Limit load in the problem of penetration of a wedge-shaped tool into the rock mass

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Abstract. The problem of penetration of an absolutely rigid round cone shaped indenter into a weighted rock (an axisymmetric plasticity problem) is being solved. Limit load, depending on the angle of internal friction of the rock, adhesion, angular opening at the top of a cone, rock weight, is determined. The dependence of loading on these parameters is given.

1. Introduction
Problems of pressing in a rigid punch into a deformable medium were considered in [1–6], problems of a rigid wedge intrusion in [7–10], and axisymmetric problems of pressing in punches were studied in [11–13]. The main features of the solutions [1–13] are: the application of mathematical models of plasticity according to the scheme of a rigid-plastic body ignoring elastic deformations and carrying out calculations with the use of characteristics method. In geoproblems, these mainly relates to calculations of bases and foundations [12, 13]. Meanwhile, calculations of rock-breaking tools, energy supply necessary for the destruction of a particular rock are of great importance for mining. At the same time, there is a need to study the influence of rock weight on the values of limit loads.

2. Mathematical model and solution
Let a rigid cone-shaped tool with the apex angle \(2\gamma\) intrudes into the rock mass. To simplify calculations, we neglect friction at the contact “tool – rock”. It is required to determine such a load, applied to the tool along the boundary \(EOA\), at which a plasticity domain is formed around the cone, as shown in the Figure 1.
The problem is solved in an axisymmetric setting. In this case, the stress and strain tensor has the form

\[
T_\sigma = \begin{pmatrix}
\sigma_r & \tau_{rz} & 0 \\
\tau_{rz} & \sigma_z & 0 \\
0 & 0 & \sigma_\phi
\end{pmatrix},
T_\varepsilon = \begin{pmatrix}
\varepsilon_r & \varepsilon_{rz} & 0 \\
\varepsilon_{rz} & \varepsilon_z & 0 \\
0 & 0 & \varepsilon_\phi
\end{pmatrix},
\]

(1)

where \(\varepsilon_r = \frac{\partial u}{\partial r}, \varepsilon_\phi = \frac{u}{r}, \varepsilon_z = \frac{\partial u}{\partial z} \), \(\varepsilon_{rz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial u}{\partial r} \right)\), \(r, z, \phi\) are axes of the cylindrical coordinate system. Main axes of tensors \(T_\sigma, T_\varepsilon\) we denote as 1, 2, 3, along with \(\sigma_1 \geq \sigma_2 \geq \sigma_3\). To describe the plasticity state of the medium, we use the Coulomb – Mohr condition:

\[
\max_n \{\tau_n + \tan \psi \sigma_n\} = k,
\]

(2)

where \(k, \psi\) are constants of rock; \(\sigma_n, \tau_n\) are normal and tangential stresses on the platform with a normal \(\vec{n}\). Further, since in state (2)

\[
\tau_n = \frac{\sigma_1 - \sigma_3}{2} \cos \psi, \quad \sigma_n = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2} \sin \psi,
\]

(3)

as follows from the Mohr diagram for stresses [2], instead of (2) we obtain the equation

\[
\frac{\sigma_1 - \sigma_3}{2 \cos \psi} + \frac{\sigma_1 + \sigma_3}{2} \tan \psi = \kappa.
\]

(4)

(4) is not affected by the second main stress \(\sigma_2\). On the other hand, in the theory of plasticity there is such a thing as a state of complete plasticity, for which the two main stresses (\(\sigma_2\) and \(\sigma_3\)) coincide. For such a state the system of equilibrium equations for stresses becomes hyperbolic [2, 14, 15]. This hypothesis is also accepted in the work. Since the stress \(\sigma_\phi\) due to (1) is the main one, the equality of the two main stresses means that

\[
\sigma_1 = \frac{\sigma_r + \sigma_\phi}{2} + \sqrt{\left(\frac{\sigma_r - \sigma_\phi}{2}\right)^2 + \tau_{rz}^2}, \quad \sigma_3 = \frac{\sigma_r + \sigma_\phi}{2} - \sqrt{\left(\frac{\sigma_r - \sigma_\phi}{2}\right)^2 + \tau_{rz}^2} = \sigma_2 = \sigma_\phi.
\]

(5)

