Partial-twuality polynomials of delta-matroids

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Abstract

Gross, Mansour and Tucker introduced the partial-twuality polynomial of a ribbon graph. Chumutov and Vignes-Tourneret posed a problem: it would be interesting to know whether the partial duality polynomial and the related conjectures would make sense for general delta-matroids. In this paper we consider analogues of partial-twuality polynomials for delta-matroids. Various possible properties of partial-twuality polynomials of set systems are studied. We discuss the numerical implications of partial-twualities on a single element and prove that the intersection graphs can determine the partial-twuality polynomials of bouquets and normal binary delta-matroids, respectively. Finally, we give a characterization of vf-safe delta-matroids whose partial-twuality polynomials have only one term.

Keywords: Set system, delta-matroid, ribbon graph, twuality, polynomial

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1. Introduction

In [16], Wilson found that the two long-standing duality operators $\delta$ (geometric duality) and $\tau$ (Petrie duality) generate a group of six ribbon graph operators, that is, every other composition of $\delta$ and $\tau$ is equivalent to one of
the five operators $\delta$, $\tau$, $\delta \tau$, $\tau \delta$, or to the identity operator. Abrams and Ellis-Monaghan [1] called the five operators twualities. The partial (geometric) dual with respect to a subset of edges of a ribbon graph was introduced by Chmutov [7] in order to unify various connections between the Jones-Kauffman and Bollobás-Riordan polynomials. Ellis-Monaghan and Moffatt [12] generalized this partial-duality construction to the other four operators, which they called partial-twualities.

Gross, Mansour and Tucker [13, 14] introduced the partial-twuality polynomial for $\delta$, $\tau$, $\delta \tau$, $\tau \delta$, and $\delta \tau \delta$. Various basic properties of partial-twuality polynomials were studied, including interpolation and log-concavity. Recently, Chumutov and Vignes-Tourneret [8] posed the following question:

**Question 1.** [8] Ribbon graphs may be considered from the point of view of delta-matroid. In this way the concepts of partial (geometric) duality and genus can be interpreted in terms of delta-matroids [9, 10]. It would be interesting to know whether the partial-$\delta$ polynomial and the related conjectures would make sense for general delta-matroids.

In [18], we showed that the partial-$\delta$ polynomials have delta-matroid analogues. We introduced the twist polynomials of delta-matroids and discussed their basic properties for delta-matroids. Chun et al. [9] showed that the loop complementation is the delta-matroid analogue of partial Petriality. In this paper we consider analogues of other partial-twuality polynomials for delta-matroids.

This paper is organised as follows. In Section 2 we recall the definition of partial-twuality polynomials of ribbon graphs. Analogously, we introduce the partial-twuality polynomials of set systems. In Section 3, various possible properties of partial-twuality polynomials of set systems are studied. In Section 4 we discuss the numerical implications of partial-twualities on a single element and the interpolation. In Section 5, we prove that the intersection graphs can determine the partial-twuality polynomials of bouquets and normal binary delta-matroids, respectively. Here we provide an answer to the question [17]: can one derive something from bouquets that could determine the partial-twuality polynomial completely. In Section 6 we give a characterization of vf-safe delta-matroids whose partial-twuality polynomials have only one term.
2. Preliminaries

2.1. Set systems and widths

A set system is a pair $D = (E, \mathcal{F})$ of a finite set $E$ together with a collection $\mathcal{F}$ of subsets of $E$. The set $E$ is called the ground set and the elements of $\mathcal{F}$ are the feasible sets. We often use $\mathcal{F}(D)$ to denote the set of feasible sets of $D$. $D$ is proper if $\mathcal{F} \neq \emptyset$, and is normal (respectively, dual normal) if the empty set (respectively, the ground set) is feasible. The direct sum of two set systems $D = (E, \mathcal{F})$ and $\tilde{D} = (\tilde{E}, \tilde{\mathcal{F}})$ with disjoint ground sets $E$ and $\tilde{E}$, written $D \oplus \tilde{D}$, is defined to be

$$D \oplus \tilde{D} := (E \cup \tilde{E}, \{F \cup \tilde{F} : F \in \mathcal{F} \text{ and } \tilde{F} \in \tilde{\mathcal{F}}\}).$$

As introduced by Bouchet in [3], a delta-matroid is a proper set system $D = (E, \mathcal{F})$ such that if $X, Y \in \mathcal{F}$ and $u \in X \Delta Y$, then there is $v \in X \Delta Y$ (possibly $v = u$) such that $X \Delta \{u, v\} \in \mathcal{F}$. Here

$$X \Delta Y := (X \cup Y) - (X \cap Y)$$

is the usual symmetric difference of sets. Note that the maximum gap in the collection of sizes of feasible sets of a delta-matroid is two [15].

For a set system $D = (E, \mathcal{F})$, let $\mathcal{F}_{\max}(D)$ and $\mathcal{F}_{\min}(D)$ be the collections of maximum and minimum cardinality feasible sets of $D$, respectively. Let $D_{\max} := (E, \mathcal{F}_{\max}(D))$ and $D_{\min} := (E, \mathcal{F}_{\min}(D))$. Let $r(D_{\max})$ and $r(D_{\min})$ denote the sizes of largest and smallest feasible sets of $D$, respectively. The width of $D$, denote by $w(D)$, is defined by

$$w(D) := r(D_{\max}) - r(D_{\min}).$$

For all non-negative integers $i \leq w(D)$, let

$$\mathcal{F}_{\max-i}(D) = \{F \in \mathcal{F} : |F| = r(D_{\max}) - i\}$$

and

$$\mathcal{F}_{\min+i}(D) = \{F \in \mathcal{F} : |F| = r(D_{\min}) + i\}.$$

2.2. Partial-twualities of set systems

We will consider the operations of twisting and loop complementation on set systems. Twisting was introduced by Bouchet in [3], and loop complementation by Brijder and Hoogeboom in [5].
Let $D = (E, \mathcal{F})$ be a set system. For $A \subseteq E$, the twist of $D$ with respect to $A$, denoted by $D^*|A$, is given by
\[
(E, \{A \Delta X : X \in \mathcal{F}\}).
\]
The $*$-dual of $D$, written $D^*$, is equal to $D^*|E$. Note that $*$-duality preserves width. Throughout the paper, we will often omit the set brackets in the case of a single element set. For example, we write $D^*|e$ instead of $D^*|\{e\}$.

