Bohr–Fourier Series on Solenoids via its Transversal Variation

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Abstract
The Bohr–Fourier series development on one-dimensional solenoids is described using invariant functions and extending Bohr’s theory through the study of transversal variation.

Keywords Solenoids · Fourier series · Invariant functions

1 Introduction

Solenoids were introduced by van Dantzig in the years 1930s, and they are an important class of groups which has been the subject of intense study during recent years, since they appear in many different contexts such as topological groups, abstract harmonic analysis, and dynamical systems, among other areas (see, e.g., [4,6,7], and [9] for a dynamical description of solenoids).

In this article, the Bohr–Fourier series development for functions on one-dimensional solenoids is done following the path traced by H. Bohr in his celebrated theory of Almost periodic functions (see [1]). Several attempts have been made to describe the theory of Fourier series on solenoids, most notably the work [3], where the authors used characters on solenoids composed with trigonometric polynomials on the unit circle to describe the theory.

Our approach uses invariant functions on an appropriate ‘covering’ space of the so-called one-dimensional universal solenoid $S$. By character theory (see, e.g., [8]),
one-dimensional solenoids are dual groups of additive subgroups of the group of rational numbers \( \mathbb{Q} \) with the discrete topology, and hence, they are homomorphic images of \( S \), which is the dual group of \( \mathbb{Q} \). The group \( S \) is a compact Abelian topological group isomorphic to the projective limit

\[
S \cong \lim_{\rightarrow n} \{ \mathbb{R}/n\mathbb{Z}, p_n \}
\]

of the projective system of unit circles \((\mathbb{R}/n\mathbb{Z})_{n \geq 1}\) with bonding maps \( p_n : z \mapsto z^n \) for each \( n \geq 1 \). The canonical projection \( S \rightarrow \mathbb{R}/\mathbb{Z} \), determined by projection onto the first coordinate, gives a locally trivial \( \hat{\mathbb{Z}} \)-bundle structure \( \hat{\mathbb{Z}} \hookrightarrow S \rightarrow \mathbb{R}/\mathbb{Z} \). Here, \( \hat{\mathbb{Z}} \) is the compact Abelian topological group called the profinite completion of \( \mathbb{Z} \). It is defined as the projective limit

\[
\hat{\mathbb{Z}} = \lim_{\rightarrow n} \{ \mathbb{Z}/n\mathbb{Z}, \rho_n \},
\]

where the subgroups \((n\mathbb{Z})_{n \geq 1}\) are partially ordered by divisibility and the surjective homomorphisms \((\rho_n)_{n \geq 1}\) are restrictions of residual classes. Since \( \hat{\mathbb{Z}} \) is also perfect and totally disconnected, it is homeomorphic to the Cantor set. Being \( \hat{\mathbb{Z}} \) the profinite completion of \( \mathbb{Z} \), it admits a canonical inclusion of \( \mathbb{Z} \) of which image is dense.

In the classical theory over the unit circle, it is very well known that there exists a one-to-one correspondence between the set

\[
\{ \mathbb{Z} - \text{invariant functions} \mathbb{R} \rightarrow \mathbb{C} \}
\]

and

\[
\{ \text{Continuous functions} \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C} \},
\]

via the universal covering map \( \pi : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \). In order to describe an analogous theory over \( S \), we introduce a convenient ‘covering’ space of \( S \) in the following way.

The group \( \mathbb{Z} \) acts freely on \( \mathbb{R} \) by covering transformations and on its profinite completion \( \hat{\mathbb{Z}} \) by right translations. This induces a properly discontinuous free action

\[
\mathbb{Z} \times (\mathbb{R} \times \hat{\mathbb{Z}}) \rightarrow \mathbb{R} \times \hat{\mathbb{Z}}, \quad (\gamma, (x, t)) \mapsto \gamma \cdot (x, t)
\]

given by

\[
\gamma \cdot (x, t) := (x + \gamma, t - \gamma) \quad (\gamma \in \mathbb{Z}).
\]

The orbit space of this action is isomorphic to the universal solenoid, i.e., \( S \cong \mathbb{R} \times_{\mathbb{Z}} \hat{\mathbb{Z}} \). Using the quotient map \( \mathbb{R} \times \hat{\mathbb{Z}} \rightarrow S \), it is established a one-to-one correspondence between

\[
\{ \mathbb{Z} - \text{invariant functions} \mathbb{R} \times \hat{\mathbb{Z}} \rightarrow \mathbb{C} \}
\]
Once this context is settled, the inspiration is Bohr’s treatment. The extension of Bohr’s notion of mean value is introduced for this class of functions (see Sect. 4): for any \( \mathbb{Z} \)-invariant function, \( \Phi : \mathbb{R} \times \hat{\mathbb{Z}} \rightarrow \mathbb{C} \), the mean value of \( \Phi \) is defined as follows:

\[
M(\Phi) := \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \int_{\hat{\mathbb{Z}}} \Phi(x, t) dx dt,
\]

whenever this limit exists with respect to the Haar product measure. Using this mean value, the Bohr–Fourier transform of \( \Phi \) is

\[
\hat{\Phi}(\chi_{\xi}, \varrho) := M(\Phi(x, t) \chi_{\xi, \varrho}(x, t)),
\]

where \( \chi_{\xi, \varrho} \) is an element of the character group \( \text{Char}(\mathbb{R} \times \hat{\mathbb{Z}}) \), which, by duality, is isomorphic to the product group \( \mathbb{R} \times \mathbb{Q}/\mathbb{Z} \), where \( \mathbb{Q}/\mathbb{Z} \) is the group of roots of unity.

