Novel Reformulations and Efficient Algorithms for the Generalized Trust Region Subproblem

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Abstract

We present a new solution framework to solve the generalized trust region subproblem (GTRS) of minimizing a quadratic objective over a quadratic constraint. More specifically, we derive a convex quadratic reformulation (CQR) via minimizing a linear objective over two convex quadratic constraints for the GTRS. We show that an optimal solution of the GTRS can be recovered from an optimal solution of the CQR. We further prove that this CQR is equivalent to minimizing the maximum of the two convex quadratic functions derived from the CQR for the case under our investigation. Although the latter minimax problem is nonsmooth, it is well-structured and convex. We thus develop two steepest descent algorithms corresponding to two different line search rules. We prove for both algorithms their global sublinear convergence rates. We also obtain a local linear convergence rate of the first algorithm by estimating the Kurdyka-Łojasiewicz exponent at any optimal solution under mild conditions. We finally demonstrate the efficiency of our algorithms in our numerical experiments.

1 Introduction

We consider the following generalized trust region subproblem (GTRS),

\[
\begin{align*}
(P) \quad \min & \quad f_1(x) := \frac{1}{2} x^\top Q_1 x + b_1^\top x \\
\text{s.t.} & \quad f_2(x) := \frac{1}{2} x^\top Q_2 x + b_2^\top x + c \leq 0,
\end{align*}
\]

where \(Q_1\) and \(Q_2\) are \(n \times n\) symmetric matrices (not necessary to be positive semidefinite), \(b_1, b_2 \in \mathbb{R}^n\) and \(c \in \mathbb{R}\).

Problem (P) is known as the generalized trust region subproblem (GTRS) \[44, 41]. When \(Q_2\) is an identity matrix \(I\) and \(b_2 = 0, c = -1/2\), problem (P) reduces to the classical trust region subproblem (TRS). The TRS first arose in the trust region method for nonlinear optimization \[15, 49]\, and has found many applications including robust optimization \[8\] and the least square problems \[50\]. As a generalization, the GTRS also admits its own applications such as time of arrival problems \[26\] and subproblems of consensus ADMM in signal processing \[29\]. Over the past two decades, numerous solution methods have been developed for TRS (see \[38, 40, 48, 42, 25, 22, 4\] and references therein).

Various methods have been developed for solving the GTRS under various assumptions (see \[37, 44, 10, 45, 16, 11, 5\] and references therein). Although it appears being nonconvex, the GTRS essentially enjoys

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its hidden convexity. The GTRS can be solved via a semidefinite programming (SDP) reformulation, due to the celebrated S-lemma [40], which was first established in [47]. However, suffering from relatively large computational complexity, the SDP algorithm is not practical for large-scale applications. To overcome this difficulty, several recent papers [30, 14, 27] demonstrated that the TRS admits a second order cone programming (SOCP) reformulation. Ben-Tal and den Hertog [7] further showed an SOCP reformulation for the GTRS under a simultaneously diagonalizing (SD) procedure of the quadratic forms. Jiang et al. [31] derived an SOCP reformulation for the GTRS when the problem has a finite optimal value and further derived a closed form solution when the SD condition fails. On the other hand, there is rich literature on iterative algorithms to solve the GTRS directly under mild conditions, for example, [37, 44, 41, 43]. Pong and Wolkowicz proposed an efficient algorithm based on minimum generalized eigenvalue of a parameterized matrix pencil for the GTRS, which extended the results in [18] and [42] for the TRS. Salahi and Taati [43] also derived a diagonalization-based algorithm under the SD condition of the quadratic forms. Recently, Adachi and Nakatsukasa [5] also developed a novel eigenvalue-based algorithm to solve the GTRS.

Our main contribution in this paper is to propose a novel convex quadratic reformulation (CQR) for the GTRS that is simpler than [7, 31] and further a minimax problem reformulation and develop an efficient algorithm to solve the minimax problem reformulation. Numerical results demonstrate that our method outperforms all the existing methods in the literature for sparse problem instances. We acknowledge that our CQR was inspired by the following CQR in Flippo and Janson [17] for the TRS,

\[
\min_x \left\{ \frac{1}{2} x^\top (Q_1 - \lambda_{\min}(Q_1) I) x + b_1^\top x + \frac{1}{2} \lambda_{\min}(Q_1) : \ x^\top x \leq 1 \right\},
\]

where \( \lambda_{\min}(Q_1) \) is the smallest eigenvalue of matrix \( Q_1 \). Unfortunately, this CQR was underappreciated in that time. Recently, people rediscovered this result; Wang and Xia [46] and Ho-Nguyen and Kilinc-Karzan [27] presented a linear time algorithm to solve the TRS by applying Nesterov’s accelerated gradient descent algorithm to (1). We, instead, rewrite the epigraph reformulation for (1) as follows,

\[
\min_{x,t} \left\{ t : \frac{1}{2} x^\top (Q_1 - \lambda_{\min}(Q_1) I) x + b_1^\top x + \frac{1}{2} \lambda_{\min}(Q_1) \leq t, \ x^\top x \leq 1 \right\}.
\]

Motivated by the above reformulation, we demonstrate that the GTRS is equivalent to exact one of the following two convex quadratic reformulations under two different conditions,

\[
(P_1) \quad \min_{x,t} \left\{ t : h_1(x) \leq t, \ h_2(x) \leq t \right\},
\]

\[
(P_2) \quad \min_{x,t} \left\{ t : h_3(x) \leq t, \ f_2(x) \leq 0 \right\},
\]

\[
(M) \quad \min H(x) := \max\{h_1(x), h_2(x)\}.
\]

We further derive efficient algorithms to solve the above minimax problem. To the best of our knowledge, the current literature lacks studies on such a problem formulation for a large scale setting except using a
black box subgradient method with an $O(1/\epsilon^2)$ convergence rate \[12\], which is really slow. Note that Section 2.3 in Nesterov’s book \[39\] presents a gradient based method with linear convergence rate for solving the minimization problem (M) under the condition that both $h_1(x)$ and $h_2(x)$ are strongly convex. However, Nesterov’s algorithms cannot be applied to solve our problem since in our problem setting at least one function of $h_1(x)$ and $h_2(x)$ is not strongly convex. By using the special structure of problem (M), we derive a steepest descent method in Section 3. More specifically, we choose either the negative gradient when the current point is smooth, or a vector in the subgradient set with the smallest norm (the steepest descent direction) when the current point is nonsmooth as the descent direction, and derive two steepest descent algorithms with two different line search rules accordingly. In the first algorithm we choose a special step size, and in the second algorithm we propose a modified Armijo line search rule. We also prove the global sublinear convergence rate for both algorithms. The first algorithm even admits a global convergence rate of $O(1/\epsilon)$, in the same order as the gradient descent algorithm, which is faster than the subgradient method. In addition, we demonstrate that the first algorithm also admits a local linear convergence rate, by a delicate analysis on the Kurdyka-Łojasiewicz (KL) \[6, 11, 34, 20\] property for problem (M). We illustrate in our numerical experiments the efficiency of the proposed algorithms when compared with the state-of-the-art methods for GTRS in the literature.

The rest of this paper is organized as follows. In Section 2, we derive an explicit CQR for problem (P) under different conditions and show how to recover an optimal solution of problem (P) from that of the CQR. In Section 3, we reformulate the CQR to a convex nonsmooth unconstrained minimax problem and derive two efficient solution algorithms. We provide convergence analysis for both algorithms. In Section 4, we demonstrate the efficiency of our algorithms from our numerical experiments. We conclude our paper in Section 5.

**Notations** We use $v(\cdot)$ to denote the optimal value of problem (\·). The matrix transpose of matrix $A$ is denoted by $A^\top$ and inverse of matrix $A$ by $A^{-1}$, respectively.

2 Convex quadratic reformulation

In this section, we derive a novel convex quadratic reformulation for problem (P). To avoid some trivial cases, we assume, w.o.l.g., the Slater condition holds for problem (P), i.e., there exists at least one interior feasible point. When both $f_1(x)$ and $f_2(x)$ are convex, problem (P) is already a convex quadratic problem. Hence, w.l.o.g., let us assume that not both $f_1(x)$ and $f_2(x)$ are convex. We need to introduce the following conditions to exclude some unbounded cases.

**Assumption 2.1.** The set $I_{PSD} := \{\lambda : Q_1 + \lambda Q_2 \succeq 0\} \cap \mathbb{R}_+$ is not empty, where $\mathbb{R}_+$ is the nonnegative orthant.

**Assumption 2.2.** The common null space of $Q_1$ and $Q_2$ is trivial, i.e., $\text{Null}(Q_1) \cap \text{Null}(Q_2) = \{0\}$.

Before introducing our CQR, let us first recall the celebrated S-lemma by defining $\tilde{f}_1(x) = f_1(x) + \gamma$ with an arbitrary constant $\gamma \in \mathbb{R}$.

**Lemma 2.3 (S-lemma \[47, 50\]).** The following two statements are equivalent:
1. The system of $\tilde{f}_1(x) < 0$ and $f_2(x) \leq 0$ is not solvable;
2. There exists $\mu \geq 0$ such that $\tilde{f}_1(x) + \mu f_2(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Using the S-lemma, the following lemma shows a necessary and sufficient condition under which problem (P) is bounded from below.
Lemma 2.4 (28). Problem (P) is bounded from below if and only if the following system has a solution for $\lambda$:

\[ Q_1 + \lambda Q_2 \succeq 0, \quad \lambda \geq 0, \quad b_1 + \lambda b_2 \in \text{Range}(Q_1 + \lambda Q_2). \]

We make Assumption 2.2 without loss of generality, because otherwise we can prove an unboundedness from below of the problem (see, e.g., 31 and 5). Under Assumption 2.2 if Assumption 2.1 fails, there exists no nonnegative $\lambda$ such that $Q_1 + \lambda Q_2 \succeq 0$ and problem (P) is unbounded from below due to Lemma 2.4. So both Assumptions 2.1 and 2.2 are made without loss of generality.

It has been shown in [37] that \(\{\lambda : Q_1 + \lambda Q_2 \succeq 0\}\) is an interval and thus \(\{\lambda : Q_1 + \lambda Q_2 \succeq 0\} \cap \mathbb{R}_+\) is also an interval (if not empty). Under Assumptions 2.1 and 2.2, we have the following three cases for \(I_{PSD}\).

Condition 2.5. The set \(I_{PSD} = [\lambda_1, \lambda_2]\) with \(\lambda_1 < \lambda_2\).

Condition 2.6. The set \(I_{PSD} = [\lambda_3, \infty)\).

Condition 2.7. The set \(I_{PSD} = \{\lambda_4\}\) is a singleton.

Note that Condition 2.6 occurs only when \(Q_2\) is positive semidefinite. Under Condition 2.7, \(Q_1\) and \(Q_2\) may not be SD and may have 2 × 2 block pairs in a canonical form under congruence 31. In this case, when $\lambda$ is given, the authors in [31] showed how to recover an optimal solution if the optimal solution is attainable, and how to obtain an $\epsilon$ optimal solution if the optimal solution is unattainable. So in the following, we mainly focus on the cases where either Condition 2.5 or 2.6 is satisfied.

Lemma 2.8. Under Condition 2.5 or 2.6, problem (P) is bounded from below.

Proof. Under Condition 2.5 or 2.6, there exists $\lambda_0$ such that $Q_1 + \lambda_0 Q_2 \succeq 0$ and $\lambda_0 \geq 0$, which further implies $b_1 + \lambda_0 b_2 \in \text{Range}(Q_1 + \lambda_0 Q_2)$ as $Q_1 + \lambda Q_2$ is nonsingular. With Lemma 2.4, we complete the proof. $\square$

### 2.1 Convex quadratic reformulation for GTRS

It is obvious that problem (P) is equivalent to its epigraph reformulation as follows,

\[(P_0) \quad \min \{t : f_1(x) \leq t, \ f_2(x) \leq 0\}.\]

To this end, we are ready to present the main result of this section.

