SHARP LOCAL SMOOTHING FOR MANIFOLDS WITH SMOOTH INFLECTION TRANSMISSION

HANS CHRISTIANSON AND JASON METCALFE

ABSTRACT. We consider a family of spherically symmetric, asymptotically Euclidean manifolds with two trapped sets, one which is unstable and one which is semi-stable. The phase space structure is that of an inflection transmission set. We prove a sharp local smoothing estimate for the linear Schrödinger equation with a loss which depends on how flat the manifold is near each of the trapped sets. The result interpolates between the family of similar estimates in [CW11]. As a consequence of the techniques of proof, we also show a sharp high energy resolvent estimate with a polynomial loss depending on how flat the manifold is near each of the trapped sets.

1. Introduction

In this paper we study the local smoothing effect for the Schrödinger equation on a class of manifolds with a trapped set that is mixed unstable and semistable, which is a version of inflection transmission. Our main result is a generalization of the local smoothing estimate

$$\int_0^T \| \langle x \rangle^{-1/2} e^{it\Delta} u_0 \|_{L^2}^2 dt \lesssim \| u_0 \|_{L^2}^2.$$ 

Such estimates first appeared in [CSS88], [Sjö87], [Veg98] and were extended to non-trapping asymptotically flat geometries in [CK95], [Doi96a]. See, e.g., [RT07], [MMT08] for some recent generalizations. The presence of trapping necessitates a loss of smoothing as was shown in [Doi96b]. If the trapping is unstable and nondegenerate, this has already been studied in [Bur04], [Chr07, Chr08, Chr10], [Dat09], [BCH09] amongst several others. Trapping that is unstable but degenerately so was the topic of [CW11]. The novel thing in this paper is the existence of semistable trapping, that is, trapping which is stable from one direction and unstable from another direction.

Let us begin by describing the geometry. Let $m_1$ and $m_2$ be positive integers, and set

$$a(x) = x^{2m_1-1}(x-1)^{2m_2/(1+x^2)}(m_1+m_2-1),$$

so that

$$a(x) \sim \begin{cases} 
  x^{2m_1-1}, & x \sim 0, \\
  (x-1)^{2m_2/(2m_1+m_2-1)} & x \sim 1, \\
  x, & |x| \to \infty.
\end{cases}$$

Set

$$A^2(x) = 1 + \int_1^x a(y) dy,$$
and notice that

\begin{equation}
A^2(x) \sim \begin{cases}
1 + x^{2m_1}, & x \sim 0, \\
C_1 + c_2(x-1)^{2m_2+1} & x \sim 1, \\
x^2, & |x| \to \infty.
\end{cases}
\end{equation}

Here $C_1 > 1$ and $c_2 < 1$ are constants which are easily computed but inessential, except for their relative sizes compared to 1.

Now let $X = \mathbb{R}_x \times \mathbb{R}_\theta / 2\pi \mathbb{Z}$, equipped with the metric

\[ ds^2 = dx^2 + A^2(x)d\theta^2, \]

so that $X$ is asymptotically Euclidean with two ends and has two trapped sets. The trapping occurs where $A'(x) = 0$, which is at $x = 0$ and $x = 1$ respectively (see Figure 1).

The metric determines the volume form

\[ d\text{Vol} = A(x)dx d\theta \]

and the Laplace-Beltrami operator acting on 0-forms

\[ \Delta f = (\partial_x^2 + A^{-2} \partial_\theta^2 + A^{-1}A' \partial_x) f. \]

We conjugate $\Delta$ and reduce to a one dimensional problem. Indeed, we set $L : L^2(X,d\text{Vol}) \to L^2(X,dx d\theta)$ to be the isometry

\[ Lu(x,\theta) = A^{1/2}(x)u(x,\theta). \]

With mild assumptions on $A$, $\tilde{\Delta} = L \Delta L^{-1}$ is (essentially) self-adjoint on $L^2(X,dx d\theta)$. More explicitly, we have

\[ -\tilde{\Delta} f = (-\partial_x^2 - A^{-2}(x)\partial_\theta^2 + V_1(x)) f \]
with
\[ V_1(x) = \frac{1}{2} A'' A^{-1} - \frac{1}{4} (A')^2 A^{-2}. \]

Given a function \( \psi \) on \( X \), we expand into its Fourier series, \( \psi(x, \theta) = \sum_k \varphi_k(x) e^{ik\theta} \), and note that
\[ (-\tilde{\Delta} - \lambda^2) \psi = \sum_k e^{ik\theta} (P_k - \lambda^2) \varphi_k(x), \]
where
\[ P_k \varphi_k(x) = \left( -\frac{d^2}{dx^2} + k^2 A^{-2}(x) + V_1(x) \right) \varphi_k(x). \]

By setting \( h = k^{-1} \), we pass to the semiclassical operator
\[ (P(h) - z) \varphi(x) = \left( -h^2 \frac{d^2}{dx^2} + V(x) - z \right) \varphi(x), \]
where the potential is
\[ V(x) = A^{-2}(x) + h^2 V_1(x) \]
and the spectral parameter is \( z = h^2 \lambda^2 \).

Our main result is the following local smoothing estimate with sharp loss. Using the common notation \( D_t = (1/i) \frac{\partial}{\partial t} \), we have:

**Theorem 1 (Local Smoothing).** Suppose \( X \) is as above with \( m_1, m_2 \geq 1 \) and assume \( u \) solves
\[ \begin{cases} (D_t - \Delta) u = 0 \text{ in } \mathbb{R} \times X, \\ u|_{t=0} = u_0 \in H^s \end{cases} \]
for some \( s > 0 \) sufficiently large. Then for any \( T < \infty \), there exists a constant \( C_T > 0 \) such that
\[ \int_0^T \left( \| \langle x \rangle^{-1} \partial_x u \|_{L^2(dVol)}^2 + \| \langle x \rangle^{-3/2} \partial_\theta u \|_{L^2(dVol)}^2 \right) dt \]
\[ \leq C_T \left( \| D_{\theta}^{\beta(m_1, m_2)} u_0 \|_{L^2(dVol)}^2 + \| D_x^{1/2} u_0 \|_{L^2(dVol)}^2 \right), \]
where
\[ \beta(m_1, m_2) = \max \left( \frac{m_1}{m_1 + 1}, \frac{2m_2 + 1}{2m_2 + 3} \right). \]

Moreover this estimate is sharp, in the sense that no polynomial improvement in regularity is true.

This theorem requires some remarks.

**Remark 1.1.** Observe that the maximum gain in regularity in the presence of inflection-transmission trapping is \( 2/(2m_2 + 3) \) derivatives. Each of these fractions lies in between sequential fractions in the numerology of [CW11], since
\[ \frac{1}{m + 1} + 1 \leq \frac{2}{2m + 3} < \frac{1}{m + 1}. \]

**Remark 1.2.** In the theorem above, the weights at infinity are different than those that appear in the standard Euclidean estimate. Standard cutoff arguments would allow us to make these match. The key new aspect of the theorem, however, is the behavior near the trapped sets, and for clarity in the proof, we do not modify the weights.
Remark 1.3. Theorem 1 and indeed also Theorem 2 below are of course true in many more situations. Of particular interest, the microlocalization step which separates the trapped sets at different energies used to prove (2.8) indicates the same result applies to a manifold with one Euclidean end and only an inflection transmission trapped set.

On the other hand, if our manifold has two Euclidean ends, a degenerate hyperbolic trapped set, and two inflection transmission trapped sets at the same semiclassical energy, it is natural to suspect that such a theorem is no longer true because the two inflection transmission sets must tunnel to each other. However, it is easy to see that the theorem still applies in this case, since the stable/unstable manifolds for the degenerate hyperbolic trapped set form a separatrix (in other words, the degenerate hyperbolic trapped set is at higher semiclassical energy). Hence the same microlocalization applies, and so does the theorem.

