(Not) weakly regular univariate bent functions

Ayça Çeşmioğlu\textsuperscript{1}, Wilfried Meidl\textsuperscript{2},

\textsuperscript{1} Department of Mathematics, Otto-von-Guericke-University, 39106 Magdeburg, Germany.
email: cesmelioglu@gmail.com

\textsuperscript{2} Sabancı University, MDBF, Orhanlı, Tuzla, 34956 İstanbul, Turkey.
email: wmeidl@sabanciuniv.edu

Abstract

In this article a procedure to construct bent functions from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$ by merging plateaued functions which are bent on $(n-2)$-dimensional subspaces of $\mathbb{F}_{p^n}$ is presented. Taking advantage of such classes of plateaued functions with a simple representation as monomials and binomials, we obtain infinite classes of bent functions with a fairly simple representation. In particular we present the first direct construction of univariate not weakly regular bent functions, and give one class explicitly in a simple representation with binomials.

Keywords: Bent function; partially bent function; Fourier transform; not weakly regular; quadratic function; polynomial.

1 Introduction

Let $V_n$ be an $n$-dimensional vector space over the prime field $\mathbb{F}_p$. A function $f : V_n \rightarrow \mathbb{F}_p$ is called a bent function if its Fourier transform $\hat{f}$ defined by

$$\hat{f}(b) = \sum_{x \in V_n} \epsilon^{f(x)-<b,x>}_p$$

satisfies $|\hat{f}(b)|^2 = p^n$ for all $b \in V_n$, where $\epsilon_p = e^{2\pi i/p}$ and $<,>$ denotes any (non-degenerate) inner product on $V_n$. Classical representations for bent functions are the multivariate representation where $V_n = \mathbb{F}_p^n$, in this case one may use the conventional dot product as inner product, and the univariate representation where $V_n = \mathbb{F}_p^n$, in which case one may use $<b,x> = \text{Tr}_n(bx)$ as inner product, where $\text{Tr}_n(z)$ denotes the absolute trace of $z \in \mathbb{F}_{p^n}$.

For $p = 2$, bent functions can only exist when $n$ is even, the Fourier coefficients $\hat{f}(b)$ are then obviously $\pm 2^{n/2}$. For $p > 2$ bent functions exist for both, $n$ even and $n$ odd. For the Fourier coefficients we then always have
where $f^*$ is a function from $V_n$ to $\mathbb{F}_p$. A bent function $f : V_n \to \mathbb{F}_p$ is called regular if for all $b \in V_n$

$$p^{-n/2} \tilde{f}(b) = \epsilon^{f^*(b)}.$$  

When $p = 2$, a bent function is trivially regular, and as can be seen from (1), for $p > 2$ a regular bent function can only exist for even $n$, and for odd $n$ when $p \equiv 1 \mod 4$. A function $f : V_n \to \mathbb{F}_p$ is called weakly regular if, for all $b \in \mathbb{F}_p^n$, we have

$$p^{-n/2} \tilde{f}(b) = \zeta^{-1} \epsilon^{f^*(b)}$$

for some complex number $\zeta$ with $|\zeta| = 1$, otherwise it is called not weakly regular. By (1), $\zeta$ can only be $\pm 1$ or $\pm i$. Note that regular implies weakly regular. All classical construction of bent functions yield (weakly) regular bent functions.

Based on earlier constructions of Boolean bent functions from near-bent functions (see [5, 9]) in [2, 3] constructions of $p$-ary bent functions have been presented. In these constructions functions in lower dimensions are merged to a bent function by adjoining variables. The resulting bent functions are given in multivariate form [2, 4], or as functions from $\mathbb{F}_{p^{n-1}} \times \mathbb{F}_p$ to $\mathbb{F}_p$ [3]. The construction turns out to be very powerful, for instance the first infinite classes of not weakly regular bent functions have been obtained. Until then only sporadic examples of not weakly regular bent functions were known, all given as univariate polynomials of the form $f(x) = \text{Tr}_n(g(x))$ for a polynomial $g(x) \in \mathbb{F}_{p^n}[x]$, see [6, 7, 13].

The objective of this paper is to develop an equivalent construction for the univariate case, i.e. for functions from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$, which is more involved as we cannot simply add variables to the finite field $\mathbb{F}_{p^n}$. Amongst others, sets of functions which are bent on $(n-2)$-dimensional subspaces of $\mathbb{F}_{p^n}$ are required. We take advantage of particularly simple representations of some classes of such functions (as monomials and binomials), and thereby obtain bent functions from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$ in a simple representation. Among those, we present the first direct construction of infinite classes of univariate not weakly regular bent functions.

In Section 2 we develop the principles of the construction and we give an explicit formula for bent functions from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$ obtained from functions which are bent on subspaces. This description of the functions enables us
also to find the representation with a unique polynomial of degree at most \( p^n - 1 \). In Section 3 we use classes of monomials and binomials to obtain univariate (not) weakly regular bent functions in a simple representation. Explicitly we describe an infinite class of not weakly regular bent functions. Some examples are given in the appendix.

2 A construction of bent polynomials

For a function \( f : V_n \rightarrow \mathbb{F}_p \) and an element \( a \in V_n \), the derivative \( D_a f : V_n \rightarrow \mathbb{F}_p \) of \( f \) in direction \( a \) is defined by \( D_a f(x) = f(x + a) - f(x) \). As well known, \( f \) is bent if and only if \( D_a f \) is balanced for all nonzero \( a \in V_n \), see [12]. An element \( a \in V_n \) for which \( D_a f \) is constant is called a linear structure of \( f \). As easily seen, the set \( \Lambda \) of the linear structures of \( f \) forms a subspace of \( V_n \), which we call the linear space of the function \( f \). We have

\[
f(x + a) = f(x) + f(a) - f(0) \text{ for all } a \in \Lambda, x \in V_n.
\]

In particular, if \( f(0) = 0 \), then equation (2) implies that \( f \) is linear on \( \Lambda \).

