Method of quantum characters in equivariant quantization

J. Donin* and A. Mudrov

Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel

Abstract

Let $G$ be a reductive Lie group, $\mathfrak{g}$ its Lie algebra, and $M$ a $G$-manifold. Suppose $\mathcal{A}_h(M)$ is a $\mathcal{U}_h(\mathfrak{g})$-equivariant quantization of the function algebra $\mathcal{A}(M)$ on $M$. We develop a method of building $\mathcal{U}_h(\mathfrak{g})$-equivariant quantization on $G$-orbits in $M$ as quotients of $\mathcal{A}_h(M)$. We are concerned with those quantizations that may be simultaneously represented as subalgebras in $\mathcal{U}_h(\mathfrak{g})$ and quotients of $\mathcal{A}_h(M)$. It turns out that they are in one-to-one correspondence with characters of the algebra $\mathcal{A}_h(M)$. We specialize our approach to the situation $\mathfrak{g} = gl(n, \mathbb{C})$, $M = \text{End}(\mathbb{C}^n)$, and $\mathcal{A}_h(M)$ the so-called reflection equation algebra associated with the representation of $\mathcal{U}_h(\mathfrak{g})$ on $\mathbb{C}^n$. For this particular case, we present in an explicit form all possible quantizations of this type; they cover symmetric and bisymmetric orbits. We build a two-parameter deformation family and obtain, as a limit case, the $\mathcal{U}(\mathfrak{g})$-equivariant quantization of the Kirillov-Kostant-Souriau bracket on symmetric orbits.

1 Introduction.

Let $G$ be a reductive Lie group and $\mathfrak{g}$ its Lie algebra. Let $M$ be a right $G$-manifold and $\mathcal{A}_h(M)$ a quantization of the function algebra $\mathcal{A}(M)$ on $M$. We suppose that the quantization is $\mathcal{U}_h(\mathfrak{g})$-equivariant, i.e. $\mathcal{A}_h(M)$ is a left $\mathcal{U}_h(\mathfrak{g})$-module algebra. We consider the problem of “restricting” $\mathcal{A}_h(M)$ to the $\mathcal{U}_h(\mathfrak{g})$-equivariant quantization on orbits in $M$. This means finding an invariant ideal in $\mathcal{A}_h(M)$, a deformation of the classical ideal specifying an orbit, such that the quotient algebra will be a flat deformation of the function algebra on the orbit. Our principal example is $M = \text{End}(V)$, where $V$ is the underlying linear space of a finite dimensional representation of $G$. We consider $\text{End}(V)$ as a right $G$-manifold with respect to the action by conjugation and study quantizations on $G$-invariant sub-manifolds in $\text{End}(V)$.

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The problem of equivariant quantization on \(\text{End}(V)\), equipped with the structure of the adjoint \(U_h(\mathfrak{g})\)-module, was considered in [D1]. It leads to the study of the so-called reflection equation (RE) algebras, [KSkl]. Torsion factored out, they are flat deformations of polynomial functions on the cone \(\text{End}^2(V)\) of matrices whose tensor square commutes with the split Casimir \(\Omega\), the image of the invariant symmetric element from \(\mathfrak{g}^{\otimes 2}\), [DM2].

It is easy to show that the RE algebra can be restricted to orbits in \(\text{End}^2(V)\) in the case \(\mathfrak{g} = gl(n, \mathbb{C})\) and \(V = \mathbb{C}^n\). The problem is to describe such a restriction explicitly, i.e., to find an appropriate ideal in the RE algebra and prove flatness of the quotient algebra as a module over \(\mathbb{C}[[h]]\). To this end, we develop a quantization method, confining ourselves to those quantizations that may be represented as subalgebras in the function algebra on the quantum group and as quotients of the RE algebra. Being simultaneously a subalgebra and a quotient algebra of flat deformations guarantees flatness of the quantization.

In the classical case, every orbit in \(M\) is realized as a subalgebra in \(A(G)\) and a quotient of \(A(M)\). The algebra \(A(M)\) is a comodule over the Hopf algebra \(A(G)\). A point \(a \in M\) defines the map \(g \rightarrow ag\) from \(G\) onto \(O_a\), the orbit passing through \(a\). It corresponds to a character \(\chi^a\) of the algebra \(A(M)\), which defines the reversed arrow from \(A(O_a)\) to \(A(G)\). Since \(O_a \subset M\), the algebra \(A(O_a)\) is also a quotient of \(A(M)\). The idea of our method is to quantize this picture. Suppose \(A(M)\) is quantized together with the \(A(G)\)-comodule structure, so that \(A_h(M)\) is a comodule over the dual Hopf algebra \(U_h^*(\mathfrak{g})\). Suppose there is a character of the algebra \(A_h(M)\) being a deformation of \(\chi^a\). Then, it defines a homomorphism from \(A_h(M)\) to \(U_h^*(\mathfrak{g})\) and the image of this homomorphism is a deformation of \(A(O_a)\). We prove that, conversely, every equivariant homomorphism \(A_h(M) \rightarrow U_h^*(\mathfrak{g})\) is of this form.

We cannot expect to quantize all orbits in such a way. Indeed, a deformed algebra has a poorer supply of characters than its classical counterpart, therefore not every orbit, in general, fits our scheme. Any character of \(A_h(M)\) corresponds to a point on \(M\) where the Poisson bracket vanishes. An open question is whether every such a point can be quantized to a character of \(A_h(M)\).

In this paper, we apply our method to the standard quantum group \(U_h(gl(n, \mathbb{C}))\) and \(M = \text{End}(\mathbb{C}^n)\). We give full classification of the Poisson brackets on semisimple orbits of \(GL(n, \mathbb{C})\) that are obtained by restriction from the RE Poisson structure on \(\text{End}(\mathbb{C}^n)\). We describe all \(U_h(gl(n, \mathbb{C}))\)-equivariant quantizations that can be obtained within our approach and present them explicitly in terms of the RE algebra generators and relations.

In particular, we build quantizations of symmetric and bisymmetric orbits\(^2\) \(GL(n, \mathbb{C})\). On symmetric orbits, the admissible Poisson brackets form a one-parameter family and we construct quantizations for all of them. On bisymmetric orbits, the admissible Poisson brackets are parameterized by a two dimensional variety. Within our approach, we quantize\(^1\) those are the orbits consisting of matrices with two and three different eigenvalues, respectively.
certain one-parameter sub-families. Also, we build quantizations on nilpotent orbits formed by matrices of zero square.

It is known that a bisymmetric orbit has the structure of a homogeneous fiber bundle over a symmetric orbit as a base. This fact is important for the Penrose transformation theory. We show that our quantization respects that structure and build the quantization of the bundle map.

We extend the quantization on symmetric orbits obtained by the method of characters to a two-parameter $U_h(gl(n, \mathbb{C}))$-equivariant family. It is a restriction of the two-parameter quantization $\mathcal{L}_{h,t}$ on End($\mathbb{C}^n$), which has the algebra $U((gl(n, \mathbb{C}))[t]$ as the limit $h \to 0$. This algebra is a $U((gl(n, \mathbb{C}))$-equivariant quantization of the Poisson-Lie bracket on $gl^*(n, \mathbb{C}) \simeq$ End$(\mathbb{C}^n)$. Taking the limit $h \to 0$ in the two-parameter deformation on symmetric orbits, we obtain explicitly the $U((gl(n, \mathbb{C}))$-equivariant quantizations of the Kirillov-Kostant-Souriau (KKS) bracket on symmetric spaces as quotients of the algebra $U(gl(n, \mathbb{C}))[t]$.

The setup of the article is as follows. The next section contains some basic information essential for our exposition. In particular, we collect some facts from the quantum group theory in Subsection 2.1 and recall definitions of modules and comodules over Hopf algebras in Subsection 2.2. Therein, we introduce the FRT algebra and RE algebras associated with the representation of $U_h(\mathfrak{g})$ on $V$ and recall their basic properties. In Section 3, we formulate the method of restricting the RE algebra to adjoint orbits in End($V$) by means of the RE algebra characters. We specialize this method to the $GL(n, \mathbb{C})$-case in Section 4. We compute the RE Poisson structures on semisimple (co)adjoint orbits of $GL(n, \mathbb{C})$ in Subsection 4.1 and present the classification of the RE algebra characters in Subsection 4.2. On this ground, we build the quantizations of symmetric and bisymmetric orbits in Subsection 4.3. In Subsection 4.4, we show that the constructed quantization respects the structure of homogeneous fiber bundles on bisymmetric orbits. In Subsection 4.5, we construct the two-parameter quantization on symmetric orbits and give the explicit quantization of the KKS bracket on them as a limit case.

2 Preliminaries.

2.1 Quantum group $U_h(\mathfrak{g})$.

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and $r \in \wedge^2 \mathfrak{g}$ a solution to the modified classical Yang-Baxter equation

\begin{equation}
[r, r] = \phi,
\end{equation}

\footnote{The FRT algebra is the quantization of the Drinfeld-Sklyanin bracket on End($V$). It was used by Faddeev, Reshetikhin, and Takhtajan for definition of Hopf algebra duals to quantum groups.}
where $[\cdot, \cdot]$ stands for the Schouten bracket and $\phi$ is an invariant element from $\wedge^3 g$. The universal enveloping algebra $\mathcal{U}(g)$ is a Hopf one, with the coproduct $\Delta_0$, counit $\varepsilon_0$, and antipode $\gamma_0$ defined by

$$\Delta_0(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon_0(X) = 0, \quad \gamma_0(X) = -X, \quad X \in g.$$  

These operations are naturally extended over $\mathcal{U}(g)[[h]]$ as a topological $\mathbb{C}[[h]]$-module. The following theorem is implied by the results of Drinfeld, [Dr2], Etingof and Kazhdan, [EK].

**Theorem 2.1.** There exists an element $F_h \in \mathcal{U}^{\otimes 2}(g)[[h]]$, 

$$F_h = 1 \otimes 1 + \frac{h}{2} r + o(h),$$

such that $\mathcal{U}(g)[[h]]$ becomes a quasitriangular Hopf algebra $\mathcal{U}_h(g)$ with the coproduct $\Delta$, counit $\varepsilon$, and antipode $\gamma$:

$$\Delta(x) = F_h^{-1} \Delta_0(x) F_h, \quad \varepsilon(x) = \varepsilon_0(x), \quad \gamma(x) = u^{-1} \gamma_0(x) u, \quad x \in \mathcal{U}_h(g).$$

The element $u$ is equal to $u = \gamma_0(F_1) F_2$, where $F_1 \otimes F_2 = F_h$ (summation implicit) and the universal R-matrix is given by

$$\mathcal{R}_h = (F_h^{-1})_{21} e^{h \omega} F_h = 1 \otimes 1 + h \left( r + \omega \right) + o(h),$$

where $\omega \in g^{\otimes 2}$ is a symmetric invariant element such that the sum $r + \omega$ satisfies the classical Yang-Baxter equation, [Dr2].

