STRUCTURE FORMATION AND THE TIME DEPENDENCE OF QUINTESSENCE

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We discuss the influence of dark energy on structure formation, especially the effects on $\sigma_8$. Our interest is particularly focused on quintessence models with time-dependent equation of state and non-negligible quintessence component in the early universe. We obtain an analytic expression for $\sigma_8$ valid for a large class of dark energy models. We conclude that structure formation is a good indicator for the history of dark energy and use our results to set constraints on quintessence models.

1 Introduction

1.1 Dark Energy

There is evidence for dark energy contributing up to about 70% of the total energy of the universe [1, 2, 3]. The nature of dark energy is an open question, a cosmological constant or a dynamical scalar field, called quintessence [4], being two major options. The interest in quintessence arises from the possibility that the enormous fine-tuning problems plaguing a cosmological constant can partially be cured by some quintessence models. However, telling the difference between a cosmological constant and quintessence or between different quintessence models is complicated because of the non-genericness of quintessence.

If quintessence couples only gravitationally to matter, the only way of detection is possibly the exploration of its time dependent energy density and equation of state, as the relative energy density fluctuations $\delta\rho_d/\rho_d$ within the horizon are negligible [5]. In order to find this time dependence, measurements of different epochs are necessary. For that reason, the interplay between quintessence and nucleosynthesis, cosmic microwave background (CMB) [6, 7, 8], weak lensing [9] and Supernovae Ia data [10, 11, 12] have recently been explored.

Also, the theory of structure formation can in principle test the history of quintessence in the large range of redshift $z \in [0, 10^4]$. As has been noticed in [11, 12], the presence of dark energy can influence the growth of structure in the universe from matter radiation equality onwards. In particular, $\sigma_8$, the rms density fluctuations averaged over $8h^{-1}$Mpc spheres, is a sensitive parameter. Until now, a quantitative understanding of the effect of quintessence on $\sigma_8$ has been missing. This paper aims to fill this gap.

Supernovae Ia observations [10, 11] indicate that the equation of state $w_d \equiv p_d/\rho_d$ of dark energy is negative today. This gives rise to the aforementioned fine-tuning problem: We have

$$\Omega_d(a) \propto a^{-3\bar{w}_d},$$

where $\bar{w}_d$ is an appropriate mean value for the equation of state. If $w_d$ has always been negative, like in the case of the cosmological constant, $\Omega_d(a)$ has been extremely small in the early universe, and its importance just today lacks a natural explanation. Scalar models with this property can be constructed for an appropriate effective scalar potential [13]. They often involve, however, an unnatural tuning of parameters [14]. The problem can be surrounded if we assume that $w_d$ became negative relatively recently and $\rho_d$ has scaled in the past like radiation or matter. We will call such models ‘models with early quintessence’ and pay particular attention to them.

The COBE [15] normalization [16] of the CMB power spectrum determines $\sigma_8$ for any given model by essentially fixing the fluctuations at decoupling. This prediction is to be compared to values of $\sigma_8$ inferred from other experiments, such as cluster abundance constraints which yield [17]

$$\sigma_8 = (0.5 \pm 0.1)\Omega_m^{\gamma},$$

where $\gamma$ is slightly model dependent and usually $\gamma \approx 0.5$. A model where these two $\sigma_8$ values do not agree can be ruled out. Standard Cold Dark Matter (SCDM) without dark
energy for instance gives $\sigma_{\text{cmb}}^8 \approx 1.5$, $\sigma_8^{\text{CMB}} \approx 0.5 \pm 0.1$ and is hence incapable of meeting both constraints.

CMB measurements suggest that the universe is flat, $\Omega \equiv \Omega_m + \Omega_r + \Omega_d = 1$ and we assume this throughout the paper.

### 1.2 Quintessence vs Cosmological Constant

Our main result is an estimate of the CMB-normalized $\sigma_8$-value for a very general class of Quintessence models $Q$ just from the knowledge of their “background solution” $[\Omega_d(a), \ w_d(a)]$ and the $\sigma_8$-value of the $\Lambda$CDM model $\Lambda$ with the same amount of dark energy today $\Omega_\Lambda^0 = \Omega_d^0(\Lambda)$:

$$\frac{\sigma_8(Q)}{\sigma_8(\Lambda)} \approx (a_{eq})^3 \frac{\Omega_d}{(1 - \Omega_\Lambda^0)^2} \left(1 + \Omega_\Lambda^0\right)^{-1/2} \frac{\tau_0(Q)}{\tau_0(\Lambda)}.$$  

(3)

