On Structural Properties of Optimal Feedback Control of Traffic Flow under the Cell Transmission Model

Saeid Jafari and Ketan Savla

Abstract

In this paper, we investigate the structure of the finite-time optimal feedback control for freeway traffic networks modeled by the Cell Transmission Model. Piecewise affine supply and demand functions are considered and optimization with respect to a general linear objective function is studied. Using the framework of multi-parametric linear programming, we show that the optimal feedback control can be represented in a closed-form by a piecewise affine function on polyhedra of the network traffic density. The resulting optimal feedback control law, however, has a centralized structure and requires instantaneous access to the state of the entire network that may lead to prohibitive communication requirements in large-scale complex networks. We subsequently examine the design of a decentralized optimal feedback controller with a one-hop information structure, wherein the optimum outflow rate from each segment of the network depends only on the density of that segment and the density of the segments immediately downstream. The decentralization is based on the relaxation of constraints that depend on state variables that are unavailable according to the information structure. The resulting decentralized control scheme has a simple closed-form representation and is scalable to arbitrary large networks; moreover, we demonstrate that, with respect to certain meaningful linear performance indexes, the performance loss due to decentralization is zero; namely, the centralized optimal controller has a decentralized realization with a one-hop information structure and is obtained at no computational cost.

I. INTRODUCTION

In infrastructure networks with wide geographical distributions, a fast, efficient, and reliable control is essential. Examples of such networks include: transportation, natural-gas, water, and crude oil networks. In recent years, due to the ever-increasing traffic demand, efficient control...
and management of transportation networks has received a great deal of attention. There has been a lot of research done on the optimal control of freeway networks based on various models for traffic systems, among which first-order models, such as the Cell Transmission Model (CTM) [1], are widely used for control design because of their analytical simplicity. The CTM is a simple macroscopic traffic model for most phenomena observed on highways including flow conservation, non-negativity, and congestion wave propagation, wherein a one-way road is partitioned into several segments called cells and the traffic flow in each segment is viewed as a homogeneous stream [2].

Since the size and complexity of transportation networks are growing, design and implementation of a control law providing an optimum operation has become more challenging and demanding. The optimal control design under the CTM dynamics using ramp metering and variable speed limits is a nonlinear non-convex optimization problem [3]–[6]. A commonly-used approach to make the computation of the optimal control trajectory tractable is to design the optimal control for a relaxed version of the problem [4]–[8]. It is, then, shown that the original and relaxed problems are equivalent in terms of the optimal solution. For instance, in [8], an exact convex-relaxation technique was proposed consisting of relaxing the supply and demand constraints, and then designing control actions (i.e., ramp metering, speed limit, and routing) to make the optimal solution of the relaxed problem feasible with respect to the original dynamics. Convexity allows one to readily use available efficient software tools such as CVX [9], [10] as well as to adapt well-known distributed algorithms for computations [11].

Another approach for optimal traffic control is the model predictive control (MPC) (also known as receding horizon control or moving horizon control) which is a model-based feedback control technique relying on a real-time optimization solver [3], [6], [12]–[14]. Although the closed-loop operation of the MPC provides a certain degree of robustness with respect to modeling uncertainties, the primary challenge of implementing MPC in real-time is computational resource; moreover, determination of the optimal control action at each time step involves centralized operations that may make its implementation for large-scale networks costly or impractical.

In order to enable effective real-time computations of the optimal control actions for large-size complex traffic networks based on the CTM, some research has focused on distributed and cooperative approaches. In [15], a cooperative distributed online algorithm is proposed for an optimal multiple-intersection traffic signal control problem. Also, a real-time traffic signal timing plan is developed in [16] by using a distributed method to optimize the network performance
with respect to a certain performance metric.

The existing results on finite-time optimal control of freeway networks based on the CTM are mainly restricted to schemes with an open-loop feedforward structure that are not robust in most actual applications. To the best of our knowledge, no research is carried out on the structural properties and decentralization of the optimal feedback control law for the finite-horizon control of freeway networks. Some insights were provided into the optimal closed-loop control design in [8] under some restrictive conditions, such as identical slope for all supply and demand functions and a specific cost function. However, extensions beyond these simple settings does not exist in the literature.

In this paper, we examine the structure of finite-time optimal closed-loop control for freeway traffic networks modeled by the CTM with respect to a general linear performance measure. We focus on a discrete-time setting, piecewise affine supply and demand functions, and cost functions that are linear in traffic densities and flows. The latter is particularly motivated by the fact that several widely-used performance metrics such as travel time, travel distance, and delay can be expressed in a linear form. Throughout the paper, we focus on the relaxed dynamics, and implicitly assume that one can apply the control design technique from [8] to make the resulting dynamics feasible.

The key enabler in this paper to find a closed-form solution for the optimal feedback control law is the framework of multi-parametric programming [17]. The main contributions of this paper are as follows: (i) by utilizing the connection between the optimal control problem for freeway networks modeled by the CTM and multi-parametric linear programming, it is shown that the optimal feedback control law is representable in a closed-form by a piecewise affine function on polyhedra of the network traffic density; (ii) a technique is proposed to design of a decentralized optimal feedback controller with a one-hop information structure, that is the optimum outflow rate from each segment of the network depends only on the density of that segment and the density of the segments immediately downstream; (iii) it is demonstrated that with respect to certain meaningful linear objective functions, the centralized feedback optimal control law has a decentralized realization with a one-hop information structure, for which the proposed decentralization approach gives a global optimal solution.

It should be highlighted that the optimal control is not necessarily unique; the primary objective of this paper is to study the existence of an optimal feedback control law in a closed form, with respect to a class of performance indexes, which has a simple structure and is suitable for
practical implementations in complex traffic networks.

The rest of the paper is organized as follows: Section II presents some preliminaries and notations used throughout the paper. The problem is defined and formulated in Section III. The main results of the paper are presented in Sections IV and V, and concluding remarks are summarized in Section VII.

II. PRELIMINARIES AND NOTATIONS

Throughout this paper, the set of integers \{1, 2, \ldots, n\} is denoted by \(\mathbb{N}_n\), and \(\{(a_i)_{i \in \mathbb{N}_n}\} = \{a_1, a_2, \ldots, a_n\}\). A convex polyhedron is the intersection of finitely many half-spaces, i.e., \(\{x \in \mathbb{R}^n | Ax \leq b\}\), for a matrix \(A \in \mathbb{R}^{m \times n}\) and a vector \(b \in \mathbb{R}^m\). A real-valued function \(f(x)\) on \(D \subseteq \mathbb{R}^n\) is said to be increasing (decreasing) if for every \(x, y \in D\) such that \(x_i \leq y_i, \forall i \in \mathbb{N}_n\), we have \(f(x) \leq f(y) (f(x) \geq f(y))\); or equivalently, if \(f(x)\) is increasing (decreasing) in every coordinate.

**Theorem 1:** [17] Consider the following multi-parametric linear program

\[
J^*(\theta) = \min_{z} c^\top z \\
\text{s.t. } Wz \leq G + S\theta, \quad \theta \in \Omega_{\theta} \subseteq \mathbb{R}^m,
\]

where \(z \in \mathbb{R}^n\) is the decision variables vector and \(\theta\) is a parameter vector, \(\Omega_{\theta}\) is a closed polyhedral set, and \(c, W, G, S\) are constant matrices. Let \(\Omega_{\theta}^*\) denote the region of parameters \(\theta\) such that (1) is feasible. Then, there exists an optimizer \(z^*(\theta) : \Omega_{\theta}^* \to \mathbb{R}^n\) which is a continuous and piecewise affine function of \(\theta\), that is

\[
z^* = pwa(\theta)
\]

\[
= L_i\theta + l_i, \quad \text{if } \theta \in \mathcal{R}_i, \quad i \in \mathbb{N}_p,
\]

where \(\mathcal{R}_i = \{\theta \in \Omega_{\theta}^* | \Pi_i\theta \leq \eta_i\}\), for \(i \in \mathbb{N}_p\), form a polyhedral partition of \(\Omega_{\theta}^*\), \(p\) is the number of polyhedral sets, \(L_i, l_i, \Pi_i, \eta_i\) are some constant matrices, and \(pwa(\cdot)\) is a generic symbol for piecewise affine functions on polyhedral sets. Moreover, the value function \(J^*(\theta) : \Omega_{\theta}^* \to \mathbb{R}\) is a continuous, convex, and piecewise affine function of \(\theta\).

The Matlab-based Multi-Parametric Toolbox [18] together with YALMIP Toolbox [19] can be used to solve multi-parametric linear programs and compute the matrices \(L_i, l_i, \Pi_i, \eta_i\) in (2).
III. PROBLEM FORMULATION

We consider the discrete-time CTM [1] to describe the evolution of the network over time. For simplicity of presentation, we first focus on a linear transportation network consisting of a long series connection of numerous segments with controllable outflow rates with no intermediate on-ramps or off-ramps, as shown in Figure 1. Extension of the results to a more general class of networks will be carried out in Section V.

Fig. 1. (a) An $n$-cell linear transportation network with no intermediate on/off-ramps, where $\rho^k_i$, $u^k_i$, $\lambda^k$, and $\ell_i$ respectively denote the traffic density of cell $i$ at time $k$, the outflow rate from cell $i$ to the downstream cell $i+1$ at time $k$, an exogenous inflow rate to the network at time $k$, and the length of cell $i$. The outflow rates, $u^k_i$'s, are to be adjusted to optimize a certain performance index. (b) The corresponding directed graph of the network topology, where edges represent cells and vertices represent interface between consecutive cells.

Let $\rho^k_i$ denote the traffic density [veh/mi] of cell $i$ at time step $k$, i.e., during the time interval $[kT_s, (k+1)T_s)$, where $T_s$ [hr] is the duration of the discrete time steps. According to the law of mass conservation, the density of cell $i$ at time step $k + 1$ satisfies [4]:

$$\rho^{k+1}_i = \rho^k_i + \frac{T_s}{\ell_i}(y^k_i - u^k_i), \quad i \in \mathbb{N}_n,$$

(3)

where $k = 0, 1, \ldots, N - 1$, $\ell_i$ is the length [mi] of cell $i$, $y^k_i$ [veh/hr] is the inflow rate to cell $i$ at time $k$, and $u^k_i$ [veh/hr] is the outflow rate from cell $i$ at time $k$. In a linear network shown in Figure 1 where the cells are increasingly numbered from upstream to downstream, we have $y^k_i = u^k_{i-1}$. For the first cell, the inflow rate is an exogenous signal denoted by $\lambda^k$. In addition, the flow rates must satisfy the following constraints at each time:

$$0 \leq u^k_i \leq \min\{v_i \rho^k_i, C_i\},$$

$$0 \leq y^k_i \leq \min\{w_i (\gamma_i - \rho^k_i), C_i\},$$

(4)

where $v_i$ is the maximum traveling free-flow speed [mi/hr] of cell $i$, $w_i$ is the backward congestion wave traveling speed [mi/hr] of cell $i$, $C_i$ is the maximum flow capacity [veh/hr] of cell $i$, and
\( \gamma_i \) is the jam traffic density [veh/mi] of cell \( i \). For the linear network shown in Figure 1 we have \( y^k_i = \lambda^k \) and \( y^k_i = u^k_{i-1} \), for \( i = 2, \ldots, n \).

