Strings, Noncommutative Geometry

and

the Size of the Target Space

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Abstract

We describe how the presence of the antisymmetric tensor (torsion) on the world sheet action of string theory renders the size of the target space a gauge non invariant quantity. This generalizes the $R \leftrightarrow 1/R$ symmetry in which momenta and windings are exchanged, to the whole $O(d,d,Z)$. The crucial point is that, with a transformation, it is possible always to have all of the lowest eigenvalues of

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the Hamiltonian to be momentum modes. We interpret this in the framework of noncommutative geometry, in which algebras take the place of point spaces, and of the spectral action principle for which the eigenvalues of the Dirac operator are the fundamental objects, out of which the theory is constructed. A quantum observer, in the presence of many low energy eigenvalues of the Dirac operator (and hence of the Hamiltonian) will always interpreted the target space of the string theory as effectively uncompactified.
1 Introduction

It has been known for some time in string theory that the size of the target space is not an invariant concept. A symmetry, called T-duality, exchanges the theory of closed strings compactified in a tiny box, of the size a small fraction of Planck length, with the theory of strings living in large universe, of size the inverse of the tiny box (times the square of the Planck length). This is a consequence \cite{1} of a symmetry of the spectrum of the Hamiltonian of the theory, which remains invariant under an exchange between the lattice, which defines a toroidal compactification, with its dual lattice. This is equivalent to an exchange between the momenta of theory (quantized in units of the inverse of the radius), with the winding modes, closed strings which stretch across the torus. As the latter are also quantized, but in units of the size of the torus, target spaces with vastly different sizes are identified. While at first sight this may seem a very curious result, the interpretation \cite{2} only refers to elementary concept of quantum mechanics. The argument of Brandenberger and Vafa is that position is just a derived concept, as the Fourier transform of momentum spaces, and in string theory a different choice is also possible, namely considering eigenstates of the winding. If the compactification radius is of the order of Planck length, the two choices are equivalent, but for a very large radius the eigenvalues of momentum are nearly continuous, while the ones of winding are far apart, the first one appearing at a very large energy, it is therefore difficult to make “localized wave packets” with the Fourier transform of winding. Conversely, with a small radius of compactification, it is the winding which gives the possibility to create localized wave packets.

The exchange between the compactification lattice and its dual is however just a part of a larger group of symmetries, namely $O(d,d,\mathbb{Z})$, and some of these transformation mix momentum and winding in a non trivial way. The purpose of this paper to argue that a quantum observer will always ”see” an uncompactified space, provided the Hamiltonian has a spectrum containing many small eigenvalues (we will make this more precise later). We will accomplish this using the tools of noncommutative geometry, that is we will consider string theory as a noncommutative geometry. We take the example of the (flat) toroidal compactification. Our aim is to just show a mechanism, and this must be understood as an example, the principles behind this mechanism are much more general. The large scale topology of the universe is of course unknown. We will be only concerned with physical interpretation, and the mathematics used will only have an ancillary purpose. The non mathematically inclined reader should bear with the mathematical parts, the main concepts should understandable even without the formal apparatus. The mathematically inclined reader should be tolerant of the (mis)use of mathematics in this paper.

The crucial observation \cite{3, 4} is that, in the noncommutative geometry of strings, the group $O(d,d,\mathbb{Z})$ is a part of the group of gauge transformations. Also important is the fact that, with a proper transformation, it is possible to have all of the lower eigenvalues to be in the momentum sector. We will
use these observations in the context of Noncommutative geometry, for which
the important feature is the spectrum of the (generalized) Dirac operator.
The action in noncommutative geometry is a spectral action, it depends es-
tially from the lower (smaller than a cutoff parameter) eigenvalues of the
Dirac operator. The noncommutative geometry of strings is given by the
vertex algebra of the vertex operators of the theory. The Dirac operator can
be used to identify the low energy (tachyonic sector) of the theory. Using
these ingredients, and the fact that the topology and metric of a space in
noncommutative geometry are described solely in terms of operators on an
Hilbert space, we will argue that a quantum observer will see the space as
effectively uncompactified.

The plan of the remainder of the paper is as follows. In section 2 we in-
troduce the string theory of interest, and the $O(d,d,\mathbb{Z})$ symmetry of interest.
In section 3 we will describe the configuration space experienced by a quan-
tum observer in the framework of noncommutative geometry. The relevant
concept of noncommutative geometry are briefly introduced. In section 4 we
introduce the noncommutative geometry of the string theory of interest, and
in particular the Dirac operator, showing also how duality symmetries are
gauge transformation leaving the spectrum of the Dirac operator invariant.
In section 5 we introduce briefly the spectral action principle and argue how
a low energy quantum observer will necessarily see an uncompactified space.
Section 6 contains conclusions and open problems.