If $Q$ is a model with ‘early quintessence’, $\Omega_d^0$ is an average value for the fraction of dark energy during the matter dominated era, before $\Omega_d$ starts growing rapidly at scale factor $a_{tr}$:

$$\Omega_d^0 \equiv \left[\ln a_{tr} - \ln a_{eq}\right]^{-1} \int_{\ln a_{eq}}^{\ln a_{tr}} \Omega_d(a) \, d\ln a.$$  

(4)

If $Q$ is a model without early quintessence, $\Omega_d^0$ is zero. The effective equation of state $\bar{w}$ for $\bar{a}_d$ during the time in which $\Omega_d$ is growing rapidly:

$$\frac{1}{\bar{w}} \equiv \frac{1}{\int_{\ln a_{eq}}^{\ln a_{tr}} \Omega_d(a) \, d\ln a}.$$  

(5)

In many cases, $w_d^0$ can be used as an approximation to $\bar{w}$ since the integrals are dominated by periods with large $\Omega_d$. The scale factor at matter-radiation equality is

$$a_{eq} \equiv \frac{\Omega_\Lambda^0}{\Omega_m^0} = \frac{4.31 \times 10^{-5}}{h^2(1 - \Omega_\Lambda^0)}.$$  

(6)

Finally, $\tau_0$ is the conformal age of the universe. Equation (3) in combination with (3) can be used to make general statements about the consistency of quintessence models with $\sigma_8$-constraints.

### 1.3 Structure Formation

In linear approximation, the theory of structure formation describes the evolution of the energy density contrast $\delta$

$$\delta(x, a) \equiv \delta \rho_m(x, a) \rho_m(a) = \frac{\rho_m(x, a) - \rho_m(a)}{\rho_m(a)},$$  

(7)

and its Fourier transform

$$\delta_k(a) \equiv V^{-1} \int V \delta(x, a) \exp(-ik \cdot x) \, d^3x.$$  

(8)

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1 and $h = 0.65$, $n = 1$, $\Omega_\Lambda h^2 = 0.021$, $\Omega_m^0 = 1$

We use here conventions where $0$ always denotes today’s value of a quantity and the subscript $m$, $r$ and $d$ denote matter, radiation and dark energy respectively.

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Figure 1: Time evolution of the growth exponent $f$ for mode $8h^{-1} \times k = 1.95$, shown as a function of $\log_{10}(a)$.

One observes horizon crossing around $\log_{10}(a) = -4$ and $f$ near unity in the matter dominated era with a drop due to the increase of dark energy in the present period. We indicate $a_{eq}$ and $a_{tr}$ as specified in the text as vertical lines. The curve is obtained for leaping kinetic term quintessence with $h = 0.65$, $\Omega_\Lambda h^2 = 0.02$, $\Omega_m^0 = 0.65$, $\Omega_d^0 = 0.045$.

Here, $V$ is the integration volume and $k$ and $x$ are the comoving wave vector and the comoving coordinate. The structure growth exponent $f$ is defined as

$$f(a) \equiv \frac{d \ln \delta_k(a)}{d \ln a},$$  

(9)

and is roughly $k$-independent for a wide range of $k$. One can use linearized General Relativity in the synchronous gauge to compute $f(a)$. For sub-horizon modes in SCDM models, one obtains $f \rightarrow 0$ in the radiation and $f = 1$ in the matter eras. 3 We define the growth factor $g$ of density perturbations between arbitrary $a_1 < a_2$ as the ratio $\delta_k(a_2)/\delta_k(a_1)$. With a suitably defined average growth exponent $f$, this is

$$g(a_1, a_2) \equiv \frac{\delta_k(a_2)}{\delta_k(a_1)} = \left(\frac{a_2}{a_1}\right)^f.$$  

(10)

The density contrast $\sigma_8$ is defined by

$$\sigma_8^2 \equiv \int_0^{\infty} \frac{dk}{2\pi^2} k^2 \delta_k^2(k) \left(\frac{3j_1(kr)}{kr}\right)^2$$  

(11)

with $r = 8h^{-1}\text{Mpc}$.

The modes with the highest weight from the $\sigma_8$ window function entered the horizon during the late radiation era. After horizon crossing $f$ decreases and starts to grow again around matter-radiation-equality (see Figure 1).

