One-dimensional topological metal

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We propose a new type of topological states of matter exhibiting topologically nontrivial edge states (ESs) within gapless bulk states (GBSs) protected by both spin rotational and reflection symmetries. A model presenting such states is simply comprised of a one-dimensional reflection symmetric superlattice in the presence of spin-orbit coupling containing odd number of sublattices per unit cell. We show that the system has a rich phase diagram including a topological metal (TM) phase where nontrivial ESs coexist with nontrivial GBSs at Fermi level. Topologically distinct phases can be reached through subband gap closing-reopening transition depending on the relative strength of inter and intra unit cell spin-orbit couplings. Moreover, topological class of the system is AI with an integer topological invariant called \( Z \) index. The stability of TM states is also analyzed against Zeeman magnetic fields and on-site potentials resulting in that the spin rotational symmetry around the lattice direction is a key requirement for the appearance of such states. Also, possible experimental realizations are discussed.

**Introduction.**—The search for exotic quantum states of matter has attracted a great deal of attention since discovery of topological insulators (TIs) \([1]\) and topological superconductors (TSs) \([2]\) in condensed matter physics. Further investigations have also revealed a novel nontrivial topological states in the so-called Weyl semimetals possessing gapless bulk and Fermi arc surface states \([3]\). In contrast, TIs and TSs have symmetry protected edge states (ESs) inside gapped bulk states. Apart from condensed matter systems, some schemes have been proposed to realize topological phases using cold atoms in optical lattices \([4]\) and employing light in photonic crystals \([5]\). Also, the exploration of topological states has been extended even to classical systems \([6]\).

In most of the TIs, the symmetry protected gapless ESs, making relevant requirement for topological quantum computations \([7]\), play dominant role only in a limited certain range of energies. Moreover, due to smallness of energy gap, ESs may also be faded by excitations of bulk states at finite temperature. On the other hand, the coexistence of ESs and bulk states occurs in a narrow energy window in three-dimensional TI candidate materials such as Bi\(_2\)Se\(_3\) and Bi\(_2\)Te\(_3\) \([8]\). However, it is intriguing to have a situation in which dominant symmetry protected ESs exist not only in the bandgap but also within the gapless bulk states (GBSs). Therefore, such system would be in turn served as topological metals (TMs) \([9]\) even by shifting Fermi level toward conduction or valence bands.

Several one dimensional models have been proposed to realize new classes of topological phases \([10]\) concerning both TIs \([11]\) and TSs \([12]\). These studies stimulate to look for new possibilities for nontrivial topological states mimicking neither TIs nor TSs. So far, however, metallic phase being quasi-degenerate with topologically protected ESs has not been reported to be nontrivial in topology. Hence, it is interesting to develop a minimal and feasible model by which TMs can be emerged easily. In the present Letter, we consider a one-dimensional spin-orbit-coupled superlattice with odd number of sublattices in each unit cell, as shown in Fig. 1(a), featuring various nontrivial phases [see Figs. 2(a) and 2(b)]. Surprisingly, we find that topologically protected ESs while keeping their non-triviality can extend into GBSs with increasing spin-orbit interaction as shown in Figs. 1(b) and 1(d). This results in nontrivial TM phase due to settling spin-orbit coupling on odd number of sublattices provided that spin rotational and reflection symmetries are not broken. It should be noted that our findings are valid in general physical grounds, independent of our specific model, and therefore may be realized in a variety of platforms such as
condensed matter systems and quantum Fermi gases.

**Model.**—We consider a one-dimensional multipartite spin-orbit-coupled superlattice along x-axis with period $T \geq 3$ described by the total tight-binding Hamiltonian as

$$
\hat{H} = \hat{H}_t + \hat{H}_{so},
$$

where kinetic ($\hat{H}_t$) and spin-orbit ($\hat{H}_{so}$) terms are given by $\hat{H}_t = \sum_{n,\sigma} \sum_{\alpha} \epsilon_{\alpha} c_{n,\sigma}^{\dagger} c_{n+1,\sigma} + h.c.$ and $\hat{H}_{so} = \sum_{n,\sigma} \sum_{\alpha} \epsilon_{\alpha} c_{n,\sigma}^{\dagger} c_{n+1,\sigma} + h.c.$, respectively. The operator $c_{n,\sigma}^{\dagger} (c_{n,\sigma})$ is fermion creation (annihilation) operator of electrons with spin $\sigma = (\uparrow, \downarrow)$ on the $\alpha$ sublattice of $n$th unit cell. In addition, $\alpha_t (\lambda_{\alpha})$ denotes the hopping (spin-orbit coupling) amplitude. Considering periodic boundary conditions and performing Fourier transformation, the Hamiltonian (1) can be written in the basis of $\psi = (\hat{c}_{\alpha,t,k}, \hat{c}_{\alpha,k,\uparrow}, \hat{c}_{\alpha,k,\downarrow})^T$ where $\hat{c}_{\alpha,k,\sigma} = (\hat{c}_{\alpha,k,\uparrow}, \hat{c}_{\alpha,k,\downarrow})$ yielding a compact form $\hat{H} = \sum_{k} \hat{\psi}^{\dagger} \hat{H}(k) \hat{\psi}$ with

