DISTINGUISHED SELF-ADJOINT EXTENSIONS OF DIRAC OPERATORS VIA HARDY-DIRAC INEQUALITIES

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Abstract. We prove some Hardy-Dirac inequalities with two different weights including measure valued and Coulombic ones. Those inequalities are used to construct distinguished self-adjoint extensions of Dirac operators for a class of diagonal potentials related to the weights in the above mentioned inequalities.

1. Introduction

In this work we deal with the problem of self-adjointness of Dirac operators. Many authors have studied this problem for Dirac operators $H_0$ coupled to an electrostatic potential $V$. Denoting $H_0 = -i\alpha \cdot \nabla + \beta$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,

$$\alpha_i := \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \quad \beta := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

$I_2$ is the identity operator on $\mathbb{C}^2$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the family of Pauli matrices.

If $V$ is a bounded function which tends to 0 at infinity, the operator $H_0 + V$ with domain $H^1(\mathbb{R}^3, \mathbb{C}^4)$ is self-adjoint, see for instance [19]. However, if $V$ has singularities, one is interested in constructing self-adjoint extensions of $H_0 + V$ originally defined on the domain $C^\infty_c(\mathbb{R}^3, \mathbb{C}^4)$. The papers [15, 14, 16, 17, 18, 13, 12, 10] treated this problem by using a different method depending on the singularity of the potential. Those works dealt exclusively with electrostatic potentials, while in [2, 3, 4, 20], Arai and Yamada consider more general matrix-valued potentials. Results on the essential self-adjointness of Dirac operators with relativistic $\delta$-sphere interactions can be found in [5, 6, 8] and similar results for the Schrödinger operator with point interactions in [1]. See notes in [19] for the complete bibliography.

We will restrict our attention to [10], the most recent work among the above mentioned ones, in which Esteban and Loss use a method based on Hardy-like inequalities. Let us explain this result in more detail. Let $V$:
$\mathbb{R}^3 \to \mathbb{R}$ be a potential such that for some constant $c(V) \in (-1, 1)$, $\Gamma := \sup_{\mathbb{R}^3} V < 1 + c(V)$ and for every $\varphi \in C_c^\infty(\mathbb{R}^3, \mathbb{C}^2)$,

$$
\int_{\mathbb{R}^3} \left( \frac{|\sigma \cdot \nabla \varphi|^2}{1 + c(V) - V} + (1 - c(V) + V)|\varphi|^2 \right) \, dx \geq 0.
$$

Then Esteban and Loss can construct a distinguished self-adjoint extension of the operator $H_0 + V$ defined on $C_c^\infty(\mathbb{R}^3, \mathbb{C}^4)$. One of the components of the operator is extended by using the Friedrichs extension and inequality (1.1), and the remaining one by choosing the right domain for the whole operator. In [11], the same authors point out that an extra condition on the potential is needed for the construction of the self-adjoint extensions mentioned in [10]. The natural condition to get the desired symmetry on the operator $H_0 + V$ is that each component of

$$
(\gamma - V)^{-2} \nabla V
$$

is square integrable, where $\gamma$ is any number in $(\Gamma, 1 + c(V))$.

In this paper we generalize, in some sense, the above mentioned result. Consider $H_0 = -i\alpha \cdot \nabla + m\beta$, using similar techniques as in [10], we construct distinguished self-adjoint extensions of Dirac operators defined as $H_V = H_0 - V$ with potentials of the type

$$
V(x) = \begin{pmatrix}
w_1(x)\|_2 & 0 \\
0 & w_2(x)\|_2
\end{pmatrix}
$$

where $w_1$ is a real function or a singular measure and $w_2$ is a function.

Assuming that $w_2$ is positive, for $w_1$ negative the proof runs quite straightforward. However, for the positive sign of $w_1$ we need to prove Hardy-Dirac inequalities such as

$$
\int_{\mathbb{R}^3} w_1|\phi|^2 \leq \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + w_2 - \lambda} + (m + \lambda) \int_{\mathbb{R}^3} |\phi|^2,
$$

for some $\lambda \in (-m, m)$.

Estimates of the type (1.3) are proved in Section 2 and we use them in Section 3 to prove the self-adjointness of the Dirac operator $H_V$. In particular, they are used to define a Hilbert space $\mathcal{H}$ with the inner product

$$
(\phi, \varphi)_\mathcal{H} := \int_{\mathbb{R}^3} (m - w_1 + \lambda)\phi \cdot \overline{\varphi} + \int_{\mathbb{R}^3} \frac{i\sigma \cdot \nabla \phi}{m + w_2 - \lambda} \cdot i\sigma \cdot \nabla \overline{\varphi}.
$$

By using the Riesz Representation Theorem we are able to extend one component of the operator. The remaining component is extended by choosing the right domain $D$. Moreover, we avoid the extra condition on the gradient of the potential equivalent to (1.2) thanks to the particular structure of the inner product, which is itself symmetric. Therefore, if we take $w_1 = w_2 = V$ such that $\sup_{x \neq 0} V(x) \leq \frac{\nu}{|x|}$, we improve the result of [10] in the sense that we can construct a distinguished self-adjoint extensions without using the condition (1.2).

Some examples of Dirac operators and Hardy-Dirac inequalities are given in the last section of the paper.
2. Hardy-Dirac estimate

**Definition 2.1.** Let \( \mathcal{A} \) be the class of potentials that contains all pairs of positive radial measurable functions, \( V_1, V_2 : \mathbb{R}^3 \to \mathbb{R}^+ \), that satisfy

\[
A_+ [V_1, V_2] := \sup_{r > 0} \left[ \frac{1}{r^2} \int_0^r (V_1(t) + V_2(t)) t^2 \, dt \right] < +\infty
\]

and

\[
A_- [V_1, V_2] := \sup_{r > 0} \left[ r^2 \int_r^\infty (V_1(t) + V_2(t)) \frac{dt}{t^2} \right] < +\infty.
\]

We can now state the main result of this section.

**Theorem 2.2.** Let \( V_1, V_2 \in \mathcal{A} \). For any \( \phi \in L^2(\mathbb{R}^3, \mathbb{C}^2) \) and any \( \gamma \geq 0 \),

\[
\int_{\mathbb{R}^3} V_1 |\phi|^2 \leq \max\{ A^2_+, A^2_\} \int_{\mathbb{R}^3} \frac{\sigma \cdot \nabla \phi}{V_2 + \gamma} + \gamma \int_{\mathbb{R}^3} |\phi|^2.
\]

The inequality holds whenever the right hand side is finite. We follow the approach of [7] for the proof. For the convenience of the reader we state the relevant results on the spectrum of \( \sigma \cdot L \) and the projections associated to the spectral space, \( X_k \), without proofs, thus making our exposition self-contained.

