Nontrivial fixed point in nonabelian models

Adrian Patrascioiu \textsuperscript{a} and Erhard Seiler\textsuperscript{b} \*  

\textsuperscript{a}Physics Department, University of Arizona, Tucson, AZ 85721, U.S.A.  
\textsuperscript{b}Max-Planck-Institut f"ur Physik, Werner-Heisenberg-Institut, F"ohringer Ring 6, 80805 Munich, Germany  

We investigate the percolation properties of equatorial strips in the two-dimensional O(3) nonlinear $\sigma$ model. We find convincing evidence that such strips do not percolate at low temperatures, provided they are sufficiently narrow. Rigorous arguments show that this implies the vanishing of the mass gap at low temperature and the absence of asymptotic freedom in the massive continuum limit. We also give an intuitive explanation of the transition to a massless phase and, based on it, an estimate of the transition temperature.

1. Introduction

This talk, though scheduled in the session on Perturbation Theory (PT), does not deal with PT as such. But one of its conclusions is that in certain nonabelian models the perturbative Callan-Symanzik $\beta$-function gives the wrong picture of the renormalization group flow.

In 1991 we developed a rigorous criterion for the existence of a massless phase in 2D spin models based on percolation properties of certain subsets of the target spin manifold \cite{1}. This was presented at the 1992 lattice conference together with nonrigorous arguments that led to the conclusion that all 2D O(N) most likely had a soft low temperature phase, contrary to prevailing expectations.

Advances in computer technology have made it feasible to tackle directly the question whether the percolation properties leading to the existence of a massless phase hold or not. In this talk I report our results giving direct numerical evidence for our old conjecture of the existence of a massless phase in models for which PT predicts mass generation and asymptotic freedom (AF); these results were first presented in \cite{2}.

For concreteness and simplicity we are dealing with the 2D O(3) model, a.k.a. classical Heisenberg model. Its standard version is defined on the square lattice, with the standard Hamiltonian (action)

$$H = - \sum_{\langle xy \rangle} s(x) \cdot s(y)$$

(1)

where $s(.)$ is a classical spin (unit 3-vector) living on the vertices $x$ of the lattice and the sum is over nearest neighbors $\langle xy \rangle$; the spins are distributed according to the Gibbs measure with density $\frac{1}{Z} \exp(-\beta H)$. For the purpose of this investigation we modify this model by introducing a constraint limiting the size of the deviation allowed between neighboring spins:

$$s(x) \cdot s(y) \geq c$$

(2)

with $-1 \leq c \leq 1$ (the so-called cut action with cut $c$), and we also replace the square by a triangular lattice; the Gibbs measure thus contains $\theta$-functions enforcing the constraint \cite{2}. These modifications are made for purely technical reasons and we have checked that they do not change the universality class \cite{3}.

2. The percolation criterion

We briefly sketch the percolation criterion developed in 1991 \cite{1}: We divide the sphere $S_2$ into three pieces:
• ‘equatorial strip’ $S_\epsilon$, defined by $|s \cdot n| < \epsilon/2$
  for some fixed unit vector $n$.
• ‘upper polar cap’ $P^+\epsilon$, defined by $s \cdot n \geq \epsilon/2$,
• ‘lower polar cap’ $P^-\epsilon$, defined by $s \cdot n \leq -\epsilon/2$.

and denote the corresponding subsets of the lattice for a given configuration by $S^\pm\epsilon$ etc.. These subsets fall into connected components called ‘clusters’; their mean size we denote by $\langle S_\epsilon \rangle$ etc.

If this mean size is finite, we say that the clusters ‘form islands’; if there is an infinite cluster, we say the subset percolates. There is a third possibility: that the clusters have divergent mean size, but none of them is infinite; this we call ‘formation of rings’.

The main result of [1] is the following

Theorem: If for a certain $c > 1 - \epsilon^2/2$ $S_\epsilon$ does not percolate, the model has no mass gap.

The reason why this is so can be understood by using the Fortuin-Kasteleyn (FK) representation [4] for the imbedded Ising spins $\sigma_x \equiv sgn(s(x))$ (which is also the basis of the cluster algorithm).

