abstract. For $\alpha, \beta > 0$ and for a locally integrable function (or, more generally, a distribution) $\varphi$ on $(0, \infty)$, we study integral operators $G_{\varphi}^{\alpha,\beta}$ on $L^2(\mathbb{R}_+)$ defined by $\left( G_{\varphi}^{\alpha,\beta} f \right)(x) = \int_{\mathbb{R}_+} \varphi(x^\alpha + y^\beta) f(y) dy$. We describe the bounded and compact operators $G_{\varphi}^{\alpha,\beta}$ and operators $G_{\varphi}^{\alpha,\beta}$ of Schatten–von Neumann class $S_p$. We also study continuity properties of the averaging projection $Q_{\alpha,\beta}$ onto the operators of the form $G_{\varphi}^{\alpha,\beta}$. In particular, we show that if $\alpha \leq \beta$ and $\beta > 1$, then $G_{\varphi}^{\alpha,\beta}$ is bounded on $S_p$ if and only if $2\beta(\beta + 1)^{-1} < p < 2\beta(\beta - 1)^{-1}$.

1. Introduction

We are going to study a class of integral operators on $L^2(\mathbb{R}_+)$ that generalizes Hankel integral operators.

For a function $\varphi$ in $L^1(\mathbb{R}_+)$, the integral Hankel operator $\Gamma_{\varphi}$ is defined on $L^2(\mathbb{R}_+)$ by

$$(\Gamma_{\varphi} f)(x) = \int_{\mathbb{R}_+} \varphi(x + y) f(y) dy.$$ 

It is easy to see that such operators are bounded on $L^2(\mathbb{R}_+)$. Operators $\Gamma_{\varphi}$ can be bounded in a much more general case. For $\Gamma_{\varphi}$ to be bounded, $\varphi$ does not have to be a function, it can be a distribution. Bounded Hankel operators $\Gamma_{\varphi}$ are unitarily equivalent to Hankel operators on $\ell^2$, i.e., operators with Hankel matrices of the form $\{\alpha_{j+k}\}_{j,k \geq 0}$. These facts can be found in [Pel4], Ch. 1, §8.

In this paper for $\alpha, \beta > 0$, we study the integral operators $G_{\varphi}^{\alpha,\beta}$ on $L^2(\mathbb{R}_+)$ defined by

$$(G_{\varphi}^{\alpha,\beta} f)(x) = \int_{\mathbb{R}_+} \varphi(x^\alpha + y^\beta) f(y) dy.$$ 

Clearly, if $\varphi$ is a locally integrable function on $\mathbb{R}_+ = (0, \infty)$, the right-hand side of (1.1) is well defined for smooth functions $f$ with compact support in $(0, \infty)$. The integral on the right-hand side of (1.1) makes sense for distributions $\varphi$ on $(0, \infty)$ and infinitely differentiable functions $\varphi$ with compact support in $\mathbb{R}_+$. We say that for a
distribution $\varphi$, the operator $G_{\varphi}^{\alpha,\beta}$ is bounded on $L^2(\mathbb{R}_+)$ if it extends by continuity to $L^2(\mathbb{R}_+)$. Integral operators $G_{\varphi}^{\alpha,\beta}$ are called \textit{distorted Hankel operators}. We are going to study boundedness, compactness and Schatten-von Neumann properties of distorted Hankel operators.

Obviously, for $\alpha = \beta = 1$, the operator $G_{\varphi}^{\alpha,\beta}$ coincides with the integral Hankel operator $\mathcal{I}_{\varphi}$. On the other hand, the limit case of the operators $G_{\varphi}^{\alpha,0}$ as $\alpha \to \infty$ are the integral operators $Q_{\varphi}$ on $L^2(\mathbb{R}_+)$ defined by

$$(Q_{\varphi}f)(x) = \int_{\mathbb{R}_+} \varphi(\max\{x, y\}) f(y)dy.$$ 

We refer the reader to [AJPR] where the operators $Q_{\varphi}$ are studied in detail. Note, however, that properties of the operators $Q_{\varphi}$ are quite different from properties of the operators $G_{\varphi}^{\alpha,\beta}$.

In §2 we collect necessary information on Schatten–von Neumann classes, weighted integral Hankel operators, and discuss properties of the averaging projection and weighted projections onto the integral Hankel operators. We state in §2 known results (Theorems A–D) that will be used in Sections 4 and 5.

In §3 we show that Theorem B, that describes the weighted integral Hankel operators $\mathcal{I}_{\varphi}^{\alpha,\beta}$ of class $\mathcal{S}_p$, does not extend to $\alpha$ and $\beta$ not satisfying the hypotheses of Theorem B.

In §4 we reduce the study of the operators $G_{\varphi}^{\alpha,\beta}$ to the study of weighted integral Hankel operators (see the definition in §2). We describe the operators $G_{\varphi}^{\alpha,\beta}$ that belong to the Schatten–von Neumann class $\mathcal{S}_p$ under a certain condition on $\alpha$, $\beta$, and $\mathcal{P}$.

In §5 we introduce the averaging projection $Q_{\alpha,\beta}$ onto the subspace of operators of the form $G_{\varphi}^{\alpha,\beta}$ and we study their metric properties. In particular, we show that if $\alpha \leq \beta$, $\beta > 1$, and

$$\frac{2\beta}{\beta + 1} < p < \frac{2\beta}{\beta - 1},$$

then $Q_{\alpha,\beta}$ is bounded on $\mathcal{S}_p$. Moreover, this result is sharp. It is interesting to observe that both $\frac{2\beta}{\beta + 1}$ and $\frac{2\beta}{\beta - 1}$ go to 2 as $\beta$ tends to $\infty$. However, in the limiting case the averaging projection on the subspace of the operators of the from $Q_{\varphi}$ is bounded for all $p \in (1, \infty)$, see [AJPR].

