MOMENT AND TAIL INEQUALITIES FOR POLYNOMIAL MARTINGALES.

The case of heavy tails.

E. Ostrovsky and L. Sirota

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan.

e-mails: eugostrovsky@list.ru; sirota3@bezeqint.net

Abstract.

In this paper we obtain the non-asymptotic exact moment and tails estimates for polynomial on martingale differences.

We give also some examples on order to show the exactness of obtained results.

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1 Introduction. Notations. Statement of problem.

Let \( (\Omega, F, P) \) be a probabilistic space, \( \Omega = \{\omega\}, \xi(i, 1), \xi(i, 2), \ldots, \xi(i, d) \) be a family of centered: \( E\xi(i, m) = 0, \ m = 1, 2, \ldots, d \) martingale differences relative some filtration \( \{F(i)\} \):

\[
F(0) = \{\emptyset, \Omega\}, F(i) \subset F(i + 1) \subset F:\

\forall k = 0, 1, \ldots, i - 1 \Rightarrow

E\xi(i, m)/F(k) = 0; E\xi(i, m)/F(i) = \xi(i, m) \ (\text{mod} \ P), \quad (1.1)

"martingale version".

In particular, the random vectors \( \vec{\xi}(m) = \{\xi(i, m)\}, i = 1, 2, \ldots, n \) for all the values \( n = 1, 2, \ldots \) may be independent; we will call this possibility " vector independent version".

A second case: the random vectors \( \vec{\xi}(m) = \{\xi(i, m)\}, i = 1, 2, \ldots, n \) may be dependent, but inside of this vectors the random variables \( \{\xi(i, m)\}, i = 1, 2, \ldots \)
are independent for all the values $m; m = 1, 2, \ldots, d$; we will denote this possibility "inside independence version". If in addition all the r.v. $\{\xi(i, m)\}$ are also independent, we will call this possibility "common (total) independent version (case)".

**Remark 1.1** In contradiction: the random vectors $\vec{\xi}(m)$ may be particularly coincide.

Further, let $I = I(d) = I(d, n) = \{(i_1, i_2, \ldots, i_d)\}$ be the set of indices of the form

$$\{(i_1, i_2, \ldots, i_d)\} : 1 \leq i_1 < i_2 < i_3 \ldots < i_{d-1} < i_d \leq n,$$

$J = J(d) = J(d, n) = \{(i_1, i_2, \ldots, i_d)\}$ be the set of indices of the form

$$J(d, n) = \{(i_1, i_2, \ldots, n)\} : 1 \leq i_1 < i_2 < i_3 \ldots < i_{d-1} \leq n - 1 < i_d = n,$$

$b(I) = b(i_1, i_2, \ldots, i_d)$ be a $d-$ dimensional numerical non-random sequence,

$$\xi(I) \overset{def}{=} \prod_{m=1}^{d} \xi(i_m, m), \xi(J) \overset{def}{=} \prod_{m=1}^{d-1} \xi(i_m, m), \sigma^2(i, m) = \text{Var}(\xi(i, m)),$$

$$Q_d = Q(d, n, \{\xi(\cdot, \cdot)\}) = Q(d, n, b) = \sum_{I \in I(d, n)} b(I) \xi(I)$$

be a homogeneous polynomial (random polynomial) of power $d$ on the variables $\{\xi(i, m)\}, m \leq d$ without diagonal members; on the other words, multiple stochastic integral on the discrete martingales measures, martingale transform, $n = 1, 2, \ldots, \infty$; in the case $n = \infty Q(d, \infty)$ should be understood as a limit $Q(d, \infty) = \lim_{n \to \infty} Q(d, n)$ if there exists with probability one.

Note that

$$\text{Var} Q(d, n, b) = \sum_{I \subset I(d, n)} b^2(I) \prod_{i=1}^{d} \sigma^2(i_m, m);$$

hence if

$$\sum_{I \subset I(d, \infty)} b^2(I) \prod_{i=1}^{d} \sigma^2(i_m, m) < \infty$$

then $Q(d, \infty)$ exists. In particular, if

$$\sup_{i, m} \text{Var} \xi(i, m) = \sup_{i, m} \sigma^2(i, m) < \infty, \sum_{I \subset I(d, \infty)} b^2(I) < \infty$$

this condition is satisfied.

We will denote further
\[ B = B(d, n) = \{ b(I) : \sum_{I \subset I(d, n)} b^2(I) = 1 \}, \quad n \leq \infty, \]
and suppose \( b \in B = B(d, n) \). Obviously, if \( b \in B \), \( \sigma^2(i, m) = 1 \), then \( \text{Var} Q_d = 1 \).

The aim of this paper is to obtain uniform tail and moment estimation for polynomial martingales.

The letter \( C, C_k, C_l(\cdot)k = 1, 2, \ldots \) with or without subscript will denote a finite positive non essential constants, not necessarily the same at each appearance.

The applications of these estimations in the theory of probability distributions in Banach spaces, method Monte-Carlo and statistics are described, e.g., in [2], [6], [15], [16], [30], [27], [31], chapter 4, [34], [35], [42], chapter 5 etc.

The one-dimensional case \( d = 1 \) see, e.g. in [1], [5], [7], [18], [19], [22], [25], [26], [29], [34], [35], [37], [39], [41]. The multidimensional case \( d \geq 2 \) is investigated in [34], see also reference therein, especially therein was considered the case light, or equally exponential tails of distributions, with correspondent lower estimations.

The offered article may be considered as a continuation of the work [34], where was obtained the exponential decreasing tail estimations for \( Q_d \) under conditions that all the summands (and multipliers) \( \xi(i, m) \) have also exponential decreasing tails.

We suppose in this article as a rule that the r.v. \( \xi(i, m) \) have heavy tails, for instance, which do not have moments of arbitrary order.

The paper is organized as follows. In the section 2 included some auxiliary facts about the random variables belonging to the so-called Grand Lebesgue Spaces (GLS); on the other words, moment spaces. In the section 3 and 4 we formulate and prove the main result: the moment bounds for distributions of polynomial martingales.

In the fifth section we generalize obtained results into the arbitrary centered polynomials from totally independent variables. The 6th section is devoted to the tails estimations for multiple martingale differences sums. In the next section we obtain a moment estimations for a reverse martingales.

The last section contains some concluding remarks.

2 Auxiliary facts.

In order to formulate and prove the main result in this case, we recall here for reader convenience some facts about so-called Grand Lebesgue Spaces [11], [12], [13], [14], [20], [21], [28] etc. or equally ”moment” spaces of random variables defined on fixed probabilistic space \((\Omega, \mathcal{A}, \mathbf{P})\); more detail description with some applications of these spaces see in [6], [24], [31], [32], [33], [34].
Notice that in the articles [12], [13], [14], [20], [21], [28], [32], [33] was considered more general case when the measure $P$ may be unbounded.

Let us consider the following norm (the so-called "moment norm") on the set of r.v. defined in our probability space by the following way: the space $G(\nu) = G(\nu; r)$ consist, by definition, on all the r.v. with finite norm $||\xi||_{G(\nu)} = \sup_{p \in (2, r)} [||\xi||_p/\nu(p)]$, $||\xi||_p := E^{1/p}[|\xi|^p]$. (2.1)

Here $r = \text{const} > 2$, $\nu(\cdot)$ is some continuous positive on the semi-open interval $[1, r)$ function such that

$$\inf_{p \in (2, r)} \nu(p) > 0, \nu(p) = \infty, p > r;$$

and as usually

$$||\xi||_p \overset{def}{=} [E|\xi|^p]^{1/p}$$

We will denote

$$\text{supp}(\nu) \overset{def}{=} [2, r) = \{p : \nu(p) < \infty\}$$

in the case when $\nu(r) = \infty$ or correspondingly if $\psi(r) < \infty$

$$\text{supp}(\nu) \overset{def}{=} [2, r] = \{p : \nu(p) < \infty\}.$$  

The case $r = +\infty$ is investigated in [24], [34], [31]; therefore, we suppose further $2 < r < \infty$.

