Universal graphs for the topological minor relation

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Abstract
A subgraph-universal graph/a topological minor-universal graph in a class of graphs \( \mathcal{G} \) is a graph in \( \mathcal{G} \) which contains every graph in \( \mathcal{G} \) as a subgraph/topological minor. We prove that the class \( \mathcal{P} \) of all countable planar graphs does not contain a topological minor-universal graph. This answers a question of Diestel and Kühn and strengthens a result of Pach stating that there is no subgraph-universal graph in \( \mathcal{P} \). Furthermore, we characterise for which subdivided stars \( T \) there is a topological minor-universal graph in the class of all countable \( T \)-free graphs.

KEYWORDS
infinite graphs, planar graphs, subdivided star, topological minor, universal graphs

1 | INTRODUCTION

Let \( R \) be either the subgraph relation, the topological minor relation, or the minor relation. We say that a graph \( \Gamma \) is \( R \)-universal in a class of graphs \( \mathcal{G} \) if \( \Gamma \in \mathcal{G} \) and \( \Gamma \) contains every graph \( G \in \mathcal{G} \) with respect to \( R \). In this paper, we investigate the existence of topological minor-universal graphs. All graphs here are countable and we do not distinguish between isomorphic graphs.

Ulam asked whether there exists a subgraph-universal planar graph (cf. [11]), which was answered negatively by Pach:
Theorem 1.1 (Pach [11]). There is no subgraph-universal graph in the class of all planar graphs.

There have been different approaches on how to weaken Ulam’s question to obtain a positive answer. For example, Huynh, Mohar, Šámal, Thomassen and Wood recently constructed a graph which contains all planar graphs as subgraphs and is not planar itself, but satisfies a number of properties which are also satisfied by planar graphs [8]. For another approach, Diestel and Kühn showed that there is a minor-universal graph in the class of all planar graphs and asked whether the same is true for topological minor-universality [6]. We answer their question negatively, strengthening Theorem 1.1:

Theorem 1.2. There is no topological minor-universal graph in the class of all planar graphs.

Lehner proved Theorem 1.2 independently in [10]. He showed that every graph $G$ which contains every planar graph as a topological minor contains arbitrarily large finite cliques as topological minors as well as an infinite clique as a minor. This implies Theorem 1.2 as such a graph $G$ clearly cannot be planar. At the same time, Lehner’s result strengthens a recent result by Huynh et al., who showed the same under the stronger premise that $G$ contains all planar graphs as subgraphs [8].

We generalise Theorem 1.2 in a different direction than Lehner:

Theorem 1.3. Any class of graphs which is defined by excluding finite topological minors of minimum degree at least 3 contains a topological minor-universal graph if and only if it contains a subgraph-universal graph.

As an application, we strengthen several known results on the nonexistence of subgraph-universal graphs. In particular, Theorem 1.2 is a direct consequence of Theorems 1.1 and 1.3 since the countable planar graphs are precisely the countable graphs without $K_{3,3}$ and $K_5$ as topological minors.

In Sections 4 and 5, we look at topological minor-universal graphs with forbidden subdivided stars. For a graph $X$, let $\text{Forb}(X)$ be the class of all graphs which do not contain $X$ as a subgraph. To decide for which graphs $X$ there is an induced subgraph-universal graph or a subgraph-universal graph in $\text{Forb}(X)$ is an ongoing quest which has been solved for many but not for all finite connected graphs $X$ (see, e.g., [1, 3, 2, 7]).

The question whether there exists a topological minor-universal graph in $\text{Forb}(X)$ will be solved in this paper for all finite connected graphs $X$. However, it is only interesting if $X$ is a subdivided star for the following reasons. First, it is trivial to find a topological minor-universal graph in $\text{Forb}(X)$ if $X$ is not a subdivided star: If $X$ contains a cycle, say of length $k$, then the infinite clique in which every edge is subdivided $k$ times is topological minor-universal in $\text{Forb}(X)$. Otherwise $X$ is a tree but not a subdivided star and thus $X$ contains two vertices of degree at least three. When we denote their distance in $X$ by $k$, the same graph as above is topological minor-universal in $\text{Forb}(X)$.

Second, the existence of a topological minor-universal graph $\Gamma$ in a class of graphs $\mathcal{G}$ is especially interesting if $\mathcal{G}$ is closed under topological minors because then $\Gamma$ yields a characterisation for the graphs in $\mathcal{G}$: Any graph $G$ is contained in $\mathcal{G}$ if and only if $G$ is a topological minor of $\Gamma$. Finally, note that $\text{Forb}(X)$ is indeed closed under topological minors for
any subdivided star $X$ (in fact, excluding a subdivided star as a subgraph or as a topological minor results in the same). On the other hand, $\text{Forb}(X)$ is not closed under topological minors for any other finite connected graphs $X$ since the universal graph in $\text{Forb}(X)$ which we found above contains $X$ as a topological minor but $X \notin \text{Forb}(X)$.

We characterise the existence of topological minor-universal graphs with forbidden subdivided stars as follows:

**Theorem 1.4.** Let $T$ be a finite subdivided star. There is a topological minor-universal graph in $\text{Forb}(T)$ if and only if at most two of the original edges of $T$ are subdivided.

A comparison to the following result by Cherlin and Shelah shows that there are fewer positive results if we look at induced subgraph- or subgraph-universal graphs, in particular the condition on the minimum degree in Theorem 1.3 cannot be omitted:

**Theorem 1.5** (Cherlin and Shelah [1]). Let $T$ be a finite tree. There is an induced subgraph-universal graph in $\text{Forb}(T)$ if and only if there is a subgraph-universal graph in $\text{Forb}(T)$ if and only if $T$ is either a path or consists of a path with an adjoined edge.

For further research on topological minor-universal graphs, we suggest the following question:

**Question 1.6.** For which finite connected graphs $X$ is there a topological minor-universal graph in the class of all graphs without $X$ as a topological minor?

Both Theorems 1.3 and 1.4 provide a partial answer.

