REGULATORS OF $K_1$ OF HYPERGEOMETRIC FIBRATIONS

MASANORI ASAKURA AND NORIYUKI OTSUBO

Abstract. We study a deformation of what we call hypergeometric fibrations. Its periods and $K_1$-regulators are described in terms of hypergeometric functions $\binom{\alpha_1, \alpha_2, \alpha_3}{\beta_1, \beta_2}; z$ in a variable given by the deformation parameter.

1. INTRODUCTION

In [1] and [2] we studied the periods and regulators for a certain class of fibrations, which we call hypergeometric fibrations (see [3] for the definition). The purpose of this paper is to extend the main results in [2].

Let $f : X \to \mathbb{P}^1$ be a hypergeometric fibration in the sense of §3.1. Let $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ be a map given by $t \mapsto t^l$ with $l \geq 1$ an integer. Let

$$
\begin{array}{cccc}
X^{(l)} & \xrightarrow{i} & X \times_{\mathbb{P}^1} \mathbb{P}^1 & \xrightarrow{f} & X \\
\downarrow f^{(l)} & & \downarrow \pi & & \downarrow f \\
\mathbb{P}^1 & & \mathbb{P}^1 & & \mathbb{P}^1
\end{array}
$$

be a Cartesian diagram with $i$ a desingularization. One of the main results in [2] is the period formula which describes the periods of $X^{(l)}$, and the other is the regulator formula which describes Beilinson’s regulator map on the motivic cohomology group $H^3_{\text{mot}}(X^{(l)}, \mathbb{Q}(2))$, especially on elements supported on certain singular fibers of $f^{(l)}$. In particular we described the regulator in terms of the special values of the generalized hypergeometric functions $\binom{\alpha_1, \alpha_2, \alpha_3}{\beta_1, \beta_2}; z$ at $z = 1$.

We extend those results in the following way. Our idea is simple, just replacing $\pi$ with a map $\pi_\lambda$ given by $t \mapsto \lambda - t^l$ for $\lambda \in \mathbb{C} \setminus \{0, 1\}$. We then obtain fibrations $f^{(l)}_\lambda : X^{(l)}_\lambda \to \mathbb{P}^1$ in the same way as above, and they are parametrized by $\lambda$. We discuss the periods and regulators for $X^{(l)}_\lambda$. Since the fibrations are parametrized by $\lambda$, the periods and regulators are no longer complex numbers but analytic functions. The main results of this paper are to describe them in terms of hypergeometric functions (Theorems 4.1, 5.1).

Taking the limits $\lambda \to 0$ of mixed Hodge structures (⇔ the nearby cycle cohomology functor $\psi_{\lambda=0}$), one can derive the main results of [2] from our main results. However we make somewhat a strong assumption “$\alpha_1 \in \mathbb{Z}$” throughout this paper, so that they do not cover all of [2].

Our another motivation is the logarithmic formula in [4] where we gave a sufficient condition for that the special value of $\binom{\alpha_1, \alpha_2, \alpha_3}{\beta_1, \beta_2}; z$ at $z = 1$ is written by a linear

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Theorem 5.1 (=a precise version of Theorem 5.9) enables us to obtain its functional version, namely we can give a sufficient condition for that \( 3F_2(z) \) is written in terms of the logarithmic functions. This will be discussed in a paper [3].

At the conference “Regulator IV” in Paris (May 2016), S. Bloch asked the first author whether results in the author’s talk gave examples to the following question of V. Golyshev.

**Question:** Let

\[
P_{HG} = D_z \prod_{i=1}^{p-1} (D_z + \beta_i - 1) - z \prod_{i=1}^{p} (D_z + \alpha_i), \quad D_z := \frac{d}{dz}
\]

be the hypergeometric differential operator and let \( M = \mathcal{D}_S/\mathcal{D}_S P_{HG} \) the \( \mathcal{D}_S \)-module on \( S = \mathbb{P}^1 \setminus \{0, 1, \infty\} \) where \( \mathcal{D}_S \) denotes the sheaf of differential operators. Suppose that \( M \) is reducible, equivalently \( \exists \alpha_i \in \mathbb{Z} \) or \( \alpha_j - \beta_k \in \mathbb{Z} \) for some \( j, k \), so that there is an exact sequence

\[
0 \to N \to M \to Q \to 0
\]

of \( \mathcal{D}_S \)-modules. Then does it underly a variation of mixed Hodge structures of geometric origin? If so, does the extension data arise from Beilinson’s regulator map on a motivic cohomology group? Moreover, is the regulator described in terms of hypergeometric functions which are solutions of \( P_{HG} \)?

See Theorem 5.8. Our regulator formula (Theorem 5.1) gives an affirmative answer in case \( p = 3 \) and \( \alpha_1 = \alpha_2 = 1 \). However we do not have a general solution to his question.

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**Notations.** For \( \alpha \in \mathbb{C} \) and an integer \( n \geq 0 \), \( (\alpha)_n = \prod_{i=0}^{n-1} (\alpha + i) \) is the Pochhammer symbol and the generalized hypergeometric function is defined by

\[
pF_{p-1} \left( \begin{array}{c} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_{p-1} \end{array} ; x \right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_n}{\prod_{j=1}^{p-1} (\beta_j)_n} \frac{x^n}{n!}.
\]

When \( p = 2 \), this is called the Gauss hypergeometric function. We use the standard notation for the product of values of the gamma function \( \Gamma(s) \)

\[
\Gamma \left( \begin{array}{c} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} \right) = \frac{\prod_{i=1}^{p} \Gamma(\alpha_i)}{\prod_{j=1}^{q} \Gamma(\beta_j)}.
\]

Throughout this paper, we fix an embedding \( \overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \), and think \( \overline{\mathbb{Q}} \) of being a subfield. For a variety \( X \) over \( \overline{\mathbb{Q}} \), \( H^n_{\text{dR}}(X) = H^n_{\text{dR}}(X/\overline{\mathbb{Q}}) \) denotes the algebraic de Rham cohomology and \( H^n(X, \mathbb{Q}) \) denotes the Betti cohomology of the analytic manifold \( X^{an} = (X \times_{\overline{\mathbb{Q}}} \mathbb{C})^{an} \).

2. **Betti-de Rham Structures, Hodge-de Rham Structures and Periods**

2.1. **Betti-de Rham structures and Hodge-de Rham structures.** Let \( k_B, k_{\text{dR}} \) be fields with fixed embeddings \( k_B \hookrightarrow \mathbb{C} \) and \( k_{\text{dR}} \hookrightarrow \mathbb{C} \). A Betti-de Rham structure over \( (k_B, k_{\text{dR}}) \) (abbreviated BdR) is a datum \( (H_B, H_{\text{dR}}, i) \) consisting of
• a finite dimensional vector space $H_B$ (resp. $H_{\text{dR}}$) over $k_B$ (resp. $k_{\text{dR}}$),
• a comparison isomorphism $\iota: \C \otimes_{k_{\text{dR}}} H_{\text{dR}} \to \C \otimes_{k_B} H_B$.

A Hodge-de Rham structure over $k_{\text{dR}}$ (abbreviated $\text{HdR}$) is a datum $(H_B, H_{\text{dR}}, F^\bullet, \iota)$ consisting of

• a finite dimensional vector space $H_B$ (resp. $H_{\text{dR}}$) over $\Q$ (resp. $k_{\text{dR}}$),
• a finite decreasing filtration $F^\bullet$ on $H_{\text{dR}}$
• a comparison isomorphism $\iota: \C \otimes_{k_{\text{dR}}} H_{\text{dR}} \to \C \otimes_{k_B} H_B$

such that $(H_B, \C \otimes_{k_{\text{dR}}} H_{\text{dR}}, \C \otimes_{k_{\text{dR}}} F^\bullet, \iota)$ is a Hodge structure in the usual sense. A mixed Hodge-de Rham structure $(H_B, W_B, H_{\text{dR}}, F^\bullet, W_{\text{dR}}, \iota)$ over $k_{\text{dR}}$ (abbreviated $\text{MHdR}$) is defined in the similar way where $W_B$ (resp. $W_{\text{dR}}$) is a finite increasing filtration on $H_B$ (resp. $H_{\text{dR}}$). The Tate twists $\Q(r) = (Q, k_{\text{dR}}, F^\bullet, \iota)$ is defined as $F^{-r}k_{\text{dR}} = k_{\text{dR}}, F^{-r+1}k_{\text{dR}} = 0$ and the comparison $\iota: k_{\text{dR}} \to \C$ given by $1 \mapsto (2\pi i)^{-r}$. The dual and tensor products of $\text{BdR}$, $\text{HdR}$ and $\text{MHdR}$ are defined in the customary way.

In this paper we usually consider the case $k_{\text{dR}} = \overline{\Q} \hookrightarrow \C$ with the fixed embedding.

A filtered Betti-de Rham structure over $(k_B, k_{\text{dR}})$ is a datum $(H_B, H_{\text{dR}}, F^\bullet, \iota)$ consisting of a Betti-de Rham structure $(H_B, H_{\text{dR}}, \iota)$ and a finite decreasing filtration $F^\bullet$ on $H_{\text{dR}}$. The category $\text{Filt-BdR} = \text{Filt-BdR}_{k_B, k_{\text{dR}}}$ of filtered Betti-de Rham structures over $(k_B, k_{\text{dR}})$ is not abelian but exact category. The Yoneda extension groups

$$\text{Ext}^\bullet_{\text{Filt-BdR}}(H, H')$$

are defined in the canonical way (cf. [5] 1.1). The following isomorphism is well-known (cf. [2] Proposition 2.1).

**Proposition 2.1** (Carlson’s isomorphism). Let $H = (H_B, H_{\text{dR}}, F^\bullet, \iota)$ be a filtered Betti-de Rham structure. Then there is a natural isomorphism

$$\text{Ext}^\bullet_{\text{Filt-BdR}}(k, H) \cong (\C \otimes_{k_{\text{dR}}} H_{\text{dR}})/(F^0 H_{\text{dR}} + \iota^{-1} H_B)$$

where $k = (k_B, k_{\text{dR}}, F^\bullet, \text{id})$ denotes the unit object which is defined as $F^0 k_{\text{dR}} = k_{\text{dR}}, F^1 k_{\text{dR}} = 0$ and the comparison is the identity.

2.2. Periods. For a Betti-de Rham structure $H = (H_B, H_{\text{dR}}, \iota)$, the period matrix of $H$ is defined to be the representation matrix of $\iota$ with respect to the $k_B, k_{\text{dR}}$-lattices $H_B, H_{\text{dR}}$, and we denote by

$$\text{Per}(H) \in \text{GL}_r(k_B)/\text{GL}_r(\C)/\text{GL}_r(k_{\text{dR}}).$$

2.3. Multiplication. A multiplication on a Betti-de Rham structure $H$ by a commutative $\Q$-algebra $R$ is defined as a ring homomorphism $R \to \text{End}_{\text{BdR}}(H)$ to the endomorphism ring of Betti-de Rham structures. The tensor product $H_1 \otimes_R H_2$ over $R$ is defined to be

$$H_1 \otimes_R H_2 = (H_{1,B} \otimes_{k_B} R H_{2,B}, H_{1,\text{dR}} \otimes_{k_{\text{dR}}} R H_{2,\text{dR}}, \iota_1 \otimes \iota_2)$$

endowed with multiplication by $R$. The multiplication on the dual Betti-de Rham structure

$$H^* = (\text{Hom}_{k_B}(H_B, k_B), \text{Hom}_{k_{\text{dR}}}(H_{\text{dR}}, k_{\text{dR}}), \iota)$$

is defined in such a way that $r \circ \phi := \phi \circ r$ for $\phi \in \text{Hom}(H_B, k_B)$ and $r \in R$.

A multiplication on a filtered $\text{BdR}$, $\text{HdR}$, $\text{MHdR}$ and its $\chi$-parts are defined in the same way as above.
Assume \( \text{Im}(k_B \to \mathbb{C}) \subset \mathbb{Q} \) and \( \text{Im}(k_{\text{dR}} \to \mathbb{C}) \subset \mathbb{Q} \) (note that \( \mathbb{Q} \to \mathbb{C} \) is fixed throughout the paper). For a homomorphism \( \chi : R \to \mathbb{Q} \), we define the \( \chi \)-part of a BdR structure \( H \) as

\[
H(\chi) := (H_B(\chi), H_{\text{dR}}(\chi), \iota)
\]

where \( k_B \otimes_R \mathbb{Q} \) and \( k_{\text{dR}} \otimes_R \mathbb{Q} \) are induced from \( \chi \) and the embeddings \( k_B \to \mathbb{Q} \) and \( k_{\text{dR}} \to \mathbb{Q} \). Then \( H(\chi) \) is a BdR over \( (\mathbb{Q}, \mathbb{Q}) \). We call its period matrix

\[
\text{Per}(H(\chi)) \in \text{GL}_r(\mathbb{Q})/\text{GL}_r(\mathbb{C})
\]

the \( \chi \)-part of the period matrix of \( H \). The \( \chi \)-part of filtered BdR, HdR and MHdR are defined in the same way.

Suppose that \( R \) is a semisimple and finite-dimensional \( \mathbb{Q} \)-algebra. Then the functor \( \text{Filt-BdR}_{k_B, k_{\text{dR}}} \to \text{Filt-BdR}_{\mathbb{Q}, \mathbb{Q}} \) given by \( H \to H(\chi) \) is exact. Composing with the forgetting functor \( \text{MHdR}_{k_{\text{dR}}} \to \text{Filt-BdR}_{\mathbb{Q}, k_{\text{dR}}} \) one has a map

\[
\text{Ext}^1_{\text{MHdR}_{k_{\text{dR}}}}(\mathbb{Q}, H) \to \text{Ext}^1_{\text{Filt-BdR}_{\mathbb{Q}, \mathbb{Q}}}(\mathbb{Q}, H(\chi)), \quad M \mapsto M(\chi)
\]

and we call \( M(\chi) \) the \( \chi \)-part of extension class \( M \).

Let \( X \) be a smooth projective variety over \( k_{\text{dR}} \). Let

\[
\text{reg} : H^i_{\text{dR}}(X, \mathbb{Q}(j)) \to \text{Ext}^1_{\text{MHdR}}(\mathbb{Q}, H^{i-1}(X, \mathbb{Q}(j))), \quad i \neq 2j
\]

be the Beilinson regulator map. Suppose that the mixed Hodge de Rham structure \( H := H^{i-1}(X, \mathbb{Q}(j)) \) over \( k_{\text{dR}} \) has a multiplication by \( R \). Then we call the composition

\[
\text{reg}(\chi) : H^i_{\text{dR}}(X, \mathbb{Q}(j)) \to \text{Ext}^1_{\text{MHdR}}(\mathbb{Q}, H) \xrightarrow{\text{Ext}^1_{\text{Filt-BdR}_{\mathbb{Q}, \mathbb{Q}}}(\mathbb{Q}, H(\chi))}
\]

the \( \chi \)-part of regulator map.

