New Cosmological Signatures from Double Field Theory

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Abstract

In cosmology, it has been a long-standing problem to establish a parameter insensitive evolution from an anisotropic phase to an isotropic phase. On the other hand, it is of great importance to construct a theory having extra dimensions as its intrinsic ingredients. We show that these two problems are closely related and can naturally be solved simultaneously in double field theory cosmology. Our derivations are based on general arguments without any fine-tuning parameters. In addition, We find that when we begin with FRW metric, the full spacetime metric of DFT totally agrees with Kaluza-Klein theory. There is a visible and invisible dimension exchange between the pre- and post-big bangs. Our results indicate that double field theory has profound physical consequences and the continuous $O(D, D)$ is a very fundamental symmetry. This observation reinforces the viewpoint that symmetries dictate physics.
1 Introduction

Double field theory (DFT) \cite{1-4} is a renewed formalism of closed string field theory. It doubles all spacetime coordinates in order to make T-duality manifest on the level of component fields. The usual coordinates $x^i$, conjugation of spacetime momenta and its dual coordinates $\tilde{x}_i$, conjugation of "winding" numbers are treated on the same footing in this theory. Then the fundamental coordinates are combined by the $O(D,D)$ index $X^M = (\tilde{x}_i, x^i)$, where $M = 1, 2, \ldots, 2D$ and $i = 1, 2, \ldots, D$. All component fields of closed string depend on doubled coordinates $\phi(\tilde{x}_i, x^i)$. Many progresses have been achieved based on DFT in recent years \cite{5 - 59}. In these works, Ref. \cite{10, 11, 30, 34, 40, 42} discussed the geometrical properties of DFT. Ref. \cite{20, 22, 24, 31, 33, 36, 47} proposed scenarios of relaxing the strong or weak constraints. Ref. \cite{21, 32, 56} are devoted to applications of DFT on cosmology. Good reviews of DFT are referred to \cite{5, 8}.
In string cosmology [60–64], four types of solutions are constructed by the scale-factor duality and the time-reversal transformation. These solutions represent contracting or expanding universes, respectively. Moreover, there are pre-big bang solutions which can not be constructed in the standard cosmology. It therefore states that the universe is not born from a singularity, but there exists a pre-big bang phase. This is the main achievement of string cosmology in contrast to the standard cosmology. However, string cosmology suffers cosmic amnesia: The pre-big bang leaves no footprints in our post-big bang universe. Furthermore, the four solutions can be grouped arbitrarily in principle. One rescuing way is to introduce dilaton potentials to the theory. Though the cosmic amnesia and solution grouping problem can be alleviated, the universe has to be expanding or contracting all along the evolution, from the pre- to post-big bangs. For both experimental and theoretical reasons, it is widely accepted that the universe is anisotropic in the early stage. As a long-standing problem, string cosmology gives some explanations at the cost of free parameters and elaborate setups [65, 66].

In DFT cosmology, however, thanks to the intrinsic $O(D,D)$ symmetry, the pre- and post-big bangs, contracting and expanding universes are naturally unified in a single line element to cover the whole spacetime [56]. With cross terms between two sets of coordinates $dxd\tilde{x}$, discussions on relations between two originally disconnected scenarios become possible. The purpose of this paper is to calculate solutions of DFT cosmology on the presence of all massless fields, and demonstrate some remarkable cosmological implications and novel features. We start with a constant Kalb-Ramond field

$$b_{ij} = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ -c_1 & 0 & b_1 & b_2 \\ -c_2 & -b_1 & 0 & b_3 \\ -c_3 & -b_2 & -b_3 & 0 \end{pmatrix}, \quad (1.1)$$

and an isotropic FRW like metric

$$g_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(\tilde{t}, t) & 0 & 0 \\ 0 & 0 & a^2(\tilde{t}, t) & 0 \\ 0 & 0 & 0 & a^2(\tilde{t}, t) \end{pmatrix}. \quad (1.2)$$

After substituting this metric into the equations of motion (EOM) of the DFT action and taking the synchronous gauge: $g_{tt} = -1, g_{t\mu} = b_{t\mu} = 0$, we get two metrics, the pre-big bang metric $dS^2_{\text{pre}}$ for $t < 0$ and the post-big bang metric $dS^2_{\text{post}}$ for $t > 0$. Each of these metrics has two solutions, representing visibly isotropic or anisotropic evolutions respectively. The two metrics are disjointed by the big bang singularity, which implies that there is no interaction between the pre- and post-big bangs, contracting and expanding universes. Therefore, there in principle are four choices of evolutions. However, it turns out that only one evolution reflects the evolution of our universe from simple physical consideration. In this physical evolution, the universe is explicitly isotropic only in the far future post-big bang region.

To explore the physics clearer, we further make use of the DFT action with an additional $O(D,D)$ invariant loop correction dilaton potential. We assume the dilaton potential takes a form $e^d$, which represents higher loop corrections. This potential is defined to include a proper volume without breaking the generalized diffeomorphisms. Using the $O(D,D)$ symmetry, we can obtain the solutions directly.
Since the big bang singularity is smoothed out by the potential, the pre-big bang metric \(ds_{\text{pre}}^2\) and the post-big bang metric \(ds_{\text{post}}^2\) are unified in a single object covering the full spacetime backgrounds. It turns out the primary features do not rely on the potential. Furthermore, after diagonalization, the metric can be rearranged as Kaluza-Klein ansatz. The dual coordinates become extra dimensions. If we compact these dimensions, the results totally agree with Kaluza-Klein compactification and the anti-symmetric Kalb-Ramond field decomposes into Kaluza-Klein vectors. Under the \(O(D,D)\) symmetry, some dimensions will expand to visible and the others will contract to invisible. These features are hidden in the traditional cosmology since there are no double coordinates. Moreover, the existence of Kaluza-Klein extra dimensions in this theory is intrinsic but not artificial.

The remainder of this paper is outlined as follows. In section 2, we present the EOM of DFT. In section 3, we clarify the \(O(D,D)\) symmetry and its applications in the traditional string cosmology and DFT. In section 4, we put forward cosmological solutions of DFT and discuss their implications. Section 5 is our conclusion and discussions. In addition, appendix is devoted to detailed calculations to get the DFT solutions with vanishing potential.

2 Equation of motion of double field theory

The spacetime action of DFT is built on the generalized \(O(D,D)\) metric

\[
\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}b^j_k \\ b^k_i g^{kj} & g_{ij} - b_{ik}b^j_l b_{lj} \end{pmatrix},
\tag{2.1}
\]

which unifies the original spacetime metric \(g_{ij}\) and the anti-symmetric Kalb-Ramond field \(b_{ij}\). The \(O(D,D)\) indices \(M\) and \(N\) are raised and lowered by

\[
\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.
\tag{2.2}
\]

The \(O(D,D)\) invariant action is based on the generalized metric

\[
S = \int d^dx d\tilde{x} e^{-2d} \left( \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_N \mathcal{H}^{KL} \partial_L \mathcal{H}_{MK} \\
- \partial_M d \partial_N \mathcal{H}^{MN} + 4\mathcal{H}^{MN} \partial_M d \partial_N d \right),
\tag{2.3}
\]

where the dilaton \(d\) is an \(O(D,D)\) scalar. The level matching condition in string theory leads to a constraint on the fields and gauge parameters \(\partial_t \tilde{\phi}(t, \tilde{t}) = 0\). The EOM of the dilaton is

\[
\frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_K \mathcal{H}_{NL} - \partial_M \partial_N \mathcal{H}^{MN}
- 4\mathcal{H}^{MN} \partial_M d \partial_N d + 4\mathcal{H}^{MN} \partial_M d \partial_N d + 4\mathcal{H}^{MN} \partial_M d \partial_N d = 0,
\tag{2.4}
\]

with \(\partial_M = \left( \tilde{\phi}, \partial_{\tilde{t}} \right)\). The EOM of the generalized metric is
\[ \mathcal{R}_{MN} \equiv K_{MN} - S^P M K_{PQ} S^Q_{\ N} = 0, \quad (2.5) \]

where

\[
K_{MN} = \frac{1}{8} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{4} \left( \partial_L - 2 \partial_L d \right) \left( \mathcal{H}^{LK} \partial_K \mathcal{H}_{MN} + 2 \partial_M \partial_N d \right) - \frac{1}{2} \partial (N \mathcal{H}^{KL} \partial_L \mathcal{H}_M)_{KL} + \frac{1}{2} \left( \partial_L - 2 \partial_L d \right) \left( \mathcal{H}^{KL} \partial (N \mathcal{H}_M)_{KL} + \mathcal{H}^K (M \partial_K \mathcal{H}^L)_{N} \right). \quad (2.6)
\]

The most natural isotropic extension of FRW metric in DFT cosmology is

\[ dS^2 = g_{ij} d\tilde{x}^i d\tilde{x}^j - g_{ik} b_{kj} d\tilde{x}^i dx^j + b_{ik} g^{kj} dx^i d\tilde{x}^j + \left( g_{ij} - b_{ik} g^{kl} b_{lj} \right) dx^i dx^j, \quad (2.7) \]

where the spacetime metric \( g_{ij} \) is isotropic, as defined in eqn. (1.2). In this paper, we set \( D = 4 \) and choose a constant Kalb-Ramond field as in eqn. (1.1).

### 3 \( O(D,D) \) Transformations

Before performing calculations, it is of importance to clarify some points about the symmetries. To our current understanding, the continuous \( O(D,D) \) in DFT is a very fundamental symmetry. Compactification of \( d = D - n \) dimensions breaks the continuous \( O(D,D) \) into an \( O(n,n) \times O(d,d) ; \mathbb{Z} \) group. The \( O(d,d) \) represents T-duality in the compactified background. Also due to the \( O(D,D) \), any background dependent on \( \tilde{x} \) alone is also a solution of \( x \) alone. This is the reason why the string low energy effective action and supergravity has the continuous \( O(D,D) \) symmetry \([67–71]\). For this reason, the scale factor duality in string cosmology is not T-duality but a realization of \( O(D,D) \) symmetry in a specific set of backgrounds. As a bonus, this interpretation provides a simple way to construct solutions of DFT: two \( O(D,D) \) paired solutions of the low energy effective action comprise a single DFT solution which respects the constraint. Moreover, in the previous works \([61, 67, 72–76]\), the various solutions can be achieved by \( O(D,D) \) transformations. There are three types of generators of \( O(D,D) \) transformations. The first one belongs to \( O(D) \times O(D) \) transformation, which is a subgroup of \( O(D,D) \). When we act the generators of \( O(D) \times O(D) \) group on solutions, we get inequivalent new solutions. The \( O(D) \times O(D) \) subgroup can be described by the following matrix

\[
\Omega = \frac{1}{2} \begin{pmatrix} S + R & R - S \\ R - S & S + R \end{pmatrix}, \quad (3.1)
\]

which satisfies \( \Omega \eta \Omega^T = \eta \) and \( S, R \) are \( O(D) \) matrices. It is a maximal compact subgroup and includes factorized duality \([77] \). The new solutions can be obtained by

\[ M \rightarrow \tilde{M} = \Omega^T M \Omega. \quad (3.2) \]

The second subgroup generates coordinate transformations and a shift of dilaton. The matrix is
\[ \Omega = \begin{pmatrix} (A^T)^{-1} & 0 \\ 0 & A \end{pmatrix}, \quad A \in GL(D), \]  
(3.3)

where the constant matrix \( A \) defines coordinate transformation \( x^i = A^j_i x^j \). When \( A^T A = 1 \), it is included in the \( O(D) \times O(D) \) transformation. The last set of generators comes from gauge transformations, defined by the matrix

\[ \Omega = \begin{pmatrix} 1 & 0 \\ \Theta & 1 \end{pmatrix}, \]  
(3.4)

where elements of the anti-symmetric matrix \( \Theta \) are constant parameters of gauge transformations \( b_{ij} \rightarrow b_{ij} + \Theta_{ij} \). If the spacetime is compact, the elements of these three generators should be integer numbers. Moreover, the last two sets of generators, which lie outside the \( O(D) \times O(D) \) group, do not generate new inequivalent solutions \(^{[72]}\) in the traditional theory.

