THE STRUCTURE OF ALGEBRAIC COVARIANT DERIVATIVE CURVATURE TENSORS

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Abstract. We use the Nash embedding theorem to construct generators for the space of algebraic covariant derivative curvature tensors.

1. Introduction

Let $M$ be an $m$ dimensional Riemannian manifold. To a large extent, the geometry of $M$ is the study of the Riemannian curvature $R \in \otimes^4 T^* M$ which is defined by the Levi-Civita connection $\nabla$ and, to a lesser extent, the study of the covariant derivative $\nabla R$. For example, $M$ is a local symmetric space if and only if $\nabla R = 0$; note that local symmetric spaces are locally homogeneous.

It is convenient to work in the algebraic context. Let $V$ be an $m$-dimensional real vector space. Let $\mathcal{A}(V) \subset \otimes^4 V^*$ and $\mathcal{A}_1(V) \subset \otimes^5 V^*$ be the spaces of all algebraic curvature tensors and all algebraic covariant derivative tensors, respectively, i.e., those tensors $A$ and $A_1$ having the symmetries of $R$ and of $\nabla R$:

$$A(x, y, z, w) = A(z, w, x, y) = -A(y, x, z, w),$$

$$A_1(x, y, z, w; v) = A_1(z, w, x, y; v) = -A_1(y, x, z, w; v),$$

$$A_1(x, y, z, w; v) + A_1(y, z, w; v) + A_1(z, x, y, w; v) + A_1(z, x, y, w; v) = 0,$$

$$A_1(x, y, z, w; v) + A_1(x, y, v, w; z) + A_1(x, y, v, w; z) = 0.$$

Let $\mathcal{S}^p(V) \subset \otimes^p V^*$ be the space of totally symmetric $p$ forms. If $\Psi \in \mathcal{S}^2(V)$ and if $\Psi_1 \in \mathcal{S}^3(V)$, define $A_\Psi \in \mathcal{A}(V)$ and $A_{1, \Psi, \Psi_1} \in \mathcal{A}_1(V)$ by:

$$A_\Psi(x, y, z, w) = \Psi(x, y) \Psi(y, z) - \Psi(x, z) \Psi(y, w),$$

$$A_{1, \Psi, \Psi_1}(x, y, z, w; v) = \Psi_1(x, y, v) \Psi_1(y, z) + \Psi_1(x, z, v) \Psi_1(y, w) - \Psi_1(x, z, v) \Psi_1(y, w, v).$$

If one thinks of $\Psi_1$ as the symmetrized covariant derivative of $\Psi$, then $A_{1, \Psi, \Psi_1}$ can be regarded, at least formally speaking, as the covariant derivative of $A_\Psi$.

Fiedler [7] used group representation theory to show:

Theorem 1.1 (Fiedler).

1. $\mathcal{A}(V) = \text{Span}_{\Psi \in \mathcal{S}^2(V)} \{A_\Psi\}$.

2. $\mathcal{A}_1(V) = \text{Span}_{\Psi \in \mathcal{S}^2(V), \Psi_1 \in \mathcal{S}^3(V)} \{A_{1, \Psi, \Psi_1}\}$.

Let $A \in \mathcal{A}(V)$ and $A_1 \in \mathcal{A}_1(V)$ be given. Choose $\nu(A)$ and $\nu_1(A_1)$ minimal so that there exist $\Psi_i \in \mathcal{S}^2(V)$, $\tilde{\Psi}_j \in \mathcal{S}^2(V)$, $\tilde{\Psi}_{1,j} \in \mathcal{S}^3(V)$, and constants $\lambda_i, \lambda_{1,j}$ so:

$$A = \sum_{1 \leq i \leq \nu(A)} \lambda_i A_{\Psi_i}, \quad A_1 = \sum_{1 \leq j \leq \nu_1(A_1)} \lambda_{1,j} A_{1, \tilde{\Psi}_{1,j}, \tilde{\Psi}_{1,j}}.$$
Set
\[ \nu(m) := \sup_{A \in \mathcal{A}(V)} \nu(A) \quad \text{and} \quad \nu_1(m) := \sup_{A_1 \in \mathcal{A}_1(V)} \nu_1(A_1). \]
The main result of this paper is the following:

**Theorem 1.2.** Let \( m \geq 2 \).

1. \( \frac{1}{2}m \leq \nu(m) \) and \( \frac{1}{2}m \leq \nu_1(m) \).
2. \( \nu(m) \leq \frac{1}{2}m(m + 1) \) and \( \nu_1(m) \leq \frac{1}{2}m(m + 1) \).

We shall establish the lower bounds of Assertion (1) in Section 2. The upper bound given in Assertion (2) for \( \nu(m) \) is due to Díaz-Ramos and García-Ríó [4] who used the Nash embedding theorem [17]; they also gave a separate argument to show \( \nu(2) = 1 \) and \( \nu(3) = 2 \). In Section 3 we shall generalize their approach to establish the following simultaneous ‘diagonalization’ result from which Theorem 1.2 (2) will follow as a Corollary:

**Theorem 1.3.** Let \( V \) be an \( m \) dimensional vector space. Let \( A \in \mathcal{A}(V) \) and let \( A_1 \in \mathcal{A}_1(V) \) be given. There exists \( \Psi_i \in S^2(V) \) and \( \Psi_{1,i} \in S^3(V) \) so that
\[ A = \sum_{1 \leq i \leq \frac{1}{2}m(m+1)} A_{\Psi_i} \quad \text{and} \quad A_1 = \sum_{1 \leq i \leq \frac{1}{2}m(m+1)} A_{1,\Psi_i,\Psi_{1,i}}. \]

The study of the tensors \( A_{\Psi} \) arose in the original instance from the Osserman conjecture and related matters; we refer to [9] [11] for a more extensive discussion than is possible here, and content ourselves with only a very brief introduction to the subject.

1.1. **The Jacobi operator.** If \( M \) is a pseudo-Riemannian manifold of signature \((p,q)\) and dimension \( m = p + q \), let \( S^+(M) \) (resp. \( S^-(M) \)) be the bundle of unit spacelike (resp. timelike) tangent vectors. The Jacobi operator \( J(x) \) for \( x \in TM \) is the self-adjoint endomorphism of \( TM \) characterized by the identity:
\[ g(J(x)y, z) = R(y, x, x, z). \]
One says that \( M \) is **spacelike Osserman** (resp. **timelike Osserman**) if the eigenvalues of \( J(\cdot) \) are constant on \( S^+(M) \) (resp. \( S^-(M) \)). It turns out these two notions are equivalent and such a manifold is simply said to be **Osserman**.

Restrict for the moment to the Riemannian setting \((p = 0)\). If \( M \) is a local rank 1 symmetric space or is flat, then the local isometries of \( M \) act transitively on the sphere bundle \( S(M) = S^+(M) \) and hence the eigenvalues of \( J(\cdot) \) are constant on \( S(M) \) and \( M \) is Osserman. Osserman [22] wondered if the converse held; this question has been called the Osserman conjecture by subsequent authors. The conjecture has been answered in the affirmative if \( m \neq 16 \) by work of Chi [3] and Nikolayevsky [13] [19] [20].

