On the Simplicity of the Eigenvalues of the Non-self-adjoint Mathieu-Hill Operators

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Abstract
We find conditions on the potential of the non-self-adjoint Mathieu-Hill operator such that the all eigenvalues of the periodic, antiperiodic, Dirichlet and Neumann boundary value problems are simple.
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1 Introduction and Preliminary Facts
Let \( P(q), A(q), D(q), N(q) \) be the operators in \( L_2[0, \pi] \) associated with the equation
\[
- y''(x) + q(x)y(x) = \lambda y(x)
\]
and the periodic
\[
y(\pi) = y(0), \quad y'(\pi) = y'(0),
\]
antiperiodic
\[
y(\pi) = -y(0), \quad y'(\pi) = -y'(0),
\]
Dirichlet
\[
y(\pi) = y(0) = 0,
\]
Neumann
\[
y'(\pi) = y'(0) = 0
\]
boundary conditions respectively.

It is well known that the spectra of the operators \( P(q) \) and \( A(q) \) consist of the eigenvalues \( \lambda_{2n} \) and \( \lambda_{2n+1} \), called as periodic and antiperiodic eigenvalues, that are the roots of
\[
F(\lambda) = 2 \quad \& \quad F(\lambda) = -2,
\]
where \( n = 0, 1, ..., F(\lambda) := \varphi'(\pi, \lambda) + \theta(\pi, \lambda) \) is the Hill discriminant and \( \varphi(x, \lambda), \theta(x, \lambda) \) are the solutions of the equation (1) satisfying the initial conditions
\[
\theta(0, \lambda) = \varphi'(0, \lambda) = 1, \quad \theta'(0, \lambda) = \varphi(0, \lambda) = 0.
\]
The eigenvalues of the operators \( D(q) \) and \( N(q) \), called as Dirichlet and Neumann eigenvalues, are the roots of
\[
\varphi(\pi, \lambda) = 0 \quad \& \quad \theta'(\pi, \lambda) = 0
\]
\( 1 \)

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Theorem 1 If $q(x)$ is an even function, then $\varphi(x, \lambda)$ is an odd function and $\theta(x, \lambda)$ is an even function. Periodic solutions are either $\varphi(x, \lambda)$ or $\theta(x, \lambda)$ unless all solutions are periodic (with period $\pi$ or $2\pi$). Moreover, the following equality holds

$$\varphi'(\pi, \lambda) = \theta(\pi, \lambda).$$

Theorem 2 For all $n$ and for any nonzero $a$ the geometric multiplicity of the eigenvalue $\lambda_n(a)$ of the operators $P(a)$ and $A(a)$ is 1 (that is, there exists one eigenfunction corresponding to $\lambda_n(a)$) and the corresponding eigenfunction is either $\varphi(x, \lambda_n(a))$ or $\theta(x, \lambda_n(a))$, where, for simplicity of the notations, the solutions of the equation

$$-y''(x) + (2a \cos 2x)y(x) = \lambda y(x)$$

satisfying (7) are denoted also by $\varphi(x, \lambda)$ and $\theta(x, \lambda)$.

In [8, 6] these theorems were proved for the real-valued potentials. However, the proofs pass through for the complex-valued potentials without any change.

The spectrum of $P(a)$, $A(a)$, $D(a)$, $N(a)$ for $a = 0$ are

$$\{(2k)^2 : k = 0, 1, \ldots\}, \{(2k+1)^2 : k = 0, 1, \ldots\}, \{k^2 : k = 1, 2, \ldots\}, \{k^2 : k = 0, 1, \ldots\}$$

respectively. All eigenvalues of $P(0)$, except 0, and $A(0)$ are double, while the eigenvalues of $D(0)$ and $N(0)$ are simple.

We use also the following result of [11].

Theorem 3 If $ab = cd$, then the Hill discriminants $F(\lambda, a, b)$ and $F(\lambda, c, d)$ (see (6)) for the operators $P(a, b)$ and $P(c, d)$ are the same.

By Theorem 2 the geometric multiplicity of the eigenvalues of $P(a)$ and $A(a)$ for any nonzero complex number $a$ is 1. However, in the non-self-adjoint case $a \in \mathbb{C}\setminus\mathbb{R}$, the multiplicity (algebraic multiplicity) of these eigenvalues, in general, is not equal to their geometric multiplicity, since the operators $P(a)$ and $A(a)$ may have associated functions (generalized eigenfunctions). Thus in the non-self-adjoint case the multiplicity (algebraic multiplicity) of the eigenvalues may be any finite number when the geometric multiplicity is 1 or 2. Therefore the investigation of the multiplicity of the eigenvalues for complex-valued potential is more complicated.
In this paper we find the conditions on $a$ such that the all eigenvalues of the operators $P(a)$, $A(a)$, $D(a)$ and $N(a)$ are simple, namely we prove the following

**Theorem 4** (Main results for the operators $P(a)$, $A(a)$, $D(a)$ and $N(a)$):

(a) If $0 < |a| \leq \frac{e}{\sqrt{6}}$, then the all eigenvalues of the operators $A(a)$ and $D(a)$ are simple.

(b) If $0 < |a| \leq \frac{3}{2}$, then the all eigenvalues of the operators $P(a)$ and $N(a)$ are simple.

This theorem with Theorem 3 implies

**Theorem 5** (Main results for the operators $A(a, b)$ and $P(a, b)$):

(a) If $0 < |ab| \leq \frac{64}{6}$, then the all eigenvalues of the operator $A(a, b)$ are simple.

(b) If $0 < |ab| \leq \frac{16}{9}$, then the all eigenvalues of the operator $P(a, b)$ are simple.

Note that there are a lot of papers about the asymptotic analyses and about the basis property of the root functions of the operators $P(a, b)$ and $A(a, b)$ (see [1-5, 7, 10] and the references in them). We do not discuss those papers, since in this paper we consider the another aspects of these operators and use only Theorems 1-3.

## 2 On the Even Potentials

In this section we analyze, in general, the even potentials. In the paper [9] the following statements about the connections of the spectra of the operators $P(q)$, $A(q)$, $D(q)$ and $N(q)$, where $q$ is an even potential, were proved.

**Lemma 1** of [9]. If $q$ is an even potential and $\lambda$ is an eigenvalue of both operators $D(q)$ and $N(q)$, then

$$ F(\lambda) = \pm 2, \quad \frac{dF}{d\lambda} = 0, \quad (14) $$

that is, $\lambda$ is a multiple eigenvalue of $L(q)$.

**Proposition 1** of [9]. Let $q$ be an even potential. Then $\lambda$ is an eigenvalue of $L(q)$ if and only if $\lambda$ is an eigenvalue of $D(q)$ or $N(q)$.