The algebra $\mathcal{U}_h(g)$ is a quantization of $\mathcal{U}(g)$ in the sense that $\mathcal{U}_h(g)/h \mathcal{U}_h(g) = \mathcal{U}(g)$ as a Hopf algebra. The coassociativity of the coproduct $\Delta$ implies the cocycle equation on $F_h$:

$$(\Delta \otimes id)(F_h)(F_h \otimes 1) = \Phi_h^{-1}(id \otimes \Delta)(F_h)(1 \otimes F_h).$$

Here, $\Phi_h$ is an invariant element from $\mathcal{U}_h^{\otimes 3}(g)$; it is called coassociator and satisfies the pentagon identity

$$(id^{\otimes 2} \otimes \Delta)(\Phi_h)(\Delta \otimes id^{\otimes 2})(\Phi_h) = (1 \otimes \Phi_h)(id \otimes \Delta \otimes id)(\Phi_h)(\Phi_h \otimes 1).$$

**Example 2.2 (Standard quantum groups).** Let $\Pi$ be the root system of $g$ and $\Pi^\pm$ the subsets of positive and negative roots. Let $e_{\pm \alpha}, \alpha \in \Pi^+$, be root vectors normalized to $(e_\alpha, e_{-\alpha}) = 1$ with respect to the Killing form. The simplest example of the classical r-matrix is

$$r = \sum_{\alpha \in \Pi^+} e_\alpha \wedge e_{-\alpha}. $$
It is called standard r-matrix and the corresponding quantum group standard or Drinfeld-Jimbo quantization of $\mathcal{U}(\mathfrak{g})$. Other possible r-matices for simple Lie algebras are listed in [BD]. They were explicitly quantized in [ESS].

By $U_h^*(\mathfrak{g})$ we mean the FRT dual to $U_h(\mathfrak{g})$, [FRT]. This is a quantized polynomial algebra on the group $G$; as a linear space, $U_h^*(\mathfrak{g})$ consists of $\text{End}^*(V)$ while $V$ runs over finite dimensional completely reducible representations of $U_h(\mathfrak{g})$.

2.2 FRT and RE algebras.

In this section, we collect some basic facts about the FRT and RE algebras and their symmetries. All $U_h(\mathfrak{g})$-modules are considered to be free over $\mathbb{C}[[h]]$. Tensor products are assumed to be completed in the $h$-adic topology.

Recall that an associative algebra $A$ over $\mathbb{C}[[h]]$ is called a left (right) $U_h(\mathfrak{g})$-module algebra if it is a left (right) module with respect to the action $\triangleright$ ($\triangleleft$) and its multiplication is consistent with the module structure:

$$x \triangleright (ab) = (x^{(1)} \triangleright a)(x^{(2)} \triangleright b), \quad (ab) \triangleleft x = (a \triangleleft x^{(1)})(b \triangleleft x^{(2)}),$$

$$1 \triangleright a = a, \quad a \triangleleft 1 = a,$$

$$x \triangleright 1_A = \varepsilon(x) 1_A, \quad 1_A \triangleleft x = \varepsilon(x) 1_A$$

for any $x \in U_h(\mathfrak{g})$ and $a, b \in A$. We adopt the standard brief Sweedler notation for the coproduct $\Delta(x) = x^{(1)} \otimes x^{(2)}$, where $x$ is an element from a Hopf algebra $\mathcal{H}$.

If $A$ is a left and right module simultaneously and the two actions commute with each other,

$$x_1 \triangleright (a \triangleleft x_2) = (x_1 \triangleright a) \triangleleft x_2, \quad x_1, x_2 \in U_h(\mathfrak{g}), \ a \in A,$$

then it is called bimodule. $A$ is a $U_h(\mathfrak{g})$-bimodule algebra if its bimodule and algebra structures are consistent with the coproduct in $U_h(\mathfrak{g})$ in the sense of (6–8).

A right $U_h^*(\mathfrak{g})$-comodule algebra is an associative algebra $A$ endowed with a homomorphism $\delta: A \rightarrow A \otimes U_h^*(\mathfrak{g})$ obeying the coassociativity constraint

$$(\text{id} \otimes \Delta) \circ \delta = (\delta \otimes \text{id}) \circ \delta$$

and the conditions

$$\delta(1_A) = 1_A \otimes 1, \quad (\text{id} \otimes \varepsilon) \circ \delta = \text{id},$$

where the identity map on the right-hand side of the second equation assumes the isomorphism $A \otimes \mathbb{C}[[h]] \simeq A$. As for the coproduct $\Delta$, we use symbolic notation $\delta(x) = x^{(1)} \otimes x^{(2)}$,.
marking the tensor component belonging to $\mathcal{A}$ with the square brackets. The subscript of the $\mathcal{U}_h^*(\mathfrak{g})$-component is concluded in parentheses. Every right $\mathcal{U}_h^*(\mathfrak{g})$-comodule $\mathcal{A}$ is a left $\mathcal{U}_h(\mathfrak{g})$-module, the action being defined through the pairing $\langle \ldots \rangle$ between $\mathcal{U}_h(\mathfrak{g})$ and $\mathcal{U}_h^*(\mathfrak{g})$:

$$x \triangleright a = a^{(1)}_1 \langle x, a^{(2)}_2 \rangle, \quad x \in \mathcal{U}_h(\mathfrak{g}), \ a \in \mathcal{A}.$$  \hspace{1cm} (12)

If $\mathcal{A}$ is finite dimensional, the converse is also true. Similarly to right $\mathcal{U}_h^*(\mathfrak{g})$-comodule algebras, one can consider left ones. They are also right $\mathcal{U}_h(\mathfrak{g})$-module algebras.

A completely reducible finite dimensional representation $\rho$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ on a linear space $V$ is naturally extended to that of $\mathcal{U}_h(\mathfrak{g})$ on $V[[h]]$. Denote by $\mathcal{M}$ the matrix space $\text{End}(V)$ and fix a basis $e_i$ in $V$. Let $e^j_i \in \mathcal{M}$ be the matrix units acting on $V$ from the right by the rule $e_i^j e_j^k = \delta^i_k e^k_i$, where $\delta^i_k$ is the Kronecker symbol. In terms of the basis $\{e^i_j\}$, the multiplication is expressed by the formula $e^j_i e^l_k = \delta^l_i e^k_j$.

As an associative algebra, $\mathcal{M}$ is a bimodule over itself with respect to the right and left regular representation. The homomorphism $\rho$ equips $\mathcal{M}[[h]]$ with the structure of a $\mathcal{U}_h(\mathfrak{g})$-bimodule. By duality, the space $\mathcal{M}^*[[h]]$ is a $\mathcal{U}_h(\mathfrak{g})$-bimodule as well.

Let $R$ denote the image of the universal R-matrix of $\mathcal{U}_h(\mathfrak{g})$ under the representation $\rho$. We shall also use the matrix $S$, which differs from $R$ by the permutation $P$ on $V \otimes V$,

$$S = PR, \quad P = \sum_{i,j} e^i_j \otimes e^j_i.$$  \hspace{1cm} (13)

**Example 2.3 (FRT algebra).** Let $\{T^i_k\} \subset \mathcal{M}^*$ be the dual basis to $\{e^k_i\}$. The associative algebra $T_R$ is generated by the matrix coefficients $\{T^i_k\} \subset \mathcal{M}^*$ subject to the FRT relations

$$\sum_{\alpha,\beta} S^{\alpha \beta}_{ij} T^m_{\alpha} T^n_{\beta} = \sum_{\alpha,\beta} T^a_{i} T^b_{j} S^{mn}_{\alpha \beta},$$  \hspace{1cm} (14)

or, in the standard compact form,

$$ST_1 T_2 = T_1 T_2 S,$$  \hspace{1cm} (15)

where $T = \sum_{i,j} T^i_j e^j_i$. The matrix elements $T^i_j$ of the representation $\rho$ may be thought of as linear functions on $\mathcal{U}_h(\mathfrak{g})$; they define an algebra homomorphism

$$T_R \rightarrow \mathcal{U}_h^*(\mathfrak{g}).$$  \hspace{1cm} (16)

**Proposition 2.4.** Let $\rho$ be a finite dimensional completely reducible representation of $\mathcal{U}_h(\mathfrak{g})$ on a module $V[[h]]$ and $T_R$ the FRT algebra associated with $\rho$. Then, $T_R$ is a $\mathcal{U}_h(\mathfrak{g})$-bimodule algebra, with the left and right actions extended from $\mathcal{M}^*$:

$$x \triangleright T = T \rho(x), \quad T \triangleright x = \rho(x)T, \quad x \in \mathcal{U}_h(\mathfrak{g}).$$  \hspace{1cm} (17)
It is a bialgebra, with the coproduct and counit being defined as
\[
\Delta(T^i_j) = \sum_{l=1}^n T^i_l \otimes T^l_i, \quad \varepsilon(T^i_j) = \delta^i_j.
\]

Composition of the coproduct with the algebra homomorphism \((\mathfrak{g})\) applied to the (left) right tensor factor makes \(T_R\) a (left) right \(U_h^*(\mathfrak{g})\)-comodule algebra.

Proof. Actions \((\mathfrak{g})\) are extended to the actions on the tensor algebra \(T(\mathcal{M}^*)[[h]]\), and the ideal generated by \((\mathfrak{g})\) turns out to be invariant. Concerning the bialgebra structure, the reader is referred to \([\text{FRT}]\). The structure of a comodule is inherited from the bialgebra one, so it is obviously coassociative. The coaction is an algebra homomorphism, being a composition of two homomorphisms.

Remark that the FRT relations \((\mathfrak{g})\) arose within the quantum inverse scattering method and were used for systematic definition of the quantum group duals in \([\text{FRT}]\).

Example 2.5 (RE algebra). Another algebra of our interest, \(L_R\), is defined as the quotient of the tensor algebra \(T(\mathcal{M}^*)[[h]]\) by the RE relations
\[
\sum_{\alpha, \beta, \mu, \nu} S^{\alpha\beta}_{ij} L^{\mu}_{\beta \mu} L^n_{\nu} = \sum_{\alpha, \beta, \mu, \nu} L_j^{\alpha} S^{\beta \mu}_{\alpha \mu} L^\nu_{\beta \nu},
\]
where \(\{L^i_j\}\) is the basis in \(\mathcal{M}^*\) that is dual to \(\{e^i_j\}\). In the compact form, \((\mathfrak{g})\) reads
\[
SL_2SL_2 = L_2SL_2S,
\]
where \(L\) is the matrix \(\sum_{i,j} L^i_j e^i_j\).

Proposition 2.6. Let \(\rho\) be a finite dimensional completely reducible representation of \(U_h(\mathfrak{g})\) on a module \(V[[h]]\) and \(T^k_i \in U^*_h(\mathfrak{g})\) its matrix coefficients. Let \(L^i_j\) be the generators of the algebra \(L_R\) associated with \(\rho\). Then, \(L_R\) is a left \(U^*_h(\mathfrak{g})\)-module algebra with the action extended from the coadjoint representation in \(\mathcal{M}^*[[h]]\):
\[
x \triangleright L = \rho(\gamma(x(1))) L(\rho(x(2))), \quad x \in U_h(\mathfrak{g}).
\]

It is a right \(U^*_h(\mathfrak{g})\)-comodule algebra with respect to the coaction
\[
\delta(L^i_j) = \sum_{l,k} L^l_k \otimes \gamma(T^k_j)T^i_l,
\]

Proof. Action \((\mathfrak{g})\) is naturally extended to \(T(\mathcal{M}^*)[[h]]\) and it preserves relations \((\mathfrak{g})\). The coassociativity of \((\mathfrak{g})\) is obvious. To prove that \(\delta\) is an algebra homomorphism, one needs to employ commutation relations \((\mathfrak{g})\) and \((\mathfrak{g})\). For details, the reader is referred to \([\text{KS}]\).
A spectral dependent version of the RE appeared first in [Cher]. In the form of (20), it may be found in articles [Skl, AFS] devoted to integrable models. The algebra \( \mathcal{L}_R \) was studied in [KSKl, KS]. Its relation to the braid group of a solid handlebody was pointed out in [K].

