RATIONALITY IN MAP AND HYPERMAP ENUMERATION BY GENUS

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Abstract. Generating functions for a fixed genus map and hypermap enumeration become rational after a simple explicit change of variables. Their numerators are polynomials with integer coefficients that obey a differential recursion, and denominators are products of powers of explicit linear functions.

1. Introduction

By a map or a ribbon graph we understand a finite connected graph with prescribed cyclic orders of half-edges at each vertex. It also can be realized as the 1-skeleton of a polygonal partition of a closed orientable surface. The genus \( g \) of a map (ribbon graph) satisfies the Euler formula

\[
2 - 2g = \#v - \#e + \#f ,
\]

where \( \#v, \#e, \#f \) are the numbers of vertices, edges and faces of the map respectively. By a hypermap we understand a bicolor map, i. e. a map whose faces are properly colored in two colors (say, white and black) so that no adjacent faces have the same color. The dual graph to a hypermap is a bipartite ribbon graph, or a Grothendieck’s “dessin d’enfant”.

We are interested in the weighted count of maps and hypermaps, where the weights are reciprocal to the orders of the corresponding automorphism groups. This is equivalent to counting rooted maps and hypermaps (i. e. those with a marked half-edge). The passage from the rooted count to the unrooted one is known, cf. [8], [9].

Denote by \( \tilde{c}_{g,n} \) (resp. \( c_{g,n} \)) the number of rooted maps (resp. hypermaps) of genus \( g \) with \( n \) edges (darts), and consider the genus \( g \) generating functions

\[
\tilde{C}_g(s) = \sum_{n=2g}^{\infty} \tilde{c}_{g,n}s^n \quad g \geq 0 , \tag{1}
\]

\[
C_g(s) = \sum_{n=2g+1}^{\infty} c_{g,n}s^n \quad g \geq 0 . \tag{2}
\]

The classical problem that goes back to Tutte [10] (or even earlier) is to compute the numbers \( \tilde{c}_{g,n} \) and \( c_{g,n} \). Effective algorithms for computing these numbers first appeared in [12] (for maps) and in [13] (for hypermaps). Recursions for the numbers \( c_{g,n} \) and \( \tilde{c}_{g,n} \), and differential equations for the generating functions \( C_g(s) \) and

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1 As observed by Grothendieck, the absolute Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) naturally acts on dessins (hypermaps); we refer the reader to [2] for details.

2 The effective enumeration of 1-vertex maps was obtained in [3], and of 1-vertex hypermaps – in [5] and, independently, in [1]. Enumeration of 1-vertex maps (or genus \( g \) gluings of a 2n-gon) was a crucial point in computing the Euler characteristic of the moduli space of algebraic curves in [3].

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\( \tilde{C}_g(s) \) were first obtained in [3] (cf. also [2] for an alternative approach to map enumeration).

In this note we show that the generating functions \( C_g(s) \) and \( \tilde{C}_g(s) \) become rational functions after simple explicit changes of variable \( s \). Their numerators are then polynomials with integer coefficients that obey a differential recursion, and denominators are products of powers of explicit linear functions.

2. Main results

We start with the case of hypermaps (Grothendieck’s dessins d’enfants).

**Theorem 1.** Under the substitution \( s = t(1 - 2t) \) we have

\[
\begin{align*}
C_0(t(1 - 2t)) &= \frac{t(1 - 3t)}{(1 - 2t)^2}, \\
C_1(t(1 - 2t)) &= \frac{t^3}{(1 - t)(1 - 4t)^2}, \\
C_g(t(1 - 2t)) &= \frac{P_g(t)}{(1 - t)^{4g - 2}(1 - 4t)^{g - 3}}, \quad g \geq 2,
\end{align*}
\]

where \( P_g(t) = \sum_{i=2g+1}^{2g} p_{g,i} t^i \) is a polynomial with integer coefficients and \( p_{g,2g+1} = \binom{2g}{g+1}. \) The polynomials \( P_g(t) \) can be computed recursively by Eq. [9].

