COMPLEX MOTIVIC $kq$-RESOLUTIONS

DOMINIC LEON CULVER AND J.D. QUIGLEY

Abstract

We analyze the $kq$-based motivic Adams spectral sequence over the complex numbers, where $kq$ is the very effective cover of Hermitian K-theory defined over $\mathbb{C}$ by Isaksen-Shkembi and over general base fields by Ananyevskiy-Röndigs-Østvær. We calculate the ring of cooperations of $kq$ modulo $v_1$-torsion, completely calculate the 0- and 1-lines of the $kq$-resolutions, completely determine the $v_1$-periodic complex motivic stable stems, and recover Andrews and Miller’s computation of the $\eta$-periodic complex motivic stable stems. As an application, we propose a motivic Telescope Conjecture and outline a program for proving the conjecture in two cases using our calculations. We also propose a model for the complex motivic stable orthogonal J-homomorphism and conjecture its location in the $kq$-resolution.

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1. Introduction

1.1. The bo-resolution. Let $E$ be a multiplicative cohomology theory. The $E$-based Adams spectral sequence is one of the most powerful tools for accessing the classical stable homotopy groups of spheres. When $E$ is flat, i.e. when $E_\ast E$ is flat as a left module over $E_\ast$, one can identify the $E_2$-page of this spectral sequence with Ext-groups in the category of $E_\ast E$-comodules

$$E_2 = \text{Ext}^{s,t}_{E, E}(E_\ast, E_\ast) \Rightarrow \pi_{t-s}(S^0_E),$$

where $(S^0)^{\wedge}_E$ is the $E$-nilpotent completion of the sphere spectrum $S^0$. When $E = HF_p$ or $E = BP$, the abutment can be identified with the $p$-local stable homotopy groups of spheres.

When $E = HF_2$, these Ext-groups are accessible in a large range by machine computation. We refer the reader to [34] for a chart depicting the range with
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t − s ≤ 70. Several patterns are visible in the $E_2$-page, but perhaps the most obvious is the $v_1$-periodic region along the line $s = \frac{1}{2}(t−s)$. It turns out that many of these periodic families survive to the $E_\infty$-page and detect nontrivial elements. The surviving families contain the image of $J$ which provides the motivating example for chromatic homotopy theory.

The computation of the image of $J$ was carried out by Adams in [1], up to a factor of two. This factor of two was resolved in the work of Quillen [50], Sullivan [55], and Friedlander [20]. The first complete calculation of the image of $J$ using exclusively homotopy theoretic-methods was given by Mahowald and Davis-Mahowald [39] [15]. It is the last approach which we focus on in this paper.

There are two issues which arise if one attempts to calculate the image of $J$ using the $HF_2$-based Adams spectral sequence:

(1) One must show that none of the relevant classes are targets of differentials. The classes can have arbitrarily high Adams filtration, so one must exclude arbitrarily long differentials.

(2) One must calculate the precise order of the 2-torsion. This is especially difficult when one has $t − s \equiv 3, 7 \mod 8$; these are the cases where the denominators of the Bernoulli numbers famously appear.

Roughly speaking, both of these issues arise from one central problem with the $HF_2$-based Adams spectral sequence: the image of $J$ is detected in arbitrarily high Adams filtration.

To circumvent this problem (and several other problems to be discussed below), Mahowald studied the $bo$-based Adams spectral sequence, or the $bo$-resolution. This resolves the issues arising from classes lying in arbitrarily high $HF_2$-Adams filtration, but gives rise to two new complications:

(1) One must show that all of the desired classes are permanent cycles.

(2) Since $bo$ is not flat, one cannot compute the $E_2$-page of the $bo$-based Adams spectral sequence using the cobar complex as one would for a flat cohomology theory.

The first issue is resolved by comparing the $bo$-resolution to the $X$-resolution, where $X$ is a certain Thom spectrum. We will not need the details of this argument in this paper, so we will not attempt to discuss them further. The second issue is resolved using the splitting

$$bo \wedge bo \simeq \bigvee_{i \geq 0} \Sigma^{4i}bo \wedge HZ_i^{cl}$$

where $HZ_i^{cl}$ is the $i$-th (classical) integral Brown-Gitler spectrum. This splitting is proven by observing that $H^*(bo) \cong A/A(1)$ decomposes as a direct sum of integral Brown-Gitler modules. One then identifies the $E_1$-page of the $bo$-resolution (modulo $v_1$-torsion) by identifying $\text{Ext}_{A(1)}(HZ_i^{cl}, i \geq 0$, with certain Adams covers of $bo$ and $bsp$.

The analysis outlined above allowed Mahowald and Davis-Mahowald to prove the following, which follows from [39 Thm. 1.1]:

**Theorem** (Mahowald, Davis-Mahowald).

1. The image of $J$ is detected in the 0- and 1-lines of the $bo$-resolution.

2. All $v_1$-periodic classes in $\pi_*(S^0)$ are detected in the 0- and 1-lines of the $bo$-resolution.
(3) The 2-torsion in the 4k − 1-stem of the image of J is cyclic of order $2^{\rho(k)}$, where $\rho(k)$ is the 2-adic valuation of $8k$.

This theorem, along with the technology of bo-resolutions, has been used to prove (or reprove) several important results from classical homotopy theory:

1. Davis and Mahowald used this and related results to precisely compute the image of J without appealing to the Adams Conjecture in [15].
2. The bo-resolution of the mod two Moore spectrum was used to verify the Telescope Conjecture at height one for $p = 2$ in [39] [13].
3. The bo-resolution for $Y = S^0/2 \wedge S^0/\eta$ to compute $v_1$-periodic unstable homotopy groups [40].

With these applications in mind, we are motivated to develop a motivic analog of the bo-resolution.

1.2. The $kq$-resolution. The analog of bo in the motivic setting is the very effective cover of Hermitian K-theory, denoted $kq$, which was defined over $\text{Spec}(C)$ by Isaksen-Shkembi in [35] and over more general base schemes by Ananyevskiy-Röndigs-Østvær in [3]. In this paper, we define and study the $kq$-resolution over $\text{Spec}(C)$, which takes the form

$$E_1^{n,t,u} = \pi^{t,u}(kq \wedge n \wedge kq) \Rightarrow \pi_{t-n,u}(S^{0,0}),$$

where $kq$ is the cofiber of the unit map $S^{0,0} \to kq$. Our main theorems are as follows:

**Theorem.** (Corollary 3.18) There is an isomorphism

$$\pi_{**}(kq \wedge kq) \cong \bigoplus_{i \geq 0} \Sigma^{4i,2i} \text{Ext}_{A(1)}^{**}(H_{Z_2}^i)$$

where $H_{Z_2}^i$ is the i-th motivic integral Brown-Gitler module.

(2) (Lemma 3.34) The right-hand side may be described explicitly, modulo $v_1$-torsion, in terms of suspensions of $kq$ and Adams covers of bo.

This gives a complete description of the ring of $kq$-cooperations modulo $v_1$-torsion over $\text{Spec}(C)$; compare with the rational computations of Ananyevskiy [2] and $\eta$-local computations of Röndigs [53]. We use this analysis to study the entire $kq$-based Adams spectral sequence. Our analysis yields the following theorem:

**Theorem (Theorem 5.1).**

1. The 0-line of the $kq$-resolution is given by

$$E_0^{0,*,*} \cong M_2[h_0, h_1, \eta_1]/(h_0 h_1, h_0 \eta_1^{4}, \tau h_1^{3})$$

where $|h_0| = (0, 0)$, $|h_1| = (1, 1)$, and $|\eta_1| = (8, 4)$.

2. The 1-line of the $kq$-resolution is given by

$$E_1^{1,*,*} \cong \bigoplus_{k \geq 0} \Sigma^{4k,2k}/2^{\rho(k)} [\tau] \oplus M_2[h_1, \eta_1^{4}]/(h_1^{3} \tau).$$

All of these classes are $v_1$-periodic.

(3) $E_{\infty}^{n,t,u} = 0$ whenever $6n > t + 7$.

We apply this theorem to two problems in motivic stable homotopy theory: the motivic Telescope Conjecture and the motivic J-homomorphism.
1.3. **The motivic Telescope Conjecture.** The \( kq \)-resolution provides a means of completely understanding certain types of periodic phenomena in the motivic stable stems over \( \text{Spec}(\mathbb{C}) \). This is well-understood in the classical [15] [52] and abelian group-equivariant [7] [8] settings, but a classification of periodicities in the motivic setting is not known over any base scheme. Partial progress has been made in the work of Andrews [4], Gheorghe [21], and Krause [37] over \( \text{Spec}(\mathbb{C}) \), and the second author over arbitrary base-fields [49] [48].

Recall that the classical Telescope Conjecture may be formulated as follows. Let \( L_n = L_{E(n)} \) denote Bousfield localization with respect to height \( n \) Johnson-Wilson theory, and let \( L^f_n \) denote the corresponding finite localization [44]. The classical Telescope Conjecture states that the natural transformation \( L^f_n \to L_n \) is an equivalence. Since both functors are smashing, it suffices to show that \( L^f_n S^{0,0} \to L_n S^{0,0} \) is an equivalence.

When \( n = 0 \), the conjecture is trivial since both functors can be identified with rationalization. When \( n = 1 \), the conjecture was verified by Bousfield using computations of Mahowald and Miller.

In Section 6, we define motivic localization functors \( L_m \) and their finite analogs \( L^f_m \) for all \( m > n \geq -1 \).

**Theorem.**

1. (Theorem 5.7) The \( v_1 \)-periodic motivic stable stems are isomorphic to the 0- and 1-lines of the \( v_1 \)-inverted \( kq \)-resolution.

2. The map

\[
L^f_{1,-1} S^{0,0} \to L_{KQ} S^{0,0}
\]

is an equivalence.

We show in Section 6 that \( \langle KQ \rangle = \langle KGL \lor KW \rangle \) and outline a program for showing that \( \langle KGL \rangle = \langle K(0) \lor K(1) \rangle \) and \( \langle KW \rangle = \langle K(w_0) \lor K(1) \rangle \).

Our calculations also apply to study “exotic periodicity” as studied in [4] [5] [21] [26] [25] [31] [17] [47] [49] [57]. Recall that \( \eta \in \pi_{1,1}^C(S^{0,0}) \) is not nilpotent. Guillou and Isaksen conjectured the form of \( \pi_{*,*}^C(S^{0,0})(\eta^{-1}) \) in [26] and Andrews and Miller proved their conjecture in [5] using the motivic Adams-Novikov spectral sequence. We recover this calculation using the \( kq \)-resolution and make progress towards the motivic Telescope Conjecture at exotic height zero.

**Theorem.**

1. (Theorem 5.3) The \( \eta \)-periodic \( C \)-motivic stable stems are detected in the 0- and 1-lines of the \( kq \)-resolution. In particular, one has \( \pi_{*,*}^C(S^{0,0})(\eta^{-1}) \) as described in [5].

2. The map

\[
L^f_{10} S^{0,0} \to L_{cKW} S^{0,0}
\]

is an equivalence, where \( cKW \simeq \eta^{-1}kq \) is connective Witt theory.

1.4. **The motivic J-homomorphism.** Finally, we are interested in understanding the image of a conjectural motivic orthogonal J-homomorphism. The image of the motivic unitary J-homomorphism was calculated over algebraically closed fields by Hu-Kriz-Ormsby in [33], where they showed that the order of the 2-torsion in the motivic J-homomorphism coincided with the order of the 2-torsion in the classical J-homomorphism. Surprisingly, calculations of Dugger-Isaksen [18] show that
the order of the 2-torsion in the relevant bidegrees is different over $\text{Spec}(\mathbb{R})$ and $\text{Spec}(\mathbb{C})$. More precisely, the generator of $\sigma$ of $\pi^+_{7,4}(S^{0,0}) \cong \pi^+_{7,4}(imJ)$ which realizes to the classical generator of $\pi_7(S^0) \cong \pi_7(imJ)$ is 16-torsion over algebraically closed fields and classically, but $16\sigma \neq 0 \in \pi^+_{7,4}(S^{0,0})$. Precisely the same disparity in the order of 2-torsion occurs when comparing the classical and $C_2$-equivariant image of $J$.\footnote{Theorem. The order of the 2-torsion in the image of the $C$-motivic orthogonal $J$-homomorphism is precisely the same as the order of the 2-torsion in the image of the classical orthogonal $J$-homomorphism.}

Using the $kq$-resolution, we are able to recover the $C$-motivic calculation of Hu-Kriz-Ormsby through purely homotopy-theoretic methods:

$$\begin{align*}
\text{Theorem.} & \quad \text{The order of the 2-torsion in the image of the $C$-motivic orthogonal J-homomorphism is precisely the same as the order of the 2-torsion in the image of the classical orthogonal J-homomorphism.}
\end{align*}$$

We believe that the $kq$-resolution will be useful over other base fields, and we plan to make a full calculation of the order of 2-torsion over arbitrary base fields in future work. We also plan to make calculations in the $C_2$-equivariant setting and compare these with calculations over $\text{Spec}(\mathbb{R})$ and Minami’s calculation of the $C_2$-equivariant image of $J$.\footnote{Theorem.} We would like to know if our calculations have number theoretic and algebro-geometric consequences. The classical $J$-homomorphism classifies stable spherical bundles over sufficiently nice spaces, so it is natural to ask if a motivic $J$-homomorphism classifies stable $\mathbb{G}_m$-torsors (equivalently, line bundles) or $\mathbb{P}^n$-bundles over sufficiently nice motivic spaces. As another geometric application, the classical $J$-homomorphism detects framings of standard spheres under the identification of the stable stems with the framed cobordism ring. An analogous identification of the motivic stable stems and the framed algebraic cobordism ring exists over perfect fields (see e.g. \cite{58}), but the role of the motivic $J$-homomorphism is not well-understood in this context.

Unfortunately, we do not know of an algebro-geometric construction of the stable orthogonal $J$-homomorphism so we cannot say much about these potential applications in geometry. Nevertheless, we propose a model for the $C$-motivic stable orthogonal $J$-homomorphism in Section 7 using work of Gheorghe-Isaksen-Krause-Ricka \cite{22} which relates the 2-complete cellular $C$-motivic stable homotopy category to the category of filtered (classical) spectra.

$$\begin{align*}
\text{Theorem.} & \quad \text{(Section 7.3) There exists a morphism of motivic spectra}
\end{align*}$$

Following Davis and Mahowald \cite{15}, it should be possible to calculate the image of $J$ in $\pi^+_{7,4}(S^{0,0})$.\footnote{1.5. Further applications. We expect the $kq$-resolution to acquire further utility as more motivic technology is developed. Mahowald used the $bo$-resolution in \cite{40} to calculate the $v_1$-periodic unstable homotopy groups of $\mathbb{R}P^2 \wedge \mathbb{C}P^2$. Given the existence of a motivic EHP sequence, we ought to be able to understand unstable $v_1$-periodicity using the $kq$-resolution.}
From a purely calculational standpoint, Lellmann-Mahowald computed the first 20 stable stems using the $bo$-resolution in [38]. Isaksen points out that the calculation of the first 24 $C$-motivic stable stems is essentially identical to the calculation in the same range classically, but that beginning in the 25-stem, the computations diverge. He suggests that this might be interesting to see from the perspective of the $kq$-resolution.

Over more general base fields, low-dimensional calculations of the stable stems appear to be quite difficult. We expect the $kq$-resolution to be a useful tool for making low-dimensional stable calculations over general bases. Indeed, Röndigs applied the $\eta$-inverted $kq$-resolution in [53] towards the study of $\eta^{-1}S^{0,0}$ over general fields of characteristic not two.

1.6. **Outline.** In Section 2 we recall the $bo$-resolution and define the $kq$-resolution. In Section 3 we analyze the $E_1$-page of the $kq$-resolution. In particular, we define motivic integral Brown-Gitler modules using a filtration of the dual motivic Steenrod algebra adapted from [39]. We show that $A/\langle A(1) \rangle$ decomposes as a sum of motivic integral Brown-Gitler modules. We then identify the cohomology of these motivic integral Brown-Gitler modules with combinations of Adams covers of $bo$, $bsp$, $kq$, and $ksp$. We use these explicit computations along with comparison to the classical case to prove that the motivic Adams spectral sequence [19] converging to $\pi_{**}(kq \wedge kq)$ collapses at $E_2$.

In Section 4 we study the differentials and vanishing regions of the $kq$-resolution. We completely avoid the use of motivic Thom spectra by using Betti realization, which forces many of relevant differentials and extensions in the $kq$-resolution. To account for classes which realize to zero, we use the $\pi_{**}(kq)$-module structure of the $kq$-resolution. We also determine a vanishing region in the $kq$-resolution following [41].

In Section 5 we state our main computational result and use it to recalculate the $\eta$-local homotopy groups of the $C$-motivic sphere; these were conjectured by Guillou-Isaksen [26, Conj. 1.3(b)] and calculated by Andrews-Miller [5] using the motivic Adams-Novikov spectral sequence. We also calculate the $\nu_1$-periodic $C$-motivic stable stems.

In Section 6 we use our calculations to motivate motivic analogs of the classical Telescope Conjecture. We relate our calculations to various finite localizations, and indicate an approach to proving certain cases of the motivic Telescope Conjecture.

Finally, we suggest a model for the $C$-motivic orthogonal $J$-homomorphism using recent work of Gheorghe-Isaksen-Krause-Ricka [22] in Section 7. We explain how this model is related to our calculations and indicate what is left to show in order to completely calculate its image in $\pi_{**}(S^{0,0})$.

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2. The $kq$-based Adams Spectral Sequence

We will always work in the category of cellular motivic spectra over $\text{Spec}(\mathbb{C})$ at the prime $p = 2$, with everything implicitly completed at 2. Let $kq$ denote the very effective cover of Hermitian K-theory $KQ$ defined in \[3\]. In this section we define $kq$-resolution and discuss the associated Adams spectral sequence. After defining the $kq$-based Adams spectral sequence, we discuss convergence.

