ENERGY DECAY OF SOME BOUNDARY COUPLED SYSTEMS INVOLVING WAVE-EULER-BERNOUlli BEAM WITH ONE LOCALLy SINGULAR FRACTIONAL KELVIN-VOIGT DAMPING

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Abstract. In this paper, we investigate the energy decay of hyperbolic systems of wave-wave, wave-Euler-Bernoulli beam and beam-beam types. The two equations are coupled through boundary connection with only one localized non-smooth fractional Kelvin-Voigt damping. First, we reformulate each system into an augmented model and using a general criteria of Arendt-Batty, we prove that our models are strongly stable. Next, by using frequency domain approach, combined with multiplier technique and some interpolation inequalities, we establish different types of polynomial energy decay rate which depends on the order of the fractional derivative and the type of the damped equation in the system.

1. Introduction.

1.1. Literature. In recent years, many researches showed interest in studying the stability and controllability of certain system. The wave equation with different kinds of damping was studied extensively. The wave is created when a vibrating source disturbs the medium. In order to restrain those vibrations, several dampings can be added such as Kelvin-Voigt damping. Many researchers were interested in problems involving this kind of damping (local or global) where different types of stability have been showed. We refer to [35, 26, 18, 27, 44, 61, 8, 46, 11, 3, 4] and the rich references therein.

The beam, or flexural member, is frequently encountered in structures and machines, and its elementary stress analysis constitutes one of the most interesting facts of mechanics of materials. For beams, there was an extensive studying, since 80’s, on the stabilization of the beam equations (see [25, 38] for the one dimensional system, and [19] for n-dimensional system). Also, the studies considered the

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linear and nonlinear boundary feedback acting through shear forces and moments [39, 41, 40] and the case control by moment has been studied in [37].

The studying of the beam equation with different types of damping was extensively considered. In 1998, the author in [44] considered the longitudinal and transversal vibrations of the Euler-Bernoulli beam with Kelvin-Voigt damping distributed locally on any subinterval of the region occupied by the beam. It was shown that when the viscoelastic damping is distributed only on a subinterval in the interior of the domain, the exponential stability holds for the transversal but not for the longitudinal motion. In [56], they considered a transmission problem for the longitudinal displacement of a Euler-Bernoulli beam, where one small part of the beam is made of a viscoelastic material with Kelvin-Voigt constitutive relation and they proved that the semigroup associated to the system is exponentially stable.

Another type of damping which was studied extensively in the past few years is the fractional damping. It is widely applied in the domain of science. The fractional-order type is not only important from the theoretical point of view but also for applications. They naturally arise in physical, chemical, biological, and ecological phenomena see for example [49], and the rich references therein. They are used to describe memory and hereditary properties of various materials and processes. For example, in viscoelasticity, due to the nature of the material microstructure, both elastic solid and viscous fluid-like response qualities are involved. Using Boltzmann’s assumption, we end up with a stress strain relationship defined by a time convolution. The viscoelastic response occurs in a variety of materials, such as soils, concrete, rubber, cartilage, biological tissue, glasses, and polymers (see [16, 59, 17, 48]). Fractional computing in modeling can improve the capturing of the complex dynamics of natural systems, and controls of fractional order type can improve performance not achievable before using controls of integer-order type. For example, systems in many quantum mechanics, nuclear physics and biological phenomena such as fluid flow are indeed fractional (see for example [59, 48, 54]).

Fractional calculus includes various extensions of the usual definition of derivative from integer to real order, including the Hadamard, Erdelyi-Kober, Riemann-Liouville, Riesz, Weyl, Grünwald-Letnikov, Jumarie and the Caputo representation. A thorough analysis of fractional dynamical systems is necessary to achieve an appropriate definition of the fractional derivative. For example, the Riemann-Liouville definition entails physically unacceptable initial conditions (fractional order initial conditions); conversely, for the Caputo representation, which is introduced by Michele Caputo [23] in 1967, the initial conditions are expressed in terms of integer-order derivatives having direct physical significance; this definition is mainly used to include memory effects. Recently, in [24] a new definition of the fractional derivative was presented without a singular kernel; this derivative possesses very interesting properties, for instance the possibility to describe fluctuations and structures with different scales. The case of wave equation with boundary fractional damping have been treated in [50, 51] where they proved the strong stability and the lack of uniform stabilization. However, the case of the plate equation or the beam equation with boundary fractional damping was treated in [1] where they showed that the energy is polynomially stable. In [6], they considered a multidimensional wave equation with boundary fractional damping acting on a part of the boundary of the domain. They established a polynomial energy decay rate for smooth solutions, under some geometric conditions. Ammari et al., in [9], studied the stabilization for
a class of evolution systems with fractional-damping. They proved the polynomial stability of the system.

Over the past few years, the coupled systems received a vast attention due to their potential applications. The coupled systems have many applications in the modeling and control of engineering, such as: aircraft, satellite antennas and road traffic (see [21] for example). Most of the work in the coupled system considers the stability of the system with various coupling, damping locations, and damping types. Many researches studied coupling systems with a Kelvin-Voigt damping such as wave-wave system, heat-wave system, Timoshenko (see [60, 4, 62]). In 2012, Tebou in [58] considered the Euler-Bernoulli equation coupled with a wave equation in a bounded domain. The frictional damping is distributed everywhere in the domain and acts through one of the equations only. For the case where the dissipation acts through the Euler-Bernoulli equation he showed that the system is not exponentially stable and that the energy decays polynomially was proved. For the case where the damping acts through the wave equation polynomial stability was proved.

Benaissa et al., in [7], considered the large time behavior of one dimensional coupled wave equations with fractional control applied at the coupled point. They showed an optimal decay result.

In [33], Hassine considered a beam and a wave equations coupled on an elastic beam through transmission conditions where the locally distributed damping acts through one of the two equations only. The systems are described as follows

\[
\begin{align*}
\begin{cases}
{u_t}_t - (u_x + D_u u_{xx})_x = 0, & \text{in } \Omega_1, \\
{u_t}_t + y_{xxx} = 0, & \text{in } \Omega_2, \\
u(x,0) = u(x,L) = 0, & t > 0, \\
u(0,t) = 0, & t > 0, \quad \text{and}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
{y_t}_t + (y_{xx} + D_b y_{xxx})_x = 0, & \text{in } \Omega_1, \\
{y_t}_t - y_{xx} = 0, & \text{in } \Omega_2, \\
y(x,0) = y(x,L) = 0, & t > 0, \\
y(0,t) = 0, & t > 0, \quad \text{and}
\end{cases}
\end{align*}
\]

where \( \Omega_1 = (0, \ell) \times \mathbb{R}^+ \), \( \Omega_2 = (\ell, L) \times \mathbb{R}^+ \), \( D_u = a(x) \chi(\epsilon, f) \) and \( D_b = b(x) \chi(\epsilon, f) \) with \( 0 < \epsilon < f < \ell < L \) and \( a(x), b(x) \geq c_0 > 0 \) in \( (\epsilon, f) \). The author proved that for the case when the dissipation acts through the wave, the energy of this coupled system decays polynomially as the time variable goes to infinity. Also, for the case where the damping acts through the beam equation polynomial stability was proved.

The case of a Euler-Bernoulli beam and a wave equations coupled via the interface by transmission conditions was considered by Hassine in [34], where he supposed that the beam equation is stabilized by a localized distributed feedback. He reached that sufficiently smooth solutions decay logarithmically at infinity even the feedback dissipation affects an arbitrarily small open subset of the interior.

In [29], the authors studied the stabilization system of a coupled wave and a Euler-Bernoulli plate equation where only one equation is supposed to be damped with a frictional damping in the multidimensional case. Under some assumption about the damping and the coupling terms, they showed that sufficiently smooth solutions of the system decay logarithmically at infinity without any geometric conditions on the effective damping domain.

In [13], Ammari and Nicaise, considered the stabilization problem for coupling the damped wave equation with a damped Kirchhoff plate equation. They proved an exponential stability result under some geometric condition. In 2018, the authors considered in [43], a system of 1-d coupled string-beam. They obtained two kinds of energy decay rates of the string-beam system with different locations of...
the frictional damping. On one hand, if the frictional damping is only actuated in the beam part, the system lacks exponential decay. Specifically, the optimal polynomial decay rate $t^{-1}$ is obtained under smooth initial conditions. On the other hand, if the frictional damping is only effective in the string part, the exponential decay of energy is presented. In 2020, the authors in [31], considered a system of two-dimensional coupled wave-plate with local frictional damping in a bounded domain. The frictional damping is only distributed in the part of the plate’s or wave’s domain, and the other is stabilized by the transmission through the interface of the plate’s and wave’s domains. They showed that the energy of the system decays polynomially under some geometric condition when the frictional damping only acts on the part of the plate, and the energy of the system is exponentially stable when the frictional damping acts only on the other part of the wave.

In 2018, Guo and Ren in [30], studied the stabilization for a hyperbolic-hyperbolic coupled system consisting of Euler-Bernoulli beam and wave equations, where the structural damping of the wave equation is taken into account. The coupling is actuated through boundary weak connection. The system is described as follows

$$
\begin{align*}
\begin{cases}
 w_{tt} + w_{xxxx} = 0, & (x, t) \in (0, 1) \times \mathbb{R}^+, \\
 u_{tt} - u_{xx} - su_{xxx} = 0, & (x, t) \in (0, 1) \times \mathbb{R}^+, \\
 w(1, t) = w_{xx}(1, t) = w(0, t) = 0, & t > 0, \\
 w_{xx}(0, t) = ru_{t}(0, t), u(1, t) = 0, & x \in (0, 1), \\
 su_{xt}(0, t) + u_x(0, t) = -rw_{xt}(0, t), & x \in (0, 1), \\
 w(x, 0) = w_0(x), w_t(x, 0) = w_1(x), & x \in (0, 1), \\
 u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (0, 1),
\end{cases}
\end{align*}
$$

(1.2)

where $(w_0, w_1, u_0, u_1)$ is the initial state and $r \neq 0, s > 0$ are constants. They concluded the Riesz basis property and the exponential stability of the system.

In [10], the authors considered a stabilization problem for a coupled string-beam system. They proved some decay results of the energy of the system. Moreover, they proved, for the same model but with two control functions, independently of the length of the beam that the energy decay with a polynomial rate for all regular initial data. In [12], the authors considered a stabilization problem for a string-beams network and proved an exponential decay result. In [14], the author considered a boundary stabilization problem for the transmission Bernoulli-Euler plate equation and proved a uniform exponential energy decay under natural conditions. In [57], a coupled system of wave-plate type with thermal effect was studied and exponentially stability was proved. In [28], the authors considered the transmission problem for a coupled system of undamped and structurally damped plate equations in two sufficiently smooth and bounded subdomains. They showed, independently of the size of the damped part, that the damping is strong enough to produce uniform exponential decay of the energy of the coupled system. In 2019, Liu and Han [32], considered a system of coupled plate equations where indirect structural or Kelvin-Voigt damping is imposed, i.e., only one equation is directly damped by one of these two damping. They showed that the associated semigroup of the system with
indirect structural damping is analytic and exponentially stable. However, with the much stronger indirect Kelvin-Voigt damping, they proved that the semigroup is even not differentiable and that the exponential stability is still maintained.

1.2. Physical interpretation of the models. In the first model (EBB)-W_{FKV}, we investigate the stability of coupled Euler-Bernoulli beam and wave equations. The coupling is via boundary connections with localized non-regular fractional Kelvin-Voigt damping, where the damping acts through the wave equation only (see Figure 1). The system that describes this model is as follows

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - (au_x + d(x)\partial_t^{\alpha,\eta}u_x)_x = 0, \\
y_{tt} + by_{xxxx} = 0, \\
u(L,t) = y(-L,t) = y_x(-L,t) = 0, & \quad (x,t) \in (0,L) \times (0,\infty), \\
u_x(0,t) + by_{x}x(0,t) = 0, y_{xx}(0,t) = 0, & \quad t \in (0,\infty), \\
u(0,t) = y(0,t), & \quad t \in (0,\infty), \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \quad x \in (0,L), \\
y(x,0) = y_0(x), y_t(x,0) = y_1(x), & \quad x \in (-L,0).
\end{align*}
\]

((EBB)-W_{FKV})

The coefficients \(a, b\) are strictly positive constant numbers, \(\alpha \in (0,1)\) and \(\eta \geq 0\).

We suppose that there exists \(0 < l_0 < l_1 < L\) and a strictly positive constant \(d_0\), such that

\[
d(x) = \begin{cases} 
  d_0, & x \in (l_0, l_1) \\
  0, & x \in (0, l_0) \cup (l_1, L).
\end{cases}
\]

The Caputo’s fractional derivative \(\partial_t^{\alpha,\eta}\) of order \(\alpha \in (0,1)\) with respect to time variable \(t\) defined by

\[
[D^{\alpha,\eta} \omega](t) = \partial_t^{\alpha,\eta} \omega(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(s-t)} \frac{d\omega}{ds}(s)ds,
\]

where \(\Gamma\) denotes the Gamma function.

In the second model W-(EBB)\(_{FKV}\), we consider a system of coupled Euler-Bernoulli beam and wave equations. These two equations are coupled through boundary connections. In this case the localized non-smooth fractional Kelvin-Voigt damping acts only on the Euler-Bernoulli beam (see Figure 2). The system
that represents this model is as follows
\[
\begin{align*}
  u_{tt} - au_{xx} &= 0, \\
  y_{tt} + (by_{xx} + d(x)\partial_t^{\alpha,\eta}y_{xx})_{xx} &= 0, \\
  u(-L,t) &= y(L,t) = y_x(L,t) = 0, \\
  au_x(0,t) + by_{xx}(0,t) &= 0, y_{xx}(0,t) = 0, \\
  u(0,t) &= y(0,t), \\
  u(x,0) &= u_0(x), u_t(x,0) = u_1(x), \\
  y(x,0) &= y_0(x), y_t(x,0) = y_1(x),
\end{align*}
\]

Next, we briefly mention three other models and their stability results. The model W-W\(_FKV\), which represents the system of coupled wave equations coupled through boundary connections with a localized non-regular fractional Kelvin-Voigt damping acting through one wave equation only (see Figure 3). The system that describes this model is as follows
\[
\begin{align*}
  u_{tt} - (au_x + d(x)\partial_t^{\alpha,\eta}u_x)_x &= 0, \\
  y_{tt} - by_{xx} &= 0, \\
  u(L,t) = y(-L,t) = 0, \\
  au_x(0,t) = by_x(0,t), \\
  u(0,t) &= y(0,t), \\
  u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\
  y(x,0) &= y_0(x), y_t(x,0) = y_1(x),
\end{align*}
\]

The model ((EBB)\(_FKV\)), where a system of Euler-Bernoulli beam with a non-regular localized fractional Kelvin-Voigt damping (see Figure 4) is considered. The system is as follows
\[
\begin{align*}
  y_{tt} + (by_{xx} + d(x)\partial_t^{\alpha,\eta}y_{xx})_{xx} &= 0, \\
  y(0,t) &= y_x(0,t) = y_{xx}(L,t) = y_{xxx}(L,t) = 0, \\
  y(x,0) &= y_0(x), y_t(x,0) = y_1(x),
\end{align*}
\]
The last model (EBB)-(EBB)$_{FKV}$, represents a system of two Euler-Bernoulli beam equations coupled through boundary connections. The localized non-smooth fractional Kelvin-Voigt damping acts only on one of the two equations (see Figure 5). The system that represents this model is as follows

\[
\begin{align*}
  &u_{tt} + au_{xxxx} = 0, & (x, t) \in (-L, 0) \times (0, \infty), \\
  &y_{tt} + (by_{xx} + d(x) \partial_t^{\alpha, \eta} y_{xx})_{xx} = 0, & (x, t) \in (0, L) \times (0, \infty), \\
  &u(-L, t) = u_x(-L, t) = y(L, t) = y_x(L, t) = 0, & t \in (0, \infty), \\
  &au_{xxx}(0, t) - by_{xxx}(0, t) = 0, u_{xx}(0) = y_{xx}(0, t) = 0, & t \in (0, \infty), \\
  &u(0, t) = y(0, t), & t \in (0, \infty), \\
  &u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (-L, 0), \\
  &y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in (0, L).
\end{align*}
\]

(EBB)-(EBB)$_{FKV}$
We give the physical meaning of the following variables.

\[ y = \text{vertical displacement}, \quad u_x = \text{the stress of the wave}. \]
\[ y_z = \text{rotation}, \]
\[ y_{xx} = \text{Bending moment}, \]
\[ y_{xxx} = \text{Shear Force}, \]
\[ y_{xxxx} = \text{Loading}. \] (1.5)

The way the beam supported is translated into conditions on the function \( y \) and its derivatives. These conditions are collectively referred to as boundary conditions. They are meaningful in physics and engineering. The boundary conditions in the model \( (\text{EBB})_{FKV} \) signifies the following

- \( y(0,t) = 0 \): This signifies that the beam is pinned to its support, which means that the beam cannot experience any deflection at \( x = 0 \).
- \( y_x(0,t) = 0 \): It signifies that the rotation at the pinned support is zero.
- \( y_{xx}(L,t) = 0 \): It means that there is no bending moment at the free end of the beam.
- \( y_{xxx}(L,t) = 0 \): This boundary conditions gives the assumption that there is no shearing force acting at the free end of the beam.

This kind of models, supported at one end with the other end free, described in the above four conditions can be referred to as a cantilever beam. A good example on the cantilever beam is a balcony, it is supported at one end only, the rest of the beam extends over the open space. Other examples are a cantilever roof in a bus shelter, car park or railway station. We give some of the advantages and disadvantages of the cantilever beam.

**Advantages:**

- Cantilever beams do not require support on the opposite side.
- The negative bending moment created in cantilever beams helps to counteract the positive bending moments created.
- Cantilever beams can be easily constructed.
- These beam enables erection with little disturbance in navigation.

**Disadvantages:**

- Cantilever beams are subjected to large deflections.
- Cantilever beams are subjected to larger moments.
- A strong fixed support or a backspan is necessary to keep the structure stable.

In a cantilever beam, the bending moment at the free end always vanishes. In fact, if we connect the beam with a wave (see \( ((\text{EBB})-\text{W}_{FKV}), (\text{W}-(\text{EBB})_{FKV}) \)) at the free end the bending moment will be zero and this induces a shear force on the end of the beam. Consequently, the fourth boundary condition above is no longer valid, and it is replaced by \( au_x(0,t) + by_{xxx}(0,t) = 0 \). This condition signifies that the shear force of the beam and the stress force of the wave are such that one cancels the other.

1.3. **Description of the paper.** In this paper, we investigate the stability results of two main models of systems with a non-smooth localized fractional Kelvin-Voigt damping where the coupling is made via boundary connections. In the first model \( ((\text{EBB})-\text{W}_{FKV}) \) we consider the coupled Euler-Bernoulli beam and wave equation with the damping acts on the wave equation only. In Subsection 2.1, we reformulate \( ((\text{EBB})-\text{W}_{FKV}) \) into an augmented model and we prove the well-posedness of the system by semigroup approach. Moreover, using a general criteria of Arendt
and Batty, we show the strong stability of our system in the absence of the compactness of the resolvent. In section 2.2, using the semigroup theory of linear operators and a result obtained by Borichev and Tomilov we show that the energy of the System ((EBB)-W\_FKV) has a polynomial decay rate of type $t^{\frac{2}{2+\alpha}}$. In the second model (W-(EBB)\_FKV), we consider Euler-Bernoulli beam and wave equations coupled through boundary connections with the damping to act through the Euler-Bernoulli beam equation only. For this model, we show that the energy of the System (W-(EBB)\_FKV) has a polynomial decay rate of type $t^{\frac{2}{3+\alpha}}$. In addition, we briefly shed the light on three other models. The model (W-W\_FKV), where we consider two wave equations coupled through boundary connections with a non-smooth localized fractional Kelvin-Voigt damping acting only on one of the two equations. We stated that we can establish a polynomial energy decay rate of type $t^{\frac{2}{2+\alpha}}$. The model ((EBB)\_FKV), we consider the Euler-Bernoulli beam with a non-smooth localized fractional Kelvin-Voigt damping. We stated that the energy of the system ((EBB)\_FKV) decays polynomially with a decay rate $t^{\frac{2}{1+\alpha}}$ where the proof is remarked to be similar to that of (W-(EBB)\_FKV). Finally, for the model ((EBB)-(EBB)\_FKV), we consider the polynomial stability of two Euler-Bernoulli beam equations coupled through boundary connections with damping acting only on one of the two equation. We remarked that we can establish a polynomial energy decay rate of type $t^{\frac{2}{2+\alpha}}$. The table below (Table 1) summarizes the decay rate of the energy for the five models. Also, it gives the decay rate of the same five models but with Kelvin-Voigt damping (as $\alpha \to 1$).

| Model           | Decay Rate   | $\alpha \to 1$ |
|-----------------|--------------|-----------------|
| (EBB)-W\_FKV    | $t^{\frac{2}{2+\alpha}}$ | $t^{-4}$         |
| W-W\_FKV        | $t^{\frac{2}{2+\alpha}}$ | $t^{-4}$         |
| W-(EBB)\_FKV    | $t^{\frac{2}{3+\alpha}}$ | $t^{-1}$         |
| (EBB)\_FKV      | $t^{\frac{2}{1+\alpha}}$ | Exponential      |
| (EBB)-(EBB)\_FKV| $t^{\frac{2}{3+\alpha}}$ | $t^{-1}$         |

Table 1. Decay Results

Some significant implications on the energy decay rate are given below:

- From the decay rate of the models ((EBB)-W\_FKV) and (W-(EBB)\_FKV) we can deduce that if we want to choose the place where the fractional Kelvin-Voigt damping acts it is better to choose the damping on the wave. Since the energy of this model decays faster compared with that of (W-(EBB)\_FKV) model.
- For the model ((EBB)-W\_FKV), if we replace the condition of the null bending moment ($y_{xx}(0)=0$) at the connecting boundary by taking the rotation to be zero ($y_t(0)=0$), we get the same decay rate. So, this result improves the work in [33] where they reached energy decay rate of type $t^{-2}$, however in our work we proved an energy decay rate of type $t^{-4}$ (as $\alpha \to 1$) (see Section 2).
- For the model ((EBB)-W\_FKV), we established an energy decay rate of type $t^{-4}$ (as $\alpha \to 1$). By comparing this energy decay rate with that in [60], where the authors considered two wave equations coupled through velocity with localized non
smooth Kelvin-Voigt and they established an energy decay rate of type $t^{-1}$. We can see that, by comparing the energy decay rate of these two systems that it is better, when considering Kelvin-Voigt damping, to consider the coupling through the boundary connection rather than through the velocity.

**Remark 1.1.** We note that in the upcoming sections, the letters used to denote the variables are independent from each other in each section.

2. (EBB)-$W_{FKV}$ model. In this section, we consider the (EBB)-$W_{FKV}$ model, where we study the stability of the system a Euler-Bernoulli and wave equations coupled through boundary connection with a localized fractional Kelvin-Voigt damping acting on the wave equation only.

2.1. Well-posedness and strong stability. In this subsection, we study the strong stability of the system (EBB)-$W_{FKV}$ in the absence of the compactness of the resolvent. First, we will study the existence, uniqueness and regularity of the solution of the system.

2.1.1. Augmented model and well-posedness. In this part, using a semigroup approach, we establish well-posedness for the system (EBB)-$W_{FKV}$. First, we recall theorem 2 stated in [50, 5].

