Geometric Algebras

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Abstract
This is the first paper in a series of eight where in the first three we develop a systematic approach to the geometric algebras of multivectors and extensors, followed by five papers where those algebraic concepts are used in a novel presentation of several topics of the differential geometry of (smooth) manifolds of arbitrary global topology. A key tool for the development of our program is the mastering of the euclidean geometrical algebra of multivectors that is detailed in the present paper.

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1 Introduction

This is the first paper in a following series of eight which have been designed in order to show how Clifford (geometric) algebra methods can be conveniently used in the study of differential geometry and geometrical theories of the gravitational field. It dispenses the use of fiber bundle theory\(^1\) and is indeed a very powerful and economic tool for performing sophisticated calculations. The first three papers\(^2\) deal with the algebraic aspects of the theory, namely Clifford algebras and the theory of extensors. Our presentation is self contained and serves besides the purpose of fixing our conventions also the one of introducing a series of “tricks of the trade” (not found easily elsewhere) necessary for quickly and efficient computations. The other five papers \[11, 12, 15, 2, 3\] develop a systematic approach to a theory of multivector and extensor calculus and their use in the differential geometry of manifolds of arbitrary topology. There are many novelties in our presentation, in particular the way we introduce the concept of deformation of geometric structures, which is discussed in detail in \[15\] and which permit us to relate \[2, 3\] some distinct geometric structures on a given manifold. Moreover, the method permit also to solve problems in one given geometry in terms of an eventually simple one, and here is a place where our theory may be a useful one for the study of geometrical theories of the gravitational field.

\(^1\)A reader interested in Clifford and spin-Clifford bundles may consult, e.g., \[17\].

\(^2\)This one and \[9, 10\].
The main issues discussed in the present paper are the constructions of the euclidean and metric geometric (or Clifford) algebras of multivectors which can be associated to a real vector space \( V \) of dimension \( n \) once we equip \( V \) with an euclidean and an arbitrary non degenerated metric of signature \((p, q)\) with \( p + q = n \). The euclidean geometrical algebra is the key tool for performing almost all calculations of the following papers. Metric geometric algebras are introduced as deformations of the euclidean geometric algebra. This is more explored in [10] where we introduce the concept of a deformation extensor associated to a given metric extensor and and by proving the remarkable golden formula. Extensors are a new kind of geometrical objects which play a crucial role in the theory presented in this series and the basics of their theory is described in [10, 11]. These objects have been apparently introduced by Hestenes and Sobczyk in [5] and some applications of the concept appears in [6], but a rigorous theory was lacking until recently\(^3\). It is important to observe that in [5] the preliminaries of the geometric calculus have been applied to the study of the differential geometry of vector manifolds. However, as admitted in [18] there are some problems with that approach. In contrast, our formulation applies to manifolds of a arbitrary topology and is free of the problems that paved the construction in [5]. There are many novelties and surprises in what follows, e.g., the concept of deformed geometries relative to a given geometry, an intrinsic Cartan calculus and other topics that are ready to be used in geometrical theories of the gravitational field, as the reader will convince himself consulting the other papers in the series.

As for the explicit contents of the present paper we introduce the concept of multivectors in Section 2 and their exterior algebra in section 3. In Section 4 we introduce the scalar product of multivectors and in section 5 the concepts of right and left contractions and interior algebras. In Section 6 we give a definition of a general real Clifford (or geometrical) algebra of multivectors and in Section 7 we study in details the relation between the euclidean and pseudo-euclidean geometrical algebras. In section 8 we present our conclusions. As additional references to several aspects of the theory of Clifford algebras, that eventually may help the interested reader, we quote\(^4\) [4, 7, 13, 14, 16, 17].

\(^3\)More details on the theory of extensors may be found in [4].

\(^4\)For a presentation of the theory of algebraic and Dirac-Hestenes spinors and spinor fields, we quote [16, 8]. Applications of the theory may be found in [17] and in references quoted there.
2 Multivectors

Let $\Lambda V$ denote the Cartesian product of all the $k$-vector spaces $\Lambda^k V$, with $0 \leq k \leq n$, i.e., $\Lambda V = \mathbb{R} \times V \times \ldots \times \Lambda^n V$. Here $V$ is $n$-dimensional vector space over the real field $\mathbb{R}$.

$$\Lambda V = \{(X_0, X_1, \ldots, X_n) \mid X_k \in \Lambda^k V, \text{ for each } k = 0, 1, \ldots, n\}. \quad (1)$$

We introduce an equality among the $(n+1)$-uples of $\Lambda V$,

$$(X_0, X_1, \ldots, X_n) = (Y_0, Y_1, \ldots, Y_n) \iff X_0 = Y_0, \ X_1 = Y_1, \ldots, X_n = Y_n. \quad (2)$$

$\Lambda V$ has a structure of a real vector space, naturally induced by the real vector space structure of all the $k$-vector spaces $\Lambda^k V$. It is realized by defining:

(i) The addition of $(X_0, X_1, \ldots, X_n) \in \Lambda V$ and $(Y_0, Y_1, \ldots, Y_n) \in \Lambda V$:

$$(X_0, X_1, \ldots, X_n) + (Y_0, Y_1, \ldots, Y_n) = (X_0 + Y_0, X_1 + Y_1, \ldots, X_n + Y_n) \in \Lambda V. \quad (3)$$

(ii) The scalar multiplication of $(X_0, X_1, \ldots, X_n) \in \Lambda V$ by $\alpha \in \mathbb{R}$:

$$\alpha(X_0, X_1, \ldots, X_n) = (\alpha X_0, \alpha X_1, \ldots, \alpha X_n) \in \Lambda V. \quad (4)$$

The vectors in $\Lambda V$ will be called multivectors over $V$. Sometimes, they will be named as multivectors of $\Lambda V$. The real vector space $\Lambda V$ is called the space of multivectors over $V$.

The zero multivector for $\Lambda V$ is just $0 = (0_0, 0_1, \ldots, 0_n)$, where $0_k$ denotes the zero $k$-vector of $\Lambda^k V$.

For each $k = 0, 1, \ldots, n$ the linear mapping $\pi_k : \Lambda V \to \Lambda^k V$ such that

if $X = (X_0, X_1, \ldots, X_n)$, then $\pi_k(X) = X_k \quad (5)$

is called the $k$-component projection operator because $\pi_k(X)$ is just the $k$-component of $X$.

For each $k = 0, 1, \ldots, n$ any multivector $X$ such that $\pi_j(X_{(k)}) = 0_j$, for $k \neq j$, is said to be an homogeneous multivector of degree $k$, or for short, a $k$-homogeneous multivector. That means that any component of $X$ which is
not its $k$-component is necessarily zero, but the $k$-component of $X$ may be zero or not. Thus, any $k$-homogeneous multivector might be denoted by

$$X_{(k)} = (\ldots, 0_{k-1}, X_k, 0_{k+1}, \ldots),$$  \hspace{1cm} (6)  

where $X_k$ is some $k$-vector belonging to $\bigwedge^k V$.

It should be noticed that $0$ is an homogeneous multivector of any degree $0, 1, \ldots, n$.

The set of homogeneous multivectors of degree $k$, i.e.,

$$\bigwedge^{(k)} V = \{X_{(k)} = (\ldots, 0_{k-1}, X_k, 0_{k+1}, \ldots) \mid X_k \in \bigwedge^k V\} \subset \bigwedge V$$  \hspace{1cm} (7)  

is a vector subspace of $\bigwedge V$. Indeed, we have that $0 \in \bigwedge^{(k)} V$ and the addition of $k$-homogeneous multivectors and the scalar multiplications of $k$-homogeneous multivectors by real numbers are themselves $k$-homogeneous multivectors.

It is easy to see that $\bigwedge^{(k)} V$ is linearly isomorphic $\bigwedge^k V$. For each $k = 0, 1, \ldots, n$ there exists a linear isomorphism between $\bigwedge^k V$ and $\bigwedge^{(k)} V$ which is realized by defining $\tau_k : \bigwedge^k V \rightarrow \bigwedge^{(k)} V$ and $\tau_k^{-1} : \bigwedge^{(k)} V \rightarrow \bigwedge^k V$ such that

$$\tau_k(X_k) = X_{(k)},$$  \hspace{1cm} (8)  

$$\tau_k^{-1}(X_{(k)}) = X_k.$$  \hspace{1cm} (9)  

We have indeed that both of $\tau_k$ and $\tau_k^{-1}$ are linear mappings, and they are inverses to each other, i.e., $\tau_k^{-1} \circ \tau_k = i_{\bigwedge^k V}$ and $\tau_k \circ \tau_k^{-1} = i_{\bigwedge^{(k)} V}$, where $i_{\bigwedge^k V}$ and $i_{\bigwedge^{(k)} V}$ are the respective identity functions for $\bigwedge^k V$ and $\bigwedge^{(k)} V$.