First using (12) and the Wronskian equality

$$ \theta(\pi, \lambda) = \pm 1 $$

we prove the following improvements of these statements.

**Theorem 6** Let $q$ be an even complex-valued function. A complex number $\lambda$ is both a Neumann and Dirichled eigenvalue if and only if it is an eigenvalue of the operator $L(q)$ with geometric multiplicity 2.

**Proof.** Suppose $\lambda$ is both a Neumann and Dirichled eigenvalue, that is, both equality in (8) hold. On the other hand, it follows from (12), (8) and (15) that

$$ \theta(\pi, \lambda) = \varphi'(\pi, \lambda) = \pm 1. \quad (16) $$

Now using (8), (16) and (7) one can easily verify that both $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ satisfy either periodic or anti-periodic boundary condition, that is, $\lambda$ is an eigenvalue of the operator $L(q)$ with geometric multiplicity 2.

Conversely, if $\lambda$ is an eigenvalue of $L(q)$ with geometric multiplicity 2, then both $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ satisfy either periodic or anti-periodic boundary condition. Therefore by (7) the equalities in (8) hold, that is, $\lambda$ is both Neumann and Dirichled eigenvalue. □
Theorem 7 Let \( q \) be an even complex-valued function. A complex number \( \lambda \) is an eigenvalue of multiplicity \( s \) of the operator \( L(q) \) if and only if it is an eigenvalue of multiplicities \( u \) and \( v \) of the operators \( D(q) \) and \( N(q) \) respectively, where \( u + v = s \) and \( u = 0 \) \((v = 0)\) means that \( \lambda \) is not an eigenvalue of \( D(q) \) \((N(q))\).

Proof. It is well-known and clear that \( \lambda_0 \) is an eigenvalue of multiplicities \( u \), \( v \) and \( s \) of the operator \( D(q), N(q) \) and \( L(q) \) respectively if and only if

\[
\varphi(\pi, \lambda) = (\lambda_0 - \lambda)^u f(\lambda), \quad \theta'(\pi, \lambda) = (\lambda_0 - \lambda)^v g(\lambda)
\]

(17)

and

\[
(F(\lambda) - 2)(F(\lambda) + 2) = (\lambda_0 - \lambda)^s h(\lambda),
\]

(18)

where \( f(\lambda_0) \neq 0, g(\lambda_0) \neq 0 \) and \( h(\lambda_0) \neq 0 \). On the other hand by (12) and (15) we have

\[
(F(\lambda) - 2)(F(\lambda) + 2) = 4\theta^2(\pi, \lambda) - 4 = 4(\theta(\pi, \lambda)\varphi'(\pi, \lambda) - 1) = 4\varphi(\pi, \lambda)\theta'(\pi, \lambda).
\]

(19)

Thus the proof of the theorem follows from (17)-(19) □

To analyze the periodic and antiperiodic eigenvalues in detail let us introduce the following notations and definitions.

Definition 1 Let \( \sigma(T) \) denotes the spectrum of the operator \( T \). A number \( \lambda \) is called PDN(q) (periodic, Dirichled and Neumann) eigenvalue if \( \lambda \in \sigma(P(q)) \cap \sigma(D(q)) \cap \sigma(N(q)) \). A number \( \lambda \in \sigma(P(q)) \cap \sigma(D(q)) \) is called PD(q) (periodic and Dirichled) eigenvalue if it is not PDN(q) eigenvalue. A number \( \lambda \in \sigma(P(q)) \cap \sigma(N(q)) \) is called PN(q) (periodic and Neumann) eigenvalue if it is not PDN(q) eigenvalue. Everywhere replacing \( P(q) \) by \( A(q) \) we get the definition of ADN(q), AD(q) and AN(q) eigenvalues.

Using Theorems 6, 7, Definition 1 and the equality \( \sigma(P(q)) \cap \sigma(A(q)) = \emptyset \) we obtain

Theorem 8 Let \( q \) be an even complex-valued function. Then

(a) The spectrum of \( P(q) \) is the union of the following three pairwise disjoint sets: \{PDN(q) eigenvalues\}, \{PD(q) eigenvalues\} and \{PN(q) eigenvalues\}.

(b) A complex number \( \lambda \) is an eigenvalue of geometric multiplicity \( 2 \) of the operator \( P(q) \) if and only if it is PDN(q) eigenvalue.

(c) A complex number \( \lambda \) is an eigenvalue of geometric multiplicity \( 1 \) of the operator \( P(q) \) if and only if it is either PD(q) or PN(q) eigenvalue.

The theorem continues to hold if \( P(q) \), PDN(q), PD(q) and PN(q) are replaced by \( A(q) \), ADN(q), AD(q) and AN(q) respectively.

Now we prove the main theorem of this section.

Theorem 9 Let \( q \) be an even complex-valued function and \( \lambda \) be an eigenvalue of geometric multiplicity \( 1 \) of the operator \( P(q) \). Then the number \( \lambda \) is an eigenvalue of multiplicity \( s \) of \( P(q) \) if and only if it is an eigenvalue of multiplicity \( s \) either of the operator \( D(q) \) (first case) or of the operator \( N(q) \) (second case). In the first case the system of the root functions of the operators \( P(q) \) and \( D(q) \) consists of the same eigenfunction \( \varphi(x, \lambda) \) and associated functions

\[
\frac{\partial \varphi(x, \lambda)}{\partial \lambda}, \frac{1}{2!} \frac{\partial^2 \varphi(x, \lambda)}{\partial \lambda^2}, \ldots, \frac{1}{(s-1)!} \frac{\partial^{s-1} \varphi(x, \lambda)}{\partial \lambda^{s-1}}.
\]

(20)

In the second case the system of the root function of the operators \( P(q) \) and \( N(q) \) consists of the same eigenfunction \( \theta(x, \lambda) \) and associated functions

\[
\frac{\partial \theta(x, \lambda)}{\partial \lambda}, \frac{1}{2!} \frac{\partial^2 \theta(x, \lambda)}{\partial \lambda^2}, \ldots, \frac{1}{(s-1)!} \frac{\partial^{s-1} \theta(x, \lambda)}{\partial \lambda^{s-1}}.
\]

(21)
The theorem continues to hold if $P(q)$ is replaced by $A(q)$.

**Proof.** Let $\lambda$ be an eigenvalue of geometric multiplicity 1 and multiplicity $s$ of the operator $P(q)$. By Theorem 1 there are two cases.

Case 1. The corresponding eigenfunction is $\varphi(x, \lambda)$.

Case 2. The corresponding eigenfunction is $\theta(x, \lambda)$.

