CMB constraint on non-Gaussianity in isocurvature perturbations

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Abstract

We study the CMB constraints on non-Gaussianity in CDM isocurvature perturbations. Non-Gaussian isocurvature perturbations can be produced in various models at the very early stage of the Universe. Since the isocurvature perturbations little affect the structure formation at late times, CMB is the best probe of isocurvature non-Gaussianity at least in the near future. In this paper, we focus on non-Gaussian curvature and isocurvature perturbations of the local-type, which are uncorrelated and in the form $\zeta = \zeta_G + \frac{3}{2}f_{\text{NL}}(\zeta_G^2 - \langle \zeta_G^2 \rangle)$ and $S = S_G + f_{\text{NL}}^{\text{ISO}}(S_G - \langle S_G^2 \rangle)$, and constrain the non-linearity parameter of isocurvature perturbations, $f_{\text{NL}}^{\text{ISO}}$, as well as the curvature one $f_{\text{NL}}$. For this purpose, we employ several state-of-art techniques for the analysis of CMB data and simulation. Assuming that isocurvature perturbations are subdominant, we apply our method to the WMAP 7-year data of temperature anisotropy and obtain constraints on a combination $\alpha^2 f_{\text{NL}}^{\text{ISO}}$, where $\alpha$ is the ratio of the power spectrum of isocurvature perturbations to that of the adiabatic ones. When the adiabatic perturbations are assumed to be Gaussian, we obtained a constraint $\alpha^2 f_{\text{NL}}^{\text{ISO}} = 40 \pm 66$ assuming the power spectrum of isocurvature perturbations is scale-invariant. When we assume that the adiabatic perturbations can also be non-Gaussian, we obtain $f_{\text{NL}} = 38 \pm 24$ and $\alpha^2 f_{\text{NL}}^{\text{ISO}} = -8 \pm 72$. We also discuss implications of our results for the axion CDM isocurvature model.
1 Introduction

The adiabaticity, or isocurvature mode of primordial density fluctuations is one of important probes of cosmology in various respects. Although current cosmological observations suggest that primordial density fluctuations are almost adiabatic and they give severe constraints on the size of isocurvature fluctuations, some fraction of their contribution is still allowed [1]. Residual isocurvature fluctuations can be generated when there exist multiple components with different origins, which are associated with dark matter, baryon and neutrino [2]. Such examples include axion [5, 6, 7, 8, 9, 10, 11, 12] and Affleck-Dine baryogenesis [13, 14, 15], where cold dark matter (CDM) and baryon isocurvature modes can be respectively generated. These isocurvature modes are basically uncorrelated with adiabatic ones, however, when one considers a scenario where a light scalar field other than the inflaton is responsible for (adiabatic) density fluctuation such as the curvaton model [16, 17, 18], isocurvature perturbations can be correlated with the adiabatic ones and be easily generated, depending on how and when CDM and baryon are created [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. In addition to above mentioned examples, a variety of models with isocurvature fluctuations has been extensively studied in the literature, hence the information on isocurvature fluctuations would give a lot of insight on various aspects of cosmology, particularly on models of dark matter and baryogenesis as well as those of the early Universe.

Although the theoretical works on non-Gaussianity from isocurvature fluctuations have been relatively well studied [32, 33, 34, 35, 36, 37], observational constraints on them have not been explored much. In particular, we cannot find any work investigating this issue by using the actual data except Ref. [48], where the constraint has been studied with Minkowski functionals using the WMAP 3 data, although there are a few papers which elaborate its future CMB constraints [49, 50, 51]. In the light that we now have precise cosmological data to explore non-Gaussianity as seen from the counterpart for adiabatic ones, non-Gaussianity in isocurvature fluctuations should be pursued more.

In this paper, we investigate the optimal constraints on the local-type non-Gaussianity in CDM (baryon) isocurvature fluctuations from CMB bispectrum estimator. Although, as mentioned above, isocurvature fluctuations can be correlated with adiabatic ones in some cases and there are also other kinds of modes such as neutrino one, we in this paper present the methodology of our analysis and concentrate to report the constraint on the CDM (baryon) uncorrelated type. Constraints on other types such as correlated ones and neutrino modes will be reported in a forthcoming paper [52].

The organization of this paper is as follows. In the next section, we give the formalism to discuss non-Gaussianity in models with isocurvature fluctuations and also set our notation. In Section 3, we describe our analysis method to obtain the constraint on non-Gaussianity from isocurvature fluctuations. Then in Section 4, we present our results.
2 Model of non-Gaussian perturbations and CMB signature

We consider primordial curvature and CDM isocurvature perturbations, respectively denoted as $\zeta$ and $S$, in the following form:

$$
\zeta(\vec{x}) = \zeta_G(\vec{x}) + \frac{3}{5} f_{NL}(\zeta_G(\vec{x})^2 - \langle \zeta_G(\vec{x})^2 \rangle),
$$

$$
S(\vec{x}) = S_G(\vec{x}) + f_{NL}^{(ISO)}(S_G(\vec{x})^2 - \langle S_G(\vec{x})^2 \rangle),
$$

where $\zeta_G$ and $S_G$ are Gaussian parts of the primordial curvature and isocurvature perturbations, respectively. $f_{NL}$ and $f_{NL}^{(ISO)}$ are the non-linearity parameters of the curvature and isocurvature perturbations, respectively. In the following, we denote these primordial perturbations with $X^A(\vec{x})$. Then Eqs. (1) and (2) can be recast into

$$
X^A(\vec{x}) = X^A_G(\vec{x}) + f_{NL}^A(X^A_G(\vec{x})^2 - \langle X^A_G(\vec{x})^2 \rangle),
$$

where $X^A_G$ is the Gaussian part of $X^A$. Here we defined a non-linearity parameter $f_{NL}^A$, which is related to the adiabatic and isocurvature ones via $f_{NL}^\zeta = \frac{3}{5} f_{NL}$ and $f_{NL}^S = f_{NL}^{(ISO)}$.

We note that the non-Gaussian primordial perturbation of Eq. (1) is of the so-called local-type, which is discussed in Refs. [53, 54] as well as many other studies. Eq. (2) would be a natural extension of this to isocurvature perturbations and hence the non-Gaussianity we consider in this paper should be regarded as an extension of the local-type one to non-adiabatic perturbations.

