KRULL DIMENSION FOR LIMIT GROUPS IV: ADJOINING ROOTS

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Abstract. This is the fourth and last paper in a sequence on Krull dimension for limit groups, answering a question of Z. Sela. In it we finish the proof, analyzing limit groups obtained from other limit groups by adjoining roots. We generalize our work on Scott complexity and adjoining roots from the previous paper in the sequence to the category of limit groups.

1. INTRODUCTION, NOTATION, THEOREMS

It will take a moment to establish the notation and define the objects needed to state our main theorem. Roughly, we are interested in solutions, in the category of limit groups, to equations of the form “adjoin a root to g.” We can give no specific characterizations of solutions, but under special circumstances arising in the second paper in this series, [Lou08b], we are able to show that most of the time solutions are unique.

The notation $Z_G(E)$ indicates the centralizer in $G$ of a subgroup $E$. The set of images of edge groups incident to a vertex group $V$ of a graph of groups decomposition is denoted by $E(V)$. The phrase “$X$ is controlled by $Y$” should be read as “there is a function $f$, defined independently of $X$ and $Y$, such that $X \leq f(Y)$”.

Let $G$ be a group. A system of equations over $G$ is a collection of words in the alphabet $\{x_i, g \mid g \in G\}$, where the $x_i$ are variables distinct from the elements of $G$. The elements of $G$ are the coefficients, and the coefficients occurring in $\Sigma$ are the coefficients of $\Sigma$. If $\Sigma$ is a system of equations over $G$ there is a canonical group $G_\Sigma$ associated to $\Sigma$ with the presentation $\langle x_i, G \mid \Sigma \rangle$, where the $x_i$ are the variables occurring in $\Sigma$. If the map $G \to G_\Sigma$ is injective then $\Sigma$ has a solution. If $G < H$ and the inclusion map extends to $G_\Sigma \to H$ then $H$ is a splitting group for $\Sigma$, and $G$ is the ground group. Splitting groups are partially ordered by the relation “maps onto.” Every pair $G < H$ is a ground-splitting pair for some (in general, many) system of equations $\Sigma(G, H)$. A tuple $(G, H, G')$ is flight if $H$ and $G'$ are both splitting groups over $G$, and $H \to G'$.

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One may ask for splitting groups in a category $\mathcal{C}$ of groups. If $H \in \mathcal{C}$ is a splitting group, then $H$ is a splitting group in $\mathcal{C}$. If $\mathcal{C}$ is the class of all groups, then there are maximal $\mathcal{C}$–splitting groups, but this is not the case for general classes.

A sequence of inclusions $\mathcal{G} = (\mathcal{G}(0) < \mathcal{G}(1) < \cdots)$ is a tower. A staircase is a pair of sequences $(\mathcal{G}, \mathcal{H})$ such that $\mathcal{G}$ is a tower and $\mathcal{G}(i)$ is a splitting group for (some system) $\Sigma(\mathcal{G}(i), \mathcal{H}(i))$, that is, $(\mathcal{G}(i), \mathcal{H}(i), \mathcal{G}(i))$ is a flight. All staircases considered in this paper have the property that all coefficients lie in $\mathcal{G}(0)$. The name staircase comes from the fact that a commutative diagram representing one looks like a staircase and walks up a tower.

**Definition 1.1 (Adjoining roots).** Let $G$ be a finitely generated group, $\mathcal{E}$ a collection of nontrivial abelian subgroups of $G$. For each $E \in \mathcal{E}$, let $\mathcal{F}(E)$ be a collection of finite index supergroups of $E$, with an inclusion map $i_{E, F}: E \hookrightarrow F$ for each $F \in \mathcal{F}(E)$, and let $\mathcal{F}(\mathcal{E})$ be the collection $\{\mathcal{F}(E)\}$. Let

$$G \left[ \frac{\mathcal{F}(\mathcal{E})}{\sqrt{\mathcal{E}}} \right] := \langle G, F \mid E = i_{E, F}(E) \rangle_{F \in \mathcal{F}(E), E \in \mathcal{E}}$$

A finitely generated group $H$ is obtained from $G$ by adjoining roots $\mathcal{F}(\mathcal{E})$ to $\mathcal{E}$ if $G < H$ and the inclusion map extends to a surjection

$$G \left[ \frac{\mathcal{F}(\mathcal{E})}{\sqrt{\mathcal{E}}} \right] \twoheadrightarrow H$$

Let $\Sigma = \Sigma(\mathcal{E}, \mathcal{F}(\mathcal{E}))$ be a system of equations corresponding to the identification of $E$ with $i_{E, F}(E)$ for all $E$ and $F \in \mathcal{F}(E)$. Then $H$ is a splitting group for $\Sigma$. We call $H$ a cyclic extension of $G$ because the relations are all of the form “adjoin a root to $G$.”

Most of the time the specific nature of $\mathcal{F}$ is immaterial, and we usually eliminate it from the notation. To further compress the language used, sometimes we simply write that $H$ is obtained from $G$ by adjoining roots.

A group is *conjugately separated abelian*, or CSA, if maximal abelian subgroups are malnormal. Let $\sim_Z$ be the relation “is conjugate into the centralizer of”. This is an equivalence relation as long as the group is CSA. Two important consequences of CSA are commutative transitivity and that every nontrivial abelian subgroup is contained in a unique maximal abelian subgroup.

Commutative transitivity can occasionally be used to simplify systems of equations. Suppose $H$ is obtained from $G$ by adjoining roots $\mathcal{F}(\mathcal{E})$ to $\mathcal{E}$. Let $\eta$ be the inclusion map. We remove some redundancy by singling out a subcollection of each of $\mathcal{E}$ and $\mathcal{F}(\mathcal{E})$, and replacing each subcollection by a single element. Fix some $\sim_Z$ equivalence class $[E]$. By conjugating we may assume that each element of $[E]$ is a subgroup of $Z_G([E])$. Replace $[E]$ by $\{Z_G([E])\}$, and replace $\bigcup_{B \in [E]} \mathcal{F}(B)$ by

$$\langle Z_G([E]), F \mid B = i_{B, F}(B) \rangle_{B \in [E], F \in \mathcal{F}(B)}^{ab}$$

Then by commutative transitivity $H$ is a quotient of

$$G \left[ \frac{\mathcal{F}(\mathcal{E})}{\sqrt{\mathcal{E}}} \right]$$
Since limit groups are CSA we make this reduction without comment. Since $\mathcal{F}(E)$ has a single element after this simplification, we will generally use the less ostentatious notation $F(E)$ or just $\sqrt{E}$. We will call a system of equations without any such redundancy reduced.

**Definition 1.2 (Staircase).** A cyclic staircase is a staircase, with tower $\mathcal{G}$, equipped with a family families $\mathcal{E}$ of subgroups $\mathcal{E}_i$ of $\mathcal{G}(i)$, $(\mathcal{G}, \mathcal{H}, \mathcal{E})$, such that

- $((\mathcal{G}(i - 1), \mathcal{H}(i), \mathcal{G}(i)))$ is a flight; $\mathcal{H}(i)$ is obtained from $\mathcal{G}(i - 1)$ by adjoining roots to $\mathcal{E}_{i-1}$
- Each $E' \in \mathcal{E}_i$ in $\mathcal{G}(i)$ centralizes, up to conjugacy, the image of an element $E$ of $\mathcal{E}_{i-1}$. If $E \in \mathcal{E}_{i-1}$ is mapped to $E' \in \mathcal{E}_i$ then we require that the image of $Z_G(E)$ in $Z_{G'}(E')$ be finite index.

To fix notation, the maps $\mathcal{G}(i) \hookrightarrow \mathcal{G}(i + 1)$, $\mathcal{G}(i) \hookrightarrow \mathcal{H}(i + 1)$, and $\mathcal{H}(i + 1) \twoheadrightarrow \mathcal{G}(i + 1)$ are denoted by $\eta_i$, $\nu_i$, and $\pi_{i+1}$, respectively. The length of $\mathcal{G}$ is denoted $\|\mathcal{G}\|$.

It will be handy to have a rough description of a staircase. A staircase of limit groups is

- freely decomposable if all $\mathcal{G}(i)$ are freely decomposable
- freely indecomposable if all $\mathcal{G}(i)$ are freely indecomposable
- QH-free if no $\mathcal{G}(i)$ has a QH subgroup
- mixed if it has both freely decomposable and freely indecomposable groups, or, if freely indecomposable, has both groups with and without QH subgroups. Otherwise it is pure.

**Definition 1.3.** Let $(i_j)$ strictly increasing sequence of indices. A staircase $(\mathcal{V}, \mathcal{W})$, such that $\mathcal{V}(j) = \mathcal{G}(i_j)$ and $\mathcal{W}(j) = \mathcal{H}(i_j)$, with maps obtained by composing maps from $(\mathcal{G}, \mathcal{H})$, is a contraction of $(\mathcal{G}, \mathcal{H})$, and is based on $(i_j)$.

To see that a contraction of a cyclic staircase is a staircase consider the following diagram:

$$
\begin{array}{cccc}
\mathcal{H}(i_j + 1) & \cdots & \mathcal{H}(i_{j+1} - 1) & \mathcal{H}(i_{j+1}) \\
\downarrow & & \downarrow & \\
\mathcal{G}(i_j) & \cdots & \mathcal{G}(i_{j+1} - 1) & \mathcal{G}(i_{j+1})
\end{array}
$$

Each $E \in \mathcal{E}_i$ has finite index image in its counterpart in $\mathcal{E}_{i+1}$, the image of $E$ in its counterpart in $\mathcal{E}_{i+1}$ is finite index. Extending an abelian group by a finite index super-group multiple times can be accomplished by extending once.

The need for contractions explains the restriction that each $E \in \mathcal{E}_i$ contain a conjugate of the image of some $E' \in \mathcal{E}_{i-1}$. If this is not the case, then there is no hope for the existence of contractions; we can’t adjoin a root to an element that isn’t there.

A segment of a staircase is a contraction whose indices are consecutive, that is $i_{j+1} - i_j = 1$ for all $j$. 
Let $\mathcal{E}$ be a collection of elements of a CSA group $G$. We denote by $\|\mathcal{E}\|$ the number of $\sim_\mathbb{Z}$ equivalence classes in $\mathcal{E}$. The complexity of $(G, \mathcal{H}, \mathcal{E})$ is the triple $\text{Comp}(\langle G, \mathcal{H}, \mathcal{E} \rangle) = (b_1(G), \text{depth}_{\text{pc}}(\mathcal{H}), \|\mathcal{E}\|)$. Complexities are not compared lexicographically: $(b', d', e') \leq (b, d, e)$ if $b' \leq b$, $d' \leq d$, and $e' \leq e + 2(d - d')b$. That this defines a partial order follows easily from the definition. The inequality is strict if one of the coordinate inequalities is strict. See Definition 2.4 and the material thereafter for a discussion of depth. Another immediate consequence of the definition of $\leq$ is that it is locally finite.\footnote{Fix $a$ and $b$. Then \( \{ x \mid a \leq x \leq b \} \) is finite.}

Let $(G, \mathcal{H}, \mathcal{E})$ be a staircase. The quantity $\text{NInj}(\langle G, \mathcal{H}, \mathcal{E} \rangle)$ is the number of indices $i$ such that $\mathcal{H}(i) \rightarrow G(i)$ is not an isomorphism.

**Theorem 1.4.** Let $(G, \mathcal{H}, \mathcal{E})$ be a staircase. There is a function $\text{NInj}(\text{Comp}(\langle G, \mathcal{H}, \mathcal{E} \rangle))$ such that $\text{NInj}(\langle G, \mathcal{H}, \mathcal{E} \rangle) \leq \text{NInj}(\text{Comp}(\langle G, \mathcal{H}, \mathcal{E} \rangle))$

**Remark 1.5.** Although it would be nice to assign a complexity $c()$ to a limit group such that if, in a flight $(G, H, G')$, $c(G) = c(G')$, then $H \rightarrow G'$ is an isomorphism, this doesn’t seem possible, and the approach taken in this paper requires that complexities be computed and compared in context.

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## 2. Complexities of sequences

The main object which enables this analysis of adjoining roots is the JSJ decomposition, a device for encoding families of splittings of groups. This exposition borrows from [BF03, RS97]. A GAD, or **generalized abelian decomposition** of a group $G$ is a finite graph of groups decomposition over abelian edge groups such that every vertex group is marked as one of rigid, abelian, or QH, where by QH we mean is the fundamental group of a compact surface with boundary possessing two intersecting essential simple closed curves. Moreover, edge groups adjacent to a QH vertex group must be conjugate to boundary components of the surface. If $A$ is an abelian vertex group, the peripheral subgroup of $A$ is the subgroup of $A$ which dies under every map $A \rightarrow \mathbb{Z}$ killing all incident edge groups.