Let $D = (E, \mathcal{F})$ be a set system and $e \in E$. Then $D^x|e$ is defined to be the set system $(E, \mathcal{F}')$, where
\[
\mathcal{F}' = \mathcal{F} \Delta \{F \cup e : F \in \mathcal{F} \text{ and } e \notin F\}.
\]
If $e_1, e_2 \in E$ then
\[
(D^x|e_1)^x|e_2 = (D^x|e_2)^x|e_1.
\]
This means that if $A = \{e_1, \cdots, e_m\} \subseteq E$ we can unambiguously define the loop complementation [5] of $D$ on $A$, by
\[
D^x|A := (\cdots (D^x|e_1)^x|e_2 \cdots )^x|e_m.
\]

It is straightforward to show that the twist of a delta-matroid is a delta-matroid [3], but the set of delta-matroids is not closed under loop complementation (see, for example, [9]). Thus, we often restrict our attention to a class of delta-matroids that is closed under loop complementation. A delta-matroid $D = (E, \mathcal{F})$ is said to be $vf$-safe [9] if the application of every sequence of twists and loop complementations results in a delta-matroid.

In [5] it was shown that twists and loop complementations give rise to an action of the symmetric group $S_3$, with the presentation
\[
S_3 \cong \mathcal{B} := < *, \times | *^2, \times^2, (\ast \times)^3 >,
\]
on set systems. If $D = (E, \mathcal{F})$ is a set system, $e \in E$ and $a = a_1 a_2 \cdots a_n$ is a word in the alphabet $\{\ast, \times\}$, then
\[
D^{a|e} := (\cdots (D^{a_1|e})^{a_2|e} \cdots )^{a_n|e}.
\]
Note that the operators $*$ and $\times$ on different elements commute [5]. If $A = \{e_1, \cdots, e_m\} \subseteq E$, we can unambiguously define
\[
D^{a|A} := (\cdots (D^{a|e_1})^{a|e_2} \cdots )^{a|e_m}.
\]
Let $D_1 = (E, \mathcal{F})$ and $D_2$ be set systems. For $\bullet \in \{\ast, \times, \ast \times, \ast \ast, \ast \ast \times\}$, we say that $D_2$ is a partial-$\bullet$ dual of $D_1$ if there exists $A \subseteq E$ such that $D_2 = D_1^{\bullet|A}$. 

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2.3. Partial-twualities of ribbon graphs

Ribbon graphs are well-known to be equivalent to cellularly embedded graphs. The reader is referred to [11, 12] for further details about ribbon graphs. A quasi-tree is a ribbon graph with one boundary component. Let $G = (V, E)$ be a ribbon graph and let

$$\mathcal{F} := \{ F \subseteq E(G) : F \text{ is the edge set of a spanning quasi-tree of } G \}.$$  

We call $D(G) =: (E, \mathcal{F})$ the delta-matroid [10] of $G$. We say a delta-matroid is ribbon-graphic if it is equal to the delta-matroid of some ribbon graph. Note that ribbon-graphic delta-matroids are vf-safe [9].

For a ribbon graph $G$ and a subset $A$ of its edge-ribbons $E(G)$, the partial dual $G^{\delta|A}$ [7] of $G$ with respect to $A$ is a ribbon graph obtained from $G$ by gluing a disc to $G$ along each boundary component of the spanning ribbon subgraph $(V(G), A)$ (such discs will be the vertex-discs of $G^{\delta|A}$), removing the interiors of all vertex-discs of $G$ and keeping the edge-ribbons unchanged.

Let $G$ be a ribbon graph and $A \subseteq E(G)$. Then the partial Petrial $G^{\tau|A}$ [11] of $G$ with respect to $A$ is the ribbon graph obtained from $G$ by adding a half-twist to each of the edges in $A$.

In [11] it was shown that the partial dual, $\delta$, and the partial Petrial, $\tau$, give rise to an action of the symmetric group $S_3$, with the presentation

$$S_3 \cong \mathcal{R} := < \delta, \tau \mid \delta^2, \tau^2, (\delta\tau)^3 >,$$

on ribbon graphs. If $G$ is a ribbon graph, $e \in E(G)$ and $a = a_1a_2\cdots a_n$ is a word in the alphabet $\{\delta, \tau\}$, then

$$G^a|e := (\cdots (G^{a_1|e})^{a_2|e} \cdots )^{a_n|e}.$$

Observe that the partial dual and the partial Petrial commute when applied to different edges [11]. If $A = \{e_1, \cdots, e_m\} \subseteq E$, we define

$$G^a|A := (\cdots (G^{a_1|e_1})^{a_2|e_2} \cdots )^{a|e_m}.$$

Let $G_1$ and $G_2$ be ribbon graphs. For $\bullet \in \{\delta, \tau, \delta\tau, \tau\delta, \delta\tau\delta\}$, we say that $G_2$ is a partial-$\bullet$ dual [11] of $G_1$ if there exists $A \subseteq E(G_1)$ such that $G_2 = G_1^{\bullet|A}$. 

2.4. Partial-twuality polynomials of ribbon graphs and set systems

Gross, Mansour and Tucker [14] introduced the concept of partial-twuality polynomials of ribbon graphs as follows.

**Definition 2 ([14])**. For \( \bullet \in \mathcal{R} \), we define the partial-\( \bullet \)-polynomial for any ribbon graph \( G \) to be the generating function

\[
\partial_{G}(z) := \sum_{A \subseteq E(G)} z^{e(G|A)}
\]

that enumerates all partial-\( \bullet \)-duals of \( G \) by Euler genus.

Analogously, we define the partial-twuality polynomials of set systems as follows.