It is also interesting that, unlike in the case of Hankel integral operators, for $\alpha, \beta \in (0, 1)$, the averaging projection $Q_{\alpha,\beta}$ is bounded on $\mathcal{S}_1$, and on the space of bounded and compact operators.
2. Preliminaries

**Schatten–von Neumann classes.** Recall that for a bounded operator \( T \) on a Hilbert space \( \mathcal{H} \) the singular values \( s_j(T) \), \( j \geq 0 \), are defined by

\[
s_j(T) = \inf \{ \| T - K \| : K \in \mathcal{B}(\mathcal{H}), \ \text{rank} \ K \leq j \}.
\]

Here \( \mathcal{B}(\mathcal{H}) \) denotes the space of bounded linear operators on \( \mathcal{H} \).

The Schatten–von Neumann class \( S_p = S_p(\mathcal{H}) \), \( 0 < p < \infty \), consists of the operators \( T \) on \( \mathcal{H} \) such that

\[
\| T \|_{S_p} = \left( \sum_{j \geq 0} (s_j(T))^p \right)^{1/p} < \infty.
\]

We denote by \( \mathcal{C}(\mathcal{H}) \) the space of compact operators on \( \mathcal{H} \).

If \( 1 \leq p < \infty \), then \( \| \cdot \|_{S_p} \) is a norm, which makes \( S_p \) a Banach space. For \( p < 1 \), \( \| \cdot \|_{S_p} \) does not satisfy the triangle inequality, it is a quasinorm (i.e., \( \| T_1 + T_2 \|_{S_p} \leq \text{const}(\| T_1 \|_{S_p} + \| T_2 \|_{S_p}) \) for \( T_1, T_2 \in S_p \)), which makes \( S_p \) a quasi-Banach space.

The linear functional trace is defined on \( S_1 \) by

\[
\text{trace} \ T = \sum_{j \geq 0} (Te_j, e_j), \quad T \in S_1,
\]

where \( \{ e_j \}_{j \geq 0} \) is an orthonormal basis in \( \mathcal{H} \). Moreover, the right-hand side does not depend on the choice of the basis.

If \( 1 < p < \infty \), the dual space \( S_p^* \) can be identified with \( S_{p'} \) with respect to the pairing

\[
\langle T, R \rangle = \text{trace} TR^*, \quad T \in S_p, \quad R \in S_{p'}.
\]

Here \( p' = p/(p - 1) \) is the dual exponent. With respect to the same pairing (2.1) one can identify \( S_1^* \) with \( \mathcal{B}(\mathcal{H}) \) and \( (\mathcal{C}(\mathcal{H}))^* \) with \( S_1 \).

In the case \( \mathcal{H} = L^2(\mathcal{X}, \mu) \) and \( \mu \) is a \( \sigma \)-finite measure, the space \( S_2 \) coincides with the set of integral operators \( \mathcal{I}_k \),

\[
(\mathcal{I}_k f)(x) = \int_\mathcal{X} k(x,y) f(y) d\mu(y), \quad f \in L^2(\mathcal{X}, \mu), \quad x, y \in \mathcal{X},
\]

with \( k \in L^2(\mathcal{X} \times \mathcal{X}, \mu \otimes \mu) \) and

\[
\| \mathcal{I}_k \|_{S_2} = \left( \int_{\mathcal{X} \times \mathcal{X}} |k(x,y)|^2 d\mu(x) d\mu(y) \right)^{1/2}.
\]
Moreover, if $T$ and $R$ are integral operators in $S_2$ with kernel functions $k$ and $\xi$, then
\[
\text{trace } TR^* = \int \int k(x,y)\overline{\xi(x,y)} \, d\mu(x) \, d\mu(y).
\]
We refer the reader to [GK] and [BS] for basic facts about Schatten–von Neumann classes.

**Besov classes.** We consider here spaces $\mathcal{B}_p^s$ of distributions on $\mathbb{R}_+ = (0, \infty)$. The space $\mathcal{B}_p^s$ can be identified with the restrictions of the Fourier transforms of functions in the Besov classes $B_p^s(\mathbb{R})$ to $\mathbb{R}_+$.

Let $v$ be a $C^\infty$ function on $\mathbb{R}$ such that\[
\text{supp } v = \left[\frac{1}{2}, 2\right] \quad \text{and} \quad \sum_{j=-\infty}^{\infty} v\left(\frac{x}{2^j}\right) = 1, \quad x > 0.
\]
Put\[
v_j(x) = v\left(\frac{x}{2^j}\right), \quad x > 0.
\]
(2.2)

For $0 < p \leq \infty$ and $s \in \mathbb{R}$, we define the space $\mathcal{B}_p^s$ as the space of distributions $\varphi$ on $\mathbb{R}_+$ such that
\[
\|\varphi\|_{\mathcal{B}_p^s} \overset{\text{def}}{=} \left(\sum_{j=-\infty}^{\infty} \left(2^{js}\|\mathcal{F}(v_j\varphi)\|_{L^p}\right)^p\right)^{1/p} < \infty, \quad p < \infty,
\]
and
\[
\|\varphi\|_{\mathcal{B}_{\infty}^s} \overset{\text{def}}{=} \sup_{-\infty < j < \infty} 2^{js}\|\mathcal{F}(v_j\varphi)\|_{L^\infty} < \infty.
\]
Here $\mathcal{F}$ denotes the Fourier transform.

