Producing flow in “racetrack” atom circuits by stirring at zero and non-zero temperature

Benjamin Eller, 1 Olatunde Oladehin, 1 Daniel Fogarty, 1 Clayton Heller, 1 Charles W. Clark, 2 and Mark Edwards 1, 2

1Department of Physics, Georgia Southern University, Statesboro, GA 30460–8031 USA
2Joint Quantum Institute, National Institute of Standards and Technology and the University of Maryland, Gaithersburg, MD 20899, USA

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We present a study of how macroscopic flow can be produced in Bose–Einstein condensate confined in a “racetrack” potential by stirring with a wide rectangular barrier. This potential consists of two half-circle channels separated by straight channels of length L and is a ring potential if L = 0. We present the results of a large set of simulations where racetrack condensates stirred with a barrier under varying conditions of barrier height, stir speed, racetrack geometry, and temperature. We found that stirring was readily able to produce circulation in ring and non–ring geometries but that the exact amount of flow produced was complicated. We therefore also studied the mechanism by which flow was produced in the stirring process. We found that circulation was induced by the swap of a vortex/anti–vortex pair that was initially created by backflow of the condensate in the region of depressed density by the barrier. When the barrier strength reached a critical value a number of these vortex–swap events occurred in rapid succession so that flow speed of the circulation produced was enough to exceed the stir speed of the barrier. Flow that was initially localized in the vortices involved in the vortex swap was converted into macroscopic flow around the racetrack by pairs of disturbances each generated during the vortex swap. Each pair consisted of a vortex/anti–vortex pair moving in the anti–stir direction and a compression wave moving in the stir direction. This picture of the mechanism for making flow will enable the design of stirring schedules that create a desired amount of flow.

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I. INTRODUCTION

Recent advances in the optical manipulation of neutral atoms 1–6 have sparked experimental and theoretical interest in systems of ultracold Bose–Einstein–condensed (BEC) atoms that are roughly analogous to electronic circuits. The difference is that the current in such ultracold–atom systems refers to the motion of neutral atoms rather than electrons. These systems are often referred to as “atom circuits” and their study is part of the emergent field of atomtronics. Interest in atom circuits derives from their potential for use in devices such as quantum simulators and sensors 6.

Atom circuits with lumped elements 6, 7 have been proposed. These include atom circuits with diodes 5 and transistors 9. Atom–circuit analogs of multiple–element electronic circuits, such as the RLC circuit have also been studied both experimentally 10, 12 and the theoretically 13.

The parameters of atom circuits can be precisely controlled and probed and so they can be used as quantum simulators. Quantum simulators are systems used to model the behavior of other analogous systems whose physical parameters are difficult to measure and/or control. Examples include modeling the early universe 14, the sonic analog of Hawking radiation 15–17, electrons in lattice potentials 18, and the fractional quantum Hall effect 19.

Atom–circuit parameters are also sensitive to the environment and can thus be used as quantum sensors. One example is rotation sensing as Sagnac interferometers 20, 21 and as analogs of Superconducting Quantum Interference Devices where rotation takes the place of magnetic flux 22, 23. Many implementations of this type include a Bose–Einstein condensate gas confined in ring geometry 24, 30.

All atom circuits require neutral–atom current for their operation. Atom circuits suitable for applications such as rotation sensing mentioned above will need to be able to make repeated measurements over time. It is likely that each measurement will result in the destruction of the initial flow state requiring that the flow be recreated. Clearly it will be advantageous to be able to create a given amount of smooth flow in the condensate on–demand. In order to be able to design atom circuits for applications a detailed understanding of how to produce such flow will be essential.

In this paper we study how current can be produced in a particular class of atom circuits by stirring. The atomtronic systems that we will focus on consist of a Bose–Einstein–condensed atomic gas confined by laser light to a horizontal plane in which an arbitrary two–dimensional potential can be created. We only considered 2D potentials that take the form of a closed channel in the shape of a racetrack. The racetrack channel takes the shape of two semi–circumferential endcaps separated by straightaways of length L, as illustrated in Fig. 1 and described more fully below.

Several methods have been used to create flow in BECs confined in ring potentials. These include transferring orbital angular momentum from a Laguerre–Gauss laser.
beams to the trapped atoms and imprinting a phase on the gas atoms using a light pulse with a tailored intensity pattern. The most popular method to-date for producing flow has been stirring the gas with a blue-detuned laser. Here we present a study of the amount of, nature of, and mechanism for creating quantized flow in racetrack BECs at zero and non-zero temperature by stirring. In Section II we present the results of a systematic set of simulations where racetrack BECs are stirred. These simulations were performed for different racetrack lengths, stirring speeds, maximum barrier energy heights, and temperatures. In Section III we describe the nature of the flow created at the end of the stirring process and present a detailed account of how stirring produces flow. In particular we discuss how the circulation of localized vortices is transferred to macroscopic flow around the ring. Finally we present conclusions in Section IV.

