On the quantum equivalence of commutative and noncommutative Chern-Simons theories at higher orders

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Abstract

We continue our investigation of the quantum equivalence between commutative and noncommutative Chern-Simons theories by computing the complete set of two-loop quantum corrections to the correlation function of a pure open Wilson line and an open Wilson line with a field strength insertion, on the noncommutative side in a covariant gauge. The conjectured perturbative equivalence between the free commutative theory and the apparently interacting noncommutative one requires that the sum of these corrections vanish, and herein we exhibit the remarkable cancellations that enforce this. From this computation we speculate on the form of a possible all-order result for this simplest nonvanishing correlator of gauge invariant observables.

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1 Introduction

In [1, 2] it was shown that the Chern-Simons action on a noncommutative spacetime given by

\[ S_{NCCS} = \frac{1}{2} \int d^3x \varepsilon^{\mu\rho\nu} \left[ A_\mu \ast \partial_\rho A_\nu - \frac{2ig}{3} A_\mu \ast A_\rho \ast A_\nu \right], \]  

(1.1)

with the standard star product

\[ f(x) \ast g(x) \equiv e^{i\frac{\theta^{\mu\nu}\partial_{\mu} \partial_{\nu}}{2}} f(y)g(z) \bigg|_{y,z \to 0} \]  

(1.2)

is a fixed-point of the Seiberg-Witten map [3]:

\[ \delta A^\mu = \frac{1}{4} \delta \theta^{\rho\sigma} \partial_{\rho} A^\sigma = -\frac{1}{4} \delta \theta^{\rho\sigma} \{ A^\rho, \partial_\sigma A_\mu + F_{\sigma\mu} \}, \]

\[ \delta F^{\mu\nu} = \frac{1}{4} \delta \theta^{\rho\sigma} \partial_{\rho} F_{\mu\nu} = \frac{1}{4} \delta \theta^{\rho\sigma} \left( 2 \{ F^{\rho\sigma}, F_{\nu\sigma} \} - \{ A^\rho, D^{\rho}_{\sigma} F^{\mu\nu} + \partial_\sigma F^{\mu\nu} \} \right). \]  

(1.3)

Here \( D_\mu \psi = \partial_\mu \psi - ig[A_\mu, \psi] \), and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \). In particular

\[ \frac{\partial S_{NCCS}(A)}{\partial \theta^{\rho\sigma}} = 0. \]  

(1.4)

This implies that noncommutative and commutative Chern-Simons theories are classically equivalent under the Seiberg-Witten map. 1 This is important for two reasons. First, in contradistinction to the usual scenario, it identifies an instance where the action on the commutative side of the map is known in closed form. Second, because the correlation functions of observables in that theory can therefore be computed explicitly (and in the \( U(1) \) case exactly, in a covariant gauge), we can test the equivalence of the theories at the quantum level by computing the correlation functions of their Seiberg-Witten transforms. 2 In turn this first requires us to identify what observables need to be computed on the noncommutative side.

It is well-known [4, 5] that there are no locally gauge-invariant observables in position space in noncommutative gauge theories because gauge transformations act as spacetime translations, as a consequence of the basic noncommutative identity

\[ e^{ik \cdot x} \ast U(x + k\theta) = U(x) \ast e^{ik \cdot x}. \]  

(1.5)

However, it was shown in [5, 6] that a complete set of gauge-invariant observables in noncommutative gauge theories which are local in momentum space are provided by the Fourier

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1This map derives from a disk computation in string theory, and is therefore itself a classical map.

2It also gives rise to the puzzle where in the \( U(1) \) case the commutative space action is free, whereas the noncommutative space action is an apparently interacting theory with a nontrivial coupling constant.
transform of open Wilson lines:

\[
W(k) = \int d^3x \, P_s \exp \left[ ig \int_0^1 d\sigma \tilde{k}^\mu A_\mu(x + \xi(\sigma)) \right] \ast e^{ik \cdot x},
\]

\[
O(k) = \int d^3x \, P_s \exp \left[ ig \int_0^1 d\sigma \tilde{k}^\mu A_\mu(x + \xi(\sigma)) \right] \ast O(x) \ast e^{ik \cdot x},
\]

where \( O(x) \) is any local operator transforming in the adjoint representation. Furthermore, a closed form of the Seiberg-Witten map for the case of a \( U(1) \) gauge group was exhibited in [7, 8, 9], which reduces in three dimensions to

\[
f_{12}(k) = \frac{1}{g\theta_{12}} \left[ W(k) - (2\pi)^3 \delta^{(3)}(k) \right]
\]

\[
f_{0i}(k) = O_{0i}(k),
\]

in a coordinate system where only \( \theta_{12} = -\theta_{21} \) is nonvanishing. Here \( f_{\mu\nu}(k) \) is the commutative field strength in momentum space, and \( O_{\mu\nu}(k) \) is the open Wilson line with a noncommutative field strength insertion at one end:

\[
O_{\mu\nu}(k) = \int d^3x \, P_s \exp \left[ ig \int_0^1 d\sigma \tilde{k}^\mu A_\mu(x + \xi(\sigma)) \right] \ast F_{\mu\nu}(x) \ast e^{ik \cdot x}.
\]

This provides the required correspondence between observables on the commutative and noncommutative sides, and by computing their correlators, allows us to test the quantum equivalence of the two theories.

On the commutative side, the simplest nonvanishing correlator of gauge invariant observables in \( U(1) \) Chern Simons theory is the two-point function of \( f_{12} \) and \( f_{0i} \), and is given exactly in momentum space by

\[
\langle f_{12}(k)f_{0i}(k') \rangle = (2\pi)^3 k_i \delta^{(3)}(k + k').
\]

Noncommutative \( U(1) \) Chern-Simons theory contains an apparently nontrivial interaction with a coupling constant \( g \), and thus the computation on the noncommutative side is at the outset nontrivial in a covariant gauge. This is further complicated by the fact that on the noncommutative side, we have to compute the correlator of composite objects, which even in free commutative theories can be nontrivial. In [15] we computed the Seiberg-Witten transform of \( \langle W(k)O_{\mu\nu}(k') \rangle \), to \( O(g^3) \) in the gauge coupling \( g \) (i.e. one-loop), and found that the \( O(g) \) term reproduced the commutative result while the one-loop or \( O(g^3) \) contributions either cancelled amongst themselves, or yielded a harmless wavefunction renormalization to the Seiberg-Witten map itself. Herein we proceed to compute the complete

\[3\]We will focus on the \( U(1) \) case in this paper which is of interest in part due to its conjectured relevance in the microscopic description fractional quantum Hall fluids [10, 11, 12, 13, 14]. The generalization to the \( U(N) \) case, which amounts to homogeneous factors of powers of \( N \), is straightforward and was discussed by us in [15].
set of two-loop, $O(g^5)$ contributions to this correlator with the expectation that it receives no quantum correction at this order, as required by the conjectured equivalence between the two theories. From this computation, and that done previously in [15], we will hopefully acquire enough insight to conjecture the complete cancellation of quantum corrections to all orders in perturbation theory.

2 Setup

In the interests of maintaining maximum transparency in this lengthy calculation and emphasizing the essentially algebraic nature of the cancellations present, we will work formally and defer discussion of regularization to the end. However we will assume that the point-splitting regulator introduced in [15], and its obvious generalization to handle the inclusion of three gauge field sources, or the field strength commutator and two gauge field sources can be applied: we assume that all operators on $W(k)$ and $O_{\mu\nu}(k')$ are separated by a path parameter spacing $\epsilon$, so as to ensure that a nonvanishing noncommutative phase is always present to regulate the diagrams with respect to the associated loop integration. For the graphs we need to explicitly evaluate that involve 'internal' loops, there are also logarithmic divergences coming from the planar parts of those loops that are of the same form found in commutative Chern-Simons theories, and which we will assume are regulated in an appropriate manner.

We will also invoke several notational simplifications to prevent the expressions from becoming too unwieldy and to hopefully make the calculation easier to follow. To this end we will absorb the homogeneous factor of $(2\pi)^{-3}$, the overall momentum conservation delta function $\delta^{(3)}(k+k')$ as well as the delta functions expressing momentum conservation at the vertices, and the integration measures into a generalized integration symbol when there is no ambiguity, with the understanding that all momenta except for $k$, the momentum carried by $W(k)$, and all path parameters are to be integrated over their appropriate ranges. Spacetime indices and (initially) momenta will be labelled according to their origin from within the expansion of $W(k)$ or $O_{\mu\nu}(k')$. Thus $p_{ij}$ refers to the momentum carried by the $i$th gauge field from the $j$th Wilson line. Specifically, $p_{i1}$ will refer to the momenta of gauge fields on $W(k)$, $p_{i2}$ to those from $O_{\mu\nu}(k')$, and since we will not require more than two gauge field sources from the path-ordered exponential part of $O_{\mu\nu}(k')$, the field strength commutator momenta will always be labelled by $p_{32}$ and $p_{42}$. Similarly the indices $\mu_{ij}$ will always be associated with the basic line element $k \equiv k\theta$ (and are hence interchangeable within tensor expressions), while indices $\alpha_i, \beta_i, \gamma_i$ and respective momenta $q_i, r_i, s_i$ are associated with contractions into internal vertices.

We will subsequently re-define or re-label the momenta to a standard set, which will make the momentum constraints manifest, homogenize the expressions, and allow us to compare diagrams by inspection of the resulting tensor expressions. Thus, throughout the
computation the indices $\rho_1, \rho_2, \ldots, \rho_7$ will always be associated with momenta $k, k - \eta, \eta, q, q + \eta, k + q,$ and $k$ respectively, where $\eta$ and $q$ will be our loop integration variables. The calculation will be presented without recourse to expanding pairs of antisymmetric symbols lacking mutual contracted indices by employing identities presented in the appendix to write all final tensor expressions in terms of $\varepsilon_{\mu\nu a},$ where $a$ is any of $\rho_1 \ldots \rho_7,$ or $\mu_{ij}$. The only unsummed indices are $\mu, \nu$ and the only unintegrated momentum is $k$.

We will freely use the fact that in Chern-Simons theory a direct contraction between any pair of gauge fields from the Wilson lines themselves vanish in the Landau gauge, as do both one-point tadpoles and contributions where the two Wilson lines are disconnected. Additionally, since the possible one-loop correction to the Chern-Simons propagator changes neither its tensor structure nor its momentum dependence as discussed in [15], the two-loop contributions involving it are necessarily a repeat of the one-loop calculation presented therein, and we will henceforth ignore them. These observations allow us to drastically reduce the number of diagrams and contractions we must consider.

A key *a posteriori* insight acquired from this calculation is to collect the contributions according to the number of gauge field sources on the pure open Wilson line $W(k),$ a fact that will be the basis for our all-orders discussion at the end. In any case, the set of cancellations we now exhibit at two-loops will be presented in this way. We label all diagrams as $[x.yy],$ where $x$ denotes the number of sources on $W(k),$ and $yy$ is a counter.

### 3 Contributions from $O(g^3)$ terms in $W(k)$

Figure 1 illustrates one of three pairs of diagrams that involve contractions involving three gauge field sources from $W(k)$ and no gauge field sources from the path-ordered exponential component of $O_{\mu\nu}(k')$. Since we will not need to perform the path parameter or the loop integrations, it suffices to consider one pair; the other two cases are identical.

For the graph with two internal vertices we have

$$[3.01] = (ig)^3(-2ig)^2 \int k^{\mu_{11}}k^{\mu_{21}}k^{\mu_{31}}\delta(3)(p_{11} + p_{21} + p_{31} - k)\delta(3)(p_{42} - k')e^{-i[p_{11} \cdot \xi_{11} + p_{21} \cdot \xi_{21} + p_{31} \cdot \xi_{31}]}$$

4Alternatively, as is well-known, the gauge and ghost graphs involved in the one-loop propagator correction formally cancel, and the one-loop shift arises from the regulator, so each of these putative contributions at two-loops pairwise cancel at the formal level.
\[
\times e^{-\frac{i}{2}p_{21} \cdot (p_{31} - k) - p_{31} \cdot k} \sin \left( \frac{31 \times 21}{2} \right) \sin \left( \frac{1 \times 2}{2} \right) \frac{P_{12}^0 P_{11}^0 P_{10}^0 P_{13}^5 P_{15}^5}{P_{12}^2 P_{11}^2 P_{10}^2 P_{13}^5 P_{15}^5} \times (-i p_{42}) \mu \epsilon \mu_{\rho_1} \epsilon \mu_{\rho_2} \epsilon \mu_{\rho_3} \delta \epsilon \mu_{\rho_4} \epsilon \mu_{\rho_5} \epsilon \alpha_1 \sigma_2 \epsilon \beta_1 \beta_2 \beta_3 \times \delta(\rho_{42} + \frac{3}{2}) \delta(\rho_{11} + \frac{2}{2}) \delta(\rho_{12} + \frac{1}{2}) \delta(\rho_{31} + \frac{1}{2}) \delta(\rho_{31} + \frac{1}{2}) - (\mu \leftrightarrow \nu). \quad (3.1)
\]

To satisfy the momentum constraints define \(p_{11} = k - \eta, p_{21} = -q, p_{31} = q + \eta, p_{42} = -k\), and \(q_3 = -\eta\). Then contracting on \(\alpha_2, \beta_3\) using (A.13) we obtain

\[
[3.01] = -4g^5 \int \tilde{k}_{11} \tilde{k}_{21} \tilde{k}_{31} \epsilon^{\left( (k \times k) \sigma_{11} + (k \times q) \sigma_{21} - k \times (q + \eta) \sigma_{31} \right) e^{-\frac{i}{2}(\eta \times q + k \times q)} \sin \left( \frac{31 \times 21}{2} \right) \sin \left( \frac{1 \times 2}{2} \right) \frac{P_{12}^0 P_{11}^0 P_{10}^0 P_{13}^5 P_{15}^5}{P_{12}^2 P_{11}^2 P_{10}^2 P_{13}^5 P_{15}^5} \times e^{-\frac{i}{2}p_{21} \cdot (p_{31} - k) - p_{31} \cdot k} \epsilon^{\left( (k \times k) \sigma_{11} + (k \times q) \sigma_{21} - k \times (q + \eta) \sigma_{31} \right) e^{-\frac{i}{2}(\eta \times q + k \times q)} \sin \left( \frac{31 \times 21}{2} \right) \sin \left( \frac{1 \times 2}{2} \right) \frac{P_{12}^0 P_{11}^0 P_{10}^0 P_{13}^5 P_{15}^5}{P_{12}^2 P_{11}^2 P_{10}^2 P_{13}^5 P_{15}^5} \times e^{-\frac{i}{2}p_{21} \cdot (p_{31} - k) - p_{31} \cdot k} \epsilon^{\left( (k \times k) \sigma_{11} + (k \times q) \sigma_{21} - k \times (q + \eta) \sigma_{31} \right) e^{-\frac{i}{2}(\eta \times q + k \times q)} \sin \left( \frac{31 \times 21}{2} \right) \sin \left( \frac{1 \times 2}{2} \right) \frac{P_{12}^0 P_{11}^0 P_{10}^0 P_{13}^5 P_{15}^5}{P_{12}^2 P_{11}^2 P_{10}^2 P_{13}^5 P_{15}^5}} \times \left\{ \epsilon_{\mu_{11} \rho_2 \mu \epsilon_{\nu_3 \rho_3} \beta_1 \epsilon_{\mu_{21} \rho_4 \beta_2} \epsilon_{\mu_{31} \rho_5 \beta_3} P_{12}^0 P_{11}^0 P_{10}^0 P_{13}^5 P_{15}^5 \times \left\{ \epsilon_{\mu_{11} \rho_2 \mu \epsilon_{\nu_3 \rho_3} \beta_1 \epsilon_{\mu_{21} \rho_4 \beta_2} \epsilon_{\mu_{31} \rho_5 \beta_3} P_{12}^0 P_{11}^0 P_{10}^0 P_{13}^5 P_{15}^5 \right\} \right\} -(\mu \leftrightarrow \nu). \quad (3.2)
\]

In the last line we have used (A.17) with \(C = k\).

There are two contractions into the commutator in the second graph:

\[
[3.02] = -(i g)^4 (-2i g) \int \tilde{k}_{11} \tilde{k}_{21} \tilde{k}_{31} \delta(\rho_{11} + \rho_{21} + \rho_{31} - k) \delta(\rho_{32} + \rho_{42} - k) e^{-i[p_{11} \cdot \xi_{11} + p_{21} \cdot \xi_{21} + p_{31} \cdot \xi_{31}]} \times e^{-\frac{i}{2}[p_{21} \cdot (p_{31} - k) - p_{31} \cdot k]} e^{-\frac{i}{2}p_{32} \cdot p_{42} \sin \left( \frac{r_3 r_4}{r_3 r_4} \right) \epsilon^{\beta_1 \beta_2 \beta_3} P_{12}^0 P_{11}^0 P_{10}^0 P_{13}^5 P_{15}^5 \times \left\{ \epsilon_{\mu_{11} \rho_2 \mu \epsilon_{\nu_3 \rho_3} \beta_1 \epsilon_{\mu_{21} \rho_4 \beta_2} \epsilon_{\mu_{31} \rho_5 \beta_3} P_{12}^0 P_{11}^0 P_{10}^0 P_{13}^5 P_{15}^5 \right\} \right\} -(\mu \leftrightarrow \nu). \quad (3.3)
\]

Again define \(p_{11} = k - \eta, p_{21} = -q, p_{31} = q + \eta\). Then contracting on \(\beta_3\) and using (A.16) we obtain

\[
[3.02] = 2i g^5 \int \tilde{k}_{11} \tilde{k}_{21} \tilde{k}_{31} \epsilon^{\left( (k \times k) \sigma_{11} + (k \times q) \sigma_{21} - k \times (q + \eta) \sigma_{31} \right) e^{-\frac{i}{2}(\eta \times q + k \times q)} \sin \left( \frac{31 \times 21}{2} \right) \sin \left( \frac{1 \times 2}{2} \right) \frac{P_{12}^0 P_{11}^0 P_{10}^0 P_{13}^5 P_{15}^5}{P_{12}^2 P_{11}^2 P_{10}^2 P_{13}^5 P_{15}^5}} \times \left\{ \epsilon_{\mu_{11} \rho_2 \mu \epsilon_{\nu_3 \rho_3} \beta_1} \epsilon_{\mu_{21} \rho_4 \beta_2} \epsilon_{\mu_{31} \rho_5 \beta_3} P_{12}^0 P_{11}^0 P_{10}^0 P_{13}^5 P_{15}^5 \right\} \times e^{-\frac{i}{2}[p_{21} \cdot (p_{31} - k) - p_{31} \cdot k]} \epsilon^{\left( (k \times k) \sigma_{11} + (k \times q) \sigma_{21} - k \times (q + \eta) \sigma_{31} \right) e^{-\frac{i}{2}(\eta \times q + k \times q)} \frac{P_{12}^0 P_{11}^0 P_{10}^0 P_{13}^5 P_{15}^5}{P_{12}^2 P_{11}^2 P_{10}^2 P_{13}^5 P_{15}^5}} \times \left\{ \epsilon_{\mu_{11} \rho_2 \mu \epsilon_{\nu_3 \rho_3} \beta_1} \epsilon_{\mu_{21} \rho_4 \beta_2} \epsilon_{\mu_{31} \rho_5 \beta_3} P_{12}^0 P_{11}^0 P_{10}^0 P_{13}^5 P_{15}^5 \right\} \right\} -(\mu \leftrightarrow \nu) \times \left\{ \epsilon_{\mu_{11} \rho_2 \mu \epsilon_{\nu_3 \rho_3} \beta_1} \epsilon_{\mu_{21} \rho_4 \beta_2} \epsilon_{\mu_{31} \rho_5 \beta_3} P_{12}^0 P_{11}^0 P_{10}^0 P_{13}^5 P_{15}^5 \right\} \right\} -(\mu \leftrightarrow \nu). \quad (3.4)
\]

We note that this cancellation is essentially the same as one of those presented in [15], because only one of the two possible terms survives in the contraction of the two sources on \(W(k)\) into the same vertex:

\[
\epsilon_{\mu_1 \rho_A \beta_2} \epsilon_{\mu_2 \rho_B \beta_3} \epsilon_{\beta_1 \beta_2 \beta_3} \tilde{k}_{11} \tilde{k}_{22} = \epsilon_{\mu_2 \rho_A \beta_2} \epsilon_{\mu_1 \rho_B \beta_3} \tilde{k}_{11} \tilde{k}_{22}. \quad (3.5)
\]
This mechanism is completely general, and therefore by induction, we can reduce any such pairs of graphs with \( n + 1 \) sources on \( W(k) \), and zero sources from the path-ordered exponential part of \( O_{\mu\nu} \) to the calculation of the \( n \) source case, where \( n \geq 2 \). (For \( n = 1 \), there is no pairing, and we merely obtain the nonvanishing tree-level result.) We will now see how this same mechanism applies to a particular subset of graphs with two sources on \( W(k) \) and with one source from the path-ordered exponential part of \( O_{\mu\nu} \), and thereby recover the other part of the \( O(g^3) \) calculation in [15]. Unfortunately, the story becomes more complicated after that because sources from \( W(k) \) will attach to distinct vertices.

