COMBINATORIAL FORMULAS FOR KAZHDAN-LUSZTIG POLYNOMIALS WITH RESPECT TO W-GRAph IDEALS

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ABSTRACT. Let \((W, S, L)\) be a weighted Coxeter system and \(J\) a subset of \(S\), Yin \([12]\) introduced the weighted \(W\)-graph ideal \(E_J\) and the weighted Kazhdan-Lusztig polynomials \(\{P_{x,y} \mid x, y \in E_J\}\). In this paper, we study the combinatorial formulas for \(P_{x,y}\), which will extend the results of Brenti \([2]\) and Deodhar \([5]\).

1. Introduction

Let \((W, S)\) be a Coxeter system, we denote by \(\ell\) and \(\leq\) the length function and the Bruhat order on \(W\), respectively.

\(W\)-graph is introduced by Kazhdan and Lusztig \([7]\). A \(W\)-graph provides a method for constructing a matrix representation of \(W\) and the entries of this matrix are called Kazhdan-Lusztig polynomials. However, these polynomials are not easy to compute.

Brenti \([2]\) provided a combinatorial algorithm for Kazhdan-Lusztig polynomials, which is depended only on \(R\)-polynomials. Later, Deodhar \([3]\) introduced the parabolic Kazhdan-Lusztig polynomials associated with a standard parabolic subgroup \(W_J\) of \(W\), and showed two analogous combinatorial formulas for these parabolic cases in \([5]\). Brenti’s result corresponds exactly to the case \(J = \emptyset\). Then, Tagawa \([10]\) generalized these formulas to weighted parabolic Kazhdan-Lusztig polynomials.

Howlett and Nguyen \([6]\) introduced the concept of \(W\)-graph ideals and showed that a \(W\)-graph can be constructed from a \(W\)-graph ideal \(E_J \subseteq W\), where \(W\) is equipped with the left weak Bruhat order \(\leq_L\). They constructed the Kazhdan-Lusztig polynomials with respect to a \(W\)-graph ideal. Deodhar’s construction exactly corresponds to the case \(E_J = D_J\), where \(D_J\) is the set of minimum coset representatives of \(W_J\).

Recently, Yin \([12]\) and \([13]\) generalized the construction of \(W\)-graph ideals to a weighted Coxeter system \((W, S, L)\). He showed that there exists a pair of dual modules \(M(E_J, L)\) and \(\tilde{M}(E_J, L)\) with respect to a given \(W\)-graph ideal \(E_J\). Similarly, Yin constructed the weighted Kazhdan-Lusztig polynomials \(\{P_{x,y} \mid x, y \in E_J\}\) and the inverse weighted Kazhdan-Lusztig polynomials \(\{Q_{x,y} \mid x, y \in E_J\}\).

In this paper, we continue the work in \([12]\) and \([13]\) with slightly different on the ground ring. We consider Hecke algebras over the ring of Laurent polynomials \(\mathbb{Z}[q^{1/2}, q^{-1/2}]\), where \(q_s = q^{L(s)}\) and \(L\) is a weight function.

This paper is organized as follows. In Section 2, we review basic concepts concerning weighted Coxeter groups and Hecke algebras. In Section 3 and 4, we modify the concept of \(W\)-graph ideals to our setting and the weighted Kazhdan-Lusztig polynomials on \(M(E_J, L)\) are also considered. In Section 5 and 6, we devote to giving combinatorial formulas for \(\{P_{x,y} \mid x, y \in E_J\}\) and coefficients of those polynomials. In Section 7, we show similar combinatorial formulas for the inverse weighted Kazhdan-Lusztig polynomials.

Key words and phrases. Hecke algebra, \(W\)-graph, \(W\)-graph ideal, Kazhdan-Lusztig polynomial.
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2. Preliminaries

In this section, we follow the conventions in [12]. Let $\Gamma$ be a totally ordered abelian group with zero element $0$ and $\leq$ the order on $\Gamma$.

Let $\mathbb{Z}[\Gamma]$ be a free $\mathbb{Z}$-module with a basis set $\{q^\gamma \mid \gamma \in \Gamma\}$ and the multiplication is given by $q^\gamma q^\zeta = q^{\gamma+\zeta}$, where $q$ is an indeterminant. For any $f \in \mathbb{Z}[\Gamma]$, we denote by $[q^\gamma]$ the coefficient of $q^\gamma$ on $[q^\gamma]$. If $f \neq 0$, then the degree of $f$ is defined to be $\deg(f) := \max \{\gamma \mid [q^\gamma] \neq 0\}$, and we set $\deg(0) = -\infty$. Then, the map $\deg : \mathbb{Z}[\Gamma] \to \Gamma \cup \{-\infty\}$ satisfies $\deg(fg) = \deg(f) + \deg(g)$.

For any reduced word $w = s_1s_2\cdots s_k$, a map $L : W \to \Gamma$ is called a weight of $W$ if

$$L(w) = L(s_1) + L(s_2) + \cdots + L(s_k).$$

Throughout, we assume that $L(s) \geq 0$ for any $s \in S$.

Let $\mathcal{H} := \mathcal{H}(W, S, L)$ be the weighted Hecke algebra corresponding to $(W, S)$ with parameters $\{q_s^{1/2} \mid s \in S\}$, where $q_s = q^{\ell(s)}$. It is well-known that $\mathcal{H}$ is a free $\mathbb{Z}[\Gamma]$-module and $\mathcal{H}$ has a basis set $\{T_w \mid w \in W\}$. In particular, the multiplication is given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ q_s T_{sw} + (q_s - 1)T_w & \text{if } \ell(sw) < \ell(w). \end{cases}$$

We denote by $- : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]$ the automorphism of $\mathbb{Z}[\Gamma]$ induced by sending $\gamma$ to $-\gamma$ for any $\gamma \in \Gamma$. This can be extended to a ring involution $- : \mathcal{H} \to \mathcal{H}$ such that

$$\sum_{w \in W} a_w \cdot T_w = \sum_{w \in W} a_w \cdot T_w = \sum_{w \in W} \overline{a_w} \cdot T_w,$$

and $\overline{T_s} = q_s^{-1}T_s + (q_s^{-1} - 1)$ for any $s \in S$.