**Theorem 2.1.** Let $\alpha \in (0, 1)$, $\eta \geq 0$ and $\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}$ be the function defined almost everywhere on $\mathbb{R}$. The relation between the ‘input’ $V$ and the ‘output’ $O$ of the following system

\[
\begin{align*}
\partial_t \omega(x, \xi, t) &+ (\xi^2 + \eta)\omega(x, \xi, t) - V(x, t)|\xi|^{\frac{2\alpha-1}{2}} = 0, \ (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}^*_+, \quad (2.1) \\
\omega(x, \xi, 0) & = 0, \ (x, \xi) \in (0, L) \times \mathbb{R}, \quad (2.2)
\end{align*}
\]

is given by

\[
O(x, t) - \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi = 0, \ (x, t) \in (0, L) \times \mathbb{R}^*_+, \quad (2.3)
\]

where

\[
[I^{1-\alpha, \eta}V](x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} V(s) ds \quad \text{and} \quad \kappa(\alpha) = \frac{\sin(\alpha \pi)}{\pi}.
\]

In the above theorem, taking the input $V(x, t) = \sqrt{d(x)}u_{xt}(x, t)$, then using Equation (1.4), we get that the output $O$ is given by

\[
O(x, t) = \sqrt{d(x)} I^{1-\alpha, \eta} u_{xt}(x, t) = \frac{\sqrt{d(x)}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \partial_s u_{x}(x, s) ds
\]
\[
= \sqrt{d(x)} \partial_t^{1-\alpha, \eta} u_{x}(x, t).
\]

Therefore, by taking the input $V(x, t) = \sqrt{d(x)}u_{xt}(x, t)$ in Theorem 2.1 and using the above equation, we get

\[
\begin{align*}
\partial_t \omega(x, \xi, t) &+ (\xi^2 + \eta)\omega(x, \xi, t) - \sqrt{d(x)}u_{xt}(x, t)|\xi|^{\frac{2\alpha-1}{2}} = 0, \ (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}^*_+, \quad (2.5)
\omega(x, \xi, 0) & = 0, \ (x, \xi) \in (0, L) \times \mathbb{R}, \quad (2.5)
\sqrt{d(x)} \partial_t^{1-\alpha, \eta} u_{x}(x, t) - \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi & = 0, \ (x, t) \in (0, L) \times \mathbb{R}^*_+.
\end{align*}
\]
Finally, by adding (2.10) and (2.11), we obtain (2.9). The proof is thus complete.

From system (2.5), we deduce that system ((EBB)-WFKV) can be recast into the following augmented model

\[
\begin{align*}
\left\{
\begin{array}{l}
{u_t} = \left( {a + \sqrt {d(\xi )} } \right)u_x + \sqrt {d(\xi )} \kappa (\alpha )\int_\mathbb{R} \left| \xi \right|^{2a - 1} \omega (\xi ,\xi ,t)d\xi \\
y_{tt} + b y_{xxx} = 0,
\end{array}
\right. \\
\omega_t (\xi ,\xi ,t) + (\left| \xi \right|^2 + \eta ) \omega (\xi ,\xi ,t) - \sqrt {d(\xi )} |u_x (\xi ,t)| |\xi |^{2a - 1} = 0,
\end{align*}
\]

with the following transmission and boundary conditions

\[
\begin{align*}
\left\{
\begin{array}{l}
u(L,t) = y(\xi ,t) = y_x (\xi ,t) = 0, \\
\kappa (\alpha )u_x (0,t) + b y_{xxx} (0,t) = 0, y_x (0,t) = 0,
\end{array}
\right. \\
u(0,t) = y(0,t),
\end{align*}
\]

and with the following initial conditions

\[
\begin{align*}
u(x,0) = u_0 (x), \\
y(x,0) = y_0 (x).
\end{align*}
\]

The energy of the system (2.6)-(2.8) is given by

\[
E_1 (t) = \frac{1}{2} \int_0^L (|u_x|^2 + a |u_x|^2) \, dx + \frac{1}{2} \int_{-L}^0 (|y_t|^2 + b |y_{xxx}|^2) \, dx \\
+ \kappa (\alpha ) \int_0^L \int_\mathbb{R} \left| \omega (\xi ,\xi ,t) \right|^2 \, d\xi \, dx.
\]

**Lemma 2.2.** Let \( U = (u, u_t, y, y_t, \omega) \) be a regular solution of the System (2.6)-(2.8). Then, the energy \( E_1 (t) \) satisfies the following estimation

\[
\frac{d}{dt} E_1 (t) = -\kappa (\alpha ) \int_0^L \int_\mathbb{R} \left| \omega (\xi ,\xi ,t) \right|^2 \, d\xi \, dx.
\]

**Proof.** First, multiplying the first and the second equations of (2.6) by \( u_t \) and \( y_t \) respectively, integrating over \((0,L)\) and \((-L,0)\) respectively, using integration by parts with (2.7) and taking the real part \( \Re \), we get

\[
\frac{1}{2} \frac{d}{dt} \left( \int_0^L |u_t|^2 + a |u_x|^2 \right) \, dx + \frac{1}{2} \frac{d}{dt} \int_{-L}^0 (|y_t|^2 + b |y_{xxx}|^2) \, dx \\
+ \Re \left( \kappa (\alpha ) \int_0^L \sqrt{d(\xi )} u_{tx} \left( \int_\mathbb{R} |\xi|^{2a-1} \omega (\xi ,\xi ,t) \, d\xi \right) \, dx \right) = 0.
\]

Now, multiplying the third equation of (2.6) by \( \kappa (\alpha ) \bar{\omega} \), integrating over \((0,L) \times \mathbb{R}\), then taking the real part, we get

\[
\frac{\kappa (\alpha )}{2} \frac{d}{dt} \int_0^L \int_\mathbb{R} \left| \omega (\xi ,\xi ,t) \right|^2 \, d\xi \, dx + \kappa (\alpha ) \int_0^L \int_\mathbb{R} (\xi^2 + \eta ) |\omega (\xi ,\xi ,t) |^2 \, d\xi \, dx \\
= \Re \left( \kappa (\alpha ) \int_0^L \sqrt{d(\xi )} u_{xt} \left( \int_\mathbb{R} |\xi|^{2a-1} \bar{\omega} (\xi ,\xi ,t) \, d\xi \right) \, dx \right).
\]

Finally, by adding (2.10) and (2.11), we obtain (2.9). The proof is thus complete. □
Remark 2.3. The condition \(|\xi|\omega(x, \xi) \in W\) is imposed to insure the existence of \(\int_{0}^{L} \int_{\mathbb{R}} (\xi^2 + \eta)|\omega(x, \xi)|^2 d\xi dx\) in (2.9) and \(\sqrt{d(x)} \int_{\mathbb{R}} |\xi|^{\frac{2n-1}{2}} \omega(x, \xi) d\xi \in L^2(0, L)\).

If \(U = (u, u_t, y, y_t, \omega)\) is a regular solution of system (2.6)-(2.8), then the system can be rewritten as evolution equation on the Hilbert space \(\mathcal{H}_1\) given by

\[
U_t = A_1 U, \quad U(0) = U_0,
\]

where \(U_0 = (u_0, u_1, y_0, y_1, 0)\).
Lemma 2.4. Let $\alpha \in (0, 1)$, $\eta \geq 0$, then the following integrals
\[
I_1(\eta, \alpha) = \kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{1 + \xi^2 + \eta} d\xi, \quad I_2(\eta, \alpha) = \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{(1 + \xi^2 + \eta)^2} d\xi
\]
and
\[
I_3(\eta, \alpha) = \int_{0}^{+\infty} \frac{\xi^{2\alpha+1}}{(1 + \xi^2 + \eta)^2} d\xi
\]
are well defined.

Proof. First, $I_1(\eta, \alpha)$ can be written as
\[
I_1(\eta, \alpha) = 2 \kappa(\alpha) \int_{0}^{+\infty} \frac{\xi^{2\alpha-1}}{1 + \xi^2 + \eta} d\xi.
\]
Thus equation (2.16) can be simplified by defining a new variable $y = 1 + \frac{\xi^2}{1+\eta}$.
Substituting $\xi$ by $(y - 1)^{-\frac{3}{2}}(1 + \eta)^{\frac{3}{2}}$ in equation (2.16), we get
\[
I_1(\eta, \alpha) = \frac{\kappa(\alpha)}{(1 + \eta)^1-\alpha} \int_{1}^{+\infty} \frac{1}{y(y - 1)^{1-\alpha}} dy.
\]
Using the fact that $\alpha \in (0, 1)$, its easy to see that $y^{-1}(y - 1)^{\alpha-1} \in L^1(1, +\infty)$, therefore $I_1(\eta, \alpha)$ is well defined. Now, for $I_2(\eta, \alpha)$, using $\eta \geq 0$ and $\alpha \in (0, 1)$, we get
\[
I_2(\eta, \alpha) < \int_{\mathbb{R}} \frac{\xi^{2\alpha-1}}{1 + \xi^2 + \eta} d\xi = \frac{I_1(\eta, \alpha)}{\kappa(\alpha)} < +\infty.
\]
Then, $I_2(\eta, \alpha)$ is well-defined.

Now, for the integral $I_3(\eta, \alpha)$, since
\[
\frac{\xi^{2\alpha+1}}{(1 + \xi^2 + \eta)^2} \sim \frac{\xi^{2\alpha+1}}{(1 + \eta)^2}, \quad \text{and} \quad \frac{\xi^{2\alpha+1}}{(1 + \xi^2 + \eta)^2} \sim \frac{1}{\xi^{3-2\alpha}},
\]
and the fact that $\alpha \in (0, 1)$, we get $I_3(\eta, \alpha)$ is well-defined.

The proof is thus complete. \qed

Proposition 2.4. The unbounded linear operator $A_1$ is $m$-dissipative in the energy space $H_1$.

Proof. For all $U = (u, v, y, z, \omega) \in D(A_1)$, one has
\[
\Re \langle (A_1 U, U) \rangle_{H_1} = -\kappa(\alpha) \int_{0}^{L} \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi)|^2 d\xi dx \leq 0,
\]
which implies that $A_1$ is dissipative. Now, let $F = (f_1, f_2, f_3, f_4, f_5) \in H_1$, we prove the existence of $U = (u, v, y, z, \omega) \in D(A_1)$, solution of the equation
\[
(I - A_1) U = F.
\]
Equivalently, one must consider the system given by
\[
u - v = f_1, \quad (2.17)
\]
\[
v - (au_x + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi)_{x} = f_2, \quad (2.18)
\]
\[
y - z = f_3, \quad (2.19)
\]
\[
z + by_{xxxx} = f_4, \quad (2.20)
\]
\[
(1 + \xi^2 + \eta)\omega(x, \xi) - \sqrt{d(x)} v_x |\xi|^{\frac{2\alpha-1}{2}} = f_5(x, \xi). \quad (2.21)
\]
Using Equations (2.17), (2.21) and the fact that $\eta \geq 0$, we get

$$\omega(x, \xi) = \frac{f_1(x, \xi)}{1 + \xi^2 + \eta} + \frac{\sqrt{d(x)u_x} |\xi|^{\frac{2\alpha-1}{2}}}{1 + \xi^2 + \eta} - \frac{\sqrt{d(x)(f_1)_x} |\xi|^{\frac{2\alpha-1}{2}}}{1 + \xi^2 + \eta}.$$

Inserting the above equation and (2.17) in (2.18) and (2.19) in (2.20), we get

$$u - \left( au_x + d(x)I_1(\eta, \alpha)u_x + d(x)I_1(\eta, \alpha)(f_1)_x - \sqrt{d(x)\kappa(\alpha)} \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} f_5(x, \xi) d\xi \right) = F_1, (2.22)$$

$$y + by_{xxx} = F_2 (2.23)$$

where $I_1(\eta, \alpha)$ is defined in Equation (2.15), $F_1 = f_1 + f_2$ and $F_2 = f_3 + f_4$. And with the following boundary conditions

$$u(L) = y(-L) = y_x(-L) = y_{xx}(0) = 0, au_x(0) + by_{xxx}(0) = 0, \text{ and } u(0) = y(0).$$

Now, we define

$$V = \{(\varphi, \psi) \in H^1_0(0, L) \times H^1_0(-L, 0); \varphi(0) = \psi(0)\}.$$ 

The space $V$ is equipped with the following inner product

$$\langle (\varphi, \psi), (\varphi_1, \psi_1) \rangle_V = a \int_{0}^{L} \varphi \overline{\varphi_1} dx + b \int_{0}^{L} \varphi_x \overline{\varphi_{1x}} dx + b \int_{-L}^{0} \varphi_x \overline{\varphi_{1x}} dx.$$

Let $(\varphi, \psi) \in V$. Multiplying equations (2.22) and (2.23) by $\overline{\varphi}$ and $\overline{\psi}$, and integrating respectively on $(0, L)$ and $(-L, 0)$, then using by parts integration, we get

$$a((u, y), (\varphi, \psi)) = L(\varphi, \psi) \quad \forall (\varphi, \psi) \in V,$$

where

$$a((u, y), (\varphi, \psi)) = \int_{0}^{L} u \varphi dx + \int_{0}^{L} u_x \varphi_x dx + \int_{0}^{L} y \psi dx + \int_{-L}^{0} y_{xx} \psi_{xx} dx$$

$$+ I_1(\eta, \alpha) \int_{0}^{L} d(x)u_x \varphi_x dx$$

and

$$L(\varphi, \psi) = \int_{0}^{L} F_1 \varphi dx + I_1(\eta, \alpha) \int_{0}^{L} d(x)(f_1)_x \varphi_x dx$$

$$- \kappa(\alpha) \int_{0}^{L} \sqrt{d(x)\varphi} \left( \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} f_5(x, \xi) d\xi \right) dx + \int_{-L}^{0} F_2 \overline{\psi} dx.$$

Using the fact that $I_1(\eta, \alpha) > 0$, we get $a$ is a sesquilinear, continuous coercive form on $V \times V$. Next, by using Cauchy-Schwartz inequality and the definition of $d(x)$, we get

$$\left| \int_{0}^{L} \sqrt{d(x)\varphi} \left( \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} f_5(x, \xi) d\xi \right) dx \right| \leq \sqrt{\frac{d_0}{\kappa(\alpha)}} \sqrt{I_2(\eta, \alpha)} \|\varphi\|_{L^2(t_0, t_1)} \|f_5\|_{W},$$

(2.26)

where $I_2(\eta, \alpha)$ is defined in Equation (2.15). Hence, $L$ is an antilinear continuous form on $V$. Then, using Lax-Milgram theorem, we deduce that there exists unique $(u, y) \in V$ solution of the variational problem (2.25). Applying the classical elliptic regularity, we deduce that $y \in H^4(-L, 0)$, and

$$\left( au_x + \sqrt{d(x)\kappa(\alpha)} \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right) \in L^2(0, L).$$
Using Lemma 2.1, the fact that $(u,f)$ well-defined. On the other hand, using the fact that $f|_{\partial L} = 0$.

It follows that $\omega(x,\xi) \in W$. From proposition 2.4, the operator $A_1$ generates a $C_0$-semigroup of contractions $(e^{tA_1})_{t \geq 0}$ following Lummer-Phillips theorem (see in [53] and [47]).

\[
\omega(x) = \frac{f_3(x,\xi)}{1 + \xi^2 + \eta} + \frac{\sqrt{d(x)u_x} |\xi|^{2\alpha-1}}{1 + \xi^2 + \eta} - \frac{\sqrt{d(x)(f_1)_x} |\xi|^{2\alpha-1}}{1 + \xi^2 + \eta}.
\]

(2.27)

It is easy to see that $(v, z) \in H^1_0(0, L) \times H^2_0(-L, 0)$ and $v(0) = z(0)$.

In order to complete the existence of $U \in D(A_1)$, we need to prove $\omega(x,\xi)$ and $|\xi|\omega(x,\xi) \in W$. From equation (2.27), we obtain

$$
\int_0^L \int_0^L |\omega(x,\xi)|^2 \, dx \leq 3 \int_0^L \int_\mathbb{R} |f_3(x,\xi)|^2 \, d\xi dx + 3d_0 I_2(\eta, \alpha) \int_0^L (|u_x|^2 + |(f_1)_x|^2) \, dx.
$$

Using Lemma 2.1, the fact that $(u, f_1) \in H^1_0(0, L) \times H^1_0(0, L)$, we obtain

$$
I_2(\eta, \alpha) \int_0^L (|u_x|^2 + |(f_1)_x|^2) \, dx < \infty.
$$

On the other hand, using the fact that $f_3 \in W$, we get

$$
\int_0^L \int_\mathbb{R} |f_3(x,\xi)|^2 \, d\xi dx \leq \frac{1}{(1 + \eta)^2} \int_0^L \int_\mathbb{R} |f_3(x,\xi)|^2 \, d\xi dx < +\infty.
$$

It follows that $\omega(x,\xi) \in W$. Next, using equation (2.27), we get

$$
\int_0^L \int_\mathbb{R} |\xi \omega(x,\xi)|^2 \, d\xi dx \leq 3 \int_0^L \int_\mathbb{R} \frac{\xi^2|f_3(x,\xi)|^2}{(1 + \xi^2 + \eta)^2} \, d\xi dx
$$

$$
+ 6d_0 I_3(\eta, \alpha) \left( \int_0^L (|u_x|^2 + |(f_1)_x|^2) \, dx \right),
$$

where $I_3(\eta, \alpha) = \int_0^{+\infty} \frac{\xi^{2\alpha+1}}{(1 + \xi^2 + \eta)^2} \, d\xi$. Using Lemma 2.1 we get that $I_3(\eta, \alpha)$ is well-defined.

Now, using the fact that $f_3(x,\xi) \in W$ and

$$
\max_{\xi \in \mathbb{R}} \frac{\xi^2}{(1 + \xi^2 + \eta)^2} = \frac{1}{4(1 + \eta)} < \frac{1}{4},
$$

we get

$$
\int_0^L \int_\mathbb{R} \frac{\xi^2|f_3(x,\xi)|^2}{(1 + \xi^2 + \eta)^2} \, d\xi dx \leq \max_{\xi \in \mathbb{R}} \frac{\xi^2}{(1 + \xi^2 + \eta)^2} \int_0^L \int_\mathbb{R} |f_3(x,\xi)|^2 \, d\xi dx
$$

$$
< \frac{1}{4} \int_0^L \int_\mathbb{R} |f_3(x,\xi)|^2 \, d\xi dx < +\infty.
$$

It follows that $|\xi|\omega \in W$. Finally, since $\omega, f_3 \in W$, we get

$$
-(|\xi|^2 + \eta) \omega(x,\xi) + \sqrt{d(x)v_x} |\xi|^{2\alpha-1} = \omega(x,\xi) - f_3(x,\xi) \in W.
$$

Therefore, there exists $U := (u, v, y, z, \omega) \in D(A_1)$ solution $(I - A_1)U = F$. The proof is thus complete.

From proposition 2.4, the operator $A_1$ is m-dissipative on $H^1_0$, consequently it generates a $C_0$-semigroup of contractions $(e^{tA_1})_{t \geq 0}$ following Lummer-Phillips theorem (see in [53] and [47]). Then the solution of the evolution Equation (2.14) admits the following representation

$$
U(t) = e^{tA_1}U_0, \quad t \geq 0,
$$

where $U_0 = (u_0, v_0, y_0, z_0, \omega_0)$.
which leads to the well-posedness of (2.14). Hence, we have the following result.

**Theorem 2.5.** Let $U_0 \in H_1$, then problem (2.14) admits a unique weak solution $U$ satisfies

$$U(t) \in C^0(\mathbb{R}^+, H_1).$$

Moreover, if $U_0 \in D(A_1)$, then problem (2.14) admits a unique strong solution $U$ satisfies

$$U(t) \in C^1(\mathbb{R}^+, H_1) \cap C^0(\mathbb{R}^+, D(A_1)).$$

2.1.2. Strong stability. This part is devoted to study the strong stability of the system. It is easy to see that the resolvent of $A$ is not compact. For this aim, we use a general criteria of Arendt-Battay in [15] (see Theorem 4.2) to obtain the strong stability of the $C_0$-semigroup $(e^{tA_1})_{t \geq 0}$. Our main result in this part is the following theorem.

**Theorem 2.6.** Assume that $\eta \geq 0$, then the $C_0$-semigroup of contractions $e^{tA_1}$ is strongly stable on $H_1$ in the sense that

$$\lim_{t \to +\infty} \|e^{tA_1}U_0\|_{H_1} = 0 \quad \forall \ U_0 \in H_1.$$

In order to prove Theorem 2.6 we need to prove that the operator $A_1$ has no pure imaginary eigenvalues and $\sigma(A_1) \cap i\mathbb{R}$ is countable, where $\sigma(A_1)$ denotes the spectrum of $A_1$. For clarity, we divide the proof into several lemmas.

**Lemma 2.2.** Let $\alpha \in (0, 1)$, $\eta \geq 0$, $\lambda \in \mathbb{R}$ and $f_5 \in W$. For $(\eta > 0$ and $\lambda \in \mathbb{R})$ or $(\eta = 0$ and $\lambda \in \mathbb{R}^+)$, we have

$$I_4(\lambda, \eta, \alpha) = i\lambda \kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1} d\xi}{i\lambda + \xi^2 + \eta} < \infty,$$

$$I_5(\lambda, \eta, \alpha) = \kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1} d\xi}{i\lambda + \xi^2 + \eta} < \infty,$$

and

$$I_6(\lambda, \eta, \alpha) := \kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1} f_5(x, \xi) d\xi}{i\lambda + \xi^2 + \eta} \in L^2(0, L).$$

**Proof.** The integrals $I_4$ and $I_5$ can be written in the following form

$I_4(\lambda, \eta, \alpha) = \lambda^2 I_7(\lambda, \eta, \alpha) + i A I_8(\lambda, \eta, \alpha)$ and $I_5(\lambda, \eta, \alpha) = -i \lambda I_7(\lambda, \eta, \alpha) + A I_8(\lambda, \eta, \alpha)$

where

$I_7(\lambda, \eta, \alpha) = \kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi$. And $I_8(\lambda, \eta, \alpha) = \kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi$.

So, we need to prove that $I_7(\lambda, \eta, \alpha), I_8(\lambda, \eta, \alpha)$ are well defined.

First, we have

$I_7(\lambda, \eta, \alpha) = 2\kappa(\alpha) \int_0^{+\infty} \frac{\xi^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi = 2\kappa(\alpha) \int_0^{1} \frac{\xi^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi + 2\kappa(\alpha) \int_{1}^{+\infty} \frac{\xi^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi$.

