The Crank-Nicholson type compact difference scheme for a loaded time-fractional Hallaire’s equation

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Abstract. In this paper, we study loaded modified diffusion equation (the Hallaire equation with the fractional derivative with respect to time). The compact finite difference scheme of Crank-Nicholson type of higher order is developed for approximating the stated problem on uniform grids. A priori estimates are obtained in difference and differential interpretations, from which there follow uniqueness, stability, and convergence of the solution of the difference problem to solution of the differential problem with the rate $O(h^4 + \tau^{2-\alpha})$. Proposed theoretical calculations are confirmed by numerical experiments on test problem.

1 Introduction

It is well known that filtration of liquids in porous media \cite{1,2}, heat transfer in a heterogeneous environment \cite{3,4}, moisture transfer in soil grounds \cite{5,13} lead to modified diffusion equations (i.e. the Hallaire equation \cite{5}). For example, the movement of water in capillary-porous media, to which the soil belongs, can occur under the influence of the most diverse driving forces. Based on the analysis of the diffusion mechanism in porous array when the occurrence of moisture flows under capillary pressure gradient a nonlinear equation is obtained \cite{5}:

$$W_t = (D(W)W_x)_x,$$

where $W$ is humidity in fractions of units, $x$ is depth, $t$ is time and $D(W)$ represents diffusivity.

Diffusion model assuming that if at the initial moment a non-uniform humidity is given, then a flow of moisture from the more wet layer into less wet layers occurs, often is not justified. To explain when and under what conditions the movement of moisture forward and backward takes place, you can use the modified diffusion equations (Hallaire equation) \cite{5}:

$$W_t = (D(W)W_x + AW_{xt})_x,$$  \hfill (1)

where $A$ is a positive constant. An equation of the form (1) is often called pseudoparabolic. Various boundary value problems for pseudoparabolic equations were studied in literature \cite{7,8,9,10,11,12,13,14,15,16}. The numerical solutions of the loaded differential equations are discussed by numerous authors \cite{17,18,19,20,21,22,23}.

Differential equations of fractional order attract more and more attention of scientists due to the fact that equations of this type can describe many physical and chemical processes, biological environments and systems, which are well interpreted as fractals (i.e soil which is most porous). Overview of the basic theory of fractional differentiation, fractional order differential equations, methods for their solution and applications can be found in \cite{24,25,26,27,28}.

Consider the modified diffusion equation (Hallaire equation) with fractional time derivative

$$\partial_t^\alpha u(x,t) = \mathcal{L}u(x,t) + \mu \partial_t \mathcal{L}u + \mathcal{I}u + f(x,t), \quad 0 < x < l, \quad 0 < t \leq T,$$ \hfill (2)

$$u(0,t) = 0, \quad u(l,t) = 0, \quad 0 \leq t \leq T, \quad u(x,0) = u_0(x), \quad 0 \leq x \leq l,$$  \hfill (3)
where

\[
\partial_0^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, \eta)}{\partial \eta} (t - \eta)^{-\alpha} d\eta, \quad 0 < \alpha < 1, \tag{4}
\]

is Caputo fractional derivative of order \( \alpha \) and

\[
\mathcal{L}u(x, t) = \frac{\partial^2 u}{\partial x^2}, \quad Iu = \sum_{k=1}^{m} q_k(x, t)u(x, t), \tag{5}
\]

\( \mu > 0, \quad |q(x, t)| \leq c_1, \quad 0 < x_1 < x_2 < \ldots < x_m < l. \)

The most common approximation to the fractional derivative \( (4) \) is the so-called \( L1 \) method \( [29, 30] \) which is defined as follows

\[
\partial_{0j+1}^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{j} \frac{u(x, t_{s+1}) - u(x, t_s)}{t_{s+1} - t_s} \int_{t_s}^{t_{s+1}} \frac{d\eta}{(t_{j+1} - \eta)^\alpha} + r^{j+1}, \tag{6}
\]

where \( 0 = \tau_0 < \tau_1 < \ldots < \tau_{j+1}, \) and \( r^{j+1} \) is approximation error. In the case of a uniform grid \( \tau = \tau_{s+1} - \tau_s, \) for all \( s = 0, 1, \ldots, j + 1 \) and it was proved that \( r^{j+1} = \mathcal{O}(\tau^{2-\alpha}) \) \( [31, 32, 33] \). In \([34, 35, 36]\) using the \( L1 \) method, difference schemes for fractional-order Caputo diffusion equations were constructed, and their stability and convergence were proved.

In contrast to the classical case, due to the non-local properties of the fractional differentiation operator, the algorithms for solving fractional-order equations are rather laborious even in the one-dimensional case. In the transition to two-dimensional and three-dimensional problems, their complexity increases significantly. In this regard, the construction of stable difference schemes of high order of accuracy is a very important task.

In \([37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]\) compact difference schemes for increasing the order of the error of approximation in spatial variables were constructed and investigated.