We consider Case 1. In the same way one can consider Case 2. In Case 1, $\theta(x, \lambda)$ is not a periodic solution, that is, it does not satisfy the periodic boundary condition (2). On the other hand, the first equality of (6) with (12) and (7) implies that

$$\theta(\pi, \lambda) = 1 = \theta(0, \lambda),$$

that is, $\theta(x, \lambda)$ satisfies the first equality in (2). Therefore $\theta(x, \lambda)$ does not satisfy the second equality of (2), that is,

$$\theta'(\pi, \lambda) \neq 0.$$  

(23)

This inequality means that $v = 0$, where $v$ is defined in Theorem 7. Therefore, by Theorem 7 we have $u = s$, that is, $\lambda$ is an eigenvalue of multiplicity $s$ of the operator $D(q)$.

Now suppose that $\lambda$ is an eigenvalue of multiplicity $s$ of $P(q)$ and $D(q)$.

$$\varphi(\pi, \lambda) = 0 = \varphi(0, \lambda).$$  

(24)

On the other hand, using the first equality of (6), (12) and (7) we get

$$\varphi'(\pi, \lambda) = 1 = \varphi'(0, \lambda).$$  

(25)

Therefore $\varphi(x, \lambda)$ is an eigenfunction of $P(q)$ corresponding to the eigenvalue $\lambda$. Then, by Theorem 1, $\theta(x, \lambda)$ is not a periodic solution. This, as we noted above, implies (23) and the equality $u = s$. Thus, by Theorem 7, $\lambda$ is an eigenvalue of multiplicity $s$ of $P(q)$.

If $\lambda$ is an eigenvalue of multiplicity $s$ of the operators $P(q)$ and $D(q)$, then

$$F(\lambda) = 2, \quad \frac{dF}{d\lambda} = 0, \quad \frac{d^2F}{d\lambda^2} = 0, \ldots, \frac{d^{s-1}F}{d\lambda^{s-1}} = 0$$  

(26)

and

$$\varphi(\pi, \lambda) = 0, \quad \frac{d\varphi(\pi, \lambda)}{d\lambda} = 0, \quad \frac{d^2\varphi(\pi, \lambda)}{d\lambda^2} = 0, \ldots, \frac{d^{s-1}\varphi(\pi, \lambda)}{d\lambda^{s-1}} = 0.$$  

(27)

Since $\varphi(0, \lambda) = 0$ and $\varphi'(0, \lambda) = 1$ for all $\lambda$, we have

$$\varphi(0, \lambda) = 0, \quad \frac{d\varphi(0, \lambda)}{d\lambda} = 0, \quad \frac{d^2\varphi(0, \lambda)}{d\lambda^2} = 0, \ldots, \frac{d^{s-1}\varphi(0, \lambda)}{d\lambda^{s-1}} = 0$$  

(28)

and

$$\varphi'(0, \lambda) = 1, \quad \frac{d\varphi'(0, \lambda)}{d\lambda} = 0, \quad \frac{d^2\varphi'(0, \lambda)}{d\lambda^2} = 0, \ldots, \frac{d^{s-1}\varphi'(0, \lambda)}{d\lambda^{s-1}} = 0.$$  

(29)

Moreover, using (26) and (12) we obtain

$$\varphi'(\pi, \lambda) = 1, \quad \frac{d\varphi'(\pi, \lambda)}{d\lambda} = 0, \quad \frac{d^2\varphi'(\pi, \lambda)}{d\lambda^2} = 0, \ldots, \frac{d^{s-1}\varphi'(\pi, \lambda)}{d\lambda^{s-1}} = 0.$$  

(30)

Thus, by (27)-(30), $\varphi(x, \lambda)$ and the functions in (20) satisfy both the periodic and Dirichlet boundary conditions. On the other hand, differentiating $s-1$ times, with respect to $\lambda$, the equation

$$-\varphi''(x, \lambda) + q(x)\varphi(x, \lambda) = \lambda\varphi(x, \lambda)$$

(31)
we obtain
\[-\left(1 \frac{1}{k!} \frac{\partial^{k} \varphi(x, \lambda)}{\partial \lambda^{k}}\right)^{\alpha} + \left(q(x) - \lambda\right) \frac{1}{k!} \frac{\partial^{k} \varphi(x, \lambda)}{\partial \lambda^{k}} = \frac{1}{(k-1)!} \frac{\partial^{k-1} \varphi(x, \lambda)}{\partial \lambda^{k-1}}\]

for \(k = 1, 2, \ldots, (s-1)\). Therefore \(\varphi(x, \lambda)\) and the functions in (20) are the root functions of the operators \(P(q)\) and \(D(q)\). Thus the first case is proved. In the same way we prove the second case. The proof of this results for \(A(q)\) are similar. 

\section{Main Results}

In this section we consider the operators \(P(a), A(a), D(a)\) and \(N(a)\) with potential
\[q(x) = 2a \cos 2x, \quad (32)\]
where \(a\) is a nonzero complex number. By Theorem 2 the geometric multiplicity of the eigenvalues of \(P(a)\) and \(A(a)\) is 1. Therefore it follows from Theorem 8 that
\[\sigma(P(a)) = \{PD(a)\ \text{eigenvalues}\} \cup \{PN(a)\ \text{eigenvalues}\}, \quad (33)\]
\[\sigma(A(a)) = \{AD(a)\ \text{eigenvalues}\} \cup \{AN(a)\ \text{eigenvalues}\}, \quad (34)\]
where \(PD(q), PN(q), AD(q)\) and \(AN(q)\) (see Definition 1) are denoted by \(PD(a), PD(a), PD(a)\) and \(PD(a)\) when the potential \(q\) is defined by (32). Moreover, Theorem 7, Theorem 9 yield the equalities
\[\sigma(D(a)) = \{PD(a)\ \text{eigenvalues}\} \cup \{AD(a)\ \text{eigenvalues}\}, \quad (35)\]
\[\sigma(N(a)) = \{PN(a)\ \text{eigenvalues}\} \cup \{AN(a)\ \text{eigenvalues}\} \quad (36)\]
and the following theorem.

\textbf{Theorem 10} For any \(a \neq 0\) the eigenvalue \(\lambda\) of the operator \(P(a)\) or \(A(a)\) is multiple if and only if it is a multiple eigenvalue either of \(D(a)\) or \(N(a)\). Moreover, the operators \(P(a), A(a), D(a)\) and \(N(a)\) have associated functions corresponding to any multiple eigenvalues.