**Remark 1.** The polynomials \( P_g(t) \) for \( g = 2, 3 \) are:

\[
\begin{align*}
P_2(t) &= 8t^5 - 92t^6 + 464t^7 - 1316t^8 + 2204t^9 - 2048t^{10} + 816t^{11}, \\
P_3(t) &= 180t^7 - 3648t^8 + 35424t^9 - 218944t^{10} + 958160t^{11} - 3102528t^{12} \\
&\quad + 7503664t^{13} - 13310768t^{14} + 16365216t^{15} - 11823680t^{16} + 117916t^{17} \\
&\quad + 6614784t^{18} - 6008320t^{19} + 1823744t^{20}.
\end{align*}
\]

In principle, they can be computed for much larger values of \( g \).

**Remark 2.** Similar result was independently obtained in [2] by a different (more complicated) method.

**Proof.** To prove the theorem, we recall a specialization of the Kadomtsev-Petviashvili (KP) equation for the hypermap count derived in [6]:

\[
(sC_g)' = 3(2s^2C_g' + sC_g) + 3s^3C_g'' + s^3(s(sC_{g-1})')''
+ s^3 \sum_{i=0}^{g} (4C_i + 6sC_i')C_{g-i}'' + 2s \delta_{g,0},
\]

(4)

where the prime ‘ stands for the derivative \( \frac{d}{dt} \). This equation is just the differential form of the recursion (11) in [6] for \( t = u = v = 1 \):

\[
(n + 1)c_{g,n} = 3(2n - 1)c_{g,n-1} + (n - 2)c_{g,n-2} + (n - 1)^2(n - 2)c_{g-1,n-2}
+ \sum_{i=0}^{g} \sum_{j=1}^{n} (4 + 6j)(n - 2 - j)c_{i,j}c_{g-i,n-2-j}.
\]

(5)

For \( g \geq 1 \) we can further rewrite Eq. (4) as the differential recursion

\[
(s - 6s^2 - 3s^3 - 4s^3C_0 - 12s^4C_0')C_g' + (1 - 3s - 4s^3C_0)C_g
= s^3C_g'' + 5s^4C_g' - 1 + 4s^3C_g' + s^3 \sum_{i=1}^{g-1} (4C_i + 6sC_i')C_{g-i}'.
\]

(6)
For \( g = 0 \) we get an ordinary differential equation that can be solved explicitly:

\[
C_0(s) = \frac{-1 + 12s - 24s^2 + (1 - 8s)^{3/2}}{32s^2}.
\]

It is easy to see that the substitution \( s = t(1 - 2t) \) considerably simplifies \( C_0 \) and makes it a rational function,

\[
C_0(t(1 - 2t)) = \frac{t(1 - 3t)}{(1 - 2t)^2}
\]

(cf. [11]). Substituting \( s = t(1 - 2t) \) in [6], we get

\[
t(1 - t)^2(1 - 2t) \dot{C}_g + (1 - t)(1 - 2t + 4t^2) C_g
= t^3(1 - 2t)^3 \left( D_t C_{g-1} + \frac{1}{(1 - 4t)^2} \sum_{i=1}^{g-1} \left( 4(1 - 4t)C_i + 6t(1 - 2t)\dot{C}_i \right) \dot{C}_{g-i} \right),
\]

where \( \dot{C}_g = \frac{d}{dt}C_g(t(1 - 2t)) \) and

\[
D_t = \frac{t^2(1 - 2t)^2}{(1 - 4t)^3} \frac{d^3}{dt^3} + \frac{t(1 - 2t)(5 - 28t + 56t^2)}{(1 - 4t)^4} \frac{d^2}{dt^2} + \frac{4(1 - 11t + 58t^2 - 144t^3 + 144t^4)}{(1 - 4t)^5} \frac{d}{dt}.
\]