2.4. Variations of Hodge-de Rham structures. Let \( S \) be a smooth variety over \( k_{\text{dR}} \). A filtered Betti-de Rham structure on \( S \) consists of a datum \( (H_B, H_{\text{dR}}, F^\bullet, \nabla, \iota) \) where

- \( H_B \) is a local system of finite dimensional \( k_B \)-modules on \( S^{an} \),
- \( H_{\text{dR}} \) is a locally free \( \mathcal{O}_S \)-module of finite rank, and \( F^\bullet \) is a finite decreasing filtration which is locally a direct summand,
- \( (H_{\text{dR}}, \nabla) \) is a connection with regular singularities on \( S \) such that \( \nabla(F^p) \subset \Omega^1_S \otimes F^{p-1} \),
- \( \iota : \mathcal{O}_S^{an} \otimes_{a^{-1}, \mathcal{O}_S} a^{-1} H_{\text{dR}} \simeq \mathcal{O}_S^{an} \otimes k_B H_B \) is a comparison isomorphism such that \( \nabla \) annihilates the lattice \( H_B \), where \( a : S^{an} \to S^{zar} \) is the canonical map from the analytic site to the Zariski site and \( \mathcal{O}_S^{an} \) denotes the sheaf of analytic functions on \( S^{an} \).

A filtered BdR on \( S \) is called a variation of Hodge-de Rham structure (abbreviated VHdR) if \( k_B = \mathbb{Q} \) and \( (H_B, \mathcal{O}_S^{an} \otimes \mathcal{O}_S H_{\text{dR}}, \mathcal{O}_S^{an} \otimes \mathcal{O}_S F^\bullet, \iota, \nabla) \) is a variation of Hodge structure in the usual sense. We also define, in a customary way, a variation of mixed Hodge-de Rham structure (abbreviated VMHdR) on \( S \) which consists of a datum \( (H_B, W^\bullet_{\text{dR}}, H_{\text{dR}}, F^\bullet, W^\bullet_{\text{dR}}, \iota, \nabla) \).

3. Hypergeometric Fibrations

In what follows we work over the base field \( k_{\text{dR}} = \mathbb{Q} \).
3.1. Definition. Let $R$ be a finite-dimensional semisimple $\mathbb{Q}$-algebra. Let $e : R \to E$ be a surjection onto a number field $E$. Let $X$ be a smooth projective variety over $k_{DR}$, and $f : X \to \mathbb{P}^1$ a surjective map. We say $f$ is a hypergeometric fibration with multiplication by $(R, e)$ if it is endowed with a multiplication on $R^1 f_*\mathbb{Q}|_U$ by $R$ where $U \subset \mathbb{P}^1$ is the maximal Zariski open set such that $f$ is smooth over $U$ and the following conditions hold. We fix an inhomogeneous coordinate $t \in \mathbb{P}^1$.

- $f$ is smooth over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$,
- $\dim E(R^1 f_*\mathbb{Q})(e) = 2$ where we write $V(e) := E \otimes_{e, R} V$ the $e$-part,
- Let $\text{Pic}_f^0 \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the Picard fibration whose general fiber is the Picard variety $\text{Pic}^0(f^{-1}(t))$, and let $\text{Pic}_f^0(e)$ be the component associated to the $e$-part $(R^1 f_*\mathbb{Q})(e)$ (this is well-defined up to isogeny). Then $\text{Pic}_f^0(e) \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ has a totally degenerate semistable reduction at $t = 1$.

The last condition is equivalent to say that the local monodromy $T$ on $(R^1 f_*\mathbb{Q})(e)$ at $t = 1$ is unipotent and the rank of log monodromy $N := \log(T)$ is maximal, namely $\text{rank}(N) = \frac{1}{2} \dim E(R^1 f_*\mathbb{Q})(e) (= [E : \mathbb{Q}]$ by the second condition).

Example 3.1 (Gauss type). Let $f : X \to \mathbb{P}^1$ be the fibration over $\overline{\mathbb{Q}}$ whose general fiber is defined by an affine equation

$$y^N = x^a(1 - x)^b(t - x)^{N-b}$$

with $0 < a, b < N$ and $\gcd(N, a, b) = 1$. Let $\mu_N \subset \overline{\mathbb{Q}}$ be the group of $N$th roots of unity. It gives automorphisms $(x, y, t) \mapsto (x, \zeta_N y, t)$ for $\zeta_N \in \mu_N$, and then it defines a multiplication by $R = \mathbb{Q}[\mu_N]$ (= group ring) on $R^1 f_*\mathbb{Q}$. Then one can show that $f$ is a hypergeometric fibration with multiplication by $(R, e)$ if and only if $d := \#\text{Ker}[e : \mu_N \to E^\times]$ satisfies $ad/N, bd/N \not\in \mathbb{Z}$.

Example 3.2 (Fermat type). Let $f : X \to \mathbb{P}^1$ be the fibration over $\overline{\mathbb{Q}}$ defined by an affine equation

$$(x^n - 1)(y^m - 1) = 1 - t.$$ 

The group ring $R = \mathbb{Q}[\mu_n, \mu_m]$ acts on $R^1 f_*\mathbb{Q}$ in a natural way. Then $f$ is a hypergeometric fibration with multiplication by $(R, e)$ if and only if $e : \mathbb{Q}[\mu_n \times \mu_m] \to E$ does not factor through the projections $\mu_n \times \mu_m \to \mu_n$ nor $\mu_n \times \mu_m \to \mu_m$. The reason why we call this “Fermat type” is the following. Letting $u = x^{-1}$, $v = y^{-1}$ and $s = tx^{-y-m}$,

$$(x^n - 1)(y^m - 1) = 1 - t \iff u^n + v^m = 1 + s.$$ 

3.2. Basic properties. We sum up some properties on our hypergeometric fibrations, which will be used in later sections. See [2] §3.3 for complete proof\footnote{The definition of “hypergeometric fibrations” in [2] §3.1 is slightly different from that in §3.3. However, the same arguments entirely work in our situation.}.

Proposition 3.3. $\dim_{\overline{\mathbb{Q}}} F^1 H^1_{\text{dr}}(X_t)(\chi) = 1$ and $\dim_{\overline{\mathbb{Q}}} G^0_{\text{Fr}} H^1_{\text{dr}}(X_t)(\chi) = 1$ where $X_t$ is a general fiber.

Proposition 3.4. $(R^1 f_*\mathbb{Q})(\chi)$ is an irreducible $\overline{\mathbb{Q}}[\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})]$-module. Moreover let $\alpha_0^N, \beta_0^N$ (resp. $\alpha^N, \beta^N$) be rational numbers such that $e^{2\pi i \alpha_0^N}, e^{2\pi i \beta_0^N}$ (resp. $e^{2\pi i \alpha^N}, e^{2\pi i \beta^N}$) are eigenvalues of the local monodromy on $(R^1 f_*\mathbb{Q})(\chi)$ at $t = 0$ (resp. $t = \infty$). Then, none of $\alpha_0^N + \alpha^N, \alpha_0^N + \beta^N, \beta_0^N + \alpha^N, \beta_0^N + \beta^N$ is an integer.
Proposition 3.5. Let $\psi_{t=1}$ denote the nearby cohomology functor at $t = 1$ and let $W_\bullet$ be the weight monodromy filtration induced by the log monodromy $N_1 = \log(T_1)$. Then there are isomorphisms

\[
\text{Gr}^W_{1} \psi_{t=1} R^1 f_* Q(e) = \text{Coker}(N_1) \cong E \otimes \mathbb{Q}(-2), \quad \text{Gr}^W_{0} \psi_{t=1} R^1 f_* Q(e) = \text{Ker}(N_1) \cong E,
\]

\[
\text{Gr}^W_{j} \psi_{t=1} R^1 f_* Q(e) = 0, \quad j \neq 0, 2
\]
of Hodge-de Rham structure with compatible $E$-action, where $E$ is endowed with a trivial Hodge-de Rham structure of type $(0, 0)$.

4. Period Formula

4.1. Setting. Let $R_0$ be a finite-dimensional semisimple $\mathbb{Q}$-algebra and $e_0 : R_0 \to E_0$ be a surjection onto a number field $E_0$. Let $f : X \to \mathbb{P}^1$ be a hypergeometric fibration over $\mathbb{Q}$ with multiplication by $(R_0, e_0)$ in the sense of [3, 4].

Let $S := \mathbb{A}^1 \setminus \{0, 1\}$ be defined over $\mathbb{Q}$ with coordinate $\lambda$. Write $\mathbb{P}^1_S := \mathbb{P}^1 \times S$. Put $\mathcal{U} := (\mathbb{A}^1 \setminus \{0, 1\} \times S) \setminus \Delta$ where $\Delta$ is the diagonal subscheme. Let $l \geq 1$ be an integer. Let $\pi : \mathbb{P}^1_S \to \mathbb{P}^1$ be a morphism over $S$ given by $(s, \lambda) \mapsto (\lambda - s^l, \lambda)$. Then we consider a variation of Hodge-de Rham structures

\[
\mathcal{M} := \pi_* \mathcal{Q} \otimes \text{pr}_1^* R^1 f_* \mathcal{Q}|_{\mathcal{U}}, \quad \text{pr}_1 : \mathbb{P}^1_S = \mathbb{P}^1 \times S \to \mathbb{P}^1
\]
on $\mathcal{U}$ and a variation of mixed Hodge-de Rham structures

\[
\mathcal{H} := R^1 \text{pr}_2_* \mathcal{M} = (\mathcal{H}_B, W_\bullet^B, \mathcal{H}_{dR}, F^*, W_\bullet^{dR}, \nabla, i), \quad \text{pr}_2 : \mathcal{U} \to S
\]
on $S$. Since $\mathcal{M}$ is a variation of Hodge-de Rham structures of pure weight 1, the weights of $\mathcal{H}$ are at most 2, 3 and 4. We have an exact sequence

\[
0 \to W_2 \mathcal{H} \to \mathcal{H} \to \mathcal{H}/W_2 \mathcal{H} \to 0
\]

(4.1) with $W_2 \mathcal{H}$ a variation of Hodge-de Rham structures of pure weight 2.

The group $\mu_l \subset \mathbb{Q}_l^\times$ of $l$th roots of unity acts on the cyclic covering $\pi$. Thus the group ring $R := R_0[\mu_l]$ acts on the sheaf $\mathcal{M}$ and hence on $\mathcal{H}$. In what follows we fix $e : R \to E$ a surjection onto a number field $E$ such that $\text{Ker}(e) \supset \text{Ker}(e_0)$. There is a unique embedding $E_0 \to E$ making the diagram

\[
\begin{array}{ccc}
R_0 & \xrightarrow{e_0} & E_0 \\
\downarrow \downarrow & & \downarrow \downarrow \\
R & \xrightarrow{e} & E
\end{array}
\]

commutative. Then the $e$-part

\[
\mathcal{M}(e) := E \otimes_{e, R} \mathcal{M} \cong E \otimes_{E_0[\mu_l]} (\pi_* \mathcal{Q} \otimes R^1 f_* \mathcal{Q}(e_0))
\]
is of rank 2 over $E$.

Let $\chi : R \to \mathbb{Q}$ be a homomorphism factoring through $e$. This also induces $R_0 \to \mathbb{Q}$ which we also write $\chi$ by abuse of notation. Define $k \in \{0, 1, \ldots, l - 1\}$ by $\chi(\zeta_l) = \zeta_l^k$ for all $\zeta_l \in \mu_l$. Put $q := k/l$. Moreover, let $\alpha^\chi, \beta^\chi$ be rational numbers in the interval $[0, 1)$ such that $e^{2\pi i \alpha^\chi}$ and $e^{2\pi i \beta^\chi}$ (resp. $e^{2\pi i \alpha}$ and $e^{2\pi i \beta}$) are eigenvalues of the local monodromy $T_0$ at $t = 0$ (resp. $T_\infty$ at $t = \infty$) on the $\chi$-part $(R^1 f_* \mathcal{Q})(\chi) = \mathbb{Q} \otimes_{\chi, R_0} R^1 f_* \mathcal{Q}$. Equivalently, these are congruent mod $Z$ to the eigenvalues of the residue $\text{Res}_{t=0}(\nabla)$ (resp. $\text{Res}_{t=\infty}(\nabla)$) of the connection.
on the \(\chi\)-part of the bundle \(R^3f_!\Omega^\bullet_X|_{\mathbb{P}^1\setminus\{0, 1, \infty\}}\). Note that the local monodromy \(T_1\) at \(t = 1\) is unipotent. Since \(T_0T_1T_\infty\) is the identity, we have
\[
\alpha_0^\chi + \beta_0^\chi + \alpha^\chi + \beta^\chi \in \mathbb{Z}.
\]

4.2. Theorem on periods. Let \(\mathcal{O}^{\text{an}}\) be the sheaf of analytic functions on \(S^{\text{an}}\), \(\mathcal{O}^{\text{zar}}\) the Zariski sheaf of rational functions (with coefficients in \(\overline{\mathbb{Q}}\)) on \(S\) with coordinate \(\lambda\). Let \(a\) be the canonical morphism from the analytic site to the Zariski site. We put \(W_2\mathcal{H}_{\text{an}}^{\text{an}} := \mathcal{O}^{\text{an}} \otimes_{\mathcal{O}^{\text{an}}} a^{-1}W_2\mathcal{H}_{\text{an}}, \quad W_2\mathcal{H}_{B}^{\text{an}} := \mathcal{O}^{\text{an}} \otimes_{\mathcal{O}^{\text{an}}} W_2\mathcal{H}_{B}\)

sheaves on the analytic site. The comparison isomorphism of \(W_2\mathcal{H}\) gives an analytic section
\[
\iota \in \Gamma(S^{\text{an}}, \text{Isom}(W_2\mathcal{H}_{\text{an}}^{\text{an}}, W_2\mathcal{H}_{B}^{\text{an}})).
\]