2 Target Space Symmetry of a String Theory

Consider the bosonic (sector of the) string, with the target space compactified
on a $d$ dimensional torus. That is, consider a linear $\sigma$ model compactified
on $\mathbb{R}^d$ quotiented by an abelian infinite group (a lattice) $\Gamma$ generated by $d$
generators $e_i$. Then the space $T_d \equiv \mathbb{R}^d/\Gamma$ is a $d$ dimensional torus. On $\Gamma$
we define an inner product of the generators, which provides a metric (of
Euclidean signature) on $T_d$:

$$\langle e_i, e_j \rangle \equiv g_{ij} . \quad (2.1)$$

The lengths of the vectors $e_i$ (here and in the following we work with units
such that Planck’s length equals unity) give the classical ‘size’ of the target
space, which is compact and periodic in all dimensions. We will see that the
presence of torsion and quantum mechanical considerations will considerably
alter this classical picture.

The dual lattice $\tilde{\Gamma}$ is spanned by the basis $e^i$ with (we implicitly complex-
ify $\Gamma$ and extend the product):

$$\langle e^i, e^j \rangle = \delta^i_j . \quad (2.2)$$

The inner products of the $e^i$'s define a metric which is the inverse of $g_{ij}$, that
is

$$\langle e^i, e^j \rangle \equiv g^{ij} . \quad (2.3)$$
Notice that, if all of the $e_i$ are quantities of order $R$, with $\det g$ of order $R^d$, then the ‘size’ of the dual lattice is (very roughly speaking) of order $1/R$. In this sense, if to a given lattice corresponds a large universe, to its dual it will correspond a small one, the dual torus $\tilde{T}_d$. This conclusion is valid however, even in this rough form, only in the absence of torsion in the action \[5\].

Classically the string is described by a two dimensional nonlinear $\sigma$ model, whose fundamental objects are the Fubini–Veneziano fields, which, for the case of a closed string\[4\], are

$$X^i(\tau, \sigma) = x^i + g^{ij}p_i\tau + g^{ij}w_i\sigma + \sum_{k \neq 0} \frac{1}{ik} \alpha_k^{(\pm)\mu} e^{i k (\tau \pm \sigma)} , \quad (2.4)$$

where $x$ represents the centre of mass of the string, $p$ its momentum and $w$ is the winding number, that is the number of times the string wraps around the direction defined by the $e_i$. Notice that because the space is compact, the momentum is quantized, and in fact it must be $p \in \tilde{\Gamma}$, while the winding number must belong to the dual lattice $w \in \Gamma$. If the size of the target space is extremely large, then the momentum will have a spectrum with very close eigenvalues, a nearly continuous spectrum, while the windings will have values far apart. But apart from these scale considerations, the role of $p$ and $w$ in (2.4) is symmetric. In the following we will concentrate on the zero modes of the string, mostly ignoring the oscillator modes. These are internal excitations of the other string, and are not sensible to the target space in which the strings live, and will therefore in general play no role in this paper. Moreover, the oscillators describe excitations which are starting at the Planck mass, while most of our considerations relate to the low energy sector of the theory.

The action of the model is that of a two dimensional nonlinear $\sigma$ model\[4\]:

$$S = \frac{1}{4\pi} \int d\sigma d\tau \left( \sqrt{\eta} \eta^{\alpha\beta} \partial_{\alpha}X^i g_{ij}\partial_{\beta}x^j + \varepsilon^{\alpha\beta} b_{ij}\partial_{\alpha}X^i \partial_{\beta}X^j \right) , \quad (2.5)$$

where $\eta$ is the world sheet two dimensional metric, $G$ is the matric defined in (2.1), and $b$ is an antisymmetric tensor which represent the ‘torsion’ of the string.

We can perform a chiral decompositions of the $X$’s defining:

$$X^i_{\pm}(\tau \pm \sigma) = x^i_{\pm} + g^{ij}p_j^{\pm}(\tau \pm \sigma) + \sum_{k \neq 0} \frac{1}{ik} c_k^{(\pm)\mu} e^{i k (\tau \pm \sigma)} . \quad (2.6)$$

The zero modes $x^i_{\pm}$ (the centre of mass coordinates of the string) and the (centre of mass) momenta $p_i^{\pm} = 2\pi p_i \pm (g - \mp b)_{ij}w^j$ are canonically conjugate variables,

$$[x^i_{\pm}, p_j^{\pm}] = -i\delta_i^j , \quad (2.7)$$

\[1\]In this paper we will consider only the simplest case of closed strings. The presence of open strings, D-branes etc. will make this structure probably much richer, but will not be considered in this paper.

\[2\]Other terms, such as a dilatons term, are possible, but we will not consider them here.
with all other commutators vanishing. The left-right momenta are
\[ p_i^\pm = \frac{1}{\sqrt{2}} (p_i \pm \langle e_i, w \rangle) \] (2.8)

The \( p^\pm \)'s belong to the lattice:
\[ \Lambda = \tilde{\Gamma} \oplus \Gamma \] (2.9)

We can therefore define the fields \( X = X_+ + X_- \), and we may equally well define \( \tilde{X} \equiv X_+ - X_- \), whose zero mode we will indicate as \( \tilde{x} \).