Clearly, the eigenfunctions corresponding to \(PN(a)\) eigenvalues, \(PD(a)\) eigenvalues, \(AD(a)\) eigenvalues and \(AN(a)\) eigenvalues have the forms
\[\Psi_{PN}(x) = \frac{a_{0}}{\sqrt{2}} + \sum_{k=1}^{\infty} a_{k} \cos 2kx, \quad (37)\]
\[\Psi_{PD}(x) = \sum_{k=1}^{\infty} b_{k} \sin 2kx, \quad (38)\]
\[\Psi_{AD}(x) = \sum_{k=1}^{\infty} c_{k} \sin(2k - 1)x, \quad (39)\]
and
\[\Psi_{AN}(x) = \sum_{k=1}^{\infty} d_{k} \cos(2k - 1)x \quad (40)\]
respectively. For simplicity of the calculating we normalize these eigenfunctions as follows

\[
\sum_{k=0}^{\infty} |a_k|^2 = 1, \quad \sum_{k=1}^{\infty} |b_k|^2 = 1, \quad \sum_{k=1}^{\infty} |c_k|^2 = 1, \quad \sum_{k=1}^{\infty} |d_k|^2 = 1.
\] (41)

Substituting the functions (37)-(40) into (13) we obtain the following equalities

\[
\lambda a_0 = \sqrt{2}a_1, \quad (\lambda - 4)a_1 = a\sqrt{2}a_0 + aa_2, \quad (\lambda - (2k)^2)a_k = ak_{k-1} + aa_{k+1},
\] (42)

\[
(\lambda - 4)b_1 = ab_2, \quad (\lambda - (2k)^2)b_k = ab_{k-1} + ab_{k+1},
\] (43)

\[
(\lambda - 1)c_1 = ac_1 + ac_2, \quad (\lambda - (2k - 1)^2)c_k = ac_{k-1} + ac_{k+1},
\] (44)

\[
(\lambda - 1)d_1 = -ad_1 + ad_2, \quad (\lambda - (2k - 1)^2)d_k = ad_{k-1} + ad_{k+1}
\] (45)

for \( k = 2, 3, \ldots \) Here \( a_k, b_k, c_k, d_k \) depend on \( \lambda \) and \( a_0, b_1, c_1, d_1 \) are nonzero constants (see [6] p. 34-35).

By Theorem 10, if the eigenvalue \( \lambda \) corresponding to one of the eigenfunctions (37)-(40), denoted by \( \Psi(x) \), is multiple then there exists associated function \( \Phi \) satisfying

\[-(\Phi(x,\lambda))'' + (q(x) - \lambda)\Phi(x,\lambda) = \Psi(x).\] (46)

Since the boundary conditions (2)-(5) are self-adjoint and \( \Psi(x) \) are eigenvalue and eigenfunction of the adjoint operator. Therefore multiplying both sides of (46) by \( \Psi \) we get \( (\Psi,\Psi) = 0 \), where \( (\cdot,\cdot) \) is the inner product in \( L_2[0,\pi] \). Thus, if the eigenvalues corresponding to the eigenfunctions (37)-(40), are multiple, then we have

\[
\sum_{k=0}^{\infty} a_k^2 = 0, \quad \sum_{k=1}^{\infty} b_k^2 = 0, \quad \sum_{k=1}^{\infty} c_k^2 = 0, \quad \sum_{k=1}^{\infty} d_k^2 = 0.
\] (47)

To prove the simplicity of the eigenvalue \( \lambda \) corresponding, say, to (40) we show that there is not a sequence \( \{d_k\} \) satisfying the above 3 equalities: (45), (41) and (47), since these equalities hold if \( \lambda \) is a multiple eigenvalue. For this we use following proposition which readily follows from (41) and (47).

**Proposition 1** If there exists \( n \in \mathbb{N} = \{1, 2, \ldots\} \) such that

\[
|d_n(\lambda)|^2 > \frac{1}{2},
\] (48)

then \( \lambda \) is a simple \( AN(a) \) eigenvalue, where \( a \neq 0 \). The statement continues to hold for \( AD(a) \), \( PD(a) \) and \( PN(a) \) eigenvalues if \( d_n \) is replaced by \( c_n, b_n \) and \( a_n \) respectively.

To apply the Proposition 1, we use following lemmas.

**Lemma 1** Suppose that \( \lambda \) is a multiple \( AN(a) \) eigenvalue corresponding to the eigenfunction (40), where \( a \neq 0 \). Then

(a) For all \( k, m \in \mathbb{N}, k \neq m \) the following inequalities hold

\[
|d_k|^2 \leq \frac{1}{2},
\] (49)

\[
|d_k \pm d_m|^2 \leq 1,
\] (50)

\[
|d_k|^2 \leq \frac{|a|^2}{|\lambda - (2k - 1)^2|^2}.
\] (51)
(b) If $\text{Re} \lambda < (2p - 1)^2 - 2 |a|$ for some $p \in \mathbb{N}$, then $|d_{k-1}| > |d_k| > 0$ and

$$|d_{k+s}| < \frac{|2a|^{s+1} |d_{k-1}|}{|\lambda - (2k - 1)^2| |\lambda - (2(k + 1) - 1)^2| ... |\lambda - (2(k + s) - 1)^2|}$$

for all $k > p$ and $s = 0, 1, ...$

(c) Let $I \subset \mathbb{N}$ and $\lambda(I) := \min_{k \in I} |\lambda - (2k - 1)^2| \neq 0$. Then

$$\sum_{k \in I} |d_k|^2 \leq \frac{4 |a|^2}{(\lambda(I))^2}.$$ 

(d) If $\lambda$ is a multiple $AD(a)$ eigenvalue corresponding to the eigenfunction $(39)$, then the inequalities $(49)$-$(53)$ continues to hold if $d_j$ is replaced by $c_j$.

**Proof.** (a) If (49) does not hold for some $k$, then by Proposition 1 $\lambda$ is a simple eigenvalue that contradicts the assumption of the lemma.

Using the last equalities of (47) and (41), we obtain

$$|(d_k \pm d_m)^2| = - \sum_{n \neq k,m} d_n^2 \pm 2d_k d_m \leq \sum_{n \neq k,m} |d_n|^2 + |d_k|^2 + |d_m|^2 = 1,$$

that is, (50) holds. Now (51) follows from (45) and (50).

(b) Suppose that $|d_k| \geq |d_{k-1}|$ for some $k > p > 0$. By (45)

$$|\lambda - (2k - 1)^2| |d_k| \leq |a| |d_{k-1}| + |a| |d_{k+1}|.$$

On the other hand, using the condition on $\lambda$ we get $|\lambda - (2k - 1)^2| > 2 |a|$. Therefore

$$|d_{k+1}| \geq 2 |d_k| - |d_{k-1}| \geq |d_k|.$$

Repeating this process $s$ times we obtain $|d_{k+s}| \geq |d_{k+s-1}|$ for all $s \in \mathbb{N}$. It means that $\{|d_{k+s}| : s \in \mathbb{N}\}$ is a nondecreasing sequence. On the other hand, $|d_k| + |d_{k+1}| \neq 0$, since if both $d_k$ and $d_{k+1}$ are zero, then using (45) we obtain that $d_j = 0$ for all $j \in \mathbb{N}$, that is, the solutions (40) is identically zero. Therefore $d_k$ does not converge to zero being the Fourier coefficient of the square integrable function $\Psi_{AN}(x)$. This contradiction shows that $\{|d_{k+s}| : s \in \mathbb{N}\}$ is a decreasing sequence. Thus $|d_k| > 0$ for all $k > p$.