Theorem 2.9. Under Assumption 2.1, by defining $h_i(x) = f_1(x) + \lambda_i f_2(x)$, $i = 1, 2, 3$, we can reformulate problem (P) to a convex quadratic problem under Conditions 2.5 and 2.6, respectively:

1. Under Condition 2.5, problem (P) is equivalent to the following convex quadratic problem,

\[ (P_1) \quad \min_{x, t} \{t : h_1(x) \leq t, \ h_2(x) \leq t\}; \]

2. Under Condition 2.6, problem (P) is equivalent to the following convex quadratic problem,

\[ (P_2) \quad \min_{x, t} \{t : h_3(x) \leq t, \ f_2(x) \leq 0\} = \min_x \{h_3(x) : f_2(x) \leq 0\}. \]

Proof. Let us first consider the case where Condition 2.5 holds. Due to Lemma 2.8, \((P_1)\) is bounded from below. Together with the assumed Slater conditions, problem \((P_1)\) admits the same optimal value as its
Due to the S-lemma, problem (P) also has the same optimal value as its Lagrangian dual [45],

\[
\text{(D)} \quad \max_{\mu \geq 0} \min_x f_1(x) + \mu f_2(x).
\]

Under Condition 2.5, i.e., \(I_{PSD} = [\lambda_1, \lambda_2]\) with \(\lambda_1 < \lambda_2\), it is easy to show that (P1) is a relaxation of (P0) since they have the same objective function and the feasible region of (P1) contains that of (P0) (note that \(f_1 \leq t\) and \(f_2 \leq 0\) imply that \(f_1(x) - t + uf_2(x) \leq 0\) for all \(u \geq 0\)). Thus,

\[
v(P_1) \leq v(P_0) = v(P).
\]

The Lagrangian dual problem of (P1) is

\[
\text{(D}_1\) \quad \max_{s_1, s_2 \geq 0} \min_{t, x} t + (s_1 + s_2)(f_1(x) - t) + (\lambda_1 s_1 + \lambda_2 s_2) f_2(x).
\]

For any primal and dual optimal solution pair \((x^*, u^*)\) of (P) and (D), due to \(u^* \in [\lambda_1, \lambda_2]\) as \(Q_1 + \mu^* Q_2 \succeq 0\) from Lemma 2.4, we can always find a convex combination \(\lambda s_1 + \lambda_2 s_2 = \mu^*\) with \(s_1 + s_2 = 1\). Hence \((x^*, s, t)\), with an arbitrary \(t\), is a feasible solution to (D1) and the objective value of problem (D1) at \((x^*, s, t)\) is the same with the optimal value of (D). This in turn implies

\[
v(D_1) \geq v(D). \tag{3}
\]

Since (P1) is convex and Slater condition is satisfied (because (P1) is a relaxation of (P) and Slater condition is assumed for (P)), \(v(P_1) = v(D_1)\). Finally, by combining (2) and (3), we have \(v(P_1) = v(D_1) \geq v(D) = v(P) = v(P_0) \geq v(P_1)\). So all inequalities above become equalities and thus (P1) is equivalent to (P).

Statement 2 can be proved in a similar way and is thus omitted.

**Remark 2.10.** Reformulation (P2) generalizes the approaches in [17, 18, 27] for the classical TRS with the unit ball constraint to the GTRS with a general convex quadratic constraint.

To our best knowledge, there is no method in the literature to compute \(\lambda_1\) and \(\lambda_2\) in Condition 2.5 for general \(Q_1\) and \(Q_2\). However, there exist efficient methods in the literature to compute \(\lambda_1\) and \(\lambda_2\) when a \(\lambda_0\) is given such that \(Q_1 + \lambda_0 Q_2 \succeq 0\) is satisfied. More specifically, the method mentioned in Section 2.4.1 in [6] gives a way to compute \(\lambda_1\) and \(\lambda_2\): first detect a \(\lambda_0\) such that \(Q_0 := Q_1 + \lambda_0 Q_2 \succ 0\), and then compute \(\lambda_1\) and \(\lambda_2\) by some generalized eigenvalues for a definite matrix pencil that are nearest to 0. Please refer to [24] for one of the state-of-the-art methods for detecting \(\lambda_0\). We can also find another iterative method in Section 5 in [37] to compute \(\lambda_0 \in \text{int}(I_{PSD})\) by reducing the length of an interval \([\lambda_1, \lambda_2] \supset I_{PSD}\). We next report our new method to compute \(\lambda_1\) and \(\lambda_2\), which is motivated by [41]. Our first step is also to find a \(\lambda_0\) such that \(Q_0 := Q_1 + \lambda_0 Q_2 \succ 0\). Then we compute the maximum generalized eigenvalues for \(Q_2 + \mu Q_0\) and \(-Q_2 + \mu Q_0\), denoted by \(u_1\) and \(u_2\), respectively. Note that both \(u_1 > 0\) and \(u_2 > 0\) due to \(Q_0 \succ 0\) and \(Q_2\) has a negative eigenvalue. So we have

\[
Q_1 + \left(\frac{1}{u_1} + \lambda_0\right) Q_2 \succeq 0 \quad \text{and} \quad Q_1 + \left(-\frac{1}{u_2} + \lambda_0\right) Q_2 \succeq 0.
\]

Thus \(Q_1 + \eta Q_2 \succeq 0\) for all \(\eta \in [\lambda_0 - \frac{1}{u_2}, \lambda_0 + \frac{1}{u_1}]\), which implies \(\lambda_1 = \lambda_0 - \frac{1}{u_2}\) and \(\lambda_2 = \lambda_0 + \frac{1}{u_1}\). In particular, when one of \(Q_1\) and \(Q_2\) is positive definite, we can skip the step of detecting the definiteness, which would save significant time in implementation.

In fact, when \(\lambda_0\) is given, we only need to compute one extreme eigenvalues, either \(\lambda_1\) or \(\lambda_2\), to obtain our convex quadratic reformulation. Define \(x(\lambda) = -(Q_1 + \lambda Q_2)^{-1}(b_1 + \lambda b_2)\) for all \(\lambda \in \text{int}(I_{PSD})\) and define \(\gamma(\lambda) = f_2(x(\lambda))\). After we have computed \(\lambda_0\) such that \(\lambda_0 \in \text{int}(I_{PSD})\), under Assumption 2.2 we further
have $Q_1 + \lambda_0 Q_2 > 0$, which makes $(Q_1 + \lambda Q_2)^{-1}$ well defined. In fact, there are Newton type methods in the literature (e.g., \cite{37}) for solving the GTRS by finding the optimal $\lambda$ through $\gamma(\lambda) = 0$. However, each step in \cite{37} involves solving a linear system $-(Q_1 + \lambda Q_2)^{-1}(b_1 + b_2)$, which is time consuming for high-dimension settings. Moreover, the Newton’s method does not converge in the so called hard case\footnote{The definition here follows \cite{37}. In fact, the definitions of hard case and easy case of the GTRS are similar to those of the TRS. More specifically, if the null space of the Hessian matrix, $Q_1 + \lambda^* Q_2$, with $\lambda^*$ being the optimal Lagrangian multiplier of problem (P), is orthogonal to $b_1 + \lambda^* b_2$, we are in the hard case; otherwise we are in the easy case.}. On the other hand, for easy case, an initial $\lambda$ in $I_{PSD}$ is also needed as a safeguard to guarantee the positive definiteness of $Q_1 + \lambda Q_2$ \cite{37}. It is shown in \cite{37} that $\gamma(\lambda)$ is either a strictly decreasing function or a constant in $\text{int}(I_{PSD})$. Following \cite{5}, we have the following three cases: if $\gamma(\lambda_0) > 0$, the optimal $\lambda^*$ locates in $[\lambda_0, \lambda_2]$; if $\gamma(\lambda_0) = 0$, $x(\lambda_0)$ is an optimal solution; and if $\gamma(\lambda_0) < 0$, the optimal $\lambda^*$ locates in $[\lambda_1, \lambda_0]$. Hence we have the following corollary, whose proof is similar to that in Theorem 2.9 and thus omitted.

**Corollary 2.11.** Assume that Assumption 2.1 holds and define $h_i(x) = f_1(x) + \lambda_i f_2(x)$, $i = 0, 1, 2$. Under Condition 2.5 the following results hold true.

1. If $\gamma(\lambda_0) > 0$, problem \((P)\) is equivalent to the following convex quadratic problem,

\[
(P_1) \min_{x,t} \{t : h_0(x) \leq t, \ h_2(x) \leq t\}.
\]

2. If $\gamma(\lambda_0) = 0$, $x(\lambda_0) = -(Q_1 + \lambda_0 Q_2)^{-1}(b_1 + \lambda_0 b_2)$ is the optimal solution.

3. If $\gamma(\lambda_0) < 0$, problem \((P)\) is equivalent to the following convex quadratic problem,

\[
(P_1) \min_{x,t} \{t : h_1(x) \leq t, \ h_0(x) \leq t\}.
\]

Since both \((P_1)\) and \((P_2)\) have a similar form to \((P)\) and can be solved in a way similar to the solution approach for \((P_1)\), we only discuss how to solve \((P_1)\) in the following.

### 2.2 Recovery of optimal solutions

In this subsection, we will discuss the recovery of an optimal solution to problem \((P)\) from an optimal solution to reformulation \((P_1)\). Before that, we first introduce the following lemma. Let us assume from now on $h_i(x) = \frac{1}{2} x^T A_i x + a_i^T x + r_i$, $i = 1, 2$.

**Lemma 2.12.** If Condition 2.5 holds, $A_1$ and $A_2$ are simultaneously diagonalizable. Moreover, we have $d^T A_1 d > 0$ for all nonzero vector $d \in \text{Null}(A_2)$.

Proof. Note that Condition 2.5 and Assumption 2.2 imply that $Q_1 + \frac{\lambda_1 + \lambda_2}{2} Q_2 > 0$, i.e., $\frac{\lambda_1 + \lambda_2}{2} > 0$. Let $A_0 = \frac{\lambda_1 + \lambda_2}{2}$ and $A_0 = L^T L$ be its Cholesky decomposition, where $L$ is a nonsingular symmetric matrix. Also let $(L^{-1})^T A_1 L^{-1} = P^T D P$ be the spectral decomposition, where $P$ is an orthogonal matrix and $D$ is a diagonal matrix. Then we have $(L^{-1} P^{-1})^T A_1 L^{-1} P^{-1} = D$ and

\[
(L^{-1} P^{-1})^T A_2 L^{-1} P^{-1} = (L^{-1} P^{-1})^T A_0 L^{-1} P^{-1} - (L^{-1} P^{-1})^T A_1 L^{-1} P^{-1} = I - D.
\]

Hence $A_1$ and $A_2$ are simultaneously diagonalizable by the congruent matrix $L^{-1} P^{-1}$.

Now let us assume $S = L^{-1} P^{-1}$ and thus $S^T A_1 S = \text{diag}(p_1, \ldots, p_n)$ and $S^T A_2 S = \text{diag}(q_1, \ldots, q_n)$ are both diagonal matrices. Define $K = \{i : q_i = 0, i = 1, \ldots, n\}$. Since $A_1 + A_2 > 0$, $p_i > 0$ for all $i \in K$. Let $e_i$ be the $n$-dimensional vector with ith entry being 1 and all others being 0s. We have $(S e_i)^T A_1 S e_i = p_i > 0$.
for all \( i \in K \). On the other hand, \( A_2 Se_i = 0 \) for all \( i \in K \). Hence \( d^\top A_1 d > 0 \) holds for all nonzero vector \( d \in \text{Null}(A_2) \).

From Lemma 2.8, Condition 2.5 implies the boundedness of problem (P) and thus the optimal solution is always attainable [31]. In the following theorem, we show how to recover the optimal solution of problem (P) from an optimal solution of problem (P1).

**Theorem 2.13.** Assume that Condition 2.5 holds and \( x^* \) is an optimal solution of problem (P1). Then an optimal solution of problem (P) can be obtained in the following ways:

1. If \( h_1(x^*) = t \) and \( h_2(x^*) \leq t \), then \( x^* \) is an optimal solution to (P);

2. Otherwise \( h_1(x^*) < t \) and \( h_2(x^*) = t \). For any vector \( v_1 \in \text{Null}(A_2) \), let \( \hat{\theta} \) be a solution of the following equation,

\[
h_1(x^* + \theta v_1) = \frac{1}{2} v_1^\top A_1 v_1 \theta^2 + (v_1^\top A_1 x^* + a_1^\top v_1) \theta + h_1(x^*) = t. \tag{4}
\]

Then \( \{ \hat{x} : \hat{x} = x^* + \hat{\theta} v_1, v_1 \in \text{Null}(A_2), \hat{\theta} \) is a solution of (4) \} forms the set of optimal solutions of (P).

**Proof.** Note that at least one of \( h_1(x^*) \leq t \) and \( h_2(x^*) \leq t \) takes equality. Then we prove the theorem for the following two cases:

1. If \( h_1(x^*) = t \) and \( h_2(x^*) \leq t \), then \( f_1(x^*) + \lambda_2 f_2(x^*) \leq f_1(x^*) + \lambda_1 f_2(x^*) \). Hence \( f_2(x^*) \leq 0 \) due to \( \lambda_2 - \lambda_1 > 0 \).