2.1. $bo$- and $kq$-resolutions. Before describing the $kq$-resolution, we review the $bo$-resolution studied by Mahowald in \[39\] (see also \[41\], \[9\], and \[38\]). Let $bo$ denote the cofiber of the unit map $S^0 \to bo$. Then the $bo$-resolution is given by the following:

$$
\begin{array}{cccccc}
S^0 & \xrightarrow{\Sigma^{-1}bo} & \Sigma^{-2}bo \wedge 2 & \cdots \\
bo & \xrightarrow{\Sigma^{-1}bo \wedge bo} & \Sigma^{-2}bo \wedge bo \wedge 2 & \\
\downarrow & & \downarrow & \\
\Sigma^{-1}bo \wedge bo & & \Sigma^{-2}bo \wedge bo \wedge 2 & \\
\end{array}
$$

where the horizontal arrows followed by vertical arrows are obtained from the cofiber sequence $\Sigma^{-1}bo \to S^0 \to bo$ are obtained by smashing with $bo$. Applying the functor $\pi_\ast$ gives a spectral sequence

$$
E_{1}^{n,t,u} = \pi_{t,u}(bo \wedge bo^{\wedge n}) \Rightarrow \pi_{t-n}(S^0)^{\wedge}_{bo}
$$

where $(S^0)^{\wedge}_{bo}$ denotes the $bo$-nilpotent completion of the sphere. It follows from a theorem of Bousfield \[13\] that $S^0_{bo} \simeq S^0$ (recall we are implicitly working in the 2-complete category).

In the motivic stable homotopy category, the role of $bo$ is played by $kq$. We define the $kq$-resolution by the analogous diagram. Let $kq$ denote the cofiber of the unit map $S^{0,0} \to kq$. Then the $kq$-resolution is given by

$$
\begin{array}{cccccc}
S^{0,0} & \xrightarrow{\Sigma^{-1,0}kq} & \Sigma^{-2,0}kq^{\wedge 2} & \cdots \\
kq & \xrightarrow{\Sigma^{-1,0}kq \wedge kq} & \Sigma^{-2,0}kq \wedge kq^{\wedge 2} & \\
\downarrow & & \downarrow & \\
\Sigma^{-1,0}kq \wedge kq & & \Sigma^{-2,0}kq \wedge kq^{\wedge 2} & \\
\end{array}
$$

and the resulting $kq$-based Adams spectral sequence takes the form

$$
E_{1}^{n,t,u} = \pi_{t,u}(kq \wedge kq^{\wedge n}) \Rightarrow \pi_{t-n,u}(kq^{\wedge 0})_{kq}
$$

where $(S^{0,0})^{\wedge}_{kq}$ denotes the $kq$-nilpotent completion. By \[43\] Theorem 1.0.1, Theorem 7.3.4], there is an equivalence $(S^{0,0})^{\wedge}_{kq} \simeq S^{0,0}$, and so the $kq$-based Adams spectral sequence converges to $\pi_{\ast}(S^{0,0})$.

We can compare these resolutions via Betti realization.

**Lemma 2.1.** \[3\] Lem. 2.13 The Betti realization of $kq$ is $bo$.

Since Betti realization preserves cofiber sequences and is strong symmetric monoidal, we obtain the following corollary which will be used frequently throughout the sequel.
Corollary 2.2. Betti realization takes the $kq$-resolution to the $bo$-resolution. In particular, Betti realization induces a multiplicative map of spectral sequences

$$E^{s,t,*}_s \to E^{s,t}_s.$$

3. Analysis of the $E_1$-page

Just as in the classical situation, the spectrum $kq$ does not satisfy Adams’ flatness condition. Consequently, the $E_2$-page of the $kq$-resolution does not have an algebraic description and we need to analyze its $E_1$-page. Since the homotopy groups $\pi_{s+*}(kq \wedge kq)$ make up the $E_1$-term of the $kq$-Adams spectral sequence, we begin by studying the homotopy groups $\pi_{s+*}(kq \wedge kq)$.

3.1. The $E_1$-page of the $bo$-resolution. To start, we recall the analysis of the $E_1$-page of the $bo$-resolution. By [39, Thm. 2.4], there is a homotopy equivalence

$$bo \wedge bo \cong \bigvee_{i=0}^{\infty} \Sigma^{4i} bo \wedge H\mathbb{Z}^{cl}_i$$

where $H\mathbb{Z}^{cl}_i$ is the $i$-th classical Brown-Gitler spectrum. Replacing the right-hand copy of $bo$ by $\overline{bo}$ gives the homotopy equivalence

$$bo \wedge \overline{bo} \cong \bigvee_{i=1}^{\infty} \Sigma^{4i} bo \wedge H\mathbb{Z}^{cl}_i.$$ 

Mahowald extends this description of the 1-line of the $E_1$-page of the $bo$-resolution to the entire $E_1$-page via the Künneth isomorphism in the proof of [39, Thm. 5.11]. In particular, one has

$$bo \wedge \overline{bo} \wedge_n \cong \bigvee_{i_1, \ldots, i_n} \Sigma^{4(i_1+\cdots+i_n)} bo \wedge H\mathbb{Z}^{cl}_{i_1} \wedge \cdots \wedge H\mathbb{Z}^{cl}_{i_n}$$

where $i_j > 0$ for all $1 \leq j \leq n$. Our goal in the remainder of the section is to obtain an analogous decomposition in the $\mathbb{C}$-motivic setting.

3.2. Motivic Brown-Gitler modules. We begin by recalling some facts about the motivic dual Steenrod algebra $A^\vee$ and the very effective cover of Hermitian $K$-theory $kq$ from [56] and [3], respectively. We then define motivic integral Brown-Gitler modules following [11].

Let $M_2 := \mathbb{F}_2[\tau], |\tau| = (0, -1)$, denote the mod two motivic homology of a point.

Theorem 3.1. [56, Sec. 12] The dual motivic Steenrod algebra $A^\vee$ is given by

$$M_2[\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\tau}_0, \bar{\tau}_1, \ldots]/(\bar{\tau}_i^2 = \tau \bar{\xi}_{i+1}),$$

where

$$|\bar{\xi}_i| = (2^{i+1} - 2, 2^i - 1), \quad |\bar{\tau}_i| = (2^{i+1} - 1, 2^i - 1).$$

The coproduct is determined by

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{2^i}, \quad \psi(\bar{\tau}_k) = \sum_{i+j=k} \bar{\tau}_i \otimes \bar{\xi}_j^{2^i} + 1 \otimes \bar{\tau}_k.$$
**Definition 3.2.** (compare with [24] Def. 3.2.1 and [35] Def. 2.4) For any $n \geq 0$, define an ideal $I(n) \subseteq A^\vee$ by

$$I(n) := (\bar{\tau}_{n+1}, \bar{\tau}_{n+2}, \ldots, \bar{\xi}_1^n, \bar{\xi}^{n-1}_2, \ldots, \bar{\xi}^n_{n}, \bar{\xi}_{n+1}, \bar{\xi}_{n+1}, \ldots).$$

We define the subalgebra $A^\vee(n) \subseteq A^\vee$ by

$$A(n)^\vee := A^\vee/I(n).$$

For $n \geq 0$, define $A(n)$ to be the $M_2$-subalgebra of $A$ defined by

$$A(n) := \langle SR^1, SR^2, \ldots, SR^{2^n} \rangle,$$

so

$$(A//A(n))^\vee \cong M_2[\bar{\xi}_1^n, \bar{\xi}^{n-1}_2, \ldots, \bar{\tau}_{n+1}, \bar{\tau}_{n+2}, \ldots]/(\bar{\tau}_i^2 = \tau \xi_{i+1}).$$

Observe that $H^{**}(HZ) \cong A//A(0)$, and dually, $H_{**}(HZ) \cong (A//A(0))^\vee$. We have the following:

**Theorem 3.3.** \[3\] The mod two motivic cohomology of $kq$ is given by

$$H^{**}(kq) \cong A//A(1).$$

We obtain the following by dualizing.

**Corollary 3.4.** The mod two motivic homology of $kq$ is given by

$$H_{**}(kq) \cong (A//A(1))^\vee \cong M_2[\bar{\xi}_1^n, \bar{\xi}^{n-1}_2, \ldots, \bar{\tau}_{n+1}, \bar{\tau}_{n+2}, \ldots]/(\bar{\tau}_i^2 = \tau \xi_{i+1}).$$

We will also need to know the structure of $H_{**}(kq)$ as a comodule over the dual Steenrod algebra. Note that $(A//A(1))^\vee$ is a subalgebra of the dual Steenrod algebra. The coaction is then obtained by restriction, yielding

$$\psi : (A//A(1))^\vee \to A^\vee \otimes (A//A(1))^\vee.$$ 

In the next subsection, we will calculate the homotopy groups of $kq \wedge kq$ via the motivic Adams spectral sequence. The $E_2$-term of this spectral sequence is given by $\text{Ext}_{A^\vee}((A//A(1))^\vee \otimes (A//A(1))^\vee)$, where our convention is that

$$\text{Ext}_{A^\vee}(M) := \text{Ext}_{A^\vee}(M, M_2).$$

A change-of-rings isomorphism yields the $E_2$-term

$$E_2 \cong \text{Ext}_{A(1)^\vee}((A//A(1))^\vee) \Rightarrow \pi_{**}(kq \wedge kq).$$

Thus we want to know $(A//A(1))^\vee$ as a comodule over $A(1)^\vee$. It follows from the definitions that there is a projection $\pi : A^\vee \to A(1)^\vee$. The $A(1)^\vee$-coaction on $A//A(1)^\vee$ is given by the composite

$$A//A(1)^\vee \stackrel{\psi}{\longrightarrow} A^\vee \otimes A//A(1)^\vee \stackrel{\pi \otimes 1}{\longrightarrow} A(1)^\vee \otimes A//A(1)^\vee.$$ 

In order to have a practical means of calculating this coaction, we need the following.

**Proposition 3.5.** As a Hopf algebra, we have

$$A(1)^\vee \cong M_2[\bar{\xi}, \tau_0, \tau_1]/(\tau_0^2 - \tau \bar{\xi}, \tau_1^2, \bar{\xi}_1^2).$$

Observe also that $A//A(1)^\vee$ is a comodule algebra (since $kq$ is an commutative ring spectrum). Thus the coaction is completely determined by its values on $\bar{\xi}$ and $\tau_j$. 

Corollary 3.6. The $A(1)^\vee$-coaction on $A//A(1)^\vee$ is completely determined by the following formulas:

$$
\begin{align*}
\psi(\xi_1^2) &= 1 \otimes \xi_1^2, \\
\psi(\xi_k) &= 1 \otimes \xi_k + \xi_1 \otimes \xi_{k-1}^2 \quad \text{for } k > 1, \\
\psi(\tau_k) &= 1 \otimes \tau_k + \tau_0 \otimes \xi_k + \tau_1 \otimes \xi_{k-1}^2 \quad \text{for } k > 1.
\end{align*}
$$

We will also need to use the $A(1)^\vee$-comodule structure of $A//A(0)^\vee$. Computing this coaction is very similar to the analysis above; we record the result here for the reader’s convenience:

Proposition 3.7. The subalgebra $A//A(0)^\vee$ is an $A(1)^\vee$-comodule algebra and its coaction is completely determined on its algebra generators. These are given by the following formulas:

$$
\begin{align*}
\psi(\xi_1) &= \xi_1 \otimes 1 + 1 \otimes \xi, \\
\psi(\xi_k) &= \xi_1 \otimes \xi_k + \xi_1 \otimes \xi_{k-1}^2 \quad k > 1, \\
\psi(\tau_1) &= \tau_1 \otimes 1 + \tau_0 \otimes \xi_1 + 1 \otimes \tau_1, \\
\psi(\tau_j) &= \tau_0 \otimes \xi_k + \tau_1 \otimes \xi_{k-1}^2 + 1 \otimes \tau_k \quad k > 1.
\end{align*}
$$

We can now define the Mahowald filtration of $(A//A(1))^\vee$.

Definition 3.8. We define the Mahowald weight of the multiplicative generators of $A^\vee$ by setting

$$
wt(\bar{\tau}_i) = wt(\bar{\xi}_i) = 2^i, \quad wt(\tau) = 0.
$$

We extend this definition to $A^\vee$ by defining $wt(x \cdot y) = wt(x) + wt(y)$. We define the Mahowald weight on the subalgebras $(A//A(n))^\vee$ in the obvious way.

Observe that for any monomial $m \in (A//A(n))^\vee$, the weight of $m$ is divisible by $2^{n+1}$. So, all monomials of $(A//A(0))^\vee$ have weight divisible by 2, and all monomials in $(A//A(1))^\vee$ have weight divisible by 4.

This definition is partially motivated by the following observation.

Lemma 3.9 (compare with [14]). The $A(1)^\vee$-coaction on $(A//A(1))^\vee$ preserves Mahowald weight.

Proof. This is a trivial consequence of Corollary 3.6. □

We conclude this subsection by defining motivic analogs of Brown-Gitler modules.

Definition 3.10. The $i$-th integral Brown-Gitler module $H\mathbb{Z}_i \subseteq (A//A(0))^\vee$ is defined by setting

$$
H\mathbb{Z}_i := \{ x \in (A//A(0))^\vee : wt(x) \leq 2i \}.
$$

The $i$-th $kq$-Brown-Gitler module $kq_i$ is defined by setting

$$
kq_i := \{ x \in (A//A(1))^\vee : wt(x) \leq 4i \}.
$$

Lemma 3.11. The integral Brown-Gitler modules $H\mathbb{Z}_i$ is an $A(1)^\vee$ subcomodule of $(A//A(0))^\vee$. 
Example 3.12. We have \( HZ_1 \cong M \{ \xi_1, \tau_1 \} \) with nontrivial \( A(1) \)-action given by \( SR^2(1) = \xi_1 \) and \( SR^1(\xi_1) = \tau_1 \). This module is realized as the motivic cohomology of a spectrum \( HZ_1 \) which can be constructed as follows. Let \( L^2 \) be the (simplicial) 4-skeleton of the geometric classifying space \( B_{gm} \mu_2 \) defined in [46]. Let \( X \) be the cofiber of the inclusion \( S^{3,2} \hookrightarrow L^2 \). Setting \( HZ_1 := \Sigma^4 L^2 F(X, S^0) \) gives a spectrum whose motivic cohomology has the desired \( A(1) \)-module structure.

Remark 3.1. In classical topology, one can construct the \( i \)-th integral Brown-Gitler spectrum \( HZ^{cl}_i \). These have the property that \( H_*(HZ^{cl}_i) \) is the \( i \)-th integral Brown-Gitler module. In the motivic case, these have been defined at odd primes by Röndigs in [53, Sec. 7] using real Betti realization.

3.3. The inductive procedure for calculating \( kq \)-cooperations. Our goal now is to compute \( \pi^{**}(kq \wedge kq) \), which will allow us to determine the 1-line of the \( kq \)-based Adams spectral sequence. We start by proving the existence of several short exact sequences which give a recursive method for calculating \( kq \)-cooperations.

Consider the motivic Adams spectral sequence
\[
\text{Ext}^{**}(A, H_*(kq \wedge kq)) \Rightarrow \pi^{**}(kq \wedge kq).
\]
By the Künneth isomorphism for motivic homology, a change-of-rings isomorphism, and Corollary 3.4, we have
\[
\text{Ext}^{**}(A, H^*(kq \wedge kq)) \cong \text{Ext}^{**}(A \wedge A/1),
\]
We will now argue that, as in the classical setting, one can decompose \( A \wedge A/1 \) as an \( A(1)^\vee \)-comodule into an infinite direct sum of suspensions the integral Brown-Gitler modules. Towards this end, we define the following.

Definition 3.13. Define \( M_1(k) \) to be the subspace of \( A \wedge A/1 \) spanned by the monomials of degree exactly equal to \( 4k \). Analogously, define \( M_0(k) \) to be the subspace of \( A \wedge A/0 \) spanned by monomials of weight \( 2k \).

Lemma 3.14. The subspaces \( M_1(k) \) are sub-comodules of \( A \wedge A/1 \). Furthermore, there is an isomorphism of \( A(1)^\vee \)-comodules
\[
A \wedge A/1 \cong \bigoplus_k M_1(k).
\]
Proof. This is an immediate consequence of Lemma 3.9.

Next, we show that the \( M_1(k) \) are isomorphic as \( A(1)^\vee \)-comodules to the integral Brown-Gitler modules (up to a suspension). This argument is an adaptation of the classical ones (cf. [11, 10, 14]).

First, there is an algebra map
\[
\varphi : (A \wedge A/1)^\vee \to (A \wedge A/0)^\vee
\]
defined on generators by
\[
\varphi(\xi_k^{2^l}) = \begin{cases} 
\xi_k^{2^l} & k > 1, \\
1 & k = 1,
\end{cases}
\]
and
\[
\varphi(\tau_k) := 7_k^{-1}.
\]

Lemma 3.15. The map \( \varphi \) is a map of ungraded \( A(1)^\vee \)-comodules.
Proof. Since both $A//A(1)^\vee$ and $A//A(0)^\vee$ are comodule algebras, it is enough to check that the map $\varphi$ commutes with the coaction on the generators. That this is true follows from direct computation using Corollary 3.6 and Proposition 3.7. □

Lemma 3.16 (compare with [10]). The map $\varphi$ maps the subspace $M_1(k)$ isomorphically onto the $A(1)^\vee$-subcomodules $H\mathbb{Z}_k$.

Proof. Let $m$ be a generic monomial in $M_1(k)$. Then $m$ is uniquely expressible as $\xi_1^{2s} x$ for some natural number $s$ and $x$ a monomial in $M_2[\xi_2, \xi_3, \tau_2, \tau_3, \ldots]$. Since $\text{wt}(m) = 4k$, we have that

$$4k = \text{wt}(m) = \text{wt}(\xi_1^{2s}) + \text{wt}(x) = 4s + \text{wt}(x).$$

Thus $\text{wt}(x) = 4(k-s)$. Hence $\varphi$ maps the subspace spanned by monomials of the form $\xi_1^{2s} x$ isomorphically onto $M_0(k-s)$. Since $H\mathbb{Z}_k$ is the direct sum of $M_0(j)$ for $0 \leq j \leq k$, we have the result. □

Corollary 3.17. The map $\varphi$ induces a graded isomorphism of $A(1)^\vee$-comodules

$$M_1(k) \cong H\mathbb{Z}_k.$$

Combining these lemmata now gives the following decomposition of $A//A(1)^\vee$.

Corollary 3.18. There is an isomorphism of $A(1)^\vee$-comodules

$$(A//A(1))^{\vee} \cong \bigoplus_{j \geq 0} \Sigma^{4j, 2j} H\mathbb{Z}_j.$$

Therefore the motivic Adams spectral sequence computing $\pi_{**}(kq \wedge kq)$ has the form

$$E_2 \cong \bigoplus_{j \geq 0} \text{Ext}_{A(1)^\vee}^{**, **}(\Sigma^{4j, 2j} H\mathbb{Z}_j) \Rightarrow \pi_{**}(kq \wedge kq).$$

Thus, we need to be able to compute the $\text{Ext}_{A(1)^\vee}$-groups of the integral Brown-Gitler modules. Following [11][14][10], we will inductively compute these groups by using the following short exact sequences.

