RANDOM WALK’S MODELS
FOR FRACTIONAL DIFFUSION EQUATION

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Abstract. Fractional diffusion equations are used for mass spreading in inhomogeneous media. They are applied to model anomalous diffusion, where a cloud of particles spreads in a different manner than the classical diffusion equation predicts. Thus, they involve fractional derivatives. Here we present a continuous variant of Grünwald-Letnikov’s formula, which is useful to compute the flux of particles performing random walks, allowing for heavy-tailed jump distributions. In fact, we set a definition of fractional derivatives yielding the operators which enable us to retrieve the space fractional variant of Fick’s law, for enhanced diffusion in disordered media, without passing through any partial differential equation for the space and time evolution of the concentration.

1. Introduction. Fractional calculus has been attracting the attention of scientists and engineers from long time ago. It is three centuries old as the conventional calculus, but not very popular among science and/or engineering community. The beauty of this subject is that fractional derivatives (and integrals) are not a local (or point) property (or quantity). Thereby this considers the history and non-local distributed effects. In other worlds, perhaps this subject translates the reality of nature better! In the last years, it has found use in studies of viscoelastic materials, as well as in many fields of science and engineering including viscoelasticity, bubbles dynamics [3] [6], fluid flow, rheology, diffusive transport [7], electrical networks, electromagnetic theory and probability [1].

In this paper, we will focus on mapping inverting (at the left) fractional integrals. In fact, left inverses to such mappings can be given by explicit formulas [9] [11] [10] [5], such as Riemann’s and Liouville’s. Marchaud’s method is more general, combines convolution and finite differences. It coincides with Riemann-Liouville’s formulas for a broad class of functions, and also with Grünwald-

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Letnikove’s definition, which is at the basis of numerical approximations to fractional derivatives. Besides, in this paper, we will establish a new expression for the left inverse of $I^\alpha_{\pm}$, defined by the limit, when $t$ tends to $0^+$, of the function for which to $f$ we associate

$$l^{-\alpha} \int_0^{+\infty} f(x \pm y) F\left(\frac{\mp y}{t}\right)dy.$$  \hspace{1cm} (1)

This point was proved in [8], for values of $\alpha$ belonging to $\mathbb{R}_+^*$. Here, we show that it holds for $\alpha \in \mathbb{C}$ such that $Re\alpha > 0$.

So, we will deal with the case of a complex order such that its real part is positive. This continuous variant involves convolution kernels which mimic essential properties of Grünwald Letnikov’s weights, but are more general. In fact, for $\alpha \in \mathbb{R}_+^*$, the weights $w^\alpha_k$ of the discrete convolution form defined sequences, proportional to $k^{-\alpha-1}$ near infinity, and all moments of integer order $r < \alpha$ are equal to zero provided $\alpha$ is not an integer. The continuous convolution kernels satisfy the hypothesis $A^1(\alpha)$ and $A^2(\alpha)$ [9]. A first application consists in computing the flux of particles spreading according to a random walk, consisting of successive jumps, independent of each other but possibly depending on the location of the point they start from, as when boundary conditions are applied, e.g. An important physical issue is in the space fractional variant of Fick’s law for enhanced diffusion in disordered media.

2. **Left inverse of Riemann-Liouville’s integrals by explicit formulas.**

2.1. **Left inverse of $I^\alpha_{\pm}$ by $D^\alpha_{\pm}$.** For $\alpha \in \mathbb{C}$ such that $Re\alpha > 0$, the left and right-sided fractional integrals of the order of $\alpha$ of function $f$ are [9] [11]

\begin{align*}
(I^\alpha_{a+}, f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \hspace{1cm} (2)

(I^\alpha_{b-}, f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt, \hspace{1cm} (3)
\end{align*}

with $-\infty \leq a < x < b \leq +\infty$.

$I^\alpha_{\pm}$ is the Riemann-Liouville’s fractional integral of the order of $\alpha$. Riemann-Liouville’s left and right-sided derivatives of the order of $\alpha$ are

\begin{align*}
(D^\alpha_{\pm} f)(x) = \frac{(\pm 1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^{+\infty} t^{n-\alpha-1} f(x \mp t) dt, \hspace{1cm} (4)
\end{align*}

with $n = \lfloor Re\alpha \rfloor + 1$. $D^\alpha_{\pm}$ is the Riemann-Liouville’s derivative of the order of $\alpha$.

Let $\varphi \in L^1((-, b), \mathbb{R})$, with $b \in \mathbb{R}$. Then $(D^\alpha_{+} I^\alpha_{-} \varphi)(x) = \varphi(x)$, for almost every $x < b$ and $0 < Re\alpha < 1$ or $\alpha = 1$. Idem for the left derivative (Corollary 4.8 of [9]).

2.2. **Left inverse of $I^\alpha_{\pm}$ by $D^\alpha_{\pm}$.** Let $L^p_{\pm} = \{\varphi \in L^p_{loc}(\mathbb{R}) ; \varphi \in L^p(\mathbb{R}^+)\}$ with $1 \leq p < +\infty$, $\varphi \in L^p_{\pm}$. We assume that $f = I^\alpha_{\pm} \varphi$ exists. Then, we can define Marchaud’s derivatives according to

$$D^\alpha_{\pm} f = \lim_{\varepsilon \to 0} D^\alpha_{\pm, \varepsilon} f = \varphi,$$

$D^\alpha_{\pm}$ is the Marchaud’s derivative of the order of $\alpha$. $(D^\alpha_{\pm, \varepsilon} f)$ itself is defined by this map

$$x \mapsto (D^\alpha_{\pm, \varepsilon} f)(x) = \frac{1}{\Gamma(\alpha)\chi_1(\alpha)} \int_0^{+\infty} \frac{(\Delta_{\pm}^l f)(x)}{t^{1+\alpha}} dt.$$
Remark 3. For $l \in \mathbb{N}$, $\Lambda_l = \{\lambda_0, \lambda_1, \ldots, \lambda_l\}$ and $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_l$, we have
\[
d_l = \det \begin{pmatrix}
1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{l-1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
1 & \lambda_l & \lambda_l^2 & \cdots & \lambda_l^{l-1}
\end{pmatrix}.
\]
And
\[
(\Delta_t^l f)(x) = \frac{1}{d_l} \det \begin{pmatrix}
f(x - \lambda_0 t) & 1 & \lambda_0 & \cdots & \lambda_0^{l-1} \\
f(x - \lambda_1 t) & 1 & \lambda_1 & \cdots & \lambda_1^{l-1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
f(x - \lambda_l t) & 1 & \lambda_l & \cdots & \lambda_l^{l-1}
\end{pmatrix}.
\]
And
\[
\psi_l(t) = \frac{1}{d_l} \det \begin{pmatrix}
e^{-\lambda_0 t} & 1 & \lambda_0 & \cdots & \lambda_0^{l-1} \\
e^{-\lambda_1 t} & 1 & \lambda_1 & \cdots & \lambda_1^{l-1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
e^{-\lambda_l t} & 1 & \lambda_l & \cdots & \lambda_l^{l-1}
\end{pmatrix},
\]
with $t \geq 0$. So we can define $\chi_l$ according to $\chi_l(z) = \int_0^{+\infty} t^{-z-1} \psi_l(t) \, dt$, $z \in \mathbb{C}$. We note that $D^\alpha_\pm$ is a left inverse of $I^\alpha_\pm$, over the following set:
\[
\{\varphi \in (I^\alpha_\pm) \mid I^\alpha_\pm \varphi \text{ exist as a Lebesgue or as an improper one}\}.
\]

**Remark 1.** Let $N \in \mathbb{R}$. The limit above exists in $L^p[-\infty, N]$, if $\varphi \in L^p$. If $\varphi$ belongs to $L^p$ with $f = I^\alpha \varphi$, then $(D^\alpha \varphi)$ tends to $\varphi$ in $L^p[N, +\infty[$, for any real $N$. If $\varphi \in L^p$, then the limit above exists in $L^p$. Moreover, the limit exists point-wise almost everywhere.

2.3. **Gr"{u}nwald-Letnikov's formulas.** Gr"{u}nwald-Letnikov's method yields approximations to the inverse of a fractional integrals for $\alpha \in \mathbb{C}$ such that $\text{Re} \alpha > 0$. It is defined according to
\[
D^\alpha_\pm f(x) = \lim_{h \to 0^+} (\pm h)^{-\alpha} \sum_{k=0}^{+\infty} w^\alpha_k f(x \mp kh).
\]

$D^\alpha_\pm$ is the Gr"{u}nwald-Letnikov’s derivative of the order of $\alpha$. The following kernel
\[
w^\alpha_k = (-1)^k \binom{\alpha}{k} = (-1)^k \frac{\Gamma(\alpha + 1)}{\Gamma(k + 1) \Gamma(\alpha - k + 1)}
\]
behaves as $k^{-\alpha - 1}$ when $k$ is large, provided $\alpha$ is not an integer [2] and $\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} = 0$ holds for $\text{Re} \alpha > 0$, and implies
\[
\sum_{k=0}^{+\infty} (-1)^k \binom{\alpha}{k} k^r = 0
\]
when $r$ is an integer satisfying $0 \leq r < \alpha$. Gr"{u}nwald-Letnikov’s derivative of the order of $\alpha$ yields a left inverse to $I^\alpha_\pm$.

**Remark 2.** Let $1 < p < \frac{1}{\text{Re} \alpha}$, i.e., for $0 < \text{Re} \alpha < 1$ and $N \in \mathbb{R}$. So, we have $D^\alpha_\pm$ and $D^\alpha_\pm$ coincide over $I^\alpha_\pm(L^p[-\infty, N] \cap L^1[-\infty, N])$. $D^\alpha_\pm$ and $D^\alpha_\pm$ coincide over $I^\alpha_\pm(L^p[N, +\infty[ \cap L^1[N, +\infty[)$. 