$$
\hat{H}(k) = \begin{pmatrix}
0 & \hat{h}_1 & \hat{h}_pe^{-ik} \\
\hat{h}_1^\dagger & \ddots & \ddots \\
\hat{h}_pe^{ik} & \hat{h}_{T-1} & 0
\end{pmatrix}_{T \times T},
$$

where $\hat{h}_\alpha = t_{\alpha} I + \lambda_{\alpha} \tau_x$ with $I$ and $\tau_x$ being identity matrix and the x-component of Pauli vector acting on spin space, respectively, and $\alpha = 1, ..., T$. When $\lambda_{\alpha} = \lambda_{T-\alpha}$ and $t_{\alpha} = t_{T-\alpha}$, our model preserves unitary reflection symmetry about a 1D mirror point (located at the middle of unit cell) as $R \hat{H}(k) R^{-1} = \hat{H}(-k)$ with $R = \delta_{i,T-1-j} \otimes \tau_x$ where $\delta_{i,j}$ is Kronecker delta. Also, because the x-component of spin is a good quantum number, the lattice has a U(1) spin rotational symmetry. So $\hat{H}(k)$ is invariant under spin rotation operator $U = I_T \otimes \tau_x$ around the x-axis where $I_T$ is an identity matrix of size $T$. In addition, effective anti-unitary time-reversal symmetry can be determined as $\overline{T} \hat{H}(k) \overline{T}^{-1} = \hat{H}(-k)$ with $\mathcal{T} = I_T \otimes \tau_y$, $K$ where $K$ is the complex conjugate. Note that the usual time-reversal symmetry is broken due to the presence of spin-orbit coupling, however, the subtle effective time-reversal symmetry is present exhibiting $\mathcal{T}^2 = 1$. Also, the reflection and spin rotation operators show the properties $\mathcal{R}^2 = U^2 = 1$. Since the symmetries of Hamiltonian are based on the conventional symmetries, thus the topology of ESs can be classified following the general classification (13, 14) of topological systems. Due to commutation relation of the reflection and time-reversal operators, i.e., $[\mathcal{R}, \mathcal{T}] = 0$, the topological classification of our model belongs to AI class with an integer topological index $\mathcal{Z}$ enumerating the number of gapless boundary pair states.

Since the spin rotation operator commutes with $\hat{H}(k)$, the Hamiltonian can be block-diagonalized into two $T \times T$ Hamiltonians as $\hat{\mathcal{H}}(k) = \hat{h}_{U=\pm}(k) \oplus \hat{h}_{U=\pm}(k)$ whose decoupled subspaces are spanned by eigenstates of $\mathcal{U}$ with eigenvalues $\pm 1$. This can be done through a unitary transformation $\hat{\mathcal{H}}(k) = \mathcal{U} \hat{\mathcal{H}}(k) \mathcal{U}^\dagger$ where $\mathcal{U}$ is constructed from the eigenspace of $\mathcal{U}$ and will be characterized below. Each block of $\hat{\mathcal{H}}(k)$ takes the form

$$
\hat{h}_{U=\pm}(k) = \begin{pmatrix}
0 & \Gamma_{\pm}^+ & \Gamma_{\pm}^e^{\pm} e^{ik} \\
\Gamma_{\pm}^+ & \Gamma_1^+ & \Gamma_{\pm}^e^{-} e^{-ik} \\
\Gamma_{\pm}^e^{\pm} e^{-ik} & \Gamma_{\pm}^e^{-} e^{ik} & 0
\end{pmatrix}_{T \times T},
$$

