SPECTRAL THEORY FOR A MATHEMATICAL MODEL OF
THE WEAK INTERACTION: THE DECAY OF THE
INTERMEDIATE VECTOR BOSONS $W^{\pm}$. I.

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ABSTRACT. We consider a Hamiltonian with cutoffs describing the weak decay
of spin 1 massive bosons into the full family of leptons. The Hamiltonian is
a self-adjoint operator in an appropriate Fock space with a unique ground
state. We prove a Mourre estimate and a limiting absorption principle above
the ground state energy and below the first threshold for a sufficiently small
coupling constant. As a corollary, we prove absence of eigenvalues and absolute
continuity of the energy spectrum in the same spectral interval.

1. Introduction

In this article, we consider a mathematical model of the weak interaction as
patterned according to the Standard Model in Quantum Field Theory (see [18, 31]).
We choose the example of the weak decay of the intermediate vector bosons $W^{\pm}$
into the full family of leptons.

The mathematical framework involves fermionic Fock spaces for the leptons and
bosonic Fock spaces for the vector bosons. The interaction is described in terms
of annihilation and creation operators together with kernels which are square inte-
grable with respect to momenta. The total Hamiltonian, which is the sum of the
free energy of the particles and antiparticles and of the interaction, is a self-adjoint
operator in the Fock space for the leptons and the vector bosons and it has an
unique ground state in the Fock space for a sufficiently small coupling constant.

The weak interaction is one of the four fundamental interactions known up to
now. But the weak interaction is the only one which does not generate bound
states. As it is well known it is not the case for the strong, electromagnetic and
gravitational interactions. Thus we are expecting that the spectrum of the Hamil-
tonian associated with every model of weak decays is absolutely continuous above
the energy of the ground state and this article is a first step towards a proof of such
a statement. Moreover a scattering theory has to be established for every such
Hamiltonian.

In this paper we establish a Mourre estimate and a limiting absorption principle
for any spectral interval above the energy of the ground state and below the mass
of the electron for a small coupling constant.

Our study of the spectral analysis of the total Hamiltonian is based on the
conjugate operator method with a self-adjoint conjugate operator. The methods
used in this article are taken largely from [4] and [13] and are based on [3] and [25].
Some of the results of this article has been announced in [8].

For other applications of the conjugate operator method see [1, 5, 6, 9, 10, 11,
12, 14, 15, 17, 21, 26].
For related results about models in Quantum Field Theory see [7] and [28] in the case of the Quantum Electrodynamics and [2] in the case of the weak interaction.

The paper is organized as follows. In section 2, we give a precise definition of the model we consider. In section 3, we state our main results and in the following sections, together with the appendix, detailed proofs of the results are given.

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2. The model

The weak decay of the intermediate bosons $W^+$ and $W^−$ involves the full family of leptons together with the bosons themselves, according to the Standard Model (see [18, Formula (4.139)] and [31]).

The full family of leptons involves the electron $e^−$ and the positron $e^+$, together with the associated neutrino $\nu_e$ and antineutrino $\bar{\nu}_e$, the muons $\mu^−$ and $\mu^+$ together with the associated neutrino $\nu_\mu$ and antineutrino $\bar{\nu}_\mu$, and the tau leptons $\tau^−$ and $\tau^+$ together with the associated neutrino $\nu_\tau$ and antineutrino $\bar{\nu}_\tau$.

It follows from the Standard Model that neutrinos and antineutrinos are massless particles. Neutrinos are left-handed, i.e., neutrinos have helicity $−1/2$ and antineutrinos are right handed, i.e., antineutrinos have helicity $+1/2$.

In what follows, the mathematical model for the weak decay of the vector bosons $W^+$ and $W^−$ that we propose is based on the Standard Model, but we adopt a slightly more general point of view because we suppose that neutrinos and antineutrinos are both massless particles with helicity $±1/2$. We recover the physical situation as a particular case. We could also consider a model with massive neutrinos and antineutrinos built upon the Standard Model with neutrino mixing [27].

Let us sketch how we define a mathematical model for the weak decay of the vector bosons $W^±$ into the full family of leptons.

The energy of the free leptons and bosons is a self-adjoint operator in the corresponding Fock space (see below) and the main problem is associated with the interaction between the bosons and the leptons. Let us consider only the interaction between the bosons and the electrons, the positrons and the corresponding neutrinos and antineutrinos. Other cases are strictly similar. In the Schrödinger representation the interaction is given by (see [18, p159, (4.139)] and [31, p308, (21.3.20)])

\[
I = \int d^3x \overline{\Psi}_e(x)\gamma^\alpha (1 - \gamma_5)\Psi_{\nu_e}(x)W_\alpha(x) + \int d^3x \overline{\Psi}_{\nu_e}(x)\gamma^\alpha (1 - \gamma_5)\Psi_e(x)W_\alpha(x)^*,
\]

where $\gamma^\alpha$, $\alpha = 0, 1, 2, 3$ and $\gamma_5$ are the Dirac matrices and $\Psi_e(x)$ and $\overline{\Psi}_{\nu_e}(x)$ are the Dirac fields for $e^−$, $e^+$, $\nu_e$ and $\bar{\nu}_e$.

