Poor man’s scaling and Lie algebras

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Abstract

We consider a general model, describing a quantum impurity with degenerate energy levels, interacting with a gas of itinerant electrons, derive general scaling equation for the model, and analyse the connection between its particular forms and the symmetry of interaction. On the basis of this analysis we write down scaling equations for the Hamiltonians which are the direct products of $su(3)$ Lie algebras and have either $SU(2) \times U(1)$ or $SU(2)$ symmetry. We also put into a new context anisotropic Coqblin—Schrieffer models proposed by us earlier.

1. Introduction

In a seminal paper, published in 1964 and entitled 'Resistance Minimum in Dilute Magnetic Alloy' [1], Kondo considered a (deceptively) simple model: magnetic impurity in a normal metal. His, and the following theoretical analysis of the problem by different authors, led to the appearance of many approaches and techniques, which became paradigms in different fields of physics [2]. One of such approaches was the so called poor man’s scaling, pioneered by Anderson [3]. Models, similar to the one mentioned above, describe magnetic ions in a crystalline electric field [4, 5], tunnelling centres [6] and system of quantum dots [7–13]. It is known that the anisotropy can substantially change the physics of the model in comparison with the isotropic case [14–21].

In our previous paper [20] we considered a general anisotropic model, describing a single quantum impurity with degenerate energy levels, interacting with a gas of itinerant electrons and derived poor man’s scaling equation to the second order of interaction for the model. We also introduced and studied the renormalizable anisotropic generalization of the Coqblin–Schrieffer (CS) model [22]. The CS model, though being well studied previously [2, 23–27], draw a lot of attention recently in connection with the studies of quantum dots [9], heavy fermions [28] and ultra-cold gases [29, 30].

In the present contribution the algebraic structure of the general scaling equation is analyzed in a more detailed way. We show the connection between the renormalizability of a Hamiltonian and the structure of the Lie algebra which is used to write down the Hamiltonian. On the basis of this analysis we introduce still another renormalizable generalization of the CS model.

The rest of the paper is constructed as follows. In section 2 we rederive the scaling equation and consider the case of the maximally symmetric interaction. In section 3 we formulate the specific algebraic problem which is of interest to us, and, as the nontrivial example of its solution, derive the scaling equation for the case, when the operator of interaction acts on $su(3)$ algebra, but has the symmetry described either by $SU(2) \times U(1)$ or by $SU(2)$ subgroup of $SU(3)$. In section 4 we put the anisotropic generalizations of the CS model (XXZ and XYZ CS models) proposed by us earlier into the new context. In section 5 we try to combine the results of the previous two section. We conclude in section 6. Some mathematical details are relegated to the Appendix.
2. Scaling equation

2.1. Scaling equation and Lie algebras
The Hamiltonian we start from is [6]

\[ H = \sum_{\kappa} c_{\kappa}^\dagger c_{\kappa} + \sum_{\alpha, \beta, ab} V_{\beta\alpha, ba} c_{\alpha}^\dagger c_{\beta}, \]  

where \( c_{\kappa}^\dagger \) and \( c_{\kappa} \) are electron creation and annihilation operators of itinerant electron with wave vector \( \kappa \) and internal quantum number \( \alpha \), \( \kappa \) is the energy of the electron; \( X_{\beta\alpha} = [b < \alpha > a] \), where \( [a] > \) \( [b] > \) are the internal states of the scattering system, is the Hubbard \( X \)-operator.

We are interested in the low energy physics, that is in the electron states in the vicinity of the Fermi energy. However, we can not just discard the energy states of the electrons at the band edges, because virtual transitions to these states influence the low energy physics. The idea of renormalization [3] consists in decreasing the band width of the itinerant electrons from \([-D_0, D_0]\) to \([-D_0 + |DD|, D_0 - |DD|] (|DD| < 0)\) and taking into account the terms which corresponded to virtual transitions through the electron states in the discarded energy intervals by renormalizing matrix elements of the perturbation connecting the states in the reduced band; this process is repeated many times, thus making the band width a running parameter. In the lowest order of perturbation theory (one loop approximation) we obtain scaling equation

\[ \frac{dV_{\beta\alpha, ba}}{d \ln \Lambda} = \rho \sum_{\gamma, \delta} [V_{\beta\gamma, bc} V_{\gamma\alpha, ca} - V_{\gamma\alpha, be} V_{\beta\gamma, ca}], \]  

where \( \rho \) is the density of states of itinerant electrons (assumed to be constant), which we further on will take being equal to 1 (more exactly, \( \rho \) can be eliminated by measuring all electronic densities in units of this constant \( \rho \)); \( \Lambda = D/D_0 \).