### 3.1 \( O(d, d) \) symmetry in string cosmology

In the traditional string cosmology, coordinates are not doubled and symmetry can be manifested in spatial parts of the metric. This is why we only consider an \( O(d, d) \) symmetry, but not an \( O(D, D) \) symmetry in the traditional theory, where \( d = D - 1 \). In the traditional spatial translation invariant string cosmology, there are two ways to manifest \( O(d, d) \) symmetry. The first method is to find the equations of motion of low energy effective action and the EOM is invariant under \( O(d, d) \) transformation. Another way is to rewrite the action to make \( O(d, d) \) symmetry manifest. In this work, we adopt the second way, since it is easy to figure out a relation between string cosmology and DFT. The first one was reviewed in our previous work \(^{[56]}\). We start by the low energy effective action

\[ S_* = \int d^D x \sqrt{-g} e^{-2\phi} \left[ R + 4 (\partial_{\mu} \phi)^2 - \frac{1}{12} H_{ijk} H^{ijk} \right], \]  
(3.5)

where \( g_{\mu\nu} \) is the string metric, \( \phi \) is the dilaton and \( H_{ijk} = 3 \partial_{[i} b_{jk]} \) is the field strength of the anti-symmetric Kalb-Ramond \( b_{ij} \) field. To consider the cosmological backgrounds, we choose synchronous gauge \( g_{tt} = -1, \ g_{ti} = b_{t\mu} = 0 \), and define the \( O(d, d) \) dilaton or shifted dilaton \( \tilde{d} \) as follows

\[ e^{-2\tilde{d}} = \sqrt{g} e^{-2\phi}. \]  
(3.6)

The low energy effective action can be rewritten as

\[ S = -\int dt e^{-2\tilde{d}} \left[ \frac{1}{8} \text{Tr} \left( \ddot{M} M^{-1} \right) + 4 d^2 \right], \]  
(3.7)

where

\[ M = \begin{pmatrix} G^{-1} & -G^{-1} B \\ BG^{-1} & G - BG^{-1} B \end{pmatrix}, \]  
(3.8)

\( G \) and \( B \) are spatial parts of \( g_{ij}(t) \) and \( b_{ij}(t) \). Therefore, \( M \) is a \( 2d \) by \( 2d \) matrix. Note that this action is invariant under the \( O(d, d) \) transformations

\[ d \rightarrow d, \quad M \rightarrow \tilde{M} = \Omega^T M \Omega, \]  
(3.9)

where \( \Omega \) is a constant matrix, satisfying

\[ \Omega^T \eta \Omega = \eta, \]  
(3.10)

\(^1\)Do not confuse the dilaton \( d \) and number of spatial dimensions \( d \)
and $\eta$ is an invariant metric of the $O(d,d)$ group
\[
\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.
\] (3.11)
If we choose $G_{ij} = \delta_{ij}a^2(t)$ and $B = 0$, the matrix (3.8) becomes
\[
M = \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix}.
\] (3.12)
After applying the $O(d) \times O(d)$ transformation (3.1), we have a new inequivalent solution
\[
\tilde{M} = \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix},
\] (3.13)
which implies that the action is invariant under $a(t) \rightarrow a^{-1}(t)$. It is the so-called scale-factor duality in the traditional string cosmology.

3.2 $O(D,D)$ symmetry in double field theory

Now, let us consider the simplification of DFT action in terms of the matrix (3.8). Recall the spacetime action of DFT
\[
S = \int d^Dx d^D\hat{x} \mathcal{L},
\] (3.14)
where
\[
\mathcal{L} = e^{-2d} \left[ \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_N \mathcal{H}^{KL} \partial_L \mathcal{H}_{MK} 
- \partial_M d \partial_N \mathcal{H}^{MN} + 4 \mathcal{H}^{MN} \partial_M d \partial_N d \right].
\] (3.15)
This action is $O(D,D)$ invariant since each of scalar terms is constructed by the contractions of $O(D,D)$ indices $M$ and $N$. It is a double formalism of the low energy effective action. To see this, we expand each term as follows
\[
\mathcal{L} = e^{-2d} \left[ \frac{1}{8} \left( g_{ij} - b_{ik}g^{kl}b_{lj} \right) \partial^i \mathcal{H}^{KL} \partial^j \mathcal{H}_{KL} - \partial^i \partial^j \left( g_{ij} - b_{ik}g^{kl}b_{lj} \right) + 4 \left( g_{ij} - b_{ik}g^{kl}b_{lj} \right) \partial^i \partial^j d 
+ \frac{1}{8} b_{ik}g^{kj} \partial^i \mathcal{H}^{KL} \partial_j \mathcal{H}_{KL} - \partial^i \partial^j \left( b_{ik}g^{kj} \right) + 4 \left( b_{ik}g^{kj} \right) \partial^i \partial^j d 
+ \frac{1}{8} \left( -g_{ik}b_{kj} \right) \partial_i \mathcal{H}^{KL} \partial^j \mathcal{H}_{KL} - \partial_i \partial^j \left( -g_{ik}b_{kj} \right) + 4 \left( -g_{ik}b_{kj} \right) \partial_i \partial^j d 
+ \frac{1}{8} g^{ij} \partial_i \mathcal{H}^{KL} \partial_j \mathcal{H}_{KL} - \partial_i \partial_j \left( g^{ij} \right) + 4 g^{ij} \partial_i \partial_j d 
\right]
- \frac{1}{2} \left( g_{ij} - b_{im}g^{mn}b_{nj} \right) \partial^i \left( g_{kl} - b_{ka}g^{ab}b_{ld} \right) \partial^j \left( g_{ik} \right) - \frac{1}{2} \left( g_{ij} - b_{im}g^{mn}b_{nj} \right) \partial^i \left( b_{ka}g^{al} \right) \partial_l \left( g^{ik} \right)
- \frac{1}{2} \left( g_{ij} - b_{im}g^{mn}b_{nj} \right) \partial^i \left( -g^{ka}b_{al} \right) \partial^j \left( -g^{ic}b_{ck} \right) - \frac{1}{2} \left( g_{ij} - b_{im}g^{mn}b_{nj} \right) \partial^i \left( b_{ka}g^{al} \right) \partial_l \left( -g^{ic}b_{ck} \right)
- \frac{1}{2} \left( b_{im}g^{mj} \partial_j \left( g_{kl} - b_{ka}g^{ab}b_{ld} \right) \partial^j \left( b_{ic}g^{ck} \right) - \frac{1}{2} b_{im}g^{mj} \partial_j \left( b_{ka}g^{al} \right) \partial_l \left( b_{ic}g^{ck} \right) \right]
- \frac{1}{2} \left( -g^{im}b_{mj} \right) \partial^i \left( g_{kl} - b_{ka}g^{ab}b_{ld} \right) \partial^j \left( b_{ic}g^{ck} \right) - \frac{1}{2} \left( -g^{im}b_{mj} \right) \partial^i \left( b_{ka}g^{al} \right) \partial_l \left( b_{ic}g^{ck} \right)
- \frac{1}{2} \left( -g^{im}b_{mj} \right) \partial^i \left( -g^{ka}b_{al} \right) \partial^j \left( -g^{ic}b_{ck} \right) - \frac{1}{2} \left( -g^{im}b_{mj} \right) \partial^i \left( b_{ka}g^{al} \right) \partial_l \left( -g^{ic}b_{ck} \right)
- \frac{1}{2} \left( -g^{im}b_{mj} \right) \partial^i \left( g_{kl} - b_{ka}g^{ab}b_{ld} \right) \partial^j \left( b_{ic}g^{ck} \right) - \frac{1}{2} \left( -g^{im}b_{mj} \right) \partial^i \left( b_{ka}g^{al} \right) \partial_l \left( b_{ic}g^{ck} \right)
\right]
\]
The calculation rules can be found in our previous work [56]. We assume all fields are double time dependent \( g(\tilde{t}, t) \) and also choose the synchronous gauge \( g_{tt} = -1, \ g_{ti} = b_{t\mu} = 0 \). The generalized Lagrangian can be simplified as

\[
\mathcal{L} = e^{-2\tilde{d}} \left[ -\frac{1}{8} \tilde{d}^t \mathcal{H}^{KL} \tilde{d}^t \mathcal{H}_{KL} - 4 \tilde{d}^t \dot{\tilde{d}}^t \dot{d} - \frac{1}{8} \partial_t \mathcal{H}^{KL} \partial_t \mathcal{H}_{KL} - 4 \partial_t \tilde{d} \partial_t d \right],
\]

which can further reduce to

\[
\mathcal{L} = e^{-2\tilde{d}} \left[ -\frac{1}{8} \tilde{d}^t M^{KL} \tilde{d}^t M_{KL} - 4 \tilde{d}^2 - \frac{1}{8} \partial_t M^{KL} \partial_t M_{KL} - 4 \tilde{d}^2 \right],
\]

with

\[
M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix},
\]

where \( \tilde{d} \equiv \frac{\partial \tilde{d}}{\partial \tilde{t}} \) and \( \dot{\tilde{d}} \equiv \frac{\partial \tilde{d}}{\partial t} \). The metrics \( G \) and \( B \) are dependent on two sets of coordinates, therefore \( G \) and \( B \) are spatial parts of \( g_{ij}(\tilde{t}, t) \) and \( b_{ij}(\tilde{t}, t) \). It is a difference to the traditional string cosmology. Furthermore, the first and third terms of Lagrangian (3.18) can be rewritten in a matrix representation

\[
S = -\int d^D x d^D \dot{x} e^{-2\tilde{d}} \left[ \frac{1}{8} \operatorname{Tr} \left( \dot{M} \dot{M}^{-1} \right) + 4 \dot{\tilde{d}}^2 + \frac{1}{8} \operatorname{Tr} \left( \dot{M} \dot{M}^{-1} \right) + 4 \tilde{d}^2 \right],
\]

where \( \dot{M}^{-1} \equiv \partial (M^{-1}) / \partial \tilde{t} \) and \( \dot{\tilde{d}} \equiv \partial (\tilde{d}^{-1}) / \partial t \). To compare with (3.7), it is obvious that this action is also invariant under \( O(d, d) \) transformations: \( M \rightarrow \Omega^T M \Omega \) and \( d \rightarrow d \). It is easy to see that solutions of the traditional string cosmology are also solutions of DFT.

4 Cosmological solutions

In this section, we find virous solutions of DFT though \( O(D, D) \) rotation. To respect the constraint, we adopt solutions dependent on only one set of coordinates, one time-like coordinate in particular in this paper. The term \( \int d\tilde{d} \) can be integrated out. Therefore, the metric only includes one time-like direction \( dt \).

4.1 Solutions of DFT \((V(d) = 0)\) with constant Kalb-Ramond field

There are two ways to approach solutions of DFT. The first method is to make use of the \( O(D, D) \) symmetry. This method is easy though, one can only obtain solutions respecting constraints. The second way is to solve the EOM. It is possible to get other solutions by the second method, especially the constraint violating solutions. The detailed calculations of the second method are put in the Appendix and we give some constraint violating solutions there.
Making use of the $O(D, D)$ symmetry, the solutions of DFT for the isotropic FRW like metric are

\[
a_\pm (t) = \left( \frac{t}{t_0} \right)^{\pm 1/\sqrt{D-1}} \,, \quad d(t) = -\frac{1}{2} \ln \frac{t}{t_0}, \quad t > 0, \\
a_\pm (-t) = \left( -\frac{t}{t_0} \right)^{\pm 1/\sqrt{D-1}} \,, \quad d(-t) = -\frac{1}{2} \ln \frac{-t}{t_0}, \quad t < 0, \tag{4.1}
\]

where $t_0$ is an initial time. In traditional string cosmology, the physical interpretations of solutions (4.1) are

| $a_\pm (t)$ | $\dot{a}(t) > 0$, expansion | $\dot{a}(t) < 0$, decelerated | $\dot{H} < 0$, decreasing curvature | post-big bang |
|------------|-----------------|-----------------|----------------|----------------|
| $a_- (t)$  | $\dot{a}(t) < 0$, contraction | $\dot{a}(t) > 0$, decelerated | $\dot{H} > 0$, decreasing curvature | post-big bang |
| $a_+ (-t)$ | $\dot{a}(t) < 0$, contraction | $\dot{a}(t) < 0$, accelerated | $\dot{H} < 0$, increasing curvature | pre-big bang |
| $a_- (-t)$ | $\dot{a}(t) > 0$, expansion | $\dot{a}(t) > 0$, accelerated | $\dot{H} > 0$, increasing curvature | pre-big bang |

Table 1. The properties of the solutions in tree level string cosmology.

To get the solutions with constant Kalb-Ramond field, we could use $O(d, d)$ transformations (3.4)

\[
\Omega = \begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix}, \tag{4.2}
\]

If we consider $D = 4$, $\Omega$ is 6 by 6 matrix and $B$ is an anti-symmetric constant matrix

\[
B = \begin{pmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & b_3 \\ -b_2 & -b_3 & 0 \end{pmatrix}. \tag{4.3}
\]

Therefore, the new solution is

\[
\ddot{M} = \Omega^T M \Omega = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \tag{4.4}
\]

where $G_{ij} = \delta_{ij} a^2_\pm (t)$. Due to the singularity at $t = 0$, the line element splits into two parts, one for the regime of $t < 0$ and the other for $t > 0$.

\[
dS^2_{pre\pm} = -dt^2 + a_\pm (-t)^{-2} (dx_3^2 + dx_3^2 + dx_4^2) + a_\pm (-t)^2 (dx_3^2 + dx_3^2 + dx_4^2) \\
-2a_\pm (-t)^{-2} b_1 d\bar{d}2 dx_3 - 2a_\pm (-t)^{-2} b_2 d\bar{d}2 dx_4 - 2a_\pm (-t)^{-2} b_3 d\bar{d}3 dx_4 \\
+2a_\pm (-t)^{-2} b_1 d\bar{d}3 dx_2 + 2a_\pm (-t)^{-2} b_2 d\bar{d}4 dx_2 + 2a_\pm (-t)^{-2} b_3 d\bar{d}4 dx_3 \\
+a_\pm (-t)^{-2} (b_1^2 + b_2^2) dx_2^2 + a_\pm (-t)^{-2} (b_1^2 + b_3^2) dx_3^2 + a_\pm (-t)^{-2} (b_2^2 + b_3^2) dx_4^2 \\
+2a_\pm (-t)^{-2} b_1 b_2 d\bar{d}4 dx_3 - 2a_\pm (-t)^{-2} b_1 b_3 d\bar{d}4 dx_2 + 2a_\pm (-t)^{-2} b_2 b_3 dx_3 dx_2, \quad t < 0 \tag{4.5}
\]
The strong constraint violated solution can be found in Appendix.