4 Contributions from \( O(g^2) \) terms in \( W(k) \)

The two gauge field sources on \( W(k) \) in the first three diagrams we consider in this section connect to the same vertex as shown in figure 2, and so we expect to find a cancellation amongst them analogous to that in the previous section and the other part of calculation at \( O(g^3) \) presented in [15].

Using (A.5) the first of these diagrams is given by

\[
[2.01] = \frac{(ig)^3}{2} (2ig)^2 \int \tilde{k}_{\mu_11} \tilde{k}_{\mu_21} \tilde{k}_{\mu_312} \delta^{(3)}(p_{11} + p_{21} - k) \delta^{(3)}(p_{12} + p_{42} - k') e^{-ip_{11} \cdot \xi_11 \cdot \eta} e^{-ip_{11} \times p_{21}} \\
e^{-ip_{12} \cdot \xi_12 \cdot \eta} e^{-ip_{12} \times p_{42} \sin(21 \times 2)} \sin(3 \times 2) \int \frac{d^4 \rho_{13} d^4 \rho_{24}}{q^2 p_{11}^2 p_{21}^2 p_{312}^2} \epsilon_{\alpha_1 \alpha_2 \alpha_3} \epsilon_{\beta_1 \beta_2 \beta_3} \left( -ip_{42} \right) \left[ \eta \mu \nu \right]_{\rho_{12} \beta_1} \epsilon_{\mu_1 \rho_{23} \epsilon_{\alpha_3 \rho_{1} \beta_1} \epsilon_{\mu_2 \rho_{4} \beta_2} \delta^{(3)}(p_{11} + q_1) \delta^{(3)}(p_{12} + q_2) \delta^{(3)}(p_{41} + r_2) \delta^{(3)}(p_{42} + r_3) \right].
\]

(4.1)

Now take \( p_{11} = k - \eta, p_{21} = \eta, q_3 = k, p_{12} = q, p_{42} = -(k + q) \), and contract on \( \alpha_2, \beta_2 \):

\[
[2.01] = -2g^5 \int \tilde{k}_{\mu_11} \tilde{k}_{\mu_21} \tilde{k}_{\mu_312} \epsilon^{i(k \times \eta)}(\sigma_{11} - \frac{1}{2}) \epsilon^{i(k \times q)}(\sigma_{12} - \frac{1}{2}) \sin(\frac{k \times q}{2}) \sin(\frac{k \times q}{2}) \frac{k^1 (k-\eta)^2 q_3 q^4 (k+q)^2}{k^2 (k-\eta)^2 q^2 q^2 (k+q)^2} \\
\times \epsilon_{\mu_1 \rho_{23} \rho_4} \epsilon_{\mu_2 \rho_1 \rho_4} (k + q) \left[ \epsilon_{\mu \nu \rho_6} (k + q)^2 \epsilon_{\rho_{12} \delta^{(3)}} - \epsilon_{\mu \nu \rho_6} \tilde{k}_{\mu_12} (k + q)^2 \right],
\]

(4.2)

where we have used \( \tilde{k}' = -\tilde{k} \), and again applied (A.17). Note this time however, the term on the right hand side of (A.17) survives because the propagator associated with \( p_{42} \) carries a loop momentum. We will need [2.03] to cancel the additional term.
Like [3.02], there are two contractions to consider in [2.02]:

\[
[2.02] = -\frac{(ig)^4}{2} \left( -2ig \right) \int \hat{k}_{\mu_1} \hat{k}_{\mu_2} \delta^{(3)}(p_{11}+p_{21}+k) \delta^{(3)}(p_{12}+p_{32}+p_{42}-k) e^{-ip_{11} \cdot \xi_{11}} e^{-\frac{1}{2} p_{11} \times p_{21}} e^{-\frac{1}{2} [p_{32} \times (p_{42}-k') - p_{42} \times k'] \sin \left( \frac{\theta_1 \times \theta_2}{2} \right) \sin \alpha_1 \alpha_2 \alpha_3 \frac{p_{11}^2 p_{32}^2 p_{12}^2}{p_{11}^2 p_{32}^2 p_{12}^2} e^{-ip_{11} \cdot \xi_{11}} e^{-\frac{1}{2} p_{11} \times p_{21}} e^{-\frac{1}{2} [p_{32} \times (p_{42}-k') - p_{42} \times k'] \sin \left( \frac{\theta_1 \times \theta_2}{2} \right) \sin \alpha_1 \alpha_2 \alpha_3 \frac{p_{11}^2 p_{32}^2 p_{12}^2}{p_{11}^2 p_{32}^2 p_{12}^2}} \times \varepsilon_{\mu_1 \rho_1 \mu_2 \nu_1 \alpha_3} \delta^{(3)}(p_{11}+q_1) \delta^{(3)}(p_{21}+q_2) \delta^{(3)}(p_{32}+q_3) + \frac{p_{11}^2}{p_{32}^2} \varepsilon_{\mu_1 \rho_1} \alpha_2 \varepsilon_{\mu_1 \rho_2} \alpha_2 \varepsilon_{\mu_2 \rho_3} \delta^{(3)}(p_{11}+q_1) \delta^{(3)}(p_{21}+q_2) \delta^{(3)}(p_{32}+q_3) \right) - (\mu \leftrightarrow \nu) .
\]

As in [2.01], set \( p_{11} = k - \eta, p_{21} = \eta, p_{32} = q, p_{42} = -(k + q) \), and contract on \( \alpha_2, \beta_2 \):

\[
[2.03] = -\frac{1}{2} \left( -2ig \right)^2 \int \hat{k}_{\mu_1} \hat{k}_{\mu_2} \delta^{(3)}(p_{11}+p_{21}+k) \delta^{(3)}(p_{32}+p_{42}+k') e^{-ip_{11} \cdot \xi_{11}} e^{-\frac{1}{2} p_{11} \times p_{21}} e^{-\frac{1}{2} [p_{32} \times p_{42} \sin \left( \frac{\theta_1 \times \theta_2}{2} \right) \sin \alpha_1 \alpha_2 \alpha_3 \varepsilon_{\beta_1 \beta_2 \beta_3} \frac{p_{11}^2 p_{32}^2 p_{12}^2}{p_{11}^2 p_{32}^2 p_{12}^2}} \times \varepsilon_{\alpha_3 \beta_1} \varepsilon_{\mu_2 \rho_3} \varepsilon_{\nu_1 \rho_3} \delta^{(3)}(p_{11}+q_1) \delta^{(3)}(p_{21}+q_2) \delta^{(3)}(p_{32}+q_3) \delta^{(3)}(p_{42}+q_3) - (\mu \leftrightarrow \eta) .
\]

Now consider [2.03]:

\[
[2.03] = -\frac{1}{2} (ig)^3 \left( -2ig \right)^2 \int \hat{k}_{\mu_1} \hat{k}_{\mu_2} \delta^{(3)}(p_{11}+p_{21}+k) \delta^{(3)}(p_{32}+p_{42}+k') e^{-ip_{11} \cdot \xi_{11}} e^{-\frac{1}{2} p_{11} \times p_{21}} e^{-\frac{1}{2} [p_{32} \times p_{42} \sin \left( \frac{\theta_1 \times \theta_2}{2} \right) \sin \alpha_1 \alpha_2 \alpha_3 \varepsilon_{\beta_1 \beta_2 \beta_3} \frac{p_{11}^2 p_{32}^2 p_{12}^2}{p_{11}^2 p_{32}^2 p_{12}^2}} \times \varepsilon_{\alpha_3 \beta_1} \varepsilon_{\mu_2 \rho_3} \varepsilon_{\nu_1 \rho_3} \delta^{(3)}(p_{11}+q_1) \delta^{(3)}(p_{21}+q_2) \delta^{(3)}(p_{32}+q_3) \delta^{(3)}(p_{42}+q_3) - (\mu \leftrightarrow \eta) .
\]

Set \( p_{11} = k - \eta, p_{21} = \eta, p_{32} = q, p_{42} = -(k + q) \), and contract on \( \alpha_2, \beta_2 \):

\[
[2.03] = 2ig^5 \int \hat{k}_{\mu_1} \hat{k}_{\mu_2} e^{i(\mathbf{k} \cdot \mathbf{n})(\sigma_1 - \frac{1}{2})} e^{-\frac{1}{2} k \times q \sin \left( \frac{\theta_1 \times \theta_2}{2} \right) \sin \alpha_1 \alpha_2 \alpha_3} \frac{\varepsilon_{\mu_1} \varepsilon_{\nu_1} \varepsilon_{\rho_2} \varepsilon_{\rho_3} \varepsilon_{\nu_4} \varepsilon_{\rho_4} (k + q)^6}{k^2 (k - \eta)^2 \eta^2 q^2 (k + q)^2} e^{-ip_{11} \cdot \xi_{11}} e^{-\frac{1}{2} p_{11} \times p_{21}} e^{-\frac{1}{2} [p_{32} \times p_{42} \sin \left( \frac{\theta_1 \times \theta_2}{2} \right) \sin \alpha_1 \alpha_2 \alpha_3 \varepsilon_{\beta_1 \beta_2 \beta_3} \frac{p_{11}^2 p_{32}^2 p_{12}^2}{p_{11}^2 p_{32}^2 p_{12}^2}} \times \varepsilon_{\alpha_3 \beta_1} \varepsilon_{\mu_2 \rho_3} \varepsilon_{\nu_1 \rho_3} \delta^{(3)}(p_{11}+q_1) \delta^{(3)}(p_{21}+q_2) \delta^{(3)}(p_{32}+q_3) \delta^{(3)}(p_{42}+q_3) - (\mu \leftrightarrow \eta) .
\]

As in [2.01], set \( p_{11} = k - \eta, p_{21} = \eta, p_{32} = q, p_{42} = -(k + q) \), and contract on \( \alpha_2, \beta_2 \):

\[
[2.03] = 4ig^5 \int \hat{k}_{\mu_1} \hat{k}_{\mu_2} e^{i(\mathbf{k} \cdot \mathbf{n})(\sigma_1 - \frac{1}{2})} \sin \left( \frac{\theta_1 \times \theta_2}{2} \right) \sin \alpha_1 \alpha_2 \alpha_3 \frac{\varepsilon_{\mu_1} \varepsilon_{\nu_1} \varepsilon_{\rho_2} \varepsilon_{\rho_3} \varepsilon_{\nu_4} \varepsilon_{\rho_4} (k + q)^6}{k^2 (k - \eta)^2 \eta^2 q^2 (k + q)^2} e^{-ip_{11} \cdot \xi_{11}} e^{-\frac{1}{2} p_{11} \times p_{21}} e^{-\frac{1}{2} [p_{32} \times p_{42} \sin \left( \frac{\theta_1 \times \theta_2}{2} \right) \sin \alpha_1 \alpha_2 \alpha_3 \varepsilon_{\beta_1 \beta_2 \beta_3} \frac{p_{11}^2 p_{32}^2 p_{12}^2}{p_{11}^2 p_{32}^2 p_{12}^2}} \times \varepsilon_{\alpha_3 \beta_1} \varepsilon_{\mu_2 \rho_3} \varepsilon_{\nu_1 \rho_3} \delta^{(3)}(p_{11}+q_1) \delta^{(3)}(p_{21}+q_2) \delta^{(3)}(p_{32}+q_3) \delta^{(3)}(p_{42}+q_3) - (\mu \leftrightarrow \eta) .
\]
In the second step we have used \((3.14)\), and in the fourth we have made the change of variables \(q \rightarrow -(k + q)\) in the second term. This allows us to compare \([2.03]\) to the first term in \([2.01]\), which is a total derivative with respect to the integration over \(\sigma_{12}\), whose measure we have suppressed. Performing the integral,

\[
\int_0^1 d\sigma_{12} \left( k \times q \right) e^{i(k \times q)(\sigma_{12} - \frac{1}{2})} = 2 \sin \left( \frac{k \times q}{2} \right),
\]

we see that the first term in \([2.01]\) is cancelled by \([2.03]\), while the second term in \([2.01]\) is cancelled by \([2.02]\) in the same way that \([3.02]\) cancels \([3.01]\).

Again, the identity \((3.3)\) ensures that this is essentially the same calculation as that presented in \([15]\) at one-loop. However, the remaining graphs we consider in this section, in which the two gauge field sources from \(W(k)\) attach to different vertices, work out quite differently.

More specifically if we think of graph \([2.01]\) as containing a tree-level four point function in the t-channel attached to the Wilson lines to form a two-loop diagram, then the graphs in figure 3 reflect the other two channels. Combining the two we obtain

\[
[2.04] = \frac{(i g)^3}{2} \frac{(-2i g)^2}{2} \int \tilde{h}^{\mu_1 11} \tilde{h}^{\mu_2 21} \tilde{h}^{\mu_3 3} \delta^{(3)}(p_{11} + p_{21} - k) \delta^{(3)}(p_{12} + p_{42} - k') e^{-i p_{11} \xi_{11}} e^{-i p_{12} \xi_{12}} \\
\times e^{-i p_{12} \times p_{42}} \sin \left( \frac{q_{12}}{2} \right) \sin \left( \frac{q_{21}}{2} \right) \int \frac{d\rho_{12} \rho_{21}}{p_{12} \rho_{21}} \frac{d\rho_{34}}{p_{34}} \frac{d\rho_{56}}{p_{56}} \left\{ e^{i k \cdot \eta} \mu \alpha_2 \alpha_3 \beta_2 \beta_3 \left( -i \rho_{12} \mu \alpha_2 \alpha_3 \beta_1 \beta_3 \right) \right\} \rightarrow -\left( \mu \leftrightarrow \nu \right).
\]

Set \(p_{11} = \eta\), \(p_{12} = q\) in the first term and \(p_{12} = -(k + q)\) in the second, and contract on \(\alpha_1, \beta_1\) and \(\alpha_3, \beta_3\) respectively, to get

\[
[2.04] = -2 g \int \tilde{h}^{\mu_1 11} \tilde{h}^{\mu_2 21} \tilde{h}^{\mu_3 3} \delta^{(3)}(k + q) \sin \left( \frac{q_{12}}{2} \right) \sin \left( \frac{q_{21}}{2} \right) \int \frac{d\rho_{12} \rho_{21}}{p_{12} \rho_{21}} \frac{d\rho_{34}}{p_{34}} \frac{d\rho_{56}}{p_{56}} \left\{ e^{i k \cdot \eta} \mu \alpha_2 \alpha_3 \beta_2 \beta_3 \left( -i \rho_{12} \mu \alpha_2 \alpha_3 \beta_1 \beta_3 \right) \right\} \\
\}

\[
= 2 g^5 \int \tilde{h}^{\mu_1 11} \tilde{h}^{\mu_2 21} \tilde{h}^{\mu_3 3} e^{i k \cdot \eta} \sin \left( \frac{q_{12}}{2} \right) \sin \left( \frac{q_{21}}{2} \right) \int \frac{d\rho_{12} \rho_{21}}{p_{12} \rho_{21}} \frac{d\rho_{34}}{p_{34}} \frac{d\rho_{56}}{p_{56}} \left\{ e^{i k \cdot \eta} \mu \alpha_2 \alpha_3 \beta_2 \beta_3 \left( -i \rho_{12} \mu \alpha_2 \alpha_3 \beta_1 \beta_3 \right) \right\}.
\]
where we have made the change of variables $\sigma_{12} \rightarrow 1 - \sigma_{12}$ in the first term. To these diagrams we pair up the ones in figure 4, which represent four sets of contractions:

\[
[2.05] = -\frac{(ig)^4}{2} (-2ig) \int \bar{k}_{\mu_1} \bar{k}_{\mu_2} k_{\mu_1} k_{\mu_2} \delta(3)(p_{11} + p_{21} - k_1) \delta(3)(p_{12} + p_{32} + p_{42} - k_2') e^{-i[p_{11} \cdot \xi_{11} + \frac{i}{2} p_{11} \times p_{21}]}
\]

\[
e^{-ip_{12} \cdot \xi_{12}} e^{-\frac{i}{4} [p_{32} \times (p_{42} - k_2') - p_{42} \times k_2']} \sin \left( \frac{q_1 \times p_2}{2} \right) \frac{\rho_{12}^{\mu \rho} \rho_{12}^{\nu \rho}}{p_{12}^4} \left\{ \rho_{12}^{\mu \rho} \varepsilon_{\mu_1 \rho_3 \mu} \varepsilon_{\nu_5 \rho_5 \alpha_2} \delta(3)(p_{11} + p_{32}) \delta(3)(p_{12} + p_{42}) + \rho_{12}^{\mu \rho} \varepsilon_{\mu_1 \rho_3 \nu} \varepsilon_{\mu_5 \rho_3 \alpha_2} \delta(3)(p_{11} + p_{42}) \delta(3)(p_{12} + p_{32}) \right\}
\]

\[
+ \left[ \rho_{12}^{\mu \rho} \varepsilon_{\mu_1 \rho_3 \alpha_1} \varepsilon_{\mu_2 \rho_4 \alpha_3} \delta(3)(p_{11} + p_{32}) \delta(3)(p_{12} + p_{42}) \right] \varepsilon_{\alpha_1 \alpha_2 \alpha_3} \varepsilon(\mu \leftrightarrow \nu) \right) .
\]