II. SURVEY STUDY OF FLOW PRODUCTION BY STIRRING

We conducted a survey study of how much flow was produced by stirring a Bose–Einstein condensate with a blue-detuned laser. The condensate initial state was assumed to be a thermal-equilibrium state at temperature, $T$, where total (condensate plus non-condensate) number of atoms was held fixed. We also assumed that the condensate was formed in a racetrack potential of length $L$. The survey consisted of a series of simulations in which the BEC was stirred with a rectangular barrier whose stir speed, $v_b$, was constant and whose energy height was ramped up to a maximum, $V_{p,\text{max}}$, held constant, and then ramped off. Each simulation in the series was uniquely specified by these parameters: $T$, $L$, $v_b$, and $V_{p,\text{max}}$.

A. Survey Study Characteristics

Here we describe the details of the ultracold–atom system modeled in the simulation, the zero– and finite–temperature models assumed to govern system behavior, the common characteristics of each simulation, and the ranges of the parameters that were varied. We begin with the system characteristics.

The BEC was assumed to be created in a potential having strong harmonic confinement in the vertical direction (z axis) and a “racetrack” potential in the $z = 0$ horizontal plane. The racetrack channel is made up of two half-circular annuli having fixed inner radius, $R_i = 12 \, \mu m$, and outer radius, $R_o = 36 \, \mu m$, separated by parallel straightaways of variable length $L$. The radius of the midline track of the $L = 0 \, \mu m$ (ring) racetrack is thus $R = 24 \, \mu m$.

After its creation, the condensate is stirred with a 2D rectangular barrier potential that always stays oriented perpendicular to the midline of the racetrack, is twice as wide as the racetrack channel width, and sweeps around the racetrack at constant linear speed. The height of the barrier is also time–dependent and this dependence is discussed below.

The full potential can be written as the sum of the harmonic vertical light–sheet potential, the racetrack potential, $V_{\text{RT}}$, and the stirring barrier potential, $V_{\text{stir}}$:

$$V_{\text{ext}}(r, t) = \frac{1}{2} M \omega_z^2 z^2 + V_{\text{RT}}(x, y) + V_{\text{stir}}(x, y, t),$$

where $M$ is the mass of a condensate atom (sodium in this study) and $\omega_z/2\pi = 320$ Hz is the frequency of the vertical harmonic confinement. The strength of the vertical confinement was not varied. The racetrack potential is illustrated in Fig. 1 and the mathematical details are presented in Appendix A.

The behavior of the condensate in zero–temperature simulations was assumed to follow the Gross–Pitaevskii equation (GPE). For non–zero temperature simulations we used the Zaremba–Nikuni–Griffin (ZNG) model.

In the ZNG model the system is assumed to have a condensate and a non-condensate. The behavior of the condensate is described by a condensate wave function, $\Phi(r, t)$, and the non-condensate is assumed to be an interacting gas described by a single–particle distribution function, $f(p, r, t)$.

The single–particle distribution function is defined so that $\int f(p, r, t) d^3r d^3p/(2\pi\hbar)^3$ is the number of particles at time $t$ having position, $r$, and momentum, $p$. The
FIG. 2. Two arrays of final phase distributions in the \( z = 0 \) plane. Each small circle with rainbow coloring displays the phase distribution at the end of a single simulation. The number above refers to the \( V_{p, \text{max}} \) for that simulation in units of the chemical potential. The heading L\_00\_TR\_09\_T\_000 appearing at the top of the left panel refers to simulations where the racetrack length was \( L = 0 \) m, the stirring speed was \( v_b = 339.3 \) m/s (TR\_09), and the temperature of the initial thermal equilibrium states was \( T = 0 \) nK. The characteristics of the right–panel distributions are the same except that \( T = 150 \) nK. The colormap of the plots runs from phase = \(-\pi\) (blue) to phase = \(0\) (green) to phase = \(2\pi\) (red).

non–condensate density, \( \tilde{n}(r, t) \) can thus be calculated as

\[
\tilde{n}(r, t) = \int \frac{d^3p}{(2\pi\hbar)^3} f(p, r, t). \tag{2}
\]

The condensate wave function follows a generalized Gross–Pitaevskii equation (GGPE) \([42]\)

\[
i\hbar \frac{\partial}{\partial t} \Phi(r, t) = \left( \hat{H}_0 + 2g\tilde{n}(r, t) - iR(r, t) \right) \Phi(r, t). \tag{3}
\]

The term \( \hat{H}_0 = \frac{-\hbar^2}{2M} \nabla^2 + V_{\text{ext}}(r, t) + gn_c(r, t) \) is the GPE Hamiltonian, \( g \) defines the strength condensate atom–atom interactions, \( n_c(r, t) = |\Phi(r, t)|^2 \) is the condensate density, \( \tilde{n}(r, t) \) is the non–condensate density and \( R(r, t) \) is a local source/sink term that describes particle exchange between condensate and non–condensate.