Remark that the algebras \( \mathcal{T}_R \) and \( \mathcal{L}_R \) may be defined for any quasitriangular Hopf algebra and its finite dimensional representation. Propositions 2.4 and 2.6 will be also valid.

It was proven in [DM2] that the FRT and RE algebras are twist-equivalent. For a detailed exposition of the twist theory, the reader is referred e.g. to [Mj]. Here we recall that the twist of a Hopf algebra \( \mathcal{H} \) with a cocycle \( \mathcal{F} \) is a Hopf algebra \( \tilde{\mathcal{H}} \) with the same multiplication as in \( \mathcal{H} \) and the coproduct

\[
\tilde{\Delta}(x) = \mathcal{F}^{-1}\Delta(x)\mathcal{F}, \quad x \in \mathcal{H}.
\]  

(23)

To ensure coassociativity of \( \tilde{\Delta} \), the element \( \mathcal{F} \) must obey the constraint

\[
(\Delta \otimes \text{id})(\mathcal{F})\mathcal{F}_{12} = (\text{id} \otimes \Delta)(\mathcal{F})\mathcal{F}_{23}.
\]  

(24)

If \( \mathcal{A} \) is a left \( \mathcal{H} \)-module algebra with the multiplication \( m \), the new associative multiplication

\[
\tilde{m}(a \otimes b) = m(\mathcal{F}_1 \triangleright a \otimes \mathcal{F}_2 \triangleright b), \quad a, b \in \mathcal{A},
\]

(25)

can be introduced on \( \mathcal{A} \). This algebra, \( \tilde{\mathcal{A}} \), is an \( \tilde{\mathcal{H}} \)-module algebra.

For example, if \( \mathcal{H} \) is quasitriangular with the universal R-matrix \( \mathcal{R} \), the coopposite Hopf algebra \( \mathcal{H}^{\text{op}} \) is a twist with \( \mathcal{F} = \mathcal{R}^{-1} \). Another example is the twisted tensor square \( \mathcal{H} \otimes \tilde{\mathcal{H}} \) of a quasitriangular Hopf algebra. This is a twist of the ordinary tensor square \( \mathcal{H} \otimes \mathcal{H} \) by the cocycle \( \mathcal{F} = \mathcal{R}_{23} \in (\mathcal{H} \otimes \mathcal{H}) \otimes (\mathcal{H} \otimes \mathcal{H}) \).

**Theorem 2.7 (DM2).** Let \( \mathcal{T}_R \) and \( \mathcal{L}_R \) be respectively the FRT and RE algebras associated with a finite dimensional representation of \( \mathcal{H} \). Consider \( \mathcal{T}_R \) as an \( \mathcal{H}^{\text{op}} \otimes \mathcal{H} \)-module algebra. Then, there exists a twist from \( \mathcal{H}^{\text{op}} \otimes \mathcal{H} \) to \( \mathcal{H} \otimes \tilde{\mathcal{H}} \) such that the induced transformation (23) converts \( \mathcal{T}_R \) to \( \mathcal{L}_R \).

This twist is performed in two steps. The first one transforms the factor \( \mathcal{H}^{\text{op}} \) to \( \mathcal{H} \) in \( \mathcal{H}^{\text{op}} \otimes \mathcal{H} \). It is carried out via the cocycle \( \mathcal{F}' = \mathcal{R}_{13} \in (\mathcal{H}^{\text{op}} \otimes \mathcal{H}) \otimes (\mathcal{H}^{\text{op}} \otimes \mathcal{H}) \). The second twist from \( \mathcal{H} \otimes \mathcal{H} \) to \( \mathcal{H} \otimes \tilde{\mathcal{H}} \) is via the cocycle \( \mathcal{F}'' = \mathcal{R}_{23} \in (\mathcal{H} \otimes \mathcal{H}) \otimes (\mathcal{H} \otimes \mathcal{H}) \). The composite transformation with the cocycle \( \mathcal{F} = \mathcal{F}'\mathcal{F}'' \) converts multiplication in \( \mathcal{T}_R \) to that in \( \mathcal{L}_R \) according to formula (23).

It follows from Theorem 2.7 that the algebras \( \mathcal{T}_R \) and \( \mathcal{L}_R \) are isomorphic as \( \mathbb{C}[[h]] \)-modules in the case \( \mathcal{H} = \mathcal{U}_h(\mathfrak{g}) \). Another consequence is that \( \mathcal{L}_R \) is a module not only over \( \mathcal{H} \) but over \( \mathcal{H} \otimes \tilde{\mathcal{H}} \) as well. The action of \( \mathcal{H} \) on \( \mathcal{L}_R \) is induced by the Hopf algebra embedding \( \mathcal{H} \to \mathcal{H} \otimes \tilde{\mathcal{H}} \) via the coproduct.
3 Quantum characters and quantization on orbits.

3.1 Invariant monoid \(\mathcal{M}_\Omega\).

Let \(V\) be a complex linear space and \(\rho\) a homomorphism of \(\mathcal{U}(\mathfrak{g})\) into the matrix algebra \(\mathcal{M} = \text{End}(V)\). An element \(\xi \in \mathfrak{g}\) generates the left and right invariant vector fields on \(\mathcal{M}\)

\[(\xi^l \triangleright f)(A) = df\left(A \rho(\xi)\right), \quad (f \triangleright \xi^r)(A) = df\left(\rho(\xi) A\right), \quad A \in \mathcal{M}, \quad f \in \mathcal{A}(\mathcal{M}),\]

defining the left and right actions of the algebra \(\mathcal{U}(\mathfrak{g})\) on functions on \(\mathcal{M}\). Given an element \(\psi \in \mathcal{U}(\mathfrak{g})\), by \(\psi^r\) and \(\psi^l\) we denote, correspondingly, its extensions to the right- and left-invariant differential operators on \(\mathcal{M}\). The left adjoint action of \(\mathcal{U}(\mathfrak{g})\) on \(\mathcal{M}\) is generated by the vector fields

\[\xi^{\text{ad}} = \xi^l - \xi^r, \quad \xi \in \mathfrak{g}.\]  \hspace{1cm} (26)

Let \(\Omega \in \mathcal{M}^\otimes 2\) be the image of the invariant symmetric tensor \(\omega \in \mathfrak{g}^\otimes 2\) participating in construction of \(\mathcal{U}_h(\mathfrak{g})\), cf. formula (3). Introduce the cone of matrices

\[\mathcal{M}_\Omega = \left\{A \in \mathcal{M} | [\Omega, A \otimes A] = 0\right\}.\]  \hspace{1cm} (27)

Evidently, \(\mathcal{M}_\Omega\) is an algebraic variety, it is closed under the matrix multiplication and invariant with respect to the two-sided action of the group \(G\). There are two remarkable Poisson structures on \(\mathcal{M}_\Omega\); they are given by the Drinfeld-Sklyanin bracket

\[r^{l,l} - r^{r,r}\]  \hspace{1cm} (28)

and the RE bracket

\[r^{\text{ad,ad}} + (\omega^{r,l} - \omega^{l,r}).\]  \hspace{1cm} (29)

**Theorem 3.1.**

1. The quotient of \(\mathcal{T}_R\) by torsion is a \(\mathcal{U}_h(\mathfrak{g})^{\text{op}} \otimes \mathcal{U}_h(\mathfrak{g})\)-equivariant quantization of Poisson bracket (28) on the cone \(\mathcal{M}_\Omega\).

2. The quotient of \(\mathcal{L}_R\) by torsion is a \(\mathcal{U}_h(\mathfrak{g})\)-equivariant quantization of Poisson bracket (29) on the cone \(\mathcal{M}_\Omega\).

**Proof.** For the proof of the first statement, the reader is referred to [DS]. The second statement is deduced from the first one using the twist from Theorem 2.7, see [DM2]. \(\square\)

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3It coincides with \(\mathcal{M} = \text{End}(V)\) for \(\mathfrak{g} = \text{sl}(n, \mathbb{C})\) and \(V = \mathbb{C}^n\).
3.2 General formulation of the method.

In the previous section, we considered two examples of equivariant quantization on the space $\mathcal{M}^\Omega$. Depending on a particular choice of symmetry, they were quotients of the algebras $T_R$ and $L_R$ by the ideal of $h$-torsion elements. Further we study $U_h(\mathfrak{g})$-invariant ideals in $L_R$ that are deformations of the classical ideals in the function algebra $A(\mathcal{M}^\Omega)$ specifying the orbits. The problem is to ensure flatness of the quotient algebras. In this section, we formulate a method realizing the quantized orbits simultaneously as quotients and subalgebras of flat deformations, and that guarantees flatness of the quantization. Briefly, we construct quantizations that are quotients of $L_R$ and subalgebras in $A_h(G) = U^*_h(\mathfrak{g})$.

Our quantization method uses analogs of points, which are one-dimensional representations or characters of quantized algebras. Let $M$ be a manifold with a right action of the group $G$. Let $a \in M$ be a point and $\chi^a$ the corresponding character of the function algebra $A(M)$. On the diagram

\[
\begin{array}{ccc}
M \times G & \rightarrow & M \\
\uparrow & & \uparrow \\
\{a\} \times G & \rightarrow & O_a \\
\chi^a \otimes \text{id} & \downarrow \\
\mathbb{C} \otimes A(G) & \leftarrow & A(O_a)
\end{array}
\]

the left square displays embedding of the orbit $O_a$ passing through $a$ into the $G$-space $M$. The induced morphisms of the function algebras are depicted on the right square. Our goal is to quantize this picture.

**Proposition 3.2.** Let $\mathcal{H}$ be a Hopf algebra over $\mathbb{C}[[h]]$ and $A$ a comodule algebra over its Hopf dual $\mathcal{H}^*$. Any character $\chi$ of $A$ defines a homomorphism $\varphi_\chi : A \rightarrow \mathcal{H}^*$ fulfilling the equivariance condition

\[
(\varphi_\chi \otimes \text{id}) \circ \delta = \Delta \circ \varphi_\chi.
\]  

(30)

Conversely, any homomorphism $\varphi : A \rightarrow \mathcal{H}^*$ obeying (30) has the form $\varphi_\chi$, where $\chi$ is a character of $A$.

**Proof.** Let $\chi$ be a character of $A$. We set $\varphi_\chi = (\chi \otimes \text{id}) \circ \delta$ and make use of the isomorphism $\mathbb{C}[[h]] \otimes \mathcal{H}^* \simeq \mathcal{H}^*$. Due to coassociativity (10) of the coaction $\delta$, condition (30) holds. Conversely, if $\varphi$ satisfies (30), then we apply the counit of the Hopf algebra $\mathcal{H}^*$ to the first tensor factor of (30) and obtain $\varphi_\chi$ with $\chi = \varepsilon \circ \varphi$. \qed

Given an associative algebra $A$ over $\mathbb{C}[[h]]$ let $\text{Char}(A)$ denote its set of characters, i.e. homomorphisms $A \rightarrow \mathbb{C}[[h]]$. Any element $\chi \in \text{Char}(A)$ defines an algebra $A_\chi$ closing up
the commutative diagram (the right-most arrow is onto)

\[
\begin{array}{ccc}
\mathcal{A} \otimes \mathcal{H}^* & \xleftarrow{\delta} & \mathcal{A} \\
\chi \otimes \text{id} & \downarrow & . \\
\mathbb{C}[[h]] \otimes \mathcal{H}^* & \xleftarrow{\varphi_{\chi}} & \mathcal{A}_\chi
\end{array}
\]

(31)

We consider \(\mathcal{H}^*\) as the left coregular module over \(\mathcal{H}\). Then, because of (30), \(\varphi_{\chi}\) is an \(\mathcal{H}\)-equivariant homomorphism of algebras, and its image \(\mathcal{A}_\chi\) is an \(\mathcal{H}\)-module algebra.

