Transport Properties of a Delta-Shell Gas with Long Scattering Lengths

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Contents

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- Scattering lengths & effective ranges
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- Transport coefficients (viscosity, thermal conductivity & diffusion)
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The Shrödinger equation

\[ \hat{H} = -\hbar^2 \frac{\Delta}{2\mu} - \nu \delta(r - a), \]

\[ \hat{H} \psi(r) Y_l m(\theta, \phi) = E \psi(r) Y_l m(\theta, \phi) \]

General solution:

\[ \psi(\rho) \equiv \frac{u(\rho)}{\rho} = A_l j_l(\rho) + B_l n_l(\rho) \]

Boundary conditions:

\[ E = \frac{\hbar^2 k^2}{2\mu}, \]
\[ \Lambda = \nu \frac{2\mu}{\hbar^2}, \]
\[ \rho = k r \]

\[ u(0) = 0 \]
\[ \psi(ka - 0) = \psi(ka + 0) \]
\[ \psi'(ka + 0) - \psi'(ka - 0) = -\frac{\Lambda}{k} \psi(ka) \]
Scattering and phase shifts

Partial wave phase shifts:
\[
\tan(\delta_l) = \frac{g x j_l^2(x)}{1 + g x j_l(x) n_l(x)}
\]

Important dimensionless variables:
\[
\begin{cases}
    x = k a \\
    g \equiv \Lambda a = \frac{2 \mu \nu}{\hbar^2 a}
\end{cases}
\]

Cross section:
\[
\sigma_l(k) \equiv 4\pi a^2 (2l + 1) \frac{\sin^2(\delta_l)}{x^2}
\]
\[
\sigma(k) = \sum_{l=0}^{\infty} \sigma_l(k)
\]
Scattering length and range

\[ k \cot(\delta_0) = -\frac{1}{a_{sl}} + r_0 \frac{k^2}{2} - Pr_0^3k^4 + O(k^6) \]

scattering length:

\[ \frac{a_{sl}}{a} = \frac{g}{g - 1}, \]

effective range:

\[ \frac{r_0}{a} = \frac{2}{3} \left(1 + \frac{1}{g}\right), \]

shape parameter:

\[ P = -\frac{3}{40} \frac{g^2(3 + g)}{(1 + g)^3}. \]

\[ k^{2l+1} \cot(\delta_l) = -\frac{1}{a^{(l)}_{sl}} + r_0^{(l)} \frac{k^2}{2} + O(k^4) \]

\[ \frac{a^{(l)}_{sl}}{a} = \left(\frac{(2l + 1)(2l + 2)}{(2l + 1)}\right)^2 \left(\frac{g}{g - (2l + 1)}\right), \]

\[ \frac{r_0^{(l)}}{a} = \frac{(2l + 1)(2l + 2)}{(2l + 3)(2l - 1)} \left(\frac{2l - 1}{g} - 1\right). \]
Bound states

\[ E = -\frac{\hbar^2 y_b^2}{2\mu a^2} \]

\[ f_l(y) \equiv -y j_l(iy) h_l(iy) = \frac{1}{g} \]
Virial Theorem for bound states with energy $E$

\[
\frac{\langle T \rangle}{\langle V \rangle} = \frac{gy^4}{N_l^2(y)} - 1
\]

\[E_\alpha = -\frac{\hbar^2}{2\mu a^2}\]

\[V(r) \propto r^n \rightarrow \frac{\langle T \rangle}{\langle V \rangle} = \frac{n}{2}\]

\[y^2 = \frac{E}{E_\alpha}\]

\[V \propto \delta(r - a)\]

\[n = -2\]
Deuteron:  

\[ a_s = 5.52 \text{ fm} \quad \text{and} \quad E = -2.2246 \text{ MeV} \]

\[ r_0 = 1.76 \text{ fm} \]
\[ r_{rms} = 1.97 \text{ fm} \]
\[ P = -0.007 \]  

Exp.

\[ a = 1.573 \text{ fm} \]
\[ g = 1.409 \]

**Delta-Shell**

**Square Well**
Some useful scales

Thermal de-Broglie wave length:

\[ \lambda(T) = \left( \frac{2\pi \hbar^2}{mk_B T} \right)^{1/2} \]

