Chapter 4

A generalization of Szász-type operators which preserves constant and quadratic test functions

Recently, Varma and Taşdelen [85] defined Szász type operators via Charlier polynomials [46] with generating function of the form:

\[ e^t (1 - \frac{t}{a})^u = \sum_{k=0}^{\infty} C^{(a)}_k (u) \frac{t^k}{k!}, \quad |t| < a \]  

(4.0.1)

and the explicit representation

\[ C^{(a)}_k (u) = \sum_{r=0}^{k} \binom{k}{r} (-u)^r \left( \frac{1}{a} \right)^r, \]

\[(\alpha)_k \text{ is the Pochhammer's symbol given by } \]

\[ (\alpha)_0 = 1, (\alpha)_k = \alpha(\alpha + 1) \ldots (\alpha + k - 1), \quad k \in \mathbb{N}. \]

For \( a > 0 \) and \( u \leq 0 \), Charlier polynomials are positive. Using these polynomials, they [85] defined the Szász-type operators as follows:

\[ L_n(f; x, a) = e^{-1} \left( 1 - \frac{1}{a} \right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C^{(a)}_k \left( - (a-1)nx \right)}{k!} f \left( \frac{k}{n} \right), \quad n \in \mathbb{N}, \]  

(4.0.2)

where \( a > 1 \) and \( x \geq 0 \). Later on, Kajla and Agrawal ([51], [52]) studied approximation properties for Szász type operators and Szász Durrmeyer type operators in various functional spaces. Now, we introduce a new sequence of Szász-type operators which preserves constant and quadratic test functions and obtain better estimates.

Let \( T_{n,a} : C[0, \infty) \rightarrow C[0, \infty) \). Then, we have

\[ T_{n,a}(f; r_{n,a}(x)) = e^{-1} \left( 1 - \frac{1}{a} \right)^{(a-1)nr_{n,a}(x)} \sum_{k=0}^{\infty} \frac{C^{(a)}_k \left( - (a-1)nr_{n,a}(x) \right)}{k!} f \left( \frac{k}{n} \right), \quad n \in \mathbb{N}, \]  

(4.0.3)

for any function \( f \in C_\beta[0, \infty) = \{ f \in C[0, \infty) : |f(x)| \leq M(1 + x)^\beta \text{ for some } M > 0 \text{ and } \beta > 0 \} \) and

\[ r_{n,a}(x) = \frac{-(3 + \frac{1}{a-1}) + \sqrt{(3 + \frac{1}{a-1})^2 + 4(n^2 x^2 - 2)}}{2n}, \]  

(4.0.4)
is a sequence of real-valued continuous functions which is defined on \([0, +\infty)\). We observe that \(r_{n,a}(x) \geq 0\) for \(x \geq \frac{\sqrt{2}}{n}\). One can notice that

(i) if \(r_{n,a}(x) \to x\) as \(n \to +\infty\), the sequence of operators defined in (4.0.3) reduces to operators (4.0.2) and

(ii) for \(r_{n,a}(x) = x, a \to +\infty\) and \(x - \frac{1}{n}\) in place of \(x\), these operators tend to classical Szász operators defined by (1.1.4).

In this chapter, we discuss rate of convergence, local approximation results and Korovkin-type approximation theorem in polynomial weighted space and obtain better estimates for the operators (4.0.3) than the operators (4.0.2).

### 4.1 Basic Estimates

**Lemma 4.1.1.** For the operators \(T_{n,a}\) defined by (4.0.3), we have

\[
T_{n,a}(1; r_{n,a}(x)) = 1,
\]
\[
T_{n,a}(t; r_{n,a}(x)) = -\left(1 + \frac{1}{a-1}\right) + \sqrt{\left(3 + \frac{1}{a-1}\right)^2 + 4(n^2x^2 - 2)} \frac{2n}{2n},
\]
\[
T_{n,a}(t^2; r_{n,a}(x)) = x^2,
\]

for \(x \geq \frac{\sqrt{2}}{n}\).

