On generalized Ramsey numbers in the sublinear regime

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Abstract

A \((p, q)\)-coloring of a graph \(G\) is an edge-coloring of \(G\) such that every \(p\)-clique receives at least \(q\) colors. In 1975, Erdős and Shelah introduced the generalized Ramsey number \(f(n, p, q)\) which is the minimum number of colors needed in a \((p, q)\)-coloring of \(K_n\). In 1997, Erdős and Gyárffás showed that \(f(n, p, q)\) is at most a constant times \(n^{\frac{p-2}{2}}\); furthermore, they determined the thresholds for where a linear and quadratic number of colors are needed, namely \(q = \left(\frac{p}{2}\right) - p + 3\) and \(q = \left(\frac{p}{2}\right) - \left\lfloor \frac{p}{2} \right\rfloor + 2\) respectively. Very recently Bennett, Dudek, and English improved this bound by a factor of \(\log n^{\frac{p-2}{2}}\) for all \(q \leq \left(\frac{p}{2}\right) - 26p + 55\), and they ask if this improvement could hold for a wider range of \(q\).

We answer this in the affirmative for the entire sublinear regime, that is where \(q < \left(\frac{p}{2}\right) - p + 3\). Furthermore, we provide a simultaneous three-way generalization as follows: where \(p\)-clique is replaced by any fixed graph \(F\) (with \(q < |E(F)| - |V(F)| + 3\)); to list coloring; and to \(k\)-uniform hypergraphs.

Our results are a new application of the Forbidden Submatching Method of the first and third authors.

1 Introduction

In 1975, Erdős and Shelah [10] proposed a generalization of Ramsey numbers as follows. A \((p, q)\)-coloring of a graph \(G\) is an edge-coloring of \(G\) such that every \(p\)-clique receives at least \(q\) colors. The generalized Ramsey number \(f(n, p, q)\) is the minimum number of colors in a \((p, q)\)-coloring of \(K_n\). This notion was further developed by Erdős and Gyárffás [11] in 1997 and has attracted much attention over the recent decades (e.g. [2], [3], [4], [7], [15], [16]). We refer the reader to the survey chapter on hypergraph Ramsey problems by Mubayi and Suk [17] as well as the recent paper of Bennett, Dudek and English [5] for more history of the area.

In 1997, Erdős and Gyárffás [11] proved the following via a simple application of the Lovász Local Lemma.

**Theorem 1.1 (Erdős and Gyárffás [11]).** For fixed positive integers \(p, q\) with \(p > 2\) and \(1 \leq q \leq \left(\frac{p}{2}\right)\), we have

\[ f(n, p, q) = O\left(n^{\frac{p-2}{2}}\right). \]

Furthermore, Erdős and Gyárffás [11] determined the linear and quadratic thresholds for this problem as follows: more specifically, if \(q = \left(\frac{p}{2}\right) - p + 3\), they showed that \(f(n, p, q) = \Theta(n)\), while Theorem 1.1 implies that if \(q = \left(\frac{p}{2}\right) - p + 2\), then \(f(n, p, q) = o(n)\); similarly if \(q = \left(\frac{p}{2}\right) - \left\lfloor \frac{p}{2} \right\rfloor + 2\), they showed that \(f(n, p, q) = \Omega(n^2)\) while if \(q = \left(\frac{p}{2}\right) - \left\lfloor \frac{p}{2} \right\rfloor + 1\), then Theorem 1.1 implies that \(f(n, p, q) = o(n^2)\).
Very recently, Bennett, Dudek and English [5] improved the bound in Theorem 1.1 by a logarithmic factor for a wide range of values of \( p \) and \( q \) in the sublinear regime as follows.

**Theorem 1.2** (Bennett, Dudek and English [5]). For fixed positive integers \( p, q \) with \( q \leq \frac{p^2 - 26p + 55}{4} \), we have

\[
f(n, p, q) = O \left( \frac{n^{p-2}}{(\log n)^{(2) - q + 1}} \right).
\]

Note this is roughly the range up to \( q \approx \frac{p^2}{2} \). Their proof is quite involved and proceeds via a random greedy process. The authors asked if the bounds could be improved even up to the \( q \approx \frac{p^2}{2} \) range.