We say that two GAD’s of a limit group are **equivalent** if they have the same elliptic subgroups. A splitting is **visible** in a GAD $\Delta$ if it corresponds to cutting a QH vertex group along a simple closed curve, a one-edged splitting of an abelian vertex group in which the peripheral subgroup is elliptic, or is a one edged splitting corresponding to an edge from an equivalent decomposition. If $\Delta$ is a GAD, then $g \in G$ is $\Delta$–elliptic if elliptic in every one-edged splitting of $G$ visible in $\Delta$. Let $\text{Ell}(\Delta)$ be the set of $\Delta$–elliptic elements.

Let $G$ be a freely indecomposable finitely generated group, and let $\mathcal{C}$ be a family of one-edged splittings of $G$ such that

- edge groups are abelian,
• noncyclic abelian subgroups are elliptic.

The main construction of JSJ theory is that given a family of splittings $\mathcal{C}$ satisfying these conditions, there is a GAD $\Delta$ such that $\text{Ell}(\Delta) = \cap_{C \in \mathcal{C}} \text{Ell}(C)$.

An abelian JSJ decomposition of $G$ is a GAD $\text{AJSJ}(G)$ such that the set of $\text{AJSJ}$–elliptic elements corresponds to the collection of all one-edged splittings satisfying the bullets above. The existence of a JSJ decomposition is somewhat subtle, as one needs to bound the size of a GAD arising in this way [Sel01, Theorem 3.9]. If $G$ is a nonelementary freely indecomposable limit group then $G$ has a nontrivial JSJ decomposition. If $G$ is elementary, the JSJ is a point.

In this paper we are interested in the principle cyclic JSJ decomposition, which is the JSJ associated to the family of principle cyclic splittings.

**Definition 2.1 ([Sel01]).** A one-edged splitting over a cyclic subgroup is *inessential* if at least one vertex group is cyclic, and is *essential* otherwise. A principle cyclic splitting of a limit group is an essential one-edged splitting $G \cong A \ast_C B$ or $G \cong A \ast_C$, over a cyclic subgroup $C$, such that either $Z_G(C)$ is cyclic or $A$ is abelian.

The principle cyclic JSJ of a freely indecomposable limit group is the JSJ decomposition corresponding to the family of principle cyclic splittings. We denote the principle cyclic JSJ by $\text{JSJ}(G)$.

Let $E \subset G$. The principle cyclic JSJ of $G$, relative to $E$ is a JSJ decomposition corresponding to the family of all principle cyclic splittings of $G$ such that each member of $E$ is elliptic. We denote the relative JSJ by $\text{JSJ}(G; E)$. A principle cyclic decomposition is simply a relative principle cyclic JSJ for some collection $E$.

The restricted principle cyclic JSJ, or restricted JSJ for short, of a freely indecomposable limit group $G$ with QH subgroups is the relative JSJ decomposition associated to the set of principle cyclic splittings whose edge groups are hyperbolic in some other principle cyclic splitting. It is obtained from the JSJ by collapsing all edges not adjacent to some QH vertex group. If $G$ doesn’t have QH vertex groups, then the restricted JSJ is just the principle cyclic JSJ. The restricted principle cyclic JSJ is denoted by $\text{rJSJ}(G)$

That limit groups have principle cyclic splittings is [Sel01, Theorem 3.2]. It need not be the case that every splitting visible in the principle cyclic JSJ is principle; for instance, a boundary component of a QH vertex group may be the only edge attached to a cyclic vertex group. The splitting corresponding to the boundary component is not essential, but is certainly visible in the principle cyclic JSJ.

In this paper we work primarily with the principle cyclic JSJ of $G$, indicated by $\text{JSJ}(G)$, and the rJSJ. If $\Delta$ is a graph of groups decomposition then $T_\Delta$ is the Bass-Serre tree corresponding to $\Delta$.

We can give a more explicit description of the principle cyclic JSJ. Consider the abelian JSJ of a limit group $G$. Clearly all QH vertex groups of $\text{AJSJ}(G)$ appear as vertex groups of $\text{JSJ}(G)$. If $A$ is an abelian vertex group of $\text{AJSJ}(G)$ with noncyclic peripheral subgroup, since there is no principle cyclic splitting of $G$ over a subgroup of $A$, the subgroup of $G$ generated by $A$ and conjugates of rigid
vertex groups having nontrivial intersection with $A$ must be elliptic in $\text{JSJ}(G)$. If $R$ is a rigid vertex group of $G$ and an edge group $E$ incident to $R$ has noncyclic centralizer in $R$, then the subgroup of $G$ generated by $R$ and any conjugate of a rigid vertex group $R'$ intersecting $R$ in a nontrivial subgroup of $E$ is also elliptic. From this we see that the principle cyclic JSJ of $G$ must have the following form:

- Every abelian vertex group has cyclic peripheral subgroup. If $R$ is adjacent to an abelian vertex group $A$, $E$ the edge group, then $R$ does not have an essential one-edged splitting over $E$ in which each element of $E(R)$ is elliptic.
- If an edge $e$ incident to a rigid vertex group $R$ has noncyclic centralizer in $R$, then the edge is attached to a boundary component of a QH vertex group.
- If two edges incident to a rigid vertex group $R$ have the same centralizer in $R$, then they are both incident to QH vertex groups, and their centralizer in $R$ is noncyclic.

The JSJ decomposition of a limit group, be it abelian or principle cyclic, is only unique up to morphisms of graphs of groups preserving elliptic subgroups. Some principle cyclic JSJ’s are more convenient to work with than others, and we assume throughout that

- Edge groups not adjacent to QH vertex groups are closed under taking roots, and edge maps of edge groups into QH vertex groups are isomorphisms with the corresponding boundary components.
- There are no inessential splittings visible in the JSJ, other than from valence one cyclic vertex groups attached to boundary components of QH vertex groups.

Let $R$ be a rigid vertex group of the full abelian JSJ of a limit group $G$, and let $\bar{R}$ be the subgroup of $G$ generated by $R$ and all elements with powers in $R$. A decomposition with the properties above can be thought of as the JSJ decomposition associated to the family of principle cyclic splittings in which all $\bar{R}$, $R$ a vertex group of the abelian JSJ, are elliptic.

In general, there are infinitely many principle cyclic decompositions of a limit group, all obtained from the principle cyclic JSJ by folding, cutting QH vertex groups along simple closed curves, and collapsing subgraphs.

**Lemma 2.2.** Let $G$ be a limit group and $E \subset G$ a collection of elements of $G$. Then there are at most $2^{|E|}$ equivalence classes of principle cyclic decompositions in which some elements of $E$ are elliptic.

**Proof.** If $E \in E$ is elliptic, then so is any $E' \in E$ such that $E \sim Z E'$.

We need to adapt the definition of the analysis lattice of a limit group given in [Sel01, §4] to the inductive proof given in section 4.4. A limit group is *elementary* if it is abelian, free, or the fundamental group of a closed surface.

**Definition 2.3** (Principle cyclic analysis lattice). The *principal cyclic analysis lattice* of a limit group $G$ is the rooted tree of groups whose levels are defined as follows:
If an elementary limit group is encountered, it is a terminal leaf of the tree.

**Definition 2.4.** The depth of a limit group $H$ is the number of levels in its principle cyclic analysis lattice, and is denoted $\text{depth}_{\text{pe}}(H)$.

The depth of a staircase $(G, \mathcal{H}, \mathcal{E})$ is $\max \{ \text{depth}_{\text{pe}}(\mathcal{H}(i)) \}$, and is denoted $\text{depth}_{\text{pe}}((G, \mathcal{H}, \mathcal{E}))$. The first betti number of $(G, \mathcal{H}, \mathcal{E})$ is the first betti number of $G(1)$.

It is not always necessary to refer to the family $\mathcal{E}$, so we suppress it from the notation when its size is irrelevant. That the depth is well defined is a consequence of Theorem 2.5.

**Theorem 2.5.** The depth of the principle cyclic analysis lattice of a limit group $L$ is controlled by its rank.

**Proof.** We only need to worry about the possibility that the principle cyclic analysis lattice contains a long branch of the form $G_0 > G_1 > \cdots$, where each $G_i$ is freely indecomposable, has no QH vertex groups, no noncyclic abelian vertex groups, and $\text{JSJ}(G_i)$ has only one nonabelian vertex group $G_{i+1}$. After observing [Lou08a, Hou08] that $L$ has a strict resolution of length at most $6 \text{rk}(L)$, the proof is identical to [Lou08b, Theorem 2.11].

We motivate our proof of Theorem 1.4 and the previous definition with an example.

**Example 2.6.** Suppose that $G(i)$ has a one-edged JSJ decomposition with two nonabelian vertices for all $i$. Since a limit group has a principle cyclic splitting, the one-edged splitting of $G(i)$ must be of the form $G_1(i) \ast_{\langle e_i \rangle} G_2(i)$. By Lemma 3.1, if $H$ is obtained from $G$ by adjoining roots, then $G$ acts hyperbolically in all splittings of $H$. In particular, every vertex group of $H$ contains a vertex group of $G$. If the JSJ of $H$ was a loop, then the map to the underlying graph kills $G$, but since the map $G \to H$ is almost onto on homology, this cannot happen. Thus each $\mathcal{H}(i)$ has a one-edged JSJ decomposition $\mathcal{H}_1(i) \ast_{\langle f_i \rangle} \mathcal{H}_2(i)$.

The triple $G(i - 1) \hookrightarrow \mathcal{H}(i) \twoheadrightarrow G(i)$ has the following form: The pairs $(G_j(i - 1), \langle e_{i-j} \rangle)$ map to the pairs $(G_j(i), \langle e_i \rangle)$ and $(\mathcal{H}_j(i), \langle f_i \rangle)$, and the maps $\eta_i$, $\nu_i$, and $\pi_i$ respect these one-edged splittings. We’ll show later that in fact $\mathcal{H}_j(i)$ is obtained from $G_j(i - 1)$ by adjoining roots. By work from [Lou08b] $G_j(i)$ is obtained from the image of $\mathcal{H}_j(i)$ by iteratively adjoining roots to the incident edge group (See Appendix A). Let $G'_j(i)$ be the image of $\mathcal{H}_j(i)$ in $G_j(i)$. Now consider the staircase $(G'_j(i), \mathcal{H}_j(i))$. The sequence $\mathcal{H}_j$ has strictly lower depth than $\mathcal{H}$.

By induction on $\text{Comp}$, there is an upper bound on the number of indices such that $\mathcal{H}_j(i) \hookrightarrow G(i)$ is not injective, and for at most twice that bound, both maps $\mathcal{H}_j(i) \twoheadrightarrow G_j(i)$ are injective. Then $\mathcal{H}(i) \twoheadrightarrow G(i)$ is strict for such indices. Since
every Dehn twist in $\langle f_i \rangle$ pushes forward to a Dehn twist in $\langle e_{i+1} \rangle$, $\pi_i$ is an isomorphism.

3. ALIGNING JSJ DECOMPOSITIONS

Let $G$ be a finitely generated group acting on a simplicial tree $T$ minimally and without inversions. It is a standard fact that the quotient $T/G$ is the underlying graph of a graph of groups decomposition of $G$. If $H < G$ is a finitely generated subgroup, there is a minimal subtree $S \subset T$ fixed (setwise) by $H$, and the action of $H$ on $S$ endows $H$ with a graph of groups decomposition. Additionally, there is an induced map of quotient graphs $S/H \to T/G$.

We are interested in the following problem: Suppose $G$, $H$, $T$, and $S$ are as above, $G$ and $H$ freely indecomposable limit groups, $T$ the Bass-Serre tree corresponding to the principle cyclic JSJ of $H$. We say that $G$ and $H$ are aligned if $S/H \to T/G$ is an isomorphism of graphs and $S/H$ is the underlying graph of the principle cyclic JSJ of $H$. Give a simple computable criterion which guarantees that $G$ and $H$ are aligned.

As long as $G^{ab} \to H^{ab}$ is virtually onto, we are able to answer this question in a reasonable way, constructing a (monotonically decreasing) complexity, equality of which will guarantee alignment of JSJs. The properties of the alignment are then used to construct graphs of spaces and maps between them which resemble Stallings’ immersions. The main idea of this section is that an inclusion as above must either “tighten up” the Grushko/JSJ becoming simpler in a quantifiable way, or can be written as a map of graphs of groups respecting the JSJ decompositions.

Let $T$ be the Bass-Serre tree corresponding to the principle cyclic JSJ of $H$, let $S$ be the minimal subtree for $G$. The quotient $S/G$ is finite and it follows from the definitions that the induced graphs of groups decomposition of $G$ is principle. For convenience, we usually conflate underlying graphs and graphs of groups decompositions. Let $\Delta_H = T/H$ and $\Delta_G = S/G$ be the underlying graphs, and let $\eta#$ be the induced map. We label a vertex $v$ of $\Delta_G$ by the corresponding label on $\eta#(v)$, unless $G_v$ is abelian, in which case we label it abelian anyway. The map $\eta#$ is well behaved:

- If $v$ is rigid then the edge groups adjacent to $v$ have nonconjugate centralizers in $G_v$ unless they are all attached to boundary components of QH vertex groups.
- Let $B$ be a maximal connected subgraph of $\Delta_G$ such that every vertex is abelian. Commutative transitivity implies that $G_B$ is abelian, and the fact that all noncyclic abelian subgroups of $H$ are elliptic in $T_H$ implies that $B$ is a tree.
- If $v$ is abelian and $\eta#(v)$ is nonabelian, then $\eta#(v)$ is rigid.
- A valence one cyclic $v$ is adjacent to a QH $w$. This follows from the assumption that the only edge groups of $H$ allowed to be not closed under taking roots are adjacent to QH vertices.
Lemma 3.1. Let $\eta: G \to H$ be a homomorphism of freely indecomposable limit groups such that $H^1(H, \mathbb{Z}) \to H^1(G, \mathbb{Z})$ is injective. Then $G$ is hyperbolic in every essential one-edged abelian splitting of $H$.