Now, realizing the product space \( \mathbb{R} \times \hat{\mathbb{Z}} \) as a trivial foliated space, the base leaf \( L_0 \) is identified with \( \mathbb{R} \times \{0\} \). By continuity of the transversal variation \( \hat{\mathbb{Z}} \rightarrow \text{C} \text{lp}(\mathbb{R}) \), to any \( t \in \hat{\mathbb{Z}} \) there corresponds a limit periodic function \( \Phi_t : \mathbb{R} \rightarrow \mathbb{C} \), where \( \text{C} \text{lp}(\mathbb{R}) \) is the set consisting of all these limit periodic functions. When \( t = 0 \), the function \( \Phi_0 \) is the corresponding function on the base leaf \( L_0 \). If \( \chi_\xi \) is any character on \( \mathbb{R} \) and \( \hat{\Phi}_0(\chi_\xi) = M(\Phi_0 \cdot \chi_\xi) \) denotes the classical Bohr–Fourier coefficient of \( \Phi_0 \), the continuous variation on the transversal leads us to prove

**Theorem 9:**

\[
\hat{\Phi}(\chi_{\xi}, \varrho) = \hat{\Phi}_0(\chi_{\xi}) \cdot \int_{\hat{\mathbb{Z}}} A_\xi(t) \cdot \varrho(t) dt,
\]

where \( A_\xi : \hat{\mathbb{Z}} \rightarrow \mathbb{R}/\mathbb{Z} \) defines a character on \( \hat{\mathbb{Z}} \) determined by the transversal variation (see Sect. 4 for details).

The Bohr–Fourier series of \( \Phi \) can now be written as follows:

\[
\Phi(x, t) = \sum_{(\xi, \varrho) \in \Omega_\Phi} \hat{\Phi}(\xi, \varrho) \chi_{\xi, \varrho}(x, t),
\]

where \( \Omega_\Phi \) is a countable subset of characters in \( \mathbb{R} \times \mathbb{Q}/\mathbb{Z} \).

Denote by \( \text{C}(\mathbb{S}) \) the set consisting of all \( \mathbb{Z} \)-invariant functions \( \Phi : \mathbb{R} \times \hat{\mathbb{Z}} \rightarrow \mathbb{C} \). Then Parseval’s identity holds

**Theorem 10:** For any \( \Phi \in \text{C}(\mathbb{S}) \),

\[
\sum_{(\xi, \varrho) \in \Omega_\Phi} |\hat{\Phi}(\xi, \varrho)|^2 = M(|\Phi|^2).
\]
The Approximation theorem follows:

**Theorem 12**: Any \( \Phi \in C(S) \) can be approximated arbitrarily by the partial sums of its Fourier series.

Further development of the theory presented here goes in two different directions: on the one hand, the full generalization to the \( L^p \) theory would be possible and, on the other, the extension of these ideas to the so-called Sullivan’s solenoidal manifolds is propitious since, according to Sullivan (see [10] and [11]), any compact one-dimensional orientable solenoidal manifold is the suspension of a homeomorphism of the Cantor set. All these themes are the subject of future research.

Section 2 presents the relevant definitions on solenoids, characters, and measures. In Sect. 3, there is a brief account of the most relevant facts to this article of the classical Bohr’s theory. Section 4 is dedicated to the description of basic ingredients of the solenoidal theory and in Sect. 5, the Bohr–Fourier series is described and compared to the classical Fourier series on \( S \), reminiscent of the series on an arbitrary compact Abelian group.

### 2 The Universal Solenoid

This section introduces the basic concepts relevant to this article: solenoids, character groups, and measures. The universal solenoid is exhibited as an orbit space, as a projective limit, and as a quotient group. The basic definitions and examples of duality theory are sketched, and the required ingredients from measure theory are introduced. A complete account of these concepts and properties can be found in the treatise [4]. More specific descriptions of most of the objects presented here can be consulted in the recent article [2].

#### 2.1 The Universal Solenoid

For every integer \( n \geq 1 \), the map \( p_n : S^1 \to S^1 \) given by \( z \mapsto z^n \) is an unbranched covering space of degree \( n \) of the unit circle \( S^1 \), identified with \( \mathbb{R}/\mathbb{Z} \) via the exponential map. If \( n, m \in \mathbb{Z}^+ \) and \( n \) divides \( m \), then there exists a unique covering map \( p_{nm} : S^1 \to S^1 \) such that \( p_n \circ p_{nm} = p_m \). This determines a projective system of covering spaces \( \{ S^1, p_n | n \geq 1 \} \) of which projective limit is the universal one-dimensional solenoid

\[
S := \lim_{\overset{\to}{n}} \{ S^1, p_n \}.
\]

\( S \) is a closed subset of the infinite torus \( \prod_{n \geq 1} S^1 \). With respect to the restricted topology, \( S \) is a compact Abelian Hausdorff topological group. The projection onto the first coordinate \( S \to S^1 \) produces a locally trivial \( \hat{\mathbb{Z}} \)-bundle structure \( \hat{\mathbb{Z}} \to S \to S^1 \), where \( \hat{\mathbb{Z}} \) is the profinite completion of \( \mathbb{Z} \), introduced before.
By considering the properly discontinuously free action of \( \mathbb{Z} \) on \( \mathbb{R} \times \hat{\mathbb{Z}} \) given by
\[
\gamma \cdot (x, t) := (x + \gamma, t - \gamma) \quad (\gamma \in \mathbb{Z}),
\]
\( S \) is identified with the orbit space \( \mathbb{R} \times \hat{\mathbb{Z}} \). Here, \( \mathbb{Z} \) is acting on \( \mathbb{R} \) by covering transformations and on \( \hat{\mathbb{Z}} \) by translations. The path-connected component of the identity element \((0, 0) \in \mathbb{R} \times \hat{\mathbb{Z}}\) is called the base leaf, and it is denoted by \( \mathcal{L}_0 \). The image of \( \mathcal{L}_0 \) under the canonical projection \( \mathbb{R} \times \hat{\mathbb{Z}} \to S \) is the path-connected component of the identity element \( 0 \in S \).