**Remark 1:** In a linear network (as shown in Figure 1), the first cell is an on-ramp and its inflow rate \( y^k_1 = \lambda^k \) is an exogenous uncontrolled variable. In order to ensure that \( \lambda^k \) is a feasible inflow rate signal, it is typically assumed that the jam traffic density of an on-ramp is infinity i.e., \( \gamma_1 = \infty \), and at each time the inflow rate satisfies \( \lambda^k \leq C_1 \).

**Assumption 1:** The length of cells \( \ell_i \) and the time interval \( T_s \) are chosen such that vehicles traveling at maximum speed \( v_i \) can not cross multiple cells in one time step, i.e., \( v_i T_s \leq \ell_i, \forall i \). Also, the backward congestion wave traveling speed \( w_i \) satisfies \( w_i T_s \leq \ell_i, \forall i \).

Assumption 1 is known as Courant-Friedrichs-L\`{e}vy condition \([8]\) which is a necessary condition for numerical stability in numerical computations. It can be easily verified that Assumption 1 and constraints \((4)\) ensure that at each time the density of each cell is non-negative and never exceeds the jam density.

It is often more convenient to express the dynamics and constraints in terms of the traffic mass of the cells. Let \( x^k_i = \ell_i \mu^k_i \) denote the traffic mass [veh] of cell \( i \) at time \( k \), then for a linear network, the dynamics of the system can be expressed in terms of \( x^k_i \) and \( u^k_i \) as follows:

\[
x^{k+1}_i = x_i^k + T_s (u^k_{i-1} - u^k_i), \quad i \in \mathbb{N}_n, \\
0 \leq u^k_i \leq \min\{(v_i/\ell_i)x^k_i, C_i\}, \quad i \in \mathbb{N}_n, \\
u^k_i \leq \min\{w_{i+1}(\gamma_{i+1} - (1/\ell_{i+1})x^k_{i+1}), C_{i+1}\}, \quad i \in \mathbb{N}_{n-1},
\]

where \( u^k_0 \equiv \lambda^k \) is an exogenous inflow rate to the network.

**Control Objective:** Consider the network dynamics \((5)\) and let \( x^k = [x^k_1, \ldots, x^k_n]^T \) be the state vector and \( u^k = [u^k_1, \ldots, u^k_n]^T \) be the control input vector of the network at time \( k \). The control objective is to design a feedback control law to generate control actions \( u^k \) such that for any initial state \( x^o \) and any exogenous inflow \( \lambda^k \), a class of cost functions of the following form over a fixed given control horizon \([0, N]\) is minimized:

\[
\min_{u^0, \ldots, u^{N-1}} J(x^0, \lambda) = \psi^N(x_N) + \sum_{k=0}^{N-1} \psi^k(x^k, u^k),
\]

subject to \((5)\), where \( N \) is a fixed final time. In this paper, we are interested in cost functions wherein \( \psi^k(x^k, u^k) \) is a linear function of \( x^k \) and \( u^k \), and \( \psi^N(x_N) \) is a linear function of \( x_N \),
that is

\[
\psi^N(x^N) = \sum_{i=1}^{n} \alpha_i^N x_i^N,
\]

\[
\psi^k(x^k, u^k) = \sum_{i=1}^{n} \alpha_i^k x_i^k + \beta_i^k u_i^k,
\]

where \(\alpha_i^k, \beta_i^k\) are cost-weighting parameters.

**Remark 2:** There are several meaningful linear performance indexes of practical interest which can be expressed in a linear form \([6], [8], [14]\); for example:

(i) minimization of the total travel time of the network is equivalent to minimization of the total number of vehicles in the entire network, then the corresponding cost is

\[
J = \sum_{k=0}^{N} \sum_{i=1}^{n} x_i^k.
\]

(ii) maximization of the total travel distance is equivalent to maximization of the flows, then the following cost should be minimized

\[
J = -\sum_{k=0}^{N-1} \sum_{i=1}^{n} u_i^k.
\]

(iii) the total congestion delay is defined as the time difference between actual travel time and the travel time in free-flow conditions whose minimization is equivalent to minimizing

\[
J = \sum_{k=0}^{N-1} \sum_{i=1}^{n} (x_i^k - (\ell_i/v_i)u_i^k).
\]

It should be emphasized that the relationships given in (5) are valid for linear transportation networks where the cells are numbered from upstream to downstream in an increasing order. Generalization to a more general class of networks is presented in Section V.

**IV. Finite-Time Optimal Feedback Control**

In this section, we consider the finite-time optimal control problem (5)-(7) for networks with a line graph as shown in Figure 1 and design a feedback optimal control law. We, subsequently, study the structural properties of the controller and design a decentralized optimal controller with a specific information structure.

**A. Centralized Control Design**

In the presence of the exogenous input \(\lambda^k\), computation of a control signal that optimizes the performance index \((6)\) and satisfies all the constraints in \(5\) at each time requires the knowledge of the sequence of \(\lambda^k\) over the entire control horizon which is often not available in advance.
Therefore, assuming that $\lambda^k$ is not known beforehand, we design a controller that optimizes the worst-case performance (i.e., for $\lambda^k = 0$, $\forall k$) that produces feasible optimal control actions for any feasible exogenous inflow rate (see Remark 1).

**Theorem 2:** The optimal control for the finite-time optimal control problem (5)-(7) can be expressed in the form of a continuous piecewise affine feedback law on polyhedra of the state vector as

$$(u^k)^* = pwa^k(x^k)$$

$$= F^k_i x^k + f^k_i, \text{ if } x^k \in R^k_i,$$

where $R^k_i = \{x \in \mathbb{R}^n | H^k_i x \leq h^k_i\}$, $i \in \mathbb{N}_{p^k}$, is the $i$th polyhedral partition of the set of feasible states, and $p^k$ is the number of polyhedral partitions at time $k$. The matrices $F^k_i, f^k_i, H^k_i, h^k_i$ are independent of $x^k$ and $\lambda^k$, $\forall k$, and can be computed offline.

**Proof:** The proof is given in the Appendix.

**Remark 3:** The parameters of the optimal feedback control law in (11) are obtained by setting the exogenous input $\lambda^k$ to zero. Then, the feasibility of the optimal control action in guaranteed for any feasible nonzero $\lambda^k$. If the trajectory of $\lambda^k$ over the control horizon is known a priori, then it can be used in control design; in this case, the parameters of the optimal controller will depend on the exogenous inflow rate to the network.

As is well known, an advantage of the closed-loop feedback control law over the open-loop control is that it can account for modeling uncertainties, noise, and disturbances as they occur. The closed-form of the control law (11) enables one to compute the controller parameters offline and stored in computer memory before the control actions are ever applied to the network. That is, there is no need to solve a large-size optimization problem at every time step for real-time implementation, unless there is a large variation in the network parameters. The optimal feedback controller (11), however, suffers from two major drawbacks that restrict its applicability to large-scale networks: (i) even though the piecewise affine form of the control law seems to be simple, as the number of cells and the control horizon increase, solving the corresponding multi-parametric linear programs may result in a very large number of polyhedral partitions making the structure of the controller complex; although applying the merging algorithms [18], [20] may considerably reduce the number of regions, in general there may be too many polyhedral sets; and (ii) determining the optimal control action at each time involves centralized operations, that is each local controller needs instantaneous access to the state of the entire network; this,
however, may not be feasible for large-size networks, as implementation of a highly reliable and fast communication system may be impractical or too costly.

It is, therefore, necessary to design an optimal feedback control law with a simple structure that requires access only to local information. Decentralized optimal control problems are often significantly more complex than the corresponding problems with centralized information. A trivial centralized optimal decision-making problem may become NP-hard under a decentralized information structure [21]. Therefore, most research has been focused on the design of meaningful suboptimal decentralized control policies and identification of tractable subclasses of problems [22], [23].

In the following subsection, we design a class of decentralized optimal control scheme and use the optimal centralized controller as a reference for performance evaluation.

B. Decentralized Control Design

In this subsection, for the problem formulated in Section [IV] we design a decentralized optimal feedback controller with a one-hop information structure as defined below.

Definition 1: A state-feedback controller is said to have an (outer) one-hop information structure, if for any \( k, i \), the control action \( u^k_i \) depends only on the state of cell \( i \), \( x^k_i \), and the state of cell(s) immediately downstream of cell \( i \).

The above definition implies that for a linear network as shown in Figure [1] a feedback control law with a one-hop information structure is of the form

\[ u^k_i = \phi^k_i(x^k_i, x^k_{i+1}) \]

In order to design a feedback controller with a one-hop information structure, we follow the same procedure as that used in the proof of Theorem [2] for the design of an optimal centralized control law, with the difference that, for each cell \( i \) and at each time step \( k \), to compute the best feasible optimum \( u^k_i \), we assume that only \( x^k_i \) and \( x^k_{i+1} \) are available for measurement and all other state variables are treated uncertain parameters. The problem is then reduced to solving an uncertain multi-parametric linear program for each \( i \) and any \( k \in [0, N-1] \). In order to improve practical applicability of the resulting controller, the design procedure is to be such that the feasibility of resulting control actions is guaranteed (i.e., the constraints in [5] are satisfied \( \forall i, k \)) and, in order to improve its practical applicability, it should lead to a closed-form solution with a simple structure for each local controller whose parameters can be computed offline.

From the proof of Theorem [2] to find an optimal action in a feedback form at time \( k \), we need to solve a multi-parametric linear program of the form [1] with \( \theta = x^k \) as the parameter
vector; however, under a decentralized information structure, $\theta$ is partially measurable. In order to determine an optimum control $u^k_i$, in the corresponding optimization problem, the inequality constraint can be written as

$$Wz \leq G + S_1\theta_1 + S_2\theta_2,$$

where $\theta_1 = [x^k_1, x^k_{i+1}]^T$ is measurable and $\theta_2 = [x^k_1, \ldots, x^k_{i-1}, x^k_{i+2}, \ldots, x^k_n]^T$ is an unknown, yet non-negative bounded parameter vector.