Delta-shell range: \( a \) \hspace{1cm} Dilution parameter: \( n a^3 \)

Delta-shell scattering length:

\[ a_{sl} = \frac{ag}{g - 1} ; \quad \text{Unitary limit: } g = 1 \]

Hard-sphere like transport (diffusion, viscosity & thermal conductivity) coefficients:

\[ \tilde{D} = \frac{3\sqrt{2}}{32} \frac{\hbar}{mna^3} , \quad \tilde{\eta} = \frac{5\sqrt{2}}{32} \frac{\hbar}{a^3} , \quad \text{and} \quad \tilde{\kappa} = \frac{75}{64\sqrt{2}} \frac{\hbar k_B}{ma^3} . \]
## Overview of transport properties

| Effect            | Flux of particle property | Gradient                        | Coefficient   | Law                      | Name of law       | Approximate expression for coefficient |
|-------------------|---------------------------|---------------------------------|---------------|--------------------------|-------------------|----------------------------------------|
| Diffusion         | Number                    | $\frac{dn}{dz}$                 | Diffusivity $D$ | $\mathbf{J}_a = -D \text{ grad } n$ | Fick’s law        | $D = \frac{1}{2} \bar{c} l$            |
| Viscosity         | Transverse momentum      | $M \frac{dv_x}{dz}$             | Viscosity $\eta$ | $\frac{F_x}{A} = J_p \times = -\eta \frac{dv_x}{dz}$ | Newtonian viscosity | $\eta = \frac{1}{2} \rho \bar{c} l$     |
| Thermal conductivity | Energy                    | $\frac{dp_a}{dz} = \bar{C}_v \frac{dT}{dz}$ | Thermal conductivity $K$ | $\mathbf{J}_a = -K \text{ grad } \tau$ | Fourier’s law     | $K = \frac{1}{2} \bar{C}_v \bar{c} l$    |
| Electrical conductivity | Charge                   | $-\frac{d\varphi}{dz} = E_z$   | Conductivity $\sigma$ | $\mathbf{J}_q = \sigma \mathbf{E}$ | Ohm’s law         | $\sigma = \frac{nq^2 l}{M \bar{c}}$     |

**Symbols:**
- $n =$ number of particles per unit volume
- $\bar{c} =$ mean thermal speed $= \langle |v| \rangle$
- $l =$ mean free path
- $\bar{C}_v =$ heat capacity per unit volume
- $\rho_a =$ thermal energy per unit volume
- $\mathbf{F}_s/A =$ shear force per unit area
- $\varphi =$ electrostatic potential
- $\mathbf{E} =$ electric field intensity
- $q =$ electric charge
- $M =$ mass of particle
- $\rho =$ mass per unit volume
- $\mathbf{p} =$ momentum

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“Thermal Physics” Ch. Kittel / H. Kroemer
Diffusion
\[ \Phi_{\text{red}} = -D \frac{dn_{\text{red}}}{dz} \]

Shear viscosity
\[ \frac{F}{A} = -\eta \frac{du}{dy} \]

Random walk of drunk sailor

Average displacement

Classical Newtonian laminar flow

Effective dragging force

Fast platform

Effective pushing force

Slow platform
Boltzmann equation for the particle distribution function: $f$ (phase space)

Equations of motion $\rightarrow$ time evolution $\rightarrow$ operator $\mathcal{D}$

$\mathcal{D}f = 0$: Liouville equation (no collisions)

Fundamental assumptions:
- Molecular chaos – no correlations
- Dilute system – encounters occur over a small fraction of molecular lifetime

$\mathcal{D}f = C(f_1 f)$: Boltzmann equation (binary collisions only)
The Boltzmann Equation

\[
\left( \frac{\partial}{\partial t} + \frac{p_1}{m} \cdot \nabla_r + F \cdot \nabla_{p_1} \right) f_1 = \int d^3p_2 \ d^3p'_1 \ d^3p'_2 \ \delta^4(P_f - P_i) \\
\times |T_{fi}|^2 (f'_2f'_1 - f_2f_1)
\]

- Nonlinear integro-differential equation for \( f_1 \)
- Except in rare cases, analytical solutions not available