Let $\xi$ be a r.v. such that there is a value $r$, $2 < r < \infty$, for which

$$\forall p > r \Rightarrow ||\xi||_p \overset{def}{=} [E|\xi|^p]^{1/p} = \infty.$$  

We can and will assume furthermore that

$$r = r_\xi = \inf\{p : ||\xi||_p = \infty\} = \sup\{p : ||\xi||_p < \infty\}.$$  

In order to emphasis the dependance of a functions on the variable $r$, we will denote the class of such a functions as $G(\nu; r)$.

The natural function $\nu_\xi(p)$ may be defined as follows:

$$\nu_\xi(p) := ||\xi||_p = [E|\xi|^p]^{1/p}. \quad (2.2)$$

Obviously,

$$||\xi||_{G(\nu_\xi)} = 1.$$  

The natural function for the family of a random variables $\{\xi(\cdot)\}$, or equally for a random vector $\vec{\xi} = \{\xi(\cdot)\}$ $\{\nu(\cdot)\} = \nu_{\xi(k)}(p) = \nu_\xi(p), k = 1, 2, \ldots, n \leq \infty$ may be defined as follows:
\{\nu_\xi(p)\} = \nu_{\{\xi\}}(p) := |\{\xi\}|_p = \sup_k [E|\xi(k)|^p]^{1/p}, \quad (2.2a)

if there exists and is finite for the values \( p \in [1, r), \ r > 1. \)

**Remark 2.0.** From the Lyapunov’s inequality follows that if for the r.v. \( \xi \ |\xi|_2 < \infty, \) then \( \forall p \in [1, 2] \ |\xi|_p \leq |\xi|_2 = C < \infty. \)

The complete description of a possible natural functions see in [31], [34], chapter 1, section 3.

**Example 2.1.** Suppose the r.v. \( \xi \) satisfies the condition

\[ T(x) = T_\xi(x) = x^{-r} \log^\gamma(x) L(\log x), \ r = \text{const} > 0, \ x > e, \quad (2.3) \]

where as ordinary the tail function \( T(x) = T_\xi(x) \) for the r.v. \( \xi \) may be defined as follows:

\[ T(x) = T_\xi(x) = \mathbb{P}(|\xi| \geq x), \ x > 0; \quad (2.4) \]

Here \( L = L(x) \) is positive continuous slowly varying as \( x \to \infty \) function, \( r = \text{const} > 1 \) and \( \gamma > -1. \) Then (see [28], [31]) for the values \( p \in [1, r), \ p \to r - 0 \)

\[ E|\xi|^p \sim C(r, \gamma, L) (r - p)^{-\gamma - 1} L(1/(r - p)). \]

Therefore, for \( 1 \leq p < r \)

\[ \nu_{\{\xi\}}(p) \asymp (r - p)^{-\gamma/p} L^{1/p}(1/(r - p)) \asymp (r - p)^{-\gamma/r} L^{1/r}(1/(r - p)). \]

Note that if \( \gamma < -1 \) or more generally

\[ \int_e^\infty x^{-1} \log^\gamma(x) L(\log x) \, dx < \infty, \]

then

\[ E|\xi|^p < \infty \iff p \in [1, r]. \]

Recall that we consider only the case \( p \geq 1. \)

This condition is equivalent to the follows:

\[ \nu_{\{\xi\}}(p) \asymp 1, \ p \in [1, r]; \ \nu_{\{\xi\}}(p) = \infty, p > r. \]

Note that an inequality \( \exists r > 0 \Rightarrow \psi(r - 0) < \infty \) is equivalent to the moment restriction \( |\xi|_r < \infty. \)

Notice that the consistent estimation of regular varying distributions is investigated, e.g. in [8], [9], [10], [36]. The properties of slowly and regular varying functions see in [40]; see also [3].

We recall now the relations between moments for r.v. \( \xi, \ \xi \in G(\nu, r) \) and its tail behavior. Namely, for \( p < r \)
\[ |\xi|_p = \left[ p \int_0^\infty u^{p-1} T_{|\xi|}(u) \, du \right]^{1/p}, \]

therefore

\[ ||\xi||G(\nu; r) = \sup_{p < r} \left\{ \left[ p \int_0^\infty u^{p-1} T_{|\xi|}(u) \, du \right]^{1/p} / \nu(p) \right\}. \]

Conversely, if the r.v. \( \xi \) belongs to the space \( G(\nu, r) \), then

\[ T_{|\xi|}(x) \leq T(||\xi||G(\nu; r))(x), \]

where by definition

\[ T^{(\nu)}(x) \overset{\text{def}}{=} \inf_{p \in (1, r)} \left[ \nu^p(p) / x^p \right], \quad x > 0. \]

Let us denote for simplicity the union of all the classes \( G(\nu, r) \), \( r > 2 \) with finite values \( r \) greater as two as \( G(\nu) : \)

\[ G(\nu) = \bigcup_{2 < r < \infty} G(\nu, r). \]

**Example 2.2.** If

\[ E|\xi|^p \leq C_2 (r - p)^{-\gamma - 1} L(1/(r - p)), \quad p < r, \]

then

\[ T_{|\xi|}(x) \leq C_4(r, \gamma, L) x^{-r} [\log x]^{\gamma + 1} L(\log x), \quad x > e. \]

Notice that there is a ”logarithmic gap” between upper and lower tail and moment relations. This gap is essential, see [28], [31]. Let us consider the following

**Example 2.3.** Let \( \zeta \) be a discrete r.v. with distribution

\[ P \left( \zeta = \exp(e^k) \right) = C_5 \exp \left( \beta rk - re^k \right), \]

\[ k = 1, 2, \ldots, \quad \beta = \text{const} > 0, \]

where obviously

\[ 1/C_5 = \sum_{k=1}^\infty \exp \left( \beta rk - re^k \right). \]

We conclude after some calculations:

\[ |\zeta|_p \asymp (r - p)^{-\beta}, \]

but for the sequence \( x(k) = \exp(\exp(k)) \)

\[ T_\zeta(x(k)) \geq c_6[\log x(k)]^\beta x(k)^{-r}. \]

**Definition 2.1.**
Let \( \nu_1 \in G(\nu; r_1) \), \( \nu_2 \in G(\nu; r_2) \), \( r_1, r_2 > 1 \) and let \( r = r(p_1, p_2) := [1/r_1 + 1/r_2]^{-1} > 1 \). We define a new operation \( \nu := \nu_1 \otimes \nu_2 \) as follows:

\[
\nu(p) = \nu_1 \otimes \nu_2(p) \overset{\text{def}}{=} \inf \left\{ \nu_1 \left( \frac{P}{a} \right) \cdot \nu_2 \left( \frac{P}{b} \right) : a, b > 0, a + b = 1 \right\}.
\] (2.6)

Note that the function \( \nu \) is correctly defined on the set \( 1 \leq p < r \).