There is a canonical map \(d := \text{rank}W_2\mathcal{H}\)
\[
\text{Isom}(W_2\mathcal{H}_{\text{an}}^{\text{an}}, W_2\mathcal{H}_{B}^{\text{an}}) \to \text{GL}_d(\mathbb{Q})\backslash\text{GL}_d(\mathcal{O}^{\text{an}})/\text{GL}_d(a^{-1}\mathcal{O}^{\text{zar}})
\]
of sheaves by associating the representation matrices with respect to the lattices \(W_2\mathcal{H}_{B}\) and \(W_2\mathcal{H}_{\text{an}}\). We call the image of \(\iota\) the period matrix of \(W_2\mathcal{H}\):
\[
\text{Per}(W_2\mathcal{H}) \in \Gamma(S^{\text{an}}, \text{GL}_d(\mathbb{Q})\backslash\text{GL}_d(\mathcal{O}^{\text{an}})/\text{GL}_d(a^{-1}\mathcal{O}^{\text{zar}})).
\]
The \(\chi\)-part of \(W_2\mathcal{H}\) defines the \(\chi\)-part of the period matrix \((r := \text{rank}W_2\mathcal{H}(\chi))
\[
\text{Per}(W_2\mathcal{H}(\chi)) \in \Gamma(S^{\text{an}}, \text{GL}_r(\mathbb{Q})\backslash\text{GL}_r(\mathcal{O}^{\text{an}})/\text{GL}_r(a^{-1}\mathcal{O}^{\text{zar}})).
\]

**Theorem 4.1** (Period formula). Assume \(\alpha_0^\chi = 0 \iff \alpha_0^\chi + \alpha^\chi + \beta^\chi \in \mathbb{Z}\) and that \(q^\chi \not\equiv 0\), \(\alpha^\chi, \beta^\chi, \alpha^\chi + \beta^\chi \pmod{\mathbb{Z}}\). Then the rank of \(W_2\mathcal{H}(\chi)\) is 2. For some \(\mu > 1, \mu \equiv q^\chi \pmod{\mathbb{Z}}\), the period matrix is locally given by
\[
\text{Per}(W_2\mathcal{H}(\chi)) = 2\pi i \begin{pmatrix}
\Theta F_\mu(\lambda) & \Theta G_\mu(\lambda)
\end{pmatrix},
\]
where we put
\[
F_\mu(\lambda) := \frac{1}{\mu}(\lambda - 1)^{\mu/2}F_1\left(\alpha^\chi, \beta^\chi ; \mu + 1 - 1 - \lambda\right),
\]
\[
G_\mu(\lambda) := (-1)^{\mu} \Gamma \left(\begin{array}{c}
\mu, \mu + 1 - \alpha^\chi - \beta^\chi \\
\mu + 1 - \alpha^\chi + \beta^\chi
\end{array}\right) 2F_1\left(\begin{array}{c}
\alpha^\chi - \mu, \beta^\chi - \mu \\
\alpha^\chi + \beta^\chi - \mu
\end{array}\right),
\]
and \(\Theta\) is a differential operator of the form
\[
\Theta = q(\lambda) + r(\lambda)\partial_\lambda, \quad q(\lambda), r(\lambda) \in \overline{\mathbb{Q}}[\lambda, 1/\lambda(\lambda - 1)].
\]

4.3. Proof of Period formula: Part 1. We first show \(\dim_{\overline{\mathbb{Q}}}W_2\mathcal{H}(\chi) = 2\). We write \(U_a := \text{pr}_{2, 1}^{-1}(a) \cong \mathbb{P}^1 \setminus \{0, 1, a, \infty\}\) for \(a \in S^{\text{an}} = \mathbb{C} \setminus \{0, 1\}\) where \(\text{pr}_{2, 1} : \mathcal{H} \to S\). Moreover we put the fibers
\[
\mathcal{M}_a := \mathcal{M}|_{\text{pr}_{2, 1}^{-1}(a)} \cong \pi_* \mathcal{Q} \otimes R^1f_*\mathcal{Q}, \quad H_a := \mathcal{H}|_{\{a\}} \cong H^1(\text{pr}_{2, 1}^{-1}(a), \mathcal{M}_a),
\]
where \(\pi_a : \mathbb{P}^1 \to \mathbb{P}^1\) is the map given by \(s \mapsto a - s^2\). When \(a \in \overline{\mathbb{Q}} \setminus \{0, 1\}\), we endow \(\mathcal{M}_a\) and \(\mathcal{H}_a\) with HodgeR structure over \(\overline{\mathbb{Q}}\) induced from the \(\mathbb{Q}\)-frames on \(\mathcal{M}_{\text{an}}\) and \(\mathcal{H}_{\text{an}}\) respectively. We then want to show \(\dim_{\overline{\mathbb{Q}}}W_2H_a(\chi) = 2\) for \(a \in \overline{\mathbb{Q}} \setminus \{0, 1\}\). The weight filtration induces an exact sequence
\[
0 \to W_2H_a(\chi) \to H_a(\chi) \to H_a(\chi)/W_2H_a(\chi) \to 0.
\]
There are canonical isomorphisms
\begin{align}
W_2H_a & \cong H^1(P^1, j_*\mathcal{M}_a), \quad j : P^1 \setminus \{0, 1, a, \infty\} \hookrightarrow P^1, \\
H_a/W_2H_a & \cong H^0(P^1, R^1j_*\mathcal{M}_a)
\end{align}
(4.2)
of mixed Hodge-de Rham structures. Let \(\varepsilon_k : Q[\mu]\rightarrow \overline{Q}\) be given by \(\varepsilon(\zeta) = \zeta^k\)
and \(\overline{Q}(\varepsilon_k) := \overline{Q} \otimes_{Q[\mu]} \pi_1Q\) a one-dimensional local system on \(P^1 \setminus \{a, \infty\}\). Then there is a natural isomorphism
\[\mathcal{M}_a(\chi) \cong \overline{Q}(\varepsilon_k) \otimes_{\overline{Q}} (R^1f_*Q)(\chi)\]
of \(\pi_1(P^1 \setminus \{0, 1, a, \infty\})\)-modules. Since \((R^1f_*Q)(\chi)\) is an irreducible \(\pi_1(P^1 \setminus \{0, 1, \infty\})\)-module (Proposition 3.3), so is \(\mathcal{M}_a(\chi)\) as a \(\pi_1(P^1 \setminus \{0, 1, a, \infty\})\)-module. In particular \(H^0(p^{-1}(a), \mathcal{M}_a(\chi)) = 0\). Hence
\[\dim_{\overline{Q}} H_a(\chi) = \dim_{\overline{Q}} H^1(p^{-1}(a), \mathcal{M}_a(\chi)) = -\chi(p^{-1}(a), \mathcal{M}_a(\chi))\]
\[= -\chi^{\text{top}}(p^{-1}(a)) \cdot \dim_{\overline{Q}} \mathcal{M}_a(\chi) = -(-2) \cdot 2 = 4.\]
Thus \(\dim_{\overline{Q}} W_2H_a(\chi) = 2 \iff \dim_{\overline{Q}} H_a(\chi)/W_2 = 2\).

Let \(T_0, T_1, T_a, T_{\infty}\) be the local monodromy on \(\mathcal{M}_a(\chi)\) at \(t = 0, 1, a, \infty\), respectively. \(T_1\) is unipotent with trivial action on \(\overline{Q}(\varepsilon_k)\), and \(T_a\) is multiplication by \(e^{2\pi i q^a}\) with trivial action on \((R^1f_*Q)(\chi)\). The eigenvalues of \(T_0\) (resp. \(T_{\infty}\)) are \(e^{2\pi i \alpha^a}, e^{2\pi i \beta^a}\) (resp. \(e^{2\pi i (-q^a+\alpha^a)}, e^{2\pi i (-q^a+\beta^a)}\)).

Recall (3.3). We then have
\[H_a/W_2 \cong H^0(P^1, R^1j_*\mathcal{M}_a)\]
\[\cong \bigoplus_{p=0, 1, a, \infty} \text{Coker}[T_p - 1 : \psi_{t=p} \mathcal{M}_a \rightarrow \psi_{t=p} \mathcal{M}_a \otimes Q(-1)]\]
where \(\psi_{t=p}\) denotes the nearby cohomology at \(t = p\). By the assumption \(q^a \neq 0, \alpha^a, \beta^a \pmod Z\), \(T_a\) and \(T_{\infty}\) have no eigenvalue 1 on the \(\chi\)-part of \(\psi_{t=p}\). Hence \(\text{Coker}[T_a - 1] = \text{Coker}[T_{\infty} - 1] = 0\). Note
\[\text{Coker}[T_p - 1 : \psi_{t=p} \mathcal{M}_a(\chi) \rightarrow \psi_{t=p} \mathcal{M}_a(\chi) \otimes Q(-1)]\]
\[\cong \overline{Q}(\varepsilon_k) \otimes_{\overline{Q}} \text{Coker}[T_p - 1 : (R^1f_*Q)(\chi) \rightarrow (R^1f_*Q)(\chi)], \quad p = 0, 1.\]
It follows from Proposition 3.3 that we have \(\dim_{\overline{Q}} \text{Coker}(T_1 - 1) = 1\). There remains to show \(\dim_{\overline{Q}} \text{Coker}(T_0 - 1) = 1\). By the assumption \(\alpha^a = 0\), one has \(\dim_{\overline{Q}} \text{Coker}(T_0 - 1) \geq 1\). If \(\dim_{\overline{Q}} \text{Coker}(T_0 - 1) = 2\) this means that \(T_0\) is trivial on \((R^1f_*Q)(\chi)\). Then \((R^1f_*Q)(\chi)\) cannot be irreducible as \(\overline{Q}[\pi_1(P^1 \setminus \{0, 1, \infty\})]\)-module. This contradicts with Proposition 3.3. We thus have \(\dim_{\overline{Q}} \text{Coker}(T_0 - 1) = 1\), and hence \(\dim_{\overline{Q}} H_a(\chi)/W_2 = 2\). This completes the proof of \(\dim_{\overline{Q}} W_2H_a(\chi) = 2\).

In the discussion above, we obtained the following.

**Proposition 4.2.** Assume \(\alpha^a \in Z\) and \(q^a \neq 0, \alpha^a, \beta^a \pmod Z\). Then there is an isomorphism
\[\mathcal{H}(e)/W_2 \cong \bigoplus_{p=0, 1} \text{Coker}[T_p - 1 : \psi_{t=p} \mathcal{M}(e) \rightarrow \psi_{t=p} \mathcal{M}(e) \otimes Q(-1)]\]
of variations of mixed Hodge-de Rham structures on \(P^1 \setminus \{0, 1, \infty\}\). Each \(\text{Coker}(T_p - 1)\) is one-dimensional over \(E\). Moreover, \(\text{Coker}(T_1 - 1)\) is endowed with a Hodge-de Rham structure of type \((2, 2)\) (Proposition 3.3).
4.4. Relative 1-form $\omega^\chi$. The $\chi$-part $(f, \Omega^1_{X/\mathbb{P}^1})(\chi)|_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}$ of the Hodge bundle has rank one (Proposition 3.3). Hence it is a trivial line bundle on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. In what follows we fix a relative 1-form

$\omega^\chi \in \mathcal{H}(\mathbb{P}^1 \setminus \{0, 1, \infty\}, (f, \Omega^1_{X/\mathbb{P}^1})(\chi))$

with coefficients in $\overline{\mathbb{Q}}$ which is everywhere nonzero (until the end of the paper). Let $X_t = f^{-1}(t)$ be the general fiber. We fix (nonzero) homology cycles

$\gamma = \gamma_t \in H_1(X_t, \overline{\mathbb{Q}}) \cap \text{Ker}(T_0 - 1), \quad \delta = \delta_t \in H_1(X_t, \overline{\mathbb{Q}}) \cap \text{Ker}(T_1 - 1)$.

Note that each $H_1(X_t, \overline{\mathbb{Q}}) \cap \text{Ker}(T_p - 1)$ is one-dimensional over $\overline{\mathbb{Q}}$ (Proposition 4.2).

**Lemma 4.3** (Key Lemma). There is a differential operator $\theta = p_0(t) + p_1(t) \frac{d}{dt}$ with $p_i(t) \in \overline{\mathbb{Q}}[t]$ such that

$$\int_\gamma \omega^\chi = B(\alpha^\chi, \beta^\chi) \cdot \theta_2 F_1 \left( \frac{\alpha^\chi, \beta^\chi}{\alpha^\chi + \beta^\chi}; t \right), \quad \int_\delta \omega^\chi = 2\pi i \cdot \theta_2 F_1 \left( \frac{\alpha^\chi, \beta^\chi}{1}; 1 - t \right).$$

(4.4)

Here $B(\alpha, \beta)$ is the beta function.

**Proof.** Since $\delta$ is $T_1$-invariant, $\int_\delta \omega^\chi$ is a single-valued meromorphic function at $t = 1$. So is $\int_\gamma \omega^\chi$ at $t = 0$ as $\gamma$ is $T_0$-invariant. Therefore (4.3) follows from [2], Lemmas 5.2, 5.3 and 5.4.

The differential equation

$$(t(D + \alpha)(D + \beta) - (D + \alpha + \beta - 1)D)f = 0, \quad D = t \frac{d}{dt}$$

has the Riemann scheme

$$\begin{align*}
\left\{ \begin{array}{ccc}
0 & 0 & \alpha \\
1 - \alpha - \beta & 0 & \beta \\
\end{array} \right. \\
t = 0 \quad t = 1 \quad t = \infty.
\end{align*}$$

Among Kummer’s 24 solutions, we used in the preceding lemma

$$f_1(t) := 2 F_1 \left( \frac{\alpha, \beta}{\alpha + \beta}; t \right), \quad f_2(t) := 2 F_1 \left( \frac{\alpha, \beta}{1}; 1 - t \right).$$

(4.5)

Later in Section 5.4, we will use the solution

$$f_3(t) := t^{1 - \alpha - \beta} 2 F_1 \left( \frac{1 - \alpha, 1 - \beta}{2 - \alpha - \beta}; t \right),$$

(4.6)

which has the characteristic exponent $1 - \alpha - \beta$ at $t = 0$. These satisfy the linear relation (cf. [6] §2.9)

$$\Gamma \left( \frac{1 - \alpha - \beta}{1 - \alpha, 1 - \beta} \right) f_1(t) - f_2(t) + \Gamma \left( \frac{\alpha + \beta - 1}{\alpha, \beta} \right) f_3(t) = 0.$$  

This can be written, using functional equations of the gamma function

$$B(\alpha, \beta) = \Gamma \left( \frac{\alpha, \beta}{\alpha + \beta} \right), \quad \Gamma(s + 1) = s \Gamma(s),$$

as

$$B(\alpha, \beta) f_1(t) + 2\pi i \frac{1 - e^{2\pi i (\alpha + \beta)}}{(1 - e^{2\pi i \alpha})(1 - e^{2\pi i \beta})} f_2(t) - B(1 - \alpha, 1 - \beta) f_3(t) = 0.$$  

(4.7)
4.5. Rational 2-forms $s^{m-1}ds \wedge \omega^X$. By taking an embedded resolution, we may assume that the reduced divisor

$$D := (f^{-1}(0) + f^{-1}(1) + f^{-1}(\infty))_{\text{red}}$$

of the singular fibers is a NCD. Recall from Lemma 4.3 (Key Lemma) the differential operator $\theta = p_0(t) + p_1(t)\frac{dt}{f}$. By replacing $\omega^X$ with $t^n(1-t)^m\omega^X$ for some $n, m \geq 0$, we may assume without loss of generality