The collision integral on the right hand side can be cast as

\[
C = \int d^3p_2 \ d\Omega \ |v_1 - v_2| \ \left( \frac{d\sigma}{d\Omega} \right) \ (f'_2f'_1 - f_2f_1)
\]

- Modifications due to Pauli supression or Bose enhancement can also be incorporated
Variables of hydrodynamics

Basic variables: \( f \equiv f(r, v, t) \)

\[
\langle A \rangle = \frac{\int d^3 p \; A f}{\int d^3 p \; f} \quad \text{(Expectation value of } A) \]

\[
v(r, t) = \langle v \rangle \quad \text{(Average velocity)}
\]

\[
\rho = m \int d^3 v \; f \quad \text{(mass density)}
\]

\[
\theta(r, t) = \frac{1}{3} m \langle \left| v - u \right|^2 \rangle \quad \text{(heat flux)}
\]

\[
P_{ij} = \rho \langle (v_i - u_i)(v_j - u_j) \rangle \quad \text{(Pressure tensor)}
\]
Dissipative hydrodynamic equations

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{(continuity)} \]

\[ \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = \frac{\mathbf{F}}{m} - \frac{1}{\rho} \left( P - \frac{\eta}{3} \nabla \cdot \mathbf{u} \right) + \frac{\eta}{\rho} \nabla^2 \mathbf{u} \]

\text{(Navier – Stokes equation)}

\[ \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \theta = -\frac{1}{c_v} (\nabla \cdot \mathbf{u}) \theta + \frac{\kappa}{\rho c_v} \nabla^2 \theta \]

\text{(Heat conduction)}
Enskog’s approximate solution of the Boltzmann equation

System is assumed to be only slightly disturbed from the equilibrium state $f^{(0)}$:

$$f = f^{(0)} + f^{(1)} + f^{(2)} + \ldots$$

Boltzmann equation: $F[f] = 0$

$$F[f] = F^{(0)}[f^{(0)}] + F^{(1)}[f^{(0)}, f^{(1)}] + F^{(2)}[f^{(0)}, f^{(1)}, f^{(2)}] + \ldots$$

$F^{(0)}[f^{(0)}] = 0 \rightarrow$ Maxwell distribution

$F^{(1)}[f^{(0)}, f^{(1)}] = 0 \rightarrow$ first approximation

$F^{(2)}[f^{(0)}, f^{(1)}, f^{(2)}] = 0 \rightarrow$ second approximation

$\ldots$
Transport integrals

Transport cross section

\[ \phi^{(n)} = 2\pi \int_{-1}^{+1} d\cos\theta (1 - \cos^n\theta) \frac{d\sigma(k, \theta)}{d\Omega} \Big|_{c.m.} \]

\[ q^{(1)} \equiv \frac{\phi^{(1)}}{4\pi a^2} = \frac{2}{x^2} \sum_l' (2l + 1) \sin^2(\delta_l), \]

\[ q^{(2)} \equiv \frac{\phi^{(2)}}{4\pi a^2} = \frac{2}{x^2} \sum_l' \frac{(l + 1)(l + 2)}{(2l + 3)} \sin^2(\delta_{l+2} - \delta_l), \]

\[ \omega^{(n,t)}(T) \equiv \int_0^\infty \gamma e^{-\gamma^2} \gamma^{2t+3} q^{(n)}(x) \]

\[ \gamma = \frac{\hbar k}{\sqrt{2\mu k_B T}} = \frac{x}{\sqrt{2\pi}} \left( \frac{\lambda(T)}{a} \right) \]
Analysis for particles with spin

\[
q^{(n)}_{(s)} = \frac{s + 1}{2s + 1} q^{(n)}_{\text{Bose}} + \frac{s}{2s + 1} q^{(n)}_{\text{Fermi}}, \quad \text{for integer } s,
\]
\[
q^{(n)}_{(s)} = \frac{s + 1}{2s + 1} q^{(n)}_{\text{Fermi}} + \frac{s}{2s + 1} q^{(n)}_{\text{Bose}}, \quad \text{for half-integer } s.
\]