The Hamiltonian of the theory (limiting ourselves to the zero mode part) is very symmetric in this chiral decomposition:
\[ H = \frac{1}{2} \left( (2\pi)^2 p_i g^{ij} p_j + w^i (g - bg^{-1}b)_{ij} w^j + 4\pi w^i b_{ik} g^{kj} p_j \right) \] (2.10)
\[ = \frac{1}{2} (p_+^2 + p_-^2) \] (2.11)

Since the momenta and the windings belong to a lattice, the spectrum is discrete.

The symmetry which exchanges the lattice with its dual is called T-duality \[1, 3\]. It corresponds to an exchange of the momentum quantum number with the winding. The zero mode corresponding to \( x \), the position of the centre of mass of the string, is exchanged with its dual \( \tilde{x} \). As heuristically discussed in the introduction (and as we will again argue below), two target spaces related by a T-dual transformation are indistinguishable at low energies. In the torsionless case \( b = 0 \) this corresponds to an exchange of \( g \) with its inverse \( g^{-1} \), and the change of size of the target space in which the radius \( R \rightarrow 1/R \). In the presence of torsion the exchange is \( g^{-1} \leftrightarrow g - bg^{-1}b \) and \( bg^{-1} \leftrightarrow -g^{-1}b \), and it depends crucially on the values of the \( b_{ij} \). In the toroidal case it is possible to exchange only some of the generators of the lattice with their duals, giving rise to a group of factorized T-dualities.

Even factorized T-duality is a subgroup of a larger group of symmetries which leaves the spectrum invariant, the full group is in fact \( O(d, d, \mathbb{Z}) \) \[4, 3\]. It is generated from three kinds of transformations:

- The factorized dualities we have already discussed.

- The changes of base of the lattices, made via a matrix which belongs to \( G(d, \mathbb{Z}) \), the group of integer valued matrices of unit determinant.

- The transformation \( b_{ij} \rightarrow b_{ij} + c_{ij} \) with \( c \) an antisymmetric tensor with integer entries.

There is a further \( \mathbb{Z}_2 \) symmetry obtained exchanging \( \sigma \) and \( \tau \) on the world sheet, but this last symmetry does not affect the target space.

Let us analyse in some details the third transformation. It changes the components of the antisymmetric second rank tensor \( b_{ij} \) by the addition of an
arbitrary integer constant. This transformation does not change the lattice $\Gamma$, as it operates only on the antisymmetric tensor $b$. It does however change the momenta conjugated to the zero modes of $X$ and $\tilde{X}$. In particular, in the spectrum (2.10), the relative contribution of the momenta conjugated to $x$ (represented by the first term,) with respect to the windings, conjugated to $\tilde{x}$, and the mixed term will change. Since this is a symmetry of the spectrum, the set of numbers which are the eigenvalues is of course unchanged, but the distribution in the three terms changes. By choosing the integers which compose the antisymmetric tensor $b$ arbitrarily large, we can make the contribution of the second and third term arbitrarily large. in other words we concentrate the lowest eigenvalues of the Hamiltonian in the momentum part. In other words the low energy spectrum is made only of the momentum eigenvalues. The lattice is still the same, but the strings are extremely twisted, and we have transferred the lowest eigenvalues of the energy from winding to momentum. This relatively simple observation is the key of our construction. In the following sections we will argue that in this case a quantum observer will observe a spacetime in which the actual radii of compactification will effectively be unobservable. Roughly speaking, a low energy strings for which in the original (small radius) lattice had a combination of momentum and winding, will now be twisted in such a way that it will appear to have just momentum, it is like the lattice “repeats itself over and over”.

Again, as in the case of the of the $R \leftrightarrow 1/R$ symmetry, we have to ask ourselves ‘what is position’? ‘How do I measure it’? And using the same heuristic arguments of [2], we can think of making wave packets using superpositions of the eigenvalues of the momentum, in the case of large torsion the eigenvalues of momentum are continuous for all practical purposes, therefore the superposition will have the character of an uncompactified space, rather than a string moving on a lattice. And this will be the situation until energies in which the new eigenvalues (coming from windings or the oscillatory modes) start to play a role. In the following section we will argue this from a quantum mechanical point of view, using the tools of noncommutative geometry.

3 Configuration Space in Quantum Mechanics.

In this section we will discuss the role of the classical configuration space in quantum mechanics. This is an extremely complicated subject, which would require a full understanding of the foundations of quantum mechanics, and its classical limit. Since this full understanding is lacking, we only point out some facts which we feel relevant. We will use the language and formalism of noncommutative geometry, but will keep the formalism to a minimum, hoping to make the relevant physical principles emerge as clearly as possible. A discussion along these line appeared in [8], a rigorous treatment of quantum
mechanics based on an algebraic approach can be found, for example, in Haag’s book \cite{Haag}, while the main reference for noncommutative geometry is the book by Connes \cite{Connes}, other useful reviews are \cite{reviews1, reviews2, reviews3}.

