Linear connections on fuzzy manifolds

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Abstract. Linear connections are introduced on a series of non-commutative geometries which have commutative limits. Quasicommutative corrections are calculated.

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1. Introduction and motivation

It is possible that the representation of spacetime by a differential manifold is only valid at length scales larger than some fundamental length and that on smaller scales the manifold must be replaced by something more fundamental. One possibility is a non-commutative geometry. If a coherent description could be found for the structure of spacetime, which was pointless on small length scales, then the ultraviolet divergences of quantum field theory could be eliminated. In fact, the elimination of these divergences is equivalent to coarse-graining the structure of spacetime over small length scales; if an ultraviolet cut-off $\Lambda$ is used then the theory does not see length scales smaller than $\Lambda^{-1}$. It is also believed that the gravitational field could serve as a universal regulator, a point of view which can be made compatible with non-commutative geometry by supposing that there is an intimate connection between (classical and/or quantum) gravity and the non-commutative structure of spacetime. To compare the two it is necessary to have a valid definition of a linear connection in non-commutative geometry. Recently such a definition has been proposed which makes full use of the bimodule structure of the space of 1-forms (Dubois-Violette and Michor 1996, Mourad 1995). It has been applied to the quantum plane (Dubois-Violette et al. 1995), the generalized quantum plane (Dimakis and Madore 1996) and to matrix geometries (Madore et al. 1995). A definition of curvature has also been proposed (Dubois-Violette et al. 1996), it is however not completely satisfactory.

In this paper we shall introduce the general notion of a fuzzy manifold and discuss linear connections on it. Some basic formulae on the subject of linear connections from previous articles are given in this section, and in section 2 a few examples of differential calculi which have been considered are briefly recalled and linear connections defined on them. In section 3 we define what we mean by a ‘fuzzy manifold’ and in section 4 we discuss linear connections on it.

Let $\mathcal{C}(V)$ be the algebra of smooth functions on $V$. For simplicity, we suppose $V$ to be parallelizable and we choose $\theta^a$ to be a globally defined moving frame on $V$. Let $(\Omega^*(V), d)$ be the ordinary differential calculus on $V$. The
definition of a connection as a covariant derivative was given an algebraic form in the Tata lectures by Koszul (1960). We shall use here the expressions ‘connection’ and ‘covariant derivative’ synonymously. A linear connection on $V$ can be defined as a linear map

$$\Omega^1(V) \xrightarrow{D} \Omega^1(V) \otimes_{C(V)} \Omega^1(V)$$

which satisfies the condition

$$D(f \xi) = df \otimes \xi + f D\xi$$

for arbitrary $f \in C(V)$ and $\xi \in \Omega^1(V)$.

The connection form $\omega^a{}_b$ is defined in terms of the covariant derivative of the moving frame:

$$D\theta^a = -\omega^a{}_b \otimes \theta^b.$$ (1.3)

Let $\pi$ be the projection of $\Omega^1(V) \otimes_{C(V)} \Omega^1(V)$ onto $\Omega^2(V)$ defined by the product in the algebra of forms. The torsion form $\Theta^a$ can be defined as

$$\Theta^a = (d - \pi \circ D)\theta^a.$$ (1.4)

The module $\Omega^1(V)$ has a natural structure as a right $C(V)$-module and the corresponding condition equivalent to (1.2) is determined using the fact that $C(V)$ is a commutative algebra:

$$D(\xi f) = D(f \xi).$$ (1.5)

Consider now a general non-commutative algebra $A$. One can then distinguish three different types of connections. A ‘left $A$-connection’ is a connection on a left $A$-module; it satisfies a left Leibniz rule. A ‘bimodule $A$-connection’ is a connection on a general bimodule $M$ which satisfies a left and right Leibniz rule. In the particular case where $M$ is the module of 1-forms we shall speak of a ‘linear connection’. A bimodule over an algebra $A$ is also a left-module over the tensor product $A^e = A \otimes_{C} A^{op}$ of the algebra with its ‘opposite’. Such a bimodule can have a bimodule $A$-connection as well as a left $A^e$-connection. These two definitions are compared in Dubois-Violette et al (1996). A linear connection over a general non-commutative algebra $A$ with a differential calculus $(\Omega^*(A), d)$ can then be defined as a linear map

$$\Omega^1(A) \xrightarrow{D} \Omega^1(A) \otimes_{A} \Omega^1(A)$$

which satisfies (1.2) for arbitrary $f \in A$ and $\xi \in \Omega^1(A)$. The module $\Omega^1(A)$ has again a natural structure as a right $A$-module, but in the non-commutative case it is impossible in general to consistently impose the condition (1.5) and a substitute must be found. We must decide how it is appropriate to define $D(\xi f)$ in terms of $D(\xi)$ and $df$. It has been proposed (Mourad 1995, Dubois-Violette and Michor 1996) to introduce as part of the definition of a linear connection a map $\sigma$ of $\Omega^1(A) \otimes_{A} \Omega^1(A)$ into itself and to define $D(\xi f)$ by the equation

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f.$$ (1.7)

We define a linear connection to be the pair $(D, \sigma)$.