### 2 | PRELIMINARIES

We repeat that all graphs in this paper are countable but from now on we do distinguish between isomorphic graphs. Let $G$ and $H$ be graphs. An embedding of $G$ in $H$ is an injective map $\gamma: V(G) \rightarrow V(H)$ that preserves adjacency. A topological embedding of $G$ in $H$ is an injective map $\gamma: V(G) \rightarrow V(H)$ such that for every edge $vw \in E(G)$ there exists a $\gamma(v)$-$\gamma(w)$ path $P_{vw}$ in $H$ with the following property: For all $e \in E(G)$, the path $P^e$ has no inner vertices in the image of $\gamma$ or in any other path $P^f$ with $e \neq f \in E(G)$. If there exists an embedding of $G$ in $H$, we write $G \leq H$ and if there exists a topological embedding of $G$ in $H$, we write $G \leq_t H$. If $\gamma: v \mapsto v$ is an embedding of $G$ in $H$ (i.e., if $G$ is a subgraph of $H$), we write $G \subseteq H$. We have $G \leq H$ if and only if $H$ contains a subgraph isomorphic to $G$ and $G \leq_t H$ if and only there is a subdivision of $G$ which is isomorphic to a subgraph of $H$. The graph $H$ is a model of $G$ if there is a partition $\{V_z: z \in V(G)\}$ of $V(H)$ into nonempty connected sets such that for all $y \neq z \in V(G)$ there is a $V_y$-$V_z$ edge in $H$ if and only if $yz \in E(G)$. We call the sets $V_z$ branch sets. We say that $G$ is a minor of $H$ and write $G \preceq H$ if $H$ has a subgraph which is a model of $G$.

Let $\mathcal{G}$ be a class of graphs and $\subseteq$ a graph relation, for example, $\subseteq \in \{\leq, \preceq, \lessdot\}$. We say that a graph $\Gamma$ is $\subseteq$-universal in $\mathcal{G}$ if $\Gamma \in \mathcal{G}$ and $G \subseteq \Gamma$ for all $G \in \mathcal{G}$. We denote by $\text{Forb}(\mathcal{G}; \subseteq)$ the class of all graphs $H$ such that there is no graph $G \in \mathcal{G}$ with $G \subseteq H$. If $\mathcal{G} = \{G_1, \ldots, G_n\}$ is finite, we also write $\text{Forb}(G_1, \ldots, G_n; \subseteq)$ for $\text{Forb}(\mathcal{G}; \subseteq)$. The graphs in $\text{Forb}(\mathcal{G}; \leq)$ will be called $\mathcal{G}$-free.

Let $G$ be a graph and $v \in V(G)$. We write $S_G(v)$ for the subgraph of $G$ with vertex set $\{v\} \cup N_G(v)$ which contains precisely all edges between $v$ and its neighbours. We write $P_n$ for a fixed path of length $n$ and $C_n$ for a fixed cycle of length $n$. Next, let $P$ be a nontrivial path in $G$.
and $X$ a subset of $V(G)$. We say that $P$ is an $X$-path if $X$ contains both endvertices of $P$ but no inner vertices of $P$.

An $\alpha$-colouring of a graph $G$ is a map $c : V(G) \to \alpha$, where $\alpha$ is an ordinal (we will always have either $\alpha = 2$ or $\alpha = \omega$). We will not explicitly mention the map $c$ and refer to $c(v)$ as the colour of $v$ in $G$. If we consider a graph $G$ together with an $\alpha$-colouring of $G$, we say that $G$ is an $\alpha$-graph. Let $G$ and $H$ be $\alpha$-graphs. A (topological) $\alpha$-embedding of $G$ in $H$ is a (topological) embedding of $G$ in $H$ which additionally preserves the colour of vertices. We write $G \leq_\alpha H$ if there is an $\alpha$-embedding of $G$ in $H$ and $G \leq_\alpha H$ if there is a topological $\alpha$-embedding of $G$ in $H$. We will also use the notions Forb($\mathcal{X}; \subseteq$) and $\subseteq$-universality for $\subseteq \in \{\leq_\alpha, \leq_\alpha\}$; all graphs that are involved in their definitions have to be $\alpha$-graphs.

## 3 FORBIDDEN TOPOLOGICAL MINORS AND FORBIDDEN MINORS

**Theorem 3.1.** If $\mathcal{X}$ is a class of finite graphs with minimum degree at least 3, then $\mathcal{G} := \text{Forb}(\mathcal{X}; \subseteq)$ contains a $\subseteq$-universal graph if and only if $\mathcal{G}$ contains a $\subseteq$-universal graph.

**Proof.** Clearly, a $\subseteq$-universal graph in $\mathcal{G}$ is also $\subseteq$-universal. Conversely, suppose that there is a $\subseteq$-universal graph $\Gamma$ in $\mathcal{G}$. We construct a $\subseteq$-universal graph $\Gamma^*$ in $\mathcal{G}$ as follows.

Let $V(\Gamma^*) := V(\Gamma)$ and

$$E(\Gamma^*) := \{vw : \text{there are infinitely many independent } v-w \text{ paths in } \Gamma\}.$$

We need to show that $\Gamma^* \in \mathcal{G}$ and that $G \leq \Gamma^*$ for every graph $G \in \mathcal{G}$.

First, suppose that $\Gamma^*$ is not a member of $\mathcal{G}$ and thus there is a graph $X \in \mathcal{X}$ and a topological embedding $\gamma$ of $X$ in $\Gamma^*$. Then $\gamma$ is also a topological embedding of $X$ in $\Gamma$, contradicting that $\Gamma \in \mathcal{G}$. We recursively find for every edge $vw \in E(X)$ a $\gamma(v)$-$\gamma(w)$ path $P^{vw}$ in $\Gamma$ whose inner vertices avoid $V(X)$ and all paths found in earlier steps. This can be done because there are infinitely many independent $\gamma(v)$-$\gamma(w)$ paths in $\Gamma$ and the inner vertices of $P^{vw}$ only need to avoid finitely many vertices.

Next, we prove that $G \leq \Gamma^*$ for every graph $G \in \mathcal{G}$. Let $G'$ be the graph obtained from $G$ by replacing every edge $e$ with infinitely many independent paths of length 2, whose inner vertices we call $v_0^e, v_1^e, v_2^e, \ldots$. We start by showing that $G' \in \mathcal{G}$. Suppose that $G'$ has a subgraph that is isomorphic to a subdivision $X'$ of a graph $X \in \mathcal{X}$, without loss of generality we assume that $X' \subseteq G'$. By suppressing every vertex of the form $v_i^e$ in $X'$ (reobtaining the edge $e$), we obtain the graph $X'' \subseteq G$. Also $X''$ is a subdivision of $X$, since the minimum degree of $X$ is at least 3 and hence we only suppressed subdividing vertices. This contradicts $G \in \mathcal{G}$ and thus we have $G' \in \mathcal{G}$.

Since $\Gamma$ is $\subseteq$-universal in $\mathcal{G}$, there is a topological embedding $\gamma$ of $G'$ in $\Gamma$. Then $\gamma_{V(G)}$ is an embedding of $G$ in $\Gamma^*$. Indeed, for any edge $vw \in G$ there are infinitely many independent $v$-$w$ paths in $G'$. Therefore, there are infinitely many independent $\gamma(v)$-$\gamma(w)$ paths in $\Gamma$ and thus $\gamma(v)\gamma(w) \in E(\Gamma^*)$. $\square$
The statement of Theorem 3.1 remains true when we exclude minors instead of topological minors:

**Corollary 3.2.** If $\mathcal{X}$ is a class of finite graphs with minimum degree at least 3, then $\text{Forb}(\mathcal{X}; \preceq)$ contains a $\leq$-universal graph if and only if it contains a $\preceq$-universal graph.