Consider, in the following, a purely quantum observer, that is somebody making experiments with a set of operators which form an algebra. For example bounded operators constructed from $p$ and $x$. Despite its historical name not necessarily all self adjoint operators on an Hilbert space can be considered to be related to an experimental procedure. In fact the programme of rigged Hilbert space, (for reviews see \cite{rigged}) is based on a particular choice of a subspace of the Hilbert space, on which some operators act in such a way to have only states with a finite energy. Here we will take a related but different point of view, based more on the algebra of operators.

The usual way to construct quantum mechanics for the motion on a manifold $M$, is to consider the Hilbert space $L^2(M)$, and an algebra of (bounded) operators acting on it. Of course the Hilbert space does not carry any information on $M$ (all separable Hilbert spaces are isomorphic), but the information on the topology of $M$ can be recovered by considering the algebra of position operator, that is, the algebra of continuous complex valued, functions on $M$, seen as operators on $L^2(M)$, with a norm given by the maximum of the modulus of the function.

The set of continuous complex valued functions on a topological space form in fact an abelian $C^*$-algebra, and according to a series of theorems due to Gel’fan and Naimark (for a review see \cite{Gelfand}), it is possible to recover the original space in an unique way from the knowledge of just the abstract algebra. The points of the topological space in this case are the set of irreducible representations of the algebra, and the topology of the space can be recovered as well. Alternatively, the space can be reconstructed as the set of maximal ideals. Ideals are subalgebras such that the product of one of their elements by any element (of the whole algebra) still belongs to the ideal. Maximal ideals are ideals which are not contained in any other ideal (except the trivial ideal of the whole algebra). The relevant example is the set of functions vanishing at a point.

There is a third way to identify the topological space corresponding to a given $C^*$-algebra algebra, it is via the pure states of the algebra. A state is a map from the algebra into complex numbers with the properties of being positive definite, of unit norm and such that:

$$\Psi : \mathcal{A} \rightarrow \mathbb{C} \ ; \ \Psi(a^*a) \geq 0 \ , \forall a \in \mathcal{A}, \ ||\Psi|| = 1 .$$

It is another result of Gel’fan that all $C^*$-algebra can be represented as bounded operator on an Hilbert space $\mathcal{H}$, then vectors $|\psi\rangle \in \mathcal{H}$ define states via the expectation values. States are however much more general. We denote the space of states by $\mathcal{S}(\mathcal{A})$. Since

$$\lambda \Psi + (1 - \lambda)\Phi \in \mathcal{S}(\mathcal{A}) , \ \forall \Psi, \Phi \in \mathcal{S}(\mathcal{A}), \ \lambda \in [0, 1]$$

In the following we will consider $M$ compact, therefore continuous functions are bounded as well.
the set of all states of an algebra $\mathcal{A}$ is a convex space. Being a convex space $\mathcal{S}(\mathcal{A})$ has a boundary whose elements are called pure states. The ‘delta-functions’, seen as maps from the algebra of continuous function into complex numbers:

$$\delta_x(f) \equiv f(x), \quad f \in \mathcal{A}$$

are examples of pure states. Namely, a state is called pure if it cannot be written as the convex combination of (two) other states. We can therefore reconstruct the space as the space of pure states, which coincides with the set of irreducible representations, and hence the space of characters. Moreover, in the commutative case, it coincides with the space of maximal ideals of $\mathcal{A}$.

The reconstruction of the underlying topological space from pure states is actually quite close in spirit to quantum physics.

Although the Hilbert space $\mathcal{H}$ has been introduced to represent our algebras, we will give it a physical meaning and see it as the space of wave functions required by quantum mechanics. To reconstruct a topological space, all we need is then an abelian subalgebra of the algebra of observables. At this, purely topological level, there are however many ambiguities (to some extent similar to the polarization choices in geometric quantization), one could choose the algebra of momentum operators, or combinations of position and momentum etc.

We will consider the configuration space of a quantum mechanical space therefore not as a set of points (and relations such as topology or a differential structure), but rather as an abelian $C^*$-algebra. This is our starting point. The Hilbert space could also be easily constructed a posteriori by giving a sesquilinear form (a scalar product) on the algebra, and completing it under the norm given by this product. Other choice for the Hilbert space are possible, a relevant one for instance is the space of spinors. A quantum observer will have at his disposal, among the bounded operators on the Hilbert space, an abelian subalgebra that he will identify with the continuous function on his space.

The “size” of this configuration space is given by a (generalized) Dirac operator $D$, a self adjoint, densely defined, compact resolvent operator on the Hilbert space. The distance between two points $x$ and $y$ is given, in terms of $D$, by Connes’ formula:

$$d(x,y) = \sup_{||D,a|| \leq 1} |a(x) - a(y)|. \quad (3.15)$$

Since we are interested in doing physics with noncommutative geometric tools, we should be able to introduce also potential and covariant derivatives as operators on $\mathcal{H}$. Also in this respect the Dirac operator plays a crucial role. With it is in fact possible to represent differential forms as bounded operators.