In this paper, we consider uncorrelated isocurvature perturbations, so that $\langle \zeta_G S_G \rangle = 0$. Thus only correlation functions which contain either $\zeta$ or $S$ are non-zero. In terms of the Fourier components of $X^A$, the bispectrum from either the primordial curvature or isocurvature perturbations is

$$
\langle X^A(\vec{k}_1)X^A(\vec{k}_2)X^A(\vec{k}_3) \rangle = 2 f_{NL}^A [P_{X^A_G}(k_1)P_{X^A_G}(k_2) + \text{(2 perms)})] (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3),
$$

where $P_{X^A_G}(k)$ is the power spectrum of the Gaussian perturbations $X^A_G$. Here we neglected loop contributions although we take those into account in Section 5.

Neglecting the secondary non-Gaussianities arising from the second or higher order cosmological perturbation theory, the harmonic coefficients of the CMB temperature anisotropy from primordial perturbations $X^A$ can be given as

$$
da_{lm}^A = 4\pi (-i)^l \int \frac{dk^3}{(2\pi)^3} g_l^A(k) X^A(\vec{k}) Y_{lm}^*(\hat{k}),
$$
where \( g_A^A(k) \) is the temperature transfer function for the primordial perturbations \( X^A \). The total CMB anisotropy is the sum of those from the curvature and isocurvature perturbations, i.e. \( a_{lm} = a^c_{lm} + a^S_{lm} \).

Since \( \zeta \) and \( S \) are uncorrelated we need to consider their polyspectra which contain only either \( \zeta \) or \( S \). Then the total polyspectra are the sum of those from each perturbation. Angular power spectrum \( C^A_l \), which is defined by 

\[
\langle a_{lm}^A a_{lm'}^A \rangle = C^A_l \delta_{ll'} \delta_{mm'},
\]

(6)

The reduced bispectrum \( b_{l_1l_2l_3} \) is defined by 

\[
\langle a_{l_1m_1} a_{l_2m_2} a_{l_3m_3} \rangle = b_{l_1l_2l_3} g_{m_1m_2m_3}^i,
\]

(7)

where \( g_{m_1m_2m_3}^i \) is the Gaunt integral, which can be written in terms of the Wigner-3j symbol as 

\[
g_{m_1m_2m_3}^i = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{array} \right) }.
\]

(8)

Given a bispectrum of primordial perturbations in Eq. (4), the reduced CMB bispectrum from each of the primordial perturbations can be given as 

\[
b_{l_1l_2l_3}^A = 2f_{NL}^A \int dr r^2 \left( \alpha_{l_1}^A(r) \beta_{l_2}^A(r) \beta_{l_3}^A(r) + (2 \text{ perms}) \right),
\]

(9)

where \( \alpha_{l}^A(r) \) and \( \beta_{l}^A(r) \) are 

\[
\alpha_{l}^A(r) = \frac{2}{\pi} \int dk k^2 g_{l}(k) j_l(kr),
\]

(10)

\[
\beta_{l}^A(r) = \frac{2}{\pi} \int dk k^2 g_{l}(k) j_l(kr) P_{X^A}(k).
\]

(11)

For later convenience, we introduce a normalized bispectrum 

\[
\hat{b}_{l_1l_2l_3}^A \equiv b_{l_1l_2l_3}^A / f_{NL}^A.
\]

(12)

The amplitude of isocurvature power spectrum is constrained from CMB angular power spectrum. In Ref. [1], it is shown that the WMAP observation of CMB gives an upper bound on the fraction of isocurvature power spectrum in the total one 

\[
\alpha \equiv \frac{P_S}{P_\zeta} < 0.15
\]

(13)

at 95% confidence level.

Note that the bispectrum is proportional to \( \alpha^2 f_{NL}^{(ISO)} \). This can be seen from Eqs. (9) and (11). Hence we report our constraint on non-Gaussianity in isocurvature fluctuations for the combination of \( \alpha^2 f_{NL}^{(ISO)} \).
3 Analysis method

Here we describe our analysis method to derive the constraints on the non-linearity parameters from CMB data. Our method is basically the same as the one given in Ref. [56].

3.1 Estimator of non-linearity parameters

In the limit of small non-linearity parameter $f_{NL}^A$, the effect of the deviation from Gaussianity manifests in the CMB bispectrum, so that the optimal estimator can be constructed from the three point function of CMB anisotropy [57]. We adopt the following cubic estimator of non-linearity parameters

$$\hat{f}_{NL}^A = \left( \frac{S_{\text{prim}}^{C}}{S_{\text{prim}}^{S}} \right)_{f_{NL}^A=1}^{-1} \left( \frac{S_{\text{prim}}^{C}}{S_{\text{prim}}^{S}} \right)_{f_{NL}^A=1},$$

(14)

where an angle bracket with subscript $f_{NL}^A=1$ indicates an ensemble average over simulations with non-zero non-linearity parameter shown in the subscript with the other non-linearity parameter being fixed to zero. $S_{\text{prim}}^A$ is the cubic statistics [57, 58, 59, 60], which is given by

$$S_{\text{prim}}^A = \frac{1}{6} \sum_{\{lm\}} \hat{b}_{lm} A_{l_1 l_2 l_3} [\hat{a}_{l_1 m_1} \hat{a}_{l_2 m_2} \hat{a}_{l_3 m_3} - 3 \hat{a}_{l_1 m_1} \langle \hat{a}_{l_2 m_2} \hat{a}_{l_3 m_3} \rangle_0],$$

(15)

where an angle bracket with subscript 0 indicates an ensemble average over Gaussian simulations. $\hat{a}_{l m}$ is a harmonic coefficient obtained from observed (or simulated) data maps with suitable weighting, which will be discussed in Section 3.3. Eq. (14) can be schematically represented as $\hat{f}_{NL}^A = \sum_{B} (S_{\text{prim}}^{A})_{f_{NL}^A=1}^{-1} S_{\text{prim}}^{B}$. In our analysis, to estimate $\langle \hat{a}_{l m} \hat{a}_{l' m'} \rangle_0$ we accumulated at least 250 Monte Carlo (MC) samples.