### 4.2 Solutions of DFT \((V(d) \neq 0)\) with constant Kalb-Ramond field

To understand the physics better, we start with the action

\[
S = \int dx^D d\hat{x}^D e^{-2d} [\mathcal{R} + V(d)],
\]

where \(V(d)\) is an \(O(D, D)\) invariant potential

\[
V(d) = V_0 e^{8d}.
\]

This non-local potential represents the backreaction of higher loop corrections and \(V_0 > 0\) [79]. It is worth noting that this dilaton has been redefined to include a proper volume which will reduce to \(d(t, \tilde{t})\) in cosmological background. It leads the dilaton potential above to be a scalar under generalized diffeomorphisms. Utilizing the \(O(D, D)\) symmetry, one can figure out a DFT cosmological solution without Kalb-Ramond field is

\[
a(t) = a_0 \left[ \frac{t}{t_0} + \left( 1 + \frac{t^2}{t_0^2} \right)^{\frac{1}{2}} \right]^3, \quad d(t) = -\frac{1}{4} \ln \left( \sqrt{V_0 t_0} \left( 1 + \frac{t^2}{t_0^2} \right) \right),
\]

where \(t_0\) and \(V_0\) are integral constants. It is of interest to note that there is one single solution of the scale factor. A single metric unifies the pre- and post- big bangs, expanding and contracting universes without any singularity. The metric is

\[
ds^2 = -dt^2 + a^{-2} (d\bar{x}_2^2 + d\bar{x}_3^2 + d\bar{x}_4^2) + a^2 (dx_2^2 + dx_3^2 + dx_4^2).
\]

To obtain the solutions with constant Kalb-Ramond field, we also choose the transformation matrix as

\[
\Omega = \begin{pmatrix}
1 & -B \\
0 & 1
\end{pmatrix}.
\]

After applying \(\tilde{M} = \Omega^T M \Omega\), the metric becomes

\[
dS^2 = -dt^2 + a(t)^{-2} (d\bar{x}_2^2 + d\bar{x}_3^2 + d\bar{x}_4^2) + a(t)^2 (dx_2^2 + dx_3^2 + dx_4^2)
\]
\[ -2a(t)^{-2}b_1d\tilde{x}_2dx_3 - 2a(t)^{-2}b_2d\tilde{x}_2dx_4 - 2a(t)^{-2}b_3d\tilde{x}_3dx_4 \\
+ 2a(t)^{-2}b_1d\tilde{x}_3dx_2 + 2a(t)^{-2}b_2d\tilde{x}_4dx_2 + 2a(t)^{-2}b_3d\tilde{x}_4dx_3 \\
+ a(t)^{-2}(b_1^2 + b_2^2)dx_3^2 + a(t)^{-2}(b_2^2 + b_3^2)dx_4^2 \\
+ 2a(t)^{-2}b_1b_2dx_3dx_4 - 2a(t)^{-2}b_1b_3dx_3dx_4 + 2a(t)^{-2}b_2b_3dx_3dx_4. \tag{4.12} \]

This metric can be rearranged as

\[
\begin{align*}
\frac{ds^2}{\lambda_0^2} &= -dt^2 + a^{-2}(d\tilde{x}_2 - b_1dx_3 - b_2dx_4)^2 + a^{-2}(d\tilde{x}_3 + b_1dx_2 - b_3dx_4)^2 \\
&+ a^{-2}(d\tilde{x}_4 + b_2dx_2 + b_3dx_3)^2 + a^2(dx_2^2 + dx_3^2 + dx_4^2), \tag{4.13}
\end{align*}
\]

One can see that coordinates $\tilde{x}$ have the same form as Kaluza-Klein extra dimensions. To make it clear, we use $\alpha, \beta = 5, 6, 7$ to denote the indices of extra dimensions and define $x^5 \equiv \tilde{x}^2$, $x^6 \equiv \tilde{x}^3$, $x^7 \equiv \tilde{x}^4$. Therefore, the metric can be recasted into the form

\[
\begin{align*}
\frac{ds^2}{\lambda_0^2} &= g_{\mu\nu}dx^\mu dx^\nu + g_{\alpha\beta}(dx^\alpha + A_\alpha^\mu dx^\mu)(dx^\beta + A_\beta^\mu dx^\mu), \tag{4.14}
\end{align*}
\]

with

\[
\begin{align*}
A_5^\mu &= (0, 0, -b_1, -b_2), \\
A_6^\mu &= (0, b_1, 0, -b_3), \\
A_7^\mu &= (0, b_2, b_3, 0), \\
g_{\mu\nu} &= \text{diag}(-1, a^2, a^2, a^2), \\
g_{\alpha\beta} &= \text{diag}(a^{-2}, a^{-2}, a^{-2}), \tag{4.15}
\end{align*}
\]

where $dx^\alpha$ are extra dimensions, all fields do not depend on $x^\alpha$, $g_{\alpha\beta}$ are radii of extra dimensions and $A_\mu^\alpha$ are Kaluza-Klein vectors. If we impose the periodic condition $x^\alpha \sim x^\alpha + 2\pi R$ and choose integer number $b_i$, this metric is in total agreement with Kaluza-Klein compactification.
The scale factors and relevant Hubble parameters are plotted in FIG. (1). It is easy to see that the visible dimensions contract and the extra dimensions expand in the pre-big bang. As time evolves, the visible dimensions will contract to invisible dimensions and extra dimensions become visible in the post-big bang. In other words, the universe evolves smoothly from an anisotropic phase to an isotropic phase, and again to anisotropy. This observation proposes important implications for future experiments. Once extra dimensions are detected, we can expect information of the pre-big bang could be extracted, which is of help for the cosmic amnesia puzzle. A more speculating application is that we can compare the size of the extra dimension with the scale of the universe. Our calculation indicates that one is the reciprocal of the other. It is reasonable that the real story is more complicated since our discussion in this work does not contain massive contributions and quantum effects. Nevertheless, we do have another way to justify string theory and the existence of the pre-big bang. Moreover, the whole process is even more parameter insensitive! Bear in mind we do not need to assume any anisotropy at the very beginning for the metric entries. Comparing with the metric of $b = 0$ (4.10), we find a duality between two distinct spacetime backgrounds.

\begin{align*}
  b = 0 & \quad b \neq 0 \\
  \tilde{x}_2 & \leftrightarrow \tilde{x}_2 - b_1 x_3 - b_2 x_4 \\
  \tilde{x}_3 & \leftrightarrow \tilde{x}_3 + b_1 x_2 - b_3 x_4 \\
  \tilde{x}_4 & \leftrightarrow \tilde{x}_4 + b_2 x_2 + b_3 x_3. \tag{4.16}
\end{align*}
does not generate new inequivalent solutions. For example, let us begin with 4–dimensional isotropic FRW metric in string frame, where
\[ ds^2 = -dt^2 + a(t)^2 (dx_2^2 + dx_3^2 + dx_4^2) . \] (4.17)
In the \( O(d,d) \) language, the matrix \( G \) is
\[ G = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix} , \quad B = 0 , \] (4.18)
After rotation \( M \rightarrow \tilde{M} = \Omega^T M \Omega \), the new solution is
\[ \tilde{G} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix} , \quad \tilde{B} = \begin{pmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & b_3 \\ -b_2 & -b_3 & 0 \end{pmatrix} . \] (4.19)
However, this new solution gives us the same spacetime background, since \( \tilde{G}_{ij} = G_{ij} = \delta_{ij} a^2(t) \). It means that the rotation gives no new inequivalent solution but is only a gauge transformation of \( B \), shifting the Kalb-Ramond field from 0 to \( \tilde{B}_{ij} \). Since \( \tilde{B} \) is a constant matrix, it is a pure gauge.

In DFT, since we double all coordinates, when we consider the spacetime background, we cannot consider the matrix \( G \) (representing one set of coordinates) only, but a full matrix \( M \). Now, we also start with same metric (4.17) as
\[ ds^2 = -dt^2 + a(t)^2 (dx_2^2 + dx_3^2 + dx_4^2) . \] (4.20)
The \( O(d,d) \) symmetry gives us the full spacetime metric
\[ ds^2 = -dt^2 + a^{-2} (dx_2^2 + dx_3^2 + dx_4^2) + a^2 (dx_2^2 + dx_3^2 + dx_4^2) . \] (4.21)
The relevant matrix \( M \) is
\[ M = \begin{pmatrix} a^{-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & a^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & a^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & a^2 \end{pmatrix} , \] (4.22)
Utilizing \( O(d,d) \) transformation by matrix \( \Omega \), a new solution is generated
\[ \tilde{M} = \begin{pmatrix} a^{-2} & 0 & 0 & 0 & 0 & 0 & -b_1 a^{-2} & -b_2 a^{-2} \\ 0 & a^{-2} & 0 & 0 & b_1 a^{-2} & 0 & 0 & 0 \\ 0 & 0 & a^{-2} & 0 & b_2 a^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & a^2 & 0 & 0 & 0 & 0 \\ 0 & b_1 a^{-2} & 0 & 0 & b_2 a^{-2} & 0 & 0 & 0 \\ b_3 a^{-2} & 0 & b_2 b_3 a^{-2} & 0 & 0 & 0 & 0 & 0 \\ -b_1 a^{-2} & 0 & b_3 a^{-2} & 0 & 0 & 0 & 0 & 0 \\ -b_2 a^{-2} & -b_3 a^{-2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \] (4.23)
Although, the part of \( G \) in \( M \) (4.22) and the part of \( \tilde{G} \) in \( \tilde{M} \) (4.23) are the same with each other (same solutions), the full matrix \( M \) is different with \( \tilde{M} \). Therefore, we can argue that this solution (4.23) has
a new inequivalent physical interpretation in cosmological background. This is a difference between the traditional string cosmology and DFT when we begin with the same metrics and utilize the same rotation.

5 Conclusion

In summary, in this paper, we first clarified that T-duality in compactified background and \(O(D,D)\) symmetry in low energy effective action are descents of the continuous \(O(D,D)\) symmetry of DFT. Next, we present the differences and relations between the traditional string cosmology and DFT in terms of \(O(d,d)\) matrix (3.8). We discussed the cosmological interpretation of DFT. To understand the physics better, we introduced an \(O(D,D)\) invariant dilaton potential, derived from the backreaction of higher loop corrections, into the DFT action to calculate isotropic FRW like solutions. Our results exhibit some remarkable novel properties:

- Extra dimensions naturally arise in DFT cosmology without any elaborate pre-assumptions, it takes a form of Kaluza-Klein compactification.
- The original equivalent solutions in the traditional string cosmology have inequivalent physical interpretation in DFT, since there are mixed terms of double coordinates.
- Though starting from an isotropic FRW like metric, surprisingly, the universe evolves smoothly from a clearly anisotropic phase all the way to an visible isotropic phase at the big bang. And again to an anisotropic phase as time goes by.
- The whole evolution is parameter insensitive. No initial condition is needed. No initial anisotropy assumption is required.
- In the region \(t \to -\infty\), there are extra dimensions hidden by the weak string coupling. While in the region \(t \sim 0\), all dimensions are revealed and visible. Visible dimensions in the pre-big bang are hidden as \(t \to \infty\).
- The visible/hidden dimension exchange sheds some light on the exploration of the pre-big bang and cosmic amnesia puzzle. It also proposes a possible justification of string theory. To fulfill these purposes, further researches are needed.

In brief, not only does DFT provide an aesthetic formalism, it also has profound physical contents. It is immediately of great interest to address the influence on particle physics by spacetime backgrounds [4,14].