The following properties of operation \( \otimes \) are obvious:

\[
(C_1 \nu_1) \otimes (C_2 \nu_2)(p) = C_1 C_2 \nu_1 \otimes \nu_2(p);
\]

\[
(C_1 p^\alpha \nu_1) \otimes (C_2 p^\beta \nu_2)(p) \leq C_3 p^{\alpha + \beta} \nu_1 \otimes \nu_2(p);
\]

\[
\nu_1 \otimes (\nu_2 + \nu_3)(p) \leq \nu_1 \otimes \nu_2(p) + \nu_1 \otimes \nu_2(p).
\]

The sense of this function: as follows from the Hölder inequality,

\[
||\xi \eta||G(\nu, r) \leq ||\xi||G(\nu_1, r_1) ||\eta||G(\nu_2, r_2).
\] (2.7)

The iteration of this operation may be defined recursively: if \( \nu_m(\cdot) \in G(\nu, r_m) \) and

\[
r := \left( \sum_{m=1}^{d} 1/r_m \right)^{-1} > 1,
\]

then

\[
\otimes_{m=1}^{d} \nu_m(p) = ((\nu_1 \otimes \nu_2) \ldots \otimes \nu_d(p)), \quad 1 \leq p < r,
\] (2.8)

and

**Proposition 2.1.**

\[
|| \prod_{m=1}^{d} \eta(m)||G(\otimes_{m=1}^{d} \nu_m) \leq \prod_{m=1}^{d} ||\eta(m)||G(\nu_m).
\]

Further, if the r.v. \( \eta(m) \) are common (total) independent and \( s = \min_m r_m > 1 \), then

\[
| \prod_{m=1}^{d} \eta(m) |_p = \prod_{m=1}^{d} |\eta(m)|_p, \quad 1 \leq p < s;
\]

therefore in this case

\[
|| \prod_{m=1}^{d} \eta(m)||G(\prod_{m=1}^{d} \nu_m, s) = \prod_{m=1}^{d} [||\eta(m)||G(\nu_m, r_m)].
\] (2.9)

Obviously, the product \( \prod_{m=1}^{d} \nu_m(p) \) is infinite iff \( p > s \).

**Example 2.4.** Let
\[ \nu_1(p) = \psi_1(r_1 - p), \ \nu_2(p) = \psi_2(r_2 - p), \]

where \( r = (1/r_1 + 1/r_2)^{-1} > 1 \) and \( \nu_j(p) = +\infty, \ p > r_j, \ j = 1, 2; \) so that the point \( z = 0+ \) is unique point of singularity for the functions \( \psi_m(z) \).

We get after simple calculations:

\[ \nu_1 \otimes \nu_2(p) \leq \psi_1 \left( \frac{r_1 + r_2}{r_2} (r - p) \right) \cdot \psi_2 \left( \frac{r_1 + r_2}{r_1} (r - p) \right), \ p < r. \quad (2.10) \]

More generally, let

\[ \nu_m(p) = \psi_m(r_m - p), \ m = 1, 2, \ldots, d, \]

\[ r := (1/r_1 + 1/r_2 + \ldots 1/r_d)^{-1} > 1; \ \psi_j(p) = +\infty, \ p > r_j, \ j = 1, 2, \ldots, d. \]

Denote

\[ z_j = \frac{\prod_{s \neq j} r_s}{\sum_{m=1}^{d} \prod_{l \neq m} r_l}. \]

We have using proposition (2.1) denoting

\[ \psi(p) = \prod_{m=1}^{d} \psi_m(z_m(r - p)) : \quad (2.11) \]

\[ || \prod_{m=1}^{d} \eta(m)||G(\psi, r) \leq \prod_{m=1}^{d} ||\eta(m)||G(\psi_m, r_m). \quad (2.12) \]

If the functions \( \psi_m(\cdot)(p) \) satisfy the \( \Delta_2 \) condition at the points \( p = 0+ \):

\[ \sup_{p \in [1, r_m]} \frac{\psi_m(z_m(r_m - p))}{\psi_m(r_m - p)} < \infty, \quad (2.13) \]

then the function \( \psi = \psi(p) \) in (2.11) may be replaced by following expression:

\[ \psi(p) \asymp \overline{\psi}(p) \overset{def}{=} \prod_{m=1}^{d} \psi_m(r - p). \]

The condition (2.13) is satisfied, e.g., for the function of a view

\[ \psi_m(p) = C \ (r_m - p)^{-\Delta_m} \ L_m(1/(r_m - p)), \ \Delta_m = \text{const} > 0, \quad (2.14) \]

\( L_m(x) \) are slowly varying as \( x \to \infty \) functions.

We obtain in the considered case

\[ \psi(p) \asymp (r - p)^{-\sum_{k=1}^{d} \Delta_k} \prod_{k=1}^{d} L_k(1/(r - p)). \quad (2.15) \]
More detail, if

\[ |\eta(m)|_p \leq C_m (r_m - p)^{-\Delta_m} L_m(1/(r_m - p)), \quad \Delta_m = \text{const} > 0, 1 \leq p < r_m, \] (2.16)

then

\[ \left| \prod_{m=1}^d \eta(m) \right|_p \leq C^{(d)} \cdot (r - p)^{-\sum_{k=1}^d \Delta_k} \cdot \prod_{k=1}^d L_k(1/(r - p)), \quad 1 \leq p < r. \] (2.17)

Recall that \( r = [\sum_{m=1}^d 1/r_m]^{-1} > 1. \)

**Example 2.5.** We prove here the precision of inequality (2.17) by means of construction of corresponding example. Let \( \epsilon \) be a positive r.v. defined on some sufficiently rich probabilistic space, such that

\[ T_\epsilon(x) = \mathbb{P}(\epsilon > x) = 1/x, \quad x > 1. \]

Put

\[ \xi = \epsilon^{1/r_1}, \quad \eta = \epsilon^{1/r_2}, \quad r_1, r_2 > 1, \quad r := (1/r_1 + 1/r_2)^{-1} > 1. \]

We have:

\[ |\xi|_p \leq \int_1^\infty x^{p-1/r_1} \, dx = (r_1 - p)^{-1}; \]

\[ |\xi|_p \sim (r_1 - p)^{-1/r_1}, \quad 1 \leq p < r_1. \]

and analogously

\[ |\eta|_p \sim (r_2 - p)^{-1/r_2}, \quad 1 \leq p < r_2. \]

Further, \( \xi \eta = \epsilon^{1/r} \),

\[ |\xi \eta|_p \sim (r - p)^{-1/r} = (r - p)^{-1/r_1 - 1/r_2}, \quad 1 \leq p < r, \]

i.e. there is an asymptotical equality in the inequality (2.17).

**Example 2.6.** Let \( r_1 = r_2 = \infty, \)

\[ |\xi|_p \leq C_1 p^{\mu_1}, \quad |\eta|_p \leq C_2 p^{\mu_2}, \quad \mu_1, \mu_2 = \text{const} > 0, \]

i.e. \( \nu_1(p) = C_1 p^{\mu_1}, \quad \nu_2(p) = C_2 p^{\mu_2}. \)

Recall, see [34], [31], that this means

\[ T_\xi(x) \leq \exp \left( -C_3 x^{1/\mu_1} \right), \quad T_\eta(x) \leq \exp \left( -C_4 x^{1/\mu_2} \right). \]

Subexamples: in the case \( \mu_1 = 1/2 \) the r.v. \( \xi \) is **subgaussian**:

\[ T_\xi(x) \leq C_1 \exp(-C_2 x^2), \quad x \geq 0. \]

In the case \( \mu_1 = 1 \) the r.v. \( \xi \) is **subexponential**: 

The r.v. $\xi$ is said to be \textit{pre-gaussian}, if $\exists \lambda_0, 0 < \lambda_0 \leq \infty \Rightarrow$ 
\[\forall \lambda, |\lambda| < \lambda_0 \Rightarrow E \exp(\lambda \xi) < \infty.\]

Evidently, if $\xi$ is pre-gaussian then 
\[\forall p \geq 0 \Rightarrow \nu_\xi(p) < \infty.\]

Further, we have:
\[\nu_1 \otimes \nu_2(p) = C_1 C_2 \mu_1^{-\mu_1} \mu_2^{-\mu_2} (\mu_2 + \mu_2)^{\mu_1+\mu_2} p^{\mu_1+\mu_2}, \quad (2.18)\]

or equally
\[T_{\xi \eta}(x) \leq \exp \left( -C_5 x^{1/(\mu_1+\mu_2)} \right). \quad (2.18a)\]

The inequality (2.18) is exact, for instance, for independent variables $\xi, \eta,$ for which $T_\xi(x) = \exp \left( -C_3 x^{1/\mu_1} \right), \ T_\eta(x) = \exp \left( -C_4 x^{1/\mu_2} \right). \quad (2.19)$

In detail, it follows from (2.19) as $p \to \infty$
\[|\xi|_p \sim C_6 p^{\mu_1}, \ |\eta|_p \sim C_7 p^{\mu_2},\]

and consequently
\[|\xi \eta|_p \sim C_8 p^{\mu_1+\mu_2}.\]

For instance, the product of two subgaussian (dependent, in general case) variables is sub-exponential.

