Reaction $\gamma\pi \to \pi\pi$ in a Confined Quark Model

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Abstract

A confined quark model study in a couple of chirally anomalous processes is presented in comparison with effective meson Lagrangian approaches of various kind. The processes considered are $\pi^0 \to \gamma\gamma$ and $\gamma \to \pi^+\pi^-\pi^0$ (or the equivalent $\pi\gamma \to \pi\pi$) for which there is a well-known low energy theorem relating the latter amplitude with the former one by a very simple algebraic relation in the zero energy (or chiral, or soft pion) limit. Our quark model naturally generates the so-called contact term in the amplitude for the second process, but with the opposite sign to what effective chiral meson models indicate. A reinterpretation of our vector pole contribution restores the consistency, however. While the first reaction is observed to serve in calibrating various models, it is found difficult, based upon the quality of the existing data in the second reaction, to single out the best model, which appears indispensable for testing the validity of the above low energy theorem. Thus the proposed experiments and their analyses should aim at attaining an optimal accuracy.

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1 Introduction

A little more than a quarter of a century ago, the chiral (axial) anomaly was found to amend the ordinary PCAC (partial conservation of axial current) relation in order to account for the physical decay of the neutral pion into two gammas, which otherwise would be too small and vanish in the strict chiral (zero meson mass) limit [1]. As has been frequently quoted, a quantitative comparison of the theoretical (anomalous PCAC) prediction with the measured decay rate lead to one of the strongest evidence for the three-colour nature endowed to the quarks. More precisely, together with a proportionality constant \( N_c \) the anomalous PCAC gives the \( \pi^0 \rightarrow \gamma\gamma \) amplitude

\[
F_{\pi\gamma\gamma} = \alpha N_c / 3\pi F_\pi = 0.025 \text{ GeV}^{-1},
\]

(1)

where \( F_\pi \approx 93 \text{ MeV} \) is the charged pion decay constant. In comparison with the corresponding quantity extracted from experiment: \( F_{\pi\gamma\gamma}^{\exp} = (0.0250 \pm 0.0003) \text{ GeV}^{-1} \), one obtains \( N_c = 3 \): the number of colours, see for example [2].

A couple of years later, a low energy theorem which relates the \( F_{\pi\gamma\gamma} \) to the amplitude for the \( \gamma \rightarrow 3\pi \) reaction at zero energy (more precisely the soft pion and soft photon limit) which we shall denote as \( F_{3\pi}(0) \equiv F_{3\pi}(0,0,0) \) was deduced based upon the anomalous PCAC and current algebra by Adler et al. [3], Terent’ev [4], and Aviv and Zee [5]: henceforth we shall call this low energy theorem as the ATA theorem for brevity. The occurrence of this \( \gamma \rightarrow 3\pi \) was also recognised as due to chiral anomaly, and the theorem claims

\[
e F_{3\pi}(0) = F_{\pi\gamma\gamma}/F_\pi^2,
\]

(2)

or

\[
F_{3\pi}(0) = \frac{e}{4\pi^2 F_\pi^2} \equiv F_{3\pi}^{\text{anom}} = 9.54 \text{ GeV}^{-3}.
\]

(3)

Note that the value of the chiral anomaly prediction denoted as \( F_{3\pi}^{\text{anom}} \) in the above equation will appear frequently in what follows. Later it was shown that, from the electromagnetically gauged Wess-Zumino-Witten (WZW) term [6][7] involving the chiral fields (ordinary pseudo-scalar octet mesons), both \( F_{\pi\gamma\gamma} \) and \( F_{3\pi}^{\text{anom}} \) can be obtained as the tree-level amplitudes explicitly satisfying the ATA theorem almost for free [6][8][9].
To test the ATA theorem, several physical processes like $\pi \to 2\pi$ in the Coulomb field of heavy nuclei (Primakoff effect), $e^+e^- \to e^+e^-3\pi$, etc. were suggested for experiments [1] [10]. In order then to either deduce the zero energy (the soft pion limit) amplitude from experiments or to predict appropriate cross sections from the theoretical side, it should be necessary to know how the appropriate physical amplitude may be extrapolated to the zero energy limit, or conversely, how the zero energy amplitude be continued to the physical region. The original (electromagnetically gauged) Wess-Zumino-Witten term alone does not really serve this objective and the standard procedure was to assume the process to go via the formation of vector mesons: $\rho$ (and $\omega$) [4][11]. Note that this procedure was not unique as will be discussed. Two experimental results have been available: one from the Primakoff effect with 40 GeV pion beam [12], see Fig.1, and the other from the $\pi^-e \to \pi^-\pi^0e$ with 300 GeV pions at the CERN SPS [13], Fig.2. For both cases, all different methods explored in those articles for extrapolation from zero energy amplitude fell inside the experimental error bars (about 20% of the central values) once the number of colours $N_c$ was fixed at three. So one might feel that at least the colour problem has found the solution, but that the ATA theorem has not yet been proven: both the theoretical models for describing the $\gamma \to 3\pi$ in the zero energy limit and at finite energies and the experiments should be improved. In fact, there have been various effective meson Lagrangian models constructed so as to be consistent with the underlying QCD (including the now fashionable Chiral Perturbation Theory [CHPT], see for example [14]). And on the experimental side (i) there are plans to repeat the pion Primakoff process with 600 GeV pions at Fermi Lab. (Fig.1), see for example [13], expecting a better statistics and making the virtual photons to become closer to the real ones than those in the previous experiment [12], and (ii) an accepted proposal (at CEBAF) for studying $\gamma\pi^+ \to \pi^+\pi^0$ in the photo double pion production: $\gamma p \to \pi^+\pi^0n$ (Fig.3), see [16] using the tagged photon beam with energies between 1 and 2 GeV (note that this experiment was investigated before [17], but the result was strongly in contradiction with the consequences of both refs. [12] and [13]). So both theoretical and experimental activities are expected to shed a new light on testing the ATA theorem with improved precisions.

In what follows we shall first discuss some salient features of various effective meson Lagrangian approaches for the $\gamma\pi \to 2\pi$ (or $\gamma \to 3\pi$) reaction and related processes in
Section 2, particularly the ways in which the vector meson contributions are implemented. This section may appear a little lengthy, but we consider it useful to give some overview of the theoretical trend in a comprehensible manner. Incidentally, by this one might see that Chiral Perturbation Theory (CHPT) is not the only relevant approach to low energy meson physics. Then in Section 3, we present our confined quark approach for the same reaction. This we think should be considered as an alternative to those effective Lagrangian approaches discussed in Section 2 where only meson degrees of freedom are the relevant quantities (with some minor exception of the appearance of current quark mass ratios, quark electric charges, etc.). Section 4 is devoted to an explicit calculation of cross section $\sigma/Z^2$ for the Primakoff process: the pion production in a nuclear Coulomb field on by an incident pion, from some models including our own, and compare with the experimental result in $[12]$. Then a short conclusion will be given at the end.