3 super-horizon modes grow as $f = 2$, $(f = 1)$ in the radiation (matter) era.
1.4 Evolution Equations

For simplicity, we set the reduced Planck mass \( M_P = (8\pi G)^{-1/2} \) to unity. The Friedmann equation for a flat universe is then

\[
3H^2 = \rho_m + \rho_r + \rho_d. \tag{12}
\]

From this and the appropriate scaling of the different energy densities due to the expansion of the Universe, we get

\[
\frac{d\ln H}{d\ln a} = -\frac{1}{2} [3 + 3\omega_d(a)\Omega_d(a) + \Omega_r(a)] \equiv -\frac{1}{2} \tilde{n}(a), \tag{13}
\]

where \( \tilde{n}(a) = 3, 4 \) for matter (radiation) domination. The sub-horizon growth of density perturbations is governed by

\[
\frac{df}{d\ln a} + f^2 + \left( 2 - \frac{1}{2} \tilde{n} \right) f - \frac{3}{2} \Omega_m = 0. \tag{14}
\]

If \( \tilde{n} \) and \( \Omega_m \) are constant (as e.g. in the Exponential Potential Model, see section 3.1 or in SCDM), \( f \) approaches the solution

\[
f = \frac{\tilde{n}}{4} - 1 + \frac{1}{2} \sqrt{\left( 2 - \frac{\tilde{n}}{2} \right)^2 + 6\Omega_m}. \tag{15}
\]

Time and conformal time are related to the scale factor by

\[
\frac{dt}{da} = \frac{d\tau}{d\ln a} = \frac{1}{aH}. \tag{16}
\]

Later on we will need an expression for \( \tau(a) = 1 \) \( \equiv \tau_0 \). Using the Friedmann equation we obtain

\[
\tau(a) = \int_0^a \frac{da'}{a'^2H(a')} = 2\sqrt{3} \sqrt{1 - \tilde{\Omega}_d(a)} \sqrt{\frac{\rho_0^{\tilde{\omega}_d a} + \rho_0^r}{\rho_0^{\tilde{\omega}_d}}} - \sqrt{\rho_0^r}. \tag{17}
\]

where

\[
\sqrt{1 - \tilde{\Omega}_d(a)} \equiv \int_0^a \frac{da'}{\sqrt{\rho_0^{\tilde{\omega}_d a'} + \rho_0^r}} / \int_0^a \frac{da'}{\sqrt{\rho_0^{\tilde{\omega}_d a'} + \rho_0^r}}. \tag{18}
\]

1.5 CMB Normalization

The CMB temperature anisotropies are related to the density perturbations at the time of decoupling. The dominant contribution on large scales is the Sachs-Wolfe \( \delta T/T \) effect:

\[
\frac{\delta T}{T}(a_{dec}) = \frac{\delta \rho}{3\rho}(a_{dec}). \tag{19}
\]

This (or indeed the refined method of \( [10] \)) is used to fix \( \delta_k(a_{dec}) \) on scales \( k < 0.01 \, h \, \text{Mpc}^{-1} \). The CMB is emitted from the surface of last scattering (SLS) which has the comoving distance \( d_{SLS} \approx \tau_0 - \tau_{dec} \) from the observer. The knowledge of this distance is necessary for the normalizing procedure because the measured angular CMB power spectrum has to be converted to a momentum power spectrum, and so the normalization depends on \( \tau_0 \) (see section 3).

2 The Influence of Dark Energy on \( \sigma_8 \)

In this section, we compare structure formation in universes with dark energy to that in SCDM and find five differences in the computation of the CMB-normalized \( \sigma_8 \)-value. The first four effects concern the growth of structure according to Equation (14), whereas the fifth affects the normalization of the matter power spectrum.

Equality shift: We have \( a_{eq} \propto (1 - \Omega_\Lambda^{1/4})^{-1} \). If \( \Omega_\Lambda \approx 0.6 \), then \( a_{eq} \) is larger than in SCDM by a factor of 2.5. Therefore \( f \) starts growing much later for the \( \sigma_8 \)-relevant modes, leading to a substantially lower \( \sigma_8 \)-value. This effect is the strongest for many dark energy models. It would be difficult to compute it analytically, because around equality too many physical processes play a role at the relevant scale (e.g. horizon entering, decoupling, damped oscillation of radiation fluctuations). We circumvent the difficulty of computing this effect analytically by comparing models with the same dark energy content today and therefore identical values of \( a_{eq} \). It is then sufficient to determine \( \sigma_8 \) numerically (by CMBFAST \( [23] \)) for one model of this class, e.g. LCDM.

Matter depletion: From Equation (14) we see that a decrease of \( \Omega_m \) leads to a decrease of \( f \). We will discuss this effect analytically in the next section.

Accelerated expansion: Also from Equation (14) we see that an accelerated expansion, i.e. a smaller value of \( \tilde{n} \), leads to a decrease of \( f \). The accelerated expansion typically affects only a recent epoch in the evolution of the universe, and hence the effect will be rather small.