Note that $\tau_k(\bigwedge^k V) = \bigwedge^{(k)} V$ and $\bigwedge^k V = \tau_k^{-1}(\bigwedge^{(k)} V)$.

Such a linear isomorphism $\tau_k$, named as the $k$-isomorphism for short in what follows, is a key piece in our theory of the geometric algebra of multivectors.

Then,

$$\dim \bigwedge^{(k)} V = \dim \bigwedge^k V = \binom{n}{k}$$  \hspace{1cm} (10)  

Any $X = (X_0, X_1, \ldots, X_n) \in \bigwedge V$ can be written as sum of all their $k$-homogeneous multivectors $\tau_k(X_k) = \tau_k \circ \pi_k(X) \in \bigwedge^{(k)} V$, i.e.,

$$X = \sum_{k=0}^{n} \tau_k(X_k) = \sum_{k=0}^{n} \tau_k \circ \pi_k(X).$$  \hspace{1cm} (11)
To prove Eq. (11) we should use Eqs. (3), (6) and (8), and Eq. (5).

If there exists \( X \in \bigwedge V \) such that \( \pi_k(X) = 0 \) for \( k \neq l \), and \( \pi_l(X) = X_l \), then

\[
X = \pi_l(X_l).
\]  

(12)

It is an immediate consequence of Eq. (11).

Eq. (11) implies that the space \( \bigwedge V \) can be written as sum\(^5\) of all their \( k \)-subspaces \( \bigwedge (k) V \), i.e.,

\[
\bigwedge V = \bigwedge (0) V + \bigwedge (1) V + \ldots + \bigwedge (n) V,
\]

(13)

or into a more suggestive form,

\[
\bigwedge V = \sum_{k=0}^{n} \tau_k \bigwedge (k) V = \sum_{k=0}^{n} \pi_k \bigwedge V.
\]

(14)

But, since

\[
\bigwedge (j) V \cap \bigwedge (k) V = \{0\}, \text{ for } j \neq k,
\]

(15)

we see that Eq. (13) can still be written as

\[
\bigwedge V = \bigwedge (0) V \oplus \bigwedge (1) V \oplus \ldots \oplus \bigwedge (n) V.
\]

(16)

As an immediate consequence\(^6\) of Eqs. (10), (13) and (15) we have that

\[
\dim \bigwedge V = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.
\]

(17)

For each \( k = 0, 1, \ldots, n \) the linear mapping \( \langle \rangle_k : \bigwedge V \to \bigwedge V \) such that

\[
\text{if } X = (X_0, X_1, \ldots, X_n), \text{ then } \langle X \rangle_k = X_k,
\]

(18)

is called the \( k \)-part operator. \( \langle X \rangle_k \) is read as the \( k \)-part of \( X \).

The \( k \)-component projection operator, the \( k \)-isomorphism and the \( k \)-part operator are involved in the following basic properties

\[
\pi_j \circ \tau_k(X_k) = \begin{cases} 
0_j, & \text{if } j \neq k \\
X_k, & \text{if } j = k
\end{cases}
\]

(19)

---

\(^5\)Recall that if \( S_1 \) and \( S_2 \) are subspaces of any space \( W \), then \( S_1 + S_2 \) is just the subspace of \( W \) defined by \( S_1 + S_2 = \{ v_1 + v_2 \mid v_1 \in S_1 \text{ and } v_2 \in S_2 \} \).

\(^6\)Recall that \( \dim(S_1 + S_2) = \dim S_1 + \dim S_2 - \dim(S_1 \cap S_2) \).
and
\[ \tau_k \circ \pi_k = \langle \rangle_k, \text{ for each } k = 0, 1, \ldots, n. \] (20)

We notice that Eq. (19) implies that
\[ \pi_k \circ \tau_k = i_{\Lambda^k V}, \text{ for each } k = 0, 1, \ldots, n, \] (21)
i.e., \( \pi_k \) is the left inverse of \( \tau_k \).

In order to prove Eq. (19) we should use Eq. (8) and take into account the definition of \( k \)-homogeneous multivectors, as set by Eq. (6).

Eq. (20) follows directly of using Eqs. (5), (8) and (18).

Note that \( X \in \bigwedge V \) is an homogeneous multivector of degree \( k \), i.e., \( X \in \bigwedge^k V \), if and only if
\[ X = \langle X \rangle_k. \] (22)
This logical equivalence is an immediate consequence of Eq. (20) whenever Eq. (5) and Eq. (8) are taken into account.

Any multivector can be written as sum of all their own \( k \)-parts, i.e.,
\[ X = \sum_{k=0}^{n} \langle X \rangle_k. \] (23)
This result follows directly from Eq. (11) and Eq. (20).

3 Exterior Product

Let \( TV = \sum_{k=0}^{\infty} T^k V \) be the tensor algebra of \( V \) and \( \mathcal{A} \) the so-called antisymmetrization operator, i.e., a linear mapping \( \mathcal{A} : T^k V \to \bigwedge^k V \) such that
(i) for all \( \alpha \in \mathbb{R} \):
\[ \mathcal{A} \alpha = \alpha, \] (24)
(ii) for all \( v \in V \):
\[ \mathcal{A} v = v, \] (25)
(iii) for all \( t \in T^k V \), with \( k \geq 2 \):
\[ \mathcal{A} t(\omega^1, \ldots, \omega^k) = \frac{1}{k!} \epsilon_{i_1 \ldots i_k} \cdot \omega^{i_1} \wedge \cdots \wedge \omega^{i_k}, \] (26)
where \( \epsilon_{i_1 \ldots i_k} \) is the so-called permutation symbol of order \( k \),

\[
\epsilon_{i_1 \ldots i_k} \equiv \epsilon^{i_1 \ldots i_k} = \begin{cases} 
1, & \text{if } i_1 \ldots i_k \text{ is even permutation of } 1 \ldots k \\
-1, & \text{if } i_1 \ldots i_k \text{ is odd permutation of } 1 \ldots k \\
0, & \text{otherwise}
\end{cases}
\]  

(27)

The exterior product of \( X_p \in \bigwedge^p V \) and \( Y_q \in \bigwedge^q V \), namely \( X_p \wedge Y_q \in \bigwedge^{p+q} V \), is defined\(^7\) by

\[
X_p \wedge Y_q = \frac{(p + q)!}{p!q!} \mathcal{A}(X_p \otimes Y_q),
\]

(28)

where \( X_p \otimes Y_q \) is the tensor product of \( X_p \) and \( Y_q \).

Let \( \{ e_j \} \) be a basis of \( V \), and \( \{ \varepsilon^j \} \) be its dual basis for \( V^* \), i.e., \( \varepsilon^j(e_i) = \delta^j_i \).

Now, let us take \( t \in T^k V \) with \( k \geq 1 \). Such a contravariant \( k \)-tensor \( t \) can be expanded onto the \( k \)-tensor basis \( \{ e_{j_1} \otimes \ldots \otimes e_{j_k} \} \) with \( j_1, \ldots, j_k = 1, \ldots, n \) by the well-known formula

\[
t = t^{j_1 \ldots j_k} e_{j_1} \otimes \ldots \otimes e_{j_k},
\]

(29)

where \( t^{j_1 \ldots j_k} = t(\varepsilon^{j_1}, \ldots, \varepsilon^{j_k}) \) are the so-called \( j_1 \ldots j_k \)-contravariant components of \( t \) with respect to \( \{ e_{j_1} \otimes \ldots \otimes e_{j_k} \} \).

From Eq.(26) it follows (non-trivially) a remarkable identity which holds for the basis 1-forms \( \varepsilon^1, \ldots, \varepsilon^n \) belonging to \( \{ \varepsilon^j \} \). It is

\[
\mathcal{A}t(\varepsilon^{j_1}, \ldots, \varepsilon^{j_k}) = \frac{1}{k!} \delta^{j_1 \ldots j_k}_{i_1 \ldots i_k} t(\varepsilon^{i_1}, \ldots, \varepsilon^{i_k}),
\]

(30)

where \( \delta^{j_1 \ldots j_k}_{i_1 \ldots i_k} \) is the so-called generalized Kronecker symbol of order \( k \),

\[
\delta^{j_1 \ldots j_k}_{i_1 \ldots i_k} = \det \begin{bmatrix} 
\delta^{j_1}_{i_1} & \ldots & \delta^{j_k}_{i_1} \\
\ldots & \ldots & \ldots \\
\delta^{j_1}_{i_k} & \ldots & \delta^{j_k}_{i_k}
\end{bmatrix}
\]

with \( i_1, \ldots, i_k \) and \( j_1, \ldots, j_k \) running from 1 to \( n \).