Approximation of the fractional derivative \( (4) \) using the \( L1 \) method gives the error of \( \mathcal{O}(\tau^{2-\alpha}) \), however a two-point approximation in the time variable of the term \( \mu \partial \mathcal{L}u \) of the right side of equation \( (2) \) at the grid points \( t = \tau_{j+1} \) gives the error \( \mathcal{O}(\tau) \). In this regard, in this paper, we propose to approximate the fractional derivative \( (4) \) on the half layers \( t = \tau_{j+\frac{1}{2}}, j = 0, 1, 2, \ldots, \) to keep the order in time \( \mathcal{O}(\tau^{2-\alpha}) \). For the numerical solution of the problem, some efficient compact difference schemes of Crank-Nicholson type (having high order accuracy) are used.

Difference methods for solving boundary value problems for pseudoparabolic fractional order equations have been studied in \([48, 49]\).

In this paper, we construct a difference analog of the fractional Caputo derivative on the half-layers, which ensures the order of approximation in the time variable no worse than \( \mathcal{O}(\tau^{2-\alpha}) \). To increase the order of approximation with respect to the spatial variable, compact difference schemes are used. Thus constructed difference scheme has order of accuracy \( \mathcal{O}(h^4 + \tau^{2-\alpha}) \). The method of energy inequalities is used to obtain a priori estimates for both differential and difference problems. Numerical experiments to verify the accuracy and efficiency of the proposed solution algorithm have been carried out.

2 A Priori Estimate in the Differential Form

Lemma 1. \([50]\) For any absolutely continuous on \([0, T]\) function \( v(t) \) the following inequality holds true

\[
v(t)\partial_0^\alpha v(t) \geq \frac{1}{2} \partial_0^\alpha v^2(t), \quad 0 < \alpha < 1.
\]
Theorem 1. If \(|q_k(x, t)| < c_1, k = 1, 2, \ldots, m\), then for the solution to problem (2)-(3) it holds true that

\[
D_{0t}^{\alpha - 1} ||u||_0^2 + ||u_x||_0^2 \leq M(T) \left( \int_0^t ||f||_0^2 d\tau + ||u_0||_0^2 + ||u_0||_0^2 \right),
\]  

(7)

where \(D_{0t}^{\alpha - 1} u = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \eta)^{-\alpha} u(x, \eta) d\eta\) is Riemann-Liouville fractional integral of order \(1 - \alpha\), \(M(T) > 0\) is a constant depending only on the input data of problem (2)-(3).

Proof. By the method of energy inequalities we find a priori estimate for solving problem (2)-(3). To do this, scalar-multiply equation (2) on \(u\):

\[
(u, \partial_t^\alpha u) = (u, \mathcal{L}u) + \mu \left( u, \frac{\partial}{\partial t} \mathcal{L}u \right) + (u, T) + (u, f),
\]  

(8)

\((u, v) = \int_0^t u v dx, (u, u) = \int_0^t u^2 dx.\)

We transform each term in (7) subject to Lemma 1 and conditions (3):

\[
(u, \partial_t^\alpha u) \geq \frac{1}{2} \partial_t^\alpha ||u||_0^2,
\]  

(9)

\((u, \mathcal{L}u) = (u_{xx}, u) = -||u_x||_0^2,
\]  

(10)

\[ \mu \left( u, \frac{\partial}{\partial t} \mathcal{L}u \right) = \mu (u_{xx}, u) = -\mu (u_x, u_{xt}) = -\frac{\mu}{2} \frac{d}{dt} ||u_x||_0^2, \]

(11)

\[(u, T) = \left( u, \sum_{k=1}^m q_k u(x_k, t) \right) = \int_0^t u(x, t) \sum_{k=1}^m q_k(u(x_k, t)) dx
\]

\[ = \sum_{k=1}^m u(x_k, t) \int_0^t q_k(x, t) u(x, t) dx \leq m c_1 l ||u||_C^2 \leq \frac{m c_1 l^3}{2} ||u_x||_0^2,
\]  

(12)

\[(u, f) \leq \frac{1}{2} ||f||_0^2 + \frac{1}{2} ||u||_0^2 \leq \frac{1}{2} ||f||_0^2 + \frac{l^2}{4} ||u_x||_0^2.
\]  

(13)

Given the transformations obtained, from (7) we find

\[
\partial_t^\alpha ||u||_0^2 + \mu \frac{d}{dt} ||u_x||_0^2 \leq (m c_1 l^3 + l^2 / 2) ||u_x||_0^2 + ||f||_0^2.
\]  

(14)

Integrate (14) from 0 to \(t\) with respect to \(\tau\) and applying the Gronwall Lemma we find inequality (7).

From a priori estimates (7) follow the uniqueness and stability of the solutions of problem (2)-(3) on the input data.