2. Otherwise, \( h_1(x^*) < t \) and \( h_2(x^*) = t \). In this case, for all \( d \in \text{Null}(A_2) \) we have \( d^\top A_1 d > 0 \) due to Lemma 2.12. We also claim that \( a_2^\top d = 0 \). Otherwise, setting \( d \) such that \( a_2^\top d < 0 \) (This can be done since we have \( a_2^\top (d - d) < 0 \) if \( a_2^\top d > 0 \)) yields

\[
h_2(x^* + d) = h_2(x^*) + \frac{1}{2} d^\top A_2 d + (x^*)^\top A_2 x^* + a_2^\top d = h_2(x^*) + a_2^\top d < t,
\]

where the second equality is due to \( d \in \text{Null}(A_2) \) and \( h_1(x^* + d) < t \) for any sufficiently small \( d \). This implies that \( (x^*, t) \) is not optimal, which is a contradiction. Equation (4) has two solutions due to the positive parameter before the quadratic term, i.e., \( v_1^\top A_1 v_1 > 0 \) and the negative constant, i.e., \( h_1(x^*) - t < 0 \). With the definition of \( \hat{\theta} \), we know \( h_1(\hat{x}) = t \) and \( h_2(\hat{x}) = t \). This further implies \( f_1(\hat{x}) = t \) and \( f_2(\hat{x}) = 0 \), i.e., \( \hat{x} \) is an optimal solution to (P).

\[\square\]

**Remark 2.14.** In Item 2 of the above proof, \( a_2^\top d = 0 \) indicates that problem (P) is in the hard case.

We next illustrate our recovery approach for the following simple example,

\[
\min \{3x_1^2 - \frac{1}{2} x_2^2 - x_2 : -x_1^2 + \frac{1}{2} x_2^2 + x_2 + 1 \leq 0 \}.
\]

Note that, for this example, Condition 2.5 holds, \( \lambda_1 = 1 \) and \( \lambda_2 = 3 \). Then we have the following CQR,

\[
\min \{ t : 2x_1^2 + 1 \leq t, x_2^2 + 2x_2 + 3 \leq t \}.
\]

An optimal solution of the CQR is \( x = (0, -1)^\top \), \( t = 2 \). However, this \( x \) is not feasible to (P). Using the approach in Theorem 2.13 we obtain an optimal solution \( \hat{x} = (\frac{\sqrt{7}}{2}, -1)^\top \) to problem (P). In fact, this
instance is in the hard case since the optimal Lagrangian multiplier, \( \lambda^* = 3 \), is at the end of the interval \( \{ \lambda : Q_1 + \lambda Q_2 \geq 0, \lambda \geq 0 \} \) and \( a - \lambda^* b \in \text{Range}(Q_1 + \lambda^* Q_2) \).

We finally point out that our method can be extended to the following variants of GTRS with equality constraint and interval constraint,

\[
\text{(EP)} \quad \min f_1(x) := \frac{1}{2} x^T Q_1 x + b_1^T x \\
\text{s.t.} \quad f_2(x) := \frac{1}{2} x^T Q_2 x + b_2^T x + c = 0,
\]

\[
\text{(IP)} \quad \min f_1(x) := \frac{1}{2} x^T Q_1 x + b_1^T x \\
\text{s.t.} \quad c_1 \leq f_2(x) := \frac{1}{2} x^T Q_2 x + b_2^T x \leq c_2.
\]

It is shown in [31] that (IP) can be reduced to (EP) with minor computation. It is obvious that all our previous results for inequality constrained GTRS hold for (EP) if we remove the non-negativity requirement for \( \lambda \) in \( I_{PSD} \), i.e., \( I_{PSD} = \{ \lambda : Q_1 + \lambda Q_2 \geq 0 \} \). We thus omit detailed discussion for (EP) to save space.

In the last part of this section, we compare the CQR in this paper with CQR for general QCQP in [19]. The authors in [19] considered the following general QCQP,

\[
\text{(QP)} \quad \min b_0^T x \quad \text{s.t.} \quad \frac{1}{2} x^T Q_i x + b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m, \quad x \in X,
\]

where \( X \) is a polyhedron. They further showed that the SDP relaxation of (QP) is equivalent to the following CQR for (QP):

\[
\text{(CQP)} \quad \min \tilde{b}_0^T x \quad \text{s.t.} \quad x \in G,
\]

where \( G = \{ x : F_s(x) \leq 0 \text{ for every } s \in T \}, F_s(x) = \sum_{i=1}^m s_i (\frac{1}{2} x^T \tilde{Q}_i x + \tilde{b}_i^T x + \tilde{c}_i) \) and

\[
T := \{ s \in \mathbb{R}^m : s \geq 0, t \in \mathbb{R}, \left( \frac{1}{2} \sum_{i=1}^m s_i \tilde{c}_i \right), \frac{1}{2} \left( \sum_{i=1}^m s_i \tilde{b}_i \right)^T, \sum_{i=1}^m s_i \tilde{Q}_i \} \succeq 0 \}.
\]

For the quadratic problem (P_1), because the variable \( t \) is linear in the objective and the constraints, we can reduce \( T \) to

\[
T := \{ s : s_1 + s_2 = 1, s \geq 0, \sum_{i=1}^2 s_i \tilde{Q}_i \succeq 0, \sum_{i=1}^2 s_i \tilde{b}_i \in \text{Range}(\sum_{i=1}^2 s_i Q_i) \},
\]

where the restriction \( s_1 + s_2 = 1 \) does not affect the feasible region \( G \) since \( F_s(x) \leq 0 \) is equivalent to \( k F_s(x) \leq 0 \) with any positive scaling \( k \) for \( s \). Note that \( h_1(x) = F_{s_1}(x) \) and \( h_2(x) = F_{s_2}(x) \) with \( s_1 = (1,0)^T \) and \( s_2 = (0,1)^T \). For any \( s \in T, h_1(x) \leq 0 \) and \( h_2(x) \leq 0 \) imply \( F_s(x) \leq 0 \) because \( h_s(x) \) is a convex combination of \( f_1(x) + \lambda_1 f_2(x) \) and \( f_1(x) + \lambda_2 f_2(x) \). Hence, by the strong duality and with analogous proof to that in Theorem [2.9], the two feasible regions of problems (P_1) and (CQP) are equivalent and we further have \( v(P_1) = v(CQP) \).

3 Efficient algorithms in solving the minimax problem reformulation of the CQR

In this section, we propose efficient algorithms to solve the GTRS under Condition [2.5]. As shown in Theorem [2.4] and Corollary [2.11], the GTRS is equivalent to (P_1) or either (P_1) or (P_1). The three problems have similar forms and can be solved by the following proposed method in this section. Hence, to save space, we only consider solution algorithms for (P_1) in this section.

The convex quadratic problem (P_1) can be cast as an SOCP problem and solved by many existing solvers, e.g., CVX [23], CPLEX [1] and MOSEK [2]. However, the SOCP reformulation is not very efficient when the dimension is large (e.g., the SOCP solver will take about 1,000 seconds to solve a problem of dimension
10,000). Fortunately, due to its simple structure, \( (P_1) \) is equivalent to the following minimax problem of two convex quadratic functions
\[
(M) \quad \min \{ H(x) := \max\{h_1(x), h_2(x)\} \}.
\]
Hence we aim to derive an efficient method to solve the above minimax problem, thus solving the original GTRS. Our method is a steepest descent method to find a critical point with \( 0 \in \partial H(x) \). It is well known that such a critical point is an optimal solution of problem \( (M) \).

The following theorem tells us how to find the steepest descent direction.

**Theorem 3.1.** Let \( g_1 = \nabla h_1(x) \) and \( g_2 = \nabla h_2(x) \). If \( g_1 \) and \( g_2 \) have opposite directions, i.e., \( g_1 = -t g_2 \) for some constant \( t > 0 \) or if \( g_1 = 0 \) and \( h_1(x) \geq h_j(x) \) for \( i \neq j, i, j \in \{1,2\} \), then \( x \) is a global optimal solution. Otherwise we can always find the steepest descent direction \( d \) in the following way:

1. when \( h_1(x) \neq h_2(x) \), \( d = -g_1 \) if \( h_1(x) > h_2(x) \) and otherwise \( d = -g_2 \);

2. when \( h_1(x) = h_2(x) \), \( d = -(\alpha g_1 + (1 - \alpha)g_2) \), where \( \alpha \) is defined in the following three cases:
   
   (a) \( \alpha = 0 \), if \( g_1^\top g_1 \geq g_1^\top g_2 \geq g_2^\top g_2 \).
   
   (b) \( \alpha = 1 \), if \( g_1^\top g_1 \leq g_1^\top g_2 \leq g_2^\top g_2 \).
   
   (c) \( \alpha = \frac{g_1^\top g_2 - g_1^\top g_2}{g_1^\top g_1 + 2g_1^\top g_2 - 2g_2^\top g_2} \), if \( g_1^\top g_2 \leq g_2^\top g_2 \) and \( g_1^\top g_2 \leq g_1^\top g_1 \).

**Proof.** If \( h_1(x) = h_2(x) \) and \( g_1 = -t g_2 \), then \( 0 \in \partial H(x) \). Hence, by the definition of subgradient, we have
\[
H(y) \geq H(x) + 0^\top (y - x) = H(x), \quad \forall y,
\]
which further implies that \( x \) is the optimal solution.

If \( g_i = 0 \) and \( h_i(x) \geq h_j(x) \) for \( i \neq j, i, j \in \{1,2\} \), then for all \( y \neq x \), we have \( H(y) \geq h_i(y) \geq h_i(x) = H(x) \), i.e., \( x \) is a global optimal solution.

Otherwise we have the following three cases:

1. When \( h_1(x) \neq h_2(x) \), (suppose, w.l.o.g., \( h_1(x) > h_2(x) \)), for all \( y \in \mathcal{B}(x, \delta) \) with \( \mathcal{B}(x, \delta) \subset \{x : h_2(x) < h_1(x)\} \), \( H(x) = h_1(x) \) and thus \( H(x) \) is differentiable at \( x \) and smooth in its neighbourhood. Hence, \( d = -g_1 \) if \( h_1(x) > h_2(x) \). Symmetrically, the case with \( h_2(x) > h_1(x) \) can be proved in the same way.

2. When \( h_1(x) = h_2(x) \), the steepest descent direction can be found by solving the following problem:
\[
\min_{\|y\|=1} \max_{g \in \partial H(x)} g^\top y.
\]
The above problem is equivalent to \( \min_{\|y\|=1} \|g\|^2 \) [13], which is exactly the following problem in minimizing a quadratic function of \( \alpha \),
\[
\min_{0 \leq \alpha \leq 1} (\alpha g_1 + (1 - \alpha)g_2)^\top (\alpha g_1 + (1 - \alpha)g_2).
\]
The first order derivative of the above objective function is \( \frac{g_1^\top g_2 - g_2^\top g_2}{g_1^\top g_1 + 2g_1^\top g_2 - 2g_2^\top g_2} \). Then if the derivative is in the interval \([0,1]\), the optimal \( \alpha \) is given by \( \frac{g_2^\top g_2 - g_1^\top g_2}{g_1^\top g_1 + 2g_1^\top g_2 - 2g_2^\top g_2} \). Otherwise, \( [5] \) takes its optimal solution on its boundary. In particular,

- when \( \frac{g_1^\top g_2 - g_2^\top g_2}{g_1^\top g_1 + 2g_1^\top g_2 - 2g_2^\top g_2} > 1 \), i.e., \( g_1^\top g_1 < g_1^\top g_2 \) and \( g_2^\top g_2 > g_1^\top g_2 \), we have \( \alpha = 1 \),

- when \( \frac{g_1^\top g_2 - g_2^\top g_2}{g_1^\top g_1 + 2g_1^\top g_2 - 2g_2^\top g_2} < 0 \), i.e., \( g_1^\top g_2 > g_2^\top g_2 \), we have \( \alpha = 0 \).
Remark 3.2. The above theorem shows that the descent direction at each point with \( h_1(x) = h_2(x) \) is either the one with the smaller norm between \( \nabla h_1(x) \) and \( \nabla h_2(x) \) or the negative convex combination \( d \) of \( \nabla h_1(x) \) and \( \nabla h_2(x) \) such that \( \nabla h_1(x)^\top d = \nabla h_2(x)^\top d \).

![Figure 1: Involving subgradient can avoid termination at nonsmooth and non-optimal points.](image)

We next present an example in Figure 1 to illustrate the necessity of involving the subgradient (in some cases, both gradients are not descent directions). Consider \( h_1(x) = x_1^2 + x_2^2 \) and \( h_2(x) = (x_1 - 1)^2 + x_2^2 \). The optimal solution of this problem is \((0, 0)\top\). The gradient method can only converge to some point in the intersection curve of \( h_1(x) = h_2(x) \), i.e., \( x_1 = 0.5 \), but not the global optimal solution. For example, when we are at \( \bar{x} = (0.5, 0.1)\top \), the gradients for \( h_1(\bar{x}) \) and \( h_2(\bar{x}) \) are \( g_1 = (1, 0.2)\top \) and \( g_2 = (-1, 0.2)\top \), respectively. Neither \(-g_1\) nor \(-g_2\) is a descent direction at \( H(\bar{x}); H(\bar{x} + \epsilon g_1) > H(\bar{x}) \) for any small \( \epsilon > 0 \), \( i = 1, 2 \), due to \( g_1^\top g_2 = -0.96 < 0 \) and \( h_1(\bar{x}) = h_2(\bar{x}) \). (The direction \(-g_1\) is a descent direction, at \( \bar{x} \), for \( h_1(x) \) but ascent for \( h_2(x) \) and thus ascent for \( H(x) \); the same analysis holds for \(-g_2\).) The way we use to conquer this difficulty is to choose the steepest descent direction in the subgradient set at points in the intersection curve. If we use the subgradient direction \( d = -\frac{1}{\epsilon}(g_1 + g_2) = -((0, 0.2)\top) \), then \( d \) is a descent direction since \( h_1(\bar{x} + \epsilon d) = H(\bar{x}) + 2\epsilon g_1^\top d + \epsilon^2 d^\top d < H(\bar{x}) \) and \( h_2(\bar{x} + \epsilon d) = H(\bar{x}) + 2\epsilon g_2^\top d + \epsilon^2 d^\top d < H(\bar{x}) \) for any \( \epsilon \) with \( 0 < \epsilon < 2 \).

