functions on the Tannaka group, and we show that evaluation at our \textit{p}-adic points is essentially given by the inverse of the Bloch-Kato exponential map.


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Introduction

For a number field $E$, one has an abelian category of mixed Tate motives $MT(E)$ \cite{DG05}. A mixed Tate motive comes equipped with a weight filtration $W$, and the associated graded pieces are sums of Tate objects. There is a natural fibre functor $\omega$ defined by

$$\omega(M) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}(\mathbb{Q}(n), \text{gr}^W_{-2n}(M));$$

we denote by $G_\omega$ the corresponding Tannaka group.

If $\mathcal{O}$ denotes the ring of integers of $E$ and $x \in \text{Spec}(\mathcal{O})$ is a closed point, then Deligne and Goncharov construct a Tannaka subcategory $MT(\mathcal{O}_x)$ of $MT(E)$ consisting of motives which are unramified at $x$ \cite[1.6]{DG05}. We will denote its group of tensor automorphisms by $G_x$.

\footnotesize
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To a mixed Tate motive $M$ we can attach its $p$-adic realization $M_p$ which is a representation of the Galois group of $E$ with coefficients in $\mathbb{Q}_p$. If the point $x$ lies over the prime $p$, then we can restrict in order to obtain a $p$-adic representation $M_{i,p}$ for the Galois group of the completion $E_x$ at $x$. We will show that $M_{i,p}$ is always semistable. Furthermore, $M_{i,p}$ is crystalline if and only if $M$ is unramified at $x$, i.e. $M \in MT(O_x)$ (Theorem 2.2.3). In fact, $p$-adic representations attached to mixed Tate motives are contained in an abelian subcategory which admits a fibre functor $\tau$ similar to $\omega$. Denoting by $H_\tau$ the corresponding Tannaka group over $\mathbb{Q}_p$, $p$-adic realization yields a group homomorphism

$$H_\tau \rightarrow G_\omega \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$  

The main purpose of this paper is to construct an $E_{x,st}$-valued point $\eta_{st}$ of $H_\tau$, where $\text{Spec}(E_{x,st})$ is a 1-dimensional affine space over the field $E_x$. The $E_x$-valued points of $\text{Spec}(E_{x,st})$ correspond naturally to the extensions of the canonical logarithm $\log : O_{E_x}^\times \rightarrow E$ to $E^\times$. Therefore, any choice of such an extension induces via $\eta_{st}$ an $E_x$-valued point of $H_\tau$ and $G_\omega$. For the Tannaka subcategory of crystalline representations the picture is simpler: if $H_{\tau,\text{cris}}$ denotes their Tannaka group and $\pi : H_\tau \rightarrow H_{\tau,\text{cris}}$ is the projection, then $\pi\circ\eta_{st}$ factors through $\text{Spec}(E_x)$ and we obtain an $E_x$-valued point $\eta$ of $H_{\tau,\text{cris}}$. We denote by $\eta^n_{ur}$ the image of $\eta$ in $G_x$.

To state our main theorem, we need to recall how extensions $M$ of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ in $MT(O_x)$ give rise to functions on $G_x$ for $n \geq 1$. The natural isomorphisms $\alpha : \mathbb{Q} \rightarrow \text{Hom}(\mathbb{Q}(n), \text{gr}_W^W M)$ and $\beta : \text{Hom}(\mathbb{Q}(0), \text{gr}_W^W M) \rightarrow \mathbb{Q}$ induce elements $\alpha^{-1} \in \omega(M)^\vee$ and $\beta^{-1} \in \omega(M)$; we set $M(\eta^n_{ur}) = \alpha^{-1}(\eta^n_{ur} : \beta^{-1}(1))$.

**Theorem** (Theorem 2.3.3). For all $n \geq 1$, the map

$$\text{Ext}^1_{MT(O_x)}(\mathbb{Q}(0), \mathbb{Q}(n)) \rightarrow E_x, \quad M \mapsto M(\eta^n_{ur}),$$

is the composition of the $p$-adic realization

$$\text{Ext}^1_{MT(O_x)}(\mathbb{Q}(0), \mathbb{Q}(n)) \rightarrow \text{Ext}^1_{\text{cris}}(\mathbb{Q}_p(0), \mathbb{Q}_p(n))$$

and the inverse of the Bloch-Kato exponential map (2.3.7).

1. **Filtered $\phi$-modules and mixed Tate filtered $\phi$-modules**

1.1. **Mixed Tate filtered $\phi$-modules.**

1.1.1. Let $K$ be a $p$-adic field with residue field $k$, i.e. $\text{char}(K) = 0$, $K$ is complete with respect to a fixed discrete valuation and the residue field $k$ is perfect of characteristic $p$. Let $W(k)$ be the ring of Witt vectors of $k$, $\sigma : W(k) \rightarrow W(k)$ the Frobenius lift and $K_0$ the field of fractions of $W(k)$.

1.1.2. We denote by $MF^\phi_K$ the category of filtered $\phi$-modules, i.e. the objects are triples $(M, \phi, F)$, where $(\hat{M}, \phi)$ is an isocrystal over $K_0$ and $F$ is a descending, exhaustive and separated filtration on $M_K = M \otimes_{K_0} K$. We denote by $MF^{\phi,N}_K$ the category of filtered $(\phi,N)$-modules, i.e. objects are tuples $(M, \phi, N, F)$ with $(\hat{M}, \phi, N) \in MF^{\phi,N}_k$ and $N : M \rightarrow M$ is a $K_0$-linear endomorphism such that $N\phi = p\phi N$. We consider $MF^{\phi}_K$ as full subcategory of $MF^{\phi,N}_K$ via the functor $(M, \phi, F) \mapsto (M, \phi, 0, F)$.  

The Dieudonné-Manin classification [Ma63, II, §4.1] implies, by descent, that every isocrystal \((M, \phi)\) over \(K_0\) admits a slope decomposition
\[
M = \bigoplus_{\lambda \in \mathbb{Q}} M_{\lambda},
\]
with \(\phi(M_{\lambda}) = M_{\lambda}\) and \((M_{\lambda}, \phi|_{M_{\lambda}})\) is isoclynic of slope \(\lambda\). From the relation \(N\phi = p\phi N\), it follows that \(N(M_{\lambda}) \subseteq M_{\lambda-1}\). In the following, we will use the notation:
\[
M_{\leq \lambda} := \bigoplus_{\lambda' \leq \lambda} M_{\lambda'}, \quad M_{\geq \lambda} := \bigoplus_{\lambda' \geq \lambda} M_{\lambda'}.
\]

**Definition 1.1.3.** We say that an object \((M, \phi, F) \in MF^\phi_K\) is a mixed Tate filtered \(\phi\)-module if the following properties are satisfied:

1. There is an isomorphism of \(\phi\)-modules
   \[
   (M, \phi) \cong \bigoplus_{i \in I} (K_0, p^{n_i} \sigma),
   \]
   for some index set \(I\), and \(n_i \in \mathbb{Z}\).

2. For all \(i \in \mathbb{Z}\) the natural map
   \[
   F^i M_K \to M_{\geq i} \otimes_{K_0} K
   \]
   is an isomorphism.

We say that \((M, \phi, N, F) \in MF^{\phi,N}_K\) is a mixed Tate filtered \((\phi, N)\)-module if \((M, \phi, F)\) is a mixed Tate filtered \(\phi\)-module.

We denote by \(MT^\phi_K\) (resp. \(MT^{\phi,N}_K\)) the full subcategory of \(MF^\phi_K\) (resp. \(MF^{\phi,N}_K\)) with mixed Tate filtered \(\phi\)-modules (resp. \((\phi, N)\)-modules) as objects. The categories \(MT^\phi_K\) and \(MT^{\phi,N}_K\) are additive. Again, we consider \(MT^\phi_K\) as full subcategory of \(MT^{\phi,N}_K\).

For \((M, \phi, N, F) \in MT^{\phi,N}_K\), it follows from Property (1) that all the slopes of \((M, \phi)\) are integers. From Property (2) we conclude that the Hodge polygon of \((M_K, F)\) equals the Newton polygon of \((M, \phi)\).

**Definition 1.1.4.** (Tate objects) Let \(n \in \mathbb{Z}\) be an integer. We define the Tate object \(K(n) \in MT^\phi_K\) by
\[
K(n) := (K_0, p^{-n} \sigma, F),
\]
with \(F\) defined by
\[
F^j = \begin{cases} 
K & \text{if } j \leq -n, \\
0 & \text{if } j > -n.
\end{cases}
\]

**Definition 1.1.5.** (Weight filtration) Let \((M, \phi, N, F) \in MT^{\phi,N}_K\). Let \(i \in \mathbb{Z}\) be an integer. We define an object \(W_{2i}(M, \phi, N, F)\) in \(MF^\phi_K\) by
\[
W_{2i}(M, \phi, N, F) := (M_{\leq i}, \phi|_{M_{\leq i}}, N|_{M_{\leq i}}, F \cap M_{\leq i}).
\]
We define an object \(gr_{2i}^W(M, \phi, N, F) \in MT^\phi_K\) by
\[
gr_{2i}^W(M, \phi, N, F) := (M_i, \phi|_{M_i}, \tilde{F}),
\]
where \(\tilde{F}\) is defined as follows:
\[
\tilde{F}^i M_i = M_i, \quad \tilde{F}^{i+1} M_i = 0.
\]
Proposition 1.1.6. Let \((M, \phi, N, F) \in MT^\phi_K\) and \(i \in \mathbb{Z}\). The following statements hold.

1. The object \(W_2(M, \phi, N, F)\) is contained in \(MT^\phi_K\).
2. There is an exact sequence

\[
0 \to W_2(i-1)(M, \phi, N, F) \to W_2(M, \phi, N, F) \to \text{gr}^W_{2i}(M, \phi, N, F) \to 0.
\]

Proof. It is sufficient to prove the statement for \((M, \phi, 0, F)\), i.e. for objects in \(MT^\phi_K\).

For (1), it is obvious that

\[
W_2(W_2(i+1)(M, \phi, F)) = W_2(M, \phi, F),
\]

for all \((M, \phi, F)\). Therefore we may reduce to the case

\[
W_2(i+1)(M, \phi, F) = (M, \phi, F).
\]

In this case \(M = M_{\leq i} \oplus M_{i+1}\), and we have to prove that for all \(j \in \mathbb{Z}\) the map

\[
F^j \cap (M_{\leq i} \otimes_{K_0} K) \to (M_{\leq i})_{\geq j} \otimes_{K_0} K
\]

is an isomorphism. Since \((M, \phi, F)\) is an object in \(MT^\phi_K\), the map is injective. In particular, the map is an isomorphism for all \(j \geq i+1\).