Now let us prove (52). Using (45) and the inequality $|d_{k-1}| > |d_k| > 0$, we get

$$|d_{k+s}| < \frac{|2a| |d_{k+s-1}|}{|\lambda - (2(k + s) - 1)^2|}$$

for all $s = 0, 1, ...$. Iterating (54) $s$ times we obtain (52).

(c) By (45) we have

$$\sum_{k \in I} |d_k|^2 \leq \sum_{k \in I} \frac{|a|^2 (|d_{k-1}| + |d_{k+1}|)^2}{(\lambda(I))^2} \leq \sum_{k \in I} \frac{2 |a|^2 (|d_{k-1}|^2 + |d_{k+1}|^2)}{(\lambda(I))^2}.$$

Note that in case $k = 1$ instead of $d_{k-1}$ we take $d_1$ (see the first equality of (45)). Now (53) follows from (41).

(d) Everywhere replacing $d_k$ by $c_k$ we get the proof of the last statement. ■

In the similar way we prove the following lemma for $P(a)$. 
Lemma 2 Suppose that $\lambda$ is a multiple PD($a$) eigenvalue corresponding to the eigenfunction (38), where $a \neq 0$. Then

(a) For all $k \in \mathbb{N}$, $m \in \mathbb{N}$, $n \in \mathbb{N}$, $n \neq m$ the following inequalities hold

$$
|b_m|^2 \leq \frac{1}{2}, \quad |b_n \pm b_m|^2 \leq 1, \quad |b_k|^2 \leq \frac{|a|^2}{|\lambda - (2k)^2|^2}.
$$

(55)

(b) If $\Re \lambda < (2p)^2 - 2|a|$ for some $p \in \mathbb{N}$, then $|b_{k-1}| > |b_k| > 0$ and

$$
|b_{k+s}| < \frac{|2a|^s+1 |b_{k-1}|}{|\lambda - (2k)^2||\lambda - (2(k+1))^2| \cdots |\lambda - (2(k+s))^2|}
$$

(56)

for all $k > p$ and $s = 0, 1, \ldots$

(c) Let $I \subset \mathbb{N}$ and $b(\lambda, I) = \min_{k \in I} |\lambda - (2k)^2| \neq 0$. Then

$$
\sum_{k \in I} |b_k|^2 \leq \frac{4|a|^2}{(b(\lambda, I))^2}.
$$

(57)

(d) If $\lambda$ is a multiple PN($a$) eigenvalue corresponding to (37) then the statements (a) and (b) continue to hold for $k > 1$, $m \geq 0$ and the statement (c) continues to hold for $I \subset \{0, 1, \ldots\}$ if $b_j$ is replaced by $a_j$.

Introduce the notation $D_n = \{ \lambda \in \mathbb{C} : |\lambda - (2n - 1)^2| \leq 2|a| \}$.

Theorem 11 (a) All eigenvalues of the operator $A(a)$ lie on the unions of $D_n$ for $n \in \mathbb{N}$.

(b) If $4n - 4 > (1 + \sqrt{2})|a|$, where $a \neq 0$, then the eigenvalues of $A(a)$ lying in $D_n$ are simple.

Proof. By (34) if $\lambda$ is an eigenvalue of the operator $A(a)$, then the corresponding eigenfunction is either $\Psi_{AN}(x)$ or $\Psi_{AD}(x)$ (see (39) or (40)). Without loss of generality, we assume that the corresponding eigenfunction is $\Psi_{AN}(x)$.

(a) Since $d_k \to 0$ as $k \to \infty$, there exists $n \in \mathbb{N}$ such that

$$
|d_n| = \max_{k \in \mathbb{N}} |d_k|.
$$

Therefore (a) follows from (45) for $k = n$.

(b) Suppose that $\lambda \in D_n$ is a multiple eigenvalue corresponding to the eigenfunction $\Psi_{AN}(x)$. By definition of $D_n$ for $k \neq n$ we have

$$
|\lambda - (2k)^2| \geq |(2n - 1)^2 - (2k - 1)^2| - |2a| \geq |(2n - 3)^2 - (2n - 1)^2| - |2a|.
$$

This together with the condition on $n$ and the definition of $d(\lambda, I)$ (see Lemma 1(c)) gives $d(\lambda, \mathbb{N}\{n\}) > 2\sqrt{2}|a|$. Thus, using (53) and (41) we get

$$
\sum_{k \neq n} |d_k|^2 < \frac{1}{2} \quad \text{and} \quad |d_n|^2 > \frac{1}{2}
$$

which contradicts Proposition 1. ■

Instead of Lemma 1 using Lemma 2 in the same way we prove the following

Theorem 12 (a) All PD($a$) eigenvalues lie in the unions of $B =: \{ \lambda : |\lambda - 4| \leq |a| \}$ and $B_n =: \{ \lambda : |\lambda - (2n)^2| \leq 2|a| \}$ for $n = 2, 3, \ldots$. All PN($a$) eigenvalues lie in the unions of $A_0 = \{ \lambda : |\lambda| \leq \sqrt{2}|a| \}$, $A_1 = \{ \lambda : |\lambda - 4| \leq (1 + \sqrt{2})|a| \}$ and $B_n$ for $n = 2, 3, \ldots$.
(b) If \( 4n - 2 > (1 + \sqrt{2})|a| \) and \( n > 1 \), where \( a \neq 0 \), then the eigenvalues of \( P(a) \) lying in \( B_n \) are simple.

Now we prove the main result for \( A(a) \).

**Theorem 13** If \( 0 < |a| \leq \frac{8}{\sqrt{6}} \), then the all eigenvalues of the operator \( A(a) \) are simple.

**Proof.** Since \( 8 > \frac{8}{\sqrt{6}}(1 + \sqrt{2}) \), by Theorem 11(b) the ball \( D_n \) for \( n > 2 \) does not contain the multiple eigenvalues of the operator \( A(a) \). Therefore we need to prove that the ball \( D_n \) for \( n = 1,2 \) also does not contain the multiple eigenvalues. Since the balls \( D_1 \) and \( D_2 \) are contained in the half plane \( \{ \lambda \in \mathbb{C} : \Re \lambda < 16 \} \) we consider the following two strips \( \{ \lambda \in \mathbb{C} : 9 < \Re \lambda < 16 \} \), \( \{ \lambda \in \mathbb{C} : 6 < \Re \lambda \leq 9 \} \) and half plane \( \{ \lambda \in \mathbb{C} : \Re \lambda \leq 6 \} \) separately. We consider the \( AN(a) \) eigenvalues, that is, the eigenvalues corresponding to the eigenfunction \((40)\). Consideration of the \( AD(a) \) eigenvalues are the same.