A metric $g$ on $V$ can be defined as a $C(V)$-linear, symmetric map of $\Omega^1(V) \otimes_{C} \Omega^1(V)$ into $C(V)$. This definition makes sense if one replaces $C(V)$ by an algebra $A$ and $\Omega^1(V)$ by any differential calculus $\Omega^1(A)$ over $A$. By analogy with the commutative case we shall say that the covariant derivative (1.6) is metric if the condition $(1 \otimes g) \circ D = d \circ g$ is satisfied.
2. Differential calculi

We shall give three examples of non-commutative geometries of increasing generality which lead naturally to the notion of a fuzzy manifold discussed in the next section. The algebra $M_n$ of $n \times n$ complex matrices will play an important role in the definition; since it is of finite dimension as a vector space, all calculations reduce to pure algebra. All of the examples we give use a differential calculus based on derivations as proposed by Dubois-Violette (1988) and developed by Dubois-Violette et al (1989, 1990). The first is based on the complete Lie algebra $\text{Der}(M_n)$ of all derivations of $M_n$, the second on a Lie subalgebra of $\text{Der}(M_n)$, the third on a subset of $\text{Der}(M_n)$, which is not necessarily a Lie algebra.

Let $\lambda^r$, for $1 \leq r \leq n^2 - 1$, be an anti-Hermitian basis of the Lie algebra $\mathfrak{su}_n$ of the special unitary group $SU_n$ in $n$ dimensions. The $\lambda^r$ generate $M_n$ as an algebra and the derivations $e_r = \text{ad} \lambda^r$ form a basis for the Lie algebra of derivations $\text{Der}(M_n)$ of $M_n$. We have lowered the index here with the Killing metric $g_{rs}$ of $SU_n$. In order for the derivations to have the correct dimensions, one must introduce a mass parameter $\mu$ and replace $\lambda^r$ by $\mu \lambda^r$, we shall set $\mu = 1$. We define $df$ for $f \in M_n$ by

$$df(e_r) = e_r(f),$$

in particular,

$$d\lambda^r(e_s) = -C_{st}^r \lambda^t.$$  \hspace{1cm} (2.1)

We define the set of 1-forms $\Omega^1(M_n)$ to be the set of all elements of the form $f \, dg$ with $f$ and $g$ in $M_n$. The set $(\Omega^*(M_n), d)$ of all differential forms with the differential $d$ is a differential algebra; it is a differential calculus over $M_n$. There is a convenient system of generators of $\Omega^1(M_n)$ as a left- or right-module completely characterized by the equations

$$\theta^r(e_s) = \delta^r_s.$$  \hspace{1cm} (2.2)

The $\theta^r$ are related to the $d\lambda^r$ by the equations

$$d\lambda^r = C_{st}^r \lambda^t \theta^s, \hspace{0.5cm} \theta^r = \lambda^s \lambda^r d\lambda^s.$$  \hspace{1cm} (2.3)

The product on the right-hand side of this formula is the product in $\Omega^*(M_n)$. We shall refer to the $\theta^r$ as a frame or Stehbein. It satisfies the identities

$$\theta^r \theta^s = -\theta^s \theta^r, \hspace{0.5cm} \theta^r f = f \theta^r, \hspace{0.5cm} f \in M_n.$$  \hspace{1cm} (2.4)

Let $\mathfrak{g}$ be the exterior algebra generated by the forms $\theta^r$. Then $\Omega^*(M_n)$ can be identified with the tensor product of $M_n$ and $\mathfrak{g}$: $\Omega^*(M_n) = M_n \otimes \mathfrak{g}$. If we define $\theta = -\lambda^r \theta^r$, we can write the differential $df$ of an element $f$ in $M_n$ as a commutator:

$$df = -[\theta, f].$$  \hspace{1cm} (2.5)