Linear programming problems with uncertain parameters have been the subject of much research and several approaches have been proposed to deal with robust optimization problems \cite{ref24} including: solving the problem for nominal values of the unknown parameters and then performing sensitivity analysis; formulating the problem as an stochastic optimization by incorporating the knowledge on the probability distribution of the uncertain parameters; and assigning a finite set of possible values to the uncertain parameters and determining a solution which is relatively good for all the scenarios \cite{ref25}. Also, some research has focused on evaluating the impact of uncertainty on the cost by computing the worst and best optimum solutions \cite{ref26} and some other work to ensure the feasibility of the optimal solution considered a worst-case approach which, in general, leads to extremely conservative solutions \cite{ref24}.

In this paper, we are interested in approaches that lead to a simple closed-form approximate optimal decentralized control law. We propose a simple method that provides a lower bound for the optimum cost value over the entire uncertainties range and leads to a decentralized control law with a simple structure that with respect to certain meaningful objective functions provides the same performance as that of the centralized controller.

Let $M$ be an upper bound for $|S_2\theta_2|$ and replace the inequality constraint in (1) by

$$Wz \leq G + S_1\theta_1 + M,$$

where the decision variable vector at time $k$ is $z = [(u^k)^T, \ldots, (u^{N-1})^T]^T$. Let us define

$$\Omega_z = \{z | Wz \leq G + S_1\theta_1 + S_2\theta_2\}$$

(14)

as the feasibility set when $\theta_2$ is perfectly known and let

$$\tilde{\Omega}_z = \{z | Wz \leq G + S_1\theta_1 + M\}.$$  

(15)

It is obvious that $\Omega_z \subseteq \tilde{\Omega}_z$, then for any vector $M \geq |S_2\theta_2|$ (element-wise), we have

$$\min_{z \in \Omega_z} c^T z \leq \min_{z \in \tilde{\Omega}_z} c^T z.$$  

(16)
That is, solving the problem over $\tilde{\Omega}_z$ provides a lower bound for the true optimum cost value. This lower bound is tight when for some $\theta_2$, $S_2\theta_2 = M$, then $z^* = \arg\min_{z \in \Omega_z} c^T z$ is referred to as the best optimum solution over the uncertainty range that provides the lowest possible cost.

The above procedure for eliminating the unknown parameter vector $\theta_2$, however, has two major drawbacks: (i) the obtained solution is not necessarily feasible; and (ii) although the dimension of the parameter space is reduced from $n$ to 2, the number of inequality constraints grows with the size of the network which in addition to increasing the computational cost associated with solving the optimization problem may lead to too many polyhedral regions.

Regarding the first issue, it should be noted that when we solve an uncertain multi-parametric linear program with parameter vector $\theta_1 = [x_i^k, x_{i+1}^k]^T$, for each $i$ and $k$, we retain and implement only $u_i^k$ from the solution vector and discard all the remaining variables. The feasibility of $u_i^k$ is guaranteed as its feasibility range at time $k$ depends only on $x_i^k$ and $x_{i+1}^k$ and is independent of the unknown parameters. Indeed, if at each time $k$, the constraints $0 \leq u_i^k \leq \min\{(v_i/\ell_i)x_i^k, C_i\}$ and $u_i^k \leq \min\{w_i^k + (1/\ell_i)x_{i+1}^k - (1/\ell_i+1)x_i^k, C_{i+1}\}$ are satisfied, the feasibility of the resulting sequence of control actions is ensured. No matter whether or not the discarded variables are feasible.

In order to address the second issue we consider the following simplification. Since the decision variables are finite (upper bounded by the maximum flow capacities $C_i$), there exists an upper bound vector $M$ for $|S_2\theta_2|$ such that all inequalities involving the uncertain parameters are always satisfied, that is every constraint that depends on the unknown parameters (elements of $\theta_2$) can be simply relaxed. This constraint relaxation makes the optimization problem separable; hence, in order to determine $u_i^k$, we do not consider the entire network, instead we consider a two-cell network with cells $i$ and $i+1$ and optimize the corresponding cost over the horizon $[k, N]$ to determine the optimum value of $u_i^k$ as a function of $x_i^k$ and $x_{i+1}^k$. Figure 2 further clarifies the decentralization process for linear networks. It can be easily verified that as the number of state variables that each cell $i$ can observe increases, the performance of the corresponding decentralized controller converges to that of the centralized one.

The above procedure for design of an approximate optimal decentralized controller with a one-hop information structure for linear networks is summarized in the following theorem.

**Theorem 3:** Let $J_{i,i+1}$ denote the cost function of the form (6), (7) over the horizon $[0, N]$ associated with a part of the network consisting of cell $i$ and its immediately downstream cell $i+1$. By following the procedure given in Section IV-A, design a centralized optimal feedback...
controller for each two-cell network with respect to the cost function $J_{i,i+1}$, which involves solving $nN$ multi-parametric linear program. The resulting controller has a one-hop information structure and is a piecewise affine function on polyhedra of the local state variables as:

$$
(u^*_i)^k = pwa^k_i(x^k_i, x^k_{i+1})
$$

$$
= F^k_{ij} \begin{bmatrix} x^k_i \\ x^k_{i+1} \end{bmatrix} + f^k_{ij}, \quad \text{if } \begin{bmatrix} x^k_i \\ x^k_{i+1} \end{bmatrix} \in R^k_{ij},
$$

where $R^k_{ij} \subseteq \mathbb{R}^2$, $j \in \mathbb{N}_{q^k}$, is the $j$th polyhedral partition of the set of feasible local states at time $k$ for the $i$th local controller. The controller parameters $F^k_{ij}, f^k_{ij}$ and the polyhedral regions $R^k_{ij}$ can be computed offline. Moreover, the feasibility of the resulting control actions is guaranteed.

The natural question that arises in connection with Theorem 3 is how to evaluate the performance and sub-optimality level of the above decentralized control scheme. As mentioned earlier, in general, performance analysis of decentralized optimal controllers is a very challenging task and no general procedure has been yet proposed to design a closed-form finite-time optimal decentralized controller with a given information structure. Although the above decentralization procedure involves relaxations that affect the conservativeness of the optimal solution, it can be shown that with respect to a class of linear cost functions, performance degradation due to decentralization (with a one-hop information structure) is zero.

**Theorem 4:** Consider the finite-time optimal control design problem (5)-(7) and assume that the cost-weighting parameters satisfy $\alpha^k_i \geq \alpha^k_{i+1} \geq 0$ and $\beta^k_i \leq \beta^k_{i+1} \leq 0$, $\forall k, i$. Then, an optimal feedback control law (with centralized information) is given by

$$
(u^*_i)^k = pwa^k_i(x^k_i, x^k_{i+1})
$$

$$
= \min \left\{ \frac{v_i x^k_i}{\ell_i}, C_i, w_{i+1} (\gamma_{i+1} - \frac{1}{\ell_{i+1}} x^k_{i+1}), C_{i+1} \right\},
$$
which can be expressed in the form of a piecewise affine function as in (17) with a one-hop information structure, wherein the controller parameters are obtained at no computational cost. Moreover, applying the decentralization procedure given in Theorem 3 gives the same control law as that of the centralized case.

Proof: The proof is given in the Appendix.

Theorem 4 implies that for certain cost functions, the centralized optimal controller has a decentralized realization with a one-hop information structure and is independent of the control horizon \( N \). In general, however, the optimal controller may need access to the state of the entire network and may depend on the control horizon. Moreover, from (18), it follows that with respect to the class of cost functions defined in Theorem 4, the optimal performance is obtained by simply setting each outflow rate \( u_i^k \) equal to its known maximum value at each time; hence, no specific control policy is needed to generate optimal control actions at each time.

We should highlight that the widely-used performance indexes (8), (9), and (10) given in Remark 2 satisfy the properties given in Theorem 4. It is also noteworthy that the conditions given in Theorem 4 are sufficient (not necessary) on a linear performance index with respect to which a centralized optimal control law has a one-hop information structure.

Remark 4: It is to be noted that the optimal control is not necessarily unique and there may be different realizations for controllers that provide the same performance level.

V. EXTENSION TO GENERAL NETWORKS

In the previous sections, we studied the optimal control design for linear transportation networks with no intermediate on/off-ramps. In this section, we extend the results to a more general class wherein junctions between cells can be of either of the three types defined below.

Definition 2: A junction with a single incoming and a single outgoing cell is called ordinary; a junction with a single incoming cell and multiple outgoing cells is called diverge; and a junction with multiple incoming cells and a single outgoing cell is called merge.

The following definitions and notations are used throughout this section.

Definition 3: Consider a network whose topology is described by directed graph \( G \). The set of edges of \( G \) corresponding to on-ramps is called the source set denoted by \( E_{on} \), and the set of edges corresponding to off-ramps is called the sink set denoted by \( E_{off} \).
Figure 3 shows a nine-cell network with all the three types of junctions. It is further assumed that at any diverge junction the traffic flow is distributed according to given turning percentages as defined below.

Fig. 3. Directed graph of the topology of a nine-cell transportation network with source set (on-ramps) $E_{on} = \{1, 2\}$ and sink set (off-ramps) $E_{off} = \{7, 8, 9\}$. The network has two ordinary junctions (labeled $o$), two diverge junctions (labeled $d$), and one merge junction (labeled $m$). It is assumed that the turning ratios of the network are known a priori, e.g., it is known that 30% of vehicles in cell 1 turn left towards cell 3 and 70% of them turn right toward cell 4, that is $R_{13} = 0.3$ and $R_{14} = 0.7$.

**Definition 4:** [8] The turning ratio (or split ratio) $R_{ij}^k \in [0, 1]$ is defined as the fraction of flow leaving cell $i \not\in E_{off}$ that is directed towards cell $j \neq i$, where $\sum_j R_{ij}^k = 1$. If cells $i$ and $j$ are not adjacent or $i = j$, the turning ratio $R_{ij}^k$ is defined to be zero.

Turning ratios may be constant or may vary with time during the control horizon and their values are estimated from historical data [27].

**Definition 5:** Let cell $i$ be an incoming cell to junction $h_i$, where $h_i$ denotes the head or the downstream junction of cell $i$. The set of all outgoing cells from junction $h_i$ is called the out-neighborhood of cell $i$ and is denoted by $E_i^+$. If $i \in E_{off}$, then $E_i^+$ is the empty set. In other words, $E_i^+$ is the set of all direct successor of cell $i$. The elements of $E_i^+$ are referred to as the out-neighbors of cell $i$ (see Figure 4).

