THE SECOND GAP ON COMPLETE SELF-SHRINKERS

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Abstract. In this paper, we study complete self-shrinkers in Euclidean space and prove that an $n$-dimensional complete self-shrinker in Euclidean space $\mathbb{R}^{n+1}$ is isometric to either $\mathbb{R}^n$, $S^n(\sqrt{n})$, or $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$, if the squared norm $S$ of the second fundamental form, $f_3$ are constant and $S$ satisfies $S < 1.83379$. We should remark that the condition of polynomial volume growth is not assumed.

1. Introduction

Let $X : M \to \mathbb{R}^{n+1}$ be a smooth $n$-dimensional immersed hypersurface in the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. One calls an immersed hypersurface $X : M \to \mathbb{R}^{n+1}$ a self-shrinker if it satisfies:

$$H = -X^\perp,$$

where $H$ denotes the mean curvature vector of $M$, $X^\perp$ denotes the orthogonal projection of $X$ onto the normal bundle of $M$.

It is well known that Huisken [12] [13] and Colding and Minicozzi [8] have proved that if $M$ is an $n$-dimensional complete embedded self-shrinker in $\mathbb{R}^{n+1}$ with $H \geq 0$ and with polynomial volume growth, then $M$ is isometric to either the hyperplane $\mathbb{R}^n$, the round sphere $S^n(\sqrt{n})$, or a cylinder $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$, $1 \leq m \leq n - 1$. For $n = 1$, see Abresch and Langer [1].

Remark 1.1. As one knows that self-shrinkers play an important role in the study of the mean curvature flow because they describe all possible blow up at a given singularity of a mean curvature flow.

On the other hand, Le and Sesum [14] and Cao and Li [2] have proved that if $M$ is an $n$-dimensional complete self-shrinker with polynomial volume growth and $S \leq 1$ in Euclidean space $\mathbb{R}^{n+1}$, then $M$ is isometric to either the hyperplane $\mathbb{R}^n$, the round sphere $S^n(\sqrt{n})$, or a cylinder $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$, $1 \leq m \leq n - 1$. Ding and Xin [9] have studied the second gap on the squared norm of the second fundamental form and they have proved that if $M$ is an $n$-dimensional complete self-shrinker with polynomial volume growth in Euclidean space $\mathbb{R}^{n+1}$, there exists a positive number $\delta = 0.022$ such that if $1 \leq S \leq 1 + 0.022$, then $S = 1$. Furthermore, Cheng and Wei [6] have proved

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Theorem 1.1. Let \( X : M \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional complete self-shrinker with polynomial volume growth in \( \mathbb{R}^{n+1} \). If the squared norm \( S \) of the second fundamental form is constant and satisfies
\[
S \leq 1 + \frac{3}{7},
\]
then \( X : M \to \mathbb{R}^{n+1} \) is isometric to one of the following:

1. \( \mathbb{R}^n \),
2. a cylinder \( S^k(\sqrt{k}) \times \mathbb{R}^{n-k} \),
3. the round sphere \( S^n(\sqrt{n}) \).

In [4], Cheng and Ogata have obtained the following results (cf. Ding and Xin [9]).

Theorem 1.2. Let \( X : M \to \mathbb{R}^3 \) be a 2-dimensional complete self-shrinker in \( \mathbb{R}^3 \). If the squared norm \( S \) of the second fundamental form is constant, then \( X : M \to \mathbb{R}^3 \) is isometric to one of the following:

1. \( \mathbb{R}^2 \),
2. a cylinder \( S^1(1) \times \mathbb{R} \),
3. the round sphere \( S^2(\sqrt{2}) \).

The following conjecture is known:

Conjecture. Let \( X : M \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional complete self-shrinker in \( \mathbb{R}^{n+1} \). If the squared norm \( S \) of the second fundamental form is constant, then \( X : M \to \mathbb{R}^{n+1} \) is isometric to one of the following:

1. \( \mathbb{R}^n \),
2. a cylinder \( S^k(\sqrt{k}) \times \mathbb{R}^{n-k} \),
3. the round sphere \( S^n(\sqrt{n}) \).

Remark 1.2. According to the result of Cheng and Ogata [4], this conjecture has been solved for \( n = 2 \). Recently, Cheng, Li and Wei [3] have solved this conjecture for \( n = 3 \) under the condition that \( f_4 \) is constant by making use of the generalized maximum principle due to Cheng and Peng [5].