**Remark 3.** For $1 \leq p < +\infty$ and $\varphi \in (L^p \cap L^1_{\text{loc}})(\mathbb{R})$, such that $f = I^\alpha_\pm \varphi$ exists and $I^{[\text{Re} \alpha]+1}_\pm \varphi$ is absolutely convergent, we have $D^\alpha_\pm f = D^\alpha_\pm f = D^\alpha_\pm \varphi = \varphi$, almost everywhere.
3. **Left inverse of a fractional integrals by a new tool.** We present a continuous variant of Grünwald-Letnikov’s formulas, with integrals instead of series. It involves a convolution kernel \( F \) which mimics the above mentioned features of Grünwald-Letnikov’s weights \( w_{\rho}^n \) by matching the following hypotheses.

- **Hypothesis** \( A^1(\alpha) \): \( F \) satisfies \( A^1(\alpha) \) if, \( \forall \rho \in \mathbb{N} \) such that \( 0 \leq \rho < \mathop{Re}\alpha, \ y^{\rho}F(y) \) is integrable in \( \mathbb{R} \) and satisfies \( \int_{0}^{\infty} F(y)y^{\rho}dy = 0 \).
- **Hypothesis** \( A^1(\alpha) \): \( F \) satisfies \( A^2(\alpha) \) if \( F(y) = F_1(y) + Cy^{-\alpha-1} \) in a neighborhood of \(+\infty\), with \( F_1(y)y^{\alpha} \) being integrable there.

**Theorem 3.1.** Let function \( F \) satisfies \( A^1(\alpha) \) and \( A^2(\alpha) \).

- **If** \( \mathop{Re}\alpha = \alpha' \notin \mathbb{N} \), then, the following points ((i), (ii) et (iii)) hold.
  - (i) \exists a constant \( \Lambda \) such that, for \( f = I_{\alpha'}^\varphi \) with \( \varphi \in L^1_r \) and \( r \geq 1 \), the limit of \( h^{-\alpha} \int_{0}^{\infty} f(x+hy)F(y)dy \) exists in \( L^1_r \) and is equal to \( \Lambda \varphi(x) \).
  - (ii) For \( f = I_{\alpha'}^\varphi \) with \( \varphi \in L^1_r \) and \( r \geq 1 \), the limit of \( h^{-\alpha} \int_{0}^{\infty} f(x-hy)F(y)dy \) exists in \( L^1_r \) and is equal to \( \Lambda \varphi(x) \).
  - (iii) The constant \( \Lambda \) in (i) and (ii) does not depend on \( r \) and \( H \) denotes Heaviside function.

- **If** \( \alpha \in \mathbb{N} \), the points (i), (ii) et (iii) persist provided \( F \) satisfies the condition: \( (\cdot)^\alpha F \) is integrable near \(+\infty\), instead of \( A^2(\alpha) \).

**Proof.** Non-integer values of \( \alpha' \) will be considered first.

**For \( \alpha' \) not an integer.**

Proving (i) is enough for (i) and (ii), and consists in comparing \( \varphi \) against the limit of \( h^{-\alpha} \int_{0}^{\infty} f(x+ht)F(t)dt \) under hypothesis \( A^1(\alpha)-A^2(\alpha) \). We have

\[
\int_{0}^{\infty} f(x+ht)F(t)dt = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} F(t) \left( \int_{x+ht}^{\infty} \varphi(y)(y-x-h\alpha^{-1}y)dt \right)dy.
\]

Setting \( y = x + hs \),

\[
h^{-\alpha} \int_{0}^{\infty} f(x+ht)F(t)dt = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} F(t) \left( \int_{t}^{\infty} \varphi(x+hs)(s-t)^{-\alpha^{-1}}ds \right)dt.
\]

Since we have

\[
\frac{1}{|\Gamma(\alpha)|} \int_{0}^{\infty} |\varphi(x+hs)| |x| \left( \int_{0}^{s} F(t)(s-t)^{\mathop{Re}\alpha^{-1}}dt \right) |ds
\]

\[
= \int_{0}^{\infty} \left| \varphi(x+hs)I_{\alpha}^\varphi(HF)(s) \right| ds
\]

\[
= \frac{1}{h} \int_{-\infty}^{0} \left| \varphi(x-t) \right| I_{\alpha}^\varphi(HF) \left( \frac{-t}{h} \right) dt
\]

\[
= h^{-1} \Phi \ast \left( I_{\alpha}^\varphi(HF) \left( \frac{-t}{h} \right) \right) \chi_{[\infty,0]}(x) = \Phi \ast \Upsilon(x),
\]

such that

\[
\Phi(t) = \begin{cases}
|\varphi(t)| & \text{si } t \in [0, +\infty[; \\
0 & \text{elsewhere.}
\end{cases}
\]

and

\[
\Upsilon(t) = h^{-1} \left| I_{\alpha}^\varphi(HF) \left( \frac{-t}{h} \right) \right| \chi_{[\infty,0]}(t).
\]
But, from the beginning, we set \( \varphi \in L^+_1 \), so that, \( \Phi \in L'(\mathbb{R}) \).

Moreover, according to 3.2 below, we have \( I^\alpha_+(HF) \in L^1(\mathbb{R}^+) \), hence \( \Upsilon \in L^1(\mathbb{R}) \).

Due to Young’s inequality, it implies that \( \Phi \) * \( \Upsilon \) \in \( L^r(\mathbb{R}) \), so that

\[
\int_0^{+\infty} \left| \varphi(x + hs) \left( \int_0^s F(t)(s-t)^{\alpha-1} \, dt \right) \right| \, ds < +\infty,
\]

almost everywhere. So that, we can apply Fubini’s theorem which implies that

\[
h^{-\alpha} \int_0^{+\infty} f(x + ht) F(t) \, dt = \int_0^{+\infty} \varphi(x + hs) I^\alpha_+(HF)(s) \, ds.
\]

Setting

\[
K(t) = \frac{I^\alpha_+(HF)(t)}{\int_0^{+\infty} I^\alpha_+(HF)(s) \, ds}.
\]

We remark that \( K \in L^1(\mathbb{R}) \) and \( \int_\mathbb{R} K(t) \, dt = 1 \). As \( \Phi \in L^r(\mathbb{R}) \), so, the theorem of approximation to identity implies that

\[
\lim_{h \to +0} \left\| \int_0^{+\infty} \Phi(\cdot + hs) I^\alpha_+(HF)(s) \, ds - \int_0^{+\infty} I^\alpha_+(HF)(s) \, ds \times \Phi \right\|_{L^r(\mathbb{R})} = 0,
\]

ie

\[
\lim_{h \to +0} h^{-\alpha} \int_0^{+\infty} f(\cdot + ht) F(t) \, dt = \int_0^{+\infty} I^\alpha_+(HF)(s) \, ds \times \varphi,
\]

in Lebesgue’s space \( L^r(\mathbb{R}^+) \). This proves (i) et (ii) if \( \alpha' \) is not an integer and \( \Lambda = \int_0^{+\infty} I^\alpha_+(HF)(s) \, ds \).

**For integer values of \( \alpha \).**

Let now \( \alpha \) be a positive integer. When \( F(x) x^\alpha \) is integrable, the Lemma 4.12 [9] implies the Theorem. \( \square \)

**Lemma 3.2.** Let \( \alpha \in \{ z \in \mathbb{C} \text{ such that } \text{Re} \, z \in \mathbb{R}^+ \setminus \mathbb{N} \} \). If \( F \) satisfies \( A^1(\alpha) \) and \( A^2(\alpha) \), then \( I^\alpha_+(HF) \) is integrable over \( \mathbb{R}^+ \).

**Proof.** Lemma 4.12 [9] (point (iii)) states that when \( x \mapsto F(x) x^\alpha \) is integrable with \( \alpha' \notin \mathbb{N} \) or \( \alpha \in \mathbb{N} \) while \( A^1(\alpha) \) holds, \( I^\alpha_+(HF) \) belongs to \( L^1 \). Hence, 3.2 will be a consequence of the Proposition below. \( \square \)

**Proposition 1.** Let \( 0 \leq m < (\text{Re} \, \alpha = \alpha') < m + 1 \) with \( m \in \mathbb{N} \), \( g^* \) such that \( g^* = g_1^* - g_2^* \), with

- \( g_1^*(x) = x^{-\alpha -1} \chi_{(A,\infty)}(x). \)
- \( g_2^*(x) = \sum_{i=0}^{m} b_i \chi_{[i,i+1]}(x). \)

Then, \( I^\alpha_+(H \, g^*) \) is integrable over \( \mathbb{R}^+ \) if and only if \( g^* \) satisfies

\[
\int_0^{+\infty} y^n \, g^*(y) \, dy = 0 \quad n \in \{ 0, 1, \cdots, m \}. \tag{5}
\]

Before showing the proposition, above, we aim to prove 3.3 and 3.4 below, which are involved in its proof.