Using the descent direction presented in Theorem 3.1, we propose two algorithms to solve the minimax problem (M), respectively, in Algorithms 1 and 2: we first compute a descent direction by Theorem 3.1, apply then two different line search rules for choosing the step size, and finally terminate the algorithm if some termination criterion is met. The advantage of our algorithms is that each iteration is very cheap, thus yielding, with an acceptable iterate number, a low cost in CPU time. The most expensive operation in each iteration is to compute several matrix vector products, which could become cheap when the matrices are sparse.

### 3.1 Line search with a special step size

In the following, we first derive a local linear convergence rate for Algorithm 1 and then demonstrate a global sublinear convergence rate for Algorithm 1. We analyze the local convergence rate by studying the growth
in the neighbourhood of any optimal solution to $H(x)$ in (M). In fact, $H(x)$ belongs to a more general class of piecewise quadratic functions. Error bound and KL property, which are two widely used techniques for convergence analysis, have been studied in the literature, for several kinds of piecewise quadratic functions, see [33, 35, 51]. However, these results are based on piecewise quadratic functions separated by polyhedral sets, which is not the case of $H(x)$. Li et al. [32] demonstrated that KL property holds for the maximum of finite polynomials, but their KL exponent depends on the problem dimension and is close to one, which leads to a very weak sublinear convergence rate. Gao et al. [20] studied the KL exponent for the TRS of finite polynomials, but their KL exponent depends on the problem dimension and is close to one, which implies that the restriction of $f$ to its domain is a continuous function. The function $f$ is said to have the KL property if for any $\forall x^* \in \{x : 0 \in \partial f(x)\}$, there exist $C, \epsilon > 0$ and $\theta \in [0, 1)$ such that

$$C \|y\| \geq |f(x) - f^*(x)|^\theta,$$

$\forall x \in B(x^*, \epsilon), \forall y \in \partial f(x),$

where $\theta$ is known as the KL exponent.

Under Condition 2.5, we know that there exists $\lambda_0 \geq 0$ such that $Q_1 + \lambda_0 Q_2 \succ 0$ and thus $b_1 + \lambda_0 b_2 \in \text{Range}(Q_1 + \lambda_0 Q_2)$ due to the non-singularity of $Q_1 + \lambda_0 Q_2$. Hence from Lemma 2.4 problem (P) (and thus problem (P1)) is bounded from below. It is shown in [31] that when the two matrices are SD and problem (P) is bounded from below, the optimal solution of problem (P) is attainable. This further implies that problem (P1) is bounded from below with its optimal solution attainable. Assuming that $x^*$ is an optimal solution, the following theorem shows that the KL inequality holds with an exponent of $1/2$ at $x^*$ under some mild conditions.

**Algorithm 1** Line search with a special step size for Problem (M)

**Input:** Parameters in the minimax problem (M)

1. Initialize $x_0$
2. for $k = 0, 1, \ldots$, do
3. if $h_1(x_k) > h_2(x_k)$ then
4. set $d_k = -\nabla h_1(x_k)$
5. else if $h_1(x_k) < h_2(x_k)$ then
6. set $d_k = -\nabla h_2(x_k)$
7. else
8. set $d_k$ corresponding to Theorem 3.1 item 2
9. end if
10. if termination criterion is met then return
11. end if
12. Choose a step size $\beta_k$ according to Theorem 3.6
13. Update $x_{k+1} = x_k + \beta_k d_k$
14. end for
Theorem 3.4. Assume that \( \min h_1(x) < \min H(x) \) and \( \min h_2(x) < \min H(x) \). Then the KL property in Definition 3.3 holds with exponent \( \theta = 1/2 \).

Proof. Note that \( \min h_1(x) < \min H(x) \) and \( \min h_2(x) < \min H(x) \) imply that, for any \( x^* \in \{ x : \partial H(x) = 0 \}, \nabla h_1(x^*) \neq 0 \) and \( \nabla h_2(x^*) \neq 0 \), respectively. Assume \( L = \max \{ \lambda_{\max}(A_1), \lambda_{\max}(A_2) \} \). We carry out our proof by considering the following two cases.

1. For any point with \( h_1(x) \neq h_2(x) \), w.l.o.g., assuming \( h_1(x) > h_2(x) \) gives rise to \( \partial H(x) = \{ \nabla h_1(x) \} \).

   Hence
   \[
   |H(x) - H(x^*)| = \frac{1}{2} (x - x^*)^\top A_1 (x - x^*) + (x^*)^\top A_1 (x - x^*) + a_1^\top (x - x^*) 
   \leq \frac{1}{2} L \|x - x^*\|^2 + \|\nabla h_1(x^*)\| \|x - x^*\|.
   \]

   On the other hand, \( \nabla h_1(x) = A_1 x + a_1 \) and
   \[
   \|\nabla h_1(x)\|^2 = \|\nabla h_1(x) - \nabla h_1(x^*) + \nabla h_1(x^*)\|^2 
   = (x - x^*)^\top A_1 (x - x^*) + \|\nabla h_1(x^*)\|^2 + 2(\nabla h_1(x^*))^\top A_1 (x - x^*) 
   \geq \|\nabla h_1(x^*)\|^2 - 2L \|\nabla h_1(x^*)\| \|x - x^*\|.
   \]

   Define \( \epsilon_0 = \min \{ 1, \frac{\|\nabla h_1(x^*)\|}{4L} \} \). As \( \nabla h_1(x^*) \neq 0 \), for all \( x \in B(x^*, \epsilon_0) \), we then have
   \[
   |H(x) - H(x^*)| \leq \frac{1}{2} L \epsilon_0^2 + \|\nabla h_1(x^*)\| \epsilon_0 \leq \frac{9}{32L} \|\nabla h_1(x^*)\|^2
   \]
   and
   \[
   \|\nabla h_1(x)\|^2 \geq \|\nabla h_1(x^*)\|^2 - 2L \|\nabla h_1(x^*)\| \epsilon_0 \geq \frac{1}{2} \|\nabla h_1(x^*)\|^2.
   \]

   Hence \( |H(x) - H(x^*)|^\theta \leq \sqrt{\frac{9}{32L}} \|\nabla h_1(x^*)\| \leq \frac{3}{4\sqrt{L}} \|\nabla h_1(x)\| \). So we have the following inequality,
   \[
   |H(x) - H(x^*)|^\theta \leq C_0 \|y\|,
   \]
   for all \( y \in \partial H(x) \) (here \( \{ \nabla h_1(x) \} = \partial H(x) \)) with \( \theta = \frac{1}{2} \), \( C_0 = \frac{3}{4\sqrt{L}} \).

2. Consider next a point \( x \) with \( h_1(x) = h_2(x) \). Define \( h_0(x) = \alpha h_1(x) + (1 - \alpha) h_2(x) \), for some parameter \( \alpha \in [0, 1] \). Let \( I = \{ i \mid \{ \nabla h_1(x^*) \}_i \neq 0 \} \). The optimality condition \( 0 \in \partial H(x^*) \) implies that there exists some \( \alpha_0 \in [0, 1] \) such that \( \alpha_0 \nabla h_1(x^*) + (1 - \alpha_0) \nabla h_2(x^*) = 0 \). Note that \( \nabla h_1(x^*) \neq 0 \) and \( \nabla h_2(x^*) \neq 0 \) as assumed and thus \( \alpha_0 \in (0, 1) \). Define \( j = \arg\max_i \{ (\nabla h_1(x^*))_i \} \), \( M_1 = \| (\nabla h_1(x^*))_j \| \) and \( M_2 = \| (\nabla h_2(x^*))_j \| \). Note that \( \partial H_{\alpha_0}(x^*) = 0 \) implies that \( \alpha_0 M_1 = (1 - \alpha_0) M_2 \). W.l.o.g., assume \( M_1 \geq M_2 \) and thus \( \alpha_0 \leq \frac{1}{2} \). Since \( A_1 x \) (\( A_2 x \), respectively) is a continuous function of \( x \), there exists an \( \epsilon_1 > 0 \) (\( \epsilon_2 > 0 \), respectively) such that for any \( x \in B(x^*, \epsilon_1) \) (\( x \in B(x^*, \epsilon_2) \), respectively), \( \frac{3}{8} M_1 \geq |(\nabla h_1(x))_j| > \frac{1}{2} M_1 \frac{3}{8} M_2 \geq |(\nabla h_2(x))_j| > \frac{1}{2} M_2 \), respectively. Let \( \epsilon_3 = \min \{ \epsilon_1, \epsilon_2 \} \). Then we have the following two subcases.

   a. For all \( x \in B(x^*, \epsilon_3) \) and \( \alpha \in [0, \frac{3}{8} \alpha_0] \), we have
   \[
   \|\nabla h_\alpha(x)\| \geq -\alpha \|\nabla h_1(x)\| + (1 - \alpha) \|\nabla h_2(x)\| 
   \geq -\frac{3}{2} \alpha M_1 + \frac{1}{2}(1 - \alpha) M_2 
   \geq -\frac{3}{8} \alpha_0 M_1 + \frac{3}{8}(1 - \alpha_0) M_2 + \left( \frac{1}{8} + \frac{1}{8} \alpha_0 \right) M_2 
   = \left( \frac{1}{8} + \frac{1}{8} \alpha_0 \right) M_2.
   \]
The third inequality is due to the fact that $\frac{3}{2} \alpha M_1 + \frac{1}{2}(1 - \alpha)M$ is a decreasing function of $\alpha$ and the last equality is due to $\alpha_0 M_1 = (1 - \alpha_0) M_2$. Symmetrically, for $\alpha \in \left[1 - \frac{1 - \alpha_0}{4}, 1\right]$, we have $|\langle \nabla h_\alpha(x) \rangle| \geq \left(\frac{3}{2} - \frac{1}{4}\alpha_0\right) M_1$. Combining these two cases and $\alpha_0 \leq \frac{1}{2}$ yields $\|\nabla h_\alpha(x)\| \geq \frac{1}{2} M_2$.

On the other hand

\[
|H(x) - H(x^*)| = \frac{1}{2} (x - x^*)^T A_1 (x - x^*) + (x^*)^T A_1 (x - x^*) + a_1^T (x - x^*) \\
\leq \frac{1}{2} L \|x - x^*\|^2 + \|(\nabla h_1(x^*))\| \|x - x^*\| \\
\leq \left(\frac{1}{2} Lc_3^2 + \|(\nabla h_1(x^*))\|\right) \|x - x^*\|.
\]

Letting $\epsilon_4 = \min\{\epsilon_3, 1\}$ leads to

\[
|H(x) - H(x^*)|^\theta \leq C_1 \|(\nabla h_\alpha(x))\|, \quad \forall \alpha \in \left[0, \frac{1}{4}\alpha_0\right] \cup \left[1 - \frac{1 - \alpha_0}{4}, 1\right], \forall x \in B(x^*, \epsilon_4)
\]

where $\theta = \frac{1}{2}$ and $C_1 = \frac{\sqrt{2Lc_3^2 + 64\|(\nabla h_1(x^*))\|}}{M_2}$.