P1: $p_i(t)$ are polynomials and $t(1-t)|p_i(t)$,

P2: $\omega^X \in \Gamma(P^1 \setminus \{\infty\}, f, \Omega^1_{X/P^1}(\log D))$, where the locally free sheaf $\Omega^1_{X/P^1}(\log D)$ is defined by the exact sequence

$$0 \longrightarrow f^*\Omega^1_{P^1}(\log(0 + 1 + \infty)) \longrightarrow \Omega^1_{X}(\log D) \longrightarrow \Omega^1_{X/P^1}(\log D) \longrightarrow 0.$$

Let $a \in \mathbb{C} \setminus \{0, 1\}$ and $\pi_a : P^1 \rightarrow P^1$ a morphism given by $s \mapsto a - s'$ as in 4.3. To distinguish the source and target of $\pi_a$, we denote by $P^1_a$ the target with inhomogeneous coordinate $t$, and by $P^1$ the source with inhomogeneous coordinate $s$. Let

$$\xymatrix{ X_a \ar[r]^i & P^1_a \times_{P^1} X \ar[d]^{f} \ar[r] & X \ar[d]^{f} \ar[l]_{f^*} \ar[r] & P^1 \ar[l]_{\pi_a} }$$

where $i$ is the desingularization such that the inverse image $D_a \subset X_a$ of $D$ is a NCD. Let $U_a \subset X_a$ (resp. $U_a \subset X_a$) be the inverse image of $P^1 \setminus \{0, 1, a, \infty\}$ (resp. $P^1 \setminus \{\infty\}$) under $\pi_a \circ f_a$. By the projection formula,

$$\mathcal{M}_a = \pi_{a*}Q \otimes R^1f_{a*}Q \cong \pi_{a*}(\pi^*_aR^1f_{a*}Q) = \pi_{a*}(R^1f_{a*}Q).$$

This implies

$$H_a = H^1(P^1 \setminus \{0, 1, a, \infty\}, \mathcal{M}_a) \cong H^1(P^1 \setminus \{s' = 0, a, a - 1, \infty\}, R^1f_{a*}Q)$$

and hence we have an exact sequence

$$0 \longrightarrow H_a \longrightarrow H^2(U_a, Q) \longrightarrow H^2(f_a^{-1}(s), Q).$$

In particular, by taking the weight 2 piece, we have a canonical isomorphism

$$W_2H_a \cong \text{Ker}[W_2H^2(U_a, Q) \rightarrow H^2(f_a^{-1}(s), Q)]$$

$$= \text{Ker}[H^2(X_a, Q)/H^2_{\overline{U}_a}(X_a) \rightarrow H^2(f_a^{-1}(s), Q)].$$

(4.8)

Consider the rational 2-form

$$s^{m-1}ds \wedge \omega^X = s^{m-1}ds \wedge \omega^X \in \Gamma(U, \Omega^2_{U/\overline{U}})$$

for $m \geq 1$. By the assumption P2,

$$s^{m-1}ds \wedge \omega^X|_{\lambda = a} \in \text{Im}[\Omega^1_{P^1_{\lambda}(\infty)} \wedge \Omega^2_{U_{\lambda}/\overline{U}_{\lambda}}(\log D) \rightarrow \Omega^2_{U_{\lambda}/\overline{U}_{\lambda}}(\log D_a)],$$

$$\subset \text{Im}[t(1-t)\Omega^1_{P^1_{\lambda}(0 + 1)} \wedge \Omega^2_{U_{\lambda}/\overline{U}_{\lambda}}(\log D) \rightarrow \Omega^2_{U_{\lambda}/\overline{U}_{\lambda}}(\log D_a)]$$

$$\subset t(1-t)\Omega^2_{U_{\lambda}/\overline{U}_{\lambda}}(\log D_a) = \Omega^2_{U_{\lambda}/\overline{U}_{\lambda}},$$

so that we have

$$s^{m-1}ds \wedge \omega^X|_{\lambda = a} \in \Gamma(U, \Omega^2_{U/\overline{U}}).$$
Let \([s^{m-1}ds \wedge \omega^\chi]|_{\lambda=a} \in H^2_{\text{dR}}(\mathcal{U}_a/\mathbb{C})\) denote the de Rham cohomology class. Obviously, its restriction to the general fiber \(f^{-1}(s)\) vanishes. Thus
\[\text{if } m \equiv k \mod l \text{ then } [s^{m-1}ds \wedge \omega^\chi]|_{\lambda=a} \in H^1(\mathbb{P}^1 \setminus \{0, 1, a, \infty\}, \mathcal{M}_a) \cap \text{Im} H^2_{\text{dR}}(\mathcal{U}_a/\mathbb{C}).\]

If \(m = k \mod l\) then \([s^{m-1}ds \wedge \omega^\chi]|_{\lambda=a}\) belongs to the \(\chi\)-part. By Proposition 4.2 together with the commutative diagram
\[
\begin{array}{ccc}
H_a & \rightarrow & H^2(U_a, \mathbb{Q}) \\
\downarrow & & \downarrow \\
H_a/W_2 & \rightarrow & H^2_{\text{dR}}(X_a, \mathbb{Q})
\end{array}
\]
with injective \(i\), we have
\[
[s^{m-1}ds \wedge \omega^\chi]|_{\lambda=a} \in W_2 H_{a,\text{dR}}(\chi) = W_2 H^1(\mathbb{P}^1 \setminus \{0, 1, a, \infty\}, \mathcal{M}_a)(\chi).
\]
This means
\[
[s^{m-1}ds \wedge \omega^\chi] \in \Gamma(S, W_2 \mathcal{H}(\chi)).
\]
Summarizing, we have:

**Lemma 4.4.** Let \(\omega^\chi\) be a relative 1-form satisfying the condition \(\text{P2} \). Then for any integer \(m \geq 1\) such that \(m \equiv k \mod l\), the rational 2-form \(s^{m-1}ds \wedge \omega^\chi \in \Gamma(\mathcal{U}, \Omega^2_{\mathcal{U}/\mathbb{Q}})\) defines a de Rham cohomology class
\[
[s^{m-1}ds \wedge \omega^\chi] \in \Gamma(S, W_2 \mathcal{H}(\chi)).
\]

As is shown in [1.3], \(W_2 H_{a,\text{dR}}(\chi)\) is two-dimensional. We shall show in below that it is spanned by \([s^{m-1}ds \wedge \omega^\chi]|_{\lambda=a}\) and \([s^{m-1}ds \wedge \omega^\chi]|_{\lambda=a}\) for some \(m\). We note that it is never obvious to show even the non-vanishing.

**4.6. Proof of Period formula: Part 2.** We compute the period matrix \(\text{Per}(W_2 \mathcal{H}(\chi))\).

Let \(A_{\text{dR}} \subset H^2_{\text{dR}}(\mathcal{U}_a)\) be the \(\mathbb{Q}\)-subspace spanned by \([s^{m-1}ds \wedge \omega^\chi]|_{\lambda=a}\) with \(m > 0\) such that \(m \equiv k \mod l\) (cf. Lemma 4.4). Consider the commutative diagram
\[
\begin{array}{ccc}
A_{\text{dR}} & \rightarrow & \text{Hom}(H^2_{\text{dR}}(\mathcal{U}_a), \mathbb{C}) \\
\downarrow & & \downarrow \\
W_2 H_{a,\text{dR}}(\chi) & \rightarrow & \text{Hom}(H^2(U_a, \mathbb{C}), \mathbb{C})
\end{array}
\]
As is easily shown,
\[
H^2(U_a)_{\text{lib}} := \text{Ker}[H^2(U_a) \rightarrow H^2(D_a \cap \mathcal{U}_a)] \rightarrow H^2(U_a)
\]
is injective (cf. [2] §6.1), and \(A_{\text{dR}}\) is obviously contained in \(H^2_{\text{dR}}(\mathcal{U}_a)_{\text{lib}}\). Our goal is to find \(m_i\) and \(Z_i \in H^2_{\text{dR}}(\mathcal{U}_a, \mathbb{Q})\) \((i = 1, 2)\) such that
\[
\int_{Z_1} s^{m_1-1}ds \wedge \omega^\chi|_{\lambda=a} - \int_{Z_2} s^{m_2-1}ds \wedge \omega^\chi|_{\lambda=a} \neq 0 \quad (4.9)
\]
and show that the entries (regarded as analytic functions of variable \(a\)) are as in Theorem 1.1. Then this gives the period matrix \(\text{Per}(W_2 \mathcal{H}(\chi))\).

Recall from [4.4] the homology cycle \(\delta \in H_1(f^{-1}(t), \mathcal{Q})(\chi)\). We think it being a homology cycle in a fiber \(f_a^{-1}(s)\). We take the Lefschetz thimble \(\Delta \subset \mathcal{U}_a\) over a segment from \(s = 0\) to \(s = \frac{1}{\sqrt{a}} - 1\) (a fixed \(l\)th root). Let \(\zeta \in \mu_l\) be a primitive \(l\)th root of unity and \(\sigma_\zeta \in \text{Aut}(\pi_a)\) be the corresponding automorphism. We denote
the automorphism of $X_a$ induced from $\sigma_\zeta \times \text{id}_X$ by the same symbol $\sigma_\zeta$. $\Delta$ has no boundary over $s = \sqrt{a} - 1$, but may have boundary over $s = 0$. Since $\sigma_\zeta$ acts on the fiber over $s = 0$ as identity, $(1 - \sigma_\zeta)\Delta$ has no boundary:

$$(1 - \sigma_\zeta)\Delta \in H_2(U_a, \overline{\mathbb{Q}}).$$

Let $T_{a=0}$ denote the local monodromy at $a = 0$ on $H_2(U_a, \overline{\mathbb{Q}})$ and we put

$$Z_1 := (1 - \sigma_\zeta)\Delta, \quad Z_2 := T_{a=0}(Z_1) \in H_2(U_a, \overline{\mathbb{Q}}). \quad (4.10)$$

Let $p_i(t) \in \overline{\mathbb{Q}}[t]$ be polynomials which satisfy $P_1$ in the beginning of 4.5. Put

$$a_i(\lambda) := \frac{(-1)^i}{i!} \partial_\lambda^i p_0(\lambda), \quad b_i(\lambda) := \frac{(-1)^i}{i!} \partial_\lambda^i p_1(\lambda), \quad (4.11)$$

so that

$$p_0(t) = \sum_{i=0}^N a_i(\lambda)(\lambda - t)^i, \quad p_1(t) = \sum_{i=0}^N b_i(\lambda)(\lambda - t)^i, \quad (4.12)$$

for a sufficiently large $N$.

**Lemma 4.5.** Let $F_\mu(\lambda)$ and $G_\mu(\lambda)$ be as defined in Theorem 4.1. Then we have, for $\mu > 1$,

- $\partial_\lambda F_\mu(\lambda) = (\mu - 1)F_{\mu-1}(\lambda)$, \quad (4.13)
- $\partial_\lambda G_\mu(\lambda) = (\mu - 1)G_{\mu-1}(\lambda)$. \quad (4.14)

**Proof.** Write $\alpha = \alpha^x$, $\beta = \beta^x$. Since

$$\partial_\lambda \alpha^x \beta^x \gamma ; \lambda = \frac{\alpha \beta}{\gamma} \frac{\alpha + 1, \beta + 1}{\gamma + 1} \quad (4.15)$$

in general, we have

$$\partial_\lambda F_\mu(\lambda) = (\lambda - 1)^{\mu-1} \left( 2F_1 \left( \frac{\alpha \beta}{\mu + 1} ; 1 - \lambda \right) + \frac{\alpha \beta}{\mu(\mu + 1)}(1 - \lambda)2F_1 \left( \frac{\alpha + 1, \beta + 1}{\mu + 2} ; 1 - \lambda \right) \right).$$

Hence (4.13) is equivalent to

$$\frac{(\alpha)_n(\beta)_n}{(\mu + 1)_n(1)_n} + \frac{\alpha \beta}{\mu(\mu + 1)} \frac{(\alpha + 1)_n - (\beta + 1)_n - 1}{(\mu + 2)_n - (1)_n - 1} = \frac{(\alpha)_n(\beta)_n}{(\mu)_n(1)_n} \quad (n \geq 1),$$

and this is easily verified. One proves (4.13) similarly, using (4.15) and the functional equation $\Gamma(s + 1) = s\Gamma(s)$. \qed

**Proposition 4.6.** For $\alpha \in \mathbb{C} \setminus \{0, 1\}$ and any positive integer $m \equiv k \pmod{t}$, put

$$P_m(\alpha) := \int_{\Delta} s^{m-1}ds \wedge \omega^\alpha|_{\lambda = \alpha}, \quad \mu = \frac{m}{t}$$

and regard it as an analytic function $P_m(\lambda)$ of variable $\lambda$. Then we have

$$P_m(\lambda) = \frac{2\pi i}{t} \sum_{i=0}^N (a_i(\lambda) + b_i(\lambda)\partial_\lambda) F_{\mu+i}(\lambda). \quad (4.16)$$

Moreover we have, if $\mu > 1$,

$$\partial_\lambda P_m(\lambda) = (\mu - 1)P_{m-1}(\lambda). \quad (4.17)$$
Proof. Letting $t = \lambda - s^i$ we have by (4.1)

$$P_m(\lambda) = \frac{2\pi i}{t} \int_{1}^{\lambda} (\lambda - t)^{\mu - 1} (p_0(t) + p_1(t)\partial_t) \times F_1 2F_1 \left( \frac{\alpha^x, \beta^x}{1} ; 1 - t \right) dt$$

$$= \frac{2\pi i}{t} \int_{1}^{\lambda} ((\lambda - t)^{\mu - 1} p_0(t) - \partial_t ((\lambda - t)^{\mu - 1} p_1(t))) \times F_1 2F_1 \left( \frac{\alpha^x, \beta^x}{1} ; 1 - t \right) dt.$$

Here the second equality follows from integration by parts and the assumption $1 - t | p_1(t)$ in $P_1$. By (4.12) and letting $1 - t = (1 - \lambda)u$, we have

$$P_m(\lambda) = \frac{2\pi i}{t} \sum_{i \geq -1} (a_i(\lambda) + (\mu + i)b_{i+1}(\lambda)) \int_{1}^{\lambda} (\lambda - t)^{\mu+i-1} \times F_1 2F_1 \left( \frac{\alpha^x, \beta^x}{1} ; 1 - t \right) dt$$

$$= \frac{2\pi i}{t} \sum_{i \geq -1} (a_i(\lambda) + (\mu + i)b_{i+1}(\lambda)) \times (\lambda - 1)^{\mu+i} \int_{0}^{1} (1 - u)^{\mu+i-1} F_1 2F_1 \left( \frac{\alpha^x, \beta^x}{1} ; (1 - \lambda)u \right) du.$$