**Lemma 3.3.** Let the function \( G \) defined by the following equation:

\[
G(X) = \int_0^X t^{-\alpha -1} \left[ (1-t)^{\alpha-1} - 1 + \sum_{k=1}^{m} \frac{(-1)^k}{k!} (\alpha-1)\cdots(\alpha-k) \, t^k \right] \, dt. \tag{6}
\]

And let \( 0 \leq m < \alpha' < m + 1 \), with \( m \in \mathbb{N} \). Then, for \( A, \) a real large enough, the function \( x \mapsto x^{-1} G \left( \frac{A}{x} \right) \) is integrable in a neighbourhood of \( +\infty \).
Proof. For \( \alpha \in \mathbb{C} \) and \( N \in \mathbb{N}^* \) and \( |x| < 1 \), we have
\[
(1+x)^\alpha \sum_{n=0}^{N} \left( \frac{\alpha}{n} \right) x^n = (N+1) \left( \frac{\alpha}{N+1} \right) \times \int_0^1 \left( \frac{1-t}{1+tx} \right)^N (1+tx)^{\alpha-1} \, dt \, x^{N+1}.
\]
Hence,
\[
[(1-t)^{\alpha-1} - \sum_{k=1}^{m} \frac{(-t)^k}{k!} (\alpha - 1) \cdots (\alpha - k - 1)] t^{-\alpha-1}
= [(1-t)^{\alpha-1} - \sum_{k=0}^{m} \left( \frac{\alpha-1}{k} \right) (-t)^k + 1 - 1] t^{-\alpha-1}
= [(-t)^{m+1} (m+1) \left( \frac{\alpha-1}{m+1} \right) \times \int_0^1 \left( \frac{1-s}{1-st} \right)^m (1-st)^{\alpha-2} ds)] t^{-\alpha-1}.
\]
Since
\[
\left| \int_0^1 \left( \frac{1-s}{1-st} \right)^m (1-st)^{\alpha-2} \, ds \right| \leq \int_0^1 |1-st|^{\alpha-2-m} \, ds \leq 1,
\]
so
\[
\left| (1-t)^{\alpha-1} - \sum_{k=1}^{m} \frac{(-t)^k}{k!} (\alpha - 1) \cdots (\alpha - k - 1) t^{-\alpha-1} \right| \leq (m+1) \left| \left( \frac{\alpha-1}{m+1} \right) \right| t^{m-\alpha}.
\]
We deduce that \( G(\frac{A}{x}) \) is the integral of a continuous function such that its module is dominated by \( \frac{|(\alpha-1)\cdots(\alpha-m-1)|}{m!} \) \( t^{m-\alpha} \). Therefore, we obtain
\[
x^{-1} G(\frac{A}{x}) \leq x^{-1} \left| \frac{(\alpha-1) \cdots (\alpha-m-1)}{(m-\alpha+1) m!} \right| t^{m-\alpha} \frac{1}{1}.
\]
\[
\Rightarrow x^{-1} |G(\frac{A}{x})| \leq \frac{|(\alpha-1) \cdots (\alpha-m-1)|}{A^{\alpha-m-1} (m-\alpha+1) m!} x^{\alpha-m-2}.
\]
\[
\Rightarrow x^{-1} G(\frac{A}{x}) \text{ is integrable in a neighborhood of } +\infty \text{ by the criterion for Riemann integrability.}
\]

Lemma 3.4.

\[
G(1) - \left( \frac{1}{\alpha} + \sum_{k=1}^{m} \frac{(-1)^k}{k!} (\alpha - 1) \cdots (\alpha - k + 1) \right) = 0.
\]

Proof. Notice that with
\[
g(p, q) = \int_0^1 ((1-t)^{q-1} - [1 + \sum_{k=1}^{m} \frac{(-1)^k}{k!} (q-1) \cdots (q-k)]) t^{p-1} \, dt.
\]
We have \( G(1) = g(-\alpha, \alpha) \) and \( \beta(p, q) = \int_0^1 (1-t)^{q-1} t^{p-1} \, dt \) (the Bernoulli Beta function).

For \( (p, q) \in \mathbb{C}^2 \), such that \( \text{Re} \, p > 0 \) and \( \text{Re} \, q > 0 \) we have \( \beta(p, q) = \Gamma(p) \Gamma(q) / \Gamma(p+q) \).

Hence
\[
g(p, q) = \beta(p, q) - \frac{1}{p} - \sum_{k=1}^{m} \frac{(-1)^k}{k! (p+k)} (q-1) \cdots (q-k)
= \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)} - \frac{1}{p} - \sum_{k=1}^{m} \frac{(-1)^k}{k! (p+k)} (q-1) \cdots (q-k).
\]
for $Re\,p > 0$ and $Re\,q > 0$.

Fixing $q = \alpha$, this equality extends to complex valued $p$ which are not negative integers and satisfy $Re(p) > -m - 1$. Let the function $F_\alpha$ defined by

$$F_\alpha : \mathbb{C} \times [0, 1] \rightarrow \mathbb{C}$$

$$(p, t) \mapsto t^{p-1} \left[ (1-t)^{\alpha-1} - (1 + \sum_{k=1}^{m} \frac{(-t)^k}{k!} \cdot (\alpha-1)\ldots(\alpha-k)) \right].$$

Let showing by the dominated convergence’s theorem that $\int_{0}^{1} F_\alpha(\cdot, t) \, dt$ is holomorphic function over a connected domain which we will set it later. Hence, we will deduce the analyticity of this function in this domain. In fact, for fixed $t \in [0, 1]$, we have $F_\alpha(\cdot, t)$ is a function of $p$, also it is holomorphic whose derivative is $\log(t) F_\alpha(\cdot, t)$. And due to the proof of 3.3, we have

$$| F_\alpha(p, t) | \leq t^{m+Re\,p} \frac{|(\alpha - 1)\ldots(\alpha - m - 1)|}{m!}.$$

Hence, for $t \in [0, 1]$, we have

$$| \frac{\partial F_\alpha}{\partial p} (p, t) | \leq -\log(t) t^{m+Re\,p} \frac{|(\alpha - 1)\ldots(\alpha - m - 1)|}{m!}.$$

$$\Rightarrow \int_{0}^{1} | \frac{\partial F_\alpha}{\partial p} (p, t) | \, dt \leq \int_{0}^{1} -\log(t) t^{m+Re\,p} \frac{|(\alpha - 1)\ldots(\alpha - m - 1)|}{m!} \, dt.$$

Since $\int_{0}^{1} \log(t) t^{m+Re\,p} \, dt$ is Bertrand’s integral, it converges if $-1 < p_0 \leq m + Re\,p$, i.e. for $-m - 1 < p_0 - m = p_1 \leq Re\,p$.

Consequently, the dominated convergence’s theorem implies that $p \mapsto g(p, \alpha)$ is holomorphic on $\Omega = \{ z \in \mathbb{C} \mid Re\,z \geq p_1 > -m - 1 \}$. It yields also that this function is analytic in the previous domain.

Moreover, the function defined by

$$p \mapsto \frac{\Gamma(p) \Gamma(\alpha)}{\Gamma(p+\alpha)} - \frac{1}{p} - \sum_{k=1}^{m} \frac{(-1)^k}{k! (p+k)} (\alpha-1)\ldots(\alpha-k)$$

and which coincide with $p \mapsto g(p, \alpha)$ for $Re\,p > 0$, is also holomorphic or analytic in the set $\Pi = \{ \Omega \setminus \{ 0, -1, \ldots, -m \} \}$. Indeed, $p \mapsto \Gamma(p)$ is holomorphic on $\Omega$, except at poles $0, -1, \ldots, -m$ of $\Gamma$. And $p \mapsto \frac{\Gamma(p) \Gamma(\alpha)}{\Gamma(p+\alpha)}$ is always holomorphic on $\mathbb{C}$, in particular on $\Pi$. Also, the map defined by

$$p \mapsto \frac{1}{p} + \sum_{k=1}^{m} \frac{(-1)^k}{k! (p+k)} (q-1)\ldots(q-k),$$

is holomorphic on $\Pi$ because $p \neq 0$ and $p + k \neq 0$.

Since $\Pi$ is a connected subset of $\{ z \in \mathbb{C} \setminus Re\,z > 0 \}$ where it has an accumulation point, the theorem of analytic continuation implies that both functions which we studied their analyticity, coincide in $\Pi$, i.e.,

$$g(p, \alpha) = \beta(p, \alpha) - \frac{1}{p} - \sum_{k=1}^{m} \frac{(-1)^k}{k! (p+k)} (\alpha-1)\ldots(\alpha-k).$$

We note that

$$(-m - 1) < (Re\,(-\alpha) = (-\alpha')) < (-m) \leq 0,$$
i.e., \((-\alpha) \in \Pi\). Thus, for \(p = -\alpha\), we have

\[ G(1) - \frac{1}{\alpha} \left[ 1 + \sum_{k=1}^{m} \frac{(-1)^k}{k!} \alpha \cdots (\alpha - k + 1) \right] = 0, \]

as \(\beta(-\alpha, \alpha) = 0\).

**Proof of Proposition 1.** Since \(g^* \in L^1_{loc}(\mathbb{R})\), \(I^\alpha_{+}(Hg^*) \in L^1_{loc}(\mathbb{R})\) and it suffices to check whether \(I^\alpha_{+}(Hg^*)\) is integrable in a neighborhood of \(+\infty\). Three preliminary steps will prepare the proof.

First, for \(x > A\), we have

\[ (5) \iff \sum_{i=0}^{m} b_i \frac{(i + 1)^{r+1} - i^{r+1}}{r + 1} = \frac{A^{\alpha+r}}{\alpha-r}, \]

for \(r \in \{0, 1, \ldots, m\}\).