**Definition 3.** For \( \bullet \in \mathcal{B} \), the partial-\( \bullet \)-polynomial of any set system \( D = (E, \mathcal{F}) \) is defined to be the generating function

\[
\partial_{D}(z) := \sum_{A \subseteq E} z^{w(D|A)}
\]

that enumerates all partial-\( \bullet \)-duals of \( D \) by width.

2.5. Binary and intersection graphs

For a finite set \( E \), let \( C \) be a symmetric \( |E| \) by \( |E| \) matrix over \( GF(2) \), with rows and columns indexed, in the same order, by the elements of \( E \). Let \( C[A] \) be the principal submatrix of \( C \) induced by the set \( A \subseteq E \). We define the set system \( D(C) = (E, \mathcal{F}) \) with

\[
\mathcal{F} := \{ A \subseteq E : C[A] \text{ is non-singular} \}.
\]

By convention \( C[\emptyset] \) is non-singular. Then \( D(C) \) is a delta-matroid [4]. A delta-matroid is said to be binary if it has a twist that is isomorphic to \( D(C) \) for some symmetric matrix \( C \) over \( GF(2) \).

Let \( D = (E, \mathcal{F}) \) be a normal binary delta-matroid. Then there exists a unique symmetric \( |E| \) by \( |E| \) matrix \( C \) over \( GF(2) \) such that \( D = D(C) \) [15, 18]. The intersection graph \( G_D \) of \( D \) is the graph with vertex set \( E \) and in which two vertices \( u \) and \( v \) of \( G_D \) are adjacent if and only if \( C_{u,v} = 1 \). A bouquet is a ribbon graph with only one vertex. If \( B \) is a bouquet, then \( D(B) \) is a normal binary delta-matroid [10]. The intersection graph \( I(B) \) of a bouquet \( B \) is the graph \( G_{D(B)} \).
Conversely, recall that a \textit{looped simple graph} \cite{15} is a graph obtained from a simple graph by adding (exactly) one loop to some of its vertices. The adjacency matrix $A(G)$ of a looped simple graph $G$ is the matrix over $GF(2)$ whose rows and columns correspond to the vertices of $G$; and where, $A(G)_{u,v} = 1$ if and only if $u$ and $v$ are adjacent in $G$ and $A(G)_{u,u} = 1$ if and only if there is a loop at $u$. Let $D$ be a normal binary delta-matroid. It obvious that $D = D(A(G_D))$.

2.6. \textit{Primal and dual types}

Let $D = (E, \mathcal{F})$ be a proper set system. An element $e \in E$ contained in no feasible set of $D$ is said to be a \textit{loop}.

\textbf{Definition 4 (\cite{10})}. Let $D = (E, \mathcal{F})$ be a set system and $e \in E$. Then

(1) $e$ is a \textit{ribbon loop} if $e$ is a loop in $D_{\text{min}}$;

(2) A ribbon loop $e$ is \textit{non-orientable} if $e$ is a ribbon loop in $D^*|\setminus e$ and is \textit{orientable} otherwise.

Let $D = (E, \mathcal{F})$ be a set system and $e \in E$. The \textit{primal type} of $e$ is $p, u$, or $t$ in $D$, if $e$ is a non-ribbon loop, an orientable loop, or a non-orientable loop, respectively, in $D$. The primal type of $e$ in $D^*$ is called the \textit{dual type} of $e$ in $D$. In combination, the primal and dual types of $e$ in $D$ are called the \textit{type} of $e$ in $D$, which is denoted by a juxtaposed pair of letters representing the primal and dual types of $e$ in $D$. For example, the type $pu$ means that the primal and dual types of $e$ are $p$ and $u$, respectively, in $D$. We observe that

(1) The primal type of $e$ is $p$ in $D$ if and only if there exists $A \in \mathcal{F}_{\text{min}}(D)$ such that $e \in A$;

(2) The dual type of $e$ is $p$ in $D$ if and only if there exists $A \in \mathcal{F}_{\text{max}}(D)$ such that $e \notin A$;

(3) The primal type of $e$ is $u$ in $D$ if and only if for any $A \in \mathcal{F}_{\text{min}}(D) \cup \mathcal{F}_{\text{min}+1}(D)$, $e \notin A$;

(4) The dual type of $e$ is $u$ in $D$ if and only if for any $A \in \mathcal{F}_{\text{max}}(D) \cup \mathcal{F}_{\text{max}-1}(D)$, $e \in A$;
The primal type of $e$ is $t$ in $D$ if and only if for any $A \in F_{\text{min}}(D)$, $e \not\in A$, and there exists $B \in F_{\text{min}+1}(D)$ such that $e \not\in B$.

The dual type of $e$ is $t$ in $D$ if and only if for any $A \in F_{\text{max}}(D)$, $e \in A$, and there exists $B \in F_{\text{max}^{-1}}(D)$ such that $e \not\in B$.

3. Some properties of partial-twuality polynomials

Various possible properties of partial-twuality polynomials of ribbon graphs were studied by Gross, Mansour and Tucker in [13, 14]. In this section we discuss the analogous results on set systems or delta-matroids.

**Proposition 5.** Let $D = (E, F)$ and $\tilde{D} = (\tilde{E}, \tilde{F})$ be set systems. Then for any $\bullet \in B$,

1. $\partial w^\bullet_D(1) = 2^{|E|}$;
2. $\partial w^\bullet_D(z)$ has degree at most $|E|$;
3. $\partial w^\bullet_{D \oplus \tilde{D}}(z) = \partial w^\bullet_D(z) \partial w^\bullet_{\tilde{D}}(z)$.