**Definition 6:** Let cell $i$ be an outgoing cell from junction $\tau_i$, where $\tau_i$ denotes the tail or the upstream junction of cell $i$. The set of all incoming cells to junction $\tau_i$ is called the in-neighborhood of cell $i$ and is denoted by $E_i^-$. If $i \in E_{on}$, then $E_i^-$ is the empty set. In other words, $E_i^-$ is the set of all direct predecessor of cell $i$. The elements of $E_i^-$ are referred to as the in-neighbors of cell $i$ (see Figure 4).
Fig. 4. The tail (or upstream junction) of a cell $i$ is denoted by $\tau_i$ and its head (or downstream junction) is denoted by $h_i$. The in-neighborhood and the out-neighborhood of cell $i$ are respectively $E_i^- = \{1, 2\}$ and $E_i^+ = \{3, 4\}$.

In a more general setting, for an $n$-cell network, the constraints can be formulated as \[8\]:

\[
\begin{align*}
  x_i^{k+1} &= x_i^k + T_i(y_i^k - u_i^k), \\
  y_i^k &= \lambda_i^k + \sum_{j=1}^n R_{ji}^k u_j^k, \\
  0 &\leq u_i^k \leq \min\{(v_i/\ell_i)x_i^k, C_i\}, \\
  y_i^k &\leq \min\{w_i(\gamma_i - (1/\ell_i)x_i^k), C_i\},
\end{align*}
\]

(19)

for any $i \in \mathbb{N}_n$, $k = 0, 1, \ldots, N - 1$, where $y_i^k$ is the total inflow in cell $i$, $\lambda_i^k \geq 0$ is an exogenous inflow rate in cell $i \in \mathcal{E}_{on}$ (if $i \not\in \mathcal{E}_{on}$, $\lambda_i^k = 0$), and $R_{ji}^k$’s are given turning ratios. Then, if $i \in \mathcal{E}_{on}$, then $y_i^k = \lambda_i^k$; and for any $i \not\in \mathcal{E}_{on}$ we have $y_i^k = \sum_{j=1}^n R_{ji}^k u_j^k$.

**Remark 5:** It is obvious that if all junctions are ordinary (i.e., a network with a line graph) and cells are increasingly numbered from upstream to downstream, then $\lambda_i^k = u_0^k$, $\lambda_i^k = 0$, for $i = 2, 3, \ldots, n$, $R_{i,(i+1)}^k = 1$, $\forall i \in \mathbb{N}_{n-1}$, and $y_i^k = u_{i-1}^k$, hence (19) reduces to (5).

Since in (19) the linearity of the dynamics and constraints are preserved, similar to the case of linear network (5), the optimization problem (19), (6), (7) is a linear program. Hence, by following the same procedure as that in the proof of Theorem 2, the **centralized** optimal feedback control law can be expressed as

\[
(u^k)^* = pwa^k(x^k) = F_i^k x^k + f_i^k, \text{ if } x^k \in \mathcal{R}_i^k,
\]

(20)

where $\mathcal{R}_i^k$ is the $i$th polyhedral partition of the set of feasible states at time $k$.

Now, we consider the design of an **approximate optimal decentralized controller** with a one-hop information structure. Let us first clarify the notion of (outer) one-hop information structure for general networks (see Definition 1), by considering the example shown in Figure 3. In this network, in a feedback control law with a one-hop information structure, $u_1^k$ is determined by knowing only $x_1^k, x_3^k, x_5^k$; and $u_4^k$ is specified by measuring only $x_4^k, x_5^k, x_8^k$, i.e., cells 8 and 5 are considered as immediate downstream cells of cell 4.
Definition 7: For a cell $i \notin E_{\text{off}}$, define $N_i^+$ as the set of all cells, other than $i$, leaving/entering the junction $h_i$, where $h_i$ denotes the downstream junction of cell $i$; also if $i \in E_{\text{off}}$, we define $N_i^+$ to be the empty set. In other words, $N_i^+$ is the set of all cells immediately downstream of cell $i$. Note that $E_i^+ \subseteq N_i^+$.

According to the above definition, a control law with a one-hop information structure is of the form

$$u^k_i = \phi_i(x^k_i, (x^k_j)_{j \in N_i^+}).$$

(21)

In order to obtain an approximate optimal decentralized control law with a one-hop information structure for a general network described by (19), we follow the same procedure as that in Section IV-B. That is, to determine an optimum value of $u^k_i$ at each time $k$, a part of the network consisting of cell $i$ and cells $j \in N_i^+$ is considered and the corresponding performance index is optimized.

We show that under certain assumptions, the centralized optimal feedback controller has a decentralized realization with a one-hop information structure. In addition, applying the aforementioned decentralization procedure gives the same control law as that of the centralized case.

**Theorem 5:** Consider the finite-time optimal control problem (6), (7) subject to (19), and assume that the cost-weighting parameters satisfy $\alpha^k_i \geq \alpha^k_j \geq 0$, $\forall k, i$ and $\forall j \in E_i^+$, and $\beta^k_i \leq \beta_{i+1}^k \leq 0$, $\forall k, i$. In addition, assume that the turning ratios are time invariant during the control horizon, i.e., $R^k_{ij} = R_{ij}$ and the network has no merge junction. Then, an optimal feedback control law (with centralized information) is given by

$$(u^k_i)^* = \text{pwa}_i(x^k_i, (x^k_j)_{j \in E_i^+})$$

(22)

$$= \min \left\{ \frac{\nu_i}{\ell_i} x^k_i, C_i, \left( \frac{w_j}{R_{ij}} (\gamma_j - \frac{1}{\ell_j} x^k_j), \frac{C_j}{R_{ij}} \right)_{j \in E_i^+} \right\},$$

which has a one-hop information structure, where the controller parameters are obtain at no computational cost. Moreover, applying the proposed decentralization procedure gives the same control law as that of the centralized case.

**Proof:** The proof is given in the Appendix.

Theorem 5 implies that for networks with no merge junction, with respect to the meaningful linear cost functions given in Remark 2, the optimal control law posses a realization with a one-hop information structure and no specific control policy is needed to generate the optimal control actions, as the optimum performance is achieved by setting each outflow rate equal
to its maximum value which is a known piecewise affine function of the state of immediately downstream cells.

Remark 6: The condition $\alpha_i^k \geq \alpha_j^k \geq 0$, $\forall j \in E_i^+$ in Theorem 5 implies that if there is a cycle in the network’s digraph, all the cycle’s cells must share the same cost weights on the states.

VI. SIMULATION

In this section through a numerical simulation we evaluate the performance of the proposed approximate optimal decentralized feedback controller and compare it with the performance of the centralized optimal controller. We assume that the cost function is a weighted sum of the traffic mass of the cells, that is

$$J = \sum_{k=0}^{N} \sum_{i=1}^{n} \alpha_i x_i^k,$$  \hspace{1em} (23)

where different weights are assigned to different segments of the network. According to Theorem 5 in a network with no merge junction with time invariant turning ratios, if $\alpha_i \geq \alpha_j \geq 0$, $\forall j \in E_i^+$, then performance loss due to decentralization is zero, as the the optimal centralized controller has a decentralized realization with a one-hop information structure. In general, however, the performance may degrade due to the constraints relaxation used in the proposed decentralization approach.

Let $J^*_{\text{cen}}$ be the optimal value of the cost function corresponding to the centralized controller and $J^*_{\text{dec}}$ be the cost when the approximate optimal decentralized control law with a one-hop information structure is applied to the system. We define the relative decentralization performance loss (as a percentage) as

$$\varepsilon = 100 \frac{J^*_{\text{dec}} - J^*_{\text{cen}}}{J^*_{\text{cen}}}.$$  \hspace{1em} (24)

In order to numerically study the performance of the approximate optimal decentralized feedback control scheme, we consider the freeway system of an area in the southern Los Angeles as shown in Figure 5(a) modeled by the CTM. The directed graph of the network of the region of interest consisting of 32 cells is shown in Figure 5(b).

For numerical simulations, the following values are considered for the parameters of the network: The sampling time is $T_s = 1/360$ hr (or 10 sec). For on-ramp cells, the jam traffic density $\gamma_i$ is assumed to be infinity and for other cells $\gamma_i = 200$ veh/mi. For all cells, the backward congestion wave traveling speed is $w_i = 13$ mi/hr. For cells 3, 4, 9, 10, 12, 16, 20, the
Fig. 5. (a) The map of an area in the southern Los Angeles. The red ellipse shows the region used in our numerical simulation. (b) The directed graph of the transportation network of the region of interest with 32 cells, where $E_{on} = \{1, 2, 7, 13, 19, 21, 22, 29, 32\}$ and $E_{off} = \{11, 15, 17, 27, 28\}$.

cell’s length is $\ell_i = 2$ mi, the free-flow speed is $v_i = 65$ mi/hr, and the maximum flow capacity is $C_i = 800$ veh/hr, and for other cells, $\ell_i = 0.5$ mi, $v_i = 25$ mi/hr, and $C_i = 400$ veh/hr. At any diverge junction, $h_i$ with incoming cell $i$, the turning ratios are time-invariant and are split uniformly between the outgoing cells, i.e., $R_{ij} = 1/n_{hi}$, where $n_{hi}$ is the number of outgoing cells from junction $h_i$.

Let $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_{32}]$ be the cos weighting vector, where $\alpha_i$ is the weight associated with the state of cell $i$ in cost function (23). We assign random integers between 1 and 6 to $\alpha_i$’s and compute the optimal cost value of the centralized controller $J_{cen}^*$ and that of the decentralized one.
(with one-hop information structure) $J_{\text{dec}}^*$, and then evaluate the relative decentralization performance loss $\varepsilon$ as defined in (24). For example, with $\alpha = [5, 1, 2, 2, 4, 1, 3, 1, 1, 5, 2, 4, 6, 3, 5, 5, 3, 4, 1, 6, 2, 2, 5, 3, 5, 3, 2, 1, 5, 3, 3, 4]$, we have $\varepsilon = 0.9942\%$. We consider 100 random weighting vectors, $\alpha$, and for each case evaluate the relative performance loss $\varepsilon$. Figure 6 shows the histogram of the relative errors, wherein for 95% of weighting vectors, the relative decentralization performance loss is less than 2%.

![Histogram of Relative Performance Loss](image)

Fig. 6. The histogram of the relative decentralization performance loss for 100 random cost-weighting vectors.

VII. CONCLUSION AND FUTURE WORK

This paper provides some structural insights into the finite-horizon optimal feedback control for freeway traffic networks. The enabling tool for the design of an optimal feedback control law is the multi-parametric linear program. It is demonstrated that the optimal controller with respect to any linear cost function is a piecewise affine function on polyhedra of the state variables. It is well known that for large-size complex networks, the prohibitive computation and computation loads makes the design and implementation of a centralized controller too costly or impractical; moreover, the effect of noise, delay, or any type of error or failure in data transmission may substantially degrade the control quality. It is, therefore, necessary to develop decentralized feedback controllers with simple structure for practical applications. A procedure is subsequently proposed to design an optimal decentralized feedback control with a “one-hop” information structure. Moreover, it is shown that the optimal feedback controller with respect
to certain linear performance indexes possesses a one-hop information structure, making the optimal controller suitable for practical implementations in large-scale networks.