The single-particle distribution function evolves according to a quantum Boltzmann equation (QBE)

\[
\frac{\partial f}{\partial t} - \nabla_r U_{\text{eff}} \cdot \nabla_p f + \frac{P}{M} \cdot \nabla_r f = C_{12}[f, \Phi] + C_{22}[f], \tag{4}
\]

where \( U_{\text{eff}}(r, t) = V_{\text{trap}}(r, t) + 2g(n_c(r, t) + \tilde{n}(r, t)) \) is an effective potential felt by the non–condensate atoms. The terms \( C_{12} \) and \( C_{22} \) describe how collisions affect the rate of change of \( f(p, r, t) \) \([42]\). In our simulation study we neglected collisions.

In our survey study simulations were conducted at \( T = 0 \) nK for seven different racetrack lengths. These ranged from \( L = 0 \) m (the ring case) up to \( L = 60 \) m in increments of \( 10 \) m. Simulations was also conducted at non–zero temperatures \( T = 100, 150, \) and \( 200 \) nK. In these simulations only lengths \( L = 0, 30, \) and \( 60 \) m were considered.

For each pair of temperature and length simulations were carried out at four different stirring speeds. The stirring speed of the barrier was kept constant throughout the entire simulation. The four speeds were chosen to be multiples of the stir speed necessary to make one revolution around the midline track of the \( L = 0 \) m (ring) racetrack in four seconds. This speed is \( v_{\text{TR}} = 2\pi R/4 \approx 37.7 \) m/s and speeds can be expressed in units of the number of “total revolutions” (TR) that would be made around the ring racetrack at that speed. The speeds used in our simulations were \( v_b = 3v_{\text{TR}}, 6v_{\text{TR}}, 9v_{\text{TR}}, \) and \( 12v_{\text{TR}} \). In our scheme for identifying the many simulations performed these speeds were denoted as TR\_03, 06, 09, and 12, respectively.

For each triple of parameters, \((T, L, v_b)\), a series of 75 simulations was conducted in which the maximum barrier height was varied between \( V_{p, \text{max}} = 0.50 \mu \) and \( V_{p, \text{max}} = 2.00 \mu \) in increments of \( 0.02 \mu \) where \( \mu \) is the chemical potential of the initial condensate. Thus a simulation is uniquely defined by specifying the four quantities: \((T, L, TR, V_{p, \text{max}})\). Approximately 4800 simulations were carried out.

In all survey simulations the energy height of the barrier was varied in the same way. The barrier was ramped linearly from zero to \( V_{p, \text{max}} \) in 500 ms, it was held constant for another 500 ms, and then ramped off linearly over a further 500 ms. In the zero–temperature simulation the system was allowed to evolve for a further 2500 ms and in the simulation at finite–temperature the extra evolution time was only 500 ms. Thus the total simulation times were 4000 ms for zero temperature and 2000 ms for non–zero temperature.

The information about the state of the condensate in each simulation was collected at 100 equally spaced times during system evolution. This information included the optical density, the \( x \) and \( y \) components of the velocity distribution in the \( z = 0 \) plane, the phase distribution in the \( z = 0 \) plane, and the \( z = 0 \) condensate density distribution. Of particular interest was the condensate phase distribution at the end of the simulation.
B. Survey Study Results

The signature of the final flow produced by stirring a condensate confined in a racetrack potential is the presence of a phase gradient in the condensate after the stirring is finished. The phase gradient can be inferred by noting the number of times the phase runs through the \(2\pi\) cycle along a path that follows the midline track of the racetrack potential. In this section we present the major findings of the study regarding the dependence on the amount of flow created on barrier strength, stir speed, and on trap geometry.

Figure 2 illustrates typical characteristics of flow production by stirring. This figure shows two arrays of false-color plots of the final phase distributions for two series of simulations. In all of these simulations the racetrack length was \(L = 0 \mu m\) (a ring potential), the stir speed of the barrier was \(v_b = 339.3 \mu m/s\) (or TR=09), and the temperature of the thermal equilibrium initial state was either \(T = 0\) nK (left panel) or \(T = 150\) nK (right panel). Each individual phase distribution plot is labeled by the value of \(V_{p,\text{max}}\) expressed in units of the chemical potential of the initial state. The colormap of the plots runs from phase = \(-\pi\) (blue) to phase = 0 (green) to phase = \(\pi\) (red). The number of units of quantized flow can be easily determined by counting the number of red spots appearing the phase distribution pattern.

The figure shows that stirring with increasingly strong barriers in the ring geometry while keeping the stir speed fixed can create flow when the barrier strength is greater than a critical fraction of the initial-state chemical potential. Furthermore the amount of flow produced saturates with increasing barrier height.

Comparing the left panel at zero temperature with the right panel at \(T = 150\) nK shows that temperature has little effect on the amount of flow produced other than that the onset of flow appears at a critical barrier height that is slightly less than the zero-temperature one. The most probable reason for this is that the total number of atoms in the system was kept fixed across all simulations in the study. Hence the \(T = 150\) nK condensate has fewer atoms than at \(T = 0\) nK.

