Gröbner bases and combinatorics for binary codes.

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Abstract

In this paper we introduce a binomial ideal derived from a binary linear code. We present some applications of a Gröbner basis of this ideal with respect to a total degree ordering. In the first application we give a decoding method for the code. By associating the code with the set of cycles in a graph, we can solve the problem of finding all codewords of minimal length (minimal cycles in a graph), and show how to find a minimal cycle basis. Finally we discuss some results on the computation of the Gröbner basis.

1 Introduction

We associate with a binary linear code a Gröbner basis for total degree compatible orderings such as degrevlex (Degree Reverse Lexicographic), for which Gröbner bases are known to be easier to compute. In our particular application the Gröbner basis has additional properties that allow us to formulate an algorithm, which has the flavour of an FGLM approach and it is especially adapted to our setting. We show how the Gröbner basis of the code can be used for decoding and solve several problems related to graphs associated with the code.

In the paper we use the term code to refer only to binary linear code even though some of our results (cf. Sections 2, 3) can be extended to the non-binary case (see also 2, 3, 4). The outline of the paper is as follows. In Section 2 we define a monoid connected to a binary linear code. An ideal associated with the code is introduced in Section 3 together with a decoding method that makes use of a Gröbner basis of the ideal. In fact, decoding is carried out using

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classical reduction to the canonical form. Some other applications are developed in Section 4 such as finding all minimal cycles in a graph and a minimal cycle basis. In Section 5 a linear algebra procedure (related to FGLM) is used to compute the Gröbner basis for the ideal associated with a code. This method is applicable in a general setting but in our setting it has additional computational advantages.

2 Binary linear codes and monoids

2.1 Binary linear codes

Let $\mathbb{F}_2$ be the finite field with 2 elements. A linear code $C$ of dimension $k$ and length $n$ is the image of a linear mapping $L : \mathbb{F}_2^k \to \mathbb{F}_2^n$, where $k \leq n$, i.e. $C = L(\mathbb{F}_2^k)$. There exists a $n \times (n-k)$ matrix $H$, called a parity check matrix, such that $cH = 0$ if and only if $c \in C$. On the other hand, there exists a $k \times n$ generator matrix $G$ such that $C = \{uG \mid u \in \mathbb{F}_2^k\}$. Normally, we consider check matrices to have linearly independent columns and generator matrices linearly independent rows. However, in some situations it is useful to regard as a check matrix any matrix whose left nullspace is the code, and as generator matrix any matrix whose row space is the code. The weight of a codeword is its Hamming distance to the word 0, and the minimum distance $d$ of a code is the minimum weight among all the non-zero codewords. The error correcting capacity of a code is $t = \left\lfloor \frac{d-1}{2} \right\rfloor$, where $\left\lfloor \cdot \right\rfloor$ is the greatest integer function. Let $B(C, t) = \{y \in \mathbb{F}_2^n \mid \exists c \in C \text{ s.t. } d(c, y) \leq t\}$, it is well known that the equation $eH = yH$ has a unique solution $e$ with weight($e$) $\leq t$ for $y \in B(C, t)$.

2.2 The monoid associated with a binary code

Let $[X]$ be the free commutative monoid generated by the $n$ variables $X = \{x_1, \ldots, x_n\}$. We have the following map from $X$ to $\mathbb{F}_2^n$:

$$\psi : X \to \mathbb{F}_2^n$$

$$x_i \mapsto e_i = (0, \ldots, 0, 1_i, 0, \ldots, 0)$$  \hspace{1cm} (1)

The map $\psi$ can be extended to a morphism from $[X]$ onto $\mathbb{F}_2^n$, where

$$\psi \left( \prod_{i=1}^{n} x_i^{\beta_i} \right) = (\beta_1 \mod 2, \ldots, \beta_n \mod 2)$$  \hspace{1cm} (2)

When no confusion arise we will use $x_i$ to refer the indeterminate in the monoid or the associated vector $e_i$ in $\mathbb{F}_2^n$. A code $C$ defines an equivalence relation $R_C$ in $\mathbb{F}_2^n$ given by

$$(x, y) \in R_C \iff x - y \in C.$$  \hspace{1cm} (3)
If we define $\xi(u) = \psi(u)H$, where $u \in [X]$, the above congruence can be translated to $[X]$ by the morphism $\psi$ as

$$u \equiv_c w \iff (\psi(u), \psi(w)) \in R_c \iff \xi(u) = \xi(w). \quad (4)$$

The morphism $\xi$ represents the transition of the syndromes from $\mathbb{F}_2^n$ to $[X]$. Thus, $\xi(w)$ is the syndrome of $w$, which is the syndrome of $\psi(w)$.

For the sake of simplicity we will use $w, u, v$ as words in $[X]$ and vectors in $\mathbb{F}_2^n$, as long as the meaning is clear from the context. The connection between the two structures can be understood from the following setting

$$w = 1 \cdot x_{i_1} \cdot \ldots \cdot x_{i_m} \in [X] \rightarrow \psi(w) = 0 + e_{i_1} + \ldots + e_{i_m} \in \mathbb{F}_2^n. \quad (5)$$

**Definition 1 (standard word).** The word $w = \prod_{i=1}^{n} x_i^{\beta_i}$ is said to be standard if $\beta_i < 2$, for every $i \in \{1, \ldots, n\}$. Given $y \in \mathbb{F}_2^n$ we say that $w$ is the standard representation of $y$ if $\psi(w) = y$ and $w$ is standard.