If $R$ is a nonabelian vertex group of a GAD $\Delta_H$ of $H$, then $G$ intersects a conjugate of $R$ in a nonabelian subgroup. If $\Delta_G$ is the induced decomposition of $G$, and there is only one nonabelian vertex group $R'$ of $\Delta_G$ mapping to $R$, then the map on underlying graphs is a submersion at $R'$.

Proof. Claim: If $G$ acts elliptically in some essential one-edged splitting then there is a map $H \rightarrow \mathbb{Z}$ which kills $G$. If the one-edged splitting is an HNN extension the claim is clear. If not, then both vertex groups of the amalgam have a map onto $\mathbb{Z}$ which kills the incident edge group.

To see the second half, suppose not, and let $\Delta'_H$ be the decomposition of $H$ obtained by conjugating edge maps to $R$ so that all incident edges either have the same or nonconjugate centralizers, and folding together edges of the conjugated decomposition which have the centralizers. Then pull all the centralizers of incident edge groups across the edge they centralize. If $T$ is the tree for $\Delta'_H$, and $S$ is the minimal $G$–invariant subtree, then the map $S/G \rightarrow T/H$ clearly misses the vertex corresponding to $R$. Let $\Delta''_H$ be the decomposition of $H$ obtained by collapsing all edges not adjacent to $R$. Then $G$ is elliptic in $\Delta''_H$. The first part provides a contradiction.

If the map is not a submersion on the level of $\Delta_H$, then the of graphs of groups $\Delta'_G \rightarrow \Delta''_H$ is not a submersion onto $R$ either, and there is an edge incident to $R$ missed by $\Delta'_G$. This edge represents an essential splitting of $H$, and so we again have a contradiction.

Definition 3.2 (Complexity of JSJs). Let $G$ be a finitely generated freely indecomposable limit group with principle cyclic decomposition $G = \Delta(\mathcal{R}, \mathcal{Q}, \mathcal{A}, \mathcal{E})$, where each $R \in \mathcal{R}$ is rigid, each $Q \in \mathcal{Q}$ is QH, each $A \in \mathcal{A}$ is finitely generated abelian, and each $E \in \mathcal{E}$ is an infinite cyclic edge group. Let

- $c_q(G) := |\sum_{Q \in \mathcal{Q}} \chi(Q)|$ is the total Euler characteristic of QH subgroups.
- $c_{bq}(G) := \sum_{Q \in \mathcal{Q}} \#\partial Q$ is the total number of boundary components of QH vertex groups.
- $\mathcal{Z}(G)$ is the collection of conjugacy classes of centralizers of edge groups of $G$. Warning: not the center of $G$.
- For a given rigid vertex $R$ of the principle cyclic JSJ decomposition, let $v(R)$ be the valence of $R$. This is the same as the number of conjugacy classes of centralizers of incident edge groups in $R$.

- $c_a(G) := \sum_{A \in \mathcal{A}} (\text{rk}(A) - 1)$
- $c_b(G) = c_a(G) + b_1(\Delta)$

The complexity of $G$ with respect to $\Delta$ is the ordered tuple

$$J\text{Comp}(G, \Delta) = (c_q(G), -c_{bq}(G), |\mathcal{Z}|, c_b(G), b_1(\Delta), |\mathcal{R}|, \sum_{R \in \mathcal{R}} v(R))$$

The “,$\Delta$” is suppressed from the notation if $\Delta$ is the principle cyclic JSJ of $G$. 
Complexities are compared lexicographically. The complexity $\text{JComp}_i$ is the restriction of $\text{JComp}$ to the first $i$ coordinates.

Throughout this section $G$ and $H$ are freely indecomposable limit groups, $\eta: G \hookrightarrow H$, and $\eta^\#: H^1(H, \mathbb{Z}) \to H^1(G, \mathbb{Z})$ is injective.

We need to be able to compare the complexity of a principle decomposition to the complexity of the JSJ.

**Lemma 3.3.** Let $G$ be a freely indecomposable limit group with principle cyclic JSJ $\Delta_G$, let $\mathcal{E}$ be a fixed family of subgroups of $G$, and let $\Delta$ be the principle cyclic decomposition of $G$ associated to the family of principle cyclic splittings in which each $E \in \mathcal{E}$ is elliptic. Then $\text{JComp}(G, \Delta) \leq \text{JComp}(G)$, with equality if and only if $\Delta$ is the JSJ.

**Proof.** We can construct $\Delta$ by cutting QH vertex groups of $\Delta_G$ along simple closed curves, folding, and collapsing subgraphs. To handle $c_b$, observe that any collection of disjoint simple closed curves on QH vertex groups of $\Delta_G$ can be completed to a collection which achieves at most $c_b(G)$.

The inequalities on $c_q$ and $c_{bq}$ are obvious, and if they are equal, then the identity map simply identifies QH vertex groups. The remaining inequalities are obvious.

We spread the proof of Theorem 3.6 across the next two lemmas.

**Lemma 3.4.** $c_q(G) \geq c_q(H)$. If equality holds then $c_{bq}(G) \leq c_{bq}(H)$.

**Proof.** Let $T$ be the Bass-Serre tree for the restricted JSJ of $H$. Since $\eta$ is injective, $G$ inherits a graph of groups decomposition $\Delta$ from its action on $T$. Let $Q$ be a vertex group of $\Delta$ conjugate into some element $Q'$ of $Q(H)$. There are two possibilities: $Q$ either has finite or infinite index in $Q'$. If $Q$ has infinite index and is nontrivial then $G$ must be freely decomposable, contrary to hypothesis. Thus $Q$ is either trivial or finite index.

Let $c$ be a simple closed curve on some element $Q'$ of $Q(H)$ giving a essential one-edged splitting $\Delta_c$ of $H$. By Lemma 3.1 $G$ acts hyperbolically in $\Delta_c$, hence there is some $Q$ which maps to a finite index subgroup of a conjugate of $Q'$. The graph of groups decomposition $\Delta$ of $G$ is obtained by slicing QH vertex groups of $G$ along simple closed curves, folding, and collapsing subgraphs of the resulting decomposition. This immediately gives $c_q(G) \geq c_q(H)$.

Suppose equality holds. Let $c_k$ be the simple closed curves cutting the QH vertex groups of JSJ$(G)$, and let $Q'_1, \ldots, Q'_m$ be the complementary components which don’t map to QH vertex groups of $H$. Since $c_q(G) = c_q(H)$, each component $Q'_j$ has Euler characteristic 0. Any such complementary component cannot be boundary parallel, thus if there are any then $c_{bq}(H) > c_{bq}(G)$. If equality holds then the QH subgroups of $G$ and those of $H$ are in one to one correspondence and the respective maps are isomorphisms.

An inclusion $G \hookrightarrow H$ as above is QH–preserving if it is a one-to-one correspondence on QH vertex groups and the maps are isomorphisms. If $H$ has an
inefficient one-edged splitting $\Delta$, then $\Delta$ corresponds to an edge connecting a valence one cyclic vertex group of $JSJ(H)$ to a QH vertex group. If $G \hookrightarrow H$ is QH–preserving then it is necessarily bijective on such valence one vertex groups.

It follows immediately from Lemma 3.1 that if $G \hookrightarrow H$ is QH–preserving then $|Z(G)| \geq |Z(H)|$.

**Lemma 3.5.** $\text{JComp}_\eta(G) \geq \text{JComp}_\eta(H)$. If equality holds then there is an induced bijection $\mathcal{A}(G) \rightarrow \mathcal{A}(H)$, and for each $\Gamma, \mathcal{A}/\mathcal{A}(\eta_\#(\Lambda))/\mathcal{A}(\eta_\#(\Lambda))$ is virtually onto.

**Proof:** We first handle $c_0$.

Let $\Delta H$ be the principle cyclic JSJ of $H$, and let $\Delta G$ be the decomposition $G$ inherits from its action on $T_{\Delta H}$. We may assume that $G \hookrightarrow H$ is QH–preserving, is bijective on conjugacy classes of centralizers of edge groups. Let

$$H_1(H; \Delta_H^0 \setminus \mathcal{A}(H)) = H_1(\Delta_H) \oplus \bigoplus_{\mathcal{A} \in \mathcal{A}(H)} A/P(A).$$

Similarly, define $H_1(G; \Delta_G^0 \setminus \mathcal{A}(G))$. The composition $G \rightarrow H \rightarrow H_1(H; \Delta_H^0 \setminus \mathcal{A}(H))$ factors through $H_1(G; \Delta_G^0 \setminus \mathcal{A}(G))$. Since $H^1(H; Z) \hookrightarrow H^1(G; Z)$ the map $H_1(G; \Delta_G^0 \setminus \mathcal{A}(G)) \rightarrow H_1(\Delta_H; \Delta_H^0 \setminus \mathcal{A}(H))$ must be virtually onto. But $\text{rk}(H_1(H; \Delta_H^0 \setminus \mathcal{A}(H))) = c_0(H)$ and $\text{rk}(H_1(G; \Delta_G^0 \setminus \mathcal{A}(G))) \leq c_0(G)$.

Let $\Delta$ be an essential one-edged splitting of $H$ in which all QH subgroups are elliptic. Let $T$ be the corresponding Bass-Serre tree. By Lemma 3.1 $G$ doesn’t fix a point in $T$ and it inherits an essential splitting $\Delta'$ from this action. Since $\eta$ is bijective of the sets of QH subgroups, and restricts to isomorphisms between them, every QH vertex group of $G$ acts elliptically in $T$. Thus there is an edge group $E'$ of $JSJ(G)$ which maps to a conjugate of the edge group of $\Delta$. Furthermore, $E'$ is an essential splitting, otherwise $G$ acts elliptically in $\Delta$.

Let $A \in \mathcal{A}(G)$ be a noncyclic abelian vertex group. If no element of $\mathcal{A}(H)$ contains the image of $A$, then $c_0(G) > c_0(H)$. If equality holds there is a well defined map $\mathcal{A}(G) \rightarrow \mathcal{A}(H)$.

Let $A$ be an abelian vertex group of $G$, and $\eta_\#(A)$ the associated vertex group of $H$. Since $H^1(H) \rightarrow H^1(G)$ is injective, the map $A/P(A) \oplus H_1(\Gamma_H) \rightarrow \eta_\#(A)/P(\eta_\#(A)) \oplus H_1(\Gamma_H)$ must be virtually onto. This map sends $A/P(A)$ to $\eta_\#(A)/P(\eta_\#(A))$ hence $b_1(\Delta_G) \geq b_1(\Delta_H)$, and if $b_1(\Delta_G) = b_1(\Delta_H)$ then $A/P(A) \rightarrow \eta_\#(A)/P(\eta_\#(A))$ must be virtually onto.

**Theorem 3.6.** $\text{JComp}_0(G) \geq \text{JComp}_0(H)$. If $\text{JComp}_0(G) = \text{JComp}_0(H)$ then

$$\sum_{R \in \mathcal{R}(G)} v(R) \geq \sum_{R \in \mathcal{R}(H)} v(R),$$

i.e., $\text{JComp}(G) \geq \text{JComp}(H)$. If $\eta: G \hookrightarrow H, \text{JComp}(G) = \text{JComp}(H)$, then $\eta$ is bijective on vertex and edge groups, maps abelian vertex, edge, and peripheral subgroups to finite index subgroups of their respective images. The map from the underlying graph of the JSJ of $G$ to the underlying graph of the JSJ of $H$ is an isomorphism.
The number of values the complexity can take is controlled by $b_1$.

**Proof of Theorem 3.6.** Assume $\text{JComp}_5(G) = \text{JComp}_5(H)$. By Lemma 3.5, the inclusion is a one-to-one correspondence on noncyclic abelian vertex groups.

Let $\Delta_H$ be the principle cyclic JSJ of $H$, let $\pi: \Delta_G \rightarrow \Delta_H$ be the induced map of underlying graphs, and let $R$ be nonabelian non-QH vertex group of $\Delta_H$. By Lemma 3.1 there is a nonabelian vertex group of $\Delta_G$ which maps to $R$. Since $\eta$ is bijective on QH subgroups, there is a rigid vertex group $R'$ of $G$ which maps to $R$. If $\text{JComp}_5(G) = \text{JComp}_5(H)$ then $R'$ is the unique such vertex group.