We answer this in the affirmative for all \( q \) below the linear threshold with our first main result as follows.

**Theorem 1.3.** For fixed positive integers \( p, q \) with \( q < \frac{(p^2)}{2} - p + 3 \), we have

\[
f(n, p, q) = O \left( \frac{n^{p-2}}{(\log n)^{(2) - q + 1}} \right).
\]

Theorem 1.3 is a new application of the Forbidden Submatching Method of the first and third author [8] (also independently introduced as conflict-free hypergraphs matchings by Glock, Joos, Kim, Kühn and Lichev [12]). Although the method has only recently been introduced, it has already been applied to resolve a number of fundamental conjectures from many different areas such as Steiner systems, Latin squares, high dimensional permutations and degenerate Turán densities. In [8] and [12], it was used to prove the existence of approximate high girth Steiner systems answering a conjecture of Glock, Kühn, Lo and Osthus [14]. In [9] and [13], it was used to prove a conjecture of Brown, Erdős and Sós [6]. In [15], it was applied to \((p, q)-coloring\) to provide a short proof that \( f(n, 4, 5) = \frac{5n^4}{6} + o(n) \).

The main contribution of this paper is to demonstrate that the machinery of the Forbidden Submatching Method can be applied to \((p, q)-coloring\) in the sublinear regime as well as to various generalizations which we will discuss in the next subsection. Whereas a number of the applications have been mostly immediate applications of the method, we note that Theorem 1.3 is not a straightforward application of the method as the natural ways of encoding the problem into the setting of hypergraph matchings do not meet the required conditions. Indeed, a main innovation of this paper is the introduction of a potential function on submatchings to define the right set of forbidden submatchings that do meet the needed conditions. Altogether, we believe that the Forbidden Submatching Method has much promise for future applications.

1.1 Generalizations and Applications

We further generalize Theorem 1.3 in three ways simultaneously as follows.

Let \( F \) be a graph. An \((F, q)-coloring\) of a graph \( G \) is an edge-coloring of \( G \) such that every copy of \( F \) in \( G \) receives at least \( q \) colors. The generalized Ramsey number \( r(G, F, q) \) is the minimum number of colors in an \((F, q)-coloring\) of \( G \).

Axenovich, Füredi, and Mubayi [2] generalized Theorem 1.3 also via a simple application of the Lovász Local Lemma as follows.

**Theorem 1.4** (Axenovich, Füredi, and Mubayi [2]). Let \( F \) be a fixed graph with \( |V(F)| > 2 \). If \( G \) is graph on \( n \) vertices and \( q \) is a positive integer with \( q \leq |E(F)| \), then

\[
r(G, F, q) = O \left( \frac{n^{|V(F)| - 2}}{n^{|E(F)| - q + 1}} \right).
\]

Thus \( r(G, F, q) \) is sublinear if \( q < |E(F)| - |V(F)| + 3 \); moreover, Axenovich, Füredi, and Mubayi further showed that if \( F \) is connected, then \( q = |E(F)| - |V(F)| + 3 \) is in fact the linear threshold when \( G = K_n \).
So our first generalization is to replace \( p \) (more specifically the implied \( K_p \)) in Theorem 1.3 by a general \( F \) with \( q < |E(F)| - |V(F)| + 3 \).

Our second generalization is to list coloring. A \( k \)-list-assignment \( L \) of the edges of a graph \( G \) is a collection of lists \( (L(e) : e \in E(G)) \) such that \( |L(e)| \geq k \) for all \( e \in E(G) \). The generalized list Ramsey number \( r_e(G,F,q) \) is the minimum \( k \) such that for every \( k \)-list-assignment \( L \) of \( E(G) \) there exists an \((F,q)\)-coloring \( \phi \) such that \( \phi(e) \in L(e) \) for all \( e \in E(G) \). Note that \( r(G,F,q) \leq r_e(G,F,q) \) for all \( G,F,q \).

Our third generalization is to hypergraphs. The definition of \( r(G,F,q) \) and \( r_e(G,F,q) \) naturally generalizes to hypergraphs \( G \) and \( F \); indeed, Erdős and Shelah’s original question concerned the generalized Ramsey number for uniform hypergraphs. We note that the Lovász Local Lemma easily yields the following generalization of Theorem 1.4 to list coloring and to \( k \)-uniform hypergraphs.