Hence in both cases where $(\eta > 0$ and $\lambda \in \mathbb{R})$ or $(\eta = 0$ and $\lambda \in \mathbb{R}^+)$, we have

$$\frac{\xi^{2\alpha-1}}{\lambda^2 + \eta^2} \sim_0 \frac{\xi^{2\alpha-1}}{\lambda^2 + \eta^2} \quad \text{and} \quad \frac{\xi^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} \sim_\infty \frac{1}{\xi^{2\alpha-2}}.$$
Since $0 < \alpha < 1$ then $I_7(\lambda, \eta, \alpha)$ is well-defined. Now, we have

$$I_8(\lambda, \eta, \alpha) = 2\kappa(\alpha) \int_0^{+\infty} \frac{\xi^{2\alpha-1}(\xi^2 + \eta)}{\lambda^2 + (\xi^2 + \eta)^2} d\xi = 2\kappa(\alpha) \int_0^{1} \frac{\xi^{2\alpha-1}(\xi^2 + \eta)}{\lambda^2 + (\xi^2 + \eta)^2} d\xi + 2\kappa(\alpha) \int_1^{+\infty} \frac{\xi^{2\alpha-1}(\xi^2 + \eta)}{\lambda^2 + (\xi^2 + \eta)^2} d\xi.$$ 

Similar to $I_7$, in the both cases where $(\eta > 0$ and $\lambda \in \mathbb{R})$ or $(\eta = 0$ and $\lambda \in \mathbb{R}^*)$, we have

$$\frac{\xi^{2\alpha-1}(\xi^2 + \eta)}{\lambda^2 + (\xi^2 + \eta)^2} \sim \frac{\xi^{2\alpha-1}(\xi^2 + \eta)}{\lambda^2 + (\xi^2 + \eta)^2} \sim \frac{1}{\xi^{3-2\alpha}}.$$ 

Since $0 < \alpha < 1$, then $I_8(\lambda, \eta, \alpha)$ is well-defined. For $I_6$, using Cauchy-Schwarz inequality and the fact that $f_5 \in W$ and that $I_7 < \infty$, we get

$$\int_0^L |I_6(x, \lambda, \eta, \alpha)|^2 dx = \kappa(\alpha)^2 \int_0^L \left| \int_\mathbb{R} \frac{|\xi|^{2\alpha-1} f_5(x, \xi)}{i\lambda + \xi^2 + \eta} d\xi \right|^2 dx$$

$$\leq \kappa(\alpha)^2 \left( \int_\mathbb{R} \frac{|\xi|^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi \right) \int_0^L \int_\mathbb{R} |f_5(x, \xi)|^2 d\xi dx < +\infty.$$ 

The proof is thus complete. \hfill \Box

**Lemma 2.3.** Let $\alpha \in (0, 1)$, $\eta \geq 0$, $\lambda \in \mathbb{R}$. For $(\eta > 0$ and $\lambda \in \mathbb{R})$ or $(\eta = 0$ and $\lambda \in \mathbb{R}^*)$, we have

$$I_{11}(\lambda, \eta, \alpha) = \int_\mathbb{R} \frac{|\xi|^{2\alpha-1}}{\sqrt{\lambda^2 + (\xi^2 + \eta)^2}} d\xi \quad \text{and} \quad I_{12}(\lambda, \eta, \alpha) = \int_\mathbb{R} \frac{|\xi|^{2\alpha+1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi$$

are well-defined.

**Proof.** We have

$$I_{11}(\lambda, \eta, \alpha) = 2 \int_0^1 \frac{\xi^{2\alpha-1}}{\sqrt{\lambda^2 + (\xi^2 + \eta)^2}} d\xi + 2 \int_1^{+\infty} \frac{\xi^{2\alpha-1}}{\sqrt{\lambda^2 + (\xi^2 + \eta)^2}} d\xi$$

Hence in the both cases where $(\eta > 0$ and $\lambda \in \mathbb{R})$ or $(\eta = 0$ and $\lambda \in \mathbb{R}^*)$, we have

$$\frac{\xi^{2\alpha-1}}{\sqrt{\lambda^2 + (\xi^2 + \eta)^2}} \sim \frac{\xi^{2\alpha-1}}{\sqrt{\lambda^2 + (\xi^2 + \eta)^2}} \sim \frac{1}{\lambda^{3-2\alpha}}.$$ 

Since $0 < \alpha < 1$ then $I_{11}(\lambda, \eta, \alpha)$ is well-defined. Now,

$$I_{12}(\lambda, \eta, \alpha) = 2 \int_0^{+\infty} \frac{\xi^{2\alpha+1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi = 2 \int_0^1 \frac{\xi^{2\alpha-1}(\xi^2 + \eta)}{\lambda^2 + (\xi^2 + \eta)^2} d\xi + 2 \int_1^{+\infty} \frac{\xi^{2\alpha-1}(\xi^2 + \eta)}{\lambda^2 + (\xi^2 + \eta)^2} d\xi.$$ 

In a similar way, in the both cases where $(\eta > 0$ and $\lambda \in \mathbb{R})$ or $(\eta = 0$ and $\lambda \in \mathbb{R}^*)$, we have

$$\frac{\xi^{2\alpha+1}}{\lambda^2 + (\xi^2 + \eta)^2} \sim \frac{\xi^{2\alpha+1}}{\lambda^2 + (\xi^2 + \eta)^2} \sim \frac{1}{\lambda^{3-2\alpha}}.$$ 

Since $0 < \alpha < 1$, then $I_{12}(\lambda, \eta, \alpha)$ is well-defined. The proof is thus complete. \hfill \Box
Lemma 2.4. Assume that $\eta \geq 0$. Then, for all $\lambda \in \mathbb{R}$, we have $i\lambda I - A_1$ is injective, i.e.
\[
\ker (i\lambda I - A_1) = \{0\}.
\]
Proof. Let $\lambda \in \mathbb{R}$, such that $i\lambda$ be an eigenvalue of the operator $A_1$ and $U = (u, v, y, z, \omega) \in D(A_1)$ a corresponding eigenvector. Therefore, we have
\[
A_1 U = i\lambda U.
\] (2.28)
Equivalently, we have
\[
v = i\lambda u, \quad \text{(2.29)}
\]
and with the boundary conditions
\[
\begin{cases}
  u(L) = y(-L) = y_x(-L) = 0, \\
  y_{xx}(0) = 0,
\end{cases}
\] (2.34)
and with the continuity transmission conditions
\[
\begin{cases}
  u(0) = y(0), \\
  au_x(0) = -by_{xxx}(0).
\end{cases}
\] (2.35)
A straightforward calculation gives
\[
0 = \Re \left( \langle i\lambda U, U \rangle_{\mathcal{H}_1} \right) = \Re \left( \langle A_1 U, U \rangle_{\mathcal{H}_1} \right) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi)|^2 d\xi dx.
\] (2.36)
Consequently, we deduce that
\[
\omega(x, \xi) = 0 \text{ a.e. in } (0, L) \times \mathbb{R}.
\] (2.37)
Inserting Equation (2.36) into (2.33) and using the definition of $d(x)$, we get
\[
v_x = 0 \text{ in } (l_0, l_1). \quad \text{(2.37)}
\]
It follows, from Equation (2.29), that
\[
\lambda u_x = 0 \text{ in } (l_0, l_1). \quad \text{(2.38)}
\]
Here we will distinguish two cases.

Case 1. If $\lambda = 0$:
From (2.29) and (2.31) we get
\[
v = z = 0 \text{ on } (0, L). \quad \text{(2.37)}
\]
Using Equations (2.30), (2.32) and (2.36) we get
\[
u_{xx} = y_{xxxx} = 0.
\]
Using the boundary conditions in (2.34) we can write $u$ and $y$ as
\[
u = c_1 (x - L) \quad \text{and} \quad y = c_3 \left( \frac{x^3}{6} - \frac{L^2}{2} x - \frac{L^3}{3} \right)
\]
where \(c_1, c_3\) are constant numbers to be determined. Now, using conditions in \((2.35)\) we get
\[
\begin{align*}
  c_1 &= \frac{L^2}{2} c_3, \\
  ac_1 &= -bc_3.
\end{align*}
\] (2.39)

Then, \(c_3(aL^2 + b) = 0\). Since \(a, b > 0\), we deduce that \(c_1 = c_3 = 0\). Then we get \(u = y = 0\). Hence, \(U = 0\). In this case the proof is complete.

**Case 2.** If \(\lambda \neq 0\):

From Equation \((2.38)\), we get
\[
u_x = 0 \quad \text{in} \quad (l_0, l_1).
\] (2.40)

Using Equations \((2.36)\) and \((2.40)\) in \((2.30)\), and using Equation \((2.29)\) we get
\[
u = 0 \quad \text{in} \quad (l_0, l_1).
\] (2.41)

Substituting equations \((2.29)\) and \((2.31)\) into Equations \((2.30)\) and \((2.32)\) and using Equation \((2.36)\), we get
\[
\begin{align*}
  \lambda^2 u + au_{xx} &= 0, \quad \text{over} \quad (0, L), \\
  \lambda^2 y - by_{xxxx} &= 0, \quad \text{over} \quad (-L, 0),
\end{align*}
\] (2.42)

From Equation \((2.42)\) and \((2.41)\), and using the unique continuation theorem (see \([42]\)) we get
\[
u = 0 \quad \text{in} \quad (0, L).
\] (2.44)

From Equation \((2.43)\), \((2.34)-(2.35)\), and using \((2.44)\) we get the following system
\[
\begin{align*}
  \lambda^2 y - by_{xxxx} &= 0, \quad \text{over} \quad (-L, 0) \\
  y(0) &= y_{xx}(0) = y_{xxx}(0) = 0, \\
  y(-L) &= y_x(-L) = 0.
\end{align*}
\] (2.45)

It’s easy to see that \(y = 0\) is the unique solution of \((2.45)\). Hence \(U = 0\). The proof is thus completed.

**Lemma 2.5.** Assume that \(\eta = 0\). Then, the operator \(-A_1\) is not invertible and consequently \(0 \in \sigma(A_1)\).

**Proof.** Let \(F = \left(x^2 - L^2, 0, \frac{-x^4}{3L^2} - \frac{4}{3}Lx - L^2, 0, 0\right) \in H_1\) and assume that there exists \(U = (u, v, y, z, \omega) \in D(A_1)\) such that \(-A_1U = F\). It follows that
\[
v_x = 2x \quad \text{in} \quad (0, L) \quad \text{and} \quad \xi^2 \omega(x, \xi) - 2x\sqrt{d(x)}|\xi|^\frac{2a-1}{2} = 0.
\]

From the above equation, we deduce that \(\omega(x, \xi) = 2|\xi|^\frac{2a-1}{2} x\sqrt{d(x)} \notin W\), therefore the assumption of the existence of \(U\) is false and consequently the operator \(-A_1\) is not invertible. The proof is thus complete.

**Lemma 2.6.** If \((\eta > 0 \text{ and } \lambda \in \mathbb{R})\) or \((\eta = 0 \text{ and } \lambda \in \mathbb{R}^*)\), then \(i\lambda I - A_1\) is surjective.
Proof. Let $F = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$, we look for $U = (u, v, y, z, \omega) \in D(A_1)$ solution of
\[
(i\lambda I - A_1)U = F.
\]
Equivalently, we have
\[
i\lambda u - v = f_1,
\]
\[
i\lambda v - \left(au_x + \sqrt{d(x)}\kappa(\alpha)\int_{\mathbb{R}} |\xi|^{2\alpha-1}\omega(x, \xi) d\xi\right) = f_2,
\]
\[
i\lambda y - z = f_3,
\]
\[
i\lambda z + by_{xxxx} = f_4,
\]
\[
(i\lambda + \xi^2 + \eta)\omega(x, \xi) - \sqrt{d(x)}v_x|\xi|^{2\alpha-1} = f_5(x, \xi).
\]
Using Equations (2.47) and (2.51) and that fact that $\eta \geq 0$ we get
\[
\omega(x, \xi) = \frac{f_5(x, \xi)}{i\lambda + \xi^2 + \eta} + \frac{\sqrt{d(x)}i\lambda u_x|\xi|^{2\alpha-1}}{i\lambda + \xi^2 + \eta} - \frac{\sqrt{d(x)}(f_1)_x|\xi|^{2\alpha-1}}{i\lambda + \xi^2 + \eta}.
\]
Substituting $v$ and $z$ in (2.47) and (2.49) into Equations (2.48) and (2.50), and using (2.52) we get
\[
\lambda^2 u + (au_x + d(x)I_4(\lambda, \eta, \alpha)u_x - g(x, \lambda, \eta, \alpha)) = f,
\]
\[
\lambda^2 y - by_{xxxx} = F,
\]
such that
\[
\begin{cases}
  f = -(f_2 + i\lambda f_1) \in L^2(0, L), \\
  g(x, \lambda, \eta, \alpha) = I_5(\lambda, \eta, \alpha)d(x)(f_1)_x - \sqrt{d(x)}I_6(x, \lambda, \eta, \alpha), \\
  F = -(f_4 + i\lambda f_3) \in L^2(-L, 0),
\end{cases}
\]
and $I_4(\lambda, \eta, \alpha)$, $I_5(\lambda, \eta, \alpha)$ and $I_6(\lambda, \eta, \alpha)$ are defined in Lemma 2.2.

Now, we distinguish two cases:

**Case 1.** $\eta > 0$ and $\lambda = 0$, then System (2.53)-(2.54) becomes
\[
\begin{align*}
(au_x - g(x, 0, \eta, \alpha)) &= -f_2, \\
by_{xxxx} &= f_4.
\end{align*}
\]
By applying Lax-Milgram theorem, and using Lemma 2.2 it is easy to see that the above system has a unique strong solution $(u, y) \in V$.

**Case 2.** $\eta \geq 0$ and $\lambda \in \mathbb{R}^*$. The system (2.53)-(2.54) becomes
\[
\begin{align*}
\lambda^2 u + (au_x + d(x)I_4(\lambda, \eta, \alpha)u_x) &= G, \\
\lambda^2 y - by_{xxxx} &= F,
\end{align*}
\]
such that
\[
G = f + g_x(x, \lambda, \eta, \alpha).
\]
We first define the linear unbounded operator $L : \mathbb{H} := H^1_0(0, L) \times H^2_L(-L, 0) \longrightarrow \mathbb{H}'$
where $\mathbb{H}'$ is the dual space of $\mathbb{H}$ by
\[
LU = \begin{pmatrix}
- (au_x + d(x)I_4(\lambda, \eta, \alpha)u_x) \\
by_{xxxx}
\end{pmatrix}, \quad \forall U \in \mathbb{H}.
\]
Thanks to Lax-Milgram theorem, it is easy to see that \( \mathcal{L} \) is isomorphism. The system \((2.55)-(2.56)\) is equivalent to

\[
(\lambda^2 \mathcal{L}^{-1} - I) U = \mathcal{L}^{-1} F, \quad \text{where} \ U = (u, y)^T \text{ and } F = (G, F)^T. \tag{2.57}
\]

Since the operator \( \mathcal{L}^{-1} \) is isomorphism and \( I \) is a compact operator from \( \mathbb{H} \) to \( \mathbb{H}' \). Then, \( \mathcal{L}^{-1} \) is compact operator from \( \mathbb{H} \) to \( \mathbb{H} \). Consequently, by Fredholm’s alternative, proving the existence of \( U \) solution of \((2.57)\) reduces to proving \( \text{ker} (\lambda^2 \mathcal{L}^{-1} - I) = \{0\} \). Indeed, if \((\tilde{u}, \tilde{y}) \in \text{ker} (\lambda^2 \mathcal{L}^{-1} - I)\), then \( \lambda^2 (\tilde{u}, \tilde{y}) - \mathcal{L} (\tilde{u}, \tilde{y}) = 0 \). It follows that,

\[
\lambda^2 \tilde{u} + (a\tilde{u}_x + d(x)I_4(\lambda, \eta, \alpha)\tilde{u}_x)_x = 0, \tag{2.58}
\]

\[
\lambda^2 \tilde{y} - b\tilde{y}_{xxxx} = 0, \tag{2.59}
\]

\[
\tilde{u}(L) = \tilde{y}(-L) = \tilde{y}_x(-L) = \tilde{y}_{xx}(0) = 0, \tag{2.60}
\]

\[
av\tilde{u}_x(0) + b\tilde{y}_{xxx}(0) = 0, \tilde{u}(0) = \tilde{y}(0). \tag{2.61}
\]

Multiplying \((2.58)\) and \((2.59)\) by \( \tilde{u} \) and \( \tilde{y} \) respectively, integrating over \((0, L)\) and \((-L, 0)\) respectively and taking the sum, then using by parts integration and the boundary conditions \((2.60)-(2.61)\), and take the imaginary part we get

\[
d_0 \Im (I_4(\lambda, \alpha, \eta)) \int_{l_0}^{l_1} |\tilde{u}_x|^2 dx = 0.
\]

From Lemma 2.2 we have \( \Im (I_4(\lambda, \alpha, \eta)) = \lambda I_8(\lambda, \eta, \alpha) \neq 0 \), we get \( \tilde{u}_x = 0 \) in \((l_0, l_1)\).

Then, system \((2.58)-(2.60)\) becomes

\[
\lambda^2 \tilde{u} + a\tilde{u}_{xx} = 0 \text{ over } (0, L), \tag{2.62}
\]

\[
\lambda^2 \tilde{y} - b\tilde{y}_{xxxx} = 0 \text{ over } (-L, 0), \tag{2.63}
\]

\[
\tilde{u}_x = 0 \text{ over } (l_0, l_1). \tag{2.64}
\]

It is now easy to see that if \((\tilde{u}, \tilde{y})\) is a solution of system \((2.62)-(2.64)\), then the vector \( \tilde{U} \) defined by \( \tilde{U} := (\tilde{u}, i\lambda\tilde{u}, \tilde{y}, i\lambda\tilde{y}, 0) \) belongs to \( D(A_1) \), and \( i\lambda \tilde{U} - A \tilde{U} = 0 \).

Therefore, \( \tilde{U} \in \text{ker} (i\lambda I_1 - A_1) \), then by using Lemma 2.4, we get \( \tilde{U} = 0 \). This implies that system \((2.57)\) admits a unique solution due to Fredholm’s alternative, hence \((2.57)\) admits a unique solution in \( V \). Thus, we define \( v := i\lambda u - f_1, z := i\lambda y - f_3 \) and

\[
\omega(x, \xi) = \frac{f_5(x, \xi)}{i\lambda + \xi^2 + \eta} + \frac{\sqrt{d(x)}i\lambda u_x |\xi|^{\frac{2\alpha - 1}{2}}}{i\lambda + \xi^2 + \eta} - \frac{\sqrt{d(x)}(f_1)_x |\xi|^{\frac{2\alpha - 1}{2}}}{i\lambda + \xi^2 + \eta}. \tag{2.65}
\]

Since \( F \in H_1 \), it is easy to see that \( v \in H^1_R(0, L), z \in H^2_R(-L, 0), v(0) = z(0) \) and

\[
(au_x + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha - 1}{2}} \omega(x, \xi, t) d\xi)_x \in L^2(0, L).
\]

It is left to prove that \( \omega \) and \( |\xi| \omega \in W \) (for the both cases). From equation \((2.65)\), we get

\[
\int_0^L \int_{\mathbb{R}} |\omega(x, \xi)|^2 d\xi dx \leq 3 \int_0^L \int_{\mathbb{R}} \frac{|f_5(x, \xi)|^2}{\lambda^2 + (\xi^2 + \eta)^2} d\xi dx
\]

\[
+ 3a_0 \left( \int_{l_0}^{l_1} (|\lambda u_x|^2 + |(f_1)_x|^2) dx \right) I_7(\lambda, \eta, \alpha).
\]
Using the fact that \( f_5 \in W \) and \((\eta > 0 \text{ and } \lambda \in \mathbb{R}) \) or \((\eta = 0 \text{ and } \lambda \in \mathbb{R}^*) \), we obtain
\[
\int_0^L \int_\mathbb{R} \frac{|f_5(x, \xi)|^2}{\lambda^2 + (\xi^2 + \eta)^2} d\xi \leq \frac{1}{\lambda^2 + \eta^2} \int_0^L \int_\mathbb{R} |f_5(x, \xi)|^2 d\xi dx < +\infty.
\]
Using Lemma 2.2, it follows that \( \omega \in W \). Next, using equation (2.65), we get
\[
\int_0^L \int_\mathbb{R} |\xi|^2 |f_5(x, \xi)|^2 d\xi dx \leq 3 \int_0^L \int_\mathbb{R} \frac{\xi^2 |f_5(x, \xi)|^2}{\lambda^2 + (\xi^2 + \eta)^2} d\xi dx + 3a_0 \int_0^L (|\lambda u_x|^2 + |(f_1)_{xj}|^2) I_{12}(\lambda, \eta, \alpha),
\]
where \( I_{12}(\lambda, \eta, \alpha) = \int_\mathbb{R} \frac{|\xi|^{2\alpha + 1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi < +\infty \) by using Lemma 2.3. Now, using the fact that \( f_5 \in W \) and
\[
\max_{\xi \in \mathbb{R}} \frac{\xi^2}{\lambda^2 + (\xi^2 + \eta)^2} = \frac{\sqrt{\eta^2 + \lambda^2}}{\lambda^2 + \left(\sqrt{\eta^2 + \lambda^2} + \eta\right)^2} = C(\lambda, \eta),
\]
we get
\[
\int_0^L \int_\mathbb{R} \frac{\xi^2 |f_5(x, \xi)|^2}{\lambda^2 + (\xi^2 + \eta)^2} d\xi dx \leq \int_0^L \max_{\xi \in \mathbb{R}} \frac{\xi^2}{\lambda^2 + (\xi^2 + \eta)^2} \int_\mathbb{R} |f_5(x, \xi)|^2 d\xi dx
\]
\[
= C(\lambda, \eta) \int_0^L \int_\mathbb{R} |f_5(x, \xi)|^2 d\xi dx < +\infty.
\]
It follows that \(|\xi|\omega \in W \). Finally, since \( \omega \in W \), we get
\[
-(\xi^2 + \eta)\omega(x, \xi) + \sqrt{d(x)} v_x |\xi|^{2\alpha + 1} = i\lambda \omega(x, \xi) - f_5(x, \xi) \in W.
\]
Thus, we obtain \( U = (u, v, y, z, \omega) \in D(\mathcal{A}_1) \) solution of \((i\lambda I - \mathcal{A}_1)U = F \). The proof is thus complete.

**Proof of Theorem 2.6.** First, using Lemma 2.4, we directly deduce that \( \mathcal{A}_1 \) has no pure imaginary eigenvalues. Next, using Lemmas 2.5, 2.6 and with the help of the closed graph theorem of Banach, we deduce that \( \sigma(\mathcal{A}_1) \cap i\mathbb{R} = \{\emptyset\} \) if \( \eta > 0 \) and \( \sigma(\mathcal{A}_1) \cap i\mathbb{R} = \{0\} \) if \( \eta = 0 \). Thus, we get the conclusion by Applying theorem 4.2 of Arendt Batt.

### 2.2. Polynomial stability in the case \( \eta > 0 \)

In this section, we study the polynomial stability of the system (2.6)-(2.8) in the case \( \eta > 0 \). For this purpose, we will use a frequency domain approach method, namely we will use Theorem 4.4. Our main result in this section is the following theorem.