2 Models for Vector Meson Contributions: a Brief History

As mentioned in the previous section, chiral anomaly predicts the zero energy $\gamma \rightarrow 3\pi$ amplitude $F_{3\pi}(0)$ which is a real constant. It must be continued to appropriate physical regions in order to be confronted with relevant quantities extracted from experiments (or the continued amplitude must be used to calculate experimentally measurable quantities). The continuation (or extrapolation) then makes the amplitude a function of several Lorentz invariant kinematical variables (the number reduces to three, of course, when all the particles are physical, of which only two are independent). It was assumed that the continuation from zero energy would arise due to the $\rho$ and $\omega$ vector meson pole contributions for reasonable values of these kinematical invariants, say up to $\sim 1$ GeV$^2$ $[4]$. For the discussion to follow, we choose the process $\gamma\pi \rightarrow \pi\pi$ to define the kinematical variables, see Figs. 1-2. In the physical region for this process, $s > 4m_{\pi}^2$ and $t < 0$ should hold. Note that in the experiments in $[12]$ $[13]$, the photon is reasonably close to the real one: $-q^2 < 2 \cdot 10^{-3}$ GeV$^2$ $[12]$, and $-q^2 \approx 10^{-3}$ GeV$^2$ $[13]$. For the proposed experiment $[16]$, the photon is real: $q^2 = 0$ but the incident pion with momentum $p_1$ is virtual: $p_1^2 < 0$. Mandelstam variables are defined as
\[ s = (p_1 + q)^2 = (p_2 + p_3)^2, \quad t = (p_1 - p_2)^2 = (p_3 - q)^2, \quad u = (p_1 - p_3)^2 = (p_2 - q)^2, \]

so that for processes of \([12]\) and \([13]\) \(s + t + u = 3m_\pi^2 + q^2\). With this preparation we shall give a review of several existing models predicting the zero energy (all the particles are soft: \(s = t = u = q^2 = 0\)) amplitude and its continuation to physical regions. As stated in the Introduction, ATA \([3][4][5]\) found \(F_{3\pi}(0) = F_{3\pi}^{\text{anom}}\) from anomalous PCAC and current algebra. Then Terent’ev \([4]\) devised a simplest continuation from the chiral anomaly prediction in the form

\[
F_{3\pi}(s, t, u) = F_{3\pi}^{\text{anom}} \left[ 1 + C_\rho e^{i\delta} \left( \frac{s}{m_\rho^2 - s} + \frac{t}{m_\rho^2 - t} + \frac{u}{m_\rho^2 - u} \right) + C_\omega e^{i\delta'} \frac{q^2}{m_\omega^2 - q^2} \right],
\]

where \(\delta\) and \(\delta'\) are unknown relative phases. This is a form inferred purely intuitively from the assumptions of (i) the vector meson dominance (hereafter denoted as VMD), and (ii) \(F_{3\pi}(s, t, u)\) satisfying a once-subtracted dispersion relation with \(F_{3\pi}^{\text{anom}}\) the subtraction constant at zero energy.

The second approach to be mentioned here is the work of Rudaz \([11]\) based upon the pole model of Gell-Mann \textit{et al.} \([18]\) in which VMD is the guiding principle: all the radiative meson processes go via productions of vector mesons. First, Rudaz obtained the \(F_{\pi\gamma\gamma}\) using the SU(3), \(\rho\) universality and the \(\omega \rightarrow \pi^0\gamma\) width and found an agreement with the prediction from chiral anomaly. Then he calculated the \(\gamma \rightarrow 3\pi\) amplitude and found that without anomalous PCAC the ATA theorem could be obtained only if the revised Kawarabayashi-Suzuki-Riazuddin-Fiazuddin (KSRF) relation \([19]\) would be adopted (à la Basdevant and Zinn-Justin (BZ) \([20]\)) to relate the vector meson and pion quantities at low energies: by defining

\[
\alpha_k \equiv \frac{g_{\rho\pi\pi}^2 F_{\pi}^2}{m_\rho^2},
\]

with \(m_\rho\) and \(g_{\rho\pi\pi}\) the \(\rho\) meson mass and universal coupling constant (the universality means, for example, \(g_{\rho\pi\pi} = g_{\rho NN}\)) respectively, the KSRF relation fixes its value as \(\alpha_k = 1/2\) (its current empirical value is about 0.53) whereas BZ obtained \(\alpha_k = 1/3\). With
this latter value the resultant Rudaz amplitude reads

\[ F_{3\pi}(s, t, u) = F_{3\pi}^{\text{anom}} \frac{m_\omega^2}{m_\omega^2 - q^2} \frac{m_\rho^2}{3} \left( \frac{1}{m_\rho^2 - s} + \frac{1}{m_\rho^2 - t} + \frac{1}{m_\rho^2 - u} \right). \]  (6)

As a function of \( s \) the ratio \( F_{3\pi}(s, 0, 0)/F_{3\pi}^{\text{anom}} \) from the Terent’ev (with unknown phases set equal to zero) and the Rudaz models are not identical but the difference is seen as at most 10%, which might roughly define what kind of experimental accuracy would be required if one wished to distinguish the model difference from given data.

The story became quite interesting when Rudaz later claimed [21] that in view of the later development the above BZ choice: \( \alpha_k = 1/3 \), would be somewhat dubious, and that the KSRF choice: \( \alpha_k = 1/2 \) must be adopted. Then in order to respect the ATA theorem he suggested to introduce a non-vanishing direct contact term for the \( \omega \to 3\pi \) coupling with its value to be fixed at \( G_\omega = -g_{\rho\pi\pi}/16\pi^2 F_\pi^3 \), in the zeroth order in \( m_\pi^2/m_\rho^2 \). This led to the revised Rudaz amplitude:

\[ F_{3\pi}^c(s, t, u) = F_{3\pi}^{\text{anom}} \frac{m_\omega^2}{m_\omega^2 - q^2} \left[ \frac{1}{2} + \frac{3}{2} \frac{m_\rho^2}{3} \left( \frac{1}{m_\rho^2 - s} + \frac{1}{m_\rho^2 - t} + \frac{1}{m_\rho^2 - u} \right) \right]. \]  (7)

Thus the naïve pole contribution in VMD model alone appeared insufficient and an empirical rather large and negative direct \( \gamma 3\pi \) contact term contribution seemed indispensable. This result was later given a support by Cohen [22] who recalled the structure of the anomalous Ward identity for \( \gamma \to 3\pi \) discussed by Aviv and Zee [5] from which the sign and magnitude of the contact and vector meson pole contributions for the above expression could be read off.

In relation to the foregoing discussion, it may be important here to touch upon the subject of the possible non-zero \( \omega 3\pi \) contact term the history of which is rather old, but still lives in modern effective meson Lagrangian approaches. Earlier, in a pole dominance approximation to the isospin one \( \rho \pi \) scattering combined with the assumption of unsubtracted dispersion relation [23], its possible existence was suggested once a consistency with Bjorken limit was imposed. Later, there was an attempt to determine its magnitude from the point of view of hard current algebra plus anomalous Ward identity for three and four point functions in describing the \( \omega \to \pi^+\pi^-\pi^0 \) and \( \gamma\pi \to 2\pi \) [24], but no definite value could be reached. Within the experimental accuracy available [25], the \( \omega \to 3\pi \) transition could be explained by the pole model of VMD [18] with no apparent need for the finite contact term. So for a while the subject stayed dormant.
From mid-80’s there arose interest in constructing effective meson Lagrangian by adding vector (pseudo-vector) meson fields to the combination of the ordinary pseudoscalar (chiral) Lagrangian and the Wess-Zumino-Witten (WZW) term. This was a revival of the once successful vector meson dominance model (VMD) in a more refined form inferred from underlying QCD. Of those we single out the works by Kaymakcalan et al. [8], Brihaye et al. [9], and Kuraev and Silagadze [26]. Kaymakcalan et al. [8] tried a non-abelian (vector and axial vector fields) gauging of the chiral plus WZW action and, after introducing the VMD (to couple the photon with vector mesons), they required to arrive at what is called the Bardeen form of the anomaly. This was found to correctly predict the $\pi^0 \to 2\gamma$ decay rate through the $\omega\rho\pi$ coupling: a kind of prerequisite for any sensible model. The model has given a definite value for the $\omega 3\pi$ contact term as a function of $\alpha_k$ as introduced before. For the KSRF choice: $\alpha_k = 1/2$, which has given about 1/4 of the Rudaz value for the contact term, this model did successfully reproduce the $\omega \to 3\pi$ rate but did not furnish the ATA theorem (in fact, Rudaz has stated [21] that for any sensible value of $\alpha_k$ the low energy theorem is not reproducible in this model). Note, however, that as an apparent converse, the Rudaz value for the contact term cannot supply the sufficient $\omega \to 3\pi$ rate consistent with experiment [26]. It should be interesting to remark in passing that the authors of ref. [26] have demonstrated that once the kaon loop correction with the WZW term for the $K^+K^- \to 3\pi$ contact interaction to the $\gamma \to 3\pi$ amplitude is added within the model of ref. [8], the ATA theorem can be restored upon suitable regularisations, the consistency condition for which makes $\alpha_k = 0.55$, very close to the empirical value (the pion loop contribution is shown to be small $O(m^2_\pi)$, thus may be disregarded).

