Model-independent effects of $\Delta$ excitation in nucleon polarizabilities

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Abstract

Model-independent effects of $\Delta(1232)$ excitation on nucleon polarizabilities are computed in a Lorentz-invariant fashion. We find a large effect of relative order $(M_{\Delta} - M)/M$ in some of the spin polarizabilities, with the backward spin polarizability receiving the largest contribution. Similar subleading effects are found to be important in the fourth-order spin-independent polarizabilities $\alpha_{E\nu}$, $\alpha_{E2}$, $\beta_{M\nu}$, and $\beta_{M2}$. Combining our results with those for the model-independent effects of pion loops we obtain predictions for spin and fourth-order polarizabilities which compare favorably with the results of a recent dispersion-relation analysis of data.

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I. INTRODUCTION

The nucleon Compton scattering ($\gamma N \rightarrow \gamma N$) amplitude, in Coulomb gauge, can be written in terms of six operator structures with coefficient functions $A_1(s, t)$ to $A_6(s, t)$:

\[
T_{fi} = \mathbf{e}' \cdot A_1(s, t) + \mathbf{e}' \cdot \mathbf{q} \cdot \mathbf{q}' A_2(s, t) + i\mathbf{\sigma} \cdot (\mathbf{e}' \times \mathbf{e}) A_3(s, t) + i\mathbf{\sigma} \cdot (\mathbf{q} \times \mathbf{q}') \mathbf{e}' \cdot \mathbf{e} A_4(s, t) + (i\mathbf{\sigma} \cdot (\mathbf{e}' \times \mathbf{q}) \mathbf{e} \cdot \mathbf{q}' - i\mathbf{\sigma} \cdot (\mathbf{e} \times \mathbf{q}) \mathbf{e}' \cdot \mathbf{q}) A_5(s, t) + (i\mathbf{\sigma} \cdot (\mathbf{e}' \times \mathbf{q}') \mathbf{e} \cdot \mathbf{q}' - i\mathbf{\sigma} \cdot (\mathbf{e} \times \mathbf{q}) \mathbf{e}' \cdot \mathbf{q}) A_6(s, t),
\]

where $\mathbf{q}$ ($\mathbf{q}'$) is the initial (final) three-momentum of the photon, and $\mathbf{e}$ ($\mathbf{e}'$) is the photon polarization vector.

**Nucleon polarizabilities** can be defined as coefficients in the low-energy expansion of this amplitude. Namely, expanding the amplitude in powers of the photon energy $\omega$, the first two terms $[O(\omega^0)$ and $O(\omega^1)]$ are fixed by the low-energy theorem (LET) \[^1\], while terms of $O(\omega^2)$ are proportional to the electric and magnetic polarizabilities $\alpha$ and $\beta$, terms of $O(\omega^3)$ define the spin polarizabilities $\gamma_1$ to $\gamma_4$, and terms of $O(\omega^4)$ define the fourth-order spin-independent polarizabilities $\alpha_{E\nu}, \beta_{M\nu}, \alpha_{E2}, \beta_{M2}$ \[^2\]. More specifically, the Breit-frame Compton amplitude—up to and including terms of order $\omega^4$—can be written using Eq. (1) and \[^2\]:

\[
A_1(s, t) = -\frac{Z^2 e^2}{M} + 4\pi \omega' (\alpha + \beta z) + 4\pi (\omega')^2 \left[ \alpha_{E\nu} + \left( \frac{\alpha_{E2} + \beta_{M\nu}}{12} \right) z + \frac{\beta_{M2}}{12} (2z^2 - 1) \right] + O(\omega^6)
\]

\[
A_2(s, t) = -4\pi \beta - 4\pi \omega' \left( \beta_{M\nu} + \frac{1}{12} \alpha_{E2} + \frac{z}{6} \beta_{M2} \right) + O(\omega^4)
\]

\[
A_3(s, t) = \frac{1}{2} (\omega + \omega') \left\{ \frac{e^2}{2M^2} \left[ Z(Z + 2\kappa) - (Z + \kappa)^2 z \right] + 4\pi \omega' (\gamma_1 + \gamma_5 z) \right\} + O(\omega^5)
\]

\[
A_4(s, t) = \frac{\omega + \omega'}{2\omega'} \left[ -\frac{e^2}{2M^2} (Z + \kappa)^2 + 4\pi \omega' \gamma_2 \right] + O(\omega^3)
\]

\[
A_5(s, t) = \frac{\omega + \omega'}{2\omega'} \left[ \frac{e^2}{2M^2} (Z + \kappa)^2 + 4\pi \omega' \gamma_4 \right] + O(\omega^3)
\]

\[
A_6(s, t) = \frac{\omega + \omega'}{2\omega'} \left[ -\frac{e^2}{2M^2} Z(Z + \kappa) + 4\pi \omega' \gamma_3 \right] + O(\omega^3),
\]

where \[^2\] $\omega = (s - M^2)/2M$, $\omega' = (M^2 - u)/2M$, and $z = (1 + t/2\omega')$. (Note that nucleon pole terms proportional to $\omega^2/M^3$ in $A_1$ and to $1/M^3$ in $A_2$ are not essential to what follows and so have been omitted here.)