**Theorem 2.7.** Assume that \( \eta > 0 \). The \( C_0 \)-semigroup \( (e^{t\mathcal{A}_1})_{t \geq 0} \) is polynomially stable; i.e. there exists constant \( C_1 > 0 \) such that for every \( U_0 \in D(\mathcal{A}_1) \), we have
\[
E_1(t) \leq \frac{C_1}{t^\frac{\ell}{\ell - \frac{\ell}{2}}} \|U_0\|^2_{D(\mathcal{A}_1)}, \quad t > 0, \forall U_0 \in D(\mathcal{A}_1).
\]  
(2.66)

According to Theorem 4.4, by taking \( \ell = 1 - \frac{\ell}{2} \), the polynomial energy decay (2.66) holds if the following conditions
\[
i\mathbb{R} \subset \rho(\mathcal{A}_1), \quad (H_1)
\]
and
\[
\limsup_{\lambda \in \mathbb{R}, |\lambda| \to \infty} \frac{1}{|\lambda|^{1 - \frac{\ell}{2}}} \left\| (i\lambda I - \mathcal{A}_1)^{-1} \right\|_{\mathcal{L}(\mathcal{H}_1)} < \infty, \quad (H_2)
\]
are satisfied. Since Condition (H1) is already proved in Lemma 2.4. We will prove condition (H2) by an argument of contradiction. For this purpose, suppose that (H2) is false, then there exists \( \{ (\lambda_n, U_n) := (u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi)) \} \subset \mathbb{R}^5 \times \mathcal{D} (A_1) \) with

\[
|\lambda_n| \to +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}_1} = \|(u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))\|_{\mathcal{H}_1} = 1, \tag{2.67}
\]

such that

\[
(\lambda_n)^{1-\frac{2}{\lambda}} (i\lambda_n I - A_1) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}(\cdot, \xi))^\top \to 0 \quad \text{in} \quad \mathcal{H}_1.
\]

For simplicity, we drop the index \( n \). Equivalently, from (2.68), we have

\[
i\lambda u - v = \frac{f_1}{\lambda^{1-\frac{2}{\lambda}}} \quad \text{in} \quad H^1_R (0, L), \tag{2.69}
\]

\[
i\lambda v - (S_d) = \frac{f_2}{\lambda^{1-\frac{2}{\lambda}}} \quad \text{in} \quad L^2 (0, L), \tag{2.70}
\]

\[
i\lambda y - z = \frac{f_3}{\lambda^{1-\frac{2}{\lambda}}} \quad \text{in} \quad H^2_L (-L, 0), \tag{2.71}
\]

\[
i\lambda z + by_{xxxx} = \frac{f_4}{\lambda^{1-\frac{2}{\lambda}}} \quad \text{in} \quad L^2 (-L, 0), \tag{2.72}
\]

\[
(i\lambda + \xi^2 + \eta) \omega (x, \xi) - \sqrt{d(x)} v_x |\xi|^{\frac{2\alpha - 1}{\lambda}} = \frac{f_5 (x, \xi)}{\lambda^{1-\frac{2}{\lambda}}} \quad \text{in} \quad W, \tag{2.73}
\]

where \( S_d = au_x + \sqrt{d(x)} \kappa (\alpha) \int_\mathbb{R} |\xi|^{\frac{2\alpha - 1}{\lambda}} \omega (x, \xi) d\xi \).

Here we will check the condition (H2) by finding a contradiction with (2.67) by showing \( \|U\|_{\mathcal{H}_1} = o(1) \). For clarity, we divide the proof into several Lemmas.

**Lemma 2.7.** Let \( \alpha \in (0, 1), \eta > 0 \) and \( \lambda \in \mathbb{R} \), then

\[
\begin{align*}
I_{12} (\lambda, \eta, \alpha) &= \int_\mathbb{R} \frac{|\xi|^{\alpha + \frac{1}{2}}}{(|\lambda| + \xi^2 + \eta)^\frac{1}{2}} d\xi = c_1 (|\lambda| + \eta)^{\frac{3}{2} - \frac{1}{\lambda}}, \\
I_{13} (\lambda, \eta) &= \left( \int_\mathbb{R} \frac{1}{(|\lambda| + \xi^2 + \eta)^\frac{1}{2}} d\xi \right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}} \frac{1}{(|\lambda| + \eta)^{\frac{1}{2}}}, \\
I_{14} (\lambda, \eta) &= \left( \int_\mathbb{R} \frac{\xi^2}{(|\lambda| + \xi^2 + \eta)^\frac{1}{2}} d\xi \right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{4}} \frac{1}{(|\lambda| + \eta)^{\frac{1}{2}}}
\end{align*}
\]

where \( c_1 = \int_1^\infty \frac{(y - 1)\frac{3}{2} - \frac{1}{2}}{y^2} dy \).

**Proof.** \( I_{12} \) can be written as

\[
I_{12} (\lambda, \eta, \alpha) = \frac{2}{(\lambda + \eta)^\frac{\alpha}{2}} \int_0^\infty \frac{\xi^{\alpha + \frac{1}{2}}}{\left( 1 + \frac{\xi^2}{|\lambda| + \eta} \right)^\frac{1}{2}} d\xi. \tag{2.74}
\]

Thus, equation (2.74) may be simplified by defining a new variable \( y = 1 + \frac{\xi^2}{|\lambda| + \eta} \). Substituting \( \xi \) by \( (y - 1)^\frac{1}{2} (\lambda + \eta)^\frac{1}{2} \) in equation (2.74), we get

\[
I_{12} (\lambda, \eta, \alpha) = (\lambda + \eta)^{\frac{3}{2} - \frac{1}{\lambda}} \int_1^\infty \frac{(y - 1)\frac{3}{2} - \frac{1}{2}}{y^2} dy.
\]

Using the fact that \( \alpha \in [0, 1] \), it is easy to see that \( y^{-2} (y - 1)\frac{3}{2} - \frac{1}{2} \in L^1 (1, +\infty) \). Hence, the last integral in the above equation is well defined. Now, \( I_{13} (\lambda, \eta) \) can be
written as
\[
(\mathcal{I}_{13}(\lambda, \eta))^2 = \frac{2}{(\lambda + \eta)^2} \int_0^\infty \frac{1}{\left(1 + \left(\frac{\xi}{\sqrt{\lambda + \eta}}\right)^2\right)^2} 2d\xi = \frac{2}{(\lambda + \eta)^\frac{3}{2}} \int_0^\infty \frac{1}{(1 + s^2)^\frac{3}{2}} ds
\]
\[
= \frac{2}{(\lambda + \eta)^\frac{3}{2}} \times \frac{\pi}{4} = \frac{\pi}{2(\lambda + \eta)^\frac{3}{2}}.
\]
Therefore, \(\mathcal{I}_{13}(\lambda, \eta) = \frac{\sqrt{\pi}}{2} \frac{1}{(\lambda + \eta)^\frac{3}{4}}\). Finally, \(\mathcal{I}_{14}(\lambda, \eta)\) can be written as
\[
(\mathcal{I}_{14}(\lambda, \eta))^2 = \frac{2}{(\lambda + \eta)^2} \int_0^\infty \frac{\xi^2}{\left(1 + \left(\frac{\xi}{\sqrt{\lambda + \eta}}\right)^2\right)^4} 4d\xi = \frac{2}{(\lambda + \eta)^\frac{3}{2}} \int_0^\infty \frac{s^2}{(1 + s^2)^\frac{3}{2}} ds
\]
\[
= \frac{2}{(\lambda + \eta)^\frac{3}{2}} \times \frac{\pi}{32}.
\]
Then \(\mathcal{I}_{14}(\lambda, \eta) = \frac{\sqrt{\pi}}{4} \frac{1}{(\lambda + \eta)^\frac{3}{4}}\). The proof has been completed. \(\square\)

**Lemma 2.8.** Assume that \(\eta > 0\). Then, the solution \((u, v, y, z, \omega) \in D(A_1)\) of system (2.69)-(2.73) satisfies the following asymptotic behavior estimations
\[
\int_0^L \int_\mathbb{R} (|\xi|^2 + \eta) |\omega(x, \xi)|^2 d\xi dx = o \left(\lambda^{-1+\frac{\alpha}{2}}\right), \quad \int_{l_0}^{l_1} |u_x|^2 dx = o \left(\lambda^{-\frac{\alpha}{2}}\right)
\]
and
\[
\int_{l_0}^{l_1} |u_x|^2 dx = o \left(\lambda^{-2-\frac{\alpha}{2}}\right).
\]

**Proof.** For clarity, we divide the proof into several steps.

**Step 1.** Taking the inner product of \(F\) with \(U\) in \(\mathcal{H}_1\), then using (2.67) and the fact that \(U\) is uniformly bounded in \(\mathcal{H}_1\), we get
\[
\kappa(\alpha) \int_0^L \int_\mathbb{R} (\xi^2 + \eta) |\omega(x, \xi)|^2 d\xi dx = -\Re \left(\langle A_1 U, U \rangle_{\mathcal{H}_1}\right) = \Re \left(\langle (i\lambda I - A) U, U \rangle_{\mathcal{H}_1}\right) = o \left(\lambda^{-1+\frac{\alpha}{2}}\right).
\]

**Step 2.** Our aim here is to prove the second estimation in (2.75).

From (2.73), we get
\[
\sqrt{d(x)} |\xi|^{2\alpha - 1} |v_x| \leq (|\lambda| + \xi^2 + \eta) |\omega(x, \xi)| + |\lambda|^{-1+\frac{\alpha}{2}} |f_5(x, \xi)|.
\]
Multiplying the above inequality by \((|\lambda| + \xi^2 + \eta)^{-2} |\xi|\), integrate over \(\mathbb{R}\), we get
\[
\sqrt{d(x)} \mathcal{I}_{12}(\lambda, \eta, \alpha) |v_x| \leq \mathcal{I}_{13}(\lambda, \eta) \left(\int_\mathbb{R} |\xi| |\omega(x, \xi)|^2 d\xi\right)^\frac{1}{2}
\]
\[
+ |\lambda|^{-1+\frac{\alpha}{2}} \mathcal{I}_{14}(\lambda, \eta) \left(\int_\mathbb{R} |f_5(x, \xi)|^2 d\xi\right)^\frac{1}{2},
\]
where \(\mathcal{I}_{12}(\lambda, \eta, \alpha), \mathcal{I}_{13}(\lambda, \eta)\) and \(\mathcal{I}_{14}(\lambda, \eta)\) are defined in Lemma 2.7. Using Young’s inequality and the definition of the function \(d(x)\) in (2.76), we get
\[
\int_{l_0}^{l_1} |v_x|^2 dx \leq 2 \frac{\mathcal{I}_{13}^2}{\mathcal{I}_{12}} \frac{o(1)}{|\lambda|^{1-\frac{\alpha}{2}}} + 2 \frac{\mathcal{I}_{14}^2}{\mathcal{I}_{12}} \frac{o(1)}{|\lambda|^{2-\alpha}}.
\]
It follows from Lemma 2.7 that
\[ \int_{l_0}^{l_1} |v_x|^2 \, dx \leq \frac{1}{c_1((\lambda + \eta)^{a-1}} \frac{o(1)}{\lambda^{1-\frac{\eta}{2}}} + \frac{\sqrt{\pi}}{4 c_1 ((\lambda + \eta)^a \lambda^{2-\alpha}} o(1). \quad (2.77) \]
Since \( \alpha \in (0, 1) \), we have \( \min(\frac{a}{2}, 2) = \frac{a}{2} \), hence from the above equation, we get the second desired estimation in (2.75).

**Step 3.** From Equation (2.69) we have
\[ i\lambda u_x = v_x - \lambda^{-1+\frac{\eta}{2}}(f_1)'. \]
It follows that
\[ \| \lambda u_x \|_{L^2([l_0,l_1])} \leq \| v_x \|_{L^2([l_0,l_1])} + |\lambda|^{-1+\frac{\eta}{2}} \| (f_1)' \|_{L^2([l_0,l_1])} \leq \frac{o(1)}{\lambda^{1-\frac{\eta}{2}}} \]
Since \( \alpha \in (0, 1) \), we have \( \min(1 + \frac{a}{2}, 2 - \frac{a}{2}) = 1 + \frac{a}{2} \), hence from the above equation, we get
\[ \int_{l_0}^{l_1} |u_x|^2 \, dx = \frac{o(1)}{\lambda^{2+\frac{\eta}{2}}}. \]
The proof is thus completed.

**Lemma 2.9.** Let \( 0 < \alpha < 1 \) and \( \eta > 0 \). Then, the solution \((u,v,y,z,\omega) \in D(A_1)\) of system (2.69)-(2.73) satisfies the following asymptotic behavior
\[ \int_{l_0}^{l_1} |S_d|^2 \, dx = \frac{o(1)}{\lambda^{1-\frac{\eta}{2}}}. \quad (2.78) \]

**Proof.** Using the fact that \( |P + Q|^2 \leq 2P^2 + 2Q^2 \), we obtain
\[
\int_{l_0}^{l_1} |S_d|^2 \, dx = \int_{l_0}^{l_1} |au_x + \sqrt{d(x)} \kappa(\alpha) \int_{[x,\xi]}^{2\alpha-1} \omega(x,\xi) d\xi|^2 \, dx \\
\leq 2a^2 \int_{l_0}^{l_1} |u_x|^2 \, dx + 2d_0 \kappa(\alpha)^2 \int_{l_0}^{l_1} \left( \int_{[x,\xi]}^{2\alpha-1} \frac{\xi^{2\alpha-1}}{\sqrt{\xi^2 + \eta}} \omega(x,\xi) d\xi \right)^2 \, dx \\
\leq 2a^2 \int_{l_0}^{l_1} |u_x|^2 \, dx + c_2 \int_{l_0}^{l_1} \left( \xi^2 + \eta \right) \omega(x,\xi)^2 \, d\xi \, dx
\]
where \( c_2 = d_0 \kappa(\alpha)^2 I_{15}(\alpha,\eta) \) and \( I_{15}(\alpha,\eta) = \int_{[x,\xi]}^{2\alpha-1} \frac{d\xi}{\xi^2 + \eta} \). We have
\[ \frac{\xi^{2\alpha-1}}{\xi^2 + \eta} \sim \frac{2\alpha-1}{\eta} \quad \text{and} \quad \frac{\xi^{2\alpha-1}}{\xi^2 + \eta} \sim \frac{1}{\xi^{3-2\alpha}}. \]
Since \( 0 < \alpha < 1 \) and \( \eta > 0 \), then \( I_{15}(\alpha,\eta) \) is well defined. Using the first and the third estimations in (2.75), we get our desired result.

**Lemma 2.10.** Assume that \( \eta > 0 \). Let \( g \in C^4([l_0,l_1]) \) such that
\[ g(l_1) = -g(l_0) = 1, \quad \max_{x \in [l_0,l_1]} |g(x)| = m_g \quad \text{and} \quad \max_{x \in [l_0,l_1]} |g'(x)| = m_g'. \]
where \( m_g \) and \( m_g' \) are strictly positive constant numbers. Then, the solution \((u,v,y,z,\omega) \in D(A_1)\) of system (2.69)-(2.73) satisfies the following asymptotic behavior
\[ |v(l_1)|^2 + |v(l_0)|^2 \leq \left( \frac{\lambda^{1-\frac{\eta}{2}}}{2} + 2m_g' \right) \int_{l_0}^{l_1} |v|^2 \, dx + \frac{o(1)}{\lambda}, \quad (2.79) \]
and
\[ |S_d(l_1)|^2 + |S_d(l_0)|^2 \leq \frac{\lambda^{1+\frac{\eta}{2}}}{2} \int_{l_0}^{l_1} |v|^2 \, dx + o(1). \quad (2.80) \]
Proof. First we will prove Equation (2.79). From Equation (2.69), we have
\[ v_x = i\lambda u_x - \lambda^{-1+\frac{\alpha}{2}}(f_1)_x. \]  
(2.81)

Multiply Equation (2.81) by \(2g\dot{v}\) and integrate over \((l_0, l_1)\), we get
\[ |v(l_1)|^2 + |v(l_0)|^2 = \int_{l_0}^{l_1} g'|v|^2dx + 2i\lambda \int_{l_0}^{l_1} v_x g\dot{v}dx - \Re \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_{l_0}^{l_1} (f_1)_x g\dot{v}dx \right). \]  
(2.82)

Then,
\[ |v(l_1)|^2 + |v(l_0)|^2 \leq m_g' \int_{l_0}^{l_1} |v|^2dx + 2\lambda m_g \int_{l_0}^{l_1} |u_x||\dot{v}|dx + 2m_g\lambda^{-1+\frac{\alpha}{2}} \int_{l_0}^{l_1} |(f_1)_x||\dot{v}|dx. \]  
(2.83)

Using Young’s inequality we have
\[ 2\lambda m_g |u_x||\dot{v}| \leq \frac{\lambda^{-1+\frac{\alpha}{2}}}{2} |v|^2 + 2\lambda^{1+\frac{\alpha}{2}} m_g^2 |u_x|^2 \]  
and \[ 2m_g \lambda^{-1+\frac{\alpha}{2}} |(f_1)_x||\dot{v}| \leq m_g' |v|^2 + \frac{m_g^2}{m_g'} \lambda^{-2+\alpha} |(f_1)_x|^2. \]  
(2.84)

Using Equation (2.84), then Equation (2.83) becomes
\[ |v(l_1)|^2 + |v(l_0)|^2 \leq \left( \frac{\lambda^{-1+\frac{\alpha}{2}}}{2} + 2m_g' \right) \int_{l_0}^{l_1} |v|^2dx + 2\lambda^{1+\frac{\alpha}{2}} m_g^2 \int_{l_0}^{l_1} |u_x|^2dx \]  
+ \[ \frac{m_g^2}{m_g'} \lambda^{-2+\alpha} \int_{l_0}^{l_1} |(f_1)_x|^2. \]  
(2.85)

Using the third estimation in Equation (2.75) and the fact that \(||(f_1)_x||_{L^2((l_0, l_1))} = o(1)\), we obtain
\[ |v(l_1)|^2 + |v(l_0)|^2 \leq \left( \frac{\lambda^{-1+\frac{\alpha}{2}}}{2} + 2m_g' \right) \int_{l_0}^{l_1} |v|^2dx + \frac{o(1)}{\lambda} + \frac{o(1)}{\lambda^{2-\alpha}}. \]  
(2.86)

Since \(\alpha \in (0, 1)\), hence
\[ |v(l_1)|^2 + |v(l_0)|^2 \leq \left( \frac{\lambda^{-1+\frac{\alpha}{2}}}{2} + 2m_g' \right) \int_{l_0}^{l_1} |v|^2dx + \frac{o(1)}{\lambda}. \]  
(2.87)

Now, we will prove (2.80). For this aim, multiply Equation (2.70) by \(-2g\ddot{S}_d\) and integrate over \((l_0, l_1)\), we get
\[ |S_d(l_1)|^2 + |S_d(l_0)|^2 = \int_{l_0}^{l_1} g'|S_d|^2dx + \Re \left( 2i\lambda \int_{l_0}^{l_1} v g\ddot{S}_d dx \right) \]  
\[ - \Re \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_{l_0}^{l_1} f_2 g S_d dx \right). \]  
(2.88)

Then,
\[ |S_d(l_1)|^2 + |S_d(l_0)|^2 \leq m_g' \int_{l_0}^{l_1} |S_d|^2dx + 2\lambda m_g \int_{l_0}^{l_1} |v||S_d|dx + 2m_g\lambda^{-1+\frac{\alpha}{2}} \int_{l_0}^{l_1} |f_2||S_d|dx. \]  
(2.89)

Using Young’s inequality and Equation (2.78) we obtain
\[ 2\lambda m_g |v||\ddot{S}_d| \leq \frac{\lambda^{1+\frac{\alpha}{2}}}{2} |v|^2 + 2m_g^2 \lambda^{-\frac{\alpha}{2}} |S_d|^2 \leq \frac{\lambda^{1+\frac{\alpha}{2}}}{2} |v|^2 + o(1). \]  
(2.90)
Using Cauchy-Schwarz inequality, Equation (2.78) and the fact that \( \|f_2\|_{L^2(I_0,l_1)} = o(1) \), we obtain
\[
2m_g\lambda^{1+\frac{\eta}{2}} \int_{l_0}^{l_1} |f_2| |S_d| dx \leq \frac{1}{\lambda^{1-\frac{\eta}{2}}} \|f_2\|_{L^2(I_0, l_1)} \|S\|_{L^2(I_0, l_1)} = \frac{o(1)}{\lambda^{\frac{2-\eta}{2}}}.
\] (2.91)

Using Equations (2.78), (2.90) and (2.91) in (2.89), and using the fact that \( \alpha \in (0, 1) \) we get
\[
|S_d(l_1)|^2 + |S_d(l_0)|^2 \leq \frac{\lambda^{1+\frac{\eta}{2}}}{2} \int_{l_0}^{l_1} |v|^2 dx + o(1).
\] (2.92)

### Lemma 2.11

Let \( 0 < \alpha < 1 \) and \( \eta > 0 \). Then, the solution \((u, v, y, z, \omega) \in D(A)\) of system (2.69)-(2.73) satisfies the following asymptotic behavior
\[
\int_{l_0}^{l_1} |v|^2 dx = \frac{o(1)}{\lambda^{1+\frac{\eta}{2}}}.
\] (2.93)

**Proof.** Multiply Equation (2.70) by \(-i\lambda^{-1}\bar{v}\) and integrate over \((l_0, l_1)\), we get
\[
\int_{l_0}^{l_1} |v|^2 dx = \Re \left( i\lambda^{-1} \int_{l_0}^{l_1} S_d \bar{v}_x dx \right) - \left[ i\lambda^{-1} S_d \bar{v} \right]_{l_0}^{l_1} + \Re \left( i\lambda^{-2+\frac{\eta}{2}} \int_{l_0}^{l_1} f_2 \bar{v} dx \right).
\] (2.94)

Estimation of the term \( \Re \left( i\lambda^{-1} \int_{l_0}^{l_1} S_d \bar{v}_x dx \right) \). Using Cauchy-Schwarz inequality, the second estimation in (2.75) and the estimation in (2.78), we get
\[
\left| \Re \left( i\lambda^{-1} \int_{l_0}^{l_1} S_d \bar{v}_x dx \right) \right| \leq \frac{1}{\lambda} \left( \int_{l_0}^{l_1} |S_d|^2 dx \right)^{\frac{1}{2}} \left( \int_{l_0}^{l_1} |v_x|^2 dx \right)^{\frac{1}{2}} = \frac{o(1)}{\lambda^{\frac{1}{2}}}.
\] (2.95)

Estimation for the term \( \Re \left( i\lambda^{-2+\frac{\eta}{2}} \int_{l_0}^{l_1} f_2 \bar{v} dx \right) \). Using Cauchy-Schwarz inequality, \( v \) is uniformly bounded in \( L^2(I_0, l_1) \) and \( \|f_2\|_{L^2(I_0, l_1)} = o(1) \), we get
\[
\left| \Re \left( i\lambda^{-2+\frac{\eta}{2}} \int_{l_0}^{l_1} f_2 \bar{v} dx \right) \right| \leq \lambda^{-2+\frac{\eta}{2}} \left( \int_{l_0}^{l_1} |f_2|^2 dx \right)^{\frac{1}{2}} \left( \int_{l_0}^{l_1} |v|^2 dx \right)^{\frac{1}{2}} = \frac{o(1)}{\lambda^{\frac{1}{2}}}.
\] (2.96)

Inserting Equations (2.95) and (2.96) in (2.94), using the fact that \( \min(\frac{3}{2}, 2-\frac{\eta}{2}) = \frac{3}{2} \), and using Young’s inequality on the second term of (2.94) we get
\[
\int_{l_0}^{l_1} |v|^2 dx \leq \frac{\lambda^{1+\frac{\eta}{2}}}{2} \left[ |v(l_1)|^2 + |v(l_0)|^2 \right] + \frac{\lambda^{-1+\frac{\eta}{2}}}{2} \left[ |S_d(l_1)|^2 + |S_d(l_0)|^2 \right] + \frac{o(1)}{\lambda^{\frac{2-\eta}{2}}}.
\] (2.97)

Now, inserting (2.79) and (2.80) in (2.97), we get
\[
\int_{l_0}^{l_1} |v|^2 dx \leq \left( \frac{1}{2} + m_g \lambda^{1+\frac{\eta}{2}} \right) \int_{l_0}^{l_1} |v|^2 dx + \frac{o(1)}{\lambda^{1+\frac{\eta}{2}}} + \frac{o(1)}{\lambda^{\frac{2-\eta}{2}}}.
\] (2.98)

Since \( \alpha \in (0, 1) \) then \( \min(1+\frac{\eta}{2}, 2-\frac{\eta}{2}) = 1+\frac{\eta}{2} \), then Equation (2.98) becomes
\[
\left( \frac{1}{2} - m_g \lambda^{1+\frac{\eta}{2}} \right) \int_{l_0}^{l_1} |v|^2 dx \leq \frac{o(1)}{\lambda^{1+\frac{\eta}{2}}}.
\] (2.99)
From Equation (2.69) we have

Assume that Lemma 2.12.