3 Difference analog of the fractional derivative

On the uniform grid \(\omega_{\tau} = \{t_j = j \tau, j = 0, \ldots, M, T = \tau M\}\) we construct a difference analog of the fractional derivative of the function \(u(t) \in C^2[0, T]\) at fixed points \(t = t_{j+\frac{1}{2}}, j = 0, 1, \ldots, M - 1.\)

\[
\partial_{0t,j+\frac{1}{2}}^\alpha u(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^{t_{j+\frac{1}{2}}} u'(\eta) (t_{j+\frac{1}{2}} - \eta)^{-\alpha} d\eta = \frac{1}{\Gamma(1 - \alpha)} \sum_{s=1}^j \int_{t_{s-1}}^{t_s} u'(\eta) (t_{j+\frac{1}{2}} - \eta)^{-\alpha} d\eta
\]  

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Proof. Considering the latter, we find
\[ \Gamma(2 - \frac{1}{2}) \left( \frac{1}{2} \right)^{\alpha} = \alpha \left( s - t - \frac{1}{2} \right) s = 1 \]
\[ \alpha = \frac{u(\xi) - u(t_{s-1})}{t_{s-1} - t} (t_{s-1} - t) \]
where \( t \in [t_{s-1}, t_s] \), \( \xi \in (t_{s-1}, t_s) \), \( u_{t,s} = \frac{u(t_{s-1}) - u(t_{s-1})}{t_{s-1} - t} \), \( u_{t,s} = \frac{u(t_{s+1}) - u(t_s)}{t_{s+1} - t} \).

Considering the latter, we find
\[
\partial^\alpha_{t\frac{j}{1+\frac{1}{2}}} u(t) \approx \frac{1}{\Gamma(1 - \alpha)} \sum_{s=1}^{\frac{j}{2}} \int_{t_{s-1}}^{t_{s+1}} \left( \frac{\Pi_{1,s} u(\eta)}{(t^{2} - \frac{1}{2}) - \eta} \right) d\eta + \frac{u_{\alpha,j}}{\Gamma(1 - \alpha)} \int_{t_j}^{t_{j+\frac{1}{2}}} \frac{1}{(t^{2} - \frac{1}{2}) - \eta} d\eta
\]
\[
= \frac{1}{\Gamma(2 - \alpha)} \sum_{s=1}^{\frac{j}{2}} \left( t^{1 - \alpha} - \frac{1}{2} \right) u_{\alpha,s} + \frac{\tau^{1 - \alpha}}{2^{1 - \alpha} \Gamma(2 - \alpha)} u_{t,j} = \frac{\tau^{1 - \alpha}}{\Gamma(2 - \alpha)} \sum_{s=0}^{\frac{j}{2}} c^{(\alpha)}_{s} u_{s} = \Delta^\alpha_{t\frac{j}{1+\frac{1}{2}}} u,
\]
where
\[ c^{(\alpha)}_{s} = \begin{cases} \frac{1}{2^{1 - \alpha}} & \text{if } j = 0, \\ (j + \frac{1}{2})^{1 - \alpha} - (j - \frac{1}{2})^{1 - \alpha}, & \text{if } j \geq 1. \end{cases} \]

Lemma 2. For all \( \alpha \in (0, 1) \) and \( u(t) \in C^2[0, t_{j+1}] \)
\[ |\partial^\alpha_{t\frac{j}{1+\frac{1}{2}}} u - \Delta^\alpha_{t\frac{j}{1+\frac{1}{2}}} u| \leq \frac{2^{\alpha} M^{j+1}_2}{\Gamma(2 - \alpha)} \left( \frac{1}{2} + 1 \right) \tau^{2 - \alpha}, \]
where \( M^{j+1}_2 = \max_{0 < t < t_{j+1}} |u''(t)|. \)

Proof. Let the following holds: \( \partial^\alpha_{t\frac{j}{1+\frac{1}{2}}} u - \Delta^\alpha_{t\frac{j}{1+\frac{1}{2}}} u = R^j_1 + R^j_2 \), where
\[
R^j_1 = \frac{1}{\Gamma(1 - \alpha)} \sum_{s=1}^{\frac{j}{2}} \int_{t_{s-1}}^{t_{s+1}} \frac{u''(\eta) d\eta}{(t^{2} - \frac{1}{2}) - \eta} = \frac{1}{\Gamma(1 - \alpha)} \sum_{s=1}^{\frac{j}{2}} \int_{t_{s-1}}^{t_{s+1}} \left( \frac{\Pi_{1,s} u(\eta)}{(t^{2} - \frac{1}{2}) - \eta} \right) d\eta
\]
\[
= \frac{1}{\Gamma(1 - \alpha)} \sum_{s=1}^{\frac{j}{2}} \int_{t_{s-1}}^{t_{s+1}} \left( u(\eta) - \Pi_{1,s} u(\eta) \right) (t^{2} - \frac{1}{2}) - \eta d\eta
\]
\[
= - \frac{\alpha}{\Gamma(1 - \alpha)} \sum_{s=1}^{\frac{j}{2}} \int_{t_{s-1}}^{t_{s+1}} \left( u(\eta) - \Pi_{1,s} u(\eta) \right) (t^{2} - \frac{1}{2}) - \eta d\eta
\]
\[
= - \frac{\alpha}{2 \Gamma(1 - \alpha)} \sum_{s=1}^{\frac{j}{2}} \int_{t_{s-1}}^{t_{s+1}} u''(\xi) (\eta - t_{s-1}) (\eta - t_{s}) (t^{2} - \frac{1}{2}) - \eta d\eta,
\]
\[
R^j_2 = \frac{1}{\Gamma(1 - \alpha)} \int_{t_j}^{t_j + \frac{1}{2}} \frac{u''(\eta) d\eta}{(t^{2} - \frac{1}{2}) - \eta} - \frac{u_{t,j}}{\Gamma(1 - \alpha)} \int_{t_j}^{t_j + \frac{1}{2}} \frac{d\eta}{(t^{2} - \frac{1}{2}) - \eta}
\]
\[
= - \frac{\alpha}{2 \Gamma(1 - \alpha)} \sum_{s=1}^{\frac{j}{2}} \int_{t_{s-1}}^{t_{s+1}} u''(\xi) (\eta - t_{s-1}) (\eta - t_{s}) (t^{2} - \frac{1}{2}) - \eta d\eta,
\]
where \( \bar{\xi}_s \in (t_{s-1}, t_s), \ s = 1, \ldots, j \) and \( \xi_{j+1}^{(1)} \in (t_j, \eta), \xi_{j+1}^{(2)} \in (\eta, t_{j+1}) \).