A function \( f : V_n \rightarrow \mathbb{F}_p \) is called partially bent if for all \( a \in V_n \) the derivative \( D_a f \) is either balanced or constant. The set of partially bent functions is a subset of the set of plateaued functions, which is the set of functions \( f : V_n \rightarrow \mathbb{F}_p \) for which \( \hat{f}(b) = 0 \) or \( |\hat{f}(b)| = p^{(n+s)/2} \) for all \( b \in V_n \) and a fixed integer \( s \), \( 0 \leq s \leq n \), depending on \( f \). This can easily be seen in the calculations below, applying the standard Welch-squaring method. In accordance with [2], we call plateaued functions \( f \) from \( V_n \) to \( \mathbb{F}_p \) for which \( s = 1 \) as near-bent functions. We remark that when \( p = 2 \), also the term semi-bent function is used for plateaued functions with \( s = 1 \) and \( n \) odd or \( s = 2 \) and \( n \) even, see [5].

Let \( f \) be a partially bent function with linear space \( \Lambda \) of dimension \( s \) as a subspace of \( V_n \). Without loss of generality we will always suppose that \( f(0) = 0 \), and hence \( f(x + a) = f(x) + f(a) \) if \( a \) is a linear structure of \( f \). We then have

\[
|\hat{f}(b)|^2 = \sum_{x,y \in V_n} e_p f(x) - f(y) - \langle b, x-y \rangle = \sum_{y,z \in V_n} e_p f(y+z) - f(y) - \langle b,z \rangle = \sum_{z \in V_n} e_p f(z) - \langle b,z \rangle \sum_{y \in V_n} e_p f(y+z) - f(y) - f(z).
\]

Using that \( f(y + z) - f(y) - f(z) \) is balanced as a function in variable \( y \) if
\[ |\hat{f}(b)|^2 = p^n \sum_{z \in \Lambda} \epsilon \frac{f(z) - <b,z>}{p} = \begin{cases} p^{n+s} & \text{if } f(z) - <b,z> \equiv 0 \text{ on } \Lambda, \\ 0 & \text{otherwise}, \end{cases} \tag{3} \]

where in the last step we use that \( f \) is linear on \( \Lambda \). Defining the support of the Fourier transform of \( f \) by \( \text{supp}(\hat{f}) := \{ b \in V_n : \hat{f}(b) \neq 0 \} \), it follows from equation (3) that \( b \in \text{supp}(\hat{f}) \) if and only if \( f(z) - <b,z> \equiv 0 \) on \( \Lambda \).

Parseval’s identity
\[ \sum_{b \in V_n} |\hat{f}(b)|^2 = p^{2n}, \]
then implies \( |\text{supp}(\hat{f})| = p^{n-s} \).

**Remark 1** As it can be seen from equation (3), \( \text{supp}(\hat{f}) \) depends on the inner product \( <,> \) which is used. Consequently, to be precise one may define the support of \( \hat{f} \) with respect to the inner product \( <,> \). As well known the absolute values appearing in the Fourier spectrum \( \{ \hat{f}(b) \mid b \in V_n \} \) of \( f \) are independent of the (non-degenerate) inner product. In particular, the property of being \( s \)-plateaued is independent from \( <,> \).

We will use the following result on partially bent functions (see [1, Theorem]).

**Lemma 1** Let \( f : V_n \to \mathbb{F}_p \) be a partially bent function with linear space \( \Lambda \) and let \( \Lambda^c \) be any complement of \( \Lambda \) in \( V_n \). Then \( f \) restricted to \( \Lambda^c \) is a bent function.

A well understood class of partially bent functions is the class of quadratic functions, see [2, 6]. For more information on partially bent functions we refer the reader to [1].

For the construction of bent functions from \( \mathbb{F}_{p^n} \) to \( \mathbb{F}_p \) we will employ partially bent functions \( f \) with a two dimensional linear space \( \Lambda = \langle \beta_1, \beta_2 \rangle \). For simplicity we fix an inner product \( <u,v> = \text{Tr}_n(\delta uv) \) on \( \mathbb{F}_{p^n} \) with respect to which the orthogonal complement \( \Lambda^\perp \) of \( \Lambda \) is a complement \( \Lambda^c \) of \( \Lambda \), and we further suppose that \( <\beta_1, \beta_2> = 0 \). Note that this implies \( <\beta_1, \beta_1> = \ell \neq 0 \) and \( <\beta_2, \beta_2> = t \neq 0 \). Conversely, the properties \( <\beta_1, \beta_2> = 0, <\beta_1, \beta_1> = \ell \neq 0 \) and \( <\beta_2, \beta_2> = t \neq 0 \) imply that \( \Lambda \cap \Lambda^\perp = \{0\} \), and hence that \( \Lambda^\perp \) is a complement of \( \Lambda \). We remark that given \( \beta_1, \beta_2 \in \mathbb{F}_{p^n} \) (linearly independent over \( \mathbb{F}_p \)), one can always find \( \delta \in \mathbb{F}_{p^n} \) such that \( \text{Tr}_n(\delta \beta_1 \beta_2) = 0, \text{Tr}_n(\delta \beta_1^2) \neq 0 \) and \( \text{Tr}_n(\delta \beta_2^2) \neq 0 \). Some properties of this inner product are used in the proof of Lemma [1] below, see also Remark [2].
For a partially bent function \( f : \mathbb{F}_p^n \to \mathbb{F}_p \) with linear space \( \Lambda = \langle \beta_1, \beta_2 \rangle \) we fix the following notation:

- \( f|_{n-2} \) is the function \( f \) restricted to \( V_{n-2} := \Lambda^c \),
- \( f|_{n-1} \) is the function \( f \) restricted to \( V_{n-1} := \langle \beta_2 \rangle + \Lambda^c \),
- \( \hat{f}|_{n-2} \) is the Fourier transform of \( f|_{n-2} \),
- \( \hat{f}|_{n-1} \) is the Fourier transform of \( f|_{n-1} \).

