Stochastic control of quantum coherence

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The application of a random modulation of a system parameter usually increases decoherence effects. Here we show how, employing an appropriate stochastic modulation, it is instead possible to preserve the quantum coherence of a system.

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Controlling quantum coherence is one of the most fundamental issues in modern information processing [1]. The most popular solution in the field of quantum information are quantum error correction codes [2] and error avoiding codes [3], both based on encoding the state into carefully selected subspaces of a larger Hilbert space involving ancillary systems. The main limitation of these strategies for combatting decoherence is the large amount of extra space resources required [4]; in particular, if fault tolerant error correction is also considered, the number of ancillary qubits enormously increases.

For this reason, other alternative approaches which do not require any ancillary resources have been pursued, and which may be divided into two main categories, according to the form of interaction with the system under study [5]. If the interaction is one way, so that the controller acts on the system without obtaining any information about its state, then the controller is called “open loop” [6]. By contrast, if the controller acts on the system on the basis of information that it obtains about the state of the system, then it is called “closed loop” [6]. In standard open loop techniques, control of quantum dynamics is achieved through the application of suitably tailored, time-dependent and deterministic, driving forces. Here we want to extend open loop control strategies by considering the possibility of using stochastic parameter modulations for quantum control.

The common wisdom is that whenever a system is subject to noise, the quality of the dynamic control is degraded, and that quantum coherence in particular is rapidly destroyed [7]. Here we show that this is not generally true and that quantum decoherence can be significantly suppressed if an appropriately tailored stochastic modulation of a system parameter is used. This fact is illustrated in this letter by considering the simple case of the dynamics of a single radiation mode in a lossy cavity, but the results can be generalized. In this open system, decoherence has a dissipative origin since it is due to photons’ leakage out of the cavity, and the stochastic control strategy will be implemented by modulating the cavity length.

Let us consider a single radiation mode with annihilation operator \(a\) within a lossy cavity, whose characteristic frequency is \(\omega = n \pi c / L\), with \(n\) an integer number, \(c\) the speed of light and \(L\) the cavity length. If photons’ leakage occurs through a partially transmitting mirror, the decay rate will be given by \(\gamma = cT/2L\), with \(T\) the mirror’s transmittivity.

In the case of optical frequencies, thermal excitation from the environment of the continuum of modes outside the cavity is negligible and the dynamics is well described by the master equation [8]

\[
\dot{\rho} = \mathcal{L}\rho = -i[\rho, H] + \gamma D[a] \rho,
\]

where \(D[A]B \equiv AB A^\dagger - \{A^\dagger A, B\}/2\) describes photon decay into the vacuum. This decay is also responsible for the rapid decay of any eventual quantum coherence generated within the cavity [9].

Let us now try to preserve the quantum coherence of the radiation mode using an appropriate stochastic control strategy. In particular, we randomly modulate the cavity length, that is, \(L \rightarrow L(t) = L_0/g(t)\) with \(g(t)\) a positive stochastic process. This is equivalent to a simultaneous random modulation of both the frequency and the decay rate of the cavity. This stochastic modulation of the cavity length moreover yields a dynamics which is indistinguishable from that driven by the constant, unmodulated, Liouvillian superoperator \(\mathcal{L}_0 = -i\omega_0 [a^\dagger a, \ldots] + \gamma_0 D[a], \ldots\), where the parameters \(\omega_0\) and \(\gamma_0\) are fixed, in the presence of a random evolution time \(t'\). In fact, one has

\[
\rho(t) = T \exp \left\{ \int_0^t ds \mathcal{L}(s) \right\} \rho(0) = \exp \{ \mathcal{L}_0 t'(t) \} \rho(0),
\]

where \(T\) denotes time ordering, and we have defined the stochastic evolution time \(t'(t) = \int_0^t ds y(s)\). This observation reminds the recently proposed model-independent approach to decoherence in quantum mechanics [10] in which the evolution time is regarded as a random variable.