Assuming the Gaussianity for $X^A$, we estimate the covariance of the estimator $\hat{f}_{NL}^A$,

$$\langle \hat{f}_{NL}^A \hat{f}_{NL}^{A'} \rangle_0 = \sum_{A' B'} (S_{\text{prim}}^{A'})_{f_{NL}^{A'}=1}^{-1} (S_{\text{prim}}^{A' B'})_{0} (S_{\text{prim}}^{B'})_{f_{NL}^{B'}=1}^{-1} \langle S_{\text{prim}}^{B'} \rangle_{f_{NL}^{B'}=1}^{-1},$$

(16)

Here we used the relation $\langle S_{\text{prim}}^{A'} \rangle_{f_{NL}^{A'}=1} = \langle S_{\text{prim}}^{A'} S_{\text{prim}}^{B'} \rangle_0$. See Appendix A for the derivation.

The estimator in Eq. (14) can be regarded as the generalization of a fast estimator in Ref. [57] to the case where the initial perturbations are mixture of non-Gaussian adiabatic and uncorrelated isocurvature perturbations. When we assume that either of adiabatic or uncorrelated isocurvature perturbations are non-Gaussian and the other is Gaussian, Eq. (14) can be reduced to

$$\hat{f}_{NL}^A = S_{\text{prim}}^{A} / \langle S_{\text{prim}}^{A} \rangle_{f_{NL}^A=1},$$

(17)
where the subscripts \( A \) indicates the perturbations which are assumed to be non-Gaussian. In this case, the variance of \( f_{NL}^A \) is given by \( 1/\langle S_{\text{prim}}^A \rangle_{f_{NL}=1} \).

We note that there is a difficulty in computing the normalization factor \( \langle S_{\text{prim}}^A \rangle_{f_{NL}=1} \). If we naively evaluate Eq. (15) by taking ensemble average over simulated non-Gaussian CMB maps with small but non-zero \( f_{NL}^A \), a substantial number of simulations are required due to a large Gaussian fluctuation. Instead, we divide \( \bar{a}_{lm} \) into its Gaussian and non-Gaussian parts and evaluate \( S_{\text{prim}}^A \) without terms which are to vanish by averaging. The details are presented in Appendix B. This treatment also removes the need for setting a non-zero fiducial \( \alpha \).

### 3.2 Non-Gaussian CMB simulation

In order to determine the normalization \( \langle S_{\text{prim}}^A \rangle_{f_{NL}=1} \), we need to simulate non-Gaussian CMB maps. With the Fourier transformation, Eq. (5) can be rewritten as

\[
a_{lm}^A = \int dr r^2 \alpha_i^A(r) \int d\hat{r} Y_{lm}^*(\hat{r}) X^A(\hat{r}).
\]  

(18)

In the case of local-type non-Gaussianity, Eq. (18) allows us to simulate the non-Gaussian CMB maps exactly [61, 62]. We adopt the fast method developed in Ref. [63], which enables simulations of the local-type non-Gaussianity with sufficient speed. The key technique in the method of Ref. [63] is that the line of sight integral in Eq. (18) is evaluated by the Gaussian quadrature with optimized nodes and weights. In our simulation, we optimized the nodes so that the mean square of the error in simulated maps \( a_{lm} \) at each multipole \( (lm) \) should be less than 0.01. For \( l_{\text{max}} = 1024 \), we found that this level of accuracy requires 42 and 15 nodes for curvature and isocurvature perturbations, respectively.

### 3.3 Optimally-weighted CMB maps

In estimating \( f_{NL}^A \), we need to suitably weight observed (and simulated) maps \( a_{lm} \) to obtain \( \tilde{a}_{lm} \), in order to take into account the sensitivity and resolution of the survey. As discussed in Ref. [56], the optimal weight is the inverse of the variance. This can be schematically represented as \( \tilde{a}_{lm} = [C^{-1}a]_{lm} \), where \( C = C_S + C_N \) is the covariance matrix, with those of signal and noise being denoted as \( C_S \) and \( C_N \), respectively. While \( C_S \) is diagonal in the harmonic space, \( C_N \) is in general complicated for real observations. In particular, WMAP and many CMB surveys have multiple channels, so that we need to take into account different beam functions, inhomogeneous noises, and a sky cut. In such a case, \( \tilde{a}_{lm} \) can be given in an implicit form as

\[
(C^{-1}_S + C^{-1}_N)C_S\tilde{a} = C^{-1}_N a,
\]

(19)

where \( C^{-1}_N = \sum_i b^{(i)} C_N^{(i)-1} b^{(i)} \) and \( C^{-1}_N a = \sum_i C_N^{(i)-1} a^{(i)} \), with the subscript \( i \) indicating a channel of the survey. \( b^{(i)} \) here is the beam function for channel \( i \).
Since the matrix \((C_S^{-1} + C_N^{-1})\) is dense in both the harmonic and pixel spaces, direct implementation of inversion \((C_S^{-1} + C_N^{-1})^{-1}\) is substantially prohibited. Instead, Eq. (19) can be solved by the conjugate gradient (CG) method with preconditioning [64]. How fast a CG method converges crucially depends on a choice of pre-conditioner. We adopt a fast method developed in Ref. [65], which makes use of the multi-grid pre-conditioning. This method also allows to marginalize over amplitude of components whose spatial templates \(\{\tau\}\) are provided. To do this, \(C_N^{-1}\) is replaced with \(C_N^{-1} - \sum_{ab} C_N^{-1} \tau_a (\tau_a C_N^{-1} \tau_b)^{-1} C_N^{-1} \tau_b\), where the subscripts \(a\) and \(b\) indicate template components. For the details of the method, we refer to Refs. [65, 56].

3.4 Data and assumption on cosmological model

In our analysis, we assume a flat power-law \(\Lambda\)CDM model as a fiducial one and adopt the mean values for the cosmological parameters from the WMAP 7-year data alone [1],

\[
(\omega_b, \omega_c, h, \tau, n_s, A_s) = (0.02249, 0.112, 0.727, 0.088, 0.967, 2.43 \times 10^{-9}),
\]

where \(\omega_b = \Omega_bh^2\) and \(\omega_c = \Omega_ch^2\) are respectively the density parameters for baryon and cold dark matter, \(h\) is the Hubble constant in units of \(100\text{km/s/Mpc}\), \(\tau\) is the optical depth of reionization, and \(n_s\) and \(A_s\) are respectively the spectral index and amplitude of the power spectrum of curvature perturbations at a reference scale \(k_* = 0.002\text{Mpc}^{-1}\), i.e.,

\[
P_\zeta(k) = \frac{2\pi^2}{k_*^3} A_s \left(\frac{k}{k_*}\right)^{n_s-1}.
\]

We also assume that the power spectrum of isocurvature perturbations is in a power-law form \(P_{SG}(k) \propto k^{n_{iso}-4}\), and with regard to the fiducial value of the spectral index \(n_{iso}\) we consider two different cases, \(n_{iso} = 0.963(= n_{adi})\) and \(n_{iso} = 1\). The transfer function of CMB is computed using the CAMB code [66].