If $p \geq 1$, $\mathcal{B}_p^s$ is a Banach space with norm $\| \cdot \|_{\mathcal{B}_p^s}$. If $p < 1$, $\mathcal{B}_p^s$ is a quasi-Banach space with quasinorm $\| \cdot \|_{\mathcal{B}_p^s}$, i.e., $\|\varphi_1 + \varphi_2\|_{\mathcal{B}_p^s} \leq \text{const} \left(\|\varphi_1\|_{\mathcal{B}_p^s} + \|\varphi_2\|_{\mathcal{B}_p^s}\right)$.

We also define the space $\mathcal{B}_\infty^s$, $s \in \mathbb{R}$, as the closed subspace of $\mathcal{B}_\infty^s$, which consists of distributions $\varphi \in \mathcal{B}_\infty^s$ such that\[
\lim_{|j| \to \infty} 2^{js}\|\mathcal{F}(v_j\varphi)\|_{L^\infty} = 0.
\]

If $\varphi$ is a distribution on $\mathbb{R}_+$ and $\psi(x) = x^\sigma \varphi(x)$ (this equality has to be understood in the distributional sense), then it is easy to see that $\varphi \in \mathcal{B}_\infty^s$ if and only if $\psi \in \mathcal{B}_{\infty}^{s-\sigma}$.

If $1 \leq p < \infty$ and $s_1, s_2 \in \mathbb{R}$, one can identify the dual space $\left(\mathcal{B}_p^{s_1}\right)^*$ with the space $\mathcal{B}_p^{s_2}$ with respect to the pairing\[
\langle \varphi, \psi \rangle = \int_0^\infty t^{s_1+s_2} \varphi(t)\overline{\psi(t)} \, dt, \quad \varphi \in \mathcal{B}_p^{s_1}, \quad \psi \in \mathcal{B}_p^{s_2}.
\]
(2.3)
Note that the integral on the right-hand side makes sense for compactly supported \( C^\infty \) functions \( \varphi \) can be understood as the value of the distribution \( \psi \) at the function \( t \mapsto t^{s_1+s_1} \varphi(t) \). The linear functional \( \varphi \mapsto \langle \varphi, \psi \rangle \) extends by continuity to the whole space \( \mathcal{B}_p^{s_1} \). As usual, \( \frac{1}{p} + \frac{1}{p'} = 1 \).

The dual space \( \mathcal{B}_1^{s_1} \) can be identified with the space \( \mathcal{B}_1^{s_1} \) with respect to the same pairing (2.3).

**Remark.** The Besov spaces \( B_p^s(\mathbb{R}) \) of functions on \( \mathbb{R} \) can be defined as the space of tempered distributions \( f \) on \( \mathbb{R} \) such that

\[
\sum_{j=-\infty}^{\infty} (2^j \| f * \xi_j \|_{LP})^p + \sum_{j=-\infty}^{\infty} (2^j \| f * \eta_j \|_{LP})^p < \infty,
\]

where \( \xi_j \) and \( \eta_j \) are functions in on \( \mathbb{R} \) such that

\[
\mathcal{F}\xi_j = v_j \quad \text{and} \quad (\mathcal{F}\eta_j)(x) = v_j(-x), \quad x \in \mathbb{R}.
\]

and the \( v_j \) are defined by (2.2). This this space contains all polynomials. It is possible to define the space \( B_p^s(\mathbb{R}) \) modulo the polynomials of degree at most \( s - 1/p \), in which case only polynomials of degree at most \( s - 1/p \) can belong to this space. In both cases

\[
\mathcal{B}_p^s = \{ \mathcal{F}f | (0, \infty) : f \in B_p^s(\mathbb{R}) \}.
\]

The subspace \( (B_p^s(\mathbb{R}))_+ \) of \( B_p^s(\mathbb{R}) \) is defined by

\[
(B_p^s(\mathbb{R}))_+ = \{ f \in B_p^s(\mathbb{R}) : \text{supp} \mathcal{F}f \subset [0, +\infty) \}.
\]

Functions in \( (B_p^s(\mathbb{R}))_+ \) can be extended analytically in a natural way to the upper half-plane. Clearly,

\[
\{(\mathcal{F}f) | (0, \infty) : f \in (B_p^s(\mathbb{R}))_+ \} = \mathcal{B}_p^s.
\]

We refer the reader to [Pee] for more information about Besov classes.

**Weighted integral Hankel operators.** For a locally integrable function \( \varphi \) on \( \mathbb{R}_+ \) the weighted integral Hankel operator \( \Gamma_{\varphi}^{\alpha,\beta} \) is defined by

\[
(\Gamma_{\varphi}^{\alpha,\beta} f)(x) = \int_0^\infty x^\alpha y^\beta \varphi(x+y) f(y) dy
\]

for smooth functions \( f \) with compact support in \( \mathbb{R}_+ \). Again, the definition makes sense for distributions \( \varphi \) on \( \mathbb{R}_+ \). The operators \( \Gamma_{\varphi}^{\alpha,\beta} \) are analogs of weighted Hankel matrices \( \Gamma_{\psi}^{\alpha,\beta} = \{(1+j)^\alpha(1+k)^\beta \hat{\psi}(j+k)\}_{j,k \geq 0} \) where \( \psi \) is a function analytic in the unit disk.

For \( \alpha = \beta = 0 \), the operator \( \Gamma_{\varphi}^{0,0} = \Gamma_{\varphi} \) is the integral Hankel operator defined in the introduction.
We need the following results.