**Lemma 2.3.** The spectrum of \( \sigma \cdot L \) is the discrete set \( \{ k \in \mathbb{Z} : k \neq -1 \} \) and \( \sigma \cdot L \) applied to a radial function is zero. Moreover, if \( \phi \) is a continuous function, then \( P_k \phi(0) = 0 \) for any \( k \in \mathbb{Z}/\{0, -1\} \).

The key points are that \( L \) commutes with all radial functions and that \(-1\) is not in the spectrum of \( \sigma \cdot L \).

**Lemma 2.4.** For any \( k, l \in Spec(\sigma \cdot L) \), \( k \neq l \), \( P_k(\sigma \cdot L)^2 P_l \equiv P_l(\sigma \cdot L)^2 P_k \equiv 0 \) in \( H^1(\mathbb{R}^3, \mathbb{C}^2) \).

**Corollary 2.5.** Any function \( \phi \in L^2(\mathbb{R}^3, \mathbb{C}^2) \) can be written

\[
\phi = \sum_{k \in \mathbb{Z}, k \neq -1} \phi_k
\]

with \( \phi_k \in X_k \) and moreover, if \( W \) is a radial function,

\[
\int_{\mathbb{R}^3} W|\phi|^2 = \sum_{k \in \mathbb{Z}, k \neq -1} \int_{\mathbb{R}^3} W|\phi_k|^2,
\]

\[
\int_{\mathbb{R}^3} W|\sigma \cdot \nabla \phi|^2 = \sum_{k \in \mathbb{Z}, k \neq -1} \int_{\mathbb{R}^3} W|\sigma \cdot \nabla \phi_k|^2.
\]

**Definition 2.6.** For \( V_1, V_2 \in \mathcal{A} \), \( A_k \) is given by

\[
A_k := \begin{cases} 
\sup_{r > 0} \frac{1}{r^{2(k+1)}} \int_0^r (V_1(s) + V_2(s)) s^{2(k+1)} \, ds, & k \in \mathbb{Z}, k \geq 0, \\
\sup_{r > 0} r^{2(k+1)} \int_r^\infty (V_1(s) + V_2(s)) \frac{ds}{s^{2(k+1)}}, & k \in \mathbb{Z}, k \leq -2.
\end{cases}
\]
We can see at once that $A_k \leq A_0$ for all $k \in \mathbb{Z}, k \geq 0$ and $A_k \leq A_{-2}$ for all $k \in \mathbb{Z}, k \leq -2$, because $V_1$ and $V_2$ are nonnegative.

**Proof of Theorem 2.2.** Let $\phi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$. From the fundamental theorem of calculus we write

$$|\phi(x)|^2 = \Re \left( \int_r^\infty -2\phi(t\omega)(\omega \cdot \nabla \phi(t\omega)) \, dt \right)$$

for $r = |x|$, $\omega = \frac{x}{|x|}$. Suppose that $W$ is a radial and real function, then

$$\int_{\mathbb{R}^3} W|\phi|^2 = -2 \Re \left( \int_{S^2} d\omega \int_0^\infty W(r\omega)r^2 \, dr \int_r^\infty \phi(t\omega)(\omega \cdot \nabla \phi(t\omega)) \, dt \right)$$

$$= -2 \Re \left( \int_{S^2} d\omega \int_0^\infty \phi(t\omega)(\omega \cdot \nabla \phi(t\omega))t^2g_W(t) \, dt \right)$$

where

$$g_W(t) := \frac{1}{t^2} \int_0^t W(r) \, r^2 \, dr.$$ 

By abuse of notation, we use the same letter $W$ for $W(x)$ and $W(r)$. Now using the identity

$$\left( (2.1) \right) \frac{x}{|x|} \cdot \nabla = \left( \sigma \cdot \frac{x}{|x|} \right) (\sigma \cdot \nabla) + \frac{1}{|x|} (\sigma \cdot L) \quad \forall x \in \mathbb{R}^3,$$

for any $\delta > 0$ and $\gamma > 0$, we obtain

$$\left( W + \frac{2}{|x|} g_W \sigma \cdot L \right) \phi, \phi \right) = -2 \Re \left( \int_{\mathbb{R}^3} \bar{\phi} \left( \sigma \cdot \frac{x}{|x|} \right) (\sigma \cdot \nabla) \phi g_W(|x|) \, dx \right)$$

$$\leq ||g_W||_{L^\infty(0, \infty)} \left[ \frac{1}{\delta} \int_{\mathbb{R}^3} \left| \sigma \cdot \nabla \phi \right|^2 \, dx + \delta \int_{\mathbb{R}^3} (V_2 + \gamma)|\phi|^2 \, dx \right].$$

Take the nonnegative spectrum of $\sigma \cdot L$, i.e., $k \in \mathbb{Z}, k \geq 0$. We want to solve

$$W_k + \frac{2k}{r} g_k = V_1 + V_2 \quad \forall r \in (0, \infty)$$

where

$$g_k = g_{W_k}(r) := \frac{1}{r^2} \int_0^r W_k(s) s^2 \, ds.$$ 

Since

$$\frac{d}{dr} \left( r^{2k} \int_0^r s^2 W_k(s) \, ds \right) = r^{2(k+1)} W_k + 2kr^{2k-1} \int_0^r s^2 W_k(s) \, ds$$

$$= r^{2(k+1)} (V_1 + V_2),$$

then

$$r^{2k} \int_0^r s^2 W_k(s) \, ds = \int_0^r s^{2(k+1)} (V_1(s) + V_2(s)) \, ds.$$ 

The equation is solved by

$$g_k(r) := \frac{1}{r^{2(k+1)}} \int_0^r (V_1(s) + V_2(s))s^{2(k+1)} \, ds.$$
and

$$W_k = V_1 + V_2 - \frac{2k}{r}g_k.$$  

By definition of $A_k$, $\|gW\|_{L^\infty(0,\infty)} = A_k$. From the above and (2.2) it follows that for $\phi = \phi_k \in C_c^\infty(\mathbb{R}^3, \mathbb{C}^2)$ such that

$$\sigma \cdot L \phi_k = k\phi_k, \quad k \in \mathbb{Z}, k \geq 0,$$

$$\int_{\mathbb{R}^3} V_1 |\phi_k|^2 + \int_{\mathbb{R}^3} V_2 |\phi_k|^2 \leq \frac{A_k}{\delta} \int_{\mathbb{R}^3} \frac{\sigma \cdot \nabla \phi_k^2}{V_2 + \gamma} + \delta A_k \int_{\mathbb{R}^3} (V_2 + \gamma) |\phi_k|^2.$$  

