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Generic section of a hyperplane arrangement and twisted Hurewicz maps

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**Notation**

A *hyperplane arrangement* is a collection

\[ \mathcal{A} = \{H_1, H_2, \ldots, H_n\} \]

of affine hyperplanes \( H_i \subset \mathbb{C}^\ell \). And denote

\[ M(\mathcal{A}) = \mathbb{C}^\ell - \bigcup_{H \in \mathcal{A}} H. \]
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3. Main results:
   Surjectivity of twisted Hurewicz maps.
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5. Corollary.
1 Randell’s result

Thm. (Randell) If $k \geq 2$, the Hurewicz map

$$h : \pi_k(M(A), x_0) \longrightarrow H_k(M(A), \mathbb{Z})$$

is the zero map.

(Proof) Let $f : S^k \rightarrow M(A)$. Consider

$$\begin{align*}
H^k(S^k) & \xrightarrow{k^*} H^k(M(A)) \\
\wedge H^1(S^k) = 0 & \xleftarrow{k^*} \wedge H^1(M(A)).
\end{align*}$$

\uparrow \text{surj.}$$
1 Randell’s result

\[ \pi_k(M) \quad h = 0 \quad H_k(M, \mathcal{L}) \]

Goal: Twisted version detects! (Sometimes)
Twisted Hurewicz maps

Generalities:
Let $\mathcal{L}$ be a local system on $X$, $C$ be a closed manifold of $\dim_{\mathbb{R}} C = k$, $f : (C, \ast) \to (X, x_0)$ a continuous map. The map $f$ and a section $t \in \Gamma(C, f^* \mathcal{L})$ determines a twisted cycle $[f] \otimes t \in H_k(X, \mathcal{L})$. 
2 Twisted Hurewicz maps

Let $\mathcal{L}$ be a rank one local system on $M(\mathcal{A})$,

$$f : (S^k, *) \to (M(\mathcal{A}), x_0)$$

a continuous map. $k \geq 2$. Since $S^k$ is simply connected, $f^* \mathcal{L}$ on $S^k$ is trivial and hence

$$\Gamma(S^k, f^* \mathcal{L}) \cong \mathcal{L}_{x_0}.$$
2 Twisted Hurewicz maps

We have

\[ h : \pi_k(S^k, *) \otimes \mathcal{L}_{x_0} \to H_k(M(A), \mathcal{L}) \]

the *twisted Hurewicz map*. (Note that it is defined only when \( k \geq 2 \).)

e.g. If \( \mathcal{L} \) is a trivial local system, then \( h \) is the classical one.
3 Main result

Def. An arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$ is called *generic-section type* if there is another arrangement $\tilde{\mathcal{A}}$ of rank $(\ell + 1)$ in $\mathbb{C}^{\ell+1}$ and a generic hyperplane $F \subset \mathbb{C}^{\ell+1}$ such that $\mathcal{A}$ is isomorphic to $F \cap \tilde{\mathcal{A}}$. 

![Diagram](image)
3 Main result

Thm. Assume $\ell \geq 2$. If $\mathcal{A}$ is generic-section type and $\mathcal{L}$ is nonresonant, then the top twisted Hurewicz map

$$h : \pi_\ell(M(\mathcal{A}), x_0) \otimes \mathcal{L}_{x_0} \longrightarrow H_\ell(M(\mathcal{A}), \mathcal{L})$$

is surjective.

Note: $H_\ell(M(\mathcal{A}), \mathcal{L}) \cong \mathbb{C}^{\left|\chi(M)\right|}$. Hence $\pi_\ell(M) \neq 0$ (Randell).
4 Proof

Proof is based on two results:

– Lefschetz Theorem on hyperplane section, (or minimality of $M(A)$).
– Nonresonance theorem for local system homology groups.
4 Proof

\[ \tilde{M} = M(\tilde{A}), \]

\[ F \subset \mathbb{C}^{\ell+1}: \text{a generic. hyperplane.} \]

Thm. (Lefschetz)

\[ \tilde{M} \cong (\tilde{M} \cap F) \cup_{\varphi} \bigcup_{i=1}^{b} D^{\ell+1} \]

attach \((\ell+1)\)-dim cells
\[ \tilde{M} \simeq (\tilde{M} \cap F) \cup_{\varphi} \bigcup_{i=1}^{b} D^{\ell+1} \]

attach \((\ell+1)\)-dim cells

\[ \partial D^{\ell+1} \simeq S^{\ell} \]

Hyperplane section
4 Proof

\[ \tilde{M} \simeq (\tilde{M} \cap F) \cup_{\varphi} \bigcup_{i=1}^{b} D^{\ell+1} \]

attach \((\ell+1)\)-dim cells

How many \((\ell + 1)\)-dim cells to attach?