In the Lorentzian setting \((p = 1)\), an Osserman manifold has constant sectional curvature [2] [5]. In the higher signature setting \((p > 1, q > 1)\) it is more natural to work with the Jordan normal form rather than just the eigenvalue structure. One says that \( M \) is **spacelike Jordan Osserman** (resp. **timelike Jordan Osserman**) if the Jordan normal form of \( J(\cdot) \) is constant on \( S^+(M) \) (resp. \( S^-(M) \)); these two notions are not equivalent. The following example is instructive. Let \((\vec{x}, \vec{y})\) for \( \vec{x} = (x_1, \ldots, x_p) \) and \( \vec{y} = (y_1, \ldots, y_p) \) be coordinates on \( \mathbb{R}^p \) where \( p \geq 3 \). Let \( f = f(\vec{x}) \in C^\infty(\mathbb{R}^p) \). Define a pseudo-Riemannian metric \( g_f \) of signature \((p,p)\) on \( \mathbb{R}^{2p} \) by setting
\[ g_f(\partial_i^x, \partial_j^x) = \partial_i^x f \cdot \partial_j^x f, \quad g_f(\partial_i^y, \partial_j^y) = 0, \quad \text{and} \quad g_f(\partial_i^x, \partial_j^y) = g_f(\partial_i^y, \partial_j^x) = \delta_{ij}. \]
Let \( \Psi \) be the Euclidean Hessian:
\[ \Psi(\partial_i^x, \partial_j^x) = \partial_i^x \partial_j^x f, \quad \Psi(\partial_i^y, \partial_j^y) = 0, \quad \text{and} \quad \Psi(\partial_i^x, \partial_j^y) = \Psi(\partial_i^y, \partial_j^x) = 0. \]
One then has that $R = A_1 \Psi$. We suppose that the restriction of $\Psi$ to $\text{Span}\{\partial_1^e, \partial_2^e\}$ is positive definite henceforth. Then $M$ is a complete pseudo-Riemannian manifold which is spacelike and timelike Jordan Osserman. Similarly set

$$\Psi_1(\partial_1^e, \partial_2^e, \partial_3^e) = \partial_1^e \partial_2^e \partial_3^e f$$

and extend $\Psi_1$ to vanish if any entry is $\partial_i^e$. One has $\nabla R = A_1, \Psi, \Psi_1$; thus if $f$ is not quadratic, $M$ is not a local symmetric space. With a bit more work one can show that for generic such $f$, $M$ is curvature homogeneous but not locally affine homogeneous. We refer to [5, 14] for further details.

1.2. The skew-symmetric curvature operator. Let $\{e_1, e_2\}$ be an orthonormal basis for an oriented spacelike (resp. timelike) 2 plane $\pi$. The skew-symmetric curvature operator $\mathcal{R}(\pi)$ is characterized by the identity

$$g(\mathcal{R}(\pi)y, z) = R(e_1, e_2, y, z);$$

it is independent of the particular orthonormal basis chosen. One says that $M$ is spacelike Ivanov-Petrova (resp. timelike Ivanov-Petrova) if the eigenvalues of $\mathcal{R}()$ are constant on the Grassmannian of oriented spacelike (resp. timelike) 2-planes; these two notions are equivalent and such a manifold is simply said to be Ivanov-Petrova. The notions spacelike Jordan Ivanov-Petrova and timelike Jordan Ivanov-Petrova are defined similarly and are not equivalent.

The Riemannian Ivanov-Petrova manifolds have been classified [10, 13, 21]; they have also been classified in the Lorentzian setting [23] if $m \geq 10$. For all these manifolds, the curvature tensors have the form $R = A_\Psi$ where $\Psi$ is an idempotent isometry and $\mathcal{R}(\pi)$ always has rank 2. Conversely, in the algebraic setting, if $R$ is a spacelike Jordan Ivanov-Petrova algebraic curvature tensor on a vector space of signature $(p, q)$ where $q \geq 5$ and where $\text{Rank}\{\mathcal{R}()\} = 2$, then there exist $\lambda$ and $\Psi$ so that $R = \lambda A_\Psi$. This once again motivates the study of these tensors. Unfortunately, the situation in the indefinite setting is again quite different. There exist spacelike Ivanov-Petrova manifolds of signature $(s, 2s)$ where $\mathcal{R}(\pi)$ has rank 4 and where the curvature tensor does not have the form $R = A_\Psi$. We refer to [15] for further details.

