On isolated singular solutions of semilinear Helmholtz equation

HUYUAN CHEN
Department of Mathematics, Jiangxi Normal University, Nanchang, Jiangxi 330022, PR China

FENG ZHOU
Center for PDEs, School of Mathematical Sciences, East China Normal University, Shanghai Key Laboratory of PMMP, Shanghai 200062, PR China

Abstract

Our purpose of this paper is to study isolated singular solutions of semilinear Helmholtz equation

\[-\Delta u - u = Q|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \quad \lim_{|x|\to 0^+} u(x) = +\infty,\]

where \(N \geq 2\), \(p > 1\) and the potential \(Q: \mathbb{R}^N \to (0, +\infty)\) is a Hölder continuous function satisfying extra decaying conditions at infinity. We give the classification of the isolated singularity in the Serrin’s subcritical case and then isolated singular solutions is derived with the form \(u_k = k\Phi + \nu_k\) via the Schauder fixed point theorem for the integral equation

\[\nu_k = \Phi \ast (Q|k\nu_\sigma + \nu_k|^{p-1}(k\nu_\sigma + \nu_k)) \quad \text{in} \quad \mathbb{R}^N,\]

where \(\Phi\) is the real valued fundamental solution \(-\Delta - 1\) and \(\nu_\sigma\) is a also a real valued solution \((-\Delta - 1)\nu_\sigma = \delta_0\) with the asymptotic behavior at infinity controlled by \(|x|^{-\sigma}\) for some \(\sigma \leq \frac{N-2}{2}\).

Keywords: Helmholtz equation, Isolated singularity.

MSC2010: 35R11, 35J75, 35R06

1 Introduction

Our purpose of this paper is to study isolated singular solutions of semilinear Helmholtz equation

\[
\begin{aligned}
-\Delta u - u &= Q|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \\
\lim_{|x|\to 0^+} u(x) &= +\infty,
\end{aligned}
\]

where \(N \geq 2\), \(p > 1\) and the potential \(Q: \mathbb{R}^N \to \mathbb{R}\) is a Hölder continuous function.

The isolated singularities of semilinear elliptic equations have been studied extensively in the last decades since the first study due to Brezis in an unpublished note (see the introduction in \cite{3}), motivated by Thomas-Fermi problem \cite{4}, where proves that the elliptic problem with absorption nonlinearity

\[-\Delta u + |u|^{q-1}u = kb_0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega\]

admits a unique distributional solution \(u_k\) for \(1 < q < N/(N-2)\) and \(k > 0\), while no solution exists when \(q \geq N/(N-2)\), where \(b_0\) is the Dirac mass concentrated at the origin. Here \(N/(N-2)\) is the Serrin’s critical exponent. More analysis on differential equations with measures could see \cite{13, 23, 26} and \cite{28} for a survey. Isolated singularity of problem

\[-\Delta u + |u|^{q-1}u = 0 \quad \text{in} \quad \Omega \setminus \{0\}, \quad u = 0 \quad \text{on} \quad \partial\Omega\]

\[\text{(1.2)}\]
has been classified by Brezis and Véron in [9], that problem (1.2) admits only the zero solution when 
$q \geq N/(N - 2)$. When $1 < q < N/(N - 2)$, Véron in [27] described all the possible singular behaviors 
of positive solutions of (1.2). In particular he proved that the singularity must be isotropic when 
$(N + 1)/(N - 1) \leq q < N/(N - 2)$, and under the positivity assumption two types of singular behaviors occur:

(i) either $u(x) \sim c_N k |x|^{2-N}$ as $|x| \to 0$ and $k$ can take any positive value; $u$ is said to have a weak singularity at 0, and actually $u = u_k$;

(ii) or $u(x) \sim c_{N,q} |x|^{-\frac{2N}{N+2}}$ as $|x| \to 0$; $u$ is said to have a strong singularity at 0, and $u = \lim_{k \to \infty} u_k$.

In contract with absorption nonlinearity, the elliptic problem with source nonlinearity

$$-\Delta u = u^p \text{ in } \Omega \setminus \{0\} \quad u = 0 \text{ on } \partial \Omega$$

(1.3)

has a different structure of the isolated singular solutions. It was classified by Lions in [19] by building the connections with the weak solutions of

$$-\Delta u = u^p + k \delta_0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.$$  \hspace{1cm} (1.4)

For $N \geq 3$ and $p \in (1, \frac{N}{N-2})$ any positive solution of (1.3) is a weak solution of (1.4) for some $k \geq 0$; when $p \geq \frac{N}{N-2}$, the parameter $k = 0$. In the subcritical case $p \in (1, \frac{N}{N-2})$, the solution of (1.3) has either the singularity of $|x|^{2-N}$ or removable singularity. For the existence, Lions in [19] showed that there exists $k^* > 0$ such that, for $k \in (0, k^*)$ problem (1.3) has at least two positive solutions including the minimal solution and a Mountain Pass type solution; for $k = k^*$ problem (1.3) has a unique solution; for $k > k^*$ there is no solution of (1.3). Setting $\Omega = B_1(0)$ and $k^*(r)$ the critical number ensuring the existence of positive solution of (1.4), it is well-known that

$$\lim_{r \to +\infty} k^*(r) = 0.$$ \hspace{1cm} (1.5)

In the super critical case, the singularity at the origin of positive solutions to

$$-\Delta u = u^p \text{ in } B_1(0) \setminus \{0\}$$ \hspace{1cm} (1.6)

is classified for $p = \frac{N}{N-2}$ by Aviles in [11], for $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ by Gidas and Spruck in [15] and for $p = \frac{N+2}{N-2}$ by Caffarelli, Gidas and Spruck in [17].

The Helmholtz equations arise from the nonlinear wave equation

$$\frac{\partial^2}{\partial t^2} \psi(t, x) - \Delta \psi(t, x) = h(x, \psi^2) \psi, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

which is used to describe wave propagation in an ambient medium with nonlinear response, where $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a real-valued function. The time-periodic ansatz $\psi(t, x) = e^{i\kappa t} u(x)$ with $\kappa > 0$ leads to the nonlinear Helmholtz equation. Motivated their great applications, the nonlinear Helmholtz equations have been attracted attentions by researchers, see [2][9][14][16][17][21][22] and the references therein. In particular, [13] applies the dual variational methods to obtain the weak real-valued solution of the integral equation of $-\Delta u - k^2 u = Q|u|^{p-1} u$ in $\mathbb{R}^N$ to avoid the strong indefiniteness of the normal energy functional works on $H^1(\mathbb{R}^N)$. For $p \in (\frac{N+2}{N-2}, \frac{N+2}{N-2})$, a weak solution of

$$u = \Phi \ast (Q\frac{|u|^{p-1}}{\sqrt{p-1}} u),$$

letting $v = Q\frac{|u|^{p-1}}{\sqrt{p-1}} u$, by obtaining the critical point of the energy functional

$$J_t(v) = \frac{1}{q^t} \int_{\mathbb{R}^N} |v|^q \, dx + \frac{1}{2} \int_{\mathbb{R}^N} (\frac{Q}{\sqrt{p-1}} v) \Phi \ast (\frac{Q}{\sqrt{p-1}} v) \, dx, \quad \forall v \in L^q(\mathbb{R}^N),$$
where $q' = \frac{p+1}{p}$ is the conjugate index of $p + 1$, $\Phi$ is a radially symmetric fundamental solution of Helmholtz operator in $\mathbb{R}^N$ verifying
\[
\Phi(x) \sim \begin{cases} 
  c_N |x|^{2-N} & \text{if } N \geq 3 \\
  -c_N \ln |x| & \text{if } N = 2
\end{cases}
as |x| \to 0^+,
\limsup_{|x| \to +\infty} |\Phi(x)||x|^{\frac{N-2}{2}} < +\infty \quad (1.7)
\]
with the normalized constant $c_N > 0$. More properties of the fundamental solution is mentioned in Section 2. In a recent paper [9], the authors studied the complex valued weak solution of
\[-\Delta u - k^2 u = Qf(x,u) \text{ in } \mathbb{R}^N,
\]
where $f$ could involve the perturbation of non-homogeneous source with the model
\[f(x,u) = |u|^{p-1}u + g(x)\]
with $g$ be bounded and decaying at infinity.

Thanks to (1.5), there is no hope to find positive solutions of Helmholtz equation (1.1). Our aim in this article is to obtain sign-changing solutions with isotropic singularity at the origin in the Serrin’s subcritical case, i.e. $p \in (1, p^*_N)$, where the Serrin’s exponent $p^*_N$ is defined by
\[
p^*_N = \begin{cases} 
  \frac{N}{N-2} & \text{if } N \geq 3, \\
  +\infty & \text{if } N = 2.
\end{cases} \quad (1.8)
\]
Our strategy of deriving isolated singular solution of (1.1) is to build the connection with weak solutions of
\[-\Delta u - u = Q|u|^{p-1}u + k\delta_0 \text{ in } \mathbb{R}^N, \quad (1.9)\]
where $k > 0$ and $\delta_0$ is the Dirac mass at the origin. Here we say that $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ is a weak solution of (1.9) if
\[
\int_{\mathbb{R}^N} [u(-\Delta)\xi - u\xi - Q|u|^{p-1}u\xi] \, dx = k\xi(0), \quad \forall \xi \in C_c^\infty(\mathbb{R}^N).
\]

Our first result is on isotropic singular solutions of (1.1) classified by Dirac source in the distributional sense.

**Theorem 1.1** Assume that $N \geq 2$, $p \in (1, p^*_N)$, $Q : \mathbb{R}^N \to \mathbb{R}$ is a Hölder continuous function in $\mathbb{R}^N$ such that $Q(0) > 0$.

If $u$ is a classical solution of (1.1), then $u \in L^1_{\text{loc}}(\mathbb{R}^N)$, $Q|u|^{p-1}u \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $u$ is a very weak solution of (1.9) for some $k > 0$ with the asymptotic behaviors that
\[
\lim_{|x| \to 0^+} \frac{u(x)}{\Phi(x)} = k,
\]
where $c_N > 0$ is a normalized constant.