The second set of models [9] [26] demanded only the U(1) electromagnetic gauging of the combined chiral plus WZW effective action, then tried to consistently introduce vector mesons in an empirically well-established principle of VMD. The models easily passed the test to reproduce the $\pi^0 \to 2\gamma$ rate. Here again the parameter $\alpha_k$ enters and with its value set equal to the KSRF one, the $\omega 3\pi$ contact term agrees with the Rudaz one, so the ATA theorem was shown to be satisfied (this means, however, that the prediction for the $\omega \to 3\pi$ turned smaller than the empirical value). Now summing up, those and other effective Lagrangian approaches may need to be confronted with other radiative meson
processes, etc. and further relevant experiments are desirable. We note in this regard that a discussion in [27] should be quite useful.

Last in line to be discussed is the chiral perturbation theory (CHPT) applied to $\pi\gamma \rightarrow 2\pi$ and $\eta \rightarrow 2\pi\gamma$ by Bijnens et al. [28]. They started with the same chiral plus WZW action like in the works discussed above [8] [9] [26] and, calculated the corrections to $F_{3\pi}^{\text{anom}}$ which, as has already been stated previously, is obtained upon simply gauging the WZW term by the photon field. The corrections include one loop diagrams involving one vertex from the WZW term, and tree diagrams from the $O(p^6)$ effective Lagrangian appropriate for the process which is required also to cancel the divergence from the loop-WZW combination. The finite part of the $O(p^6)$ tree coefficients were fixed by assuming their saturation by the nonet vector meson contributions and VMD, then by integrating over the vector meson fields (so, needless to say, no explicit vector mesons but only the original pseudoscalar octets appear in the effective Lagrangian). The resultant correction term is shown to have the following form,

$$F_{3\pi}^{\text{CHPT}}(s, t, u) = F_{3\pi}^{\text{anom}} \left( 1 + C_{\text{loops}}^{\pi}(s, t, u) + \frac{3m_\pi^2}{2m_\rho^2} \right),$$

where $C_{\text{loops}}^{\pi}(s, t, u)$ is the loop correction:

$$C_{\text{loops}}^{\pi}(s, t, u) = \frac{m_\pi^2}{32\pi^2 F_\pi^2} \left\{ \ln \frac{m_\rho^2}{m_\pi^2} - 7 - \frac{s - 4}{3} \sqrt{\frac{s - 4}{s}} \ln \frac{1 + \sqrt{1 - 4/s}}{1 - \sqrt{1 - 4/s}} \right. $$

$$+ \frac{t - 4}{3} \ln \frac{1 - 4/t + 1}{\sqrt{1 - 4/t - 1}} + \frac{u - 4}{3} \ln \frac{1 - 4/u - 1}{\sqrt{1 - 4/u + 1}} $$

$$- \frac{i\pi}{3} \sqrt{\frac{s - 4}{s}} \right\}.$$ 

Here $\bar{s} = s/m_\pi^2$, $\bar{t} = t/m_\pi^2$, and $\bar{u} = u/m_\pi^2$. The appearance of the $m_\rho$ in logarithm is a standard feature arising from the mass scale needed for dimensional regularisation adopted: a common practice in CHPT is to set this scale equal to the $\rho$ mass. Needless to say that the $O(p^6)$ correction should vanish in the chiral limit. It may be of use to mention that the origin of the third term in the big bracket in Eq.(8) due to the $O(p^6)$ tree contribution is basically the second term in the linearised vector meson propagator.
contributions, viz.

$$\frac{m^2}{m^2 - s} \approx 1 + \frac{s}{m^2} + O[(s/m^2)^2],$$

together with $s + t + u = 3m^2_{\pi}$, ($q^2 = 0$ for the physical photon). This approximation may work typically up to $|s, t$ or $u| \approx 8m^2_{\pi}$: roughly the validity limit of the CHPT expansion. For a physical $\gamma\pi \to 2\pi$ process (where $s \geq 4m^2_{\pi}$) this linearised version is therefore expected to behave differently from the VMD results discussed above beyond this limit for $s$, even without considering the loop contribution.

3 A Confined Quark Approach to the $\gamma \to 3\pi$ Amplitude

In the previous section we have seen effective meson Lagrangian approaches which were used not only to find the relation between $F_{\pi\gamma\gamma}$ and $F_{3\pi}(0)$: to establish the ATA low energy theorem, but to continue the latter quantity into appropriate physical regions. All modern versions of these approaches are based upon the sum of the lowest order chiral Lagrangian and the corresponding Wess-Zumino-Witten (WZW) term in order to ensure the possibility of retaining chiral anomaly which allows transitions: even number of pseudoscalar mesons $\leftrightarrow$ odd number of pseudoscalar mesons. The way in which the photon and vector (and axial vector) meson fields are introduced to this combination differs in different models: some are rather close to each other while others are not. But the basic guideline (often employed in implementing the vector-axial vector fields) is that once these fields are formally integrated out, the resulting action functional is just the original chiral plus WZW term gauged electromagnetically. Thus in a way those effective meson Lagrangians are consistent with chiral anomaly by construction and in one way or another (recall the kaon loop correction [26] for the model of ref. [8]) should reproduce the ATA theorem. In this regard we have been curious about if there may be a way to approach our chirally anomalous processes from the quark model point of view. Our interest in this regard is not in those models like in [29] or [30] in which either specific quark models or QCD Lagrangian is adopted in attempting to derive the effective (non-renormalisable) meson Lagrangian. Rather, we want to regard mesons as explicit quark-antiquark bound states, so photons couple to mesons through quarks, and various electromagnetic and
mesonic transitions proceed explicitly through quark loops. Thus our vector mesons are not of Yang-Mills (non-abelian gauge) type. Besides, no VMD assumption is present. This may well give different form of amplitudes for the processes of our interest here as compared with those from the effective meson Lagrangians discussed so far. In principle, any quark model which may be able to describe low energy meson properties will do, and here we shall adopt a confined quark model: Quark Confinement Model developed at Dubna (QCM). There is a detailed monograph [31] on this approach so we only supply here the minimum prerequisite for the present purpose.

