Simple Euclidean arrangements with one \((\geq 5)\)-gon

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August 24, 2010

Abstract. Let \(\mathcal{L}\) be a simple Euclidean arrangement of \(n\) pseudolines. It is shown that if \(\mathcal{L}\) has exactly one \((\geq 5)\)-gon \(P\), and \(k\) is the number of edges of \(P\) that are adjacent to an unbounded cell of the subarrangement of \(\mathcal{L}\) induced by the pseudolines in \(P\), then \(\mathcal{L}\) has exactly \(n-k\) triangles and \(k+n(n-5)/2\) quadrilaterals. We also prove that if each pseudoline of \(\mathcal{L}\) is adjacent to \(P\) then \(\mathcal{L}\) is stretchable.

1 Introduction

Recently, Leaños et al. \(^8\) proved that if a simple Euclidean arrangement of \(n\) pseudolines has no \((\geq 5)\)-gons, then it is stretchable and has exactly \(n-2\) triangles and \((n-2)(n-3)/2\) quadrilaterals. Our goal in this paper is to study the same problems for the case when all the bounded cells of the arrangement under consideration are, except one, either triangles or quadrilaterals.

We recall that a simple noncontractible closed curve in the projective plane \(\mathbb{P}\) is a pseudoline, and an arrangement of pseudolines is a collection \(\mathcal{B} = \{p_0, p_1, \ldots, p_n\}\) of pseudolines that intersect (necessarily cross) pairwise exactly once. Since \(\mathbb{P} \setminus p_0\) is homeomorphic to the Euclidean plane \(\mathbb{E}\), we may regard \(\{p_1, \ldots, p_n\}\) as an arrangement of pseudolines in \(\mathbb{E}\) (and regard \(p_1, \ldots, p_n\) as pseudolines in \(\mathbb{E}\)). An arrangement is simple if no point belongs to more than two pseudolines. The cell complex of an Euclidean arrangement in \(\mathbb{P}\) has both bounded and unbounded cells. As in \(^4\) and \(^8\), we are only interested in bounded cells (whose interiors are the polygons or faces). Thus it is clear what is meant by a triangle, a quadrilateral, or, in general, an \(n\)-gon of the arrangement. In this work we are interested in arrangements in which every bounded cell is, except one, either a triangle or a quadrilateral; for this reason we say that such an arrangement has one \((\geq 5)\)-gon.

One of the most interesting and widely studied problems concerning arrangements of lines and pseudolines is the determination of upper and lower bounds for the number \(p_k\) of \(k\)-gons. An extensive amount of research in such problems followed Grünbaum’s seminal work.

Theorem 1. Let \(\mathcal{L}\) be a simple Euclidean arrangement of \(n\) pseudolines with exactly one \((\geq 5)\)-gon \(P\), and let \(k\) be the number of edges of \(P\) that are adjacent to an unbounded cell of the subarrangement of \(\mathcal{L}\) induced by the pseudolines in \(P\). Then \(\mathcal{L}\) has exactly \(n-k\) triangles.

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\(^{3}\)Rivera–Martínez was supported by PROMEP (SEP), grant UAZ-PTC-103
As an immediate consequence of Theorem 1 and the fact that the number of bounded cells in a simple Euclidean arrangement of \( n \) pseudolines is \( 1 + n(n - 3)/2 \), we have the following corollary.

**Corollary 2.** Let \( \mathcal{L} \) be a simple Euclidean arrangement of \( n \) pseudolines with exactly one \((\geq 5)\)-gon \( P \), and let \( k \) be the number of edges of \( P \) that are adjacent to an unbounded cell of the subarrangement of \( \mathcal{L} \) induced by the pseudolines in \( P \). Then \( \mathcal{L} \) has exactly \( n - k \) triangles, \( k + n(n - 5)/2 \) quadrilaterals, and one \((\geq 5)\)-gon.

**Proof.** Let \( p_3 \) and \( p_4 \) be the number of triangles and quadrilaterals of \( \mathcal{L} \), respectively. Since the number of bounded cells in a simple Euclidean arrangement of \( n \) pseudolines is \( 1 + n(n - 3)/2 \) and all the bounded cells of \( \mathcal{L} \) are, except one, either triangles or quadrilaterals, then \( p_3 + p_4 = n(n - 3)/2 \). By Theorem 1, \( p_3 = n - k \). This implies \( p_4 = k + n(n - 5)/2 \), as desired. \( \square \)

In this work, we denote by \( \mathfrak{S} \) the set of simple Euclidean arrangements of pseudolines with one \((\geq 5)\)-gon in which every pseudoline is adjacent to the (unique) \((\geq 5)\)-gon. Every arrangement shown in Figure 1 is an element of \( \mathfrak{S} \).

![Figure 1: Three arrangements of \( \mathfrak{S} \).](image)

Our first aim in this work is to prove Theorem 1 for the case where the arrangement in question belongs to \( \mathfrak{S} \) (namely, Theorem 8). This is the content of Section 2. In Section 3 we show Theorem 1 by induction on the number of pseudolines of \( \mathcal{L} \) that are not in \( P \) (Theorem 6 is the base case of the induction in the proof of Theorem 1).

Although the main aim of all the lemmas and propositions in Sections 2 and 3 is to prove Theorem 1, some of them are interesting in themselves. This is the case of Proposition 9, which says that every pseudoline in a simple Euclidean arrangement is adjacent to at least one triangle.

Propositions 6 and 7 (in Section 2) have applications not only in the proof of Theorem 1 but also in the proof of our main result on stretchability (Theorem 3). Together, these propositions reveal a structural property of the arrangements of \( \mathfrak{S} \): if \( Q \) is the \((\geq 5)\)-gon of \( M \in \mathfrak{S} \) and \( T \) is a triangle of \( M \), then \( Q \) and \( T \) have a common edge; see the examples in Figure 1. In the proof of Theorem 3 (Section 4), we use this structural property to guarantee that every arrangement in \( \mathfrak{S} \) has 3 pseudolines with certain properties. The properties of such 3 pseudolines play a crucial role in the proof of Theorem 3.
An arrangement of lines in \( \mathbb{E} \) is a collection of straight lines, no two of them parallel. Thus, every arrangement of lines is an arrangement of pseudolines. On the other hand, not every arrangement of pseudolines is stretchable, that is, equivalent to an arrangement of lines, where two arrangements are equivalent if they generate isomorphic cell decompositions of \( \mathbb{E} \). Every arrangement of eight pseudolines is stretchable \([6]\), but there is a simple non-stretchable arrangement of nine pseudolines \([11]\) (unique up to isomorphism; see \([6]\)). Stretchability questions are typically difficult: deciding stretchability is NP-hard \([15]\), even for simple arrangements \([2]\). The concept of stretchability is particularly relevant because of the close connection between arrangements of pseudolines and rank 3 oriented matroids: on this ground, the problem of stretchability of arrangements is equivalent to the problem of realizability for oriented matroids (see \([1, 10]\)).

The following theorem is the main result of this work with respect to stretchability and establishes that every arrangement of \( \mathfrak{I} \) is stretchable.

**Theorem 3.** If \( \mathcal{L} \) is an arrangement of \( \mathfrak{I} \), then \( \mathcal{L} \) is stretchable.

The proof of Theorem 3 will be given in Section 4. Finally, in Section 5 we show that there are exponentially many non-isomorphic arrangements of \( \mathfrak{I} \).