Take $\delta = \frac{1}{A_k}$,

$$\int_{\mathbb{R}^3} V_1 |\phi_k|^2 \leq A_k^2 \int_{\mathbb{R}^3} \frac{\sigma \cdot \nabla \phi_k^2}{V_2 + \gamma} + \gamma \int_{\mathbb{R}^3} |\phi_k|^2.$$

Since $A_k \leq A_0 = A_+$ for all $k \in \mathbb{Z}, k \geq 0$,

$$\int_{\mathbb{R}^3} V_1 |\phi_k|^2 \leq A_+^2 \int_{\mathbb{R}^3} \frac{\sigma \cdot \nabla \phi_k^2}{V_2 + \gamma} + \gamma \int_{\mathbb{R}^3} |\phi_k|^2.$$  

We now apply the same argument for the negative spectrum of $\sigma \cdot L$. Let $\phi \in C_c^\infty(\mathbb{R}^3, \mathbb{C}^2)$ and write

$$|\phi(x)|^2 = |\phi(0)|^2 + 2 \Re \left( \int_0^r \phi(t\omega)(\omega \cdot \nabla \phi(t\omega)) \, dt \right)$$

for $r = |x|$, $\omega = \frac{x}{|x|}$. Using the same notation as before and assuming that $\phi(0) = 0$,

$$\int_{\mathbb{R}^3} W|\phi|^2 = 2 \Re \left( \int_{S^2} d\omega \int_0^\infty \phi(t\omega)(\omega \cdot \nabla \phi(t\omega)) t^2 h_W(t) \, dt \right)$$

where

$$h_W(t) := \frac{1}{t^2} \int_t^\infty W(r) r^2 \, dr.$$  

By (2.1) and for any $\delta > 0$ and $\gamma > 0$, we obtain

$$\left\langle \left( W - \frac{2}{|x|} h_W \sigma \cdot L \right) \phi, \phi \right\rangle \leq \|hW\|_{L^\infty(0,\infty)} \left[ \frac{1}{\delta} \int_{\mathbb{R}^3} \frac{\sigma \cdot \nabla \phi^2}{V_2 + \gamma} \, dx + \delta \int_{\mathbb{R}^3} (V_2 + \gamma) |\phi|^2 \, dx \right].$$  

For the negative spectrum of $\sigma \cdot L$, i.e., $k \in \mathbb{Z}, k \leq -2$ we solve

$$W_k - \frac{2k}{r} h_k = V_1 + V_2 \quad \forall r \in (0, \infty)$$

where

$$h_k = h_{W_k}(r) := \frac{1}{r^2} \int_r^\infty W_k(s) s^2 \, ds.$$
Since
\[ \frac{d}{dr} \left( r^{2k} \int_r^\infty s^2 W_k(s) \, ds \right) = r^{2(k+1)} W_k + 2kr^{2k-1} \int_r^\infty s^2 W_k(s) \, ds = -r^{2(k+1)} (V_1 + V_2). \]
Equation (2.5) can be solved by taking
\[ h_k(r) := \frac{1}{r^{2(k+1)}} \int_r^\infty (V_1(s) + V_2(s)) s^{2(k+1)} \, ds \]
and
\[ W_k = V_1 + V_2 + \frac{2k}{r} h_k. \]
By definition \( \|h_W\|_{L^\infty(0, \infty)} = A_k. \) Let \( \phi = \phi_k \in C_\infty^\infty(\mathbb{R}^3, C^2) \) such that
\[ \sigma \cdot L \phi_k = k \phi_k, \quad k \in \mathbb{Z}, k \leq -2. \]
By Lemma (2.3), \( \phi_k(0) = 0. \) Now using (2.4) and the above estimates we obtain
\[ \int_{\mathbb{R}^3} |V_1| |\phi_k|^2 + \int_{\mathbb{R}^3} |V_2| |\phi_k|^2 \leq \frac{A_k}{\delta} \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi_k|^2}{V_2 + \gamma} + \delta A_k \int_{\mathbb{R}^3} (V_2 + \gamma) |\phi_k|^2. \]
Take \( \delta = \frac{1}{A_k} \)
\[ \int_{\mathbb{R}^3} |V_1| |\phi_k|^2 \leq A_k^2 \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi_k|^2}{V_2 + \gamma} + \gamma \int_{\mathbb{R}^3} |\phi_k|^2. \]
Since \( A_k \leq A_{-2} = A_- \) for all \( k \in \mathbb{Z}, k \leq -2, \)
\[ (2.6) \quad \int_{\mathbb{R}^3} |V_1| |\phi_k|^2 \leq A^2_\delta \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi_k|^2}{V_2 + \gamma} + \gamma \int_{\mathbb{R}^3} |\phi_k|^2. \]
By (2.3), (2.6) and using a density argument we conclude that
\[ (2.7) \quad \int_{\mathbb{R}^3} V_1 |\phi_k|^2 \leq \max\{A^2_+, A^2_-\} \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi_k|^2}{V_2 + \gamma} + \gamma \int_{\mathbb{R}^3} |\phi_k|^2 \]
for any \( \phi_k \in L^2(\mathbb{R}^3, C^2) \) such that \( \sigma \cdot L \phi_k = k \phi_k, k \in \mathbb{Z}, k \neq -1. \) Sum on \( k \in \mathbb{Z}, k \neq -1 \) and use Corollary 2.5 to complete the proof. \( \square \)

Remark 2.7. (i) The constants \( A_+ \) and \( A_- \) are scaling invariant. Let \( V_1^\alpha(x) = \alpha V_1(\alpha x) \) and \( V_2^\alpha(x) = \alpha V_2(\alpha x) \) for \( \alpha \in \mathbb{R}, \) then \( A_+[V_1^\alpha, V_2^\alpha] = A_+[V_1, V_2] \) and \( A_-[V_1^\alpha, V_2^\alpha] = A_-[, V_2]. \)
(ii) Notice also that we could put two different scalings by taking
\[ \tilde{A}_+[V_1, V_2] := \sup_{r > 0} \frac{1}{r^2} \int_0^r V_1(t) \, t^2 \, dt + \sup_{r > 0} \frac{1}{r^2} \int_0^r V_2(t) \, t^2 \, dt \]
and
\[ \tilde{A}_-[V_1, V_2] := \sup_{r > 0} \frac{r^2}{2} \int_0^\infty V_1(t) \, dt + \sup_{r > 0} \frac{r^2}{2} \int_r^\infty V_2(t) \, dt. \]
In this case, since \( A_+ \leq \tilde{A}_+ \) and \( A_- \leq \tilde{A}_-, \) the constant in the inequality of Theorem 2.2 is worst, however, we gain on freedom.
Corollary 2.8. Let $V_1, V_2 \in A$, $c_1$ and $c_2$ positive constants such that 
\[ c_1 c_2 \leq \frac{1}{\max\{A_+^2, A_-^2\}} , \]
and $m \in \mathbb{R}^+$. Then there exists a $\lambda \in (0, m)$ such that
\[ \int_{\mathbb{R}^3} c_1 V_1 |\phi|^2 \leq \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + c_2 V_2 - \lambda} + (m + \lambda) \int_{\mathbb{R}^3} |\phi|^2 . \]