### 2.2 Characters

For any locally compact Abelian group \( A \), the group of characters or dual group of \( A \) is the Abelian group consisting of all continuous homomorphism \( \text{Hom}_{\text{cont}}(A, \mathbb{S}^1) \) with the compact-open topology. This group is usually denoted by \( \hat{A} \), or by \( \text{Char}(A) \). By the classical theory, the group of characters of a compact Abelian group is a discrete Abelian group, and vice versa, the character group of a discrete Abelian group is a compact Abelian group. Also, the character group of a product of two Abelian groups is isomorphic to the product of the character groups.

The following examples and facts are relevant for this work and they are very well known:

1. \( \text{Char}(\mathbb{R}) \cong \mathbb{R} \),
2. \( \text{Char}(\hat{\mathbb{Z}}) \cong \mathbb{Q}/\mathbb{Z} \), where \( \mathbb{Q}/\mathbb{Z} \) is the group of roots of unity,
3. \( \text{Char}(\mathbb{R} \times \hat{\mathbb{Z}}) \cong \text{Char}(\mathbb{R}) \times \text{Char}(\hat{\mathbb{Z}}) \cong \mathbb{R} \times \mathbb{Q}/\mathbb{Z} \).

Statement (3) says that any character of \( \mathbb{R} \times \hat{\mathbb{Z}} \) has the form
\[
\chi_{\xi, \varrho} = \chi_{\xi} \cdot \chi_{\varrho},
\]
for some \( \xi \in \mathbb{R} \) and \( \varrho \in \mathbb{Q}/\mathbb{Z} \).

An important character group for this development is

**Remark 1** \( \text{Char}(S) \cong \mathbb{Q} \).

Classically, this isomorphism is deduced from the fact that there is an isomorphism of topological groups between the solenoid \( S \) and the so-called *adèle class group* of the rational numbers \( \mathbb{A}_\mathbb{Q}/\mathbb{Q} \), where \( \mathbb{A}_\mathbb{Q} \) is the adèle group of \( \mathbb{Q} \) and \( \mathbb{Q} \hookrightarrow \mathbb{A} \) is a discrete cocompact subgroup. However, for the purposes of this article, it is convenient to calculate the character group of \( S \) in an alternative way as follows.

The solenoid \( S \) can also be realized as the quotient group \( (\mathbb{R} \times \hat{\mathbb{Z}})/\mathbb{Z} \), where \( \mathbb{Z} \) is immersed diagonally as a discrete subgroup by
\[
\mathbb{Z} \hookrightarrow \mathbb{R} \times \hat{\mathbb{Z}}, \quad n \mapsto (-n, n).
\]

In order to be able to compute the dual group of a quotient group, the duality theory establishes the isomorphism
\[
\text{Char}( (\mathbb{R} \times \hat{\mathbb{Z}})/\mathbb{Z} ) \cong \text{Ann}(\mathbb{Z}),
\]
where $\text{Ann}(\mathbb{Z})$ is the annihilator subgroup of $\mathbb{Z}$ in $\text{Char}(\mathbb{R} \times \hat{\mathbb{Z}})$. It happens that the characters in $\mathbb{R} \times \hat{\mathbb{Z}}$ which annihilate the generator $(-1, 1)$ of $\mathbb{Z}$ in the product group are precisely the characters determined by elements in $\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$.

By duality theory, the surjective homomorphism $\mathbb{R} \times \hat{\mathbb{Z}} \to S$ induces a monomorphism between the dual groups $\mathbb{Q} \to \mathbb{R} \times \mathbb{Q}/\mathbb{Z}$ whose image is isomorphic to the subgroup $\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$. This identification is very important in this article:

**Remark 2** There is a one-to-one correspondence between the discrete Abelian groups $\mathbb{Q}$ and $\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$.

### 2.3 Haar Measure

Denote by $dx$ the usual Haar measure on $\mathbb{R}$ and by $dt$ the Haar measure on $\hat{\mathbb{Z}}$ normalized in such a way that

$$\int_{\hat{\mathbb{Z}}} dt = 1.$$ 

So, the Haar measure on $\mathbb{R} \times \hat{\mathbb{Z}}$ is the product measure $dx \times dt = dx \, dt$, and it induces the normalized Haar measure $d\mu$ on $S$, i.e.,

$$\int_S \phi \, d\mu = \int_{\hat{\mathbb{Z}}} \int_{\mathbb{R}} \Phi \, dx \, dt,$$

for any lifting $\Phi : \mathbb{R} \times \hat{\mathbb{Z}} \to \mathbb{C}$ of $\phi : S \to \mathbb{C}$.

### 3 Classical Bohr’s Theory

This section is a brief summary of Bohr’s theory of almost periodic functions. We follow closely Bohr’s seminal work [1].

Let $C(\mathbb{R})$ be the space of complex-valued continuous functions on $\mathbb{R}$ with the uniform norm. Define the action by translations of $\mathbb{R}$ on $C(\mathbb{R})$ by

$$\mathbb{R} \times C(\mathbb{R}) \to C(\mathbb{R}), \quad (t, \phi) \mapsto \phi^t,$$

where $\phi^t : \mathbb{R} \to \mathbb{R}$ is given by $\phi^t(x) := \phi(x + t)$.

Denote by $\Omega_{\mathbb{R}}(\phi)$ the orbit of $\phi$ under this action and by $\text{Hull}(\phi)$ the closed convex hull of $\Omega_{\mathbb{R}}(\phi)$ in $C(\mathbb{R})$.

Given $\phi \in C(\mathbb{R})$ and $\varepsilon > 0$, the number $\tau = \tau(\varepsilon) \in \mathbb{R}$ is called a translation number of $\phi$ (corresponding to $\varepsilon$), whenever

$$\|\phi^{\tau(\varepsilon)} - \phi\|_{\text{sup}} \leq \varepsilon.$$

**Definition 1** $\phi \in C(\mathbb{R})$ is called almost periodic if given $\varepsilon > 0$, there exists a relatively dense set of translation numbers of $\phi$ corresponding to $\varepsilon$, i.e., for all $\varepsilon$, there exists a
length $L = L(\varepsilon)$ such that each interval of length $L$ contains at least one translation number corresponding to $\varepsilon$.