(31)

Let us take \( X \in \bigwedge^k V \) with \( k \geq 2 \). By definition \( X \in T^k V \) and is completely skew-symmetric, hence, it must be \( X = \mathcal{A}X \). Then, by using Eq.(30) we get a combinatorial identity which relates the \( i_1 \ldots i_k \)-components to the \( j_1 \ldots j_k \)-components for \( X \). It is

\[
X^{j_1 \ldots j_k} = \frac{1}{k!} \delta^{j_1 \ldots j_k}_{i_1 \ldots i_k} X^{i_1 \ldots i_k},
\]

(32)

\(^7\)Take notice that other definitions with other factors before the antisymmetrization operator are possible.
From Eq. (26) by using a well-known property of the antisymmetrization operator, namely: \( \mathcal{A}(At \otimes u) = \mathcal{A}(t \otimes Au) = \mathcal{A}(t \otimes u) \), a noticeable formula for expressing simple \( k \)-vectors in terms of the tensor products of \( k \) vectors can be easily deduced. It is

\[
v_1 \wedge \ldots \wedge v_k = \epsilon^{i_1 \ldots i_k} v_{i_1} \otimes \ldots \otimes v_{i_k}.
\]

(33)

If \( \omega^1, \ldots, \omega^k \in V^* \), then

\[
v_1 \wedge \ldots \wedge v_k (\omega^1, \ldots, \omega^k) = \epsilon^{i_1 \ldots i_k} \omega^1(v_{i_1}) \ldots \omega^k(v_{i_k}).
\]

(34)

Eq. (33) implies (non-trivially) a remarkable identity which holds for the basis vectors \( e_1, \ldots, e_n \) belonging to any basis \( \{e_j\} \) of \( V \). It is

\[
e_{i_1} \wedge \ldots \wedge e_{i_k} = \delta_{i_1 \ldots i_k}^{j_1 \ldots j_k} e_{j_1} \otimes \ldots \otimes e_{j_k}.
\]

(35)

Once again let us take \( X \in \bigwedge^k V \) with \( k \geq 2 \). Since \( X \in T^k V \) and is completely skew-symmetric, the use of Eq. (32) and Eq. (35) in Eq. (29) allows us to obtain the expansion formula

\[
X = \frac{1}{k!} X^{i_1 \ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k}.
\]

(36)

We recall now the basic properties of the elementary exterior product of \( p \)-vector and \( q \)-vector.

For any \( V_p, W_p \in \bigwedge^p V \) and \( X_q, Y_q \in \bigwedge^q V \)

\[
(V_p + W_p) \wedge X_q = V_p \wedge X_q + W_p \wedge X_q, \quad V_p \wedge (X_q + Y_q) = V_p \wedge X_q + V_p \wedge Y_q \quad \text{(distributive laws)}.
\]

(37)

(38)

For any \( X_p \in \bigwedge^p V \), \( Y_q \in \bigwedge^q V \) and \( Z_r \in \bigwedge^r V \)

\[
(X_p \wedge Y_q) \wedge Z_r = X_p \wedge (Y_q \wedge Z_r) \quad \text{(associative law)}.
\]

(39)

For any \( X_p \in \bigwedge^p V \) and \( Y_q \in \bigwedge^q V \)

\[
X_p \wedge Y_q = (-1)^{pq} Y_q \wedge X_p.
\]

(40)
3.1 Exterior Product of Multivectors

The exterior product of \( X, Y \in \bigwedge V \), namely \( X \wedge Y \in \bigwedge V \), is defined by

\[
X \wedge Y = \sum_{k=0}^{n} \sum_{j=0}^{k} \tau_k(\pi_j(X) \wedge \pi_{k-j}(Y)).
\] (41)

Note that on the right side there appears the exterior product of \( j \)-vectors and \((k-j)\)-vectors, as defined by Eq. (28), which means that

\[
\pi_k(X \wedge Y) = \sum_{j=0}^{k} \pi_j(X) \wedge \pi_{k-j}(Y), \text{ for each } k = 0, 1, \ldots, n
\] (42)

and, if \( X = (X_0, \ldots, X_k, \ldots, X_n) \) and \( Y = (Y_0, \ldots, Y_k, \ldots, Y_n) \), then

\[
X \wedge Y = (X_0 Y_0, \ldots, \sum_{j=0}^{k} X_j \wedge Y_{k-j}, \ldots, \sum_{j=0}^{n} X_j \wedge Y_{n-j}).
\] (43)

This exterior product is an internal law on \( \bigwedge V \). It is associative and satisfies the usual distributive laws (on the left and on the right).

In order to prove that the exterior product of multivectors, as defined by Eq. (41), satisfies the associative law we should use the summation identity

\[
\sum_{i=0}^{j} \sum_{j=0}^{k} X_i \wedge (Y_{j-i} \wedge Z_{k-j}) = \sum_{j=0}^{k} \sum_{i=0}^{j} X_j \wedge (Y_i \wedge Z_{k-j-i}) \text{ and Eq. (39)}
\]

Thus, by using Eq. (42) a straightforward calculation gives

\[
\pi_k((X \wedge Y) \wedge Z) = \sum_{j=0}^{k} \pi_j(X \wedge Y) \wedge Z_{k-j} = \sum_{j=0}^{k} (\sum_{i=0}^{j} X_i \wedge Y_{j-i}) \wedge Z_{k-j}
\]

\[
= \sum_{j=0}^{k} \sum_{i=0}^{k-j} (X_i \wedge Y_{j-i}) \wedge Z_{k-j} = \sum_{j=0}^{k} \sum_{i=0}^{j} X_j \wedge (Y_{j-i} \wedge Z_{k-j})
\]

\[
= \sum_{j=0}^{k} \sum_{i=0}^{j} X_j \wedge (Y_i \wedge Z_{k-j-i}) = \sum_{j=0}^{k} X_j \wedge (\sum_{i=0}^{k-j} Y_i \wedge Z_{k-j-i})
\]

\[
= \sum_{j=0}^{k} X_j \wedge \pi_{k-j}(Y \wedge Z) = \pi_k(X \wedge (Y \wedge Z)).
\]
The distributive laws on the left and on the right are immediate consequences of the Eq. (37) and Eq. (38), respectively.

The space of multivectors $\bigwedge V$ endowed with this exterior product $\wedge$ is an associative algebra called the exterior algebra of multivectors.

Let us take $X_p \in \bigwedge^p V$ and $Y_q \in \bigwedge^q V$ with $0 \leq p + q \leq n$. By using Eq. (42) and Eq. (19), we have that

$$\pi_k(\tau_p(X_p) \wedge \tau_q(Y_q)) = \sum_{j=0}^{k} \pi_j \circ \tau_p(X_p) \wedge \pi_{k-j} \circ \tau_q(Y_q)$$

$$= \sum_{j=0}^{k} \left\{ 0_j, \ j \neq p \ \wedge \ \left\{ 0_{k-j}, \ k - j \neq q \right\} X_p, \ j = p \ \wedge \ \left\{ 0_{k-j}, \ k - j = q \right\} Y_q, \ k - j = q.\right.$$  

But, the sum on the right side of the equation above is $0_k = \sum_{j=0}^{k} 0_j \wedge 0_{k-j}$, for each $k = 0, 1, \ldots, n$ unless there exists some $k_0$ with $0 \leq k_0 \leq n$ such that $k_0 - p = q$. Whence, we see that $k_0 = p + q$ is the unique number which can satisfy the required conditions. Then, from the above equation it follows that

$$\pi_k(\tau_p(X_p) \wedge \tau_q(Y_q)) = 0_k, \text{ for } k \neq p + q$$

$$\pi_{p+q}(\tau_p(X_p) \wedge \tau_q(Y_q)) = X_p \wedge Y_q.$$

Hence, using Eq. (12) we finally get

$$\tau_p(X_p) \wedge \tau_q(Y_q) = \tau_{p+q}(X_p \wedge Y_q). \quad (44)$$

**4 Metric Structure**

Let us equip $V$ with a metric tensor, i.e., a symmetric and non-degenerate covariant 2-tensor over $V$, $G : V \times V \rightarrow \mathbb{R}$ such that

$$G(v, w) = G(w, v) \text{ for all } v, w \in V. \quad (45)$$

If $G(v, w) = 0$ for all $w \in V$, then $v = 0$. \quad (46)

As usual we write

$$G(v, w) \equiv v \cdot w, \quad (47)$$
and call \( v \cdot w \) the scalar product of the vectors \( v, w \in V \).

The pair \((V, G)\) is called a metric structure for \( V \). Sometimes, \( V \) is said to be a scalar product vector space.

Let \( \{e_k\} \) be any basis of \( V \), and \( \{\varepsilon^k\} \) be its dual basis for \( V^* \). As we know, \( \{\varepsilon^k\} \) is the unique basis of \( V^* \) which satisfies \( \varepsilon^k(e_j) = \delta^k_j \).

Let \( G_{jk} = G(e_j, e_k) \), since \( G \) is non-degenerate, it follows that \( \det [G_{jk}] \neq 0 \). Then, there exist the \( jk \)-entries for the inverse matrix of \( [G_{jk}] \), namely \( G^{jk} \), i.e., \( G^{ks}G_{sj} = G_{js}G^{sk} = \delta^k_j \).