In other words, after the Compton amplitude is expanded in powers of photon energy:

\[
T_{fi} = \sum_k T^{(k)} \omega^k,
\]

where the coefficients $T^{(k)}$ are operator valued, Eq. (1) can be presented schematically as:

\[
T^{(0)} \sim \frac{(eZ)^2}{M}, \quad T^{(1)} \sim \frac{\kappa}{M^2}, \quad T^{(2)} \sim (\alpha, \beta), \quad T^{(3)} \sim (\gamma_1, \gamma_2, \gamma_3, \gamma_4), \quad T^{(4)} \sim (\alpha_{E\nu}, \beta_{M\nu}, \alpha_{E2}, \beta_{M2}).
\]
According to the LET, the Born graphs give the full result for $T^{(0)}$ and $T^{(1)}$. All other effects (e.g., meson loops) which contribute to $T^{(0)}$ and $T^{(1)}$ can only result in renormalizations of the charge and magnetic moment.

In contrast, $T^{(2)}$, $T^{(3)}$, etc., can be influenced by a number of effects. Most significant are those that are proportional to a negative power of light hadronic scales, the lightest such scale being, of course, the pion mass, $m_\pi \simeq 139$ MeV. Contributions which scale with negative powers of $m_\pi$ are due to the long-range effects of the pion cloud, see Fig. 1. These contributions to $\alpha$, $\beta$, and the $\gamma_i$’s are known from chiral perturbation theory [5, 6, 7].

The next lightest hadronic scale is the excitation energy of the $\Delta(1232)$-isobar:

$$\Delta = M_\Delta - M \simeq 293 \text{ MeV.}$$

The most significant effects of $\Delta$ excitation contribute inverse powers of $\Delta$ to the nucleon polarizabilities, and hence, diverge in the large-$N_c$ limit. In the real world where $\Delta$ is not vanishing but given by Eq. (5), these effects are still potentially important, since $\Delta$ is heavier than $m_\pi$ by only about a factor of two.

The leading effect of $\Delta$ excitation was computed for $\alpha$, $\beta$, and $\gamma_1$–$\gamma_4$ by Hemmert et al. [8, 9], and for the fourth-order polarizabilities by Holstein et al. [4]. Here we shall compute the complete effect of the $\Delta$-excitation contribution (Fig. 2) in a manifestly covariant fashion. While we find agreement with previous calculations for the leading contributions, the subleading ones, suppressed by $\Delta/M$ relative to leading, bring a sizable correction to some polarizabilities. The biggest correction is in the backward spin polarizability, $\gamma_\pi$. In fact, there the subleading $\Delta$ effect exceeds the leading one by at least a factor of two. We stress that this result is a model-independent consequence of Lorentz invariance and the existence of a light P33-resonance. Indeed, any model of Compton scattering should give the same answer for all pieces of polarizabilities which scale with negative powers of $m_\pi$ and $\Delta$.

The paper is organized as follows. In Sec. II we perform naive dimensional analysis on the operators $T^{(k)}$ of Eq. (3), in order to estimate the relative importance of $\pi N$ loops, $\Delta$ excitation, $\pi \Delta$ loops, and other mechanisms in these quantities. Having established that polarizabilities receive contributions which scale with negative powers of $m_\pi$ and $\Delta$, we compute the $1/\Delta$ pieces...

FIG. 1: The one $\pi N$-loop contributions to $\gamma N$ scattering (crossed graphs are not shown).

FIG. 2: The $\Delta$-excitation graphs.
arising from $\Delta$ excitation in Sec. [III]. In Sec. [LV] we briefly review the results of Refs. [4, 5, 6, 7, 8, 9] for the pieces of $\alpha, \beta, \gamma_1-\gamma_4,$ and $\alpha_{E\nu}, \alpha_{E2}, \beta_{M\nu}, \beta_{M2},$ which are due to $\pi N$ and $\pi \Delta$ loops. We sum the model-independent contributions discussed in Secs. [III] and [LV] and compare to results of a dispersion-relation analysis as well as to recent experimental values for the forward and backward spin polarizabilities in Sec. [V].

II. NAIVE DIMENSIONAL ANALYSIS FOR POLARIZABILITIES

Contributions to $T^{(k)}$ in Eq. (3) can be classified according to whether they are generated by pion physics, by the excitation of $\Delta$ degrees of freedom, or by physics at a higher-energy scale, $\Lambda$. The minimum value of $\Lambda$ is set by the mass of the next light meson or by the next $N^*$-resonance excitation-energy. $\Lambda$ can also take values of the other heavy-mass scales in the theory, such as $M, M\Delta,$ and $4\pi f_\pi$.

At tree level pions can only contribute to Compton scattering through the chiral Wess-Zumino-Witten (WZW) anomaly. The leading (in negative powers of $m_\pi$) contribution of the WZW-anomaly to the amplitude $T^{(k)}$ ($k \geq 3$) scales as:

$$T^{(k)}(\text{Anomaly}) \sim \frac{1}{\Lambda^2 m_\pi^{k-1}}.$$  \hspace{1cm} (6a)

In the case of loop graphs the dominant contribution of a pion-nucleon $L$-loop graph to the amplitude $T^{(k)}$, with $k \geq L + 1$, behaves according to:

$$T^{(k)}(\pi N \text{ loop}) \sim \frac{1}{\Lambda^{2L} m_\pi^{k+1-2L}}.$$  \hspace{1cm} (6b)