Let the matrix \( V_{\beta\alpha, ba} \) be a sum of direct products of Hermitian matrices, acting in \( ab \) and \( \alpha\beta \) spaces respectively

\[ V = 2 \sum_{\rho \rho'} c_{\rho\rho'} G^\rho \otimes \Gamma^\rho', \]  

where the set of matrices \( \{ G^\rho \} \) is closed with respect to commutation and hence generates some Lie algebra \( g \). We assume that the set of matrices \( \{ \Gamma^\rho \} \) generates the algebra which is isomorphic to \( g \). (The indices \( a, b \) of the matrix \( G^\rho \) and the indices \( \alpha, \beta \) of the matrix \( \Gamma^\rho \) further on will be suppressed for brevity). Actually further on we will not distinguish these two algebras. (The multiplier 2 is added to avoid the appearance of the multiplier 1/2 later).

With the help of equation (3) we can write down equation (2) in a more transparent form

\[ \frac{d\epsilon_{\rho\rho'}}{d \ln \Lambda} G^\rho \otimes \Gamma^\rho' = \sum_{\alpha\beta\gamma\delta} [G^\alpha, G^\beta] \otimes [\Gamma^\alpha, \Gamma^\beta] \epsilon_{\alpha\beta\gamma\delta}. \]  

Introducing structure constants of the Lie algebras \( g \)

\[ [G^\alpha, G^\beta] = i \sum_p f^p_{\alpha\beta} G^p, \quad [\Gamma^\alpha, \Gamma^\beta] = i \sum_p f^p_{\alpha\beta} \Gamma^p, \]  

we can write down equation (2) in an even more transparent form [20]

\[ \frac{d\epsilon_{\rho\rho'}}{d \ln \Lambda} = - \sum_{\alpha\beta\gamma\delta} f^p_{\alpha\beta} f^p_{\gamma\delta} \epsilon_{\alpha\beta\gamma\delta}. \]  

From the point of view of calculus, equation (6) is (one of) the simplest scaling equation, one can consider. However, it has an interesting algebraic structure, which is the subject of the present communication.

2.2. Maximally symmetric interaction
The algebra \( g \) defines the group \( G \). If the symmetry of the interaction is maximal (corresponds to the group \( G \)), it (the interaction) is proportional to the quadratic invariant of the group [31]

\[ V = 2i \sum_p G^p \otimes \Gamma^p. \]  

Notice that the Hamiltonian of free electrons has a maximal symmetry (in the electron internal quantum number space it is proportional to unit matrix), so we’ll interchangeably talk about the symmetry of the Hamiltonian and the symmetry of the interaction.
Substituting $c_{pp'} = J \delta_{pp'}$ into the r.h.s. of equation (6) we obtain
\[
\frac{dc_{pp'}}{d \ln \Lambda} = -J^2 \sum_s f_{st}^p f_{st}^{p'}.
\] (8)

Assuming that the generators are self-dual
\[
\text{Tr} (G^p G^{p'}) = \frac{1}{2} \delta_{p,p'}
\] (same for $\Gamma$'s), we have
\[
\sum_s f_{st}^p f_{st}^{p'} = N \delta_{p,p'},
\] (10)

where $N$ is the dimensionality of the matrices $G^p$ [32]. Thus the interaction (7) is renormalizable, and the scaling equation for such an interaction is
\[
\frac{df}{d \ln \Lambda} = -NJ^2.
\] (11)

(The results of this Subsection, being written down in mutually dual bases, one can find in appendix A).

Note that if we take $g = su(2)$, the generators being $\{G^p\} = \{S^x, S^y, S^z\}$ and $\{\Gamma^p\} = \{\frac{1}{2} \sigma_x, \frac{1}{2} \sigma_y, \frac{1}{2} \sigma_z\}$, the maximally symmetric interaction is
\[
V = JS \cdot \sigma,
\] (12)

which corresponds to the isotropic Kondo model. And if we take $g = su(N)$ and return from the generators notation to Hubbard operators and electron creation and annihilation operators notation, the maximally symmetric interaction is
\[
V = \sum_{mn} X_{mm'} a_{mm'}^\dagger a_{mm'},
\] (13)

which corresponds to the CS model [22].