**Definition 3.3.** Let \(\mathcal{A}\) be a right \(\mathcal{H}^*\)-comodule algebra. Two characters \(\chi_1, \chi_2 \in \text{Char}(\mathcal{A})\) are called gauge-equivalent, \(\chi_1 \sim \chi_2\), if there exists an element \(\eta \in \text{Char}(\mathcal{H}^*)\) such that

\[
\chi_2 = (\chi_1 \otimes \eta) \circ \delta.
\]

(32)

This is an equivalence relation. Indeed, \(\chi_1 \sim \chi_2\) and \(\chi_2 \sim \chi_3\) implies \(\chi_1 \sim \chi_3\), due to coassociativity of the coaction. Also, \(\chi_1 \sim \chi_2 \Rightarrow \chi_2 \sim \chi_1\), since (32) implies \(\chi_1 = (\chi_2 \otimes \eta \circ \gamma) \circ \delta\). Obviously \(\chi_1 \sim \chi_1\) via the counit \(\varepsilon \in \text{Char}(\mathcal{H}^*)\) of the Hopf algebra \(\mathcal{H}^*\). In the classical situation \(\mathcal{A} = \mathcal{A}(M)\) and \(\mathcal{H}^* = \mathcal{A}(G)\), two characters are gauge-equivalent if and only if the corresponding points lie on the same orbit of \(G\).

**Proposition 3.4.** Let \(\mathcal{A}\) be a right \(\mathcal{H}^*\)-comodule algebra and \(\chi_1 \sim \chi_2 \in \text{Char}(\mathcal{A})\). There exist Hopf algebra automorphisms \(f_H: \mathcal{H} \to \mathcal{H}\), \(f_{\mathcal{H}^*}: \mathcal{H}^* \to \mathcal{H}^*\) and an algebra automorphism \(f_A: \mathcal{A} \to \mathcal{A}\) such that the diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\varphi_{\chi_1}} & \mathcal{H}^* \\
\downarrow f_A & \downarrow & \downarrow f_{\mathcal{H}^*} \\
\mathcal{A} & \xrightarrow{\varphi_{\chi_2}} & \mathcal{H}^*
\end{array}
\]

is commutative and \(\mathcal{H}\)-equivariant with respect to the left \(\mathcal{H}\)-actions

\[
\begin{array}{ccc}
\mathcal{H} \otimes \mathcal{A} & \to & \mathcal{A} \\
\downarrow f_H \otimes f_A & & \downarrow f_H \otimes f_{\mathcal{H}^*} \\
\mathcal{H} \otimes \mathcal{A} & \to & \mathcal{H} \otimes \mathcal{H}^* \to \mathcal{H}^*
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{H} \otimes \mathcal{H}^* & \to & \mathcal{H}^* \\
\downarrow f_H \otimes f_{\mathcal{H}^*} & & \downarrow f_H \\
\mathcal{H} \otimes \mathcal{H}^* & \to & \mathcal{H}^*
\end{array}
\]

**Proof.** Let \(\eta\) be the character of \(\mathcal{H}^*\) realizing the equivalence \(\chi_1 \sim \chi_2\) in (32). It may be thought of as a group-like element from \(\mathcal{H}\), i.e. the one whose coproduct is equal to \(\Delta(\eta) = \eta \otimes \eta\). The similarity transformation with such an element is an automorphism of

\footnote{This is obviously true for finite dimensional Hopf algebras. In the quantum group case, \(\mathcal{H}\) is complete in the \(h\)-adic topology. Then \(\mathcal{H}^*\) consists of continuous linear functionals on \(\mathcal{H}\). It is equipped with the weak topology, in which continuous linear functionals on \(\mathcal{H}^*\) form \(\mathcal{H}\).}
the Hopf algebra $\mathcal{H}$. We set

$$f_{\mathcal{H}^*}(x) = \eta(x_{(1)})x_{(2)}\eta(\gamma(x_{(3)})), \quad x \in \mathcal{H}^*, \quad (33)$$

$$f_{\mathcal{H}}(y) = \gamma(\eta)y\eta, \quad y \in \mathcal{H}, \quad (34)$$

$$f_{\mathcal{A}}(a) = a_{[1]}\eta(\gamma(a_{(2)})), \quad a \in \mathcal{A}. \quad (35)$$

A straightforward verification using coassociativity of $\delta$ and $\Delta$ shows that these maps possess the required properties. \hfill \Box

Specifically for the deformation quantization situation, we suppose that $\mathcal{A}_h(M)$ is a quantization of $\mathcal{A}(M)$ and it is a comodule algebra for $\mathcal{A}_h(G) \simeq U_h^*(g)$. The coaction $\delta: \mathcal{A}_h(M) \to \mathcal{A}_h(M) \otimes U_h^*(g)$ is assumed to be a deformation of the classical map $\mathcal{A}(M) \to \mathcal{A}(M) \otimes \mathcal{A}(G)$. This implies that $\mathcal{A}_h(M)$ is an equivariant quantization of $\mathcal{A}(M)$ because every $U_h^*(g)$-comodule is a $U_h(g)$-module via action (12). In connection with Proposition 3.2, there arises the problem of describing the set $\text{Char}(\mathcal{A}_h(M))$.

The following statement is elementary.

**Proposition 3.5.** Let $M$ be a Poisson manifold and $\mathcal{A}_h(M)$ a quantization of the function algebra $\mathcal{A}(M)$. Let $a \in M$ and suppose $\chi_h$ is a character of the algebra $\mathcal{A}_h(M)$ such that $\chi_h(f) = f(a) \mod h$, $f \in \mathcal{A}(M)$. Then, the Poisson bracket vanishes at the point $a$.

**Proof.** By definition, $\chi_h(m_h(f, g)) = \chi_h(f)\chi_h(g)$, for $f, g, \in \mathcal{A}(M)$. Expanding this equality in the deformation parameter and collecting the terms before $h$, we come to the condition

$$m_1(f, g)(a) + \chi_1(fg) = f(a)\chi_1(g) + \chi_1(f)g(a). \quad (36)$$

Here $m_1$ and $\chi_1$ are the infinitesimal terms of the deformed multiplication $m_h$ and character $\chi_h$. Skew-symmetrization of (36) proves the statement. \hfill \Box

Given a Poisson manifold $M$, we denote by $\text{Char}_0(M)$ the subset of points where the Poisson bracket vanishes.

**Question 1.** Let $M$ be a Poisson manifold and $\mathcal{A}_h(M)$ a quantization of its function algebra. Given a point $a \in \text{Char}_0(M)$, does there exist a character $\chi_h \in \text{Char}(\mathcal{A}_h(M))$ such that $\chi_h(f) = f(a) \mod h$ for all $f \in \mathcal{A}(M)$?

### 3.3 Application to the quantum matrix algebra $\mathcal{L}_R$.

We specialize the construction suggested in the previous subsection, to the situation when $\mathcal{H}$ is the quantum group $U_h(g)$, $\mathcal{H}^*$ the quantized function algebra $\mathcal{A}_h(G) \simeq U_h^*(g)$ on the group $G$, and $M$ the matrix cone $M^{12}$ relative to a given representation of $U_h(g)$. The equivariant
quantization \( A_h(M) \) is the RE algebra \( L_R \). It is a \( \mathcal{U}_h^*(\mathfrak{g}) \)-comodule algebra, and we may apply Proposition 3.2 in order to obtain quantization of orbits in \( M^\Omega \) as quotients of \( L_R \). Elements of \( \text{Char}(L_R) \) are defined by matrices \( A \in M[[h]] \) solving the numerical reflection equation

\[
SA_2SA_2 = A_2SA_2S.
\]

**Definition 3.6.** We say that a matrix \( A_h \in M[[h]] \) belongs to the orbit \( O_A \) if \( A_h = A \circ u \) for some invertible element \( u \in \mathcal{U}(\mathfrak{g})[[h]] \).

**Theorem 3.7.** Let \( A_h \in M[[h]] \) be a solution of (37) and \( \chi = \chi^{A_h} \) the corresponding character of the algebra \( L_R \). Suppose the matrix \( A_h \) belongs to \( O_A \subset M^\Omega \). Then, the algebra \( A_\chi \) closing up the commutative diagram (the right-most arrow is onto)

\[
\begin{array}{c}
\mathcal{L}_R \otimes \mathcal{U}_h^*(\mathfrak{g}) & \leftarrow & \mathcal{L}_R \\
id \otimes \chi^{A_h} & \downarrow & \\
\mathbb{C}[[h]] \otimes \mathcal{U}_h^*(\mathfrak{g}) & \leftarrow & A_\chi
\end{array}
\]

is a \( \mathcal{U}_h(\mathfrak{g}) \)-equivariant quantization of the polynomial algebra on \( O_A \). Its embedding into the left coregular \( \mathcal{U}_h(\mathfrak{g}) \)-module \( \mathcal{U}_h^*(\mathfrak{g}) \) is equivariant.

**Proof.** The algebra \( A_\chi \) is simultaneously defined by the commutative diagram (38) as a quotient and a subalgebra of flat \( \mathbb{C}[[h]] \)-modules, therefore it is flat. \( A_\chi \) is invariant under the left action \( x \triangleright T = T\rho(x), x \in \mathcal{U}_h(\mathfrak{g}) \), and we should check that it is isomorphic to \( \mathcal{A}(O_A)[[h]] \) as a \( \mathcal{U}(\mathfrak{g})[[h]] \)-module. Let \( u \) be an element from \( \mathcal{U}_h(\mathfrak{g}) \), such that \( A_h = A \circ u \). The coboundary twist of \( \mathcal{U}_h(\mathfrak{g}) \) with the element \( \Delta(u)(u^{-1} \otimes u^{-1}) \) converts \( A_\chi \) into \( \tilde{A}_\chi \) for which \( A \) is a character. The matrix \( A \) defines an equivariant embedding \( \tilde{A}_\chi \) into the twisted algebra \( \tilde{\mathcal{U}}_h^*(\mathfrak{g}) \), which is generated by the matrix elements of \( \tilde{T} = \rho(u)T\rho(u^{-1}) \). Its image is a subalgebra \( \tilde{A}_\chi \) in \( \tilde{\mathcal{U}}_h^*(\mathfrak{g}) \). Since \( \mathcal{U}_h(\mathfrak{g}) \) itself is a twist of \( \mathcal{U}(\mathfrak{g})[[h]] \), the algebras \( \tilde{A}_\chi \) and \( A_\chi \) are isomorphic as \( \mathcal{U}(\mathfrak{g})[[h]] \)-modules. Obviously, the subalgebra \( \tilde{A}_\chi \) coincides with \( \mathcal{A}(O_A) \) modulo \( h\mathcal{A}(O_A) \).