Remark 1.4. We briefly discuss why we have chosen to call this type of smooth trapping “inflection-transmission” type trapping. The inflection part refers to the fact that the effective potential after separating variables has an inflection point at the trapped set. We have also included transmission in our name because this kind of trapping bears some resemblance to the traditional transmission problem.

The traditional transmission problem concerns a wave equation in a medium for which the speed of propagation is distinct in different regions. For example, one might study solutions to the equation

\[
\begin{align*}
\left(\partial_t^2 - \Delta\right)u &= 0 \text{ for } |x| < 1, \\
\left(\partial_t^2 - c^2 \Delta\right)u &= 0 \text{ for } |x| > 1,
\end{align*}
\]

where \(c \neq 1\). Of course one also needs to indicate appropriate boundary conditions at the interaction surface where \(|x| = 1\) (see, for example, [CPV01, CPV99, CV10]).

On the other hand, if we consider a surface of revolution given by a specific generating curve \(C\) in the \((x_1, x_3)\) plane, rotated around the \(x_3\) axis, we get a similar looking picture. Let

\[C = \{(x_1, 0, x_3) = (A(r), 0, B(r)), r \geq 0\},\]

where

\[
A(r) = \begin{cases} 
  r, & \text{for } 0 \leq r \leq 1, \\
  \frac{1}{2}r, & \text{for } r \geq 3,
\end{cases}
\]

and assume \(0 \leq A'(r) \leq 1\) and \(A\) has an inflection point at, say, \(r = 2\). The function \(A(r)\) is sketched schematically in Figure 2.
Figure 3. The manifold obtained by rotating the curve $C$ about the $x_3$ axis.

We suppose that $B'$ is compactly supported in the region $1 \leq r \leq 3$ and fix $B(0) = 0$. The function $B(r)$ is also depicted in Figure 2.

Rotating the curve $C$ about the $x_3$ axis in $\mathbb{R}^3$ yields a manifold which is flat near 0 and flat outside a compact set, and changes “height” in between (see Figure 3). Moreover, if we compute the Laplacian on this surface of revolution, we see that

$$\Delta_g = \frac{1}{r} \partial_r r \partial_r + \frac{1}{r^2} \partial_\theta^2, \quad 0 \leq r \leq 1$$

but

$$\Delta_g = 4 \left( \frac{1}{r} \partial_r r \partial_r + \frac{1}{r^2} \partial_\theta^2 \right), \quad r \geq 3.$$

See, e.g., [Boo11] where such a computation has been carried out in much detail.

As in [CW11], once we prove Theorem 1 we can obtain a resolvent bound. For simplicity, say that our surface of revolution is Euclidean at infinity. That is, assume $A(x) = x$ for $|x| \gg 0$. Alternatively, we could require dilation analyticity at infinity, which would permit asymptotically conic spaces as were treated in [WZ00].

We let

$$R(\lambda) = (-\Delta_g - \lambda^2)^{-1}$$

denote the resolvent on $X$ (where it exists), and take $\text{Im} \lambda < 0$ as our physical sheet. With a choice of appropriate branch cut, $\chi R(\lambda) \chi$ extends meromorphically to $\{\lambda \in \mathbb{R} : \lambda \gg 0\}$ for any $\chi \in C_c^\infty(X)$. See, e.g., [SZ91]. And, in the degenerate inflection point setting, we have
Theorem 2. For any $\chi \in C^\infty_c(X)$, there exists a constant $C = C_{m_1, m_2, \chi} > 0$ such that for $\lambda \gg 0$,

$$\|\chi R(\lambda - i0)\chi\|_{L^2 \to L^2} \leq C \max\{\lambda^{-2/(m_1+1)}, \lambda^{-4/(2m_2+3)}\}.$$ 

Moreover, this is the nut estimate, in the sense that no better polynomial rate of decay holds true.

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2. Local smoothing estimates

In this section, we shall prove the main local smoothing estimate. In the subsequent section, we shall saturate the inequality, thus showing that the loss is sharp.

The proof of the estimate uses a positive commutator argument. On Euclidean space, such a proof of local smoothing is well-known, though the interested reader can see [CW11, Section 2.1] for an exposition which is quite akin to what follows.

We first show

Proposition 2.1. Suppose $u$ solves

$$\begin{cases} (D_t - \tilde{\Delta})u = 0, \\ u(0, x, \theta) = u_0. \end{cases}$$

Then for any $T < \infty$, there exists a constant $C_T > 0$ such that

$$\int_0^T \left( \| \langle x \rangle^{-1} \partial_x u \|^2_{L^2(dx d\theta)} + \| \langle x \rangle^{-3/2} \partial_\theta u \|^2_{L^2(dx d\theta)} \right) dt \leq C_T \left( \| \langle D_\theta \rangle^\beta(m_1, m_2) u_0 \|^2_{L^2(dx d\theta)} + \| \langle D_x \rangle^{1/2} u_0 \|^2_{L^2(dx d\theta)} \right),$$

where $\beta(m_1, m_2)$ is as in (1.2).

The equation (2.1) is obtained by conjugating the original equation by the operator $L$. Upon conjugating back, Proposition 2.1 shows that the estimate of Theorem 1 holds.

2.1. Proof of Proposition 2.1

The proof will be broken into three steps. The first is to use a positive commutator argument to prove full smoothing away from the periodic orbits at $x = 0$ and $x = 1$. We then expand into a Fourier series to reduce to a one dimensional problem, and we reduce the problem to understanding the high frequency part. Using a $TT^*$ argument, gluing techniques and a semiclassical rescaling, we show that the high frequency estimate follows from a cutoff resolvent estimates near each instance of trapping and subsequently prove those.

2.1.1. The estimate away from $x = 0$ and $x = 1$. For a self-adjoint operator $\tilde{\Delta}$ and a time-independent, self-adjoint multiplier $B$, we have

$$\frac{d}{dt} \langle u, Bu \rangle = -2 \text{Im} \langle (D_t - \tilde{\Delta})u, Bu \rangle + i \langle [-\Delta, B]u, u \rangle.$$

In particular, if

$$B = \frac{1}{2} \arctan(x) D_x + \frac{1}{2} D_x \arctan(x),$$

then

$$\frac{d}{dt} \langle u, Bu \rangle = -2 \text{Im} \langle (D_t - \tilde{\Delta})u, Bu \rangle + i \langle [-\Delta, B]u, u \rangle.$$
and satisfies 
\[ i[-\Delta, B] = 2D_x \langle x \rangle^{-2} D_x + 2D_\theta A' A^{-3} \arctan(x) D_\theta - \frac{3x^2 - 1}{\langle x \rangle^6} - V'_1 \arctan(x). \]

Upon integrating (2.2) over \([0, T]\), we obtain
\[ \int_0^T \langle i[-\Delta, B] u, u \rangle \, dt = i \langle u, \arctan(x) \partial_x u \rangle \bigg|_0^T - \frac{1}{2} \langle u, \langle x \rangle^{-1} u \rangle \bigg|_0^T \]
for a solution \( u \) to (2.1). Using energy estimates, the right side is controlled by \( \|u_0\|_{\dot{H}^{1/2}}^2 \). Noting also that energy estimates permit the control
\[ \left| \int_0^T \left( \frac{3x^2 - 1}{\langle x \rangle^6} u + V'_1 \arctan(x) u, u \right) \, dt \right| \leq CT \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^2}^2 \leq C_T \|u_0\|_{\dot{H}^{1/2}}^2, \]

it now follows from integration by parts that we have established
\[ \int_0^T \left( \|\langle x \rangle^{-1} \partial_x u\|_{L^2}^2 + \|A' A^{-3} \arctan(x) \partial_\theta u\|_{L^2}^2 \right) \, dt \leq C_T \|u_0\|_{\dot{H}^{1/2}}^2. \]