We have

\[
\Psi_e(x) = \left(\frac{1}{2\pi}\right)^{3/2} \sum_{s = \pm 1/2} \int d^3p \left( b_{e,+} (p, s) \frac{u(p, s)}{\sqrt{p_0}} e^{ip\cdot x} + b_{e,-}^* (p, s) \frac{v(p, s)}{\sqrt{p_0}} e^{-ip\cdot x} \right),
\]

\[
\overline{\Psi}_e(x) = \Psi_e(x)^\dagger \gamma^0.
\]
Here \( p_0 = (|p|^2 + m_e^2)^{1/2} \) where \( m_e > 0 \) is the mass of the electron and \( u(p, s) \) and \( v(p, s) \) are the normalized solutions to the Dirac equation (see [18] Appendix).

The operators \( b_{e,+}(p, s) \) and \( b_{e,-}^*(p, s) \) (respectively \( b_{e,-}(p, s) \) and \( b_{e,+}^*(p, s) \)) are the annihilation and creation operators for the electrons (respectively the positrons) satisfying the anticommutation relations (see below).

Similarly we define \( \Psi_{\nu_e}(x) \) and \( \overline{\Psi}_{\nu_e}(x) \) by substituting the operators \( c_{\nu_e,\pm}(p, s) \) and \( c_{\nu_e,\pm}^*(p, s) \) for \( b_{e,\pm}(p, s) \) and \( b_{e,\pm}^*(p, s) \) with \( p_0 = |p| \). The operators \( c_{\nu_e,+}(p, s) \) and \( c_{\nu_e,-}^*(p, s) \) (respectively \( c_{\nu_e,-}(p, s) \) and \( c_{\nu_e,+}^*(p, s) \)) are the annihilation and creation operators for the neutrinos associated with the electrons (respectively the antineutrinos).

For the \( W_\alpha \) fields we have (see [30] §5.3).

\[
W_\alpha(x) = \left( \frac{\sqrt{2}}{2\pi} \right)^{\frac{3}{2}} \sum_{\lambda = -1, 0, 1} \int \frac{d^3k}{\sqrt{2k_0}} (\epsilon_\alpha(k, \lambda)a_+(k, \lambda)e^{ik.x} + \epsilon_\alpha^*(k, \lambda)a_-^*(k, \lambda)e^{-ik.x}) .
\]

Here \( k_0 = (|k|^2 + m_W^2)^{1/2} \) where \( m_W > 0 \) is the mass of the bosons \( W^\pm \). \( W^+ \) is the antiparticle of \( W^- \). The operators \( a_+(k, \lambda) \) and \( a_\lambda^*(k, \lambda) \) (respectively \( a_-^*(k, \lambda) \) and \( a_-(k, \lambda) \)) are the annihilation and creation operators for the bosons \( W^- \) (respectively \( W^+ \)) satisfying the canonical commutation relations. The vectors \( \epsilon_\alpha(k, \lambda) \) are the polarizations of the massive spin 1 bosons \( W^\pm \) (see [30] Section 5.2).

The interaction (2.1) is a formal operator and, in order to get a well defined operator in the Fock space, one way is to adapt what Glimm and Jaffe have done in the case of the Yukawa Hamiltonian (see [16]). For that sake, we have to introduce a spatial cutoff \( g(x) \) such that \( g \in L^1(\mathbb{R}^3) \), together with momentum cutoffs \( \chi(p) \) and \( \rho(k) \) for the Dirac fields and the \( W_\mu \) fields respectively.

Thus when one develops the interaction \( I \) with respect to products of creation and annihilation operators, one gets a finite sum of terms associated with kernels of the form

\[
\chi(p_1) \chi(p_2) \rho(k) \hat{g}(p_1 + p_2 - k) ,
\]

where \( \hat{g} \) is the Fourier transform of \( g \). These kernels are square integrable.

In what follows, we consider a model involving terms of the above form but with more general square integrable kernels.

We follow the convention described in [30] section 4.1 that we quote: “The state-vector will be taken to be symmetric under interchange of any bosons with each other, or any bosons with any fermions, and antisymmetric with respect to interchange of any two fermions with each other, in all cases, whether the particles are of the same species or not”. Thus, as it follows from section 4.2 of [30], fermionic creation and annihilation operators of different species of leptons will always anticommute.

Concerning our notations, from now on, \( \ell \in \{1, 2, 3\} \) denotes each species of leptons. \( \ell = 1 \) denotes the electron \( e^- \) the positron \( e^+ \) and the neutrinos \( \nu_e, \bar{\nu}_e \). \( \ell = 2 \) denotes the muons \( \mu^-, \mu^+ \) and the neutrinos \( \nu_\mu, \bar{\nu}_\mu \). and \( \ell = 3 \) denotes the tau-leptons and the neutrinos \( \nu_\tau, \bar{\nu}_\tau \).

