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Hybrid delta modulator: stability analysis using sliding mode theory

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ABSTRACT
The present study proposes a new dynamic two-level quantizer, called as hybrid delta modulator ($\Delta H\!-\!M$), which combines the features of both delta-modulator and delta-sigma modulator. In the transient state, the $\Delta H\!-\!M$ exhibits the dynamical behaviour of delta modulator ($\Delta\!-\!M$) while in steady state, its behaviour is similar to delta-sigma modulator ($\Delta\Sigma\!-\!M$). This study investigates about various dynamics of the proposed $\Delta H\!-\!M$ in both continuous and discrete-time domains. The stability conditions of $\Delta H\!-\!M$ are derived using the theory of sliding and quasi-sliding mode. The theoretical results are validated through extensive simulations.

Key words: $\Delta\!-\!M$: delta modulator; $\Delta H\!-\!M$: hybrid delta modulator; $\Delta\Sigma\!-\!M$: delta sigma modulator; ADC: analogue to digital converter; CT: continuous-time; DAC: digital to analogue converter; DT: discrete-time; NCS: networked control system; QSM: quasi-sliding model; QSMD: quasi-sliding mode domain; SS: steady state; TP: transient process

1. Introduction

During the past few decades, quantizers and modulators have been playing a vital role in the field of mixing signals, communications, power electronics, control systems, etc. Two-level quantizers, also known as single-bit converters, such as delta-sigma modulators ($\Delta\Sigma\!-\!M$) are becoming more popular compared to other types of quantizers due to their simplicity of implementation and robustness (Gai, Xia, & Chen, 2006; Gomez-Estern, de Wit, & Rubio, 2011; de Wit, Gomez-Estern, & Rubio, 2009; Xia, 2007; Xia & Chen, 2007; Xia, Chen, Gai, & Zinobier, 2008; Xia & Zinobier, 2004, 2006) and have extensively been used in power converters (Al-Makhles, Patel, & Swain, 2013c, 2014; Chen, Chen, & Wu, 2010; Siraramirez, 2003). These systems are well known for their ability in reduction of the data rate, latency and large numbers of wires in networked control systems (NCS) where the bandwidth of communication channel is limited and hardware resources are constrained (Al-Makhles, Patel, & Swain, 2013a, 2013b; Al-Makhles et al., 2014; Almakhles, Swain, Nasiri, & Patel, 2017; Almakhles, Swain, & Patel, 2015; Gomez-Estern et al., 2011; Karimirreddy & Zhang, 2017; Kermani & Sakly, 2014; Li, Dong, Han, Hou, & Li, 2017; Li & Fujimoto, 2008; Li, Dong, Han, & Li, 2018; Mantz, 2015; Premeratane, Halgamuge, & Mareels, 2013; de Wit et al., 2009).

It is worth to note that $\Delta\Sigma\!-\!M$ can be implemented both in continuous and discrete time domains. The choice between continuous-time (CT) $\Delta\Sigma\!-\!M$ and discrete-time (DT) $\Delta\Sigma\!-\!M$ depends on the specific application (i.e. ADC CT $\Delta\Sigma\!-\!M$, DAC DT $\Delta\Sigma\!-\!M$). For example, in applications such as NCSs, power converters etc., $\Delta\Sigma\!-\!M$ can either be implemented in CT or DT domain depending on the convenience of the designer (Almakhles, Swain, & Nasiri, 2017).

The $\Delta\Sigma\!-\!M$ (also referred to as differential modulator) mainly consists of a transmitter (encoder or modulator) followed by a receiver (decoder or demodulator) where the dynamics of the input variables can be found directly. Note that the input signal is equivalent to the signals (single-bit) between encoder and decoder. However, it requires an integrator at the demodulation side to reconstruct the input signals (Haykin, 2000). The $\Delta\Sigma\!-\!M$ is also considered as dynamic quantizer and inherently contains relay components (i.e. two-level quantizers) which introduce nonlinearity and more complexity to these modulators/systems (Almakhles et al., 2017; Azuma & Sugie, 2008; de Wit et al., 2009).