Denote by $\text{C}_{ap}(\mathbb{R})$ the complex vector space consisting of all almost periodic functions on $\mathbb{R}$.

**Example 1** Any periodic function is obviously an almost periodic function.

Some important properties of almost periodic functions are summarized in

**Properties 1** The following properties are satisfied:

a. If $\varphi \in \text{C}_{ap}(\mathbb{R})$, then $\varphi$ is a uniformly continuous function.

b. The sum of almost periodic functions is an almost periodic function.

c. The uniform limit of almost periodic functions is an almost periodic function.

Since the sum of periodic functions is an almost periodic function, particularly, trigonometric polynomials are almost periodic functions. An interesting observation is that every function $\varphi$ which can be approximated uniformly by trigonometric polynomials is an almost periodic function (see Theorem 4).

Our main interest is the subspace of all limit periodic functions $\text{C}_{lp}(\mathbb{R}) \subset \text{C}_{ap}(\mathbb{R})$, which consists of all functions $\varphi$ such that $\varphi$ is the uniform limit of periodic functions.

**Definition 2** For every almost periodic function, there exists the mean value

$$M(\varphi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi(x)dx.$$  

It is clear that if $\varphi \in \text{C}_{ap}(\mathbb{R})$ and $t \in \mathbb{R}$, then $\varphi^t \in \text{C}_{ap}(\mathbb{R})$, and therefore, there exists $M(\varphi^t)$.

**Theorem 2** [1, §50, pp. 39–44] $M: \text{C}_{ap}(\mathbb{R}) \to \mathbb{C}$ is a continuous linear functional which is invariant under right translations. That is,

a. $M(\varphi + \psi) = M(\varphi) + M(\psi)$, for any $\varphi, \psi \in \text{C}_{ap}(\mathbb{R})$.

b. $M(\varphi^t) = M(\varphi)$, for any $\varphi \in \text{C}_{ap}(\mathbb{R})$ and $t \in \mathbb{R}$.

c. If $\varphi$ is the uniform limit of a sequence $(\varphi_n)_{n \in \mathbb{N}}$ then

$$M(\varphi) = \lim_{n \to \infty} M(\varphi_n).$$

Now recall the concept of the Fourier series of an almost periodic function. A normalized orthogonal system $\{e^{2\pi i \xi x} : \xi \in \mathbb{R}\}$ satisfies the relation:

$$M(e^{2\pi i \xi_1 x} e^{-2\pi i \xi_2 x}) = \delta(\xi_1, \xi_2),$$

where $\delta(\xi_1, \xi_2) = 1$ if $\xi_1 = \xi_2$ and 0 in other case. The elements of this system are called basic elements, and this set can be identified with $\text{Char}(\mathbb{R})$.  

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Consider $\varphi \in \mathcal{C}_{ap}(\mathbb{R})$ and $\xi \in \mathbb{R}$. The function $\varphi(x)e^{-2\pi i \xi x}$ is the product of an almost periodic function and a purely periodic function, so it is an almost periodic function and its mean value

$$M(\varphi(x)e^{-2\pi i \xi x}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi(x)e^{-2\pi i \xi x} \, dx$$

exists. The next theorem is of fundamental importance for the theory.

**Theorem 3** ([1], §55, p. 48) The function $a(\xi) := M(\varphi(x)e^{-2\pi i \xi x})$ is zero for all values of $\xi$ with the exception of at most a countable set of numbers $\xi$. These values are called the Fourier exponents of the function $\varphi$.

This theorem allows us to carry the theory of Bohr–Fourier series into the theory of almost periodic functions, in the sense that it is possible to associate to an almost periodic function $\varphi$ its unique Bohr–Fourier series

$$\sum_{n \in \mathbb{N}} a(\xi_n)e^{2\pi i \xi_n x}.$$

**Remark 3** Parseval’s identity holds for any almost periodic function $\varphi$:

$$\sum_{n \in \mathbb{N}} |a(\xi_n)|^2 = M(|\varphi|^2).$$

The main result of the theory reads as follows.

**Theorem 4** ([1], §84, p. 81) Every almost periodic function $\varphi$ can be uniformly approximated by finite sums $S_N(x) = \sum_{n=1}^N a_n e^{2\pi i \xi_n x}$. The exponents in the approximating sums $S_N(x)$ can be chosen to be precisely the Fourier exponents $\xi_n$ of the function $\varphi$.

### 4 Solenoidal Bohr–Fourier Theory

This section presents the basic elements required for the development of the theory of the solenoidal Bohr–Fourier series. First, we will analyze the relevant spaces of continuous functions, both on $S$ and on $\mathbb{R} \times \hat{\mathbb{Z}}$, and the continuous variation of the functions with respect to the transversal variable. This allows us to define the appropriate notion of mean value and to describe its transversal variation.

#### 4.1 Continuous Invariant Functions on $S$

Recall that $C_{lp}(\mathbb{R})$ is the space of limit periodic functions $\mathbb{R} \to \mathbb{C}$ in the sense of Bohr. Let $C(S)$ be the space of continuous functions $\phi : S \to \mathbb{C}$. It is well known that there is a one-to-one correspondence between $C(S)$ and the space $C_{\mathbb{Z}}(\mathbb{R} \times \hat{\mathbb{Z}})$ of
continuous functions $\Phi : \mathbb{R} \times \hat{\mathbb{Z}} \rightarrow \mathbb{C}$ satisfying the invariant condition under the action of $\mathbb{Z}$ given by

$$\Phi(\gamma \cdot (x, t)) = \Phi(x + \gamma, t - \gamma) = \Phi(x, t), \quad ((x, t) \in \mathbb{R} \times \hat{\mathbb{Z}}, \gamma \in \mathbb{Z}).$$

**Remark 4** In order to develop the Bohr–Fourier theory for $C(S)$, we will work on the space $C_\mathbb{Z}(\mathbb{R} \times \hat{\mathbb{Z}})$, which after projection provides us with the Bohr–Fourier theory on $C(S)$ described at the end of Sect. 5. From now on, we will write $C(S)$ for both spaces.