To prove the simplicity of the eigenvalues lying in the above strips, we assume that \( \lambda \) is a multiple eigenvalue. Using Lemma 1 by direct calculating (see Estimation 1 and Estimation 2 in Appendix) we show that \((48)\) for \( n = 2 \) holds that contradicts Proposition 1.

Investigation the half plane \( \Re \lambda \leq 6 \) is more complicated. Here we use the first two equalities of \((45)\)

\[
(\lambda - 1)d_1 = -ad_1 + ad_2, \quad (\lambda - 9)d_2 = ad_1 + ad_3.
\]

By direct calculating we get (see Estimation 3 and Estimation 4 in the Appendix)

\[
\sum_{k=3}^{\infty} |d_k|^2 < 0.03415, \quad \frac{|d_3|}{|d_2|} < 0.17432
\]

Then by \((41)\) we have

\[
|d_1|^2 + |d_2|^2 > 1 - \varepsilon,
\]

where \( \varepsilon = 0.03415 \). On the other hand, by \((49)\), \( |d_1|^2 \leq \frac{1}{2} \), \( |d_2|^2 \leq \frac{1}{2} \). These inequalities and \((47)\) imply that

\[
|d_1|^2 = \frac{1}{2} - \varepsilon_1, \quad |d_2|^2 = \frac{1}{2} - \varepsilon_2, \quad d_2 = -d_1 + \varepsilon_3,
\]

where \( \varepsilon_1 \geq 0 \), \( \varepsilon_2 \geq 0 \), \( \varepsilon_1 + \varepsilon_2 = \varepsilon \), \( |\varepsilon_3| < 0.03415 \). Now, one can easily see that

\[
\left( \frac{d_2}{d_1} \right)^2 = -1 + \alpha, \quad \frac{d_2}{d_1} = \pm(i + \delta),
\]

where \( |\alpha| < \frac{0.03415}{0.5 - 0.03415} < 0.074 \), \( |\delta| < \frac{1}{2} |0.074| + \frac{1}{4} |0.074|^2 < 0.04 \). Therefore we have

\[
\frac{d_2}{d_1} = \pm \frac{(i + \delta)^2 - 1}{i + \delta} = \pm \frac{2i(i + \delta) + \delta^2}{i + \delta} = \pm 2i + \gamma,
\]

where \( |\gamma| < \frac{0.04i^2}{0.04} < 0.002 \). On the other hand, dividing the first equality of \((58)\) by \( d_1 \) and the second by \( d_2 \) and then subtracting second from the first and taking into account \((61)\) we get

\[
\frac{8}{a} = \pm 2i - 1 + \gamma - \frac{d_3}{d_2},
\]

where by assumption \( |\frac{8}{a}| \geq \sqrt{6} \). Therefore using the second estimation of \((59)\) in \((62)\) we get the contradiction

\[
2.4495 < \sqrt{6} \leq |\frac{8}{a}| < \sqrt{6} + 0.17432 + 0.002 < 2.4125 \quad \blacksquare
\]
In the same way we consider the simplicity of the eigenvalues of the operators \( P(a), D(a) \) and \( N(a) \). First let us investigate the eigenvalues of \( D(a) \). Since the eigenvalues of \( D(a) \) is the union of \( PD(a) \) and \( AD(a) \) eigenvalues and the \( AD(a) \) eigenvalues are investigated in Theorem 13, we investigate the \( PD(a) \) eigenvalue.

**Theorem 14** If \( 0 < |a| \leq 5 \), then all \( PD(a) \) eigenvalues are simple. Moreover, if \( 0 < |a| \leq \frac{5}{\sqrt{6}} \), then the all eigenvalues of the operator \( D(a) \) are simple.

**Proof.** The second statement follows from the first statement and Theorem 13. Therefore we need to prove the first statement by using (43). Since \( 14 > 5(1 + \sqrt{2}) \), by Theorem 12, the \( PD(a) \) eigenvalues lying in the ball \( B_n \) for \( n > 3 \) are simple.

If \( \lambda \in B_3 \), then \( 26 \leq \text{Re} \lambda \leq 46 \). Using Lemma 2 and (41) we obtain the estimations (see Estimation 5 in Appendix)

\[
\sum_{k \neq 3} |b_k|^2 < \frac{1}{2}, \quad |b_3|^2 > \frac{1}{2}
\]

which, by Proposition 1, proves the simplicity of the \( PD(a) \) eigenvalues lying in \( B_3 \).

Now we need to prove that the balls \( B_2 \) and \( B_2 \) does not contain the multiple \( PD(a) \) eigenvalues. Since these balls are contained in the strip \( \{ \lambda \in \mathbb{C} : \text{Re} \lambda \leq 26 \} \) we consider the following cases: \( 16 < \text{Re} \lambda \leq 26 \), \( 12 < \text{Re} \lambda \leq 16 \) and \( \text{Re} \lambda \leq 12 \).

In the first two cases using Lemma 2 we get the inequality (see Estimation 6 and Estimation 7) obtained from (48) for \( n = 2 \) by replacing \( d_n \) with \( b_n \) which proves, by Proposition 1, the simplicity of the eigenvalues.

Now consider the third case \( \text{Re} \lambda \leq 12 \). Using Lemma 2 we obtain (see Estimation 8 and Estimation 9 in Appendix)

\[
\sum_{k=3}^{\infty} |b_k|^2 < \frac{1}{15}, \quad \frac{|b_3|}{|b_2|} < 0.2131
\]

(63)

The first inequality of (63) with (41) implies that

\[
|b_1|^2 + |b_2|^2 > 1 - \beta,
\]

where \( \beta < \frac{1}{15} \). Instead of (60) using (64) and repeating the proof of (61) we obtain

\[
\frac{b_2}{b_1} - \frac{b_1}{b_2} = \frac{(i + \delta)^2 - 1}{i + \delta} = \frac{2i(i + \delta) + \delta^2}{i + \delta} = \pm 2i + \gamma_1,
\]

(65)

where \( |\gamma_1| < 0.01 \). Now dividing the first equality of (43) by \( b_1 \) and the second equality of (43) for \( k = 2 \) by \( b_2 \) and then subtracting second from the first and using (65) we get

\[
\frac{12}{a} = \pm 2i + \gamma_1 - \frac{b_3}{b_2},
\]

(66)

where by assumption \( \frac{|12|}{a} \geq 2.4 \). Thus, using (63) in (66) we get the contradiction

\[
2.4 \leq \frac{|12|}{a} < 2 + 0.2131 + 0.01 = 2.2231
\]

**Theorem 15** If \( 0 < |a| \leq \frac{4}{3} \), then the all eigenvalues of the operators \( P(a) \) and \( N(a) \) are simple.