By the integral representation of $\beta F_2$ (cf. [7], (4.1.2))

$$\int_{0}^{1} F_1 2F_1 \left( \frac{a,b}{d} ; xt \right) t^{c-1}(1-t)^{e-1} dt = B(c, e - c) \times F_2 \left( \frac{a,b,c}{d,e} ; x \right),$$

we have

$$\int_{0}^{1} (1 - u)^{\mu+i-1} F_1 2F_1 \left( \frac{\alpha^x, \beta^x}{1} ; (1 - \lambda)u \right) du
= B(1, \mu + i) \times F_2 \left( \frac{\alpha^x, \beta^x}{1, \mu + i + 1} ; 1 - \lambda \right)
= \frac{1}{\mu + i} F_1 2F_1 \left( \frac{\alpha^x, \beta^x}{\mu + i + 1} ; 1 - \lambda \right).$$

Hence we obtain

$$P_m(\lambda) = \frac{2\pi i}{t} \sum_{i \geq -1} (a_i(\lambda) + (\mu + i)b_{i+1}(\lambda)) F_{\mu+i}(\lambda).$$

Now (4.10) follows using (4.13), from which (4.11) follows using

$$\partial_\lambda a_i(\lambda) = -(i + 1)a_{i+1}(\lambda), \quad \partial_\lambda b_i(\lambda) = -(i + 1)b_{i+1}(\lambda),$$

and (4.13). □

Proposition 4.7. Let the notations be as in Proposition 4.6. There is a differential operator $\Theta = q(\lambda) + r(\lambda)\partial_\lambda$ with $q(\lambda), r(\lambda) \in \mathbb{Q}[\lambda, 1/\lambda(1 - \lambda)]$, such that

$$P_m(\lambda) = 2\pi i \cdot \Theta F_\mu(\lambda).$$

Proof. By (4.13), we have $F_{\mu+i}(\lambda) = \frac{(\mu)_i}{(\mu)_N} \partial_\lambda^{N-i} F_{\mu+N}(\lambda)$ ($i = 0, 1, \ldots, N$). Hence by Proposition 4.6 we have $P_m(\lambda) = 2\pi i \cdot \Theta_1 F_{\mu+N}(\lambda)$ with

$$\Theta_1 = \frac{1}{t} \sum_{i=0}^{N} \left( \frac{(\mu)_i}{(\mu)_N} \times (a_i(\lambda)\partial^{N-i}_\lambda + b_i(\lambda)\partial^{N+1-i}_\lambda) \right).$$

Recall that $F_\mu(\lambda)$ is a solution of the differential equation satisfied by

$$2F_1 \left( \frac{\alpha^x - \mu, \beta^x - \mu}{\alpha^x + \beta^x - \mu} ; \lambda \right),$$
i.e., \( \mathcal{D}F_\mu(\lambda) = 0 \) with
\[
\mathcal{D} := \lambda(1-\lambda)\partial_\lambda^2 + \{ \alpha^x + \beta^x - \mu - (\alpha^x + \beta^x - 2\mu + 1)\lambda \} \partial_\lambda - (\alpha^x - \mu)(\beta^x - \mu).
\] (4.19)
Hence we have
\[
(\alpha^x - \mu)(\beta^x - \mu)F_\mu = \left\{ (\alpha^x + \beta^x - \mu - (\alpha^x + \beta^x - 2\mu + 1)\lambda) - \lambda(1-\lambda)\partial_\lambda \right\}(\mu-1)F_{\mu-1}.
\]
Applying this iteratively, we obtain a differential operator \( \Theta_2 \) of degree \( N \) such that
\[
F_{\mu+N}(\lambda) = \Theta_2 F_\mu(\lambda).
\]
By reducing the degree of \( \Theta_1\Theta_2 \) using (4.19), we obtain the proposition. \( \square \)

To compute the degree along \( Z_2 \) (see (4.10)), we prepare the following.

**Proposition 4.8.** Assume \( m > l, \ m \equiv k \) (mod \( l \)), and that \( \mu := m/l \) satisfies \( \mu \neq 0, \alpha^x, \beta^x, \alpha^x + \beta^x \) (mod \( \mathbb{Z} \)). Let \( \Theta \) be as in Proposition 4.7. Then we have
\[
\left( \int_{Z_2} s^{m-l-1}ds \omega^x \right) \left( \int_{Z_1} s^{m-l-1}ds \omega^x \right) = 2\pi i (1 - \xi^m) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Theta F_\mu(\lambda) & \Theta G_\mu(\lambda) \\ \partial_\lambda \Theta F_\mu(\lambda) & \partial_\lambda \Theta G_\mu(\lambda) \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 - \xi \end{pmatrix}.
\]
Here \( \int_{Z_i} s^{m-l-1}ds \omega^x \) is also regarded as an analytic function of variable \( \lambda \) as in Proposition 4.7.

**Proof.** Firstly, since \( \sigma_\zeta \) acts on \( s^{m-l}ds \) as multiplication by \( \zeta^m \), we have, by Proposition 4.7,
\[
\int_{Z_1} s^{m-l}ds \omega^x = (1 - \xi^m)F_m(\lambda) = 2\pi i (1 - \xi^m)\Theta F_\mu(\lambda).
\]
Secondly, \( G_\mu(\lambda) \) is a solution of the differential equation (4.19), and its monodromy at \( \lambda = 0 \) is given by
\[
T_{\lambda=0}(F_\mu, G_\mu) = (F_\mu, G_\mu) \begin{pmatrix} \xi & 0 \\ 1 - \xi & 1 \end{pmatrix}, \quad \xi := e^{2\pi i (\mu - \alpha^x - \beta^x)}
\]
(see [4, §2.9 (43)]). Note that \( \xi \) depends only on \( \mu \) mod \( \mathbb{Z} \) and \( \xi \neq 1 \) by the assumption. Hence we have
\[
\int_{Z_2} s^{m-l}ds \omega^x = 2\pi i (1 - \xi^m)\Theta T_{\lambda=0}F_\mu(\lambda) = 2\pi i (1 - \xi^m)\Theta (\xi F_\mu(\lambda) + (1 - \xi)G_\mu(\lambda)).
\]
Then the computation for \( \int_{Z_i} s^{m-l-1}ds \omega^x \) (i = 1, 2) follows by Proposition 4.7. \( \square \)

Now we finish the proof of Theorem 4.1. By Proposition 4.8, it suffices to show
\[
\begin{vmatrix}
\Theta F_\mu(\lambda) & \Theta G_\mu(\lambda) \\
\partial_\lambda \Theta F_\mu(\lambda) & \partial_\lambda \Theta G_\mu(\lambda)
\end{vmatrix} \neq 0
\]
for some \( m \). This is equivalent to that \( \Theta F_\mu(\lambda)/\Theta G_\mu(\lambda) \) is non-constant. Suppose that \( \Theta F_\mu(\lambda) = C\Theta G_\mu(\lambda) \) for some constant \( C \). Then \( F_\mu(\lambda) = CG_\mu(\lambda) \) is a solution of both \( \theta f = 0 \) and (4.19). Since (4.19) is irreducible by the assumption that \( \mu \neq \alpha, \beta, \alpha + \beta \) (mod \( \mathbb{Z} \)), and the order of \( \Theta \) is one, we have \( \Theta = 0 \). Suppose that this is the case for any \( m > l \) with \( m \equiv k \) (mod \( l \)). Then \( P_m(\lambda) = \int_{\Delta} s^{m-l}ds \omega^x = 0 \) for any such \( m \). Applying the elementary lemma below (replace \( s \) with \( x^{1/l} \)), it follows that \( \int_{\Delta} \omega^x = 0 \), hence a contradiction. \( \square \)
Lemma 4.9. Let $f$ be a continuous function on the closed interval $[0, 1]$ whose zeros have no accumulation point. If $\int_0^1 f(x)x^{n-1}dx = 0$ for all $n \in \mathbb{Z}_{>0}$, then $f \equiv 0$.

Proof. By replacing $f(x)$ with $(x-1)f(x)$ if necessary, we can assume $f(1) = 0$. Suppose that $f \not\equiv 0$. By replacing $f$ with $-f$ if necessary, there exists $a \in (0, 1)$ such that $f(x) > 0$ for any $x \in (a, 1)$. Put $M = \max_{x \in [0, a]} |f(x)|$ and choose $b, c \in (a, 1)$ ($b < c$) and $m > 0$ such that $f(x) \geq m$ for any $x \in [b, c]$. Then

$$\int_0^1 f(x)x^{n-1}dx > -M \int_0^a x^{n-1}dx + m \int_b^c x^{n-1}dx = \frac{m c^n}{n} \left(1 - \left(\frac{b}{c}\right)^n\right) - \frac{M}{m} \left(\frac{a}{c}\right)^n.$$ 

Since the right-hand side is positive for sufficiently large $n$, this contradicts the assumption. \qed

5. Regulator Formula

We keep the setting and the notations in \[\ref{Gr'}\]. Put

$$C := \text{Gr}^W_2 \psi_{t=1} \mathcal{M} \cong \pi_* \mathcal{Q}[1]_S \otimes (\text{Gr}^W_2 \psi_{t=1} R^1 f_* \mathcal{Q}) \ (5.1)$$

a VHdR on $S$. In this section we discuss the exact sequences

$$0 \rightarrow W_2 \mathcal{H}(e) \rightarrow \mathcal{H}(e) \rightarrow (\mathcal{H}/W_2 \mathcal{H})(e) \rightarrow 0 \ (5.2)$$

$$0 \rightarrow W_2 \mathcal{H}(e) \rightarrow \mathcal{H}'(e) \rightarrow C(e) \otimes \mathcal{Q}(-1) \rightarrow 0$$

of mixed Hodge-de Rham structures on $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ arising from \[\ref{Gr'}\], where the right vertical inclusion is as in Proposition \[\ref{HdR}]. Since $\text{Gr}^W_2 \psi_{t=1} (R^1 f_* \mathcal{Q})(e_0) \cong E_0$ is a constant VHdR of type $(1, 1)$, $C(e)$ is one-dimensional over $E$ and endowed with Hodge type $(1, 1)$ (however the monodromy is nontrivial).

5.1. Setting. Let $Q : R^1 f_* \mathcal{Q} \otimes R^1 f_* \mathcal{Q} \rightarrow \mathcal{Q}(-1)$ be a polarization form which also induces a polarization on the $e_0$-part $(R^1 f_* \mathcal{Q})(e_0)$. It naturally extends to a non-degenerate pairing $Q : \mathcal{M} \otimes \mathcal{M} \rightarrow \mathcal{Q}(-1)$ which is compatible with the action of $\text{Aut}(\pi) \cong \mu_1$, namely $Q(\sigma x, \sigma y) = Q(x, y)$ for $\sigma \in \text{Aut}(\pi)$. This also induces a polarization on the $e$-part $\mathcal{M}(e)$. We have isomorphisms

$$(\mathcal{M})^* \cong \mathcal{M} \otimes \mathcal{Q}(1), \quad (\mathcal{M}(e))^* \cong \mathcal{M}(e) \otimes \mathcal{Q}(1) \ (5.3)$$

induced from $Q$ where $(\cdot)^*$ denotes the dual sheaf. Let $j : \mathcal{H} \rightarrow \mathbb{P}^1$ and $\text{pr}_2 : \mathbb{P}^1_S = \mathbb{P}^1 \times S \rightarrow S$. Then there are isomorphisms

$$(\mathcal{H})^* = (R^1 \text{pr}_2)_! Rj_* \mathcal{M}^* \cong R^1 \text{pr}_2)_! j_* \mathcal{M}^* \otimes \mathcal{Q}(1) \cong R^1 \text{pr}_2)_! j_* \mathcal{M} \otimes \mathcal{Q}(2) \ (5.4)$$

induced from the Verdier duality and \[\ref{HdR}\]. We show that \[\ref{HdR}\] induces an isomorphism

$$(W_2 \mathcal{H})^* \cong W_2 \mathcal{H} \otimes \mathcal{Q}(2). \quad (5.5)$$

Let $i : Z = \mathbb{P}^1_S \setminus \mathcal{H} \hookrightarrow \mathbb{P}^1_S$ be the complement. Note that $Z$ is finite etale over $S$. There is an exact sequence

$$0 \rightarrow i_! i^* j_* \mathcal{M} \rightarrow j_* \mathcal{M} \rightarrow j_* \mathcal{M} \rightarrow 0$$
of mixed Hodge modules where \( j_* \mathcal{M} = R^0 j_* \mathcal{M} \). Applying \( Rpr_{j*} \), one has

\[
\begin{array}{c}
0 \rightarrow i_* j_* \mathcal{M} \rightarrow R^1 pr_{j*} \mathcal{M} \rightarrow R^1 p r_{j*} \mathcal{M} \rightarrow 0
\end{array}
\]

because \( (R^0 pr_{j*} j_* \mathcal{M})_a = H^0(U_a, \mathcal{M}_a) = 0 \) for \( a \in \mathbb{C} \setminus \{ 0, 1 \} \) (see 3.3 for the notation). The mixed Hodge-de Rham structure

\[
i_* j_* \mathcal{M} = \bigoplus_{p=0,1,a,\infty} \ker [T_p-1 : \psi_{t=p} \mathcal{M}_a \rightarrow \psi_{t=p} \mathcal{M}_a \otimes \mathbb{Q}(-1)]
\]

has weight \( \leq 1 \). This implies \( \text{Gr}^W_2 R^1 pr_{j*} j_* \mathcal{M} = R^1 pr_{j*} j_* \mathcal{M} = W_2 \mathcal{H} \). We thus have \((5.5)\) by taking the graded piece of \((5.4)\) of weight \(-2\).