**Theorem A.** Suppose that $\alpha > 0$ and $\beta > 0$. Then $\Gamma^\alpha_\varphi$ is bounded on $L^2(\mathbb{R}_+)$ if and only if $\varphi \in \mathcal{B}^{\alpha+\beta}_\infty$ and $\Gamma^{\alpha,\beta}_\varphi$ is compact if and only if $\varphi \in b^{\alpha+\beta}_\infty$.

We refer the reader to [Pel2] for the corresponding result for weighted Hankel matrices and to [JP] for more general results than Theorem A.

**Theorem B.** Let $0 < p < \infty$. Suppose that $\min\{\alpha, \beta\} > \max\{-\frac{1}{2}, -\frac{1}{p}\}$. Then $\Gamma^{\alpha,\beta}_\varphi \in S_p$ if and only if $\varphi \in \mathcal{B}^{1/p+\alpha+\beta}_p$.

For $p \geq 1$ the description of the weighted Hankel matrices $\Gamma^{\alpha,\beta}_\psi$ of class $S_p$ was obtained in [Pel1] for $\alpha = \beta = 0$ and in [Pel2] in the case $\min\{\alpha, \beta\} > \max\{-\frac{1}{2}, -\frac{1}{p}\}$. For integral Hankel operators and $p \geq 1$ see [CR], [R], and [JP]. In the case $p < 1$ we refer the reader to [Pel1] for weighted Hankel matrices and to [S] for the weighted integral Hankel operators. See also [Pel4], Ch. 6.

**Averaging projection onto the integral Hankel operators.** Consider the orthogonal projection $\mathcal{P}$ on the Hilbert–Schmidt class $S_2$ onto the subspace of Hankel integral operators. Clearly, if $k \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$, then

$$\mathcal{P}J_k = \Gamma_\varphi,$$

where

$$\varphi(x) = \frac{1}{x} \int_0^x k(t, x-t)dt.$$

For $2 < p < \infty$, the averaging projection $\mathcal{P}$ can be defined on the dense subset $S_p \cap S_2$ of of $S_p$. As usual, we say that $\mathcal{P}$ is bounded on $S_p$ if it extends by continuity to a bounded operator on $S_p$.

**Theorem C.** Let $1 < p < \infty$. Then the averaging projection $\mathcal{P}$ is bounded on $S_p$.

The same result for the averaging projection onto the space of Hankel matrices was obtained in [Pel1], see also [Pel4]. The same proof works for the averaging projection onto the set of integral Hankel operators.

Note that $\mathcal{P}$ is unbounded on $S_1$ and on the spaces of bounded and compact operators (see [Pel1], [Pel4]). However, there are bounded projections on $S_1$ onto the subspace of integral Hankel operators. Indeed, for $\alpha, \beta > 0$ we define the weighted averaging projection $\mathcal{P}_{\alpha,\beta}$ by

$$\mathcal{P}_{\alpha,\beta}J_k = \Gamma_\varphi,$$

where

$$\varphi(x) = \frac{1}{x} \int_0^x k(t, x-t)dt.$$
where
\[ \varphi(x) = \frac{\int_0^x t^\alpha (x - t)^\beta k(t, x - t) dt}{\int_0^x t^\alpha (x - t)^\beta dt}. \]

Theorem D. Let \( \alpha, \beta > 0 \). Then \( \mathcal{P}_{\alpha,\beta} \) is a bounded operator on \( S_1 \).

The corresponding result for weighted averaging projection on the subspace of Hankel matrices was found by the first author, see ([Pel2] and [Pel4]). The same proof works in the case of integral operators.

3. Theorem B is sharp

When we study properties of the averaging projection \( \mathcal{Q}_{\alpha,\beta} \) onto the space of operators of the form \( \mathcal{G}^\alpha_{\varphi} \), we will need the fact that Theorem B cannot be extended to other values of \( \alpha \) and \( \beta \).

Theorem 3.1. Let \( \alpha, \beta \in \mathbb{R} \) and \( 0 < p < \infty \). Suppose that \( \min\{\alpha, \beta\} \leq \max\{-\frac{1}{2}, -\frac{1}{p}\} \). Then there are functions \( \psi \in \mathbb{B}^{1/p+\alpha+\beta}_p \) such that \( \Gamma^\alpha_{\varphi,\beta} \notin S_p \).

Proof. Consider first the case \( p \geq 2 \). For a positive integer \( n \) we define the function \( \varphi_n \) on \( \mathbb{R}_+ \) by
\[ \varphi_n(x) = \begin{cases} 1, & x \in (1, 1 + \frac{2}{n}) , \\ 0, & \text{otherwise}. \end{cases} \]

It follows easily from the definition of \( \mathbb{B}^{1/p+\alpha+\beta}_p \) that
\[ \|\varphi_n\|_{\mathbb{B}^{1/p+\alpha+\beta}_p} \leq \text{const} \cdot \|\mathcal{F}\varphi_n\|_{L^p} \leq \text{const} \cdot n^{-1/p'}. \]