### 1.1 Notation

An \( n \)-gonal region \( \mathcal{Q} \) of an Euclidean arrangement of pseudolines \( \mathcal{L} \) is any \( n \)-gon of a subarrangement of \( \mathcal{L} \) (thus, every \( n \)-gon of \( \mathcal{L} \) is an \( n \)-gonal region of \( \mathcal{L} \) but not conversely).

Let \( \mathcal{Q} \) and \( p \), respectively, an \( n \)-gonal region and a pseudoline of a simple Euclidean arrangement of pseudolines \( \mathcal{L} \). If \( p \) is adjacent to \( \mathcal{Q} \), then we say that \( p \) is in \( \mathcal{Q} \) or that \( \mathcal{Q} \) is in \( p \). If \( p \) is in \( \mathcal{Q} \), then we denote the edge of \( \mathcal{Q} \) in \( p \) as \( e_{p, \mathcal{Q}} \). If \( \mathcal{Q} \) is formed by \( p_1, \ldots, p_n \), then the subarrangement of \( \mathcal{L} \) whose pseudolines are \( p_1, \ldots, p_n \) is the subarrangement of \( \mathcal{L} \) induced by \( \mathcal{Q} \). See Figure 2.

If \( p \) and \( q \) are pseudolines in an Euclidean arrangement of pseudolines, then the crossing between \( p \) and \( q \) will be denoted by \( v_{p,q} \).

In the interest of clarity, and without loss of generality, we assume frequently in our arguments that some pseudolines of Euclidean arrangement under consideration are directed. If \( p \) is a directed pseudoline of an Euclidean arrangement, we denote by \( p^+ \) (respectively, \( p^- \)) the semiplane to the right of \( p \) (respectively, to the left). Note that if \( p \) is a directed pseudoline, \( \{p, p^+, p^-\} \) is a partition of \( \mathbb{E} \).

Let \( \mathcal{Q} \) be a polygon of an Euclidean arrangement of pseudolines \( \mathcal{L} \) and let \( e \) be an edge of \( \mathcal{Q} \). We say that \( e \) is a critical edge of \( \mathcal{Q} \) if \( e \) is adjacent to an unbounded cell of \( \mathcal{L} \). If \( \mathcal{L} \) has only one \( (\geq 5) \)-gon \( P \), and \( k \) is the number of edges of \( P \) that are adjacent to an unbounded cell of the subarrangement of \( \mathcal{L} \) induced by the pseudolines in \( P \), then we say that \( \mathcal{L} \) is \( k \)-critical. Note that the criticality of \( \mathcal{L} \) is defined by the criticality of the subarrangement of \( \mathcal{L} \) induced by its \( (\geq 5) \)-gon. See Figure 2.
Figure 2: Let \( L := \{a, b, c, d, e, f\} \) be the arrangement in (i) and let \( L' \) be the arrangement in (ii). Note that \( P \) is the only \((\geq 5)\)-gon of \( L \), and that \( L' \) is the subarrangement of \( L \) induced by the pseudolines in \( P \). Since \( e_a, P \) and \( e_f, P \) are the only edges of \( P \) that are adjacent to an unbounded cell of \( L' \), then \( L' \) is 2–critical, and so \( L \) is 2–critical (remember, the criticality of \( L' \) defines the criticality of both \( L' \) and \( L \)).

2 Triangles in arrangements of \( \Im \): proof of Theorem 8

Before proceeding with the proof of Theorem 8 we need to establish some results.

Lemma 4. Let \( L \) be a simple Euclidean arrangement of pseudolines. Let \( p, q \) and \( r \) be distinct pseudolines of \( L \) and let \( T \) be the triangular region formed by \( p, q \) and \( r \). If none of the pseudolines of \( L \) crosses \( e_{r, T} \), then \( \ell \in \{p, q\} \) is in a triangle of \( L \) and such a triangle is contained in \( T \).

Proof. We prove the theorem with \( \ell = p \) (the case \( \ell = q \) is analogous). For brevity of notation, let \( r' := e_{r, T}, p' := e_{p, T} \) and \( q' := e_{q, T} \). We proceed by induction on the number of pseudolines of \( L \). As \( p, q \) and \( r \) are distinct pseudolines of \( L \), it follows that \(|L| \geq 3\).

It is readily checked that the statement holds for the unique (up to isomorphism) simple Euclidean arrangement with 3 pseudolines (in this case, \( T \) is the required triangle). Thus for some integer \( n \geq 3 \) we assume (a) the statement holds for every simple Euclidean arrangement of \( k \leq n \) pseudolines, and (b) that \(|L| = n + 1\).

Case 1. There exists a pseudoline \( s \in L \setminus \{p, q\} \) such that \( s \) does not cross \( T \). Since \( s \notin \{p, q, r\} \), then \( p, q \) and \( r \) are in \( L' := L \setminus \{s\} \). As \( L' \) is a subarrangement of \( L \), it follows that \( L' \) is simple and none of the pseudolines of \( L' \) crosses \( r' \). Thus, by the inductive hypothesis, there exists a triangle of \( L' \), say \( T' \), such that \( p \) is in \( T' \) and \( T' \) is contained in \( T \). As \( s \) does not cross \( T \) and \( T' \subseteq T \), then \( T' \) is a triangle of \( L \). Moreover, \( T' \) is the required triangle.

Case 2. Every pseudoline of \( L \setminus \{p, q, r\} \) crosses \( T \). Since none of the pseudolines of \( L \) crosses \( r' \), then every pseudoline of \( L \setminus \{p, q, r\} \) crosses \( T \) through \( p' \) and \( q' \). Note that \(|L| \geq 4 \) implies \( L \setminus \{p, q, r\} \) \( \neq \emptyset \). Let us assume (without loss of generality) that \( p \) is directed from \( v_{p, q} \) to \( v_{p, r} \). Now, we label the pseudolines of \( L \setminus \{p, q, r\} \) with \( p_1, \ldots, p_{n-2} \) according to the order in which they are crossed by \( p \). This labeling is well defined because \( L \) is simple.

It follows from \( p_1 \notin \{p, q, r\} \) that \( p, q \) and \( r \) are in \( L^* := L \setminus \{p_1\} \). Since \( L^* \) is a
subarrangement of \( \mathcal{L} \), then \( \mathcal{L}^* \) is simple and none of the pseudolines of \( \mathcal{L}^* \) crosses \( r' \). Thus, by inductive hypothesis, there exists a triangle of \( \mathcal{L}^* \), say \( T^* \), such that \( p \) is in \( T^* \) and \( T^* \) is contained in \( T \).

Note that if \( \mathcal{L} = \{p, q, r, p_1\} \), then \( T^* = T \) and the triangle formed by \( p, q, \) and \( p_1 \) is the required polygon. We may therefore assume that \( |\mathcal{L}| \geq 5 \). We also observe that if \( p_1 \) does not cross \( T^* \), then \( T^* \) is the required triangle of \( \mathcal{L} \). So we assume that \( p_1 \) crosses \( T^* \).

**Subcase 2.1.** \( q \) is in \( T^* \). As \( p, q \) are in \( T^* \), and \( p_1 \notin \mathcal{L}^* \) and \( \mathcal{L}^* \) is simple, then \( p_2 \) is the third pseudoline in \( T^* \). Now, observe that if \( v_{p_1, p_2} \) is in \( T \), then the triangle formed by \( p, p_1 \) and \( p_2 \) is the required polygon, and in the other case, the triangle formed by \( p, q, \) and \( p_1 \) is the required polygon.