We need to show the surjectivity for \(j \leq i\). By assumption, for every \(m \in (M_{\leq i})_{\geq j} \otimes_{K_0} K\) there exists a preimage \(m' \in F^j M_K\). By definition, the projection of \(m'\) to \(M_{i+1} \otimes_{K_0} K\) vanishes, thus \(m' \in F^j \cap (M_{\leq i} \otimes_{K_0} K)\).

For (2), there is an obvious morphism \(W_2(i-1)(M, \phi, F) \to W_2(M, \phi, F)\) in \(MT^\phi_K\). The morphism \(W_2(M, \phi, F) \to \text{gr}^W_{2i}(M, \phi, N, F)\) is defined by the projection \(M_{\leq i} \to M_i\). Since \(F^{j+1} \cap (M_{\leq i} \otimes_{K_0} K) = 0\), the projection is compatible with the filtrations. Therefore the sequence \((1.1.1)\) is well-defined.

In order to prove that the sequence is exact we need to show that it is an exact sequence of \(\phi\)-modules and an exact sequence of filtered \(K\)-vector spaces. The first statement is obvious. For the second statement we note that all members in the sequence \((1.1.1)\) are objects in \(MT^\phi_K\), thus the Hodge polygons equal the Newton polygons. In particular,

\[
\dim(F^j \cap M_{\leq i}) = \dim(F^j \cap M_{\leq i-1}) + \dim \tilde{F}^j,
\]

for all \(j \in \mathbb{Z}\). This immediately implies the claim.

\[\square\]

Corollary 1.1.7. The category \(MT^\phi_K\) is contained in the category of weakly admissible filtered \((\phi, N)\)-modules.

Proof. We use the fact that weakly admissible filtered \((\phi, N)\)-modules are stable under extensions. Therefore the claim follows from Proposition 1.1.6 provided we prove that \(\text{gr}^W_{2i}(M, \phi, N, F)\) is weakly admissible for all \((M, \phi, N, F) \in MT^\phi_K\) and all \(i \in \mathbb{Z}\). By Definition 1.1.5 \(\text{gr}^W_{2i}(M, \phi, N, F)\) is isomorphic to a direct sum of Tate objects \(K(-i)\). Since Tate objects are (weakly) admissible, we are done.

\[\square\]

In contrast to the category \(MF^\phi_K\), the category of weakly admissible filtered \((\phi, N)\)-modules \(MF^\phi_K\) is an abelian category.
Proposition 1.1.8. Let $f : (M, \phi_M, N_M, F_M) \to (M', \phi_{M'}, N_{M'}, F_{M'})$ be a morphism in $MT^\phi_K$. We denote by $\ker(f)$ and $\coker(f)$ the kernel of $f$ and the cokernel of $f$ in $MT^\phi_K,\omega$, respectively. Then $\ker(f)$ and $\coker(f)$ are contained in $MT^\phi_K$. In particular, $MT^\phi_K$ is an abelian category.

Proof. First, consider the full subcategory $C$ of isocrystals over $K_0$ with objects $(M, \phi)$ such that there exists an isomorphism

$$(M, \phi) \cong \bigoplus_{i \in I} (K_0, p^{ni} \sigma).$$

It is easy to see that $C$, as subcategory of the category of isocrystals, contains all the kernels and cokernels of morphisms in $C$.

We denote by $f_0$ the induced morphism $(M, \phi_M) \to (M', \phi_{M'})$. Then

$$\ker(f) = (\ker(f_0), \phi |_{\ker(f_0)}, N |_{\ker(f_0)}, F \cap (\ker(f_0) \otimes K)).$$

We know that $\ker(f_0) \in C$ and thus satisfies Property (1) of Definition 1.1.3. It remains to show that

$$F^i_M \cap (\ker(f_0) \otimes K_0, K) \to \ker(f_0)_{\geq i} \otimes K_0, K$$

is an isomorphism. We have a commutative diagram

$$
\begin{array}{ccc}
0 & \to & F^i_M \cap (\ker(f_0) \otimes K_0, K) \\
& \downarrow & \downarrow \cong \\
0 & \to & \ker(f_0)_{\geq i} \otimes K_0, K
\end{array}
\begin{array}{ccc}
& \to & F^i_M \\
& \downarrow & \downarrow \cong \\
& \to & M_{\geq i} \otimes K_0, K
\end{array}
\begin{array}{ccc}
& \to & M'_{\geq i} \otimes K_0, K.
\end{array}
$$

Moreover, both rows are exact, which implies Property (2) of Definition 1.1.3.

The claim for the cokernel follows dually.

\[\square\]

1.1.9. The categories $MT^\phi_K$ and $MT^\phi_K$ are $\mathbb{Q}_p$-linear rigid $\otimes$-categories.

Lemma 1.1.10. The functor

$$\hat{\omega} : MT^\phi_K \to (\mathbb{Q}_p\text{-vector spaces}), \quad (M, \phi, N, F) \mapsto \bigoplus_{n \in \mathbb{Z}} \hat{\omega}_n(M, \phi, F),$$

with

$$\hat{\omega}_n(M, \phi, F) = \text{Hom}_{MT^\phi_K}(K(n), \text{gr}^W_{-2n}(M, \phi, F)),$$

is a fibre functor. In particular, $(MT^\phi_K, \hat{\omega})$ and $(MT^\phi_K, \hat{\omega})$ are Tannaka categories.

Proof. It is easy to see that $\hat{\omega}$ is a $\otimes$-functor. In order to see that $\hat{\omega}$ is exact and faithful we will prove the existence of an isomorphism

$$\hat{\omega}_{K_0} \cong (\gamma : (M, \phi, N, F) \mapsto M),$$

where $\hat{\omega}_{K_0}(M, \phi, N, F) = \hat{\omega}(M, \phi, N, F) \otimes_{\mathbb{Q}_p} K_0$ and $\gamma$ forgets about $\phi, N$ and $F$. Since $\gamma$ is exact and faithful, this will imply the claim.

In order to construct (1.1.3), we observe that there is a functorial isomorphism

$$\text{Hom}_{MT^\phi_K}(K(n), \text{gr}^W_{-2n}(M, \phi, N, F)) \otimes_{\mathbb{Q}_p} K_0 \to M_{-n},$$

$$\phi \otimes a \mapsto a \cdot \phi(1).$$

\[\square\]
**Proposition 1.1.11.** An object \((M, \phi, N, F) \in MF^{\phi,N,\text{wa}}_K\) belongs to \(MT^{\phi,N}_K\) if and only if there exists an increasing exhaustive separated filtration \(W\) by subobjects of \((M, \phi, N, F) \in MF^{\phi,N,\text{wa}}_K\) such that \(W_i/W_{i-1}\) vanishes if \(i\) is odd, and is a sum of Tate objects \(K(-\frac{i}{2})\) if \(i\) is even.

**Proof.** For \((M, \phi, N, F) \in MT^{\phi,N}_K\), such a filtration exists by Definition 1.1.5, Proposition 1.1.6, and the fact that \(\text{gr}^W_2(M, \phi, N, F)\) is a sum of Tate objects \(K(-\frac{i}{2})\).

Suppose now that \((M, \phi, N, F) \in MF^{\phi,N,\text{wa}}_K\) admits a filtration \(W\) satisfying the assumptions. It is easy to see that \((M, \phi)\) satisfies Property (1) of Definition 1.1.3.

In general, if \(0 \to M_1 \to M \to M_2 \to 0\) is an exact sequence in \(MF^{\phi,N,\text{wa}}_K\), and \(M_1, M_2\) satisfy Property (2), then \(M\) satisfies Property (2). By induction on \(i\) we conclude that \(W_i \in MT^{\phi,N}_K\) for all \(i\).

It is clear that any filtration as in Proposition 1.1.11 has to coincide with the weight filtration, and that any morphism between two objects in \(MT^{\phi,N}_K\) has to be strict with respect to the weight filtrations on these objects.

1.2. The crystalline logarithmic point.

1.2.1. Recall from (1.1.2) that we have a fibre functor \(\tilde{\omega}\) equipping \(MT^\phi_K\) and \(MT^{\phi,N}_K\) with the structure of Tannaka categories (Lemma 1.1.10). Let \(G_{\tilde{\omega}}\) and \(G_{\tilde{\omega}}^{\text{st}}\) denote the pro-algebraic groups which represent tensor automorphisms of \(\tilde{\omega}\) on \(MT^\phi_K\) and \(MT^{\phi,N}_K\), respectively. In other words, we have \(G_{\tilde{\omega}} = \text{Aut}^\otimes_{MT^\phi_K} \tilde{\omega}\) and \(G_{\tilde{\omega}}^{\text{st}} = \text{Aut}^\otimes_{MT^{\phi,N}_K} \tilde{\omega}\). The goal of this section is to construct a non-trivial \(K\)-valued point \(\eta\) of \(G_{\tilde{\omega}}^{\text{st}}\).

**Definition 1.2.2.** For \((M, \phi, F) \in MT^\phi_K\) we define \(\eta(M, \phi, F) : M_K \to M_K\) to be the unique endomorphism rendering the following diagram commutative:

\[
\begin{array}{ccc}
M^{i_1 \oplus K_0} K & \to & M^{i_2 \oplus K_0} K \\
\eta(M, \phi, F) & \downarrow & \downarrow (i_1 \otimes \pi_i)^{-1} \\
M_K & \to & \bigoplus_{i \in \mathbb{Z}} F^i M_K,
\end{array}
\]

where \(i_i : M_i \to M_{i+1}\) is the obvious inclusion, \(\pi_i : F^i M_K \to M_{i+1} \otimes K_0\) is the projection and therefore by definition an isomorphism (Definition 1.1.3.2), and \(\sum_{i \in \mathbb{Z}}\) is the sum over the obvious inclusions.

**Lemma 1.2.3.** The morphisms \(\eta\) from Definition 1.2.2 define a tensor automorphism of the fibre functor \(\tilde{\omega}_K = \tilde{\omega} \otimes_{\mathbb{Q}_p} K\).

**Proof.** Via the \(\otimes\)-isomorphism (1.3.3) we may identify \(\tilde{\omega} \otimes_{\mathbb{Q}_p} K_0\) with the forgetful functor \((M, \phi, F) \to M\). After tensoring with \(K\) we obtain \(\tilde{\omega}_K(M, \phi, F) = M_K\).

First, let us prove that \(\eta(M, \phi, F)\) is an automorphism. We denote by \(\eta(M, \phi, F)[i, j] : M_j \otimes K_0 \to M_i \otimes K_0\)
the the composition with the inclusion $M_i \otimes K \to M_K$ and the projection $M_K \to M_i \otimes K$. It is easy to see from the definitions that

$$\eta(M, \phi, F)[i, j] = \begin{cases} 0 & \text{if } i > j, \\ id_{M_i} & \text{if } i = j. \end{cases}$$

Therefore $\eta(M, \phi, F)$ is an automorphism.