**Proof.** Let $\mathcal{Y}$ be the class of all models of graphs in $\mathcal{X}$ that are finite and have minimum degree at least 3. We will show that $\text{Forb}(\mathcal{X}; \preceq) = \text{Forb}(\mathcal{Y}; \preceq)$, then the claim follows from Theorem 3.1. $\text{Forb}(\mathcal{X}; \preceq) \subseteq \text{Forb}(\mathcal{Y}; \preceq)$ is clear. For the proof of the converse inclusion, let $G$ be a graph that is not in $\text{Forb}(\mathcal{X}; \preceq)$. Thus there is a subgraph $X'$ of $G$ which is a model of some $X \in \mathcal{X}$. We choose $X'$ so that it has no vertices of degree 1, the branch sets induce finite trees in $X'$, and there is at most one edge connecting two distinct branch sets. By suppressing all vertices of degree 2 in $X'$, we obtain a topological minor $Y$ of $X'$ with minimum degree at least 3. Since also the minimum degree of $X$ is at least 3, every branch set in $X'$ contains a vertex of degree at least 3 which was not suppressed in the construction of $Y$. Therefore, $X$ is a minor of $Y$. Additionally, $Y$ contains no loops or double edges because every cycle in $X'$ contains at least three vertices of degree at least 3 in $X'$, one for each branch set it traverses. Thus, $Y \in \mathcal{Y}$. Since $Y \succeq X' \subseteq G$, it follows that $G$ is not contained in $\text{Forb}(\mathcal{Y}; \preceq)$.

**Corollary 3.3.** The following classes of graphs do not contain $\preceq$-universal graphs:

1. $\text{Forb}(K^5, K_{3,3}; \preceq)$ (the planar graphs),
2. $\text{Forb}(K^n; \preceq)$ for $n \geq 5$,
3. $\text{Forb}(K_{n,m}; \preceq)$ for $n, m \geq 3$,
4. $\text{Forb}(K^n; \preceq)$ for $n \geq 5$.

**Proof.** The claim follows from Theorem 3.1, Corollary 3.2 and the fact that none of these classes contains a $\leq$-universal graph: Pach [11] showed that there is no $\leq$-universal planar graph, Diestel, Halin and Vogler [5] showed the same for the classes in (2) and (4) and Diestel [4] for the classes in (3).

## 4 FORBIDDEN SUBDIVIDED STARS—POSITIVE RESULTS

In this section we prove Theorem 4.2, which states that there is a $\preceq$-universal $T$-free graph if $T$ is a subdivided star in which at most 2 of the original edges are subdivided. We use the following notation for subdivided stars:

**Definition 4.1.** Let $k \in \mathbb{N}$ and $p_1 \leq \cdots \leq p_k \in \mathbb{N}$ and let $P^1, \ldots, P^k$ be disjoint paths such that the length of $P^i$ is $p_i$ for all $i \leq k$.

We write $T(p_1, \ldots, p_k)$ for the graph obtained from the paths $P^1, \ldots, P^k$ by adding a vertex $c$ and identifying one endvertex of each path $P^i$ with $c$. We call $c$ the centre of $T(p_1, \ldots, p_k)$. If $p_1 = \cdots = p_k = 1$, then we also write $S_k$ for $T(p_1, \ldots, p_k)$. 
**Theorem 4.2.** Let \( k \in \mathbb{N} \) and \( p_1 \leq \cdots \leq p_k \in \mathbb{N} \). If \( k \leq 2 \) or if \( k \geq 3 \) and \( p_{k-2} = 1 \), then there exists a \( \preceq \)-universal graph in \( \text{Forb}(T(p_1, ..., p_k); \preceq) \).

For the rest of this section, we write \( T \) for the graph \( T(p_1, ..., p_k) \) from Theorem 4.2. For paths \( T \), Komjáth, Mekler and Pach [9] showed that there exists a \( \preceq \)-universal graph in \( T \in \text{Forb}(T; \preceq) \), which is also \( \preceq \)-universal. In the following, we will therefore assume that \( k \geq 3 \) and \( p_{k-2} = 1 \).

We begin by proving the existence of a \( \preceq \)-universal \( S_k \)-free graph or, equivalently, a \( \preceq \)-universal graph of maximum degree \( k - 1 \). (In Lemma 4.3, we prove that there is a \( \preceq \)-universal connected \( S_k \)-free graph. By taking countably many disjoint copies of that universal graph, we obtain a \( \preceq \)-universal graph for all \( S_k \)-free graphs that are not necessarily connected.)

**Lemma 4.3.** There is a \( \preceq_\omega \)-universal \( \omega \)-graph \( \Gamma^* \) in the class \( \mathcal{G} \) of all connected \( S_k \)-free \( \omega \)-graphs on at least three vertices, such that \( \Gamma^* \) satisfies the following stronger assertion: For every \( \omega \)-graph \( G \in \mathcal{G} \) there is a topological \( \omega \)-embedding \( \gamma^* \) of \( G \) in \( \Gamma^* \) which is degree preserving, that is, \( d_G(v) = d_{\Gamma^*}(\gamma^*(v)) \) for all \( v \in V(G) \).

**Proof.** Let \( [\mathbb{N}]^{<k} \) be the (countable) set of all subsets of \( \mathbb{N} \) of size less than \( k \) and choose a function \( f : [\mathbb{N}]^{<k} \rightarrow [\mathbb{N}]^{<k} \) such that for every \( A \in [\mathbb{N}]^{<k} \) there are infinitely many \( n \in \mathbb{N} \) with \( f(n) = A \). Furthermore, let \( (R_i : i \in \mathbb{N}) \) be a family of pairwise disjoint rays. Now we construct \( \Gamma^* \) from the graph \( \bigcup_{i \in \mathbb{N}} R_i \) by adding for all \( j \in \mathbb{N} \) a vertex \( v_j \), which we join to the \( j \)th vertex of each ray \( R_i \) with \( i \in f(j) \) (see Figure 1). We define an \( \omega \)-colouring of \( \Gamma^* \) in such a way that for every set \( A \in [\mathbb{N}]^{<k} \) and for every colour \( c \in \mathbb{N} \), there is an integer \( j \in f^{-1}(A) \) such that \( v_j \) has colour \( c \). This is possible because \( f^{-1}(A) \) is infinite for all \( A \in [\mathbb{N}]^{<k} \). The vertices of \( \bigcup_{i \in \mathbb{N}} R_i \) may be coloured arbitrarily.