(b) Next let us consider the case with $\alpha \in \left[\frac{\alpha_0}{4}, 1 - \frac{1 - \alpha_0}{4}\right]$. In this case, defining $A_\alpha = \alpha A_1 + (1 - \alpha) A_2$ and $a_\alpha = \alpha a_1 + (1 - \alpha) a_2$ gives rise to

\[
\|\nabla h_\alpha(x)\|^2 = \|\nabla h_\alpha(x) - \nabla h_\alpha(x^*) + \nabla h_\alpha(x^*)\|^2 \\
= (x - x^*)^T A_\alpha A_\alpha (x - x^*) + \|\nabla h_\alpha(x^*)\|^2 + 2(\nabla h_\alpha(x^*))^T A_\alpha (x - x^*)
\]

and since $h_1(x) = h_2(x)$ and $h_1(x^*) = h_2(x^*)$,

\[
|H(x) - H(x^*)| = \frac{1}{2} (x - x^*)^T A_\alpha (x - x^*) + (x^*)^T A_\alpha (x - x^*) + a_\alpha^T (x - x^*) \\
= \frac{1}{2} (x - x^*)^T A_\alpha (x - x^*) + (\nabla h_\alpha(x^*))^T (x - x^*).
\]

Define $\mu_0 = \lambda_{\min}(A_\alpha)$. Then

\[
\|\nabla h_\alpha(x)\|^2 - 2\mu_0 |H(x) - H(x^*)| \\
= (x - x^*)^T A_\alpha (A_\alpha - \mu_0 I) (x - x^*) + \|\nabla h_\alpha(x^*)\|^2 \\
+ 2(\nabla h_\alpha(x^*))^T (A_\alpha - \mu_0 I) (x - x^*) \\
\geq \|(A_\alpha - \mu_0 I) (x - x^*) + \nabla h_\alpha(x^*)\|^2 + \mu_0 (x - x^*)^T (A_\alpha - \mu_0 I) (x - x^*) \geq 0.
\]

We next show that $\mu_0$ is bounded from below. Define $\mu_1 (\mu_2$, respectively) as the smallest nonzero eigenvalue of $A_1 (A_2$, respectively). Note that $\alpha A_1 + (1 - \alpha) A_2$ is positive definite for all $\alpha \in \left[\frac{\alpha_0}{4}, 1 - \frac{1 - \alpha_0}{4}\right]$ as assumed in Condition (2.12). Then $A_1$ and $A_2$ are simultaneously diagonalizable as shown in Lemma (2.12). Together with the facts that $A_1 \succeq 0$ and $A_2 \succeq 0$, there exists a nonsingular matrix $P$ such that $P^T A_1 P = D_1 \succeq \mu_1 \diag(\delta)$ and $P^T A_2 P = D_2 \succeq \mu_2 \diag(\delta)$, where $\delta_i = 1$ if $D_{1i} > 0$ and $\delta_i = 0$ otherwise. Since $\alpha \in \left[\frac{\alpha_0}{4}, 1 - \frac{1 - \alpha_0}{4}\right]$, $\lambda_{\min}(A_\alpha) \geq \min\{\alpha \mu_1, (1 - \alpha) \mu_2\} \geq \min\{\frac{\alpha_0}{4} \mu_1, \frac{1 - \alpha_0}{4} \mu_2\} > 0$. From $\|\nabla h_\alpha\|^2 - 2\mu_0 |H(x) - H(x^*)| \geq 0$, we know $\|\nabla h_\alpha\|^2 - \mu_0 |H(x) - H(x^*)| \geq 0$.

Let $\theta = \frac{1}{2}$, $C_2 = \sqrt{\frac{1}{2\mu_0}}$. We have

\[
C_2 \|\nabla h_\alpha(x)\| \geq |H(x) - H(x^*)|^\theta, \quad \forall \alpha \in \left[\frac{\alpha_0}{4}, 1 - \frac{1 - \alpha_0}{4}\right], \forall x \in B(x^*, \epsilon_4).
\]

Combining cases (a) and (b) gives rise to

\[
|H(x) - H(x^*)|^\theta \leq C_3 \|\nabla h_\alpha(x)\|
\]

with $\theta = \frac{1}{2}$, $C_3 = \max\{C_1, C_2\}$, for all $x \in B(x^*, \epsilon_4)$. 

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Combining cases 1 and 2 yields that the KL inequality holds with \( \theta = \frac{1}{2} \) and \( C = \max\{C_0, C_3\} \) for all \( x \in B(x^*, \epsilon) \) with \( \epsilon = \min\{\epsilon_0, \epsilon_4\} \).

Note that the assumption \( \min h_1(x) < \min H(x) \) and \( \min h_2(x) < \min H(x) \) means that we are in the easy case of GTRS as in this case \( \lambda^* \) is an interior point of \( I_{PSD} \) and \( Q_1 + \lambda^* Q_2 \) is nonsingular, where \( \lambda^* \) is the optimal Lagrangian multiplier of the GTRS [37]. However, there are two situations for the hard case. Let us consider the KL property for \( H(x) \) at the optimal solution \( x^* \). When \( h_i(x^*) > h_j(x^*) \), for \( i = 1 \) or 2 and \( j = \{1, 2\}/\{i\} \), in the neighbourhood \( x^* \), \( H(x) \) is just \( h_i(x) \), and the KL is also 1/2 [9]. In such a case, our algorithm performs asymptotically like the gradient descent method for unconstrained quadratic minimization. However, when \( h_i(x^*) = h_j(x^*) \) (note that \( \min h_j(x) < H(x^*) \) can still hold in this situation), the KL exponent is not always 1/2 for \( H(x) \). Consider the following counterexample with \( h_1(x) = x_1^2 \) and \( h_2(x) = (x_1 + 1)^2 + x_2^2 - 1 \). The optimal solution is \((0, 0)\) and is attained by both \( h_1 \) and \( h_2 \). Let \( x_2 = -\epsilon \), where \( \epsilon \) is a small positive number. Consider the curve where \( h_1(x) = h_2(x) \), which further implies \( x_1 = -\epsilon^2/2 \). Then we have

\[
(1 - \beta) \nabla h_1 + \beta \nabla h_2 = 2 \left( 1 - \beta \right) \frac{\epsilon^2}{\beta \epsilon} + \beta \left( -\frac{\epsilon^2}{\beta \epsilon} + 1 \right) = 2 \left( -\frac{\epsilon^2}{\beta \epsilon} + \beta \right),
\]

and thus

\[
\min_{y \in \partial H(x)} \|y\|^2 = \min_{\beta} 4 \left( \beta^2 \epsilon^2 + \beta^2 - \epsilon^2 \beta + \frac{\epsilon^4}{4} \right).
\]

\[
= \min_{\beta} 4 \left( (1 + \epsilon^2) \left( \beta - \frac{\epsilon^2}{2(1 + \epsilon^2)} \right)^2 - \frac{\epsilon^4}{4(1 + \epsilon^2)} + \frac{\epsilon^4}{4} \right)
\]

\[
= \frac{\epsilon^6}{2(1 + \epsilon^2)} = \frac{\epsilon^6}{2} + O(\epsilon^8).
\]

Thus, \( \min_{y \in \partial H(x)} \|y\| = O(\epsilon^4) \). On the other hand,

\[
H(x) - H(x^*) = x_1^2 = \frac{\epsilon^4}{4},
\]

The KL inequality cannot hold with \( \theta = 1/2 \), but it holds with \( \theta = 3/4 \) since \( \min_{y \in \partial H(x)} \|y\| = O(\epsilon^3) \) and \( H(x) - H(x^*) = O(\epsilon^4) \).

**Remark 3.5.** It is interesting to compare our result with a recent result on KL exponent of the quadratic sphere constrained optimization problem [20]. In [20], the authors showed that the KL exponent is 3/4 in general and 1/2 in some special cases, for the following problem,

\[
(T) \quad \min \frac{1}{2} x^\top A x + b^\top x \quad \text{s.t. } x^\top x = 1.
\]

The above problem is equivalent to the TRS when the constraint of the TRS is active, which is the case of interest in the literature. For the TRS, the case that the constraint is inactive is trivial: Assuming \( x^* \) being the optimal solution, \( (x^*)^\top x^* < 1 \) if and only if the objective function is convex and the optimal solution of the unconstrained quadratic function \( \frac{1}{2} x^\top A x + b^\top x \) locates in the interior of the unit ball. The authors in [20] proved that the KL exponent is 3/4 in general and particularly the KL exponent is 1/2 if \( A - \lambda^* I \) is nonsingular, where \( \lambda^* \) is the optimal Lagrangian multiplier. The later case is a subcase of the easy case for the TRS and the case that the KL exponent equals 3/4 only occurs in some special situations of the hard case.

On the other hand, our result shows the KL exponent is 1/2 for the minimax problem when the associated GTRS is in the easy case. So our result can be seen as an extension of the results on KL exponent for problem \((T)\) in [20]. One of our future research is to verify if the KL exponent is 3/4 for \( H(x) \) when the associated GTRS is in the hard case.
For convergence analysis with error bound or KL property, we still need a sufficient descent property to achieve the convergence rate. We next propose an algorithm with such a property. We further show that our algorithm converges locally linearly with the descent direction chosen in Theorem 3.4 and the step size specified in the following theorem.

**Theorem 3.6.** Assume that the conditions in Theorem 3.4 hold and that the initial point \( x^0 \in B(x^*, \epsilon) \). Assume that \( h_i \) is the active function when \( h_1(x_k) \neq h_2(x_k) \) and \( h_j, j = \{1, 2\}/\{i\}, \) is thus inactive. Let the descent direction be chosen in Theorem 3.4 and the associated step size be chosen as follows.

1. When \( h_1(x_k) = h_2(x_k) \),
   - if there exists \( g_\alpha = \alpha \nabla h_1(x_k) + (1 - \alpha) \nabla h_2(x_k) \) with \( \alpha \in [0, 1] \) such that \( \nabla h_1(x_k)^T g_\alpha = \nabla h_2(x_k)^T g_\alpha \), then set \( d_k = -g_\alpha \) and \( \beta_k = 1/L \), where \( L = \max\{\lambda_{\max}(A_1), \lambda_{\max}(A_2)\} \);
   - otherwise set \( d_k = -\nabla h_i(x_k) \) for \( i \) such that \( \nabla h_1(x_k)^T \nabla h_2(x_k) \geq \nabla h_i(x_k)^T \nabla h_i(x_k), \) \( i = 1, 2 \), and \( \beta_k = 1/L \).

2. When \( h_1(x_k) \neq h_2(x_k) \) and the following quadratic equation for \( \gamma \),

\[
ax^2 + bx + c = 0, \quad (6)
\]

where \( a = \frac{1}{2} \gamma^2 \nabla h_1(x_k)^T (A_i - A_j) \nabla h_i(x_k), \) \( b = (\nabla h_i(x_k)^T - \nabla h_j(x_k)^T) \nabla h_i(x_k) \) and \( c = h_i(x_k) - h_j(x_k) \), has no positive solution or any positive solution \( \gamma \geq 1/L \), set \( d_k = -\nabla h_i(x_k) \) with and \( \beta_k = 1/L \);

3. When \( h_1(x_k) \neq h_2(x_k) \) and the quadratic equation (6) has a positive solution \( \gamma < 1/L \), set \( \beta_k = \gamma \) and \( d_k = -\nabla h_i(x_k) \).

Then the sequence \( \{x_k\} \) generated by Algorithm 7 satisfies, for any \( k \geq 1 \),

\[
H(x_k) - H(x^*) \leq \left( \frac{2C^2 L^* - 1}{2C^2 L} \right)^{k-1} (H(x^0) - H(x^*)), \quad (7)
\]

and

\[
\text{dist}(x_k, \mathbf{x})^2 \leq \frac{2}{L} H(x_k) - H(x^*) \leq \frac{2}{L} \left( \frac{2C^2 L^* - 1}{2C^2 L} \right)^{k-1} (H(x^0) - H(x^*)).
\]

Proof. For simplicity, let us denote \( g_i = \nabla h_i(x_k) \) for \( i = 1, 2 \). We claim the following sufficient descent property for steps 1, 2 and 3:

\[
H(x_k) - H(x_{k+1}) \geq \frac{L}{2} \|x_k - x_{k+1}\|^2.
\]

Hence, if the step size is 1/L (i.e., steps 1 and 2), we have

\[
H(x_l) - H(x^*) \leq C \|d_l\|^2 = C^2 L^2 \|x_l - x_{l+1}\|^2 \leq 2C^2 L \left( H(x_l) - H(x_{l+1}) \right),
\]

where the first inequality is due to the KL inequality in Theorem 3.4, the second equality is due to \( x_{l+1} = x_l - \frac{1}{L} d_l \) and the last inequality is due to the sufficient descent property. Rearranging the above inequality yields

\[
H(x_{l+1}) - H(x^*) \leq \frac{2C^2 L - 1}{2C^2 L} (H(x_l) - H(x^*)).
\]

And since our method is a descent method, we have \( H(x_{l+1}) - H(x^*) \leq H(x_l) - H(x^*) \) for all iterations. Suppose that there are \( p \) iterates of step size 1, \( q \) iterates of step size 2, and \( r \) iterates of step size 3. From
the definitions of the steps, every step 3 is followed by a step 1 and thus \( r \leq p + 1 \) if we terminate our algorithm at step 1 or 2. So for all \( k \geq 1 \), after \( k = p + q + r \) steps, we have

\[
H(x_k) - H(x^*) \leq \left( \frac{2C^2L - 1}{2C^2L} \right)^{p+q} (H(x^0) - H(x^*)) \leq \left( \frac{2C^2L - 1}{2C^2L} \right)^{\frac{k-1}{2}} (H(x^0) - H(x^*)).
\]

The sufficient descent property further implies that

\[
\frac{L}{2} \sum_{k=0}^{\infty} \|x_k - x_{k+1}\|^2 \leq H(x_k) - H(x^*).
\]