We combine the foreground-cleaned maps of V and W bands of the WMAP 7-year data [67, 68]#3 with a resolution \(N_{side} = 512\) of the HEALPix pixelization scheme [69]#4. We adopt the KQ75y7 mask [68], which cuts 28.4 % of the sky. We also set the maximum multipole \(l_{max}\) to 1024 in our analysis. We marginalize the amplitudes of the monopole \(l = 0\) and dipoles \(l = 1\) as default. We also optionally marginalize the amplitudes of Galactic foreground components at large angular scales using the templates for the synchrotron, free-free and dust emissions from Ref. [68].

4 Result

Our results of constraints on the non-linearity parameters are summarized in Table 1.

To check our analysis method, we evaluate constraints on the non-linearity parameter for adiabatic perturbations, \(f_{NL}\), assuming isocurvature perturbations being absent, and

#3http://lambda.gsfc.nasa.gov
#4http://healpix.jpl.nasa.gov
setups | $f_{\text{NL}}$ | $\alpha^2 f_{\text{NL}}^{\text{(ISO)}}$
--- | --- | ---
$n_{\text{iso}} = 0.963$  
 w/o template marginalization | 31 ± 21 | 5 ± 63
(36 ± 23) | ($-39 \pm 69$) | w/ template marginalization | 32 ± 21 | 40 ± 63
(32 ± 23) | (0 ± 70)

$n_{\text{iso}} = 1$  
 w/o template marginalization | 31 ± 21 | 19 ± 62
(34 ± 23) | ($-22 \pm 70$) | w/ template marginalization | 32 ± 21 | 40 ± 66
(38 ± 24) | ($-8 \pm 72$)

Table 1: Constraints on $f_{\text{NL}}$ and $\alpha^2 f_{\text{NL}}^{\text{(ISO)}}$ for different setups. We adopted four different setups regarding the fiducial value of $n_{\text{iso}}$ and template marginalization of the Galactic foregrounds. Constraints without parenthesis are estimated by fixing the other non-linearity parameter to zero. On the other hand, ones with parenthesis are estimated by marginalizing over the other non-linearity parameter.

compare them with those in a previous study. We obtain $f_{\text{NL}} = 31 \pm 21$ at 1σ level without template marginalization of the Galactic foregrounds. The central value by about 0.5σ deviates from that of Ref. [1], which gives $f_{\text{NL}} = 42 \pm 21$ from the foreground-cleaned V+W band data with a resolution $N_{\text{side}} = 1024$. Since there are substantial differences between our analysis and that of Ref. [1], such as fiducial cosmological models and resolutions of the maps used, we believe that this level of difference is acceptable and our result is consistent with the previous study. With template marginalization, we obtain $f_{\text{NL}} = 32 \pm 21$, which is exactly the same as the one given by the WMAP group [1]. We found that template marginalization little affects constraints on $f_{\text{NL}}$.

Now we present constraints on non-Gaussianity in uncorrelated isocurvature perturbations. We first assume that the adiabatic perturbations are Gaussian and fix $f_{\text{NL}}$ to be zero. For the cases of $n_{\text{iso}} = 0.963$ and $n_{\text{iso}} = 1$, we respectively obtain $\alpha^2 f_{\text{NL}}^{\text{(ISO)}} = 5 \pm 63$ and $19 \pm 62$ at 1σ level without template marginalization. With template marginalization, these change to $40 \pm 63$ and $40 \pm 66$. We found that the constraints on $\alpha^2 f_{\text{NL}}^{\text{(ISO)}}$ are not strongly affected by the fiducial value of $n_{\text{iso}}$. On the other hand, the constraints are more or less dependent on the treatment of the Galactic foregrounds. The central values can change by 0.5σ while the error remains almost unchanged. However, this shows that the effects of residual foregrounds are not severe.

When we assume that both adiabatic and isocurvature perturbations can be non-Gaussian, we obtain a joint constraint on $(f_{\text{NL}}, \alpha^2 f_{\text{NL}}^{\text{(ISO)}})$. For the cases of $n_{\text{iso}} = 0.963$ and $n_{\text{iso}} = 1$, we respectively obtain $(f_{\text{NL}}, \alpha^2 f_{\text{NL}}^{\text{(ISO)}}) = (36 \pm 23, -39 \pm 69)$ and $(34 \pm 23, -22 \pm 70)$ without template marginalization. With template marginalization, these changes to $(f_{\text{NL}}, \alpha^2 f_{\text{NL}}^{\text{(ISO)}}) = (32 \pm 23, 0 \pm 70)$ and $(38 \pm 24, -8 \pm 72)$. The error for each non-linearity parameter here is estimated by marginalizing over the other non-linearity parameter. In
Fig. 1, we show 2D constraints in the $f_{NL}$-$\alpha^2 f_{NL}^{(ISO)}$ plane for the cases with template marginalization. Due to the correlation of $\alpha^2 f_{NL}^{(ISO)}$ with $f_{NL}$, a simultaneous fit for both of these variables changes the central value of $\alpha^2 f_{NL}^{(ISO)}$, which are not so constrained as $f_{NL}$. We found that the dependence on $n_{iso}$ is weak and the central values of $\alpha^2 f_{NL}$ can change by 0.5σ by the treatment of the Galactic foregrounds.

In our analysis, we omitted effects of unresolved point sources, which may bias our constraints on $f_{NL}$ and $\alpha^2 f_{NL}^{(ISO)}$. For the case of purely adiabatic perturbations, Ref. [1] studies effects of unresolved point sources and concludes that $f_{NL}$ can be biased by 2 when the WMAP 7-year data is used. Because effects of unresolved point sources should be quite small at large angular scales $l < O(100)$, where uncorrelated isocurvature perturbations can be prominent, we expect bias in $\alpha^2 f_{NL}^{(ISO)}$ induced by unresolved point sources should be even smaller. We thus conclude that our constraints should be little affected by unresolved point sources.