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A Appendix

A.1 Some notations and calculation rules

The generalized metric and its inverse are given by

\[
\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}b^j_k \\ b^k_i g^{kj} & g^{ij} \end{pmatrix}, \quad \mathcal{H}^{MN} = \begin{pmatrix} g_{ij} - b_{ik}g^{kl}b_{lj} & b_{ik}g^{kj} \\ -g^{ik}b_{kj} & g^{ij} \end{pmatrix}.
\]

(A.1)

The inner product of coordinates and their duals are defined as

\[
\mathcal{H}_{11} \left( \frac{\partial}{\partial \tilde{x}_i}, \frac{\partial}{\partial \tilde{x}_j} \right) = g^{ij}, \quad \mathcal{H}_{12} \left( \frac{\partial}{\partial \tilde{x}_i}, \frac{\partial}{\partial x^j} \right) = -g^{ik}b^j_k,
\]

\[
\mathcal{H}_{21} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial \tilde{x}_j} \right) = b^k_i g^{kj}, \quad \mathcal{H}_{22} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = g_{ij} - b_{ik}g^{kl}b_{lj}.
\]

(A.2)

Making use of following new notations to represent the block matrix of generalized metric

\[
\mathcal{H}_1^{1(i)} g^{1(j)} = G^{ij}, \quad \mathcal{H}_1^{1(i)} g^{2(j)} = B_i^j, \quad \mathcal{H}_2^{1(i)} g^{1(j)} = C^i_j, \quad \mathcal{H}_2^{1(i)} g^{2(j)} = D_{ij},
\]

(A.3)

and

\[
\mathcal{H}_1^{M_1} = \mathcal{H}_2^{M_2} = \eta_{21} = \mathcal{H}_1^{M_2} = \mathcal{H}_2^{M_1}.
\]

(A.4)

we find

\[
\mathcal{H}_1^{1(i)} g^{1(j)} = \mathcal{H}_1^{1(i)} g^{2(j)} = B_i^j, \quad \mathcal{H}_2^{1(i)} g^{1(j)} = \mathcal{H}_1^{1(i)} g^{1(j)} = G^{ij},
\]

\[
\mathcal{H}_1^{2(i)} g^{1(j)} = \mathcal{H}_2^{2(i)} g^{2(j)} = D_{ij}, \quad \mathcal{H}_2^{2(i)} g^{1(j)} = \mathcal{H}_2^{2(i)} g^{1(j)} = C^i_j.
\]

(A.5)

Then the background metrics are forms

\[
\tilde{g}^{ij} = \begin{pmatrix} -1 & \tilde{a}^2 \\ \tilde{a}^2 & \tilde{a}^2 \end{pmatrix}, \quad g_{ij} = \begin{pmatrix} -1 & a^2 \\ a^2 & a^2 \end{pmatrix},
\]

(A.6)

where

\[
\tilde{a} = a^{-1}
\]

(A.7)

In calculations, we only have two combinations of derivatives and metrics. The detailed calculation rules can be read in [56]

\[
g^{\bullet\bullet} \partial g_{\bullet\bullet}, \quad \tilde{g}^{\bullet\bullet} \partial \tilde{g}_{\bullet\bullet}.
\]

(A.8)

where we choose the simplest $b$ field
with $b$ is a non-zero constant.

### A.2 Equation of motion of the dilaton

\[
\begin{align*}
\frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_K \mathcal{H}_{NL} \\
- \partial_M \partial_N \mathcal{H}^{MN} - 4 \mathcal{H}^{MN} \partial_M d \partial_N d + 4 \partial_M \mathcal{H}^{MN} \partial_N d + 4 \mathcal{H}^{MN} \partial_M \partial_N d \\
= \frac{1}{8} \mathcal{H}^{(m)}{1(n)} \partial_{1(m)} \mathcal{H}^{(i)1(j)} \partial_{1(n)} \mathcal{H}_{1(i)1(j)} + \frac{1}{8} \mathcal{H}^{(m)}{1(n)} \partial_{1(m)} \mathcal{H}^{(i)2(j)} \partial_{1(n)} \mathcal{H}_{1(i)2(j)} \\
+ \frac{1}{8} \mathcal{H}^{(m)}{2(n)} \partial_{2(m)} \mathcal{H}^{(i)1(j)} \partial_{2(n)} \mathcal{H}_{1(i)1(j)} + \frac{1}{8} \mathcal{H}^{(m)}{2(n)} \partial_{2(m)} \mathcal{H}^{(i)2(j)} \partial_{2(n)} \mathcal{H}_{1(i)2(j)} \\
+ \frac{1}{8} \mathcal{H}^{(m)}{2(n)} \partial_{2(m)} \mathcal{H}^{(i)1(j)} \partial_{2(n)} \mathcal{H}_{2(i)1(j)} + \frac{1}{8} \mathcal{H}^{(m)}{2(n)} \partial_{2(m)} \mathcal{H}^{(i)2(j)} \partial_{2(n)} \mathcal{H}_{2(i)2(j)} \\
+ \frac{1}{8} \mathcal{H}^{(m)}{2(n)} \partial_{2(m)} \mathcal{H}^{(i)1(j)} \partial_{2(n)} \mathcal{H}_{2(i)1(j)} + \frac{1}{8} \mathcal{H}^{(m)}{2(n)} \partial_{2(m)} \mathcal{H}^{(i)2(j)} \partial_{2(n)} \mathcal{H}_{2(i)2(j)} \\
- \frac{1}{2} \mathcal{H}^{(m)}{1(n)} \partial_{1(m)} \mathcal{H}^{(i)1(j)} \partial_{1(n)} \mathcal{H}_{1(i)1(j)} - \frac{1}{2} \mathcal{H}^{(m)}{1(n)} \partial_{1(m)} \mathcal{H}^{(i)2(j)} \partial_{1(n)} \mathcal{H}_{1(i)2(j)} \\
- \frac{1}{2} \mathcal{H}^{(m)}{1(n)} \partial_{1(m)} \mathcal{H}^{(i)1(j)} \partial_{1(n)} \mathcal{H}_{2(i)1(j)} - \frac{1}{2} \mathcal{H}^{(m)}{1(n)} \partial_{1(m)} \mathcal{H}^{(i)2(j)} \partial_{1(n)} \mathcal{H}_{2(i)2(j)} \\
- \frac{1}{2} \mathcal{H}^{(m)}{2(n)} \partial_{2(m)} \mathcal{H}^{(i)1(j)} \partial_{2(n)} \mathcal{H}_{1(i)1(j)} - \frac{1}{2} \mathcal{H}^{(m)}{2(n)} \partial_{2(m)} \mathcal{H}^{(i)2(j)} \partial_{2(n)} \mathcal{H}_{1(i)2(j)} \\
- \frac{1}{2} \mathcal{H}^{(m)}{2(n)} \partial_{2(m)} \mathcal{H}^{(i)1(j)} \partial_{2(n)} \mathcal{H}_{2(i)1(j)} - \frac{1}{2} \mathcal{H}^{(m)}{2(n)} \partial_{2(m)} \mathcal{H}^{(i)2(j)} \partial_{2(n)} \mathcal{H}_{2(i)2(j)} \\
- \partial_{1(i)} \partial_{1(j)} \mathcal{H}^{(i)1(j)} - \partial_{1(i)} \partial_{2(j)} \mathcal{H}^{(i)2(j)} - \partial_{2(i)} \partial_{1(j)} \mathcal{H}^{(i)1(j)} - \partial_{2(i)} \partial_{2(j)} \mathcal{H}^{(i)2(j)}
\end{align*}
\]
Substituting the metric, one gets the result

\[ -4H^{1(i)(j)} \partial_{1(i)}d \partial_{1(j)}d - 4H^{1(2(i)} \partial_{1(i)}d \partial_{2(j)}d - 4H^{2(1)(j)} \partial_{2(i)}d \partial_{1(j)}d - 4H^{2(2(i)} \partial_{2(i)}d \partial_{2(j)}d \]

\[ 4\partial_{1(i)}H^{1(1)(j)} \partial_{1(j)}d + 4\partial_{1(i)}H^{1(2)(j)} \partial_{2(j)}d + 4\partial_{2(i)}H^{2(1)(j)} \partial_{1(j)}d + 4\partial_{2(i)}H^{2(2)(j)} \partial_{2(j)}d \]

\[ 4H^{1(i)(j)} \partial_{1(i)}d \partial_{1(j)}d + 4H^{1(2)(j)} \partial_{1(i)}d \partial_{2(j)}d + 4H^{2(1)(j)} \partial_{2(i)}d \partial_{1(j)}d + 4H^{2(2)(j)} \partial_{2(i)}d \partial_{2(j)}d. \] (A.10)

Substituting the metric, one gets the result

\[ 3\left( \frac{\ddot{a}}{a^2} \right) + 3\left( \frac{\dot{a}^2}{a^2} \right) + 4\dot{d}^2 + 4\ddot{d}^2 - 4\dddot{d} - 4\ddot{d} = 0. \] (A.11)

We define the Hubble parameters as

\[ H = \frac{\partial_a}{a}, \quad \bar{H} = \frac{\bar{\partial}_a}{a}. \] (A.12)

The EOM of the dilaton becomes

\[ \left( 3H^2 - 4\dot{d} + 4\ddot{d} \right) + \left( 3\bar{H}^2 - 4\ddot{d} + 4\bar{\dot{d}}^2 \right) = 0. \] (A.13)

**A.3 Equation of motion of the generalized metric**

Recall the definition of the Ricci-like tensor

\[ \mathcal{R}_{MN} = \mathcal{K}_{MN} - S^P_M \mathcal{K}_{PQS^Q_N} = 0. \] (A.14)

To make calculations simpler, we separate this equation into two parts

\[ \mathcal{K}_{MN} = \ast \mathcal{K}_{MN} + \ast \mathcal{K}_{MN}, \] (A.15)

where the first part only includes a generalized metric

\[ \ast \mathcal{K}_{MN} \equiv \frac{1}{8} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{4} \partial_L \left( \mathcal{H}^{LK} \partial_K \mathcal{H}_{MN} \right) \]

\[ -\frac{1}{2} \partial_L \left( \mathcal{H}^{KL} \partial_L \mathcal{H}_{MN} \right) + \frac{1}{2} \partial_L \left( \mathcal{H}^{KL} \partial_L \mathcal{H}_{MN} \right), \] (A.16)

and the second includes the dilaton

\[ \ast \mathcal{K}_{MN} \equiv \frac{1}{2} \partial_L d \left( \mathcal{H}^{KL} \partial_K \mathcal{H}_{MN} \right) + 2 \partial_M \partial_N d - \partial_L d \left( \mathcal{H}^{KL} \partial_L \mathcal{H}_{MN} \right) + \mathcal{H}^{KL} \partial_L \mathcal{H}_{MN} \] (A.17)

Therefore, the Ricci-like tensor can be rewritten as

\[ \mathcal{R}_{MN} = \ast \mathcal{R}_{MN} + \ast \mathcal{R}_{MN} = 0. \] (A.18)
A.3.1 Calculation of $\star \mathcal{R}_{MN}$

Expanding every term in $\star \mathcal{R}_{MN}$, we have

$$
\star \mathcal{R}_{MN} = \star \mathcal{K}_{MN} - S^P M \star \mathcal{K}_{PQS} S^Q N
$$

$$
= \frac{1}{8} \partial_M \mathcal{H}^{1(i)1(j)} \partial_N \mathcal{H}_{1(i)1(j)} + \frac{1}{8} \partial_M \mathcal{H}^{1(i)2(j)} \partial_N \mathcal{H}_{1(i)2(j)} + \frac{1}{8} \partial_M \mathcal{H}^{2(i)1(j)} \partial_N \mathcal{H}_{2(i)1(j)} + \frac{1}{8} \partial_M \mathcal{H}^{2(i)2(j)} \partial_N \mathcal{H}_{2(i)2(j)}
$$

$$
- \frac{1}{4} \partial_1(i) \left( \mathcal{H}^{1(i)1(j)} \partial_1(j) \mathcal{H}_{MN} \right) - \frac{1}{4} \partial_1(i) \left( \mathcal{H}^{1(i)2(j)} \partial_2(j) \mathcal{H}_{MN} \right) - \frac{1}{4} \partial_2(i) \left( \mathcal{H}^{2(i)1(j)} \partial_1(j) \mathcal{H}_{MN} \right) - \frac{1}{4} \partial_2(i) \left( \mathcal{H}^{2(i)2(j)} \partial_2(j) \mathcal{H}_{MN} \right)
$$

$$
- \frac{1}{2} \partial_{(N} \mathcal{H}^{1(i)1(j)} \partial_{(j)} \mathcal{H}_{M)1(i)} - \frac{1}{2} \partial_{(N} \mathcal{H}^{1(i)2(j)} \partial_{(j)} \mathcal{H}_{M)1(i)} - \frac{1}{2} \partial_{(N} \mathcal{H}^{2(i)1(j)} \partial_{(j)} \mathcal{H}_{M)2(i)} - \frac{1}{2} \partial_{(N} \mathcal{H}^{2(i)2(j)} \partial_{(j)} \mathcal{H}_{M)2(i)}
$$

$$
+ \frac{1}{2} \partial_1(j) \left( \mathcal{H}^{1(i)1(j)} \partial_{[N} \mathcal{H}_{M]1(i)} + \frac{1}{2} \partial_2(j) \left( \mathcal{H}^{1(i)2(j)} \partial_{[N} \mathcal{H}_{M]1(i)} \right) \right)
$$