For every $t \in \hat{\mathbb{Z}}$, the function $\Phi_t : \mathbb{R} \rightarrow \mathbb{C}$ defined by

$$\Phi_t(x) = \Phi(x, t)$$

is continuous. The invariant condition can be written as follows:

$$\Phi_{t-\gamma}(x + \gamma) = \Phi_t(x), \quad ((x, t) \in \mathbb{R} \times \hat{\mathbb{Z}}, \gamma \in \mathbb{Z}). \quad (1)$$

**Remark 5** According to Theorem 2.4 in [5], for every $t \in \hat{\mathbb{Z}}$, the function $\Phi_t : \mathbb{R} \rightarrow \mathbb{C}$ is limit periodic.

A nice consequence of this remark is the following:

**Proposition 1** For each $\Phi \in C(S)$, the map

$$\hat{\mathbb{Z}} \rightarrow C_{lp}(\mathbb{R}), \quad t \rightarrow \Phi_t$$

is uniformly continuous. That is, if $(t_n)_{n \geq 1}$ is a sequence of points in $\mathbb{Z} \subset \hat{\mathbb{Z}}$ which converges to $t \in \hat{\mathbb{Z}}$ in the profinite topology, then the sequence $(\Phi_{t_n})_{n \geq 1}$ in $C_{lp}(\mathbb{R})$ converges to $\Phi_t \in C_{lp}(\mathbb{R})$ in the uniform topology of $C_{lp}(\mathbb{R})$.

This Proposition implies that there is a one-to-one correspondence between the space $C(S)$ and $C(\hat{\mathbb{Z}}, C_{lp}(\mathbb{R}))$.

**Remark 6** As a matter of notation, it is important to notice that the function $\Phi_t$ does not correspond exactly with the usual definition of right translations on $C(\mathbb{R})$ which is denoted by $\Phi^t$ here. This notation emphasizes the dependence of $\Phi$ on the transversal variable. However, when $t = 0$ in $\hat{\mathbb{Z}}$, the invariant condition restricted to $\mathcal{L}_0$ implies that

$$\Phi_s^0(x) = \Phi_0(x + s)
= \Phi(x + s, 0)
= \Phi(x + s + (-s), 0 - (-s))
= \Phi(x, s)
= \Phi_s(x),$$
for any \( s \in \mathbb{Z} \) and \( x \in \mathcal{L}_0 \). Furthermore, for any \( t, s \in \mathbb{Z} \subseteq \hat{\mathbb{Z}} \) and \( x \in \mathcal{L}_t \) (the leaf through \( t \)), the relation above is written as follows:

\[
\Phi^s_t(x) = \Phi_t(x + s) = \Phi(x + s, t) = \Phi(x + s + (-s), t - (-s)) = \Phi(x, t + s) = \Phi_{t+s}(x).
\]

### 4.2 The Mean Value

For any function \( \Phi \in \mathbb{C}(S) \), the mean value of \( \Phi \) is given by

\[
\mathcal{M}(\Phi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\mathbb{Z}} \Phi(x, t) dx dt,
\]

whenever this limit exists.

**Theorem 5** \( \mathcal{M}(\Phi) = M(\Phi_t) \), for any choice of \( t \in \hat{\mathbb{Z}} \) fixed.

**Proof** If \( t, s \in \mathbb{Z} \), Remark 6 implies that \( \Phi_{t+s} = \Phi^s_t \). By translation invariance of the mean value (see Theorem 2(b)), it follows that

\[
M(\Phi_{t+s}) = M(\Phi^s_t) = M(\Phi_t) \quad (t \in \mathbb{Z}).
\]

Now, by what had been said before, if \( (t_n)_{n \geq 1} \) is a sequence of points in \( \mathbb{Z} \subseteq \hat{\mathbb{Z}} \) which converges to \( t \in \hat{\mathbb{Z}} \) in the profinite topology, then the sequence \( (\Phi_{t_n})_{n \geq 1} \) converges to \( \Phi_t \). By properties of the mean value (see Theorem 2(c)), \( M(\Phi_t) = \lim_{n \to \infty} M(\Phi_{t_n}) \).

This means that, for any \( t \in \hat{\mathbb{Z}} \) fixed, the mean value is constant and equal to \( M(\Phi_t) \) in \( \hat{\mathbb{Z}} \). Therefore,

\[
\mathcal{M}(\Phi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\mathbb{Z}} \Phi(x, t) dx dt = \int_{\mathbb{Z}} M(\Phi_t) dt = M(\Phi_t).
\]

\[\square\]

**Theorem 6** The invariant mean \( \mathcal{M} : \mathbb{C}(S) \to \mathbb{C} \) is a continuous linear functional which is invariant under right translations. That is,

\[\begin{align*}
a. \quad & \mathcal{M}(\Phi + \Psi) = \mathcal{M}(\Phi) + \mathcal{M}(\Psi), \text{ for any } \Phi, \Psi \in \mathbb{C}(S). \\
b. \quad & \mathcal{M}(\Phi^s) = \mathcal{M}(\Phi), \text{ for any } \Phi \in \mathbb{C}(S) \text{ and } s \in \mathbb{R}.
\end{align*}\]
c. If $\Phi$ is the uniform limit of a sequence $(\Phi_n)_{n\in\mathbb{N}}$, then

$$\mathcal{M}(\Phi) = \lim_{n\to\infty} \mathcal{M}(\Phi_n).$$

### 4.3 Bohr–Fourier Transform

Given any function $\Phi \in C(S)$ and any character $\chi_{\xi,\varrho} \in \text{Char}(\mathbb{R} \times \hat{\mathbb{Z}})$, the Bohr–Fourier transform of $\Phi$ in the mean sense is given by

$$\hat{\Phi}(\chi_{\xi,\varrho}) = \mathcal{M}(\Phi(x, t) \overline{\chi_{\xi,\varrho}(x, t)})$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{\mathbb{Z}} \int_{0}^{T} \Phi(x, t) \overline{\chi_{\xi,\varrho}(x, t)} \, dx \, dt.$$