Assuming that \( C_0, \ldots, C_{g-1} \) are known, we can think of [7] as an ODE for \( C_g \). The integrating factor for this equation is

\[
\frac{1}{(1 - 2t)^2},
\]

so that we get from [7]

\[
\frac{d}{dt} \left( \frac{t(1 - t)^3}{(1 - 2t)^2} C_g \right)
= t^3(1 - t) \left( D_t C_{g-1} + \frac{1}{(1 - 4t)^2} \sum_{i=1}^{g-1} \left( 4(1 - 4t)C_i + 6t(1 - 2t)\dot{C}_i \right) \dot{C}_{g-i} \right),
\]

or, equivalently,

\[
C_g(t(1 - 2t)) = \frac{(1 - 2t)^2}{t(1 - t)^5}
\]

\[
\times \int t^3(1 - t) \left( D_t C_{g-1} + \frac{1}{(1 - 4t)^2} \sum_{i=1}^{g-1} \left( 4(1 - 4t)C_i + 6t(1 - 2t)\dot{C}_i \right) \dot{C}_{g-i} \right) dt.
\]

Since by definition \( C_g(0) = 0 \) for all \( g \geq 0 \), Eq. [9] determines \( C_g \) uniquely in terms of \( C_0, \ldots, C_{g-1} \). In particular, this equation immediately yields

\[
C_1(t(1 - 2t)) = \frac{t^3}{(1 - t)(1 - 4t)^2},
\]

\[
C_2(t(1 - 2t)) = \frac{8t^5 - 92t^6 + 464t^7 - 1316t^8 + 2204t^9 - 2048t^{10} + 816t^{11}}{(1 - t)^3(1 - 4t)^7}.
\]

Let us show that \( C_g(t(1 - 2t)) \) has the form [3] for any \( g \geq 3 \). We will use the elementary formula

\[
\frac{d}{dt} \left( \frac{t^\alpha}{(1 - t)^\beta(1 - 4t)^\gamma} \right) = \frac{\alpha t^{\alpha-1} + (-5\alpha + \beta + 4\gamma)t^\alpha + 4(\alpha - \beta - \gamma)t^{\alpha+1}}{(1 - t)^{\beta+1}(1 - 4t)^{\gamma+1}}.
\]

Then we have

\[
D_t C_{g-1} = \frac{(2g - 1)(2g)^2 \beta_{g-1, 2g-1} - t(2g-2) + \ldots - 256 \beta_{g-1, 2g-16} + 16g^{9g-9}}{(1 - t)^{4g-4}(1 - 4t)^{9g-2}}
\]

