Large Universe as the initial condition for a Collapsing Universe

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Abstract

Using the extension of the standard Hawking-Hartle prescription for defining a wave function for the universe, we show that it is possible, given a suitable form for the scalar field potential, to have the universe begin at its largest size and thereafter contract, with the growth of perturbations proceeding from small, at the largest size, to largest when the universe is small. This solution would dominate the wave function by an exponentially large amount if one chooses the Hartle Hawking prescription for the wave-function, but is exponentially sub-dominant for the Linde-Vilenkin prescription.
I. LARGE INITIAL UNIVERSE

In a surprising result, Hartle and Hawking twenty years ago showed 1 how, at least in the semi-classical theory of quantum gravity, one could give an intuitively appealing definition for a preferred wave function for the universe. If one believes that nature selects this definition, then they argued that this wave function explains various features which we observe the universe to have. One of these features is that the universe is simple when it is small 2 and becomes more and more complex as it grows in size. While Hawking originally believed that the universe must always be simple when small 3, Page and Laflamme 4 pointed out that this wave-function would be expected to also contain probabilities for the universe to be complex when small. This was interpreted as indicating a classical history of the universe which began simple when small, expanded, with the gravitational instability causing growth in the quantum fluctuations present when small. The universe would eventually re-collapse with the fluctuations continuing to grow. If one wrote the HH wave-function in terms of a basis which contained such semi-classical components, one would interpret them to say that the thermodynamic arrow of time would be driven by these fluctuations and that one should interpret those solutions such that the low fluctuation end should be interpreted as the beginning and the high as the end of the evolution.

However, there is an alternative interpretation of the HH wave-function (see for example Jheeta and Unruh 5). The prescription bears a strong resemblance to the prescription of a semi-classical WKB wave-function near a classical turning point of a potential. The classical solution for the HH models is that of a De Sitter space with the the cosmological constant dominating the evolution – ie the classical solution looks like De Sitter space in spherical coordinates near the point where the spherical space reaches it minimum size.

Interpreted in this way there is another place where the universe could and does bounce; namely if a closed universe has matter such that the energy density of the matter drops as the universe expands. In this case, the curvature terms in the FRW equations eventually dominates and halts the expansion and the universe re-collapses. The question which we address is whether the HH prescription could be used to place boundary conditions at this bounce point rather than at the point where the universe achieves it minimum size?

We will examine a simple model, with a homogeneous scalar field driving the dynamics of the universe. The space-time will be assumed to be a closed spherical homogeneous and isotropic FRW cosmology with metric

$$ds^2 = N^2dt^2 - a(t)^2 \left(dr^2 + \sin(r)^2 \left(d\theta^2 + \sin(\theta)^2 d\phi^2\right)\right)$$  \hspace{1cm} (1)

The mini-superspace action for this space-time becomes

$$S = \int Na^3\left(-6\frac{\dot{a}^2}{N^2a^2} + \frac{6}{a^2} + \frac{1}{2N^2}\dot{\phi}^2 - V(\phi)\right) dt$$  \hspace{1cm} (2)

which has as semi-classical equations of motion

$$\frac{d}{dt} \frac{a^3 d\phi}{N dt} + Na^3 \frac{dV(\phi)}{d\phi} = 0$$  \hspace{1cm} (3)

$$12a \frac{d}{N dt} \frac{da}{N dt} + 6 \frac{da}{N dt}^2 + 6 + 3a^2\left(\frac{1}{2N^2} \dot{\phi}^2 - V\right) = 0$$  \hspace{1cm} (4)
\[
\left( \frac{\dot{a}^2}{2N^2a^2} + \frac{6}{a^2} - \frac{1}{2N^2\dot{\phi}^2} - V(\phi) \right) = 0 \tag{5}
\]

where \( \dot{\cdot} = \frac{d}{dt} \). (The third is the constraint equation, and is an integral of motion of the other two).

The semi-classical HH prescription is that the path integral will be dominated by a solution of the equations which has only a final boundary given by the classical state on which one wishes to evaluate the wave function, i.e., the single boundary is defined by the values of \( a \) and \( \phi \) (or \( \dot{\phi} \)) which one wants on the final boundary. We must find solutions to these equations which have no initial boundary— which have \( a(t) \) going to zero and \( \phi(t) \) regular as \( a(t) \) approaches zero. This is impossible if \( N \) (and \( a \)) are real. We can however find such solutions if \( N \) is chosen to be complex, and in particular if we choose \( N \) to become purely imaginary as \( a \) approaches zero. (For a generalisation to arbitrary complex \( N \) see Jheeta and Unruh \[5\]). One choice, if it works, is to take \( N \) and \( a \) to be purely real over some range of times (the Lorentzian regime), and then where \( a(t) \) has zero derivative, to take \( N \) to be purely imaginary. This allows one to smoothly join \( a(t) \) across the surface where \( N \) discontinuously.