To establish a connection between the randomized time evolution of Refs. [10] and the model of a cavity mode with a stochastically modulated cavity length we assume that the statistical properties of the cavity length modulation factor \(y(t)\) are determined just by the Gamma distribution \(P(t, t')\) for the random evolution time \(t'\) of Ref. [10]. To be more specific, we assume that at discrete times \(t_n\) separated by a time interval \(\Delta t\),
independent random variables \( y(t_n) \) are generated, e.g. by a computer, according to the distribution

\[
P(y) = \left( \frac{\Delta t}{\tau^2} \right)^{\Delta t/\tau} \frac{\Gamma(\Delta t/\tau)}{\Gamma(\Delta t/\tau - 1)} \exp(-y\Delta t/\tau).
\] (2)

The random number \( y(t_n) \) determines the “instantaneous” cavity length \( L(t_n) = L_0/y(t_n) \). Equation (2) is a Gamma probability distribution \([13]\), with parameter \( \Delta t/\tau \), where \( \tau \) quantifies the strength of the fluctuations. In fact it is \( \langle y(t_n) \rangle = 1 \) and \( \langle (y(t_n) - 1)^2 \rangle = \tau/\Delta t \). Choosing the probability distribution (2) means choosing a specific, uncommon way of modulating the cavity length. In fact it is easy to see \([13]\) that it implies a strongly non-Gaussian cavity length modulation, which assumes Gaussian properties \( P(y) \approx \exp \left[ -(y - 1)^2 \Delta t/2\tau \right] / \sqrt{2\pi \tau/\Delta t} \) in the limit \( \Delta t/\tau \gg 1 \) only.

The above introduced discreteness in time, is dictated by the unavoidably finite rate of random number generation. Nevertheless, the time interval \( \Delta t \) can be much smaller than the typical time scale upon which one observes the system dynamics (this is essentially determined by \( \gamma_0^{-1} \) times the inverse of the mean photon number). Hence, we can consider the independent cavity length rescalings \( y(t_n) \) as occurring continuously in time, i.e. \( \Delta t \to 0 \). In the continuous approximation, \( y(t) \) becomes a white, non-Gaussian stochastic process, which can be rewritten as \( y(t) = 1 + \xi(t) \), where \( \xi(t) \) is a zero-mean stochastic process, such that \( \langle \xi(t)\xi(t') \rangle = \tau \delta(t-t') \).

As noted above, the evolution in the presence of the cavity length modulation can be reinterpreted as the dynamics of a cavity with fixed length and with a random evolution time \( t'(t) \). However, since the sum (integral) of independent Gamma-distributed processes is again a Gamma-distributed process with the parameter given by the sum (integral) of the single parameters \([13]\), the probability distribution of the effective random time \( t'(t) \) is just the Gamma distribution of Refs. \([13,12]\).

\[
P(t, t') = \frac{e^{-t'/\tau} [t'/\tau]^{t'/\tau - 1}}{\Gamma(t'/\tau)} \quad t' \geq 0,
\] (3)

with parameter \( t/\tau \). This fact simplifies the study of the dynamics of the dissipative radiation mode in the presence of the stochastic modulation of the cavity length. In fact any dynamical quantity can be obtained by first evaluating it in the absence of modulation and then averaging it over the random time distribution (3).

A first interesting quantity is the time evolution of the intracavity mean photon number \( \langle a(t) a(t) \rangle = \langle n(t) \rangle \) (the time average is denoted by the overbar). In the absence of any stochastic modulation one has \( \langle n(t) \rangle = \langle n(0) \rangle = \exp(-\gamma_0 t) \), showing the energy decay due to photon leakage through the partially transmitting mirror. In the presence of the cavity length modulation one has instead

\[
\langle n(t) \rangle = \int dt' P(t, t')(n(t')) = \langle n(0) \rangle e^{-\gamma_0 t},
\]

with the new effective energy decay rate \( \tilde{\gamma}_e \) given by

\[
\tilde{\gamma}_e = \tau^{-1} \log \left( 1 + \gamma_0 \tau \right).
\] (4)

It is always \( \tilde{\gamma}_e \leq \gamma_0 \) and therefore cavity length modulation always yields inhibition of dissipation. What is relevant is that the suppression of energy damping increases for increasing fluctuation strength parameter \( \tau \) and that one has perfect inhibition, \( \tilde{\gamma}_e = 0 \), in the limit \( \gamma_0 \tau \to \infty \), when the modulation becomes strongly non-Gaussian.