Again, by Lemma 3.1, since there is only one vertex group $R'$ mapping to $R$, the map $E(R') \rightarrow E(R)$ is onto and $v(R') \geq v(R)$.

Let $Z$ be an essential cyclic abelian vertex group of $\Delta_H$, and let $Z_1, \ldots, Z_k$ be the vertex groups of $\Delta_G$ mapping to $Z$. Since $\eta$ is bijective on nonabelian vertex groups, and since all vertex groups adjacent to $Z$ are nonabelian, the induced map $\eta^\#: \sqcup E(Z_i) \rightarrow E(Z)$ is bijective. Arguing as in Lemma 3.1, $k = 1$ and the map $E(Z_1) \rightarrow E(Z)$ is bijective. The same observation shows that if $A$ is noncyclic abelian, then there is a unique $A'$ mapping to $A$ and that the map on the link is onto. The map is also injective, again because $\eta$ is a bijective on nonabelian vertex groups.

Thus, if the complexities are equal, then the inclusion must induce a homeomorphism of underlying graphs. By construction, the map is label preserving, and it automatically respect all incidence and conjugacy data from the respective JSJ decompositions.

This shows that $\text{JComp}(G, \Delta_G) \geq \text{JComp}(H)$, and if equality holds, then the morphism $\Delta_G \rightarrow \Delta_H$ is of the correct form. By Lemma 3.3 $\text{JComp}(G) \geq \text{JComp}(G, \Delta_G)$, and if $\text{JComp}(G) = \text{JComp}(H)$, then $\Delta_G$ is just the principle cyclic JSJ of $G$. This gives the first half of the theorem.

The bound on the number of values the complexity can take follows from either acylindrical accessibility [Sel97] plus the bound on the rank of a limit group with complexity $b_0$, or [Lou08b, Lemma 2.7], which gives a bound on the complexity of the principle cyclic JSJ in terms of the first betti number. Those arguments bound the number of essential vertex groups. Adjoining roots doesn’t increase the first betti number, so if $b_1$ and $b_2$ are boundary components of a QH vertex group adjacent to inessential vertex groups, then a simple closed curve cutting off a pair of pants with $b_1, b_2$ as the two other boundary components makes a contribution of one to $b_1(G)$; $n$ nonintersecting simple closed curves as above make a contribution of $n$ to $b_1(G)$, thus each QH vertex group is attached to at most $2b_1(G)$ inessential vertex groups. Since $b_1(G)$ controls the number of QH vertex groups, there are boundedly many inessential abelian vertex groups. 

In light of Theorem 3.6, if $\text{JComp}(G) = \text{JComp}(H)$, then we say that $G$ and $H$ are aligned. Before representing injections of limit groups topologically, we devote a section to proving Theorem 1.4, assuming the material from section 5.
4. Proof of Theorem 1.4

The bound implicitly computed in the proof of Theorem 1.4 can be made slightly better if we show that nonabelian limit groups with first betti number 2 are free. The next lemma is not necessary, but we record it here for lack of a better place to put it. In [FGM+98], Fine, et al., classify limit groups with rank at most three. The next lemma shows that in rank two the rank can be relaxed to first betti number.

Lemma 4.1. Let \( G \) be a limit group with first betti number 2. Then \( G \cong \mathbb{F}_2 \) or \( \mathbb{Z}^2 \).

Proof. We may assume \( G \) is nonabelian and freely indecomposable. If \( G \) is abelian it satisfies the theorem trivially, and if freely decomposable, the free factors are limit groups with first betti number one, and must be infinite cyclic.

The proof is by induction on the depth of the cyclic analysis lattice. All essential cyclic splittings of \( G \) are HNN extensions, otherwise there is a one-edged cyclic splitting such that each vertex group has betti number at least two, and \( G \) therefore has first betti number at least three. By a simple variation of the proof of Theorem 2.5 the depth of the cyclic analysis lattice of \( G \) is finite. Suppose that \( G \) has a QH vertex group \( Q \). Then any essential simple closed curve on \( Q \) must correspond to an HNN extension of \( G \): \( G = G' \ast_E \). Since the splitting comes from a QH vertex group, \( G' \) must be freely decomposable, hence is \( \mathbb{F}_2 \). If \( G \) has no QH vertex groups it’s principle cyclic JSJ decomposition must be a bouquet of circles.

Let \( G = G_0 > G_1 > G_2 > \cdots > G_n \) be a sequence of vertex groups of cyclic JSJ decompositions such that \( G_i, i < n - 1 \), is freely indecomposable and has a bouquet of circles as its principle cyclic JSJ, terminating at the first index \( n \) such that such that \( G_n \) is freely decomposable, hence free. This chain must have finite length since the cyclic analysis lattice is finite. We argue that \( G_n \) free implies that \( G_{n-1} \) is free.

Let \( f : G_{n-1} \rightarrow \mathbb{F} \) be a homomorphism such that \( f(G_n) \) has nonabelian image. Since \( G_{n-1} \) is an HNN extension of \( G_n \), by Corollary 1.6 of [Lou08c], the images of the incident edge groups in \( G_n \) can be conjugated to a basis for \( G_n \) and \( G_{n-1} \) is freely decomposable, contrary to hypothesis. \( \square \)

Definition 4.2 (Extension). An extension of a pure staircase \((G, \mathcal{H})\) is a staircase \((G', \mathcal{H}')\) such that the diagrams in Figure 1 commute. An extension is admissible if one of the following mutually exclusive conditions holds.

- \( G \) is freely decomposable, and the freely indecomposable free factors of \( \mathcal{H}'(i) \) embed in \( \mathcal{H}(i) \) under \( \sigma_i \).
- \( G \) is freely indecomposable, has QH subgroups, and the vertex groups of the decomposition of \( \mathcal{H}'(i) \) obtained by collapsing all edges not adjacent to QH vertex groups embed in \( \mathcal{H}(i) \) for all \( i \). (This is just the restricted principle cyclic JSJ.)
- \( G \) is freely indecomposable, QH–free, and for all \( i \), vertex groups of the (restricted) principle cyclic JSJ of \( \mathcal{H}'(i) \) embed in vertex groups of \( \mathcal{H}(i) \) under \( \sigma_i \).

An admissible extension has the property that each \( \sigma_i \) is strict, surjective, and maps elliptic subgroups of a decomposition of \( \mathcal{H}'(i + 1) \) to elliptic subgroups of...
The relation “mapps onto” partially orders the collection of extensions, and if $\mathcal{H}''$ is an extension of $\mathcal{H}'$ then $\mathcal{H}'' \succeq \mathcal{H}'$. For some $i$, if $\sigma_i$ is not one-to-one on the sets of vertex groups or edge groups then the inequality is strict. The envelope of a rigid vertex group of the principle cyclic JSJ is just the vertex group, hence if $\sigma_i$ is one-to-one on the sets of vertex groups and edge groups then it is an isomorphism.

We work with staircases which are maximal with respect to $\succeq$, rather than arbitrary staircases. To do this we have to pay a penalty, but not too large of one.

**Lemma 4.3.** For all $K$ there exists $C = C(K, \text{Comp}(\ ))$ such that if $(\mathcal{G}, \mathcal{H}, \mathcal{E})$ is a staircase and $\text{NInj}(\text{Comp}((\mathcal{G}, \mathcal{H}, \mathcal{E}))) = C(K, \text{Comp}((\mathcal{G}, \mathcal{H}, \mathcal{E})))$, then there is a $\succeq$-maximal extension of a contraction $(\mathcal{G}', \mathcal{H}', \mathcal{E}')$ of $(\mathcal{G}, \mathcal{H}, \mathcal{E})$ with $\text{NInj}((\mathcal{G}', \mathcal{H}', \mathcal{E}')) \geq K$ and $\text{Comp}((\mathcal{G}', \mathcal{H}', \mathcal{E}')) \leq \text{Comp}((\mathcal{G}, \mathcal{H}, \mathcal{E}))$.

The constants in this lemma do not depend on $\|\mathcal{E}\|$, and its proof is formally identical to the proof of [Lou08b, Theorem 4.2]. To adapt the proof, we need to show that the strict resolutions arising in an extension have bounded length. This follows from [Lou08b, Lemma 2.7], bounding the rank of $\mathcal{H}(i)$ from above by a function of $\text{Comp}((\mathcal{G}, \mathcal{H}))$, but a proof more in the spirit of this paper goes as follows: If $\mathcal{H}^{(n)}(i) \rightarrow \cdots \rightarrow \mathcal{H}(i)$ is a strict resolution appearing in a sequence of extensions, then $\text{JComp}((\mathcal{G})) \geq \text{JComp}((\mathcal{H}^{(m)}(i)))$ (See Lemma 3.6 and Definition 3.2.), moreover, if $\mathcal{H}^{(m+1)}(i) \rightarrow \mathcal{H}^{(m)}(i)$ is not injective on sets of vertex or edge spaces, or collapses subsurface groups of QH vertex groups, the complexity must decrease. By Theorem 3.6 the number of values the complexity takes is controlled by $b_1$, and the resolutions have length controlled by $\text{Comp}((\mathcal{G}, \mathcal{H}, \mathcal{E}))$.

Each pure kind of staircase is handled in turn over the next three subsections. In all cases the strategy is the same: either there is compatibility between (collapses of) rJSJ decompositions/Grushko factorizations, the complexity decreases, or proper extensions exist.

4.1. **Freely decomposable.** This is the most singular case in that the arguments work for nearly all finitely generated groups, not just limit groups.

The complexity for freely indecomposable groups is used to show that base sequences of freely indecomposable staircase can be divided into segments such that
ADJOINING ROOTS

the base groups of a segment have the same JSJ decompositions, in the sense of Theorem 3.6. There is a similar complexity for freely decomposable groups which accomplishes the same thing but with regard to Grushko decompositions. The following theorem from [Lou08c] shows how the complexity for freely decomposable groups is useful.

**Definition 4.4** (Scott complexity). Let $G$ be a finitely generated group with Grushko decomposition $G = G_1 \ast \cdots \ast G_p \ast F_q$. The Scott complexity of $G$ is the lexicographically ordered pair $\text{sc}(G) := (q - 1, p)$.

The number of Scott complexities of limit groups with $b_1 = b$ is bounded by $b^3$.

**Theorem 4.5** (Scott complexity and adjoining roots to groups). Suppose that $\phi: G \hookrightarrow H$ and $H$ is a quotient of $G' = G[\sqrt[\gamma_i]1]$, $\gamma_i$ a collection of distinct conjugacy classes of indivisible elements of $G$ such that $\gamma_i \neq \gamma_j^{-1}$ for all $i, j$ and $\gamma_i \in \gamma_i$. Then $\text{sc}(G) \geq \text{sc}(H)$. If equality holds and $H$ has no $\mathbb{Z}_2$ free factors, then there are presentations of $G$ and $H$ as

$$G \cong G_1 \ast \cdots \ast G_p \ast F_q^G, \quad H \cong H_1 \ast \cdots \ast H_p \ast F_q^H$$

a partition of $\{\gamma_i\}$ into subsets $\gamma_{j,i}$, $j = 0, \ldots, p$, $i = 1, \ldots, i_p$, representatives $\gamma_{j,i} \in G_j \cap \gamma_{j,i}$, $i \geq 1$, $\gamma_{0,i} \in F_q \cap \gamma_{0,i}$, such that with respect to the presentations of $G$ and $H$:

- $\phi(G_i) < H_i$
- $G_j[\sqrt[\gamma_{j,i}^1]] \twoheadrightarrow H_j$
- $\phi(F_q^G) < F^H_q$
- $F^G_q = \langle \gamma_{0,1} \rangle \ast \cdots \ast \langle \gamma_{0,i_0} \rangle \ast F$
- $F^H_q = \langle \sqrt[\gamma_{0,1}1] \rangle \ast \cdots \ast \langle \sqrt[\gamma_{0,i_0}1] \rangle \ast F$
- $G' \cong G_1[\sqrt[\gamma_{1,i_1}1]] \ast \cdots \ast G_p[\sqrt[\gamma_{p,i_1}1]] \ast \langle \sqrt[\gamma_{0,1}1] \rangle \ast \cdots \ast \langle \sqrt[\gamma_{0,i_0}1] \rangle \ast F$

All homomorphisms are those suggested by the presentations, and the maps on $F$ are the identity.

This is [Lou08c, Theorem 1.2].

**Remark 4.6.** Theorem 4.5 is stated in terms of adjoining roots to cyclic subgroups of a group, whereas Definition 1.1 refers to collections of abelian subgroups. This difference is immaterial to the discussion here since adjoining roots to a noncyclic abelian group can be accomplished by adjoining roots to a suitable collection of cyclic subgroups. By passing from a noncyclic abelian subgroup to cyclic subgroups, the measure $\| \|$ is unchanged.

**Definition 4.7** (Free products). Let $(G_i, H_i)$ be a collection of staircases on the same index set $I$. Then the graded free product $((G_i), (H_i))$, with the obvious maps, is also a sequence of adjunctions of roots.