The gauge-equivalence between characters of \( \mathcal{L}_R \) are realized by means of elements from \( \text{Char}(\mathcal{U}_h^*(\mathfrak{g})) \), which are described by the following proposition.

**Proposition 3.8.** For the Drinfeld-Jimbo quantum group \( \mathcal{U}_h(\mathfrak{g}) \), the set \( \text{Char}(\mathcal{U}_h^*(\mathfrak{g})) \) consists of the elements \( e^\eta \in \mathcal{U}_h(\mathfrak{g}) \), where \( \eta \) belongs to the Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \).

**Proof.** The standard r-matrix (3) is of zero weight, so the subset \( \text{Char}_0(G) \) coincides with the maximal torus corresponding to the Cartan subalgebra. As an associative algebra, \( \mathcal{U}_h^*(\mathfrak{g}) \) is isomorphic to \( \mathcal{U}(\mathfrak{g}^*)[[h]] \), \[\text{Dr1}\]. Its characters are parameterized by the dual space to \( \mathfrak{g}^*/[\mathfrak{g}^*, \mathfrak{g}^*] = \mathfrak{h}^* \). On the other hand, the elements \( e^\eta, \eta \in \mathfrak{h} \), are group-like because all \( \eta \) are primitive with respect to the coproduct in \( \mathcal{U}_h(\mathfrak{g}) \), \[\text{Dr1}\].
In the next section we specialize our consideration to the Drinfeld-Jimbo quantum group \( \mathcal{U}_h(gl(n, \mathbb{C})) \), its representation in \( \text{End}(\mathbb{C}^n) \), and the related RE algebra \( \mathcal{L}_R \).

4 The \( GL(n, \mathbb{C}) \)-case.

From now on, we concentrate on the case \( g = gl(n, \mathbb{C}) \), \( G = GL(n, \mathbb{C}) \) and its representation in \( \mathcal{M} = \text{End}(\mathbb{C}^n) \). Let us fix the Cartan subalgebra \( \mathfrak{h} \subset g \) as the subspace of diagonal matrices in \( \mathcal{M} \). As above, we denote by \( \Pi_g, \Pi^\pm_g \) the sets of all, positive, and negative roots with respect to \( \mathfrak{h} \). It is customary, in the \( gl(n, \mathbb{C}) \)-case under consideration, to take the trace pairing for the invariant scalar product \( (\cdot, \cdot) \) on \( g \). Let \( e_\alpha, e_{-\alpha}, \alpha \in \Pi^+_g \) be the root vectors normalized to \( (e_\alpha, e_{-\alpha}) = 1 \). Put \( h_i \) to be diagonal matrix idempotents \( h_i = e_i^i, i = 1, \ldots, n \); they form an orthonormal basis in \( \mathfrak{h} \). The standard classical r-matrix and the invariant symmetric 2-tensor \( \omega \) for \( gl(n, \mathbb{C}) \) are

\[
    r = \sum_{\alpha \in \Pi^+_g} (e_{-\alpha} \otimes e_\alpha - e_\alpha \otimes e_{-\alpha}),
\]

\[
    \omega = \sum_{i=1}^n h_i \otimes h_i + \sum_{\alpha \in \Pi^+_g} (e_{-\alpha} \otimes e_\alpha + e_\alpha \otimes e_{-\alpha}).
\]

Quantization of these data yields the Yang-Baxter operator

\[
    R = q \sum_{i=1,\ldots,n} e_i^i \otimes e_i^i + \sum_{i,j=1,\ldots,n \atop i \neq j} e_i^j \otimes e_j^i + (q - q^{-1}) \sum_{i,k=1,\ldots,n \atop i < k} e_k^i \otimes e_i^k,
\]

where \( q = e^h \). This is the image of the universal R-matrix \( \mathcal{R} \in \mathcal{U}_h^{\otimes 2}(gl(n, \mathbb{C})) \). The corresponding matrix \( S \), defined by \( [13] \), satisfies the Hecke condition

\[
    S^2 - (q - q^{-1})S = 1 \otimes 1.
\]

We start our quantization programme with computing the relevant Poisson structures.

4.1 RE Poisson structures on adjoint orbits of \( GL(n, \mathbb{C}) \).

Let \( M \) be a right \( G \)-space and \( r_M \) the bivector field on \( M \) obtained from the classical r-matrix \( r \in \wedge^2 \mathfrak{g} \) by the group action. Recall, \([DGS]\), that if \( \mathcal{A}_h(M) \) is a \( \mathcal{U}_h(\mathfrak{g}) \)-equivariant quantization on a right \( G \)-space \( M \), the corresponding Poisson structure has the form

\[
    p = r_M + f,
\]

where \( f \) is a homogeneous function of degree \( 1 \) on \( M \). We use slightly different definition of the quantum group than in \([DGS]\); this results in the different sign before \( r_M \).
where $r_M$ is the bivector field on $M$ generated by the r-matrix via the group action; $f$ is a $G$-invariant bivector field on $M$ whose Schouten bracket is equal to

$$[f, f] = -[r_M, r_M] = -\phi_M.$$ 

Here $\phi \in \wedge^3 \mathfrak{g}$ is an ad-invariant element, see (1). The group $G$ is equipped with the Poisson-Lie structure related to the classical r-matrix $r$, $[Dr_1]$. Admissible Poisson brackets on $M$ are such that the action $M \times G \to M$, where $M \times G$ is the Cartesian product of Poisson manifolds, is a Poisson map. The bivector $f$ defines a skew-symmetric bilinear operation on $\mathcal{A}(M)$ called a $\phi$-bracket. Specifically for the case $M = \text{End}(V)$, where the latter is considered as the right adjoint $G$-module, the invariant part of the RE Poisson bracket is given by the expression within parentheses in (29).

Classification of $\phi$-brackets on semisimple orbits of semisimple Lie groups was done in [DGS] and [Kar]. In this subsection, we compute those obtained by restriction of the RE bracket (29) to semisimple orbits of the group $GL(n, \mathbb{C})$ in $\text{End}(\mathbb{C}^n)$. Semisimple are orbits consisting of diagonalizable matrices. They are characterized by eigenvalues and their multiplicities. As abstract homogeneous spaces, they are specified by ordered sets of multiplicities which fix the stabilizer subgroups $H \subset G$. Eigenvalues specify a Poisson structure on the right coset space $H \backslash G$ by embedding it into the Poisson manifold $\text{End}(\mathbb{C}^n)$. We shall show that the RE Poisson structures on $H \backslash G$ form a variety whose dimension coincides with the rank of the orbit.

**Lemma 4.1.** Let $X, Y$ be linear functions from $\mathcal{M}^*$ identified with $\mathcal{M}$ by the trace pairing. The $\mathcal{U}(\mathfrak{g})$-invariant part of the reflection equation Poisson bracket (24) on $\mathcal{M}^\Omega$ is given by

$$f(X, Y)(A) = (A^2, [X, Y]) = \text{Tr}(A^2[X, Y]), \quad A \in \mathcal{M}^\Omega. \tag{44}$$

**Proof.** We identify $gl(n, \mathbb{C})$ with $\mathcal{M}$, as well as the tangent space at the point $A \in \mathcal{M}$. Right- and left-invariant vector fields on $\mathcal{M}$ take the form $\xi^l \triangleright X = \xi X$, $X \triangleleft \xi^r = X \xi$, $\xi \in \mathfrak{g}$, $X \in \mathcal{M}^* \sim \mathcal{M}$. Calculating the invariant part of the RE bracket (24) with the invariant element $\omega$ from (10), we find

$$((X \triangleleft \omega_1^r)(\omega_2^l \triangleright Y) - (\omega_1^r \triangleright X)(Y \triangleleft \omega_2^r))|_A = (X\Omega_1, A)(\Omega_2 Y, A) - (\Omega_1 X, A)(Y\Omega_2, A).$$

Here, we used the conventional notation with implicit summation, $\omega = \omega_1 \otimes \omega_2$ and $\Omega = \Omega_1 \otimes \Omega_2$. The $\mathcal{U}(gl(n, \mathbb{C}))$-invariant element $\Omega$ coincides with the matrix permutation $P$, see (13). Using the identity $(\Omega, X \otimes Y) = (X, Y)$, which is valid for any $X, Y \in \mathcal{M}$, we come to formula (44). 

Introduce notation $G_m = GL(m)$, $m \in \mathbb{N}$, and put $G_{[n_1, \ldots, n_k]} = G_{n_1} \times \ldots \times G_{n_k}$, a Levi subgroup in $G_n$. To every set $\{n_i\}$ of $k$ positive integers such that $\sum_{i=1}^k n_i = n$
corresponds the right coset space $O_{[n_1,\ldots,n_k]} = G_{[n_1,\ldots,n_k]} \backslash G_n$. This becomes a one-to-one correspondence between classes of isomorphic homogeneous manifolds and sets $\{n_i\}$ provided they are ordered.

An abstract homogeneous space, $O_{[n_1,\ldots,n_k]}$, is realized by orbits in $\text{End}(\mathbb{C}^n)$. Every such realization induces a Poisson structure on it via restriction of the RE bracket (29), and different orbits give different Poisson structures. Consider the direct sum decomposition $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_k}$ and set $P_{n_i} : \mathbb{C}^n \to \mathbb{C}^{n_i}$ be the diagonal projector of rank $n_i$, $i = 1,\ldots,k$. We define the orbit $O_{[n_1,\ldots,n_k;\lambda_1,\ldots,\lambda_k]}$ as that passing through the point $\sum_{i=1}^k \lambda_i P_{n_i}$, where $\lambda_i$ are pairwise distinct complex numbers. This correspondence between orbits and diagonal matrices becomes one-to-one if we require some linear ordering among those parameters $\lambda_i$ that correspond to equal $n_i$. We choose the lexicographic ordering on $\mathbb{C}$: $\lambda_1 > \lambda_2 \iff \text{Re}\lambda_1 > \text{Re}\lambda_2$ or $\text{Re}\lambda_1 = \text{Re}\lambda_2$ and $\text{Im}\lambda_1 > \text{Im}\lambda_2$. To summarize, the abstract homogeneous manifolds with the RE Poisson structures are in one-to-one correspondence with ordered sets of pairs $(n_i, \lambda_i) \in \mathbb{N} \times \mathbb{C}$ such that $n_i > 0$ and $\sum_i n_i = n$. The ordering on $\mathbb{N} \times \mathbb{C}$ is defined as

$$(n_1, \lambda_1) > (n_2, \lambda_2) \iff n_1 > n_2 \text{ or } n_1 = n_2 \text{ and } \lambda_1 > \lambda_2. \quad (45)$$

Our further goal is to compute RE Poisson brackets, which are distinguished by their invariant parts, on the abstract homogeneous manifolds $O_{[n_1,\ldots,n_k]}$. Let $\mathfrak{l} \subset \mathfrak{g} = \text{gl}(n,\mathbb{C})$ be the Lie algebra of the stabilizer $G_{[n_1,\ldots,n_k]}$ and $\Pi_\mathfrak{l} \subset \Pi_\mathfrak{g}$ its root system. The set of quasi-roots $\Pi_{\mathfrak{g}/\mathfrak{l}}$ corresponding to a given Levi subalgebra $\mathfrak{l}$ consists of equivalence classes, $\Pi_{\mathfrak{g}/\mathfrak{l}} = (\Pi_\mathfrak{g} - \Pi_\mathfrak{l}) \mod \mathbb{Z} \Pi_\mathfrak{l}$ [DGS]. For any pair of quasi-roots $\vec{\alpha}$ and $\vec{\beta}$, it is possible to choose such representatives $\alpha$ and $\beta$ that $\alpha + \beta = \vec{\alpha} + \vec{\beta}$. The root vectors $e_\alpha, \alpha \in \vec{\alpha} \in \Pi_{\mathfrak{g}/\mathfrak{l}}$ form a basis of the tangent space $\mathfrak{g}/\mathfrak{l}$ at the origin of $O_{[n_1,\ldots,n_k]}$. When $\mathfrak{g} = \text{gl}(n,\mathbb{C})$, we can consider $\mathfrak{h}^*$ as a subspace in $\mathcal{M}$, using the invariant trace pairing. All roots of $\mathfrak{g}$ are parameterized by pairs of integers $i,j = 1,\ldots,n$, $i \neq j$, and can be represented by the elements $\alpha_{ij} = e_i^j - e_j^i$. By definition, the orbit $O_{[n_1,\ldots,n_k;\lambda_1,\ldots,\lambda_k]}$ passes through the point $\sum_{i=1}^k \lambda_i P_{n_i}$. The corresponding quasi-roots are labeled by the pairs of integers $i,j = 1,\ldots,k-1$, $i \neq j$. Every quasi-root $\vec{\alpha}_{ij}$ is the set of elements $h_i - h_j$, where $h_i$ are diagonal matrix idempotents of rank one such that $h_i P_{n_j} = \delta_{ij} h_i = P_{n_j} h_i$.