To describe the states (4), (5), we introduce the following notations
\[ \sigma = \frac{\sigma_r + \sigma_z}{2}, \quad \tau = \sqrt{\left(\frac{\sigma_r - \sigma_z}{2}\right)^2 + r_z^2}, \quad \tan 2\theta = \frac{2\tau_z}{\sigma_r - \sigma_z}, \]

where \( \theta \) is the angle between the first direction for the tensor \( T_\sigma \) and the axis \( r \) in the Figure 1. The following system of equations is solved

\[
\begin{align*}
\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_z}{\partial z} + \frac{\sigma_r - \sigma_z}{r} &= 0, \\
\frac{\partial \tau_z}{\partial z} + \frac{\tau_z}{r} + \frac{\partial \sigma_z}{\partial z} &= -\gamma_a, \\
\tau + \sigma \tan \psi &= k, \quad \sigma_r = \sigma + \tau \cos 2\theta, \quad \sigma_z = \sigma - \tau \cos 2\theta, \quad \tau_z = \tau \sin 2\theta, \quad \sigma_\phi = \sigma - \tau,
\end{align*}
\]

where \( \gamma_a \) is a specific weight of the medium.

As a result, the characteristics of the system of differential equations of equilibrium are found in the form:

\[ \lambda_1 = \left( \frac{dr}{d\theta} \right)_1 = \tan(\theta + \mu)m, \quad \lambda_2 = \left( \frac{dr}{d\theta} \right)_2 = \tan(\theta - \mu), \]

where angles \( \mu \) and \( \psi \) are connected by the relation

\[ \mu = \pi / 4 + \psi / 2. \]

Besides, on each of the characteristics (7) the relations connecting the differentials \( d\theta, \ d\sigma, \ dr \) of the functions \( \theta = \theta(r, z), \ \sigma = \sigma(r, z) \) and the coordinate \( r \) are determined. For characteristic \( \lambda_1 \) the relation is obtained in the form

\[ 2d\theta + \frac{\sin 2\mu}{\sigma \cos 2\mu + a} d\sigma + \left[ \frac{2\cos \theta \sin \mu}{r \cos(\theta + \mu)} + \frac{\cos(\theta - \mu)}{\sigma \cos 2\mu + a} \right] d\tau = 0, \]

for characteristic \( \lambda_2 \):

\[ 2d\theta - \frac{\sin 2\mu}{\sigma \cos 2\mu + a} d\sigma + \left[ -\frac{2\cos \theta \sin \mu}{r \cos(\theta - \mu)} + \frac{\cos(\theta - \mu)}{\sigma \cos 2\mu + a} \right] d\tau = 0, \]

where \( a = k \cos \psi \).

The further task is to solve the problem applying the relations (7) – (10). In order to do this we move from the points on the boundary \( AD \) to the points on the boundary \( OA \) (along the line \( PQRS \)). At the boundary \( AD \) stresses \( \sigma_r, \ \tau_z \) are equal to zero, the circuit is stress-free. On the other hand, when the wedge \( EOA \) is inserted into the rock mass, the triangle \( ACD \) will be compressed from the sides, i.e. the tangential stress \( \sigma_z \) at the boundary \( AD \) will be compressive. This means that the first principal stress on \( AD \) will be the stress \( \sigma_z \) and the angle between the first principal direction for tensors \( T_\sigma \) and \( T_\varepsilon \) becomes equal \( \pi / 2 \), i.e.

\[ \theta_{AD} = \pi / 2, \]

In this case, the characteristics forming an acute angle with an axis \( Or \) will be the characteristics of the \( \lambda_2 \) kind; they form an angle \( \pi / 2 - \mu = \pi / 4 - \psi / 2 \) with the axis \( Or \). At the boundary \( AD \), the plasticity condition (4), where \( \sigma_1 = \sigma_z = 0, \ \sigma_3 = \sigma, \) is satisfied, so

\[ \sigma_3 = -\frac{2k \cos \psi}{1 - \sin \psi} = -2k \tan \mu, \quad \sigma|_{AD} = -k \tan \mu. \]
Let us consider the triangle $OAB$. Here, the directions of the main axes $1, 3$ are indicated in the Figure 1 (along the segment $OA$ the tangential stress $\sigma_t$ will be negative, but its absolute value is less than the absolute value of the compressive stress $\sigma_3$ that coincides with the normal stress $\sigma_n$ on $OA$). The value of the angle $\theta$ on $OA$, due to the specified location of the main axes, is equal to

$$\theta|_{OA} = \pi / 2 - \gamma,$$

where $2\gamma$ — an angular opening at the top $O$ of the cone shaped tool inserted into the rock mass.