**Lemma 4.8.** Suppose Theorem 1.4 holds for all staircases with complexity less than $(b_0, d_0, e_0)$. Then Theorem 1.4 holds for pure freely decomposable staircases of complexity $(b_0, d_0, e_0)$. 

Proof. Let \((G, H, E)\) be a staircase with complexity \((b_0, d_0, e_0)\). Since limit groups are torsion free, no \(G(i)\) has a \(\mathbb{Z}/2\) free factor, and by Theorem 4.5 for all but \(b_1(G(1))^3\) indices \(i_j\), the subsequences \(G(i) \hookrightarrow G(i + 1) \hookrightarrow \cdots \hookrightarrow G(i + 1 - 1)\) can be decomposed into free products of freely indecomposable groups staircases. Moreover, elements of \(E_i\) are either part of a basis of a free free factor of \(G(i)\) or are conjugate into a freely indecomposable free factor of \(G(i)\). Write \(G(i)\) as the free product 

\[ G(i)_1 \cdots G(i)_p F_i \]

given by the lemma, where \(F_i\) is a free group of rank \(q\) and \(sc(G(i)) = (q - 1, p)\) for all \(i\). Let \(E^j_i\) be the subset of \(E_i\) consisting of elements conjugate into \(G(i)_j\), and rearrange indices so that \(G(i)_j\) maps to \(G(i + 1)_j\) for all \(j\). Let \(E^0_i\) be the elements of \(E\) which are conjugate into \(F_i\). By Theorem 4.5 there are decompositions 

\[ G(i) \sqrt{E_i} \cong \left( G(i)_1 \left[ \sqrt{E^1_i} \right] \cdots G(i)_p \left[ \sqrt{E^p_i} \right] \right) \ast F_i \left[ \sqrt{E^0_i} \right] \]

where the last factor is free. Let \(H(i + 1) := \text{Im}_{H(i + 1)} (G(i) \left[ \sqrt{E^1_i} \right])\). The sequence \(H'\) defined by 

\[ H'(i + 1) := (\ast_j H(i + 1)_j) \ast F_i \left[ \sqrt{E^0_i} \right] \]

is an extension of \(H\). Then \((G, H', E)\) splits as a free product, the freely indecomposable free factors of which are \((G_j, H'_j, E^j_i)\). These free factors have strictly lower \(b_1\) than \(G\), depth at most \(d_0 = \text{depth}_{H}(H)\), hence have \(\text{NInj}((G_j, H'_j, E^j_i)) \leq \text{NInj}(\text{Comp}(b_0 - 1, d_0, e_0)) =: B\). If \(\|G\| > B \cdot b_1(G) \geq B \cdot p\), then, for some index \(l\), the map \(H'(l) \hookrightarrow G(l)\) is visibly an isomorphism. Since this map factors through \(H(l), H(l) \hookrightarrow G(l)\) is an isomorphism as well.

We finish this subsection by proving the base case of the induction. Let \((G, H, E)\) be a maximal staircase of complexity \((b, 2, e)\). By the proof of Lemma 4.5, the staircase splits as a free product of freely indecomposable staircases \((G_i, H_i, E^i)\), and such that each \(H_i(j)\) is elementary. If \(G_i\) is abelian, then clearly \(H_i(j) \hookrightarrow G_i(j)\) is an isomorphism, and if nonabelian, \(H_i(j)\) is the fundamental group of a closed surface. Since \(G_i\) is freely indecomposable, it is also the fundamental group of a closed surface. Divide \(G_i\) into segments such that the Euler characteristic is constant on each segment. Then \(G_i(j) \hookrightarrow G_i(j + 1)\) is an isomorphism on each segment and \(H_i(j)\) is a trivial extension of \(G_i(j - 1)\) for all \(j\) on each segment, thus \(H_i(j) \hookrightarrow G_i(j)\) is an isomorphism.

4.2. Freely indecomposable, QH. Lemma 5.3 allows us to handle injections \(G \hookrightarrow H, J\text{Comp}(G) = J\text{Comp}(H)\), and such that \(G\) has a QH subgroup.

Lemma 4.9. Suppose Theorem 1.4 holds for all staircases with strictly lower complexity than \((b_0, d_0, e_0)\). Then Theorem 1.4 holds for staircases with QH subgroups and complexity \((b_0, d_0, e_0)\).
The strategy is to find an extension \((G, \mathcal{H}', \mathcal{E})\) of \((G, \mathcal{H}, \mathcal{E})\) such that the QH subgroups of \(\mathcal{H}'\) are the “same” as those from \(G\). See Figure 2. The group \(\mathcal{H}(i)\) may be a total mess, but luckily it is a homomorphic image of a limit group which shares its restricted JSJ with \(G(i)\) and \(G(i+1)\).

To do this an auxiliary lemma which follows immediately from Lemma 5.3 is needed.

**Lemma 4.10.** Let \(G'\) be obtained from \(G\) by adjoining roots to a collection of abelian subgroups \(\mathcal{E}\). If \(J\text{Comp}(G) = J\text{Comp}(G')\) then every element \(E \in \mathcal{E}\) such that \([E : F(E)] > 1\) is conjugate into a non-QH vertex group of rJSJ\((G)\).

We use the immersion representing \(G \hookrightarrow G'\) constructed in subsection 5.1.

**Proof:** Fix \(E\) as in the statement of the lemma. We are done if we show that \(E\) is elliptic in every one edged splitting of \(G\) obtained by cutting a QH subgroup along an essential simple closed curve which doesn’t cut off a Möbius band. Start with an immersion representing the rJSJ decompositions of \(G\) and \(G'\), and let \(\Sigma_Q\) be the surface which contains \(c\). There is a unique element \(\eta_\#(Q)\) containing the image of \(Q\), and the map \(Q \to \eta_\#(Q)\) is surjective. Since \(Q \to \eta_\#(Q)\) is represented by a homeomorphism \(\Sigma_Q \to \Sigma_{\eta_\#(Q)}\) there is a simple closed curve \(\eta_\#(c)\) contained in \(\Sigma_{\eta_\#(Q)}\) and a closed annular closed neighborhood \(A\) of \(c\) mapping homeomorphically to a neighborhood of \(\eta_\#(c)\). Use these neighborhoods to construct new graphs of spaces \(Y_G\) and \(Y_{G'}\) representing \(G\) and \(G'\) by regarding the annulus as a new edge space and collapsing all but the newly introduced edges. By construction, the map \(Y_G \hookrightarrow Y_{G'}\) is an immersion. By Lemma 5.3, if some element of \(\mathcal{E}\) crosses \(c\), then \(c\) maps to a power of \(\eta_\#(c)\). \(\square\)

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**Figure 2.** Illustration of Lemma 4.9

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\(^2\)We could have instead redefined an essential curve as one which gives a principle cyclic splitting and isn’t boundary parallel.
Proof of Lemma 4.9. Suppose \((G, \mathcal{H})\) is a staircase such that \(sc(G(i))\) is the constant sequence and \(c_i(G(1)) \neq 0\). Let \(\Delta_i\) be the rJSJ of \(G(i)\). Every edge of \(\Delta_i\) is infinite cyclic and connects a vertex group to a boundary component of a QH vertex group. Since the inclusions \(G(i) \hookrightarrow G(i + 1)\) respected graphs of spaces, by the first part of Lemma 4.10, every element of \(\mathcal{E}_i\) is conjugate into some non-QH vertex group of \(\Delta_i\). Let \(V_0^i, \ldots, V_n^i\) be the non-QH vertex groups of \(\Delta_i\). We regard \(G(i)\) as a graph of groups \(\Gamma(V_j^i, Q_k, E_l)\), where \(G(i) \hookrightarrow G(i + 1)\) is compatible with the decomposition \(\Gamma\) in the sense that \(V_j^i\) maps to a conjugate of \(V_{j+1}^i\), the map respects edge group incidences, and the inclusion is the identity on the QH vertex groups \(Q_k\).

Let \(\mathcal{E}_i^j\) be the elements of \(\mathcal{E}_i\) conjugate into \(V_j^i\), and arrange that each \(E \in \mathcal{E}_i^j\) is contained in \(V_j^i\) by conjugating if necessary. Let \(W_j^{i+1}\) be the image of \(V_j^i \bigcup \mathcal{E}_i^j\) in \(\mathcal{H}(i + 1)\) and let \(\mathcal{H}'(i + 1) = \Gamma(W_j^{i+1}, Q_k, E_l)\). Then \((G, \mathcal{H}', \mathcal{E})\) is an extension of \((G, \mathcal{H}, \mathcal{E})\): The map implicit map \(\sigma_i : \mathcal{H}'(i) \rightarrow \mathcal{H}(i)\) is strictly, therefore the sequence \(\mathcal{H}'\) consists of limit groups. By definition, \(V_j^{i+1}\) is obtained from \(V_j^i\) by adjoining roots. Let \(V_j\) be the sequence \(V_j^i = V_j^i\) and let \(W_j^i = W_j^i\).

The staircases \((V_j, W_j, \mathcal{E}_j)\) all have lower first betti number than \((G, \mathcal{H}, \mathcal{E})\). Let \(B(b_0)\) be the maximal number of vertex groups of a limit group with first betti number \(b_0\) [Lou08b, Lemma 2.7]. If \(\|G\| > \text{Nilj}(b_0 - 1, d_0, c_0) \cdot B(b_0)\) then for at least one index \(l\) all \(W_j(l) \rightarrow V_j(l + 1)\) are injective. Thus \(\pi'_l = \text{Mod}(\mathcal{H}'(l), \text{rJSJ})\) strict. Since all modular automorphisms of \(\mathcal{H}'(l)\) are either inner, Dehn twists in boundary components of QH vertex groups, or induced by boundary respecting homeomorphisms of surfaces representing QH vertex groups, by construction, every element of \(\text{Mod}(\mathcal{H}'(l), \text{rJSJ})\) pushes forward to a modular automorphism of \(\mathcal{G}(l)\). An easy exercise shows that \(\mathcal{H}'(l) \rightarrow \mathcal{G}(l)\) is an isomorphism. Since \(\pi'_l = \pi_l \circ \sigma_l\), \(\pi_l\) is an isomorphism. 

\[\square\]

4.3. Freely indecomposable, no QH. The neighborhood of a vertex group \(V\) of a graph of groups decomposition is the subgroup generated by \(V\) and conjugates of adjacent vertex groups which intersect \(V\) nontrivially, and is denoted \(\text{Nbhd}(V)\). Let \((G, H, G')\) be a flight and suppose \(G\) is freely indecomposable, has no QH vertex groups, and that \(J\text{Comp}(G) = J\text{Comp}(G')\). Let \(\eta : G \rightarrow G'\) be the inclusion map. An abelian vertex group \(A\) of \(G\) is \(H\)-elliptic if \(H\) doesn’t have a principle cyclic splitting over a subgroup of \(Z_H(\nu(A))\).

Let \(A_H\) be the collection of abelian vertex groups of \(G\) which are \(H\)-elliptic. Suppose that \(H\) is obtained from \(G\) by adjoining roots to the collection \(\mathcal{E}\). Let \(\mathcal{E}_H^{\text{ell}}\) be the sub-collection of \(\mathcal{E}\) consisting of elements of \(E\) which are hyperbolic in the principle cyclic JSJ of \(G\) but which have elliptic image in the principle cyclic JSJ of \(H\). Let \(\text{JSJ}_H(G)\) be the JSJ decomposition of \(G\) with respect to the collection of principle cyclic splittings in which all \(\text{Nbhd}(A), A \in A_H, E \in \mathcal{E}_H^{\text{ell}}\) are elliptic:

\[
\text{JSJ}_H(G) := \text{JSJ}(G; \{\text{Nbhd}(A), E \mid A \in A_H, E \in \mathcal{E}_H^{\text{ell}}\})
\]
Let \( JSJ^*_H(G') \) be the JSJ decomposition of \( G' \) associated to the collection of all principle cyclic splittings of \( G' \) in which all \( \eta#(A_H), A \in A_H, \) and \( \eta#(E), E \in E_H^{ell} \), are elliptic. That is

\[
JSJ^*_H(G') := JSJ(G', \{ Nbd(\eta#(A)), \eta#(E) \mid A \in A_H, E \in E_H^{ell} \})
\]

The main lemma is that the decompositions of \( G \) and \( G' \) induced by \( H \) are intimately related to the principle cyclic JSJ of \( H \) as long as the flight admits no proper extensions. Let \( V \) be a vertex group of \( JSJ_H(G) \). There is a vertex group \( \eta#(V) \) of \( JSJ^*_H(G') \) which contains the image of \( V \). Let \( \mathcal{E}_V \) be the collection of elements of \( E \) which are conjugate into \( V \), along with the collection of incident edge groups. Likewise for \( \eta#(V) \), let \( \mathcal{E}(\eta#(V)) \) be the set of centralizers of images of elements of \( \mathcal{E}_V \).