**Subcase 2.2.** \( q \) is not in \( T^* \). Since none of the pseudolines of \( \mathcal{L} \setminus \{p, q, r\} \) crosses \( r' \), then \( r \) is not in \( T^* \). Thus, there are distinct integers \( i, j \in \{2, \ldots, n-2\} \) such that \( T^* \) is formed by \( p, p_i \) and \( p_j \). Without loss of generality, we assume that \( i < j \). Since \( T^* \) is a triangle of \( \mathcal{L}^* \) and \( p_1 \) crosses \( T^* \) and \( \mathcal{L}^* \) is simple, then when we add \( p_1 \) to \( \mathcal{L}^* \) to obtain \( \mathcal{L} \), a triangle and a quadrilateral are generated. Both are contained in \( T^* \) and are polygons of \( \mathcal{L} \). Let \( Q \) be such a quadrilateral. As \( v_{p, p_1} \) is not in the closed arc of \( p \) with endpoints \( v_{p, p_i} \), and \( v_{p, p_j} \), it follows that \( p_1 \) crosses \( T^* \) through \( e_{p_i, T^*} \) and \( e_{p_j, T^*} \). Thus \( Q \) is formed by \( p, p_1, p_i \) and \( p_1 \). Let \( p'_i := e_{p_i, Q} \), and let \( \mathcal{L}_i \) be the subarrangement of \( \mathcal{L} \) with pseudolines \( p, p_1, \ldots, p_i \) and let \( T^i \) be the triangular region formed by \( p, p_1 \) and \( p_i \). Observe that \( T^i \) is contained in \( T \). Because \( Q \) is a polygon of \( \mathcal{L} \), none of the pseudolines of \( \mathcal{L} \) crosses \( p'_i \). In particular none of the pseudolines of \( \mathcal{L}_i \) crosses \( p'_i \). By the induction hypothesis, \( \mathcal{L}_i \) has a triangle \( T^{**} \), such that \( p \) is in \( T^{**} \) and \( T^{**} \) is contained in \( T^i \). Since none of the pseudolines of \( \mathcal{L} \) crosses \( p'_i \), and \( p_2, p_3, \ldots, p_{i-1} \) are all the pseudolines of \( \mathcal{L} \) that cross \( p \) between \( v_{p, p_i} \) and \( v_{p, p_1} \), it follows that none of the pseudolines of \( \mathcal{L} \setminus \{p, p_1, p_2, \ldots, p_i\} \) crosses \( T^i \). Thus, \( T^{**} \) is a triangle of \( \mathcal{L} \) such that \( p \) is in \( T^{**} \) and \( T^{**} \) is contained in \( T^i \) (and hence in \( T \)), as desired. □

**Observation 5.** If \( \mathcal{L} \) is a simple Euclidean arrangement of pseudolines and \( P \) is a (≥ 4)–gon of \( \mathcal{L} \), then \( P \) has at most 2 critical edges.

**Proof.** We prove the observation by contradiction. Suppose that \( P \) has at least 3 critical edges. Let \( \mathcal{L}' \) be a subarrangement of \( \mathcal{L} \) induced by 4 of the pseudolines in \( P \), of which at least 3 contain critical edges of \( P \). Figure 3 shows \( \mathcal{L}' \), the unique (up to isomorphism) simple Euclidean arrangement with 4 pseudolines.

![Figure 3](image)

Figure 3: The unique (up to isomorphism) simple Euclidean arrangement with 4 pseudolines.
Since $\mathcal{L}'$ is a subarrangement of $\mathcal{L}$, then $P$ is contained in one of the eleven regions defined by $\mathcal{L}'$ (see Figure 3). Since every pseudoline of $\mathcal{L}'$ is in $P$, and $A, B$ are the only regions of $\mathcal{L}'$ that are adjacent to every pseudoline of $\mathcal{L}'$, then $P$ must be contained in one of $A$ or $B$. Observe that in both cases, neither $p$ nor $q$ can contain a critical edge of $P$. This contradicts our assumption that $\mathcal{L}'$ has at least 3 critical edges of $P$. □

**Proposition 6.** If $\mathcal{L} \in \mathcal{S}$ then every non-critical edge of the $(\geq 5)$–gon of $\mathcal{L}$ is also an edge of a triangle of $\mathcal{L}$.

**Proof.** Let $P$ be the $(\geq 5)$–gon of $\mathcal{L}$. Let $e'_1$ be any non-critical edge of $P$ and let $e'_2, e'_3$ be the edges of $P$ that are adjacent to $e'_1$. For $i = 1, 2, 3$ let $e_i$ be the pseudoline of $\mathcal{L}$ that contains $e'_i$. Without loss of generality we may assume that each $e_i$ is directed in such a way that $P$ lies to the right of $e_i$.

Since $e'_1$ is a non-critical edge of $P$, then the region to the left of $e'_1$ is a polygon of $\mathcal{L}$. We denote by $Q$ such a polygon. $Q$ must be a triangle or a quadrilateral, because $P$ is the only $(\geq 5)$–gon of $\mathcal{L}$. Assume, for the purpose of contradiction, that $Q$ is a quadrilateral.

Before proceeding, note that $e_j$ ($j = 2, 3$) contains an edge of $Q$ because $\mathcal{L}$ is simple. Thus, as $Q$ is quadrilateral, $\mathcal{L} \setminus \{e_1, e_2, e_3\}$ contains a pseudoline $x$, such that $Q$ is formed by $x, e_1, e_2$ and $e_3$. Let $x' := e_{x,Q}$ and let $x_2$ and $x_3$ be the two unbounded subarcs of $x$ obtained by deleting $x'$. Without loss of generality, we assume that $x_j$ ($j = 2, 3$) is the unbounded subarc of $x$ that has endpoint at $v_{x,e_j}$. Since $v_{x,e}$ is the unique intersection point between $x$ and $e_j$, then $x_j$ is contained in $e_j^-$. Thus, $x_j$ is not in $P$. On the other hand, is clear that $P$ cannot be adjacent to $x'$, and hence $P$ is not in $x$, which contradicts the fact that every pseudoline of $\mathcal{L}$ is in $P$. □

**Proposition 7.** If $\mathcal{L} \in \mathcal{S}$ then every triangle of $\mathcal{L}$ has a common edge with the $(\geq 5)$–gon of $\mathcal{L}$.

**Proof.** Let $T$ be any triangle of $\mathcal{L}$ and let $p, q$ and $r$ be the pseudolines in $T$. We denote by $P$ the $(\geq 5)$–gon of $\mathcal{L}$. Let $\mathcal{L}'$ be the subarrangement of $\mathcal{L}$ induced by $T$. Figure 4 shows $\mathcal{L}'$, the unique (up to isomorphism) simple Euclidean arrangement with 3 pseudolines.

