Logarithmic embeddings and logarithmic semistable reductions

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Abstract

In this paper, we give a criterion for the existence of logarithmic embeddings — which was first introduced by Steenbrink — for general normal crossing varieties. Using this criterion, we also give a new proof of the theorem of Kawamata–Namikawa which gives a criterion for the existence of the log structure of semistable type.

1 Introduction

Let $X$ be a connected, geometrically reduced algebraic scheme over a field $k$. Then $X$ is said to be a normal crossing variety of dimension $n-1$ if there exists an isomorphism of $k$-algebras

$$\hat{O}_{X,x} \xrightarrow{\sim} k(x)[[T_1, \ldots, T_n]]/(T_1 \cdots T_n)$$

for each closed point $x \in X$, where $\hat{O}_{X,x}$ denotes the completion of the local ring $O_{X,x}$ along its maximal ideal (Definition 2.1). Normal crossing varieties usually appear in contexts of algebraic geometry via degenerations and normal crossing divisors. In the first case, they appear as a specialization of a family of smooth varieties. Normal crossing varieties are usually considered and expected to be limits of smooth varieties, and — as is well–known — they are important to the theory of moduli. As for the second situation, a normal crossing divisor is a divisor of a smooth variety which itself is a normal crossing variety. Normal crossing divisors play important roles in various fields of algebraic geometry. For example, a pair of smooth variety and its normal crossing divisor is usually called a log variety. Considering log varieties instead of smooth varieties — or usually admitting some mild singularities — alone, some algebro geometric theories (e.g., minimal model theory, etc.) are well generalized.

Relating with a normal crossing variety $X$, there are two problems, smoothings and embeddings, in light of degenerations and normal crossing divisors, respectively.

The smoothing problem is a problem to find a Cartesian diagram

$$
\begin{array}{ccc}
X & \rightarrow & \mathfrak{x} \\
\downarrow & & \downarrow \\
0 & \rightarrow & \Delta,
\end{array}
$$

for a normal crossing variety $X$ (in this situation, we should assume that $X$ is proper over $k$), where $\Delta$ is a one-dimensional regular scheme, $\mathfrak{x}$ is a regular scheme proper flat
and generically smooth over $\Delta$, and $0$ is a closed point of $\Delta$ whose residue field is $k$. We usually take, as the base scheme $\Delta$, the spectrum of a discrete valuation ring, e.g., the ring of formal power series over $k$ or — in case $k$ is perfect — the ring of Witt vectors over $k$. In the complex analytic situation, Friedmann [2] studied the smoothing problem generally, and solve it for degenerated K3 surfaces. Recently, Kawamata–Namikawa [9] approached this problem by introducing a new method; the logarithmic method. The Cartesian diagram as above with $\Delta$ a spectrum of an Artinian local ring $A$ is called an infinitesimal smoothing, if it is étale locally isomorphic to the diagram

$$
\begin{array}{ccc}
\text{Spec} k[Z_1, \ldots, Z_n]/(Z_1 \cdots Z_l) & \to & \text{Spec} A[Z_1, \ldots, Z_n]/(Z_1 \cdots Z_l - \pi) \\
\downarrow & & \downarrow \\
\text{Spec} k & \to & \text{Spec} A,
\end{array}
$$

where $\pi$ is an element of the maximal ideal of $A$ and $\pi \neq 0$. The central problem to find such an infinitesimal smoothing is to compute the obstruction class of $X$ to have such a diagram and to show vanishing or non–vanishing of it.

The embedding problem is a problem to find a closed embedding $X \hookrightarrow V$ over $k$ of $X$ as a normal crossing divisor, where $V$ is a smooth variety over $k$. If $X$ is smoothable in the above sense with $\Delta$ a smooth algebraic variety over $\text{Spec } k$, the smoothing family $X \hookrightarrow \mathfrak{x}$ gives an embedding of this sense. If $X$ is smooth, this problem becomes trivial, since we can take as $V$ the product of $X$ and, for example, $\mathbb{P}^1$. But for a general normal crossing variety, this problem seems far from satisfactory solutions. Similarly to the smoothing problem, we can consider this problem in the infinitesimal sense.

In this paper, we consider the above problems in a logarithmic sense. We consider logarithmic generalizations of smoothings and embeddings of normal crossing varieties according to Kajiwara [3], Kawamata–Namikawa [4] and Steenbrink [11], and we solve their existence problems. These generalizations are done in terms of logarithmic geometry of Fontaine, Illusie and Kazuya Kato.

**Logarithmic geometry** — or log geometry — was first founded by Fontaine and Illusie based on their idea of, so–called, log structures; afterwards, it was established as a generally organized theory and applied to various fields of algebraic and arithmetic geometry by Fontaine, Illusie and Kazuya Kato (cf. [7], [8]). In various kinds of geometries including algebraic geometry, we usually consider local ringed spaces, i.e., the pairs of topological spaces — possibly in the sense of Grothendieck topologies — and sheaves of local rings over them. The basic idea of Fontaine and Illusie is that, instead of local ringed spaces alone, they consider local ringed spaces equipped with some additional structure — which they call the logarithmic structures — written in terms of sheaves of commutative and unitary monoids (see [7] for the precise definition). In algebro geometric situations, these log structures usually represent “something” of the underlying local ringed spaces, e.g., divisors or the structure of torus embeddings, etc. Through these foundations, they suggested to generalize the “classical” geometries by considering “log objects” — such as log schemes — which are the pairs of local ringed spaces and log structures on them.

In the present paper, we recall and generalize the logarithmic embedding (Definition 4.1) introduced by Steenbrink [11]. A logarithmic embedding — which is regarded as a logarithmic generalization of a log variety — is a certain log scheme $(X, \mathcal{M}_X)$ with $X$ a normal crossing variety. Then we prove the following theorem which gives a criterion for the existence of logarithmic embeddings:
Theorem (Theorem 4.4) For a normal crossing variety $X$, a logarithmic embedding of $X$ exists if and only if there exists a line bundle $L$ on $X$ such that $L \otimes \mathcal{O}_X \mathcal{O}_D \sim T^1_X$, where $D$ is the singular locus of $X$.

Here, $T^1_X$ is an invertible $\mathcal{O}_D$-module, called the infinitesimal normal bundle (cf. [2]), which is naturally isomorphic to $\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$; we recall the construction of it in §3.

A normal crossing variety $X$ is said to be $d$-semistable if $T^1_X$ is a trivial bundle on $D$ (cf. [2]). By the above theorem, any $d$-semistable normal crossing variety $X$ has a logarithmic embedding.

As for the smoothing problem, we recall and generalize the concept, the logarithmic semistable reduction (Definition 5.1) introduced by Kajiwara [5] (in one dimensional case) and Kawamata–Namikawa [9] (by a different but essentially the same method). Using the above theorem, we get a criterion for the existence of logarithmic semistable reductions, which was first proved by Kawamata–Namikawa [9] in the complex analytic situation, as follows:

Theorem (Theorem 5.4) (cf. [9]) For a normal crossing variety $X$, the log structure of semistable type on $X$ exists if and only if $X$ is $d$-semistable.

The composition of this paper is as follows. In §2, we study the geometry of normal crossing varieties in general. In particular, we define good étale local charts on normal crossing varieties, and prove the existence of them. In §3, we recall the basic construction of the tangent complex of a normal crossing variety, and introduce the invertible sheaf $T^1_X$ on $D$. We introduce the logarithmic embedding in §4. This section also contains the proof of our main theorem. The logarithmic semistable reduction is studied in §5.

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Conventions: All sheaves are considered with respect to étale topology. By a monoid, we mean — as usual in the contexts of log geometry — a set with a commutative and associative binary operation and the neutral element. For such a monoid $M$, we denote by $M^{gp}$ the Grothendieck group of $M$. We denote by $\mathbb{N}$ the monoid of non-negative integers.

2 Normal crossing varieties

Throughout this paper, we always work over a fixed base field $k$. As usual, an algebraic $k$-scheme is, by definition, a separated scheme of finite type over $k$. Let $X$ be an algebraic $k$-scheme and $x \in X$ a point. We denote the residue field at $x \in X$ by $k(x)$.

Definition 2.1 Let $X$ be a connected and geometrically reduced algebraic $k$-scheme. Then $X$ is said to be a normal crossing variety over $k$ of dimension $n - 1$ if the following condition is satisfied: For any closed point $x \in X$, there exists an isomorphism

$$\hat{O}_{X,x} \sim k(x)[[T_1, \ldots, T_n]]/(T_1 \cdots T_{l_x})$$

of $k$-algebras, where $l_x$ is an integer ($1 \leq l_x \leq n$) depending on $x$. Here, we denote by $\hat{O}_{X,x}$ the completion of the local ring $O_{X,x}$ by its maximal ideal.