where $\Gamma_{\pm}^\mp = t_{\alpha} \pm \lambda_{\alpha}$. Note that $\hat{h}_{U=\pm}$ has both reflection and time-reversal symmetries because of $[\mathcal{U}, \mathcal{R}] = 0$ and $[\mathcal{U}, \mathcal{T}] = 0$. Now, we define the topological invariant $\mathcal{Z}$ as follows [15]. Each sector of transformed Hamiltonian $\hat{\mathcal{H}}(k)$ commutes with reflection operator at reflection symmetric momenta $k_{ref} = (0, \pi)$, thus, the eigenstates of $\hat{h}_{U=\pm}$ have a well-defined parity $\mathcal{G}_{\Xi=\pm}(k_{ref}) = \pm 1$ at those points. This, subsequently, allows for specifying an integer invariant to classify $\hat{h}_{U=\pm}$ as $N_{\mathcal{G}_{\Xi=\pm}} = |n_{1,i,U=\pm} - n_{2,i,U=\pm}|$. Here, we have defined $n_{1,i,U=\pm}$ and $n_{2,i,U=\pm}$ as the number of negative parities related to the energy bands of $\hat{h}_{U=\pm}(k)$ in the $i$th bandgap at $k_{ref}$ equals 0 and $k_{ref} = \pi$, respectively. So, the meaningful $\mathcal{Z}$ topological invariant for multi-band and multi-subspace structure of the system can be defined as

$$
\mathcal{Z} := \sum_{j=\pm}^{T-1} \sum_{i=1} N_{\mathcal{G}_{\Xi=\pm}},
$$

giving the number of localized ESs under open boundary conditions.

Interestingly, each subsystem, described by $\hat{h}_{U=\pm}$, is similar to a one-dimensional spinless system consisting of $T$ "super-sublattices" per unit cell. Each super-sublattice is comprised of a sublattice with opposite spin species so that the new hopping amplitude between two adjacent super-sublattices in the subsystem labelled by $U = \pm$ is $\Gamma_{\pm}^\mp$, i.e., a linear combination of the spin-orbit coupling and usual hopping amplitude. This can be illuminated by a transformation...
from the old basis to the new one through the unitary matrix 
$$U_{2T \times 2T} \Psi_i = \sum_{j=1}^{2T} U_{ij} \psi_j$$
where \( i = 1, \ldots, 2T \). The non-zero matrix elements of \( U \) are

$$U_{\alpha,2(T-\alpha+1)} = \frac{1}{\sqrt{2}}, \quad U_{2T-\alpha+1,2\alpha-1} = \frac{1}{\sqrt{2}}$$

$$U_{\alpha,2(T-\alpha)+1} = -\frac{1}{\sqrt{2}}, \quad U_{2T-\alpha+1,2\alpha} = -\frac{1}{\sqrt{2}}$$

Therefore, the new basis is a vector column \( \hat{\Psi} = (\hat{\Psi}^-, \hat{\Psi}^+) \)
where \( \hat{\Psi}^- (\hat{\Psi}^+) \) corresponds to the basis of eigenspace of \( \mathcal{U} \) with eigenvalues \(-1 (1)\) whose entries are super-sublattices given by

$$\hat{\Psi}^{\pm}_n = 1/\sqrt{2}(\hat{c}_{T-\alpha+1,k,n} \pm \hat{c}_{T-\alpha+1,k,n})$$

On the other hand, it is also easy to obtain real-space operator for the spin rotational symmetry as

$$\mathcal{U} = I_{TN} \times \tau_x$$

where \( I_{TN} \) is an identity matrix of size \( TN \) with \( N \) being the number of unit cells. Therefore, the real-space Hamiltonian \( (1) \) can be brought into two block-diagonal matrices in the
eigenspace of \( \mathcal{U} = \pm \) as \( \hat{T} = \hat{h}_{\mathcal{U}=\pm} \oplus \hat{h}_{\mathcal{U}=\pm} \). In order to study the localization of the states of \( \hat{T} \), we calculate normalized logarithm of the inverse participation ratio (IPR) of an
eigenvector \( |\psi_i\rangle \) associated with eigenenergy \( E \) as defined by \( I_E = \text{Ln}(|\langle i|\psi_i\rangle|^4)/\text{Ln}(2TN) \) where \( |i\rangle \) is the basis elements \([16]\). Here, \( I_E = -1 \) denotes
delocalized states, whereas much more localized ones tend to have bigger values up to \( I_E = 0 \).