**Remark 1.1** We remark that the regularity of $Q$ could be weakened to be $L^\infty$, Moreover, the asymptotic behavior at the origin could be replaced by
\[Q(x) \geq Q_0|x|^\theta \text{ in } B_r(0) \setminus \{0\}\]
for some $Q_0 > 0$, $r > 0$ and $\theta > -2$. In this case, the Serrin’s exponent $p^*_N$ should be
\[
\frac{N + \theta_0}{N-2} \quad \text{if } N \geq 3, \quad +\infty \quad \text{if } N = 2.
\]

For simplicity, we in this paper deal with the case that $Q$ is continuous and $Q(0) > 0$. 


For a given number \( \alpha \) and \( \sigma \), the set of all nontrivial solutions of homogeneous Helmholtz equation (1.10) with the asymptotic behavior controlled by \((1 + |x|)^{-\sigma}\), i.e.

\[
S_\sigma = \left\{ u \text{ is Helmholtz harmonic : } \sup_{x \in \mathbb{R}^N} (|u(x)|(1 + |x|)^\sigma) < +\infty \right\}.
\]

**Proposition 1.1** Let \( N \geq 2 \) and \( S_\sigma \) be the solutions’ set defined by (1.11), then

\[
S_\sigma = \emptyset \quad \text{for } \sigma > \frac{N - 1}{2}
\]

and the mapping: \( \sigma \in (-\infty, \frac{N-1}{2}] \to S_\sigma \) is strictly decreasing, i.e. for \(-\infty < \sigma_2 < \sigma_1 < \frac{N-1}{2}\), we have that

\[
\emptyset \neq S_{\frac{N-1}{2}} \subset S_{\sigma_1} \subsetneq S_{\sigma_2}.
\]

**Remark 1.2** Note that the typical non-vanishing Helmholtz functions are \( \psi(x) = \sin x_i, \psi(x) = \cos x_i \) for \( x = (x_1, \ldots, x_N), i = 1, \ldots, N \).

For the existence of weak solution of (1.9), we need more restrictions on \( Q \) as following:

\((Q_\alpha)\) \( Q : \mathbb{R}^N \to \mathbb{R} \) is a Hölder continuous function in \( L^\infty_\alpha(\mathbb{R}^N) \) with \( \alpha \in \mathbb{R} \) such that \( Q(0) > 0 \), where \( L^\infty_\alpha(\mathbb{R}^N) \) is the space of functions \( w \) such that

\[
||w||_{L^\infty_\alpha} := \text{esssup}_{x \in \mathbb{R}^N} |w(x)|(1 + |x|)^\alpha < +\infty.
\]

For a given number \( \alpha \in \mathbb{R} \), we denote

\[
p_\alpha^\# = 1 + \frac{2}{N - 1} \left( \frac{N + 1}{2} - \alpha \right).
\]

Note that the Serrin exponent \( p_N^* \) arises from the isolated singularity related to the fundamental solution at the origin and while the critical exponent \( p_\alpha^\# \) is the one to control the decay at infinity. For the existence of singular solution, it requires \( p_\alpha^\# < p_N^* \), which holds if

\[
\alpha \in \mathbb{R} \text{ for } N = 2, \quad \alpha > 0 \text{ for } N = 3 \quad \text{and } \quad \alpha > \frac{N(N-3)}{2(N-2)} \text{ if } N \geq 4.
\]

**Theorem 1.2** Assume that \( N \geq 2 \), the potential \( Q \) verifies \((Q_\alpha)\) with \( \alpha \) satisfying (1.13). Let \( p_\alpha^\#, p_N^* \) be defined in (1.8) and (1.12) respectively.

Then for

\[
p \in (1, +\infty) \cap [p_\alpha^\#, p_N^*)
\]

and \( \psi_\sigma \in S_\sigma \) with

\[
\sigma \in \left[ \frac{1}{p - 1} \left( \frac{N + 1}{2} - \alpha \right), \frac{N - 1}{2} \right],
\]

there exists \( k^* > 0 \) such that for \( k \in (0, k^*) \), problem (1.9) admits a solution \( u_k \) in an integral form

\[
u_k = \Phi * (Q|u_k|^{p-2}u_k) + k(\Phi + \psi_\sigma) \quad \text{in } \mathbb{R}^N.
\]
Moreover, (i) $u_k$ is a classical solution of (1.1) and satisfy
\[ \lim_{x \to 0} \frac{u_k(x)}{\Phi(x)} = k \quad \text{and} \quad \limsup_{x \to 0} |u_k(x)||x|^\sigma < +\infty; \tag{1.15} \]

(ii) if $\sigma \in \left( \frac{1}{p-1} \left( \frac{N+1}{2} - \alpha \right), \frac{N-1}{2} \right)$,
\[ \limsup_{|x| \to +\infty} |u_k(x) - k(\Phi + \psi)(x)||x|^{\sigma_p} < +\infty, \tag{1.16} \]

where $\sigma_p := \min \{ \frac{N-1}{2}, \alpha + \sigma_p - \frac{N+1}{2} \} > \sigma$.

Remark 1.3 (i) $u_k$ is disturbed from the singular function $kw_\sigma := k(\Phi + \psi)$, where $\psi \in S_\sigma$, $\Phi$ is the radially symmetric fundamental solution of Helmholtz operator and $w_\sigma$ is also a weak solution of
\[-\Delta u - u = \delta_0 \quad \text{in} \quad \mathbb{R}^N. \]

(ii) If $\alpha \geq \frac{N+1}{2}$, then for any $p \in (1, p^*_N)$ the solution could be disturbed by functions in $S_0$. This means that problem (1.1) admits some singular solutions, which don't have to vanish uniformly at infinity.

Moreover, when $p > p^*_N$, our choice of $\sigma$ verifies that $\sigma < \sigma_p$, so for $\psi \in S_\sigma \setminus S_\sigma$, the singular solutions could be differed by the asymptotic behaviors of $\psi$. This means, if $\psi_{\sigma_1}, \psi_{\sigma_2} \in S_\sigma \setminus S_\sigma$ such that $\psi_{\sigma_1} - \psi_{\sigma_2} \in S_\sigma \setminus S_\sigma$, then corresponding solutions are different.

(iii) When $N = 2$, we can deal with the potential $Q \equiv 1$.

Our approach for singular solutions of (1.9) is to use the Schauder fixed point theorem of (1.14) for the integral inequation
\[ v_k = \Phi * (Q|kw_\sigma + v_k|^{p-2}(kw_\sigma + v_k)) \quad \text{in} \quad \mathbb{R}^N, \]
where $w_\sigma = \Phi + \psi$. This method also could be applied to obtain the complex valued solutions of (1.1), which is discussed in Section 5.

The rest of this paper is organized as follows. In Section 2, we introduce the fundamental solution of Helmholtz operator and classify the the set of Helmholtz harmonic functions. Section 3 is devoted to the classification of isolated singularity of classical solution of (1.1). In Section 4, we prove the existence of distributional solution of (1.9) by Schauder fixed point theorem.

2 Preliminary

2.1 Fundamental solutions

In polar coordinates, we have that
\[ \Delta = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{N-1}}, \]

where $\Delta_{S^{N-1}}$ is the Beltrami Laplacian and $S^{N-1} = \{ x \in \mathbb{R}^N : |x| = 1 \}$. It is known that the eigenvalue $\mu_j = j(j + N - 2)$ and corresponding eigenfunctions $v_j$ for $\Delta_{S^{N-1}}$, i.e.
\[-\Delta_{S^{N-1}} v_j = \mu_j v_j \quad \text{in} \quad S^{N-1}. \]

In fact, the multiplicity of eigenfunctions relative to $\mu_j$ equals the dimension of the space of homogeneous, harmonic polynomials of degree $j$. 
Let
\[ J_{\lambda_j}(t) = \sum_{i=0}^{+\infty} \frac{(-1)^i}{i!\Gamma(i+\lambda_j+1)} \left( \frac{t}{2} \right)^{2i+\lambda_j} \] (2.1)
which is the Bessel function of first kind with order
\[ \lambda_j := \sqrt{\frac{(N-2)^2}{4} + \mu_j} , \] (2.2)
that is,
\[ \frac{d^2}{dr^2} J_{\lambda_j} + \frac{1}{r} \frac{d}{dr} J_{\lambda_j} + \left( 1 - \frac{\lambda_j^2}{r^2} \right) J_{\lambda_j} = 0. \]

For \( j \in \mathbb{N} \), denote
\[ \psi_j(x) = |x|^{-\frac{N-2}{2}} J_{\lambda_j}(|x|) \psi\left( \frac{x}{|x|} \right), \quad \forall x \in \mathbb{R}^N, \] (2.3)
then \( \psi_j \) is a classical solution of Homogeneous Helmholtz equation (1.10). Particularly, we set
\[ \Psi = \psi_0 \quad \text{in} \quad \mathbb{R}^N. \] (2.4)

It is known that
\[ \lim_{r \to 0^+} \psi_j(r)r^{-\frac{N-2}{2}-\lambda_j} = \frac{2^{-\lambda_j}}{\Gamma(\lambda_j + 1)} \] (2.5)
and \( \psi_j \) is oscillated at \(+\infty\) with the order \( r^{-\frac{N+1}{2}} \).