We observe that
\[ A' A^{-3} \arctan(x) \geq 0 \]
and satisfies
\[ A' A^{-3} \arctan(x) \sim \begin{cases} x^{2m_1}, & x \sim 0, \\ c'_{2} (x - 1)^{2m_2}, & x \sim 1, \\ |x|^{-3}, & |x| \to \infty. \end{cases} \]

Thus,
\[ \|x|^{m_1} |x - 1|^{m_2} \langle x \rangle^{-m_1 - m_2 - 3/2} \partial_\theta u\|_{L^2} \leq C \|A' A^{-3} \arctan(x) \partial_\theta u\|_{L^2}, \]
and hence we have the estimate
(2.3)
\[ \int_0^T \left( \|\langle x \rangle^{-1} \partial_x u\|_{L^2}^2 + \|x|^{m_1} |x - 1|^{m_2} \langle x \rangle^{-m_1 - m_2 - 3/2} \partial_\theta u\|_{L^2}^2 \right) \, dt \leq C_T \|u_0\|_{\dot{H}^{1/2}}^2. \]

That is, we have perfect smoothing in the radial direction and in the \( \theta \) direction away from \( x = 0 \) and \( x = 1 \), which is precisely where the trapped sets reside.

2.1.2. Fourier decomposition. To get an estimate in the directions tangential to the trapping, we decompose into Fourier series
\[ u(t, x, \theta) = \sum_k e^{ik\theta} u_k(t, x) \]
and
\[ u_0(x, \theta) = \sum_k e^{ik\theta} u_{0,k}(x). \]

By Plancherel’s theorem, it suffices to show
\[ \int_0^T \left( \|\langle x \rangle^{-1} \partial_x u_k\|_{L^2(dx)}^2 + k^2 \|\langle x \rangle^{-3/2} u_k\|_{L^2(dx)}^2 \right) \, dt \]
\[ \leq C_T \left( \|k|\beta(m_1, m_2) u_{0,k}\|_{L^2(dx)}^2 + \|D_x\|^{1/2} u_{0,k}\|_{L^2(dx)}^2 \right). \]
We note that as $\partial_0 u_0 = 0$, where in an abuse of notation $u_0$ here stands for the zero mode, the estimate when $k = 0$ follows trivially from \[2.3\]. Thus, it remains to show

$$
\int_0^T \| \chi(x)k u_k \|^2_{L^2(\mathbb{R})} \, dt \leq C_T \left( \| \langle \chi \rangle^{\beta(m_1,m_2)} u_{0,k} \|^2_{L^2} + \| u_{0,k} \|^2_{H^{1/2}} \right), \quad |k| \geq 1
$$

for some $\chi \in C^\infty_c(\mathbb{R})$ with $\chi(x) \equiv 1$ in a neighborhood of $x = 0$ and also in a neighborhood of $x = 1$.

In the sequel, we shall be working with a fixed $k$ and as such shall drop the subscript notation. Set

$$
P_k = D_k^2 + A^{-2}(x)k^2 + V_1(x).
$$

Notice that $P_k$ is merely $-\Delta$ applied to the $k$th mode. We fix an even function $\psi \in C^\infty_c(\mathbb{R})$ which is 1 for $|r| \leq \epsilon$ and vanishes for $|r| \geq 2\epsilon$ where $\epsilon > 0$ will be determined later. Then let

$$
u = u_{h_0} + u_{i_0}, \quad u_{h_1} = \psi(D_x/k)u.
$$

2.1.3. Low frequency estimate. We first examine $u_{i_0}$ and reduce estimating it to understanding a bound for $u_{h_1}$. We observe that $u_{i_0} = (1 - \psi(D_x/k))u$ solves

$$
(D_k + P_k)u_{i_0} = -P_k \psi(D_x/k)|u = k \langle x \rangle^{-1}L_k(x)^{-1}\tilde{\psi}(D_x/k)u
$$

where $L_k$ is $L^2$ bounded uniformly in $k$ and $\tilde{\psi} \in C^\infty_c$ which is identity on the support of $\psi$.

Choosing the same multiplier $B$, replacing $-\Delta$ with the self-adjoint $P_k$, and integrating \[2.2\] yields

$$
\int_0^T \langle \nu, B \rangle_{L^2} \, dt \leq C \left( \int_0^T (\langle u_{i_0}, \arctan(x)\partial_x u_{i_0} \rangle + |\langle u_{i_0}, \langle(\gamma)^{-1} u_{i_0} \rangle|) \right)
$$

Continuing to argue as above shows that

$$
\int_0^T \| \langle \gamma \rangle^{-1}\partial_x u_{i_0} \|^2_{L^2} \, dt \leq C_T \left( \| u_{i_0} \|^2_{H^{1/2}} + \int_0^T \langle k \rangle^{-1}kL_k \langle \gamma \rangle^{-2}\tilde{\psi}(D_x/k)u, B u_{i_0} \rangle \, dt \right).
$$

Applying the Schwarz inequality to the last term and bootstrapping, we obtain

$$
\int_0^T \| \langle \gamma \rangle^{-1}\partial_x u_{i_0} \|^2_{L^2} \, dt \leq C_T \left( \| u_{i_0} \|^2_{H^{1/2}} + \int_0^T \| k \langle \gamma \rangle^{-2}\tilde{\psi}(D_x/k)u \|^2_{L^2} \, dt \right).
$$

The frequency cutoff guarantees that

$$
\int_0^T \| \langle \gamma \rangle^{-1}k u_{i_0} \|^2_{L^2} \, dt \leq C \int_0^T \| \langle \gamma \rangle^{-1}\partial_x u_{i_0} \|^2_{L^2} \, dt.
$$

As \[2.3\] provides control on the last term in \[2.5\] away from $x = 0$ and $x = 1$, it suffices to prove

$$
\int_0^T \| \chi k\tilde{\psi}(D_x/k)u \|^2_{L^2} \, dt \leq C_T \| k^{\beta(m_1,m_2)} u_0 \|^2_{L^2}.
$$

Here $\chi$ is a cutoff which is 1 in a neighborhood of the trapped geodesics at $x = 0$ and $x = 1$. The desired bound, thus, will follow once $u_{h_1}$ is controlled as the precise choice of cutoff $\psi$ is inessential.
2.1.4. High frequency estimate. It remains to estimate \( u_{hi} \) in the vicinity of \( x = 0 \) and \( x = 1 \). We fix a cutoff \( \chi \in C_0^\infty(\mathbb{R}) \) which is 1 in a neighborhood of \( x = 0 \) and in a neighborhood of \( x = 1 \). Let

\[
F(t)g = \chi(x)\psi(D_x/k)k^r e^{-itP_k}g,
\]

where the constant \( r > 0 \) will be determined later. We seek to determine \( r \) so that \( F : L^2 \to L^2([0,T];L^2_{x}) \) as

\[
\|k^{1-r}F(t)u_0\|_{L^2([0,T],L^2_x)} \leq C_T \|k^{1-r}u_0\|_{L^2}
\]

is a local smoothing estimate. \( F \) is such a mapping if and only if \( FF^* : L^2 L^2 \to L^2 L^2 \), where we have abbreviated \( L^2([0,T];L^2_x) = L^2 L^2 \). A straightforward computation shows that

\[
FF^* f(t,x) = \chi(x)\psi(D_x/k)k^{2r} \int_0^T e^{-i(t-s)P_k} \psi(D_x/k) \chi(x)f(s,x)ds,
\]

and

\[
\|FF^* f\|_{L^2 L^2} \leq C_T \|f\|_{L^2 L^2}
\]

is the desired estimate. We write \( FF^* f(t,x) = \chi(x)\psi(D_x/k)(v_1 + v_2) \) where

\[
v_1 = k^{2r} \int_0^t e^{-i(t-s)P_k} \psi(D_x/k) \chi(x)f(s,x)ds,
\]

\[
v_2 = k^{2r} \int_t^T e^{-i(t-s)P_k} \psi(D_x/k) \chi(x)f(s,x)ds.
\]