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*We are assuming at this point that the Hilbert space is a space of spinors.*
Given an abstract algebra of $p$-forms:

$$
\omega = \sum_i a_0 da_1 \ldots da_p,
$$

$$
d\omega = \sum_i da_0 da_1 \ldots da_p
$$

we define a linear representation of $\omega$ and $d\omega$ as bounded operators:

$$
\pi(\omega) = \sum_i a_0 [D, a_1] \ldots [D, a_p],
$$

$$
\pi(d\omega) = \sum_i [D, a_0] [D, a_1] \ldots [D, a_p].
$$

Since it may occur that $\sum_i a_0 da_1 \ldots da_p = 0$ while $\sum_i da_0 da_1 \ldots da_p \neq 0$, care has to be taken in quotienting out these forms (the so called “junk” forms), for details see for example [10, 11]. For instance, in the usual commutative case, in which the algebra is the one of complex valued functions on an an manifold, the Hilbert space is the one of spinors on spacetime and the Dirac operator is the usual one $D = \gamma^\mu \partial_\mu$, the forms $dx^\mu$ are represented by $\pi(dx^\mu) = \gamma^\mu$. In going to higher forms one has to retain only the anti-symmetric part of the product of gamma matrices. In the following we will omit to explicitly indicate the symbol $\pi$ when we talk of forms, which we will assume always represented on $\mathcal{H}$. Once we have defined forms we can then define connections (generic hermitean one forms):

$$
A = \sum_i a_i [D, b_i]
$$

and a covariant Dirac operator

$$
D_A = D + A
$$

The curvature also can be defined.

$$
F_A = [D, A] + A^2
$$

Connections and curvature transforms properly under a gauge group. In fact in noncommutative geometry gauge transformation have a nice characterization as the unitary transformation of the algebra into itself (or the inner automorphisms). If in fact we conjugate all of the element of the algebra by an unitary element $A \rightarrow U^{-1}AU$, the physics must remain unchanged. This means that the differential forms defined in (3.17) have to transform properly. In fact for a matrix algebras this invariance is exactly the invariance for a unitary gauge group.

A final ingredient is the integral, since we are constructing a formalism based on algebra of operators, without the geometrical concepts of points etc, the best characterization of integration is as a trace. In general for commutative algebras is possible to show that the integral of a function is
the (properly regularized) trace of this function (represented as an operator in the Hilbert space) times $|D|^{-d}$. We define the Dixmier trace $\text{tr}_\omega L$ of an operator $L$, with eigenvalues $\lambda_n$, with $\lambda_{n+1} \geq \lambda_n$, to be:

$$\text{tr}_\omega L = \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \lambda_n$$

(3.21)

For the algebra of continuous functions on a $d$-dimensional compact manifold $M$, this definition yields

$$\int_M f(x) \, dp(x) = \text{tr}_\omega f |D|^{-d}.$$  

(3.22)

We have therefore equipped our quantum observer with a series of tools suited to him: algebras of operators, traces etc. In the commutative case these tools reconstruct the usual differential geometry, but we have defined them in such a way that they can be used without any reference to an underlying “commutative” geometry. If we are in a commutative case, the quantum observer has therefore at his disposal an algebra of observables, in this algebra he recognizes an abelian subalgebra, that he calls the space on which he lives, and with formula (3.15) he calculates distances, metric etc. The set of an Hilbert space $\mathcal{H}$, a $C^*$-algebra realized as operators on $\mathcal{H}$ and a Dirac operator is called the Spectral Triple.

There is the possibility that the quantum observer finds himself on a non-commutative space. That is, among his set of quantum observable he does not identify an abelian algebra giving him the configuration space, he can however define (in an approximate sense perhaps) some sort of “noncommutative” space, the algebra corresponding to it is however non abelian, usually a deformation, governed by a small parameter, of an abelian algebra. This is, for example, the situation envisaged in [16], in which the algebra of position operators is noncommutative.

There is also an intermediate possibility, suppose for example that the Hilbert space the observer has at his disposal is that of spinors with an index, which transform under the fundamental representation of $SU(n)$. In this case the algebra of “position” operators is actually made of functions from the manifold to $n \times n$ matrices. This is obviously a noncommutative algebra, so the Gel’fand–Naimark theorem (at least in the commutative form we have enunciated) does not apply, it is nevertheless obvious that the configuration space is the manifold $M$ all the same. The choice of abelian subalgebras (diagonal matrices) would create various identical copies of the manifold. The question is easily resolved noticing that the algebra of $n \times n$ matrices has only one nontrivial irreducible representation, or rather, all representations are unitarily equivalent. Therefore the set of irreducible representations (up to unitary transformations) of the algebra is still in a one to one correspondence with the underlying manifold $M$. The noncommutativity of the algebra however makes it impossible the identification of points with pure states, there is in fact a $n$ dimensional sphere of pure states at each point.
of space (corresponding to the various unitarily equivalent representations) and they can be seen as an "internal" space, points connected by a gauge transformation.