In fact, we have the following:

**Theorem 7** If $\Phi$ is any function in $C(S)$ and $\chi_{\xi,\varrho}$ is any element in $\text{Char}(\mathbb{R} \times \hat{\mathbb{Z}})$, then

$$\hat{\Phi}(\chi_{\xi,\varrho}) = \int_{\mathbb{Z}} M(\Phi_t e^{-2\pi i \xi x}) \overline{\chi_{\varrho}(t)} \, dt.$$

**Proof**

$$\hat{\Phi}(\chi_{\xi,\varrho}) = \lim_{T \to \infty} \frac{1}{T} \int_{\mathbb{Z}} \int_{0}^{T} \Phi(x, t) \overline{\chi_{\xi,\varrho}(x, t)} \, dx \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{\mathbb{Z}} \int_{0}^{T} \Phi(x, t) \chi_{\xi}(x) \overline{\chi_{\varrho}(t)} \, dx \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{\mathbb{Z}} \int_{0}^{T} \Phi(x, t) e^{-2\pi i \xi x} \overline{\chi_{\varrho}(t)} \, dx \, dt$$

$$= \int_{\mathbb{Z}} \left[ \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \Phi_t(x) e^{-2\pi i \xi x} \, dx \right] \overline{\chi_{\varrho}(t)} \, dt$$

$$= \int_{\mathbb{Z}} M(\Phi_t e^{-2\pi i \xi x}) \overline{\chi_{\varrho}(t)} \, dt.$$

Since $\Phi_t$ is limit periodic for all $t \in \hat{\mathbb{Z}}$, Hull($\Phi_t$) is a quotient group of the solenoid (see [5], Theorem 2.2). By duality, Char(Hull($\Phi_t$)) is a (discrete) subgroup of the group Char(S) $\cong \mathbb{Q}$.

**Remark 7** The function $M(\Phi_t e^{-2\pi i \xi x})$ is zero for all values of $\xi$ with the exception of at most a countable subset $\Omega_{\Phi_t}$ of $\mathbb{Q}$ (Compare to Theorem 3).

Theorem 7 tells us that the study of the variation of $M(\Phi_t e^{-2\pi i \xi x})$ with respect to the transversal variable $t$ must be done.
4.4 Transversal Variation

First, fix $t = 0$, the identity element in $\hat{\mathbb{Z}}$. The function $\Phi_0 \in \text{C}_{lp}(\mathbb{R})$ is a limit periodic function defined on the base leaf $\mathcal{L}_0 = \mathbb{R} \times \{0\} \subset \mathbb{R} \times \hat{\mathbb{Z}}$. According to Bohr’s theory:

- The frequency module of $\Phi_0$ is a countable subset of rational numbers $\Omega_{\Phi_0} \subset \mathbb{R}$,
- the invariant mean $M(\Phi_0)$ defined as
  \[
  M(\Phi_0) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi_0(x)dx
  \]
  exists, and,
- the $\xi^{th}$ Fourier coefficient of $\Phi_0$,
  \[
  \hat{\Phi}_0(\xi) = M(\Phi_0(x)e^{-2\pi i \xi x}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi_0(x)e^{-2\pi i \xi x}dx,
  \]
is well defined.

Remark 8 Using the fact that $\mathbb{R}$ is self-dual, sometimes we will also write

\[
\hat{\Phi}_0(\chi_\xi) = M(\Phi_0(x)\chi_\xi(x)) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi_0(x)\chi_\xi(x)dx,
\]
emphasizing the use of the character $\chi_\xi \in \text{Char}(\mathbb{R})$ associated with $\xi$.

The Fourier series of $\Phi_0$ is written as follows:

\[
\Phi_0(x) = \sum_{\xi \in \Omega_{\Phi_0}} \hat{\Phi}_0(\xi)\chi_\xi(x) = \sum_{\xi \in \Omega_{\Phi_0}} \hat{\Phi}_0(\xi)e^{2\pi i \xi x}.
\]

Theorem 8 If $\Phi \in C(S)$ then

\[
M(\Phi_t e^{-2\pi i \xi x}) = A_\xi(t)M(\Phi_0(x)e^{-2\pi i \xi x}),
\]

where $A_\xi : \hat{\mathbb{Z}} \longrightarrow S^1$ is the character associated in $\text{Char}(\hat{\mathbb{Z}})$ to $\xi \in \mathbb{Q}$.

Proof The first part of Remark 6 implies that for any $t \in \mathbb{Z} \subset \hat{\mathbb{Z}}$, the identity $\Phi_0(x + t) = \Phi_t(x)$ holds for every $x \in \mathcal{L}_0$. The mean value of $\Phi_0$ is precisely invariant under these translations, i.e., $M(\Phi_0(x + t)) = M(\Phi_0(x))$. Hence,

\[
M(\Phi_t(x)e^{-2\pi i \xi x}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi_t(x)e^{-2\pi i \xi x}dx
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi_0(x + t)e^{-2\pi i \xi x}dx
\]
\[
e^{2\pi i \xi t} \lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi_0(x) e^{-2\pi i \xi x} \, dx = e^{2\pi i \xi t} M(\Phi_0(x) e^{-2\pi i \xi x}).
\]

This means that for any \( t \in \mathbb{Z} \subset \hat{\mathbb{Z}} \), the mean value \( M(\Phi_t(x) e^{-2\pi i \xi x}) \) is transformed into \( e^{2\pi i \xi t} M(\Phi_0(x) e^{-2\pi i \xi x}) \). This calculation, together with the continuous variation, can be used to determine the mean value \( M(\Phi_t(x) e^{-2\pi i \xi x}) \) for any \( t \in \hat{\mathbb{Z}} \).