As stated in Section 3.4, our constraints are derived with other cosmological parameters being fixed. However, these parameters themselves have uncertainties, which can bias and/or weaken our constraints. Following the method of Ref. [70], we here discuss size of errors on the nonlinearity parameters coming from uncertainties in cosmological parameters. According to the study, given a difference in a cosmological parameters $\Delta p_i$, bias in a nonlinearity parameter $\Delta f_{NL}^A$ can be approximately given by:

$$\Delta f_{NL}^A \approx \sum_{BC} [F^{-1}]_{AB} \frac{\partial F}{\partial p_i}_{BC} f_{NL}^C \Delta p_i,$$

where $F_{AB}$ is the Fisher matrix for non-linearity parameters,

$$F_{AB} = \frac{1}{6} \sum_{l_1 l_2 l_3} \frac{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{array} \right)^2 \frac{\hat{b}^A_{l_1 l_2 l_3}}{C_{l_1} C_{l_2} C_{l_3}} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{array} \right) \frac{\hat{b}^B_{l_1 l_2 l_3}}{C_{l_1} C_{l_2} C_{l_3}}.$$

We estimate $\Delta f_{NL}^A$ for each of the cosmological parameters, whose uncertainties $\Delta p_i$ are taken from the constraint from the WMAP 7-year data alone in Ref. [1]. For the fiducial values of nonlinearity parameters, we here consider two extreme cases, $(f_{NL}, \alpha^2 f_{NL}^{(ISO)}) = (70, 0)$ and $(0, 70)$, which are at 1σ deviation with current constraints. For the case of $(f_{NL}, \alpha^2 f_{NL}^{(ISO)}) = (70, 0)$, $\Delta f_{NL}$ can be as large as 5 from uncertainties in $n_s$ and $A_s$, while $\Delta(\alpha^2 f_{NL}^{(ISO)})$ can be as large as 8 from those in $\omega_c$ and $h$. On the other hand, for the case of $(f_{NL}, \alpha^2 f_{NL}^{(ISO)}) = (0, 70)$, $\Delta f_{NL}$ can be as large as 0.3 from uncertainties in $\omega_c$ and $n_s$, while $\Delta(\alpha^2 f_{NL}^{(ISO)})$ can be as large as 7 from that in $\omega_c$. From these estimates, we conclude

Eq. (21) is not exactly the same as Ref. [70], in which only derivatives of bispectra, $\partial \hat{b}_{l_1 l_2 l_3}^A / \partial p_i$ are taken into account, but those of the covariance matrix or power spectrum $\partial C_l / \partial p_i$ should not. However, since there should not be severe cancelation between $\partial \hat{b}_{l_1 l_2 l_3}^A / \partial p_i$ and $\partial C_l / \partial p_i$ for any of cosmological parameters we consider here, we believe Eq. (21) should give rough estimates of bias in the non-linearity parameters.
Figure 1: Joint constraints on $f_{NL}$ and $\alpha^2 f_{NL}^{(ISO)}$ at 1 and 2\(\sigma\) levels with template marginalization. Solid red and dashed green lines respectively show the cases for $n_{iso} = 0.963$ and $n_{iso} = 1$, with the central values being indicated by the red plus and green cross. Blue star corresponds to the case where the perturbations are purely Gaussian.

that uncertainties in the cosmological parameters can affect the limits on $f_{NL}$ and $\alpha^2 f_{NL}^{(ISO)}$ by about twenty and ten percents, respectively.

Having all these results, we conclude that CMB data is consistent with Gaussianity at 2\(\sigma\) level, even if the uncorrelated CDM isocurvature perturbations are included.

Let us compare our results with other studies. While our method is optimal based on the bispectrum, constraints on the same type of non-Gaussianity is studied in Ref. [48] based on the Minkowski functionals, which gives $\alpha^2 f_{NL}^{(ISO)} = -15 \pm 60$, when the adiabatic perturbations are assumed to be Gaussian and $n_{iso} = 1$. We found our constraint is consistent with the previous study. On the other hand, we cannot find any improvement in the constraint, although our method is based on the optimal estimator and should be better than suboptimal ones. This suggests that the Minkowski functional method is almost optimal for uncorrelated CDM isocurvature perturbations. This can also be expected from the Fisher matrix analysis in Ref. [49], which gives a Cramér-Rao bound $\Delta(\alpha^2 f_{NL}) = 60$.

As a joint constraint on $f_{NL}$ and $f_{NL}^{(ISO)}$, our results are the first one obtained from observed data. On the other hand, the same constraint is forecasted using the Fisher matrix analysis in Ref. [49]. We note that 1\(\sigma\) errors of our results are consistent with the forecast.
5 Application to the axion model

In this section, we apply the constraints on the isocurvature non-Gaussianity to the axion isocurvature model. First, we shortly describe how non-Gaussian isocurvature perturbations in axion CDM arise in the inflationary Universe based on the $\delta N$-formalism \[71, 72\]. The baseline of our derivation is the same as in Ref. \[32\] (See also Ref. \[48\]). Throughout this section, we denote the energy density of a component $i$ on the uniform density hyper-surface of the total matter with $\rho_i(\vec{x})$. We adopt the following non-linear definition of isocurvature perturbations in a component $i$,

$$S_i(\vec{x}) = 3(\zeta_i - \zeta)(\vec{x}),$$  \hspace{1cm} (23)

where $\zeta_i$ and $\zeta$ are respectively the curvature perturbations on the uniform density hyper-surfaces of the component $i$ and the total matter. According to the $\delta N$-formalism, $\rho_i(\vec{x})$ should be given by $\rho_i(\vec{x}) = \bar{\rho}_i e^{(1+w_i)S_i(\vec{x})}$, where $\bar{\rho}_i$ is the mean of $\rho_i(\vec{x})$ and $w_i$ is the equation of state of the component $i$.

The axion is a pseudo Nambu-Goldstone boson of the Peccei-Quinn (PQ) U(1) symmetry, which solves the strong CP problem in quantum chromodynamics (QCD). If the PQ symmetry is broken during inflation, the axion has a classical field value $a_i = F_a \theta$ and a vacuum fluctuation $\delta a$, where $F_a$ is the axion decay constant and $\theta \in [-\pi, \pi]$ is the initial misalignment angle during inflation. At high temperature, the axion is massless. As temperature $T$ decreases the QCD phase transition takes place. At this moment, the axion becomes massive and starts the coherent oscillation around the true vacuum. This oscillation of the axion contributes to the energy density of CDM. We assume that CDM in the Universe is a mixture of the axion and other CDM components which are adiabatic.

