 Barrier Penetration for Supersymmetric Shape-Invariant Potentials

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Abstract

Exact reflection and transmission coefficients for supersymmetric shape-invariant potentials barriers are calculated by an analytical continuation of the asymptotic wave functions obtained via the introduction of new generalized ladder operators. The general form of the wave function is obtained by the use of the \( \mathbf{F}(-\infty, +\infty) \)-matrix formalism of Fröman and Fröman which is related to the evolution of asymptotic wave function coefficients.

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I. INTRODUCTION

Quantum tunneling through a potential barrier governs many interesting phenomena in physics ranging from fusion reactions in stars [1] to the study of transitions from metastable states [2]. There are very few exactly solvable examples of barrier penetration. Supersymmetric quantum mechanics has been shown to be a useful technique to explore exactly solvable problems in quantum mechanics [3]. An integrability condition called shape-invariance was introduced by Gendenshtein [4] and was cast into an algebraic form by Balantekin [5]. Reflection and transmission coefficients for a large class of shape-invariant potentials was given by Cooper et al [6]. A general operator method for calculating scattering amplitudes for supersymmetric shape-invariant potentials was introduced by Khare and Sukhatme [7]. Even though an approximate method in the context of supersymmetric semiclassical approximation [8] to calculate tunneling through one-dimensional potential barriers was presented in [9] exact tunneling probabilities for shape-invariant barriers was not explicitly derived. We cover this latter subject in this article.

Introducing the superpotential function

\[ W(x) \equiv -\frac{\hbar}{\sqrt{2m}} \left[ \frac{\Psi'_0(x)}{\Psi_0(x)} \right], \]  

(1.1)

where \( \Psi_0(x) \) is the ground-state wave function of the Hamiltonian \( \hat{H} \), and defining the operators

\[ \hat{A} \equiv W(x) + \frac{i}{\sqrt{2m}} \hat{p}, \]  

(1.2)

\[ \hat{A}^\dagger \equiv W(x) - \frac{i}{\sqrt{2m}} \hat{p}, \]  

(1.3)

we can show that

\[ \hat{H} - E_0 = \hat{A}^\dagger \hat{A}. \]  

(1.4)

Since the ground-state wave function satisfies the condition

\[ \hat{A} \Psi_0(x) = 0 \]  

(1.5)

the supersymmetric partner potentials

\[ \hat{H}_1 = \hat{A}^\dagger \hat{A} \quad \hat{H}_2 = \hat{A} \hat{A}^\dagger \]  

(1.6)

have the same energy spectra except the ground state of \( \hat{H}_1 \) which has no corresponding state in the spectra of \( \hat{H}_2 \). The corresponding potentials are given by

\[ V_1(x) = [W(x)]^2 - \frac{\hbar}{\sqrt{2m}} \frac{dW}{dx}, \]  

(1.7)

\[ V_2(x) = [W(x)]^2 + \frac{\hbar}{\sqrt{2m}} \frac{dW}{dx}. \]  

(1.8)
The shape-invariance condition [4]
\[ V_2(x, a_1) = V_1(x, a_2) + R(a_1) \]  \hspace{1cm} (1.9)
can also be written as [3]
\[ \hat{A}(a_1) \hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2) \hat{A}(a_2) + R(a_1), \] \hspace{1cm} (1.10)
where \( a_{1,2} \) are a set of parameters that specify space-independent properties of the potentials (such as strength, range, and diffuseness). The parameter \( a_2 \) is a function of \( a_1 \) and the remainder \( R(a_1) \) is independent of \( \hat{x} \) and \( \hat{p} \). Not all exactly solvable potentials are shape-invariant [10]. In the cases studied so far the parameters \( a_1 \) and \( a_2 \) are either related by a translation [10,11] or a scaling [12]. Introducing the similarity transformation that replaces \( \hat{a}_1 \) with \( \hat{a}_2 \) in a given operator
\[ \hat{T}(a_1) \hat{O}(a_1) \hat{T}^\dagger(a_1) = \hat{O}(a_2) \] \hspace{1cm} (1.11)
and the operators
\[ \hat{B}_+ = \hat{A}^\dagger(a_1) \hat{T}(a_1) \] \hspace{1cm} (1.12)
\[ \hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1) \hat{A}(a_1), \] \hspace{1cm} (1.13)
the Hamiltonian takes the form
\[ \hat{H} - E_0 = \hat{B}_+ \hat{B}_-. \] \hspace{1cm} (1.14)
Using Eq. (1.10) one can easily prove the commutation relation
\[ [\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger(a_1) R(a_1) \hat{T}(a_1) \equiv R(a_0), \] \hspace{1cm} (1.15)
where we used the identity
\[ R(a_n) = \hat{T}(a_1) R(a_{n-1}) \hat{T}^\dagger(a_1), \] \hspace{1cm} (1.16)
valid for any \( n \). Equation (1.15) suggests that \( \hat{B}_+ \) and \( \hat{B}_- \) are the appropriate creation and annihilation operators provided that their non-commutativity with \( R(a_1) \) is taken into account. In this paper we extend the use of \( \hat{B}_+ \) and \( \hat{B}_- \) operators for calculating the asymptotic behaviors of the wave functions related with incidence of a particle on a supersymmetric shape-invariant potential barrier and obtain the exact transmission and reflection coefficients.