In fact, we have

\[
\int_0^{+\infty} y^n g^*(y)\,dy = \int_0^{+\infty} y^n (g_1^* - g_2^*)\,dy
\]

\[
= \int_0^{+\infty} y^n [y^{-\alpha-1} \chi_{[A, +\infty]}(y) - \sum_{i=0}^{m} b_i \chi_{[i,i+1]}(y)]\,dy
\]

\[
= \int_A^{+\infty} y^{n-\alpha-1}\,dy - \sum_{i=0}^{m} b_i \left( \int_i^{i+1} y^n\,dy \right)
\]

\[
= \frac{y^{n-\alpha}}{n-\alpha} \bigg|_{y=A}^{+\infty} - \sum_{i=0}^{m} b_i \frac{(i + 1)^{n+1} - i^{n+1}}{n + 1}.
\]

Since \(0 \leq n \leq m < \alpha'\),

\[
\int_0^{+\infty} y^n g^*(y)\,dy = A^{n-\alpha} - \frac{n}{n-\alpha} + \lim_{y \to +\infty} \left( \frac{y^{n-\alpha}}{n-\alpha} - \sum_{i=0}^{m} b_i \frac{(i + 1)^{n+1} - i^{n+1}}{n + 1} \right).
\]

And

\[
\lim_{y \to +\infty} \left| \frac{y^{n-\alpha}}{n-\alpha} \right| = \lim_{y \to +\infty} \left| \frac{y^{n-Re\alpha}}{n-\alpha} \right| = 0.
\]

Then,

\[ (5) \iff A^{n-\alpha} - \frac{n}{n-\alpha} - \sum_{i=0}^{m} b_i \frac{(i + 1)^{n+1} - i^{n+1}}{n + 1} = 0. \]

Second, for \(x > A\), we have

\[ \Gamma(\alpha) I^\alpha_{+}(Hg^*) (x) = \int_A^x (x-y)^{\alpha-1} y^{-\alpha-1} \,dy - \Gamma(\alpha) I^\alpha_{+}(Hg_2^*) (x). \]

And

\[ \Gamma(\alpha) I^\alpha_{+} H\chi_{[i,i+1]}(x) = \int_i^{i+1} (x-y)^{\alpha-1} \,dy. \]

We set \(f(y) = (x-y)^{\alpha-1}\), then, we apply the expansion of Taylor-Young at 0 of ordre \(m\). Thus, we obtain

\[
f(y) = \sum_{k=0}^{m} \frac{y^k}{k!} f^{(k)}(0) + R_m(y).
\]
such that $R_m(x) = \int_0^x \frac{(x-t)^m}{m!} f^{(m+1)}(t) \, dt$.

$$f(y) = \sum_{k=0}^{m} \frac{y^k}{k!} [(x - y)^{\alpha - 1}]^k(0) + R_m(y).$$

Since

$$[(x - y)^{\alpha - 1}]^k(0) = (x - y)^{\alpha - 1 - k} (\alpha - 1) \ldots (\alpha - k) (-1)^k.$$

We have

$$f(y) = \sum_{k=0}^{m} \frac{y^k}{k!} x^{\alpha - 1 - k} (\alpha - 1) \ldots (\alpha - k) (-1)^k + R_m(y).$$

Thus,

$$\Gamma(\alpha) I_+^\alpha \chi_{[k,i+1]}(x) = \sum_{k=0}^{m} \frac{(-1)^k}{k!} \int_i^{i+1} y^k dy + \int_i^{i+1} R_m(y) dy,$$

$$= \frac{x^\alpha}{\alpha} \sum_{k=1}^{m+1} x^{-k} \alpha (\alpha - 1) \ldots (\alpha - k + 1) \frac{(-1)^{k-1}}{(k-1)!},$$

$$= \left( \frac{1}{\alpha} \right) R_m(x).$$

Let $\int_i^{i+1} x^{-\alpha} R_m(y) dy = x^{-m-2} B_i(x)$ with $B_i$ is bounded near $+\infty$. So, it leads to the fact that $x \mapsto B_i(x) x^{\alpha - m - 2}$ is integrable in neighbourhood of $+\infty$, as $m + 2 - \alpha > 1$.

Third,

$$\int_A (x - y)^{\alpha - 1} y^{-\alpha - 1} dy = \int_A (1 - \frac{y^2}{x^2})^{\alpha - 1} x^{\alpha - 1} (\frac{y}{x})^{\alpha - 1} x^{-\alpha - 1} dy.$$

Setting $z = \frac{y}{x}$,

$$\int_A (x - y)^{\alpha - 1} y^{-\alpha - 1} dy$$

$$= \int_0^{1} (1 - z)^{\alpha - 1} z^{-\alpha - 1} x^{-1} dz = \int_0^{1} (1 - z)^{\alpha - 1} z^{-\alpha - 1} x^{-1} dz$$

$$- x^{-1} \int_0^{1} z^{-\alpha - 1} (1 + \sum_{k=1}^{m} \frac{(-1)^k}{k!} (\alpha - 1) \ldots (\alpha - k) z^k) dz]$$

$$- x^{-1} \int_0^{1} (1 - z)^{\alpha - 1} z^{-\alpha - 1} x^{-1} dz$$

$$- x^{-1} \int_0^{1} z^{-\alpha - 1} (1 + \sum_{k=1}^{m} \frac{(-1)^k}{k!} (\alpha - 1) \ldots (\alpha - k) z^k) dz]$$

$$+ x^{-1} \int_0^{1} z^{-\alpha - 1} (1 + \sum_{k=1}^{m} \frac{(-1)^k}{k!} (\alpha - 1) \ldots (\alpha - k) z^k) dz,$$
such that
\[ x^{-1} \int_{\frac{A}{x}}^{1} z^{-\alpha - 1} \left( 1 + \sum_{k=1}^{m} \frac{(-1)^k}{k!} (\alpha - 1)\ldots(\alpha - k) z^k \right) dz \]
\[ = x^{-1} \frac{x^{-\alpha}}{\alpha A^\alpha} + x^{-1} \sum_{k=1}^{m} \frac{(-1)^k}{k!} (\alpha - 1)\ldots(\alpha - k) \left( \frac{A}{x} \right)^{k-\alpha - 1} \left( \frac{1}{\alpha-k} \right). \]

Consequently, we have
\[ \int_{A}^{x} (x-y)^{\alpha - 1} y^{-\alpha - 1} dy \]
\[ = x^{-1} \left[ G(1) - \left( \frac{1}{\alpha} + \sum_{k=1}^{m} \frac{(-1)^k}{k!} (\alpha - 1)\ldots(\alpha - k + 1) \right) \right] - x^{-1} G\left( \frac{A}{x} \right) \]
\[ + x^{-1} \frac{1}{\alpha} \left( \frac{x}{A} \right)^\alpha + \sum_{k=1}^{m} \frac{(-1)^k}{k!} (\alpha - 1)\ldots(\alpha - k + 1) \left( \frac{x}{A} \right)^{\alpha-k}, \]

We now looking at the integrability of the following map near $+\infty$:
\[ x \mapsto x^{-1} \left[ \frac{1}{\alpha} \left( \frac{x}{A} \right)^\alpha + \sum_{k=1}^{m} \frac{(-1)^k}{k!} (\alpha - 1)\ldots(\alpha - k + 1) \left( \frac{x}{A} \right)^{\alpha-k} \right] \]
\[ - \Gamma(\alpha) I_+^\alpha (Hg^\alpha)(x). \]

For this aim we consider that
\[ = x^{\alpha-m-1} \alpha \left[ A^{-\alpha} x^m + \sum_{r=0}^{m} x^{m-r} \frac{(-1)^r}{(\alpha-r) r!} \times \right. \]
\[ \left. \alpha (\alpha-1)\ldots(\alpha-r) A^{-\alpha} - A^{-\alpha} x^m \right]. \]

And
\[ \Gamma(\alpha) I_+^\alpha (Hg^\alpha)(x) = x^{\alpha-m-1} \alpha \sum_{i=0}^{m} b_i \sum_{k=1}^{m+1} \frac{(i)^k - (i + 1)^k}{k!} \]
\[ \times \alpha \cdots (\alpha + 1 - k) (-1)^k x^{m+1-k} + \frac{1}{\alpha} \sum_{i=0}^{m} b_i B_i(x) x^{\alpha-m-2} \]
\[ = x^{\alpha-m-1} \alpha \sum_{i=0}^{m} b_i \sum_{r=0}^{m} \frac{(i)^{r+1} - (i + 1)^{r+1}}{(r+1)!} \]
\[ \times \alpha \cdots (\alpha-r) (-1)^{r+1} x^{m-r} + \frac{1}{\alpha} \sum_{i=0}^{m} b_i B_i(x) x^{\alpha-m-2}. \]

Thus, we obtain
\[ \Gamma(\alpha) I_+^\alpha (Hg^\alpha)(x) = x^{\alpha-m-1} \alpha \left[ P(x) - (x^{-1} G\left( \frac{A}{x} \right) + \frac{1}{\alpha} \sum_{i=0}^{m} b_i B_i(x) x^{\alpha-m-2} \right) \]
\[ \in L^1[A, +\infty[, \text{ thanks to 3.3} \]
\[ +x^{-1} \left[ G(1) - \left( \frac{1}{\alpha} + \sum_{k=1}^{m} \frac{(-1)^k}{k!} (\alpha - 1) \cdots (\alpha - k + 1) \right) \right] = 0, \text{ thanks to 3.4} \]

Such that
\[ P(x) = \sum_{r=0}^{m} x^{m-r} \alpha (\alpha - 1) \cdots (\alpha - r) (-1)^{r+1} \]
\[ \times \left( - \sum_{i=0}^{m} b_i (i)^{r+1} - b_i (i+1)^{r+1} \frac{A^{\alpha}}{r+1} \right). \]

Since \( \frac{x^2}{\alpha} P(x) \sim \frac{C}{x^{m+1-\alpha}} \) and \( m + 1 - \alpha - d < 1 - d \leq 1 \), the polynome \( P \) of degree \( d \) is integrable in neighbourhood of \( +\infty \) if and only if \( P \equiv 0 \). But this condition is satisfied if and only if
\[ \sum_{i=0}^{m} b_i (i+1)^{r+1} - i^{r+1} \frac{A^{\alpha}}{r+1} = 0. \]

This proves the Proposition.

4. Porous and heterogeneous medium and fractional Fick’s law. For particles performing a Markovian random walk, the flux depends on the concentration and on the transition probability density function. It also depends on the geometry of the medium and on the physical properties of the boundaries, if there are, as we will see. After having set these points, we will focus on what happens to the flux when time and length scales of the random walk are made small, compared with those, characterizing the variations of the density of the cloud of particles.

4.1. The flux of particles performing Lévy flights. For a given particle, the location after \( n \)th jump is \( \sum_{i=0}^{n} X_i \), and it happens at time \( \sum_{i=0}^{n} T_i \).

- Jump amplitudes are independent variables, with density \( \varphi_l(x) = \frac{1}{l} L_\alpha (\frac{x}{l}) \).
  Here, \( L_\alpha \) denotes the density of a normalized Lévy law of stability index \( \alpha \) \( (1 < \alpha < 2) \) and skewness parameter \( \theta \). \( l \) is a length scale.
- Waiting times are also independent, and for the sake of simplicity we assume that they are distributed according to the Poisson density \( \psi_\tau(t) = \frac{1}{\tau} e^{-\tau} \), such that \( \tau \) is the mean waiting time.