For general \( n \), since this problem is too difficult, one can consider the special case:

Problem. Let \( X : M \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional complete self-shrinker in \( \mathbb{R}^{n+1} \). If the squared norm \( S \) of the second fundamental form is constant, then \( X : M \to \mathbb{R}^{n+1} \) is isometric to one of the following:

1. \( \mathbb{R}^n \),
2. a cylinder \( S^k(\sqrt{k}) \times \mathbb{R}^{n-k} \),
3. the round sphere \( S^n(\sqrt{n}) \),
4. \( S \geq 2 \).

In this paper, we prove the following:

Theorem 1.3. Let \( X : M \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional complete self-shrinker in \( \mathbb{R}^{n+1} \). If the squared norm \( S \) of the second fundamental form and \( f_3 \) are constants and \( S \) satisfies
\[
S \leq 1.83379,
\]
then $M$ is isometric to one of the following:
(1) $\mathbb{R}^n$,
(2) a cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$,
(3) the round sphere $S^n(\sqrt{n})$,
where $f_3 = \sum_{j=1}^n \lambda_j^3$ and $\lambda_j$’s are principal curvatures of $X : M \to \mathbb{R}^{n+1}$.

**Remark 1.3.** In our theorem 1.3, we do not assume that complete self-shrinkers $X : M \to \mathbb{R}^{n+1}$ have polynomial volume growth and it is known that there are many complete self-shrinkers without polynomial volume growth.

## 2. Preliminaries

In this section, we give some notations and formulas. Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional self-shrinker in $\mathbb{R}^{n+1}$. Let $\{e_1, \ldots, e_n, e_{n+1}\}$ be a local orthonormal basis along $M$ with dual coframe $\{\omega_1, \ldots, \omega_n, \omega_{n+1}\}$, such that $\{e_1, \ldots, e_n\}$ is a local orthonormal basis of $M$ and $e_{n+1}$ is normal to $M$. Then we have

$$\omega_{n+1} = 0, \quad \omega_{in+1} = \sum_{j=1}^n h_{ij} \omega_j, \quad h_{ij} = h_{ji},$$

where $h_{ij}$ denotes the component of the second fundamental form of $M$. $H = \sum_{j=1}^n h_{jj} e_{n+1}$ is the mean curvature vector field, $|H| = \sum_{j=1}^n h_{jj}$ is the mean curvature and $II = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j e_{n+1}$ is the second fundamental form of $M$. The Gauss equations and Codazzi equations are given by

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}, \quad (2.1)$$

$$h_{ijk} = h_{ikj}, \quad (2.2)$$

where $R_{ijkl}$ is the component of curvature tensor, the covariant derivative of $h_{ij}$ is defined by

$$\sum_{k=1}^n h_{ijk} \omega_k = dh_{ij} + \sum_{k=1}^n h_{kj} \omega_{ki} + \sum_{k=1}^n h_{ik} \omega_{kj}.$$  

Let $$F_i = \nabla_i F, \quad F_{ij} = \nabla_j \nabla_i F, \quad h_{ijk} = \nabla_k h_{ij}, \quad \text{and} \quad h_{ijkl} = \nabla_l \nabla_j h_{ij},$$
where $\nabla_j$ is the covariant differentiation operator, we have

$$h_{ijkl} - h_{ijlk} = \sum_{m=1}^n h_{im} R_{mjk} + \sum_{m=1}^n h_{mj} R_{mikl}.$$  

The following elliptic operator $\mathcal{L}$ is introduced by Colding and Minicozzi in [8]:

$$\mathcal{L} f = \Delta f - \langle X, \nabla f \rangle, \quad (2.4)$$

where $\Delta$ and $\nabla$ denote the Laplacian and the gradient operator on the self-shrinker, respectively and $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^{n+1}$. By a direct calculation, we have

$$\mathcal{L} h_{ij} = (1 - S) h_{ij}, \quad \mathcal{L} H = H(1 - S), \quad \mathcal{L} X_i = -X_i, \quad \mathcal{L} |X|^2 = 2(n - |X|^2). \quad (2.5)$$
(2.6) \[ \frac{1}{2} \mathcal{L}S = \sum_{i,j,k} h^2_{ijk} + S(1 - S). \]

If \( S \) is constant, then we obtain from (2.6)

(2.7) \[ \sum_{i,j,k} h^2_{ijk} = S(S - 1), \]

hence one has either

(2.8) \[ S = 0, \quad \text{or} \quad S = 1, \quad \text{or} \quad S > 1. \]

We can choose a local field of orthonormal frames on \( M^n \) such that, at the point that we consider,

\[ h_{ij} = \begin{cases} 
\lambda_i, & \text{if } i = j, \\
0, & \text{if } i \neq j,
\end{cases} \]

then

\[ S = \sum_{i,j} h^2_{ij} = \sum_i \lambda_i^2, \]

where \( \lambda_i \) is called the principal curvature of \( M \). From (2.1) and (2.3), we get