\textbf{Example 2.7.} Let now $r = \text{const} > 1,$
\[|\xi|_p \leq C_1 (r-p)^{-\gamma} L_1(1/(r-p)) =: \nu_1(p), \ 1 \leq p < r;\]
\[|\eta|_p \leq C_2 p^\mu L_2(p) =: \nu_2(p), \ p \geq 1,\]

where $L_1(z), L_2(z)$ are slowly varying as $z \to \infty$ positive continuous functions.

We obtain substituting in the formula (2.6) the values $a = 1/(1+C(r-p)), \ b = 1 - a :$
\[\nu_1 \otimes \nu_2(p) = |\xi \eta|_p \leq C_3 (r-p)^{-\gamma-m} L_1(1/(r-p)) L_2(1/(r-p)), \ 1 \leq p < r. \quad (2.20)\]

We show now the exactness of an estimation (2.20). Let $\Omega = \{\omega\} = [0, 1] \text{ with Lebesgue measure } P$ and let
\[ \xi = \omega^{-1/r} \mid \log \omega \mid^\kappa L(\mid \log \omega \mid), \ r = \text{const} > 1, \]

where \( L = L(x) \) is as before is positive continuous slowly varying as \( x \to \infty \) function,

\[ \eta = \mid \log \omega \mid^\mu, \ \mu = \text{const} > 0. \]

We find by the direct calculation, see [32]:

\[ \mid \xi \mid_p \approx (r - p)^{-\kappa - 1/r} L(1/(r - p)), p \in [1, r); \mid \eta \mid_p \approx p^\mu, p \geq 1; \]

\[ \mid \xi \eta \mid_p \approx (r - p)^{-\mu - \kappa - 1/r} L(1/(r - p)), p \in [1, r). \]

**Example 2.8.** Define the following \( \nu \) function \( \psi(r(p)) \):

\[ \psi(r(p)) = 1, \ p \leq r; \ \psi(r(p)) = +\infty, p > r, \ r = \text{const} > 1. \]

It is easy to verify using the Lyapunov’s inequality that the \( G_{\psi(r)}(\cdot) \) space coincides with the classical Lebesgue-Riesz space \( L(r) \):

\[ \|\xi\|_{G_{\psi(r)}(\cdot)} = |\xi|. \]  \tag{2.21}

Further, let \( \xi(i), \ i = 1, 2, \ldots, d \) be some r.v. from the spaces \( G_{\psi(r(i))} = L(r(i)) \), where as before \( r(i) > 1, \ r := 1/(\sum 1/r(i)) \geq 1 \). We infer:

\[ \|\prod_i \xi(i)\|_{G_{\psi(r)}} \leq \prod_i \|\xi(i)\|_{G_{\psi(r(i))}}, \]  \tag{2.22a}

or equally

\[ \|\prod_i \xi(i)\|_{L(r)} \leq \prod_i |\xi(i)|_{L(r(i))}. \]  \tag{2.22b}

For instance, \( |\xi \eta| \leq |\xi|_2 \cdot |\eta|_2; \ |\xi \eta \zeta| \leq |\xi|_3 \cdot |\eta|_3 \cdot |\zeta|_3 \).

Evidently, the estimates (2.22a) or (2.22b) are essentially non-improvable.

### 3 Main result: martingale version

We recall before formulating the main result some useful for us moment inequalities for the sums of centered martingale differences and independent r.v., [33]. Namely, let \( \{\theta(i)\} \) be a sequence of centered martingale differences relative any filtration; then

\[ \sup_n \sup_{b \in B} \left\| \sum_{i=1}^n b(i) \theta(i) \right\|_p \leq K_M(p) \sup_i |\theta(i)|_p, \]  \tag{3.0}

where for the optimal value of the constant \( K_M = K_M(p) \) is true the inequality
\[ K_M(p) \leq p \sqrt{2}, \quad p \geq 2. \]

Note that the upper bound in (3.0)
\[ K_I(p) \leq 0.87p / \log p, \quad p \geq 2 \]
is true for the independent centered r.v. \{\theta(i)\}, see also [33].

We denote the natural function for the sequences \( \bar{\xi}(m) = \{\xi(i, m)\}, i = 1, 2, \ldots, n \) as \( \nu_m = \nu_m(p) : \)
\[ \nu_m(p) = \sup_{k=1,2,\ldots,n} \|\xi(k, m)\|_p, \quad m = 1, 2, \ldots, d, \]
and suppose that for all the values \( 1 \leq m \leq d \) the function \( \nu_m(\cdot) \) belongs to the set \( G(\nu) \). More detail, we assume
\[ \nu_m(\cdot) \in G(\nu, r_m), \quad r_m > 2. \]

Define the sequence of a functions \( \zeta_m = \zeta_m(p), \quad m = 1, 2, \ldots, d \) by the following recursion:
\[ \zeta_1(p) = K_M(p) \nu_1(p), \]  
(initial condition) and
\[ \zeta_{m+1}(p) = K_M(p) \cdot [\{\zeta_m \otimes \nu_{m+1}\}(p)], \quad m = 1, 2, \ldots, d - 1. \]

**Theorem 3.1 (martingale version).**
Let us denote as before
\[ r = \left( \sum_{m=1}^{d} \frac{1}{r_m} \right)^{-1} \]
and suppose \( r > 1 \). Proposition:
\[ \sup_{b \in B} \|Q(d, n, b)\| G(\zeta_d) \leq \prod_{m=1}^{d} \|\bar{\xi}(m)\| G(\nu(\cdot), r_m). \]

**Proof** is at the same as in [34]; it used the estimate (3.0) and induction over \( d \). Namely, the one-dimensional case \( d = 1 \) coincides directly with inequality (3.0).

Let now \( d \geq 2 \). The sequence \( (Q_d(n), F(n)) \) is again martingale with correspondent martingale-differences
\[ \beta(n) := Q_d(n) - Q_d(n - 1) = \xi(n, d) \sum_{I \in J(d, n)} b(I) \xi(I) = \]
\[ \xi(n, d) \sum_{1 \leq i_1 < i_2 < \ldots < i_{d-1} \leq n-1} b(i_1, i_2, \ldots, i_{d-1}, n) \prod_{m=1}^{d-1} \xi(i_m, m) = \]

\[ \frac{12}{12} \]
\[
\left[ \xi(n, d) \cdot \sqrt{\sum_{I \in J(d,n)} b^2(I)} \right] \times \nabla
\left[ \frac{\sum_{I \in J(d,n)} b(I) \xi(I)}{\sqrt{\sum_{I \in J(d,n)} b^2(I)}} \right] \overset{\text{def}}{=} \{ \eta(n, d) \} \cdot \{ \tau(n, d) \}, \quad (3.6)
\]

where

\[
\tau(n, d) := \left[ \frac{\sum_{I \in J(d,n)} b(I) \xi(I)}{\sqrt{\sum_{I \in J(d,n)} b^2(I)}} \right],
\]

\[
\eta(n, d) := \left[ \xi(n, d) \cdot \sqrt{\sum_{I \in J(d,n)} b^2(I)} \right]. \quad (3.7)
\]

It follows from the induction statement

\[
|\tau(n, d)|_p \leq \zeta_{d-1}(p). \quad (3.8)
\]

It remains to use the induction statement and the definition of the operation \( \otimes \):

\[
|Q_d(n)|_p \leq \nu_d \otimes \zeta_{d-1}(p), \quad (3.9)
\]

**Example 3.1.** Let

\[
\nu_m(p) \leq (r_m - p)^{-\Delta_m} L_m(1/(r_m - p)), \quad 2 < r_m < \infty, \quad \Delta_m = \text{const} > 0,
\]

\[
r := \left( \sum_{m=1}^d \frac{1}{r_m} \right)^{-1} > 1, \quad 1 \leq p < r.
\]

\(L_m(x)\) be positive continuous slowly varying as \(x \to \infty\) functions.