(11)
and
\[
\frac{1}{(1-4t)^2} \sum_{i=1}^{g-1} \left( 4(1-4t)C_i + 6((1-2t)\dot{C}_i) \right) \dot{C}_{g-1} = \frac{r_g t^{2g+1} + \ldots + 256r_{g-1}g_{g-1}t^{g+2}}{(1-t)^{4g-4}(1-4t)^{5g-2}} ,
\]
where \( r_g \) is some constant. Notice that the top degree term in the numerator on the right hand side of (12) comes entirely from the product \( C_1 \dot{C}_{g-1} \). Multiplying both sides of (11) and (12) by \( t^3(1-t) \) and taking their sum we get that the integrand in (9) has the form
\[
\frac{Q_g(t)}{(1-t)^{4g-5}(1-4t)^{5g-2}} ,
\]
where \( Q_g(t) = \sum_{i=2g+1}^{3g-7} q_{g,i} t^i \) is a polynomial with \( q_{g,2g+1} = (2g-1)(2g)^2 p_{g-1,2g-1} \). Therefore, we can rewrite (9) as
\[
C_g(t(1-2t)) = \frac{(1-2t)^2}{t(1-t)^3} \int \frac{Q_g(t)}{(1-t)^{4g-5}(1-4t)^{5g-2}} dt .
\]
To perform integration in (13) we decompose the integrand into the sum
\[
\frac{Q_g(t)}{(1-t)^{4g-5}(1-4t)^{5g-2}} = a + \sum_{i=2}^{4g-5} \frac{a_i}{(1-t)^i} + \sum_{j=2}^{5g-2} \frac{b_j}{(1-4t)^j} .
\]
Note that no terms of the form \( \frac{dt}{1-t} \) or \( \frac{1}{1-4t} \) can appear in the right hand side of (13) because the Taylor series expansion of the left hand side of (14) has integer coefficients. Integrating we obtain
\[
\int \frac{Q_g(t)}{(1-t)^{4g-5}(1-4t)^{5g-2}} dt = at + b + \sum_{i=1}^{4g-6} \frac{a_{i+1}}{i} \frac{1}{(1-t)^i} + \sum_{j=1}^{5g-3} \frac{b_{j+1}}{4j} \frac{1}{(1-4t)^j} ,
\]
where the condition \( C_g(0) = 0 \) implies
\[
b = -\sum_{i=1}^{4g-6} \frac{a_i}{i} - \sum_{j=2}^{5g-3} \frac{b_{j+1}}{4j} .
\]
Multiplying the right hand side of (16) by \( (1-t)^{4g-6}(1-4t)^{5g-3} \) we get a polynomial of the form \( R_g(t) = \sum_{i=2g+2}^{9g-8} \sum_{j} r_{i,j} t^i \). To complete the proof we put \( P_g(t) = \frac{(1-2t)^2}{t} R_g(t) \) and notice that \( p_{g,2g+1} = (2g-1)(2g)^2 p_{g-1,2g-1} \). Moreover, we see that \( t = 1/2 \) is a root of \( P_g(t) \) of multiplicity 2 provided \( g \geq 2 \). □

Now we continue with map enumeration.

**Theorem 2.** Under the substitution \( s = t(1-3t) \) we have
\[
\widetilde{C}_0(t(1-3t)) = \frac{1-4t}{(1-3t)^2} ,
\]
\[
\widetilde{C}_1(t(1-3t)) = \frac{t^2}{(1-2t)(1-6t)^2} ,
\]

\(^3\)We owe this observation to F. Petrov.
\(^4\)Numerically, we also have \( p_{g,2g-7} \neq 0 \). \( P_g(1) \neq 0 \), \( P_g(1/4) \neq 0 \). In principle, this can be verified along the same lines as above, but computations become too cumbersome to reproduce them here.
\[ \tilde{C}_g(t(1 - 3t)) = \frac{\tilde{P}_g(t)}{(1 - 2t)^{3g/2}(1 - 6t)^{3g/3}}, \quad g \geq 2, \quad (17) \]

where \( \tilde{P}_g(t) = \sum_{i=2}^{g-6} \tilde{p}_{g,i} t^i \) with \( \tilde{p}_{g,2g} = \frac{(4g-11)!}{2g+1} \). The polynomials \( \tilde{P}_g(t) \) can be computed recursively by Eq. (23).

Remark 3. The polynomials \( \tilde{P}_g(t) \) for \( g = 2, 3 \) are:

\[ \tilde{P}_2(t) = 21t^4 - 336t^5 + 2334t^6 - 91081t^7 + 211776t^8 - 277566t^9 + 15876610, \]
\[ \tilde{P}_3(t) = 1485t^6 - 41184t^7 + 539073t^8 - 4483458t^9 + 268938910 - 12423004t^{11} \]
\[ + 453861279t^{12} - 130735312t^{13} + 2897271774t^{14} - 4737605112t^{15} + 5355443952t^{16} - 3723895296t^{17} + 1197496224t^{18}. \]

Like in the case of hypermaps, they can be computed for much larger values of \( g \).