**Results and discussion.**—Without loss of generality, we focus on the case of three sublattices per unit cell, \( T = 3 \), as the model illustrated in Fig. 1(a). Reflection symmetry requires \( \lambda_1 = \lambda_2 \equiv \lambda' \) and \( \tau_1 = \tau_2 \equiv \tau' \). Here, the intra and inter unit cell hoppings, respectively, are \( \tau' = t - \delta t \) and \( \tau_3 = t + \delta t \) where \( t (\delta t) \) stands for hopping energy (hopping modulation strength). Energy spectrum of \( \hat{T} \) and its IPR as a function of intra unit cell spin-orbit coupling strength \( \lambda' \) under open boundary conditions is shown in Fig. 1(b) with inter unit cell spin-orbit coupling strength \( \lambda_3 = -\lambda' \) and \( \delta t < 0 \). The solid lines and hexagons represent the eigenvalues of \( \hat{h}_{\mathcal{U}=\pm} \) and \( \hat{h}_{\mathcal{U}=\pm} \), respectively. As \( \lambda' \) increasing, two gap closings occur simultaneously in the subbands of \( \hat{h}_{\mathcal{U}=\pm} \) away from Fermi energy at \( \lambda' = 0.5t \) and then two degenerate localized ESs emerge in the bandgaps. Interestingly, with further increase in \( \lambda' \), the pairs of ESs enter to the GBSs of \( \hat{h}_{\mathcal{U}=\pm} \) at \( \lambda' = t \). These topological ESs meet each other at Fermi energy with \( \lambda' = \tau' = 1.5t \) leading to the appearance of strongly localized fourfold degenerate states. Finally, they stay in the GBSs while preserving their localization with an extra increase of \( \lambda' \). The corresponding topological \( Z \) integer is plotted in Fig. 1(c). In the parameter region where the ESs appear, \( Z \) integer takes value 2 demonstrating the existence of two pairs of localized ESs steaming from \( \langle \hat{N}_1, \hat{U}_{\mathcal{L}=\pm} \rangle = \langle \hat{N}_2, \hat{U}_{\mathcal{L}=\pm} \rangle = (1, 1) \) whereas \( \langle \hat{N}_1, \hat{U}_{\mathcal{L}=\pm}, \hat{N}_2, \hat{U}_{\mathcal{L}=\pm} \rangle = (0, 0) \). As a result, topologically protected ESs of an eigenspace could penetrate into trivial GBSs of the other one.

In particular, both of the eigenspaces may host nontrivial topological phases by choosing appropriate spin-orbit cou-
pling values in a way that topological phase transitions occur in both subspaces. This will result in appearance of four pairs of ESs. In Figs. 1(d) and 1(e), respectively, the dependence of energy spectrum \( \hat{T} \) and topological invariant \( Z \) on \( \lambda' \) is presented for \( \delta t = 0 \) and \( \lambda_3 = \lambda' + 2.5t \). As shown in Fig. 1(d), the first topological phase transition happens in the \( \hat{h}_{\mathcal{U}=\pm} \) spectrum at \( \lambda' = -2.25t \) leading to appearance of highly localized ESs in the bandgaps (GBSs) for the parameter space \( \lambda' \in (-2.25t, -0.75t) \). After taking place of the second topological phase transition in the \( \hat{h}_{\mathcal{U}=\pm} \) spectrum at \( \lambda' = -2.25t \), two new ESs are emerged in addition to the former ones. Therefore, the system hosts four topological ESs \( Z \) = 4, as shown in Fig. 1(e). When \( \lambda' \) further increases, surprisingly, topologically protected ESs of the \( \hat{h}_{\mathcal{U}=\pm} \) spectrum reside inside nontrivial topological bulk states of the \( \hat{h}_{\mathcal{U}=\pm} \) and system re-enters to the TM phase. This is in contrast to the case of topological bound states embedded in non-topological continuous spectrum \([17]\). Moreover, the electron-like and hole-like ESs intersect each other at \( \lambda' = \tau' = \pm 0.5t \) [see Fig. 1(d)]. Remarkably, from both Figs. 1(b) and 1(d), one can see that ESs of an eigenspace at Fermi energy are quasi-degenerate not only with their own highly degenerate bulk states but also with continuum states of the other eigenspace establishing TM phase at such level. Note also that the values of \( Z \) change at which the subband gap closing/reopening occurs [see Figs. 1(c) and 1(e)].

In order to shed light on the mechanisms underlying the above-mentioned behaviors, we focus on understanding the effect of spin and sublattice degrees of freedom on band structure. In fact, the odd number of sublattices provides bulk metallic ground states resulting in breaking particle-hole and chiral symmetries. Therefore, possible bandgaps can only occur away from Fermi surface. Now, exploiting spin-orbit coupling breaks spin degeneracy and subsequently each band splits into two subbands corresponding to two different helical components. The resulting spin helical subbands retain the nontrivial ESs and at the same time push them into the metallic bulk states as a consequence of U(1) symmetry. As such,
if the system is in a topologically nontrivial phase then the
Fermi level crosses at some of the topological ESs embedded
in GBSs.