Lemma 3.19. (compare with [11] Lem. 2.5) There are short exact sequences of $A(1)^\vee$-comodules

$$0 \to \Sigma^{4j, 2j} H\mathbb{Z}_j \to H\mathbb{Z}_{2j} \to kq_{j-1} \otimes (A//A(0))^\vee \to 0,$$

$$0 \to \Sigma^{4j, 2j} H\mathbb{Z}_j \otimes H\mathbb{Z}_{2j+1} \to H\mathbb{Z}_{2j+1+1} \to kq_{j-1} \otimes (A//A(0))^\vee \to 0.$$

Before delving into the proof of this theorem, we need some preliminaries. The following is inspired by the technical work in [10] and [14]. Recall that

$$A(1)//A(0)^\vee = E(\xi_1, \tau_1).$$

Note that there is an isomorphism of $\mathbb{M}_2$-modules

$$\kappa : A//A(0)^\vee \to A//A(1)^\vee \otimes A(1)//A(0)^\vee$$

where $m$ is a monomial in $A//A(1)^\vee$. While this map is an isomorphism of $\mathbb{M}_2$-modules, it is not an isomorphism of $A(1)^\vee$-comodules (where we endow the right hand side with the diagonal comodule structure), as the following example illustrates.
Example 3.20. On the left-hand side, we have the element $\bar{\xi}_2$, whose coaction is given by
\[
\alpha(\tau_1) = 1 \otimes \tau_1 + \tau_0 \otimes \bar{\xi}_1 + \tau_1 \otimes 1.
\]
Under $\kappa$ this is sent to
\[
\kappa \alpha(\bar{\xi}_2) = 1 \otimes 1 \otimes \tau_1 + \tau_0 \otimes 1 \otimes \bar{\xi}_1 + \tau_1 \otimes 1 \otimes 1.
\]
On the other hand, $\kappa(\tau_1) = 1 \otimes \tau_1$, the coaction of which is
\[
\alpha(1 \otimes \tau_1) = 1 \otimes 1 \otimes \tau_1 + \tau_1 \otimes 1 \otimes 1
\]
since $\tau_1$ is primitive in $A(1)\backslash A(0)^\vee$.

While $\kappa$ is not a comodule map, there is a filtration on $A\backslash A(0)^\vee$ which induces a comodule map on the associated graded. Define a filtration $F^j A\backslash A(0)^\vee$ on $A\backslash A(0)^\vee$ by
\[
F^j A\backslash A(0)^\vee := \kappa^{-1} \left( \bigoplus_{k \geq j} M_1(k) \otimes A(1)\backslash A(0)^\vee \right).
\]
Clearly, this is a multiplicative filtration on $A\backslash A(0)^\vee$, and furthermore induces a map on the associated graded
\[
E_0 \kappa : E_0 A\backslash A(0)^\vee \to A\backslash A(1)^\vee \otimes A(1)\backslash A(0)^\vee.
\]
Since $\kappa$ is clearly an isomorphism of $F_2$-vector spaces, the map $E_0 \kappa$ is also an isomorphism of $F_2$-vector spaces.

Proposition 3.21. The map $E_0 \kappa$ is an algebra map.

Proof. Consider two monomials $x, y \in A\backslash A(0)^\vee$ and write them as $x = m \bar{\xi}_1 \tau_1^p$ and $y = m' \bar{\xi}_1 \tau_1^q$. Then under $E_0 \kappa$, we have
\[
x \mapsto m \otimes \bar{\xi}_1 \tau_1^p
\]
and
\[
y \mapsto m' \otimes \bar{\xi}_1 \tau_1^q.
\]
If, say, $\epsilon = \epsilon' = 1$, then $xy$ lives in higher filtration than expected, and so $xy = 0$ in the associated graded. This shows that the map $E_0 \kappa$ is an algebra map. \qed

Proposition 3.22. The map $E_0 \kappa$ is a comodule map.

Proof. Since we know that $E_0 \kappa$ is an algebra map, and since $E_0 A\backslash A(0)^\vee$ and $A\backslash A(1)^\vee \otimes A(1)\backslash A(0)^\vee$ are comodule algebras, it is enough to show that the coaction commutes with $E_0 \kappa$ on the set of algebra generators
\[
\{\bar{\xi}_1, \bar{\xi}_2, \ldots; \tau_1, \tau_2, \ldots\}.
\]
An easy inspection of the formulas for the $A(1)$-coaction gives the result. \qed

Following \[10\] \[11\] \[14\], we define
\[
R^j A\backslash A(0) := A\backslash A(0)^\vee / F^{j+1} A\backslash A(0)^\vee.
\]
This clearly has an induced filtration and so we get induced maps
\[
E_0 \kappa : E^0 R^j A\backslash A(0)^\vee \to kq_j \otimes A(1)\backslash A(0)^\vee.
\]
Remark 3.23. There is a finite filtration on $R^j A//A(0)$. Thus we get an associated spectral sequence

$$\text{Ext}_{A(1)^{\vee}}(E_0 R^j A//A(0)) \to \text{Ext}_{A(1)^{\vee}}(R^j A//A(0)).$$

The $E_1$-term is isomorphic to

$$\text{Ext}_{A(1)^{\vee}}(E_0 R^j A//A(0)) \cong \text{Ext}_{A(0)^{\vee}}(kq).$$

We have

$$\text{Ext}_{A(0)^{\vee}}(kq) \cong \mathbb{M}_2[v_0] \otimes M_*(kq); Q_0 \oplus (v_0\text{-torsion concentrated in Adams filtration 0}).$$

The $Q_0$-Margolis homology of $H_* kq$ is

$$M_*(A//A(1)^{\vee}; Q_0) = \mathbb{M}_2[\xi_1^2].$$

The Mahowald weight filtration induces a filtration on the $Q_0$-Margolis homology and so we have

$$M_*(kq; Q_0) \cong \mathbb{M}_2[\xi_1^2, \ldots, \xi_{2j}].$$

Since the spectral sequence is linear over $\text{Ext}_{A(1)^{\vee}}(\mathbb{M}_2)$, and since the $v_0$-towers are concentrated in even stems, the spectral sequence collapses.

Proof of 3.19 Consider the composite

$$H\mathbb{Z}_{2j} \to A//A(0)^{\vee} \to R^{j-1} A//A(0)^{\vee}.$$

We claim this is a surjective morphism. Let $x = m\xi_1^{\tau_1} \eta$ where $m$ is a monomial in $A//A(1)^{\vee}$. Suppose that this determines a nonzero class in $R^{j-1} A//A(0)^{\vee}$. Recall that

$$R^{j-1} A//A(0) = A//A(0)/F^j A//A(0)^{\vee}.$$

For this to be a nonzero class, the Mahowald weight of $m$ must be no more than $4(j - 1) = 4j - 4$. Since we have $\text{wt}(\xi_1) = \text{wt}(\tau_1) = 2$, the monomial $m\xi_1^{\tau_1} \eta$ has Mahowald weight bounded by $4j$. Thus $x \in H\mathbb{Z}_{2j}$, and hence the map $H\mathbb{Z}_{2j} \to R^{j-1} A//A(0)^{\vee}$ is surjective.

We need to calculate the kernel. Consider an element $x = m\xi_1^{\tau_1} \eta$ with $m \in A//A(1)^{\vee}$. Suppose this element projects to 0 in $R^{j-1} A//A(0)^{\vee}$. Then the Mahowald weight of $m$ must be 4. So the kernel of the map is

$$M_4(2j) \cong \sigma^{4j, 2j} H\mathbb{Z}_{2j}.$$

This completes the proof for $H\mathbb{Z}_{2j}$. The case for odd indices follows similarly. □

3.4. $\text{Ext}_{A(1)^{\vee}}$ of Brown-Gitler modules. Lemma 3.19 allows us to compute $\text{Ext}^{**}_{A(1)^{\vee}}(H\mathbb{Z}_2)$ from $\text{Ext}^{**}_{A(1)^{\vee}}(H\mathbb{Z}_2^{\otimes j})$. We can compute $\text{Ext}^{**}_{A(1)^{\vee}}(H\mathbb{Z}_2)$ using the algebraic Atiyah-Hirzebruch spectral sequence (aAHSS),

$$E_1^{**} = \bigoplus_{x \in B} \text{Ext}^{**}_{A(1)^{\vee}}(x; A(1)^{\vee}(\mathbb{M}_2) \Rightarrow \text{Ext}^{**}_{A(1)^{\vee}}(H\mathbb{Z}_2))$$

where $B = \{1, \xi_1, \tau_1\}$. By [39 Thm. 6.6], we have

$$\text{Ext}^{**}_{A(1)^{\vee}}(\mathbb{M}_2) \cong \mathbb{M}_2[h_0, h_1, \alpha, \beta]$$

with $|h_0| = (1, 0, 0), |h_1| = (1, 1, 1), |\alpha| = (3, 4, 2)$, and $|\beta| = (4, 8, 4)$.

The $E_1$-page of the aAHSS is depicted in Figure 3.24, the $E_2$-page of the aAHSS is depicted in Figure 3.25, and the $E_3 = E_{\infty}$-page of the aAHSS is depicted in Figure.
Differentials can be read off from the $A^V$-comodule structure of $H^*_{\mathbb{Z}}$ as in the classical setting. The hidden extensions shown in Figure 3.26 may be deduced by applying the Betti realization functor $R_{\mathbb{C}} : SH_{\mathbb{C}} \to SH$ and comparing to the classical computation of $\text{Ext}^{**}_{A(1)c}(H^*_{\mathbb{Z}})$. 

Figure 3.24. The $E_1$-page of the aAHSS converging to $\text{Ext}^{**}_{A(1)c}(H^*_{\mathbb{Z}})$ with $d_1$-differentials. A black $\bullet$ represents $M_2$, a red $\bullet$ represents $\mathbb{F}_2$, and a black $\Box$ represents $M_2[h_0]$. Differentials are blue and $\tau$-linear.

Figure 3.25. The $E_2$-page of the aAHSS converging to $\text{Ext}^{**}_{A(1)c}(H^*_{\mathbb{Z}})$ with $d_2$-differentials. Notation is as in Figure 3.24, with dashed blue arrows representing differentials where either the source, target, or both are $\tau$-torsion.

Figure 3.26. The $E_3 = E_\infty$-page of the aAHSS converging to $\text{Ext}^{**}_{A(1)c}(H^*_{\mathbb{Z}})$. Notation is as in Figure 3.24. Vertical green lines indicate hidden $h_0$-extensions while a green $\bullet$ indicates that two classes are connected by a hidden $h_0$- or $\tau$-extension.
Definition 3.27. Let $ksp$ be the very effective cover of $\Sigma^{4,2}KQ$.

Corollary 3.28. There is an isomorphism
\[ \text{Ext}^{***}(H\mathbb{Z}_1) \cong \pi_*(ksp). \]

In [39] (see also [11]), Mahowald expresses $\text{Ext}^{***}(H\mathbb{Z}_1^\otimes i)$ in terms of the Adams covers of $bo$ and $bsp$. The motivic analog of this result is slightly more complicated; as an example, we begin by computing $\text{Ext}^{***}(H\mathbb{Z}_1 \otimes H\mathbb{Z}_1)$. One way to approach this calculation is via the aAHSS for the functor $\text{Ext}^{***}(H\mathbb{Z}_1 \otimes -)$. This is a spectral sequence taking the form
\[ \text{Ext}^{***}(H\mathbb{Z}_1 \otimes M_2 \{1, \xi, \tau\}) \Rightarrow \text{Ext}^{***}(H\mathbb{Z}_1^2). \]

The differentials and extensions can be determined as in the previous example. We present the calculation in the following figures.

Figure 3.29. $E_1$-page of the aAHSS for $\text{Ext}^{***}(H\mathbb{Z}_1 \otimes H\mathbb{Z}_1)$ with $d_1$-differentials. A black $\bullet$ represents $M_2$, a red $\bullet$ represents $F_2$. Differentials are blue and $\tau$-linear.

Figure 3.30. The $E_2$-page of the aAHSS for $\text{Ext}^{***}(H\mathbb{Z}_1 \otimes H\mathbb{Z}_1)$ with $d_2$-differentials. A black $\bullet$ represents $M_2$, a red $\bullet$ represents $F_2$. Differentials are
blue and $\tau$-linear, with a dashed differential indicating that the target, source, or both are $\tau$-torsion.

Figure 3.31. The $E_3 = E_\infty$-page of the aAHSS for $\text{Ext}^{\bullet}_{A(1)^\vee}(H\mathbb{Z}_1 \otimes H\mathbb{Z}_1)$. A black $\bullet$ represents $M_2$, a red $\bullet$ represents $F_2$. Vertical green lines indicate hidden $h_0$-extensions while a green $\bullet$ indicates that two classes are connected by a hidden $h_0$- or $\tau$-extension.

Already, we see that the classical description of $\text{Ext}^{\bullet\bullet\bullet}_{A(1)^\vee}(H\mathbb{Z}_1 \otimes H\mathbb{Z}_1)$ in terms of the Adams covers of $kq$ and $ksp$ fails when $i = 2$. In particular, there would be an infinite $\tau$-torsion $h_1$-tower beginning in bidegree $(2,0)$ of Figure 3.31 if this were the case. However, one may express the above computation as

$$\text{Ext}^{\bullet\bullet\bullet}_{A(1)^\vee}(H\mathbb{Z}_1 \otimes H\mathbb{Z}_1) \cong (M_2 \otimes \tau_{<4} \text{Ext}^{\bullet\bullet}_{A(1)^\vee}(bo^{(2)}) \oplus \tau_{\geq 4} \text{Ext}^{\bullet\bullet\bullet}_{A(1)^\vee}(kR^{(2)}))$$

where $\text{Ext}^{\bullet\bullet}_{A(1)^\vee}(bo^{(2)})$ is made into a trigraded object in an appropriate way.

More generally, we see that $\text{Ext}^{\bullet\bullet\bullet}_{A(1)^\vee}(H\mathbb{Z}_1 \otimes H\mathbb{Z}_1)/(v_1 - \text{tor})$ may be expressed as a combination of a classical Adams cover of $bo$ or $bsp$ and a suspension of $kq$ or $ksp$. 
Recall from [11] Lem. 3.2 that there are isomorphisms
\[
\frac{\Ext^{**}(H_2^0)}{v_1 - \text{tor}} \cong \begin{cases} 
\Ext^{**}(b_0(i)), & \text{if } i \equiv 0 \mod 2, \\
\Ext^{**}(b_{sp}(i-1)), & \text{if } i \equiv 1 \mod 2.
\end{cases}
\]
If \( i \equiv 0 \mod 4 \), then \( i = 4k \) and the Ext-groups on the right-hand side above can be expressed as
\[
\Ext^{**}(b_0(i)) \cong \bigoplus_{j=0}^{2k-1} \Sigma^j F_2[h_0] \oplus \Sigma^{4k} \Ext^{**}(b_0).
\]
Similar decompositions hold for other congruence classes of \( i \); these decompositions lead us to the following definition.

**Definition 3.32.** Let \( Z_i \) be the trigraded \( A(1)^{\gamma} \)-comodule defined by \( Z_i := \)
\[
\begin{cases} 
\bigoplus_{j=0}^{i/2-1} \Sigma^j M_2[h_0] \oplus \Sigma^{2j} \Ext^{**}(kq)[0], & \text{if } i \equiv 0 \mod 4, \\
\bigoplus_{j=0}^{i-1/2} \Sigma^j M_2[h_0] \oplus \Sigma^{2j+2} \Ext^{**}(kq)[1], & \text{if } i \equiv 1 \mod 4, \\
\bigoplus_{j=0}^{i/2-1} \Sigma^j M_2[h_0] \oplus \Sigma^{2j-2} M_2 \oplus \Sigma^{2j} \Ext^{**}(ksp)[1], & \text{if } i \equiv 2 \mod 4, \\
\bigoplus_{j=0}^{i-1/2} \Sigma^j M_2[h_0] \oplus \Sigma^{2j-1} M_2[h_1]/(h_1^2) \oplus \Sigma^{2j+1} \Ext^{**}(ksp)[2], & \text{if } i \equiv 3 \mod 4.
\end{cases}
\]

**Lemma 3.33.** There is an isomorphism of \( A(1)^{\gamma} \)-comodules
\[
\frac{\Ext^{**}(H_2^2)}{v_1 - \text{tor}} \cong Z_i.
\]

**Proof.** Regarding \( \Ext^{**}(H_2^0) \odot H_2^i \) as a homology theory in the category of \( A(1)^{\gamma} \)-comodules, we may inductively compute \( \Ext^{**}(H_2^0) \odot H_2^i \) via the aHSS associated to the filtration of \( H_2^i \) by topological dimension. Thus we obtain a convergent spectral sequence
\[
\Ext^{**}(H_2^0) \odot H_2^i \otimes \bigoplus M_2 \{ x[0], x[2], x[3] \} \Rightarrow \Ext^{**}(H_2^0).
\]
Here, an expression \( \alpha x[k] \) where \( \alpha \in \Ext^{**}(H_2^0) \) denotes the element of \( \Ext(H_2^0) \) on the cell of dimension \( k \). In particular, this spectral sequence arises from applying the Ext-functor to the following filtration by comodules of \( H_2^0 \). Let \( F^k H_2^0 \) denote the subspace spanned by generators in degrees \( \leq k \). This is clearly a filtration by comodules. Thus we get
\[
0 \longrightarrow F^0 H_2^0 \longrightarrow F^1 H_2^0 \longrightarrow F^2 H_2^0 \longrightarrow F^3 H_2^0 = H_2^0
\]
and our spectral sequence arises from the exact couple coming from applying Ext. Here the differentials are obtained by taking a cobar representative for a class \( \alpha x[k] \), lifting them to the cobar complex for \( F^k H_2^0 \), applying the cobar differential, and then projecting on to the highest filtration forms. Consider an element \( \alpha x[3] \) on \( E_1 \). This is represented in the (normalized) cobar complex for \( F^3/F^2 \) by \( a \otimes \tau_1 \), where \( a \) is a cocycle of the cobar complex for \( H_2^0 \) representing \( \alpha \). Since \( a \) is a cocycle, the cobar differential yields
\[
d(\alpha \otimes \tau_1) = \alpha \otimes \tau_0 \xi_1.
\]
Thus we derive for each $\alpha \in \text{Ext}(H\mathbb{Z}_2^{\otimes i})$
\[ d_1(\alpha x[3]) = a h_0 x[2]. \]
Similar considerations show that there is a $d_2$-differential
\[ d_2(\alpha x[2]) = a h_1 x[0]. \]
Degree considerations show that these are the only differentials which occur.