**Theorem 1.5.** Let \( k \geq 2 \) be a positive integer. Let \( F \) be a fixed \( k \)-uniform hypergraph with \( |V(F)| > k \) and let \( q \) be a positive integer with \( q \leq |E(F)| \). If \( G \) is a \( k \)-uniform hypergraph on \( n \) vertices, then

\[
r_e(G,F,q) = O\left(n^{\frac{|V(F)|-k}{|E(F)|-q+1}}\right).
\]

Here then is our general main result which improves the bound in Theorem 1.5 by a logarithmic factor in the sublinear regime.

**Theorem 1.6.** Let \( k \geq 2 \) be a positive integer. Let \( F \) be a fixed \( k \)-uniform hypergraph with \( |V(F)| > k \) and let \( q \) be a positive integer with \( q < |E(F)| - |V(F)| + k + 1 \). If \( G \) is a \( k \)-uniform hypergraph on \( n \) vertices, then

\[
r_e(G,F,q) = O\left(n^{\frac{|V(F)|-k}{\log n}}\right).
\]

As Bennett, Dudek, and English noted in [5], upper bounds on generalized Ramsey numbers have implications for the lower bounds of a problem of Brown, Erdős, and Sós [6] as follows. Let \( F^{(k)}(n;j,i) \) denote the minimum number of edges \( m \) such that every \( k \)-uniform hypergraph on \( n \) vertices and \( m \) edges has a set of \( j \) vertices spanning at least \( i \) edges. In this notation, the first and third authors [9] recently showed the following, confirming a conjecture of Brown, Erdős, and Sós [6] for 3-uniform hypergraphs.

**Theorem 1.7.** For all \( j \geq 4 \),

\[
\lim_{n \to \infty} n^{-2} F^{(3)}(n;j,j-2)
\]

exists.

For general \( i,j,k \), Bennett, Brown, Erdős, and Sós [6] showed that \( F^{(k)}(n;j,i) = \Omega\left(n^{\frac{k+j}{i-1}}\right) \). For graphs, Bennett, Dudek, and English [5] observed that \( F^{(2)}(n;j,i) \geq \frac{\binom{n}{j}}{\binom{n}{j-1}} \), since a \( (j,\binom{j}{2} - i + 2) \)-coloring of \( K_n \) contains no set of \( j \) vertices which spans \( i \) edges of the same color (and consequently every color class has at most \( F^{(2)}(n;j,i) \) edges). This together with Theorem 1.2 improves the lower bound of \( F^{(2)}(n;j,i) \) to \( \Omega\left((n^{2i-j} \log n)^{\frac{1}{i-1}}\right) \) for all \( i \geq \frac{j^2 + 24j - 47}{4} \).

Similarly for \( k \)-uniform hypergraphs, one can observe that for general \( i,j \),

\[
F^{(k)}(n;j,i) \geq \frac{\binom{n}{k}}{r(K_n^{(k)},K_j^{(k)};\binom{j}{2} - i + 2)}.
\]

Hence, Theorem 1.6 implies the following.

**Theorem 1.8.** For all fixed \( i,j,k \) with \( i > j - k + 1 \),

\[
F^{(k)}(n;j,i) = \Omega\left(n^{k+i-j} \log n^{\frac{1}{i-1}}\right).
\]
2 Proof of Theorem 1.6

As mentioned in the introduction, Theorem 1.6 will follow as an application of the main theorem of [8] by the first and third authors where they developed a very general theory of finding matchings in hypergraphs avoiding forbidden submatchings. In order to state that theorem, we first need a few definitions.

**Definition 2.1 (Configuration Hypergraph).** Let \( J \) be a (multi)-hypergraph. We say a hypergraph \( H \) is a configuration hypergraph for \( J \) if \( V(H) = E(J) \) and \( E(H) \) consists of a set of matchings of \( J \) of size at least two. We say a matching of \( J \) is \( H \)-avoiding if it spans no edge of \( H \).