In QCM [31], generally there are contributions both from quark loops and meson pole diagrams. The gauge invariance of the Lagrangian under electromagnetic interaction leads to the gauge invariant form of the transitions $V \rightarrow \gamma$ ($V$: vector meson) that gives a vanishing contribution with real photons. The model is specified by the interaction Lagrangian describing the meson transition into quarks, for example, for $\pi$ and $\rho$ mesons one can write

$$L_I(x) = \frac{ig_\pi}{\sqrt{2}} \bar{\pi}(x) q(x) \gamma^5 q(x) + \frac{g_\rho}{\sqrt{2}} \bar{\rho}(x) q(x) \gamma^\mu q(x) + eA^\mu(x) \bar{q}(x) \gamma^\mu q(x)$$

(9)

with the meson-quark-quark coupling constants $g_M$ determined by the *compositeness condition*: that the renormalisation constants of the meson fields are equal to zero so no elementary meson:

$$Z_M = 1 - g_M^2 \Pi'_M(m_M^2) = 0,$$

(10)

where the empirical meson mass is the input. Here $\Pi'_M$ is the derivative of the meson mass operator. This way of determining the quark-meson coupling strength is one of the differences as compared with other approaches using quark loops like ref. [32] in which actual processes like $\pi^0 \rightarrow 2\gamma$ is fit to determine the coupling. Note that the above equation provides the right normalization of the charge form factor $F_M(0) = 1$. This could be readily seen from the Ward identity

$$g_M^2 \Pi'_M(p^2) = g_M^2 \frac{1}{2p^2} \frac{\partial \Pi_M(p^2)}{\partial p^\mu} = g_M^2 \frac{1}{2p^2} p^\mu T^\mu_M(p, p) = F_M(p) = 1.$$
where $T_{M}(p, p')$ is the three-point function describing the meson electromagnetic charge form factor. It may be of use to add that the compositeness condition ensures that unlike some of the quark-meson hybrid models, we have no danger of double counting.

Mesonic interactions (meson-photon, meson-meson, etc.) in QCM proceed only through closed quark loops:

\[
\int d\sigma_v \text{tr} \left[ M(x_1) S_v(x_1 - x_2) \cdots M(x_n) S_v(x_n - x_1) \right].
\]  

(11)

Here,

\[
S_v(x_1 - x_2) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x_1 - x_2)} \frac{1}{v\Lambda - p}
\]  

(12)

is the quark propagator with the scale parameter $\Lambda$ characterising the size of the confinement (or equivalent to something like the constituent quark mass), and the measure $d\sigma_v$, which is essential for quark confinement, is defined to provide the absence of the propagator singularities in quark loops, prohibiting the physical quark production:

\[
\int \frac{d\sigma_v}{v - z} = G(z) = a(-z^2) + zb(-z^2).
\]  

(13)

The shapes of the confinement functions $a(u)$ and $b(u)$, and the scale parameter $\Lambda$ have been determined from a reasonable model description of the low-energy hadronic quantities, see [31] for more details:

\[
a(u) = a_0 \exp(-u^2 - a_1 u) \quad b(u) = b_0 \exp(-u^2 + b_1 u). \quad (14)
\]

Here we have fixed $a_0 = b_0 = 2, a_1 = 1, b_1 = 0.4$ and $\Lambda = 460$ MeV, which describe various basic meson constants quite well, some examples of which may be found in Table 1, for which various integrals defined in Table 2 are used. See also other quantities in [31].

Note that in Table 2 and in the rest of this work the dimensionless meson masses defined as $\mu_M = m_M/\Lambda$ appear. We also note that some of the functions in the latter table define the meson-quark-antiquark coupling strength through the compositeness condition in Eq.(10): for example

\[
\frac{3g_{\pi}^2}{4\pi^2} = \frac{2}{R_{PP}(\mu_{\pi}^2)}, \quad \frac{3g_{\rho}^2}{4\pi^2} = \frac{3}{R_{VV}(\mu_{\rho}^2)}.
\]  

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Naturally, the coupling constants $g_{\rho\pi\pi}$, $g_{\rho\pi\gamma}$, and $g_{\rho\gamma}$ obtained in the model describe the decay widths of the $\rho$ successfully, which read

$$\Gamma(\rho \to \pi\pi) = m_\rho \frac{g_{\rho\pi\pi}^2}{48\pi} \left[ 1 - \frac{4m_\pi^2}{m_\rho^2} \right]^{3/2},$$

$$\Gamma(\rho \to \pi\gamma) = m_\rho \frac{g_{\rho\pi\gamma}^2}{96\pi} \left[ 1 - \frac{m_\pi^2}{m_\rho^2} \right]^3,$$

$$\Gamma(\rho \to e^+e^-) = \frac{4\pi\alpha^2}{3} m_\rho g_{\rho\gamma}^2.$$  

Also it may be useful here to mention that from Table 1 and Eq.(5) our model gives $\alpha_k = 0.53$ (by adopting the standard value: $m_\rho = 770$ MeV), identical to the empirical value and quite close to the KSRF value of 1/2.

For our present objective we first calculate the $\pi^0 \to \gamma\gamma$ amplitude: $F_{\pi\gamma\gamma}$. The contribution comes only from the triangular quark loop (Fig.4): no vector meson pole. From Table 1 one can read off the expression

$$F_{\pi\gamma\gamma} = \alpha_{\pi F_{\pi}} \left[ \frac{2R_{PVV}(\mu_\pi^2)R_P(\mu_\pi^2)}{R_{PP}(\mu_\pi^2)} \right] = 0.94 \frac{\alpha}{\pi F_{\pi}} = 0.024 \text{ GeV}^{-1},$$

in excellent agreement with the experimental value quoted in the introduction and with the chiral anomaly prediction: $\alpha/\pi F_{\pi} = 0.025$ GeV$^{-1}$ (here again $\mu_M = m_M/\Lambda$ is used).

We note that in the chiral limit: $m_\pi = 0$, the value in the bracket of Eq.(15) becomes 0.97 even somewhat closer to the chiral anomaly prediction. Thus we may conclude that the model is well calibrated to have successfully passed the first hurdle.

The next step is to obtain $F_{3\pi}(s,t,u)$ for $\gamma \to 3\pi$ which, in the context of the present model, is described by two types of diagrams in Fig.5. We will consider an alternative physical process $\pi\gamma \to \pi\pi$. The invariant matrix element consists of two parts and reads

$$M^\mu[\pi(p_1) + \gamma(q) \to \pi(p_2) + \pi(p_3)] = \varepsilon^{\mu\nu\eta\kappa}(p_1)_\nu(p_2)_\eta(p_3)_\kappa F_{3\pi}(s,t,u),$$

where

$$F_{3\pi}(s,t,u) = F_{3\pi}^{box}(s,t,u) + F_{3\pi}^{pole}(s,t,u)$$
\[ F_{3\pi}^{\text{box}}(s, t, u) = \frac{e}{\Lambda^3} \left( \frac{g_{\pi}(s, t, u)}{\sqrt{2}} \right)^3 \frac{1}{4\pi^2} \left\{ R_\square(s, t) + R_\square(t, u) + R_\square(u, s) \right\}, \]  
\hline
\[ F_{\text{pole}}(s, t, u) = \frac{1}{m_\rho} \left[ \frac{2R_{\rho\pi\pi}(s)g_{\rho\pi\gamma}(s)}{m_\rho^2 - s} + (s \to t) + (s \to u) \right], \] 
\hline

The function defining the box-diagram (corresponding to the first diagram in Fig.5 which, however, is not drawn as a proper box) is written as

\[ R_\square(s, t) = 2 \int d^4\alpha \delta(1 - \sum_{i=1}^{4} \alpha_i) \left[ -\alpha'(-D_4(\alpha)) \right], \] 

where \( \alpha' \) is the derivative of function \( \alpha(u) \) in Eq.(14), and \( D_4 \) is equal to

\[ D_4(\alpha) = (\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4)m_\pi^2/\Lambda^2 + \alpha_2\alpha_4 s/\Lambda^2 + \alpha_1\alpha_3 t/\Lambda^2. \]