In order to get the desired hidden extensions, we use the isomorphism $H\mathbb{Z}_2^{\otimes i}[\tau^{-1}] \cong (H\mathbb{Z}_2^{cl})^{\otimes i}[\tau^{\pm 1}]$ which induces a ring homomorphism
\[ \text{Ext}_{A(1)^{\vee}}(H\mathbb{Z}_2^{\otimes i}) \to \text{Ext}_{A(1)^{\vee}}((H\mathbb{Z}_2^{cl})^{\otimes i}). \]
Comparison with the classical computation (cf. [11]) yields the desired extensions. □

This gives us the input to calculate the Ext-groups for the comodules $H\mathbb{Z}_n$ using long exact sequences in Ext arising from the short exact sequences of Lemma 3.19. The following should be compared with [39, Prop. 2.6] and [11, Prop. 3.3].

**Lemma 3.34.** There is an isomorphism of $A(1)^{\vee}$-comodules
\[ \frac{\text{Ext}^{\ast\ast\ast}_{A(1)^{\vee}}(H\mathbb{Z}_n)}{v_1 - \text{tor}} \cong \mathbb{Z}_{2n - \alpha(n)}, \]
where $\alpha(n)$ is the number of 1’s in the dyadic expansion of $n$.

**Proof.** For this proof only, all Ext-groups are implicitly calculated modulo $v_1$-torsion. We proceed by induction on $n$. The lemma is clear for $n = 1$ by Corollary 3.28. Assume now that the lemma holds for all $H\mathbb{Z}_i$ with $i < n$.

If $n$ is even, then $H\mathbb{Z}_n$ fits into the short exact sequence
\[ 0 \to \Sigma^{2n,n} H\mathbb{Z}_n/2 \to H\mathbb{Z}_n \to \overline{kq}_{n/2-1} \otimes (A(1)/A(0))^{\vee} \to 0 \]
by Lemma 3.19. Applying $\text{Ext}^{\ast\ast\ast}_{A(1)^{\vee}}(-)$, we see that $\text{Ext}^{\ast\ast\ast}_{A(1)^{\vee}}(H\mathbb{Z}_n)$ decomposes into the Ext-groups of the left-hand side and right-hand side. By Remark 3.23 the Ext-groups of the right-hand side are isomorphic to
\[ \text{Ext}^{\ast\ast\ast}_{A(1)^{\vee}}(\overline{kq}_{n/2-1} \otimes (A(1)/A(0))^{\vee}) \cong \bigoplus_{j=0}^{n/2-1} \Sigma^{4j,2j} M_2[h_0]. \]

The Ext-groups of the left-hand side are given by
\[ \text{Ext}^{\ast\ast\ast}_{A(1)^{\vee}}(\Sigma^{2n,n} H\mathbb{Z}_n/2) \cong \Sigma^{2n,n} \mathbb{Z}_{n-\alpha(n/2)} \]
by the induction hypothesis, so we have
\[ \text{Ext}^{\ast\ast\ast}_{A(1)^{\vee}}(H\mathbb{Z}_n) \cong \bigoplus_{j=0}^{n/2-1} \Sigma^{4j,2j} M_2[h_0] \oplus \Sigma^{2n,n} \mathbb{Z}_{n-\alpha(n/2)} \cong \mathbb{Z}_{2n-\alpha(n)} \]
(note that $\alpha(n) = \alpha(n/2)$).

On the other hand, if $n$ is odd, then $H\mathbb{Z}_n$ fits into the short exact sequence
\[ 0 \to \Sigma^{2(n-1),n-1} H\mathbb{Z}_{(n-1)/2} \otimes H\mathbb{Z}_{1} \to H\mathbb{Z}_n \to \overline{kq}_{(n-1)/2-1} \otimes (A(1)/A(0))^{\vee} \to 0 \]
by Lemma 3.19. The Ext-groups of the right-hand side are given by
\[ \text{Ext}^{\ast\ast\ast}_{A(1)^{\vee}}(\overline{kq}_{(n-1)/2-1} \otimes (A(1)/A(0))^{\vee}) \cong \bigoplus_{j=0}^{(n-1)/2-1} \Sigma^{4j,2j} M_2[h_0] \]
by Remark 3.23 It remains to calculate
\[ \text{Ext}^{\ast \ast \ast}_{A(1)^{\vee}} \left( \Sigma^{2(n-1),n-1} H \mathbb{Z}_{(n-1)/2} \otimes H \mathbb{Z}_1 \right). \]

As in the previous proof, we analyze the aAHSS associated to the homology theory \( \text{Ext}^{\ast \ast \ast}_{A(1)^{\vee}} (\_ \otimes H \mathbb{Z}_1) \) in the category of \( A(1)^{\vee} \)-comodules. This spectral sequence takes the form
\[ \text{Ext}^{\ast \ast \ast}_{A(1)^{\vee}} (H \mathbb{Z}_{(n-1)/2} \otimes M_2 \{x[0], x[2], x[3]\}) \Rightarrow \text{Ext}^{\ast \ast \ast}_{A(1)^{\vee}} (H \mathbb{Z}_{(n-1)/2} \otimes H \mathbb{Z}_1). \]

By the induction hypothesis, the left-hand side is isomorphic as \( A(1)^{\vee} \)-comodules to
\[ Z_{n-1-\alpha((n-1)/2)} \otimes_{M_2} M_2 \{x[0], x[2], x[3]\}. \]

Suppose that \( n-1-\alpha((n-1)/2) \equiv 0 \mod 4 \). Then \( Z_{n-1-\alpha((n-1)/2)} \) decomposes as a direct sum
\[ \bigoplus\limits_{j=0}^{(n-\alpha(n))/2-1} \Sigma^{4j,2j} M_2[h_0] \oplus \Sigma^{2(n-1-\alpha(\frac{n-1}{2})),n-1-\alpha(\frac{n-1}{2})} \text{Ext}^{\ast \ast \ast}_{A(1)^{\vee}} (kq)[0] \]
\[ \cong \bigoplus\limits_{j=0}^{(n-\alpha(n))/2-1} \Sigma^{4j,2j} M_2[h_0] \oplus \Sigma^{2n-2\alpha(n),n-\alpha(n)} \text{Ext}^{\ast \ast \ast}_{A(1)^{\vee}} (kq)[0] \]
where we have used the relation \( \alpha\left(\frac{n-1}{2}\right) = \alpha(n) - 1 \) to rewrite the upper bound of the direct sum and the bidegree of the suspension. This splitting gives rise to an analogous decomposition of the aAHSS. On the left-hand summand, the aAHSS collapses at \( E_2 \) and is identical to the \( E_1 \)-page (recall that we are working modulo \( v_1 \)-torsion). The aAHSS for the right-hand summand was calculated in Corollary 3.28 to be \( Z_1 \). Therefore we obtain an isomorphism of \( A(1)^{\vee} \)-comodules
\[ \text{Ext}^{\ast \ast \ast}_{A(1)^{\vee}} (H \mathbb{Z}_{(n-1)/2} \otimes H \mathbb{Z}_1) \cong \bigoplus\limits_{j=0}^{(n-\alpha(n))/2-1} \Sigma^{4j,2j} M_2[h_0] \oplus \Sigma^{2n-2\alpha(n),n-\alpha(n)} Z_1[0]. \]

Altogether, we see that \( \text{Ext}^{\ast \ast \ast}_{A(1)^{\vee}} (H \mathbb{Z}_n) \) is isomorphic to
\[ \bigoplus\limits_{j=0}^{(n-1)/2-1} \Sigma^{4j,2j} M_2[h_0] \oplus \Sigma^{2(n-1),n-1} \bigoplus\limits_{k=0}^{(n-\alpha(n))/2-1} \Sigma^{4k,2k} M_2[h_0] \oplus \Sigma^{2n-2\alpha(n),n-\alpha(n)} Z_1[0] \]
which can be rewritten as
\[ \bigoplus\limits_{j=0}^{(2n-\alpha(n))/2-1} \Sigma^{4j,2j} M_2[h_0] \oplus \Sigma^{4n-2\alpha(n),2n-\alpha(n)} \text{Ext}^{\ast \ast \ast}_{A(1)^{\vee}} (kq) \cong Z_{2n-\alpha(n)}. \]

The calculations for other congruence classes of \( n-1-\alpha((n-1)/2) \mod 4 \) are similar. The only subtlety arises in the case \( n-1-\alpha((n-1)/2) \equiv 2 \mod 4 \), in which case the copy of \( M_2 \) suspended by \( 2i-2 \) is related to the copy of \( M_2[h_0] \) beginning in cohomological filtration 1 by a hidden \( h_0 \)-extension. This can be seen by comparing to the classical case.

We can now compute the ring of \( kq \)-cooperations \( \pi_\ast(kq \wedge kq) \) using the motivic Adams spectral sequence. Recall the following theorem of Mahowald:

**Theorem 3.35.** [39] Thm. 2.9 The Adams spectral sequence converging to \( \pi_\ast(bo \wedge bo) \) collapses at \( E_2 \), i.e. \( E_2 = E_\infty \).
Corollary 3.36. The motivic Adams spectral sequence converging to $\pi_{**}(kq \land kq)$ collapses at $E_2$.

Proof. Since Betti realization is obtained by inverting $\tau$ and then setting $\tau = 1$, there can be no differentials in the motivic Adams spectral sequence where both the source and target are $\tau$-free. On the other hand, the only $\tau$-torsion classes are $h_1$-torsion free. Therefore the only possible differentials are between $\tau$-torsion $h_1$-towers, but these are not possible for tridegree reasons. More precisely, the beginning of each $\tau$-torsion $h_1$-tower is separated from another by stem $4i$ and motivic weight $2i$, $i \geq 1$. Since $\text{stem}(h_1) = 1$ and $\text{wt}(h_1) = 1$, the possible targets of a $kq$-Adams differential are in too high of a weight. □

Figure 3.37. The $E_2 = E_\infty$-page of the motivic Adams spectral sequence converging to $\pi_{**}(kq \land kq)$ with $v_1$-torsion classes suppressed. The horizontal axis indicates stem, and the vertical axis indicates which $i$ in the decomposition above is used. Adams filtration and motivic weight are suppressed. A □ represents $\mathbb{M}_2[h_0]$, a black • represents $\mathbb{M}_2$, and a red • indicates $\mathbb{F}_2$. Horizontal lines indicate multiplication by $h_1$, with a red horizontal line indicating that the target is simple $\tau$-torsion. A red horizontal arrow represents an infinite $h_1$-tower.

3.5. A vanishing line of slope $1/3$ on $E_1$. We now show that the $E_1$-page of the $kq$-resolution has a vanishing line of slope $1/3$. Recall that we have the cofiber sequence

$$S^{0,0} \to kq \to \overline{kq}.$$  

The long exact sequence in homotopy groups shows that $\pi_{t,*}(\overline{kq}) = 0$ for $t \leq 3$. Thus, the smash powers $kq \land \overline{kq}^s$ are $4s - 1$-connected. Thus we can conclude the following.

Lemma 3.38. We have $E_1^{s,t,*} = 0$ whenever $t < 4s$.

In other words, on the $E_1$-page of the $kq$-resolution, the region above the line of slope 1/3 passing through the origin consists only of trivial groups. This result will be used in the sequel to prove that there is a vanishing line of slope 1/5 on the $E_2$-page.

4. Differentials and a vanishing line of slope 1/5 on the $E_2$-page

4.1. Review of the $E_\infty$-page of the bo-resolution. We begin by reviewing the analysis of the $kq$-resolution from [39] and [41]. Mahowald’s main theorem consists of two parts: the calculation of the 0- and 1-lines of the $E_\infty$-page of the bo-resolution ([39 Thm. 1.1.(a-b)]) and the vanishing line of slope 1/5 on the $E_2$-page of the bo-resolution ([39 Thm. 1.1.(c)]).
Both parts depend on understanding differentials in the \( bo \)-resolution. The first set of differentials is discussed in the following theorem, which is also called the “Bounded Torsion Theorem” in [39 Cor. 3.7].

**Theorem 4.1.** [39 Thm. 5.11][41 (c’)] Each homotopy class in \( \pi_* (bo \wedge bo^{\wedge n}) \) is either of Adams filtration zero or one, or in the image of \( d_n \), or is mapped essentially under \( d_n \), or \( n = 0 \) or \( n = 1 \) and the homotopy can be identified with \( \Ext_{A(1)}^{\infty} (H_{Z_2}^{\wedge n}) \).

The second set of differentials produces the relevant 2-torsion in the 1-line of the \( bo \)-resolution.

**Theorem 4.2.** [39 Pg. 380] Let \( k \geq 1 \). The \( d_1 \)-differential \( d_1 : E_1^{0, 4k} \cong \mathbb{Z} \rightarrow \mathbb{Z} \cong E_1^{1, 4k} \) is given by multiplication by \( 2^\rho(k) \), where \( \rho(k) \) is the 2-adic valuation of \( 4k \).

**Remark 4.3.** Both theorems follow from an explicit comparison with the \( X \)-resolution, where \( X := Th(\Omega S^5 \rightarrow BO) \). In the next section, we will produce analogous differentials in the \( kq \)-resolution. Instead of defining a motivic analog of \( X \), we produce differentials in the \( kq \)-resolution by comparing to the \( bo \)-resolution using Betti realization. Over more general base fields, we expect that a combination of equivariant Betti realization and base-change functors will suffice for producing these differentials.

The vanishing line requires an additional argument which we sketch now. Let \( A_1 \) be a spectrum with \( H^*(A_1) \cong A(1) \). Then the \( E_2 \)-page of the \( bo \)-resolution for \( A_1 \) may be identified with the Adams spectral sequence for \( A_1 \). The latter has a vanishing line of slope \( 1/5 \) on its \( E_2 \)-page which can be proven using the May spectral sequence. If one takes \( A_1 = S^0/(2, \eta, v_1) \), then one might hope to produce the vanishing line in the \( bo \)-resolution for \( S^0 \) through a series of Bockstein spectral sequences. However, this runs into the issue that one only has a Bockstein spectral sequence at the level of \( E_1 \)-pages in general.

Instead, Mahowald relates the \( E_2 \)-page of the \( bo \)-resolution for the mod two Moore spectrum \( S^0/2 \) to the \( E_2 \)-page of the \( bo \)-resolution for \( A_1 \) above a line of slope \( 1/5 \). This leads to a vanishing line of slope \( 1/5 \) in the \( E_2 \)-page of the \( bo \)-resolution for \( S^0/2 \). To obtain the vanishing line for the sphere, Mahowald then uses the cofiber sequence

\[
S^0 \rightarrow S^0 \rightarrow S^0/2
\]

along with the explicit calculation of 2-torsion and periodicity in the \( E_2 \)-page above a line of slope \( 1/5 \). This explicit calculation essentially follows from the expression for the \( E_1 \)-page in terms of Adams covers of \( bo \) and \( bsp \).

### 4.2. Differentials

In this section, we prove the motivic analog of Mahowald’s Bounded torsion theorem.

The differentials in the relevant portion of the \( kq \)-resolution can be determined via Betti realization, \( \tau \)-linearity, and \( \eta \)-linearity. Recall that \( \pi_*(X(\mathbb{C})) \otimes \mathbb{F}_2[\tau, \tau^{-1}] \cong \pi_* (X)[\tau^{-1}] \), so the realization of a \( \tau \)-torsion class is zero and the realization of a \( \tau \)-free class is nonzero. This fact will allow us to deduce many of the relevant differentials in the \( kq \)-resolution from the differentials calculated by Mahowald in the \( bo \)-resolution [39]. We obtain the remaining differentials using the \( \pi_* (kq) \)-module structure of the \( kq \)-resolution.
Recall the isomorphism
\[ \pi_*(kq \wedge \overline{kq}^{\wedge n})[\tau^{-1}] \cong \pi_*(bo \wedge \overline{bo}^{\wedge n}) \otimes \mathbb{F}_2[\tau, \tau^{-1}]. \]
We have shown above that \( \pi_*(kq \wedge kq) \) is a direct sum of combinations of \( \pi_*(bo^{(j)}) \) (in appropriate bidegrees) and \( \pi_*(kR^{(j)}) \) for various values of \( j \), together with \( \mathbb{Z}_2 \)-summands in Adams filtration zero. The same holds when we consider \( \pi_*(kq \wedge \overline{kq}^{\wedge n}) \) for any \( n \) by a similar analysis; compare with the proof of [39, Thm. 5.11].

We may compare the \( kq \)-resolution and \( bo \)-resolution precisely since \( kq(\mathbb{C}) \simeq bo \).

The following statements are clear from the above observations:

1. Any class in \( \pi_*(kq \wedge \overline{kq}^{\wedge n}) \) coming from \( \pi_*(bo^{(j)}) \) (placed in the correct bidegrees) is \( \tau \)-torsion free. In particular, these classes are in one-to-one correspondence with a subset \( S \) of classes in \( \pi_*(bo \wedge \overline{bo}^{\wedge n}) \).
2. The classes in \( \pi_*(kq \wedge \overline{kq}^{\wedge n}) \) coming from \( \pi_*(\Sigma^{3}\Sigma^{3}kq) \) satisfy precisely one of the following:
   a) The class is \( \tau \)-torsion free. The subset of such \( \tau \)-torsion free classes is in one-to-one correspondence with the complement of \( S \) in \( \pi_*(bo \wedge \overline{bo}^{\wedge n}) \).
   b) The class is simple \( \tau \)-torsion. Any such class is \( \eta \)-torsion free.

**Proposition 4.4.**

1. Each \( \tau \)-torsion free homotopy class in \( \pi_*(kq \wedge \overline{kq}^{\wedge n}) \) is either of Adams filtration zero or one, or in the image of \( d_n \), or is mapped essentially under \( d_n \), or \( n = 0 \) or \( n = 1 \) and the homotopy can be identified with \( \pi_*(HZ_1 \wedge kq) \).
2. Each \( \tau \)-torsion class in \( \pi_*(kq \wedge \overline{kq}^{\wedge n}) \) is in the image of \( d_n \), or is mapped essentially under \( d_n \), or \( n = 0 \) or \( n = 1 \) and the homotopy can be identified with \( \pi_*(HZ_1 \wedge kq) \).

**Proof.**

1. This follows immediately from Theorem 4.1 and Betti realization.
2. The only \( \tau \)-torsion classes contain \( \eta^j \) for \( j \geq 3 \) as a factor. The multiplication map \( kq \wedge kq \to kq \) equips the \( kq \)-resolution with a \( \pi_*(kq) \)-module structure, so the fate of \( \eta^j \) for \( j \geq 3 \) is the same as the fate of \( \eta \) and \( \eta^j \).