The isomorphisms \((5.3)\) and \((5.5)\) are \textit{not} compatible with respect to the multiplication by \( R \). Here the multiplication on the left hand side of \((5.3)\) or \((5.5)\) is given as in \(2.3\). For \( r \in E \), we denote by \( t^r \) the multiplication on \( \mathcal{M}(e) \) such that

\[
Q(rx, y) = Q(x, t^r y), \quad \forall x, y.
\]

The multiplication by \( r \) on the left corresponds to \( t^r \) in the right of \((5.3)\). Note \( t^\sigma = \sigma^{-1} \) for \( \sigma \in \text{Aut}(\pi) \). For \( \chi : R \rightarrow \mathbb{Q} \), we denote \((-)(t^\chi)\) the subspace on which \( t^r \) acts by multiplication by \( \chi(r) \) for all \( r \in E \). Then \((5.3)\) and \((5.5)\) induce

\[
(\mathcal{M}(\chi))^* \cong \mathcal{M}(t^\chi), \quad (W_2 \mathcal{H}(\chi))^* \cong W_2 \mathcal{H}(t^\chi).
\]

5.2. \textbf{Theorem on regulators.} Let \( \mathcal{O}_{\text{zar}} \) be the Zariski sheaf of polynomial functions (with coefficients in \( \overline{\mathbb{Q}} \)) on \( S = \mathbb{A}^1_{\mathbb{C}} \setminus \{ 0, 1 \} \) with coordinate \( \lambda \). Let \( \mathcal{O}_{an} \) be the sheaf of analytic functions on \( S_{an} = \mathbb{C}^\infty \setminus \{ 0, 1 \} \). Let \( a \) be the canonical morphism from the analytic site to the Zariski site. Set

\[
\mathcal{J} := \text{Coker}[a^{-1} F^2 W_2 \mathcal{H}_{dR} \otimes \mathcal{O}_{an} \otimes a^{-1} \mathcal{O}_{zar} \mathcal{O}_{an} W_2 \mathcal{H}_{dR}],
\]

\[
\mathcal{J}^* := \text{Coker}[a^{-1} \text{Hom}(W_2 \mathcal{H}_{dR}/F^1, \mathcal{O}_{zar}) \otimes a^{-1} W_2 \mathcal{H}_{dR}^* \rightarrow \text{Hom}(a^{-1} W_2 \mathcal{H}_{dR}, \mathcal{O}_{an})]
\]

sheaves on the analytic site \( C_{an} \setminus \{ 0, 1 \} \). Note \( \mathcal{J}^* \cong \mathcal{J} \) by \((5.3)\).

Let \( C \) be the VHdR on \( S \) as defined in \((5.1)\). Let \( h : \bar{S} \rightarrow S \) be a generically finite and dominant map such that \( \sqrt{\lambda - 1} \in \overline{\mathbb{Q}}(\bar{S}) \). Then \( h^* C \) is a constant VHdR of type \((1,1)\). Let

\[
\delta : h^* C(e) \otimes \mathbb{Q}(1) = \text{Hom}_{\text{VHdR}}(Q, h^* C \otimes \mathbb{Q}(1)) \rightarrow \text{Ext}^1_{\text{VHdR}}(Q, h^* W_2 \mathcal{H}(e) \otimes \mathbb{Q}(2)).
\]

be the connecting homomorphism arising from the exact sequence of \((5.2)\). Let \( \rho \) be the composition of maps

\[
h^* C(e) \otimes \mathbb{Q}(1) \xrightarrow{\delta} \text{Ext}^1_{\text{VHdR}}(Q, h^* W_2 \mathcal{H} \otimes \mathbb{Q}(2)) \rightarrow \Gamma(S_{an}, h^* \mathcal{J}(e)) \xrightarrow{\sim} \Gamma(\bar{S}_{an}, h^* \mathcal{J}^*(e))
\]
where the second arrow is constructed in a similar way to the proof of Proposition 2.1. Let $\rho(\chi)$ be $t\chi$-part of $\rho$, namely the composition of maps

$$\bar{Q} \cong (h^*C(e) \otimes \mathbb{Q}(1))(t\chi) \rightarrow \text{Ext}^1_{\text{Filh-BdR}}(\bar{Q}, h^*W_2\mathcal{H}(t\chi) \otimes \mathbb{Q}(2))$$

Here $\mathcal{J}^*(\chi)$ is defined by replacing $W_2\mathcal{H}_{\text{dr}}$ with $W_2\mathcal{H}_{\text{B}}(\chi)$, and $W_2\mathcal{H}_{\text{B}}$ with $W_2\mathcal{H}_{\text{B}}(\chi) = \text{Hom}_{\mathcal{Q}}(W_2\mathcal{H}_{\text{B}}(\chi), \mathcal{Q})$.

**Theorem 5.1 (Regulator formula).** Let the assumptions, $\mu$ and $\Theta$ be as in Theorem 4.7. Let $\rho(\chi)(1) \in (h^*\mathcal{O}^{\text{an}})^2 \cong \text{Hom}(\alpha^{-1}W_2\mathcal{H}_{\text{dr}}(\chi), h^*\mathcal{O}^{\text{an}})$ be a local lifting where the isomorphism is with respect to a $\mathbb{Q}$-frame of $W_2\mathcal{H}_{\text{dr}}(\chi)$. Then we have

$$\rho(\chi)(1) \equiv (\Theta H_\mu(\lambda), (\mu - 1)^{-1}\partial_\lambda \Theta H_\mu(\lambda)) \pmod{\mathbb{Q}(\lambda)^2},$$

where we put

$$H_\mu(\lambda) := \frac{1}{(1 - \alpha \lambda)(1 - \beta \lambda)}(\lambda - 1)^{\mu - 1} \binom{1, 1, -1}{2 - \alpha \lambda, 2 - \beta \lambda, 1 - \lambda}. $$

The map $\rho$ is related to Beilinson's regulator map in the following way. Let $\mathbb{P}^1_{\bar{S}} : = \mathbb{P}^1 \times \bar{S}$ and $\pi : \mathbb{P}^1_{\bar{S}} \rightarrow \mathbb{P}^1$ given by $(s, \lambda) \mapsto \lambda - h(s)^{t\chi}$. Let

\[
\begin{array}{c}
\begin{array}{ccc}
X_{\bar{S}} & \xrightarrow{i} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\
\downarrow f_{\bar{S}} & & \downarrow f \\
\mathbb{P}^1_{\bar{S}} & \xrightarrow{\pi} & \mathbb{P}^1 \\
\end{array}
\end{array}
\]

with $i$ desingularization. Let $g := \text{pr}_2 \circ f_{\bar{S}} : X_{\bar{S}} \rightarrow \bar{S}$ be a projective smooth map. Let

$$\text{reg} : H^3_{\text{Filh}}(X_{\bar{S}}, \mathbb{Q}(2)) \rightarrow \text{Ext}^1_{\text{VMHdR}}(\mathbb{Q}, R^2 g_* \mathbb{Q}(2))$$

be the Beilinson regulator map. Letting $(R^2 g_* \mathbb{Q})_0 := \text{Ker}[R^2 g_* \mathbb{Q} \rightarrow \text{pr}_2^* R^2 f_{\bar{S}}^* \mathbb{Q}]$, then there is a canonical surjective map $(R^2 g_* \mathbb{Q})_0 \rightarrow h^*W_2\mathcal{H}$ (cf. (4.3)), so that we have

$$\text{reg}_0 : H^3_{\text{Filh}}(X_{\bar{S}}, \mathbb{Q}(2))_0 \rightarrow \text{Ext}^1_{\text{VMHdR}}(\mathbb{Q}, h^*W_2\mathcal{H} \otimes \mathbb{Q}(2))$$

where $H^3_{\text{Filh}}(X_{\bar{S}}, \mathbb{Q}(2))_0 \subset H^3_{\text{Filh}}(X_{\bar{S}}, \mathbb{Q}(2))$ is the inverse image of $\text{Ext}^1_{\text{VMHdR}}(\mathbb{Q}, (R^2 g_* \mathbb{Q})_0)$. Let $D_{\bar{S}} \subset X_{\bar{S}}$ be the inverse image of singular fibers $D \subset X$ of $f$. Then there is a canonical map

$$H^3_{\text{Filh}, D_{\bar{S}}}(X_{\bar{S}}, \mathbb{Q}(2)) \rightarrow H^3_{B, D_{\bar{S}}}(X_{\bar{S}}, \mathbb{Q}(2)) \cap H^{0,0} \rightarrow \text{Hom}_{\text{VMHdR}}(\mathbb{Q}, h^*\mathcal{H}/W_2 \otimes \mathbb{Q}(2))$$
from the motivic cohomology group with support in $D_{\tilde{S}}$. Then the following diagram is commutative

\[
\begin{array}{ccccccccc}
H^3_{\mathcal{X}, D_{\tilde{S}}} (X_{\tilde{S}}, \mathbb{Q}(2)) & \longrightarrow & \text{Hom}_{\text{MHDM}} (\mathbb{Q}, h^* \mathcal{H}/W_2 \otimes \mathbb{Q}(2)) \\
\downarrow & & \downarrow \\
H^3_{\mathcal{X}} (X_{\tilde{S}}, \mathbb{Q}(2))_0 & \longrightarrow & \text{Ext}_{\text{MHDM}}^1 (\mathbb{Q}, h^* W_2 \mathcal{H} \otimes \mathbb{Q}(2)) \\
\end{array}
\]

5.3. **Proof of Regulator formula**: Part 1. Let

\[
\begin{array}{ccc}
X_a & \overset{i}{\longrightarrow} & \mathbb{P}^1_a \times_{\mathbb{P}^1} X \\
\downarrow f_a & & \downarrow f \\
\mathbb{P}^1_a & \longrightarrow & \mathbb{P}^1 \\
\end{array}
\]

be as in [4.5] Let $D_a \subset X_a$ be the inverse image of the singular fibers of $f$. We denote the coordinate of $\mathbb{P}^1$ (resp. $\mathbb{P}^1_a$) by $t$ (resp. $s$). We also use the notation in [4.3] Note $\mathcal{A}_a \cong \pi_{a*} R^1 f_{a*} \mathbb{Q}$ is endowed with de Rham structure induced from a $\mathbb{Q}$-frame on $\mathcal{A}_{\text{dR}}$. The distinguished triangle

\[
0 \longrightarrow j_! \pi_{a*} f_{a*} \mathbb{Q} \longrightarrow j_! \tau_{\leq 1} \pi_{a*} Rf_{a*} \mathbb{Q} \longrightarrow j_! \pi_{a*} R^1 f_{a*} \mathbb{Q} [-1] \longrightarrow 0
\]

and the fact that

\[
H^2 (\mathbb{P}^1_a, j_! \pi_{a*} \mathbb{Q}) = H^0 (\mathbb{P}^1_a, Rf_{a*} \pi_{a*} \mathbb{Q})^* = H^0 (\mathbb{P}^1_a \setminus \{0, 1, a, \infty\}, \pi_{a*} \mathbb{Q})^* = 0
\]

implies

\[
H^2 (\mathbb{P}^1_a, j_! \pi_{a*} \tau_{\leq 1} Rf_{a*} \mathbb{Q}) \cong H^1 (\mathbb{P}^1_a, j_! \mathcal{A}_a).
\]

Hence we have an injective map

\[
H^1 (\mathbb{P}^1_a, j_! \mathcal{A}_a) \rightarrow H^2 (\mathbb{P}^1_a, j_! \pi_{a*} Rf_{a*} \mathbb{Q}) = H^2 (\mathbb{P}^1, \pi_{a*} Rf_{a*} j_! \mathbb{Q}) = H^2 (X_a, D_a; \mathbb{Q}).
\]

We have a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & W_2 H_a (2) & \longrightarrow & H_a (2) & \longrightarrow & H_a / W_2 (2) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & (W_2 H_a)^* & \longrightarrow & H^1 (\mathbb{P}^1, j_! \mathcal{A})^* & \longrightarrow & H^1 (\mathbb{P}^1, j_! \mathcal{A})^*/W_{-2} & \longrightarrow & 0 \\
\uparrow a_1 & & \uparrow a_2 & & \uparrow a_3 & & \uparrow a_4 & & \uparrow a_5 \\
0 & \longrightarrow & H_2 (X_a, \mathbb{Q}) / H_2 (D_a) & \longrightarrow & H_2 (X_a, D_a; \mathbb{Q}) & \longrightarrow & H_1 (D_a) & \longrightarrow & 0 \\
\uparrow b_1 & & \uparrow b_2 & & \uparrow b_3 & & \uparrow b_4 & & \uparrow b_5 \\
0 & \longrightarrow & H_2 (\mathcal{U}_a, \mathbb{Q}) / H_2 (D_0^\circ) & \longrightarrow & H_2 (\mathcal{U}_a, D_0^\circ; \mathbb{Q}) & \longrightarrow & H_1 (D_0^\circ) & \longrightarrow & 0
\end{array}
\]

where $D_0^\circ := D_a \cap \mathcal{U}_a$. Since $a_2$ is surjective, so are $a_1$ and $a_3$. Moreover $a_3$ is bijective (local invariant cycle theorem).
Let $Z_1, Z_2$ be the homology cycles $(1, 1)$, and $(m_1, m_2) = (l_1, l_\mu - l)$ where $\mu$ is as in Theorem 5.1. Note that $(W_2 H_a)^\ast(\chi)$ is spanned by the images of $Z_1$ and $Z_2$. Hence

\[
H_{a,B}(\chi)/W_2 \cap H^{0,0} \cong \mathbb{Q}
\]

\[
\to H_1^B(D_a^\circ)(\chi) \cap H^{0,0}
\]

\[
\to \text{Ext}^1_{\text{Fib-BDR}}(\mathbb{Q}, H_2(U_a, \mathbb{Q})/H_2(D_a^\circ)(\chi))
\]

\[
\to \text{Ext}^1_{\text{Fib-BDR}}(\mathbb{Q}, (W_2 H_a)^\ast(\chi))
\]

\[
\cong \text{Coker}[(W_2 H_{a,B})^\ast(\chi) \oplus \text{Hom}(W_2 H_{a,\text{dR}}/F^1, \mathbb{Q}) \to \text{Hom}(W_2 H_{a,\text{dR}}(\chi), \mathbb{C})]
\]

\[
\cong \text{Coker}[\text{Hom}(W_2 H_{a,\text{dR}}(\chi)/F^1, \mathbb{Q}) \to \text{Hom}(W_2 H_{a,\text{dR}}(\chi), \mathbb{C})/\text{Im} H_2(U_a, \mathbb{Q})]
\]

and the composition of the above coincides with the restriction $\rho(\chi)|_{\lambda=a}$. Moreover the image of $H_2(U_a, \mathbb{Q})$ is given by the period matrix

\[
\text{Per}(W_2 \mathcal{H}(\chi))(\chi)|_{\lambda=a} = \left(\int_{Z_1} s^{m_1-1}d\omega^1 \int_{Z_2} s^{m_2-1}d\omega^2 \right)|_{\lambda=a} : \mathbb{Q}^2 \to \mathbb{C}^2
\]

under the isomorphism

\[
\text{Hom}(W_2 H_{a,\text{dR}}(\chi), \mathbb{C}) \cong \mathbb{C}^2
\]

given by the $\mathbb{Q}$-basis $\{s^{m_1-1}d\omega^1, s^{m_2-1}d\omega^2\}$ of $W_2 H_{a,\text{dR}}(\chi)$. Let $D_a^{ss} \subset D_a$ be the inverse image of $f^{-1}(1)$. Note $D_a^{ss} \subset U_a$. Then for $1 \in \mathbb{Q} \cong H_1(D_a^{ss})(\chi) \subset H_1(D_a^\circ)(\chi) \cap H^{0,0}$, we want to compute a lifting

\[
\rho(\chi)(1) = (\phi_1(a), \phi_2(a)) \in \mathbb{C}^2.
\]

**Lemma 5.2.** Let $\Gamma(a) \in H_2^B(U_a, (\pi_a f_a)^{-1}(1); \mathbb{Q})$ be a lifting of the homology cycle $1 \in \mathbb{Q} \cong H_1(D_a^{ss})(\chi)$. Then there is an algebraic function $R(\lambda) \in \mathbb{Q}(\lambda)$ of variable $\lambda$ such that

\[
\phi_i(a) = \int_{\Gamma(a)} s^{m_i-1}d\omega^i|_{\lambda=a} + R(a)
\]

for a in a small neighbourhood of $\mathbb{C}^n \setminus \{0, 1\}$.