The value of \( R_\square \) changes very slowly: less than 1% when \( s \) and \( t \) are varied within and near the physical region for the process under consideration. We may therefore safely approximate the box contribution by setting \( s = t = u = 0 \) neglecting all the mass and momentum dependence with quite a good accuracy. Then one trivially finds \( R_\square(0, 0) = 2 \) leading to

\[ F_{3\pi}^{\text{box}}(0, 0, 0) \approx \frac{6R_P(0)}{R_{P P}(0)^3} \frac{e}{4\pi^2 F_\pi^3} = 0.22 F_{3\pi}^{\text{anom}}. \]  
\hline

The vector meson pole (or resonance) contributions: the second diagram in Fig.5, are defined by \( F_{\text{pole}}(s, t, u) \) in Eq.(19) which contains the \( \rho\pi\pi \) and \( \rho\pi\gamma \) vertices which are not constants but are explicitly dependent upon the Mandelstam variables due to their quark loop structure,

\[ g_{\rho\pi\pi}(s) = \frac{2\pi}{R_{P P}(s/\Lambda^2)} \sqrt{\frac{2}{R_{V V}(\mu^2_\rho)}}, \]

\[ g_{\rho\pi\gamma}(s) = \frac{e m_\rho}{\Lambda} \sqrt{\frac{2}{3}} \frac{R_{P V V}(s/\Lambda^2)}{\sqrt{R_{P P}(\mu^2_\rho)R_{V V}(\mu^2_\rho)}}. \]

Fig.6 demonstrates their dependence on the virtual \( \rho \) meson mass squared. Apparently the variations are not quite small. This may provide a rather interesting effect on which we shall discuss later. Note that the \( \rho \)-meson decay widths as presented before are defined

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by the coupling strengths of Eq.(22) with the $s$-variable set equal to the square of the $ho$-meson mass: $g_{\rho\pi\pi} = g_{\rho\pi\pi}(m_\rho^2)$, $g_{\rho\pi\gamma} = g_{\rho\pi\gamma}(m_\rho^2)$, and $g_{\rho\gamma} = g_{\rho\gamma}(m_\rho^2)$. As is well known, there are relations between these constants (see, for example, [33])

$$g_{\rho\gamma} = 1/g_{\rho\pi\pi}, \quad g_{\rho\pi\gamma} = \frac{1}{2}g_{\rho\pi\pi} \cdot \frac{m_\rho}{e} F_{\pi\gamma\gamma} = e m_\rho \frac{g_{\rho\pi\pi}}{8 \pi^2 F_\pi},$$

(23)

which are obtained based upon an assumption on the VMD and PCAC relation. In our model the corresponding relations are found numerically as

$$g_{\rho\gamma} = 1.08/g_{\rho\pi\pi}, \quad g_{\rho\pi\gamma} = \frac{0.89}{2}g_{\rho\pi\pi} \cdot \frac{m_\rho}{e} F_{\pi\gamma\gamma} = 0.84 e m_\rho \frac{g_{\rho\pi\pi}}{8 \pi^2 F_\pi},$$

(24)

where our result in Eq.(15) has been taken into account. One can see that the relations (23) are reproduced within the commonly accepted accuracy of the VMD assumption although our model is not based upon it. In this respect we should stress that with empirical values for $g_{\rho\pi\pi}$ etc. one obtains the result very similar to ours in Eq.(24).

Upon combining the box and $\rho$-pole contributions, we find for null momenta ($s=t=u=0$, soft pions and photon)

$$F_{3\pi}(0) \equiv F_{3\pi}(0, 0, 0) = F_{3\pi}^{\text{box}}(0, 0, 0) + \frac{6g_{\rho\pi\pi}(0)g_{\rho\pi\gamma}(0)}{m_\rho^2} \tag{25}$$

$$= \frac{e}{4\pi^2 F_\pi^3}[0.22 + 0.83] = 1.05 F_{3\pi}^{\text{anom}},$$

which is very close to the chiral anomaly prediction (the ATA derivation) or, equivalently, to the one from the electromagnetically gauged Wess-Zumino-Witten term as discussed in the previous section although we have borrowed nothing from those approaches.

A particularly noticeable feature of our result is the dominant rôle played by the $\rho$ meson pole contribution, similar to the VMD approaches. In this respect, it may be useful to make here a close comparison of our result with the VMD result. Within the VMD models (see, for example) [11] the zero energy $\gamma \to 3\pi$ amplitude from the vector meson pole contribution may be obtained as

$$F_{3\pi}^{\text{VM}}(0) = 3\alpha_k \cdot F_{3\pi}^{\text{anom}},$$

(26)
the derivation of which requires the relations in Eq.(23). A possible presence of the direct contact term \[21\] introduces an additional contribution of the form

\[
F_{3\pi}^{\text{con}}(0) = \gamma(\alpha_k) \cdot F_{3\pi}^{\text{anom}},
\]

thus

\[
F_{3\pi}(0) = F_{3\pi}^{\text{con}} + F_{3\pi}^{\text{VM}} = [\gamma(\alpha_k) + 3\alpha_k] \cdot F_{3\pi}^{\text{anom}}.
\]

While the form of the \(\rho\) meson pole contribution is identical in any VMD approaches, that of the contact contribution differs in different models: for example, Refs. \[9\] \[21\] find \(\gamma(\alpha_k) = 1 - 3\alpha_k\), whereas \[8\] obtained \(\gamma(\alpha_k) = 1 - 3\alpha_k + 3/2\alpha_k^2\) (we recall that the first form works well with the ATA theorem while the second can be made consistent with the \(\omega \rightarrow 3\pi\) width, both with reasonable values of \(\alpha_k\)).

Regarding our result Eq.(25), it may appear quite natural to identify the box contribution as that coming from the contact term in the effective meson Lagrangian. Then although, as we repeat, our model is not based in any way upon the VMD assumption, it would be tempting to assume (although somewhat dubious !) that the relations in Eq.(23) may well hold among the coupling strengths evaluated at zero energy, viz. among \(g_{\rho \pi \pi}(0), g_{\rho \pi \gamma}(0),\) etc. Then from Eqs.(25-28) this seemingly natural but somewhat naive assumption leads to \(\alpha_k = 0.28\) (from \(3\alpha_k = 0.83\)). We note that since the form of the contact term is not unique, it is wiser not to use it to extract \(\alpha_k\). Clearly, the thus obtained value of \(\alpha_k\) appears quite smaller as compared with both the empirical and the KSRF values. Does this imply that our model is not consistent with the Vector Dominance Model quantitatively, or qualitatively, or even is there anything wrong with our approach?

In order to investigate this problem, we want to stress that the key point is the fact that, as is clear in Eq.(22) in our quark approach, the couplings to the \(\rho\) meson are functions of the virtual mass square of the \(\rho\), and that , as stated before, the mass square dependence is not small (see Fig.6). As a consequence, the \(\rho\)-meson contribution in Eq.(25) evaluated at zero \(\rho\) mass square is considerably smaller than the ordinary value obtained in VMD models with fixed coupling strengths (corresponding to those strength functions evaluated at the physical \(\rho\) mass). This then leads to the extraction of small \(\alpha_k\) mentioned above as long as we do not question the appropriateness of such an extraction and identification as \(\alpha_k\). Now if one insists that somehow a consistency with VMD is
mandatory in order our approach to be acceptable, it may be possible to reinterpret our result for that objective as follows: Eq.(19) which is the $\rho$-pole contribution to the $\gamma \to 3\pi$ amplitude in our model may be decomposed into the pure $\rho$ meson pole and the rest, the latter being analytic at the $\rho$ meson pole,

$$
\frac{2 \cdot g_{\rho\pi\pi}(m_\rho^2)g_{\rho\pi\gamma}(m_\rho^2)}{m_\rho(m_\rho^2 - s)} + R(s) + (s \to t) + (s \to u).
$$

(29)
The pure $\rho$-pole contribution in the above expression is the one appearing as the $\rho$ meson pole contribution in the VMD models. The analytic part is beyond the VMD approach.