**Proof.** Using \(t = 1, u = -(a-1)n r_{n,a}(x)\) in (4.0.1) and by simple differentiation, we get

\[
\sum_{k=0}^{\infty} \frac{C_k(u)}{k!} \frac{-(a-1)n r_{n,a}(x)}{k!} = e^{\left(1 - \frac{1}{a}\right)}^{-(a-1)n r_{n,a}(x)},
\]
\[
\sum_{k=0}^{\infty} \frac{C_k(u)}{k!} \frac{-(a-1)n r_{n,a}(x)}{k!} = e^{\left(1 - \frac{1}{a}\right)}^{-(a-1)n r_{n,a}(x)} (1 + n r_{n,a}(x)),
\]
\[
\sum_{k=0}^{\infty} \frac{C_k(u)}{k!} \frac{-(a-1)n r_{n,a}(x)}{k!} = e^{\left(1 - \frac{1}{a}\right)}^{-(a-1)n r_{n,a}(x)} \left(2 + \left(3 + \frac{1}{a-1}\right) n r_{n,a}(x) + n^2 r_{n,a}(x)^2\right).
\]

Using these equalities and operators (4.0.3), we can easily prove Lemma 4.1.1. \(\square\)
4.2. Rate of convergence

Lemma 4.1.2. Let \( \psi^i_x(t) = (t-x)^i \), \( i = 0, 1, 2 \). For the operators (4.0.3), we have

\[
\begin{align*}
T_{n,a}(\psi^0_x, r_{n,a}(x)) &= 1, \\
T_{n,a}(\psi^1_x, r_{n,a}(x)) &= -\frac{(1+\frac{1}{a-1}) + \sqrt{(3+\frac{1}{a-1})^2 + 4(n^2x^2 - 2)}}{2n} - x, \\
T_{n,a}(\psi^2_x, r_{n,a}(x)) &= 2x^2 + \frac{(1+\frac{1}{a-1})}{n} x - x \frac{\sqrt{(3+\frac{1}{a-1})^2 + 4(n^2x^2 - 2)}}{n}.
\end{align*}
\]

Proof. Using Lemma 4.1.1, we can easily prove Lemma 4.1.2.

Lemma 4.1.3. For the operators \( T_{n,a} \), we obtain

\[
\begin{align*}
\lim_{n \to \infty} n T_{n,a}(\psi^1_x, r_{n,a}(x)) &= -\frac{\left(1+\frac{1}{a-1}\right)}{2}, \\
\lim_{n \to \infty} T_{n,a}(\psi^2_x, r_{n,a}(x)) &= \left(1+\frac{1}{a-1}\right)x.
\end{align*}
\]

Proof. Proof of this Lemma is obvious.

4.2 Rate of convergence

Theorem 4.2.1. Let \( f \in C^\beta_\beta[0, +\infty) \) and \( x \geq \sqrt{x} \). Then, for the operators \( T_{n,a} \) defined by (4.0.3), we have

\[
|T_{n,a}(f; r_{n,a}(x), a) - f(x)| \leq 2 \omega(f; \delta_n),
\]

where \( \delta_n = (T_{n,a}(\psi^2_x, r_{n,a}(x)))^{\frac{1}{2}} \).

Proof. We have the difference

\[
\begin{align*}
|T_{n,a}(f; r_{n,a}(x)) - f(x)| &\leq e^{-1} \left(1-\frac{1}{a}\right)^{(a-1)n r_{n,a}(x)} \sum_{k=0}^{\infty} \frac{C^a_k}{k!} \frac{(-a-1)n r_{n,a}(x)}{x - \frac{k}{n}} \\
&\leq \left\{1+\frac{1}{\delta_n} e^{-1} \left(1-\frac{1}{a}\right)^{(a-1)n r_{n,a}(x)} \sum_{k=0}^{\infty} \frac{C^a_k}{k!} \frac{(-a-1)n r_{n,a}(x)}{x - \frac{k}{n}}\right\} \omega(f; \delta_n)
\end{align*}
\]
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\[
\leq \left\{ 1 + \frac{1}{\delta_n} \sqrt{e^{-1} \left( 1 - \frac{1}{a} \right)^{(a-1)nr_n(x)}} \sum_{k=0}^{\infty} \frac{C_k^{(a)} (- (a-1)nr_n(x))}{k!} \right\} \times \sqrt{\left( \frac{k - x}{n} \right)^2} \omega(f; \delta_n) \\
\leq \left\{ 1 + \frac{1}{\delta_n} \sqrt{T_n,a(\psi_2^2; x)} \right\} \omega(f; \delta_n),
\]

which proves the Theorem 4.2.1. \qed

**Remark 4.1.** For the Szász type operators $L_n$ given by (4.0.2) and for every $f \in C[0, \infty) \cap E$, we have

\[
|L_n(f; x) - f(x)| \leq \left\{ 1 + \sqrt{x \left( 1 + \frac{1}{a-1} \right) + \frac{2}{n}} \right\} \omega\left( f; \frac{1}{\sqrt{n}} \right),
\]