**Proof.** For (1), the value $\partial w^\bullet_D(1)$ counts the total number of partial-$\bullet$ duals of $D$, which is $2^{|E|}$. For any subset $A \subseteq E$, if $B \in F(D^*|A)$, then $\emptyset \subseteq B \subseteq E$. We have $r(D^*|A_{\text{min}}) \geq 0$ and $r(D^*|A_{\text{max}}) \leq |E|$. Thus $0 \leq w(D^*|A) \leq |E|$ and (2) then follows. For any subset $C \subseteq E \cup \tilde{E}$, we have

$$(D \oplus \tilde{D})^*|C = D^*|{(C \cap E)} \oplus \tilde{D}^*|{(C \cap \tilde{E})}.$$ 

Then

$$\partial w^\bullet_{D \oplus \tilde{D}}(z) = \partial w^\bullet_D(z) \partial w^\bullet_{\tilde{D}}(z),$$

by the additivity of width over the direct sum, from which (3) follows. \qed

**Proposition 6.** Let $D = (E, F)$ be a set system and $A \subseteq E$. Then

$$\partial w^\bullet_D(z) = \partial w^\bullet_{D^*|A}(z)$$

for $\bullet \in \{*, \times, \times \times \}$.

**Proof.** This is because the set of all loop complementations of $D$ is the same as that of $D^*|A$. The same reasoning applies to the operators $*$ and $\times \times \times$. \qed
Remark 7. Proposition 6 is not true for the operators \( \times \times \) and \( \times \ast \). For example, let \( D = (E, \mathcal{F}) \) with \( E = \{1\} \) and \( \mathcal{F} = \{\emptyset, \{1\}\} \). Then \( D^{\times \times |1} = (\{1\}, \{\emptyset\}) \) and \( D^{\times \ast |1} = (\{1\}, \{\{1\}\}) \). We have

\[
\frac{\partial}{\partial w} w^{\times \times}_D(z) = \frac{\partial}{\partial w} w^{\times \ast}_D(z) = 1 + z
\]

and

\[
\frac{\partial}{\partial w} w^{\times \times}_{D^\ast \times|1}(z) = \frac{\partial}{\partial w} w^{\times \ast}_{D^\ast \times|1}(z) = 2.
\]

Obviously, \( \frac{\partial}{\partial w} w^{\times \times}_D(z) \neq \frac{\partial}{\partial w} w^{\times \times}_{D^\ast \times|1}(z) \) and \( \frac{\partial}{\partial w} w^{\times \ast}_D(z) \neq \frac{\partial}{\partial w} w^{\times \ast}_{D^\ast \times|1}(z) \).

Lemma 8 ([5]). Let \( D = (E, \mathcal{F}) \) be a set system and \( A \subseteq E \). Then

\[ F_{\text{min}}(D) = F_{\text{min}}(D^\times |A) \]

and

\[ F_{\text{max}}(D) = F_{\text{max}}(D^{\times \times |A}) = F_{\text{max}}(D^{\times \ast |A}). \]

Proposition 9. Let \( D = (E, \mathcal{F}) \) be a set system and \( A \in F_{\text{min}}(D), B \in F_{\text{min}}(D^\ast) \). Then

(1) \( D^{\bullet |A} \) is normal for \( \bullet \in \{\ast, \times, \ast \times, \ast \times \ast\} \);

(2) \( D^\times |B \) is dual normal.

Proof. (1) We may assume that \( A \neq \emptyset \), otherwise the conclusion is trivial. For any \( e \in A \), since \( A \in F_{\text{min}}(D) \), it follows that \( A \in F_{\text{min}}(D^\times |e) \) by Lemma 8 and \( A - e \in F_{\text{min}}(D^\times |e) \). Thus \( A - e \in F_{\text{min}}(D^{\times \times |e}) \). Then \( A - e \in F_{\text{min}}(D^{\times \ast |e}) \) by Lemma 8. From the above, we have \( A - e \in F_{\text{min}}(D^{\bullet |e}) \) for \( \bullet \in \{\ast, \times, \ast \times, \ast \times \ast\} \). In the same manner we can see that \( \emptyset \in F_{\text{min}}(D^{\bullet |A}) \) for \( \bullet \in \{\ast, \times, \ast \times, \ast \times \ast\} \) and conclusion (1) then follows.

(2) Since \( B \in F_{\text{min}}(D^\ast) \), it follows that \( E - B \in F_{\text{max}}(D) \). Then \( E - B \in F_{\text{max}}(D^{\times \times |B}) \) by Lemma 8, that is, \( E - B \in F(D^{\times \times |B}) \). Thus \( E \in F(D^{\times \times |B}) \). Obviously,

\[ E \in F_{\text{max}}(D^{\times \times \times |B}) = F_{\text{max}}((D^{\times |B})^{\times \times |B}). \]

Then \( E \in F_{\text{max}}(D^{\times |B}) \) by Lemma 8, that is, \( E \in F(D^{\times |B}) \). Thus \( D^\times |B \) is dual normal.
Remark 10. For investigation of partial-\(\bullet\) polynomials of set systems for \(\bullet \in \{\ast, \ast \times \ast\}\), Propositions 6 and 9 motivate us to focus on normal set systems, and for \(\ast \times \) or \(\times \ast\), we cannot just focus on normal set systems. For example, let \(D = ((1), \{\{1\}\})\). We have \(\partial w^\ast_{\ast}(z) = 2\). Observe that all normal set systems with ground set \(\{1\}\) are \(D_1 = ((1), \emptyset)\) and \(D_2 = ((1), \emptyset, \{1\})\). Since \(\partial w^\ast_{\ast}(z) = \partial w^\ast_{\ast}(z) = 1 + z\), it follows that there is no normal set system \(D'\) such that \(\partial w^\ast_{\ast}(z) = \partial w^\ast_{\ast}(z)\).

The following theorem provides a link between partial-\(\ast \bullet \ast\) and partial-\(\bullet\) polynomials of set systems.