Using the fact that $|\lambda| \to +\infty$, we can take $\lambda \geq \frac{4\varepsilon_0 + \frac{1}{2}}{m}$, and we get our desired result. The proof is thus complete.

**Lemma 2.12.** Assume that $\eta > 0$. Let $h \in C^1([0, L])$ and $\varphi \in C^2([-L, 0])$, then the solution $(u, v, \gamma, z, \omega) \in D(A_1)$ of system (2.69)-(2.73) satisfies the following estimation

$$
\int_0^L h' \left( |v|^2 + a^{-1}|S_d|^2 \right) dx + \int_0^L \varphi' \left( |\gamma|^2 + 3b|\gamma_x|^2 \right) dx + 2b \int_0^L y_{xx} \varphi'' y_x dx
$$

$$
b\varphi(-L)|y_{xx}(-L)|^2 + h(0)|v(0)|^2 - ah(L)|u_x(L)|^2 + ah(0)|u_x(0)|^2
$$

$$
+ \Re(2b\gamma_{xx}(0)\varphi(0)\bar{y}_x(0)) - \Re(\varphi(0)|z(0)|^2) = o(1).
$$

(2.100)

**Proof.** The proof is divided into several steps.

**Step 1.** Multiplying Equation (2.70) by $2a^{-1}h\bar{S}_d$ and integrating over $(0, L)$, we get

$$
\Re \left( 2a^{-1}i\lambda \int_0^L vh\bar{S}_d dx \right) + a^{-1} \int_0^L h' |S|^2 dx - a^{-1}h(L)|S_d(L)|^2
$$

$$
+ a^{-1}h(0)|S_d(0)|^2 = \Re \left( 2a^{-1}\lambda^{-1+\frac{\alpha}{2}} \int_0^L f_2 h\bar{S}_d dx \right).
$$

(2.101)

From Equation (2.69) we have

$$
i\lambda \bar{u}_x = -\bar{v}_x - \lambda^{-1+\frac{\alpha}{2}}(\bar{f}_1)_x.
$$

Then

$$
i\lambda a^{-1}\bar{S}_d = -\bar{v}_x - \lambda^{-1+\frac{\alpha}{2}}(\bar{f}_1)_x + i\lambda a^{-1}\sqrt{d(x)}\kappa(\alpha) \int_\mathbb{R} |\xi|^{\frac{2\alpha-1}{2}} \bar{\omega}(x, \xi) d\xi.
$$

(2.102)

Then the first term of (2.101), become

$$
\Re \left( 2a^{-1}i\lambda \int_0^L vh\bar{S}_d dx \right) = \int_0^L h'|v|^2 dx - h(L)|v(L)|^2
$$

$$
+ h(0)|v(0)|^2 - \Re \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_0^L vh(\bar{f}_1)_x dx \right)
$$

$$
+ \Re \left( 2i\lambda a^{-1}\kappa(\alpha) \int_0^L hv\sqrt{d(x)} \left( \int_\mathbb{R} |\xi|^{\frac{2\alpha-1}{2}} \bar{\omega}(x, \xi) d\xi \right) dx \right).
$$

(2.103)

Inserting Equation (2.103) into (2.101), and using the fact that $v$ and $S_d$ are uniformly bounded in $L^2(0, L)$ and $\|f_2\|_{L^2(0, L)} = o(1)$ and $\|f_1\|_{H^1_{L}(0, L)} = o(1)$, we obtain

$$
\int_0^L h' \left( |v|^2 + a^{-1}|S_d|^2 \right) dx + h(0)|v(0)|^2 - h(L)|u_x(L)|^2 + h(0)a|u_x(0)|^2
$$

$$
+ \Re \left( 2i\lambda a^{-1}\kappa(\alpha) \int_0^L hv\sqrt{d(x)} \left( \int_\mathbb{R} |\xi|^{\frac{2\alpha-1}{2}} \bar{\omega}(x, \xi) d\xi \right) dx \right) = o(1).
$$

(2.104)

Estimation of the term $\Re \left( 2i\lambda a^{-1}\kappa(\alpha) \int_0^L hv\sqrt{d(x)} \left( \int_\mathbb{R} |\xi|^{\frac{2\alpha-1}{2}} \bar{\omega}(x, \xi) d\xi \right) dx \right).$

Using the definition of $d(x)$ and Cauchy-Schwarz inequality, the fact that $0 < \alpha < 1$
and \( \eta > 0 \), and using the first estimation in (2.75) and Equation (2.93), we obtain
\[
\left| \Re \left( 2i\lambda a^{-1}\kappa(\alpha) \int_0^L h v \sqrt{\lambda(x)} \left( \int_\mathbb{R} |\xi|^{2\frac{\eta}{2}} \bar{\omega}(x, \xi) d\xi \right) dx \right) \right| = o(1) \tag{2.105}
\]
Inserting Equation (2.105) in Equation (2.104), and using the fact that \( \alpha \in (0, 1) \) we obtain
\[
\int_0^L h' \left( |v|^2 + a^{-1}|S_x|^2 \right) dx + h(0)|v(0)|^2 - ah(L)|u_x(L)|^2 + ah(0)|u_x(0)|^2 = o(1) \tag{2.106}
\]
**Step 2.** Multiplying Equation (2.73) by \( 2\varphi_y \) and integrating over \((-L, 0)\), we get
\[
\Re \left( 2i\lambda \int_{-L}^0 z \varphi_y dx \right) - 2b \int_{-L}^0 y_{xxx} \varphi_y dx - 2b \int_{-L}^0 y_{xxx} \phi_{yy} dx + \Re \left( 2b y_{xxx} \phi_{yy} \right)_{-L}^0 = \Re \left( 2 \lambda^{-1+\frac{\eta}{2}} \int_{-L}^0 f_4 \phi_y dx \right) \tag{2.107}
\]
Integrating by parts the second and third terms of the above equation we get
\[
\Re \left( 2i\lambda \int_{-L}^0 z \varphi_y dx \right) + 2b \int_{-L}^0 \varphi'|y_{xx}|^2 dx + 2b \int_{-L}^0 y_{xxx} \varphi'' y_x dx - \left[ 2b y_{xxx} \varphi'' y_x \right]_{-L}^0 + \Re \left( 2b y_{xxx}(0) \varphi(0) y_x(0) \right)
= \Re \left( 2 \lambda^{-1+\frac{\eta}{2}} \int_{-L}^0 f_4 \varphi_y dx \right) \tag{2.108}
\]
From Equation (2.71), we have
\[
i\lambda \varphi_x = -\bar{z}_x - \lambda^{-1+\frac{\eta}{2}} (\bar{f}_3)_x \tag{2.109}
\]
By inserting Equation (2.109) into the first term of (2.108), we get
\[
\Re \left( 2i\lambda \int_{-L}^0 \varphi_y dx \right) = \Re \left( -2 \int_{-L}^0 \varphi z \bar{z} dx - 2 \lambda^{-1+\frac{\eta}{2}} \int_{-L}^0 (\bar{f}_3)_x \varphi z dx \right)
= \int_{-L}^0 \varphi|z|^2 dx - \varphi(z) \left( \varphi(0) |z(0)|^2 \right) - \Re \left( 2 \lambda^{-1+\frac{\eta}{2}} \int_{-L}^0 (\bar{f}_3)_x \varphi z dx \right) \tag{2.110}
\]
Inserting Equation (2.110) into (2.108), and using the boundary conditions we get
\[
\int_{-L}^0 \varphi' (|z|^2 + 3b|y_{xx}|^2) dx + 2b \int_{-L}^0 \varphi'' y_{xx} \bar{y}_x dx + b\varphi(-L)|y_{xx}(-L)|^2
+ \Re \left( 2 b \varphi(0) \bar{y}_{xx}(0) \bar{y}_x(0) - \varphi(0)|z(0)|^2 \right)
= \Re \left( 2 \lambda^{-1+\frac{\eta}{2}} \int_{-L}^0 \varphi_4 \bar{y}_x dx \right) + \Re \left( 2 \lambda^{-1+\frac{\eta}{2}} \int_{-L}^0 \varphi(\bar{f}_3)_x z dx \right) \tag{2.111}
\]
Estimation of the term \( \Re \left( 2 \lambda^{-1+\frac{\eta}{2}} \int_{-L}^0 \varphi_4 \bar{y}_x dx \right) \). Using Poincare inequality, Cauchy-Schwarz inequality, the definition of \( \varphi \), and \( y_{xx} \) is uniformly bounded in
Now, summing Equations (2.106) and (2.114), we get our desired result.

\[ \Re \left( 2\lambda^{-1+\frac{2}{p}} \int_{-L}^{0} \varphi f_{4} \tilde{y}_{x} \, dx \right) \leq \lambda^{-1+\frac{2}{p}} \| y_{x} \|_{L^{2}(0, L)} \| f_{4} \|_{L^{2}(0, L)} \leq \lambda^{-1+\frac{2}{p}} c_{p} \| y_{xx} \|_{L^{2}(0, L)} \| f_{4} \|_{L^{2}(0, L)} = \frac{o(1)}{\lambda^{1-\frac{2}{p}}}. \] (2.112)

Estimation of the term \( \Re \left( 2\lambda^{-1+\frac{2}{p}} \int_{-L}^{0} \varphi (\tilde{f}_{3})_{x} \, z \, dx \right) \). Using Cauchy-Schwarz inequality, the definition of \( \varphi \), and \( z \) is bounded in \( L^{2}(-L, 0) \), and that \( \| (f_{3})_{x} \|_{L^{2}(-L, 0)} = o(1) \) we get

\[ \left| \Re \left( 2\lambda^{-1+\frac{2}{p}} \int_{-L}^{0} \varphi (\tilde{f}_{3})_{x} \, z \, dx \right) \right| \leq \lambda^{-1+\frac{2}{p}} \left( \int_{0}^{L} |z|^{2} \, dx \right)^{\frac{1}{2}} \left( \int_{0}^{L} |(f_{3})_{x}|^{2} \, dx \right)^{\frac{1}{2}} = \frac{o(1)}{\lambda^{1-\frac{2}{p}}}. \] (2.113)

Inserting (2.112) and (2.113) into (2.111), we get

\[ \int_{-L}^{0} \varphi' (|z|^{2} + 3b|y_{xx}|^{2}) \, dx + 2b \int_{-L}^{0} \varphi'' y_{xx} \dot{y}_{x} \, dx + b \varphi(-L)|y_{xx}(-L)|^{2} + \Re \left( 2b \varphi(0)y_{xxx}(0)\tilde{y}_{x}(0) - \varphi(0)|z(0)|^{2} \right) = \frac{o(1)}{\lambda^{1-\frac{2}{p}}}. \] (2.114)

Now, summing Equations (2.106) and (2.114), we get our desired result. \( \square \)

**Lemma 2.13.** Assume that \( \eta > 0 \). The solution \((u, v, y, z, \omega) \in D(A_{1}) \) of system (2.69)-(2.73) satisfies the following estimation

\[ \| U \|_{H_{1}} = o(1). \] (2.115)

**Proof.** The proof of this Lemma is divided into several steps.

**Step 1.** In this step we will prove that \( \| v \|_{L^{2}(0, L)} = o(1) \) and \( \| u_{x} \|_{L^{2}(0, L)} = o(1) \).

Taking \( h(x) = x \theta_{1}(x) + (x - L) \theta_{2}(x) \) and \( \varphi(x) = 0 \) in Equation (2.100), where \( \theta_{1}, \theta_{2} \in C^{1}([0, L]) \) are defined as follows

\[ \theta_{1}(x) = \begin{cases} 1 & \text{if } x \in [0, l_{0}], \\ 0 \leq \theta_{1} \leq 1 & \text{if } x \in [l_{1}, L], \\ \text{elsewhere}, \end{cases} \] (2.116)

and

\[ \theta_{2}(x) = \begin{cases} 1 & \text{if } x \in [l_{1}, L], \\ 0 \leq \theta_{2} \leq 1 & \text{if } x \in [0, l_{0}], \\ \text{elsewhere}. \end{cases} \] (2.117)

we get

\[ \int_{0}^{L} (|x\theta_{1} + (x - L)\theta_{2}| + |S_{d}|^{2}) \, dx + \int_{0}^{L} (x\theta'_{1} + (x - L)\theta'_{2}) \left( |v|^{2} + a^{-1} |S_{d}|^{2} \right) \, dx = o(1). \] (2.118)

Using Equations (2.78) and (2.93) and the definition of \( \theta_{1} \) and \( \theta_{2} \), we get

\[ \int_{0}^{L} (\theta_{1} + \theta_{2}) \left( |v|^{2} + a^{-1} |S_{d}|^{2} \right) \, dx = o(1). \] (2.119)

Hence, we deduce that

\[ \int_{0}^{l_{0}} |v|^{2} \, dx = o(1) \quad \text{and} \quad a \int_{0}^{l_{0}} |u_{x}|^{2} \, dx = o(1). \] (2.120)
and
\[ \int_0^L |v|^2 \, dx = o(1) \quad \text{and} \quad a \int_0^L |u_x|^2 \, dx = o(1). \] (2.121)

Using (2.120), (2.121), (2.78) and (2.93), we get the desired result of Step 1.

**Step 2.** Taking \( h(x) = x - L \) and \( \varphi(x) = 0 \) in Equation (2.100), we get
\[ \int_0^L \left( |v|^2 + a^{-1}|S|^2 \right) \, dx - L|v(0)|^2 - aL|u_x(0)|^2 = o(1). \] (2.122)

By using Step 1, we get
\[ |v(0)|^2 = o(1) \quad \text{and} \quad |u_x(0)|^2 = o(1). \] (2.123)

**Step 3.** The aim of this step is to prove that \( \|z\|_{L^2(-L,0)} = o(1) \) and \( \|y_{xx}\|_{L^2(-L,0)} = o(1) \).

Taking \( h(x) = 0 \) and \( \varphi(x) = x + L \) in Equation (2.100), we get
\[ \int_{-L}^0 \left( |z|^2 + 3b|y_{xx}|^2 \right) \, dx + \Re(2Ly_{xx}(0)\bar{y}_x(0)) - L|z(0)|^2 = o(1). \] (2.124)

Using (2.123) and the transmission conditions we have
\[ b|y_{xxx}(0)| = a|u_x(0)| = o(1) \quad \text{and} \quad |z(0)|^2 = |v(0)|^2 = o(1). \] (2.125)

Inserting Equation (2.125) in Equation (2.124) and using the fact that \( |y_{x}(0)| \leq \sqrt{L} \|y_{xx}\|_{L^2(-L,0)} = O(1) \) we get
\[ \int_{-L}^0 |z|^2 \, dx = o(1) \quad \text{and} \quad b \int_{-L}^0 |y_{xx}|^2 \, dx = o(1). \] (2.126)

Finally, using Equations (2.11), (2.75), (2.126), and Step 1., we get that \( \|U\|_{\mathcal{H}_1} = o(1) \). \( \Box \)

**Proof of Theorem 2.7.** From Lemma 2.13 we get that \( \|U\|_{\mathcal{H}_1} = o(1) \), which contradicts (2.67). Consequently, condition (H2) holds. This implies, from Theorem 4.4, the energy decay estimation (2.66). The proof is thus complete.

3. **W-(EBB)\(_{FKV}\) model.** This section is devoted to study the stability of the model (W-(EBB)\(_{FKV}\)), where we consider the Euler-Bernoulli beam and wave equations coupled through boundary connection. We take the fractional Kelvin-Voigt damping to be a localized internal damping acting on the Euler-Bernoulli beam only.

3.1. **Well-posedness and strong stability.** In this section, we give the strong stability results of the system (W-(EBB)\(_{FKV}\)). First, using a semigroup approach, we establish well-posedness result for the system (W-(EBB)\(_{FKV}\)).

In Theorem 2.1, taking the input \( V(x,t) = \sqrt{d(x)} y_{xxt}(x,t) \), then using (1.4), we get the output \( O \) is given by
\[
O(x,t) = \sqrt{d(x)} I^{1-\alpha,\eta} y_{xxt}(x,t) = \frac{\sqrt{d(x)}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \partial_s y_{xx}(x,s) \, ds = \sqrt{d(x)} \partial_x^{\alpha,\eta} y_{xx}(x,t).
\]
Therefore, by taking the input $V(x,t) = \sqrt{\alpha(x,y)}y_{ext}(x,t)$ in Theorem 2.1 and using the above equation, we get

$$\partial_t \omega(x,\xi,t) + (\xi^2 + \eta)\omega(x,\xi,t) - \sqrt{\alpha(x,y)}y_{ext}(x,t)\xi^{2}\partial_x \omega = 0, \quad (x,\xi,t) \in (0,L) \times \mathbb{R} \times \mathbb{R}^*_+, \quad (x,\xi,t) \in (0,L) \times \mathbb{R} \times \mathbb{R}^*_+,$$

From system (3.1), we deduce that system (W-(EBB)\text{FKV}) can be recast into the following augmented model

$$\begin{align*}
\begin{cases}
\alpha & \frac{\partial \omega}{\partial t} + \alpha \frac{\partial y}{\partial x} = 0, & (x,t) \in (-L,0) \times \mathbb{R}^*_+,
\frac{\partial y}{\partial t} + b \frac{\partial y}{\partial x} = 0, & (x,t) \in (0,L) \times \mathbb{R}_+,
\end{cases}
\end{align*}$$

with the following transmission and boundary conditions

$$\begin{align*}
\begin{cases}
u(t,0) = y(t,0) = y_x(t,0) = 0, & t (0,\infty),
\partial_u (0,0) + b \partial y_{x}(0,0) = 0, & t (0,\infty),
u(t,0) = y(t,0), & t (0,\infty),
\end{cases}
\end{align*}$$

and with the following initial conditions

$$\begin{align*}
u(x,0) = u_0(x), & x (-L,0)
y(x,0) = y_0(x), & x \in (0,L), \xi \in \mathbb{R}.
\end{align*}$$

The energy of the system (3.2)-(3.4) is given by

$$E_3(t) = \frac{1}{2} \int_{-L}^{0} (|\nu_x|^2 + a|\nu_x|^2) \, dx + \frac{1}{2} \int_{0}^{L} (|\nu_x|^2 + b|y_{x}|^2) \, dx$$

$$+ \frac{\kappa(\alpha)}{2} \int_{0}^{L} \int_{\mathbb{R}} |\omega(x,\xi,t)|^2 \, d\xi \, dx.$$ 

By similar computation to Lemma 2.2, it is easy to see that the energy $E_3(t)$ satisfies the following estimation

$$\frac{d}{dt} E_3(t) = -\kappa(\alpha) \int_{0}^{L} \int_{\mathbb{R}} |\omega(x,\xi,t)|^2 \, d\xi \, dx.$$ 

Since $\alpha \in (0,1)$, then $\kappa(\alpha) > 0$, and therefore $\frac{d}{dt} E_3(t) \leq 0$. Thus, system (3.2)-(3.4) is dissipative in the sense that its energy is a non-increasing function with respect to time variable $t$. Now, we define the following Hilbert energy space $\mathcal{H}_3$ by

$$\mathcal{H}_3 = \left\{ \begin{array}{l}
u, v, y, z, \omega \in H_1^0(-L,0) \times L^2(-L,0) \times H^2_0(0,L) \times L^2(0,L) \times W, \\
such that \quad u(0) = y(0)
\end{array} \right\},$$

where $W = L^2 ((0,L) \times \mathbb{R})$ and

$$\begin{align*}
\begin{cases}
H_1^0(-L,0) = \{ u \in H^1(-L,0); u(-L) = 0, \}
H^2_0(0,L) = \{ y \in H^2(0,L); y(L) = y_x(L) = 0. \}
\end{cases}
\end{align*}$$

(3.6)
The energy space $\mathcal{H}_3$ is equipped with the inner product defined by

$$
\langle U, U \rangle_{\mathcal{H}_3} = \int_{-L}^{0} v\bar{u} \, dx + a \int_{-L}^{0} u_x(\bar{u}_1) \, dx + \int_{0}^{L} z\bar{z}_1 \, dx + b \int_{0}^{L} y_{xx}(\bar{y}_1) \, dx
$$

$$
+ \kappa(\alpha) \int_{0}^{L} \omega(x, \xi) \bar{\omega}(x, \xi) \, d\xi,
$$

for all $U = (u, v, y, z, \omega)$ and $U_1 = (u_1, v_1, y_1, z_1, \omega_1)$ in $\mathcal{H}_3$. We use $\|U\|_{\mathcal{H}_3}$ to denote the corresponding norm. We define the unbounded linear operator $A_3 : D(A_3) \subset \mathcal{H}_3 \to \mathcal{H}_3$ by

$$
D(A_3) = \left\{ U = (u, v, y, z, \omega) \in \mathcal{H}_3; \ (v, z) \in H_0^1(-L, 0) \times H_0^2(0, L), \ u \in H^2(-L, 0), \ b y_{xx} + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2n-4}{2}} \omega(x, \xi) \, d\xi \in L^2(0, L), \right. \\

\left. - (|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2n-4}{2}}, \ |\xi| \omega(x, \xi) \in W; \right. \\

au_t(0) + by_{xx}(0) = 0, \ y_{xx}(0) = 0, \ \text{and} \ v(0) = z(0)
$$

and for all $U = (u, v, y, z, \omega) \in D(A_3)$,

$$
A_3(u, v, y, z, \omega)^T = \begin{pmatrix}
v \\
a u_{xx} \\
z \\
- \left( b y_{xx} + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2n-4}{2}} \omega(x, \xi) \, d\xi \right) \\
- (|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2n-4}{2}}
\end{pmatrix}.
$$

If $U = (u_t, y_t, y_t, \omega)$ is a regular solution of system (3.2)-(3.4), then the system can be rewritten as evolution equation on the Hilbert space $\mathcal{H}_3$ given by

$$
U_t = A_3 U, \quad U(0) = U_0,
$$

(3.7)

where $U_0 = (u_0, u_1, y_0, y_1, 0)$.

Similar to Proposition 2.4, the operator $A_2$ is m-dissipative on $\mathcal{H}_3$, consequently it generates a $C_0$-semigroup of contractions $(e^{tA_3})_{t \geq 0}$ following Lummer-Phillips theorem (see in [47] and [53]). Then the solution of the evolution Equation (3.7) admits the following representation

$$
U(t) = e^{tA_3} U_0, \quad t \geq 0,
$$

which leads to the well-posedness of (3.7). Hence, we have the following result.

**Theorem 3.1.** Let $U_0 \in \mathcal{H}_3$, then problem (3.7) admits a unique weak solution $U$ satisfies

$$
U(t) \in C^0 \left( \mathbb{R}^+, \mathcal{H}_3 \right).
$$

Moreover, if $U_0 \in D(A_3)$, then problem (3.7) admits a unique strong solution $U$ satisfies

$$
U(t) \in C^1 \left( \mathbb{R}^+, \mathcal{H}_3 \right) \cap C^0 \left( \mathbb{R}^+, D(A_3) \right).
$$

**Theorem 3.2.** Assume that $\eta \geq 0$, then the $C_0$-semigroup of contractions $e^{tA_3}$ is strongly stable on $\mathcal{H}_3$ in the sense that 

$$
\lim_{t \to +\infty} \|e^{tA_3} U_0\|_{\mathcal{H}_3} = 0.
$$
Proof of Theorem 3.2. The proof of this theorem follows by proceeding with similar arguments as in Subsection 2.1, and using the Arendt Batty Theorem (see Theorem 4.2 in Appendix).

3.2. Polynomial stability in the case \( \eta > 0 \). The aim of this part is to study the polynomial stability of system (3.2)-(3.4) in the case \( \eta > 0 \). As the condition \( i\mathbb{R} \subset \rho(A_3) \) is already checked in the subsection 3.1, it remains to prove that condition (4.1) holds (see Theorem 4.4 in Appendix). This is established by using specific multipliers, some interpolation inequalities and by solving differential equations of order 4. Our main result in this part is the following theorem.