The following lemma shows that \( f \) given as above is near-bent with linear space \( \langle \beta_2 \rangle \) when restricted to \( V_{n-1} = \langle \beta_2 \rangle + \Lambda^c \). We consider the case of \( \Lambda^c = \Lambda^\perp \) with an inner product defined as above.

**Lemma 2** Let \( f : \mathbb{F}_p^n \to \mathbb{F}_p \) be a partially bent function with linear space \( \Lambda = \langle \beta_1, \beta_2 \rangle \), and let \( V_{n-1} = \langle \beta_2 \rangle + \Lambda^c \). Then \( f|_{n-1} : V_{n-1} \to \mathbb{F}_p \) is near-bent with linear structure \( \langle \beta_2 \rangle \).

**Proof:** Supposing that \( \langle, \rangle \) satisfies the above described conditions, we observe that \( \langle, \rangle \) also defines a non-degenerate inner product on \( V_{n-1} \) and on \( V_{n-2} = \Lambda^\perp = \Lambda^c \). Further we know that \( \langle \beta_2, \beta_2 \rangle = t \neq 0 \). Obviously the linear structure \( \beta_2 \) of \( f \) is also a linear structure of \( f|_{n-1} \). For an element \( b = b_2 \beta_2 + y \in V_{n-1} \), \( b_2 \in \mathbb{F}_p \), \( y \in V_{n-2} = \Lambda^c \), with \([2]\) we then have

\[
\hat{f}|_{n-1}(b) = \sum_{c \in \mathbb{F}_p} \sum_{x \in \Lambda^c} \epsilon_p^{f(c\beta_2 + x) - \langle b_2 \beta_2 + y, c\beta_2 + x \rangle} = \sum_{c \in \mathbb{F}_p} \epsilon_p^{f(c\beta_2) - b_2ct} \sum_{x \in \Lambda^c} \epsilon_p^{f(x) - \langle x, y \rangle} = \hat{f}|_{n-2}(y) \sum_{c \in \mathbb{F}_p} \epsilon_p^{f(c\beta_2) - b_2ct} = \begin{cases} 0 & \text{if } b_2 \neq f(\beta_2)/t, \\ p\hat{f}|_{n-2}(y) & \text{if } b_2 = f(\beta_2)/t. \end{cases}
\]

Consequently by Lemma \([1]\) \( f|_{n-1} \) is near-bent. \qed

With the following proposition we can generate sets of \( p \) near-bent functions on \( V_{n-1} \) such that every element of \( V_{n-1} \) is in the support of the Fourier transform for exactly one function in the set.

**Proposition 1** Let \( f_0, \ldots, f_{p-1} \) be partially bent functions from \( \mathbb{F}_p^n \) to \( \mathbb{F}_p \), all with the same linear space \( \Lambda = \langle \beta_1, \beta_2 \rangle \), and for \( 0 \leq k \leq p - 1 \) let \( \gamma_k \in V_{n-1} \) be such that

\[
f_k(\beta_2) + \langle \gamma_k, \beta_2 \rangle = f_0(\beta_2) + k. \tag{4}\]

The functions \( g_{k|_{n-1}} : V_{n-1} \to \mathbb{F}_p \) defined by

\[
g_k(x) = f_k(x) + \langle \gamma_k, x \rangle, \quad k = 0, 1, \ldots, p - 1 \tag{5}\]
(restricted to \(V_{n-1}\)) form a set of near-bent functions with supp(\(\widehat{g}_j\)) \(\cap\)

\(\text{supp}(\widehat{g}_k) = \emptyset\) if \(j \neq k\).

**Proof**: By the above discussion it is guaranteed that \(f_{k|n-1}\), i.e. \(f_k\) restricted to \(V_{n-1}\), is near-bent with linear space \(\langle \beta_2 \rangle\) for all \(k = 0, \ldots, p-1\). We have to show that the addition of the linear functions \(\langle \gamma_k, x \rangle\) to the functions \(f_k\) separates the supports of the Fourier transforms of the corresponding functions on \(V_{n-1}\). By equation (3), \(b \in V_{n-1}\) is an element of \(\text{supp}(\widehat{g}_k)\) (with respect to the inner product \(\langle \cdot, \cdot \rangle\)) if and only if \(g_k(\beta_2) - \langle b, \beta_2 \rangle = 0\). Suppose that \(b \in \text{supp}(\widehat{g}_k) \cap \text{supp}(\widehat{g}_j)\) for some \(0 \leq k, j \leq p-1\). Then

\[
0 = f_k(\beta_2) + \langle \gamma_k, \beta_2 \rangle - \langle b, \beta_2 \rangle = f_0(\beta_2) + k - \langle b, \beta_2 \rangle
\]

\[
= f_j(\beta_2) + \langle \gamma_j, \beta_2 \rangle - \langle b, \beta_2 \rangle = f_0(\beta_2) + j - \langle b, \beta_2 \rangle,
\]

and hence \(j = k\). Since as a consequence of Parseval’s identity we have \(|\text{supp}(\widehat{g}_k)\| = p^{n-2}\), the result follows. \(\Box\)

We remark that though the support of a near-bent function depends on the considered inner product, by the method of Proposition 1 one obtains a set of \(p\) near-bent functions such that every element \(b\) of \(V_{n-1}\) is in the support of the Fourier transform for exactly one function, independent of the inner product used.