$$
+ \frac{1}{2} \partial_1(j) \left( \mathcal{H}^{2(i)1(j)} \partial_{[N} \mathcal{H}_{M]2(i)} + \frac{1}{2} \partial_2(j) \left( \mathcal{H}^{2(i)2(j)} \partial_{[N} \mathcal{H}_{M]2(i)} \right) \right)
$$

$$
+ \frac{1}{2} \partial_1(j) \left( \mathcal{H}^{1(i)} (M \partial_1(i) \mathcal{H}^{1(j)} N) + \frac{1}{2} \partial_2(j) \left( \mathcal{H}^{1(i)} (M \partial_1(i) \mathcal{H}^{2(j)} N) \right) \right)
$$

$$
+ \frac{1}{2} \partial_1(j) \left( \mathcal{H}^{2(i)} (M \partial_2(i) \mathcal{H}^{1(j)} N) + \frac{1}{2} \partial_2(j) \left( \mathcal{H}^{2(i)} (M \partial_2(i) \mathcal{H}^{2(j)} N) \right) \right)
$$

$$
- \frac{1}{8} S^{1(i)} M \partial_1(i) \mathcal{H}^{1(k)1(l)} \partial_1(j) \mathcal{H}_{1(k)1(l)} S^{1(j)} N - \frac{1}{8} S^{1(i)} M \partial_1(i) \mathcal{H}^{1(k)2(l)} \partial_1(j) \mathcal{H}_{1(k)2(l)} S^{1(j)} N
$$

$$
- \frac{1}{8} S^{1(i)} M \partial_1(i) \mathcal{H}^{2(k)1(l)} \partial_1(j) \mathcal{H}_{2(k)1(l)} S^{1(j)} N - \frac{1}{8} S^{1(i)} M \partial_1(i) \mathcal{H}^{2(k)2(l)} \partial_1(j) \mathcal{H}_{2(k)2(l)} S^{1(j)} N
$$

$$
- \frac{1}{8} S^{1(i)} M \partial_1(i) \mathcal{H}^{1(k)1(l)} \partial_2(j) \mathcal{H}_{1(k)1(l)} S^{2(j)} N - \frac{1}{8} S^{1(i)} M \partial_1(i) \mathcal{H}^{1(k)2(l)} \partial_2(j) \mathcal{H}_{1(k)2(l)} S^{2(j)} N
$$

$$
- \frac{1}{8} S^{1(i)} M \partial_1(i) \mathcal{H}^{2(k)1(l)} \partial_2(j) \mathcal{H}_{2(k)1(l)} S^{2(j)} N - \frac{1}{8} S^{1(i)} M \partial_1(i) \mathcal{H}^{2(k)2(l)} \partial_2(j) \mathcal{H}_{2(k)2(l)} S^{2(j)} N
$$

$$
- \frac{1}{8} S^{2(i)} M \partial_2(i) \mathcal{H}^{1(k)1(l)} \partial_1(j) \mathcal{H}_{1(k)1(l)} S^{1(j)} N - \frac{1}{8} S^{2(i)} M \partial_2(i) \mathcal{H}^{1(k)2(l)} \partial_1(j) \mathcal{H}_{1(k)2(l)} S^{1(j)} N
$$

$$
- \frac{1}{8} S^{2(i)} M \partial_2(i) \mathcal{H}^{2(k)1(l)} \partial_1(j) \mathcal{H}_{2(k)1(l)} S^{1(j)} N - \frac{1}{8} S^{2(i)} M \partial_2(i) \mathcal{H}^{2(k)2(l)} \partial_1(j) \mathcal{H}_{2(k)2(l)} S^{1(j)} N
$$

$$
- \frac{1}{8} S^{2(i)} M \partial_2(i) \mathcal{H}^{1(k)1(l)} \partial_2(j) \mathcal{H}_{1(k)1(l)} S^{2(j)} N - \frac{1}{8} S^{2(i)} M \partial_2(i) \mathcal{H}^{1(k)2(l)} \partial_2(j) \mathcal{H}_{1(k)2(l)} S^{2(j)} N
$$

$$
- \frac{1}{8} S^{2(i)} M \partial_2(i) \mathcal{H}^{2(k)1(l)} \partial_2(j) \mathcal{H}_{2(k)1(l)} S^{2(j)} N - \frac{1}{8} S^{2(i)} M \partial_2(i) \mathcal{H}^{2(k)2(l)} \partial_2(j) \mathcal{H}_{2(k)2(l)} S^{2(j)} N
$$

$$
+ \frac{1}{4} S^{1(i)} M \partial_1(l) \left( \mathcal{H}^{1(i)1(k)} \partial_1(k) \mathcal{H}_{1(i)1(j)} \right) S^{1(j)} N + \frac{1}{4} S^{1(i)} M \partial_1(l) \left( \mathcal{H}^{1(i)2(k)} \partial_2(k) \mathcal{H}_{1(i)1(j)} \right) S^{1(j)} N
$$

24
\[
+ \frac{1}{4} S^{1(i)}_{M} \partial_{2(l)} \left( H^{2(l)}(1(k)) \partial_{1(k)} H_{1(i)}(1(j)) \right) S^{1(j)}_{N} + \frac{1}{4} S^{1(i)}_{M} \partial_{2(l)} \left( H^{2(l)}(2(k)) \partial_{2(k)} H_{1(i)}(1(j)) \right) S^{1(j)}_{N}
+ \frac{1}{4} S^{1(i)}_{M} \partial_{1(l)} \left( H^{1(l)}(1(k)) \partial_{1(k)} H_{1(i)}(2(j)) \right) S^{2(j)}_{N} + \frac{1}{4} S^{1(i)}_{M} \partial_{1(l)} \left( H^{1(l)}(2(k)) \partial_{2(k)} H_{1(i)}(2(j)) \right) S^{2(j)}_{N}
+ \frac{1}{4} S^{2(i)}_{M} \partial_{2(l)} \left( H^{2(l)}(1(k)) \partial_{1(k)} H_{2(i)}(1(j)) \right) S^{1(j)}_{N} + \frac{1}{4} S^{2(i)}_{M} \partial_{2(l)} \left( H^{2(l)}(2(k)) \partial_{2(k)} H_{2(i)}(1(j)) \right) S^{1(j)}_{N}
+ \frac{1}{4} S^{2(i)}_{M} \partial_{2(l)} \left( H^{2(l)}(1(k)) \partial_{1(k)} H_{2(i)}(2(j)) \right) S^{2(j)}_{N} + \frac{1}{4} S^{2(i)}_{M} \partial_{2(l)} \left( H^{2(l)}(2(k)) \partial_{2(k)} H_{2(i)}(2(j)) \right) S^{2(j)}_{N}
+ \frac{1}{2} S^{1(i)}_{M} \partial_{1(l)} \left( H^{1(k)}(1(k)) \partial_{1(k)} H_{1(i)}(1(k)) \right) S^{1(j)}_{N} + \frac{1}{2} S^{1(i)}_{M} \partial_{1(l)} \left( H^{1(k)}(2(k)) \partial_{2(k)} H_{1(i)}(1(k)) \right) S^{1(j)}_{N}
+ \frac{1}{2} S^{1(i)}_{M} \partial_{2(l)} \left( H^{1(k)}(1(k)) \partial_{1(k)} H_{1(i)}(2(k)) \right) S^{2(j)}_{N} + \frac{1}{2} S^{1(i)}_{M} \partial_{2(l)} \left( H^{1(k)}(2(k)) \partial_{2(k)} H_{1(i)}(2(k)) \right) S^{2(j)}_{N}
+ \frac{1}{2} S^{1(i)}_{M} \partial_{2(l)} \left( H^{2(k)}(1(k)) \partial_{1(k)} H_{2(i)}(1(k)) \right) S^{1(j)}_{N} + \frac{1}{2} S^{1(i)}_{M} \partial_{2(l)} \left( H^{2(k)}(2(k)) \partial_{2(k)} H_{2(i)}(1(k)) \right) S^{1(j)}_{N}
+ \frac{1}{2} S^{2(i)}_{M} \partial_{1(l)} \left( H^{1(k)}(1(k)) \partial_{1(k)} H_{2(i)}(1(k)) \right) S^{2(j)}_{N} + \frac{1}{2} S^{2(i)}_{M} \partial_{1(l)} \left( H^{1(k)}(2(k)) \partial_{2(k)} H_{2(i)}(1(k)) \right) S^{2(j)}_{N}
+ \frac{1}{2} S^{2(i)}_{M} \partial_{2(l)} \left( H^{2(k)}(1(k)) \partial_{1(k)} H_{2(i)}(2(k)) \right) S^{2(j)}_{N} + \frac{1}{2} S^{2(i)}_{M} \partial_{2(l)} \left( H^{2(k)}(2(k)) \partial_{2(k)} H_{2(i)}(2(k)) \right) S^{2(j)}_{N}

- \frac{1}{2} S^{1(i)}_{M} \partial_{1(l)} \left( H^{1(k)}(1(k)) \partial_{1(j)} H_{1(i)}(1(k)) \right) S^{1(j)}_{N} - \frac{1}{2} S^{1(i)}_{M} \partial_{2(l)} \left( H^{1(k)}(2(k)) \partial_{1(j)} H_{1(i)}(1(k)) \right) S^{1(j)}_{N}
- \frac{1}{2} S^{1(i)}_{M} \partial_{1(l)} \left( H^{2(k)}(1(k)) \partial_{1(j)} H_{1(i)}(2(k)) \right) S^{1(j)}_{N} - \frac{1}{2} S^{1(i)}_{M} \partial_{2(l)} \left( H^{2(k)}(2(k)) \partial_{1(j)} H_{1(i)}(2(k)) \right) S^{1(j)}_{N}
- \frac{1}{2} S^{1(i)}_{M} \partial_{1(l)} \left( H^{1(k)}(1(k)) \partial_{2(j)} H_{1(i)}(1(k)) \right) S^{2(j)}_{N} - \frac{1}{2} S^{1(i)}_{M} \partial_{2(l)} \left( H^{1(k)}(2(k)) \partial_{2(j)} H_{1(i)}(1(k)) \right) S^{2(j)}_{N}
- \frac{1}{2} S^{1(i)}_{M} \partial_{2(l)} \left( H^{1(k)}(1(k)) \partial_{1(j)} H_{2(i)}(2(k)) \right) S^{1(j)}_{N} - \frac{1}{2} S^{1(i)}_{M} \partial_{2(l)} \left( H^{1(k)}(2(k)) \partial_{1(j)} H_{2(i)}(2(k)) \right) S^{1(j)}_{N}
- \frac{1}{2} S^{2(i)}_{M} \partial_{1(l)} \left( H^{1(k)}(1(k)) \partial_{2(j)} H_{2(i)}(1(k)) \right) S^{1(j)}_{N} - \frac{1}{2} S^{2(i)}_{M} \partial_{2(l)} \left( H^{1(k)}(2(k)) \partial_{2(j)} H_{2(i)}(1(k)) \right) S^{1(j)}_{N}
- \frac{1}{2} S^{2(i)}_{M} \partial_{2(l)} \left( H^{2(k)}(1(k)) \partial_{2(j)} H_{2(i)}(2(k)) \right) S^{2(j)}_{N} - \frac{1}{2} S^{2(i)}_{M} \partial_{2(l)} \left( H^{2(k)}(2(k)) \partial_{2(j)} H_{2(i)}(2(k)) \right) S^{2(j)}_{N}
- \frac{1}{2} S^{2(i)}_{M} \partial_{2(l)} \left( H^{2(k)}(1(k)) \partial_{2(j)} H_{2(i)}(2(k)) \right) S^{2(j)}_{N} - \frac{1}{2} S^{2(i)}_{M} \partial_{2(l)} \left( H^{2(k)}(2(k)) \partial_{2(j)} H_{2(i)}(2(k)) \right) S^{2(j)}_{N}
\]
Using the calculation rules to simplify the equations, we get

