Point-curve incidences in the complex plane

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Abstract

We prove an incidence theorem for points and curves in the complex plane. Given a set of $m$ points in $\mathbb{R}^2$ and a set of $n$ curves with $k$ degrees of freedom, Pach and Sharir proved that the number of point-curve incidences is $O(m^2/k + n^{2/3} + m + n)$. We establish the slightly weaker bound $O_{\varepsilon}(m^2/k + n^{2/3} + m + n)$ on the number of incidences between $m$ points and $n$ (complex) algebraic curves in $\mathbb{C}^2$ with $k$ degrees of freedom. We introduce a new tool to the study of geometric incidences by combining polynomial partitioning with foliations. Specifically, we rely on Frobenius’ theorem on integrable distributions. We also apply various algebraic geometry machinery such as Chevalley’s upper semi-continuity theorem.

1 Introduction

Given a set $\mathcal{P}$ of points and a set $\mathcal{V}$ of geometric objects (for example, one might consider lines, circles, or planes) in a vector space $F^d$ over a field $F$, an incidence is a pair $(p, V) \in \mathcal{P} \times \mathcal{V}$ such that the point $p$ is contained in the object $V$. In incidence problems, one is usually interested in the maximum number of incidences in $\mathcal{P} \times \mathcal{V}$, taken over all possible sets $\mathcal{P}, \mathcal{V}$ of a given size. For example, the well-known Szemerédi-Trotter Theorem states that any set of $m$ points and $n$ lines in $\mathbb{R}^2$ must have $O(m^{2/3}n^{2/3} + m + n)$ incidences.

Incidence theorems have a large variety of applications. For example, in the last few years they have been used by Guth and Katz to almost completely settle Erdős’ distinct distances problem; by Bourgain and Demeter to study restriction problems in harmonic analysis; by Raz, Sharir, and Solymosi to study expanding polynomials; by Farber, Ray, and Smorodinsky to study properties of totally positive matrices.

1.1 Previous work

We will be concerned with the number of incidences between points and various classes of plane curves.

Definition 1.1. Let $\mathcal{C}$ be a set of simple plane curves and let $\mathcal{P} \subset \mathbb{R}^2$ be a set of points. We say that the arrangement $(\mathcal{P}, \mathcal{C})$ has $k$ degrees of freedom and multiplicity-type $s$ if

- For any $k$ points from $\mathcal{P}$, there are at most $s$ curves from $\mathcal{C}$ passing through all of them.

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• Any pair of curves from $\mathcal{C}$ intersect in at most $s$ points.

The current best known bound for incidences between points and general curves in $\mathbb{R}^2$ is the following (better bounds are known for some specific types of curves, such as circles and parabolas).

**Theorem 1.2** (Pach and Sharir [18]). Let $\mathcal{P}$ be a set of $m$ points in $\mathbb{R}^2$ and let $\mathcal{C}$ be a set of $n$ simple plane curves. Suppose that $(\mathcal{P}, \mathcal{C})$ has $k$ degrees of freedom and multiplicity type $s$. Then

$$I(\mathcal{P}, \mathcal{C}) = O_{k,s}(m^{\frac{k}{2k-1}}n^{\frac{2k-2}{2k-1}} + m + n).$$

If the curves are algebraic, then we can drop the requirement that the curves are simple (however, the implicit constant will now depend on the degree of the curves). This special case was proved several years earlier than Theorem 1.2, and we will state it separately here.

**Theorem 1.3** (Pach and Sharir [18]; Clarkson, Edelsbrunner, Guibas, Sharir, Welzl [5]). Let $\mathcal{P}$ be a set of $m$ points in $\mathbb{R}^2$ and let $\mathcal{C}$ be a set of $n$ algebraic curves of degree at most $D$. Suppose that $(\mathcal{P}, \mathcal{C})$ has $k$ degrees of freedom and multiplicity type $s$. Then

$$I(\mathcal{P}, \mathcal{C}) = O_{k,s,D}(m^{\frac{k}{2k-1}}n^{\frac{2k-2}{2k-1}} + m + n).$$

Less is known about point-curve incidences in the complex plane. If we add the additional requirement that pairs of curves must intersect transversely, then an analogue of Theorem 1.3 can be proved using the techniques of Solymosi-Tao from [22], although these methods introduce an $\varepsilon$ loss in the exponent. Similar bounds without the $\varepsilon$ loss in the exponent were also proved by the second author in [27]; however, in addition to the requirement that curves intersect transversely, the results of [27] have an additional restriction on the relative sizes of $\mathcal{P}$ and $\mathcal{C}$. Finally, Tóth [24] proved the important special case where the curves in $\mathcal{C}$ are lines.

Asking for the curves to intersect transversely is rather restrictive; some of the simplest cases such as incidences with circles or parabolas do not satisfy this requirement. If we do not require that pairs of curves intersect transversely, then much less is known. Very recently, Solymosi and de Zeeuw [21] proved a complex analog of Theorem 1.3 but only for the special case where the points form a lattice. This bound was already used to prove several results in the complex plane—see [19] [25]. Finally, a very recent result by Dvir and Gopi [8] considers incidences between points and lines in $\mathbb{C}^d$, for any $d \geq 3$.

### 1.2 New results

We obtain a complex analogue of Theorem 1.3, although our version introduces an $\varepsilon$ loss in the exponent.

**Theorem 1.4.** For each $k \geq 1, D \geq 1$, $s \geq 1$, and $\varepsilon > 0$, there is a constant $C = C_{k,D,s,k}$ so that the following holds. Let $\mathcal{P} \subset \mathbb{C}^2$ be a set of $m$ points and let $\mathcal{C}$ be a set of $n$ complex algebraic curves of degree at most $D$. Suppose that $(\mathcal{P}, \mathcal{C})$ has $k$ degrees of freedom and multiplicity type $s$. Then

$$I(\mathcal{P}, \mathcal{C}) \leq C(m^{\frac{k}{2k-1} + \varepsilon}n^{\frac{2k-2}{2k-1}} + m + n).$$

The new improvement is that Theorem 1.4 does not require the curves to intersect transversely; to do this we need several new ideas which are discussed below.

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1. That is, whenever two complex curves intersect at a smooth point of both curves, their tangent lines at the point of intersection are distinct.
1.3 Proof sketch

Each point of $\mathcal{P}$ can be regarded as a point in $\mathbb{R}^4$, and each curve of $\mathcal{C}$ can be regarded as a 2-dimensional variety in $\mathbb{R}^4$. Thus the problem is reduced to bounding the number of incidences between a set $\mathcal{Q}$ of points in $\mathbb{R}^4$ and a set $\mathcal{S}$ of two-dimensional surfaces in $\mathbb{R}^4$. If every pair of surfaces $S, S' \in \mathcal{S}$ intersect transversely, then the bound (1) can be obtained by using the techniques of Solymosi and Tao from [22]. However, if many pairs of surfaces $S, S'$ fail to intersect transversely, then the techniques from [22] do not apply.

Luckily, the surfaces in $\mathcal{S}$ are special—they come from complex curves in $\mathbb{C}^2$. In particular, the surfaces $S \in \mathcal{S}$ are defined by pairs of polynomials that satisfy the Cauchy-Riemann equations. As we will show below, this means that the only way that many surfaces in $\mathcal{S}$ can lie in a common low-degree hypersurface $Z$ is if the surfaces are leafs of a foliation of $Z$. Of course, if the surfaces $S \in \mathcal{S}$ are leafs of a foliation then they are disjoint, so the total number of point-surface incidences is small. Sections 2–5 are devoted to making this statement precise.

2 Tools from algebraic geometry

2.1 $\mathbb{R}$ and $\mathbb{C}$

In this paper we work over the fields $\mathbb{R}$ and $\mathbb{C}$. If $\zeta \in \mathbb{R}^d$, we define $\zeta^* \subset \mathbb{C}^d$ to be the image of $\zeta$ under the standard embedding $\mathbb{R}^d \to \mathbb{C}^d$. We also identify $\mathbb{C}$ with $\mathbb{R}^2$ (and more generally $\mathbb{C}^d$ with $\mathbb{R}^{2d}$) using the map $x + iy \mapsto (x, y)$. We use $\iota : \mathbb{C}^d \to \mathbb{R}^{2d}$ to express this map. For example, if $\zeta \in \mathbb{C}^2$, $\iota(\zeta) \in \mathbb{R}^4$, $\iota(\zeta)^* \in \mathbb{C}^4$, and $\iota(\iota(\zeta)^*) \in \mathbb{R}^8$. If $\mathcal{C}$ is a collection of curves in $\mathbb{C}^2$, we define $\iota(\mathcal{C}) = \{\iota(\gamma) : \gamma \in \mathcal{C}\}$.