We estimate the errors \( R_j^i \) and \( R_j^{i+1/2} \):

\[
|R_j^i| \leq \frac{\alpha M_j^2 + 1}{2\Gamma(1-\alpha)} \sum_{s=1}^{t_s} \left( \eta - t_{s-1} \right) (t_s - \eta) (t_{j+1/2} - \eta)^{-\alpha-1} d\eta
\]

\[
\leq \frac{\alpha M_j^2 + 1}{8\Gamma(1-\alpha)} \sum_{s=1}^{t_s} \left( t_{j+1/2} - \eta \right)^{-\alpha-1} d\eta = \frac{\alpha M_j^2 + 1}{2\Gamma(1-\alpha)} \int_0^{t_{j+1/2}} (t_{j+1/2} - \eta)^{-\alpha-1} d\eta
\]

\[
= \frac{M_j^2 \alpha \tau \eta}{8\Gamma(1-\alpha)} \left( \frac{2\alpha}{\tau^\alpha} - \frac{1}{(j+1/2)^\alpha \tau^\alpha} \right) \leq \frac{2\alpha M_j^2 + 1}{8\Gamma(1-\alpha)} \tau^{2-\alpha},
\]

\[
|R_j^{i+1/2}| \leq \frac{M_j^2 + 1}{2\Gamma(1-\alpha)} \int_{t_j}^{t_{j+1/2}} \frac{d\eta}{(t_{j+1/2} - \eta)\alpha} = \frac{2\alpha M_j^2 + 1}{4\Gamma(2-\alpha)} \tau^{2-\alpha}.
\]

Lemma 2 is proved.

**Lemma 3.** Let \( v \in C^6[x_i-1, x_{i+1}], \xi_i \in (x_i-1, x_{i+1}), \) then

\[
\frac{v''(x_{i+1}) + 10v''(x_i) + v''(x_{i-1})}{12} = \frac{v(x_{i+1}) - 2v(x_i) + v(x_{i-1})}{h^2} + \frac{h^4}{240}v^{(6)}(\xi_i).
\]

### 4 Stability and Convergence of Compact Finite-Difference Scheme

To solve the problem, we apply the compact finite difference method. For this purpose, on uniform grid \( \bar{\omega}_{h, \tau} \) to differential problem (2)-(3) we put in correspondence the difference scheme of the order of approximation \( O(h^4 + \tau^{2-\alpha}) \):

\[
\Delta_{0, j+1/2}^h \mathcal{H} y_i = Y_{xx,i} + \mu y_{xx,i} + \mathcal{H}_i \bar{Y} + \mathcal{H}_i \varphi_i^{j+1/2}, \ i = 1, \ldots, N-1, \ j = 0, \ldots, M-1, (15)
\]

\[
y_i^{j+1} = y_N^{j+1} = 0, \ j = 0, \ldots, M-1, \ y_i^0 = u_0(x_i), \ i = 0, \ldots, N, (16)
\]

where

\[
\mathcal{H}_i y_i^j = \frac{1}{12} \left( y_{i+1}^j + 10y_i^j + y_{i-1}^j \right) = y_i^j + \frac{h^2}{12} y_{xx,i}^j, \ i = 1, \ldots, N-1, \ Y^{j+1} = \frac{1}{2} \left( y^{j+1} + y^j \right),
\]

5
\[ y_{ik} = \frac{(x_k - x_{ik})(x_k - x_{ik+1})(x_k - x_{ik+2})}{-6h^3} y_{ik-1} + \frac{(x_k - x_{ik-1})(x_k - x_{ik})(x_k - x_{ik+2})}{2h^3} y_{ik} + \frac{(x_k - x_{ik-1})(x_k - x_{ik})(x_k - x_{ik+1})}{6h^3} y_{ik+1} + \frac{(x_k - x_{ik})(x_k - x_{ik+1})(x_k - x_{ik+2})}{2h^3} y_{ik+1} + \frac{(x_k - x_{ik})(x_k - x_{ik+1})(x_k - x_{ik+2})}{6h^3} y_{ik+2}, \]

where \( x_{ik} \leq x_k \leq x_{ik+1} \), \( d_{i}^{j+1/2} = q(x_i, t_{j+1/2}) \), \( \varphi_i^{j+1/2} = f(x_i, t_{j+1/2}) \).

In the sequel we will assume that \( h < \min\{x_1, 1 - x_m\} \).