One may then add the analytic part to the box contribution and regard the sum as the contact term introduced phenomenologically in the effective chiral vector meson Lagrangian approaches stated in the last section. With this rearrangement, Eq.(25) may be rewritten as

$$
F_{3\pi}(0) = F_{3\pi}^\text{anom} \left( F_{\text{con}} + 3 \cdot \frac{2g_{\rho\pi\pi}(m_\rho^2)g_{\rho\pi\gamma}(m_\rho^2)}{m_\rho^2 F_{3\pi}^\text{anom}} \right),
$$

(30)
where the first term in the above bracket is the contact term contribution defined as

$$
F_{\text{con}} = \frac{F_{3\pi}^\text{box}(0, 0, 0)}{F_{3\pi}^\text{anom}} - 3 \cdot \frac{2\left[g_{\rho\pi\pi}(m_\rho^2)g_{\rho\pi\gamma}(m_\rho^2) - g_{\rho\pi\pi}(0)g_{\rho\pi\gamma}(0)\right]}{m_\rho^2 F_{3\pi}^\text{anom}},
$$

(31)
whereas the second term is the standard $\rho$ pole contribution. It is easy to show that this second term may be identified as $3\alpha_k$ of Eq.(5) when relations in Eq.(23) are strictly obeyed. In our model where, instead, Eq.(24) holds, the effective value of $\alpha_k$ is obtained as

$$
\alpha_k^{\text{eff}} = \frac{2g_{\rho\pi\pi}(m_\rho^2)g_{\rho\pi\gamma}(m_\rho^2)}{m_\rho^3 F_{3\pi}^\text{anom}} = 0.84 \cdot \left(\frac{g_{\rho\pi\pi}F_3}{m_\rho}\right)_{\text{empirical}}^{\text{2}} = 0.44.
$$

(32)
Clearly, this is consistent with the KSFR prediction ($\alpha_k = 0.5$) within the accuracy of the VMD approach. Then from Eq.(31) the corresponding contact term is equal to $F_{\text{con}} = 1.05 - 3\alpha_k^{\text{eff}} = -0.27$, which is consistent with the one in effective Lagrangian approaches both in sign and magnitude. Thus our result has been shown not in conflict with the VMD models.

Our amplitude with non-vanishing Mandelstam variables may be conveniently rewritten in terms of $F_{3\pi}^\text{anom}$,

$$
F_{3\pi}(s, t, u) = F_{3\pi}^\text{anom} \left[ 0.22 + C_\rho(s)D_\rho(s) + C_\rho(t)D_\rho(t) + C_\rho(u)D_\rho(u) \right],
$$

(33)
where
\[ C_\rho(s) = \frac{2g_{\rho\pi\gamma}(s)}{m_\rho^3 F_{3\pi}^{\text{anom}}}, \quad \text{and} \quad D_\rho(s) = \frac{m_\rho^2}{m_\rho^2 - s}. \]

4 Comparison of Models With Data from Existing Primakoff Process

Here we first discuss the value of the zero energy $\gamma \to 3\pi$ amplitude from various models. As already discussed, the chiral anomaly prediction is 9.5 (in units of GeV$^{-3}$). Of those models incorporating vector mesons, some have reproduced (or have been tailored to reproduce) the chiral anomaly value [9] [21] [26] while some other [8] gives a larger value: about 13.0 due to the smaller magnitude of the $\omega 3\pi$ contact term. The chiral perturbation calculation of ref.[28] to $O(p^6)$ as quoted by Moinester [15] offers the number 10.7 ± 0.5 (it should be mentioned here that at the zero energy [or the soft pion limit] this approach should reproduce the anomaly prediction so the quoted value should be taken with some caution). Our QCM yields 10.0 for its value. These values should be compared with the one extracted from the Primakoff process [12] is 12.9 ± 0.9 (stat) ± 0.5 (sys). Based upon this experimental value it is hard to rule out (or give preference to) any of the prediction.

Next, the model amplitude will be used to predict the integrated Primakoff cross section for the pion production by incoming pions as in Fig.1. We adopt the Mandelstam variables as introduced previously and go to the centre of mass system for the two final state pions. Then for $\gamma\pi \to \pi\pi$ (see, Fig.1) noting that $s + t + u = 3m_\pi^2 + q^2$, we have
\[ \vec{p}_1 + \vec{q} = \vec{p}_2 + \vec{p}_3 = 0, \quad \cos\theta = \frac{(\vec{p}_1 \cdot \vec{p}_2)}{||\vec{p}_1|| ||\vec{p}_2||}, \quad p_1^0 = E, \]

and taking into account that the photon is almost real ($q^2 \approx 0$), one may write
\[ t = \frac{1}{2}[3m_\pi^2 - s + (s - m_\pi^2)\sqrt{1 - 4m_\pi^2/s \cos\theta}], \]
\[ u = \frac{1}{2}[3m_\pi^2 - s - (s - m_\pi^2)\sqrt{1 - 4m_\pi^2/s \cos\theta}]. \]

The physical region for $s$ is $4m_\pi^2 \leq s \leq 16m_\pi^2$. We obtain various model amplitudes for $F_{3\pi}(s, t, u)$ in units of the chiral anomaly prediction: $F_{3\pi}^{\text{anom}} \equiv e/4\pi^2 F_\pi^3$ as a function of $s$ for the forward scattering angle: Fig.7 (the angular dependence has been found rather
weak so we only give only one angular value). The plotted curves are the original Rudaz VMD model result [11], then his modified (improved) amplitude [21], the amplitude from Kaymakcalan et al.’s model [8], chiral perturbation (CHPT) prediction [28], and our present quark model result. Except for the amplitudes from [8] and [28], other curves are mutually rather close. Kaymakcalan et al.’s [8] result is due to its smaller sized of the contact term which has given a large zero energy amplitude. On the other hand the CHPT amplitude above $s \geq 8m_{\pi}^2$ has come out quite differently from the others. The validity for the CHPT expansion is up to about this value of $s$, so the resultant difference may be natural. If future experiments may be able to extract this $s$ and scattering angle dependent behaviour of the amplitude, it would be of quite help. At present, however, we should be satisfied with their prediction on the available data [12].

The cross section for pion production by the pion incident in the nuclear Coulomb field (the Primakoff process)

$$\pi^- + (Z, A) \rightarrow \pi^- + \pi^0 + (Z, A)$$

as in Fig.1 may be expressed in terms of the $\gamma\pi \rightarrow 2\pi$ cross section through the equivalent photon method:

$$\frac{d\sigma}{ds dt dq^2} = \frac{Z^2 \alpha}{\pi} \left[ \frac{q^2 - q_{\min}^2}{q^4} \right] \frac{1}{s - m_{\pi}^2} \frac{d\sigma_{\gamma\pi \rightarrow \pi\pi}}{dt},$$

(35)

where

$$q_{\min}^2 = \left( \frac{s - m_{\pi}^2}{2E} \right)^2,$$

and

$$\frac{d\sigma_{\gamma\pi \rightarrow \pi\pi}}{dt} = \frac{|F_{3\pi}(s, t, u)|^2}{512\pi} (s - 4m_{\pi}^2) \sin^2 \theta$$

(36)

with the vertex $F_{3\pi}(s, t, u)$ defined by the dynamics of the $\gamma \rightarrow 3\pi$ transtion.

