Trace Operators for Modulation, $\alpha$-Modulation and Besov Spaces

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Abstract. In this paper, we consider the trace theorem for modulation spaces $M_{p,q}^s$, $\alpha$-modulation spaces $M_{p,q}^{s,\alpha}$ and Besov spaces $B_{p,q}^s$. For the modulation space, we obtain the sharp results.

Key words and phrases. Trace theorems; modulation spaces; $\alpha$-modulation spaces; Besov spaces.

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1 Introduction

The $\alpha$-modulation spaces $M_{p,q}^{s,\alpha}$, introduced by Gröbner in [10] are a class of function spaces that contain Besov spaces $B_{p,q}^s$ ($\alpha = 1$) and modulation spaces $M_{p,q}^s$ ($\alpha = 0$) as special cases.

There are two kinds of basic coverings on Euclidean $\mathbb{R}^n$ which is very useful in the theory of function spaces and their applications, one is the uniform covering $\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}^n} Q_k$, where $Q_k$ denote the unit cube with center $k$; another is the dyadic covering $\mathbb{R}^n = \bigcup_{k \in \mathbb{N}} \{\xi : 2^{k-1} \leq |\xi| < 2^k\} \bigcup \{\xi : |\xi| \leq 1\}$. Roughly speaking, these decompositions together with the frequency-localized techniques yield the frequency-uniform decomposition operator $\Box_k \sim \mathcal{F}^{-1} \chi_{Q_k} \mathcal{F}$ and the dyadic decomposition operator $\Delta_k \sim \mathcal{F}^{-1} \chi_{\{\xi : |\xi| \sim 2^k\}} \mathcal{F}$, respectively. The tempered distributions acted on these decomposition operators and equipped with the $\ell^q (L^p(\mathbb{R}^n))$ norms, we then obtain Feichtinger’s modulation spaces and Besov spaces, respectively.

During the past twenty years, the third covering was independently found by Feichtinger and Gröbner [3, 4, 10], and Päivärinta and Somersalo [12]. This covering, so called $\alpha$-covering has a moderate scale which is rougher than that of the uniform covering and is thinner than that of the dyadic covering. Applying the $\alpha$-covering...
to the frequency spaces, in a similar way as the definition of Besov spaces, Gröbner [10] introduced the notion of \( \alpha \)-modulation spaces.

Let \( n \geq 2 \). For any \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), we denote \( \bar{x} = (x_1, \ldots, x_{n-1}) \). Given a Banach function space \( X(\mathbb{R}^n) \) defined on \( \mathbb{R}^n \) and \( f \in X \), we ask for the trace of \( f \) on the hyperplane \( \{ x : x = (\bar{x}, 0) \} \). For the sake of convenience, this hyperplane will be written as \( \mathbb{R}^{n-1} \). It is clear that a clarification of this problem is of importance for the boundary value problems of the partial differential equations. Now we exactly describe the trace operators.

**Definition 1.1** Let \( X \) and \( Y \) be quasi-Banach function spaces defined on \( \mathbb{R}^n \) and \( \mathbb{R}^{n-1} \), respectively. Denote
\[
\mathcal{T} : f(x) \rightarrow f(\bar{x}, 0).
\]

If \( \mathcal{T} : X \rightarrow Y \) and there exists a constant \( C > 0 \) such that
\[
\|\mathcal{T} f(x)\|_Y \leq C \|f\|_X, \quad \forall f \in X,
\]
and there exists a continuous linear operator \( \mathcal{T}^{-1} : Y \rightarrow X \) such that \( \mathcal{T} \mathcal{T}^{-1} \) is identical operator, then \( \mathcal{T} \) is said to be a retraction from \( X \) onto \( Y \).

If \( \mathcal{T} \) is a retraction from \( Y \) onto \( X \), we see that the trace of \( f \in X \) is well behaved in \( Y \). The trace theorems in modulation spaces and Besov spaces have been extensively studied. Feichtinger [5] considered the trace theorem for the modulation space \( M^{s_{p,q}} \) in the case \( 1 \leq p, q \leq \infty \), \( s > 1/q' \) and he obtained that \( \mathcal{T} M^{s_{p,q}}(\mathbb{R}^n) = M^{s_{p,q}/q'}(\mathbb{R}^{n-1}) \). Frazier and Jawerth [8] proved that \( \mathcal{T} B^{s_{p,q}}(\mathbb{R}^n) = B^{s_{p,q}/p}(\mathbb{R}^{n-1}) \) in the case \( 0 < p, q \leq \infty \) and \( s - 1/p > \max((n-1)(1/p - 1), 0) \).

In this paper, we will show the following:

**Theorem 1.2** Let \( n \geq 2 \), \( 0 < p, q \leq \infty \), \( s \in \mathbb{R} \). Then
\[
\mathcal{T} : f(x) \rightarrow f(\bar{x}, 0), \quad \bar{x} = (x_1, \cdots, x_{n-1})
\]
is a retraction from \( M^{s_{p,q}}_{p,q \land 1}(\mathbb{R}^n) \) onto \( M^{s_{p,q}}_{p,q}(\mathbb{R}^{n-1}) \).

Theorem 1.2 is sharp in the sense that \( \mathcal{T} : M^{s_{p,q,r}}_{p,q,r}(\mathbb{R}^n) \not\rightarrow M^{s_{p,q}}_{p,q}(\mathbb{R}^{n-1}) \) for some \( r > 1 \), \( p, q \geq 1 \). In view of the basic embedding \( M^{s_{p,q}}_{p,q} \subset M^{s_{2,q}}_{p,q,q_2} \) for \( s - s_2 > 1/q - 1/q_2 > 0 \), \( s \geq 0 \), we immediately have
Corollary 1.3 Let \( n \geq 2, 0 < p, q \leq \infty, s \geq 0 \). Let \( \mathcal{Tr} \) be as in (1.3). Then for any \( \varepsilon > 0 \),
\[
\mathcal{Tr} : M^{s+\frac{1}{q'}}_{p,q} (\mathbb{R}^n) \to M^s_{p,q} (\mathbb{R}^{n-1}) .
\]

One may ask if Corollary 1.3 holds for the limit case \( s = \varepsilon = 0 \), we can give a counterexample to show that \( \mathcal{Tr} : M^{1/q'}_{p,q} (\mathbb{R}^n) \not\to M^0_{p,q} (\mathbb{R}^{n-1}) \) in the case \( p, q > 1 \). Write
\[
s_p = (n-1)(1/(p \wedge 1) - 1).
\]
It is easy to see that \( s_p = 0 \) for \( p \geq 1 \) and \( s_p = (n-1)(1/p - 1) \) for \( p < 1 \). For the trace of \( \alpha \)-modulation spaces, we have the following result.

Theorem 1.4 Let \( n \geq 2, 0 < p, q \leq \infty, s \geq \alpha(n-1)/q + \alpha s_p \). Let \( \mathcal{Tr} \) be as in (1.3). Then
\[
\mathcal{Tr} : M^{s+\alpha/p}_{p,p\wedge q\wedge 1} (\mathbb{R}^n) \to M^s_{p,q} (\mathbb{R}^{n-1}) .
\]

The case \( s < \alpha(n-1)/q + \alpha s_p \) is more complicated. We have the following

Remark 1.5 Let \( n \geq 2, 0 < p, q \leq \infty, s < \alpha(n-1)/q + \alpha s_p \). Let \( \mathcal{Tr} \) be as in (1.3). Then
\[
\mathcal{Tr} : M^{s+\sigma_{\alpha,p,q}}_{p,p\wedge q\wedge 1} (\mathbb{R}^n) \to M^s_{p,q} (\mathbb{R}^{n-1}) ,
\]

where
\[
\sigma_{\alpha,p,q} = \begin{cases} 
\alpha/p + (1-\alpha)[\alpha(n-1)/q + \alpha s_p - s] , & q s + (n-1)(1-\alpha) - q \alpha s_p > 0 , \\
\alpha/p + \alpha s_p - s + \varepsilon , & q s + (n-1)(1-\alpha) - q \alpha s_p = 0 , \\
\alpha/p + \alpha s_p - s , & q s + (n-1)(1-\alpha) - q \alpha s_p < 0 . 
\end{cases}
\]

Theorem 1.3 is sharp in the case \( s \geq 0, p = q = 1 \). As the end of this paper, we consider the trace of Besov spaces. If \( s > s_p \), the corresponding result has been obtained in [8]. If \( s \leq s_p \), we have the following trace theorem for Besov spaces:

Theorem 1.6 Let \( n \geq 2, 0 < p, q \leq \infty, s \leq s_p \). Let \( \mathcal{Tr} \) be as in (1.3). Then we have
\[
\mathcal{Tr} : \tilde{B}^{s+1/p}_{p,p\wedge q\wedge 1} (\mathbb{R}^n) \to B^s_{p,q} (\mathbb{R}^{n-1}) ,
\]

and
\[
\mathcal{Tr} : B^{s+1/p}_{p,p\wedge q\wedge 1} (\mathbb{R}^n) \to B^s_{p,q} (\mathbb{R}^{n-1})
\]
in the case \( s < s_p \). Moreover, when \( 1 < p < \infty \), we have
\[
\mathcal{Tr} : \tilde{B}^{1/p,1/p}_{p,q\wedge 1} (\mathbb{R}^n) \to B^0_{p,q} (\mathbb{R}^{n-1}) ,
\]
\[
\mathcal{Tr} : B^{1/p,s+1/p}_{p,q\wedge 1} (\mathbb{R}^n) \to B^s_{p,q} (\mathbb{R}^{n-1}), \quad s < 0 .
\]
The following are some notations which will be frequently used in this paper: \( \mathbb{R}, \mathbb{N}, \text{and} \mathbb{Z} \) will stand for the sets of reals, positive integers and integers, respectively. \( \mathbb{R}_+ = [0, \infty), \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). \( c < 1, C > 1 \) will denote positive universal constants, which can be different at different places. \( a \lesssim b \) stands for \( a \leq Cb \) for some constant \( C > 1 \), \( a \sim b \) means that \( a \lesssim b \) and \( b \lesssim a \). We write \( a \wedge b = \min(a, b), a \vee b = \max(a, b) \). We denote by \( p' \) the dual number of \( p \in [1, \infty] \), i.e., \( 1/p + 1/p' = 1 \). We will use Lebesgue spaces \( L^p := L^p(\mathbb{R}^n), \| \cdot \|_p := \| \cdot \|_{L^p} \), We denote by \( \mathcal{S} := \mathcal{S}(\mathbb{R}^n) \) and \( \mathcal{S}' := \mathcal{S}'(\mathbb{R}^n) \) the Schwartz space and tempered distribution space, respectively. \( B(x, R) \) stands for the ball in \( \mathbb{R}^n \) with center \( x \) and radius \( R \), \( Q(x, R) \) denote the cube in \( \mathbb{R}^n \) with center \( x \) and side-length \( 2R \). \( \mathcal{F} \) or \( \mathcal{F}^{-1} \) denotes the Fourier transform; \( \mathcal{F}^{-1} \) denotes the inverse Fourier transform. For any set \( A \) with finite elements, we denote by \( \#A \) the number of the elements of \( A \).