We see therefore that there are noncommutative algebras which give (at least at the topological level) the same geometry, this concept is captured by the concept of Morita equivalence \[1\]. Two Morita equivalent C*-algebras have equivalent representation theories, therefore at the topological level they will describe the same set of 'points', with the same topology. The "physics" they would describe (if we interpret them as the algebra on space) differ therefore only in the "internal space". Quantum observers using Morita equivalent algebras of operators will therefore conclude they are describing different theories on the same manifold. In the fifth section we give a criterion based on the spectrum of the Dirac operator, with which he can identify an algebra which will give the quantum observer his space, commutative or noncommutative.

4 The Noncommutative Geometry of Strings

In this section we will describe the noncommutative geometry corresponding to the string theory of section 2. As we have seen in the preceding section we need three ingredients for this purpose, let us construct them in turn. We will be necessarily brief and details (and further references) can be found in [18, 19, 11, 20, 21]. We start the construction from the Hilbert space \( \mathcal{H} \) on which the \( X^\pm \) act as quantum operators:

\[
\mathcal{H} = L^2\left( \mathcal{T}_d \times \tilde{\mathcal{T}}_d, \prod_{i=1}^{d} \frac{dx^i d\tilde{x}^i}{(2\pi)^2} \right) \otimes \mathcal{F}^+ \otimes \mathcal{F}^-
\]

(4.23)

The \( \mathcal{F}^\pm \) are Fock spaces on which the creation and annihilation oscillatory modes act (and therefore will not be relevant at low energies), while the \( L^2 \) space of spinors in (4.23) is generated by the canonical pairs of position and momenta zero modes (2.7), and can be expressed in several isomorphic ways:

\[
L^2\left( \mathcal{T}_d \times \tilde{\mathcal{T}}_d, \prod_{i=1}^{d} \frac{dx^i d\tilde{x}^i}{(2\pi)^2} \right) \cong L^2\left( \mathcal{T}_d, \prod_{i=1}^{d} \frac{dx^i}{2\pi} \right) \otimes \mathbb{C} L^2\left( \tilde{\mathcal{T}}_d, \prod_{i=1}^{d} \frac{d\tilde{x}^i}{2\pi} \right)
\]

\[
\cong \bigoplus_{w \in L} L^2\left( \mathcal{T}_d, \prod_{i=1}^{d} \frac{dx^i}{2\pi} \right)
\]

\[
\cong \bigoplus_{p \in L^*} L^2\left( \tilde{\mathcal{T}}_d, \prod_{i=1}^{d} \frac{d\tilde{x}^i}{2\pi} \right)
\]

(4.24)

That is, we can consider the zero modes part of \( \mathcal{H} \) as copies of the spinors of the torus (a copy for each winding), or as copies of spinors on the dual torus, or as tensor product of two spinor spaces, reflecting the various choices of the spectrum.

We wish to describe interacting strings, and therefore we must use operators which describe strings splitting, joining etc. Such operators are called vertex operators, they form an algebra and have a distinguished position in...
mathematics as well as in physics (for reviews see for example [22]), although we will need very little of the beautiful mathematical formalism of these algebras. The most important feature is the operator state correspondence. This means that, to each state on the string Hilbert space, it is possible to associate an operator acting on the vacuum:

\[ V_\psi |0\rangle = |\psi\rangle \quad (4.25) \]

The fundamental vertex operator is the so called “tachyonic” vertex operator (this can actually be a misnomer since, according to the string theory one is considering, tachyons can be absent):

\[ V_{q^+, q^-}(z_+, z_-) \equiv V(e^{-iq_+^\mu x_+^\mu - iq_-^\mu x_-^\mu} \otimes \underline{1}; z_+, z_-) = (-1)^{q_+ w^\mu} : e^{-iq_+^\mu X_+^\mu(z_+) - iq_-^\mu X_-^\mu(z_-)} : (4.26) \]

where \((q^+, q^-), (r^+, r^-) \in \Lambda, z_\pm = -i(\tau \pm \sigma)\), and \(: \cdot :\) denotes normal ordering.

Thus we take as second ingredient the vertex operator algebra. A word of caution, in general a vertex operator algebra is not a \(C^\ast\)-algebra. In fact, operators defined as in (4.26) are not bounded. To overcome this one can smear them [18], however also smeared operators are not necessarily bounded [19, 23]. Alternatively one can consider a cutoff on the oscillators (effectively an ultraviolet cutoff on the world sheet) [21]. The algebra, seen as bounded operators on \(H\), will then have to be completed to give a \(C^\ast\)-algebra.