2.3. Reduction to principal axes

If the generators are chosen to be Hermitian, the matrix $c_{pp'}$ is real. If the matrix is in addition symmetric, it can be diagonalized by an orthogonal transformation of the generators. (In appendix B we consider a particular model when the matrix $G_{pp'}$ is not symmetric). Note that an orthogonal transformation (even a more general unitary one) does not change the commutation relations among the generators. Thus equation (3) can be ‘reduced to the principal axes’, that is to the form
\[
V = 2 \sum_p J_p G^p \otimes \Gamma^p.
\] (14)

By substituting $J_{\alpha'} = J_{\alpha} \delta_{\alpha,\alpha'}$ into the r.h.s. of equation (6) we obtain
\[
\frac{dc_{pp'}}{d \ln \Lambda} = -\sum_s f_{st}^p f_{st}^{p'} J_s.
\] (15)

If
\[
\sum_s f_{st}^p f_{st}^{p'} J_s = \delta_{p,p'} \sum_s (f_{st}^p)^2 J_s,
\] (16)

the interaction (14) is renormalizable.

The condition (16) is fulfilled for the algebra $su(2)$, due to the fact that for a given pair $s, t$ the structure constant $f_{st}^p$ is different from zero for only one value of $p$, that is the sum in the r.h.s. of equation (5) consists of only a single summand. Thus the general interaction can be presented as
\[
V = J_1 S^x \otimes \sigma^x + J_2 S^y \otimes \sigma^y + J_3 S^z \otimes \sigma^z.
\] (17)

We call the model with the interaction (17) the XYZ Kondo model.

Studying commutation relations for the algebra $su(3)$ presented in appendix C shows that for most pairs $s, t$, $f_{st}^p$ is different from zero only for one value of $p$. The pairs 4, 5 and 6, 7 result in two summands. However, if we impose constraint $J_4 J_5 = J_6 J_7$ equation (16) is still valid. We will assume that the constraint is imposed further on. Thus both for the $su(2)$ and for the $su(3)$ algebras the scaling equation becomes
\[
\frac{df}{d \ln \Lambda} = -\sum_s (f_{st}^p)^2 J_s.
\] (18)

Further on in this paper equations (14) and (18) will be our starting point.
For the Kondo model (taking into account the universally known $su(2)$ algebra commutation relations), equation (18) takes the form of the system of three equations

$$\frac{df_i}{d \ln \Lambda} = -2J_i J_k,$$

where $\{i, j, k\}$ is an arbitrary permutation of the Cartesian indices $x, y, z$.

3. Symmetry as the guiding principle

3.1. A good symmetry is a (partially) broken symmetry

Following the previous section, let us ask ourselves what are the other particular renormalizable cases of the general interaction (14) (or, even more general, (3)), that is what are the constraints imposed on the interaction, which are respected by the renormalization? We suggest symmetry as a guiding principle to find the answers.

A (too) simple case of the maximally symmetric interaction considered above corresponded to the symmetry group of the Hamiltonian $G$ being the group defined by the algebra on which it acts. Let us now decrease the symmetry of the interaction from the group $G$ to some its subgroup $G'$. It is obvious that the general interaction having this symmetry will be renormalizable. It is also obvious that such an interaction will be an arbitrary linear combination of all the invariant quadratic elements of the subgroup adjacently acting on the algebra of the group.