(4.2.1)

where $E = \{ f : [0, \infty) \to R, |f(x)| \leq Me^{Ax}, A \in R \text{ and } M \in (0, \infty) \}$. We observe that the operators $T_{n,a}$ have better approximation than the operators $L_n$. Since $2x = \sqrt{4x^2n^2}$ and $(3 + \frac{1}{a-1})^2 - 8 > 0$ for all values of $a > 1$, then $2x^2 < x\sqrt{(3 + \frac{1}{a-1})^2 + 4(3x^2n^2 - 2)}$. This implies that $2x^2 - x\sqrt{(3 + \frac{1}{a-1})^2 + 4(3x^2n^2 - 2)} + \frac{(1 + \frac{1}{a-1})}{n} < (1 + \frac{1}{a-1})x + \frac{2}{n^2}$. Hence

\[
\sqrt{T_{n,a}(\psi_2^2; r_n,a(x))} < \sqrt{L_n(\psi_2^2; x)}.
\]

**Example 4.1.** For $f(x) = \sin^2 x \in C[0,4]$ and $a = 2$, there are two figure for two different set of values of $n$ which shows better approximation of $T_{n,a}$ than the operators $L_n$.

![Figure 4.1: Convergence of the operators $T_{n,a}$ and $L_n$ for $n \in \{2,5,8\}$](image-url)
Example 4.2. For \( f(x) = \frac{1}{\sqrt{1+x^4}} \), \( a = 2 \), we estimate error for different power of 3 by using modulus of continuity for \( T_{n,a} \) and \( L_n \).

| Value of n | Estimates by operators \( T_{n,a} \) | Estimates by operators \( L_n \) |
|------------|-------------------------------------|----------------------------------|
| 3          | 0.0261                              | 0.47953                          |
| \( 3^2 \)  | 0.026017                            | 0.276994                         |
| \( 3^3 \)  | 0.025851                            | 0.159961                         |
| \( 3^4 \)  | 0.025187                            | 0.092296                         |
| \( 3^5 \)  | 0.023446                            | 0.05319                          |
| \( 3^6 \)  | 0.019881                            | 0.030672                         |
| \( 3^7 \)  | 0.014659                            | 0.017645                         |
| \( 3^8 \)  | 0.009521                            | 0.010185                         |
| \( 3^9 \)  | 0.005712                            | 0.005877                         |
| \( 3^{10} \)| 0.003311                            | 0.003393                         |

Figure 4.3: Error estimates by the operators \( T_{n,a} \) and \( L_n \).

4.3 Local Approximation Results

Theorem 4.3.1. Let \( f \in C_B^2[0,\infty) \). Then, for all \( x \geq \frac{\sqrt{7}}{n} \) there exists a constant \( C > 0 \) such that

\[
| T_{n,a}(f; r_{n,a}(x)) - f(x) | \leq C \omega_2(f; \sqrt{\gamma_{n,a}(x)}) + \omega(f; T_{n,a}(\psi; r_{n,a}(x))),
\]

where \( \gamma_{n,a}(x) = T_{n,a}(\psi^2; r_{n,a}(x)) + (T_{n,a}(\psi; r_{n,a}(x)))^2 \).
Proof. First, we consider the auxiliary operators as follows:

\[ \hat{T}_{n,a}(f; r_{n,a}(x)) = T_{n,a}(f; r_{n,a}(x)) + f(x) - f(\eta_{n,a}(x)), \quad (4.3.1) \]

where \( \eta_{n,a}(x) = T_{n,a}(\psi_t; r_{n,a}(x)) + x \). By the equation (4.3.1), we get

\[ \hat{T}_{n,a}(1; r_{n,a}(x)) = 1, \quad \hat{T}_{n,a}(\psi_t(t); r_{n,a}(x)) = 0, \quad |\hat{T}_{n,a}(f; r_{n,a}(x))| \leq 3 \| f \|. \quad (4.3.2) \]

For any \( g \in C^2_B[0, \infty) \) and by the Taylor’s theorem, we have

\[ g(t) = g(x) + (t - x)g'(x) + \int_x^t (t - v)g''(v)dv. \quad (4.3.3) \]