If $Z \subset \mathbb{R}^d$ is a variety, let $Z^* \subset \mathbb{C}^d$ be the smallest complex variety containing $Z$; i.e., $Z^*$ is obtained by embedding $Z$ into $\mathbb{C}^d$ and then taking the Zariski closure. If $Z \subset \mathbb{C}^d$, let $Z(\mathbb{R}) \subset \mathbb{R}^d$ be the set of real points of $Z$.

2.2 Quantitative results on varieties and ideals

Let $K = \mathbb{R}$ or $\mathbb{C}$. If $X \subset K^d$ is a set, let $\overline{X}$ be the Zariski closure of $X$; this is the smallest variety in $K^d$ that contains $X$ (we do not require varieties to be irreducible). If $Z \subset K^d$ is a variety, let $I(Z)$ be the ideal of polynomials in $K[x_1, \ldots, x_d]$ that vanish on $Z$. If $I \subset K[x_1, \ldots, x_d]$ is an ideal, let $Z(I) \subset K^d$ be the intersection of the zero-sets of all polynomials in $I$. Sometimes it will be ambiguous whether an ideal $I$ is a subset of $\mathbb{R}[x_1, \ldots, x_d]$ or $\mathbb{C}[x_1, \ldots, x_d]$. To help resolve this ambiguity, we will write $Z_\mathbb{R}(I)$ or $Z_\mathbb{C}(I)$. If $P \in K[x_1, \ldots, x_d]$ is a polynomial, we abuse notation and write $Z(P)$ instead of $Z((P))$. If $I \subset \mathbb{C}[x_1, \ldots, x_d]$ is an ideal, we use $\sqrt{I} = I(Z(I))$ to denote the radical of $I$.

Definition 2.1. Let $Z \subset \mathbb{C}^d$ be an irreducible variety. We say a point $\zeta \in Z$ is generic if $\zeta \in Z \setminus Y$; here $Y$ is a proper subvariety of $Z$ that depends only on the points and curves from the statement of Theorem 1.3 and any previously defined objects.

We define a generic real point of $Z$ to be a real point of $Z \setminus Y$. If the set of real points of $Z$ is Zariski dense in $Z$, then $Z$ always contains a generic real point.

Definition 2.2. Let $I \subset K[x_1, \ldots, x_d]$ be an ideal. We define the complexity of $I$ as complexity($I$) = $\min(\deg f_1 + \ldots + \deg f_\ell)$, where the minimum is taken over all collections $\{f_1, \ldots, f_\ell\}$ with $I = (f_1, \ldots, f_\ell)$. By Hilbert's basis theorem, complexity($I$) is always finite.

Lemma 2.3 (Varieties and their defining ideals). Let $Z \subset \mathbb{C}^d$ be a variety of degree $C$. Then complexity($I(Z)$) = $O_{d,C}(1)$.
Proof. This is essentially [1] Theorem A.3]. In [1], the authors prove the weaker statement that there exists a set of polynomials \(g_1, \ldots, g_t\) such that \(\sum \deg g_j = O_{d,C}(1)\) and \(I(Z) = \sqrt{(g_1, \ldots, g_t)}\). However, a set of generators for \(\sqrt{(g_1, \ldots, g_t)}\) can then be computed using Gröbner bases (see e.g. [1] for an introduction to Gröbner bases). The key result is due to Dubé [7], which says that a reduced Gröbner basis for \((g_1, \ldots, g_t)\) can be found (for any monomial ordering) such that the sum of the degrees of the polynomials in the basis is \(O_{d,C}(1)\), where \(C' = \sum \deg g_j\). Since \(C' = O_{d,C}(1)\), we conclude that the sum of the degrees of the polynomials in the Gröbner basis is \(O_{d,C}(1)\). Once a Gröbner basis for \((g_1, \ldots, g_t)\) has been obtained, a set of generators for \(\sqrt{(g_1, \ldots, g_t)}\) can then be computed (see e.g. [10] Section 9). 

### 2.3 Regular points, singular points, and smooth points

**Definition 2.4.** Let \(X \subset \mathbb{R}^d\) be a variety of dimension \(d'\) and let \(\zeta \in X\). We say that \(\zeta\) is a smooth point of \(X\) if there is a Euclidean neighborhood \(U \subset \mathbb{R}^d\) containing \(\zeta\) such that \(X \cap U\) is a \(d'\)-dimensional smooth manifold; for example, see [11] Section 3.3. In this work we only consider smooth manifolds, and for brevity we refer to these simply as manifolds. Let \(X_{\text{smooth}}\) be the set of smooth points of \(X\); then \(X_{\text{smooth}}\) is a \(d'\)-dimensional smooth manifold.

Similarly, let \(X \subset \mathbb{C}^d\) be a variety of dimension \(d'\) and let \(\zeta \in X\). We say that \(\zeta\) is a smooth point of \(X\) if there is a Euclidean neighborhood \(U \subset \mathbb{C}^d\) containing \(x\) such that \(X \cap U\) is a \(d'\)-dimensional complex manifold. Again, let \(X_{\text{smooth}}\) be the set of smooth point of \(X\); then \(X_{\text{smooth}}\) is a \(d'\)-dimensional complex manifold.

Let \(X \subset \mathbb{C}^d\) be a variety of pure dimension \(d'\) (i.e. all irreducible components of \(X\) have dimension \(d'\)), and let \(f_1, \ldots, f_{\ell}\) be polynomials that generate \(I(X)\). We say that \(\zeta \in X\) is a regular point of \(X\) if

\[
\text{rank} \begin{bmatrix} \nabla f_1(\zeta) \\ \vdots \\ \nabla f_{\ell}(\zeta) \end{bmatrix} = d - d'.
\]

Let \(X_{\text{reg}}\) be the set of regular points of \(X\). If \(\zeta \in X\) is not a regular point of \(X\), then \(\zeta\) is a singular point of \(X\). Let \(X_{\text{sing}}\) be the set of singular points of \(X\).

**Lemma 2.5 ([17], Corollary 1.26).** Let \(X \subset \mathbb{C}^d\) be a variety of pure dimension \(d'\). Then \(X_{\text{smooth}} = X_{\text{reg}}\).

**Lemma 2.6.** Let \(X \subset \mathbb{C}^d\) be a variety of degree \(C\). Then \(X_{\text{sing}}\) is a variety of dimension strictly smaller than \(\dim(X)\), and \(\deg(X_{\text{sing}}) = O_{C,d}(1)\).

**Proof.** By Lemma 2.3 there exist polynomials \(f_1, \ldots, f_{\ell}\) such that \((f_1, \ldots, f_{\ell}) = I(X)\) and \(\sum_{i=1}^{\ell} \deg f_i = O_{C,d}(1)\). We have

\[
X_{\text{sing}} = \left\{ \zeta \in X : \text{rank} \begin{bmatrix} \nabla f_1(\zeta) \\ \vdots \\ \nabla f_{\ell}(\zeta) \end{bmatrix} < d - d' \right\}.
\]

Equation 3 shows that \(X_{\text{sing}}\) can be written as the zero locus of \(O_{\ell}(1) = O_{d,C}(1)\) polynomials, each of degree \(O_{d,C}(1)\). Thus \(X_{\text{sing}}\) is a variety of degree \(O_{d,C}(1)\). It remains to prove that \(X_{\text{sing}}\) has dimension strictly smaller than \(\dim(X)\). This property can be found, for example, in [13] Chapter I, Theorem 5.3. 

\[\square\]
2.4 Constructible sets

A constructible set is a set $X \subset \mathbb{C}^d$ of the form

$$X = \left( ((X_1 \backslash X_2) \cup X_3) \backslash X_4 \ldots \right),$$

where $X_1, \ldots, X_\ell$ are non-empty varieties in $\mathbb{C}^d$, and $\dim X_{j+1} < \dim X_j$ for each index $j$. See [12 Chapter 3] for further details. We define $\dim(X) = \dim(\overline{X}) = \dim(X_1)$.

We define the complexity of $X$ to be $\min(\deg(X_1) + \deg(X_2) + \ldots + \deg(X_\ell))$, where the minimum is taken over all representations of $X$ of the form (4). Note that this definition is not standard. However, since we are interested only in constructible sets of bounded complexity, any reasonable definition of complexity would work equally well.