The performance loss due to the proposed decentralization scheme for general transportation networks should be examined analytically, and the effect of parameters such as the network size and control horizon on the conservativeness of the solutions should be investigated. We plan to extend our formulation to the extended version of the CTM to include features like capacity drop, and also to second-order macroscopic models as well as to other physical networks such as natural gas and water networks. Our ultimate objective is to develop a principled approach for distributed optimal control of physical infrastructure networks under given information constraints.

**APPENDIX**

**Proof of Theorem 2** The proof follows by following similar procedure as that in [28, §2]. Although the objective function considered in [28] is not of the form of (7) (as cost-weighting parameters can be negative in (7)), the procedure given in [28, §2] is still applicable. Let \( J^{[0,N]} \) denote the performance index over the entire control horizon \([0, N]\) as defined in (6)-(7). The closed-form solution to the first equation in (5) starting from initial state \( x_0 \) is given by

\[
x_k^i = x_0^i + T_s \sum_{j=0}^{k-1}(u_{i,j}^j - u_{i,j}^{j-1}), \quad i \in \mathbb{N}_n.
\]  

(25)

By substituting (25) into (6)-(7), the cost function can expressed as a linear combination of \( u_i^k \)'s, \( k \in [0, N-1], \ i \in \mathbb{N}_n \), as follows:

\[
J = \sum_{i=1}^{n} \sum_{k=0}^{N} \alpha_i^k x_0^i + T_s \sum_{i=1}^{n} \sum_{k=1}^{N-1} \alpha_i^k u_{i-1}^j
\]

\[
- T_s \sum_{i=1}^{n} \sum_{k=1}^{N-1} \sum_{j=0}^{k-1} \alpha_i^k u_i^j + \sum_{i=1}^{n} \sum_{k=0}^{N-1} \beta_i^k u_i^k
\]

\[
= \sum_{i=1}^{n} \sum_{k=0}^{N} \alpha_i^k x_0^i + T_s \sum_{k=1}^{N-1} \sum_{j=0}^{k-1} \alpha_1^k \lambda^j
\]

\[
- T_s \sum_{k=1}^{N-1} \sum_{j=0}^{k-1} \left( \sum_{i=1}^{n} \alpha_i^k u_i^j - \sum_{i=2}^{n} \alpha_i^k u_i^{j-1} \right) + \sum_{i=1}^{n} \sum_{k=0}^{N-1} \beta_i^k u_i^k
\]

\[
= \sum_{i=1}^{n} \sum_{k=0}^{N} \alpha_i^k x_0^i + T_s \sum_{k=1}^{N} \sum_{j=0}^{k-1} \alpha_1^k \lambda^j + \sum_{i=1}^{n} \sum_{k=0}^{N-1} \beta_i^k u_i^k
\]

\[
- \sum_{k=1}^{N} \sum_{j=0}^{k-1} T_s (\alpha_i^k - \alpha_{i+1}^k) u_i^j - \sum_{k=1}^{N} \sum_{j=0}^{k-1} T_s \alpha_i^k u_i^j.
\]
Then, the cost function can be written as
\[ J = g(x^0, \lambda) - \sum_{k=0}^{N-1} \left( \mu^k_1 u^k_1 + \mu^k_2 u^k_2 + \ldots + \mu^k_n u^k_n \right), \] (26)
where
\[ g(x^0, \lambda) = \sum_{i=1}^{n} \sum_{k=0}^{N} \alpha^k_i x^0_i + T_s \sum_{k=1}^{N} \sum_{j=0}^{k-1} \alpha^k_j \lambda^j \] (27)
is a function of the initial state \( x^0 \) and the external input \( \lambda \) over the entire control horizon and is independent of the decision variables \( u^k_i \)'s, and the coefficients of the decision variables in (26) are given by
\[ \mu^k_q = -\beta^k_q + \sum_{j=k+1}^{N} T_s (\alpha^j_q - \alpha^j_{q+1}). \] (28)
No need to mention that for \( q = n \), the cost-weighting parameter \( \alpha^j_{q+1} \) does not exist, then \( \mu^k_n = -\beta^k_n + \sum_{j=k+1}^{N} T_s \alpha^j_n \).

Similarly, from (5) and (25), the constraints can be written in terms of decision variables \( u^k_i \)'s and the external input \( u^k_0 = \lambda^k \). In a linear network, the state of the first cell (on-ramp), \( x^k_1 \), is the only state variable that depends on the exogenous input \( \lambda^k \), and the only constraint involving \( x^k_1 \) is
\[ u^k_1 \leq \left( \frac{v_1}{\ell_1} \right) x^k_1 = \left( \frac{v_1}{\ell_1} \right) x^0_1 + \left( \frac{v_1}{\ell_1} \right) T_s \sum_{j=0}^{k-1} \lambda^j - \left( \frac{v_1}{\ell_1} \right) T_s \sum_{j=0}^{k-1} u^j_1, \]
which can be written as
\[ \frac{v_1}{\ell_1} T_s \sum_{j=0}^{k-1} u^j_1 + u^k_1 \leq \frac{v_1}{\ell_1} x^0_1 + \frac{v_1}{\ell_1} T_s \sum_{j=0}^{k-1} \lambda^j \] (29)
Since the trajectory of the exogenous input \( \lambda^k \) is not known beforehand, the second term in the right-hand side of (29) is unknown. In order to ensure that for any feasible \( \lambda^k \), the solutions to the optimization problem are feasible, we set \( \lambda^k = 0, \forall k \), and optimize the worst-case performance.

Due to the linearity of the objective function and constraints, the optimization problem can be expressed as a multi-parametric linear program of the form (1), wherein the state vector at time \( k = 0 \), i.e., \( \theta = x^0 \), is treated as a varying parameter vector in the optimization problem, and the decision variable vector contains the control actions for \( k = 0, \ldots, N - 1 \), i.e., \( z = [(u^0)^\top, \ldots, (u^{N-1})^\top]^\top \). From Theorem 1, we have
\[ z^* = L_i x^0 + l_i, \text{ if } \Pi_i x^0 \leq \eta_i, \]
which can be expressed as
\[
\begin{aligned}
(u^0)^* &= L_{0i}x^0 + l_{0i}, \\
(u^1)^* &= L_{1i}x^0 + l_{1i}, \\
&\quad \vdots \\
(u^{N-1})^* &= L_{(N-1)i}x^0 + l_{(N-1)i},
\end{aligned}
\]
where \(L_{ji}\) is the \(j\)th row of matrix \(L_i\) and \(l_{ji}\) is the \(j\)th element of vector \(l_i\). The above results imply that when optimizing the performance index starting at \(k = 0\) over the control horizon \([0, N]\), i.e., \(J^{[0,N]}\) with parameter vector \(\theta = x^0\), then the optimal solution (30) provides a state-feedback optimal control law only at the initial time \(k = 0\), i.e.,
\[
(u^0)^* = L_{0i}x^0 + l_{0i}, \quad \text{if} \quad \Pi_i x^0 \leq \eta_i.
\]
Hence, to design a feedback control law, we retain only the first equation in (30) and discard the rest of them. Therefore, in (11), the parameters of the optimal feedback controller at time \(k = 0\) are given by
\[
F^0_i = L_{0i}, \quad f^0_i = l_{0i}, \quad H^0_i = \Pi_i, \quad h^0_i = \eta_i.
\] (31)
The optimal value of \(u^0\) when is applied to the system gives an optimal value of \(x^1\), then at the next time step by repeating the same procedure starting at the initial time \(k = 1\) over the control horizon \([1, N]\) with \(x^1\) as a parameter vector, we can express the optimal value of \(u^1\) as a piecewise affine function of \(x^1\). Therefore, in general, optimizing the performance index over the time interval \([j, N]\), i.e., \(J^{[j,N]}\) with parameter vector \(\theta = x^j\) and decision variable vector \(z = [(u^j)^\top, \ldots, (u^{N-1})^\top]^\top\), provides a state-feedback optimal control law at time step \(j\) in the form of a piecewise affine function on polyhedra of \(x^j\), for any \(j = 0, 1, \ldots, N - 1\). Hence, by solving \(N\) multi-parametric linear programs, the optimal feedback controller can be expressed as (11).

**Proof of Theorem 4** Considering the constraints in (5), in any linear network, each decision variable \(u^k_i\) satisfies the following bound constraint at each time \(k\):

\[
0 \leq u^k_i \leq \bar{u}^k_i,
\]
\[
\bar{u}^k_i = \min \left\{ \frac{v_i}{\ell_i}x^k_i, C_i, w_{i+1}(\gamma_{i+1} - \frac{1}{\ell_{i+1}}x_{i+1}^k), C_{i+1} \right\}.
\] (32)
Then at each time $k$, given $x^k$, the upper limit $\bar{u}_i^k$ is known. From (26), the sequence of optimal control actions can be obtained as

$$u^* = \arg \max_{s.t.} \left\{ \sum_{k=0}^{N-1} \left( \mu_1^k u_1^k + \mu_2^k u_2^k + \ldots + \mu_n^k u_n^k \right) \right\}. \tag{33}$$

Under the assumptions $\alpha_i^k \geq \alpha_{i+1}^k \geq 0$ and $\beta_i^k \leq \beta_{i+1}^k \leq 0$, the coefficients of the decision variables $\mu_i^k$’s as defined in (28) satisfy $\mu_i^k \geq \mu_{i+1}^k \geq 0$, $\forall k, i$. By using the dynamic programming approach [29, §6.2], we show that under the given assumptions, an optimum solution to (33) is obtained when each variable $u_i^k$ is set equal to its upper limit $\bar{u}_i^k$ defined in (32).

Consider the maximization problem (33) and define the objective function over the time interval $[k, N]$ starting at time $k$ from initial state $x^k$ as

$$I^k(x^k) = \sum_{j=k}^{N-1} \left( \mu_1^j u_1^j + \ldots + \mu_n^j u_n^j \right). \tag{34}$$

Then, the corresponding functional equation of dynamic programming is given by

$$I^k(x^k) = \max_{u^k} \left\{ Q^k \right\}, \text{ s.t. (32), given } x^k, \tag{35}$$

where $I^k(x^k)$ denotes the optimal value of the objective function over the time interval $[k, N]$ from initial state $x^k$. Hence, the optimization over the horizon $[0, N]$ is converted into optimization over only one control vector $u^k$ at a time by working backward in time for $k = N - 1, N - 2, \ldots, 0$. The optimization problem (35) is a bound constrained optimization problem, hence if we show that $Q^k$ is an increasing function of $u^k$ (i.e., increasing in every coordinate $u_1^k, u_2^k, \ldots, u_n^k$, $\forall k$, then $(u^k)^* = \bar{u}_i^k$ is an optimum solution.

