On the equation $N_{K/k}(\Xi) = P(t)$

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February 21, 2012

Abstract

For varieties given by an equation $N_{K/k}(\Xi) = P(t)$, where $N_{K/k}$ is the norm form attached to a field extension $K/k$ and $P(t)$ in $k[t]$ is a polynomial, three topics have been investigated:

(i) computation of the unramified Brauer group of such varieties over arbitrary fields;

(ii) rational points and Brauer-Manin obstruction over number fields (under Schinzel’s hypothesis);

(iii) zero-cycles and Brauer-Manin obstruction over number fields.

In this paper, we produce new results in each of three directions. We obtain quite general results under the assumption that $K/k$ is abelian (as opposed to cyclic in earlier investigation).

MSC classification : 11G35, 14G05

Keywords : Brauer group, Brauer-Manin obstruction, rational point, zero-cycle.
Introduction

A class of geometrically integral varieties defined over a number field \( k \) satisfies the Hasse principle if a variety in this class has a \( k \)-rational point as soon as it has rational points in all the completions \( k_v \) of the field \( k \). For example, quadrics, Severi-Brauer varieties are known to satisfy this principle. However, counterexamples to the Hasse principle are also known even in the class of rational varieties. In 1970, Manin ([20], [21]) showed that an obstruction based on the Brauer group of varieties, now referred to as the Brauer-Manin obstruction, can often explain failures of the Hasse principle. Further work (see [1] for a survey) has shown that for some classes of rational varieties the Brauer-Manin obstruction is the only obstruction to the Hasse principle.

For some classes of varieties for which the Hasse principle has been proved, weak approximation is also known: namely, given a variety \( X \) over a number field \( k \), and given a \( k_v \)-rational point \( P_v \) of \( X \) for each \( v \) in a finite set of places of \( k \), one may find a \( k \)-rational point on \( X \) as close as one wishes to each \( P_v \) (for the \( v \)-adic topology). However, for more general unirational varieties, counterexamples to weak approximation are known, and one may define a Brauer-Manin obstruction to weak approximation ([7], [8], [11]) and ask whether it is the only obstruction in the class of geometrically unirational varieties.

For general varieties, it seems quite unreasonable to hope for such statements. A more reasonable conjecture relates to zero-cycles of degree 1. A variety defined over a field \( k \) has a zero-cycle of degree 1 over \( k \) if and only if the degrees of the finite field extensions \( K/k \) over which it acquires a rational point are globally coprime. There also exist counterexamples to the Hasse principle for zero-cycles of degree 1. Similarly, one may define a Brauer-Manin obstruction and ask whether this is the only obstruction for arbitrary smooth projective varieties over a number field ([2]).

The present paper focuses on varieties defined over the ground field \( k \) by an equation

\[ N_{K/k}(\Xi) = P(t), \]

where \( K/k \) is a finite field extension, \( N_{K/k} \) denotes the norm map, \( \Xi \) is a “variable” in \( K \) and \( P(t) \in k[t] \) is a nonconstant polynomial.

To compute the Brauer-Manin obstruction, one must compute the Brauer group of a smooth projective model of the variety under study. That group is sometimes referred to as the unramified Brauer group of the variety.
In 2003, Colliot-Thélenne, Harari and Skorobogatov ([4]) discussed the unramified Brauer group for varieties defined by an equation $N_{K/k}^{\Xi} = P(t)$. In fact, they defined a partial compactification of the smooth locus of these varieties. For this partial compactification, they gave a formula for its vertical Brauer group and the quotient of its Brauer group by the vertical Brauer group. They pointed out that sometimes the unramified Brauer group can be calculated by the formula. However, it is still open how to determine the unramified Brauer group for more general cases. They also raised questions about the unramified Brauer group of two special examples ([4], p. 82 and p. 83).

In §2, building upon the formula for $\text{Br}(X)$ in [4], we compute $\text{Br}(X^c)$ in several new cases.

In §2.1, for $P(t)$ irreducible and $K/k$ “general”, we show $\text{Br}(X^c) = \text{Br}_0(X^c)$ (Theorem 2.2).

In §2.2, for $P(t)$ irreducible and $K/k$ abelian, we show that the quotient $\text{Br}(X)/\text{Br}(X^c)$ is 2-torsion, and we show that the quotient is trivial in many cases (Theorem 2.5).

In §2.3, motivated by the question in [4], we give a formula for $\text{Br}(X^c)/\text{Br}_0(X^c)$ when $K/k$ is bicyclic and $P(t)$ has all roots in $k$, possibly multiple.

In a series of earlier papers, rational points and zero-cycles of degree 1 have been studied on smooth projective models of varieties defined by an equation $N_{K/k}^{\Xi} = P(t)$, and more generally on varieties fibred over the projective line whose general fibre is birationally a principal homogeneous space under a torus.

Under Schinzel’s hypothesis (H), the question was first studied by Colliot-Thélène and Sansuc ([9]). Further work under (H) is due to Swinnerton-Dyer, Serre, Colliot-Thélène, Skorobogatov ([27],[26],[14],[13]).

As explained in [14] and [13], a device due to Salberger [23] enables one to transform some of the conditional proofs for the existence of rational points obtained under (H) into unconditional proofs for the existence of zero-cycles of degree 1.

In [13], restrictions are made on the fibres: on the one hand one assumes that the Hasse principle and weak approximation hold in the smooth fibres, on the other hand one requires some abelianity condition for the splitting fields associated to the components of the singular fibres.

In this paper, we prove results of the above type in cases not covered by [13]: either the Hasse principle need not hold in the fibres, or the abelian splitting condition is not fulfilled.
We consider three types of varieties over the ground field $k$:

1. Varieties defined by an equation $N_{K/k}(\Xi) = P(t)$, where $K/k$ is an abelian extension and $\mathbb{A}^2_P(\widehat{T})P = \mathbb{A}^2_P(\widehat{T})$ (see §1 for definition).

2. Varieties defined by an equation $(x_1^2 - ax_2^2)(y_1^2 - by_2^2)(z_1^2 - abz_2^2) = P(t)$ where $a, b \in k^*$.

3. Varieties defined by an equation $N_{K/k}(\Xi) = P(t)$ where $K/k$ is of degree 3 (non-Galois).

In §3, we prove:

**Theorem 0.1.** Let $V$ be the smooth locus of one of the above three varieties. Assume Schinzel’s hypothesis holds. Then the Brauer-Manin obstruction to the Hasse principle and weak approximation for rational points is the only obstruction for any smooth proper model of $V$.

In §4, we prove:

**Theorem 0.2.** Let $V$ be the smooth locus of one of the above three varieties. If there is no Brauer-Manin obstruction to the existence of a zero-cycle of degree 1 on a smooth proper model $V^c$ of $V$, then there is a zero-cycle of degree 1 on $V^c$ (defined over $k$).

1 Some recollections from [4], and some complements

Let us recall some results from [4]. Let $k$ be a field of characteristic zero. Let $P(t) \in k[t]$ be a nonzero polynomial. We consider the affine variety over $k$ defined by

$$N_{K/k}(\Xi) = P(t)$$

where $K/k$ is a finite product of finite separable field extensions of $k$, $\Xi$ is a variable in $K$ and $N_{K/k}$ is the formal norm associated with the extension $K/k$. In [4] $K/k$ is a separable finite field extension. In fact much of the theory holds for $K/k$ a product of fields. This point will be usually used in late sections.

Let $V \subset \mathbb{A}^{n+1}_k \simeq \mathbb{A}^1_k \times R_{K/k}(\mathbb{A}^1_k)$ be the smooth locus of the affine hypersurface defined by the equation (1). If $P(t)$ is separable, $V$ is the hypersurface defined by (1). The projection $(\Xi,t) \mapsto t$ defines a surjective morphism $p : V \to \mathbb{A}^1_k$. Let $U_0 \subset \mathbb{A}^1_k$ be the open subset defined by $P(t) \neq 0$ and
$U = p^{-1}(U_0) \subset V$. Let $E/k$ be the minimal Galois extension which splits $P(t)$ and contains $K$. Let $T = R^1_{K/k}(\mathbb{G}_m)$ and $\hat{T}$ be the character group of $T$. The Picard group of $U_0 \times_k E$ is zero and there exists an isomorphism of $E$-varieties

$$U_E \simeq U_{0,E} \times E T_E \simeq U_{0,E} \times E \mathbb{G}_m^{n-1}.$$ 

Therefore $\text{Pic}(U_E) = 0$ and the quotient $E[U]^*/E^*$ is generated by the linear factors of $P(t)$ over $E$ and the characters of $T_E$. Let $T_c = \mathbb{R}_{1,k}(\mathbb{G}_m)$ and $\hat{T}$ be the character group of $T$. The Picard group of $U_0 \times_k E$ is zero and there exists an isomorphism of $E$-varieties

$$U_E \simeq U_{0,E} \times E T_E \simeq U_{0,E} \times E \mathbb{G}_m^{n-1}.$$ 

Therefore $\text{Pic}(U_E) = 0$ and the quotient $E[U]^*/E^*$ is generated by the linear factors of $P(t)$ over $E$ and the characters of $T_E$. Let $T_c$ be a smooth $T$-equivariant compactification (as in [5]). The contracted product $U \times^T T^c$ is a partial compactification of $U$ and proper and smooth over $U_0$. Let $X$ be the smooth $k$-variety by gluing $V$ and $U \times^T T^c$ along $U$. In this paper $X$ is called the CHS partial compactification of $V$.

In [4] the following results are established: The natural map $k^* \to k^*[X]$ is an isomorphism, $\text{Pic}(X)$ is finitely generated and torsionfree, $\text{Br}(X) = 0$. By Hironaka, one may find a smooth proper compactification $X_c \subset X$ with a morphism $X^c \to \mathbb{P}^1_k$ extending the morphism $X \to \mathbb{P}^1_k$. There are natural inclusions

$$\text{Br}(X_c) \hookrightarrow \text{Br}(X).$$

One is interested in knowing when these are equalities. Recall that for any $k$-variety $V$, the image of the natural map $\text{Br}(k) \to \text{Br}(V)$ is denoted $\text{Br}_0(V)$.

There are also natural inclusions

$$\text{Br}_{vert}(X) \hookrightarrow \text{Br}(X)$$

and

$$\text{Br}_{vert}(X_c) \hookrightarrow \text{Br}_{vert}(X^c)$$

where $\text{Br}_{vert}(X)$ denotes the subgroup of $\text{Br}(X)$ whose image in the Brauer group $\text{Br}_{X_0}$ of the generic fibre of $X \to \mathbb{P}^1$ is in the image of $\text{Br}(k(\mathbb{P}^1)) \to \text{Br}(X_0)$, and similarly for $\text{Br}_{vert}(X^c)$.

One is interesting in deciding which groups contribute to $\text{Br}(X)$ and to $\text{Br}(X^c)$, and ultimately in computing $\text{Br}(X^c)$.

Let $Z[K/k]$ denote the $\text{Gal}(E/k)$-module $Z[\text{Gal}(E/k)/\text{Gal}(E/K)]$. Write $P(t) = cp_1(t)^{e_1} \cdots p_m(t)^{e_m}$ where $p_i(t)$ is irreducible. Let $Z_P$ be the permutation $\text{Gal}(E/k)$-module associated with the polynomial $P(t)$, which is a direct sum of $Z[L_i/k]$ where $L_i = k[t]/(p_i(t))$. Let $N_i = N_{L_i/k} \in Z[L_i/k]$ be $\sum_{\sigma} \sigma$ with $\sigma$ running over all embeddings of $L_i$ into $E$. Similarly we denote $N' = N_{K/k} \in Z[K/k]$. It is easy to verify that as $\text{Gal}(E/k)$-modules

$$E[U]^*/E^* \simeq (Z_P \oplus Z[K/k])/Z(e_1 N_1 + \cdots + e_m N_m + N').$$
Let $F/k$ be a Galois extension and $M$ be a finitely generated $\text{Gal}(F/k)$-module. For simplicity, in this paper we always denote 

$$H^i(F/k, M) =: H^i(\text{Gal}(F/k), M).$$

If $F = \bar{k}$, we denote it simply by $H^i(k, M)$. We define

$$X^i_\omega(F/k, M) = \bigcap_{g \in \text{Gal}(F/k)} \ker[H^i(F/k, M) \to H^i(\langle g \rangle, M)]$$

where $\langle g \rangle$ is the cyclic subgroup of $\text{Gal}(F/k)$ generated by $g$. If $F = \bar{k}$, we also simply denote $X^i_\omega(k, M)$ by $X^i_\omega(M)$. Let $M$ be a $\text{Gal}(F/k)$-module, it can be viewed as a $\text{Gal}(\bar{k}/k)$-module naturally. If $M$ is torsion-free of finite type, we have

$$X^2_\omega(F/k, M) = X^2_\omega(M).$$

Let $j_P : Z \to Z_P$ be defined by sending $1$ to $-(e_1N_1 + \cdots + e_mN_m)$. Let $F/k$ be a Galois extension and $F \supset E$. For any $\text{Gal}(F/k)$-module $M$, $j_P$ induces a morphism $M \to M \otimes \mathbb{Z}_P$. We define

$$X^2_\omega(F/k, M) = \ker[X^2_\omega(F/k, M) \to X^2_\omega(F/k, M \otimes \mathbb{Z}_P)].$$

If $F = \bar{k}$, we simply denote it by $X^2_\omega(M)_P$.

With the similar notation as above, Colliot-Thélène, Harari and Skorobogatov proved the following theorem:

**Theorem 1.1.** [4, Proposition 2.5] Let $X/k$ be the CHS partial compactification of $V$ as above. Then:

(a) There is the following exact sequence

$$0 \to H^1(k, \hat{T} \otimes \mathbb{Z}_P)/j_{P*}H^1(k, \hat{T}) \to H^1(k, \text{Pic}(\overline{X})) \to X^2_\omega(\hat{T})_P \to 0.$$ (2)

(b) The elements of $\text{Br}(X)$ whose image in $H^1(k, \text{Pic}(\overline{X}))$ come from $H^1(k, \hat{T} \otimes \mathbb{Z}_P)$ are precisely the elements of $\text{Br}_{\text{vert}}(X)$.

Under the assumption $H^3(k, \hat{k}^*) = 0$, which is satisfied if $k$ is a number field, the above exact sequence (2) identifies with

$$0 \to \text{Br}_{\text{vert}}(X)/\text{Br}_0(X) \to \text{Br}(X)/\text{Br}_0(X) \to X^2_\omega(\hat{T})_P \to 0.$$ (3)
As in [4], we also have the following exact sequence

\[ 0 \rightarrow H^1(E/k, \hat{T} \otimes \mathbb{Z}_P) / j_{P *} H^1(E/k, \hat{T}) \rightarrow H^1(E/k, \hat{T}) \rightarrow \mathbb{H}^1(E/k, \hat{T}), \]

We also have the following commutative diagram:

\[
\begin{array}{c}
0 \rightarrow H^1(E/k, \hat{T} \otimes \mathbb{Z}_P) / j_{P *} H^1(E/k, \hat{T}) \rightarrow H^1(E/k, \hat{T}) \rightarrow \mathbb{H}^1(E/k, \hat{T}) \rightarrow 0 \\
\downarrow \cong \quad \downarrow \rho \quad \downarrow \cong
\end{array}
\]

So we can see the morphism \( \rho \) is an isomorphism.

Let us mention the easy application of the Theorem 1.1.

**Proposition 1.2.** Let \( k, K, P(t), X \) be as above. Assume that \( K \) is a field, and that the extension \( K/k \) is abelian, and that \( P(t) \) is irreducible. Then

(a) \( H^1(k, \hat{T} \otimes \mathbb{Z}_P) / j_{P *} H^1(k, \hat{T}) = 0 \).

(b) \( \text{Br}_0(X) \cong \text{Br}_{\text{vert}}(X) \).

(c) If \( H^3(k, \mathbb{K}) = 0 \), then \( \text{Br}(X) / \text{Br}_0(X) \cong \mathbb{I}^2_{\omega}(\hat{T})_P \).

(d) If \( K/k \) is cyclic, then \( H^1(k, \text{Pic}(X)) = 0 \), hence \( \text{Br}_0(X) = \text{Br}(X) \).

**Proof.** Let \( L = k[t](P(t)) \). We know

\[
H^1(k, \hat{T} \otimes \mathbb{Z}_P) = H^1(L, \hat{T}) = \text{Ker}[H^1(L, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(L.K, \mathbb{Q}/\mathbb{Z})] = \text{Hom}(\text{Gal}(L.K/L), \mathbb{Q}/\mathbb{Z}).
\]

Since the extension \( K/k \) is abelian, the restriction morphism

\[
\text{Hom}(\text{Gal}(K/k), \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}(\text{Gal}(L.K/L), \mathbb{Q}/\mathbb{Z})
\]

is surjective.

So \( H^1(k, \hat{T} \otimes \mathbb{Z}_P) / j_{P *} H^1(k, \hat{T}) = 0 \).