Meanwhile, the leading contribution to $T^{(k)}$ due to $\Delta$ excitation, i.e. the one with most powers of $1/\Delta$, is a tree-level graph with the $\Delta$ in either the s- or u-channel (Fig. 2):

$$T^{(k)}(\Delta) \sim \frac{1}{\Lambda^{2} \Delta^{k-1}}.$$  \hspace{1cm} (6c)

as long as $k \geq 2$. The contribution of a $\pi \Delta$ $L$-loop graph has as its leading piece:

$$T^{(k)}(\pi \Delta \text{ loop}) \sim \begin{cases} \frac{1}{\Lambda^{2L} m_\pi^{k-2L-1} \Delta}, & \text{even } k; \\ \frac{1}{\Lambda^{2L} m_\pi^{k-2L-1} \Delta^2}, & \text{odd } k; \end{cases}$$  \hspace{1cm} (6d)

provided that $k \geq L + 1$ and we assume $m_\pi$ is significantly smaller than $\Delta$.
Finally, all the higher energy (“short-range”) effects scale as:

\[ T^{(k)}(\text{short-range}) \sim \frac{1}{\Lambda^{k+1}}. \]  

(6e)

Therefore, if \( \Lambda \) is significantly above \( m_\pi \) and \( \Delta \), the short-range physics cannot affect the contributions which scale with negative powers of \( m_\pi \) and \( \Delta \). Hence the latter contributions are not only dominant at low energy but also model-independent. Any theory of Compton scattering which obeys chiral, gauge, and Lorentz symmetries, includes pion loops, and has a light \( \Delta \)-resonance should give the same answer for the contributions which scale with negative powers of \( m_\pi \) and \( \Delta \).

III. CALCULATION OF \( \Delta \)-EXCITATION EFFECTS

To compute the effect due to \( \Delta \) excitation, we assume the following form of the electromagnetic \( N\Delta \) transition Lagrangian [10]:

\[
\mathcal{L}_{\gamma N\Delta} = \frac{3e}{4MM_+} \bar{\not{N}} T^\dagger_3 \left( ig_M F^{\mu \nu} - g_E \gamma_5 F^{\mu \nu} \right) \partial_\mu \Delta_\nu + \text{H.c.},
\]  

(7)

where \( M_+ = \frac{1}{2}(M + M_\Delta) \), and \( T_3 \) is the isospin \( N\Delta \) transition matrix, with normalization \( T^\dagger_3 T_3 = \frac{2}{3} \).

This \( \gamma N\Delta \) coupling is invariant under electromagnetic gauge transformations (to the order to which we work), as well as under the spin-3/2 gauge transformation:

\[
\Delta_\mu(x) \rightarrow \Delta_\mu(x) + \partial_\mu \varepsilon(x),
\]  

(8)

with \( \varepsilon \) a spinor field. Invariance under (8) ensures the correct spin-degrees-of-freedom counting [11]. Other forms of this coupling, such as the conventional \( G_1-G_2 \) representation with off-shell parameters [12], may result in different “short-range” pieces of polarizabilities, however contributions proportional to negative powers of \( \Delta \) will be the same.

In the Delta’s rest frame (where \( \Delta_0 = 0 \), \( \partial_0 \Delta_i = -iM_\Delta \Delta_i \), and \( \partial_i \Delta_j = 0 \)) the coupling (7) becomes

\[
\mathcal{L}_{\gamma N\Delta} = -\frac{3eM_\Delta}{4MM_+} \bar{N} T^\dagger_3 \left( g_M B^i + g_E \gamma_5 E^i \right) \Delta_i + \text{H.c.},
\]  

(9)

where \( B^i \) is the magnetic and \( E^i \) the electric field. Thus, the two terms correspond to \( N\Delta \) magnetic and electric transitions, respectively. The precise relation of these couplings to the conventional helicity and multipole amplitudes is given in the Appendix.

Computing the sum of the \( s \)- and \( u \)-channel \( \Delta \) contributions, Fig. 2 to the polarizabilities we obtain (see Ref. [10] for more details):

\[
(\alpha, \beta) = e^2 \frac{1}{4\pi 2M_+^4} \left( -\frac{g_E^2}{2M_+}, \frac{g_M^2}{\Delta} \right),
\]  

(10a)

\[
(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = e^2 M \frac{M_+}{4\pi 4M_+^3 \Delta} \left( 4M_+^2 g_M^2 + \frac{g_E^2}{2M_+}, \frac{g_M^2}{\Delta}, -\frac{2M_+}{M_+^2} g_M^2, \frac{M_+^2 M_+^2 \Delta}{M_+^2} \right). \]  

(10b)
This is an exact result for the $\Delta$-excitation contribution. The result of Hemmert et al. \cite{hemmert, kem} corresponds to the leading term in a $\Delta/M$ expansion of these expressions. We choose to expand in $\Delta/M_+$, then $\beta$ is entirely of leading order (LO) while $\alpha$ is of next-to-leading order (NLO):

$$\alpha = -\frac{e^2}{4\pi} \frac{g_E^2}{4M_+^3} = O(1), \quad (11a)$$

$$\beta = \frac{e^2}{4\pi} \frac{g_M^2}{2M_+^3 \Delta} = O(1/\Delta). \quad (11b)$$

The spin polarizabilities to, respectively, LO and NLO in this expansion are given by:

$$O(1/\Delta^2) : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \frac{e^2}{4\pi} \frac{M}{4M_+^3 \Delta^2} (0, -g_M^2, 0, g_M^2). \quad (12a)$$

$$O(1/\Delta) : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \frac{e^2}{4\pi} \frac{1}{4M_+^2 \Delta} \left( 4g_M^2 + \frac{M^2}{2M_+^2} g_E^2, 0, -2g_M^2, 2g_M^2 \right). \quad (12b)$$