The topological phase diagram of the system in the plane
\((\lambda', \lambda_3)\) is depicted in Figs. 2(a) and 2(b) for \(\delta t = -t/2\)
and \(\delta t = +t/2\), respectively. The black solid lines denote
the border between topologically different phases where
the bulk subband gap closes. Also, the dashed lines corres-
dpond to TM states at Fermi level. The topological phase
diagrams are categorized into three distinct phases according
to \(Z = 0, 2,\) and \(4\) implying none (trivial phase shown by
blue color), two, and four pairs of symmetry protected ESs,
respectively. The regions \(Z = 2\) and \(4\) are divided into
subcategories depending on the appearance of ESs either
within GBSs or in the bandgaps. In both diagrams, there
are three possibilities for the region \(Z = 4\) indicated by red,
green, and cyan colors corresponding to the simultaneous
presence of two pairs of ESs in the bandgaps and the other
two pairs of ESs within GBSs, four pairs of ESs in the
bandgaps, and four pairs of ESs within GBSs, respectively.
We can witness that for \(\delta t = +t/2\) the presence of four pairs
of ESs in the bandgaps increases dramatically in comparison
with \(\delta t = -t/2\). In addition, the regions of \(Z = 2\) have
two possibilities displayed by yellow and brown colors
corresponding to the two pairs of ESs within GBSs and
within bandgaps, respectively. The region of ESs within
GBSs with \(Z = 2\) for \(\delta t = -t/2\) is more dominant than that of
\(\delta t = +t/2\).

**Stability of ESs.**—The existence of topological ESs that are
quasi-degenerate with GBSs is ensured by the presence of spin
rotational symmetry. To illustrate this feature, let us investi-
gate ESs stability against perturbations originated from on-site
potential and Zeeman magnetic field. We add the term \(\hat{H}' = \hat{H}_V + \hat{H}_B\) to total Hamiltonian (1) including on-site Hamiltonian
\(\hat{H}_V = \sum_{n,\sigma} \sum_{i} V_{n,\alpha}(c_{\alpha,n,\sigma}^\dagger c_{\alpha,n,\sigma})\) and Zeeman Hamiltonian
\(\hat{H}_B = \sum_{n,\sigma,\sigma'} \sum_{i} \delta t V_{n,\alpha}(M_{n,\alpha,\sigma}^\sigma)\). Here, \(V_{n,\alpha}\) defines the amplitude of on-site potential, \(\sigma\) is the Pauli
vector acting on the spin subspace and the Zeeman field vector
is \(M_{n,\alpha} = (M_{n,x,\alpha}, M_{n,y,\alpha}, M_{n,z,\alpha})\). For concreteness, we
will inspect the effects of on-site potential and Zeeman mag-
netic field on the topological properties of system separately.
First, we assume that the system is in the presence of Zeem-
man field and absence of on-site potential. Since the lattice is
invariant under rotations about the \(x\)-axis, we apply Zeeman
field along the \(y\)-axis. The \(y\)-component of Zeeman field vi-
olates lattice \(U(1)\) symmetry and, in consequence, \(\hat{H}(k)\) can
not be block diagonalized. Accordingly, the topological in-
variant (4) is no longer valid. To discriminate the role of
\(U(1)\) symmetry from that of reflection symmetry, we break
\(U(1)\) symmetry such that the reflection symmetry remains un-
touched. To do so, we need to set \(M_{y,3} = M_{y,1}\). Under such
situation, the energy spectrum of Figs. 1(b) and 1(d) is re-
calculated and plotted in Figs. 3(a) and 3(b), respectively, in
the presence of \(y\)-component of staggered Zeeman field with
\(M_{y,1} = -M_{y,2} = t/2\). Interestingly, one can observe that
ESs within GBS can not survive resulting in termination of
ESs from one end with no gap closure. While in-gap states
preserve their degeneracies owing to the reflection symme-
try. Although this effect is obtained for staggered Zeeman
field but similar results hold for uniform Zeeman field.
Otherwise, preserving the spin rotational symmetry and violating
reflection one lead to destroying ESs and making the system
topologically trivial (not shown). As a result, the TM phase
manifests itself whenever both reflection and \(U(1)\) symmetries
are present simultaneously and an odd number of sublattices
is taken into account.