Suppose that \( \xi = p/q \) where \((p, q) = 1\). Choose the neighborhood base \( \mathcal{B} \) of \( 0 \in \hat{\mathbb{Z}} \), where any element \( B_n = n! \mathbb{Z} \in \mathcal{B} \) is the ball centered at 0 with respect to the metric associated to the profinite topology. Given a sequence \( (t_n)_{n\in\mathbb{N}} \) in \( \mathbb{Z} \subset \hat{\mathbb{Z}} \) such that \( t_n \to t \), for any \( k \in \mathbb{N} \), there exists \( N \in \mathbb{N} \) large enough such that if \( m, n \geq N \), then \( t_n - t_m \in B_k \). Note that if \( k \geq q \), then \( q \mid k! \), and therefore, \( t_n \equiv t_m \mod q \). Hence, \( e^{2\pi i \xi (t_n - t_m)} = 1 \), which means that the values \( e^{2\pi i \xi t_n} \) are eventually constant.

Finally, it does not depend on the choice of the sequence \( (t_n)_{n\in\mathbb{N}} \), and they determine a continuous function that actually is a character. If we call \( A_\xi \) this map, we have established that

\[
M(\Phi_t(x) e^{-2\pi i \xi x}) = A_\xi(t) M(\Phi_0(x) e^{-2\pi i \xi x}).
\]

\[\Box\]

**Remark 9** Note that \( A_\xi \) can be written as follows:

\[
A_\xi(t) = \frac{M(\Phi_t(x) e^{-2\pi i \xi x})}{M(\Phi_0(x) e^{-2\pi i \xi x})} = \frac{\hat{\Phi}_t(\chi_\xi)}{\hat{\Phi}_0(\chi_\xi)}.
\]

The results proved before, Theorems 7, 8, and Remark 9, can be used to compute the Fourier transform of any function \( \Phi \in C(\mathbb{S}) \) in the following way: for any character \( \chi_{\xi, \Theta} \in \text{Char}(\mathbb{R} \times \hat{\mathbb{Z}}) \),

\[
\hat{\Phi}(\chi_{\xi, \Theta}) = \mathcal{M}(\Phi(x, t) \overline{\chi_{\xi, \Theta}(x, t)}) = \lim_{T \to \infty} \frac{1}{T} \int_{\mathbb{Z}} \int_0^T \Phi(x, t) \overline{\chi_{\xi, \Theta}(x, t)} \, dx \, dt = \int_{\mathbb{Z}} M(\Phi_t e^{-2\pi i \xi x}) \overline{\chi_{\Theta}(t)} \, dt \]

\[
= M(\Phi_0 \cdot \overline{\chi_\xi}) \cdot \int_{\mathbb{Z}} A_\xi(t) \cdot \overline{\chi_\Theta(t)}. 
\]

According to Remark 7, the mean value \( M(\Phi_0 \cdot \overline{\chi_\xi}) \) is zero for all values of \( \xi \) with the exception of at most a countable subset of \( \mathbb{Q} \).

The integral expression in the last equality is evaluated by integration of characters of \( \hat{\mathbb{Z}} \):

\[
\int_{\mathbb{Z}} A_\xi(t) \cdot \overline{\chi_\Theta(t)} \, dt = 1
\]

\[\Box\ Springer\]
if and only if \( A_\xi(t) = \chi_\varrho(t) \), and 0 in other case. Also, \( A_\xi \) and \( \chi_\varrho \) define the same character if and only if \( \varrho = \xi \mod \mathbb{Z} \).

So, the final form of the Fourier coefficient of any function \( \Phi \in C(S) \) is given in the following:

**Theorem 9**

\[
\hat{\Phi}(\chi_\xi, \varrho) = \hat{\Phi}_0(\chi_\xi) \cdot \int_{\mathbb{Z}} A_\xi(t) \cdot \overline{\chi_\varrho(t)} dt,
\]

where \( \hat{\Phi}_0(\chi_\xi) = M(\Phi_0 \cdot \overline{\chi_\xi}) \) when \( \varrho = \xi \mod \mathbb{Z} \), and 0 in other case. In particular, the mean value is zero except at most a countable set \( \Omega \Phi \simeq \Omega \Phi_0 \).

## 5 Solenoidal Bohr–Fourier Series

This final section describes the Bohr–Fourier series for a complex-valued continuous function \( \phi \) on the solenoid \( S \) through the associated continuous \( \mathbb{Z} \)-invariant function on \( \mathbb{R} \times \hat{\mathbb{Z}} \). It also presents the solenoidal version of Parseval’s identity and the Approximation theorem. Finally, this theory is compared to the classical theory on \( S \) viewed as a compact Abelian group.

### 5.1 Classical Fourier Series on \( S \)

According to the classical harmonic analysis on the compact Abelian topological group \( S \), given a continuous function \( \phi : S \rightarrow \mathbb{C} \), the Fourier series can be defined as follows:

\[
\sum_{q \in \mathbb{Q}} \hat{\Phi}(\chi_q) \chi_q(z),
\]

where \( \chi_q \) is the character of \( S \) associated to \( q \in \mathbb{Q} \) and

\[
\hat{\Phi}(\chi_q) = \int_S \phi(z) \overline{\chi_q(z)} dz.
\]

In what follows, we describe the corresponding Bohr–Fourier series through the theory developed previously.

### 5.2 Solenoidal Bohr–Fourier Series

Denote by \( \overline{\Phi} \) the Bohr–Fourier series associated to a given function \( \Phi \in C(S) \):

\[
\overline{\Phi}(x, t) = \sum_{(\xi, \varrho) \in \Omega \Phi} \hat{\Phi}(\xi, \varrho) \chi_{\xi, \varrho}(x, t).
\]
Remark 10 Since $\chi_{\xi, \varrho} = \chi_{\xi} \cdot \chi_{\varrho}$, when $t = 0$, $\chi_{\varrho}(0) = 1$ for any $\varrho$. By Theorem 9, it follows that $\hat{\Phi}(\xi, \varrho) = \hat{\Phi}_0(\xi)$ and in consequence $\Omega_{\Phi} \cong \Omega_{\Phi_0}$. This allows us to identify the Fourier series introduced here with the usual Bohr–Fourier series when restricted to the base leaf $L_0$:

$$\overline{\Phi}_0(x) = \sum_{\xi \in \Omega_{\Phi_0}} \hat{\Phi}_0(\xi) \chi_{\xi}(x).$$

Following the order of ideas presented by Bohr (see [1, Sects. 70 and 84]), the solenoidal version of the main results of Bohr’s theory, such as Parseval’s identity, the uniqueness theorem and the approximation theorem, is now discussed.