Take $p_{11} = \eta$. In the first two terms set $p_{12} = -(k + q)$, and in the second two set $p_{12} = q$. Then contracting on $\alpha_3$ we obtain

\[
[2.05] = -ig^5 \int \bar{k}_{\mu_1} \bar{k}_{\mu_2} k_{\mu_1} k_{\mu_2} e^{i k \times q \left( \frac{1}{2} - \sigma_{11} \right)} \sin \left( \frac{q_1 \times q_2}{2} \right) \frac{\rho_{12}^{\mu \rho} \rho_{12}^{\nu \rho}}{p_{12}^4} \left\{ \sin \left( \frac{q_1 \times q_2}{2} \right) \frac{(k + q)^2}{(k - q)^2} \right\}
\]

\[
\times e^{i k \times q \left( \frac{1}{2} - \sigma_{12} \right)} \varepsilon_{\mu_1 \rho_3 \alpha_1} \varepsilon_{\mu_2 \rho_3 \alpha_3} \left[ \varepsilon_{\mu_1 \rho_3 \mu} \varepsilon_{\nu_5 \rho_5 \alpha_2} + e^{-\frac{i}{2} q \times q} \varepsilon_{\mu_1 \rho_3 \nu} \varepsilon_{\mu_5 \rho_5 \alpha_2} \right] +
\]

\[
+ \sin \left( \frac{q_1 \times q_2}{2} \right) \frac{\rho_{12}^{\mu \rho} \rho_{12}^{\nu \rho}}{q^4} e^{i k \times q \left( \frac{1}{2} - \sigma_{12} \right)} \varepsilon_{\mu_1 \rho_3 \alpha_1} \varepsilon_{\mu_2 \rho_3 \alpha_3} \left[ e^{i \frac{1}{2} q \times q - k \times k} \varepsilon_{\mu_1 \rho_3 \mu} \varepsilon_{\nu_5 \rho_5 \alpha_2} + e^{-\frac{1}{2} q \times q + k \times k} \varepsilon_{\mu_2 \rho_3 \mu} \varepsilon_{\nu_5 \rho_5 \alpha_2} \right]
\]

\[
+ e^{-\frac{i}{2} q \times q - k \times k} \varepsilon_{\mu_1 \rho_3 \mu} \varepsilon_{\mu_2 \rho_3 \mu} \} - (\mu \leftrightarrow \nu)
\]

\[
= 2g^5 \int \bar{k}_{\mu_1} \bar{k}_{\mu_2} k_{\mu_1} k_{\mu_2} e^{i k \times q \left( \frac{1}{2} - \sigma_{11} \right)} \sin \left( \frac{q_1 \times q_2}{2} \right) \frac{\rho_{12}^{\mu \rho} \rho_{12}^{\nu \rho}}{p_{12}^4} \left\{ \sin \left( \frac{q_1 \times q_2}{2} \right) \frac{(k - q)^2}{(k + q)^2} \right\}
\]

\[
\times e^{i k \times q \left( \frac{1}{2} - \sigma_{12} \right)} \varepsilon_{\mu_1 \rho_3 \alpha_1} \varepsilon_{\mu_2 \rho_3 \alpha_3} \left[ \varepsilon_{\mu_1 \rho_3 \mu} \varepsilon_{\nu_5 \rho_5 \alpha_2} - \frac{(k + q)^2}{q^4} \varepsilon_{\mu_1 \rho_3 \nu} \varepsilon_{\mu_5 \rho_5 \alpha_2} \right] +
\]

\[
+ \sin \left( \frac{q_1 \times q_2}{2} \right) \frac{\rho_{12}^{\mu \rho} \rho_{12}^{\nu \rho}}{q^4} e^{i k \times q \left( \frac{1}{2} - \sigma_{12} \right)} \varepsilon_{\mu_1 \rho_3 \alpha_1} \varepsilon_{\mu_2 \rho_3 \alpha_3} \left[ e^{i \frac{1}{2} q \times q - k \times k} \varepsilon_{\mu_1 \rho_3 \mu} \varepsilon_{\nu_5 \rho_5 \alpha_2} - \frac{(k - q)^2}{q^4} \varepsilon_{\mu_2 \rho_3 \mu} \varepsilon_{\nu_5 \rho_5 \alpha_2} \right]
\]

\[
\right) ,
\]

where we have used (A.16) in the last line. Again, [2.04] and [2.05] do not cancel, but it is natural to combine them using (A.17) to obtain a total derivative term with respect to the $\sigma_{12}$ integration:

\[
[2.04] + [2.05] = 2g^5 \int k_{\mu_1} k_{\mu_2} e^{i k \times q \left( \frac{1}{2} - \sigma_{11} \right)} \sin \left( \frac{q_1 \times q_2}{2} \right) \frac{\rho_{12}^{\mu \rho} \rho_{12}^{\nu \rho}}{q^4} \times
\]

\[
(k \times q) e^{i k \times q \left( \frac{1}{2} - \sigma_{12} \right)} \left[ \varepsilon_{\mu_1 \rho_3 \mu} \varepsilon_{\mu_2 \rho_3 \mu} \varepsilon_{\nu_5 \rho_5} + \varepsilon_{\mu_1 \rho_3 \mu} \varepsilon_{\mu_2 \rho_3 \mu} \varepsilon_{\nu_5 \rho_4} \right].
\]

9
In analogy to [2.01]-[2.03] we now hope to cancel these terms with the contribution in figure 5, which represents two sets of contractions (with the same natural diagrammatic representation). However, the fact that the two sources from the commutator contract into distinct vertices will seriously complicate the problem as we shall now see:

\[
[2.06] = -\frac{(ig)^3}{2} (-2ig)^2 \int \tilde{k}^{\mu_11} \tilde{k}^{\mu_21} \xi^{\text{(3)}}(p_{11}+p_{21}-k)\xi^{\text{(3)}}(p_{32}+p_{42}-k') \epsilon^{-p_{11}+p_{21}} \epsilon^{-p_{32}+p_{42}} \\
\frac{\rho^{p_3}_{\rho^{q_3}_{\rho^{k_3}_{\rho^{\eta_3}_{\rho^{5_3}}}}}}{\tilde{k}_{11}^{\rho^{1_1}_{\rho^{2_1}_{\rho^{3_1}}}}} \sin \left(\frac{2\pi x}{z}\right) \sin \left(\frac{2\pi y}{z}\right) \epsilon^{\alpha_1 \alpha_2 \alpha_3} \epsilon^{\beta_1 \beta_2 \beta_3} \epsilon_{\mu_11 \rho_3 \rho_1} \epsilon_{\mu_21 \rho_3 \rho_2} \epsilon_{\alpha_3 \rho_5 \rho_1} \delta^{(3)}(p_{11}+q_1) \\
\delta^{(3)}(p_{21}+q_2) \delta^{(3)}(q_1+r_1) \left\{ \frac{\rho^{p_3}_{\rho^{q_3}_{\rho^{k_3}_{\rho^{\eta_3}_{\rho^{5_3}}}}}}{\tilde{k}_{32}^{\rho^{1_2}_{\rho^{2_2}_{\rho^{3_2}}}}} \epsilon_{\mu_3 \rho_6 \beta_3} \epsilon_{\nu_3 \rho_6 \beta_3} \epsilon^{\delta^{(3)}}(p_{32}+q_3) \delta^{(3)}(p_{42}+r_3) \\
+ \frac{\rho^{p_3}_{\rho^{q_3}_{\rho^{k_3}_{\rho^{\eta_3}_{\rho^{5_3}}}}}}{\tilde{k}_{32}^{\rho^{1_2}_{\rho^{2_2}_{\rho^{3_2}}}}} \epsilon_{\mu_3 \rho_6 \beta_3} \epsilon_{\nu_3 \rho_6 \beta_3} \delta^{(3)}(p_{32}+q_3) \delta^{(3)}(p_{42}+q_2) \right\} - (\mu+\nu). \quad (4.13)
\]

Set \( p_{11} = \eta \) and \( p_{32} = q \) in the first term, and \( p_{32} = -(k+q) \) in the second. Then contracting on \( \alpha_1, \beta_2 \), we get

\[
[2.06] = 2ig^5 \int \tilde{k}^{\mu_11} \tilde{k}^{\mu_21} e^{ik \times n} (\frac{1}{2} - \sigma_1) \frac{(k-q)^2 (q+k)^2}{(k-q)^2 (q+k)^2} \sin \left(\frac{2\pi x}{z}\right) \sin \left(\frac{2\pi y}{z}\right) \epsilon^{\alpha_1 \alpha_2 \alpha_3} \\
\epsilon^{\beta_1 \beta_2 \beta_3} \epsilon_{\alpha_3 \rho_5 \beta_1} \epsilon_{\mu_11 \rho_3 \alpha_1} \epsilon_{\mu_21 \rho_3 \beta_2} \left\{ \epsilon_{\mu_4 \rho_5 \beta_3} \epsilon_{\nu_4 \rho_5 \beta_3} \epsilon^{\delta^{(3)}}(p_{32}+q_3) \delta^{(3)}(p_{42}+q_2) \right\} - (\mu+\nu) \\
= -4g^5 \int \tilde{k}^{\mu_11} \tilde{k}^{\mu_21} e^{ik \times n} (\frac{1}{2} - \sigma_1) \frac{(k-q)^2 (q+k)^2}{(k-q)^2 (q+k)^2} \sin \left(\frac{2\pi x}{z}\right) \sin \left(\frac{2\pi y}{z}\right) \epsilon^{\alpha_1 \alpha_2 \alpha_3} \\
\epsilon_{\mu_21 \rho_3 \rho_5} \epsilon_{\mu_4 \rho_5 \beta_3} \epsilon_{\nu_4 \rho_5 \beta_3} \epsilon_{\nu_4 \rho_5 \beta_3} \delta^{(3)}(p_{32}+q_3) \delta^{(3)}(p_{42}+q_2) - (\mu+\nu) \\
= -4g^5 \int \tilde{k}^{\mu_11} \tilde{k}^{\mu_21} e^{ik \times n} (\frac{1}{2} - \sigma_1) \frac{(k-q)^2 (q+k)^2}{(k-q)^2 (q+k)^2} \sin \left(\frac{2\pi x}{z}\right) \sin \left(\frac{2\pi y}{z}\right) \epsilon^{\alpha_1 \alpha_2 \alpha_3} \\
\times \left\{ \epsilon_{\mu_21 \rho_3 \rho_5} \epsilon_{\mu_4 \rho_5 \beta_3} \epsilon_{\nu_4 \rho_5 \beta_3} \epsilon_{\mu_4 \rho_5 \beta_3} \delta^{(3)}(p_{32}+q_3) \delta^{(3)}(p_{42}+q_2) \\
+ \epsilon_{\mu_11 \rho_3 \rho_5} \epsilon_{\mu_21 \rho_3 \rho_5} \epsilon_{\mu_4 \rho_5 \beta_3} \epsilon_{\nu_4 \rho_5 \beta_3} - \epsilon_{\mu_3 \rho_4 \rho_5} \epsilon_{\mu_4 \rho_5 \beta_3} \right\} ; \\
(4.14)
\]

where we have used (A.15) and (A.16). Using the interchangeability of the \( \mu_{11} \) and \( \mu_{21} \) indices, we now repeatedly apply (A.14) to the first, third and fifth terms as follows:

\[
\begin{align*}
&\left\{ \epsilon_{\mu_21 \rho_3 \rho_5} \epsilon_{\mu_4 \rho_5 \beta_3} \epsilon_{\nu_4 \rho_5 \beta_3} \epsilon_{\mu_4 \rho_5 \beta_3} \delta^{(3)}(p_{32}+q_3) \delta^{(3)}(p_{42}+q_2) \\
&+ \epsilon_{\mu_11 \rho_3 \rho_5} \epsilon_{\mu_21 \rho_3 \rho_5} \epsilon_{\mu_4 \rho_5 \beta_3} \epsilon_{\nu_4 \rho_5 \beta_3} - \epsilon_{\mu_3 \rho_4 \rho_5} \epsilon_{\mu_4 \rho_5 \beta_3} \right\} \tilde{k}^{\mu_11} \tilde{k}^{\mu_21} \\
&= \left\{ \epsilon_{\mu_11 \rho_3 \rho_5} \epsilon_{\mu_21 \rho_3 \rho_5} + \epsilon_{\mu_11 \rho_3 \rho_5} \epsilon_{\mu_21 \rho_3 \rho_5} + \epsilon_{\mu_11 \rho_3 \rho_5} \epsilon_{\mu_21 \rho_3 \rho_5} + \epsilon_{\mu_11 \rho_3 \rho_5} \epsilon_{\mu_21 \rho_3 \rho_5} - \epsilon_{\mu_11 \rho_3 \rho_5} \epsilon_{\mu_21 \rho_3 \rho_5} \\
&- \epsilon_{\mu_11 \rho_3 \rho_5} \epsilon_{\mu_21 \rho_3 \rho_5} \right\} \tilde{k}^{\mu_11} \tilde{k}^{\mu_21} \\
&= \left\{ \epsilon_{\mu_21 \rho_3 \rho_5} \epsilon_{\mu_11 \rho_3 \rho_5} + \epsilon_{\mu_21 \rho_3 \rho_5} \epsilon_{\mu_11 \rho_3 \rho_5} + \epsilon_{\mu_21 \rho_3 \rho_5} \epsilon_{\mu_11 \rho_3 \rho_5} \right\} \tilde{k}^{\mu_11} \tilde{k}^{\mu_21} \\
&= \epsilon_{\mu_11 \rho_3 \rho_5} \epsilon_{\mu_11 \rho_3 \rho_5} \epsilon_{\mu_11 \rho_3 \rho_5} \epsilon_{\mu_11 \rho_3 \rho_5} \tilde{k}^{\mu_11} \tilde{k}^{\mu_21} .
\end{align*}
\]

(4.15)
Thus diagram [2.06] is given by

\[ [2.06] = \frac{(ig)^2}{2} \left( -2ig \right)^3 \int k^{\mu_1} k^{\mu_2} e^{ik \times \eta(\frac{1}{2} - \sigma_{11})} \frac{k^{\mu_1} k^{\mu_2} (q_1 + q_2) (q_3 + q_4) (k + q)^{26}}{k^2 (q_1 + q_2 + q_3 + q_4 + k + q)^2} \right. \]

\[ \times \left. \left( \delta_{\mu_1 \mu_2} - \delta_{\mu_1 \rho_2} \right) \left( \delta_{\mu_2 \rho_1} - \delta_{\mu_2 \rho_2} \right) \right) e^{i \gamma_1 \gamma_2 \gamma_3} e^{i \beta_3 \beta_4 \beta_5} e^{i \beta_6} e^{i \beta_7} e^{i \beta_8} \right. \]

\[ \left. \left( \delta_{\rho_1 \rho_2} - \delta_{\rho_1 \rho_4} \right) \left( \delta_{\rho_2 \rho_1} - \delta_{\rho_2 \rho_4} \right) \right) \right] \left[ \delta_{\mu_1 \mu_1} \rho_1 \gamma_2 \right] \left[ \delta_{\mu_2 \mu_2} \rho_1 \gamma_2 \right] \]

and so evaluating the surface terms in (4.12) with respect to the \( \sigma_{12} \) integration, and adding [2.06], we obtain

\[ [2.04] + [2.05] + [2.06] = \frac{(ig)^2}{2} \left( -2ig \right)^3 \int k^{\mu_1} k^{\mu_2} e^{ik \times \eta(\frac{1}{2} - \sigma_{11})} \frac{k^{\mu_1} k^{\mu_2} (q_1 + q_2) (q_3 + q_4) (k + q)^{26}}{k^2 (q_1 + q_2 + q_3 + q_4 + k + q)^2} \right. \]

\[ \times \left. \left( \delta_{\mu_1 \mu_2} - \delta_{\mu_1 \rho_2} \right) \left( \delta_{\mu_2 \rho_1} - \delta_{\mu_2 \rho_2} \right) \right) e^{i \gamma_1 \gamma_2 \gamma_3} e^{i \beta_3 \beta_4 \beta_5} e^{i \beta_6} e^{i \beta_7} e^{i \beta_8} \right. \]

\[ \left. \left( \delta_{\rho_1 \rho_2} - \delta_{\rho_1 \rho_4} \right) \left( \delta_{\rho_2 \rho_1} - \delta_{\rho_2 \rho_4} \right) \right) \right] \left[ \delta_{\mu_1 \mu_1} \rho_1 \gamma_2 \right] \left[ \delta_{\mu_2 \mu_2} \rho_1 \gamma_2 \right] \]

This does not obviously vanish, and it is not immediately clear what can be used to cancel this residual piece. We will now show that the contributions in diagram 6, which contain one-loop vertex corrections, precisely cancel this piece.

Evaluating diagram [2.07], we obtain

\[ [2.07] = 4g^5 \int k^{\mu_1} k^{\mu_2} e^{ik \times \eta(\frac{1}{2} - \sigma_{11})} \frac{k^{\mu_1} k^{\mu_2} (q_1 + q_2) (q_3 + q_4) (k + q)^{26}}{k^2 (q_1 + q_2 + q_3 + q_4 + k + q)^2} \right. \]

\[ \times \left. \left( \delta_{\mu_1 \mu_2} - \delta_{\mu_1 \rho_2} \right) \left( \delta_{\mu_2 \rho_1} - \delta_{\mu_2 \rho_2} \right) \right) e^{i \gamma_1 \gamma_2 \gamma_3} e^{i \beta_3 \beta_4 \beta_5} e^{i \beta_6} e^{i \beta_7} e^{i \beta_8} \right. \]

\[ \left. \left( \delta_{\rho_1 \rho_2} - \delta_{\rho_1 \rho_4} \right) \left( \delta_{\rho_2 \rho_1} - \delta_{\rho_2 \rho_4} \right) \right) \right] \left[ \delta_{\mu_1 \mu_1} \rho_1 \gamma_2 \right] \left[ \delta_{\mu_2 \mu_2} \rho_1 \gamma_2 \right] \]