For a one-stage process with initial state $x^{N-1}$, the $Q$-function is

$$Q^{N-1} = \mu_1^{N-1} u_1^{N-1} + \ldots + \mu_n^{N-1} u_n^{N-1}. \tag{36}$$

Since $\mu_i^k \geq 0$, $\forall i, k$, then $Q^{N-1}$ is an increasing function of $u_i^{N-1}$, $\forall i$, then $(u_i^{N-1})^* = \bar{u}_i^{N-1}$.

For a two-stage process with initial state $x^{N-2}$, the $Q$-function is

$$Q^{N-2} = (\mu_1^{N-2} u_1^{N-2} + \ldots + \mu_n^{N-2} u_n^{N-2}) + (\mu_1^{N-1} \bar{u}_1^{N-1} + \ldots + \mu_n^{N-1} \bar{u}_n^{N-1}) \tag{37}$$

$$= \sum_{i=1}^{n} (\mu_i^{N-2} u_i^{N-2} + \mu_i^{N-1} \bar{u}_i^{N-1}).$$
From (32) and that \(x_i^{N-1} = x_i^{N-2} + T_s(u_{i-1}^{N-2} - u_{i}^{N-2})\), we have

\[
\begin{align*}
\bar{u}_i^{N-1} &= \min \left\{ \frac{u_i}{\ell_i}x_i^{N-1}, w_{i+1}x_i - \frac{w_{i+1}}{\ell_{i+1}}x_i^{N-1}, C_i, C_{i+1} \right\} \\
&= \min \left\{ \frac{u_i}{\ell_i}x_i^{N-2} + \sigma_i u_i^{N-2} - \sigma_i u_i^{N-2}, \\
&\quad w_{i+1}x_i - \frac{w_{i+1}}{\ell_{i+1}}x_i^{N-1} - \kappa_{i+1} u_i^{N-2} \\
&\quad + \kappa_{i+1} u_i^{N-2}, C_i, C_{i+1} \right\},
\end{align*}
\]

(38)

where \(\sigma_i = (v_i/\ell_i)T_s \in [0, 1], \kappa_i = (w_i/\ell_i)T_s \in [0, 1]\) (see Assumption 1). Then, by multiplying both sides of (38) by \(\mu_i^{N-1}\) and adding \(\mu_i^{N-2}u_i^{N-2}\) to the both sides we obtain

\[
\begin{align*}
\mu_i^{N-2}u_i^{N-2} + \mu_i^{N-1}u_i^{N-1} &= \\
&= \min \left\{ \mu_i^{N-1} \frac{u_i}{\ell_i}x_i^{N-2} + \mu_i^{N-1} \sigma_i u_i^{N-2} + s_i^{N-2} u_i^{N-2}, \\
&\quad \mu_i^{N-1} w_{i+1}x_i - \frac{w_{i+1}}{\ell_{i+1}}x_i^{N-2} + t_i^{N-2} u_i^{N-2} \\
&\quad + \mu_i^{N-1} \kappa_{i+1} u_i^{N-2}, \mu_i^{N-1} C_i + \mu_i^{N-2} u_i^{N-2}, \mu_i^{N-1} C_{i+1} \\
&\quad + \mu_i^{N-2}u_i^{N-2} \right\},
\end{align*}
\]

(39)

where \(s_i^{N-2} = \mu_i^{N-2} - \mu_i^{N-1}\sigma_i\) and \(t_i^{N-2} = \mu_i^{N-2} - \mu_i^{N-1}\kappa_{i+1}\). Since \(\mu_i^k \geq \mu_i^{k+1} \geq 0\) and \(\sigma_i, \kappa_i \in [0, 1]\), then \(s_i^{N-2}, t_i^{N-2} \geq 0, \forall i\). From (37) and (39), and that the coefficients of \(u_i^{N-2}\) are non-negative \(\forall i\), it follows that \(Q^{N-2}\) is an increasing function of \(u_i^{N-2}, \forall i\), then \((u_i^{N-2})^* = \bar{u}_i^{N-2}\), where we have used the fact that the minimum and the sum of increasing functions are also increasing.

Similarly, for a \(k\)-stage process with initial state \(x^{N-k}\), assuming that \(u_i^j = \bar{u}_i^j\), for \(j = N-k+1, \ldots, N-2, N-1\), the \(Q\)-function is given by

\[
Q^{N-k} = \sum_{i=1}^{n} (\mu_i^{N-k} u_i^{N-k} + \mu_i^{N-k+1} \bar{u}_i^{N-k+1} + \ldots + \mu_i^{N-2} \bar{u}_i^{N-2} + \mu_i^{N-1} \bar{u}_i^{N-1}).
\]

(40)

From (32) and that \(x_i^{N-1} = x_i^{N-k} + T_s \sum_{j=N-k}^{N-1} (u_j^{j-1} - u_j^j)\), we have

\[
\begin{align*}
\bar{u}_i^{N-1} &= \min \left\{ \frac{u_i}{\ell_i}x_i^{N-1}, w_{i+1}x_i - \frac{w_{i+1}}{\ell_{i+1}}x_i^{N-1}, C_i, C_{i+1} \right\} \\
&= \min \left\{ \frac{u_i}{\ell_i}x_i^{N-k} + \sigma_i \sum_{j=N-k}^{N-2} u_j^{j-1} - \sigma_i \sum_{j=N-k}^{N-2} u_j^j, \\
&\quad w_{i+1}x_i - \frac{w_{i+1}}{\ell_{i+1}}x_i^{N-2} - \kappa_{i+1} \sum_{j=N-k}^{N-2} u_j^j \\
&\quad + \kappa_{i+1} \sum_{j=N-k}^{N-2} u_j^{j+1}, C_i, C_{i+1} \right\},
\end{align*}
\]

(41)
where $u_i^j = \bar{u}_i^j$, for $j \geq N - k + 1$. By multiplying both sides of (41) by $\mu_i^{N-1}$ and then adding $\sum_{j=N-k}^{N-2} \mu_i^j u_i^j$ to the both sides (wherein $u_i^j = \bar{u}_i^j$, for $j \geq N - k + 1$), we obtain

$$
\sum_{j=N-k}^{N-1} \mu_i^j u_i^j = \min \left\{ \mu_i^{N-1} \frac{u_i}{\ell_i} x_{i-k} + \mu_i^{N-1} \sigma_i \sum_{j=N-k}^{N-2} u_i^j, \mu_i^{N-1} u_i + \sum_{j=N-k}^{N-2} s_i^j u_i^j, \mu_i^{N-1} \gamma_i + u_i^{N-1} - \sum_{j=N-k}^{N-2} \ell_i \bar{u}_i^j + \sum_{j=N-k}^{N-2} t_i^j u_i^j \right\} + \mu_i^{N-1} \kappa_i + \sum_{j=N-k}^{N-2} j^j u_i^j, \mu_i^{N-1} C_i + \sum_{j=N-k}^{N-2} j^j u_i^j \right\}
$$

where $s_i^j = \mu_i^j - \mu_i^{N-1} \sigma_i$ and $t_i^j = \mu_i^j - \mu_i^{N-1} \kappa_i + 1$. Since $\mu_i^{k+1} \geq 0$ and $\sigma_i, \kappa_i \in [0, 1]$, then $s_i^j \geq s_i^{j+1} \geq 0$ and $t_i^j \geq t_i^{j+1} \geq 0, \forall i$ and any $j = N - k, \ldots, N - 2$.

Let $\delta_i^j$ be a generic symbol for a sequence of parameters satisfying $\delta_i^j \geq \delta_i^{j+1} \geq 0, \forall i, j$. Due to the non-negativity of the coefficients of $u_i^j$ in the right-hand side of (42), $\sum_{j=N-k}^{N-2} \delta_i^j u_i^j$ is maximized if the terms $\sum_{j=N-k}^{N-2} \delta_i^{j-1} u_i^j$, $\sum_{j=N-k}^{N-2} \delta_i^j u_i^j$, and $\sum_{j=N-k}^{N-2} \delta_i^{j+1} u_i^j$ are maximized, $\forall i$, and so on. Finally, we have a two-stage process, i.e., maximization of $\delta_i^j u_i^N - k + 1 \bar{u}_i^N - k + 1$, $\forall i$, which implies that $(u_i^{N-k})^* = \bar{u}_i^{N-k}, \forall i$.

Therefore, under the given assumptions on the cost-weighting parameters, the optimum control is independent of the control horizon $N$ and is obtained at no computational cost by setting each $u_i^k$ equal to its known upper limit $\bar{u}_i^k, \forall i, k$. It is easy to verify that the upper limit $\bar{u}_i^k$ is in the form of a piecewise affine function as (17); for example, for the last cell $n$, where $u_n^k \leq \min\{(v_n/\ell_n)x_n^k, C_n\}$, we have

$$(u_n^k)^* = \begin{cases} (v_n/\ell_n)x_n^k, & \text{if} \ (v_n/\ell_n)x_n^k \leq C_n, \\ C_n, & \text{if} \ -(v_n/\ell_n)x_n^k \leq -C_n. \end{cases}$$

From the expression for $\bar{u}_i^k$, it follows that to implement an optimal outflow rate $u_i^k$, we need to measure only $x_i^k$ and $x_{i+1}^k$. The above arguments are applicable to linear networks of any size; hence, the decentralization procedure given in Theorem 4, which is based on optimization of the performance indexes corresponding to two-cell networks at each time, gives the same solution as that of the centralized one.