Let \( \Lambda_N = \frac{N-2}{2} \) and denote
\[ \Phi(x) = c_0|x|^{-\frac{N+2}{2}} \lim_{\Lambda \to \Lambda_N} \frac{\cos(\Lambda \pi) J_\Lambda(|x|) - J_{-\Lambda}(|x|)}{\sin \Lambda \pi}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \]
where \( c_0 > 0 \) depends on \( N \), \( J_{\frac{N-2}{2}} \) is defined by (2.1) replaced \( \Lambda_k \) by \( \frac{N-2}{2} \) and
\[ J_{-\Lambda}(t) = \sum_{m=0}^{+\infty} \frac{(-1)^m t^{2m-\Lambda}}{m!\Gamma(m - \Lambda + 1)} \left( \frac{1}{2} \right)^{2m-\Lambda}, \quad \forall t > 0. \]

Note that the related asymptotic behaviors for Bessel functions show that as \( |x| \to 0^+ \)
\[ \Phi(x) = \begin{cases} 
   c_N|x|^{2-N} + O(|x|^{3-N}) & \text{if} \ N \geq 3, \\
   -c_N \ln |x| + O(|x|^{3-N}) & \text{if} \ N = 2,
\end{cases} \]
and \( -\Delta \Phi - \Phi = \delta_0 \) in the distributional sense. Moreover, we have \( -\Delta \Phi - \Phi = 0 \) in \( \mathbb{R}^N \setminus \{0\} \) pointwisely. As a conclusion, \( \Phi \) is a real fundamental solution for the Helmholtz equation, so if \( v \) is, say, a Schwartz function on \( \mathbb{R}^N \) the convolution \( \Phi \ast v \) satisfies
\[ -\Delta (\Phi \ast v) - \Phi \ast v = v \quad \text{in} \quad \mathbb{R}^N. \] (2.6)
The function \( \Phi \) is radially symmetric, has the singularity \( c_N|x|^{2-N} \) near the origin and oscillates at infinity controlled by \( |x|^{\frac{2-N}{2}} \).

On the other hand, a complex-valued fundamental solution \( \Phi_c \) could be involved as
\[ \Phi_c(x) := (2\pi)^{-\frac{N}{2}} \mathcal{F}^{-1}((|\xi|^2 - 1 - i0)^{-1})(x) = \frac{i}{4}(2\pi|x|)^{\frac{2-N}{2}} H_{\frac{N-2}{2}}^{(1)}(|x|), \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \]
where \( i \) is the imagine unit, \( H_{\frac{N-2}{2}}^{(1)} \) is the Hankel function of the first kind of order \( \frac{N-2}{2} \) and \( \mathcal{F}^{-1}((|\xi|^2 - 1 - i0)^{-1}) \) is approached by \( [-\Delta - (1 + \epsilon i)]^{-1} \) as \( \epsilon \to 0^+ \). In fact, the operator
\[-\Delta - (1 + i\epsilon) : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)\] is an isomorphism. Moreover, for any \( f \) from the Schwartz space \( \mathcal{S} \) its inverse is given by

\[
[-\Delta - (1 + i\epsilon)]^{-1} f(x) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{ix\xi} \frac{\hat{f}(\xi)}{|\xi|^2 - (1 + i\epsilon)} d\xi.
\]

For \( H^{(1)}_{\frac{N-2}{2}} \), we have the asymptotic expansions

\[
H^{(1)}_{\frac{N-2}{2}}(s) = \begin{cases} 
\frac{\sqrt{\frac{2}{\pi s}} e^{i(s-\frac{N-1}{4}) \pi} [1 + O(s^{-1})]}{\pi} & \text{as } s \to \infty, \\
-\frac{i\Gamma(s-\frac{N-2}{2}) (\frac{2}{s})^\frac{N-2}{4}}{\pi} [1 + O(s)] & \text{as } s \to 0^+,
\end{cases}
\]

(see e.g. [20 Formulas (5.16.3)]), so there exists a constant \( c_0 > 0 \) such that

\[
|\Phi_c(x)| \leq c_0 \max \left\{ |x|^{2-N}, 1 + (-\ln|x|)_+, (1 + |x|)^{-\frac{N-2}{2}} \right\} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\},
\]

where \( t_+ = \max\{0, t\} \). Moreover, \( \Phi_c \) satisfies the equation \(-\Delta \Phi_c - \Phi_c = \delta_0\) together with Sommerfeld’s outgoing radiation condition

\[
|\nabla \Phi_c(x) - i\Phi_c(x)\hat{x}| = o(|x|^{-\frac{N}{2}}) \quad \text{as } |x| \to \infty,
\]

where \( \hat{x} = \frac{x}{|x|} \). Furthermore, one has that

\[
\Phi_c = \Phi + i\Psi,
\]

where \( \Psi \) is a real Helmholtz harmonic function such that, by (2.8),

\[
|\nabla \Phi(x) + \Psi(x)\hat{x}| = o(|x|^{-\frac{N}{2}}), \quad |\nabla \Psi(x) - \Phi(x)\hat{x}| = o(|x|^{-\frac{N}{2}}) \quad \text{as } |x| \to \infty.
\]

### 2.2 Sets of Helmholtz harmonic functions

In this subsection, we will show the existence of the various homogeneous Helmholtz functions.

**Lemma 2.1** For \( \sigma \in \mathbb{R} \), denote

\[
\mathcal{S}_\sigma^\alpha = \left\{ u \text{ is Helmholtz harmonic : } \limsup_{|x| \to +\infty} (|u(x)|(1 + |x|)^\sigma) \in (0, +\infty) \right\}.
\]

Then for \(-\infty < \sigma < \frac{N-1}{2}\), we have that

\[
\mathcal{S}_\sigma^\alpha \neq \emptyset.
\]

**Proof.** Particular, for any positive integer \( M \leq N \), we can set \( \Phi_M \) defined in (2.1) replacing \( N \) by \( M \), is also a Helmholtz harmonic in \( \mathbb{R}^N \) with the decay at infinity controlled by \(|x|^{-\frac{M-1}{2}}\).

Generally, we next show when \( \sigma \in (-\frac{N-1}{2}, +\infty) \), there is a Helmholtz harmonic function \( u \) such that

\[
\limsup_{|x| \to +\infty} |u(x)|(1 + |x|)^{-\sigma} \in (0, +\infty).
\]

Since

\[
\Psi(x) = |x|^{-\frac{N-1}{2}} \sin(|x| - \theta_0) + O(|x|^{-\frac{N+1}{2}}) \quad \text{near the infinity},
\]

where \( \theta_0 = \frac{\Delta \pi}{4} + \frac{\pi}{4} = \frac{(N-1)\pi}{4} \) > 0, then there exist \( n_0 \in \mathbb{N} \), \( t_n \in [2^n, 2^n + 2\pi] \) such that, letting \( x_n = t_ne_1 \) with \( e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^N \), for some \( n \geq n_0 \),

\[
\begin{align*}
\frac{3}{4} 2^{-\frac{N-1}{2}} < \Psi(x_n) := \max_{t \in [2^n, 2^n + 2\pi]} \Psi(te_1) < \frac{5}{4} 2^{-\frac{N-1}{2}}.
\end{align*}
\]
Note that
\[ \lim_{n \to +\infty} (t_n - t_{n-1})2^{-n+1} = 1. \]

Now we denote
\[ w_1(x) = \sum_{n=n_0}^{+\infty} a_n \Psi(x - x_n), \quad \forall x \in \mathbb{R}^N, \tag{2.10} \]
where \( \{a_n\}_n \) is a sequence of nonnegative numbers \( a_n = 2^{\sigma n} \).

For \( \sigma \in (-\frac{N-1}{2}, \frac{N-1}{2}) \), we have that \( \sum_{n=n_0}^{+\infty} a_n2^{-\frac{N-1}{2}n} < +\infty \) and
\[ w_1(0) = \sum_{n=n_0}^{+\infty} a_n \Psi(x_n) < \frac{5}{4} \sum_{n=n_0}^{+\infty} a_n2^{-\frac{N-1}{2}n} < +\infty. \]

Similarly, we get that \( w_1(x) \) is bounded for any point \( x \) in \( \mathbb{R}^N \), so \( w_1 \) is well-defined in \( \mathbb{R}^N \).

Note that \( c|x_n|^\sigma \leq a_n \Psi(0) < w_1(x_n) \)
\[ < a_n \Psi(0) + \sum_{m=n_0}^{n-1} a_m 2^{-\frac{N-1}{2}(n-m)} + \sum_{m=n+1}^{+\infty} a_m 2^{-\frac{N-1}{2}m} \]
\[ < c|x_n|^\sigma + \sum_{m=n_0}^{n-1} a_m 2^{-\frac{N-1}{2}(n-m)} + \sum_{m=n+1}^{+\infty} a_m 2^{-\frac{N-1}{2}m}, \]
where
\[ \sum_{m=n_0}^{n-1} a_m 2^{-\frac{N-1}{2}(n-m)} \leq 2^{-\frac{N-1}{2}n} \sum_{m=n_0}^{n-1} 2^m(\frac{N-1}{2} + \sigma) \leq C2^{\sigma n} = C|x_n|^\sigma \]
and
\[ \sum_{m=n+1}^{+\infty} a_m 2^{-\frac{N-1}{2}m} \leq \sum_{m=n+1}^{+\infty} 2^{(\sigma - \frac{N-1}{2})m} \]
\[ \leq C2^{(\sigma - \frac{N-1}{2})n} = C|x_n|^\sigma - \frac{N-1}{2}. \]

Therefore, we have that
\[ \lim_{|x| \to +\infty} |w_1(x)|(1 + |x|)^\sigma \in (0, +\infty). \tag{2.11} \]

For \( \sigma > \frac{N-1}{2} \), let
\[ w_2(x) = \sum_{n=n_0}^{+\infty} a_n [\Psi(x - x_n) - \Psi(x + x_n)], \quad \forall x \in \mathbb{R}^N, \tag{2.12} \]
where \( \{a_n\}_n \) is a sequence of nonnegative numbers \( a_n = 2^{\sigma n} \). Note that \( w_2(0) = 0 \), so \( w_2 \) is well-defined in \( \mathbb{R}^N \) and \( (2.11) \) holds. We complete the proof. \( \Box \)

**Proof of Proposition 1.1** Let \( \psi \) be a solution of \( (1.10) \) satisfying that for some \( \sigma > \frac{N-1}{2} \) and \( c_1 > 0 \),
\[ |\psi(x)| \leq c_1 (1 + |x|)^{-\sigma}, \quad \forall x \in \mathbb{R}^N, \]
then
\[ \lim_{r \to +\infty} \frac{1}{r} \int_{B_r(0)} |\psi(x)|^2 dx \leq \lim_{r \to +\infty} c_2 r^{N-1-2\sigma} = 0. \]
From Rellich uniqueness, we have that $\psi = 0$. This means $S_{\sigma} = \emptyset$ for $\sigma > \frac{N-1}{2}$.

For $\sigma = \frac{N-1}{2}$, $S_{\sigma}$ contains the functions $\psi_k$ with $k \in \mathbb{N}$, defined by (2.3).