Of special interest are the forward ($\gamma_0 = \gamma_1 - \gamma_2 - 2\gamma_4$) and backward ($\gamma_\pi = \gamma_1 + \gamma_2 + 2\gamma_4$) spin polarizabilities. From Eq. (12a) and Eq. (12b) it is easy to derive the Delta contribution to these quantities:

$$O(1/\Delta^2) : (\gamma_0, \gamma_\pi) = \frac{e^2}{4\pi} \frac{M}{4M_+^3 \Delta^2} (-g_M^2, g_M^2), \quad (13a)$$

$$O(1/\Delta) : (\gamma_0, \gamma_\pi) = \frac{e^2}{4\pi} \frac{M}{4M_+^2 \Delta} \left( -\frac{1}{2} g_E^2, \frac{8M^2}{M_+^2} g_M^2 + \frac{1}{2} g_E^2 \right). \quad (13b)$$

Our expansion parameter $\Delta/M_+ \sim \frac{1}{3}$ is relatively small. However, the subleading result for the spin polarizabilities contains large coefficients. In fact, in the backward direction, the subleading effect is always larger than the leading one by at least a factor of

$$\frac{g_{\pi}^{NLO}}{g_{\pi}^{LO}} = 8 \frac{M_+}{M^2} \approx 2. \quad (14)$$

Non-zero values of $g_E$ only serve to increase this factor.

Next we compute the $\Delta$-excitation contribution to the fourth-order polarizabilities at leading and subleading order. Details are given in Appendix B. The result is:

$$O(1/\Delta^3) : (\alpha_{E \nu}, \beta_{M \nu}, \alpha_{E2}, \beta_{M2}) = \frac{e^2}{4\pi} \frac{1}{M_+^2 \Delta^3} (0, g_M^2, 0, 0), \quad (15a)$$

$$O(1/\Delta^2) : (\alpha_{E \nu}, \beta_{M \nu}, \alpha_{E2}, \beta_{M2}) = \frac{e^2}{4\pi} \frac{1}{4M_+^3 \Delta^2} \times \left( -\frac{13}{2} g_M^2 - g_E^2 + 2g_E g_M, g_M(g_M - g_E), 0, -6g_M^2 \right). \quad (15b)$$

IV. REVIEW OF MODEL-INDEPENDENT PION-LOOP CONTRIBUTIONS

All the $1/m_\pi$ terms which scale as in Eq. (6a) and Eq. (6b) have already been computed \cite{hirai, wagner}. The leading non-analytic (LNA) behavior in $m_\pi$ of $\alpha$ and $\beta$ comes entirely from one-loop
graphs, Fig. 1 in chiral perturbation theory [3]:

\[
O(1/m_\pi) : (\alpha, \beta) = \frac{e^2}{4\pi} \left( \frac{g_A}{4\pi f_\pi} \right)^2 \frac{1}{m_\pi} \left( \frac{5\pi}{6}, \frac{\pi}{12} \right),
\]

where \( e^2/4\pi \approx 1/137, \) \( g_A \approx 1.26, \) \( f_\pi \approx 93 \text{ MeV}. \) Corrections to \( \alpha \) and \( \beta \) suppressed by \( m_\pi/M \) relative to leading have the same scaling as short-range effects [6e], and so a model-independent result for them is less useful.

For the fourth-order spin polarizabilities the situation is different, since the leading contribution has an additional power of \( 1/m_\pi \) [5]:

\[
O\left(\frac{1}{m_\pi}^2\right) : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \frac{e^2}{4\pi} \left[ \frac{1}{3m_\pi^2} \left( \frac{g_A}{4\pi f_\pi} \right)^2 \left( \frac{2}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) + \frac{g_A(2Z - 1)}{(2\pi f_\pi)^2 m_\pi^2 \pi} \left( -1, 0, \frac{1}{2}, -\frac{1}{2} \right) \right].
\]

In this case \( \pi N \) loops and the WZW anomaly graph all contribute.

 Corrections of \( O(m_\pi/M) \) relative to these leading effects are then also model-independent predictions of chiral perturbation theory. These were computed recently in [7]:

\[
O(1/m_\pi) : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \frac{e^2}{4\pi} \left( \frac{g_A}{4\pi f_\pi} \right)^2 \frac{\pi}{12M m_\pi} \times (3 + 10Z, 8 + \kappa_v + 3(1 + \kappa_s)(2Z - 1), \frac{5}{2} + Z, -\frac{15}{2} - 2\kappa_v - 2(1 + \kappa_s)(2Z - 1))
\]

and in [8]:

\[
O(1/m_\pi) : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) = -\frac{e^2}{4\pi} \left( \frac{g_A}{4\pi f_\pi} \right)^2 \frac{\pi}{12M m_\pi} \times (3 + 10Z, 6 - \kappa_v + (1 + \kappa_s)(2Z - 1), \frac{5}{2} - Z, -\frac{11}{2}).
\]

Although the results of [4] and [8] are different, the computation of all one-loop graphs in both papers agree, as do the predictions for all directly-observable experimental quantities [13]. The difference lies in the definition of spin polarizabilities.