$\Delta\!-\!M$ has the ability to directly reveal the dynamic of the input variables, where the single-bit signals between the encoder and decoder are ideally equivalent to the rate of change of the input signals. $\Delta\Sigma\!-\!M$ contains
all the useful information of the input signal in the low-frequency range and suppresses noise at the high-frequency range. This feature is known as noise shaping. The present study combines these features of the $\Delta$-M and $\Delta \Sigma$-M. In the transient state, the $\Delta H$-M exhibits the dynamical behaviour of $\Delta$-M while in steady state, its behaviour is similar to $\Delta \Sigma$-M. Thus the stability region is increased and the noise effect is significantly reduced.

The $\Delta \Sigma$-M can roughly be classified into two categories; namely fixed step-size and adaptive step-size. When the step size of $\Delta \Sigma$-M is adaptive, it is called as $\Delta H$-M (Haykin, 2000). It has been shown that the stability of $\Delta \Sigma$-M is critically dependent on quantizer gain (Almakhles et al., 2017; Xia & Chen, 2007; Xia & Zinober, 2006). The objective of this study is to investigate if the stability performance can be improved by making the step-size of the quantizer to be adaptive. Recently, in Almakhles et al. (2017), the authors have investigated the performance of data-driven $\Delta$-M and $\Delta \Sigma$-M. The present study investigates some inherent dynamical properties of data-driven hybrid $\Delta H$-M in CT and DT domains and establishes that the stability region of this modulator is larger compared to that of $\Delta \Sigma$-M. The main contributions of this study are: (i) Derivation of stability conditions and periodicity for both data-driven fixed and hybrid $\Delta H$-M using QSM analysis; (ii) Accurate estimation of hitting-step is derived for both data-driven fixed and hybrid $\Delta H$-M; (iii) The quantizer gain for fixed $\Delta \Sigma$-M which guarantees the stability, is derived using dynamics and bounds of the input signals.

The rest of the paper is organized as follows. Sections 2 and 3 discuss CT $\Delta H$-M and DT $\Delta H$-M, respectively. The effectiveness of the proposed $\Delta H$-M is demonstrated using simulations in Section 4 with conclusions in Section 5.

2. Continuous-time hybrid delta modulator ($\Delta H$-M)

The schematic of a continuous-time $\Delta H$-M is shown in Figure 1. From this figure, the following relations can be established:

\[
\dot{s}(t) = x(t) - \tilde{x}(t),
\]

\[
s(t) = \min \left\{ \int_{-\infty}^{t} \dot{s}(\lambda) \, d\lambda, \, \phi \delta_{t} \right\},
\]

\[
\dot{\tilde{x}}(t) = (E^X_{\Delta H} \cdot D^X_{\Delta H}) \ast x(t)
\]

\[
= \tilde{x}(t) + \phi \delta_{t},
\]

where $x(t) \in \mathbb{R}^n, \tilde{x}(t) \in \mathbb{R}^n, \phi \in \mathbb{R}_+ \text{ and } \tilde{x}(t) \in \mathbb{R}^n$ denote respectively the input of the encoder $E^X_{\Delta H}$, output of decoder $D^X_{\Delta H}$, two-level quantizer gain and estimated input. Furthermore, $\delta_{t} = \text{sgn}(s(t))$ and $\tilde{x}(t)$ is defined as

\[
\tilde{x}(t) = \phi \varphi(t),
\]

where $\varphi(t) = \delta_{t} - \tau (d\varphi(t)/dt) \in [-1, 1]$. Note that due to the limited integrator in (2), $s(t)$ varies in the range $[-\phi, \phi]$.

The estimated error $e(t)$ is defined as $e(t) = x(t) - \tilde{x}(t)$. The operating regions of $\Delta H$-M (defined in (1)–(4)) can be divided into three regions depending on the values of $e(t)$ and $\phi$. This is explained using binary sequence of the two-level quantizer as follows.

1. Steady state region, ($\Omega_{SS}$: $|e(t)| \leq \phi$) where $\delta_{t} = \{-1, +1, \ldots, -1, +1\}$ $\Rightarrow$ $\varphi(t) \approx 0$.
2. Transient process region-1, ($\Omega_{I}^{+}$: $e(t) > \phi$) where $\delta_{t} = \{+1, +1, \ldots, +1, +1\}$ $\Rightarrow$ $\varphi(t) \approx +1$.
3. Transient process region-2, ($\Omega_{I}^{-}$: $e(t) < -\phi$) where $\delta_{t} = \{-1, -1, \ldots, -1, -1\}$ $\Rightarrow$ $\varphi(t) \approx -1$.