**Proposition 4.2.** Consider an ordered set of $k$ positive integers $n_1 \geq n_2 \geq \ldots \geq n_k$. The invariant part of RE Poisson bracket (29) on the homogeneous space $O_{[n_1,\ldots,n_k]}$ is induced by the right-invariant bivector field

$$
\sum_{\vec{\alpha}_{ij} \in \Pi_{\mathfrak{g}/\mathfrak{l}}} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \sum_{\alpha \in \vec{\alpha}_{ij}} (e_\alpha \otimes e_{-\alpha} - e_{-\alpha} \otimes e_\alpha), \quad (46)
$$

where $k$ complex numbers $\{\lambda_i\}$ form a decreasing sequence of pairs $(n_i, \lambda_i)$, $i = 1,\ldots,k$. 

Proof. As was shown in [DGS], an invariant bracket on $O_{[n_1,\ldots,n_k]}$ must have the form
\[
\sum_{\bar{\alpha}_{ij} \in \Pi^+_1} c(\bar{\alpha}_{ij}) \sum_{\alpha \in \bar{\alpha}_{ij}} (e_{-\alpha} \otimes e_{-\alpha} - e_{-\alpha} \otimes e_{\alpha}).
\] (47)

To find the coefficients $c(\bar{\alpha}_{ij})$, let us evaluate (47) on the linear functions $e_{-\alpha}, e_{+\alpha} \in M^* \simeq M$, $\alpha \in \bar{\alpha}_{ij}$, at a diagonal matrix $A$:
\[
c(\bar{\alpha}_{ij})e_{-\alpha}(A)e_{\alpha}([A,e_{-\alpha}]) = c(\bar{\alpha}_{ij})([e_{\alpha}, e_{-\alpha}], A)([e_{-\alpha}, e_{\alpha}], A) = -c(\bar{\alpha}_{ij})(\alpha(A))^2.
\]
On the other hand, formula (44) gives
\[
f(e_{-\alpha}, e_{\alpha})|_A = (A^2, [e_{-\alpha}, e_{\alpha}]) = -\alpha(A^2).
\]
Substituting $A = \sum_{i=1}^k \lambda_i P_{n_i}$, we obtain the coefficients $c(\bar{\alpha}_{ij})$:
\[
c(\bar{\alpha}_{ij}) = \frac{\alpha(A^2)}{(\alpha(A))^2} = \frac{\lambda_i^2 - \lambda_j^2}{(\lambda_i - \lambda_j)^2}.
\]

Remark that the RE Poisson structures on $O_{[n_1,\ldots,n_k]}$ form a $k-1$-dimensional variety because bivector (46) is stable under the dilation transformation $\lambda_i \to \nu \lambda_i$, $i = 1, \ldots, k$, $\nu \neq 0$.

4.2 Characters of the RE algebra relative to the standard quantum group $U_h(gl(n, \mathbb{C}))$.

In this subsection, we formulate the classification theorem for characters of the algebra $L_R$ associated with the representation of the standard $U_h(gl(n, \mathbb{C}))$ in $\text{End}(\mathbb{C}^n)$. To do so, we need the following data.

**Definition 4.3.** An admissible pair $(Y, \sigma)$ consists of a subset $Y \subset I = \{1, \ldots, n\}$ and a decreasing injective map $\sigma: Y \to I$.

Clearly such a map is uniquely determined by its image $\sigma(Y)$. Introduce the subsets $Y_+ = \{i \in Y | i > \sigma(i)\}$ and $Y_- = \{i \in Y | i < \sigma(i)\}$. Let $b_- = \max\{Y_- \cup \sigma(Y_+)\}$ and $b_+ = \min\{Y_+ \cup \sigma(Y_-)\}$. Because $\sigma$ is a decreasing map, one has $b_- < b_+$.

**Theorem 4.4 ([M]).** For the standard $gl(n, \mathbb{C})$ R-matrix (44), the numerical solutions to the RE equation (37) fall into the following two classes.
1. Let \((Y, \sigma)\) be the admissible pair such that \(Y = [1, m] \cup [l + 1, l + m]\), where \(m\) and \(l\) are non-negative integers such that \(l + m \leq n\) and \(m \leq l\); then \(\sigma(i) = l + m + 1 - i\) for \(i \in Y\). Solutions of type A are gauge equivalent to

\[
A(l, m; \lambda, \mu) = \mu \sum_{i=1}^{m} e_i + \lambda \sum_{i=1}^{l} e_i + \sqrt{\lambda \mu} \sum_{i=1}^{m} (e_{\sigma(i)}^i - e_i^\sigma),
\]

where \(\mu, \lambda\) are arbitrary complex numbers. The matrix \(A(l, m; \lambda, \mu)\) has eigenvalues \(\mu, \lambda, 0\) with multiplicities \(m, l, n - m - l\), correspondingly. It is semisimple if and only if \(\lambda \neq \mu\).

2. Let \((Y, \sigma)\) be an admissible pair such that \(\text{card}(Y) \leq \frac{n}{2}\) and \(\sigma(Y) \cap Y = \emptyset\). Let \(l\) be an integer from the semiclosed interval \([b_-, b_+]\) and \(\lambda \in \mathbb{C}\). Solutions to the numerical RE of type B are gauge equivalent to

\[
B(Y, \sigma, l; \lambda) = \lambda \sum_{i=1}^{l} e_i^l + \sum_{i \in Y} e_i^{\sigma(i)}.
\]

The matrix \(B(Y, \sigma, l; \lambda)\) has eigenvalue \(\lambda\) and \(0\) of multiplicities \(l\) and \(n - l\). It is semisimple if and only if \(\lambda \neq 0\), otherwise it is nilpotent of nilpotence degree two.

The layout of solutions to the numerical RE is as follows. Generic RE matrix of type A is obtained by embedding \(A(l, m; \lambda, \mu)|_{l+m=n}\) as the left top block extended with zeros to the entire matrix. The matrix \(A(l, m; \lambda, \mu)|_{l+m=n}\) itself has the form

\[
A(l, m; \lambda, \mu)|_{l+m=n} = \begin{bmatrix}
\star & \star & \star & \star \\
\star & \mu + \lambda & \sqrt{\lambda \mu} & \star \\
\star & \star & \lambda & \star \\
\star & \star & \star & \star \\
\end{bmatrix}.
\]

The \((l-m) \times (l-m)\) square in the middle is located in the center of the matrix and disappears when \(l = m\).

Solutions of type B are decomposed into the direct sum of matrices \(\lambda e_i^l, i \notin Y \cup \sigma(Y), i < l, \lambda e_i^l + e_i^{\sigma(i)}, i \in Y_-,\) and \(\lambda e_i^{\sigma(i)} + e_i^{\sigma(i)}, i \in Y_+\).

4.3 Quantization of symmetric and bisymmetric orbits.

In this subsection, we apply the method of characters to quantization of the RE bracket (29) restricted to adjoint orbits of \(GL(n, \mathbb{C})\) in \(\text{End}(\mathbb{C}^n)\).
Proposition 4.5. The sub-variety $O_{[n_1,\ldots,n_k;\lambda_1,\ldots,\lambda_k]} \subset \mathcal{M}$ is determined by the system of equations

\[(A - \lambda_1) \cdots (A - \lambda_k) = 0,\]  

\[\text{Tr}(A^m) = \sum_{i=1}^{k} n_i \lambda_i^m, \quad m = 0, \ldots, k - 1.\]  

Proof. Reduction to the canonical Jordan form. \[\square\]

Relations (48) and (49) may be generalized to the quantum case, and one can try to use them for building quantization of semisimple orbits. However, the problem is how to ensure the quotient by those relations be a flat module over $\mathbb{C}[[h]]$. The method of quantum character enables us to do that for certain types of orbits.

Relation (48) makes sense in the algebra $L_R$, because the entries of any polynomial in generating matrix $L$ form a $\mathcal{U}_h(gl(n, \mathbb{C}))$-module of the same type as $L$ itself. There is a q-analog of the matrix trace, too. Let $D$ be a matrix $D \in \mathcal{M}[[h]]$ such that the linear functional $A \to \text{Tr}(DA)$ on $\mathcal{M}[[h]]$ is invariant with respect to the action $A \to \rho(\gamma(x_{(1)}))A\rho(x_{(2)})$ of the quantum group $\mathcal{U}_h(gl(n, \mathbb{C}))$. It is unique up to a scalar factor and we take it in the form

\[D = \sum_{i=1}^{n} q^{-2i+2} e_i.\]  

Definition 4.6. Quantum trace of a matrix $A$ with entries in an associative algebra $A$ is the element

\[\text{Tr}_q(A) = \sum_{i=1}^{n} q^{-2i+2} A_i = \text{Tr}(DA) \in A.\]  

When $A = L_R$ and $A = L$, the matrix of generators, this is an invariant element belonging to the center of $L_R$. Moreover, the quantum traces $\text{Tr}_q(L^k)$ of the powers in $L$ are invariant and central as well, $\text{[AFS, KS]}$.

Lemma 4.7. Let $L$ be an RE matrix with coefficients in an associative algebra $A$. Suppose the matrix coefficients $T_n^m \in \mathcal{U}_h^*(\mathfrak{g})$ commute with all $L_i$. Then, the quantum trace is invariant under the similarity transformation with the matrix $T$,

\[\text{Tr}_q(\gamma(T)L) = \text{Tr}_q(L).\]  

For any polynomial $\mathcal{P}$ in one variable,

\[\mathcal{P}(\gamma(T)L) = \gamma(T)\mathcal{P}(L)T.\]
Proof. Formula (53) is an immediate corollary of the equality $\gamma(T) = T^{-1}$. Verification of (52) is less simple and uses relations (15) and (20) in the algebras $U^*_h(g)$ and $L_R$. The prove can be found, e.g., in [KS].