![Figure 4: The unique (up to isomorphism) simple Euclidean arrangement with 3 pseudolines.](image)

Since $\mathcal{L}'$ is a subarrangement of $\mathcal{L}$, then $P$ must be contained in one of the six unbounded regions defined by $\mathcal{L}'$ (see Figure 4). By symmetry, we may assume that $P$ is contained in $A$ or $B$. Since $\mathcal{L} \in \mathcal{S}$, $p$ is in $P$ and hence $P$ cannot be contained in $B$. So $P \subset A$. Let $r' := e_{r,T}$. Because $P \subset A$ and $r$ is in $P$, there exists a subarc $r'' \subseteq r'$ that is an edge of $P$. On the other hand, none of the pseudolines of $\mathcal{L}$ crosses $r'$ because $T$ is a polygon of $\mathcal{L}$. Thus, $r'' = r'$, as desired. □
**Theorem 8.** If $L \in \mathcal{L}$ and $L$ is $k$–critical, then the number of triangles of $L$ is $|L| - k$.

**Proof.** Let $P$ be the $(\geq 5)$–gon of $L$ and $n := |L|$. Since the pseudolines of $L$ intersect pairwise exactly once, it follows that distinct edges of $P$ come from distinct pseudolines of $L$. This implies that $P$ has at most $n$ edges. On the other hand, since every pseudoline of $L$ is in $P$, then $P$ has at least $n$ edges. Thus, $P$ is an $n$–gon and the assertion follows from Propositions 6 and 7. □

### 3 Triangles in simple Euclidean arrangements with one $(\geq 5)$–gon: proof of Theorem 1

We begin the proof of Theorem 1 with the Proposition 9.

**Proposition 9.** Let $L$ be a simple Euclidean arrangement of $n \geq 3$ pseudolines. Then every pseudoline of $L$ is adjacent to at least one triangle.

**Proof.** We apply induction on $|L|$. The assertion is trivial for the unique (up to isomorphism) simple Euclidean arrangement with 3 pseudolines. Thus for some integer $n \geq 3$ we assume (a) the statement holds for every simple Euclidean arrangement of $k \leq n$ pseudolines, and (b) that $|L| = n + 1$.

Let $t$ be any pseudoline of $L$ and let $s$ be a pseudoline of $L \setminus \{t\}$. By the inductive hypothesis, $L' := L \setminus \{s\}$ has a triangle $T$, such that $t$ is in $T$. If $s$ does not cross $T$, then $T$ is a triangle of $L$ and $T$ is the required triangle. So we assume that $s$ crosses $T$. Let $x, y \in L'$ be the other two pseudolines in $T$. Observe that if $s$ crosses $T$ through $t$ and $p \in \{x, y\}$, then the triangle formed by $t, s$ and $p$ is the required triangle. We may therefore assume that $s$ crosses $T$ through $x$ and $y$. This implies that none of the pseudolines of $L$ crosses the subarc of $x$ with endpoints $v_{x,t}$ and $v_{x,s}$. Hence, by Lemma 4, there exists a triangle of $L$, say $T_t$, such that 1) $T_t$ is contained in the triangular region formed by $x, s$ and $t$, and 2) $t$ is in $T_t$, as desired. □

The construction in Figure 5 shows that Proposition 9 is best possible.

![Figure 5: Pseudoline $p$ is adjacent to exactly one triangle.](image)

**Lemma 10.** Let $L$ be a simple Euclidean arrangement of pseudolines and let $M$ be a subarrangement of $L$. If $Q$ is a $(\geq 5)$–gon of $M$ and $w$ is an edge of $Q$ such that none of the pseudolines of $L$ crosses $w$, then every edge of $Q$ that is adjacent to $w$ contains a subarc that is an edge of a $(\geq 5)$–gon of $L$. Moreover, $Q$ contains such a $(\geq 5)$–gon.
Proof. Let \( q \) be any of the two edges of \( Q \) that are adjacent to \( w \). Let us assume (without loss of generality) that \( q \) is directed in such a way that \( Q \) lies to the right of \( q \). Let \( p \neq w \) (respectively \( r \neq q \)) the edge of \( Q \) that is adjacent to \( q \) (respectively \( w \)). Since \( Q \) is a \((\geq 5)\)-gon, \( p \) is not adjacent to \( r \).

Let \( C \) be the set of all pseudolines of \( L \) that cross \( q \), and let \( C' \) be the set of all pseudolines of \( L \setminus (M \cup C) \). Let \( M' \) be the subarrangement of \( M \) obtained by deleting the pseudolines in \( C \). Let \( Q' \) be the polygon of \( M' \) that is adjacent to \( q \) and lies to the right of \( q \). Clearly, \( Q' \) is contained in \( Q \). Note that both \( q \) and \( w \) are edges of \( Q' \). This implies that \( v_{q,r} \) and \( v_{w,r} \) are vertices of \( Q' \). Since \( L \) is simple and \( q \) (respectively \( w \)) is an edge of \( Q' \), then there exists a subarc \( p' \subseteq p \) (respectively \( r' \subseteq r \)) with an endpoint at \( v_{q,r} \) (respectively \( v_{w,r} \)) such that \( p' \) (respectively \( r' \)) is an edge of \( Q' \). Since \( p \) is not adjacent to \( r \), \( p' \) is not adjacent to \( r' \). We denote by \( v_{r'q} \) (respectively \( v_{qrp} \)) the vertex of \( Q' \) that is an endpoint of \( r' \) (respectively \( p' \)) but not an endpoint of \( w \) (respectively \( q \)).

Since the edges of \( Q' \) form a cycle, then there is a path \( \Gamma \), formed by edges of \( Q' \) with endpoints \( v_{p'} \) and \( v_{r'} \), such that \( \Gamma \cap \{w,q\} = \emptyset \). As \( p' \) and \( r' \) are not adjacent, then \( \Gamma \) contains at least one edge, and hence \( Q' \) is a \((\geq 5)\)-gon.

If |\( C \)| = 0 then \( Q' \) is a \((\geq 5)\)-gon of \( L \), and \( q \) and \( Q' \) are as desired. So we assume that |\( C \)| = \( n > 0 \).

We label the pseudolines of \( L \) with \( q_1, \ldots, q_n \) according to the order in which they are found when \( q \) is walked from \( v_{q,w} \) to \( v_{q,p} \). This labeling is well defined because \( L \) is simple.

For \( i = 1, \ldots, n - 1 \) let \( \alpha_i \) be the subarc of \( q \) with endpoints \( v_{q,q_i} \) and \( v_{q,q_{i+1}} \). Let \( \alpha_0 \) be the subarc of \( q \) with endpoints \( v_{q,w} \) and \( v_{q,q_1} \), and let \( \alpha_n \) be the subarc of \( q \) with endpoints \( v_{q,q_n} \) and \( v_{q,p} \). For \( j = 0, \ldots, n \) (without loss of generality we assume that \( \alpha_i \) has the same direction as \( q \)) let \( Q'_j \) be the polygon to the right of \( \alpha_j \) (\( \alpha_j \) is an edge of \( Q'_j \)). Clearly, \( Q'_j \) is contained in \( Q' \). From the definition of \( Q'_j \) it is not difficult to see that \( Q'_j \) is a polygon of \( L \). Thus, if \( Q'_j \) is a \((\geq 5)\)-gon for some \( j \in \{0, \ldots, n-1\} \) we are done. So assume that every element of \( \{Q'_0, \ldots, Q'_{n-1}\} \) is a \((\leq 4)\)-gon.