This was determined in the first part of the proposition.

\( \square \)

**Proposition 4.5.** Let \( k \geq 1 \). The \( d_1 \)-differential \( d_1 : E_1^{0,4k,\ell} \cong \mathbb{Z} \to \mathbb{Z} \cong E_1^{1,4k,\ell} \) is given by multiplication by \( 2^{\rho(k)} \) for any \( \ell \leq 2k \), where \( \rho(k) \) is the 2-adic valuation of \( 4k \).

**Proof.** This follows immediately from Theorem 4.2 and Betti realization. \( \square \)

These propositions completely determine the 0- and 1-lines of the \( kq \)-resolution; see Parts (a) and (b) of Theorem 5.1.

### 4.3. A vanishing line of slope 1/5

As we saw in the last section, there is a vanishing line of slope 1/3 in the \( E_1 \)-page of the \( kq \)-resolution. It turns out that this naïve vanishing line suffices to calculate \( \pi_*(S^{0,0})[\eta^{-1}] \), but in order to calculate \( \pi_*(S^{0,0})[v_1^{-1}] \), we will need a stronger vanishing result. To prove this stronger vanishing statement, we generalize Mahowald’s arguments [39, pp. 380-381] and [11 (c)]

We begin by defining some motivic analogs of classical finite complexes.
Definition 4.6. Let $V(0) := \text{cofib}(S^{0,0} \rightarrow \Sigma S^{0,0})$ be the motivic mod two Moore spectrum. Using the long exact sequence in homotopy, one can show that there exists a unique lift of $\eta \in \pi_{1,1}(S^{0,0})$ to a map $\tilde{\eta} : \Sigma^{1,1}V(0) \rightarrow V(0)$.

Let $Y := \text{cofib}(\Sigma^{1,1}V(0) \rightarrow \Sigma V(0))$. Then $Y \simeq V(0) \wedge C\eta$ where $C\eta := \text{cofib}(S^{1,1} \rightarrow S^{0,0})$. Using the Atiyah-Hirzebruch spectral sequence, one can show that $Y$ admits a self-map $v_1 : \Sigma^{2,1}Y \rightarrow Y$. Finally, define $A_1$ by $A_1 := \text{cofib}(\Sigma^{2,1}Y \rightarrow Y)$.

Remark 4.7. The Betti realization of $Y$ is the classical spectrum $Y$, and the motivic self-map $v_1 : \Sigma^{2,1}Y \rightarrow Y$ realizes to the classical self-map $\psi^{2,1} : \Sigma Y \rightarrow Y$. Since the classical self-map is non-nilpotent and the motivic self-map is $\tau$-torsion free, the motivic self-map is non-nilpotent.

Lemma 4.8. The motivic spectrum $A_1$ realizes $A(1)$. In other words, there is an isomorphism of $A(1)$-modules $H^{**}(A_1) \cong A(1)$.

Proof. This follows from the long exact sequences in motivic cohomology associated to the cofiber sequences defining $A_1$, along with the facts that $SR^1$ detects $2$, $SR^2$ detects $\eta$, and $Q_1$ detects $v_1$. \hfill \square

Lemma 4.9. The $kq$-resolution for $A_1$ coincides with the motivic $HF_2$-Adams resolution for $A_1$. In particular, we have

$$kqE_2(A_1) \cong E_{2\text{mass}}^n(A_1) \cong \operatorname{Ext}^{**}_A(A(1)).$$

Proof. The motivic Adams spectral sequence

$$E_2 = \operatorname{Ext}^{**}_A(H^{**}(A_1 \wedge kq)) \cong \operatorname{Ext}^{**}_A(A(1)) \Rightarrow \pi_{**}(A_1 \wedge kq)$$

collapses to show that $A_1 \wedge kq \simeq HF_2$. The result then follows by definition of the $kq$- and $HF_2$-Adams resolutions. \hfill \square

Proposition 4.10. We have $\operatorname{Ext}^{n,t,*}_A(A(1)) = 0$ for $6n > t + 5$.

Proof. Let $I = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i > 0, j \geq 0\} \setminus \{(1, 0), (1, 1), (2, 0)\}$. The motivic May spectral sequence [19] converging to $\operatorname{Ext}^{**}_A(A(1))$ has $E_1$-page $M_2[h_{i,j} | (i, j) \in I]$. Define $s(i, j) = 1/\text{stem}(h_{i,j})$ to be the slope of the line spanned by $h_{i,j}$ in Adams grading, i.e. with Adams filtration on the vertical axis and stem on the horizontal axis. Then it is easy to check that

$$\max_{(i,j) \in I} s(i, j) = 1/3, \quad \text{and} \quad \max_{(i,j) \in I \setminus \{(1,2)\}} s(i, j) = 1/5.$$

Moreover, the maximal $s(i, j)$ for $(i, j) \in I$ is realized by $(i, j) = (1, 2)$. Since $h_{1,2}^4 = h_{2,1}^2 h_3^2 = 0$ by [19] Tables 2-3, we conclude that aside from a finite number of $h_{1,2}$ multiples, the May spectral sequence is generated by classes on or below a line of slope $1/5$. Since $h_{1,2}^4 = 0$, it follows that if $n > \frac{1}{5}(t - n) + 1$ then $\operatorname{Ext}^{n,t,*}_A(A(1)) = 0$. Thus we conclude that the Ext-group is trivial when $6n > t + 5$. \hfill \square

Combining the previous proposition with Lemma 4.9, we obtain the following vanishing region in the $kq$-resolution for $A_1$.

Corollary 4.11. We have $kqE_2^{n,t,*}(A_1) = 0$ for $6n > t + 5$. 
We claim that the same vanishing region holds in $kqE_{2}^{*,*,*}(S^{0,0})$. In [41], Mahowald first argues that there is a vanishing line of slope $1/5$ in the $E_2$-page of the bo-resolution for the mod 2 Moore spectrum. He then deduces the vanishing line for the sphere from the cofiber sequence
$$S^{0} \to S^{0} \to M.$$  
We will prove that the $E_2$-page of the $kq$-resolution for the motivic mod 2 Moore spectrum has a vanishing line of slope $1/5$, and from there argue that the $E_2$-page for the motivic sphere spectrum has a vanishing line of slope $1/5$ in a similar manner to Mahowald. However, our argument for the vanishing line for $M$ differs from the one in [41]. The essence of our argument is to bootstrap up from $A_1$ to $M$ via a sequence of “Bockstein spectral sequences.” Properly speaking, there are no such Bockstein spectral sequences for reasons we will highlight below.

The cofiber sequence
$$\Sigma^{2,1}Y \xrightarrow{v_1} Y \to A_1,$$  
gives the following diagram:
$$\begin{array}{cccc}
Y & \xleftarrow{\Sigma^{2,1}Y} & \Sigma^{4,2}Y & \cdots \\
\downarrow & & \downarrow & \\
A_1 & \Sigma^{2,1}A_1 & \Sigma^{4,2}A_1 & \\
\end{array}$$  
Since $H^*(A_1) = A(1)$, we get short exact sequences of $kq$-Adams spectral sequence $E_1$-pages
$$0 \to kqE_1^{n,t+2,w+1}(Y) \to kqE_1^{n,t,w}(Y) \to kqE_1^{n,t,w}(A_1) \to 0.$$  
Indeed, we have cofiber sequences
$$kq \wedge kq^{\wedge n} \wedge \Sigma^{2,1}Y \to kq \wedge kq^{\wedge n} \wedge Y \to kq \wedge kq^{\wedge n} \wedge A_1,$$  
and applying cohomology gives a split short exact sequence. Therefore we get a short exact sequence in Ext; since the Adams spectral sequences collapse, we get short exact sequences of $kq$-Adams spectral sequence $E_1$-terms. Thus we get a long exact sequence on $E_2$-terms,
$$\cdots \to kqE_2^{n,t+2,w+1}(Y) \to kqE_2^{n,t,w}(Y) \to kqE_2^{n,t,w}(A_1) \to kqE_2^{n+1,t+2,w+1}(Y) \to \cdots,$$  
This gives an exact couple of the form
$$E_1^{n,t,w,\alpha} = kqE_2^{n,t,w}(A_1) \cdot \{v_1^\alpha\} \Rightarrow E_2^{n,t+2w,\alpha}(Y)$$  
from which we deduce the following vanishing result.

**Proposition 4.12.** The $E_2$-term of $kq$-based Adams spectral sequence for $Y$ has a vanishing line of slope $1/5$. In particular, if $t + 5 < 6n$, then $kqE_2^{n,t,w}(Y) = 0$.

Recall next that there is a cofiber sequence
$$\Sigma^{1,1}M \xrightarrow{\eta} M \to Y.$$  
We would like produce an “$\eta$-Bockstein spectral sequence” as we did above in order to deduce a vanishing line. However, this doesn’t quite work as above because the cohomology of $Y$ is not free over $A(1)$. In particular, there is not a short exact sequence on $kqE_1$ and hence there is not the desired long exact sequence on $kqE_2$.  

\[\]
We will show that there is a short exact sequence on $kq E_1$ in the region above a line of slope $1/5$, from which we will be able to produce an $\eta$-BSS in a region.

Observe that applying $kq E_1^{n,t,*}$ to the above cofiber sequence gives a long exact sequence

$$\cdots \to kq E_1^{n,t-1,w-1}(M) \xrightarrow{\eta} kq E_1^{n,t,w}(M) \xrightarrow{\eta} kq E_1^{n,t,w}(Y) \xrightarrow{\partial} kq E_1^{n,t-2,w-1}(M) \xrightarrow{\eta} kq E_1^{n,t-1,w}(M) \to \cdots$$

(2)

Whenever multiplication by $\eta$ is 0, this long exact sequence breaks up into short exact sequences which give rise to long exact sequences on $kq E_2$. The following lemma says that this occurs above a line of slope $1/5$.

Lemma 4.13. In $kq E_1^{n,t,w}(M)$, multiplication by $\eta$ is trivial so long as $t < 6n - 4$.

Proof. Consider the cofiber sequence for the mod 2 Moore spectrum. This induces a long exact sequence

$$\cdots \to kq E_1^{n,t,w}(S^0) \to kq E_1^{n,t,w}(S^0) \to kq E_1^{n,t,w}(M) \to kq E_1^{n-1,t,w}(S^0) \to \cdots$$

Recall that

$$kq E_1^{n,t,w}(S^0) \cong \bigoplus_{\ell(I) = n} \text{Ext}_{A(1)}^{*,*}(\Sigma^{|I|} H\mathbb{Z}_I)$$

where the sum runs over multi-indices $I$ of positive integers whose length is $n$, and where

$$H\mathbb{Z}_I := H\mathbb{Z}_{i_1} \otimes \cdots \otimes H\mathbb{Z}_{i_n}.$$  

It follows from Lemmas 3.33 and 3.34 that when

$$t - s + 4 < \kappa(I) := \begin{cases} 4|I| + 2|I| - \alpha(I) & |I| \equiv 0 \mod 2 \\ 4|I| + 2|I| - \alpha(I) - 1 & |I| \equiv 1 \mod 2 \end{cases}$$

where

$$\alpha(I) = \sum_{j=1}^n \alpha(i_j)$$

that

$$\text{Ext}_{A(1)}^{s,t,w}(\Sigma^{|I|} H\mathbb{Z}_I)$$

consists only of $h_0$-towers and $v_1$-torsion classes in Adams filtration zero. Thus multiplication by 2 is injective in this region. Hence the long exact sequence above becomes short exact sequence

$$0 \to \text{Ext}_{A(1)}^{s,t,w}(\Sigma^{|I|} H\mathbb{Z}_I) \to \text{Ext}_{A(1)}^{s,t,w}(\Sigma^{|I|} H\mathbb{Z}_I) \to \text{Ext}_{A(1)}^{s,t,w}(A(0) \otimes \Sigma^{|I|} H\mathbb{Z}_I) \to 0.$$  

for $s > 0$. This shows that for $s > 0$ and $t - s < 6|I| - \alpha(I)$, the modules $\text{Ext}_{A(1)}^{s,t,w}(A(0) \otimes H\mathbb{Z}_I)$ consist only of simple $\eta$-torsion. Since the evil classes are also simple $\eta$-torsion, we can actually relax the condition so that $s \geq 0$.

Since the Adams spectral sequence computing $kq E_1^{n,t,w}(M)$ collapses at $E_2$, we conclude from the above, that $kq E_1^{n,t,w}(M)$ consists only of simple $\eta$-torsion whenever

$$t < \min\{\kappa(I) \mid \ell(I) = n\}.$$
Note that the lowest norm multi-index is
\[ I = (1,1,\ldots,1). \]

It follows from Lemma 3.33 that
\[ 6n < \min \{ \kappa(I) \mid \ell(I) = n \}. \]

This proves the lemma. \[\square\]

We can now obtain a vanishing line for \( M \).

**Proposition 4.14.** For \( t < 6n - 1 \), we have short exact sequences
\[ 0 \to kqE^{n,t,w}_1(M) \to kqE^{n,t,w}_1(Y) \to kqE^{n,t-2,w-1}_1(M) \to 0. \]

**Proof.** From the cofiber sequence (1), we have the long exact sequence (2). In the case \( t + 1 < 6n \), it follows from the previous lemma that multiplication by \( \eta \) is trivial. The result follows. \[\square\]

Since the maps in the short exact sequence fo the proposition commute with the \( d_1 \)-differential, we obtain the following corollary.

**Corollary 4.15.** For \( t < 6n - 1 \), we have a short exact sequence of cochain complex
\[ 0 \to kqE^{n,t,w}_1(M) \to kqE^{n,t,w}_1(Y) \to kqE^{n,t-2,w-1}_1(M) \to 0 \]
where the differential is the Adams \( d_1 \)-differential. Thus, in this region, we get a long exact sequence
\[ \cdots \to kqE^{n,t,w}_2(M) \to kqE^{n,t,w}_2(Y) \to kqE^{n,t-2,w-1}_2(M) \to kqE^{n+1,t,w}_2(M) \to \cdots. \]
Moreover, the connecting homomorphism detects multiplication by \( \eta \) in \( \pi_+ \).

We will use this to prove that the \( E_2 \)-page of the \( kq \)-resolution for the mod 2 Moore spectrum has a vanishing line of slope 1/5. Fix a pair \((t,w)\). Then from the long exact sequence (3) and the vanishing line of slope 1/5 for \( Y \), we see that \( kqE^{n,t,w}_2(Y) = 0 \) whenever \( t + 5 < 6n \). Hence there is a sequence of isomorphisms
\[ kqE^{n,t-2,w-1}_2(M) \cong kqE^{n+1,t,w}_2(M) \cong \cdots \cong kqE^{n+j,t+2(j-1),w+j-1}_2(M) \cong \cdots \]
for all \( j \geq 0 \). Eventually, these terms will lie in the region above the naïve vanishing line of slope 1/3. Indeed, for \( j \gg 0 \), we have
\[ t + 2(j - 1) < 4(n + j). \]

Therefore all of the groups in this sequence vanish. This shows that whenever \( t + 5 < 6n \), we have that
\[ kqE^{n+j,t+2(j-1),w+j-1}_2(M) = 0 \]
for all \( j \geq 0 \). Thus, we have shown the following.

**Corollary 4.16.** For \( t + 7 < 6n \), we have that
\[ kqE^{n,t,w}_2(M) = 0. \]
We are now in a position to prove the vanishing line for the sphere. In light of Proposition 4.4, the only non-trivial classes on $E^n_{\infty,t,w}(S^0)$ are detected by elements of Adams filtration 0 or 1. For this reason, it suffices to show a vanishing line for the algebraic $kq$-resolution for the sphere. This is a spectral sequence which is obtained by applying the functor $\text{Ext}^{s,t,w}_{A_*}$ to the $kq$-Adams resolution. It is a convergent spectral sequence of the form

$$alg E^n_{1,*,t,w}(X) = \text{Ext}^{s,t,w}_{A_*}(H_*(kq \wedge kq^n \wedge X)) \Rightarrow \text{Ext}^{s+n,t,w}_{A_*}(X).$$

Our argument is an adaption of the one found in [41].

In [41] Mahowald defines a function $a(j)$ whose values are given by the following table,

| $j \mod 4$ | 0 | 1 | 2 | 3 |
|------------|---|---|---|---|
$\begin{align*}
a(j) & \equiv 0 & -2 & -2 & -1
\end{align*}$

**Proposition 4.17** (Compare with Prop 3.6 of [41]). If $t < 6j + a(j)$, then:

1. If $a \in \text{Ext}^{0,t,*}(H_*(S_j))$, then either $h_0^i a \neq 0$ for all $i$ or $h_0 a = 0$;
2. If $a \in \text{Ext}^{s,t,*}(H_*(S_j))$, $a \neq 0$, then $a = h_0^i a'$ for some $a'$.

**Proof.** We can conclude this from [41] Prop. 3.6 by Betti realization and from our calculations in Section 3.3.

**Definition 4.18.** Let $\text{TExt}^{s,t,*}_{A(1)}(H_*(S_j))$ denote the subspace spanned by the classes $a$ for which $h_0^i a = 0$ for some $i$.

**Proposition 4.19.** If $t < 6j + a(j)$, then

$$\text{Ext}^{0,t+j,*}(S_j \wedge M) = \text{Ext}^{0,t+j,*}(S_j) \oplus \text{TExt}^{0,t,*}(S^{0,1} \wedge S_j).$$

Furthermore, for $t < 6j + a(j)$ we get a short exact sequence of cochain complexes

$$0 \rightarrow alg E_1^{0,t,*}(S^0) \rightarrow alg E_1^{1,0,t,*}(M) \rightarrow alg E_1^{1,0,t,*}(S^1) \rightarrow 0$$

**Proof.** The proof is the same as the proof of [41] Prop. 3.7. The cofiber sequence

$$S^0 \rightarrow M \rightarrow S^1$$

induces a long exact sequence in $\text{Ext}(S_j \wedge -)$. In particular, we have

$$\text{Ext}^{0,t,*}(S_j) \rightarrow \text{Ext}^{0,t,*}(S_j \wedge M) \rightarrow \text{Ext}^{0,t,*}(S_j \wedge S^1) \rightarrow \text{Ext}^{1,t,*}(S_j) \rightarrow \cdots.$$}

The connecting homomorphism is given by multiplication by $h_0$. By the previous proposition, in the range $t < 6j + a(j)$, the group $\text{TExt}$ is precisely the kernel of multiplication by $h_0$. The proposition follows.

Let $C_{j,t,*}^{s,t,*}(S^0) = \text{TExt}^{s,t,*}(S_j)$.