1.3. The Szabó operator. There is an analogous operator to the Jacobi operator which is defined by $\nabla R$. The Szabó operator $J_1(x)$ is the self-adjoint endomorphism of $TM$ characterized by $g(J_1(x)y, z) = \nabla R(y, x, x, z; x)$. One says that $M$ is spacelike Szabó (resp. timelike Szabó) if the eigenvalues of $J_1(\cdot)$ are constant on $S^+(M)$ (resp. $S^-(M)$); these notions are equivalent and such a manifold is simply said to be Szabó. The notion spacelike (resp. timelike) Jordan Szabó is defined similarly.

In his study of 2 point symmetric spaces, Szabó [23] gave a very lovely topological argument showing that any Riemannian Szabó manifold is necessarily a local symmetric space—i.e. $\nabla R = 0$. This result was subsequently extended to the Lorentzian case [16]. In the higher signature setting, again the situation is unclear. The metric $g_1$ described in Display [16] defines a Szabó pseudo-Riemannian manifolds of signature $(p, p)$.

Even in the algebraic setting, there are no known non-zero elements $A_1 \in \mathcal{A}(V)$ which are spacelike Jordan Szabó. It has been shown [12] that if $A_1$ is a spacelike Jordan Szabó algebraic covariant derivative curvature tensor on a vector space of signature $(p, q)$, where $q \equiv 1 \mod 2$ and $p < q$ or where $q \equiv 2 \mod 4$ and $p < q - 1$, then $A_1 = 0$. This algebraic result yields an elementary proof of the geometrical fact that any pointwise totally isotropic pseudo-Riemannian manifold with such a signature $(p, q)$ is locally symmetric. The general question of finding non-trivial spacelike Jordan Szabó covariant algebraic curvature tensors, or conversely showing non exist, remains open.
The examples discussed above motivate consideration of the tensors $A_{1,\Psi,\Psi}$, and more generally of tensors which are combinations of these. We hope that Theorems 1.2 and 1.3, although of interest in their own right, will play a central role in these investigations.

2. A LOWER BOUND FOR $\nu(m)$ AND FOR $\nu_1(m)$

Let $V$ be an $m$ dimensional vector space, let $A \in \mathcal{A}(V)$, and let $A_1 \in \mathcal{A}_1(V)$. Give $V$ a positive definite inner product $\langle \cdot, \cdot \rangle$. The associated curvature operators are then defined by the identities:

$$\langle \mathcal{R}_A(\xi_1, \xi_2)z, w \rangle = A(\xi_1, \xi_2; z, w),$$

and

$$\langle \mathcal{R}_{A_1}(\xi_1, \xi_2, \xi_3)z, w \rangle = A_1(\xi_1, \xi_2, z; \xi_3).$$

Theorem 1.2 (1) will follow from the following Lemma:

**Lemma 2.1.** Let $V$ be a vector space of dimension $m = 2m$ or $m = 2n+1$.

1. If $\Psi \in S^2(V)$ and if $\Psi_1 \in S^3(V)$, then for any $\xi_1, \xi_2, \xi_3 \in V$ one has:

   $$\text{Rank}\{\mathcal{R}_{A_1}(\xi_1, \xi_2)\} \leq 2 \quad \text{and} \quad \text{Rank}\{\mathcal{R}_{A_1,\Psi,\Psi_1}(\xi_1, \xi_2, \xi_3)\} \leq 2.$$

2. If $A \in \mathcal{A}(V)$ and $A_1 \in \mathcal{A}_1(V)$, then for any $\xi_1, \xi_2, \xi_3 \in V$ one has:

   $$\text{Rank}\{\mathcal{R}_A(\xi_1, \xi_2)\} \leq 2\nu(A) \quad \text{and} \quad \text{Rank}\{\mathcal{R}_{A_1}(\xi_1, \xi_2, \xi_3)\} \leq 2\nu(A_1).$$