### 2.3 Binary codes and the set of cycles in a graph

Let $G = (V, E)$ an undirected 2-connected graph without loops or multiple edges, where $V$ is the set of vertices and $E$ the set of edges. An edge is denoted by an unordered pair of vertices $(x, y)$. A cycle is a subgraph such that any vertex degree is even. Therefore, a cycle can be written as either a set of edges $\{(x_1, x_2), (x_2, x_3), \ldots\}$ or as a closed path $(x_1, x_2, x_3, \ldots, x_1)$. The length of a cycle is the number of edges it contains.

The sum of two cycles is defined as the symmetric set difference $C + C' = (C \cup C') \setminus (C \cap C')$. With this sum the set of cycles forms an $\mathbb{F}_2^m$-vector space which is a subspace of $\mathbb{F}_2^m$, where $m = |E|$ (the number of edges). Therefore, the set $C$ of cycles in a graph can be considered as a binary code of length $m$. A basis of this vector space is called a cycle basis, and its dimension is well-known to be the Betti number $\dim(C) = m - |V| + 1$ (see, for example, [12, 13, 14]). We define the length of a basis as the total length of the cycles in it.

### 3 The ideal associated with a code

In this section we define a particular ideal associated with a code. Ideals associated with soft-decision maximum likelihood decoding can be found in [11], and these in turn are related to ideals arising in integer programming using Gröbner basis [2].

Consider the polynomial ring $K[X]$, where $K$ is a field. Let $\ll < \ll_T$ be a fixed, total degree compatible term order with $x_1 < x_2 \cdots < x_n$ on $[X]$. We use $<$ for this term order as the meaning of the symbol will always be clear from the context, and write $>$ where appropriate. As usual, $T(f)$ denotes the maximal term of a polynomial $f$ with respect to the order $<$ and $Td(f)$ the total degree of the maximal term $T(f)$ of $f$. The set of maximal terms of the set $F \subseteq K[X]$ is denoted $T\{F\}$ and $T(F)$ denotes the semigroup ideal generated by $T\{F\}$. Finally, $\langle F \rangle$ is the polynomial ideal generated by $F$. 
Definition 2. Let $C$ be a code and $R_C$ the equivalence relation defined in equation (3). The ideal $I(C)$ associated with $C$ is

$$I(C) = \langle \{w - v \mid (\psi(w), \psi(u)) \in R_C\} \rangle \subseteq K[X].$$

(6)

Let be $\{w_1, \ldots, w_k\}$ be the row vectors of a generator matrix for a code (more generally any matrix whose rows span the code $C$), i.e., a basis (spanning set) of the code as subspace of $F_n^2$. Let

$$I = \langle \{w_1 - 1, \ldots, w_k - 1\} \cup \{x_i^2 - 1 \mid i = 1, \ldots, n\} \rangle \subseteq K[X].$$

(7)

be the ideal generated by the set of binomials $\{w_1 - 1, \ldots, w_k - 1\} \cup \{x_i^2 - 1 \mid i = 1, \ldots, n\} \in K[X]$. Since $\{w_1, \ldots, w_k\}$ generate $C$ it is clear that $I = I(C)$.

3.1 Error-correcting reduced Gröbner basis

Let $G_T$ be the reduced Gröbner basis of the ideal $I(C)$ with respect to $<$. Note that $G_T$ can be computed by Buchberger’s algorithm starting with the initial set $\{w_1 - 1, \ldots, w_k - 1\} \cup \{x_i^2 - 1 \mid i = 1, \ldots, n\}$. However, there are some computational advantages in this case. The coefficient field is $F_2$ (and therefore there is no coefficient growth), and the maximal length of a word appearing in the computation is $n$ (the binomials $x_i^2 - 1$ prevent the length being greater than $n$). Thus the two principal disadvantages of Gröbner basis computations are not valid for this case. In addition, total degree compatible term orders are among the most efficient for the computation of Gröbner bases.

Although the usual reduction could be carried out with the same result, we introduce a special reduction in order to have a more efficient process.

Definition 3 (One step reduction). Reduction in one step ($\rightarrow$) using $G_T$ is defined as follows. For any $w \in [X]$: 

1. reduce $w$ to its standard form $w'$ using the relations $x_i^2 \rightarrow 1$, for all $x_i \in X$.

2. reduce $w'$ with respect to $G_T$ by the usual one step reduction.

This reduction process is well defined since it is confluent and noetherian. Thus, it will end after a finite number of one step reductions with a unique irreducible element corresponding to the starting element. Moreover, if we denote by $\text{Can}(w, G_T)$ the canonical form of $w$ with respect to $G_T$ we have the following result.

Theorem 1 (Canonical forms of the vectors in $B(C, t)$). Let $C$ be a code and let $G_T$ be the reduced Gröbner basis with respect to $<$. If $w \in [X]$ satisfies the condition $\text{weight}(\psi(\text{Can}(w, G_T))) \leq t$ then $\psi(\text{Can}(w, G_T))$ is the error vector corresponding to $\psi(w)$. On the other hand, if $\text{weight}(\psi(\text{Can}(w, G_T))) > t$ then $\psi(w)$ contains more than $t$ errors.
Proof. The uniqueness of the canonical form is guaranteed by its definition, and thus we only need to prove that the standard representation of the error vector associated with a vector $y$ satisfies the condition to be the canonical form of $y$.