Since the diagram (1.2.1) is functorial, $\eta$ defines a natural transformation. The compatibility with the tensor product is obvious. \qed

1.2.4. Let us explain the construction of $\eta$ in the formalism of [Del94]. For $(M, \phi, F) \in M_{\phi}^{\phi}$ there are three filtrations on $M_K$:

1. the weight filtration:

$$W_i M_K = \begin{cases} M_{\geq \frac{i}{2}} \otimes K & \text{if } i \text{ is even} \\ M_{\frac{i}{2} + \frac{1}{2}} \otimes K & \text{if } i \text{ is odd.} \end{cases}$$

2. The Hodge filtration $F$.

3. The filtration

$$F^i M_K := M_{\geq i} \otimes K \quad \text{for all } i \in \mathbb{Z}. $$

The three filtration $W, F, \bar{F}$ satisfy the condition

$$\text{Gr}^i_F \text{Gr}^j_W M_K = 0 \quad \text{for } n \neq p + q,$n

of [Del94] §1.1. Induced by $F, \bar{F}$, we obtain maps

$$a_F : M_K = \bigoplus_{i \in \mathbb{Z}} F^i \cap W_{2i} \to \bigoplus_{i \in \mathbb{Z}} \text{Gr}^W_{2i} M_K,$n$$

$$a_{\bar{F}} : M_K = \bigoplus_{i \in \mathbb{Z}} \bar{F}^i \cap W_{2i} \to \bigoplus_{i \in \mathbb{Z}} \text{Gr}^W_{2i} M_K,$n

where $F^i \cap W_{2i} \to \text{Gr}^W_{2i} M_K$ is the natural map (and similarly for $\bar{F}$). We obtain a unipotent automorphism $d = a_F a_{\bar{F}}^{-1}$ of $\bigoplus_{i \in \mathbb{Z}} \text{Gr}^W_{2i} M_K$.

It is easy to see that we have the equality

$$\eta(M, \phi, F) = a_{\bar{F}}^{-1} \circ d \circ a_F.$n

1.2.5. Let us see in explicit terms how $\eta$ compares the crystalline structure with the Hodge filtration. For $(M, \phi, F) \in M_{\phi}^{\phi}$ we say that $v_1, \ldots, v_d \in \tilde{\omega}(M, \phi, F) \otimes_{\mathbb{Q}_p} K$ is a homogeneous basis if it is a basis of $\tilde{\omega}(M, \phi, F) \otimes_{\mathbb{Q}_p} K$, and for every $v_i$ there is an integer $n_i$ with $v_i \in \tilde{\omega}_{n_i}(M, \phi, F) \otimes_{\mathbb{Q}_p} K$; we set $\text{deg}(v_i) = n_i$.

Recall that for all integers $i$ we have isomorphisms

$$a_i : M_{-i} \otimes K \to \tilde{\omega}_i(M, \phi, F) \otimes_{\mathbb{Q}_p} K,$n$$

$$b_i : F^{-i} \cap W_{-2i} M_K \to \tilde{\omega}_i(M, \phi, F) \otimes_{\mathbb{Q}_p} K.$n

The first map is the inverse of (1.1.3) and the second map is given by the composition

$$b_i : F^{-i} \cap W_{-2i} M_K \to M_K \xrightarrow{\text{projection}} M_{-i} \otimes K \to \tilde{\omega}_i(M, \phi, F) \otimes_{\mathbb{Q}_p} K.$n

For a homogeneous basis $\{v_j\}$ we set

$$v_j^{\text{cryst}} := a_{\text{deg}(v_j)}^{-1}(v_j), \quad v_j^{\text{Hodge}} := b_{\text{deg}(v_j)}^{-1}(v_j).$$
We denote by \( \{ v_j^{\text{crys, } V} \} \) the basis dual to \( \{ v_j^{\text{crys}} \} \). By definition of \( \eta \) we have
\[
v_j^{\text{crys, } V}(\eta(v_j)) = v_j^{\text{crys, } V}(v_j^{\text{Hodge}}).
\]

1.2.6. By Lemma 1.2.3 we obtain a \( K \)-valued point \( \eta \in G_{\omega}(K) \); we call this point the logarithmic point. Let us check that \( \eta \) is not the identity.

**Proposition 1.2.7.** Let \( n \in \mathbb{Z} \) be an integer. We have
\[
\text{Ext}^1_{MT_K}(K(0), K(n)) \cong \begin{cases} K & \text{if } n > 0 \\ 0 & \text{if } n \leq 0. \end{cases}
\]

Let
\[
0 \to K(n) \xrightarrow{\iota} (E, \phi, F) \xrightarrow{\pi} K(0) \to 0
\]
be an extension. For \( n \neq 0 \), there are unique sections \( f : E \to K_0 \) and \( v : K_0 \to E \) of the underlying maps of \( K_0 \)-isocrystals of \( \iota \) and \( \pi \), respectively. The isomorphism \( \text{Ext}^1_{MT_K}(K(0), K(n)) \), for \( n \neq 0 \), is given by the formula
\[
E \mapsto f(\eta(E, \phi, F)(v(1))).
\]

**Proof.** First, we consider the case \( n = 0 \). Let \( (E, \phi, F) \) be as in (1.2.4). We have \( F^0(E_K) = E_K \) and \( F^1(E_K) = 0 \) by Definition 1.1.3(2). In view of Definition 1.1.3(2) there is an isomorphism \( (E, \phi) \cong (K_0, \sigma) \oplus (K_0, \sigma) \), thus there is a section of \( \pi \) in \( MT_K^0 \).

For \( n \neq 0 \): From the slope decomposition we obtain natural sections \( f, v \) as \( \phi \)-modules. If \( n < 0 \) then \( F^1E_K = \iota(K) \) which means \( (E, \phi, F) = K(0) \oplus K(n) \).

For \( n > 0 \), we can uniquely write \( F^{n+1}E_K = K(a \cdot \iota(1) + v(1)) \) with \( a \in K \). Obviously,
\[
f(\eta(E, \phi, F)(v(1))) = a.
\]

It is clear that \( F^{n+1}E_K \) is the only invariant for extensions. \( \square \)

1.2.8. Recall that we have a fibre functor \( \bar{\omega} \) to the category of \( \mathbb{Q}_p \)-vector spaces. In the obvious way \( \bar{\omega} \) factors through the category of graded \( \mathbb{Q}_p \)-vector spaces. Furthermore, we have an automorphism \( \eta \) of \( \bar{\omega}_K \) (Lemma 1.2.3).

**Definition 1.2.9.** We define \( \mathcal{C}_\eta \) to be the category of pairs \( (V, \eta) \), where \( V \) is a finite dimensional graded \( \mathbb{Q}_p \)-vector space and \( \eta : V \otimes K \to V \otimes K \) is a \( K \)-linear map such that for all \( n \in \mathbb{Z} \):
\[
(\eta - id)(V_n \otimes K) \subset \bigoplus_{i > n} V_i \otimes K.
\]

Morphisms \( (V_1, \eta_1) \to (V_2, \eta_2) \) are \( \mathbb{Q}_p \)-linear morphisms \( \tau : V_1 \to V_2 \) which respect the grading and commute with the endomorphisms \( \eta_i \), i.e. \( \eta_2 \circ (\tau \otimes id_K) = (\tau \otimes id_K) \circ \eta_1 \).

The category \( \mathcal{C}_\eta \) is a \( \otimes \)-category with
\[
(V_1, \eta_1) \otimes (V_2, \eta_2) = (V_1 \otimes V_2, \eta_1 \otimes \eta_2).
\]

**Proposition 1.2.10.** The functor
\[
\Psi : MT_K^0 \to \mathcal{C}_\eta
\]
\[
(M, \phi, F) \mapsto \left( \bigoplus_{n \in \mathbb{Z}} \omega_n(M, \phi, F), \eta(M, \phi, F) \right)
\]
is an equivalence of \( \otimes \)-categories.

**Proof.** By Lemma 1.2.3, \( \eta \) is functorial and \( \Psi \) is a \( \otimes \)-functor. It follows from (1.2.2) that

\[
(\eta - id)(\tilde{\omega}_n \otimes K) \subset \bigoplus_{i > n} \tilde{\omega}_i \otimes K.
\]

We define a functor

\[
\Phi : C_{\eta} \to MT_{K}^{\phi}
\]

\[
(\oplus_{n \in \mathbb{Z}} V_n, \eta) \mapsto \left( \oplus_{n \in \mathbb{Z}} (V_{-n} \otimes_{\mathbb{Q}_p} K_0, p^{-n} \otimes \sigma), F \right),
\]

with the following filtration:

\[
F^i := \eta \left( \bigoplus_{j \geq i} V_{-j} \otimes_{\mathbb{Q}_p} K \right),
\]

for all \( i \). Property (1.2.3) implies that \( \Phi \) is well-defined. From Definition 1.2.2 it easily follows that \( \Psi \circ \Phi = id_{C_{\eta}} \).

On the other hand, we have \( \Phi \circ \Psi \cong id_{MT_{K}^{\phi}} \) via

\[
\Phi \circ \Psi(M, \phi, F) \to (M, \phi, F)
\]

\[
\bigoplus_{n \in \mathbb{Z}} \tilde{\omega}_{-n}(M, \phi, F) \otimes_{\mathbb{Q}_p} K_0 \quad \text{via (1.1.4)} \to M.
\]

\[\square\]

**Remark 1.2.11.** Via the dictionary of Section 1.2.4, Proposition 1.2.10 is a variant of [Del94, Proposition 1.2].

1.3. The semistable logarithmic point.

1.3.1. Let \( K \) be as in \( \text{(1.1.1)} \) with residue field \( k \). We denote by \( \nu_K \) the valuation of \( K \).