\[
\begin{align*}
&-\frac{1}{2} S^{(i)}_{\text{M} J} \partial_1 (H^{(1)}_{\text{M} J} \partial_1 H^{(1)}_{\text{M} J}) S^{(j)}_{\text{N}} - \frac{1}{2} S^{(i)}_{\text{M} J} \partial_2 (H^{(1)}_{\text{M} J} \partial_1 H^{(2)}_{\text{M} J}) S^{(j)}_{\text{N}} \\
&- \frac{1}{2} S^{(i)}_{\text{M} J} \partial_1 (H^{(2)}_{\text{M} J} \partial_1 H^{(1)}_{\text{M} J}) S^{(j)}_{\text{N}} - \frac{1}{2} S^{(i)}_{\text{M} J} \partial_2 (H^{(2)}_{\text{M} J} \partial_1 H^{(2)}_{\text{M} J}) S^{(j)}_{\text{N}} \\
&- \frac{1}{2} S^{(i)}_{\text{M} J} \partial_1 (H^{(1)}_{\text{M} J} \partial_2 H^{(1)}_{\text{M} J}) S^{(j)}_{\text{N}} - \frac{1}{2} S^{(i)}_{\text{M} J} \partial_2 (H^{(1)}_{\text{M} J} \partial_2 H^{(2)}_{\text{M} J}) S^{(j)}_{\text{N}} \\
&- \frac{1}{2} S^{(i)}_{\text{M} J} \partial_1 (H^{(2)}_{\text{M} J} \partial_2 H^{(1)}_{\text{M} J}) S^{(j)}_{\text{N}} - \frac{1}{2} S^{(i)}_{\text{M} J} \partial_2 (H^{(2)}_{\text{M} J} \partial_2 H^{(2)}_{\text{M} J}) S^{(j)}_{\text{N}} \\
&- \frac{1}{2} S^{(i)}_{\text{M} J} \partial_1 (H^{(1)}_{\text{M} J} \partial_2 H^{(1)}_{\text{M} J}) S^{(j)}_{\text{N}} - \frac{1}{2} S^{(i)}_{\text{M} J} \partial_2 (H^{(1)}_{\text{M} J} \partial_2 H^{(2)}_{\text{M} J}) S^{(j)}_{\text{N}} \\
&- \frac{1}{2} S^{(i)}_{\text{M} J} \partial_1 (H^{(2)}_{\text{M} J} \partial_2 H^{(1)}_{\text{M} J}) S^{(j)}_{\text{N}} - \frac{1}{2} S^{(i)}_{\text{M} J} \partial_2 (H^{(2)}_{\text{M} J} \partial_2 H^{(2)}_{\text{M} J}) S^{(j)}_{\text{N}}
\end{align*}
\]
\[-\frac{1}{8} S^{1(j)}_{\mu} \tilde{\partial}^i D^{kl} \tilde{\partial}^j D_{kl} S^{1(j)}_{\mu N} - \frac{1}{8} S^{1(i)}_{\mu} \tilde{\partial}^i D^{kl} \tilde{\partial}_j D_{kl} S^{2(j)}_{\mu N} - \frac{1}{8} S^{2(i)}_{\mu} \partial_i D^{kl} \tilde{\partial}^j D_{kl} S^{1(j)}_{\mu N} \]

\[-\frac{1}{8} S^{2(j)}_{\mu} \partial_i D^{kl} \partial_j D_{kl} S^{2(j)}_{\mu N} \]

\[+ \frac{1}{4} S^{1(j)}_{\mu} \tilde{\partial}^i \left( g_{ik} \tilde{\partial}^k g^{ij} \right) S^{1(j)}_{\mu N} + \frac{1}{4} S^{1(i)}_{\mu} \tilde{\partial}^i \left( B^k_i \partial_k g^{ij} \right) S^{1(j)}_{\mu N} + \frac{1}{4} S^{1(i)}_{\mu} \partial_i \left( C^k_i \tilde{\partial}^k g^{ij} \right) S^{1(j)}_{\mu N} \]

\[+ \frac{1}{4} S^{1(j)}_{\mu} \tilde{\partial}^j \left( g_{ik} \tilde{\partial}^k B^i_j \right) S^{2(j)}_{\mu N} + \frac{1}{4} S^{1(i)}_{\mu} \tilde{\partial}^j \left( B^i_k \partial_k B^i_j \right) S^{2(j)}_{\mu N} + \frac{1}{4} S^{1(i)}_{\mu} \partial_i \left( C^k_i \tilde{\partial}^k B^i_j \right) S^{2(j)}_{\mu N} \]

\[+ \frac{1}{4} S^{1(j)}_{\mu} \tilde{\partial}^j \left( g_{ik} \tilde{\partial}^k C^i_j \right) S^{1(j)}_{\mu N} + \frac{1}{4} S^{2(j)}_{\mu} \tilde{\partial}^j \left( B^i_k \partial_k C^i_j \right) S^{1(j)}_{\mu N} + \frac{1}{4} S^{2(j)}_{\mu} \partial_i \left( C^k_i \tilde{\partial}^k C^i_j \right) S^{1(j)}_{\mu N} \]

\[+ \frac{1}{4} S^{1(j)}_{\mu} \partial_i \left( D^{ik} \partial_k g^{ij} \right) S^{1(j)}_{\mu N} + \frac{1}{4} S^{1(i)}_{\mu} \partial_i \left( D^{ik} \partial_k B^i_j \right) S^{2(j)}_{\mu N} + \frac{1}{4} S^{2(i)}_{\mu} \partial_i \left( D^{ik} \partial_k C^i_j \right) S^{1(j)}_{\mu N} \]

\[+ \frac{1}{4} S^{2(j)}_{\mu} \partial_i \left( D^{ik} \partial_k D_{ij} \right) S^{2(j)}_{\mu N} \]
\[-\frac{1}{4} S^{2(i)}_{M} \partial^j \left( S_{kij} \partial^i C_i^k \right) S^{1(j)}_{N} - \frac{1}{4} S^{2(i)}_{M} \partial^l \left( B^l_{kij} \partial^j C_i^k \right) S^{1(j)}_{N} - \frac{1}{4} S^{2(i)}_{M} \partial^j \left( C_i^k \partial^j D_{ik} \right) S^{1(j)}_{N} \]

\[-\frac{1}{4} S^{2(i)}_{M} \partial^j \left( G_{kij} \partial^i \partial^j B^i_k \right) S^{1(j)}_{N} - \frac{1}{4} S^{2(i)}_{M} \partial^l \left( B^l_{kij} \partial^j B^i_k \right) S^{1(j)}_{N} - \frac{1}{4} S^{2(i)}_{M} \partial^j \left( C_i^k \partial^j B^i_k \right) S^{1(j)}_{N} \]

Our aim is to calculate the components of $\star \mathcal{R}_{MN}$, which can be expressed as

\[\star \mathcal{R}_{MN} = \begin{pmatrix} \star \mathcal{R}_{11} & \star \mathcal{R}_{12} \\ \star \mathcal{R}_{21} & \star \mathcal{R}_{22} \end{pmatrix}.\]

We find

\[
\begin{align*}
\star \mathcal{R}_{2(\rho)2(q)} &= \frac{1}{8} \partial^\rho G_{ij} \partial^\rho G^{ij} + \frac{1}{8} \partial^\rho B^i_j \partial^\rho B^i_j + \frac{1}{8} \partial^\rho C^i_j \partial^\rho C^i_j + \frac{1}{8} \partial^\rho D^{ij} \partial^\rho D^{ij} \\
&- \frac{1}{4} \partial^j \left( G_{ij} \partial^j D_{pq} \right) - \frac{1}{4} \partial^j \left( B^i_j \partial^j D_{pq} \right) - \frac{1}{4} \partial^j \left( C^i_j \partial^j D_{pq} \right) - \frac{1}{4} \partial^j \left( D^{ij} \partial^j D_{pq} \right)
\end{align*}
\]
\[
\begin{align*}
-\frac{1}{4} \partial_q G_{ij} \partial^2 C^i_p - \frac{1}{4} \partial_q B^i_j \partial_j C^i_p - \frac{1}{4} \partial_q C^i_j \partial^2 D_{pi} - \frac{1}{4} \partial_q D^{ij} \partial_j D_{pi} \\
-\frac{1}{4} \partial_q G_{ij} \partial^2 C^i_q - \frac{1}{4} \partial_q B^i_j \partial_j C^i_q - \frac{1}{4} \partial_q C^i_j \partial^2 D_{qi} - \frac{1}{4} \partial_q D^{ij} \partial_j D_{qi} \\
+ \frac{1}{4} \partial^i (G_{ij} \partial_j C^i_p) + \frac{1}{4} \partial_j (B^i_j \partial_j C^i_p) + \frac{1}{4} \partial^i (C^i_j \partial_q D_{pi}) + \frac{1}{4} \partial_j (D^{ij} \partial_q D_{pi}) \\
+ \frac{1}{4} \partial^i (G_{ij} \partial_j C^i_q) + \frac{1}{4} \partial_j (B^i_j \partial_j C^i_q) + \frac{1}{4} \partial^i (C^i_j \partial_q D_{qi}) + \frac{1}{4} \partial_j (D^{ij} \partial_q D_{qi}) \\
+ \frac{1}{4} \partial^i (G_{ip} \partial^j G_{jq}) + \frac{1}{4} \partial_j (G_{ip} \partial^j C^i_q) + \frac{1}{4} \partial^i (C^i_j \partial_q G_{jq}) + \frac{1}{4} \partial_j (C^i_j \partial_q C^i_q) \\
+ \frac{1}{4} \partial^i (G_{ip} \partial^j G_{jp}) + \frac{1}{4} \partial_j (G_{ip} \partial^j C^i_p) + \frac{1}{4} \partial^i (C^i_j \partial_q G_{jp}) + \frac{1}{4} \partial_j (C^i_j \partial_q C^i_p) \\
-\frac{1}{8} G_{ip} \partial^j G_{kl} \partial^i G^{k^l} G_{jq} - \frac{1}{8} G_{ip} \partial^j B^l_k \partial^i B^k_j G_{jq} - \frac{1}{8} G_{ip} \partial^j C^k_i \partial^i G_{jq} - \frac{1}{8} G_{ip} \partial^j D^{kl} \partial^i D_{kl} G_{jq} \\
-\frac{1}{8} G_{ip} \partial^j G_{kl} \partial^i G^{k^l} C^q_j - \frac{1}{8} G_{ip} \partial^j B^l_k \partial^i B^k_j C^q_j - \frac{1}{8} G_{ip} \partial^j C^k_i \partial^i C^q_j - \frac{1}{8} G_{ip} \partial^j D^{kl} \partial^i D_{kl} C^q_j \\
-\frac{1}{8} C^i_p \partial^q G_{kl} \partial^i G^{k^l} C^q_j - \frac{1}{8} C^i_p \partial^q B^l_k \partial^i B^k_j C^q_j - \frac{1}{8} C^i_p \partial^q C^k_i \partial^i C^q_j - \frac{1}{8} C^i_p \partial^q D^{kl} \partial^i D_{kl} C^q_j \\
-\frac{1}{8} C^i_p \partial^q G_{kl} \partial^i G^{k^l} C^q_j - \frac{1}{8} C^i_p \partial^q B^l_k \partial^i B^k_j C^q_j - \frac{1}{8} C^i_p \partial^q C^k_i \partial^i C^q_j - \frac{1}{8} C^i_p \partial^q D^{kl} \partial^i D_{kl} C^q_j \\
+ \frac{1}{4} G_{ip} \partial^j (G_{ik} \partial^k G^{ij}) G_{jq} + \frac{1}{4} G_{ip} \partial^j (B^l_k \partial^i G^{ij}) G_{jq} + \frac{1}{4} G_{ip} \partial^j (C^k_i \partial^i G^{ij}) G_{jq} + \frac{1}{4} G_{ip} \partial^j (D^{kl} \partial^i G^{ij}) G_{jq} \\
+ \frac{1}{4} G_{ip} \partial^j (G_{ik} \partial^k B^j) C^q_j + \frac{1}{4} G_{ip} \partial^j (B^l_k \partial^i B^j) C^q_j + \frac{1}{4} G_{ip} \partial^j (C^k_i \partial^i B^j) C^q_j + \frac{1}{4} G_{ip} \partial^j (D^{kl} \partial^i B^j) C^q_j \\
+ \frac{1}{4} C^i_p \partial^j (G_{ik} \partial^k C^j) G_{jq} + \frac{1}{4} C^i_p \partial^j (B^l_k \partial^i C^j) G_{jq} + \frac{1}{4} C^i_p \partial^j (C^k_i \partial^i C^j) G_{jq} + \frac{1}{4} C^i_p \partial^j (D^{kl} \partial^i C^j) G_{jq} \\
+ \frac{1}{4} C^i_p \partial^j (G_{ik} \partial^k D_{ij}) C^q_j + \frac{1}{4} C^i_p \partial^j (B^l_k \partial^i D_{ij}) C^q_j + \frac{1}{4} C^i_p \partial^j (C^k_i \partial^i D_{ij}) C^q_j + \frac{1}{4} C^i_p \partial^j (D^{kl} \partial^i D_{ij}) C^q_j \\
+ \frac{1}{4} G_{ip} \partial^j G_{kl} \partial^i G^{k^l} G_{jq} + \frac{1}{4} G_{ip} \partial^j B^l_k \partial^i G^{k^l} G_{jq} + \frac{1}{4} G_{ip} \partial^j C^k_i \partial^i G^{k^l} G_{jq} + \frac{1}{4} G_{ip} \partial^j D^{kl} \partial^i G^{k^l} G_{jq} \\
+ \frac{1}{4} G_{ip} \partial^j G_{kl} \partial^i G^{k^l} G_{jq} + \frac{1}{4} G_{ip} \partial^j B^l_k \partial^i G^{k^l} G_{jq} + \frac{1}{4} G_{ip} \partial^j C^k_i \partial^i G^{k^l} G_{jq} + \frac{1}{4} G_{ip} \partial^j D^{kl} \partial^i G^{k^l} G_{jq} \\
+ \frac{1}{4} G_{ip} \partial^j G_{kl} \partial^i G^{k^l} C^q_j + \frac{1}{4} G_{ip} \partial^j B^l_k \partial^i G^{k^l} C^q_j + \frac{1}{4} G_{ip} \partial^j C^k_i \partial^i G^{k^l} C^q_j + \frac{1}{4} G_{ip} \partial^j D^{kl} \partial^i G^{k^l} C^q_j \\
+ \frac{1}{4} G_{ip} \partial^j G_{kl} \partial^i G^{k^l} C^q_j + \frac{1}{4} G_{ip} \partial^j B^l_k \partial^i G^{k^l} C^q_j + \frac{1}{4} G_{ip} \partial^j C^k_i \partial^i G^{k^l} C^q_j + \frac{1}{4} G_{ip} \partial^j D^{kl} \partial^i G^{k^l} C^q_j \\
+ \frac{1}{4} G_{ip} \partial^j G_{kl} \partial^i G^{k^l} C^q_j + \frac{1}{4} G_{ip} \partial^j B^l_k \partial^i G^{k^l} C^q_j + \frac{1}{4} G_{ip} \partial^j C^k_i \partial^i G^{k^l} C^q_j + \frac{1}{4} G_{ip} \partial^j D^{kl} \partial^i G^{k^l} C^q_j \\
+ \frac{1}{4} G_{ip} \partial^j G_{kl} \partial^i G^{k^l} C^q_j + \frac{1}{4} G_{ip} \partial^j B^l_k \partial^i G^{k^l} C^q_j + \frac{1}{4} G_{ip} \partial^j C^k_i \partial^i G^{k^l} C^q_j + \frac{1}{4} G_{ip} \partial^j D^{kl} \partial^i G^{k^l} C^q_j \\
\end{align*}
\]
\[-\frac{1}{4}G_{ip}\partial^j\left(G_{kl}\partial^iG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial^jB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial^jB_{kl}^i\right)G_{jq}\]

