Comments on orientifold projection in the conifold
and $SO \times USp$ duality cascade

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Abstract

We study the O3-plane in the conifold. On the D3-brane world-volume we obtain $SO \times USp$ gauge theory that exhibits a duality cascade phenomenon. The orientifold projection is determined on the type IIB string side, and corresponds to that of O4-plane on the dual type IIA side. We show that SUGRA solutions of Klebanov-Tseytlin and Klebanov-Strassler survive under the projection. We also investigate the orientifold projection in the generalized conifolds, and verify desired features of the O4-projection in the type IIA picture.

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1 Introduction

In the past years, an extension of AdS/CFT correspondence \cite{1,2,3} has been investigated away from conformality. Especially, type IIB SUGRA solutions that describe D3-branes at the conifold singularity beautifully reproduce phenomena of field theories, such as RG flow, duality cascade, chiral symmetry breaking and confinement.

When $N$ D3-branes are placed at the conifold singularity, $N = 1$ superconformal field theory which is $SU(N) \times SU(N)$ gauge theory with $2(N, \overline{N}) \oplus 2(\overline{N}, N)$ is realized on the branes \cite{4}. The addition of $M$ fractional D3-branes changes the gauge groups to $SU(N + M) \times SU(N)$ and breaks the conformal invariance. As we flow to IR, the gauge coupling constant of $SU(N + M)$ diverges and Seiberg duality must be performed for better description of the field theory. As the dual theory has similar gauge groups $SU(N - M) \times SU(N)$ and matter content as the original theory, this process repeats successively. This is called “RG cascade” or “duality cascade” \cite{5}. At the bottom of this cascade, Affleck-Dine-Seiberg superpotential is dynamically generated \cite{8}. The moduli space of vacua is deformed and chiral symmetry is broken by gaugino condensation. The type IIB SUGRA solution of Klebanov-Tseytlin (KT solution) \cite{6,7} describes D3-branes at the conifold singularity and incorporates this cascade. The NS-NS $B$ field that corresponds to the gauge couplings $1/g_1^2 - 1/g_2^2$ has logarithmic radial dependence. And 5-form fluxes which corresponds the rank of the gauge group suitably decrease. The SUGRA solution found by Klebanov-Strassler (KS solution) \cite{5} furthermore reproduces far IR phenomena as well as duality cascade. It has asymptotically the same form as Klebanov-Tseytlin \cite{6} solution, while near the origin, the singularity of the conifold is deformed and the branes are replaced with fluxes. So it signals confinement in the gauge theory \cite{5}.

In this paper, we extend these results to the $SO \times USp$ gauge theory. In the type IIA brane configurations, there are two possibilities to obtain $SO$ or $USp$ gauge group. One is with O6-planes \cite{10}. Another is with an O4-plane \cite{11,12}. Taking T-duality to the conifold with D3-branes, we have brane configurations with a NS5-brane along 012345, a NS5'-brane along 012389 and D4-branes along 01236 \cite{13}. To obtain the $SO \times USp$ gauge groups, only the O4-plane along 01236 is allowed in the case. We consider the corresponding orientifold projection in type IIB theory. Such projection is also discussed in \cite{14}. But we give another projection by studying symmetries of the type IIB conifold. Our projection gives the correct field theory. Other models with O6-planes have been well studied and corresponding orientifold projections in type IIB theory are given in \cite{15,16,17,18,19}. We also comment on KT/KS solutions. They still solve equations of motion under the projection. Moreover we generalize the projection in the conifold to one in the generalized conifolds. In the type IIA picture, we have some NS5-branes and NS5'-branes with the O4-plane. The orientifold projection is consistent with the feature of the O4-plane such that the gauge groups must be $(SO \times USp)^n$ \cite{11,12} and the total number of NS5 and NS5'-branes requires to be even \cite{11,12,20}. 

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This paper is organized as follows. In section 2, as a heuristic step, we analyze type IIA brane configurations. In section 3, we determine the orientifold projection in gauge theory language. Then, we analyze the field theory and observe similar phenomena as in the $SU \times SU$ case. In section 4, we give the O3-plane interpretation to our orientifold projection. We also comment on the SUGRA solutions and the duality cascade. In section 5, we determine the orientifold projection in the generalized conifold. Section 6 is devoted to conclusion.

2 Preliminary Observation

2.1 Expectation from type IIA brane configuration

The duality cascade phenomenon is most easily seen in the type IIA elliptic model picture. $N$ D3-branes at the conifold singularity is T-dual to type IIA brane configurations: one NS5-brane along the 012345 directions, the other NS5'-brane along the 012389 directions and $N$ D4-branes along the 01236 directions. The $x^6$ direction is compactified and four-dimensional $\mathcal{N} = 1$ gauge theory is realized on D4-branes along the 0123 directions.

Adding $M$ fractional D3-branes on the type IIB side corresponds to adding $M$ D4-branes stretched between NS5 and NS5’ as depicted in Fig 1 (a). We call them fractional D4-branes. The four-dimensional field theory has gauge groups $SU(N + M) \times SU(N)$. $SU(N + M)$ factor comes from the NS-NS’ interval and $SU(N)$ factor comes from the other interval. Imbalance of D4-brane tension causes logarithmic bending of NS5-branes world-volume and positions of two NS5-branes depend on the energy scale. This is conveniently described by moving the NS5’-brane. When the NS5’-brane crosses the NS5-brane, $M$ fractional D4-branes in the NS-NS’ interval shrink and re-grow on the other side (Fig 1 (b)). This process changes the orientation of fractional D4-branes. $M$ of $N$ D4-branes are annihilated. Then $N - M$ fractional D4-branes in the NS’-NS interval remain. After all, this brane crossing process changes the gauge groups to $SU(N) \times SU(N - M)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Brane configurations for $SU \times SU$ duality cascade. The NS5’-brane moves from right to left in this figure. Fig (a) and (b) are before and after brane crossing. Two copies of the fundamental region are shown.}
\end{figure}

$SO \times USp$ gauge theory also exhibits duality cascade phenomenon. It can be also easily seen in IIA picture. Let us put the O4-plane on top of D4-branes: the 01236 directions. As the O4-plane changes its sign of R-R
charge across the NS5-brane \[12\], the O4\(^{-}\)-plane in the NS-NS' interval becomes the O4\(^{+}\)-plane in the NS'-NS interval. When we put \(N + M + 2\) fractional D4 branes in the NS-NS' interval and \(N\) fractional D4 branes in the NS'-NS interval, gauge groups become \(SO(N + M + 2) \times USp(N)\) (Fig. 2).

As opposed to the previous case, when the NS5'-brane crosses the NS5-brane from right to left, \(M + 2\) fractional D4-branes shrink and \(M - 2\) fractional anti-D4-branes emerges in NS'-NS interval (Fig. 3). The number of D4-branes is determined by conservation of D4-brane charge flowing into NS5-branes \[21\]. We must remember that when the O4-plane is crossed by NS5'-brane, O4\(^{+}\)(O4\(^{-}\)) in the NS'-NS (NS-NS') interval becomes O4\(^{-}\)(O4\(^{+}\)) in the NS'-NS (NS-NS') interval respectively. After pair annihilation process, gauge groups change to \(USp(N) \times SO(N - M + 2)\) (Fig. 4).

\[\text{Figure 2: Brane configuration for } SO(N + M + 2) \times USp(N). \text{ Two copies of fundamental region are shown. Here } N = 2n, \ M = 2m.\]

\[\text{Figure 3: The NS5'-brane has crossed the NS5-brane. Notice that D4-brane charges flowing away from NS5'-brane is always } M = 2m.\]

Further brane crossing changes \(M - 2\) D4-branes in the NS-NS' interval into \(M + 2\) anti-D4-branes in the NS'-NS interval. After pair annihilation there are \(N - M + 2\) D4-branes and the O4\(^{-}\)-plane in the NS-NS'

* We comment on our convention. We count the R-R-charges including mirrors. For example, O9\(^{-}\) has \(-32\) D9-brane charge, and O4\(^{-}\) has \(-1\) D4-charge. When \(M = 0\), D4-brane tension between both sides of NS5-branes balances.
interval, and $N - 2M$ D4-branes and the O$^{+}$-plane in the NS'NS interval. So the gauge groups become $SO(N - M + 2) \times USp(N - 2M)$.

The brane configuration gives us a good understanding for RG cascade, however, identification of gauge groups and matter contents is rather heuristic. More detailed discussion is desirable to compare with explicit formulation of the orientifold projection.

### 2.2 Expectation from gauge theory

Before detailed analysis, we comment on the duality cascade in terms of field theories.

Ignoring cumbersome restriction on the rank of gauge groups and number of flavors, Seiberg dual to $SO(N_c)$ gauge theory with $N_f$ flavors is $SO(N_f - N_c + 4)$ gauge theory with $N_f$ flavors and singlets [22]. And the dual to $USp(N_c)$ gauge theory with $N_f$ flavors is $USp(N_f - N_c - 4)$ gauge theory with $N_f$ flavors and singlets [23].