Lemma 2.7. Let $d' < d$; let $Y, Z \subset \mathbb{C}^d$ and $W \subset \mathbb{C}^{d'}$ be constructible sets of complexity at most $C$. Let $\pi: \mathbb{C}^d \rightarrow \mathbb{C}^{d'}$ be the projection to the last $d'$ coordinates.

a) $\overline{Z}$ is an affine variety of degree $O_{C,d}(1)$.

b) $Y \cup Z$ and $Y \backslash Z$ are constructible sets of complexity $O_{C,d}(1)$.

c) $Y \times W \subset \mathbb{C}^{d+d'}$ is a constructible set of complexity $O_{C,d,d'}(1)$.

d) $\pi(Y)$ is a constructible set of complexity $O_{C,d}(1)$.

e) $\pi^{-1}(W)$ is a constructible set of complexity $O_{C,d}(1)$.

Proof sketch. Parts (a), (b) and (c) are straightforward from the representation (4) of $Y$ and $Z$. For part (d) see [12 Theorem 3.16] or [16 Chapter 2.6, Theorem 6]. Part (e) follows from (c). $\square$

Definition 2.8 (Constructible sets of polynomials). Let $\mathbb{C}[z_1, \ldots, z_d]_{\leq D}$ be the vector space of polynomials of degree $\leq D$. We can identify this vector space with the variety $\mathbb{C}^{D+d}$. We say a set $X \subset \mathbb{C}[z_1, \ldots, z_d]_{\leq D}$ is constructible of complexity $\leq C$ if the corresponding subset of $\mathbb{C}^{D+d}$ is constructible of complexity $\leq C$.

2.5 Dominant maps and Chevalley’s upper semi-continuity theorem

Definition 2.9. Let $X \subset \mathbb{C}^d$ be a constructible set and let $Y \subset \mathbb{C}^{d'}$ be an irreducible variety. Let $\pi: \mathbb{C}^d \rightarrow \mathbb{C}^{d'}$ be the projection to the last $d'$ coordinates. We say that $\pi: X \rightarrow Y$ is dominant if $\pi(X)$ is Zariski dense in $Y$.

Given a variety $X$, a function $f: X \rightarrow Z$ is upper semi-continuous if for every $a \in Z$ the set $\{x \in X: f(x) \geq a\}$ is Zariski closed. The following is a corollary of Chevalley’s upper semi-continuity theorem.

Theorem 2.10. Let $X \subset \mathbb{C}^d$ and $Y \subset \mathbb{C}^{d'}$ be irreducible varieties and suppose $\pi: X \rightarrow Y$ is dominant. Then at a generic (with respect to $X$ and $Y$) point $\zeta \in Y$, we have that $\pi^{-1}(\zeta)$ is a constructible set of dimension $\dim X - \dim Y$.

For example, see [12 Corollary 11.13] and the paragraph following it; for the claim that the set is constructible, see also [12 Theorem 3.16]).

\*\*The results stated in [12 Theorem 3.16] and [16 Chapter 2.6, Theorem 6] simply say that the projection of a constructible set is constructible. However, the proof is constructive and thus gives us a bound on the complexity of the projection.\*\*
Lemma 2.11. Let $X \subset \mathbb{C}^d$, $Y \subset \mathbb{C}^d$ be irreducible varieties. Suppose that $\pi: X \to Y$ is dominant and $\dim X > \dim Y$. Let $H \subset \mathbb{C}^d$ be a generic linear variety of dimension $\dim(X) - \dim(Y)$. Then $\pi: (H \cap X) \to Y$ is dominant.

Proof. The proof is based on the two following observations.

- If $A \subset \mathbb{C}^d$ is a constructible set, and if $H \subset \mathbb{C}^d$ is a generic linear variety with $\dim(H) + \dim(A) = d$, then $H \cap A \neq \emptyset$.
- Let $A \subset \mathbb{C}^d$ be an irreducible variety and fix $D \geq 1$. Then there is a number $N_0$ with the following property: If $Q \subset A$ is a finite set of points in general position (i.e., each point is selected generically with respect to $A$ and the previously chosen points) and if $|Q| \geq N_0$, then no proper subvariety of $A$ of degree at most $D$ can contain all the points in $Q$.

By Lemma 2.11(d), there exists $D \geq 1$ such that $\deg(\pi(H \cap X)) \leq D$ for any linear variety $H$ of dimension $d - (\dim X - \dim Y)$. Let $Q \subset \pi(X) \subset Y$ be a finite set of points such that any proper subvariety of $Y$ of degree at most $D$ cannot contain every point of $Q$.

By Theorem 2.10 for each $\zeta \in Q$ the inverse $\pi^{-1}(\zeta)$ is a constructible set of dimension at least $\dim(X) - \dim(Y)$. If we let $H$ be a generic (with respect to $X, Y$, and $Q$) linear variety of dimension $d - (\dim X - \dim Y)$, then $H$ intersects every set $\{\pi^{-1}(\zeta): \zeta \in Q\}$. In particular, $Q \subset \pi(X \cap H)$. But since $\deg(\pi(H \cap X)) \leq D$, this implies that $\pi(X \cap H) = Y$; that is, $\pi: (X \cap H) \to Y$ is dominant.

Lemma 2.12. Let $X \subset \mathbb{C}^d$ be a constructible set of complexity $\leq C$, and let $Y \subset \mathbb{C}^d$ be an irreducible variety. Suppose that the projection map $\pi: X \to Y$ is dominant. Then there is a constructible set $X' \subset X$ of complexity $O_{C,d}(1)$ such that $\dim(X') = \dim(Y)$ and the projection map $\pi: X' \to Y$ is dominant.

Proof. Let $Z_1, \ldots, Z_j$ be the irreducible components of $X$. Without loss of generality, we may assume that $\pi: (Z_1 \cap X) \to Y$ is dominant. We have $\deg Z_1 = O_{C,d}(1)$, and since $Z_1 \cap X$ is Zariski dense in $Z_1$, $\dim(Z_1 \cap X) = \dim Z_1$.

Consider a generic linear variety $H$ of dimension $\dim(Z_1) - \dim(Y)$. We have that $\dim(Z_1 \cap H) = \dim Y$ and $\dim((Z_1 \cap H) \cap H) < \dim Y$.

By Lemma 2.11 the projection $\pi: (Z_1 \cap H) \to Y$ is dominant. We have

$$\dim((Z_1 \cap H) \cap H) \leq \dim((Z_1 \cap H)) < \dim Y,$$

so in particular, $\pi: (Z_1 \cap X \cap H) \to Y$ is dominant. Let $X' = Z_1 \cap X \cap H$; by Lemma 2.7 this is a constructible set of complexity $O_{C,d}(1)$.

Lemma 2.13. Let $X \subset \mathbb{C}^d$, $Y \subset \mathbb{C}^{d'}$ be irreducible varieties of degree $\leq D$, where $d > d'$. Let $\pi: \mathbb{C}^d \to \mathbb{C}^{d'}$ be the projection to the last $d'$ coordinates. Suppose that $\dim(X) = \dim(Y)$, that $\pi(X) \subset Y$, and that the projection $\pi: X \to Y$ is dominant. Then there exists a proper sub-variety $X' \subset X$ of degree $O_{d,D}(1)$ so that if we define $\tilde{X} = X \setminus X'$ and $\tilde{Y} = \pi(\tilde{X})$, then $\tilde{Y}$ is dense in $Y$, and the Jacobian of $\pi: \tilde{X} \to \tilde{Y}$ is non-zero at every point $\zeta \in \tilde{X}$.

Proof. The main observation is that if $X_1 \subset \mathbb{C}^d$ and $Y_1 \subset \mathbb{C}^{d'}$ are complex manifolds, then the Jacobian of $\pi: X_1 \to Y_1$ is non-zero at the point $\zeta \in X_1$ if $\dim(T_{\zeta}) = \dim(\pi(T_{\zeta}))$. 

\[ \]
By Lemma 2.13 we can write \( I(X) = (f_1, \ldots, f_\ell) \), where \( \sum_{i=1}^\ell \deg f_i = O_{d,D}(1) \). For \( \zeta \in \mathbb{C}^d \), let \( M[f_1, \ldots, f_\ell](\zeta) \) be the \( \ell \times d \) matrix
\[
M[f_1, \ldots, f_\ell](\zeta) = \begin{bmatrix}
\nabla f_1(\zeta) \\
\vdots \\
\nabla f_\ell(\zeta)
\end{bmatrix},
\]
and let \( M'[f_1, \ldots, f_\ell](\zeta) \) be the \( \ell \times d' \) sub-matrix obtained by deleting the first \( d - d' \) columns.