If \( K/k \) is cyclic, then \( \mathbb{I}^2_{\omega}(\hat{T}) = 0 \). By the sequence [2], we have

\( H^1(k, \text{Pic}(X)) = 0 \). \( \square \)

Let \( T' \) be the torus over \( k \) defined by

\[
\prod_{i=1}^{m} N_{L_i/k}(\Xi_i)^{e_i} \cdot N_{K/k}(\Upsilon) = 1.
\]
We can see
\[ E[U]^*/E^* \simeq \hat{T}' \]
as Gal(E/k)-modules, where \( \hat{T}' \) is the character group of \( T' \). Therefore we have
\[ \text{III}^2_\omega(E/k, E[U]^*/E^*) \simeq \text{III}^2_\omega(\hat{T}'). \]

**Proposition 1.3.** Let \( X \) and \( X^c \) be as in section 1. The natural surjective map \( \text{Pic}(X^c) \rightarrow \text{Pic}(X) \) induces an injection
\[ H^1(k, \text{Pic}(X^c)) \hookrightarrow H^1(k, \text{Pic}(X)) \]
and isomorphisms
\[ \text{III}^1_\omega(\text{Pic}(X^c)) \simeq \text{III}^1_\omega(\text{Pic}(X)) \simeq \text{III}^2_\omega(\hat{T}'). \]

**Proof.** The open set \( U \subset X \subset X^c \) is the smooth affine variety defined by
\[ N_{K/k}(\Xi) = P(t) \] and \( P(t) \neq 0 \).

We have the following exact sequence
\[ 0 \rightarrow \bar{k}[U]^*/\bar{k}^* \rightarrow \text{Div}_{X \setminus U}(X^c) \rightarrow \text{Pic}(X^c) \rightarrow \text{Pic}(U) \rightarrow 0 \]
We already know \( \text{Pic}(U) = 0 \). Since \( \text{Div}_{X \setminus U}(X^c) \) is a torsion-free permutation Gal(\( \bar{k}/k \))-module, we deduce \( H^1(k, \text{Div}_{X \setminus U}(X^c)) = 0 \). Then we have the following exact sequence
\[ 0 \rightarrow H^1(k, \text{Pic}(X^c)) \rightarrow H^2(k, \bar{k}[U]^*/\bar{k}^*) \rightarrow H^2(k, \text{Div}_{X \setminus U}(X^c)). \]

Thus we obtain the exact sequence
\[ 0 \rightarrow \text{III}^1_\omega(\text{Pic}(X^c)) \rightarrow \text{III}^2_\omega(\bar{k}[U]^*/\bar{k}^*) \rightarrow \text{III}^2_\omega(\text{Div}_{X \setminus U}(X^c)). \]

Since \( \text{Div}_{X \setminus U}(X^c) \) is a permutation Gal(\( \bar{k}/k \))-module, we deduce
\[ \text{III}^2_\omega(\text{Div}_{X \setminus U}(X^c)) = 0. \]

Thus \( \text{III}^1_\omega(\text{Pic}(X^c)) \simeq \text{III}^2_\omega(\bar{k}[U]^*/\bar{k}^*) \). And \( \bar{k}[U]^*/\bar{k}^* \simeq \hat{T}' \) as Gal(\( \bar{k}/k \))-modules. Thus we have \( \text{III}^1_\omega(\text{Pic}(X^c)) \simeq \text{III}^2_\omega(\hat{T}'). \)
Since $\bar{k}[X]^* = \tilde{k}^*$, we have the exact sequence

$$0 \to \bar{k}[U]^*/\tilde{k}^* \to \text{Div}_{X\setminus U}(\bar{X}) \to \text{Pic}(\bar{X}) \to \text{Pic}(U) \to 0.$$  

With similar arguments as above, we have $\text{III}_1^1(\text{Pic}(\bar{X})) \simeq \text{III}_2^2(\tilde{k}[U]^*/\tilde{k}^*)$. Furthermore we have the following exact sequence

$$\text{III}_1^1(\text{Pic}(\bar{X})) \xrightarrow{\cong} \text{III}_2^2(\tilde{k}[U]^*/\tilde{k}^*)$$

Thus $\text{III}_1^1(\text{Pic}(\bar{X})) \simeq \text{III}_2^1(\text{Pic}(X))$. \qed

**Remark 1.4.** Proposition 1.3 gives a description of a subgroup of $H^1(k, \text{Pic}(\bar{X}))$. Using this proposition, we can give some nontrivial varieties $X$ (defined as above) such that there exists a nonzero element in $\text{III}_2^2(\tilde{k}[U]^*/\tilde{k}^*)$ which comes from $\text{Br}(X^c)$. For example, suppose $k$ is a field satisfying $H^3(k, \bar{k}^*) = 0$, $X$ is the CHS partial compactification of the smooth affine $k$-variety defined by such equation $N_{K/k}(\Xi) = P(t)$ which satisfies:

(a) $K = k(\sqrt{a}, \sqrt{b}, \sqrt{c})$ and $[K : k] = 8$, where $a, b, c \in \tilde{k}^*$;

(b) $P(t)$ is an irreducible polynomial over $k$ and suppose $k[t]/(P(t)) = k(\sqrt{a}, \sqrt{b})$.

## 2 Calculation of the Brauer group

In this section, we keep notation as in [4] and as in §1 above. In [4], Colliot-Thélène, Harari and Skorobogatov gave a formula for the vertical Brauer group $\text{Br}_{\text{vert}}(X)$ and the quotient $\text{Br}(X)/\text{Br}_{\text{vert}}(X)$. They pointed out that the vertical unramified Brauer group $\text{Br}_{\text{vert}}(X^c)$ can be calculated by the formula. However, it is still open how to determine the unramified Brauer group $\text{Br}(X^c)$. The aim of this section is to investigate the unramified Brauer group $\text{Br}(X^c)$.

### 2.1 The case $P$ is irreducible and some linear independence condition is satisfied

**Lemma 2.1.** Let $P(t)$ be an irreducible polynomial and let $L = k[t]/(P(t))$. Let $K^{cl}$ (resp. $L^{cl}$) be the Galois closure of $K$ (resp. $L$) over $k$. If $L \cap K^{cl} = k$, then $\text{III}_2^2(\hat{T})_P = 0$.  

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Proof. The field $E$ is the composite $K^{cd}L^{cd}$. We know $\text{III}_2(\hat{T}) = \text{III}_2(E/k, \hat{T})$. Then

$$\text{III}_2(\hat{T})_P = \text{Ker}[j_{P*} : \text{III}_2(\hat{T}) \rightarrow \text{III}_2(E/k, \hat{T} \otimes \mathbb{Z}_P)]$$

$$= \text{Ker}[j_{P*} : \text{III}_2(E/k, \hat{T}) \rightarrow \text{III}_2(E/k, \hat{T} \otimes \mathbb{Z}_P)]$$

$$= \text{Ker}[j_{P*} : \text{III}_2(E/k, \hat{T}) \rightarrow H^2(E/k, \hat{T} \otimes \mathbb{Z}_P)].$$

By Shapiro’s lemma, we have

$$H^2(E/k, \hat{T} \otimes \mathbb{Z}_P) \simeq H^2(E/L, \hat{T}).$$

Thus

$$\text{III}_2(\hat{T})_P = \text{Ker}[\text{Res}_{L/k} : \text{III}_2(E/k, \hat{T}) \rightarrow H^2(E/L, \hat{T})].$$

We have the following commutative diagram

$$\text{III}_2(E/k, \hat{T}) \quad \xrightarrow{f} \quad H^2(E/L, \hat{T})$$

$$\downarrow \quad \quad \quad \quad \downarrow g$$

$$\text{III}_2(K^{cd}/k, \hat{T}) \quad \xrightarrow{g} \quad H^2(L.K^{cd}/L, \hat{T}).$$

Since $\text{Gal}(E/K^{cd})$ acts trivially on $\hat{T}$, $f$ is an isomorphism. By the Hochschild-Serre spectral sequence one obtains the exact sequence

$$H^1(E/L.K^{cd}, \hat{T})^{\text{Gal}(E/L)} \rightarrow H^2(L.K^{cd}/L, \hat{T}) \rightarrow H^2(E/L, \hat{T}).$$

We know $\hat{T}$ is a torsion-free module with trivial $\text{Gal}(E/L.K^{cd})$-action. Thus $H^1(\text{Gal}(E/L.K^{cd}), \hat{T}) = 0$. Then $g$ is injective. So we have

$$\text{III}_2(\hat{T})_P = \text{Ker}[\text{III}_2(K^{cd}/k, \hat{T}) \rightarrow H^2(L.K^{cd}/L, \hat{T})].$$

Since $L \cap K^{cd} = k$, we have a natural group isomorphism between $\text{Gal}(L.K^{cd}/L)$ and $\text{Gal}(K^{cd}/k)$. Therefore

$$H^2(K^{cd}/k, \hat{T}) \simeq H^2(L.K^{cd}/L, \hat{T}).$$

Directly we have $\text{III}_2(\hat{T})_P = 0$. □

Remark. In fact, we can prove a more general result. Let $P(t) = p_1(t) \cdots p_m(t)$, where each $p_i(t)$ is an irreducible polynomial and $p_i(t) \neq p_j(t)$ for any $i \neq j$. Let $L_i = k[t]/(p_i(t))$. If there exists $i_0$ such that $L_{i_0} \cap K^{cd} = k$, then $\text{III}_2(\hat{T})_P = 0$. 10
Theorem 2.2. Let $P(t)$ be an irreducible polynomial over $k$ and $L = k[t]/(P(t))$. Let $V$ be the smooth affine variety over $k$ defined by $N_{K/k}(\Xi) = P(t)$. Let $V^c$ be a smooth compactification of $V$ and $K^{cl}$ the Galois closure of $K$ over $k$. If $L \cap K^{cl} = k$ then $\text{Br}(V^c)/\text{Br}_0(V^c) = 0$.

Proof. Since

$$\text{Br}(V^c)/\text{Br}_0(V^c) \cong \text{Br}(X^c)/\text{Br}_0(X^c) \subset \text{Br}(X)/\text{Br}_0(X) \hookrightarrow H^1(k, \text{Pic}(X)),$$

we only need to show $H^1(k, \text{Pic}(X)) = 0$.

Since $L \cap K^{cl} = k$, we have $\Pi_2^\mathbb{Z}(\hat{T})_P = 0$ by Lemma 2.1. So we only need to show $H^1(k, \hat{T} \otimes \mathbb{Z}_P)/j_{P*}H^1(k, \hat{T}) = 0$ by the basic sequence (2).

Obviously

$$H^1(k, \hat{T} \otimes \mathbb{Z}_P)/j_{P*}H^1(k, \hat{T}) \cong H^1(E/k, \hat{T} \otimes \mathbb{Z}_P)/j_{P*}H^1(E/k, \hat{T}).$$

By Shapiro’s lemma, we have

$$H^1(E/k, \hat{T} \otimes \mathbb{Z}_P) \cong H^1(E/L, \hat{T}).$$

Since $L \cap K^{cl} = k$, there is a natural isomorphism between $\text{Gal}(L.K^{cl}/L)$ and $\text{Gal}(K^{cl}/k)$. Thus

$$H^1(L.K^{cl}/L, \hat{T}) \cong H^1(K^{cl}/k, \hat{T}).$$

Therefore we have the following commutative diagram

$$
\begin{array}{ccc}
H^1(E/k, \hat{T}) & \xrightarrow{j_{P*}} & H^1(E/k, \hat{T} \otimes \mathbb{Z}_P) \\
\parallel & & \parallel \\
\simeq & & \simeq \\
\xrightarrow{\text{Res}_{L/k}} & & \xrightarrow{\text{Res}_{L/k}} \\
H^1(K^{cl}/k, \hat{T}) & \xrightarrow{\simeq} & H^1(L.K^{cl}/L, \hat{T})
\end{array}
$$

Directly we deduce

$$H^1(E/k, \hat{T} \otimes \mathbb{Z}_P)/j_{P*}H^1(E/k, \hat{T}) = 0.$$
Corollary 2.3. Let \( L \) and \( K \) be extensions over \( k \). Let \( T' \) be the torus over \( k \) defined by \( N_{L/k}(\Xi_1)N_{K/k}(\Xi_2) = 1 \). If \( L \cap K^{\text{cl}} = k \) or \( K/k \) is cyclic, then \( \text{III}^2_\omega(\hat{T}') = 0 \), in particular, if \( k \) is a number field, principal homogeneous spaces of \( T' \) satisfy the Hasse principle and weak approximation.

Proof. Let \( P(t) \) be an irreducible polynomial over \( k \) such that \( L \cong k[t]/(P(t)) \). Let \( V \) be the smooth affine variety over \( k \) defined by \( P(t) = N_{K/k}(\Xi) \). By Proposition 1.3, we have \( \text{III}^2_\omega(\hat{T}') \subset H^1(k, \text{Pic}(\overline{V})) \). The statement follows from Proposition 1.2 and Theorem 2.2.

Suppose \( k \) is a number field. Since \( \text{III}^2_\omega(\hat{T}') = 0 \), the result follows from the fact that the Brauer-Manin obstruction is the unique obstruction to the Hasse principle and weak approximation for principal homogeneous spaces of tori (Theorem 8.12 in [25] or Theorem 5.2.1 in [28]). \( \square \)

Remark. The case where \( K/k \) is cyclic in Corollary 2.3 was proved in an unpublished paper of Sansuc.

2.2 The case \( P(t) \) irreducible and \( K/k \) an abelian field extension, under no linear independence condition

In this section, we investigate the case where \( P(t) \) is irreducible and \( K/k \) abelian. The main result in Theorem 2.5, which in many situation gives a good control on the quotient \( \text{Br}(X)/\text{Br}(X^c) \).

Let \( P(t) \) be an irreducible polynomial over \( k \) and \( K/k \) an abelian extension. Let \( L = k[t]/(P(t)) \). Recall \( E/k \) the minimal Galois extension which splits \( P(t) \) and contains \( K \). Let \( G = \text{Gal}(E/k) \). Let \( g \) be an element of \( G \) and \( E_g \) the fixed field of \( \langle g \rangle \) in \( E \). We have the following commutative diagram:

\[
\begin{array}{ccccccccc}
\text{H}^1(E/k, \hat{T} \otimes \mathbb{Z}_P)/j_{P*}\text{H}^1(E/k, \hat{T}) & \text{H}^1(E/k, \text{Pic}(X_E)) & \text{III}^2_\omega(E/k, \hat{T})_P & 0 \\
0 & \text{H}^1(E/E_g, \hat{T} \otimes \mathbb{Z}_P)/j_{P*}\text{H}^1(E/E_g, \hat{T}) & \text{H}^1(E/E_g, \text{Pic}(X_E)) & \text{III}^2_\omega(E/E_g, \hat{T}_I) & 0 \\
0 & & & & & & 0
\end{array}
\]

Since \( \text{Gal}(E/E_g) \) is cyclic, one has \( \text{III}^2_\omega(E/E_g, \hat{T}) = 0 \). From the above commutative diagram, we have a natural map

\[
f_g : \text{H}^1(E/k, \text{Pic}(X_E)) \rightarrow \text{H}^1(E/E_g, \hat{T} \otimes \mathbb{Z}_P)/j_{P*}\text{H}^1(E/E_g, \hat{T}).
\]
Let $K_g = E_g \cap K$ and $H = \text{Gal}(E/K_g)$. Since $K/k$ is an abelian extension, $K_g/k$ is also an abelian extension. Then $H$ is a normal subgroup of $G$. And $K/K_g$ is cyclic. Then

$$\Pi^2_K(E/K_g, \hat{T}) = \Pi^2_K(K/K_g, \hat{T}) = 0.$$ 

Similarly, we have the commutative diagram:

$$\begin{array}{ccc}
0 & \longrightarrow & H^1(E/k, \hat{T} \otimes \mathbb{Z}_p)/j_p^* H^1(E/k, \hat{T}) \\
& \downarrow & \downarrow \text{Res} \\
0 & \longrightarrow & H^1(E/K_g, \hat{T} \otimes \mathbb{Z}_p)/j_p^* H^1(E/K_g, \hat{T}) \\
& \downarrow & \downarrow \text{Res} \\
0 & \longrightarrow & H^1(E/E_g, \hat{T} \otimes \mathbb{Z}_p)/j_p^* H^1(E/E_g, \hat{T}) \\
& \downarrow \cong & \downarrow \\
& & H^1(E/E_g, \hat{T}).
\end{array}$$

Since the image of $H^1(E/k, \text{Pic}(X_E))$ in $H^1(E/K_g, \text{Pic}(X_E))$ is $G/H$-invariant, the map $f_g$ is equal to the composite map

$$H^1(E/k, \text{Pic}(X_E)) \to [H^1(E/K_g, \hat{T} \otimes \mathbb{Z}_p)/j_p^* H^1(E/K_g, \hat{T})]^{G/H} \to H^1(E/E_g, \hat{T} \otimes \mathbb{Z}_p)/j_p^* H^1(E/E_g, \hat{T})$$

To investigate $H^1(E/k, \text{Pic}(X_E))$, we will use its image in $H^1(E/E_g, \hat{T} \otimes \mathbb{Z}_p)/j_p^* H^1(E_g, \hat{T})$. In fact its image is contained in the subgroup $[H^1(E/K_g, \hat{T} \otimes \mathbb{Z}_p)/j_p^* H^1(E/K_g, \hat{T})]^{G/H}$. In the following, we will calculate this subgroup.