The set of derivations $\text{Der}(M_n)$ of $M_n$ is the natural analogue of the set of all smooth vector fields $\text{Der}(\mathcal{C}(V))$ on a manifold $V$. If $V$ is parallelizable then $\text{Der}(\mathcal{C}(V))$ is a free module over $\mathcal{C}(V)$. In the non-commutative case the derivations form a module only over the centre of the algebra. However, $\Omega^1(M_n)$ is a free left- or right-module over $M_n$ of rank $n^2 - 1$. It is worth noting that, because of (2.7), it is generated by $\theta$ alone as a bimodule. We refer, for example, to Dubois-Violette et al (1996) for a discussion of this point. If $V$ is a parallelizable manifold then $\text{Der}(\mathcal{C}(V))$ has a set of generators $e_a$, which is closed under
the Lie bracket and which has the property that if \( e_{\alpha} f = 0 \) for all \( e_{\alpha} \) then \( f \) is a constant function. We have supposed that the \( \lambda^a \) are elements of the fundamental representation of \( SU_n \). We could equally well have assumed that they lie in an \( n \)-dimensional representation of \( SU_n \) for some \( m < n \). The corresponding set of derivations \( \text{Der}_m(M_n) \) of \( M_n \) will be a Lie subalgebra of \( \text{Der}(M_n) \) and if the representation is irreducible the \( \lambda^a \) will generate \( M_n \) as an algebra. The derivations will have the property that if they all annihilate a matrix \( f \) then \( f \) will be proportional to the identity. Such matrices correspond to the constant functions in the ordinary case. The smallest value of \( m \) is \( m = 2 \). The matrices \( \lambda^a \) then generate the irreducible \( n \)-dimensional representation of \( SU_n \).

With a restricted set of derivations, one can define the exterior differential exactly as before using (2.1). However, now the set of \( e_{\alpha} \) is a basis of \( \text{Der}_m(M_n) \subseteq \text{Der}(M_n) \). The derivations are taken, so to speak, only along the preferred directions. Equation (2.2) remains valid, the only change being that the structure constants are those of the Lie algebra of derivations. The difference lies in the fact that the forms are of course multilinear maps on the preferred derivations and are not defined for all elements of \( \text{Der}(M_n) \). The formula (2.3), which defines the dual forms, is as before but the meaning of the expression \( \theta^a \) changes. If we choose the complete set \( \text{Der}(M_n) \) of derivations then \( 1 \leq a \leq n^2 - 1 \) and each \( \theta^a \) is a \((n^2 - 1) \times n^2\) matrix. It takes the vector space \( \text{Der}(M_n) \) into \( M_n \). If we choose \( \text{Der}_m(M_n) \) as the derivations then \( 1 \leq a \leq m^2 - 1 \) and each \( \theta^a \) is a \((m^2 - 1) \times n^2\) matrix. It takes the vector space \( \text{Der}_m(M_n) \) into \( M_n \) and it is not defined on the remaining generators of \( \text{Der}(M_n) \). The \( \theta^a \) satisfy the same structure equations as the components of the Maurer–Cartan form on the special unitary group \( SU_m \); equation (2.5) is satisfied with the \( SU_m \) structure constants.

Using the basis \( e_{\alpha} \) of \( \text{Der}_m(M_n) \) and its dual we can write the differential of an element \( f \in M_n \) as

\[
\text{d} f = e_{\alpha} f \theta^a. \quad (2.8)
\]

The complete differential is given by \( \text{d} f = e_{\alpha} f \theta^a \). If \( \text{d} f(e_{\alpha}) = 0 \) then \( e_{\alpha} f = 0 \). This means that \( f \) is proportional to the unit element and therefore that \( \text{d} f = 0 \). However if \( \alpha \) is a general 1-form in \( \Omega^1(M_n) \) then the condition \( \alpha(e_{\alpha}) = 0 \) does not imply that \( \alpha = 0 \). When we consider the restricted set \( \text{Der}_m(M_n) \) of derivations we shall choose the algebra of forms to be the differential algebra generated by the forms \( \theta^a \) considered as \((n^2 - 1) \times n^2\) matrices. In this case if \( \alpha \) is a 1-form which satisfies the condition \( \alpha(e_{\alpha}) = 0 \) then \( \alpha = 0 \). The algebra of forms constructed from \( M_n \) using \( \text{Der}_m(M_n) \) as derivations will be designated \( \Omega^*_m(M_n) \). Each vector space \( \Omega^*_m(M_n) \) can be canonically embedded in \( \Omega^1(M_n) \). An element of the former is a \((m^2 - 1) \times n^2\) matrix which can be extended by adding zeros to a \((n^2 - 1) \times n^2\) matrix. Let \( \wedge^*_m \) be the exterior algebra generated by the forms \( \theta^a \) defined using the derivations \( \text{Der}_m(M_n) \). Then \( \Omega^*_m(M_n) \) can be again identified with the tensor product of \( M_n \) and \( \wedge^*_m \): \( \Omega^*_m(M_n) = M_n \otimes \wedge^*_m \). More details of the construction of \( \Omega^*_m(M_n) \) can be found in Madore (1995).