If \( \zeta \in X_{\text{reg}} \), then \( M[f_1, \ldots, f_\ell](\zeta) \) has rank \( \dim(X) \), and the Jacobian of the projection \( \pi: X_{\text{reg}} \to Y \) is zero precisely if \( M'[f_1, \ldots, f_\ell](\zeta) \) has rank smaller than \( \dim(X) \). If this occurs, then \( \zeta \) is a critical point of the projection \( \pi: X_{\text{reg}} \to Y_{\text{reg}} \).

Let
\[
W = \{ \zeta \in X: \text{rank } (M'[f_1, \ldots, f_\ell](\zeta)) < \dim(X) \}.
\]

Let \( X_1 = X_{\text{reg}} \cap \pi^{-1}(Y_{\text{reg}}) \). Thus \( W \cap X_1 \) is contained in the set of critical points of \( \pi: X_1 \to Y_{\text{reg}} \), and \( \pi(W \cap X_1) \) is contained in the set of critical values. By Sard’s theorem, this set cannot be a (Zariski) dense open subset of \( Y \). But since \( \pi(W \cap X_1) \) is a constructible set, we conclude that it must be contained in a proper subvariety \( Y' \subset Y \). Since \( X_1 \) is dense in \( X \), this implies that \( \pi(W) \) is not dense in \( Y \); i.e., it is contained in a proper subvariety of \( Y \); this subvariety has degree \( O_{d,D}(1) \).

To complete the proof, let \( X' = X_{\text{sing}} \cup \pi^{-1}(Y_{\text{sing}}) \cup \pi^{-1}(Y') \).

**Lemma 2.14.** Let \( X \subset \mathbb{C}^d \) be a constructible set of complexity \( \leq C \), and let \( Y \subset \mathbb{C}^d \) be an irreducible variety of degree \( \leq D \), where \( d > d' \). Let \( \pi: \mathbb{C}^d \to \mathbb{C}^{d'} \) be the projection to the last \( d' \) coordinates. Suppose that \( \dim(X) = \dim(Y) \), that \( \pi(X) \subset Y \), and that the projection \( \pi: X \to Y \) is dominant.

Then there is a proper subvariety \( Y' \subset Y \) of degree \( O_{d,C,D}(1) \) with the following properties. Let \( \bar{Y} = Y \setminus Y' \) and let \( \bar{X} = \pi^{-1}(\bar{Y}) \). Then \( \bar{X} \) and \( \bar{Y} \) are complex manifolds, and the Jacobian of the projection \( \pi: \bar{X} \to \bar{Y} \) is non-zero at every point \( \zeta \in \bar{X} \). In particular, if we regard \( \bar{X} \) and \( \bar{Y} \) as smooth manifolds in \( \mathbb{R}^{2d} \), then \( \pi: \bar{X} \to \bar{Y} \) is a local diffeomorphism.

**Proof.** Let \( X_1 \cup \ldots \cup X_\ell \) be the irreducible components of \( \bar{X} \) of dimension \( \dim(X) \), and let \( X_0 \) be the union of all irreducible components of \( \bar{X} \) of dimension strictly less than \( \dim(X) \). For every \( 1 \leq i \leq \ell \), if \( \pi: X_i \to Y \) is dominant, apply Lemma 2.13 and let \( X'_i \) be the resulting set (that is, the set \( X' \) from Lemma 2.13).

If \( \pi: X_i \to Y \) is not dominant, let \( X'_i = \pi(X_i) \).

Let \( X^* = X \setminus \bar{X} \). Then \( X \subset \bar{X} \setminus X^* \), and \( X^* \) has dimension at most \( \dim(X) - 1 \) and degree \( O_{d,C}(1) \).

Let \( X' = \bar{X}_{\text{sing}} \cup X_0 \cup X^* \cup \bigcup_{i=1}^\ell X'_i \). Then \( X' \) has dimension at most \( \dim(X) - 1 \) and degree \( O_{C,D}(1) \). If \( \zeta \in X = \bar{X} \setminus X' \), then \( \zeta \) lies in a unique component \( X_j \) and \( \zeta \in X_j \setminus Y_j \). Thus by Lemma 2.13 the Jacobian of \( \pi: \bar{X} \to \bar{Y} \) is non-zero at \( \zeta \). Recall that \( \pi: \bar{X} \to \bar{Y} \) is a map of complex manifolds. Thus if we regard \( \iota(\bar{X}) \) and \( \iota(\bar{Y}) \) as real manifolds, then \( \iota: \bar{X} \to \bar{Y} \) is a local diffeomorphism.

**3 The Cauchy-Riemann Equations**

Let \( K = \mathbb{R} \) or \( \mathbb{C} \). Let \( u, v: K^4 \to K \) be functions. We will write \( u = u(x_1, y_1, x_2, y_2) \), \( v = v(x_1, y_1, x_2, y_2) \). If \( K = \mathbb{R} \), then we require that \( u, v \in C^2(\mathbb{R}^4) \). If \( K = \mathbb{C} \), then we require that \( u \) and \( v \) be holomorphic. We say that the pair \((u, v)\) satisfies the Cauchy-Riemann equations (CR equations) at the point \((x_1, y_1, x_2, y_2)\) if
\[ \frac{\partial u}{\partial x_k} = \frac{\partial v}{\partial y_k}, \quad \frac{\partial u}{\partial y_k} = -\frac{\partial v}{\partial x_k}, \quad k = 1, 2. \] (5)

Here the partial derivatives are either real derivatives (if \( K = \mathbb{R} \) and \( u, v \) are in \( C^2(\mathbb{R}^4) \)), or complex derivatives (if \( K = \mathbb{C} \) and \( u, v \) are holomorphic). If (5) holds at every point of \( K^4 \), then we say that the pair \((u, v)\) satisfies the CR equations.

One important property of the CR equations is that when \( K = \mathbb{R} \) and \( u \) and \( v \) are in \( C^2(\mathbb{R}^4) \), then \((u, v)\) satisfies the Cauchy-Riemann equations if and only if \( f = u + iv \) is holomorphic.

**Lemma 3.1** (Surfaces are either transverse or tangent). Let \( u, v, u', v' \in \mathbb{C}[x_1, y_1, x_2, y_2] \). Suppose that both \((u, v)\) and \((u', v')\) satisfy the CR equations at the point \( \zeta = (x_1, y_1, x_2, y_2) \in \mathbb{C}^4 \). Let \( X = Z(u) \cap Z(v) \), \( X' = Z(u') \cap Z(v') \), and suppose \( \zeta \in X_{\text{smooth}} \cap X'_{\text{smooth}} \). Then either \( T_{\zeta}(X) = T_{\zeta}(X') \), or \( T_{\zeta}(X) \cap T_{\zeta}(X') = 0 \). In particular, it is impossible to have \( \dim(T_{\zeta}(X) \cap T_{\zeta}(X')) = 1 \).

**Proof.** Suppose \((u, v)\) satisfy the CR equations at \( \zeta \). Let \((a_1, a_2, a_3, a_4) \in \mathbb{C}^4 \). If
\[
(a_1, a_2, a_3, a_4) \cdot \nabla u(\zeta) = 0 \quad \text{and} \quad (a_1, a_2, a_3, a_4) \cdot \nabla v(\zeta) = 0,
\]
then by (5), we have
\[
(-a_2, a_1, -a_4, a_3) \cdot \nabla u(\zeta) = 0 \quad \text{and} \quad (-a_2, a_1, -a_4, a_3) \cdot \nabla v(\zeta) = 0. \] (6)

Now, suppose \( T_{\zeta}(X) \cap T_{\zeta}(X') \neq 0 \), so in particular \( \dim(T_{\zeta}(X) \cap T_{\zeta}(X')) \geq 1 \). Note that \( T_{\zeta}(X) = \langle \nabla u(\zeta), \nabla v(\zeta) \rangle \perp \) (i.e., \( T_{\zeta}(X) \) is a two-dimensional subspace of \( \mathbb{C}^4 \) that is orthogonal to both \( \nabla u(\zeta) \) and \( \nabla v(\zeta) \)), and similarly for \( T_{\zeta}(X') \). Thus, if \((a_1, a_2, a_3, a_4) \in T_{\zeta}(X) \cap T_{\zeta}(X') \subset \mathbb{C}^4 \), then
\[
(a_1, a_2, a_3, a_4) \cdot \nabla u(\zeta) = 0, \quad (a_1, a_2, a_3, a_4) \cdot \nabla v(\zeta) = 0,
\]
\[
(a_1, a_2, a_3, a_4) \cdot \nabla u'(\zeta) = 0, \quad (a_1, a_2, a_3, a_4) \cdot \nabla v'(\zeta) = 0.
\]