Since $SO(N_1) \times USp(N_2)$ theory is obtained by projection from $SU(N_1) \times SU(N_2)$ with $2(N_1, N_2) \oplus 2(N_1, N_2)$, we have $2(N_1, N_2)$. The number of matters are reduced to half compared to $SU \times SU$ theory. Then duality cascade occurs as following.

\[
SO(N + M + 2) \times USp(N) \\
\Rightarrow SO((N - M) + 2) \times USp((N - M) + M) \\
\Rightarrow SO((N - 2M) + M + 2) \times USp((N - 2M)) \\
\ldots
\]

which is expected from the type IIA picture.

\[\text{\small{† For } USp \text{ gauge theory } N_f \text{ must be even for the absence of global anomaly.}}\]
3 Determination of Orientifold Projection

In this section, we determine the orientifold projection in the conifold in terms of gauge theory on D3-branes. From the string theory point of view an orientifold projection is product of space-time orbifold projection \( R \) and world-sheet parity \( \Omega \) or \( \Omega(-)^F \). Because these are symmetries of type IIB string theory, there exist counterparts in the world-volume gauge theory of D3-branes at the conifold singularity. Luckily, Klebanov and Witten have already identified the space-time symmetry and \( \Omega(-)^F \) as the global symmetry of the gauge theory \([4]\). We can determine the projection from minor extension of their results.

3.1 Symmetry of \( SU(N_1) \times SU(N_2) \) theory

The world-volume theory of \( N \) D3-branes and \( M \) fractional D3-branes on the conifold singularity is \( \mathcal{N} = 1 \) supersymmetric \( SU(N + M) \times SU(N) \) gauge theory with two chiral multiplets \( A_1, A_2 \) in \((N + M, N)\) representation and two chiral multiplets \( B_1, B_2 \) in \((N + M, N)\). This theory has the superpotential

\[
W = \lambda \text{tr} (A_i B_j A_k B_l) \epsilon^{ik} \epsilon^{jl}.
\] (3.1)

For convenience, we sometimes denote \( N_1 = N + M \) and \( N_2 = N \).

In the following, we briefly review the results on the dictionary of symmetries of the conifold and gauge theory in the \( M = 0 \) case \([4]\). The moduli space of vacua is the conifold since D3-branes can freely move on the conifold. To see this, suppose that we have diagonal vev of \( A_i = a_i = \text{diag}(a_i^{(1)}, \ldots, a_i^{(N)}) \), \( B_i = b_i = \text{diag}(b_i^{(1)}, \ldots, b_i^{(N)}) \).

Then F-flatness conditions \( A_1 B_i A_2 = A_2 B_i A_1 \) and \( B_1 A_i B_2 = B_2 A_i B_1 \) are trivially satisfied. Gauge equivalence \( a_i \sim e^{i\alpha} a_i, b_i \sim e^{-i\alpha} b_i \) and D-flatness conditions

\[
|a_1^{(r)}|^2 + |a_2^{(r)}|^2 - |b_1^{(r)}|^2 - |b_2^{(r)}|^2 = 0
\] (3.2)
defines the conifold as symplectic quotient.

Another way to see the moduli space as the conifold is to form gauge invariant quantities \( z_{ij}^{(r)} := a_i^{(r)} b_j^{(r)} \), which satisfy the defining equation of the conifold \( \det z_{ij}^{(r)} = 0 \). If we denote (we omit the superscript \( (r) \) henceforth)

\[
z_{ij} = \begin{pmatrix}
z_3 + iz_4 & z_1 - iz_2 \\
z_4 - iz_3 & z_1 + iz_2
\end{pmatrix},
\] (3.3)

the equation is recast into \( z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \).

Symmetries of the \( SU(N_1) \times SU(N_2) \) theory is summarized in Table 1, where \( \Lambda_1 \) and \( \Lambda_2 \) are dynamical scale of two gauge groups. \( b \) and \( \tilde{b} \) are one-loop beta coefficients and \( \lambda \) is the coupling constant in the superpotential.
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
& SU(N_1) & SU(N_2) & SU(2) & SU(2) & U(1)_B & U(1)_A & U(1)_R \\
\hline
A_i & N_1 & N_2 & 2 & 1 & 1/2N_1N_2 & 1/2N_1N_2 & 1/2 \\
B_i & N_1 & N_2 & 1 & 2 & -1/2N_1N_2 & 1/2N_1N_2 & 1/2 \\
A^b_i & 0 & 2/N_1 & 2(N_1 - N_2) & 0 & 2/N_2 & -2(N_1 - N_2) & 0 \\
\lambda & 0 & -2/N_1N_2 & 0 & -2/N_1N_2 & 0 & -1 & \\
\theta & & & & & & & \\
\hline
\end{array}
\]

Table 1: Quantum numbers of \(SU(N_1) \times SU(N_2)\) theory.

From the above relation between the conifold and gauge theory, we can obtain symmetries which are needed for the orientifold projection. The R-symmetry in gauge theory acts on the fields as

\[
G_\epsilon : \begin{cases} 
\theta & \mapsto e^{-i\epsilon}\theta, \\
A_i & \mapsto e^{i\epsilon}A_i, \\
B_i & \mapsto e^{i\epsilon}B_i.
\end{cases}
\tag{3.4}
\]

We denote the generator of \(\mathbb{Z}_2\) subgroup of this R-symmetry as \(G(=G_\pi)\). Although \(G^2\) changes sign of \(A_i, B_i\), this is gauge equivalent to \(A_i, B_i\). Hence, it is \(\mathbb{Z}_2\) generator.

From the parameterization \(z_{ij} = a_i b_j\), we can read off transformation rule in SUGRA side

\[
z_i \mapsto e^{i\epsilon}z_i.
\tag{3.5}
\]

Under this transformation the holomorphic 3-form rotates as

\[
\Omega = \frac{dz_1 \wedge dz_2 \wedge dz_3}{z_4} \mapsto e^{2i\epsilon}\Omega.
\tag{3.6}
\]

Because the holomorphic 3-form can be constructed from covariant constant spinor as \(\Omega_{mnl} = \gamma^m \Gamma_{mnl} \eta\), it transforms as

\[
\eta \mapsto e^{i\epsilon}\eta.
\tag{3.7}
\]

Note that covariantly constant spinor and chiral superspace coordinates rotate oppositely.

In the same way, the space-time reflection \(R_4 : z_4 \mapsto -z_4\) changes the sign of the holomorphic 3-form so covariantly constant spinor transforms as \(\eta \mapsto -i\eta\). On the gauge theory side, corresponding \(\mathbb{Z}_2\) transformation becomes

\[
S_1 : \begin{cases} 
\mathcal{W}_1 & \mapsto \gamma_1 \mathcal{W}_2 \gamma_1^{-1} \\
\mathcal{W}_2 & \mapsto \gamma_2 \mathcal{W}_1 \gamma_2^{-1} \\
A_i & \mapsto \epsilon_1 \gamma_1 B_i \gamma_2^{-1} \\
B_i & \mapsto (\epsilon^{-1})_2 \gamma_2 A_i \gamma_1^{-1}
\end{cases}
\tag{3.8}
\]

where \(\mathcal{W}_1\) and \(\mathcal{W}_2\) are field strength multiplets of each gauge group and \(\gamma_1, \gamma_2\) act on Chan-Paton factor to relate D-branes and their mirror images. \(A_i\) and \(B_i\) are exchanged because \(A_i\) and \(B_i\) are spinors of opposite chirality under \(SO(4) \sim SU(2) \times SU(2)/\mathbb{Z}_2\), and reflection \(R_4\) acts as gamma matrix \(\gamma_4\). We can replace \(\epsilon\) by
| Gauge Theory side | IIB SUGRA side |
|-------------------|----------------|
| \( G \in \mathbb{Z}_2 \subset U(1)_R \) | \( R_{1234} : \text{Reflection} \) |
| \( \theta \mapsto -\theta \) | \( z_\mu \mapsto -z_\mu \) |
| \( A_i \mapsto iA_i \) | \( \eta \mapsto -\eta \) |
| \( B_i \mapsto iB_i \) | &nbsp; |

| \( S_1 \) | \( R_4 : \text{Reflection} \) |
| \( \theta \mapsto i\theta \) | &nbsp; |
| \( W_1 \mapsto \gamma_1 W_2 \gamma_1^{-1} \) | \( (z_1, z_2, z_3, z_4) \mapsto (z_1, z_2, z_3, -z_4) \) |
| \( W_2 \mapsto \gamma_2 W_1 \gamma_2^{-1} \) | \( \eta \mapsto -i\eta \) |
| \( A_i \mapsto g_i^j \gamma_1 B_j \gamma_2^{-1} \) | &nbsp; |
| \( B_i \mapsto (g^{-1}_i)^j \gamma_2 A_j \gamma_1^{-1} \) | &nbsp; |

| \((SU(2) \times SU(2))/\mathbb{Z}_2 \) | \( SO(4) \) |
| \( A_i \mapsto g_i^j A_j \) | \( z_\mu \mapsto M_\mu z_\nu \) |
| \( B_i \mapsto h_i^j B_j \) | &nbsp; |

Table 2: Correspondence between symmetries of gauge theory side and SUGRA side.