2 \hspace{1cm} \alpha\text{-modulation spaces}

2.1 Definition

A countable set \( Q \) of subsets \( Q \subset \mathbb{R}^n \) is said to be an admissible covering if \( \mathbb{R}^n = \bigcup_{Q \in Q} Q \) and there exists \( n_0 < \infty \) such that \( \#\{Q' \in Q : Q \cap Q' \neq \emptyset\} \leq n_0 \) for all \( Q \in Q \). Denote

\[
\begin{align*}
    r_Q &= \sup\{r \in \mathbb{R} : B(c_r, r) \subset Q\}, \\
    R_Q &= \inf\{R \in \mathbb{R} : Q \subset B(c_R, R)\}.
\end{align*}
\]

(2.1)

Let \( 0 \leq \alpha \leq 1 \). An admissible covering is called an \( \alpha \)-covering of \( \mathbb{R}^n \), if \( |Q| \sim (x)^{\alpha n} \) (uniformly) holds for all \( Q \in Q \) and for all \( x \in Q \), and \( \sup_{Q \in Q} R_Q/r_Q \leq K \) for some \( K < \infty \).

Let \( Q \) be an \( \alpha \)-covering of \( \mathbb{R}^n \). A corresponding bounded admissible partition of unity of order \( p \) (\( p \)-BAPU) \( \{\psi_Q\}_{Q \in Q} \) is a family of smooth functions satisfying

\[
\begin{align*}
    &\psi_Q : \mathbb{R}^n \rightarrow [0, 1], \supp \psi_Q \subset Q, \\
    &\sum_{Q \in Q} \psi_Q(x) \equiv 1 \quad \forall x \in \mathbb{R}^n, \\
    &\sup_{Q \in Q} |Q|^{1/(p \wedge 1) - 1} \| \mathcal{F}^{-1} \psi_Q \|_{L^{p \wedge 1}} < \infty.
\end{align*}
\]

Definition 2.1 Let \( 0 < p, q, s \leq \infty, s \in \mathbb{R}, 0 \leq \alpha \leq 1 \). Let \( Q \) be an \( \alpha \)-covering of \( \mathbb{R}^n \) with the \( p \)-BAPU \( \{\psi_Q\}_{Q \in Q} \). We denote by \( M^{s, \alpha}_{p,q} \) the space of all tempered distributions \( f \) for which the following is finite:

\[
\|f\|_{M^{s, \alpha}_{p,q}} = \left( \sum_{Q \in Q} \langle \xi \rangle^{qs} \| \mathcal{F}^{-1} \psi_Q \mathcal{F} f \|_{L^p(\mathbb{R}^n)}^q \right)^{1/q},
\]

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where $\xi_Q \in Q$ is arbitrary. For $q = \infty$, we have a usual substitution for the $\ell^q$ norm with the $\ell^\infty$ norm.

We now give an exact equivalent norm on $M_{p,q}^{s,\alpha}$. Denote

$$Q_k = Q(|k|^{\frac{\alpha}{1-\alpha}} k, r \langle k \rangle^{\frac{\alpha}{1-\alpha}}), \quad k \in \mathbb{Z}^n.$$ 

It is known that, there exists a constant $r_1 > 0$ such that for any $r > r_1$, $\{Q_k\}_{k \in \mathbb{Z}^n}$ is an $\alpha$-covering of $\mathbb{R}^n$, i.e., $\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}^n} Q_k$ and there exists $n_0 \in \mathbb{N}$ such that $# \{l \in \mathbb{Z}^n : Q_k \cap Q_{k+l} \neq \emptyset\} \leq n_0$. Moreover, $|Q_k| \sim \langle k \rangle^{\frac{\alpha}{1-\alpha}}$. Let $\eta : \mathbb{R} \to [0,1]$ be a smooth bump function satisfying

$$\eta(\xi) := \begin{cases} 
1, & |\xi| \leq 1, \\
\text{smooth,} & 1 < |\xi| \leq 2, \\
0, & |\xi| > 2.
\end{cases} \quad (2.2)$$

We write for $k = (k_1, ..., k_n)$ and $\xi = (\xi_1, ..., \xi_n)$,

$$\phi_k(\xi) = \eta \left( \frac{\xi - |k|^{\frac{\alpha}{1-\alpha}} k}{r \langle k \rangle^{\frac{\alpha}{1-\alpha}}} \right).$$

Put

$$\psi_k(\xi) = \frac{\phi_k(\xi_1) ... \phi_k(\xi_n)}{\sum_{k \in \mathbb{Z}^n} \phi_k(\xi_1) ... \phi_k(\xi_n)}, \quad k \in \mathbb{Z}^n. \quad (2.3)$$

We have

**Lemma 2.2** Let $0 \leq \alpha < 1$, $0 < p \leq \infty$ and $\{\psi_k\}_{k \in \mathbb{Z}^n}$ be as in (2.3). Then $\{\psi_k\}_{k \in \mathbb{Z}^n}$ is a $p$-BAPU for $r > r_1$. In the case $\alpha = 0$, we can take $r_1 = 1/2$.

**Proposition 2.3** Let $0 \leq \alpha < 1$, $0 < p, q \leq \infty$, then

$$\|f\|_{M_{p,q}^{s,\alpha}} = \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{\frac{\alpha}{1-\alpha}} \|\mathcal{F}^{-1} \psi_k \mathcal{F} f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}$$

is an equivalent norm on $\alpha$-modulation space with the usual modification for $q = \infty$.

**Proof.** See [1].
2.2 Equivalent norm via a New $p$-BAPU

We now construct a new covering, which is of importance for the proof of Theorem 1.4. Let $j \in \mathbb{Z} \setminus \{0\}$. We divide $[-|j|^{\frac{1}{1-\alpha}}, |j|^{\frac{1}{1-\alpha}}]$ into $2|j|/\{r_1\} = 2N_j$ intervals with equal length:

$$[-|j|^{\frac{1}{1-\alpha}}, |j|^{\frac{1}{1-\alpha}}] = [r_{j-N_j}, r_{j-N_j+1}] \cup \ldots \cup [r_{j-N_j-1}, r_{j-N_j}].$$

Denote

$$\mathcal{R} = \{r_{j,s} : j \in \mathbb{N}, s = -N_j, \ldots, N_j\}.$$

We further write

$$\mathcal{K}_j^n = \{k = (k_1, \ldots, k_n) : k_i \in \mathcal{R}, \max_{1 \leq i \leq n} |k_i| = |j|^{\frac{1}{1-\alpha}}\}.$$

For any $k \in \mathcal{K}_j^n$, we write

$$Q_{kj} = Q(k, r|j|^{\frac{\alpha}{1-\alpha}}), \quad Q_{k0} = Q(0, 2).$$

We will write $\mathcal{K}_j = \mathcal{K}_j^n$ if there is no confusion.

**Proposition 2.4** There exists $r_1 > 0$ such that for any $r > r_1$, $\{Q_{kj}\}_{k \in \mathcal{K}_j, j \in \mathbb{Z}_+}$ is an $\alpha$-covering of $\mathbb{R}^n$.

**Proof.** Let $j \in \mathbb{N} \cup \{0\}$. We see that there exists $r_1 > 0$ such that for any $r > r_1$, $\{Q(|j|^{\frac{\alpha}{1-\alpha}}j, r|j|^{\frac{\alpha}{1-\alpha}})\}_j$ is an $\alpha$-covering of $\mathbb{R}$. Hence we easily see that

$$\mathbb{R} \subset \bigcup_{k \in \mathcal{K}_j, j \in \mathbb{Z}_+} Q_{kj}, \quad |Q_{kj}| \sim |j|^{\frac{\alpha}{1-\alpha}} \sim \langle \xi_{Q_{kj}} \rangle_{n_0}^{\alpha}, \forall \xi_{Q_{kj}} \in Q_{kj},$$

$$\#\{Q_{kj} : Q_{kj} \cap Q_{kj'} \neq \emptyset\} \leq n_0 < \infty.$$  

Now, on the basis of the $\alpha$-covering constructed above, we further construct a $p$-BAPU. Let $j$ be fixed. Denote for $i = 1, \ldots, n$,

$$\phi_{kj}(\xi_i) = \phi \left( \frac{\xi_i - k_i}{r(j)^{\frac{\alpha}{1-\alpha}}} \right), \quad k = (k_1, \ldots, k_n) \in \mathcal{K}_j.$$

$$\phi_{kj}(\xi) = \phi_{kj}(\xi_1) \ldots \phi_{kj}(\xi_n).$$

We put

$$\psi_{kj}(\xi) = \frac{\phi_{kj}(\xi)}{\sum_{k \in \mathcal{K}_j, j \in \mathbb{Z}_+} \phi_{kj}(\xi)}.$$

(2.4)
Figure 1: $\alpha$-covering, the case of $n = 2$, $\alpha = 1/2$, $r_1 = 1$.