Thus,

\[
(D_t + P_k)v_l = (-1)^l i k^{2r} \psi(D_x/k) \chi(x)f, \quad l = 1, 2
\]

and

\[
\|\chi \psi v_l\|_{L^2 L^2} \leq C_T \|f\|_{L^2 L^2}
\]

would imply the desired estimate. By Plancherel’s theorem, this is equivalent to showing

\[
\|\hat{\chi} \hat{\psi} \hat{v}_l\|_{L^2 L^2} \leq C_T \|\hat{f}\|_{L^2 L^2}
\]

where \( \hat{f} \) denotes the Fourier transform of \( f \) in the time variable. I.e., we are required to show that

\[
\|\chi \psi k^{2r} (\tau \pm i0 + P_k)^{-1} \psi\|_{L^2 \to L^2} = O(1)
\]

uniformly in \( \tau \). Setting, as above, \(-z = \tau k^{-2}, h = k^{-1}, \) and \( V = A^{-2}(x) + h^2 V_1(x) \), we need

\[
\|\chi(x)\psi(hD_x)(-z \pm i0 + (hD_x)^2 + V)^{-1} \psi(hD_x)\chi(x)\|_{L^2 \rightarrow L^2} \leq C h^{-2(1-r)}.
\]

We recall

\[
P = (hD_x)^2 + V
\]

and shall use gluing techniques to reduce proving

\[
\|\rho_\delta(P - z) u\|_{L^2} \geq ch^{-2\beta(m_1, m_2)} \|\rho_\delta u\|_{L^2}, \quad s < -1/2
\]

which implies \( \|u\| \) with \( 1 - r = \beta(m_1, m_2) \) as desired, to proving microlocal invertibility estimates near the trapped sets. In \( \|u\| \), \( \rho_\delta \) is a smooth function that is identity on a large compact set and is equivalent to \( \langle x \rangle^s \) near infinity.

The gluing techniques that we shall employ are outlined in [Chr13]. See also [Chr08, Proposition 2.2] and [DV].
Recall that we are working in $T^*\mathbb{R}$ with principal symbol $p = \xi^2 + V(x)$ where the potential $V(x)$ is a short range perturbation of $x^{-2}$ and has critical points at precisely $x = 0, 1$. The critical point at $x = 0$ is a maximum with value 1 while the critical point at 1 is an inflection point with potential value $C_1^{-1}$. This means that in terms of the Hamiltonian vector field, $H_p$, the level set $\{p = 1\}$ contains the critical point $(0, 0)$ and the level set $\{p = C_1^{-1}\}$ contains the critical point $(1, 0)$. Furthermore, $\pm V'(x) \leq 0$ for $\pm x \geq 0$ with equality only at these critical points.

As in [Chr13], we fix a few cutoffs. Let $M > 1$ be sufficiently large so that there is a symbol $p_0$ such that $p_0 = p$ for $|x| \geq M - 1$ and the operator $P_0$ associated to symbol $p_0$ satisfies

$$
\|\rho_{-s}(P_0 - z)u\|_{L^2} \geq C \frac{\hbar}{\log(1/\hbar)} \|\rho_s u\|_{L^2}.
$$

Such a $P_0$ is, e.g., the $m = 1$ case of [CW11] and such bounds follow from [Chr07, Chr10]. Here $\rho_s$ is a smooth function such that $\rho_s > 0$, $\rho_s(x) \equiv 1$ on a neighborhood of $\{|x| \leq 2M\}$, and $\rho_s \equiv \langle x \rangle^s$ for $x$ sufficiently large. We choose $\Gamma \in C_c^\infty(\mathbb{R})$ with $\Gamma \equiv 1$ on $\{|x| \leq M - 1\}$ with support in $\{|x| \leq M\}$. In particular, $p = p_0$ on $\text{supp}(1 - \Gamma)$.

Let

$$
(2.9) \quad \Lambda(r) := \int_0^r \langle t \rangle^{-1-\epsilon_0} \, dt,
$$

for some fixed $\epsilon_0 > 0$, which is a function chosen to be globally bounded with positive derivative and $\Lambda(r) \sim r$ near $r = 0$. Then let

$$
a(x, \xi) = \Lambda(x)\Lambda(\xi),
$$

so that

$$
H_p a = (2\xi \partial_x - V'(x)\partial_\xi) a
= 2\xi\Lambda(x)\Lambda'(x) - V'(x)\Lambda(x)\Lambda'(\xi).
$$

Since $\pm V'(x) < 0$ for $\pm x > 0$, $x \neq 1$, for any $\epsilon > 0$ we have

$$
(2.10) \quad H_p a \geq c_0 > 0, \quad |x| \in [\epsilon/2, 1 - (\epsilon/2)] \cup [1 + (\epsilon/2), M].
$$

We further have

$$
(2.11) \quad H_p a \geq c_0' > 0, \quad |\xi| \geq \delta > 0 \text{ and } |x| \leq M.
$$

For $j = 0, 1$, let $\Gamma_j(x)$ be equal to 1 for $|x - j| \leq \epsilon/2$ with support in $\{|x - j| \leq \epsilon\}$. And set

$$
\Gamma_2 = \Gamma - \Gamma_0 - \Gamma_1
$$

so that $\Gamma_2$ is supported in $\{|x| \in [\epsilon/2, 1 - (\epsilon/2)] \cup [1 + (\epsilon/2), M]\}$.

We may use a commutator argument to prove the necessary microlocal black box estimate for $\Gamma_2$. Indeed, for any $z \in \mathbb{R}$, using (2.10),

$$
2 \operatorname{Im} ((P - z)\Gamma_2 u, a^n\Gamma_2 u) = -i ((P - z)\Gamma_2 u, a^n\Gamma_2 u) + i (a^n\Gamma_2 u, (P - z)\Gamma_2 u)
= i (P, a^n)\Gamma_2 u, \Gamma_2 u) \geq c_1 \hbar \langle \Gamma_2 u, \Gamma_2 u \rangle
$$

for some $c_1 > 0$. Then

$$
c_1 \hbar \|\Gamma_2 u\|^2 \leq 2 \|(P - z)\Gamma_2 u\| a^n u \| \leq C \|(P - z)\Gamma_2 u\| \|\Gamma_2 u\|,
$$

so that

$$
\|\Gamma_2 u\| \leq C' \hbar^{-1} \|(P - z)\Gamma_2 u\|,
$$
We now choose two microlocal cutoffs. For $j = 0, 1$, let $\psi_j = \psi_j(\xi^2 + V(x))$ be functions of the principal symbol $p$. For some $\delta > 0$ to be fixed momentarily, assume $\psi_0 \equiv 1$ for $\{|p - 1| \leq \delta\}$ with slightly larger support and similarly $\psi_1 \equiv 1$ for $\{|p - C_1^{-1}| \leq \delta\}$ with slightly larger support. The parameter $\delta > 0$ may now be fixed, depending on $\varepsilon > 0$, so that $\psi_1 \equiv 1$ on $\text{supp}(\Gamma_1) \cap \{\xi \equiv 0\}$. These cutoffs are depicted in Figure 4. Repeating the commutator argument above but instead using (2.11) allows us to conclude

$$\|\Gamma_0(1 - \psi_0)u\| \leq C h^{-1} \| (P - z)\Gamma_0(1 - \psi_0)u\|$$

and

$$\|\Gamma_1(1 - \psi_1)u\| \leq C h^{-1} \| (P - z)\Gamma_1(1 - \psi_1)u\|.$$ 

We also include a separate figure (Figure 5) that illustrates that such a microlocalization can be carried out in the case described in Remark 1.3.