For $-\infty < \sigma_1 < \sigma_2 \leq \frac{N-1}{2}$, it is easy to see that $S_{\sigma_2} \subsetneq S_{\sigma_1}$. From Lemma 2.1 we have that for $-\infty < \sigma_2 < \sigma_1$, $S_{\frac{N-1}{2}} \subsetneq S_{\sigma_1} \subsetneq S_{\sigma_2}$.

We complete the proof. □

3 Classification

In this section, we build the connection between the singular solutions of (1.1) and the very weak solutions of (1.9).

**Lemma 3.1** Assume that $N \geq 2$, $p > 1$, $Q : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative Hölder continuous function such that $Q(0) > 0$ and $u$ is a classical solution of (1.1).

Then $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $Q|u|^{p-1}u \in L^1_{\text{loc}}(\mathbb{R}^N)$.

**Proof.** From the assumptions that continuous $Q(0) > 0$ and $\lim_{|x| \to 0^+} u(x) = +\infty$, there exists $r_0 > 0$ such that

$$Q > \frac{Q(0)}{2}, \quad u > 0 \quad \text{in } B_{r_0} \setminus \{0\}. \quad (3.1)$$

In fact, $u$ is a positive classical solution of

$$\begin{cases} -\Delta u - u = Q|u|^{p-1}u & \text{in } B_{r_0} \setminus \{0\}, \\ \lim_{|x| \to 0^+} u(x) = +\infty. \end{cases} \quad (3.2)$$

By contradiction, we assume that $f := Q|u|^{p-1}u \not\in L^1(B_{r_0})$. Since $f$ is positive and continuous in $B_{r_0} \setminus \{0\}$.

Let $\{r_n\}_n \subset (0, r_0)$ be a sequence of strictly decreasing positive numbers converging to zero and for any $r_n$, we have that

$$\lim_{r \to 0^+} \int_{B_{r_n}(0) \setminus B_r(0)} f(x) dx = +\infty,$$

then there exists $R_n \in (0, r_n)$ such that

$$\int_{B_{r_n}(0) \setminus B_{R_n}(0)} f dx = n,$$

*Case of $\mu \geq 0$. Let $\delta_n = \frac{1}{n} \Gamma \mu \int \chi_{B_{r_n}(0) \setminus B_{R_n}(0)}$, then the problem

$$\begin{cases} -\Delta v = \delta_n & \text{in } B_{r_0}, \\ v = 0 & \text{on } \partial B_{r_0} \end{cases}$$

has a unique positive solution $v_n$ satisfying (in the usual sense)

$$\int_{B_{r_0}} v_n(-\Delta) \xi dx = \int_{B_{r_0}} \delta_n \xi dx, \quad \forall \xi \in C^{1,1}_0(B_{r_0}).$$

For any $\xi \in C^{1,1}_0(B_{r_0})$, we have that

$$\int_{B_{r_0}} w_n(-\Delta) \xi dx = \int_{\Omega} \delta_n \xi dx \to \xi(0) \quad \text{as } n \to +\infty.$$
Therefore, for any compact set $\mathcal{K} \subset B_{r_0} \setminus \{0\}$
\[
|v_n - \frac{1}{cN} G_{r_0}|_{C^1(\mathcal{K})} \to 0 \quad \text{as} \quad n \to +\infty.
\]

So we fix a point $x_0 \in \Omega \setminus \{0\}$, let $d_0 = \frac{\min(|x_0|, r_0 - |x_0|)}{2}$ and $\mathcal{K} = B_{d_0}(x_0)$, then there exists $n_0 > 0$ such that for $n \geq n_0$,
\[
w_n \geq \frac{1}{2c_\mu} G_{r_0} \quad \text{in} \quad \mathcal{K},
\]
(3.3)

where $G_{r_0}$ is the weak solution of
\[
-\Delta u = \delta_0 \quad \text{in} \quad B_{r_0}, \quad u = 0 \quad \text{on} \partial B_{r_0}.
\]

Let $w_n$ be the solution of
\[
\begin{cases}
-\Delta u = n\delta_n & \text{in} \quad B_{r_0}, \\
u = 0 & \text{on} \partial B_{r_0},
\end{cases}
\]
then the comparison principle implies that $w_n \geq n v_n$ in $B_{r_0}$. Together with (3.3), we derive that
\[
v_n \geq \frac{n}{2cN} G_{r_0} \quad \text{in} \quad \mathcal{K}.
\]

Then by comparison principle again, we have that
\[
u(x_0) \geq \frac{n}{2cN} G_{r_0} \to +\infty \quad \text{as} \quad n \to +\infty,
\]
which contradicts that $u$ is classical solution of (1.1).

Therefore, we have that $Q |u|^{p-1} u \in L^1(B_{r_0})$ and $u \in L^1(B_{r_0})$ by the fact that $Q > \frac{Q(0)}{2}$ and $p > 1$. \hfill \Box

The following estimates is important for our analysis of singularity and regularity.

**Proposition 3.1** [24], Chapter V (Proposition 2.2 in [8] with $\alpha = 1$) Let $h \in L^s(\Omega)$ with $s \geq 1$, then there exists $c_3 > 0$ such that

(i) if $\frac{1}{s} < \frac{2}{N}$,
\[
g_{r_0}[h] \|_{G^{\min(1, 2 - N/s)}}(B_{r_0}) \leq c_3 \|h\|_{L^s(B_{r_0})};
\]
\[
(3.4)
\]
(ii) if $\frac{1}{s} \leq \frac{1}{r} + \frac{2}{N}$ and $s > 1$,
\[
g_{r_0}[h] \|_{L^r(B_{r_0})} \leq c_3 \|h\|_{L^r(B_{r_0})};
\]
\[
(3.5)
\]
(iii) if $1 < \frac{1}{r} + \frac{2}{N}$,
\[
g_{r_0}[h] \|_{L^r(B_{r_0})} \leq c_3 \|h\|_{L^1(B_{r_0})}.
\]
\[
(3.6)
\]

Now we are in the position to prove Theorem 1.1.

**Proof of Theorem 1.1** Let $r_0 > 0$ satisfy (3.1) and then $u$ is a positive classical solution of
\[
\begin{cases}
-\Delta u = u + Q |u|^{p-1} u & \text{in} \quad B_{r_0} \setminus \{0\}, \\
\lim_{|x| \to 0^+} u(x) = +\infty.
\end{cases}
\]

Define the operator $L$ by the following
\[
L(\xi) := \int_{B_{r_0}} [u(-\Delta \xi - \xi) - Q |u|^{p-1} u \xi] \, dx, \quad \forall \xi \in C^\infty_c(B_{r_0}).
\]

First we claim that for any $\xi \in C^\infty_c(\mathbb{R}^N)$ with the support in $B_{r_0} \setminus \{0\}$,
\[
L(\xi) = 0.
\]
In fact, since $\xi \in C^\infty_c(\mathbb{R}^N)$ has the support in $B_{r_0} \setminus \{0\}$, then there exists $r \in (0, r_0/2)$ such that $\xi = 0$ in $B_r(0)$ and then

$$L(\xi) = \int_{B_{r_0}\setminus B_r} [u(-\Delta \xi - \xi) - Q|u|^{p-1}u\xi] \, dx$$

$$= \int_{B_{r_0}\setminus B_r} (-\Delta u - u - Q|u|^{p-1}u)\xi \, dx$$

$$= 0.$$  

Now let $\eta_0 : \mathbb{R}^N \to [0, 1]$ be a smooth radially symmetric function such that $\eta_0 = 1$ in $B_1$ and $\eta_0 = 0$ in $B_2$.

By Lemma 3.4 we have that $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $Q|u|^{p-1}u \in L^1_{\text{loc}}(\mathbb{R}^N)$, then from Theorem 1 in [5], it implies that

$$L = k\delta_0 \quad \text{for some} \quad k \geq 0,$$

that is,

$$L(\xi) = \int_{B_{r_0}} [u(-\Delta \xi - \xi) - Q|u|^{p-1}u\xi] \, dx = k\xi(0), \quad \forall \xi \in C^1_{c}(B_{r_0}). \quad (3.7)$$

Let $\Gamma$ be a Harmonic function such that $\Gamma = u$ on $\partial B_{r_0}$ then $\Gamma \in C^2(B_{r_0})$ and $u_0 = u - \Gamma$ is a weak nonnegative solution of

$$\begin{cases}
-\Delta w = w + \Gamma + Q|w + \Gamma|^{p-1}(w + \Gamma) + k\delta_0 & \text{in} \quad B_{r_0}, \\
 w = 0 & \text{on} \quad \partial B_{r_0}
\end{cases}$$

for some $k \geq 0$.

When $k = 0$, then

$$w = G_{r_0}[f(w)],$$

where

$$f(w) = w + \Gamma + Q|w + \Gamma|^{p-1}(w + \Gamma).$$

Here we have that for some $c_4 > 0$

$$|f(w)| \leq c_4(w^p + 1).$$

Let $N \geq 3$ and we infer by Proposition 3.4 that $u^p \in L^a(B_{r_0})$ with $t_0 = \frac{1}{2}[1 + \frac{N}{pN}] > 1$, then use Proposition 6.1 again $u \in L^{1p}(B_{r_0})$ and $u^p \in L^{11}(B_{r_0})$ with

$$t_1 = \frac{1}{p N - 2t_0} t_0 > t_0.$$

If $t_1 > \frac{1}{2} Np$, by Proposition 3.1 $u \in L^\infty(B_{r_0})$ and then it could be improved that $u$ is a classical solution of

$$\begin{cases}
-\Delta w = w + \Gamma + Q|w + \Gamma|^{p-1}(w + \Gamma) & \text{in} \quad B_{r_0}, \\
 w = 0 & \text{on} \quad \partial B_{r_0}
\end{cases}$$

If $t_1 < \frac{1}{2} Np$, we proceed as above. By Proposition 6.4 $u \in L^{12p}(B_{r_0})$, where

$$t_2 = \frac{1}{p N - 2t_1} > \frac{1}{p N - 2t_0} t_1 = \left(\frac{1}{p N - 2t_0}\right)^2 t_0.$$

Inductively, let us define

$$t_m = \frac{1}{p N - 2t_{m-1}} > \left(\frac{1}{p N - 2t_0}\right)^m t_0 \to +\infty \quad \text{as} \quad m \to +\infty.$$
Then there exists $m_0 \in \mathbb{N}$ such that
\[ t_{m_0} > \frac{1}{2} N p \]
and by Proposition 3.1 part (i),
\[ w \in L^\infty(B_{r_0}), \]
then $u \in L^\infty$, which contradicts the assumption that $\lim_{|x| \to 0^+} u(x) = +\infty$. When $N = 2$, we can choose $t_1 > p$ and a contradiction is derived.