Theorem 10 (Parseval’s identity) For any $\Phi \in C(S)$

$$\sum_{(\xi, \varrho) \in \Omega_{\Phi}} |\hat{\Phi}(\xi, \varrho)|^2 = M(|\Phi|^2).$$

Proof According to Theorem 9,

$$|M(\Phi(\xi, \varrho))|^2 = |M(\Phi_0(\xi))|^2.$$

Therefore, considering the Bohr–Fourier series of $\Phi_0$, the classical Parseval’s identity (see Theorem 3) and Theorem 5 imply that

$$\sum_{(\xi, \varrho) \in \Omega_{\Phi}} |\hat{\Phi}(\xi, \varrho)|^2 = \sum_{\xi \in \Omega_{\Phi_0}} |\hat{\Phi}_0(\xi)|^2$$

$$= M(|\Phi_0|^2)$$

$$= M(|\Phi|^2).$$

\[\square\]

Theorem 11 (Uniqueness) Any $\Phi \in C(S)$ is uniquely determined by its Fourier series.

Proof Uniqueness follows from Parseval’s identity as in Bohr (see [1, Sect. 71]). By Theorem 5, if $M(|\Phi|^2) = 0 = M(|\Phi_0|^2)$, then the second equality implies that $\Phi_0 = 0$. Finally, $\Phi_0 \equiv 0$ implies $\Phi_t \equiv 0$ for every $t \in \widehat{\mathbb{Z}}$, and therefore, $\Phi \equiv 0$. \[\square\]

Remark 11 As was established by Bohr, these theorems are equivalent and play a fundamental role in the development of the theory.

Another implication of the theory developed here is that since any function can be approximated on the base leaf by the Fourier series in the classical sense, and it coincides with the restriction of the solenoidal version, we can extend the argument to the solenoid by limits and the approximation theorem follows immediately.

Theorem 12 (Approximation theorem) Any $\Phi \in C(S)$ can be approximated arbitrarily by the partial sums of its Fourier series.
5.3 Invariance of the Bohr–Fourier Series

To conclude the analysis on the Bohr–Fourier series, it should be verified that the theory just developed projects naturally to the universal solenoid. This is done as follows. First recall that the invariance of any $\Phi \in C(S)$ under the action of $\mathbb{Z}$ reads as:

$$\Phi_{t-\gamma}(x + \gamma) = \Phi_t(x), \quad ((x, t) \in \mathbb{R} \times \hat{\mathbb{Z}}, \gamma \in \mathbb{Z}).$$

This implies that the mean value $M(\Phi(x, t) \chi_{\xi, \varrho}(x, t))$ is invariant under the $\mathbb{Z}$-action. Hence, the Bohr–Fourier coefficients are invariant under the action of $\mathbb{Z}$. If $dz$ denotes the Haar measure on $S$, the Fourier coefficients of the induced function $\phi$ are determined by $\hat{\Phi}$ in the following way:

$$\hat{\Phi}(\chi_{\xi, \varrho}) = M(\Phi(x, t) \chi_{\xi, \varrho}(x, t)) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\mathbb{Z}} \Phi(x, t) \chi_{\xi, \varrho}(x, t) dx dt = \int_S \phi(z) \chi_q(z) dz = \hat{\phi}(q),$$

where $q = \xi + \varrho$. Finally, this allows us to ‘project’ the Bohr–Fourier series of any $\mathbb{Z}$-invariant function $\Phi : \mathbb{R} \times \hat{\mathbb{Z}} \to \mathbb{C}$ to the classical Fourier series of a function $\phi : S \to \mathbb{C}$ as follows:

$$\sum_{q \in \mathbb{Q}} \hat{\phi}(\chi_q) \chi_q(z).$$

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References

1. Bohr, H.: Almost Periodic Functions. Chelsea, New York (1947)
2. Cruz-López, M., López-Hernández, F.J., Verjovsky, A.: Some aspects of Rotation Theory on Compact Abelian Groups. Colloq. Math. 161(1), 131–155 (2020)
3. Hewitt, E., Ritter, G.: Fourier series on certain solenoids. Math. Ann. 257, 61–83 (1981)
4. Hewitt, E., Ross, K.A.: Abstract Harmonic Analysis I, 2nd edn. Springer, Berlin (1979)
5. López-Hernández, F.J.: Dynamics of induced homeomorphisms of one-dimensional solenoids. Discret. Contin. Dyn. Syst. 38(9), 4243–4257 (2018)
6. Mihăilescu, E.: Unstable directions and fractal dimension for skew products with overlaps in fibers. Math. Z. 269, 733–750 (2011)
7. Markus, L., Meyer, K.R.: Periodic orbits and solenoids in generic Hamiltonian dynamical systems. Am. J. Math. 102(1), 25–92 (1980)
8. Pontryagin, L.: Topological Groups. Gordon and Breach, New York (1966)
9. Robinson, C.: Dynamical Systems. Stability, Symbolic Dynamics, and Chaos. Taylor & Francis, New York (1999)
10. Sullivan, D.: Solenoidal manifolds. Proceedings of algebraic methods in geometry, Guanajuato, 2011. J. Singular. 9, 203–205 (2014)

11. Verjovsky, A.: Commentaries on the paper solenoidal manifolds by Dennis. Proceedings of algebraic methods in geometry, Guanajuato, 2011. J. Singular. 9, 245–251 (2014)

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