**Proof.** See [2], Proposition 7.2; the situation there is slightly different but the same discussion works. \hfill \square

5.4. **Proof of Regulator formula : Part 2.** Recall from (4.4) the homology cycles $\delta, \gamma \in H_1(X_t, \mathbb{Q})(\chi)$. By the local invariant cycle theorem, there is an exact sequence

\[
H_1(X_t, \mathbb{Q}) \xrightarrow{T_0-1} H_1(X_t, \mathbb{Q}) \to H_1(f^{-1}(0), \mathbb{Q}) \to 0,
\]

(5.6)

where $T_0$ is the local monodromy at $t = 0$. We note that $H_1(f^{-1}(0), \mathbb{Q})$ has multiplication by $R_0$ induced from (5.6) (recall from (4.4) that $R_1 f_0, \mathbb{Q}$ has multiplication by $R_0$). An element $\gamma' \in H_1(X_t, \mathbb{Q})(\chi)$ vanishes as $t \to 0$ if and only if it belongs to the one-dimensional space

\[
\text{Ker}[H_1(X_t, \mathbb{Q})(\chi) \to H_1(f^{-1}(0), \mathbb{Q})] = \text{Im}[T_0-1 : H_1(X_t, \mathbb{Q})(\chi) \to H_1(X_t, \mathbb{Q})(\chi)].
\]

**Lemma 5.3.** Put

\[
\gamma' := \gamma + \frac{1 - e^{2\pi i(\alpha^x + \beta^x)}}{(1 - e^{2\pi i(\alpha^x)})(1 - e^{2\pi i(\beta^x)})}\delta.
\]
Then \( \gamma' \) is a basis of \( \text{Ker}[H_1(X_t,q)](\chi) \to H_1(f^{-1}(0),q) \), and we have

\[
\int_{\gamma'} \omega^\chi = B(1 - \alpha^\chi, 1 - \beta^\chi) \theta \left( t^{1 - \alpha^\chi - \beta^\chi} \binom{1 - \alpha^\chi, 1 - \beta^\chi}{2 - \alpha^\chi - \beta^\chi} \right).
\]

**Proof.** If \( \alpha^\chi + \beta^\chi = 1 \), then \( T_0 \) is unipotent and \( \text{Im}(T_0 - 1) = \text{Ker}(T_0 - 1) \), for which \( \gamma' = \gamma \) belongs. The assertion about the period is Lemma 5.3. Suppose \( \alpha^\chi + \beta^\chi \neq 1 \). Recall the notations (4.5), (4.6) and the relation (4.7). The assertion about the period follows from (4.7). Since \( f_1(t) \) and \( f_3(t) \) form a basis of the local solutions near \( t = 0 \), and \( T_0 - 1 \) annihilates \( f_1(t) \) but not \( f_3(t) \), the \( \gamma' \) generates \( \text{Im}(T_0 - 1) \).

We regard \( \gamma' \) as a homology cycle in a general fiber of \( f_a \). Fix \( \ell \)th roots \( \sqrt[\ell]{a} \) and \( \sqrt[\ell]{a - 1} \). Let \( \Gamma_a \subset \mathcal{U}_a \) be the Lefschetz thimble over a path from \( s = \sqrt[\ell]{a - 1} \) to \( s = \sqrt[\ell]{a} \) with fiber \( \gamma' \) (\( s \) is the coordinate of \( \mathbb{P}^1_\mathbb{C} \)). Note that \( \delta \) vanishes as \( t \to 1 \) but \( \delta \) does not. Therefore \( \Gamma_a \) has a nontrivial boundary supported on \( f_a^{-1}(\sqrt[\ell]{a - 1}) \cong f^{-1}(1) \):

\[
\Gamma_a \in H^B_2(U_a, f_a^{-1}(\sqrt[\ell]{a - 1}); \mathbb{Q}), \quad \partial \Gamma_a \neq 0 \in H^B_1(f_a^{-1}(\sqrt[\ell]{a - 1})) \cong H^B(f^{-1}(1)).
\]

Note that \( H^B(f^{-1}(1), \mathbb{Q}) \) has multiplication by \( R_0 \) via the local invariant cycle theorem (cf. (5.6)). The \( \chi |_{R_0} \)-part \( H^B(f^{-1}(1), \mathbb{Q})(\chi |_{R_0}) \) is one-dimensional, spanned by \( \partial \Gamma_a \). Hence the \( \chi \)-part

\[
H^B_1((\pi_a f_a)^{-1}(1), \mathbb{Q})(\chi) \cong H^B_1(f^{-1}(1), \mathbb{Q})(\chi |_{R_0}) \otimes H^B_0((\pi_a^{-1}(1))(\chi |_{\mu})),
\]

is one-dimensional, spanned by the sum \( \sum_{\sigma \in \mu} \chi(\sigma)^{-1} \cdot \sigma(\partial \Gamma_a) \). Therefore, in Lemma 5.2 we may take \( \Gamma(a) \) to be the sum \( \sum_{\sigma \in \mu} \chi(\sigma)^{-1} \cdot \sigma \Gamma_a \). Then we have

\[
\int_{\Gamma(a)} s^{m-1} \omega^\chi |_{\lambda = a} = \sum_{\sigma \in \mu} \chi(\sigma)^{-1} \int_{\sigma \Gamma} s^{m-1} \omega^\chi |_{\lambda = a} = l \int_{\Gamma} s^{m-1} \omega^\chi |_{\lambda = a}.
\]

**Lemma 5.4.** Let \( H_\mu(\lambda) \) be as defined in Theorem 5.7. Then we have

\[
H_\mu(\lambda) = B(1 - \alpha^\chi, 1 - \beta^\chi) \int_0^1 (\lambda - t)^{\alpha-1} t^{1-\alpha^\chi - \beta^\chi} \binom{1 - \alpha^\chi, 1 - \beta^\chi}{2 - \alpha^\chi - \beta^\chi} dt,
\]

and

\[
\partial_\lambda H_\mu(\lambda) = (\mu - 1) H_{\mu - 1}(\lambda).
\]

**Proof.** Recall the integral representation of \( 2F_1 \) (cf. (7), (4.1.2))

\[
B(b, c - b) 2F_1 \left( \begin{array}{c} a, b \\ c \end{array} ; x \right) = \int_0^1 (1 - xt)^{-a} t^{b-1} (1 - t)^{c-b-1} dt.
\]

Applying this, we have, writing \( \alpha = \alpha^\chi \), \( \beta = \beta^\chi \),

\[
B(1 - \alpha, 1 - \beta) 2F_1 \left( \begin{array}{c} 1 - \alpha, 1 - \beta \\ 2 - \alpha - \beta \end{array} ; t \right) = \int_0^1 (1 - ts)^{-\alpha} s^{-\beta} (1 - s)^{-\alpha} ds.
\]
Letting $u = 1 - t$, $v = 1 - st$, we have

$$
\int_0^1 \int_0^1 (\lambda - t)^{\mu-1} t^{1-\alpha-\beta} (1 - ts)^{\alpha-1} s^\beta (1 - s)^{-\alpha} ds \, dt \\
= (\lambda - 1)^{\mu-1} \int_0^1 \int_0^v \left( 1 - \frac{u}{1 - \lambda} \right)^{\mu-1} u^{\alpha-1} (v - u)^{-\alpha} (1 - v)^{-\beta} \, du \, dv \\
= (\lambda - 1)^{\mu-1} \int_0^1 \int_0^1 \left( 1 - \frac{uv}{1 - \lambda} \right)^{\mu-1} (1 - w)^{-\alpha} (1 - v)^{-\beta} \, dw \, dv.
$$

Then, using (5.9) and (4.18), we obtain (5.7). Since

$$
For a positive integer $m$ satisfying $m > 0$, put

$$
Q_m(a) := \int_{\Gamma_a} s^{m-1} \, ds |_{\lambda = a}, \quad \mu = \frac{m}{t}
$$

regarded as an analytic function for $a$. Then we have

$$
Q_m(\lambda) = \frac{1}{l} \sum_{i=0}^{N} (a_i(\lambda) + b_i(\lambda) \partial_\lambda) H_{\mu+i}(\lambda), \quad (5.10)
$$

where $a_i(\lambda), b_i(\lambda)$ are as defined in (4.11). Moreover we have, if $\mu > 1$,

$$
\partial_\lambda Q_m(\lambda) = (\mu - 1) Q_{m-1}(\lambda). \quad (5.11)
$$

Proof. Since $\frac{ds}{\lambda} = \frac{dt}{l} \frac{ds}{\lambda}$, we have by Lemma 5.5

$$
Q_m(\lambda) = \frac{1}{l} B(1 - \alpha, 1 - \beta) \int_0^1 (\lambda - t)^{\mu-1} \theta \left( t^{1-\alpha-\beta} 2F_1 \left( \frac{1 - \alpha}{2 - \alpha, 2 - \beta}; t \right) \right) \, dt.
$$

By Lemma 5.3 the same argument as in the proof of Proposition 4.10 works to prove the proposition. □

Lemma 5.6. Let the differential operator $\mathcal{D}$ be as defined in (4.19). Then we have

$$
\mathcal{D} H_\mu(\lambda) = - (\lambda - 1)^{\mu-1}.
$$

Proof. Put $x = \frac{1}{1 - \lambda}$, $F(x) = 3F_2 \left( \frac{1,1,1-\mu}{2-\alpha,2-\beta}; x \right)$ and $D = x \frac{d}{dx}$. Using $Dx^n = nx^n$, one easily verifies

$$
(x(D + 1) + \mu - (D + 1 - \alpha)(D + 1 - \beta))F = - (1 - \alpha)(1 - \beta).
$$

So $H_\mu$ satisfies $\mathcal{D} H_\mu = -1$ with

$$
\mathcal{D} = (x(D + 1) + \mu - (D + 1 - \alpha)(D + 1 - \beta))(x(D + 1))^{-\mu-1} = (-x)^{\mu-1} (x(D + \mu) - (D - \alpha + \mu)(D - \beta + \mu)).
$$
Since $D = (1 - \lambda)\partial_\lambda$, where $\partial_\lambda = \frac{d}{d\lambda}$, one obtains $\mathcal{D}_1 = (\lambda - 1)^{1-\mu}\mathcal{D}$. Hence the lemma follows.

Now we finish the proof of Theorem 5.1. Let $\mu = m/l$ be as in Theorem 4.1. By Lemma 5.6, we have

$$\phi_1(\lambda) \equiv Q_m(\lambda) = \frac{1}{l} \sum_{i=0}^{N} (a_i(\lambda) + b_i(\lambda))\partial_\lambda H_{\mu+i}(\lambda), \mod \mathbb{Q}(\lambda), \quad (5.12)$$

$$\phi_2(\lambda) \equiv Q_{m-l}(\lambda) = (\mu - 1)^{-1}\partial_\lambda Q_m(\lambda), \mod \mathbb{Q}(\lambda). \quad (5.13)$$

By Lemma 5.8, we have similarly as in the proof of Proposition 4.7 that $Q_m(\lambda) = \Theta_1 H_{\mu+N}(\lambda)$ where $\Theta_1$ is the same differential operator as in the proof of Proposition 4.7. By Lemma 5.9, we have

$$H_{\mu+N}(\lambda) \equiv \Theta_2 H_{\mu}(\lambda) \mod \mathbb{Q}(\lambda)$$

where $\Theta_2$ is the same differential operator in the proof of Proposition 4.7. Hence $\phi_1(\lambda) \equiv \Theta_1 \Theta_2 H_{\mu}(\lambda) = \Theta H_{\mu}(\lambda)$ as desired. \hfill \Box

5.5. Question of Golyshev. We give an affirmative answer to the question of Golyshev in a special case.

Lemma 5.7. Let

$$P_{HG} := D_\lambda(D_\lambda - \mu + \alpha^x + \beta^x - 1) - \lambda(D_\lambda + \alpha^x - \mu)(D_\lambda + \beta^x - \mu)$$

be the hypergeometric differential operator. Put

$$Q_{HG} := \theta_\lambda P_{HG}, \quad \theta_\lambda := (1 - \lambda)\partial_\lambda + (\mu - 1)\lambda,$$

and local systems of $\mathbb{C}$-modules on $S := \mathbb{P}^1 \setminus \{0, 1, \infty\}$

$$V_P := \text{Sol}(D_S/D_{S}P_{HG}), \quad V_Q := \text{Sol}(D_S/D_{S}Q_{HG}),$$

where $D_S$ denotes the ring of differential operators on $S$. Let

$$0 \rightarrow V_P \rightarrow V_Q \rightarrow V_Q/V_P \rightarrow 0$$

be the exact sequence obtained by applying the solution functor $\text{Sol}(\bullet) := \text{Hom}_{D_S}(\bullet, \mathcal{O}_S)$ on

$$0 \rightarrow D_S/D_{S}\theta_\lambda \rightarrow D_S/D_{S}Q_{HG} \rightarrow D_S/D_{S}P_{HG} \rightarrow 0.$$  

Then, for any generically finite dominant map $h : T \rightarrow S$, the exact sequence

$$0 \rightarrow h^*V_P \rightarrow h^*V_Q \rightarrow h^*(V_Q/V_P) \rightarrow 0 \quad (5.14)$$

of $\mathbb{C}[\pi_1(T)]$-modules does not split.