Let us now estimate from below \( \|\Gamma^\alpha_{\varphi,\beta}\|_{S_p} \). Define the function \( k \) on \( \mathbb{R}^2_+ \) by
\[ k(x, y) = \begin{cases} x^\alpha y^\beta, & (x, y) \in \bigcup_{k=0}^n \Delta_j, \\ 0, & (x, y) \notin \bigcup_{k=0}^n \Delta_j, \end{cases} \]

where
\[ \Delta_j = \left( \frac{j}{n}, \frac{j + 1}{n} \right) \times \left( \frac{n-j}{n}, \frac{n-j+1}{n} \right), \quad 0 \leq j \leq n. \]
Since the squares $\Delta_j$ have disjoint projections onto the coordinate axes, it is easy to see that

$$\| I_{\varphi_n}^{\alpha,\beta} \|_{S_p} \geq \| \mathcal{I}_k \|_{S_p} = \left( \sum_{j=0}^{n} \| \mathcal{I}_{k_j} \|_{S_p}^p \right)^{1/p},$$

where $k_j = k \chi_{\Delta_j}$ and $\chi_{\Delta_j}$ is the characteristic function of $\Delta_j$. Clearly, $\mathcal{I}_{k_j}$ is a rank one operator and

$$\| \mathcal{I}_{k_j} \|_{S_p} \geq \frac{j^\alpha (n - j)^\beta}{n^{\alpha+\beta+1}}.$$

Without loss of generality, we may assume $\alpha \leq \beta$. It is easy to verify that

$$\| I_{\varphi_n}^{\alpha,\beta} \|_{S_p} \geq \left( \sum_{j=0}^{n} \frac{j^\alpha (n - j)^\beta}{n^{\alpha+\beta+1}} \right)^{1/p} \geq \text{const} \left\{ \begin{array}{ll}
\frac{n^{-\alpha-1}}{n^{\alpha+\beta+1}}, & \alpha < -1/p, \\
\frac{n^{-\alpha-1} (\log(1+n))^{1/p}}{n^{\alpha+\beta+1}}, & \alpha = -1/p.
\end{array} \right.$$

Clearly,

$$\frac{\| I_{\varphi_n}^{\alpha,\beta} \|_{S_p}}{\| \varphi_n \|_{\mathcal{B}_{p}^{1/p+\alpha+\beta}}} \to \infty \text{ as } n \to \infty,$$

which completes the proof in the case $p \geq 2$.

Suppose now that $p < 2$. Again, we assume that $\alpha \leq \beta$. We prove that if $\alpha \leq -\frac{1}{2}$, the condition $\psi \in \mathcal{B}_{p}^{1/p+\alpha+\beta}$ does not even imply that $I_{\varphi_n}^{\alpha,\beta} \in S_2$. Let $\psi$ be a nonzero smooth function with support in $[1, 2]$. Clearly, $\mathcal{F} \psi \in L^p$, and so $\psi \in \mathcal{B}_{p}^{1/p+\alpha+\beta}$. On the other hand,

$$\| I_{\varphi_n}^{\alpha,\beta} \|_{S_2}^2 = \int_1^2 |\varphi(t)|^2 \int_0^t x^{2\alpha} (t - x)^{2\beta} dx \, dt = \infty,$$

since, clearly,

$$\int_0^t x^{2\alpha} (t - x)^{2\beta} dx = \infty.$$

4. Boundedness, Compactness, and Schatten–von Neumann Properties

Let $k$ be a function on $\mathbb{R}_+^2$ such that the integral operator on $L^2(\mathbb{R}_+)$ with kernel function $k$ is bounded on $L^2(\mathbb{R}_+)$. As in §2, denote this integral operator by $\mathcal{I}_k$:

$$(\mathcal{I}_k f)(x) = \int_{\mathbb{R}_+} k(x,y) f(y) dy.$$
We say that \( k \in S_p(\mathbb{R}^2_+ \mathbb{K}) \) if the operator \( J_k \) belongs to the Schatten–von Neumann class \( S_p \) and we write \( \|k\|_{S_p} \overset{\text{def}}{=} \|J_k\|_{S_p} \). Let now \( \alpha \) and \( \beta \) be nonzero real numbers. We put
\[
k_{\alpha,\beta}(x,y) \overset{\text{def}}{=} x^{\frac{1}{2\alpha}-\frac{1}{2}} y^{\frac{1}{2\beta}-\frac{1}{2}} k \left( \frac{x}{\alpha}, y \right).
\]
(4.1)

We introduce the unitary operator \( U_\alpha \) on \( L^2(\mathbb{R}_+) \) defined by
\[
(U_\alpha f)(x) = \frac{1}{\sqrt{|\alpha|}} x^{\frac{1}{2\alpha}-\frac{1}{2}} f \left( \frac{x}{\alpha} \right), \quad f \in L^2(\mathbb{R}_+).
\]

It is easy to see that
\[
U_\alpha J_k = \frac{1}{\sqrt{\alpha \beta}} J_{k_{\alpha,\beta}} U_\beta, \quad \alpha, \beta \in \mathbb{R} \setminus \{0\},
\]
and so
\[
\|k_{\alpha,\beta}\|_{S_p} = \sqrt{\alpha \beta} \|k\|_{S_p}.
\]

**Theorem 4.1.** Suppose that \( \alpha, \beta \in (0,1) \). Then \( G_{\alpha,\beta} \) is bounded if and only if \( \varphi \in \mathcal{B}_{\frac{1}{2\alpha} + \frac{1}{2\beta}} - 1 \). The operator \( G_{\alpha,\beta} \) is compact if and only if \( \varphi \in \mathcal{B}_{\frac{1}{2\alpha} + \frac{1}{2\beta}} - 1 \).