On the other hand, observe that for \( i = 1, \ldots, n-1 \), \( q_i \) and \( q_{i+1} \) are in \( Q'_j \). Moreover, note that \( e_{q_i, q_{i+1}} = e_{q_i, q'_j} \). For brevity of notation let \( q_i' := e_{q_i, Q'_{i+1}} = e_{q_i, Q'_i} \).

Since none of the pseudolines of \( L \) crosses \( w \) and \( L \) is simple, then \( w, \alpha_0, q'_1 \) and a subarc \( r_0 \) of \( r' \), which has a common endpoint with \( w \), belong to the edge set of \( Q'_0 \). Thus, \( Q'_0 \) is a quadrilateral with edge set \( \{w, \alpha_0, q'_1, r_0\} \). Since \( L \) is simple, \( r_0 \) is a proper subarc of \( r' \) and consequently \( q_1 \) must cross \( r' \). Let \( v_1 \) be the crossing between \( r' \) and \( q_1 \) (\( v_1 \) is common endpoint of both \( r_0 \) and \( q'_1 \)). On the other hand, since \( Q'_0 \) is a quadrilateral of \( L \), then none of the pseudolines of \( L \) crosses \( q'_1 \).

The argument in the next paragraph works for \( i = 1, \ldots, n-1 \) if we apply it repeatedly in ascending order.

As \( L \) is simple and none of the pseudolines of \( L \) crosses \( q'_i \), then \( q'_i, \alpha_i, q'_{i+1} \) and a subarc \( r_i \) of \( r' \setminus (r_0 \cup \ldots \cup r_{i-1}) \), which has an endpoint at \( v_i \), belong to the edge set of \( Q'_{i+1} \). Thus, \( Q'_{i+1} \) is a quadrilateral with edge set \( \{q'_i, \alpha_i, q'_{i+1}, r_i\} \). As \( L \) is simple, then \( r_i \) is a proper subarc of \( r' \setminus (r_0 \cup \ldots \cup r_{i-1}) \) and consequently \( q_{i+1} \) must cross \( r' \setminus (r_0 \cup \ldots \cup r_{i-1}) \). Let \( v_{i+1} \) be the intersection point between \( r' \setminus (r_0 \cup \ldots \cup r_{i-1}) \) and \( q_{i+1} \) (\( v_{i+1} \) is common endpoint of both \( r_i \) and \( q'_{i+1} \)). On the other hand, as \( Q'_i \) is a quadrilateral of \( L \), then none of the pseudolines of \( L \) crosses \( q'_{i+1} \).

From the previous paragraph we know that \( Q'_{n-1} \) is a quadrilateral with edges \( q'_{n-1}, \alpha_{n-1}, q'_n \)
and \( r_{n-1} \). Since \( \mathcal{L} \) is simple and \( r_{n-1} \) is a proper subarc of \( r' \setminus (r_0 \cup \ldots \cup r_{n-2}) \), which has an endpoint at \( v_n \), then the subarc \( r_n := r' \setminus (r_0 \cup \ldots \cup r_{n-1}) \) whose endpoints are \( v_n \) and \( v_{r'} \) is an edge of \( Q'_n \). Thus, \( q'_n, \alpha_n, p', r_n \), all the edges of \( \Gamma \), and \( r_n \) are the edges of \( Q'_n \). As \( \Gamma \) has at least one edge and \( \Gamma \cap \{ q'_n, \alpha_n, p', r_n \} = \emptyset \), then \( Q'_n \) is a \((\geq 5)\)-gon, as desired. \( \square \)

**Proof of Theorem 2.** We proceed by induction on the number of pseudolines of \( \mathcal{L} \) that are not in \( P \). If the number of pseudolines of \( \mathcal{L} \) that are not in \( P \) is zero, then \( \mathcal{L} \in \mathcal{A} \) and we are done by Theorem 8. So we assume that \( \mathcal{L} \) has \( \eta \geq 1 \) pseudolines that are not in \( P \) and that the assertion holds for every simple Euclidean arrangement \( \mathcal{N} \) with one \((\geq 5)\)-gon such that \( \mathcal{N} \) has \( j < \eta \) pseudolines that are not in the \((\geq 5)\)-gon of \( \mathcal{N} \).

Let \( p \) be a pseudoline of \( \mathcal{L} \) that is not in \( P \) and let \( \mathcal{L}' \) be the subarrangement of \( \mathcal{L} \) obtained by deleting \( p \). It is not difficult to see that \( \mathcal{L}' \) is a simple Euclidean arrangement with one \((\geq 5)\)-gon. In fact, note that \( P \) is the \((\geq 5)\)-gon of \( \mathcal{L}' \). Then both \( \mathcal{L} \) and \( \mathcal{L}' \) are \( k \)-critical and by the inductive hypothesis \( \mathcal{L}' \) has \(|\mathcal{L}'| - k = (|\mathcal{L}| - 1) - k = n - 1 - k \) triangles.

Now we partition the set of triangles of \( \mathcal{L} \) into two subsets. The first subset is denoted by \( A \) and consists of all the triangles that have an edge in \( p \). The second subset is denoted by \( B \) and contains the rest of the triangles of \( \mathcal{L} \). So the number of triangles of \( \mathcal{L} \) is \(|A| + |B|\).

It follows from the definition of \( \mathcal{L}' \) that every triangle of \( B \) is a triangle of \( \mathcal{L}' \). On the other hand, it is clear that if \( Q'_1, \ldots, Q'_\ell \) are the polygons of \( \mathcal{L}' \) that are crossed by \( p \), and \( \alpha \) of them are triangles, then the number of triangles of \( \mathcal{L}' \) is \(|B| + \alpha \). To complete the proof, it suffices to show that \(|A| - 1 = \alpha \).

Let \( A_1 \) be the subset of \( A \) consisting of all the triangles that are contained in a polygon of \( \mathcal{L}' \), and let \( A_2 := A \setminus A_1 \). Note that \( A_2 \) is the set of all triangles of \( \mathcal{L} \) that have a critical edge in \( p \). Without loss of generality we may assume that \( p \) is directed. For \( j \in \{1, \ldots, \ell\} \) we denote by \( p_j \) the open subarc of \( p \) that intersects \( Q'_j \) and by \( Q_{l,j} \) (respectively, \( Q_{r,j} \)) the polygon of \( \mathcal{L} \) to the left (respectively, right) of \( p_j \) such that \( p_j = e_{p,Q_{l,j}} = e_{p,Q_{r,j}} \). Observe that \( \{ Q_{l,j}, Q_{r,j}, p_j \} \) is a partition of \( Q'_j \). From the definitions of \( A \), \( Q_{l,j} \), \( Q_{r,j} \) and \( A_1 \), it is not difficult to see that \( A_1 = A \cap \{ Q_{l,1}, Q_{r,1}, \ldots, Q_{l,\ell}, Q_{r,\ell} \} \). Because \( P \) is the unique \((\geq 5)\)-gon of \( \mathcal{L} \) and \( p \) is not in \( P \), every \( Q_{d,j} \) is a triangle or a quadrilateral for \( d \in \{l, r\} \). Moreover, since \( \mathcal{L} \) is simple and the pseudolines of \( \mathcal{L} \) intersect pairwise exactly once, \( Q_{l,j} \) and \( Q_{r,j} \) are not both triangles. Thus, \( Q'_j \) is a triangle if and only if exactly one of \( Q_{l,j} \) or \( Q_{r,j} \) is a triangle. This implies that \(|A_1| = \alpha \). Thus, all we need to show is that \(|A_2| = 1 \).