Proof. Take $\gamma = \frac{m - \lambda}{c_2}$ in Theorem 2.2. Hence
\[ \int_{\mathbb{R}^3} V_1 |\phi|^2 \leq \max\{A_+^2, A_-^2\} \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + V_2 - \frac{\lambda}{c_2}} + \frac{m - \lambda}{c_2} \int_{\mathbb{R}^3} |\phi|^2 . \]

By assumption,
\[ \int_{\mathbb{R}^3} V_1 |\phi|^2 \leq \frac{1}{c_1 c_2} \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + V_2 - \frac{\lambda}{c_2}} + \frac{m - \lambda}{c_2} \int_{\mathbb{R}^3} |\phi|^2 \]
\[ = \frac{1}{c_1} \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + c_2 V_2 - \lambda} + \frac{m - \lambda}{c_2} \int_{\mathbb{R}^3} |\phi|^2 . \]

Now, if
\[ (2.8) \quad \frac{c_1}{c_2} \leq \frac{m + \lambda}{m - \lambda} , \]
the corollary follows. Note that we can choose $\lambda \in (0, m)$ close enough to $m$ such that (2.8) holds. □

Remark 2.9. The same results hold for $V_1$ nonnegative radial Radon measure which, in what follows, we will denote by $\mu$. In this case, we have to redefine $A$ as the class of pairs $\mu, V_2$ such that $\mu$ is a singular positive radial measure supported in $\mathbb{R}^3 \setminus \{0\}$ and $V_2$ is a positive radial measurable function bounded in a neighborhood of the support of $\mu$ that satisfy
\[ A_+ [\mu, V_2] := \sup_{r > 0} \left[ \frac{1}{r^2} \left( \int_0^r t^2 d\mu + \int_0^r V_2(t) \frac{dt}{t^2} \right) \right] < +\infty \]
and
\[ A_- [\mu, V_2] := \sup_{r > 0} \left[ r^2 \left( \int_r^{\infty} \frac{1}{t^2} d\mu + \int_r^{\infty} V_2(t) \frac{dt}{t^2} \right) \right] < +\infty . \]

The proof of Theorem 2.2 for $V_1$ a measure can be handled in much the same way, the only difference being in the definition of $\int_{\mathbb{R}^3} |\phi|^2 d\mu$, i.e., we have to assure that the expression makes sense.

From [9] we know that if $\mu$ is a positive radial measure, then
\[ \int_{\mathbb{R}^3} |\phi|^2 d\mu \leq C ||\phi||_{L^2} ||\nabla \phi||_{L^2} \]
holds for some $C$ if and only if $\mu(B(0, r)) \leq Br^2$ for some constant $B$ and all $r > 0$. Since $\mu \in A$, it satisfies the inequality. Let $\Omega$ be the support of $\mu$ and $\Omega_\epsilon := \{ x : d(x, \Omega) < \epsilon \}$. It suffices to show that $\phi, \nabla \phi \in L^2_{\Omega_\epsilon}$. 
Define a smooth cut-off function $\eta$ as

$$
\eta := \begin{cases} 
1 & \text{if } x \in \Omega_{\epsilon/2} \\
0 & \text{if } x \notin \Omega_{\epsilon/2}.
\end{cases}
$$

Assume that

$$
(2.9) \quad C \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + V_2 - \lambda} + c \int_{\mathbb{R}^3} |\phi|^2 < +\infty \quad \text{for } C, c \geq 0,
$$

then $\phi \in L^2(\mathbb{R}^3, \mathbb{C}^2)$, in particular, it is in $L^2(\Omega_{\epsilon}, \mathbb{C}^2)$. On the other hand,

$$
\int_{\mathbb{R}^3} \nabla (\eta \phi)^2 = \int_{\mathbb{R}^3} |\sigma \cdot \nabla \phi|^2 = \int_{\mathbb{R}^3} |(\sigma \cdot \nabla \eta_2) \phi + \eta \sigma \cdot \nabla \phi|^2 \\
\leq 2 \int_{\mathbb{R}^3} |\nabla \eta|^2 |\phi|^2 + 2 \int_{\mathbb{R}^3} \eta^2 |\sigma \cdot \nabla \phi|^2.
$$

The first term on the right side is finite, because $\nabla \eta$ is bounded and $\phi \in L^2(\mathbb{R}^3, \mathbb{C}^2)$. Let us show that so is the second one. Since $w_2$ is bounded in $\Omega_{\epsilon}$, then

$$
\int_{\mathbb{R}^3} \eta^2 |\sigma \cdot \nabla \phi|^2 \leq \int_{\Omega_{\epsilon/2}} |\sigma \cdot \nabla \phi|^2 \leq C \int_{\Omega_{\epsilon/2}} \frac{|\sigma \cdot \nabla \phi|^2}{m + V_2 - \lambda} < +\infty.
$$

Therefore, if (2.9) holds $\int_{\mathbb{R}^3} |\phi|^2 d\mu$ is well-defined.

Corollary 2.8 and Remark 2.9 will be very useful in the next section.

3. Self-adjointness and Essential Self-adjointness

Let $V$ be a potential such that

$$
V(x) = \begin{pmatrix} w_1(x) I_2 & 0 \\
0 & w_2(x) I_2 \end{pmatrix}
$$

where $w_1$ is a real function or a measure, $w_2$ is a real function and $I_2$ is the identity operator on $\mathbb{C}^2$. The Dirac operator coupled to the potential $V$ takes the form

$$
H_V := -i\alpha \cdot \nabla + m\beta - V.
$$

**Proposition 3.1.** Let $w_1, w_2$ real functions such that $w_1(x) \leq 0$ and $w_2(x) \geq 0$ and locally integrable. Then, the space

$$
\mathcal{H} := \left\{ \phi \in L^2(\mathbb{R}^3, \mathbb{C}^2) : \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{1 + w_2} + \int_{\mathbb{R}^3} (1 - w_1)|\phi|^2 < \infty \right\}
$$

is a Hilbert space with the norm

$$
||\phi||^2_\mathcal{H} = \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{1 + w_2} + \int_{\mathbb{R}^3} (1 - w_1)|\phi|^2.
$$

Moreover, for any $a, b > 0$ the $\mathcal{H}$-norm is equivalent to

$$
||\phi||^2_\mathcal{H} = \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{b + w_2} + \int_{\mathbb{R}^3} (a - w_1)|\phi|^2.
$$

In particular, $a = m + \lambda$, $b = m - \lambda$ if $\lambda \in (-m, m)$. 