(2.9) \[ h_{ijij} - h_{jiji} = (\lambda_i - \lambda_j)\lambda_i\lambda_j. \]

By a direct calculation, we obtain

(2.10) \[ \sum_{i,j,k,l} h^2_{ijkl} = S(S - 1)(S - 2) + 3(A - 2B), \]

where \( A = \sum_{i,j,k} \lambda_i^2 h^2_{ijk}, \quad B = \sum_{i,j,k} \lambda_i \lambda_j h^2_{ijk}. \)

We define two functions \( f_3 \) and \( f_4 \) as follows:

\[ f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki} = \sum_{j=1}^n \lambda_j^3, \quad f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li} = \sum_{j=1}^n \lambda_j^4, \]

Then, the following formulas can be found in [6]:

**Lemma 2.1.** Let \( X : M \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional self-shrinker in \( \mathbb{R}^{n+1} \). Then

(2.11) \[ \frac{1}{3} \mathcal{L}f_3 = (1 - S)f_3 + 2C, \]

(2.12) \[ \frac{1}{4} \mathcal{L}f_4 = (1 - S)f_4 + (2A + B). \]

(2.13) \[ A - B \leq \frac{1}{3}(\lambda_1 - \lambda_2)^2 t S^2, \]

(2.14) \[ C^2 \leq \frac{1}{3}(A + 2B)t S^2, \]

where \( C = \sum_{i,j,k} \lambda_i^2 h^2_{ijk}. \)
3. Estimates for geometric invariants

In this section, we will give some estimates which are needed to prove our theorem. From now on, we denote

\[ S - 1 = tS, \]

where \( t \) is a positive constant and \( t < \frac{1}{2} \) if we assume that \( S \) is constant and \( S > 1 \), then

\[ (1 - t)S = 1, \quad \sum_{i,j,k} h^2_{ijk} = tS^2. \]

Defining

\[ u_{ijkl} := \frac{1}{4} (h_{ijkl} + h_{jikl} + h_{klji} + h_{iijk}), \]

we have

\[ \sum_{i,j,k,l} h^2_{ijkl} \geq \sum_{i,j,k,l} u^2_{ijkl} + \frac{3}{2} (Sf - f^2_3) \]

according to Gauss equations (2.1).

Let

\[ (3.1) \quad Sf \equiv Sf_4 - (f_3)^2 = S \sum \lambda^4_i - \left( \sum \lambda^3_i \right)^2 = \frac{1}{2} \sum (\lambda_i - \lambda_j)^2 \lambda_i^2 \lambda_j^2. \]

Since \( S \) and \( f_3 \) are constant, we have

\[ \sum \lambda_i h_{iik} = 0, \quad \sum \lambda^2_i h_{iik} = 0, \text{ for any } k \]

and

\[ \sum \lambda_i h_{ijkl} = -\sum h_{ijj} h_{ijkl}, \quad \sum \lambda^2_i h_{iikl} = -2 \sum \lambda_i h_{ijk} h_{ijl}, \text{ for any } k, l. \]

Hence, one has

\[ \sum_{i,j} h_{ijj} \lambda_i \lambda_j = -C, \quad \sum_{i,j} h_{ijj} \lambda^2_i \lambda_j = -2B. \]

Defining

\[ a_{ij} = \sum_{i,j} h_{ij}, \]

with \( y = \frac{f_3}{S} \), we have

\[ \sum_{i,j,k,l} u_{ijkl} h_{ijh_{kl}} = \sum_{i,j} \frac{1}{2} (h_{ijj} + h_{jjj}) \lambda_i \lambda_j = -C. \]

\[ \sum_{i,j,k,l} u_{ijkl} a_{ij} h_{kl} = \sum_{i,j} \frac{1}{2} (h_{ijj} + h_{jjj})(\lambda^2_i - y \lambda_i) \lambda_j = -B - \frac{1}{2} A + yC. \]