Applying auxiliary operators defined by (4.3.1) in equation (4.3.3), we get

\begin{align*}
\hat{T}_{n,a}(g; r_{n,a}(x)) - g(x) &= g'(x)\hat{T}_{n,a}(t-x; r_{n,a}(x)) \\
&+ \hat{T}_{n,a}\left(\int_x^t (t-v)g''(v)dv; r_{n,a}(x)\right) \\
&= \hat{T}_{n,a}\left(\int_x^t (t-v)g''(v)dv; r_{n,a}(x)\right) \\
&= T_{n,a}\left(\int_x^t (t-v)g''(v)dv; r_{n,a}(x)\right) \\
&- \int_x^t (\eta_{n,a}(x)-v)g''(v)dv.
\end{align*}

Therefore,

\[ |\hat{T}_{n,a}(g; r_{n,a}(x)) - g(x)| \leq \left| T_{n,a}\left(\int_x^t (t-v)g''(v)dv; r_{n,a}(x)\right) \right| \\
+ \left| \int_x^t (\eta_{n,a}(x)-v)g''(v)dv \right|. \quad (4.3.4) \]

Since,

\[ \left| \int_x^t (t-v)g''(v)dv \right| \leq (t-x)^2 \| g'' \|, \quad (4.3.5) \]

then,

\[ \left| \int_x^t (\eta_{n,a}(x)-v)g''(v)dv \right| \leq (\eta_{n,a}(x)-x)^2 \| g'' \|. \quad (4.3.6) \]
Using (4.3.4), (4.3.5) and (4.3.6), we have
\[
|\tilde{T}_{n,a}(g; r_{n,a}(x)) - g(x)| \leq \left\{ T_{n,a}((t-x)^2; r_{n,a}(x)) + (\eta_{n,a}(x) - x)^2 \right\} ||g''||
\leq \gamma_{n,a}(x)||g''||.
\] (4.3.7)

Now, we have
\[
|T_{n,a}(f; r_{n,a}(x)) - f(x)| \leq |\tilde{T}_{n,a}(f - g; r_{n,a}(x))| + |(f - g)(x)|
+ |\tilde{T}_{n,a}(g; r_{n,a}(x)) - g(x)| + |f(\eta_{n,a}(x)) - f(x)|,
\]
using (4.3.7), we get
\[
|T_{n,a}(f; r_{n,a}(x)) - f(x)| \leq 4||f - g|| + |\tilde{T}_{n,a}(g; r_{n,a}(x)) - g(x)|
+ |f(\eta_{n,a}(x)) - f(x)|
\leq 4||f - g|| + \gamma_{n,a}(x)||g''|| + \omega(f; T_{n,a}(\psi_k; r_{n,a}(x))).
\]

By the definition of Peetre’s K-functional, we find
\[
|T_{n,a}(f; r_{n,a}(x)) - f(x)| \leq C\omega_2\left(f; \sqrt[2]{\gamma_{n,a}(x)}\right) + \omega(f; T_{n,a}(\psi_k; r_{n,a}(x))).
\]

This proves Theorem 4.3.1.

Now, we introduce a local result in Lipschitz class as

**Theorem 4.3.2.** For \( x \in \left[ \frac{\sqrt{7}}{n}, +\infty \right) \) and \( f \in Lip^*_M(\alpha) \), we have
\[
|T_{n,a}(f; r_{n,a}(x)) - f(x)| \leq M \left[ \frac{\Theta_{n,a}(x)}{x} \right]^{\frac{\alpha}{2}},
\]
where \( \Theta_{n,a}(x) = T_{n,a}((t-x)^2; r_{n,a}(x)) \).

**Proof.** Let \( \alpha = 1 \) and \( x \in (0, \infty) \). Then, for \( f \in Lip^*_M(1) \), we have
\[
|T_{n,a}(f; r_{n,a}(x)) - f(x)| \leq e^{-1} \left( 1 - \frac{1}{a} \right)^{(a-1)nr_{n,a}(x)} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nr_{n,a}(x))}{k!} \times \left| f\left(\frac{k}{n}\right) - f(x) \right| dt
\leq Me^{-1} \left( 1 - \frac{1}{a} \right)^{(a-1)nr_{n,a}(x)} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nr_{n,a}(x))}{k!} \times \frac{|k/n - x|}{\sqrt{\frac{k}{n} + x}}.
\]
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\[
\leq \frac{M}{\sqrt{x}} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nr_{n,a}(x)} \sum_{k=0}^{\infty} C_k^{(a)} \frac{-(a-1)nr_{n,a}(x)}{k!}
\times \left|\frac{k}{n} - x\right|^\alpha
\leq \frac{M}{\sqrt{x}} T_{n,a}(|t-x|; r_{n,a}(x))
\leq \frac{M}{\sqrt{x}} \sqrt{T_{n,a}((t-x)^2; r_{n,a}(x))}
\leq M \left(\frac{\Theta_{n,a}(x)}{x}\right)^{\frac{1}{2}}.
\]

Thus, the assertion holds for \( \alpha = 1 \).