The integer $l_x$ is called the multiplicity at $x \in X$. We sometimes denote it by $l^X_x$ if we want to emphasize the scheme $X$. The Zariski closure of the set of closed points whose multiplicity is greater than 1 is the singular locus of $X$, which we denote by $D$. 


A standard example of normal crossing varieties is an affine scheme

\[ \text{Spec } k[T_1, \ldots, T_n]/(T_1 \cdots T_l) \quad (1 \leq l \leq n). \]

This scheme consists of \( l \) irreducible components which intersect transversally along the singular locus

\[ \text{Spec } k[T_1, \ldots, T_n]/(T_1 \cdots \widehat{T_j} \cdots T_l : 1 \leq j \leq l). \]

Each irreducible component is isomorphic to the affine \((n-1)\)-space over \( k \). In general, a normal crossing variety \( X \) is said to be simple if each irreducible component of \( X \) is smooth over \( k \). For example, a smooth \( k \)-variety is a simple normal crossing variety.

Let \( V \) be a smooth \( k \)-variety of dimension \( n \). A reduced divisor \( X \) on \( V \) is called a normal crossing divisor if \( X \) itself is a normal crossing variety of dimension \( n - 1 \). In this case, the closed embedding \( X \hookrightarrow V \) is called a NCD embedding of \( X \). For example, the affine normal crossing variety (2) is a normal crossing divisor in the affine \( n \)-space over \( k \).

The proof of the following proposition is straightforward and is left to the reader.

**Proposition 2.2** Let \( Y \) be a connected scheme étale over a connected algebraic \( k \)-scheme \( X \). If \( X \) is a normal crossing variety, then so is \( Y \). The converse is also true if the étale morphism \( Y \to X \) is surjective.

It is clear that an étale morphism leaves invariant the multiplicity at every closed point, i.e., if \( \varphi: Y \to X \) is an étale morphism of normal crossing varieties and \( y \in Y \) is a closed point, then we have \( l^Y_y = l^X_{\varphi(y)} \).

In the following paragraphs of this section, we shall study the local nature of normal crossing varieties for the later purpose. In the subsequent sections, we need to take a good étale neighborhood around every closed point. We require that these étale neighborhoods have good coordinate systems which serve for several explicit calculations. To clarify the notion of “good” étale neighborhoods, we define them as follows:

**Definition 2.3** Let \( X \) be a normal crossing variety and \( x \in X \) a closed point. Let \( \varphi: U \to X \) be an étale morphism with \( U \) a simple normal crossing variety and \( z_1, \ldots, z_{l_x} \in \Gamma(U, \mathcal{O}_U) \), where \( l_x \) is the multiplicity at \( x \). Then \( (\varphi: U \to X; z_1, \ldots, z_{l_x}) \) is said to be a local chart around \( x \) if the following conditions are satisfied:

(a) There exists a unique point \( y \in U \) such that \( \varphi(y) = x \).

(b) There exists a closed immersion \( i: U \hookrightarrow V \), where \( V \) is an affine smooth \( k \)-scheme.

(c) There exist \( Z_1, \ldots, Z_n \in \Gamma(V, \mathcal{O}_V) \) which form a regular parameter system at \( i(y) \in V \) such that \( z_i = i^*Z_i \) for \( 1 \leq i \leq l_x \), and \( U \) is defined as a closed subset in \( V \) by the ideal \( (Z_1 \cdots Z_{l_x}) \).

(d) each ideal \( (z_i) \) is prime and the irreducible components of \( U \) are precisely the closed subsets of \( U \) corresponding to the ideals \( (z_1), \ldots, (z_{l_x}) \).

Note that \( i: U \hookrightarrow V \) is, due to (c), a NCD embedding. Moreover, due to (d), all the irreducible components intersect and contain the point \( y \).

The following theorem assures the existence of local chart around every closed point of normal crossing variety \( X \). We prove this theorem later in this section.
Theorem 2.4 Let $X$ be a normal crossing variety and $x \in X$ a closed point. Then there exists a local chart $(\varphi: U \to X; z_1, \ldots, z_n)$ around $x$.

Since any étale open set of $X$ is again a normal crossing variety, we have the following:

Corollary 2.5 Let $X$ be a normal crossing variety. Then the set of all local charts forms an open basis with respect to the étale topology on $X$.

Remark 2.6 Theorem 2.4 implies that any normal crossing variety is realized as a simple normal crossing divisor on some smooth $k$-variety étale locally. But a normal crossing variety, in general, cannot be a normal crossing divisor globally on a smooth $k$-variety. In the next section, we will see a necessary condition for a normal crossing variety to be a normal crossing divisor (Proposition 3.2).

For the proof of Theorem 2.4, we need one lemma:

Lemma 2.7 Let $q$ be a height zero prime ideal in $K[T_1, \ldots, T_n]/(T_1 \cdots T_l)$ $(1 \leq l \leq n)$, where $K$ is a field. Then $q = (T_j)$ for some $j$ $(1 \leq j \leq l)$.

Proof. By Krull’s principal ideal theorem, any non-zero element in $q$ is a zero factor. Hence any element in $q$ is a multiple of $T_j$’s $(1 \leq j \leq l)$. Since $q$ is a prime ideal, $q$ must contain $T_j$ for some $j$ $(1 \leq j \leq l)$, i.e., $(T_j) \subseteq q$. But since the height of $q$ is zero and $(T_j)$ is a prime ideal, we have $q = (T_j)$. □

Proof of Theorem 2.4. The complete local ring $\mathcal{O}_{X,x}$ is isomorphic to the complete local ring $k(x)[T_1, \ldots, T_n]/(T_1 \cdots T_l)$ which is a completion of the local ring $(k(x)[T_1, \ldots, T_n]/(T_1 \cdots T_l))_0$. Then due to [1, Corollary (2.6)], there exist a scheme $U$ and étale morphisms $\varphi: U \to X$ and $\phi: U \to \text{Spec} k(x)[T_1, \ldots, T_n]/(T_1 \cdots T_l)$ such that $\varphi(y) = x$ and $\phi(y) = 0$ for some $y \in U$. We fix this closed point $y \in U$. Since $\varphi$ is étale, we may assume — replacing $U$ by its Zariski open subset if necessary — that $y$ is the only point which is mapped to $x$ by $\varphi$. Obviously we may assume that $U$ is connected and affine. We can remove all the irreducible components which do not contain $y$. Then we may assume that all the irreducible components of $U$ contain $y$. We set $U = \text{Spec} A$ and $B: = k(x)[T_1, \ldots, T_n]/(T_1 \cdots T_l)$.

Since $U$ is étale over a reduced $k$-scheme $\text{Spec} B$, the $k$-algebra $A$ is reduced. Take a minimal prime factorization

$$(0) = p_1 \cap \cdots \cap p_l,$$

of the ideal $(0) = \sqrt{(0)}$. Since each $p_i$ is minimal in the set of all prime ideals, the height of each $p_i$ is zero. Obviously the prime decomposition corresponds to the decomposition of $U$ into irreducible components. Set $q_i = \phi(p_i)$ which is a prime ideal of height zero in $B$ for $1 \leq i \leq l$. Due to Lemma 2.4, we have $q_i = (T_{j_i})$ for some $j_i$ $(1 \leq j_i \leq l)$, i.e., any generic point of a irreducible component of $U$ is mapped by $\phi$ to a generic point of a irreducible component of $\text{Spec} B$.

Let us suppose that the map $i \mapsto j_i$ is not injective, i.e., there exist $i$ and $j$ $(i \neq j)$ such that $q_i = q_j$. Consider the Cartesian diagram

$$\begin{array}{ccc}
\{q_i\} \times \text{Spec} B & \longrightarrow & U \\
\phi_i \downarrow & & \phi \\
\{q_i\} & \longrightarrow & \text{Spec} B,
\end{array}$$

where $\phi_i: q_i \to \{q_i\}$ is the inclusion.
where the horizontal arrows are closed immersions and the vertical ones are étale. The scheme $U_i = \{q_i\} \times_{\text{Spec } B} U$ is also a normal crossing variety. Since $\{p_i\} \cap \{q_j\}$ is a closed subscheme (which contains $y$) of $U_i$, the multiplicity $l^U_i$ at $y$ in $U_i$ is greater than 1. But since the irreducible component $\{q_i\}$ is smooth, we have $l_{\phi_i(y)} = 1$. This is a contradiction since $l^U_i = l_{\phi_i(y)}$. Thus, the map $i \mapsto j_i$ is injective, i.e., there is at most one component over each component of Spec $B$. Moreover, in this case, we have $U_i = \{p_i\}$. Then the irreducible component $U_i$ is étale over a smooth scheme $\{q_i\}$, and hence the normal crossing variety $U$ is simple. Moreover, the prime ideal $p_i$ is a principal ideal $(z_i)$, where $z_i = \phi^* T_i$ for $1 \leq i \leq l$, since $\{p_i\} = \{q_i\} \times_{\text{Spec } B} U$ implies that $p_i = q_i \otimes_B A$.