II. EXACT WAVE FUNCTIONS

The wave functions for all currently known supersymmetric shape-invariant potential barriers can be calculated analytically using supersymmetric operator techniques [13,14]. The final result can be expressed by single operators \( \hat{B}_+ \) and \( \hat{B}_-^{-1} \) [15] or by couples of these operators. In the first case we can use the two additional commutation relations
\[
[\hat{B}_+ \hat{B}_-, \hat{B}_+] = \sum_{k=1}^{n} R(a_k) \hat{B}_+^n, \tag{2.1}
\]
and
\[
[\hat{B}_+ \hat{B}_-, \hat{B}_-] = \sum_{k=1}^{n} R(a_k) \hat{B}_-^n, \tag{2.2}
\]
obtained by induction using the relations
\[
R(a_n) \hat{B}_+ = \hat{B}_+ R(a_{n-1}), \tag{2.3}
\]
\[
R(a_n) \hat{B}_- = \hat{B}_- R(a_{n+1}), \tag{2.4}
\]
that readily follow from Eqs. (1.11), (1.12) and (1.13). Considering that the Schrödinger equation can be written as
\[
\hat{B}_+ \hat{B}_- \Psi(x) = \Lambda \Psi(x), \tag{2.5}
\]
then the Eqs. (2.1) and (2.2) imply that \( \hat{B}_+ \) and \( \hat{B}_- \) can be used as ladder operators to solve the Eq. (2.5) \[5,16\]. To this end we introduce \( \Psi(0) \) as the solution of the equation
\[
\hat{A}_-(a_1) \Psi(0) = 0 = \hat{B}_-(a_1) \Psi(0), \tag{2.6}
\]
which implies
\[
\Psi(0)(x, a_1) \sim \exp \left( -\frac{\sqrt{2m}}{\hbar} \int_{x}^{\infty} d\xi W(\xi, a_1) \right). \tag{2.7}
\]
If the function
\[
f(n) = \sum_{k=1}^{n} R(a_k) \tag{2.8}
\]
can be analytically continued so that the condition
\[
f(\mu) = \Lambda \tag{2.9}
\]
is satisfied for a particular (in general complex) value of \( \mu \), then the Eq. (2.1) implies that one possible form for the solution of Eq. (2.5) is \( \hat{B}_+^\mu \Psi(0)(x, a_1) \). Similarly if \( \Psi(0)(x) \) satisfies the equation
\[
\hat{B}_+(a_1) \Psi(0)(x) = 0, \tag{2.10}
\]
which implies that
\[
\hat{T}(a_1) \Psi(0)(x) \sim \exp \left( \frac{\sqrt{2m}}{\hbar} \int_{x}^{\infty} d\xi W(\xi, a_1) \right), \tag{2.11}
\]
or
\[
\Psi^{(0)}_+(x, a_0) \sim \exp \left( \frac{\sqrt{2m}}{\hbar} \int^x d\xi W(\xi, a_0) \right),
\] (2.12)

then the Eq. (2.2) implies that other possible form for the solution of Eq. (2.5) is
\[
\hat{B}^{-\mu-1}\Psi^{(0)}_+(x, a_0).
\]

At this point we conclude that the components of the wave functions, written in terms of the singles operators \( \hat{B}_+ \) and \( \hat{B}^{-1} \), for supersymmetric shape-invariant potential barriers can be written down as
\[
\Psi_-(x) = \beta \hat{B}_+^\mu \Psi^{(0)}(x, a_1)
\] (2.13a)
\[
\Psi_+(x) = \gamma \hat{B}^{-\mu-1}_- \Psi^{(0)}_+(x, a_0),
\] (2.13b)

where \( \mu \) is obtained by the relation
\[
\Lambda = \sum_{k=1}^{\mu} R(a_k)
\] (2.14)

where \( \beta \) and \( \gamma \) are constants and
\[
\Psi^{(0)}_{\pm}(x, a_\mu) = \exp \left( \pm \frac{\sqrt{2m}}{\hbar} \int^x d\xi W(\xi, a_\mu) \right).
\] (2.15)

With each value of \( \mu \) we obtain several possible expressions for the components \( \Psi_{\pm}(x) \) and the general expression for the wave function can be obtained with these components or a combination of them.

It is also possible to express the components of the wave functions using couples of the operators \( \hat{B}_+ \) and \( \hat{B}^{-1} \). In this case we can use the equations (2.1), (2.2) and the relations (2.3), (2.4) to show by induction that
\[
[\hat{B}_+ \hat{B}_-, (\hat{B}_+ \hat{B}^{-1})^n] = \sum_{k=1}^{2n} R(a_k) (\hat{B}_+ \hat{B}^{-1})^n,
\] (2.16)

and
\[
[\hat{B}_- \hat{B}_+, (\hat{B}^{-1} \hat{B}_+)^n] = \sum_{k=1}^{2n} R(a_k) (\hat{B}^{-1} \hat{B}_+)^n.
\] (2.17)

Using these last two equations and the same conditions (2.6) and (2.10) we can show that the components of the wave functions, written in terms of couples of the operators \( \hat{B}_+ \) and \( \hat{B}^{-1} \), for supersymmetric shape-invariant potential barriers can be written down as
\[
\Psi_-(x) = \beta (\hat{B}_+ \hat{B}^{-1})^\nu \Psi^{(0)}(x, a_1)
\] (2.18a)
\[
\Psi_+(x) = \gamma (\hat{B}^{-1} \hat{B}_+)^\nu \hat{B}^{-1} \Psi^{(0)}_+(x, a_0)
\] (2.18b)

and where \( \nu \) is obtained by the relation
\[
\Lambda = \sum_{k=1}^{2\nu} R(a_k).
\] (2.19)

Note that in a given problem either Eqs. (2.13) or Eqs. (2.18) could be used, but not both.
III. ASYMPTOTIC WAVE FUNCTIONS

The formal expressions for the components of the wave functions can be express in explicit forms if we evaluate them asymptotically. In the case of single operators expressions we first note that using Eqs. (1.12) and (1.13) the results (2.13) can be written as

\[ \Psi_-(x) = \beta \hat{A}_+(a_1) \hat{A}_+(a_2) \cdots \hat{A}_+(a_m) \Psi_-(0, x, a_{\mu+1}) \]  
(3.1a)

\[ \Psi_+(x) = \gamma \hat{A}_-(a_1) \hat{A}_-(a_2) \cdots \hat{A}_-(a_{\mu+1}) \Psi_+(0, x, a_{\mu+1}) \]  
(3.1b)

In this point we need to consider the two basic asymptotic behavior for the superpotential: i) \( W(x \to \pm \infty, a_{\mu}) \) is constant (i.e., the potential barrier goes to a constant); and ii) \( W(x \to \pm \infty, a_{\mu}) \to \pm \infty \) (i.e., the potential barrier goes to \(-\infty\)). In the former limit the commutator

\[ \left[ \frac{\partial}{\partial x}, W(x, a_{\mu}) \right] = W'(x, a_{\mu}) \]  
(3.2)

vanishes. In the ladder case this commutator can be ignored as

\[ W(x, a_n) W(x, a_k) + W'(x, a_n) = W(x, a_n) W(x, a_k) \left( 1 + \frac{W'(x, a_n)}{W(x, a_n) W(x, a_k)} \right) \]
\[ \to W(x, a_n) W(x, a_k) , \]  
(3.3)

provided that \( W'(x, a_n)/W(x, a_n) \) remains finite, which is the case for all realistic superpotentials. Hence in both limits we can write Eqs. (3.1) as

\[ \Psi_-(x) = \beta W_1 + W_{\mu+1}) \Psi_-(0, x, a_{\mu+1}) \]  
(3.4a)

\[ \Psi_+(x) = \gamma (W_1 + W_{\mu+1})^{-1}(W_2 + W_{\mu+1})^{-1} \cdots (W_{\mu+1} + W_{\mu+1})^{-1} \Psi_+(0, x, a_{\mu+1}) . \]  
(3.4b)

In these equations the quantity \( W_m \) is the short-hand notation for \( W(x, a_m) \).