Let $y \in B(C,t)$ and denote by $e = c_y$ be the error vector corresponding to $y$. Then $eH = yH$ and weight($e$) $\leq t$. If $w_e$ is the standard representation of $e$ then weight($e$) coincides with the total degree of $w_e$. Accordingly, $Td (w_e) \leq t$. It is clear that there cannot be another word $u$ such that $Td (u) \leq t$ and $\xi(u) = yH$, since this would mean that there are two solutions for the linear system with weight at most $t$, and this is not possible because $y \in B(C,t)$. Therefore, it is clear that $w_e$ is the minimal element with respect to $<$ having the same syndrome as $y$.

We see later that the error-correcting capability $t$ of the code can be computed from $G_T$ (see Remark $\ref{1}$).

Example 1 (Decoding a binary code using its associated Gröbner basis). Let be $G$ be a generator matrix of the $[6,2,3]$ binary code $C$ over $\mathbb{F}_2$ defined as

$$G = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

From this matrix we obtain a set of generating polynomials for $I(C)$ as in equation $\eqref{7}$ as follows:

$$I(C) = (x_1x_2x_4x_5 - 1, x_2x_3x_6 - 1, x_1x_3x_4x_5x_6 - 1, x_1^2 - 1, x_1^2 - 1, x_3^2 - 1, x_4^2 - 1, x_5^2 - 1, x_6^2 - 1).$$

A Gröbner basis of $I(C)$ with respect to the degrevlex is

$$G_T = \{x_1^2 - 1, x_2^2 - 1, x_3^2 - 1, x_4^2 - 1, x_5^2 - 1, x_6^2 - 1, x_2x_3 - x_6, x_2x_4 - x_1x_5, x_2x_5 - x_1x_4, x_2x_6 - x_3, x_3x_6 - x_2, x_4x_5 - x_1x_2, x_3x_4 - x_5x_6, x_1x_3x_5 - x_6x_4, x_5x_1x_4 - x_3x_5, x_1x_5x_6 - x_3x_4 \}.$$ 

The decoding process consists of obtaining the errors as a common reduction process modulo the Gröbner basis $G_T$. Suppose the word $w = x_1x_2x_3x_4x_5$ is received. The canonical form of $w$ modulo $G_T$ is $x_3$, since $x_3$ has weight 1 and the code is 1-error correcting (see Theorem $\ref{4}$), then the corresponding codeword is $x_1x_2x_4x_5$ or 110110.

4 Further Applications.

We will show that the Gröbner basis $G_T$ for a code $C$ can be used to solve some other problems in coding theory and graph theory. In general, let $c_g$ be the codeword associated to the binomial $g = w - v \in I(C)$, so that $c_g = \psi(w) + \psi(v)$, and let $w_e$ be the standard word corresponding to $c$. 


Theorem 2 (Reduction of a codeword). Let $c$ be a codeword such that $\text{weight}(c) = d'$. Then there exists $g_1 \in G_T$ such that:

1. $\text{Td}(g_1) \leq t' = [(d' - 1)/2] + 1$.
2. $w_c \xrightarrow{g_1} w_2$, such that $\text{Td}(w_2) \leq d'$ and $w_2 < w_c$.
3. $c = c_{g_1} + c_{w_2}$, where $\text{weight}(c_{g_1}) \leq d'$ and $\text{weight}(c_{w_2}) \leq d'$.

Proof. Let $w_{c_1}$ and $u_{c_1}$ be such that $w_c = w_{c_1}u_{c_1}$ with weight($w_{c_1}$) = $t'$ and $u_{c_1} < w_{c_1}$. It is clear that $w_{c_1} - u_{c_1} \in I(\mathcal{C})$ since $w_{c_1}$ and $u_{c_1}$ have the same syndrome. Therefore, $w_{c_1} \in T(G_T)$. Let $g_1 = w_{1} - v_1 \in G_T$ where $v_1 < w_1$ satisfy $w_{c_1} = w_1v_1$ for some $v_1 \in [X]$. Then $\text{Td}(g_1) = \text{Td}(w_1) \leq t'$ and hence $g_1$ satisfies condition (1).

Now, $w_c \xrightarrow{g_1} w_2 = u_{c_1}w_1v_1$. Note that $\text{Td}(w_{c_1}) = d' - t'$ and $\text{Td}(u_1v_1) \leq t'$, which implies $\text{Td}(w_2) \leq d'$. Also, $v_1 < w_1$ implies $v_1u_1u_{c_1} < w_1u_1u_{c_1}$, that is, $w_2 < w_c$, and (2) follows.

In order to prove (3), we observe that $\text{Td}(w_1) \leq \text{Td}(w_{c_1})$ and by the construction of $w_{c_1}$ and $u_{c_1}$ and $v_1$ being a canonical form, we have also that $\text{Td}(v_1) \leq \text{Td}(u_{c_1})$. Thus, $\text{Td}(w_1v_1) \leq d'$ and weight($c_{g_1}$) $\leq d'$. It is easy to see that $c = c_{g_1} + c_{w_2}$ since $c = \psi(w_1) + \psi(u_1) + \psi(u_{c_1})$, $c_{g_1} = \psi(w_1) + \psi(v_1)$, and $c_{w_2} = \psi(u_{c_1}) + \psi(u_1) + \psi(v_1)$.

Remark 1. As a consequence of (1.) in this Theorem, given $G_T$ and $g \in G_T$ a binomial such that

$$\text{Td}(g) = \min \{ \text{Td}(f) \mid f \in G_T \setminus \{x_i^2 - 1 \mid i = 1, \ldots, n\} \}$$

we have $t = \text{Td}(g) - 1$. Moreover, in order to find such a $g$ it is not necessary to compute the whole Gröbner basis $G_T$ (see Theorem 5 in [3] or Remark 1 in Section 3).