Since the map $i \mapsto j_i$ is injective, we have $l \leq l_x$. Note that the multiplicity $l_y$ at $y$ in $U$ equals to $l_x$. Since the simple normal crossing variety $U$ consists of $l$ irreducible components, we have $l_y = l_x \leq l_x$. Hence we have $l_x = l_x$.

The scheme Spec $B$ is a normal crossing divisor in the $n$ dimensional affine space over $k(x)$. Hence, due to [3, Exposé 1. Proposition 8.1], any point in $U$ has a Zariski open neighborhood which is embedded in a smooth $k(x)$-variety as a normal crossing divisor. This implies that, replacing $U$ by its Zariski open neighborhood of $y$, we may assume that $U$ can be embedded in an affine smooth $k(x)$-scheme $V = \text{Spec } R$ of dimension $n$ as a normal crossing divisor. Let $\iota : U \hookrightarrow V$ be the closed immersion.

Finally, consider the Cartesian diagram

$$
\begin{array}{ccc}
U & \hookrightarrow & V \\
\phi \downarrow & & \Phi \downarrow \\
\text{Spec } B & \hookrightarrow & \text{Spec } k[T_1, \ldots, T_n],
\end{array}
$$

where $\Phi$ is an étale morphism. Set $Z_i = \Phi^* T_i \in \Gamma(V, \mathcal{O}_V)$ for $1 \leq i \leq n$. Then $Z_1, \ldots, Z_n$ form a regular parameter system at $\iota(y) \in V$. We also have $z_i = \iota^* Z_i$ for $1 \leq i \leq l_x$. It is clear that the closed subscheme $U$ in $V$ is defined by an ideal $(Z_1 \cdots Z_n)$. Then the proof of the theorem is completed. $\square$

The following lemma will be needed in the later arguments.

**Lemma 2.8** Let $(\varphi : U' \to X; z'_1, \ldots, z'_l)$ be a local chart on $X$ around some closed point and $(\psi : U \to U'; z_1, \ldots, z_l)$ a local chart on $U'$ around some closed point. Then $\psi : U \to U'$ is injective in codimension zero, i.e., it maps the generic points of irreducible components on $U$ injectively to those of $U'$.

**Proof.** Let $\eta \in U'$ be a codimension zero point. Since $U'$ is simple, $\{\eta\}$ is regular and so is $\{\eta\} \times_{U'} U$ whenever it is not empty. Then each connected component of $\{\eta\} \times_{U'} U$ is irreducible and its generic point is of codimension zero. Hence each connected component of $\{\eta\} \times_{U'} U$ is an irreducible component of $U$. Since any two of irreducible components of $U$ intersect, $\{\eta\} \times_{U'} U$ itself is an irreducible component of $U$. Hence, if $\xi \in U$ is a codimension zero point such that $\psi(\xi) = \eta$, we have $\{\xi\} = \{\eta\} \times_{U'} U$. In particular, there exists at most one such $\xi$. $\square$

For a normal crossing variety $X$, the normalization $\nu : \tilde{X} \to X$ of $X$ is defined as usual: The scheme $\tilde{X}$ is defined by the disjoint union of the normalizations of irreducible components of $X$ and $\nu : \tilde{X} \to X$ is the natural morphism. The normalization $\tilde{X}$ is a smooth $k$-scheme due to Theorem 2.4 and the following lemma.
Lemma 2.9 Let \( U \to Z \) be a étale morphism of \( k \)-varieties. Let \( \tilde{U} \to U \) and \( \tilde{Z} \to Z \) be normalizations of \( U \) and \( Z \), respectively. Then there exists a natural isomorphism \( \tilde{U} \sim U \times_Z \tilde{Z} \). In particular, the natural morphism \( U \to \tilde{Z} \) is étale.

Proof. Since \( U \times_Z \tilde{Z} \to \tilde{Z} \) is étale and \( \tilde{Z} \) is normal, \( U \times_Z \tilde{Z} \) is a normal variety. Hence there exists a unique morphism \( \phi: U \times_Z \tilde{Z} \to \tilde{U} \) which factors the morphism \( U \times_Z \tilde{Z} \to U \). Moreover \( \phi \) also factors the morphism \( U \times_Z \tilde{Z} \to \tilde{Z} \) since the last morphism is the unique morphism determined by the morphism \( U \times_Z \tilde{Z} \to Z \). Hence the natural morphism \( \phi: \tilde{U} \to U \times_Z \tilde{Z} \) is the inverse morphism of \( \phi \). \( \square \)

For a local chart \( (\varphi: U \to X; z_1, \ldots, z_l) \), the normalization of \( U \) is given by the disjoint union of all irreducible components and the natural morphism, i.e.,

\[ \nu_U: \tilde{U} = \prod_{i=1}^l U_i \to U, \]

where \( U_i \) is the irreducible component of \( U \) corresponding to the ideal \( (z_i) \).

Set \( \mathcal{D} = D \times_X \tilde{X} \), which is a divisor of \( \tilde{X} \).

Lemma 2.10 \( \mathcal{D} \) is a normal crossing divisor of \( \tilde{X} \).

Proof. Let \( (\varphi: U \to X; z_1, \ldots, z_l) \) be a local chart on \( X \). Then \( D_U := D \times_X U \) is nothing but the singular locus of \( U \) and is étale over \( D \). Consider the normalization \( \nu_U: \tilde{U} \to U \) as above. Set

\[ \overline{D_U} := D_U \times_U \tilde{U}. \]

Clearly, \( \overline{D_U} \) is a normal crossing divisor of \( \tilde{U} \) defined by an ideal \( (z_1 \cdots \hat{z}_i \cdots z_l) \) on \( U_i \).

There exists a natural morphism \( \overline{D_U} \to \mathcal{D} \). Since one can easily see that there exists a natural isomorphism

\[ \overline{D_U} \sim \mathcal{D} \times \tilde{X} \tilde{U} \]

and the morphism \( \tilde{U} \to \tilde{X} \) is étale due to Lemma 2.9, the morphism \( \overline{D_U} \to \mathcal{D} \) is étale. Then, considering all the local charts on \( X \), \( \mathcal{D} \) is a normal crossing divisor on \( \tilde{X} \) due to Proposition 2.2. \( \square \)

3 Tangent complex on a normal crossing variety

In this section, we recall the tangent complex and the infinitesimal normal bundle \( \mathcal{T}_X^1 \) of a normal crossing variety \( X \) which will play important roles in the subsequent sections.

Let \( X \) be a normal crossing variety over a field \( k \). For a local chart \( (\varphi: U = \text{Spec} A \to X; z_1, \ldots, z_l) \) of \( X \) around some closed point, we use the following notation in this and subsequent sections: Let \( V = \text{Spec} R \) and \( Z_1, \ldots, Z_l \) be as in Definition 2.3. Set \( I_j := (Z_j) \) and \( J_j := (Z_1 \cdots \hat{Z}_j \cdots Z_l) \) for \( 1 \leq j \leq l \). (If \( l = 1 \), we set \( J_1 = R \) for the convention.) Then \( A = R/I \) where \( I := I_1 \cdots I_l \). Moreover, the ideal \( I_j/I \subset A \) is generated by \( z_j = (Z_j \text{ mod } I) \) and is prime of height zero. Set \( J := J_1 + \cdots + J_l \). Then the singular locus \( D_U := D \times_X U \) of \( U \) is the closed subscheme defined by \( J \). We set \( Q := R/J \).