But first one has to notice that in 2D it is not possible that two disjoint clusters both percolate, and therefore, if $c > 1 - \epsilon^2/2$, the union of the polar caps cannot percolate, because then both of them would percolate. But if also $S_\epsilon$ does not percolate, as assumed, a lemma of Russo [5] assures us that the clusters of each of the three sets have divergent mean size.

But due to the inequality $c > 1 - \epsilon^2/2$, each of the clusters of $P_\epsilon^+$ or $P_\epsilon^-$ has to be contained in its entirety inside a FK cluster, and hence also the FK clusters have divergent mean size.

Since the mean size of the FK clusters is equal to the susceptibility of the imbedded Ising spins, this divergent mean size is incompatible with exponential clustering, and thus there is no mass gap.

3. Percolation properties: numerical study

We investigated numerically the percolation properties of the equatorial strip for the special case $\beta = 0$. In this case the parameter $c$ replaces the temperature in determining how ordered or disordered the system is.

The results of our investigation [2] are summarized in the ‘percolation phase diagram’ Fig.1. The diagram is semiquantitative, but qualitatively correct, as we will explain.

Figure 1. Phase diagram of the $O(3)$ model on the T lattice

In this figure the solid line is the curve $c = 1 - \epsilon^2/2$; above that line the two polar caps cannot ‘touch’ and therefore their union cannot percolate. The dashed line separates a regime in which the strip $S_\epsilon$ percolates (above) and one in which it does not (below). For small $\epsilon$, the strip forms islands for $\epsilon$ below that line. Around $c = 0.4$ a dotted line branches off; below it the strip still forms islands, whereas between the dotted and the dashed lines the clusters of the strip have mean infinite size without percolating (‘formation of rings’). The interesting region is the one between the solid and the dashed lines: here the strip does not percolate but the inequality of the theorem holds. So in this regime our theorem can be applied and allows us to conclude that for $c > c_o$, $c_o \approx 0.7$ there is no mass gap.

Fig.1 also shows that for $\epsilon < \epsilon_o$, with $\epsilon_o \approx 0.76$ the equatorial strip does not percolate for any $c$. [Allèes et al [3] published a study showing that for $\epsilon = 1.05$ and $\beta = 2.0$ (standard action) the equatorial strip percolates. This is correct, but since their choice of parameters is such that they are
both in the massive phase and in the percolation region of the strip, it is not very relevant for our problem.

Let me now explain from which facts our ‘phase diagram’ was derived: In Fig. 2 we show that ratios \( r \equiv \langle P_\epsilon \rangle / \langle S_\epsilon \rangle \) as a function of \( c \) for several values of \( \epsilon \) between .78 and .91 and for lattice sizes \( L \) from 160 to 1280.

It is seen that for small \( c \) (depending on \( \epsilon \)) \( r \) increases sharply with \( L \) (we have data which are off the scale of this figure and show that the increase continues). This expresses the fact that in this regime \( S_\epsilon \) percolates, whereas its complement \( P_\epsilon \) forms islands of finite size. At a certain value of \( c \) the curves for different \( L \) intersect and \( r \) becomes scale invariant; this is the critical point of percolation for the chosen value of \( \epsilon \) in which both sets form rings. The pair \((c, \epsilon)\) defines a point on the dashed curve in Fig. 1.

If we increase \( c \) beyond the intersection point, the size dependence of \( r \) is reversed, indicating that we are now in the regime of percolation of \( P_\epsilon \).

Increasing \( c \) still further, the curves come together again and remain together, indicating that for all \( c \) in that regime (depending on \( \epsilon \)) we are in the regime of rings formation of both sets, i.e. we have entwined the regime between the dashed and dotted curves in Fig. 1. For \( c \to 1 \) the ratio \( r \) converges to the geometric ratio \((2 - \epsilon) / \epsilon\) of the sets \( P_\epsilon \) and \( S_\epsilon \).