For any $J \subseteq S$, let $W_J := \langle J \rangle$ be the corresponding parabolic subgroup of $W$. We denote the set of minimum coset representatives of $W_J$ by

$$D_J = \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in J\}.$$

Let $\leq_L$ be the left weak Bruhat order on $W$. We say $x \leq_L y$ if and only if $y = zx$ and $\ell(y) = \ell(z) + \ell(x)$ for some $z \in W$. If this is the case, then $x$ is said to be a suffix of $y$.

Definition 2.1. ([12] Definition 2.3) Let $E \subseteq W$, we define

$$\text{Pos}(E) := \{s \in S \mid \ell(xs) > \ell(x) \text{ for all } x \in E\}.$$

Obviously, $\text{Pos}(E)$ is the largest subset $J$ of $S$ satisfying $E \subseteq D_J$. Let $E$ be an ideal in poset $(W, \leq_L)$, that is, $E$ is a subset of $W$ such that if $u \in W$ is a suffix of $v \in E$, then $u = v$. This implies that $\text{Pos}(E) = S/E = \{s \in S \mid s \notin E\}$.

Let $J$ be a subset of $\text{Pos}(E)$ such that $E \subseteq D_J$. In this context, we shall denote by $E_J$ the set $E$ with respect to $J$. For each $y \in E_J$, we define the following subsets of $S$.

$$SD(y) := \{s \in S \mid \ell(sy) < \ell(y)\},$$

$$SA(y) := \{s \in S \mid \ell(sy) > \ell(y) \text{ and } sy \in E_J\},$$

$$WD(y) := \{s \in S \mid \ell(sy) > \ell(y) \text{ and } sy \notin D_J\},$$

$$WA(y) := \{s \in S \mid \ell(sy) > \ell(y) \text{ and } sy \in D_J/E_J\}. $$
It is obvious that for each \( y \in E_J \), each \( s \in S \) appears in exactly one of the four sets defined above (since \( E_J \subseteq D_J \)).

## 3. W-graph ideal modules

We recall from [13] the definition of W-graph ideals with some modifications.

**Definition 3.1.** ([13] Definition 1.1, Modified) A W-graph consists of the following data.

1. A vertex set \( \Lambda \) together with a map \( I \) which assigns a subset \( I(x) \subseteq S \) to each vertex \( x \in \Lambda \).
2. For each \( s \in S \) with \( L(s) = 0 \), there is an edge: \( x \to sx \). For each \( s \in S \) with \( L(s) > 0 \), there is a collection of edges such that \( \mu^s_{x,y} = \mu^s_{y,x} \) and \( \{ \mu^s_{x,y} \in \mathbb{Z}[\Gamma] \mid x, y \in \Lambda, s \in I(x), s \notin I(y) \} \).
3. Let \([A]_{\mathbb{Z}[\Gamma]}\) be a free \( \mathbb{Z}[\Gamma] \)-module with basis \( \{b_y \mid y \in \Lambda\} \). We require that the map \( T_s \to \tau_s \) gives an \( \mathcal{H} \)-representation of \( \mathcal{H} \) with the following multiplication.

\[
\tau_s(b_y) = \begin{cases} 
   b_{sy} & \text{if } L(s) = 0; \\
   -b_y & \text{if } L(s) > 0, s \in I(y); \\
   q_s b_y + q_s^{1/2} \sum_{x \in \Lambda, s \in I(x)} \mu^s_{x,y} b_x & \text{if } L(s) > 0, s \notin I(y).
\end{cases}
\]

**Definition 3.2.** ([13] Definition 2.4, Modified) Let \((W,S,L)\) be a weighted Coxeter system and \( E_J \subseteq W \), we call \( E_J \) a W-graph ideal if the following holds.

1. There is a \( \mathbb{Z}[\Gamma]-\)free \( \mathcal{H} \)-module \( M(E_J, L) \) with a basis \( \{\Gamma_y \mid y \in E_J\} \) such that

\[
T_s \Gamma_y = \begin{cases} 
   q_s \Gamma_{sy} + (q_s - 1) \Gamma_y & \text{if } s \in SD(y), \\
   \Gamma_{sy} & \text{if } s \in SA(y), \\
   -\Gamma_y & \text{if } s \in WD(y), \\
   q_s \Gamma_y - \sum_{z < y, z \in E_J} r^s_{z,y} \Gamma_z & \text{if } s \in WA(y),
\end{cases}
\]

for some polynomials \( r^s_{z,y} \in q_s \mathbb{Z}[\Gamma] \).

2. The \( \mathcal{H} \)-module \( M(E_J, L) \) admits a \( \mathbb{Z}[\Gamma] \)-semilinear involution \( \Gamma_y \to \widetilde{\Gamma}_y \) satisfying \( \widetilde{\Gamma}_1 = \Gamma_1 \) and \( \widetilde{T_w \Gamma_y} = \widetilde{T_w} \widetilde{\Gamma}_y \) for all \( T_w \in \mathcal{H}, \Gamma_y \in M(E_J, L) \).

We call \( M(E_J, L) \) the W-graph ideal module and refer to [13] for the definition of another W-graph ideal module \( \widetilde{M}(E_J, L) \).

There exists an algebra homomorphism \( \Phi : \mathcal{H} \to \mathcal{H} \) given by \( \Phi(q_s) = -q_s \) and \( \Phi(T_w) = \epsilon_w q_w \widetilde{T_w} \) for all \( s \in S, w \in W \), where the bar is the standard involution on \( \mathcal{H} \). Furthermore, \( \Phi^2 = \text{id} \) and \( \Phi \) commutes with the bar involution.