**Definition 2.2 (Bipartite Hypergraph).** We say a hypergraph \( J = (A, B) \) is bipartite with parts \( A \) and \( B \) if \( V(J) = A \cup B \) and every edge of \( J \) contains exactly one vertex from \( A \). We say a matching of \( J \) is \( A \)-perfect if every vertex of \( A \) is in an edge of the matching.

**Definition 2.3.** Let \( H \) be a hypergraph. The \( i \)-degree of a vertex \( v \) of \( H \), denoted \( d_{H,i}(v) \), is the number of edges of \( H \) of size \( i \) containing \( v \). The maximum \( i \)-degree of \( H \), denoted \( \Delta_i(H) \), is the maximum of \( d_{H,i}(v) \) over all vertices \( v \) of \( H \).

The \( s \)-codegree of a set of vertices \( S \) of \( H \), denoted \( d_{H,s}(S) \) is the number of edges of \( H \) of size \( s \) containing \( S \). The maximum \((s,t)\)-codegree of \( H \) is

\[
\Delta_{s,t}(H) := \max_{S \in \binom{V(H)}{s}} d_{H,s}(S).
\]

We define the common 2-degree of distinct vertices \( u,v \in V(H) \) as \( |\{w \in V(H) : uw,vw \in E(H)\}| \). Similarly, we define the maximum common 2-degree of \( H \) as the maximum of the common 2-degree of \( u \) and \( v \) over all distinct pairs of vertices \( u,v \) of \( H \).

**Definition 2.4.** Let \( J \) be a hypergraph and let \( H \) be a configuration hypergraph of \( J \). We define the \( i \)-codegree of a vertex \( v \in V(J) \) and \( e \in E(J) = V(H) \) with \( v \notin e \) as the number of edges of \( H \) of size \( i \) who contain \( e \) and an edge incident with \( v \). We then define the maximum \( i \)-codegree of \( J \) with \( H \) as the maximum \( i \)-codegree over vertices \( v \in V(J) \) and edges \( e \in E(J) = V(H) \) with \( v \notin e \).

We are now ready to state the main theorem from [8].

**Theorem 2.5.** [Theorem 1.16 from [8]] For all integers \( r_1, r_2 \geq 2 \) and real \( \beta \in (0,1) \), there exist an integer \( D_\beta \geq 0 \) and real \( \alpha > 0 \) that follow for all \( D \geq D_\beta \):

Let \( J = (A, B) \) be a bipartite \( r_1 \)-bounded (multi)-hypergraph with 2-degrees at most \( D^{1-\beta} \) such that every vertex in \( A \) has degree at least \( (1 + D^{-\alpha})D \) and every vertex in \( B \) has degree at most \( D \). Let \( H \) be an \( r_2 \)-bounded configuration hypergraph of \( J \) with \( \Delta_i(H) \leq \alpha \cdot D^{-1} \log D \) for all \( 2 \leq i \leq r_2 \) and

\[
\Delta_{s,t}(H) \leq D^{s-t-\beta} \quad \text{for all} \ 2 \leq t < s \leq r_2.
\]

If the maximum 2-degree of \( J \) with \( H \) and the maximum common 2-degree of \( H \) are both at most \( D^{1-\beta} \), then there exists an \( H \)-avoiding \( A \)-perfect matching of \( J \) and indeed even a set of \( D \) disjoint \( H \)-avoiding \( A \)-perfect matchings of \( J \).

Theorem 2.5 is very general: namely, it simultaneously generalizes the famous matching theorem of Pippenger and Spencer [18] (in the case when \( H \) is empty albeit with weaker codegree assumptions) and (vertex) independent set results for girth five hypergraphs or hypergraphs with small codegrees, for example the celebrated result of Ajtai, Komlós, Pintz, Spencer, and Szemerédi [1] (in the case when \( J \) is a matching).

Indeed, the bipartite hypergraph \( A \)-perfect matching formulation of Theorem 2.5 is stronger than all of the above as it even implies the coloring and list coloring versions of those results. Moreover, we will need the full strength of this formulation to prove Theorem 1.6 as will become clear from the proof.