We introduce the scalar product of 1-forms \( \omega, \sigma \in V^* \) by

\[
\omega \cdot \sigma = G^{jk} \omega(e_j) \sigma(e_k).
\] (48)

It should be noticed that the real number given by Eq.(48) does not depend on the choice of \( \{e_k\} \).

Now, we can define the so-called reciprocal bases of \( \{e_k\} \) and \( \{\varepsilon^k\} \). Associated to \( \{e_k\} \) we introduce the well-defined basis \( \{e^k\} \) by

\[
e^k = G^{ks} e_s, \quad \text{for each } k = 1, \ldots, n.
\] (49)

Such \( e^1, \ldots, e^n \in V \) are the unique basis vectors for \( V \) which satisfy

\[
e^k \cdot e_j = \delta^k_j.
\] (50)

Associated to \( \{\varepsilon^k\} \), we can also introduce a well-defined basis \( \{\varepsilon_k\} \) by

\[
\varepsilon_k = G_{ks} \varepsilon^s, \quad \text{for each } k = 1, \ldots, n.
\] (51)

Such \( \varepsilon_1, \ldots, \varepsilon_n \in V^* \) are the unique basis 1-forms for \( V^* \) which satisfy

\[
\varepsilon_j \cdot \varepsilon^k = \delta^k_j.
\] (52)

The bases \( \{e^k\} \) and \( \{\varepsilon_k\} \) are respectively called the reciprocal bases of \( \{e_k\} \) and \( \{\varepsilon^k\} \) (relatives to the metric tensor \( G \)).

Note that \( \{\varepsilon_k\} \) is the dual basis of \( \{e^k\} \), i.e.,

\[
\varepsilon_k(e^l) = \delta^l_k,
\] (53)

an immediate consequence of Eqs. (51) and (49).

From Eqs. (51) and (52), Eqs. (49) and (50) taking into account Eq. (48), we easily get that

\[
\varepsilon_j \cdot \varepsilon_k = e_j \cdot e_k,
\] (54)

\[
e^j \cdot e^k = G^{jk} = \varepsilon^j \cdot \varepsilon^k.
\] (55)
Using Eq. (50) we get two expansion formulas for \( v \in V \)
\[
v = v \cdot e^k e_k = v \cdot e_k e^k.
\] (56)

Using Eq. (52) we have that for all \( \omega \in V^* \)
\[
\omega = \omega \cdot \varepsilon_k e^k = \omega \cdot \varepsilon^k e_k.
\] (57)

Let us take \( X \in \bigwedge^k V \) with \( k \geq 2 \). By following analogous steps to those which allowed us to get Eq. (36) we can now obtain another expansion formula for \( k \)-vectors, namely
\[
X = \frac{1}{k!} X_{j_1 \ldots j_k} e^{j_1} \wedge \ldots \wedge e^{j_k},
\] (58)
where \( X_{j_1 \ldots j_k} = X(\varepsilon_{j_1}, \ldots, \varepsilon_{j_k}) \) are the so-called \( j_1 \ldots j_k \)-covariant components of \( X \) (with respect to the \( k \)-tensor basis \( \{e^{j_1} \otimes \ldots \otimes e^{j_k}\} \) with \( j_1, \ldots, j_k = 1, \ldots, n \)).

Next, we will obtain a relation between the \( i_1 \ldots i_k \)-covariant components of \( X \) and the \( j_1 \ldots j_k \)-contravariant components of \( X \). By using Eq. (57) and Eq. (54), a straightforward calculation yields
\[
X(\varepsilon_{i_1} \ldots \varepsilon_{i_k}) = X(e_{i_1} \cdot e_{s_1} \varepsilon^{s_1}, \ldots, e_{i_k} \cdot e_{s_k} \varepsilon^{s_k})
= X(\varepsilon^{s_1}, \ldots, \varepsilon^{s_k})(e_{i_1} \cdot e_{s_1}) \ldots (e_{i_k} \cdot e_{s_k})
= \frac{1}{k!} X(\varepsilon^{j_1}, \ldots, \varepsilon^{j_k}) \delta_{j_1 \ldots j_k}^{s_1 \ldots s_k} (e_{i_1} \cdot e_{s_1}) \ldots (e_{i_k} \cdot e_{s_k}),
\]
hence,
\[
X_{i_1 \ldots i_k} = \frac{1}{k!} X^{j_1 \ldots j_k} \det \begin{bmatrix} e_{i_1} \cdot e_{j_1} & \cdots & e_{i_1} \cdot e_{j_k} \\ \vdots & \ddots & \vdots \\ e_{i_k} \cdot e_{j_1} & \cdots & e_{i_k} \cdot e_{j_k} \end{bmatrix}.
\] (59)

### 4.1 Scalar Product

Once a metric structure \((V, G)\) has been given we can equip \( \bigwedge^p V \) with a scalar product of \( p \)-vectors. \( \bigwedge V \) can then be endowed with a scalar product of multivectors. This is done as follows.

The scalar product of \( X_p, Y_p \in \bigwedge^p V \), namely \( X_p \cdot Y_p \in \mathbb{R} \), is defined by the axioms:
\textbf{Ax-i} For all $\alpha, \beta \in \mathbb{R}$:
\[ \alpha \cdot \beta = \alpha \beta \text{ (real product of } \alpha \text{ and } \beta). \] 

\textbf{Ax-ii} For all $X_p, Y_p \in \wedge^p V$, with $p \geq 1$:
\[ X_p \cdot Y_p = \frac{1}{p!} X_p(\varepsilon^{i_1}, \ldots, \varepsilon^{i_p}) Y_p(\varepsilon_{i_1}, \ldots, \varepsilon_{i_p}), \]
\[ = \frac{1}{p!} X_p(\varepsilon_{i_1}, \ldots, \varepsilon_{i_p}) Y_p(\varepsilon^{j_1}, \ldots, \varepsilon^{j_p}), \tag{61} \]

where $\{\varepsilon_i\}$ is the reciprocal basis of $\{\varepsilon^i\}$, as defined by Eq.(51).

It is not difficult to realize that the real number defined by Eq.(61) does not depend on the bases $\{\varepsilon_i\}$ and $\{\varepsilon^i\}$ for calculating it. Indeed, by using the expansion formulas for 1-forms: $\omega = \omega \cdot \varepsilon_j \varepsilon^j$ (relative to any pair of reciprocal bases $\{\varepsilon^j\}$ and $\{\varepsilon_j\}$), and $\omega = \omega \cdot \varepsilon^\nu \varepsilon'_i$ (relative to any pair of reciprocal bases $\{\varepsilon^\nu\}$ and $\{\varepsilon'_i\}$), we have that
\[ X_p(\varepsilon^{i_1'}, \ldots, \varepsilon^{i_p'}) Y_p(\varepsilon_{i_1}, \ldots, \varepsilon_{i_p}) \]
\[ = X_p(\varepsilon^{i_1'} \cdot \varepsilon_{j_1} \varepsilon^{j_1}, \ldots, \varepsilon^{i_p'} \cdot \varepsilon_{j_p} \varepsilon^{j_p}) Y_p(\varepsilon_{i_1}, \ldots, \varepsilon_{i_p}) \]
\[ = X_p(\varepsilon^{j_1}, \ldots, \varepsilon^{j_p}) Y_p(\varepsilon_{i_1}, \ldots, \varepsilon_{i_p}, \varepsilon^{j_1'} \varepsilon'_{i_1}, \ldots, \varepsilon_{j_p} \varepsilon'_{i_p}) \]
\[ = X_p(\varepsilon^{j_1}, \ldots, \varepsilon^{j_p}) Y_p(\varepsilon_{j_1}, \ldots, \varepsilon_{j_p}). \]

It is a well-defined scalar product on $\wedge^p V$, since it is symmetric, satisfies the distributive laws, has the mixed associativity property and is non-degenerate i.e., if $X_p \cdot Y_p = 0$ for all $Y_p$, then $X_p = 0$.

We prove here only the non-degeneracy property. Choose, e.g., $Y_p \equiv \varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_p}$. Thus, all that must proved is that if $X_p \cdot (\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_p}) = 0$, then $X_p = 0$.

Using Eq.(35), more precisely, its version using the reciprocal basis vectors $e^1, \ldots, e^n$, the duality condition of $\{e^k\}$ and $\{\varepsilon_k\}$, and taking into account Eq.(34), we have that
\[ X_p \cdot (\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_p}) = 0 \Rightarrow \frac{1}{p!} X_p(\varepsilon^{i_1}, \ldots, \varepsilon^{i_p}) \delta^{j_1 \ldots j_p}_{s_1 \ldots s_p} \varepsilon_{s_1} (e^{s_1}) \ldots \varepsilon_{i_p} (e^{s_p}) = 0 \]
\[ \Rightarrow \frac{1}{p!} X_p(\varepsilon^{i_1}, \ldots, \varepsilon^{i_p}) \delta^{j_1 \ldots j_p}_{s_1 \ldots s_p} \delta_{s_1}^{i_1} \ldots \delta_{s_p}^{i_p} = 0 \]
\[ \Rightarrow X_p(\varepsilon^{j_1}, \ldots, \varepsilon^{j_p}) = 0, \]
i.e., $X_p = 0$.