**Theorem 1** Let \(\langle \cdot, \cdot \rangle\) be an inner product on \(\mathbb{F}_p^n\) and let \(\beta_1 \in \mathbb{F}_p^n\) be such that \(\langle \beta_1, \beta_1 \rangle = \ell \neq 0\), let \(g_0, \ldots, g_{p-1}\) be functions from \(\mathbb{F}_p^n\) to \(\mathbb{F}_p\) with linear structure \(\beta_1\), and suppose that the restrictions \(g_k|_{n-1}\) to \(V_{n-1} = \langle \beta_1 \rangle^\perp\) are near-bent and \(\text{supp}(\widehat{g}_{j|n-1}) \cap \text{supp}(\widehat{g}_k|_{n-1}) = \emptyset\) if \(j \neq k\). With \(\gamma = \ell^{-1} \beta_1\), the function \(F : \mathbb{F}_p^n \to \mathbb{F}_p\)

\[
F(x) = -\sum_{k=0}^{p-1} \prod_{j=0, j\neq k}^{p-1} (\langle \gamma, x \rangle - j)g_k(x)
\]

(6)

is bent.

**Proof**: Since \(\langle \beta_1, \beta_1 \rangle = \ell \neq 0\), the orthogonal complement \(V_{n-1} = \langle \beta_1 \rangle^\perp\) is a complement of \(\langle \beta_1 \rangle\), and \(\langle \cdot, \cdot \rangle\) is a non-degenerate inner product on \(V_{n-1}\). Let \(x = c_1\beta_1 + y, c_1 \in \mathbb{F}_p, y \in V_{n-1}\), be the unique representation of \(x \in \mathbb{F}_p^n\) as a sum of elements of \(V_{n-1} = \langle \beta_1 \rangle\) and \(V_{n-1}\). Then using \(\langle \gamma, \beta_1 \rangle = \ell^{-1} \langle \beta_1, \beta_1 \rangle = 1\) and \(\langle \beta_1 \rangle = V_{n-1}^\perp\) we obtain

\[
\langle \gamma, x \rangle - j = c_1 \langle \gamma, \beta_1 \rangle + \langle \gamma, y \rangle - j = c_1 - j.
\]
Hence $\prod_{j=0}^{p-1}(<\gamma, x > -j)$ vanishes if $k \neq c_1$ and for $k = c_1$ we have $\prod_{j=0}^{p-1}(<\gamma, x > -j) = -1$. Consequently,

$$F(x) = g_{c_1}(x) = g_{c_1}(c_1\beta_1 + y) = g_{c_1}(c_1\beta_1) + g_{c_1}(y),$$

where in the last step we use that $\beta_1$ is a linear structure of $g_{c_1}$.

Let $b = b_1\beta_1 + z$, $b_1 \in \mathbb{F}_p$, $z \in V_{n-1}$. Again using $V_{n-1} = \langle \beta_1 \rangle^\perp$ we then get

$$\hat{F}(b_1\beta_1 + z) = \sum_{y \in V_{n-1}} e_p^{g_{c}(c\beta_1)+y} - <b_1\beta_1+z,c\beta_1+y>$$

$$= \sum_{c \in \mathbb{F}_p} e_p^{g_{c}(c\beta_1)} \sum_{y \in V_{n-1}} e_p^{g_{c}(y) - \text{Tr}_n(xy)} = \sum_{c \in \mathbb{F}_p} e_p^{g_{c}(c\beta_1) - b_1c\ell} g_{c|n-1}(z).$$

Since every $z \in V_{n-1}$ is in the support of $\hat{g}_{c|n-1}$ of exactly one $c$, for this $c$ we then have

$$\hat{F}(b_1\beta_1 + z) = e_p^{g_{c}(c\beta_1) - b_1c\ell} g_{c|n-1}(z),$$

and therefore $|\hat{F}(b_1\beta_1 + z)| = p^{n/2}$.

The near-bent functions on $V_{n-1}$ constructed in Proposition 1 together with the inner product $<,>$ considered in the proposition satisfy the assumptions of Theorem 1 and we can suggest the following procedure for constructing bent polynomials:

- Choose $p$ partially bent functions $f_k : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$, $0 \leq k \leq p - 1$, all with the same 2-dimensional linear space $\Lambda = \langle \beta_1, \beta_2 \rangle$.

- Choose an inner product $< u, v > = \text{Tr}_n(\delta uv)$ on $\mathbb{F}_p^n$ such that $< \beta_1, \beta_2 > = 0$, and $\Lambda^\perp$ with respect to this inner product is a complement of $\Lambda$. We remark that therefore it is sufficient that $\delta$ satisfies $\text{Tr}_n(\delta \beta_1 \beta_2) = 0$, $\text{Tr}_n(\delta \beta_1^2) = \ell \neq 0$ and $\text{Tr}_n(\delta \beta_2^2) = t \neq 0$.

- For $k = 0, \ldots, p - 1$ choose $\gamma_k \in V_{n-1} = \langle \beta_1 \rangle^\perp$ which satisfy equation (4), to obtain the functions $g_k$ defined as in equation (5). We emphasize that such elements $\gamma_k$ always exist.

- With $\gamma = \ell^{-1} \beta_1$ construct the function $F : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ given as in equation (6).