An accurate estimate of the number of condensate atoms at this temperature can be obtained by using the ZNG model to compute the condensate fraction versus temperature for fixed total number atoms for various racetrack geometries. These curves can then be fitted using the function:

\[
\frac{N_c}{N} = 1 - \left(\frac{T}{T_c}\right)^a,
\]

with \(T_c\) and \(a\) as fitting parameters. We have done this and find the following values. For \(L = 0 \mu m, T_c = 264.4\) nK and \(a = 2.097\); for \(L = 30 \mu m, T_c = 249.3\) nK and \(a = 2.558\); and for \(L = 60 \mu m, T_c = 235.8\) nK and \(a = 2.464\). Thus, for \(L = 30 \mu m\), the condensate fraction at \(T = 150\) nK is about 73% or roughly 364,000 atoms as compared to the 500,000 atom condensate at \(T = 0\) nK. More details about the ZNG initial states can be found in Appendix B.

Stirring can also produce flow when the confining channel shape is not a ring. This is shown in Fig. 3 which contains two phase-distribution plot arrays for the \(L = 30 \mu m\) racetrack. The conditions for these simulations are the same as for those in Fig. 2 except for the length of the racetrack. Comparing these two figures we can see that more flow is produced in the longer-length racetrack.

The non-ring racetrack case does exhibit one interesting feature not present in the ring case shown in Fig. 2. This is that the amount of flow produced in not a monotonic function of the barrier strength. This is illustrated in the second row of the left panel of Fig. 3. Starting with the leftmost graph on this row, where \(V_{p,\text{max}}/\mu = 0.70\), the number of units of flow produced in this and the following graphs progress as \(4, 4, 5, 4, 5, 4, 4, 4\). Oscillations persist in a seemingly random pattern as the barrier strength increases. Similar oscillations of flow versus barrier strength are present at non-zero temperature as can be seen in the right-hand panel of this figure.

The oscillations in the flow produced as the barrier strength is increased is further illustrated in Fig. 4. Figure 4(a) displays the final flow versus \(V_{p,\text{max}}\) for each of
the four temperatures considered in the survey study for the \( L = 0 \mu \text{m} \) racetrack. Figure 4(b) shows the flow produced for the \( L = 30 \mu \text{m} \) racetrack. It is important to note that all winding number values are integers and the different temperature curves in the figure have been given a slight vertical offset for clarity.

The \( L = 0 \mu \text{m} \) racetrack case shows no oscillations except at the highest temperature. The \( L = 30 \mu \text{m} \) case on the other hand shows many oscillations at all temperatures. These graphs also show that there is no flow until \( V_{p, \text{max}}/\mu \) exceeds a critical value. The flow then rises rapid to a plateau after which it oscillates around an average value. This average value can be estimated by determining the number of units of flow speed needed to reach the speed of the stirrer. One unit of average flow speed can be approximated as \( \bar{h}/M \) times the phase gradient around the racetrack midline:

\[
v_{\text{flow}} = \frac{\bar{h}}{M} \left( \frac{2\pi}{2\pi R + 2L} \right)
\]

where \( R = (R_i + R_o)/2 \) is the average radius of the racetrack endcaps. The stir speed in units of the flow speed, \( v_b/v_{\text{flow}} \), appears as the solid black line in Figs. 4(a) and (b). This ratio provides a rough estimate of the amount of flow that can be produced by stirring.

The full story of the amount of flow produced is more complicated and depends on the details of the time dependence of the barrier turn-on and the shape of the racetrack. These things can be understood by studying the mechanism of how stirring produces flow within the Gross–Pitaevskii model. We discuss this in the next section.

III. HOW STIRRING PRODUCES FLOW

In this section we describe how stirring the condensate with a barrier produces flow within the Gross–Pitaevskii model. It is well-established that flow is accompanied by production and motion of vortices in the condensate \[31, 32, 44, 45\] and that is the case here. Here we address how and where vortices form, how they move thereafter. We also discuss how circulation in the form of a localized vortex is converted into macroscopic flow around the entire racetrack. Finally we describe what conditions lead to the final amount of flow that results from stirring.

A. Creation of a single unit of flow

Each unit of flow that is produced begins by the creation of a vortex, whose circulation is the same as that of the stirring, near the outside of the channel in the barrier region. An accompanying antivortex is also created near the inside of the channel in the barrier region. These vortices form during the stirring process because, as the height of the moving barrier increases, the condensate density at the barrier location decreases causing a backflow (i.e., flow in the anti-stir direction) form in the region of depressed density.

This backflow is illustrated in Figs. 5(a) and (e). In the full figure we have plotted a series of snapshots of the velocity distribution shortly before until shortly after the creation of the first unit of flow when stirring the ring racetrack condensate. The conditions for this simulation were the same as for those shown in Fig. 2 and where \( V_{p, \text{max}} = 0.98\mu \approx 57.0 \text{nK} \). The top row shows the velocity distributions of a ring condensate in the barrier
FIG. 5. Velocity distributions of the $L = 0 \mu m$ ring (top row in red) and $L = 30 \mu m$ racetrack (bottom row in blue) condensates near the barrier region during the ramp–up of the stirring. Stir direction is counterclockwise. (a),(e) backflow develops in the barrier region; (b),(f) vortex/anti–vortex swap; (c),(g) two disturbances form: vortex/anti–vortex move in the anti–stir direction and compression wave moves in the stir direction; (d),(h) disturbances propagate away at the average speed of sound.

region at various times while the bottom row shows the same thing for a racetrack condensate.