Here, we will present results for the case of spin-1/2 particles only.
Shear viscosity

\[ \tilde{\eta} = \frac{5h}{32\sqrt{2\pi} a^3}, \]

\[ \frac{[\eta]_1}{\tilde{\eta}} = \left( \frac{a}{\lambda(T)} \right) \frac{1}{\omega^{(2,2)}(T)}, \]

\[ \frac{[\eta]_2}{[\eta]_1} = 1 + \frac{3(7 \omega^{(2,2)}(T) - 2 \omega^{(2,3)}(T))^2}{2 \left( \omega^{(2,2)}(T) (77 \omega^{(2,2)}(T) + 6 \omega^{(2,4)}(T)) - 6 (\omega^{(2,3)}(T))^2 \right)}, \]

and symmetry correction: \[ \times \left( 1 - n\lambda^3(T)\epsilon(T) \right) \]

- For constant cross sections, the \( \omega \)– integrals are \( T \)–independent; as a result \[ [\eta]_{1,2} \propto T^{1/2} \] as \[ \lambda(T) \propto T^{-1/2}. \]
Viscosity vs inverse scattering length

\[ \log_{10}(\eta/\eta_1/\tilde{\eta}) \]

\[ a/a_{sl} = (g-1)/g \]

Hard Spheres $g \rightarrow -\infty$
Asymptotic trends of viscosity

\[
\frac{\eta}{\tilde{\eta}} \to \begin{cases} 
\left(\frac{1-g}{g}\right)^2 \left(\frac{T}{\tilde{T}}\right)^{1/2} & \text{for } g \neq 1, 3 \\
6\pi \left(\frac{T}{\tilde{T}}\right)^{3/2} & \text{for } g = 1 \\
\frac{16}{111} \left(\frac{T}{\tilde{T}}\right)^{1/2} & \text{for } g = 3.
\end{cases}
\]

Characteristic temperature:

\[\tilde{T} \equiv \frac{2\pi \hbar^2}{k_B ma^2} \quad \text{or} \quad \frac{T}{\tilde{T}} = \left(\frac{a}{\bar{\lambda}}\right)^2\]
Self-diffusion

\[
\tilde{\mathcal{D}} = \frac{3h}{32\sqrt{2\pi}} \frac{1}{ma^3 n},
\]

\[
\frac{[\mathcal{D}]_1}{\tilde{\mathcal{D}}} = \left( \frac{a}{\lambda(T)} \right) \frac{1}{\omega^{(1,1)}(T)},
\]

\[
\frac{[\mathcal{D}]_2}{[\mathcal{D}]_1} = 1 + \frac{(5\omega^{(1,1)}(T) - 2\omega^{(1,2)}(T))^2}{\omega^{(1,1)}(T) (30\omega^{(1,1)}(T) + 4\omega^{(1,3)}(T) + 8\omega^{(2,2)}(T)) - 4(\omega^{(1,2)}(T))^2},
\]

and symmetry correction: \( \times \left( 1 - n\lambda^3(T)\varepsilon(T) \right) \)

- For constant cross sections, the \( \omega \)– integrals are \( T \)–independent; as a result \( [\mathcal{D}]_{1,2} \propto T^{1/2} \) as \( \lambda(T) \propto T^{-1/2} \).
Diffusion vs temperature

\[ \log_{10}\left(\frac{[D]\_1}{\tilde{D}}\right) \]

\[ \log_{10} \frac{T}{\tilde{T}} \]

asymptotic function

\[ g = -3 \]

\[ 3.2 \]

\[ 3 \]

\[ 1 \]
Asymptotic trends of diffusion

\[ \frac{\mathcal{D}}{\mathcal{D}} \to 2 \left( \frac{1 - g}{g} \right)^2 \sqrt{\frac{T}{\tilde{T}}} \quad \text{for} \quad g \neq 1, 3. \]

\[ \frac{\mathcal{D}}{\mathcal{D}} \to \begin{cases} 8\pi \left( \frac{T}{\tilde{T}} \right)^{3/2} & \text{for } g = 1 \\ \frac{6}{13} \left( \frac{T}{\tilde{T}} \right)^{1/2} & \text{for } g = 3 \end{cases} \]