Let \( \xi_1 = (p_1, s_1) \) be the quantum variables of a massive lepton, where \( p_1 \in \mathbb{R}^3 \) and \( s_1 \in \{-1/2, 1/2\} \) is the spin polarization of particles and antiparticles. Let \( \xi_2 = (p_2, s_2) \) be the quantum variables of a massless lepton where \( p_2 \in \mathbb{R}^3 \) and \( s_2 \in \{-1/2, 1/2\} \) is the helicity of particles and antiparticles and, finally, let \( \xi_3 = (k, \lambda) \) be the quantum variables of the spin 1 bosons \( W^+ \) and \( W^- \) where \( k \in \mathbb{R}^3 \) and \( \lambda \in \{-1, 0, 1\} \) is the polarization of the vector bosons (see [30].
section 5). We set \( \Sigma_1 = \mathbb{R}^3 \times \{-1/2, 1/2\} \) for the leptons and \( \Sigma_2 = \mathbb{R}^3 \times \{-1, 0, 1\} \) for the bosons. Thus \( L^2(\Sigma_1) \) is the Hilbert space of each lepton and \( L^2(\Sigma_2) \) is the Hilbert space of each boson. The scalar product in \( L^2(\Sigma_j), j = 1, 2 \) is defined by

\[
(f, g) = \int_{\Sigma_j} \bar{f}(\xi) g(\xi) d\xi, \quad j = 1, 2.
\]

Here

\[
\int_{\Sigma_1} d\xi = \sum_{s=+\frac{1}{2}, \frac{1}{2}} \int dp \quad \text{and} \quad \int_{\Sigma_2} d\xi = \sum_{\lambda=0,1,-1} \int dk, \quad (p, k \in \mathbb{R}^3).
\]

The Hilbert space for the weak decay of the vector bosons \( W^+ \) and \( W^- \) is the Fock space for leptons and bosons that we now describe.

Let \( \mathcal{S} \) be any separable Hilbert space. Let \( \otimes_a^n \mathcal{S} \) (resp. \( \otimes_a^o \mathcal{S} \)) denote the antisymmetric (resp. symmetric) \( n \)-th tensor power of \( \mathcal{S} \). The fermionic (resp. bosonic) Fock space over \( \mathcal{S} \), denoted by \( \mathcal{F}_a(\mathcal{S}) \) (resp. \( \mathcal{F}_s(\mathcal{S}) \)), is the direct sum

\[
\mathcal{F}_a(\mathcal{S}) = \bigoplus_{n=0}^{\infty} \bigotimes_{a}^{n} \mathcal{S} \quad \text{(resp. } \mathcal{F}_s(\mathcal{S}) = \bigoplus_{n=0}^{\infty} \bigotimes_{s}^{n} \mathcal{S} )
\]

where \( \otimes_a^o \mathcal{S} = \otimes_s^o \mathcal{S} = \mathbb{C} \). The state \( \Omega = (1, 0, 0, \ldots, 0, \ldots) \) denotes the vacuum state in \( \mathcal{F}_a(\mathcal{S}) \) and in \( \mathcal{F}_s(\mathcal{S}) \).

For every \( \ell \), \( \mathcal{F}_\ell \) is the fermionic Fock space for the corresponding species of leptons including the massive particle and antiparticle together with the associated neutrino and antineutrino, i.e.,

\[
\mathcal{F}_\ell = \bigotimes_{n=0}^{4} \mathcal{F}_a(L^2(\Sigma_1)) \quad \ell = 1, 2, 3.
\]

We have

\[
\mathcal{F}_\ell = \bigoplus_{q_\ell \geq 0, \bar{q}_\ell \geq 0, r_\ell \geq 0, \bar{r}_\ell \geq 0} \mathcal{F}_{\ell}^{(q_\ell, \bar{q}_\ell, r_\ell, \bar{r}_\ell)},
\]

with

\[
\mathcal{F}_{\ell}^{(q_\ell, \bar{q}_\ell, r_\ell, \bar{r}_\ell)} = (\otimes_a^o L^2(\Sigma_1)) \otimes (\otimes_s^o L^2(\Sigma_1)) \otimes (\otimes_a^s L^2(\Sigma_1)) \otimes (\otimes_s^s L^2(\Sigma_1)) \cdot
\]

Here \( q_\ell \) (resp. \( \bar{q}_\ell \)) is the number of massive particle (resp. antiparticles) and \( r_\ell \) (resp. \( \bar{r}_\ell \)) is the number of neutrinos (resp. antineutrinos). The vector \( \Omega_\ell \) is the associated vacuum state. The fermionic Fock space denoted by \( \mathcal{F}_L \) for the leptons is then

\[
\mathcal{F}_L = \bigotimes_{\ell=1}^{3} \mathcal{F}_\ell,
\]

and \( \Omega_L = \bigotimes_{\ell=1}^{3} \Omega_\ell \) is the vacuum state.