\( \Gamma^* \) does not contain a copy of \( S_k \) because the vertices \( v_j \) for \( j \in \mathbb{N} \) have degree less than \( k \) and all other vertices have degree at most 3. It remains to find a degree preserving topological \( \omega \)-embedding \( \gamma^* \) of \( G \) in \( \Gamma^* \) for every \( \omega \)-graph \( G \in \mathcal{G} \). We enumerate \( E(G) =: \{e_0, e_1, ...\} \). For every vertex \( v \in V(G) \), we pick an integer \( j \in f^{-1}(\{i \in \mathbb{N} : v \text{ is incident to } e_i \text{ in } G\}) \)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The graph \( \Gamma^* \) when \( f(1) = \{0, 1\}, f(2) = \{0, 2\}, f(3) = \{0, 1\}, ... \) drawn without regarding its vertex colouring.}
\end{figure}
such that \( v \) has the same colour in \( G \) as \( v_j = :\gamma^*(v) \) in \( \Gamma \). Then clearly, \( \gamma^* \) is degree preserving. Next, \( \gamma^* \) is injective since in a connected graph on at least three vertices, the set of incident edges is different for every vertex. Finally, for all edges \( e_i = vw \in E(G) \) we have to find internally disjoint \( \gamma^*(v)\)\(\gamma^*(w)\) paths \( P_{e_i}^* \) in \( \Gamma^* \) which avoid the image of \( \gamma^* \). Each path \( P_{e_i}^* \) can be built using the ray \( R_i \) together with the \( \gamma^*(v)\)\(\gamma^*(w)\) edge and the \( \gamma^*(w)\)\(R_i\) edge in \( \Gamma^* \).

We remark that we can also use the construction of \( \Gamma \) in the proof of Lemma 4.3 for building a \( \preceq \)-universal graph in the class of all locally finite graphs if we replace \( [\mathbb{N}]^{<k} \) by \( [\mathbb{N}]^{<\omega} \). A \( \preceq \)-universal locally finite graph however does not exist; this is an easy observation which was first made by de Brujn, see [12].

Our next aim in the proof of Theorem 4.2 is Lemma 4.6, a result on the structure of \( T \)-free graphs. In the proof of Lemma 4.6, we analyse the block structure of \( T \)-free graphs (a block of a graph \( G \) is a maximal connected subgraph of \( G \) without a cutvertex). We need the following lemmas:

**Lemma 4.4** (Diestel [4, Chap. 1, Exercise 3]). Let \( n \in \mathbb{N} \). Every 2-connected graph \( G \) containing a path of length \( n^2 \) contains a cycle of length at least \( n \).

We recall that \( k \geq 3 \), \( p_{k-2} = 1 \) and \( T = T(p_1, ..., p_k) \).

**Lemma 4.5.** Let \( G \) be a \( T \)-free graph and let \( B \) be a block of \( G \) that contains a vertex \( v \in V(B) \) with \( d_G(v) \geq k \). Then there is no path of length \( m := (k + 1)^2(2p_k)^2 \) in \( B \).

**Proof.** Suppose for a contradiction that \( B \) contains a path of length \( m \). Therefore, \( |B| \geq 3 \) and it follows that \( B \) is 2-connected. By Lemma 4.4, there is a cycle \( C \) in \( B \) of length at least \( \sqrt{m} = (k + 1) \cdot 2p_k \). First we consider the case that \( S_B(v) \cap C = \emptyset \). By 2-connectedness of \( B \) there are two disjoint \( S_B(v)\)\(C\) paths \( P^1 \) and \( P^2 \) in \( B \). Denote the endvertex of \( P^1 \) in \( C \) by \( x_1 \). We can find an \( x_1\)\(x_2\) path \( Q \) in \( C \) of length at least \( 2p_k + 1 \) since \( |C| \geq 8p \geq 2(2p_k + 1) \). However, there is a copy of \( T \) in \( S_B(v) \cup P^1 \cup P^2 \cup Q \), a contradiction.

If \( S_B(v) \cap C = \{w\} \) for some vertex \( w \), then let \( P^1 = w \) be a path of length 1. Note that \( w \neq v \). Since \( B \) is 2-connected, there is an \( S_B(v)\)\(C\) path \( P^2 \) in \( B - w \). Now we can prove that \( T \leq G \) as in the case \( S_B(v) \cap C = \emptyset \).

Finally, suppose that \( |V(S_B(v)) \cap V(C)| \geq 2 \). There are at most \( |S_B(v)| = k + 1 \) many distinct \( S_B(v)\)\(P\)s contained in \( C \). Therefore, there must be an \( S_B(v)\)\(P\) \( P \) in \( C \) of length at least \( 2p_k \) since \( |C| \geq (k + 1) \cdot 2p_k \). This is a contradiction as \( S_B(v) \cup P \) contains a copy of \( T \).

**Lemma 4.6.** Let \( m \) be as in Lemma 4.5 and let \( G \) be a connected \( T \)-free graph with \( P_{p_{k,m}} \leq G \). Then there is a connected induced subgraph \( G^* \) of \( G \) and for every vertex \( v \in V(G^*) \) there is a connected induced subgraph \( G_v \) of \( G \) such that the following properties hold:

1. \( G = G^* \cup \bigcup_{v \in V(G^*)} G_v \),
2. \( G^* \cap G_v = \{v\} \) and \( G_v \cap G_w = \emptyset \) for all \( v \neq w \in V(G^*) \),
3. \( d_G(v) < k \) for all \( v \in V(G^*) \),
4. \( P_{2p_k} \leq G^* \) and
(5) \( P_{4p_k + 2m} \nsubseteq G_v \) for all \( v \in V(G) \).

Proof. Let \( A \) be the set of cutvertices in \( G \) and \( B \) the set of blocks of \( G \) and \( K \) the block graph of \( G \). Recall that \( V(K) = A \cup B \) and \( E(G) = \{ aB : a \in A, B \in B, a \in B \} \) and that \( K \) is a tree. Since \( G \) contains a path of length \( 4p_km \), there is either a block \( B \in B \) containing a path of length \( m \) or there is a path \( P = B_1a_1B_2a_2 \ldots B_{4p_k-1}a_{4p_k-1}B_{4p_k} \) in \( K \) containing \( 4p_k \) blocks. In the first case let \( H^* := B \) and in the second case let

\[
H^* := \bigcup_{p_k + 1 \leq i \leq 3p_k} B_i.
\]

In both cases, there is clearly a path of length \( 2p_k \) in \( H^* \). We also have \( d_G(v) < k \) for all \( v \in V(H^*): \) In the first case, this holds by Lemma 4.5. For the second case, let us suppose that there are a block \( B_i \) with \( p_k + 1 \leq i \leq 3p_k \) and a vertex \( v \in B_i \) with \( d_G(v) \geq k \). Since \( B_i \) is 2-connected, there is an \( S_B(v) - a_{i-1} \) path \( P^1 \) and an \( S_B(v) - a_i \) path \( P^2 \) in \( B_i \) such that \( P^1 \) and \( P^2 \) are disjoint. Now we extend \( S_G(v) \cup P^1 \cup P^2 \) to a copy of \( T \) in \( G \) by extending the path \( P^1 \) into the blocks \( B_j \) with \( j < i \) and extending \( P^2 \) into the blocks \( B_j \) with \( j > i \). This contradicts \( G \) being \( T \)-free. Hence \( d_G(v) < k \) for all \( v \in V(H^*) \).