For the special case of vectors Eq. (61) reduces to

$$v \cdot w = \varepsilon^i(v) \varepsilon_i(w) = \varepsilon_i(v) \varepsilon^i(w),$$

(62)
i.e., $G = \varepsilon^i \otimes \varepsilon_i = \varepsilon_i \otimes \varepsilon^i$.

Note that Eq. (62) is consistent with Eq. (47). We have indeed that

$$\varepsilon^i(v) \varepsilon_i(w) = \varepsilon^i(v) G_{ij} \varepsilon^j(w) = \varepsilon^i(v) G(e_i, e_j) \varepsilon^j(w) = G(v, w).$$

The well-known formula for the scalar product of simple $k$-vectors can be easily deduced from Eq. (61). It is:

$$(v_1 \wedge \ldots \wedge v_k) \cdot (w_1 \wedge \ldots \wedge w_k) = \det \begin{bmatrix} v_1 \cdot w_1 & \ldots & v_1 \cdot w_k \\ \ldots & \ldots & \ldots \\ v_k \cdot w_1 & \ldots & v_k \cdot w_k \end{bmatrix}.$$  

(63)

Proof

By using Eq. (44) and Eq. (62), and recalling the $k \times k$ determinant formula,

$$\det [a_{ij}] = \frac{1}{k!} \varepsilon^{i_1 \ldots i_k} \varepsilon^{j_1 \ldots j_k} a_{i_1 j_1} \ldots a_{i_k j_k}. $$

A straightforward calculation gives

$$(v_1 \wedge \ldots \wedge v_k) \cdot (w_1 \wedge \ldots \wedge w_k)$$

$$= \frac{1}{k!} v_1 \wedge \ldots \wedge v_k (\varepsilon^{s_1}, \ldots, \varepsilon^{s_k}) w_1 \wedge \ldots \wedge w_k (\varepsilon_{s_1}, \ldots, \varepsilon_{s_k})$$

$$= \frac{1}{k!} \varepsilon^{i_1 \ldots i_k} \varepsilon^{s_1} (v_{i_1}) \ldots \varepsilon^{s_k} (v_{i_k}) \varepsilon^{j_1 \ldots j_k} \varepsilon_{s_1} (w_{j_1}) \ldots \varepsilon_{s_k} (w_{j_k})$$

$$= \frac{1}{k!} \varepsilon^{i_1 \ldots i_k} \varepsilon^{j_1 \ldots j_k} \varepsilon^{s_1} (v_{i_1}) \varepsilon_{s_1} (w_{j_1}) \ldots \varepsilon^{s_k} (v_{i_k}) \varepsilon_{s_k} (w_{j_k})$$

$$= \frac{1}{k!} \varepsilon^{i_1 \ldots i_k} \varepsilon^{j_1 \ldots j_k} (v_{i_1} \cdot w_{j_1}) \ldots (v_{i_k} \cdot w_{j_k})$$

$$= \det [v_i \cdot w_j].$$

Now, we can generalize Eq. (60) in order to get the expected expansion formulas for $k$-vectors. For all $X \in \bigwedge^k V$ it holds two expansion formulas

$$X = \frac{1}{k!} X \cdot (e^{j_1} \ldots e^{j_k}) \cdot (e_{j_1} \wedge \ldots \wedge e_{j_k}), $$

(64)

Proof
For $X \in \bigwedge^k V$ with $k \geq 2$, by recalling Eq. (36) and Eq. (58) there are unique real numbers $X_{i_1...i_k}$ with $i_1, \ldots, i_k = 1, \ldots, n$ such that
\[
X = \frac{1}{k!} X_{i_1...i_k} e_{i_1} \wedge \ldots \wedge e_{i_k} = \frac{1}{k!} X_{i_1...i_k} e^{i_1} \wedge \ldots \wedge e^{i_k}.
\]
Indeed, take the scalar products $X \cdot (e^{j_1} \wedge \ldots \wedge e^{j_k})$. Using Eq. (63), Eq. (50) and Eq. (32) we have that
\[
X \cdot (e^{j_1} \wedge \ldots \wedge e^{j_k}) = \frac{1}{k!} X_{i_1...i_k} (e_{i_1} \cdot e_{j_1} \ldots e_{i_k} \cdot e_{j_k}) = X_{i_1...i_k} \delta_{j_1...j_k} = X \cdot (e^{j_1} \wedge \ldots \wedge e^{j_k}).
\]
Analogously, we can prove that $X_{j_1...j_k} = X \cdot (e_{j_1} \wedge \ldots \wedge e_{j_k})$.

### 4.2 Scalar Product of Multivectors

The scalar product of $X, Y \in \bigwedge V$, namely $X \cdot Y \in \mathbb{R}$, is defined by
\[
X \cdot Y = \sum_{k=0}^{n} \pi_k(X) \cdot \pi_k(Y).
\]  
(65)

Note that on the right side there appears the scalar products of $k$-vectors with $0 \leq k \leq n$, as defined by Eqs. (60) and (61). It means that if $X = (X_0, X_1, \ldots, X_n)$ and $Y = (Y_0, Y_1, \ldots, Y_n)$, then
\[
X \cdot Y = \sum_{k=0}^{n} X_k \cdot Y_k.
\]  
(66)

By using Eqs. (60) and (61) we can easily note that Eq. (65) can still be written as
\[
X \cdot Y = X_0 Y_0 + \sum_{k=1}^{n} \frac{1}{k!} X_k(\varepsilon^{i_1}, \ldots, \varepsilon^{i_k}) Y_k(\varepsilon_{i_1}, \ldots, \varepsilon_{i_k})
= X_0 Y_0 + \sum_{k=1}^{n} \frac{1}{k!} X_k(\varepsilon_{i_1}, \ldots, \varepsilon_{i_k}) Y_k(\varepsilon^{i_1}, \ldots, \varepsilon^{i_k}).
\]  
(67)
It is important to observe that the operation defined by Eq. (65) is indeed a well-defined scalar product on $\wedge V$, since it is symmetric, satisfies the distributive laws, has the mixed associative property and is not degenerate, i.e., if $X \cdot Y = 0$ for all $Y$, then $X = 0$.

Let us take $X_p \in \wedge^p V$ and $Y_q \in \wedge^q V$. By using Eq. (65) and Eq. (19) we have that

$$\tau_p(X_p) \cdot \tau_q(Y_q) = \sum_{k=0}^{n} \pi_k \circ \tau_p(X_p) \cdot \pi_k \circ \tau_q(Y_q)$$

$$= \sum_{k=0}^{n} \begin{cases} 0_k, & k \neq p \\ X_p, & k = p \end{cases} \cdot \begin{cases} 0_k, & k \neq q \\ Y_q, & k = q \end{cases}.$$

The sum on the right side of this equation is $0 = \sum_{k=0}^{n} 0_k \cdot 0_k$ unless there exists some $k_0$ with $0 \leq k_0 \leq n$ such that $k_0 = p$ and $k_0 = q$. Whence, we see that for $p \neq q$ there is no number $k_0$ to satisfy the required conditions. But, for $p = q$ we have that $k_0 = p = q$ trivially satisfies them. Thus, from the above equation it follows that

$$\tau_p(X) \cdot \tau_q(Y_q) = \begin{cases} 0, & \text{if } p \neq q \\ X_p \cdot Y_p, & \text{if } p = q \end{cases}.$$

5 Contracted Products

The left contracted product of $X_p \in \wedge^p V$ and $Y_q \in \wedge^q V$ with $0 \leq p \leq q \leq n$, namely $X_p \cdot Y_q \in \wedge^{q-p} V$, is defined by the following axioms

Ax-i For all $X_p, Y_p \in \wedge^p V$:

$$X_p \cdot Y_p = \tilde{X}_p \cdot \tilde{Y}_p = \tilde{X}_p \cdot \tilde{Y}_p.$$

Ax-ii For all $X_p \in \wedge^p V$ and $Y_q \in \wedge^q V$ with $p < q$:

$$X_p \cdot Y_q = \frac{1}{(q-p)!} (X_p \wedge e^{i_1} \wedge \ldots \wedge e^{i_{q-p}}) \cdot Y_q e^{i_1} \wedge \ldots \wedge e^{i_{q-p}}$$

$$= \frac{1}{(q-p)!} (X_p \wedge e^{i_1} \wedge \ldots \wedge e_{i_{q-p}}) \cdot Y_q e^{i_1} \wedge \ldots \wedge e^{i_{q-p}}. \quad (70)$$

The right contracted product of $X_p \in \wedge^p V$ and $Y_q \in \wedge^q V$ with $n \geq p \geq q \geq 0$, namely $X_p \cdot Y_q \in \wedge^{p-q} V$, is defined by the following axioms
Ax-i For all \(X, Y \in \bigwedge^p V\):
\[
X_p \wedge Y_p = \widetilde{X}_p \cdot Y_p = X_p \cdot \tilde{Y}_p.
\] (71)

Ax-ii For all \(X \in \bigwedge^p V\) and \(Y \in \bigwedge^q V\) with \(p > q\):
\[
X_p \wedge Y_q = \frac{1}{(p-q)!} X_p \cdot (e_{i_1} \wedge \ldots \wedge e_{i_{p-q}} \wedge \tilde{Y}_q) e_{i_1} \wedge \ldots \wedge e_{i_{p-q}}\]
\[
= \frac{1}{(p-q)!} X_p \cdot (e_{i_1} \wedge \ldots \wedge e_{i_{p-q}} \wedge \tilde{Y}_q) e_{i_1} \wedge \ldots \wedge e_{i_{p-q}}.\] (72)

It should be noticed that the \((q-p)\)-vector defined by Eq. (70) and the \((p-q)\)-vector defined by Eq. (72) do not depend on the choice of the reciprocal bases \(\{e_i\}\) and \(\{e^i\}\) used for calculating them.