The bosonic Fock space for the vector bosons \( W^+ \) and \( W^- \), denoted by \( \mathcal{F}_W \), is then

\[
\mathcal{F}_W = \mathcal{F}_s(L^2(\Sigma_2)) \otimes \mathcal{F}_s(L^2(\Sigma_2)) \simeq \mathcal{F}_a(L^2(\Sigma_2) \oplus L^2(\Sigma_2)) .
\]

We have

\[
\mathcal{F}_W = \bigoplus_{t \geq 0, \bar{t} \geq 0} \mathcal{F}_{W}^{(t, \bar{t})},
\]

where \( \mathcal{F}_{W}^{(t, \bar{t})} = (\otimes_s^o L^2(\Sigma_2)) \otimes (\otimes_s^s L^2(\Sigma_2)). \) Here \( t \) (resp. \( \bar{t} \)) is the number of bosons \( W^- \) (resp. \( W^+ \)). The vector \( \Omega_W \) is the corresponding vacuum.
The Fock space for the weak decay of the vector bosons $W^+$ and $W^-$, denoted by $\mathfrak{F}$, is thus

$$\mathfrak{F} = \mathfrak{F}_L \otimes \mathfrak{F}_W$$

and $\Omega = \Omega_L \otimes \Omega_W$ is the vacuum state.

For every $\ell \in \{1, 2, 3\}$ let $\mathcal{D}_\ell$ denote the set of smooth vectors $\psi_\ell \in \mathfrak{F}_\ell$ for which $\psi_\ell^{(q_\ell,q_\ell,r_\ell,\bar{r}_\ell)}$ has a compact support and $\psi_\ell^{(q_\ell,q_\ell,r_\ell,\bar{r}_\ell)} = 0$ for all but finitely many $(q_\ell,q_\ell,r_\ell,\bar{r}_\ell)$. Let

$$\mathcal{D}_L = \bigotimes_{\ell=1}^3 \mathcal{D}_\ell.$$

Here $\otimes$ is the algebraic tensor product.

Let $\mathcal{D}_W$ denote the set of smooth vectors $\phi \in \mathfrak{F}_W$ for which $\phi^{(t,\bar{t})}$ has a compact support and $\phi^{(t,\bar{t})} = 0$ for all but finitely many $(t,\bar{t})$.

Let

$$\mathfrak{D} = \mathcal{D}_L \otimes \mathcal{D}_W.$$

The set $\mathfrak{D}$ is dense in $\mathfrak{F}$.

Let $A_\ell$ be a self-adjoint operator in $\mathfrak{F}_\ell$ such that $\mathcal{D}_\ell$ is a core for $A_\ell$. Its extension to $\mathfrak{F}_L$ is, by definition, the closure in $\mathfrak{F}_L$ of the operator $A_1 \otimes 1_2 \otimes 1_3$ with domain $\mathcal{D}_L$ when $\ell = 1$, of the operator $1_1 \otimes A_2 \otimes 1_3$ with domain $\mathcal{D}_L$ when $\ell = 2$, and of the operator $1_1 \otimes 1_2 \otimes A_3$ with domain $\mathcal{D}_L$ when $\ell = 3$. Here $1_\ell$ is the operator identity on $\mathfrak{F}_\ell$.

The extension of $A_\ell$ to $\mathfrak{F}_L$ is a self-adjoint operator for which $\mathcal{D}_L$ is a core and it can be extended to $\mathfrak{F}$. The extension of $A_\ell$ to $\mathfrak{F}$ is, by definition, the closure in $\mathfrak{F}$ of the operator $A_\ell \otimes 1_W$ with domain $\mathfrak{D}$, where $\mathfrak{D}$ is the extension of $\mathcal{D}_\ell$ to $\mathfrak{F}_L$.

The extension of $A_\ell$ to $\mathfrak{F}$ is a self-adjoint operator for which $\mathfrak{D}$ is a core.

Let $B$ be a self-adjoint operator in $\mathfrak{F}_W$ for which $\mathcal{D}_W$ is a core. The extension of the self-adjoint operator $A_\ell \otimes B$ is, by definition, the closure in $\mathfrak{F}$ of the operator $A_1 \otimes 1_2 \otimes 1_3 \otimes B$ with domain $\mathfrak{D}$ when $\ell = 1$, of the operator $1_1 \otimes A_2 \otimes 1_3 \otimes B$ with domain $\mathfrak{D}$ when $\ell = 2$, and of the operator $1_1 \otimes 1_2 \otimes A_3 \otimes B$ with domain $\mathfrak{D}$ when $\ell = 3$. The extension of $A_\ell \otimes B$ to $\mathfrak{F}$ is a self-adjoint operator for which $\mathfrak{D}$ is a core.