**Proposition 3.3.** ([13] Theorem 3.1) There exists a unique homomorphism \( \eta \) from \( M(E_J, L) \) to \( \widetilde{M}(E_J, L) \) such that \( \eta(\Gamma_1) = \widetilde{\Gamma}_1 \) and \( \eta(T_w \Gamma_y) = \Phi(T_w) \eta(\Gamma_y) \) for all \( T_w \in \mathcal{H} \) and \( \Gamma_y \in M(E_J, L) \). Moreover,

1. \( \eta \) is a bijection and the inverse \( \eta^{-1} \) satisfies properties of \( \eta \).
2. \( \eta \) commutes with the involution on \( M(E_J, L) \) and \( \widetilde{M}(E_J, L) \).

**Proof.** Let \( \eta(\Gamma_y) = \epsilon_y q_y \Gamma_y \), then it is easy to see the proposition. \( \square \)
4. Kazhdan-Lusztig polynomials with respect to $W$-graph ideals

In this section, we recall the weighted Kazhdan-Lusztig polynomials introduced by Yin [12] and [13]. First, let $\Gamma' := \{L(s) \mid s \in S\}$ and we consider the totally order $\leq$ on $\Gamma$. Then, we have $\mathbb{Z}[\Gamma'] \subseteq \mathbb{Z}[\Gamma]$ and $\mathbb{Z}[\Gamma'] = \{ \prod q_i^{n_i} \mid s_i \in S, n_i \in \mathbb{Z} \}$.

4.1. The weighted $R$-polynomials on $M(E_J, L)$. We recall from [12] the construction of $\{R_{x,y} \mid x, y \in E_J\}$ and $\{\tilde{R}_{x,y} \mid x, y \in E_J\}$ with some modifications.

**Definition 4.1.** There is a unique family of polynomials $R_{x,y} \in \mathbb{Z}[\Gamma]$ satisfying

$$\Gamma^y = \sum_{x \in E_J} \epsilon_x \epsilon_y q_y^{-1} R_{x,y}^x \Gamma_x,$$

we call them the weighted $R$-polynomials on $M(E_J, L)$.

We assume that $R_{x,y} = 0$ if $x \notin E_J$ or $y \notin E_J$. Then, we have

**Proposition 4.2.** Let $x, y \in E_J$, then $R_{x,y} = 0$ if $x \neq y$, $R_{x,y} = 1$ if $x = y$.

**Proposition 4.3.** For any $x, y \in E_J$, we have $R_{x,y} \in \mathbb{Z}[\Gamma']$ and

$$0 \leq \deg (R_{x,y}) \leq L(y) - L(x).$$

There are some further properties for $R_{x,y}$ as follows.

**Lemma 4.4.** ([12] Lemma 4.3, [13] Corollary 3.2, Modified) Let $x, y \in E_J$, then

1. $R_{x,y} = \epsilon_x \epsilon_y q_y^{-1} \tilde{R}_{x,y}$.
2. $\sum_{x \in t \subseteq y, t \in E_J} \epsilon_t \epsilon_y R_{x,t} \tilde{R}_{t,y} = \delta_{x,y}$ (Kronecker delta).

**Proof.** To prove the first statement, we apply the function $\eta$ to both sides of the formula

$$\Gamma^y = \sum_{x \in E_J} \epsilon_x \epsilon_y q_y^{-1} R_{x,y}^x \Gamma_x,$$

and use the fact that $\eta$ commutes with the involution to compare with the formula

$$\tilde{\Gamma}^y = \sum_{x \in E_J} \epsilon_x \epsilon_y q_y^{-1} \tilde{R}_{x,y} \tilde{\Gamma}_x,$$

then we can see the result.

To prove the second statement, we use (1) and the equality $\tilde{\Gamma}_y = \Gamma_y$,

$$\Gamma_y = \sum_{x \in E_J} \epsilon_x \epsilon_y q_y R_{x,y} \Gamma_x = \sum_{t \subseteq y, x \in t} \epsilon_x \epsilon_t R_{x,t} \tilde{R}_{t,y} \Gamma_x = \sum_{x \in y} \left( \sum_{x \in t \subseteq y} \epsilon_x \epsilon_t R_{x,t} \tilde{R}_{t,y} \right) \Gamma_x,$$

Therefore, we have

$$\sum_{x \in t \subseteq y} \epsilon_t \epsilon_y R_{x,t} \tilde{R}_{t,y} = \epsilon_x^{-1} \epsilon_y \sum_{x \in t \subseteq y} \epsilon_x \epsilon_t R_{x,t} \tilde{R}_{t,y} = \epsilon_x^{-1} \epsilon_y \delta_{x,y} = \delta_{x,y}.$$

**Corollary 4.5.** Let $x, y \in E_J$, then

$$\sum_{x < t \subseteq y, t \subseteq E_J} \epsilon_x \epsilon_t \tilde{R}_{x,t} \tilde{R}_{t,y} = \delta_{x,y} - R_{x,y}.$$

We can easily find the following relation between $R_{x,y}$ and other kinds of $R$-polynomials.
Remark 4.6. With the above conventions, we have

1. If $E = D$ and $x, y \in D$, then
   \[ R_{x,y} = R'_{x,y}(q) \psi, \quad u_s = -1, \]
   \[ \tilde{R}_{x,y} = R'_{x,y}(q) \psi, \quad u_s = q_s, \]
   where $R'_{x,y}(q) \psi$ is a weighted parabolic $R$-polynomial ([10]), $u_s$ is a solution of the equation $u_s^2 = q_s + (q_s - 1) u_s$ and $\psi$ is a weight of $W$ into $\mathbb{Z}[\Gamma]$.