The equations we have given above are with respect to an arbitrary basis \( \lambda^a \), but they are all tensorial in character with respect to a change of basis

\[
\lambda^a \mapsto \tilde{\lambda}^a = \Lambda^a_b \lambda^b, \quad (\Lambda^a_b) \in GL_{m^2 - 1}.
\]

We find

\[
C^a_{bc} = \Lambda^a_b \Lambda^{-1} e^b \Lambda^{-1} e^e C^d_{ef},
\]

and the Killing metric \( g_{ab} \) transforms as

\[
g'_{ab} = \Lambda^c_a \Lambda^d_b g_{cd}, \quad (2.10)
\]
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From the expression for $\theta^a$, we see that it transforms as

$$\theta^a \mapsto \theta^a = \Lambda^a_b \theta^b.$$  

(2.11)

The equivalent of a global map of the manifold $V$ onto itself is an automorphism of $M_n$, given by

$$\lambda^a \mapsto \lambda'^a = g^{-1} \lambda^a g, \quad g \in U_n.$$  

(2.12)

We have written $U_n$ instead of $GL_n$ because of the condition that the matrices $\lambda^a$ be anti-Hermitian. The set $\lambda^a$ remains invariant only if $m = n$. Equation (2.9) then defines a map of $U_n$ into $GL_{n-1}$.

We shall show now that it is possible to construct a differential calculus using an arbitrary set of derivations, which do not necessarily form a Lie subalgebra of $\text{Der}(M_n)$. Of the many differential calculi which can be constructed over the algebra $M_n$, the largest is the differential envelope or universal differential calculus $(\Omega^1(M_n), d)$; every other differential calculus can be considered as a quotient of it. For the definitions we refer the reader to, for example, Connes (1994). Let $(\Omega^p(M_n), d)$ be another differential calculus over $M_n$. The choice of distinguishing subscript will become evident in the next section. There exists a unique $d_a$-homomorphism $\phi$ of $\Omega^1(M_n)$ into $\Omega^1(M_n)$. It is given by

$$\phi(d_a f) = df.$$  

(2.14)

The restriction $\phi_p$ of $\phi$ to each $\Omega^p(M_n)$ is defined by

$$\phi_p(f_0 df_1 \cdots df_p) = f_0 df_1 \cdots df_p.$$  

Suppose now that we know how to construct an $M_n$-module $\Omega^1_c(M_n)$ and an application

$$M_n \longrightarrow \Omega^1_c(M_n).$$  

(2.15)

Then using (2.13) there is a method of constructing $\Omega^p_c(M_n)$ for $p \geq 2$ as well as the extension of the differential. Since we know $\Omega^1(M_n)$ and $\Omega^1_c(M_n)$, we can suppose that $\phi_1$ is given. We must construct $\Omega^2_c(M_n)$. The simplest consistent choice would be to set

$$\Omega^2_c(M_n) = \Omega^2_c(M_n)/d_a \ker \phi_1,$$  

(2.16)

where $d_a \ker \phi_1$ is the bimodule generated by $d_a \ker \phi_1$. This is the largest differential calculus consistent with the constraints on $\Omega^1_c(M_n)$. The map $\phi_2$ is defined to be the projection of $\Omega^2_c(M_n)$ onto $\Omega^2_c(M_n)$ so defined and $d$ is defined by $d(f dg) = df dg$. This procedure can be continued by iteration to arbitrary order in $p$.

To initiate the above construction, we shall define the 1-forms using a set of derivations. For each integer $d$ let $\lambda^a$ be a set of $d$ linearly independent anti-Hermitian elements of $M_n$ which generate $M_n$ as an algebra. Introduce the vector space $\text{Der}(M_n)$ spanned by the derivations $e_a = \text{ad} \lambda_a$. We have lowered the index here with the Killing metric $g_{ab}$ of an $SU_m$ to be chosen shortly. In general, the $e_a$ do not form a Lie algebra but they do, however, satisfy commutation relations as a consequence of the commutation relations of $M_n$. Define the map (2.15) by

$$df(e_a) = e_a f.$$  

(2.17)
The $\lambda^a$ can be completed to form a set $\lambda^a$ which generates an $n$-dimensional irreducible representation of $\mathfrak{su}_m$ for some $m \leq n$ and the operator $d$ can be extended so that $df$ acts on all of the corresponding derivations $e_a$. We can use this extension to construct a set of $d$ elements $\theta^a$ of $\Omega^1(M_n)$ such that