Therefore, $k > 0$.

Now we show the singularity of $u$ for $k > 0$. Note that
\[ w = \mathcal{G}_{r_0}[f(w)] + k \mathcal{G}_{r_0}[\delta_0], \quad (3.8) \]
letting
\[ w_1 = \mathcal{G}_{r_0}[f(w)] \quad \text{and} \quad \Upsilon_0 = k \mathcal{G}_{r_0}[\delta_0], \]
then by Young's inequality,
\[ f(w) \leq c_5 \left(1 + w_1^p + \Upsilon_0^p\right). \quad (3.9) \]
By the definition of $w_1$ and (3.9), we obtain that
\[ w_1 \leq c_5 \mathcal{G}_{r_0}[w_1^p] + \Upsilon_1, \quad (3.10) \]
where $w_1 \in L^s(B_{r_0})$ for any $s \in (1, \frac{N}{N-2\alpha})$ and
\[ \Upsilon_1 = c_5 \mathcal{G}_{r_0}[\Upsilon_0^p]. \]

Denoting $\mu_1 = 2 + (2 - N)p$, then for $0 < |x| < \frac{r_0}{2}$,
\[ \Upsilon_1(x) \leq \begin{cases} 
  c_6 |x|^\mu_1 & \text{if } \mu_1 < 0, \\
  -c_6 \log |x| & \text{if } \mu_1 = 0, \\
  c_6 & \text{if } \mu_1 > 0.
\end{cases} \]
If $\mu_2 \leq 0$ ( $\mu_1 > 0$ for $N = 2$), denote
\[ w_2 = c_6 \mathcal{G}_{r_0}[w_1^p]. \]
Then $w_2 \in L^s(\Omega)$ with $s \in \left(1, \frac{N}{N-2}\right)$, $w_1 \leq w_2 + \Upsilon_1$ and
\[ w_2 \leq c_6 \left(\mathcal{G}_{\alpha}[w_1^p] + \mathcal{G}_{\alpha}[\Upsilon_1^p]\right). \]

Let $\mu_2 = \mu_1 p + 2$, then $\mu_2 > \mu_1$ and for $0 < |x| < \frac{r_0}{2}$,
\[ \Upsilon_2(x) := c_6 \mathcal{G}_{r_0}[\Upsilon_1^p](x) \leq \begin{cases} 
 c_7 |x|^\mu_2 & \text{if } \mu_2 < 0, \\
 -c_7 \log |x| & \text{if } \mu_2 = 0, \\
 c_7 & \text{if } \mu_2 > 0.
\end{cases} \]

Inductively, we assume that
\[ w_{n-1} \leq c_{n-1} \mathcal{G}_{r_0}[w_{n-1}^p] + c_{n-1} \mathcal{G}_{\alpha}[\Upsilon_{n-2}], \]
where $c_{n-1} > 0$, $w_{n-1} \in L^s(B_{r_0})$ for $s \in \left[1, \frac{N}{N-2\alpha}\right)$, $\Gamma_{n-2}(x) \leq |x|^\mu_{n-2}$ for $\mu_{n-2} < 0$.

Let
\[ w_n = c_{n-1} \mathcal{G}_{r_0}[w_{n-1}^p], \quad \Upsilon_{n-1} = c_{n-1} \mathcal{G}_{r_0}[\Upsilon_{n-2}]. \]
and

$$\mu_{n-1} = \mu_{n-2}p + 2,$$

then \( u_n \in L^s(B_{r_0}) \) for \( s \in [1, \frac{N}{N-2}) \) and for \( 0 < |x| < \frac{r_0}{2} \) and \( c_n > 0 \),

$$\Upsilon_{n-1}(x) := \mathcal{G}_{r_0}[\Upsilon_{n-2}^p](x) \leq \begin{cases} 
    c_n |x|^\mu_{n-1} & \text{if } \mu_{n-1} < 0, \\
    -c_n \log |x| & \text{if } \mu_{n-1} = 0, \\
    c_n & \text{if } \mu_{n-1} > 0.
\end{cases}$$

We observe that

$$\mu_{n-1} - \mu_{n-2} = p(\mu_{n-2} - \mu_{n-3}) = p^{n-3}(\mu_2 - \mu_1) \to +\infty \text{ as } n \to +\infty.$$

Then there exists \( n_2 \geq 1 \) such that

$$\nu_{n_2-1} > 0 \text{ and } \nu_{n_2-2} \leq 0$$

and

$$w \leq w_{n_2} + \sum_{i=1}^{n_2-1} \Upsilon_i + \Upsilon_0, \quad (3.11)$$

where \( \Upsilon_i \leq c|x|^\mu_i \) and

$$w_{n_2} \leq c_{n_2}(\mathcal{G}_{r_0}[w_{n_2}^p] + 1).$$

We next claim that \( w_{n_2} \in L^\infty(\Omega) \). Since \( w_{n_2} \in L^s(\Omega) \) for \( s \in [1, \frac{N}{N-2}) \), letting

$$t_0 = \frac{1}{2}(1 + \frac{1}{p - \frac{N}{N-2}}) \in (1, \frac{N}{N-2}),$$

then \( \frac{N}{p - \frac{N}{2}t_0} > 1 \) and by Proposition 3.1 we have that \( w_{n_2} \in L^{t_1}(B_{r_0}) \) with

$$t_1 = \frac{Nt_0}{p - \frac{N}{2}t_0}.$$

Inductively, it implies by \( w_{n_2} \in L^{t_{n-1}}(B_{r_0}) \) that \( w_{n_2} \in L^{t_n}(B_{r_0}) \) with

$$t_n = \frac{1}{p - \frac{N}{2}t_{n-1}} > \left( \frac{N}{p - \frac{N}{2}t_0} \right)^n t_0 \to +\infty \text{ as } n \to \infty.$$

Then there exists \( n_3 \in \mathbb{N} \) such that

$$s_{n_3} > \frac{Np}{2}$$

and by Proposition 3.1 part (i), it infers that

$$w_{n_2} \in L^\infty(B_{r_0}).$$

Therefore, it implies by \( w \geq \Upsilon_0 \) and (3.11) that

$$\lim_{|x| \to 0^+} w(x)|x|^{N-2} = c_N k \text{ if } N \geq 3, \quad \lim_{|x| \to 0^+} \frac{w(x)}{-\ln |x|} = c_N k \text{ if } N = 2,$$

where \( c_N > 0 \) is the normalized constant.

For \( C_\infty^c(\mathbb{R}^N) \), we can divided \( \xi \) into \( \xi_1 + \xi_2 \), where \( \xi_1, \xi_2 \) are smooth function such that

$$\text{supp}(\xi_1) \subset B_{r_0}, \quad \text{supp}(\xi_2) \subset B_{\frac{r_0}{2}}.$$
Since $u$ is a classical solution of (1.1), then we have that

$$\mathcal{L}(\xi_2) = \int_{\mathbb{R}^N} [u(-\Delta)\xi_2 - u\xi_2 - Q|u|^{p-1}u\xi_2] \, dx = 0.$$  

which, together with (3.7) replaced $\xi$ by $\xi_1$, implies that

$$\int_{\mathbb{R}^N} [u(-\Delta)\xi - u\xi - Q|u|^{p-1}u\xi] \, dx = k\xi(0), \quad \forall \xi \in C_c^\infty(\mathbb{R}^N).$$

Since any $\xi \in C^{1,1}_c(\mathbb{R}^N)$ could be approximated by a sequence of functions in $C_c^\infty(\mathbb{R}^N)$, so (3.7) holds for any $\xi \in C^{1,1}_c(\mathbb{R}^N)$.

## 4 Existence of weak solutions

We first provide some important estimates from the convolution by fundamental solution of Helmholtz operator.

**Lemma 4.1** For

$$N \geq 2, \quad \tau > \frac{N + 1}{2}, \quad \theta \in (-1, N - 2) \setminus \{0\},$$

let $U_{\theta,\tau} : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ be a function in $L^1_{loc}(\mathbb{R}^N)$ such that for some $c_8 > 0$

$$|U_{\theta,\tau}| \leq c_8 |x|^{-\theta - 2}(1 + |x|)^{-\tau + \theta + 2}, \quad \text{a.e. in } \mathbb{R}^N \setminus \{0\}. \quad (4.1)$$

Then there exists $c_9 > 0$ such that

$$|\Phi * U_{\theta,\tau}(x)| \leq c_9 (1 + |x|)^{\theta}(1 + |x|)^{\max\left\{-\frac{N+1}{2}, 1 + \frac{N}{2}\right\}} \theta, \quad \forall x \in \mathbb{R}^N \quad (4.2)$$

and

$$|\nabla \Phi * U_{\theta,\tau}(x)| \leq c_9 |x|^{-\theta - 1}(1 + |x|)^{\max\left\{-\frac{N+1}{2}, 1 + \frac{N}{2}\right\} + 1} \theta, \quad \forall x \in \mathbb{R}^N. \quad (4.3)$$

**Proof.** In the following, the letter $c_j > 0$ always denotes constants which only depends on $N$, $\alpha$ and $k$. We observe that

$$|\Phi(x)| \leq \begin{cases} 
  c_{10} |x|^{2-N} & \text{if } N \geq 3, \\
  c_{10} \log \frac{2}{|x|} & \text{if } N = 2, \\
  c_{11} |x|^{1-N} & \text{for } 0 < |x| \leq 1
\end{cases}$$

and

$$|\Phi(x)|, \ |
abla \Phi| \leq c_{12} |x|^{-\frac{N-2}{\theta}} \quad \text{if } |x| > 1.$$  