Note that, for \( 1 \leq j \leq l \), \( I_j/I J_j \) is a free \( A \)-module of rank one and is generated by \( \zeta_j := (Z_j \text{ mod } I J_j) \). There exists a natural isomorphism \( I_j/I J_j \otimes_A Q \sim I_j/J J_j \) of \( Q \)-modules which maps \( \zeta_j \otimes 1 \) to \( \zeta_j := (Z_j \text{ mod } J J_j) \). Moreover, there exists a natural isomorphism

\[ I/I^2 \sim I_1/I J_1 \otimes_A \cdots \otimes_A I_l/I J_l \]
of $A$-modules, and hence, the $A$-module $I/I^2$ is free of rank one and is generated
by $\zeta_1 \otimes \cdots \otimes \zeta_t$. We denote by $\pi_j$ the natural projection $I_j/I_j \rightarrow I_j/I \subset A$.

The cotangent complex of the morphism $k \rightarrow A$ is given by

$$L : 0 \rightarrow R \otimes_R A \xrightarrow{\delta} \Omega^1_{R/k} \otimes_R A \rightarrow 0,$$

where $\delta$ is defined by $R \rightarrow F : R \xrightarrow{d} \Omega^1_{R/k}$ with $F := Z_1 \cdots Z_l$ (cf. [4]). Then the tangent complex of $U$ is the complex

$$\text{Hom}_A(L', A) : 0 \rightarrow \Theta_{R/k} \otimes_R A \xrightarrow{\delta^*} \text{Hom}_A(R \otimes_R A, A) \rightarrow 0,$$

where $\Theta_{R/k} := \text{Hom}_R(\Omega^1_{R/k}, R)$.

We define

$$T^1_A = \text{Hom}_A(R \otimes_R A, A)/\delta^*(\Theta_{R/k} \otimes_R A).$$

Lemma 3.1 We have the natural isomorphism

$$T^1_A \cong \text{Hom}_A(I/I^2, A) \otimes_A Q.$$

Proof. Consider the exact sequence

$$0 \rightarrow I/I^2 \rightarrow \Omega^1_{R/k} \otimes_R A \rightarrow \Omega^1_{A/k} \rightarrow 0.$$

By definition, we have $T^1_A = \text{Coker}(\text{Hom}_A(\Omega^1_{R/k} \otimes_R A, A) \rightarrow \text{Hom}_A(I/I^2, A))$. Then one can show — by direct calculations — that $\text{Hom}_A(I/I^2, A) \rightarrow T^1_A$ is nothing but the “tensoring” morphism $\otimes_A Q$. Moreover, we have $T^1_A \cong \text{Ext}^1_{A/k}(\Omega^1_{A/k}, A)$. □

Considering all the local charts $U$ on $X$, these modules $T^1_X$ glue to an invertible $O_D$-module on $X$, which is denoted by $T^1_X$; this is well–known (cf. [1]) but, for the later purpose, we prove it in the following.

Suppose we have two local charts $(\varphi : U \rightarrow X; z_1, \ldots, z_l)$ and $(\varphi' : U' \rightarrow X; z'_1, \ldots, z'_{l'})$ and an étale morphism $\psi : U \rightarrow U'$ such that $\varphi = \varphi' \circ \psi$. (Because we are interested in the singular locus, we shall assume $l > 1$ and $l' > 1$.) For these local charts, we use all the notation as above. (For $U'$, we denote them by $A', I', J', \zeta'_j$, etc.) Let $f : A' \rightarrow A$ be the ring homomorphism corresponding to $\psi$. We shall show that the morphism $\psi$ induces naturally an isomorphism $T^1_{A'} \otimes_{Q'} Q \cong T^1_A$ of $Q$-modules.

Let $U_j$ (resp. $U'_j$) be the irreducible component of $U$ (resp. $U'$) corresponding to $I_j/I$ (resp. $I'_j/I'$) for $1 \leq j \leq l$ (resp. $1 \leq j \leq l'$). Since $\psi$ is étale and injective in codimension zero (Lemma 2.3), we may assume that the generic point of $U_j$ is mapped to that of $U'_j$ by $\psi$ for $1 \leq j \leq l$. In particular, we have $l \leq l'$. Then one sees easily that $U \times_U U'_j \cong U_j$ for $1 \leq j \leq l$. This implies that $A/(I_j/I) \cong (A'/(I'_j/I')) \otimes_{A'} A \cong A/((I'_j/I') \otimes_{A'} A)$, and hence,

$$I_j/I = (I'_j/I') \otimes_{A'} A, \quad (1 \leq j \leq l)$$

as ideals in $A$. For $1 \leq j \leq l$, we can set $f(z'_j) = u_j z_j$ for some $u_j \in A$. Here, each $u_j$ is determined up to modulo $J_j/I$. Due to (8), $u_j z_j$ generates the ideal $I_j/I$, and hence, $u_j$ is a unit in $A/(J_j/I)$ (and, of course, in $A/(J/I)$). (Note that $u_j$ is not necessarily a unit in $A$, since $A$ is not an integral domain for $l > 1$.) Then there exists an isomorphism (naturally induced by $f$) of $Q$-modules

$$\tau_j : I'_j/I' I'_j \otimes_{A'} Q \cong I_j/I I_j \otimes_{A} Q$$
by \( \xi_j \mapsto (u_j \mod J/I) \xi_j \). The natural projection \( \pi_i': I_i'/I_i' \to I_i'/I' \subset A' \) \((1 \leq i \leq l')\) and \( f \) induce an \( A \)-module morphism

\[
(10) \quad \bar{\rho}_i: I_i'/I_i' \otimes_{A'} A \to A.
\]

For \( 1 \leq j \leq l \), \( \bar{\rho}_j \) maps \( I_i'/I_i' \otimes_{A'} A \) surjectively onto \( I_j/I \), and for \( i > l \), \( \bar{\rho}_i \) is an isomorphism; because, for \( i > l \), one sees that \( \bar{\rho}_i(\xi_i' \otimes 1) = f(\xi_i') \) is an invertible element of \( A \) as follows: Since \( \psi \) is injective in codimension zero, the point \( I_i'/I' \) does not belong to \( \psi(U) \); hence \( \psi \) maps \( U = \text{Spec} A \) to \( \text{Spec} A'_{(I_i'/I')} \), and this implies the image of elements in \( I_i'/I' \) under \( f \) is invertible. Set \( \rho_i = \bar{\rho}_i \otimes_A Q \). Then these isomorphisms induce

\[
(11) \quad \tau = \tau_1 \otimes_Q \cdots \otimes_Q \tau_l \otimes_Q \rho_{l+1} \otimes_Q \cdots \otimes_Q \rho_P: I'/I^2 \otimes_{A'} Q \sim I/I^2 \otimes_A Q.
\]

The \( Q \)-dual of \( \tau \) is the desired isomorphism (cf. Lemma 3.1). One can easily check that this isomorphism \( \tau \) does not depend on parameters \( z_j' \), \( z_j \); it is canonically induced by \( f: A' \to A \). Hence, for any sequence of étale morphisms of local charts \( U \xrightarrow{\psi} U' \xrightarrow{\psi'} U'' \), we obviously have \( \tau'' = \psi' (\tau \otimes_Q Q) \), where \( \tau': I'/I^2 \otimes_{A'} Q \sim I/I^2 \otimes_A Q \), \( \tau''': I''/I'^2 \otimes_{A''} Q' \sim I'/I'^2 \otimes_{A'} Q' \), and \( \tau'''': I''/I'^2 \otimes_{A''} Q \sim I/I^2 \otimes_A Q \) are the isomorphisms defined as above with respect to \( \psi, \psi' \), and \( \psi'' \). Then one sees easily that there exists a unique \( \mathcal{O}_D \)-module whose restriction to each \( U \) is the \( \mathcal{O}_{D_U} \)-module corresponding to \( \mathcal{T}^1_\mathcal{X} \); and it is nothing but our desired \( \mathcal{O}_D \)-module \( \mathcal{T}^1_\mathcal{X} \). Note that there exists a natural isomorphism \( \mathcal{T}^1_\mathcal{X} \sim \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X) \).

Suppose \( X \) has a global NCD embedding \( X \hookrightarrow V \). Then by Lemma 3.1, the restriction of the normal bundle \( \mathcal{N}_{\mathcal{X} | V} \) to the singular locus \( D \) is isomorphic to \( \mathcal{T}^1_\mathcal{X} \). Hence we have the following:

**Proposition 3.2** If a normal crossing variety \( X \) over \( k \) is embedded into a smooth \( k \)-variety as a normal crossing divisor, then there exists a line bundle \( \mathcal{L}_\mathcal{X} \) on \( X \) such that \( \mathcal{L}_\mathcal{X} \otimes_{\mathcal{O}_X} \mathcal{O}_D \sim \mathcal{T}^1_\mathcal{X} \).