$$\theta^a(e_b) = \delta^a_b.$$

If the ordering is such that the $\lambda^a$ are the first $d$ elements of the $\lambda^a$, then we choose the $\theta^a$ to be the first $d$ elements of the corresponding $\theta^a$ constructed above. We shall again refer to the set of $\theta^a$ as a frame or Stehbein. By construction the $\theta^a$ again form an anticommuting set and because of (2.18) they commute with the elements of $M_n$. Therefore as a left- or right-module, $\Omega^1(M_n)$ is again free, of rank $n$, and $\Omega^r(M_n)$ can be again identified with the tensor product of $M_n$ and the exterior algebra $\wedge^r \ast$ generated by the forms $\theta^a$: $\Omega^r(M_n) = M_n \otimes \wedge^r \ast$. The vector space $\Omega^r(M_n)$ can be canonically embedded in $\Omega^1(M_n)$. An element of the former is a $d \times n^2$ matrix which can be extended by adding zeros to a $(n^2 - 1) \times n^2$ matrix. Because the 2-forms are generated by products of the $\theta^a$ one has

$$d \theta^a = -\frac{1}{2} C^a_{\beta \gamma} \theta^\beta \theta^\gamma.$$

The differential here is defined by (2.13) and it does not necessarily coincide with the restriction of the differential of (2.5). Because of the relation $f \theta^a = \theta^a f$, the structure elements $C^a_{\beta \gamma}$ are necessarily complex numbers. Define $\theta = -\lambda_a \theta^a$. Then one sees that (2.7) is still valid and it again follows that as a bimodule $\Omega^1_{(V)}(M_n)$ is generated by one element.

A differential manifold $V$ can always be embedded in a flat Euclidean space of sufficiently high dimension and a metric linear connection on the manifold can be considered as defined by the embedding in terms of the standard flat connection in the enveloping space. We have a similar situation here. The $\lambda^a$ are the analogues of the coordinates of $V$ and the $\lambda'$ of (2.4) are the analogues of the coordinates of the higher-dimensional Euclidean space. Since the algebra $M_n$ is the analogue of the algebra of functions $\mathcal{C}(V)$, the $\lambda'$ can also be considered as the analogue of a basis of $\mathcal{C}(V)$ if we add $\lambda^0 = 1$. Both of these analogies are enlightening but neither is perfect.

Each $\lambda'$ of (2.4) can be written as a polynomial $\lambda' = \lambda'(\lambda^a)$ in the $\lambda^a$. Therefore using the Leibniz rule we have

$$d \lambda' = A'_\alpha (d \lambda^\alpha)$$

where $A'_\alpha (d \lambda^\alpha)$ is a polynomial in $\lambda^a$ and $d \lambda^\alpha$, which is linear in the latter. On the right-hand side $d \lambda^\alpha$ can by restriction be considered as elements of $\Omega^1_{(V)}(M_n)$. The $\lambda'$ defined in (2.3) can be expressed in terms of the $d \lambda'$ by (2.4) and, similarly, $d \lambda^\alpha$ can be expressed in terms of the $\theta^a$: $d \lambda^\alpha = e_\beta \lambda^\alpha \theta^\beta$, provided of course the differential is considered as being within the differential calculus $\Omega^*_{(V)}(M_n)$. Therefore, there exists a relation $\lambda' = \Lambda'_\alpha \theta^a$ analogous to (2.20), where now the coefficients, elements of $M_n$, can all be put on the left since they commute with the Stehbein. We have seen, however, that this relation is quite trivial since $\lambda' \equiv 0$ for $d < r \leq n^2 - 1$ and $\Lambda'_\beta = \delta^\beta_\beta$. Although for $r > d$ the $\lambda'$ do not vanish, as $(n^2 - 1) \times n^2$ matrices their first $d$ columns are equal to zero. Dual to the embedding of $\text{Der}_C(M_n)$ into $\text{Der}_m(M_n)$ there is a projection of $\Omega^1_{(V)}(M_n)$ onto $\Omega^1_{(C)}(M_n)$, and we have

$$\Omega^1_{(V)}(M_n) = \Omega^1_{(C)}(M_n) \oplus \mathcal{N}.$$

We have shown that there is a natural choice of complement $\mathcal{N}$.