Shift in horizon crossing: Due to the different expansion history, a mode \( k \) enters the horizon at a different scale factor than in SCDM. As the equality shift, this is difficult to calculate analytically. Once again, we partially evade this difficulty by comparing models with the same \( \Omega_\Lambda \). Numerically, we find the residual effect to be small compared to the other effects and hence we will neglect it.

Normalization shift: A universe with dark energy is typically about 30 to 60% older than a SCDM universe. This means that the distance \( \tau_0 - \tau_{dec} \) to the SLS is larger than in SCDM. Thus, the measured angular temperature correlations correspond to momentum space correlations with smaller \( k \): \( \langle \delta T \rangle_k = \langle \delta T \rangle_k(\text{SCDM}) \),

\[
\frac{k}{k'} = \frac{\tau_0 - \tau_{dec}}{\tau_0(\text{SCDM}) - \tau_{dec}(\text{SCDM})} \approx \frac{\tau_0}{\tau_0(\text{SCDM})}. \tag{20}
\]

From the Sachs-Wolfe effect, the CMB temperature perturbations are proportional to the density perturbations which on super-horizon scales are determined by the initial power spectrum

\[
P(k) = \frac{\delta_k^2}{k^3} = A_k k^n = A_k k^{n} \times (k/k')^n = A_{k0}(k')^n \tag{21}
\]
with spectral index $n$. Hence, we get for the ratio of perturbations

$$\frac{\delta_k(a_{dec})}{\delta_k(a_{dec}, \text{SCDM})} = \left(\frac{k'}{k}\right)^{n/2} \approx \left(\frac{\tau_0}{\tau_0(\text{SCDM})}\right)^{n/2}. \quad (22)$$

With $n \approx 1$ this accounts for the last factor in Equation (1).

### 3 Effective Models

In this section we show how generic dark energy models can effectively be described as an appropriate combination of quintessence with an exponential potential (for small $a$) and a dark energy model with constant equation of state $w_d$ (for large $a$). We start by investigating the two 'pure cases' separately.

#### 3.1 The Exponential Potential

Quintessence with an Exponential Potential (EP) $V(\phi) = e^{-\lambda \phi}$ (and standard kinetic term) has been investigated in [1, 4, 24, 8]. For $\lambda > 2$, the quintessence field is forced into an attractor solution with $\Omega_d = 4/\lambda^2$ during radiation domination and $\Omega_d = 3/\lambda^2$ during matter domination. So $\Omega_d$ is constant during structure formation. The expansion history of the universe (and hence $\tau$) is almost unchanged compared to SCDM, and during matter domination we have $w_d = w_m = 0$. Present observations on $\Omega_b^0$ and $\Omega_d^0$ suggest that this model needs to be modified for large $a$, at least around the present epoch $a \approx 1$. **Matter depletion** is the dominant effect on structure formation. The structure growth exponent (see [6])

$$f = -1 + \frac{\sqrt{25 - 24 \Omega_d}}{4} = 1 - \frac{3}{5} \Omega_d + O(\Omega_d^2), \quad (23)$$

is smaller than in SCDM and reduces $\sigma_8$ correspondingly. For small $\Omega_d$ this amounts to a change in $\sigma_8$ by a factor of

$$\frac{\sigma_8(\text{EP})}{\sigma_8(\text{SCDM})} \approx \left(\frac{1}{a_{eq}}\right)^{-3 \Omega_d / 5}. \quad (24)$$

The Equality shift lowers $\sigma_8$ even more. For the EP model, it follows that 10% Quintessence lowers $\sigma_8$ by about 50%. If a Quintessence model is different from the EP, but $\Omega_d$ is relatively small all the time and does not vary too fast, its effect on structure formation will be almost like an EP model. According to Equations (3), (23) we have to replace $f$ and therefore $\Omega_d$ in this case by its logarithmic mean value

$$\Omega_d^d \equiv - (\ln a_{eq})^{-1} \int_{\ln a_{eq}}^0 \Omega_d(a) d \ln a, \quad (25)$$

for structure formation. Thus, Equation (24) remains valid if we substitute $\Omega_d^d$ for $\Omega_d$.

#### 3.2 Dark Energy with Constant Negative Equation of State

Dark energy models with constant equation of state (CES) $w_d < 0$ have been investigated e.g. in [1, 20]. We wish to extend the analytical discussion by making some simplifications. The CES models have the property that the dark energy becomes important just in the present epoch. The matter depletion is not as important as in the EP case, because the decrease of $f$ just started recently. If $w$ is closer to zero, the matter depletion becomes stronger, because the dark energy component became important earlier in the past. Current data favors models with $\Omega_d^0 \in [0.5, 0.7]$, so the equality shift is very strong. The expansion history of the universe is changed giving rise to the accelerated expansion and normalization shift effects.