We may now conclude (2.8) provided that we can establish such microlocal invertibility estimates for $\Gamma_j^\psi_j u$.

The invertibility estimate near $(0, 0)$ has been proved in [Chr07, Chr10] for $m_1 = 1$ and in [CW11] for $m_1 > 1$. For convenience, this is restated.
Lemma 2.4. For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(\mathbb{T}^*\mathbb{R})$ have compact support in $\{|(x,\xi)| \leq \epsilon\}$. Then there exists $C_\epsilon > 0$ such that
\begin{equation}
\|(P - z)\varphi^w u\| \geq C_\epsilon h^{2m_1/(m_1 + 1)}\|\varphi^w u\|, \quad z \in [1 - \epsilon, 1 + \epsilon],
\end{equation}
if $m_1 > 1$. If $m_1 = 1$, then $h^{2m_1/(m_1 + 1)}$ is replaced by $h/(\log(1/h))$.

We need only prove the corresponding estimate near $(1,0)$. This is also used to prove Theorem 2.

Lemma 2.3. For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(\mathbb{T}^*\mathbb{R})$ have compact support in $\{|(x - 1,\xi)| \leq \epsilon\}$. Then there exists $C_\epsilon > 0$ such that
\begin{equation}
\|(P - z)\varphi^w u\| \geq C_\epsilon h^{(4m_2 + 2)/(2m_2 + 3)}\|\varphi^w u\|, \quad z \in [C_1^{-1} - \epsilon, C_1^{-1} + \epsilon] .
\end{equation}

The proof of this estimate proceeds through several steps. First, we rescale the principal symbol of $P$ to introduce a calculus of two parameters. We then quantize in the second parameter which eventually will be fixed as a constant in the problem. This technique has been used in [SZ07], and we shall employ the generalizations proved in [CW11].

2.2. The two parameter calculus. Before we proceed to the proof of Lemma 2.3 we shall first review some facts about the two parameter calculus. These ideas were introduced in [SZ07], and we shall employ the generalizations proved in [CW11].

We set
\[ S_{\alpha,\beta}^{k,m,\tilde{m}}(\mathbb{T}^*\mathbb{R}^n) := \bigg\{ a \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n)^* \times (0,1]^2) : \bigg| \partial_x^\alpha \partial_{\xi}^\gamma a(x,\xi;h,\tilde{h}) \bigg| \leq C_{\rho,\gamma} h^{-m} \tilde{h}^{-\tilde{m}} \bigg( \frac{\tilde{h}}{h} \bigg)^{\alpha |\rho| + \beta |\gamma|} \langle \xi \rangle^{k - |\gamma|} \bigg\} , \]
with $\alpha \in [0,1]$ and $\beta \leq 1 - \alpha$. Throughout, we take $\tilde{h} \geq h$. We abbreviate $S_{\alpha,\beta}^{0,0,0}$ by $S_{\alpha,\beta}$. The focus shall be on the marginal case $\alpha + \beta = 1$. In particular, even in this marginal case, we have that $a \in S_{\alpha,\beta}^{k,m,\tilde{m}}$ and $b \in S_{\alpha,\beta}^{k',m',\tilde{m}'}$ implies that
\[ \text{Op}_N^w(a) \circ \text{Op}_N^w(b) = \text{Op}_N^w(c) \]
for some symbol $c \in S_{\alpha,\beta}^{k+k',m+m',\tilde{m}+\tilde{m}'}$.

We also have the following expansion. This is from [SZ07, Lemma 3.6] in the case that $\alpha = \beta = 1/2$ and from [CW11] in the more general case.

Lemma 2.4. Suppose that $a, b \in S_{\alpha,\beta}$, and that $c^w = a^w \circ b^w$. Then
\begin{equation}
\sum_{k=0}^N \frac{1}{k!} \left( i\frac{h}{2} \sigma(D_x, D_\xi; D_y, D_\eta) \right)_k a(x,\xi) b(y,\eta) \big|_{x=y,\xi=\eta} + e_N(x,\xi) ,
\end{equation}
where for some $M$
\begin{equation}
|\partial^\tau e_N| \leq C_N h^{N+1}
\end{equation}
\begin{equation}
\sup_{\gamma_1 + \gamma_2 = \gamma} \sup_{(x,\xi) \in \mathbb{T}^*\mathbb{R}^n} \sup_{|\rho| \leq M, |\rho| \in \mathbb{N}^{m_1}} |\Gamma_{\alpha,\beta,\rho,\gamma}(D)(\sigma(D))^{N+1} a(x,\xi) b(y,\eta) | ,
\end{equation}
where $\sigma(D) = \sigma(D_x, D_\xi; D_y, D_\eta)$ as usual, and
\[ \Gamma_{\alpha,\beta,\rho,\gamma}(D) = (h^\alpha \partial_{(x,y)}, h^\beta \partial_{(\xi,\eta)})^{\rho} \partial_{(x,\xi)}^{\gamma_1} \partial_{(y,\eta)}^{\gamma_2} .\]
With the scaling of coordinates
\[(x, \xi) = B(X, \Xi) = ((h/\tilde{h})^\alpha X, (h/\tilde{h})^\beta \Xi),\]
it follows that if \(a \in S^{k,m,\tilde{m}}_{\alpha,\beta} \) then \(a \circ B \in S^{k,m,\tilde{m}}_{0,0,\tilde{m}} \). Moreover, the unitary operator
\[T_{h,\tilde{h}}u(X) = \left( \frac{h}{\tilde{h}} \right)^{\frac{m}{2}} u \left( \left( \frac{h}{\tilde{h}} \right)^\alpha X \right),\]
relates the quantizations
\[(2.17) \quad Op_h^w(a \circ B)T_{h,\tilde{h}}u = T_{h,\tilde{h}}Op_h^w(a)u.\]

2.3. **Proof of Lemma 2.3** Due to the cutoff \(\varphi^w\), we are working microlocally in \(\{|(x-1, \xi)| \leq \epsilon\}\). We notice that it suffices to demonstrate (2.13) for \(P = \tilde{h}^2 V_1 - z\) as \(V_1\) is bounded in this region and \((4m_2 + 2)/(2m_2 + 3) < 2\).

Let 
\[q_1 = \xi^2 + A^{-2} - z\]
be the principal symbol of \(Q_1\). Applying Taylor’s theorem about \(x = 1\) to \(A^{-2}\), we have
\[q_1 = \xi^2 - \frac{c_2}{C_1^2} (x - 1)^{2m_2 + 1}(1 + \tilde{a}(x)) - z_1\]
where \(z_1 = z - C_1^{-1} \in [-\epsilon, \epsilon]\) and \(\tilde{a}(x) = O(|x-1|^{2m_2+1})\). The Hamilton vector field \(H\) associated to the symbol \(q_1\) is
\[H = 2\xi \partial_x + \left( 2m_2 + 1 \right) \frac{c_2}{C_1^2} (x - 1)^{2m_2 + 1} + O(|x-1|^{4m_2+1}) \partial_\xi.\]