As long as a multiplier \(p \sqrt{2}\) is bounded in considered case: \(p \sqrt{2} \leq r \sqrt{2}\), we conclude by virtue of theorem 3.1:

\[
\sup_{b \in B} ||Q(d, n, b)|| G(\zeta_d) \leq \prod_{m=1}^d ||\tilde{\xi}(m)|| G(\nu_m, r_m),
\]

where

\[
\zeta_d(p) = 2^{d/2} r^d (r - p)^{-\Delta_m} \prod_{m=1}^d L_m(z_m/(r - p)),
\]

or equally

\[
\sup_{b \in B} |Q_d|_p \leq 2^{d/2} r^d (r - p)^{-\Delta_m} \prod_{m=1}^d L_m(z_m/(r - p)), \quad 1 \leq p < r. \quad (3.10)
\]
Example 3.2. As long as the case when $\forall m \ r_m = \infty$ was investigated in [34], we consider here the mixed possibility, when

$$3k \in [2, d - 1], \ r_1, r_2, \ldots, r_k \in (2, \infty); \ r_{k+1} = r_{k+2} = \ldots = r_d = \infty.$$ 

In detail, suppose

$$\nu_m(p) = (r_m - p) - \Delta_m L_m(1/(r_m - p)), \ m = 1, 2, \ldots, k, \ 2 \leq p < r_m;$$

$$\nu_m(p) = p^\mu_m L_m(p), \ m = k + 1, \ldots, d, \ p \in [1, \infty),$$

where as before all the functions $L_m(x)$ be positive continuous slowly varying as $x \to \infty$ functions.

Put

$$\tau = (\sum_{m=1}^{k} 1/r_m)^{-1} > 1, \ \mathcal{L}(p) = \prod_{m=1}^{k} L_m(z_m/(\tau - p)) \prod_{m=k+1}^{d} L_m(1/(\tau - p), \ 1 \leq p < \tau.$$ 

From theorem 3.1 follows:

$$\zeta_d(p) = 2^{k/2} \tau^k (\tau - p)^{-\sum_{m=1}^{k} \Delta_m - \sum_{m=k+1}^{d} \mu_m - (d-k)} \mathcal{L}(p). \quad (3.11)$$

Example 3.3. Let us illustrate the exactness of the assertions of examples (3.1) and (3.2). It is sufficient thereto to consider the case $n = 1, d = 2$ and refer to the examples 2.4, 2.5 and 2.6.

Example 3.4. Let the source centered martingale differences $\xi(i, m), \ i = 1, 2, \ldots, n$ be r.v. from the fixed finite ball of the spaces $G_{\psi(r(i))} = L(r(i))$:

$$\sup_i |\xi(i, m)|_{r(m)} =: D(m) < \infty,$$

where as before $r(i) > 1$, $r := 1/(\sum_m 1/r(m)) \geq 1$. We deduce by virtue of the example 2.8 and the boundedness of the value $p\sqrt{2}$:

$$\sup_{b \in B} |Q_{d,b}| \leq C_3(d; \{r(m)\}) \prod_{m=1,2,\ldots,d} \sup_{i=1,2,\ldots,n} |\xi(i, m)|_{r(m)}. \quad (3.12)$$

Obviously, the estimate (3.12) is unprovable; it is sufficient to show this to consider the case $d = 2, 3, \ldots; \ n = 1$ and refer to the example 2.8.

4 Main result: independent versions

Analogously may be considered all the "independent cases".
A. "Common independent version".

Recall that in this case all the random variables \( \{\xi(i,m)\}, i = 1, 2, \ldots, n; \ m = 1, 2, \ldots, d \) are mean zero and independent.

**Theorem 4.1.**

Define the sequence of a functions \( \zeta_m = \zeta_m(p), \ m = 1, 2, \ldots, d \) by the following way:

\[
\zeta_1(p) = K_I(p) \nu_1(p),
\]

(initial condition) and by recursion

\[
\zeta_{m+1}(p) = K_M(p) \zeta_m(p) \times \nu_{m+1}(p), \ m = 1, 2, \ldots, d - 1;
\]

with the explicit solution

\[
\zeta_d(p) = K_I(p) K_M^{d-1}(p) \prod_{m=1}^d \nu_m(p).
\]

**Proposition:**

\[
\sup_{b \in B} ||Q(d, n, b)||G(\zeta_d) \leq \prod_{m=1}^d ||\zeta(m)||G(\nu_m, r_m).
\]

**Example 4.1.** Let the all the r.v. \( \xi(i,m) \) are mean zero and common (total) independent, \( \xi(i,m) \in G(\nu_m, r_m) \), such that

\[
\sup_{i=1,2,\ldots,n} ||\xi(i,m)||G(\nu_m, r_m) < \infty,
\]

and \( \max_m r_m < \infty, \ s = \min_m r_m > 1 \), then

\[
\zeta_d(p) \leq C \prod_{m=1}^d \sup_{i=1,2,\ldots,n} |\xi(i,m)|_p, \ 1 \leq p < s;
\]

therefore in this case

\[
\sup_{b \in B} ||Q_d||G(\prod_{m=1}^d \nu_m, s) \leq C \cdot \prod_{m=1}^d \sup_{i=1,2,\ldots,n} ||\xi(i,m)||G(\nu_m, r_m).
\]

Evidently, the product \( \prod_{m=1}^d \nu_m(p) \) is infinite iff \( p > s \).

B. "Inside independent version".

**Theorem 4.2.**

Define the sequence of a functions \( \zeta_m = \zeta_m(p), \ m = 1, 2, \ldots, d \) by the following way:

\[
\zeta_1(p) = K_I(p) \nu_1(p),
\]
(initial condition) and
\[ \zeta_{m+1}(p) = K_M(p) \left[ \zeta_m(p) \otimes \nu_{m+1}(p) \right], \quad m = 1, 2, \ldots, d - 1; \quad (4.6) \]
(recursion). Proposition:
\[ \sup_{b \in B} ||Q(d, n, b)||G(\zeta_d) \leq \prod_{m=1}^{d} ||\tilde{\zeta}(m)||G(\nu_m, r_m). \quad (4.7) \]

**Example 4.2** is a slight modification of Example 3.1. Indeed, let under condition of the theorem 4.2
\[ \nu_m(p) \leq (r_m - p)^{-\Delta_m} L_m(1/(r_m - p)), \quad 2 < r_m < \infty, \quad \Delta_m = \text{const} > 0, \]
\[ r := \left( \sum_{m=1}^{d} 1/r_m \right)^{-1} > 1, \quad 1 \leq p < r. \]
\( L_m(x) \) be positive continuous slowly varying as \( x \to \infty \) functions.