To illustrate the approach to renormalizability based on symmetry principle, let us return to the Kondo model and assume that the symmetry of the interaction is $U(1)$; we take $S^z$ as the generator of the subgroup. Due to decrease of symmetry, in addition to the existing for $SU(2)$ symmetry invariant element $S^x \otimes \sigma^x + S^y \otimes \sigma^y + S^z \otimes \sigma^z$, there appears a new invariant element: $S^z \otimes \sigma^z$. For our purposes it is better to say that we have two invariants of the group $U(1): S^z \otimes \sigma^z + S^y \otimes \sigma^y$ and $S^z \otimes \sigma^z$. Thus the most general interaction with the symmetry $U(1)$ can be written as an arbitrary linear combination of these invariant elements

$$V = I_x (S^x \otimes \sigma^x + S^y \otimes \sigma^y) + I_z S^z \otimes \sigma^z.$$

We call the model with the interaction (20) the XXZ Kondo model. For the interaction (20) the scaling equation is [3]

$$\frac{df_x}{d \ln \Lambda} = -2J_x J_z,$$

$$\frac{df_z}{d \ln \Lambda} = -2J^2_z.$$

Notice that $T^z$ generates Cartan subalgebra of the $su(2)$ algebra. This statement, though seeming simultaneously trivial and not very relevant, is actually the seed out of which section 4 will grow from.

Further on we consider several less simple cases, when we can explicitly realize the program, formulated above. The scaling equations we obtain, to the best of our knowledge were not written down before. Everywhere, apart from the last Subsection, $g = su(3)$.

3.2. $U(2)/SU(2)$ symmetry of the interaction

Let the symmetry of the interaction is either $SU(2) \times U(1) = U(2)$ or $SU(2)$. In any case we have three invariant elements (see appendix C). Thus we can write down the most general interaction with the prescribed symmetry as

$$V = 2I_A \sum_{p=1}^{3} G^p \otimes \Gamma^p + 2I_B \sum_{p=4}^{7} G^p \otimes \Gamma^p + 2I_C G^8 \otimes \Gamma^8,$$

where the generators $G^p$ and $\Gamma^p$ are just the $T^p$ operators from the appendix C in the appropriate spaces. From equations (22) and (18) we obtain the scaling equation as

$$\frac{dI_A}{d \ln \Lambda} = -2J_A^2 - J_B^2,$$

$$\frac{dI_B}{d \ln \Lambda} = -\frac{3}{2}I_A J_B - \frac{3}{2}I_B J_C,$$

$$\frac{dI_C}{d \ln \Lambda} = -3J_B^2,$$

where the coefficients in the r.h.s. of equation (23) are: (the number of generators of $V_A$ minus one) times 1 and the number of generators of $V_B$ times $\frac{1}{2}$; the coefficients in the r.h.s. of equation (24) are: twice the number of
generators of $V_A$ times $\frac{3}{4}$ and twice the number of generators of $V_C$ times $\frac{1}{4}$; the coefficient in the r.h.s. of equation (25) is the number of generators of $V_B$ times $\frac{5}{4}$.

3.3. Mixing of $U(2)$ ($SU(2)$) and $U(1)$ symmetries

The symmetry of the Hamiltonian considered in the previous Subsection being $U(2) \times U(1)$ or $U(2)$, the $su(3)$ algebra was reduced, to the direct sum of the subspaces (see appendix C). (Actually, here and further on we should have talked not about the subspaces, but about the direct products of two isomorphic subspaces. But we prefer being brief to being rigorous.) Thus the Hamiltonian is block diagonal.

Let us decrease the symmetry of the part the Hamiltonian acting on the subspace $V_A$ to $U(1)$. Obviously, instead of the single invariant element of the larger symmetry group on the subspace $V_A$ we had previously $\sum_{p=1}^{L}(T^p)^2$, now there are two invariant elements: $\sum_{p=1}^{L}(T^p)^2$ and $(T^3)^2$. Hence, the most general interaction, satisfying these relaxed symmetry demands, is

$$V = 2J_{AI} \sum_{p=1}^{2} G^p \otimes \Gamma^p + 2J_{AII} G^3 \otimes \Gamma^3 + 2J_B \sum_{p=4}^{7} G^p \otimes \Gamma^p + 2J_C G^8 \otimes \Gamma^8. \quad (26)$$

The scaling equation in this case is

$$\frac{dJ_{AI}}{d\ln \Lambda} = -2J_{AI} J_{AII} - J_B^2,$$

$$\frac{dJ_{AII}}{d\ln \Lambda} = -2J_A^2 - J_B^2,$$

$$\frac{dJ_B}{d\ln \Lambda} = -\left( J_{AI} + \frac{1}{2} J_{AII} \right) J_B - \frac{3}{2} J_B J_C,$$

$$\frac{dJ_C}{d\ln \Lambda} = -3J_B^2. \quad (27)$$

If we put $J_{AI} = J_{AII} = J_A$ we obviously return to equations (23)–(25).