2. If $E = W$ (i.e. $J = \emptyset$) and $x, y \in W$, then
   \[ R_{x,y} = \tilde{R}_{x,y} = R'_{x,y}, \]
   where $R'_{x,y}$ is a weighted $R$-polynomial ([9]). If further $q_s = q$ for all $s \in S$, then
   \[ R_{x,y}(q) = \tilde{R}_{x,y}(q) = R_{x,y}(q), \]
   where $R_{x,y}(q)$ is a $R$-polynomial ([7]).

4.2. The weighted Kazhdan-Lusztig polynomials on $M(E_J, L)$.

Proposition 4.7. There exists a unique family of polynomials \( \{ P_{x,y} \in \mathbb{Z}[\Gamma] \mid x, y \in E_J \} \) satisfying the following condition:

\[ q_x^{-1} q_y P_{x,y} = \sum_{x \leq t \leq y, t \in E_J} R_{x,t} P_{t,y}. \]

We call them the weighted Kazhdan-Lusztig polynomials on $M(E_J, L)$.

Proof. First, we show the existence of $P_{x,y}$ by induction on $\ell(y) - \ell(x)$. If $\ell(y) - \ell(x) = 0$, then $P_{x,x} = P_{x,x} \iff P_{x,x} = 1$. If $\ell(y) - \ell(x) \neq 0$, then it follows our assumption and Corollary 4.5 that

\[ F := \sum_{x < t \leq y, t \in E_J} R_{x,t} P_{t,y} \]

\[ = q_x q_y^{-1} \sum_{x < t \leq y, t \in E_J} \epsilon_x \epsilon_t \tilde{R}_{x,t} \left( \sum_{t \leq z \leq y, z \in E_J} R_{t,z} P_{z,y} \right) \]

\[ = q_x q_y^{-1} \sum_{x < t \leq y, z \in E_J} \delta_{x,t} P_{t,y} - q_x q_y^{-1} \sum_{x < t \leq y, t \in E_J} R_{x,t} P_{t,y} \]

\[ = -q_x q_y^{-1} \sum_{x < t \leq y, t \in E_J} R_{x,t} P_{t,y} \]

This implies

\[ F + P_{x,y} = q_x q_y^{-1} (F + P_{x,y}) + \mathbb{F} = q_x q_y^{-1} P_{x,y}, \]

Therefore, the existence of $P_{x,y}$ is shown by replacing $F$. The uniqueness of $P_{x,y}$ is obvious by the method of the proof of the existence above. \qed

Similarly, we set $P_{x,y} = 0$ if $x \notin E_J$ or $y \notin E_J$. 

Corollary 4.8. Assume that $x < y \in E_J$, then $P_{x,y} \in \mathbb{Z}[\Gamma']$ and
\[ 0 \leq \deg(P_{x,y}) < \frac{L(y) - L(x)}{2}. \]

Proof. It follows Proposition 4.3 and Proposition 4.7 that $P_{x,y} \in \mathbb{Z}[\Gamma']$ and the inequality on the left is true. Let $d := \deg(P_{x,y})$ and note that
\[ q_x^{-1/2} q_y^{1/2} P_{x,y} - q_x^{1/2} q_y^{-1/2} P_{x,y} = q_x^{1/2} q_y^{-1/2} \sum_{x < t \leq y, i \in E_J} R_{x,t} P_{t,y}, \]
we have
\[ \frac{L(y) - L(x)}{2} - d \leq \deg\left(q_x^{-1/2} q_y^{1/2} P_{x,y}\right) \leq \frac{L(y) - L(x)}{2}, \]
\[ \frac{L(x) - L(y)}{2} \leq \deg\left(q_x^{1/2} q_y^{-1/2} P_{x,y}\right) \leq \frac{L(x) - L(y)}{2} + d. \]

Then, by the uniqueness of $P_{x,y}$,
\[ \frac{L(x) - L(y)}{2} + d < 0 < \frac{L(y) - L(x)}{2} - d. \]

Therefore, the inequality on the right is also true. \qed

Remark 4.9. It is worth mentioning that the $\mathcal{H}$-module $M(E_J, L)$ has a unique basis \{ $C_y \mid y \in E_J$ \} such that $C_y = C_y$ for any $y \in E_J$, and
\[ C_y = \sum_{x \leq y \in E_J} \epsilon_x \epsilon_y q_x^{-1/2} q_y^{1/2} P_{x,y} \Gamma_x. \]

This is the so-called Kazhdan-Lusztig basis.

The relation between $P_{x,y}$ and other kinds of Kazhdan-Lusztig polynomials is very similar to Remark 4.6, we do not show details.

5. The combinatorial formulas for $P_{x,y}$

In this section, we define $\mathcal{H}$-polynomials on $M(E_J, L)$ and show combinatorial formulas for $\{P_{x,y} \mid x, y \in E_J\}$, which extend the results of Tagawa [10] and Deodhar [5].

For any $\gamma \in \Gamma$, we define the following truncation functions.
\[ U_\zeta \left( \sum_{\gamma \geq 0} [q^\gamma] q^\gamma \right) = \sum_{\gamma \geq 0} [q^\gamma] q^\gamma \text{ and } L_\zeta \left( \sum_{\gamma < 0} [q^\gamma] q^\gamma \right) = \sum_{\gamma < 0} [q^\gamma] q^\gamma. \]

Definition 5.1. Assume that $E_J$ is a $W$-graph ideal.

1. Let $\mathcal{J}_k(x, y) = \{ \varphi : x = x_0 < x_1 < \cdots < x_{k+1} = y \mid x_i \in E_J \}$ be the set of all $E_J$-chains of length equal to $k + 1$, where $x_0$ is called the initial element of $\varphi$ and $x_{k+1}$ is called the final element of $\varphi$. Let $x < y$, we denote by $\mathcal{J}(x, y) = \bigcup_{k \geq 0} \mathcal{J}_k(x, y)$ the set of all $E_J$-chains.