**Matter depletion and accelerated expansion:** Numerically we find that the approximation $f(a) \approx 1 - \frac{3}{5} \Omega_d(a)$ is still valid to about 5% in CES models, even when $\Omega_d$ is as large as 0.6. As $\Omega_d(a)/\Omega_m(a) \propto a^{-3w}$, there is no dark energy contribution at early times when the radiation component is significant. Conversely, radiation is negligible when dark energy contributes and hence

$$\Omega_d(a) = \frac{\Omega_d^0}{(1 - \Omega_d^0)(1 - 3w) + \Omega_d^0}. \quad (26)$$

Now, to quantify the difference in structure growth compared to SCDM we fix an $a_{tr}$ which lies in the matter era in both the SCDM and the CES model and at which dark energy contribution is small. The appropriate averaged growth exponent from $a_{tr}$ to today is then given by

$$\bar{f} = - \left[\ln (a_{tr})\right]^{-1} \int_0^{a_{tr}} \left[1 - \frac{3}{5} \Omega_d(a)\right] d \ln a \approx 1 + \frac{3}{5} \int_0^1 \Omega_d \frac{d a}{a} / \ln (a_{tr}) \approx 1 + \frac{1}{5w} \ln (1 - \Omega_d^0) / \ln (a_{tr}). \quad (27)$$

Using Equation (3) we find that the change in the growth factor

$$g(a_{tr}, a = 1; \text{CES}) / g(a_{tr}, a = 1; \text{SCDM}) = a_{tr}^{-1} (1 - \Omega_d^0)^{-1/(5w)} \quad (28)$$

is independent of $a_{tr}$.

**Normalization shift:** According to Equation (22), we must compute the conformal age of the universe $\tau_0 = \tau(a = 1)$. Neglecting the radiation density, we obtain from Equations (13) and (22)

$$\tau_0 = \frac{2 \sqrt{3}}{\sqrt{\rho_0}} \left(\frac{1}{2} \frac{1 - \frac{1}{6w}}{1 - \frac{1}{6w} - \frac{\Omega_d^0}{1 - \Omega_d^0}}\right), \quad (29)$$

where $F$ is the hypergeometric function $2F_1$. We would now like to compare two CES models $A$ and $B$ which have the same $\Omega_d^0$ but different $w$. Because $\Omega_d^0$ is the same, the equality shift is the same in both cases and cancels out. From the other effects we get (cf. Equations (23), (26))

$$\frac{\sigma_8(A)}{\sigma_8(B)} \approx \left(1 - \Omega_d^0(w_B^{-1} - w_A^{-1})/5\right) \sqrt{\frac{F(w_A)}{F(w_B)}}. \quad (30)$$

For realistic values of $\Omega_d^0$ and $w$, this approximation is precise to about 5%.
We next consider models where \( w_d \) is time-dependent but always negative. If \( w \) does not vary rapidly, the difference does not become relevant as long as \( \Omega_d \) is substantially larger than zero, and we can take today’s value \( w_d(a = 1) \) as an approximation and consider the model as an CES model. If \( w \) varies rapidly, we can instead use an average value of \( w_d \) defined via
\[
\frac{1}{\bar{w}} = \frac{\int_{\ln\bar{a}}^{0} \Omega_d(a)/w(a) \, d\ln a}{\int_{\ln\bar{a}}^{0} \Omega_d(a) \, d\ln a}.
\]

### 3.3 General Models

We will now consider models with negative \( w \) today but non-negligible quintessence in the early universe, i.e. those with \( w_d(a) \geq 0 \) for small \( a \). We call them models with early quintessence (EQ). Such models are particularly interesting because they combine the naturalness properties of EP models with the realistic late cosmology of CES models. The difference between EQ and the CES-like models is relevant for structure formation only in the case that \( w_d(a) \geq 0 \) in an \( a \)-interval after equality (unlike in k-essence where this is only the case in the radiation era) where \( \Omega_d(a) \) is non-negligible. For the phenomenology of structure formation, we will describe these models as a combination of EP- and CES-models. For any early quintessence model EQ, we pick a certain scale factor \( a_{tr} \) at which the dark energy’s equation of state falls below \(-0.25 \) (although the precise value is not essential to our results). For the effects related to the growth factor (matter depletion and accelerated expansion), we consider the periods before and after \( a_{tr} \) separately and multiply the growth factors arising from both epochs. The time history of the growth exponent \( f(a) \) for a typical model in this class is shown in figure [1].