Let $L_g = L \cap K_g$. Let $P(t) = c(t - \xi_1) \cdots (t - \xi_n), \xi_i \in E$. We can see $\text{Gal}(E/L_g)$ can act on the set $\{\xi_1 : 1 \leq i \leq n\}$. Thus the set will split into some orbits $O_1, \cdots, O_m$. Let $p_i(t) = \prod_{\xi \in O_i} (t - \xi)$. Then we have $P(t) = c p_1(t) \cdots p_m(t)$, where $p_i(t) \in L_g[t]$. Since $L_g/k$ is Galois (in fact it is abelian), the group $\text{Gal}(L_g/k)$ acts transitively on the set $\{p_i(t) : 1 \leq i \leq m\}$. Let $L_i = L_g[t]/(p_i(t))$. We can see $L_i \cong L$ and there is the natural isomorphism

$$H^1(E/K_g, \hat{T} \otimes \mathbb{Z}_p) \cong \bigoplus_{i=1}^m \text{Hom}(\text{Gal}(L_i.K/L_i.K_g), \mathbb{Q}/\mathbb{Z})$$

$$= \bigoplus_{i=1}^m \text{Hom}(\text{Gal}(K/K \cap (L_i.K_g)), \mathbb{Q}/\mathbb{Z}).$$

Since $\text{Gal}(K/k)$ is abelian, the group $\text{Gal}(E/k)$ trivially acts on $\text{Hom}(\text{Gal}(K/K \cap (L_i.K_g)), \mathbb{Q}/\mathbb{Z})$. Let

$$R = \text{Hom}(\text{Gal}(K/K \cap (L_i.K_g)), \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(\text{Gal}(K/K \cap (L.K_g)), \mathbb{Q}/\mathbb{Z}).$$

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Then we have

\[ H^1(E/K_g, \hat{T} \otimes \mathbb{Z}_p) \simeq R[L_g/k] \]

as \( \text{Gal}(E/k) \)-modules.

Denote \( \Delta = \text{Gal}(L_g/k) \). Let \( N = \sum_{\sigma \in \Delta} \sigma \). The following diagram

\[
\begin{array}{ccc}
H^1(E/K_g, \hat{T} \otimes \mathbb{Z}_p) & \xrightarrow{j_{P^*}} & H^1(E/K_g, \hat{T} \otimes \mathbb{Z}_p) \\
\text{Res} & & \downarrow \cong \\
R[L_g/k] & & \\
\end{array}
\]

is commutative and the image of \( \text{Res} \) in \( R[L_g/k] \) is \( R \cdot N \). Then we have an isomorphism as \( \text{Gal}(E/k) \)-modules

\[ H^1(E/K_g, \hat{T} \otimes \mathbb{Z}_p) / j_{P^*} H^1(E/K_g, \hat{T}) \simeq R[\Delta]/(R \cdot N). \]

Then

\[ [H^1(E/K_g, \hat{T} \otimes \mathbb{Z}_p) / j_{P^*} H^1(E/K_g, \hat{T})]^{G/H} \simeq [R[\Delta]/(R \cdot N)]^\Delta. \]

By the exact sequence

\[ 0 \to R \to R[\Delta] \to R[\Delta]/(R \cdot N) \to 0, \]

we have an exact sequence

\[ 0 \to R \xrightarrow{\cong} R[\Delta]/(R \cdot N) \xrightarrow{\Delta} \text{Hom}(\Delta, R) \to 0. \]

Therefore \( [R[\Delta]/(R \cdot N)]^\Delta \simeq \text{Hom}(\Delta, R) \).

Let \( \psi \in \text{Hom}(\Delta, R) \), we define \( f(\psi) = \sum_{\sigma \in \Delta} \psi(\sigma) \sigma \). Let \( \sigma_0 \in \Delta \), we have

\[
\sigma_0(f(\psi)) - f(\psi) = \sum_{\sigma} \psi(\sigma) \sigma_0 \sigma - \sum_{\sigma} \psi(\sigma) \sigma \\
= \sum_{\sigma} (\psi(\sigma) - \psi(\sigma_0)) \sigma - \sum_{\sigma} \psi(\sigma) \sigma \\
= -\psi(\sigma_0) \sum_{\sigma} \sigma \in R \cdot N.
\]

Then \( f(\psi) \in [R[\Delta]/(R \cdot N)]^\Delta \).
Assume $f(\psi_1) = f(\psi_2)$. Then $\sum \sigma(\psi_1(\sigma) - \psi_2(\sigma))\sigma \in R \cdot N$. So we have $\psi_1(\sigma) - \psi_2(\sigma)$ is a constant for all $\sigma \in \Delta$. Let $e$ be the unit of $\Delta$, we have $\psi_1(e) - \psi_2(e) = 0$. Then $\psi_1 = \psi_2$. So we have

$$[R[\Delta]/(R \cdot N)]^\Delta = \{f(\psi) \mid \psi \in \text{Hom}(\Delta, R)\}.$$ 

Since $K/K_g$ is cyclic, we can choose $\chi$ to be a primitive character of $\text{Gal}(K/K_g)$. And $\text{Res}(\chi)$ is also a primitive character $\chi_i$ of $\text{Gal}(K/K \cap (L_i.K_g))$.

**Lemma 2.4.** Any element in the image of $\text{Br}(X)$ in $\text{Br}(X_{E_g})$ has the form

$$A = \rho + \sum_{\sigma \in \text{Gal}(L_g/k)} \psi(\sigma)\text{Cores}_{L(t)/k(t)}(t - \eta_i, \text{Res}(\chi))$$

$$= \rho + \sum_{\sigma \in \text{Gal}(L_g/k)} \psi(\sigma)(p(t)^{\sigma}, \chi),$$

where $\rho \in \text{Br}(E_g)$, $\psi \in \text{Hom}(\text{Gal}(L_g/k), \mathbb{Z}/b)$ and $b = \#\text{Gal}(K/K \cap (L.K_g))$.

**Proof.** It follows from the above argument and the remark on p. 76 of [4]. \qed

**Theorem 2.5.** Suppose $\text{char}(k) = 0$. Let $P(t)$ be an irreducible polynomial over $k$ and $K/k$ an abelian extension. Let $L = k[t]/(P(t))$. Let $X$ be the CHS partial compactification defined in [4], as recalled in §1, $X_c$ the compactification of $X$. Then:

(a) The quotient $\text{Br}(X)/\text{Br}(X_c)$ is 2-torsion.

(b) $\text{Br}(X_c) = \text{Br}(X)$ if one of the following conditions holds:

(1) $\text{Gal}(K/k) \cong \mathbb{Z}/2^i \times A$, where $1 \geq 0$ and $A$ is of odd order;

(2) $[L \cap K : k]$ is odd;

(3) $[L : L \cap K]$ is even;

(4) There exists $s \geq 1$ such that $2^s \mid [L : k]$ and the cokernel of the multiplication by $2^{s-1}$ on $\text{Gal}(K/k)$ is of odd order;

(5) $L/k$ contains an abelian subfield $L'/k$ with $\text{Gal}(L'/k) \cong (\mathbb{Z}/2)^3$. 

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Proof. Let \( k(X) \) be the function field of \( X \). For each discrete valuation ring \( A \) which contains \( k \) and with fraction field \( k(X) \) and residue field \( \kappa_A \), there is a residue map

\[
\partial_A : \text{Br}(k(X)) \to H^1(\kappa_A, \mathbb{Q}/\mathbb{Z}).
\]

Since \( \text{char}(k) = 0 \), Grothendieck’s purity theorem (cf. Thm1.3.2 [14]) gives

\[
\text{Br}(X^c) = \bigcap_A \text{Ker}(\partial_A) \subset \text{Br}(k(X)),
\]

where \( A \) runs through all discrete valuation rings.

Let \( B \in \text{Br}(X) \), firstly we will prove \( 2B \in \text{Br}(X^c) \). Assume it is not, then there is a discrete valuation ring \( A \) such that

\[
2\partial_A(B) \neq 0 \in H^1(\kappa_A, \mathbb{Q}/\mathbb{Z}).
\]

Then there is an element \( g \in \text{Gal}(\bar{k}_A/\kappa_A) \), such that

\[
2\partial_A(B)(g) \neq 0 \in \mathbb{Q}/\mathbb{Z}.
\]

Since \( k \subset \kappa_A \), we can fix an embedding \( \bar{k} \hookrightarrow \bar{k}_A \). Let \( E_g \) be the fixed field of \( g \) in \( E \). Let \( K_g = K \cap E_g \) and \( L_g = L \cap K_g \). There is a natural map \( f : \text{Br}(X) \to \text{Br}(X_{E_g}) \).

Let \( b = \#\text{Gal}(K/K \cap (L.K_g)) \). By Lemma 2.2, there exists \( \psi_B \in \text{Hom}(\text{Gal}(L_g/k), \mathbb{Z}/b) \) such that

\[
f(B) = \rho + \sum_{\sigma \in \text{Gal}(L_g/k)} \psi_B(\sigma)(p_1(t)^\sigma, \chi)
\]

where \( \rho \in \text{Br}(E_g) \) and \( \chi \) is a primitive character of \( \text{Gal}(K/K_g) \). Since \( E_g(X)/k(X) \) is a finite extension, there is a discrete valuation ring \( A_g \in E_g(X) \) which extends \( A \). Since \( A_g/A \) is unramified, we have \( \kappa_{A_g} = \kappa_A.K_g \). So \( g \in \text{Gal}(\bar{k}_A/\kappa_{A_g}) \). By Proposition 1.1.1 in [14], we have

\[
\partial_A(B)(g) = \partial_{A_g}(f(B))(g).
\]

Then \( 2\partial_{A_g}(f(B)) \neq 0 \).

Let \( m = \#\text{Gal}(K/K_g) \). We will discuss the question by two cases.

(a) The case \( \text{ord}_{A_g}(t) \geq 0 \).

Since \( \partial_{A_g}(f(B)) \neq 0 \), there is a \( \sigma_0 \) such that \( \text{ord}_{A_g}(p_1(t)^{\sigma_0}) > 0 \). And

\[
\text{ord}_{A_g}(p_1(t)^{\sigma}) = 0 \text{ for } \sigma \neq \sigma_0
\]
since $p_1(t)^\sigma$ and $p_1(t)^{\sigma_0}$ are relatively prime. Then we have

$$\operatorname{ord}_{A_g}(p_1(t)^{\sigma_0}) = \operatorname{ord}_{A_g}(P(t)).$$

Therefore

$$\partial_{A_g}(f(B)) = \psi_B(\sigma_0)\operatorname{ord}_{A_g}(p_1(t)^{\sigma_0}) \cdot \bar{\chi} = \psi_B(\sigma_0)\operatorname{ord}_{A_g}(P(t)) \cdot \bar{\chi} = 0$$

by the equation $N_{K/k}(\Xi) = P(t)$. A contradiction is derived.

(b) The case $\operatorname{ord}_{A_g}(t) = \delta < 0$.

Denote $\operatorname{deg}(p_1(t)) = d$. Then

$$\operatorname{ord}_{A_g}(p_1(t)^\sigma) = \delta \cdot \operatorname{deg}(p_1(t)^\sigma) = \delta \cdot \operatorname{deg}(p_1(t)) = \delta d.$$

Therefore

$$\partial_{A_g}(f(B)) = \sum\limits_\sigma \psi_B(\sigma)\operatorname{ord}_{A_g}(p_1(t)^\sigma) \cdot \bar{\chi} = \delta d \sum\limits_\sigma \psi_B(\sigma) \cdot \bar{\chi}.$$ 

Let $\psi(\operatorname{Gal}(L_g/k)) = \langle u \rangle \subseteq \mathbb{Z}/b$ where $b = \#\operatorname{Gal}(K/K \cap (L.K_g))$ and $u \mid b$. Then we have

$$\partial_{A_g}(f(B)) = \delta d \cdot \#\operatorname{Ker}(\psi_B) \cdot u(1 + 2 + \cdots + b/u - 1) \cdot \bar{\chi}$$

$$= \delta db \cdot \#\operatorname{Ker}(\psi_B) \cdot (b/u - 1)/2 \cdot \bar{\chi}.$$

And

$$m = [K : K_g] = [K : K \cap (L.K_g)] \cdot [K \cap (L.K_g) : K_g]$$

$$= b \cdot [K \cap (L.K_g) : K_g].$$

Since

$$[L.K_g : K_g] = [L : L_g] = d$$

and $[K \cap (L.K_g) : K_g] \mid [L.K_g : K_g]$,

one has $[K \cap (L.K_g) : K_g] \mid d$. Let

$$d' = d/[K \cap (L.K_g) : K_g] = [L.K_g : K \cap (L.K_g)]$$

$$= [L.K : K] = [L : L \cap K].$$

So we have

$$\partial_{A_g}(f(B)) = \delta md' \cdot \#\operatorname{Ker}(\psi_B) \cdot (b/u - 1)/2 \cdot \bar{\chi}.$$
Since \( m \cdot \chi = 0 \), we have
\[
2\partial_A(g(f(B))) = 0,
\]
it is a contradiction to \( 2\partial_A(g(f(B))) = 0 \). Then we prove \( 2B \in \text{Br}(X^c) \).

By the above proof, if \( 2 \mid \delta d' \cdot \#\text{Ker}(\psi_B) \cdot (b/u - 1) \) for each cyclic subgroup \( \langle g \rangle \subset \text{Gal}(\bar{k}/k) \), then \( \partial_A(B) = 0 \) for every discrete valuation ring \( A \) containing \( k \) and with fraction field \( k(X) \), therefore \( B \in \text{Br}(X^c) \).

Case (1). If \( i = 0 \), then the order \( \text{Gal}(K/k) \) is odd. Therefore \( 2 \mid (b/u - 1) \).

Suppose \( i > 0 \), then \( \text{Gal}(K/k) = H^1 \times H^2 \), where \( H^1 \) is cyclic and of order \( 2^i \) and \( H^2 \) is of odd order. By the Künneth formula (p. 96 in [22]), we have
\[
H^2(\text{Gal}(K/k), \mathbb{Q}/\mathbb{Z}) = \bigoplus_{i+j=2} H^i(H^1, H^j(H^2, \mathbb{Q}/\mathbb{Z})).
\]
Since the order of \( H^1 \) and \( H^2 \) is relatively prime, we have
\[
H^1(H^1, H^1(H^2, \mathbb{Q}/\mathbb{Z})) = 0.
\]
And \( H^2(H^1, \mathbb{Q}/\mathbb{Z}) = 0 \) since \( H^1 \) is cyclic. Then we have
\[
H^2(\text{Gal}(K/k), \mathbb{Q}/\mathbb{Z}) = H^2(H^2, \mathbb{Q}/\mathbb{Z}).
\]
Let \( h = \#H^2 \). Since \( H^1(k, \text{Pic}(X)) = \prod_{\nu}(K/k, \hat{T})_P \subset H^2(\text{Gal}(E/k), \mathbb{Q}/\mathbb{Z}) \), we have \( h \cdot H^1(k, \text{Pic}(X)) = 0 \).

Let \( B \in \text{Br}(X) \). Since \( \text{Br}(X)/\text{Br}_0(X) \cong H^1(k, \text{Pic}(X)) \) and \( \partial_A(\text{Br}_0(X)) = 0 \), one has
\[
h \partial_A(B) = 0 \in H^1(\bar{k}_A/\kappa_A, \mathbb{Q}/\mathbb{Z}).
\]
On the other hand we have
\[
2\partial_A(B) = 0 \in H^1(\bar{k}_A/\kappa_A, \mathbb{Q}/\mathbb{Z})
\]
by the above argument. Since \( (2, h) = 1 \), we have \( \partial_A(B) = 0 \in H^1(\bar{k}_A/\kappa_A, \mathbb{Q}/\mathbb{Z}) \). So \( B \in \text{Br}(X^c) \).

Case (2). We have \( \psi_B \) is of odd order. So \( b/u \) is odd, then \( 2 \mid (b/u - 1) \).

Case (3). Obviously \( 2 \mid d' = [L : L \cap K] \).

Case (4). If \( 2^s \mid [L \cap K : k] \), then \( [L : L \cap K] \) is even. We have \( 2 \mid d' \). So we only need to consider the case \( 2^s \mid [L \cap K : k] \).

Let \( 2^i \mid [L \cap K : L_g] \) and \( 2^{i+1} \mid [L \cap K : L_g] \). Since \( [(L \cap K), K_g : K_g] = [L \cap K : L_g] \), we have
\[
[K : (L \cap K), K_g] = [K : K_g]/[L \cap K : L_g].
\]
Since \((L \cap K).K_g \subset K \cap (L.K_g)\), we have \([K : K \cap (L.K_g)]\) is a factor of \([K : (L \cap K).K_g]\). Since \(K/K_g\) is cyclic and the cokernel of the multiplication by \(2^{s-1}\) on \(\text{Gal}(K/k)\) is of odd order, one has \(K/K \cap (L.K_g)\) is of degree \(2^a b\) with \(a \leq s - i - 1\) and \(b\) is odd. On the other hand \(L_g/k\) is of degree \(2^a b\) where \(a' \geq s - i\) and \(b'\) is odd. Then the kernel of the map

\[
\psi_B : \text{Gal}(L_g/k) \to H^1(\text{Gal}(K/K \cap (L.K_g)), \mathbb{Q}/\mathbb{Z})
\]

is of degree divided by 2. That is \(2 \mid \# \text{Ker}(\psi_B)\).

Case (5). If \(L' \not\subset L \cap K\), then \(2 \mid [L : L \cap K]\). We have \(2 \mid d'\). So only need to consider the case \(L' \subset L \cap K\).

Since \(K/K_g\) is cyclic, \(L \cap K/L_g\) is also cyclic. Then there is a subfield \(L''\) of \(L_g\) (which is also a subfield of \(L'\)) with \(\text{Gal}(L''/k) \cong \mathbb{Z}/2 \times \mathbb{Z}/2\). Then \(2 \mid \# \text{Ker}(\psi_B)\).