**Proposition 4.20.** For $t < 6j + a(j) - 4$, we have $H^*(C_{j,t,*}^{0,t,w}) = alg E_{2}^{0,t,w}$ and $alg E_{2}^{*,s,t+1,w} = 0 = H^*(C_{j,t,*}^{s,t+1,w})$ for $s > 0$.

**Proof.** In this range of dimensions, the $M_2[h_0]$-towers form an acyclic complex. This is because in Proposition 4.19 the left hand side has vanishing homology in this range. Also, when $s > 0$, then $C_{j,t,*}^{s,t,u} = 0$ for $s > 0$ in this range.

We deduce the following.
Proposition 4.21 (compare with Prop 3.9[1]). If \( t < 6j + a(j) - 4 \), then there is an exact sequence

\[
\cdots \to H^j(C_\bullet^{0,t,*}(S^0)) \to \text{alg} \ E_2^{j,0,t,*}(M) \to \text{alg} \ E_2^{j,0,t,*}(S^1) \to \text{alg} \ E_2^{j+1,0,t,*}(S^0) \to \cdots
\]

Proof. From Proposition 4.19 we have the short exact sequence of cochain complexes

\[
0 \to \text{alg} \ E_1^{0,0,t,*}(S^0) \to \text{alg} \ E_1^{0,0,t,*}(M) \to \text{alg} \ E_1^{0,0,t,*}(S^1) \to 0
\]

and from this it follows that we get a long exact sequence in cohomology. In the range \( t < 6n + a(n) - 4 \), it follows from Proposition 4.20 we can remove or impose a bar. This gives the desired long exact sequence. \( \square \)

From these, it follows that \( kq \ E_\infty(S^0) \) has a vanishing line of slope 1/5.

Theorem 4.22. In \( kq \ E_\infty^n(t,*) \) \( S^0 \) = 0 if \( t + 7 < 6n \).

Proof. The previous result implies that if \( \text{alg} \ E_2^{n',0,t,*}(M) = 0 \) for all \( n' \geq n \), then \( \text{alg} \ E_2^{n',0,t,*}(S^1) \cong \text{alg} \ E_2^{n'+1,0,t,*}(S^0) \) for all \( n' > n \). In particular, if we have fixed \( t \), then in the long exact sequence of the previous proposition, the groups \( \text{alg} \ E_2^{n',0,t,*}(M) \) are eventually all zero. For example, it follows from Corollary 4.16 that once \( t < 6n - 7 \), these groups are all 0 for later \( n' \). In other words, we have

\[
\text{alg} \ E_2^{n',0,t-1,*}(S^0) \cong \text{alg} \ E_2^{n'+1,0,t,*}(S^0)
\]

so long as \( t < 6n' - 7 \).

Now from the 1/3-vanishing line, we know that \( \text{alg} \ E_2^{n,0,t,*}(S^0) = 0 \) whenever \( t < 4n \). Combining these observations, we can “push” the vanishing of \( \text{alg} \ E_2^{n,0,t,*}(S^0) \). For example, consider \( t = 4n \). Clearly \( 4n < 6n - 7 \), and so we have the isomorphism

\[
\text{alg} \ E_2^{0,4n,*}(S^0) \cong \text{alg} \ E_2^{n+1,0,4n-1,*}(S^0)
\]

and the latter group is trivial as \( 4n-1 < 4n+4 \). More generally, if \( 4n \leq t < 6n-14 \), then we have the string isomorphisms

\[
\text{alg} \ E_2^{n,0,t,*}(S^0) \cong \text{alg} \ E_2^{n+1,0,t-1,*} \cong \cdots
\]

and at some point the topological degree falls within the naïve vanishing region above a line of slope 1/3, and so all of these groups are trivial.

From Proposition 4.4(1) above, we know that when \( 6n > t + 7 \) the group \( kq \ E_\infty^n(t,*) \) could only possibly possess a class which was detected by an element of Adams filtration 0 or 1. Now a class in Adams filtration 1 necessarily must be connected to one in Adams filtration 0 by multiplication by \( h_0 \) or \( h_1 \). But the argument above shows that an element of Adams filtration 0 cannot appear in the region \( 6n > t + 7 \). \( \square \)

5. Main results

In this section, we state our main result on the \( kq \)-resolution and state its two main applications to motivic periodicity.
5.1. The main theorem. We state our main computational result in the following theorem.

**Theorem 5.1.** We have the following:

1. The 0-line of the $kq$-resolution is given by
   \[ E_{\infty}^{0,*} \cong M_2[h_0, h_1, v_1^4] / (h_0 v_1^4, h_1^2 h_1) \]
   where $|h_0| = (0, 0)$, $|h_1| = (1, 1)$, and $|v_1^4| = (8, 4)$.

2. The 1-line of the $kq$-resolution is given by
   \[ E_{\infty}^{1,*} \cong \bigoplus_{k \geq 0} \Sigma^{4k} \mathbb{Z} / 2^{\mu(k)} [\tau] \oplus M_2[h_1, v_1^4] / (h_1^3 \tau). \]
   All of these classes are $v_1$-periodic.

3. $E_n^{t,u}_{\infty} = 0$ whenever $6n > t + 7.$

**Proof.** Parts (1) and (2) follow from Propositions 4.4 and 4.5. Part (3) is Theorem 4.22.

5.2. The $\eta$-local sphere. Recall that $\eta : S^{1,1} \to S^{0,0}$ is not nilpotent.

**Definition 5.2.** Let $X$ be a motivic spectrum with a non-nilpotent self-map $v : X \to \Sigma^{-r,-s} X$. We define the $v$-telescope of $X$ to be the colimit
   \[ v^{-1} X := \text{colim}(X \xrightarrow{v} \Sigma^{-r,-s} X \xrightarrow{v} \Sigma^{-2r,-2s} X \xrightarrow{v} \ldots). \]
   If $X = S^{0,0}$, we will refer to a $v$-telescope $v^{-1} S^{0,0}$ as the $v$-local sphere.

The motivic homotopy groups of $\pi_* (\eta^{-1} S^{0,0})$ were conjectured by Guillou-Isaksen in [26, Conj. 1.3(2)]. This conjecture was confirmed by Andrews-Miller in [5]. This serves as an “exotic” motivic analog of the $v_1$-periodic computation of the previous subsection.

**Theorem 5.3.** The motivic stable homotopy groups of the $\eta$-local sphere are
   \[ \pi_* (\eta^{-1} S^{0,0}) \cong F_2[h_1^{\pm 1}, \mu, \epsilon] / (\epsilon^2) \]
   where $|\eta| = (1, 1)$, $|\mu| = (9, 5)$, and $|\epsilon| = (8, 5)$.

**Proof.** We claim that $\pi_* (S^{0,0}[\eta^{-1}])$ is detected in the 0- and 1-lines of the $kq$-resolution. If this holds, then the corollary is clear from inspection of the 0- and 1-lines described in Theorem 5.1.

By part (2) of Proposition 4.4, every $\tau$-torsion class in $kq$-Adams filtration greater than one has Adams filtration zero or one, is killed by a differential, or supports a nontrivial differential. The classes in Adams filtration zero correspond to the classes in the zero line of the Adams spectral sequence for a connective cover of $bo$ or $bsp$ (placed in appropriate tridegrees). Within a fixed filtration, then, there can be at most two powers of $\eta$ detected. Therefore an infinite $\eta$-tower above $kq$-Adams filtration one would be detected along a line of slope at least 1/2 in the $kq$-resolution, contradicting the vanishing line of slope 1/3 proven in Lemma 3.38.

**Remark 5.4.** In the terminology of Andrews [4] and Gheorghe [21], the computation of $\pi_* (\eta^{-1} S^{0,0})$ completely identifies all $w_0$-periodic classes in the motivic stable stems.
C-motivic $kq$-resolutions

Gheorghe-Isaksen-Krause-Ricka have constructed a C-motivic modular forms spectrum $mmf$ [22] which serves as a computational analog of the classical topological modular forms spectrum $tmf$ [17]. The $mmf$-based Adams spectral sequence might serve as useful tool for understanding the $v_1$-periodic and $w_1$-periodic motivic stable stems.

5.3. The $v_1$-periodic C-motivic stable stems. Our goal in this section is to identify the $v_1$-periodic part of the motivic stable stems. We begin with some definitions following [39, Sec. 6].

**Definition 5.5.** Let $\gamma : S^{j,k} \to X$ and let $Y := S^{0,0}/(2,\eta)$. Then there are potentially four maps

1. $\gamma^\#_i : \Sigma^{j-3,k-1}Y \xrightarrow{p_1} S^{j,k} \xrightarrow{\gamma} X$ where $p_1$ is the collapse onto the top cell,
2. $\Sigma^{j-2,k-1}Y \xrightarrow{p_2} \Sigma^{j,k}S/2 \xrightarrow{\gamma^\#} X$ where $p_2$ is the collapse onto the top two cells and $\gamma^\#$ is an extension of $\gamma$ (if it exists),
3. $\Sigma^{j-1,k}Y \xrightarrow{p_3} \Sigma^{j-2,k-1}B^2_4 \xrightarrow{\gamma^\#} X$ where $B^2_4$ is the simplicial 2-coskeleton of the 4-skeleton of $B_{gm\mu}^2$ and $p_3$ and $\gamma^\#$ are analogously defined, and
4. $\Sigma^{j,k}Y \xrightarrow{\gamma^\#} Y$.

If a map of type $(i,h)$ exists and the composite

$$\Sigma^{j-4+i+2\ell,k-1+h+\ell}Y \xrightarrow{u_\ell} \Sigma^{j-4+i,k-1+h}Y \xrightarrow{\gamma^\#} X$$

is essential for all $\ell \geq 0$ for all $\gamma^\#$, then we say that $\gamma$ is $v_1$-periodic of type $i$.

**Example 5.6.** (compare with [39, Exm. 6.2] The Atiyah-Hirzebruch spectral sequence for $Y$ and Betti realization imply that $\eta \in \pi_{1,1}(S^{0,0})$ is $v_1$-periodic of type 2. Similarly, one can show that $\nu \in \pi_{3,2}(S^{0,0})$ is $v_1$-periodic of type 3. The generator of the image of the two torsion in the image of the motivic unitary J-homomorphism (see [33] and below) in $\pi_{4k-1,4k}(S^{0,0})$ is $v_1$-periodic of type 1.

**Theorem 5.7.** The only $v_1$-periodic classes in $\pi_{**}(S^{0,0})$ are those described in parts (1) and (2) of Theorem 5.1.

**Proof.** The proof is identical to the proof of [39, Thm. 6.3]. Above $kq$-Adams filtration one, every class is $v_1$-torsion. There can be at most two classes in a $v_1$-periodic family detected in a fixed filtration, so any $v_1$-periodic family is detected on or above a line of slope $1/4$. This contradicts the vanishing line of slope $1/5$ in part (c) of Theorem 5.1.

6. Motivic Telescope Conjectures

In this section, we place the computations of Theorem 5.3 and Theorem 5.7 in the context of chromatic motivic homotopy theory.

6.1. Localization functors. We begin with some motivic analogs of the results from [44]. Let $E$ be any motivic spectrum.

**Definition 6.1.** (compare with [44])

1. A motivic spectrum $W$ is $E$-local if and only if $[T,W] = 0$ for every $E$-acyclic spectrum $T$.
2. A map $f : X \to Y$ is an $E$-equivalence if and only if $E_*f$ is an isomorphism.
Theorem 6.2. \cite{13} \cite{28} For any motivic spectra \(E\) and \(X\), there is a unique (up to canonical equivalence) \(E\)-equivalence \(\eta : X \to L_E X\) where \(L_E X\) is an \(E\)-local motivic spectrum.

The motivic spectrum \(L_E X\) is called the Bousfield localization of \(X\) with respect to \(E\).

Definition 6.3. (compare with \cite{44} Def. 3) Let \(A\) be a set of motivic spectra.

1. A motivic spectrum \(W\) is finitely \(A\)-local if and only if \([\Sigma^m \cdot A, W] = 0\) for every \(A \in A\) and every \(m, n \in \mathbb{Z}\).
2. A motivic spectrum \(Z\) is finitely \(A\)-acyclic if and only if \([Z, W] = 0\) for every finitely \(A\)-local motivic spectrum \(W\).
3. A map \(f : X \to Y\) is a finite \(A\)-equivalence if and only if its mapping cone is finitely \(A\)-acyclic.

The proof of \cite{44} Thm. 4 carries over to the motivic setting without change to prove the following:

Theorem 6.4. For any set \(A\) of finite motivic spectra and any motivic spectrum \(X\), there is a unique (up to canonical equivalence) finite \(A\)-equivalence \(\eta : X \to L^f_A X\) where \(L^f_A X\) is an \(A\)-local spectrum.

The motivic spectrum \(L^f_A X\) is called the finite localization of \(X\) with respect to \(A\). If \(A\) is the set of finite \(E\)-acyclic spectra for some motivic spectrum \(E\), then we will write \(L^f_E\) for \(L^f_A\). Any \(E\)-local spectrum is finitely \(E\)-local, so we obtain a unique morphism \(L^f_E X \to L_E X\) under \(X\). This gives rise to a natural transformation \(L^f_E \to L_E\), and the discussion from \cite{44} Sec. 2 carries over to the motivic setting without alteration to give the following results.

Corollary 6.5. The natural transformation \(L^f_E \to L_E\) is an equivalence if and only if \(E\) is smashing (i.e. the map \(L_E X \cong L_E (X \wedge S^{0,0}) \to X \wedge L_ES^{0,0}\) is an equivalence for all \(X\)) and the natural map \(L^f_ES^{0,0} \to L_ES^{0,0}\) is an equivalence.

Corollary 6.6. Finite \(A\)-localization is Bousfield localization with respect to the spectrum \(L^f_A S^{0,0}\).

We conclude our discussion of localization with the following definition and lemma.

Definition 6.7. The Bousfield class of \(E\), denoted \(\langle E \rangle\), is the set of spectra \(X\) such that \(E_+(X) = 0\).

We record one more lemma, which follows immediately from the definitions.

Lemma 6.8. If \(\langle E \rangle = \langle F \rangle\), then \(L_E \simeq L_F\) and \(L^f_E \simeq L^f_F\).

6.2. Recollection of the classical Telescope Conjecture. In order to motivate the motivic Telescope Conjecture, we recall three equivalent formulations of the classical Telescope Conjecture. Let \(K(n)^{cl}\) denote the classical \(n\)-th Morava K-theory and let \(E(n)^{cl}\) denote the classical \(n\)-th Johnson-Wilson theory. Recall that a classical finite complex \(X\) is of type \(n\) if \(K(i)^{cl}_n(X) = 0\) for \(i < n\) and \(K(n)^{cl}_n(X) \neq 0\). If \(X\) is of type \(n\), then there exists a non-nilpotent \(v_n\)-self-map \(v : \Sigma^k X \to X\) for some \(k > 0\) which induces an isomorphism in \(K(n)^{cl}\)-homology.

The classical Telescope Conjecture first appeared in \cite{51} 10.5, where it had the following form:
Conjecture 6.9 (Classical Telescope Conjecture (Telescopic Formulation)). Let \( X \) be any finite complex of type \( n \) with non-nilpotent \( v_n \)-self-map \( v : \Sigma^k X \to X \). Then \( \langle v^{-1} X \rangle \) depends only on \( n \), and \( \langle v^{-1} X \rangle = \langle K(n)^{cl} \rangle \).

Let \( K(\leq n)^{cl} = \bigvee_{i=0}^{n} K(i)^{cl} \). Miller provided two new formulations of the classical Telescope Conjecture in [44, Sec. 3].

Conjecture 6.10 (Classical Telescope Conjecture (Localization Formulation)). The natural transformation \( L^f_{K(\leq n)^{cl}} \to L_{K(\leq n)^{cl}} \) is an equivalence.

Conjecture 6.11 (Classical Telescope Conjecture (Smashing Formulation)). The map \( L^f_{K(\leq n)^{cl}} S^0 \to L_{K(\leq n)^{cl}} S^0 \) is an equivalence.

These three formulations were shown to be equivalent in [44]. There are two key ideas in this identification. Using the fact that all finite localizations are smashing plus the fact that \( L_{K(\leq n)^{cl}} \) is smashing [52], Miller showed that the Localization and Smashing Formulations are equivalent. Then, Miller used the periodicity theorem, asymptotic uniqueness of \( v_n \)-self maps, and a thick subcategory argument to identify the Telescopic and Localization Formulations. In particular, one needs the following identification.

Proposition 6.12. If \( X \) is a \( K(n-1)^{cl} \)-acyclic finite complex with \( v_n \)-self-map \( v : \Sigma^k X \to X \), then the map \( X \to v^{-1} X \) is a finite \( K(n)^{cl} \)-localization.

6.3. The motivic Telescope Conjecture. Our goal now is to propose motivic analogs of the three formulations of the classical Telescope Conjecture described above. We begin with some background from chromatic motivic homotopy theory.

First, recall that Borghesi defined \( \text{C-motivic} \) \( K \)-theories \( K(n) \) in [12] satisfying

\[
K(n)_{**} \cong \mathbb{M}_2[v_n^{\pm 1}]
\]

where \( |v_n| = (2^{n+1} - 2, 2^n - 1) \). These were also defined by Hornbostel in [28].

Definition 6.13. A finite motivic spectrum \( X \) is of classical type \( n \) if \( K(i)_{**}(X) = 0 \) for \( i < n \) and \( K(n)_{**}(X) \neq 0 \).

Lemma 6.14. If \( X \) is of classical type \( n \) and \( H_{**}(X) \) is \( \tau \)-torsion free, then \( X \) admits a non-nilpotent \( v_n \)-self map \( v : X \to \Sigma^{-r} X \), i.e. a self-map \( v \) which induces an isomorphism on \( K(n) \)-homology. Moreover, any two such \( v \) coincide after raising them to suitable powers.

Proof. The Betti realization \( Re(X) \) is a classical spectrum of (classical) type \( n \), so it admits a non-nilpotent \( v_n \)-self-map \( v^{cl} : X \to \Sigma^{-r} X \) which induces an isomorphism on \( K(n)^{cl} \)-homology. Betti realization is strong symmetric monoidal, so the map

\[
\pi_{s,t}(X) \to \pi_{s}(Re(X))
\]

is a homomorphism of graded rings. In particular, there exists some non-trivial element \( v \in \pi_{s,t}(X) \) which maps to \( v^{cl} \). Since \( v^{cl} \) is non-nilpotent, the element \( v \) must also be non-nilpotent. The isomorphism \( H_{**}(X)[\tau^{-1}] \cong H_{*}(Re(X))[\tau^{\pm 1}] \) and the universal coefficient theorem show that \( v \) induces an isomorphism in \( K(n) \)-homology. Asymptotic uniqueness follows similarly from Betti realization.