But by (5), we must also have
\[
(-a_2, a_1, -a_4, a_3) \cdot \nabla u(\zeta) = 0, \quad (-a_2, a_1, -a_4, a_3) \cdot \nabla v(\zeta) = 0,
\]
\[
(-a_2, a_1, -a_4, a_3) \cdot \nabla u'(\zeta) = 0, \quad (-a_2, a_1, -a_4, a_3) \cdot \nabla v'(\zeta) = 0.
\]

Therefore \((-a_2, a_1, -a_4, a_3) \in T_{\zeta}(X) \cap T_{\zeta}(X')\), and thus \(\dim(T_{\zeta}(X) \cap T_{\zeta}(X')) \geq 2 \). We conclude that \( T_{\zeta}(X) = T_{\zeta}(X') \). \( \Box \)

### 3.1 Pairs of conjugate polynomials

Let \( f \in \mathbb{C}[z_1, z_2] \leq D \). If we identify \( \mathbb{C}^2 \) with \( \mathbb{R}^4 \) then the functions
\[
(x_1, y_1, x_2, y_2) \mapsto \text{Re} f(x_1 + iy_1, x_2 + iy_2),
\]
\[
(x_1, y_1, x_2, y_2) \mapsto \text{Im} f(x_1 + iy_1, x_2 + iy_2)
\] (7)

are polynomials in \( \mathbb{R}[x_1, y_1, x_2, y_2] \leq D \). We will call these polynomials \( \text{Re}[f] \) and \( \text{Im}[f] \), respectively.

**Lemma 3.2.** Let \( f \in \mathbb{C}[z_1, z_2] \) and let \( \zeta \in \mathbb{C}^2 \). If \( \nabla f(\zeta) \neq 0 \), then \( \iota(\zeta)^* \in \mathbb{C}^4 \) is a regular point of \( Z_\mathbb{C}(\text{Re}[f]) \cap Z_\mathbb{C}(\text{Im}[f]) \).
Lemma 3.3. Let $Z \subset C^4$ be an irreducible variety of degree at most $C$. Then the set
\[ Z^{inc} = \{(f, \zeta) \in \mathbb{C}[z_1, z_2]_{\leq D} \times \mathbb{C}^2 : f(\zeta) = 0, \nabla f(\zeta) \neq 0, \iota(Z_C(f)) \subset Z\} \] (8)
is a constructible set of complexity $O_{C,D}(1)$.

Proof. Define
\[ S = \{(f, \zeta) \in \mathbb{C}[z_1, z_2]_{\leq D} \times \mathbb{C}^2 : f(\zeta) = 0, \iota(\zeta) \notin Z\}. \]

Notice that $S$ is a constructible set of complexity $O_{C,D}(1)$. Let $\pi$ be the projection
\[ \pi : \mathbb{C}[z_1, z_2]_{\leq D} \times \mathbb{C}^2 \to \mathbb{C}[z_1, z_2]_{\leq D}. \]

Then
\[ Z^{inc} = \{(f, \zeta) \in \mathbb{C}[z_1, z_2]_{\leq D} \times \mathbb{C}^2 : f(\zeta) = 0, \nabla f(\zeta) \neq 0\} \setminus (\pi(S) \times \mathbb{C}^2). \]

\[
4 \quad \text{Foliations and Frobenius' Theorem}
\]

Definition 4.1. We will often refer to the point $0 \in \mathbb{R}^d$. Sometimes, however, the dimension of the underlying vector space may be ambiguous. Where it is helpful, we write $0_{\ell}$ to remind the reader that the point 0 belongs to the vector space $\mathbb{R}^\ell$.

Let $M \subset \mathbb{R}^N$ be a $d$-dimensional real manifold and let $0 < d' < d$. Let $\mathcal{F}$ be a set of disjoint $d'$-dimensional manifolds, such that $\bigcup_{L \in \mathcal{F}} L = M$. We say that $\mathcal{F}$ is a foliation of $M$ if for every $\zeta \in M$, there exists an open Euclidean neighborhood $U \subset \mathbb{R}^N$ containing $\zeta$, an open Euclidean neighborhood $V \subset \mathbb{R}^N$ containing 0, and a diffeomorphism $\varphi : U \to V$ such that (i) $\varphi(M \cap U) = (\mathbb{R}^d \times \{0_{N-d}\}) \cap V$, and (ii) for each $L \in \mathcal{F}$, $\varphi(L \cap U) = (\mathbb{R}^{d'} \times \{\alpha_L\} \times \{0_{N-d}\}) \cap V$, where $\alpha_L \in \mathbb{R}^{d-d'}$. The manifolds of $\mathcal{F}$ are called the leaves of the foliation. A nice introduction to foliations can be found in [14] Chapter 19.

Let $M$ be a $d$-dimensional real manifold and let $TM$ be the tangent bundle of $M$. Let $E \subset TM$ be a $(d + d')$-dimensional sub-manifold of $TM$. We say that $E$ is a $d'$-dimensional sub-bundle of $TM$ if for every $\zeta \in M$, we have $(\{\zeta\} \times T_\zeta(M)) \cap E = \{\zeta\} \times V$, where $V$ is a $d'$-dimensional vector subspace of $\mathbb{R}^d$. We will call this subspace $E(\zeta) \subset \mathbb{R}^d$. Intuitively, the vector space $E(\zeta)$ varies smoothly as the base-point $\zeta$ changes.

If $E$ is a sub-bundle of $TM$ and $X : M \to TM$ is a vector field, we say that $X$ takes values in $E$ if $X(\zeta) \in E$ for all $\zeta \in M$. Given two smooth vector fields $X, Y : M \to TM$, their Lie bracket $[X, Y]$ is the smooth vector field that satisfies
\[ [X, Y](f) = X(Y(f)) - Y(X(f)) \quad \text{for each } f \in C^\infty(M). \]

Notice that this is a coordinate-free definition. If $M \subset \mathbb{R}^N$ and we choose coordinates $x_1, \ldots, x_N$ for $\mathbb{R}^N$, then an equivalent definition that is based on coordinates is
\[
[X, Y](f) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left( X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}. \quad (9)
\]
A sub-bundle $E \subset T(M)$ is integrable (or involutive) if for every two vector fields $X, Y$ that take values in $E$, the Lie bracket $[X, Y]$ takes values in $E$. 

We say that $E$ arises from a foliation of $M$ if there exists a foliation $\mathcal{F}$ of $M$ with the following property: if $\zeta \in M$ and $L \subset M$ is the leaf passing through $\zeta$, then $E(\zeta) = T\zeta(L)$. Similarly, we say that the foliation $\mathcal{F}$ corresponds to $E$.

**Theorem 4.2 (Frobenius).** $E \subset TM$ is integrable if and only if $E$ arises from a foliation of $M$.

A proof of Theorem 4.2 can be found, for example, in [14, Chapter 19]. Similarly, the following lemma can be found in [14, Theorem 19.21].

**Lemma 4.3.** Let $M$ be a $d$-dimensional manifold and let $\mathcal{F}$ be a foliation of $M$ into $d'$-dimensional leafs. Let $E \subset TM$ be the sub-bundle that arises from $\mathcal{F}$. Let $A$ be a maximal connected $d'$-dimensional sub-manifold of $M$, such that for every point $\zeta \in A$, $T\zeta(A) = E(\zeta)$. Then $A$ is a leaf of the foliation $\mathcal{F}$.

**Corollary 4.4.** Let $M$ be a $d$–dimensional manifold and let $E \subset TM$ be a $d'$-dimensional sub-bundle. Suppose that for every $\zeta \in M$, there exists a $d'$-dimensional connected sub-manifold $W(\zeta) \subset M$ such that $T\zeta(W) = E(\zeta)$. Suppose furthermore that we have the following “consistency condition”:

- If $\zeta, \zeta' \in M$ such that $\zeta' \in W(\zeta)$, then $T\zeta'(W(\zeta')) = T\zeta'(W(\zeta)) = E(\zeta')$.

Then the sub-manifolds $W(\zeta)$ form a foliation of $M$.