**Lemma 4.** \([51] \text{p. 171}\) If \( f_j \) is a non-decreasing function, \( f_j \geq f_{j-1} \) for all \( j = 1, 2, \ldots \), then, from inequality

\[ g_{j+1} = c_0 \sum_{k=1}^{j} \tau g_k + f_j, \quad j = 0, 1, 2, \ldots, \quad g_1 \leq f_0. \]

an evaluation follows:

\[ g_{j+1} = e^{c_0 \tau_{j+1}} f_j, \quad j = 0, 1, 2, \ldots. \]

**Lemma 5.** \([38]\) If \( g_{j+1}^{1} > g_{j-1}^{1} > \ldots > g_{0}^{1} \), \( j = 0, 1, \ldots, M - 1 \), then for any function \( v(t) \), defined on the grid \( \omega_{r} \), the following inequality holds

\[ v^{j+1} g^{\alpha}_{\omega_{j+1}} v \geq \frac{1}{2} g^{\alpha}_{\omega_{j+1}}(v^2) + \frac{1}{2} g^{j+1}_{g^j}(v^2) - \frac{1}{2} g^{j+1}_{g^j} v \left( g^{\alpha}_{\omega_{j+1}} v \right)^2, \]

where \( g^{\alpha}_{\omega_{j+1}} v_i = \sum_{s=0}^{j} (s_i^{j+1} - v_i) g_{s}^{j+1}, \quad g_{s}^{j+1} > 0, \quad g_{-1}^{j+1} = 0. \)

It is obvious that the condition \( c_0^{(1)} > c_1^{(1)} > c_2^{(1)} > c_3^{(1)} > \ldots > c_j^{(1)} \) is fulfilled in the case, when \( \alpha > \alpha_0 = \log_{3} \frac{3}{2} \), since for \( \alpha \leq \alpha_0 \), \( c_0^{(1)} \leq c_1^{(1)} \). In order to use Lemma 5, we transform the resulting difference analog of the fractional derivative to the following form

\[ \Delta^{\alpha}_{\omega_{j+1}} y = \tilde{\Delta}^{\alpha}_{\omega_{j+1}} y - \gamma y_{\omega_{j+1}}^{1-\alpha}, \quad (17) \]

where

\[ \tilde{\Delta}^{\alpha}_{\omega_{j+1}} y = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{j} \tilde{c}_s^{j-\alpha} y_s, \]

\[ \{ \tilde{c}_0^{(a)} = 1, \quad \tilde{c}_j^{(a)} = c_j^{(a)}, \quad c_0^{(a)} = c_1^{(a)} > c_2^{(a)} > \ldots > c_j^{(a)} \}, \quad j = 1, 2, \ldots, \quad \gamma = \frac{2^{1-\alpha} - 1}{2^{1-\alpha} \Gamma(2-\alpha)} > 0. \]

**Theorem 2.** Let conditions \( |q_k(x, t)| < c_1, \quad k = 1, 2, \ldots, m \) be fulfilled, then is such \( \tau_0 \), that if \( \tau \leq \tau_0 \), then for the solution of difference problem \([12], [16] \) the following a priori estimate is valid

\[ ||y_{\omega_{j+1}}^{1-\alpha}||_{0} \leq M(T) \left( \sum_{j'=0}^{j} ||H_{\omega} \varphi||_{0}^{2} \tau + ||\dot{y}^{0}||_{0}^{2} + ||y^{0}_{\omega}||_{0}^{2} \right), \quad (18) \]

where \( M(T) > 0 \) is a constant which does not depend on \( h \) and \( \tau \).

**Proof.** Multiply the equation \([15]\) scalar by \( y^{j+1} \):

\[ \left( y^{j+1}, \Delta^{\alpha}_{\omega_{j+1}} (H_{\omega} y) \right) = \left( y^{j+1}, Y_{\omega} \right) + \mu \left( y^{j+1}, y_{\omega} \right) + \]
We transform the sums entering into \( (19) \), taking into account Lemma 5, inequality \( \|y\|_C^2 \leq \frac{\ell^2}{2\|y_2\|^2} \), and the conditions \( (16) \):

\[
\left( y^{j+1}, \Delta^\alpha_{\overline{\omega}, j+\frac{1}{2}} \mathcal{H}_h y \right) = \left( y^{j+1}, \Delta^\alpha_{\overline{\omega}, j+\frac{1}{2}} \mathcal{H}_h y \right) - \gamma \tau^{-1-\alpha} \left( y^{j+1}, \mathcal{H}_h y^{j+1} \right) \\
= \left( y^{j+1}, \Delta^\alpha_{\overline{\omega}, j+\frac{1}{2}} \mathcal{H}_h y \right) - \gamma \tau^{-1-\alpha} \left( y^{j+1}, y^{j+1}_t \right) - \frac{\gamma \tau^{-1-\alpha} \|h\|^2}{12} \left( y^{j+1}, y^{j+1}_t \right) \\
= \left( y^{j+1}, \Delta^\alpha_{\overline{\omega}, j+\frac{1}{2}} \mathcal{H}_h y \right) - \frac{\gamma \tau^{-1-\alpha}}{2} \left( 1, (y^2)_t + \tau y^2 \right) + \frac{\gamma \tau^{-1-\alpha} \|h\|^2}{12} \left( y^{j+1}, y^{j+1}_t \right) \\
\geq \frac{1}{2} \Delta^\alpha_{\overline{\omega}, j+\frac{1}{2}} \|y\|_{\mathcal{H}_h}^2 - \frac{\gamma \tau^{-1-\alpha}}{2} \left( \|y\|_0^2 \right)_t + \tau \|y\|_0^2 + \frac{\gamma \tau^{-1-\alpha} \|h\|^2}{12} \left( \|y\|_0^2 \right)_t, \quad (20)
\]