However, in general, both \( \dot{a} \) and \( \dot{\phi} \) cannot be taken to be zero at the same time \( t \) and still find a solution with only a single boundary. Since the equations of motion for \( a \) and \( \phi \) are smooth second order equations, the derivatives \( \frac{d}{dt} \) must also be smooth. While we can keep \( a(t) \) real across this boundary, since going from real zero to imaginary zero is a continuous transformation, we cannot do the same for \( \phi \). Thus, \( \phi \) must thus become complex as one crosses this boundary.

We are going to try to model the universe such that at time \( t = 0 \) we have \( \dot{a} = 0 \), and for \( t > 0 \), the universe contracts from the value of \( a \) at this time. I.e., we take \( t = 0 \) to be the time at which the universe achieves its maximum value in the Lorentzian regime.

Examining the constraint equation for \( a \), let us examine this boundary where \( a(t) = 0 \). Since \( \dot{a} \) is, by assumption less, than zero (the universe is contracting) for larger values of \( t \), the potential term must be sub-dominant. The curvature term \( 6a \) cancels with the kinetic energy. If \( V \) were to dominate, then \( a^2V \) decreases as \( a \) decreases, and \( \dot{a}^2 \) would become imaginary. Thus, in order that we have such a maximum in the radius of the universe, the term \( a^3\dot{\phi}^2 \) must be dominating the energy, and \( \dot{\phi}^2 \) must be decreasing at least as fast as \( 1/a^2 \)

But for very small \( V \) this is precisely how \( \dot{\phi} \) is doing. (For small \( \frac{dV}{d\phi} \), \( \phi \) obeys

\[
\frac{d}{dt} a^3 \frac{d\phi}{Ndt} \approx 0 \tag{6}
\]

or

\[
\frac{d\phi}{Ndt} \approx \frac{1}{a^3} \tag{7}
\]

I.e., in order that the universe re-collapse, the potential term cannot dominate the evolution of the universe—it must be dominated by the kinetic energy of the matter.

Now, \( \frac{1}{N} \dot{\phi} \) must be continuous across that boundary where \( N \) goes from real to imaginary. Since we are taking \( N \) to be purely imaginary for \( t < 0 \), \( \dot{\phi} \) must also be imaginary for times just before zero. I.e, \( \phi \) must pick up an imaginary part.

Let us now examine the evolution of this model in this realm where \( N \) is purely imaginary. Let us assume that the potential \( V(\phi) \) is such that the potential remains real for \( \phi = \phi_0 + i\Phi \).
where \( \phi_0 \) is the value of \( \phi \) at the point where \( \dot{a} = 0 \). Since \( N \) is arbitrary, we will take \( N = 1 \) for \( t > 0 \) and \( N = i \) for \( t < 0 \). The equation of motion for \( \Phi \) will be

\[
\frac{d}{d\tau}a^3 \frac{d\Phi}{dt} = \frac{dV}{d\Phi} \tag{8}
\]

and for \( a \), the constraint equation gives

\[
\left( \frac{da}{dt} \right)^2 = -1 + \frac{1}{6}a^2 (\frac{1}{2} \frac{d\Phi^2}{dt} + V(\Phi)) \tag{9}
\]

The second order Einstein equation for \( a \) shows that through \( t = 0 \), \( \ddot{a} \) reverses sign. Thus, while the universe contracts toward the future in the Lorentzian regime, it must expand into the past at this turning point.

Thus \( \frac{da}{dt} \) is less than zero for \( t < 0 \) near 0, and the universe expands as go to negative \( t \). This raises the question of how one could get \( a \) to go to zero in this \( N \)-imaginary region. The kinetic energy which dominates is driving the universe larger. The answer is that as \( a \) increases, \( \dot{\Phi} \) keeps decreasing. If the potential \( V(\Phi) \) remains non-zero, eventually it will dominate once again. Once it dominates, the potential term will keep increasing until it becomes of the same order as the curvature term (the 1 in the equation for \( \dot{a} \)) and \( a \) has another turning point. The universe will then collapse as a function of \( t \), the kinetic term will again grow, and the danger is that one will hit another turning point. If however the potential is arranged so that during this collapse, the potential increases, and thus \( \dot{\Phi} \) decreases, one can arrange the system so that \( a \) can collapse all the way to zero smoothly.