**Proposition 2.5** Let $0 < p < \infty$, $\psi_{kj}$ be as in (2.4). Then $\{\psi_{kj}\}_{k \in \mathcal{K}_j, j \in \mathbb{Z}^+}$ is a $p$-BAPU.

Noticing that $|\xi| \sim |j|^{1/(1-\alpha)}$ if $\xi \in \mathcal{K}_j$, $j \neq 0$, we immediately have

**Proposition 2.6** Let $0 < \alpha < 1$, $0 < p, q \leq \infty$, then

$$
\|f\|_{M_{p,q,\alpha}^*} = \left( \sum_{j \in \mathbb{Z}^+} \sum_{k \in \mathcal{K}_j} \| \mathcal{F}^{-1} \psi_{kj} \mathcal{F} f \|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}
$$

is another equivalent norm on $\alpha$-modulation space.

**2.3 Modulation spaces**

In the case $\alpha = 0$, we get an equivalent norm on modulation spaces $M_{p,q}^*$:

$$
\|f\|_{M_{p,q}^*} = \left( \sum_{k \in \mathbb{Z}^n} \| \mathcal{F}^{-1} \psi_k \mathcal{F} f \|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}.
$$

(2.5)
The modulation spaces $M_{p,q}$ in the case $0 < p, q < 1$ was studied in [17, 18, 19] by using the norm $(2.5)$. Soon after, Kobayashi [11] independently considered such a generalization in the case $0 < p, q < 1$.

Recalling that $\bar{x} = (x_1, \ldots, x_{n-1})$, we also define the following anisotropic modulation spaces $M_{p,q,r}^s$ for which the norm is defined as

$$\|f\|_{M_{p,q,r}^s} = \left( \sum_{k_n \in \mathbb{Z}} \left( \sum_{\bar{k} \in \mathbb{Z}^{n-1}} \langle \bar{k} \rangle^{qs} \|\mathcal{F}^{-1} \psi_k \mathcal{F} f\|_{L^p_\mathbb{R}^n} \right)^{r/q} \right)^{1/r}.$$ 

This anisotropic version is of importance for the trace of modulation spaces.

### 2.4 Besov spaces

Write $\varphi(\cdot) = \eta(\cdot) - \eta(2\cdot)$ and $\varphi_k := \varphi(2^{-k} \cdot)$ for $k \geq 1$. $\varphi_0 := 1 - \sum_{k \geq 1} \varphi_k$. For simplicity, we write $\Delta_k = \mathcal{F}^{-1} \varphi_k \mathcal{F}$. The norm on Besov spaces $B_{p,q}^s(\mathbb{R}^n)$ are defined as follow:

$$\|f\|_{B_{p,q}^s} = \left( \sum_{j=0}^{\infty} 2^{sjq} \|\Delta_j f\|_{L^p_\mathbb{R}^n}^q \right)^{1/q}.$$ 

For our purpose, we also need the following

$$\tilde{B}_{p,q}^s(\mathbb{R}^n) = \left( \sum_{k=0}^{\infty} k\cdot 2^{skq} \|\Delta_k f\|_{L^p_\mathbb{R}^n}^q \right)^{1/q}.$$ 

In the case $1 < p < \infty$, using Lizorkin’s decomposition of $\mathbb{R}^n$, we have an equivalent quasi-norm on $B_{p,q}^s(\mathbb{R}^n)$. Let

$$K_k = \{x : |x_j| < 2^k, j = 1, 2, \ldots, n\} \setminus \{x : |x_j| < 2^{k-1}, j = 1, 2, \ldots, n\}$$

where $k \in \mathbb{Z}^+$ and

$$K_0 = \{x : |x_j| \leq 1, j = 1, 2, \ldots, n\}$$

Subdivide $K_k$ with $k = 1, 2, 3, \ldots$, by the $3n$ hyper-planes $\{x : x_m = 0\}$ and $\{x : x_m = \pm 2^{k-1}\}$, where $m = 1, \ldots, n$, into cubes $P_{k,t}$. If $k$ is fixed, we obtain $T = 4^n - 2^n$ cubes. The cubes near the $n$-th axis are numbered by $t = 1, \ldots, 2^n$ in an arbitrary way and the others are numbered by $t = 2^n + 1, \ldots, T$. Let $P_{0,t} = K_0$, if $t = 1, \ldots, T$. Then

$$\mathbb{R}^n = \bigcup_{k=0}^{\infty} K_k = \bigcup_{k=0}^{\infty} \bigcup_{t=1}^{T} P_{k,t}.$$
Let $\chi_{k,t}$ be a characteristic function on $P_{k,t}$. Then

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n)} \asymp \left( \sum_{k=0}^{\infty} \sum_{t=1}^{T} 2^{skq} \|\mathcal{F}^{-1}\chi_{k,t} \mathcal{F} f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}.$$  

We construct two new norms. For simplicity, we write

$$\Delta_{k,t} = \mathcal{F}^{-1}\chi_{k,t} \mathcal{F}.$$  

Define

$$B_{p,q}^{s_1,s_2} = \left( \sum_{k=0}^{\infty} \left( \sum_{t=1}^{2^n} 2^{s_1kq} \|\Delta_{k,t} f\|_{L^p(\mathbb{R}^n)}^q + \sum_{t=2^n+1}^{T} 2^{s_2kq} \|\Delta_{k,t} f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \right),$$

$$\tilde{B}_{p,q}^{s_1,s_2} = \left( \sum_{k=0}^{\infty} \left( \sum_{t=1}^{2^n} k2^{s_1kq} \|\Delta_{k,t} f\|_{L^p(\mathbb{R}^n)}^q + \sum_{t=2^n+1}^{T} 2^{s_2kq} \|\Delta_{k,t} f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \right).$$

3 Proof of Theorem 1.2

If there is no explanation, we always assume $r = 1/2$ in the $p$-BAPU for the case of modulation spaces. To show our main theorem, we will use the following
Lemma 3.1 (Triebel, [14]) Let $\Omega$ be a compact subset of $\mathbb{R}^n$ and $0 < p \leq \infty$. Denote $L^p_\Omega = \{ f \in L^p : \text{Supp} \hat{f} \subset \Omega \}$. Let $0 < r < p$. Then
\[
\left\| \sup_{z \in \mathbb{R}^n} \frac{f(\cdot - z)}{1 + |z|^{n/r}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \| f \|_{L^p(\mathbb{R}^n)},
\]
holds for any $f \in L^p_\Omega$.

Assume that $\text{Supp} \hat{f} \subset B(\xi_0, R)$. It is easy to see that for $g = e^{ix\xi_0} f(R^{-1})$, $\hat{g} = R^n \hat{f}(R(\xi - \xi_0))$. It follows that $\text{Supp} \hat{g} \subset B(0, 1)$. Taking $\Omega = B(0, 1)$ in Lemma 3.1, we find that
\[
\left\| \sup_{z \in \mathbb{R}^n} g(\cdot - z) \right\|_{L^p(\mathbb{R}^n)} \lesssim \| g \|_{L^p(\mathbb{R}^n)},
\]
By scaling, we have
\[
\left\| \sup_{z \in \mathbb{R}^n} \frac{f(\cdot - z)}{1 + |Rz|^{n/r}} \right\|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)} \tag{3.1}
\]
Note that the constant $C$ in (3.1) is independent of $f \in L^p_{B(\xi_0, R)} = \{ f \in L^p : \text{Supp} \hat{f} \subset B(\xi_0, R) \}$. It is also independent of $\xi_0 \in \mathbb{R}^n$.

For convenience, we write
\[
\Box_k = \mathcal{F}^{-1} \psi_k \mathcal{F}, \quad k \in \mathbb{Z}^n.
\]

We define the maximum function $M^*_k f$ as follows:
\[
M^*_k f = \sup_{y \in \mathbb{R}^n} \frac{\Box_k f(x - y)}{1 + |y|^{n/r}}. \tag{3.2}
\]

Taking $y_1 = \ldots = y_{n-1} = 0$, $y_n = x_n$ in (3.2), we have for $|x_n| \leq 1$,
\[
|(\Box_k f)(\bar{x}, 0)| \lesssim |M^*_k f(x)|, \quad \bar{x} = (x_1, \ldots, x_{n-1})
\]
Hence
\[
\|(\Box_k f)(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \lesssim \|M^*_k f(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})}, \tag{3.3}
\]
Integrating (3.3) over $x_n \in [0, 1]$, one has that
\[
\|(\Box_k f)(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \lesssim \int_{\mathbb{R}} \|M^*_k f(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})}^p dx_n,
\]
Hence
\[
\|(\Box_k f)(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \lesssim \|M^*_k f\|_{L^p(\mathbb{R}^n)}. \tag{3.4}
\]
We denote by \( \mathcal{F}_\xi (\mathcal{F}_\xi^{-1}) \) the partial (inverse) Fourier transform on \( \bar{x} (\bar{\xi}) \). Write \( \psi_k(\bar{x}) \) as the \( p \)-BAPU functions in \( \mathbb{R}^{n-1} \) as in (2.3), i.e.,

\[
\psi_k(\bar{\xi}) = \frac{\phi_{k_1}(\xi_1) \cdots \phi_{k_{n-1}}(\xi_{n-1})}{\sum_{\bar{k} \in \mathbb{Z}^{n-1}} \phi_{k_1}(\xi_1) \cdots \phi_{k_{n-1}}(\xi_{n-1})}, \quad \bar{k} \in \mathbb{Z}^{n-1}.
\]  

(3.5)