The coherent oscillation of the axion synchronously starts on the uniform energy density hyper-surface of the total matter at around $m_{\text{axion}}(T) \simeq 3H(T)$ \#6, where $m_{\text{axion}}$ and $H$ is the mass of axion and Hubble expansion rate, respectively. The energy density of the coherent oscillation is proportional to the square of its initial amplitude, $\rho_{\text{axion}}(\vec{x}) \propto (a_i + \delta a(\vec{x}))^2$, which leads

$$e^{S_{\text{axion}}(\vec{x})} = 1 + 2 \frac{a_i \delta a(\vec{x})}{a_*^2} + \frac{\delta a^2(\vec{x}) - \langle \delta a^2 \rangle}{a_*^2},$$ \hspace{1cm} (24)

where

$$a_*^2 \equiv a_i^2 + \langle \delta a^2 \rangle.$$ \hspace{1cm} (25)

Since other CDM components are assumed to be adiabatic, their energy density is uniform on the uniform density hyper-surface of the total matter. Therefore $\rho_{\text{CDM}}(\vec{x})$ should be given by

$$\rho_{\text{CDM}}(\vec{x}) = \bar{\rho}_{\text{CDM}} [(1 - r) + r e^{S_{\text{axion}}}],$$ \hspace{1cm} (26)

\#6 Here we assume that there are no isocurvature perturbations in neutrinos or, if any, extra radiations [3]. Otherwise, the coherent oscillation does not start synchronously on the uniform density hyper-surface and as a consequence, additional isocurvature perturbations in the axion can be induced.
where \( r = \bar{\rho}_{\text{axion}}/\bar{\rho}_{\text{CDM}} \) is the energy fraction of the axion in CDM and \((1 - r)\) is that of other CDM components. Thus we finally obtain the isocurvature perturbations in CDM,

\[
S_{\text{CDM}} = \ln \left[ \frac{\rho_{\text{CDM}}(x)}{\bar{\rho}_{\text{CDM}}} \right] = \ln [1 + r e^{S_{\text{axion}}}]
\]

\[
\simeq 2r \frac{a_i \delta a}{a_i^2} + \left[ \frac{1}{4r a_i} \right] \left[ \left( 2r \frac{a_i \delta a}{a_i^2} \right)^2 - \left( \frac{2a_i \delta a}{a_i^2} \right)^2 \right] + \cdots (27)
\]

The last equality is approximately valid for \( r \ll 1 \). In the following, we keep terms up to the second order.

According to Refs. [73, 32], \( r \) is given by

\[
r = 0.2 \omega_c^{-1} \left( \frac{F_a}{10^{12} \text{GeV}} \right)^{-0.82} \left[ \left( \frac{F_a}{10^{12} \text{GeV}} \right)^2 \theta^2 + \left( \frac{H_{\text{inf}}/2\pi}{10^{12} \text{GeV}} \right)^2 \right], \quad (28)
\]

where the first and second terms in the square bracket correspond to contributions from the classical field \( a_i \) and the fluctuation \( \delta a \), respectively. The fluctuation of axion \( \delta a \) is almost scale-invariant (See also Ref. [12]) and its root mean square is given by \( \sqrt{\langle \delta a^2 \rangle} = H_{\text{inf}}/2\pi \), where \( H_{\text{inf}} \) is the Hubble scale during inflation. Then power spectrum and bispectrum of CDM isocurvature perturbations are given by\(^\#7\)

\[
\langle S_{\text{CDM}}(\vec{k}_1)S_{\text{CDM}}(\vec{k}_2) \rangle = 4r^2 \left( \frac{F_a}{H_{\text{inf}}/2\pi} \right)^2 \frac{2\pi^2}{k_1^3} (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2), \quad (29)
\]

\[
\langle S_{\text{CDM}}(\vec{k}_1)S_{\text{CDM}}(\vec{k}_2)S_{\text{CDM}}(\vec{k}_3) \rangle = 8r^3 \left[ \left( \frac{F_a}{F_a \theta} + \frac{H_{\text{inf}}/2\pi}{H_{\text{inf}}/2\pi} \right)^2 \right] \times \left[ \frac{(2\pi)^3}{k_1^3 k_2^3} + (2 \text{ perms}) \right] (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3). \quad (30)
\]

Having all these results, constraints should be

\[
0.2 \left( \frac{F_a}{10^{12} \text{GeV}} \right)^{-0.82} \left[ \left( \frac{F_a}{10^{12} \text{GeV}} \right)^2 \theta^2 + \left( \frac{H_{\text{inf}}/2\pi}{10^{12} \text{GeV}} \right)^2 \right] < \omega_c, \quad (31)
\]

\[
0.16 \left( \frac{F_a}{10^{12} \text{GeV}} \right)^{-1.64} \left[ \left( \frac{F_a}{10^{12} \text{GeV}} \right)^2 \theta^2 + \left( \frac{H_{\text{inf}}/2\pi}{10^{12} \text{GeV}} \right)^2 \right] \left( \frac{H_{\text{inf}}/2\pi}{10^{12} \text{GeV}} \right)^2 < 0.15 A_s \omega_c^2, \quad (32)
\]

\[
0.063 \left( \frac{F_a}{10^{12} \text{GeV}} \right)^{-2.46} \left[ \left( \frac{F_a}{10^{12} \text{GeV}} \right)^2 \theta^2 + \left( \frac{H_{\text{inf}}/2\pi}{10^{12} \text{GeV}} \right)^2 \right] \left( \frac{H_{\text{inf}}/2\pi}{10^{12} \text{GeV}} \right)^4 < 140 A_s^2 \omega_c^3, \quad (33)
\]

From top to bottom, these three equations correspond to the constraints on the abundance \( r < 1 \), the isocurvature power spectrum \( \alpha < 0.15 \) (95% CL) [1], and non-Gaussianity

\(^\#7\) Although, in the analysis in the previous sections, we omitted the loop contributions, we include them here. We also assume that logarithmic factors appear in the expression are considered to be \( \mathcal{O}(1) \) and we here set them to be unity.
\( |\alpha^2 f_{NL}^{(ISO)}| < 140 \) (2\( \sigma \) level) from our results, respectively. We should, however, note that since loop contributions are omitted in deriving the constraints in the previous sections, thus our results cannot in a rigorous manner be applied to cases for \( F_a \theta \lesssim H_{\text{inf}}/2\pi \), where loop contributions dominate polyspectra. Therefore, in the parameter regions with \( F_a \theta \lesssim H_{\text{inf}}/2\pi \), our constraints should be deemed as rough estimate.