In addition, the \(x\)-component of Zeeman field or on-site
potential conserve \(U(1)\) symmetry. One readily finds that
these terms have diagonal entries in each block of transformed
Hamiltonian \(\mathcal{H}(k) + (h_{\mu=-} + h_{\mu=+}) \bigoplus (h_{\mu=-} + h_{\mu=+})\) with \(h_{\mu=\pm} = \mu_{\mu}^\pm \delta_{i,j}\) where \(\mu_{\mu}^\pm = V_{\mu,T} \pm M_{\mu,T-i+1}\). Here, reflection symmetry requires \(V_{\mu} = V_{-\mu,1}\)
and \(M_{x,i} = M_{x,T-i+1}\), for which we set \(V_{1} = V_{3}\) and
\(M_{x,1} = M_{x,3}\) in the case of \(T = 3\). Obviously, the uni-
form on-site potential (x-component of Zeeman field) shifts
the energy levels of each block of \(\mathcal{H}(k)\) to the same (oppo-
site) direction. These enable us to shift the energies of ESs
appearing away from Fermi level toward \(E = 0\) while their
quasi-degeneracy with GBSs remain intact. Therefore, the
TM phase will be accessible for much larger range of spin-
orbit couplings. Moreover, interestingly, either alternating on-
site potential or \(x\)-component of Zeeman splitting can impose
gap closing in each block of \(\mathcal{H}(k)\) independently. So, the
number of ESs would be asymmetric about Fermi level re-
sulting in inducing odd number of ESs. The energy spectra
of a finite chain as functions of \(\lambda'\) in the presence of a uni-

![FIG. 4. (Color online) Energy spectrum of \(\hat{H}\) with its IPR versus (a)(c) \(\lambda' (\bar{V}_1)\) with \(\delta t = -t/2\). \(\lambda_3 = \lambda'\) and \(M_{x,1} = M_{x,2} = M_{x,3} = t/2\) \(\lambda_3 = 2.5t\). \(V_3 = V_1 = V_2 = -t\) for 30 (60) unit cells. Bottom panels (b) and (d) are the corresponding topological integer \(Z\) of the top panels.](image-url)
form Zeeman field along x-axis, and $V_1$ are depicted in Figs. 4(a) and 4(c), respectively. One can see from Fig. 4(a) that the x-component of Zeeman field splits the energy spectra of $\hat{h}_{y=+}$ and $\hat{h}_{y=-}$ relative to each other, as already mentioned. Also, three pairs of ESs are appeared in parameter space $V_1 \in (-2.87t, -1.92t)$ and $V_2 \in (-0.08t, 0.87t)$ in the energy spectrum of Fig. 4(c). The bottom panels of Fig. 4 show the topological invariant of top panels.

**Experimental proposal.**—Recent experimental achievements make it possible to realize SSH model by fabricating heterostructures of alternating thin films of band insulators and TIs [18] relying on condensed matter physics. Also, modulated spin-orbit coupling can be implemented either by applying local electric field [19] or by using cluster of heavy atoms [20]. In the latter case, due to proximity effect, spin-orbit coupling can be transferred from the bands of atoms that are in contact with heterostructures to the bands of the system so that other properties of the structure itself remain unaffected. On the other hand, using cold atoms in optical lattices provide an excellent playground with the easy tunability of control parameters to simulate topological bands [21] in artificial quantum systems like SSH chain [22]. In the cold atoms experimental method, we suggest to employ superposition of retroreflected laser beams or to imprint superlattice with a spatial light modulator producing extended SSH model with three number of sublattices [23]. Furthermore, it is possible to engineer spin-orbit interaction in tripartite lattices [24] even for neutral cold-atomic gases with current experimental status [25]. Using spatially resolved radio-frequency spectroscopy [26], the ESs within GBSs can be recognized from the local density of states.

**Conclusions.**—We revealed a new kind of exotic topological states characterized by the coexistence of topologically nontrivial ESs and nontrivial states either at or far from the Fermi level. The main ingredient of these exotic metallic states arises from the coupling of odd number of sublattices to spin degree of freedom in a 1D periodic arrays. We found that such systems undergo a topological phase transition under subband gap closure conditions. The effects of Zeeman magnetic fields and on-site potentials on the topological phases indicate that TM states are protected by both reflection and spin rotational symmetries. The concept of nontrivial TM phase may be generalized to higher invariants and non-hermitian case, as well as including interaction effect.

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