![Figure 1](image_url)

**Figure 1.** Continuous-time $\Delta H$-M (a) $\Delta H$-M based encoder ($E^X_{\Delta H}$), (b) communication channel, (c) $\Delta H$-M based decoder ($D^X_{\Delta H}$).
Remark 2.1: The behaviour of the $\Delta_{H-M}$ in the region $\Omega_{ss}$ is similar to that of $\Delta_{\Sigma-M}$ and in the regions $\Omega_{tp}^I$ and $\Omega_{tp}^H$, it behaves like $\Delta-M$.

2.1. Stability analysis of continuous-time $\Delta_{H-M}$

For the all three operating regions, the stability of CT $\Delta_{H-M}$ is proven in following proposition.

Proposition 2.1: For the system described in (1)–(4), following condition is valid for the sliding mode condition to be reachable within finite time:

$$|\delta(t)| \leq \Delta X_t < \phi, \quad \Delta X_t \in \mathbb{R}_+.$$ (5)

Proof: The proof is given considering three cases.

- **Case-1: Stability analysis of continuous-time $\Delta_{H-M}$ in the region $\Omega_{ss}$.**

  Consider a Lyapunov function $V(t)$ as

  $$V(t) = \frac{1}{2} s(t)^2.$$ (6)

  Using (1) and (3), it follows that

  $$\dot{s}(t) = e(t) - \phi \delta_t.$$ (7)

  The derivative of (6) can be expressed as

  $$\dot{V}(t) = s(t)e(t) - \phi |s(t)|,$$ (8)

  using (7) and $\delta_t$ definition, $\delta_t = |s(t)|/s(t)$.

  It is evident that $\dot{V}(t) < 0$, since $|e(t)| \leq \phi$ in $\Omega_{ss}$.

- **Case-2: Stability analysis of continuous-time $\Delta_{H-M}$ in the region $\Omega_{tp}^I$.**

  If the increase of $x(t)$ is such that $\dot{x}(t) < 0$, then this will result in $x(t) > \dot{x}(t)$ at the beginning. As a result of this, the error $e(t) \rightarrow \phi$. However, any further increase of $x(t)$ ($e(t) > \phi$) will force the operating regions to shift (from $\Omega_{ss}$ to $\Omega_{tp}^I$) which implies $e(t) > \phi, \forall e(t) \in \Omega_{tp}^I$.

  In the proceeding, the stability of the $\Delta_{H-M}$ in the region $\Omega_{tp}^I$ is studied using equivalent control-based sliding mode. Let us replace the fast discontinuous component $\delta_t$ in (7) by its equivalent slow components $\phi(t)$. The shifting of the operating regions from $\Omega_{ss}$ to $\Omega_{tp}^I$ implies that

  $$\psi(t) = 1 \Rightarrow \frac{d}{dt} \psi(t) = 0.$$ (9)

  Let us consider a Lyapunov function as

  $$V_{tp}(t) = \frac{1}{2} \delta(t)^2.$$ (10)

  Using (7) and $\delta_t \Rightarrow \psi(t)$ (i.e. equivalent control-based sliding mode) yields

  $$\dot{s}(t) = e(t) - \phi \psi(t).$$ (11)

  Furthermore, by using (1)–(4) and (9), the following equation can be derived:

  $$\dot{V}_{tp}(t) = \dot{s}(t)s(t) = (e(t) - \phi \psi(t))(\dot{x}(t) - \phi).$$ (12)

  Note that in the region $\Omega_{tp}^I, e(t) > \phi$ and therefore, $e(t) - \phi > 0$. Furthermore, since (5) is true, then $V_{tp}(t) < 0$. Furthermore, when (5) is true, the rate of monotonous increment of $\dot{x}(t)$ in (4) is higher compared to the rate of increment of $x(t)$. As a result of this, the conditions required for the existence of sliding mode is fulfilled; as $\dot{x}(t)$ will force the operating regions to shift back from $\Omega_{tp}^I$ to $\Omega_{ss}$ within finite time. This is defined as hitting time $t_f$ which will be estimated in the next proposition.

- **Case-3: Stability analysis of continuous-time $\Delta_{H-M}$ in the region $\Omega_{tp}^H$.**

  The proof of Case-3 is similar to the proof of Case-2. ■

2.2. Computation of the hitting time ($t_f$) for continuous-time $\Delta_{H-M}$

In this section, we estimate the hitting time $t_f$, which equals to the time needed for the trajectory of the $\Delta_{H-M}$ to hit the sliding manifold.