**Proof.** By Theorem 13 and Theorem 14 we need to prove that if \( |a| \leq \frac{4}{3} \), then all \( PN(a) \) eigenvalues are simple. Since \( 6 > (1 + \sqrt{2}) \frac{4}{3} \), by Theorem 12, the \( PN(a) \) eigenvalues lying in the ball \( B_n \) for \( n > 1 \) are simple.
Now we prove that the balls $A_0$ and $A_1$ do not contain the multiple $PN(a)$ eigenvalues. Since these balls are contained in $\{ \lambda \in \mathbb{C} : \text{Re}\lambda < 8 \}$ we consider the following cases:

Case 1: $3 \leq \text{Re}\lambda < 8$. Using (42) and Lemma 2 (see Estimation 10 in Appendix) we obtain $|a_1|^2 > 1/2$, which, by Proposition 1, proves the simplicity of the eigenvalues.

Case 2: $\text{Re}\lambda < 3$. Using Lemma 2 we obtain (see Estimations 11 and 12 in Appendix)

$$\sum_{k=2}^{\infty} |a_k|^2 < \frac{1}{58} \frac{|a_2|}{|a_1|} < 0.10301$$ (67)

The first inequality of (67) with (41) implies that

$$|a_0|^2 + |a_1|^2 > 1 - \rho,$$

where $\rho < 1/58$. Instead of (60) using (68) and repeating the proof of (61) we obtain

$$\frac{a_1}{a_0} - \frac{a_0}{a_1} = \pm 2i + \gamma,$$

where $|\gamma| < 0.0006$. Now dividing the first equality of (42) by $a_0$ and the second by $a_1$ and then subtracting second from the first and taking into account (69) we get

$$\frac{4}{a} = \pm 2\sqrt{2}i + \sqrt{2}\gamma - \frac{a_2}{a_1},$$

where by assumption $|a| > 3$. Therefore using (67) we get the contradiction

$$3 \leq \frac{4}{a} < \sqrt{2}(2 + 0.0006) + 0.10301 = 2.9323.$$

### 4 Appendix

**Estimation 1**: Let $9 < \text{Re}\lambda < 16$. By (51) we have

$$|d_1|^2 \leq \frac{|a|^2}{|\lambda - 1|^2} \leq \frac{8\sqrt{6}}{|8|^2} = \frac{1}{6}, \quad |d_3|^2 \leq \frac{|a|^2}{|\lambda - 25|^2} \leq \frac{8\sqrt{6}}{|9|^2} = \frac{32}{243}.$$  

Since $d(\lambda, \{4, 5, \ldots\}) < 33$ using (53) we get

$$\sum_{k=4}^{\infty} |d_k|^2 < \frac{4 |\frac{8}{33}|}{|33|^2} = \frac{128}{3267}.$$  

These inequalities imply that

$$\sum_{k \neq 2} |d_k|^2 < \frac{128}{3267} + \frac{32}{243} + \frac{1}{6} = \frac{19849}{58806} < \frac{1}{2}.$$  

**Estimation 2**: Let $6 < \text{Re}\lambda \leq 9$. By (51)

$$|d_1|^2 \leq \frac{8\sqrt{6}}{|5|^2} = \frac{32}{75}, \quad |d_3|^2 \leq \frac{8\sqrt{6}}{|16|^2} = \frac{1}{24}.$$
Now using the obvious equality $d(\lambda, \{4, 5, \ldots\}) \leq 40$ and (53), we get

$$\sum_{k=3}^{\infty} |d_k|^2 \leq 4 \left(\frac{8 \sqrt{6}}{40}\right)^2 = \frac{2}{75}, \quad \sum_{k \neq 2} |d_k|^2 \leq 32 \frac{8}{75} + \frac{1}{24} + \frac{2}{75} = \frac{99}{200} < \frac{1}{2}. $$

**Estimation 3.** Let $\Re \lambda \leq 6$. By (52) and (49) we have

$$|d| \leq \frac{2 \times 8 \sqrt{6}}{|43| |19|^2}, \quad |d| \leq \frac{2 \times 8 \sqrt{6}}{|75| |43| |19|} \leq \frac{2 \times 8 \sqrt{6}}{|115|^2} \leq \frac{2 \times 8 \sqrt{6}}{|115|^2}. \quad (71)$$

Now using (51) and (53) and taking into account $d(\lambda, \{6, 7, \ldots\}) \leq 115$ we obtain

$$|d_3|^2 \leq \frac{2 \times 8 \sqrt{6}}{|19|^2} = \frac{32}{1083} \& \sum_{k=6}^{\infty} |d_k|^2 \leq \frac{2 \times 8 \sqrt{6}}{|115|^2}. $$

These inequalities imply that

$$\sum_{k=3}^{\infty} |d_k|^2 = \frac{32}{1083} + \left(\frac{2 \times 8 \sqrt{6}}{|43| |19|^2}\right)^2 + \left(\frac{2 \times 8 \sqrt{6}}{|75| |43| |19|}\right)^2 + \frac{4 \times 8 \sqrt{6}}{|115|^2} < 0.03415 $$

**Estimation 4.** Now we estimate $\frac{|d_3|}{|d_2|}$ for $\Re \lambda \leq 6$. Iterating (45) for $k = 3$, we get

$$d_3 = \frac{ad_2 + ad_4}{\lambda - 25} = \frac{ad_2}{\lambda - 25} + \frac{a}{\lambda - 25} \left(\frac{ad_3 + ad_5}{\lambda - 49}\right). \quad (72)$$

Therefore, dividing both sides of (72) by $d_2$ and using (52) we obtain

$$\frac{|d_3|}{|d_2|} \leq \frac{8 \sqrt{6}}{19} + \frac{8 \sqrt{6}}{|43| |19|^2} + \frac{4 \times 8 \sqrt{6}}{|43|^2 |19|^3} + \frac{8 \times 8 \sqrt{6}}{|43|^2 |19|^2} \leq 0.17432 $$