**Proposition 2.6.** Suppose \(\text{char}(k) = 0\) and \(H^3(k, \bar{k}^*) = 0\). Let \(n\) be an integer and let \(L/k\) be cyclic of order \(n\). Let \(K/k\) be an abelian extension with \(\text{Gal}(K/k) = \mathbb{Z}/n \times \mathbb{Z}/n\). Suppose \(L \subset K\). Let \(P(t)\) be an irreducible polynomial over \(k\) such that \(L \cong k[t]/(P(t))\). Let \(V\) be the smooth affine variety over \(k\) defined by \(N_K/k(\Xi) = P(t)\). Then

\[
\text{Br}(V^c)/\text{Br}_0(V^c) = \begin{cases} 
\mathbb{Z}/(n/2) & \text{if } n \text{ is even}, \\
\mathbb{Z}/n & \text{if } n \text{ is odd}.
\end{cases}
\]

**Proof.** Let \(X\) be the CHS partial compactification of \(V\) (see §1 for definition). We have \(H^1(k, \hat{T} \otimes \mathbb{Z}_p)/j_P. H^1(k, \hat{T}) = 0\) by Proposition 1.2. Since \(L \subset K\), we have

\[
\text{III}_2^2(k, \hat{T})_P = \text{III}_2^2(k, \hat{T}) \simeq H^3(\mathbb{Z}/n \times \mathbb{Z}/n, \mathbb{Z}) \simeq \mathbb{Z}/n,
\]

the last equation follows from the Künneth formula (p. 96 in [22]). By the basis sequence [21], we have

\[
H^1(k, \text{Pic}(X)) \simeq \text{III}_2^2(k, \hat{T})_P \simeq \mathbb{Z}/n.
\]

Since \(H^3(k, \bar{k}^*) = 0\), we have

\[
\text{Br}(X)/\text{Br}_0(X) \simeq H^1(k, \text{Pic}(X)) \simeq \mathbb{Z}/n.
\]

If \(n\) is odd, the result immediately follows from the case (a) in Theorem 2.5. Then we only need to consider the case \(n\) is even. Write \(n = 2^s n_0\) where
Let $T$ by Proposition 1.3 and Corollary 2.3, we have $f \in \text{Br}(X)/\text{Br}_0(X) \cong \text{H}^1(K/k, \text{Pic}(X_K))$ of order 2, and then the order of $f$ is of odd order. Since $\Theta = \text{Br}(X)$ is odd and $s \geq 1$. By Theorem 2.4, we can show that an element $A$ in $\text{Br}(X)/\text{Br}_0(X) \cong \text{H}^1(K/k, \text{Pic}(X_K))$ of order 2 is not contained in $\text{Br}(X)/\text{Br}_0(X)$.

Let $\langle g \rangle$ be a cyclic subgroup of $\text{Gal}(K/k)$. Let $K_g$ be the fixed field of $\langle g \rangle$ in $K$. We have the following morphism

$$f : \text{H}^1(K/k, \text{Pic}(X_K)) \to \prod_{\langle g \rangle} \text{H}^1(K/K_g, \text{Pic}(X_K)).$$

Let $T'$ be the $k$-torus defined by $N_{L/k}(\Xi_1) \cdot N_{K/k}(\Xi_2) = 1$. Since

$$\text{Ker}(f) = \Pi_{\omega}(K/k, \text{Pic}(X_K)) = \Pi_2(K/k, \widehat{T}') = 0$$

by Proposition 1.3 and Corollary 2.3, we have $f(A)$ is of order 2.

Let $f = \prod_{\langle g \rangle} f_g$ and $L_g = L \cap K_g$. Let $L'/k$ be the unique subfield of $L/k$ of degree 2. If $\langle g \rangle$ does not fix $L'$, then $2^s \nmid [L_g : k]$. By Lemma 2.4, the order of $f_g(A)$ is not divisible by 2. If the order of $\langle g \rangle$ is not divisible by 2, then the order of $f_g(A)$ is also not divisible by 2 by Lemma 2.4. However, there exists $\langle g_0 \rangle$ such that $f_{g_0}(A)$ is of order divisible by 2 since $f(A)$ is of order 2. By the above argument, we know $\langle g_0 \rangle$ fixes $L'$ and $2^s \mid \# \langle g_0 \rangle$.

Let $\Theta_0/k$ be the unique subfield of $K$ with $\text{Gal}(\Theta_0/k) = \mathbb{Z}/n_0 \times \mathbb{Z}/n_0$. Let $\Theta = K_{g_0}/\Theta_0$. Then we can see $\Theta = L'/\Theta_0$.

By the inflation-restriction sequence, we have the kernel of the map

$$\text{H}^1(K/K_{g_0}, \text{Pic}(X_K)) \to \text{H}^1(K/\Theta, \text{Pic}(X_K))$$

is of odd order. Since $f_g(A) \in \text{H}^1(K/K_{g_0}, \text{Pic}(X_K))$ is of order 2, we have $f_{g_0}(A) \in \text{H}^1(K/\Theta, \text{Pic}(X_{\Theta}))$ is also of order 2.

Assume there is $A \in \text{Br}(X)$ which lifts $A$. Let $f_{\Theta} : \text{Br}(X) \to \text{Br}(X_{\Theta})$ be the natural map. Then $f_{\Theta}$ is reduced by $f'_{\Theta}$. Since $f'_{\Theta}(\text{Br}(X_{\Theta})) \subset \text{Br}(X_{\Theta})$, we have $f'_{\Theta}(A) \in \text{Br}(X_{\Theta})$.

By Lemma 2.4, we have

$$f'_{\Theta}(A) = \rho + \sum_{\sigma \in \text{Gal}(L/k)} \psi_A(\sigma)(t - \sigma(\eta), \chi)$$

where $\rho \in \text{Br}(\Theta)$, $\eta$ is a root of $P(t)$ in $L$, $\psi_A \in \text{Hom}(\text{Gal}(L/k), \mathbb{Z}/2^s)$ and $\chi \in \text{Hom}(\text{Gal}(K/\Theta), \mathbb{Z}/2^s)$. Since $f_{g_0}(A)$ is of order 2, we deduce $\psi_A$ is of order divisible by 2 and $\chi$ is a primitive character.
The variety $X_{\Theta}$ contains an open affine $\Theta$-subvariety $U$ defined by

$$\prod_j N_{L_j/k}(\Xi_j) = P(t) \text{ and } t \neq 0.$$ 

And $L_j/k$ is of degree $2^s$. Let $Q(u) = u^n P(1/u)$. Let $W$ be the smooth affine $\Theta$-variety defined by

$$\prod_j N_{L_i/k}(\Xi'_j) = Q(u).$$

Obviously the open subvariety of $W$ defined by $u \neq 0$ is isomorphic to $U$ by the map $u \mapsto \frac{1}{t}, \Xi'_1 \mapsto \Xi_1/t^\mu, \Xi'_j \mapsto \Xi_j$ for $j \geq 2$, where $\mu = n/2^s$. Let $D$ be the divisor of $W/\Theta$ defined by $u = 0$. It is easy to see that the divisor $D$ is geometrically irreducible. So we have $\bar{\Theta} \cap \kappa_D = \Theta$, where $\kappa_D$ is the function field of $D$. The local ring $A_D$ associated with $D$ is a discrete valuation ring, $\text{ord}_{A_D}(t) = -1$ and $\kappa_D = \kappa_{A_D}$. So $\text{ord}_{A_D}(t - \sigma(\eta)) = -1$ for all $\sigma$. Therefore

$$\partial_{A_D}(f'_{\Theta} (A)) = - \sum_{\sigma \in \text{Gal}(L/k)} \psi_A(\sigma) \cdot \chi.$$

We know $\text{Gal}(L/k)$ is cyclic of order $n$, $\psi_A$ is of order $2^s$. We have

$$\sum_{\sigma} \psi_A(\sigma) = n_0 \cdot 2^{s-1}(2^s - 1) \equiv -n_0 \cdot 2^{s-1} \in \mathbb{Z}/2^s.$$

Since $\chi$ is a generator of $\text{Hom}(\text{Gal}(K/\Theta), \mathbb{Z}/2^s)$ and $\#\text{Gal}(K/\Theta) = \mathbb{Z}/2^s$, we have

$$\sum_{\sigma} \psi_A(\sigma) \cdot \chi = 2^{s-1} \cdot \chi \neq 0 \in \text{Hom}(\text{Gal}(K/\Theta), \mathbb{Q}/\mathbb{Z}).$$

Since $\bar{\Theta} \cap \kappa_{A_D} = \Theta$, we have

$$\partial_{A_D}(f'_{\Theta} (A)) \neq 0 \in H^1(\kappa_{A_D}, \mathbb{Q}/\mathbb{Z}).$$

On the other hand we have $f'_{\Theta} (A) \in \text{Br}(X_{\bar{\Theta}})$, then the residue

$$\partial_{A_D}(f'_{\Theta} (A)) = 0 \in H^1(\kappa_{A_D}, \mathbb{Q}/\mathbb{Z}).$$

A contradiction is derived. □

**Remark.** In the case $n = 2$, we get $\text{Br}(V^c) = \text{Br}_0(V^c)$, this answers the final question in [4, Questions on p. 82 and p. 83].
2.3 A case with $K/k$ an abelian field extension and $P(t)$ with multiple rational roots

The part of this section is to prove Proposition 2.8, which is motivated by the Question (a) in [4, p. 82]. Let $G$ be a finite group and $M$ a $G$-module. Define

$$\Pi^2_\omega(G, M) = \bigcap_{g \in G} \ker[H^2(G, M) \to H^2(\langle g \rangle, M)].$$

Lemma 2.7. Let $p$ be a prime and $G = \mathbb{Z}/p \times \mathbb{Z}/p$. Then

$$\Pi^2_\omega(G, \mathbb{Z}/p) = \begin{cases} 0 & \text{if } p = 2, \\ \mathbb{Z}/p & \text{if } p \text{ is odd.} \end{cases}$$

Proof. Let $R = \mathbb{Z}/p$. Let $H_1 = \langle g_1 \rangle, H_2 = \langle g_2 \rangle$ and $G = H_1 \times H_2$. Let $N = N_1 + N_2$ with $N_i = \sum_{\sigma \in G/H_i} \sigma$ for $i = 1, 2$. Denote

$$\Delta = (R[G/H_1] \oplus R[G/H_2])/(R \cdot N).$$

Then we have the following exact sequence

$$0 \to R \to R[G/H_1] \oplus R[G/H_2] \to \Delta \to 0.$$ 

Since the restriction map $H^1(G, \mathbb{Z}/p) \to H^1(H_1, \mathbb{Z}/p) \times H^1(H_2, \mathbb{Z}/p)$ is surjective, we have the following exact sequence

$$0 \to H^1(G, \Delta) \to H^2(G, \mathbb{Z}/p) \to H^2(H_1, \mathbb{Z}/p) \times H^2(H_2, \mathbb{Z}/p).$$

Let $H_3$ be a non-trivial cyclic subgroup of $G$ and $H_3 \neq H_1, H_2$. It is easy to verify that $R[G/H_i] \simeq R[H_3]$ as $H_3$-modules for $i = 1, 2$. Then

$$H^1(H_3, R[G/H_1] \times R[G/H_2]) = 0 \text{ if } i \geq 1.$$ 

Therefore we have the following commutative diagram

$$\begin{array}{ccc}
0 & \to & H^1(G, \Delta) \to H^2(G, \mathbb{Z}/p) \to H^2(H_1, \mathbb{Z}/p) \oplus H^2(H_2, \mathbb{Z}/p) \\
\downarrow & & \downarrow \\
0 & \to & H^1(H_3, \Delta) \to H^2(H_3, \mathbb{Z}/p) \to 0
\end{array}$$

Therefore we have

$$\Pi^2_\omega(G, \mathbb{Z}/p) \cong \bigcap_{H_3 \neq H_1, H_2} \ker[H^1(G, \Delta) \to H^1(H_3, \Delta)].$$

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By the inflation-restriction sequence, we have
\[ H^1(G/H_3, \Delta^{H_3}) = \text{Ker}[H^1(G, \Delta) \to H^1(H_3, \Delta)]. \]

Since \( G/H_3 \) is cyclic and \( \Delta \) is finite, one has
\[ |H^1(G/H_3, \Delta^{H_3})| = |\hat{H}^0(G/H_3, \Delta^{H_3})| = |\Delta^G / N_{G/H_3}(\Delta^{H_3})|. \]

It is easy to verify that \( \Delta^G \simeq \mathbb{Z}/p \) by the exact sequence
\[ 0 \to \mathbb{Z}/p \to R[G/H_1] \oplus R[G/H_2] \to \Delta \to 0. \]

Furthermore, we have
\[ 0 \to \mathbb{Z}/p \to \mathbb{Z}/p \oplus \mathbb{Z}/p \to \Delta^{H_3} \to H^1(H_3, \mathbb{Z}/p) \to 0. \]

So \( \Delta^{H_3} \) is generated by
\[ R \cdot N_1 \oplus R \cdot N_2 \text{ and } \{ \sum_{g \in H_3} \chi(g)(\bar{g}, \bar{g}) \mid \chi \in H^1(H_3, \mathbb{Z}/p)\}, \]

where \( (\bar{g}, \bar{g}) \in G/H_1 \times G/H_2 \). Since \( G/H_3 \) trivially acts on \( R \cdot N_1 \oplus R \cdot N_2 \), we have
\[ N_{G/H_3}(R \cdot N_1 \oplus R \cdot N_2) = p(R \cdot N_1 \oplus R \cdot N_2) = 0. \]

And \( H_1 \simeq G/H_3 \) since \( G = \mathbb{Z}/p \times \mathbb{Z}/p \) and \( p \) is a prime. Let
\[ u = \sum_{g \in H_3} \chi(g)(\bar{g}, \bar{g}) \in \Delta, \text{ where } \chi \text{ is a primitive character}. \]

Then
\[ \sigma(u) = \sum_{g \in H_3} \chi(g)(\bar{g}, \sigma \bar{g}) \text{ where } \sigma \in H_1. \]

Then we have
\[ N_{G/H_3}(u) = \sum_{\sigma \in H_1} \sum_{g \in H_3} \chi(g)(\bar{g}, \sigma \bar{g}) = \sum_{g \in H_3} \chi(g) \sum_{\sigma \in H_1} (\bar{g}, \sigma \bar{g}) \]
\[ = \sum_{g \in H_3} \chi(g)(0, \sum_{\sigma \in H_1} \sigma \bar{g}) = (0, \sum_{g \in H_3} \chi(g)N_2) \]
\[ = \begin{cases} (0, N_2) & \text{if } p = 2, \\ 0 & \text{if } p \text{ is odd.} \end{cases} \]
Therefore we have

\[
| H^1(G/H_3, \Delta) | = | \Delta^G/N_{G/H_3}(\Delta^H_3) | = \begin{cases} 
0 & \text{if } p = 2, \\
p & \text{if } p \text{ is odd}.
\end{cases}
\]

If \( p = 2 \), we have \( \text{III}_2^2(G, \mathbb{Z}/p) = 0 \).

By the Künneth formula, we have

\[
H^2(G, \mathbb{Z}/p) \simeq \bigoplus_{i+j=2} H^i(H_1, H^j(H_2, \mathbb{Z}/p)).
\]

Therefore

\[
H^1(G, \Delta) \simeq \text{Ker}[H^2(G, \mathbb{Z}/p) \to H^2(H_1, \mathbb{Z}/p) \oplus H^2(H_2, \mathbb{Z}/p)] \simeq \mathbb{Z}/p.
\]

Suppose \( p \) is odd. By the above argument,

\[
\text{Ker}[H^1(G, \Delta) \to H^1(H_3, \Delta)] \simeq H^1(G/H_3, \Delta^H_3) \simeq \mathbb{Z}/p.
\]

Therefore the restriction map \( H^1(G, \Delta) \to H^1(H_3, \Delta) \) is a zero map. So we have \( \text{III}_2^2(G, \mathbb{Z}/p) \simeq H^1(G, \Delta) \simeq \mathbb{Z}/p \). \( \square \)

**Proposition 2.8.** Suppose \( H^3(k, \bar{k}^*) = 0 \). Let \( P(t) = c \prod_{i=1}^m (t - e_i)^{d_i} \) with \( (d_1, \ldots, d_m) = d \). Let \( K/k \) be an abelian extension with \( \text{Gal}(K/k) = \mathbb{Z}/n \times \mathbb{Z}/n \). Suppose \( n \mid d \). Let \( V \) be the smooth locus of the affine \( k \)-variety defined by \( N_{K/k}(\Xi) = P(t) \). Let \( V^c \) be the smooth compactification of \( V \). Then

\[
\text{Br}(V^c)/\text{Br}_0(V^c) \cong \text{III}_2^2(K/k, \mathbb{Z}/d).
\]

**Proof.** Let \( X \) be the CHS partial compactification of \( V \) (see §1 for definition). Let \( \langle g \rangle \subset \text{Gal}(K/k) \) be a cyclic subgroup. Let \( K_g \) be the fixed field of \( \langle g \rangle \) in \( K \). We have the following map

\[
f : \text{Br}(X)/\text{Br}_0(X) \to \prod_{\langle g \rangle} \text{Br}(X_{K_g})/\text{Br}_0(X_{K_g}).
\]

Let \( f = \prod_{\langle g \rangle} f_g \). Since \( n \mid d \), the \( K_g \)-variety \( X_{K_g} \) is \( K_g \)-birationally isomorphic to \( \mathbb{A}^1 \times Y \), where \( Y \) is a variety defined by the equation \( c = N_{K/K_g}(\Xi) \). Then

\[
\text{Br}(X_{K_g}^c)/\text{Br}_0(X_{K_g}^c) \cong H^1(K_g, \text{Pic}(Y^c)) = 0
\]

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since $K/K_g$ is cyclic. If $A \in Br(X^c)/Br_0(X^c)$, we have $f_\varphi(A) \in Br(X_{K_g}^c)/Br_0(X_{K_g})$. Then $f_\varphi(A) = 0 \in Br(X_{K_g})/Br_0(X_{K_g})$. Therefore
\[ Br(X^c)/Br_0(X^c) \subset \text{Ker}(f). \]

Let $T'$ be the torus over $k$ defined by
\[ t_1^{d_1} \cdots t_m^{d_m} N_{K/k}(\Xi) = 1. \]

By Proposition 1.3, we know $\text{Ker}(f) \simeq \Sha^2(K/k, \hat{T'})$ is contained in $Br(X^c)/Br_0(X^c)$. Then we only need to show
\[ \Sha^2(K/k, \hat{T'}) \simeq \Sha^2(K/k, \mathbb{Z}/d). \]

Let $N = \sum_{\sigma \in \text{Gal}(K/k)} \sigma$. We know
\[ \hat{T'} = (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}[K/k])/\mathbb{Z} \cdot ((d_1, \cdots, d_m) + N). \]

Let
\[ M = (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z})/\mathbb{Z} \cdot (d_1, \cdots, d_m). \]

Obviously there is the following exact sequence
\[ 0 \to \mathbb{Z}[K/k] \to \hat{T'} \to M \to 0. \]

Then we have $\Sha^2(K/k, \hat{T'}) \simeq \Sha^2(K/k, M)$. And $M \cong \mathbb{Z}/d \times M'$ where $M'$ is a free $\mathbb{Z}$-module. We have $\Sha^2(K/k, M) \cong \Sha^2(K/k, \mathbb{Z}/d)$ since $\Sha^2(K/k, M') = 0$. \qed

By Lemma 2.7 and Proposition 2.8, we have the following result.