If we assume that the superpotential satisfies the condition

\[ W(x, a_n) = W(x, a_1) + (n-1) \zeta(x) , \]  
(3.5)

then these asymptotic equations can be in a form suitable for analytic continuation, that is

\[ \Psi_-(x) = \beta \zeta^\mu \frac{\Gamma(2z + 2\mu)}{\Gamma(2z + \mu)} \Psi_-(0, x, a_{\mu+1}) \]  
(3.6a)

\[ \Psi_+(x) = \gamma \zeta^{-\mu-1} \frac{\Gamma(2z + \mu)}{\Gamma(2z + 2\mu + 1)} \Psi_+(0, x, a_{\mu}) , \]  
(3.6b)

where \( z = W(x, a_1)/\zeta(x) \). The condition given by Eq. (3.3) is satisfied for a number of superpotentials and in the last section we give some examples. If this condition is not satisfied, the analytic continuation may still be done, but will be more complicated.

When the asymptotic behavior of the superpotential is \( W(x, a_n) \to \pm \infty \) we can use the identity
\[
\lim_{y \to \pm \infty} \frac{1}{y^{\mu} \Gamma(y + \mu)} = 1 \quad (3.7)
\]

to express Eqs. (3.6) in the simple form
\[
\Psi_-(x) = \beta (2W_1)^{\mu} \Psi^{(0)}_-(x, a_{\mu+1}) \quad (3.8a)
\]
\[
\Psi_+(x) = \gamma (2W_1)^{-\mu-1} \Psi^{(0)}_+(x, a_{\mu+1}). \quad (3.8b)
\]

We can repeat the same procedure used above in the case of couple of operators. Again, using Eqs. (1.12) and (1.13) the results (2.18) can be written as
\[
\Psi_-(x) = \beta \prod_{k=1}^{\nu} \hat{A}_+(a_{2k-1}) \hat{A}^{-1}_+(a_{2k}) \Psi^{(0)}_-(x, a_{2\nu+1}) \quad (3.9a)
\]
\[
\Psi_+(x) = \gamma \prod_{k=1}^{\nu} \hat{A}_{-1}(a_{2k-1}) \hat{A}_+(a_{2k}) \hat{A}^{-1}_+(a_{2\nu+1}) \Psi^{(0)}_+(x, a_{2\nu+1}). \quad (3.9b)
\]

Considering the superpotential asymptotic simplifications given by Eqs. (3.2), (3.3) and the analytic continuation condition (3.5) we can write the result for the components of the asymptotic wave functions in this case as
\[
\Psi_-(x) = \beta \frac{\Gamma(1 - z - \nu)}{\Gamma(1 - z - 2\nu) \Gamma(\nu + \frac{1}{2})} \Psi^{(0)}_-(x, a_{2\nu+1}) \quad (3.10a)
\]
\[
\Psi_+(x) = \gamma \frac{\Gamma(-z - 2\nu) \Gamma(\nu + \frac{1}{2})}{\Gamma(1 - z - \nu)} \Psi^{(0)}_+(x, a_{2\nu+1}). \quad (3.10b)
\]

IV. GENERAL ASYMPTOTIC WAVE FUNCTIONS AND THE TRANSMISSION AND REFLECTION COEFFICIENTS

Using the formalism developed in the Ref. [17] we can write two possible asymptotic solutions for the one-dimensional time-independent Schrödinger equation in the form
\[
\Psi_1(x \to \pm \infty) = A_{11}(\pm \infty) f_1(x) + A_{21}(\pm \infty) f_2(x) \quad (4.1a)
\]
\[
\Psi_2(x \to \pm \infty) = A_{12}(\pm \infty) f_1(x) + A_{22}(\pm \infty) f_2(x), \quad (4.1b)
\]

where
\[
f_1(x) = \frac{\exp(+i\chi(x))}{\sqrt{q(x)}} \quad \text{and} \quad f_2(x) = \frac{\exp(-i\chi(x))}{\sqrt{q(x)}}, \quad (4.2)
\]

with
\[
\chi(x) = \int_x^\infty q(\xi) d\xi \quad \text{and} \quad q(x) = \frac{\sqrt{2m}}{\hbar} W(x, a_{\mu+1}). \quad (4.3)
\]

If we define the vectors
\[
| \Psi(x) \rangle = \left[ \begin{array}{c} | \Psi_1(x) \rangle \\ | \Psi_2(x) \rangle \end{array} \right] \tag{4.4}
\]

and
\[
\langle f(x) \mid = \left[ \langle f_1(x) \mid \langle f_2(x) \mid \right] \tag{4.5}
\]

then we can write
\[
\langle \Psi(x \to \pm \infty) \mid = \langle f(x) \mid A(\pm \infty), \tag{4.6}
\]

where the asymptotic coefficients matrix is given by
\[
A(\pm \infty) = \left[ \begin{array}{cc} A_{11}(\pm \infty) & A_{12}(\pm \infty) \\ A_{21}(\pm \infty) & A_{22}(\pm \infty) \end{array} \right] \tag{4.7}
\]

Considering that the space evolution of the coefficients matrix \( A \), given by an iteration process, can be written as
\[
A(x) = F(x, x_0) A(x_0), \tag{4.8}
\]

then if we know the asymptotic coefficients in \(-\infty\) and \(+\infty\) we can obtain the evolution matrix
\[
F(-\infty, +\infty) = A(-\infty) A^{-1}(+\infty). \tag{4.9}
\]

Using the Eq. (4.9) we can show two basics properties of the \( F \) matrix:
\[
det F(-\infty, +\infty) = 1 \tag{4.10a}
\]
\[
F(-\infty, +\infty) = F^{-1}(+\infty, -\infty). \tag{4.10b}
\]