\[-\frac{1}{4}G_{ip}\partial^j\left(G_{kl}\partial^i\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial^j\partial^lG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial^j\partial^lB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial^j\partial^lB_{kl}^i\right)G_{jq}\]

\[-\frac{1}{4}G_{ip}\partial^j\left(G_{kl}\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial^jB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial^jB_{kl}^i\right)G_{jq}\]

\[-\frac{1}{4}G_{ip}\partial_j\left(G_{kl}\partial^iG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial_jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial_jB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial_jB_{kl}^i\right)G_{jq}\]

\[-\frac{1}{4}G_{ip}\partial^j\left(G_{kl}\partial^iG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial^jB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial^jB_{kl}^i\right)G_{jq}\]

\[-\frac{1}{4}G_{ip}\partial^j\left(G_{kl}\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial^jB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial^jB_{kl}^i\right)G_{jq}\]

\[-\frac{1}{4}G_{ip}\partial^j\left(G_{kl}\partial^iG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial^jB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial^jB_{kl}^i\right)G_{jq}\]

\[-\frac{1}{4}G_{ip}\partial^j\left(G_{kl}\partial^iG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial^jB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial^jB_{kl}^i\right)G_{jq}\]

\[-\frac{1}{4}G_{ip}\partial^j\left(G_{kl}\partial^iG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial^jB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial^jB_{kl}^i\right)G_{jq}\]

\[-\frac{1}{4}G_{ip}\partial^j\left(G_{kl}\partial^iG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial^jB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial^jB_{kl}^i\right)G_{jq}\]

\[-\frac{1}{4}G_{ip}\partial^j\left(G_{kl}\partial^iG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial^jB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial^jB_{kl}^i\right)G_{jq}\]

\[-\frac{1}{4}G_{ip}\partial^j\left(G_{kl}\partial^iG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial^jB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial^jB_{kl}^i\right)G_{jq}\]

\[-\frac{1}{4}G_{ip}\partial^j\left(G_{kl}\partial^iG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial^jB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial^jB_{kl}^i\right)G_{jq}\]

\[-\frac{1}{4}G_{ip}\partial^j\left(G_{kl}\partial^iG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(B_{k}^i\partial^jG^{ik}\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(C_{k}^i\partial^jB_{k}^i\right)G_{jq} - \frac{1}{4}G_{ip}\partial_l\left(D_{kl}\partial^jB_{kl}^i\right)G_{jq}\]

When \(p = q = 1\), we find

\[\star R_{2(1)2(1)}\]

\[= \frac{1}{8}\partial_tG_{ii}\partial_tG_{ii} + \frac{1}{8}\partial_tB_{ij}^j\partial_tB_{ij}^i + \frac{1}{8}\partial_tC_{ij}^j\partial_tC_{ij}^i + \frac{1}{8}\partial_tD_{ii}\partial_tD_{ii} - \frac{1}{8}G_{tt}\partial_t\partial_tG_{kk}G_{kk} - \frac{1}{8}G_{tt}\partial_t\partial_tB_{k}^i\partial_tB_{k}^i - \frac{1}{8}G_{tt}\partial_t\partial_tC_{k}^i\partial_tC_{k}^i - \frac{1}{8}G_{tt}\partial_t\partial_tD_{kk}D_{kk} - \frac{1}{8}G_{tt}\partial_t\partial_tD_{kk}^j\partial_tD_{kk}^j\]

\[= -\frac{3}{4}\frac{\dot{a}^2}{a^2} + 3\frac{\ddot{a}^2}{a^2} \tag{A.23}\]

When \(p = q = 2\), we find

\[\star R_{2(2)2(2)}\]

\[= -\frac{1}{4}\partial_t(D_{ii}\partial_tD_{ii}) + \frac{1}{4}G_{22}\partial_t(D_{22}\partial_tG_{22}) \tag{A.24}\]
Finally, we have

\[
\begin{align*}
\mathcal{R}_{2(3,4)2(3,4)} & = -\frac{1}{4} \tilde{\partial}_t (D^{tt} \partial_t C_3) \mathcal{G}_{33} + \frac{1}{4} \tilde{\partial}_t (D^{tt} \partial_t B_4) \mathcal{C}_3^4 \\
& + \frac{1}{4} \tilde{\partial}_t (D^{tt} \partial_t C_4) \mathcal{G}_{33} + \frac{1}{4} \tilde{\partial}_t (D^{tt} \partial_t D_{44}) \mathcal{C}_4^4 \\
& - \frac{1}{4} \tilde{\partial}_t (G_{tt} \tilde{\partial} \mathcal{G}_{33}) + \frac{1}{4} \tilde{\partial}_t (G_{tt} \tilde{\partial} \mathcal{G}_{33}) \mathcal{G}_{33} + \frac{1}{4} \tilde{\partial}_t (D^{tt} \tilde{\partial} \mathcal{B}_4) \mathcal{C}_3^4 \\
& + \frac{1}{4} \tilde{\partial}_t (G_{tt} \tilde{\partial} \mathcal{C}_4^3) \mathcal{G}_{33} + \frac{1}{4} \tilde{\partial}_t (G_{tt} \tilde{\partial} D_{44}) \mathcal{C}_4^4 \\
& = (1 - a^{-4} b^2) (-\tilde{a}^2 + a\tilde{a}) + (\tilde{a}^{-4} - b^2) (\tilde{a}^2 - a\tilde{a}). \quad (A.25)
\end{align*}
\]

Finally, we have

\[
\begin{align*}
\mathcal{R}_{2(1)2(1)} & = - \left[ 3 \frac{\tilde{a}^2}{a^2} \right] + \left[ 3 \frac{\tilde{a}^2}{a^2} \right], \\
\mathcal{R}_{2(2)2(2)} & = - \left[ \tilde{a}^2 - a\tilde{a} \right] + a^{-4} \left[ \tilde{a}^2 - a\tilde{a} \right], \\
\mathcal{R}_{2(3,4)2(3,4)} & = - (1 - a^{-4} b^2) \left[ \tilde{a}^2 - a\tilde{a} \right] + (\tilde{a}^{-4} - b^2) \left[ \tilde{a}^2 - a\tilde{a} \right]. \quad (A.26)
\end{align*}
\]

A.3.2 Calculation of \( \mathcal{R}_{MN} \)

\[
\begin{align*}
\mathcal{R}_{MN} & = *\mathcal{K}_{MN} - S^P \mathcal{K}_{PM} S^Q_N \\
& = 1 \tilde{\partial}^i \partial_{(N} \mathcal{M}_{i j}) + \tilde{\partial}^j \partial_{(N} \mathcal{M}_{i j)} \\
& + \frac{1}{2} \tilde{\partial}_d \left( C_{j}^i \partial_j \mathcal{M}_{MN} \right) + \frac{1}{2} \tilde{\partial}_d \left( D_{j}^i \partial_j \mathcal{M}_{MN} \right) \\
& + 2 \tilde{\partial}_{MN} \tilde{\partial}_d \mathcal{M} \\
& - \tilde{\partial}^i \partial_{(N} \mathcal{M}_{i(j)} \mathcal{H}_{1(1)(j)}) - \tilde{\partial}^j \partial_{(N} \mathcal{M}_{i(j)} \mathcal{H}_{1(1)(j)}) \\
& - \tilde{\partial}_d \left( B_{j}^i \partial_{(N} \mathcal{M}_{i(j)} \mathcal{H}_{1(1)(j)}) \right) - \tilde{\partial}_d \left( D_{j}^i \partial_{(N} \mathcal{M}_{i(j)} \mathcal{H}_{1(1)(j)}) \right) \\
& - \tilde{\partial}^i \partial_{(N} \mathcal{M}_{i(j)} \mathcal{H}_{1(1)(j)}) - \tilde{\partial}^j \partial_{(N} \mathcal{M}_{i(j)} \mathcal{H}_{1(1)(j)}) \\
& - \tilde{\partial}_d \left( \mathcal{H}_{1(1)(i)} \mathcal{M}_{(i(j)} \mathcal{H}_{1(1)(j)}) \right) - \tilde{\partial}_d \left( \mathcal{H}_{1(1)(i)} \mathcal{M}_{(i(j)} \mathcal{H}_{1(1)(j)}) \right) \\
& - \frac{1}{2} S^1_{M} \tilde{\partial}_{1(i)} d \left( \mathcal{H}^{1(1)(i)} \mathcal{M}_{1(1)(i)} \mathcal{H}_{1(1)(i)}) \right) - \frac{1}{2} S^1_{M} \tilde{\partial}_{1(i)} d \left( \mathcal{H}^{1(1)(i)} \mathcal{M}_{1(1)(i)} \mathcal{H}_{1(1)(i)}) \right) \\
& - \frac{1}{2} S^1_{M} \tilde{\partial}_{2(i)} d \left( \mathcal{H}^{2(1)(i)} \mathcal{M}_{1(1)(i)} \mathcal{H}_{1(1)(i)}) \right) - \frac{1}{2} S^1_{M} \tilde{\partial}_{2(i)} d \left( \mathcal{H}^{2(1)(i)} \mathcal{M}_{1(1)(i)} \mathcal{H}_{1(1)(i)}) \right)
\end{align*}
\]

31
\[
- \frac{1}{2} S_{M_0}^{1(i)} \partial_{1(0)} \partial_{1(0)} d \left( H_{1(0)} \partial_{1(0)} H_{1(0)} \right) S_{N}^{2(j)} - \frac{1}{2} S_{M_0}^{1(i)} \partial_{1(0)} d \left( H_{1(0)} \partial_{2(0)} H_{1(0)} \right) S_{N}^{2(j)}
\]

\[
- \frac{1}{2} S_{M_0}^{1(i)} \partial_{2(0)} d \left( H_{1(0)} \partial_{1(0)} H_{1(0)} \right) S_{N}^{2(j)} - \frac{1}{2} S_{M_0}^{1(i)} \partial_{2(0)} d \left( H_{1(0)} \partial_{2(0)} H_{1(0)} \right) S_{N}^{2(j)}
\]

\[
- \frac{1}{2} S_{M_0}^{2(i)} \partial_{1(0)} d \left( H_{1(0)} \partial_{1(0)} H_{2(0)} \right) S_{N}^{2(j)} - \frac{1}{2} S_{M_0}^{2(i)} \partial_{1(0)} d \left( H_{1(0)} \partial_{2(0)} H_{2(0)} \right) S_{N}^{2(j)}
\]

\[
- \frac{1}{2} S_{M_0}^{2(i)} \partial_{2(0)} d \left( H_{1(0)} \partial_{1(0)} H_{2(0)} \right) S_{N}^{2(j)} - \frac{1}{2} S_{M_0}^{2(i)} \partial_{2(0)} d \left( H_{1(0)} \partial_{2(0)} H_{2(0)} \right) S_{N}^{2(j)}
\]

\[
-2 S_{M_0}^{1(i)} \partial_{1(i)} d S_{N}^{1(j)} + 2 S_{M_0}^{1(i)} \partial_{1(i)} d S_{N}^{2(j)} - 2 S_{M_0}^{2(i)} \partial_{2(i)} d S_{N}^{1(j)} + 2 S_{M_0}^{2(i)} \partial_{2(i)} d S_{N}^{2(j)}
\]
Using our notations, we find

\[+ \frac{1}{2} S_{M}^{1(i)} \partial_{(1)} d \left( H_{1(k)}^{1(i)} \partial_{1(k)} H_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( H_{2(k)}^{1(i)} \partial_{2(k)} H_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( H_{1(k)}^{2(i)} \partial_{1(k)} H_{2(l)}^{2(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( H_{2(k)}^{2(i)} \partial_{2(k)} H_{2(l)}^{2(i)} \right) S_{N}^{1(j)}
\]