2. Let $\mathcal{M}_k(x, y) = \{ \varphi : x = x_0 \leq x_1 \leq \cdots \leq x_{k+1} = y \mid x_i \in E_J \}$ be the set of all $E_J$-multichains of length equal to $k + 1$, and we denote by $\mathcal{M}(x, y) = \bigcup_{k \geq 0} \mathcal{M}_k(x, y)$ the set of all $E_J$-multichains.

Note that an $E_J$-chain is an $E_J$-multichain as well. Conversely, for each $E_J$-multichain, there exists a unique $E_J$-chain which is obtained by removing the repetitions.

We define a family of polynomials which is depended only on the weighted $R$-polynomials $R_{x,y}$. We call them $\mathcal{H}$-polynomials on $M(E_J, L)$. 

Definition 5.2. Assume that \( x \leq y \in E_J \) and \( \varphi : x = x_0 \leq x_1 \leq \cdots \leq x_{r+1} = y \) is a \( E_J \)-multichain, let
\[
\mathcal{R}_\varphi = \begin{cases} 
 q_{x_1}^{-1} q_y R_{x,y} & \text{if } r = 0, \\
 R_{x_1,1} U_{L(x)-L(x_1)} \left( q_{x_1}^{-1} q_y \mathcal{R}_{\varphi'} \right) & \text{if } r \geq 1.
\end{cases}
\]
where \( \varphi' : x_1 \leq x_2 \leq \cdots \leq y \in \mathcal{M}(x_1, y) \).

Then, we show some properties of \( \mathcal{R} \)-polynomials.

Proposition 5.3. Assume that \( x \leq y \in E_J \), \( \varphi \in \mathcal{M}(x, y) \), \( \varphi' \in \mathcal{M}(x_1, y) \) and \( \mathcal{R}_\varphi \neq 0 \).

1. Let \( d(\varphi) \) be the maximum power of \( q \) that divides \( \mathcal{R}_\varphi \), then
\[
\frac{L(y)-L(x_1)}{2} < \deg(\mathcal{R}_\varphi) \leq L(y) - L(x) - d(\varphi').
\]

2. For all \( i \in \{2, 3, \ldots, \ell(\varphi)\} \), we have \( x_{i-1} < x_i \).

Proof. (1) The inequality on the left follows the definition of \( \mathcal{R}_\varphi \). Since
\[
\deg(\mathcal{R}_\varphi) = \deg(\mathcal{R}_{x_1}) + \deg \left( U_{L(x)-L(x_1)} \left( q_{x_1}^{-1} q_y \mathcal{R}_{\varphi'} \right) \right)
\]
\[
\leq L(x_1) - L(x) + \deg \left( q_{x_1}^{-1} q_y \mathcal{R}_{\varphi'} \right) = L(y) - L(x) - d(\varphi'),
\]
the inequality on the right is also true.

(2) We show this by induction on \( \ell(\varphi) \). If \( \ell(\varphi) = 2 \), then \( \varphi = (x_0, x_1, x_2) \) and \( x_1 < x_2 \) (there is a contradiction between \( \mathcal{R}_{x_0,x_1,x_2} = 0 \) and our assumption if \( x_1 = x_2 \)).

Suppose that the inequality holds when \( \ell(\varphi) \leq k \), \( (k \geq 2) \), then we show it for \( \ell(\varphi) = k + 1 \). Since \( \mathcal{R}_{x_0,x_1,\cdots,x_{k+1}} \neq 0 \),
\[
\deg \left( q_{x_1}^{-1} q_{x_{k+1}} \mathcal{R}_{x_2,\cdots,x_{k+1}} \right) > \frac{L(y)-L(x_1)}{2}.
\]
Then, by statement (1), we have \( L(x_1) < L(x_2) \). This is equivalent to \( x_1 < x_2 \). Therefore,
\[
x_1 < x_2 < x_3 < \cdots < x_{k+1}
\]
follows our inductive hypothesis. \( \square \)

The following corollaries are obvious.

Corollary 5.4. Let \( x \leq y \in E_J \) and \( \varphi \in \mathcal{M}(x, y) \). If \( \ell(y) - \ell(x) < \ell(\varphi) - 1 \), then \( \mathcal{R}_\varphi = 0 \).

Corollary 5.5. For any \( x \leq z \in E_J \) and \( k \in \mathbb{N} \), let \( z^{(k)} := (z, z, \ldots, z) \in (E_J)^k \). Then, \( \mathcal{R}_{z^{(k)}} = 0 \) if \( k \geq 3 \), \( \mathcal{R}_{x,z^{(k)}} = 0 \) if \( k \geq 2 \).

Corollary 5.6. For any \( x, y \in E_J \), we have
\[
\mathcal{R}_{x,y} = \epsilon_x \epsilon_y q_x q_y^{-1} \mathcal{R}_{x,y},
\]
where \( \mathcal{R}_{x,y} \) is the \( \mathcal{R} \)-polynomials on \( \mathcal{M}(E_J, L) \).

In order to prove our main result in this section, we require the following lemmas.

Lemma 5.7. Let \( x < y \in E_J \), then
\[
P_{x,y} = \mathcal{R}_{x,y} = \sum_{x \leq t \leq y, t \in E_J} q_t^{-1} q_y \mathcal{R}_{x,t} \mathcal{P}_{t,y}.
\]

Proof. By applying the involution \( \mathcal{T} \) on \( q_x^{-1} q_y \mathcal{P}_{x,y} = \sum_{x \leq t \leq y, t \in E_J} R_{x,t} \mathcal{P}_{t,y} \), one can get the required result. \( \square \)
Lemma 5.8. For any $x < y \in E_J$, we have

$$[q^0] \left( \sum_{\varphi \in \mathcal{I}(x,y)} \mathcal{R}_\varphi \right) = \begin{cases} [q_x^{-1}q_y] R_{x,y} & \text{if } \ell(\varphi) = 1, \\ 0 & \text{if } \ell(\varphi) \geq 2, \end{cases}$$

Proof. It is easy to prove by the definition of $\mathcal{R}_\varphi$. □

We now have the following result as described in the introduction.