The length scale \( l \) and mean waiting time \( \tau \) satisfy \( \frac{l}{\tau} = K \).

a) Flux of particles performing Lévy flights in an infinite medium.

Let us denote by \( \pm \int_{|x|} F_{\alpha, \theta} \left( \frac{x-y}{l} \right) d\mu(y, t) \frac{dt}{t} \) the probability of crossing \( x \) to the right or to the left, such that \( \mu(\cdot, t) \) is the measure giving the probability \( \mu(I, t) \) that the particle be in interval \( I \) at time \( t \). Let \( P(\cdot, t) \) be the density of the previous measure. And \( \int_{|x|} F_{\alpha, \theta} (\pm \frac{x}{\tau}) = \pm \int_{|x|} L_\alpha (z) \frac{dz}{z} \) is the probability for a jump to have an amplitude of more than \( y \) to the right or to the left. Also, we set that the probability of making one jump during infinitesimal time interval \([t, t + dt]\) is \( \frac{dt}{\tau} \). So, the flux is the probability rate, hence the following difference:
\[ \infty \mathcal{W}_{\alpha,\theta}^\tau P(x,t) = \tau^{-1} \left[ \int_0^x P(x-y,t) F_{\alpha,\theta}^+(\frac{y}{l}) \, dy - \int_0^{+\infty} P(x+y,t) F_{\alpha,\theta}^-(\frac{y}{l}) \, dy \right]. \] (10)

b) Flux of particles performing Lévy flights in semi-infinite domain; limited by a reflecting boundary.

The expression giving the flux may be modified more or less deeply by the presence of a boundary. In fact, imagine that each particle hitting a purely reflecting wall located at \( x = 0 \) bounces and finally flies the length of the jump, which had been assigned to it before the shock. In this case, we have to take account of two points. First, if the amplitude of the jumps directed to the left and starting from \( x+y \) \((y > 0)\) is larger than \( 2x+y\), then arrives at the right of \( x \) hence do not enter the balance. Second, jumps directed to the left and starting from \( x-y\), with \( 0 < y < x\), may cross \( x \) to the right if the amplitude is of more than \( 2x - y\).

Therefore, the flux is given by this equation:

\[ 0 W_{\alpha,\theta}^\tau (x,t) P = \tau^{-1} \left[ \int_0^x P(x-y,t) [F_{\alpha,\theta}^+(\frac{y}{l}) + F_{\alpha,\theta}^-(\frac{y-2x}{l})] \, dy - \int_0^{+\infty} P(x+y,t) [F_{\alpha,\theta}^-(\frac{-y}{l}) - F_{\alpha,\theta}^-(\frac{-2x-y}{l})] \, dy \right]. \]

Let \( P^*(x) = P(x)\), for \( x > 0 \) and \( P^*(x) = P(-x)\), for \( x < 0 \), we obtain

\[ 0 W_{\alpha,\theta}^\tau (x,t) P = \tau^{-1} \int_0^{+\infty} P^*(x-y,t) F_{\alpha,\theta}^+\left(\frac{y}{l}\right) \, dy \\
- \tau^{-1} \int_0^{+\infty} P^*(x+y,t) F_{\alpha,\theta}^-\left(\frac{-y}{l}\right) \, dy \] (11)

In fact, the previous equation is given by the following variable exchange:

\[
\begin{align*}
Y &= 2x - y \\
s &= 2x + y
\end{align*}
\]

\[
\tau^{-1} \int_0^x P(x-y,t) F_{\alpha,\theta}^-\left(\frac{y-2x}{l}\right) \, dy + \tau^{-1} \int_0^{+\infty} P(x+y,t) F_{\alpha,\theta}^-\left(\frac{-2x-y}{l}\right) \, dy \\
= \tau^{-1} \int_0^{2x} P(Y-x,t) F_{\alpha,\theta}^-\left(\frac{-Y}{l}\right) \, dY + \tau^{-1} \int_{2x}^{+\infty} P(s-x,t) F_{\alpha,\theta}^-\left(\frac{-s}{l}\right) \, ds \\
= \tau^{-1} \int_x^{+\infty} P(h-x,t) F_{\alpha,\theta}^-\left(\frac{-h}{l}\right) \, dh \\
= \tau^{-1} \int_x^{h} P^*(x-h,t) F_{\alpha,\theta}^-\left(\frac{-h}{l}\right) \, dh,
\]

because \((x-h) < 0 \Rightarrow P^*(x-h,t) = P(h-x,t)\).

The integral transform \( P \rightarrow l^\circ \int_0^{+\infty} P(x \pm y,t) F_{\alpha,\theta}^\circ\left(\mp \frac{y}{l}\right) \, dy \) are present on the right-hand sides of (11) and (10) giving the flux. Nevertheless, cumulated probabilities \( F_{\alpha,\theta}^\circ(\pm \cdot) \) satisfy \( A_2(\alpha - 1) \), but of course not \( A_1(\alpha - 1) \). In fact, let \( X \) a random variable \( \alpha \)-stable. And we denote \( \sigma \) the scale parameter, \( \theta \) the
skewness parameter and $P$ the measure of probability. Then, we have

$$\lim_{t \to +\infty} t^\alpha P(X > t) = \sigma C(\alpha) \frac{1 + \theta}{2}$$

and

$$\lim_{t \to +\infty} t^\alpha P(X < -t) = \sigma C(\alpha) \frac{1 - \theta}{2}.$$  

This result is obtained due to the asymptotic behavior of the density of a stable law given for $x > A > 0$, (with $A$ large enough). In fact, we have

- $L_{\theta}^\alpha(-x) = L_{\theta}^\alpha(x)$.
- For $1 < \alpha < 2$, $\alpha - 2 < \theta \leq 2 - \alpha$ and $x > A > 0$ with $A$ large enough, we have

$$L_{\theta}^\alpha(x) = \frac{1}{\pi x} \sum_{n=1}^{+\infty} \frac{(-x^{-\alpha})^n \Gamma(1 + n\alpha)}{n!} \sin \frac{n\pi}{2} (\theta - \alpha).$$

The coefficient of the leading term is

$$C_{\theta, \alpha} = \frac{-1}{\pi} \cos \frac{\pi}{2} \theta.$$

$c)$ Involving the new expression for left inverse of Riemann-Liouville’s integral in computing the microscopic flux.

$F_{\alpha, \theta}^\pm(\pm \cdot)$ satisfies both $A_1(\alpha - 1)$ and $A_2(\alpha - 1)$. So, we can apply the previous theorem witch yields the following equations:

$$\lambda D_{\alpha}^{\alpha - 1} P(x, t) = \lim_{l \to 0^+} t^{-\alpha} \int_0^{+\infty} P(x - y, t) \frac{\hat{F}_{\alpha, \theta}^+(y)}{l} dy \times D_{\alpha}^{\alpha - 1} P(x, t).$$
And
\[
\lim_{l \to 0^+} l^{-\alpha} \int_0^{+\infty} P(x + y, t) \tilde{F}_{\alpha,\theta}^{-}(\frac{-y}{l}) dy = (\int_0^{+\infty} I_{+}^{\alpha-1} H \tilde{F}_{\alpha,\theta}^{-}(\frac{-y}{l}) dy) \\
\times D_{-}^{\alpha-1} P(x, t) \\
= \lambda_{\alpha} D_{-}^{\alpha-1} P(x, t).
\]

besides, an appropriate choice of \(f_{\alpha,\theta}\) yields that the third expression in (15) tend to right and left-sided local fractional derivatives of order \(\alpha - 1\). The choice is \(f_{\alpha,\theta}(t) = (2 - \alpha) \chi_{[0,1]}(t)(1 - t)^{1-\alpha} \lambda_{\alpha,\theta}\).

Then, we have
\[
\lim_{l \to 0^+} l^{-\alpha} \int_0^{+\infty} (P(x \pm y, t) - P(x, t)) f_{\alpha,\theta}(\frac{y}{l}) dy = \pm \Gamma(3 - \alpha) \lambda_{\alpha,\theta} D_{\pm}^{KG,\alpha-1} P(x, t),
\]
such that \(D_{\pm}^{KG,\alpha} f (x) = \lim_{h \to 0^+} l^{-\alpha} I_{\pm}^{\alpha-2} \pm [(f(\cdot) - f(x))] (x \pm h)\)
\[
= \lim_{h \to 0^+} \frac{1}{\Gamma(1 - q)} h^{-q} \int_0^1 \pm (f(x + th) - f(x))(1 - t)^{-q} dt.
\]

For sufficiently well-behaved functions \((in \ L^p(\mathbb{R}) \cap H^{\alpha-1,\epsilon})\) the Kolwankar and Gangal’s derivative exists and is identically zero.

The value of \(\lambda_{\pm}\) can be given by comparing \(D_{\pm}^{KG,\alpha-1} f\) against the limit of
\[
l^{-\alpha} \int_0^{+\infty} (f(x + y) - f(x)) \tilde{F}_{\alpha,\theta}^{-}(\frac{-y}{l}) dy
\]
for some particular function \(f\). The comparising will be simpler with functions \(f\) whose local derivative of order \(\alpha - 1\) is identically zero in neighborhood of infinity.

In fact, set \(f(x) = \chi_{[1,2]}(x)\) and \(x \in [1, 2]\), then we have
\[
D_{\pm}^{KG,\alpha-1} f(x) = D_{\pm}^{KG,\alpha-1} \chi_{[1,2]}(x) = 0,
\]
because \(f\) is a differentiable function on \(\mathbb{R}\).