In Fig. 5(e) it is easy to see that the backflow speed is greatest at the inner and outer edges of the channel. At these points the racetrack plus barrier potential has its largest value. As the barrier moves the condensate in front of the barrier must migrate to the back of the barrier. The atoms at the inner and outer channel edges must travel farther and thus move faster to avoid the regions of high potential. This is the mechanism whereby vortices are formed by stirring with a barrier that is significantly wider than the stirred condensate.

When the height of the barrier reaches a critical value, the vortex migrates from the outside to the inside of the channel as can be seen in Figs. 5(b) and (f). At the same time antivortex also moves from the inside to outside the channel. This event marks the onset of a jump in the circulation as computed by

$$\Gamma \equiv \oint_{C} \mathbf{v} \cdot d\mathbf{l} = \frac{\hbar}{M} (2\pi n)$$

where $C$ is the circular path along the midline track of the racetrack oriented in the stir direction and $n$ is the winding number (i.e., the number of times the phase cycles through $2\pi$ while traversing the path $C$.)

Shortly after this vortex swap two disturbances are generated. The first is the vortex/antivortex pair, located on the inside and outside of the channel respectively, move away from the barrier in the anti–stir direction. This vortex pair causes atoms on the anti–stir side of the barrier to flow in the stir direction. The second disturbance is a compression wave that propagates away from barrier region in the stir direction. This compression wave also moves atoms in the stir direction. Both disturbances move at a speed that is approximately the local speed of sound ($c(r) = \sqrt{g_m(r)/m}$) averaged over the cross section of the condensate. As we will see these disturbances are the mechanism by which the localized circulation in the form of a vortex is converted into macroscopic flow around the entire racetrack.

B. Final flow production: ring case

In this subsection we describe the dynamics of flow production in the racetrack during the stirring process and address the question of what total amount of flow is produced. Here we address the $L = 0 \mu m$ ring racetrack case. The reasons for considering the ring first is that the ring flow dynamics is simpler than for the racetrack. It will also enable us to identify effects that are due solely to the ring and those that are caused by the extra complications of the racetrack.

In Fig. 6(a) the blue curve shows the winding number around the midline for the ring case illustrated in Fig. 5 as a function of time during the stirring. The vertical axis on the left side of the graph is measured in units of the quantized flow speed. The red curve indicates the barrier
height normalized to its maximum value. The cyan curve depicts the speed of the stirring barrier in units of the quantized flow speed around the midline track.

The behavior of the circulation depicted here is simple: below a critical value of the barrier height there is no circulation, at the critical value three units of flow are created in rapid succession. With each new unit of flow, the velocity of the stirring barrier relative to the flowing condensate decreases by one unit of flow speed. The figure shows that speed of the flowing condensate overtakes or nearly matches the speed of the barrier. In this case backflow that developed when stirring a stationary condensate becomes a forward flow. Thus the behavior described earlier that led to the creation of the new units of flow can be reversed and units flow of can be lost.

After the three units of circulation are created there are three vortex/anti–vortex pairs traveling in the anti–stir direction and three compression waves traveling in the stir direction. The behavior of these disturbances during the stirring is depicted in Fig. 4. The top panel in this figure is a topographical plot where the color represents the condensate density. The horizontal axis is the time elapsed since the beginning of the stirring and the vertical axis locates points along the midline track by their arc length as measured counterclockwise from the six o’clock position of the ring (for reference, see Fig. 5A)). The bottom panel has the same layout except that the color depicts the component of the local condensate velocity tangent to the midline track.

In the density plot (top panel) the main feature is the path of the barrier which appears as a series of diagonal (upper left to lower right) stripes. It is important to note that the top and bottom of the panel both represent the same location in the condensate. Thus when the leftmost diagonal strip hits the bottom at around 300 ms, it immediately reappears at the top again. The stripe color evolves from a beige color at $t = 0$ ms representing the bulk condensate density to dark blue representing zero density as the barrier strength ramps up until it return to the beige color as barrier ramps off again.

Also present are light–brown and dark–brown stripes. Three dark/light–brown stripe pairs emanate from the barrier path in rapid succession starting at around $t = 325$ ms (compare with Figs. 5(d),(h)). Each dark–brown stripe represents a disturbance were the density is greater than the bulk density and that moves in the stir direction at the speed of sound. The dark–brown stripes represent compression–wave disturbances that circle the ring many times during the stirring period.

The light–brown stripes represent the vortex/anti–vortex disturbances. These also circle the ring many times during the stir period but move in the anti–stir direction. Both of these disturbances also appear as yellow stripes in the tangential–velocity plot shown in the bottom panel. Here we can see that all of the stripes represent an increase in the tangential velocity component above the initial zero value. The blue color of the barrier stripe indicates a backflow but we can see that, each time one of the disturbances (of either kind) sweeps through the barrier region, the (blue) backflow turns into a (red) forward flow. Note that during the ramp–off period of the barrier ($t > 1000$ ms) the latticework of stripes seen in the bottom panel smears out to a nearly even average value.