\[
(y^{j+1}, Y_{xx}) = -(y^{j+1}, y^{j+1}_{xx}) = -\frac{1}{2} \left( y^{j+1}_x, y^{j+1}_x + y^{j+1}_x \right) \\
= -\frac{1}{4} \left( (y^{j+1}_x + y^{j+1}_x)^2 + (y^{j+1}_x)^2 - (y^{j+1}_x)^2 \right) = -\frac{1}{4} \left( \|y^{j+1}\|_0^2 - \frac{\tau}{4} \left( \|y\|_0^2 \right)_t \right) \quad (21)
\]

\[
\mu \left( y^{j+1}, x_{xx} \right) = -\mu \left( y^{j+1}_x, x_{xx} \right) = -\frac{\mu}{2} \left( (y^2)_x + \tau y^2 \right) = -\frac{\mu}{2} \left( \|y\|_0^2 \right)_t + \|y\|_0^2 \quad (22)
\]

\[
(I\bar{Y}, y^{j+1}) \leq \sum_{k=1}^{m} \bar{Y} \left( a_k^{j+\frac{1}{2}}, y^{j+1} \right) \leq mc_1 \|y^{j+1}\|_C \|y^{j+1}\|_C \\
\leq mc_1 \left( \|y^{j+1}\|_C^2 + \|y^{j+1}_x\|_2 \|y\|_C \right) \leq \frac{3mc_1^3}{4} \left( \|y^{j+1}_x\|_0^2 + \|y^2\|_0^2 \right), \quad (23)
\]

\[
(y^{j+1}, \mathcal{H}_h \varphi) \leq \frac{1}{2} \|y^{j+1}\|_0^2 + \frac{1}{2} \|\mathcal{H}_h \varphi\|_0^2 \leq \frac{1}{4} \|y^{j+1}\|_0^2 + \frac{1}{2} \|\mathcal{H}_h \varphi\|_0^2. \quad (24)
\]

Taking into account the obtained estimates \( (20) - (24) \), from \( (19) \) we find

\[
\Delta^\alpha_{\overline{\omega}, j+\frac{1}{2}} \|y\|_{\mathcal{H}_h}^2 + \left( \mu + \frac{\tau}{2} + \frac{\gamma \tau^{-1-\alpha} \|h\|^2}{6} \right) \left( \|y\|_0^2 \right)_t + \frac{1}{2} \|y^{j+1}\|_0^2 + \frac{1}{2} \|y^{j+1}_x\|_0^2 + \tau \left( \mu - \frac{\gamma \tau^{-1-\alpha} \|h\|^2}{2} \right) \|y\|_0^2 \\
\leq \gamma \tau^{-1-\alpha} \left( \|y\|_0^2 \right)_t + \frac{3mc_1^3}{2} + \frac{1}{2} \left( \|y^{j+1}\|_0^2 + \|y^{j+1}_x\|_0^2 \right) + \|\mathcal{H}_h \varphi\|_0^2. \quad (25)
\]

Summing \( (25) \) from 0 to \( j \) multiply \( \tau \), we get

\[
\sum_{s=0}^{j} \Delta^\alpha_{\overline{\omega}, j+\frac{1}{2}} \|y^{s+1}\|_{\mathcal{H}_h}^2 + \left( \mu + \frac{\tau}{2} + \frac{\gamma \tau^{-1-\alpha} \|h\|^2}{6} \right) \|y^{j+1}\|_0^2 + \sum_{s=0}^{j} \|Y^{s+1}\|_0^2 \tau
\]
where \( M_1 = M_1(\alpha, m, c_1, l, \gamma) \) is a known positive constant.

We estimate the first term on the right-hand side of (26) as follows:

\[
\sum_{s=0}^{j} \left( \|y_{x}^{s+1}\|^2_0 + \|y_{x}^{s}\|^2_0 \right) \tau \leq \sum_{s=0}^{j} \|y_{x}^{s+1}\|^2_0 + \|y_{x}^{s}\|^2_0 \leq 2 \sum_{s=0}^{j} \|y_{x}^{s+1}\|^2_0 + \|y_{x}^{s}\|^2_0 + \tau \|y_{x}^{0}\|^2_0.
\]

Considering the latter, from (26) we find

\[
\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{j} \alpha_s \|y_{x}^{s+1}\|^2_{H_h} + \left( \mu + \frac{\tau}{2} + \frac{\gamma \tau^{1-\alpha} h^2}{6} - M_1 \lambda \right) \|y_{x}^{j+1}\|^2_0 + \sum_{s=0}^{j} \|y_{x}^{s+1}\|^2_0 \tau + \tau \left( \mu + \frac{\gamma \tau^{1-\alpha}}{2} \right) \sum_{s=0}^{j} \|y_{x}^{s+1}\|^2_0 \tau \leq 2 M_1 \sum_{s=1}^{j} \|y_{x}^{s}\|^2_0 + M_2 \left( \sum_{s=0}^{j} \|H_h \varphi\|^2_0 \tau + \|y^{0}\|^2_0 + \|y_{x}^{0}\|^2_0 \right),
\]

where \( M_2 \) is a known positive constant.