Proof. It is easy to check that
\[
(\phi, \varphi)_\mathcal{H} := \int_{\mathbb{R}^3} (1 - w_1) \phi \cdot \varphi + \int_{\mathbb{R}^3} \frac{i \sigma \cdot \nabla \phi}{1 + w_2} \cdot i \sigma \cdot \nabla \varphi
\]
is an inner product.

We have to see that \(\mathcal{H}\) is complete. Let \(\phi_n\) be a Cauchy sequence in \(\mathcal{H}\), then so is in \(L^2(1 - w_1)\) and \(\sigma \cdot \nabla \phi_n\) in \(L^2\left(\frac{1}{1 + w_2}\right)\). Hence, there exist a function \(\phi \in L^2(1 - w_1)\) such that
\[
\lim_{n \to \infty} ||\phi_n - \phi||_{L^2(1-w_1)} = 0
\]
and a function \(\psi \in L^2\left(\frac{1}{1 + w_2}\right)\) such that
\[
\lim_{n \to \infty} ||\sigma \cdot \nabla \phi_n - \psi||_{L^2\left(\frac{1}{1 + w_2}\right)} = 0.
\]
We claim that \(\psi = \sigma \cdot \nabla \phi\). Since
\[
\int_{\mathbb{R}^3} |\phi_n - \phi|^2 \leq \int_{\mathbb{R}^3} (1 - w_1)|\phi_n - \phi|^2,
\]
\(\phi_n\) tends to \(\phi\) in \(L^2(\mathbb{R}^3, \mathbb{C}^2)\) when \(n \to \infty\). Now, let \(\varphi\) be a test function, then
\[
\left| \int_{\mathbb{R}^3} (\sigma \cdot \nabla \phi_n - \psi) \varphi \right| \leq \left( \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi_n - \psi|^2}{1 + w_2} \right)^{1/2} \left( \int_{\mathbb{R}^3} (1 + w_2)|\varphi|^2 \right)^{1/2}.
\]
Notice that since \(w_2\) is locally integrable the second term on the right side is bounded. Moreover, the first term on the right tends to zero, thus, \(\sigma \cdot \nabla \phi_n\) tends to \(\psi\) in the sense of distributions. Now recalling that if \(\phi_n\) tends to \(\phi\) in \(L^2(\mathbb{R}^3, \mathbb{C}^2)\) when \(n\) tends to \(\infty\), then
\[
\lim_{n \to \infty} \frac{\partial \phi_n}{\partial x_j} = \frac{\partial \phi}{\partial x_j}
\]
in the distributional sense, it follows that
\[
\lim_{n \to \infty} \sigma \cdot \nabla \phi_n = \sigma \cdot \nabla \phi,
\]
which completes the proof.

Moreover, since there exist a constant \(c\) such that \(c \geq a\) and \(c \geq \frac{1 + w_2}{b + w_2}\)
and another constant \(C\) such that \(C \geq \frac{1}{a}\) and \(C \geq \frac{b + w_2}{1 + w_2}\), it is easy to check that \(\mathcal{H}\) and \(\tilde{\mathcal{H}}\) norms are equivalent. \(\square\)

Proposition 3.2. Let \(V_1, V_2 \in A\) and \(A_+, A_-\) given by Definition 2.1. Let \(w_1\) and \(w_2\) such that
\[
(3.1) \quad 0 \leq w_1, w_2, w_1(x) \leq c_1 V_1(\|x\|) \text{ and } w_2(x) \leq c_2 V_2(\|x\|),
\]
and $c_1c_2 \leq \frac{1}{\max\{A_+^2, A_-^2\}}$. Then the space

$$
\mathcal{H} := \left\{ \phi \in L^2(\mathbb{R}^3, \mathbb{C}^2) : \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{1 + w_2} + \int_{\mathbb{R}^3} |\phi|^2 < \infty \right\}
$$

is a Hilbert space with the norm

$$
||\phi||^2_{\mathcal{H}} = \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{1 + w_2} + \int_{\mathbb{R}^3} |\phi|^2.
$$

For any $a, b > 0$ the $\mathcal{H}$-norm is equivalent to

$$
||\phi||^2_{\tilde{\mathcal{H}}} = \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{b + w_2} + a \int_{\mathbb{R}^3} |\phi|^2.
$$

In particular, we can take $a = m + \lambda$, $b = m - \lambda$ if $\lambda \in (-m, m)$. Moreover, if we take $\lambda$ such that the condition $(2.8)$ holds,

$$
||\phi||^2_{\mathcal{H}_{w_1}} = \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + w_2 - \lambda} + \int_{\mathbb{R}^3} (m - w_1 + \lambda)|\phi|^2
$$

also defines an equivalent norm.

**Proof.** The fact that $\mathcal{H}$ is Hilbert is the particular case $w_1 = 0$ in Proposition 3.1. To complete the proof we only need to see the equivalence between the $\tilde{\mathcal{H}}$ and $\mathcal{H}_{w_1}$ norms. However, before doing that we need a previous result.

Since $c_1c_2 \leq \frac{1}{\max\{A_+^2, A_-^2\}}$, then there exists $\epsilon > 0$ such that

$$
(1 + \epsilon)c_1c_2 \leq \frac{1}{\max\{A_+^2, A_-^2\}}.
$$

Hence, $(1 + \epsilon)c_1$, $c_2$, $V_1$ and $V_2$ satisfy the hypotheses in Corollary 2.8. Therefore, for $\lambda$ satisfying

$$
(1 + \epsilon)c_1 \leq \frac{m + \lambda}{m - \lambda},
$$

we have

$$
\int_{\mathbb{R}^3} (1 + \epsilon)c_1 V_1 |\phi|^2 \leq \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + c_2 V_2 - \lambda} + (m + \lambda) \int_{\mathbb{R}^3} |\phi|^2.
$$

By (3.1) we get

$$
\int_{\mathbb{R}^3} (1 + \epsilon)w_1 |\phi|^2 \leq \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + w_2 - \lambda} + (m + \lambda) \int_{\mathbb{R}^3} |\phi|^2.
$$

Hence,

$$
(3.2) \quad \epsilon \int_{\mathbb{R}^3} w_1 |\phi|^2 \leq \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + w_2 - \lambda} + \int_{\mathbb{R}^3} (m - w_1 + \lambda)|\phi|^2.
$$
We will use inequality (3.2) to prove the first part of the equivalence. We have
\[ ||\phi||_{H}^{2} = \int_{\mathbb{R}^{3}} \frac{|\sigma \cdot \nabla \phi|^{2}}{m + w_{2} - \lambda} + (m + \lambda) \int_{\mathbb{R}^{3}} |\phi|^{2} \]
\[ = \int_{\mathbb{R}^{3}} \frac{|\sigma \cdot \nabla \phi|^{2}}{m + w_{2} - \lambda} + \int_{\mathbb{R}^{3}} (m - w_{1} + \lambda)|\phi|^{2} + \int_{\mathbb{R}^{3}} w_{1}|\phi|^{2} \]
\[ \leq ||\phi||_{H_{w_{1}}}^{2} + \frac{1}{\epsilon} ||\phi||_{H_{w_{1}}}^{2} \leq C ||\phi||_{H_{w_{1}}}^{2}. \]