**Lemma 4.11.** Let \( (G, H, G') \) be a flight without any proper extensions, and suppose \( G \) is freely indecomposable, has no QH vertex groups, and that \( J\text{Comp}(G) = J\text{Comp}(G') \). Let \( \eta: G \to G' \) be the inclusion map. Then the following hold:

- For each vertex group \( W \) of the principle cyclic JSJ of \( H \) there are unique vertex groups \( V \) and \( \eta#(V) \) of \( JSJ_H(G) \) and \( JSJ^*_H(G') \), respectively, such that \( \nu(V) \prec W, \pi(W) \prec \eta#(V) \).
- \( W \) is obtained from \( V \) by adjoining roots to \( \mathcal{E}_V \) and \( ||\mathcal{E}_V|| \leq ||\mathcal{E}|| + 2b_1(G) \).
- \( \eta#(V) \) is obtained from \( \pi(W) \) by adjoining roots to the images of \( \mathcal{E}(V) \) (the edge groups incident to \( V \))
- If \( \pi \) is injective on vertex groups then it is an isomorphism.

The proof of Lemma 4.11 is contained in section 5, where graphs of spaces \( X_G, X_H \), representing \( JSJ_H(G) \) and \( JSJ(H) \), respectively, and an immersion \( X_G \to X_H \) representing \( G \to H \), such that if the immersion is not one-to-one on edge spaces, then there must be a nontrivial extension, are constructed. The remainder of the lemma is largely formal, and relies on a simplification of the construction of strict homomorphisms from [Lou08b].

### 4.4. Finishing the argument.

In this section we prove Theorem 1.4, postponing the proofs of lemmas used in the previous section until section 5. Let \( (G, H) \) be a staircase with complexity \((b_0, d_0, e_0)\), such that no contraction has any proper extensions, and suppose that Theorem 1.4 holds for staircases with complexity less than \((b_0, d_0, e_0)\). By Theorem 3.6 there is some constant \( B(b_0) \) such that \( (G, H) \) can be divided into \( B(b_0) \) staircases of constant Scott complexity: (To maintain uniformity of the exposition, some sequences are allowed to be empty.)

\[
(G, H) \mapsto \{(G_i, H_i)\}_{i=1, \ldots, B(b_0,d_0)}
\]

\[
G_i(1) = G(j_i), \ldots, H_i(2) = H(j_i + 1), \ldots
\]

Only the last of these can consist of freely indecomposable groups. Each staircase \( (G_i, H_i), i < B(b_0) \), by Theorem 4.8, has \( N\text{Inj} \) bounded above by \( b_0 \cdot N\text{Inj}(b_0 - 1, d_0, e_0) \), since there are at most \( b_0 \) freely indecomposable free factors.
Thus we may confine our analysis to freely indecomposable staircases. By Theorem 3.6, we may divide the staircase \((\mathcal{G}, \mathcal{H}, \mathcal{E})\) into boundedly many segments, the number depending only on the complexity of \(b_1(\mathcal{G})\), exhausting the tower, such that \(J\text{Comp}\) is constant on each segment. By Lemma 4.3 we may assume that each segment is maximal with respect to \(\leq\).

Like the case when each \(\mathcal{G}(i)\) is freely decomposable, if \(\mathcal{G}(i)\) has a QH vertex group, by Lemma 4.9 such staircases have bounded \(\text{NInj}\).

The only possibility left is that the contractions of \((\mathcal{G}, \mathcal{H})\) are QH–free. Let \(I\) be the index set for \(\mathcal{G}\), and color the triple \(i < j < k\) red if \(\text{JSJ}^*_\mathcal{H}(j)(\mathcal{G}(j)) \cong \text{JSJ}^*_\mathcal{H}(k)(\mathcal{G}(j))\), and blue otherwise. Then by Ramsey’s theorem for hypergraphs, for all \(K\) there exists an \(L\) such that if \(\|\mathcal{G}\| > L\) then there is a subset \(I' \subset I\) of size at least \(K\) such that all triples whose elements are in \(I'\) have the same color.

**Lemma 4.12.** There is an upper bound to the size of blue subsets which depends only on \(b_1(\mathcal{G})\) and \(\|\mathcal{E}\|\).

**Proof:** By Lemma 2.2, there are at most \(2\|\mathcal{E}\|\) equivalence classes of principle cyclic decomposition of \(G\) in which some element of \(\mathcal{E}\) is elliptic. (There may be none.) Suppose \(|I'| > 2\|\mathcal{E}\|\), and consider the collection of principle cyclic decompositions \(\{\text{JSJ}^*_\mathcal{H}(l)(\mathcal{G}(i))\}\). Thus, for some \(i < j < k\), \(\text{JSJ}^*_\mathcal{H}(j)(\mathcal{G}(i))\) and \(\text{JSJ}^*_\mathcal{H}(k)(\mathcal{G}(i))\) have the same elliptic subgroups. Then \(\text{JSJ}^*_\mathcal{H}(k)(\mathcal{G}(j)) \cong \text{JSJ}^*_\mathcal{H}(j)(\mathcal{G}(j))\) since a JSJ decomposition is determined up to equivalence solely by its elliptic subgroups. □

We are now on the home stretch. Suppose again that \((b_0, d_0, e_0)\) is the lowest complexity for which Theorem 1.4 doesn’t hold. By Lemma 4.12 and the prior discussion, there must be staircases \((\mathcal{G}, \mathcal{H}, \mathcal{E})\) of arbitrary \(\text{NInj}\), which have complexity \((b_0, d_0, e_0)\), are maximal, pure, and have no QH vertex groups.

Let \((\mathcal{G}, \mathcal{H})\) be such a staircase. Let \(\mathcal{V}^j(i)\) be the nonabelian vertex groups of \(\mathcal{G}(i)\), indexed such that \(\mathcal{V}^j(i)\) maps to \(\mathcal{V}^j(k)\) for all \(k > i\). Let \(\mathcal{W}^j(i)\) be the corresponding rigid vertex group of \(\mathcal{H}(i)\). By the second bullet of Lemma 4.11, \(\mathcal{W}^j(i + 1)\) is obtained from \(\mathcal{V}^j(i)\) by adjoining roots to \(\mathcal{E}_i^{j'}\), the set of elements of \(\mathcal{E}_i\) which are conjugate into \(\mathcal{V}^j(i)\), along with the incident edge groups.

Let \(\mathcal{V}^j(i) < \mathcal{V}^j(i)\) be the image of \(\mathcal{W}^j(i)\) in \(\mathcal{G}(i)\). By the third bullet of Lemma 4.11, \(\mathcal{V}^j(i)\) is obtained from \(\mathcal{V}^j(i)\) by adjoining roots to the images of the edge groups incident to \(\mathcal{W}^j(i)\). Let

\[
\mathcal{E}_i^{j'} := \left\{ E \cap \mathcal{V}^j(i) \mid E \in \mathcal{E}_i^{j'} \right\} \cup \left\{ E \cap \mathcal{V}^j(i) \mid E \in \mathcal{E}(\mathcal{V}^j(i)) \right\}
\]

The incident edge groups are cyclic, and we can build \(\mathcal{W}^j(i)\) by simply adjoining roots to \(\mathcal{E}_i^{j'}\) in \(\mathcal{V}^j(i)\). Then \(\mathcal{E}_i^{j'}\) is larger than \(\mathcal{E}\) by at most the number of edge groups incident to \(\mathcal{V}^j(i)\), which is at most \(2b_1(\mathcal{G})\). That is,

\[
\|\mathcal{E}_i^{j'}\| \leq \|\mathcal{E}\| + 2b_1(\mathcal{G}) \leq \|\mathcal{E}\| + 2b_1(\mathcal{G})(\text{depth}_{\text{pc}}(\mathcal{H}) - \text{depth}_{\text{pc}}(\mathcal{W}^j))
\]

Given a sufficiently long QH–free staircase \((\mathcal{G}, \mathcal{H}, \mathcal{E})\), we passed to a maximal extension (which we will also call \((\mathcal{G}, \mathcal{H}, \mathcal{E})\)) of a substaircase of prescribed length,
such that the sequences of vertex groups \((V_j, W_j, E_j)\) of the extension were cyclic staircases. The vertex groups of the extension are subgroups of the vertex groups of \(\mathcal{H}\), hence the depth of \(W_j(i)\) is strictly less than the depth of \(\mathcal{H}\). Moreover, the first betti number of \(W_j(i)\) is at most \(b_1(\mathcal{H})\) and by Lemma 4.11, \(\text{Comp}((G, \mathcal{H}, E)) > \text{Comp}((V_j, W_j, E_j))\). There is an upper bound \(B(b_0)\) to the number of vertex groups of the principle cyclic JSJ of a limit group with first betti number \(b_0\). If \(\|G\| > B(b_0) \cdot \text{NInj}(b_0, d_0 - 1, e_0 + 2b_0)\) there is some index \(l\) such that \(\mathcal{H}(l) \rightarrow G(l)\) is injective on all vertex groups. By the last bullet of Lemma 4.11, \(\mathcal{H}(l) \rightarrow G(l)\) is an isomorphism.

5. Hyperbolic to Elliptic

5.1. Graphs of spaces and immersions. In this section we are given a fixed flight \((G, G', H)\) of limit groups. By a graph of spaces representing a principle cyclic decomposition of a limit group \(G\) we mean a graph of spaces of the following form:

- For each rigid vertex group \(R\) a space \(X_R\). Let \(E(R)\) be the edge groups incident to \(R\), and for each \(E \in E(R)\) let \(\sqrt{E}\) be the maximal cyclic subgroup of \(R\) containing the image of \(E\). For each \(E \in E\) there is an embedded copy \(S_E\) of \(S^1\) in \(X_R\) representing the conjugacy class of \(\sqrt{E}\).
- For each edge \(E\), a copy \(T_E\) of \(S^1\), with basepoint \(b_E\) and an edge space \(T_E \times I\). On occasion we confuse \(T_E\) with \(T_E \times \frac{1}{2}\), and sometimes refer to \(T_E\) as the edge space. The interval \(b_E \times I\) is denoted \(t_E\), and we choose an arbitrary orientation for \(t_E\). The end of the edge space associated to \(E\) is attached via the covering map \(T_E \cong S_E\) representing \(E \hookrightarrow \sqrt{E}\).
- For each abelian vertex group a torus \(T_A\). If \(A\) is infinite cyclic then \(T_A\) has a basepoint \(b_A\) and the incident edge maps are simply covering maps which send \(b_E\) to \(b_A\). These covering maps are isomorphisms unless the edge is adjacent to a QH vertex group, in which case they may be proper. For each edge space edge \(E\) adjacent to \(A\), an edge space \(T_E \times I\) and an embedded copy of \(T_E\) in \(T_A\). This assumes edge groups not adjacent to QH vertex groups are primitive. Though there may be QH vertex groups, the cases which this definition is designed to handle do not, and we let this inconsistency slide. Unlike the rigid case, the embedded \(T_E\) need not be disjoint, though if they meet, they coincide. We require that any two embedded \(T_E\), differ by an element of \(T_A\), treated now as a group.
- For each QH vertex group \(Q\) a surface with boundary \(\Sigma_Q\).
- If an edge group \(E\) is incident to a QH vertex group \(Q\) then \(T_E\) is identified with a boundary component of \(\Sigma_Q\).
- The resulting graph of spaces has the fundamental group of \(G\).

Let \(\eta: G \hookrightarrow H\) be an inclusion of limit groups, and let \(\Pi_G\) and \(\Pi_H\) be principle cyclic decompositions of \(G\) and \(H\), respectively, such that \(\eta\) maps vertex groups to vertex groups, edge groups to edge groups, and respects edge data, i.e., if \(E \hookrightarrow V\), \(\eta_\#(E) \hookrightarrow \eta_\#(V)\), then the obvious square commutes. If this is the case then \(\eta\) respects \(\Pi_G\) and \(\Pi_H\). Let \(X_H\) be a graph of spaces representing \(\Pi_H\). Then there
is a principle cyclic decomposition $\Pi_G$ of $G$, a space $X_G$ representing $\Pi_G$, and an immersion $\psi: X_G \to X_H$, inducing $\eta$, of the following form:

- For each abelian vertex group $A$ there is a finite sheeted covering map $\psi|_{T_A}: T_A \to T_{\eta\#(A)}$. The inclusions of incident edge spaces are respected by $\eta$:
  \[
  \psi|_{\text{Im}(T_E)} = (T_{\eta\#(E)} \hookrightarrow T_{\eta\#(A)}) \circ \psi|_{T_E}
  \]

- For each $E$ there is a finite sheeted product-respecting covering map $T_E \times I \to T_{\eta\#(E)} \times 1$ which maps $t_E$ to $t_{\eta\#(E)}$. If $E$ is adjacent to a QH vertex group then the degree of the covering map is one.

- For each $R$ there is a map $X_R \to X_{\eta\#(R)}$ such that for each edge group $E$ incident to $R$ the following diagram commutes:
  \[
  \begin{array}{ccc}
  T_E \times \{0\} & \rightarrow & X_R \\
  \downarrow & & \downarrow \\
  T_{\eta\#(E)} \times \{0\} & \rightarrow & X_{\eta\#(R)}
  \end{array}
  \]
  Likewise for $\times \{1\}$.