The following propositions provides important properties of $T \{G_T\}$.

Proposition 1. (Relation between $\text{Td}(g)$ and weight($c_g$)) Let $g \in G_T$ satisfy weight($c_g$) = $d'$, and let $t' = [(d' - 1)/2] + 1$. Then $\text{Td}(g) = t'$ or $\text{Td}(g) = t' + 1$.

Proof. It is clear that if $t' = 1$ (which would imply that the code $\mathcal{C}$ has 0 error-correcting capability) the result is true. We may assume that $t' \geq 2$.

Obviously $\text{Td}(g) \geq t'$, otherwise weight($c_g$) $< d'$. Suppose that $\text{Td}(g) > t' + 1$, and let $T(g) = xw$, $\text{Can}(g, G_T) = v$ (where $x$ is any variable belonging to the support of $T(g)$). Observe that $\text{Td}(w) \geq t' + 1$ and $\text{Td}(v) \leq t' - 2$. As a consequence, $w > xv$ and thus $w \in T(G_T \setminus \{g\})$ (note that $w - xv \in I(\mathcal{C})$) which cannot happen because $G_T$ is a reduced Gröbner basis. This completes the proof.

Proposition 2. (Codewords of minimal weight) Let $c$ be a codeword of minimal weight $d$. If $d$ is odd then there exists $g \in G_T$ such that $c = c_g$ and $\text{Td}(g) = t + 1$. If $d$ is even then either there exists $g \in G_T$ such that $c = c_g$ and $\text{Td}(g) = t + 1$.
or there exist \( g_1, g_2 \in G_T \) such that \( c = c_{g_1} + c_{g_2} = \psi(w_1) + \psi(w_2) \), where \( g_1 = w_1 - v \), \( g_2 = w_2 - v \) \((w_1 = T(g_1), w_2 = T(g_2), v = \text{Can}(g_1, G_T) = \text{Can}(g_2, G_T))\), with \( t + 1 = Td(g_1) = Td(g_2) \).

**Proof.** Let \( f = w_c - v_c \), where \( Td(w_c) = t + 1 \), \( Td(v_c) = d - t - 1 \) and \( c_f = c \).

If \( d \) is odd, then, by Theorem 1, \( v_c \) is a canonical form \((\text{weight}(v_c) = t)\) and \( \text{Can}(w_c, G_T) = v_c \). By Theorem 2, there exists \( g_1 \in G \) that satisfies the conditions of the theorem. In this case, part (1) implies that \( Td(g_1) = t + 1 \). (By Proposition 1, there are no maximal terms of degree less than \( t + 1 \), apart from the monomials with support size 1). Consequently, \( T(g_1) = w_c \) and therefore, \( f = g_1 \).

If \( d \) is even then \( \text{weight}(v_c) = t + 1 \) and it is not necessarily a canonical form. If it is a canonical form then we are in the same case as before, that is, there exists \( g_1 \in G \) such that \( c = c_{g_1} \) and \( Td(g_1) = t + 1 \). If \( v_c \) is not a canonical form then there exist \( g_1, g_2 \in G \), such that \( Td(g_1) = Td(g_2) = t + 1 \), \( T(g_1) = w_c \), \( T(g_2) = v_c \), and \( \text{Can}(g_1, G_T) = \text{Can}(g_2, G_T) = v \). It is easy to check that these two binomials satisfy \( c = c_{g_1} + c_{g_2} \) \((c = \psi(w_c) + \psi(v_c) = c_{g_1} + c_{g_2} \text{ because the term } \psi(v) \text{ appears twice and therefore vanishes})\). □

Using the connection between cycles in graph and binary codes (see Section 2.3), the previous theorem enables us to obtain all the minimal cycles of a graph according to their lengths. We will use \( G_T \) to compute a minimal cycle basis (see [12]), that is, a basis of the set of cycles considered as vector space which has minimal length. First, we have the following result, whose proof is a straightforward application of Theorem 2.

**Proposition 3.** (Decomposition of a codeword) Any codeword \((\text{or cycle in the corresponding graph})\) can be decomposed as a sum of the form \( c = \sum_{i=1}^{l} c_{g_i} \), where \( g_i \in G_T \), \( \text{weight}(c_{g_i}) \leq \text{weight}(c) \), and

\[
Td(g_i) \leq \left\lceil \frac{\text{weight}(c) - 1}{2} \right\rceil + 1, \text{ for all } i = 1, \ldots, l.
\]

By Theorem 2, \( c \in C \) can be reduced in one step while the weight of \( c_{g_1} \) and \( c_{g_2} \) remains less than or equal to \( \text{weight}(c) \). It is sufficient to carry this out finitely many times because the reduction process must arrive at the canonical form 1 (the empty word) after finitely many steps \((c_{\text{emptyword}} = (0, \ldots, 0))\).

A minimal cycle basis can be obtained as a certain subset \( G' \) of \( G_T \). The computation of \( G_T \) guarantees steps similar to those in Horton’s Algorithm for computing a minimal cycle basis (see [13]). A greedy algorithm can be used to extract a cycle basis from the set \( \{c_g \mid g \in G_T \} \setminus \{0, \ldots, 0\} \), which turns out to be a minimal cycle basis. This is made explicit in the following theorem.