4. Anisotropic Coqblin—Schrieffer models

4.1. XXZ Coqblin—Schrieffer model

In our previous paper [20] we introduced a generalization of the CS model defined by the interaction

$$V = J_x \sum_{m,m'} X_{mm'} c_m^+ c_{m'} + J_z \sum_{m} X_{mm} c_m^+ c_m - \frac{J_z}{N} \sum_{m} X_{mm} c_m^+ c_m. \quad (28)$$

We called the Hamiltonian the XXZ CS model, because it was obtained by analogy with the XXZ Heisenberg model. The Hamiltonian turned out to be renormalizable, and scaling equation was

$$\frac{dJ_x}{d\ln \Lambda} = -(N - 2)J_x^2 - 2J_x J_z,$$

$$\frac{dJ_z}{d\ln \Lambda} = -NJ_z^2. \quad (29)$$

To connect the previous result with the new ones, let us put $N = 3$ in equation (28) and, after a bit of algebra, write down the equation as

$$V = 2J_x \sum_{p=1}^{8} G^p \otimes \Gamma^p + 2J_x (G^3 \otimes \Gamma^3 + G^8 \otimes \Gamma^8), \quad (30)$$

where prime means that the terms corresponding to $p = 3$ and $p = 8$ are excluded from the sum. Note that the operators $T^3$ and $T^8$ generate Cartan subalgebra of the $su(3)$ algebra [33].

Equation (30) can be easily generalized. Looking at $N - 1$ generators of Cartan subalgebra of the $su(N)$ algebra (equation (D1) from appendix D) we realize that the equality

$$V = 2J_x \sum_{p} G^p \otimes \Gamma^p + 2J_x \sum_{q} G^q \otimes \Gamma^q, \quad (31)$$

where in the first sum the summation is with respect to $p$ enumerating the generators of the algebra $su(N)$ excluding the generators of Cartan subalgebra, and in the second sum the summation is with respect to the values of $q$ enumerating the generators of Cartan subalgebra (the latter being given by equation (D1)) is valid for any $N$. Thus for $N = 4$ there are three terms in the second sum in equation (31), corresponding to $q = 3, 8, 15$ in the notation of generators of [33].
4.2. XYZ Coqblin—Schrieffer model

In our previous publication \[20\] we also introduced a generalization of the CS model defined by the interaction

\[
V = \frac{J_x}{2} \sum_{m,m'} X_{mnm'}(e_n^+ e_m + e_m^+ e_n) + \frac{J_y}{2} \sum_{m,m'} X_{mnm'}(e_n^+ e_m - e_m^+ e_n) \\
+ J_z \sum_m X_{mm} e_m^+ e_m - \frac{J}{N} \sum_{mnm'} X_{mnm'} e_m^+ e_n.
\]

We called the model the XYZ CS model, because it was obtained by analogy with the XYZ Heisenberg model. The Hamiltonian also turned out to be renormalizable, and scaling equation for it was written as

\[
\frac{d l_x}{d \ln \Lambda} = -(N-2)J_x J_y - 2J_y J_z \\
\frac{d l_y}{d \ln \Lambda} = -(N-2)J_x J_y - 2J_x J_z \\
\frac{d l_z}{d \ln \Lambda} = -N J_x J_y.
\]

The \(J_z\) term in the interaction (32) is identical to that in equations (28) and (31). For \(N = 3\), the interaction, being expressed through the generators, is

\[
V = 2J_x \sum_{p=1,4,6} G^p \otimes \Gamma^p + 2J_y \sum_{p=2,5,7} G^p \otimes \Gamma^p + 2J_z (G^3 \otimes \Gamma^3 + G^8 \otimes \Gamma^8).
\]

Equation (C1) shows that \(\lambda_1, \lambda_4, \lambda_6\) are all similar to each other (and similar to \(\sigma_x\)), and \(\lambda_2, \lambda_5, \lambda_7\) are all similar to each other (and similar to \(\sigma_y\)). So the similarity of the XYZ CS model (for \(N = 3\)) and the XYZ Kondo model is obvious. For \(N = 4\), the interaction (32), being expressed through the generators, is

\[
V = 2J_x \sum_{p=1,4,6,9,11,13} G^p \otimes \Gamma^p + 2J_y \sum_{p=2,5,7,10,12,14} G^p \otimes \Gamma^p + 2J_z (G^3 \otimes \Gamma^3 + G^8 \otimes \Gamma^8 + G^{15} \otimes \Gamma^{15})
\]

(again we used the notation of generators of \([33]\)).