For \( a < a_{tr} \), we consider this EQ model as an effective exponential model EP with
\[
\hat{\Omega}_d^E(EP) = \frac{\int_{\ln a_{eq}}^{\ln a_{tr}} \Omega_d(a) \, d\ln a}{\ln a_{tr} - \ln a_{eq}},
\]
while for \( a > a_{tr} \), we treat it as an effective CES model
\[
\frac{1}{\bar{w}(CES)} = \frac{\int_{\ln a_{eq}}^{0} \Omega_d(a)/w(a) \, d\ln a}{\int_{\ln a_{eq}}^{0} \Omega_d(a) \, d\ln a}.
\]

The effective total growth factor is obtained by multiplying the expressions (32) and (33) with appropriate modification of (2), replacing \( 1/a_{eq} \) by \( a_{tr}/a_{eq} \) and \( \Omega_d \) by \( \Omega_d^E \). In general, \( a_{tr} \) will be relatively close to unity if \( \Omega_d^E \) is non-negligible.

We are now in the position to derive Equation (8) by combining the results for EP and CES models of the previous sections. For two general models A and B with the same \( \Omega_d^A \) (and hence \( a_{eq} \)) but different histories i.e. different \( w \) and \( \Omega_d^B \), we find
\[
\frac{\sigma_8(A)}{\sigma_8(B)} \approx \left( \frac{a_{tr}}{a_{eq}} \right)^{-3(\Omega_d^A(a_{tr})-\Omega_d^B(a_{tr}))/5} \left(1-\Omega_d^B(\bar{a}) \right)^{(\bar{w}-\bar{w}_A)/5} \times \sqrt{\frac{\tau_0(A)}{\tau_0(B)}}.
\]

Inserting the values relevant for the cosmological constant model, \( \Omega_d^A = 0, \bar{w}(\Lambda) = -1 \) and replacing \( a_{tr} \) with \( a_0 = 1 \) in (8) (thus neglecting the cutoff of the EP part), we obtain Equation (8).

The usefulness of this Equation (which is precise to about 5%) lies in the fact that we need to numerically compute the \( \sigma_8 \)-value of only one model - e.g. \( \Lambda \)CDM - for a given \( \Omega_d^B \). From this, \( \sigma_8 \) of all other models with the same \( \Omega_d^A \) can be estimated from their background solution only. We see that \( \sigma_8 \) depends on the three quantities \( \Omega_d^A, \Omega_d^B \) and \( \bar{w}_A \). Here, the dark energy today, \( \Omega_d^B \), mainly contributes through the equality shift and a small amount through the normalization shift. The amount of dark energy during structure formation, \( \Omega_d^B \) enters through matter depletion in the EP-part of the model, and the late-time equation of state \( \bar{w} \) through matter depletion and normalization shift in the CES-part of the model. Hence \( \sigma_8 \) is a very promising quantity for constraining these quantities.

### 4 Dependence of \( \sigma_8 \) on other Parameters

The density contrast \( \sigma_8 \) can be a sensitive indicator of the detailed properties of dark energy once the other cosmological parameters are known. For the present, our ability to constrain dark energy models using \( \sigma_8 \) is limited by the imperfect knowledge of these parameters. We observe a certain degeneracy arising in particular from \( h \) and the spectral index \( n \).

The Hubble parameter \( h \) appears in the denominator of Equation (8) and hence strongly affects the equality shift. A higher value of \( h \) leads to a smaller \( \sigma_8 \) and so to a higher value of \( \sigma_8 \). On the other hand a higher value of \( h \) gives a smaller \( \tau_0 \), hence a smaller CMB normalization. The first effect is much stronger than the second one.

The spectral index \( n \) appears in the CMB normalization procedure, when the measured large scale anisotropies (\( k \approx 10^{-2}h \text{ Mpc}^{-1} \)) are extrapolated towards larger \( k \) (\( \approx 1h \text{ Mpc}^{-1} \)). As \( \delta_k \propto k^{n/2} \), the effect on \( \sigma_8 \) is
\[
\frac{\sigma_8(n)}{\sigma_8(n = 1)} \approx 10^{n-1}.
\]