\( g_i^j \in SU(2) \) in eq. (3.8), and this corresponds to the \( SO(3) \) degrees of freedom of \(-1\) eigen-vector of reflection. Note that this is also R-symmetry. Because superpotential \( W \) changes its sign under eq. (3.8), \( \theta \) must rotates \( \theta \mapsto i\theta \) for superpotential \( W \) not to vanish.

Lastly, the world sheet parity \( \Omega(-)^F_L \) corresponds to \( \begin{pmatrix} -1 \\ -1 \end{pmatrix} \in SL(2, \mathbb{Z}) \) duality group. In particular, it acts on unbroken SUSY parameter in the presence of D3-brane as \( \epsilon_L \mapsto \Gamma_{0123}\epsilon_L \). On the gauge theory side \( \Omega(-)^F_L \) acts as

\[
S_2 : \begin{cases}
W_1 &\mapsto iW_2, \\
W_2 &\mapsto iW_1, \\
A_i &\mapsto iA_i, \\
B_i &\mapsto iB_i.
\end{cases}
\tag{3.9}
\]

This also changes sign of the superpotential \( W \), hence is R-symmetry : \( \theta \mapsto i\theta \).

The dictionary of symmetries on the gauge theory side and on the type IIB SUGRA side is summarized in Table 2.

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\* In fact, we must pay attention to \( \pm \) sign in transformation law for spinors. We use the notation \( R_4 \) and \( \tilde{R}_4 \) to distinguish two elements of \( Spin(6) \) uplifted from \( SO(6) \). For example, \( R_{1234} = R_1 R_2 R_3 R_4 : \eta \mapsto +\eta \) is different from \( \tilde{R}_{1234} : \eta \mapsto -\eta \).  
- The signs for \( \theta \) in \( S_1, S_2 \) may differ relatively. but it is taken as the same sign in \( R_{1234} \).  
- Transformation law for holomorphic 3-form decides that for covariant constant spinor \( \eta \) only up to sign. But the sign in \( \tilde{R}_{1234} \) is determined by continuity as \( U(1)_R \).  
- The sign in \( R_4 \) is determined by \( S_1 \) on the gauge theory side.

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3.2 Determination of projection

Now we can determine the orientifold projection. Because we expect the resulting theory to posses $\mathcal{N} = 1$ supersymmetry, the orientifold projection leave the chiral super space coordinate $\theta$ invariant, otherwise gauge fields and gauginos acquire opposite parity. Therefore a possible choice for the orientifold projection for this theory will be $GS_2S_1$.

$$GS_2S_1: \begin{cases} 
\theta \mapsto \theta, \\
W_1 \mapsto \gamma_1 W_1 \gamma_1^{-1}, \\
W_2 \mapsto \gamma_2 W_2 \gamma_2^{-1}, \\
A_i \mapsto ig_i \gamma_1 B_j \gamma_2^{-1}, \\
B_i \mapsto i(g^{-1})_i^j \gamma_2 A_j \gamma_1^{-1}.
\end{cases} \quad (3.10)$$

On the IIB SUGRA side, the space-time part of this projection is

$$R_{123} : (z_1, z_2, z_3, z_4) \mapsto (-z_1, -z_2, -z_3, +z_4). \quad (3.11)$$

In the case of $N_1 \neq N_2$, although we don’t know how to separate world sheet parity $\Omega(-)^{F_L}$ and reflection $R_4$ on the gauge theory side, it is not necessary for our purpose. Since $GS_2S_1$ corresponds to $R_{123}\Omega(-)^{F_L}$ on the SUGRA side and does not exchange two gauge groups, we may expect it has the same form as the $N_1 = N_2$ case.

Compatibility of two relations $A_i = ig_i \gamma_1 B_j \gamma_2^{-1}$ and $B_i = i(g^{-1})_i^j \gamma_2 A_j \gamma_1^{-1}$ requires

$$\gamma_1 t_{\gamma_1^{-1}} = -\gamma_2 t_{\gamma_2^{-1}} = \pm 1. \quad (3.12)$$

The solution to these conditions is essentially

$$\gamma_1 = \gamma_{SO} = \begin{pmatrix} 0 & i1 \\ i1 & 0 \end{pmatrix}, \quad \gamma_2 = \gamma_{Sp} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.13)$$

In particular, combination of $SO$ and $USp$ projection is allowed and agrees with the expectation from the type IIA picture.

3.3 Analysis of the resulting field theory

In this subsection, we briefly analyze the field theory after the orientifold projection in similar manner to the case of $SU \times SU$ gauge theory [5]. In the previous section, we have obtained

$$W_1 \mapsto \gamma_{SO} W_1 \gamma_{SO}^{-1},$$
$$W_2 \mapsto \gamma_{Sp} W_2 \gamma_{Sp}^{-1},$$
$$A_i \mapsto ig_i \gamma_{SO} B_j \gamma_{Sp}^{-1},$$

In particular, combination of $SO$ and $USp$ projection is allowed and agrees with the expectation from the type IIA picture.
\[ B_i \rightarrow i(g^{-1})_i^j \gamma_{Sp} t A_j \gamma_{SO}^{-1}, \]  
\[ (3.15) \]

as the orientifold projection.

This correctly produces \( SO(N_1) \times USp(N_2) \) gauge groups, and matters are reduced to half by the relation \( B_i = -i(g^{-1})_i^j \gamma_{Sp} t A_j \gamma_{SO} \) as expected from the type IIA picture.

The superpotential becomes

\[ W = \lambda \text{tr}(A_i B_j A_k B_l) \epsilon^{ik} \epsilon^{jl} \]
\[ \sim \lambda \text{tr}(A_i \gamma_{Sp} t A_m \gamma_{SO} A_k \gamma_{Sp} t A_n \gamma_{SO})(g^{-1})_j^m (g^{-1})_l^n \epsilon^{ik} \epsilon^{jl} \]
\[ = \lambda \text{tr}(A_i \gamma_{Sp} t A_j \gamma_{SO} A_k \gamma_{Sp} t A_l \gamma_{SO}) \epsilon^{ik} \epsilon^{jl}. \]  
\[ (3.16) \]

F-flatness conditions are

\[ t A_1 \gamma_{SO} A_i \gamma_{Sp} t A_2 - t A_2 \gamma_{SO} A_i \gamma_{Sp} t A_1 = 0. \]  
\[ (3.17) \]

This can be obtained simply replacing \( B_i \) by \(-i(g^{-1})_i^j \gamma_{Sp} t A_j \gamma_{SO}\) in F-flatness conditions of the \( SU \times SU \) case.

If we take the vev to be block diagonal form

\[ A_i = \begin{pmatrix} a_i & 0 \\ 0 & \tilde{a}_i \end{pmatrix} := \begin{pmatrix} a_i^{(1)} \\ \vdots \\ \vdots \\ 0 \\ \cdots \\ 0 \\ a_i^{(N_2/2)} \end{pmatrix}, \]
\[ B_i = \begin{pmatrix} b_i & 0 \\ 0 & \tilde{b}_i \end{pmatrix} := \begin{pmatrix} b_i^{(1)} \\ \vdots \\ \vdots \\ 0 \\ \cdots \\ 0 \\ b_i^{(N_2/2)} \end{pmatrix}, \]  
\[ (3.18) \]

\( \tilde{a}_i \) and \( \tilde{b}_i \) can be removed due to the projection \( (3.13) \). So, we have

\[ A_i = \begin{pmatrix} a_i & 0 \\ 0 & g_i^j b_j \end{pmatrix}, \quad B_i = \begin{pmatrix} b_i & 0 \\ 0 & -(g^{-1})_i^j a_j \end{pmatrix}. \]  
\[ (3.19) \]

These automatically satisfy F-flatness conditions. In our basis, Cartan subalgebra of both \( SO(N_1) \) and \( USp(N_2) \) are \( \begin{pmatrix} X & X \\ -X & -X \end{pmatrix} \) with \( X \) being diagonal. The D-flatness condition \( |a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2 = 0 \) and
gauge equivalence \( a_i \sim e^{i \epsilon} a_i \), \( b_i \sim e^{-i \epsilon} b_i \) are reproduced by using only \( A_i \). The moduli space of vacua is still the conifold.