Theorem 5.9. Assume that $E_J$ is a W-graph ideal and $x, y \in E_J$.

1. If $x < y$, then $P_{x,y} = L_{\frac{L(y) - L(x)}{2}} \left( \sum_{\varphi \in \mathcal{I}(x,y)} \mathcal{R}_\varphi \right)$.

2. If $x \leq y$, then $P_{x,y} = \sum_{\varphi \in \mathcal{I}(x,y)} \mathcal{R}_\varphi$.

Proof. According to Corollary 5.4, the sum on the right side of (1) and (2) is finite. We show (1) by induction on $\ell(y) - \ell(x)$. If $\ell(y) - \ell(x) = 1$, then we use the involution on two sides of the equation introduced in Proposition 4.7, and we get $P_{x,y} - q_x^{-1}q_y P_{x,y} = q_x^{-1}q_y P_{x,y}$. It follows the definition of $\mathcal{R}_\varphi$ and Proposition 4.8 that $P_{x,y} = L_{\frac{L(y) - L(x)}{2}} (\mathcal{R}_{x,y})$.

Assume that (1) holds when $\ell(y) - \ell(x) < k$ and we show it for $\ell(y) - \ell(x) = k$. First, by Lemma 5.7, we have

$$[q^0] (P_{x,y}) - [q^0] (\mathcal{R}_{x,y}) = [q^0] \left( \sum_{x \leq t < y, t \in E_J} q_t^{-1}q_y \mathcal{R}_{x,t} P_{t,y} \right).$$

Since $\deg (P_{x,y}) < \frac{L(y) - L(x)}{2}$ and Lemma 5.8, it is easy to check that

$$[q^0] (P_{x,y}) = [q^0] \left( L_{\frac{L(y) - L(x)}{2}} \left( \sum_{\varphi \in \mathcal{I}(x,y)} \mathcal{R}_\varphi \right) \right).$$

Assume again that the following holds for all pairs $x' < y'$ and $\gamma' < \gamma$ ($q^\gamma \in \mathbb{Z}[\Gamma]$),

$$[q^\gamma] (P_{x,y}) - [q^\gamma] (q_x^{-1}q_y P_{x,y}) = [q^\gamma] \left( \sum_{\varphi \in \mathcal{I}(x,y)} \mathcal{R}_\varphi \right).$$

Similarly, by Lemma 5.7, we have

$$[q^\gamma] (P_{x,y}) - [q^\gamma] (\mathcal{R}_{x,y}) - [q_x^{-1}q_y P_{x,y}] (P_{x,y})$$

$$= \sum_{x < t < y, t \in E_J} \sum_{0 \leq t \leq \gamma} [q^{\gamma - t}] (\mathcal{R}_{x,t}) [q^\xi] (q_t^{-1}q_y P_{t,y}),$$

and

$$[q^\gamma] (q_t^{-1}q_y P_{t,y}) = \begin{cases} 0 & \text{if } \xi \leq \frac{L(y) - L(t)}{2}, \\ [q_t^{-1}q_y P_{t,y}] (P_{t,y}) & \text{if } \xi > \frac{L(y) - L(t)}{2}. \end{cases}$$

Then, we substitute the latter one into the former one,
prove by Corollary 4.8.

and Corollary 5.4, we have

This completes the proof of (2).

For any \( x < t < y \) and \( \frac{L(y) - L(t)}{2} < \xi \leq \gamma \), we have \( L(y) - L(t) - \xi < \xi < \gamma \). Then, following our assumption,

\[
[q^{-\xi}] (P_{t,y}) - [q^{-\xi}] (P_{t,y}) = [q^{-\xi}] (P_{t,y}) \left( \sum_{\varphi' \in \mathcal{F}(t,y)} R_{\varphi'} \right).
\]

However, \([q^{\xi}] (P_{x,y}) = 0 \) (since \( \xi \geq \frac{L(y) - L(t)}{2} \)). By the definition of \( R_{\varphi} \), Proposition 5.3 and Corollary 5.4, we have

\[
[q^{-\xi}] (P_{x,y}) - [q^{-\xi}] (R_{x,y}) - [q^{-\xi}] (P_{x,y}) = \sum_{\varphi \in \mathcal{F}(x,y)} [q^{-\xi}] (R_{\varphi}) - \sum_{\varphi \in \mathcal{F}(x,y)} [q^{-\xi}] (R_{\varphi}) = [q^{-\xi}] (R_{x,y})
\]

The above is equivalent to \( P_{x,y} - q^{-1} q_y P_{x,y} = \sum_{\varphi \in \mathcal{F}(x,y)} R_{\varphi} \). Therefore, (1) is easy to prove by Corollary 4.8.

Now, we can show the statement (2). If \( x = y \), then (2) is true since \( P_{x,x} = 1 = R_{x,x} \). If \( x < y \), then by (1), Proposition 5.3, Corollary 5.4, Corollary 5.5 and the fact \( U_{a+b} (q^n P) = q^n L_{a+b}(P) \), we have

\[
P_{x,y} = \sum_{\varphi \in \mathcal{F}(x,y)} R_{\varphi} + q^{-1} q_y L_{\frac{L(y) - L(x)}{2}} (\sum_{\varphi \in \mathcal{F}(x,y)} R_{\varphi})
\]

\[
= \sum_{\varphi \in \mathcal{F}(x,y)} R_{\varphi} + \sum_{\varphi \in \mathcal{F}(x,y)} U_{L(y) - L(x)} (q^{-1} q_y R_{\varphi})
\]

\[
= \sum_{\varphi \in \mathcal{F}(x,y)} R_{\varphi} + \sum_{\varphi \in \mathcal{F}(x,y)} R_{x,\varphi} = \sum_{x < t < y, \varphi \in \mathcal{F}(x,y), t \in E_J} R_{x,\varphi} + \sum_{\varphi \in \mathcal{F}(x,y)} R_{x,\varphi} + \sum_{\varphi \in \mathcal{F}(x,y)} R_{\varphi}
\]

\[
= \sum_{x < t < y, \varphi \in \mathcal{F}(x,y), t \in E_J} R_{x,\varphi} + \sum_{\varphi \in \mathcal{F}(x,y)} R_{\varphi} = \sum_{x < t < y, \varphi \in \mathcal{F}(x,y), t \in E_J} R_{x,\varphi} + \sum_{\varphi \in \mathcal{F}(x,y)} R_{\varphi}
\]