Then we have

\[
(\mathcal{F}_\xi^{-1} \psi_k \mathcal{F}_\xi)(\bar{x}, 0) = \sum_{\bar{l} \in \mathbb{Z}^n} (\mathcal{F}_\xi^{-1} \psi_k \mathcal{F}_\xi^{-1} \psi_l \mathcal{F}) (\bar{x}, 0) = \sum_{\bar{l} \in \mathbb{Z}^n} (\mathcal{F}_\xi^{-1} \psi_k) \ast (\mathcal{F}_\xi^{-1} \mathcal{F}_\xi \mathcal{F}_l) (\bar{x}, 0)
\]

From the support property of \( \psi_l \) as in (2.3), we find that

\[
\psi_k \psi_l = 0, \quad \text{if } |\bar{l} - \bar{k}| \geq C.
\]

Hence

\[
(\mathcal{F}_\xi^{-1} \psi_k \mathcal{F}_\xi)(\bar{x}, 0) = \sum_{\bar{l} \in \mathbb{Z}^n, |\bar{l} - \bar{k}| \leq C} (\mathcal{F}_\xi^{-1} \psi_k) \ast ((\mathcal{F}_\xi^{-1} \mathcal{F}_\xi \mathcal{F}_l) (\cdot, 0))
\]

Case 1. \( 1 \leq p \leq \infty \). Using Young’s inequality, (3.1) and (3.4), we obtain

\[
\| \mathcal{F}_\xi^{-1} \psi_k \mathcal{F}_\xi f (\bar{x}, 0) \|_{L^p(\mathbb{R}^{n-1})} \lesssim \sum_{\bar{l} \in \mathbb{Z}^n, |\bar{l} - \bar{k}| \leq C} \| \mathcal{F}_\xi^{-1} \psi_k \|_{L^1(\mathbb{R}^{n-1})} \| \mathcal{F}_\xi^{-1} \psi_l \mathcal{F}_\xi f \|_{L^p(\mathbb{R}^{n-1})} \lesssim \sum_{\bar{l} \in \mathbb{Z}^n, |\bar{l} - \bar{k}| \leq C} \| M^*_{l} f \|_{L^p(\mathbb{R}^n)} \lesssim \sum_{\bar{l} \in \mathbb{Z}^n, |\bar{l} - \bar{k}| \leq C} \| \Box_l f \|_{L^p(\mathbb{R}^n)}.
\]

Hence,

\[
\| f (\bar{x}, 0) \|_{M^{p,q}_{\Box} (\mathbb{R}^{n-1})} \lesssim \left( \sum_{\bar{k} \in \mathbb{Z}^{n-1}} \langle \bar{k} \rangle^{sq} \left( \sum_{\bar{l} \in \mathbb{Z}^n, |\bar{l} - \bar{k}| \leq C} \| \Box_l f \|_{L^p(\mathbb{R}^n)} \right)^q \right)^{1/q}.
\]

If \( 0 < q \leq 1 \), then

\[
\| f (\bar{x}, 0) \|_{M^{p,q}_{\Box} (\mathbb{R}^{n-1})} \lesssim \left( \sum_{\bar{l} \in \mathbb{Z}^n} \sum_{\bar{k} \in \mathbb{Z}^{n-1}} \langle \bar{k} \rangle^{sq} \chi_{(|\bar{l} - \bar{k}| \leq C)} \| \Box_l f \|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \lesssim \left( \sum_{\bar{l} \in \mathbb{Z}^n} \| \Box_l f \|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} = \| f \|_{M^{p,q}_{\Box} (\mathbb{R}^n)}.
\]
If $1 \leq q \leq \infty$, using Minkowski’s inequality together with Hölder’s inequality,

$$\|f(\cdot, 0)\|_{M^p_{q,1}(\mathbb{R}^{n-1})} \lesssim \left( \sum_{\vec{k}, \vec{l} \in \mathbb{Z}^{n-1}} \left( \sum_{k \in \mathbb{Z}} \chi_{(|\vec{k} - \vec{l}| \leq C)} \|\Box_l f\|_{L^p(\mathbb{R}^n)} \right)^q \right)^{1/q} \lesssim \sum_{k \in \mathbb{Z}} \left( \sum_{l \in \mathbb{Z}^{n-1}} \langle \vec{k} \rangle^q \chi_{(|\vec{k} - \vec{l}| \leq C)} \|\Box_l f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} = \|f\|_{M^p_{q,1}}.$$ 

To begin with the proof for the case $0 < p < 1$, we need the following lemma:

**Lemma 3.2** Let $0 < p \leq 1$. Suppose that $f, g \in L^p_{B(x_0, R)}$, then there exists a constant $C > 0$ which is independent of $x_0 \in \mathbb{R}^n$ and $R > 0$ such that

$$\|f * g\|_p \leq CR^{n(\frac{1}{p} - 1)}\|f\|_p \|g\|_p.$$ 

**Proof.** In the case $f, g \in L^p_{B(0, 1)}$, we have

$$\|f * g\|_p \lesssim \|f\|_p \|g\|_p.$$ 

Taking $f_\lambda = f(\lambda \cdot)$ and $g_\lambda = g(\lambda \cdot)$, we see that

$$f_{R^{-1}}, g_{R^{-1}} \in L^p_{B(0, 1)}, \quad \text{if } f, g \in L^p_{B(0, R)}.$$ 

Hence, for any $f, g \in L^p_{B(0, R)}$,

$$\|f_{R^{-1}} * g_{R^{-1}}\|_p \lesssim \|f_{R^{-1}}\|_p \|g_{R^{-1}}\|_p.$$ 

By scaling, we have

$$\|f * g\|_p \lesssim R^{n(\frac{1}{p} - 1)}\|f\|_p \|g\|_p.$$ 

By a translation $e^{2\pi x_0 \xi} f = \hat{f}(\xi - x_0)$, we immediately have the result, as desired. \[\square\]

**Case 2.** $0 < p < 1$. By Lemma 3.2, (3.1) and (3.4),

$$\|\mathcal{F}_\xi^{-1} \psi_k \mathcal{F}_x f(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \lesssim \sum_{l \in \mathbb{Z}^{n-1}, |\vec{k} - \vec{l}| \leq C} \|\mathcal{F}_\xi^{-1} \psi_k\|_{L^p(\mathbb{R}^{n-1})} \|\mathcal{F}_\xi^{-1} \psi_l \mathcal{F}_x f(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})}.$$ 

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\[
\lesssim \sum_{l \in \mathbb{Z}^n, |k-l| \leq C} \| (\mathcal{F}^{-1} \psi_l \mathcal{F} f)(\cdot, 0) \|_{L^p(\mathbb{R}^{n-1})}^p \lesssim \sum_{l \in \mathbb{Z}^n, |k-l| \leq C} \| \Box_t f \|_{L^p(\mathbb{R}^n)}^p.
\]

It follows that
\[
\| f(\bar{x}, 0) \|_{M^{s,0}_{p,q}} \lesssim \left( \sum_{k \in \mathbb{Z}^n} \langle \vec{k} \rangle^{sq} \left( \sum_{l \in \mathbb{Z}^n} \| \Box_t f \|_{L^p(\mathbb{R}^n)}^q \chi(|k-l| \leq C) \right) \right)^{q/p} \frac{1}{q}
\]

If \( q \leq p \), one has that
\[
\| f(\bar{x}, 0) \|_{M^{s,0}_{p,q}(\mathbb{R}^{n-1})} \lesssim \left( \sum_{l \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} \langle \vec{k} \rangle^{sq} \| \Box_t f \|_{L^p(\mathbb{R}^n)}^q \chi(|k-l| \leq C) \right)^{1/q}
\]
\[
\lesssim \| f \|_{M^{s,0}_{p,q}}.
\]

If \( q \geq p \), using Minkowski’s inequality, we have
\[
\| f(\bar{x}, 0) \|_{M^{s,0}_{p,q}(\mathbb{R}^{n-1})} \lesssim \left( \sum_{l \in \mathbb{Z}^n} \left( \sum_{k \in \mathbb{Z}^n} \langle \vec{k} \rangle^{sq} \| \Box_t f \|_{L^p(\mathbb{R}^n)}^q \chi(|k-l| \leq C) \right) \right)^{p/q} \frac{1}{p}
\]
\[
\lesssim \left( \sum_{l \in \mathbb{Z}^n} \left( \sum_{k \in \mathbb{Z}^n} \langle \vec{k} \rangle^{sq} \| \Box_t f \|_{L^p(\mathbb{R}^n)}^q \right) \right)^{1/p}
\]
\[
= \| f \|_{M^{s,0}_{p,q,p}}.
\]

In order to show \( \mathcal{T} \mathcal{R} \) is a retraction, we need to show the existence of \( \mathcal{T} \mathcal{R}^{-1} \). Let \( \eta \) be as in (2.2) satisfying \( \langle \mathcal{F}^{-1} \eta \rangle(0) = 1 \). For any \( f \in M^{s,0}_{p,q}(\mathbb{R}^{n-1}) \), we define
\[
g(x) = [(\mathcal{F}^{-1} \eta)(x_n)] f(\bar{x}) := (\mathcal{T} \mathcal{R}^{-1} f)(x).
\]

It is easy to see that \( g(\bar{x}, 0) = f(\bar{x}) \) and \( \Box_k g = 0 \) for \( |k_n| \geq 3 \). Hence,
\[
\| g \|_{M^{s,0}_{p,q,\wedge q \land 1}(\mathbb{R}^n)} \lesssim \left( \sum_{k_n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}^{n-1}} \langle \vec{k} \rangle^{sq} \| \Box_k g \|_{L^p(\mathbb{R}^n)}^q \right) \right)^{p \wedge q \land 1} \frac{1}{p \wedge q \land 1}
\]
\[
= \sum_{|k_n| \leq 2} \left( \sum_{k \in \mathbb{Z}^{n-1}} \langle \vec{k} \rangle^{sq} \| \Box_k f \|_{L^p(\mathbb{R}^n)}^q \| \mathcal{F}^{-1} \eta \|_{L^p(\mathbb{R})}^q \right)^{1/q}
\]

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\[ \lesssim \|f\|_{M^s_{p,q}(\mathbb{R}^{n-1})}. \]