Take \( p_{11} = \eta \), and \( s_3 = q \). Then contract on \( \alpha_2 \), \( \beta_2 \):

\[ [2.07] = 4g^5 \int k^{\mu_1} k^{\mu_2} e^{ik \times \eta(\frac{1}{2} - \sigma_{11})} \frac{k^{\mu_1} k^{\mu_2} (q_1 + q_2) (q_3 + q_4) (k + q)^{26}}{k^2 (q_1 + q_2 + q_3 + q_4 + k + q)^2} \right. \]

\[ \times \left. \left( \delta_{\mu_1 \mu_2} - \delta_{\mu_1 \rho_2} \right) \left( \delta_{\mu_2 \rho_1} - \delta_{\mu_2 \rho_2} \right) \right) e^{i \gamma_1 \gamma_2 \gamma_3} e^{i \beta_3 \beta_4 \beta_5} e^{i \beta_6} e^{i \beta_7} e^{i \beta_8} \right. \]

\[ \left. \left( \delta_{\rho_1 \rho_2} - \delta_{\rho_1 \rho_4} \right) \left( \delta_{\rho_2 \rho_1} - \delta_{\rho_2 \rho_4} \right) \right) \right] \left[ \delta_{\mu_1 \mu_1} \rho_1 \gamma_2 \right] \left[ \delta_{\mu_2 \mu_2} \rho_1 \gamma_2 \right] \]
\begin{align*}
&+ \varepsilon_{\mu_{21}\rho_{3}\rho_{5}\varepsilon_{\rho_{2}\rho_{3}\rho_{4}}k^1_{\rho}\varepsilon_{\mu_{1}\rho_{1}} - \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}\rho_{11}}k^1_{\rho}\varepsilon_{\mu_{1}\rho_{1}}p_4} \\
&= 4g^5 \int \tilde{k}^{\mu_{11}} \tilde{k}^{\mu_{21}} e^{ik \times \eta \left( \frac{1}{2} - \sigma_{11} \right)} \frac{\varepsilon_{\rho_{2}\rho_{3}\rho_{4}}p^2_{(k-q)p^5_{(k+q)p}}}{k \times \eta \left( \frac{1}{2} - \sigma_{11} \right)} \sin \left( \frac{k \times q}{2} \right) \sin \left( \frac{k \times q}{2} \right) \sin \left( \frac{k \times q}{2} \right) \sin \left( \frac{k \times q}{2} \right) \\
&\times \left[ -\varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}}\varepsilon_{\mu_{11}\rho_{1}} - \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}\rho_{11}}k^{\rho_{1}}_{\rho}\varepsilon_{\mu_{1}\rho_{1}}p_{6} - \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}\rho_{11}}k^{\rho_{1}}_{\rho}\varepsilon_{\mu_{1}\rho_{1}}p_{4} \\
&+ \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}}\varepsilon_{\mu_{11}\rho_{1}} - \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}\rho_{11}}k^{\rho_{1}}_{\rho}\varepsilon_{\mu_{1}\rho_{1}}p_{6} - \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}\rho_{11}}k^{\rho_{1}}_{\rho}\varepsilon_{\mu_{1}\rho_{1}}p_{4} \right],
\end{align*}
where we have used (A.17) on the terms containing $\varepsilon_{\rho_{1}\rho_{1}}$. Collecting the terms proportional to $k^2$, and applying (A.14) we have
\begin{align*}
&\left[ -\varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}}\varepsilon_{\mu_{11}\rho_{1}} - \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}\rho_{11}}k^{\rho_{1}}_{\rho}\varepsilon_{\mu_{1}\rho_{1}}p_{6} - \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}\rho_{11}}k^{\rho_{1}}_{\rho}\varepsilon_{\mu_{1}\rho_{1}}p_{4} \right] \\
&\times \left( \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}}\varepsilon_{\mu_{11}\rho_{1}} - \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}\rho_{11}}k^{\rho_{1}}_{\rho}\varepsilon_{\mu_{1}\rho_{1}}p_{6} - \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}\rho_{11}}k^{\rho_{1}}_{\rho}\varepsilon_{\mu_{1}\rho_{1}}p_{4} \right],
\end{align*}
\begin{align*}
&\text{Inserting this back into (4.19) we obtain finally:} \\
&\left[ \frac{2.07}{4g^5} \int \tilde{k}^{\mu_{11}} \tilde{k}^{\mu_{21}} e^{ik \times \eta \left( \frac{1}{2} - \sigma_{11} \right)} \frac{\varepsilon_{\rho_{2}\rho_{3}\rho_{4}}p^2_{(k-q)p^5_{(k+q)p}}}{k \times \eta \left( \frac{1}{2} - \sigma_{11} \right)} \sin \left( \frac{k \times q}{2} \right) \sin \left( \frac{k \times q}{2} \right) \sin \left( \frac{k \times q}{2} \right) \sin \left( \frac{k \times q}{2} \right) \\
&\times \left[ \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}}\varepsilon_{\mu_{11}\rho_{1}} - \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}\rho_{11}}k^{\rho_{1}}_{\rho}\varepsilon_{\mu_{1}\rho_{1}}p_{6} - \varepsilon_{\mu_{21}\rho_{3}\rho_{5}}\varepsilon_{\rho_{2}\rho_{6}\rho_{11}}k^{\rho_{1}}_{\rho}\varepsilon_{\mu_{1}\rho_{1}}p_{4} \right].
\end{align*}

To this contribution, we must add the ghost loop diagram from figure 6. Remembering that once we fix the contractions among the gauge fields, there are two sets of contractions among the ghost fields corresponding to the two directions of ghost number flow, we have:
\begin{align*}
&\left[ \frac{2.08}{-\frac{(i g/2)^3}{2}} \right] \int \tilde{k}^{\mu_{11}} \tilde{k}^{\mu_{21}} \delta^{(3)}(p_{11} + p_{21} - k) \delta^{(3)}(p_{42} - k)e^{-i\rho_{11}^{\mu_{11}}}e^{-i\rho_{11}^{\mu_{21}}}e^{-i\rho_{11}^{\mu_{11}}}e^{-i\rho_{11}^{\mu_{21}}} \\
&\times \left( \sin \frac{k \times q}{2} \right)^{\nu} \left( \sin \frac{k \times q}{2} \right)^{\nu} \left( \sin \frac{k \times q}{2} \right)^{\nu} \left( \sin \frac{k \times q}{2} \right)^{\nu} \left( \sin \frac{k \times q}{2} \right)^{\nu} \left( \sin \frac{k \times q}{2} \right)^{\nu} \\
&\times \frac{(k \times q)^{\nu}}{2} \left( \sin \frac{k \times q}{2} \right)^{\nu} \left( \sin \frac{k \times q}{2} \right)^{\nu} \left( \sin \frac{k \times q}{2} \right)^{\nu} \left( \sin \frac{k \times q}{2} \right)^{\nu} \left( \sin \frac{k \times q}{2} \right)^{\nu} \left( \sin \frac{k \times q}{2} \right)^{\nu} \left( \sin \frac{k \times q}{2} \right)^{\nu} \\
&\times \delta^{(3)}(p_{21} + p_{42}) \delta^{(3)}(p_{11} + p_{21} - k) \left\{ \delta^{(3)}(q_{31} + q_{31}) \delta^{(3)}(q_{31} + q_{31}) \delta^{(3)}(q_{31} + q_{31}) + \delta^{(3)}(q_{31} + q_{31}) \delta^{(3)}(q_{31} + q_{31}) \delta^{(3)}(q_{31} + q_{31}) \right\},
\end{align*}
where we have included a leading minus sign for the fermion loop. As usual we take $p_{11} = q$, and now we take $s_3 = q$ in the first term, and $s_1 = q$ in the second, from which we immediately get:
\begin{align*}
&\left[ \frac{2.08}{4g^5} \int \tilde{k}^{\mu_{11}} \tilde{k}^{\mu_{21}} e^{ik \times \eta \left( \frac{1}{2} - \sigma_{11} \right)} \frac{\varepsilon_{\rho_{2}\rho_{3}\rho_{4}}p^2_{(k-q)p^5_{(k+q)p}}}{k \times \eta \left( \frac{1}{2} - \sigma_{11} \right)} \sin \left( \frac{k \times q}{2} \right) \sin \left( \frac{k \times q}{2} \right) \sin \left( \frac{k \times q}{2} \right) \sin \left( \frac{k \times q}{2} \right) \\
&\times \left[ \varepsilon_{\mu_{11}\rho_{3}\rho_{4}}\varepsilon_{\rho_{2}\rho_{3}\rho_{4}}k^{\rho_{1}}_{\rho}\varepsilon_{\mu_{1}\rho_{1}}p_{6} + \varepsilon_{\mu_{11}\rho_{3}\rho_{4}}\varepsilon_{\rho_{2}\rho_{6}\rho_{11}}k^{\rho_{1}}_{\rho}\varepsilon_{\mu_{1}\rho_{1}}p_{4} \right].
\end{align*}
Figure 7: Diagrams [1.01] and [1.02]. Compare with [2.01] and [2.02].

The ghost graph [2.08] cancels the last two terms in [4.21], while the first term in [4.21] is exactly the negative of the residual piece from diagrams [2.04]-[2.06]. In summary we have found the cancellation:

$$[2.04] + [2.05] + [2.06] + [2.07] + [2.08] = 0. \quad (4.24)$$

Combining this with the cancellation amongst [2.01], [2.02] and [2.03], we conclude that the sum of contributions from $O(g^2)$ terms in $W(k)$ to the correlator $\langle W(k)O_{\mu
u}(k') \rangle$ at $O(g^5)$ vanish.

## 5 Contributions from $O(g)$ terms in $W(k)$

We now turn to the most difficult case, where we have up to four sources on $O_{\mu
u}(k')$. All of the contributions in this section have one gauge field source from $W(k)$, and it will require all of the graphs to produce a final cancellation. We will further absorb the common factor $\delta^{(3)}(p_{11} - k)$ associated with this source, into the integration symbol.

First consider the contributions from figure 7. The first is evaluated as

$$[1.01] = (ig)^3(-2ig)^2 \int \bar{k}^{\mu_11} k^{\mu_12} k^{\mu_22} \delta^{(3)}(p_{12} + p_{22} + p_{42} - k') e^{-i[p_{12} \cdot \xi_{12} + p_{22} \cdot \xi_{22}]} e^{-\frac{i}{2}[p_{22} \cdot (p_{42} - k') - p_{42} \cdot k']}$$

$$\cdot \sin \left( \frac{2\pi q \cdot k}{2\pi} \right) \sin \left( \frac{\pi r \cdot k}{2\pi} \right) \epsilon^{\alpha_1 \alpha_2 \alpha_3} \epsilon^{\beta_1 \beta_2 \beta_3} \frac{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\beta_4}}{\frac{1}{p_{11}^2} \frac{1}{p_{22}^2} \frac{1}{p_{42}^2} \frac{1}{p_{32}^2}} (-i p_{42}) \mu \epsilon_{\mu_1 \mu_11} \epsilon_{\nu_\rho \sigma_\sigma_2} \epsilon_{\alpha_3 \beta_4} \epsilon_{\mu_2 \rho_3 \beta_2}$$

$$\cdot \epsilon_{\mu_2 \rho_5 \beta_3} \delta^{(3)}(p_{11} + q_1) \delta^{(3)}(p_{42} + q_2) \delta^{(3)}(q_3 + r_1) \delta^{(3)}(p_{12} + r_2) \delta^{(3)}(p_{42} + r_3) - (\mu \leftrightarrow \nu). \quad (5.1)$$

Setting $p_{12} = -\eta$, and $p_{22} = q + \eta$, and contracting on $\alpha_1, \beta_3$ we obtain

$$[1.01] = 4g^5 \int \bar{k}^{\mu_11} k^{\mu_12} k^{\mu_22} e^{-i[k \cdot \eta \sigma_1 - k \cdot (q + \eta) \sigma_2]} \epsilon^{\nu \rho \sigma} e^{-\frac{i}{2}[\eta \cdot q - k \cdot q]} \sin \left( \frac{2\pi q \cdot k}{2\pi} \right) \sin \left( \frac{\pi r \cdot k}{2\pi} \right) \frac{1}{k \cdot \eta \cdot \eta + (\eta \cdot q) \cdot (\eta \cdot q) \cdot (q \cdot k) \cdot (q \cdot k) \cdot (q \cdot q) \cdot (\eta \cdot \eta)}$$

$$\cdot \epsilon_{\mu_2 \rho_5} \epsilon_{\mu_2 \rho_3} \epsilon_{\rho_3 \rho_5} (k + q) [\mu \epsilon_{\nu \rho}] \rho_{\mu_11}. \quad (5.2)$$

The second diagram in figure 7 represents two sets of contractions and is evaluated as

$$[1.02] = -(ig)^4(-2ig) \int \bar{k}^{\mu_11} k^{\mu_12} k^{\mu_22} \delta^{(3)}(p_{12} + p_{22} + p_{32} + p_{42} - k') e^{-i[p_{12} \cdot \xi_{12} + p_{22} \cdot \xi_{22}]} e^{-\frac{i}{2}[p_{22} \cdot (p_{32} + p_{42} - k') + p_{32} \cdot p_{42} - k'] \frac{p_{11}^{\alpha_1} p_{32}^{\alpha_3} p_{42}^{\alpha_4}}{p_{11}^{\rho_3} p_{22}^{\rho_4} p_{32}^{\rho_5} p_{42}^{\rho_6}} \epsilon_{\mu_3 \rho_3} \epsilon_{\mu_2 \rho_5} \epsilon_{\rho_5 \beta_3}$$

$$\cdot \epsilon_{\mu_2 \rho_3 \beta_2} \epsilon_{\mu_2 \rho_5 \beta_3} \delta^{(3)}(p_{12} + r_2) \delta^{(3)}(p_{42} + r_3) \frac{p_4^{\beta_4}}{p_3^{\beta_4}} + \epsilon_{\mu_1 \mu_11} [\nu \epsilon_{\mu_1} \epsilon_{\mu_1 \rho_4} \beta_4 \delta^{(3)}(p_{11} + p_{42}) \delta^{(3)}(p_{22} + r_1) \frac{p_{42}^{\beta_4}}{p_{32}^{\beta_4}}] . \quad (5.3)$$
Again take $p_{12} = -\eta$, $p_{22} = q + \eta$ and in the first term take $p_{42} = -q$, while in the second take $p_{32} = -q$:

$$\begin{align*}
[1.02] &= -2ig^5 \int \tilde{k}^{\mu_11} \tilde{k}^{\mu_12} \tilde{k}^{\mu_{22}} e^{-i[k \times \eta \sigma_{12} - k \times (q + \eta) \sigma_{22}]} e^{\frac{1}{2} [\eta \times q - k \times q]} \sin \left( \frac{\mu_{\nu} \rho_{\mu_1} \rho_{\mu_2} \rho_{\nu} (q + \eta)^{\rho_\delta}}{k^2 \eta^2 q^2 (q + \eta)^2} \right) \\
&= 4g^5 \int \tilde{k}^{\mu_11} \tilde{k}^{\mu_12} \tilde{k}^{\mu_{22}} e^{-i[k \times \eta \sigma_{12} - k \times (q + \eta) \sigma_{22}]} e^{\frac{1}{2} [\eta \times q - k \times q]} \sin \left( \frac{\mu_{\nu} \rho_{\mu_1} \rho_{\mu_2} \rho_{\nu} (q + \eta)^{\rho_\delta}}{k^2 \eta^2 q^2 (q + \eta)^2} \right) \\
&\quad \times \varepsilon_{\mu_11 1 \mu_2 2} e^{\frac{1}{2} [\eta \times q - k \times q]} \sin \left( \frac{\mu_{\nu} \rho_{\mu_1} \rho_{\mu_2} \rho_{\nu} (q + \eta)^{\rho_\delta}}{k^2 \eta^2 q^2 (q + \eta)^2} \right) \\
&\quad \times \varepsilon_{\mu_{32} \rho_{3} \rho_{4}} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \\
&\quad \times \left( \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \right) & (\mu + \nu). \tag{5.4}
\end{align*}$$

where we have used \[ A.16 \] in the last step. Combining \[ 1.01 \] and \[ 1.02 \] using \[ A.17 \], we obtain

$$\begin{align*}
[1.01] + [1.02] &= 4g^5 \int \tilde{k}^{\mu_11} \tilde{k}^{\mu_12} \tilde{k}^{\mu_{22}} e^{-i[k \times \eta \sigma_{12} - k \times (q + \eta) \sigma_{22}]} e^{\frac{1}{2} [\eta \times q - k \times q]} \sin \left( \frac{\mu_{\nu} \rho_{\mu_1} \rho_{\mu_2} \rho_{\nu} (q + \eta)^{\rho_\delta}}{k^2 \eta^2 q^2 (q + \eta)^2} \right) \\
&\quad \times \varepsilon_{\mu_{32} \rho_{3} \rho_{4}} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \\
&\quad \times \left( \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \right) & (\mu + \nu). \tag{5.5}
\end{align*}$$

Now consider the diagrams in figure 8, which, like the similar diagrams in the previous section, correspond to the other two channels with respect to \[ 1.01 \]:

$$\begin{align*}
[1.03] &= (ig)^3 (-2ig)^2 \int \tilde{k}^{\mu_11} \tilde{k}^{\mu_12} \tilde{k}^{\mu_{22}} e^{\frac{3}{4} (p_{12} \times p_{22} - k') e^{-i[p_{12} \cdot \xi_{12} + p_{22} \cdot \xi_{22}]} e^{-\frac{1}{2} [p_{22} \times (p_{42} - k') - p_{42} \times k']} \\
&\quad \times \varepsilon_{\nu \rho_{3} \rho_{4}} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \\
&\quad \times \left( \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \varepsilon_{\mu_{12} \rho_1 \rho_2} \right) & (\mu + \nu). \tag{5.6}
\end{align*}$$

Set $p_{42} = q + \eta$, and $p_{22} = -\eta$ in the first term, $p_{12} = -\eta$ in the second. Contracting on $\alpha_1, \beta_1$ yields

$$\begin{align*}
[1.03] &= 4g^5 \int \tilde{k}^{\mu_11} \tilde{k}^{\mu_12} \tilde{k}^{\mu_{22}} e^{\frac{3}{4} (p_{12} \times p_{22} - k') e^{-i[p_{12} \cdot \xi_{12} + p_{22} \cdot \xi_{22}]} e^{-\frac{1}{2} [p_{22} \times (p_{42} - k') - p_{42} \times k']} \\
&\quad \times \varepsilon_{\mu_11 1 \mu_2 2} e^{\frac{1}{2} [\eta \times q + k \times q]} \sin \left( \frac{\mu_{\nu} \rho_{\mu_1} \rho_{\mu_2} \rho_{\nu} (q + \eta)^{\rho_\delta}}{k^2 \eta^2 q^2 (q + \eta)^2} \right) \\
&\quad \times \left\{ e^{-i[k \times \eta \sigma_{12} + k \times q \sigma_{22}]} e^{\frac{1}{2} [\eta \times q + k \times q]} + e^{-i[k \times \eta \sigma_{12} + k \times q \sigma_{22}]} e^{\frac{1}{2} [-\eta \times q + k \times q + 2k \times \eta]} \right\}. \tag{5.7}
\end{align*}$$

As usual, we pair these together with the diagrams in figure 9 which we denote by \[ 1.04a \] and \[ 1.04b \]. Writing out the four sets of contractions they represent, we get

$$\begin{align*}
[1.04] &= -(ig)^4 (-2ig)^2 \int \tilde{k}^{\mu_11} \tilde{k}^{\mu_12} \tilde{k}^{\mu_{22}} e^{\frac{3}{4} (p_{12} \times p_{22} + p_{32} + p_{42} - k') e^{-i[p_{12} \cdot \xi_{12} + p_{22} \cdot \xi_{22}]} e^{-\frac{1}{2} [p_{22} \times (p_{42} - k') - p_{42} \times k']} \\
&\quad \times \varepsilon_{\mu_11 1 \mu_2 2} e^{\frac{1}{2} [\eta \times q + k \times q]} \sin \left( \frac{\mu_{\nu} \rho_{\mu_1} \rho_{\mu_2} \rho_{\nu} (q + \eta)^{\rho_\delta}}{k^2 \eta^2 q^2 (q + \eta)^2} \right) \\
&\quad \times \left\{ e^{-i[k \times \eta \sigma_{12} + k \times q \sigma_{22}]} e^{\frac{1}{2} [\eta \times q + k \times q]} + e^{-i[k \times \eta \sigma_{12} + k \times q \sigma_{22}]} e^{\frac{1}{2} [-\eta \times q + k \times q + 2k \times \eta]} \right\}. \tag{5.8}
\end{align*}$$
Figure 9: Diagrams [1.04a] and [1.04b]. Compare with [2.05a] and [2.05b].