1.3.2. Recall that we have a homomorphism

\[
[.] : k^\times \to \mathcal{O}_K^\times, \quad x \mapsto [x],
\]

by taking the Teichmüller lift. Denoting by \( U_K := \{ x \in \mathcal{O}_K^\times; x \in 1 + m_K \} \) the 1-units, we obtain a decomposition

\[
\mathcal{O}_K^\times = k^\times \times U_K.
\]

The logarithm

\[
(1.3.1) \quad \log : \mathcal{O}_K^\times \to \mathcal{O}_K
\]

is by definition trivial on the factor \( k^\times \) and is given by

\[
\log(u) = \sum_{n \geq 1} (-1)^{n+1} \frac{(u - 1)^n}{n} \quad \text{for all } u \in U_K.
\]
1.3.3. We consider \( \mathcal{O}_K^\times \otimes \mathbb{Z} \mathbb{Q} \) and \( K_\mathbb{Q}^\times := \mathcal{O}_K^\times \otimes \mathbb{Z} \mathbb{Q} \) as \( \mathbb{Q} \)-vector spaces, therefore we may form the symmetric algebras \( \text{Sym}_\mathbb{Q}(\mathcal{O}_K^\times, \mathbb{Q}) \) and \( \text{Sym}_\mathbb{Q}(K_\mathbb{Q}^\times) \). The exact sequence
\[
0 \to \mathcal{O}_K^\times \to K_\mathbb{Q}^\times \nu_K \to \mathbb{Q} \to 0
\]
implies that \( \text{Spec}(\text{Sym}_\mathbb{Q}(K_\mathbb{Q}^\times)) \) is a 1-dimensional affine space over the scheme \( \text{Spec}(\text{Sym}_\mathbb{Q}(\mathcal{O}_K^\times)) \). In other words, for \( x \in K_\mathbb{Q}^\times \) with \( \nu_K(x) \neq 0 \), the map
\[
\text{Sym}_\mathbb{Q}(\mathcal{O}_K^\times)[X] \to \text{Sym}_\mathbb{Q}(K_\mathbb{Q}^\times), \quad X \mapsto x,
\]
is an isomorphism.

The logarithm (1.3.1) induces a ring homomorphism
\[
\text{Sym}_\mathbb{Q}(\mathcal{O}_K^\times) \to K,
\]
because \( K \) is torsion free.

**Definition 1.3.4.** We define the \( K \)-algebra \( K_{st} \) by
\[
K_{st} := \text{Sym}_\mathbb{Q}(K_\mathbb{Q}^\times) \otimes \text{Sym}_\mathbb{Q}(\mathcal{O}_K^\times) K.
\]

By base change, \( \text{Spec}(K_{st}) \) is a 1-dimensional affine space over \( K \). We have a natural logarithm
(1.3.3) \[
\log_{st} : K_\mathbb{Q}^\times \to K_{st}, \quad x \mapsto x \otimes 1.
\]
The \( K \)-valued points of \( \text{Spec}(K_{st}) \) admit the following description:
(1.3.4) \[
\text{Spec}(K_{st})(K) = \{ \text{extensions } \log : K_\mathbb{Q}^\times \to K \text{ of (1.3.1)} \}
\]
\[
f \mapsto f^* \circ \log_{st}.
\]

By an extension \( \log : K_\mathbb{Q}^\times \to K \) we mean a homomorphism such that the restriction to \( \mathcal{O}_K^\times \) equals (1.3.1).

1.3.5. The \( p \)-adic Hodge theory for \( K \) (and fixed valuation \( \nu_K \)) depends for semistable representations on the choice of a logarithm
\[
\log : K_\mathbb{Q}^\times \to K.
\]
It will be important for us that our constructions do not depend on a particular choice, and for this we have to recall the basic constructions of \( p \)-adic Hodge theory.

We denote by \( R \) the ring
\[
R := \varprojlim \mathcal{O}_K/p\mathcal{O}_K,
\]
where the maps are given by rising to the \( p \)-th power \( x \mapsto x^p \). Denoting by \( C_K = \hat{K} \) the \( p \)-adic completion of \( K \), we have a multiplicative bijection
\[
\varprojlim \mathcal{O}_C \to R,
\]
where the projective system is defined by rising to the \( p \)-th power again. In other words, we can represent every element \( x \) in \( R \) by \( (x^{(0)}, x^{(1)}, \ldots) \) with \( x^{(n)} \in \mathcal{O}_C \) and \( x^{(n-1)} = (x^{(n)})^p \).

Let \( \nu_K \) (resp. \( \nu_C \)) be the extension of \( \nu_K \) (resp. \( \nu_K \)) to \( \hat{K} \) (resp. \( C_K \)). The map
\[
\nu_R : R \setminus \{0\} \to \mathbb{Q}, \quad x \mapsto \nu_C(x^{(0)})
\]
can be extended to a valuation
\[
\nu_R : \text{Frac}(R)^\times \to \mathbb{Q}
\]
with valuation ring $R$.

Let $B_{cris}$ be the crystalline period ring; we define

$$B_{st} = \text{Sym}_Q(\text{Frac}(R) \otimes \mathbb{Z} \otimes \text{Sym}_Q(R^\times \otimes \mathbb{Q}) B_{cris},$$

where $\text{Sym}_Q(R^\times \otimes \mathbb{Q}) \to B_{cris}$ is induced by the crystalline logarithm

$$\log_{cris} : R^\times \to B_{cris}.$$  

Again, $R^\times = \bar{k}^\times \times (1 + m_R); \log_{cris}$ is trivial on $\bar{k}^\times$ and given by

$$\log_{cris}(u) = \sum_{n \geq 1} (-1)^{n+1} \frac{([u] - 1)^n}{n}$$

for $u \in 1 + m_R$, where $[u]$ denotes the Teichmüller lift of $u$ in the Witt ring $W(R)$ of $R$.

By construction we have a natural logarithm

$$\log_{st} : \text{Frac}(R) \to B_{st}, \ x \mapsto x \otimes 1.$$  

The ring $B_{st}$ has the following properties.

1. We have a $\text{Gal}(\bar{K}/K)$-action on $B_{st}$ extending the action on $B_{cris}$.
2. We have a Frobenius map $\phi : B_{st} \to B_{st}$ extending the Frobenius map on $B_{cris}$. Moreover,

$$\phi \circ \log_{st} = p \log_{st}.$$  

3. We have a $B_{cris}$-linear derivation $N : B_{st} \to B_{st}$ such that

$$N(\log_{st}(x)) = \nu_R(x) \quad \text{for all } x \in \text{Frac}(R)^\times.$$  

After choosing a logarithm

$$\log : K^\times \to K,$$

which extends (1.3.4), we obtain a morphism of $B_{cris}$-algebras

$$\gamma_{log} : B_{st} \to B_{dR}.$$  

The morphism depends on the choice of log, and the filtration on $B_{st}$ induced by the filtration on $B_{dR}$ via $\gamma_{log}$ depends on log. In order to simplify the comparison between different logarithms we will restrict ourselves to logarithms log such that $\log(K_0^\times) \subset K_0$. In other words, we will only consider $K_0$-valued points of $\text{Spec}(K_{0,st})$.

**Proposition 1.3.6.** For $\log, \log' \in \text{Spec}(K_{0,st})(K_0)$ there is a unique ring homomorphism

$$\delta_{log,log'} : B_{st} \to B_{st}$$

such that $\gamma_{log} \circ \delta_{log,log'} = \gamma_{log}$. The map $\delta_{log,log'}$ is given by

$$\delta_{log,log'} = \exp \left( \frac{\log(x) - \log'(x)}{\nu_K(x)} N \right)$$

for every $x \in K_0 \setminus O_{K_0}^\times$.

**Proof.** Uniqueness follows from the fact that $\gamma_{log}$ is injective.

Choose $\bar{p} \in R$ with $\bar{p}^{(0)} = p$. By definition we have

$$\gamma_{log}(\log_{st}(\bar{p})) = \log_{dR}(\bar{p}\bar{p})/p + \log(p),$$
where \( \log_{dR} \) is defined by the usual series since \([\bar{p}] / p \) is a 1-unit in \( B_{dR} \). Since \( \text{Spec}(B_{st}) \) is a 1-dimensional affine space over \( \text{Spec}(B_{cris}) \), there exists a unique morphism of \( B_{cris} \)-algebras \( \delta_{\log, \log'} \) such that
\[
\delta_{\log, \log'}(\log_{st}(\bar{p})) = \log_{st}(\bar{p}) + \log(p) - \log'(p).
\]
Obviously, \( \delta_{\log, \log'} \) satisfies \( \gamma_{\log} \circ \delta_{\log, \log'} = \gamma_{\log} \) and the equality \([1.3.5]\). \( \square \)

By using \( \gamma_{\log} \) we obtain a filtration on \( B_{st} \). The \( p \)-adic Hodge theory \([CF00]\) Thm. A] asserts that the functor \( \overline{\text{forget}} \) satisfies
\[
\overline{\text{forget}}(\mathcal{M}, \phi, N, F) = (\mathcal{M}, \phi, N),
\]
implies that for the filtrations we have the following comparison:
\[
\overline{\text{forget}}_P \circ \delta_{\log} = \delta_{\log} \circ \overline{\text{forget}}_P,
\]
\( \overline{\text{forget}}_P \) being the full subcategory of \( \text{st} \)-adic \( \mathbb{Q}_p \)-representations of \( \text{Gal}(\bar{K}/K) \) which admit an increasing exhaustive filtration \( W \) by subrepresentations of \( V \) such that \( W_i/W_{i-1} \) vanishes if \( i \) is odd, and is a sum of Tate objects \( \mathbb{Q}_p(\frac{-i}{2}) \) if \( i \) is even. We call an object of \( \text{MT}_{G_K} \) a mixed Tate representation of \( \text{Gal}(\bar{K}/K) \).

**Proposition 1.3.8.** Let \( \log \in \text{Spec}(K_{0, st})(K_0) \). Then
\[
\text{MT}_{G_K} = D_{\text{st, log}}^{-1}(\mathcal{M}^\phi_{K, W, a}).
\]
In particular, every mixed Tate representation is semistable.

**Proof.** From Proposition 1.1.11 it follows that every object in \( D_{\text{st, log}}^{-1}(\mathcal{M}^\phi_{K, W, a}) \) admits a filtration \( W \) satisfying the properties of Definition 1.3.7.

Now, suppose that \( V \) is a \( p \)-adic representation of \( \text{Gal}(\bar{K}/K) \) which admits a filtration \( W \) as in Definition 1.3.7. If we know that \( V \) is semistable then clearly \( D_{\text{st, log}}(V) \in \mathcal{M}^\phi_{K, W, a} \) by Proposition 1.1.11 again. Therefore it suffices to prove that \( V \) is semistable.

We use induction on the length of the filtration \( W \) of \( V \). If the filtration \( W \) has length \( \leq 1 \), semistability of \( V \) follows from those of \( \mathbb{Q}_p(n) \). In general, let \( n \) be the smallest integer such that \( W_{2n} V = V \). Then we have an exact sequence
\[
0 \to (W_{2n-2} V) \otimes \mathbb{Q}_p(n) \to V \otimes \mathbb{Q}_p(n) \to (V/W_{2n-2} V) \otimes \mathbb{Q}_p(n) \to 0.
\]
By the induction hypothesis the terms on the left and right are semistable. Moreover, since the weights of the term on the left are \( \leq -2 \) and the term on the right

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has weight 0, we have

\[ F^0_dR((W_{2n-2}V) \otimes \mathbb{Q}_p(n)) = 0 \]

\[ F^0_dR((V/W_{2n-2}V) \otimes \mathbb{Q}_p(n)) = D_{dR}((V/W_{2n-2}V) \otimes \mathbb{Q}_p(n)). \]

Therefore [Nek93, Proposition 1.28] shows that the middle term is also semistable. □

Obviously,

\[ \tau = \tilde{\omega} \circ D_{st,\log} \]

is independent of \( \log \), and \((MT_{G_K}, \tau)\) is a Tannaka category (by Lemma 1.1.10).