Let us examine a specific model. We take the potential to be

\[
V(\phi) = V_0 + \alpha \phi^4. \tag{10}
\]

We are going to assume that at \( t=0 \), \( \phi(0) = \phi_0 = 0 \), \( a(0) \) is large, and \( \dot{a}(0) = 0 \). We will assume that \( V_0 \) is smaller than \( 1/a(0)^2 \) and thus \( \dot{\phi}^2 \approx \frac{12}{a(0)^2} \).

If the solution is such that \( a \) approaches zero smoothly at \( t = t_0 \) (so that \( \Phi \) and \( \dot{\Phi} \) are both finite there) then near \( t = t_a \) we have

\[
\Phi(t) \approx \Phi(t_0)(1 - \frac{\alpha}{8}(t - t_0)^2 + \ldots) \tag{11}
\]

\[
a(t) = (t - t_0)(1 - (V_0 - \frac{\alpha}{4}\Phi(t_0)^4)(t - t_0)^2 + \ldots) \tag{12}
\]
The evolution of $\phi$ and $a$ in time in the $N = i$ regime. We have $V_0 = 10^{-8}$, $\alpha$ the quartic coefficient is $2 \times 10^{-11}$. The value of $a$ at $t = 0$ is carefully tuned.

In figure 1 we plot a solution with $V_0 = 10^{-8}$ and $\alpha = 2 \times 10^{-11}$. The lower axis is Note that the field $\Phi$ approaches a constant for most of the evolution.

If the potential in the imaginary direction is sufficiently flat for a large enough value of $\Phi$, the universe can oscillate between the the kinetic $\dot{\Phi}$ dominated bounce at smaller values of $a$ and the potential dominated one at large values of $a$ for many bounces. In Figure 2 we have a solution with $V_0 = 10^{-8}$ and $\alpha = 10^{-13}$, where we have three bounces before the radius eventually goes to zero.

In the $N$ real regime, the potential $a$ will decrease, and the kinetic dominated term will grow roughly as $\dot{\phi} = 1/a^3$. If the kinetic energy dominates, the solution for $a$ goes as $t^1/3$, and $\phi$ grows logarithmically. If the potential in the real $\phi$ direction grows, it may eventually dominate the matter again, and the universe would suffer a De Sitter like bounce at small values of $a$.

Figure 2 is the solution for $N$ and $\phi$ real into the past from the large radius bounce which matches to the solution for Figure 1. In this particular case, because the potential eventually grows so rapidly with $\phi$, the smaller bounce radius is not that much smaller than the largest size. However this depends entirely on the behaviour of the potential for large real $\phi$. Thus, if our potential were say $V_0 \cos(\phi(t)/\phi_a)$, with $\phi_a$ very large, our potential along the real $\phi$ axis would remain bounded by $\phi_0$, and the dynamics of $a$ would be kinetic dominated for all times to the past of the large bounce. In the imaginary direction, $V(\phi)$ would have the required confining shape and would, once $\phi$ had grown to of order $\phi_a$, allow for the radius to go smoothly to zero. In the real $\phi$ and $N$ regime, the solution would be kinetic dominated all the way to a singularity.
Here alpha is taken as $10^{-13}$, with the rest of the parameters the same as in figure 1. The initial value of $\phi$ is carefully tuned to give the multiple oscillations for $a$.

In the spirit of HH, we would interpret this universe as beginning in the Euclidean regime with $a = 0$ and $\phi = i \Phi$ and $\dot{\phi}$ regular. The universe would ”expand” and bounce between the potential and kinetic dominated bounce points for as long as desired, until it suddenly makes the transition to real $N$ and real $\phi$ at the lower kinetic dominated bounce point. The universe would then collapse into the future, until finally, either the potential dominated and the universe suffers a De Sitter, inflationary type bounce, or until the the universe finally collapses to a singularity.

II. SIGN OF EUCLIDEAN ACTION

Since these solutions are possible, how would they enter into the path integral for the “wave function of the universe”? There has of course been much heat expended on the various schools as to the sign one should take for the action in the Euclidean regime. These amount to either a choice of sign for $N$ in the imaginary regime, or a choice in the sign of $a$ (since $a$ enters only quadratically in all expressions of physical relevance). Thus the contributions of these imaginary $N$ parts of the action will contribute either $e^S$ or $e^{-S}$ where $S$ is the integrated action for the imaginary $N$ regime (or is the imaginary part of the action in the case of a generic complex metric and field dominating the action). For our purposes, this will be the value of

\[ S_I = \int \left( -6a + aa^{\dot{a}} + a^3 \left( \frac{1}{2} \Phi^2 + V(I\Phi) \right) \right) dt \]  

