Non-uniform beam vibration using Differential Transform Method

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Abstract. The paper focuses on the vibration characteristics of non-uniform Euler-Bernoulli beam using Differential Transform Method (DTM). DTM is a numerical method to solve differential equations where the governing equations are reduced into a set of polynomials. Non-uniformity is considered corresponding to linear variation in breadth and height of the beam. The effect of taper ratio on the fundamental frequency of tapered beams is also analysed. The method has proved to be accurate, simple and effective for eigenvalue analysis. For the two cases of non-uniform beam analysed, the frequency computed by the method of differential transform is found to be comparable with the previously available results.

1. Introduction

For the design applications in the field of architecture, aeronautics and robotics, non prismatic beams have a great importance due to their economic, aesthetic and weight to strength optimisation considerations. The knowledge of the vibration properties, particularly natural frequency is significant in these applications. Free vibration analysis of tapered beams was carried out by several researchers using analytic or approximate methods. Goel [1] obtained characteristic equations for linearly tapering beams with elastically restrained ends analytically. The effect of a concentrated mass at free end of a cantilever beam was also analysed. Naguleswaran [2] presented a direct analytical solution based on Frobenius method for the frequency and modeshape of a complete beam and truncated beams of constant slant depth and linearly varying breadth. Abrate [3] obtained exact solution of non-uniform rods and beams. Grossi [4] used Rayleigh-Schmidt method for the study of tapered beam with elastically restrained ends. DTM was employed to solve the free and forced vibration problems of non-uniform beams resting on a non homogeneous elastic foundation by Chen et al [5]. Vibration
characteristics of a rotating tapered beam was analysed by Özdemir [6] using DTM. Closed form series solution of free vibration was obtained by Adomian modified decomposition method (AMDM) by Chen, et al [7]. Rosa et al [8] studied the vibration of tapered beams in presence of rotationally and axially flexible supports using the cell discretisation method.

In the present paper the free vibration analysis is done for a tapered beam with wedge beam and cone beam condition using DTM. The dimensionless frequencies are obtained for different taper ratios and the effect of taperness in the frequency is discussed. The results obtained are compared with the reference values which are found to be in very good agreement.

2. Differential Transform Method

Differential Transform Method was first used by Zhou in 1986 for the solution of initial value problem in electric circuit analysis. DTM is a transformation technique to obtain approximate solutions of the ordinary and partial differential equations. The method is based on the Taylor series expansion where the governing differential equation and boundary conditions are transformed into a set of algebraic equations according to certain transformation rules and the solution of these set of equations gives the required solution. Thus DTM provides an iterative procedure to obtain higher order series in contrast with the Taylor series method where calculation of higher derivatives becomes difficult.

Consider the function \( f(x) \) which is analytic in domain \( D \), let \( x=x_0 \) represent any point within domain \( D \), then, the differential transform of \( f(x) \) is given by:

\[
F[k] = \frac{1}{k!} \left( \frac{d^k f}{dx^k} \right)_{x=x_0} \tag{1}
\]

Where \( f(x) \) is the original function and \( F[k] \) is the differential transform. The function \( f(x) \) can be defined by the inverse differential transform as,

\[
f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{d^k f}{dx^k} \right)_{x=x_0} (x-x_0)^k = \sum_{k=0}^{\infty} F[k] (x-x_0)^k \tag{2}
\]

For practical problems \( f(x) \) is represented by a finite series as

\[
f(x) = \sum_{k=0}^{N} F[k] (x-x_0)^k \tag{3}
\]

Which implies that \( \sum_{k=N+1}^{\infty} F[k] (x-x_0)^k \) is negligibly small. \( N \) is decided on the convergence of the eigenvalues. Fundamental theorems of differential transform are listed in table 1.
Table 1 Fundamental transformation theorems of DTM.

| Original function                           | Transformed function                  |
|---------------------------------------------|---------------------------------------|
| $f(x) = g(x) \pm h(x)$                      | $F[k] = G[k] \pm H[k]$                 |
| $f(x) = c \cdot g(x)$                       | $F[k] = c \cdot G[k]$                  |
| $f(x) = g(x) \cdot h(x)$                    | $F[k] = \sum_{l=0}^{k} G[l] \cdot H[k-l]$ |
| $f(x) = \frac{d^n g(x)}{dx^n}$              | $F[k] = (k+1)(k+2)\ldots(k+n)G[k+n]$  |
| $f(x) = x^n$                                | $F[k] = \delta (k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$ |