Let \( \mathcal{X} \to \Delta \) be a semistable reduction of schemes, i.e., a flat and generically smooth morphism between regular schemes with \( \Delta \) one-dimensional and every closed fiber is a normal crossing variety. Suppose \( X \to \text{Spec} k \) is isomorphic to a closed fiber of this family. Then one sees that the normal bundle \( \mathcal{N}_{\mathcal{X} | V} \) is trivial on \( X \), and so is \( \mathcal{T}^1_\mathcal{X} \).

**Definition 3.3** (cf. [2]) A normal crossing variety \( X \) is said to be \( d \)-semistable if \( \mathcal{T}^1_\mathcal{X} \) is the trivial line bundle on \( D \).

Due to the above observation, we have the following:

**Proposition 3.4** (cf. [2]) The \( d \)-semistability is a necessary condition for the existence of global smoothings of \( X \).

## 4 Logarithmic embeddings

In this section, we define the logarithmic embedding of a normal crossing varieties (cf. [31]). This concept is defined in terms of log geometry of Fontaine, Illusie, and Kazuya Kato (cf. [7]).
Let $X$ be a normal crossing variety over a field $k$. Suppose that $X$ has a NCD embedding $\iota: X \hookrightarrow V$. We denote the open immersion $V \setminus X \hookrightarrow V$ by $j$. We define a log structure on $X$ by

$$\iota^* (O_V \cap j_* O_{V \setminus X}) \rightarrow O_X,$$

where $\iota^*$ denotes the pull–back of log structures (cf. [7, (1.4)]). We call this the log structure associated to the NCD embedding $\iota: X \hookrightarrow V$. For a general normal crossing variety $X$, we cannot define the log structure of this type on $X$, because $X$ may not have a NCD embedding. But, as we have seen in Remark 2.6, $X$ has étale locally a NCD embedding. Then we can consider the log strcuture of this type for a general $X$ defined as follows:

**Definition 4.1** (cf. [11]) A log structure $M_X \rightarrow O_X$ is said to be of embedding type, if the following condition is satisfied: There exists an étale covering $\{\varphi_{\lambda}: U_{\lambda} \rightarrow X\}_{\lambda \in \Lambda}$ by local charts — with the NCD embeddings $\iota_{\lambda}: U_{\lambda} \hookrightarrow V_{\lambda}$ as in Definition 2.3 — such that, for each $\lambda \in \Lambda$, the restriction $M_{U_{\lambda}} := \varphi_{\lambda}^* M_X \rightarrow O_{U_{\lambda}}$ is isomorphic to the log structure associated to the NCD embedding $\iota_{\lambda}$. If $M_X \rightarrow O_X$ is a log structure of embedding type of $X$, we call the log scheme $(X, M_X)$ the logarithmic embedding.

Let $(X, M_X)$ be a logarithmic embedding. We can explicitly write this log structure $M_X$ étale locally. Let $\nu: \tilde{X} \rightarrow X$ be a normalization of $X$. Take a local chart $\varphi: U \rightarrow X$ with parameters $z_1, \ldots, z_l$ such that $M_U := \varphi^* M_X \rightarrow O_U$ is the log structure associated to the NCD embedding $\iota: U \hookrightarrow V$. Let $U = \bigcup_{i=1}^{l} U_i$ be the decomposition into irreducible components, where $U_i$ is the irreducible component corresponding to the ideal $(z_i)$. The normalization $\tilde{U} = \coprod_{i=1}^{l} U_i \rightarrow U$ is denoted by $\nu_U$. Note that, due to Lemma 2.9, we have $U \times_X \tilde{X} \cong \tilde{U}$. Define a homomorphism of monoids

$$\alpha: (\nu_U)_* N_{\tilde{U}} \rightarrow O_U$$

by $\alpha(e_{U_i}) = z_i$ for $i = 1, \ldots, l$, where $(e_{U_i})$ is the standard base of $(\nu_U)_* N_{\tilde{U}} = \bigoplus_{i=1}^{l} N_{U_i}$. Then $\alpha$ induces a log structure

$$O_{\tilde{U}} \bigoplus (\nu_U)_* N_{\tilde{U}} \rightarrow O_U.$$ 

**Proposition 4.2** The log structure $M_U \rightarrow O_U$ is isomorphic to $\{13\}$.

**Proof.** Let $Z_1, \ldots, Z_l \in \Gamma(V, O_V)$ be as in Definition 2.3. By definition of the log structure associated to the embedding $\iota: U \hookrightarrow V$, these $Z_1, \ldots, Z_l$ are sections of the sheaf $M_U$. Define a morphism

$$\psi: (\nu_U)_* N_{\tilde{U}} \rightarrow M_U$$

by $\psi(e_{U_i}) = Z_i$ for $1 \leq i \leq l$. Let

$$\tilde{\psi}: (\nu_U)_* N_{\tilde{U}} \rightarrow M_U/O_U^*$$

...
be the composition of \( \psi \) followed by the natural projection \( \mathcal{M}_U \to \mathcal{M}_U/\mathcal{O}_U^\times \). It is easy to see that \( \tilde{\psi} \) is injective. Since sections of \( \mathcal{O}_V \cap j_*\mathcal{O}_V^\times \) are precisely those of \( \mathcal{O}_V \) which may take zeros along \( \iota(U) \), these are written in the form \( uZ_1^{a_1} \cdots Z_l^{a_l} \) where \( u \in \mathcal{O}_V^\times \) and \( a_1, \ldots, a_l \in \mathbb{N} \). This implies that the morphism \( \tilde{\psi} \) is an isomorphism. Then, consider the exact sequence of sheaves of monoids

\[
1 \to \mathcal{O}_U^\times \to \mathcal{M}_U \to \mathcal{M}_U/\mathcal{O}_U^\times \to 1,
\]

where the second arrow is injective. This exact sequence splits since \( \tilde{\psi} \) is an isomorphism and \( \psi \) defines a cross section \( \mathcal{M}_U/\mathcal{O}_U^\times \to \mathcal{M}_U \). By this, we can easily obtain the desired result. \( \square \)

Thus, a log structure of embedding type is determined by the morphism \( \alpha: (\nu_U)_*\mathbb{N}_{\tilde{U}} \to \mathcal{O}_U \) such that \( \alpha_1(e_{U_i}) \) is a local defining function of the component \( U_i \) for each \( i = 1, \ldots, l \). Let \( \alpha' \) be another such homomorphism. Then — replacing \( U \) by sufficiently small Zariski open subset — we can take \( u_i \in \Gamma(U, \mathcal{O}_X^\times) \) such that \( \alpha'(e_{U_i}) = u_i\alpha(e_{U_i}) \) for each \( i \). Then the isomorphism of log structures of embedding type determined by \( \alpha \) and \( \alpha' \) is described by the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_U^\times \oplus (\nu_U)_*\mathbb{N}_{\tilde{U}} & \overset{\phi}{\longrightarrow} & \mathcal{O}_U^\times \oplus (\nu_U)_*\mathbb{N}_{\tilde{U}} \\
\phi \downarrow & & \phi \downarrow \\
\mathcal{O}_X & & \mathcal{O}_X,
\end{array}
\]

where \( \phi \) is defined by \( \phi(1, e_{U_i}) = (u_i, e_{U_i}) \) for each \( i = 1, \ldots, l \). In particular, the log structure of embedding type exists étale locally, and is unique up to isomorphisms.

**Corollary 4.3** For any logarithmic embedding \((X, \mathcal{M}_X)\), we have an exact sequence of abelian sheaves

\[
1 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathcal{M}_X^{gp} \longrightarrow \nu_*\mathbb{Z}_X \longrightarrow 0.
\]

**Proof.** Due to the local expression (13). \( \square \)

In the rest of this section, we prove the following theorem, which is the main theorem of this paper.

**Theorem 4.4** For a normal crossing variety \( X \), the logarithmic embedding of \( X \) exists if and only if there exists a line bundle \( L \) on \( X \) such that \( L \otimes \mathcal{O}_X \mathcal{O}_D \cong \mathcal{T}_X^1 \).

For the proof of this theorem, we shall prove some lemmas as follows. Let \((\varphi: U = \text{Spec} A \to X; z_1, \ldots, z_l)\) be a local chart on \( X \). Let the NCD embedding \( U \to V = \text{Spec} R \) and the ideals \( I_j, I, J_j, J \) of \( R \) be as in the previous section.