**Theorem 11.** Let \(D = (E, F)\) be a set system. Then for any \(\bullet \in B\),

\[ \partial w^\bullet_{\ast \ast}(z) = \partial w^\bullet_{\ast}(z). \]

**Proof.** For any \(A \subseteq E\), we observe that doing partial-\(\ast \bullet \ast\) on \(A\) is the same as first doing \(\ast\) to \(E\), then doing \(\bullet\) to \(A\), and then doing \(\ast\) to \(E\) again, that is,

\[ D^\bullet_{\ast \ast}|^A = ((D^\ast)^{\bullet}|^A)^{\ast}. \]

Since \(\ast\)-duality preserves width, it follows that

\[ w(D^\bullet_{\ast \ast}|^A) = w((D^\ast)^{\bullet}|^A)^{\ast}) = w((D^\ast)^{\bullet}|^A). \]

Thus the partial-\(\ast \bullet \ast\) polynomial of \(D\) is identical to the partial-\(\bullet\) polynomial of \(D^\ast\).

**4. Partial-twuality for a single element**

In this section, we discuss the numerical implications of partial-twualities on a single element \(e\), depending on the type of \(e\).

**Lemma 12 ([6]).** Let \(D = (E, F)\) be a delta-matroid and \(e \in E\) such that \(r(D_{\text{min}}) = r(D^\ast_{\text{min}})^{\ast | e})\). Then \(F_{\text{min}}(D) = F_{\text{min}}(D^\ast_{| e}).\)

**Remark 13.** Lemma 12 is not true for set systems. For example, let

\[ D = ((1, 2, 3), \{\{1\}, \{2, 3\}\}). \]

We know \(r(D_{\text{min}}) = r(D^\ast_{\text{min}}|2) = 1\). But

\[ F_{\text{min}}(D) = \{\{1\}\} \]

and

\[ F_{\text{min}}(D^\ast_{\text{min}}) = \{\{3\}\} . \]
Lemma 14. Let $D = (E, \mathcal{F})$ be a delta-matroid and $e \in E$. If $e$ is a non-orientable loop, then for any $A \in \mathcal{F}_{\text{min}}(D)$, $A \cup e \in \mathcal{F}(D)$.

Proof. Since the primal type of $e$ is $t$ in $D$, it follows that $e \notin A$ and there exists $B \in \mathcal{F}_{\text{min}+1}(D)$ such that $e \in B$. Then $B - e \in \mathcal{F}_{\text{min}}(D|e)$. We have $r(D_{\text{min}}) = r(D_{\text{min}}|e)$ and hence $\mathcal{F}_{\text{min}}(D) = \mathcal{F}_{\text{min}}(D|e)$ by Lemma 12. Then $A \in \mathcal{F}_{\text{min}}(D|e)$, that is, $A \in \mathcal{F}(D|e)$. Thus, $A \cup e \in \mathcal{F}(D)$. \hfill $\Box$

Lemma 15 ([2]). If $X$ is any feasible set in a delta-matroid $D$, then there exist $A \in \mathcal{F}_{\text{min}}(D)$ and $B \in \mathcal{F}_{\text{max}}(D)$ such that $A \subseteq X \subseteq B$.

Theorem 16. Let $D = (E, \mathcal{F})$ be a vf-safe delta-matroid and $e \in E$. Table 1 gives the value of $w(D_{\text{min}}|e) - w(D)$ for any $\bullet \in \mathcal{B}$.

Proof. The three possible primal types (and dual types) of $e$ in $D$ are as follows:

Case 1. If the primal type of $e$ is $p$ in $D$, there exists $A \in \mathcal{F}_{\text{min}}(D)$ such that $e \in A$. Then $A - e \in \mathcal{F}_{\text{min}}(D|e)$. Thus

$$r(D_{\text{min}}|e) = r(D_{\text{min}}) - 1$$

and the primal types of $e$ are $u$ and $p$ in $D_{\text{min}}|e$ and $D_{\text{min}}|e$, respectively.

Case 2. If the primal type of $e$ is $u$ in $D$, then for any $A \in \mathcal{F}_{\text{min}}(D) \cup \mathcal{F}_{\text{min}+1}(D)$, $e \notin A$. Thus

$$r(D_{\text{min}}|e) = r(D_{\text{min}}) + 1$$

and the types of $e$ are $p$ and $t$ in $D_{\text{min}}|e$ and $D_{\text{min}}|e$, respectively.
Table 2: A summary of Cases 1, 2 and 3.

| Primal type of $e$ | $D$ | $D_e^\times$ | $D_e^*$ | $r(D_e^\times_{\text{min}})$ | $r(D_e^*_{\text{min}})$ |
|--------------------|-----|--------------|--------|-----------------------------|-----------------------------|
| $p$                | $u$ | $p$          |        | $r(D_{\text{min}})$ - 1     |                             |
| $u$                | $p$ | $t$          |        | $r(D_{\text{min}}) + 1$     |                             |
| $t$                | $t$ | $u$          |        |                             | $r(D_{\text{min}})$        |

**Case 3.** If the primal type of $e$ is $t$ in $D$, then for any $A \in \mathcal{F}_{\text{min}}(D)$, $e \notin A$, and there exists $B \in \mathcal{F}_{\text{min}+1}(D)$ such that $e \in B$. Thus

$$r(D_e^\times_{\text{min}}) = r(D_{\text{min}})$$

and the primal types of $e$ is $t$ in $D_e^\times$. By Lemma 14, we have $A \cup e \in \mathcal{F}_{\text{min}+1}(D)$ for any $A \in \mathcal{F}_{\text{min}}(D)$. Then $A \cup e \notin \mathcal{F}(D_e^\times)$. Furthermore, we know that for any $B \in \mathcal{F}_{\text{min}+1}(D)$ containing $e$, $B - e \in \mathcal{F}_{\text{min}}(D)$, otherwise there is no $A' \in \mathcal{F}_{\text{min}}(D)$ such that $A' \subseteq B$, contradicting Lemma 15. Since $\mathcal{F}_{\text{min}}(D_e^\times) = \mathcal{F}_{\text{min}}(D)$, it follows that there is no $B' \in \mathcal{F}_{\text{min}}(D_e^\times) \cup \mathcal{F}_{\text{min}+1}(D_e^\times)$ such that $e \in B'$. Then the primal type of $e$ is $u$ in $D_e^\times$.

Here, we give a summary of Cases 1, 2 and 3 as shown in Table 2.