Let us take \(X \in \bigwedge^p V\) and \(Y \in \bigwedge^q V\) with \(p \leq q\). For all \(Z \in \bigwedge^{q-p} V\) the following identity holds
\[
(X_p \wedge Y_q) \cdot Z_{q-p} = Y_q \cdot (\widetilde{X}_p \wedge Z_{q-p}).
\] (73)

For \(p < q\) Eq. (73) follows directly from Eq. (70) and Eq. (64). But, for \(p = q\) it trivially follows by taking into account Eq. (65), etc.

Let us take \(X \in \bigwedge^p V\) and \(Y \in \bigwedge^q V\) with \(p \geq q\). For all \(Z \in \bigwedge^{p-q} V\) the following identity holds
\[
(X_p \wedge Y_q) \cdot Z_{p-q} = X_p \cdot (Z_{p-q} \wedge \tilde{Y}_q).
\] (74)

For \(p > q\) Eq. (74) follows directly from Eq. (72) and Eq. (63). For \(p = q\) it follows from Eq. (71).

We present now the basic properties of the elementary contracted products of \(p\)-vector with \(q\)-vector.

For any \(V, W \in \bigwedge^p V\) and \(X, Y \in \bigwedge^q V\) with \(p \leq q\)
\[
(V_p + W_p) \wedge X_q = V_p \wedge X_q + W_p \wedge X_q,
\]
\[
V_p \wedge (X_q + Y_q) = V_p \wedge X_q + V_p \wedge Y_q \text{ (distributive laws)}.\] (75)

For any \(V, W \in \bigwedge^p V\) and \(X, Y \in \bigwedge^q V\) with \(p \geq q\)
\[
(V_p + W_p) \wedge X_q = V_p \wedge X_q + W_p \wedge X_q,
\]
\[
V_p \wedge (X_q + Y_q) = V_p \wedge X_q + V_p \wedge Y_q \text{ (distributive laws)}.\] (76)
These distributive laws are immediate consequence of the distributive laws for the exterior product and the scalar product.

For any $X_p \in \bigwedge^p V$ and $Y_q \in \bigwedge^q V$ with $p \leq q$

$$X_p \cdot Y_q = (-1)^{p(q-p)} Y_q \cdot X_p.$$  \hfill (77)

Indeed, by using Eq. (73), Eq. (40) and Eq. (74) we have that

$$(X_p \cdot Y_q) \cdot Z_{q-p} = Y_q \cdot (\tilde{X}_p \wedge Z_{q-p}) = (-1)^{p(q-p)} Y_q \cdot (Z_{q-p} \wedge \tilde{X}_p) = (-1)^{p(q-p)} (Y_q \cdot X_p) \cdot Z_{q-p},$$

hence, by non-degeneracy of scalar product, the required result follows.

5.1 Contracted Product of Multivectors

The left and right contracted products of $X, Y \in \bigwedge V$, namely $X \cdot Y \in \bigwedge V$ and $X \cdot Y \in \bigwedge V$, are defined by

$$X \cdot Y = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \tau_k (\pi_j (X) \cdot \pi_{k+j} (Y)).$$ \hfill (78)

$$X \cdot Y = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \tau_k (\pi_{k+j} (X) \cdot \pi_j (Y)).$$ \hfill (79)

Note that on the right side of Eq. (78) there appear the left contracred products of $j$-vectors with $(k+j)$-vectors, as defined by Eqs. (69) and (70). On the right side of Eq. (79) there appear the right contracted product of $(k+j)$-vectors with $j$-vectors, as defined by Eqs. (71) and (72). It means that

$$\pi_k (X \cdot Y) = \sum_{j=0}^{n-k} \pi_j (X) \cdot \pi_{k+j} (Y),$$ \hfill (80)

$$\pi_k (X \cdot Y) = \sum_{j=0}^{n-k} \pi_{k+j} (X) \cdot \pi_j (Y) \text{ for each } k = 0, 1, \ldots, n.$$ \hfill (81)

And, if $X = (X_0, \ldots, X_k, \ldots, X_n)$ and $Y = (Y_0, \ldots, Y_k, \ldots, Y_n)$, then

$$X \cdot Y = (\sum_{j=0}^{n} X_j \cdot Y_j, \ldots, \sum_{j=0}^{n-k} X_j \cdot Y_{k+j}, \ldots, X_0 Y_n),$$ \hfill (82)

$$X \cdot Y = (\sum_{j=0}^{n} X_j \cdot Y_j, \ldots, \sum_{j=0}^{n-k} X_{k+j} \cdot Y_j, \ldots, X_n Y_0).$$ \hfill (83)
These contracted products are internal laws on $\Lambda V$. Each of $\triangleright$ and $\triangleleft$ satisfies the distributive laws, as easily verify by recalling Eq. (75) and Eq. (76), but both interior products are non associative. $\Lambda V$ endowed with each one of these contracted products is a non-associative algebra which will be called an "interior algebra of multivectors.

Let us take $X_p \in \Lambda^p V$ and $Y_q \in \Lambda^q V$. By using Eq. (80) and Eq. (19) we have that

$$
\tau_k (\tau_p(X_p) \triangleright \tau_q(Y_q)) = \sum_{j=0}^{n-k} \pi_j \circ \tau_p(X_p) \triangleright \pi_{k+j} \circ \tau_q(Y_q)
$$

$$
= \sum_{j=0}^{n-k} \left\{ \begin{array}{ll}
0, & j \neq p \\
X_p, & j = p \\
Y_q, & k + j = q 
\end{array} \right.
$$

The sum on the right side of the equation above is $0_k = \sum_{j=0}^{n-k} 0_j \cdot 0_{k+j}$, for each $k = 0, 1, \ldots, n$ unless there exists some $k_0$ with $0 \leq k_0 \leq n$ such that $k_0 + p = q$. Whence, we see that for $p > q$ there is no number $k_0$ which can satisfy the required conditions. But, for $p \leq q$ we have that $k_0 = q - p$ is the unique number to satisfy them. Then, from the equation above it follows that

$$
\tau_k (\tau_p(X_p) \triangleright \tau_q(Y_q)) = 0_k, \text{ for } p > q
$$

$$
\tau_{q-p} (\tau_p(X_p) \triangleright \tau_q(Y_q)) = X_p \triangleright Y_q, \text{ for } p \leq q.
$$

Hence, by taking into account Eq. (12) we finally obtain

$$
\tau_p(X_p) \triangleright \tau_q(Y_q) = \left\{ \begin{array}{ll}
0, & \text{if } p > q \\
\tau_{q-p}(X_p \triangleright Y_q), & \text{if } p \leq q 
\end{array} \right. \quad (84)
$$

By following analogous steps to those which allowed us to arrive to Eq. (84) we can also obtain

$$
\tau_p(X_p) \triangleleft \tau_q(Y_q) = \left\{ \begin{array}{ll}
0, & \text{if } p < q \\
\tau_{p-q}(X_p \triangleleft Y_q), & \text{if } p \geq q 
\end{array} \right. \quad (85)
$$

We finalize this section by presenting two noticeable formulas involving the contracted products and the scalar product, and two other remarkable
formulas relating the contracted products to the exterior product and scalar product.

For any $X, Y, Z \in \bigwedge V$

$$(X \Leftrightarrow Y) \cdot Z = Y \cdot (\bar{X} \wedge Z), \quad (86)$$

$$(X \Leftrightarrow Y) \cdot Z = X \cdot (Z \wedge \bar{Y}). \quad (87)$$

**Proof**

We only give the proof for the first statement, the other being analogous.