**Lemma 4.5** The natural morphism

\[
\bigoplus_{j=1}^l J_j/I \longrightarrow J/I
\]

of \( A \)-modules, induced by \( J_j \hookrightarrow J \), is an isomorphism.
Proof. The surjectivity is clear. We are going to show the injectivity. Take \( a_1Z_1 \cdots \hat{Z}_j \cdots Z_l \in J_j \) — where \( Z_1, \ldots, Z_l \) are as in the previous section — for \( 1 \leq j \leq l \) such that

\[
\sum_{j=1}^{l} a_j Z_1 \cdots \hat{Z}_j \cdots Z_l = b \cdot Z_1 \cdots Z_l,
\]

where \( a_j, b \in R \). Since \( R \) is an integral domain, \( a_j \) is divisible by \( Z_j \), and hence, we have \( a_j Z_1 \cdots \hat{Z}_j \cdots Z_l \equiv 0 \) (mod \( I \)). \( \square \)

Let \( \pi_j: I_j/II_j \to I_j/I \) and \( q_j: I_j/I \to I_j/JI_j(\cong I_j/II_j \otimes_A Q) \) where \( Q = R/J \) be the natural projections and set \( p_j := q_j \circ \pi_j \). Let \( q: I/I^2 \to I/I(\cong I/I^2 \otimes_A Q) \) be the natural projection.

**Lemma 4.6** Let \( M_1, \ldots, M_l \) be free \( A \)-modules of rank one and set \( M := M_1 \otimes_A \cdots \otimes_A M_l \). Suppose we are given an \( A \)-module isomorphism \( \bar{g}: M \cong I/I^2 \) and \( A \)-module homomorphisms \( g_j: M_j \to I_j/I \), for \( 1 \leq j \leq l \), such that

1. for each \( j \), there exists a free generator \( \delta_j \) of \( M_j \) such that \( g_j(\delta_j) = z_j \),
2. \( (q_1 \circ g_1) \otimes \cdots \otimes (q_l \circ g_l) = q \circ \bar{g} \).

Then there exists a unique collection \( \{ \bar{g}_j: M_j \cong I_j/II_j \}_{j=1}^l \) of \( A \)-isomorphisms such that \( \pi_j \circ \bar{g}_j = g_j \) for each \( j \) and \( \bar{g}_1 \otimes \cdots \otimes \bar{g}_l = \bar{g} \).

Proof. We fix the free generators \( \delta_j \) of \( M_j \) as above. Then \( M \) is generated by \( \delta_1 \otimes \cdots \otimes \delta_l \). Set \( \bar{g}(\delta_1 \otimes \cdots \otimes \delta_l) = v \zeta_1 \otimes \cdots \otimes \zeta_l \) where \( v \in A^\times \). By the second condition, we have \( v \equiv 1 \) (mod \( J/I \)), i.e.,

\[
v = 1 + \sum_{j=1}^{l} a_j z_1 \cdots \hat{z}_j \cdots z_l
\]

for \( a_j \in A \). We set \( u_j = 1 + a_j z_1 \cdots \hat{z}_j \cdots z_l \) and define \( \bar{g}_j \) by \( \bar{g}_j(\delta_j) = u_j \zeta_j \) for \( 1 \leq j \leq l \). Then, since \( v = u_1 \cdots u_l \), each \( u_j \) is a unit in \( A \) and \( \bar{g}_j \) is an isomorphism. Moreover, we have \( \bar{g}_1 \otimes \cdots \otimes \bar{g}_l = \bar{g} \) as desired. The uniqueness follows from Lemma 4.7. \( \square \)

**Proof of Theorem 4.4** We first prove the “if” part. This part is divided into four steps.

**Step 1:** Here, we shall describe the log structure of embedding type by another étale local expression. Let \( (\varphi: U = \text{Spec } A \to X; z_1, \ldots, z_l) \) be a local chart. For \( m = (m_1, \ldots, m_l) \in \mathbb{N}^l \), define an \( A \)-module \( P_m \) by

\[
P_m := (I_1/II_1)^{\otimes m_1} \otimes_A \cdots \otimes_A (I_l/II_l)^{\otimes m_l}.
\]

Each \( P_m \) is a free \( A \)-module of rank one and \( P_{(1, \ldots, 1)} \cong I/I^2 \). The natural projections \( \pi_j \) induce a natural \( A \)-homomorphism

\[
\sigma_m: P_m \to A.
\]

Define a monoid

\[
M := \left\{ (m, a) \mid m \in \mathbb{N}^l, \quad a : \text{a generator of } P_m \right\}.
\]
and a homomorphism $M \to A$ of monoids by $(m, a) \mapsto \sigma_m(a)$. Then the associated log structure $\alpha_U : \mathcal{M}_U \to \mathcal{O}_U$ of the pre-log structure $M \to A$ is that of embedding type on $U$.

**Step 2:** Now, we assume that we are given a line bundle $\mathcal{L}$ on $X$ satisfying $\mathcal{L} \otimes \mathcal{O}_X \cong (\mathcal{T}_X)^\vee$. Suppose we have two local charts $(\varphi : U \to X; z_1, \ldots, z_l)$ and $(\varphi' : U' \to X; z'_1, \ldots, z'_{l'})$ and an étale morphism $\psi : U \to U'$ such that $\varphi = \varphi' \circ \psi$. For these local charts, we use the notation as in the previous section; such as $U = \text{Spec} A \hookrightarrow V = \text{Spec} R$, $U' = \text{Spec} A' \hookrightarrow V' = \text{Spec} R'$, $f : A' \to A$, $I$, $I'$, etc. As in the previous section, we may assume $(I_j') \otimes A' A = I_j/I$ as ideals in $A$ for $1 \leq j \leq l$, and set $f(z'_j) = u_jz_j$ (each $u_j$ is determined up to modulo $J_j/I$). To give the line bundle $\mathcal{L}$ as above is equivalent to giving a compatible system of isomorphisms

$$\tilde{\tau} : I'/I^{l'} \otimes A' A \to I/I^l,$$

for all such $U \to U'$, with $\tilde{\tau} \otimes A Q = \tau$, where $\tau$ is defined as in (11). Then we shall show that $\tilde{\tau}$ induces canonically an isomorphism of log structures $\psi^* \mathcal{M}_{U'} \xrightarrow{\sim} \mathcal{M}_U$, and prove that these isomorphisms form a compatible system that the log structures $\mathcal{M}_U$ glue to a log structure of embedding type on $X$. Moreover — since local charts form an étale open basis (Corollary 2.3) — we can pass through this procedure replacing $U$ by its Zariski open subset if necessary. In particular, we may assume that each $u_j$ as above is a unit in $A$, because $(u_j \mod J/I)$ is a unit in $A/(J/I)$ (in case $l > 1$). Fix a locally constant section $w \in H^0(D, \mathcal{O}_D^\times)$ (Actually, we can take $w$ as any global section in $H^0(D, \mathcal{O}_D^\times)$ but, if we do so, the following argument have to be modified slightly.)

**Step 3:** (i) If $l = l' = 1$, $i.e.$, $I_1 = I$ and $I'_1 = I'$, then we set $\tilde{\tau}_1 : I'_1/I^{l'}_1 \otimes A' A \xrightarrow{\sim} I_1/I^l_1$ by $\tilde{\tau}_1 = \tilde{\tau}$.

(ii) If $l = 1$ and $l' > 1$, we define $\tilde{\tau}_1 : I'_1/I^{l'}_1 \otimes A' A \xrightarrow{\sim} I_1/I^l_1$ as follows: Suppose $\tilde{\tau}$ maps $\zeta'_1 \otimes \cdots \otimes \zeta'_{l'} \otimes 1$ to $v\zeta_1$, where $v \in A^\times$. Let $\tilde{\rho}_j : I'_j/I^{l'}_j \otimes A' A \to A$ be as (10), for $1 \leq i \leq l'$. Suppose, moreover, for each $\tilde{\rho}_i$, $i > 1$, maps $\zeta'_1 \otimes 1$ to $v_i \in A^\times$. Then, define $\tilde{\tau}_1$ by $\tilde{\tau}_1(\zeta'_1 \otimes 1) = w_Uv_2v_1\cdots v_{l'}\zeta_1$, where $w_\mathcal{U}$ is a non-zero scalar which coincides with $w$ restricted to $D_U$.