3. Non-uniform Euler-Bernoulli Beam

The equation of motion for transverse vibration of a tapered beam is given by

$$\frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 y(x, t)}{\partial x^2} \right] + \rho A(x) \frac{\partial^2 y(x, t)}{\partial t^2} = 0, \quad 0 < x < L \quad (5)$$

Where $y(x, t)$ is transverse deflection, $A(x)$ and $I(x)$ are area of cross section and moment of inertia at a position $x$ respectively, $\rho$ is the mass density and $E$ is Young’s modulus of the material. The free vibration equation of the beam can be reduced as,

$$\frac{d^2}{dx^2} \left[ EI(x) \frac{d^2 Y(x)}{dx^2} \right] - \rho A(x) \omega^2 Y(x) = 0, 0 < x < L \quad (6)$$

With the boundary conditions for a fixed-free beam stated as;

$$y = 0; \quad \frac{dy}{dx} = 0; \quad \text{at } x = 0$$

$$\frac{d^2 y}{dx^2} = 0; \quad \frac{d^3 y}{dx^3} = 0; \quad \text{at } x = L \quad (7)$$

Consider a non-uniform beam with the breadth $b(x)$ and height $h(x)$. Assuming both to be varying linearly according to the taper ratio defined by $c_b = \frac{b_1}{b_0}$ and $c_h = \frac{h_1}{h_0}$ where $b_0$, $b_1$ and $h_0$, $h_1$ are the breadth and height of the beam at the free end and fixed end respectively. Hence we get,

$$b(x) = b_0 \left[ 1 + \left( c_b - 1 \right) \frac{x}{L} \right]$$

$$h(x) = h_0 \left[ 1 + \left( c_h - 1 \right) \frac{x}{L} \right] \quad (8)$$

The area $A(x)$ and moment of inertia $I(x)$ can be expressed as,
Where

\[
A(x) = A_0 \left[ 1 - \gamma_b \frac{x}{L} \right]^{\frac{1}{3}} \left[ 1 - \gamma_h \frac{x}{L} \right]^{\frac{1}{3}}
\]

\[
I(x) = I_0 \left[ 1 - \gamma_b \frac{x}{L} \right]^{\frac{1}{3}} \left[ 1 - \gamma_h \frac{x}{L} \right]^{\frac{1}{3}}
\]

3.1 Non-dimensionalising the governing equations and boundary conditions

Introducing the dimensionless parameters,

\[
\xi = \frac{x}{L}; \quad Y(\xi) = \frac{Y(x)}{L}; \quad A(\xi) = \frac{A(x)}{A_0}; \quad I(\xi) = \frac{I(x)}{I_0}; \quad \Omega^2 = \frac{\rho A_0 \omega^2 L^4}{E_0 I_0}
\]

The equation (6) becomes

\[
\frac{d^2 I(\xi)}{d\xi^2} \left[ I(\xi) \frac{d^2 Y(\xi)}{d\xi^2} \right] - \frac{A(\xi)}{A_0} \Omega^2 Y(\xi) = 0
\]

Where \(\Omega\) is the dimensionless frequency.

The boundary conditions in equation (7) can be rewritten as

\[
Y(\xi) = 0; \quad \frac{dY(\xi)}{d\xi} = 0 \quad \text{for} \quad \xi = 0
\]

\[
\frac{d^3 Y(\xi)}{d\xi^3} = 0; \quad \frac{d^3 Y(\xi)}{d\xi^3} = 0 \quad \text{for} \quad \xi = 1
\]

4. Free Vibration Analysis of tapered beams using DTM

The dimensionless frequencies are determined for tapered beams with taper ratios corresponding to wedge beam and cone beam using Differential Transform Method. Once the governing equation and boundary conditions are transformed as per the transformation rules in DTM, a set of algebraic equations are obtained, the solution of these set of equations gives the respective frequencies.