Stress $\sigma_3 = -p$, where the pressure value $p$ will be determined from the problem solution. On the boundary $OA$ (4) is true, and at $\sigma_3 = -p$ it follows that

$$\sigma_1 = 2k \cot \mu - p \cot^2 \mu , \quad \sigma_1|_{OA} = k \cot \mu - \frac{p}{2\sin^2 \mu}. \quad (14)$$

The angle, formed by the characteristic of the $\lambda_2$ kind with the axis $Or$ on the basis of (7), (13), is equal to $\pi / 2 - \gamma - \mu$. That is why $\angle BOA \leq \mu$. Since the triangle $OAB$ is isosceles, the angle $\angle OAB$ also coincides with $\mu$. Therefore, the angle $\angle EAB$ is equal to $\pi / 2 - \gamma + \mu$. Calculating $\angle EAC$, we find its value $\pi - \angle CAD$, but $\angle CAD = \angle ADC = \pi / 2 - \mu$ and so the angle $\angle EAC = \pi / 2 + \mu$. Comparing $\angle EAB$ and $\angle EAC$ we find that $\angle BAC = \gamma$. Further, in triangles $OAB$ and $ACD$ we assume that the angle $\theta$ is constant.

Let us consider the centered field $BAC$. To describe it, we use polar coordinates $\rho, \chi$:

$$r - r_A = \rho \cos \chi, \quad z - z_A = \rho \sin \chi. \quad (15)$$

Substituting (15) into (7), we find the dependences

$$\left\{ \begin{array}{l}
(\lambda_1): (d \rho \sin \chi + \rho \cos \chi d \chi \cos(\theta + \mu)) = (d \rho \cos \chi - \rho \sin \chi d \chi \sin(\theta + \mu)), \\
(\lambda_2): (d \rho \sin \chi + \rho \cos \chi d \chi \cos(\theta + \mu)) = (d \rho \cos \chi - \rho \sin \chi d \chi \sin(\theta + \mu)),
\end{array} \right. \quad (16)$$

or

$$\left\{ \begin{array}{l}
(\lambda_1): d\rho(\sin \chi \cos(\theta + \mu) + \cos \chi \sin(\theta + \mu)) = -\rho(\cos \chi \cos(\theta + \mu) + \sin \chi \sin(\theta + \mu))d\chi, \\
(\lambda_2): d\rho(\sin \chi \cos(\theta - \mu) - \cos \chi \sin(\theta - \mu)) = -\rho(\cos \chi \cos(\theta - \mu) + \sin \chi \sin(\theta - \mu))d\chi.
\end{array} \right. \quad (17)$$

Assuming that the characteristics of the $\lambda_1$ kind are straight lines satisfying the equation $\chi = \text{const}$ coming out of point A, then for these lines we have equality $d\chi = 0$. And it follows from the first equation (16) that the requirement $\sin(\chi - \theta - \mu) = 0$ or the dependence $\chi - \theta - \mu = \pi n, \quad n \in Z, \quad Z$ — the set of integers, must be fulfilled.

Analyzing the Figure 1, we notice that on the boundary $AB$ the angle $\theta = \theta|_{AB} = \pi / 2 - \gamma$ and the polar angle $\chi = -(\pi / 2 + \gamma - \mu)$, hence, in the previous expression, we should put $n = -1$, that is $\chi = \theta + \mu - \pi$. Substituting this condition into the second equation (16), we notice that

$$d\rho \sin(\chi - \theta + \mu) = -\rho \cos(\chi - \theta + \mu)d\chi,$$

or

$$d\rho \sin 2\mu = -\rho \cos 2\mu d\chi. \quad (17)$$

From (17) it follows that

$$\frac{d\rho}{\rho d\chi} = -\cot 2\mu.$$
\[ \rho = \rho_0 e^{-(\chi - \chi_0) \cot 2\mu}. \]  \hspace{1cm} (18)

We assume that \( \chi_0 = -\pi / 2 + \mu \). Then for \( \chi = \chi_0 \) we have \( \rho = \rho_0 = AQ \), for \( \chi = -\pi / 2 + \mu - \gamma \), \( \rho = \rho_0 e^{\cot 2\mu} = AR \), thus \( \rho < \rho_0 \) or \( AR < AQ \) as shown in the Figure 1.

Then, to solve the problem we should, with the help of (10), connect all these three states distributed in triangles \( ACD \), \( OAB \) and sector \( BAC \) into one whole. We start the calculation by moving from the boundary \( AD \), assuming that the angle \( \theta = \pi / 2 \) in the entire triangle \( ACD \).