We have
\[
l^{-\alpha} \int_0^{+\infty} (f(x + y) - f(x)) \tilde{F}_{\alpha,\theta}^{-}(\frac{-y}{l}) dy
\]
\[
= l^{-\alpha} \int_0^{+\infty} (f(x + y) - f(x)) (\tilde{F}_{\alpha,\theta}^{-}(\frac{-y}{l}) + f_{\alpha,\theta}(\frac{y}{l})) dy
\]
\[
= l^{-\alpha} \int_0^{+\infty} (f(x + y) - f(x)) f_{\alpha,\theta}(\frac{y}{l}) dy + l^{-\alpha} \int_0^{+\infty} f(x + y) \tilde{F}_{\alpha,\theta}^{-}(\frac{-y}{l}) dy
\]
\[
- \int_0^{+\infty} f(x) \tilde{F}_{\alpha,\theta}^{-}(\frac{-y}{l}) dy.
\]
As we know that
\[
\int_0^{+\infty} f(x) \tilde{F}_{\alpha,\theta}^{-}(\frac{-y}{l}) dy = 0
\]
and
\[
l^{-\alpha} \int_0^{+\infty} (f(x + y) - f(x)) f_{\alpha,\theta}(\frac{y}{l}) dy = \Gamma(3 - \alpha) \lambda_{\alpha,\theta} D_{\pm}^{KG,\alpha-1} f(x) = 0.
\]
Then,
\[
\lim_{l \to 0^+} l^{-\alpha} \int_0^{+\infty} (f(x + y) - f(x)) F_{\alpha, \theta}^{-\left(\frac{-y}{l}\right)} dy = \lambda_- D^{\alpha-1} f(x).
\]

It yields this equation:
\[
\lambda_- = \lim_{l \to 0^+} l^{-\alpha} \int_0^{+\infty} (f(x + y) - f(x)) F_{\alpha, \theta}^{-\left(\frac{-y}{l}\right)} dy.
\]

On one hand, the compute of the numerator gives for \( x \in ]1, 2[ \)
\[
\lim_{l \to 0^+} l^{-\alpha} \int_0^{+\infty} (x|1, 2|(x + y) - x|1, 2|(x)) F_{\alpha, \theta}^{-\left(\frac{-y}{l}\right)} dy
\]
\[
= -\frac{C_{\alpha, -\theta}}{\alpha (\alpha - 1)} (2 - x)^{1-\alpha}.
\]

On the other hand, the compute of the denominator gives for \( x \in ]1, 2[ \),
\[
D^{\alpha-1} x|1, 2|(x) = \frac{(2 - x)^{1-\alpha}}{\Gamma(2 - \alpha)}.
\]

Then, we have
\[
\lambda_- = -\frac{\Gamma(2 - \alpha)}{(\alpha - 1) \alpha} (C_{\alpha, -\theta}) = -\frac{\Gamma(2 - \alpha)}{(\alpha - 1) \alpha} \frac{1}{\pi} \Gamma(1 + \alpha) \sin \frac{\pi}{2} (\theta + \alpha)
\]
\[
= \frac{(-\alpha)(1 - \alpha) \Gamma(-\alpha)}{\pi (\alpha - 1) \alpha} \Gamma(1 + \alpha) \sin \frac{\pi}{2} (\theta + \alpha)
\]
\[
= \frac{\Gamma(-\alpha) \Gamma(1 + \alpha)}{\pi} \sin \frac{\pi}{2} (\theta + \alpha)
\]
\[
= \frac{\sin \frac{\pi}{2} (\theta + \alpha)}{\sin (\pi \alpha)}.
\]

And similarly we have
\[
\lambda_+ = \frac{\sin \frac{\pi}{2} (\alpha - \theta)}{\sin (\pi \alpha)}.
\]

4.2. **Fractional Fick’s law.** When \( \tau \) and \( l \) tend to zero while satisfying the scaling \( \frac{\tau}{l^2} = K \), the limit \( \infty Q(x, t) \) of mapping \( \infty W_{1, \tau}^{\alpha, \theta}(x, t) \) witch denote the flux through \( x \) for random walks in unbounded domains, is such that
\[
\lim_{l \to 0^+} \infty W_{1, \tau}^{\alpha, \theta} = \infty Q(x, t).
\]

\[
\infty Q(x, t) : P \mapsto K \left( \lambda_+ D^{\alpha-1}_+ P(x) - \lambda_- D^{\alpha-1}_- P(x) \right)
\]
\[
- K \Gamma(3 - \alpha) L_{\alpha, \theta} [D^{KG, \alpha-1}_+ P(x, t) + D^{KG, \alpha-1}_- P(x, t)].
\]

To compute the diffusive limit of the flux in a semi-infinite domain limited by a reflective barrier, we have to rewrite the third term of (11) in an improved form. So we have
\[
\int_{x}^{t} \int_0^{+\infty} P^*(x - y, t) \left( F_{\alpha, \theta}^+ \left( \frac{y}{l} \right) - F_{\alpha, \theta}^{-\left(\frac{-y}{l}\right)} \right) dy
dy
\]
\[
= \int_{x}^{t} \int_0^{+\infty} P^*(x - y, t) \left( F_{\alpha, \theta}^+ \left( \frac{y}{l} \right) - F_{\alpha, \theta}^{-\left(\frac{-y}{l}\right)} \right) dy
dy
\]
\[
- \int_{x}^{t} \int_0^{+\infty} P^*(x - y, t) \left( F_{\alpha, \theta}^+ \left( \frac{y}{l} \right) - F_{\alpha, \theta}^{-\left(\frac{-y}{l}\right)} \right) dy
dy
\]
the macroscopic flux \( Q \) because

Moreover, we have

4.3. Space-fractional diffusion equation. When the density of particles \( P \) and the macroscopic flux \( Q \) are derivable, mass conservation without sources implies

\[
\partial_t P(x, t) = -\partial_x Q(x, t).
\]

Moreover, we have

\[
\partial_x D_x^{\alpha-1} = \pm D_x^\alpha,
\]

because

\[
\partial_x D_x^{\alpha-1} \varphi(x) = \partial_x \left(-\frac{d}{dx}\right)^{[\alpha-1]+1} f^{1-\{\alpha-1\} - \frac{d}{dx}}^{[\alpha]+1} f^{1-\{\alpha\}} \varphi(x) = -D_x^\alpha \varphi(x) .
\]

Also, the local Kolwankar-Gangal derivatives with order of less than 1 are identically zero. Hence, in an infinite medium, we deduce from (16) the space-fractional diffusion equation in an infinite domain given by this equation

\[
\partial_t P(x, t) = - K \left[ \lambda_+ D_x^\alpha P(x) + \lambda_- D_x^\alpha P(x) \right]
\]

\[
= - K \frac{1}{\Gamma(2-\alpha)} \partial_x^2 \left[ \frac{\sin \frac{\pi}{2} (\alpha-\theta)}{\sin \pi \alpha} \int_{-\infty}^x P(y) (x-y)^{1-\alpha} dy \right]
\]

\[
+ \left(\frac{\sin \frac{\pi}{2} (\alpha+\theta)}{\sin \pi \alpha}\right) \int_0^{+\infty} P(y) (y-x)^{1-\alpha} dy \right] = K \nabla_x^{\alpha,\theta} P(x, t),
\]

Hence

\[
\int_0^{1+\infty} [I - H] P^*(x-y,t)] (F_{x,\theta}^+(\frac{y}{t})) - F_{x,\theta}^-(\frac{-y}{t})) dy.
\]

With

\[
H P^*(x-y) = \begin{cases} 0 & \text{if } x < y; \\ P^*(x-y) & \text{if } x > y. \end{cases}
\]

According to the Theorem, we have

\[
\lim_{l \to 0^+} l^{-\alpha} \int_0^{1+\infty} (I - H) P^*(x-y,t)] (F_{x,\theta}^+(\frac{y}{t})) - F_{x,\theta}^-(\frac{-y}{t})) dy
\]

\[
= (\lambda_+ - \lambda_-) D_x^{\alpha-1}((I - H) P^*) (x).
\]

Then, the diffusive limit of the flux in semi-infinite medium; limited by a reflecting boundary, is such that

\[
\lim_{l \to 0^+} l^{-\alpha} \int_0^{1+\infty} (I - H) P^*(x-y,t)] (F_{x,\theta}^+(\frac{y}{t})) - F_{x,\theta}^-(\frac{-y}{t})) dy
\]

\[
= (\lambda_+ - \lambda_-) D_x^{\alpha-1}((I - H) P^*) (x).
\]

4.3. Space-fractional diffusion equation. When the density of particles \( P \) and the macroscopic flux \( Q \) are derivable, mass conservation without sources implies

\[
\partial_t P(x, t) = -\partial_x Q(x, t).
\]

Moreover, we have

\[
\partial_x D_x^{\alpha-1} = \pm D_x^\alpha,
\]

because

\[
\partial_x D_x^{\alpha-1} \varphi(x) = \partial_x \left(-\frac{d}{dx}\right)^{[\alpha-1]+1} f^{1-\{\alpha-1\} - \frac{d}{dx}}^{[\alpha]+1} f^{1-\{\alpha\}} \varphi(x) = -D_x^\alpha \varphi(x) .
\]

Also, the local Kolwankar-Gangal derivatives with order of less than 1 are identically zero. Hence, in an infinite medium, we deduce from (16) the space-fractional diffusion equation in an infinite domain given by this equation

\[
\partial_t P(x, t) = - K \left[ \lambda_+ D_x^\alpha P(x) + \lambda_- D_x^\alpha P(x) \right]
\]

\[
= - K \frac{1}{\Gamma(2-\alpha)} \partial_x^2 \left[ \frac{\sin \frac{\pi}{2} (\alpha-\theta)}{\sin \pi \alpha} \int_{-\infty}^x P(y) (x-y)^{1-\alpha} dy \right]
\]

\[
+ \left(\frac{\sin \frac{\pi}{2} (\alpha+\theta)}{\sin \pi \alpha}\right) \int_0^{+\infty} P(y) (y-x)^{1-\alpha} dy \right] = K \nabla_x^{\alpha,\theta} P(x, t),
\]
such that $\nabla_x^{\alpha,\beta}$ is the Riesz-Feller derivative of order $\alpha$ and skewness parameter $\theta$. In a medium, limited by a reflective barrier, we deduce from (18) that $P$ evolves according to

\[
\partial_t P(x, t) = -K \left[ \lambda_+ (D_+^{\alpha} P^*)(x, t) + \lambda_- (D_-^{\alpha} P^*)(x, t) \right] + (\lambda_+ - \lambda_-) D_0^{\alpha} ((I - H) P^*)(x, t)
\]

\[
= K \nabla_x^{\alpha,\beta} H P(x, t)
\]

\[
- \frac{K}{2} \frac{\lambda_+}{\Gamma(2-\alpha)} \partial_x \int_0^{+\infty} P(y, t) (x+y)^{1-\alpha} dy
\]

\[
+ \frac{K}{2} \frac{\lambda_-}{\Gamma(2-\alpha)} \int_0^{+\infty} (x+y)^{1-\alpha} P(y, t) dy.
\]

\[\tag{20}\]