The last ingredient we need is a Dirac operator. In this case we can in fact naturally define two of them:

\[ D^\pm(\tau \pm \sigma) = \sum_{k=-\infty}^{\infty} D_k^\pm e^{ik(\tau \pm \sigma)} \quad (4.27) \]

where

\[ D_k^\pm = \sqrt{2} \gamma_\mu^\pm \otimes \alpha_{k}^{\pm \mu} \quad (4.28) \]

with \(\alpha_0^\pm = p^\pm\) and the \(\gamma^\pm\) are the appropriate gamma matrices in the \(\pm\) sectors. In analogy with the definition of momentum and winding from the \(p^\pm\) we can define two operators:

\[ D = D^+ + D^- \quad (4.29) \]
\[ \tilde{D} = D^+ - D^- \quad (4.30) \]

The algebra of vertex operators \(\mathcal{A}\), the Hilbert space (4.23) and \(D\) are the three elements which form the Fröhlich-Gawędzki spectral triple, which describes the noncommutative geometry of interacting strings. The algebra of vertex operators is of course non abelian and very complicated, and therefore the geometry of the stringy spacetime is highly nontrivial. This is not surprising, we know the extreme richness and beauty which lies behind string theory, and the intricacies (and beauty) of vertex operator algebras are its algebraic counterpart.
We note that, if we construct “vertex operators” only with \( x \), we have multiplicative operators of the kind \( e^{i k x} \), which are the Fourier basis of the algebra of position operators, which can be therefore seen as a subalgebra of the vertex operator algebra\(^1\). The spaces of the zero modes \( x \) or \( \tilde{x} \) can be recovered using the Dirac operators as follows \(^1\). We first define the Hilbert space obtained from the Hilbert space (4.23) simply eliminating the oscillators Fock spaces:

\[
\mathcal{H}_0 = \mathcal{P}\mathcal{H}
\]  

(4.31)

We then define the subalgebra \( \mathcal{A}_0 \) to be the commutant of the Dirac operator \( \tilde{D} \), restricted to the Hilbert space \( \mathcal{H}_0 \),

\[
\mathcal{A}_0 = \mathcal{P}_0 (\text{comm } D) \mathcal{P}_0 \equiv \{ V \in \mathcal{A} \mid [D, V] \mathcal{P}_0 = 0 \}
\]

(4.32)

It is the largest subalgebra of \( \mathcal{A} \) with the property

\[
\mathcal{A}_0 \mathcal{H}_0 = \mathcal{H}_0
\]

(4.33)

The essence of T-duality form the noncommutative geometry point of view is that we could have substituted the role of \( D \) in (4.32) with \( \tilde{D} \) as first suggested in \(^{18}\). In fact, together with \( \{ \mathcal{A}, \mathcal{H}, D \} \), there is a totally equivalent dual spectral triple \( \{ \mathcal{A}, \mathcal{H}, \tilde{D} \} \). That is, one can define an algebra \( \tilde{\mathcal{A}} \) dual of \( \mathcal{A} \), which commutes with \( D \). This explain the symmetry. Another crucial aspect \(^3\) is that this transformation is a gauge transformation of the theory.

As we said the gauge group of a spectral triple is given by the unitary elements of the algebra. The unitary element of a vertex operator algebra form a very complicated group, which is the gauge group of the string. Under a the gauge transformation the Dirac operator transforms as \( D \to U^{-1}DU \). The point is that there are several unitary elements of the algebra \(^4\) such that

\[
\tilde{D} = D \to U^{-1}DU
\]

(4.34)

This shows that \( D \) and \( \tilde{D} \) have the same spectrum. The gauge transformation (4.34) only relabels the eigenvalues of the operators, calling momentum eigenvalues the ones which were winding eigenvalues and viceversa for example.

To conclude we note that, although the construction described here refers to the bosonic string, generalizations to other strings based on the toroidal compactification, such as the heterotic string, are possible \(^{24}\).

5 Spectral Action and Low Energy Theories

We want to discuss some “low energy” limit in string theory, and therefore we have to specify better what we mean. In this section we will briefly introduce

\(^{1}\)We stress again that here by vertex operator algebra we need the algebra opportunely regularized and completed.
the Chamseddine–Connes spectral action principle [25], and will argue that
low energy means considering a theory in which only the low part of the
spectrum of $D$ is considered. This is possible because in the framework of
Noncommutative geometry one constructs a spectral geometry, in which the
information is stored in the spectrum of $D$. And low energy refers to an
action in which only the lower part of the spectrum is excited.

The spectral action principle is based on the covariant Dirac operator,
and on the variation of its eigenvalues. The action must be read in a Wilson
renormalization scheme sense, and it depends on an ultraviolet cutoff $m_0$:

$$S_{m_0} = \text{Tr} \chi \left( \frac{D_A^2}{m_0^2} \right)$$  \hspace{1cm} (5.35)

where $D_A$ is the covariant derivative defined in (3.19) and $\chi(x)$ is a
function which is 1 for $x \leq 1$ and then goes rapidly to zero (some smoothened
characteristic function). The action (5.35) effectively counts the eigenvalues
of the covariant Dirac operator up to the cutoff. Considering, in fact, the
eigenvalues of $D_A$ as sequences of numbers, and these sequences as dynamical
variables of euclidean gravity, the spectral action is then the action of “gen-
eral relativity” in this space [26]. The trace in the action can be calculated
using known heat kernel techniques [25, 27], and the resulting theory con-
tains a cosmological constant, the Einstein–Hilbert and Yang–Mills actions,
plus some terms quadratic in the Riemann tensor. Chamseddine [28] has
used the Dirac operators (4.29) in the spectral action principle and shown
that they lead to the low energy effective string action.