The reverse inequality is immediate. \( \square \)

**Proposition 3.3.** Let \( V_{1} \) be a singular, radial and positive measure supported in \( \mathbb{R}^{3} \setminus \{0\} \) and \( V_{2} \) a function that satisfy the conditions in Remark 2.9. Let \( w_{1} = c_{1}V_{1} \), which we denote by \( \mu \), \( w_{2} \geq 0 \) such that \( w_{2}(x) \leq c_{2}V_{2}(|x|) \) and \( c_{1}c_{2} < \max\{A_{1}^{2}, A_{2}^{2}\} \). If we take \( \lambda \) such that the condition (2.8) holds and \( w_{2} \) is bounded in a neighborhood of the support of \( \mu \),
\[ ||\phi||_{H_{\mu}} = \int_{\mathbb{R}^{3}} \frac{|\sigma \cdot \nabla \phi|^{2}}{m + w_{2} - \lambda} + (m + \lambda) \int_{\mathbb{R}^{3}} |\phi|^{2} - \int_{\mathbb{R}^{3}} |\phi|^{2} d\mu \]
defines an equivalent norm in the Hilbert space \( H \) given in Proposition 3.2.

The proof runs as in Proposition 3.2, the only difference being in the definition of \( \int_{\mathbb{R}^{3}} |\phi|^{2} d\mu \). However, since \( V_{2} \) satisfies the conditions in Remark 2.9, it is well-defined.

**Remark 3.4.** The same result holds for \( w_{1} \) a measure with regular and singular parts, as long as the singular part satisfies the conditions in Proposition 3.2 and the regular part satisfies the ones in Proposition 3.2.

We fix a value \( \lambda \) satisfying the condition (2.8). In what follows we use the notation of this inner product
\[ (\phi, \varphi)_{H} := \int_{\mathbb{R}^{3}} (m - w_{1} + \lambda)\phi \cdot \varphi + \int_{\mathbb{R}^{3}} \frac{i\sigma \cdot \nabla \phi}{m + w_{2} - \lambda} \cdot \frac{i\sigma \cdot \nabla \varphi}{m + w_{2} - \lambda}. \]

Define \( D \) the domain of the Dirac operator containing all pairs \((\phi, \chi) \in H \times L^{2}(\mathbb{R}^{3}, \mathbb{C}^{2})\) such that
\[ (m - w_{1} + \lambda)\phi - i\sigma \cdot \nabla \chi, -i\sigma \cdot \nabla \phi + (m - w_{2} + \lambda)\chi \in L^{2}(\mathbb{R}^{3}, \mathbb{C}^{2}). \]

We understand the last two expressions in the following sense; the linear functional \((\eta, (m - w_{2} + \lambda)\chi) + (-i\sigma \cdot \nabla \eta, \phi)\), which is defined for all test functions, extends uniquely to a bounded linear functional on \( L^{2}(\mathbb{R}^{3}, \mathbb{C}^{2}) \). Likewise for \((\eta, (m - w_{1} + \lambda)\phi) + (-i\sigma \cdot \nabla \eta, \chi)\).

We can now state our main result.

**Theorem 3.5.** Under the hypotheses of Proposition 3.1, 3.2 or 3.3 the Dirac operator \( H_{V} \) defined on \( D \) is self-adjoint. Furthermore, it is the unique self-adjoint extension of \( H_{V} \) on \( C_{c}^{\infty}(\mathbb{R}^{3}, \mathbb{C}^{4}) \) such that the domain is contained in \( H \times L^{2}(\mathbb{R}^{3}, \mathbb{C}^{2}) \).
Proof. Here we follow the approach of [10]. The self-adjointness is proved by showing that \( H_V \) is symmetric and that \( H_V + \lambda \) is a bijection.

We start by showing that \( H_V + \lambda \) is a bijection from \( \mathcal{D} \) to \( L^2(\mathbb{R}^3, \mathbb{C}^2) \). To prove that the operator is onto pick \( (F_1, F_2) \in L^2(\mathbb{R}^3, \mathbb{C}^2) \) and define the linear functional \( T : \mathcal{H} \rightarrow \mathbb{C} \) such that

\[
T(\eta) = (F_1, \eta)_{L^2(\mathbb{R}^3, \mathbb{C}^2)} + \left( \frac{F_2}{m + w_2 - \lambda}, -i\sigma \cdot \nabla \eta \right)_{L^2(\mathbb{R}^3, \mathbb{C}^2)}, \quad \eta \in \mathcal{H}.
\]

Let us see that \( T \) is bounded. By Cauchy-Schwarz,

\[
|T(\eta)| \leq ||F_1||_{L^2} ||\eta||_{L^2} + ||F_2||_{L^2} \left| \left| \frac{-i\sigma \cdot \nabla \eta}{m + w_2 - \lambda} \right| \right|_{L^2}.
\]

Since \( F_1 \in L^2(\mathbb{R}^3, \mathbb{C}^2) \) and \( \eta \in \mathcal{H} \), the first term on the right side is well defined and bounded. Since \( F_2 \in L^2(\mathbb{R}^3, \mathbb{C}^2) \) and

\[
(3.3) \left| \left| \frac{-i\sigma \cdot \nabla \eta}{m + w_2 - \lambda} \right| \right|^2 \leq \frac{1}{m - \lambda} \int_{\mathbb{R}^3} \frac{\sigma \cdot \nabla \eta^2}{m + w_2 - \lambda} \, dx \leq \frac{c}{m - \lambda} ||\eta||_{\mathcal{H}},
\]

the second term is also bounded.