**Theorem 3 (Finding a minimal cycle basis).** Given the set \( C' = \{c_g \mid g \in G_T \} \setminus \{0, \ldots, 0\} \), where the elements of \( C' \) are ordered so that \( c_{g_1} < c_{g_2} \) when one of the following conditions holds:

1. \( Td(g_1) < Td(g_2) \).
Then the cycle basis obtained by applying a greedy algorithm to \( C' \) is a minimal cycle basis.

**Remark 2.** When \( G_T \) is computed it is close to being ordered according to \( \prec \). The only changes necessary are to reorder elements of the same maximal term degree, by considering first the weights of the corresponding codewords.

**Proof.** There are two things to show in order to prove the result.

1. The set \( C' \) contains a minimal cycle basis.

2. The ordering \( \prec \) used to order the set \( C' \) is weight compatible with the goal of obtaining a basis of minimal length.

If these conditions hold then it is clear that a minimal cycle basis will be obtained by applying a greedy algorithm to extract a basis from \( C' \). Since the set \( C' \) is a generating set of \( C \), it does contain a basis.

Proof of (1.): Let \( B = \{c_1, \ldots, c_l\} \) be a minimal cycle basis. By applying Proposition \( 3 \) we can decompose any \( c_i \) as

\[
c_i = \sum_{j=1}^{n_i} c_{g_{ij}}, \quad \text{where} \quad \text{weight}(c_{g_{ij}}) \leq \text{weight}(c_i) \quad \text{for all} \quad j = 1, \ldots, n_i.
\]

Let \( C(B) = \{c_{g_{ij}} \mid i = 1, \ldots, l; j = 1, \ldots, n_i\} \). Is clear that \( C(B) \) is a generating set of \( C \). Moreover, the basis \( B' \) obtained by applying a greedy algorithm to \( C(B) \) has length at most the length of \( B \). Thus, \( B' \) is a minimal cycle basis. Note that \( C(B) \subseteq C' \).

Proof of (2.): Let \( g_1, g_2 \in G_T \) satisfy \( d_1 = \text{weight}(c_{g_1}) < \text{weight}(c_{g_2}) = d_2 \). Let \( t_1 = [(d'_1 - 1)/2] + 1 \) and \( t_2 = [(d'_2 - 1)/2] + 1 \), so that \( t_1 \leq t_2 \). The only conflict between \( \prec \) and the weights occurs when \( \text{Td}(g_1) > \text{Td}(g_2) \) and this is possible only if \( \text{Td}(g_1) > t_1 \) (due to Proposition \( 1 \) and the inequality \( t_1 \leq t_2 \)). By Proposition \( 3 \) we can find a set \( \{c_{f_i} \mid i = 1, \ldots, l\} \) such that \( c_{g_1} = \sum_{i=1}^l c_{f_i} \), where \( f_i \in G_T \), weight(\( c_{f_i} \)) \leq d_1 \) and \( \text{Td}(f_i) \leq t_1 \), for all \( i = 1, \ldots, l \). This means that, in this case, \( c_{g_1} \) is already a linear combination of elements in \( C' \) that occur earlier according to \( \prec \). When \( \text{Td}(g_1) \leq \text{Td}(g_2) \) (and \( d_1 < d_2 \)) we have \( c_{g_1} \prec c_{g_2} \). This completes the proof. \( \Box \)

**Example 2.** Given a graph \((V, U)\) of five vertices \( V = \{1, 2, 3, 4, 5\} \) and six edges \( U = \{(1, 2), (1, 4), (1, 5), (2, 3), (3, 4), (4, 5)\} \), the corresponding vector space is \( \mathbb{F}_2^3 \) (the length of codewords is the number of edges). It is easy to form a check matrix \( H \) (whose columns are not, in general, linearly independent). Then \( c \in \mathbb{F}_2^3 \) is a cycle if and only if \( cH = 0 \). Each row of \( H \) corresponds to the representation of one of the edges such that there are exactly two ones in the positions corresponding to the vertices of the edge, so the matrix is as follows:  

\[
\begin{array}{cccc}
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{array}
\]
Application of Theorem 2
Let us consider the codeword (i.e. the cycle)
w_0 = x_1x_3x_4x_5x_6. It is easy to see that there are just three cycles, which are those of Example 1.
From this matrix one can compute a generator matrix \( G \) of that example is a generator matrix and we have already computed the G\( \ddot{\text{r}} \)ber basis \( G_T \) for this code.

Application of Proposition 2
In this case the minimum distance is \( d = 3 \), then all codewords (cycles) of minimal weight (minimal length) can be obtained as certain \( c_g \) where \( g \in G_T \). In this case there is just one, namely, \( w_c = x_2x_3x_6 \) \( (c = (0, 1, 1, 0, 0, 1)) \).

Application of Proposition 3
Let \( g_1 = x_3x_6 - x_2, g_2 = x_2x_4 - x_1x_5 \), we have that \( g_1, g_2 \in G_T, c = c_{g_1} + c_{g_2}, \) and all the conditions for weight(\( \cdot \)) and Td(\( \cdot \)) are satisfied.