**Remark 2** For the construction of bent functions it is sufficient to assume that $< \beta_1, \beta_1 > = \ell \neq 0$ and $< \beta_1, \beta_2 > = 0$. For $V_{n-1} = \langle \beta_2 \rangle^\perp \Lambda^\perp$ in Theorem

7
one can then take $\langle \beta_1 \rangle^\perp$. In this case, all properties needed in the proof of Theorem 1 hold but $\Lambda^\perp$ may not be a complement of $\Lambda$ with respect to this inner product. Then a different inner product has to be considered for the proof of Lemma 2 where the orthogonality of $\beta_2$ and $\Lambda^c$, and $\langle \beta_2, \beta_2 \rangle = t \neq 0$ is needed.

The representation of the bent function in Theorem 1 with a closed formula enables also the determination of the corresponding unique polynomial in $F_{p^n}[x]$ of degree at most $p^n - 1$. As one may expect, in general this representation does not look simple at all (see the examples in Section 3 and in the appendix). As pointed out in the following corollary, we have a rather simple representation for the functions defined in (6) originated in their construction principle.

**Corollary 1.** Let $f_k : F_{p^n} \rightarrow F_p$, $0 \leq k \leq p - 1$, be partially bent functions all with the same 2-dimensional linear space $\Lambda = \langle \beta_1, \beta_2 \rangle$ and let $\langle u, v \rangle = \text{Tr}_n(\delta uv)$ for an element $\delta \in F_{p^n}$ such that $\langle \beta_1, \beta_1 \rangle = l \neq 0$, and $\langle \beta_1, \beta_2 \rangle = 0$. For $k = 0, \ldots, p - 1$, let $\gamma_k \in V_{n-1} = \langle \beta_1 \rangle^\perp$ be such that $f_k(\beta_2) + \langle \gamma_k, \beta_2 \rangle = f_0(\beta_2) + k$. Then with $g_k(x) := f_k(x) + \langle \gamma_k, x \rangle$, $k = 0, \ldots, p - 1$, and $\gamma = l^{-1} \beta_1$ the function

$$F(x) = g_{\langle \gamma, x \rangle}(x)$$

is a bent function.

**Proof:** Observing that all requirements for the function defined as in (6) to be bent are satisfied, the statement follows since (6) reduces to $g_k(x)$ if $\langle \gamma, x \rangle = k$. $\square$

### 3 (Not) weakly regular bent polynomials from quadratic monomials and binomials

The simplicity of the representation of the bent functions in Corollary 1 equals the simplicity of the ingredient partially bent functions $f_k$. Hence the above procedure for the construction of bent functions motivates the study of partially bent functions from $F_{p^n}$ to $F_p$ with a 2-dimensional linear space and a simple representation. Since all quadratic functions are partially bent, it is natural to analyze elements of this class of functions (see e.g. 5, 11) starting with monomials and binomials (in trace form).

For quadratic monomials $f(x) = \text{Tr}_n(\alpha x^{p^r+1})$ the Fourier coefficients are known. The subsequent lemma gives the conditions under which we have a
linear space of dimension 2. For a proof for $p$ odd we refer to \([3, \text{Theorem 1}]\), the case $p = 2$ follows straightforward with the same approach.

**Lemma 3** Let $f(x) = \text{Tr}_n(\alpha x^{p^r+1})$, let $g$ be a primitive element of $\mathbb{F}_{p^n}$, and suppose that $\alpha = g^c$.

1. If $p = 2$, then $f$ is 2-plateaued if and only if
   (i) $n \equiv 0 \pmod{4}$, $r$ is odd and $3$ divides $c$, or 
   (ii) $n \equiv 2 \pmod{4}$, and $r$ is even or $3$ divides $c$.

2. If $p$ is odd, then $f$ is 2-plateaued if and only if $n$ is even, $r$ is odd, and $c$ satisfies the equation $y(p^2 - 1) + c(p^r - 1) = (p^n - 1)/2$ for some integer $y$.

The following proposition presents an infinite class of quadratic binomials with a 2-dimensional linear space.

**Proposition 2** Let $p$ be an odd prime, $n$ a positive integer divisible by 3, and let $r = 2\kappa$ for an integer $\kappa \geq 0$. The quadratic binomial $f(x) = \text{Tr}_n(\frac{p+1}{2}x^2 + x^{p^r+1})$ has a 2-dimensional linear space if and only if $\kappa = 0$, or $\kappa \geq 1$ and $n$ is odd.

**Proof:** With the standard Welch-squaring method we see that the linear space $\Lambda$ of $f$ is the kernel of the linearized polynomial (cf. \([2, \text{Equation (3.2)}])$

$$L(x) = x^{p^r} + x^{p^{2\kappa}} + x \in \mathbb{F}_{p^n}[x].$$

The dimension of $\Lambda$ is then the degree of $\gcd(x^n - 1, A(x)) = a(x)$ where $A(x) = x^r + x^{2\kappa} + 1$ is the associate of $L(x)$, if $a(x) = \sum a_i x^i$, then the kernel of $L(x)$ is the set of all solutions of $\sum a_i x^i$, see \([10, \text{p.118}]\). We recall that when $n = n_1 p^v$, $\gcd(n_1, p) = 1$, then the polynomial $x^n - 1 \in \mathbb{F}_{p^n}[x]$ can be factored as

$$x^n - 1 = \prod_{d|n_1} (\Phi_d(x))^{p^v},$$

where $\Phi_d(x) \in \mathbb{F}_{p^n}[x]$ is the $d$th cyclotomic polynomial which has degree $\varphi(d)$. Using that $\Phi_6(x) = x^2 - x + 1$ and $\Phi_6(x^{2^j}) = \Phi_6(2)(x)$ for $j \geq 0$, we see that for $r = 2^{\kappa}$, $\kappa \geq 0$, the polynomial $A(x)$ factors as