**Corollary 2.9.** Suppose $H^3(k, \bar{k}^*) = 0$. Let $p$ be a prime. Let $P(t) = c \prod_{i=1}^m (t-e_i)^p$. Let $K/k$ be an abelian extension with $\text{Gal}(K/k) = \mathbb{Z}/p \times \mathbb{Z}/p$. Let $V$ be the smooth locus of the affine $k$-variety defined by $N_{K/k}(\Xi) = P(t)$. Let $V^c$ be the smooth compactification of $V$. Then
\[ Br(V^c)/Br_0(V^c) = \begin{cases} 0 & \text{if } p = 2, \\ \mathbb{Z}/p & \text{if } p \text{ is odd.} \end{cases} \]

**Remark.** For $p = 2$, this corollary answers Question (a) in [4, p. 82].
3 Rational points under Schinzel’s hypothesis

Let \( k \) be a number field. Let \( \Omega_k \) be the set of all places of \( k \) and \( S_{\text{inf}} \subset \Omega_k \) all infinite places. Let \( P(t) \) be a polynomial over \( k \). In this section we mainly consider the question whether the Brauer-Manin obstruction is the only obstruction to the Hasse principle and weak approximation for a smooth and proper model of the \( k \)-variety defined by

\[
N_{K/k}(\Xi) = P(t)
\]

where \( \Xi \) is a variable in \( K \), \( K/k \) is a finite extension and \( N_{K/k} \) is a formal norm associated with the extension for the variable \( \Xi \).

A positive answer was given by Colliot-Thélène, Sansuc and Swinnerton-Dyer in their remarkable paper ([12]) when the degree of \( P(t) \) is 4 and \( K/k \) is a quadratic extension. A positive answer is also known when the extension \( K/k \) is of degree 3 and the polynomial \( P(t) \) is of degree at most 3 ([6]). For \( k = \mathbb{Q} \) and \( K/\mathbb{Q} \) arbitrary, and \( P(t) \) having just two roots, each of them rational, we also have a positive answer ([4],[16]).

Conditional results have been obtained under Schinzel’s hypothesis (H). Under this hypothesis, Colliot-Thélène, Skorobogatov and Swinnerton-Dyer ([13, Thm. 1.1 (e)]), proved that the Brauer-Manin obstruction is the only obstruction to the Hasse principle and weak approximation for smooth projective models of varieties defined by an equation \( N_{K/k}(\Xi) = P(t) \) when the extension \( K/k \) is abelian and norm equations \( N_{K/k}(\Xi) = c \) (for any \( c \in k^* \)) satisfy the Hasse principle and weak approximation, for instance when \( K/k \) is cyclic. Their result is more general, they consider smooth projective varieties \( X \) over \( k \) which admit a fibration over the projective line such that

(a) A certain abelianity condition on the splitting field of the singular fibres holds (Condition (i) in [13, Thm. 1.1]).

(b) The Hasse principle and weak approximation hold in the smooth fibres.

In this section we shall handle three new classes of fibrations over the projective line whose generic fibre is birationally a principal homogeneous space under a torus. In each of these classes one of conditions (a) or (b) is not in general fulfilled.

**Lemma 3.1.** Let \( k \) be a number field and \( K/k \) a finite field extension. Let \( P(t) \) be a polynomial over \( k \) and \( T \) the torus \( R^1_{K/k}(\mathbb{G}_m) \). Let \( Y \) be the smooth
projective variety over $k$ defined by the equation

$$N_{K/k}(\Xi) = P(t).$$

Suppose $S$ is a finite set of places of $k$ which contains all archimedean places $S_{\text{inf}}$ of $k$. For $v \in S$, let $P_v \in Y(k_v)$ and let $V_v$ be an open neighborhood of $P_v$ with respect to the $v$-adic topology. Then there is a smooth affine variety $W$ over $k$ defined by

$$N_{K/k}(\Xi) = P^{(1)}(t) \neq 0$$

which is $k$-isomorphic to an open subvariety of $Y$. Let $p : W \to \mathbb{A}^1 \subset \mathbb{P}^1$ be the projection by $(\Xi, t) \mapsto t$ and $\infty$ be the infinite point of $\mathbb{P}^1$. The affine variety $W$ has the following properties:

1. There exists $P'_v \in W(k_v)$ such that $P'_v \in V_v$ for each $v \in S \setminus S_{\text{inf}}$.
2. The fiber $W_{\infty}$ is smooth over $k$ and contains a $k_v$-point $P'_v \in V_v$ for all $v \in S_{\text{inf}}$.
3. If $\mathcal{III}_{\omega}(\tilde{T}) = \mathcal{III}_{\omega}(\tilde{T})$ (see §1 for definition), then $\mathcal{III}_{\omega}(\tilde{T})_{P(v)} = \mathcal{III}_{\omega}(\tilde{T})$.

Proof. Let $p : Y \to \mathbb{P}^1$ be the natural projection. Let $U_0 \subset \mathbb{A}^1_k$ be a non-empty Zariski open set defined by $P(t) \neq 0$. By implicit function theorem, since $Y$ is smooth and geometrically integral over $k$, any $v$-adic neighborhood of $P_v \in Y(k_v)$ is Zariski dense on $Y$. Therefore we can choose a point $\tilde{P}_v \in p^{-1}(U_0)(k_v)$ for each $v \in S$ such that $\tilde{P}_v \in V_v$. If $v \in S \setminus S_{\text{inf}}$, we let $P'_v = \tilde{P}_v$.

Let $\tilde{Q}_v = p(\tilde{P}_v) \in U_0(k_v) \subset \mathbb{A}^1(k_v)$. The fiber $Y_{\tilde{Q}_v}$ is smooth over $k_v$ for each $v \in S_{\text{inf}}$. Then we can choose a $k$-point $Q_0 \in U_0(k)$ which is different from all $\tilde{Q}_v$, close enough to each $\tilde{Q}_v$ for each $v \in S_{\text{inf}}$, such that the fiber $Y_{Q_0}$ contains a $k_v$-point $P'_v \in V_v$.

Let $t_0$ be the coordinate of $Q_0$. We choose a change of coordinate by $u = 1/(t - t_0)$. Let $R(u) = u^{\deg(P)}P(1/u + t_0)$ and $n = [K : k]$. Let $P^{(1)}(u) = u^{ln - \deg(P)}R(u)$. Then we get the new smooth affine variety $W$ defined by

$$N_{K/k}(\Xi) = P^{(1)} \neq 0,$$

where $l$ big enough such that $ln - \deg(P) \geq 0$. Obviously $W$ is $k$-isomorphic to the open subvariety of $p^{-1}(U_0)$ defined by $t \neq t_0$, which contains all $\tilde{P}_v$ for $v \in S$. Then $W$ contains all $P'_v$ for $v \in S \setminus S_{\text{inf}}$. And the fiber $W_{\infty} \cong Y_{Q_0}$. So $W_{\infty}$ is smooth over $k$ and contains a $k_v$-point $P'_v \in V_v$ for all $v \in S_{\text{inf}}$.  

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Write $P(t) = cp_1(t)^{e_1} \cdots p_m(t)^{e_m}$, where $p_i(t)$ is irreducible over $k$. Let $L_i = k[t]/(p_i(t))$ and $L_i^d$ the Galois closure of $L_i$ over $k$. By the assumption, we know the map

$$e_i \text{Res}_{L_i/k} : \text{III}^2_\omega(k/k, \hat{T}) \to \text{III}^2_\omega(k/L_i, \hat{T})$$

is zero. So

$$e_i[L_i : k] \text{III}^2_\omega(\hat{T}) = e_i \text{Cores}_{K_i/k} \circ \text{Res}_{K_i/k} \text{III}^2_\omega(\hat{T}) = 0.$$  

Then $e_i \text{deg}(L_i) \text{III}^2_\omega(K/k, \hat{T}) = 0$. Therefore $\text{deg}(P(t)) \text{III}^2_\omega(K/k, \hat{T}) = 0$. Obviously $n \text{III}^2_\omega(\hat{T}) = 0$. And we have $\text{III}^2_\omega(\hat{T})_Q = \text{III}^2_\omega(\hat{T})_P$ since $u \nmid Q(u)$. Then

$$\text{III}^2_\omega(\hat{T})_{P(t)} = \text{III}^2_\omega(\hat{T}).$$

\[
\square
\]

Let $k$ be a number field and $K/k$ a field extension. Let $P(t)$ be a polynomial over $k$. Let $X$ be the CHS partial compactification of the smooth locus of the affine variety over $k$ defined $N_{K/k}(\Xi) = P(t)$. Let $E/k$ be the minimal Galois extension which splits $P(t)$ and contains $K$ with the Galois group $G = \text{Gal}(E/k)$. Let $g$ be an element of $G$ and Let $E_g$ be the fixed field of $g$ in $E$. Let $f_g : \text{Br}(X) \to \text{Br}(X_{E_g})$ be the natural map.

**Lemma 3.2.** Assume $K/k$ is an abelian extension. For any $A \in \text{Br}(X)$, then $f_g(A) \in \text{Br}(X_{E_g})$ has the following form

$$\sum_{i,j} (q_i(t), \chi_j) + \delta_{B,g},$$

where $q_i(t) | P(t)$ is irreducible over $E_g$, $\delta_{B,g} \in \text{Br}(E_g)$ and $\chi_j$ is a character of $\text{Gal}(K.E_g/E_g)$.

**Proof.** We have the following commutative diagram:

$$H^1(E/k, \hat{T} \otimes Z_P)/j_P H^1(E/k, \hat{T}) \to H^1(E/k, \text{Pic}(X_E)) \to \text{III}^2_\omega(E/k, \hat{T})_P$$

$$H^1(E/E_g, \hat{T} \otimes Z_P)/j_P H^1(E/E_g, \hat{T}) \to H^1(E/E_g, \text{Pic}(X_E)) \to \text{III}^2_\omega(E/E_g, \hat{T})_P$$

Since $\text{Gal}(E/E_g)$ is cyclic, we have $\text{III}^2_\omega(E/E_g, \hat{T}) = 0$. Then

$$\text{Br}(X_{E_g})/\text{Br}_0(X_{E_g}) \cong H^1(E/E_g, \text{Pic}(X_E)) \cong H^1(E/E_g, \hat{T} \otimes Z_P)/j_P H^1(E/E_g, \hat{T}).$$
Write $P(t) = cq_1(t)^{e_1} \cdots q_r(t)^{e_r}$ where $q_i(t)$ is irreducible over $E_g$ for $1 \leq i \leq r$. Let $L_i = E_g[t]/(q_i(t))$ and $\xi_i$ the residue class of $t$ in $L_i$. By Shapiro’s lemma,

$$H^1(E/E_g, \hat{T} \otimes \mathbb{Z}_P) = \oplus_{i=1}^{r} H^1(E/E_g, \hat{T} \otimes \mathbb{Z}_{q_i}) = \oplus_{i=1}^{r} H^1(E/L_i, \hat{T}).$$

Let $K_g = K \cap E_g$ and $K_i = K \cap L_i$. By the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}[G] \to \hat{T} \to 0$,

we have $H^1(E/L_i, \hat{T}) = H^1(K/K_i, \hat{T}) = \text{Hom}(\text{Gal}(K/K_i), \mathbb{Q}/\mathbb{Z})$. By the remark on p. 76 of [4], we have $f_g(A)$ has the form

$$\sum_{i,j} \text{Core}_{L_i/E_g}(t - \xi_i, \hat{\chi}_j) + \delta_{A,g},$$

where $\delta_{A,g} \in \text{Br}(E_g)$ and $\hat{\chi}_j$ is a character of $\text{Gal}(K.L_i/L_i)$. Since $\text{Gal}(K/K_g)$ is abelian, then $\hat{\chi}_j = \text{Res}_{K_g/K_i}(\chi_j)$ where $\chi_j \in \text{Hom}(\text{Gal}(K/K_g), \mathbb{Q}/\mathbb{Z})$. Therefore $f_g(A)$ has the form

$$\sum_{i,j} (q_i(t), \chi_j) + \delta_{A,g}.$$

□

**Theorem 3.3.** Let $k$ be a number field and $K/k$ an abelian extension. Let $P(t)$ be a polynomial over $k$. Let $T = R^1_{K/k}(\mathbb{G}_m)$. Suppose $\text{III}_1^2(\hat{T})_P = \text{III}_2^2(\hat{T})$ (see §1 for definition). Assume Schinzel’s hypothesis holds. Then the Brauer-Manin obstruction to the Hasse principle and weak approximation for rational points is the only obstruction for any smooth proper model of the variety over $k$ defined by the equation

$$N_{K/k}(\Xi) = P(t).$$

**Proof.** It is sufficient to prove the statement for any given model. Let $V$ be the smooth locus of the affine $k$-variety defined by $N_{K/k}(\Xi) = P(t)$. Let $Y$ be a smooth compactification of $V$ with a projection $p : Y \to \mathbb{P}^1_k$ defined by $(\Xi, t) \mapsto t$.

We will prove both the Hasse principle part and the weak approximation part of the theorem at one stroke. We assume that $Y$ has points in all
completions of $k$, and we are given a finite set $S$ of places of $k$ which contains all archimedean places $S_{inf}$, and points $P_v \in Y(k_v)$ for $v \in S$. We assume that there is no Brauer-Manin obstruction to weak approximation for $(P_v)_{v \in S}$. This means that we may complete the family $(P_v)_{v \in S}$ to a family $(P_v)_{v \in \Omega}$ such that
\[
\forall \mathcal{A} \in \text{Br}(Y), \sum_{v \in \Omega_k} inv_v(\mathcal{A}(P_v)) = 0 \in \mathbb{Q}/\mathbb{Z}. \tag{4}
\]

From this, we want to deduce that there exists $P \in Y(k)$ as close as we wish to each $P_v \in Y(k_v)$ for $v \in S$.

Since $\text{Br}(Y)/\text{Br}(k)$ is finite and the pairing of $Y(k_v)$ with elements of the Brauer group is continuous, we can choose a neighborhood $V_v$ of $P_v$ for each $v \in S$ which is small enough such that the value of every $\mathcal{B} \in \text{Br}(Y)$ on $V_v$ is constant. By Lemma 3.11, we can find the smooth affine open subvariety $W$ of $Y$ defined by
\[
N_{K/k}(\Xi) = P^{(1)}(t) \neq 0,
\]
which satisfies the following properties:

1. There exists $P'_v \in W(k_v)$ such that $P'_v \in V_v$ for each $v \in S \setminus S_{inf}$.
2. The fiber $W_\infty$ (over the infinite point) is smooth over $k$ and contains a $k_v$-point $P'_v \in V_v$ for all $v \in S_{inf}$.
3. $\text{III}^2_\omega(\tilde{T})_{P^{(1)}} = \text{III}^2_\omega(\tilde{T})$.

Then we can replace all $P_v$ by $P'_v$ for all $v \in S$. There is also no Brauer-Manin obstruction to weak approximation for $(P_v)_{v \in S}$. Let $W_0$ be the open subvariety of $\mathbb{A}^1$ defined by $P^{(1)} \neq 0$. We are now looking for a point $Q \in W_0(k)$ with associated coordinate $\lambda \in k$, such that $\lambda$ is very close to each $Q_v$ for $v \in S$, $v$ finite, $\lambda$ is big enough at each of the archimedean completions of $k$ ($Q$ is close enough to the infinite point), and such that the fibre $W_Q$ has a $k$-rational point.