Note that

$$U_{\theta,\tau}(x) \leq 2|x|^{-2+\theta} \chi_{B_1}(x) + 2|x|^{-\tau} \chi_{B_1}(x), \quad x \in \mathbb{R}^N \setminus \{0\},$$

where $B_r(z)$ is the ball centered at $z$ with radius $r > 0$, $B_r(0) = B_r$, $A^c = \mathbb{R}^N \setminus A$ and $\chi_A$ is characteristic function of $A$, i.e. $\chi_A(x) = 1$ for $x \in A$ and $\chi_{A^c}(x) = 0$ for $x \in A^c$. Let

$$U_1(x) = |x|^{-2+\theta} \chi_{B_1}(x), \quad U_2(x) = |x|^{-\tau} \chi_{B_1}(x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$

From the fact that $-\Delta(|x|^{-\theta}) = \theta(N - 2 + \theta)|x|^{-2-\theta}$ in $\mathbb{R}^N \setminus \{0\}$, we can deduce

$$C_N \int_{\mathbb{R}^N} \frac{|y|^{-\theta-2}}{|x - y|^{N-2}} \, dy = \frac{1}{\theta(N - 2 - \theta)} |x|^{-\theta}$$

and

$$C_2 \int_{\mathbb{R}^2} |y|^{-\theta-2} (-\ln |x - y|) \, dy \leq c_{13}.$$
For \(x \in B_2 \setminus \{0\}\), we have that for \(N \geq 3\) and \(\theta \in (0, N-2)\)
\[
|\langle \Phi * U_1 \rangle(x)| \leq C_N \int_{B_1} \frac{|y|^{-\theta-2}}{|x-y|^{N-2} \tau} dz \leq \frac{1}{\theta(N-2-\theta)} |x|^{-\theta},
\]
and for \(N = 2\) and \(\theta \in (-1, 0)\)
\[
|\langle \Phi * U_1 \rangle(x)| \leq C_N \int_{B_1} |y|^{-\theta-2} (-\ln |x-y|) dz \leq c_{14}.
\]
When \(x \in \mathbb{R}^N \setminus B_2(0)\),
\[
|\langle \Phi * U_1 \rangle(x)| \leq c(|x| - 1)^{\frac{1-N}{2}} \int_{B_1} |y|^{-\theta-2} dy \leq c_{15} |x|^{\frac{1-N}{2}}.
\]
Thus
\[
|\langle \Phi * U_1 \rangle(x)| \leq c_{16} |x|^{-\theta} (1 + |x|)^{\frac{1-N}{2} + \theta}, \quad \forall x \in \mathbb{R}^N \text{ for } N \geq 3 \tag{4.4}
\]
and
\[
|\langle \Phi * U_1 \rangle(x)| \leq c_{17} (1 + |x|)^{-\frac{1}{2}}, \quad \forall x \in \mathbb{R}^2. \tag{4.5}
\]
Note that for \(x \in B_4 \setminus \{0\}\), we have that for \(N \geq 3\)
\[
|\langle \Phi * U_2 \rangle(x)| \leq C_N \int_{B_1 \setminus B_1} \frac{|y|^{-\tau}}{|x-y|^{N-2} \tau} dy + C_N \int_{B_1} \frac{|y|^{-\tau}}{|x-y|^{\frac{N-1}{2}}} dy
\]
\[
\leq c_{18} \left(1 + \int_{B_1} \frac{|y|^{-\tau-N-1}}{2} dz \right)
\]
and for \(N = 2\),
\[
|\langle \Phi * U_2 \rangle(x)| \leq C_N \int_{B_1 \setminus B_1} |y|^{-\tau} (-\ln |x-y|) dy + C_N \int_{B_1} \frac{|y|^{-\tau}}{|x-y|^2} dy \leq c_{18},
\]
thus,
\[
|\langle \Phi * U_2 \rangle(x)| \leq c_{19}.
\]
For \(x \in B_3^1\), there holds for \(N \geq 3\),
\[
|\langle \Phi * U_2 \rangle(x)| \leq \int_{B_1(x)} \frac{|y|^{-\tau}}{|x-y|^{N-2} \tau} dy + \int_{B_{1/2}(x) \setminus B_1(x)} \frac{|y|^{-\tau}}{|x-y|^{\frac{N-1}{2}}} dy
\]
\[
+ \int_{B_{1/2}(0) \setminus B_1(0)} \frac{|y|^{-\tau}}{|x-y|^{\frac{N-1}{2}}} dy + \int_{(B_{1/2}(0) \cup B_{1/2}(x))} |y|^{-\tau-\frac{N-1}{2}} dy
\]
\[
\leq c_{20} (|x|^{-\tau} + |x|^{-\tau+\frac{N+1}{2}})
\]
for \(N = 2\),
\[
|\langle \Phi * U_2 \rangle(x)| \leq \int_{B_1(x)} |y|^{-\tau} \ln |x-y| dy + \int_{B_{1/2}(x) \setminus B_1(x)} \frac{|y|^{-\tau}}{|x-y|^2} dy
\]
\[
+ \int_{B_{1/2}(0) \setminus B_1(0)} \frac{|y|^{-\tau}}{|x-y|^2} dy + \int_{(B_{1/2}(0) \cup B_{1/2}(x))} |y|^{-\tau-\frac{1}{2}} dy
\]
\[
\leq c_{21} (|x|^{-\tau} + |x|^{-\tau+\frac{3}{2}}),
\]
where we used the fact that \(\tau > \frac{3}{2}\).
Thus there exists $c_{22} > 0$ such that

$$|(Φ * U_2)(x)| \leq c_{22}(1 + |x|)^{\frac{N+1}{2}-\tau}, \quad \forall x \in \mathbb{R}^N.$$  \hspace{1cm} (4.6)

Therefore, (4.2) follows by (4.4) and (4.6) for $N \geq 3$, and by (4.5) and (4.6) for $N = 2$.

Similarly, we can have the following estimates: For $N \geq 2$, we have that

$$|(\nabla Φ * U_1)(x)| \leq c_{23}|x|^{-\theta - 1}(1 + |x|)^{\frac{N+1}{2} + \theta + 1}, \quad \forall x \in \mathbb{R}^N.$$  \hspace{1cm} (4.7)

and

$$|(\nabla Φ * U_2)(x)| \leq c_{24}(1 + |x|)^{\frac{N}{2} - \tau}, \quad \forall x \in \mathbb{R}^N,$$  \hspace{1cm} (4.8)

which imply (4.3).

Now we are in the position to show the existence of singular solution of (1.1).

**Proof of Theorem 1.2.** Denote

$$w_\sigma(x) := Φ(x) + φ_\sigma(x), \quad x \in \mathbb{R}^N \setminus \{0\},$$

where $φ_\sigma \in S_\sigma$ and

$$\sigma \in \left[\frac{1}{p-1} \left(\frac{N+1}{2} - \alpha\right), \frac{N-1}{2}\right],$$

which is a nonempty interval by the assumption of $p$.

Then $w_\sigma$ is a solution weak solution of

$$-\Delta w_\sigma - w_\sigma = \delta_0 \quad \text{in} \quad \mathbb{R}^N.$$  \hspace{1cm} (4.9)

Note that there is function $ψ_\sigma \in S_\sigma$ such that $w_\sigma = Φ + ψ_\sigma$ is smooth in $\mathbb{R}^N \setminus \{0\}$ and

$$\lim_{|x| \to 0} w_\sigma(x) \cdot |x|^{N-2} = C_N \quad \text{and} \quad \limsup_{|x| \to +\infty} |w_\sigma(x)||x|^\sigma = c_{25}.$$  \hspace{1cm} (4.10)

To obtain a solution of (1.9), we turn to obtain the weak solution of

$$-\Delta v - v = Q|kw_\sigma + v|^{p-1}(kw_\sigma + v) \quad \text{in} \quad \mathbb{R}^N$$

by considering the equivalent equation

$$v = Φ * \left(Q|kw_\sigma + v|^{p-1}(kw_\sigma + v)\right).$$  \hspace{1cm} (4.11)

For $p < p_N^*$, we have that $(2-N)p + 2 > 2 - N$, and now we fix

$$\theta_p = \begin{cases} \frac{2-N}{2} + \frac{(2-N)p+2}{2} & \text{if} \quad 2 - (N-2)p \leq 0, \\ 0 & \text{if} \quad 2 - (N-2)p > 0 \end{cases}$$

and denote

$$W_p(x) = |x|^\theta_p(1 + |x|)^{-\sigma - \theta_p} \quad \text{for} \quad x \in \mathbb{R}^N \setminus \{0\}.$$  \hspace{1cm} (4.12)

It is worth noting that

$$W_p(x) = (1 + |x|)^{-\sigma} \quad \text{if} \quad 2 - (N-2)p > 0.$$  \hspace{1cm} (4.13)

Now let’s denote

$$D_{p,k} = \left\{ v \in L^1(\mathbb{R}^N) : |v| \leq kW_p \quad \text{a.e. in} \quad \mathbb{R}^N \right\}$$

and

$$\mathcal{T} v = Φ * [Q|v + kw_\sigma|^{p-1}(v + kw_\sigma)], \quad \forall v \in D_{p,k}.$$  \hspace{1cm} (4.14)
Observe that for $N \geq 3$

$$Q|v + kw_\sigma|^p \leq \begin{cases} c_{26}k^p|x|^{(2-N)p} & \text{for } 0 < |x| \leq 1 \\ c_{26}k^p|x|^{-\alpha - \sigma_p} & \text{for } |x| > 1 \end{cases}$$

and for $N \geq 3$

$$Q|v + kw_\sigma|^p \leq \begin{cases} c_{26}k^p(1 - \ln |x|)^p & \text{for } 0 < |x| \leq 1 \\ c_{26}k^p|x|^{-\alpha - \sigma_p} & \text{for } |x| > 1, \end{cases}$$

then we apply Lemma 4.1 to obtain that

$$\left| \Phi^* [Q|v + kw_\sigma|^{p-1}(v + kw_\sigma)] \right| \leq \begin{cases} c_{27}k^p|x|^\theta_p & \text{for } 0 < |x| \leq 1, \\ c_{27}k^p|x|^{-\sigma_p} & \text{for } |x| > 1, \end{cases} \quad (4.11)$$

where $c' > 0$ is independent of $k$,

$$\sigma_p = \min \left\{ \frac{N - 1}{2}, \alpha + \sigma_p - \frac{1 + N}{2} \right\}.$$ 

In order to show $TD_{p,k} \subset D_{p,k}$, we need

$$\sigma \leq \sigma_p,$$

that is,

$$\sigma \leq \alpha + \sigma_p - \frac{1 + N}{2}. \quad (4.12)$$

Note that (4.12) holds by our choice of $\sigma$ i.e.

$$\sigma \in \left[ \frac{1}{p - 1} \left( \frac{N + 1}{2} - \alpha \right), \frac{N - 1}{2} \right]$$

and

$$\frac{1}{p - 1} \left( \frac{N + 1}{2} - \alpha \right) \leq \frac{N - 1}{2},$$

which is guaranteed by $p \geq p^\#$.