$$A(x) = x^{2^{\kappa+1}} + x^{2^{\kappa}} + 1 = (x^2 + x + 1) \prod_{j=1}^{\kappa} (x^{2^j} - x^{2^{j-1}} + 1)$$

$$= \Phi_3(x) \prod_{j=0}^{\kappa-1} \Phi_6(x^{2^j}) = \Phi_3(x) \prod_{j=0}^{\kappa-1} \Phi_{6,2^j}(x) = \prod_{j=0}^{\kappa} \Phi_{3,2^j}(x).$$
In view of Corollary 1, $F$ is described as $F(x) = \text{Tr}_6(gx^5)$ if $\text{Tr}_6((g^3 + g^2 + 1)x) = 0$ and $F(x) = \text{Tr}_6(g^{22}x^5)$ otherwise.

Expanding the trace terms we get the unique representation of $F(x)$ as a
polynomial of degree at most $2^6 - 1$, as one expects, as a rather complicated expression

$$F(x) = g_{51}x^{56} + g_{27}x^{52} + g_{12}x^{50} + g_{30}x^{49} + g^2x^{48} + g^3x^{44} + x^{42} + g^4x^{41} + g^{24}x^{40} + g^{27}x^{38} + g^{24}x^{37} + g^{15}x^{35} + g^{33}x^{34} + g^4x^{33} + g^7x^{32} + g^{57}x^{28} + g^{45}x^{26} + g^{6}x^{25} + g^x^{24} + g^{33}x^{22} + g^{12}x^{21} + g^{12}x^{20} + g^{45}x^{19} + g^{48}x^{17} + g^{15}x^{16} + g^{60}x^{14} + g^{54}x^{13} + g^{32}x^{12} + g^{48}x^{11} + g^6x^{10} + g^{49}x^8 + g^{30}x^7 + g^{16}x^6 + g^3x^5 + g^{56}x^4 + g^8x^3 + g^{28}x^2 + g^{14}x.$$  

The Fourier spectrum of $F$ is the multiset $\{-8^{28}, 8^{36}\}$, where the integer in the exponent denotes the multiplicity of the corresponding Fourier coefficient in $\{\hat{F}(b) \mid b \in F_p^n\}$. The algebraic degree of $F$ is 3.

We are particularly interested in a first direct construction of not weakly regular bent functions in the framework of finite fields. Taking advantage of the binomial description of 2-plateaued partially bent functions in Proposition 2 in the subsequent corollary we present an infinite class of not weakly regular bent functions in arbitrary odd characteristic with a simple description.

**Corollary 2** For an odd integer $n$ divisible by 3, let $\beta_1, \beta_2$ be solutions in $F_{p^n}$ of $x^{p^2} + x^p + x$ (which are linearly independent over $F_p$), let $\delta \in F_{p^n}$ such that $\Tr_n(\delta \beta_2^2) = l \neq 0$ and $\Tr_n(\delta \beta_1 \beta_2) = 0$ and let $\Gamma \in F_n$ such that $\Tr_n(\delta \Gamma \beta_2) = r \neq 0$. Let $r = 2^\kappa$, $\kappa \geq 1$, $a_0 = 1$, $c_0 = 0$, and for $a_k \in F_p$ let $c_k \in F_p$, $1 \leq k \leq p - 1$, be given as

$$c_k = t^{-1}[(1 - a_k)\Delta + k], \quad \text{where} \quad \Delta = \Tr_n\left(\frac{p+1}{2} \beta_2^2 + \beta_2^{p+1}\right). \quad (7)$$

Then the function

$$F(x) = a_{\Tr_n(\delta \Gamma x)}\Tr_n\left(\frac{p+1}{2} x^2 + x^{p^2+1}\right) + c_{\Tr_n(\delta \Gamma x)}\Tr_n(\delta \Gamma x) \quad (8)$$

is bent. It is not weakly regular if and only if $a_k$ is a nonsquare in $F_p$ for some $1 \leq k \leq p - 1$.

**Proof:** By the proof of Proposition 2 the 2-dimensional linear space of the partially bent function $f(x) = \Tr_n\left(\frac{p+1}{2} x^2 + x^{p^2+1}\right)$ consists of the solutions of $x^{p^2} + x^p + x$. Hence with this choice of $\beta_1, \beta_2$ and $\delta$ the requirements in Corollary 1 are satisfied. Let $f_k(x) = a_k x f(x)$, then with the definition of $c_k$ in (7), the partially bent functions $f_k$ satisfy $f_k(\beta_2) + \Tr_n(\delta \Gamma_k \beta_2) = f_0(\beta_2) + k$, ...
i.e. the functions $g_k(x) := f_k(x) + c_k \text{Tr}_n(\delta \Gamma x)$ satisfy $\text{supp}(\hat{g}_k) \cap \text{supp}(\hat{g}_j) = \emptyset$ for $0 \leq l \neq k \leq p - 1$. With Corollary 1 the function $F(x)$ is bent.

By Theorem 1 in [4], since $n$ is odd, the nonzero Fourier coefficients of $f(x)$ change the sign if we multiply $f(x)$ by a nonsquare in $\mathbb{F}_p$ (see also [2, Theorem 4.3]). Since we choose $g_0(x) = f_0(x) = f(x)$, we combine partially bent functions with Fourier coefficients of opposite sign if at least for one $1 \leq k \leq p - 1$ the coefficient $a_k$ is a nonsquare. For details on this technique of obtaining not weakly regular bent functions we refer to [2, 4].