The knowledge of \( F(-\infty, +\infty) \) permits the determination of the exact transmission and reflection barrier coefficients. If we consider a wave incidence from \(-\infty\) to \(+\infty\) then we can write the asymptotic wave function in the form
\[
\Psi(x \to -\infty) = f_1(x) + C_R f_2(x) \tag{4.11a}
\]
\[
\Psi(x \to +\infty) = C_T f_2(x), \tag{4.11b}
\]

or
\[
\Psi(x \to -\infty) = \langle f(x) \mid a(-\infty) \rangle \tag{4.12a}
\]
\[
\Psi(x \to +\infty) = \langle f(x) \mid a(+\infty) \rangle, \tag{4.12b}
\]

where
\[
| a(-\infty) \rangle = \left[ \begin{array}{c} 1 \\ C_R \end{array} \right] \quad \text{and} \quad | a(+\infty) \rangle = \left[ \begin{array}{c} 0 \\ C_T \end{array} \right]. \tag{4.13}
\]

Considering that
\[
| a(-\infty) \rangle = F(-\infty, +\infty) | a(+\infty) \rangle \tag{4.14}
\]
we can conclude that

\[ C_T = \frac{1}{F_{12}(-\infty, +\infty)} \]  
\[ C_R = \frac{F_{22}(-\infty, +\infty)}{F_{12}(-\infty, +\infty)} , \]  

and the transmission and reflection coefficients are given by \( T = |C_T|^2 \) and \( R = |C_R|^2 \). At this point if we consider the conservation of probability, the time reversal invariance, and the invariance under space reflection it is possible to show the additional \( F \)-matrix properties:

\[ F_{11}(-\infty, +\infty) = -F_{22}(-\infty, +\infty) \]  
\[ F_{21}(-\infty, +\infty) = -F_{12}(-\infty, +\infty) \]  
\[ |F_{12}(-\infty, +\infty)| = |F_{12}(+\infty, -\infty)| \geq 1 \]  
\[ |F_{22}(-\infty, +\infty)| = |F_{22}(+\infty, -\infty)| \leq |F_{12}(-\infty, +\infty)| . \]

V. APPLICATIONS

A. Parabolic Barrier

For a parabolic potential barrier \([18]\]

\[ V_1(x) = V_0 - \frac{1}{2}m\Omega^2 x^2 , \]  

the corresponding superpotential, obtained by using the Eq. (1.7), is given by

\[ W(x, a_1) = a_1 x \]  

where \( a_1 = \pm i\sqrt{\frac{m}{2}\Omega} \). The shape invariance condition (1.9) imply that

\[ R(a_n) = \pm 2i\varepsilon_0 \]  

where \( \varepsilon_0 = \frac{1}{2}h\Omega \). Using the Eq. (2.19) we can conclude that

\[ \sum_{k=1}^{\mu} R(a_k) = \pm i2\mu\varepsilon_0 = \Lambda = E - V_0 \mp i\varepsilon_0 \]  

or

\[ \mu = -\frac{1}{2} \pm i\frac{\lambda}{2} \]  

where \( \lambda = (V_0 - E)/\varepsilon_0 \). The asymptotic form for the components of the wave function can be obtained using these results in the Eqs. (3.8)
\[ \Psi_-(x) = \beta (\pm i)^{-\frac{3}{2} \mp i \frac{\lambda x}{4}} \frac{\exp (\pm i g^2)}{\sqrt{x}} \] (5.6a)

\[ \Psi_+(x) = \gamma (\pm i)^{-\frac{1}{2} \mp i \frac{\lambda x}{2}} \frac{\exp (\pm i g^2)}{\sqrt{x}} \] (5.6b)

or

\[ \Psi_-(x) = \beta \left\{ e^{\mp i \frac{(k+\frac{3}{4})\pi}{\lambda}} e^{-(k+\frac{3}{4})\pi \lambda} \right\} \frac{\exp (\mp i g^2)}{\sqrt{\varrho}} \] (5.7)

and

\[ \Psi_+(x) = \gamma \left\{ e^{\mp i \frac{(k+\frac{3}{4})\pi}{\lambda}} e^{(k+\frac{3}{4})\pi \lambda} \right\} \frac{\exp (\pm i g^2)}{\sqrt{\varrho}} \] (5.8)

where \( \varrho = \sqrt{m\Omega/2 \hbar x} \) and \( k = 0, 1, 2, \ldots \). Using these results it is possible to write two asymptotic solutions of the Schrödinger equation when \( x \to +\infty \) in the form

\[ \Psi_1(x \to +\infty) = e^{i \frac{\pi}{4}} e^{-\frac{n \lambda x}{4}} \varrho^{-\frac{1}{2}} \left[ \exp (-i g^2) \right] + e^{-i \frac{\pi}{4}} e^{-\frac{n \lambda x}{4}} \varrho^{-\frac{1}{2}} \left[ \exp (+i g^2) \right] \] (5.9a)

\[ \Psi_2(x \to +\infty) = e^{-i \frac{\pi}{4}} e^{-\frac{n \lambda x}{4}} \varrho^{-\frac{1}{2}} \left[ \exp (-i g^2) \right] + e^{i \frac{\pi}{4}} e^{-\frac{n \lambda x}{4}} \varrho^{-\frac{1}{2}} \left[ \exp (+i g^2) \right] \] (5.9b)

therefore, if we identify

\[ f_1(x) = \frac{\exp (-i g^2)}{\sqrt{\varrho}} \quad \text{and} \quad f_2(x) = \frac{\exp (+i g^2)}{\sqrt{\varrho}} , \] (5.10)

we can conclude that the elements of \( A(+\infty) \)-matrix will be

\[ A_{11}(+\infty) = e^{i \frac{\pi}{4}} e^{-\frac{n \lambda x}{4}} \varrho^{-\frac{1}{2}} \] (5.11a)

\[ A_{12}(+\infty) = e^{-i \frac{\pi}{4}} e^{-\frac{n \lambda x}{4}} \varrho^{-\frac{1}{2}} \] (5.11b)

\[ A_{21}(+\infty) = e^{-i \frac{\pi}{4}} e^{-\frac{n \lambda x}{4}} \varrho^{-\frac{1}{2}} \] (5.11c)

\[ A_{22}(+\infty) = e^{i \frac{\pi}{4}} e^{-\frac{n \lambda x}{4}} \varrho^{-\frac{1}{2}} . \] (5.11d)