Finding a minimal cycle basis
We observe that
\[
C' = \{c_{x_2x_3 - x_6}, c_{x_2x_6 - x_3}, c_{x_2x_6 - x_2}, c_{x_2x_4 - x_1x_5}, c_{x_2x_5 - x_1x_4}, c_{x_2x_5 - x_1x_2}, c_{x_1x_3x_4 - x_5x_6}, c_{x_1x_3x_5 - x_2x_4}, c_{x_2x_6x_4 - x_3x_5}, c_{x_1x_5x_6 - x_3x_4}\}
\]
where \( \prec \) has been used to reorder the binomials at the same level according to Td(\( \cdot \)). Applying a greedy algorithm to \( C' \) we first choose \( c_1 = (0, 1, 1, 0, 0, 1) \), the next two binomials correspond also to \( c_1 \), and then the second linearly independent vector, corresponding to \( g = x_2x_4 - x_1x_5 \), is \( c_2 = (1, 1, 0, 1, 1, 0) \). Since the dimension of the vector space is 2, we already have a basis which is a minimal cycle basis by Theorem 3.

5 Computation of the Gröbner basis.
In this section we present a linear algebraic procedure that allows us to compute the Gröbner basis associated with a code. The background to this technique can be found in [9, 10].

Given a set \( F = \{f_1, f_2, \ldots, f_r\} \) of polynomials in \( K[X] = K[x_1, \ldots, x_n] \) generating an ideal \( I \) let compute a basis for the syzygy module \( M \) in \( K[X]^{r+1} \)

\[
H = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}
\]

From this matrix one can compute a generator matrix \( G \), although for this example it is easy to see that there are just three cycles, which are those of Example 1.
of the generator set \( F' = \{-1, f_1, f_2, \ldots, f_r\} \). Each of the syzygies corresponds to a solution
\[
f = \sum_{i=1}^{r} b_i f_i \quad b_i \in K[X], i = 1, \ldots, r
\]
and thus points to an element \( f \) in the ideal \( I \) generated by \( F \).

The main idea is that the set
\[
\begin{align*}
f_1 &= (f_1, 1, 0, 0, \ldots, 0) \\
f_2 &= (f_2, 0, 1, 0, \ldots, 0) \\
\vdots \\
f_r &= (f_r, 0, 0, 0, \ldots, 1)
\end{align*}
\]
is a basis of the syzygy module \( M \), and moreover it is a Gröbner basis with respect to a position over term (POT) ordering \( <_w \) induced from an ordering \( < \) in \( K[X] \) and the weight vector \( w = (1, T_<(f_1), \ldots, T_<(f_r)) \). Also, the leading term of \( f_i \) is \( e_{i+1} \) with respect the ordering \( <_w \) where \( e_i \) denotes the unit vector of length \( r + 1 \) (see [1] for an introduction to Gröbner bases of modules).

Now we use the FGLM idea [8] and run through the terms of \( K[X]^{r+1} \) in the order determined by \( < \) and \( e_i < e_j \) if \( i < j \), using a term over position (TOP) ordering. At each step the canonical form of the term with respect to the original basis is 0 apart from the first component so the determination of the linear relations takes place in that component. This provides a convenient representation for the canonical form with respect to the initial Gröbner basis as a \( K \)-vector space, and any linear relation obtained as a consequence of reduction of the first component in \( K[X] \) will give a corresponding relation for the elements of the module.

**Example 3.** Let \( I = \langle x^2 + x + 1, xy + x + 1 \rangle \) in \( \mathbb{F}_2[x, y] \) and take \( < \) to be the deglex order with \( x < y \). Displaying only the first component we have

| \( \quad \) | 1 | \( x \) | \( y \) | \( x^2 \) | \( xy \) | \( y^2 \) | \( x^3 \) | \( x^2y \) | \( xy^2 \) | \( y^3 \) |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| \( (1, 0, 0) \) | 1 | \( y \) | \( x^2 \) | \( xy \) | \( y^2 \) | \( x^3 \) | \( x^2y \) | \( xy^2 \) | \( y^3 \) | \( y \) |
| \( (0, 1, 0) \) | \( 0 \) | \( 1 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) |
| \( (0, 0, 1) \) | \( 0 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) |

**after reduction**

| \( \quad \) | 1 | \( y \) | \( x^2 \) | \( xy \) | \( y^2 \) | \( x^3 \) | \( x^2y \) | \( xy^2 \) | \( y^3 \) | \( y \) |
| \( (1, 0, 0) \) | 1 | \( y \) | \( x^2 \) | \( xy \) | \( y^2 \) | \( x^3 \) | \( x^2y \) | \( xy^2 \) | \( y^3 \) | \( y \) |
| \( (1, 1, 0) \) | 0 | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) |
| \( (0, 1, 1) \) | 0 | \( 0 \) | \( 0 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) |

**introduce** \( x \)

| \( \quad \) | 1 | \( y \) | \( x^2 \) | \( xy \) | \( y^2 \) | \( x^3 \) | \( x^2y \) | \( xy^2 \) | \( y^3 \) | \( y \) |
| \( (x, 0, 0) \) | 1 | \( y \) | \( x^2 \) | \( xy \) | \( y^2 \) | \( x^3 \) | \( x^2y \) | \( xy^2 \) | \( y^3 \) | \( y \) |
| \( (x, x, 0) \) | 0 | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) |
| \( (0, x, x) \) | 0 | \( 0 \) | \( 0 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) |