**Proof of Theorem 5** The solution to the first equation in (19) starting from initial state $x^0$ is given by

$$x_i^k = x_i^0 + T_i \sum_{j=0}^{k-1} (u_i^j - u_i^j), \ i \in N_n.$$  (43)
By substituting (43) into (6)-(7) we have
\[
J = \sum_{i=1}^{N} \sum_{k=0}^{N} \alpha^k_{i}x^0_{i} + T_s \sum_{i=1}^{N} \sum_{k=0}^{N} \sum_{j=0}^{k-1} \alpha^k_{i}y^j_{i} \\
- T_s \sum_{i=1}^{N} \sum_{k=0}^{N} \sum_{j=0}^{k-1} \alpha^k_{i}u^j_{i} + \sum_{i=1}^{N} \sum_{k=0}^{N-1} \beta^k_{i}u^k_{i}.
\]

Using the relation
\[
y^j_{i} = \begin{cases} 
\lambda^j_{i}, & \text{if } i \in \mathcal{E}_\text{on} \\
\sum_{q=1}^{n} R^j_{qi} u^j_{i}, & \text{if } i \not\in \mathcal{E}_\text{on}
\end{cases}
\]
the cost function can be written as
\[
J = \sum_{i=1}^{N} \sum_{k=0}^{N} \alpha^k_{i}x^0_{i} + T_s \sum_{k=1}^{N} \sum_{j=0}^{k-1} \alpha^k_{i} \lambda^j_{i} \\
- T_s \sum_{k=1}^{N} \sum_{j=0}^{k-1} \left( \sum_{i=1}^{n} \alpha^k_{i} u^j_{i} - \sum_{i=1}^{n} \alpha^k_{i} R^j_{qi} u^j_{i} \right) \\
+ \sum_{i=1}^{n} \sum_{k=0}^{N-1} \beta^k_{i} u^k_{i} \\
= \sum_{i=1}^{N} \sum_{k=0}^{N} \alpha^k_{i}x^0_{i} + T_s \sum_{k=1}^{N} \sum_{j=0}^{k-1} \alpha^k_{i} \lambda^j_{i} \\
- T_s \sum_{q=1}^{n} \sum_{k=1}^{N} \sum_{j=0}^{k-1} \left( \alpha^k_{q} - \sum_{i=1}^{n} \alpha^k_{i} R^j_{qi} \right) u^j_{q} + \sum_{i=1}^{n} \sum_{k=0}^{N-1} \beta^k_{i} u^k_{i}.
\]

Then, we can express the cost function as
\[
J = g(x^0, \lambda) - \sum_{k=0}^{N-1} (\mu^k_{1} u^k_{1} + \mu^k_{2} u^k_{2} + \ldots + \mu^k_{n} u^k_{n}),
\]
where the coefficients of the decision variables are
\[
\mu^k_{q} = -\beta^k_{q} + \sum_{j=k+1}^{N} T_s (\alpha^j_{q} - \sum_{i \notin \mathcal{E}_\text{on}} R^j_{qi} \alpha^j_{i}),
\]
and
\[
g(x^0, \lambda) = \sum_{i=1}^{N} \sum_{k=0}^{N} \alpha^k_{i}x^0_{i} + T_s \sum_{k=1}^{N} \sum_{j=0}^{k-1} \alpha^k_{i} \lambda^j_{i}.
\]

From the definition of turning ratios (see Definition 4), \(\sum_{i \notin \mathcal{E}_\text{on}} R^j_{qi} \alpha^j_{i}\) is a convex combination of \(\alpha^j_{i}\)'s, for all \(i \in \mathcal{E}_\text{on}^+\), and under the assumptions on the cost-weighting parameters, i.e.,
\(\alpha^k_q \geq \alpha^k_i \geq 0, \forall k, q \) and \(\forall i \in \mathcal{E}_q^+,\) and \(\beta^k_q \leq \beta^{k+1}_q \leq 0, \forall k, q,\) and that the turning ratios are time invariant, it follows that \(\mu^k_q \geq \mu^{k+1}_q \geq 0, \forall k, q.\) Therefore, in order to find a sequence of optimal control actions, we need to solve

\[
\mathbf{u}^* = \arg \max_{s.t. \ \{19\}} \left\{ \sum_{k=0}^{N-1} \left( \mu^k_1 u^k_1 + \mu^k_2 u^k_2 + \ldots + \mu^k_n u^k_n \right) \right\}
\]  

(48)

Let us now consider the constraints in (19) with time-invariant turning ratios. The inflow rate to cell \(i, y^k_i,\) can be written as

\[
y^k_i = \begin{cases} 
\lambda^k_i, & \text{if } i \in \mathcal{E}_{on}, \\
R_{ji} u^k_j, & \text{if } i \notin \mathcal{E}_{on} \text{ and } \tau_i \text{ is either } \\
\text{diverge or ordinary,} \\
\sum_{j \in \mathcal{E}_i^-} u^k_j & \text{if } i \notin \mathcal{E}_{on} \text{ and } \tau_i \text{ is merge,}
\end{cases}
\]  

(49)

where in the second case, \(j\) is the only in-neighbor of cell \(i.\) In particular, if the tail of cell \(i, \tau_i,\) is an ordinary junction, we have \(R_{ji} = 1;\) hence, \(y^k_i = u^k_j,\) where \(j\) is the only in-neighbor of cell \(i.\) From (19) and (49), it follows that if the head of cell \(i, h_i,\) is either a diverge or an ordinary junction, then \(u^k_i\) satisfies the following bound constraint:

\[
0 \leq u^k_i \leq \bar{u}^k_i,
\]

(50)

where the upper limit \(\bar{u}^k_i\) is known, provided that \(x^k_i \) and \(x^k_j, \forall j \in \mathcal{E}_i^+,\) are given. However, if there exists a merge junction \(h_i \) with \(\mathcal{E}_i^+ = \{q\},\) the outflow rates \(u^k_i, \) for any \(i \in \mathcal{E}_q^+ \) must satisfy

\[
0 \leq u^k_i \leq \min \left\{ \frac{v_i}{\ell_i} x^k_i, C_i, \left( \frac{w_j (\gamma_j - 1 \ell_j)}{R_{ij} \ell_j}, C_j \right) \right\}, \forall i \in \mathcal{E}_q^-
\]

(51)

By using the dynamic programming approach, we show that in a network without any merge junction, under the given assumptions on the cost-weighting parameters and the turning ratios, an optimum solution to (48) is obtained when each variable \(u^k_i\) is set equal to its upper limit \(\bar{u}^k_i\) defined in (50).
Consider the maximization problem (48), then similar to the proof of Theorem 4, the objective function over the time interval \([k, N]\) starting at time \(k\) from initial state \(x^k\) is defined as (34) and the corresponding functional equation of dynamic programming is given by

\[
I^k(x^k) = \max_{u^k} \{Q^k\}, \quad \text{s.t. (50), given } x^k,
\]

\[
Q^k = (\mu^k_1 u^k_1 + \ldots + \mu^k_n u^k_n) + I^{k+1}(x^{k+1}),
\]

which is a bound-constrained optimization problem. Similar to the proof of Theorem 4, for a one-stage process with initial state \(x^{N-1}\), the \(Q\)-function is

\[
Q^{N-1} = \mu^{N-1}_1 u^{N-1}_1 + \ldots + \mu^{N-1}_n u^{N-1}_n.
\]

Since \(\mu^k_i \geq 0, \forall i, k\), then \(Q^{N-1}\) is an increasing function of \(u^{N-1}_i, \forall i\), then \((u^{N-1}_i)^* = \bar{u}^{N-1}_i\).

For a two-stage process with initial state \(x^{N-2}\), the \(Q\)-function is

\[
Q^{N-2} = \sum_{i=1}^n (\mu^{N-2}_i u^{N-2}_i + \mu^{N-1}_i \bar{u}^{N-1}_i).
\]

From (50) and that \(x^{N-1}_i = x^{N-2}_i + T_s(y^{N-2}_i - u^{N-2}_i)\) and \(y^k_i = R_{ri} u^k_i\), where \(r\) is the only in-neighbor of cell \(i\) (see (49)), we have

\[
\bar{u}^{N-1}_i = \min \left\{ \frac{v_i}{\ell_i} x^{N-1}_i, \left( \frac{w_j}{R_{ij}} \left( \gamma_j - \frac{1}{\ell_j} x^{N-1}_j \right) \right)_{j \in \mathcal{E}_i^+} \right\},
\]

\[
C_i, \left( \frac{C_j}{R_{ij}} \right)_{j \in \mathcal{E}_i^+}
\]

\[
= \min \left\{ \frac{v_i}{\ell_i} x^{N-2}_i + \sigma_i R_{ri} u^{N-2}_r - \sigma_i u^{N-2}_i, \right. \]

\[
\left. \left( \frac{w_j}{R_{ij}} \frac{R_{ij}}{\ell_j} x^{N-2}_j - \kappa_j u^{N-2}_j + \frac{\kappa_j}{R_{ij}} u^{N-2}_j \right)_{j \in \mathcal{E}_i^+} \right\},
\]

\[
C_i, \left( \frac{C_j}{R_{ij}} \right)_{j \in \mathcal{E}_i^+}
\]

where \(\sigma_i = (v_i/\ell_i) T_s \in [0, 1]\), \(\kappa_i = (w_i/\ell_i) T_s \in [0, 1]\). By multiplying both sides of (55) by
where $\mu_i^{N-1}$ and adding $\mu_i^{N-2}u_i^{N-2}$ to the both sides we obtain
\[
\mu_i^{N-2}u_i^{N-2} + \mu_i^{N-1}u_i^{N-1} = \\
\min\left\{ \mu_i^{N-1} \frac{\ell_i}{\ell_i} x_i^{N-1} + \mu_i^{N-1} \sigma_i R_{ri} u_i^{N-2} + s_i^{N-2} u_i^{N-2}, \right. \\
\left( \frac{\mu_i^{N-1} w_j \gamma_j}{R_{ij}} - \mu_i^{N-1} \frac{w_j}{R_{ij}} x_j^{N-2} + t_{ij}^{N-2} u_i^{N-2} + \mu_i^{N-1} \frac{\kappa_j}{R_{ij}} u_j^{N-2} \right)_{j \in \mathcal{E}_i^+}, \\
\mu_i^{N-1} C_i + \mu_i^{N-2} u_i^{N-2}, \\
\left( \frac{\mu_i^{N-1} C_j}{R_{ij}} + \mu_i^{N-2} u_i^{N-2} \right)_{j \in \mathcal{E}_i^+} \right\},
\]
(56)
where $s_i^{N-2} = \mu_i^{N-2} - \mu_i^{N-1} \sigma_i$ and $t_{ij}^{N-2} = \mu_i^{N-2} - \mu_i^{N-1} \kappa_j$, $j \in \mathcal{E}_i^+$. Since $\mu_i^k \geq \mu_i^{k+1} \geq 0$ and $\sigma_i, \kappa_i \in [0, 1]$, then $s_i^{N-2}, t_{ij}^{N-2} \geq 0$, $\forall i$. From (54) and (56), and that the coefficients of $u_i^{N-2}$ are non-negative $\forall i$, and using the fact that the minimum and the sum of increasing functions are also increasing, it follows that $Q^{N-2}$ is an increasing function of $u_i^{N-2}$, then $(u_i^{N-2})^* = \bar{u}_i^{N-2}$, $\forall i$, is an optimal control action.