We now define the creation and annihilation operators. For each $\ell = 1, 2, 3$, $b_{\epsilon,\ell}(\xi_1)$ (resp. $b_{\epsilon,\ell}^*(\xi_1)$) is the annihilation (resp. creation) operator for the corresponding species of massive particle when $\epsilon = +$ and for the corresponding species of massive antiparticle when $\epsilon = -$. Similarly, for each $\ell = 1, 2, 3$, $c_{\epsilon,\ell}(\xi_2)$ (resp. $c_{\epsilon,\ell}^*(\xi_2)$) is the annihilation (resp. creation) operator for the corresponding species of neutrino when $\epsilon = +$ and for the corresponding species of antineutrino when $\epsilon = -$. The operator $a_{\epsilon}(\xi_3)$ (resp. $a_{\epsilon}^*(\xi_3)$) is the annihilation (resp. creation) operator for the boson $W^-$ when $\epsilon = +$ and for the boson $W^+$ when $\epsilon = -$.

Let $\Psi \in \mathfrak{D}$ be such that

$$\Psi = \left( \Psi^{(Q)} \right)_Q,$$

with $Q = \left( (q_\ell,q_\ell,r_\ell,\bar{r}_\ell)_{\ell=1,2,3}, (t,\bar{t}) \right)$, and

$$\Psi^{(Q)} = \left( \bigotimes_{\ell=1}^3 \Psi^{(q_\ell,q_\ell,r_\ell,\bar{r}_\ell)} \right) \otimes \phi^{(t,\bar{t})},$$

where $(q_\ell,q_\ell,r_\ell,\bar{r}_\ell,t,\bar{t}) \in \mathbb{N}^6$. Here, $(\Psi^{(q_\ell,q_\ell,r_\ell,\bar{r}_\ell)})_{q_\ell \geq 0, q_\ell \geq 0, r_\ell \geq 0, \bar{r}_\ell \geq 0} \in \mathcal{D}_\ell$, and $(\phi^{(t,\bar{t})})_{t \geq 0, \bar{t} \geq 0} \in \mathcal{D}_W$. 
We define

\[
Q_{\ell,+} = \left( (q_\ell, \tilde{q}_\ell, r_\ell, \tilde{r}_\ell)_{\ell<\ell}, (q_\ell + 1, \tilde{q}_\ell, r_\ell, \tilde{r}_\ell), (q_\ell, \tilde{q}_\ell, r_\ell, \tilde{r}_\ell)_{\ell>\ell}, (t, \tilde{t}) \right),
\]

\[
Q_{\ell,-} = \left( (q_\ell, \tilde{q}_\ell, r_\ell, \tilde{r}_\ell)_{\ell<\ell}, (q_\ell, \tilde{q}_\ell + 1, r_\ell, \tilde{r}_\ell), (q_\ell, \tilde{q}_\ell, r_\ell, \tilde{r}_\ell)_{\ell>\ell}, (t, \tilde{t}) \right),
\]

\[
\tilde{Q}_{\ell,+} = \left( (q_\ell, \tilde{q}_\ell, r_\ell, \tilde{r}_\ell)_{\ell<\ell}, (q_\ell, \tilde{q}_\ell, r_\ell + 1, \tilde{r}_\ell), (q_\ell, \tilde{q}_\ell, r_\ell, \tilde{r}_\ell)_{\ell>\ell}, (t, \tilde{t}) \right),
\]

\[
\tilde{Q}_{\ell,-} = \left( (q_\ell, \tilde{q}_\ell, r_\ell, \tilde{r}_\ell)_{\ell<\ell}, (q_\ell, \tilde{q}_\ell, r_\ell, \tilde{r}_\ell + 1), (q_\ell, \tilde{q}_\ell, r_\ell, \tilde{r}_\ell)_{\ell>\ell}, (t, \tilde{t}) \right),
\]

and

\[
Q_{\ell_0,+} = \left( (q_\ell, \tilde{q}_\ell, r_\ell, \tilde{r}_\ell)_{\ell=1,2,3}, (t + 1, \tilde{t}) \right),
\]

\[
Q_{\ell_0,-} = \left( (q_\ell, \tilde{q}_\ell, r_\ell, \tilde{r}_\ell)_{\ell=1,2,3}, (t, \tilde{t} + 1) \right).
\]

We define

\[
(b_{\ell,+(\xi_1)} \Psi)^{(Q)}(\ldots; \xi_1^{(1)}, \xi_1^{(2)}, \ldots, \xi_1^{(q_\ell)}; \ldots) = \sqrt{q_\ell + 1} \prod_{\ell' < \ell} (-1)^{\eta_{\ell'} + \tilde{q}_{\ell'}} \Psi^{(Q_{\ell',+})}(\ldots; \xi_1^{(1)}, \xi_1^{(2)}, \ldots, \xi_1^{(q_\ell)}; \ldots)
\]

\[
(b_{\ell,-(\xi_1)} \Psi)^{(Q)}(\ldots; \xi_1^{(1)}, \xi_1^{(2)}, \ldots, \xi_1^{(q_\ell)}; \ldots) = \sqrt{q_\ell + 1} \prod_{\ell' < \ell} (-1)^{\eta_{\ell'} + \tilde{q}_{\ell'}} \Psi^{(Q_{\ell',-})}(\ldots; \xi_1^{(1)}, \xi_1^{(2)}, \ldots, \xi_1^{(q_\ell)}; \ldots)
\]