**Case 4.** If the dual type of $e$ is $p$ in $D$, there exists $A \in \mathcal{F}_{\text{max}}(D)$ such that $e \notin A$. Then $A \cup e \in \mathcal{F}_{\text{max}}(D_e^\times) \cap \mathcal{F}_{\text{max}}(D_e^\times)$. Thus

$$r(D_e^\times_{\text{max}}) = r(D_e^\times_{\text{max}}) = r(D_{\text{max}}) + 1$$

and the dual types of $e$ are $u$ and $t$ in $D_e^\times$ and $D_e^\times$, respectively.

**Case 5.** If the dual type of $e$ is $u$ in $D$, then for any $A \in \mathcal{F}_{\text{max}}(D) \cup \mathcal{F}_{\text{max}-1}(D)$, $e \in A$. Thus

$$r(D_e^\times_{\text{max}}) = r(D_{\text{max}}) - 1$$

and

$$r(D_e^\times_{\text{max}}) = r(D_{\text{max}})$$

and the dual types of $e$ are $p$ and $u$ in $D_e^\times$ and $D_e^\times$, respectively.
Table 3: A summary of Cases 4, 5 and 6

| Dual type of $e$ | $D$ | $D^{|e}$ | $D^{x|e}$ | $r(D^{|e}_{\max})$ | $r(D^{x|e}_{\max})$ |
|------------------|-----|---------|---------|-------------------|-------------------|
| $p$              | $u$ | $t$     | $r(D_{\max}) + 1$ | $r(D_{\max}) + 1$ |
| $u$              | $p$ | $u$     | $r(D_{\max}) - 1$ | $r(D_{\max})$     |
| $t$              | $t$ | $p$     | $r(D_{\max})$    | $r(D_{\max}) - 1$ |

Table 4: The the widths of $D^{|e}$ and $D^{x|e}$

| Type of $e$ | $r(D^{|e}_{\min})$ | $r(D^{|e}_{\max})$ | $r(D^{x|e}_{\max})$ | $w(D^{|e})$ | $w(D^{x|e})$ |
|-------------|--------------------|--------------------|----------------------|-------------|-------------|
| $pp$        | $r(D_{\min}) - 1$  | $r(D_{\max}) + 1$  | $r(D_{\max}) + 1$   | $w(D) + 2$  | $w(D) + 1$  |
| $uu$        | $r(D_{\min}) + 1$  | $r(D_{\max}) - 1$  | $r(D_{\max})$       | $w(D) - 2$  | $w(D)$      |
| $pu$        | $r(D_{\min}) - 1$  | $r(D_{\max}) - 1$  | $r(D_{\max})$       | $w(D)$      | $w(D)$      |
| $up$        | $r(D_{\min}) + 1$  | $r(D_{\max}) + 1$  | $r(D_{\max}) + 1$   | $w(D)$      | $w(D) + 1$  |
| $tp$        | $r(D_{\min})$      | $r(D_{\max}) + 1$  | $r(D_{\max}) + 1$   | $w(D) + 1$  | $w(D) + 1$  |
| $tu$        | $r(D_{\min})$      | $r(D_{\max}) - 1$  | $r(D_{\max}) - 1$   | $w(D) - 1$  | $w(D)$      |
| $pt$        | $r(D_{\min}) - 1$  | $r(D_{\max})$      | $r(D_{\max}) - 1$   | $w(D) + 1$  | $w(D) - 1$  |
| $ut$        | $r(D_{\min}) + 1$  | $r(D_{\max})$      | $r(D_{\max}) - 1$   | $w(D) - 1$  | $w(D) - 1$  |
| $tt$        | $r(D_{\min})$      | $r(D_{\max})$      | $r(D_{\max}) - 1$   | $w(D)$      | $w(D) - 1$  |

**Case 6.** If the dual type of $e$ is $t$ in $D$, then for any $A \in \mathcal{F}_{\max}(D)$, $e \in A$. Thus $E-A \in \mathcal{F}_{\min}(D^*)$ and $(E-A) \cup e \in \mathcal{F}(D^*)$ by Lemma 14. It follows that $A-e \in \mathcal{F}_{\max-1}(D)$. We have

$$r(D^{|e}_{\max}) = r(D_{\max})$$

and the dual type of $e$ is $t$ in $D^{|e}$. Moreover, we observe that for any $B \in \mathcal{F}_{\max-1}(D)$ not containing $e$, $B \cup e \in \mathcal{F}_{\max}(D)$, otherwise there is no $B' \in \mathcal{F}_{\max}(D)$ such that $B \subseteq B'$, contradicting Lemma 15. It follows that

$$r(D^{x|e}_{\max}) = r(D_{\max}) - 1$$

and the dual type of $e$ is $p$ in $D^{x|e}$, respectively.

Here, we provide a summary of Cases 4, 5 and 6 as shown in Table 3. Then the the widths of $D^{|e}$ and $D^{x|e}$ can be calculated by Tables 2 and 3 as shown in Table 4. Hence, the columns 2 and 3 of Table 1 are computed. If the type of $e$ is $pp$ in $D$, then

$$w(D^{|e}) = w(D) + 2$$
and

$$w(D^{\times\{e\}}) = w(D) + 1,$$

and the types of $e$ are $uu$ and $pt$ in $D^{\times\{e\}}$ and $D^{\times\{e\}}$, respectively. Thus

$$w(D^{\times\times\{e\}}) = w((D^{\times\{e\}})^{\times}) = w(D^{\times\{e\}}) = w(D) + 2,$$

and

$$w(D^{\times\times\{e\}}) = w((D^{\times\{e\}})^{\times\{e\}}) = w(D^{\times\{e\}}) + 1 = w(D) + 2,$$

and the type of $e$ is $tu$ in $D^{\times\times\{e\}}$. We have

$$w(D^{\times\times\times\{e\}}) = w((D^{\times\times\{e\}})^{\times\{e\}}) = w(D^{\times\times\{e\}}) + 1 = w(D) + 1.$$

The other entries in columns 4, 5 and 6 of Table 1 are computed similarly. \(\square\)

The polynomial $p(z) = \sum_{i=0}^{n} c_i z^i$ is said to have a gap of size $k$ \([14]\) at coefficient $c_i$ if $c_{i-1} c_{i+k} \neq 0$ but $c_i = c_{i+1} = \ldots = c_{i+k-1} = 0$. If the polynomial $p(z)$ is nonzero and has no gaps, we call it interpolating.