1.3.9. Recall from Lemma 1.1.10 that \((MT_{K^0}, \tilde{\omega})\) is a Tannaka category. We will use the ring \( K_{st} \) (Definition 1.3.4) and \( \log_{st} \) (1.3.3).

**Definition 1.3.10.** For a logarithm \( \log \in \text{Spec}(K_{0, st})(K_0) \) and \((M, \phi, N, F) \in MT_{K^0,N} \) we define \( \eta_{st, \log}(M, \phi, N, F) \in \text{End}_{K_{st}}(M \otimes_{K_0} K_{st}) \) by

\[ \eta_{st, \log}(M, \phi, N, F) := \exp \left( \frac{\log(x) - \log_{st}(x)}{\nu_K(x)} N \right) \eta(M, \phi, F), \]

for \( x \in K_0^1 \otimes_{\mathcal{O}_{K_0}} \mathbb{Q}_p \). For the definition of \( \eta(M, \phi, F) \) we refer to Definition 1.2.2.

Obviously, \( \eta_{st, \log} \) does not depend on the choice \( x \in K_0^1 \otimes_{\mathcal{O}_{K_0}} \mathbb{Q}_p \), but it depends on \( \log \).

**Lemma 1.3.11.** Let \( \log \in \text{Spec}(K_{0, st})(K_0) \). The morphisms \( \eta_{st, \log} \) from Definition 1.3.10 define a tensor automorphism of the fibre functor \( \tilde{\omega}_{K_{st}} = \tilde{\omega} \otimes_{\mathbb{Q}_p} K_{st} \). In other words, \( \eta_{st, \log} \in G_{st}(K_{st}) \) with \( G_{st} = \text{Aut}_{MT_{K^0,N}} \).

**Proof.** Via the \( \otimes \)-isomorphism (1.1.3) we may identify \( \tilde{\omega} \otimes_{\mathbb{Q}_p} K_0 \) with the forgetful functor \((M, \phi, N, F) \mapsto M\). After tensoring with \( K_{st} \) we obtain \( \tilde{\omega}_{K_{st}}(M, \phi, N, F) = M \otimes_{K_0} K_{st} \). Lemma 1.2.3 implies that \( \eta(M, \phi, F) \) is a tensor automorphism, thus it suffices to prove the statement for \( \exp \left( \frac{\log(x) - \log_{st}(x)}{\nu_K(x)} N \right) \). The functoriality follows immediately. The compatibility with the \( \otimes \)-structure follows from

\[ N_{M_1 \otimes M_2} = N_{M_1} \otimes 1 + 1 \otimes N_{M_2}. \]

□

**Lemma 1.3.12.** The \( K_{st} \)-valued point

\[ \eta_{st} = \eta_{st, \log} \circ D_{st, \log} \]

of \( \text{Aut}_{MT_{G_K}} \) \( \tau \) is independent of the choice of \( \log \in \text{Spec}(K_{0, st})(K_0) \).

**Proof.** Let \( \log, \log' \in \text{Spec}(K_{0, st})(K_0) \) and \( V \in MT_{G_K} \). In view of (1.3.7) we get

\[ \eta(\text{forget}_N D_{st, \log}(V)) = \exp \left( \frac{\log(x) - \log'(x)}{\nu_K(x)} N \right) \eta(\text{forget}_N D_{st, \log}(V)), \]

(1.3.9)
for very $x \in K_{0}\backslash \mathcal{O}_{K_{0}}^\times$, and forget$_N(M, \phi, N, F) = (M, \phi, F)$. Thus
\[
\eta_{st, \log'}D_{st, \log}^\prime(V) = \exp\left(\frac{\log'(x) - \log_{st}(x)}{\nu_K(x)}N\right)\eta(\text{forget}_N D_{st, \log}(V)) \\
= \exp\left(\frac{\log(x) - \log_{st}(x)}{\nu_K(x)}N\right)\eta(\text{forget}_N D_{st, \log}(V)) \quad \text{by (1.3.9)} \\
= \eta_{st, \log}D_{st, \log}(V).
\]
\[\square\]

**Example 1.3.13.** By Kummer theory any $q \in K^\times$ defines an extension $V$ of the $\text{Gal}(\bar{K}/K)$-representation $\mathbb{Q}_p(0)$ by $\mathbb{Q}_p(1)$. This in turn gives via $D_{st, \log}$ an extension of $K(0)$ by $K(1)$ in $MT^\phi_N$: $$0 \to K(1) \to M \to K(0) \to 0,$$
which may be described as follows. The underlying $K_0$-space of $M$ has a basis $e_0, e_1$ such that the following conditions are satisfied:

1. the action of $\phi$ is given by $\phi(e_i) = p^{-i}e_i$ for $i = 0, 1$,
2. $e_1$ is the image of $1 \in K(1)$,
3. $e_0$ maps to $1 \in K(0)$.

The filtration is given by $F^{-1}M_K = M_K$, $F^0M_K = K \cdot \langle \log(q)e_1 + e_0 \rangle$ and $F^1M_K = 0$. Finally $N$ is given by $Ne_0 = -\nu_K(q) \cdot e_1$ and $Ne_1 = 0$. Then we easily compute
\[
\eta_{st}(V) = \begin{pmatrix}
1 & 0 \\
-\nu_K(q)(\log(x) - \log_{st}(x)) & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\log(q) & 1
\end{pmatrix}
\]
for the obvious basis of
\[
\tau(V) = \text{Hom}(\mathbb{Q}(0), \mathbb{Q}(0)) \oplus \text{Hom}(\mathbb{Q}(1), \mathbb{Q}(1)),
\]
and every $x \in K_{0}\backslash \mathcal{O}_{K_{0}}^\times$, or, equivalently, for every $x \in K\backslash \mathcal{O}_{K}^\times$. If $\nu_K(q) \neq 0$ then we can take $q = x$ in order to see that
\[
\eta_{st}(V) = \begin{pmatrix}
1 & 0 \\
\log_{st}(q) & 1
\end{pmatrix}
\]
holds for all $q \in K^\times$.

**1.4. Tannaka group scheme of mixed Tate filtered $\phi$-modules.**

**Definition 1.4.1.** We define $\mathcal{L}$ to be the graded $\mathbb{Q}_p$-Lie algebra freely generated by $K^\times$ (i.e. the dual of $K$, where $K$ is considered as $\mathbb{Q}_p$-vector space) in each degree $i > 0$. In other words, $\mathcal{L}$ is defined by
\[\text{(1.4.1)}\]
\[
\text{Hom}_{\text{(graded $\mathbb{Q}_p$-Lie algebras)}}(\mathcal{L}, \mathcal{T}) = \text{Hom}_{\text{(graded $\mathbb{Q}_p$-vector spaces)}}(\oplus_{i \in \mathbb{Z}_{>0}} K^\times, \mathcal{T}),
\]
for every graded $\mathbb{Q}_p$-Lie algebra $\mathcal{T}$. Via (1.4.1) we get $\mathbb{Q}_p$-linear maps
\[
a_i : K^\times \to \mathcal{L}_i, \quad \text{for all } i > 0.
\]

Obviously, $\mathcal{L}$ is concentrated in positive degrees and each $\mathcal{L}_i$ is a finite dimensional $\mathbb{Q}_p$-vector space.
\textbf{Definition 1.4.2.} For all $n > 0$, we define a $\mathbb{Q}_p$-Lie algebra $\mathcal{L}_{\leq n}$ by

$$
\mathcal{L}_{\leq n} = \mathcal{L} / \left( \oplus_{i > n} \mathcal{L}_i \right).
$$

We set $\hat{\mathcal{L}} = \varprojlim_n \mathcal{L}_{\leq n}$. For every field extension $K \supseteq \mathbb{Q}_p$ we define $\hat{\mathcal{L}}_K := \varprojlim_n \left( \mathcal{L}_{\leq n} \otimes_{\mathbb{Q}_p} K \right)$.

We will be only interested in the finite dimensional graded representations of $\mathcal{L}$, which can be identified with the finite dimensional graded representations of $\hat{\mathcal{L}}$.

1.4.3. There is natural element $\epsilon \in \hat{\mathcal{L}}_K$ defined as follows:

$$
\epsilon := \sum_{i > 0} (a_i \otimes \text{id}_K)(\text{id}),
$$

with $\text{id} \in K^\vee \otimes_{\mathbb{Q}_p} K$ the canonical element. After choosing a $\mathbb{Q}_p$-basis $v_1, \ldots, v_d$ of $K$, we see that

$$
\epsilon = \sum_{i > 0} \sum_{j=1}^d a_i (v_j^\vee) \otimes v_j.
$$

Let $V$ be a finite dimensional graded representation of $\mathcal{L}$ then $\exp(\epsilon)$ is a unipotent automorphism of $V \otimes_{\mathbb{Q}_p} K$.

\textbf{Proposition 1.4.4.} The $\otimes$-functor

$$
\Psi : (\text{finite dim. graded } \mathcal{L}\text{-modules}) \to \mathcal{C}_\eta
$$

$$
V \mapsto (V, \exp(\epsilon))
$$

is an equivalence of categories.

\textit{Proof.} Note first that $\Psi$ is well-defined, because $(\exp(\epsilon) - \text{id})(V_n) \subset \oplus_{i > n} V_i$, for all $n$, thus (1.2.5) is satisfied.

Let us prove that $\Psi$ is essentially surjective. Let $(\oplus_n V_n, \eta)$ be an object of $\mathcal{C}_\eta$. Since $1 - \eta$ is nilpotent we can define

$$
\tilde{\epsilon} = \log(\eta) = \log(1 - (1 - \eta)).
$$

For every $i > 0$ we define a $\mathbb{Q}_p$-linear map $\beta_i : K^\vee \rightarrow \text{End}(V)_i$ by

$$
\beta_i(f) = \sum_n (id_{V_{n+i}} \otimes f) \circ \text{proj}_{V_{n+i} \otimes K} \circ \tilde{\epsilon} \circ \text{incl}_{V_n},
$$

where $\text{incl}_{V_n} : V_n \rightarrow V$ and $\text{proj}_{V_{n+i} \otimes K} : V \otimes K \rightarrow V_{n+i} \otimes K$ is the inclusion and the projection, respectively. Via (1.4.1) we obtain a graded representation $\rho : \mathcal{L} \rightarrow \text{End}(V)$. We need to show that $\exp(\rho(\epsilon)) = \eta$, or equivalently $\rho(\epsilon) = \tilde{\epsilon}$. This is a straightforward computation which we leave to the reader.