\[
(c_{\ell,+(\xi_2)} \Psi)^{(Q)}(\ldots; \xi_2^{(1)}, \xi_2^{(2)}, \ldots, \xi_2^{(r_\ell)}; \ldots) = \sqrt{r_\ell + 1} \prod_{\ell' < \ell} (-1)^{\eta_{\ell'} + \tilde{q}_{\ell'}} \prod_{\ell' < \ell} (-1)^{\eta_{\ell'} + \tilde{r}_{\ell'}} \Psi^{(Q_{\ell',+})}(\ldots; \xi_2^{(1)}, \xi_2^{(2)}, \ldots, \xi_2^{(r_\ell)}; \ldots)
\]

\[
(c_{\ell,-(\xi_2)} \Psi)^{(Q)}(\ldots; \xi_2^{(1)}, \xi_2^{(2)}, \ldots, \xi_2^{(r_\ell)}; \ldots) = \sqrt{r_\ell + 1} \prod_{\ell' < \ell} (-1)^{\eta_{\ell'} + \tilde{q}_{\ell'}} \prod_{\ell' < \ell} (-1)^{\eta_{\ell'} + \tilde{r}_{\ell'}} \Psi^{(Q_{\ell',-})}(\ldots; \xi_2^{(1)}, \xi_2^{(2)}, \ldots, \xi_2^{(r_\ell)}; \ldots)
\]

and

\[
(a_{+(\xi_3)} \Psi)^{(Q)}(\ldots; \xi_3^{(1)}, \xi_3^{(2)}, \ldots, \xi_3^{(t_\ell)}; \ldots) = \sqrt{t_\ell + 1} \Psi^{(Q_{\ell_0,+})}(\ldots; \xi_3^{(1)}, \xi_3^{(2)}, \ldots, \xi_3^{(t_\ell)}; \ldots),
\]

\[
(a_{-(\xi_3)} \Psi)^{(Q)}(\ldots; \xi_3^{(1)}, \xi_3^{(2)}, \ldots, \xi_3^{(t_\ell)}; \ldots) = \sqrt{t_\ell + 1} \Psi^{(Q_{\ell_0,-})}(\ldots; \xi_3^{(1)}, \xi_3^{(2)}, \ldots, \xi_3^{(t_\ell)}; \ldots).
\]

As usual, \(b_{\ell,+(\xi_1)}\) (resp. \(c_{\ell,+(\xi_2)}\)) is the formal adjoint of \(b_{\ell,+(\xi_1)}\) (resp. \(c_{\ell,+(\xi_2)}\)). For example, we have

\[
(b_{\ell,+(\xi_1)} \Psi)^{(Q_{\ell_0,+})}(\ldots; \xi_1^{(1)}, \xi_1^{(2)}, \ldots, \xi_1^{(q_\ell+1)}; \ldots) = \frac{1}{\sqrt{q_\ell + 1}} \prod_{\ell' < \ell} (-1)^{\eta_{\ell'} + \tilde{q}_{\ell'}} ^{q_\ell+1} \sum_{i=1} (1)^{q_\ell+1} \delta(\xi_1 - \xi_1^{(i)}) \Psi^{(Q)}(\ldots; \xi_1^{(1)}, \xi_1^{(2)}, \ldots, \xi_1^{(q_\ell+1)}; \ldots),
\]
where $\widetilde{\gamma}$ denotes that the $i$-th variable has to be omitted, and $\delta(\xi_1 - \xi_3^{(i)}) = \delta_{s_1 s_1'} \delta(p_1 - p_1^{(i)})$. The operator $a_s^*(\xi_3)$ is the formal adjoint of $a_s(\xi_3)$ and we have

$$
(a_s^*(\xi_3)\Psi)^{(Q_0, +)}(\cdots, \xi_3^{(1)}, \xi_3^{(2)}, \cdots, \xi_3^{(r + 1)}, \cdots) = \frac{1}{\sqrt{t + 1}} \sum_{i=1}^{t+1} \delta(\xi_3 - \xi_3^{(i)})\Psi(Q)(\cdots, \xi_3^{(1)}, \cdots, \xi_3^{(r + 1)}, \cdots)
$$

where $\delta(\xi_3 - \xi_3^{(i)}) = \delta_{\lambda\lambda'(i)} \delta(k - k^{(i)})$.