Hence, with \( \sum_{k=0}^{\infty} \|x_k - x_{k+1}\|^2 \geq \text{dist}(x_k, \mathbf{x})^2 \), we have \( \frac{L}{2} \text{dist}(x_k, \mathbf{x})^2 \leq H(x_k) - H(x^*) \). Thus

\[
\text{dist}(x_k, \mathbf{x})^2 \leq \frac{2}{L} (H(x_k) - H(x^*)).
\]

By noting \( g_i = A_i x_k + a_i \), we have

\[
h_i(x_{k+1}) - h_i(x_k) = \frac{1}{2} (x_k + d_k)^\top A_i (x_k + d_k) + a_i^\top (x_k + d_k) - \left[ \frac{1}{2} (x_k)^\top A_i x_k + a_i^\top x_k \right]
\]

\[
= \frac{1}{2} d_k^\top A_i d_k + (A_i x_k + a_i)^\top d_k
\]

\[
= \frac{1}{2} g_i^\top d_k + g_i \top d_k.
\]

We next prove our claim (1) according to the three cases in our updating rule:

1. When \( h_1(x_k) = h_2(x_k) \), noting that \( h_i \) is active at \( x_{k+1} \) as assumed, we have

\[
H(x_k) - H(x_{k+1}) = h_i(x_k) - h_i(x_{k+1}).
\]

- If there exists an \( \alpha \) such that \( g_\alpha^\top g_1 = g_\alpha^\top g_2 \), we have \( g_\alpha^\top g_i = g_\alpha^\top g_\alpha \). And by noting that \( d_i = -g_\alpha \), we further have

\[
h_i(x_{k+1}) - h_i(x_k) = \frac{1}{2L^2} d_k^\top A_i d_k + \frac{1}{L} g_i^\top d_k
\]

\[
\leq \frac{1}{2L} g_\alpha^\top g_\alpha - \frac{1}{L} g_\alpha^\top g_\alpha
\]

\[
= -\frac{1}{2L} g_\alpha^\top g_\alpha.
\]

Substituting \( g_\alpha = L(x_k - x_{k-1}) \) to the above expression, we have the following sufficient descent property,

\[
H(x_k) - H(x_{k+1}) = h_i(x_k) - h_i(x_{k+1}) \geq \frac{L}{2} \|x_k - x_{k+1}\|^2.
\]

- If there does not exist an \( \alpha \) such that \( g_\alpha^\top g_1 = g_\alpha^\top g_2 \), then we must have \( g_1^\top g_2 > 0 \). And thus we must have \( g_1^\top g_1 \geq g_1^\top g_2 \geq g_2^\top g_2 \) or \( g_2^\top g_2 \geq g_1^\top g_2 \geq g_1^\top g_1 \). If \( g_1^\top g_i \geq g_i^\top g_j \geq g_j^\top g_j \), we set \( d_k = -g_j \). Then

\[
H(x_{k+1}) - H(x_k) \leq \max\{h_i(x_{k+1}) - h_i(x_k), h_j(x_{k+1}) - h_j(x_k)\}
\]

\[
\leq \max\{\frac{1}{2L^2} g_j^\top A_i g_j - \frac{1}{L} g_j^\top g_j, \frac{1}{2L^2} g_j^\top A_i g_j - \frac{1}{L} g_j^\top g_j\}
\]

\[
\leq \max\{\frac{1}{2L^2} g_j^\top A_i g_j - \frac{1}{L} g_j^\top g_j, \frac{1}{2L^2} g_j^\top A_i g_j - \frac{1}{L} g_j^\top g_j\}
\]

\[
= -\frac{1}{2L} g_j^\top g_j = -\frac{L}{2} \|x_k - x_{k+1}\|^2.
\]

Symmetrically, if \( g_i^\top g_j > 0 \) and \( g_i^\top g_j \geq g_j^\top g_j \geq g_i^\top g_i \), setting \( d_k = -g_i \) yields the same sufficient descent property.
2. When \( h_1(x_k) \neq h_2(x_k) \) and the quadratic equation \( \gamma \) for \( \gamma \) has no positive solution or has a positive solution \( \gamma \geq 1/L \), we have \( h_i(x_{k+1}) > h_j(x_{k+1}) \) for \( x_{k+1} = x_k + \beta_k d_k \), where \( d_k = -\nabla h_i(x_k) \) and \( \beta_k = \frac{1}{L} \). Moreover,

\[
H(x_{k+1}) - H(x_k) = h_i(x_{k+1}) - h_i(x_k)
= \frac{1}{2L} g_i^\top A_i g_i - \frac{1}{L} g_i^\top g_i
\leq -\frac{1}{2L} g_i^\top g_i.
\]

Hence \( H(x_k) - H(x_{k+1}) \geq \frac{1}{2L} g_i^\top g_i \geq \frac{L}{2} \| x_k - x_{k+1} \|^2 \).

3. When \( h_1(x_k) \neq h_2(x_k) \) and the quadratic equation \( \gamma \) has a positive solution \( \gamma < 1/L \). With \( \beta_k = \gamma \) and \( d_k = -\nabla h_i(x_k) \), it is easy to see that the step size \( \gamma \) makes \( h_1(x_{k+1}) = h_2(x_{k+1}) \). Then we have

\[
H(x_{k+1}) - H(x_k) = h_i(x_{k+1}) - h_i(x_k)
= \frac{1}{2} \gamma^2 d_k^\top A_i d_k + \gamma g_i^\top d_k
\leq \frac{1}{2} L \gamma^2 g_i^\top g_i - \gamma g_i^\top g_i
= \left( \frac{L}{2} - \frac{1}{\gamma} \right) \| x_k - x_{k+1} \|^2,
\]

which further implies \( H(x_k) - H(x_{k+1}) \geq \frac{L}{2} \| x_k - x_{k+1} \|^2 \) due to \( \gamma \leq \frac{1}{L} \).

\[ \square \]

**Remark 3.7.** It is worth to note that Step 3 in our algorithm is somehow similar to the retraction step in manifold optimization \([3]\). In manifold optimization, in every iteration, each point is retracted to the manifold. In Step 3, every point is drawn to the curve that \( h_1(x) = h_2(x) \).

We will next show that in general a global sublinear convergence rate, in the same order with the gradient descent algorithm, can also be theoretically guaranteed for Algorithm 1.

**Theorem 3.8.** Assume that \( x^* \) is an optimal solution. Then we have

\[
H(x_N) - H(x^*) \leq \frac{L}{N} \| x_0 - x^* \|^2.
\]

That is, the required iterate number for \( H(x_N) - H(x^*) \leq \epsilon \) is at most \( O(1/\epsilon) \).

**Proof.** From the proof in Theorem 3.6, for any step size \( \gamma \leq 1/L \), we have

\[
H(x_{k+1}) - H(x_k) \leq -\gamma g^\top g + \frac{1}{2} L \gamma^2 g^\top g \leq -\frac{\gamma}{2} g^\top g.
\]

From the convexity of \( H(x) \) and \( g \in \partial H(x_k) \), we have

\[
H(x_{k+1}) \leq H(x_k) - \frac{\gamma}{2} g^\top g
\leq H(x^*) + g^\top (x_k - x^*) - \frac{\gamma}{2} g^\top g
= H(x^*) + \frac{1}{2\gamma} \left( \| x_k - x^* \|^2 - \| x_k - x^* - \gamma g \|^2 \right)
= H(x^*) + \frac{1}{2\gamma} \left( \| x_k - x^* \|^2 - \| x_{k+1} - x^* \|^2 \right).
\]
Since \( H(x_{k+1}) \geq H(x^*) \), we have \( \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 \geq 0 \). Let us use indices \( i_k, k = 0, \ldots, K \) to denote the indices in Steps 1 and 2. By noting that \( \gamma = 1/L \), we have

\[
H(x_{i_{k+1}}) \leq H(x^*) + \frac{L}{2} \left( \|x_{i_k} - x^*\|^2 - \|x_{i_{k+1}} - x^*\|^2 \right).
\]

Note that every Step 3 is followed by Step 1. Hence \( N \leq 2K + 1 \). Adding the above inequalities from \( i_0 \) to \( i_K \), we have

\[
\sum_{k=0}^{K} H(x_{i_k}) - H(x^*) \leq \frac{L}{2} \sum_{k=0}^{K} \left( \|x_{i_k} - x^*\|^2 - \|x_{i_{k+1}} - x^*\|^2 \right) \\
\leq \frac{L}{2} \left( \|x_{i_0} - x^*\|^2 - \|x_{i_K} - x^*\|^2 \right) + \sum_{k=1}^{K} - \|x_{i_{k-1}} - x^*\|^2 + \|x_{i_k} - x^*\|^2 \right) \\
\leq \frac{L}{2} \|x_{i_0} - x^*\|^2 \\
\leq \frac{L}{2} \|x_0 - x^*\|^2,
\]

where in the second inequality we use the fact,

\[
- \|x_{i_{k-1}} - x^*\|^2 + \|x_{i_k} - x^*\|^2 \leq - \|x_{i_{k-1}} - x^*\|^2 + \|x_{i_k} - x^*\|^2 \leq \cdots \leq 0.
\]

Since \( H(x_k) \) is non-increasing, by noting that \( N \leq 2K + 1 \), we have

\[
H(x_N) - H(x^*) \leq \frac{1}{K+1} \sum_{k=0}^{K} H(x_{i_k}) - H(x^*) \\
\leq \frac{L}{N} \|x_0 - x^*\|^2.
\]

\[
\square
\]

### 3.2 Line search with the modified Armijo rule

An alternative way to choose the step size in the classical gradient descent type methods is the line search with the Armijo rule. A natural thought is then to extend the Armijo rule in our minimax problem \((M)\) as in the proposed Algorithm \(2\). In particular, we set the following modified Armijo rule to choose the smallest nonnegative integer \( k \) such that the following inequality holds for the step size \( \beta_k = \xi s^k \) with \( 0 < \xi \leq 1 \) and \( 0 < s < 1 \),

\[
f(x_k + \beta_k p_k) \leq f(x_k) + \sigma \beta_k p_k^\top g,
\]

where \( 0 \leq \sigma \leq 0.5 \), \( g = -d \) and \( d \) is the steepest descent direction defined in Theorem \(3.1\). Particularly, we set the search direction \( p_k = d \) at iterate \( k \). Our numerical result in the next section shows that Algorithm \(2\) has a comparable performance when compared with (or even better than) Algorithm \(1\). For the sake of completeness, we present the convergence result for Algorithm \(2\) in the following. Before that, we generalize the definition of a critical point to a \((\rho, \delta)\) critical point.

**Definition 3.9.** A point \( x \) is called a \((\rho, \delta)\) critical point of \( H(x) = \max\{h_1(x), h_2(x)\} \) if \( \exists \|g\| < \delta \), for some \( g \in \partial H_\rho(x) \), where \( \partial H_\rho(x) \) is defined as follows:
The following proposition shows the relationship of a critical point and a \((\rho, \delta)\) critical point. As this result is pretty obvious, we omit its proof.

**Proposition 3.10.** Assume that \(\{x_k\}\) is a sequence in \(\mathbb{R}^n\) and that \((\rho^t, \delta^t) \to (0,0)\), for \(t \to \infty\) and that there exists a positive integer \(K(t)\), such that \(x_k\) is a \((\rho^t, \delta^t)\) critical point of \(H(x)\) for all \(k \geq K(t)\) and \(t \geq 1\). Then, every accumulation point of the sequence \(\{x_k\}\) is a critical point of \(H(x)\).

Slightly different from Algorithm 1, our goal in Algorithm 2 is to find a \((\rho, \delta)\) critical point. With Proposition 3.10 we conclude that Algorithm 2 outputs a solution that is sufficiently close to a critical point of \(H(x)\).

**Theorem 3.11.** Assume that i) \(d = \arg\min_{y \in \partial H_{\rho}(x_k)} \|y\|\) with \(\rho > 0\), ii) the termination criterion is \(\|d\| < \delta\) for some \(\delta > 0\) and iii) \(x^*\) is an optimal solution. Then for any given positive numbers \(\rho\) and \(\delta\), Algorithm 2 generates a \((\rho, \delta)\) critical point in at most

\[
\frac{H(x_0) - H(x^*)}{\sigma s \min\{1/L, \xi, \frac{\rho}{2\nu}\} \delta^2}
\]

iterations, where \(G\) is some positive constant only depending on the initial point and problem setting.

Proof. Consider the following different cases with \(\|d\| \geq \delta\).