5. Discussion

One may ask what is the connection between sections 3 and 4? The answer is that both these sections follow from section 2. More specifically, looking at equations (22), (26), (31), and (34) we realize that all renormalizable interactions found by us, using either the symmetry reasoning, or the analogies with the Heisenberg model, turn out to be generalizations of equation (7). In fact, all the above mentioned equations can be presented as

\[
V = 2J_x \sum_{|p_1|} G^p \otimes \Gamma^p + 2J_y \sum_{|p_2|} G^p \otimes \Gamma^p + \ldots,
\]

where \(|p_1|, |p_2|, \ldots\) present the partition of all the generators of the appropriate (\(su(3)\) for equations (22), (26), and (34), \(su(4)\) for equation (31)) Lie algebra. It would be interesting to try to look at the general equation (36) from the point of view of the theory of invariants.

6. Conclusions

We studied algebraic structure of the scaling equation for a general model, describing a quantum impurity with degenerate energy levels, interacting with a gas of itinerant electrons. More specifically we studied the connection between the explicit form of the scaling equation and the symmetry of interaction. On the basis of this analysis we have written down the scaling equations for the Hamiltonians having \(U(2)(SU(2))\) symmetry. We also presented a new representation for the anisotropic CS models proposed by us earlier.

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Appendix A. Maximally symmetric interaction analysed in mutually dual bases

In section 2 we assumed that the generators are self-dual. However, the results may be written down in the basis independent form. In fact, analysing equation (3) we realise that for the matrices $\Gamma$ we can use the basis different from that used for the matrices $G$. Distinguishing the former by the sign tilde above the generators we can rewrite equations (3) and (6) respectively as

$$ V = 2 \sum_{pp'} \epsilon_{pp'} G^p \otimes \tilde{\Gamma}^{p'} $$  \hspace{1cm} (A1) 

and

$$ \frac{dc_{pp'}}{d \ln \Lambda} = - \sum_{st'} f^p_{st} \gamma^{p'}_{t'} c_{st} c_{st'} $$ \hspace{1cm} (A2) 

If we chose the tilde basis as the dual basis to that without the tilde, whatever the former is, the invariant element of the algebra can be written in the form [34]

$$ \text{inv} = \sum_{p} G^p \otimes \tilde{\Gamma}^p, $$ \hspace{1cm} (A3) 

and equation (10) becomes

$$ \sum_{st} f^p_{st} \gamma^{p'}_{t'} = N \delta_{pp'}. $$ \hspace{1cm} (A4) 

Hence the interaction having the full symmetry $G$ is

$$ V = 2J \sum_{p} G^p \otimes \tilde{\Gamma}^p, $$ \hspace{1cm} (A5) 

and we regain equations (11), but this time written down in an arbitrary basis.

Appendix B. Non-symmetric XXZ Kondo model

Everywhere in the main body of the paper we assume that the interaction matrix $\epsilon_{pp'}$ is symmetric. Here we want to show, using the XXZ Kondo model as an example, what the removal of this limitation can lead to. In distinction to the treatment of the model in the section 3.1, there appears now an additional invariant: $S^x \otimes \sigma^y - S^y \otimes \sigma^x$, and the most general interaction with the symmetry $U(1)$ can be written as an arbitrary linear combination of the three invariant elements

$$ V = I_x (S^x \otimes \sigma^x + S^y \otimes \sigma^y) + I_y S^y \otimes \sigma^x + I_{DM} (S^x \otimes \sigma^y - S^y \otimes \sigma^x) $$ \hspace{1cm} (B1) 

(the index DM standing for Dzyaloshinskii-Moriya). Using equation (6), we get the scaling equation as

$$ \frac{dI_x}{d \ln \Lambda} = -2J_x I_x $$ 

$$ \frac{dI_{DM}}{d \ln \Lambda} = -2J_{DM} I_x $$ 

$$ \frac{dI_y}{d \ln \Lambda} = -2J_x^2 - 2J_{DM}^2. $$ \hspace{1cm} (B2) 