It follows that \( \text{Tr}^{-1} : M^s_{p,q}(\mathbb{R}^{n-1}) \to M^s_{p,q,p\wedge 1}(\mathbb{R}^{n}) \). \[ \square \]

As the end of this section, we show that Theorem 1.2 and Corollary 1.3 are sharp conclusions. First, we show that \( \text{Tr} : M^0_{p,q,r}(\mathbb{R}^{n}) \nrightarrow M^0_{p,q}(\mathbb{R}^{n-1}) \) if \( r > 1 \). Let \( \eta \) be as in \((2.2)\), \( f = \mathcal{F}^{-1}(\eta(2\xi_1)\ldots\eta(2\xi_n)) \). For \( k = (k_1, \ldots, k_n) \), we denote

\[ F(x) = \sum_{|k_n| \leq 2^N} \langle k_n \rangle^{-1} e^{i k_n x} f(x). \]

It is easy to see that \( \mathcal{F} F(\xi) = \sum_{|k_n| \leq 2^N} \langle k_n \rangle^{-1} \eta(2\xi_1)\ldots\eta(2(\xi_n - k_n)). \)

Hence, \( \Box_k F = 0 \) if \( \max_{i=1,\ldots,n-1} |k_i| > 2 \) or \( |k_n| > 2^N + 1 \). In view of the definition

\[ \|F\|_{M^0_{p,q,r}(\mathbb{R}^{n})} \lesssim \left( \sum_{|k_n| \leq 2^N + 1} \left( \sum_{|k_i| \leq 2, 1 \leq i \leq n-1} \|\Box_k F\|_{L^p(\mathbb{R}^{n})}^q \right)^{r/q} \right)^{1/r} \]

\[ \lesssim \sum_{|k_i| \leq 2, 1 \leq i \leq n-1} \left( \sum_{|k_n| \leq 2^N + 1} \|\Box_k F\|_{L^p(\mathbb{R}^{n})}^r \right)^{1/r} \]

\[ \lesssim \left( \sum_{|k_n| \leq 2^N + 1} \langle k_n \rangle^{-r} \right)^{1/r} \lesssim 1. \]

On the other hand, we may assume that \( (\mathcal{F}_{\xi}^{-1}\eta(2\cdot))(0) = 1 \). We have

\[ F(\bar{x},0) = \left( \sum_{|k_n| \leq 2^N} \langle k_n \rangle^{-1} \right) \mathcal{F}_{\xi}^{-1}[\eta(2\xi_1)\ldots\eta(2(\xi_n - 1))]. \]

So,

\[ \|F\|_{M^0_{p,q}(\mathbb{R}^{n-1})} \gtrsim \left( \sum_{|k_n| \leq 2^N} \langle k_n \rangle^{-1} \right) \left( \sum_{|k_i| \leq 2, 1 \leq i \leq n-1} \|\mathcal{F}_{\xi}^{-1}\psi_\xi \eta(2\xi_1)\ldots\eta(2\xi_{n-1})\|_{L^p(\mathbb{R}^{n-1})}^q \right)^{1/q} \]

\[ \gtrsim N. \]

Let \( N \to \infty \), we have \( \text{Tr} : M^0_{p,q,r}(\mathbb{R}^{n}) \nrightarrow M^0_{p,q}(\mathbb{R}^{n-1}) \).
Next, we show that $T_{\frac{1}{D}}: M^{1/q'}_{p,q}(\mathbb{R}^n) \not\rightarrow M^0_{p,q}(\mathbb{R}^{n-1})$ as $q > 1$. For $k = (k_1, \ldots, k_n)$, we denote
\[ F(x) = \sum_{|k_n| \leq 2^N} \frac{1}{\langle k_n \rangle \ln \langle k_n \rangle} e^{ik_n x_n} f(x). \]
Similarly as in the above, we have
\[
\|F\|_{M^{1/q'}_{p,q}(\mathbb{R}^n)} \lesssim \left( \sum_{|k_n| \leq 2^{N+1}} \sum_{|k_i| \leq 2, 1 \leq i \leq n-1} \langle k_n \rangle^{q-1} \|\Box_k F\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \lesssim \left( \sum_{|k_n| \leq 2^{N+1}} \frac{1}{\langle k_n \rangle \ln \langle k_n \rangle} \right)^{1/q} \lesssim 1.
\]
On the other hand,
\[
\|F\|_{M^0_{p,q}(\mathbb{R}^{n-1})} \gtrsim \left( \sum_{|k_n| \leq 2^N} \frac{1}{\langle k_n \rangle \ln \langle k_n \rangle} \right) \rightarrow \infty, \quad N \rightarrow \infty.
\]

4 Proof of Theorem 1.4 and Remark 1.5

For convenience, we write
\[
\Box_{k,j}^\alpha = \mathcal{F}^{-1} \psi_{k,j} \mathcal{F}, \quad k \in \mathcal{K}_j, \; j \in \mathbb{Z}_+.
\]
We define the maximum function $M^*_{k,j} f$ as follows:
\[
M^*_{k,j} f = \sup_{y \in \mathbb{Z}^n} \frac{|\Box_{k,j}^\alpha f(x - y)|}{1 + |(j)^{\alpha/(1-\alpha)} y|^n/r}. \tag{4.1}
\]
Taking $y_1 = \ldots = y_{n-1} = 0$, $y_n = x_n$ in (4.1), we have for $(j)^{-\alpha/(1-\alpha)} \leq |x_n| \leq 2(j)^{-\alpha/(1-\alpha)}$,
\[
|(\Box_{k,j}^\alpha f)(\bar{x},0)| \lesssim |M^*_{k,j} f(x)|, \quad \bar{x} = (x_1, \ldots, x_{n-1}).
\]
Hence
\[
\|((\Box_{k,j}^\alpha f)\cdot,0)\|_{L^p(\mathbb{R}^{n-1})} \lesssim \|M^*_{k,j} f(\cdot,x_n)\|_{L^p(\mathbb{R}^{n-1})}, \tag{4.2}
\]
Integrating (4.2) over $x_n \in [(j)^{-\alpha/(1-\alpha)}, 2(j)^{-\alpha/(1-\alpha)}]$, one has that
\[
\|((\Box_{k,j}^\alpha f)\cdot,0)^p\|_{L^p(\mathbb{R}^{n-1})} \lesssim (j)^{\alpha/(1-\alpha)} \int_{\mathbb{R}} \|M^*_{k,j} f(\cdot,x_n)\|_{L^p(\mathbb{R}^{n-1})}^p dx_n.
\]
Hence
\[ \| (\Box_{k,j}^\alpha f) (\cdot, 0) \|_{L^p(\mathbb{R}^{n-1})} \lesssim (j)^{\alpha/p(1-\alpha)} \| M_{k,j}^\alpha f \|_{L^p(\mathbb{R}^n)}. \] (4.3)

We denote by \( \mathcal{F}_x (\mathcal{F}_x^{-1}) \) the partial (inverse) Fourier transform on \( \bar{x} = (x_1, \ldots, x_{n-1}) \) \( (\bar{\xi} = (\xi_1, \ldots, \xi_{n-1})) \). Write \( \psi_{m,l}(\bar{x}) \) as the \( p \)-BAPU functions in \( \mathbb{R}^{n-1} \) as in (2.4). So, by the definition,
\[ \| f \|_{M_{p,q}^{\alpha}(\mathbb{R}^{n-1})} = \left( \sum_{l \in \mathbb{Z}^n} \sum_{m \in \mathcal{X}_l^{n-1}} \langle l \rangle^{\frac{aq}{p}} \| \mathcal{F}_x^{-1} \psi_{m,l}(\bar{x}) \mathcal{F}_x f \|_{L^q(\mathbb{R}^{n-1})} \right)^{1/q}. \] (4.4)

In order to have no confusion, we always denote by \( \psi_{m,l} \) the \( p \)-BAPU function in \( \mathbb{R}^{n-1} \) and by \( \psi_{k,j} \) the \( p \)-BAPU function in \( \mathbb{R}^n \). From the support property of \( \psi_{k,j} \), we find that
\[ (\mathcal{F}_x^{-1} \psi_{m,l} \mathcal{F}_x f)(\bar{x}, 0) = \sum_{j \geq |l-C, k \in \mathcal{X}_j^n} (\mathcal{F}_x^{-1} \psi_{m,l} \mathcal{F}_x^{-1} \psi_{k,j} \mathcal{F} f)(\bar{x}, 0). \] (4.5)

For our purpose we further decompose \( \mathcal{X}_j^n \). Denote
\[ \mathcal{X}_{j,\lambda} = \{ k \in \mathcal{X}_j^n : \max_{1 \leq i \leq n-1} | k_i | = \lambda \}, \quad \lambda = r_{j0}, r_{j1}, \ldots, r_{jN_j}, \quad r_{j0} = 0. \]

We easily see that \( \sum_{k \in \mathcal{X}_j^n} = \sum_{\lambda=0}^{r_{j1} \ldots r_{jN_j}} \sum_{\lambda \in \mathcal{X}_{j,\lambda}^n}, N_j \sim \langle j \rangle \). Now we divide our discussion into the following four cases.

**Case 1.** \( 1 \leq p \leq \infty \) and \( 0 < q \leq 1 \). By (4.4) and (4.5),
\[ \| f (\cdot, 0) \|_{M_{p,q}^{\alpha}(\mathbb{R}^{n-1})} \lesssim \sum_{j \in \mathbb{Z}^n} \sum_{\lambda=0}^{r_{j1}, \ldots, r_{jN_j}} \sum_{k \in \mathcal{X}_{j,\lambda}^n} \sum_{l \leq j+C} \sum_{m \in \mathcal{X}_l^{n-1}} \langle l \rangle^{\frac{aq}{p}} \times \| (\mathcal{F}_x^{-1} \psi_{m,l}) * (\mathcal{F}_x^{-1} \psi_{k,j} \mathcal{F} f)(\cdot, 0) \|_{L^p(\mathbb{R}^n)}. \] (4.6)

In order to control (4.6) by \( \| f \|_{M_{p,q}^{\alpha/p,\alpha}} \), we need to bound the sum \( \sum_{l \leq j+C} \sum_{m \in \mathcal{X}_l^{n-1}} \).