Proposition 2.2: When (5) is true and $|x(0)| \gg 0$, the upper bound of the hitting time $t_f$ of $\Delta_{H-M}$ is given by

$$t_f \leq \frac{|x(0)|}{\phi - \Delta X_t}.$$ (13)

Proof: In the following, the maximum value of the hitting time $t_f$ is estimated which is defined as the time required for the trajectory of the $\Delta_{H-M}$ to be forced back into the steady state region $\Omega_{ss}$, from either of the operating regions $\Omega_{tp}^I$ or $\Omega_{tp}^H$. 
This means that in the combined region $\Omega_{\text{tp}} = \Omega_{\text{tp}}^l \cup \Omega_{\text{tp}}^u$, the rate of change of $\tilde{x}(t)$ is higher compared to the rate of change of $x(t)$. This will force $\Delta H$-M to change its operating region from $\Omega_{\text{tp}}$ to $\Omega_{\text{tp}}^H$ within finite time which is less than the maximum hitting time $t_f$ in (16).

### 3. Discrete-time hybrid delta modulator ($\Delta H$-M)

The schematic of the DT $\Delta H$-M is shown in Figure 2. From this figure, the following relations are obvious:

$$s(k + 1) = x(k) + 2\phi \text{ sat}(s(k)) - \hat{x}(k),$$

and

$$\hat{x}(k) = \tilde{x}(k) + \phi \delta_k,$$

where $x(k) \in \mathbb{R}$ is the input of the DT $\Delta H$-M, $\delta_k = \text{sgn}(s(k))$ and

$$2\phi \text{ sat}(s(k)) = \begin{cases} s(k), & |s(k)| \leq 2\phi, \\ 2\phi \delta_k, & |s(k)| > 2\phi. \end{cases}$$

Moreover, let

$$\tilde{x}(k) = \tilde{x}(k - 1) + \phi \psi(k),$$

where $\psi(k)$ is defined as the DT function related with the equivalent control-based sliding mode and is defined as

$$\psi(k) = \frac{1}{\kappa} \sum_{r=0}^{\kappa-1} \delta_{k-r},$$

where $\psi(k) \in \{-1, -(\kappa - 2)/\kappa, \ldots, 0, \ldots, (\kappa - 2)/\kappa, +1\}$ and $\kappa \in \mathbb{N}\setminus\{0\}$. For convenience and simplicity of computation only the case of $\kappa = 2$ is considered:

$$\psi(k) = \frac{1}{2} (\delta_k + \delta_{k-1}) \in \{-1, 0, +1\}. \quad (21)$$

The error is defined as $e(k) = x(k) - \tilde{x}(k)$, the main task of the function (21) is to maintain $e(k)$ in the neighbourhood of 0. The DT system (17)–(21) is in QSM if $|e(k)| < \varepsilon$. 

![Figure 2. Discrete-time $\Delta H$-M (a) $\Delta H$-M based encoder ($E^\chi_{\Delta H}$), (b) communication channel and (c) $\Delta H$-M based decoder ($D^\chi_{\Delta H}$).](image)
3.1. Stability analysis of discrete-time $\Delta_{H-M}$

The stability of DT $\Delta_{H-M}$ in all the operating regions is proven in the following proposition.

**Proposition 3.1:** For a system described by (17)–(21), if

$$|\Delta x(k)| \leq \Delta X_k < \phi, \quad (22)$$

then the system trajectory will converge, from any initial state $e(0)$, to quasi-sliding mode domain (QSMD) which is defined by $|e(k)| \leq \varepsilon$, where $\varepsilon$ is bounded such that $\varepsilon \leq 2\phi$, $\forall k > k^*$. Once the system trajectory enters into the quasi-sliding mode domain, it will remain in QSMD for all the subsequent time.

**Proof:** Stability of the $\Delta_{H-M}$ (17)–(21) will be studied in the two regions: $\Omega_{ss}$ and $\Omega_{tp} = \Omega_{tp}^I \cup \Omega_{tp}^II$.

- **Case-1:** Stability analysis of discrete-time $\Delta_{H-M}$ in the region $\Omega_{ss}$.