To prove Theorem 1.6 we will apply Theorem 2.5 to certain well-chosen \( J \) and \( H \) to find an \( H \)-avoiding \( A \)-perfect matching of \( J \) which will correspond to an \((F,q)\)-coloring of \( G \). The main idea is to set \( A \) to be the edges of \( G \) and define the edges of \( J \) to correspond to edge-color pairs \((e,c)\) where \( c \in L(e) \) in such a
way that \( J \) is a graph that is the disjoint union of stars. Thus an \( A \)-perfect matching of \( J \) will correspond to an \( L \)-coloring of the edges of \( G \).

To enforce that the coloring is an \((F, q)\)-coloring we would naturally define our edges of \( H \) to be the sets of edge-color pairs which would correspond to a coloring of a copy of \( F \) with at most \( q - 1 \) colors. By setting these to be the forbidden configurations, it follows that an \( H \)-avoiding \( A \)-perfect matching of \( J \) corresponds to an \((F, q)\)-coloring of \( G \).

It turns out that the gained logarithmic factor in Theorem 1.6 precisely results from the log \( D \) factor in Theorem 2.5. Unfortunately, this natural definition of \( H \) does not satisfy the small codegree conditions of Theorem 2.5, namely that \( \Delta_{s,t}(H) \leq D^{s-t-\beta} \). Indeed, there can be subconfigurations of the bad configurations whose codegrees are too large.

Thus the main innovation in our proof of Theorem 1.6 is to define a potential function over all possible subconfigurations and then define \( H \) to be the set of minimal subconfigurations with potential at least that of the \((q - 1)\)-colored copies of \( F \). The small codegree condition will then follow by the minimality of \( H \). Importantly, we are thus using that \( H \) in Theorem 2.5 is allowed to have mixed uniformities. Intriguingly, the only place we use that \( q < |E(F)| - |V(F)| + k + 1 \) is to show that \( H \) has no configurations of size two (and hence we do not need to check the maximum 2-codegree of \( J \) with \( H \) or the common 2-degree of \( H \)).

We are now prepared to prove Theorem 1.6 as follows.

**Proof of Theorem 1.6.** For ease of notation, let \( p := |V(F)| \) and \( r := |E(F)| \). Set \( T := \left\lceil C \cdot \left( \frac{n^{p-k}}{\log n} \right)^{r-q+1} \right\rceil \).

Note we will choose \( n, C \) sufficiently large to satisfy various inequalities that appear throughout the proof. Let \( L \) be a \( T \)-list-assignment of \( E(G) \). We assume without loss of generality that \(|L(e)| = T\) for all \( e \in E(G) \).

We define a bipartite graph \( J = (A, B) \) as follows:

- \( A := E(G) \)
- \( B := \{(e, c) : e \in E(G), c \in L(e)\} \)
- \( E(J) := \{(e, (e, c)) : e \in E(G), c \in L(e)\} \).

For convenience of notation, we will refer to the edge \((e, (e, c))\) as simply \((e, c)\).

Let \( \beta := \frac{1}{2(p-k)}, r_1 := 2 \) and \( r_2 := r \). Recall that \( p > k \) by assumption and hence \( \beta > 0 \). Let \( D_\beta \) and \( \alpha \) be as in Theorem 2.5 for \( r_1, r_2 \) and \( \beta \). Let \( D := \frac{T}{2} \). Since \( n \) and \( C \) are large enough, we have that \( D \geq D_\beta \).

Note that \( J \) is a bipartite 2-bounded hypergraph with 2-codegrees at most \( 1 \leq D^{1-\beta} \) such that every vertex in \( A \) has degree at least \( T = 2D \geq (1 + D^{-\alpha})D \) (since \( \alpha > 0 \)) and every vertex in \( B \) has degree at most \( 1 \leq D \).