In the case of \( x \to -\infty \) if we consider that

\[ \varrho^{\pm i \frac{\pi}{2}} = e^{\mp i \frac{(n+\frac{3}{4})\pi \lambda}{\varrho}} \varrho^{\pm i \frac{\pi}{2}} , \quad n = 0, 1, 2, \ldots , \] (5.12)

in the Eqs. (5.8) then we can write two asymptotic solutions of the Schrödinger equation when \( x \to -\infty \) in the form

\[ \Psi_1(x \to -\infty) = \left[ (e^{i \frac{\pi}{4}} + e^{i \frac{\pi}{4}}) e^{\frac{n \lambda x}{4}} + e^{-i \frac{\pi}{4}} e^{-\frac{n \lambda x}{4}} \right] \varrho^{\frac{1}{2}} \left[ \exp (-i g^2) \right] + \]

\[ + \left[ (e^{i \frac{\pi}{4}} + e^{i \frac{\pi}{4}}) e^{\frac{n \lambda x}{4}} - e^{i \frac{\pi}{4}} e^{-\frac{n \lambda x}{4}} \right] \varrho^{-\frac{1}{2}} \left[ \exp (+i g^2) \right] \] (5.13)
and
\[
\Psi_2(x \to -\infty) = \left[ (e^{i\frac{3\pi}{4}} + e^{i\frac{\pi}{4}})e^{-\frac{x}{4}} + e^{i\frac{\pi}{4}}e^{-\frac{3x\lambda}{4}} \right] | \varphi \rangle | i\frac{\lambda}{2} \left[ \frac{\exp(-ig^2)}{\sqrt{\varrho}} \right] + \\
\left[ (e^{i\frac{3\pi}{4}} + e^{i\frac{\pi}{4}})e^{-\frac{x}{4}} - e^{-i\frac{\pi}{4}}e^{-\frac{3x\lambda}{4}} \right] | \varphi \rangle | -i\frac{\lambda}{2} \left[ \frac{\exp(i g^2)}{\sqrt{\varrho}} \right], \tag{5.14}
\]
therefore we can identify the elements of \(A(-\infty)\)-matrix as
\[
A_{11}(-\infty) = \left[ (e^{i\frac{3\pi}{4}} + e^{i\frac{\pi}{4}})e^{\frac{x\lambda}{4}} + e^{-i\frac{\pi}{4}}e^{-\frac{3x\lambda}{4}} \right] | \varphi \rangle | i\frac{\lambda}{2} \tag{5.15a}
\]
\[
A_{12}(-\infty) = \left[ (e^{i\frac{3\pi}{4}} + e^{i\frac{\pi}{4}})e^{-\frac{x}{4}} + e^{i\frac{\pi}{4}}e^{-\frac{3x\lambda}{4}} \right] | \varphi \rangle | i\frac{\lambda}{2} \tag{5.15b}
\]
\[
A_{21}(-\infty) = \left[ (e^{i\frac{3\pi}{4}} + e^{i\frac{\pi}{4}})e^{\frac{x\lambda}{4}} - e^{i\frac{\pi}{4}}e^{-\frac{3x\lambda}{4}} \right] | \varphi \rangle | -i\frac{\lambda}{2} \tag{5.15c}
\]
\[
A_{22}(-\infty) = \left[ (e^{i\frac{3\pi}{4}} + e^{i\frac{\pi}{4}})e^{-\frac{x}{4}} - e^{-i\frac{\pi}{4}}e^{-\frac{3x\lambda}{4}} \right] | \varphi \rangle | -i\frac{\lambda}{2}. \tag{5.15d}
\]

In the choice of the two asymptotic wave functions for \(x \to \pm\infty\) we considered the set of properties given by the Eqs. (4.10) and (4.16) that the \(F\)-matrix need to satisfies. Using the results for the \(A(-\infty)\) and \(A(+\infty)\) in the Eqs. (4.9) we can show that the evolution matrix can be written as
\[
F(-\infty, +\infty) = \left[ \frac{ie^{\frac{x\lambda}{2}}}{(-1 + ie^{\frac{x\lambda}{2}})} | \varphi \rangle | -i\lambda \right] \left[ 1 + i e^{\frac{x\lambda}{2}} \right] | \varphi \rangle | i\lambda \right], \tag{5.16}
\]
and the exact transmission and reflection coefficients are given by
\[
T = \frac{1}{|F_{12}(-\infty, +\infty)|^2} = \frac{1}{1 + e^{\pi\lambda}}, \tag{5.17}
\]
and
\[
R = \frac{|F_{22}(-\infty, +\infty)|^2}{|F_{12}(-\infty, +\infty)|^2} = \frac{e^{\pi\lambda}}{1 + e^{\pi\lambda}}. \tag{5.18}
\]

### B. Morse Barrier

For a Morse potential barrier \[19\]
\[
V_1(x) = V_0 \left( 2e^{x/b} - e^{2x/b} \right), \tag{5.19}
\]
the corresponding superpotential, obtained by the Eq. (1.7), is given by
\[
W(x, a_1) = a_1 + \alpha e^{x/b} \tag{5.20}
\]
where
\[
\begin{cases}
a_1 = \sqrt{\varepsilon}(1 \mp is) \\
\alpha = \pm i\sqrt{V_0} \end{cases}, \tag{5.21}
\]
with $\varepsilon = h^2/(8mb^2)$ and $s = \sqrt{V_0/\varepsilon}$. The shape invariance condition (1.9) imply that

$$R(a_n) = a_n^2 - a_{n+1}^2,$$  \hspace{1cm} (5.22)

where $a_{n+1} = a_n + 2\sqrt{\varepsilon}$. Using the Eq. (2.19) we can conclude that

$$\sum_{k=1}^\mu R(a_k) = a_1^2 - a_{\mu+1}^2 = \Lambda = E + a_1^2$$  \hspace{1cm} (5.23)

or

$$a_{\mu+1} = \pm i\sqrt{\varepsilon}.\hspace{1cm} (5.24)$$

If we remember that

$$\mu = \frac{a_{\mu+1} - a_1}{2\sqrt{\varepsilon}},$$  \hspace{1cm} (5.25)

we can use the Eqs. (5.21) and (5.24) to show that

$$\mu = -\frac{1}{2} \pm \frac{s}{2} \pm \frac{r}{2},$$  \hspace{1cm} (5.26)

where $r = \sqrt{E/\varepsilon}$.