We introduce the new variables
\[X - 1 = \frac{x - 1}{(h/\tilde{h})^\alpha}, \quad \Xi = \frac{\xi}{(h/\tilde{h})^\beta},\]
where
\[\alpha = \frac{2}{2m_2 + 3}, \quad \beta = \frac{2m_2 + 1}{2m_2 + 3},\]
and, as above, we shall use \(B\) to denote the map \(B(X - 1, \Xi) = (x - 1, \xi)\). In these new coordinates, we record that
\[(2.18) \quad H = (h/\tilde{h})^{\frac{2m_2 - 1}{2m_2 + 3}} \left( 2\Xi \partial_X + (2m_2 + 1) \frac{c_2}{C_1^2} (X - 1)^{2m_2} \partial_\Xi \right.\]
\[\left. + O((h/\tilde{h})^{2m_2 + 1}\alpha |X - 1|^{4m_2 + 1}) \partial_\Xi \right).\]

We recall the definition \(\Lambda(r)\) of \(\Lambda(r)\), and we similarly set
\[\Lambda_2(r) = 1 + \int_{-\infty}^r (t)^{-1-\rho_0} dt.\]
Then, for a cutoff function \(\chi(s)\) which is identity for \(|s| < \delta_1\) and vanishes for \(|s| > 2\delta_1\), we introduce
\[a(x, \xi; h, \tilde{h}) = \Lambda(\Xi) \Lambda_2(X - 1) \chi_1(x - 1) \chi_1(\xi),\]
where \(\delta_1 > 0\) is another parameter which will be fixed shortly. As \(\tilde{h} \geq h\), we have that
\[\left| \partial_X^\alpha \partial_\Xi^\beta a \right| \leq C_{\alpha,\beta}.\]
We compute
\[H(a) = (h/\tilde{h})^{\frac{2m_2 - 1}{2m_2 + 3}} g(x, \xi; h, \tilde{h}) + r(x, \xi; h, \tilde{h})\]
where

\begin{equation}
\tag{2.19}
g = \chi(x-1)\chi(\xi) \left(2\Lambda(\xi)(X-1)^{-1-c_0} + (2m_2 + 1)\frac{c_0}{C_1}(X-1)^{2m_2}(\xi)^{-1-c_0} \Lambda_2(X-1)(1 + O(|x-1|^{2m_2 + 1})) \right)
\end{equation}

and

\[ \text{supp } r \subset \{|x-1| > \delta_1\} \cup \{ |\xi| > \delta_1\}. \]

We first seek to show that the following lemma from \([CW11]\) may be applied to \(g\):

**Lemma 2.5.** Let a real-valued symbol \(\tilde{g}(x, \xi; h)\) satisfy

\[ \tilde{g}(x, \xi; h) = \begin{cases} 
  c(\xi^2 + x^{2m})(1 + r_2), & \xi^2 + x^2 \leq 1 \\
  b(x, \xi; h), & \xi^2 + x^2 \geq 1,
\end{cases} \]

where \(c > 0\) is constant, \(r_2 = \mathcal{O}_{\mathcal{S}_{a, \beta}}(\delta_1)\), and \(b > 0\) is elliptic. Then there exists \(c_0 > 0\) such that

\[ (\text{Op}_h^w(\tilde{g})u, u) \geq c_0 h^{2m/(m+1)}\|u\|_{L^2}^2 \]

for \(h\) sufficiently small.

In the sequel, we shall only be applying the above to functions which are microlocally cutoff to the set where \(\chi(x-1)\chi(\xi) \equiv 1\). As the errors off this set will be \(\mathcal{O}(h^\infty)\), we shall assume that \(|x-1| \leq \delta_1\) and \(|\xi| \leq \delta_1\) throughout this discussion.

Over \(|(x-1, \xi)| \leq 1\), we have \(\Lambda(\xi) \sim \xi, \Lambda_2(X-1) \sim 1,\) and \((X-1)^{-1-c_0} \sim 1\). Thus, the term \(g\), given in (2.19), of \(H(a)\) is bounded below by a multiple of \(\xi^2 + (X-1)^{2m_2}\).

We next consider \(|(X-1, \xi)| \geq 1\). Since \(\text{sgn} \Lambda(\xi) = \text{sgn} (s),\) when \(|\xi| \geq \max(|X-1|^{1+c_0}, 1/4)\), then

\[ g \geq 2\Lambda(\xi)(X-1)^{-1-c_0}\xi \geq \frac{|\xi|}{\langle \xi \rangle} \geq C > 0. \]

For \(|X-1|^{1+c_0} \geq \max(|\xi|, 1/4)\), we have

\[ g \geq C'\xi^2 \Lambda_2(X-1)(X-1)^{2m_2} \geq |X-1|^{-1-c_0} |X-1|^{1+c_0} |X-1|^{2m_2} \geq C'' > 0, \]

provided \((1 + c_0)^2 < 2m_2\). In the region of interest \(|(X-1, \xi)| \geq 1\), the larger of \(|\xi|\) and \(|X-1|^{1+c_0}\) is assuredly greater than \(1/4\) if \(c_0 > 0\) is sufficiently small. Hence, we have shown that

\[ g \geq C > 0 \quad \text{in } \{\xi^2 + (X-1)^2 \geq 1\}. \]

Recapping, we have found that

\[ H(a) = (h/\hat{h})^{\frac{2m_2-1}{2m_2}} g + r \]

with

\[ r = \mathcal{O}_{\mathcal{S}_{a, \beta}}((h/\hat{h})^{(2m_2-1)/(2m_2+3)}((h/\hat{h})^\alpha |\xi| + (h/\hat{h})^{\beta}|X-1|^{2m_2})) \]

supported as above and

\[ g(X, \xi; h) = \begin{cases} 
  c(\xi^2 + (X-1)^{2m_2})(1 + r_2), & \xi^2 + (X-1)^2 \leq 1 \\
  b, & \xi^2 + (X-1)^2 \geq 1,
\end{cases} \]

where \(c > 0\) is a constant, \(r_2 = \mathcal{O}_{\mathcal{S}_{a, \beta}}(\delta_1)\), and \(b > 0\) is elliptic.
By translating, using the blowdown map $\mathcal{B}$, and relating the quantizations as in the previous section, we may use Lemma 2.5 to obtain a similar bound on $g$.

**Lemma 2.6.** For $g$ given by (2.19) and $\hbar > 0$ sufficiently small, there exists $c > 0$ such that

$$
\|\text{Op}_h^w(g \circ \mathcal{B}^{-1})\|_{L^2 \to L^2} > c\hbar^{2m_2/(m_2+1)},
$$

uniformly as $h \downarrow 0$.

The proof of this lemma follows exactly as that in [CW11] and is, thus, omitted.

Before completing the proof of Lemma 2.3, we need the following lemma about the lower order terms in the expansion of the commutator of $Q_1$ and $a^w$.

**Lemma 2.7.** The symbol expansion of $[Q_1, a^w]$ in the $h$-Weyl calculus is of the form

$$
[Q_1, a^w] = \text{Op}_h^w \left( \frac{i\hbar}{2} \sigma(D_x, D_{\xi}; D_y, D_\eta) \left( q_1(x, \xi)a(y, \eta) - q_1(y, \eta)a(x, \xi) \right) \right)_{x=y, \xi=\eta} + e(x, \xi) + r_3(x, \xi),
$$

where $r_3$ is supported in $\{(x, \xi) \geq \delta_1\}$ and $e$ satisfies

$$
\|\text{Op}_h^w(e)\|_{L^2 \to L^2} \leq C\hbar^{2m_2+7 - \frac{2m_2+4}{2m_2+1}} \left( \|\text{Op}_h^w(g \circ \mathcal{B}^{-1})\|_{L^2 \to L^2} + \mathcal{O}(\hbar^{2m_2+7-\frac{2m_2+4}{2m_2+1}}) \right),
$$

with $g$ given by (2.19).