This completes the proof of (2). \( \square \)
Moreover, we have the following corollary immediately. Let \( \Psi = \bigcup_{k \leq \ell(y)-\ell(x)} \mathcal{J}_k(x,y) \) and \( \Upsilon = \bigcup_{k \leq \ell(y)-\ell(x)} \mathcal{M}_k(x,y) \).

**Corollary 5.10.** Let \( x, y \in E_I \).

1. If \( x < y \), then \( P_{x,y} = L_{L(y)-L(x)} \left( \sum_{\varphi \in \Psi} \mathcal{R}_\varphi \right) \).
2. If \( x \leq y \), then \( P_{x,y} = \sum_{\varphi \in \Upsilon} \mathcal{R}_\varphi \).

**Proof.** It follows Proposition 5.4 that \( \mathcal{R}_\varphi = 0 \) if \( \varphi \in \bigcup_{k \geq \ell(y)-\ell(x)+1} \mathcal{M}_k(x,y) \). Then, the corollary follows Theorem 5.9. \( \square \)

### 6. The Coefficients of \( P_{x,y} \)

The purpose in this section is to obtain explicit formulas for the coefficients of \( P_{x,y} \). Let

\[ \Gamma'' := \{ \sum n_i L(s_i) \mid s_i \in S, n_i \in \mathbb{Z}, i \in \mathbb{N} \} \]

Then, by Theorem 5.9 and Corollary 5.10, we immediately have the following results.

**Corollary 6.1.** Assume that \( x, y \in E_I \) and \( \gamma \in \Gamma'' \).

1. If \( x < y \), then \( [q^\gamma] (P_{x,y}) = [q^\gamma] \left( \sum_{\varphi \in \Psi} \mathcal{R}_\varphi \right) \).
2. If \( x \leq y \), then \( [q^\gamma] (P_{x,y}) = [q^\gamma] \left( \sum_{\varphi \in \Upsilon} \mathcal{R}_\varphi \right) \).

Before to show the formulas, we have to show the following.

**Lemma 6.2.** Assume that \( x \leq y \in E_I \), \( r \in \mathbb{N} \), \( \varphi \in \mathcal{M}(x,y) \) and \( \gamma \in \Gamma'' \), then

\[ [q^\gamma] (\mathcal{R}_\varphi) = \sum_{S \in \mathcal{F}_\varphi} \prod_{i=0}^r \left[ q_{x_{i+1}} q_y q_0 q_{x_i} q_y q_1 \right] (R_{x_{i+1}}) \]

where we set \( S := (\lambda_0, \lambda_1, \ldots, \lambda_{r+1}) \) and

\[ \mathcal{F}_\varphi := \{ (a_0, a_1, \ldots, a_{r+1}) \in (\Gamma'')^{r+2} \mid a_0 = L(y) - L(x) - \gamma, \]

\[ \gamma \geq a_1 > a_2 > \cdots > a_r > a_{r+1} = 0, \]

\[ a_i > L(y) - L(x_i) - a_i \geq a_{i+1} \ for \ i \in \{1, 2, \ldots, r\} \} \]

**Proof.** It follows the definition of \( \mathcal{R}_\varphi \) and \( \lambda_0 = L(y) - L(x) - \gamma \) that

\[ [q^\gamma] (\mathcal{R}_\varphi) = \sum_{\gamma > \lambda_1 \geq 0} [q^{-\lambda_1}] (\mathcal{R}_{x_{i+1}}) [q_{x_{i+1}} q_y q_1] (\mathcal{R}_\varphi) \]

\[ = \sum_{S' \in \mathcal{F}_{L(y)-L(x_1)-\lambda_1} (\varphi')} \left( \prod_{i=1}^r \left[ q_{x_{i+1}} q_y q_1 q_0^{-1} q_{x_i} q_y q_{x_{i+1}} q_y q_{x_i} \right] (R_{x_{i+1}}) \right) \]

\[ = \sum_{S \in \mathcal{F}_\varphi} \prod_{i=0}^r \left[ q_{x_{i+1}} q_y q_0^{-1} q_{x_i} q_y q_{x_{i+1}} \right] (R_{x_{i+1}}) \],
Lemma 7.2. Let
\[ \mathcal{F}_{L(y)-L(x_1)-1}(\varphi') = \{ (a_1, a_2, \cdots, a_{r+1}) \in (\Gamma')^{r+1} | \]
\[ a_1 = \lambda_1, \]
\[ L(y) - L(x_1) - \lambda_1 \geq a_2 > a_3 > \cdots > a_r > a_{r+1} = 0, \]
\[ a_i > L(y) - L(x_i) - a_i \geq a_{i+1} \text{ for } i \in \{2, 3, \cdots, r\} \}. \]

This completes the proof of the lemma. \[\square\]