The two first integrals of equation (B2) are obvious

$$ I_x^2 - I_y^2 - I_{DM}^2 = \text{const}, \quad \frac{I_{DM}}{I_x} = \text{const}. $$ \hspace{1cm} (B3) 

So the equation can be easily integrated in terms of trigonometric (hyperbolic) functions.
Appendix C. Group SU(3) and its subgroups

The generators of su(3) algebra can be chosen as \( T^p = \lambda^p / 2 \), where the Gell-Mann matrices

\[
\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},
\]

\[
\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}
\]

(C1)

and

\[
\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\]

(C2)

are the su(3) analogs of the Pauli matrices [32]. With this choice of generators, the structure constants \( f_{ij}^k \) are totally antisymmetric with respect to the interchange of any pair of indices, and are given by [32]

\[
f_{12}^3 = 1,
\]

(C3)

\[
f_{14}^7 = f_{16}^5 = f_{24}^6 = f_{25}^7 = f_{34}^8 = f_{37}^8 = \frac{1}{2},
\]

(C4)

\[
f_{64}^5 = f_{86}^7 = \frac{\sqrt{3}}{2},
\]

(C5)

while all other \( f_{ij}^k \) not related to these by permutation are zero. It will be important for us in section 3.2 that there are only three different squares of structure constants: \( \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \).

The group SU(3) acts adjointly on the su(3) algebra. The matrix elements of the generators of the adjoint representation \( D \) are [33]

\[
(D(T^p))_l^i \equiv 2 \text{Tr}(T^p [T^l, T^i]) = i f_{pl}^i,
\]

(C6)

The algebra \( su(3) \) realises irreducible representation of the group SU(3). Consider now the adjoint action of the subgroup SU(2), generated by \( [T^1, T^2, T^3] \), on the same algebra. In this case the \( su(3) \) algebra, considered as just a vector space, can be reduced to a direct sum of three subspaces: \( V_A \) with three generators \( [T^1, T^2, T^3] \), \( V_B \) with four generators \( [T^1, T^5, T^6, T^7] \), and \( V_C \) with a single generator \( T^8 \)

\[
su(3) \to V_A \oplus V_B \oplus V_C.
\]

(C7)

Looking at equations (C3)–(C5) we realise that each subspace is closed under the adjoint action of SU(2), thus we have decomposed the representation of the group SU(2), realised on the algebra su(3), into three representations.

For each of the subspaces there obviously exist an invariant (with respect to algebra generated by \( [T^1, T^2, T^3] \)) element, which is equal to the sum of all the subspace generators. In each case it is the only such element, because the representations realised on each of subspace are irreducible.

Now consider the adjoint action of the group \( U(2) = SU(2) \times U(1) \) on the algebra su(3). The group has additional, in comparison with the group SU(2), symmetry generator \( T^8 \), which commutes with each of the generators \( [T^1, T^2, T^3] \). [36] Because the generator \( T^8 \) do not mix the subspaces introduced above, the representation decomposition is still valid. Also, because all the representations considered above were irreducible for the group SU(2), they remain as such after the upgrading the symmetry. One can check up that all the previous invariant elements are the invariant elements of \( T^8 \) also. So finally, both the group \( SU(2) \times U(1) \) and the group \( U(2) \), acting on the algebra \( su(3) \) has three and only three quadratic invariant elements: \( \sum_{p=1}^{3} (T^p)^2 \), \( \sum_{p=4}^{7} (T^p)^2 \), and \( (T^8)^2 \).
Appendix D. Generators of Cartan subalgebra of the $su(N)$ algebra

$N - 1$ generators of Cartan subalgebra of the $su(N)$ algebra are [33]

$$\begin{pmatrix}
\frac{1}{2} & -1 & 0 \\
-1 & \frac{1}{2} & \sqrt{3} \\
0 & -\sqrt{3} & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & \frac{2}{\sqrt{3}} \\
0 & \frac{2}{\sqrt{3}} & 0
\end{pmatrix}, \quad \ldots, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -\sqrt{2N(N-1)} \\
0 & -\sqrt{2N(N-1)} & 1
\end{pmatrix} \quad (D1)$$

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