If \( K' \) is a subtree of \( K \), we write \( \cup K' \) for \( \bigcup \{ B \in B : B \in V(K') \} \). Let \( \tilde{K}^* \) be the maximal subtree of \( K \) that contains all vertices of \( K \) which are blocks of \( H^* \), such that there is no vertex \( v \in \cup \tilde{K}^* \) with \( d_G(v) \geq k \). We delete all leaves of \( \tilde{K}^* \) that are cutvertices in \( G \) and call the resulting graph \( K^* \). Then

\[
G^* := \bigcup K^*
\]

satisfies property (3) by definition of \( K^* \) and property (4) since already \( H^* \) contains a path of length \( 2p_k \).

For all components \( H \) of \( K - K^* \) we have \( G^* \cap H = \langle v \rangle \) for some vertex \( v \in V(G^*) \cap A \). Let \( K_v := H \) and \( G_v := \cup K_v \). For the proof of property (5), we begin by decomposing \( G_v \) into a family \( \mathcal{G}_v \) of graphs that only intersect in \( v \). Let \( \mathcal{K}_v \) be the set of components of \( K_v - v \) and \( \mathcal{G}_v := \{ \cup K' : K' \in \mathcal{K}_v \} \). For all \( K' \in \mathcal{K}_v \) we show that the graph \( G' := \bigcup K' \in \mathcal{G}_v \) (Figure 2) is \( P_{4p_k + m} \)-free, which implies that \( G_v \) is \( P_{4p_k + 2m} \)-free and thus proves property (5).

Let \( D \) be the block of \( G' \) containing \( v \). By maximality of \( K^* \), we know that \( D \) contains a vertex \( w \) with \( d_G(w) \geq k \). Therefore, Lemma 4.5 implies that \( D \) does not contain a path of length \( m \). Additionally, for every component \( K'' \) of \( K' - D \) the graph \( G'' := \cup K'' \) does not contain a path of length \( 2p_k \) (Figure 2): Suppose that \( G'' \) contains a path \( \langle x \rangle \) of length \( 2p_k \). We denote the unique vertex in \( D \cap G'' \) by \( u \). Let \( P_x \) be an \( S_D(w) - x \) path in \( D \) for \( x \in \{ v, u \} \) such that \( P_v \) and \( P_u \) are disjoint. Further, let \( P^* \) be a path of length \( 2p_k \) in \( G'' \), which exists by property (4), and let \( Q_v \) be a \( v - P^* \) path in \( G^* \) and \( Q_u \) a \( u - P^* \) path in \( G'' \). However, \( S_G(w) \cup P_v \cup P_u \cup Q_v \cup Q_u \cup P^* \cup P^* \subseteq G \)

contains a copy of \( T \). Therefore, \( G'' = \bigcup K'' \) does not contain a path of length \( 2p_k \) for every choice of \( K'' \). Together with \( D \) being \( P_m \)-free, this implies that there is no path of length \( 4p_k + m \) in \( G' \).
Finally, for all \( v \in V(G^*) \) for which \( G_v \) has not been defined yet, let \( G_v := \{v\} \). Then properties (1) and (2) are clearly satisfied. □

For the proof of Theorem 4.2, we need the following lemma by Cherlin and Tallgren:

**Lemma 4.7** (Cherlin and Tallgren [3]). Let \( \alpha \in \mathbb{N} \) and let \( \mathcal{X} \) be a finite set of finite connected \( \alpha \)-graphs. Suppose that for some \( n \in \mathbb{N} \), every \( \alpha \)-coloured version of \( P_n \) is contained in \( \mathcal{X} \). Then there is a \( \leq \alpha \)-universal \( \alpha \)-graph in \( \leq T\text{Forb}(\mathcal{X}; \leq \alpha) \).

The idea for our proof of Theorem 4.2 is to build a \( \leq \)-universal graph \( \Gamma \) in \( \leq T\text{Forb}(T; \leq) \) by starting with the \( \leq \)-universal \( S_k \)-free graph \( \Gamma^* \) from Lemma 4.3 and attaching to every vertex \( v \in V(\Gamma^*) \) a \( \leq \)-universal \( P_{8p_k+2m} \)-free graph \( \Gamma_v \) that exists by Lemma 4.7. If \( G \) is any connected \( T \)-free graph, we want to decompose \( G \) as in Lemma 4.6 and find a topological embedding \( \gamma \) of \( G \) in \( \Gamma \) by embedding \( G^* \) in \( \Gamma^* \) and each graph \( G_v \) in \( \Gamma_{\gamma(v)} \).

**Proof of Theorem 4.2.** Recall that we have already seen that the theorem holds for \( k = 2 \), so we can assume that \( k \geq 3 \). We will build a graph \( \Gamma \) that is \( \leq \)-universal in the class of all connected \( T \)-free graphs containing \( P_{8p_k+2m} \). By Lemma 4.7 (applied with \( \alpha = 1 \)), there exists a \( \leq \)-universal graph \( \Gamma' \) in \( \leq \text{Forb}(T, P_{8p_k+2m}; \leq) \). Then the disjoint union of \( \Gamma' \) with countably many copies of \( \Gamma \) is \( \leq \)-universal in \( \leq T\text{Forb}(T; \leq) \).

If \( H \) is a 2-graph with exactly one vertex \( v \) of colour 1, then we write \( \Pi \) for the graph obtained from \( H \) by attaching a path of length \( p_k \) to \( v \). Let

**FIGURE 2** The graph \( G \) and its subgraphs.
• $\mathcal{H}_1$ be the set of all connected 2-graphs $H$ with $V(H) \subseteq \{1, \ldots, |T|\}$ such that $H$ contains exactly one vertex of colour 1 and $T \subseteq \Pi$,
• $\mathcal{H}_2$ be the set of all 2-coloured versions of the graph $P_{8p_k + 2m}$ and
• $\mathcal{H}_3$ be the set of all 2-coloured versions $P$ of the paths $P_i$ for $i < 8p_k + 2m$ such that all the inner vertices of $P$ have colour 0 and the endvertices have colour 1.

For all $n < k$, let

• $\mathcal{H}_n^a$ be the one-element set containing the 2-graph $S_{n+1}$ so that the centre of $S_{n+1}$ has colour 1 and all other vertices have colour 0 and
• $\mathcal{H}_n := \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3 \cup \mathcal{H}_n^a$.