- For each $\Sigma_Q$ a homeomorphism $\Sigma_Q \to \Sigma_{\eta\#(Q)}$. The maps $X_R \to X_{\eta\#(R)}$ (similarly for $T_A$’s) respect attaching maps of boundary components of surfaces.

- If $E_1$ and $E_2$ are incident to $A$ and $T_{E_1}$ and $T_{E_2}$ have the same image in $T_A$, then $\eta\#(E_1) \neq \eta\#(E_2)$.

The existence of immersions as above is an easy variation on Stallings’s folding. One way to construct immersions of graphs representing subgroups is to pass to the cover of a graph representing a subgroup and trimming trees. There is an analogous construction in this context.

### 5.2. Roots, immersions, and resolving

We need to be able to represent conjugacy classes of elements of limit groups as nice paths in graphs of spaces.

**Definition 5.1** (Edge path). Let $X_G$ be a graph of spaces representing a principle cyclic decomposition of $G$. The zero skeleton of $X_G$, denoted $X_G^0$, is the union of vertex spaces.

An edge path in a graph of spaces $X_G$ is a map $p: [0, 1] \to X_G$ such that $p^{-1}(X_G^0)$ contains $\{0, 1\}$ and is a disjoint collection of closed subintervals. Let $[a, b]$ be the closure of a complementary component of $p^{-1}(X_G^0)$. Then $p$ maps $[a, b]$ homeomorphically to some $t_E$.

Let $X_R$ be a vertex space. Set $\partial X_R$ be the union of copies of edge spaces contained in $X_R$. An edge path $p$ is reduced if every restriction $p|_{[a, b]}([a, b]; \{a, b\}) \rightarrow (X_R, \partial X_R)$ does not represent the relative homotopy group $\pi_1(X_R, \partial X_R)$.

A continuous map $\gamma: S^1 \to X_G$ is cyclically reduced if all edge-path restrictions of $\gamma$ to subintervals $I \subset S^1$ are reduced edge paths.

The following lemma is standard and follows easily from Stallings folding [Sta65, Sta83] and the definitions.
Lemma 5.2. If \( g \in G \) then there is a cyclically reduced edge path \( \gamma : S^1 \to X_G \) representing the conjugacy class \([g]\).

Let \( \psi : X_G \to X_H \) be an immersion representing \( G \hookrightarrow H \). If \( \gamma : S^1 \to X_G \) is a reduced edge path then \( \psi \circ \gamma \) is a reduced edge path in \( X_H \).

For each edge \( E \) of \( X_G \), we introduced a subset \( t_E \) of the edge space \( T_E \times I \). We think of \( t_E \) as a formal element representing the path \( I \to b_E \times I \) with a fixed but arbitrary orientation. Let \( \tau(t_E) \) be the image of the basepoint of \( T_E \) in the vertex space of \( X_G \) at the terminal end of \( T_E \times I \), and let \( \iota(t_E) \) be the image of the basepoint in the vertex space at the initial end of \( T_E \). Then every nonelliptic element represented by a cyclically reduced path can be thought of as a composition \( t_E \)'s, their inverses, and elements of relative homotopy groups of vertex spaces. Moreover, if the subword \( t_E \eta t_E^{-1} \) appears then \( g \) is not contained in the image of \( E \).

Let \( \gamma \in G \) be represented by a cyclically reduced edge path \( \gamma ; \psi \circ \gamma \) is an edge path in \( X_H \), and if it is not cyclically reduced, then for some sub-path \( t_E \eta t_E^{-1} \) of \( \gamma \) (we may need to reverse the orientation of \( t_E \)), the image of this subpath is homotopic into \( \eta_\#(T_E) \), which means that \([h] \in \eta_\#(E)\). Since \( \gamma \) is reduced, \([h] \notin E \), and since the image of \( E \) in \( \eta_\#(E) \) is finite index, for some \( l > 0 \), \([h]^l \in E \). Since edge groups are primitive unless adjacent to QH vertex groups, \( E \) must be attached to a boundary component of a QH vertex. This implies that \( \eta_\#(E) \) is also attached to a boundary component of a QH vertex group, but this means \( E \to \eta_\#(E) \) is an isomorphism, contradicting the fact that \([h] \notin E \).

Let \( G \) and \( H \) be freely indecomposable limit groups, \( H \) obtained from \( G \) by adjoining roots to \( \mathcal{E} \), \( \eta : G \hookrightarrow H \). Let \( \Pi_G \) and \( \Pi_H \) be principle cyclic decompositions and suppose that if \( K \) is elliptic in \( \Pi_G \) if and only if \( \eta(K) \) is elliptic in \( \Pi_H \). Let \( \psi : X_G \to X_H \) be an immersion representing the inclusion.

Without loss of generality, suppose that all elements of \( \mathcal{E} \) are self-centralized and nonconjugate. Let \( \mathcal{E}_e \) be the elements of \( \mathcal{E} \) which are elliptic in \( \Pi_G \) and let \( \mathcal{E}_h \) be the elements of \( \mathcal{E} \) which are hyperbolic in \( \Pi_G \).

For each \( E \in \mathcal{E} \) let \( T_E \) be a torus representing \( E \), \( T_{F(E)} \) a torus representing \( F(E) \), and let \( T_E \to T_{F(E)} \) be the covering map corresponding to the inclusion \( E \to F(E) \). Let \( M_E \) be the mapping cylinder of the covering map. If \( \langle \gamma \rangle \in \mathcal{E} \) we abuse notation and refer to \( M_{\langle \gamma \rangle} \) as \( M_\gamma \). The copy of \( T_{F(E)} \) in \( M_E \) is the core of \( M_E \), and if \( E \) is infinite cyclic, it is the core curve. The copy of \( T_E \) in \( M_E \) is the boundary, and is denoted \( \partial M_E \).

For each element \( E \in \mathcal{E}_e \), let \( f_E : T_E \to X_G \) be a map representing the inclusion \( E \hookrightarrow G \) which has image in a vertex space of \( X_G \). If \( E \) is an abelian vertex group of \( \Pi_G \) then we identify \( T_E \) with the torus \( T_A \subset X_G \). For each \( \langle \gamma \rangle \in \mathcal{E}_h \), \(^3\) represent \( \gamma \) by a reduced edge path, abusing notation, \( \gamma : \partial M_\gamma \to X_G \).

Build a space \( X'_G \) by attaching the \( M_E \) and \( M_\gamma \) to \( X_G \) along \( T_E \) and \( \text{Im}(\gamma) \) by the maps \( f_E \) and \( \gamma \), respectively.

By hypothesis there is a \( \pi_1 \)-surjective map \( \psi' : X'_G \to X_H \). We choose this map carefully: For \( E \in \mathcal{E}_e \), \( F(E) \) has elliptic image in \( \Pi_H \). Choose a map \( T_{F(E)} \to \)

\(^3\) All elements of \( \mathcal{E}_h \) are infinite cyclic.
proper one-edged, and that the edge group is freely indecomposable. Let \( \rho \) be a 4-connected representative of \( \psi/\gamma: S^1 \to X_H \) and let the map on the core curve agree with this representative.

The restriction of \( \psi' \), defined thus far, to the disjoint union of \( X_G \) and the core curves of the \( M_\gamma \), is transverse to the subsets \( T_{\eta#}(E) \times \{1/2\} \). Extend \( \psi' \) to \( X_G \) so the composition \( M_\gamma \hookrightarrow X'_G \xrightarrow{\psi'} X_H \) is transverse to all \( T_{\eta#}(E) \times \{1/2\} \). Let \( \Lambda \) be the preimage

\[
\psi'^{-1}\left(\bigcup_{E \in \mathcal{E}(G)} \left(T_{\eta#}(E) \times \left\{1/2\right\}\right)\right)
\]

Suppose some component of \( \Lambda \) is a circle which misses the boundary and core of some \( M_\gamma \). By transversality this component of \( \Lambda \) is a one manifold without boundary, and is therefore a circle. If this circle bounds a disk then there is a map homotopic \( \psi' \), which agrees with \( \psi' \) on the core curves and \( X_G \) such that the number of connected components of the preimage is strictly lower. If the circle doesn’t bound a disk in \( M_\gamma \) then it is boundary parallel. If this is the case then \( \gamma \) is elliptic and we have a contradiction.

Fix a mapping cylinder \( M_\gamma \) and consider the preimage of \( \Lambda \) under the map \( M_\gamma \to X'_G \). The preimage is a graph all of whose vertices are contained in the core curve of \( M_\gamma \) or in the boundary of \( M_\gamma \). If any component of the preimage of \( \Lambda \) doesn’t connect the boundary of \( M_\gamma \) to the core curve, then it is an arc and there is an innermost such arc which can be used to show that one of either \( \gamma \) or \( \gamma' \) is not reduced. Thus the preimages of arcs connect the core curve to the boundary.

Let \( b \) be a point of intersection of \( \Lambda \) with the core curve of \( M_\gamma \). There are \( k_\gamma \) arcs, where \( k_\gamma \) is the degree of the root added to \( \gamma, s_1, \ldots, s_{k_\gamma} \) (cyclically ordered by traversing \( \partial M_\gamma \)) in \( \Lambda \) connecting \( b \) to \( \partial M_\gamma \). Now consider the arcs as paths \( s_j: [0,1] \to M_\gamma \). The composition \( p_\gamma := s_2^{-1}s_1 \) is a path in \( M_\gamma \) from \( \partial M_\gamma \) to \( \partial M_\gamma \). Let \( D_\gamma \) be the sub-path of \( \gamma \) obtained by traversing \( \partial M_\gamma \) from \( * := s_1 \cap \partial M_\gamma \) to \( * := s_2 \cap \partial M_\gamma \). The path \( D_\gamma p_\gamma \) is homotopic, relative to \( * \), to \( s_1^{-1} k_\gamma \sqrt{7}s_1 \). In particular,

\[
(D_\gamma p_\gamma)^{k_\gamma} \simeq \gamma
\]

A possible neighborhood of a component of \( \Lambda \) is illustrated in Figure 3.

Three interrelated lemmas.

**Lemma 5.3.** Suppose \( \eta: G \hookrightarrow H, H \) obtained from \( G \) by adjoining roots to \( \mathcal{E}, G \) freely indecomposable. Let \( \Pi_H \) be a one-edged splitting of \( G \) over a cyclic edge group \( E_H \). Let \( \Pi_G \) be the splitting \( G \) inherits from its action via \( \eta \) on the Bass-Serre tree for \( \Pi_H \). Represent \( \Pi_H \) by a graph of spaces \( X_H \), and choose a graph of spaces \( X_G \) and an immersion \( \psi: X_G \hookrightarrow X_H \) representing \( \eta \). Suppose that \( \Pi_G \) is one-edged, and that the edge group is \( E \). If \( \mathcal{E}_h \) is nonempty then \( E \hookrightarrow \eta#(E) \) is a proper finite index inclusion.
Proof. Let \( \langle \gamma \rangle \in E_h \), and represent \( \gamma \) by a reduced edge path crossing \( t_E \). Since \( \psi \)
is one-to-one on edge spaces, \( p_\gamma \) is a closed path. As such, it represents an element of the fundamental group of \( X'_G \).

Then \( \psi \circ p_\gamma \) is homotopic in \( T_{\eta \#(E)} \), relative to the image of \( * \), to \( \psi \circ p_\gamma \). Let \( \alpha = D_\gamma p_\gamma \). Then \( \psi \circ \alpha \) is homotopic rel the image of \( * \) to \( \psi \circ D_\gamma p_\gamma \). But then \( [\alpha]^{k_{\gamma}} = \gamma \) contradicting indivisibility of \( \gamma \).

Lemma 5.4. Let \( G \hookrightarrow G' \) be an adjunction of roots. Let \( \Pi_{G'} \) be a principle cyclic splitting of \( G' \) with one abelian vertex group \( A \), let \( \Pi_G \) be the associated splitting of \( G \), and represent \( G \hookrightarrow G' \) by an immersion \( \eta: X_G \hookrightarrow X_{G'} \) reflecting \( \Pi_G \) and \( \Pi_{G'} \). Suppose there is a unique vertex group \( A' \) of \( \Pi_G \) mapping to \( A \), and that there is at most one element of \( E \) conjugate into \( A' \). If \( \eta \) is one-to-one on edges adjacent to \( A' \) then the induced map \( F(A') \to A/P(A) \) is onto.