As an easy example of this construction, one can consider the case $m = 2$ with the $\lambda^a$ as the generators of the $n$-dimensional representation of $\mathfrak{su}_2$. On can then choose $d = 2$...
and the $\lambda^\alpha$ as the first two elements of the set of $\lambda^a$. The corresponding $\theta^\alpha$ will be the first two elements of the $\theta^a$ which generate $\Omega_1^c(M_n)$. To each of them can be added any linear combination of $\theta^3$, which would constitute an ambiguity within the calculus $\Omega_2(M_n)$. However, since
\[ \theta^3(e_1) = 0, \quad \theta^3(e_2) = 0 \]
within the calculus $\Omega_1^c(M_n)$ one has $\theta^3 \equiv 0$. The $\theta^a$ are two elements of $\Omega_1^1(M_n)$ restricted to the derivations $e_\alpha$. Their product is therefore antisymmetric. The 2-forms $d\theta^a$ must be calculated using (2.4) and (2.20) as well as the rule we have given for extending the $d$ of the calculus $\Omega_1^c(M_n)$. More details of this construction in a more general setting can be found in the article by Dimakis and Madore (1996).

3. Fuzzy manifolds

To discuss the commutative limit it is convenient to change the normalization of the generators $\lambda^\alpha$. Recall that the $\lambda^\alpha$ have dimensions of mass. We introduce the parameter $\bar{k}$ with the dimensions of (length$)^2$ and define 'coordinates' $x^\alpha$ by
\[ x^\alpha = i\bar{k}\lambda^\alpha. \quad (3.1) \]
In the commutative limit the mass parameter $\mu$ must tend to infinity, since otherwise the derivations $e_\alpha$ would tend to zero and the dual 1-forms diverge. Therefore if $x^a$ is to remain bounded we must have $\bar{k} \to 0$ in the commutative limit. If we were considering a non-commutative model of spacetime then we would be tempted to identify $\bar{k}$ with the inverse of the square of the Planck mass, $\bar{k} = \mu^{-2}$, and consider spacetime as fundamentally non-commutative in the presence of gravity. The 1-form $\theta$ diverges in the commutative limit as it must because of the relation (2.7) and because of the fact that it generates the entire set of 1-forms as a bimodule.

We define matrices $L^{\alpha\beta}$ by the equations
\[ [x^\alpha, x^\beta] = i\bar{k}L^{\alpha\beta}. \quad (3.2) \]
By taking higher-order commutators of the $\lambda^\alpha$ the algebra will eventually close as a Lie algebra to form an irreducible $n$-dimensional representation $\lambda^a$ of the Lie algebra of $SU_m$ for some $m \leq n$:
\[ [\lambda_a, \lambda_b] = C^{\, c}_{\, ab}\lambda_c. \]
The structure constants must have the dimensions of mass. By assumption, $m^2 - 1 - d$ must be at least as large as the number of Casimir relations of $SU_m$. Set $x^a = i\bar{k}\lambda^a$, then
\[ [x_a, x_b] = i\bar{k}C^{\, c}_{\, ab}x_c. \quad (3.3) \]
We can order the $x^a$ as the sequence
\[ x^a = (x^a, L^{a\beta}, [x^a, L^{\beta\gamma}], \ldots). \]
Because of the quadratic Casimir relation of $SU_m$ we have $g_{ab}x^ax^b = r^2$ for some 'radius' $r$, therefore
\[ \bar{k} \sim \frac{r^2}{n}, \quad (3.4) \]
and so $\bar{k} \to 0$ as $n \to \infty$; this is the commutative limit. Globally, the limit manifold $V$ will then be a submanifold of the sphere of radius $r$ in $\mathbb{R}^{m^2-1}$. A metric on it would necessarily have Euclidean signature.
For each $l \geq 1$ let $C_l$ be the vector space of $l$th-order symmetric polynomials in the $x^a$ and $L_l$ the vector space of $l$th-order symmetric polynomials in the $x^a$. Then we have

$$C_l \subset L_l, \quad C_l \subset C_{l+1}, \quad L_l \subset L_{l+1},$$

(3.5)

and the set $\{L_l\}$ is a filtration of $M_n$. We have

$$C \equiv \bigcup_l C_l \subseteq M_n, \quad L \equiv \bigcup_l L_l = M_n.$$  

(3.6)

For fixed $l$ the set $C_l$ tends to the set of $l$th-order polynomials in the $x^a$ in the limit $n \to \infty$. The $\{C_l\}$ do not form a graded algebra, but from the definition of the $\{L_l\}$ we have

$$C_k C_l \subseteq C_{k+l} + \bar{k} L_k + l - 1.$$  

A specific example is the fuzzy 2-sphere (Madore 1992). Consider $R^3$ with coordinates $x^a$, $1 \leq a \leq 3$, and Euclidean metric $g_{ab}$. Let $V$ be the sphere $S^2$ defined by

$$g_{ab} x^a x^b = r^2.$$  

(3.7)

Consider the algebra $P$ of polynomials in the $x^a$ and let $I$ be the ideal generated by the relation (3.7). That is, $I$ consists of elements of $P$ with $g_{ab} x^a x^b - r^2$ as a factor. Then the quotient algebra $P/I$ is dense in the algebra $C(S^2)$. Any element of $P/I$ can be represented as a finite multipole expansion of the form

$$f(x^a) = f_0 + f_a x^a + \frac{1}{2} f_{ab} x^a x^b + \cdots,$$  

(3.8)

where the $f_{a_1 \ldots a_k}$ are completely symmetric and trace free. We obtain a vector space of dimension $n^2$ if we consider only polynomials of order $n - 1$. We can redefine the product of the $x^a$ to make this vector space into the algebra of $n \times n$ matrices.