We see that a higher value of \( n \) leads to a higher \( \sigma_8 \). We emphasize that a value \( n = 1 \) is not a prediction of all models of inflation. In particular, a value of \( n \approx 1.15 \) has been suggested in [2] for a natural explanation of the smallness of density fluctuations by the long duration of inflation. In this proposal, the density fluctuations on very small scales (that have left the horizon just at the end of inflation) are of order unity. The smallness of the inhomogeneities on galactic or even larger scales is then explained by the slope in the spectrum and the ‘long lever arm’ once the corresponding scales left the horizon more than 50 e-foldings before the end of inflation. A spectral index \( n > 1 \) typically arises if inflation happens not too far from the Planck scale where effective couplings to higher order curvature terms are still relevant. Such a scenario can typically be found in dimensional models of inflation [3].
5 The Maximum of the Power Spectrum

As well as $\sigma_b$, the presence of dark energy influences also other features of the power spectrum, such as the location of its maximum or the slope of decrease towards large $k$. In principle, appropriate quantities related to these features could also be used to detect quintessence. At the moment it is not possible to locate the position of the maximum of the power spectrum, which we denote here as $k_{\text{max}}$. The spectrum is too flat over a wide region, and the error bars are too large, however observational data is improving.

The $k$-value at which the power spectrum peaks, $k_{\text{max}}$, is a very good indicator for early quintessence but is rather insensitive to the recent history of $a$ (in contrast to $\sigma_b$ which is sensitive to both). In the following, we will assume that $\Omega_k$ is almost constant during the early matter era, and we will identify the $\Omega_k$ at that time with $\Omega_k^0$. We define here $k_0$ as the wave number of the mode which enters the horizon just at matter-radiation equality. From Equations (1) and (17) we get

$$k_0 = \frac{2\pi}{\tau_{eq}} = 0.165 \text{ Mpc}^{-1} \times \left( \frac{h}{0.65} \right)^2 \frac{1 - \Omega_d^0}{\sqrt{1 - \Omega_d(\tau_{eq})}},$$

(36)

where $\tau_{eq} = \Omega_d(\tau_{eq})$ is given by Equation (18). We find that $k_{\text{max}}$ is smaller than $k_0$ by a factor of more than four, hence it enters the horizon during the early matter era. We define $\kappa$, the ratio of $k_0$ to $k_{\text{max}}$

$$\kappa \equiv \frac{k_0}{k_{\text{max}}}$$

(37)

The slope of the power spectrum is roughly given by

$$\frac{d \ln P(k)}{d \ln k} \approx n - 4 \left( 1 - \frac{f_{\text{sub}}(\Omega_{\text{hor}})}{f_{\text{sup}}(\Omega_{\text{hor}})} \right),$$

(38)

where $f_{\text{sub}}$, is the growth exponent on sub-horizon scales, $f_{\text{sup}}$ the one on super-horizon scales, and $\tau_{eq}$ is the scale factor when the mode $k$ enters the horizon. $f_{\text{sub}}$ and $f_{\text{sup}}$ depend only on the relative energy densities of the $m$, $r$ and $d$ components, but $\rho_r/\rho_m$ is always at the same time $a/a_{\text{eq}}$. We conclude that $\kappa$ is only sensitive to $\Omega_d$ and the spectral index $n$. We find that there is also a slight dependence on the other parameters $h$, $\Omega_d^0$ and $\Omega_b$. This is due to the baryons, which partially suppress $f_{\text{sub}}$ as long as they are coupled to the photons. Hence the ratio $a_{\text{eq}}/a_{\text{dec}}$ affects $\kappa$. One can easily find fitting formulas for $\kappa$. For instance, models with $\Omega_d^0 \in [0.6, 0.7]$, $h = 0.65$, $n = 1$ and $\Omega_b h^2 = 0.02$ are well described by

$$\ln \kappa = 1.57 + 2.70 \Omega_d^0.$$  

(39)

Approximating $\Omega_d(\tau_{eq}) \approx \Omega_d^0$ we find a strong dependence of the maximum of the power spectrum on early quintessence as for the above parameters

$$k_{\text{max}} \propto \left( 1 - \Omega_d^0 \right)^{-1/2} \exp(-2.7 \Omega_d^0) \approx (1 - 2.2 \Omega_d^0).$$

(40)

6 Specific Models

As a typical model with early quintessence we choose the Leaping Kinetic Term (LKT) model from [17]. It has the Lagrangian

$$L = \frac{1}{2} k^2 (\phi) \partial_{\mu} \phi \partial^{\mu} \phi + e^{-\phi},$$

(41)

with $k_0$ directly related to $\Omega_d^{sf}$ by $\Omega_d^{sf} = 3k_0^2$. Figure 2 shows the $\Omega_d^{sf}$ dependence of the CMB normalized $\sigma_b$ in this model obtained by a numerical solution using a modified cmbfast code. We find good agreement in comparison with the analytic estimate Equation (3), the agreement is excellent. We observe a strong dependence of $\sigma_b$ on the amount of dark energy during structure formation, $\Omega_d^{sf}$. For the parameters used in figure 2 ($h = 0.7$, $n = 1$, $\Omega_b h^2 = 0.02$, $\Omega_d^{sf} = 0.6$), a non-zero amount of early quintessence $\Omega_d^{sf} \approx 0.05 \pm 0.04$ would be favored (see also figure 3).