Characteristic temperature:

\[ \tilde{T} \equiv \frac{2\pi \hbar^2}{k_B m a^2} \quad \text{or} \quad \frac{T}{\tilde{T}} = \left( \frac{a}{\chi} \right)^2 \]
Effective physical volumes

| g     | $mn\mathcal{D}$       | $\eta$         | $mn\mathcal{D}/\eta$ |
|-------|------------------------|----------------|----------------------|
| 1     | $\frac{3\sqrt{2}\pi}{4} \frac{\hbar}{\lambda^3}$ | $\frac{15\sqrt{2}\pi}{16} \frac{\hbar}{\lambda^3}$ | $\frac{4}{5} = 0.80$ |
| 3     | $\frac{9\sqrt{2}\pi}{104} \frac{\hbar}{\lambda a^2}$ | $\frac{5\sqrt{2}\pi}{111} \frac{\hbar}{\lambda a^2}$ | $\frac{999}{520} = 1.92$ |
| $\neq 1, 3$ | $\frac{3\sqrt{2}\pi}{8} \frac{\hbar}{\lambda a_{sl}^2}$ | $\frac{5\sqrt{2}\pi}{16} \frac{\hbar}{\lambda a_{sl}^2}$ | $\frac{6}{5} = 1.20$ |

Table 1: First order coefficients of diffusion (times $mn$), shear viscosity, and their ratios for $T \ll \tilde{T}$ for select $g$'s.
Diffusion to viscosity ratio

\[
\frac{mn [\mathcal{D}]}{[\eta]}_1
\]

1.5

\[
\frac{1.7 \times 10^{-3}}{4.5 \times 10^{-2}}
\]

6/5

\[
\frac{6}{5}
\]

1

\[
\frac{4}{5}
\]

0.5

\[
\frac{1}{\tilde{\tau}} = 1
\]

\[
a / a_{sl} = (g - 1) / g
\]

Hard Spheres

-1.5 -1.0 -0.5 0.5 1.0 1.5
Viscosity, $\eta$, to entropy density, $s$, ratio

- Is there a lower limit to $\eta/s$?

- First proposal: $\eta/s \geq (4\pi)^{-1} (\hbar/k_B)$.
  Kovtun, Son & Starinets (2005)

- Recent works indicate even lower limits!
  Brigante et al. (2008), Buchel et al. (2008),
  Kats & Petrov (2009)

- What does the dilute delta-shell gas yield?

- Is there anything deep in such a limit?
Entropy density of a dilute delta-shell gas

\[
  s = (5/2 - \ln(n\lambda^3) + \delta s(T) \, n\alpha^3) \, n k_B ,
\]

\[
  \delta s(T) = \left( \frac{a_2(T)}{2} - T \frac{d a_2(T)}{dT} \right) \left( \frac{\lambda}{\alpha} \right)^3 ,
\]

The second virial coefficient (that includes interactions)

\[
a_2(T) = \mp 2^{-5/2} - 2^{3/2} \sum_{l} (2l + 1)
\]

\[
  \times \left( e^{-E_l/(k_B T)} + \frac{1}{\pi} \int_{0}^{\infty} dx \, \frac{\partial \delta_l}{\partial x} \, e^{-\xi(T)x^2} \right) ,
\]

where the prime indicates summation over even \( l \)'s for Bosons (-) and odd \( l \)'s for Fermions (+), \( E_l \) is the energy of the bound state with angular momentum \( l \) and \( \xi(T) = (\lambda/\alpha)^2/(2\pi) \).
Viscosity to entropy density ratio
Lessons learned from the delta-shell gas

- Our analysis is restricted to the dilute gas limit, in which two particle interactions dominate but with scattering lengths that can take various values including infinity.

- Even at the two-body level, a rich structure in the temperature dependence and the effective physical volume responsible for the overall behavior of the transport coefficients are evident.

- The role of resonances in reducing the transport coefficients are amply delineated.

- Improved estimates of $\eta$ and $s$ have large roles on the ratio $\eta/s$!

- In the dilute gas limit, $\eta/s$ for the delta-shell gas remains above $(4\pi)^{-1}\hbar/k_B$.

- Matching our results to those of intermediate and extreme degeneracies which highlight the additional roles of superfluidity and superconductivity reveals the extent to which many-body effects play a crucial role.