5. The fractional dispersion equation via Laplace-Fourier analysis and the generalized master equation in a medium limited by a reflective barrier. In the theory of CTRWs (Continuous Time Random Walks), it is assumed that waiting times are independent and identically distributed random variables with density $\psi$. Let $\psi(t) = \frac{\beta(t+1)}{\Gamma(\gamma+1)}$ such that $g(t) = -\frac{d}{dt} E_\beta(-t^\beta)$, and $E_\beta = E_{\beta,1}$ is the Mittag-Leffler function of order $\beta \in [0, 1]$. We denote

\[
E_{\alpha,\beta}(z) = \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad \alpha > 0, \beta \in \mathbb{C} \text{ and } z \in \mathbb{C}.
\]

Let

\[
e_\alpha(t, \lambda) = e_{\alpha,1}(t, \lambda) = E_\alpha(\lambda t^\alpha)
\]

and

\[
e_{\alpha,\beta}(t, \lambda) = t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha).
\]

Since

\[\forall \gamma > 0 \quad D_0^\alpha(e_{\alpha}(\cdot, \lambda))(x) = \lambda e_{\alpha,\alpha-\gamma+1}(x, \lambda),\]

then, if we take $\lambda = -1$ and $\gamma = 1$, we will have

\[
g(t) = -\frac{d}{dt} E_\beta(-t^\beta) = -D_0^1(e_\beta(\cdot,-1))(t) = e_{\beta,\beta}(t, -1).
\]

Furthermore the Laplace transform of $e_{\alpha,\beta}(\cdot, \lambda)$ (symbolized by the notation $\sim$) can be written as

\[
\hat{e}_{\alpha,\beta}(\cdot, \lambda)(u) = \frac{u^{\alpha-\beta}}{u^\alpha + \lambda}, \quad \text{for} \quad |u| > |\lambda|^{\frac{1}{\beta}}, \quad \text{Re}(u) > 0.
\]

So, we have

\[
\hat{g}(u) = \frac{u^{\beta-\beta}}{u^\beta + 1} = \frac{1}{u^\beta + 1}.
\]

Hence

\[
\hat{w}(u) = \frac{1}{\tau} \hat{g}(u \tau) = \frac{1}{1 + (u \tau)^\beta} = 1 - u^\beta \tau^\beta + o(\tau^\beta).
\]

To proceed further we assume that the transition probability follows an $\alpha-$ stable $1 < \alpha < 2$ law. Let $C(x, t)$ be the density of the probability of finding a walker in $[x, x+dx]$ at time $t$. And let $C$ satisfies the initial condition $C(x, 0) = \sigma(x-x_0)$, with $x_0 > 0$. Also, we may imagine a purely reflecting wall located at $x = 0$.

Let $\Lambda(x, x')$ the probability density that a walker go through $x$, given that the particle is at position $x'$. We assume that

\[
\Lambda(x, x') = \rho_d(x, x') + \rho_r(x, x'),
\]
with $\rho_d(x, x')$ is the direct transition probability and $\rho_r(x, x')$ is the transition probability after reflection by the elastic wall. We denote $\rho$ an even function and $f$ the density of the $\alpha-$stable law such that

$$\rho(x) = \frac{1}{T} f\left(\frac{x}{T}\right),$$

$$\rho_d(x, x') = \rho(x - x'),$$

and

$$\rho_r(x, x') = \rho(-x - x')$$

(due to the interaction between Lévy flights and reflecting wall) [4]. The Fourier transform of $f$ can be written as

$$\hat{f}(k) = e^{-|k|^{\alpha}}.$$ 

Then, we have

$$\hat{\rho}(k) = \hat{f}(kl) = e^{-l^{\alpha}|k|^{\alpha}}.$$ 

So, the density has the asymptotic expansion

$$\hat{\rho}(k) = 1 - l^{\alpha} |k|^{\alpha} + o(l^{\alpha}), \; \text{in V} \; (0 +).$$

The master equation is given by this equation:

$$C(x, t) = \sigma_{x_0}(x) \left(1 - \int_0^t w(t') dt'\right) + \int_{x' = 0}^{+\infty} \int_0^t C(x', t') [\rho(x - x') + \rho(-x - x')] w(t - t') dt' dx'.$$

Setting $C^*(x, t) = C(x, t)$ for $x > 0$ and $C^*(x, t) = C(-x, t)$ for $x < 0$ (even extension of $C$). We obtain

$$C^*(x, t) = \int_{-\infty}^x \int_{t' = 0}^t C^*(x', t') \rho(x - x') w(t - t') dt' dx'$$

$$+ (\sigma_{x_0}(x) + \sigma_{-x_0}(x)) \int_t^{+\infty} w(t') dt'.$$

$$C^*(x, t) = \int_{-\infty}^x \rho(x - x') \left(\int_{t' = 0}^t C^*(x', t') w(t - t') dt'\right) dx'$$

$$+ (\sigma_{x_0}(x) + \sigma_{-x_0}(x)) \int_t^{+\infty} w(t') dt'$$

$$= \int_{-\infty}^x \rho(x - x') (C^*(x', .) * w)(t) dx'$$

$$+ (\sigma_{x_0}(x) + \sigma_{-x_0}(x)) \int_t^{+\infty} w(t') dt',$$

then

$$C^*(x, t) = [\rho * (C^* * w)(t)](x) + (\sigma_{x_0}(x) + \sigma_{-x_0}(x)) \int_t^{+\infty} w(t') dt'.$$

The Fourier transform of $C^*(\cdot, t)$ yields

$$\hat{C^*}(\cdot, t)(k) = \hat{\rho}(k) [(C^* \ast w)(t)](k) + (\sigma_{x_0} + \sigma_{-x_0})(k) \int_t^{+\infty} w(t') dt'.$$
that holds, we have

\[ \int_{t}^{+\infty} w(t') \, dt' = \hat{\rho}(k) \hat{C}^\star(k, u) w(t) + (\sigma_{x_0} + \sigma_{x_0})(k) \int_{t}^{+\infty} w(t') \, dt'. \]

The Laplace transform yields

\[ \hat{C}^\star(k, u) = \hat{\rho}(k) \hat{C}^\star(k, u) \hat{\tilde{w}}(u) + (\sigma_{x_0} + \sigma_{x_0})(k) \left( \int_{0}^{+\infty} w(t') \, dt' \right)(u) \]

and

\[ = \hat{\rho}(k) \hat{C}^\star(k, u) \hat{\tilde{w}}(u) + \left( e^{-ikx_0} + e^{ikx_0} \right) \int_{0}^{+\infty} w(t') \, dt' - \hat{\tilde{w}}(u). \]

Hence

\[ \hat{C}^\star(k, u) = \hat{\rho}(k) \hat{C}^\star(k, u) \hat{\tilde{w}}(u) + 2 \cos(k x_0) \frac{1 - \hat{\tilde{w}}(u)}{u}, \]

because of \( \int_{0}^{+\infty} w(t') \, dt' = 1 \). Then, we have

\[ \hat{C}^\star(k, u) = \frac{1 - \hat{\tilde{w}}(u)}{\rho(k)} \frac{2 \cos(k x_0)}{\hat{\tilde{w}}(u)}. \]

If we substitute \( \hat{\rho} \) and \( \hat{\tilde{w}} \) for their asymptotic expansion, we obtain

\[ \hat{C}^\star(k, u) = \frac{2 \cos(k x_0) u^{\beta-1} \tau^\beta}{1 - \left( 1 - 1^{\alpha} \right) \left( 1 - u^\beta \tau^\beta \right)} \]

\[ = \frac{2 \cos(k x_0) u^{\beta-1} \tau^\beta}{\tau^\beta + k^\alpha + \frac{u^\beta \tau^\beta}{\frac{1}{\alpha} |k|^\alpha}}. \]

In the diffusive limit and for fixed \((k, u) \neq (0, 0)\), and since the scaling

\[ \begin{align*}
  l & \to 0 \\
  \tau & \to 0 \\
  \frac{l^{\alpha}}{\tau^\beta} & = \lambda_{\alpha, \beta}
\end{align*} \]

holds, we have

\[ \hat{C}^\star(k, u) = \frac{2 \cos(k x_0) u^{\beta-1} (\lambda_{\alpha, \beta})^{-1}}{(\lambda_{\alpha, \beta})^{-1} u^\beta + |k|^\alpha} = \frac{2 \cos(k x_0) u^{\beta-1}}{u^\beta + |k|^\alpha} \]

We denote the fractional derivative of Caputo of order \( \alpha \) by \( \alpha D^\alpha_{x,t} f(x) \), such that

\[ \alpha D^\alpha_{x,t} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-u)^{m-\alpha-1} f^{(m)}(u) \, du, \]

with \( m - 1 < \alpha < m \). Its Laplace transform is as follows:

\[ \alpha D^\alpha_{x,t} f(s) = s^\alpha f(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k} f^{(k)}(0+). \]

Hence, for \( \beta \in [0, 1] \), we have

\[ \alpha D^\alpha_{x,t} \hat{C}^\star(x, u)(u) = u^{\beta} \hat{C}^\star(x, u)(u) - (u^{\beta-1} C^\star(x, u), 0). \]

Then

\[ \hat{\alpha D^\alpha_{x,t} C^\star}(k, u) = u^{\beta} \hat{\tilde{C}}^\star(k, u) - 2 u^{\beta-1} \cos(k x_0) \]

\[ = u^{\beta} \hat{\tilde{C}}^\star(k, u) - \hat{\tilde{C}}^\star(k, u) (u^\beta + |k|^\alpha \lambda_{\alpha, \beta}) \]

\[ = -\hat{\tilde{C}}^\star(k, u) |k|^\alpha \lambda_{\alpha, \beta}. \]
Lemma 5.1. Let $g_\alpha(x) = \frac{1}{\|x\|^\alpha}$, with $x \in \mathbb{R}^d$ and $0 < \alpha < d$.