**Estimation 5.** Let $26 \leq \Re \lambda \leq 46$. Using (56) and (58) we obtain

$$|b_1|^2 \leq \frac{|a|^2}{\lambda - 4}|^2 \leq \frac{|5|^2}{|2|^2} = \frac{25}{484}, \quad |b_2|^2 \leq \frac{|a|^2}{\lambda - 16}|^2 \leq \frac{|5|^2}{|10|^2} = \frac{1}{4}. $$

$$|b_4|^2 \leq \frac{|a|^2}{\lambda - 64}|^2 \leq \frac{|5|^2}{|18|^2} = \frac{25}{324}, \quad \sum_{k=5}^{\infty} |b_k|^2 \leq \frac{4 |5|^2}{|54|^2} = \frac{25}{729} $$

Thus

$$\sum_{k \neq 3} |b_k|^2 \leq \frac{25}{484} + \frac{1}{4} + \frac{25}{324} + \frac{25}{729} = \frac{145759}{352836} < \frac{1}{2}. $$
Estimation 6. Let $16 < \Re \lambda \leq 26$. By (55) and (57) we have
\[
|b_1|^2 \leq \frac{|a|^2}{|\lambda - 4|^2} \leq \frac{|5|^2}{12^2} = \frac{25}{144}, \quad |b_3|^2 \leq \frac{|a|^2}{|\lambda - 36|^2} \leq \frac{|5|^2}{10^2} = \frac{1}{4},
\]
\[
\sum_{k=4}^{\infty} |b_k|^2 \leq \frac{4 |5|^2}{38} = \frac{25}{361}, \quad \sum_{k \neq 2} |b_k|^2 \leq \frac{25}{144} + \frac{1}{4} + \frac{25}{51984} < \frac{1}{2}.
\]

Estimation 7. Let $12 < \Re \lambda \leq 16$. By (55) and (57)
\[
|b_1|^2 \leq \frac{|5|^2}{|8|^2} = \frac{25}{64}, \quad |b_3|^2 \leq \frac{|5|^2}{|20|^2} = \frac{1}{16}, \quad |b_4|^2 \leq \frac{|5|^2}{|48|^2} = \frac{25}{2304},
\]
\[
\sum_{k=5}^{\infty} |b_k|^2 \leq \frac{4 |5|^2}{|88|^2} = \frac{25}{1936}, \quad \sum_{k \neq 2} |b_k|^2 \leq \frac{25}{2704} + \frac{1}{16} + \frac{25}{465088} < \frac{1}{15}.
\]

Estimation 8. Let $\Re \lambda \leq 12$. By (55) and (57) we have
\[
|b_4|^2 \leq \frac{|5|^2}{|52|^2} = \frac{25}{2704}, \quad |b_3|^2 \leq \frac{|5|^2}{|24|^2} = \frac{25}{576},
\]
\[
\sum_{k=5}^{\infty} |b_k|^2 \leq \frac{4 |5|^2}{|88|^2} = \frac{25}{1936}, \quad \sum_{k \neq 3} |b_k|^2 \leq \frac{25}{2704} = \frac{25}{1935} = 0.1327 < \frac{1}{15}.
\]

Estimation 9. Here we estimate $\frac{|b_2|}{|b_1|}$ for $\Re \lambda \leq 12$. Iterating (43) for $k = 3$, we get
\[
b_3 = \frac{ab_2 + ab_4}{\lambda - 36} = \frac{ab_2}{\lambda - 36} + \frac{a(36 + ab_5)}{\lambda - 36} = \frac{ab_2}{\lambda - 36} + \frac{a^3 b_3}{(\lambda - 36)^2 (\lambda - 64)} + \frac{a^2 b_5}{(\lambda - 36)(\lambda - 64)}.
\]
Now dividing both sides of (73) by $b_2$ and using (56) we obtain
\[
\frac{|b_3|}{|b_2|} \leq \frac{\sqrt{\frac{2}{52}} a b_1}{\lambda - 36} \leq \frac{\frac{4 |a|^2}{|\lambda|^2}}{\frac{4 |a|^2}{|\lambda - 36|^2}} \leq \frac{16}{81} \leq \frac{1}{16} \leq \frac{1}{16} \leq \frac{1}{36}, \quad \sum_{k=3}^{\infty} |a_k|^2 \leq \frac{4 |a|^2}{|32|^2} = \frac{4}{441}, \quad \sum_{k \neq 1} |a_k|^2 \leq \frac{16}{81} + \frac{1}{36} + \frac{4}{441} < \frac{1}{2}.
\]

Estimation 10. Let $3 \leq \Re \lambda < 8$. By (42), Lemma 2(d) and (55)
\[
|a_0|^2 \leq \frac{|\sqrt{2} a a_1|^2}{|\lambda|^2} \leq \frac{|4|^2}{|3|^2} = \frac{16}{81}, \quad |a_2|^2 \leq \frac{|a|^2}{|\lambda - 16|^2} \leq \frac{|4|^2}{|8|^2} = \frac{1}{36},
\]
\[
\sum_{k=3}^{\infty} |a_k|^2 \leq \frac{4 |a|^2}{|28|^2} = \frac{4}{441}, \quad \sum_{k \neq 1} |a_k|^2 \leq \frac{16}{81} + \frac{1}{36} + \frac{4}{441} < \frac{1}{2}.
\]

Estimation 11. Let $\Re \lambda < 3$. By Lemma 2(d), (55) and (57) we have
\[
|a_2|^2 \leq \frac{|a|^2}{|\lambda - 16|^2} \leq \frac{|3|^2}{|13|^2} = \frac{16}{1521}, \quad \sum_{k=3}^{\infty} |a_k|^2 \leq \frac{4 |a|^2}{|33|^2} = \frac{64}{9801}.
\]
\[ \sum_{k=2}^{\infty} |a_k|^2 \leq \frac{16}{1521} + \frac{64}{9801} < \frac{1}{58}. \]

**Estimation 12.** Here we estimate \( \frac{a_2}{a_1} \) for \( \text{Re} \lambda < 3 \). Iterating (42) for \( k = 2 \), we get

\[
a_2 = \frac{aa_1 + aa_3}{\lambda - 16} = \frac{aa_1}{\lambda - 16} + \frac{a}{\lambda - 16} \left( \frac{aa_2 + aa_4}{\lambda - 36} \right) \]

\[
= \frac{aa_1}{\lambda - 16} + \frac{a^3a_1}{(\lambda - 16)^2(\lambda - 36)} + \frac{a^3a_3}{(\lambda - 16)^2(\lambda - 36)} + \frac{a^2a_4}{(\lambda - 16)(\lambda - 36)}.\]

Now dividing both sides of (74) by \( a_1 \) and using Lemma 2(d), (56) we obtain

\[
\frac{|a_2|}{|a_1|} \leq \frac{4}{13} + \frac{1}{33|13|^2} + \frac{41}{33^2|13|^3} + \frac{8}{61|33|^2|13|^2} < 0.10301
\]

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