It is easy to see that for fixed \( k, j \),
\[ \# \{ m \in \mathcal{X}_l^{n-1} : \text{supp } \psi_{m,l} \cap \text{supp } \psi_{k,j} (\cdot, 0) \neq \emptyset \} \lesssim \min \left( \langle l \rangle^{n-2}, \langle j \rangle^{\alpha/n-2} \right) \left( \langle l \rangle^{\alpha/n-1}/\langle l \rangle^{\alpha/n-1} \right). \] (4.7)

Moreover, \( k \in \mathcal{X}_{j,\lambda}^{n-1} \) means that \( \text{supp } \psi_{m,l} \cap \text{supp } \psi_{k,j} (\cdot, 0) \neq \emptyset \) only if \( a^{1-\alpha} (\langle j \rangle^{\alpha} \lesssim l \lesssim (1+a)^{1-\alpha} \langle j \rangle^{\alpha} \). Hence, in view of Young’s inequality, (4.3),
\[ \Delta_{ja} := \sum_{k \in \mathcal{X}_{j,\lambda}^{n-1}} \sum_{m \in \mathcal{X}_l^{n-1}} \langle l \rangle^{\frac{aq}{p}} \| (\mathcal{F}_x^{-1} \psi_{m,l}) * (\mathcal{F}_x^{-1} \psi_{k,j} \mathcal{F} f)(\cdot, 0) \|_{L^p(\mathbb{R}^n)}^q \]
We discuss the following four subcases.

Case 1A. \( \alpha(n - 1) \leqs q s \). If \( a = 0 \), one has that

\[
\Delta_{ja} \les \sum_{k \in \mathcal{X}_{j,ja}} \sum_{0 \le l \le (j)^{a}} \langle l \rangle^{\frac{aq}{p} + (a - 2)} \langle j \rangle^{\frac{aq}{p(1 - a)} + \frac{a(n - 2)}{1 - a}} \| \Box_{k,j} f \|_{L^p(R^n)}^q
\]

\[
\les \sum_{k \in \mathcal{X}_{j,ja}} \langle j \rangle^{\frac{aq}{p} + (a - 2)} \langle n^{a + \frac{a(n - 1)}{2}} \rangle \| \Box_{k,j} f \|_{L^p(R^n)}^q
\]

(4.9)

If \( a \geq 1 \), we have

\[
\Delta_{ja} \les \sum_{k \in \mathcal{X}_{j,ja}} \sum_{a1 - \alpha(j)^{a} \le l \le (1 + a)1 - \alpha(j)^{a}} \langle l \rangle^{\frac{aq}{p} + (a - 2)} \langle j \rangle^{\frac{aq}{p(1 - a)} + \frac{a(n - 2)}{1 - a}} \| \Box_{k,j} f \|_{L^p(R^n)}^q
\]

\[
\les \sum_{k \in \mathcal{X}_{j,ja}} a^{q_{s} - a(n - 1)} \langle j \rangle^{\frac{aq}{p} + (a - 2)} \langle n^{a + \frac{a(n - 1)}{2}} \rangle \| \Box_{k,j} f \|_{L^p(R^n)}^q.
\]

(4.10)

It follows from \( qs \geq \alpha(n - 1) \) that \( \Delta_{ja} \) takes the maximal value as \( a = N_j \sim \langle j \rangle \).

Hence,

\[
\Delta_{ja} \les \sum_{k \in \mathcal{X}_{j,ra}} \langle j \rangle^{\frac{aq}{p} + (s + \frac{a}{p})} \| \Box_{k,j} f \|_{L^p(R^n)}^q.
\]

(4.11)

Inserting the estimates of \( \Delta_{ja} \) as in (4.9) and (4.11) into (4.6), we have

\[
\| f(\cdot,0) \|_{M_{p,q}^s(R^n)}^{q_{s} - a(n - 1)} \les \sum_{j \in \mathbb{Z}_{+}} \sum_{k \in \mathcal{X}_{j,n}} \langle j \rangle^{\frac{aq}{p} + (s + \frac{a}{p})} \| \Box_{k,j} f \|_{L^p(R^n)}^q = \| f \|_{M_{p,q}^{s + a/p,\alpha}}^q.
\]

Case 1B. \( \alpha(n - 1) > qs \) and \( q s + (1 - \alpha)(n - 1) > 0 \). If \( a \geq 1 \), from (4.10) and \( \alpha(n - 1) > qs \) we have

\[
\Delta_{ja} \les \sum_{k \in \mathcal{X}_{j,ra}} \langle j \rangle^{\frac{aq}{p} + (a - 2)} \langle j \rangle^{\frac{aq}{p(1 - a)} + \frac{a(n - 2)}{1 - a}} \| \Box_{k,j} f \|_{L^p(R^n)}^q.
\]

(4.12)
Since $qs + (1 - \alpha)(n - 1) > 0$, similar to (4.9), we see that (4.12) also holds for the case $a = 0$. It follows that

$$\|f(\cdot, 0)\|_{M_{p,q}^{r,0}(\mathbb{R}^{n-1})} \lesssim \|f\|_{M_{p,q}^{\alpha + (1 - \alpha)\frac{n(n-1)}{q} + \alpha/p, \alpha}(\mathbb{R}^n)}.$$  

**Case 1C.** $qs = -(n - 1)(1 - \alpha)$. Using the first estimate as in (4.9), we have for $a = 0$,

$$\Delta_j a \lesssim \sum_{k \in \mathcal{X}_{j,r}^n} \langle j \rangle^{\frac{qa}{1 - \alpha} + \frac{q\alpha}{p(1 - \alpha)} + \alpha(n - 1)} \|\Box_k^a f\|_{L^p(\mathbb{R}^n)}^q. \quad (4.13)$$

For $a \geq 1$,

$$\Delta_j a \lesssim \sum_{k \in \mathcal{X}_{j,r}^n} \langle j \rangle^{\frac{qa}{1 - \alpha} + \frac{q\alpha}{p(1 - \alpha)} + \alpha(n - 1)} \|\Box_k^a f\|_{L^p(\mathbb{R}^n)}^q. \quad (4.14)$$

This implies that

$$\|f(\cdot, 0)\|_{M_{p,q}^{r,0}(\mathbb{R}^{n-1})} \lesssim \|f\|_{M_{p,q}^{r,0}(\mathbb{R}^{n-1})}.$$  

**Case 1D.** $qs < -(n - 1)(1 - \alpha)$. It is easy to see that $\ln(j)$ can be removed in (4.13). So, we have the result, as desired.

**Case 2.** $1 \leq p, q \leq \infty$. Using Minkowski’s inequality, we have

\[
\begin{align*}
\|f(\cdot, 0)\|_{M_{p,q}^{r,0}(\mathbb{R}^{n-1})} & \lesssim \left( \sum_{l \in \mathbb{Z}_+} \sum_{m \in \mathcal{X}_l^{n-1}} \langle l \rangle^{\frac{qa}{1 - \alpha}} \left( \sum_{j \in \mathbb{Z}_+, k \in \mathcal{X}_j^n} \|(\mathcal{F}_l^{-1} \psi_{m,l}) * (\mathcal{F}_l^{-1} \psi_{k,j} \mathcal{F} f)(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \right) \right)^{1/q} \\
& \lesssim \left( \sum_{j \in \mathbb{Z}_+} \sum_{a=0}^{N_j} \sum_{k \in \mathcal{X}_{j,r}^n} \left( \sum_{l \in \mathbb{Z}_+} \sum_{m \in \mathcal{X}_l^{n-1}} \langle l \rangle^{\frac{qa}{1 - \alpha}} \|(\mathcal{F}_l^{-1} \psi_{m,l}) * (\mathcal{F}_l^{-1} \psi_{k,j} \mathcal{F} f)(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \right)^q \right)^{1/q}.
\end{align*}
\]

(4.15)

Using the same way as in Case 1, we can get that for any $k \in \mathcal{X}_{j,r}^n$,

\[
\begin{align*}
\sum_{l \in \mathbb{Z}_+} \sum_{m \in \mathcal{X}_l^{n-1}} \langle l \rangle^{\frac{qa}{1 - \alpha}} \|(\mathcal{F}_l^{-1} \psi_{m,l}) * (\mathcal{F}_l^{-1} \psi_{k,j} \mathcal{F} f)(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})}^q & \lesssim \sum_{a \geq (1 + \alpha)j} \langle l \rangle^{\frac{qa}{1 - \alpha}} \langle j \rangle^{\frac{q\alpha}{p(1 - \alpha) - \alpha(1 - \alpha)}} \langle j \rangle^{\frac{q\alpha}{p(1 - \alpha) - \alpha(1 - \alpha)}} \langle j \rangle \langle l \rangle^{\frac{qa}{1 - \alpha}} \|\Box_k^a f\|_{L^p(\mathbb{R}^n)}^q.
\end{align*}
\]

(4.16)
For convenience, we write

\[
\|F^{-1}\psi_{m,l}F f(\cdot,0)\|_{L^p_{\mathbb{R}^n-1}}^p \lesssim \sum_{j \geq l-C, k \in \mathcal{X}_j^n} \|F^{-1}\psi_{m,l}F^{-1}\psi_{k,j}F f(\cdot,0)\|_{L^p_{\mathbb{R}^n-1}}^p.
\]

(4.17)

It follows that

\[
\|f(\cdot,0)\|_{M^{a,j}_{p,q}'(\mathbb{R}^{n-1})}
\lesssim \left( \sum_{l \in \mathbb{Z}^+} \sum_{m \in \mathcal{X}_l^{n-1}} \|l\|^{1-\alpha} \sum_{j \in \mathbb{Z}^+} \sum_{k \in \mathcal{X}_j^n} \|F^{-1}\psi_{m,l} \ast (F^{-1}\psi_{k,j}F f)(\cdot,0)\|_{L^p_{\mathbb{R}^n-1}}^p \right)^{1/q}.
\]