Neglecting the $q^2$-dependence in $F_{3\pi}(s, t, u)$, viz. setting $q^2 \approx 0$ (thus $s + t + u = 3m_{\pi}^2$),

the integrated cross section per unit proton charge in the nucleus is given as

$$\frac{\sigma}{Z^2} = \frac{\alpha}{1024\pi^2} \int_{4m_{\pi}^2}^{s_{\text{max}}} ds \left( \frac{s - 4m_{\pi}^2}{\sqrt{s}} \right)^{3/2} \left\{ \ln \frac{q_{\text{max}}^2}{q_{\text{min}}^2} - 1 + \frac{q_{\text{min}}^2}{q_{\text{max}}^2} \right\} \int_0^\pi d\theta \sin^3 \theta |F_{3\pi}(s, t, u)|^2.$$

(37)
The condition for the Primakoff effect is the following \cite{1} \cite{12}:

\[ E^2 >> m^2_\pi \sim s >> q^2. \]

This condition is better satisfied when the energy of the incident pion beam becomes higher. The Serpukhov experiment \cite{12} has been carried out with the 40 GeV pion beam. The maximum momentum transfer was \( q^2_{\text{max}} = 2 \cdot 10^{-3} \text{GeV}^2 \) and \( s_{\text{max}} = 10m^2_\pi \).

The result of the calculation of the cross section for various approaches are shown in Table 3. One can see that the results are rather close to each other, except for the one from the model of ref. \cite{8}, which could be understood because its prediction for \( F_{3\pi}(0) \) is about 30% larger than the chiral anomaly prediction, as discussed in Section 3. We note that a somewhat larger value (about 10%) of the cross section in our QCM model as compared with those from those of refs. \cite{21} and \cite{28} is also largely the reflection that our \( F_{3\pi}(0) \) is about 5% larger than the chiral anomaly prediction. In Table 3 the row denoted as \textit{Chiral Anomaly} has been obtained from replacing \( F_{3\pi}(s,t,u) \) simply by \( F_{3\pi}^{\text{anom}} \equiv e/4\pi^2 F^3_\pi \), thus disregarding the energy dependence (and angular variations) in calculating \( \sigma/Z^2 \) above. This is found to reduce the integrated cross section by more than 10%. But at present, the extracted values from the experiment may only be able to say that the result from ref.\cite{8} may be inadequate (note, however, that as we have stated before, one K-meson loop correction seems to have restored the value of the contact term to the Rudaz one \cite{21}). Improved experiments are definitely in need.

5 Discussion and Conclusion

It may sound repetitive, but we make a pedagogical review at the beginning of the discussion.

Only the \textit{anomalous} (but not the ordinary) PCAC combined with current algebra was able to correctly predict the observed \( \pi^0 \rightarrow 2\gamma \) decay rate \cite{1}, and the zero energy \( \gamma \rightarrow 3\pi \) amplitude \( F_{3\pi}(0) \) was predicted from the former amplitude: we have termed this as the ATA theorem \cite{3} \cite{4} \cite{5}. This theorem was rederived quite naturally later from the electromagnetically gauged Wess-Zumino-Witten action (WZW) \cite{6} \cite{7}. Modern versions of the vector dominance models (VMD) \cite{8} \cite{9} \cite{21} \cite{26} were used in an attempt to reach
this theorem with the help of the KSRF relation [19] (and its more relaxed form). In order to be consistent with the ATA theorem the vector dominance models have been required to introduce a non-zero constant four-point coupling for \( \omega \to 3\pi \) leading to a contact \( \gamma \to 3\pi \) coupling which has an opposite sign as compared with the vector meson pole contribution. While the vector meson pole contribution in these models for the zero energy \( \gamma \to 3\pi \) amplitude is 1.5 times the value obtained from the ATA theorem, the value of the contact term contribution differs: -1/2 times the ATA value from refs. [9] [21] [26], and about 4 times smaller in ref. [8] (when evaluated at the tree level), and the former is only consistent with the ATA theorem. We have calculated above the Primakoff cross section for these models and found that improved experiments may be able to say which group of the VMD model (regarding the contact term value) should be more relevant to the \( \gamma \to 3\pi \) amplitude.

We have devised a confined quark model (QCM) to study the same subject without implementing any explicit chiral anomaly. The result has come out quite consistent with the ATA theorem and the consequence of the electromagnetically gauged WZW action. The obtained amplitude \( F_{3\pi}(s, t, u) \) is fairly close to the VMD prediction of refs. [9] [21] [26]. However, if we regard our quark box diagram as the contact term contribution, it is found to have the same sign as the vector meson pole contribution, and counts just 22% of the ATA theorem (or the chiral anomaly) prediction: \( F_{3\pi}^{\text{anom}} \equiv e/4\pi^2 F_{3\pi}^3 \). This then implies (as it has eventually turned out) that our vector meson pole contribution accounts for about 80% of the chiral anomaly prediction. This should be contrasted to the result of the VMD prediction discussed above which provides \( 1.5 F_{3\pi}^{\text{anom}} \) upon employing the KSRF relation: \( \alpha_k = 1/2 \). At this point we want to remark that, as in Fig.6 both our \( \rho \pi \pi \) and \( \rho \gamma \pi \) coupling vertices are functions of the \( \rho \) meson mass squared arising from the quark loop, and that their variations are not negligible, especially the former. Their values at \( m_\rho^2 \) are practically the same as those in the VMD models, see Table 1, but at zero mass they are smaller. Those smaller values have been used to find the value of \( \approx 80\% \) for the \( \rho \)-pole contribution mentioned above. When those coupling constants are evaluated at \( m_\rho^2 \), one finds that the vector meson contribution to \( F_{3\pi}(0) \) becomes 1.3 times the chiral anomaly prediction, not very different from the VMD result of 1.5 times (recall Eqs.(30,32)).
As for the difference in sign and magnitude of the contact term between our quark model prediction and those from the VMD models, it does not appear easy to find the answer to this discrepancy as long as we stick to regarding the contact term as our box contribution: within our confined quark model approach the loop integral for the box diagram is completely convergent and gauge invariant. In the approach in ref.[5], the loop integral generates a constant term breaking the gauge invariance so it must be subtracted out (it is related to the so-called surface term arising from the shifting of the integration variable in a divergent integral, see for example section 6.2 of ref.[34]). This subtraction appears to be the origin of the sign change which we do not have in QCM. So what could be the possible compromise? There may be some effects coming from the excited vector meson(s) like $\rho'$ which was expected to cure the trouble with the $\omega \rightarrow 3\pi$ rate with the larger contact term: $-0.5F_{3\pi}^{\text{anom}}[27]$. Or the effect of the Regge trajectory contributions might be included[26]. Either of these effects might be included in the vector meson pole contributions. These lines of considerations might work as a bridge between our and the VMD pictures: one might suspect that some type of hadron-quark duality (for example, quark loops vs. meson loops) be operative somewhere.