We have also used the LKT model to explore the constraints on quintessence [18]. We emphasize that according to our analytic discussion the use of specific LKT models is not a restriction of generality. Other quintessence models with the same values of $\Omega_d^{sf}$, $\Omega_d^{sf}$ and $\bar{w}$ will lead to the same results. In Figure 4 we have plotted the allowed range for early time quintessence $\Omega_d^{sf}$ and the spectral index $n$ for different choices of $\Omega_d^{sf}$ and $h$. A spectral index $n$ near unity favors the absence of early quintessence if $\Omega_d^{sf}$ is large and $\Omega_d^{sf}$ is small.

4 Another example of an early quintessence model has been proposed in [19].
$h$ is small. On the contrary, for small $\Omega_d^0$ and large $h$, a few percent quintessence during structure formation are favored. This holds, in particular if the spectral index is somewhat larger than one, e.g. $n \approx 1.15$ [26].

An overall bound for early quintessence can be obtained from Equation (3) if we assume $h < 0.75$, $\Omega_d^0 > 0.5$ and $n < 1.2$. One finds

$$\Omega_d^0 \lesssim 0.2 \quad (43)$$

This is of the same order as the bound from big bang nucleosynthesis, $\Omega_d^{bbs} < 0.2$ [24,25]. This bound can be substantially improved by more precise determinations of $h$, $\Omega_d^0$ and $n$.

7 Summary

We have analyzed the effects of dark energy on structure formation. We found that $\sigma_8$ - and possibly $k_{max}$ - are extremely promising indicators for constraining the present amount of dark energy $\Omega_d^0$, the present equation of state $w_d$ and especially the amount of early quintessence $\Omega_d^q$. The CMB-normalized value of $\sigma_8$ depends on all cosmological parameters. As a rough guide for the strength of these dependencies around standard values $\Omega_d^0 = 0.65$, $h = 0.65$, $n = 1$, $\Omega_b h^2 = 0.02$ with $-1 < \bar{w} < -0.5$ we get

- Increasing $h$ by 0.1 ⇒ Increase of $\sigma_8$ by 20% 
- Increasing $\Omega_d^0$ by 0.1 ⇒ Decrease of $\sigma_8$ by 20% 
- Increasing $n$ by 0.1 ⇒ Increase of $\sigma_8$ by 25% 
- Increasing $\bar{w}$ by 0.1 ⇒ Decrease of $\sigma_8$ by 5-10% 
- Increasing $\Omega_b h^2$ by 0.01 ⇒ Decrease of $\sigma_8$ by 10% 
- Increasing $\Omega_d^q$ by 0.1 ⇒ Decrease of $\sigma_8$ by 50%

Comparing with observation, the dependencies listed can be used for a quick check of viability for a given quintessence model and parameter set. If $\Omega_d^0$ is increased by 0.1, cluster abundances according to Equation 3 yield an approx. 20% higher value of $\sigma_8^{\text{cluster}}$. In combination with the corresponding decrease of $\sigma_8^{\text{cmb}}$, the net effect on the ratio $\sigma_8^{\text{cmb}}/\sigma_8^{\text{cluster}}$ is therefore a decrease by 33%. For a $\Lambda$CDM universe with standard values as above one has $\sigma_8^{\text{cmb}} = 0.90$ and $\sigma_8^{\text{cmb}}/\sigma_8^{\text{cluster}} = 1.01 \pm 0.2$. Compatibility of the cosmological scenario requires this ratio to be close to unity.

Once a subset of cosmological parameters such as $h$, $\Omega_d^0$, $\Omega_b h^2$ and $n$ are accurately determined by other measurements e.g. CMB anisotropies, the quantitative understanding of structure formation may become a central ingredient for the distinction between various forms of dark energy. In particular, by distinguishing quintessence from a cosmological constant it could serve as an indicator for a new field, the cosmos, mediating a new force with similar strength as gravity. If this field does not couple to ordinary matter, cosmology will be the only way for proving or disproving its existence.

Acknowledgments

We would like to thank Gert Aarts, Matthew Lilley and Eduard Thommes for helpful discussions.

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Figure 3: Allowed range of early quintessence and spectral index $n$ for various values of the Hubble parameter $h$ and present dark energy $\Omega_d^0$.

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