Form the summation identity

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} A_{k+j} \cdot (B_{j} \wedge C_{k}) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} A_{k} \cdot (B_{j} \wedge C_{k-j})$$

and by using Eqs. (80), (82), (73) and (42), a straightforward calculation yields

$$(X \Leftrightarrow Y) \cdot Z = \sum_{k=0}^{n} \pi_{k} (X \Leftrightarrow Y) \cdot Z_{k} = \sum_{k=0}^{n} \sum_{j=0}^{n-k} X_{j+k} \cdot (\bar{X}_{j} \wedge Z_{k})$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{n-k} (X_{j+k} \cdot Z_{k}) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} Y_{j+k} \cdot (\bar{X}_{j} \wedge Z_{k-j})$$

$$= \sum_{k=0}^{n} Y_{k} \cdot \pi_{k} (\bar{X} \wedge Z) = Y \cdot (\bar{X} \wedge Z). \blacksquare$$

For any $X, Y, Z \in \bigwedge V$

$$X \Leftrightarrow (Y \Leftrightarrow Z) = (X \wedge Y) \cdot Z, \quad (88)$$

$$(X \Leftrightarrow Y) \Leftrightarrow Z = X \cdot (Y \wedge Z). \quad (89)$$

**Proof**

We prove only Eq. (88). The proof of Eq. (89) is analogous and will be left to the reader.

Let $W \in \bigwedge V$. By using Eq. (86) and the associative law for the exterior product of multivectors, we have that

$$(X \Leftrightarrow (Y \Leftrightarrow Z)) \cdot W = (Y \Leftrightarrow Z) \cdot (\bar{X} \wedge W) = Z \cdot ((\bar{Y} \wedge \bar{X}) \wedge W)$$

$$= Z \cdot ((\bar{X} \Leftrightarrow \bar{Y}) \wedge W) = ((X \Leftrightarrow Y) \Leftrightarrow Z) \cdot W. \blacksquare$$
Hence, by the non-degeneracy of the $G$-scalar product, it follows the required result.

6 Clifford Product and $\mathcal{C}l(V,G)$

The two interior algebras together with the exterior algebra allow us to define a Clifford product of multivectors which is also an internal law on $\bigwedge V$. The Clifford product of $X, Y \in \bigwedge V$, denoted by juxtaposition $XY \in \bigwedge V$, is defined by the following axioms

Ax-ci For all $\alpha \in \mathbb{R}$ and $X \in \bigwedge V$:
\[
\tau_0(\alpha)X = \alpha X \quad \text{(scalar multiplication of } X \text{ by } \alpha). \tag{90}
\]

Ax-cii For all $v \in V$ and $X \in \bigwedge V$:
\[
\tau_1(v)X = \tau_1(v)_\downarrow X + \tau_1(v) \wedge X, \tag{91}
\]
\[
X \tau_1(v) = X_L \tau_1(v) + X \wedge \tau_1(v). \tag{92}
\]

Ax-ciii For all $X, Y, Z \in \bigwedge V$:
\[
(XY)Z = X(YZ). \tag{93}
\]

The Clifford product is distributive and associative. $\bigwedge V$ endowed with this Clifford product is an associative algebra which will be called the geometric algebra of multivectors associated to a metric structure $(V, G)$. It will be denoted by $\mathcal{C}l(V,G)$.

Now, due to the existence of a linear isomorphism between the $k$-vectors and the $k$-homogeneous multivector as given by Eqs. (8) and (9, and by recalling the remarkable propositions about the exterior product, the scalar product and the contracted products of multivectors as given by Eq. (44), Eq. (68) and Eqs. (84) and (85), we can introduce a notational convention which are more convenient for performing calculations with multivectors using the geometric algebra.

In whichever addition or products of multivectors all $k$-homogeneous multivectors $\tau_k(X_k)$ will be identified with its corresponding $k$-vector $X_k$.

i. Any addition of multivectors $\tau_p(X_p) + \tau_q(Y_q)$, $\tau_p(X_p) + Y$ and $X + \tau_q(Y_q)$ will be respectively denoted by $X_p + Y_q$, $X_p + Y$ and $X + Y_q$.

ii. Any product of multivectors $\tau_p(X_p) * \tau_q(Y_q)$, $\tau_p(X_p) * Y$ and $X * \tau_q(Y_q)$, where $*$ means either $(\wedge)$, $(\cdot)$, $(\downarrow, \uparrow)$ or (Clifford product), will be respectively denoted by $X_p * Y_q$, $X_p * Y$ and $X * Y_q$.
7 Euclidean and pseudo-Euclidean Geometric Algebras

Let us equip $V$ with an arbitrary (but fixed once for all) euclidean metric $G_E$, i.e., a metric tensor on $V$ with the strong condition of being positive definite, i.e.,

$$G_E(v, v) \geq 0 \text{ for all } v \in V \text{ and if } G_E(v, v) = 0, \text{ then } v = 0. \quad (94)$$

$V$ endowed with an euclidean metric $G_E$, i.e., $(V, G_E)$, is called an euclidean metric structure for $V$. Sometimes, $(V, G_E)$ is said to be an euclidean space.

Associated to $(V, G_E)$ an euclidean scalar product of vectors $v, w \in V$ is given by

$$v \cdot G_E w = G_E(v, w). \quad (95)$$

We introduce also an euclidean scalar product of $p$-vectors $X_p, Y_p \in \wedge^p V$ and euclidean scalar product of multivectors $X, Y \in \wedge V$, namely $X_p \cdot G_E Y_p \in \mathbb{R}$ and $X \cdot G_E Y \in \mathbb{R}$, using respectively the Eqs..(60) and (61), and Eq.(65).

Let us take any metric tensor $G$ on the vector space $V$. Associated to the metric structure $(V, G)$ a scalar product of vectors $v, w \in V$ is represented by

$$v \cdot G w = G(v, w). \quad (96)$$

Of course, the corresponding scalar product of $p$-vectors $X_p, Y_p \in \wedge^p V$ and scalar product of multivectors $X, Y \in \wedge V$, namely $X_p \cdot G Y_p \in \mathbb{R}$ and $X \cdot G Y \in \mathbb{R}$, are defined respectively by Eqs..(60) and (61) and Eq.(65).

We will find a relationship between $(V, G)$ and $(V, G_E)$, thereby showing how an arbitrary $G$-scalar product on $\wedge^p V$ and $\wedge V$ is related to a $G_E$-scalar products on $\wedge^p V$ and $\wedge V$.

Choose once and for all a fundamental euclidean metric structure $(V, G_E)$. For any metric tensor $G$ there exists an unique linear operator $g$ such that for all $v, w \in V$

$$v \cdot G w = g(v) \cdot G_E w. \quad (97)$$

Such $g$ is given by

$$g(v) = (v \cdot G e_k) e^k \cdot G_E = (v \cdot G e_k) e_k, \quad (98)$$

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where \( \{e_k\} \) is any basis of \( V \), and \( \{e^k_{G_E}\} \) is its reciprocal basis with respect to \( (V,G_E) \), i.e., \( e_k \cdot e^l_{G_E} = \delta^l_k \). Note that the vector \( g(v) \) does not depend on the basis \( \{e_k\} \) chosen for calculating it.

We prove now that \( g(v) \) as given by Eq.(98) satisfies Eq.(97). Using Eq.(56) we have,

\[
g(v) \cdot w = (v \cdot e_k)(e^k_{G_E} \cdot w) = (v \cdot (e^k_{G_E} \cdot e_k)) = v \cdot w.
\]

Now, suppose that there is some \( g' \) which satisfies Eq.(97), i.e., \( v \cdot w = g'(v) \cdot w \). Then, by using once again Eq.(56) we have that

\[
g'(v) = (g'(v) \cdot e_k) e^k_{G_E} = (v \cdot e_k) e^k_{G_E} = g(v),
\]

i.e., \( g' = g \).

So the existence and the uniqueness of such a linear operator \( g \) are proved.

Since \( G \) is a symmetric covariant 2-tensor over \( V \), i.e., \( G(v,w) = G(w,v) \) \( \forall v,w \in V \), it follows from Eq.(97) that \( g \) is an adjoint symmetric linear operator with respect to \( (V,G_E) \), i.e.,

\[
g(v) \cdot w = v \cdot g(w),
\]

(99)

The property expressed by Eq.(99) is coded by the equation \( g = g^{\dagger(G_E)} \).

Since \( G \) is a non-degenerate covariant 2-tensor over \( V \) (i.e., if \( G(v,w) = 0 \) \( \forall w \in V \), then \( v = 0 \)) it follows that \( g \) is a non-singular (invertible) linear operator. Its inverse linear operator is given by the noticeable formula

\[
g^{-1}(v) = G^{jk}(v \cdot e_j)e_k,
\]

(100)

where \( G^{jk} \) are the \( jk \)-entries of the inverse matrix of \( [G_{jk}] \) with \( G_{jk} \equiv G(e_j,e_k) \). Note that the vector \( g^{-1}(v) \) does not depend on the basis \( \{e_k\} \) chosen for its calculation.