**Proof.** Suppose first that \( \varphi \) is a locally integrable function on \( \mathbb{R}_+ \). Consider the kernel function \( \kappa \) of \( G_{\alpha,\beta} \):
\[
\kappa(x,y) = \varphi(x^{\alpha} + y^{\beta}), \quad x, y > 0.
\]
Obviously, by (4.1)
\[
\kappa_{\alpha,\beta}(x,y) = x^{\frac{1}{2\alpha}-\frac{1}{2}} y^{\frac{1}{2\beta}-\frac{1}{2}} \varphi(x+y),
\]
(4.2)
i.e., the integral operator with kernel function \( \kappa_{\alpha,\beta} \) is a weighted integral Hankel operator. Hence, \( G_{\alpha,\beta} \) is bounded (or compact) if and only if \( \Gamma_{\varphi}^{-\frac{1}{2\alpha} - \frac{1}{2\beta} - \frac{1}{2}} \) is bounded (or compact). By Theorem A (see §2), this is equivalent to the fact that \( \varphi \in \mathcal{B}_{\frac{1}{2\alpha} + \frac{1}{2\beta}} - 1 \) (or \( \varphi \in \mathcal{B}_{\frac{1}{2\alpha} + \frac{1}{2\beta}} - 1 \)).

If \( \varphi \) is a distribution, it is easy to verify that the formula
\[
\Gamma_{\varphi}^{-\frac{1}{2\alpha} - \frac{1}{2\beta} - \frac{1}{2}} = \sqrt{\alpha \beta} U_\alpha G_{\alpha,\beta} U_\beta^*
\]
still holds, which implies the result. \( \blacksquare \)

**Theorem 4.2.** Let \( p, \alpha, \) and \( \beta \) be positive numbers such that
\[
\max\{\alpha, \beta\}(p-2) < p.
\]
Then \( G_{\alpha,\beta} \in S_p \) if and only if \( \varphi \in \mathcal{B}_{\frac{1}{2\alpha} + \frac{1}{2\beta}} - 1 \).
Proof. Again, apply formula (4.3). Obviously, \( \alpha \) and \( \beta \) satisfy the hypotheses of Theorem B. The result follows from Theorem B. ■

It turns out that in the case \( 1 \leq p < \infty \) the necessity of the condition \( \varphi \in \mathfrak{B}_{1}^{\frac{1}{2p} + \frac{1}{2q} - 1} \) holds for any positive \( \alpha \) and \( \beta \).

Theorem 4.3. Let \( 1 \leq p < \infty \), and let \( \alpha \) and \( \beta \) be positive numbers. Suppose that \( G_{\alpha,\beta} \varphi \in \mathcal{S}_{p} \). Then \( \varphi \in \mathfrak{B}_{1}^{\frac{1}{2p} + \frac{1}{2q} - 1} \).

Proof. As we have already observed, the integral operator \( I_{\kappa_{\alpha,\beta}} \) with kernel \( \kappa_{\alpha,\beta} \) defined by (4.2) must belong to \( \mathcal{S}_{p} \).

Suppose first that \( p > 1 \). We apply to \( I_{\kappa_{\alpha,\beta}} \) the averaging projection \( \mathcal{P} \) (see §2). It is easy to verify that

\[
\mathcal{P} I_{\kappa_{\alpha,\beta}} = \text{const} \, \Gamma_{\psi},
\]

where

\[
\psi(x) = x^{\frac{1}{2p} + \frac{1}{2q} - 1} \varphi(x).
\]

By Theorem C, \( \Gamma_{\psi} \in \mathcal{S}_{p} \). Now by Theorem B, \( \psi \in \mathfrak{B}_{1}^{1/p} \), which is equivalent to the fact that \( \varphi \in \mathfrak{B}_{p}^{\frac{1}{2p} + \frac{1}{2q} + \frac{1}{p} - 1} \) (see Section 2).

Let now \( p = 1 \). We apply to the operator \( I_{\kappa_{\alpha,\beta}} \) the weighted projection \( \mathcal{P}_{1,1} \), which is bounded on \( \mathcal{S}_{1} \) (see Theorem D). It is easy to verify that

\[
\mathcal{P}_{1,1} I_{\kappa_{\alpha,\beta}} = \text{const} \, \Gamma_{\psi},
\]

where \( \psi \) is defined by (4.4). By Theorem B, \( \psi \in \mathfrak{B}_{1}^{1} \), which is equivalent to the fact that \( \varphi \in \mathfrak{B}_{1}^{\frac{1}{2p} + \frac{1}{2q}} \) (see §2). ■

5. The Averaging Projection onto the Operators \( G_{\alpha,\beta} \)

In this section we study metric properties of the averaging projection on the class of operators of the form \( G_{\alpha,\beta} \). Consider the orthogonal projection \( Q_{\alpha,\beta} \) from the Hilbert–Schmidt class \( \mathcal{S}_{2} \) onto the subspace of \( \mathcal{S}_{2} \) of operators of the form \( G_{\alpha,\beta} \). Clearly, \( Q_{1,1} \) is just the averaging projection \( \mathcal{P} \) onto the Hankel integral operators. If we identify the Hilbert–Schmidt operators on \( L^{2}(\mathbb{R}+) \) with the space \( L^{2}(\mathbb{R}_{+}^{2}) \), we find that \( Q_{\alpha,\beta} \) is the orthogonal projection onto the subspace of functions that are constant on the sets \( \{(x, y) \in \mathbb{R}_{+}^{2} : x^{\alpha} + y^{\beta} = c\} \), \( c > 0 \).