1. If \(|h_1(x_k) - h_2(x_k)| < \rho\), then as assumed \(\|d\| > \delta\) and from Theorem 3.11 we know that \(d = \arg\min_{\alpha \in [0,1]} \|\alpha \nabla h_1(x_k) + (1-\alpha) \nabla h_2(x_k)\|\) is just the parameter \(\alpha\) which we choose in Algorithm 2. It suffices to show that the step size \(\beta_k\) is bounded from below such that

\[
H(x_{k+1}) - H(x_k) \leq -\sigma \beta_k d^\top d.
\]
This further suffices to show that $\beta_k$ is bounded from below such that for $i = 1$ or 2,

$$h_i(x_{k+1}) - h_i(x_k) = -\beta_k \nabla h_i(x_k)^T d + \frac{1}{2} \beta_k^2 d^T A_i d \leq -\sigma \beta_k d^T d. \quad (9)$$

By noting that $\nabla h_i^T d \geq d^T d$ from Remark 3.2 the second inequality in (9) holds true for all $\beta_k \leq 2(1 - \sigma)/L$. Then the step size chosen by the modified Armijo rule satisfies $\beta_k \geq s \min\{2(1 - \sigma)/L, \xi\}$, which further implies that

$$H(x_k) - H(x_{k+1}) \geq \sigma \beta_k g^T g = \sigma \beta_k \|g\|^2 \geq \sigma s \min\{2(1 - \sigma)/L, \xi\} \delta^2.$$ 

2. If $h_1(x_k) - h_2(x_k) > \rho$ and $\|\nabla h_1(x_k)\| > \delta$, we have $g = \nabla h_1(x_k)$. Because $H(x_k)$ is decreasing, under Condition 2.1 $h_1(x_k) + h_2(x_k) = \frac{1}{2} x_k^T (A_1 + A_2) x_k + (a_1 + a_2)^T x_k \leq 2h_1(x_k) = 2H(x_k) \leq 2H(x_0)$ and thus $x_k$ is bounded due to $A_1 + A_2 = 2(Q_1 + \frac{\lambda + \lambda_2}{2} Q_2) > 0$. This further implies that $\nabla h_i(x_k) = A_i x_k + b_i$ is bounded for all $k$. So there exists some positive constant only depending on the initial point and problem parameters such that $\|\nabla h_i(x_k)\| \leq G$, $i = 1, \ldots, n$. Hence $\|d\| \leq G$ because $d$ is a convex combination of $\nabla h_1(x_k)$ and $\nabla h_2(x_k)$. Then we have

$$h_1(x_{k+1}) - h_1(x_k) \leq -\beta_k \nabla h_1(x_k)^T d + \frac{1}{2} \beta_k^2 d^T A_1 d$$

and for any $\beta_k \leq 1/L$,

$$h_2(x_{k+1}) - h_2(x_k) \leq -\beta_k \nabla h_2(x_k)^T g + \frac{1}{2} \beta_k^2 g^T A_1 g$$

$$\leq \beta_k G \|g\| + \frac{1}{2} \beta_k^2 L \|g\|^2$$

$$\leq \beta_k G^2 (1 + \frac{3}{2} \beta_k L)$$

$$\leq \frac{3}{2} \beta_k G^2.$$

On the other hand, when $\beta_k \leq 1/L$,

$$h_1(x_{k+1}) - h_1(x_k) \leq -\beta_k \nabla h_1(x_k)^T g + \frac{1}{2} \beta_k^2 g^T A_1 g \leq -\beta_k g^T g + \frac{1}{2} \beta_k^2 L g^T g \leq -\frac{1}{2} \beta_k g^T g.$$

Note that for all $\beta_k \leq \frac{\rho}{2G^2}$, $\frac{3}{2} \beta_k G^2 + \frac{3}{2} \beta_k g^T g \leq \rho$. Thus for $\beta_k \leq \min\{1/L, \frac{\rho}{2G^2}\}$, we have

$$h_1(x_{k+1}) \leq h_1(x_k) - \frac{1}{2} \beta_k g^T g,$$

$$h_2(x_{k+1}) \leq h_2(x_k) + \frac{3}{2} \beta_k G^2 \leq h_1(x_k) - \rho + \frac{3}{2} \beta_k G \leq h_1(x_k) - \frac{1}{2} \beta_k g^T g.$$

Hence we have

$$H(x_{k+1}) - H(x_k) = \max\{h_1(x_{k+1}), h_2(x_{k+1})\} - h_1(x_k)$$

$$= \max\{h_1(x_{k+1}) - h_1(x_k), h_2(x_{k+1}) - h_1(x_k)\}$$

$$\leq \max\{h_1(x_{k+1}) - h_1(x_k), h_2(x_{k+1}) - h_2(x_k)\}$$

$$\leq -\frac{1}{2} \beta_k g^T g.$$

So the Armijo rule implies $\beta_k \geq s \min\{1/L, \xi, \frac{\rho}{2G^2}\}$, i.e., $\beta_k$ is lower bounded. Then according to the modified Armijo rule, we have

$$H(x_k) - H(x_{k+1}) \geq \sigma \beta_k g^T g \geq \sigma s \min\{1/L, \xi, \frac{\rho}{2G^2}\} \delta^2. \quad (10)$$

3. Symmetrically, the case with $h_2(x_k) - h_1(x_k) > \rho$ yields the same result as in (10).
The above three cases show that $H(x_k) - H(x_{k+1}) \geq \sigma s \min\{1/L, \xi, \frac{\rho}{2G}\} \delta^2$ (as $1 - \sigma \geq 1/2$, the decrease in case 1 also admits this bound). Since the decrease in each iterate is larger than $\sigma s \min\{1/L, \xi, \rho G^2\} \delta^2$, the total iterate number is bounded by

$$H(x_0) - H(x^*) \leq \sigma s \min\{1/L, \xi, \rho G^2\} \delta^2.$$

□

At the current stage, we cannot demonstrate a theoretical convergence rate for Algorithm 2 as good as the sublinear rate $O(1/\rho)$ for Algorithm 1 in Theorem 3.6. But our numerical tests show that Algorithm 2 converges as fast as Algorithm 1. Proposition 3.10 and Theorem 3.11 offer our main convergence result for Algorithm 2 as follows.

Theorem 3.12. Assume that $(\phi_k, \psi_k) \to 0$ and that $\{x^{(k)}\}$ is a sequence of solutions generated by Algorithm 2 with $\rho = \phi_k$ and $\delta = \psi_k$. Then any accumulation point of $\{x^{(k)}\}$ is an optimal solution of problem (M).

4 Numerical tests

In this section, we illustrate the efficiency of our algorithm with numerical experiments. All the numerical tests were implemented in Matlab 2016a, 64bit and were run on a Linux machine with 48GB RAM, 2600MHz cpu and 64-bit CentOS release 7.1.1503. We compare both Algorithms 1 and 2 with the ERW algorithm in [41]. We disable the parallel setting in the Matlab for fair comparison. If the parallel setting is allowed, our algorithm has a significant improvement, while the ERW algorithm does not.

We use the following same test problem as [41] to show the efficiency of our algorithms,

(IP) $\min x^\top Ax - 2a^\top x$

s.t $c_1 \leq x^\top Bx \leq c_2$,

where $A$ is an $n \times n$ positive definite matrix and $B$ is an $n \times n$ (nonsingular) symmetric indefinite matrix. We first reformulate problem (IP) to a formulation of problem (P) in the following procedure, which is motivated from [41] (the proof in [41] is also based on the monotonicity of $\gamma(\lambda)$, which is defined in Section 2.1), in order to apply the CQR for problem (P) and then invoke Algorithms 1 and 2 to solve the CQR.

Theorem 4.1. Let $x_0 = -A^{-1}a$. Then the followings hold.

1. If $x_0^\top Bx_0 < c_1$, problem (IP) is equivalent to

(IP$_1$) $\min\{x^\top Ax - 2a^\top x : \text{s.t. } c_1 \leq x^\top Bx\}$.

2. Else if $c_1 \leq x_0^\top Bx_0 \leq c_2$, problem (IP) admits an interior solution $x_0$;

3. Otherwise $c_2 < x_0^\top Bx_0$, problem (IP) is equivalent to

(IP$_2$) $\min\{x^\top Ax - 2a^\top x : \text{s.t. } x^\top Bx \leq c_2\}$.

Proof. Item 2 is obvious. Item 1 and Item 3 are symmetric. So in the following, we only prove Item 1.

In our problem set, matrix $A$ is positive definite and $B$ is indefinite. Hence, in the definition $I_{PSD} = \{\lambda : Q_1 + \lambda Q_2 \succeq 0\}$, we have $\lambda_1 < 0$, $\lambda_2 > 0$. Thus from Case 1 in Section 2.2.2 in [41] we know, when $x_0^\top Bx_0 < c_1$, problem (IP) is equivalent to

(EP$_1$) $\min\{x^\top Ax - 2a^\top x : \text{s.t. } c_1 = x^\top Bx\}$.
Since \( x_0^T B x_0 < c_1 \), the optimal solution of (IP_1) must be at its boundary \([37]\). This further yields that problem (IP) is equivalent to (IP_1).

Theorem 4.1 helps us solve problem (IP) as an inequality constrained GTRS instead of solving two GTRS with equality constraints. Before showing the numerical results, let us illustrate some functions used in our initialization. To obtain the CQR, the generalized eigenvalue problem is solved by \texttt{eigifp} in Matlab, which was developed in [21] for computing the maximum generalized eigenvalues for sparse definite matrix pencils. In our numerical setting \texttt{eigifp} is usually faster than the Matlab function \texttt{eigs}, though \texttt{eigs} will outperform \texttt{eigifp} when the condition number is large or the density is low. We use the Matlab command \texttt{sprandsym(n,density,cond,2)} and \texttt{sprandsym(n,density)} to generate \( Q_1 \) and \( Q_2 \). We set the density of matrices at 0.01 and use three levels of condition number for matrix \( Q_1 \), i.e., 10, 100 and 1000 and, in such settings, \texttt{eigifp} always dominates \texttt{eigs} (this may be because \texttt{eigs} is developed for computing extreme generalized eigenvalues for arbitrary matrices and does not utilize the definiteness and symmetry properties of the matrix pencils in our problem setting). In general, the main cost in estimating \( L \) is to compute the maximum eigenvalues of matrices \( A_1 \) and \( A_2 \), which may be time consuming for large-scale matrices. To conquer this difficulty, we can estimate a good upper bound with very cheap cost instead. Specially, we can run the function \texttt{eigifp} with precision 0.1, which is much more efficient than computing the true maximum eigenvalue with \texttt{eigifp} and, assuming \( M \) is the output, \( M + 0.1 \) is then a good upper bound for \( L \). In our numerical tests, we just use \texttt{eigifp} to estimate \( L \) since our main goal is to illustrate the efficiency of Algorithm 2. In Algorithm 1 to avoid some numerical accuracy problem, we approximate \( h_1(x_k) = h_2(x_k) \) by \( |h_1(x_k) - h_2(x_k)|/(|h_1(x_k)| + |h_2(x_k)|) \leq \epsilon_1 \). Also we use \( |h_1(x_k) - h_2(x_k)|/(|h_1(x_k)| + |h_2(x_k)|) \leq \epsilon_1 \) instead of \( |h_1(x_k) - h_1(x_k)| \leq \rho \) in Algorithm 2 for stableness consideration. In our numerical tests for both Algorithms 1 and 2 we use the following termination criteria (if any one of the following three conditions is met, we terminate our algorithm), which are slightly different from the presented algorithms for robust consideration:

1. \( H(x_{k-1}) - H(x_k) < \epsilon_2 \),
2. \( |h_1(x_k) - h_2(x_k)|/(|h_1(x_k)| + |h_2(x_k)|) \leq \epsilon_1, \|\alpha \nabla h_1(x_k) + (1 - \alpha) \nabla h_2(x_k)\| \leq \epsilon_3 \),
3. \( \|\nabla h_i(x_k)\| \leq \epsilon_4 \) and \( |h_1(x_k) - h_2(x_k)|/(|h_1(x_k)| + |h_2(x_k)|) > \epsilon_1 \), where \( i \neq j \) and \( i, j \in \{1, 2\} \),

where \( \epsilon_1, \epsilon_2 \) and \( \epsilon_3 > 0 \) are some small positive numbers for termination of the algorithm. Particularly, we set \( \epsilon_1 = 10^{-8}, \epsilon_2 = 10^{-11} \) and \( \epsilon_3 = 10^{-8} \) in Algorithm 1, and \( \epsilon_1 = 10^{-8}, \epsilon_2 = 10^{-11}, \epsilon_3 = 10^{-8}, \sigma = 10^{-4} \) and \( \xi = 1 \) (for the modified Armijo rule) in Algorithm 2.