For convenience, we write

\[
\Upsilon_{ja} := \sum_{k \in \mathcal{X}_j^n, l \leq j+C, m \in \mathcal{X}_l^{n-1}} \sum_{r \neq j} \sum_{\alpha \leq a} \langle l \rangle^{\frac{sq}{1-\alpha}} \|F^{-1}\psi_{m,l} \ast (F^{-1}\psi_{k,j}F f)(\cdot,0)\|_{L^p_{\mathbb{R}^n-1}}^q.
\]

(4.19)

It follows from Lemma 3.2, the property of $p$-BAPU and (4.17) that

\[
\Upsilon_{ja} \lesssim \sum_{k \in \mathcal{X}_j^n, r \neq j} a^{1-\alpha} \langle r \rangle^{\alpha} \sum_{\alpha \leq a} \sum_{m \in \mathcal{X}_l^{n-1}} \langle l \rangle^{\frac{sq}{1-\alpha}} \|F^{-1}\psi_{m,l}\|_{L^p_{\mathbb{R}^{n-1}}}^q \|F^{-1}\psi_{k,j}F f(\cdot,0)\|_{L^p_{\mathbb{R}^n-1}}^q
\]

\[
\times \min \left( \langle l \rangle^{n-2}, \langle j \rangle^{\frac{\alpha(a-n)}{1-\alpha}}, \langle l \rangle^{\frac{\alpha(n-2)}{1-\alpha}}, \langle j \rangle^{\frac{qa}{p(1-\alpha)}} \right) \|\square_{k,j} f\|_{L^p_{\mathbb{R}^n}}^q.
\]

(4.20)

If $a \geq 1$, then we have

\[
\Upsilon_{ja} \lesssim \sum_{k \in \mathcal{X}_j^n, r \neq j} a^{sq-\alpha(a-1)q(a-n)(1-1)} \langle j \rangle^{\frac{qa}{p(1-\alpha)}} \|\square_{k,j} f\|_{L^p_{\mathbb{R}^n}}^q.
\]

(4.21)
If $a = 0$,
\[
\Upsilon_{ja} \lesssim \sum_{k \in \mathcal{K}^n_{j,ja}} \sum_{0 \leq l \leq \langle j \rangle^\alpha} \langle l \rangle^{\frac{aq}{n} + \frac{a(n-1)}{1-p} + (n-2)} \langle j \rangle^{\frac{a(n-1)}{1-p} + \frac{aq}{p}} \| \Box_{k,j} f \|_{L^p(\mathbb{R}^n)}^q.
\]

(4.22)

Now we divide our discussion into the following four subcases.

**Case 3A.** $qs - \alpha(n-1) - qa(n-1)(1/p - 1) \geq 0$. If $a \geq 1$, we see that the upper bound in (4.21) will be attained at $a \sim \langle j \rangle$. If $a = 0$, the summation on $l$ in (4.22) can be easily controlled. Anyway, we have
\[
\Upsilon_{ja} \lesssim \sum_{k \in \mathcal{K}^n_{j,ja}} \langle j \rangle^{\frac{aq}{n} + \frac{a(n-1)}{1-p} + \frac{aq}{p}} \| \Box_{k,j} f \|_{L^p(\mathbb{R}^n)}^q.
\]

(4.23)

Combining (4.18) and (4.23), we immediately have the result, as desired.

**Case 3B.** $q\alpha s_p - (1 - \alpha)(n-1) < qs < q\alpha s_p + \alpha(n-1)$. In this case, using (4.21) and (4.22), we can repeat the procedure as in Case 1B to get the result and we omit the details of the proof.

**Case 3C.** $q\alpha s_p - (1 - \alpha)(n-1) = qs$. This case is similar to Case 1C.

**Case 3D.** $q\alpha s_p - (1 - \alpha)(n-1) > qs$. We can deal with this case by following the same way as in Case 1D.

**Case 4.** $0 < p < 1, q > p$. By Minkowski’s inequality, we have
\[
\|f(\cdot, 0)\|_{M^{\alpha}_{p,q}(\mathbb{R}^{n-1})} \lesssim \left( \sum_{l \in \mathbb{Z}_+} \sum_{m \in \mathcal{K}^n_l} \langle l \rangle^{\frac{aq}{n}} \left( \sum_{j \in \mathbb{Z}_+} \sum_{k \in \mathcal{K}^n_j} \| (\mathcal{F}^{-1}_{\xi} \psi_{m,l}) \ast (\mathcal{F}^{-1}_{\psi_{k,j}} f)(\cdot, 0) \|_{L^p(\mathbb{R}^{n-1})}^p \right)^{q/p} \right)^{1/q}.
\]

(4.24)

Then we can repeat the procedures as in the proof of Theorem 1.2 and the above techniques in Case 3 to have the result, as desired. The details of the proof are omitted.
5 Proof of Theorem 1.6

Now we prove Theorem 1.6. Now we define the maximum function \( M^*_k f \) as follows:

\[
M^*_k f = \sup_{y \in \mathbb{Z}^n} \frac{|\Delta_k f(x - y)|}{1 + |2^k y|^n/r}.
\]  

(5.1)

Taking \( y_1 = \ldots = y_{n-1} = 0, \ y_n = x_n \) in (5.1), we have for \( 2^{-k-1} \leq |x_n| \leq 2^{-k} \),

\[
|\Delta_k f(x, 0)| \lesssim |M^*_k f(x)|, \quad x = (x_1, \ldots, x_{n-1})
\]

Hence

\[
\|\Delta_k f(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \lesssim \|M^*_k f(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})},
\]  

(5.2)

Integrating (5.2), one has that

\[
\|\Delta_k f(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \lesssim 2^k \int_{\mathbb{R}} \|M^*_k f(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})} \, dx_n,
\]

Hence

\[
\|\Delta_k f(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \lesssim 2^{k/p} \|M^*_k f\|_{L^p(\mathbb{R}^n)}.
\]  

(5.3)

Write \( \varphi'_k(\bar{x}) \) as the BAPU functions in \( \mathbb{R}^{n-1} \). Then for fixed \( k \), we have

\[
(\mathcal{F}_\xi^{-1} \varphi'_k \mathcal{F}_\bar{x}) (\bar{x}, 0) = \sum_{l=k-1}^{\infty} (\mathcal{F}_\xi^{-1} \varphi'_k \mathcal{F}_\bar{x} \mathcal{F}^{-1} \varphi_l \mathcal{F})(\bar{x}, 0) = \sum_{l=k-1}^{\infty} (\mathcal{F}_\xi^{-1} \varphi'_k) * (\mathcal{F}^{-1} \varphi_l \mathcal{F})(\bar{x}, 0)
\]

Case 1. \( 1 \leq p \leq \infty \). Using Young’s inequality, (5.1) and (5.3), we obtain

\[
\|\mathcal{F}_\xi^{-1} \varphi'_k \mathcal{F}_\bar{x} f(\bar{x}, 0)\|_{L^p(\mathbb{R}^{n-1})} \lesssim \sum_{l=k-1}^{\infty} \|\mathcal{F}_\xi^{-1} \varphi'_k\|_{L^1(\mathbb{R}^{n-1})} \|\mathcal{F}^{-1} \varphi_l \mathcal{F} f\|_{L^p(\mathbb{R}^{n-1})}
\]

\[
\lesssim \sum_{l=k-1}^{\infty} \|M^*_l f\|_{L^p(\mathbb{R}^n)}
\]

\[
\lesssim \sum_{l=k-1}^{\infty} 2^{l/p} \|\Delta_l f\|_{L^p(\mathbb{R}^n)}.
\]

Hence,

\[
\|f(\bar{x}, 0)\|_{B^{p,q}_{\mathbb{R}^{n-1}}} \lesssim \left( \sum_{k=0}^{\infty} 2^{skq} \left( \sum_{l=k-1}^{\infty} 2^{l/p} \|\Delta_l f\|_{L^p(\mathbb{R}^n)} \right)^q \right)^{1/q}.
\]

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If $0 < q \leq 1$, then
\[
\|f(\bar{x}, 0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \left( \sum_{l=0}^{\infty} \sum_{k=0}^{l+1} 2^{skq} 2^{lq/p} \|\Delta_l f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}.
\]

If $s = 0$, then
\[
\|f(\bar{x}, 0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \left( \sum_{l=0}^{\infty} 2^{lq/p} \|\Delta_l f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \lesssim \|f\|_{B^{1/p}_{p,q}(\mathbb{R}^n)}.
\]

In the case $s < 0$,
\[
\|f(\bar{x}, 0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \left( \sum_{l=0}^{\infty} 2^{lq/p} \|\Delta_l f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \lesssim \|f\|_{B^{1/p}_{p,q}(\mathbb{R}^n)}.
\]

If $1 \leq q \leq \infty$, using Minkowski's inequality,
\[
\|f(\bar{x}, 0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \sum_{l=0}^{\infty} \left( \sum_{k=0}^{l+1} 2^{skq} 2^{lq/p} \|\Delta_l f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}
\]
then
\[
\|f(\bar{x}, 0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \begin{cases} 
\|f\|_{B^{1/p}_{p,q}(\mathbb{R}^n)} & s = 0, \\
\|f\|_{B^{1/p}_{p,q}(\mathbb{R}^n)} & s < 0.
\end{cases}
\]