We are going to characterize those \( \alpha, \beta, \) and \( p \), for which the projection \( Q_{\alpha,\beta} \) is bounded on \( \mathcal{S}_{p} \). Clearly, \( Q_{\alpha,\beta} T \) is well-defined for \( T \in \mathcal{S}_{p} \) if \( p \leq 2 \). If \( p > 2 \), we say that \( Q_{\alpha,\beta} \) is bounded on \( \mathcal{S}_{p} \) if it extends to a bounded operator from \( \mathcal{S}_{p} \cap \mathcal{S}_{2} \). Since \( Q_{\alpha,\beta} \) is self-adjoint on \( \mathcal{S}_{2} \), it follows that \( Q_{\alpha,\beta} \) is bounded on \( \mathcal{S}_{p} \) if and only if it is bounded on \( \mathcal{S}_{p_{0}} \), \( 1 < p < \infty \), and \( Q_{\alpha,\beta} \) is bounded on \( \mathcal{S}_{1} \) if and only if it is bounded on \( \mathcal{B}(L^{2}(\mathbb{R}_{+})) \).
Lemma 5.1. Let \( k \in L^2(\mathbb{R}_+^2) \) and let \( \alpha, \beta > 0 \). Put
\[
\varphi(x) \overset{\text{def}}{=} \int_0^\frac{\pi}{2} k\left(x^{\frac{1}{\alpha}} \cos^2 t, x^{\frac{1}{\beta}} \sin^2 t\right) \cos^{\frac{2}{\alpha} - 1} t \sin^{\frac{2}{\beta} - 1} t \, dt 
\]
where
\[
A(\alpha, \beta) \overset{\text{def}}{=} \int_0^\frac{\pi}{2} \cos^{\frac{2}{\alpha} - 1} t \sin^{\frac{2}{\beta} - 1} t \, dt
\]
Then
\[
Q_{\alpha, \beta} k = g_{\varphi}^{\alpha, \beta}.
\]  

Proof. The result follows from the following easily verifiable formula:
\[
\iint_{\mathbb{R}_+^2} f(x, y) \, dx \, dy = \frac{2}{\alpha \beta} \int_0^\infty \int_0^{\frac{\pi}{2}} f\left(r^{\frac{1}{\alpha} + \frac{1}{\beta} - 1}\cos^2 t, r^{\frac{1}{\alpha} + \frac{1}{\beta} - 1}\sin^2 t\right) \cos^{\frac{2}{\alpha} - 1} t \sin^{\frac{2}{\beta} - 1} t \, dt \, dr,
\]
which holds for any nonnegative measurable function \( f \).

Remark. Note that \( A(\alpha, \beta) = \frac{1}{2} B\left(\frac{1}{\alpha}, \frac{1}{\beta}\right) \), where \( B \) is the Euler Beta function.

Theorem 5.2. Let \( 1 \leq p < \infty \), and let \( \alpha \) and \( \beta \) be positive numbers. Suppose that
\[
-p < \max\{\alpha, \beta\}(p - 2) < p.
\]  
Then \( Q_{\alpha, \beta} \) is bounded on \( S_p \).

Note that if \( \max\{\alpha, \beta\} \leq 1 \), the rightmost inequality in (5.4) holds for any \( p \). If \( \max\{\alpha, \beta\} > 1 \), then (5.4) is equivalent to the inequalities
\[
\frac{2 \max\{\alpha, \beta\}}{\max\{\alpha, \beta\} + 1} < p < \frac{2 \max\{\alpha, \beta\}}{\max\{\alpha, \beta\} - 1}.
\]
Clearly, \( \frac{2 \max\{\alpha, \beta\}}{\max\{\alpha, \beta\} + 1} \) and \( \frac{2 \max\{\alpha, \beta\}}{\max\{\alpha, \beta\} - 1} \) are dual exponents, i.e., the sum of their reciprocals is equal to one.

Proof. Consider first the case \( p > 1 \). Let \( T_{\alpha, \beta} \) be the operator on \( S_p \) defined by
\[
T_{\alpha, \beta} = \varphi, \quad \text{where} \quad Q_{\alpha, \beta} T = g_{\varphi}^{\alpha, \beta},
\]
see Lemma 5.1. By Theorem 4.2, we have to show that \( T_{\alpha, \beta} \) is a bounded operator from \( S_p \) to \( B_{L^p}^\frac{1}{\alpha} + \frac{1}{\beta} - 1 \).
Consider the dual exponent $p' = p/(p - 1)$. Define the operator
\[
\mathcal{R} : B_{p'}^{1 + \frac{1}{2\alpha} + \frac{1}{2\beta} - 1} \to S_{p'}(\mathbb{R}_+^2)
\]
by
\[
(\mathcal{R}\varphi)(x, y) = \varphi(x^\alpha + y^\beta), \quad x, y > 0.
\]
By Theorem 1.2, $\mathcal{R}$ is a bounded operator.

Let $k \in S_1(\mathbb{R}_+^2)$ and let $Q_{\alpha,\beta}I_k = G_{\psi}(\alpha, \beta)$ (see (5.1) and (5.3)). We have
\[
\int_{\mathbb{R}_+^2} \int (\mathcal{R}\varphi)(x, y) k(x, y) \, dx \, dy = \int_{\mathbb{R}_+^2} \int (\mathcal{R}\varphi)(x, y) \psi(x^\alpha + y^\beta) \, dx \, dy
\]
\[
= \frac{2A(\alpha, \beta)}{\alpha\beta} \int_0^\infty r^{\frac{1}{2\alpha} + \frac{1}{2\beta} - 1} \varphi(r) \psi(r) \, dr. \tag{5.6}
\]
It follows that $T_{\alpha,\beta} = \mathcal{R}^*$ if we identify \(B_{p'}^{1 + \frac{1}{2\alpha} + \frac{1}{2\beta} - 1}\) with \(B_{p'}^{1 + \frac{1}{2\alpha} + \frac{1}{2\beta} - 1}\) with respect to the pairing
\[
\langle f, g \rangle = \int_{\mathbb{R}_+} r^{\frac{1}{2\alpha} + \frac{1}{2\beta} - 1} f(r) \overline{g(r)} \, dr \tag{5.7}
\]
(see §2). Hence, $T_{\alpha,\beta}$ is a bounded.