To improve the accuracy of the solution, we apply the Newton refinement process in Section 4.1.2 in [5]. More specifically, assuming \( x^* \) is the solution returned by our algorithm, we update \( x^* \) by

\[
\delta = \frac{(x^*)^T B x^*}{2 \| B x^* \|^2} B x^*, \quad x^* = x^* - \delta.
\]

In general, the ERW algorithm can achieve a higher precision than our method (after the Newton refinement process); the precision in their method is about \( 10^{-14} \), while ours is slightly less precise than theirs. Letting \( v_1 \) denote the optimal value of ERW algorithm and \( v_2 \) denote the optimal value of our algorithm, we have at least \( |v_2 - v_1|/v_1 \approx 10^{-10} \) for most cases. The iteration number reduces to 1/5 if we reduce the precision of from \( \epsilon_1 = 10^{-8}, \epsilon_2 = 10^{-11}, \epsilon_3 = 10^{-8} \) to \( \epsilon_1 = 10^{-5}, \epsilon_2 = 10^{-8}, \epsilon_3 = 10^{-5} \). This observation seems reasonable as our method is just a first order method.

We report our numerical results in Table 1. We use “Alg1” and “Alg2” to denote Algorithms 1 and 2 respectively. For each \( n \) and each condition number, we generate 10 Easy Case and 10 Hard Case 1 examples.
Please refer to Table 1 in [41] for the detailed definitions of Easy Case and Hard Cases 1 and 2. There is a little difference about the definitions of easy and hard cases between [41] and [37]. Our analysis in the above sections uses the definitions in [37]. In fact, the Easy Case and Hard Case 1 are the easy case and Hard Case 2 is the hard case mentioned in the above sections and [37]. We use the notation “time” to denote the average CPU time (in unit of second) and “iter” to denote the average iteration numbers for all the three algorithms. For “Alg1” and “Alg2”, “time” is just the time for Algorithms 1 and 2, respectively. The notation “time_eig” denotes the average CPU time for computing the generalized eigenvalue for our algorithm. So the total time for solving problem (P) should be the summation of the time of reformulate (P) into (M) and the time of Algorithm 1 or 2, whose main cost is just “time” + “time_eig”. And “fail” denotes the failure times in the 10 examples in each case for the ERW algorithm. One reason of the failures may be that the ERW algorithm terminates in 10 iterations even when it does not find a good approximated solution. We point out that for randomly generated test examples, our method always succeeds in finding an approximated solution to prescribed precision while the ERW algorithm fails frequently in Hard Case 1. Another disadvantage of the ERW algorithm is the requirement of an efficient prior estimation of the initialization, which is unknown in general. In our numerical test, we assume that such an initialization is given as the same as [41] does.

We also need to point out that in the Hard Case 2, our algorithms do not outperform the ERW algorithm which uses the shift and deflation technique. The main time cost of shift and deflate operation is the computation of the extreme generalized eigenvalue of the matrix pencil \((A, B)\) and its corresponding generalized eigenvectors. In the test instances, as the dimension of the eigenspace of the extreme generalized eigenvalue is one, the shift and deflation technique directly finds the optimal solution by calling \texttt{eigfip} once. Our algorithm reduces to an unconstrained quadratic minimization in Hard Case 2. However, the condition number of this unconstrained quadratic minimization is so large that our algorithm performs badly as the classical gradient method. To remedy this disadvantage, we can add a step with almost free-time cost that claims that either we are in Hard Case 2 and output an optimal solution or we are in Easy Case or Hard Case 1. Recall that the hard case (or equivalently, Hard Case 2) states that \(b_1 + \lambda^*b_2\) is orthogonal to the null space of \(Q_1 + \lambda^*Q_2\) which means that \(\lambda^*\) must be a boundary point of \(I_{PSD}\). Suppose \(\lambda_i = \lambda^*\). Then we must have that \(x^* = \arg\min H(x)\) and \(H(x^*) = h_i(x^*)\) for some \(i = 1\) or 2. In fact, if \(\nabla h_i(x) = 0\) and \(h_i(x) \geq h_j(x), j \in \{1, 2\}/\{i\}\) for some \(x\), then \(x\) is optimal and we are in the hard case. So \(\nabla h_i(x) = 0\) and \(h_i(x) \geq h_j(x)\) is sufficient and necessary for \(x\) to be optimal to (M) and be in the hard case. Hence we can construct an optimal solution for problem (M) as \(\hat{x} = (Q_1 + \lambda_iQ_2)^\dagger(b_1 + \lambda_i b_2) + \sum_k \alpha_j v_j\) (where \(A^\dagger\) denotes the Moore–Penrose pseudoinverse of \(A\)) if \(v_j, j = 1, \ldots, k\) are the generalized eigenvectors of matrix pencil \((Q_1, Q_2)\) with respect to the generalized eigenvalue \(\lambda_i\) such that \(h_i(\hat{x}) \geq h_j(\hat{x})\) and \(\alpha \geq 0\). This equals to identifying if a small dimensional convex quadratic programming problem (with variable \(\alpha\)) has an
optimal value less than \( h_1((Q_1 + \lambda_1 Q_2)\mathbf{1}|(b_1 + \lambda_1 b_2)) \). And if such \( \alpha \) does not exist, we are in the easy case (or equivalently, Easy Case or Hard Case 1). This technique is very similar to the shift and deflation technique in [13, 41, 31, 14]. Hence we can solve Hard Case 2 within almost the same CPU time as the ERW algorithm. So we do not make further comparison for Hard Case 2.

Our numerical tests show that both Algorithms 1 and 2 are much more efficient than the ERW algorithm in Easy Case and for most cases in Hard Case 1. The efficiency of our algorithms is mainly due to that we only call the generalized eigenvalue solver once and every iteration only involves several matrix vector products (which are very cheap for sparse matrices). We also note that, in Easy Case, Algorithm 1 is faster than Algorithm 2 when the condition number is small and slower than Algorithm 2 when the condition number is large. This may be because that Algorithm 2 is equipped with the modified Armijo rule, which makes it more aggressive in choosing the step size and thus yields a faster convergence. In Hard Case 1, Algorithm 2 is still much more efficient than the ERW algorithm while Algorithm 1 is slower than the ERW algorithm in about half the cases. This is because Algorithm 2 has a moderate iterate number due to the aggressiveness in choosing the step size and Algorithm 1 has a much large iterate number for these cases. Moreover, our algorithms always succeed, while the ERW algorithm fails frequently in Hard Case 1. A more detailed analysis with condition number for Algorithm 1 will be given in the following.

We note that several examples (of the 10 examples) in Easy Cases admit a much larger iterate number than average. This motivates us to analyze the main factor that affects the convergence rate (reflected by the iteration number) of Algorithm 1 (the analysis for Algorithm 2 seems hard due to the non-smoothness of the problem). We then find that the main factor is \( \sqrt{\max_\alpha/\min_{\text{nnz}}^{\lambda \alpha}} \), as evidenced by the fact that examples in Easy Case and Hard Case 1 with more iterates all have a larger \( \sqrt{\max_\alpha/\min_{\text{nnz}}^{\lambda \alpha}} \) where \( \lambda_{\max \alpha} \) denotes the maximum eigenvalue of matrix \( \alpha A_1 + (1 - \alpha)A_2 \) and \( \lambda_{\min \text{nnz}}^{\lambda \alpha} \) denotes the smallest nonzero eigenvalue of matrix \( \alpha A_1 + (1 - \alpha)A_2 \) with \( \alpha \) being defined in Theorem 3.1 in the last iteration. In fact, when \( x^k \rightarrow x^* \in \{ x : \partial H(x) = 0 \} \) (in our examples, the optimal solution is unique), let the value of \( \alpha \) at iterate \( k \) be \( \alpha^k \), then \( \alpha^k \rightarrow \alpha^* \), where \( \alpha^* \) is the solution of \( \alpha \nabla h_1(x^*) + (1 - \alpha) \nabla h_2(x^*) = 0 \). From the definition of KL exponent, we have

\[
C \times \min_\alpha \| \alpha \nabla h_1(x^k) + (1 - \alpha) \nabla h_2(x^k) \| \geq |H(x^k) - H(x^*)|^{1/2}.
\]

Intuitively, the smallest value of \( C \) should be at least

\[
\min_\alpha \| \alpha \nabla h_1(x^k) + (1 - \alpha) \nabla h_2(x^k) \|^{1/2} = \frac{|(\alpha h_1(x^k) - h_2(x^k)) + (1 - \alpha)(h_1(x^k) - h_2(x^k))|^{1/2}}{\min_\alpha \| \alpha \nabla h_1(x^k) + (1 - \alpha) \nabla h_2(x^k) \|^{1/2}}
\]

which is upper bounded by \( \sqrt{\max_\alpha/\min_{\text{nnz}}^{\lambda \alpha}} \). Thus, the asymptotic value of \( C \) can be roughly seen as \( \sqrt{\lambda_{\max \alpha}/2\lambda_{\min \text{nnz}}^{\lambda \alpha}} \). Hence both Easy Case and Hard Case 1 admit local linear convergence and the convergence rate is

\[
(\sqrt{1 - \frac{1}{2C^2L}})^k = \left(\sqrt{1 - \frac{\lambda_{\min \text{nnz}}^{\lambda \alpha}}{L_{\lambda_{\max \alpha}}}}\right)^k
\]

from Theorem 3.6. We also observe from our numerical tests that in most cases the values of \( \lambda_{\max \alpha} \) are similar and that \( \lambda_{\min \text{nnz}}^{\lambda \alpha} \) in Easy Case is much larger than \( \lambda_{\min \text{nnz}}^{\lambda \alpha} \) in Hard Case 1 and \( \lambda_{\max \alpha} \) in Easy Case is very close to \( \lambda_{\max \alpha} \) in Hard Case 1. Hence, \( \sqrt{1 - \frac{\lambda_{\min \text{nnz}}^{\lambda \alpha}}{L_{\lambda_{\max \alpha}}}} \) in Easy Case is usually much smaller than that in Hard Case 1. (As \( Q_2 \) is random in our setting, the larger the condition number of \( Q_1 \) is, the larger expectation of \( \sqrt{1 - \frac{\lambda_{\min \text{nnz}}^{\lambda \alpha}}{L_{\lambda_{\max \alpha}}}} \) is.) This explains why the condition number of matrix \( Q_1 \) measures, to a large degree, the hardness of our algorithms in solving problem (M). Since Easy Case has a smaller \( \sqrt{1 - \frac{\lambda_{\min \text{nnz}}^{\lambda \alpha}}{L_{\lambda_{\max \alpha}}}} \) than Hard Case 1 for the same condition number and problem dimension, Easy Case can be solved faster than Hard Case 1. This coincides with our numerical results, i.e., Easy Case admits a smaller iterate number than Hard Cases 1.
We also tried to apply MOSEK \cite{2} to solve the CQR. But our numerical results showed that MOSEK is much slower than both our algorithms and the ERW algorithm, which took about 833 seconds for Easy Case and 960 second for Hard Case 1 with \( n = 10000 \) and \( \text{cond} = 10 \). So we do not run further numerical experiments with MOSEK. We also tested the SOCP reformulation \cite{7} under the simultaneous digonalization condition of the quadratic forms of the GTRS and the DB algorithm in \cite{3} based on the simultaneous digonalization condition of the quadratic forms. The simultaneous digonalization condition naturally holds for problem (IP) when \( A \) is positive definite. Our preliminary result shows that our method is much more efficient than the two methods based on simultaneous diagonalization when \( n \geq 10000 \) and \( \text{density} = 0.01 \) and thus we also do not report the numerical comparison in this paper. We believe this is mainly because the simultaneously diagonalization procedure of the matrices involves matrix inverse, matrix matrix product, a full Cholesky decomposition and a spectral decomposition (of a dense matrix), which is more time consuming than the operations of matrix vector products in our algorithm. Hence we do not report the numerical results based on the simultaneous diagonalization technique.

5 Concluding remarks

In this paper, we have derived a simple convex quadratic reformulation for the GTRS, which only involves a linear objective function and two convex quadratic constraints under mild assumption. We further reformulate the CQR to an unconstrained minimax problem under Condition 2.5, which is the case of interest. The minimax reformulation is a well structured convex, albeit non-smooth, problem. By investigating its inherent structure, we have proposed two efficient matrix-free algorithms to solve this minimax reformulation. Moreover, we have offered a theoretical guarantee of global sublinear convergence rate for both algorithms and demonstrate a local linear convergence rate for Algorithm 1 by proving the KL property for the minimax problem with an exponent of 1/2 under some mild conditions. Our numerical results have demonstrated clearly out-performance of our algorithms over the state-of-the-art algorithm for the GTRS.

As for our future research, we would like to show whether the CQR and the minimax reformulation and the algorithms for the minimax problem can be extended to solve GTRS with additional linear constraints. As the analysis in numerical section indicates that our algorithms have similar performance with unconstrained quadratic minimization, i.e., both algorithms admit a locally linear convergence rate with the steepest descent method, we would like to generalize existing algorithms that are efficient in solving unconstrained quadratic minimization to solve our minimax reformulation, e.g., the conjugate gradient method or Nesterov’s accelerated gradient descent algorithm. Another line of future research is to investigate whether our algorithm can be extended to general minimax problems with more (finite number of) functions. It is also interesting to verify whether the KL property still holds and whether the KL exponent is still 1/2 when more functions are involved.

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