We divide the remainder of the proof into two parts.

**Part I.** In this part we shall show that \(|A_2| \geq 1 \). To obtain a contradiction (that \( P \) is in \( p \)) we assume that \( A_2 \) is empty. By Proposition 3 \( p \) is in at least one triangle, say \( T \), of \( \mathcal{L} \). \( T \) must be an element of \( A_1 \) because \( A_2 \) is empty. Thus, there exists an \( m \in \{1, \ldots, \ell\} \) such that \( T = Q_{l,m} \) or \( T = Q_{r,m} \). Without loss of generality (changing the direction of \( p \) if necessary) we may assume that \( T := Q_{l,m} \). So \( Q := Q_{r,m} \) is a quadrilateral. Figure 6 shows the unique (up to isomorphism) simple Euclidean arrangement with 4 pseudolines. So we may assume (without loss of generality) that \( p \) and the three pseudolines in \( Q'_{m} \) are directed and labeled as in Figure 6.

**Case 1.** \( R \) is a triangle of \( \mathcal{L} \). This implies that none of the pseudolines of \( \mathcal{L} \) crosses \( e_{p,R} \). Analogously, since \( T \) is a triangle of \( \mathcal{L} \), none of the pseudolines of \( \mathcal{L} \) crosses \( e_{q,T} \). Since \( A_2 \) is empty, the region \( S \) to the left of \( e_{p,R} \) must be a polygon of \( \mathcal{L} \). Because \( \mathcal{L} \) is simple, \( v_{1,p}, v_{q,p} \) and \( v_{q,t} \) are vertices of \( S \) (and hence \( e_{p,R} \) and \( e_{q,T} \) are edges of \( S \)). In this
circumstances, it is easy to see that $S$ must be a $(\geq 5)$–gon. But $P$ is the unique $(\geq 5)$–gon of $\mathcal{L}$. So $P = S$, and therefore $P$ is in $p$.

Case 2. $R$ is not a triangle of $\mathcal{L}$. This means that at least one of the pseudolines of $\mathcal{L}^* := \mathcal{L} \setminus \{p, q, r, t\}$ crosses $R$. Since $Q$ is a polygon of $\mathcal{L}$ and $\mathcal{L}$ is simple, none of the pseudolines of $\mathcal{L}$ crosses $e_{q,R} \cup \{v_{q,t}, v_{p,q}, v_{p,t}\}$. Then $x \in \mathcal{L}^*$ crosses $R$ if and only if $x$ crosses $e_{p,R}$ and $e_{t,R}$. Let $a \in \mathcal{L}^*$ be the first pseudoline that crosses $p$ when $p$ is walked from $v_{p,t}$ to $v_{p,q}$ (a is well-defined because $\mathcal{L}$ is simple).

Subcase 2.1. $\mathcal{L}^*$ contains at least one element, say $b$, such that the intersection point between $a$ and $b$ is in $R$. In such a case, note that a pentagon $S$, is formed by $a, b, p, q$ and $t$. Thus, $S$ is a $(\geq 5)$–gon of the subarrangement of $\mathcal{L}$ induced by $a, b, p, q$ and $t$, and $e_{q,Q}$ is an edge of $S$. By Lemma 10, $e_{p,S}$ contains a subarc that is an edge of a $(\geq 5)$–gon of $\mathcal{L}$, say $S'$. But $P$ is the unique $(\geq 5)$–gon of $\mathcal{L}$. So $P = S'$, and therefore $P$ is in $p$.

Subcase 2.2. None of the pseudolines of $\mathcal{L}^*$ crosses $a$ in $R$. This implies that the triangular region $Z$, formed by $a, p,$ and $t$ is a triangle of $\mathcal{L}$ (see Figure 7), and hence none of the pseudolines of $\mathcal{L}$ crosses $e_{p,z}$. Since $A_2$ is empty, the region $Z'$ to the left of $e_{p,z}$ must be a polygon of $\mathcal{L}$. Then $Z'$ is a triangle or a quadrilateral, because $P$ is the unique $(\geq 5)$–gon of $\mathcal{L}$ and $p$ is not in $P$. On the other hand, since in a simple Euclidean arrangement there are no triangles with a common edge, $Z'$ is a quadrilateral. Because $\mathcal{L}$ is simple, $Z'$ has an edge in each of $a$ and $t$. Let $z$ be the fourth pseudoline in $Z'$. As $v_{q,t}, v_{r,t} \in p^+$ and $v_{z,t} \in p^-$, then $z \notin \{q, r\}$. Observe that if $v_{p,z} \in e_{p,R}$ then $v_{t,z} \in e_{t,R}$, because $e_{q,R} = e_{q,Q}$ cannot be crossed and $\mathcal{L}$ is simple. But $v_{t,z} \in p^-$ and $e_{t,R} \in p^+$, so $v_{p,z} \notin e_{p,R}$. Moreover, since $\mathcal{L}$ is simple and $Q$ is a polygon of $\mathcal{L}$, if $p' := e_{p,R} \cup e_{p,Q} \cup \{v_{p,r}, v_{p,q}, v_{p,t}\}$, then $v_{p,z} \notin p'$.

The last two cases from Part I now arise, depending on whether the intersection point between $r$ and $z$ occurs in $p^+$ or $p^-$.

Subcase 2.2.1 $v_{r,z} \in p^+$. Because $\mathcal{L}$ is simple and $T$ is a polygon of $\mathcal{L}$, $v_{z,q} \notin e_{q,T} \cup \{v_{p,q}, v_{q,r}\}$. So a pentagon $S$, is formed by $a, p, q, r$ and $z$. Moreover, $S$ is a $(\geq 5)$–gon of the subarrangement of $\mathcal{L}$ induced by $a, p, q, r$ and $z$, and $e_{q,T} = e_{q,S}$, see Figure 8). Since $T$ is a polygon of $\mathcal{L}$, none of the pseudolines of $\mathcal{L}$ crosses $e_{q,S}$. Thus, by Lemma 10 $e_{p,S}$ contains a subarc that is an edge of a $(\geq 5)$–gon of $\mathcal{L}$, say $S'$. But $P$ is the unique $(\geq 5)$–gon of $\mathcal{L}$. So $P = S'$, and therefore $P$ is in $p$.

Subcase 2.2.2 $v_{r,z} \in p^-$. Then $v_{p,z} \in t^+$ because $v_{p,z} \notin p'$. Together, $v_{r,z} \in p^+$ and

Figure 6: The unique (up to isomorphism) simple Euclidean arrangement with 4 pseudolines.
Figure 7: The form of the subarrangement of $\mathcal{L}$ induced by $a, p, q, r, t$ and $z$ if no pseudoline of $\mathcal{L}^*$ crosses $a$ in $R$.