As long as a multiplier \( p \sqrt{2} \) is bounded in considered case: \( p \sqrt{2} \leq r \sqrt{2} \), we conclude by virtue of theorem 3.1:
\[ \sup_{b \in B} ||Q(d, n, b)||G(\zeta_d) \leq C(d, \{r_m\}) \prod_{m=1}^{d} ||\tilde{\zeta}(m)||G(\nu_m, r_m), \]
where
\[ \zeta_d(p) = (r - p)^{-\sum_{m=1}^{d} \Delta_m} \prod_{m=1}^{d} L_m(z_m/(r - p)), \]
or equally
\[ \sup_{b \in B} |Q_d| \leq C(d, \{r_m\}) (r - p)^{-\sum_{m=1}^{d} \Delta_m} \prod_{m=1}^{d} L_m(z_m/(r - p)), \quad 1 \leq p < r. \]

**C. "Vector independent version"**.

**Theorem 4.3.**

Define the sequence of a functions \( \zeta_m = \zeta_m(p), \quad m = 1, 2, \ldots, d \) by the following way:
\[ \zeta_1(p) = K_M(p) \nu_1(p), \quad (4.8) \]
(initial condition) and
\[ \zeta_{m+1}(p) = K_M(p) \zeta_m(p) \times \nu_{m+1}(p), \quad m = 1, 2, \ldots, d - 1; \quad (4.9) \]
(recursion); with the explicit solution:

\[ \zeta_d(p) = K^d_M(p) \cdot \prod_{m=1}^{d} \nu_m(p), \]

Proposition:

\[
\sup_{b \in B} ||Q(d, n, b)||G(\zeta_d) \leq \prod_{m=1}^{d} ||\vec{\zeta}(m)||G(\nu_m, r_m). \tag{4.10}
\]

5 Arbitrary centered polynomial from independent variables

We generalize here the results of the last section on the arbitrary centered polynomial from common independent variables.

In detail: let all the random variables \( \{\xi(i, m)\}, 1 = 1, 2, \ldots, n; m = 1, 2, \ldots, d \) be centered and common independent. Arbitrary centered polynomial \( R_d = R_d(n) = R_d(n, \{\xi(i, m)\} \) of degree \( d \) may be written in the form of

\[
R_d = \sum_{1 \leq i_1 < i_2 \ldots < i_d \leq n} b(i_1, i_1, \ldots, i_1; i_2, i_2, \ldots, i_2; \ldots, i_d, i_d, \ldots, i_d) \times \\
\prod_{l=1}^{d} \left[ \xi^{k(l)}(i_l, l) - m(k(l), i_l, l) \right], \tag{5.1}
\]

where \( m(k, i, l) = E\xi^k(i, l), k(l) \in \{0, 1, \ldots, d\}, \) so that \( \sum_l k(l) = d, \)

\[
k(l) = \text{card}\{i_l \text{ in } b(i_1, i_1, \ldots, i_1; i_2, i_2, \ldots, i_2; \ldots, i_d, i_d, \ldots, i_d)\},
\]

and if some \( k(l) = 0 \) then by definition

\[
\xi^{k(l)}(i_l, l) - m(k(l), i_l, l) = 1.
\]

We again suppose \( b \in B; \) the multi-sequence \( b(I) \) may be not symmetric. The convergence of the series for \( R_d \) is investigate in [34].

For instance, if \( d = 2, \) then \( R_d \) has a view:

\[
R_d = \sum_{1 \leq i < j \leq n} b(i, j)\xi(i)\xi(j) + \sum_{i=1}^{n} b(i, i)(\xi^2(i) - E\xi^2(i)) + \\
\sum_{1 \leq i < j \leq n} c(i, j)\eta(i)\eta(j) + \sum_{i=1}^{n} c(i, i)(\xi^2(i) - E\xi^2(i)) + \\
\sum_{1 \leq i < j} a_1(i, j)\xi(i)\eta(j) + \sum_{1 \leq i < j} a_2(i, j)\eta(i)\xi(j).
\]
Theorem 5.1. Suppose that
\[ T_{\xi(i,m)}(x) \leq x^{-r(m)} \log^{\gamma(m)}(x) L_m(\log x), \quad x \geq e, \tag{5.2} \]
where \( r(m), \gamma(m) = \text{const} \) such that \( r(m) < \infty, L_m(z) \) are continuous slowly varying as \( z \to \infty \) positive functions,

\[ \underline{r} \overset{\text{def}}{=} \min_m r(m) > d. \]

We define:
\[ M(r) = \{ m : r(m) = \underline{r} \}, \quad \gamma = \max_{m \in M(r)} \gamma(m), \]
\[ M_\gamma = \{ m \in M(r), \gamma(m) = \gamma \}, \]
\[ \overline{L}(z) = \max_{m \in M_\gamma} L_m(z), \quad z \geq 1. \]

Proposition: for the values \( p, 1 \leq p < r/d \) and in the case when \( \gamma > -1 \)

\[ \sup_{b \in B} E|R_d|^p \leq C(\{r(m)\}, \{\gamma(m)\}, \{L_m(\cdot)\}) \cdot (r/d - p)^{-\gamma - 1} \overline{L}(1/((r/d - p))). \tag{5.3a} \]

When \( \gamma < -1 \) or more generally

\[ \max_{m_0 \in M_\gamma} \sup_{j=1,2,...,n} E|\xi^d(j,m_0)|^p < \infty, \]
then

\[ \sup_{b \in B} E|R_d|^p \leq C_1(\{r(m)\}, \{\gamma(m)\}) < \infty. \tag{5.3b} \]

Proof. Let \( m_0 \) be arbitrary number from the set \( M_\gamma \); it is easy to verify that the main member in the expression for \( R_d \) has a view:

\[ R_M = \sum_{j=1}^n b(j)(\xi^d(j,m_0) - E\xi^d(j,m_0)), \tag{5.4} \]
where \( \sum_j b^2(j) = 1. \)

We have from the condition \( (5.2) \)

\[ T_{\xi^d(j,m_0) - E\xi^d(j,m_0)}(x) \leq C x^{-z/d} \log^{\gamma}(x) \overline{L}(\log x), \quad x \geq e, \tag{5.5} \]
therefore

\[ E|\xi^d(j,m_0) - E\xi^d(j,m_0)|^p \leq C_2 (r/d - p)^{-\gamma - 1} \overline{L}(1/((r/d - p))). \tag{5.6} \]

It remains to use the Rosenthal’s inequality for the r.v. \( R_M. \)
Note that in connection with common independence
\[
|\xi(1,i_1)\xi(2,i_2)\ldots\xi(d,i_d)|_p = |\xi(1,i_1)|_p|\xi(2,i_2)|_p\ldots|\xi(d,i_d)|_p \leq
\]
\[
C_3 (r(1) - p)^{-\gamma(1)-1} L_1((r(1) - p)) (r(2) - p)^{-\gamma(2)-1} L_2((r(2) - p)))\ldots \times
\]
\[
(r(d) - p)^{-\gamma(d)-1} L_d((r(d) - p)), \ p < \min_m r(m), \quad (5.7)
\]
which is significantly smallest as the right-hand side of inequality (5.6).

Remark 5.1. The assertion of theorem 5.1. may be rewritten as follows. Let us denote
\[
\rho_d(p) = (r/d - p)^{-\gamma-1} L(1/(r/d - p)), \ 1 \leq p < r/d;
\]
\[
\theta_m(p) = (r(m) - p)^{-\gamma(m)-1} L_m(1/(r(m) - p)), \ 1 \leq p < r(m),
\]
then
\[
\sup_{b\in B} ||R_d||G(\rho_d) \leq C(\{r(m)\}, \{\gamma(m)\}, \{L_m(\cdot)\}) \prod_{m=1}^d \left[ \sup_{i=1}^n |\xi(i,m)| G(\theta_m) \right]. \quad (5.8)
\]
Note that our results improve and generalize ones obtained in [38].

6 Tail estimations

In many practical cases is convenient to operate by tails estimations for sums of random variables instead moments ones, for example in the method Monte-Carlo and statistics, [35], see also [5].

Let \( \nu = \nu(p) \) be any function from the set \( G(\nu) \), for instance, from the set \( G(\nu, r) \); we denote
\[
T^{(\nu)}(x) = \exp[-(p \log \nu(p))^\ast (\log x)], \ x > e, \quad (6.1)
\]
where
\[
f^\ast(y) = \sup_{z, z > 0, f(z) < \infty} (yz - f(z)) \quad (6.2)
\]
is ordinary Young-Fenchel, or Legendre transform of the function \( f = f(z) \).

Further, let \( \zeta_d = \zeta_d(p) \) (or \( \rho_d = \rho_d(p) \)) be one of the functions \( \zeta_d(p), \rho_d(p) \) introduced before.