4.1. DTM formulation for a wedge beam.

In the case of a wedge beam, the taper ratios are \(c_b = 1; \quad c_h = \alpha\) which corresponds to beam with a constant breadth and linearly varying height. Hence, \(\gamma_b = 0; \quad \text{and} \quad \gamma_h = (1 - \alpha)\). The area of cross section and the moment of inertia vary according to the relation given as;

\[
A(\xi) = \left[ 1 - \gamma_h \frac{\xi}{L} \right]
\]

\[
I(\xi) = \left[ 1 - \gamma_h \frac{\xi}{L} \right]^3
\]
\[
\sum_{r=0}^{k} I (k-r) (r+1) (r+2) (r+3) (r+4) Y(r+4) + \\
2 \sum_{r=0}^{k} I (k-r+1) (k-r+1) (r+1) (r+2) (r+3) Y(r+3) \\
+ \sum_{r=0}^{k} (k-r+1)(k-r+2) I(k-r+2) (r+1)(r+2)Y(r+2) - \sum_{r=0}^{k} \Omega^2 A(k-r) Y(r) = 0
\] (14)

Where
\[
A(k) = \delta(k) - \gamma_h \delta(k-1) \quad \text{and} \quad I(k) = \delta(k) - 3\gamma_h \delta(k-1) + 3\gamma_h^2 \delta(k-2) - \gamma_h^3 \delta(k-3)
\]

Hence the DTM equation (14) becomes
\[
\sum_{r=0}^{k} \left[ \delta(k-r) - 3\gamma_h \delta(k-1-r) + 3\gamma_h^2 \delta(k-r-2) - \gamma_h^3 \delta(k-r-3) \right] Y(r+4) \\
2 \sum_{r=0}^{k} \left[ (k-r+1) \delta(k-r+1) - 3\gamma_h \delta(k-r) + 3\gamma_h^2 \delta(k-r-1) - \gamma_h^3 \delta(k-r-2) \right] Y(r+3) \\
+ \sum_{r=0}^{k} \left[ (k-r+1)(k-r+2) \delta(k-r+2) - 3\gamma_h \delta(k-r+1) + 3\gamma_h^2 \delta(k-r) - \gamma_h^3 \delta(k-r-1) \right] Y(r+2) \\
= \sum_{r=0}^{k} \Omega^2 \left[ \delta(k-r) - \gamma_h \delta(k-r-1) \right] Y(r)
\] (15)

Subjected to the boundary conditions in equation (12) transformed to DTM as
\[
Y[0] = 0 \\
Y[1] = 0 \\
\sum_{k=0}^{n} k(k-1) Y(k) = 0 \\
\sum_{k=0}^{n} k(k-1)(k-2) Y(k) = 0
\] (16)

4.2. DTM formulation for a cone beam

In the case of a cone beam the tapering is considered such that the breadth and the height of the beam vary linearly across the span. Therefore, the ratios taken are of the form \( c_h = c_h \); hence we take \( \gamma_h = \gamma_h = \gamma \)

The variation of area and moment of inertia considered are
\[
A(\xi) = (1-\gamma \xi)^2 \\
I(\xi) = (1-\gamma \xi)^4
\] (17)
Application of differential transformation result in the equation (14) stated as,

\[
\sum_{r=0}^{k} I(k-r)(r+1)(r+2)(r+3)(r+4)Y(r+4) + 2 \sum_{r=0}^{k} I(k-r+1)(k-r+1)(r+1)(r+2)(r+3)Y(r+3) + \sum_{r=0}^{k} (k-r+2)I(k-r+2)(r+1)(r+2)Y(r+2) - \sum_{r=0}^{k} \Omega^2 A(k-r)Y(r) = 0
\]

Where in the case of cone beam,

\[
A(k) = \delta(k) - 2\gamma\delta(k-1) + \gamma^2 \delta(k-2) \quad \text{and} \quad I(k) = \delta(k) - 4\gamma\delta(k-1) + 6\gamma^2\delta(k-2) - 4\gamma^3\delta(k-3) + \gamma^4\delta(k-4)
\]