Then 
$$\hat{g}_\alpha(k) = C_{\alpha,d} \|k\|^{\alpha-d},$$

such that $C_{\alpha,d} = \frac{2^{d-\alpha}}{\Gamma(d/2)} \Gamma(\frac{d-\alpha}{2}) \pi^{\frac{d}{2}}$.

Let the functions $\Phi$ and $\Psi$ which satisfying the following equation:
$$\hat{\Phi}(k) = \|k\|^{-s} \hat{\Psi}(k).$$

So thanks to 5.1 we have
$$\hat{\Phi}(k) = \frac{1}{C_{d-s,d}} \hat{g}_{d-s}(k) \hat{\Phi}_s(k) = (2\pi)^{-d} C_{s,d} \|k\|^{-s-d}(k) \hat{\Phi}(k).$$

And, by applying the inverse Fourier transform, we obtain
$$\Phi(x) = (2\pi)^{-d} C_{s,d} \frac{1}{\|k\|^{-s}} \ast \Phi_s(x).$$

The inverse Laplace transform of $(0D_t^\beta C^*) \hat{(k, u)}$, yields
$$[0D_t^\beta C^*(., t)](k) = -[\hat{C^*}(., t)](k) = (2\pi)^{-d} C_{s,d} \|k\|^{-s-d}(k) \hat{\Phi}(k),$$

with $-1 < \alpha - 2 < 0$, due to $1 < \alpha < 2$.

Let $\gamma = 2 - \alpha$, then
$$[0D_t^\beta C^*(., t)](k) = -\lambda_{\alpha,\beta} \|k\|^2 (2\pi)^{-1} C_{\gamma,1} \times (C^*(., t) \ast \frac{1}{|.|^{1-\gamma}})(k),$$

with 
$$C_{\gamma,1} = \frac{2^{1-\gamma}}{\Gamma(\frac{\gamma}{2})} \Gamma\left(\frac{1-\gamma}{2}\right) \pi^{\frac{d}{2}} = \frac{2^{\alpha-1}}{\Gamma(1-\frac{\alpha}{2})} \Gamma\left(\frac{\alpha-1}{2}\right) \pi^{\frac{d}{2}}.$$

We rewrite $C_{\gamma,1}$ in a suitable form by using the following functional equation:
$$\Gamma(1-z) \Gamma(z) = \frac{\pi}{\sin(\pi z)}.$$

Then, we have
$$\Gamma\left(\frac{\alpha-1}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi \alpha}{2} - \frac{\pi}{2}\right)} \frac{1}{\Gamma(1-\frac{\alpha}{2})} = \frac{\pi}{\cos\left(\frac{\pi \alpha}{2}\right)} \frac{1}{\Gamma(1-\frac{\alpha}{2}) \Gamma(\frac{\alpha}{2})}.$$

Hence,
$$(2\pi)^{-1} C_{\gamma,1} = \frac{-2^{\alpha-1}}{(1-\alpha) \Gamma(1-\frac{\alpha}{2}) \Gamma(\frac{\alpha}{2}) \cos\left(\frac{\pi \alpha}{2}\right)}.$$

Besides, the following duplication formula (Legendre formula)
$$\Gamma(a) = \frac{2^{a-1}}{\sqrt{\pi}} \Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{a+1}{2}\right),$$

yields, by setting $a = 1 - \alpha$,
$$\frac{2^{a-1}}{\Gamma(1-\frac{\alpha}{2}) \Gamma(\frac{\alpha}{2})} = \frac{2^{-a+1}}{2 \Gamma(\frac{\alpha+1}{2}) \Gamma(\frac{\alpha}{2})} = \frac{1}{2 \sqrt{\pi} \Gamma(\alpha)} = \frac{1}{2 \sqrt{\pi} \Gamma(1-\alpha)}.$$
Therefore, we have

\[
[0 D^\alpha_t C^*(.,t)](k) = -\lambda_{\alpha,\beta} \mid k \mid^2 (2 \pi)^{-1} C_{\gamma,1} \times (C^*(.,t) \ast \frac{1}{\mid \cdot \mid^{1-\gamma}})(k) \\
= - \mid k \mid^2 \lambda_{\alpha,\beta} (C^*(.,t) \ast \frac{1}{\mid \cdot \mid^{1-\gamma}})(k) \times \frac{\cos(\frac{\pi \alpha}{2}) \sqrt{\pi}}{\Gamma(2 - \alpha)} \\
= [\frac{\partial}{\partial x^2} C^*(.,t) \ast \frac{1}{\mid \cdot \mid^{1-\gamma}}](k) \frac{-\lambda_{\alpha,\beta}}{2 \cos(\frac{\pi \alpha}{2}) \Gamma(2 - \alpha)}.
\]

by applying the inverse Fourier transform, we obtain

\[
o D^\alpha_t C^*(x, t) = \frac{-\lambda_{\alpha,\beta}}{2 \cos(\frac{\pi \alpha}{2}) \Gamma(2 - \alpha)} \times \frac{\partial^2}{\partial x^2} \int_{-\infty}^{+\infty} C^*(y, t) \mid x - y \mid^{1-\alpha} \ dy \\
= \frac{-\lambda_{\alpha,\beta}}{2 \cos(\frac{\pi \alpha}{2}) \Gamma(2 - \alpha)} \times \frac{\partial^2}{\partial x^2} \bigg( \int_{0}^{+\infty} C(y, t) \mid x - y \mid^{1-\alpha} \ dy \\
+ \int_{-\infty}^{0} C(-y, t) (x + y)^{1-\alpha} \ dy \bigg).
\]

Setting the variable exchange \(-y = s\). Then

\[
o D^\alpha_t C^*(x, t) = \frac{-\lambda_{\alpha,\beta}}{2 \cos(\frac{\pi \alpha}{2}) \Gamma(2 - \alpha)} \times \frac{\partial^2}{\partial x^2} \bigg( \int_{0}^{+\infty} C(y, t) \mid x - y \mid^{1-\alpha} \ dy \\
+ \int_{0}^{+\infty} C(s, t) (x + s)^{1-\alpha} \ ds \bigg).
\]

We deduce that for fixed \(x\) and \(t\), such that \(x > 0\), the density \(C(x, t)\) satisfies the following equation:

\[
o D^\alpha_t C(x, t) = \frac{-\lambda_{\alpha,\beta}}{2 \cos(\frac{\pi \alpha}{2}) \Gamma(2 - \alpha)} \times \frac{\partial^2}{\partial x^2} \bigg( \int_{0}^{+\infty} C(y, t) \mid x - y \mid^{1-\alpha} + (x + y)^{1-\alpha} \ dy \bigg).
\]

**Remark 4.** Let \(\beta = 1\), then waiting times are distributed according to the Poisson density \(\psi(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}\). And we retrieve the previous result obtained in the fourth section by calculating the macroscopic flux of a cloud of particles performing skewed \((\theta = 0)\) Lévy flights. We suppose of course that the medium is limited by a purely reflecting wall located in \(x = 0\) and is without any source or sink. In fact, we have

\[
o D^1_t C(x, t) = \partial_t C(x, t) \ \text{et} \ \lambda_{\alpha,1} = \frac{l^\alpha}{\tau} = K.
\]

And

\[
\frac{-\lambda_{\alpha,\beta}}{2 \cos(\frac{\pi \alpha}{2}) \Gamma(2 - \alpha)} \times \frac{\partial^2}{\partial x^2} \bigg( \int_{0}^{+\infty} C(y, t) \mid x - y \mid^{1-\alpha} \ dy \bigg) = K \nabla_x^{\alpha,0} HP(x, t).
\]

Hence

\[
o D^1_t C(x, t) = \partial_t C(x, t) = K \nabla_x^{\alpha,0} HP(x, t) \\
- \frac{K}{2 \cos(\frac{\pi \alpha}{2}) \Gamma(2 - \alpha)} \times \frac{\partial^2}{\partial x^2} \bigg( \int_{0}^{+\infty} C(y, t) (x + y)^{1-\alpha} \ dy \bigg).
\]
6. **Conclusion.** The new expression for the left inverse of $I^{\alpha}_{\pm}$, which can be used for complex values of the order $\alpha$ of the derivation such that $\Re \alpha = \alpha' > 0$, generalizes Grünwald-Letnikov’s, with integrals instead of series. It combines convolution, contraction (multiplication by $l$) or dilatation (multiplication by $\frac{1}{l}$) of the argument of one among the two involved functions, and dilatation (multiplication by $l^{-\alpha}$) of the issue. Then, the limit “$l$ tending to zero” yields a fractional derivative, according to our theorem. In order to satisfy $A^1(\alpha)$, the kernel $F$ has to oscillate such that all moments of integer order smaller than $\alpha'$ be equal to zero. Except when $\alpha$ is an integer, $\alpha + 1$ represents the first power of $x^{-1}$ in the expansion, near infinity, of the kernel. Some improvements should now allow us to extend the theorem to higher dimensions. Our theorem is not only a mathematical result, but it also serves in physics, since it allowed us to represent flux of particles performing Markovian random walks in a very natural way. Moreover, the theorem can be applied in the field of signal processing.

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