The general GPE mechanism for flow production in the ring by stirring with a rectangular barrier can thus be summarized as follows. The stirring barrier both moves and increases in strength. This generates a backflow in the region of depressed density. This backflow is fastest at the inner and outer channel edges in this region since the channel plus barrier potential is greatest at these locations. This flow causes a vortex/anti–vortex pair to form at the outer and inner edges, respectively. Eventually these two vortices swap locations and this generates...
FIG. 7. Topographic spacetime plots of the density (top) and the tangential velocity component along the racetrack midline (bottom) of the \( L = 0 \mu m \) (ring) racetrack BEC during the stirring process. The dark stripe indicates the path of the stirring barrier and the steeper yellow stripes appearing in the density and velocity plots represent the vortex/anti–vortex and compression–wave disturbances shown in Fig. 5(d) that convert circulation confined near a localized vortex into into macroscopic flow around the racetrack.

a moving vortex (now on the inner channel edge)/anti–vortex (now at the outer edge) pair that moves away from the barrier in the anti–stir. At the same times a compression wave moves away from the barrier in the stir direction. These disturbances both move at the average speed of sound.

This creation of a pair of disturbances repeats in rapid succession until the flow generated overtakes the speed of the stirring barrier. The generated disturbances cycle rapidly around the ring and thereby convert the large, localized circulation into evenly distributed flow. If the generated flow is larger than the barrier speed, the backflow in the barrier region becomes a forward flow and this can cause loss of a unit of flow. Thus the circulation can oscillate during the stirring period and the final flow amount will depend on how long the stirring period lasts.

All of these features are present when flow is created in the non–ring racetrack case. However, there are some features which only take place for \( L \neq 0 \) racetrack potentials. We consider this case next.

C. Final flow production: racetrack case

The circulation as a function of time for a non–ring \( (L = 30 \mu m) \) case is exemplified in Fig. 6(b) where all other conditions are the same as the ring case discussed in the previous section. This graph shows that, as in the ring case, the circulation (shown as the blue curve) is zero until a succession of vortex–swap events produces enough flow so that it is greater than the barrier stir speed (shown as the cyan horizontal line in the figure).

In this case, five units of flow exceed the barrier speed by almost a full flow speed unit. When the disturbance pair generated by the first vortex swap travel around the racetrack and come back and sweep through the barrier region they cause a forward flow to develop in the barrier region. This can be seen in the middle panel of Fig. 8 which displays a color–map plot of the component of the condensate velocity along the midline track. If we follow the path of the barrier from \( t = 0 \) we first encounter the five vortex–swap events. Up to this point, the color of this track is blue indicating a backflow. Shortly after the fifth vortex swap there is a red patch corresponding to the time when the disturbances from the first swap event return to the barrier region.

At the time when the disturbances from the first swap event return to the barrier region (after the fifth vortex swap) a red patch appears corresponding to forward flow. During this red patch at about \( t = 375 \) ms the circulation drops from five to four units. See Fig. 6(b). In this way the circulation can oscillate around the number of units that makes the flow closest to the stir speed of the barrier.

Another circulation–changing mechanism that is only present in the non–ring racetrack case occurs when the barrier transitions from straight parts of the racetrack to curved parts or vice–versa. The times when the barrier is on straight or curved parts are indicated in Fig. 6(b) by the red– and black–colored curve that depicts the barrier height. The graph is colored red for times when the barrier is on the curved parts of the racetrack and black–colored when it is on the straightaways. Careful examination of the circulation graph shows that, when the barrier transitions from curved to straight (red to black) race-
track parts, the circulation increases by one unit. When the barrier transitions from straight to straight parts the circulation decreases by one unit. We also note that this only happens when the barrier strength is above a certain strength.

IV. SUMMARY

We have presented a study of flow production by stirring Bose–Einstein condensates confined in atomtronic racetrack potentials. We performed a series of simulations under conditions in which the racetrack geometry, initial–state temperature, stir speed, and maximum barrier height were varied. The study also included an investigation into the mechanism of how flow is produced under the Gross–Pitaevskii model.

We found that stirring is an effective way of creating flow and that there is no difficulty in creating smooth flow in a condensate confined in a non–ring potential. We also found that flow was readily created when stirring systems initially at finite temperature and that the main effect was a decrease in the condensate atom number when the total number of atoms was held fixed.

The final amount of flow created by stirring these systems depended on the racetrack geometry, the temperature of the initial state, the stir speed, and the maximum energy height of the stirring barrier. We found that no flow was produced until the maximum strength of the stirring barrier exceeded a critical value. Above this value generally the number created was such that the speed of flow created was nearest the stir speed but could vary by one up or down depending on the stirring schedule.

Flow is produced in the stirring process by a vortex/anti–vortex pair that form on the outside and inside of the channel, respectively, due to condensate backflow that developed in the barrier region. Circulation is generated when these vortices swap positions. This is accompanied by the creation of two disturbances, the vortex/anti–vortex pair move away from the barrier in the anti–stir direction and a compression wave that moves in the stir direction. Both of these disturbances promote flow in the stir direction.