Considering the asymmetry of the Morse potential barrier the wave function will have a different behavior in $+\infty$ and $-\infty$. Therefore the asymptotic form of the components of the wave function for $x \rightarrow +\infty$ can be obtained using the last results in the Eqs. (3.8)

$$\Psi_-(x \rightarrow +\infty) = \beta e^{\mp i\Psi} e^{-\Psi(s+\mp r)} \left[ \frac{\exp(\mp i\frac{\Psi}{2}\exp(x/b))}{\sqrt{\exp(x/b)}} \right],$$  \hspace{1cm} (5.27a)

$$\Psi_+(x \rightarrow +\infty) = \gamma e^{\mp i\Psi} e^{\Psi(s-\mp r)} \left[ \frac{\exp(\pm i\frac{\Psi}{2}\exp(x/b))}{\sqrt{\exp(x/b)}} \right].$$  \hspace{1cm} (5.27b)

Using these results it is possible to write two asymptotic solutions of the Schrödinger equation when $x \rightarrow +\infty$ in the form

$$\Psi_1(x \rightarrow +\infty) = e^{i\frac{\Psi}{2}} e^{-\frac{s-\Psi}{4}(s+\mp r)} \left[ \frac{\exp(-i\frac{\Psi}{2}\exp(x/b))}{\sqrt{\exp(x/b)}} \right]$$  \hspace{1cm} (5.28a)

$$+ e^{-i\frac{\Psi}{2}} e^{\frac{s}{4}(s-r)} \left[ \frac{\exp(i\frac{\Psi}{2}\exp(x/b))}{\sqrt{\exp(x/b)}} \right].$$

$$\Psi_2(x \rightarrow +\infty) = e^{-i\frac{\Psi}{2}} e^{-\frac{s-r}{4}(s-r)} \left[ \frac{\exp(-i\frac{\Psi}{2}\exp(x/b))}{\sqrt{\exp(x/b)}} \right]$$  \hspace{1cm} (5.28b)

$$+ e^{i\frac{\Psi}{2}} e^{\frac{s+r}{4}(s+r)} \left[ \frac{\exp(i\frac{\Psi}{2}\exp(x/b))}{\sqrt{\exp(x/b)}} \right].$$
therefore, if we identify
\[
 f_1(x \to +\infty) = \frac{\exp\left(-i\frac{s}{2}\exp(x/b)\right)}{\sqrt{\exp(x/b)}} \quad \text{and} \quad f_2(x \to +\infty) = \frac{\exp\left(+i\frac{s}{2}\exp(x/b)\right)}{\sqrt{\exp(x/b)}},
\]
we can conclude that the elements of \( A(+\infty) \)-matrix will be
\[
\begin{align*}
 A_{11}(+\infty) &= e^{i\frac{s}{2}}e^{-\frac{i}{3}\pi(s+r)} \\
 A_{12}(+\infty) &= e^{-i\frac{s}{2}}e^{-\frac{i}{3}\pi(s-r)} \\
 A_{21}(+\infty) &= e^{-i\frac{s}{2}}e^{\frac{i}{3}\pi(s-r)} \\
 A_{22}(+\infty) &= e^{i\frac{s}{2}}e^{\frac{i}{3}\pi(s+r)}.
\end{align*}
\]  

(5.29)

In the case of \( x \to -\infty \) we can substitute for \( s \) and \( r \) and using the Eq. (3.6) find
\[
\begin{align*}
 \Psi_- (x \to -\infty) &= \beta \frac{\sqrt{r}}{\Gamma\left(\frac{1}{2} \pm i\frac{s}{2} \pm i\frac{r}{2}\right)} e^{\mp ikx} \\
 \Psi_+ (x \to -\infty) &= \gamma \frac{\sqrt{r}}{\Gamma\left(\frac{1}{2} \pm i\frac{s}{2} \pm i\frac{r}{2}\right)} e^{\pm ikx},
\end{align*}
\]

(5.30)

where \( k = \sqrt{2mE/\hbar} \). Using these results it is possible to write two asymptotic solutions of the Schrödinger equation when \( x \to -\infty \) in the form
\[
\Psi_1(x \to -\infty) = \left[ \frac{e^{i\frac{3\pi}{4}} e^{-\frac{\pi}{2}}}{\Gamma\left(\frac{1}{2} + i\frac{s}{2} + i\frac{r}{2}\right)} + \frac{e^{i\frac{\pi}{4}} e^{\frac{s}{2}}}{\Gamma\left(\frac{1}{2} + i\frac{s}{2} - i\frac{r}{2}\right)} \right] \sqrt{r} \Gamma(i\pi) e^{ikx}
\]
\[
+ \left[ \frac{e^{i\frac{3\pi}{4}} e^{\frac{s}{4}(s-r)}}{\Gamma\left(\frac{1}{2} - i\frac{s}{2} - i\frac{r}{2}\right)} + \frac{e^{i\frac{\pi}{4}} e^{-\frac{s}{2}}}{\Gamma\left(\frac{1}{2} - i\frac{s}{2} + i\frac{r}{2}\right)} \right] \sqrt{r} \Gamma(i\pi) e^{-ikx},
\]

(5.31)

and
\[
\Psi_2(x \to -\infty) = \left[ \frac{e^{i\frac{3\pi}{4}} e^{-\frac{s}{2}}}{\Gamma\left(\frac{1}{2} + i\frac{s}{2} + i\frac{r}{2}\right)} + \frac{e^{i\frac{3\pi}{4}} e^{\frac{s}{2}(s+r)}}{\Gamma\left(\frac{1}{2} + i\frac{s}{2} - i\frac{r}{2}\right)} \right] \sqrt{r} \Gamma(-i\pi) e^{ikx}
\]
\[
+ \left[ \frac{e^{i\frac{3\pi}{4}} e^{\frac{s}{2}r}}{\Gamma\left(\frac{1}{2} - i\frac{s}{2} - i\frac{r}{2}\right)} + \frac{e^{i\frac{3\pi}{4}} e^{\frac{s}{2}r}}{\Gamma\left(\frac{1}{2} - i\frac{s}{2} + i\frac{r}{2}\right)} \right] \sqrt{r} \Gamma(-i\pi) e^{-ikx},
\]

(5.32)

therefore, if we identify
\[
 f_1(x \to -\infty) = \frac{\exp(+ikx)}{\sqrt{k}} \quad \text{and} \quad f_2(x \to -\infty) = \frac{\exp(-ikx)}{\sqrt{k}},
\]
we can conclude that the elements of \( A(-\infty) \)-matrix will be