**Proof.** We will show that $E$ is integrable. For the convenience of the reader, we will work in coordinates. We can assume that $M \subset \mathbb{R}^N$ for some $N \geq d$. Fix $\zeta \in M$ and let $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ be a diffeomorphism with the following properties:

- $\varphi(\zeta) = 0_N$ (see Definition 4.11 for the definition of $0_N$).
- There exists a neighborhood $U \subset \mathbb{R}^N$ of $\zeta$ and a neighborhood $V \subset \mathbb{R}^N$ of $0_N$ such that $\varphi(U \cap M) = \mathbb{R}^d \times \{0_{N-d}\} \cap V$.
- For the same neighborhoods $U, V$, we have $\varphi(U \cap W(\zeta)) = \mathbb{R}^d \times \{0_{N-d'}\} \cap V$.

That is, in a neighborhood of $\zeta$, the image of $M$ under $\varphi$ is a copy of $\mathbb{R}^d$ and the image of $W(\zeta)$ is a copy of $\mathbb{R}^d \subset \mathbb{R}^d$.

Let $X, Y$ be vector-fields in $E$. Let $X', Y'$ be the push-forward of $X, Y$ under $\varphi$. Recall that $X'(0), Y'(0) \in T\mathbb{R}^d \subset T\mathbb{R}^d$. It suffices to show that

$$[X', Y'](0) \in T_0(\mathbb{R}^d \times \{0_{N-d'}\}) \quad (10)$$

(e.g., see [14, Corollary 8.31]). By the consistency condition, for every point $\zeta' \in W(\zeta) \cap U$ we have $E(\zeta') = T\zeta'(W(\zeta))$. This means that for every point $y \in \mathbb{R}^d \times \{0_{N-d'}\} \cap V$, we have

$$X'(y) \in T_y(\mathbb{R}^d \times \{0_{N-d'}\}) \quad \text{and} \quad Y'(y) \in T_y(\mathbb{R}^d \times \{0_{N-d'}\}).$$

This in turn implies that for every point $y \in \mathbb{R}^d \times \{0_{N-d'}\} \cap V$, we have

$$[X', Y'](y) \in T_y(\mathbb{R}^d \times \{0_{N-d'}\}).$$

In particular, $(10)$ holds and $E$ is integrable. By Theorem 4.2 there is a foliation of $M$ that is consistent with $E$. By Lemma 4.3 the leafs of $E$ are precisely the manifolds $\{W(x), x \in M\}$. \ □
5  Real varieties that contain many complex curves

**Theorem 5.1.** Let $D \geq 1$. Let $P \in \mathbb{R}[x_1, y_1, x_2, y_2]$ be an irreducible polynomial of degree at most $C$. Then there exists a variety $Y \subset \mathbb{C}^4$ of dimension $\leq 2$ and degree $O_{C,D}(1)$ such that for every $\zeta \in \mathbb{Z}_\mathbb{R}(P) \setminus Y(\mathbb{R})$, there is at most one irreducible complex curve $\gamma \subset \mathbb{C}^2$ of degree $\leq D$ such that $\zeta \in \iota(\gamma_{\text{reg}})$ and $\iota(\gamma) \subset \mathbb{Z}_\mathbb{R}(P)$.

**Proof.** Let $Z = \mathbb{Z}_\mathbb{R}(P)^*$; by [26, Lemma 7], $Z$ is irreducible, and by Lemma 3.3, $Z^{\text{inc}}$ is constructible of complexity $O_{C,D}(1)$. Let $\pi$ be the projection $Z^{\text{inc}} \to Z$. Then $\zeta^* \in \pi(Z^{\text{inc}})$ if $\zeta \in \mathbb{Z}_\mathbb{R}(P)$ and there exists a curve $\gamma \subset \mathbb{C}^2$ of degree at most $D$ such that $\zeta \in \iota(\gamma_{\text{reg}})$ and $\iota(\gamma) \subset \mathbb{Z}_\mathbb{R}(P)$ (and thus $\iota(\gamma)^* \subset Z$). Without loss of generality, we can assume that $\gamma$ is irreducible. If not, we can replace it by the (unique) irreducible component containing $\zeta$.

If $\dim(\pi(Z^{\text{inc}})) \leq 2$, let $Y = \pi(Z^{\text{inc}})$ and we are done. If $\dim(\pi(Z^{\text{inc}})) = 3$, then $\pi: Z^{\text{inc}} \to Z$ is dominant. By Lemma 2.12, there exists a constructible set $W \subset Z^{\text{inc}}$ of complexity $O_{C,D}(1)$ such that $\dim(W) = \dim(Z)$ and $\pi: W \to Z$ is dominant. By Lemma 2.14, there exists a set $Y \subset Z$ of dimension $\leq 2$ and complexity $O_{C,D}(1)$ with the following properties: Let $\tilde{Z} = Z \setminus Y$ and let $W' = W \cap \pi^{-1}(\tilde{Z})$. Then $\pi: W' \to \tilde{Z}$ is a local diffeomorphism. We will show that the conclusions of Theorem 5.1 hold with this choice of $Y$.

5.1  Defining the foliation.

Let $\text{Gr}(2,4; \mathbb{C})$ be the Grassmannian of 2-dimensional vector spaces in $\mathbb{C}^4$. This is a smooth algebraic variety (i.e., it has no singular points). For each pair $(f, \zeta) \in \mathbb{C}[z_1, z_2]_{\leq 2} \times \mathbb{C}^2$ with $f(\zeta) = 0$, $\nabla f(\zeta) \neq 0$, we associate the element $\langle (\nabla \text{Re}[f](\iota(\zeta))^*), (\nabla \text{Im}[f](\iota(\zeta))^*), (\nabla \text{Im}[f](\iota(\zeta))^*) \rangle \in \text{Gr}(2,4; \mathbb{C})$. By Lemma 3.2, $\langle (\nabla \text{Re}[f](\iota(\zeta))^*), (\nabla \text{Im}[f](\iota(\zeta))^*) \rangle$ is indeed a 2-dimensional vector space. Consider the map $\psi: \tilde{Z}^{\text{inc}} \to \text{Gr}(2,4; \mathbb{C})$ given by

$$
(f, \zeta) \mapsto \langle (\nabla \text{Re}[f](\zeta)), (\nabla \text{Im}[f](\zeta)), (\nabla \text{Im}[f](\zeta)) \rangle.
$$

(11)

This is a $C^\infty$ map of manifolds. Let $\sigma: \text{Gr}(2,4; \mathbb{C}) \to \text{Gr}(4,8; \mathbb{R})$ be the map $V \subset \mathbb{C}^4 \mapsto \iota(V) \subset \mathbb{R}^8$. Thus $\sigma \circ \psi: \tilde{Z}^{\text{inc}} \to \text{Gr}(4,8; \mathbb{R})$.

Note that if $(f, \zeta) \in W'$ then $\sigma \circ \psi(f, \zeta) \subset T_{\iota(\zeta)} \iota(\tilde{Z})$; here $\iota(\tilde{Z}) \subset \mathbb{R}^8$ is a six-dimensional manifold, and $T_{\iota(\zeta)} \iota(\tilde{Z})$ is its tangent space at the point $\iota(\zeta)^* \subset \mathbb{R}^8$. We now define a sub-bundle of $T(\iota(\tilde{Z}))$ by defining sub-bundles of $T(U)$ for a collection of open sets $\{U\}$ that form an open cover of $\iota(\tilde{Z})$.