We can form two kinds of meson operators with respect \( SO(N_1) \) and \( USp(N_2) \).

\[
M_{ij}^{SO} := t A_i \gamma_i (g^{-1})_j A_l = \begin{pmatrix} a_i b_j \\ -(g b)_i (g^{-1}) a_j \end{pmatrix},
\]

\[
M_{ij}^{Sp} := A_i \gamma_i (g^{-1})_j t A_l = \begin{pmatrix} a_i b_j \\ +(g b)_i (g^{-1}) a_j \end{pmatrix}.
\] (3.20)

Raising the flavor index by \( \gamma_i \) or \( \gamma_j \), we can see these two mesons have the same eigen values,

\[
Z_{ij} \sim M_{ij}^{SO} (-i \gamma_i) \sim M_{ij}^{Sp} (\gamma_j) \sim \begin{pmatrix} a_i b_j \\ -(g b)_i (g^{-1}) a_j \end{pmatrix}.
\] (3.21)

Note that the positions of mirror D-branes can be read off from the Chan-Paton index structure,

\[
\tilde{a}_i = g_{ij} b_j,
\]

\[
\tilde{b}_i = -(g^{-1})_i a_j,
\]

\[
\tilde{z}_{ij} = -g_{ik} z_{lk} (g^{-1})_j.
\] (3.22)

If we take \( g = \epsilon \) in particular, the effects of the projection to the conifold is

\[
R : (z_1, z_2, z_3, z_4) \mapsto (-z_1, -z_2, -z_3, +z_4).
\] (3.23)

3.3.1 Symmetry

Chiral operators are also obtained from those of the \( SU \times SU \) case by replacing \( A_i \) with \( B_i \),

\[
\mathcal{O}_n := C^{(k_1 \ldots k_n)}_{L} C^{(l_1 \ldots l_n)}_{R} \text{Tr}(A_{k_1} (\gamma_{Sp} t A_{l_1} \gamma_{SO}) \ldots A_{k_n} (\gamma_{Sp} t A_{l_n} \gamma_{SO})).
\] (3.24)

Global symmetries are reduced to \( SU(2) \), \( U(1)_A \) and \( U(1)_R \) which are summarized in Table 3

| \( A_i \) | \( SO(N_1) \) | \( USp(N_2) \) | \( SU(2) \) | \( U(1)_A \) | \( U(1)_R \) |
|---|---|---|---|---|---|
| \( A_i \) | \( N_1 \) | \( N_2 \) | 2 | \( 1/2 N_1 N_2 \) | \( 1/2 \) |
| \( \Lambda_{SO}^b \) | \( N_1 \) | \( N_2 \) | \( 2/N_1 \) | \( 2(N_1 - N_2 - 2) \) |
| \( \Lambda_{USp}^b \) | \( N_1 \) | \( N_2 \) | \( 1/N_2 \) | \( -(N_2 - N_1 + 2) \) |
| \( \lambda \) | \( N_1 \) | \( N_2 \) | \( -2/N_1 N_2 \) | \( 0 \) |
| \( \mathcal{O}_n \) | \( (n + 1) \otimes (n + 1) \) | \( 2n/N_1 N_2 \) | \( n \) |

Table 3: Quantum numbers of \( SO \times USp \) theory

The dynamical scale of \( USp \) gauge group always appears through \( \Lambda_{Sp}^b \), hence the anomaly free \( R \) symmetry is \( \mathbb{Z}_{2M} \) when we take \( N_1 = N + M + 2 \) and \( N_2 = N \).
3.3.2 RG cascade

From the “Novikov-Shifman-Vainshtein-Zakharov beta function”\cite{24}, we obtain\cite{24}

\[ \frac{d}{d \log(\Lambda/\mu)} \frac{8\pi^2}{g_{SO}^2} = 3(N_1 - 2) - 2 \times N_2 \times 1(1 - \gamma_A), \tag{3.25} \]

\[ \frac{d}{d \log(\Lambda/\mu)} \frac{8\pi^2}{g_{Sp}^2} = \frac{3}{2}(N_2 + 2) - 2 \times N_1 \times 1(1 - \gamma_A), \tag{3.26} \]

where \( \gamma_A \) denotes the anomalous dimension of the matter \( A_i \)'s. If we impose conformal invariance, \( \gamma_A = -1/2 \) and \( N_1 - N_2 = 2 \) are required. These conditions agree with R-R force balance in the type IIA brane configuration picture. Away from the conformality, \( \gamma_A \sim -1/2 + O(N_1 - N_2/N_1 + N_2) \), we obtain

\[ \frac{d}{d \log(\Lambda/\mu)} \frac{8\pi^2}{g_{SO}^2} \sim 3(N_1 - N_2 - 2) + O(N_1 - N_2, N_1 + N_2), \tag{3.27} \]

\[ \frac{d}{d \log(\Lambda/\mu)} \frac{8\pi^2}{g_{Sp}^2} \sim \frac{3}{2}(N_2 - N_1 + 2) + O(N_1 - N_2, N_1 + N_2). \tag{3.28} \]

Hence two gauge couplings flow opposite way.

When \( N_1 = N + M + 2 \) and \( N_2 = N \), the \( SO \) gauge group becomes strong coupling and we must perform Seiberg duality transformation for reliable description. We have already verified in sec 2.2 gauge groups become \( SO(N - M + 2) \times USp(N) \). Upon this duality transformation, we have dual quarks \( \tilde{A}_i \) and extra singlets \( M_{ij}^{SO} \) which are mesons of the original theory. The superpotential of the dual theory becomes

\[ W = \lambda \text{tr} (M_{ij}^{SO} \gamma^{Sp} M_{kl}^{SO} \gamma^{Sp}) \epsilon^{ik} \epsilon^{jl} + \frac{1}{\mu} \text{tr} (M_{ij} \gamma^{Sp} \epsilon^{ik} \epsilon^{jl}). \tag{3.29} \]

Since singlets are massive, we may integrate them out and then we have a superpotential of the same form as original theory (eq. (3.14)).

When \( N_2 > N_1 - 2 \) above analysis applies in the same way. And we find the \( SO \times USp \) duality cascade as in eq. (2.1).

3.3.3 Deformed conifold as quantum moduli space

Now we want to show that the quantum moduli space is deformed as in the \( SU \times SU \) case at the bottom of the cascade. We may suppose \( N_1 \gg N_2 \) (\( M \gg N \), \( N_1 = N + M + 2, N_2 = N \)) or \( N_2 \gg N_1 \) (\( M \gg N \), \( N_1 = N + 2, N_2 = N + M \)) as a result of successive cascade.

\[ \text{Here, we have used NSVZ beta function } \cite{24} \text{ of the form} \]

\[ \frac{d}{d \log(\Lambda/\mu)} \frac{8\pi^2}{g_{SO}^2} = 3T(G) - \sum_i T(R_i)(1 - \gamma_i) \]

where \( T(R) \) denotes the index of the representation \( R \) (it is defined as the normalization of generators \( T(R)\delta^{ab} = \text{tr} T_R^a T_R^b \)). The summation is taken over the representation to which the \( i \)-th matter belongs.

This is not the standard form of the NSVZ beta function. In duality cascade literature, normalization of the gauge coupling is chosen so that denominator of NSVZ beta function is not needed, which is commented in \cite{25, 26}. The relation with normalization of the gauge coupling and the exact expression of the beta function was found in \cite{27}. A simple exposition is given for example in \cite{28}.
Firstly when $N_1 \gg N_2$, $SO(N_1)$ gets strong coupling and $USp(N_2)$ may be treated as flavor symmetry ( $\Lambda_{Sp}$ can be ignored). Due to a strong coupling effect Affleck-Dine-Seiberg superpotential $W_{ADS} = \left( \frac{\Lambda_{Sp}^8}{\det M_{ij}^{SO}} \right)^{1/(N_1-2N_2-2)}$ is generated, where the determinant is taken to $SU(2)$ and $USp(N_2)$ as one flavor index. Let us take the diagonal form $A_i = \begin{pmatrix} a_i \\ (gb_i) \end{pmatrix}$ with first $N_2/2$ nonzero elements $a_i$ taking the same value and also $b_i$. Hence the $2N_2$ by $2N_2$ $SO$ meson matrix is brought to $N_2/2$ by $N_2/2$ block diagonal form with each diagonal entry as

$$
\begin{pmatrix}
(g t^z t^{-1} g^{-1})_{11} & z_{11} \\
(g t^z t^{-1} g^{-1})_{21} & z_{21}
\end{pmatrix}
\begin{pmatrix}
(g t^z t^{-1} g^{-1})_{12} \\
(g t^z t^{-1} g^{-1})_{22}
\end{pmatrix}.
$$

(3.30)

We have $\det M_{ij}^{SO} = ((\det z_{ij})^2)^{N_2/2}$, where the small determinant is taken to $SU(2)$ index. On the other hand $W_{tree} \sim \tr (M_{ij}^{SO}) \gamma_{Sp} M_{kl}^{SO} \gamma_{Sp} e^{ik\epsilon l} \sim \lambda \det z_{ij}$. The supersymmetric vacuum condition $\partial(W_{tree} + W_{ADS}) = 0$ is

$$
0 = \left( \lambda - \left[ \frac{\Lambda_{Sp}^8}{(\det z_{ij})^{N_1-N_2-2}} \right]^{1/(N_1-2N_2-2)} \right) z_{ij}.
$$

(3.31)

Hence, the quantum moduli space becomes the deformed conifold and has $M$ branches ($N_1 = N + M + 2, N_2 = N$).