Figure 10: Diagrams [1.05] and [1.06]. Compare with [2.03] and [2.06].

\[ e^{-\frac{g}{4}\left[p_{22} \times (p_{32} + p_{42} - k') + p_{32} \times (p_{42} - k') - p_{42} \times k'\right] \sin \left(\frac{v_1}{2}\right) e^{\delta_1} \delta_3 \left(\frac{p_{11} + r_1}{P_{11}}\right) \varepsilon_{\mu_{11} \rho_1 \beta_1} \delta(\mu_1 + r_1) \]

\[ \times \left\{ \frac{\rho_3 \rho_6}{P_{12}^2} \varepsilon_{\mu_{22} \rho_6} \delta_3 \left(p_{22} + r_2\right) \left[ \frac{\rho_4}{P_{12}} \varepsilon_{\mu_{12} \rho_4} \varepsilon_{\rho_4} \right] \delta_3 \left(p_{12} + p_{32}\right) \right\} + \frac{\rho_4}{P_{12}} \varepsilon_{\mu_{12} \rho_4} \varepsilon_{\rho_4} \delta_3 \left(p_{22} + p_{32}\right) \]

\[ \times \delta_3 \left(p_{42} + r_3\right) + \frac{\rho_4}{P_{12}} \varepsilon_{\mu_{22} \rho_3} \varepsilon_{\rho_3} \delta_3 \left(p_{42} + p_{42}\right) \delta_3 \left(p_{32} + r_1\right) \left\} \right\} - (\mu + \nu). \]  

(5.8)

Set \( p_{12} = -\eta, p_{22} = -(k + q) \) in the first two terms and \( p_{12} = -(k + q), p_{22} = -\eta \) in the second two. Then contract on \( \beta_1 \), and use the symmetry under \( \mu_{12} \leftrightarrow \mu_{22} \) to obtain:

\[ [1.04] = -2ig^5 \int k_{\mu_{11}} k_{\mu_{12}} \tilde{k}_{\mu_{22}} \sin \left(\frac{k \times q}{2}\right) \sin \left(\frac{k \times q}{2}\right) \delta \left(p_{12} + p_{32}\right) \delta \left(p_{12} + p_{42}\right) \]

\[ \times \left[ \left[ e^{2i[k \times q + 2k \times \eta + k \times q]} \varepsilon_{\mu_{12} \rho_4} \varepsilon_{\rho_4} + e^{2i[k \times q + 2k \times q]} \varepsilon_{\mu_{12} \rho_4} \varepsilon_{\rho_4} \right] \varepsilon_{\mu_{12} \rho_4} \varepsilon_{\rho_4} + e^{2i[k \times q + 2k \times \eta + k \times q]} \varepsilon_{\mu_{12} \rho_4} \varepsilon_{\rho_4} \right] \]

\[ = 4g^5 \int k_{\mu_{11}} k_{\mu_{12}} \tilde{k}_{\mu_{22}} \sin \left(\frac{k \times q}{2}\right) \sin \left(\frac{k \times q}{2}\right) \delta \left(p_{12} + p_{32}\right) \delta \left(p_{12} + p_{42}\right) \]

\[ \times \left\{ e^{2i[k \times q + 2k \times \eta + k \times q]} e^{2i[k \times q + 2k \times \eta]} + e^{2i[k \times q + 2k \times q]} e^{2i[k \times q + 2k \times \eta]} \right\}, \]

(5.9)

where we have used (A.16) in the final step.

Summing the contributions in figures 8 and 9, and invoking (A.17) as usual we obtain

\[ [1.03] + [1.04] = 4g^5 \int k_{\mu_{11}} k_{\mu_{12}} \tilde{k}_{\mu_{22}} \sin \left(\frac{k \times q}{2}\right) \sin \left(\frac{k \times q}{2}\right) \delta \left(p_{12} + p_{32}\right) \delta \left(p_{12} + p_{42}\right) \]

\[ \times \left\{ e^{2i[k \times q + 2k \times \eta + k \times q]} e^{2i[k \times q + 2k \times \eta]} + e^{2i[k \times q + 2k \times q]} e^{2i[k \times q + 2k \times \eta]} \right\}, \]

(5.10)

Analogous to [2.03] and [2.06], we now have the diagrams given in figure 10, and which we denote by [1.05] and [1.06] respectively. However, in the last section [2.03] precisely cancelled
[2.01] and [2.02], in a calculation very similar to that presented in [15]. On the other hand we will find that [1.05] by itself does not cancel the equivalent graphs here, [1.01] and [1.02], but serves to ‘correct’ a noncommutative phase in other diagrams.

Evaluating [1.05], we find that

\[
[1.05] = -(ig)^3 (-2ig)^2 \int \tilde{k}^{B12} \tilde{k}^{B122} e^{i p_{12} x_{12} e^{2} e^{-1/2 [p_{32} \times (p_{42} - k') - p_{42} \times k']}} e^{-i p_{12} \xi_{12} e^{2} e^{-1/2 [p_{32} \times (p_{42} - k') - p_{42} \times k']} e^{-1/2 [p_{32} \times (p_{42} - k') - p_{42} \times k']}}
\]

\[
\sin \left( \frac{1}{2} x_{32} \right) \sin \left( \frac{1}{2} x_{42} \right) e^{-1/2 \left( \eta \times q - k \times q \right) \sin \left( \frac{1}{2} x_{42} \right) \sin \left( \frac{1}{2} x_{42} \right) e^{-1/2 \left( \eta \times q - k \times q \right)}}
\]

\[
\alpha \beta \gamma \delta \epsilon \zeta \eta \theta \varphi \chi \psi \omega
\]

\[
\delta^{(3)}(p_{11} + q_{1}) \delta^{(3)}(p_{12} + q_{2}) \delta^{(3)}(p_{32} + r_{2}) \delta^{(3)}(p_{42} + r_{3}) \delta^{(3)}(q_{3} + r_{1}) - \left( \mu \leftrightarrow \nu \right).
\]

(5.11)

So as not to introduce a propagator carrying the momentum \( k - \eta \), set \( p_{12} = -(k + q) \), \( p_{42} = -\eta \), and contract on \( \alpha_{1}, \beta_{1} \):

\[
[1.05] = 4ig^{5} \int \tilde{k}^{B11} \tilde{k}^{B12} e^{-i k \times \eta \sigma_{12} e^{-1/2 \left( \eta \times q - k \times q \right) \sin \left( \frac{1}{2} x_{42} \right) \sin \left( \frac{1}{2} x_{42} \right) e^{-1/2 \left( \eta \times q - k \times q \right)}}
\]

\[
\times \left\{ \varepsilon_{\mu_{1} \rho_{1} \rho_{3}} \varepsilon_{\rho_{3} \rho_{4} \mu_{1} \varepsilon_{\mu_{1} \rho_{1} \rho_{3}}} + \varepsilon_{\mu_{12} \rho_{1} \rho_{3}} \varepsilon_{\rho_{3} \rho_{4} \mu_{1} \varepsilon_{\mu_{1} \rho_{1} \rho_{3}}} \right\}
\]

\[
\times \varepsilon_{\mu_{1} \rho_{1} \rho_{3}} \left\{ \varepsilon_{\mu_{12} \rho_{1} \rho_{3}} \varepsilon_{\rho_{3} \rho_{4} \mu_{1} \varepsilon_{\mu_{1} \rho_{1} \rho_{3}}} \right\}
\]

\[
\times \varepsilon_{\mu_{11} \rho_{1} \rho_{3}} \varepsilon_{\mu_{12} \rho_{1} \rho_{3}} \varepsilon_{\mu_{13} \rho_{1} \rho_{3}},
\]

(5.12)

where we have invoked (A.15) twice to obtain the second line and cancelled the resulting cross term proportional to \( \varepsilon_{\mu_{1} \mu_{12}} \), and where we have changed variables \( \eta \to -(q + \eta) \) in the second term on the second line, which effectively interchanges the \( \rho_{3} \) and \( \rho_{5} \) indices.

Now consider [1.06], which is similar to [2.06] and represents two sets of contractions with one natural diagrammatic representation. Evaluating it we obtain

\[
[1.06] = -(ig)^3 (-2ig)^2 \int \tilde{k}^{B11} \tilde{k}^{B122} e^{i p_{12} x_{12} e^{2} e^{-1/2 [p_{32} \times (p_{42} - k') - p_{42} \times k']}} e^{-i p_{12} \xi_{12} e^{2} e^{-1/2 [p_{32} \times (p_{42} - k') - p_{42} \times k']} e^{-1/2 [p_{32} \times (p_{42} - k') - p_{42} \times k']}}
\]

\[
\sin \left( \frac{1}{2} x_{32} \right) \sin \left( \frac{1}{2} x_{42} \right) e^{-1/2 \left( \eta \times q - k \times q \right) \sin \left( \frac{1}{2} x_{42} \right) \sin \left( \frac{1}{2} x_{42} \right) e^{-1/2 \left( \eta \times q - k \times q \right)}}
\]

\[
\left\{ \varepsilon^{\rho_{5} \rho_{5}} \varepsilon_{\mu_{p_{4}} \alpha_{2} \rho_{5}} \varepsilon_{\rho_{5} \rho_{3} \rho_{5}} \delta_{(3)}(p_{12} + q_{2}) \delta_{(3)}(p_{32} + r_{2}) \delta_{(3)}(p_{42} + r_{3}) \delta_{(3)}(q_{3} + r_{1}) - \left( \mu \leftrightarrow \nu \right) \right\}
\]

Set \( p_{12} = -\eta \), and \( p_{32} = -(k + q) \) in the first term and \( p_{42} = -(k + q) \) in the second. Contracting on \( \alpha_{1}, \beta_{2} \) we get

\[
[1.06] = 4ig^{5} \int \tilde{k}^{B11} \tilde{k}^{B12} e^{-i k \times \eta \sigma_{12} e^{2} e^{-1/2 \left( \eta \times q - k \times q \right) \sin \left( \frac{1}{2} x_{42} \right) \sin \left( \frac{1}{2} x_{42} \right) e^{-1/2 \left( \eta \times q - k \times q \right)}}
\]

\[
\times \left\{ \varepsilon^{\rho_{5} \rho_{5}} \varepsilon_{\mu_{p_{4}} \alpha_{2} \rho_{5}} \varepsilon_{\rho_{5} \rho_{3} \rho_{5}} \delta_{(3)}(p_{12} + q_{2}) \delta_{(3)}(p_{32} + r_{2}) \delta_{(3)}(p_{42} + r_{3}) \delta_{(3)}(q_{3} + r_{1}) - \left( \mu \leftrightarrow \nu \right) \right\}
\]

\[
(\delta^{(3)}(p_{12} + q_{2}) \delta^{(3)}(p_{32} + r_{2}) \delta^{(3)}(p_{42} + r_{3}) \delta^{(3)}(q_{3} + r_{1}) - \left( \mu \leftrightarrow \nu \right)).
\]

(5.13)
\[ \begin{align*}
8g^5 \int k_{\mu_1 - 1} k_{\mu_2} e^{ik \times \eta (\frac{1}{2} - \sigma_1)} \sin (\frac{\pi x_4}{\sqrt{2}}) & \sin (\frac{k \times q}{2}) \sin (\frac{k \times q - k \times q}{2}) e^{ik_1 \eta x^0 \eta x^0 (q + \eta) \mu_5 (k + q) \eta^6} \\
\left[ \varepsilon_{\mu_1 \rho_6} \varepsilon_{\mu_2 \rho_3} \varepsilon_{\rho_1 \rho_4 \mu_2} - \varepsilon_{\mu_1 \rho_6} \varepsilon_{\mu_2 \rho_3} \varepsilon_{\rho_1 \rho_4 \mu_2} - \varepsilon_{\mu_1 \rho_6} \varepsilon_{\mu_2 \rho_3} \varepsilon_{\rho_1 \rho_4 \mu_2} \right] \\
\end{align*} \]

Examining the graphs we have computed so far in (5.5), (5.10), (5.12) and (5.14), we find ourselves in a similar situation as that after having computed [2,06]. However, while the tensor structures are now in the same form across these contributions, the noncommutative phases are not manifestly the same, as they were in the equivalent point in the previous section, and a little more work is required. Denote the sum of these graphs, [1.01]-[1.06], by $S$. First examine terms that are proportional to $\varepsilon_{\mu \nu \rho \delta}$, which occur in (5.5), and (5.14):

\[ [1.01] + [1.02] = 4g^5 \int k_{\mu_1 - 1} k_{\mu_2} e^{-i(k \times \eta \sigma_1 - k \times (q + \eta) \sigma_2)} e^{i\eta x \eta x} \frac{[\eta x \eta x]}{2} (k \times q) \varepsilon_{\mu_2 \rho_1 \mu_4} \varepsilon_{\mu_1 \rho_3 \rho_5} \varepsilon_{\mu \nu \rho \delta} \sin (\frac{\pi x_4}{\sqrt{2}}) \sin (\frac{k \times q}{2}) \sin (\frac{k \times q - k \times q}{2}) e^{ik_1 \eta x^0 \eta x^0 (q + \eta) \mu_5 (k + q) \eta^6} \\
\]

\[ [1.06] \supset 8g^5 \int k_{\mu_1 - 1} k_{\mu_2} e^{ik \times \eta (\frac{1}{2} - \sigma_1)} \sin (\frac{\pi x_4}{\sqrt{2}}) \sin (\frac{k \times q}{2}) \sin (\frac{k \times q - k \times q}{2}) \varepsilon_{\mu_2 \rho_1 \mu_4} \varepsilon_{\mu_1 \rho_3 \rho_5} \varepsilon_{\mu \nu \rho \delta} \sin (\frac{\pi x_4}{\sqrt{2}}) \sin (\frac{k \times q}{2}) \sin (\frac{k \times q - k \times q}{2}) e^{ik_1 \eta x^0 \eta x^0 (q + \eta) \mu_5 (k + q) \eta^6} \\
\]

To proceed, perform the change of variables $\eta \rightarrow -(q + \eta)$ in each of these terms, and then average the original form and the new forms. Then we may write:

\[ [1.01] + [1.02] = 2g^5 \int k_{\mu_1 - 1} k_{\mu_2} \sin (\frac{\pi x_4}{\sqrt{2}}) \sin (\frac{k \times q}{2}) e^{ik_1 \eta x^0 \eta x^0 (q + \eta) \mu_5 (k + q) \eta^6} \varepsilon_{\mu_2 \rho_1 \mu_4} \varepsilon_{\mu_1 \rho_3 \rho_5} \varepsilon_{\mu \nu \rho \delta} (k \times q) \left\{ e^{i(k \times \eta \sigma_1 - k \times (q + \eta) \sigma_2)} e^{\frac{1}{2} \eta x \eta x} + e^{i(k \times (q + \eta) \sigma_1 - k \times \eta \sigma_2)} e^{\frac{1}{2} \eta x \eta x} \right\}, \]

while the term from [1.06] can be written as:

\[ [1.06] \supset 4g^5 \int k_{\mu_1 - 1} k_{\mu_2} \sin (\frac{\pi x_4}{\sqrt{2}}) \sin (\frac{k \times q}{2}) e^{ik_1 \eta x^0 \eta x^0 (q + \eta) \mu_5 (k + q) \eta^6} \varepsilon_{\mu_2 \rho_1 \mu_4} \varepsilon_{\mu_1 \rho_3 \rho_5} \varepsilon_{\mu \nu \rho \delta} \\
\left\{ e^{ik \times \eta (\frac{1}{2} - \sigma_1)} \sin (\frac{x_4 \eta x^0}{\sqrt{2}}) + e^{ik \times (q + \eta) (\sigma_1 - \frac{1}{2})} \sin (\frac{\eta x \eta x^0}{\sqrt{2}}) \right\}. \]

By performing the integrals over $p_{12}$ and $p_{22}$, and doing some trigonometry, we can show that these are the negatives of each other and so cancel. Denoting $\eta \times q = A$, $k \times q = B$ and $k \times \eta = C$ we obtain:

\[ 2k \times q \left\{ e^{-i(k \times \eta \sigma_1 - k \times (q + \eta) \sigma_2)} e^{\frac{1}{2} \eta x \eta x} + e^{i(k \times (q + \eta) \sigma_1 - k \times \eta \sigma_2)} e^{-\frac{1}{2} \eta x \eta x} \right\} \]
The terms in \( S \) proportional to \( \varepsilon_{\mu
u\rho_5} \), which occur in (5.10), (5.12), and (5.14):

\[
[1.03] + [1.04] = 4g^5 \int \tilde{k}_{\mu_1 \mu_2} \tilde{k}_{\mu_3 \mu_4} \sin \left( \frac{k \times q}{2} \right) \sin \left( \frac{\mu_1 \rho_3 \rho_4 (q + \eta) \rho_5 (k + q) \rho_6}{k \times (q + \eta)} \right) \varepsilon_{\mu_1 \rho_3 \rho_4} \varepsilon_{\mu_2 \rho_3 \rho_4} \varepsilon_{\mu_5 \rho_5} \varepsilon_{\mu_6 \rho_6} \varepsilon_{\mu_7 \rho_7} \varepsilon_{\mu_8 \rho_8} \varepsilon_{\mu_9 \rho_9},
\]

\[
[1.05] = -8g^5 \int \tilde{k}_{\mu_1 \mu_2} \tilde{k}_{\mu_3 \mu_4} \sin \left( \frac{k \times q}{2} \right) \left[ \sin \left( \frac{\mu_1 \rho_3 \rho_4 (q + \eta) \rho_5 (k + q) \rho_6}{k \times (q + \eta)} \right) \right] \varepsilon_{\mu_1 \rho_3 \rho_4} \varepsilon_{\mu_2 \rho_3 \rho_4} \varepsilon_{\mu_5 \rho_5} \varepsilon_{\mu_6 \rho_6} \varepsilon_{\mu_7 \rho_7} \varepsilon_{\mu_8 \rho_8} \varepsilon_{\mu_9 \rho_9},
\]

\[
[1.06] \supset 8g^5 \int \tilde{k}_{\mu_1 \mu_2} \tilde{k}_{\mu_3 \mu_4} \sin \left( \frac{k \times q}{2} \right) \sin \left( \frac{\mu_1 \rho_3 \rho_4 (q + \eta) \rho_5 (k + q) \rho_6}{k \times (q + \eta)} \right) \varepsilon_{\mu_1 \rho_3 \rho_4} \varepsilon_{\mu_2 \rho_3 \rho_4} \varepsilon_{\mu_5 \rho_5} \varepsilon_{\mu_6 \rho_6} \varepsilon_{\mu_7 \rho_7} \varepsilon_{\mu_8 \rho_8} \varepsilon_{\mu_9 \rho_9}. \tag{5.19}
\]

We need not apply the trick that we applied for the \( \varepsilon_{\mu\nu\rho_0} \) terms here, and can directly evaluate the phases. The phase from \([1.03] + [1.04] \) is evaluated to be

\[
4k \times (q + \eta) \int e^{-i[k \times q \sigma_1 + k \times q \sigma_2]} e^{\frac{i}{2} [\eta q + k x q + 2k x q]} e^{-i[k \times q \sigma_1 + k \times q \sigma_2]} e^{\frac{i}{2} [-\eta q + 2k x q]}
\]

\[
= \frac{4}{BC} \left\{ B e^{\frac{i}{4} (A+B)} - B e^{\frac{i}{4} (A-B-2C)} + C e^{\frac{i}{4} (-A+B+2C)} - C e^{\frac{i}{4} (A-B)} \right\}
\]

\[
= \frac{4}{BC} \left\{ 4C \sin \left( \frac{A}{2} \right) \sin \left( \frac{B}{2} \right) + 2B \left[ \cos \left( \frac{A-B}{2} \right) - \cos \left( \frac{A-B-2C}{2} \right) \right] \right\}. \tag{5.20}
\]
Thus the terms in \([1.05]\) is given by
\[
-8 \int e^{ik\times q(\frac{1}{2}-\sigma_{12})}\sin\left(\frac{\pi q\cdot s}{2}\right) = -\frac{16}{B}\sin\left(\frac{B}{2}\right)\sin\left(\frac{A}{2}\right),
\]
(5.21)
and the phase from \([1.06]\) is given by
\[
8 \int e^{ik\times \eta(\frac{1}{2}-\sigma_{12})}\sin\left(\frac{\pi q\cdot s}{2}\right) = \frac{16}{C}\sin\left(\frac{C}{2}\right)\sin\left(\frac{A-B-C}{2}\right)
= \frac{8}{C}\left[\cos\left(\frac{A-B-2C}{2}\right) - \cos\left(\frac{A-B}{2}\right)\right].
\]
(5.22)
Thus the terms in \(S\) proportional to \(\varepsilon_{\mu\nu\rho_5}\) also cancel.