(13)

During the bouncing, all the terms are of the same order as the first term, and we have

\[ S_I \propto \int a dt \]  

(14)

By hypothesis both $a$ and $t$ are large, and furthermore, $t$ in general will be of the same order as $na$, where $n$ is the number of bounces in the imaginary $N$ regime, making $S_I$ of order of
na^2. a here will be of order its maximum where \( a^2 \propto \frac{1}{V_0} \). Ie, the action will be of order \( \frac{V}{V_0} \). For small \( V_0 \), and for the positive choice of the \( S \) in the action, this would completely dominate the path integral. The theory would predict with overwhelming probability that the universe began large and then collapsed down to a small size, rather than the more traditional way around. Furthermore, the fluctuations would grow during the collapse phase. The universe would grow more and more chaotic as its size decreased, with the simplest being when the universe was at its largest diameter. This sign is the sign usually attributed to Hartle and Hawking [1].

The alternative sign, usually attributed to Linde and Vilenkin [3], [4] (although derivable as simply one of two possibilities in defining the Hartle Hawking wave function) would on the other hand make the contributions of this particular solution to the semi-classical equations, completely sub-dominant.

One of the interesting features of this analysis is the realization that in order to maintain continuity of the scalar field equation through the transition from the real to imaginary \( N \), one must also allow \( \phi \) to become complex. In particular, if the field and scale factor are real in the real \( N \) regime, then the derivative of the field in general must be pure imaginary in the imaginary \( N \) regime. Thus, we note that the sign of the kinetic term \( a^3 \frac{1}{2N^2} \dot{\phi}^2 \) does not change on the transition from real to imaginary \( \phi \).

### III. FLUCTUATIONS

Let us examine the action for the inhomogeneous minimally coupled scalar field \( \psi \) in the geometry. The action for this field is

\[
S = \int Na^3 \left( \frac{1}{2N^2} \dot{\psi}^2 - \frac{1}{a^2} (\partial \psi)^2 \right) dtd\Omega^3
\]

(15)

where the \( d\Omega^3 \) is the spatial integral, and \( (\partial \phi)^2 \) represents the spatial derivative part of the action. Again at the surface where \( N \) makes the transition from real to imaginary, and \( \dot{a} \) is zero, \( \psi \) must again be complex to keep \( \dot{\psi}/N \) continuous. Ie, again we can write \( \psi = \psi_0 + i\Psi \). The action then becomes, taking \( N = i \),

\[
S_e = i \int a^3 \left( \frac{1}{2} \dot{\Psi}^2 + \frac{1}{a^2} (\partial \Psi)^2 \right) dtd\Omega^3
\]

(16)

Taking the semi-classical approximation to the path integral with the VL sign we have \( e^{iS_e} \) as the contribution to the wave-function. All the terms in the action are positive, indicating that we want to minimize the action to find the dominant terms. Using the apherical eigenstates for the spatial modes of the field we have the wave function factor as

\[
e^{-\int a^3 \left( \frac{1}{2} \dot{\Psi}^2 + \frac{k^2}{a^2} \Psi_k^2 \right) dt}
\]

(17)

near \( a = 0 \), \( \psi \) will again be linear in \( t \), and the solution for the field which is regular near \( t = t_a \) is

\[
\Psi \approx \Psi_0 (1 + \frac{1}{8} t^2 + ...)
\]

(18)

The action exponent can be written as

\[
\int a^3 \left( \frac{1}{2} \dot{\Psi}_k^2 + \frac{k^2}{a^2} \Psi_k^2 \right) dt = a^3 \Psi_k(t) \dot{\Psi}_k(t) \bigg|_{t_a=0}
\]

(19)
The lower limit is zero. As long as \( \frac{1}{a^2} \frac{d^2 a^3}{dt^2} > \frac{k^2}{a^2} \Psi_k \) will “slow roll” and

\[
\dot{\Psi}_k \approx \frac{k^2}{3a^2} \Psi_k \quad (20)
\]

while in the other limit,

\[
\dot{\Psi}_k \approx \left( \frac{k}{a} \right) \Psi_k \quad (21)
\]

Thus, at the transition point, the exponent will be proportional to \(-\Psi_k^2\) (ie quadratic) and will be concentrated near zero with a width proportional to either \(\sqrt{\frac{a}{k}}\) for short wavelengths, or \(\frac{\sqrt{\Pi}}{k}\) for long wavelengths.

Thus, for the large initial conditions, the universe will start in its vacuum state at the largest turning point of the theory.

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