Using our notations, we find

\[+ \frac{1}{2} S_{M}^{1(i)} \partial_{1(I)} d \left( \mathcal{H}_{1(k)}^{1(i)} \partial_{1(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{2(k)}^{1(i)} \partial_{2(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{1(k)}^{2(i)} \partial_{1(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{2(k)}^{2(i)} \partial_{2(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)}
\]

Using our notations, we find

\[+ \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{1(k)}^{1(i)} \partial_{1(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{2(k)}^{1(i)} \partial_{2(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{1(k)}^{2(i)} \partial_{1(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{2(k)}^{2(i)} \partial_{2(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)}
\]

Using our notations, we find

\[+ \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{1(k)}^{1(i)} \partial_{1(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{2(k)}^{1(i)} \partial_{2(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{1(k)}^{2(i)} \partial_{1(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{2(k)}^{2(i)} \partial_{2(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)}
\]

Using our notations, we find

\[+ \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{1(k)}^{1(i)} \partial_{1(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{2(k)}^{1(i)} \partial_{2(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{1(k)}^{2(i)} \partial_{1(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{2(k)}^{2(i)} \partial_{2(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)}
\]

Using our notations, we find

\[+ \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{1(k)}^{1(i)} \partial_{1(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{2(k)}^{1(i)} \partial_{2(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{1(k)}^{2(i)} \partial_{1(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{2(k)}^{2(i)} \partial_{2(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)}
\]

Using our notations, we find

\[+ \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{1(k)}^{1(i)} \partial_{1(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{2(k)}^{1(i)} \partial_{2(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{1(k)}^{2(i)} \partial_{1(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{2(k)}^{2(i)} \partial_{2(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)}
\]

Using our notations, we find

\[+ \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{1(k)}^{1(i)} \partial_{1(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{2(k)}^{1(i)} \partial_{2(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{1(k)}^{2(i)} \partial_{1(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{2(k)}^{2(i)} \partial_{2(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)}
\]

Using our notations, we find

\[+ \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{1(k)}^{1(i)} \partial_{1(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{2(k)}^{1(i)} \partial_{2(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{1(k)}^{2(i)} \partial_{1(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{2(k)}^{2(i)} \partial_{2(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)}
\]

Using our notations, we find

\[+ \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{1(k)}^{1(i)} \partial_{1(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{1(l)} d \left( \mathcal{H}_{2(k)}^{1(i)} \partial_{2(k)} \mathcal{H}_{1(l)}^{1(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{1(k)}^{2(i)} \partial_{1(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)} + \frac{1}{2} S_{M}^{1(i)} \partial_{2(l)} d \left( \mathcal{H}_{2(k)}^{2(i)} \partial_{2(k)} \mathcal{H}_{2(l)}^{2(i)} \right) S_{N}^{1(j)}
\]
\[-\frac{1}{2} S_{M}^{1(i)} \partial \mathcal{I} d \left( G_{ik} \partial^{k} B_{j}^{i} \right) S_{N}^{2(j)} - \frac{1}{2} S_{M}^{1(i)} \partial \mathcal{I} d \left( B_{i}^{k} \partial_{k} B_{j}^{i} \right) S_{N}^{2(j)} - \frac{1}{2} S_{M}^{1(i)} \partial \mathcal{I} d \left( C_{i}^{j} \partial^{k} B_{j}^{i} \right) S_{N}^{2(j)}\]

\[-\frac{1}{2} S_{M}^{2(i)} \partial \mathcal{I} d \left( G_{ik} \partial^{k} C_{j}^{i} \right) S_{N}^{1(j)} - \frac{1}{2} S_{M}^{2(i)} \partial \mathcal{I} d \left( B_{i}^{k} \partial_{k} C_{j}^{i} \right) S_{N}^{1(j)} - \frac{1}{2} S_{M}^{2(i)} \partial \mathcal{I} d \left( C_{i}^{j} \partial^{k} C_{i}^{j} \right) S_{N}^{1(j)}\]

\[-\frac{1}{2} S_{M}^{2(i)} \partial \mathcal{I} d \left( G_{ik} \partial^{k} D_{ij} \right) S_{N}^{2(j)} - \frac{1}{2} S_{M}^{2(i)} \partial \mathcal{I} d \left( B_{i}^{k} \partial_{k} D_{ij} \right) S_{N}^{2(j)} - \frac{1}{2} S_{M}^{2(i)} \partial \mathcal{I} d \left( C_{i}^{j} \partial^{k} D_{ij} \right) S_{N}^{2(j)}\]

\[-\frac{1}{2} S_{M}^{1(i)} \partial \mathcal{I} d \left( D^{ik} \partial_{k} G_{ij} \right) S_{N}^{1(j)} - \frac{1}{2} S_{M}^{1(i)} \partial \mathcal{I} d \left( D^{ik} \partial_{k} B_{j}^{i} \right) S_{N}^{2(j)} - \frac{1}{2} S_{M}^{1(i)} \partial \mathcal{I} d \left( D^{ik} \partial_{k} C_{j}^{i} \right) S_{N}^{1(j)}\]

\[-\frac{1}{2} S_{M}^{2(i)} \partial \mathcal{I} d \left( D^{ik} \partial_{k} B_{j}^{i} \right) S_{N}^{2(j)}\]

\[-2 S_{M}^{1(i)} \partial \mathcal{I} d S_{N}^{1(j)} - 2 S_{M}^{1(i)} \partial \mathcal{I} \partial_{j} d S_{N}^{2(j)} - 2 S_{M}^{2(i)} \partial \mathcal{I} \partial_{j} d S_{N}^{1(j)} - 2 S_{M}^{2(i)} \partial \mathcal{I} \partial_{j} d S_{N}^{2(j)}\]
\[ + \frac{1}{2} S^{(i)}_{M} \partial d \left( G_{ki} \partial k G_{li} \right) S^{(j)}_{N} + \frac{1}{2} S^{(i)}_{M} \partial d \left( C_{k} \partial k G_{li} \right) S^{(j)}_{N} + \frac{1}{2} S^{(i)}_{M} \partial d \left( G_{ki} \partial k C_{j} \right) S^{(j)}_{N} \]

\[ + \frac{1}{2} S^{(i)}_{M} \partial d \left( G_{kj} \partial k G_{li} \right) S^{(j)}_{N} + \frac{1}{2} S^{(i)}_{M} \partial d \left( C_{j} \partial k G_{li} \right) S^{(j)}_{N} + \frac{1}{2} S^{(i)}_{M} \partial d \left( G_{kj} \partial k C_{i} \right) S^{(j)}_{N} \]

\[ + \frac{1}{2} S^{(i)}_{M} \partial d \left( D^{kj} \partial k D^{li} \right) S^{(j)}_{N} + \frac{1}{2} S^{(i)}_{M} \partial d \left( D^{kj} \partial k D^{li} \right) S^{(j)}_{N} + \frac{1}{2} S^{(i)}_{M} \partial d \left( D^{kj} \partial k C_{j} \right) S^{(j)}_{N} \]

\[ + \frac{1}{2} S^{(i)}_{M} \partial d \left( C_{j} \partial k D^{li} \right) S^{(j)}_{N} + \frac{1}{2} S^{(i)}_{M} \partial d \left( C_{j} \partial k D^{li} \right) S^{(j)}_{N} + \frac{1}{2} S^{(i)}_{M} \partial d \left( C_{j} \partial k C_{l} \right) S^{(j)}_{N} \]

\[ \frac{1}{2} S^{(i)}_{M} \partial d \left( C_{j} \partial k C_{i} \right) S^{(j)}_{N} + \frac{1}{2} S^{(i)}_{M} \partial d \left( C_{j} \partial k C_{i} \right) S^{(j)}_{N} + \frac{1}{2} S^{(i)}_{M} \partial d \left( C_{j} \partial k C_{i} \right) S^{(j)}_{N} \]

\[ = 2 \tilde{d} \left( \tilde{a} - a \right) - 2 da \tilde{a}. \]

(a.31)

Finally, we have

\[ * R_{2(1),2(1)} = 2 \tilde{d} - 2 \tilde{d}, \]

\[ * R_{2(2),2(2)} = 2 \tilde{d} \left( \tilde{a} - a \right) - 2 da \tilde{a}, \]

(a.32)
Using the definitions of Hubble parameters, EOM of the generalized metric can be rewritten as

\[ * R_{2(3,4)2(3,4)} = -(1 - a^{-4}b^2) \left[ 2a\dot{a} \right] + (\dot{a}^{-4} - b^2) \left[ 2\ddot{a} \right]. \] (A.33)

Recall that

\[ * R_{2(t)2(t)} = - \left[ 3\dot{a}^2 + \frac{a^2}{2} \right] + \left[ 3\dot{a}^2 - 2\ddot{a} \right], \]
\[ * R_{2(2)2(2)} = - \left[ \dot{a}^2 - a\ddot{a} + 2\dot{a}\ddot{a} \right] + \dot{a}^{-4} \left[ \dot{a}^2 - a\ddot{a} + 2\dot{a}\ddot{a} \right], \]
\[ * R_{2(3,4)2(3,4)} = -(1 - a^{-4}b^2) \left[ \dot{a}^2 - a\ddot{a} + 2\dot{a}\ddot{a} \right] + (\dot{a}^{-4} - b^2) \left[ \dot{a}^2 - a\ddot{a} + 2\dot{a}\ddot{a} \right]. \] (A.34)

We get the equation of motion of the generalized metric

\[ R_{2(t)2(t)} = - \left[ 3H^2 - 2\ddot{a} \right] + \left[ 3\dot{H}^2 - 2\ddot{a} \right], \]
\[ R_{2(2)2(2)} = - a^2 \left[ -\dot{H} + 2d\dot{H} \right] + \dot{a}^{-4} \left[ -\dot{H} + 2d\dot{H} \right], \]
\[ R_{2(3,4)2(3,4)} = -(a^2 - a^{-2}b^2) \left[ -\dot{H} + 2d\dot{H} \right] + (\dot{a}^{-2} - a^2b^2) \left[ -\dot{H} + 2d\dot{H} \right]. \] (A.35)

After lengthy calculations, we also find there exist symmetries

\[ R_{1(p)1(q)} \rightarrow H^{\bullet\bullet} \leftrightarrow H^{\bullet\bullet}, \quad \dot{\partial}^\bullet \leftrightarrow \partial^\bullet \rightarrow R_{2(p)2(q)}, \]
\[ R_{1(p)2(q)} \rightarrow H^{\bullet\bullet} \leftrightarrow H^{\bullet\bullet}, \quad \dot{\partial}^\bullet \leftrightarrow \partial^\bullet \rightarrow R_{2(p)1(q)}. \] (A.37)

It means that we do not need to calculate other terms. Finally, the EOM of DFT are

\[ \left[ 3H^2 - 4\ddot{d} + 4d^2 \right] + \left[ 3\dot{H}^2 - 4\ddot{d} + 4d^2 \right] = 0, \]
\[ - \left[ 3H^2 - 2\ddot{d} \right] + \left[ 3\dot{H}^2 - 2\ddot{d} \right] = 0, \]
\[ - \left[ -\dot{H} + 2d\dot{H} \right] + \left[ -\dot{H} + 2d\dot{H} \right] = 0. \] (A.38)

Clearly, the barred part and unbarred parts are identical, being the same as the EOM in string cosmology. This indicates that if \( a(t, \bar{t}) \) is a solution, an \( O(D, D) \) rotation of \( a(t, \bar{t}) \) is also a solution, as one can expect from the explicit \( O(D, D) \) invariance in the action. One can easily check that

\[ a_{\pm}(\bar{t}, t) = \left| \frac{t}{\bar{t}} \right|^{\pm 1/\sqrt{D-1}}, \quad d(\bar{t}, t) = -\frac{1}{2} \ln |\bar{t}| t, \]
\[ a_\pm (\tilde{t}, t) = |t \tilde{t}|^{\pm 1/\sqrt{D-1}}, \quad d (\tilde{t}, t) = -\frac{1}{2} \ln |t \tilde{t}|, \] (A.39)

are solutions of the EOM.