Since $\mathcal{H}_2 \subseteq \mathcal{H}_n$, there exists a $\leq 2$-universal $\mathcal{H}_n$-free 2-graph $\Gamma^n$ by Lemma 4.7. We fix an enumeration $\Gamma^n_0, \Gamma^n_1, \Gamma^n_2, \ldots$ of all components of $\Gamma^n$ which contain a vertex of degree at least $n$ that has colour 1. Note that each component $\Gamma^n_i$ contains exactly one vertex $u$ of colour 1 because $\Gamma^n_i$ is connected and $\mathcal{H}_2 \cup \mathcal{H}_3$-free. The degree of $u$ must be $n$ since $\Gamma^n_i$ is $\mathcal{H}_n^a$-free.

Let $\Gamma^*$ be the $\omega$-graph from Lemma 4.3. For every vertex $v \in V(\Gamma^*)$, let $\Gamma_v := \Gamma^*_{c(v)}$ where

$$n(v) := k - d_{\Gamma^*}(v) - 1$$

and $c(v)$ is the colour of $v$ in $\Gamma^*$. Notice that $n(v) \geq 0$ since $\Gamma^*$ is $S_k$-free. By renaming vertices of each $\Gamma_v$, we obtain for all $v \neq w \in V(\Gamma^*)$ that $\Gamma_v$ and $\Gamma_w$ are disjoint, the unique vertex of colour 1 in $\Gamma_v$ is $v$, and $\Gamma^* \cap \Gamma_v = \{v\}$. We define (the uncoloured graph)

$$\Gamma := \Gamma^* \cup \bigcup_{v \in V(\Gamma^*)} \Gamma_v.$$

We need to show that $\Gamma$ is $T$-free and that $G \unlhd \Gamma$ for every connected $T$-free graph $G$ with $P_{4p_k m} \leq G$.

Suppose for a contradiction that there is an embedding $\gamma$ of $T$ in $\Gamma$ and let $z$ be the centre of $T$. We have $\gamma(z) \in \Gamma_v - \Gamma^*$ for some $v \in V(\Gamma^*)$ since any vertex $u \in V(\Gamma^*)$ satisfies $d_{\Gamma^*}(u) = d_{\Gamma^*}(u) + d_{\Gamma_v}(u) = k - 1$. Therefore, $\gamma(T) \cap \Gamma^*$ is either empty or a path of length at most $p_k$ with endvertex $v$, which contradicts $\Gamma_v$ being $\mathcal{H}_1$-free.

Now let $G$ be an arbitrary connected $T$-free graph with $P_{4p_k m} \leq G$ and consider the graph $G^*$ and the graphs $G_v$ from Lemma 4.6. We furnish each $G_v$ with a 2-colouring by colouring $v$ with 1 and all other vertices with 0. Then $G_v$ is $\mathcal{H}_1$-free: Suppose that there is a graph $H \in \mathcal{H}_1$ and a 2-embedding $f$ from $H$ to $G_v$. Since $G^*$ is connected with $P_{2p_k} \leq G^*$ by Lemma 4.6(4), there is a path $P$ of length $p_k$ in $G^*$ with endvertex $v$. This is a contradiction because $G[f(H)] \cup P \subseteq G$ contains a copy of $T$. Hence $G_v$ is $\mathcal{H}_1$-free. $G_v$ is also $\mathcal{H}_3$-free by Lemma 4.6(5) and $\mathcal{H}_3$-free since $v$ is the only vertex of colour 1 in $G_v$.

Lastly, we have $d_{G}(v) < k$ by Lemma 4.6(3) and therefore

$$d_{G_v}(v) = d_{G}(v) - d_{G^*}(v) \leq k - d_{G^*}(v) - 1 = n'(v),$$
which shows that \( G_v \) is \( \mathcal{A}_4^{n'(v)} \)-free. Therefore, there is a 2-embedding of \( G_v \) in \( \Gamma^{n'(v)} \), and since \( G_v \) is connected, there is an integer \( c'(v) \in \mathbb{N} \) such that the image of this 2-embedding is contained in \( \Gamma^{n'(v)}_{c'(v)} \).

Next, define an \( \omega \)-colouring of \( G^* \) by giving each vertex \( v \in V(G^*) \) the colour \( c'(v) \). Since \( G^* \) is \( S_k \)-free by Lemma 4.6(3), Lemma 4.3 implies that there is a topological \( \omega \)-embedding \( \gamma^* \) of \( G^* \) in \( \Gamma^* \) with \( d_{c^*(v)} = d_{c'(v)}(\gamma^*(v)) \) and hence \( n'(v) = n(\gamma^*(v)) \) for all \( v \in V(G^*) \). For every vertex \( v \in V(G^*) \), we have \( c'(v) = c(\gamma^*(v)) \) because \( \gamma \) respects colouring. This means that \( \Gamma^*_{c'(v)} \) is a copy of \( \Gamma^*_{c'(v)} \). Therefore, there exists a 2-embedding \( \gamma_v \) of \( G_v \) in \( \Gamma_v \) and we have \( \gamma_v(v) = \gamma^*(v) \) since \( \gamma_v \) respects colouring. Thus

\[
\gamma^* \bigcup_{v \in V(G^*)} \gamma_v
\]

is a topological embedding of \( G \) in \( \Gamma \).

5  FORBIDDEN SUBDIVIDED STARS—NEGATIVE RESULTS

In this section, we prove that there is no \( \preceq \)-universal \( T \)-free graph if \( T \) is a subdivided star in which at least 3 of the original edges are subdivided. The proof relies on ideas from Cherlin and Shelah in [1, Proposition 3.3].

**Theorem 5.1.** Let \( k \in \mathbb{N} \) and \( p_1 \leq \cdots \leq p_k \in \mathbb{N} \). If \( k \geq 3 \) and \( p_{k-2} \geq 2 \), then there exists no \( \preceq \)-universal graph in \( \text{Forb}(T(p_1, \ldots, p_k); \leq) \).

**Proof.** Let \( T := T(p_1, \ldots, p_k) \) and let \( P^1, \ldots, P^k \) be the corresponding paths from Definition 4.1. Let \( m \) be minimal with \( p_m > 1 \) and define \( n := k - m + 1 \). Then exactly \( n \) of the paths \( P^1, \ldots, P^k \) have length at least \( p_m \) and the other paths have length 1.