We must prove that indeed \( g^{-1} \circ g = g \circ g^{-1} = i_V \), where \( i_V \) is the identity function for \( V \).

By using Eq.(100), Eq.(97), Eq.(99) for \( (V,G) \) and Eq.(56) for \( (V,G) \), we have that

\[
g^{-1} \circ g(v) = G^{jk} (g(v) \cdot e_j)e_k = G^{jk} (v \cdot e_j)e_k = (v \cdot e_j)e_k = v.
\]
i.e., $g^{-1} \circ g = i_V$.

By using Eq. (100), Eq. (56) for $(V, G_E)$, Eq. (97), Eq. (49) for $(V, G)$, Eq. (50) for $(V, G)$ and Eq. (56) for $(V, G_E)$, we have that

$$g \circ g^{-1}(v) = G_{jk} \left( v \cdot e_j \cdot g(e_k) \cdot e_l \right)_{G_E} = (v \cdot e_j)(e_j^l \cdot G_E)_{G_E} = (v \cdot e_j)\delta^l_j e_l = v,$$

i.e., $g \circ g^{-1} = i_V$.

It should be remarked that such $g$ only depends on the choice of the fundamental euclidean structure $(V, G_E)$. However, $g$ codifies all the geometric information contained in $G$. Such $g$ will be called the metric operator for $G$.

Now, we show how the scalar product $X_p \cdot Y_p$ is related to the scalar product $X_p \cdot G_E \cdot Y_p$.

For any simple $k$-vectors $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k V$ and $w_1 \wedge \ldots \wedge w_k \in \bigwedge^k V$ it holds

$$ (v_1 \wedge \ldots v_k) \cdot (w_1 \wedge \ldots w_k) = g(v_1 \wedge \ldots v_k) \cdot (w_1 \wedge \ldots w_k), \quad (101) $$

where $g$ is the so-called outermorphism (or exterior power)\(^8\) of $g$.

**Proof**

We will use Eq. (63) for $(V, G)$ and $(V, G_E)$. By using Eq. (94) and a fundamental property of any outermorphism, a straightforward calculation yields

$$ (v_1 \wedge \ldots v_k) \cdot (w_1 \wedge \ldots w_k) = \det \begin{bmatrix} v_1 \cdot w_1 & \cdots & v_1 \cdot w_k \\ \vdots & \ddots & \vdots \\ v_k \cdot w_1 & \cdots & v_k \cdot w_k \end{bmatrix} = \det \begin{bmatrix} G(v_1) \cdot w_1 & \cdots & G(v_1) \cdot w_k \\ \vdots & \ddots & \vdots \\ G(v_k) \cdot w_1 & \cdots & G(v_k) \cdot w_k \end{bmatrix} = (g(v_1) \wedge \ldots g(v_k)) \cdot (w_1 \wedge \ldots w_k)$$

$$ = g(v_1 \wedge \ldots v_k) \cdot (w_1 \wedge \ldots w_k). \blacksquare$$

---

\(^8\) $g$ is the unique linear operator on $\bigwedge V$ which satisfies the following conditions: (i) $\alpha \in \mathbb{R}: g(\alpha) = \alpha$, (ii) $v \in V: g(v) = g(v)$ and (iii) $X, Y \in \bigwedge V: g(X \wedge Y) = g(X) \wedge g(Y)$. 25
For any $k$-vectors $X_k, Y_k \in \bigwedge^k V$ it holds
\[
X_k \cdot Y_k = g(X_k) \cdot G E Y_k. \tag{102}
\]

\textbf{Proof}

From the distributive laws and the mixed associative property for the scalar product of $k$-vectors with $k \geq 1$, we get using Eq.(36) and Eq.(101) that
\[
X_k \cdot G Y_k = \left( \frac{1}{k!} \right)^2 X_{i_1 \ldots i_k} Y_{j_1 \ldots j_k} (e_{i_1} \wedge \ldots e_{i_k}) \cdot (e_{j_1} \wedge \ldots e_{j_k})
\]
\[
= \left( \frac{1}{k!} \right)^2 X_{i_1 \ldots i_k} Y_{j_1 \ldots j_k} \left( \frac{1}{k!} \right) Y_{i_1 \ldots i_k} \wedge \ldots \wedge e_{i_k} \wedge (e_{j_1} \wedge \ldots e_{j_k})
\]
\[
= \left( \frac{1}{k!} \right)^2 X_{i_1 \ldots i_k} Y_{j_1 \ldots j_k} \left( \frac{1}{k!} \right) Y_{i_1 \ldots i_k} \wedge \ldots \wedge e_{i_k} \wedge (e_{j_1} \wedge \ldots e_{j_k})
\]
\[
= \left( \frac{1}{k!} \right)^2 X_{i_1 \ldots i_k} Y_{j_1 \ldots j_k} \left( \frac{1}{k!} \right) Y_{i_1 \ldots i_k} \wedge \ldots \wedge e_{i_k} \wedge \wedge (e_{j_1} \wedge \ldots e_{j_k})
\]
\[
= \left( \frac{1}{k!} \right)^2 X_{i_1 \ldots i_k} Y_{j_1 \ldots j_k} \left( \frac{1}{k!} \right) Y_{i_1 \ldots i_k} \wedge \ldots \wedge e_{i_k} \wedge \wedge (e_{j_1} \wedge \ldots e_{j_k})
\]
\[
= g(X_k) \cdot G E Y_k.
\]

Note that for the special case of the scalar product of scalars, Eq.(102) trivially holds by using Eq.(60) and taking into account a fundamental property of any outermorphism.\[ \square \]

Next, we show how the scalar product $X \cdot Y$ is related to the scalar product $X \cdot G Y$. For any multivectors $X, Y \in \bigwedge V$ it holds
\[
X \cdot G Y = g(X) \cdot G E Y. \tag{103}
\]

\textbf{Proof}

We will use Eq.(63) for $(V, G)$ and $(V, G_E)$. By using Eq.(102) and taking into account that $g \circ \pi_k = \pi_k \circ g$ (the grade-preserving property for the outermorphism), a straightforward calculation yields
\[
X \cdot G Y = \sum_{k=0}^{n} \pi_k(X) \cdot \pi_k(Y) = \sum_{k=0}^{n} g \circ \pi_k(X) \cdot G E Y_k
\]
\[
= \sum_{k=0}^{n} g \circ \pi_k(Y) = g(X) \cdot G E Y. \square
\]

The $G$-contracted products are related to the $G_E$-contracted products by two noticeable formulas.
For any $X, Y \in \bigwedge V$

\begin{align*}
X \downarrow Y &= g(X) \downarrow g^{-1}(Y), \quad (104) \\
X \uparrow Y &= X \uparrow g(Y). \quad (105)
\end{align*}

**Proof**

We only give the proof of the first statement, the other is analogous. By using Eq. (86) for the metric structure $(V, G)$ we have that for all $X, Y, Z \in \bigwedge V$

\[(X \downarrow Y) \cdot G^{-1}(Z) = Y \cdot (\tilde{X} \wedge g^{-1}(Z)),\]

by using Eq. (103)

\[g(X \downarrow Y) \cdot G^{-1}(Z) = g(Y) \cdot (\tilde{X} \wedge g^{-1}(Z)).\]

Since the outermorphism of an adjoint symmetric linear operator on $V$ is itself an adjoint symmetric linear operator on $\bigwedge V$, then recalling a fundamental property of the outermorphism, it follows that

\[(X \downarrow Y) \cdot G^{-1}(Z) = Y \cdot (g(X) \wedge g^{-1}(Z)),\]

now, by using Eq. (86) for the euclidean metric structure $(V, G_E)$ we have that for all $X, Y, Z \in \bigwedge V$

\[(X \downarrow Y) \cdot G_E^{-1}(Z) = (g(X) \downarrow g(Y)) \cdot G_E^{-1}(Z),\]

hence, by non-degeneracy of the $G_E$-scalar product of multivectors, it follows that

\[(X \downarrow Y) = g(X) \downarrow g(Y).\]

**8 Conclusions**

In this paper, the first in a series of eight we start setting the algebraic basis of the geometrical calculus. We introduced the concept of multivectors in

\[g^{-1} \equiv (g^{-1}) = (g)^{-1}.\]
section 2 and their exterior algebra in section 3. In section 4 we introduced the key concept of the scalar product of multivectors and in section 5 the concepts of right and left contractions and interior algebras. In section 6 we give a definition of a general real Clifford (or geometrical) algebra of multivectors and in section 7 we study in details the euclidean and pseudo-euclidean geometrical algebras. We emphasize that our presentation of the geometric (or Clifford) algebras has been devised in order to give to the reader a powerful computational tool and to permit the introduction of the theory of extensors in a natural and simple way. To better achieve our objective we gave in this paper and in the following ones many details of the calculations (tricks of the trade). To readers interested in aspects of the general theory of Clifford algebras not covered here we strongly recommend the excellent textbooks [7, 13, 14, 1].

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