Substituting these expressions in equation (14), the DTM equation of cone beam can be written as,

\[
\sum_{r=0}^{k} \left[ (\delta(k-r) - 4\gamma\delta(k-r-1) + 6\gamma^2\delta(k-r-2) - 4\gamma^3\delta(k-r-3) + \gamma^4\delta(k-r+4)) \right] + \sum_{r=0}^{k} \left[ (\delta(k-r+1) - 4\gamma\delta(k-r) + 6\gamma^2\delta(k-r-1) - 4\gamma^3\delta(k-r-2) + \gamma^4\delta(k-r+3)) \right] + \sum_{r=0}^{k} \left[ (\delta(k-r+2) - 4\gamma\delta(k-r+1) + 6\gamma^2\delta(k-r) - 4\gamma^3\delta(k-r-1) + \gamma^4\delta(k-r+2)) \right] - \sum_{r=0}^{k} \left[ (\delta(k-r) - 2\gamma\delta(k-r-1) + \gamma^2\delta(k-r-2)) \Omega^2 Y(r) \right]
\]

subjected to the same set of boundary conditions corresponding to cantilever beam given by the equation (16).

5. Solution Procedure

The transformed differential equations as stated in equation (15) and equation (18) are solved for the non-dimensional frequency \( \Omega \) by rearranging the set of algebraic equations into an eigenvalue problem. From the boundary conditions corresponding to \( \xi = 0 \), we have \( Y[0] = 0 \) and \( Y[1] = 0 \) and

Let \( Y[2] = D_1 \); and \( Y[3] = D_2 \) where \( D_1 \) and \( D_2 \) are constants. For \( k = 0,1,2,3, \ldots \), the subsequent values, \( Y[5], Y[6], \ldots \) etc. can be determined in terms of \( D_1, D_2, \) and \( \Omega \). Substituting all \( Y[k] \) into the second set of boundary conditions: i.e. corresponding to \( \xi = 1 \), results in two simultaneous equations in \( \Omega \) corresponding to the \( N^{th} \) term. These equations can be rearranged to the matrix form as

\[
[A][D] = [0]
\]

Where \([A] = \begin{bmatrix} a_{11}(\Omega) & a_{12}(\Omega) \\ a_{21}(\Omega) & a_{22}(\Omega) \end{bmatrix}\) and \([D] = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}\)

The system of equations in equation (19) which is linear in \( D_1 \) and \( D_2 \) will have a non-trivial solution only if determinant of \( A \) is zero. Hence we can write,
\[
\begin{vmatrix}
a_{11}(\Omega) & a_{12}(\Omega) \\
a_{21}(\Omega) & a_{22}(\Omega)
\end{vmatrix} = 0
\] (21)

The \( \Omega \) thus obtained after solving equation (21) is the non-dimensional frequency of the beam. Therefore we get \( \Omega = \Omega_j^N \) which corresponds to the frequency of \( j^{th} \) mode. The value of \( N \) is decided based on the desirable accuracy that is required. \( |\Omega_j^N - \Omega_j^{N-1}| \leq \varepsilon \) Where \( \varepsilon \) is a small value taken as 0.001 in the present analysis.

6. Numerical Results

Free vibration analysis of a tapered wedge beam and tapered cone beam with cantilever support condition is done using DTM. The dimensionless frequencies are determined for different taper ratios. The derived algebraic equations are solved in Mathematica.

6.1 Non-uniform cantilever wedge beam (\( \gamma_b = 0, \gamma_h = \gamma \))

In the case of a wedge beam with \( \gamma=0.5 \), the area and moment of inertia can be taken as;
\[
A(\xi) = A_0 (1 - 0.5\xi) \\
I(\xi) = I_0 (1 - 0.5\xi)^3
\]

According to the boundary conditions at \( \xi=0 \), we have \( Y[0] = 0, Y[1] = 0 \). Assuming \( Y[3] = D_1, \ Y[4] = D_2 \), the values of subsequent \( Y[k] \) are determined from the recursive equation (15). Hence we get \( Y[4] = -0.125D_1 + 0.75D_2 \), \( Y[5] = -0.1D_1 + 0.45D_2 \), \( Y[6] = 0.0028\Omega^2D_1 \). Substituting these values of \( Y[k] \) into the equation (16) which are the boundary conditions corresponding to \( \xi=1 \), and rearranging the terms the determinant of \( A \) can be formed. For the nontrivial solutions, the determinant is solved which gives the required non-dimensional frequencies. For a value of \( N =30 \), we get the solution of the determinant as;
\[
\Omega_1^{30} = 3.8237 \\
\Omega_2^{30} = 18.3185 \\
\Omega_3^{30} = 47.2672
\]

Hence the natural frequency of the beam can be derived as \( \omega = 3.8238\sqrt{\frac{EI_0}{\rho A_0 L^2}} \).