Substituting this value into (10), we obtain the equation

\[ -\sin 2\mu \frac{d\sigma}{dr} = \gamma_a, \]

after integrating it we find expression for \( \sigma \) in the form

\[ \sigma = -\frac{\gamma_a}{\sin 2\mu} r + C, \]  \hspace{1cm} (19)

where \( C \) is the integration constant. At the point \( P \) shown in the Figure 1 values \( r = r_p \), wherein \( r_A \leq r_p \leq r_D \), where \( r_A = H \tan \gamma \), \( H \) – immersion depth of the tool, \( r_B = H \tan \mu e^{\cot 2\mu} \cos \gamma \). The value \( \sigma \) at the point \( P \) according to (12) is equal to \( \sigma = -\kappa \tan \mu \). Based on these data, the integration constant is as follows

\[ C = -\kappa \tan \mu + \gamma_a r_p / \sin 2\mu, \]

That is why the value \( \sigma \) along the line \( PQ \) in the triangle \( ACD \) is determined by the expression

\[ \sigma = -\kappa \tan \mu + \gamma_a (r_p - r) / \sin 2\mu, \]

in particular when \( r = r_Q \)

\[ \sigma_Q = -\kappa \tan \mu + \gamma_a (r_p - r_Q) / \sin 2\mu. \]  \hspace{1cm} (20)

As \( r_p < r_Q \), than \( |\sigma_Q| \geq |\sigma_p| \).

Let us consider a segment \( SR \) of a polyline \( PQRS \). Here, just as with the triangle \( ACD \), it is assumed that in the triangle \( OAB \) the angle \( \theta \) also is a constant value (13), therefore \( d\theta = 0 \) and the equation (10) is transformed into

\[ -\tan 2\mu d(\sigma \cos 2\mu + a) + \left[ -\frac{2\sin \gamma \sin \mu}{r \sin(\gamma + \mu)} (\sigma \cos 2\mu + a) + \gamma_a \frac{\sin(\gamma - \mu)}{\sin(\gamma + \mu)} \right] dr = 0. \]  \hspace{1cm} (21)

This is a linear differential equation of the first order with respect to the unknown function \( \sigma \cos 2\mu + a \). Its solution has the form:

\[ \sigma \cos 2\mu + a = \gamma_a \frac{\cos 2\mu \sin(\gamma - \mu) \cos \mu}{\sin \gamma \cos 2\mu + \cos \mu \sin(\gamma + \mu)} + \frac{C}{r \cos \mu \sin(\gamma + \mu)} \].  \hspace{1cm} (22)

The integration constant \( C \) in (22) is obtained from (14):

\[ C = \left( \frac{\kappa \sin 2\mu - p}{2 \sin^2 \mu} \cos 2\mu - \gamma_a \right) \frac{\cos 2\mu \sin(\gamma - \mu) \cos \mu}{\sin \gamma \cos 2\mu + \cos \mu \sin(\gamma + \mu)} r_S \sin \gamma \cos 2\mu \cos \mu \sin(\gamma + \mu), \]  \hspace{1cm} (23)

where \( p \) is an unknown pressure at the point \( S \) shown in the Figure 1, \( r_S \) – value of the polar radius at the point \( S \). Applying (22), (23) we find the value at the point \( R \).

The movement along the arc \( RQ \) shown in the Figure 1 is examined. For this case
\[ dr = d(r_A + \rho \cos \chi) = d \rho \cos \chi - \rho \sin \chi d \chi = \rho \theta e^{\epsilon_1 \cot 2\mu} e^{-\chi \cot 2\mu} (-\cot 2\mu \cos \chi - \sin \chi) d \chi = -\rho \cos(\theta - \mu) \sin 2\mu d \chi \]  

(24)

Substituting (24) into (10) we get the equation to determine the function \( t = \sigma \cos 2\mu + a \)

\[-\tau g 2\mu \frac{dt}{d\theta} + t \left( 2 + \frac{\cos \theta}{e^{(\theta - \pi/2)\cot 2\mu} + \cos(\theta + \mu)} \right) = \gamma a \rho_0 e^{-(\theta - \pi/2)\cot 2\mu} \cos(\theta + \mu) \sin 2\mu \]

(25)

Equation (25) is a linear differential equation of the first order, its solution is found in the form \( t = u v \), to determine functions \( u \) and \( v \) we get the system of equations

\[
\begin{align*}
-\tau g 2\mu v' + v & \left( 2 + \frac{\cos \theta}{(r_A / \rho_0) e^{(\theta - \pi/2)\cot 2\mu} + \cos(\theta + \mu)} \right) = 0, \\
-\tau g 2\mu u' v & = \gamma a \rho_0 e^{-(\theta - \pi/2)\cot 2\mu} \cos(\theta + \mu) \sin 2\mu.
\end{align*}
\]

(26)

The function \( v \) is restored from the first equation (26) and \( u \) from the second one [16]. Thus, the function \( t = u v \) on the segment \( QR \) is determined.