Fix a point $\zeta \in \iota(\tilde{Z}) \subset \mathbb{R}^8$ and let $U \subset \iota(\tilde{Z})$ be a Euclidean open set containing $\zeta$ such that for every connected component $V \subset \pi^{-1}(U) \subset \tilde{Z}^{\text{inc}}$, $\pi: V \to U$ is a diffeomorphism. Since $\pi$ is a local diffeomorphism, we can always find an open set $U$ with this property. Fix a connected component $V \subset \pi^{-1}(U)$ and set $\rho_V = (\pi|_V)^{-1}$ (where $\pi|_V$ is $\pi$ with its domain restricted to $V$). Thus $\rho_V: U \to V$ is a diffeomorphism. Consider the map $\alpha_V: U \to \text{Gr}(4,8; \mathbb{R})$ given by

$$
\alpha_V: \zeta \mapsto \sigma \circ \psi \circ \rho_V(\zeta).
$$

(12)

**Lemma 5.2.** The sub-bundle given by (12) is well-defined. That is, $E(\zeta)$ does not depend on which connected component $V \subset \pi^{-1}(U)$ is chosen.
Proof. Consider two points \((f, \zeta), (f', \zeta)\) \(\in Z^{\text{inc}}\). By Lemma 3.1 either
\[
(T_{i(\zeta)}(\mathbb{Z}_C(\text{Re}[f]) \cap \mathbb{Z}_C(\text{Im}[f]))) \cap (T_{i(\zeta')}((\mathbb{Z}_C(\text{Re}[f']) \cap \mathbb{Z}_C(\text{Im}[f']))) = 0,
\]
or
\[
T_{i(\zeta)}(\mathbb{Z}_C(\text{Re}[f]) \cap \mathbb{Z}_C(\text{Im}[f])) = T_{i(\zeta')}(\mathbb{Z}_C(\text{Re}[f']) \cap \mathbb{Z}_C(\text{Im}[f'])).
\]
The former is impossible, since both \(i(T_{i(\zeta)}(\mathbb{Z}_C(\text{Re}[f]) \cap \mathbb{Z}_C(\text{Im}[f])))\) and \(i(T_{i(\zeta')}(\mathbb{Z}_C(\text{Re}[f']) \cap \mathbb{Z}_C(\text{Im}[f'])))\) are four-dimensional sub-spaces of the six-dimensional tangent space \(T_{i(\zeta)}(V)\).

Therefore, if \(V, V' \subset \pi^{-1}(U)\), we have \(\alpha_V = \alpha_{V'}\).

Lemma 5.2 implies that for every \(\zeta \in \iota(\hat{Z})\) we have a uniquely defined sub-bundle \(E_\zeta \subset T(U_\zeta)\), where \(U_\zeta\) is a sufficiently small neighborhood of \(\zeta\). The sets \(\{U_\zeta\}\) cover \(\iota(\hat{Z})\), and for all pairs of points \(\zeta, \zeta' \in \iota(\hat{Z})\), the sub-bundles satisfy the consistency condition
\[
E_\zeta \cap TU_{\zeta'} = E_{\zeta'} \cap TU_\zeta.
\]
This means that there exists a sub-bundle \(E \subset T(\iota(\hat{Z}))\) such that for every \(\zeta \in \hat{Z}, E \cap T(\iota(U_\zeta)) = E(\zeta)\).

By Corollary 4.4, \(E\) corresponds to a foliation. Thus, if \(\zeta \in \hat{Z}\) and \((f, \zeta) \in W'\), then in a small (Euclidean) neighborhood \(U\) of \(\iota(U(\zeta))\), \(\iota(U(\mathbb{Z}_C(f))) \cap U\) is the unique leaf of the foliation passing through \(\iota(U(\zeta))\). In particular, if \((f', \zeta) \in Z^{\text{inc}}\), then \(\iota(U(\mathbb{Z}_C(f))) \cap U\) and \(\iota(U(\mathbb{Z}_C(f')) \cap U\) intersect in a set that is relatively open in the induced topology of \(\iota(U(\mathbb{Z}_C(f)))\) (and also relatively open in the induced topology of \(\iota(U(\mathbb{Z}_C(f'))\)). Thus \(\iota(U(\mathbb{Z}_C(f)))\) and \(\iota(U(\mathbb{Z}_C(f'))\) intersect in a set that contains \(\iota(\zeta')\) and is Zariski dense in \(\iota(U(\mathbb{Z}_C(f)))\) and \(\iota(U(\mathbb{Z}_C(f'))\). In particular \(\mathbb{Z}_C(f)\) and \(\mathbb{Z}_C(f')\) must intersect in a one-dimensional curve (i.e. the intersection cannot be finite). If \(\mathbb{Z}_C(f)\) and \(\mathbb{Z}_C(f')\) are irreducible, we conclude \(\mathbb{Z}_C(f) = \mathbb{Z}_C(f')\), i.e. there is a unique irreducible complex curve \(\gamma\) of degree \(\leq D\) such that \(\zeta\) is a regular point of \(\gamma\).

6 Proof of Theorem 1.4

We are now ready to prove Theorem 1.4. For the reader’s convenience, we restate it here

**Theorem 1.4.** For each \(k \geq 1, D \geq 1, s \geq 1, \) and \(\epsilon > 0\), there is a constant \(C = C_{\epsilon, D, s, k}\) such that the following holds. Let \(\mathcal{P} \subset \mathbb{C}^2\) be a set of \(m\) points and let \(\mathcal{C}\) be a set of \(n\) complex algebraic curves of degree at most \(D\). Suppose that \((\mathcal{P}, \mathcal{C})\) has \(k\) degrees of freedom and multiplicity type \(s\). Then
\[
I(\mathcal{P}, \mathcal{C}) \leq C\left(m^{\frac{k}{D^2} + \epsilon} n^{\frac{2k}{D^2} + 1} + m + n\right).
\]

**Proof.** We will make crucial use of the Guth-Katz polynomial partitioning technique from [11, Theorem 4.1].

**Theorem 6.1.** Let \(\mathcal{P}\) be a set of \(m\) points in \(\mathbb{R}^d\). For each \(r \geq 1\), there exists a polynomial \(P\) of degree \(\leq r\) such that \(\mathbb{R}^d \setminus \mathbb{Z}(P)\) is a union of \(O(r^d)\) connected components (cells), and each cell contains \(O(m/r^d)\) points of \(\mathcal{P}\).

Since the curves of \(\mathcal{C}\) have \(k\) degrees of freedom, the Kővári-Sós-Turán theorem (e.g., see [15, Section 4.5]) implies
\[
I(\mathcal{P}, \mathcal{C}) = O(m n^{1/k} + n).
\]

When \(m = O(n^{1/k})\), this implies the bound
\[
I(\mathcal{P}, \mathcal{C}) = O(n)\]

Thus, we may assume that
\[
n = O\left(m^k\right)\quad (13)\]
We will prove by induction on \(m+n\) that

\[
I(\mathcal{P}, \mathcal{C}) \leq \alpha_1 m^{\frac{k}{2k-1} + \varepsilon} n^{\frac{2k-2}{2k-1}} + \alpha_2 (m + n),
\]

where \(\alpha_1, \alpha_2\) are sufficiently large constants. The base case where \(m + n\) is small can be handled by choosing sufficiently large values of \(\alpha_1\) and \(\alpha_2\). In practice, we will bound \(I(\iota(\mathcal{P}), \iota(\mathcal{C}))\). Since \(\iota : \mathbb{C}^2 \to \mathbb{R}^4\) is a bijection, \(I(\mathcal{P}, \mathcal{C}) = I(\iota(\mathcal{P}), \iota(\mathcal{C}))\).

### 6.1 Partitioning \(\mathbb{R}^4\)

Let \(P\) be a partitioning polynomial of degree \(r\), as described in Theorem 6.1. The constant \(r\) will be chosen later. The asymptotic relations between the various constants in the proof are

\[
2^{1/\varepsilon} \ll r \ll \alpha_2 \ll \alpha_1.
\]

By definition, we have \(\deg P \leq r\). Let \(\Omega_1, \ldots, \Omega_\ell\) be the cells of the partition; we have \(\ell = O(r^4)\). Let \(\mathcal{V}_i\) be the set of varieties from \(\iota(\mathcal{C})\) that intersect the interior of \(\Omega_i\) and let \(\mathcal{P}_i\) be the set of points \(p \in \mathcal{P}\) such that \(\iota(p) \in \Omega_i\). Let \(m_i = |\mathcal{P}_i|, m' = \sum_{i=1}^{k} m_i, \text{ and } n_i = |\mathcal{V}_i|\). By definition, for every \(1 \leq i \leq \ell\) we have \(m_i = O(m/r^4)\).