Proof. The proof of Theorem 2 is very similar to that of Theorem 1, so we will outline only its main steps. We recall a specialization of the Kadomtsev-Petviashvili (KP) equation for the map count derived in [6]:

\[ (s\tilde{C}_g)' = 4(2s^2\tilde{C}'_g + s\tilde{C}_g) + 2s^3(2s(s\tilde{C}_g - 1)' + s\tilde{C})' + s^3(2s(s\tilde{C}_g - 1)' + s\tilde{C})' + 3s^2 \sum_{i=0}^{g} (\tilde{C}_i + 2s\tilde{C}_i')(\tilde{C}_{g-i} + 2s\tilde{C}_{g-i}) + \delta_{g,0}, \quad (18) \]

where the prime \( ' \) stands for the derivative \( \frac{d}{dt} \). This equation is just a differential form of the recursion (16) in [6] for \( t = u = 1 \):

\[ (n + 1)\tilde{c}_{g,n} = 4(2n - 1) \tilde{c}_{g,n-1} + (2n - 1)(2n - 3)(n - 1) \tilde{c}_{g-1,n-2} + 3 \sum_{i=0}^{g} (2j + 1)(2n - 2 - j) + 1 \tilde{c}_{i,j} \tilde{c}_{g-i,n-2-j}. \quad (19) \]

For \( g \geq 1 \) we can further rewrite Eq. (18) as the differential recursion

\[ (s - 8s^2 - 12s^3\tilde{C}_0 - 24s^4\tilde{C}_0')(\tilde{C}_g' + (1 - 4s - 6s^2\tilde{C}_0 - 12s^3\tilde{C}_0') \tilde{C}_g = 4s^5\tilde{C}_{g-1} + 24s^4\tilde{C}_g + 27s^3\tilde{C}_{g-1} + 3s^2\tilde{C}_{g-1} + 3s^2 \sum_{i=0}^{g} (\tilde{C}_i + 2s\tilde{C}_i')(\tilde{C}_{g-i} + 2s\tilde{C}_{g-i}). \quad (20) \]

For \( g = 0 \) we get an ordinary differential equation that can be solved explicitly:

\[ \tilde{C}_0(s) = \frac{-1 + 18s + (1 - 12s)^{3/2}}{54s^2}. \]

It is easy to see that the substitution \( s = t(1 - 3t) \) considerably simplifies \( \tilde{C}_0 \) and makes it a rational function, namely

\[ \tilde{C}_0(t(1 - 3t)) = \frac{1 - 4t}{(1 - 3t)^2} \]

(cf. [11]). Substituting \( s = t(1 - 3t) \) in (20), we get

\[ t(1 - 2t)(1 - 3t) \tilde{C}_g + (1 - 4t + 6t^2) \tilde{C}_g = t^2(1 - 3t)^2 \]
\[ \times \left( \tilde{D}_t \tilde{C}_{g-1} + 3 \sum_{i=1}^{g-1} \left( \tilde{C}_i + \frac{t(1 - 3t)}{1 - 6t} \tilde{C}_i \right) \left( \tilde{C}_{g-i} + \frac{t(1 - 3t)}{1 - 6t} \tilde{C}_{g-i} \right) \right), \quad (21) \]

where \( \tilde{C}_g = \frac{d}{dt} \tilde{C}_g(t(1 - 3t)) \) and

\[ \tilde{D}_t = \frac{4t^3(1 - 3t)^3}{(1 - 6t)^3} \frac{d^3}{dt^3} + \frac{24t^2(1 - 3t)^2(1 - 9t + 27t^2)}{(1 - 6t)^4} \frac{d^2}{dt^2}. \]
\[
+ \frac{9t(1 - 3t)(3 - 56t + 456t^2 - 1728t^3 + 2592t^4)}{(1 - 6t)^5} \cdot \frac{d}{dt} + 3. \tag{22}
\]