When the barrier exceeds a critical strength during the stirring a number of vortex swap events occur in rapid succession each producing a pair of counter propagating disturbances that repeatedly move around the racetrack at approximately the average speed of sound. These disturbances serve to convert localized circulation into macroscopic flow around the racetrack. The number of swap events is such that the speed of the flow produced overtakes the speed of the stirring barrier.

The circulation around the racetrack can oscillate because, when one or more of these disturbances encounter the barrier region, the backflow is converted into a forward flow. In this case it is possible to have a vortex swap in the opposite sense as described above and the total circulation can be decreased by one.

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Appendix A: Racetrack and Barrier potentials

The full potential used in simulating the stirring of a racetrack Bose–Einstein condensate is given by

\[ V_{\text{tot}}(x,y,t) = \frac{1}{2} M \omega_0^2 z^2 + V_{\text{RT}}(x,y) + V_{\text{stir}}(x,y,t). \quad (A1) \]

The first term represents the vertical harmonic confinement used to restrict the gas to a quasi–two–dimensional horizontal plane. The second term is the racetrack potential that confines the condensate to a racetrack–shaped channel within this plane. The last term is the potential of the stirring barrier. We assume that only the first two terms are present for the purposes of defining the initial state.

The racetrack potential is written as a sum of step–up and step–down functions using hyperbolic tangents as follows.

\[
V_{\text{RT}}(x,y) = V_{\text{rt}} \left\{ \frac{1}{2} \tanh \left( \frac{\rho(x,y) - R_0}{\sigma} \right) + \frac{1}{2} \tanh \left( \frac{R_1 - \rho(x,y)}{\sigma} \right) + \tanh \left( \frac{R_2 - R_1}{2\sigma} \right) \right\},
\]

where \( R_1 = 12 \mu m \) and \( R_2 = 36 \mu m \) are the inner and outer radii of the semicircular endcaps. The factor \( \sigma = 24 \mu m \) measures the steepness of the step functions. The last hyperbolic tangent term is present above so that the minimum value of the potential is zero.

The factor \( \rho(x,y) \) places the jump–up and jump–down sites of the potential thus defining the location of the channel. It is defined as

\[
\rho(x,y) = \begin{cases} \sqrt{(x - L/2)^2 + y^2} & x > L/2 \\ \sqrt{(x + L/2)^2 + y^2} & x < -L/2 \\ |y| & |x| \leq L/2 \end{cases}
\]

where \( L \) is the length of the straightaways.

The stir potential is a 2D rectangular barrier whose center coordinates, orientation, and energy height can all have arbitrary time dependence. The actual potential is most expressed in terms of step–up and step–down functions defined as

\[
V_{\text{up}}(x,x_{up},\sigma) = \frac{1}{2} \left[ 1 + \tanh \left( \frac{x - x_{up}}{\sigma} \right) \right]
\]

\[
V_{\text{dn}}(x,x_{dn},\sigma) = \frac{1}{2} \left[ 1 + \tanh \left( \frac{x_{dn} - x}{\sigma} \right) \right]
\]

where \( x_{up} \) and \( x_{dn} \) denote the places where the step functions equal one–half and \( \sigma \) is the steepness of the step.

Using these functions we can write the stir potential as

\[
V_{\text{stir}}(x,y,t) = V_p(t) \left\{ V_{\text{up}}(x_p(x,y,t), -L_p/2, \sigma) \times V_{\text{dn}}(x_p(x,y,t), L_p/2, \sigma) \times V_{\text{up}}(y_p(x,y,t), -W_p/2, \sigma) \times V_{\text{dn}}(y_p(x,y,t), W_p/2, \sigma) \right\}
\]

where \( x_p \) and \( y_p \) are barrier coordinates

\[
x_p(x,y,x_c(t),\theta_p(t)) = (x - x_c(t)) \cos(\theta_p(t)) + (y - y_c(t)) \sin(\theta_p(t))
\]

\[
y_p(x,y,x_c(t),\theta_p(t)) = -(x - x_c(t)) \sin(\theta_p(t)) + (y - y_c(t)) \cos(\theta_p(t)).
\]

Here \( x_c(t) \) and \( y_c(t) \) are the time–dependent barrier center coordinates and \( \theta_p(t) \) is the time–dependent angle that the long dimension of the rectangle makes with the \( x \) axis. The parameters \( L_p = 48 \mu m \) and \( W_p = 3 \mu m \) are the length and width of the barrier, respectively. The barrier steepness is \( \sigma = 0.3 \mu m \).

The barrier center coordinates follow the midline track of the racetrack are parameterized using the arc length, \( s \), which is measured from the left end of the bottom straightaway:

\[
s(t) = s_0 + v_b t \mod s_{\text{total}} \quad (A5)
\]

where \( v_b \) is the stir speed, \( s_0 = L + \pi R/2 \) is the start point of the barrier stirring, and \( s_{\text{total}} = 2L + 2\pi R \) is total arc length of the midline track and where \( R = (R_0 + R_1)/2 \).