To summarize thus far, we have shown the cancellation of terms proportional to \(\varepsilon_{\mu\nu\rho_5}\) and \(\varepsilon_{\mu\nu\rho_6}\) in \(S\). As in the previous section, this leaves a term proportional to \(\varepsilon_{\mu\nu\mu_12}\):
\[
S = \sum_{i=1}^{6} [1.0i] = -8g^5 \int \tilde{k}^{\mu_11} \tilde{k}^{\mu_12} e^{ik\times \eta(\frac{1}{2}-\sigma_{12})}\sin\left(\frac{\pi q\cdot s}{2}\right)\sin\left(\frac{\pi q\cdot k}{2}\right)\sin\left(\frac{\pi q\cdot k}{2}\right)
\]
\[
\times \varepsilon_{\nu\rho_2 \sigma_2} \delta^{3}(q_3+q_1) \delta^{3}(r_3+q_1) \delta^{3}(s_3+q_1) \delta^{3}(p_1+q_2) \delta^{3}(p_1+p_2) - (\mu \leftrightarrow \nu).
\]
(5.23)
Thus, in analogy with the previous section, we now compute the diagrams in figure 11, which contain the one-loop vertex correction. Evaluating the diagram with the internal gauge loop we obtain
\[
\int_{1.07} = (ig)^2 (-2ig)^3 \int \tilde{k}^{\mu_11} \tilde{k}^{\mu_12} \delta^{3}(p_1+p_2-p_3) e^{-ip_12-\xi_{12}} e^{-ip_12 \times p_2} \sin\left(\frac{\pi q\cdot s}{2}\right)\sin\left(\frac{\pi q\cdot k}{2}\right)\sin\left(\frac{\pi q\cdot k}{2}\right)
\]
\[
\times (k - \eta) \mu \varepsilon_{\nu\rho_2 \sigma_2} \delta^{3}(q_3+q_1) \delta^{3}(r_3+q_1) \delta^{3}(s_3+q_1) \delta^{3}(p_1+q_2) \delta^{3}(p_1+p_2) - (\mu \leftrightarrow \nu).
\]
(5.24)
Set \(p_{12} = -\eta, q_3 = -q\), and contract on \(\alpha_2, \beta_2\) to get:
\[
\int_{1.07} = -8g^5 \int \tilde{k}^{\mu_11} \tilde{k}^{\mu_12} e^{ik\times \eta(\frac{1}{2}-\sigma_{12})}\sin\left(\frac{\pi q\cdot s}{2}\right)\sin\left(\frac{\pi q\cdot k}{2}\right)\sin\left(\frac{\pi q\cdot k}{2}\right)
\]
\[
\times (k - \eta) \mu \varepsilon_{\nu\rho_2 \sigma_2} \delta^{3}(q_3+q_1) \delta^{3}(r_3+q_1) \delta^{3}(s_3+q_1) \delta^{3}(p_1+q_2) \delta^{3}(p_1+p_2) - (\mu \leftrightarrow \nu).
\]
we compute \( \sigma \) to create new surface terms with respect to the piece in \( \text{1.08} \).

The calculation of the ghost loop graph \( \text{1.08} \) is very similar to \( \text{2.07} \). Remembering the minus sign for the closed internal fermion loop, and the two sets of contractions for the two ghost number flow directions we have

\[
\text{1.08} = -g^2 (2g)^3 \int \frac{d^2 q}{2\pi} \frac{d^2 k}{2\pi} \frac{d^3 p_{12}}{2\pi} \frac{d^3 p_{14}}{2\pi} \epsilon_{\mu \nu \rho \sigma} \epsilon_{\mu \nu \rho \sigma} \sin \left( \frac{\sqrt{2} \mu - \sqrt{2} \nu}{2} \right) \sin \left( \sqrt{2} \rho \right) \sin \left( \sqrt{2} \sigma \right)
\]

Set \( p_{12} = -\eta \), and take \( r_1 = q \) in the first contraction, and \( r_3 = q \) in the second to obtain

\[
\text{1.08} = -g^5 \int \frac{d^2 k}{2\pi} \frac{d^2 q}{2\pi} \frac{d^3 p_{12}}{2\pi} \frac{d^3 p_{14}}{2\pi} \epsilon_{\mu \nu \rho \sigma} \epsilon_{\mu \nu \rho \sigma} \sin \left( \frac{\sqrt{2} \mu - \sqrt{2} \nu}{2} \right) \sin \left( \sqrt{2} \rho \right) \sin \left( \sqrt{2} \sigma \right) \]

Thus the ghost graph \( \text{1.08} \) cancels the last two terms in \( \text{1.07} \). Adding these last two contributions to \( S \), we can summarize our results thus far as

\[
\sum_{i=1}^{8} \text{1.08} = 8g^5 \int \frac{d^2 k}{2\pi} \frac{d^2 q}{2\pi} \frac{d^3 p_{12}}{2\pi} \frac{d^3 p_{14}}{2\pi} \epsilon_{\mu \nu \rho \sigma} \epsilon_{\mu \nu \rho \sigma} \sin \left( \frac{\sqrt{2} \mu - \sqrt{2} \nu}{2} \right) \sin \left( \sqrt{2} \rho \right) \sin \left( \sqrt{2} \sigma \right) \]

where we have used (A.17) to arrive at the last line. The second term cancels the residual piece in \( S \). Notice that unlike the case of \( \text{2.07} \), the right hand side of (A.17) again survives to create new surface terms with respect to the \( \sigma_{12} \) integration, which we will deal with after we compute \( \text{1.08} \).
Figure 12: Diagrams [1.09] and [1.10], which cancel the surface terms generated by [1.07] and [1.08].

Finally, we must add the contributions shown in figure 12. In anticipation of the result we will write [1.09] as follows:

\[
[1.09] = - (i g)^2 (2 i \eta)^3 \int \hat k^{\mu 11} \delta^{(3)}(p_{32} + p_{42} - k') e^{-\frac{1}{2} p_{32} \times p_{42}} \sin (\frac{\eta x_{12}}{2}) \sin (\frac{\eta y_{12}}{2}) \sin (\frac{\eta z_{12}}{2}) \xi^{\alpha_1 2 \alpha_3} \\
\varepsilon^{123} \varepsilon^{\alpha_3 \beta_3 \gamma_3} \varepsilon^{\alpha_3 \rho_3 \beta_1} \varepsilon^{\beta_3 \rho_3 \gamma_1} \varepsilon^{\gamma_3 \rho_3 \alpha_1} \delta^{(3)}(q_3 + r_1) \delta^{(3)}(r_3 + s_1) \delta^{(3)}(s_3 + q_1) e^{\frac{1}{2} \hat p_{11}^2 \hat p_{32}^2 \hat p_{42}^2 \hat p_{53}^2 \hat p_{52}^2 \hat p_{53}^2} \\
\varepsilon^{\mu_1 \rho_1 \alpha_2} \delta^{(3)}(p_1 + q_2) \xi^{\rho_1 \beta_3} \varepsilon^{\rho_1 \rho_2} \delta^{(3)}(p_3 + q_2) \delta^{(3)}(p_2 + s_2) \delta^{(3)}(p_4 + r_2) \delta^{(3)}(p_3 + s_2) \delta^{(3)}(p_4 + r_2) \delta^{(3)}(p_3 + s_2) \delta^{(3)}(p_4 + r_2) \\
\varepsilon^{\mu_1 \rho_1 \alpha_2} \delta^{(3)}(p_1 + q_2) \xi^{\rho_1 \beta_3} \varepsilon^{\rho_1 \rho_2} \delta^{(3)}(p_3 + q_2) \delta^{(3)}(p_2 + s_2) \delta^{(3)}(p_4 + r_2) \delta^{(3)}(p_3 + s_2) \delta^{(3)}(p_4 + r_2) \\
- (\mu \leftrightarrow \nu), 
\]

where we have written the contractions in two equivalent ways in the second line. They are equivalent because the contractions of the two commutator gauge field sources into the two remaining (i.e. after the contraction from the source on \(W(k)\) into an internal vertex is fixed) internal vertices are arbitrary: we are averaging again by pre-emptively performing the variable change \(\eta \rightarrow k - \eta\) in the second term.

Then choosing \(q_3 = -q\), and \(p_{32} = -\eta\) in the first term, \(p_{42} = -\eta\) in the second, and contracting on \(\alpha_3, \beta_3, \gamma_3\) we obtain

\[
[1.09] = 4 i g^5 \int \hat k^{\mu 11} \frac{k_1^{\mu 1} (k - k')^{\mu 2} \eta^{\mu 3} \eta^{\mu 4} (q + \eta)^{\mu 5} (q + k - \eta)^{\mu 6}}{k^2 (k - k')^2 (q - \eta)^2 (q + k - \eta)^2} \sin (\frac{\eta x_{12}}{2}) \sin (\frac{\eta y_{12}}{2}) \sin (\frac{\eta z_{12}}{2}) \xi^{\alpha_1 2 \alpha_3} \\
\varepsilon^{\gamma_1 \gamma_2 \gamma_3} \varepsilon^{\alpha_3 \rho_3 \beta_1} \varepsilon^{\beta_3 \rho_3 \gamma_1} \varepsilon^{\gamma_3 \rho_3 \alpha_1} \varepsilon^{\mu_1 \rho_1 \alpha_2} \left[ e^{\frac{1}{2} \hat k \times \eta} \varepsilon^{\mu_3 \rho_3 \beta_2} \xi^{\nu_2 \gamma_2} + e^{\frac{1}{2} \hat k \times \eta} \varepsilon^{\mu_3 \rho_3 \beta_2} \xi^{\nu_3 \gamma_2} \right] \\
-8 g^5 \int \hat k^{\mu 11} \frac{k_1^{\mu 1} (k - k')^{\mu 2} \eta^{\mu 3} \eta^{\mu 4} (q + \eta)^{\mu 5} (q + k - \eta)^{\mu 6}}{k^2 (k - k')^2 (q - \eta)^2 (q + k - \eta)^2} \sin (\frac{\eta x_{12}}{2}) \sin (\frac{\eta y_{12}}{2}) \sin (\frac{\eta z_{12}}{2}) \xi^{\alpha_1 2 \alpha_3} \\
\varepsilon^{\beta_1 \beta_2 \beta_3} \varepsilon^{\gamma_1 \gamma_2 \gamma_3} \varepsilon^{\alpha_3 \rho_3 \beta_1} \varepsilon^{\beta_3 \rho_3 \gamma_1} \varepsilon^{\gamma_3 \rho_3 \alpha_1} \varepsilon^{\mu_1 \rho_1 \alpha_2} \left[ e^{\mu \nu \rho_2} \varepsilon^{\rho_3 \beta_2} \xi^{\nu_2 \gamma_2} + e^{\mu \nu \rho_2} \varepsilon^{\rho_3 \beta_2} \xi^{\mu_2 \gamma_2} \right] \\
-8 g^5 \int \hat k^{\mu 11} \frac{k_1^{\mu 1} (k - k')^{\mu 2} \eta^{\mu 3} \eta^{\mu 4} (q + \eta)^{\mu 5} (q + k - \eta)^{\mu 6}}{k^2 (k - k')^2 (q - \eta)^2 (q + k - \eta)^2} \sin (\frac{\eta x_{12}}{2}) \sin (\frac{\eta y_{12}}{2}) \sin (\frac{\eta z_{12}}{2}) \xi^{\alpha_1 2 \alpha_3} \\
\left[ \varepsilon^{\alpha_3 \rho_3 \beta_1} \varepsilon^{\beta_3 \rho_3 \gamma_1} \varepsilon^{\gamma_3 \rho_3 \alpha_1} \varepsilon^{\mu_1 \rho_1 \alpha_2} \left[ e^{\mu \nu \rho_2} \varepsilon^{\rho_3 \beta_2} \xi^{\nu_2 \gamma_2} + e^{\mu \nu \rho_2} \varepsilon^{\rho_3 \beta_2} \xi^{\mu_2 \gamma_2} \right] \right] \\
-8 g^5 \int \hat k^{\mu 11} \frac{k_1^{\mu 1} (k - k')^{\mu 2} \eta^{\mu 3} \eta^{\mu 4} (q + \eta)^{\mu 5} (q + k - \eta)^{\mu 6}}{k^2 (k - k')^2 (q - \eta)^2 (q + k - \eta)^2} \sin (\frac{\eta x_{12}}{2}) \sin (\frac{\eta y_{12}}{2}) \sin (\frac{\eta z_{12}}{2}) \xi^{\alpha_1 2 \alpha_3} \\
\left[ \varepsilon^{\mu \nu \rho_2} \varepsilon^{\rho_3 \beta_2} \xi^{\nu_2 \gamma_2} + e^{\mu \nu \rho_2} \varepsilon^{\rho_3 \beta_2} \xi^{\mu_2 \gamma_2} \right] \\
\left[ \varepsilon^{\mu_1 \rho_1 \rho_4} \varepsilon^{\rho_3 \rho_4 \rho_1} - \varepsilon^{\mu_1 \rho_1 \rho_5} \varepsilon^{\rho_3 \rho_4 \rho_6} - \varepsilon^{\mu_1 \rho_1 \rho_6} \varepsilon^{\rho_3 \rho_4 \rho_1} - \varepsilon^{\mu_1 \rho_1 \rho_4} \varepsilon^{\rho_3 \rho_4 \rho_1} \right]
\]
\[\begin{align*}
- [\varepsilon_{\mu_1 \rho_1 \rho_2} &\varepsilon_{\rho_3 \rho_4 \rho_5} \varepsilon_{\rho_5 \rho_6} + \varepsilon_{\mu_1 \rho_1 \rho_5} \varepsilon_{\rho_3 \rho_4 \rho_6} - \varepsilon_{\mu_1 \rho_1 \rho_5} \varepsilon_{\rho_3 \rho_4 \rho_6}], \\
&= -8g^5 \int \tilde{k}_{\mu_1 \nu_1} k^{\alpha \beta} (q_3 + p_2 - k^\alpha - k^\beta) e^{-\frac{i}{2} k_{\mu_1} p^\nu} \sin \left( \frac{2\pi x_{\mu_1}}{2} \right) \sin \left( \frac{2\pi y_{\mu_1}}{2} \right) \left( \varepsilon_{\mu_1 \rho_1 \rho_5} \varepsilon_{\rho_3 \rho_4 \rho_6} + \varepsilon_{\mu_1 \rho_1 \rho_5} \varepsilon_{\rho_3 \rho_4 \rho_6} + \varepsilon_{\mu_1 \rho_1 \rho_5} \varepsilon_{\rho_3 \rho_4 \rho_6} + \varepsilon_{\mu_1 \rho_1 \rho_5} \varepsilon_{\rho_3 \rho_4 \rho_6} \right)
&= -8g^5 \int \tilde{k}_{\mu_1 \nu_1} k^{\alpha \beta} (q_3 + p_2 - k^\alpha - k^\beta) e^{-\frac{i}{2} k_{\mu_1} p^\nu} \sin \left( \frac{2\pi x_{\mu_1}}{2} \right) \sin \left( \frac{2\pi y_{\mu_1}}{2} \right) \left( \varepsilon_{\mu_1 \rho_1 \rho_5} \varepsilon_{\rho_3 \rho_4 \rho_6} + \varepsilon_{\mu_1 \rho_1 \rho_5} \varepsilon_{\rho_3 \rho_4 \rho_6} + \varepsilon_{\mu_1 \rho_1 \rho_5} \varepsilon_{\rho_3 \rho_4 \rho_6} + \varepsilon_{\mu_1 \rho_1 \rho_5} \varepsilon_{\rho_3 \rho_4 \rho_6} \right),
&\quad \text{where we have repeatedly applied (A.13) in conjunction with the following momentum identity:}
\varepsilon_{\rho_2 \rho_5 \rho_6} (k - \eta)^\rho (q + \eta)^\rho (k + q)^\rho = 0.
\end{align*}\]