In Fig. 2, we plotted the constraints of Eqs. (31)-(33) in the \( H_{\text{inf}}-\theta \) plane with several fixed values of \( F_a \), with \( \omega_c \) and \( A_s \) being fixed to the values given in Eq. (20). Constraints have critical points at \( F_a \theta = H_{\text{inf}}/2\pi \), where contributions of the classical field value and the fluctuation are comparable. We found that the constraint on non-Gaussianity from our results gives an upper bound on \( H_{\text{inf}} \) comparable to that from one on the isocurvature power spectrum.

When \( H_{\text{inf}}/2\pi > F_a \), the PQ symmetry restores during inflation\(^\#8\) and in that case CDM axions are produced from the system of axionic string-wall system as well as the coherent oscillation with an initial alignment angle \( \theta = \mathcal{O}(1) \). This excludes \( H_{\text{inf}}/2\pi > F_a \) for \( F_a \gtrsim 10^{11}\text{GeV} \) in Fig. 2 (For details, we refer to e.g. Refs. [75, 76]).

These constraints lead that even if \( \theta \) can be small by chance, large \( F_a \gtrsim 10^{11}\text{GeV} \) is not allowed unless the inflation scale is low \( H_{\text{inf}} \lesssim 10^{11}\text{GeV} \).

We note that in deriving the constraints above, we fixed the values of \( \omega_c \) and \( A_s \), which in principle have uncertainties themselves. Since these parameters are precisely determined to an accuracy of several percents from current data [1], their uncertainties can be safely omitted in the right hand sides of Eqs. (31)-(33). On the other hand, as is discussed in the previous section, more significant are the effects of uncertainties in the cosmological parameters on the estimation of \( \alpha^2 f_{NL}^{(ISO)} \), which can increase the right hand side of Eq. (33) by about ten percents. Still, Fig. 2 is not affected very much, because the bounds on \( H_{\text{inf}} \) and \( \theta \) are proportional to some fractional powers of the right hand side of Eq. (33).

6 Conclusion

We studied constraints on non-Gaussianity in a mixture of adiabatic and uncorrelated CDM isocurvature perturbations, which should be regarded as an extension of the adiabatic local-type one to non-adiabatic primordial perturbations. We adopted the optimal bispectrum estimator for the non-linearity parameters, and a fast method for non-Gaussian CMB simulation and the optimal weighting of observed maps are integrated in our analysis. Using the WMAP 7-year data of CMB temperature maps with template marginalization of the Galactic foregrounds, we obtained a constraint \( \alpha^2 f_{NL}^{(ISO)} = 40 \pm 62 \) at 1\( \sigma \) level for the scale invariant isocurvature power spectrum when the adiabatic perturbations are assumed to be Gaussian. Under the same setup, we also obtained a joint constraint on the non-linearity parameters \((f_{NL}, \alpha^2 f_{NL}^{(ISO)}) = (38 \pm 24, -8 \pm 72) \). The constraints weakly depend on the fiducial value of the isocurvature spectral index. Effects of the Galactic

\(^\#8\)Here we assumed the reheating temperature \( T_{\text{reh}} \) is below \( H_{\text{inf}} \). If \( T_{\text{reh}} > H_{\text{inf}} \), the PQ symmetry can restore for smaller \( H_{\text{inf}} \) after inflation.
Figure 2: Constraints on axion isocurvature model in the $H_{\text{inf}}$-$\theta$ plane for several fixed values of $F_a$. Shaded regions are excluded by cosmological considerations. At small $\theta$, the constraint on $H_{\text{inf}}$ from the non-Gaussianity is slightly better than one from the power spectrum. See text for details.
foregrounds at large angular scales are not severe. We found no statistically significant deviation from Gaussianity and the current WMAP observation of CMB is consistent with Gaussianity even when we include this type of isocurvature perturbations. We applied our results to the axion model.

Since the CDM and baryon isocurvature modes affect the CMB anisotropy in the same way except for the overall amplitude, our constraint can be translated into the baryon isocurvature perturbations. This can be easily done by substituting \( \left( \frac{\Omega_b}{\Omega_c} \right)^3 \alpha^2 f_{NL}^{(\text{ISO})} \) for \( \alpha^2 f_{NL}^{(\text{ISO})} \), where \( \Omega_b \) and \( \Omega_c \) are the density parameters of baryon and CDM, respectively. Although we restrict ourselves to uncorrelated CDM isocurvature perturbations in this paper, our method can be generalized to the cases of non-Gaussianity in isocurvature perturbations correlated with adiabatic ones and other types such as neutrino ones. These will be studied in a forthcoming paper [52].

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A  Equivalence between \( \langle S_{\text{prim}}^A \rangle_{f_{NL}^B=1} \) and \( \langle S_{\text{prim}}^A S_{\text{prim}}^B \rangle_0 \)

In this appendix, we show that the normalization factor \( \langle S_{\text{prim}}^A \rangle_{f_{NL}^B=1} \) and the covariance \( \langle S_{\text{prim}}^A S_{\text{prim}}^B \rangle_0 \) is equivalent.

From Eq. (15), we obtain

\[
\langle S_{\text{prim}}^A \rangle_{f_{NL}^B=1} = \frac{1}{6} \sum_{\{lm\}} \hat{b}_{l_1 l_2 l_3} G^{l_1 l_2 l_3}_{m_1 m_2 m_3} \langle \tilde{a}_{l_1 m_1} \tilde{a}_{l_2 m_2} \tilde{a}_{l_3 m_3} \rangle_{f_{NL}^B=1},
\]