$v_{p,z} \in t^+$ imply that the triangular region $H$ formed by $r, t$ and $z$ is contained in $t^+$. Because both $a$ and $p$ cross $t$ in $e_{t,H}$, $a$ and $p$ cross $H$. Since $v_{a,z} \in t^-$, it follows that $a$ crosses $H$ through $e_{t,H}$ and $e_{r,H}$. Analogously, $p$ crosses $H$ through $e_{t,H}$ and $e_{z,H}$ because $v_{p,r} \in t^-$. It follows from the last two assertions and the fact that $v_{a,p}$ is not in $H$ that $a, p, r, t$ and $z$ form a pentagon $S$. Moreover, $S$ is a $(\geq 5)$–gon of the subarrangement of $\mathcal{L}$ induced by $a, p, r, t$ and $z$, and $e_{t,Z} = e_{t,S}$, see Figure 8 ii). Since $Z$ is a triangle of $\mathcal{L}$, none of the pseudolines of $\mathcal{L}$ crosses $e_{t,S}$. Thus, by Lemma 10, $e_{p,S}$ contains a subarc that is an edge of a $(\geq 5)$–gon of $\mathcal{L}$, say $S'$. But $P$ is the unique $(\geq 5)$–gon of $\mathcal{L}$. So $P = S'$, and therefore $P$ is in $p$.

Figure 8: i) illustrates the case where the intersection point between $r$ and $z$ is in $p^-$; and ii) illustrates the case where the intersection point between $r$ and $z$ is in $p^+$. 

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Part II. In this part we shall prove that $|A_2| \leq 1$. Suppose not. Let $M, N$ be distinct triangles of $A_2$. It is not difficult to see that the configurations shown in Figure 9 are all (up to isomorphism) the possible configurations where both $M$ and $N$ are in $p$; note that it is impossible for $M$ and $N$ to be on the same side of $p$ and share a vertex, since $L$ is simple.

Assume that $M, N$ and $p$ are placed as in Figure 9 (i) or (ii). Since $L$ is simple, $v_{x,y} \in p^+$ or $v_{x,y} \in p^-$. If $v_{x,y} \in p^+$ (respectively, $v_{x,y} \in p^-$) then $e_{p,M}$ (respectively, $e_{p,N}$) is not a critical edge of $M$ (respectively, $N$), since, e.g., $N$ is contained in the triangle formed by $u, x$ and $y$. However, this contradicts the fact that every triangle in $A_2$ has critical edge in $p$.

So we may assume that $M, N$ and $p$ are placed as in Figure 9 (iii). Because $e_{p,M}$ and
\(e_{p,N}\) are, respectively, critical edges of \(M\) and \(N\), \(v_{x,y} \in p^-\). This implies that a pentagon, \(S\), is formed by \(p, x, y, u, \) and \(v\) (see Figure 10).

Moreover, \(S\) is a \((\geq 5)\)-gon of the subarrangement of \(\mathcal{L}\) induced by \(p, x, y, u, \) and \(v\), and \(e_{u,S} = e_{u,M}\). Since \(M\) is a triangle of \(\mathcal{L}\), none of the pseudolines of \(\mathcal{L}\) crosses \(e_{u,S}\). Thus, by Lemma 10 \(e_{p,S}\) contains a subarc that is an edge of a \((\geq 5)\)-gon of \(\mathcal{L}\), say \(S'\). But \(P\) is the unique \((\geq 5)\)-gon of \(\mathcal{L}\). So \(P = S'\), and therefore \(P\) is in \(p\) \(\Box\).

4 Arrangements in \(\mathfrak{S}\) are stretchable: proof of Theorem 3

Recall that Theorem 3 asserts that if \(\mathcal{L}\) is an arrangement in \(\mathfrak{S}\), then \(\mathcal{L}\) is stretchable. The proof is by induction on \(|\mathcal{L}|\). In [6], Goodman and Pollack showed that any Euclidean arrangement with 8 pseudolines is stretchable. So we may assume that \(|\mathcal{L}| = n \geq 9\) and that every arrangement of \(\mathfrak{S}\) with \(j < n\) pseudolines is stretchable.

Let \(P\) be the \((\geq 5)\)-gon of \(\mathcal{L}\). Since \(\mathcal{L} \in \mathfrak{S}\) and \(|\mathcal{L}| = n\), \(P\) is an \(n\)-gon. By Observation 5 we know that \(P\) has at most two critical edges. Hence, \(P\) has three consecutive non-critical edges, say \(a', b'\) and \(c'\). Let \(a, b\) and \(c\) be, respectively, the pseudolines containing \(a', b'\) and \(c'\). Without loss of generality we assume that \(a, b\) and \(c\) are directed in such a way that \(P\) lies to the left of each of them (see Figure 11).

![Figure 11: \(a', b'\) and \(c'\) are consecutive non-critical edges of \(P\).](image)

By Proposition 6 each of \(a', b'\) and \(c'\) is an edge of a triangle of \(\mathcal{L}\). Let \(A, B\) and \(C\) be, respectively, the triangles of \(\mathcal{L}\) that are adjacent to \(a', b'\) and \(c'\). See Figure 11.

For \(x \in \{a, b, c\}\) and \(\ell = 1, \ldots, n-1\) let \(x_\ell\) be the \(\ell\)-th pseudoline of \(\mathcal{L} \setminus \{x\}\) that is crossed by \(x\). Since \(\mathcal{L}\) is simple, the labels: \(a_1, \ldots, a_{n-1}; b_1, \ldots, b_{n-1};\) and \(c_1, \ldots, c_{n-1}\) are well-defined. In fact, \(\{a_1, \ldots, a_{n-1}\} \setminus \{c\} = \{c_1, \ldots, c_{n-1}\} \setminus \{a\} = \mathcal{L} \setminus \{a, c\}\). From now on, we assume that \(a_k = c, b_t = a\) and \(c_r = a\). Then \(i)\) \(A\) is formed by \(a, b = a_{k-1}\) and \(a_{k-2}\); \(ii)\) \(B\) is formed by \(b, a = b_t\) and \(c = b_{t+1}\) and \(iii)\) \(C\) is formed by \(c, b = c_{r+1}\) and \(c_{r+2}\).

See Figure 11 An easy consequence of \(i)\), \(ii)\) and \(iii)\) is, respectively, that \(3 \leq k \leq n - 1\), \(2 \leq t \leq n - 3\), and \(1 \leq r \leq n - 3\).

Let \(R_1 := a^- \cap c^-\), \(R_2 := a^+ \cap c^-\), \(R_3 := a^+ \cap c^\) and \(R_4 := a^- \cap c^\) be the four unbounded regions defined by \(a\) and \(c\) (see Figure 12). Note that \(P \subset R_1\).

Before we apply the inductive hypothesis, we need to establish some of the structural
properties of $L$. Specifically, we shall prove that the pseudolines of $L \setminus \{a, b, c\} = \{a_1, \ldots, a_k-2, a_{k+1}, a_{k+2}, \ldots, a_n\}$ may be placed in $R_2$ and $R_4$ as shown in Figure 13.

For $l', l'' \in \{1, \ldots, n-1\}$ and $w \in \{a, b, c\}$ let $w[l', l''] := \{w[l', l''], \ldots, w[l''+1]\}$. By convention, $w[l', l''] = \emptyset$ if $l' > l''$. Since $P$ has an edge in every pseudoline of $L$ (since $L \in \mathcal{I}$) and $P \subset R_1$, every pseudoline of $L \setminus \{a, b, c\}$ crosses $R_1$.

Since every pseudoline of $L \setminus \{a, b, c\}$ intersects $R_1$ and crosses exactly once each of $a, b$ and $c$, $c[1, r-1] \subseteq b[1, t-1] \subseteq a[1, k-2]$ and $a[k+1, n-1] \subseteq b[t+2, n-1] \subseteq c[r+2, n-1]$. This implies that $r \leq t \leq k-1$.