As the value of \( N \) increases, the convergence of higher modes occurs. Following the same procedure, the natural frequencies for other values of taper ratios are evaluated. The non-dimensional frequencies of wedge beam with \( c_h \) corresponding to 0.2, 0.4, 0.6, 0.8 and 1 are listed in table 2.
Table 2 Non-dimensional Frequency of Wedge Beam with fix-free support condition for different taper ratio $c_h$.

| $c_h$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ |
|-------|-------------|-------------|-------------|
|       | DTM         | Ref[7]      | DTM         | Ref[7]      | DTM         | Ref[7]      |
| 0.2   | 4.3027      | 4.2925      | 15.8077     | 15.7427     | 37.0784     | 36.8843     |
| 0.4   | 3.9343      | 3.9343      | 17.4879     | 17.4879     | 44.0257     | 44.0248     |
| 0.5   | 3.8237      | 3.8238      | 18.318      | 18.3172     | 47.2585     | 47.2648     |
| 0.6   | 3.7371      | 3.7371      | 19.1139     | 19.1138     | 50.3559     | 50.35366    |
| 0.8   | 3.6083      | 3.6083      | 20.6210     | 20.6210     | 56.1923     | 56.1923     |
| 1     | 3.516       | 3.516       | 22.0345     | 22.0345     | 61.6973     | 61.69644    |

6.2 Non-uniform Cantilever Cone Beam
The results of the non-dimensional frequencies of cone beam with cantilever support condition with different taper ratio $c_h$ are listed in table 3.

Table 3 Non-dimensional Frequency of Wedge Beam with fix-free support condition for different taper ratio $c_h$.

| $c_h$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ |
|-------|-------------|-------------|-------------|
|       | DTM         | Ref[7]      | DTM         | Ref[7]      | DTM         | Ref[7]      |
| 0.2   | 6.2156      | 6.1965      | 18.5125     | 18.3855     | 40.1917     | 39.8337     |
| 0.4   | 5.0091      | 5.0091      | 19.0649     | 19.0649     | 45.7394     | 45.7384     |
| 0.5   | 4.6252      | 4.6252      | 19.5477     | 19.5476     | 48.58       | 48.5789     |
| 0.6   | 4.3196      | 4.3188      | 20.0793     | 20.05       | 52.2712     | 52.2712     |
| 0.8   | 3.8551      | 3.8551      | 21.0567     | 21.0567     | 56.6303     | 56.6303     |
| 1     | 3.516       | 3.516       | 22.0345     | 22.0345     | 61.6973     | 61.69644    |
The values of dimensionless frequency for wedge beam and cone beam are found to be matching with the values presented in the reference [7]. However, the frequencies calculated by DTM for a taper ratio of 0.2 is deviating slightly from the reference values as it requires more number of terms to be included in the DTM analysis. It is found that the rate of convergence of frequencies is slow when the taper ratio decreases which is shown in figure 1. Generally when the value of $C_h$ is equal to 1, it represents the uniform Euler-Bernoulli beam and the values of frequency computed by DTM formulation agrees well with the exact values of the uniform beam.

![Convergence plot of eigenvalues (Non-Dimensional Frequency) of wedge beam for taper ratio 0.2 and 0.5 using DTM](image)

**Figure 1** The Convergence plot of eigenvalues (Non-Dimensional Frequency) of wedge beam for taper ratio 0.2 and 0.5 using DTM

### 6.3 Effect of Taper ratio on Non-dimensional Frequency

The variation of frequencies with the taper ratio is as shown in figure 2. Both the beams follow the same trend of frequency variation with the taper ratios and it can also be seen that for higher modes the frequencies increases with the taper ratio and the first mode frequency is reducing with the taperness of the beam.
7. Conclusions

The method of analysing the free vibration of non-uniform beams using DTM is presented. The computation is done for a beam with constant breadth and linearly varying height, and also for beams with breadth and height varying linearly. Even though for the cone beam, the area and moment of inertia varies by a power of 2 and 4 respectively, method proposed proves to be accurate and easy to implement. The effect of taper ratio of beams on the fundamental frequencies is studied. The results are in very good agreement with other methods in literature.

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