Proof. Let \( E_1, \ldots, E_n \) be the edges adjacent to \( A' \), and set \( F(E_i) = \eta \#(E_i) = P(A) \). Let \( H \) be the limit group defined as follows: Let \( \Delta = \Delta(R_j, E_i, A') \) be a graphs of groups representation of \( \Pi_G \). Let \( E_{R_j} \) be the subcollection of \( E \) consisting of elements conjugate into \( R_j \). Let

\[
S_j := \text{Im}_{G'} \left( \langle R_j \left[ E_{R_j} \right], gF(E_i)g^{-1} \rangle \right)_{gE_i} \text{ for } g < R_j
\]

and

\[A'' := \text{Im}_{G'}(F(A'), P(A))\]

Let \( H := \Delta(S_j, F(E_i), A'') \) There are maps \( G \hookrightarrow H \hookrightarrow G' \). We now show that \( H \hookrightarrow G' \) is actually surjective. To do this we need to show that every element \( \langle \gamma \rangle \) of \( E_h \) has a \( k_{\gamma} \)-th root in \( H \). This is precisely the argument given at the end of Lemma 5.3. Let \( G' \to A/P(A) \) be the map which kills all vertex, edge groups, and stable letters, other than \( A \). The quotient map clearly kills everything except \( A \) and \( F(A') \), giving the desired surjection. \( \square \)
Lemma 5.5. Let $(G, H, G')$ be a flight without any proper extensions. Suppose $G$ is freely indecomposable, has no OH vertex groups, and $\text{JComp}(G) = \text{JComp}(G')$. Represent the $G \leftrightarrow H$ by an immersion $X_G \leftrightarrow X_H$, representing $\text{JSJ}_H(G)$, and $\text{rJSJ}_H(G)$, respectively. Then $\nu$ is one-to-one on edge spaces.

Every vertex group $W$ of $H$ is obtained from a vertex group $V$ of $\text{JSJ}_H(G)$ by adjoining roots to the elements of $E$ which are conjugate into $V$, along with edge groups incident to $V$.

**Proof of Lemma 5.5.** Represent $G \leftrightarrow H$ by an immersion $X_G \rightarrow X_H$, where $X_G$ represents $\text{JSJ}_H(G)$ and $X_H$ represents the principle cyclic JSJ of $H$. For each edge $E_i$ of $X_G$, let $e_i$ be a generator, let $k_i$ be the largest degree of a root of $e_i$ in $H$, let $F(E) = \langle f_i \rangle$, and let $E \leftrightarrow F(E)$ be the map which sends $e_i$ to $f_i^{k_i}$. Let $E'$ be the collection of elements of $E$ which are elliptic in $H$ along with all edge groups of $\text{JSJ}_H(G)$.

Consider the group $G \left[ \sqrt{E} \right]$. Let $\Delta = \Delta(R_i, A_j, E_k)$ be a graph of groups representation of $\text{JSJ}_H(G)$. Let $E_{R_i}$ be the set of elements of $E'$ which are conjugate into $R_i$. Likewise, let $E_{A_j}$ be the set of elements of $E'$ which are conjugate into $A_j$. Let

$$S_i = \text{Im}_H(\langle S_i, gBg^{-1} \rangle gBg^{-1} \leq S_i, B \in E_{S_i})$$

where $gBg^{-1} < S_i$, and where $S_i$ is either some rigid vertex group $R_i$ or abelian vertex group $A_j$. Let

$$H' = \Delta(R'_i, A'_j, F(E_k))$$

and choose a graph of spaces $X_{H'}$ representing this decomposition of $H'$. There is a pair of maps of graphs of spaces $X_G \rightarrow X_{H'}, X_{H'} \rightarrow X_H$, and there is an epimorphism $G \left[ \sqrt{E} \right] \rightarrow H'$. The map $\psi': X_G \rightarrow X_{H'}$ is one-to-one on edge spaces. Moreover, $H'$ is a limit group since the map $H' \rightarrow H$ is clearly strict.

The proof of the lemma will be complete if we can show that $\psi'$ extends to $X'_G$, that is, if $H'$ contains all roots of elements adjoined to $E$. Then the image of $G \left[ \sqrt{E} \right]$ (with the induced graph of groups decomposition) in $H'$ is a nontrivial extension of $H$.

Consider the paths $D_\gamma$ and $p_\gamma$ defined previously through resolving. We defined $p_\gamma := s_2^{-1}s_1$ and set $* = s_1 \cap \partial M_\gamma$. Let $*_2 := s_2 \cap \partial M_\gamma$. To show that $H'$ has a $k_\gamma$-th root of $\gamma$ we need to show that $X_{H'}$ has a path $p'_\gamma$ from the image of $*_2$ to the image of $*$ whose image under $X_{H'} \rightarrow X_{G'}$ is homotopic rel endpoints to the image of $p_\gamma$.

Suppose that $T_{E_1} \times \frac{1}{2}$ and $T_{E_2} \times \frac{1}{2}$ are the midpoints of edge spaces containing $*$ and $*_2$, respectively, and suppose, without loss of generality, that $D_{\gamma}$ starts and ends by traversing the second and first halves of $T_{E_1}$ and $T_{E_2}$, respectively, in the positive direction. The first key observation to make is that we can choose the orientations of $t_{E_1}$ so that the terminal endpoints of $t_{E_1}$ and $t_{E_2}$ are both contained in some $T_A$: $E_1$ and $E_2$ are conjugate in $H$, must therefore be conjugate in $G$ since $\text{JComp}(G') = \text{JComp}(G)$, and cannot both be adjacent to a rigid vertex group of $G$, otherwise there is a rigid vertex group $R$ of $G$ such that $v(R) > v(\eta_#(R))$. 

The only other possibility is that they are both adjacent to an abelian vertex group $A$, as claimed.

Let $t^+_\varphi(E_i)$ be the half of $t^+_{\varphi}(E_i)$ obtained by traversing $t^+_{\varphi}(E_i)$ from the mid-point to the terminal endpoint. By Lemma 5.4, $H' \rightarrow H$ is surjective on abelian vertex groups, and by construction, the terminal endpoints of $t^+_{\varphi}(E_i)$ agree. Let $p'_p := t^+_{\varphi}(E_2)(t^+_{\varphi}(E_1))^{-1}$. Then $p'_p$ is a path from $\varphi(\ast_2)$ to $\varphi(\ast)$ whose image in $X_H$ is homotopic rel endpoints into $\psi_{\#}(E_1) (= \psi_{\#}(E_2))$. Since $H' \rightarrow H$ is surjective on edge groups, there is a closed path $h_\gamma$ in $(T_{\varphi}(E_i), \varphi(\ast))$ which maps to the image of $p_\gamma$. Set $p'_\gamma := h_\gamma p'_p$. The image of $p'_\gamma$ is homotopic rel endpoints to the image of $p_\gamma$ in $X_H$. Arguing as in Lemma 5.3, $(\varphi \circ D_\gamma)p'_\gamma$ is a $k_\gamma$-th root of $\varphi \circ \gamma$ and the map $X'_G \rightarrow X_H$ is one-to-one on edges and the situation above never occurs.

Thus there is a map $X'_G \rightarrow X_{H'}$. Since $H$ has no proper extensions, $\text{Im}_{H'} \left(G[\sqrt{E}]\right) \rightarrow H$ is an isomorphism. In particular, $X_G \rightarrow X_H$ is one-to-one on edges and the situation above never occurs.

Consider the construction of $H'$. Now that we know that $H' \cong H$, $\Delta$ must be the principle cyclic JSJ of $H$. If there is a principle cyclic splitting of $H$ not visible in $\Delta$ then it must be a cyclic splitting inherited from the relative (to incident edge groups) principle cyclic JSJ decomposition of some vertex group of $\Delta$. On the other hand, all vertex groups of $\Delta$ must be elliptic in the principle cyclic JSJ of $H$ since they are obtained by adjoining roots to subgroups of $G$ which are guaranteed to be elliptic in the principle cyclic JSJ of $H$.

This nearly completes the proof of Lemma 4.11. We need to prove that the vertex groups of $\text{JSJ}_H^*(G')$ are obtained from the images of the vertex groups of $\text{JSJ}(H)$ by adjoining roots, and that $\pi$ is injective if its restrictions to vertex groups are injective.

Let $\Delta = \Delta(R_i, A_j, E_k)$ be a graph of groups decomposition representing the principle cyclic JSJ of $H$. We know that all vertex and edge groups of $\Delta$ map to vertex and edge groups of $\text{JSJ}_H^*(G')$. Let $\Phi_\ast(\pi) : \Phi_\ast(H) \rightarrow G'$ be the strict homomorphism constructed in [Lou08b, § 5], and also in the appendix of this paper, and let $\Phi_{\ast}(\Delta)$ be the principle cyclic decomposition of $\Phi_\ast(H)$ in which all images of vertex groups of $\Delta$ are elliptic. Clearly $\Phi_\ast(\pi)$ maps elliptic subgroups of $\Phi_\ast(\Delta)$ to elliptic subgroups of $\text{JSJ}_H^*(G')$. Moreover, if $A$ is a noncyclic abelian vertex group of $H$, then by construction, $\Phi_\ast(\pi)$ maps $A/P(A)$ onto $\pi_{\#}(A)/P(\pi_{\#}(A))$. Thus all modular automorphisms of $\Phi_\ast(H)$ supported on abelian vertex groups of $\Delta$ push forward to modular automorphisms of $G'$. Another consequence of the hypothesis $\text{JComp}(G) = \text{JComp}(G')$ is that $\Phi_\ast(H) \rightarrow G'$ is one to one on the set of edge groups adjacent to every vertex group, hence every Dehn twist of $\Phi_\ast(H)$ pushes forward to a Dehn twist of $G'$. A strict map which allows all modular automorphisms to push forward is an isomorphism, therefore $\Phi_\ast(\pi)$ is an isomorphism.

The third bullet follows immediately from the construction of $\Phi_\ast$.
APPENDIX A. CONSTRUCTING STRICT HOMOMORPHISMS

We give here a description of the process of constructing strict homomorphisms of limit groups. Let $G$ be a group with a one-edged splitting $\Delta$ with nonabelian vertex groups of the form $G \cong R \ast \langle e \rangle S$, and suppose there is a map $\varphi : G \to L$, $L$ a limit group, which embeds $R$ and $S$. Then $R$ and $S$ are limit groups. Suppose further that $\varphi$ embeds $R \ast \langle e \rangle Z_S(\langle e \rangle)$ and $Z_R(\langle e \rangle) \ast \langle e \rangle S$. Then $\varphi$ is strict, and $G$ is a limit group. There is a process, whose output is a limit group $\Phi_s(G)$, which takes the data $(G, \Delta, L, \varphi)$ and produces a triple $G \to \Phi_s(G) \to L$, such that the composition is $\varphi$, $\Phi_s(G)$ splits over the centralizer of $\langle e \rangle$, and $\Phi_s(G) \to L$ is strict.

The process is one of pulling centralizers and passing to images of vertex groups in a systematic way. The reader should compare this to the more general construction detailed in [Lou08b], and a formally identical version in the proof of [BF03, Lemma 7.9]. Let $G = G_0$. Define for

- odd $i$: $G_i = R_{i-1} \ast Z_{R_{i-1}}(\langle e \rangle) S_i$, where $S_i := \text{Im}_L(Z_{R_{i-1}}(\langle e \rangle) \ast Z_{S_{i-1}}(\langle e \rangle) S_{i-1})$

- even $i$: $G_i = R_i \ast Z_{S_{i-1}}(\langle e \rangle) S_{i-1}$, where $R_i := \text{Im}_L(R_{i-1} \ast Z_{S_{i-1}}(\langle e \rangle) Z_{S_i}(\langle e \rangle))$

We claim that this process terminates in finite time. The sequence of quotients $G_0 \to G_1 \to \ldots$ embeds edge groups at every step. Since abelian subgroups of limit groups are finitely generated and free, and since finitely generated free abelian groups satisfy the ascending chain condition the assertion holds. The direct limit $G_\infty$ is called $\Phi_s(G)$.

This discussion is relevant to the proof of Lemma 5.5, but we must vary the construction a little. Let $\overline{H}$ be the quotient of $H$ obtained by passing to the images in $G'$ of vertex groups of the (restricted) principle cyclic JSJ of $H$, with the induced graph of groups decomposition $\Delta(R_i, A_j, E_k)$. The core of $H$, Core($H$) is the group obtained by replacing each abelian vertex group $A$ by its peripheral subgroup. Consider the situation in Lemma 5.5. There is a homomorphism Core($\overline{H}$) $\to G'$, and each group is equipped with a principle cyclic decomposition $\Delta_{\text{Core}(\overline{H})}$ and $\Delta_{G'}$, respectively. Moreover, the nonabelian vertex groups of $\Delta_{\text{Core}(\overline{H})}$ map to nonabelian vertex groups of $G'$, and the edge groups of Core($\overline{H}$) map to edge groups of $\Delta_{G'}$. The centralizers of edges incident to nonabelian vertex groups of $G'$ are infinite cyclic, and this implies that in the process of pulling centralizers in Core($\overline{H}$), the pulled group is always infinite cyclic. Each vertex group of Core($\overline{H}$) has elliptic image in $G'$, and since $G'$ is principle, centralizers are cyclic in the relevant vertex groups of $G'$. Iteratively adjoining roots to an infinite cyclic subgroup and passing to quotients multiple times can be accomplished in one step, thus the vertex groups of Core($\overline{H}$) are obtained from the vertex groups of Core($\overline{H}$) by adjoining roots to incident edge groups. There are surjective maps $H \to \Phi_s(H) := \text{Core}(\overline{H}) \ast Z(P(A_j)) (Z(P(A_j)) \oplus A/P(A_j)) \to L$. 
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