Lemma 6.15. Let \( X \) and \( Y \) be finite motivic spectra of classical type \( n \) with \( \tau \)-torsion free homology with non-nilpotent \( v_n \)-self-maps \( \psi \) and \( \phi \), respectively, and
let \( f : X \to Y \) be any map. Then there are positive integers \( i \) and \( j \) for which the diagram
\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow \phi^i \quad \quad \quad \quad \downarrow \phi^j \\
\Sigma^{-r,-s}X \xrightarrow{\Sigma^{-r,-s}f} \Sigma^{-r,-s}Y
\end{array}
\]
commutes.

**Proof.** The result follows from Betti realization and the analogous classical result of Hopkins and Smith \cite{HopkinsSmith}.

**Remark 6.16.** As a first guess, one might take \( K(\leq n) := \bigvee_{i=0}^{n} K(i) \) and conjecture that \( L_{f}K(\leq n) \to L_{K(\leq n)} \) is an equivalence. The previous lemma allows one to identify \( L_{f}K(\leq n)X \simeq v^{-1}X \) for any finite motivic spectrum \( X \) of classical type \( n \) with \( \tau \)-torsion free homology, where \( v \) is a non-nilpotent \( v_{n} \)-self map of \( X \). This provides part of the input needed to relate the Telescopic and Localization Formulations of the resulting Telescope Conjectures.

However, there are still several issues with this approach. We do not know if \( L_{K(\leq n)} \) is smashing, and we do not expect a thick subcategory argument to hold in this context. Moreover, our calculations in Section 6.5 suggest that the motivic Telescope Conjecture at height one should carry more information that that which is detected by \( K(0) \vee K(1) \).

Gheorghe constructed exotic \( \mathbb{C} \)-motivic Morava K-theory spectra \( K(w_{n}) \) in \cite[Cor. 3.14]{Gheorghe} with
\[
K(w_{n})^{**} \cong \mathbb{F}_{2}[w_{n}^{\pm 1}]
\]
where \( |w_{n}| = (2^{i+2} - 3, 2^{i+1} - 1) \). These detect certain “exotic” forms of periodicity in the \( \mathbb{C} \)-motivic stable stems. For example, the spectrum \( K(w_{0}) \) detects \( \eta \)-periodicity, and \( K(w_{1}) \) detects the non-nilpotent self-map \( w_{1}^{2} : \Sigma^{20,12}S^{0.0}/\eta \to S^{0.0}/\eta \) which Andrews used to produce various exotic periodic families in \cite{Andrews}.

Gheorghe’s exotic Morava K-theories fit into a larger family of exotic Morava K-theories \( K(\beta_{ij}) \) defined by Krause in \cite{Krause}. The following proposition specializes \cite[Prop. 6.9]{Krause} to the case \( p = 2 \).

**Proposition 6.17.** \cite{Krause} For each \( i > j \geq 0 \), there is a \( \mathbb{C} \)-motivic 2-complete cellular \( C\tau \)-module \( K(\beta_{ij}) \) with
\[
K(\beta_{ij})^{**} \cong \mathbb{F}_{2}[\alpha_{ij}, \beta_{ij}^{\pm 1}]/(\alpha_{ij}^{2} = \beta_{ij}),
\]
with \( |\alpha_{ij}| = (2^{i+1}(2^{i} - 1) - 1, 2^{i}(2^{i} - 1)) \) and \( |\beta_{ij}| = (2^{j+2}(2^{i} - 1) - 2, 2^{j+1}(2^{i} - 1)) \).

For \( j = 0 \), they admit an \( E_{\infty} \)-ring structure.

Krause observes that \( K(w_{n-1}) \simeq K(\beta_{n,0}) \). Following \cite[Prop. 4.35]{Krause} and \cite[Pg. 124]{Krause}, we define
\[
d_{ij} := \begin{cases} 
1 & \text{if } j = -1, \\
\frac{1}{2^{i+1}(2^{i} - 1)} & \text{else}.
\end{cases}
\]

We set \( K(\beta_{i,-1}) := K(i) \) for \( i \geq 0 \).
Definition 6.18. A finite motivic spectrum $X$ is of type $(m, n)$ if $H_{**}(X)$ is $\tau$-torsion free, and $K(\beta_{ij})_{**}(X) = 0$ for all $i > j \geq -1$ with $d_{ij} > d_{mn}$ and $K(\beta_{mn})_{**}(X) \neq 0$.

Proposition 6.19. Suppose that $X$ is a finite motivic spectrum of type $(m, n)$. Then there is a non-nilpotent self-map $v : \Sigma^{0}X \rightarrow X$ of slope $d_{mn}$ which induces an isomorphism on $K(\beta_{mn})_{**}(X)$. Moreover, any two such $v$ coincide after raising them to suitable powers.

Proof. This follows from the combination of Lemma 6.14 and [37, Thm. 6.12]. □

Definition 6.20. We will refer to a non-nilpotent self-map as described in Proposition 6.19 as a self-map of type $(m, n)$.

Example 6.21.

(1) A non-nilpotent self-map of type $(m, -1)$ is a $v_{m}$-self-map.

(2) A non-nilpotent self-map of type $(m, 0)$ is a $w_{m-1}$-self-map.

Applying Lemma 6.15 and [37, Thm. 6.12] gives the following corollary.

Corollary 6.22. Let $X$ and $Y$ be finite motivic spectra of type $(m, n)$ with non-nilpotent self-maps of type $(m, n)$ $\psi$ and $\phi$, respectively, and let $f : X \rightarrow Y$ be any map. Then there are positive integers $i$ and $j$ for which the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow\psi^i & & \downarrow\phi^j \\
\Sigma^{-r,-s}X & \xrightarrow{f} & \Sigma^{-r,-s}Y
\end{array}
\]

commutes.

The proof of [44, Prop. 14] now carries over to prove the following:

Corollary 6.23. If $X$ is a finite motivic spectrum of type $(m, n)$ with non-nilpotent self-map $v : X \rightarrow \Sigma^{-r,-s}X$ of type $(m, n)$, then the map $X \rightarrow v^{-1}X$ is a finite $K(\beta_{mn})$-localization.

In other words, we can identify the finite $K(\beta_{mn})$-localization of a type $(m, n)$ spectrum $X$ with the telescope of its non-nilpotent self-map of type $(m, n)$. This leads to the following conjectures.

Conjecture 6.24 (Motivic Telescope Conjecture (Telescopic Formulation)). Let $X$ be a finite motivic spectrum of type $(m, n)$ with non-nilpotent self-map $v : \Sigma^{-r,-s}X \rightarrow X$ of type $(m, n)$. Then $\langle v^{-1}X \rangle$ depends only on $m$ and $n$, and $\langle v^{-1}X \rangle = \langle K(\beta_{ij}) \rangle$.

Let $K(\leq \beta_{mn}) := \bigvee_{(i,j) \in S_{mn}} K(\beta_{ij})$ where $S_{mn}$ is the set of $(i, j)$ such that $d_{ij} > d_{mn}$. We define $L_{mn}^{f} := L_{K(\leq \beta_{mn})}^{f}$ and $L_{mn} := L_{K(\leq \beta_{mn})}$.

Conjecture 6.25 (Motivic Telescope Conjecture (Localization Formulation)). The natural transformation $L_{mn}^{f} \rightarrow L_{mn}$ is an equivalence.

Conjecture 6.26 (Motivic Telescope Conjecture (Smashing Formulation)). The map $L_{mn}^{f}S_{0,0} \rightarrow L_{mn}S_{0,0}$ is an equivalence.
The equivalence of these three formulations would follow from Corollary 6.23 along with the following conjecture which may be viewed as a motivic analog of Ravenel’s Smashing Conjecture [51, 10.6].

**Conjecture 6.27** (Motivic Smashing Conjecture). For each \((m, n)\), the localization functor \(L_{mn} \) is smashing, i.e. \(L_{mn} X \simeq X \wedge L_{mn} S^{0,0}\).

6.4. **The motivic Telescope Conjecture for \(K(\leq \beta_{10})\).** Our goal in this section is to relate our calculations to Conjecture 6.26 in the case where \((m, n) = (1, 0)\). In Proposition 6.28, we show that \(L^{f}_{10} S^{0,0} \simeq \eta^{-1} S^{0,0}\). We then show that \(\eta^{-1} S^{0,0} \simeq L_{cKW} S^{0,0}\) in Proposition 6.29 where \(cKW \simeq \eta^{-1} kq\) is connective Witt theory. Finally, we conjecture that \(cKW = \langle K(0) \vee K(w_0) \rangle\) in Conjecture 6.30 which would imply that \(L_{cKW} S^{0,0} \simeq L_{10} S^{0,0}\). In summary, we will show

\[
L^{f}_{10} S^{0,0} \simeq \eta^{-1} S^{0,0} \simeq L_{cKW} S^{0,0} \simeq L_{10} S^{0,0}.
\]

**Proposition 6.28.** There is an equivalence of motivic spectra \(L^{f}_{10} S^{0,0} \simeq \eta^{-1} S^{0,0}\).

Note that \(S^{0,0}\) is of type \((0, -1)\) and not \((1, 0)\), so this lemma is not automatic from Definition 6.18 and Corollary 6.23.

**Proof.** Recall that \(S^{0,0}\) has (at least) two non-nilpotent self-maps:

1. The degree 2 self-map \(2 : S^{0,0} \to S^{0,0}\) is a non-nilpotent self-map of type \((0, -1)\).
2. The first Hopf map \(\eta : S^{1,1} \to S^{0,0}\) is a non-nilpotent self-map of type \((1, 0)\).

In particular, \(S^{0,0}\) is of type \((0, -1)\). In order to apply Corollary 6.23, we realize \(S^{0,0}\) as a homotopy limit of finite spectra of type \((1, 0)\), then verify that this homotopy limit commutes with \(\eta\)-localization.

Let \(S^{0,0}/2^k := cofib(S^{0,0} \to S^{0,0})\) be the mod \(2^k\) Moore spectrum. Then we have

\[
S^{0,0} = (S^{0,0})^\wedge_2 \simeq \lim_k S^{0,0}/2^k,
\]

where the first equality is a reminder that everything is implicitly 2-complete. The long exact sequence in homotopy arising from the cofiber sequence defining \(S^{0,0}/2^k\) implies that \(2 : S^{0,0}/2^k \to S^{0,0}/2^{k+1}\) is nilpotent and that \(\eta : \Sigma S^{0,0}/2^k \to S^{0,0}/2^k\) is non-nilpotent. Therefore \(S^{0,0}/2^k\) is of type \((1, 0)\) for all \(k \geq 1\), so by Corollary 6.23 we have \(L^{f}_{10}(S^{0,0}/2^k) \simeq \eta^{-1}(S^{0,0}/2^k)\) and thus

\[
S^{0,0} \simeq \lim_k \eta^{-1}(S^{0,0}/2^k).
\]

The lemma will follow if we can show that

\[
\lim_k \eta^{-1}(S^{0,0}/2^k) \simeq \eta^{-1}(\lim_k S^{0,0}/2^k) \simeq \eta^{-1} S^{0,0}.
\]

The homotopy groups of the left-hand side can be computed via an inverse limit of localized motivic Adams spectral sequences

\[
E_2 = \lim_k h^{-1} Ext^{***}_A(S^{0,0}/2^k) \Rightarrow \pi_*(\lim_k \eta^{-1}(S^{0,0}/2^k))
\]

and the homotopy groups of the right-hand side can be computed via a localized inverse limit motivic Adams spectral sequence

\[
E'_2 = h^{-1} \lim_k Ext^{***}_A(S^{0,0}/2^k) \Rightarrow \pi_*(\eta^{-1} S^{0,0}).
\]
Convergence of the relevant inverse limit spectral sequences follows from [24], Prop. 4.2.22], and convergence of the relevant localized spectral sequences follows from adapting [42, Thm. 2.13].

Explicit calculation using the algebraic Atiyah-Hirzebruch spectral sequence and calculations of Guillou-Isaksen [26, Thm. 1.1] shows that the colimit-limit interchange map

$$\eta^{-1}S^{0,0} \to \lim_k \eta^{-1}S^{0,0}/2^k$$

induces an isomorphism of spectral sequences beginning with $E_2 \cong E_2$. The lemma follows by our previous remarks on convergence of both spectral sequences. \(\square\)

In particular, the calculation of Theorem 6.30 may be interpreted as a calculation of $\pi_*(L_{10}^{f}S^{0,0})$.

**Proposition 6.29.** There is an equivalence of motivic spectra

$$\eta^{-1}S^{0,0} \simeq L_{cKW}S^{0,0}.$$  

**Proof.** The left-hand side may be computed by inverting $\eta : S^{1,1} \to S^{0,0}$ in the $E_\infty$-page of the $kq$-resolution. On the other hand, the right-hand side may be computed using the $cKW$-based motivic Adams spectral sequence

$$E_1^{n,t,u} = \pi_{t,u}(cKW^\wedge_n \wedge cKW) \Rightarrow \pi_*(L_{cKW}S^{0,0}),$$

where convergence follows from [43, Thm. 7.3.4]. We have

$$cKW = KW_{\geq 0} \simeq \eta^{-1}kq$$

by definition of the homotopy t-structure. Therefore we may identify the $cKW$-based motivic Adams spectral sequence with the $\eta$-localized $kq$-resolution, where $\eta \in \pi_{1,1}(kq)$. This converges to $\eta^{-1}\pi_*(S^{0,0})$ by the obvious motivic analog of [42, Thm. 2.13]; the proposition follows. \(\square\)

**Conjecture 6.30.** There is an equivalence of Bousfield classes

$$\langle cKW \rangle = \langle K(0) \lor K(w_0) \rangle.$$

We do not know how to prove this conjecture. However, Lemma 6.8 and Conjecture 6.30 would imply that $L_{cKW} \simeq L_{10}$. Applying our previous computations, this would imply that $L_{10}^{f}S^{0,0} \to L_{10}S^{0,0}$ is an equivalence. If Conjecture 6.27 also holds in the case $(m,n) = (1,0)$, then all three formulations of the motivic Telescope Conjecture would hold for $(m,n) = (1,0)$.

**Remark 6.31.** It is also be possible to formulate an “exotic motivic Telescope Conjecture” using only wedges of Krause’s exotic Morava K-theories (but not Borghesi’s motivic Morava K-theories). Indeed, if one sets $K(\leq \beta_{mn})ex := \bigvee_{(i,j) \in S_{mn}^{ex}} K(\beta_{ij})$ where $S^{ex}_{mn} := S_{mn} \setminus \{(i,j) : S_{mn} : j = -1\}$, then [37, Thm. 6.12] identifies $L^{f}_{K(\leq \beta_{mn})ex}X$ as the telescope of a non-nilpotent self-map of type $(m,n)$ for any $X$ such that $K(\beta_{ij})_*(X) = 0$ for $(i,j) \in S_{mn}^{ex} \setminus \{(m,n)\}$. One can therefore relate the various formulations of such an exotic motivic Telescope Conjecture. The case $(m,n) = (1,0)$ would then follow from our calculations if one can show that $\langle cKW \rangle = \langle K(w_0) \rangle$ and that $L_{K(w_0)}$ is smashing.

**Remark 6.32.** An exotic motivic Telescope Conjecture as suggested above may be possible to study in the category of cellular $C^{\tau}$-modules using [24]. For example, one sees that $\pi_*(\eta^{-1}S^{0,0})$ consists entirely of simple $\tau$-torsion classes. Therefore
an exotic motivic Telescope Conjecture might be equivalent to a conjecture in the category of $C\tau$-modules. The latter is equivalent to the derived category of $BP, BP$-comodules by [33]. Telescope conjectures always hold in the derived category of a commutative Hopf algebra by [30 Sec. 6.3], so it seems plausible that this exotic analog of the Telescope Conjecture holds.

6.5. The motivic Telescope Conjecture for $K(\leq \beta_{1,-1})$. We now turn to Conjecture [6, 26] in the case where $(m, n) = (1, -1)$. In Proposition [6.36] we show that $L_{1,-1}^f S_{0,0} \simeq v_1^{-1} S_{0,0}$ using the inverse limit of localized Adams spectral sequences. We then show that $v_1^{-1} S_{0,0} \simeq L_{KQ} S_{0,0}$ in Proposition [6.37] by comparing localizations of the $kq$-resolution. Using the Wood cofiber sequence and the isotropy separation sequence for $KGL$, we then prove an equivalence of Bousfield classes $\langle KQ \rangle = (KGL \vee KW)$ in Proposition [6.38] which implies $L_{KQ} S_{0,0} \simeq L_{KGL \vee KW} S_{0,0}$. Finally, we outline the steps needed to prove Conjecture [6.40] which would imply $\langle KGL \vee KW \rangle = (K(0) \vee K(0) \vee K(1))$ and thus $L_{KGL \vee KW} S_{0,0} \simeq L_{1,-1} S_{0,0}$. In summary, we will show

$$L_{1,-1}^f S_{0,0} \simeq v_1^{-1} S_{0,0} \simeq L_{KQ} S_{0,0} \simeq L_{KGL \vee KW} S_{0,0} \simeq L_{1,-1} S_{0,0}.$$ 

We begin by defining a sequence of finite motivic spectra of type $(1, -1)$ whose homotopy limit is $S_{0,0}$.

**Definition 6.33.** Recall from the previous section that $S_{0,0}/2^k$ admits a non-nilpotent self-map $\eta : S_{1,1}/2^k \to S_{0,0}/2^k$. Let $Y_k := S_{0,0}/(2^k, \eta^k)$.

We observe that $\lim_k Y_k \simeq (S_{0,0})_{2,\eta} \simeq (S_{0,0})_{2} \simeq S_{0,0}$, where the second equivalence follows from [31 Thm. 1] and the last equality is a reminder that we have implicitly 2-completed everything.

**Lemma 6.34.** For each $k \geq 1$, the motivic spectrum $Y_k$ admits a non-nilpotent self-map of type $(1, -1)$

$$v_1 : \Sigma^{2,1} Y_k \to Y_k.$$

**Proof.** The map can be constructed using the long exact sequence in homotopy groups associated to the cofiber sequence

$$S_{1,1}/2^k \xrightarrow{\eta^k} S_{0,0}/2^k$$

as in the classical case. It is straightforward to verify that the map is $\tau$-torsion free, so it realizes to the analogous self-map of $S^0/(2^k, \eta^k)$. Since the classical self-map is non-nilpotent and Betti realization is strong symmetric monoidal, we conclude that $v_1$ is non-nilpotent motivically.