Let $E/k$ be the minimal Galois extension which splits $P^{(1)}(t)$ and contains $K$. Let $G = \text{Gal}(E/k)$. Let $g$ be an element of $G$. Let $E_g$ be the fixed field of $\langle g \rangle$ in $E$ and $K_g = K \cap E_g$. Let $X$ be the CHS partial compactification (see §1 for definition) of the affine $k$-variety defined by
\[
N_{K/k}(\Xi) = P^{(1)}(t).
\]
We know $H^1(k, \text{Pic}(X_E)) \to \Sha^2(E/k, \hat{T})$ is surjective. Then we can choose a finite subset $B \subset \text{Br}(X)$, such that the image of $B$ by the composite map

$$\text{Br}(X) \to H^1(E/k, \text{Pic}(X_E)) \to \Sha^2(E/k, \hat{T})$$

is $\Sha^2(E/k, \hat{T})$. By Lemma 3.2, the restriction of $B \in B$ to $X_{E_g}$ has the form

$$\sum_{i,j} (q_i(t), \chi_j) + \delta_{B,g},$$

where $q_i(t) | P^{(1)}(t)$ is irreducible over $E_g$, $\delta_{B,g} \in \text{Br}(E_g)$ and $\chi_j$ is a character of $\text{Gal}(K/K_g)$.

Let $\{p_i(t) | 1 \leq i \leq m\}$ be all irreducible terms of $P^{(1)}(t)$ over $k$. Let

$$A = \{(p_i(t), \chi) \in \text{Br}(W) | 1 \leq i \leq m, \chi \in \text{Hom}(\text{Gal}(K/k), \mathbb{Q}/\mathbb{Z})\}.$$
(ii) For each irreducible term \( q_i(t) \) of the polynomial \( P^{(1)}(t) \), there exists a place \( v_i \) such that \( p_i(\lambda) \) is a uniformizer at \( v_i \) and is a unit in \( k_v \) if \( v \not\in S_1 \) and \( v \neq v_i \).

For each \( A \in A \cup B \), we have

\[
0 = \sum_{v \in S_1} \text{inv}_v(A(P_v)) = \sum_{v \in S_1} \text{inv}_v(A(P'_v)).
\]

In particular, for each \( A = (p_i(\lambda), \chi) \in A \), we have

\[
\sum_{v \in S_1} \text{inv}_v((p_i(\lambda), \chi)) = 0.
\]

Since the sum of all local invariants \( \text{inv}_v \) of \( (p_i(\lambda), \chi) \) over \( k \) vanishes (global class field theory), we deduce

\[
\sum_{v \not\in S_1} \text{inv}_v((p_i(\lambda), \chi)) = 0.
\]

We have \( \text{inv}_v((p_i(\lambda), \chi)) = 0 \) for \( v \not\in S_1 \) and \( v \neq v_i \), since \( p_i(\lambda) \) is a unit at \( v \) by condition (ii) above. Then \( \text{inv}_v((p_i(\lambda), \chi)) = 0 \).

Since \( p_i(\lambda) \) is a uniformizer at \( v_i \) and \( \chi \) runs through all characters of \( \text{Gal}(K/k) \) and \( K/k \) is abelian, we know \( K/k \) is totally split at \( v_i \). Therefore the fibre \( W_\lambda \) contains a \( k_{v_i} \)-point \( P'_v \) for all places \( v \) of \( k \).

Let \( v \not\in S_1 \) and \( v \neq v_i \) for \( 1 \leq i \leq m \). Let \( \langle g \rangle = \text{Gal}(E_w/k_v) \subset \text{Gal}(E/F) \) where \( w \) is a place of \( E \) over \( v \). Let \( E_g \) be the fixed field of \( g \) in \( E \). We fix an embedding \( E_g \to k_v \).

Then we have

\[
\mathcal{B}(P'_v) = f_g(\mathcal{B})(P'_v) \in \text{Br}(k_v)
\]

where \( f_g \) is the natural restriction from \( \text{Br}(X) \) to \( \text{Br}(X_{E_g}) \). By Lemma 3.2, we know \( f_g(\mathcal{B}) \) has the form \( \sum_{i,j}(q_i(t), \chi_j) + \delta_{B,g} \) with \( q_i(t) | P^{(1)}(t) \). Since \( q_i(\lambda) \) has valuation 0 at \( v \), we have

\[
\text{inv}_v(\mathcal{B}(P'_v)) = 0 + \text{inv}_v(\delta_{B,g}) = 0.
\]

Let \( v \) be some \( v_i \). Let \( \langle g \rangle = \text{Gal}(E_w/k_v) \subset \text{Gal}(E/k) \). By the above argument, we know that \( K_w/k_v \) is totally split, then \( \langle g \rangle \) fixes \( K \). We know

\[
\mathcal{B}(P'_v) = f_g(\mathcal{B})(P'_v) \in \text{Br}(k_v),
\]

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and $f_g(B)$ has the form $\sum_{i,j}(q_i(t),\chi_j) + \delta_{B,g}$ where $\chi_j$ is a character of $\text{Gal}(K.E_g/E_g)$. Obviously $\chi_j = 0$. So

$$inv_v(B(P_v')) = 0 + inv_v(\delta_{B,g}) = 0.$$ 

Therefore we have

$$\sum_{v \in \Omega_k} inv_v(B(P_v')) = \sum_{v \in S_1} inv_v(B(P_v')) = 0.$$ 

The map $H^1(k, \text{Pic}(X)) \to \text{III}^2_\omega(\widehat{T})_{P(1)}$ in diagram (3) is reduced by the map $\text{Pic}(\bar{X}) \to \text{Pic}(X_\eta)$, where $\eta$ is the generic point of $\mathbb{P}^1$. We have the following commutative diagram

$$\begin{align*}
\text{Pic}(X_{k[t]_{(t-\lambda)}}) & \longrightarrow \text{Pic}(X_{k(t)}) \\
\text{Pic}(X_{\bar{k}[t]_{(t-\lambda)}}) & \longrightarrow \text{Pic}(X_{\lambda}).
\end{align*}$$

Therefore we have the following commutative diagram

$$\begin{align*}
\text{Pic}(\bar{X}) & \longrightarrow \text{Pic}(X_{k(t)}) \\
\text{Pic}(\bar{X}) & \longrightarrow \text{Pic}(X_{\lambda}).
\end{align*}$$

By Lemma 2.1([1]), we know the morphism $i : \text{Pic}(X_{k(t)}) \to \text{Pic}(X_\lambda)$ is an isomorphism as $\text{Gal}(\bar{k}/k)$-module and $\text{Gal}(k(t)/\bar{k}(t))$ acts trivially on $\text{Pic}(X_\eta)$. And $\text{Pic}(X_\eta)$ is torsion-free. Therefore we have the isomorphism

$$H^1(k(t), \text{Pic}(X_\eta)) \cong H^1(k, \text{Pic}(X_{k(t)}) \cong H^1(k, \text{Pic}(X_\lambda)).$$

Therefore we have the following commutative diagram

$$\begin{align*}
H^1(k, \text{Pic}(\bar{X})) & \longrightarrow H^1(k(t), \text{Pic}(X_\eta)) \\
H^1(k, \text{Pic}(\bar{X})) & \longrightarrow H^1(k, \text{Pic}(X_\lambda)).
\end{align*}$$

Then the image of $B$ by the induced map $\text{Br}(X)/\text{Br}(k) \to \text{Br}(X_\lambda)/\text{Br}(k)$ is surjective. Therefore there is no Brauer-Manin obstruction on $X_\lambda$ for $(P_v')_{v \in \Omega_k}$. Thus there is a $k$-point on $W_\lambda (= X_\lambda)$ which is close enough to $(P_v')_{v \in S}$ by the property of the principal homogeneous space of algebraic groups (Theorem 8.12 in [25] or Theorem 5.2.1 in [28]).
Remark. The condition $\prod_2^2(\hat{T})_P = \prod_2^2(\hat{T})$ is equivalent to the condition that the natural morphism

$$\text{Br}(X)/\text{Br}(k) \rightarrow \text{Br}(X_\eta)/\text{Br}(k(\eta))$$

is surjective, where $X$ is the CHS partial compactification with the projection $p : X \rightarrow \mathbb{A}^1$, and $\eta$ is the generic point of $\mathbb{A}^1$.

As a direct application of this theorem, we have the following corollary.

Corollary 3.4. Let $n$ be a positive integer and let $K/k$ be an abelian extension with $\text{Gal}(K/k) = \mathbb{Z}/n \times \mathbb{Z}/n$. Let $P(t)$ be an irreducible polynomial over $k$ and $L = k[t]/(P(t))$. Assume that $L$ contains a cyclic subfield of $K$ with degree $n$. Let $V$ be the smooth affine variety over $k$ defined by

$$N_{K/k}(\Xi) = P(t).$$

Assume Schinzel’s hypothesis holds, then the Brauer-Manin obstruction to the Hasse principle and weak approximation is the only obstruction for any smooth projective model of $V$.

Proof. Let $L = k[t]/(P(t))$ and $T$ the torus $R^1_{K/k}(\mathbb{G}_m)$. Then $\text{Gal}(K/L \cap K)$ is cyclic. By the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[\text{Gal}(K/k)] \rightarrow \hat{T} \rightarrow 0,$$

we have

$$\prod_2^2(\hat{T}) = H^3(K/k, \mathbb{Z}) \text{ and } \prod_2^2(\hat{T})_P = \text{Ker}[H^3(K/k, \mathbb{Z}) \rightarrow H^3(K/L \cap K, \mathbb{Z})].$$

Since $K/L \cap K$ is cyclic, we have $H^3(K/L \cap K, \mathbb{Z}) = H^1(K/L \cap K, \mathbb{Z}) = 0$. Then

$$\prod_2^2(\hat{T})_P = \prod_2^2(\hat{T}).$$

□

In the following theorem the condition $\prod_2^2(\hat{T})_P = \prod_2^2(\hat{T})$ is not in general fulfilled.

Theorem 3.5. Let $k$ be a number field and $P(t)$ a polynomial over $k$. Assume Schinzel’s hypothesis holds. Then the Brauer-Manin obstruction to the Hasse principle and weak approximation for rational points is the only obstruction for any smooth proper model of the variety over $k$ defined by the equation

$$(x_1^2 - ax_2^2)(y_1^2 - by_2^2)(z_1^2 - abz_2^2) = P(t),$$

where $a, b \in k^*$. 

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Proof. It is sufficient to prove the statement for any given model. Let $V$ be the smooth locus of the affine $k$-variety defined by

$$(x_1^2 - ax_2^2)(y_1^2 - by_2^2)(z_1^2 - abz_2^2) = P(t).$$

Let $Y$ be a smooth compactification of $V$ with a projection $p : Y \rightarrow \mathbb{P}^1$. If one of the three numbers $a, b, ab$ is a square in $k^*$, this theorem is obvious. Then we only need to consider the case all numbers $a, b, ab$ are not contained in $k^*$.

We assume that $Y$ has points in all completions of $k$, and we are given a finite set $S \supset S_{inf}$ of places of $k$, and points $P_v \in Y(k_v)$ for $v \in S$. We assume that there is no Brauer-Manin obstruction to weak approximation for $(P_v)_{v \in S}$. Since $\text{Br}(Y)/\text{Br}(k)$ is finite and the pairing of $Y(k_v)$ with elements of the Brauer group is continuous, we can choose a neighborhood $V_v$ of $P_v$ for each $v \in S$ which is small enough such that the value of every $B \in \text{Br}(Y)$ on $V_v$ is constant. With a similar argument as Lemma 3.1 we can find an open smooth affine subvariety $U$ of $Y$ defined by

$$(x_1^2 - ax_2^2)(y_1^2 - by_2^2)(z_1^2 - abz_2^2) = P(1)(t)$$

which satisfies the following properties:

1. There exists $P'_v \in U(k_v)$ such that $P'_v \in V_v$ for each $v \in S \setminus S_{inf}$.

2. The fiber $U_\infty$ is smooth over $k$ and contains a $k_v$-point $P'_v \in V_v$ for all $v \in S_{inf}$.

Then we can replace all $P_v$ by $P'_v$ for all $v \in S$. There is also no Brauer-Manin obstruction to weak approximation for $(P_v)_{v \in S}$. Let $U_0$ be the open subvariety of $\mathbb{A}^1$ defined by $P^{(1)}(t) \neq 0$. We are now looking for a point $Q \in U_0(k)$ with associated coordinate $\lambda \in k$, such that $\lambda$ is very close to each $Q_v$ for $v \in S$, $v$ finite, $\lambda$ is big enough at each of the archimedean completions of $k$ (i.e., $Q$ is close enough to the infinite point), and such that the fibre $Y_\lambda$ has a $k$-rational point.

Let $P^{(1)}(t) = cp_1(t)^{e_1} \cdots p_m(t)^{e_m}$, where $p_i(t)$ is irreducible over $k$. Let

$$A = \{(p_i(t), b) \in \text{Br}(U) \mid 1 \leq i \leq m\} \cup \{(x_1^2 - ax_2^2, b)\}.$$

We know $(x_1^2 - ax_2^2, b)$ is the unique generator of the Brauer group of the smooth fibre of $Y$ (see [3]). Obviously $A \subset \text{Br}(U)$. We enlarge $S$ so that it
contains all the original places at which we want to approximate and that it also contains the places associated in Hypothesis $H_1$ ([14, p. 71]) to the polynomials of $P^{(1)}(t)$ and all ramified places of $K/k$, where $K = k(\sqrt{a}, \sqrt{b})$.

There is no Brauer-Manin obstruction to weak approximation for $(P_v)_{v \in S}$. According to Harari’s formal lemma (see [15]), we may find a finite set $S_1$ of places of $k$, containing $S$, and points $P_v \in U(k_v)$, $v \in S_1$, and which extend the given family

$$P_v \in U(k_v), \; v \in S,$$

such that for each $\mathcal{A} \in A$

$$\sum_{v \in S_1} \text{inv}_v(\mathcal{A}(P_v)) = 0.$$

Applying Hypothesis $(H_1)$ ([14, p. 72, Proposition 4.1]), we thus find $\lambda \in k$ close enough to each $\lambda_v = p(P_v)$ for the finite places $v \in S_1$, $\lambda$ integral away from $S_1$, and $\lambda$ as large as we wanted at the archimedean places, such that:

(i) The fibre $Y_\lambda$ of $p$ contains a $k_v$-point $P'_v$ which is as close as we wish to $P_v$ for all places $v \in S_1$, and such that

$$\text{inv}_v(\mathcal{A}(P'_v)) = \text{inv}_v(\mathcal{A}(P_v))$$

for each $\mathcal{A} \in A$ and $v \in S_1$.

(ii) For each irreducible term $p_i(t)$ of the polynomial $P^{(1)}(t)$, there exists a place $v_i$ such that $p_i(\lambda)$ is a uniformizer at $v_i$ and is a unit in $k_v$ if $v \not\in S_1$ and $v \neq v_i$.

If $v \not\in S_1$, then $K/k$ is unramified at $v$. Then one of $a, b, ab$ is a square in $k_v^*$. Then the fibre $Y_\lambda$ of $p$ contains a $k_v$-point $P'_v$ for all places $v \not\in S_1$.

For each $\mathcal{A} \in A$, we have

$$0 = \sum_{v \in S_1} \text{inv}_v(\mathcal{A}(P_v)) = \sum_{v \in S_1} \text{inv}_v(\mathcal{A}(P'_v)).$$

In particular, for each $\mathcal{A} = (p_i(t), b) \in A$, we have

$$\sum_{v \in S_1} \text{inv}_v((p_i(\lambda), b)) = 0.$$
By the global class field theory, we have
\[ \sum_{v \notin S_1} inv_v((p_i(\lambda), b)) = 0. \]

With condition (ii) above, we have \( inv_v((p_i(\lambda), b)) = 0 \) for \( v \notin S_1 \) and \( v \neq v_i \), since \( p_i(\lambda) \) is a unit at \( v \). Then \( inv_{v_i}((p_i(\lambda), b)) = 0 \). Since \( p_i(\lambda) \) is a uniformizer at \( v_i \), we have \( k(\sqrt{b})/k \) is totally split at \( v_i \).

Let \( B = (x_1^2 - ax_2^2, b) \). Let \( v \notin S_1 \) and \( v \neq v_i \) for \( 1 \leq i \leq m \). If \( k(\sqrt{b})/k \) is split at \( v \), then \( inv_v(B(P')_v) = 0 \).

Let \( v \) be some \( v_i \). Since \( b \) is a square in \( k_v^* \), we have \( inv_{v_i}(B(P'_{v_i})) = 0 \). Therefore

One has
\[ inv_v(B(P'_{v_i})) = (x_1^2 - ax_2^2, b)_v = (P^{(1)}(\lambda), b)_v = 0 \]
since \( P^{(1)}(\lambda) \) is a unit at \( v \). Therefore we have \( inv_v(B(P'_{v_i})) = 0 \).

Let \( v \) be some \( v_i \). Since \( b \) is a square in \( k_v^* \), we have \( inv_{v_i}(B(P'_{v_i})) = 0 \). Therefore
\[ \sum_{v \in \Omega_k} inv_v(B(P'_{v_i})) = \sum_{v \in S_1} inv_v(B(P'_{v_i})) = 0. \]

Note that \( B \) generates the Brauer group \( Br(Y_\lambda)/Br(k) \), there is no Brauer-Manin obstruction on \( Y_\lambda \) for \( (P'_{v_i})_{v \in \Omega_k} \). There is a \( k \)-point on \( Y_\lambda \) which is close enough to \( (P'_{v_i})_{v \in S} \) by Theorem 8.12 in [25]. □

In the above we considered the case that the extension \( K/k \) is abelian. In the following we will consider the case that \( K/k \) is non-abelian. The original idea owes to Colliot-Thélène and the further development owes to Wittenberg.