Theorem 6.1. We have correspondingly for the martingale, independent and polynomial cases and for the values \( x > e \) :
\[\sup_{b \in B} T_{Q_d}(x) \leq T^{(\xi_d)}(x/C_1), \quad (6.3a),\]

\[\sup_{b \in B} T_{V_d}(x) \leq T^{(\xi_d)}(x/C_2), \quad (6.3b),\]

\[\sup_{b \in B} T_{R_d}(x) \leq T^{(\rho_d)}(x/C_3), \quad (6.3c).\]

**Proof** follows immediately from the obtained moment estimations and from the following implication [28], [32]: if for the r.v. \(\eta\) there holds

\[||\eta||_{\nu} \leq 1\] or equally
\[|\eta|_p \leq \nu(p), \quad p : \nu(p) < \infty,\]

then

\[T_{\eta}(x) \leq T^{(\nu)}(x), \quad x > e. \quad (6.4)\]

**Example 6.1.** If for some \(r > 1\)

\[\zeta_d^p(p) \leq C \, (r - p)^{-\gamma - 1} \, L(1/(r - p)), \quad 1 \leq p < r,\]

where as before \(L = L(z)\) be positive continuous slowly varying as \(z \to \infty\) function, then

\[\sup_{b \in B} T_{Q_d}(x) \leq C_4(r, \gamma, L) \, x^{-r} \, [\log x]^{\gamma + 1} \, L(\log x), \quad x > e. \quad (6.5)\]

**Example 6.2.** Let the source centered martingale differences \(\xi(i, m), \quad i = 1, 2, \ldots, n\) be r.v. from the fixed finite ball of the spaces \(G_{\psi(r(i))} = L(r(i)):\)

\[\sup_{i} |\xi(i, m)|_{r(m)} := D(m) < \infty,\]

where as before \(r(i) > 1, \quad r := 1/(\sum_{m} 1/r(m)) \geq 1.\) We deduce using the Tchebychev’s inequality and inequality (3.12)

\[\sup_{b \in B} \mathbf{P}(|Q_d| > x) \leq C_3(d; \{r(m)\}) \prod_{m=1,2,\ldots,d} \sup_{i=1,2,\ldots,n} |\xi(i, m)|_{r(m)} \cdot x^{-r}, \quad x > 0. \quad (6.6)\]

### 7 Reverse martingales

We investigate *in this section* the case when all the sequences \(\tilde{\xi}(m) = \{\xi(i, m)\}, \quad m = 1, 2, \ldots, d; \quad i = n, n+1, \ldots, N,\) where \(1 \leq N \leq \infty\) are mean zero reverse martingales differences *relative some reverse filtration* \(F(i).\)

This means that
\[
F(i+1) \subset F(i) \subset F,
\]
and \(\forall k = i+1, i+2, \ldots \Rightarrow\)
\[
E\xi(i,m)/F(k) = 0; \ E\xi(i,m)/F(i) = \xi(i,m) \pmod{P}, \tag{7.1}
\]
"reverse martingale version".

Analogously to the first section may be defined "vector independent version", "inside independent version" and "common independent version" for reverse martingales.

An example: let \(\xi(i,m)\) be a sequence of independent centered r.v. Set
\[
S_m(n) = \sum_{i=n}^{N} \xi(i,m), \ F(i) = \sigma(\xi(i,m), \xi(i+1,m), \ldots), \tag{7.2}
\]
where we suppose in the case \(N = \infty\) that the series in (7.2) converge with probability one and in the \(L_2(\Omega)\) sense.

For all the values \(m\) the pair \((S_m(n), F(n))\) is a reverse martingale. The correspondent reverse martingales differences are
\[
S_m(n) - S_m(n+1) = \xi(n,m).
\]

More facts about martingales and reverse martingales see in the classical book [17]. For instance, for the reverse martingale \((S(n), F(n))\) there exists a limit \(\lim_{n \to \infty} S(n)\) with probability one. Further, let \(N < \infty\).

**Proposition 7.1.**

The sequence \((Z_i, F(i))\), \(i = 1, 2, \ldots, N\) is reverse martingale iff the sequence
\[
(Z_{N-i+1}, F_{N-i+1}) \tag{7.3}
\]
is ordinary martingale, see [17], chapter 1, section 1.

Let \(I = I_n^{(N)} = I_n^{(N)}(d) = \{(i_1, i_2, \ldots, i_d)\}\) be the set of indices of the form
\[
\{(i_1, i_2, \ldots, i_d)\} : n \leq i_1 < i_2 < \ldots < i_{d-1} < i_d \leq N,
\]
\[ \]
\[J = J_n^{(N)}(d) = \{(i_1, i_2, \ldots, i_d)\}\) be the set of indices of the form
\[
J_n^{(N)}(d) = \{(i_1, i_2, \ldots, N)\} : n \leq i_1 < i_2 < i_3 \ldots < i_{d-1} \leq N - 1 < i_d = N,
\]
\[ \]
b(I) = b(i_1, i_2, \ldots, i_d)\) be a \(d-\) dimensional numerical non-random sequence,
\[
\xi(I) \overset{\text{def}}{= \prod_{m=1}^{d} \xi(i_m, m), \xi(J) \overset{\text{def}}{= \prod_{m=1}^{d-1} \xi(i_m, m), \sigma^2(i,m) = \text{Var}(\xi(i,m))},
\]

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\[ V_d = V_n^{(N)}(d, n, \{\xi(\cdot, \cdot)\}) = \sum_{I \in I_n^{(N)}(d, n)} b(I)\xi(I) \quad (7.4) \]

be a homogeneous polynomial (random polynomial) of power \(d\) on the variables \(\{\xi(i, m)\}, \ m \leq d\) without diagonal members.

Let us denote
\[ \nu_m^{(n)}(p) = \sup_{i: n \leq i \leq N} |\xi(i, m)|_p, \quad (7.5) \]
where in the case \(N < \infty\) the symbol "sup" must be replaced by "min";
\[ \zeta_d(p) = \zeta_d^{(n)}(p) = \sup_{b \in B} |V_n^{(N)}(d, n, \{\xi(\cdot, \cdot)\})|_p = \sup_{b \in B} |V_d|_p. \quad (7.6) \]

**Remark 7.1.** It will be presumed that for some \(b(m), 2 < b(m) \leq \infty\) the value \(\nu_m^{(n)}(p)\) is finite for \(2 \leq p < b(m)\).

**Remark 7.2.** It is naturally to wait that in the case \(N = \infty\) \(\forall m = 1, 2, \ldots, d \Rightarrow \lim_{n \to \infty} \sum_{i=n}^{N} \xi(i, m) = 0\) and hence
\[ \lim_{n \to \infty} \nu_m^{(n)}(p) = 0, \ 2 \leq p < b(m), \]
or equally that there exist a sequences \(\epsilon_n^{(m)}\) tending to null as \(n \to \infty\) such that
\[ \nu_m^{(n)}(p) \leq \epsilon_n^{(m)} \cdot \nu_m^{(0)}(p), \quad (7.7) \]
where the sequences \(\nu_m^{(0)}(p)\) are bounded for all the values \(p, \ 2 \leq p < b(m)\).