**Definition 6.35.** We define the $v_1$-inverted motivic sphere spectrum $v_1^{-1} S_{0,0}$ by setting

$$v_1^{-1} S_{0,0} := \lim_k v_1^{-1} Y_k.$$ 

**Proposition 6.36.** There is an equivalence of motivic spectra

$$L_{1,-1}^f S_{0,0} \simeq v_1^{-1} S_{0,0}.$$ 

**Proof.** It suffices to show that $v_1^{-1} S_{0,0}$ is finitely $K(\leq \beta_{1,-1})$-local. Since $Y_k$ is of type $(1, -1)$, we have $L_{1,-1}^f Y_k \simeq v_1^{-1} Y_k$ for all $k \geq 1$. Let $W$ a finitely $K(\leq \beta_{1,-1})$-acyclic motivic spectrum. Then if $f \in \{W, v_1^{-1} S_{0,0}\} = \{W, \lim_k v_1^{-1} Y_k\}$ is nontrivial,
we must have $f : W \to v_1^{-1} Y_N$ nontrivial for some $N \geq 1$. But this is a contradiction since $v_1^{-1} Y_N \simeq L_{1,-1}^1 Y_N$ is finitely $K(\leq \beta_{1,-1})$-acyclic.

\begin{proposition}
There is an equivalence of motivic spectra
\[ v_1^{-1} S^{0,0} \simeq L_{KQ} S^{0,0} . \]
\end{proposition}

\begin{proof}
The proof is similar to the proof of Proposition 6.29. The homotopy groups of the left-hand side may be obtained by inverting $v_1$ in the $E_\infty$-page of the $kq$-resolution. The homotopy groups of the right-hand side may be obtained using the $KQ$-based motivic Adams spectral sequence. Since $KQ \simeq v_1^{-1} kq$, we may identify the $v_1$-localized $kq$-resolution with the $KQ$-based motivic Adams spectral sequence. The result follows.
\end{proof}

\begin{proposition}
There is an equivalence of Bousfield classes $\langle KQ \rangle = \langle KGL \vee KW \rangle$.
\end{proposition}

We thank the motivic homotopy theory group in Osnabrück for help with the following proof.

\begin{proof}
We start by showing that $\langle KQ \rangle \subseteq \langle KGL \vee KW \rangle$. Let $X$ be a motivic spectrum with $KQ_{**}(X) = 0$, so $X \in \langle KQ \rangle$. The Wood cofiber sequence \[ \Sigma^{1,1} KQ \xrightarrow{\eta} KQ \to KGL \]
implies that $KGL_{**}(X) = 0$. Since $KW \simeq \eta^{-1} KQ$, we also see that $KW_{**}(X) = 0$. Therefore $X \in \langle KGL \vee KW \rangle$.

Now suppose $X$ satisfies $(KGL \vee KW)_{**}(X) = 0$. Then $KGL_{**}(X) = 0$ and $KW_{**}(X) = 0$. Recall the cofiber sequence \[ KGL_{hC2} \to KGL^{C2} \to \Phi^{C2} KGL. \]
which can be further identified with
\[ KGL_{hC2} \to KQ \to KW. \]
Since homotopy orbits is a filtered colimit, it commutes with homology and smash products. Therefore $KGL_{**}(X) = 0$ implies that $(KGL_{hC2})_{**}(X) = 0$. Since $KW_{**}(X) = 0$, we conclude that $KQ_{**}(X) = 0$ and therefore $X \in \langle KQ \rangle$.
\end{proof}

\begin{remark}
More generally, Proposition 6.38 follows from the motivic analog of \[51\] Lem. 1.34 by taking $X = KQ$ and $g = \eta$, so $Y \simeq KGL$ and $X \simeq KW$.
\end{remark}

\begin{conjecture}
There are equivalences of Bousfield classes
\[ \langle KGL \rangle = \langle K(0) \vee K(1) \rangle \quad \text{and} \quad \langle KW \rangle = \langle K(w_0) \vee K(1) \rangle. \]
\end{conjecture}

\begin{remark}
The first equivalence would follow from a motivic analog of \[51\] Thm. 2.1.(d), which in turn would rely on a motivic analog of Johnson and Wilson’s formula for $K(n)_{cl}^d(X)$ in \[30\]. We do not know how to approach the second equivalence.
\end{remark}

If Conjecture 6.40 holds, then we would have $L_{KQ} \simeq L_{KGL \vee KW} \simeq L_{K(\leq \beta_{1,-1})}$. In particular, we would have $L_{KGL \vee KW} S^{0,0} \simeq L_{(1,-1)} S^{0,0}$. If we also knew that $L_{1,-1}$ (or equivalently, $L_{KQ}$ or $L_{KGL \vee KW}$) is smashing, then all three formulations of the motivic Telescope Conjecture for $(m,n) = (1,-1)$ would hold.
7. Motivic \( J \)-homomorphisms

We conclude by discussing the relation of our calculations to the motivic \( J \)-homomorphism. We begin by recalling the work of Hu-Kriz-Ormsby on the motivic unitary \( J \)-homomorphism in Section 7.1. In Section 7.2, we recall and build upon recent work of Gheorghe-Isaksen-Krause-Ricka \([22]\) which provides a topological model for the category of \( \mathbb{C} \)-motivic cellular spectra. Using these results, we propose a model for the motivic stable orthogonal \( J \)-homomorphism in Section 7.3 and speculate on the location of its image in the \( kq \)-resolution.

7.1. Recollection of the motivic unitary \( J \)-homomorphism. Recall that Hu-Kriz-Ormsby defined a motivic unitary \( J \)-homomorphism

\[
J_u : \pi_k(K^{alg}(K)) \to \pi^{K, \ell}_{2k+1 \ell}(S^0,0)
\]

in \([33\text{ Sec. 1]}\) (see also \([6\text{ Sec. 16.2]}\)), where \( k+\ell > 1 \) and \( K \) is any algebraically closed field of characteristic zero. They computed its image using the motivic Adams-Novikov spectral sequence:

**Theorem 7.1.** \([33\text{ Thm. 3]}\) The image of \( J_u \) is isomorphic to the 2-primary component of the image of the classical unitary \( J \)-homomorphism in dimension \( 2\ell + k - 1 \).

In particular, the image of the unitary \( J \)-homomorphism only contains \( \tau \)-free classes in \( \pi_*(S^0,0) \). The corresponding elements in the classical stable stems are contained in the image of the classical orthogonal \( J \)-homomorphism since the unitary \( J \)-homomorphism factors through the orthogonal \( J \)-homomorphism. By \([15]\), the image of the orthogonal \( J \)-homomorphism is detected in the 0- and 1-lines of the \( b_0 \)-resolution. Translating back to the motivic setting using Betti realization and \( \tau \)-inversion, we obtain the following corollary:

**Corollary 7.2.** The image of the motivic unitary \( J \)-homomorphism \( J_u \) is detected in the 0- and 1-lines of the \( kq \)-resolution. More precisely, it is contained in the cokernel of the nontrivial \( d_1 \)-differentials from \( kq \)-Adams filtration zero to one.

7.2. The Gheorghe-Isaksen-Krause-Ricka equivalence. Let \( Sp^\text{zop} = \text{Fun}(\mathbb{Z}^\text{op},Sp) \), where \( \mathbb{Z} \) is regarded as a poset, be the category of filtered spectra \([22\text{ Def. 2.1]}\). In \([22]\), Gheorghe-Isaksen-Krause-Ricka construct a functor

\[
\Gamma_* : Sp \to \text{Mod}_{\Gamma_* S^0}
\]

from the category of classical spectra to the category of left modules over \( \Gamma_* S^0 \) \([22\text{ Def. 3.10]}\). Concretely, this functor is defined by

\[
\Gamma_* X : w \mapsto Tot(\tau_{\geq 2w}(X \wedge MU^*+1))
\]

where \( MU^*+1 \) is the usual cosimplicial resolution of the \( E_\infty \)-ring spectrum \( MU \), \( \tau_{\geq 2w} \) is the \( 2w \)-connective cover, and \( Tot \) is totalization.

The category of \( \Gamma_* S^0 \)-modules is equivalent to the category of cellular \( \mathbb{C} \)-motivic spectra \([22\text{ Thm. 6.12]}\) after 2-completion. The equivalence is induced by adjoint functors

\[
- \otimes S^{0,*} : \text{Mod}_{\Gamma_* S^0} \rightleftarrows \text{Mot}_\mathbb{C} : \Omega^0_*
\]

The right adjoint is defined by

\[
\Omega^0_{w}(X) := F_s(S^{0,w},X)
\]
where \( F_\ast(S^0,w,X) \) is the (classical) mapping spectrum between motivic spectra. The left-adjoint is given by

\[
\operatorname{hocolim}_{i\geq j} Y_i \otimes S^{0,j} \]

where \( Y_i \otimes - \) is defined using the fact that \( \mathcal{M}ot_C \) is enriched over classical spectra.

The following lemma and construction are due to Mark Behrens.

**Lemma 7.3.** Let \( \operatorname{Re} : \mathcal{M}ot_C \to \operatorname{Sp} \) denote Betti realization and let \( E_\ast \) be a filtered spectrum with \( E_j = \operatorname{colim}_{i\to\infty} E_{-i} \) for all \( j \ll 0 \). The functor \( \hat{\operatorname{Re}} : \operatorname{Mod}_{\Gamma^\ast S^0} \to \operatorname{Sp} \) in the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}ot_C & \xrightarrow{\operatorname{Re}} & \operatorname{Mod}_{\Gamma^\ast S^0} \\
\downarrow & & \downarrow \\
\operatorname{Sp} & \xleftarrow{\hat{\operatorname{Re}}} & \operatorname{Sp}
\end{array}
\]

is given by

\[
\hat{\operatorname{Re}}(E_\ast) \operatorname{colim}_{i\to\infty} E_{-i}.
\]

**Proof.** Betti realization commutes with colimits and is strong symmetric monoidal, so we have

\[
\operatorname{Re}(E_\ast \otimes S^0,\ast) = \operatorname{Re}(\operatorname{colim}_{i\geq j} E_i \otimes S^{0,j}) = \operatorname{colim}_{i\geq j} \operatorname{Re}(E_i) \wedge S^0 = \operatorname{colim}_{i\to\infty} E_{-i}.
\]

\( \square \)

**Construction 7.4.** Suppose that \( E_\ast \in \operatorname{Mod}_{\Gamma^\ast S^0} \) satisfies

\[
\operatorname{Tot}(\tau_{\geq 2w}(\operatorname{colim}_{i\to\infty} E_{-i} \wedge MU^{*+1})) \simeq \operatorname{Tot}(\tau_{\geq 2w} E_w \wedge MU^{*+1})
\]

for all \( w \in \mathbb{Z}^{\text{op}} \) and that under this equivalence, the filtered spectrum structure maps factor as

\[
\tau_{\geq 2w}(E_w \wedge MU^{*+1}) \to \operatorname{Tot}(\tau_{\geq 2w-2}(E_w \wedge MU^{*+1})) \to \operatorname{Tot}(\tau_{\geq 2w-2}(E_{w-1} \wedge MU^{*+1})).
\]

Suppose further that the Adams-Novikov spectral sequence for \( E_w \) converges for all \( w \in \mathbb{Z}^{\text{op}} \). Then we define a map

\[
\Gamma_\ast(\operatorname{colim}_{i\to\infty} E_{-i}) \to E_\ast
\]

which is given for \( \ast = w \) by

\[
\operatorname{Tot}(\tau_{\geq 2w}(\operatorname{colim}_{i\to\infty} E_{-i} \wedge MU^{*+1})) \simeq \operatorname{Tot}(\tau_{\geq 2w}(E_w \wedge MU^{*+1})) \to \operatorname{Tot}(E_w \wedge MU^{*+1}) \simeq E_w.
\]

The filtered spectrum structure maps on the left-hand side are given by Equation 4 and the filtered spectrum structure maps on the right-hand side are given by

\[
\operatorname{Tot}(E_w \wedge MU^{*+1}) \to \operatorname{Tot}(E_{w-1} \wedge MU^{*+1}),
\]

so the map above commutes with the filtered spectrum structure maps.

**Corollary 7.5.** If \( E \) is a motivic spectrum such that \( \Omega_\ast^{10}(E) \) satisfies the above conditions, then there exists a map of motivic spectra

\[
(\Gamma_\ast(\operatorname{Re}(E))) \otimes S^{0,\ast} \to E.
\]

**Lemma 7.6.** Suppose that \( E \) is a motivic spectrum for which the motivic Adams-Novikov spectral sequence converges and for which there exists a \( u \in \mathbb{Z} \) such that \( \pi_{\ast,u}(E) \cong \pi_{\ast,u-n}(E) \) for all \( n \geq 0 \). Then \( \Omega_\ast^{10}(E) \) satisfies the conditions of Construction 7.4.
Proof. This is immediate from the properties of the function spectrum $F_s(-, -)$. □

**Corollary 7.7.** There are equivalences of motivic spectra

$$HF_2 \simeq \Gamma_* (HF_2^c), \quad HZ \simeq \Gamma_* (HZ^{cl}), \quad kgl \simeq \Gamma_* (bu), \quad kq \simeq \Gamma_* (bo).$$

Proof. The Betti realization of each motivic spectrum on the left-hand side of the equivalence coincides with the classical spectrum that $\Gamma_*$ is being applied to on the right-hand side. Each motivic spectrum satisfies the hypotheses of Lemma 7.6 so we obtain maps from the right-hand side of each equivalence to the left-hand side by Construction 7.4. These maps induce an isomorphism between motivic Adams-Novikov spectral sequences by construction. The motivic Adams-Novikov spectral sequence converges for each motivic spectrum above, so the map is an equivalence of motivic spectra. □

**Corollary 7.8.** There is an isomorphism between the $kq$-resolution and the $\Gamma_* (bo)$-resolution.

**Remark 7.9.** In other words, we could have replaced $kq$ by $\Gamma_* (bo)$ in all of the analysis above and arrived at the same conclusions. On the other hand, we do not claim that $\Gamma_*$ applied to the $bo$-resolution is equivalent to the $kq$-resolution. The functor $\Gamma_*$ is not generally strong symmetric monoidal [22, Rmk. 3.8], and we cannot apply [22, Prop. 3.25] to obtain an equivalence $\Gamma_* (bo) \wedge \Gamma_* (S0) \simeq \Gamma_* (bo \wedge bo)$.

### 7.3. A model for the motivic stable orthogonal J-homomorphism

Following the philosophy that $\Gamma_*$ of a classical spectrum gives the correct motivic analog, we propose the following model for the $C$-motivic orthogonal J-homomorphism. The classical J-homomorphism is obtained by taking the colimit of maps $O(n) \to \Omega^n S^n$ to obtain

$$J^{cl}_o : O \to \Omega^\infty \Sigma^\infty S^0 = QS^0.$$

This is an infinite loop map by construction, so we can deloop once and take the corresponding map of $\Omega$-spectra to obtain

$$J^{cl}_o : \Sigma^{-1}bo \to S^0.$$

**Definition 7.10.** We define the $C$-motivic orthogonal J-homomorphism by applying $\Gamma_*$ to the classical orthogonal J-homomorphism:

$$J_o := \Gamma_*, J^{cl} : \Gamma_*(\Sigma^{-1}bo) \to \Gamma_*(S^0) \simeq S^{0,0}.$$

**Lemma 7.11.** The groups $MU_*(bo)$ are concentrated in even degrees.

We learned the following proof from Mark Behrens.

Proof. Consider the cofiber sequence

$$\Sigma bo \wedge MU \to bo \wedge MU \to bo \wedge MU \wedge C\eta.$$

Since multiplication by $\eta$ is trivial in $MU_*$, turning triangles gives a short exact sequence

$$0 \to \pi_*(bo \wedge MU) \to \pi_*(bo \wedge MU \wedge C\eta) \to \pi_*(\Sigma^2 bo \wedge MU).$$

Recall that $bo \wedge C\eta \simeq bu$. Since $MU_*(bu)$ is concentrated in even degrees, the result follows. □
Since $MU_*(bo)$ is concentrated in even degrees, we have
\[ \Gamma_*(\Sigma^{-1}bo) \simeq \Sigma^{-1,-1} \Gamma_*(bo) \simeq \Sigma^{-1,-1} kq \]
by [22 Lem. 3.14]. Therefore we may rewrite the $\mathbb{C}$-motivic orthogonal J-homomorphism as
\[ J_0 := \Sigma^{-1,-1} kq \to S^{0,0}. \]

**Remark 7.12.** We do not now how to explain the difference in motivic weight between our model and the unitary model given in [33]. For example, one would expect \( J_0 : \pi_7(\Sigma^{-1,-1} kq) \to \pi_7(S^{0,0}) \) to be an isomorphism. However, we have \( \pi_7(\Sigma^{-1,-1} kq) \cong \pi_{8,5}(kq) = 0 \). Since our model \( J_0 \) is constructed using “just classical homotopy theory”, it seems plausible that \( \text{im}(J_0) \) consists of the $\tau$-torsion free classes in the image of a more “geometrically defined” motivic orthogonal J-homomorphism. In particular, the analogous construction over $\mathbb{R}$ would fail to completely miss the class $16\sigma \in \pi_8(S^{0,0})$ mentioned in the Introduction since $\tau \cdot 16\sigma = 0 \in \pi_7(S^{0,0})$.

By [39], the image of $J_0^{cl}$ is detected in the 0- and 1-lines of the $bo$-resolution. Since we have defined $J_0$ by applying $\Gamma_*$ to everything in sight and the $kq$-resolution is isomorphic to the $\Gamma_*(bo)$-resolution, we would like to conclude that the image of $J_0$ is detected in the 0- and 1-lines of the $kq$-resolution. However, we can only conclude the following:

**Corollary 7.13.** The image of the $\mathbb{C}$-motivic orthogonal J-homomorphism is detected in $\Gamma_*$ applied to the 0- and 1-lines of the $bo$-resolution.

**Remark 7.14.** In order to identify the image of $J_0$ with the 0- and 1-lines of the $kq$-resolution, it would suffice to show that
\[ \Gamma_*(bo \wedge bo) \simeq \Gamma_*(bo) \wedge \Gamma_*(bo). \]

Such an equivalence would follow from [22 Prop. 3.25] if one could loosen the restriction that $X$ be an even-cell complex.

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University of Illinois, Urbana-Champaign
E-mail address: dculver@illinois.edu

University of Notre Dame
E-mail address: jquigle2@nd.edu