  Assume that the system start from the initial point, $e(0)$ where $|e(0)| \leq \phi$ (e.g. $|x(0) - \hat{x}(0)| < \phi$) which implies that $e(0) \in \Omega_{ss}$. For the discrete-time $\Delta_{H-M}$, the dynamics described in (17)–(21), assume $s(0) = 0$. Let us consider two possible cases ($s(k) \geq 0$ and $s(k) < 0$) within $\Omega_{ss}$.

  1. Consider the case when $s(k) \geq 0$, in (17) and $\psi(k) = 0$ in (20) and (21). Then (17) can be expressed as

     $$s(k + 1) = s(k) + e(k) - \phi.$$ 

    Since $|e(k)| \leq \phi$ in the region $\Omega_{ss}$, then $s(k + 1) < s(k)$.

  2. Consider the case when $s(k) < 0$, in (17) and $\psi(k) = 0$ in (20) and (21). Then (17) can be expressed as

     $$s(k + 1) = s(k) + e(k) + \phi.$$ 

    Since $|e(k)| \leq \phi$ in the region $\Omega_{ss}$, then $s(k + 1) > s(k)$.

   From the above two cases ($s(k) \geq 0$ and $s(k) < 0$), it can be concluded that $s(k)$ decreases monotonically when $s(k) \geq 0$ and it increases monotonically when $s(k) < 0$. This implies that $|s(k + 1)| < |s(k)|$.

- **Case-2:** Stability analysis of discrete-time $\Delta_{H-M}$ in the region $\Omega_{tp}^I$.

In the following section, the stability condition for the scenario when the system trajectory is outside QSMD ($e(0) > \varepsilon$) is investigated. Note that, in this case, $e(0) > \varepsilon$, $x(0) > \hat{x}(0)$ and as a result of that $s(k) > 0$, $\forall 0 \leq k \leq k^*$.

Using iterations,

$$e(k) = \sum_{j=1}^{k} \Delta x(i) + e(0) - k\phi$$

$$\leq k(\Delta X_k - \phi) + e(0)$$

$$\leq e(k - 1), \quad \forall \Delta x(k) \leq \Delta X_k < \phi. \quad (23)$$

Furthermore (17) can also be expressed as

$$s(k + 1) = e(k) + \phi, \quad \forall s(k) \geq 2\phi.$$ 

Note that when $e(0) > \varepsilon$ and $s(k) > 2\phi$, $\forall 0 \leq k \leq k^*$, then $e(k) < e(k - 1)$, $s(k + 1) < s(k)$. In other words, this means that $s(k)$ decreases monotonically and QSM is reached within finite number of steps $k^*$ such that $s(k) \leq \varepsilon$, $\forall k \geq k^*$.

- **Case-3:** Stability analysis of discrete-time $\Delta_{H-M}$ in the region $\Omega_{tp}^II$.

Proof of Case-3 is similar to the proof of Case-2. \hfill \blacksquare

3.2. Computation of the hitting step $k^*$ for discrete-time $\Delta_{H-M}$

In this section, we estimate the hitting step $k^*$. The hitting step $k^*$ is defined as the maximum number of steps required by the trajectory of the $\Delta_{H-M}$ to hit the sliding manifold.

**Proposition 3.2:** For the system described in (17)–(21), if (22) is true, then the number of steps required for the trajectory of discrete-time $\Delta_{H-M}$ to cross the switching manifold $s(k) = 0$ (from any initial value $s(0)$), equals to $k^*$ where $k^* = |m|$, and

$$m \leq \frac{|e(0)|}{\phi - \Delta X_k}. \quad (24)$$

The floor operation is denoted by $|m|$.

**Proof:** In the succeeding section the hitting step $k^*$, where the system trajectory of the discrete-time $\Delta_{H-M}$ crosses the hyper-plane $e(k) = 0$, is estimated. Note that the results are derived under the assumption of $e(0) > \varepsilon$. The $m$th iteration of (23) gives

$$e(m) = e(0) + \sum_{k=0}^{m-1} \Delta x(k) - m\phi. \quad (25)$$

From Proposition 3.1, it is known that there exists a step where $\delta_k$ changes from $+1$ to $-1$. Consider the case of maximum $m$ where $\Delta x(k) = \Delta X_k$. When $e(|m|) \leq 0$,
Figure 3. Continuous-time $\Delta_H$ input, output and state response.