Second when $N_2 \gg N_1$, $USp(N_2)$ becomes strong coupling. Affleck-Dine-Seiberg superpotential is $W_{ADS} = \left( \frac{\Lambda_{Sp}^8}{\text{Pf} M_{ij}^{USp}} \right)^{2/(N_2-2N_1+2)}$ where Pfaffian is taken to $SU(2)$ and $SO(N_1)$ as one flavor index. Putting $A_i$ “diagonal” as in the $SO$ case, $USp$ mesons are brought to $N_1/2$ by $N_1/2$ block diagonal form with each diagonal entry as

$$
\begin{pmatrix}
-(g t^z t^{-1} g^{-1})_{11} & z_{11} \\
-(g t^z t^{-1} g^{-1})_{21} & z_{21}
\end{pmatrix}
\begin{pmatrix}
-(g t^z t^{-1} g^{-1})_{12} \\
-(g t^z t^{-1} g^{-1})_{22}
\end{pmatrix}.
$$

(3.32)

We have $\text{Pf} M_{ij}^{SO} = ((\det z_{ij})^{N_1/2}$. The supersymmetric vacuum condition is

$$
0 = \left( \lambda - \left[ \frac{\Lambda_{Sp}^{25}}{(\det z_{ij})^{N_2-N_1+2}} \right]^{1/(N_2-2N_1+2)} \right) z_{ij}.
$$

(3.33)

Again, the quantum moduli space becomes the deformed conifold and has $M$ branches ($N_1 = N + 2, N_2 = N + M$).

### 3.4 Support for our argument

We give some comments on our orientifold projection.

Our projection is different from the one proposed by [14]. But we claim our projection is the correct one for the Klebanov-Strassler model from the following reason. The resulting $SO \times USp$ theory must have an
\( \mathcal{N} = 1 \) SUSY as can be expected from the IIA picture. The holomorphic coordinates of the conifold \( z_i \) are constructed as chiral superfields on the gauge theory side. Since their projection relates chiral superfield \( z_i \) and anti-chiral superfield \( \bar{z}_i \), it is not compatible with \( \mathcal{N} = 1 \) supersymmetry. On the other hand, our projection is determined from \( \mathcal{N} = 1 \) supersymmetry as one of the requirement, and the resulting field theory exhibits duality cascade.

As we have noted in the footnote of sec 3.1, there are some sign ambiguities in the transformation property of chiral superspace coordinate \( \theta \). But these ambiguities only affects whether the projection for \( (z_1, z_2, z_3, z_4) \) is \((-,-,+,-)\) or \((+,+,-,-)\). First projection corresponds to the O3-plane, because the fixed point of this projection is located only at the tip of the conifold. On the other hand, second one has (real) four dimensional fixed point set in the conifold. Therefore it corresponds to the O7-plane.

For later convenience, let us relabel coordinates in eq. (3.3) as \( x = z_{11}, \ y = z_{22}, \ z = z_{12}, \ w = z_{21} \). Then our projection acts as \( (x, y, z, w) \mapsto (y, x, -z, -w) \). If we take the T-duality, the conifold becomes intersecting NS5-branes located at \( zw = 0 \). With our choice of coordinates in the IIA picture (sec 2.1), correspondence of the coordinates become \( z = x_4 + ix_5 \) and \( w = x_8 + ix_9 \). Therefore the projection \( (z, w) \mapsto (-z, -w) \) implies that it gives the O4-plane in the type IIA picture. At this stage, \( (x, y) \) and \( (z, w) \) seems to be on equal footing. So one might suppose that \( xy = 0 \) is also allowed as positions of intersecting NS5-branes after T-duality. But as we will see in sec 3, \( (z, w) \) is suitable as the locus of NS5-branes when the conifold is viewed as one in the series of the generalized conifolds. So our choice of projection is consistently extended to the O4-plane in the generalized NS5-brane configurations.

Note that our projection cannot be imposed on the resolved conifold. The D-flatness condition of the \( SU(N) \times SU(N) \) theory is solved as

\[
|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = \left( |a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2 \right) = \xi 1 \quad (3.34)
\]

where \( \xi \) is a constant. Here we consider the \( N_1 = N_2 \) case because if \( N_1 \neq N_2 \), \( \xi \) is zero. In the type IIA picture, \( N_1 \neq N_2 \) means that \( |N_1 - N_2| \) fractional D4-branes are suspended between the NS5-brane and the NS5'-brane. Hence it is impossible to pull a single NS5-brane away with keeping supersymmetry. Non zero constant \( \xi \) means that the conifold singularity is resolved. If we introduce the orientifold, \( \tilde{a} \) and \( \tilde{b} \) relate to \( b \) and \( a \) by eq. (3.22). Then \( \xi \) must vanish. This is consistent with the type IIA picture where NS5-branes cannot be pulled away from the O4-plane.

4 IIB SUGRA solution

In this section we investigate the space-time aspects of our orientifold projection in more detail and show that KT/KS solutions \( [3, 7] \) survive with some shifts of boundary conditions.
4.1 Fixed points of orientifold projection

The singular conifold \( z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \) is a cone over \( T^{1,1} \) space \( \sim (SU(2) \times SU(2))/U(1) \). This is easy to see in gauge theory or symplectic quotient construction.

Block diagonal elements \( a_i, b_i \) of chiral superfields \( A_i, B_i \) is identified as \( a_i \sim e^{i\epsilon}a_i \) and \( b_i \sim e^{-i\epsilon}b_i \).

We introduce vector and matrix notation, \( \mathbf{a} := \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \), \( \mathbf{b} := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \), \( \mathbf{a} := \begin{pmatrix} a_1 \\ -a_2 \end{pmatrix} \), \( \mathbf{b} := \begin{pmatrix} b_1 \\ -b_2 \end{pmatrix} \).

From D-flatness condition, \( |a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2 = 0 \), the radial coordinate of the cone is defined as \( \rho^2 = |\mathbf{a}|^2 = |\mathbf{b}|^2 \). When \( \rho = 1 \), \( \mathbf{a} \) and \( \mathbf{b} \) belong to \( SU(2) \).

\( U(1) \) identification now reads

\[
a \sim a \begin{pmatrix} e^{i\epsilon} \\ e^{-i\epsilon} \end{pmatrix}, \quad b \sim b \begin{pmatrix} e^{i\epsilon} \\ e^{-i\epsilon} \end{pmatrix}. \tag{4.1}\]

On the other hand, \( T^{1,1} \) space is topologically \( S^3 \times S^2 \). We can manifestly construct gauge invariant \( S^3 \sim SU(2) \) coordinates

\[
c = c_\mu \tau_\mu := a^\dagger b, \tag{4.2}\]

and \( S^2 \) coordinate by moment map

\[
n^i := a^\dagger \tau^i a. \tag{4.3}\]

In this notation the orientifold projection is

\[
\mathbf{a} \mapsto g \mathbf{b}, \quad a \mapsto g b, \\
\mathbf{b} \mapsto -g^{-1} a, \quad b \mapsto -g^{-1} a. \tag{4.4}\]

For \( S^3 \) coordinates

\[
c = a^\dagger b \mapsto gb^\dagger (-g^{-1} a) = -g^\dagger c^\dagger g^{-1}. \tag{4.5}\]

When \( g = \epsilon \), this acts like quaternionic conjugation,

\[
(c_1, c_2, c_3, c_4) \mapsto (-c_1, -c_2, -c_3, c_4). \tag{4.6}\]

For \( S^2 \) coordinates

\[
a^\dagger \tau^i a \mapsto (g^\dagger \mathbf{a})^\dagger \tau^i (g^\dagger \mathbf{a}) = -a^\dagger c^\dagger c^\dagger a. \tag{4.7}\]

Here we have used \( g = \epsilon \) in the last step. This is a combined operation of reflection and rotation that depends on \( S^3 \) coordinates, \( n_i \mapsto -R(c)^j_i n^j \). Direct calculation shows that it has eigen value \((-,-,+)+(,-,-,+)\) on the equator of \( S^3 \) \((c_4 = 0)\) and \((-,-,-)\) away from the equator. At the equator \((c_4 = 0)\), \( S^2 \) part has fixed point set \( S^1 \). But the projection acts as anti-podal map on the equator of \( S^3 \), so it has no fixed point. The projection also
has no fixed point away from the equator, since it act as anti-podal map on both $S^2$ and constant $c_4$ section of $S^3$. Therefore, the orientifold projection on the singular conifold has a fixed point only at the apex, where both $S^2$ and $S^3$ collapse.