Proof. We first note that $P_{HG} = \lambda \mathcal{D}$ where $\mathcal{D}$ is the differential operator in Theorem 4.19. Let $F_\mu(\lambda), G_\mu(\lambda)$ be as in Theorem 5.1 and $H_\mu(\lambda)$ as in in Theorem 5.1. Then the solutions of $P_{HG}$ are $F_\mu(\lambda), G_\mu(\lambda)$ (cf. the proof of Propositions 4.7 and 4.8), and the solutions of $Q_{HG}$ are $F_\mu(\lambda), G_\mu(\lambda), H_\mu(\lambda)$ (cf. Lemma 5.6).

$$V_P = \langle F_\mu(\lambda), G_\mu(\lambda) \rangle_{\mathbb{C}}, \quad V_Q = \langle F_\mu(\lambda), G_\mu(\lambda), H_\mu(\lambda) \rangle_{\mathbb{C}}.$$  

Since $\text{Ext}_{\pi_1(S)}(V_Q/V_P, V_P) \rightarrow \text{Ext}_{\pi_1(T)}(h^*(V_Q/V_P), h^*V_P)$ is injective, we may assume $T = S$. Assume that the sequence (5.13) splits. This means that there are $c_1, c_2 \in \mathbb{C}$ such that $\mathcal{H}(H_\mu(\lambda) + c_1 F_\mu(\lambda) + c_2 G_\mu(\lambda))$ is stable under the action of $\pi_1(S, \lambda)$. The eigenvalues of the local monodromy $T_\infty$ at $\lambda = \infty$ on $V_P$ are $e^{2\pi i(\alpha^x - \mu)}, e^{2\pi i(\beta^x - \mu)}$. On the other hand $H_\mu(\lambda)$ is the eigenvector with eigenvalue $e^{-2\pi i\mu}$. Therefore $c_1 = c_2 = 0$, namely $H_\mu(\lambda)$ is stable under the action of
\(\pi_1(S, \lambda)\). The eigenvalues of the local monodromy \(T_0\) at \(\lambda = 0\) on \(V_Q\) (resp. \(V_P\)) are \(1, 1, e^{2\pi i(\mu - \alpha^x - \beta^x)}\) (resp. \(1, e^{2\pi i(\mu - \alpha^x - \beta^x)}\)). Therefore the eigenvalue of \(T_0\) on \(V_Q/V_P \cong \mathbb{C}\) is 1, namely the trivial action. Therefore \(T_1 = T_{\infty}^{-1}\) acts on \(H_\mu(\lambda)\) by multiplication by \(e^{2\pi i\mu}\). Thus the function

\[
(1 - \alpha^x)(1 - \beta^x)(\lambda - 1)^{1-\mu}H_\mu(\lambda) = \binom{1, 1, 1 - \mu}{2 - \alpha^x, 2 - \beta^x ; (1 - \lambda)^{-1}}
\]

has the trivial monodromy, and this means that this is a rational function. This is impossible. Indeed let \(\sum n a_n z^n\) be the Laurent expansion of the above with respect to variable \(z = 1 - \lambda\). Then this satisfies a differential equation

\[
Q = (D_z - 1)(D_z - 1/(D_z - 1 + \mu)) - z(D_z - 1 + \alpha^x)(D_z - 1 + \beta^x).
\]

Hence

\[
(n - 1)^2(n - 1 + \mu)a_n - (n - 1)(n - 2 + \alpha^x)(n - 2 + \beta^x)a_{n-1} = 0, \quad \forall n.
\]

Then \(a_n = 0\) for all \(n \leq 0\) as \(a_n = 0\) for \(n < 0\). Moreover

\[
a_n = \frac{(n - 2 + \alpha^x)(n - 2 + \beta^x)}{(n - 1)(n - 1 + \mu)}a_{n-1} = \frac{(\alpha^x)^{n-1}(\beta^x)^{n-1}}{(n - 1)!}(1 + \mu)^{n-1}a_1, \quad n \geq 1.
\]

We thus have

\[
\sum_n a_n z^n = a_1 z \cdot \binom{\alpha^x, \beta^x}{1 + \mu, z}.
\]

The right hand side has nontrivial monodromy unless \(a_1 = 0\).

**Theorem 5.8.** Let the notation and assumption be as in Theorem 5.1. Then the dual of the exact sequence

\[
0 \rightarrow W_2 \mathcal{H}_{\text{dR}}(t; \chi) \rightarrow \mathcal{H}_{\text{dR}}'(t; \chi) \rightarrow C(t; \chi) \rightarrow 0 \quad (5.15)
\]

of connections which underlies \((5.2)\) is isomorphic to

\[
0 \rightarrow DS/\theta_\chi DS \rightarrow DS/DSQ_{\text{BG}} \rightarrow DS/DS_{\text{HG}} \rightarrow 0. \quad (5.16)
\]

In particular, the extension \((5.15)\) is nontrivial by Lemma 5.7. In other words, the regulator \(\rho(t; \chi)\) in Theorem 5.1 does not vanish.

**Proof.** By the Riemann-Hilbert correspondence, it is enough to show that there is an isomorphism

\[
0 \rightarrow W_2 \mathcal{H}_{\text{B}}(t; \chi) \rightarrow \mathcal{H}_{\text{B}}'(t; \chi) \rightarrow C(t; \chi) \rightarrow 0 \quad (5.17)
\]

where the top sequence is the underlying local systems of \(\mathbb{C}\)-modules.

We know that the local system \(W_2 \mathcal{H}_{\text{B}}(t; \chi) \cong (W_2 \mathcal{H}_{\text{B}}(\chi))^*\) are spanned by \(Z_1, Z_2,\) and \(C(t; \chi)\) by the boundary of \(\Gamma(\lambda)\) (see \((4.10)\) and Lemma 5.2 for the notation). It is not hard to see that the monodromy representation of \(C(t; \chi)\) is isomorphic to
that of $V_Q/V_P$ (cf. (5.11)). The monodromy representation of $\langle Z_1, Z_2 \rangle_C$ is isomorphic to that of
\[
\left\langle \int_{Z_1} s^{m_1-1} ds_1, \int_{Z_2} s^{m_1-1} ds_2 \right\rangle = (\Theta F_\mu(\lambda), \Theta G_\mu(\lambda))
\]
\[
\cong (F_\mu(\lambda), G_\mu(\lambda)) = V_P
\]
by Theorem 4.1 (Period formula).

To do this we need to check that the above integrals are linearly independent over $\mathbb{C}$. The first and second integrals are spanned by $\Theta F_\mu(\lambda)$ and $\Theta G_\mu(1 - \lambda)$ and they are linearly independent by Theorem 4.1 (Period formula). The 3rd integral is equal to $\Theta H_\mu(\lambda)$ modulo an algebraic function by Theorem 5.1 (Regulator formula). Suppose that the 3rd integral is a linear combination of the 1st and 2nd integrals. Then
\[
\Theta H_\mu(\lambda) = c_1 \Theta F_\mu(\lambda) + c_2 \Theta G_\mu(\lambda) + (\text{an algebraic function}), \quad (\exists c_i \in \mathbb{C}).
\]
Let $T_\infty$ be the local monodromy at $\lambda = \infty$. Then the eigenvalues on $V_P = \langle \Theta F_\mu(\lambda), \Theta G_\mu(\lambda) \rangle$ are $e^{2 \pi i (\alpha^\vee - \mu)}$, $e^{2 \pi i (\beta^\vee - \mu)}$ and $\Theta H_\mu(\lambda)$ is the eigenvector with eigenvalue $e^{-2 \pi i \mu}$. Applying $(T_\infty - e^{2 \pi i (\alpha^\vee - \mu)})(T_\infty - e^{2 \pi i (\beta^\vee - \mu)})$ to the above, we have that $\Theta H_\mu(\lambda)$ is an algebraic function. However since (5.14) is a nontrivial extension (Lemma 5.7), there is some $g \in \mathbb{C}[\pi_1(S, \lambda)]$ such that $g \Theta H_\mu(\lambda) = c_1 \Theta F_\mu(\lambda) + c_2 \Theta G_\mu(\lambda) \neq 0$. Applying $\Theta$, we have that $f(\lambda) := c_1 \Theta F_\mu(\lambda) + c_2 \Theta G_\mu(\lambda) = g \Theta H_\mu(\lambda)$ is also an algebraic function. Since $\Theta F_\mu(\lambda)$ and $\Theta G_\mu(\lambda)$ are linearly independent over $\mathbb{C}$, $f(\lambda) \neq 0$. It generates the 2-dimensional space $\langle \Theta F_\mu(\lambda), \Theta G_\mu(\lambda) \rangle \cong V_P$ as $\mathbb{C}[\pi_1(S)]$-module since $V_P$ is irreducible. Hence the monodromy representation of $V_P$ factors through a finite quotient. This is a contradiction. Hence the three integrals (5.18) are linearly independent over $\mathbb{C}$.

We now have that the monodromy representation of $\mathcal{H}_B/(\lambda)$ is isomorphic to that of
\[
\Theta F_\mu(\lambda), \Theta G_\mu(\lambda), \Theta H_\mu(\lambda) + (\text{an algebraic function}).
\]
Therefore letting $h : T \to S$ be a finite covering which trivializing the monodromy of the algebraic function, we have an isomorphism $h^* \mathcal{H}_B/(\lambda) \cong h^* V_P$ in a canonical way. Thus the extension data $[h^* V_Q] \in \text{Ext}^1_{\pi_1(T)}(h^* (V_Q/V_P), h^* V_P)$ coincides with $[h^* \mathcal{H}_B/(\lambda)] \in \text{Ext}^1_{\pi_1(T)}(h^* \text{Coker}(T_1 - 1)(\lambda), h^* \mathcal{H}_B/(\lambda))$ under the natural isomorphisms $V_P \cong W_2 \mathcal{H}_B/(\lambda)$ and $V_Q/V_P \cong \text{Coker}(T_1 - 1)(\lambda)$. Now the assertion follows from the injectivity of $\text{Ext}^1_{\pi_1(S)}(V_Q/V_P, V_P) \to \text{Ext}^1_{\pi_1(T)}(h^* (V_Q/V_P), h^* V_P)$. This completes the proof. \hfill \Box

5.6. Complement : Precise formula of Regulators. Applying the 3-term relation on $F_2$ (e.g. 2 Lemma 7.5) to (5.12) and (5.13), one can obtain a more explicit description of $\phi_\mu(\lambda)$ as $\overline{Q}(\lambda)$-linear combinations of $H_\mu(\lambda)$ and $H_{\mu-1}(\lambda)$:
\[
H_\mu(\lambda) := (1 - \alpha^\vee)^{-1}(1 - \beta^\vee)^{-1}(\lambda - 1)^{\mu-1} \cdot F_2 \left( \frac{1, 1, 1 - \mu}{2 - \alpha^\vee, 2 - \beta^\vee, (1 - \lambda)^{-1}} \right), \quad (5.19)
\]
\[
H_{\mu-1}(\lambda) := (1 - \alpha^x)^{-1}(1 - \beta^x)^{-1}(\lambda - 1)^{\mu-2} \binom{1,1,2-\mu}{2-\alpha^x,2-\beta^x} (1-\lambda)^{-1}
\]
where \(\mu := m/l\) (cf. Prop. 4.8). The following theorem is used in \([3]\).

**Theorem 5.9** (Regulator formula – precise version). Let the notation and assumption be as in Theorem 5.1. Let \(\mu = m/l\) be as in Theorem 4.1. Let \(a_1(\lambda), b_1(\lambda)\) be as in \([3]\). Put \(a := 2 - \alpha^x, b := 2 - \beta^x\), and

\[
e_i(s) := (-1)^i(a_1(\lambda) + (s + i)b_1(\lambda))(1 - \lambda)^i
\]

\[
= \begin{cases} 
\binom{d^i p_i(\lambda)}{d_{i+1}} + (s + i) \frac{d^{i+1} p_i(\lambda)}{d_{i+1}} \frac{(1-\lambda)^i}{i!} & i \geq 0, \\
-(s - 1)p_1(\lambda)/(1 - \lambda) & i = -1,
\end{cases}
\]

\(A(s) := s(a + b + 2s - 3 - s(1 - \lambda)^{-1})/(a + s - 1)(b + s - 1), \quad B(s) := s(1 - s)\lambda/(a + s - 1)(b + s - 1)\)

with indeterminate \(s\). Define \(C_i(s)\) and \(D_i(s)\) by

\[
\begin{pmatrix}
C_{i+1}(s) \\
D_{i+1}(s)
\end{pmatrix} = \begin{pmatrix}
A(s) & (\lambda - 1)^{-1} \\
B(s) & 0
\end{pmatrix} \begin{pmatrix}
C_i(s + 1) \\
D_i(s + 1)
\end{pmatrix}, \quad \begin{pmatrix}
C_{-1}(s) \\
D_{-1}(s)
\end{pmatrix} := \begin{pmatrix}0 \\
1\end{pmatrix}.
\]

Put

\[
E_1^{(r)}(s) := \sum_{i \geq -1} e_i(s + r)C_{r+i}(s), \quad E_2^{(r)}(s) := \sum_{i \geq -1} e_i(s + r)D_{r+i}(s).
\]

Then

\[
\phi_1(\lambda) := C_1(1 - \lambda)^n [E_1^{(n)}(\mu)H_\mu(\lambda) + E_2^{(n)}(\mu)H_{\mu-1}(\lambda)],
\]

\[
\phi_2(\lambda) := C_2(1 - \lambda)^{n-1} [E_1^{(n-1)}(\mu)H_\mu(\lambda) + E_2^{(n-1)}(\mu)H_{\mu-1}(\lambda)]
\]

modulo \(\overline{\mathbb{Q}(\lambda)}\) with some \(C_1, C_2 \in \overline{\mathbb{Q}}[x]\). Here we note that \(E_i^{(r)}(\mu) \in \overline{\mathbb{Q}(\lambda)}\) are rational functions of variable \(\lambda\).

**Proof.** The 3-term relation on \(3F_2\) implies that \(C_i\) and \(D_i\) satisfy

\[
3F_2 \begin{pmatrix} 1,1,1-s-i; & x \end{pmatrix}_{a,b} \equiv C_i(s,x)3F_2 \begin{pmatrix} 1,1,1-s; & x \end{pmatrix}_{a,b} + D_i(s,x)3F_2 \begin{pmatrix} 1,1,2-s; & x \end{pmatrix}_{a,b}
\]

modulo \(\mathbb{Q}(s,x)\). Hence

\[
(1-\lambda)^{m/l+i-1}3F_2 \begin{pmatrix} 1,1,1-m/l-i; & 2-\alpha^x,2-\beta^x; & (1-\lambda)^{-1} \end{pmatrix}
\]

\[
\equiv (1-\alpha^x)(1-\beta^x)(1-\lambda)^{i+r}(C_{i+r}(\mu,x)H_\mu(\lambda) + D_{i+r}(q^x,x)H_{\mu-1}(\lambda))
\]

for \(m = k + lr, r \in \mathbb{Z}\). Apply this to \([15, 12]\) and \([5, 13]\). The rest is a direct computation (left to the reader). \(\square\)

**References**

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Department of Mathematics, Hokkaido University, Sapporo, 060-0810 Japan
E-mail address: asakura@math.sci.hokudai.ac.jp

Department of Mathematics and Informatics, Chiba University, Chiba, 263-8522 Japan
E-mail address: otsubo@math.s.chiba-u.ac.jp