**Proposition 17.** For any vf-safe delta-matroid $D$, the following statements hold:

1. $\partial w^*_{D}(z)$ is interpolating for $\bullet = \times$ or $\times \times$;
2. $\partial w^*_{D}(z)$ has no gaps of size 2 or more for any $\bullet \in B$.

**Proof.** For any element $e$ and subset $A$ of $E$, we observe that $w(D^{\bullet \Delta\{e\}})$ and $w(D^{\bullet \{A\}})$ differ by at most one for $\bullet \in \{\times, \times \times\}$, and by at most two for $\bullet \in \{\times, \times \times, \times \} \times \{\times\}$ by Theorem 16. This yields statements (1) and (2). \(\square\)

**Remark 18.** There exists a vf-safe delta-matroid $D$ such that $\partial w^*_{D}(z)$ is not interpolating for $\bullet \in \{\times, \times \times, \times \}$. For example, let

$$D_1 = (\{1, 2\}, \emptyset, \{1, 2\})$$

and

$$D_2 = (\{1, 2\}, \emptyset, \{1\}, \{1, 2\}).$$

We have

$$\partial w^*_{D_1}(z) = 2 + 2z^2$$

and

$$\partial w^*_{D_2}(z) = \partial w^*_{D_2}(z) = 1 + 3z^2.$$
5. Partial-twuality polynomials and intersection graphs

In [17], we showed that two bouquets with the same intersection graph have the same partial-δ polynomial. In this section, we prove that the intersection graphs can determine the partial-twuality polynomials of bouquets and normal binary delta-matroids, respectively. Let \( \eta : \mathcal{R} \to \mathcal{B} \) be the group isomorphism induced by \( \eta(\delta) = * \), and \( \eta(\tau) = \times \).

**Lemma 19** ([10]). If \( G \) is a ribbon graph, \( A \subseteq E \) and \( \bullet \in \mathcal{R} \). Then
\[
D(G^{\bullet|A}) = D(G)^{\eta(\bullet)|A}
\]
and
\[
\varepsilon(G) = w(D(G)).
\]

**Proposition 20.** Let \( G = (V, E) \) be a ribbon graph and \( \bullet \in \mathcal{R} \). Then
\[
\partial w_{D(G)}^{\eta(\bullet)}(z) = \partial \varepsilon_G^\bullet(z).
\]

**Proof.** By Lemma 19, for any \( A \subseteq E \),
\[
w(D(G)^{\eta(\bullet)|A}) = w(D(G^{\bullet|A})) = \varepsilon(G^{\bullet|A}).
\]
Hence \( \partial w_{D(G)}^{\eta(\bullet)}(z) = \partial \varepsilon_G^\bullet(z) \).

**Theorem 21.** If two normal binary delta-matroids \( D \) and \( \tilde{D} \) have the same intersection graph, then \( \partial w_D^\bullet(z) = \partial w_{\tilde{D}}^\bullet(z) \) for any \( \bullet \in \mathcal{B} \).

**Proof.** Since \( G_D = G_{\tilde{D}}, \ D = D(A_G), \) and \( \tilde{D} = D(A_{\tilde{D}}), \) we have \( D = \tilde{D} \). Thus \( \partial w_D^\bullet(z) = \partial w_{\tilde{D}}^\bullet(z) \) for any \( \bullet \in \mathcal{B} \).

**Theorem 22.** Let \( B \) and \( \tilde{B} \) be two bouquets. If \( G_{D(B)} = G_{D(\tilde{B})} \), then \( \partial \varepsilon_B^\bullet(z) = \partial \varepsilon_{\tilde{B}}^\bullet(z) \) for any \( \bullet \in \mathcal{R} \).

**Proof.** Since \( G_{D(B)} = G_{D(\tilde{B})} \), it follows that \( D(B) = D(\tilde{B}) \). For any \( A \subseteq E(B) \), we denote its corresponding subset of \( E(\tilde{B}) \) by \( \tilde{A} \), then
\[
D(B^{\bullet|A}) = D(B)^{\eta(\bullet)|A} = D(\tilde{B})^{\eta(\bullet)|\tilde{A}} = D(\tilde{B}^{\bullet|\tilde{A}}),
\]
by Lemma 19. We have
\[
w(D(B^{\bullet|A})) = w(D(\tilde{B}^{\bullet|\tilde{A}})).
\]
Since \( w(D(B^{\bullet|A})) = \varepsilon(B^{\bullet|A}) \) and \( w(D(\tilde{B}^{\bullet|\tilde{A}})) = \varepsilon(\tilde{B}^{\bullet|\tilde{A}}) \), it follows that \( \varepsilon(B^{\bullet|A}) = \varepsilon(\tilde{B}^{\bullet|\tilde{A}}) \). Thus \( \partial \varepsilon_B^\bullet(z) = \partial \varepsilon_{\tilde{B}}^\bullet(z) \).
6. Partial-twuality monomials

We [18, 19] showed that a normal binary delta-matroid whose partial-* polynomials have only one term if and only if each connected component of the intersection graph of the delta-matroid is either a complete graph of odd order or a single vertex with a loop. In this section, we give a characterization of vf-safe delta-matroids whose partial-× and *×* polynomials have only one term.

Lemma 23 ([5]). Let \( D = (E, \mathcal{F}) \) be a set system and \( X, Y \subseteq E \). We have \( Y \in \mathcal{F}(D^{|X|}) \) if and only if \( |\{Z \in \mathcal{F}(D) \mid Y - X \subseteq Z \subseteq Y\}| \) is odd.