Next, we need to prove that $\Psi$ is fully faithful. Clearly, $\Psi$ is faithful. Let $V, U$ be two graded representations of $\mathcal{L}$, and suppose $\tau : V \rightarrow U$ is a morphism which respects the grading and commutes with $\exp(\epsilon)$. Then $\tau$ commutes with $\epsilon$. Fix a
basis \( v_1, \ldots, v_d \) of \( K \). For \( v \in V \) we get

\[
\tau v = \tau(\sum_{i>0} \sum_j a_i(v_j) \otimes v_j) = \sum_{i>0} \sum_j \tau(a_i(v_j))(v) \otimes v_j, \\
\epsilon \tau = \sum_{i>0} \sum_j a_i(v_j) \otimes v_j.
\]

Therefore \( \tau \circ a_i(v_j) = a_i(v_j) \circ \tau \) for all \( i, j \). Since \( L \) is generated by the elements \( \{a_i(v_j)\}_{i,j} \) we see that \( \tau \) is a morphism of \( L \)-representations.

**Corollary 1.4.5.** There is an equivalence of \( \otimes \)-categories

\[
\Theta : \text{MT}^\phi_K \to (\text{finite dim. graded } L\text{-modules})
\]

such that

- \( \text{forg} \circ \Theta = \tilde{\omega} \), where \( \text{forg} \) forgets about the \( L \)-action.
- \( \exp(\epsilon) |_{\Theta(M,\phi,F)} = \eta(M,\phi,F) \).

**Proof.** Follows immediately from Proposition 1.2.10 and Proposition 1.4.4. \( \square \)

**Corollary 1.4.6.** Let \( U \) be the pro-algebraic group \( U = \lim_{\leftarrow n} \exp(L/(\oplus_{i>n} L_i)) \).

Let \( G_{\tilde{\omega}} \) be the Tannaka group attached to the fibre functor \( \tilde{\omega} \) (Lemma 1.1.10).

There is an isomorphism \( G_{\tilde{\omega}} \cong \mathbb{G}_m \ltimes U \)

such that \( \eta \in G_{\tilde{\omega}}(K) \) corresponds to \( \exp(\epsilon) \in U(K) \).

**Proof.** In view of Corollary 1.4.5 the statement follows from the fact that \( \mathbb{G}_m \ltimes U \) is the Tannaka group of the fibre functor:

\[
\text{forg} : (\text{graded finite dimensional } L\text{-modules}) \to (\mathbb{Q}_p\text{-vector spaces}).
\]

The action of \( \mathbb{G}_m \) on \( U \) is induced by the action of \( \mathbb{G}_m \) on \( L \) given by the grading:

\[
\mathbb{G}_m \times L_i \to L_i; \quad (t,x) \mapsto t^i \cdot x.
\]

\( \square \)

2. **Mixed Tate motives over a number field and logarithmic points**

2.1. **Mixed Tate motives.**

2.1.1. Let \( E \) be a number field and \( S \) a set of finite places. Let \( \mathcal{O} \) be the ring of integers of \( E \), and \( |\text{Spec}(\mathcal{O})| \) the maximal spectrum of \( \mathcal{O} \). We denote by

\[
\mathcal{O}_S := \bigcap_{x \in \text{Spec}(\mathcal{O}) \setminus S} \mathcal{O}_x
\]

the ring of \( S \)-integers of \( E \); the elements of \( \mathcal{O}_S \) are integral outside of \( S \). We will be mainly interested in two cases for \( S \). In the first case, we have \( S = |\text{Spec}(\mathcal{O})| \) and \( \mathcal{O}_S = E \). In the second case, we have \( S = |\text{Spec}(\mathcal{O})| \setminus \{x\} \), for a point \( x \in |\text{Spec}(\mathcal{O})| \), and \( \mathcal{O}_S = \mathcal{O}_x \) is the local ring at \( x \).
2.1.2. Deligne and Goncharov defined in [DG05, 1.6] an abelian category of mixed Tate motives \( \text{MT}(\mathcal{O}_S) \). By definition it is the full subcategory of \( \text{MT}(E) \) consisting of objects which are unramified outside \( S \) in the following sense. Let \( x \in \{|\text{Spec}(\mathcal{O})|\} \) be a point lying over a prime \( p \); then we say that \( M \in \text{MT}(E) \) is unramified at \( x \) if for all primes \( \ell \neq p \) the corresponding Galois representation \( M_\ell \) is unramified at \( x \), i.e. the inertia subgroup \( I_x \) (which is only well-defined up to conjugation) acts trivially at \( M_\ell \)[DG05, Proposition 1.8].

2.1.3. For extensions of Tate objects we know that:

\[
\text{Ext}^1_{\text{MT}(\mathcal{O}_S)}(\mathbb{Q}(0), \mathbb{Q}(1)) = \mathcal{O}_S^* \otimes \mathbb{Q},
\]

\[
\text{Ext}^1_{\text{MT}(\mathcal{O}_S)}(\mathbb{Q}(0), \mathbb{Q}(n)) = \begin{cases} 0 & \text{if } n \leq 0, \\ \text{Ext}^1_{\text{MT}(E)}(\mathbb{Q}(0), \mathbb{Q}(n)) & \text{if } n \neq 1, \\ \end{cases}
\]

\[
\text{Ext}^2_{\text{MT}(\mathcal{O}_S)}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0 \quad \text{for all } n \in \mathbb{Z},
\]

(see [DG05, Proposition 1.9]).

2.1.4. Every object of \( \text{MT}(\mathcal{O}_S) \) comes equipped with a finite increasing functorial weight filtration, indexed by even integers. For all \( n \in \mathbb{Z} \) the graded pieces \( \text{gr}_{W_n}(M) \) are sums of copies of \( \mathbb{Q}(-n) \).

In view of [DG05, 1.1] the \( \otimes \)-functor

\[
\omega : \text{MT}(\mathcal{O}_S) \to (\mathbb{Q}\text{-vector spaces}),
\]

\[
\omega(M) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}(\mathbb{Q}(n), \text{gr}_{W_{2n}}(M)),
\]

is a fibre functor, therefore \( \text{MT}(\mathcal{O}_S) \) is a Tannaka category. We denote by \( G_{S,\omega} \) the group scheme of \( \otimes \)-automorphisms of \( \omega \). By [DG05, 2.1] we can write \( G_{S,\omega} \) as a semi-direct product:

\[
G_{S,\omega} = G_m \rtimes U_{S,\omega},
\]

where \( U_{S,\omega} \) is a unipotent group and \( G_{S,\omega} \to G_m \) is induced by the obvious grading of \( \omega \). If \( S = |\text{Spec}(\mathcal{O})| \) then we simply write \( G_\omega = G_{S,\omega} \).

2.1.5. Functor to \( p \)-adic representations. Let \( x \in |\text{Spec}(\mathcal{O})| \) be a point lying over a prime \( p \). Let \( K = E_x \) be the completion of \( E \) at the place \( x \). Choose algebraic closures \( \bar{E}, \bar{K} \), and an embedding \( \iota : \bar{E} \to \bar{K} \).

To \( M \in \text{MT}(E) \) we can attach a Galois representation \( M_p \) of \( \text{Gal}(\bar{E}/E) \) with coefficients in \( \mathbb{Q}_p \), which is called the \( p \)-adic realization of \( M \). By using \( \iota \), we get a continuous homomorphism

\[
\text{Gal}(\bar{K}/K) \to \text{Gal}(\bar{E}/E),
\]

and we can restrict \( M_p \) in order to obtain a \( p \)-adic representation \( M_{x,p} \) of \( \text{Gal}(\bar{K}/K) \).

**Proposition 2.1.6.** The assignment \( M \mapsto M_{x,p} \) defines a functor

\[
(\_)_p : \text{MT}(E) \to \text{MT}_{G_K}.
\]

See Definition [1.3.7] for \( \text{MT}_{G_K} \).

**Proof.** The \( p \)-adic realization is functorial. Thus we only need to show that \( M_{x,p} \in \text{MT}_{G_K} \), which follows immediately from the existence of the weight filtration of \( M \) and Definition [1.3.7]. \( \square \)
The set \( \{ \iota : \bar{E} \to \bar{E}_x \} \) of embeddings over \( E \) is a torsor under the Galois group \( \text{Gal}(\bar{E}/E) \), and for every \( g \in \text{Gal}(\bar{E}/E) \) there is a natural transformation:

\[
\alpha_g : (\cdot)_{i,p} \xrightarrow{\cong} (\cdot)_{\log, p}.
\]

**Lemma 2.1.7.** For the fibre functor \( \tau \) (defined in \( 1.3.8 \)) and the fibre functor \( \omega \) defined in \( 2.1.1 \) we have a canonical isomorphism

\[
\tau \circ (\cdot)_{i,p} \cong \omega \otimes \mathbb{Q}_p.
\]

For every \( g \in \text{Gal}(\bar{E}/E) \), the diagram

\[
\begin{array}{ccc}
\tau \circ (\cdot)_{i,p} & \xrightarrow{\cong} & \tau \circ (\cdot)_{\log, p} \\
\downarrow \downarrow & & \downarrow \downarrow \\
\omega \otimes \mathbb{Q}_p & & \omega \otimes \mathbb{Q}_p
\end{array}
\]

is commutative.

**Proof.** Straightforward.

2.1.8. Recall that we have constructed a \( K_{st} \)-valued \( \eta_{st} \) of \( \text{Aut} \otimes \tau \) (Lemma 1.3.12).

**Proposition 2.1.9.** For every embedding \( \iota, \eta_{st} := \eta_{st} \circ (\cdot)_{i,p} \) defines a \( K_{st} \)-valued point of \( \text{Aut} \otimes \omega \) which is independent of the choice of \( \iota \).

**Proof.** Since \( \tau \circ (\cdot)_{i,p} = \omega \otimes \mathbb{Q}_p \) by Lemma 2.1.7 \( \eta_{st} \circ (\cdot)_{i,p} \) is a \( K_{st} \)-valued point of \( \text{Aut} \otimes \omega \).