**Proof.** Since everything is in the Weyl calculus, only the odd terms in the exponential composition expansion are non-zero. In accordance with Lemma 2.4, we set

$$
e(x, \xi) = \chi(x-1)\chi(\xi)
\times \sum_{k=1}^{m_2-1} \frac{2}{(2k+1)!} \left( \frac{i\hbar}{2} \sigma(D) \right)^{2k+1} q_1(x, \xi)\Lambda((\tilde{h}/\hbar)^{\beta}\eta)\Lambda_2((\tilde{h}/\hbar)^{\alpha}(y-1)) \bigg|_{x=y, \xi=\eta}
+ \chi(\xi)\chi(x-1)e_{2m_2}(x, \xi).
$$

Here we have extracted the terms in the expansion where derivatives fall on the cutoff $\chi(\eta)$ of $a$ as these terms have supports compatible with $r_3$. For convenience, however, $e_{2m_2}$ denotes the full error in the expansion of $[Q_1, a^w]$.

Recalling that $q_1(x, \xi) = \xi^2 - (x-1)^{2m_2+1}(1 + \tilde{a}(x))$, it follows that

$$
\tilde{e}_k := \hbar^{2k+1}\chi(x-1)\chi(\xi)\sigma(D)^{2k+1} q_1(x, \xi)\Lambda((\tilde{h}/\hbar)^{\beta}\eta)\Lambda_2((\tilde{h}/\hbar)^{\alpha}(y-1)) \bigg|_{x=y, \xi=\eta}
+ \hbar^{2k+1}\chi(x-1)\chi(\xi) D_\xi^{2k+1} q_1(x, \xi) D_{\eta}^{2k+1} \Lambda((\tilde{h}/\hbar)^{\beta}\eta)\Lambda_2((\tilde{h}/\hbar)^{\alpha}(y-1)) \bigg|_{x=y, \xi=\eta}
= \hbar^{2k+1}(x-1)^{2m_2+1-(2k+1)}(1 + \mathcal{O}((x-1)^{2m_2+1}))
\times (\tilde{h}/\hbar)^{(2k+1)\beta}\Lambda^{(2k+1)}((\tilde{h}/\hbar)^{\beta}\xi)
\times \Lambda_2((\tilde{h}/\hbar)^{\alpha}(x-1))\chi(x-1)\chi(\xi)
.$$
for $1 \leq k \leq m_2 - 1$.

In order to estimate $e$, we first estimate each $\tilde{e}_k$, $1 \leq k \leq m_2 - 1$, using conjugation to the 2-parameter calculus. We have

$$
\|\text{Op}_h^w(\tilde{e}_k)u\|_{L^2} = \|T_{h,\tilde{h}}\text{Op}_h^w(\tilde{e}_k)T_{h,\tilde{h}}^{-1}T_{h,\tilde{h}}u\|_{L^2} \leq \|T_{h,\tilde{h}}\text{Op}_h^w(\tilde{e}_k)T_{h,\tilde{h}}\|_{L^2 \rightarrow L^2}\|u\|_{L^2}
$$

since $T_{h,\tilde{h}}$ is unitary. We recall that $T_{h,\tilde{h}}\text{Op}_h^w(\tilde{e}_k)T_{h,\tilde{h}}^{-1} = \text{Op}_h^w(\tilde{e}_k \circ B)$ and note that

$$
\tilde{e}_k \circ B = ch^{2k+1}(h/\tilde{h})^{(2m_2+1)-(2k+1)}(X - 1)^{(2m_2+1)-(2k+1)}
$$

$$
\times (1 + \mathcal{O}((x - 1)^{(2m_2+1)}))\Lambda((\Xi))\Lambda_2(X - 1)\chi(x - 1)\chi(\xi),
$$

which can be estimated by

$$
Ch^{4m_2+2}h^{-2m_2+1/2}h^{2(k-1)}(X - 1)^{(2m_2+1)-(2k+1)}\Lambda((\Xi))\Lambda_2(X - 1)\chi(x - 1)\chi(\xi).
$$

On $|X - 1| \leq 1$, we have that

$$
k = (X - 1)^{(2m_2+1)-(2k+1)}\Lambda((\Xi))\Lambda_2(X - 1)\chi(x - 1)\chi(\xi)
$$

is bounded, and thus,

$$
\|\text{Op}_h^w(k)\|_{L^2 \rightarrow L^2} \leq C\tilde{h}^{-2m_2/(m_2+1)}\|\text{Op}_h^w(g \circ B^{-1})\|_{L^2 \rightarrow L^2}
$$

by Lemma 2.6. While on $|X - 1| \geq 1$, we have $k \leq g$, and thus

$$
\|\text{Op}_h^w(k)\|_{L^2 \rightarrow L^2} \leq \|\text{Op}_h^w(g)\|_{L^2 \rightarrow L^2} + O(\tilde{h}^2) \leq \|\text{Op}_h^w(g \circ B^{-1})\|_{L^2 \rightarrow L^2} + O(\tilde{h}^2).
$$

For $e_{2m_2}$, by the standard $L^2$ continuity theorem of $h$-pseudodifferential operators, it suffices to estimate a finite number of derivatives of the error $e_{2m_2}$. We note the bound of Lemma 2.4

$$
|\partial^n e_{2m_2}| \leq Ch^{2m_2+1}\sum_{\gamma_1 + \gamma_2 = \gamma}(x,\xi) \in T^*\mathbb{R}^n\sup_{(y,\eta) \in T^*\mathbb{R}^n}(\eta,\rho) \in \mathbb{N}^n, |\rho| \leq M |\Gamma_{\alpha,\beta,\rho,\gamma}(D)(\sigma(D))^{2m_2+1}q_1(x,\xi)a(y,\eta)|.
$$

We have

$$(\sigma(D))^{2m_2+1}q_1(x,\xi)a(y,\eta) = c(1 + \mathcal{O}(x - 1)^{(2m_2+1)}\chi(y - 1)\Lambda((\tilde{h}/h)^\alpha(y - 1))\Lambda((\tilde{h}/h)^\beta)\chi(\eta)).$$

The last factor is $\mathcal{O}((\tilde{h}/h)^{(2m_2+1)\beta})$. Moreover, the derivatives $h^\beta \partial_\eta$ and $h^\alpha \partial_y$ preserve the order of $h$ and increase the order of $\tilde{h}$, while the other derivatives lead to higher powers of $\tilde{h}/h \ll 1$. It, thus, follows that $|\partial^n (\chi(x - 1)\chi(\xi)e_{2m_2})|$ is

$$
\mathcal{O}(h^{4m_2+2}/(2m_2+3)\tilde{h}^{(2m_2+1)^2/(2m_2+3)}),
$$

and thus, when also combined with Lemma 2.4 satisfies the given bound.  

We now complete the proof of Lemma 2.3. We set $v = \varphi^w u$ where $\varphi$ has support where $\chi(x)\chi(\xi) = 1$, and in particular, away from the support of $r_3$.  

\[\square\]
Then Lemmas 2.6 and 2.7 yield
\[
\langle [Q_1, a^w] v, v \rangle = h \langle \text{Op}_h^w (g \circ B^{-1}) v, v \rangle + \langle \text{Op}_h^w (e) u, u \rangle + O(h^\infty) \|v\|_{L^2}^2
\]
\[
= h^{\frac{4m+2}{2m+3}} \left( \frac{h^2}{\tilde{h}} - \frac{2m+1}{2m+3} + \frac{2m+1}{2m+3} \right) \langle [Q_1, a^w] v, v \rangle
\]
\[
+ (O(h^\infty) + O(\tilde{h}^{2+\frac{2m+1}{2m+3}})) \|v\|_{L^2}^2
\]
\[
\geq Ch^{\frac{4m+2}{2m+3}} \tilde{h}^{1+ \frac{2m+1}{2m+3}} \|v\|_{L^2}^2,
\]
for \( \tilde{h} \) sufficiently small. The Schwarz inequality and the \( L^2 \) continuity theorem for \( h \)-pseudodifferential operators guarantees
\[
|\langle [Q_1, a^w] v, v \rangle| \leq C \|Q_1 v\|_{L^2} \|v\|_{L^2},
\]
and thus the desired bound with \( 1 \gg \tilde{h} > 0 \) fixed.

\[ \square \]

3. Quasimodes

We end by constructing quasimodes near \((1, 0)\) in phase space and use these to saturate the estimate of Proposition 2.1 and hence that of Theorem 1. The proofs follow from straightforward modifications of those in [CW11]. We, thus, only provide a terse description.