Theorem 24. Let \( D = (E, \mathcal{F}) \) be a vf-safe delta-matroid. Then

1. \( \partial w_D^x(z) = cz^m \) if and only if \( \mathcal{F}(D) = \{E\} \);

2. \( \partial w_D^* (z) = cz^m \) if and only if \( \mathcal{F}(D) = \{\emptyset\} \).

Proof. (1) Suppose that \( \partial w_D^x(z) = cz^m \). Then for any \( e \in E \), the dual type of \( e \) is \( u \) in \( D \), otherwise applying \( \times|e \) changes the width according to Theorem 16. Then for any \( A \in \mathcal{F}_{\text{max}}(D) \cup \mathcal{F}_{\text{max}-1}(D) \), we have \( e \in A \). Thus \( \mathcal{F}_{\text{max}}(D) = \{E\} \) and \( \mathcal{F}_{\text{max}-1}(D) = \emptyset \). Suppose \( \mathcal{F}_{\text{max}-2}(D) \neq \emptyset \). Let \( B \in \mathcal{F}_{\text{max}-2}(D) \) and \( f \in E - B \). Then \( B \cup f, E \in \mathcal{F}(D^{|f|}) \) by Lemma 23. Observe that \( B \cup f \in \mathcal{F}_{\text{max}-1}(D^{|f|}) \) and \( E \in \mathcal{F}_{\text{max}}(D^{|f|}) \). Let \( g \in E - (B \cup f) \). Then there exists \( B \cup f \in \mathcal{F}_{\text{max}-1}(D^{|f|}) \cup \mathcal{F}_{\text{max}}(D^{|f|}) \) such that \( g \notin B \cup f \). Thus the dual type of \( g \) is not \( u \) in \( D^{|f|} \). We have \( w(D^{|f|}) \neq w((D^{|f|})^{|g|}) \) by Theorem 16. Then \( \partial w_D^x|f(z) \neq cz^m \). Note that \( \partial w_D^x|f(z) = \partial w_D^x(z) \) by Proposition 6. It follows that \( \partial w_D^x(z) \neq cz^m \), a contradiction. Then \( \mathcal{F}_{\text{max}-2}(D) = \emptyset \). Since the maximum gap in the collection of sizes of feasible sets of a delta-matroid is two, it follows that \( \mathcal{F}(D) = \{E\} \).

Conversely, for any \( X \subseteq E \),

\[
\mathcal{F}_{\text{min}}(D^{|X|}) = \mathcal{F}_{\text{min}}(D) = \{E\}
\]

by Lemma 8. Then \( \mathcal{F}(D^{|X|}) = \{E\} \). Thus \( w(D^{|X|}) = 0 \) and \( \partial w_D^x(z) = 2^{|E|} \).

(2) For *×*, by Theorem 11, \( \partial w^{*\times*}_D(z) = \partial w^{\times*}_D(z) = cz^m \) if and only if \( \mathcal{F}(D^*) = \{E\} \) if and only if \( \mathcal{F}(D) = \{\emptyset\} \).

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References

[1] L. Abrams and J. Ellis-Monaghan, New dualities from old: generating geometric, Petrie, and Wilson dualities and trialities of ribbon graphs, *Combin. Probab. Comput.* 31 (2022) 4: 574–597.

[2] J. E. Bonin, C. Chun and S. D. Noble, Delta-matroids as subsystems of sequences of Higgs lifts, *Adv. in Appl. Math.* 126 (2021) 101910.

[3] A. Bouchet, Greedy algorithm and symmetric matroids, *Math. Program.* 38 (1987) 147–159.

[4] A. Bouchet, Representability of Δ-matroids, *Colloq. Math. Soc. János Bolyai* (1987) 167–182.

[5] R. Brijder and H. Hoogeboom, The group structure of pivot and loop complementation on graphs and set systems, *European J. Combin.* 32 (2011) 1353–1367.

[6] R. Brijder and H. Hoogeboom, Nullity and loop complementation for delta-matroids, *SIAM J. Discrete Math.* 27 (2013) 492–506.

[7] S. Chmutov, Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial, *J. Combin. Theory Ser. B* 99 (2009) 617–638.

[8] S. Chmutov and F. Vignes-Tourneret, On a conjecture of Gross, Mansour and Tucker, *European J. Combin.* 97 (2021) 103368.

[9] C. Chun, I. Moffatt, S. D. Noble and R. Rueckriemen, On the interplay between embedded graphs and delta-matroids, *Proc. London Math. Soc.* 118 (2019) 3: 675–700.

[10] C. Chun, I. Moffatt, S. D. Noble and R. Rueckriemen, Matroids, delta-matroids and embedded graphs, *J. Combin. Theory Ser. A* 167 (2019) 7–59.

[11] J. A. Ellis-Monaghan and I. Moffatt, Twisted duality for embedded graphs, *Trans. Amer. Math. Soc.* 364 (2012) 1529–1569.

[12] J. A. Ellis-Monaghan and I. Moffatt, Graphs on surfaces, Springer New York, 2013.
[13] J. L. Gross, T. Mansour and T. W. Tucker, Partial duality for ribbon graphs, I: Distributions, European J. Combin. 86 (2020) 103084.

[14] J. L. Gross, T. Mansour and T. W. Tucker, Partial duality for ribbon graphs, II: Partial-twuality polynomials and monodromy computations, European J. Combin. 95 (2021) 103329.

[15] I. Moffatt, Surveys in Combinatorics, 2019: Delta-matroids for graph theorists, 2019.

[16] S. Wilson, Operators over regular maps, Pacific J. Math. 81 (1979) 559–568.

[17] Q. Yan and X. Jin, Partial-dual genus polynomials and signed intersection graphs, Forum Math. Sigma 10 (2022) e69.

[18] Q. Yan and X. Jin, Twist polynomials of delta-matroids, Adv. in Appl. Math. 139 (2022) 102363.

[19] Q. Yan and X. Jin, Twist monomials of binary delta-matroids, Preprint arXiv: 2205.03487v1 [math.CO].