For the fourth-order polarizabilities only the LNA contribution of \( \pi N \) loops is known at present [3, 4, 5]:

\[
O(1/m_\pi^3) : (\alpha_{E\nu}, \beta_{M\nu}, \alpha_{E2}, \beta_{M2}) = \frac{e^2}{4\pi} \left( \frac{g_A}{4\pi f_\pi} \right)^2 \frac{\pi}{10m_\pi^3} \left( \frac{7}{4}, \frac{7}{4}, 7, -3 \right).
\]

Contributions that are proportional to negative powers of \( \Delta \) can also come from the \( \pi \Delta \)-loop graphs, Fig. 3. The scaling [6d] means that \( \pi \Delta \) loops contribute to the \( O(1/\Delta) \) term in \( \alpha \) and \( \beta \), and to both the \( 1/\Delta^2 \) and \( 1/\Delta \) term in the spin polarizabilities. The contribution to the spin-independent polarizabilities was computed in Ref. [8]:

\[
O(1/\Delta) : (\alpha, \beta) = \frac{e^2}{4\pi} \frac{2}{3} \left( \frac{h_A}{4\pi f_\pi} \right)^2 \frac{1}{\Delta} \left( \frac{1 + \frac{1}{9} \ln f(\Delta m_\pi)}{1 + \frac{1}{9} \ln f(\Delta m_\pi)} \right)^2,
\]

where \( h_A = g_A/2f_\pi \).
with \( f(\xi) = \xi + \sqrt{\xi^2 - 1} \) and \( h_A \) the \( \pi N \Delta \) coupling. The LNA behavior of the \( \pi \Delta \) loops for the \( \gamma \)'s has been computed in Ref. [9]. It is:

\[
O(1/\Delta^2) : (\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \frac{e^2}{4\pi} \frac{1}{27\Delta^2} \left( \frac{h_A}{4\pi f_\pi} \right)^2 \left( -2, 2 - 2 \ln f(\frac{\Delta}{m_\pi}), 1 - \ln f(\frac{\Delta}{m_\pi}), -1 + \ln f(\frac{\Delta}{m_\pi}) \right). \tag{21}
\]

Unfortunately, the \( 1/\Delta \) piece of the \( \pi \Delta \) loop effect on \( \gamma_1-\gamma_4 \) does not yet exist in the literature. Since the leading contribution of \( \pi \Delta \) loops (21) is numerically small, we expect the \( 1/\Delta \) piece to be small as well. Nevertheless, a future computation of this contribution is important as it will complete the analysis of the model-independent effects in spin polarizabilities.

For the fourth-order polarizabilities situation is even less satisfactory. There we know completely just the \( O(1/m_\pi^2 \Delta) \) and \( O(1/\Delta^3) \) contributions due to \( \pi \Delta \) loops [1]:

\[
O(1/m_\pi^2 \Delta) : (\alpha_{EV}, \beta_{\mu\nu} , \alpha_{E2}, \beta_{M2}) = \frac{e^2}{4\pi} \frac{2}{45m_\pi^2 \Delta} \left( \frac{h_A}{4\pi f_\pi} \right)^2 \left( -\frac{29}{18}, \frac{3}{2}, \frac{22}{3}, -6 \right), \tag{22a}
\]

\[
O(1/\Delta^3) : (\alpha_{EV}, \beta_{\mu\nu} , \alpha_{E2}, \beta_{M2}) = \frac{e^2}{4\pi} \frac{2}{45\Delta^3} \left( \frac{h_A}{4\pi f_\pi} \right)^2 \left( \frac{56}{18} - \frac{1}{6} \ln f(\frac{\Delta}{m_\pi}), -1 + \frac{1}{6} \ln f(\frac{\Delta}{m_\pi}), -12 + 6 \ln f(\frac{\Delta}{m_\pi}), 6 - 6 \ln f(\frac{\Delta}{m_\pi}) \right). \tag{22b}
\]

One-loop graphs with insertions from \( \mathcal{L}_{\pi \Delta N}^{(2)} \) can generate effects in the fourth-order polarizabilities scaling like \( 1/(m_\pi \Delta M_+) \). Results for these sub-leading effects do not presently exist in the literature. However, we expect them to be comparable to the small \( 1/\Delta^3 \) effects calculated in Ref. [4], since \( \Delta/M_+ \) is of roughly the same size as \( m_\pi^2/\Delta^2 \). Once again though, checking this expectation in a full calculation of these “relativistic” \( \pi \Delta \)-loop effects is an important future step.

V. DISCUSSION AND CONCLUSION

Combining the results of previous two sections, we now have all the \( 1/m_\pi \) and \( 1/\Delta \) pieces of nucleon spin polarizabilities, except one — the subleading [i.e., \( O(1/\Delta) \)] contribution of the \( \pi \Delta \) loops. This is expected to be small, so here we focus on the model-independent contributions which are already known.

The numerical values for these pieces of \( \gamma_1-\gamma_4 \) are presented in Table III. The sum of these contributions can be compared to the results of the dispersion-relation (DR) analyses shown in the last two columns. The differences between the two DR analyses can be regarded as indicative of the size of their theoretical uncertainty. For the \( \pi N \)-loop \( O(m_\pi^{-1}) \) contribution we quote two results: the first is due to Ref. [3] and the second (in brackets) is due to Ref. [7]. The total sum also is given as two numbers in the cases where [4] and [3] disagree.