For a matching \( M \) of \( J \), denote by \( E(M) \) the set of edges \( e \) of \( G \) such that there exists \( f \in M \) with \( e \in f \). Note that \(|M| = |E(M)| \) since \( M \) is a matching. We let \( V(M) \) denote the set of vertices \( v \) of \( G \) spanned by \( E(M) \), and \( C(M) \) denote the set of colors \( c \) such that there exists \( f = (e, c) \in M \) for some edge \( e \) of \( G \). We then define the potential of a matching \( M \) of \( J \) as

\[
\rho(M) := |M| - |C(M)| - (|V(M)| - k) \cdot \left( \frac{r - q + 1}{p - k} \right).
\]

Let \( \kappa_F \) denote the set of subgraphs of \( G \) that are isomorphic to \( F \). The configuration hypergraph \( H \) of \( J \) is defined as follows. We let \( V(H) = E(J) \) and \( E(H) \) be the set of matchings \( M \) of \( J \) such that

- \(|M| \geq 2, \)
- \( E(M) \subseteq E(K) \) for some \( K \in \kappa_F, \)
- \( \rho(M) \geq 0, \)
- and subject to those conditions \( M \) is inclusion-wise minimal.
Note that $H$ is $r$-bounded since for every $K \in \kappa_F$, we have $|E(K)| \leq r$. Furthermore, if $M$ is a matching of $J$ with $|M| = r$, $|V(M)| = p$ and $|C(M)| \leq q - 1$, then

$$\rho(M) \geq r - (q - 1) - (p - k) \cdot \left(\frac{r - q + 1}{p - k}\right) = 0. \quad (1)$$

Suppose $Z$ is an $H$-avoiding $A$-perfect matching of $J$. Let $\phi$ be an edge coloring of $G$ defined as $\phi(e) := c$ where $(e, c) \in Z$. Since $Z$ is an $A$-perfect matching of $J$, we have that $\phi$ is well-defined for all edges $e$ of $G$. Since $(e, \phi(e)) \in E(J)$, it follows that $\phi(e) \in L(e)$ for every edge $e$ of $G$. Furthermore, since $Z$ is $H$-avoiding, it follows from (1) that $\phi$ is an $(F, q)$-coloring of $G$ (since otherwise it would contain a matching $M$ with $|M| = r$, $|V(M)| = p$ and $|C(M)| \leq q - 1$ and hence would contain an edge of $H$ contradicting that $Z$ is $H$-avoiding).

Thus it suffices to prove that there exists an $H$-avoiding $A$-perfect matching of $J$. To that end, we verify that $J$ and $H$ satisfy the hypotheses of Theorem 2.5 where $\beta, r_1, r_2$ and $D$ are defined as above.

Let $I$ denote the set of triples of integers $(m, v, \ell)$ with $2 \leq m \leq r$, $k + 1 \leq v \leq p$, $1 \leq \ell \leq r$ where

$$m \geq \ell + (v - k) \cdot \left(\frac{r - q + 1}{p - k}\right). \quad (2)$$

Note that $|I|$ is a constant that depends only on $k$, $p$, $q$ and $r$. For a triple $(m, v, \ell) \in I$, we let $H_{m,v,\ell}$ be the $m$-uniform configuration hypergraph of $J$ where $E(H_{m,v,\ell})$ is the set of edges $M$ of $H$ with $|M| = m$, $|V(M)| = v$ and $|C(M)| = \ell$. Note that by definition of $H$, we have $H = \bigcup_{(m, v, \ell) \in I} H_{m,v,\ell}$.

We first claim that $H$ contains no edges of size exactly 2. Suppose for a contradiction that there exists $M \in E(H)$ with $|M| = 2$. Then $|V(M)| \geq k + 1$ (as $M$ is a matching of $J$) and $|C(M)| \geq 1$, and hence

$$\rho(M) \leq 2 - 1 - 1 \cdot \left(\frac{r - q + 1}{p - k}\right) < 0,$$

where the last inequality follows since $q < r - p + k + 1$. This contradicts the definition of $H$, which proves the claim. It follows then that the maximum 2-codegree of $J$ with $H$ and the maximum common 2-degree of $H$ are both zero and hence are at most $D^{1-\beta}$.