Definition 4.8 (Quantum integers). By quantum integer $\hat{k}$, $k \in \mathbb{Z}$, we mean the quantity $\hat{k} = \frac{1-q^{-2k}}{1-q^{-2}}$.

Obviously, $\hat{k}$ is equal to the quantum trace, $\hat{k} = \sum_{i=1}^{k} q^{-2i+2}$, of the unit endomorphism of the space $\mathbb{C}^k$, for $k \in \mathbb{N}$.

Recall that we consider the set $\mathbb{N} \times \mathbb{C}$ lexicographically ordered, see (45).

Theorem 4.9. Consider two pairs $(l, \lambda)$ and $(m, \mu)$ from $\mathbb{N} \times \mathbb{C}$ and assume $(l, \lambda) \succ (m, \mu)$, $l + m = n$. The quotient of the algebra $L_R$ by the relations

\begin{align*}
(L - \lambda)(L - \mu) &= 0, \quad (54) \\
\text{Tr}_q(L) &= \lambda \hat{\lambda} + \mu \hat{\mu}, \quad (55)
\end{align*}

is the $U_h(gl(n, \mathbb{C}))$-equivariant quantization of the manifold $M = O[l,m]$ with the Poisson bracket

$$r_M + \zeta \sum_{\alpha \in \Delta_{12}} (e_\alpha \otimes e_{-\alpha} - e_{-\alpha} \otimes e_\alpha),$$

where $\zeta = \frac{\lambda + \mu}{\lambda - \mu}$.

Proof. The Poisson structure on $O[l,m]$ is induced by embedding $O[l,m]$ in $\mathcal{M}$ as the orbit $O[l,m;\lambda,\mu]$. The RE matrix $A_h = A(l, m; \lambda, \mu)|_{l+m=n}$ does not depend on the deformation parameter and belongs to $O[l,m;\lambda,\mu]$. Applying Theorem 4.4, we quantize this orbit by the quantum character corresponding to $A_h$. As a subalgebra in $U^*_h(gl(n, \mathbb{C}))$ the algebra $A_h(O[l,m])$ is generated by entries of the RE matrix $\gamma(T)AT$, which fulfills conditions (54) and (55), by Lemma 4.7.

Note that if one of the eigenvalues $\mu, \lambda$ turns to zero, there are several numerical RE matrices giving quantization of the same orbit. We can take, e.g., the matrix $B(Y, \sigma, l; \lambda)$ for $A_h$, with arbitrary set $Y$ such that $\max\{Y_- \cup \sigma(Y_+)\} < l$. It has eigenvalues $\lambda$ and 0 of multiplicities $l$ and $n-l$. For example, one can pick $Y = \emptyset$ and consider the RE matrix $\lambda \sum_{i=1}^{l} e_i^j$. This solution to the numerical RE and the corresponding quantization was built in [DM1].

Theorem 4.10. Consider two pairs $(l, \lambda)$ and $(m, \mu)$ from $\mathbb{N} \times \mathbb{C}$ and assume $(l, \lambda) \succ (m, \mu)$, $n - (l + m) = k > 0$. The quotient of the algebra $L_R$ by the relations

\begin{align*}
L(L - \lambda)(L - \mu) &= 0, \quad (56) \\
\text{Tr}_q(L) &= \lambda \hat{\lambda} + \mu \hat{\mu}, \quad (57) \\
\text{Tr}_q(L^2) &= (\lambda + \mu)(\lambda \hat{\lambda} + \mu \hat{\mu}) - \lambda \mu \hat{l} + \hat{m} \quad (58)
\end{align*}
is the $\mathcal{U}_h(gl(n, \mathbb{C}))$-equivariant quantization of the following manifolds:

1. $M = O_{[l,m,k]}$ with the Poisson bracket

\[ r_M + \zeta \sum_{\alpha \in \tilde{\alpha}_{12}} (e_{\alpha} \otimes e_{-\alpha} - e_{-\alpha} \otimes e_{\alpha}) + \sum_{\alpha \in \tilde{\alpha}_{13}} (e_{\alpha} \otimes e_{-\alpha} - e_{-\alpha} \otimes e_{\alpha}), \]

if $(l, \lambda) > (m, \mu) > (k, 0)$, 

2. $M = O_{[l,k,m]}$ with the Poisson bracket

\[ r_M + \zeta \sum_{\alpha \in \tilde{\alpha}_{13}} (e_{\alpha} \otimes e_{-\alpha} - e_{-\alpha} \otimes e_{\alpha}) + \sum_{\alpha \in \tilde{\alpha}_{12} \cup \tilde{\alpha}_{32}} (e_{\alpha} \otimes e_{-\alpha} - e_{-\alpha} \otimes e_{\alpha}), \]

if $(l, \lambda) > (k, 0) > (m, \mu)$, 

3. $M = O_{[k,l,m]}$ with the Poisson bracket

\[ r_M + \zeta \sum_{\alpha \in \tilde{\alpha}_{23}} (e_{\alpha} \otimes e_{-\alpha} - e_{-\alpha} \otimes e_{\alpha}) + \sum_{\alpha \in \tilde{\alpha}_{21} \cup \tilde{\alpha}_{31}} (e_{\alpha} \otimes e_{-\alpha} - e_{-\alpha} \otimes e_{\alpha}), \]

if $(k, 0) > (l, \lambda) > (m, \mu)$,

where $\zeta = \frac{\lambda + \mu}{\lambda - \mu}$.

Proof. As Poisson manifolds, all the three possibilities are realized by the bisymmetric orbit passing through the point $A_h = A(l, m; \lambda, \mu)$ (cf. Theorem 4.4), which satisfies the RE equation and defines a character of the RE algebra. By Theorem 4.9, the quantization of the orbit with this character is the quotient of the RE algebra and the subalgebra in $\mathcal{U}_h^*(gl(n, \mathbb{C}))$ generated by entries of the RE matrix $\gamma(T)A_hT$. Since the matrix $A_h$ satisfies conditions (56–58), so does the matrix $\gamma(T)A_hT$, by Lemma 4.7.

Remark 4.11. There is a one-parameter family of RE Poisson structures on symmetric orbits, as follows from Proposition 4.12, and their quantization is described by Theorem 4.9. On bisymmetric orbits, the RE Poisson brackets form a two-parameter family. Theorem 4.10 provides quantization for only special one-parameter sub-families. This is a limitation of the method of characters, which gives those and only those quantizations that can be represented as subalgebras in $\mathcal{U}_h^*(g)$ and quotients of $\mathcal{L}_R$.

Remark 4.12. Theorem 4.4 provides two classes of non-semisimple numerical RE matrices obtained by the limits $A(l, m; \lambda, \mu)|_{\lambda \to \mu}$ and $B(Y, \sigma, l; \lambda)|_{\lambda \to 0}$. They belong to orbits that are limits of the semisimple ones. The orbit passing through $B(Y, \sigma, l; \lambda)|_{\lambda = 0}$ is nilpotent. Using Theorem 3.4, one can quantize all nilpotent orbits of nilpotence degree two.

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4.4 Quantizing fiber bundle $O_{[l,m,k]} \to O_{[l+m,k]}$.

In this subsection we consider the following problem. The group embedding $G_{[l,m,k]} \to G_{[l+m,k]}$ defines a map of the right coset spaces,

$$O_{[l,m,k]} \to O_{[l+m,k]}, \quad (59)$$

which is a bundle with the fiber $O_{[l,m]} = G_{[l,m,k]} \setminus G_{[l+m,k]} \simeq G_{[l,m]} \setminus G_{[l+m]}$. Map (59) is equivariant with respect to the action of the group $G_{l+m+k}$ and the bundle is homogeneous. Suppose the total space and the base of the bundle are Poisson manifolds and map (59) is Poisson. The problem is to quantize the diagram

$$\mathcal{A}(O_{[l,m,k]}) \leftarrow \mathcal{A}(O_{[l+m,k]}), \quad (60)$$
i.e., to build $U_h(gl(l + m + k))$-equivariant quantizations of the function algebras and an equivariant monomorphism

$$\mathcal{A}_h(O_{[l,m,k]}) \leftarrow \mathcal{A}_h(O_{[l+m,k]}). \quad (61)$$

The spaces $O_{[l,m,k]}$ and $O_{[l+m,k]}$ can be realized as the orbits $O_{[l,m;k,\mu,\lambda,0]}$ and $O_{[l+m,k;\lambda,\mu,0]}$ (here we do not assume ordering (15)) with the induced RE Poisson structures. We will show that their quantizations built in the previous subsection admit an equivariant morphism, a quantization of (61). This will imply that the projection $O_{[l,m;k;\lambda,\mu,0]} \to O_{[l+m,k;\lambda,\mu,0]}$ is a Poisson map, because of the flatness of the quantizations.

**Remark 4.13.** The fiber over the origin $A(l, m; \lambda, \mu) \in O_{[l,m;k,\mu,\lambda,0]}$ is realized as a Poisson manifold $O_{[l,m;\lambda,\mu]}$. It can be shown that its embedding to $O_{[l,m;k;\lambda,\mu,0]}$ is a Poisson map as well and this map can be lifted to a homomorphism of quantized algebras that is equivariant with respect to the quantum group embedding $U_h(gl(l + m)) \to U_h(gl(l + m + k))$. The proof of this statement is rather straightforward, and we do not concentrate on this subject here.

Before formulating the main result of this subsection, let us prove an algebraic statement. Let $P(x)$ be a polynomial in one variable with coefficients in a commutative ring $\mathbb{K}$. Suppose $P(0) = 0$. Fix two scalars $\alpha, \beta \in \mathbb{K}$ and consider an associative unital algebra $\mathcal{A}(e, s)$ over $\mathbb{K}$ generated by the elements $\{e, s\}$ subject to relations

$$eses = sese \quad ("reflection equation"), \quad s^2 - \beta s = 1 \quad ("Hecke condition"). \quad (62)$$

**Lemma 4.14.** The correspondence $s \to s, e \to P(e)$ gives a homomorphism of the algebra $\mathcal{A}(e, s)$ to the quotient algebra $\mathcal{A}(e, s)/(eP(e) - \alpha e)$. This homomorphism is factored through the ideal $(e^2 - \alpha e)$.
Proof. First let us note that the last statement is an immediate corollary of the condition \( P(0) = 0 \). We should verify that, modulo the ideal \((eP(e) - \alpha e)\), the following relation holds true in the algebra \( A(e, s) \):

\[
P(e)sP(e)s = sP(e)sP(e).
\] (63)

We will prove a stronger assertion; namely, for any \( m = 1, 2, \ldots \), the identity

\[
P(e)se^m s = se^m sP(e)
\] (64)

is valid modulo the ideal \((eP(e) - \alpha e)\). This will imply (63), because \( P(0) = 0 \) by the hypothesis of the lemma. We assume \( \beta \neq 0 \) since otherwise \( se^m s \) can be replaced by \((ses)^m\), and the prove becomes immediate. For \( m = 1 \) this is a consequence of the "reflection equation" relation (62). Suppose (64) is proven for some integer \( m = l \geq 1 \). Using the "Hecke condition" (62), we rewrite (64) for \( m = l + 1 \) as

\[
P(e)se^l(s^2 - \beta s)es = se^l(s^2 - \beta s)esP(e).
\]

The terms without \( \beta \) compensate each other, by the induction assumption. The problem reduces to checking the equality

\[
P(e)se^l ses = se^l sesP(e).
\]

Employing the induction assumption, rewrite the left-hand side as

\[
P(e)se^l ses = se^{l-1} sesP(e) = se^{l-1} sesP(e) = \alpha se^{l-1} ses = \alpha se^l ses.
\]

The right hand side can be transformed as

\[
se^l sesP(e) = se^{l-1} esesP(e) = se^{l-1} esesP(e) = \alpha se^{l-1} eses = \alpha se^l ses.
\]

Here, we used the "reflection equation" (62).