Finally we compute the ghost graph [1.10] paired with [1.09], and show that it cancels the last three terms in (5.30):

\[\begin{align*}
[1.10] &= + (ig)^2 \cdot (-2i g)^3 \int \tilde{k}_{\mu_1 \nu_1} \delta^3 (p_{12} + p_2 + k - k_e) e^{-\frac{i}{2} k_{\mu_1} p^\nu} \sin \left( \frac{2\pi x_{\mu_1}}{2} \right) \sin \left( \frac{2\pi y_{\mu_1}}{2} \right) (-i)^3 q_1^\alpha r_1^\beta s_1^\gamma \\
&\quad \times \left[ \delta^{\dot{3}} (p_{12} + p_2 + k - k_e) e^{-\frac{i}{2} k_{\mu_1} p^\nu} \sin \left( \frac{2\pi x_{\mu_1}}{2} \right) \sin \left( \frac{2\pi y_{\mu_1}}{2} \right) (-i)^3 q_1^\alpha r_1^\beta s_1^\gamma \\
&\quad \times \left[ \delta^{\dot{3}} (p_{12} + p_2 + k - k_e) e^{-\frac{i}{2} k_{\mu_1} p^\nu} \sin \left( \frac{2\pi x_{\mu_1}}{2} \right) \sin \left( \frac{2\pi y_{\mu_1}}{2} \right) (-i)^3 q_1^\alpha r_1^\beta s_1^\gamma \\
&\quad \times \left[ \delta^{\dot{3}} (p_{12} + p_2 + k - k_e) e^{-\frac{i}{2} k_{\mu_1} p^\nu} \sin \left( \frac{2\pi x_{\mu_1}}{2} \right) \sin \left( \frac{2\pi y_{\mu_1}}{2} \right) (-i)^3 q_1^\alpha r_1^\beta s_1^\gamma \right]
\end{align*}\]

where again we have written each of the terms in the first in two different ways, and averaged their contributions. Thus for the first two terms take \( p_{32} = -\eta \), and for the second two \( p_{42} = -\eta \). Furthermore take \( r_1 = q \) for the first and third terms, and \( r_3 = q \) for the second and fourth terms. Then we have

\[\begin{align*}
[1.10] &= 4i g^5 \int \tilde{k}_{\mu_1 \nu_1} \sin \left( \frac{2\pi x_{\mu_1}}{2} \right) \sin \left( \frac{2\pi y_{\mu_1}}{2} \right) \left[ \delta^{\dot{3}} (p_{12} + p_2 + k - k_e) e^{-\frac{i}{2} k_{\mu_1} p^\nu} \sin \left( \frac{2\pi x_{\mu_1}}{2} \right) \sin \left( \frac{2\pi y_{\mu_1}}{2} \right) (-i)^3 q_1^\alpha r_1^\beta s_1^\gamma \\
&\quad \times \left[ \delta^{\dot{3}} (p_{12} + p_2 + k - k_e) e^{-\frac{i}{2} k_{\mu_1} p^\nu} \sin \left( \frac{2\pi x_{\mu_1}}{2} \right) \sin \left( \frac{2\pi y_{\mu_1}}{2} \right) (-i)^3 q_1^\alpha r_1^\beta s_1^\gamma \\
&\quad \times \left[ \delta^{\dot{3}} (p_{12} + p_2 + k - k_e) e^{-\frac{i}{2} k_{\mu_1} p^\nu} \sin \left( \frac{2\pi x_{\mu_1}}{2} \right) \sin \left( \frac{2\pi y_{\mu_1}}{2} \right) (-i)^3 q_1^\alpha r_1^\beta s_1^\gamma \right]
\end{align*}\]
\[ 
+ e^{-\frac{x}{2}k\times\eta} \left[ \varepsilon_{\mu_1\mu_2\rho_5} \varepsilon_{\nu_3\rho_3\rho_4} + \varepsilon_{\mu_1\mu_2\rho_4} \varepsilon_{\nu_3\rho_3\rho_5} \right] - (\mu \leftrightarrow \nu) 
= 8g^5 \int k^{\mu_1} \sin \left( \frac{2\pi \eta}{2} \right) \sin \left( \frac{2\pi q \times k \times \eta}{2} \right) \sin \left( \frac{2\pi q \times k \times \eta}{2} \right) 
\times \left\{ \varepsilon_{\mu_1\mu_2\rho_6} \varepsilon_{\rho_5\rho_3\rho_4} - \varepsilon_{\rho_5\rho_4\rho_3} \varepsilon_{\mu_1\mu_2\rho_6} + \varepsilon_{\mu_1\mu_2\rho_4} \varepsilon_{\rho_5\rho_3\rho_6} - \varepsilon_{\rho_5\rho_3\rho_6} \varepsilon_{\mu_1\mu_2\rho_4} \right\} 
\] 
where we have used (A.14) and the momentum identity (5.31). Thus the sum of the diagrams in figure 12 is given by

\[ 
\sum_{i=9}^{10} [i] = -8g^5 \int k^{\mu_1} \frac{k^\mu (k-q)^\nu (q+\eta)^\rho (k+q)^\sigma}{k^\nu (k-q)^\rho (q+\eta)^\sigma (k+q)^\nu} \sin \left( \frac{2\pi \eta}{2} \right) \sin \left( \frac{2\pi q \times k \times \eta}{2} \right) \sin \left( \frac{2\pi q \times k \times \eta}{2} \right) 
\times \left\{ \varepsilon_{\mu_1\mu_2\rho_6} \varepsilon_{\rho_5\rho_3\rho_4} + \varepsilon_{\rho_5\rho_4\rho_3} \varepsilon_{\mu_1\mu_2\rho_6} + \varepsilon_{\mu_1\mu_2\rho_4} \varepsilon_{\rho_5\rho_3\rho_6} + \varepsilon_{\rho_5\rho_3\rho_6} \varepsilon_{\mu_1\mu_2\rho_4} \right\}. 
\] 

It turns out that the two terms in brackets are the same. To see this make the following changes of variables in the second term: \( \eta \rightarrow k - \eta, \) and \( q \rightarrow -(k + q). \) We obtain the identity

\[ 
\int \sin \left( \frac{2\pi \eta}{2} \right) \sin \left( \frac{2\pi q \times k \times \eta}{2} \right) \sin \left( \frac{2\pi q \times k \times \eta}{2} \right) 
\times \frac{k^\mu (k-q)^\nu (q+\eta)^\rho (k+q)^\sigma}{k^\nu (k-q)^\rho (q+\eta)^\sigma (k+q)^\nu} \varepsilon_{\mu_1\mu_2\rho_6} \varepsilon_{\rho_5\rho_3\rho_4} 
= \int \sin \left( \frac{2\pi \eta}{2} \right) \sin \left( \frac{2\pi q \times k \times \eta}{2} \right) \sin \left( \frac{2\pi q \times k \times \eta}{2} \right) 
\times \frac{k^\mu (k-q)^\nu (q+\eta)^\rho (k+q)^\sigma}{k^\nu (k-q)^\rho (q+\eta)^\sigma (k+q)^\nu} \varepsilon_{\mu_1\mu_2\rho_4} \varepsilon_{\rho_5\rho_3\rho_6}. 
\] 

Thus comparing with (5.28), we obtain finally

\[ 
\sum_{i=9}^{10} [i] = -16g^5 \int \frac{k^\mu (k-q)^\nu (q+\eta)^\rho (k+q)^\sigma}{k^\nu (k-q)^\rho (q+\eta)^\sigma (k+q)^\nu} \sin \left( \frac{2\pi \eta}{2} \right) \sin \left( \frac{2\pi q \times \eta}{2} \right) \sin \left( \frac{2\pi q \times k \times \eta}{2} \right) 
\times \varepsilon_{\mu_1\mu_2\rho_6} \varepsilon_{\rho_5\rho_3\rho_4} \varepsilon_{\rho_4\rho_5\rho_6} 
= -8 \sum_{i=1}^{8} [i], 
\] 

after performing the trivial \( \sigma_{12} \) integration in (5.28). That is

\[ 
\sum_{i=1}^{10} [i] = 0. 
\] 

Except for the two-loop propagator corrections, this completes the contributions from \( O(g) \) terms in \( W(k) \) to the correlator \( \langle W(k)O_{\mu\nu}(k') \rangle. \) The set of one-particle irreducible diagrams contributing to the two-loop propagator corrections which do not themselves contain one-loop propagator corrections within them are shown in figure 13. Each of these graphs will have the same noncommutative phases and propagators, and we will demonstrate that the
The most complicated is the pure gauge loop, diagram [1.11]:

\[
[1.11] = (ig)(-2ig)^4 \int \tilde{k}^{\mu_1 11} \delta^{(3)}(p_{42} - k') \sin \left( \frac{\pi x_2}{2} \right) \sin \left( \frac{\pi x_3}{2} \right) \sin \left( \frac{\pi x_4}{2} \right) \sin \left( \frac{\pi x_5}{2} \right) \epsilon^{\alpha_1 \alpha_2 \alpha_3} \epsilon^{\delta_1 \beta_2 \delta_3} \\
\times \gamma_1 \gamma_2 \gamma_3 \epsilon^{\delta_1 \delta_2 \delta_3} p_1^{\mu_1} p_2^{\rho} p_3^{\sigma} p_4^{\tau} \left( -i p_{42} \right) \left[ \mu \nu \rho \tau \right] \gamma_{\mu_1} \rho_3 p_{21} \epsilon_{\alpha_1 \alpha_2 \alpha_3} \epsilon_{\beta_1 \beta_2 \beta_3} \\

\epsilon_{\gamma_1 \gamma_2 \gamma_3} \epsilon^{\delta_1 \delta_2 \delta_3} (p_{11} + q_2) \delta^{(3)}(p_{42} + t_2) \delta^{(3)}(q_1 + s_3) \delta^{(3)}(q_3 + t_1) \delta^{(3)}(r_2 + s_2) \delta^{(3)}(r_3 + t_1) \delta^{(3)}(s_1 + t_3)
\]

(5.38)

Setting \( s_3 = q \), and \( s_1 = \eta \), and contracting on \( \alpha_1, \beta_1, \gamma_1 \), and \( \delta_1 \) we obtain

\[
[1.11] = 16g^5 \int \tilde{k}^{\mu_1 11} \sin \left( \frac{k \cdot q}{2} \right) \sin \left( \frac{\pi x_2}{2} \right) \sin \left( \frac{\pi x_3}{2} \right) \sin \left( \frac{\pi x_4}{2} \right) \sin \left( \frac{\pi x_5}{2} \right) \frac{k^{\mu_1} (k - q)^{\nu} p_3^{\rho} p_4^{\sigma} p_5^{\tau}}{k \cdot (k - q) p_3^{\rho} p_4^{\sigma} p_5^{\tau}} \epsilon_{\rho_3\rho_4\rho_5} \eta^{\rho_3} q^{\rho_4} (q + \eta)^{\rho_5} = 0,
\]

\[(5.40)\]

and changed variables \( \eta \to k - \eta, q \to -(k + q) \) in the last term on the first line to cancel the penultimate term there. Diagram [1.12] is given by

\[
[1.12] = -ig(-2ig)^4 \int \tilde{k}^{\mu_1 11} \delta^{(3)}(p_{42} - k') \sin \left( \frac{\pi x_2}{2} \right) \sin \left( \frac{\pi x_3}{2} \right) \sin \left( \frac{\pi x_4}{2} \right) \sin \left( \frac{\pi x_5}{2} \right) \epsilon^{\delta_1 \delta_2 \delta_3} p_1^{\mu_1} p_2^{\rho} p_3^{\sigma} p_4^{\tau} \left( -i p_{42} \right) \left[ \mu \nu \rho \tau \right] \gamma_{\mu_1} \rho_3 p_{21} \epsilon_{\delta_1 \delta_2 \delta_3} \\
\times \gamma_1 \gamma_2 \gamma_3 \epsilon^{\delta_1 \delta_2 \delta_3} (p_{12} + q_2) \delta^{(3)}(p_{42} + t_2) \delta^{(3)}(p_{11} + q_2) \delta^{(3)}(r_2 + t_1) \delta^{(3)}(s_2 + t_3) \delta^{(3)}(s_3 + q_1) \delta^{(3)}(q_1 + q_2) \delta^{(3)}(s_3 + q_1) \delta^{(3)}(s_3 + q_1) \\
\times \epsilon_{\delta_1 \delta_2 \delta_3} \epsilon_{\gamma_1 \gamma_2 \gamma_3} \epsilon^{\delta_1 \delta_2 \delta_3} (q_1 + r_1) \delta^{(3)}(r_3 + s_1) \delta^{(3)}(s_3 + q_1) + \delta^{(3)}(q_3 + q_1) \delta^{(3)}(s_3 + q_1) \delta^{(3)}(r_3 + q_1) \delta^{(3)}(r_3 + q_1).
\]

(5.41)
In the first term set $s_2 = \eta, s_3 = q$, and in the second set $s_2 = \eta, s_1 = q$. Then contract on $\delta_1$ to obtain:

\[
[1.12] = -16g^5 \int \tilde{k}_{11}^\mu \sin \left(\frac{k \times x}{2}\right) \sin \left(\frac{n \times k - k \times q}{2}\right) \sin \left(\frac{n \times x}{2}\right) \sin \left(\frac{k \times x}{2}\right) \frac{k_{\mu 1}^\mu (k - q)^2 \eta^\alpha \eta^\beta (q + n)^5 (k + q)^6 k^\kappa}{k^2 (k - q)^2 (q + n)^2 (k + q)^2} \\
\times k_{\mu} \left[ \varepsilon_{\mu 1 \rho 4} \left( \varepsilon_{\nu \rho \tau \rho \delta} \varepsilon_{\tau \rho \delta \mu} - \varepsilon_{\nu \rho \tau \rho \delta} \varepsilon_{\tau \rho \delta \mu} \right) + \varepsilon_{\mu 1 \rho 4} \varepsilon_{\rho 4 \rho 3 \rho 4} \right] - (\mu \leftrightarrow \nu),
\]

where we have used (A.14) several times in conjunction with (5.40), and

\[
\varepsilon_{\mu 1 \rho 2 \rho 3} k_{\mu 1}^\mu (k - q)^2 \eta^\rho = \varepsilon_{\mu 1 \rho 4} \varepsilon_{\rho 4 \rho 3 \rho 4} k_{\mu 1}^\mu (k + q)^6 = 0.
\]

Diagram [1.13] evaluates similarly:

\[
[1.13] = -ig(-2i)^4 \int \tilde{k}_{11}^\mu \delta^{(3)}(p_{42} - k') \sin \left(\frac{q_1 \times q_2}{2}\right) \sin \left(\frac{q_1 \times q_2}{2}\right) \sin \left(\frac{q_1 \times q_2}{2}\right) \sin \left(\frac{q_1 \times q_2}{2}\right) \frac{k_{\mu 1}^\mu (k - q)^2 \eta^\alpha \eta^\beta (q + n)^5 (k + q)^6 k^\kappa}{k^2 (k - q)^2 (q + n)^2 (k + q)^2} \\
\times \left[ \varepsilon_{\mu 1 \rho 4} \varepsilon_{\rho 4 \rho 3 \rho 4} \right] - (\mu \leftrightarrow \nu),
\]

\[
\delta^{(3)}(q_3 + \mathbf{r}_1) \delta^{(3)}(q_3 + \mathbf{r}_1) \delta^{(3)}(q_3 + \mathbf{r}_1) \delta^{(3)}(q_3 + \mathbf{r}_1) + \delta^{(3)}(q_3 + \mathbf{r}_1) \delta^{(3)}(q_3 + \mathbf{r}_1) \delta^{(3)}(q_3 + \mathbf{r}_1)
\]

Take $r_2 = q, r_1 = \eta$ in the first term, and $r_2 = q, r_3 = \eta$ in the second term, and contract on $\alpha_1$ to get

\[
[1.13] = 16g^5 \int \tilde{k}_{11}^\mu \sin \left(\frac{k \times x}{2}\right) \sin \left(\frac{n \times k - k \times q}{2}\right) \sin \left(\frac{k \times x}{2}\right) \sin \left(\frac{k \times x}{2}\right) \frac{k_{\mu 1}^\mu (k - q)^2 \eta^\alpha \eta^\beta (q + n)^5 (k + q)^6 k^\kappa}{k^2 (k - q)^2 (q + n)^2 (k + q)^2} \\
\times \left[ \varepsilon_{\mu 1 \rho 4} \varepsilon_{\rho 4 \rho 3 \rho 4} \right] - (\mu \leftrightarrow \nu),
\]

This leaves finally the ghost loop [1.14]:

\[
[1.14] = -ig(-2i)^4 \int \tilde{k}_{11}^\mu \delta^{(3)}(p_{42} - k') \sin \left(\frac{q_1 \times q_2}{2}\right) \sin \left(\frac{q_1 \times q_2}{2}\right) \sin \left(\frac{q_1 \times q_2}{2}\right) \sin \left(\frac{q_1 \times q_2}{2}\right) \frac{k_{\mu 1}^\mu (k - q)^2 \eta^\alpha \eta^\beta (q + n)^5 (k + q)^6 k^\kappa}{k^2 (k - q)^2 (q + n)^2 (k + q)^2} \\
\times \left[ \varepsilon_{\mu 1 \rho 4} \varepsilon_{\rho 4 \rho 3 \rho 4} \right] - (\mu \leftrightarrow \nu),
\]

Taking $s_1 = \eta, s_3 = q$ in the first term, and $s_1 = q, s_3 = \eta$ in the second term, we obtain

\[
[1.14] = -16g^5 \int \tilde{k}_{11}^\mu \sin \left(\frac{k \times x}{2}\right) \sin \left(\frac{n \times k - k \times q}{2}\right) \sin \left(\frac{n \times x}{2}\right) \sin \left(\frac{k \times x}{2}\right) \frac{k_{\mu 1}^\mu (k - q)^2 \eta^\alpha \eta^\beta (q + n)^5 (k + q)^6 k^\kappa}{k^2 (k - q)^2 (q + n)^2 (k + q)^2} \\
\times \left[ \varepsilon_{\mu 1 \rho 4} \varepsilon_{\rho 4 \rho 3 \rho 4} \right] - (\mu \leftrightarrow \nu),
\]

\[
-32g^5 \int \tilde{k}_{11}^\mu \sin \left(\frac{k \times x}{2}\right) \sin \left(\frac{n \times k - k \times q}{2}\right) \sin \left(\frac{n \times x}{2}\right) \sin \left(\frac{k \times x}{2}\right) \frac{k_{\mu 1}^\mu (k - q)^2 \eta^\alpha \eta^\beta (q + n)^5 (k + q)^6 k^\kappa}{k^2 (k - q)^2 (q + n)^2 (k + q)^2} \\
\times \left[ \varepsilon_{\mu 1 \rho 4} \varepsilon_{\rho 4 \rho 3 \rho 4} \right] - (\mu \leftrightarrow \nu),
\]
using (A.14) repeatedly, or equivalently changing variables \( \eta \to (k - \eta), q \to -(k + q) \) in the second term, which again effectively interchanges the \( \rho_2 \leftrightarrow \rho_3 \) and \( \rho_4 \leftrightarrow \rho_6 \) indices.

Comparing the results from [1.11] – [1.14], we see that

\[
\sum_{i=11}^{14} [1,i] = 0. \tag{5.48}
\]

The remaining contributions to the two-loop propagator all involve the linearly divergent one-loop corrections to the gauge or ghost propagator as subgraphs, and either formally cancel pairwise or vanish, or need to be carefully regulated as per our discussion in section 2, and that in [15]. As discussed in the latter, the tensor structure and the momentum dependence of the propagators are not modified by their one-loop corrections. Thus the presence of them as subgraphs in the remaining two-loop graphs, will essentially reduce these two-loop graphs to the one-loop propagator correction case. Since the one-loop corrections to the propagator occurred in our calculation as a quantum correction to the single nonvanishing graph at tree-level that reflects the equivalence between the commutative and noncommutative theories, and the net effect of these corrections to our calculation was to induce a harmless finite renormalization to the Seiberg-Witten map itself (if the equivalence is to be maintained), any such effect at two-loops can be similarly absorbed into a two-loop renormalization of Seiberg-Witten map. We will not consider them further here.

To summarize this section, we conclude that the sum of the \( O(g) \) contributions from \( W(k) \) to the correlator \( \langle W(k)O_{\mu\nu}(k') \rangle \) at \( O(g^5) \) vanishes.

6 Discussion and Conclusions

To summarize the results from sections 3 to 5, we have demonstrated the complete cancellation amongst all of the order \( g^5 \) contributions to the correlator \( \langle W(k)O_{\mu\nu}(k') \rangle \). Thus far our discussion has been almost entirely formal to keep the calculations and the cancellations transparent, and so let us now discuss the issue of regularization. In [15] we showed that a point-splitting regulator separating gauge field sources on the Wilson lines or the gauge fields of the field strength commutator, and natural [16] from the point of view of the computation of disk amplitudes in string theory where noncommutative gauge theory in spacetime arises from a point-splitting regularization of operators on the worldsheet boundary [17], was sufficient to regularize the divergences arising from the graphs studied therein. Specifically, the path parameter integrations along the Wilson lines had their integration ranges restricted so that operators were never closer than \( \epsilon_{\tilde{t}k} \) together. The operators making up the commutator in the field strength were similarly separated from each other, and in the case of one graph, from an operator along the path-ordered exponential part of \( O_{\mu\nu}(k) \). In the naïve limit \( \epsilon \to 0 \), the unregularized composite operators are recovered. The essential effect of this regulator is to ensure that the noncommutative phases do not vanish in each graph: putative
planar components that are superficially divergent are regulated by noncommutative phases of the form $e^{ia\xi k \cdot x}$, where $p$ is a loop momentum, and $a$ is some half-integer. It has the explicit advantage of not modifying the propagators or vertices, and so the tensor algebra in each of the graphs is unaltered. The expense paid is that the noncommutative phases between graphs become somewhat complicated relative to each other, and graphs that formally cancel now differ by terms of order $\epsilon$. Furthermore, it obviously cannot regulate divergences coming from graphs with loops that are internal (i.e. do not connect to the Wilson lines themselves).