Eventually, an analytical solution of the problem was obtained in all strain areas: in the triangle \( ACD \) shown in the Figure 1 in the form (20), in the centered field \( BAC \) in the form (26), in the triangle \( OAB \) in the form (22), (26) Solutions in each domain depend on their "own" constants, which are found from the conditions of continuity of solutions at the boundaries \( AC \), \( AB \). The constant in the triangle \( ACD \) is determined by the boundary conditions at the boundary \( AD \). The load on \( AO \) corresponds to the solution in the triangle \( OAB \).

There is a remark: from the solutions (20), (22), (26) given above, it is clear that the pressure at the point \( S \) shown in the Figure 1 depends on the coordinate \( r \) of the point \( P \). Due to this fact the pressure at each point on the edge of the wedge will be "its own". To find the full load (or force) on \( OA \), it will be necessary to integrate all normal stresses \( \sigma_3 \) along \( OA \).

The paper presents the results of calculating the ultimate load depending on the input parameters.

3. Conclusions
For a weighty medium, the relations on the characteristics of a hyperbolic system of inelastic deformation of the rock mass around a cone-shaped tool are integrated. It is shown that these relations are linear differential equations of the first order. The stress field is investigated in the case of simple stress states with a single loading, as well as in the case of a centered field.

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References
[1] Hill R 1956 The Mathematical Theory Of Plasticity (Moscow: GITTL) 529 p (in Russian)
[2] Sokolovsky VV 1969 Theory of Plasticity (Moscow: High School) 608 p (in Russian)
[3] Kachanov M 1969 Fundamentals of the Theory of Plasticity (Moscow: Science) 420 p (in Russian)
[4] Hill R, Lee EH and Tupper SJ 1947 The theory of wedge indentation of ductile materials Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences Vol 188 pp 273–289
[5] Ivlev DD and Maksimova LA 2000 On the indentation of indenter in ideal rigid-plastic strip, Proc. RAS. Mechanics are Solid Body Vol 3 pp 131–136. (in Russian)
[6] Soloviev YuI 1960 To the problem of plastic state of a material under a rigid rough stamp with out-of-center loading Applied Mechanics and Technical Physics Vol 1 pp 110–116 (in Russian)

[7] Ivlev DD 1957 Indentation of a thin blade into a plastic medium Proc. USSR Academy of Sciences. Department of Technical Sciences No 10 pp 89 – 93 (in Russian)

[8] Anisimov AN and Khromov AI 2007 The introduction of a wedge into half-space subject to the Coulomb-Mohr fluidity, Bulletin of the Samara State Technical University. Series: Physics and Mathematics Vol 1 pp 44–49 (in Russian)

[9] Davydov DV and Myasninkin YuM 2009 On the introduction of bodies into a rigid plastic medium Bulletin of Voronezh State University. Series: Physics. Mathematics Vol 1 pp 94–100 (in Russian)

[10] Chanyshev AI and Abdulin IM 2018 The problem solution on wedge penetration in an initially anisotropic medium within the rigid-plastic scheme IOP Conference Series: Earth and Environmental Science Vol 134 pp 012011.

[11] Chanyshev AI, Abdulin IM, Gutarova IV, Efimenko LL, Frolova IV and Lukyashko OA 2019 Complete and incomplete plasticity states for initially anisotropic media Fundamental and Applied Questions of Mining Sciences Vol 6 No 1 pp 244–249 (in Russian)

[12] Soloviev YuI and Karaulov AM 1991 The Static-kinematic Method in the Theory of Limiting Equilibrium of Soils and the Prandtl Problem, Proc. Universities. Construction No 11 pp 100–106 (in Russian)

[13] Karaulov AM 2008 Ultimate Pressure of an Annular Foundation on a Foundation with a Hard Underlying Layer, Proc. Universities. Construction Vol 5 pp 14–18 (in Russian)

[14] Berezantsev VG 1970 Calculation of the Foundations of Structures (Leningrad: Stroyizdat) 208 p (in Russian)

[15] Berezantsev VG 1952 The Axisymmetric Problem of the Theory of Limit Equilibrium of a Granular Medium (Moscow: GITL) 120 p (in Russian)

[16] Korn GA and Korn TM 1974 Math Reference Book for Scientists and Engineers (Moscow: Nauka) 832 p (in Russian)