Now, we will prove for \( \alpha \in (0,1) \). From the Hölder’s inequality with \( p = \frac{1}{\alpha} \) and \( q = \frac{1}{1-\alpha} \), we have

\[
|T_{n,a}(f; r_{n,a}(x)) - f(x)| = e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nr_{n,a}(x)} \sum_{k=0}^{\infty} C_k^{(a)} \frac{-(a-1)nr_{n,a}(x)}{k!}
\times \left|\frac{k}{n} - f(x)\right|^{\frac{1}{\alpha}}
\times \left|\frac{k}{n} - f(x)\right|^{\frac{1}{a}}
\leq e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nr_{n,a}(x)} \sum_{k=0}^{\infty} C_k^{(a)} \frac{-(a-1)nr_{n,a}(x)}{k!}
\times \left|\frac{k}{n} - f(x)\right|^{\frac{1}{\alpha}}
\times \left|\frac{k}{n} - f(x)\right|^{\frac{1}{a}}.
\]

Since, \( f \in Lip_M^*(\alpha) \), we obtain

\[
|T_{n,a}(f; r_{n,a}(x)) - f(x)| \leq M \left( e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nr_{n,a}(x)} \sum_{k=0}^{\infty} C_k^{(a)} \frac{-(a-1)nr_{n,a}(x)}{k!}
\times \left|\frac{k}{n} - x\right|^{\alpha}
\times \sqrt{\frac{k}{n} + x}
\leq \frac{M}{\sqrt{x}} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nr_{n,a}(x)} \sum_{k=0}^{\infty} C_k^{(a)} \frac{-(a-1)nr_{n,a}(x)}{k!}
\times \left|\frac{k}{n} - x\right|^{\alpha}\right).
\]

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4.4 Weighted Korovkin type theorem

Theorem 4.4.1. Let \( T_{n,a} \) be the sequence of linear positive operators defined by (4.0.3). Then, for \( f \in C^k_\rho \), we have

\[
\lim_{n \to \infty} \| T_{n,a}(f; r_{n,a}(x)) - f(x) \|_\rho = 0.
\]

Proof. To prove the theorem, it is sufficient to show that

\[
\lim_{n \to \infty} \| T_{n,a}(t_i; x) - x_i \|_\rho = 0, \quad \text{for } i = 0, 1, 2.
\]

It is obvious that \( \lim_{n \to \infty} \| T_{n,a}(1; x) - 1 \|_\rho = 0 \) and \( \lim_{n \to \infty} \| T_{n,a}(x^2; x) - x^2 \|_\rho = 0 \). Now, from the Lemma 4.1.2, we have

\[
\sup_{x \in [0, \infty)} \frac{|T_{n,a}(t; x) - x|}{1 + x^2} \leq \frac{1}{2n} \sup_{x \in [0, \infty)} \left( 1 + \frac{a-1}{x} \right) \frac{1}{(1 + x^2)} \\
+ \frac{\sqrt{(1 + \frac{1}{a-1})^2 + 4(n^2 x^2 - 2)}}{2n(1 + x^2)}.
\]

Since \( \sqrt{a+b} < \sqrt{a} + \sqrt{b} \), then

\[
\sup_{x \in [0, \infty)} \frac{|T_{n,a}(t; x) - x|}{1 + x^2} \leq \frac{1}{2n} \sup_{x \in [0, \infty)} \left( 1 + \frac{a-1}{x} \right) \frac{1}{(1 + x^2)} \\
+ \frac{\sqrt{(1 + \frac{1}{a-1})^2 + 6(a^{-1})} + \sqrt{4n^2 x^2 - 2n x}}{2n(1 + x^2)}.
\]
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\[ \leq \frac{(1 + \frac{1}{a-1})}{2n} \sup_{x \in [0,\infty)} \frac{1}{1+x^2} + \sup_{x \in [0,\infty)} \sqrt{1 + \frac{1}{(a-1)^2} + \frac{6}{a-1}} \frac{1}{2n(1+x^2)}, \]

which shows that as \( n \to \infty \), \( ||T_{n,a}(t;x) - x||_\rho \to 0 \).

Hence, the theorem is proved. \( \square \)