We remark that $\beta_1, \beta_2$, the solutions of the linearized polynomial $x^{p^2} + x^p + x$ (which can be determined with standard methods using linear systems) are independent of $n$. For $x \in \mathbb{F}_{p^n}$, the value of the linear term $k = \text{Tr}_n(\delta \Gamma x)$ decides on the coefficient $a_k$ for the binomial part in (8) and the coefficient $c_k$ for the linear part in $F(x)$ in (7).

4 Conclusion

Based on earlier constructions of Boolean bent functions from near-bent functions (see [5, 9]), in [2, 3] constructions of bent functions in arbitrary characteristic have been presented. Amongst others, with these constructions the first infinite classes of not weakly regular bent functions were obtained. The functions are given in multivariate form or as functions from $\mathbb{F}_{p^{n-1}} \times \mathbb{F}_p$ to $\mathbb{F}_p$. Until then only sporadic examples of not weakly regular bent functions have been found via computer search, [6, 7, 13]. All of these sporadic examples were given in univariate form, i.e. as functions from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$ (represented as polynomials in trace form). In this article an equivalent but more involved procedure in the framework of functions from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$ is developed. In particular we obtain the first direct construction of not weakly regular bent functions in univariate form. We take advantage of some infinite classes of partially bent monomials and binomials with a 2-dimensional linear space, to construct (not) weakly regular bent functions from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$ with a simple representation.

5 Appendix

Example 2 Let $g$ be a root of the primitive polynomial $x^4 + 2x^3 + 2 \in \mathbb{F}_3[x]$. Then by Lemma 2 the functions

$$f_0(x) = f_2(x) = \text{Tr}_4(g^4x^{28}) \text{ and } f_1(x) = \text{Tr}_4(2g^4x^{28})$$
from $\mathbb{F}_3$ to $\mathbb{F}_3$ are 2-plateaued. For both functions the linear space is $\Lambda = \langle \beta_1, \beta_2 \rangle$ with $\beta_1 = g^2, \beta_2 = g^3 + 2g + 1$. We then have $\text{Tr}_4(\beta_1^2) = 2$, $\text{Tr}_4(\beta_1 \beta_2) = 0$, $\text{Tr}_4(\beta_2^2) \neq 0$. With Proposition 7 we get the near-bent functions from $\mathbb{F}_3$ to $\mathbb{F}_3$

$g_{0|3}(x) = \text{Tr}_4(g^4x^{28})$, $g_{1|3}(x) = \text{Tr}_4(2g^4x^{28} + x)$, $g_{2|3}(x) = \text{Tr}_4(g^4x^{28} + 2x)$

with $\text{supp}(\widehat{g}_{j|3}) \cap \text{supp}(\widehat{g}_{k|3}) = \emptyset$ if $j \neq k$. With Theorem 7 we obtain the weakly regular bent function

$F(x) = 2[(\text{Tr}_4(\gamma x) - 1)(\text{Tr}_4(\gamma x) - 2)g_0(x) + \text{Tr}_4(\gamma x)(\text{Tr}_4(\gamma x) - 2)g_1(x) + \text{Tr}_4(\gamma x)(\text{Tr}_4(\gamma x) - 1)g_2(x)]$

or alternatively

$F(x) = \begin{cases} 
\text{Tr}_4(g^4x^{28}) & \text{if } \text{Tr}_4(\gamma x) = 0 \\
\text{Tr}_4(2g^4x^{28} + x) & \text{if } \text{Tr}_4(\gamma x) = 1 \\
\text{Tr}_4(g^4x^{28} + 2x) & \text{if } \text{Tr}_4(\gamma x) = 2,
\end{cases}$

with Fourier spectrum $\{-9^{21}, -9\epsilon_3^{30}, -9\epsilon_3^{20}\}$.

**Example 3** Let $g$ be a root of the primitive polynomial $x^3 + 2x + 1 \in \mathbb{F}_3[x]$. According to Proposition 2 the functions

$f_0(x) = \text{Tr}_3(2x^2 + x^{10})$, $f_1(x) = \text{Tr}_3(2x^2 + x^4)$

are partially bent functions from $\mathbb{F}_3$ to $\mathbb{F}_3$. For each function, the linear space is $\Lambda = \langle \beta_1, \beta_2 \rangle$ with $\beta_1 = g$ and $\beta_2 = 1$. Again we have $\text{Tr}_3(\beta_1^2) = 2 \neq 0$ and $\text{Tr}_6(\beta_1 \beta_2) = 0$, but now $\text{Tr}_6(\beta_2^2) = 0$, see Remark 2.

**A**: Put $f_2(x) = f_1(x)$, then with Proposition 1 we obtain the near-bent functions from $\mathbb{F}_2$ to $\mathbb{F}_3$

$g_{0|2}(x) = \text{Tr}_3(2x^2 + x^{10})$, $g_{1|2}(x) = \text{Tr}_3(2x^2 + x^4 + 2g^2x)$, $g_{2|2}(x) = \text{Tr}_3(2x^2 + x^4 + g^2x)$

with $\text{supp}(\widehat{g}_{j|2}) \cap \text{supp}(\widehat{g}_{k|2}) = \emptyset$ if $j \neq k$. With $\gamma = 2g$, Theorem 7 yields the weakly regular bent function

$F(x) = 2[(\text{Tr}_3(2gx) - 1)(\text{Tr}_3(2gx) - 2)g_0(x) + \text{Tr}_3(2gx)(\text{Tr}_3(2gx) - 2)g_1(x) + \text{Tr}_3(2gx)(\text{Tr}_3(2gx) - 1)g_2(x)]$

$= \begin{cases} 
\text{Tr}_3(2x^2 + x^{10}) & \text{if } \text{Tr}_3(gx) = 0 \\
\text{Tr}_3(2x^2 + x^4 + 2g^2x) & \text{if } \text{Tr}_3(gx) = 2 \\
\text{Tr}_3(2x^2 + x^4 + g^2x) & \text{if } \text{Tr}_3(gx) = 1,
\end{cases}$
with the Fourier spectrum \{(-3^{3/2}/2i)^9, (-3^{3/2}i\epsilon_3)^{12}, (-3^{3/2}i\epsilon_3^2)^6}\.