Hence, because of

\[ \sum_{i,j,k,l} \left\{ u_{ijkl} + \alpha (a_{ij} h_{kl} + h_{ij} a_{kl}) + \beta h_{ij} h_{kl} \right\}^2 \geq 0, \]
we obtain
\[
\sum_{i,j,k,l} u_{ijkl}^2 \geq -2\alpha \sum_{i,j,k,l} u_{ijkl}(a_{ij}h_{kl} + h_{ij}a_{kl}) - \alpha^2 \sum_{i,j,k,l} (a_{ij}h_{kl} + h_{ij}a_{kl})^2 \\
- 2\beta \sum_{i,j,k,l} u_{ijkl}h_{ij}h_{kl} - \beta^2 \sum_{i,j,k,l} (h_{ij}h_{kl})^2 - 2\alpha\beta \sum_{i,j,k,l} (a_{ij}h_{kl} + h_{ij}a_{kl})h_{ij}h_{kl} \\
= 2\alpha(2B + A - 2yC) - 2\alpha^2 Sf + 2\beta C - \beta^2 S^2 \\
\geq 2\alpha(2B + A - 2yC) - 2\alpha^2 Sf + \frac{C^2}{S^2}
\]
by taking \( \beta = \frac{C}{S^2} \). Since \( f_3 \) is constant, we have from Lemma 2.1
\[
tSf_3 = 2C
\]
and
\[
Sf = Sf_4 - f_3^2 = \frac{1}{2} \sum_{i,j} (\lambda_i - \lambda_j)^2 \lambda_i^2 \lambda_j^2
\]
(3.2)
\[
= \frac{1}{2} \sum_{i,j} (h_{ii} - h_{jj})(\lambda_i - \lambda_j)\lambda_i \lambda_j \\
= A - 2B,
\]
that is,
(3.3)
\[
Sf = Sf_4 - f_3^2 = A - 2B.
\]
From
(3.4)
\[
\sum_{i,j,k,l} h_{ijkl}^2 = S(S - 1)(S - 2) + 3(A - 2B)
\]
and
\[
\sum_{i,j,k,l} h_{ijkl}^2 = \sum_{i,j,k,l} u_{ijkl}^2 + \frac{3}{2}(Sf_4 - f_3^2),
\]
we have
(3.5)
\[
S(S - 1)(S - 2) = \sum_{i,j,k,l} u_{ijkl}^2 - \frac{3}{2}(A - 2B).
\]
Since \( S \) is constant, by making use of the generalized maximum principle due to Cheng and Peng \([5]\), we have that there exist a sequence \( \{p_k\} \) in \( M \) such that
(3.6)
\[
tSf_4 \geq 2A + B,
\]
it follows from (3.2) that
\[
tf_3^2 \geq (2 - t)A + (1 + 2t)B.
\]
Thus, for \( z \geq 0 \), from Lemma 2.1, one has
\[
C^2 \leq \frac{1}{3}(A + 2B)tS^2.
\]
\[
\sum_{i,j,k,l} u_{ijkl}^2 - \frac{3}{2} (A - 2B) \\
\geq 2\alpha(2B + A - 2yC) - 2\alpha^2 Sf + \frac{C^2}{S^2} - \frac{3}{2} (A - 2B) \\
= 2\alpha(2B + A) - (2\alpha^2 + \frac{3}{2})(A - 2B) + (-2\alpha t + (1 + z)\frac{t^2}{4})f_3^2 - \frac{C^2}{S^2} \\
\geq 2\alpha(2B + A) - (2\alpha^2 + \frac{3}{2})(A - 2B) \\
+ (-2\alpha + (1 + z)\frac{t}{4}) \{(2 - t)A + (1 + 2t)B\} - z\frac{t}{3}(A + 2B) \\
= \{2\alpha - 2\alpha^2 - \frac{3}{2} + (-2\alpha + (1 + z)\frac{t}{4})(2 - t) - z\frac{t}{3}\} A \\
+ \{4\alpha + 4\alpha^2 + 3 + (-2\alpha + (1 + z)\frac{t}{4})(1 + 2t) - z\frac{2t}{3}\} B,
\]
where \(-2\alpha + (1 + z)\frac{t}{4} \geq 0\). By taking
\[
tz = \frac{8\alpha^2 - 8t\alpha + 6 + t(t + 3)}{1 - t},
\]
we have
\[
4\alpha + 4\alpha^2 + 3 + (-2\alpha + (1 + z)\frac{t}{4})(1 + 2t) - z\frac{2t}{3} \\
= -2\alpha + 2\alpha^2 + \frac{3}{2} - (-2\alpha + (1 + z)\frac{t}{4})(2 - t) + z\frac{t}{3} \\
= 2(1 - t)\alpha + 2\alpha^2 + \frac{3}{2} - \frac{t(2 - t)}{4} + tz\frac{3t - 2}{12} \\
= 2(1 - t)\alpha + 2\alpha^2 + \frac{3}{2} - \frac{t(2 - t)}{4} + \frac{8\alpha^2 - 8t\alpha + 6 + t(t + 3) 3t - 2}{1 - t} \frac{3t - 2}{12} \\
= \frac{4\alpha^2 + 4(3 - 4t)\alpha + 3 + 2t(4t - 3)}{6(1 - t)}.
\]
We take \(\alpha\) such that
\[
4\alpha^2 + 4(3 - 4t)\alpha + 3 + 2t(4t - 3) = 0
\]
if \(t \leq \frac{9 - \sqrt{33}}{8}\). Thus, we have from (3.5), (3.7) and (3.9) that \(t \geq \frac{1}{2}\). It is impossible. Hence, we have
\[
t > \frac{9 - \sqrt{33}}{8}.
\]
In this case, taking \(\alpha = -\frac{3 - 4t}{2}\), we obtain
\[
2\alpha^2 + 2(3 - 4t)\alpha + \frac{3}{2} + t(4t - 3) = -\left(\frac{(3 - 4t)^2}{2}\right) + \frac{3}{2} + t(4t - 3) = -4t^2 + 9t - 3,
\]
\[ tS^2(2t - 1)S \geq \sum_{i,j,k,l} u_{ijkl}^2 - \frac{3}{2}(A - 2B) \]
\[ \geq \frac{4t^2 - 9t + 3}{3(1-t)}(A - B). \]