**after reduction**

| \( \quad \) | 1 | \( y \) | \( x^2 \) | \( xy \) | \( y^2 \) | \( x^3 \) | \( x^2y \) | \( xy^2 \) | \( y^3 \) | \( y \) |
| \( (x + 1, 0, 1) \) | 1 | \( y \) | \( x^2 \) | \( xy \) | \( y^2 \) | \( x^3 \) | \( x^2y \) | \( xy^2 \) | \( y^3 \) | \( y \) |
| \( (1, x + 1, 0) \) | 0 | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) |
| \( (1, 1, x) \) | 0 | \( 0 \) | \( 0 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) |
Hence \((y + x, y + 1, x + 1)\) is a syzygy and therefore \(y + x \in I\) and it is the first element in deglex order; we can now omit all the multiples of \(y \cdot (1, 1, 0)\) from consideration. Continuing the computation we find

\[
\begin{array}{c|cccccccc}
\text{introduce } y & 1 & x & y & x^2 & xy & y^2 & x^2y & xy^2 & y^3 \\
(y, 0, 0) & & & 1 & & & 1 & & & \\
(y, y, 0) & & & & & & & 1 & 1 & \\
(0, y, y) & & & & & & & & & \\
\hline
\text{after reduction} & & & & & & & & & \\
(y, 0, 0) & & & & & & & & & 1 \\
(y + x, y + 1, x + 1) & & & & & & & & & \\
(1, y + 1, x + y) & & & & & & & & & \\
\end{array}
\]

Thus \((x^2 + x + 1, 1, 0)\) is a syzygy and \(x^2 + x + 1\) is the second basis element in \(I\) relative to deglex. We can omit all multiples of \(x^2(1, 0, 0)\). It follows that \(\{y + x, x^2 + x + 1\}\) is the required Gröbner basis.

Note that the above procedure is completely general and can be used for any base field. Although the general construction uses only straightforward linear algebra it has a major drawback in that to determine that a polynomial \(f\) belongs to the ideal (in which case \(f\) will be an element of the Gröbner basis), one must compute the minimal representation \(f = \sum h_i f_i\) where the \(f_i\) are the initial generators. It is known that the degrees of the \(f_i\) can be doubly exponential in \(n\), the number of variables. This is usually called the Nullstellensatz problem [6].

Remark 3. However, the particular properties of our setting allow us to use this algorithm for computing the Gröbner basis associated to a binary code:

1. Since the words in our initial generating set are of the form \(w_i - 1\), after the first reduction we always have only elements in \([X]\) as the representative elements for canonical forms (i.e. coordinate vectors in the vector space \(K[X]\)).

2. Because of \(\square\) above, in our case, a vector of \(K[X]^{r+1}\) introduced a row either reduces to zero or else it represents a new irreducible element. Therefore, an element does not reduce to one of lesser degree apart from
to degree zero (in which case we have obtained a new syzygy and a new element of the reduced basis).

3. We use a total degree compatible ordering $< \text{ on } [X]$ and the new ordering in the module is a TOP ordering, which looks first for the maximal terms in any position, and after that takes into account that $e_i < e_j$ if $i < j$.

4. From (4) and (6) above we find that the degrees of all components in the vectors are the same, which implies that the degrees of the cofactors (the $h_i$) are at most the degree of the new element $g$ of the basis. For this element $g$, the leading term $T(g)$ is in standard form (otherwise it would be a multiple of some $x_i^2$ which contradicts $g \in G \setminus \{x_i^2 - 1 \mid i = 1, \ldots, n\}$).

The maximal length of a standard form is $n$.

Remark 4. Note that since the terms are added in the ordering used for computing the Gröbner basis associated to the code then the first syzygy we find so that it corresponds to a binomial $g$ whose maximal term is in standard form, satisfies $t = Td(g) - 1$ (see Remark 7).

Example 4. Consider as a “toy example” the binary code $C$ with generator matrix

$$G = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}.$$

We find that

$$I(C) = \langle f_1 = x_1x_3 - 1, f_2 = x_2x_3 - 1, f_3 = x_1^2 - 1, f_4 = x_2^2 - 1, f_5 = x_3^2 - 1 \rangle.$$

In the associated syzygy computation the rows corresponding to the binomials $x_i^2 - 1$ are considered as implicit in the computations: see, for example, in the Table below when the syzygy corresponding to $x_3 - x_1$ is obtained.

| $-1$ | $x_1$ | $x_2$ | $x_3$ | $x_1x_2$ | $x_1x_3$ | $x_2x_3$ | $x_1x_2x_3$ | multiples of $x_1^2$ |
|------|-------|-------|-------|--------|--------|--------|------------|------------------|
| $(1,0,0,0,0,0)$ | 1 | | | | | | | |
| $(1,1,0,0,0,0)$ | | | | | | | | |
| $(1,0,1,0,0,0)$ | | | | | | | | |
| introduce $x_1$ | | | | | | $x_1^2x_3$ | | |
| $(x_1,0,0,0,0,0)$ | 1 | | | | | | | |
| $(x_1,x_1,0,0,0,0)$ | | | | | | | | |
| $(x_1,x_0,x_1,0,0,0)$ | | | | | | | | |
| introduce $x_2$ | | | | | | $x_2^2x_3$ | | |
| $(x_2,0,0,0,0,0)$ | 1 | | | | | | | |
| $(x_2,x_2,0,0,0,0)$ | | | | | | | | |
| $(x_2,x_2,0,0,0,0)$ | | | | | | | | |
| reduction | | | | | | $x_2^2x_3$ | | |
| $(x_2,0,0,0,0,0)$ | | | | | | | | |
| $(x_2 - x_1,x_2,x_1,0,0,0)$ | | | | | | | | |
| $(x_2,0,x_2,0,0,0)$ | | | | | | | | |
Thus \(x_2 - x_1 = x_1 f_2 - x_2 f_1\), \(x_2 - x_1\) belongs to the Gröbner basis and we can now omit all the multiples of \(x_2(1, 1, 0, 0, 0, 0)\) from our computation. Continuing we find