**Theorem 3.6.** Let \( k \) be a number field and \( P(t) \) a polynomial over \( k \). Let \( K/k \) be of degree 3 and non-cyclic. Assume Schinzel’s hypothesis holds. Then the Brauer-Manin obstruction to the Hasse principle and weak approximation for rational points is the only obstruction for any smooth proper model of the variety over \( k \) defined by the equation
\[ N_{K/k}(\Xi) = P(t). \]
Proof. Let $V$ be the smooth locus of the affine $k$-variety defined by $N_{K/k}(\Xi) = P(t)$. Let $Y$ be a smooth compactification of $V$ with a projection $p : Y \to \mathbb{P}^1_k$ defined by $(\Xi, t) \mapsto t$.

Let $p(t)^e | P(t)$ and $p(t)^{e+1} \nmid P(t)$ with $e > 0$ and $p(t)$ monic and irreducible over $k$. Let $L = k[t]/(p(t))$. Assume there is an embedding $K \hookrightarrow L$. Let $\xi$ be a root of $p(t)$ in $L$ and let $\hat{Y}$ be a smooth proper $k$-variety defined by $N_{K/k}(\Xi) = P(t)/p(t)^e$.

(i) Suppose $P(t)/p(t)^e = c \in k^\times$. Then there is a birational isomorphism

$$Y \to \hat{Y} \times \mathbb{P}^1_k, \ (\Xi, t) \mapsto (\Xi \cdot N_{L/K}(t - \xi)^e, t).$$

Then $\hat{Y}$ is a principal homogeneous space of a torus. Therefore the Brauer-Manin obstruction to the Hasse principle and weak approximation is the only obstruction for $Y$ by Theorem 5.2.1 in [28].

(ii) Suppose $P(t)/p(t)^e \notin k^\times$. Then there is a birational isomorphism

$$Y \to \hat{Y}, \ (\Xi, t) \mapsto (\Xi \cdot N_{L/K}(t - \xi)^e, t).$$

Applying the above argument inductively, we can assume that there does not exist an embedding $K \hookrightarrow L$, where $L = k[t]/(p(t))$ and $p(t)$ is any irreducible factor of $P(t)$.

We assume that $Y$ has points in all completions of $k$, and we are given a finite set $S$ of places of $k$, and points $P_v \in Y(k_v)$ for $v \in S$. We assume that there is no Brauer-Manin obstruction to weak approximation for $(P_v)_{v \in S}$. Since $Br(Y)/Br(k)$ is finite and the pairing of $Y(k_v)$ with elements of the Brauer group is continuous, we can choose a neighbourhood $V_v$ of $P_v$ for each $v \in S$ which is small enough such that the value of every $B \in Br(Y)$ on $V_v$ is constant. By Lemma [3.1] we can find an open smooth affine subvariety $U$ of $Y$ defined by

$$N_{K/k}(\Xi) = P^{(1)}(t) \text{ and } P^{(1)}(t) \neq 0,$$

which satisfies the following properties:

(1) There exists $P'_v \in U(k_v)$ such that $P'_v \in V_v$ for each $v \in S \setminus S_{\text{inf}}$.

(2) The fiber $U_\infty$ is smooth over $k$ and contains a $k_v$-point $P'_v \in V_v$ for all $v \in S_{\text{inf}}$.

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Then we can replace all $P_v$ by $P'_v$ for all $v \in S$. There is also no Brauer-Manin obstruction to weak approximation for $(P_v)_{v \in S}$. Let $U_0$ be the open subvariety of $\mathbb{A}^1$ defined by $P^{(1)} \neq 0$. We are now looking for a point $Q \in U_0(k)$ with associated coordinate $\lambda \in k$, such that $\lambda$ is very close to each $Q_v$ for $v \in S$, $v$ finite, $\lambda$ is big enough at each of the archimedean completions of $k$ ($Q$ is close enough to the infinite point), and such that the fibre $U_\lambda$ has a $k$-rational point.

Let $P^{(1)} = cp_1(t)^{e_1} \cdots p_m(t)^{e_m}$, where $p_i(t)$ is monic and irreducible over $k$. Let $k(\sqrt{d})$ be the unique quadratic field contained in the Galois closure $K^{cl}$ of $K/k$. Let $L_i = k[t]/(p_i(t))$. Let $T_1 = \{1 \leq i \leq m \mid k(\sqrt{d}) \not\subset L_i\}$ and $T_2 = \{1 \leq i \leq m \mid k(\sqrt{d}) \subset L_i\}$. For $i \in T_2$, $p_i(t) = l_i(t)l_i(t)^\sigma$, where $l_i(t) \in k(\sqrt{d})[t]$ is irreducible and $\sigma$ is the generator of Gal($k(\sqrt{d})/k$). Denote

$$A_1 = \{(p_i(t), d) \in \text{Br}(U) \mid i \in T_1\}$$

and

$$A_2 = \{\text{Cores}_{k(\sqrt{d})/k}(l_i(t), \chi) \in \text{Br}(U) \mid i \in T_2\},$$

where $\chi$ is a fixed primitive character of Gal($K^{cl}/k(\sqrt{d})$).

We enlarge $S$ so that it contains all the original places at which we want to approximate and that it also contains the places associated in Hypothesis $H_1$ ([$14$, p. 71]) to the polynomials of $P^{(1)}(t)$ and all ramified places of $K^{cl}/k$. There is no Brauer-Manin obstruction to weak approximation for $(P_v)_{v \in S}$. According to Harari’s formal lemma (see [15]), we may find a finite set $S_1$ of places of $k$, containing $S$, and points $P_v \in U(k_v)$, $v \in S_1$, which extend the given family

$$P_v \in U(k_v), v \in S,$$

such that for each $A \in A_1 \cup A_2$

$$\sum_{v \in S_1} inv_v(A(P_v)) = 0.$$
If \( i \in T_1 \), we know \( k(\sqrt{d}) \not\subset L_i \). Let \( E_i = L_i^{cl}K^{cl} \) and \( F_i = L_i.K \). We claim that there exists \( g \in \text{Gal}(E_i/F_i) \) such that \( g \) does not fix \( F_i(\sqrt{d}) \). Otherwise \( F_i = F_i(\sqrt{d}) \). Then \( K^{cl} \subset F_i \). Therefore \( L_i.K^{cl} = L_i.K \). So we have \( L_i \cap K^{cl} \neq k \). Since \( k(\sqrt{d}) \) is the unique quadratic subfield of \( K^{cl} \) and \( k(\sqrt{d}) \not\subset L_i \), we have \( [L_i \cap K^{cl} : k] = 3 \). All 2-Sylow subgroup of \( \text{Gal}(K^{cl}/k) = S_3 \) are conjugate, there is an embedding \( K \hookrightarrow L_i \) by Galois theory. A contradiction is derived to the assumption \( K \not\hookrightarrow L_i \).

There are infinitely many places \( v \) of \( k \), such that \( g \) is contained in the conjugation class of the Frobenius of \( v \) by Chebotarev’s density theorem. We can choose such a place \( v_i \) and \( v_i \not\in S' \). Therefore \( p_i(t) = \pi_{v_i} \) has a solution \( \lambda_{v_i} \in \mathfrak{o}_{k_{v_i}} \) and \( p_j(\lambda_{v_i}) \in \mathfrak{o}_{k_{v_i}}^{\times} \) for \( j \neq i \), where \( \pi_{v_i} \) is a uniformizer of \( k_{v_i} \). And obviously \( U_{\lambda_{v_i}} \) has a \( k_{v_i} \)-point \( P_{v_i} \) since \( K \) can embed into \( k_{v_i} \). Therefore we have

\[
\text{inv}_{v_i}((p_i(\lambda_{v_i}), d)) \neq 0 \quad \text{and} \quad \text{inv}_{v_i}(\mathcal{A}(P_{v_i})) = 0
\]

for \( \mathcal{A} \in A_1 \) and \( \mathcal{A} \neq (p_i(t), d) \). Suppose \( \mathcal{A} \in A_2 \). Since \( p_j(\lambda_{v_i}) \) is a unit, we have \( \text{inv}_{v_i}(\mathcal{A}(P_{v_i})) = 0 \). We replace \( S_1 \) by \( S_1 \cup \{ v_i \mid i \in T_1 \} \). Then we have

\[
\sum_{v \in S_1} \text{inv}_v(\mathcal{A}(P_v)) \neq 0 \quad \text{for} \quad \mathcal{A} \in A_1
\]

and

\[
\sum_{v \in S_1} \text{inv}_v(\mathcal{A}(P_v)) = 0 \quad \text{for} \quad \mathcal{A} \in A_2. \tag{5}
\]

Now apply Hypothesis \((H_1) \) \cite[p. 72, Proposition 4.1]{14}, we thus find \( \lambda \in k \) close enough to each \( \lambda_v = p(P_v) \) for the finite places \( v \in S_1 \), \( \lambda \) integral away from \( S_1 \), and \( \lambda \) as large as we wanted at the archimedean places, such that:

(i) The fibre \( Y_\lambda \) of \( p \) contains a \( k_v \)-point \( P'_v \) which is as close as we wish to \( P_v \) for all places \( v \in S_1 \), and such that

\[
\text{inv}_v(\mathcal{A}(P'_v)) = \text{inv}_v(\mathcal{A}(P_v))
\]

for each \( \mathcal{A} \in A_1 \cup A_2 \) and \( v \in S_1 \).

(ii) For each irreducible term \( p_i(t) \) of the polynomial \( P^{(1)}(t) \), there exists a place \( v'_i \) such that \( p_i(\lambda) \) is a uniformizer at \( v'_i \) and is a unit in \( k_v \) if \( v \not\in S_1 \) and \( v \neq v'_i \).
For each $A \in A_1$, we have

$$0 \neq \sum_{v \in S_1} \text{inv}_v(A(P_v)) = \sum_{v \in S_1} \text{inv}_v(A(P'_v)).$$

Then we have

$$\sum_{v \in S_1} \text{inv}_v((p_i(\lambda), d)) \neq 0 \text{ for } i \in T_1.$$ 

By the global class field theory, we have

$$\sum_{v \not\in S_1} \text{inv}_v((p_i(\lambda), d)) \neq 0.$$ 

With condition (ii) above, we have $\text{inv}_v((p_i(\lambda), d)) = 0$ for $v \not\in S_1$ and $v \neq v'_i$, since $p_i(\lambda)$ is a unit in $k_v$. Then

$$\text{inv}_{v'_i}(p_i(\lambda), d)) \neq 0.$$ 

Since $p_i(\lambda)$ is a uniformizer at $v'_i$, we have $k(\sqrt{d})/k$ is inert at $v'_i$. The Galois group $\text{Gal}(K^d_{w'_i}/k_{v'_i}) \subset \text{Gal}(K^d/k) = S_3$ is cyclic and of degree divided by 2, where $w'_i$ is a place of $K^d$ over $v'_i$. Then we have $\text{Gal}(K^d_{w'_i}/k_{v'_i}) \cong \mathbb{Z}/2$. Therefore $K$ can embed into $k_{v'_i}$. Then $Y_\lambda$ has a $k_{v'_i}$-point for $i \in T_1$.

Let $i \in T_2$. By equation (5) and class field theory, we have

$$0 = \sum_{v \not\in S_1} \text{inv}_v(\text{Cores}_{k(\sqrt{d})/k}(l_i(\lambda), \chi)) = \text{inv}_{v'_i}(\text{Cores}_{k(\sqrt{d})/k}(l_i(\lambda), \chi)).$$ 

Since

$$\text{ord}_{v'_i}(p_i(\lambda)) = \text{ord}_{v'_i}(l_i(\lambda)l_i(\lambda)^{\sigma}) = 1,$$

we have $\chi$ is trivial. Then $Y_\lambda$ has a $k_{v'_i}$-point for $i \in T_2$. Since $\text{Br}(Y_\lambda) = \text{Br}_0(Y_\lambda)$ (Proposition 9.1 in [10]), there is a $k$-point on $Y_\lambda$ which is close enough to $(P'_v)_{v \in S}$ by Theorem 8.12 in [25].

4 Brauer-Manin properties for zero-cycles of degree 1

This section is devoted to the proof, for zero-cycles of degree 1, of unconditional versions of the theorems in [13] using Salberger’s device, as in [13].
The same general comments as made in the beginning of the previous section may be made here. In particular most of the results in the present section are not covered by [13, Theorem 4.1].

**Theorem 4.1.** Let $k$ be a number field and $K/k$ an abelian extension. Let $P(t)$ be a polynomial over $k$. Let $V$ be the smooth locus of the affine $k$-variety defined by

$$N_{K/k}(\Xi) = P(t).$$

Let $T = \mathbb{R}^1_{K/k}(\mathbb{G}_m)$. Suppose $\mathbb{III}^2_\omega(\hat{T})_P = \mathbb{III}^2_\omega(\hat{T})$. If there is no Brauer-Manin obstruction to the existence of a zero-cycle of degree 1 on a smooth proper model $V^c$ of $V$, then there is a zero-cycle of degree 1 on $V^c$ (defined over $k$).

**Proof.** Let $U$ be the smooth affine variety over $k$ defined by

$$N_{K/k}(\Xi) = P(t) \neq 0.$$ 

Let $U^c$ be a smooth compactification of $U$ with a projection $p : U^c \to \mathbb{P}^1$.

Let $E/k$ be the minimal Galois extension which splits $P(t)$ and contains $K$. Let $G = \text{Gal}(E/k)$. Let $g$ be an element of $G$ and let $E_g$ be the fixed field of $\langle g \rangle$ in $E$ and $K_g = K \cap E_g$. By the assumption, we know $\mathbb{III}^2_\omega(E/k, \hat{T})_P = \mathbb{III}^2_\omega(E/k, \hat{T})$. Then we can choose a finite subset $B \subset \text{Br}(X)$, such that the image of $B$ by the composite map

$$\text{Br}(X) \to H^1(E/k, \text{Pic}(X_E)) \to \mathbb{III}^2_\omega(E/k, \hat{T})$$

is $\mathbb{III}^2_\omega(E/k, \hat{T})$. By Lemma 3.2, the restriction of $B \in B$ to $X_{E_g}$ has the form

$$\sum_{i,j}(q_i(t), \chi_j) + \delta_{B,g},$$

where $\delta_{B,g} \in \text{Br}(E_g)$, $q_i(t) \mid P^{(1)}(t)$ is a polynomial over $E_g$ and $\chi_j$ is a character of $\text{Gal}(K/K_g)$.

Let $P(t) = cp_1(t)^{e_1} \cdots p_m(t)^{e_m}$, where $p_i(t)$ is an irreducible polynomial over $k$. Let $S_0$ be a finite set of places of $k$ containing all the archimedean places and all the bad finite places in sight: finite places where one $p_i(t)$ is not integral, finite places where all $p_i(t)$ are integral but the product $\prod_i p_i(t)$ does not remain separable when reduced modulo $v$, places ramified in the extension $K/k$ and all places $v$ such that some $\text{inv}_w(\delta_{B,g}) \neq 0$ where $w$ is a
place of \( E \) over \( v \). Let \( N_0 \) be a closed point of \( U \) such that \( k(N_0) = K \). Let 
\[ d = [K : k]. \]

Let 
\[ A = \{(p_i(t), \chi) \in \text{Br}(U) \mid 1 \leq i \leq m, \chi \in \text{Hom}(\text{Gal}(K/k), \mathbb{Q}/\mathbb{Z})\}. \]

Since we assume that there is no Brauer-Manin obstruction to the existence of 
a zero-cycle of degree 1, by an obvious variant of Harari’s result ([14, Theorem 3.2.2]) we may find a finite set \( S_1 \) of places of \( k \) containing \( S_0 \) and for each 
\( v \in S_1 \) a zero-cycle \( z_v \) of degree 1 with support in \( U \times_k k_v \) such that

\[ \sum_{v \in S_1} \text{inv}_v(\langle A, z_v \rangle) = 0 \text{ for all } A \in A \cup B. \quad (6) \]

Let \( s \) be the least common multiple of the orders of \( A \in A \cup B \). Let us write the zero-cycle \( z_v \) as \( z_v^+ + z_v^- \), with \( z_v^+ \) and \( z_v^- \) effective cycles. Let \( z_v^1 \) be the effective cycle \( z_v^+ + (ds - 1)z_v^- \). We have \( z_v = z_v^1 - dsz_v^- \), hence 
\( \langle A, z_v \rangle = \langle A, z_v^1 \rangle \) since each \( A \) is killed by \( s \). We thus have

\[ \sum_{v \in S_1} \text{inv}_v(\langle A, z_v^1 \rangle) = 0 \text{ for all } A \in A \cup B. \quad (7) \]

Similarly \( \langle A, sN_0 \rangle = 0 \). The degree of \( z_v^1 \) is congruent to 1 modulo \( ds \).
The cycle \( sN_0 \) has degree \( ds \). Adding suitable multiples of \( sN_0 \) to each \( z_v^1 \) for 
\( v \) in the finite set \( S_1 \), we then find effective cycles \( z_v^2 \), all of the same degree 
\( 1 + Dsd \) for some \( D > 0 \), and such that

\[ \sum_{v \in S_1} \text{inv}_v(\langle A, z_v^2 \rangle) = 0 \text{ for all } A \in A \cup B. \quad (8) \]

We claim that in (3), for each \( v \in S_1 \), each effective cycle \( z_v^2 \) may be assumed to be a sum of distinct closed points (i.e. there are no multiplicities) whose 
images under \( p_{k_v} : U_{k_v} \to \mathbb{A}^1_{k_v} \) are also distinct. Indeed, if \( P \) is a closed 
point of \( U_{k_v} \), with residue field \( F = k_v(P) \), and \( A \) is a class in \( \text{Br}(U) \), the 
map \( U(F) \to \text{Br}(F) \subset \mathbb{Q}/\mathbb{Z} \) given by evaluation of \( A \) is continuous. Since 
\( U \) is smooth, the point \( P \) defines a non-singular \( F \)-point of \( U \times_k F \). The 
statement follows from the implicit function theorem.