To corroborate that for \( \epsilon \) less than about 0.76 \( S_\epsilon \) does not percolate, no matter what \( c \) is, we also measured the ratio \( r' \equiv \langle P_0^+ \rangle / \langle S_\epsilon \rangle \), where \( \epsilon' \) is chosen in such a way that the two sets have equal area, i.e. \( \epsilon' = 2 - 2\epsilon = 0.5 \). Fig. 3 shows that for \( c < 0.4 \) \( r' \) increases sharply with \( L \). Since the
polar cap cannot percolate, this means that $P_\epsilon'$ forms rings of arbitrary size, whereas $S_\epsilon$ forms islands of finite size. The behavior changes drastically around $c = 0.4$, the $L$ dependence of the ratio $r'$ becomes a much milder increase, compatible with a power law behavior of both numerator and denominator, but clearly ruling out percolation of $S_\epsilon$. The only possible interpretation is that both sets form rings of arbitrary size.

The crucial fact for our conclusion is obviously that a polar cap can form rings of arbitrary size even though it is smaller than a hemisphere. This ring formation then prevents percolation of the corresponding equatorial strip, and thus allows the application of our theorem.

Our approach has been by necessity in the spirit of finite size scaling, studying how various quantities change with increasing size. So is it conceivable that we are deceived by finite size behavior that changes its character at some astronomical lattice size? Obviously in the truly interesting region near the critical point (which is around or slightly below $c = 0.7$) we cannot work on lattices of thermodynamic size. But we did an additional test at $c = 0$, where the correlation length is about $53$ and we can easily go to thermodynamic lattices: We measured directly the mean cluster size $\langle P_0,95 \rangle$ as a function of $L$ for $L$ up to $1280$. The results displayed in Fig. 4 show a linear dependence of $\ln \langle P_0,95 \rangle$ on $\ln (L)$, indicating a powerlike increase of $\langle P_0,95 \rangle$ for lattices much larger than the correlation length. This is in sharp contrast with the behavior at $c = -1$, also shown in Fig. 4, where one can clearly see $\langle P_0,95 \rangle$ leveling off.

If we combine this with another fact, which is plausible and which we also checked, namely that $\langle P_\epsilon \rangle$ is a monotonically increasing function of $c$, we reach the conclusion that this ring formation must persist for all $c < 1$.

4. Concluding remarks

Our main conclusion is that the $2D\ O(3)$ has a transition to a massless phase at low temperature, contrary to standard lore, which derives from the PT calculation of the Callan-Symanzik $\beta$-function. We have questioned the validity of PT in the models showing perturbative asymptotic freedom in various publications and talks at lattice conferences (see for instance [3]), but our percolation study makes the conclusion unavoidable that PT does not give the correct asymptotic expansion for the $\beta$-function.

A question that has been asked is what is ‘driving the transition’ to the massless phase. This is also answered in our recent paper [2]: it can be understood as the transition from gas of instantons to a gas (or liquid) of super-instantons, which are the dominant excitations at low temperature and create the disorder required by the Mermin-Wagner theorem. In [3] we used a simple-minded energy-entropy argument to give a rough estimate of the transition temperature for the standard action model; the result ($\beta_{crt} \approx \pi$), is consistent with known numerical results.

REFERENCES

1. A. Patrascioiu and E. Seiler, J. Stat. Phys. 69 (1992) 573; Nucl. Phys. (Proc. Suppl.), B30 (1993) 184.
2. A. Patrascioiu and E. Seiler, Absence of Asymptotic Freedom in Non-Abelian Models, hep-th/0002153.
3. A. Patrascioiu and E. Seiler, paper in preparation.
4. P. W. Kasteleyn and C. M. Fortuin, J. Phys. Soc. Jpn. 26 (Suppl.) (1969) 11; C. M. Fortuin and P. W. Kasteleyn, Physica 57 (1972) 536.
5. L. Russo, Z. Wahrsch. verw. Gebiete, 43 (1978) 39.
6. B. Allès, J. J. Alonso, C. Criado and M. Pepe, Phys. Rev. Lett. 83 (1999) 3669.
7. A. Patrascioiu and E. Seiler, Phys. Rev. Lett. 74 (1995) 1920; Nucl. Phys. (Proc. Suppl.) B 42 (1995) 826.