Case 2. $0 < p < 1$.
\[
\|\mathcal{F}_{\xi}^{-1} \varphi_k \mathcal{F}_{\xi} f(\bar{x}, 0)\|_{L^p(\mathbb{R}^{n-1})} \lesssim \sum_{l=k-1}^{\infty} 2^{l(n-1)(1/p-1)} \|\mathcal{F}_{\xi}^{-1} \varphi_k\|_{L^p(\mathbb{R}^{n-1})} \|\mathcal{F}_{\xi} f\|_{L^p(\mathbb{R}^{n-1})}^p \lesssim \sum_{l=k-1}^{\infty} 2^{l(n-1)(1-p)} 2^{k(n-1)(p-1)} 2^l \|M^n f\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{l=k-1}^{\infty} 2^{l(n-1)(1-p)} 2^{k(n-1)(p-1)} 2^l \|\Delta_l f\|_{L^p(\mathbb{R}^n)}.
\]

It follows that
\[
\|f(\bar{x}, 0)\|_{B^s_{p,q}(\mathbb{R}^{n-1})} \lesssim \left( \sum_{k=0}^{\infty} 2^{skq} \left( \sum_{l=k-1}^{\infty} 2^{l(n-1)(1-p)} 2^{k(n-1)(p-1)} 2^l \|\Delta_l f\|_{L^p(\mathbb{R}^n)}^p \right)^{q/p} \right)^{1/q}.
\]
If \( q \leq p \), one has that
\[
\| f(\bar{x}, 0) \|_{B^{s}_{p,q}(\mathbb{R}^{n-1})}^q 
\lesssim \sum_{k=0}^{\infty} \sum_{l=k-1}^{\infty} 2^{l(n-1)(1/p-1)q} 2^{k(s+(n-1)(1-1/p))q} 2^{l/q/p} \| \Delta_l f \|_{L^p(\mathbb{R}^n)}^q.
\]
Therefore,
\[
\| f(\bar{x}, 0) \|_{B^{s}_{p,q}(\mathbb{R}^{n-1})} \lesssim \left\{ \begin{array}{ll}
\| f \|_{B^{s}_{p,q}(\mathbb{R}^{n-1})}^s & s = s_p, \\
\| f \|_{B^{s}_{p,q}(\mathbb{R}^{n-1})}^s & s < s_p.
\end{array} \right.
\]

If \( q \geq p \), using Minkowski’s inequality, we have
\[
\| f(\bar{x}, 0) \|_{B^{s}_{p,q}(\mathbb{R}^{n-1})} 
\lesssim \left\{ \begin{array}{ll}
\sum_{l=1}^{\infty} \left( \sum_{k=0}^{\infty} (\chi(k < l))^2 \left( 2^{l(n-1)(1/p-1)q} 2^{k(s+(n-1)(1-1/p))q} 2^l \right) (\| \Delta_l f \|_{L^p(\mathbb{R}^n)}^p)^{q/p} \right)^{1/p} \\
\sum_{l=1}^{\infty} \sum_{k=0}^{\infty} 2^{l(n-1)(1/p-1)p} 2^{k(s+(n-1)(1-1/p)p)q} 2^l \| \Delta_l f \|_{L^p(\mathbb{R}^n)}^q \right\}^{1/p}
\]

Therefore,
\[
\| f(\bar{x}, 0) \|_{B^{s}_{p,q}(\mathbb{R}^{n-1})} \lesssim \left\{ \begin{array}{ll}
\| f \|_{B^{s}_{p,q}(\mathbb{R}^{n-1})}^s & s = s_p, \\
\| f \|_{B^{s}_{p,q}(\mathbb{R}^{n-1})}^s & s < s_p.
\end{array} \right.
\]

In the case \( 1 < p < \infty \), we define the maximum function \( M_{k,t}^* f \) as follows:
\[
M_{k,t}^* f = \sup_{y \in \mathbb{Z}^n} \frac{|\Delta_{k,t} f(x-y)|}{1 + |2^k y|^{n/p}}.
\] (5.4)

Taking \( y_1 = \ldots = y_{n-1} = 0, y_n = x_n \) in (5.4), we have for \( 2^{-k-1} \leq |x_n| \leq 2^{-k},
\[
| (\Delta_{k,t} f)(\bar{x}, 0) | \lesssim | M_{k,t}^* f (x) |, \quad \bar{x} = (x_1, \ldots, x_{n-1})
\]

Hence
\[
\| (\Delta_{k,t} f)(\cdot, 0) \|_{L^p(\mathbb{R}^{n-1})} \lesssim \| M_{k,t}^* f (\cdot, x_n) \|_{L^p(\mathbb{R}^{n-1})},
\] (5.5)

Integrating (5.5), one has that
\[
\| (\Delta_{k,t} f)(\cdot, 0) \|_{L^p(\mathbb{R}^{n-1})}^p \lesssim 2^k \int_{\mathbb{R}} \| M_{k,t}^* f (\cdot, x_n) \|_{L^p(\mathbb{R}^{n-1})}^p dx_n,
\]
Hence

\[ \| (\Delta_{k,t} f)(\cdot, 0) \|_{L^p(\mathbb{R}^{n-1})} \lesssim 2^{k/p} \| M_{k,t} f \|_{L^p(\mathbb{R}^n)}. \]  

(5.6)

Let \( \chi'_{k,t}(\bar{x}) \) as the characteristic functions in \( \mathbb{R}^{n-1} \). Then for fixed \( k \) and \( t \), we have

\[
(F^{-1}_y \chi'_{k,t} \mathcal{F}_x)(\bar{x}, 0) = \sum_{l=k}^{\infty} (F^{-1}_y \chi'_{k,t} \mathcal{F}_x F^{-1}_y \chi_{l,t} \mathcal{F})(\bar{x}, 0)
\]

\[
= \sum_{l=k}^{\infty} (F^{-1}_y \chi'_{k,t}) \ast (F^{-1}_y \chi_{l,t} \mathcal{F})(\bar{x}, 0)
\]

Using Young’s inequality, (3.1) and (5.6), we obtain

\[
\| F^{-1}_y \chi'_{k,t} \mathcal{F}_x f(\bar{x}, 0) \|_{L^p(\mathbb{R}^{n-1})}
\]

\[
\lesssim \sum_{l=k}^{\infty} \| F^{-1}_y \chi'_{k,t} \|_{L^1(\mathbb{R}^{n-1})} \| F^{-1}_y \chi_{l,t} \mathcal{F} f \|_{L^p(\mathbb{R}^{n-1})}
\]

\[
\lesssim \sum_{l=k}^{\infty} \| M_{l,t} f \|_{L^p(\mathbb{R}^n)}
\]

\[
\lesssim \sum_{l=k}^{\infty} 2^{l/p} \| \Delta_{l,t} f \|_{L^p(\mathbb{R}^n)}.
\]

Hence,

\[
\| f(\bar{x}, 0) \|_{B^{s,q}_{p,q}(\mathbb{R}^{n-1})} \lesssim \left( \sum_{k=0}^{\infty} \sum_{t=1}^{T} 2^{skq} \left( \sum_{l=k}^{\infty} 2^{l/p} \| \Delta_{l,t} f \|_{L^p(\mathbb{R}^n)} \right)^q \right)^{1/q}.
\]

If \( 0 < q \leq 1 \), then

\[
\| f(\bar{x}, 0) \|_{B^{s,q}_{p,q}(\mathbb{R}^{n-1})}
\]

\[
\lesssim \left( \sum_{l=0}^{\infty} \sum_{k=0}^{2^n} \sum_{t=1}^{2^n} 2^{skq} 2^{lq/p} \| \Delta_{l,t} f \|_{L^p(\mathbb{R}^n)}^q + \sum_{l=0}^{\infty} \sum_{k=1}^{2^n+1} \sum_{t=2^{n+1}}^{T} 2^{skq} 2^{lq/p} \| \Delta_{l,t} f \|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}.
\]

If \( s = 0 \), then

\[
\| f(\bar{x}, 0) \|_{B^{s,q}_{p,q}(\mathbb{R}^{n-1})} \lesssim \left( \sum_{l=0}^{\infty} \sum_{t=1}^{2^n} 2^{lq/p} \| \Delta_{l,t} f \|_{L^p(\mathbb{R}^n)}^q + \sum_{l=0}^{\infty} \sum_{t=2^{n+1}}^{T} 2^{lq/p} \| \Delta_{l,t} f \|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}.
\]

In the case \( s < 0 \),

\[
\| f(\bar{x}, 0) \|_{B^{s,q}_{p,q}(\mathbb{R}^{n-1})} \lesssim \left( \sum_{l=0}^{\infty} \sum_{t=1}^{2^n} 2^{lq/p} \| \Delta_{l,t} f \|_{L^p(\mathbb{R}^n)}^q + \sum_{l=0}^{\infty} \sum_{t=2^{n+1}}^{T} 2^{skq} 2^{lq/p} \| \Delta_{l,t} f \|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}.
\]
\[ \| f \|_{B_{p,q}^{1/p,1/p}(R^n)} \leq \| f \|_{\tilde{B}_{p,q}^{1/p,1/p}(R^n)} \]

If \( 1 \leq q \leq \infty \), using Minkowski’s inequality,

\[ \| f(\bar{x}, 0) \|_{B_{p,q}^{s}(R^{n-1})} \leq \sum_{l=0}^{\infty} \left( \sum_{k=0}^{T} \sum_{t=1}^{2^n} 2^{skq_2^{lq/p}} \| \Delta_l,t f \|_{L^q(R^n)} \right)^{1/q} \]

\[ \leq \sum_{l=0}^{\infty} \left( \sum_{k=0}^{l-1} \sum_{t=1}^{2^n} 2^{skq_2^{lq/p}} \| \Delta_l,t f \|_{L^q(R^n)} + \sum_{k=l}^{T} \sum_{t=2^{n+1}}^{2^n} 2^{skq_2^{lq/p}} \| \Delta_l,t f \|_{L^q(R^n)} \right)^{1/q} \]

then

\[ \| f(\bar{x}, 0) \|_{B_{p,q}^{s}(R^{n-1})} \leq \begin{cases} \| f \|_{\tilde{B}_{p,q}^{1/p,1/p}(R^n)} & s = 0, \\ \| f \|_{B_{p,q}^{1/p,1/p}(R^n)} & s < 0. \end{cases} \]

\[ \square \]

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