Here we are not to be concerned with the details of (5.35), nor with
its results for the description of gravity and the standard model. What we
wish to stress is that such an action comes from a spectral principle, that
is, the starting point is the spectrum of an operator, and its variations as
the backgrounds fields (the one–form $A$ in this case) change. One can ask,
in fact, what is the role of the algebra in the spectral action, as the latter
depends just on the trace of the Dirac operator. Of course the role of the
algebra is in the fact that in (5.35) appears the covariant Dirac operator.
And the form $A = \sum a_i[D,b_i]$ depends on the algebra chosen. Let us now
apply these considerations to the Fröhlich-Gawędzki spectral triple.

The spectrum of $D$ and $\tilde{D}$, or of any operator obtained from them with an
$O(d, d, \mathbb{Z})$ unitary transformations are the same. Let us call $D$ for convenience
the one for which the lower eigenvalues are the one relative to momentum.
Here by lower we mean the ones which are lower than the energy of the
oscillatory modes (of the order of the Planck mass $m_p$). If the cutoff $m_0$ is
lower than $m_p$, the cutoff function $\chi$ causes the projection of the operator
on the Hilbert space $\mathcal{H}$, defined in (4.31). Elements of the algebra which
commutes with $D$ (such as the elements of $\tilde{A}$) will not contribute to the
variations of the action, and will therefore be unobservable. This algebra
can be constructed as the commutant of the T-dual operator $\tilde{D}$. This means
that the winding modes degrees of freedom are unobservable. Since the Dirac
operator has a near continuous spectrum, the tachyonic, low energy, algebra
is spanned by operators of the kind

\[ V_p = e^{ipq} \]

(5.36)

can be considered the Fourier modes describing an uncompactified space.

In fact, in the spirit of section 3 of this paper, a quantum observer with a spectral action, will have, as potentials at his disposal, only the elements of the algebra which give low energy perturbations of the lowest eigenvalues of \( D \), always with the assumption of the cutoff \( m_0 < m_p \) so that oscillatory modes do not play a role. This is the abelian algebra of functions on some space time. If, as we have seen, there are many low eigenvalues, the observer will experience an effectively decompactified space time. The algebra which he will measure will be composed of the operators which will create low energy perturbation to \( D \). At this point we have to make the sole assumption that \( D \) has a spectrum with several small eigenvalues. In this way the quantum observer will experience a (nearly) continuous spectrum of the momentum, the sign of an uncompactified space.

The strings could still be seen as compactified on a “small” lattice, but the presence of a very large torsion term \( b \) has drastically changed the operator content of the theory, and this has rendered space effectively uncompactified.

6 Conclusions

The usual geometric notions of points, distances etc. are basically classical. They will need to be redefined a quantum observer, dealing with states on an operator algebra. The claim of this paper is that the torsion term makes the observable radius of the target space a non invariant concept, since, with a gauge transformation, it is possible to render it arbitrarily large (under the fairly mild assumption that there are many low energy eigenvalues of the Hamiltonian). A quantum observer will measure in fact the Fourier transform of a near continuous spectrum, that is a uncompactified space. We have argued this with the simplest possible example of bosonic theory. The framework in which we investigated this has been noncommutative geometry, and in particular the spectral action principle.

The presence of open strings, and in particular of D-branes will probably enrich this picture even more. In fact a theory of D-branes has a matrix action in the large mass limit, and in turn this theory can be compactified on a noncommutative torus \[29\], a genuine noncommutative space \[30\] generated by \( d \) unitary generators with the relation

\[ U_i U_j = e^{i\omega_{ij}} U_j U_i \]

(6.37)

The noncommutative Torus is also instrumental in the construction of the T-dual action in \[21\]. Different theories of branes related by \( O(d,d,\mathbb{Z}) \) transformation give rise to Morita equivalent noncommutative tori \[31, 32\].
Needless to say, there are several unanswered questions. For example, in the formalism, there is no reason at present for which there are four uncompactified dimensions. Or rather, in the spirit of this paper, why the eigenvalues of the Dirac operator are such that only for four dimension the spectrum is nearly continuous. The formalism we have used is euclidean, therefore question such has the expansion of the universe cannot even asked. Moreover we passed dangerously close to several fundamental aspects of quantum mechanics. We attempted on purpose to stay away from issues such as the semiclassical limit and the quantum theory of measurement. Those issues are of course of paramount importance, but we decided not to concentrate on them, giving priority to more concrete aspects of string theory. It is our strong feeling that the tools noncommutative geometry are the right ones to enable the description of a stringy geometry. In this paper we have seen just an example, others still wait to be investigated.

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