The independence of the choice of \( \iota \) follows from the commutative diagram

\[
\begin{array}{ccc}
(\tau \circ (\cdot)_{i,p}) \otimes \mathbb{Q}_p K_{st} & \xrightarrow{\eta_{st}} & (\tau \circ (\cdot)_{i,p}) \otimes \mathbb{Q}_p K_{st} \\
\downarrow \downarrow & & \downarrow \downarrow \\
\omega \otimes \mathbb{Q} K_{st} & & \omega \otimes \mathbb{Q} K_{st} \\
\downarrow & & \downarrow \\
(\tau \circ (\cdot)_{\log, p}) \otimes \mathbb{Q}_p K_{st} & \xrightarrow{\eta_{st}} & (\tau \circ (\cdot)_{\log, p}) \otimes \mathbb{Q}_p K_{st}
\end{array}
\]

where the triangles are commutative by Lemma 2.1.7 and the square is commutative because \( \eta_{st} \) is functorial.

2.2. Crystalline characterization of unramified motives.

2.2.1. Let \( E \) be a number field, and let \( M \) be a mixed Tate motive over \( E \), i.e. an object in \( \text{MT}(E) \). Let \( \nu \) be a finite place of \( E \), \( M \) is unramified at \( \nu \) [DG05, Definition 1.4, §1.7] if the coaction [DG05, (1.2.2)]

\[
e_M : \omega(M) \to \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) \otimes \omega(M)
\]

of \( \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) = E^\times \otimes_{\mathbb{Z}} \mathbb{Q} \) on \( \omega(M) \) factors through a coaction of \( \mathcal{O}_\nu^\times \otimes_{\mathbb{Z}} \mathbb{Q} \).
2.2.2. Recall from Proposition 2.1.6 that $M_{i,p}$ is a mixed Tate Galois representation of $G_K = \text{Gal}(\bar{K}/K)$ for the completion $K = E_\nu$ at $\nu$. In particular, $M_{i,p}$ is semistable (Proposition 1.3.8). In the following we will simply write $M_p = M_{p,i}$.

We call $M_p$ crystalline if the monodromy operator $N$ of $D_{st}(M_p)$ is trivial, or equivalently if

$$(B_{\text{cris}} \otimes_{Q_p} M_p)^{G_K} \to (B_{\text{st}} \otimes_{Q_p} M_p)^{G_K}$$

is an isomorphism.

**Theorem 2.2.3.** Let $M$ be a mixed Tate motive over $E$ and $\nu$ a finite place of $E$. Then $M$ is unramified at $\nu$ if and only if $M_p$ is crystalline.

**Proof.** First note that the statement that $M$ is unramified at $\nu$ is equivalent to the statement that for every subquotient $N$ of $M$ which is of the form

$$0 \to \mathbb{Q}(n+1) \to N \to \mathbb{Q}(n) \to 0,$$

for some $n$, the extension class $\text{Ext}^1(\mathbb{Q}(n), \mathbb{Q}(n+1)) = \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) = E^\times \otimes \mathbb{Q}$ lies in $O_\nu^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ [DG05, §1.4].

Also in the category of $p$-adic representations of a $p$-adic field $K$, a representation in $\text{Ext}^1_{G_K}(\mathbb{Q}_p(0), \mathbb{Q}_p(1))$ that is associated to some $q \in K^\times \otimes \mathbb{Q} \subseteq \lim_{\leftarrow n}(K^\times/(K^\times)^{p^n}) \otimes \mathbb{Q} = \text{Ext}^1_{G_K}(\mathbb{Q}_p(0), \mathbb{Q}_p(1))$ is crystalline if and only if $q \in O_K^\times \otimes \mathbb{Q}$ [Liu02, Example 2.3.2].

First, suppose that $M_p$ is crystalline, then every subquotient of $M_p$ is crystalline. So in order to prove that $M$ is unramified at $\nu$ we may assume that $M = N$, where $N$ is as above with $n = 0$ (after Tate twist). Therefore we have an extension in $\text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))$, defined by some $q \in E^\times \otimes \mathbb{Q}$, whose $p$-adic realization is crystalline at $\nu$. Then the above remark implies that the image of $q$ in $E^\times_\nu \otimes \mathbb{Q}$ lies in $O_\nu^\times \otimes \mathbb{Q}$, hence $q \in O_\nu^\times \otimes \mathbb{Q}$ and $M$ is unramified at $\nu$.

Suppose conversely that $M_p$ is unramified at $\nu$. We have to show that the monodromy operator $N$ on $D_{st}(M_p) =: D(M)$ vanishes. Note that $N$ maps the slope $\lambda$ piece of $D(M)$ to the slope $\lambda - 1$ piece. Therefore, if $N$ is nonzero on $D(M)$ then there exists an $n$ such that $N$ is nonzero on $D(W_{2n}M/W_{2n-4}M) = D((W_{2n}M/W_{2n-4}M)_p)$. Replacing $M$ by $(W_{2n}M/W_{2n-4}M)\otimes \mathbb{Q}(n)$ we may assume that $M$ is defined by a class in $\text{Ext}^1(\mathbb{Q}(0)^{\otimes r}, \mathbb{Q}(1)^{\otimes s}) = \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))^{\otimes rs}$, $M$ is unramified, and $N$ is nonzero on $D(M)$. By passing to a subquotient we may further assume that $r = s = 1$. This gives an extension in $\text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))$ which is unramified at $\nu$ (and hence defined by some $q \in O_\nu^\times \otimes \mathbb{Q}$) and whose $p$-adic realization is not crystalline at $\nu$. This is a contradiction. \qed

2.2.4. Recall the notation of Section 2.1.1. Let $x \in |\text{Spec}(O)|$ be a point; in the following we will work with $S = |\text{Spec}(O)| \setminus \{x\}$, thus $O_S = O_x$.

Let $p$ be the prime lying under $x$. In view of Theorem 2.2.3 we know that $MT(O_x)$ is the full subcategory of $MT(E)$ consisting of motives $M$ such that the $p$-adic realization $M_p$ is crystalline at $x$.

We denote by $G_x$ the group scheme of $\otimes$-automorphisms of the fibre functor (see (2.1.1))

$$\omega : MT(O_x) \to (\mathbb{Q}\text{-vector spaces}).$$

The group scheme $G_x$ is a quotient of $G_\omega = \text{Aut}_{MT(E)}^{\otimes \omega}$. 
Lemma 2.2.5. The morphism \( \text{Spec}(E_x, st) \xrightarrow{\eta_x} G_x \) factors through the structure morphism \( \text{Spec}(E_x, st) \to \text{Spec}(E_x) \) and thus defines a point \( \eta^\ur_x \in G_x(E_x) \).

Proof. The point \( \eta_x \) was defined in Proposition 2.1.9. If \( M \in MT(O_x) \) then \( D_{st}(M_{t,p}) \) has vanishing monodromy operator \( N \) and \( \eta_{st}(M_{t,p}) = \eta_{st,log}D_{st}(M_{t,p}) \) takes values in \( E_x \) by Definition 1.3.10. \( \square \)

2.3. Main theorem.

2.3.1. Let \( x \in |\text{Spec}(O)| \) and let \( E_x \) be the completion of \( E \) at \( x \). Bloch and Kato [BK90, Definition 3.10] define an exponential map

\[
(2.3.1) \quad \text{exp} : E_x \to \text{Ext}^1(Q_p(0), Q_p(n)), \quad \text{for all } n \geq 1,
\]

where \( \text{Ext}^1 \) is computed in the category of \( p \)-adic representation of \( \text{Gal}(\overline{E}_x/E_x) \).

Note that, in fact, the image of the exponential map lies among the crystalline representations \( \text{Ext}^1_{\text{cryst}}(Q_p(0), Q_p(n)) \) [BK90, Example 3.9]. Via \( p \)-adic Hodge theory, we obtain a map

\[
(2.3.2) \quad E_x \to \text{Ext}^1_{\text{cryst}}(Q_p(0), Q_p(n)) \cong \text{Ext}^1_{MT_{E_x}}(E_x(0), E_x(n)),
\]

which, by abuse of notation, will also be called the Bloch-Kato exponential map.

2.3.2. For an extension \( M \in \text{Ext}^1_{MT(O_x)}(Q(0), Q(n)) \) with \( n \geq 1 \), there are natural maps \( v_0 : Q \to \omega(M) \) and \( f_n : \omega(M) \to Q \) defined as follows. By definition, there are isomorphisms \( \alpha : Q(n) \to \text{gr}^W_{2n}M \) and \( \beta : \text{gr}^W_{0}M \to Q(0) \); we define

\[
\begin{align*}
v_0 : Q &= \text{Hom}(Q(0), Q(0)) \xrightarrow{\beta^{-1}} \omega_0(M) \to \omega(M), \\
f_n : \omega(M) &\to \omega_n(M) \xrightarrow{\alpha^{-1}} \text{Hom}(Q(n), Q(n)) = Q.
\end{align*}
\]

Therefore, we can attach to \( M \) a function in \( \mathbb{A}^1(G_x) \), defined by

\[
M(t) := f_n(t \cdot v_0),
\]

for every point \( t : T \to G_x \).

Theorem 2.3.3. Let \( E \) be a number field and \( O \) be the ring of integers. Let \( x \in |\text{Spec}(O)| \) be a closed point over a prime \( p \). For the Tannaka category \( (MT(O_x), \omega) \) of mixed Tate motives we denote by \( G_x \) the group scheme of \( \otimes \)-automorphisms of \( \omega \). For all \( n \geq 1 \), the map

\[
\text{Ext}^1_{MT(O_x)}(Q(0), Q(n)) \to E_x, \quad M \mapsto M(\eta^\ur_x),
\]

induced by \( \eta^\ur_x \in G_x(E_x) \) (see Lemma 2.2.5), is the composition of the \( p \)-adic realization

\[
\text{Ext}^1_{MT(O_x)}(Q(0), Q(n)) \to \text{Ext}^1_{\text{cryst}}(Q_p(0), Q_p(n))
\]

and the inverse of the Bloch-Kato exponential map (2.3.1).

Proof. Let us prove that evaluation at the point \( \eta^\ur_x \) has the desired compatibility with the Bloch-Kato exponential map (see (2.3.2))

\[
\text{exp} : E_x \to \text{Ext}^1_{\text{cryst}}(Q_p(0), Q_p(n)) \xrightarrow{\beta} \text{Ext}^1_{MT_{E_x}}(E_x(0), E_x(n)).
\]

For this we need to recall the construction of the exponential map. For the rest of the proof let \( K := E_x \). First there is an exact sequence [BK90, Proposition 1.17]:

\[
(2.3.3) \quad 0 \to Q_p \to B_{crys}^{e=1} \oplus B_{dR}^+ \to B_{dR} \to 0,
\]
where the first map sends $x$ to $(x, x)$ and the second one sends $(x, y)$ to $x - y$.