Next, we check the degrees of $H$. Let $(e, c) \in V(H)$, and assume $e = \{u_1, u_2, \ldots, u_k\}$. Fix an arbitrary triple $(m, v, \ell) \in I$. In particular, by the choices of $v, k, p, q, r$ and (2), we have

$$m - \ell > 1. \quad (3)$$

We now calculate an upper bound on the number of $M \in E(H_{m,v,\ell})$ such that $(e, c) \in M$ as follows. First, there are at most $n^{v-k}$ choices for the $v - k$ vertices of $V(M) \setminus \{u_1, u_2, \ldots, u_k\}$. Second, there are at most $T^{\ell-1}$ choices for the $\ell - 1$ colors in $C(M) \setminus \{c\}$. Once we have fixed $V(M)$ and $C(M)$, the number of choices for $M$ is at most $O(1)$. Therefore, there exists a constant $C_1$ such that

$$\frac{d_{H_{m,v,\ell}}((e, c))}{D^{m-1} \log D} \leq C_1 \cdot \frac{n^{v-k} \cdot T^{\ell-1}}{T^{m-1} \log n} \leq C_1 \cdot C^{\ell-m} \cdot \frac{n^{v-k}}{\log n} \left(\frac{n^{p-k}}{\log n}\right)^{\frac{\ell-m}{r-q+1}}.$$

Since $n$ and $C$ are chosen to be sufficiently large and since $p > k$, we have $C, \frac{n^{p-k}}{\log n} > 1$. Therefore by (2), (3), and the monotonicity of the function $a^x$ (where $a > 1$), we have

$$\frac{d_{H_{m,v,\ell}}((e, c))}{D^{m-1} \log D} \leq \frac{C_1}{C} \cdot \frac{n^{v-k}}{\log n} \left(\frac{n^{p-k}}{\log n}\right)^{\frac{v-k}{p-k}} = \frac{C_1}{C} \cdot (\log n)^{v-k} \leq \alpha \left(\frac{\ell}{|I|}\right),$$

where the last inequality follows since $C$ is chosen to be large enough and $v \leq p$. Therefore, for any $2 \leq i \leq r$ and $(e, c) \in V(H)$, we have

$$\sum_{(i,v,\ell) \in I} d_{H_{i,v,\ell}}((e, c)) \leq \alpha \cdot D^{i-1} \log D.$$
We now check the codegrees of $H$. Fix integers $s, t$ with $2 \leq t < s \leq r_2$. Let $S$ be a subset of $V(H)$ of size $t$. We desire to show that $d_{H, s}(S) \leq D^{s-t-\beta}$. If $S$ is not strictly contained in at least one edge of $H$, then $d_{H, s}(S) = 0$ as desired. So we assume that $S$ is strictly contained in some edge $f$ of $H$. By definition of $H$, we have that $E(f) \subseteq E(K)$ for some $K \in \kappa_F$ and $\rho(f) \geq 0$. Since $|S| = t \geq 2$ and $S \subseteq f$, we have by the minimality of $H$ that $\rho(S) < 0$.

Recall that $|S| = t$, and let $v' := |V(S)|$ and $\ell' := |C(S)|$. Since $\rho(S) \cdot (p - k)$ is an integer, it follows that $\rho(S) \leq -\frac{1}{p-k}$ and hence

$$t \leq \ell' + (v' - k) \left( \frac{r - q + 1}{p - k} \right) - \frac{1}{p - k}. \quad (4)$$

Fix an arbitrary triple $(s, v, \ell) \in I$ with $s > t$, $v \geq v'$ and $\ell \geq \ell'$. Let $\mu := (\ell - s) - (\ell' - t)$, and recall that $\beta = \frac{1}{2(p-k)}$. Then by (2) and (4), we have

$$\mu \leq -(v - v') \left( \frac{r - q + 1}{p - k} \right) - 2\beta, \quad (5)$$

and in particular, $\mu + \beta < 0$.

We now calculate an upper bound on the number of $M \in E(H_{s,v,\ell})$ such that $S \subseteq M$ as follows. Similarly as before, there are at most $n^{v-v'}$ choices for the $v - v'$ vertices of $V(M) \setminus V(S)$, and at most $T^{t-\ell'}$ choices for the $\ell - \ell'$ colors in $C(M) \setminus C(S)$. Once we have fixed $V(M)$ and $C(M)$, the number of choices for $M$ is at most $O(1)$. Therefore, there exists a constant $C_2$ such that

$$\frac{d_{H_{s,v,\ell}}(S)}{D^{s-t-\beta}} \leq C_2 \cdot \frac{n^{v-v'} \cdot T^{t-\ell'}}{T^{s-t-\beta} \cdot C^{n+\beta \cdot n^{v-v'} \left( \frac{n^{p-k}}{\log n} \right)^{\frac{\mu+\beta}{r-q+1}}}}.$$ 