Choosing \( \tau \leq \tau_0 = \min \left\{ \frac{\mu}{2 M_1}, \left( \frac{2 \mu}{\gamma \tau^{1-\alpha}} \right)^{1/\alpha} \right\} \) and applying Lemma 4 to inequality (27), we obtain a priori estimate (18).

From a priori estimate (18), uniqueness and stability follow, and also the convergence of the solution of the difference problem to the solution of differential problem with the rate \( \mathcal{O}(h^4 + \tau^{2-\alpha}) \).

**Remark.** Obviously, the results obtained in this paper also hold in the case when

\[
I u = \int_0^t q(x, t) u(x, t) dx,
\]

if \( |q(x, t)| < c_1 \).

In this case, in order to preserve the order of approximation of a compact difference scheme, it suffices to apply the Simpson’s rules.

## 5 Results of a numerical experiment

Consider the following test case.

\[
\frac{\partial^\alpha u}{\partial \nu^\alpha} = u_{xx} + \mu u_{xxt} + \sum_{s=1}^{3} q_s(x, t) u(x_s, t) + f,
\]

\[
\begin{align*}
&u(0, t) = 0, \quad u(1, t) = 0, \\
&u(x, 0) = u_0(x),
\end{align*}
\]

where the right side \( f(x, t) \) is chosen so that the function \( u(x, t) = (t^3 + t^{2+\alpha} + 1) \sin(3\pi x) \) is the exact solution of the original equations. Wherein

\[
\mu = 1, \quad x_1 = 0.2, \quad x_2 = 0.5, \quad x_3 = 0.8, \quad T = 1.
\]
\[ q_1(x, t) = e^{x + t}, \quad q_2(x, t) = \sin(x + t), \quad q_3(x, t) = \cos(x + t). \]

The tables 1 and 2, for different values \( \alpha = 0.1; 0.5; 0.9 \) and reducing the size of the grid, shows the maximum value of the error \( (z = y - u) \) and the convergence order (CO) in the norms \( \| \cdot \|_{C(\bar{\omega}_h \tau)} \) and \( \| \cdot \|_0 \), where \( \| y \|_{C(\bar{\omega}_h \tau)} = \max_{(x, \tau, x') \in \bar{\omega}_h \tau} |y| \). The error decreases in accordance with the order of approximations of \( \mathcal{O}(h^4 + \tau^{2-\alpha}) \). The convergence order is determined by the following formula: \( \text{CO} = \log_{\frac{h_1}{h_2}} \frac{\| z_1 \|}{\| z_2 \|} \), where \( z_i \) is the error corresponding to \( h_i \).

Table 1: The error and the convergence order in the norms \( \| \cdot \|_0 \) and \( \| \cdot \|_{C(\bar{\omega}_h \tau)} \) when decreasing time-grid size for different values of \( \alpha = 0.1; 0.5; 0.9 \) on \( t = 1, \tau = 1/10000 \).

| \( \alpha \) | \( h \) | \( \| z \|_{C(\bar{\omega}_h \tau)} \) | CO | \( \max_{0 \leq j \leq M} \| z_j \| \) | CO | \( \max_{0 \leq j \leq M} \| z_j \|_0 \) | CO |
|---|---|---|---|---|---|---|---|
| 0.1 | 1/6 | 9.757379e-2 | 3.5958 | 4.2632 | 4.2999 | 4.132291e-1 | 3.9232 |
| | 1/12 | 3.007053e-3 | 4.202591e-4 | 2.014845e-4 | 4.2579 | 4.2222 | 4.0143 |
| | 1/24 | 4.661932e-2 | 3.6085 | 2.120782e-4 | 4.2201 | 1.680156e-3 | 3.9311 |
| 0.5 | 1/6 | 9.661932e-2 | 3.6085 | 4.202591e-4 | 2.014845e-4 | 4.2222 | 4.0143 |
| | 1/12 | 7.921449e-3 | 3.92770e-3 | 2.12782e-4 | 4.2201 | 1.680156e-3 | 3.9311 |
| | 1/24 | 4.140444e-4 | 4.2579 | 2.120782e-4 | 4.2201 | 1.680156e-3 | 3.9311 |
| 0.9 | 1/6 | 9.561826e-2 | 3.6216 | 2.095680e-4 | 4.2105 | 1.673201e-3 | 4.0077 |
| | 1/12 | 7.768228e-3 | 3.879776e-3 | 2.095680e-4 | 4.2105 | 1.673201e-3 | 4.0077 |
| | 1/24 | 4.075689e-4 | 4.2524 | 2.095680e-4 | 4.2105 | 1.673201e-3 | 4.0077 |

6 Conclusion

In this paper, a difference analogue of the fractional Caputo derivative is constructed on the half-layers, which most effectively approximates the problem under consideration in a time variable. Compact difference schemes are used to increase the order of approximation in the spatial variable. The difference scheme constructed in this paper has order of accuracy \( \mathcal{O}(h^4 + \tau^{2-\alpha}) \). The stability and convergence of difference schemes with a speed equal to the order of approximation error are proved. The proposed theoretical calculations are confirmed by numerical experiments on test problems. All numerical calculations were performed using Julia (version 1.0.1).