Once the action on $S^3 \times S^2$ is known, we can extend this to the deformed and resolved conifold. When the conifold is deformed, only $S^2$ shrinks and $S^3$ remains finite volume at the apex. From eq (4.6) it has two fixed points at the north and south pole of $S^3(c_4 = \pm 1)$ at the apex of the conifold (Fig 5). Physically, we have two O3-planes located at the north pole and the south pole of the apex. This is in contrast with D3-branes, which are replaced by fluxes far in the IR of gauge theory [5, 9]. When the conifold is resolved, $S^2$ remains finite volume and $S^3$ shrinks. Since $\mathbb{Z}_2$ action on $S^2$ depends on $S^3$ coordinates, it is not well-defined at the apex. This is also consistent with the previous analysis.

4.2 IIB SUGRA solution

Construction of SUGRA solutions in the presence of orientifold planes takes two steps. As we will show, all the fields in the ansatz of KT/KS solutions have even parity with respect to the orientifold projection $\Omega(\mathcal{O})(-)^F_L \mathcal{O}$. Hence KT/KS solution survives the projection. All we have to do is simply giving suitable boundary conditions.

In KT/KS solutions, 2-form gauge fields $B_{NS}$, $B_{RR}$ and their field strengths are expressed by linear combination of the following 2-forms

$$\frac{1}{2}(g^1 \wedge g^2 + g^3 \wedge g^4), \quad (4.8)$$
$$g^1 \wedge g^2 - g^3 \wedge g^4, \quad (4.9)$$
$$g^1 \wedge g^3 + g^2 \wedge g^4 \quad (4.10)$$

and 3-forms

$$\frac{1}{2}g^5 \wedge (g^1 \wedge g^2 + g^3 \wedge g^4), \quad (4.11)$$
$$-g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4), \quad (4.12)$$
\[ g^5 \wedge (g^1 \wedge g^2 - g^3 \wedge g^4). \] (4.13)

See Appendix \[A\] for our conventions. These 2-forms and 3-forms have odd parity under space-time part of the projection, \( R : (z_1, z_2, z_3, z_4) \mapsto (-z_1, -z_2, -z_3, z_4). \) On the other hand, the metric of the singular/deformed conifold and five form field strength which is proportional to the volume form of \( T^{1,1} \) space have even parity under \( R. \) As for \( \Omega(-)^F \), 2-forms \( B_{\text{NS}} \) and \( B_{\text{RR}} \) have odd parity. Therefore, all fields in KT/KS solution are even under the whole projection \( \Omega(-)^F \).

Next we consider proper boundary conditions for the background which corresponds to \( SO(N + M + 2) \times USp(N) \) gauge theory. In order to find this, we use the type IIA brane picture. In \( SU \times SU \) case, whole D4-branes contribute to D3-charges in type IIB theory and fractional D4-branes contribute to D5-charges. At first sight there might appear \( M + 2 \) fractional D4-branes. But two of the fractional D4-branes and the O4\(^-\)-plane in the NS-NS' interval and the O4\(^+\)-plane in the NS'-NS interval give one unit of D3-charge in type IIB picture. So D5-charges of this configuration will be \( M! \).

In the case of \( SO(N + M + 2) \times USp(N) \) theory, we propose the following boundary condition in covering space,

\[
\frac{1}{4\pi^2\alpha'} \int_{S^3} F_3 = M, \quad \frac{1}{(4\pi^2\alpha')^2} \int_{T^{1,1}} F_5 = N + 1. \quad (4.14)
\]

The corresponding KT/KS solution is obtained by only replacing \( N \) with \( N + 1 \).

### 4.3 O3\(^+\) or O3\(^-\)? — discussion —

In the deformed conifold which captures correct IR nature of the gauge theory, there are two fixed points at the north and south poles of \( S^3 \). Are these fixed points O3\(^-\)-plane or O3\(^+\)-plane? From T-dualized type IIA picture, we expect that one is O3\(^+\)-plane and the other O3\(^-\)-plane. But boundary conditions in eq. (4.14) are only aware of overall fluxes and seem to smear such microscopic input.

As investigated in [29], the O3\(^+\)-plane can be interpreted as the O3\(^-\)-plane wrapped by an \( \mathbb{RP}^2 \)-shaped NS5-brane. So our boundary condition might be understood as O3\(^-\)-planes placed at both north and south poles of \( S^3 \) and the wrapped NS5-brane is smeared.

Aside from the interpretation of our boundary condition, the orientifolded conifold includes \( \mathbb{RP}^2 \) in interesting way. In the coordinate \( c \) and \( n^i \), the orbifold part of the orientifold projection is \((c_1, c_2, c_3, c_4) \mapsto (-c_1, -c_2, -c_3, +c_4)\) and \( n^i \mapsto -R(c)^{ij}n^j \). Consider \( S^2 \times S^2 \) obtained by setting \( c_4 \) to be constant. Let the \( S^2 \) in \( S^3 \) shrink toward the north pole as we increase radial coordinate from the apex \( \rho = \epsilon \) (Fig. 3). In this way, we obtain \( S^5 \) in the conifold that surround the north pole \((c_4 = 1)\) as “join” of two \( S^2 \)'s. \( \mathbb{Z}_2 \) acts on the \( S^5 \) as anti-podal map, hence \( \mathbb{RP}^5 \) is obtained. Direct computation shows that \((n^1, n^2, n^3) = (c_1, c_2, c_3)\) is \(-1\) eigen vector of \(-R(c)^{ij}\). So diagonal \( S^2 \) becomes \( \mathbb{RP}^2 \). As for the south pole, \( \mathbb{RP}^5 \) and \( \mathbb{RP}^2 \) are also obtained.
in the same way. These two $\mathbb{R}P^2$ are continuously moved to each other by changing $c_4$. This implies that configuration of the $O3^+$-plane at the north pole and the $O3^-$-plane at the south pole is equivalent to the $O3^-$-plane at the north pole and the $O3^+$-plane at the south pole. At the bottom of duality cascade, either of the gauge groups might be regarded as flavor symmetry, this might be interpreted $SO$ and $USp$ gauge theory can be continuously interpolated in the IR as found in [30].

![Figure 6: The north pole of $S^3$ at the apex is surrounded by $S^5$ in the conifold.](image)

The $\mathbb{R}P^2$ might be essential to explain the duality cascade phenomenon on the type IIB side. Firstly, let us interpret $N + 1$ units of D3-charge as $N + 2 - 1$. $N + 2$ of them come from D3-branes and $-1$ comes from two $O3^-$-planes. For convenience sake, let us suppose $N + M + 2$ fractional D3-branes are stuck on the $O3^-$-plane at the north pole and $N + 2$ fractional D3-branes are stuck on the $O3^-$-plane at the south pole. As mentioned above the $O3^+$-plane can be regarded as the $O3^-$-plane wrapped by NS5-brane. Two units of fractional D3-charge that convert $O3^-$-plane to $O3^+$-plane come from Chern-Simons-like coupling on the NS5-brane [29]. Let the NS5-brane enclose the $O3^-$-plane at the south pole. Then we may interpret the $O3^-$-plane and 2 fractional D3-charges as the $O3^+$-plane. So the gauge group can be identified as $SO(N + M + 2) \times USp(N)$. On each step of duality cascade, D3-charges decrease by $M$ units. Hence we have $M + (N - M) + 2$ fractional D3-charges at the north pole and $(N - M) + 2$ at the south pole. This time we propose that the NS5-brane encloses the $O3^-$-plane at the north pole. So we have $USp(N) \times SO(N - M + 2)$ gauge groups which agree with duality cascade eq. (2.1).

To reproduce the correct cascade, the NS5-brane has to bounce between the north and the south pole during the cascade steps. Above proposal is natural from the view point of the type IIA picture, since as mentioned in sec 2.1, the $O4^+$-plane in the NS'-NS interval becomes the $O4^-$-plane after brane crossing.
5 Orientifold in Generalized conifolds

In section 3, we have succeeded to determine the projection on the gauge theory side, which corresponds to the O3-plane in the conifold on the type IIB side and the O4-plane in type IIA elliptic models. It is tempting to generalize the orientifold projection to the case of generalized conifolds \( xy = z^n w^m \). Again type IIA models are most illustrative pictures where the generalized conifolds are realized as transversal \( n \) NS5-branes and \( m \) NS5’-branes [31]. In these pictures the O4-plane is still allowed when \( n + m \) is even [20].

Above brane configurations are obtained from \( \bar{N} \) parallel NS5-branes by rotating some of the NS5-branes. Since the O4-projection is well-defined through all the interpolating theory, it is sufficient to consider the case of parallel NS5-branes. This configuration corresponds to \( \mathcal{N} = 2, \mathbb{C}^2/\mathbb{Z}_N \times \mathbb{C} \) quiver gauge theory in the type IIB picture.