Quasimodes were already constructed near \((0, 0)\) in [CW11]. We focus only on the construction near the inflection point. We let
\[
\tilde{P} = -h^2 \partial_x^2 - c_2 (x - 1)^{2m_2+1}
\]
near \( x = 1 \) and construct quasimodes that are localized to a small neighborhood of \( x = 1 \).

We set
\[
\gamma = \frac{4m + 2}{2m + 3},
\]
\[
E = (\alpha + i\beta) h^\gamma \text{ where } \alpha, \beta > 0 \text{ and are independent of } h, \text{ and}
\]
\[
\varpi(x) = \int_1^x (E + c_2 (y - 1)^{2m_2+1})^{1/2} dy,
\]
where the branch of the square root is chosen to have positive imaginary part. Letting
\[
u(x) = (\varpi')^{-1/2} e^{i\varpi/h},
\]
we see that
\[
(hD)^2 u = (\varpi')^2 u + fu,
\]
where
\[
f = -h^2 \left( \frac{3}{4} (\varpi')^{-2} (\varpi'')^2 - \frac{1}{2} (\varpi')^{-1} \varpi'' \right).
\]

Straightforward modifications of the proof contained in [CW11, Lemma 3.1] yield the following:

Lemma 3.1. The phase function \( \varpi \) satisfies the following properties:
(i): There exists $C > 0$ independent of $h$ such that

$$|\text{Im } \varpi| \leq Ch.$$  

In particular, if $|x - 1| \leq C h^{\gamma/(2m_2+1)}$, $|\text{Im } \varpi| \leq C'$ for some $C' > 0$ independent of $h$.

(ii): There exists $C > 0$ independent of $h$ such that if $\delta > 0$ is sufficiently small and $|x - 1| \leq \delta h^{\gamma/(2m_2+1)}$, then

$$C^{-1} h^{\gamma/2} \leq |\varpi'(x)| \leq Ch^{\gamma/2}. $$

(iii):

$$\varpi' = (E + c_2(x - 1)^{2m_2+1})^{1/2},$$

$$\varpi'' = \frac{1}{2} c_2(2m_2 + 1)(x - 1)^{2m_2} (\varpi')^{-1},$$

$$\varpi''' = \left( \frac{1}{2} c_2(2m_2 + 1)(2m_2)(E(x - 1)^{2m_2-1} + c_2(x - 1)^{4m_2}) \right. \left. - \frac{1}{4} c_2^2(2m_2 + 1)^2(x - 1)^{4m_2} \right) (\varpi')^{-3}. $$

In particular, there are constants $C_{m_2,1}, C_{m_2,2}$ such that

$$f = -h^2 (C_{m_2,1}(x - 1)^{4m_2} + C_{m_2,2} E(x - 1)^{2m_2-1}) (\varpi')^{-4}. $$

From this, we obtain that $|u(x)| \sim |\varphi'|^{-1/2}$ for all $x$. We localize $u$ by setting

$$\mu = \delta h^{\gamma/(2m_2+1)}, \ 0 < \delta \ll 1$$

fixing $\chi(s) \in C^\infty_c(\mathbb{R})$ so that $\chi \equiv 1$ for $|s| \leq 1$ and supp $\chi \subset [-2, 2]$, and letting

$$\tilde{u}(x) = \chi((x - 1)/\mu) u(x).$$

More calculations, which are again in the spirit of those contained in [CWTI], show that $\|\tilde{u}\|_{L^2}^2 \sim h^{(1-2m_2)/2+3}$ and

$$(hD)^2 \tilde{u} = (\varpi')^2 \tilde{u} + R,$$

where

$$R = f \tilde{u} + [(hD)^2, \chi((x - 1)/\mu)] u.$$ 

Moreover, the remainder satisfies

$$(3.1) \quad \|R\|_{L^2} = O(h^\gamma) \|\tilde{u}\|_{L^2}. $$

This quasimode can then be used to saturate the local smoothing estimates near the inflection point. We, again, refer the interested reader to the proof in [CWTI] Theorem 3], which provides the following.

**Theorem 3.** Let $\varphi_0(x, \theta) = e^{ik\theta} \tilde{u}(x)$, where $\tilde{u} \in C^\infty_c(\mathbb{R})$ was constructed above. We let $h = |k|^{-1}$, where $|k|$ is taken sufficiently large. Suppose $\psi$ solves

$$\begin{cases} (D_t - \bar{A}) \psi = 0, \\ \psi|_{t=0} = \varphi_0. \end{cases}$$

Then for any $\chi \in C^\infty_c(\mathbb{R})$ such that $\chi \equiv 1$ on supp $\tilde{u}$ and $A > 0$ sufficiently large, independent of $k$, there exists a constant $C_0 > 0$ independent of $k$ such that

$$(3.2) \quad \int_0^{|k|^{-1/(2m_2+1)} / A} \| (D_\theta) \chi \psi \|_{L^2} dt \geq C_0^{-1} \| (D_\theta)^{(2m_2+1)/(2m_2+3)} \varphi_0 \|_{L^2}^2.$$
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REFERENCES

[BGH09] Nicolas Burq, Colin Guillarmou, and Andrew Hassell. Strichartz estimates without loss on manifolds with hyperbolic trapped geodesics. *preprint*, arXiv:0907.3345v1 [math.AP], 2009.

[Boo11] Robert Booth. Energy estimates on asymptotically flat surfaces of revolution. *Masters Project*, *University of North Carolina*, 2011.

[Bur04] N. Burq. Smoothing effect for Schrödinger boundary value problems. *Duke Math. J.*, 123(2):403–427, 2004.

[Chr07] Hans Christianson. Semiclassical non-concentration near hyperbolic orbits. *J. Funct. Anal.*, 246(2):145–195, 2007.

[Chr08] Hans Christianson. Dispersive estimates for manifolds with one trapped orbit. *Comm. Partial Differential Equations*, 33:1147–1174, 2008.

[Chr10] Hans Christianson. Quantum monodromy and non-concentration near a closed semi-hyperbolic orbit. *Trans. Amer. Math. Soc. to appear*, 2010.

[Chr13] Hans Christianson. High-frequency resolvent estimates on asymptotically Euclidean warped products. *in preparation*, 2013.

[CKS95] Walter Craig, Thomas Kappeler, and Walter Strauss. Microlocal dispersive smoothing for the Schrödinger equation. *Comm. Pure Appl. Math.*, 48(8):769–860, 1995.

[CPV99] Fernando Cardoso, Georgi Popov, and Georgi Vodev. Distribution of resonances and local energy decay in the transmission problem. II. *Math. Res. Lett.*, 6(3-4):377–396, 1999.

[CPV01] Fernando Cardoso, Georgi Popov, and Georgi Vodev. Asymptotics of the number of resonances in the transmission problem. *Comm. Partial Differential Equations*, 26(9-10):1811–1859, 2001.

[Dat09] Kiril Datchev. Local smoothing for scattering manifolds with hyperbolic trapped sets. *Comm. Math. Phys.*, 286(3):837–850, 2009.

[Doi98b] Shin-ichi Doi. Remarks on the Cauchy problem for the Schrödinger equation. *Proc. Amer. Math. Soc.*, 102(4):874–878, 1998.