Assuming that \(\tilde{C}_0, \ldots, \tilde{C}_{g-1}\) are known, we can think of (21) as an ODE for \(\tilde{C}_g\). The integrating factor for this equation is \(t(1 - 3t)/1 - 2t\), so that we get from (21)

\[
\tilde{C}_g(t(1 - 3t)) = \frac{1 - 3t}{t(1 - 2t)} \times \int t^2 \left( \tilde{D}_t \tilde{C}_{g-1} + 3 \sum_{i=1}^{g-1} \left( \tilde{C}_i + \frac{t(1 - 3t)}{1 - 6t} \tilde{C}_i \left( \tilde{C}_{g-i} + \frac{t(1 - 3t)}{1 - 6t} \tilde{C}_{g-i} \right) \right) \right) dt.
\tag{23}
\]

Since by definition \(\tilde{C}_g(0) = 0\) for all \(g \geq 1\), Eq. (23) determines \(\tilde{C}_g\) uniquely in terms of \(\tilde{C}_0, \ldots, \tilde{C}_{g-1}\). In particular, this equation immediately yields

\[
\tilde{C}_1(t(1 - 3t)) = \frac{t^2}{(1 - 2t)(1 - 6t)^2}.
\]

Let us show that \(\tilde{C}_g(t(1 - 3t))\) has the form (17) for any \(g \geq 2\). Using an analogue of (10) we obtain, after some cancellations, that the integrand in (23) has the form

\[
\tilde{Q}_g(t) = \frac{1 - 3t}{(1 - 2t)^{3g - 2}(1 - 6t)^{5g - 2}},
\tag{24}
\]

where \(\tilde{Q}_g(t) = \sum_{i=2g}^{8g-6} \tilde{a}_{g,t} t^i\) is a polynomial with \(\tilde{a}_{g,2g} = (2g - 1)(4g - 1)(4g - 3) p_{g-1,2g-2}\). It can be further decomposed into the sum

\[
\tilde{Q}_g(t) = \frac{1 - 3t}{(1 - 2t)^{3g - 2}(1 - 6t)^{5g - 2}} = \sum_{i=2}^{3g-2} \tilde{a}_i \frac{(1 - 2t)^i}{(1 - 2t)^i} + \sum_{j=2}^{5g-2} \tilde{b}_j \frac{1}{(1 - 6t)^j}.
\tag{25}
\]

Integrating it, we obtain

\[
\int (1 - 2t)^{3g - 3}(1 - 6t)^{5g - 3} dt = \tilde{b} + \sum_{i=1}^{3g-3} \tilde{a}_{i+1} \frac{1}{2i} \cdot 1 + \sum_{j=2}^{5g-3} \tilde{b}_{j+1} \frac{6j}{6j},
\tag{26}
\]

where the condition \(\tilde{C}_g(0) = 0\) implies

\[
\tilde{b} = - \sum_{i=1}^{3g-3} \frac{\tilde{a}_{i+1}}{2i} = \sum_{j=1}^{5g-3} \frac{\tilde{b}_{j+1}}{6j}.
\]

Multiplying the right side of (26) by \((1 - 2t)^{3g - 3}(1 - 6t)^{5g - 3}\) we get a polynomial of the form \(\tilde{R}_g(t) = \sum_{i=2g+1}^{8g-6} \tilde{r}_{g,t} t^i\). To complete the proof we put \(\tilde{P}_g(t) = \frac{1 - 3t}{(1 - 2t)^{3g - 3}(1 - 6t)^{5g - 3}} \tilde{R}_g(t)\) and notice that \(\tilde{P}_{g,2g} = (2g - 1)(4g - 1)(4g - 3) p_{g-1,2g-2}\). Moreover, we see that \(t = 1/3\) is a root of \(\tilde{P}_g(t)\). \(\square\)

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5 Numerically, we also have \(p_{g,8g-6} \neq 0, \tilde{P}_g(1/2) \neq 0, \tilde{P}_g(1/6) \neq 0\).
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