Another interesting quantity is the behavior of the cavity field, which is essentially expressed by the average \( \langle a(t) \rangle \). In the absence of any stochastic modulation one has \( \langle a(t) \rangle = \langle a(0) \rangle e^{-i\omega_0 t - \gamma_0 t/2} \). This quantity shows the effect of photon leakage on the phase of the intracavity field and it may provide some information on the possibility to control decoherence using cavity length modulation. In fact, since in this model decoherence is just caused by photon leakage, we expect that any control exerted on the field decay rate will reflect itself into a control of quantum decoherence. In the presence of the cavity length modulation one has \( \langle a(t) \rangle = \langle a(0) \rangle e^{-i\omega_0 t - \gamma_0 t/2} \), where the renormalized field decay rate \( \tilde{\gamma} \) is

\[
\tilde{\gamma} = \tau^{-1} \log \left[ (1 + \gamma_0 \tau/2)^2 + \omega_0^2 \tau^2 \right],
\] (5)

and the renormalized oscillation frequency is \( \tilde{\omega} = \tau^{-1} \tan^{-1} \left[ \omega_0 \tau/(1 + \gamma_0 \tau/2) \right] \). The effective field decay rate \( \tilde{\gamma} \) of Eq. (5) is different from the effective energy decay rate \( \tilde{\gamma}_e \), because it is sensitive not only to the decay rate modulation induced by the cavity length modulation (as \( \tilde{\gamma}_e \)), but also to the simultaneously induced frequency modulation, which may provide phase damping effects. The behavior of the effective field decay rate \( \tilde{\gamma}_e \) is, plotted in Fig. 1 as a function of the modulation strength parameter \( \gamma_0 \tau \). Curve (a) of Fig. 1 refers to Eq. (5) and, at variance with what happens for \( \tilde{\gamma}_e \) of Eq. (4), shows an initial increase of the effective cavity decay rate for increasing modulation amplitude \( \tau \). This means that not only cavity length increases the cavity field decay rate. This decay acceleration reaches a maximum around \( \omega_0 \tau \approx 1 \) and then starts to decrease for increasing \( \tau \). What is interesting is that the ratio \( \tilde{\gamma}/\gamma_0 \) becomes less than one and tends to zero for larger \( \tau \). This means that not only dissipation, but also field decay can be completely inhibited by the cavity length modulation at a sufficiently large \( \tau \) parameter, when the stochastic modulation assumes strongly non-Gaussian properties. The threshold value \( \tau_{th} \) for decay inhibition, \( \tilde{\gamma} < \gamma_0 \), depends on the cavity quality factor \( Q = \omega_0/\gamma_0 \), and it can be calculated iteratively, obtaining

\[
\gamma_0 \tau_{th} \approx 2 \log Q + 2 \log \left[ 2 \log \tau \right].
\] (6)
In the absence of modulation, Eq. (6) leads to the simple result \( \mathcal{V} = \exp \{-2|\alpha|^2 [1 - \eta(t)]\} \). In the presence of the stochastic modulation of the cavity length, the corresponding visibility can be evaluated by performing an appropriate average of the dynamical quantities over the probability distribution \( P(t, t') \). In particular, we have to consider the following replacements in Eq. (6):

\[
e^{-2|\alpha|^2 \tau (1 - \eta(t))} \langle X | \alpha(t) \rangle \langle -\alpha(t) | X \rangle \rightarrow e^{-2|\alpha|^2 \tau (1 - \eta(t))} \langle X | \alpha(t) \rangle \langle -\alpha(t) | X \rangle, \quad \langle X | \alpha(t) \rangle \langle \alpha(t) | X \rangle \rightarrow \langle X | \alpha(t) \rangle \langle \alpha(t) | X \rangle, \quad \langle X | -\alpha(t) \rangle \langle -\alpha(t) | X \rangle \rightarrow \langle X | -\alpha(t) \rangle \langle -\alpha(t) | X \rangle
\]