5.1 Comparison to O6-plane case

Before analyzing the O4-plane case in the type IIA picture, let us recall what takes place in the O6-plane case [15, 16, 17]. We simply review the \( O6^- - O6^- \) case with \( \mathcal{N} = 2 \) SUSY in four dimensions [15, 16]. The brane configurations are \( N \) NS5-branes along 012345, D4-branes along 01236, and 2 \( O6^-\)-planes along 0123789. We consider the case that two NS5-branes intersect the O6-planes (Fig. 7 (a)). Therefore \( N \) is even.

Taking the T-duality along the \( x^6 \) direction, we have D3-branes on the fixed point of \( \mathbb{C}^2/\mathbb{Z}_N \times \mathbb{C} \) singularity with an O7-plane. In the first place, let us see the spectrum on the D3-branes world-volume theory without the O7-plane. This theory is obtained from \( \mathcal{N} = 4 \) SU gauge theory by orbifold projection [32]. In \( \mathcal{N} = 1 \) language, this theory has 3 adjoint chiral superfields \((X,Y,Z)\) which correspond to the transverse directions to the D3-branes. And the superpotential is \( W = \text{tr}(Z[X,Y]) \). \( \mathbb{Z}_N \) orbifold projection \( \theta : (x,y,z) \mapsto (e^{2\pi i/N}x, e^{-2\pi i/N}y, z) \) acts on each Chan-Paton sector as

\[
\begin{align*}
X_{ij} &= e^{\frac{2\pi i}{N}(i-j+1)}X_{ij}, \\
Y_{ij} &= e^{\frac{2\pi i}{N}(i-j-1)}Y_{ij}, \\
Z_{ij} &= e^{\frac{2\pi i}{N}(i-j)}Z_{ij}, \\
W_{ij} &= e^{\frac{2\pi i}{N}(i-j)}W_{ij}.
\end{align*}
\]

(5.1)

Gauge fields surviving the orbifold projection are

\[
W = \begin{pmatrix}
W_{11} & W_{12} & \cdots & W_{1N} \\
W_{21} & W_{22} & \cdots & W_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
W_{N1} & W_{N2} & \cdots & W_{NN}
\end{pmatrix}
\]

(5.2)

\footnote{We need D6-branes for the cancellation of tadpoles, however, they are irrelevant to our discussion. So we omit them.}
which give $SU_1 \times SU_2 \times \cdots \times SU_N$ gauge groups. Surviving matter fields are

$$X = \begin{pmatrix} X_{12} & \cdots & X_{N-1,N} \\ \vdots & \ddots & \vdots \\ X_{N1} & & X_{N-1,N} \end{pmatrix}, \quad Y = \begin{pmatrix} Y_{21} & \cdots & Y_{1N} \\ \vdots & \ddots & \vdots \\ Y_{N1} & & Y_{N,N-1} \end{pmatrix}$$

(5.3)

and

$$Z = \begin{pmatrix} Z_{11} & \cdots & Z_{22} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & Z_{NN} \end{pmatrix}.$$  

(5.4)

$W_{i,i}$ and $Z_{i,i}$ are a vectormultiplet of the $i$-th $SU$ gauge group. $X_{i,i+1}$ and $Y_{i+1,i}$ are ($\bar{\square}_i, \square_{i+1}$) and ($\square_i, \bar{\square}_{i+1}$) representation in $SU_i \times SU_{i+1}$ groups. Hence they are combined into hypermultiplets.

In the type IIA picture, $i$-th gauge group is on $i$-th D4-branes which suspended between $(i-1)$-th and $i$-th NS5-brane. Therefore $X_{i,i+1}$ correspond to open strings which run from $i$-th D4-branes to $(i+1)$-th D4-branes, and $Y_{i+1,i}$ are ones from $(i+1)$-th D4-branes to $i$-th D4-branes.

The O7 projections for Chan-Paton matrices $^1$ are given by

$$X = \gamma_{\Omega^t} t X \gamma_{\Omega^t}^{-1}$$

(5.5)

where $\Omega^t = \Omega(-)^F z R_z$ and

$$\gamma_{\Omega^t} = \begin{pmatrix} \cdots & 1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & 1 \\ -1 & \cdots & \cdots \end{pmatrix}.$$  

(5.6)

The projections to other fields are similar to $X$. These projections give the following relations,

$$W_{i,i} = -i W_{N-i+1,N-i+1},$$  

(5.7)

$$Z_{i,i} = -i Z_{N-i+1,N-i+1},$$  

(5.8)

$$X_{i,i+1} = i X_{N-i,N-i+1} \text{ for } i \neq \frac{N}{2}, N,$$  

(5.9)

$$X_{\frac{N}{2}+1,\frac{N}{2}+1} = -i X_{\frac{N}{2},\frac{N}{2}},$$  

(5.10)

$$Y_{i+1,i} = i Y_{N-i+1,N-i} \text{ for } i \neq \frac{N}{2}, N,$$  

(5.11)

$$Y_{\frac{N}{2}+1,\frac{N}{2}} = -i Y_{\frac{N}{2},\frac{N}{2}+1}.$$  

(5.12)

$^1$For $W$ and $Z$, we need the minus sign in RHS of Eq (5.5).
Gauge groups $SU_i$ for $N/2 < i \leq N$ are related to $SU_{N-i+1}$. Hence the resulting gauge theory is $SU_1 \times SU_2 \times \cdots \times SU_{N/2-1} \times SU_{N/2}$ with matters in $\square_1 \oplus (\square_1, \square_2) \oplus (\square_2, \square_3) \oplus \cdots \oplus (\square_{N/2-1}, \square_{N/2}) \oplus \square_{N/2}$ representation.

In the type IIA picture, that orientifold projection nicely matches with the brane configuration. We take $N/2$-th and $N$-th NS5-branes as intersecting with O6-planes (Fig. 7(a)). The $i$-th D4-branes are mirrors to $(N-i+1)$-th D4-branes by the O6-planes. Open strings corresponding to $X_{i,i+1}$ and $Y_{i+1,i}$ relate to the mirror open strings $X_{N-i,N-i+1}$ and $Y_{N-i+1,N-i}$ respectively.

The O6-planes relate the D4-branes to ones in the different Chan-Paton sector. In the case of the O4-plane, the D4-branes are related to ones in the same Chan-Paton sector (Fig. 7(b)). So the orientifold projection $\gamma_{\Omega'}$ becomes the (block) diagonal matrix and relates the open strings to ones in the same sector as we will see in the next subsection.

![Figure 7: Brane configuration for $SU \times SU \times SU$ model and $(SO \times USp)^3$ model. Open strings connecting $i$-th D4-brane to $(i+1)$-th D4-brane correspond to $X_{i,i+1}$ and $Y_{i+1,i}$.](image)

5.2 Determination of the projection in $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{C}$ case

From the type IIA picture, the projection in the type IIB $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{C}$ orbifold relates open strings that connect adjacent fractional D4 branes with opposit orientation in the same Chan-Paton sector. Therefore, since $X$ and $Y$ are related, it is natural to expect the orientifold projection takes following form.

$$\chi_{\Omega'} : X^I \mapsto -\chi^I_{\gamma_{\Omega'}}X^J_{\gamma_{\Omega'}}^{-1}. \quad (5.13)$$

where $X^I$, $I = 1, 2, 3$ are $(X,Y,Z)$. This projection is combined operation of usual O3 projection $\Omega' = \Omega(-)^F L R_{XYZ}$ and rotation $\chi : (X,Y,Z) \mapsto (Y,-X,Z)$. This kind of generalization of the O3 projection is allowed in our case [32].

Let us take $\gamma_{\Omega'} = \text{block.diag}(\gamma_1, \gamma_2, ..., \gamma_N)$. From the conditions

$$X = -\gamma_{\Omega'} Y \gamma_{\Omega'}^{-1},$$

we have

\begin{align*}
\text{Tr}(X) &= -\text{Tr}(\gamma_{\Omega'} Y \gamma_{\Omega'}^{-1}), \\
\text{Tr}(Y) &= -\text{Tr}(X \gamma_{\Omega'}^{-1} \gamma_{\Omega'}), \\
\text{Tr}(XY) &= -\text{Tr}(X Y \gamma_{\Omega'}^{-1} \gamma_{\Omega'}),
\end{align*}
gauge groups have alternating structure
\(SO \oplus\) in consistent with the results found by [33] in the context of O5-D5 systems. We take \(N\) even below. O4-plane in the type IIA picture [11, 20]! So we have the O3-plane in the orbifold theory. Our projection is which give adjoint matters of \(N\) \(\gamma\) with second relations eq. (5.16) requires

First relations eq. (5.15) are rewritten as

\[ Y = Y_{i+1} + Y_i \gamma_{i+1}^{-1} \]

\[ Z = Z_{i+1} + Z_i \gamma_{i+1}^{-1} \]

for even \(i\), for odd \(i\).