By \cite{ref} Theorem A.2], every variety from \(\mathcal{V}\) intersects \(O(r^2)\) cells of \(\mathbb{R}^4 \setminus \mathcal{Z}(P)\). Therefore, \(\sum_{i=1}^{\ell} n_i = O(nr^2)\). Combining this with Hölder’s inequality implies

\[
\sum_{i=1}^{\ell} n_i^{2k-2} = O \left( (nr^2)^{\frac{2k-2}{2k-1} r^{\frac{k}{2k-1}}} \right) = O \left( n^{\frac{2k-2}{2k-1} r^{\frac{k}{2k-1}}} \right).
\]

By the induction hypothesis, we have

\[
\sum_{i=1}^{\ell} I(\mathcal{P}_i, \mathcal{V}_i) \leq \sum_{i=1}^{\ell} \left( \alpha_1 m_i^{\frac{k}{2k-1} + \varepsilon} n_i^{\frac{2k-2}{2k-1}} + \alpha_2 (m_i + n_i) \right)
\]

\[
\leq O \left( \alpha_1 m^{\frac{k}{2k-1} + \varepsilon} r^{\frac{k}{2k-1}} - \frac{4k}{2k-1} \sum_{i=1}^{\ell} n_i^{2k-2} \right) + \sum_{i=1}^{\ell} \alpha_2 (m_i + n_i)
\]

\[
\leq O \left( \alpha_1 r^{-\varepsilon} m^{\frac{k}{2k-1} + \varepsilon} n^{\frac{2k-2}{2k-1}} + \alpha_2 (m' + O(nr^2)) \right).
\]

By \cite{ref}, we have \(n^{\frac{k}{2k-1}} = O \left( m^{\frac{k}{2k-1}} \right)\), which in turn implies \(n = O \left( m^{\frac{k}{2k-1} + \frac{2k-2}{2k-1}} \right)\). Thus, when \(\alpha_1\) is sufficiently large with respect to \(r\) and to \(\alpha_2\), we have

\[
\sum_{i=1}^{\ell} I(\mathcal{P}_i, \mathcal{V}_i) = O \left( \alpha_1 r^{-\varepsilon} m^{\frac{k}{2k-1} + \varepsilon} n^{\frac{k}{2k-1}} \right) + \alpha_2 m'.
\]

By taking \(r\) to be sufficiently large with respect to \(\varepsilon\) and to the constant of the \(O\)-notation, we have

\[
\sum_{i=1}^{\ell} I(\mathcal{P}_i, \mathcal{V}_i) \leq \frac{\alpha_1}{2} m^{\frac{k}{2k-1} + \varepsilon} n^{\frac{2k-2}{2k-1}} + \alpha_2 m',
\]

i.e.,

\[
I(\iota(\mathcal{P}) \setminus \mathcal{Z}_\mathbb{R}(P), \iota(\mathcal{C})) \leq \frac{\alpha_1}{2} m^{\frac{k}{2k-1} + \varepsilon} n^{\frac{2k-2}{2k-1}} + \alpha_2 m'.
\]
6.2 Incidences on the partitioning hypersurface

It remains to bound incidences with points that are on the partitioning hypersurface \(Z(P)\). To do this, we will make use of the point-curve bound from Theorem 5.1.

**Lemma 6.2.** Let \(P \subset \mathbb{C}^2\). Let \(C\) be a collection of complex curves of degree at most \(C_0\) such that \((P, C)\) has \(k\) degrees of freedom and multiplicity type \(s\). Let \(Y \subset \mathbb{C}^4\) be an algebraic variety of degree at most \(C_1\). Suppose that for each \(\gamma \in C\), \(\iota(\gamma) \cap Y(\mathbb{R})\) is a real algebraic set of dimension at most one. Then

\[
I(\iota(P) \cap Y(\mathbb{R}), \ i(C)) = O(|P|^{(k)}|C|^{2(2k-2)/2k-1}) + |P| + |C|, \tag{15}
\]

where the implicit constant depends on \(k\), \(s\), \(C_0\), and \(C_1\).

**Remark 1.** See [1] for the definition of the dimension of a real algebraic variety. Informally, a real algebraic variety \(V\) has dimension at most one if there does not exist a subset \(V' \subset V\) that is homeomorphic to the open two-dimensional cube \((0,1)^2 \subset \mathbb{R}^2\).

**Proof.** Let \(\pi: \mathbb{R}^4 \to \mathbb{R}^2\) be a generic linear transformation (see Definition 2.1). Then for each \(\gamma \in C\), \(\pi(\iota(\gamma) \cap Y(\mathbb{R})) \subset \mathbb{R}^2\) is the zero set of a non-zero polynomial of degree \(O_{C_0,C_1}(1)\) (see [22] Section 5.1); These sets are either plane curves or collections of points.

Let \(\Gamma = \{\pi(\iota(\gamma) \cap Y(\mathbb{R})) : \gamma \in C\}\). Then \(\Gamma\) is a set of (not necessarily irreducible) plane algebraic curves, and \(\{\pi(\iota(P)), \Gamma\}\) has \(k\) degrees of freedom and multiplicity type \(O_{s,C_0,C_1}(1)\).

By Theorem 6.3

\[
I(\pi(\iota(P)), \Gamma) = O(|P|^{(k)}|C|^{2(2k-2)/2k-1}) + |P| + |C|, \tag{16}
\]

where the implicit constant depends on \(k\), \(s\), \(C_0\), and \(C_1\). But since each incidence in \(I(\iota(P) \cap Y(\mathbb{R}), \ i(C))\) appears as an incidence in \(I(\pi(\iota(P)), \Gamma)\), (16) implies (15).

We are now ready to bound the incidences involving points lying on \(Z(\mathbb{R})(P)\). Factor \(P = P_1 \cdots P_\ell\) into irreducible components, and let \(Z_1 \cdots Z_\ell\) be the corresponding irreducible components of \(Z(\mathbb{R})(P)\). Let \(P_0 = \iota(P) \cap Z(\mathbb{R})(P)\) and \(m_0 = |P_0| = m - m'\).

For each irreducible component \(Z_i\), let \(P'_i = (P_0 \cap Z_i) \cup \bigcup_{j < i} P'_j\). Let \(C_i = \{\gamma \in C : \iota(\gamma) \subset Z_i\}\). Note that for each \(\gamma \in \Gamma\), we have \(\gamma_{\text{sing}} = O_D(1)\). Thus

\[
\left|\{\left(p, \gamma\right) : \gamma \in \gamma_{\text{sing}}\}\right| = O_D(n).
\]

Apply Theorem 5.3 to each surface \(Z(P_j)\), and let \(Y_j \subset \mathbb{C}^4\) be the resulting variety. By Theorem 5.1 for each index \(1 \leq i \leq \ell\), we have

\[
\left|\{\left(p, \gamma\right) \in P'_i \times C_i : \iota(p) \in Z(\mathbb{R})(P_i) \setminus Y_i(\mathbb{R}), \ p \in \gamma_{\text{smooth}}\}\right| \leq |P'_i|, \tag{17}
\]

Since \(Y_i\) is an algebraic variety of degree \(O_{\ast}(1)\) and dimension at most two, at most \(O_{\ast}(1)\) varieties of the form \(\iota(\gamma)\) can be contained in \(Y_i(\mathbb{R})\). Thus for each \(1 \leq i \leq \ell\),

\[
\left|\{\left(p, \gamma\right) \in P'_i \times C_i : \iota(p) \in Y_i(\mathbb{R}), \ p \in \gamma_{\text{smooth}}, \ i(\gamma) \subset Y_i(\mathbb{R})\}\right| = O_{\ast}\left(|P'_i|\right). \tag{18}
\]

It remains to control the size of the sets

\[
\{(p, \gamma) \in P'_i \times C : \iota(\gamma) \not\subset Z(\mathbb{R})(P_i)\}
\]

and

\[
\{(p, \gamma) \in P'_i \cap Y_i \times C : \iota(\gamma) \not\subset Y_i(\mathbb{R})\}.
\]
By Lemma 6.2, these sets have size

$$O(|\mathcal{P}_i|^{k/(2k-1)} n^{(2k-2)/(2k-1)} + |\mathcal{P}_i| + n).$$

(19)

Combining (17), (18), and (19), we have that for each index $i$,

$$I(\mathcal{P}_i, \iota(\gamma)) = O(|\mathcal{P}_i'|^{k/(2k-1)} n^{(2k-2)/(2k-1)} + |\mathcal{P}_i'| + n).$$

Since there are $\ell \leq r = O(1)$ indices, we have

$$I(\mathcal{P}_0, \iota(\mathcal{C})) = O(m_0^{k/(2k-1)} n^{(2k-2)/(2k-1)} + m_0 + n).$$

Taking $\alpha_1, \alpha_2$ to be sufficiently large with respect to the constant of the $O$-notation, we have

$$I(\iota(\mathcal{P}) \cap \mathcal{Z}_\mathbb{R}(P), \iota(\mathcal{C})) \leq \frac{\alpha_1}{2} m^{k/(2k-1)} n^{(2k-2)/(2k-1)} + \alpha_2(m_0 + n).$$

(20)

Combining (20) and (14) completes the induction.

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