to get the corresponding, averaged, \( \mathcal{V} \). A cumbersome analytic expression can be obtained \([14]\) and the corresponding behavior of \( \mathcal{V} \) as a function of time for different values of the modulation strength parameter \( \tau \) is shown in Fig. 2. The relevant result is that the visibility, i.e., the quantum coherence properties of the system, behaves in the same way as the field decay rate. In particular we see either an acceleration, or, more importantly, a deceleration of decoherence according to the value of the parameter \( \tau \). The usual decay of the visibility in the absence of modulation (\( \tau = 0 \)) is shown with a dashed curve. When \( \gamma_0 \tau \neq 0 \) we observe an acceleration of the decay of the visibility (lower curve) when the modulation strength \( \tau \) is not too large (\( \gamma_0 \tau = 1.5 \) in the figure) or a slowing down of the decay (upper curves) when \( \tau \) becomes sufficiently large (\( \gamma_0 \tau = 20, 100 \) in Fig. 2). The threshold value between the two behaviors coincides with that for decay inhibition \( \tau_{\text{th}} \) of Eq. (6).

![FIG. 1. Log-log plot of the ratio \( \tilde{\gamma}/\gamma_0 \) as function of \( \gamma_0 \tau \). Curve (a) refers to the Gamma stochastic modulation of Eq. (2), and curve (b) refers to a Gaussian stochastic cavity length modulation. We have also set \( Q = 10^7 \).](image1)

Let us now directly address the decoherence control issue. We consider as initial state of the cavity field a linear superposition state. In order to control the coherence in the continuous variable case we shall consider the well known Schrödinger cat state, a superposition of two coherent states of the form \( |\alpha\rangle + |-\alpha\rangle \) and see what happens by employing the above stochastically modulated dynamics. The same could eventually be done for a superposition of Fock states.

The time evolution of the Schrödinger cat state in the absence of any modulation is determined by the usual Liouvillian and it can be described in the following way \([11]\): \( \rho(t) = N^2 \left( |\alpha(t)\rangle \langle \alpha(t)| + | -\alpha(t)\rangle \langle -\alpha(t)| + \exp[-2|\alpha|^2(1 - \eta(t))] |\alpha(t)\rangle \langle -\alpha(t)| + | -\alpha(t)\rangle \langle \alpha(t)| \right) \), where we have introduced \( \alpha(t) = \alpha \exp[-(i\omega_0 + \gamma_0/2)t] \) and \( \eta(t) = e^{-\gamma_0 t}. \) A good characterization of the time development of the quantum coherence of the state of the cavity mode is provided by the visibility with respect to an observable \([11]\). For the quadrature observable \( X = (a + a^\dagger)/\sqrt{2} \), the quantum visibility is given by

\[
\mathcal{V} = \frac{|e^{-2|\alpha|^2(1 - \eta(t))} \langle X | \alpha(t) \rangle \langle -\alpha(t) | X \rangle|}{\sqrt{|\langle X | \alpha(t) \rangle \langle \alpha(t) | X \rangle| |\langle X | -\alpha(t) \rangle \langle -\alpha(t) | X \rangle|}}, \quad (7)
\]

where \( \langle X | \alpha \rangle = (\frac{1}{2})^{1/4} \exp \left[ -\frac{|\alpha|^2}{2} - \frac{X^2}{2} - \frac{X^2}{2} + \sqrt{2}X \alpha \right] \).
the above decoherence control results with respect to small changes of the stochastic modulation. As suggested above, the Gamma stochastic modulation of the cavity length could be experimentally implemented using a cavity with a computer-controlled length, and a fast random number generator. Any eventual imperfect stochastic modulation of the cavity length can be described in terms of an additional, zero-mean, white, stochastic process \( e(t) \), such that \( \langle e(t)e(t') \rangle = \sigma \delta(t-t') \). The stochastic process \( e(t) \) describes the “error” in the modulation at time \( t \), so that the actual fluctuating cavity length is \( L(t) = L_0/(y(t) + e(t)) \). We have checked that the above decay and decoherence inhibition results are not significantly changed as long as the strength \( \sigma \) of the additional, undesired, modulation \( e(t) \) is not too large. More precisely, it is possible to see with a perturbative treatment that the results are stable if \( \sigma \gamma_0 Q^2 \ll 1 \). This condition can be explicitly seen for example in the case of the renormalized field decay rate \( \tilde{\gamma} \) and it is shown in Fig. 3 in the case of a Gaussian stochastic process \( e(t) \).