For any \( i, j \), we have
\[ -\lambda_i \lambda_j \leq \frac{1}{4}(\lambda_i - \lambda_j)^2. \]
Hence, we get, for \( \lambda_i \lambda_j \leq 0 \) and \( \lambda_i \lambda_k \leq 0 \),
\[ \|\lambda_i \lambda_j\| + \|\lambda_i \lambda_k\| \leq \|\lambda_i \lambda_j \| + \|\lambda_i \lambda_k\| \leq (Sf)^\frac{1}{4}, \]
and
\[ -2\lambda_i \lambda_j \leq 2(\lambda_i \lambda_j)^\frac{1}{4} \leq 2\left(\frac{1}{4}(\lambda_i - \lambda_j)^2\lambda_i^2 \lambda_j^2\right)^\frac{1}{4} \leq 2\left(\frac{1}{4}Sf\right)^\frac{1}{4}. \]

Since
\[ 3(A - B) = \sum_{i,j,k} (\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - \lambda_i \lambda_j - \lambda_i \lambda_k - \lambda_j \lambda_k)h_{ijk}^2 \]
\[ = \sum_{i,j,k \neq i} (\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - \lambda_i \lambda_j - \lambda_i \lambda_k - \lambda_j \lambda_k)h_{ijk}^2 \]
\[ + 3 \sum_{i \neq j} (\lambda_i^2 + \lambda_j^2 - 2\lambda_i \lambda_j)h_{ijj}^2, \]
we conclude from (3.12) and (3.13),
\[ 3(A - B) \leq \left( S + 2\left(\frac{1}{4}Sf\right)^\frac{1}{4}\right) \sum_{i,j,k} h_{ijk}^2 = \left( S + 2\left(\frac{Sf}{4}\right)^\frac{1}{4}\right) tS^2. \]

From Lemma 2.1 and (3.6), one gets
\[ Sf \leq \frac{A - B}{3(1-t)}. \]
Since \( S - 1 = tS \), we infer from (3.15) and (3.16)
\[ 3(A - B) \leq \left( S + 2\left(\frac{A - B}{12(1-t)}\right)^\frac{1}{3}\right) tS^2. \]
If we assume that \( 3(A - B) \leq a_k tS^3 \), then we obtain from (3.17) that
\[ 3(A - B) \leq a_{k+1} tS^3, \]
where
\[ a_{k+1} = \left( 1 + 2\left(\frac{a_k t}{36(1-t)}\right)^\frac{1}{3}\right). \]
Let $a_1 = 2$ and $t < 0.454682$, we get from the above equations
\begin{equation}
(3.19) \quad a_7 \leq 1.67738,
\end{equation}
then
\begin{equation}
(3.20) \quad 3(A - B) \leq 1.67738tS^3.
\end{equation}
From (3.11) and (3.20), we have
\begin{equation}
(3.21) \quad (2t - 1) \geq \frac{4t^2 - 9t + 3}{3(1 - t)} \times \frac{1.67738}{3}.
\end{equation}
Then, we get
\[(t - 0.454682)(t - 1.24897) \leq 0,
\]
it follows that $t \geq 0.454682$. It is a contradiction. Hence, we get $t \geq 0.454682$ and $S \geq 1.83379$. We complete the proof of Theorem 1.3.

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