Once we adopt a somewhat more relaxed definition of our contact term, however, the resolution of the apparent discrepancy in sign and the size of the contact term in our and VMD models may seem feasible. This could proceed along the line we exploited in Section 3 when extracting $\alpha^{eff}_k$ from our prediction of $F_{3\pi}(0)$. Namely, we decompose our $\rho$-pole contribution into the pure pole and analytic parts. Then the analytic part is added to the box contribution which one may regard as the effective contact term. Now, this effective contact term is no longer a mere constant, but a function of the Mandelstam variables: $s$, $t$ and $u$. It is then easy to see, as found in Section 3, that in the zero energy limit this term obtains a negative sign with magnitude about half the Rudaz value[11] making our $F_{3\pi}(0)$ prediction very close to the chiral anomaly one. At finite energies (finite $s$) one may be able to see that the magnitude of this term decreases, and this is consistent with the apparent need for small (or zero) contact term contribution in the $\omega \rightarrow 3\pi$ width within the context of the effective chiral vector meson models. We note in this respect that both experimental and model investigations, for example, on $e^+e^- \rightarrow 3\pi$ (including $\omega \rightarrow 3\pi$) should be quite helpful.
From the point of view of the proposed $\gamma\pi \rightarrow \pi\pi$ experiments (either the photon or pion in the initial state is virtual, or at best close to real) \cite{15} \cite{16} to extract $F_{3\pi}(0)$ or more ambitiously to deduce $F_{3\pi}(s, t, u)$ for different $s$ and c.m. scattering angle (for example, to clarify the distinction between our quark model and VMD predictions: notably the rôle played by either the constant or non-constant contact term, the latter arising from the Mandelstam variable dependent $\rho$ couplings), they will be required to attain a very high precision in data acquisition and analyses. In this respect it may be useful to be reminded that if some sort of analytic extrapolation to a pole (either for the pion or photon) is needed in analysing the data, one must not forget the fact that assumed analyticity property (of the relevant amplitude) and the statistical nature of the data are mutually exclusive. So instead of a naïve straightforward extrapolation, some optimal mapping may have to be sought beforehand in order to minimise the unwanted (and in most cases the dominant) background contributions, see for example refs. \cite{35} \cite{36} concerning this procedure. Finally, in view of the closeness of the predictions from some VMD models, based upon the chiral plus the WZW action which basically implements the chiral anomaly, and our confined quark model approach (which, at least, is not explicitly built upon chiral anomaly like the WZW term), may it be of some use to ask ourselves if the proposed experiments will be really testing the ATA theorem as a consequence of chiral anomaly?

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Fig.1. Pion production by an incident pion in the Coulomb field of a heavy nucleus (the Primakoff process) [12, 15].

Fig.2. Pion electroproduction by an incident pion [13].

Fig.3. Photo double pion production on the proton [16].

Fig.4. The $\pi^0 \to 2\gamma$ decay through the quark triangle.

Fig.5. The $\gamma \to 3\pi$ in QCM. The first diagram is called the Box contribution while the second type is termed as the vector meson pole contribution.

Fig.6. The squared mass dependence of the $\rho \to \pi\gamma$ and $\rho \to \pi\pi$ coupling vertices in QCM normalised to the values at the $\rho$ mass.

Fig.7. The $\gamma \to 3\pi$ amplitude for the physical process $\gamma\pi \to 2\gamma$ for zero scattering angle, normalised to the chiral anomaly prediction: $F_{3\pi}^{\text{anom}} \equiv e/4\pi F_\pi^3$. Various model predictions drawn are (1) the VMD model of Rudaz [11], (2) the modified Rudaz model implementing the KSRF relation [21], (3) chiral Lagrangian with Wess-Zumino-Witten term gauged by massive vector-axial vector Yang-Mills fields [8], (4) the Chiral Perturbation calculation of Bijnens et al. ref. [28] (since the amplitude is complex, its absolute value is plotted for this approach), and (5) the present confined quark model (QCM).
List of Tables

Table 1. Some QCM results for low energy mesonic quantities as compared with their experimental counterparts.

Table 2. Various functions entering the QCM calculation of meson vertices (coupling strengths), masses (self energies), etc. Functions $a(u)$ and $b(u)$ characterise the confined quark propagator as discussed in Section 3.

Table 3. Comparison of several model predictions and the experimental result of the cross section for $\pi^-\gamma \rightarrow \pi^-\pi^0$. Experiment was done in the Coulomb field of a nucleus of charge $Z$ (Primakoff effect), thus the appropriate quantity to be compared with model results is $\sigma/Z^2$. 
Table 1. Several low energies quantities obtained in QCM.

| Quantity | QCM | Expt. |
|----------|-----|-------|
| $F_\pi$  | $\frac{\Lambda R_P(\mu_\pi^2)}{2\pi} \sqrt{\frac{3}{R_{PP}(\mu_\pi^2)}}$ | 93 MeV | 93 MeV |
| $F_K$    | $\frac{\Lambda R_P(\mu_K^2)}{2\pi} \sqrt{\frac{3}{R_{PP}(\mu_K^2)}}$ | 112 MeV | 111 MeV |
| $F_{\pi\gamma\gamma}$ | $\frac{e^2}{\pi} \frac{1}{\Lambda} \frac{R_{PVV}(\mu_\pi^2)}{\sqrt{3R_{PP}(\mu_\pi^2)}}$ | 0.024 GeV$^{-1}$ | 0.025 GeV$^{-1}$ |
| $g_{\rho\pi\pi}$ | $\frac{2\pi\sqrt{2}R_{PPP}(\mu_\rho^2)}{R_{PP}(\mu_\pi^2)\sqrt{2}}$ | 6.0 | 6.0 |
| $g_{\rho\pi\gamma}$ | $\frac{em_\rho}{\Lambda} \sqrt{\frac{2}{3}} \frac{R_{PVV}(\mu_\rho^2)}{\sqrt{R_{PP}(\mu_\rho^2)R_{VV}(\mu_\pi^2)}}$ | 0.16 | 0.17 |
| $g_{\rho\gamma}$ | $\frac{1}{2\pi} \frac{R_V(\mu_\rho^2)}{\sqrt{2R_{VV}(\mu_\pi^2)}}$ | 0.18 | 0.20 |

Table 2. Structural integrals.

\[
R_{PP}(x) = B_0 + \frac{x}{4} \int_0^1 du \frac{1}{\sqrt{1-u}} \left(\frac{1}{4}xu \right)^{1-u/2}
\]

\[
R_{VV}(x) = B_0 + \frac{x}{4} \int_0^1 du \left(\frac{1}{4}xu \right)^{1-u/2+u^2/4} \sqrt{1-u}
\]

\[
R_{P}(x) = A_0 + \frac{x}{4} \int_0^1 du \frac{1}{\sqrt{1-u}} \left(\frac{1}{4}xu \right)^{1-u/2}
\]

\[
R_{PVV}(x) = \frac{1}{4} \int_0^1 du \ln \left(\frac{1+\sqrt{1-u}}{1-\sqrt{1-u}}\right)
\]

\[
R_{PPP}(x) = B_0 + \frac{x}{4} \int_0^1 du \frac{1}{\sqrt{1-u}} \left(\frac{1}{4}xu \right)^{1-u/2}
\]

\[
R_{V}(x) = B_0 + \frac{x}{4} \int_0^1 du \left(\frac{1}{4}xu \right)^{1-u/2} \sqrt{1-u}
\]

\[
A_0 = \int_0^\infty du \frac{1}{\sqrt{1-u}} = 1.09 \quad B_0 = \int_0^\infty du \left(\frac{1}{4}xu \right)^{1-u/2} = 2.26
\]
| Model                  | $\sigma/Z^2$ (nb) | Expt. | Ref. |
|------------------------|-------------------|-------|------|
| Chiral anomaly         | 0.95              |       |      |
| Modified Rudaz         | 1.17              | 1.78 ± 0.47 (C) | 21   |
| Bijnens et al.         | 1.12              | 1.54 ± 0.34 (Al) | 28   |
| Kaymacalan et al.      | 2.10              | 1.64 ± 0.37 (Fe) | 8    |
| QCM                    | 1.27              |       |      |