B: Put \( f_2(x) = 2f_1(x) = \text{Tr}_3(x^2 + 2x^4) \), such that the nonzero Fourier coefficients of \( f_1 \) and \( f_2 \) have opposite signs. Again with Proposition 1, we obtain the near-bent functions from \( V_2 \) to \( \mathbb{F}_3 \)

\[ g_{0|2}(x) = \text{Tr}_3(2x^2 + x^{10}), \quad g_{1|2}(x) = \text{Tr}_3(2x^2 + x^4 + 2g^2x), \quad g_{2|2}(x) = \text{Tr}_3(x^2 + 2x^4 + g^2x) \]

with \( \text{supp}(\hat{g}_{j|2}) \cap \text{supp}(\hat{g}_{k|2}) = \emptyset \) if \( j \neq k \), and then with Theorem 1 the not weakly regular bent function

\[
F(x) = 2[(\text{Tr}_3(2gx) - 1)(\text{Tr}_3(2gx) - 2)g_0(x) + \text{Tr}_3(2gx)(\text{Tr}_3(2gx) - 2)g_1(x) \\
+ \text{Tr}_3(2gx)(\text{Tr}_3(2gx) - 1)g_2(x)]
\]

\[
= \begin{cases} 
\text{Tr}_3(2x^2 + x^{10}) & \text{if } \text{Tr}_3(gx) = 0 \\
\text{Tr}_3(2x^2 + x^4 + 2g^2x) & \text{if } \text{Tr}_3(gx) = 2 \\
\text{Tr}_3(x^2 + 2x^4 + g^2x) & \text{if } \text{Tr}_3(gx) = 1,
\end{cases}
\]

where again \( \gamma = 2g \). As one would expect, the unique polynomial representation of this not weakly regular bent function does not look very simple:

\[
F(x) = g^8x^{24} + g^3x^{22} + x^{21} + x^{20} + gx^{19} + g^1x^{18} + g^{11}x^{16} + 2x^{15} + g^{16}x^{14} \\
+ g^3x^{13} + g^8x^{12} + g^9x^{11} + g^5x^{10} + g^3x^9 + g^2x^8 + x^7 + g^3x^6 + 2x^5 \\
+ g^{21}x^4 + gx^3 + g^9x^2 + 2x.
\]

The Fourier spectrum of \( F \) is

\{(-3^{3/2}/2i)^3, (-3^{3/2}/2i)^6, (3^{3/2}i\epsilon_3)^3, (-3^{3/2}i\epsilon_3)^9, (3^{3/2}i\epsilon_3^2)^3, (-3^{3/2}i\epsilon_3^2)^3\}. The algebraic degree of \( F \) is 4. We remark that \( F \) attains the upper bound on the algebraic degree of bent functions, see [3, 8].

References

[1] C. Carlet, Partially-bent functions. Designs, Codes, Cryptogr. 3 (1993), 135–145.

[2] A. Çeşmelioğlu, G. McGuire, W. Meidl, A construction of weakly and non-weakly regular bent functions. J. Comb. Theory, Series A 119 (2012), 420–429.

[3] A. Çeşmelioğlu, W. Meidl, Bent functions of maximal degree. IEEE Trans. Inform. Theory 58 (2012), 1186–1190.
[4] A. Çeşmelioğlu, W. Meidl, A Construction of bent functions from plateaued functions. Designs, Codes, Cryptogr. 66 (2013), 231–242.

[5] P. Charpin, E. Pasalic, C. Tavernier, On bent and semi-bent quadratic Boolean functions. IEEE Trans. Inform. Theory 51 (2005), 4286–4298.

[6] T. Helleseth, A. Kholosha, Monomial and quadratic bent functions over the finite fields of odd characteristic. IEEE Trans. Inform. Theory 52 (2006), 2018–2032.

[7] T. Helleseth, A. Kholosha, New binomial bent functions over the finite fields of odd characteristics. IEEE Trans. Inform. Theory 56 (2010), 4646–4652.

[8] X.D. Hou, p-ary and q-ary versions of certain results about bent functions and resilient functions. Finite Fields Appl. 10 (2004), 566–582.

[9] G. Leander, G. McGuire, Construction of bent functions from near-bent functions. Journal of Combinatorial Theory, Series A 116 (2009), 960–970.

[10] R. Lidl, H. Niederreiter, Finite Fields, 2nd ed., Encyclopedia Math. Appl., vol. 20, Cambridge Univ. Press, Cambridge, 1997.

[11] W. Meidl, A. Topuzoğlu, Quadratic functions with prescribed spectra. Designs, Codes, Cryptogr. 66 (2013), 257–273.

[12] K. Nyberg, Perfect nonlinear S-boxes. In: Proceedings of Advances in Cryptology - Eurocrypt’ 91, Lecture Notes in Computer Science 547, Springer-Verlag, Berlin 1991, 378–386.

[13] Y. Tan, J. Yang, X. Zhang, A recursive approach to construct p-ary bent functions which are not weakly regular. In: Proceedings of IEEE International Conference on Information Theory and Information Security, Beijing, 2010, to appear.