where we used the fact that \( \langle \tilde{a}_{l m} \rangle_{f_{NL}^A=1} = 0 \). As discussed in Section 3.3, we take \( \tilde{a}_{l m} = \)
\[
\sum_{l'm'} [C^{-1}]_{lm,l'm'} a_{l'm'} f_{NL}^B = 1
\]
\[
\langle \tilde{a}_{l_1 m_1} \tilde{a}_{l_2 m_2} \tilde{a}_{l_3 m_3} \rangle = 1
\]
\[
= \sum_{\{l'm'\}} [C^{-1}]_{l_1 m_1, l_1'm_1} [C^{-1}]_{l_2 m_2, l_2'm_2} [C^{-1}]_{l_3 m_3, l_3'm_3} \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle f_{NL}^B = 1
\]
\[
= \sum_{\{l'm'\}} [C^{-1}]_{l_1 m_1, l_1'm_1} [C^{-1}]_{l_2 m_2, l_2'm_2} [C^{-1}]_{l_3 m_3, l_3'm_3} G_{m_1 m_2 m_3}^{\nu \nu} \hat{b}_{l_1 l_2 l_3}^{B} \hat{b}_{l_1' l_2' l_3'}. 
\]
Combining Eqs. (34) and (36), we obtain
\[
\langle S_{\text{prim}}^{A} \rangle f_{NL}^B = 1 = \frac{1}{6} \sum_{\{lm'l'm'\}} \hat{b}_{l_1 l_2 l_3}^{A} \hat{b}_{l_1' l_2' l_3'}^{B} G_{m_1 m_2 m_3}^{\nu \nu} G_{m_1'm_2'm_3'}^{\nu \nu} \left[ \langle \tilde{a}_{l_1 m_1} \tilde{a}_{l_2 m_2} \tilde{a}_{l_3 m_3} \tilde{a}_{l_1'm_1} \tilde{a}_{l_2'm_2} \tilde{a}_{l_3'm_3} \rangle \right]_0
\]
\[
- 6 \langle \tilde{a}_{l_1 m_1} \tilde{a}_{l_2 m_2} \tilde{a}_{l_3 m_3} \rangle_0 \langle \tilde{a}_{l_2 m_2} \tilde{a}_{l_3 m_3} \rangle_0 + 9 \langle \tilde{a}_{l_1 m_1} \tilde{a}_{l_2 m_2} \tilde{a}_{l_3 m_3} \rangle_0 \langle \tilde{a}_{l_2 m_2} \tilde{a}_{l_3 m_3} \rangle_0 \langle \tilde{a}_{l_1 m_1} \tilde{a}_{l_2 m_2} \tilde{a}_{l_3 m_3} \rangle_0 .
\]
By taking the Wick's contraction, the terms in the square bracket are reduced into
\[
[C^{-1}]_{l_1 m_1, l_1'm_1} [C^{-1}]_{l_2 m_2, l_2'm_2} [C^{-1}]_{l_3 m_3, l_3'm_3} + \text{ (5 perms)},
\]
where we used the fact that \( \langle a_{lm} a_{l'm'} \rangle_0 = C_{lm,l'm'} \). Thus, we obtain
\[
\langle S_{\text{prim}}^{A} S_{\text{prim}}^{B} \rangle_0 = \frac{1}{6} \sum_{\{lm'l'm'\}} \hat{b}_{l_1 l_2 l_3}^{A} \hat{b}_{l_1' l_2' l_3'}^{B} G_{m_1 m_2 m_3}^{\nu \nu} G_{m_1'm_2'm_3'}^{\nu \nu} [C^{-1}]_{l_1 m_1, l_1'm_1} [C^{-1}]_{l_2 m_2, l_2'm_2} [C^{-1}]_{l_3 m_3, l_3'm_3}. 
\]
From Eqs. (36) and (39), we finally see that \( \langle S_{\text{prim}}^{A} \rangle f_{NL}^B = 1 = \langle S_{\text{prim}}^{A} S_{\text{prim}}^{B} \rangle_0 \).

B Estimation of normalization

Here we show how we can estimate the normalization factor \( \langle S_{\text{prim}}^{A} \rangle f_{NL}^B = 1 \) from simulation.

First we divide the CMB anisotropy \( a_{lm}^A \) into Gaussian and non-Gaussian parts
\[
a_{lm}^A = a_{G,lm}^A + f_{NL} a_{NG,lm}^A,
\]
where \( a_{G,lm}^A \) and \( a_{NG,lm}^A \) respectively are the Gaussian and non-Gaussian contributions, which are given by
\[
a_{G,lm}^A = \int dr \int dr' Y_{lm}^* (\hat{r}) X_G^A (\hat{r}), \quad a_{NG,lm}^A = \int dr \int dr' Y_{lm}^* (\hat{r}) (X_G^A (\hat{r})^2 - \langle X_G^A (\hat{r})^2 \rangle). 
\]
Since $\tilde{a}_{lm}$ is a linear combination of $a_{lm}$, we can also divide the weighted map into its Gaussian and non-Gaussian parts, $\tilde{a}_{lm}^A = \tilde{a}_{lm}^G + f_{NL}^A \tilde{a}_{lm}^A$. Then $\langle S^A_{\text{prim}} \rangle_{f_{NL}=1}$ in Eq. (34) can be rewritten as

$$
\langle S^A_{\text{prim}} \rangle_{f_{NL}=1} = \frac{1}{6} \sum_{\{lm\}} \tilde{b}_{1l1l_3}^A \tilde{G}_{m_1m_2m_3}^{l_1l_2l_3} \langle (\tilde{a}_{G,l_1m_1} + \tilde{a}_{B,NG,l_1m_1}^B) (\tilde{a}_{G,l_2m_2} + \tilde{a}_{B,NG,l_2m_2}^B) (\tilde{a}_{G,l_3m_3} + \tilde{a}_{B,NG,l_3m_3}^B) \rangle_{f_{NL}=1}
$$

$$
= \frac{1}{6} \sum_{\{lm\}} \tilde{b}_{1l1l_3}^A \tilde{G}_{m_1m_2m_3}^{l_1l_2l_3} \left( 3 \langle \tilde{a}_{B,NG,l_1m_1}^B \tilde{a}_{G,l_2m_2}^B \tilde{a}_{G,l_3m_3}^B \rangle + \langle \tilde{a}_{NG,l_2m_2}^B \tilde{a}_{NG,l_2m_2}^B \tilde{a}_{NG,l_3m_3}^B \rangle \right)_{f_{NL}=1},
$$

(43)

where the last equality follows from the Wick’s theorem. As long as non-Gaussianity is not too large, the second term in the last line can be omitted. To evaluate $\langle \tilde{a}_{NG,l_1m_1}^B \tilde{a}_{G,l_2m_2}^B \tilde{a}_{G,l_3m_3}^B \rangle$, we checked that $O(10)$ MC samples are enough for convergence. In addition, since the terms in the last line of Eq. (43) consists of only anisotropy $\tilde{a}_{lm}^B$ from a single perturbation mode $X^B$, we can estimate $\langle S^A_{\text{prim}} \rangle_{f_{NL}=1}$ without assuming a fiducial value for the fraction of the isocurvature contribution $\alpha$.

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