For a $k$-stage process with initial state $x^{N-k}$, assuming that $u_i^j = \bar{u}_i^j$, for $j = N - k + 1, \ldots, N - 2, N - 1$, the $Q$-function is given by
\[
Q^{N-k} = \sum_{i=1}^{n} \left( \mu_i^{N-k} u_i^{N-k} + \mu_i^{N-k+1} \bar{u}_i^{N-k+1} + \ldots + \mu_i^{N-2} \bar{u}_i^{N-2} + \mu_i^{N-1} \bar{u}_i^{N-1} \right).
\]
(57)
From (50) and that $x_i^{N-1} = x_i^{N-k} + T_s \sum_{l=N-k}^{N-2} (y_i^l - u_i^l)$, we have
\[
\bar{u}_i^{N-1} = \min\left\{ \frac{\ell_i}{\ell_i} x_i^{N-1}, \left( \frac{w_j}{R_{ij}} \gamma_j - \frac{1}{\ell_j} x_j^{N-1} \right)_{j \in \mathcal{E}_i^+}, \right. \\
C_i, \left( \frac{C_j}{R_{ij}} \right)_{j \in \mathcal{E}_i^+} \right\} \\
= \min\left\{ \frac{\ell_i}{\ell_i} x_i^{N-k} + \sigma_i R_{ri} \sum_{l=N-k}^{N-2} u_i^l - \sigma_i \sum_{l=N-k}^{N-2} u_i^l, \right. \\
\left( \frac{w_j \gamma_j}{R_{ij}} - \frac{w_j}{R_{ij}} x_j^{N-k} - \kappa_j \sum_{l=N-k}^{N-2} u_i^l + \frac{\kappa_j}{R_{ij}} \sum_{l=N-k}^{N-2} u_j^l \right)_{j \in \mathcal{E}_i^+}, \left( \frac{C_j}{R_{ij}} \right)_{j \in \mathcal{E}_i^+} \right\},
\]
(58)
where $r$ is the only in-neighbor of cell $i$ and $u_i^j = \bar{u}_i^j$, for $j \geq N - k + 1$. By multiplying both sides of (58) by $\mu_i^{N-1}$ and then adding $\sum_{l=N-k}^{N-2} \mu_i^{N-2} u_i^l$ to the both sides (wherein $u_i^l = \bar{u}_i^l$, for
\( l \geq N - k + 1 \), we obtain

\[
\sum_{t=N-k}^{N-1} \mu_i^t u_i^t = \min \left\{ \mu_i^{N-1} y_i^{N-k} + \mu_i^{N-1} N \sigma_i \sum_{t=N-k}^{N-2} u_i^t + \mu_i^{N-1} s_i^t u_i^t, \right. \\
\left( \mu_i^{N-1} \frac{w_j}{R_{ij}} - \mu_i^{N-1} \frac{w_j}{R_{ij}} x_j^N + \sum_{t=N-k}^{N-2} t_{ij}^t u_i^t + \mu_i^{N-1} \frac{\kappa_j}{R_{ij}} \sum_{t=N-k}^{N-2} u_i^t \right)_{j \in E^+} + \mu_i^{N-1} C_i + \right. \\
\left. \sum_{t=N-k}^{N-2} \mu_i^t u_i^t \left( \mu_i^{N-1} C_j \frac{1}{R_{ij}} + \sum_{t=N-k}^{N-2} \mu_i^t u_i^t \right)_{j \in E^+} \right\},
\]

where \( s_i^t = \mu_i^t - \mu_i^{N-1} \sigma_i \) and \( t_{ij}^t = \mu_i^t - \mu_i^{N-1} \kappa_j, j \in E^+ \). Since \( \mu_i^k \geq \mu_i^{k+1} \geq 0 \) and \( \sigma_i, \kappa_i \in [0, 1] \), then \( s_i^t \geq s_i^{t+1} \geq 0 \) and \( t_{ij}^t \geq t_{ij}^{t+1} \geq 0, \forall i \) and any \( l = N - k, \ldots, N - 2 \).

Due to the non-negativity of the coefficients of \( u_i^j \) in the right-hand side of (59), \( \sum_{t=N-k}^{N-1} \delta_i^t u_i^t \) is maximized if the terms \( \sum_{l=N-k}^{N-2} \delta_i^{l+1} u_i^{l+1}, \sum_{l=N-k}^{N-2} \delta_i^{l+1} u_i^l, \) and \( \sum_{l=N-k}^{N-2} \delta_i^{l+1} u_i^{l+1} \) are maximized, \( \forall i, l \). The above recursion implies that \( u_i^{N-k} = \bar{u}_i^{N-k}, \forall i \), is an optimal solution.

Therefore, the optimum control is independent of the control horizon \( N \) and is obtained at no computational cost by setting each \( u_i^k \) equal to its known upper limit \( \bar{u}_i^k, \forall i, k \).

From the expression for \( \bar{u}_i^k \), it follows that to implement an optimal outflow rate \( u_i^k \), we need to measure only \( x_i^k \) and \( (x_j^k)_{j \in E_i^+} \). This arguments are applicable to any network of any size without merge junctions; hence, the proposed decentralization procedure with one-hop information structure gives the same solution as that of the centralized one. 

**References**

[1] C. Daganzo, “The cell transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory,” *Transportation Research Part B*, vol. 28, no. 4, pp. 269–287, 1994.

[2] L. Adacher and M. Tiriolo, “A macroscopic model with the advantages of microscopic model: A review of cell transmission model’s extensions for urban traffic networks,” *Simulation Modelling Practice and Theory*, vol. 86, pp. 102–119, 2018.

[3] A. Hegyi, B. D. Schutter, and H. Hellendoorn, “Model predictive control for optimal coordination of ramp metering and variable speed limits,” *Transportation Research Part C: Emerging Technologies*, vol. 13, no. 3, pp. 185–209, 2005.

[4] G. Gomes and R. Horowitz, “Optimal freeway ramp metering using the asymmetric cell transmission model,” *Transportation Research Part C*, vol. 14, no. 4, pp. 244–262, 2006.

[5] A. Muralidharan and R. Horowitz, “Optimal control of freeway networks based on the link node cell transmission model,” in *2012 American Control Conference (ACC)*, June 2012, pp. 5769–5774.

[6] ——, “Computationally efficient model predictive control of freeway networks,” *Transportation Research Part C: Emerging Technologies*, vol. 58, pp. 532–553, 2015.
[7] A. K. Ziliaskopoulos, “A linear programming model for the single destination system optimum dynamic traffic assignment problem,” Transportation Science, vol. 34, no. 1, pp. 37–49, 2000.
[8] G. Como, E. Lovisari, and K. Savla, “Convexity and robustness of dynamic traffic assignment and freeway network control,” Transportation Research Part B: Methodological, vol. 91, pp. 446–465, 2016.
[9] M. Grant and S. Boyd, “CVX: Matlab software for disciplined convex programming, version 2.1.” http://cvxr.com/cvx, Mar. 2014.
[10] ——, “Graph implementations for nonsmooth convex programs,” in Recent Advances in Learning and Control, ser. Lecture Notes in Control and Information Sciences. Springer-Verlag. 2008, pp. 95–110.
[11] Q. Ba and K. Savla, “On distributed computational approaches for optimal control of traffic flow over networks,” in Allerton Conference on Communication, Control and Computing, 2016.
[12] I. Papamichail, A. Kotsialos, I. Margonis, and M. Papageorgiou, “Coordinated ramp metering for freeway networks – a model-predictive hierarchical control approach,” Transportation Research Part C: Emerging Technologies, vol. 18, no. 3, pp. 311–331, 2010.
[13] M. Hadiuzzaman and T. Z. Qiu, “Cell transmission model based variable speed limit control for freeways,” Canadian Journal of Civil Engineering, vol. 40, no. 1, pp. 46–56, 2013.
[14] Y. Han, A. Hegyi, Y. Yuan, S. Hoogendoorn, M. Papageorgiou, and C. Roncoli, “Resolving freeway jam waves by discrete first-order model-based predictive control of variable speed limits,” Transportation Research Part C: Emerging Technologies, vol. 77, pp. 405–420, 2017.
[15] S. Timotheou, C. G. Panayiotou, and M. M. Polycarpou, “Distributed traffic signal control using the cell transmission model via the alternating direction method of multipliers,” IEEE Transactions on Intelligent Transportation Systems, vol. 16, no. 2, pp. 919–933, 2015.
[16] P. Shao, L. Wang, W. Qian, Q. Wang, and X. Yang, “A distributed traffic control strategy based on cell-transmission model,” IEEE Access, vol. 6, pp. 10 771–10 778, 2018.
[17] F. Borrelli, Constrained Optimal Control of Linear and Hybrid Systems. Springer, 2003.
[18] M. Herceg, M. Kvasnica, C. Jones, and M. Morari, “Multi-Parametric Toolbox 3.0,” in Proc. of the European Control Conference, Zürich, Switzerland, July 17–19 2013, pp. 502–510. [Online]. Available: http://control.ee.ethz.ch/~mpt
[19] J. Lofberg. YALMIP: A toolbox for modeling and optimization in MATLAB. [Online]. Available: https://yalmip.github.io/download/
[20] M. Baotic, F. J. Christophersen, and M. Morari, “A new algorithm for constrained finite time optimal control of hybrid systems with a linear performance index,” in 2003 European Control Conference (ECC), 2003, pp. 3323–3328.
[21] J. Tsitsiklis and M. Athans, “On the complexity of decentralized decision making and detection problems,” IEEE Transactions on Automatic Control, vol. 30, no. 5, pp. 440–446, May 1985.
[22] R. Cogill, M. Rotkowitz, B. Van Roy, and S. Lall, “An approximate dynamic programming approach to decentralized control of stochastic systems,” in Control of Uncertain Systems: Modelling, Approximation, and Design, B. A. Francis, M. C. Smith, and J. C. Willems, Eds. Springer Berlin Heidelberg, 2006, pp. 243–256.
[23] H. Lakshmanan and D. P. de Farias, “Decentralized approximate dynamic programming for dynamic networks of agents,” in 2006 American Control Conference (ACC), June 2006, pp. 1648–1653.
[24] D. Bertsimas and A. Thiele, “A robust optimization approach to inventory theory,” Operations Research, vol. 54, no. 1, pp. 150–168, 2006.
[25] V. Gabrel, C. Murat, and N. Remli, “Linear programming with interval right hand sides,” International Transactions in Operational Research, vol. 17, no. 3, pp. 397–408, 2010.
[26] J. W. Chinneck and K. Ramadan, “Linear programming with interval coefficients,” *The Journal of the Operational Research Society*, vol. 51, no. 2, pp. 209–220, 2000.

[27] J. Krumm, “Where will they turn: predicting turn proportions at intersections,” *Personal and Ubiquitous Computing*, vol. 14, no. 7, pp. 591–599, 2010.

[28] F. Borrelli, “Discrete time constrained optimal control,” Ph.D. dissertation, Swiss Federal Institute of Technology (ETH) Zurich, 2002.

[29] F. L. Lewis, D. L. Vrabie, and V. L. Syrmos, *Optimal Control*. John Wiley & Sons, Inc., 2012.