We use the Riesz Representation Theorem to conclude that there exists a unique \( \phi \in \mathcal{H} \) such that

\[
(\phi, \eta)_{\mathcal{H}} = T(\eta) \quad \forall \eta \in \mathcal{H}
\]

i.e.,

\[
((m - w_1 + \lambda)\phi, \eta)_{L^2} + \left( \frac{-i\sigma \cdot \nabla \phi}{m + w_2 - \lambda}, -i\sigma \cdot \nabla \eta \right)_{L^2} = (F_1, \eta)_{L^2} + \left( \frac{F_2}{m + w_2 - \lambda}, -i\sigma \cdot \nabla \eta \right)_{L^2}.
\]

Equivalently,

\[
((m - w_1 + \lambda)\phi, \eta)_{L^2} + \left( \frac{F_2 + i\sigma \cdot \nabla \phi}{m - w_2 + \lambda}, -i\sigma \cdot \nabla \eta \right)_{L^2} = (F_1, \eta)_{L^2}.
\]

Define

\[
\chi = \frac{F_2 + i\sigma \cdot \nabla \phi}{m - w_2 + \lambda}
\]

which is in \( L^2(\mathbb{R}^3, \mathbb{C}^2) \), because \( F_2 \in L^2(\mathbb{R}^3, \mathbb{C}^2) \) and \( \phi \in \mathcal{H} \). Now by definition,

\[
((m - w_1 + \lambda)\phi, \eta)_{L^2} + (\chi, -i\sigma \cdot \nabla \eta)_{L^2} = (F_1, \eta)_{L^2}.
\]

This holds for all test function \( \eta \), but since \( F_1 \in L^2(\mathbb{R}^3, \mathbb{C}^2) \), the functional

\[
\eta \rightarrow ((m - w_1 + \lambda)\phi, \eta)_{L^2} + (\chi, -i\sigma \cdot \nabla \eta)_{L^2}
\]

extends uniquely to a continuous functional on \( L^2(\mathbb{R}^3, \mathbb{C}^2) \) which implies

\[
(m - w_1 + \lambda)\phi - i\sigma \cdot \nabla \chi = F_1.
\]

Now since \( \chi \) is a function in \( L^2(\mathbb{R}^3, \mathbb{C}^2) \), from its definition we have

\[
(-m - w_2 + \lambda)\chi = F_2 + i\sigma \cdot \nabla \phi \quad \text{a.e.}
\]

so that

\[
(-m - w_2 + \lambda)\chi - i\sigma \cdot \nabla \phi = F_2 \quad \text{a.e.}
\]
The injection is trivial, because the Riesz Representation Theorem tells that for each \((F_1, F_2)\) there exists a unique \(\phi\) such that \((\phi, \eta)_H = T(\eta)\) for all \(\eta \in H\). For \((F_1, F_2) = (0, 0)\), \(\phi = 0\) satisfies the equation, thus, \(\phi\) must be zero. And, in consequence, \(\chi = 0\).

To prove the symmetry let \((\phi, \chi), (\tilde{\phi}, \tilde{\chi}) \in D\) and

\[
(H_V + \lambda) \begin{pmatrix} \phi \\ \chi \end{pmatrix} + \begin{pmatrix} -i \sigma \cdot \nabla \phi \\ -m - w_2 + \lambda \end{pmatrix} = (m - w_1 + \lambda) \phi - i \sigma \cdot \nabla \chi, (\tilde{\phi}, \tilde{\chi}) = ((-m - w_2 + \lambda) \chi - i \sigma \cdot \nabla \phi, \tilde{\chi}).
\]

Take

\[
(\phi, \tilde{\phi})_H + \left((-m - w_2 + \lambda) \left[\chi + \frac{-i \sigma \cdot \nabla \phi}{-m - w_2 + \lambda}, \frac{-i \sigma \cdot \nabla \phi}{-m - w_2 + \lambda}\right]_{L^2} \right.
\]

\[
\left.+ \left((-m - w_2 + \lambda) \chi, \frac{-i \sigma \cdot \nabla \phi}{-m - w_2 + \lambda}\right)_{L^2} + \left(-i \sigma \cdot \nabla \phi, \frac{-i \sigma \cdot \nabla \phi}{-m - w_2 + \lambda}\right)_{L^2}\right) = ((m - w_1 + \lambda) \phi, \tilde{\phi}) + (\chi, -i \sigma \cdot \nabla \tilde{\phi}).
\]

Observe that

\[
(3.4) \quad (\phi, \tilde{\phi})_H + \left((-m - w_2 + \lambda) \left[\chi + \frac{-i \sigma \cdot \nabla \phi}{-m - w_2 + \lambda}, \frac{-i \sigma \cdot \nabla \phi}{-m - w_2 + \lambda}\right]_{L^2} \right.
\]

equals to

\[
(3.5) \quad ((m - w_1 + \lambda) \phi - i \sigma \cdot \nabla \chi, \tilde{\phi})
\]

for \(\tilde{\phi} \in C_c^\infty(\mathbb{R}^3, \mathbb{C}^2)\). Note also that the first term of \(3.4\) makes sense because \(\phi, \tilde{\phi} \in H\) and the second one because, since \((\phi, \chi) \in D\),

\[
(-m - w_2 + \lambda) \left[\chi + \frac{-i \sigma \cdot \nabla \phi}{-m - w_2 + \lambda}\right] \in L^2(\mathbb{R}^3, \mathbb{C}^2)
\]

and since \(\tilde{\phi} \in H\),

\[
\frac{-i \sigma \cdot \nabla \tilde{\phi}}{-m - w_2 + \lambda} \in L^2(\mathbb{R}^3, \mathbb{C}^2)
\]

as we proved in \(3.3\). \(3.5\) makes sense by definition of the domain. We next show that \(3.4\) and \(3.5\) are continuous in \(\tilde{\phi}\) with respect to the \(H\)-norm. By definition of the domain,

\[
((m - w_1 + \lambda) \phi - i \sigma \cdot \nabla \chi, \tilde{\phi}) \leq c||\tilde{\phi}||_H
\]
and

\[
(\phi, \tilde{\phi})_H + \left( (-m - w_2 + \lambda) \left[ \chi + \frac{-i\sigma \cdot \nabla \phi}{m - w_2 + \lambda}, \frac{-i\sigma \cdot \nabla \tilde{\phi}}{m - w_2 + \lambda} \right] \right)_{L^2}
\]

\[
= \left( (m - w_1 + \lambda) \phi, \tilde{\phi} \right) + \left( \frac{-i\sigma \cdot \nabla \phi}{m + w_2 - \lambda}, \frac{-i\sigma \cdot \nabla \tilde{\phi}}{m - w_2 + \lambda} \right)_{L^2}
\]

\[
+ (\chi, -i\sigma \cdot \nabla \tilde{\phi})_{L^2} + \left( \frac{-i\sigma \cdot \nabla \phi}{-m - w_2 + \lambda} \right)
\]

\[
= \left( (m - w_1 + \lambda) \phi, \tilde{\phi} \right) + (\chi, -i\sigma \cdot \nabla \tilde{\phi}) \leq c \| \tilde{\phi} \|_{L^2} \leq c \| \tilde{\phi} \|_H,
\]

where \( c \) is a constant. In short, for \( \tilde{\phi} \) chosen to be in \( C_\infty^c(\mathbb{R}^3, \mathbb{C}^2) \), we have two expressions that are continuous in \( \tilde{\phi} \) with respect to \( H \)-norm that coincide in \( C_\infty^c(\mathbb{R}^3, \mathbb{C}^2) \). Then, by the Hahn-Banach Theorem, each one has a unique extension to a bounded linear transformation defined on \( H \). Hence, they coincide on the domain. Therefore, we get that

\[
\left( (H_V + \lambda) \left( \begin{array}{c} \phi \\ \chi \end{array} \right), \left( \begin{array}{c} \tilde{\phi} \\ \tilde{\chi} \end{array} \right) \right)
\]

equals

\[
(\phi, \tilde{\phi})_H + \left( (-m - w_2 + \lambda) \left[ \chi + \frac{-i\sigma \cdot \nabla \phi}{m - w_2 + \lambda}, \frac{-i\sigma \cdot \nabla \tilde{\phi}}{m - w_2 + \lambda} \right] \right)_{L^2}
\]

which is symmetric in \( (\phi, \chi) \) and \( (\tilde{\phi}, \tilde{\chi}) \).