**Theorem 4.15.** Let \( E \) and \( B \) be the RE matrices whose entries generate the algebras \( A_h(O[l, m, k; \lambda, \mu, 0]) \) and \( A_h(O[l+m, k; -\lambda\mu, 0]) \), respectively. The matrix correspondence \( \pi(B) = E^2 - (\lambda + \mu)E \) is extended to the \( U_h(\mathfrak{gl}(l + m + k)) \)-equivariant algebra morphism

\[
A_h(O[l, m, k; \lambda, \mu, 0]) \leftarrow A_h(O[l+m, k; -\lambda\mu, 0]).
\] (65)

**Proof.** The matrix \( \pi(B) \) satisfies the equations \( \pi(B)(\pi(B) + \lambda\mu) = 0 \) and

\[
\text{Tr}_q(\pi(B)) = (\lambda + \mu)(\lambda\tilde{m} + \mu\tilde{m}) - \lambda\mu l + m - (\lambda + \mu)(\lambda\tilde{m} + \mu\tilde{m}) = -\lambda\mu l + m.
\]

Taking into account Theorem 4.13, it remains to show that \( \pi(B) \) is an RE matrix. The matrix \( E \) fulfills polynomial relation (56), and the braid matrix \( S \) matrix satisfies the Hecke condition (62). It remains to apply Lemma 4.14, setting \( e = E_2, s = S, \ P = \pi, \ \alpha = -\lambda\mu, \ \beta = q - q^{-1} \).
4.5 Quantizing the Kirillov-Kostant-Souriau bracket on symmetric orbits.

It is known that there is a two-parameter quantization $L_{h,t}$ on $\text{End}(\mathbb{C}^n)$ that is equivariant with respect to the adjoint action of $U_h(gl(n, \mathbb{C}))$ \cite{D}. The corresponding Poisson structure is obtained from (29) by adding the Poisson-Lie bracket with arbitrary overall factor $t$. The algebra $L_{h,t}$ can be obtained from $L_R$ by the substitution $L = E + \frac{t}{1-q^{-2}}$, $q = e^h$, of the matrix of generators. Relations (20) go over into

$$SE_2S E_2 - E_2S E_2 S = q t \ [E_2, S].$$

(66)

This is true for any matrix $S$ satisfying the Hecke condition (42). In the limit $h \to 0$, the matrix $S$ tends to the permutation operator $P$ on $\mathbb{C}^n \otimes \mathbb{C}^n$ and relations (66) turns into those of the classical universal enveloping algebra $U(gl(n, \mathbb{C})) [t]$. Indeed, the substitution $P \to S$ in (66) gives explicitly

$$E^i_j E^m_n - E^m_n E^i_j = t (\delta^m_n E^i_j - \delta^i_n E^m_j).$$

(67)

The algebra $U(gl(n, \mathbb{C})) [t]$ is a quantization of the Poisson-Lie bracket on $\text{End}^*(\mathbb{C}^n)$ identified with $gl(n, \mathbb{C})$ by the invariant trace pairing. Its restriction to orbits is the Kirillov-Kostant-Souriau bracket.

**Theorem 4.16.** Let $\mu_1$, $\mu_2$ be two distinct complex numbers and $n_1$, $n_2$ two positive integers such that $n_1 + n_2 = n$. Let $\{E^i_j\}_{i,j=1}^n$ be the set of generators of the algebra $L_{h,t}$ subject to relations (66). The quotient of $L_{h,t}$ by the relations

$$(E - \mu_1)(E - \mu_2) = 0,$$

(68)

$$\text{Tr}_q(E) = \hat{n}_1 \mu_1 + \hat{n}_2 \mu_2 + t \hat{n}_1 \hat{n}_2.$$  

(69)

is a two-parameter quantization on the symmetric orbit $O_{[n_1, n_2; \mu_1, \mu_2]}$. It is equivariant with respect to the adjoint action of $U_h(gl(n, \mathbb{C}))$.

**Proof.** Consider the quantization of $O_{[l, m; \lambda, \mu]}$ as a quotient of the RE algebra $L_R$, along the line of Theorem 4.9. The substitution

$$L = E + \frac{t}{1-q^{-2}}, \quad \lambda = \mu_1 + \frac{t}{1-q^{-2}}, \quad \mu = \mu_2 + \frac{t}{1-q^{-2}}, \quad l = n_1, \quad m = n_2$$

transforms relations (20) into (66) while (54–55) into (68–69). \hfill \Box

**Corollary 4.17.** Let $\mu_1$, $\mu_2$ be two distinct complex numbers and $n_1$, $n_2$ two positive integers such that $n_1 + n_2 = n$. Let $\{E^i_j\}_{i,j=1}^n$ be the set of generators of the algebra $U(gl(n, \mathbb{C})) [t]$
subject to relations (67). The quotient of \( \mathcal{U}(\mathfrak{gl}(n, \mathbb{C}))[t] \) with respect to the ideal generated by

\[
(E - \mu_1)(E - \mu_2) = 0,
\]
\[
\text{Tr}(E) = n_1\mu_1 + n_2\mu_2 + tn_1n_2,
\]

is the quantization of the KKS bracket on the symmetric orbit \( O_{[n_1, n_2; \mu_1, \mu_2]} \). It is equivariant with respect to the adjoint action of \( \mathcal{U}(\mathfrak{gl}(n, \mathbb{C})) \).

Proof. Taking the limit \( h \to 0 \) in the two-parameter quantization of Theorem 4.16.

\[\square\]

Remark 4.18. Theorem 4.16 gives a two-parameter generalization of the quantum sphere, \[\text{GS}\], to symmetric orbits. The explicit description of the quantized KKS bracket in Corollary 4.17 is an especially interesting result that could not be otherwise obtained than extending deformation to the q-domain. This is a remarkable application of the quantum group theory.

Example 4.19 (Complex sphere). Now we illustrate Corollary 4.17 on \( \mathcal{U}(\mathfrak{gl}(2, \mathbb{C})) \)-equivariant quantization of the complex sphere \( O_{[\mu_1, \mu_2; 1, 1]} \subset \text{End}(2, \mathbb{C}) \). Our goal is to demonstrate on this simple example that the system of conditions (67), (70), and (71) is self-consistent. It is known that \( O_{[\mu_1, \mu_2; 1, 1]} \) is specified, as a maximal orbit, by values of two invariant functions, the traces of a matrix and its square. We shall show that matrix equation (70) boils down to a condition on the second Casimir of \( \mathcal{U}(\mathfrak{gl}(2, \mathbb{C})) \) only when the first Casimir is fixed as in (71).

Consider the \( 2 \times 2 \)-matrix \( E = ||E_j^i|| \) of generators of \( \mathcal{U}((2, \mathbb{C}))[t] \) obeying the commutation relations

\[
\begin{align*}
[E_1^1, E_2^1] &= tE_2^1, & [E_1^2, E_1^1] &= -tE_1^2, \\
[E_2^2, E_1^2] &= tE_2^2, & [E_2^1, E_2^1] &= -tE_1^1, \\
[E_1^1, E_2^2] &= 0, & [E_1^2, E_1^2] &= t(E_1^1 - E_2^2),
\end{align*}
\]

which are obtained by specialization of system (57) to the \( gl(2, \mathbb{C}) \)-case. Put \( n_1 = n_2 = 1, \sigma_1 = \mu_1 + \mu_2, \sigma_2 = \mu_1\mu_2 \) in (70–71) and write (70) out explicitly:

\[
(E_1^1)^2 + E_1^2E_2^1 - \sigma_1E_1^1 + \sigma_2 = 0,
\]
\[
(E_2^2)^2 + E_2^1E_1^2 - \sigma_1E_2^2 + \sigma_2 = 0,
\]
\[
E_1^1E_1^1 + E_2^2E_2^1 - \sigma_1E_1^1 = 0,
\]
\[
E_1^1E_2^2 + E_2^1E_2^1 - \sigma_1E_1^2 = 0.
\]
Equations (73–76) are equivalent to the system

\begin{align*}
(E_1^1)^2 + (E_2^2)^2 &+ E_2^1 E_1^1 + E_2^2 E_1^2 - \sigma_1 (E_1^1 + E_2^2) + 2\sigma_2 = 0, \quad (77) \\
(E_1^1)^2 - (E_2^2)^2 - t(E_1^1 - E_2^2) - \sigma_1 (E_1^1 - E_2^2) &= 0, \quad (78) \\
E_1^1 E_2^1 + E_2^2 E_1^1 - (\sigma_1 + t) E_2^1 &= 0, \quad (79) \\
E_1^1 E_2^2 + E_2^2 E_1^2 - (\sigma_1 + t) E_1^2 &= 0. \quad (80)
\end{align*}

To obtain it, we pulled the diagonal generators to the left in (75–76) and took the sum and difference of (73–74) using the commutation relations (72).

Equations (78–80) are satisfied modulo the condition

\[ E_1^1 + E_2^2 - (\sigma_1 + t) = 0. \tag{81} \]

which is the specialization of (71) for \( O_{[\mu_1, \mu_2; 1, 1]} \). Equations (77) and (81) can be rewritten in terms of the elements \( \text{Tr}(E) = \sum_i E_i^i, \text{Tr}(E^2) = \sum_{i,j} E_j^j E_i^i \) that generate the center of \( \mathcal{U}(gl(2, \mathbb{C})) \). The resulting relations for the quantized orbit \( O_{[\mu_1, \mu_2; 1, 1]} \) are

\begin{align*}
\text{Tr}(E^2) &= \sigma_1 (\sigma_1 + t) - 2\sigma_2, \quad (82) \\
\text{Tr}(E) &= (\sigma_1 + t). \quad (83)
\end{align*}

Together with (72), this is a \( \mathcal{U}((2, \mathbb{C})) \)-equivariant quantization of the KKS bracket on the two-dimensional complex sphere.

5 Conclusion.

The method of quantum characters formulated in this paper is designed for building \( \mathcal{U}_h(g) \)-equivariant quantizations on a \( G \)-manifold \( M \) that are representable as subalgebras in \( \mathcal{U}_h^*(g) \) and quotients of \( \mathcal{A}_h(M) \). Despite its simplicity, it allows to obtain new and interesting results, for example, the two-parameter quantization on semisimple orbits. Analyzing the quantizations built within the present approach, one may come to the following conclusion. The two-parameter quantization on a semisimple orbit of \( GL(n, \mathbb{C}) \) may be sought for in the form of a matrix polynomial equation on the generators \( E_i^j \in L_{h,t} \) with additional conditions on the quantum traces \( \text{Tr}_q(E^k), k \in \mathbb{N} \). This conjecture turns out to be true. The proof is based on a different technique than that used in the present paper. It is the subject of our forthcoming publication, \( [DM3] \), as well as the explicit equations defining quantized semisimple orbits of \( GL(n, \mathbb{C}) \), including the special case of the KKS bracket.

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e-mail: donin@macs.biu.ac.il
e-mail: mudrova@macs.biu.ac.il;