Once again, since $n$ and $C$ are chosen to be sufficiently large and $p > k$, we have $C, \frac{n^{p-k}}{\log n} > 1$. Therefore by (5) and the monotonicity of the function $a^x$ (where $a > 1$), we have

$$\frac{d_{H_{s,v,\ell}}(S)}{D^{s-t-\beta}} \leq C_2 \cdot n^{v-v'} \left( \frac{n^{p-k}}{p-k} \right)^{-\frac{v-v'}{r-q+1}} = C_2 \cdot n \cdot \frac{\beta(p-k)}{r-q+1} \leq \frac{1}{|I|},$$

where the last inequality follows since $n$ is chosen to be large enough and $p > k$.

Hence, for any $2 \leq t < s \leq r_2$, we have

$$\Delta_{s,t}(H) = \max_{S \in \binom{V(H)}{t}} d_{H, s}(S) \leq \max_{S \in \binom{V(H)}{t}} \sum_{(s, v, \ell) \in I} d_{H_{s, v, \ell}}(S) \leq D^{s-t-\beta}.$$

Finally, applying Theorem 2.5 on $J$ and $H$, we obtain that there exists an $H$-avoiding $A$-perfect matching of $J$, which completes the proof.

3 Concluding Remarks

We note that if $F$ is connected, then Theorem 1.6 holds with $n$ replaced by $(k - 1)\Delta(G)$. This is a natural bound in the “sparse” regime (when $\Delta(G)$ is much smaller than $n$). Unfortunately our proof of Theorem 1.6 does not immediately carry over to provide a logarithmic improvement over this simple bound in the sparse regime. The reason is that, while $F$ is connected, the forbidden configurations (the edges of $H$) are simply subsets of $F$ and hence may induce a disconnected graph. So we are unsure what the correct bound is in the sparse regime. We think it would be interesting to determine these numbers more precisely.

As mentioned in the introduction, Glock, Joos, Kim, Kühn and Lichev [12] independently developed a similar theory of avoiding forbidden submatchings as in the Forbidden Submatching Method of [8]. The proof of Glock et. al. proceeds via a random greedy process while the proof in [8] uses the nibble method. One of the main results of [12] is a weaker form of Theorem 2.5 which essentially has two limitations
compared to Theorem 2.5. First it only holds for dense graphs (where $D \geq \text{polylog} |V(J)|$); second, it only finds an almost perfect matching in a general hypergraph $J$. While not immediately obvious, the bipartite $A$-perfect formulation in Theorem 2.5 actually implies the coloring and list coloring versions of their results and more. Intriguingly, even though the main results of this paper, Theorems 1.3 and 1.6 are in the dense regime, the theorems do not follow from the results of Glock et al. since the proof requires the $A$-perfect formulation of Theorem 2.5 to actually find an $(F,q)$-coloring of all edges of $G$ instead of just for almost all edges of $G$.

In this paper, we also defined the generalized list Ramsey number. While we believe this notion is quite natural, we had not seen it in the literature to date. As seen in Theorem 1.5, the upper bounds from Theorems 1.1 and 1.4 hold for list coloring. Of course, the lower bounds for coloring also carry over to list coloring. Thus the linear and quadratic thresholds for $r_{\ell}(K_n, K_p, q)$ are the same as for $r(K_n, K_p, q)$. Nevertheless, we wonder if the list version may be interesting in its own right. In particular, could there be better lower bound constructions when one is allowed to choose a list assignment? So we ask the following.

**Question 3.1.** Do $r(K_n, K_p, q)$ and $r_{\ell}(K_n, K_p, q)$ differ in order of magnitude for some $p$ and $q$?

**Question 3.2.** More generally for a fixed positive integer $k \geq 2$, do $r(K_n^{(k)}, K_p^{(k)}, q)$ and $r_{\ell}(K_n^{(k)}, K_p^{(k)}, q)$ differ in order of magnitude for some $p$ and $q$?

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