**Theorem 3.3.** Assume that \( \eta > 0 \). The \( C_0 \)-semigroup \( (e^{tA_3})_{t \geq 0} \) is polynomially stable; i.e. there exists constant \( C_3 > 0 \) such that for every \( U_0 \in D(A_3) \), we have

\[
E_3(t) \leq \frac{C_2}{t^{\frac{3}{2}}\alpha} \| U_0 \|_{D(A_3)}^2, \quad t > 0, \forall U_0 \in D(A_3).
\]

According to Theorem 4.4, by taking \( \ell = 3 - \alpha \), the polynomial energy decay (3.8) holds if the following conditions

\[
i\mathbb{R} \subset \rho(A_3), \quad (R_1)
\]

and

\[
\limsup_{\lambda \in \mathbb{R}, |\lambda| \to \infty} \frac{1}{|\lambda|^{\frac{3}{2}-\alpha}} \left\| (i\lambda I - A_3)^{-1} \right\|_{\mathcal{L} (\mathcal{H}_3)} < \infty, \quad (R_2)
\]

are satisfied. Since condition (R_1) is already proved (see Subsection 3.1), we still need to prove condition (R_2). For this purpose we will use an argument of contradiction. Suppose that (R_2) is false, then there exists \( \{ (\lambda_n, U_n := (u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))^\top) \} \subset \mathbb{R}^* \times D(A_3) \) with

\[
|\lambda_n| \to +\infty \quad \text{and} \quad \| U_n \|_{\mathcal{H}_3} = \| (u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi)) \|_{\mathcal{H}_3} = 1,
\]

such that

\[
(\lambda_n^{\frac{3}{2}-\alpha}) (i\lambda_n I - A_3) U_n = F_n := (f_1, f_2, f_3, f_4, f_5, \omega_n(\cdot, \xi))^\top \to 0 \quad \text{in} \quad \mathcal{H}_3.
\]

For simplicity, we drop the index \( n \). Equivalently, from (3.10), we have

\[
i\lambda u - v = \frac{f_1}{\lambda^{\frac{3}{2}-\alpha}} \quad \text{in} \quad H^1_L(-L, 0), \quad (3.11)
\]

\[
i\lambda v - au_{xx} = \frac{f_2}{\lambda^{\frac{3}{2}-\alpha}} \quad \text{in} \quad L^2(-L, 0), \quad (3.12)
\]

\[
i\lambda y - z = \frac{f_3}{\lambda^{\frac{3}{2}-\alpha}} \quad \text{in} \quad H^2_H(0, L), \quad (3.13)
\]

\[
i\lambda z + S_{xx} = \frac{f_4}{\lambda^{\frac{3}{2}-\alpha}} \quad \text{in} \quad L^2(0, L), \quad (3.14)
\]

\[
(i\lambda + \xi^2 + \eta)\omega(x, \xi) - \sqrt{d(x)}z_{xx}\xi^{\frac{\alpha-1}{2}} = \frac{f_5(x, \xi)}{\lambda^{\frac{3}{2}-\alpha}} \quad \text{in} \quad W, \quad (3.15)
\]

where \( S = by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{\alpha-1}{2}} \omega(x, \xi) d\xi \). Here we will check the condition (R_2) by finding a contradiction with (3.9) by showing \( \| U \|_{\mathcal{H}_3} = o(1) \). For clarity, we divide the proof into several Lemmas.
Lemma 3.1. Assume that $\eta > 0$. Then, the solution $(u, v, y, z, \omega) \in D(A_3)$ of system (3.11)-(3.15) satisfies the following asymptotic behavior estimations

$$
\begin{align*}
\int_0^L \int_{\mathbb{R}} \left( |\xi|^2 + \eta \right) |\omega(x, \xi)|^2 d\xi dx &= o\left( \frac{1}{\lambda^{1-\alpha}} \right), \\
\int_0^{l_1} |y_{xx}|^2 dx &= o\left( \frac{1}{\lambda^4} \right) \quad \text{and} \quad \int_0^{l_1} |S|^2 dx = o(1).
\end{align*}
$$

(3.16)

Proof. For the clarity of the proof, we divide the proof into several steps.

Step 1. Taking the inner product of $F$ with $U$ in $\mathcal{H}_3$, then using (3.9) and the fact that $U$ is uniformly bounded in $\mathcal{H}_3$, we get

$$
\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi)|^2 d\xi dx = -\Re\left( \langle A_3 U, U \rangle_{\mathcal{H}_3} \right) = \Re\left( \langle (i\lambda I - A_3) U, U \rangle_{\mathcal{H}_3} \right) = o \left( \lambda^{-3+\alpha} \right).
$$

Step 2. Our aim here is to prove the second estimation in (3.16).

From (3.15), we get

$$
\sqrt{d(x)} |\xi^{2+\alpha-1} z_{xx}| \leq (|\lambda| + \xi^2 + \eta) |\omega(x, \xi)| + |\lambda|^{-3+\alpha} |f_5(x, \xi)|.
$$

Multiplying the above inequality by $(|\lambda| + \xi^2 + \eta)^{-2} |\xi|$, integrating over $\mathbb{R}$ and proceeding in a similar way as in Lemma 2.8 (Section 2.2), we get the second desired estimation in (3.16).

Step 3. From Equation (3.13) we have that

$$
\|\lambda u_{xx}\|_{L^2(l_0, l_1)} \leq \|z_{xx}\|_{L^2(l_0, l_1)} + |\lambda|^{-3+\alpha} \|f_5\|_{L^2(l_0, l_1)}.
$$

Using Step 2, the fact that $\|f_5\|_{H^3(0, L)} = o(1)$, and that $\alpha \in (0, 1)$, we get the third estimation in (3.16).

Step 4. Using the fact that $|P + Q|^2 \leq 2P^2 + 2Q^2$, we obtain

$$
\int_0^{l_1} |S|^2 dx = \int_0^{l_1} \left| b_{u_{xx}} + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{2+\alpha-1} \omega(x, \xi) d\xi \right|^2 dx \\
\leq 2b_1 \int_0^{l_1} |y_{xx}|^2 dx + 2d_0 \kappa(\alpha)^2 \int_0^{l_1} \left( \int_{\mathbb{R}} |\xi|^{2+\alpha-1} \sqrt{\xi^2 + \eta} \omega(x, \xi) d\xi \right)^2 dx \\
\leq 2b_2 \int_0^{l_1} |y_{xx}|^2 dx + c_2 \int_0^{l_1} (\xi^2 + \eta) |\omega(x, \xi)|^2 d\xi dx
$$

where $c_2 = -d_0 \kappa(\alpha)^2 \Gamma_{15}(\alpha, \eta)$ is defined in Lemma 2.9. Thus, we get the last estimation in (3.16). Hence, the proof is complete.

□

Lemma 3.2. Assume that $\eta > 0$. The solution $(u, v, y, z, \omega) \in D(A_3)$ of system (3.11)-(3.15) satisfies the following asymptotic behavior

$$
\|z\|_{H^2(l_0, l_1)} = o\left( \frac{1}{\lambda} \right), \quad \|y\|_{H^2(l_0, l_1)} = o\left( \frac{1}{\lambda^2} \right), \quad \text{and} \quad \|S_x\|_{L^2(l_0, l_1)} = o\left( \frac{1}{\lambda^{3+\alpha}} \right).
$$

(3.17)

Proof. The proof of this Lemma will be divided into two steps.

Step 1. Let $0 < \varepsilon < \frac{l_1 - l_0}{2}$. We define $h \in C^\infty([0, L])$, $0 \leq h \leq 1$ on $[0, L]$, $h = 1$ on $(l_0 + \varepsilon, l_1 - \varepsilon)$, and $h = 0$ on $(0, l_0) \cup (l_1, L)$. Also, we define $\max_{\varepsilon \in [0, L]} |h'(x)| = m'_h$
and \( \max_{x \in [0, L]} |h''(x)| = m''_h \), where \( m'_h \) and \( m''_h \) are strictly positive constant numbers.

Multiply equation (3.14) by \(-i\lambda^{-1}h\bar{z}\) and integrate over \((l_0, l_1)\), we get

\[
\int_{l_0}^{l_1} h|z|^2 dx = i\lambda^{-1} \int_{l_0}^{l_1} S(h''\bar{z} + 2h'z_x + h_{xxx})dx - i\lambda^{-4+\alpha} \int_{l_0}^{l_1} h f \bar{z} dx. \tag{3.18}
\]

Using Nirenberg inequality Theorem (see [52]), Equations (3.9) and (3.16), we have

\[
\|z_x\|_{L^2(l_0, l_1)} \leq \|z_{xx}\|_{L^2(l_0, l_1)} \|z\|^{1/2}_{L^2(l_0, l_1)} + \|z\|_{L^2(l_0, l_1)} \leq O(1). \tag{3.19}
\]

Estimation of the term \(i\lambda^{-1} \int_{l_0}^{l_1} h''S\bar{z}\). Using Cauchy Schwarz inequality, last estimation in (3.16) and the fact that \(z\) is uniformly bounded in \(L^2(0, L)\) and that \(0 < \alpha < 1\), we get

\[
\left| i\lambda^{-1} \int_{l_0}^{l_1} h''S\bar{z} dx \right| \leq \frac{m''_h}{\lambda} \int_{l_0}^{l_1} |S| |z| dx \leq \frac{o(1)}{\lambda^{2-\alpha}}. \tag{3.20}
\]

Estimation of the term \(i\lambda^{-1} \int_{l_0}^{l_1} hSz_{xx} dx\). Using Cauchy Schwarz inequality, the second and the last estimations in (3.16), we get

\[
\left| i\lambda^{-1} \int_{l_0}^{l_1} hS\bar{z}_{xx} dx \right| \leq \frac{m'_h}{\lambda} \left( \int_{l_0}^{l_1} |S|^2 dx \right)^{1/2} \left( \int_{l_0}^{l_1} |z_{xx}|^2 dx \right)^{1/2} = \frac{o(1)}{\lambda^{2-\alpha}}. \tag{3.21}
\]

Estimation of the term \(i\lambda^{-1} \int_{l_0}^{l_1} h'Sz_x dx\). Cauchy Schwarz inequality, the last estimation in (3.16), Equation (3.19) and the fact that \(0 < \alpha < 1\), we get

\[
\left| i\lambda^{-1} \int_{l_0}^{l_1} h'S\bar{z}_x dx \right| \leq \frac{m'h}{\lambda} \int_{l_0}^{l_1} |S| |z_x| dx \leq \frac{o(1)}{\lambda^{2-\alpha}}. \tag{3.22}
\]

Estimation of the term \(i\lambda^{-4+\alpha} \int_{l_0}^{l_1} h f \bar{z} dx\). Using Cauchy-Schwarz inequality, the fact that \(\|f\|_{L^2(0, L)} = O(1)\) and the fact that \(z\) is uniformly bounded in \(L^2(0, L)\), we get

\[
\left| i\lambda^{-4+\alpha} \int_{l_0}^{l_1} h f \bar{z} dx \right| = \frac{o(1)}{\lambda^{4-\alpha}}. \tag{3.23}
\]

Thus, using Equations (3.20)-(3.23) in (3.18), we get

\[
\int_{l_0}^{l_1} h|z|^2 dx = \frac{o(1)}{\lambda^{2-\alpha}} \quad \text{and} \quad \int_{l_0+\epsilon}^{l_1-\epsilon} |z|^2 dx = \frac{o(1)}{\lambda^{2-\alpha}}. \tag{3.24}
\]

**Step 2.** Applying the interpolation theorem involving compact subdomain ([2], Theorem 4.23), we obtain

\[
\|z_x\|_{L^2(l_0, l_1)} \leq \|z_{xx}\|_{L^2(l_0, l_1)} + \|z\|_{L^2(l_0+\epsilon, l_1-\epsilon)}.
\]

Then, by using (3.24) and that \(0 < \alpha < 1\), we get

\[
\|z_x\|_{L^2(l_0, l_1)} = \frac{o(1)}{\lambda}.
\]
Also, using Theorem yields that
\[ \|z\|_{L^2(t_0, t_1)} = \frac{o(1)}{\lambda}. \]  
(3.26)
Thus, using the first estimation in (3.16), (3.25) and (3.26), we obtain the first estimation in Lemma 3.2.

Now, from Equation (3.13) we have \( i\lambda y - z = \lambda^{-3+\alpha}f_3 \), then
\[ \|y\|_{H^2(t_0, t_1)} \leq \frac{1}{\lambda} \|z\|_{H^2(t_0, t_1)} + \frac{1}{\lambda^{3-\alpha}} \|f_3\|_{H^2(t_0, t_1)}, \]
using the fact that \( \alpha \in (0, 1) \), \( \|f_3\|_{H^2(t_0, t_1)} = o(1) \), and the first estimation in Lemma 3.2, we obtain the second estimation of Lemma 3.2. Using Nirenberg inequality Theorem (see [52]), Lemma 3.16 and (3.26), we get
\[ \|S_x\|_{L^2(t_0, t_1)} \leq \|S_{xx}\|_{L^2(t_0, t_1)} \|S\|_{L^2(t_0, t_1)} + \|S\|_{L^2(t_0, t_1)} = \frac{o(1)}{\lambda^{\frac{3}{2}}}. \]  
(3.27)
The proof has been completed.

**Remark 3.4.** It is easy to see the existence of \( h(x) \). For example, we can take
\[ h(x) = \cos \left( \frac{\pi(l_1 - x)}{l_1 - l_0} \right) \]
and we get that \( h(l_1) = -h(l_0) = 1 \), \( h \in C^\infty([0, L]) \), \( |g(x)| \leq 1 \) and \( |g'(x)| \leq \frac{\pi}{l_1 - l_0} \).

**Lemma 3.3.** Assume that \( \eta > 0 \). The solution \( (u, v, y, z, \omega) \in D(A_3) \) of system (3.11)-(3.15) satisfies the following asymptotic behavior
\[ |y(l_0)| = \frac{o(1)}{\lambda^2}, \|y_x(l_0)\| = \frac{o(1)}{\lambda^2}, |y(l_1)| = \frac{o(1)}{\lambda^2}, \text{ and } |y_x(l_1)| = \frac{o(1)}{\lambda^2}. \]  
(3.28)
Moreover,
\[ \frac{1}{\lambda^2} \|y_{xx}(l_0^-)\| = \frac{o(1)}{\lambda^{\frac{3}{2}}} \quad \text{and} \quad \frac{1}{\lambda^2} |y_{xx}(l_1^+)\| = \frac{o(1)}{\lambda^{\frac{3}{2}}}. \]  
(3.29)
**Proof.** For the proof of (3.28). Since \( y, z \in H^2(0, L) \), Sobolev embedding theorem implies that \( y, z \in C^1[0, L] \). Then, using the second estimation in Lemma 3.2 we get (3.28).

Define
\[ J(z)(x) = \frac{1}{\lambda^{1-\alpha}} \int_x^L \int_s^L z(\tau) d\tau ds \]  
(3.30)
and
\[ X = \frac{1}{i\lambda} [-S + J(f_4)]. \]  
(3.31)
From (3.13) and (3.31), we get
\[ X_{xx} = z. \]  
(3.32)
From (3.31), and using the fact that \( \|f_4\|_{L^2(0, L)} = o(1) \), we have that
\[ \|\lambda X\|_{L^2(t_0, t_1)} \leq \|S\|_{L^2(t_0, t_1)} + \|J(f_4)\|_{L^2(t_0, t_1)} \leq \frac{o(1)}{\lambda^{\frac{3}{2}}}. \]  
(3.33)
It follows that,
\[ \|X\|_{L^2(t_0, t_1)} = \frac{o(1)}{\lambda^{\frac{3}{2}}}. \]  
(3.34)
Proof. We note that
\[
S \quad \text{from the interpolation inequality (see Theorem 4.17 in [2]), we have}
\]
\[
\|X_{xx}\|_{L^2(\{0, t\} \cup \{a, t\})} = o(1) \frac{1}{\lambda}, \quad \|X_{xxx}\|_{L^2(\{0, t\} \cup \{a, t\})} = o(1) \frac{1}{\lambda^3/4}.
\]
Now, using the interpolation inequality theorem [52], and \(\alpha \in (0, 1)\), we get
\[
\|X\|_{L^2(\{0, t\} \cup \{a, t\})} \leq \|X_{xx}\|_{L^2(\{0, t\} \cup \{a, t\})}^\frac{1}{2} \|X\|_{L^2(\{0, t\} \cup \{a, t\})} + \|X\|_{L^2(\{0, t\} \cup \{a, t\})} = o(1) \frac{1}{\lambda^{3/4}}.
\]
By using Equations (3.34)-(3.36), we get
\[
\|X\|_{H^4(\{0, t\} \cup \{a, t\})} = o(1) \frac{1}{\lambda}.
\]
From the interpolation inequality (see Theorem 4.17 in [2]), we have
\[
\|\lambda^{1/2}X\|_{H^4(\{0, t\} \cup \{a, t\})} \leq \|\lambda^{1/2}X\|_{L^2(\{0, t\} \cup \{a, t\})} \cdot \|\lambda^{1/2}X\|_{L^2(\{0, t\} \cup \{a, t\})} = o(1) \frac{1}{\lambda^{3/4}}.
\]
We note that \(S = b_{yx} \) on \((0, a) \cup \{a, L\}\). Also, we have that \(y \in H^4(0, a)\) and \(y \in H^4(1, L)\).

From (3.31), we have
\[
b_{yx} = (J(f_4) - i\lambda X)_x(0) \quad \text{and} \quad b_{yx} = (J(f_4) - i\lambda X)_x(1) \quad (3.39)
\]
Dividing Equation (3.39) by \(\lambda^{3/2}\), and using Equation (3.38) and the fact that \(\|f_4\|_{L^2(0, L)} = o(1)\), we get Equation (3.29). Thus, the proof of the Lemma is complete. \(\square\)

Lemma 3.4. Assume that \(\eta > 0\). The solution \((u, v, x, z, \omega) \in D(A)\) of system (3.11)-(3.15) satisfies the following asymptotic behavior for every \(h \in C^2([0, L])\) and \(h(0) = h(L) = 0\), and for every \(g \in C^1[-L, L]\)
\[
\int_{-L}^{0} g'(\nu^2 + a|u_x|^2) \nu d\nu - g(0)|v(0)|^2 - ag(0)|u_x(0)|^2 + ag(-L)|u_x(-L)|^2 = o(1) \frac{1}{\lambda^{3-\alpha}}.
\]

Proof. Multiply Equation (3.14) by \(2h\bar{y}_x\) and integrate over \((0, L)\) we get
\[
\Re \left(2i\lambda \int_{0}^{L} zh\bar{y}_x dx\right) + \Re \left(2 \int_{0}^{L} hS_{xx} \bar{y}_x dx\right) = \Re \left(2\lambda^{-3+\alpha} \int_{0}^{L} f_4 h\bar{y}_x dx\right). \quad (3.42)
\]
From Equation (3.13) we have
\[
i\lambda \bar{y}_x = -\bar{v}_x - \lambda^{-3+\alpha}(f_4)_x.
\]
Then
\[
\Re \left(2i\lambda \int_{0}^{L} zh\bar{y}_x dx\right) = \int_{0}^{L} h'|v|^2 dx - \Re \left(2\lambda^{-3+\alpha} \int_{0}^{L} hz(f_4)_x dx\right). \quad (3.43)
\]
For the second term of Equation (3.42), integrating by parts we get
\[ \left| \Re \left( 2\lambda^{-3+\alpha} \int_0^L h\bar{z}(f_3)_x dx \right) \right| \lesssim \frac{1}{|\lambda|^{3-\alpha}} \left( \int_0^L |z|^2 dx \right)^{1/2} \left( \int_0^L |(f_3)_x|^2 dx \right)^{1/2} \]
\[ = \frac{o(1)}{\lambda^{3-\alpha}}. \] (3.44)

Then, using Equation (3.44) in Equation (3.43) we get
\[ \Re \left( 2\lambda \int_0^L z\bar{y}_x dx \right) = \int_0^L h' |z|^2 dx + \frac{o(1)}{\lambda^{3-\alpha}}. \] (3.45)

Estimation of the term \( \Re \left( 2\lambda^{-3+\alpha} \int_0^L f_4 h\bar{y}_x dx \right) \). Using Cauchy-Schwarz inequality, the definition of \( h \), the fact that \( \|f_4\|_{L^2(0,L)} = o(1) \), and that \( z \) is uniformly bounded in \( L^2(0,L) \), we get
\[ \Re \left( 2\lambda^{-3+\alpha} \int_0^L f_4 h\bar{y}_x dx \right) \lesssim \frac{1}{\lambda^{3-\alpha}} \left( \int_0^L |f_4|^2 dx \right)^{1/2} \left( \int_0^L |y_{xx}|^2 dx \right)^{1/2} \leq \frac{o(1)}{\lambda^{3-\alpha}}. \] (3.46)

For the second term of Equation (3.42), integrating by parts we get
\[ \Re \left( 2 \int_0^L h_S_{xx}\bar{y}_x dx \right) = \Re \left( 2 \int_0^L h'' S\bar{y}_x dx \right) + \Re \left( 2 \int_0^L h' S\bar{y}_{xx} dx \right) \]
\[ - \Re \left( 2 \int_0^L h_S \bar{y}_{xx} dx \right). \] (3.47)

For the term \( \Re \left( 2 \int_0^L h' S\bar{y}_{xx} dx \right) \), we have
\[ \Re \left( 2 \int_0^L h' S\bar{y}_{xx} dx \right) = 2b \int_0^L h' |y_{xx}|^2 dx \]
\[ + \Re \left( 2 \int_0^L h'\bar{y}_{xx} \sqrt{d(x)\kappa(x)} \int_{\mathbb{R}} |\xi|^{\frac{2a-1}{2}} \omega(x,\xi) d\xi dx \right). \] (3.48)

Estimation of the term \( \Re \left( 2 \int_0^L h' \bar{y}_{xx} \sqrt{d(x)\kappa(x)} \int_{\mathbb{R}} |\xi|^{\frac{2a-1}{2}} \omega(x,\xi) d\xi dx \right) \). Using Cauchy-Schwarz inequality, the definition of the functions \( d(x) \) and \( h \), and using Lemma 3.1, we get
\[ \left| \Re \left( 2 \int_0^L h' \bar{y}_{xx} \sqrt{d(x)\kappa(x)} \int_{\mathbb{R}} |\xi|^{\frac{2a-1}{2}} \omega(x,\xi) d\xi dx \right) \right| = \frac{o(1)}{\lambda^{3-\alpha}}. \] (3.49)
This yields that,
\[ \Re \left( 2 \int_{0}^{L} h' S \tilde{y}_{xx} dx \right) = 2b \int_{0}^{L} h' |y_{xx}|^2 dx + \frac{o(1)}{\lambda^{3-\alpha}}, \quad (3.50) \]

For the term \(-\Re \left( 2 \int_{0}^{L} h S_{xx} \tilde{y} dx \right)\). We have
\[ -\Re \left( 2 \int_{0}^{L} h S_{xx} \tilde{y} dx \right) = -\Re \left( 2b \int_{0}^{l_{10}} h y_{xxx} \tilde{y}_{xx} dx + 2b \int_{l_{1}}^{L} h y_{xxx} \tilde{y}_{xx} dx \right) \]
\[ - \Re \left( 2 \int_{l_{10}}^{l_{1}} h S_{xx} \tilde{y} dx \right). \quad (3.51) \]

Integrating the above Equation by parts, we get
\[ -\Re \left( 2 \int_{0}^{L} h S_{xx} \tilde{y} dx \right) = b \int_{0}^{l_{10}} h' |y_{xx}|^2 dx + b \int_{l_{1}}^{L} h' |y_{xx}|^2 dx - bh(l_{0}) |y_{xx}(l_{0})|^2 \]
\[ + bh(l_{1}) |y_{xx}(l_{1})|^2 - \Re \left( 2 \int_{l_{10}}^{l_{1}} h S_{xx} \tilde{y} dx \right). \quad (3.52) \]

Then, substituting Equation (3.50) and (3.52) in Equation (3.47), we get
\[ \Re \left( 2 \int_{0}^{L} h S_{xx} \tilde{y} dx \right) = 2b \int_{0}^{L} h' |y_{xx}|^2 dx + b \int_{0}^{l_{10}} h' |y_{xx}|^2 dx + b \int_{l_{1}}^{L} h' |y_{xx}|^2 dx \]
\[ - bh(l_{0}) |y_{xx}(l_{0})|^2 + bh(l_{1}) |y_{xx}(l_{1})|^2 - \Re \left( 2 \int_{l_{10}}^{l_{1}} h S_{xx} \tilde{y} dx \right) \]
\[ + \Re \left( 2 \int_{0}^{L} h'' S \tilde{y}_{xx} dx \right) + \frac{o(1)}{\lambda^{3-\alpha}}. \quad (3.53) \]

Therefore, substituting Equations (3.45), (3.46), and (3.53) into (3.42), we get our desired result.