Suppose now that $p = 1$. Clearly, (5.4) means that $0 < \alpha, \beta < 1$. Consider the operator
\[
\mathcal{R} : b_{\infty}^{\frac{1}{2\alpha} + \frac{1}{2\beta} - 1} \to C(L^2(\mathbb{R}_+))
\]
defined by (5.3). It is bounded by Theorem A. Again, it is easy to see from (5.6) that $T_{\alpha,\beta}^* = \mathcal{R}$ if we identify \(b_{\infty}^{\frac{1}{2\alpha} + \frac{1}{2\beta} - 1}\) with \(b_{1}^{\frac{1}{2\alpha} + \frac{1}{2\beta} - 1}\) with respect to the pairing (5.7). □

Now suppose that $\alpha, \beta \in (0, 1)$. By Theorem 5.2, the averaging projection $Q_{\alpha,\beta}$ is bounded on $S_1$. We can consider the conjugate operator $Q_{\alpha,\beta}^*$ on the space $\mathcal{B}(L^2(\mathbb{R}_+))$ with respect to the standard pairing between $S_1$ and $\mathcal{B}(L^2(\mathbb{R}_+))$ (see §2). Since $Q_{\alpha,\beta}$ is a self-adjoint operator on $S_2$, it follows that $Q_{\alpha,\beta}^* T = Q_{\alpha,\beta} T$ for $T \in S_2$. We can now extend the averaging projection $Q_{\alpha,\beta}$ to the space $\mathcal{B}(L^2(\mathbb{R}_+))$ by
\[
Q_{\alpha,\beta} T = Q_{\alpha,\beta}^* T, \quad T \in \mathcal{B}(L^2(\mathbb{R}_+)).
\]
It is easy to show that if $T$ is a bounded integral operator on $L^2(\mathbb{R}_+)$ with kernel function $k$, then $Q_{\alpha,\beta} T$ can be defined as in (5.1) and (5.2).
Theorem 5.3. Let $\alpha, \beta \in (0, 1)$. Then $Q_{\alpha,\beta}$ is a bounded operator on the space $\mathcal{B}(L^2(\mathbb{R}_+))$ of bounded operators and on the space $\mathcal{C}(L^2(\mathbb{R}_+))$ of compact operators.

Proof. We have already explained the fact that $Q_{\alpha,\beta}$ is bounded on $\mathcal{B}(L^2(\mathbb{R}_+))$. The boundedness of $Q_{\alpha,\beta}$ on $\mathcal{C}(L^2(\mathbb{R}_+))$ follows immediately from the fact that $Q_{\alpha,\beta}S_2 \subset S_2$ and the fact that $S_2$ is dense in $\mathcal{C}(L^2(\mathbb{R}_+))$. □

Let us prove now that Theorems 5.2 and 5.3 are sharp.

Theorem 5.4. Suppose that $\alpha$ and $\beta$ are positive numbers such that the averaging projection $Q_{\alpha,\beta}$ is bounded on $S_p$, $0 < p < \infty$. Then $p \geq 1$ and (5.4) holds.

Theorem 5.5. Suppose that $\alpha$ and $\beta$ are positive numbers such that the averaging projection $Q_{\alpha,\beta}$ is bounded on the space $\mathcal{B}(L^2(\mathbb{R}_+))$ or on the space $\mathcal{C}(L^2(\mathbb{R}_+))$. then $\alpha, \beta < 1$.

In fact for $p < 1$ the following much stronger result holds.

Theorem 5.6. Let $0 < p < 1$ and $\alpha, \beta > 0$. Then there is no bounded projection from $S_p$ onto the subspace of operators of the form $\mathcal{G}_{\varphi}^{\alpha,\beta}$.

Let us first prove Theorem 5.6.

Proof of Theorem 5.6. The result follows from Theorem 4.2 and the Kalton theorem [K], which says, in particular, that if $X$ is a complemented subspace of $S_p$, $0 < p < 1$, such that $X$ can be imbedded isomorphically to an $L^p$ space, then the $S_p$ quasinorm and the $S_1$ norm on $X$ are equivalent. Indeed, let $X$ be the subspace of $S_p$ of operators of the form $\mathcal{G}_{\varphi}^{\alpha,\beta}$. By Theorem 1.2, the $S_1$ norm and the $S_p$ quasinorm on $X$ are not equivalent. It follows easily from the definition of the spaces $\mathcal{B}_p^s$ given in §2 that $X$ can be imbedded isometrically in an $L^p$ space. Thus by the Kalton theorem $X$ is not a complemented subspace of $S_p$. □

Proof of Theorem 5.4. Suppose that $1 \leq p < \infty$. The reasoning given in the proof of Theorem 5.2 shows that if the averaging projection $Q_{\alpha,\beta}$ is bounded on $S_p$, then the condition $\varphi \in \mathcal{B}_p^{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{p} - 1}$ implies $\mathcal{G}_{\varphi}^{\alpha,\beta} \in S_p$. The result now follows from (4.2) and from Theorem 5.1. □

Proof of Theorem 5.3. If $Q_{\alpha,\beta}$ is bounded on $\mathcal{B}(L^2(\mathbb{R}_+))$ or on $\mathcal{C}(L^2(\mathbb{R}_+))$, then, by duality, $Q_{\alpha,\beta}$ is bounded on $S_1$. The result follows now from Theorem 5.4. □

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