The following canonical anticommutation and commutation relations hold.

$$
\{b_{\ell, \epsilon}(\xi_1), b_{\ell', \epsilon'}(\xi_1')\} = \delta_{\ell\ell'} \delta_{\epsilon\epsilon'} \delta(\xi_1 - \xi_1')
,$$

$$
\{a_{s}(\xi_3), a_{s'}^*(\xi_3')\} = \delta_{ss'} \delta(\xi_3 - \xi_3')
,$$

$$
\{b_{\ell, \epsilon}(\xi_1), c_{\ell', \epsilon'}(\xi_1')\} = \delta_{\ell\ell'} \delta_{\epsilon\epsilon'} \delta(\xi_1 - \xi_1')
$$

and

$$
\{a_{s}(\xi_3), a_{s'}(\xi_3')\} = 0
,$$

$$
\{b_{\ell, \epsilon}(\xi_1), a_{s'}(\xi_3)\} = 0
$$

are bounded operators in $\mathfrak{F}$ such that

$$
\|b_{\ell, \epsilon}^s(\varphi)\| = \|c_{\ell, \epsilon}^s(\varphi)\| = \|\varphi\|_{L^2}
$$

where $b^\epsilon$ (resp. $c^\epsilon$) is $b$ (resp. $c$) or $b^*$ (resp. $c^*$).

The operators $b_{\ell, \epsilon}^s(\varphi)$ and $c_{\ell, \epsilon}^s(\varphi)$ satisfy similar anticommutation relations (see e.g. [29]).

The free Hamiltonian $H_0$ is given by

$$
H_0 = H_0^{(1)} + H_0^{(2)} + H_0^{(3)}
$$

$$
= \sum_{\ell=1}^{3} \sum_{s=\pm} \int w_{\ell}^{(1)}(\xi_1)b_{\ell, s}(\xi_1)b_{\ell, s}(\xi_1) d\xi_1 + \sum_{\ell=1}^{3} \sum_{s=\pm} \int w_{\ell}^{(2)}(\xi_2)c_{\ell, s}(\xi_2)c_{\ell, s}(\xi_2) d\xi_2
$$

$$
+ \int w^{(3)}(\xi_3)a_{s}(\xi_3)a_{s}(\xi_3) d\xi_3
$$

where

$$
w_{\ell}^{(1)}(\xi_1) = (|p_1|^2 + m_1^2)^{\frac{\ell}{2}}, \quad \text{with} \quad 0 < m_1 < m_2 < m_3
,$$

$$
w_{\ell}^{(2)}(\xi_2) = |p_2|,
$$

$$
w^{(3)}(\xi_3) = (|k|^2 + m_W^2)^{\frac{3}{2}}
$$

where $m_W$ is the mass of the bosons $W^+$ and $W^-$ such that $m_W > m_3$. 
In what follows, we approximate the singular kernels by square integrable functions.

The kernels $G_{I,\ell,\epsilon,\epsilon'}$, computed in theoretical physics, contains a $\delta$-distribution because of the conservation of the momentum (see [18] [30, section 4.4]).

Because of the bare vacuum will not be an eigenvector of the total Hamiltonian for every $g > 0$ as we expect from the physics.

In what follows, we approximate the singular kernels by square integrable functions.

Thus, from now on, the kernels $G_{I,\ell,\epsilon,\epsilon'}$ are supposed to satisfy the following hypothesis .

**Hypothesis 2.1.** For $\alpha = 1, 2$, $\ell = 1, 2, 3$, $\epsilon, \epsilon' = \pm$, we assume

\[ G_{I,\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3) \in L^2(\Sigma_1 \times \Sigma_1 \times \Sigma_2) . \]

**Remark 2.2.** A similar model can be written down for the weak decay of pions $\pi^-$ and $\pi^+$ (see [18] section 6.2).\]

**Remark 2.3.** The total Hamiltonian is more general than the one involved in the theory of weak interaction because, in the Standard Model, neutrinos have helicity $-1/2$ and antineutrinos have helicity $1/2$.

In the physical case, the Fock space, denoted by $\mathfrak{F}$, is isomorphic to $\mathfrak{F}_L \otimes \mathfrak{F}_W$, with

\[ \mathfrak{F}_L = \bigotimes_{\ell=1}^3 \mathfrak{F}_\ell' , \]
and

$$\mathfrak{H}' = (\otimes_{\alpha}^2 L^2(\Sigma_1)) \otimes (\otimes_{\alpha}^2 L^2(\mathbb{R}^3)).$$

The free Hamiltonian, now denoted by \( H'_0 \), is then given by

$$H'_0 = \sum_{\ell = 1}^{3} \sum_{c = \pm} w^{(1)}_{\ell}(\xi_1)b^*_{\ell,c}(\xi_1)b_{\ell,c}(\xi_1)d\xi_1 + \sum_{\ell = 1}^{3} \sum_{\vartheta = \pm} \int_{\mathbb{R}^3} |p_2|c^*_{\ell,\vartheta}(p_2)c_{\ell,\vartheta}(p_2)dp_2$$

and

$$+ \sum_{\ell = 1}^{3} \int w^{(3)}_{\ell}(\xi_3)a^*_{\ell}(\xi_3)a_{\ell}(\xi_3)d\xi_3,$$

and the interaction, now denoted by \( H' \), is the one obtained from \( H_1 \) by supposing that \( G^{(1)}(\xi_1, (p_2, s_2), \xi_3) = 0 \) if \( s_2 = \frac{1}{2} \). The total Hamiltonian, denoted by \( H' \), is then given by \( H' = H'_0 + g H' \). The results obtained in this paper for \( H \) hold true for \( H' \) with obvious modifications.