In generating Table III for the \( \gamma N \Delta \) couplings we used the values extracted from our recent analysis of Compton scattering data [10]: \( g_M = 2.6, \ g_E = -6 \). The value of \( g_M \) is consistent with the large-\( N_c \) value \( g_M \simeq 2.63 \). The value of \( g_E \) is unusually large, however, combined with the rather small \( g_M \) value, it leads to a reasonable radiative width of the \( \Delta \) resonance. Also, here \( g_E \) affects only \( \gamma_1 \) at \( O(\Delta^{-1}) \) — and that in a fairly mild way: the magnetic transition still
dominates over the electric one in spin polarizabilities. For the $\pi N \Delta$ coupling we have used the large-$N_c$ estimate: $h_A = \frac{3}{\sqrt{2}} g_A \approx 2.7$, which is about 5% smaller than the value inferred from the width of the $\Delta$-resonance.

From this table it is clear that adopting the results of Ref. [6] for the $O(m_\pi^{-1})$ corrections makes the comparison to the DR analysis much more favorable. It is also clear that the $O(\Delta^{-1})$ effect plays a crucial role in achieving agreement with the DR result.

Table II shows the model-independent contributions for the forward and backward spin polarizabilities of the nucleon. The sum of all the presented contributions can again be compared to DR analyses and also to the recent experimental results obtained at the LEGS (BNL) and MAMI (Mainz) facilities.

The $O(1/\Delta)$ effect of the $\Delta$-excitation plays a very significant role in the backward spin polarizability $\gamma_\pi$. Because of this large and positive correction the sum of all model-independent pieces for the proton is $\gamma_\pi^{(p)} = -34 \times 10^{-4}$ fm$^4$, which lies in between the mutually-inconsistent LEGS and MAMI measurements. The prediction for the neutron is consistent with the recent MAMI measurement.

In Table III we show the results for some of the model-independent contributions to the fourth-order polarizabilities. Their sum can then be compared with two recent dispersion analyses. While the leading $\Delta$-excitation effect is non-vanishing only for $\beta_{M\nu}$, the subleading contribution is substantial in $\alpha_{E\nu}$, $\beta_{M\nu}$, and $\beta_{M2}$. In all three cases it helps produce better agreement with the DR results.

Admittedly, although the expressions for the polarizabilities presented here are model independent, the particular values of the parameters used depend on the method of their extraction. For instance, assuming that $\Delta$ excitation completely dominates the $E2$ and $M1$ multipoles of pion photoproduction at the $\Delta$ peak, one can use the Particle Data Group values for the helicity amplitudes [18] and the relations given in Appendix A to find $g_M \approx 3$ and $g_E \approx -1$. The prediction for all the polarizabilities then changes accordingly, as shown in Table IV where we present results for the polarizabilities when the PDG $\gamma N \Delta$ parameters are adopted in preference to those found in the fit of Ref. [10]. The subleading effect of $\Delta$ excitation are important for this set of parameters too.

We close with a note of caution. It is only fair to point out that, while the sum of the model-independent contributions to the spin polarizabilities presented here compares favorably with dispersion relations and experiments, the situation in the spin-independent $\alpha$ and $\beta$ polarizabilities is not as pleasing. There the $O(m_\pi^{-1})$ effect alone is in good agreement with experiment. This agreement is spoiled when $O(\Delta^{-1})$ corrections due to $\Delta$ excitation and $\pi\Delta$ loops are included.

Regardless of whether the agreement shown in Tables II and III is coincidental or not, Eqs. (12a) and (12b) derived here are model-independent results for the pieces of the nucleon polarizabilities arising from magnetic and electric excitation of the $\Delta$. They exhibit a large correction (of order $1/\Delta$) to the leading result (of order $1/\Delta^2$) for the spin polarizabilities and also produce substantial effects of $O(1/\Delta^2)$ in the fourth-order polarizabilities.

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APPENDIX A: HELICITY AND MULTIPOLE $\gamma N \rightarrow \Delta$ AMPLITUDES

Here we relate $g_E$ and $g_M$ to conventional $\gamma N \Delta$ amplitudes. The relation to the helicity amplitudes is:

\begin{align}
A_{1/2} &= -\frac{e}{8M^{3/2}} \sqrt{\frac{\Delta}{M_+}} [2M_+ g_M + \Delta g_E], \quad (A1a) \\
A_{3/2} &= -\frac{\sqrt{3}e}{8M^{3/2}} \sqrt{\frac{\Delta}{M_+}} [2M_+ g_M - \Delta g_E], \quad (A1b)
\end{align}

where $\Delta = M_\Delta - M$, $M_+ = \frac{1}{2}(M_\Delta + M)$. The inverse relation is:

\begin{align}
g_E &= -\frac{1}{e} \left( \frac{2M_\Delta}{\Delta} \right)^{3/2} \sqrt{2M_+} \left( A_{1/2} - \frac{1}{\sqrt{3}} A_{3/2} \right), \quad (A2a) \\
g_M &= -\frac{1}{e} \frac{(2M_\Delta)^{3/2}}{\sqrt{2M_+ \Delta}} \left( A_{1/2} + \frac{1}{\sqrt{3}} A_{3/2} \right). \quad (A2b)
\end{align}