![Log-log plot of the ratio \( \tilde{\gamma}/\gamma_0 \) as function of \( \gamma_0 \tau \) for different values of the “error strength” \( \sigma \). Curves are for \( \sigma \gamma_0 = 0 \) (a), \( 10^{-5} \) (b), \( 10^{-4} \) (c). We have also set \( Q = 10^2 \).](image)

In conclusion, we have studied the possibility of a stochastic control of (dissipative) decoherence by tailoring suitable random modulations of a system parameter. Against the widespread opinion that “noise” is detrimental for quantum effects, we have shown that if the statistical properties of the modulation are appropriately chosen, this stochastic control strategy could be used in principle to control decoherence. Here we have considered the specific model of a single cavity mode with a randomly modulated cavity length. We have seen that, when the modulation is stochastic, with strongly non-Gaussian properties, decoherence and dissipation can be inhibited and that the scheme can even tolerate some imperfection in the modulation. Although we have considered a specific model, our results can be generalized to dissipative systems in Ohmic environments where the damping rate and the frequency have the same fluctuations [14]. That allows us to recast the above described treatment. Finally, our approach shares some similarities with the inhibition of atomic decay through random ac-Stark shift discussed in Ref. [13]. However our proposal is different since it strongly depends on the statistical properties of the random modulation and it is especially suited to the control of quantum decoherence. Another analogy occurs with the use of kicks to prevent the decay of a system [14]. In this latter case, dephasing introduced by kicks were deterministic processes well defined in time. Instead, the present approach is merely probabilistic, so it would be more manageable.

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[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, (Cambridge University Press, 2000).
[2] P. W. Shor, Phys. Rev. A 52, 2493 (1995), A. M. Steane, Proc. R. Soc. London A 452, 2551 (1995); E. Knill and R. Laflamme, Phys. Rev. A 55, 900 (1997).
[3] P. Zanardi and M. Rasetti, Phys. Rev. Lett. 79, 3306 (1997); L. M. Duan and G. C. Guo, Phys. Rev. Lett. 79, 1953 (1997); D. A. Lidar, I. L. Chuang, and K. B. Whaley, Phys. Rev. Lett. 81, 2594 (1998).
[4] A. M. Steane, Nature (London) 399, 124 (1999).
[5] S. Lloyd, Phys. Rev. A 62, 022108 (2000).
[6] L. Viola and S. Lloyd, Phys. Rev. A 58, 2733 (1998); D. Vitali and P. Tombesi, ibid. 59, 4178 (1999); L. Viola, et al. Phys. Rev. Lett. 82, 2417 (1998).
[7] H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett. 70, 548 (1993); Phys. Rev. A 49, 1350 (1994); P. Tombesi and D. Vitali, Phys. Rev. A 51, 4913 (1995); P. Goetsch et al., ibid. 54, 4519 (1996); D. Vitali et al., Phys. Rev. Lett. 79, 2442 (1997).
[8] D. Giulini et al., *The Appearance of Classical World in Quantum Mechanics* (Springer, Berlin, 1996), and references therein.
[9] R. Bonifacio, et al., Phys. Rev. A 61, 053802 (2000).
[10] C. W. Gardiner, *Quantum Noise*, (Springer, Berlin, 1991).
[11] D. F. Walls and G. J. Milburn, Phys. Rev. A 31, 2403 (1985); G. J. Milburn and D. F. Walls, ibid. 38, 1087 (1988).
[12] R. Bonifacio, Il Nuovo Cimento 114B, 473 (1999).
[13] L. Mandel, and E. Wolf, *Optical Coherence and Quantum Optics*, (Cambridge University Press, Cambridge, 1995), p. 30.
[14] S. Mancini, D. Vitali, R. Bonifacio, and P. Tombesi, in preparation.
[15] G. Harel et al., Opt. Exp. 2, 355 (1998).