Minimality (Dimca-Papadima-Randell-Suciu)

\[ \implies b = b_{\ell+1}(\tilde{M}). \]
4 Proof

\[ \tilde{M} = M(\tilde{\mathcal{A}}) \text{ and } M = M(\mathcal{A}) = \tilde{M} \cap F. \]

\[ \tilde{M} \simeq M \cup_{\varphi} \bigcup_{i=1}^{b_{\ell+1}} D^{\ell+1} \]

attach \((\ell+1)\)-dim cells

Associated twisted chain complexes:

\[ C_{\bullet}(\tilde{M}) = C_{\bullet}(M) \oplus C^{b_{\ell+1}}. \]
4 Proof

\[ C_{\ell+1}(\tilde{M}) \xrightarrow{\partial_L} C_\ell(\tilde{M}) \rightarrow \cdots \rightarrow C_0(\tilde{M}) \]

Nonresonance Theorem:
Suppose \( \mathcal{L} \) is a generic local system. Then only \( H_{\ell+1}(C_\bullet(\tilde{M})) \) and \( H_\ell(C_\bullet(M)) \) survive.
4 Proof

\[ C_{\ell+1}(\tilde{M}) \xrightarrow{\partial_{\mathcal{L}}} C_{\ell}(\tilde{M}) \rightarrow \cdots \rightarrow C_0(\tilde{M}) \]

Only \( H_{\ell+1}(C_{(*)}(\tilde{M})) \) and \( H_{\ell}(C_{(*)}(M)) \) survive.

Observation 1:

\[ \partial_{\mathcal{L}} : C_{\ell+1}(\tilde{M}) \rightarrow H_{\ell}(M, \mathcal{L}) \] is surjective.
Recall the decomposition

\[ \tilde{M} \simeq M \cup_{\varphi} \bigcup_{i=1}^{b_{\ell+1}} D_{\ell+1} \]

is defined by attaching maps

\[ \varphi_i : \partial(D_{\ell+1}) = S^{\ell} \longrightarrow M. \]
Observation 2: The twisted boundary map splits

\[ C_{\ell+1}(\tilde{M}) \xrightarrow{\partial_{\mathcal{L}}} C_{\ell}(M) \]

\[ \pi_{\ell}(M) \otimes \mathcal{L}_{x_0} \xrightarrow{h} H_{\ell}(M, \mathcal{L}) \]

to the twisted Hurewicz map \( h \). Thus

\[ h : \pi_{\ell}(M) \otimes \mathcal{L}_{x_0} \longrightarrow H_{\ell}(M, \mathcal{L}) \]

is surjective. \(\square\)
Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^\ell$ of generic section type and $\mathcal{L}$ be a nonresonant rank one local system. Then every twisted $\ell$-cycle is represented by an $\ell$-dimensional sphere.
(Falk) Let $\mathcal{A}$ be a line arrangement in $\mathbb{C}^2$ with $|\mathcal{A}| \geq 3$. Suppose no two lines are parallel. Then $\pi_2(M(\mathcal{A})) \neq 0$. 
(Falk) Let $\mathcal{A}$ be a line arrangement in $\mathbb{C}^2$. Let $F$ be a generic line. Then $\mathcal{A} \cup \{F\}$ is not $K(\pi, 1)$. 
6 Reference

M. Falk, $K(\pi, 1)$ arrangements. *Topology*, 34 (1995) 141–154.

R. Randell, Homotopy and group cohomology of arrangements. *Topology and its applications*, 78 (1997) 201–213.

M. Yoshinaga, Generic section of a hyperplane arrangement and twisted Hurewicz maps. *Topology and its applications*, 2008 (to appear).