Theorem 6.3. Assume that \( x, y \in E_J, \varphi \in \mathcal{M}(x, y), r \in N \) and \( \gamma \in \Gamma'' \), then
\[ [q^\gamma] (P_{x,y}) = [q^\gamma] (R_{x,y}) \]
\[ + \sum_{r=1}^{\ell(y)-\ell(x)} \sum_{\varphi \in \mathcal{M}(x,y)} \sum_{S \in \mathcal{F}_{y}(\varphi)} \prod_{i=1}^{r} [q_{x,i+1}q_{y}^{-1}q_{i}^{\lambda_i}q_{i+1}^{\lambda_i}] (R_{x_i,x_{i+1}}). \]

Proof. Following Corollary 5.4 and Theorem 5.9, one can easily check that
\[ [q^\gamma] (P_{x,y}) = [q^\gamma] (R_{x,y}) + \sum_{r=1}^{\ell(y)-\ell(x)} \sum_{\varphi \in \mathcal{M}(x,y)} [q^\gamma] (R_{\varphi}). \]

The result is a straightforward consequence of Lemma 6.2 and the definition of \( R_{\varphi} \). \[\square\]

7. The inverse weighted Kazhdan-Lusztig polynomials

In this section, we recall from [13] the construction of \( \{Q_{x,y} \mid x, y \in E_J\} \) and give combinatorial formulas for those polynomials, which are similar to \( \{P_{x,y} \mid x, y \in E_J\} \). This also extends the results of [11] and [5].

Let \( y \in E_J \), the formula for \( C_y \) introduced in Remark 4.9 may be rewritten as
\[ q_y^{1/2} C_y = \sum_{x \leq y, x \in E_J} \epsilon_x \epsilon_y P_{x,y} q_x^y \Gamma_x, \]
and inverting this gives
\[ q_y^y \Gamma_y = \sum_{x \leq y, x \in E_J} Q_{x,y} q_x^{1/2} C_x, \]
where \( Q_{x,y} \) is given recursively by
\[ \sum_{x \leq t \leq y, t \in E_J} \epsilon_t \epsilon_y Q_{x,t} P_{t,y} = \delta_{x,y}. \]

Proposition 7.1. There exists a unique family of polynomials \( \{Q_{x,y} \in \mathbb{Z}[\Gamma'] \mid x, y \in E_J\} \) satisfying \( Q_{x,y} = 0 \) if \( x \not\succeq y \), \( Q_{x,x} = 1 \) and
\[ 0 \leq \deg (P_{x,y}) < \frac{L(y)-L(x)}{2}. \]

The following is similar to [9] Section 10 and [13] Subsection 3.3. We omit the proof.

Lemma 7.2. Let \( x \leq y \in E_J \), then
\[ q_x^{-1} q_y Q_{x,y} = \sum_{x \leq t \leq y, t \in E_J} Q_{x,t} \tilde{R}_{t,y}. \]

Next, we will show some results which can be proved similar to Section 5 and Section 6. Therefore, we describe only the statement of results and the proofs are omitted.

Definition 7.3. Assume that \( x \leq y \in E_J \) and \( \varphi \in \mathcal{M}(x, y) \), we define
\[ R_\varphi = \begin{cases} 
    q_x^{-1} q_y \tilde{R}_{x,y} & \text{if } \ell(\varphi) = 1, \\
    U(L(x_r) - L(x))/2 \left( q_x^{-1} q_y \tilde{R}_\varphi^* \right) \tilde{R}_{x_r,y}^* & \text{if } \ell(\varphi) \geq 2.
\end{cases} \]

where \( \varphi' : x = x_0 < x_1 \leq \cdots < x_r \in \mathcal{M}(x, x_r) \).

**Theorem 7.4.** Assume that \( x, y \in E_J \).

1. If \( x < y \), then \( Q_{x,y} = L(x,y) - L(x)(x,y) \sum_{\varphi \in \mathcal{M}(x,y)} \tilde{R}_\varphi \).
2. If \( x \leq y \), then \( Q_{x,y} = \sum_{\varphi \in \mathcal{M}(x,y)} \tilde{R}_\varphi^* \).

**Theorem 7.5.** Assume that \( x, y \in E_J \) and \( \gamma \in \Gamma'' \).

1. If \( x < y \), then \([q^\gamma] \left( Q_{x,y} \right) = [q^\gamma] \left( L(x,y) - L(x) \sum_{\varphi \in \mathcal{M}(x,y)} \tilde{R}_\varphi \right) \).
2. If \( x \leq y \), then \([q^\gamma] \left( Q_{x,y} \right) = [q^\gamma] \left( \sum_{\varphi \in \mathcal{M}(x,y)} \tilde{R}_\varphi^* \right) \).

**Theorem 7.6.** Assume that \( x, y \in E_J, \varphi \in \mathcal{M}(x, y), r \in N \) and \( \gamma \in \Gamma'' \), then
\[ [q^\gamma] \left( Q_{x,y} \right) = [q^\gamma] \left( \ell(y) - \ell(x) \right) + \sum_{r=1}^{\ell(y)-\ell(x)} \sum_{\varphi \in \mathcal{M}(x,y)} \sum_{\psi \in \mathcal{F}_\gamma^*(\varphi)} \prod_{i=1}^{r} \left[ q_x q_{x_{r-i}}^{-1} q_{\lambda_i}^{-1} q_{\lambda_{i+1}}^{-1} \right] \left( \tilde{R}_{x_{r-i}, x_{r-i+1}} \right), \]

where we set \( S : = (\lambda_0, \lambda_1, \cdots, \lambda_r) \) and
\[ \mathcal{F}_\gamma^*(\varphi) := \{(a_0, a_1, \cdots, a_{r+1}) \in (\Gamma'^{\gamma})^{r+2} | \]
\[ a_0 = L(y) - L(x) - \gamma, \]
\[ \gamma \preceq a_1 \preceq a_2 \preceq \cdots \preceq a_r \preceq a_{r+1} = 0, \]
\[ a_i \geq L(x_{r-i}) - L(x) - a_i \geq a_{i+1} \text{ for } i \in \{1, 2, \cdots, r\} \} \}.

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