(5.33)
\[ A_{11}(\infty) = \begin{bmatrix} e^{\frac{\pi s}{4}} e^{-\frac{\pi r}{2}} \\
\Gamma \left( \frac{1}{2} + i\frac{s}{2} + i\frac{r}{2} \right) \end{bmatrix} + \begin{bmatrix} e^{\frac{\pi s}{4}} e^{\frac{\pi r}{2}} \\
\Gamma \left( \frac{1}{2} + i\frac{s}{2} - i\frac{r}{2} \right) \end{bmatrix} \sqrt{r} \Gamma (ir) \] (5.35a)

\[ A_{12}(\infty) = \begin{bmatrix} e^{\frac{\pi s}{4}} \\
\Gamma \left( \frac{1}{2} + i\frac{s}{2} + i\frac{r}{2} \right) \end{bmatrix} + \begin{bmatrix} e^{\frac{\pi s}{4}} e^{\frac{\pi r}{2}} \end{bmatrix} \sqrt{r} \Gamma (-ir) \] (5.35b)

\[ A_{21}(\infty) = \begin{bmatrix} e^{\frac{\pi s}{4}} e^{\frac{\pi r}{2}} \\
\Gamma \left( \frac{1}{2} - i\frac{s}{2} - i\frac{r}{2} \right) \end{bmatrix} + \begin{bmatrix} e^{\frac{\pi s}{4}} e^{-\frac{\pi r}{2}} \\
\Gamma \left( \frac{1}{2} - i\frac{s}{2} + i\frac{r}{2} \right) \end{bmatrix} \sqrt{r} \Gamma (ir) \] (5.35c)

\[ A_{22}(\infty) = \begin{bmatrix} e^{\frac{\pi s}{4}} e^{\frac{\pi r}{2}} \\
\Gamma \left( \frac{1}{2} - i\frac{s}{2} - i\frac{r}{2} \right) \end{bmatrix} + \begin{bmatrix} e^{\frac{\pi s}{4}} e^{-\frac{\pi r}{2}} \\
\Gamma \left( \frac{1}{2} - i\frac{s}{2} + i\frac{r}{2} \right) \end{bmatrix} \sqrt{r} \Gamma (-ir) \] (5.35d)

Again, in the choice of the two asymptotic wave functions for \( x \to \pm \infty \) we have considered the properties of the \( F \)-matrix. Using the results for the \( A(\infty) \) and \( A(+\infty) \) in the Eqs. (4.9) we can show that the evolution matrix can be written as

\[ F(-\infty, +\infty) = \begin{bmatrix} i g & i h \\
i h^* & i g^* \end{bmatrix} \] (5.36)

where

\[ g = \frac{e^{\frac{\pi s-r}{2}} \sqrt{\Gamma(\frac{3}{2})}}{\Gamma \left( \frac{1}{2} + i\frac{s}{2} + i\frac{r}{2} \right)} \] (5.37)

and

\[ h = \frac{e^{\frac{\pi s+r}{2}} \sqrt{\Gamma(\frac{3}{2})}}{\Gamma \left( \frac{1}{2} + i\frac{s}{2} - i\frac{r}{2} \right)} \] (5.38)

In this case the exact transmission and reflection coefficients are given by

\[ T = \frac{1}{|F_{12}(-\infty, +\infty)|^2} = \frac{e^{-\frac{\pi s-r}{2}} \sinh(\pi r)}{\cosh \left[ \frac{\pi}{2} (s-r) \right]}, \] (5.39)

and

\[ R = \frac{|F_{22}(-\infty, +\infty)|^2}{|F_{12}(-\infty, +\infty)|^2} = \frac{e^{-\pi r} \cosh \left[ \frac{\pi}{2} (s+r) \right]}{\cosh \left[ \frac{\pi}{2} (s-r) \right]} . \] (5.40)

**C. Eckart Barrier**

For an Eckart potential barrier \[ V_1(x) = V_0 \text{sech}^2(x/2b), \] (5.41)

the correspondent superpotential, obtained by the Eq. (1.7), is given by
\[ W(x, a_1) = a_1 \tanh \left( \frac{x}{2b} \right) \]  

(5.42)

where

\[ a_1 = \sqrt{\varepsilon} \left( -1 \pm i s \right) , \]  

(5.43)

with \( \varepsilon = h^2/(32mb^2) \) and \( s = \sqrt{V_0/\varepsilon - 1} \). The shape invariance condition (1.3) imply that

\[ R(a_n) = a_n^2 - a_{n+1}^2 , \]  

(5.44)

where \( a_{n+1} = a_n - 2\sqrt{\varepsilon} \). Using the Eq. (2.19) we can conclude that

\[ \sum_{k=1}^{2\nu} R(a_k) = a_1^2 - a_{2\nu+1}^2 = \Lambda = E + a_1^2 \]  

(5.45)

or

\[ a_{2\nu+1} = \pm i \sqrt{E} . \]  

(5.46)

If we remember that

\[ 2\nu = \frac{a_{2\nu+1} - a_1}{-2\sqrt{\varepsilon}} , \]  

(5.47)

we can use the Eqs. (5.43) and (5.46) to show that

\[ 2\nu = -\frac{1}{2} \pm \frac{s}{2} \mp \frac{r}{2} , \]  

(5.48)

where \( r = \sqrt{E/\varepsilon} \).

The asymptotic form of the components of the wave function can be obtained using these results in the Eqs. (3.10)

\[ \Psi_-(x) = \beta \frac{\Gamma \left( \frac{3}{4} \pm i \frac{s}{2} \pm i \frac{r}{4} \right)}{\Gamma \left( 1 \pm i \frac{s}{4} \right) \Gamma \left( \frac{1}{4} \pm i \frac{s}{4} \mp i \frac{r}{4} \right)} e^{\mp ikx} \]  

(5.49a)

\[ \Psi_+(x) = \gamma \frac{\Gamma \left( i \frac{s}{2} \right) \Gamma \left( \frac{1}{4} \pm i \frac{s}{4} \pm i \frac{r}{4} \right)}{\Gamma \left( \frac{3}{4} \pm i \frac{s}{4} \pm i \frac{r}{4} \right)} e^{\pm ikx} , \]  

(5.49b)

where \( k = \sqrt{2mE/\hbar} \). Using these results and the relation

\[ \Gamma \left( \frac{1}{4} \pm iy \right) \Gamma \left( \frac{3}{4} \mp iy \right) = \frac{\sqrt{2\pi}}{\cosh (\pi y) \pm \sinh (\pi y)} , \]  