The relation to the multipole amplitudes is:

\begin{align}
E2 &= -\frac{1}{2} \left( A_{1/2} - \frac{1}{\sqrt{3}} A_{3/2} \right) = \frac{e}{8M^{3/2}} \sqrt{\frac{\Delta}{M_+}} \Delta g_E, \quad (A3a) \\
M1 &= -\frac{1}{2} \left( A_{1/2} + \sqrt{3} A_{3/2} \right) = \frac{e}{8M^{3/2}} \sqrt{\frac{\Delta}{M_+}} [4M_+ g_M - \Delta g_E]. \quad (A3b)
\end{align}

Therefore the $E2/M1$ ratio is given by

\begin{equation}
\frac{E2}{M1} = \frac{\Delta g_E}{g_M - \frac{\Delta}{4M_+} g_E}. \quad (A4)
\end{equation}

It is interesting to note here that the correct large $N_\tau$ limit of this ratio is trivially reproduced by Eq. (A4) provided that $g_E$ and $g_M$ do not depend on $N_\tau$. Indeed, then only the masses depend on $N_\tau$: $\Delta = O(1/N_\tau)$, $M_+ = O(N_\tau)$, and hence, from Eq. (A4), the ratio behaves as

\begin{equation}
\frac{E2}{M1} = O(1/N_\tau^2), \quad (A5)
\end{equation}

in agreement with the recent result of Jenkins, Ji and Manohar [19].

APPENDIX B: ON CALCULATION OF FOURTH-ORDER POLARIZABILITIES

Since the representation of the Compton amplitude given by Eqs. 1 and 2 is manifestly invariant under crossing ($s \leftrightarrow u$), we compute only the s-channel contributions $A_i^s(\omega, t)$. The full amplitudes are then obtained by adding the crossed partner as follows:

\begin{align}
A_i(\omega, t) &= A_i^s(\omega, t) + A_i^s(-\omega', t), \quad \text{for } i = 1, 2; \\
A_i(\omega, t) &= A_i^s(\omega, t) - A_i^s(-\omega', t), \quad \text{for } i = 3, \ldots, 6;
\end{align}
Thus, if the s-channel amplitude has a low-energy expansion,
\[ A_s^i(\omega, t) = \sum_{nl} a_{nl}^{(i)} \omega^n t^l \]  
(B1)
the full amplitude expands as
\[ A_i(\omega, t) = \sum_l \left( \sum_{\text{even } n} a_{nl}^{(i)} F_n + \sum_{\text{odd } n} a_{nl}^{(i)} G_n \right) t^l, \text{ for } i = 1, 2 \]
\[ A_i(\omega, t) = \sum_l \left( \sum_{\text{odd } n} a_{nl}^{(i)} F_n + \sum_{\text{even } n} a_{nl}^{(i)} G_n \right) t^l, \text{ for } i = 3, \ldots, 6 \]
where \( F_n \equiv \omega + \omega' \) and \( G_n \equiv \omega - \omega' \) satisfy the following recursion relations
\[ F_n = \omega \omega' (F_{n-2} + \tau G_{n-1}), \]  
(B2a)
\[ G_n = \omega \omega' (G_{n-2} + \tau F_{n-1}), \]  
(B2b)
with \( \tau = -t/2\omega \omega' = 1 - z \). The coefficients \( a_{nl} \) can then be straightforwardly related to the polarizabilities defined by Eq. (2). For example, in fourth order we find the following relations:
\[ \alpha_{E2} + \beta_{M2} + \frac{1}{12} (\alpha_{E2} + \beta_{M2}) = 2a_{10}^{(1)}, \]
\[ \beta_{M2} + \frac{1}{12} \alpha_{E2} + \frac{1}{3} \beta_{M2} = 4a_{21}^{(1)} - 3a_{30}^{(1)}, \]
\[ \beta_{E2} - \frac{1}{12} \alpha_{E2} + \frac{1}{6} \beta_{M2} = -2a_{20}^{(2)}, \]
\[ \frac{1}{6} \beta_{M2} = 8a_{12}^{(1)} + a_{20}^{(1)} - 2a_{11}^{(2)} = a_{10}^{(2)} - 4a_{01}^{(2)}. \]  
(B3)
Our results for \( A_s^i(\omega, t) \) due to \( \Delta \) excitation can be found in Ref. [10]. The resulting expressions for the fourth-order polarizabilities are given above, in Eqs. (15a) and (15b).

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[20] In Eq. (2), $M$ is the nucleon mass, $Z = 1$ for the proton, $Z = 0$ for the neutron, $e^2/4\pi \simeq 1/137$, and $\kappa = (\kappa_s + \kappa_v 3)/2$, with $\kappa_s \simeq -0.12$ and $\kappa_v \simeq 3.71$ the isoscalar and isovector anomalous magnetic moments of the nucleon.
[21] The quantities $\omega$, $\omega'$, and $z$ are defined in an invariant fashion. In the lab frame they can be interpreted as the initial and final photon energy and the scattering angle.
[22] Note that in general, whether the leading contribution is due to a magnetic or electric transition will be determined by the parity of the resonance.
TABLE I: Pieces of nucleon spin polarizabilities which scale with negative powers of light hadronic scales \(m_\pi\) and \(\Delta\) compared to the dispersion-relation analysis of Ref. [14]. The numbers in round brackets in columns “\(O(m_\pi^{-1})\)” and “Sum” are obtained using Eq. (18a), while the numbers outside the brackets are obtained using Eq. (18b). All values are in units of \(10^{-4}\) fm\(^4\).