Let \( H_1 \) be the graph consisting of two vertices \( x_1 \) and \( x_2 \) together with infinitely many independent \( x_1-x_2 \) paths of length \( p_m \) and let \( H_2 \) be the graph with vertex set \( \{x_1, x_2, y_1, y_2, \ldots, y_{p_m}\} \) where every two vertices are adjacent except for \( x_1 \) and \( x_2 \).

\[\text{FIGURE 3} \quad \text{The graph } G_\alpha \text{ when } p_m = 2, n = 4 \text{ and } \alpha = (1, 1, 2, \ldots). \text{ The fat edges form a subdivision of } U_\alpha \text{ in } G_\alpha.\]
We use $H_1$ and $H_2$ to construct an uncountable family of $T$-free graphs. Consider the uncountable set $A \subseteq \{1, 2\}^\mathbb{N}$ of all sequences that alternatingly contain one or two 1’s and one 2, beginning with a 1. For all $\alpha \in A$, let $U_\alpha$ be an $(n - 1)$-regular tree with root $r_\alpha$. For each edge $vw \in E(U_\alpha)$, we denote the minimum of $d_{U_\alpha}(r_\alpha, v)$ and $d_{U_\alpha}(r_\alpha, w)$ by $\ell(vw)$. For all $\alpha \in A$, we define a graph $G_\alpha$ as follows: We begin with $U_\alpha$, replace every edge $vw \in E(U_\alpha)$ with a copy of $H_{\alpha(\ell(vw))}$, and identify $x_1$ with $v$ and $x_2$ with $w$ (see Figure 3). Note that for every vertex $v \in V(U_\alpha)$ there is at least one edge $e \in E(U_\alpha)$ which was replaced by a copy of $H_1$. Hence $v$ has infinite degree in $G_\alpha$.

We now prove two crucial properties of $G_\alpha$ for all $\alpha \in A$:

**Claim 1.** $G_\alpha$ is $T$-free.

**Proof of Claim 1.** Suppose that there is a copy of $T$ in $G_\alpha$ and let $v$ be its centre. It is impossible that $v \in V(G_\alpha) \setminus V(U_\alpha)$: If $v$ is contained in a copy of $H_1$, then the degree of $v$ in $G_\alpha$ is 2 and therefore $v$ cannot be the centre of a copy of $T$. Now suppose that $v$ is contained in a copy of $H_2$. The copy of $T$ contains three paths of length at least $p_m$ starting in $v$. Each of them has at least $p_m + 1$ vertices and must therefore use one of the vertices corresponding to $x_1$ or $x_2$ in the copy of $H_2$ which is a contradiction. Therefore, $v$ must be contained in $V(U_\alpha)$. However, $v$ is only contained in $n - 1$ copies of $H_1$ or $H_2$ and each of them can only contain inner vertices of at most one of the paths $P^i$ for $i \geq m$, a contradiction.

**Claim 2.** Every subdivision $G'_\alpha$ of $G_\alpha$ with at least one subdividing vertex $v$ contains a subgraph isomorphic to $T$.

**Proof of Claim 2.** Let $H$ be the subdivision of a copy of $H_i$ for $i = 1$ or $i = 2$ in $G'_\alpha$ which contains $v$ and let $\gamma$ be a topological embedding of $H_i$ in $H$. Then we can find an embedding of $T$ in $G'_\alpha$ as follows. We map the centre of $T$ to $\gamma(x_1)$, and $p_m$ to a path of length $p_m$ in $H$ containing $v$ but not $\gamma(x_2)$, and $P^i$ for $i > m$ to paths each containing one neighbour of $\gamma(x_1)$ in $U_\alpha$. Since $\gamma(x_1)$ is contained in $V(U_\alpha)$, it has infinite degree in $G_\alpha$ and thus also in $G'_\alpha$. Therefore it is easy to embed the paths $P^i$ for $i < m$ in $G'_\alpha$.

Suppose for a contradiction that $\Gamma$ is a $\preceq$-universal graph in $Forb(T; \preceq)$. For all $\alpha \in A$, we have $G_\alpha \preceq \Gamma$ by Claim 1 and therefore $G_\alpha \preceq \Gamma$ by Claim 2. We will show that this is impossible.

We define a symmetric binary relation $R$ on the vertices of $\Gamma$ such that $R(v, w)$ if and only if $\Gamma$ contains a subgraph isomorphic to $H_1$ or $H_2$ where $v$ and $w$ play the roles of $x_1$ and $x_2$, respectively.

**Claim 3.** Let $\alpha \in A$ and suppose that $G_\alpha \subseteq \Gamma$. If $u \in V(G_\alpha)$ is a vertex that is contained in $V(U_\alpha)$ and $v \in V(\Gamma)$ is any vertex, then $R(u, v)$ implies that also $v \in V(U_\alpha)$ and that $u$ and $v$ are adjacent in $U_\alpha$.

**Proof of Claim 3.** Let $\alpha \in A$ be given and write $G := G_\alpha$ and $U := U_\alpha$. First, note that $G$ contains a subdivision $U'$ of $U$ (see Figure 3). We claim that the following is true:
(1) If $a \neq b \in V(U)$ are adjacent in $G$, then $a$ and $b$ are also adjacent in $U$ and we have $\alpha(e(a, b)) = 1$ (which means that there is a copy of $H_1$ in $G$ between $a$ and $b$).

(2) If $a \neq b \in V(U)$, $c \subset V(\Gamma) \setminus V(U)$, and $ac, cb \in E(\Gamma)$, then $a$ and $b$ are adjacent in $U$.

For the proof of (1), first suppose that $a$ and $b$ are not adjacent in $U$. Then $G + ab \subset \Gamma$ contains a copy of $T$ with centre $a$ which is a contradiction: Let $a'$ be the neighbour of $a$ in $U$ which lies on the $a$-$b$ path in $U$. We embed $P_m$ in the copy of $H_2$ or $H_2$ which we inserted for the edge $aa'$ in $G$ and $P_i$ for $i > m$ in $U' + ab$. It is easy to embed $P_i$ for $i < m$ since $a$ has infinite degree in $G$. Thus $a$ and $b$ are adjacent in $U$. Now suppose that $\alpha(e(a, b)) = 2$ and let $\gamma$ be a topological embedding of $H_2$ in $G$ with $\gamma(x_a) = a$ and $\gamma(x_b) = b$. However, then we can find a copy of $T$ in $\Gamma$ with centre $a$ by embedding $P_m$ in the path $ay(y_1) \ldots \gamma(y_{p_m})$ and $P_i$ for $i > m$ into $U' + ab$. Again it is easy to embed $P_i$ for $i < m$. The proof of (2) is similar, we only have to choose $U'$ so that $c \notin V(U')$.