(5.50)

it is possible to write two asymptotic solutions of the Schrödinger equation when \( x \to +\infty \) in the form

\[ \Psi_1(x \to +\infty) = C_1 e^{-ikx} + C_1^* e^{ikx} \]  

(5.51a)

\[ \Psi_2(x \to +\infty) = C_2^* e^{-ikx} - C_2 e^{ikx} , \]  

(5.51b)
\[ C_1 = \sqrt{2\pi} \left\{ \cosh \left( \frac{\pi}{4} (r + s) \right) + i \sinh \left( \frac{\pi}{4} (r + s) \right) \right\}^{-1} \times \left\{ \Gamma \left( 1 + \frac{r}{2} \right) \Gamma \left( \frac{1}{4} \pm \frac{s}{4} \right) \Gamma \left( \frac{1}{4} \pm \frac{s}{4} - \frac{i}{4} \right) \right\}^{-1} \] (5.52)

and

\[ C_2 = \sqrt{2\pi} \Gamma \left( i \frac{r}{2} \right) \left\{ \cosh \left( \frac{\pi}{4} (r + s) \right) - i \sinh \left( \frac{\pi}{4} (r + s) \right) \right\}^{-1} \times \left\{ \Gamma \left( \frac{3}{4} \pm i \frac{s}{4} + \frac{r}{4} \right) \Gamma \left( \frac{3}{4} \pm i \frac{s}{4} + i \frac{r}{4} \right) \right\}^{-1}. \] (5.53)

Therefore, if we identify in the Eq. (5.51)

\[ f_1(x \to +\infty) = \frac{e^{-ikx}}{\sqrt{k}} \quad \text{and} \quad f_2(x \to +\infty) = \frac{e^{ikx}}{\sqrt{k}}, \] (5.54)

we can conclude that the elements of \( A(+\infty) \)-matrix will be

\[ A_{11}(+\infty) = C_1 \] (5.55a)
\[ A_{12}(+\infty) = C^*_2 \] (5.55b)
\[ A_{21}(+\infty) = C^*_1 \] (5.55c)
\[ A_{22}(+\infty) = -C_2. \] (5.55d)

Also we can write two asymptotic solutions of the Schrödinger when \( x \to -\infty \) in the form

\[ \Psi_1(x \to -\infty) = C_1 e^{ikx} - C^*_1 e^{-ikx} \] (5.56a)
\[ \Psi_2(x \to -\infty) = -C^*_2 e^{ikx} - C_2 e^{-ikx}, \] (5.56b)

and identifying

\[ f_1(x \to -\infty) = \frac{e^{ikx}}{\sqrt{k}} \quad \text{and} \quad f_2(x \to -\infty) = \frac{e^{-ikx}}{\sqrt{k}}, \] (5.57)

we can conclude that the elements of \( A(-\infty) \)-matrix will be

\[ A_{11}(-\infty) = C_1 \] (5.58a)
\[ A_{12}(-\infty) = -C^*_2 \] (5.58b)
\[ A_{21}(-\infty) = -C^*_1 \] (5.58c)
\[ A_{22}(-\infty) = -C_2. \] (5.58d)

Using the results for the \( A(-\infty) \) and \( A(+\infty) \) in the Eq. (4.9) we can show that the evolution matrix can be written as

\[ \mathbf{F}(-\infty, +\infty) = \begin{bmatrix} g & h \\ -h^* & -g^* \end{bmatrix} \] (5.59)
where

$$g = \frac{C_1 C_2 - C_1^* C_2^*}{C_1 C_2 + C_1^* C_2^*}$$  \hspace{1cm} (5.60)$$

and

$$h = \frac{2C_1 C_2^*}{C_1 C_2 + C_1^* C_2^*}. \hspace{1cm} (5.61)$$

On using the Eqs. (5.52), (5.53) after a considerable amount of algebra we can show that the exact transmission and reflection coefficients are given by

$$T = \frac{1}{|F_{12}(-\infty, +\infty)|^2} = \frac{1}{4} \left| \frac{C_2}{C_2^*} + \frac{C_1^*}{C_1} \right|^2 = \frac{\sinh^2 \left( \frac{\pi r}{2} \right)}{\sinh^2 \left( \frac{\pi s}{2} \right) + \cosh^2 \left( \frac{\pi s}{2} \right)}, \hspace{1cm} (5.62)$$

and

$$R = \frac{|F_{22}(-\infty, +\infty)|^2}{|F_{12}(-\infty, +\infty)|^2} = \frac{1}{4} \left| \frac{C_2}{C_2^*} - \frac{C_1^*}{C_1} \right|^2 = \frac{\cosh^2 \left( \frac{\pi s}{2} \right)}{\sinh^2 \left( \frac{\pi r}{2} \right) + \cosh^2 \left( \frac{\pi r}{2} \right)}. \hspace{1cm} (5.63)$$

VI. CONCLUDING REMARKS

In conclusion we note that the present technique is a powerful and an elegant prescription to obtain exact reflection and transmission coefficients. This method may also be used for all supersymmetric shape invariant potential barriers that satisfy the analytic continuation condition (5.3).

One possible application of shape-invariance formalism is to multidimensional quantum tunneling. In nuclear physics applications multidimensional quantum tunneling can be visualized as the tunneling of a quantum mechanical system (such as a nucleus with internal excitation) instead of a structureless particle through a one-dimensional barrier. The nucleus typically taken to enter the barrier in its ground state and may emerge either in the ground state or in an excited state at the other side of the barrier. The interaction between the penetrating quantum system and the barrier also needs to be specified based on the physical conditions of the problem. It has been known for some time that, if the excitation energies are neglected, the penetration probability of an $N$-dimensional system can be reduced to a sum of probabilities of $N$ one-dimensional suitably defined barriers \[21\]. The eigenchannel formulation remains valid even for finite excitation energies as long as the energy-dependence of the weight factors is taken into account \[22\]. Our formulation would be applicable in such cases if the eigenpotentials are shape-invariant. A simpler limit would assume factorization of the interaction between the barrier and the quantum system into a product of two quantities which are functions of the barrier and internal degrees of freedom respectively. Such a factorization approach was already applied to a coupled system of equations for bound states \[23\].

Our formulation also casts the tunneling problem in an algebraic basis \[5,24\]. If the internal system can be described by an algebraic model such as the interacting boson model \[25\] then it may be possible to cast the entire problem into an algebraic framework. A group-theoretical formulation can be a starting point of systematic approximations such as those given in Ref. \[26\]. A detailed study of such aspects is deferred to later work.
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