Now, we will prove Equation (3.41). For this aim, multiply Equation (3.12) by \(2g \tilde{u}_{x}\) and integrate over \((-L,0), \) we get
\[ \Re \left( 2i \lambda \int_{-L}^{0} v g \tilde{u}_{x} dx \right) - \Re \left( 2a \int_{-L}^{0} g u_{xx} \tilde{u}_{x} dx \right) = \Re \left( 2\lambda^{-3+\alpha} \int_{-L}^{0} f_{2} g \tilde{u}_{x} dx \right). \quad (3.54) \]

We have \(i \lambda \tilde{u}_{x} = -\tilde{v}_{x} - \lambda^{-3+\alpha}(\tilde{f}_{1})_{x}. \) Then,
\[ \Re \left( 2i \lambda \int_{-L}^{0} v g \tilde{u}_{x} dx \right) = \int_{-L}^{0} g' |v|^2 dx - g(0) |v(0)|^2 - \Re \left( 2\lambda^{-3+\alpha} \int_{-L}^{0} v g(\tilde{f}_{1})_{x} dx \right). \quad (3.55) \]

Estimiation of the term \(\Re \left( 2\lambda^{-3+\alpha} \int_{-L}^{0} v g(\tilde{f}_{1})_{x} dx \right). \) Using Cauchy-schwarz inequality, \(\|\tilde{f}_{1}\|_{H_{2}^{-1}(-L,0)} = o(1), \) and the fact that \(v\) is bounded in \(L^2(0,L), \) we get
\[ \left| \Re \left( 2\lambda^{-3+\alpha} \int_{-L}^{0} v g(\tilde{f}_{1})_{x} dx \right) \right| \leq \frac{o(1)}{\lambda^{3-\alpha}}. \quad (3.56) \]
Then, using the above equation, (3.55) becomes
\[
\Re \left( 2i\lambda \int_{-L}^{0} v g \bar{u}_x dx \right) = \int_{-L}^{0} g' |v|^2 dx - g(0) |v(0)|^2 + \frac{o(1)}{\lambda^{3-\alpha}}.
\] (3.57)

Integrating the second term of Equation (3.54), we get
\[
- \Re \left( 2a \int_{-L}^{0} g_{xx} \bar{u}_x dx \right) = a \int_{-L}^{0} g' |u_x|^2 dx - g(0) |u_x(0)|^2 + a \lambda v(-L)^2.
\] (3.58)

Estimation of the term \( \Re \left( 2\lambda^{-3+\alpha} \int_{-L}^{0} f_2 g \bar{u}_x dx \right) \). Using Cauchy-Schwarz inequality, \( \|f_2\|_{L^2(-L,0)} = o(1) \), and the fact that \( u_x \) is bounded in \( L^2(-L,0) \), we get
\[
\left| \Re \left( 2\lambda^{-3+\alpha} \int_{-L}^{0} f_2 g \bar{u}_x dx \right) \right| = \frac{o(1)}{\lambda^{3-\alpha}}.
\] (3.59)

Thus, summing Equations (3.57), (3.58) and (3.59) we obtain our desired result. \( \square \)

**Lemma 3.5.** Assume that \( \eta > 0 \). The solution \( (u,v,y,z,\omega) \in D(A_3) \) of system (3.11)-(3.15) satisfies the following asymptotic behavior
\[
|\lambda y(0)| = O(1), \quad |y_{xx}(l_0^-)| = O(1) \quad \text{and} \quad |y_{xx}(l_1^+)| = O(1).
\] (3.60)

**Proof.** Take \( g(x) = x + L \) in Equation (3.41), we get
\[
\int_{-L}^{0} (|v|^2 + a |u_x|^2 dx) - L |v(0)|^2 - aL |u_x(0)|^2 = \frac{o(1)}{\lambda^{3-\alpha}}.
\] (3.61)

From the above Equation and using the fact that \( v \) and \( u_x \) are uniformly bounded in \( L^2(-L,0) \), we get
\[
|v(0)| = O(1) \quad \text{and} \quad |u_x(0)| = O(1).
\] (3.62)

From Equation (3.13) we have
\[
i\lambda u = v + \lambda^{-3+\alpha} f_1.
\]

Then, using (3.62) and the fact that \( \|f_1\|_{L^2(-L,0)} = o(1) \), we get
\[
|\lambda u(0)| \leq |v(0)| + \lambda^{-3+\alpha} |f_1(0)| \leq |v(0)| + \lambda^{-3+\alpha} \sqrt{L} \|f_1\|_{L^2(-L,0)} = O(1).
\] (3.63)

From the continuity condition \( (u(0) = y(0)) \), we deduce that \( |\lambda y(0)| = O(1) \). In order to prove the second term in the Equation (3.60) we proceed as follows. From Equation (3.13) we have
\[
z = i\lambda y - \lambda^{-3+\alpha} f_3.
\]

Substituting the above Equation into Equation (3.14), we get
\[
- \lambda^2 y + b y_{xxxx} = F \quad \text{on} \quad (0,l_0) \cup (l_1,L),
\] (3.64)

where \( F = \lambda^{-3+\alpha} (f_4 + i\lambda f_3) \).

Multiply Equation (3.64) by \( 2\zeta \bar{y}_x \), where \( \zeta \in C^2[0,l_0], \zeta(0) = \zeta'(l_0) = 0 \) and \( \zeta(l_0) = 1 \), and integrate over \( (0,l_0) \), we get
\[
- \Re \left( 2\lambda^2 \int_{0}^{l_0} \zeta y \bar{y}_x dx \right) + \Re \left( 2b \int_{0}^{l_0} \zeta y_{xxxx} \bar{y}_x dx \right) = \Re \left( 2\lambda^{-3+\alpha} \int_{0}^{l_0} (f_4 + i\lambda f_3) \zeta \bar{y}_x dx \right).
\] (3.65)
Estimation of the term $\Re \left( 2\lambda^2 \int_0^{l_0} \zeta y y_x dx \right)$. Integrating by parts and using Equation (3.28), we get

$$
- \Re \left( 2\lambda^2 \int_0^{l_0} \zeta y y_x dx \right) = \int_0^{l_0} \zeta' |\lambda y|^2 dx + \frac{o(1)}{\lambda^2}. \tag{3.66}
$$

Integrating by parts the second term in Equation (3.65), and using Lemma 3.3, we get

$$
\Re \left( 2b \int_0^{l_0} \zeta y_{xxx} y_x dx \right) = -\Re \left( 2b \int_0^{l_0} \zeta' y_{xxx} y_x dx \right) + \Re \left( 2b \int_0^{l_0} \zeta y_{xxx} y_x dx \right)
+ \Re \left( 2b y_{xxx}(l_0^+) y_x(l_0) \right)
+ \Re \left( 2b \int_0^{l_0} \zeta'' y_{xxx} y_x dx \right)
+ 3b \int_0^{l_0} \zeta' |y_{xx}|^2 dx
+ b\zeta(l_0) |y_{xx}(l_0)|^2 + \frac{o(1)}{\lambda^{1-\alpha}}.
\tag{3.67}
$$

Integrating by parts the last term of Equation (3.65) and using (3.28) and the fact that $|f_3(l_0)| \lesssim \|f_3\|_{H^2_{\bar{H}}(0,L)} = o(1)$, we get

$$
\Re \left( 2\lambda^{-3+\alpha} \int_0^{l_0} (f_4 + i\lambda f_3) \zeta y_x dx \right) = \Re \left( 2\lambda^{-3+\alpha} \int_0^{l_0} f_4 \zeta y_x dx \right)
- \Re \left( 2i\lambda^{-3+\alpha} \int_0^{l_0} f_3 \zeta' \lambda y dx \right)
+ \Re \left( 2i\lambda^{-3+\alpha} \int_0^{l_0} (f_3)_x \zeta \lambda y dx \right) + \frac{o(1)}{\lambda^{1-\alpha}}. \tag{3.68}
$$

Substituting Equations (3.66), (3.67), and (3.68) in Equation (3.65), and using the fact that $\|U\|_{H^1} = 1$, $\|y_x\|_{L^2(0,L)} \leq c_p \|y_{xx}\|_{L^2(0,L)} = O(1)$ and $\|f_3\|_{H^2_{\bar{H}}(0,L)} = o(1)$, we obtain our desired term. For the last term in Equation (3.60), we proceed in a similar way as above and thus the proof of the Lemma is complete. \hfill \square

**Lemma 3.6.** Assume that $\eta > 0$. The solution $(u,v,y,z,\omega) \in D(A_3)$ of system (3.11)-(3.15) satisfies the following asymptotic behavior

$$
|y_{xx}(l_0^+)| = \frac{o(1)}{\lambda} \quad \text{and} \quad |y_{xx}(l_1^+)| = \frac{o(1)}{\lambda}.
\tag{3.69}
$$

Proof. Equation (3.64) can be written as

$$
\left( \frac{\partial}{\partial x} - i\mu \right) \left( \frac{\partial}{\partial x} + i\mu \right) \left( \frac{\partial^2}{\partial x^2} - \mu^2 \right) y = \frac{1}{b} F, \text{ on } (0,L) \cup (l_1, L). \tag{3.70}
$$

where $\mu = \sqrt{\lambda / \sqrt{b}}$.

On the interval $(0, l_0)$:

Let $Y_{l_0} = \left( \frac{\partial}{\partial x} + i\mu \right) \left( \frac{\partial^2}{\partial x^2} - \mu^2 \right) y$. Solving the on the interval $(0, l_0)$ the following Equation

$$
\left( \frac{\partial}{\partial x} - i\mu \right) Y_{l_0} = \frac{1}{b} F
$$
we get
\[ Y^1_{l_0} = K_1 e^{i\mu(x-l_0)} + \frac{1}{b} \int_{l_0}^x e^{i\mu(x-z)} F(z) \, dz, \]  
(3.71)
where
\[ K_1 = y_{xx}(l_0^-) - \mu^2 y_x(l_0) + i\mu y_{xx}(l_0^-) - i\mu^3 y(l_0). \]

Let \( Y^2_{l_0} = \left( \frac{\partial^2}{\partial x^2} - \mu^2 \right) y. \) We will solve the following differential equation
\[ \left( \frac{\partial}{\partial x} + i\mu \right) Y^2_{l_0} = Y^1_{l_0}. \]  
(3.72)

By using the solution \( Y^1_{l_0}, \) we obtain the solution of the differential equation (3.72)
\[ Y^2_{l_0} = K_2 e^{-i\mu(x-l_0)} + \frac{K_1}{\mu} \sin((x - l_0)\mu) + \frac{1}{b\mu} \int_{l_0}^x \int_{l_0}^s e^{i\mu(2s-x-z)} F(z) \, dz \, ds. \]  
(3.73)

Integrating by parts the last term of the above Equation, we get
\[ \frac{1}{b\mu} \int_{l_0}^x \int_{l_0}^s e^{i\mu(2s-x-z)} F(z) \, dz \, ds = \frac{1}{b\mu} \int_{l_0}^x \sin((x - z)\mu) F(z) \, dz. \]

Inserting the above Equation in (3.73), we get
\[ Y^2_{l_0} = K_2 e^{-i\mu(x-l_0)} + \frac{K_1}{\mu} \sin((x - l_0)\mu) + \frac{1}{b\mu} \int_{l_0}^x \sin((x - z)\mu) F(z) \, dz \]  
(3.74)
where
\[ K_2 = y_{xx}(l_0^-) - \mu^2 y(l_0). \]

Let \( Y^3_{l_0} = \left( \frac{\partial}{\partial x} - \mu \right) y. \) The solution of the following differential equation
\[ \left( \frac{\partial}{\partial x} + \mu \right) Y^3_{l_0} = Y^2_{l_0} \]  
(3.75)
is given by
\[ Y^3_{l_0} = K_3 e^{-\mu(x-l_0)} + \frac{K_2}{\mu(1-i)} \left[ e^{-i\mu(x-l_0)} - e^{-\mu(x-l_0)} \right] \]
\[ + \frac{1}{b\mu} \int_{l_0}^x \int_{l_0}^s e^{i\mu(s-z)} \sin((s - z)\mu) F(z) \, dz \, ds \]  
(3.76)
\[ - \frac{K_1}{2\mu^2} \left[ \cos((x - l_0)\mu) - \sin((x - l_0)\mu) - e^{-\mu(x-l_0)} \right] \]
where
\[ K_3 = y_x(l_0) - \mu y(l_0). \]
Take \( x = 0 \) in (3.76) and multiply the equation by \( \mu e^{-\mu t_0} \), we get

\[
\sqrt{\frac{\alpha}{4}} |y_{xx}(l_0)| \leq |\mu y_x(0)| e^{-\mu t_0} + |\mu^2 y(0)| e^{-\mu t_0} + |y_x(l_0)| + |\mu y(l_0)|
+ \frac{1}{2\mu} |y_{xxx}(l_0)| \left[ 2e^{-\mu t_0} + 1 \right] + \frac{1}{2} |\mu^2 y(l_0)| \left( 2e^{-\mu t_0} + 1 \right)
+ \frac{1}{\sqrt{2}} |\mu y_x(l_0)| \left( 2e^{-\mu t_0} + 1 \right) + \frac{1}{\sqrt{2}} |\mu^2 y(l_0)| \left( e^{-\mu t_0} + 1 \right)
\]

(3.77)

For the integral in the above Equation, we have

\[
\frac{1}{b} \int_0^{l_0} \int_0^s e^{\mu(s-t_0)} \sin((s-z)\mu)F(z)dzds
= \frac{1}{b\lambda^{3-\alpha}} \int_0^{l_0} \int_0^s e^{\mu(s-t_0)} \sin((s-z)\mu)f_4(z)dzds
+ \frac{1}{b\lambda^{3-\alpha}} \int_0^{l_0} \int_0^s e^{\mu(s-t_0)} \sin((s-z)\mu)i\lambda f_3(z)dzds.
\]

Estimation of the first integral in the right side of Equation (3.78).

\[
\left| \frac{1}{b\lambda^{3-\alpha}} \int_0^{l_0} \int_0^s e^{\mu(s-t_0)} \sin((s-z)\mu)f_4(z)dzds \right| \leq \frac{l_0^{3/2}}{\lambda^{3-\alpha}} \int_0^{l_0} |f_4(z)|^2dz = o(1).
\]

(3.79)

Estimation of the second term in the second side of Equation (3.78). Integrating by parts, and using the fact that \( |f_3(0)| \leq \|f_3\|_{L^2(0,L)} = o(1) \) and \( |f_3(l_0)| \leq \|f_3\|_{L^2(0,L)} = o(1) \), we get

\[
\left| \frac{1}{b\lambda^{3-\alpha}} \int_0^{l_0} \int_0^s e^{\mu(s-t_0)} \sin((s-z)\mu)i\lambda f_3(z)dzds \right|
= \left| \frac{i\lambda e^{-\mu t_0}}{b\lambda^{3-\alpha}} \int_0^{l_0} \left( \int_0^z \sin((s-z)\mu)ds \right) f_3(z)dz \right|
= \left| \frac{i\lambda e^{-\mu t_0}}{2b\mu\lambda^{3-\alpha}} \int_0^{l_0} \left( \cos(\mu z) + \sin(\mu z) - e^{\mu z} \right) f_3(z)dz \right|
\leq \frac{\lambda}{2e\mu e^{\mu t_0} \lambda^{3-\alpha}} \left[ 2 \int_0^{l_0} |f_3(z)|dz + \int_0^{l_0} e^{\mu z} f_3(z)dz \right]
\leq \frac{\lambda}{2e\mu e^{\mu t_0} \lambda^{3-\alpha}} \left[ 2 \int_0^{l_0} |f_3(z)|dz + \frac{1}{\mu} (|f_3(0)| + |f_3(l_0)|) e^{\mu t_0} \right]
+ \frac{e^{\mu t_0}}{\mu} \int_0^{l_0} |f_3'(z)|dz = o(1).
\]

(3.80)
Hence,

\[
\left| \frac{1}{b} \int_0^{l_0} \int_{l_0}^x e^{j(s-l_0)} \sin((s-z)\mu)F(z)dzds \right| = o(1) \frac{1}{\lambda^{3-\alpha}}. \tag{3.81}
\]

Since \( e^{\sqrt{\zeta}} \geq \frac{1}{5} \zeta^2 \), and taking \( \zeta = \frac{\lambda b}{v b} \forall \lambda \in \mathbb{R}_+ \), and using Lemmas 3.3 and 3.5, and Equation (3.81) in Equation (3.77), we get that

\[
|y_{xx}(l_0)| = o(1) \frac{1}{\lambda}. \tag{3.82}
\]

On the interval \((l_1, L)\):

Proceeding with a similar computation as on \((0, l_0)\), we get

\[
Y_{l_1}^2 = K_2 e^{-i\mu(x-l_1)} + \frac{K_1}{\mu} \sin((x-l_1)\mu) + \frac{1}{b\mu} \int_{l_1}^x \sin((x-z)\mu)F(z)dz \tag{3.83}
\]

where

\[
K_1 = y_{xx}(l_1^+) - \mu^2 y_x(l_1) + i\mu y_{xx}(l_1^+) - i\mu^3 y(l_1)
\]

and

\[
K_2 = y_{xx}(l_1^+) - \mu^2 y(l_1).
\]

We will solve the following differential Equation

\[
\left( \frac{\partial}{\partial x} - \mu \right) Y_{l_1}^3 = Y_{l_1}^2 \tag{3.84}
\]

where \( Y_{l_1}^3 = \left( \frac{\partial}{\partial x} + \mu \right) y \). The solution of (3.83) is

\[
Y_{l_1}^3 = K_3 e^{\mu(x-l_1)} - \frac{K_2}{1 + i} \left[ e^{-i\mu(x-l_1)} - e^{i\mu(x-l_1)} \right]
- \frac{K_1}{2\mu^2} \left[ \cos((x-l_1)\mu) + \sin((x-l_1)\mu) - e^{i\mu(x-l_1)} \right] + \frac{1}{b\mu} \int_{l_1}^x \int_{l_1}^z e^{\mu(x-z)} \sin((s-z)\mu)F(z)dzds
\]

where

\[
K_3 = y_x(l_1) + \mu y(l_1).
\]

Taking \( x = L \) in Equation (3.84) and multiplying by \( \mu e^{-\mu(L-l_1)} \), we get

\[
\left| \frac{1}{\sqrt{2}} |y_{xx}(l_1^+)| \leq |\mu y_x(l_1)| + |\mu^2 y(l_1)| + \frac{e^{-\mu(l_1)}}{\sqrt{2}} |y_{xx}(l_1^+)| + \frac{1}{\sqrt{2}} |\mu^2 y_x(l_1)| \left( e^{-\mu(L-l_1)} + 1 \right) + \frac{1}{\mu} |y_{xx}(l_1^+)| \left( 2e^{-\mu(L-l_1)} + 1 \right) + |\mu y_x(l_1)| \left( 2e^{-\mu(L-l_1)} + 1 \right) + |\mu^2 y(l_1)| \left( 2e^{-\mu(L-l_1)} + 1 \right) + \frac{1}{b} \int_{l_1}^L \int_{l_1}^z e^{-\mu(z-l_1)} \sin((s-z)\mu)F(z)dzds \right|
\]

\[
\int_{l_1}^x \int_{l_1}^z e^{-\mu(s-l_1)} \sin((s-z)\mu)F(z)dzds = o(1) \frac{1}{\lambda^{3-\alpha}}. \tag{3.85}
\]

Using the same computation in Equation (3.78), we get

\[
\left| \frac{1}{b} \int_{l_1}^x \int_{l_1}^z e^{-\mu(s-l_1)} \sin((s-z)\mu)F(z)dzds \right| = o(1) \frac{1}{\lambda^{3-\alpha}}. \tag{3.86}
\]
Since \( e^{\sqrt{\zeta}} \geq \frac{1}{5} s^2 \) and taking \( \zeta = \lambda(L - l_1)^2 \), and using Lemmas 3.3, 3.5, and Equation (3.86) in (3.85), we get

\[
|y_{xx}(l_1^+)| = o\left(\frac{1}{\lambda}\right).
\]

Thus the proof of this Lemma is complete.