Suppose that we suppress the terms $n$th order in the expansion (3.8) of every function $f$. The resulting set is a vector space $A_n$ of dimension $n^2$. We can introduce a new product in the $x^a$, which will make it into the algebra $M_n$. Let $J^a$ generate the $n$-dimensional irreducible representation of $su_2$ normalized so that $[J_a, J_b] = i \epsilon_{abc} J^c$. They then satisfy the quadratic Casimir relation $J_a J^a = (n^2 - 1)/4$; we choose $\lambda_a = (1/n) J_a$. The structure constants are then given by $C_{abc} = r^{-1} \epsilon_{abc}$ and from (3.1) we have

$$x^a = \frac{k}{r} J^a.$$  

(3.9)

The $x^a$ satisfy the commutation relations (3.3) and from the Casimir relation we obtain the equation

$$4r^4 = (n^2 - 1) k^2,$$  

(3.10)

in particular, (3.4) is satisfied. The space $L_l$ is the space of symmetric polynomials of order $l$ in the $x^a$. Define $x^a$ as the first two of the $x^a$, then $L^{12} = r^{-1} x^3$. Because of the Casimir relation we have

$$\bigcup_l C_l = \bigcup_l L_l = M_n.$$  

(3.11)

For $n \gg l$ $L_l$ can be identified as the space of polynomials of order $l$ on $S^2$ and $C_l$ as the space of polynomials of order $l$ on the coordinate patch.

It is known (Weyl 1931) that the algebra $M_n$ can be also be generated by two matrices $u$ and $v$, which satisfy the relations

$$u^a = 1, \quad v^a = 1, \quad uv = q vu, \quad q = e^{2 \pi i/n}.$$  

(3.12)

The space $C_l$ then becomes the space of symmetric polynomials of order $l$ in $u$ and $v$. For $n \gg l$ it can be identified as the space of polynomials of order $l$ on the torus. One sees from
these two examples that the structure of the limit manifold is determined by the filtration. The dimension of the manifold is encoded in the dimension of \( C^1 \). The manifolds differ in global topology because the vector spaces \( C^l \) differ. A polynomial in the \( x^a \) of order \( l \), with \( n \gg l \), can of course always be written as a polynomial in \( u \) and \( v \), but will then in general be of order \( n \). The transformation in no way respects the filtration. This corresponds to the fact that a map from the torus onto the sphere is necessarily singular. A physical theory expressed in terms of the matrix approximation would detect the difference between the topologies through the dependence of the action on the different derivations \( e_a = \text{ad} x_a \).

We shall refer to the algebra \( M_n \) with the set \( C \) as a fuzzy manifold. The point-like structure which distinguishes an ordinary manifold has been replaced by a cellular structure much like the Bohr-cell structure of quantized phase space. For finite \( n \), the dimension of \( M_n \) is finite and so the number of degrees of freedom of a quantum field on \( M_n \) is also finite. A quantum field theory over a filtered \( M_n \) is necessarily finite. Explicit constructions of such field theories have been given (Madore 1992, Grosse and Madore 1992, Grosse et al 1996). There is an abundant literature on field theories defined on the algebra defined by the relations (3.12) for arbitrary values of \( q \). Any arbitrary algebra \( A \) with generators \( x^a \) which satisfy commutation relations of the form (3.2) where \( \bar{k} \) is ‘small’ could also be considered as a fuzzy manifold provided the right-hand side is sufficiently non-degenerate. Of more physical relevance for relativistic physics are non-compact manifolds, which can support metrics of Minkowski signature. The first example along the lines indicated by (3.2) was given by Snyder (1947). See also Madore (1988, 1995). Doplicher et al (1995) have given an analysis of several possible non-commutative extensions of Minkowski space within the context of relativistic quantum field theory. We refer to Dimakis et al (1996) for a discussion of the relation between the non-commutative structure of spacetime and classical gravity.