The center coordinates are written in terms of the arc length as

\[
x_c(s) = \begin{cases} s - \frac{L}{2} & 0 \leq s < L \\ \frac{L}{2} + R \sin \left( \frac{s L}{R} \right) & L \leq s < s_1 \\ \frac{L}{2} + \pi R - s & s_1 \leq s < s_2 \\ \frac{L}{2} - R \sin \left( \frac{s - s_{\text{total}} + \pi R}{R} \right) & s_2 \leq s < s_{\text{total}} \end{cases}
\]

and

\[
y_c(s) = \begin{cases} -R & 0 \leq s < L \\ -R \cos \left( \frac{s L}{R} \right) & L \leq s < s_1 \\ R & s_1 \leq s < s_2 \\ R \cos \left( \frac{s - s_{\text{total}} + \pi R}{R} \right) & s_2 \leq s < s_{\text{total}} \end{cases}
\]

Here \( s_1 = s_{\text{total}}/2 \) and \( s_2 = s_{\text{total}} - \pi R \).

The time dependence of the orientation angle is given by

\[
\theta_p(s) = \begin{cases} -\frac{\pi}{2} & 0 \leq s < L \\ -\frac{\pi}{2} + \frac{s - L}{R} & L \leq s < s_1 \\ \frac{\pi}{2} & s_1 \leq s < s_2 \\ \frac{\pi}{2} + \frac{s - s_{\text{total}} + \pi R}{R} & s_2 \leq s < s_{\text{total}} \end{cases}
\]

This dependence orients the barrier so that it is always perpendicular to the midline of the racetrack.

Finally the dependence of the energy height of the barrier on time is written as

\[
V_p(t) = \begin{cases} \frac{(t/T_1)V_{p_{\max}}}{T_1} & 0 \leq t < T_1 \\ \frac{V_{p_{\max}}}{T_2} & T_1 \leq t < T_2 \\ (3 - t/T_1)V_{p_{\max}} & T_2 \leq t < T_3 \\ 0 & t \geq T_3 \end{cases}
\]

where \( T_1 = 500 \text{ ms}, T_2 = 1000 \text{ ms}, \) and \( T_3 = 1500 \text{ ms} \). This ramps the barrier linearly up to its maximum value, \( V_{p_{\max}} \), over a time \( T_1 \), keeps it constant at this value for another time interval \( T_1 \), and ramps it down linearly to zero over yet another time \( T_1 \), and is zero thereafter.
FIG. 9. ZNG condensate fraction versus temperature for three different racetrack geometries. The blue + symbols indicate initial-state condensate fraction as computed by the ZNG model while the solid red line is a fit to the function $N_c/N = 1 - (T/T_c)\alpha$. (a) $L = 0 \, \mu m$, $T_c = 264.4 \, nK$, $\alpha = 2.697$; (b) $L = 30 \, \mu m$, $T_c = 249.3 \, nK$, $\alpha = 2.558$; and (c) $L = 60 \, \mu m$, $T_c = 235.8 \, nK$, $\alpha = 2.464$.

Appendix B: ZNG initial states

Initial states for the ZNG model are thermal equilibrium states defined by the temperature, $T$, the total number of atoms in the system, $N$, the external potential, $V_{\text{ext}}(r)$ (here vertical harmonic plus racetrack), and the atom–atom interaction strength, $g$. The result of the calculation of the ZNG initial state is a condensate wave function, $\Phi_0(r)$, and a non–condensate density, $\tilde{n}_0(r)$. From these, the number of condensate atoms, $N_c$ and the chemical potential, $\mu_0$ can be obtained.

The iterative method we used to compute these quantities was to start with an initial guess that the non–condensate density was zero, so that $N_c = N$, and solve Eq. 3 with $R$ and $\tilde{n}$ set to zero. This yielded a condensate wave function. This wave function was then used to construct a first guess at the single–particle distribution function. In thermal equilibrium, this function has the form

$$f^0(p, r) = \frac{1}{e^{\tilde{n}_0[p^2/2m + U_0(r) - \mu_0]} - 1} \quad (B1)$$

where, in general,

$$U_0(r) = V_{\text{ext}}(r) + 2g \left[|\Phi(r)|^2 + \tilde{n}(r)\right]. \quad (B2)$$

The single–particle distribution function is used to compute a new guess for the non–condensate density using Eq. 2. This density is integrated over all position space to obtain a new guess at the number of non–condensate atoms. This is subtracted from the total number of atoms in the system, $N$, to obtain a new guess at the number of condensate atoms. This procedure then repeats alternately finding a new condensate wave function and then a new non–condensate density until convergence is achieved. This procedure is described in more detail in Ref. [42].

This procedure was carried out for the three racetrack geometries $L = 0 \, \mu m$, $L = 30 \, \mu m$, and $L = 60 \, \mu m$ for the three different temperatures considered in the survey simulation study, $T = 100 \, nK$, $T = 150 \, nK$, and $T = 200 \, nK$. The results of these calculations for the condensate fraction versus temperature are shown in Fig. 9. The data calculated from the ZNG were fit to the function given in Eq. 5 and these curves are shown in red.

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