By the assumption $p > 1$, there exists $k^* > 0$ such that for $k \in (0, k^*)$

$$TD_{p,k} \subset D_{p,k}. \quad (4.13)$$

Note that for $v \in D_{p,k}$, applying Lemma 4.1 again we that

$$\left| \nabla \Phi^* [Q|v + kw_\sigma|^{p-1}(v + kw_\sigma)] \right| \leq \begin{cases} c_{28}k^p|x|^\theta_p & \text{for } 0 < |x| \leq 1, \\ c_{28}k^p|x|^{-\sigma_p} & \text{for } |x| > 1. \end{cases}$$

So we have that $Tv \in W^{1,\infty}_L(\mathbb{R}^N \setminus \{0\})$, by the embeddings $W^{1,\infty}(B_R \setminus B_1) \hookrightarrow L^1(B_R \setminus B_1)$ are compact and together with upper bound in $D_{p,k}$, we obtain that $T$ is a compact operator in $D_{p,k}$.

Observing that $D_{p,k}$ is a closed and convex set in $L^1(\mathbb{R}^N)$, we may apply Schauder fixed point theorem to derive that there exists $v_k \in D_{p,k}$ such that

$$Tv_k = v_k.$$ 

Since $|v_k| \leq kW_p$, so $v_k$ is locally bounded in $\mathbb{R}^N \setminus \{0\}$, then $u_k := v_k + kw_\sigma$ satisfies

$$\limsup_{|x| \to +\infty} |u(x)||x|^{\sigma} < +\infty.$$
Moreover, if \( \sigma_p > \sigma \), then (1.11) implies that
\[
|v_k(x)| = \Phi \ast |Qv_k + kw_\sigma|^{p-1}(v_k + kw_\sigma)| \leq c_{29}k^p|x|^{-\sigma_p} \quad \text{for } |x| > 1
\]
and then
\[
\lim_{|x| \to +\infty} \sup |u_k(x) - kw_\sigma(x)||x|^{-\sigma_p} \in [0, +\infty).
\]
By the standard interior regularity results, \( u_k \) is a positive classical solution of (1.1). From Theorem 1.1 it implies that \( u_k \) is a distributional solution of (1.1). \( \square \)

5 Complex valued solutions

Let \( u \) be a complex value solutions of (1.1), then it could be written as
\[
u = u_1 + iu_2,\]
where \( u_1, u_2 \) are the isolated singular solution of system in real valued framework
\[
\begin{cases}
-\Delta u_1 - u_1 = Q|u|^{p-1}u_1 & \text{in } \mathbb{R}^N \setminus \{0\}, \\
-\Delta u_2 - u_2 = Q|u|^{p-1}u_2 & \text{in } \mathbb{R}^N \setminus \{0\}, \\
\lim_{|x| \to 0^+} u_1(x) = +\infty,
\end{cases}
\]
where \( |u| = (u_1^2 + u_2^2)^{\frac{1}{2}} \).

Moreover, the complex valued weak solution of (1.9) could be written
\[
\begin{cases}
-\Delta u_1 - u_1 = Q|u|^{p-1}u_1 + k\delta_0 & \text{in } \mathcal{D}'(\mathbb{R}^N), \\
-\Delta u_2 - u_2 = Q|u|^{p-1}u_2 & \text{in } \mathcal{D}'(\mathbb{R}^N).
\end{cases}
\]

It is worthing noting that, in the complex valued framework, the isolated singular solution of (1.1) is no longer to be classified by the Dirac mass in (1.1) in the Serrin’s subcritical case. In fact, it should be classified by
\[
-\Delta u - u = Q|u|^{p-1}u + k_1\delta_0 + k_2i\delta_0 \quad \text{in } \mathbb{R}^N
\]
or in the form of system (in the real valued framework)
\[
\begin{cases}
-\Delta u_1 - u_1 = Q|u|^{p-1}u_1 + k_1\delta_0 & \text{in } \mathcal{D}'(\mathbb{R}^N), \\
-\Delta u_2 - u_2 = Q|u|^{p-1}u_2 + k_2\delta_0 & \text{in } \mathcal{D}'(\mathbb{R}^N).
\end{cases}
\]

However, the weak solutions of (1.9) are classical isolated singular solutions of (1.11). Therefore, our interest in this section is to obtain weak solutions (1.9) in the Serrin’s subcritical case and the result states as follows.

**Theorem 5.1** Assume that \( N \geq 2 \), the potential \( Q \) verifies (Q\(\alpha\)) with \( \alpha \) satisfying (1.13). Let \( p_\alpha^\#, p_\alpha^* \) be defined in (1.8) and (1.12) respectively.

Then for
\[
p \in (1, +\infty) \cap [p_\alpha^\#, p_\alpha^*]
\]
and \( \psi_\sigma \in \mathcal{S}_\sigma \) with
\[
\sigma \in \left[\frac{1}{p - 1} \left(\frac{N + 1}{2} - \alpha\right), \frac{N - 1}{2}\right],
\]
there exists \( k^* > 0 \) such that for \( k \in (0, k^*) \), problem (1.9) admits a complex valued solution \( u_k \) in an integral form
\[
u_k = \Phi_c \ast (Q|u_k|^{p-1}u_k) + k(\Phi_c + \psi_\sigma) \quad \text{in } \mathbb{R}^N.
\]
In order to obtain the complex valued solution, we need the following estimate.

**Lemma 5.1** Let

\[ N \geq 2, \quad \tau > \frac{N + 1}{2}, \quad \theta \in (-1, N - 2) \setminus \{0\}, \]

then for \( U_{\theta, \tau} \in L^1(\mathbb{R}^N, \mathbb{C}) \) verifying

\[ |U_{\theta, \tau}(x)| \leq |x|^{-\theta - 2}(1 + |x|)^{-\tau + \theta + 2}, \quad \text{a.e. for } x \in \mathbb{R}^N \setminus \{0\}, \]

there holds for \( c_{30} > 0 \)

\[ |(\Phi_c * U_{\theta, \tau})(x)| \leq c_{30}(1 + |x|^{-\theta})(1 + |x|)\max\{-\frac{N-1}{2} , \frac{1+\theta}{2} - \tau\} + \theta, \quad \forall x \in \mathbb{R}^N \quad (5.4) \]

and

\[ |(\nabla \Phi_c * U_{\theta, \tau})(x)| \leq c_{30}|x|^{-\theta - 1}(1 + |x|)\max\{-\frac{N-1}{2} , \frac{1+\theta}{2} - \tau\} + \theta + 1, \quad \forall x \in \mathbb{R}^N. \quad (5.5) \]

**Proof.** Note that \( \Phi_c = \Phi + i\Psi, \) \( U_{\theta, \tau} = U_{\theta, \tau, 1} + iU_{\theta, \tau, 2} \)

\[ \Phi * U_{\theta, \tau} = \Phi * U_{\theta, \tau, 1} + i\Phi * U_{\theta, \tau, 2} \]

and

\[ \nabla \Phi * U_{\theta, \tau} = \nabla \Phi * U_{\theta, \tau, 1} + i\nabla \Phi * U_{\theta, \tau, 2}, \]

then the related estimates could see Lemma [4.1]. So we only need to prove that there exists \( c_{32} > 0 \) such that for \( j = 1, 2 \)

\[ |(\Psi * U_{\theta, \tau, j})(x)| \leq c_{32}(1 + |x|)^{\frac{N+1}{2} - \tau}, \quad \forall x \in \mathbb{R}^N \quad (5.6) \]

and

\[ |(\nabla \Psi * U_{\theta, \tau, j})(x)| \leq c_{32}(1 + |x|)^{\frac{N+1}{2} - \tau}, \quad \forall x \in \mathbb{R}^N. \quad (5.7) \]

Indeed, note that \( \Psi \) and \( \nabla \Psi \) are smooth in \( \mathbb{R}^N \) and have the asymptotic behavior at infinity controlled by \( (1 + |x|)^{-\frac{N-1}{2}} \). Let

\[ U_1(x) = |x|^{-2 - \theta} \chi_{B_1}(x), \quad U_2(x) = |x|^{-\tau} \chi_{B_1}(x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}. \]

Similar computations in Lemma [4.1] we have that

\[ |(\Psi * U_1)(x)| \leq c_{32}(1 + |x|)^{\frac{1-N}{2}}, \quad \forall x \in \mathbb{R}^N \quad (5.8) \]

and

\[ |(\Psi * U_2)(x)| \leq c_{32}(1 + |x|)^{\frac{1+N}{2} - \tau}, \quad \forall x \in \mathbb{R}^N, \quad (5.9) \]

which imply (5.6).