**Lemma 3.7.** Assume that \( \eta > 0 \). The solution \((u, v, y, z, \omega) \in D(A_3) \) of system (3.11)-(3.15) satisfies the following asymptotic behavior

\[
\int_0^L |z|^2 dx = o\left(\frac{1}{\lambda^2}\right) \quad \text{and} \quad b \int_0^L |y_{xx}|^2 dx = o\left(\frac{1}{\lambda^2}\right). \tag{3.87}
\]

**Proof.** Taking \( h(x) = x \theta_1(x) \) in Equation (3.40), where \( \theta_1 \in C^1([0, L]) \) is defined as follows

\[
\theta_1(x) = \begin{cases} 
1 & \text{if } x \in [0, l_0], \\
0 & \text{if } x \in [l_1, L], \\
0 \leq \theta_1 \leq 1 & \text{elsewhere.}
\end{cases} \tag{3.88}
\]

Estimation of the term \( \Re \left( 2 \int_0^L h'' S \bar{y}_x dx \right) \). Using Cauchy-Schwarz inequality, the definition of \( h \), Lemmas 3.1 and 3.2, we get

\[
\left| \Re \left( 2 \int_0^L h'' S \bar{y}_x dx \right) \right| = o\left(\frac{1}{\lambda^2}\right). \tag{3.89}
\]

Estimation of the term \( \Re \left( 2 \int_{l_0}^{l_1} h S_x \bar{y}_{xx} dx \right) \). Using Cauchy-Schwarz inequality, the definition of \( h \), Lemmas 3.1 and 3.2, we get

\[
\left| \Re \left( 2 \int_{l_0}^{l_1} h S_x \bar{y}_{xx} dx \right) \right| \leq \left( \int_{l_0}^{l_1} |S_x|^2 dx \right)^{1/2} \left( \int_{l_0}^{l_1} |y_{xx}|^2 dx \right)^{1/2} = o\left(\frac{1}{\lambda^2}\right). \tag{3.90}
\]

Using Lemma 3.6, Equations (3.89) and (3.90) in (3.40), we get

\[
\int_0^L h'|z|^2 dx + 2b \int_0^L h'|y_{xx}|^2 dx + b \int_0^{l_0} h'|y_{xx}|^2 dx + b \int_{l_1}^L h'|y_{xx}|^2 dx = o\left(\frac{1}{\lambda^2}\right). \tag{3.91}
\]

By using the definition of \( h \), we obtain

\[
\int_{l_0}^{l_1} |z|^2 dx = o\left(\frac{1}{\lambda^2}\right) \quad \text{and} \quad b \int_{l_0}^{l_1} |y_{xx}|^2 dx = o\left(\frac{1}{\lambda^2}\right). \tag{3.92}
\]

Now, taking \( h(x) = x \theta_2(x) \) in Equation (3.40), where \( \theta_2 \in C^1([0, L]) \) is defined as follows

\[
\theta_2(x) = \begin{cases} 
1 & \text{if } x \in [l_1, L], \\
0 & \text{if } x \in [0, l_0], \\
0 \leq \theta_2 \leq 1 & \text{elsewhere.}
\end{cases} \tag{3.93}
\]

Proceeding in a similar way as above we get

\[
\int_{l_1}^L |z|^2 dx = o\left(\frac{1}{\lambda^2}\right) \quad \text{and} \quad b \int_{l_1}^L |y_{xx}|^2 dx = o\left(\frac{1}{\lambda^2}\right). \tag{3.94}
\]

Therefore, using the third estimation of (3.16), Lemma 3.2 and combining Equations (3.92), and (3.94) we get our desired result. \( \square \)
Lemma 3.8. Assume that $\eta > 0$. The solution $(u, v, y, z, \omega) \in D(\mathcal{A}_3)$ of system (3.11)-(3.15) satisfies the following asymptotic behavior

$$|y_{xxx}(0)| = o(1).$$  \hspace{1cm} (3.95)

Proof. From the interpolation inequality Theorem (see [52]), and using the fact that $y \in H^4(0, l_0)$, $\|f_4\|_{L^2(0, l_0)} = o(1)$, Equation (3.14) and Lemma 3.7, we get

$$\|y_{xxx}\|_{L^2(0, l_0)} \leq \|y_{xxx}\|_{H^2(0, l_0)}^{1/2} \|y_{xxx}\|_{H^4(0, l_0)}^{1/2} + \|y_{xxx}\|_{L^2(0, l_0)}$$

$$\leq \left[ \frac{1}{\lambda^2} \|y\|_{L^2(0, l_0)} + \frac{1}{\lambda^2} \|f_4\|_{L^2(0, l_0)}^2 \right] \|y_{xxx}\|_{L^2(0, l_0)} + \|y_{xxx}\|_{L^2(0, l_0)}$$

$$= o(1).$$ \hspace{1cm} (3.96)

Using Equation (3.14) and the definition of $S$ on $(0, l_0)$ and (3.7), we get

$$\|y_{xxx}\|_{L^2(0, l_0)} = o(1).$$ \hspace{1cm} (3.97)

Then, from (3.96) and (3.97), we get

$$\|y_{xxx}\|_{H^1(0, l_0)} = o(1).$$

Since $H^1(0, l_0) \subset C([0, l_0])$, then we get the desired result. Thus, the proof is complete. \Box

Lemma 3.9. Assume that $\eta > 0$. The solution $(u, v, y, z, \omega) \in D(\mathcal{A}_3)$ of system (3.11)-(3.15) satisfies the following asymptotic behavior

$$\int_{-L}^{0} (|v|^2 + a|u_x|^2) \, dx = o(1).$$ \hspace{1cm} (3.98)

Proof. Using the transmission condition and Lemma 3.8, we have

$$|u_x(0)| = |y_{xxx}(0)| = o(1).$$ \hspace{1cm} (3.99)

Now, let $q \in C^2([0, l_0])$ such that $q(l_0) = q_x(l_0) = 0$, $q(0) = 1$. Multiply Equation (3.64) by $q_y$ and integrate over $(0, l_0)$, and using the fact that $\|f_3\|_{H^1(0, L)} = o(1)$, $\|f_4\|_{L^2(0, L)} = o(1)$, and $\|y_x\|_{L^2(0, L)} \leq \|y_x\|_{L^2(0, l_0)} \leq \|y_{xxx}\|_{L^2(0, l_0)} = O(1)$, we get

$$\frac{1}{2} \int_{0}^{l_0} q' (|y|^2 + 3|y_{xx}|^2) \, dx + b \int_{0}^{l_0} q'' y_{xx} y_x \, dx + \frac{1}{2} |\lambda y(0)|^2 - \Re \{by_{xxx}(0)\bar{y}_x(0)\} = o(1).$$ \hspace{1cm} (3.100)

By using Lemmas 3.7, 3.8 and the fact that $|y_x(0)| \lesssim \|y_{xxx}\|_{L^2(0, L)} = O(1)$ and $|y_x|_{L^2(0, L)} \lesssim \|y_x\|_{L^2(0, l_0)} \lesssim \|y_{xxx}\|_{L^2(0, l_0)} = O(1)$ in Equation (3.100), we obtain

$$|\lambda y(0)|^2 = o(1).$$ \hspace{1cm} (3.101)

From the transmission conditions we have

$$|\lambda u(0)| = |\lambda y(0)| = o(1).$$

It follows from the above Equation and Equation (3.11) that

$$|v(0)| = o(1).$$ \hspace{1cm} (3.102)

Thus, by taking $g(x) = x + L$ in Equation (3.41), and using Equations (3.99) and (3.102) we get our desired result. \Box
Proof of Theorem 2.7. From Lemmas 3.7 and 3.9, we get that \(\|U\|_{H^3} = o(1)\), which contradicts (3.9). Consequently, condition \((R_2)\) holds. This implies, from Theorem 4.4, the energy decay estimation (3.8). The proof is thus complete.

Remark 3.5. • The result in [33] can be improved. Indeed, in [33], Fathi considered a Euler-Bernoulli beam and wave equations coupled through transmission conditions. The damping is locally distributed and acts on the wave equation, and the rotation vanishes at the connecting point \((y_x(\ell) = 0)\). The system is given in the left side of Equation (1.1). He proved the polynomial stability with energy decay rate of type \(t^{-2}\). By using similar computations as in Section 2 by taking \(\alpha = 1\), and by solving the ordinary differential equations in Section 3 we can reach that \(|y_{xx}(\ell)| = o(1)|. Thus, with the same technique of the proof of Section 2, we can reach that energy of the system (the left system in Equation (1.1)) of the mentioned paper satisfies the decay rate \(t^{-4}\).

• In [33], when the damping acts on the beam equation the energy decay rate reached was \(t^{-2}\) (the left system in Equation (1.1)), when taking the condition \(y_x(\ell) = 0\) at the connecting point. However, in this paper, we proved in section 3 the polynomial energy decay rate of type \(t^{-1}\), when taking \(y_{xx}(0) = 0\) at the connecting point. From this comparison, we see that the boundary conditions play a critical role in the energy decay rate for the system (3.2).

Remark 3.6. For the model \((W-W_{FKV})\), after recasting the system into an augmented model and by using similar arguments and steps of proof as in Section 2, we can reach that the system is polynomially stable and that the energy satisfies the following estimate
\[
E(t) \leq \frac{C_4}{t^{\frac{\alpha}{2}}} \|U_0\|^2_{D(\mathcal{A})}, \quad t > 0, \forall U_0 \in D(\mathcal{A}), C_4 > 0. \tag{3.104}
\]

Remark 3.7. By recasting the System \((EBB_{FKV})\) into an augmented model and by proceeding in a similar way as in Section 3, we can prove the polynomial stability of the system and that the energy satisfies the following estimate
\[
E(t) \leq \frac{C_5}{t^{\frac{\alpha}{2}}} \|U_0\|^2_{D(\mathcal{A})}, \quad t > 0, \forall U_0 \in D(\mathcal{A}), C_5 > 0. \tag{3.105}
\]

Remark 3.8. For the model \((EBB)-(EBB_{FKV})\), following a similar argument of the proof in Section 3, we can obtain that the system is polynomially stable and that the energy satisfies the following estimation
\[
E(t) \leq \frac{C_6}{t^{\frac{\alpha}{2}}} \|U_0\|^2_{D(\mathcal{A})}, \quad t > 0, \forall U_0 \in D(\mathcal{A}), C_6 > 0. \tag{3.106}
\]

4. Appendix. In this section, we introduce the notions of stability that we encounter in this work.

Definition 4.1. Assume that \(A\) is the generator of a \(C_0\)-semigroup of contractions \((e^{tA})_{t \geq 0}\) on a Hilbert space \(\mathcal{H}\). The \(C_0\)-semigroup \((e^{tA})_{t \geq 0}\) is said to be

1. strongly stable if
\[
\lim_{t \to +\infty} \|e^{tA}x_0\|_H = 0, \quad \forall x_0 \in H;
\]

2. exponentially (or uniformly) stable if there exist two positive constants \(M\) and \(\epsilon\) such that
\[
\|e^{tA}x_0\|_H \leq Me^{-\epsilon t}\|x_0\|_H, \quad \forall t > 0, \forall x_0 \in H;
\]
3. polynomially stable if there exists two positive constants $C$ and $\alpha$ such that 
$$
\|e^{tA}x_0\|_H \leq Ct^{-\alpha}\|x_0\|_H, \quad \forall \ t > 0, \ \forall \ x_0 \in D(A).
$$

In that case, one says that the semigroup $(e^{tA})_{t \geq 0}$ decays at a rate $t^{-\alpha}$. The $C_0$-semigroup $(e^{tA})_{t \geq 0}$ is said to be polynomially stable with optimal decay rate $t^{-\alpha}$ (with $\alpha > 0$) if it is polynomially stable with decay rate $t^{-\alpha}$ and, for any $\varepsilon > 0$ small enough, the semigroup $(e^{tA})_{t \geq 0}$ does not decay at a rate $t^{-(\alpha-\varepsilon)}$.

To show the strong stability of a $C_0$-semigroup of contraction $(e^{tA})_{t \geq 0}$ we rely on the following result due to Arendt-Batty [15].

**Theorem 4.2.** Assume that $A$ is the generator of a $C_0$-semigroup of contractions $(e^{tA})_{t \geq 0}$ on a Hilbert space $H$. If

1. $A$ has no pure imaginary eigenvalues,
2. $\sigma(A) \cap i\mathbb{R}$ is countable,

where $\sigma(A)$ denotes the spectrum of $A$, then the $C_0$-semigroup $(e^{tA})_{t \geq 0}$ is strongly stable.

Concerning the characterization of exponential stability of a $C_0$-semigroup of contraction $(e^{tA})_{t \geq 0}$ we rely on the following result due to Huang [36] and Prüss [55].

**Theorem 4.3.** Let $A : D(A) \subset H \to H$ generate a $C_0$-semigroup of contractions $(e^{tA})_{t \geq 0}$ on $H$. Assume that $i\lambda \in \rho(A), \ \forall \lambda \in \mathbb{R}$. Then, the $C_0$-semigroup $(e^{tA})_{t \geq 0}$ is exponentially stable if and only if

$$
\lim_{\lambda \in \mathbb{R}, \ |\lambda| \to +\infty} \|(i\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < +\infty.
$$

Also, concerning the characterization of polynomial stability of a $C_0$-semigroup of contraction $(e^{tA})_{t \geq 0}$ we rely on the following result due to Borichev and Tomilov [22] (see also [45] and [20]).

**Theorem 4.4.** Assume that $A$ is the generator of a strongly continuous semigroup of contractions $(e^{tA})_{t \geq 0}$ on $H$. If $i\mathbb{R} \subset \rho(A)$, then for a fixed $\ell > 0$ the following conditions are equivalent

$$
\sup_{\lambda \in \mathbb{R}} \left\| (i\lambda I - A)^{-1} \right\|_{\mathcal{L}(H)} = O \left( |\lambda|^\ell \right),
$$

$$
\left\| e^{tA}U_0 \right\|^2_H \leq \frac{C}{t^{\ell}} \left\| U_0 \right\|^2_{D(A)}, \ \forall \ t > 0, \ U_0 \in D(A), \ for \ some \ C > 0.
$$

5. **Conclusion.** We have studied the stabilization of five models of systems. We considered a Euler-Bernoulli beam equation and a wave equation coupled through boundary connections with a localized non-regular fractional Kelvin-Voigt damping that acts through the wave equation only. We proved the strong stability of the system using Arendt-Batty criteria. In addition, we established a polynomial energy decay rate of type $t^{-\frac{\alpha}{2}}$. Moreover, we studied the system of Euler-Bernoulli beam and wave equations coupled through boundary connections where the dissipation acts through the beam equation. We proved a polynomial energy decay rate of type $t^{-\frac{\alpha}{2}}$. In addition, we shed the light on the the case where we consider two wave equations coupled via boundary connections with localized non-smooth
fractional Kelvin-Voigt damping and that we can obtain a polynomial energy decay rate of type $t^{-\frac{\alpha}{2}}$. Also, we considered the system of Euler-Bernoulli beam alone with the same localized non-smooth damping where we remarked that we can establish a polynomial energy decay rate of type $t^{-\frac{\alpha}{2}}$. Finally, for a system of two Euler-Bernoulli beam equations coupled through boundary conditions with a localized non-regular fractional Kelvin-Voigt damping acting only on one of the two equations. We remarked that we can reach a polynomial energy decay rate of type $t^{-\frac{3\alpha}{2}}$.

REFERENCES

[1] Z. Achouri, N. E. Amroun and A. Benaissa, The Euler-Bernoulli beam equation with boundary dissipation of fractional derivative type, Math. Methods Appl. Sci., 40 (2017), 3837–3854.
[2] R. A. Adams, Sobolev Spaces / Robert A. Adams, Academic Press New York, 1975.
[3] M. Akil, H. Badawi, S. Nicaise and A. Wehbe, On the stability of Bresse system with one discontinuous local internal kelvin–voigt damping on the axial force, Z. Angew. Math. Phys., 72 (2021), Paper No. 126, 27 pp.
[4] M. Akil, H. Badawi and A. Wehbe, Stability results of a singular local interaction elastic/viscoelastic coupled wave equations with time delay, Commun. Pure Appl. Anal., 20 (2021), 2991–3028.
[5] M. Akil, Y. Chitour, M. Ghader and A. Wehbe, Stability and exact controllability of a timoshenko system with only one fractional damping on the boundary, Asymptotic Analysis, 119 (2020), 221–280.
[6] M. Akil and A. Wehbe, Stabilization of multidimensional wave equation with locally boundary fractional dissipation law under geometric conditions, Math. Control Relat. Fields, 9 (2019), 97–116.
[7] H. Allouni, M. Kesri and A. Benaissa, On the asymptotic behaviour of two coupled strings through a fractional joint damper, Rend. Circ. Mat. Palermo (2), 69 (2020), 613–640.
[8] M. Alves, J. M. Rivera, M. Sepúlveda and O. V. Villagrán, The lack of exponential stability in certain transmission problems with localized kelvin–voigt dissipation, SIAM J. Appl. Math., 74 (2014), 345–365.
[9] K. Ammari, H. Fathi and L. Robbiano, Fractional-feedback stabilization for a class of evolution systems, J. Differential Equations, 268 (2020), 5751–5791.
[10] K. Ammari, M. Jellouli and M. Mehrenberger, Feedback stabilization of a coupled string-beam system, Netw. Heterog. Media, 4 (2009), 19–34.
[11] K. Ammari, Z. Liu and F. Shel, Stability of the wave equations on a tree with local Kelvin-Voigt damping, Semigroup Forum, 100 (2020), 364–382.
[12] K. Ammari and M. Mehrenberger, Study of the nodal feedback stabilization of a string-beams network, J. Appl. Math. Comput., 36 (2011), 441–458.
[13] K. Ammari and S. Nicaise, Stabilization of a transmission wave/plate equation, J. Differential Equations, 249 (2010), 707–727.
[14] K. Ammari and G. Vodev, Boundary stabilization of the transmission problem for the Bernoulli-Euler plate equation, Cubo, 11 (2009), 39–49.
[15] W. Arendt and C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, Trans. Amer. Math. Soc., 306 (1988), 837–852.
[16] R. L. Bagley and P. J. Torvik, Fractional calculus - a different approach to the analysis of viscoelastically damped structures, AIAA Journal, 21 (1983), 741–748.
[17] R. L. Bagley and P. J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, Journal of Rheology, 27 (1983), 201–210.
[18] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, SIAM J. Control Optim., 30 (1992), 1024–1065.
[19] J. Bartolomeo and R. Triggiani, Uniform energy decay rates for Euler-Bernoulli equations with feedback operators in the Dirichlet/Neumann boundary conditions, SIAM J. Math. Anal., 22 (1991), 46–71.
[20] C. J. K. Batt and T. Duyckaerts, Non-uniform stability for bounded semi-groups on Banach spaces, J. Evol. Equ., 8 (2008), 765–780.
[21] S. K. Biswas and N. U. Ahmed, Optimal control of large space structures governed by a coupled system of ordinary and partial differential equations, *Math. Control. Signals Syst.*, 2 (1989), 1–18.

[22] A. Borichev and Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, *Math. Ann.*, 347 (2010), 455–478.

[23] M. Caputo, Linear models of dissipation whose Q is almost frequency independent-II, *Geophysical Journal*, 13 (1967), 529–539.

[24] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, *Prog. Fract. Differ. Appl.*, 1 (2015), 73–85.

[25] G. Chen, M. C. Delfour, A. M. Krall and G. Payre, Modeling, stabilization and control of serially connected beams, *SIAM J. Control Optim.*, 25 (1987), 526–546.

[26] G. Chen, S. A. Fulling, F. J. Narcowich and S. Sun, Exponential decay of energy of evolution equations with locally distributed damping, *SIAM J. Appl. Math.*, 51 (1991), 266–301.

[27] S. Chen, K. Liu and Z. Liu, Spectrum and stability for elastic systems with global or local kelvin–voigt damping, *SIAM J. Appl. Math.*, 59 (1999), 651–668.

[28] R. Denk and F. Kämmelrander, Exponential stability for a coupled system of damped-undamped plate equations, *IMA J. Appl. Math.*, 83 (2018), 302–322.

[29] X. Fu and Q. Lu, Stabilization of the weakly coupled wave-plate system with one internal damping, 2017.

[30] B.-Z. Guo and H.-J. Ren, Stability and regularity transmission for coupled beam and wave equations through boundary weak connections, *ESAIM Control Optim. Calc. Var.*, 26 (2020), Paper No. 73, 29 pp.

[31] Y.-P. Guo, J.-M. Wang and D.-X. Zhao, Energy decay estimates for a two-dimensional coupled wave-plate system with localized frictional damping, *ZAMM Z. Angew. Math. Mech.*, 100 (2020), e201900030, 14 pp.

[32] Z.-J. Han and Z. Liu, Regularity and stability of coupled plate equations with indirect structural or Kelvin-Voigt damping, *ESAIM Control Optim. Calc. Var.*, 25 (2019), Paper No. 51, 14 pp.

[33] F. Hassine, Energy decay estimates of elastic transmission wave/beam systems with a local kelvin-voigt damping, *International Journal of Control*, (2015), 1–29.

[34] F. Hassine, Asymptotic behavior of the transmission Euler-Bernoulli plate and wave equation with a localized Kelvin-Voigt damping, *Discrete Contin. Dyn. Syst. Ser. B*, 21 (2016), 1757–1774.

[35] F. L. Huang, On the mathematical model for linear elastic systems with analytic damping, *SIAM J. Control Optim.*, 26 (1988), 714–724.

[36] F. L. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. Differential Equations*, 1 (1985), 43–56.

[37] G. Ji and I. Lasiecka, Nonlinear boundary feedback stabilization for a semilinear Kirchhoff plate with dissipation acting only via moments-limiting behavior, *J. Math. Anal. Appl.*, 229 (1999), 452–479.

[38] J. U. Kim and Y. Renardy, Boundary control of the Timoshenko beam, *SIAM J. Control Optim.*, 25 (1987), 1417–1429.

[39] J. E. Lagnese, Uniform boundary stabilization of homogeneous isotropic plates, Part of the *Lecture Notes in Control and Information Sciences*, (1987), 204–215.

[40] I. Lasiecka, Stabilization of wave and plate-like equations with nonlinear dissipation on the boundary, *J. Differential Equations*, 79 (1989), 340–381.

[41] I. Lasiecka, Asymptotic behavior of solutions to plate equations with nonlinear dissipation occurring through shear forces and bending moments, *Appl. Math. Optim.*, 21 (1990), 167–189.

[42] J. Le Rousseau and G. Lebeau, On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations, *ESAIM Control Optim. Calc. Var.*, 18 (2012), 712–747.

[43] Z.-J. Han and Z. Liu, Exponential decay of energy of the Euler–Bernoulli beam with locally distributed Kelvin–Voigt damping, *SIAM J. Control Optim.*, 36 (1998), 1086–1098.

[44] Z. Liu and B. Rao, Characterization of polynomial decay rate for the solution of linear evolution equation, *Z. Angew. Math. Phys.*, 56 (2005), 630–644.
Z. Liu and Q. Zhang, Stability of a string with local Kelvin–Voigt damping and nonsmooth coefficient at interface, SIAM J. Control Optim., 54 (2016), 1859–1871.

Z. Liu and S. Zheng, Semigroups Associated with Dissipative Systems, volume 398 of Chapman & Hall/CRC Research Notes in Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 1999.

M. Mainardi and E. Bonetti, The application of real-order derivatives in linear viscoelasticity, In H. Giesekus and M. F. Hibberd, editors, Progress and Trends in Rheology II, pages 64–67, Heidelberg, 1988. Steinkopff.

D. Matignon and C. Prieur, Asymptotic stability of Webster-Lokshin equation, Math. Control Relat. Fields, 4 (2014), 481–500.

B. Mbodje, Wave energy decay under fractional derivative controls, IMA J. Math. Control Inform., 23 (2006), 237–257.

B. Mbodje and G. Montseny, Boundary fractional derivative control of the wave equation, IEEE Trans. Automat. Control, 40 (1995), 378–382.

L. Nirenberg, An extended interpolation inequality, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3), 20 (1966), 733–737.

A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, volume 44 of Applied Mathematical Sciences, Springer-Verlag, New York, 1983.

I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications, Mathematics in Science and Engineering, Academic Press, London, 1999.

J. Prüss, On the spectrum of $C_0$-semigroups, Trans. Amer. Math. Soc., 284 (1984), 847–857.

C. A. Raposo, W. D. Bastos and J. A. J. Avila, A transmission problem for Euler-Bernoulli beam with Kelvin-Voigt damping, Appl. Math. Inf. Sci., 5 (2011), 17–28.

M. L. Santos and J. E. Muñoz Rivera, Analytic property of a coupled system of wave-plate type with thermal effect, Differential and Integral Equations, 24 (2011), 965–972.

L. Tebou, Energy decay estimates for some weakly coupled Euler-Bernoulli and wave equations with indirect damping mechanisms, Math. Control Relat. Fields, 2 (2012), 45–60.

P. J. Torvik and R. L. Bagley, On the appearance of the fractional derivative in the behavior of real materials, J. Appl. Mech., 51 (1984), 294–298.

A. Wehbe, I. Issa and M. Akil, Stability results of an elastic/viscoelastic transmission problem of locally coupled waves with non smooth coefficients, Acta Appl. Math., 171 (2021), Paper No. 23, 46 pp.

Q. Zhang, Exponential stability of an elastic string with local Kelvin-Voigt damping, Z. Angew. Math. Phys., 61 (2010), 1009–1015.

X. Zhang and E. Zuazua, Long-time behavior of a coupled heat-wave system arising in fluid-structure interaction, Arch. Ration. Mech. Anal., 184 (2007), 49–120.

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