This regulator can be naturally extended to the case where there are three or more sources along $W(k)$, as well as the case where there are two sources on $O_{\mu\nu}(k')$ in addition to those from the commutator. For example we take

$$\int_0^1 d\sigma_1 \int_0^{\sigma_1} d\sigma_2 A(x+\xi(\sigma_1)) \ast A(x+\xi(\sigma_2)) \ast A(x) \ast e^{ik \cdot x} \rightarrow \int_{-\epsilon}^{1-\epsilon} d\sigma_1 \int_{\epsilon}^{1-\epsilon} d\sigma_2 A(x+\xi(\sigma_1)) \ast A(x+\xi(\sigma_2)) \ast A(x) \ast e^{ik \cdot x},$$

(6.1)

where both upper and lower limits of integration are modified because of (1.5), and the cyclicity property of the star product under the integration over spacetime. As long as the operators are separated along the Wilson lines, a nonvanishing noncommutative phase will protect loop momenta integrations associated with gauge sources along $W(k)$ or $O_{\mu\nu}(k')$ carrying those momenta.

It should be clear that the computations in [3.01] – [3.02], and [2.01] – [2.03], being essentially identical to those in [15], will go through rigorously with this regulator. The graphs [2.04] – [2.06], which are crossings of those in [2.01] – [2.03], should also be made rigorously finite, so that we can legitimately sum those contributions and apply (A.17). The $q$ integration in that sum is finite by power counting since $\epsilon_{\rho_4\rho_5\rho_6} q^{\rho_4}(q + \eta)^{\rho_5}(k + q)^{\rho_6}$ is linear in $q$, and the (subsequent) $\eta$ integration is protected by a nonvanishing noncommutative phase with the regulator applied. On the other hand, [2.07] and [2.08] contain, in addition, logarithmically divergent terms with respect to the $q$ integration: the integration associated with the internal loop in those graphs. As discussed above, these graphs, which contain the one-loop vertex corrections as subgraphs, need to be regulated separately. The similarity between graphs [2.06] and [2.07] however, suggests the existence of a regulator which would retain the compatibility between [2.04] – [2.06] and [2.07] – [2.08], and make the final cancellation with $\sum_{i=1}^{6}[2,i]$ more rigorous. Specifically we might regulate [2.07] – [2.08] by point-splitting the fields in the internal noncommutative vertices. Then, phases which combine to produce planar, phase-independent pieces would be supplemented by additional, $\epsilon$ dependent phases which survive to regulate the momentum integrals. It would be interesting to carry out this construction, as it might have wider applicability to noncommutative field theories in general: the standard commutative divergences that survive as the planar pieces of noncommutative graphs might be controlled in a way analogous their nonplanar counterparts.

Presumably, the graphs [1.01] – [1.06] are also made rigorously finite by the extension of the point-splitting regulator. However, in order to exhibit the intricate cancellations that
we found amongst them, we found it necessary to explicitly evaluate their path parameter integrations, and perform some trigonometric gymnastics with the unregulated phases. It is less obvious therefore, that the cancellations found there will be automatically preserved by careful application of this regulator. It is possible that identities involving $\star_n$ products\cite{17,18} might simplify this computation, since the complication of having to evaluate these integrals to find the cancellations arises intrinsically from the higher order expansion of the Wilson lines. Finally, the graphs [1.07] − [1.10] are similar to [2.07] − [2.08]$^5$ in that they also contain the one-loop vertex correction graphs as subgraphs, and so once the latter are properly regularized, the former will be too.

Let us now consider the possible generalization of our results to higher orders. Clearly, proceeding as we have done to higher orders, even formally, would quickly become prohibitive. However, our calculations here in conjunction with those in [15] strongly suggest that the cancellation of quantum corrections to the correlator holds at any order in perturbation theory, and that the mechanisms which enforce such cancellations are already present at the orders we have studied. Specifically, we have found that we can organize the calculation according to the number of sources on the pure Wilson line $W(k)$. Our work suggests that it is sufficient to consider the simpler correlators at $O(g^{2n-1})$ given by

$$\left\langle \prod_{i=1}^{m} k^{\mu i} A_{\mu i}(p_{1i}) O_{\mu\nu}(k') \right\rangle \delta^{(3)}(\sum_{j=1}^{m} p_{1j} - k) , \hspace{1em} m = 1...n. \quad (6.2)$$

Moreover, we have seen that the identity (A.17),

$$\left[ C^\rho C_{[\mu} \varepsilon_{\nu]} \rho \mu i + \varepsilon_{\mu \nu \mu i} C^2 \right] k^{\mu i} = (k \times C) \varepsilon_{\nu \rho \nu} C^\rho , \quad (6.3)$$

lies at the heart of all of the cancellations, and allows us to compare graphs with different number of propagators. The first term on the left hand side arises from graphs that involve the derivative term in the field strength, while the second arises from the commutator term, and is associated with similar graphs with one less propagator and vertex. The right hand side yields a surface term with respect to a path parameter integration in $O_{\mu\nu}(k')$, and is also associated with a graph involving the field strength commutator, but with one more propagator and with one less source from the path-ordered exponential part of $O_{\mu\nu}(k')$.

These observations suggest how to find the cancellations at a given order $O(g^{2n-1})$. Consider any graph with $i \leq n$ sources on $W(k)$, $j < n$ sources on the path-ordered exponential component of $O_{\mu\nu}(k')$ (henceforth denoted $W_2$), and the one source from the derivative term in the field strength. Call this graph $I$, and fix $i$ for the remainder of this discussion. To graph $I$ is always paired a graph also with $(i,j)$ sources on $W(k)$ and $W_2$ respectively, but with two sources from the field strength commutator, one less vertex, one less propagator and otherwise identical to $I$. Call this graph $II$. Colloquially, we can think of forming it

\footnote{Certainly it would be surprising if the cancellation we found between [1.07] − [1.08] and [1.09] − [1.10] did not hold rigourously, since this directly generalizes the one-loop cancellation.}
from graph I by collapsing the propagator joining the field strength derivative term with an internal vertex, into the field strength insertion, and removing the vertex. The other two propagators entering into the internal vertex now connect directly to the field strength commutator (and the two ways of doing this compensate for the sinusoidal phase that we lose by removing the vertex). To each pair of such graphs, we apply (A.17). In the case where \( j = 0 \), the right hand side is zero and we are done. Otherwise build graph III, formed from graph I by ‘curling’ one of the sources on \( W_2 \) into the field strength insertion (see [2.01] \( \rightarrow \) [2.03] for example). In doing so, we lose a path parameter integration and source along \( W_2 \), so \( j \rightarrow j - 1 \), and contract with the field strength commutator as opposed to the field strength derivative. In the \( i = 1, j = 1 \) case studied in [15], and in the \( i = 2, j = 1 \) case in [2.01] \( \rightarrow \) [2.03], this provides the right hand side of (A.17), and hence a cancellation with graphs I and II.

Otherwise we have to consider the set of graphs \{Ia, Ib, ...\} with fixed \( i \) and \( j \), and their partners \{IIa, IIb, ...\}. We then build all of the graphs III which can be formed from the I’s by curling a source on \( W_2 \) into the field strength to obtain a graph with \( (i, j - 1) \) sources on \( W(k) \) and \( W_2 \) respectively, but with the same number of propagators, as in [1.05] and [1.06]. These will cancel all of the I and II pairs via (A.17), but as we saw in [2.06] and [1.06], will generally leave residual terms proportional to \( \varepsilon_{\mu \nu \mu_1 j} \tilde{k}^{\mu_1} \). These will occur in the graphs III where the propagators from the field strength commutator sources connect to distinct vertices. To each such graph, we then invert the process we used in forming II from I, inserting a propagator which joins the derivative term in the field strength to a new vertex, and creating an internal loop in the process to form a new graph. Thus, for example, we obtain [2.07] and [1.07] from [2.06] and [1.06]. Applying (A.17) to the terms proportional to \( C_{[\mu \varepsilon_{\nu \rho_a \mu_1 j]} \tilde{k}^{\mu_1}} \) in these new graphs, we cancel the aforementioned residual pieces, but generate new surface terms from the right hand side of (A.17), if \( C \) depends on a loop momentum, or equivalently if \( j > 0 \), while the ghost graph cancels the terms not proportional to \( C_{[\mu \varepsilon_{\nu \rho_a \mu_1 j]} \tilde{k}^{\mu_1}} \). Treating this new graph as one of type I, we then iterate the process of curling a source on \( W_2 \) into the field strength insertion, and create a new graph of type III, thereby reducing \( j \) further. Thus we obtain [1.09] from [1.07].\(^6\) This process terminates when \( j = 0 \), since \( C = k \) in (A.17) and the right hand side vanishes.

Thus far this discussion has been essentially independent of the other structure that might be present in the graphs, because the cancellations we found primarily occur between the derivative (plus noncommutative Chern-Simons vertex) and commutator terms in the field strength, in conjunction with the Wilson line expansion and the noncommutative phases it generates. Chern-Simons theory is such that quantum corrections to the correlators of the basic fields, and the one-particle irreducible functions are essentially trivial. While this has been explored in a covariant gauge at only one-loop in the noncommutative case (see

\(^6\) Alternatively note that [1.07] and [1.09], along with [1.06], are like the graphs we considered in [15], with one-loop vertex corrections in place of the vertex itself. Thus by reducing \( j \), we are really setting up a form of induction.
For example), and partially at two-loops here, there is no reason as yet to suspect that the well-known commutative results at higher orders do not extend to the noncommutative case. (See however [21,22] for axial gauge results involving basic fields in noncommutative Chern-Simons theory.) If we assume this to be true, then we expect to be able to effectively neglect or reduce the internal loops in our graphs, and apply the above arguments to the ones whose loops are formed solely with the Wilson line noncommutative structure to establish cancellations at an arbitrary order.

To conclude, we have explicitly shown that the order $g^5$, two-loop quantum corrections to the correlator of an open Wilson line $W(k)$, and an open Wilson line with a field strength insertion $O_{\mu\nu}(k')$ in noncommutative Chern-Simons theory cancel amongst themselves. This lends further support to the conjecture that noncommutative and commutative Chern-Simons theories are perturbatively equivalent. It would be interesting both to employ the regulator introduced in [15] and extended above to make the results presented here rigorous, as well as make explicit the heuristic argument just discussed to establish an all-orders result at least formally. We have focussed on $\langle W(k)O_{\mu\nu}(k') \rangle$ because it is the simplest nontrivial correlator of composite, gauge-invariant objects, but it would also be interesting to examine other correlators at higher orders in the theory, such as the pure three point function of open Wilson lines $\langle W(k_1)W(k_2)W(k_3) \rangle$ we studied at the lowest nontrivial order in [15]. Here the metric independence of the non-gauge fixed theory does not forbid nontrivial dependence on $k_1 \times k_2$, but equivalence with the commutative theory does; a preliminary investigation into the correlator at $O(g^6)$ has not revealed an obvious set of complete cancellations. Most importantly however, the equivalence or inequivalence of the two theories at a nonperturbative level needs to be established.

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Appendix A. Conventions and Feynman rules

The action of noncommutative Chern-Simons theory in terms of a canonically normalized gauge field is given by

$$S_{NCCS} = \frac{1}{2} \int d^3x \ e^{\mu\rho\nu} \left[ A_\mu \ast \partial_\rho A_\nu - \frac{2ig}{3} A_\mu \ast A_\rho \ast A_\nu \right], \quad (A.1)$$

while the standard (noncommutative) ghost action is given by

$$S_{\text{ghost}} = \int d^3x \ \partial^\mu \bar{c} \ast D_\mu c, \quad (A.2)$$

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where $D_\mu(c) \equiv \partial_\mu c - ig(A_\mu * c - c * A_\mu)$. Our Fourier transform convention is

$$A_\mu(x) = \int \frac{d^3 p}{(2\pi)^3} e^{-ikx} A_\mu(k).$$  \hspace{1cm} (A.3)

We use the standard covariant gauge-fixing term proportional to $(\partial \cdot A)^2$ and then take the Landau gauge, which is known to be infrared safe, at least in perturbative commutative Chern-Simons theory.

Our path ordering convention (with respect to the star product) for the open Wilson lines puts larger path parameter values on the left:

$$P_\star \exp \left[ ig \int_0^1 d\sigma \tilde{k}^\mu A_\mu(x + \xi(\sigma)) \right] = 1 + ig \int_0^1 d\sigma_1 \tilde{k}^\mu A_\mu(x + \xi(\sigma_1))$$

$$+ (ig)^2 \int_0^1 d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \tilde{k}^\mu A_{\mu_1}(x + \xi(\sigma_1)) * \tilde{k}^\nu A_{\mu_2}(x + \xi(\sigma_2)) + O(g^3), \hspace{1cm} (A.4)$$

where $\xi^\mu_1 = \xi^\nu(\sigma_i) = \tilde{k}^\mu \sigma_i \equiv k_{\nu} \theta^\nu \mu \sigma_i$. In the expansion of the pure open Wilson line, one of the path integrations at each order is redundant, so that we will use for example the fact that

$$\int d^3 x \int_0^1 d\sigma_1 \int_0^{\sigma_1} d\sigma_2 (k\theta) \cdot A(x + \xi(\sigma_1)) * (k\theta) \cdot A(x + \xi(\sigma_2)) * e^{ik\cdot x}$$

$$= \frac{1}{2} \int d^3 x \int_0^1 d\sigma (k\theta) \cdot A(x + \xi(\sigma)) * (k\theta) \cdot A(x) * e^{ik\cdot x}. \hspace{1cm} (A.5)$$

Since we always work in momentum space, the following identities, which use the momentum and index conventions discussed in section 2, will be useful for our calculations:

$$\int d^3 x \int_0^1 d\sigma_1 \int_0^{\sigma_1} d\sigma_2 A_{\mu_1}(x + \xi(\sigma_1)) * A_{\mu_2}(x + \xi(\sigma_2)) * e^{ik\cdot x}$$

$$= \frac{1}{2} \int_0^1 d\sigma_1 \int \frac{d^3 p_{11} d^3 p_{21}}{(2\pi)^3} \delta^{(3)}(p_{11} + p_{21} - k) e^{-i(k \times p_{11})\sigma_1} e^{-i(p_{21} - p_{11}) A_{\mu_1}(p_{11}) A_{\mu_2}(p_{21}), \hspace{1cm} (A.6)}$$

$$\int d^3 x \int_0^1 d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \ A_{\mu_{12}}(x + \xi(\sigma_1)) * A_{\mu_{22}}(x + \xi(\sigma_2)) * \partial_\mu A_\nu(x) * e^{ik\cdot x}$$

$$= \int_0^1 d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int \frac{d^3 p_{12} d^3 p_{22} d^3 p_{42}}{(2\pi)^9} \delta^{(3)}(p_{12} + p_{22} + p_{42} - k') \ e^{-i[(k' \times p_{12})\sigma_1 + (k' \times p_{22})\sigma_2]}$$

$$\times e^{-i[p_{22} \times (p_{42} - k')] - h(p_{42}) A_{\mu_{12}}(p_{12}) A_{\mu_{22}}(p_{22}) A_\nu(p_{42}), \hspace{1cm} (A.7)}$$

$$\int d^3 x \int_0^1 d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \ A_{\mu_{12}}(x + \xi(\sigma_1)) * A_{\mu_{22}}(x + \xi(\sigma_2)) * A_\mu(x) * A_\nu(x) * e^{ik\cdot x}$$

$$= \int_0^1 d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \int \frac{d^3 p_{12} d^3 p_{22} d^3 p_{32} d^3 p_{42}}{(2\pi)^9} \delta^{(3)}(p_{12} + p_{22} - k') \ e^{-i[(k' \times p_{12})\sigma_1 + (k' \times p_{22})\sigma_2]}$$

$$\times e^{-i[p_{32} \times (p_{42} + k') - h(p_{42}) A_{\mu_{12}}(p_{12}) A_{\mu_{22}}(p_{22}) A_\mu(p_{42}) A_\nu(p_{42}), \hspace{1cm} (A.8)}$$
where \( k \times p \equiv k \theta \cdot p = \vec{k} \cdot p = k_\mu \theta^{\mu\nu} p_\nu \).

The momentum space Feynman rules for the gauge field and ghost propagators respectively are

\[
p, \mu \quad q, \nu = (2\pi)^3 \delta^{(3)}(p + q) \varepsilon_{\mu\nu\rho} \frac{p^\rho}{p^2}
\]

(A.9)

\[
p \quad q = (2\pi)^3 \delta^{(3)}(p + q) \frac{i}{p^2},
\]

(A.10)

while the Feynman rules for the triple gauge, and the ghost-antighost-gauge vertices are given respectively by

\[
q, 2, \alpha_2 = -2ig\delta^{(3)}(\sum q_i) \sin \left( \frac{q_1 \times q_2}{2} \right) \varepsilon^{\alpha_1\alpha_2\alpha_3}
\]

(A.11)

\[
q_1, \alpha_1 \quad q_3, \alpha_3 = -2ig\delta^{(3)}(\sum q_i) \sin \left( \frac{q_1 \times q_2}{2} \right) (-iq_1)^\mu,
\]

(A.12)

where the momentum \( q_1 \) appearing in the latter rule is associated with the antighost.

We will extensively use the following identities involving the antisymmetric tensor:

\[
\varepsilon^{aAB} \varepsilon_{aCD} = \delta^A_C \delta^D_B - \delta^D_C \delta^A_B \equiv \delta^{AB}_{CD} - \delta^{AB}_{DC}
\]

(A.13)

\[
\varepsilon_{ABC} \varepsilon_{DEF} = \varepsilon_{ABD} \varepsilon_{CEF} + \varepsilon_{ABE} \varepsilon_{CFD} + \varepsilon_{ABF} \varepsilon_{CDE},
\]

(A.14)

the second of which yields

\[
\varepsilon_{\rho_a \rho_b [\mu \varepsilon^{\nu]}_{\rho_c \rho_{i j}}] = \varepsilon_{\mu \nu \rho_c} \varepsilon_{\mu \rho_a \rho_{i j}} - \varepsilon_{\nu \rho_c \rho_a} \varepsilon_{\mu \rho_{i j}}.
\]

(A.15)

Contracting with two factors of \( \tilde{k}^{\mu_i} \) gives

\[
\varepsilon_{\mu \rho_a \rho_c} \varepsilon_{\nu \rho_b \rho_{i j}} \tilde{k}^{\mu_1} \tilde{k}^{\mu_2} - (\mu \leftrightarrow \nu) = \varepsilon_{\mu \nu \rho_c} \varepsilon_{\rho_a \rho_b \rho_{i j}} \tilde{k}^{\mu_i} \tilde{k}^{\mu_j}.
\]

(A.16)

Finally, the central identity that underlies most of the cancellations between Feynman diagrams we will exhibit is given by

\[
\left[ C^\rho C_{\mu \varepsilon^{\nu}_{\rho \mu_i}} + \varepsilon_{\mu \nu \rho} C^2 \right] \tilde{k}^{\mu_i} = (k \times C) \varepsilon_{\mu \nu \rho} C^\rho.
\]

(A.17)
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