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Table 2: The error and the convergence order in the norms $\| \cdot \|_0$ and $\| \cdot \|_{C(\omega_h)}$ when decreasing space-grid size for different values of $\alpha = 0.1; 0.5; 0.9$ on $t = 1$, $h = 1/1000$.

| $\alpha$ | $\tau$ | $\| \bar{z} \|_{C(\omega_h)}$ | CO | $\max_{0 \leq j \leq M} \| z^j \|_0$ | CO | $\max_{0 \leq j \leq M} \| z^j \|_{z^j}$ | CO |
|----------|--------|-------------------------------|-----|-----------------------------------|-----|-----------------------------------|-----|
| 0.1      | 1/10   | 7.046545e-3                   |     | 4.429421e-3                      |     | 4.137926e-2                      |     |
|          | 1/20   | 1.764596e-3                   |     | 1.110134e-3                      |     | 1.037209e-2                      |     |
|          | 1/40   | 4.181103e-4                   |     | 2.781883e-4                      |     | 2.599478e-3                      |     |
|          | 1/80   | 1.106090e-4                   |     | 6.970348e-5                      |     | 6.514120e-4                      |     |
|          | 1/160  | 2.768977e-5                   |     | 1.746323e-5                      |     | 1.632212e-4                      |     |
|          | 1/320  | 6.931562e-6                   |     | 4.378509e-6                      |     | 4.089443e-5                      |     |
|          | 1/640  | 1.734385e-6                   |     | 1.095364e-6                      |     | 1.032998e-5                      |     |
|          | 1/1280 | 4.327143e-7                   | 2.01| 2.732386e-7                      | 2.01| 2.554293e-6                      | 2.01|
|          | 1/2560 | 1.066077e-7                   | 2.03| 6.675703e-8                      | 2.03| 6.232795e-7                      | 2.03|
|          | 1/5120 | 2.604856e-8                   | 2.04| 1.650392e-8                      | 2.04| 1.546747e-7                      | 2.04|
| 0.5      | 1/10   | 8.224209e-3                   | 3.78| 5.239804e-3                      | 3.78| 4.904309e-2                      | 3.78|
|          | 1/20   | 2.067613e-3                   | 3.80| 1.319392e-3                      | 3.80| 1.235161e-2                      | 3.80|
|          | 1/40   | 5.204621e-4                   | 3.82| 3.327375e-4                      | 3.82| 3.115679e-3                      | 3.82|
|          | 1/80   | 1.312926e-4                   | 3.84| 8.412851e-5                      | 3.84| 7.897983e-4                      | 3.84|
|          | 1/160  | 3.322563e-5                   | 3.86| 2.135228e-5                      | 3.86| 2.000616e-4                      | 3.86|
|          | 1/320  | 8.445963e-6                   | 3.88| 5.448625e-6                      | 3.88| 5.107308e-5                      | 3.88|
|          | 1/640  | 2.160852e-6                   | 3.90| 1.401153e-6                      | 3.90| 1.314085e-3                      | 3.90|
|          | 1/1280 | 5.571053e-7                   | 3.92| 3.635032e-7                      | 3.92| 3.411806e-6                      | 3.92|
|          | 1/2560 | 1.457657e-7                   | 3.94| 9.614532e-8                      | 3.94| 9.030448e-7                      | 3.94|
|          | 1/5120 | 3.709931e-8                   | 3.96| 2.484624e-8                      | 3.96| 2.332695e-7                      | 3.96|
| 0.9      | 1/10   | 9.385612e-3                   | 5.74| 6.006000e-3                      | 5.74| 5.625169e-2                      | 5.74|
|          | 1/20   | 2.369760e-3                   | 5.76| 1.519540e-3                      | 5.76| 1.423377e-2                      | 5.76|
|          | 1/40   | 6.028051e-4                   | 5.78| 3.879506e-4                      | 5.78| 3.635528e-3                      | 5.78|
|          | 1/80   | 1.554828e-4                   | 5.80| 1.001797e-4                      | 5.80| 9.445138e-4                      | 5.80|
|          | 1/160  | 4.109394e-5                   | 5.82| 2.691906e-5                      | 5.82| 2.527009e-4                      | 5.82|
|          | 1/320  | 1.130943e-5                   | 5.84| 7.542754e-6                      | 5.84| 7.089996e-5                      | 5.84|
|          | 1/640  | 3.310073e-6                   | 5.86| 2.659464e-6                      | 5.86| 2.132475e-5                      | 5.86|
|          | 1/1280 | 1.051882e-6                   | 5.88| 7.436887e-7                      | 5.88| 7.000605e-6                      | 5.88|
|          | 1/2560 | 3.876684e-7                   | 5.90| 2.690409e-7                      | 5.90| 2.535322e-6                      | 5.90|
|          | 1/5120 | 1.578844e-7                   | 5.92| 1.079825e-7                      | 5.92| 1.015036e-6                      | 5.92|

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