For $n \geq 1$, the Bloch-Kato construction gives a map

$$K = (\mathbb{Q}_p(n) \otimes B_{dR})^{G_K} \to \text{Ext}_{\text{crys}}^1(\mathbb{Q}_p(0), \mathbb{Q}_p(n)).$$

This map is obtained as follows. First tensor the above exact sequence with $\mathbb{Q}_p(n)$:

$$0 \to \mathbb{Q}_p(n) \to (\mathbb{Q}_p(n) \otimes B_{crys}^{n-1}) \oplus (\mathbb{Q}_p(n) \otimes B_{dR}^+) \to \mathbb{Q}_p(n) \otimes B_{dR} \to 0.$$ 

Then an element $a$ in $K(n) = (\mathbb{Q}_p(n) \otimes B_{dR})^{G_K}$ gives a map $\mathbb{Q}_p \to \mathbb{Q}_p(n) \otimes B_{dR}$, pulling back the above exact sequence via this map gives the extension we were looking for.

More explicitly, for $a \in K$ the extension constructed above is:

$$0 \to V_n \to V \to V_0 \to 0,$$

where $V_0 = \mathbb{Q}_p \cdot t^n \otimes at^{-n}$, $V_n = \mathbb{Q}_p \cdot t^n$, and $V$ is a 2-dimensional representation of $G_K$ with basis which can be described as follows. By the exact sequence (2.3.3), there exists $x \in B_{crys}^{n-1}$ and $y \in B_{dR}^+$ such that $at^{-n} = x - y$. Then $V$ has basis \{(t^n \otimes x, t^n \otimes y), (t^n \otimes 1, t^n \otimes 1)\}. For $\sigma \in G_K$,

$$\sigma(t^n \otimes x, t^n \otimes y) = (t^n \otimes x, t^n \otimes y) + \gamma(\sigma)(t^n \otimes 1, t^n \otimes 1),$$

for some $\gamma(\sigma) \in \mathbb{Q}_p$. Therefore

$$\chi_{cyc}(\sigma)^n \sigma(x) = x + \gamma(\sigma)$$

and

$$\chi_{cyc}(\sigma)^n \sigma(y) = y + \gamma(\sigma).$$

Let us now try to find what this extension corresponds to after we apply the functor $(\cdot \otimes B_{crys})^{G_K}$. First note that $(V \otimes B_{crys})^{G_K}$ has basis $e_n := (t^n \otimes 1, t^n \otimes 1) \otimes t^{-n}$ and $e_0 := (t^n \otimes x, t^n \otimes y) \otimes 1 - (t^n \otimes 1, t^n \otimes 1) \otimes x$.

That $e_n$ is invariant under the Galois action is clear. In order to see that $e_0$ is $G_K$ invariant let $\sigma \in G_K$. Then

$$\sigma(e_0) = ((t^n \otimes (x + \gamma(\sigma)) \otimes (y + \gamma(\sigma))) \otimes 1 - (t^n \otimes 1, t^n \otimes 1) \otimes (x + \gamma(\sigma))) = e_0.$$

Now note that $\varphi(e_n) = p^{-n}e_n$ and $\varphi(e_0) = e_0$. Furthermore $e_n$ is the image of $1 \in K(n)$ and $e_0$ maps to $1 \in K(0)$ in the exact sequence (note that $\mathbb{Q}_p(0)$ is identified with $V_0$ via the map that sends $1$ to $t^n \otimes at^{-n}$):

$$0 \to K(n) \to (V \otimes B_{crys})^{G_K} \to K(0) \to 0.$$ 

Therefore in order to compare Bloch-Kato’s construction we need only compute the filtration on $(V \otimes B_{crys})^{G_K} \otimes_{K_0} K$. So we need to compute the 0-th piece of the filtration on $(V \otimes B_{dR})^{G_K}$.

We claim that $ae_n + e_0 \in \text{Fil}^0(V \otimes B_{dR})^{G_K}$. This follows immediately from

$$ae_n + e_0 = (t^n \otimes x, t^n \otimes y) \otimes 1 - (t^n \otimes 1, t^n \otimes 1) \otimes y,$$

and the fact that $y \in B_{dR}^+$. Now Proposition 1.2.27 implies the claim.

2.4. Archimedean places. In this section we recall the story for archimedean places; our reference is [Del94] and [BD94].
2.4.1. Let $E$ be a number field and $\sigma : E \to \mathbb{C}$ an embedding. To $M \in MT(E)$ we can attach a real mixed Tate Hodge structure $M_\sigma$. Recall that a real mixed Tate Hodge structure $(H, W, F)$ consists of an $\mathbb{R}$-vector space $H$, an increasing filtration $W$ of $H$, and a decreasing filtration $F$ of $H \otimes_\mathbb{R} \mathbb{C}$ such that

$$\text{Gr}_F^n \text{Gr}_F^q \text{Gr}_n^W (H \otimes \mathbb{C}) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even and } (p, q) \neq (\frac{n}{2}, \frac{n}{2}). \end{cases}$$

Induced by $F, \bar{F}$, we obtain maps

$$a_F: H \otimes \mathbb{C} = \bigoplus_{i \in \mathbb{Z}} F^{-i} \cap W_{-2i} \to \bigoplus_{i \in \mathbb{Z}} \text{Gr}_F^i H \otimes \mathbb{C},$$

$$a_{\bar{F}}: H \otimes \mathbb{C} = \bigoplus_{i \in \mathbb{Z}} \bar{F}^{-i} \cap W_{-2i} \to \bigoplus_{i \in \mathbb{Z}} \text{Gr}_{\bar{F}}^i H \otimes \mathbb{C},$$

where $F^i \cap W_{2i} \to \text{Gr}_F^{2i} H \otimes \mathbb{C}$ is the natural map (and similarly for $\bar{F}$). For the automorphism $d = a_{\bar{F}} a_F^{-1}$ of $\bigoplus_{i \in \mathbb{Z}} \text{Gr}_F^{-2i} H \otimes \mathbb{C}$ we know that $(d - 1)(\text{Gr}_F^{-2i} H \otimes \mathbb{C}) \subset \bigoplus_{j > i} \text{Gr}_F^{-2j} H \otimes \mathbb{C},$ by [Del94, p.510], and $\bar{d} = d^{-1}$ [Del94, p.513].

2.4.2. Let $C$ be the category of pairs $(\bigoplus_i H_i, d)$ where $\bigoplus_i H_i$ is a graded $\mathbb{R}$-vector space and $d: \bigoplus_i H_i \otimes \mathbb{C} \to \bigoplus_i H_i \otimes \mathbb{C}$ is an automorphism satisfying the conditions $\bar{d} = d^{-1}$ and $(d - 1)(H_i) \subset \bigoplus_{j > i} H_j$, for all $i$.

**Proposition 2.4.3.** [Del94 p.514] The functor

$$(\text{Real mixed Tate Hodge structures}) \to C$$

$$(H, W, F) \mapsto (\bigoplus_{i \in \mathbb{Z}} \text{Gr}_F^{-2i} H, d)$$

is an equivalence of categories.

The maps $d$ define a $C$-valued $\otimes$-automorphism for the fibre functor

$$\tilde{\omega}: (\text{Real mixed Tate Hodge structures}) \to (\mathbb{R}\text{-vector spaces}),$$

$$\tilde{\omega}(H, W, F) = \bigoplus_{i \in \mathbb{Z}} \text{Gr}_F^i H.$$

2.4.4. Recall the definition of $\omega$ in (2.1.1). For the functor

$$\mathcal{R}_\sigma : MT(E) \to (\text{Real mixed Tate Hodge structures}), \quad M \mapsto M_\sigma,$$

we have an isomorphism

$$(2.4.1) \quad \omega \otimes_{\mathbb{Q}} \mathbb{R} \cong \tilde{\omega} \circ \mathcal{R},$$

depending on the choice $(2\pi i)^n$ as a generator for the real vector space underlying $\mathbb{R}(n)$, in other words we have to choose a square root of $-1$ in $\mathbb{C}$. In order to avoid this choice one can define

$$\tilde{\omega}(H, W, F) = \bigoplus_{n \in \mathbb{Z}} i^n \cdot \text{Gr}_F^{-2n} H,$$

as in [BD94 p.111], but we won’t do that.

Via Equation (2.4.1), $d$ defines a $C$-valued point of $G_\omega$, the $\otimes$-automorphisms of the fibre functor $\omega$. We define $\epsilon = \log(d)$, $\epsilon$ defines a $C$-valued point of $\text{Lie}(G_\omega)$. 


The dictionary for the notation of [BD94, p.111] is 
\[ d = b^{-1}, \quad \epsilon = -2 \cdot N, \]
and \( N \) is purely imaginary.

2.4.5. For \( z \in E \setminus \{0, 1\} \) there is a polylogarithm motive \( \{z\} \in MT(E) \) (strictly speaking it is a pro-object). The motive \( \{z\} \) is defined as a subquotient of the motivic paths from the tangent vector \( t_0 = z \), in the tangent space at 0, to \( z \) (see [DG05, Theorem 4.4]). The \( \mathbb{Q} \)-Hodge realization of \( \{z\} \) is described in [BD94, p.98] and uniquely determines \( \{z\} \).

For every \( k \in \mathbb{Z}_{\geq 0} \) we have natural isomorphisms 
\[ \alpha_k : \text{gr}^{W_{-2k}} \{z\} \xrightarrow{\cong} \mathbb{Q}(k); \]
we define \( v_0 \in \omega(\{z\}) \) and \( f_k \in \omega(\{z\})^\vee \) by 
\[ v_0 : \mathbb{Q} = \text{Hom}(\mathbb{Q}(0), \mathbb{Q}(0)) \xrightarrow{\alpha_0^{-1}} \omega_0(\{z\}) \]
\[ f_k : \omega(\{z\}) \xrightarrow{\omega_k} \omega_k(\{z\}) \xrightarrow{\alpha_k} \text{Hom}(\mathbb{Q}(k), \mathbb{Q}(k)) = \mathbb{Q}. \]

We denote by \((v_0, \{z\}, f_k) \in A^1(\text{Lie } G_\omega)\) the function 
\[ X \mapsto f_k(X \cdot v_0). \]

By [BD94, Proposition 2.7] we have 
\[ (v_0, \{z\}, f_k)(\epsilon) = \begin{cases} 
2i \sum \ell b_{\ell} \log(z^\ell) \text{Im}(\text{Li}_{k-\ell}(z)) & \text{if } k \text{ is even}, \\
2i \sum \ell b_{\ell} \log(z^\ell) \text{Re}(\text{Li}_{k-\ell}(z)) & \text{if } k \text{ is odd}. 
\end{cases} \]

Here, \( \{b_{\ell}\} \) are the Bernoulli numbers and \( \text{Li} \) is the polylogarithm. For \( k \) even, the result does not depend on the choice of the square root of \(-1\); for \( k \) odd it is independent up to a sign.

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