Another consequence of the fact that every pseudoline of $L \setminus \{a, b, c\}$ is in $P \subset R_1$ is the following: if $x, y \in a[1, k-2]$ or $x, y \in c[r+2, n-1]$ then $v_{x,y} \in R_1$.

Because $R_2$ contains no crossings between pseudolines of $a[1, k-2]$, and $c[1, r-1] \subseteq b[1, t-1] \subseteq a[1, k-2]$,

$$b_{t-j} = \begin{cases} a_{k-1-j} = c_{r-j} & \text{if } j = 1, 2, \ldots, r-1, \\ a_{k-1-j} & \text{if } j = r, r+1, \ldots, t-1. \end{cases} \quad (1)$$

Analogously, because $R_4$ contains no crossings between pseudolines of $c[r+2, n-1]$, and $a[k+1, n-1] \subseteq b[t+2, n-1] \subseteq c[r+2, n-1]$. 

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\[ b_{t+1+i} = \begin{cases} 
 c_{r+1+i} = a_{k+i} & \text{if } i = 1, 2, \ldots, n-k-1, \\
 c_{r+1+i} = b_{1+t-r} & \text{if } i = n-k, n-k+1, \ldots, n-t-2. 
\end{cases} \]

(2)

It follows from Equations (1) and (2) that the pseudolines of \( \mathcal{L} \setminus \{a, b, c\} \) are placed in \( R_2 \) and \( R_4 \) as shown in Figure 14.

Figure 14: Since \( R_2 \) contains no crossings between pseudolines of \( a[1, k-2], \{a[1, k-t-1], b[1, t-r]\}, c[1, r-1] \) is a partition of \( a[1, k-2] \). Analogously, \( \{a[k+1, n-1], b[n+t-k+1, n-1], c[n+r-t, n-1]\} \) is a partition of \( c[r+2, n-1] \) because \( R_4 \) contains no crossings between pseudolines of \( c[r+2, n-1] \).

From Eqs. (1) and (2) we deduce that \( a[k-r, k-2] = c[1, r-1] \) and \( a[k+1, n-1] = c[r+2, n+r-k] \), respectively. Let \( H := a[1, k-r-1] \). From these equations and the fact that \( a[1, k-2] \cup a[k+1, n-1] = c[1, r-1] \cup c[r+2, n-1] = \mathcal{L} \setminus \{a, b, c\} \) it follows that \( H = c[n+r-k+1, n-1] \) (see Figure 14). Let \( a_{i_1} = c_{j_1} \) and \( a_{i_2} = c_{j_2} \) be pseudolines of \( H \). If \( i_1 < i_2 \) then \( j_1 < j_2 \) (respectively, if \( i_1 > i_2 \) then \( j_1 > j_2 \)) because \( v_{a_{i_1}, a_{i_2}} \) is in \( R_1 \). Thus, \( c_{n+r-k+\ell} = a_{\ell} \) for \( \ell = 1, \ldots, k-r-1 \). From this equation and Figure 14 it follows that the pseudolines of \( \mathcal{L} \setminus \{a, b, c\} \) are placed in \( R_2 \) and \( R_4 \) as shown in Figure 13 as desired.

On the other hand, it is not difficult to see that the subarrangement of \( \mathcal{L} \) obtained by deleting \( b \) belongs to \( \mathcal{L} \). By the induction hypothesis \( \mathcal{L} \setminus \{b\} \) is stretchable. Let \( L_i^* \) be an arrangement of straight lines, which is equivalent to \( \mathcal{L} \setminus \{b\} \). If \( \theta \) denotes an element of \( \mathcal{L} \setminus \{b\} \) (for example, a pseudoline, a region, etc.), we denote by \( \theta^* \) the corresponding element in \( L_i^* \). For \( s = 1, \ldots, k-r-1 \) let \( m_s \) be the slope of \( a^*_s \). Since any two lines of \( L_i^* \) intersect exactly once, \( L_i^* \) has no lines with equal slopes, moreover, since the crossing of any two lines of \( H^* \) is in \( R_1^* \), \( m_0 < m_1 < \cdots < m_{k-r-1} < m_{k-r} \), where \( m_0 \) and \( m_{k-r} \) are, respectively, the slopes of \( a^* \) and \( c^* \). Let \( d^* \) be the line with slope \((m_{k-r-1} + m_{k-r})/2\) through \( v_{a^*, c^*} \). Since \( L_i^* \) and \( \mathcal{L} \setminus \{b\} \) are equivalent and \( m_0 < m_1 < \cdots < m_{k-r} \), \( d^* \) crosses the lines of \( L_i^* \setminus \{a^*, c^*\} \) in the exact same order in which \( b \) crosses the pseudolines of \( \mathcal{L} \setminus \{a, b, c\} \). Also, note that \( d^* \) can be moved in such a way that (i) the order in which \( d^* \) crosses the lines of \( L_i^* \setminus \{a^*, c^*\} \) is preserved and (ii) \( d^* \) intersects \( R_1^* \) (conditions (i) and (ii) can be ensured by a sufficiently small motion, keeping the slope, of \( d^* \)). Finally, note that the arrangement of lines obtained by such a motion of \( d^* \) is equivalent to \( \mathcal{L} \), as desired. □
5 On the number of non–isomorphic simple Euclidean arrangements with one \((\geq 5)\)–gon.

In this section we shall show that there are exponentially many non–isomorphic arrangements of \(\mathcal{I}\). We recall that two Euclidean arrangements are isomorphic (or combinatorially equivalent) if there is an incidence and dimension–preserving bijection between their induced cell decompositions (see [3]).

A 2-colored necklace with \(2m\) beads in which opposite beads have different colors is a self-dual necklace. An example of a self-dual necklace is shown in Figure 15 i). Two self-dual necklaces are isomorphic if one can be obtained from the other by rotation or reflection or both. In [9] was proved that the number of non-isomorphic self-dual necklaces is given by

\[
Q(m) = 2^{\lfloor (m-1)/2 \rfloor} + \frac{1}{2m} \sum_{k|m, k \text{ odd}} \phi(k)2^{m/k},
\]

where \(\phi(k)\) is the Euler totient function.

Since \(Q(m)\) grows exponentially with \(m\), it is enough to exhibit a one-to-one correspondence between the set of self-dual necklaces and a subset of \(\mathcal{I}\).

Let \(C\) be a self-dual necklace with \(2m \geq 6\) beads colored 0 and 1, and let \(P\) be the regular polygon of \(2m\) sides. Now we extend every edge of \(P\) to both sides in such a way that each pair of non-parallel segments intersect as shown in Figure 15 ii). Finally, we intersect every pair of parallel segments according to color 1 of \(C\) as shown in Figure 15 iii). By construction, the resultant arrangement \(P_C\) belongs to \(\mathcal{I}\). It is not difficult to see that (using this procedure) distinct necklaces generate distinct arrangements.

Figure 15: How to associate an arrangement of \(\mathcal{I}\) to each self-dual necklace with \(2m\) beads. i) \(C\) is a self-dual necklace with 8 beads. ii) The edges of the regular 8–gon \(P\) are extended in such a way that any pair of non-parallel segments intersect. iii) \(P_C\) is the arrangement of \(\mathcal{I}\) associated with \(C\). We use the beads with color 1 of \(C\) to determine the side in that every pair of parallel segments intersect.
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