We give some remarks. Combined with \(Z_N\) orbifold group, the orientifield projection group has the structure \(\mathbb{Z}_N \oplus \mathbb{Z}_N \chi \Omega\). \(\chi \Omega\) is required to be order 4. We can verify it explicitly

\[ (\chi \Omega)^4 : X^I : (\gamma \gamma_i^{-1})^2 \]

Since our solution satisfies \((\gamma \gamma_i^{-1} = \text{block.diag} (1, -1, 1, -1, ..., 1, -1))\), it has exactly order 4.
5.3 Generalized conifold case

As remarked before, once the orientifold projection is obtained in the orbifold $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{C}$, we can extend this result to the generalized conifold case.

The orbifold $xy = w^N$ is deformed to the generalized conifold $xy = z^m w^n$ ($n+m = N$) by complex structure deformation with the interpolating equation $xy = \prod_{i=1}^{N}(w - \alpha_i z)$, where $\alpha_i$'s are deformation parameters. In terms of type IIA theory, this corresponds to arbitrary rotated NS5-branes. In the gauge theory, this corresponds to mass deformation $W_m = \text{tr} (MZ^2)$ where $M$ is a certain mass matrix.

To investigate the effect of the orientifold projection on the space-time, we see the relation between the coordinates $(x,y,z,w)$ and matter fields $(X,Y,Z)$. The moduli space of vacua of the $\mathcal{N} = 2$ gauge theory can be identified as $xy = w^N$ as follows. For this purpose, it is sufficient to assume that each field has diagonal expectation values.

The F-flatness condition $[X,Z] = 0$ requires $Z_{i,i}X_{i,i+1} = X_{i,i+1}Z_{i+1,i+1}$, hence the solution is $Z_{11} = Z_{22} = \ldots = Z_{NN}$. $[X,Y] = 0$ requires $X_{i,i+1} Y_{i+1,i} = Y_{i,i-1} X_{i-1,i}$, hence $X_{12} Y_{21} = X_{23} Y_{32} = \ldots = X_{N1} Y_{1N}$. $[Y,Z] = 0$ requires no further constraint. Then gauge invariant operators modulo F-flatness conditions are

$$
\begin{align*}
  x &= X_{12} X_{23} \cdots X_{N1}, \\
  y &= Y_{21} Y_{32} \cdots Y_{1N}, \\
  w &= X_{12} Y_{21} = X_{23} Y_{32} = \cdots = X_{N1} Y_{1N}, \\
  z &= Z_{11} = Z_{22} = \cdots = Z_{NN}.
\end{align*}
$$

These operators obey one constraint $xy = w^N$ which is the defining equation of $\mathbb{C}^2/\mathbb{Z}_N$ as hypersurface and $z$ parameterizes a complex plane $\mathbb{C}$.

Our projection eqs (5.15), (5.16) and (5.17) acts on $x, y, z, w$ as

$$
(x, y, z, w) \mapsto (y, x, -z, -w). \tag{5.22}
$$

Here we used the fact that $N$ is even. In our projection $z$ and $w$ have the same parity. Therefore we can deform the orbifold $xy = w^N$ to the generalized conifolds $xy = z^m w^n$ through $xy = \prod_{i=1}^{N}(w - \alpha_i z)$ under our projection. This is consistent with the type IIA picture in which NS5-branes can be freely rotated.

This is in contrast to the O6-projections in \[17\], in which $z$ and $w$ have opposite parity. So the defining equation is deformed only pairwise $xy = \prod_{i=1}^{N/2}(z - \alpha_i w)(z + \alpha_i w)$.

6 Conclusion

We have determined the orientifold projection in the conifold in type IIB theory. This has been obtained by analysis of the correspondence between symmetries of the field theory realized on the world-volume of
D3-branes and type IIB SUGRA following Klebanov and Witten [4].

In the type IIB SUGRA picture, the spacetime reflection of the orientifold projection maps the coordinates of the conifold \((z_1, z_2, z_3, z_4)\) to \((-z_1, -z_2, -z_3, z_4)\). This orientifold projection has been identified as the O3-plane. In the T-dualized type IIA theory, this becomes the O4-plane. Our orientifold projection freezes the parameter of blowing up the singularity of the conifold since FI-parameter must vanish under the projection. This is consistent with the type IIA picture where we can not pull a single NS5-brane away from the O4-plane.

In terms of the field theory, the projection reduces the \(SU \times SU\) gauge theory to the \(SO \times USp\) gauge theory. From the field theory analysis, we have found that duality cascade phenomenon occurs in RG flow like \(SU \times SU\) theory [5]. This has been also expected from the type IIA brane configuration. At the bottom of the cascade the singularity of the conifold is deformed by Affleck-Dine-Seiberg superpotential as in the \(SU \times SU\) case again.

We have also found that the corresponding SUGRA solutions to the \(SO \times USp\) gauge theory can be obtained by only modifying the boundary conditions for the R-R-charges in KS and KT solutions [5]. This is better understood in type IIA picture, since if we focus on the R-R-charges the combination of \(O4^+, O4^-\)-planes and two fractional D4-branes gives one whole D4-brane charge. The boundary condition has matched with the duality cascade phenomenon.

We have extended the orientifold projection to the case of generalized conifolds. The projection requires the total number of NS and NS’-branes to be even. Moreover the gauge groups become \(SO \times USp \times \cdots \times SO \times USp\). These properties are consistent with the features of the O4-plane [11, 20]. The projection agrees with one which we have obtained by analysis to the conifold.

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**Appendix**
A Orientifold projection on popular parameterization

The conifold and deformed conifold metrics are given in [34]. In the literature $Z = z_{\mu} \tau_{\mu}$ is often parameterized by two $SU(2)$ matrices

$$L_i = \begin{pmatrix} \cos \frac{\theta_i}{2} e^{i(\psi_i + \phi_i)/2} & -\sin \frac{\theta_i}{2} e^{-i(\psi_i - \phi_i)/2} \\ \sin \frac{\theta_i}{2} e^{i(\psi_i - \phi_i)/2} & \cos \frac{\theta_i}{2} e^{-i(\psi_i + \phi_i)/2} \end{pmatrix},$$

(A.1)
as

$$Z = L_1 Z_0 L_2^\dagger,$$ (A.2)

where

$$Z_0 = \begin{cases} \begin{pmatrix} 1 \\ e^{\tau/2} \end{pmatrix} & \text{for the singular conifold,} \\ \begin{pmatrix} e^{-\tau/2} \\ e^{\tau/2} \end{pmatrix} & \text{for the deformed conifold.} \end{cases}$$ (A.3)

Basis of 1-forms on $T^{1,1}$ are

$$g^1 = \frac{1}{\sqrt{2}} (e^1 - e^3), \quad g^2 = \frac{1}{\sqrt{2}} (e^2 - e^4),$$

$$g^3 = \frac{1}{\sqrt{2}} (e^1 + e^3), \quad g^4 = \frac{1}{\sqrt{2}} (e^2 + e^4),$$

$$g^5 = e^5,$$ (A.4)

where

$$e^1 := -\sin \theta_1 d\phi_1, \quad e^2 := d\theta_1,$$

$$e^3 := \cos \psi \sin \theta_2 d\phi_2 - \sin \psi d\theta_2,$$

$$e^4 := \sin \psi \sin \theta_2 d\phi_2 + \cos \psi d\theta_2,$$

$$e^5 := d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2.$$ (A.5)

Here $\psi_1$ and $\psi_2$ appear only through the combination $\psi = \psi_1 + \psi_2$ which has period $4\pi$.

Once the orientifold projection written in $z_{ij}$, this can be also used for the projection on the deformed conifold. In fact, there are two-ways to write space-time $Z_2$ in terms of $L_i$’s. But this ambiguity is artifact of $U(1)$ redundancy. A convenient choice will be $L_1 \mapsto g \bar{L}_2 e^{i\tau_1}$, $L_2 \mapsto -g \bar{L}_1 e^{i\tau_1}$. When $g = \epsilon$, it is written in angular coordinate as

$$\theta_1 \leftrightarrow \theta_2, \quad \phi_1 \leftrightarrow \phi_2, \quad \psi \leftrightarrow \psi + 2\pi$$ (A.6)

One-forms transform

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \mapsto \begin{pmatrix} -\cos \psi & -\sin \psi \\ -\sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} e_3 \\ e_4 \end{pmatrix}, \quad \begin{pmatrix} e_3 \\ e_4 \end{pmatrix} \mapsto \begin{pmatrix} -\cos \psi & -\sin \psi \\ -\sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$ (A.7)
It is easy to verify various 2-forms and 3-forms in eqs. (4.8), (4.11) is odd under the space-time part of the projection. It can be more easily verified $SO(4)$ invariant expression found in [25].

Note that $SU(2) \times SU(2)$ to which $(L_1, L_2)$ belong is different from $SU(2) \times SU(2)$ to which $(a, b)$ belong in section 4.1.

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