3. There exist $A \in \mathcal{A}(V)$, $A_1 \in \mathcal{A}_1(V)$, and $\xi_1, \xi_2, \xi_3 \in V$ so:

   $$\text{Rank}\{\mathcal{R}_A(\xi_1, \xi_2)\} = 2m \quad \text{and} \quad \text{Rank}\{\mathcal{R}_{A_1}(\xi_1, \xi_2, \xi_3)\} = 2\bar{m}.$$

**Proof.** If $\Psi \in S^2(V)$ and $\Psi_1 \in S^3(V)$, let $\psi$ and $\psi_1(\cdot)$ be the associated self-adjoint endomorphisms characterized by the identities

$$\langle \psi x, y \rangle = \Psi(x, y) \quad \text{and} \quad \langle \psi_1(z)x, y \rangle = \Psi_1(x, y, z).$$

Assertion (1) follows from the expression:

$$\mathcal{R}_{A_1}(\xi_1, \xi_2) = \{\Psi(\xi_2, y)\psi_1 - \{\Psi_1(\xi_2, y)\psi_1\} \xi_2, \quad \text{and} \quad \mathcal{R}_{A_1,\Psi,\Psi_1}(\xi_1, \xi_2, \xi_3) = \{\Psi(\xi_2, y)\psi_1 - \{\Psi_1(\xi_2, y)\psi_1\} \xi_2, \xi_3\}.$$

Let $A_i := A_{\psi_i}$, $A_{1,i} := A_{1,\psi_i,\psi_i}$, $R_i := R_{A_i}$, and $R_{1,i} := R_{A_{1,i}}$. Set

$$A = \sum_{1 \leq i \leq \nu(A)} A_i \quad \text{and} \quad A_1 = \sum_{1 \leq i \leq \nu(A_1)} A_{1,i}.$$

Assertion (2) follows from Assertion (1) as

$$\text{Rank}\{\mathcal{R}_A(\cdot)\} = \text{Rank}\{\sum_{1 \leq i \leq \nu(A)} \mathcal{R}_i(\cdot)\} \leq \sum_{1 \leq i \leq \nu(A)} \text{Rank}\{\mathcal{R}_i(\cdot)\} \leq 2\nu(A),$$

$$\text{Rank}\{\mathcal{R}_{A_1}(\cdot)\} = \text{Rank}\{\sum_{1 \leq i \leq \nu(A_1)} \mathcal{R}_{1,i}(\cdot)\} \leq \sum_{1 \leq i \leq \nu(A_1)} \text{Rank}\{\mathcal{R}_{1,i}(\cdot)\} \leq 2\nu(A_1).$$

If $\dim(V) = 2m$, let $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ be an orthonormal basis for $V$; if $\dim(V)$ is odd, the argument is similar and we simply extend $A$ and $A_1$ to be trivial on the additional basis vector. Define the non-zero components of $\Psi_i \in S^2(V)$ and $\Psi_{1,i} \in S^3(V)$ by:

$$\Psi_{i}(e_j, e_k) = \Psi_{i}(f_j, f_k) = \delta_{ij}\delta_{ik},$$

$$\Psi_{1,i}(e_j, e_k, e_l) = \Psi_{1,i}(f_j, f_k, f_l) = \delta_{ij}\delta_{ik}\delta_{il};$$

$\Psi_{i}(\cdot, \cdot)$ and $\Psi_{1,i}(\cdot, \cdot, \cdot)$ vanish if both an ‘$e$’ and an ‘$f$’ appear. Let

$$A := \sum_{1 \leq i \leq \bar{m}} A_i, \quad A_1 := \sum_{1 \leq i \leq \bar{m}} A_{1,i},$$

$$\xi_1 := e_1 + \ldots + e_{\bar{m}}, \quad \xi_2 := f_1 + \ldots + f_{\bar{m}}, \quad \xi_3 := \xi_1 + \xi_2.$$
We may then complete the proof of Assertion (3) by computing:

\[ \mathcal{R}_A(\xi_1, \xi_2)e_i = \mathcal{R}_1(e_i, f_i)e_i = -f_i, \]
\[ \mathcal{R}_A(\xi_1, \xi_2)f_i = \mathcal{R}_1(e_i, f_i)f_i = e_i, \]
\[ \mathcal{R}_A(\xi_1, \xi_2, \xi_3)e_i = \mathcal{R}_{1, i}(e_i, f_i, e_i + f_i)e_i = -2f_i \]
\[ \mathcal{R}_A(\xi_1, \xi_2, \xi_3)f_i = \mathcal{R}_{1, i}(e_i, f_i, e_i + f_i)f_i = 2e_i. \]

3. GEOMETRIC REALIZABILITY

Henceforth, let \( \langle \cdot, \cdot \rangle \) be a non-singular innerproduct on an \( m \)-dimensional vector space \( V \), let \( A \in \mathcal{A}(V) \) and let \( A_1 \in \mathcal{A}(V) \).

Although the following is well-known, see for example Belger and Kowalski [1] where a more general result is established, we shall give the proof to keep the development as self-contained as possible and to establish notation needed subsequently.

**Lemma 3.1.**

1. If \( g \) is a pseudo-Riemannian metric on \( \mathbb{R}^m \) with \( \partial g_{jk}(0) = 0 \), then:
   \( R_{ijk}(0) = \frac{1}{2} \{ \partial_k g_{ijl} + \partial_j g_{kil} - \partial_l g_{kij} - \partial_i g_{klj} \}(0) \).
   \( R_{ijk}(0) = \left( \partial_k \Gamma_{i j l} - \partial_l \Gamma_{i j k} \right)(0) \).
   \( R_{ikjn}(0) = \{ \partial_n R_{ijk}(0) \} \).

2. There exists the germ of a pseudo-Riemannian metric \( g \) on \( (\mathbb{R}^m, 0) \) and an isomorphism \( \Xi \) from \( T_0(\mathbb{R}^m) \) to \( V \) so that
   \( \Xi^* \langle \cdot, \cdot \rangle = g|_{T_0(\mathbb{R}^m)} \).
   \( \Xi^* A = R|_{T_0(\mathbb{R}^m)} \).
   \( \Xi^* A_1 = \nabla R|_{T_0(\mathbb{R}^m)} \).

*Proof.* Since the 1 jets of the metric vanish at the origin, we have

\[ \Gamma_{ijk} := g(\nabla_{\partial_i} \partial_j, \partial_k) = \frac{1}{2} (\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}) = O(|x|), \]
\[ R_{ijk}(0) = \{ \partial_k \Gamma_{i j l} - \partial_l \Gamma_{i j k} \}(0), \quad \text{and} \]
\[ R_{ikjn}(0) = \{ \partial_n R_{ijk}(0) \} \).

Assertion (1) now follows; see, for example, [11] [cf Lemma 1.1.1] for further details. To prove the second assertion, choose an orthonormal basis \( \{ e_1, \ldots, e_m \} \) for \( V \) so that \( \langle e_i, e_j \rangle = \pm \delta_{ij} \); we use this orthonormal basis to identify \( V = \mathbb{R}^m \). Let \( A_{ijk} \) and \( A_{i, jkl} \) denote the components of \( A \) and of \( A_1 \), respectively. Define

\[ g_{ik} = \langle e_i, e_k \rangle = \frac{1}{3} \sum_{j} A_{ijk} x_j x_l - \frac{1}{6} \sum_{j} A_{i, jkl} x_j x_l x_n. \]