The proof is completed by showing the uniqueness part of the theorem. Assume that there exists another self-adjoint extension such that for any \((\phi, \chi) \in D' \supset C_\infty^c(\mathbb{R}^3, \mathbb{C}^4)\), then \((\phi, \chi) \in H \times L^2(\mathbb{R}^3, \mathbb{C}^2)\). Since \( H_V \) is self-adjoint on \( D' \),

\[
(\phi, (m - w_1) \phi - i\sigma \cdot \nabla \chi) + (\tilde{\chi}, (m - w_2) \chi - i\sigma \cdot \nabla \phi)
\]

\[
= \left( (m - w_1) \phi - i\sigma \cdot \nabla \tilde{\chi}, \phi \right) + (\tilde{\chi}, (m - w_2) \chi - i\sigma \cdot \nabla \phi, \chi)
\]

for all \((\phi, \chi) \in C_\infty^c(\mathbb{R}^3, \mathbb{C}^4)\). This means that the expressions \((m - w_1 + \lambda) \phi - i\sigma \cdot \nabla \chi\) and \((m - w_2 + \lambda) \chi - i\sigma \cdot \nabla \phi\) belong to \( L^2(\mathbb{R}^3, \mathbb{C}^2)\) in the distributional sense. Thus, \((\phi, \chi) \in D, i.e., D' \subset D \) and \( D^* \subset (D')^* \). Now since \( H_V \) is self-adjoint in \( D \) and \( D', D = D' \).

\[\Box\]

4. Some Examples

4.1. Let \( w_1, w_2 \) such that \( 0 \leq w_1(x) \leq \frac{\nu_1}{|x|} \) and \( 0 \leq w_2(x) \leq \frac{\nu_2}{|x|} \). Since

\[
A_+\left[\frac{1}{|x|}, \frac{1}{|x|}\right] = A_-\left[\frac{1}{|x|}, \frac{1}{|x|}\right] = 1, \text{ Theorem 3.5 holds for } \nu_1 \nu_2 < 1.
\]

Observe that we gain freedom on the constants \( \nu_1, \nu_2 \) with respect to \( |x| \). While they obtain essentially self-adjointness for \( sup V(x) \leq \frac{\nu_1}{|x|}, \nu < 1, \) we have \( \nu_1, \nu_2 < 1 \). Therefore, we can take one of the constants large as long as we decrease the other one.
4.2. Let $w_1(x) = a \delta_{|x|=R}$, $a > 0$ and $0 \leq w_2(x) \leq \frac{\nu}{|x|}$. For $V_1(x) = \delta_{|x|=R}$ and $V_2(x) = \frac{1}{|x|}$ we obtain

$$
\int_{|x|=R} |\phi|^2 d\sigma(x) \leq \frac{9}{4} \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + \frac{1}{|x|} - \lambda} \, dx + (m - \lambda) \int_{\mathbb{R}^3} |\phi|^2 \, dx,
$$

where $d\sigma(x)$ is the measure in the sphere of radius $R$, and $A_+ = A_- = \frac{3}{2}$.

If $a \nu < \frac{4}{9}$, then

$$
a \int_{|x|=R} |\phi|^2 d\sigma(x) < \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + \frac{1}{|x|} - \lambda} \, dx + (m + \lambda) \int_{\mathbb{R}^3} |\phi|^2 \, dx,
$$

and therefore Theorem 3.5 holds.

**Remark 4.1.** (i) Note that the right hand side of (4.1) does not depend on $R$, so, we can take the supremum and get

$$
\sup_{R>0} \int_{|x|=R} |\phi|^2 d\sigma(x) \leq \frac{9}{4} \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + \frac{1}{|x|} - \lambda} \, dx + (m - \lambda) \int_{\mathbb{R}^3} |\phi|^2 \, dx.
$$

(ii) Observe that if $a$ tends to zero, $\nu$ can be as large as we want. This coincides with the self-adjointness result for

$$
V = \begin{pmatrix} 0 & \nu \\ 0 & \frac{1}{|x|} \end{pmatrix}
$$

which holds for $\nu \in [0, +\infty)$.

**Remark 4.2.** Let $w_1(x) = c_1 \delta_{|x|=R}$ and $0 \leq w_2(x) \leq c_2 \frac{1}{\epsilon} \eta \left( \frac{|x|-1}{\epsilon} \right)$ for $\epsilon > 0$ and $c_1, c_2 > 0$. The inequality we obtain in this case is

$$
\int_{|x|=R} |\phi|^2 d\sigma(x) \leq \max\{A_+^2, A_-^2\} \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + \frac{1}{\epsilon} \eta \left( \frac{|x|-1}{\epsilon} \right) - \lambda} \, dx
$$

$$
+ (m + \lambda) \int_{\mathbb{R}^3} |\phi|^2 \, dx.
$$

If $c_1 c_2 < \frac{1}{\max\{A_+^2, A_-^2\}}$, 

$$
\int_{|x|=R} c_1 |\phi|^2 d\sigma(x) < \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + \frac{1}{\epsilon} \eta \left( \frac{|x|-1}{\epsilon} \right) - \lambda} \, dx + (m + \lambda) \int_{\mathbb{R}^3} |\phi|^2 \, dx
$$

$$
\leq \frac{1}{m - \lambda} \int_{|x| \geq 1+\epsilon} |\sigma \cdot \nabla \phi|^2 dx
$$

$$
+ \int_{|x| \leq 1-\epsilon} \frac{|\sigma \cdot \nabla \phi|^2}{m + \frac{1}{\epsilon} - \lambda} \, dx + (m + \lambda) \int_{\mathbb{R}^3} |\phi|^2 \, dx.
$$

If $\epsilon$ tends to zero we do not recover the Dirac delta function, thus we cannot consider the case that $w_2$ is a measure.
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