We claim that while keeping (3) we can moreover assume that, for each 
\( z_v^2 \) and each closed point \( P \) in the support of \( z_v^2 \), the field extension map 
\( k_v(f(P)) \subset k_v(P) \) is an isomorphism. Once more, it is enough to replace \( P \)
by a suitable and close enough $k_v(P)$-rational point on $U_{k_v}(P)$: this follows from Lemma 6.2.1 on p. 89 of [14].

Each of the zero-cycles $p(z_v^2)$ is now given by a separable monic polynomial $G_v[t] \in k_v[t]$ of degree $1 + Dds$, prime to $P(t)$ and with the property that the smooth fibres of $p$ above the roots of $G_v$ have rational points over their field of definition. By Krasner’s lemma, any monic polynomial $H(t)$ close enough to $G_v(t)$ for the $v$-adic topology on the coefficients will be separable, with roots ‘close’ to those of $G_v$. Thus the fibres above the roots of the new polynomial are still smooth and still possess rational points over their field of definition.

An irreducible polynomial $G(t)$ of degree $1 + Dds$ defines a closed point $M$ of degree $1 + Dds$ on $A_1^{k_v}$. Let $F = k(M) = k[t]/(G(t))$. Let $\theta$ be the residue class of $t$ in $F$. We now choose the irreducible polynomial $G(t)$ as given by Theorem 3.1 on p. 15 of [14] with the field $L$ in this theorem is $K$ and $V'$ ($V$ in this theorem) is the places of $k$ lying over places in $\mathbb{Q}$ which are totally split in $K$, such that

(i) For each place $v \in S_1$, $G(t)$ is close enough to $G_v(t)$, such that the fibre $U_\theta$ contains an $F_w$-point $P_w$ for each place $w$ of $F$ over $v$, and such that $\sum_{w|v} P_w$ is ‘close’ enough to $z_v^2$ satisfying

$$
\sum_{w|v} inv_v(Cores_{F_w/k_v}(A(P_w))) = inv_v(\langle A, z_v^2 \rangle)
$$

for each $A \in A \cup B$.

(ii) For each irreducible term $p_i(t)$ of the polynomial $P(t)$, there exists a place $w_i$ of $F$ such that $p_i(\theta)$ is a uniformizer at $w_i$ and is a unit in $F_w$ if $w$ is not over $S_1 \cup V'$ and $w \neq w_i$.

We claim that the fibre $U_\theta/F$ has points in all completions of $F$.

At a place $w$ of $F$ over a place $v \notin S_1$ and $w \neq w_i$ for all $i$, the existence of an $F_w$-point $P_w$ on $U_\theta$ is clear since either $P(\theta)$ is a unit or $K.F/F$ is totally split.

Let $v_i$ be the unique place of $k$ reduced by $w_i$. With condition (i), (ii) and the global reciprocity law, we have

$$
inv_{v_i}(Cores_{F/k}(p_i(\theta), \chi)) = 0.
$$

Since $\chi$ runs through all characters of $\text{Gal}(K/k)$ and condition (ii), we have $K.F/F$ is totally split at $w_i$. Therefore the fibre $X_\theta$ of $p$ contains an $F_{w_i}$-point $P_{w_i}$.

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Let $B \in B$. Let $w$ be a place of $F$ over $v \notin S_1$ and $w \neq w_i$ for $1 \leq i \leq m$. Let

$$\langle g \rangle = \text{Gal}((E.F)_{w'}/F_w) \subset \text{Gal}(E.F/F) \hookrightarrow \text{Gal}(E/k),$$

where $w'$ is a place of $E.F$ over $w$. Let $E_g$ be the fixed field of $\langle g \rangle$ in $E$. We know

$$B(P_w) = f_g(B)(P_w) \in \text{Br}(F_w),$$

where $f_g$ is the natural restriction from $\text{Br}(X)$ to $\text{Br}(X_{E_g})$. And $f_g(B)$ has the form $\sum_{j,l}(q_j(t), \chi_l) + \delta_{B,g}$ with $h_j(t) \mid P(t)$. Since either $q_j(\theta)$ has valuation 0 at $w$ or $K.F/F$ is totally split at $w$, we have

$$\text{inv}_w(B(P_w)) = 0 + \text{inv}_w(\delta_{B,g}) = 0.$$

Let $w$ be some $w_i$. Let

$$\langle g \rangle = \text{Gal}((E.F)_{w'}/F_w) \subset \text{Gal}(E.F/F) \hookrightarrow \text{Gal}(E/k).$$

By the above argument, we know $(K.F)_{w''}/F_w$ is totally split, then $K.F \subset E_g.F$. We also know

$$B(P_w) = f_g(B)(P_w) \in \text{Br}(F_w)$$

and $f_g(B)$ has the form $\sum_{j,l}(q_j(t), \chi_l) + \delta_{B,g}$, where $\chi_l$ is a character of $\text{Gal}(K/K_g)$ and $K_g = K \cap E_g$. Since $K.F \subset E_g.F$, we have

$$f_g(B) = 0 + \delta_{B,g} = \delta_{B,g} \in \text{Br}(X_{E_g,F}).$$

So we have

$$\text{inv}_w(B(P_w)) = 0 + \text{inv}_w(\delta_{B,g}) = 0.$$

Let $w$ be a place of $F$ over $v \in S_1$. With condition (i), we have

$$\sum_{w \mid v} \text{inv}_v(\text{Cores}_{F_w/k_v}B(P_w)) = \text{inv}_v(\langle B, z_{w}^2 \rangle).$$

Therefore we have

$$\sum_{w \in \Omega_F} \text{inv}_v(\text{Cores}_{F_w/k_v}B(P_w)) = \sum_{v \in S_1} \text{inv}_v(\langle B, z_{w}^2 \rangle) = 0.$$

We know

$$\text{inv}_v(\text{Cores}_{F_w/k_v}B(P_w)) = \text{inv}_w(B(P_w)).$$
by the property of local fields. Then we have

$$\sum_{w \in \Omega_F} inv_w(B(P_w)) = 0.$$ 

Since $[F : k]$ and $[K : k]$ are relatively prime, we have $F \cap K = k$. Therefore the natural map $Br(T^c)/Br_0(T^c) \to Br(T_F^c)/Br_0(T_F^c)$ is an isomorphism. With a similar argument as the last part in the proof of Theorem 3.3 we have $B$ generates the Brauer group $Br(U^c_{\theta})/Br_0(U^c_{\theta})$. Therefore there is no Brauer-Manin obstruction on $U^c_\theta$ for $(P_w)_{w \in \Omega_F}$. Hence $U^c_\theta/F$ possesses an $F$-point. □

With the help of Salberger’s device, by similar argument as above we have the following result which corresponds to Theorem 3.5.

**Theorem 4.2.** Let $k$ be a number field and $P(t)$ a polynomial over $k$. Let $V$ be the smooth locus of the affine $k$-variety defined by

$$(x_1^2 - ax_2^2)(y_1^2 - by_2^2)(z_1^2 - abz_2^2) = P(t)$$

where $a, b \in k^*$. If there is no Brauer-Manin obstruction to the existence of a zero-cycle of degree 1 on a smooth proper model $V^c$ of $V$, then there is a zero-cycle of degree 1 on $V^c$ (defined over $k$).

**Proof.** Let $U$ be the smooth affine variety over $k$ defined by

$$(x_1^2 - ax_2^2)(y_1^2 - by_2^2)(z_1^2 - abz_2^2) = P(t) \neq 0.$$ 

Let $U^c$ be a smooth compactification of $U$ with a projection $p : U^c \to \mathbb{P}^1$. If one of the three numbers $a, b, ab$ is a square in $k^*$, this theorem is obvious. Then we only need to consider the case all numbers $a, b, ab$ are not contained in $k^{*2}$.

Let $P(t) = cp_1(t)^{e_1} \cdots p_m(t)^{e_m}$. Let $S_0$ be a finite set of places of $k$ containing all the archimedean places and all the bad finite places in sight: finite places where one $p_i(t)$ is not integral, finite places where all $p_i(t)$ are integral but the product $\prod_i p_i(t)$ does not remain separable when reduced modulo $v$, places ramified in the extension $K/k$ where $K = k(\sqrt{a}, \sqrt{b})$. Let $N_0$ be a closed point of $U$ such that $k(N_0) = k(\sqrt{a})$.

Let

$$A = \{(p_i(t), b) \in Br(U) \mid 1 \leq i \leq m\} \cup \{(x_1^2 - ax_2^2, b)\}.$$ 

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We know \((x_1^2 - ax_2^2, b)\) is the unique generator of the Brauer group of the smooth fibre of \(U\) (see [3]).

Since we assume that there is no Brauer-Manin obstruction to the existence of a zero-cycle of degree 1, by an obvious variant of Harari’s result ([14, Theorem 3.2.2]) we may find a finite set \(S_1\) of places of \(k\) containing \(S_0\) and for each \(v \in S_1\) a zero-cycle \(z_v\) of degree 1 with support in \(U \times_k k_v\) such that

\[
\sum_{v \in S_1} \text{inv}_v(\langle \mathcal{A}, z_v \rangle) = 0 \quad \text{for all } \mathcal{A} \in A.
\]

With the similar argument as in the proof of Theorem 4.1 for each \(v \in S_1\), we can assume that:

(i) Each cycle \(z_v\) is effective and has the same degree \(D\), where \(D\) is odd.

(ii) Each cycle \(z_v\) is a sum of distinct closed points \(P_v\) whose images under \(p_{k_v}: U_{k_v} \to \mathbb{A}^1_{k_v}\) are also distinct, and the field extension map \(k_v(p(P_v)) \subset k_v(P_v)\) is an isomorphism.

Each of the zero-cycles \(p(z_v)\) is now given by a separable monic polynomial \(G_v[t] \in k_v[t]\) of degree \(D\), prime to \(P(t)\) and with the property that the (smooth) fibres of \(p\) above the roots of \(G_v(t)\) have rational points over their field of definition.

An irreducible polynomial \(g(t)\) of degree \(D\) defines a closed point \(M\) of degree \(D\) on \(\mathbb{A}^1_k\). Let \(F = k(M) = k[t]/(G(t))\). Let \(\theta\) be the residue class of \(t\) in \(F\). We now choose the irreducible polynomial \(G(t)\) as given by Theorem 3.1 on p. 15 of [14] with the field \(L\) in this theorem is \(K\) and \(V'\) (\(V\) in this theorem) is the places of \(k\) lying over places in \(Q\) which are totally split in \(K\), such that

(i) For each place \(v \in S_1\), \(G(t)\) is close enough to \(G_v(t)\), such that the fibre \(U_\theta\) contains an \(F_w\)-point \(P_w\) for each place \(w\) of \(F\) over \(v\), and such that \(\sum_{w|v} P_w\) is close enough to \(z_v^2\) satisfying

\[
\sum_{w|v} \text{inv}_v(\text{Cores}_{F_w/k_v}\mathcal{A}(P_w)) = \text{inv}_v(\langle \mathcal{A}, z_v^2 \rangle)
\]

for each \(\mathcal{A} \in A\).

(ii) For each irreducible term \(p_i(t)\) of the polynomial \(P(t)\), there exists a place \(w_i\) of \(F\) such that \(p_i(\theta)\) is a uniformizer at \(w_i\) and is a unit in \(F_w\) if \(w\) is not over \(S_1 \cup V'\) and \(w \neq w_i\).
If \( w \) is not over \( S_1 \), then \( K.F/F \) is unramified at \( w \). Then one of \( a, b, ab \) is a square in \( F_w^* \). Then the fibre \( U_\theta \) of \( p \) contains an \( F_w \)-point \( P_w \).

For each \( A = (p_i(t), b) \in A \), we have
\[
\sum_{v \in S_1} inv_v(Cores_{F_w/k_v}(p_i(\theta), b)) = 0.
\]

By the global class field theory, one has
\[
\sum_{v \notin S_1} inv_v(Cores_{F_w/k_v}(p_i(\theta), b)) = 0.
\]

Let \( v_i \) be the unique place of \( k \) reduced by \( w_i \). With condition (ii) above, we deduce
\[
inv_{v_i}(Cores_{F_{w_i}/k_{v_i}}(p_i(\theta), b)) = 0.
\]

Since \( p_i(\theta) \) is a uniformizer at \( w_i \), we have \( F(\sqrt{b})/F \) is totally split at \( w_i \).

Let \( B = (x_1 - ax_2^2, b) \). Suppose \( w \) is not over \( S_1 \) and \( w \neq w_i \) for \( 1 \leq i \leq m \).

If \( F(\sqrt{b})/F \) is split at \( w \), then \( inv_w(B(P_w)) = 0 \).

If \( F_w(\sqrt{b}) = F_w(\sqrt{a}) \) is inert over \( F_w \), then \( inv_w(B(P_w)) = 0 \).

If \( F_w(\sqrt{b})/F_w \) is inert and \( F_w(\sqrt{a})/F_w \) is split, then
\[
(y_1^2 - by_2^2, b) = (z_1^2 - abz_2^2, b) = 0.
\]

One has
\[
inv_w(B(P_w)) = (x_1^2 - ax_2^2, b) = (P(\theta), b) = 0
\]

since \( P(\theta) \) is a unit at \( w \). Therefore we have \( inv_w(B(P_w)) = 0 \).

Let \( w \) be some \( w_i \). Since \( b \) is a square in \( F_w^* \). So we have \( inv_{w_i}(B(P_{w_i})) = 0 \). Therefore we have
\[
\sum_{w \in \Omega_F} inv_w(B(P_w)) = \sum_{w \in \Omega_F} \sum_{w \in S_1} inv_v(Cores_{F_w/k_v}B(P_w))
= \sum_{w \in S_1} inv_v(Cores_{F_w/k_v}B(P_w))
= \sum_{w \in S_1} inv_v(\langle B, z_v \rangle) = 0.
\]

Since \( F \cap K = k \), we deduce that \( B \) generates the unramified Brauer group of \( U_{\theta}^c/F \). Therefore there is no Brauer-Manin obstruction on \( U_{\theta}^c \) for \( (P_w)_{w \in \Omega_F} \).

Hence \( U_{\theta}^c/F \) possesses an \( F \)-point. \( \square \)
For all primes $p$ (not only $p = 3$), we can prove the following result (which corresponds to Theorem 3.6).

**Theorem 4.3.** Let $k$ be a number field and $P(t)$ a polynomial over $k$. Let $p$ be a prime. Let $K/k$ be of degree $p$ and not cyclic. Let $V$ be the smooth locus of the affine $k$-variety defined by

$$N_{K/k}(\Xi) = P(t).$$

If there is no Brauer-Manin obstruction to the existence of a zero-cycle of degree 1 on a smooth proper model $V^c$ of $V$, then there is a zero-cycle of degree 1 on $V^c$ (defined over $k$).

**Proof.** Obviously $V$ has a closed point $N_0$ with $k(N_0) = K$. Then we only need to show that there is a field $F/k$ with degree $[F : k]$ prime to $p$ such that $V^c(F) \neq \emptyset$.

Let $K^{cl}$ be the Galois closure of $K/k$ with the Galois group $G = \text{Gal}(K^{cl}/k)$. Then $G$ is a subgroup of the symmetric group $S_p$. Let $H$ be the $p$-Sylow subgroup of $G$. We can see $H$ is cyclic and of order $p$. Let $\Theta$ be the fixed field of $H$. Then $[\Theta : k]$ is relative prime to $p$.

One has a family $\{z_v\}_v$ of zero-cycles of degree 1 which is orthogonal to $\text{Br}(V^c)$. Then one pushes $\{z_v\}_v$ to $V^c_{\Theta}$, one gets a family of zero-cycles of degree 1 on $V^c_{\Theta}$. A projection formula for the Brauer pairing shows the family of zero-cycles on $V^c_{\Theta}$ is orthogonal to $\text{Br}(V^c_{\Theta})$. Since $K^{cl}/\Theta = K, \Theta/\Theta$ is cyclic (of degree $p$), one gets a zero-cycle of degree 1 on $V^c_{\Theta}$ by Theorem 4.1 (or Theorem 4.1 in [13]). So there is a field $F/\Theta$ with the degree $[F : \Theta]$ prime to $p$ such that $V^c(F) \neq \emptyset$. Since $[\Theta : k]$ is prime to $p$, we have $[F : k] = [F : \Theta] \cdot [\Theta : k]$ is also prime to $p$. \hfill \Box

**Acknowledgment** The author would like to thank Colliot-Thélène for helpful discussions and valuable suggestions. The work is supported by the Morningside Center of Mathematics and NSFC, grant # 10901150.

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