(iii) Suppose $l > 1$ and $l' > 1$. We claim that, under the conditions

\begin{align}
(16) \quad \pi_j \circ \tilde{\tau}_j = \tilde{\rho}_j, \quad (1 \leq j \leq l) \\
(17) \quad \tilde{\tau}_1 \otimes A \cdot \cdots \otimes A \tilde{\tau}_l \otimes A \tilde{\rho}_{l+1} \otimes A \cdots \otimes A \tilde{\rho}_l = \tilde{\tau},
\end{align}

the $A$-isomorphisms

$$\tilde{\tau}_j : I'_j/I^{l'}_j \otimes A' A \xrightarrow{\sim} I_j/I^l_j$$

exist uniquely for $1 \leq j \leq l$. Set $M_j : = I'_j/I^{l'}_j \otimes A' A$ and $g_j := \tilde{\rho}_j$ for $1 \leq j \leq l$. Define $\tilde{g}$ by $\tilde{g} \otimes A \tilde{\rho}_{l+1} \otimes A \cdots \otimes A \tilde{\rho}_l = \tilde{\tau}$ (this is possible since $\tilde{\rho}_i(\zeta'_1 \otimes 1)$ is a unit element in $A$ for $i > l$), which is obviously an isomorphism. Then — since we assumed each $u_i$ to be a unit in $A — M_j : = I'_j/I^{l'}_j \otimes A' A$, $g$, and $g_j$ satisfy the conditions in Lemma 4.6. Hence our claim follows from this lemma.

Note that, in any cases, we have the following commutative diagram:

\[
\begin{array}{ccc}
I'_j/I^{l'}_j & \to & I_j/I^l_j \\
\pi'_j \downarrow & & \downarrow \pi_j \\
A' & \xrightarrow{f} & A,
\end{array}
\]
for \(1 \leq j \leq l\); this follows from (16) in case \(l, l' > 1\), and is quite obvious in the other cases.

**Step 4:** These morphisms \(\tilde{\tau}_j\) induce the morphisms

\[
\gamma_m' : P_{m'} \to P_m,
\]

where \(m = (m_1, \ldots, m_l)\) for \(m' = (m_1, \ldots, m_{l'}) \in \mathbb{N}^{l'}.\) Then these \(\gamma_m'\) induce naturally a morphism of monoids \(M' \to M\) compatible with \(M' \to A', M \to A\) and \(f.\)

By the construction of these morphisms, the induced morphism of sheaves of monoids \(\gamma : \psi^*M_{U'} \to M_U\) is an isomorphism. By the commutative diagram (18), this isomorphism commutes the following diagram:

\[
\begin{align*}
\psi^*M_{U'} & \xrightarrow{\sim} M_U \\
\psi^*\alpha_{U'} & \downarrow \alpha_U \\
\mathcal{O}_U & \xrightarrow{\sim} \mathcal{O}_U;
\end{align*}
\]

hence \(\gamma\) is an isomorphism of log structures. Our construction of the isomorphism \(\gamma\) is canonical in the following sense: Suppose we are given a sequence of \(\text{étale}\) morphisms \(U \to U' \to U''\) of local charts (with \(U\) and \(U''\) sufficiently small), we have \(\gamma'' = \gamma \circ (\psi^*\gamma'),\) where \(\gamma : \psi^*M_{U'} \to M_{U}, \gamma' : \psi^*M_{U''} \to M_{U'}\) and \(\gamma'' : \psi^*\psi^*M_{U''} \to M_U\) are the isomorphisms of log structures defined as above corresponding to \(\psi, \psi'\) and \(\psi' \circ \psi,\) respectively. This follows from the naturality of \(\tau_j\) and \(\tilde{\rho}_j,\) and the compatibility of \(\tilde{\tau}'s.\)

Then there exists a unique log structure \(M_X\) on \(X\) which is of embedding type. Hence the “if” part is now proved.

Conversely, suppose we are given a log structure \(M_X\) of embedding type. Then we have an exact sequence (15) of abelian sheaves. Considering the cohomology exact sequence, we obtain a morphism

\[
\delta : H^0(X, \nu_*Z_X) \to H^1(X, \mathcal{O}_X^\times) (\cong Pic X).
\]

In \(H^0(X, \nu_*Z_X),\) we consider the element \(\delta\) which is defined by the image of \(1 \in Z_X\) under the diagonal morphism \(Z_X \to \nu_*Z_X.\) Then \(\delta(\delta)\) defines a line bundle \(\mathcal{L} = C_{M_X}\) on \(X.\) We shall show that this line bundle satisfies \(\mathcal{L} \otimes \mathcal{O}_D \sim (T_X^1)^\vee.\)

The line bundle \(\mathcal{L}\) is constructed as follows: the inverse image of \(\delta\) under \(M_X^{gp} \to \nu_*Z_X\) defines a principally homogeneous space over \(\mathcal{O}_X^\times\) and hence defines a line bundle, which is nothing but \(\mathcal{L}.\) Let \(U = \text{Spec} A\) be a local chart as above. Then the inverse image of \(\delta\) restricted to \(U\) gives a generator of an \(A\)-module \(I/I^2\) which is — due to Lemma 3.1 — a local lifting of \(T_X^1\) restricted to \(U.\) Hence \(\mathcal{L}\) satisfies the desired condition. \(\square\)

**Remark 4.7 1.** As we have seen above, the log structure of embedding type exists locally and is unique up to isomorphisms. The sheaf of germs of automorphisms of such a log structure is naturally isomorphic to \(\mathcal{K},\) where \(\mathcal{K}\) is defined by the exact sequence

\[
1 \to \mathcal{K} \to \mathcal{O}_X^\times \to \mathcal{O}_D^\times \to 1.
\]

This can be shown by the following steps: (i) any automorphism over a sufficiently small local chart \(U\) is given by \(\phi\) in the diagram (14) with \(\alpha = \alpha';\) (ii) \(\phi\) is determined by \(\{u_i\}\) with \(u_i \in \Gamma(U, \mathcal{O}_X^\times)\) such that \(z_i = u_i \cdot z_i\) for each \(i;\) (iii) hence such \(u_i\)'s are written in the form of \(u_i = 1 + a_i \cdot z_1 \cdots z_i \cdots z_l;\) (iv) due to Lemma 1.5, to give a system \(\{u_i\}\) is
equivarent to give \( u = u_1 \cdots u_i \) which is a section of \( K \). Hence the obstruction for the existence of log structures of embedding type lies in \( H^2(X,K) \). The proof of Theorem 4.4 shows that this class coincides with the obstruction class for a lifting of \((T^1_X)^\vee\) on \( X \), i.e., the image of \((T^1_X)^\vee\) under \( H^1(D,\mathcal{O}_D^\times) \to H^2(X,K) \).

2. One sees easily — by the proof of Theorem 4.4 — that there exists a natural surjective map

\[
\{ \text{isom. class of log structures of embedding type on } X \} \longrightarrow \{ \mathcal{L} \in \text{Pic } X \mid \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_D \cong (T^1_X)^\vee \} \tag{20}
\]

by \( \mathcal{M} \mapsto \mathcal{L}_\mathcal{M} \), where \( \mathcal{L}_\mathcal{M} \) is defined as in the proof of Theorem 4.4. If \( \mathcal{M}_X \) is associated to a global NCD embedding \( X \hookrightarrow V \), then \( \mathcal{L}_\mathcal{M}_X \) is nothing but the conormal bundle of \( X \) in \( V \). The set of isomorphism classes of log structures of embedding type on \( X \), is a principally homogeneous space over \( H^1(X,K) \). Then one sees easily that the map (20) is equivariant to \( H^1(X,K) \to \ker(\text{Ker}(H^1(X,\mathcal{O}_X^\times) \to H^1(D,\mathcal{O}_D^\times))) \); in this case, the logarithmic embeddings are determined by their “normal bundles.”

3. By the exact sequence (15), a log structure of embedding type \( \mathcal{M}_X \) on \( X \) defines an extension class in \( \text{Ext}^1_{\mathcal{Z}_X}(\nu_*\mathcal{Z}_{\tilde{X}},\mathcal{O}_X^\times) \). Under the morphism \( \text{Ext}^1_{\mathcal{Z}_X}(\nu_*\mathcal{Z}_{\tilde{X}},\mathcal{O}_X^\times) \to \text{Ext}^1_{\mathcal{Z}_X}(\mathcal{Z}_X,\mathcal{O}_X^\times) \), induced by the diagonal morphism \( \mathcal{Z}_X \to \nu_*\mathcal{Z}_{\tilde{X}} \), and the natural identification \( \text{Ext}^1_{\mathcal{Z}_X}(\mathcal{Z}_X,\mathcal{O}_X^\times) \cong \text{Pic } X \), this class is mapped to the class corresponding to the line bundle \( \mathcal{L}_\mathcal{M}_X \) defined as above. (The proof is straightforward and left to the reader.)