Figure 4. Continuous-time $\Delta_H$ sliding manifold.
then \( m \leq e(0)/(\phi - \Delta X_k) \). Similar results can be derived for the case when \( e(0) < -\epsilon \). Hence from the both scenarios, \(|e(0)| > \epsilon\), it can be concluded that the system trajectory of the discrete-time \( \Delta_H-M \) requires \( k^* \) number of steps to cross the surface \( e(k) = 0 \) where

\[
m \leq \frac{|e(0)|}{\phi - \Delta X_k},
\]

and \( k^* = \lfloor m \rfloor \). This completes the proof of Proposition 3.2.

4. Simulation results

In this section, the dynamical properties of \( \Delta_H-M \) in both CT and DT domains are investigated. In the simulation, the input, output and trajectory of the modulators are computed and their behaviour in various operating regions are studied considering the signal

\[
y(t) = \begin{cases} 
3r(t) - r(t - 5) - 2r(t - 10) & 0 \leq t \leq 15, \\
\sin(0.7t) + \sin(0.256t) + 5.89 \sin(0.385t + 2.5), & 15 < t \leq 30,
\end{cases}
\]

where \( r(t) \) is the ramp signal.

4.1. Simulation results of continuous-time \( \Delta_H \)

Consider the CT \( \Delta_H-M \), described in (1)–(4) with \( x(t) = y(t) \) (defined in (26)) as the input signal and \( s(0) = 0 \). The trajectories of this modulator are shown in Figures 3–5.

According to the stability condition (5), the CT \( \Delta_H-M \), with quantizer gain \( \phi = 2 \), is stable for all \( |\dot{x}(t)| \leq X_{\Delta t}, \forall X_{\Delta t} = 2 \). It can be seen that when \( t \in [0, 5) \), \( \dot{x}(t) > 2 \). Therefore, the modulator becomes unstable (see Figure 3). Note that if this condition persists for a longer period, it will eventually lead to instability. For \( t \in [5, 10) \), the derivative of the input signal becomes equal to the quantizer gain, i.e. \( \dot{x}(t) = \phi \). Therefore, the system is marginally stable during this time (see Figures 4 and 5). In this region, both the input \( x(t) \) and the output \( \hat{x}(t) \) increase with the same rate which is evident from Figure 3. For \( t \in [10, 5) \), the operating region is switched to the stability region. Note that stability region consists of sliding mode and equivalent mode. The hitting time \( t_f \) is calculated from (13) and has been found to be 12.5 s. The trajectory in the phase plane, shown in Figure 4, further confirms the results presented in Figures 3 and 5.

4.2. Simulation results of discrete-time \( \Delta_H \)

Consider the DT equivalent of the signal in (26) (denoted as \( y(k) \)), \( \forall t \in [\Delta k, (k + 1)\Delta] \), where \( k \) and \( \Delta \) denote the sampling step and period, respectively. Let the input of the \( \Delta_H \) (17) be \( x(k) = y(k) \). The trajectories of \( \Delta_H \) system

![Figure 5. Continuous-time \( \Delta_H \).](image-url)
are shown in Figure 6 for $\phi = 0.4$, $h = 0.2$ and $s(0) = 0$. According to the stability condition (22), it can be seen that $\Delta_H$ with $\phi = 0.4$, is stable for all $|x(k)| \leq 0.4$. Thus, the system output $\hat{x}(k)$ diverges from the system input $x(k)$ when $\Delta x(k) = 0.6$, $\forall 0 \leq k < 25$ and it converges when $\Delta x(k) \leq 0.4$, $\forall 50 \leq k < 65$; since it satisfies stability conditions. According to (24), the number of steps which the trajectory takes to make transition from marginal mode (which starts at $k = 51$) to equivalent mode, is 15. This gives $k_f = 66$.

5. Conclusion

The stability conditions and accurate estimation of hitting-step for both data-driven $\Delta_H$-M are derived analytically using QSM analysis. It is found that the stability and the upper bound of hitting-time, for $\Delta_H$-M, are critically dependent on some properties of the input signals, quantizer gain, as well as the adaptive parameters in $\Delta_H$-M. The simulation results confirm theoretical findings. Future research includes using $\Delta_H$-M in applications such as in Power Electronics, in event-triggered NCS, load frequency control of multi-area interconnected power systems (Lu, Zhou, Zeng, & Zheng, 2019) and in the internet of things (IoT). Also, this adaptive algorithm can further be improved to tackle many challenges that we face in the aforementioned applications.

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