| introduce \(x_3\)                | 1 | \(x_1\) | \(x_2\) | \(x_3\) | \(x_1 x_2\) | \(x_1 x_3\) | \(x_2 x_3\) | \(x_1 x_2 x_3\) | multiples of \(x_1^2\) |
|---------------------------------|---|---------|---------|---------|------------|------------|------------|-----------------|----------------------|
| \((x_3, 0, 0, 0, 0, 0)\)         |   |         |         |         |            |            |            |                 |                      |
| \((x_3, x_3, 0, 0, 0, 0)\)       |   |         |         |         |            |            |            |                 |                      |
| \((x_3, 0, x_3, 0, 0, 0)\)       |   |         |         |         |            |            |            |                 |                      |
| reduction                       | 1 |         |         |         |            |            |            |                 |                      |
| \((x_3, 0, 0, 0, 0, 0)\)         |   |         |         |         |            |            |            |                 |                      |
| \((x_3 - x_1, x_3, 0, 0, 0, 0)\) |   |         |         |         |            |            |            |                 |                      |
| \((x_3 - x_2, 0, x_3, 0, 0, 0)\) |   |         |         |         |            |            |            |                 |                      |

We have the syzygy \(x_3 - x_1 = x_1 f_5 - x_3 f_1\) and \(x_3 - x_1\) belongs to the Gröbner basis (note that we can make reductions of terms \(T \cdot x_i^2\), \(T\) a term, as soon as we have introduced \(T\) since \(x_i^2 - 1\) is a generator). The result of the computation is the Gröbner basis \(\{x_2 - x_1, x_3 - x_1, x_1^2 - 1\}\)

**Remark 5.** In recording the computations we need only to keep the first components on the left and pointers to those places with a 1 in the rest of the table. This gives the following adapted FGLM basis conversion algorithm.

### 5.1 Adapted FGLM algorithm

The algorithm computes a Gröbner basis for the syzygy module, but we are interested only in the first component which is the Gröbner basis \(G_T\) for \(I(C)\).

For theoretical reasons we will denote by \(G(M)\) the set which is constructed by the algorithm, which on termination is a Gröbner basis for the module, but we will just compute the first component \(G\) of this set.

In the algorithm we use two main structures. One is \(List\), which has the form \((v_1, v_2)\), where \(v_1\) represents the first component of the corresponding vector in the module, and \(v_2\) is the representative element in \(K[X]\) (in our case an element of \([X]\) – see \(\Box\) of Remark \(\Box\)). If \(w = (v_1, v_2) \in List\) then we write \(w[1] = v_1\) and \(w[2] = v_2\). The second structure is the list \(N\) that stores the first components of the elements of \(List\) that are canonical forms. The third structure is the list \(V\) whose \(r\)-th element \(v_r\) is the representative element in \(K[X]\) of the \(r\)-th element of \(N\) (the second component of the pairs in \(List\)).

Subroutines of the algorithm:

- **InsertNexts**(*w, List*) inserts the products \(wx\) (for \(x \in X\)) in \(List\) and sorts it by increasing order with respect to \(<\), with account being taken first of the first component, and, in case these are equal, then by comparison of the second components. The reader should note that **InsertNexts** could count the number of times that an element \(w\) is inserted in \(List\), so \(w[1] \in N_<(I) \cup T_<(G)\) if and only if this coincides with the number of variables in the support of \(w[1]\) (if not, this would means that
$w[1] \in T_<(I) \setminus T_<(G)$, see [8]). This criterion can be used to determine
the boolean value of the test condition in Step 4 of the Algorithm II

- **NextTerm(List)** removes the first element from List and returns it.

- **Member($v, [v_1, \ldots, v_r]$)** returns $j$ if $v = v_j$ or false otherwise.

**Algorithm 1.**

**Input** $F = \{w_1 - 1, w_2 - 1, \ldots, w_r - 1\}$ the set of binomials associated with a generating set of a binary code

$<_T$ a total degree compatible ordering

**Output** The reduced Gröbner basis $G_T$ of the ideal

$\langle F \cup \{x_i^2 - 1 \mid i = 1, \ldots, n\} \rangle$ w.r.t. $<_T$

1. List := [(1,1), (1,w_i)_{i=1,\ldots,n}, (1,x_i^2)_{i=1,\ldots,n}] (the elements should be ordered following $<_T$ in the second component of the pairs),

   $G_T := \{ \}, N := [\]$.

2. While List $\neq \emptyset$ do

3. $w :=$ NextTerm(List);

4. If $w \notin T(G(M))$;

5. $v' := w[2]$;

6. $j :=$ Member($v', [v_1, \ldots, v_r]$);

7. If $j \neq$ false then $G := G \cup \{w[1] - w_j\}$;

8. else $r := r + 1$;

9. $v_r := v'$;

10. $w_r := w[1]$, $N := N \cup \{w_r\}$;

11. List := InsertNexts($w_r, List$);

12. Return[G]

Note that this algorithm for computing the Gröbner basis $G_T$ associated to the code $C$ is especially well suited in our setting since all the elements in the basis (respectively codewords, cycles) appear in an increasing term ordering (respectively increasing ordering on the weight or the length) during the computation. Moreover, the computation can be stopped when a desired weight of the codewords (respectively length of the cycles) is obtained which is usefull for finding many combinatorial properties of the code (respectively the graph) such that the minimal distance (see Remarks 1, 4) or finding the minimal codewords (see Proposition 2).
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