5 Logarithmic semistable reductions

**Definition 5.1** (cf. [6], [9]) A log structure of embedding type \( \mathcal{M}_X \to \mathcal{O}_X \) is said to be of *semistable type*, if there exists a homomorphism \( \mathcal{Z}_X \to \mathcal{M}_X^{\text{gp}} \) of abelian sheaves on \( X \) such that the diagram

\[
\mathcal{M}_X^{\text{gp}} \leftarrow \mathcal{Z}_X \quad \nu_* \mathcal{Z}_{\tilde{X}} \quad \mathcal{M}_X \quad \mathcal{M}_X^{\text{gp}} \to \nu_* \mathcal{Z}_{\tilde{X}}
\]

commutes, where \( \nu: \mathcal{Z}_X \to \nu_* \mathcal{Z}_{\tilde{X}} \) is the diagonal homomorphism, and \( \mathcal{M}_X^{\text{gp}} \to \nu_* \mathcal{Z}_{\tilde{X}} \) is the projection in (13).

If \( \mathcal{M}_X \) is a log structure of semistable type, the homomorphism \( \mathcal{Z}_X \to \mathcal{M}_X^{\text{gp}} \) induces the homomorphism \( \mathcal{N}_X \to \mathcal{M}_X \) of monoids by the following Cartesian diagram:

\[
\mathcal{N}_X \longrightarrow \mathcal{M}_X \quad \mathcal{Z}_X \longrightarrow \mathcal{M}_X^{\text{gp}}
\]

this follows easily from the local expression (13). This morphism defines a morphism of log schemes

\[
(X, \mathcal{M}_X) \longrightarrow (\text{Spec } K, \mathcal{N})
\]
Here, \((\text{Spec} \ k, N)\) is the standard point defined by \(N \rightarrow k\) which maps \(m \in N\) to \(0^m\). We call this morphism of log schemes the **logarithmic semistable reduction**. Logarithmic semistable reductions are log smooth in the sense of [7].

**Remark 5.2** Let \(f: X \rightarrow \Delta\) be a semistable reduction of schemes; i.e., a proper flat generically smooth morphism \(f\) with \(X\) a regular scheme and \(\Delta\) a one-dimensional regular local scheme, with the closed fiber \(X \rightarrow 0 = \text{Spec} \ k\) a normal crossing variety. Then this morphism induces canonically a logarithmic semistable reduction \((X, M_X) \rightarrow (\text{Spec} \ k, N)\) on the closed fiber as follows: We define a log structure \(M_X = \mathcal{O}_X \bigcap j_\ast \mathcal{O}_{X \setminus X} \hookrightarrow \mathcal{O}_X\) where \(j: X \setminus X \hookrightarrow X\) is an open immersion. Take a local parameter \(t \in \mathcal{O}_\Delta\) around \(0 = \text{Spec} \ k\). Then \(f^{-1}(t)\) belongs to \(M_X\). We define a homomorphism of monoids \(N_X \rightarrow M_X\) by \(1 \mapsto f^{-1}(t)\). Then this homomorphism extends to a morphism of log schemes \((X, M_X) \rightarrow (\Delta, 0)\), (21)

where the log structure on \(\Delta\) is the associated log structure of

\[ N \rightarrow \mathcal{O}_\Delta \quad \text{by} \quad m \mapsto t^m. \]

Taking the pull–back of (21) to the closed fiber, we get a logarithmic semistable reduction. Note that the monoid morphism \(N_X \rightarrow M_X\) induces \(Z_X \rightarrow M_X^{\text{gp}}\) which satisfies the condition in Definition 5.1. Hence, such a morphism \(Z_X \rightarrow M_X^{\text{gp}}\) for a general log structure of semistable type can be regarded as a “parametrization.”

**Remark 5.3** The logarithmic semistable reduction induced by a semistable reduction family, as in Remark 5.2, is regarded as the “closed fiber” of the morphism (21) of log schemes. Then, conversely, one can consider the theory of deformations which deal with liftings of the logarithmic semistable reductions. This is nothing but the **logarithmic deformation** of Kawamata–Namikawa [4], and also a part of the **log smooth deformation** developed in [6].

Using Theorem 4.4 — which is proved in the previous section — we get a new proof of the theorem of Kawamata–Namikawa as follows:

**Theorem 5.4** (cf. [4]) For a normal crossing variety \(X\), the log structure of semistable type on \(X\) exists if and only if \(X\) is \(d\)-semistable.

To prove the theorem, we need the following lemma:

**Lemma 5.5** Let \((X, M_X)\) be a logarithmic embedding. Consider the exact sequence (13) of abelian sheaves and the induced morphism

\[ \text{Hom}_{Z_X}(Z_X, \nu_{\ast}Z_{\hat{X}}) \xrightarrow{\delta} \text{Ext}^1_{Z_X}(Z_X, \mathcal{O}_{\hat{X}}) \]

Let \(\delta \in \text{Hom}_{Z_X}(Z_X, \nu_{\ast}Z_{\hat{X}})\) be the diagonal morphism. Then, under the natural identification \(\text{Ext}^1_{Z_X}(Z_X, \mathcal{O}_{\hat{X}}) \sim \text{Pic} X\), we have

\[ \delta(\delta) = [\mathcal{L}_{M_X}], \]

where \(\mathcal{L}_{M_X}\) is the line bundle defined in the previous section.
Proof. This lemma follows from the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_X, \nu_* \mathbb{Z}_X) & \xrightarrow{\delta} & \text{Ext}^1_{\mathbb{Z}_X}(\mathbb{Z}_X, \mathcal{O}_X^\times) \\
\cong & & \cong \\
H^0(X, \nu_* \mathbb{Z}_X) & \rightarrow & H^1(X, \mathcal{O}_X^\times)
\end{array}
\]

where the vertical morphisms are natural isomorphisms and the definition of the line bundle \(L_{M_X}\). \(\square\)

Proof of Theorem 5.4. Suppose \(M_X\) is a log structure of semistable type. Consider the exact sequence

\[
\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_X, M_{X}^{gp}) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_X, \nu_* \mathbb{Z}_X) \xrightarrow{\delta} \text{Ext}^1_{\mathbb{Z}_X}(\mathbb{Z}_X, \mathcal{O}_X^\times)
\]

induced by (15). The “parametrization” morphism \(\mathbb{Z}_X \rightarrow M_{X}^{gp}\) is mapped to \(\delta\) by \(\pi\). This implies that the line bundle \(L_{M_X}\) is trivial. Then so is \((T_X^1)^{\vee}\) because \(L_{M_X} \otimes \mathcal{O}_X \mathcal{O}_D\) is isomorphic to \((T_X^1)^{\vee}\).

Conversely, if \(X\) is \(d\)-semistable, there exists at least one log structure of embedding type on \(X\) due to Theorem 4.3. Since \(L_{M_X} \otimes \mathcal{O}_X \mathcal{O}_D\) is trivial, we can take the log structure \(M_X\) of embedding type such that the corresponding line bundle \(L_{M_X}\) is trivial (due to the natural surjection (20)). Since the obstruction for the existence of a morphism \(\mathbb{Z}_X \rightarrow M_{X}^{gp}\) which is mapped to \(\delta\), is nothing but the class \([L_{M_X}]\), we deduce that \(M_X\) is of semistable type. \(\square\)

As is shown in the above proof, the log structure of semistable type on \(X\) is — considering the natural surjection (20) — the log structure of embedding type which is mapped to the trivial bundle on \(X\). Hence we have the following:

**Corollary 5.6** Let \(X\) be a proper, \(d\)-semistable normal crossing variety, and assume that the singular locus \(D\) is connected. Then, the log structure of semistable type on \(X\) exists uniquely.

**Example 5.7** Let \(X := X_0 \cup \cdots \cup X_N\) be a chain of surfaces defined as follows: Each \(X_i\) is the Hirzebruch surface of degree \(a_i \leq 0\). The surfaces \(X_{i-1}\) and \(X_i\) are connected by identifying the section \(s'_{i-1}\) on \(X_{i-1}\) and the one \(s_i\) on \(X_i\), where \((s'_{i-1})^2 = a_{i-1}\) and \((s_i)^2 = -a_i\), for \(1 \leq i \leq N\). Then \(X\) has a log structure of embedding type if and only if \(a_i | (a_{i-1} + a_{i+1})\) for \(1 \leq i \leq N - 1\), while \(X\) has a log structure of semistable type if and only if \(a_0 = a_1 = \cdots = a_N\).

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