| \(\gamma_i^{(N)}\) | WZW | \(\pi N\) loops | \(\Delta\) excitation | \(\pi\Delta\) loops | Sum | Disp. relation |
|---|---|---|---|---|---|---|
| \(\gamma_1^{(p)}\) | -22.1 | 4.4 | -3.4 | 0 | 3.5 | 18.1 | Ref. [14] Ref. [15] |
| \(\gamma_2^{(p)}\) | 0.2 | 2.2 | -0.8 | 1.8 | 0 | -0.2 | -6.0 | -2.5 |
| \(\gamma_3^{(p)}\) | 11.0 | 1.1 | -0.4 | 0 | 1.2 | 10.5 | (10.0) |
| \(\gamma_4^{(p)}\) | -11.0 | -1.1 | 1.4 | 1.8 | 1.2 | 7.6 | -4.6 |
| \(\gamma_1^{(n)}\) | 22.1 | 4.4 | -0.8 | 0 | 3.5 | 28.7 | |
| \(\gamma_2^{(n)}\) | 0.2 | 2.2 | -0.4 | 1.8 | 0 | -0.2 | -0.2 | -2.2 |
| \(\gamma_3^{(n)}\) | -11.0 | 1.1 | -0.7 | 0 | 1.2 | 11.8 | |
| \(\gamma_4^{(n)}\) | 11.0 | -1.1 | 1.4 | 1.8 | 1.2 | 14.6 | |

TABLE II: Pieces of nucleon forward and backward spin polarizabilities which scale with negative powers of \(m_\pi\) and \(\Delta\) compared to a dispersion-relation analysis and experimental values. (For the \(O(m_\pi^{-1})\) contribution the result of Ref. [14] was used here.) All values are in units of \(10^{-4}\) fm\(^4\).

| \(\gamma_i^{(N)}\) | WZW | \(\pi N\) loops | \(\Delta\) excitation | \(\pi\Delta\) loops | Sum | DR [14] | Experiment |
|---|---|---|---|---|---|---|---|
| \(\gamma_1^{(p)}\) | 0 | 4.4 | -5.4 | 1.8 | 12 | -0.5 | -2.1 | -1.1 | -1.55 \(\pm\) 0.18 | -1.0 \(\pm\) 0.2 |
| \(\gamma_2^{(p)}\) | -44.1 | 4.4 | -1.3 | 1.8 | 5.8 | -0.5 | -34.0 | -33.7 | -27.2 \(\pm\) 3.1 | -38.7 \(\pm\) 1.8 |
| \(\gamma_1^{(n)}\) | 0 | 4.4 | -3.3 | -1.8 | 1.2 | -0.5 | 0.0 | 0.2 |
| \(\gamma_2^{(n)}\) | 44.1 | 4.4 | 1.7 | 1.8 | 5.8 | -0.5 | 57.4 | 57.0 |

TABLE III: Pieces of fourth-order spin-independent nucleon polarizabilities which scale with negative powers of \(m_\pi\) and \(\Delta\) together with a comparison to dispersion-relation analyses. All values are in units of \(10^{-4}\) fm\(^5\).

| \(\alpha_{E_V}\) | \(\alpha_{E_2}\) | \(\beta_{M_V}\) | \(\beta_{M_2}\) |
|---|---|---|---|
| 2.2 | 3.5 | 20.8 | -8.9 |
| 0 | -5.5 | -1.1 | 0 |
| -1.5 | 1.4 | 6.7 | 0 |
| 0.7 | -0.2 | -0.8 | 0.5 |
| -4.1 | 8.5 | 26.7 | 15.9 |
| -4.0 | 9.3 | 29.1 | -24.1 |
| -3.8 | 9.1 | 27.5 | -22.4 |

TABLE III: Pieces of fourth-order spin-independent nucleon polarizabilities which scale with negative powers of \(m_\pi\) and \(\Delta\) together with a comparison to dispersion-relation analyses. All values are in units of \(10^{-4}\) fm\(^5\).
\[ \Delta \text{ excitation} \]

| \( \Delta \) excitation | Sum       |
|--------------------------|-----------|
| \( \Delta \) excitation  | \( \gamma_1 \) | 0 | 3.1 | -18.4 | 28.3 |
|                          | \( \gamma_2 \) | -2.5 | 0 | -1.3 | -0.8 |
|                          | \( \gamma_3 \) | 0 | -1.5 | 10.1 | -12.2 |
|                          | \( \gamma_4 \) | 2.5 | 1.5 | -6.7 | 15.4 |
|                          | \( \gamma_0 \) | -2.5 | 0.1 | -3.7 | -1.7 |
|                          | \( \gamma_\pi \) | 2.5 | 6.2 | -33.0 | 58.4 |
|                          | \( \alpha_{E\nu} \) | 0 | -3.3 | -1.8 | -1.8 |
|                          | \( \beta_{M\nu} \) | 6.6 | -1.1 | 10.7 | 10.7 |
|                          | \( \alpha_{E2} \) | 0 | 0 | 26.7 | 26.7 |
|                          | \( \beta_{M2} \) | 0 | -2.7 | -16.6 | -16.6 |

**TABLE IV:** Results for \( \Delta \)-excitation pieces of the spin and fourth-order polarizabilities using an alternative set of \( \gamma N \Delta \) parameters: \( g_M = 3, g_E = -1 \). The last two columns indicate how the total prediction is changed.