Clearly \( g_{ik} = g_{ki} \). As \( g|_{T_0 \mathbb{R}^m} = \langle \cdot, \cdot \rangle \), \( g \) is non-degenerate on some neighborhood of \( 0 \). Since the 1 jets of the metric vanish at 0 we have by Assertion (1) that

\[ R_{ijk}(0) = \frac{1}{6} \{ -A_{jk,l} - A_{jkl} - A_{i,jlk} - A_{j,lk} + A_{i,jl} + A_{ijkl} \} \]
\[ = \frac{1}{6} \{ 4A_{ijkl} - 2A_{il,jk} - 2A_{iklj} \} = A_{ijkl}, \]
\[ R_{ikjn}(0) = \frac{1}{12} \{ -A_{jkln} + A_{jkl} + A_{jnkl} - A_{jkn} - A_{jnl} - A_{jnk} - A_{jnk} \}
- A_{ijkn} + A_{i,jkn} - A_{i,nkj} + A_{i,nkj} - A_{i,jkn} - A_{i,jkn}
+ A_{i,jkn} + A_{jkn} + A_{jln} + A_{jln} + A_{jln} + A_{jln} + A_{jln} + A_{jln}
+ A_{i,jkn} + A_{jkn} + A_{jln} + A_{jln} + A_{jln} + A_{jln} + A_{jln} + A_{jln}
\]
We suppose the inner product $\langle \cdot , \cdot \rangle$ is positive definite henceforth. We apply the Nash embedding theorem \[17\] to find an embedding $f : \mathbb{R}^m \rightarrow \mathbb{R}^{m+\kappa}$ realizing the metric $g$ constructed in Lemma 3.1. By writing the submanifold as a graph over its tangent plane, we can choose coordinates $(x, y)$ on $\mathbb{R}^{m+\kappa}$ where $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_\kappa)$ so that

$$f(x) = (x, f_1(x), \ldots, f_\kappa(x)) \quad \text{where} \quad df_{\nu}(0) = 0 \quad \text{for} \quad 1 \leq \nu \leq \kappa.$$  

Since $f_\nu(\partial_x^i) = (0, \ldots, 1, \ldots, 0, \partial_x^i f_1, \ldots, \partial_x^i f_\kappa)$, we have

$$g_{ij}(x) = \delta_{ij} + \sum_{1 \leq \sigma \leq \kappa} \partial^i f_\sigma \cdot \partial^j f_\sigma.$$  

Let $\Psi^\sigma_{ij} := \partial^i \partial^j f_\sigma(0)$ and $\Psi^\sigma_{ik} := \partial^i \partial^k f_\sigma(0)$. As $dg_{ij}(0) = 0$, by Lemma 3.1

$$R_{ijkl}(0) = \frac{1}{2} \sum_{1 \leq \sigma \leq \kappa} \{ (\Psi^\sigma_{ij} \Psi^\sigma_{kl} + \Psi^\sigma_{il} \Psi^\sigma_{kj}) + (\Psi^\sigma_{ji} \Psi^\sigma_{lk} + \Psi^\sigma_{jk} \Psi^\sigma_{li}) \}$$

$$- \{ (\Psi^\sigma_{ij} \Psi^\sigma_{lk} + \Psi^\sigma_{il} \Psi^\sigma_{kj}) - (\Psi^\sigma_{ji} \Psi^\sigma_{lk} + \Psi^\sigma_{jk} \Psi^\sigma_{li}) \}$$

$$= \sum_{1 \leq \sigma \leq \kappa} \{ \Psi^\sigma_{ij} - \Psi^\sigma_{il} \Psi^\sigma_{jk} - \Psi^\sigma_{ik} \Psi^\sigma_{jl} \} = \sum_{1 \leq \sigma \leq \kappa} A_{\Psi^\sigma},$$

Consequently, $\nu(A) \leq \kappa$ and $\nu(A_1) \leq \kappa$. Theorem 1.3 follows from the Nash embedding theorem as in the analytic category we may take $\kappa \leq \frac{m}{2}(m+1)$.

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\section*{Dedication}

11 de Marzo de 2004 Madrid: En memoria de todas las víctimas inocentes. Todos íbamos en ese tren. (In memory of all these innocent victims. We were all on that train.)

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