For the proof of Claim 3, we begin with the case that $R(u, v)$ witnessed by a subgraph of $\Gamma$ isomorphic to $H_2$ such that $u$ and $v$ play the roles of $x_1$ and $x_2$ in $H_2$, respectively. Then $v$ must lie in $V(U')$ because otherwise we can find a copy of $T$ in $\Gamma$ with centre $u$: We embed all paths $P_i$ with $i > m$ in $U'$ and $P_m$ in one of the infinitely many $u$-$v$ paths which avoids $P_i - u$ for all $i > m$. Therefore, we can assume that $v \in V(U')$ and that we cannot find a different subdivision $U''$ of $G$ with $v \notin V(U'')$. Hence $v \in V(U)$. Suppose for a contradiction that $u$ and $v$ are not adjacent in $U$ and choose $U'$ so that $d_{U'}(u, v) > p_m$. However, then we can find a copy of $T$ in $\Gamma$ with centre $u$ by embedding $P_i$ for $m \leq i < k$ in $U' - v$ and $P_k$ in the union of $U'$ with a $u$-$v$ path in $\Gamma$ avoiding $P_i - u$ for all $m \leq i < k$.

Otherwise we have $R(u, v)$ because $H_2 \subset \Gamma$, witnessed by an embedding $f$ of $H_2$ in $\Gamma$ such that $f(x_1) = u$ and $f(x_2) = v$. Additionally, we may assume that there is no such embedding of $H_2$ in $\Gamma$. We begin by showing that we can assume that $V(U)$ contains none of the vertices $f(y_1), \ldots, f(y_{p_m})$. Suppose that there are $i \neq j$ with $f(y_i), f(y_j) \in V(U)$. Then $uf(y_i), uf(y_j), f(y_i)f(y_j) \in E(U)$ by (1), which is impossible because $U$ is a tree. Therefore at most one of the vertices $f(y_1), \ldots, f(y_{p_m})$ can lie in $V(U)$, without loss of generality $f(y_1), \ldots, f(y_{p_m-1}) \notin V(U)$.

If $v \in V(U)$ we can show that $u$ and $v$ are adjacent in $U$, as required in Claim 3: Since $f(y_1) \notin V(U)$ and the edges $uf(y_1)$ and $f(y_1)v$ are contained in $E(\Gamma)$, it follows from (2) that $u$ and $v$ are adjacent in $U$. It is left to show that $v \notin V(U)$ is impossible.

Finally, we demonstrate that $v, f(y_1), \ldots, f(y_{p_m-1}) \notin V(U)$ leads to a contradiction. We begin by showing that we can always choose $U'$ so that it contains none of the vertices $v, f(y_1), \ldots, f(y_{p_m-1})$. There is only one case in which this is impossible: We must have $H_2 \subset G$, witnessed by an embedding $g$ of $H_2$ in $G$ such that $\{v, f(y_1), \ldots, f(y_{p_m-1})\} = \{g(y_1), \ldots, g(y_{p_m})\}$. First we notice that $f(y_{p_m}) \in V(U)$, as otherwise we can find a copy of $T$ in $\Gamma$ with centre $g(x_1)$ by embedding $P_m$ in the path $g(x_1)g(y_2)g(y_3)\ldots g(y_{p_m})g(y_{p_m})$ and $P_i$ for $i > m$ into a subdivision of $U$ in $G$ containing $g(y_1)$. Furthermore, since $uf(y_{p_m}) \in E(\Gamma)$, (1) implies that there is a vertex $w \in \{u, f(y_{p_m})\}$ which is not an element of $\{g(x_1), g(x_2)\}$. The edges $wf(y_1)$ and $f(y_1)g(x_1)$ are contained in $E(\Gamma)$ for $i = 1, 2$ and thus (2) implies that $w$ and $g(x_1)$ are adjacent in $U$. However, it follows that $g(x_1), g(x_2)$ and $w$ induce a triangle in the tree $U$. Hence we can assume that $U'$ does not contain $v, f(y_1), \ldots, f(y_{p_m-1})$. But now we can find a copy of $T$ with centre $u$ in the union of $U'$ with the path $uf(y_1)\ldots f(y_{p_m-1})v$ in $\Gamma$, a contradiction. \qed
For all $\alpha \in A$ let $\gamma_{\alpha}$ be an embedding of $G_{\alpha}$ in $\Gamma$ and define $D_{\alpha}^{i} := \{ v \in V(U_{\alpha}) : d_{U_{\alpha}}(r_{\alpha}, v) = i \}$ for all $i \in \mathbb{N}$. Since $A$ is uncountable but $\Gamma$ is countable, there exist $\alpha \neq \alpha' \in A$ such that $\gamma_{\alpha}(r_{\alpha}) = \gamma_{\alpha'}(r_{\alpha'})$. Without loss of generality, we assume that $G_{\alpha}, G_{\alpha'} \subseteq \Gamma$ (and not just $G_{\alpha}, G_{\alpha'} \trianglelefteq \Gamma$). By Claim 3, we have $D_{\alpha}^{i} = D_{\alpha'}^{i}$ for all $i \in \mathbb{N}$. Since $\alpha \neq \alpha'$, it follows that there are embeddings $f : H_{1} \to G_{\alpha} \subseteq \Gamma$ and $g : H_{2} \to G_{\alpha'} \subseteq \Gamma$ such that $f(x_{1}) = g(x_{1})$ and $f(x_{2}) = g(x_{2})$ (we might have to swap the roles of $\alpha$ and $\alpha'$ for that).

The vertices $g(y_{1}), ... , g(y_{p_{m}})$ are not contained in any copy of $H_{2}$ in $G_{\alpha}$: Suppose for a contradiction that, for example, $g(y_{1})$ is contained in a copy $H'_{2}$ of $H_{2}$ in $G_{\alpha}$. Note that $g(y_{1}), ... , g(y_{p_{m}}) \notin V(U_{\alpha'}) = V(U_{\alpha})$. Furthermore, $g(y_{1})$ is adjacent to $g(x_{1}) = f(x_{1})$ and $g(x_{2}) = f(x_{2})$ and to vertices $y, z$ that play the role of $x_{1}, x_{2}$ in $H'_{2}$. Hence $\{f(x_{1}), f(x_{2})\} \neq \{y, z\}$ and therefore $g(y_{1})$ is adjacent to at least three distinct vertices from $V(U_{\alpha})$. Then by (2) we can find a triangle in $U_{\alpha}$, a contradiction.

Let $U' \subseteq G_{\alpha}$ be a subdivision of $U_{\alpha}$. Since $g(y_{1}), ... , g(y_{p_{m}})$ are not contained in any copy of $H_{2}$ in $G_{\alpha}$ and $g(y_{1}), ... , g(y_{p_{m}}) \notin V(U_{\alpha})$, we can choose $U'$ so that it contains none of the vertices $g(y_{1}), ... , g(y_{p_{m}})$. However, the union of $U'$ with the path $g(x_{1})g(y_{1}) ... g(y_{p_{m}})$ contains a copy of $T$. Thus a $\preceq$-universal graph $\Gamma$ in $\text{Forb}(T; \preceq)$ cannot exist.

**Proof of Theorem 1.4.** We combine Theorems 4.2 and 5.1. □

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