Note that (5.8) and (5.9) hold true replacing \( \Psi \) by \( \nabla \Psi \), then (5.7) holds true. \( \Box \)

**Proof of Theorem 5.1.** Set

\[ w_\sigma(x) := \Phi_c(x) + \phi_\sigma(x), \quad x \in \mathbb{R}^N \setminus \{0\}, \]

where \( \phi_\sigma \in \mathcal{S}_\sigma \) and

\[ \sigma \in \left[ \frac{1}{p-1} \left( \frac{N+1}{2} - \alpha \right), \frac{N-1}{2} \right]. \]

It is a complex valued solution weak solution of

\[ -\Delta w_\sigma - w_\sigma = \delta_0 \quad \text{in } \mathbb{R}^N. \]
To obtain a complex valued solution of (1.9), we would consider the complex valued solution of
\[ \mathbf{v} = \Phi_c \ast \left( Q|k\mathbf{w}_\sigma + \mathbf{v}|^{p-1}(k\mathbf{w}_\sigma + \mathbf{v}) \right). \] (5.10)

Recall that
\[ \theta_p = \begin{cases} \frac{2-N}{2} + \frac{(2-N)p+2}{2} & \text{if } 2 - (N-2)p \leq 0, \\ 0 & \text{if } 2 - (N-2)p > 0 \end{cases} \]

and
\[ W_p(x) = |x|^{-\theta_p}(1 + |x|)^{-\sigma + \theta_p} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}. \]

Re-denote
\[ D_{p,k} = \{ \mathbf{v} \in L^1(\mathbb{R}^N; \mathbb{C}) : |\mathbf{v}| \leq kW_p \text{ a.e. in } \mathbb{R}^N \} \]

and
\[ T_c\mathbf{v} = \Phi_c \ast [Q|\mathbf{v} + k\mathbf{w}_\sigma|^{p-1}(\mathbf{v} + k\mathbf{w}_\sigma)], \quad \forall \mathbf{v} \in D_{p,k}. \]

Observe that
\[ |Q|\mathbf{v} + k\mathbf{w}_\sigma|^p \leq \begin{cases} c_{33}k^p|x|^{(2-N)p} & \text{for } 0 < |x| \leq 1, \\ c_{33}k^p|x|^{-\alpha - \sigma p} & \text{for } |x| > 1, \end{cases} \]

then we apply Lemma 5.1 to obtain that
\[ |\Phi_c \ast [Q|\mathbf{v} + k\mathbf{w}_\sigma|^{p-1}(\mathbf{v} + k\mathbf{w}_\sigma)]| \leq \begin{cases} c_{34}k^p|x|^\theta_p & \text{for } 0 < |x| \leq 1, \\ c_{34}k^p|x|^{-\sigma p} & \text{for } |x| > 1, \end{cases} \] (5.11)

where \( c_{34} > 0 \) is independent of \( k \),
\[ \sigma_p = \min \left\{ \frac{N - 1}{2}, \alpha + \sigma p - \frac{1 + N}{2} \right\}. \]

In order to show \( T_cD_{p,k} \subset D_{p,k} \), we need
\[ \sigma \leq \sigma_p, \]
that is,
\[ \sigma \leq \alpha + \sigma p - \frac{1 + N}{2}. \] (5.12)

Note that the first inequality of (4.12) holds by our choice of \( \sigma \) i.e.
\[ \sigma \in \left[ \frac{1}{p-1} \left( \frac{N + 1}{2} - \alpha \right), \frac{N - 1}{2} \right] \]

by \( p \geq p_0^\# \).

By the assumption \( p > 1 \), there exists \( k^* > 0 \) such that for \( k \in (0,k^*) \)
\[ T_cD_{p,k} \subset D_{p,k}. \]

Note that for \( \mathbf{v} \in D_{p,k} \), applying Lemma 4.1 again we that
\[ |\nabla \Phi_c \ast [Q|\mathbf{v} + k\mathbf{w}_\sigma|^{p-1}(\mathbf{v} + k\mathbf{w}_\sigma)]| \leq \begin{cases} c_{34}k^p|x|^\theta_p^{-1} & \text{for } 0 < |x| \leq 1, \\ c_{34}k^p|x|^{-\sigma p} & \text{for } |x| > 1. \end{cases} \]

So we have that \( T_c\mathbf{v} \in W^{1,\infty}_{loc}(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) \), by the embeddings \( W^{1,\infty}(B_R \setminus B_{R^\#}, \mathbb{C}) \hookrightarrow L^1(B_R \setminus B_{R^\#}, \mathbb{C}) \) are compact and together with upper bound in \( D_{p,k} \), we obtain that \( T_c \) is a compact operator in \( D_{p,k} \).
Observing that $\mathcal{D}_{p,k}$ is a closed and convex set in $L^1(\mathbb{R}^N, \mathbb{C})$, we may apply Schauder fixed point theorem to derive that there exists $v_k \in \mathcal{D}_{p,k}$ such that

$$\mathcal{T}_c v_k = v_k.$$ 

Since $|v_k| \leq k W_p$, so $v_k$ is locally bounded in $\mathbb{R}^N \setminus \{0\}$, then $u_k := v_k + kw_\sigma$ satisfies

$$\limsup_{|x| \to +\infty} |u_k(x)||x|^{\sigma} < +\infty.$$ 

Moreover, if

$$\alpha + \sigma p - \frac{1 + N}{2} > \sigma,$$

then (5.11) implies that

$$|v_k(x)| = |\Phi_c \ast [Q|v_k + kw_\sigma|^{p-1}(v_k + ku_\sigma)]| \leq c_{36} k^p |x|^{-\sigma p} \quad \text{for } |x| > 1$$

and then

$$\limsup_{|x| \to +\infty} |u_k(x) - kw_\sigma(x)||x|^{\sigma} \in [0, +\infty).$$

By the standard interior regularity results for system (5.2), $u_k$, is a positive classical solution of (1.1). From Theorem 1.1 it implies that $u_k$ is a distributional solution of (1.1). \qed

Acknowledgements: This work is is supported by NNSF of China, No: 12071189 and 12001252, by the Jiangxi Provincial Natural Science Foundation, No: 20202BAB201005, 20202ACBL201001. F.Z. is supported by Science and Technology Commission of Shanghai Municipality (STCSM), grant No. 18dz2271000 and also supported by NSFC (No. 11431005).

References

[1] P. Aviles, Local behaviour of the solutions of some elliptic equations, Comm. Math. Phys. 108, 177-192 (1987).

[2] S. Agmon, A representation theorem for solutions of the Helmholtz equation and resolvent estimates for the Laplacian, Analysis, et cetera, 39-76 (1990).

[3] Ph. Bénilan and H. Brezis, Nonlinear problems related to the Thomas-Fermi equation, J. Evolution Eq. 3, 673-770 (2003).

[4] H. Brezis, Some variational problems of the Thomas-Fermi type. Variational inequalities and complementarity problems, Proc. Internat. School, Erice, Wiley, Chichester, 53-73 (1980).

[5] H. Brezis and P.L. Lions; A note on isolated singularities for linear elliptic equations, in Mathematical Analysis and Applications, Acad. Press, 263-266 (1981).

[6] H. Brezis and L. Véron, Removable singularities of some nonlinear elliptic equations, Arch. Rational Mech. Anal. 75, 1-6 (1980).

[7] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. 42, 271-297 (1989).

[8] H. Chen, A. Quaas, Classification of isolated singularities of nonnegative solutions to fractional semi-linear elliptic equations and the existence results, J. London Math. Soc. 97(2): 196-221 (2018).
[9] H. Chen, G. Evéquoz and T. Weth, Complex solutions and stationary scattering for the nonlinear Helmholtz equation, *SIAM J. Math. Anal.* 53(2), 2349-2372 (2021).

[10] A. Enciso, D. Peralta-Salas, Bounded solutions to the Allen-Cahn equation with level sets of any compact topology, *Analysis and PDE* 9(6), 1433-1446 (2016).

[11] G. Evéquoz, A dual approach in Orlicz spaces for the nonlinear Helmholtz equation, *Z. Angew. math. Phys.* 66, 2995-3015 (2015).

[12] G. Evéquoz and T. Weth, Real solutions to the nonlinear Helmholtz equation with local nonlinearity, *Arch. Rational Meth. Anal.* 211, 359-388 (2014).

[13] G. Evéquoz and T. Weth, Dual variational methods and nonvanishing for the nonlinear Helmholtz equation, *Adv. math.* 280, 690-728 (2015).

[14] G. Evéquoz and T. Yesil, Dual ground state solutions for the critical nonlinear Helmholtz equation, *arXiv: 1707.00959* (2017).

[15] B. Gidas and J. Spruck, Global and local behaviour of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* 34, 525-598 (1981).

[16] S. Guitiérrez, Non trivial $L^q$ solutions to the Ginzburg-Landau equation, *Math. Ann. 328*, 1-25 (2004).

[17] E. Jalade, Inverse problem for a nonlinear Helmholtz equation, *Ann. l’IHP Anal. non linéaire.* 21(4), 517-531 (2004).

[18] T. Klimsiak, A. Rozkosz, On semilinear elliptic equations with diffuse measures. *NoDEA Nonlinear Differential Equations Appl.* 25(4) No. 35, 23 pp. (2018).

[19] P. Lions, Isolated singularities in semilinear problems, *J. Diff. Eq.* 38(3), 441-450 (1980).

[20] N. Lebedev, Special functions and their applications, *Dover Publications Inc., New York*, 1972.

[21] R. Mandel, The limiting absorption principle for periodic differential operators and applications to nonlinear Helmholtz equations, *Comm. Math. Phys.* 368(2), 799-842(2019).

[22] R. Mandel, E. Montefusco and B. Pellacci, Oscillating solutions for nonlinear Helmholtz equations, *Z. Angew. Math. Phys.* 68-121(2017)

[23] A. C. Ponce and N. Wilmet, Schrödinger operators involving singular potentials and measure data, *J. Diff. Eq.* 263, 3581-3610 (2017).

[24] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, *Princeton University Press*, (1970).

[25] Y. Wang, Isolated singularities of solutions of defocusing Hartree equation. *Nonlinear Anal.* 156, 70-81 (2017).

[26] L. Véron, Weak and strong singularities of nonlinear elliptic equations, *Proc. Symp. Pure Math.* 45, 477-495 (1986).

[27] L. Véron, Singular solutions of some nonlinear elliptic equations, *Nonlinear Anal. T. M. & A.* 5, 225-242 (1981).

[28] L. Véron, Elliptic equations involving Measures, *Stationary Partial Differential equations, Vol. I, 593-712, Handb. Differ. Equ., North-Holland, Amsterdam*