**Theorem 7.1.** (Reverse martingale case.)

We define the sequence \(\zeta_d(p)\) by the following "reverse" recurrent relation
\[ \zeta_m(p) = K_M(p) \cdot \left\{ \left[ \nu_m^{(n)} \otimes \zeta_{m-1} \right] (p) \right\}, \ m = d - 1, d - 2, \ldots, 1 \quad (7.8) \]
with the following endpoint condition:
\[ \zeta_d(p) = K_M(p) \cdot \nu_d^{(n)}(p). \quad (7.9) \]

Let us denote as before
\[ r = \left( \sum_{m=1}^{d} \frac{1}{r_m} \right)^{-1} \]
and suppose \(r > 1\). Proposition:
\[ \sup_{b \in B} ||V_n^{(N)}(d, n, \{\xi(\cdot, \cdot)\})||G(\zeta_i) \leq \prod_{m=1}^{d} ||\zeta(m)||G(\nu_m^{(n)}). \quad (7.10) \]

**Theorem 7.2 ("Total independent version").**

Define the sequence of a functions \(\zeta_m = \zeta_m(p), \ m = 1, 2, \ldots, d\) by the following way:
\[ \zeta_d(p) = K_I(p) \cdot \nu_d^{(n)}(p), \quad (7.11) \]
(endpoint condition) and by reverse recursion

\[ \zeta_m(p) = K_M(p) \zeta_{m+1}(p) \times \nu_m^{(n)}(p), \quad m = d - 1, d - 2, \ldots, 1 \]  \hspace{1cm} (7.12)

with the explicit solution

\[ \zeta_1(p) = K_I(p) K_M^{d-1}(p) \prod_{m=1}^{d} \nu_m(p). \]  \hspace{1cm} (7.13)

Proposition:

\[ \sup_{b \in B} ||V_n^{(N)}(d, n, \{\xi(\cdot, \cdot)\})||G(\zeta_1) \leq \prod_{m=1}^{d} ||\tilde{\zeta}(m)||G(\nu_m^{(n)}, r_m). \]  \hspace{1cm} (7.14)

**Theorem 7.3** ("Inside independent version").

Define the sequence of a functions \( \zeta_m = \zeta_m(p), \ m = 1, 2, \ldots, d \) by the following way:

\[ \zeta_d(p) = K_I(p) \nu_d^{(n)}(p), \]  \hspace{1cm} (7.15)

(endpoint condition) and

\[ \zeta_m(p) = K_M(p) \cdot \left\{ \left[ \zeta_{m+1} \otimes \nu_{m+1}^{(n)}(p) \right](p) \right\}, \quad m = d - 1, d - 2, \ldots, 1 \]  \hspace{1cm} (7.16)

(reverse recursion). Proposition:

\[ \sup_{b \in B} ||V_n^{(N)}(d, n, \{\xi(\cdot, \cdot)\})||G(\zeta_1) \leq \prod_{m=1}^{d} ||\tilde{\zeta}(m)||G(\nu_m^{(n)}, r_m). \]  \hspace{1cm} (7.17)

**Theorem 7.4** ("Vector independent version").

Define the sequence of a functions \( \zeta_m = \zeta_m(p), \ m = 1, 2, \ldots, d \) by the following way:

\[ \zeta_d(p) = K_M(p) \nu_d^{(n)}(p), \]  \hspace{1cm} (7.18)

(endpoint condition) and

\[ \zeta_m(p) = K_M(p) \zeta_{m+1}(p) \times \nu_m^{(n)}(p), \quad m = d - 1, d - 2, \ldots, 1; \]  \hspace{1cm} (7.19)

(reverse recursion); with the explicit solution:

\[ \zeta_1(p) = K_M^d(p) \cdot \prod_{m=1}^{d} \nu_m^{(n)}(p). \]
Proofs are at the same as in the sections 3 and 4 and may omitted.

8 Concluding remarks

1. A ”good λ inequality”.

Let $X$ and $Y$ be two non-negative r.v. such that there exist a constants $\beta > 1$, $\delta > 0$, $\epsilon > 0$ for which

$$\forall \lambda > 0 \ P(X > \beta \lambda, Y \leq \delta \lambda) \leq P(X > \lambda).$$

The following inequality (implication) is called A ”good λ inequality”:

$$EX^p \leq \beta^p \delta^{-p}(1 - \epsilon \beta^p)^{-1}EY^p, \ 0 < p < |\log_{\beta} \epsilon|.$$  

This assertion is proved, e.g. in [17], chapter 2, lemma 2.4 and is used widely in the theory of martingales.

Let us denote $r = |\log_{\beta} \epsilon|$ and suppose $r > 1$,

$$Y \in G(\psi, r) \iff |Y|^p \leq \psi(p), \ p < r.$$  

For the function

$$\psi_r(p) := \psi(p) (r - p)^{-1/r}$$

we deduce:

$$||X||G(\psi_r, r) \leq C \ ||Y||G(\psi, r).$$

2. Maximal inequality for polynomial martingales.

As long as all the sequences $n \to Q(d, n, b)$ and $n \to V^{(N)}(d, N - n, b)$ are martingales relative the source filtration $F(i)$ (or $F(N - i)$), we can apply the famous Doob’s inequality under before formulated conditions:

$$\sup_{b \in B} \left| \sup_{n \leq N} Q(d, n, b) \right|_p \leq C \zeta_d(p) \cdot \frac{p}{p - 1},$$

$$\sup_{b \in B} \left| \sup_{n \leq N} V^{(N)}(d, n, b) \right|_p \leq C \zeta_d(p) \cdot \frac{p}{p - 1}.$$  

3. A new norm of a random vectors.

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Let $X = X(\omega) = \vec{X} = (X_1, X_2, \ldots, X_D)$ be a $D$-dimensional random vector and $b \in S^{D-1}$ be arbitrary non-random normed vector with the same dimension as the vector $X$:

$$b \in S^{D-1} \iff b \in B \iff \sum_{i=1}^{D} b^2(i) = 1.$$ 

We define the $L(p)$, $p \geq 1$ and correspondingly $G(\psi)$ norm of the random vector (vector $L_p(\Omega)$ norm) $X = \vec{X} = (X_1, X_2, \ldots, X_D)$ as follows:

$$|\vec{X}|_p \overset{\text{def}}{=} \sup_{b \in B} |(X, b)|_p,$$

$$||\vec{X}||G(\psi) = ||\vec{X}||G(\psi; a, b) \overset{\text{def}}{=} \sup_{p \in (a, b)} \left[ ||\vec{X}|/\psi(p) \right].$$

Note that in the one-dimensional case $D = 1$ this definition coincides with ordinary.

If for example all the components $X_i$ are centered, independent and identical distributed, then

$$|X_1|_p \leq |\vec{X}|_p \leq K_1(p) |X_1|_p.$$

Our main results, for instance, theorem 3.1, may be reformulated in the terms of vector $L_p$ and vector $G(\psi)$ norms as follows:

$$||\xi(I)||G(\zeta_d) \leq \prod_{m=1}^{d} ||\vec{\xi}(m)||G(\nu^{(m)}, r_m) = 1.$$ 

Here

$$D = \frac{n(n-1)(n-2) \ldots (n-d+1)}{d!}, \quad n > d + 1.$$ 

4. Rosenthal’s approach.

H.P. Rosenthal in [37] proved that each sequence of centered independent r.v. generates in $L_p(\Omega)$, $2 \leq p < \infty$ the subspace isomorphic to $l_2$. See also [5].

Our main results imply that this is true for the multiple sequences of mean zero polynomial martingales $\xi(I)$:

$$|b(\cdot)|^2 = \sum_{I} b^2(I).$$

5. Normed multiple sums estimates.

It is no hard to obtain the moment and tail estimates for normed multiple sums

$$\hat{Q}_d = \frac{Q_d}{\sqrt{\text{Var}(\{Q_d\})}}$$

in the terms of normed summands
\[ \hat{\xi}(i, m) = \frac{\xi(i, m)}{\sqrt{\text{Var}\{\xi(i, m)\}}} = \frac{\xi(i, m)}{\sigma(i, m)}. \]

They are at the same as before; when we write instead the functions \( \nu_m(p) \) the functions

\[ \hat{\nu}_m(p) = \sup_{i=1,2,...,n} |\hat{\xi}(i, m)|_p = \sup_{i=1,2,...,n} \left[ \frac{\xi(i, m)}{\sigma(i, m)} \right]_p \]

and correspondingly \( b \in \hat{B} \), where

\[ \hat{B} = \{ b = \vec{b} : \sum_{I \subset I(d,n)} b^2(I) \prod_{m=1}^{d} \sigma^2(i_m, m) = 1 \}. \]

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