If we rewrite (2.20) in terms of \( x^a \) we see that in the commutative limit

\[
A'_a(dx^a) = \frac{\partial x^r}{\partial x^a} dx^a + o(\bar{k}).
\]

(3.13)

This gives the differential of an arbitrary function in terms of the differential of the coordinates.

The commutative limit of the Stehbein is a delicate question and not in general clear. It could not exist as a moving frame on the limit manifold unless the latter is parallelizable. Consider the case of \( M_n \) with the filtrations which define the fuzzy sphere and the fuzzy torus. The torus is parallelizable but the 2-sphere is not and the \( \theta^a \) constructed in the previous section tend to the Maurer–Cartan forms on the Hopf bundle \( S^3 \) over \( S^2 \), a space which is parallelizable. The first two \( \theta^a \) of the set of \( \theta^a \) have a regular limit when considered as elements of \( \Omega^1(M_n) \) but the limit does not define a frame on \( S^2 \).

4. Linear connections

From (2.5) we see that the linear connection defined by

\[
D\theta^r = -\omega^r_s \otimes \theta^s, \quad \omega^r_s = -\frac{1}{2} C^r_{st} \theta^t
\]

(4.1)

has vanishing torsion. It is the unique torsion-free metric connection on \( \Omega^1(M_n) \) (Madore et al 1995). With this connection the geometry of \( M_n \) looks like the invariant geometry of the group \( SU_n \) and it has been used in the construction of non-commutative generalizations of Kaluza–Klein theories. We refer the reader to Madore and Mourad (1996) for a recent
discussion. Since the elements of the algebra commute with the frame \( \theta^r \), we can define \( D \) on all of \( \Omega^r(M_n) \) using (1.2) or (1.7). The map \( \sigma \) is given by

\[
\sigma(\theta^r \otimes \theta^s) = \theta^s \otimes \theta^r. \tag{4.2}
\]

The map (1.6) has a natural extension (Koszul 1960)

\[
\Omega^r(A) \otimes_A \Omega^1(A) \xrightarrow{D} \Omega^r(A) \otimes_A \Omega^1(A) \tag{4.3}
\]
given, for \( \alpha \in \Omega^r(A) \) and \( \xi \in \Omega^1(A) \) by

\[
D(\alpha \xi) = d\alpha \otimes \xi + (-1)^r \alpha D\xi. \tag{4.6}
\]

Because of the relation \( D(f \theta^r) = D(\theta^r f) \), the coefficients \( \omega^r_{\beta \gamma} \) must be complex numbers. The connection (4.6) is induced from the connection (4.1) in much the same way that the linear connection on \( V \) mentioned above is induced from the standard flat connection on the embedding space. It depends directly on the choice of \( \lambda^a \). We cannot claim that it is the only connection associated to the differential calculus \( \Omega^r(M_n) \). The unicity problem has been discussed in a different context by Dimakis and Madore (1996).

Consider (4.6), in the limit \( \lambda \to 0 \) we have

\[
\theta^a = \theta^a_{(0)} + \kappa \theta^a_{(1)} + o(\kappa^2), \quad \omega^a_{\beta \gamma} = \omega^a_{(0)\beta \gamma} + \kappa \omega^a_{(1)\beta \gamma} + o(\kappa^2)
\]

and so from (4.6) we obtain a coupled set of algebraic equations for \( \omega^a_{(0)\beta \gamma} \) and \( \omega^a_{(1)\beta \gamma} \), the unknowns, in terms of \( \theta^a_{(0)\beta} \) and \( \theta^a_{(1)\beta} \), the knowns. If we neglect the correction terms of order \( \kappa \) and consider the limiting values of the \( \theta^a \) as a moving frame on \( V \), then we can say that the non-commutative structure determines a metric in the commutative limit. It is possible that there is a one-to-one correspondence between the non-commutative structure and the limiting value of the metric. In any case the limiting metric must be very special because of the restrictions we have found on the connection. This point is discussed in more detail in Dimakis and Madore (1996) and in Dimakis et al (1996).
In the commutative limit the commutator defines quite generally a Poisson bracket on the manifold $V$. If we set
\[
\{ x^\alpha, x^\beta \} = \lim_{k \to 0} \frac{1}{i k} [ x^\alpha, x^\beta ]
\]
then from (3.2) we see that
\[
\{ x^\alpha, x^\beta \} = \lim_{k \to 0} L^{\alpha \beta}.
\] (4.7)
The right-hand side is not necessarily an element of $\mathcal{C}(V)$. It belongs, however, to a commutative extension $\mathcal{A}_0$ of $\mathcal{C}(V)$. We refer to Dubois-Violette et al (1996) for a discussion of this point in a more general context. A study of the relation of the Poisson structure and the linear connection has not been made.

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