Under Hypothesis [24], a well defined operator on \( D \) corresponds to the formal interaction \( H_1 \) as it follows.

The formal operator

$$\int G^{(1)}_{\ell,\ell', c}(\xi_1, \xi_2, \xi_3)b^*_{\ell, c}(\xi_1)c^*_{\ell', c}(\xi_2)a_{\ell}(\xi_3)d\xi_1 d\xi_2 d\xi_3$$

is defined as a quadratic form on \((D_\ell \otimes D_W) \times (D_\ell \otimes D_W)\) as

$$\int (c_{\ell, c}(\xi_2)b_{\ell, c}(\xi_1)\psi, G^{(1)}_{\ell,\ell', c}a_{\ell}(\xi_3)\phi)d\xi_1 d\xi_2 d\xi_3,$$

where \( \psi, \phi \in D_\ell \otimes D_W \).

By mimicking the proof of [24, Theorem X.44], we get a closed operator, denoted by \( H^{(1)}_{1,\ell,\ell', c} \), associated with the quadratic form such that it is the unique operator in \( \mathfrak{H}_\ell \otimes \mathfrak{H}_W \) such that \( D_\ell \otimes D_W \subset D(H^{(1)}_{1,\ell,\ell', c}) \) is a core for \( H^{(1)}_{1,\ell,\ell', c} \) and

$$H^{(1)}_{1,\ell,\ell', c} = \int G^{(1)}_{\ell,\ell', c}(\xi_1, \xi_2, \xi_3)b^*_{\ell, c}(\xi_1)c^*_{\ell', c}(\xi_2)a_{\ell}(\xi_3)d\xi_1 d\xi_2 d\xi_3$$

as quadratic forms on \((D_\ell \otimes D_W) \times (D_\ell \otimes D_W)\).

The formal operator

$$\int G^{(1)}_{\ell,\ell', c}(\xi_1, \xi_2, \xi_3)c_{\ell, c}(\xi_2)b^*_{\ell, c}(\xi_1)a^*_{\ell}(\xi_3)d\xi_1 d\xi_2 d\xi_3$$

is similarly associated with \((H^{(1)}_{1,\ell,\ell', c})^*\) and

$$(H^{(1)}_{1,\ell,\ell', c})^* = \int G^{(1)}_{\ell,\ell', c}(\xi_1, \xi_2, \xi_3)c_{\ell, c}(\xi_2)b^*_{\ell, c}(\xi_1)a^*_{\ell}(\xi_3)d\xi_1 d\xi_2 d\xi_3$$

as quadratic forms on \((D_\ell \otimes D_W) \times (D_\ell \otimes D_W)\). Moreover, \( D_\ell \otimes D_W \subset D((H^{(1)}_{1,\ell,\ell', c})^*) \) is a core for \((H^{(1)}_{1,\ell,\ell', c})^*\).

Again, there exists two closed operators \( H^{(2)}_{1,\ell,\ell', c} \) and \((H^{(2)}_{1,\ell,\ell', c})^*\) such that \( D_\ell \otimes D_W \subset D(H^{(2)}_{1,\ell,\ell', c}) \), \( D_\ell \otimes D_W \subset D((H^{(2)}_{1,\ell,\ell', c})^*) \) and \( D_\ell \otimes D_W \) is a core for \( H^{(2)}_{1,\ell,\ell', c} \) and \((H^{(2)}_{1,\ell,\ell', c})^*\) such that

$$H^{(2)}_{1,\ell,\ell', c} = \int G^{(2)}_{\ell,\ell', c}(\xi_1, \xi_2, \xi_3)b^*_{\ell, c}(\xi_1)c_{\ell', c}(\xi_2)a^*_{\ell}(\xi_3)d\xi_1 d\xi_2 d\xi_3.$$
\[(H_{1,ℓ,ε,ε'}^{(2)})^* = \int G_{1,ℓ,ε,ε'}^{(2)}(ξ_1, ξ_2, ξ_3) a_ε(ξ_3) c_{ε'}(ξ_2) b_{ℓ,ε}(ξ_1) dξ_1 dξ_2 dξ_3\]

as quadratic forms on \(\mathcal{D}_L \times \mathcal{D}_W\).

We shall still denote \(H_{I,ℓ,ε,ε'}^{(α)}\) and \((H_{I,ℓ,ε,ε'}^{(α)})^*\) (\(α = 1, 2\)) their extensions to \(\mathfrak{F}\).

The set \(\mathcal{D}\) is then a core for \(H_{I,ℓ,ε,ε'}^{(α)}\) and \((H_{I,ℓ,ε,ε'}^{(α)})^*\)

Thus
\[H = H_0 + g \sum_{α=1,2,}\sum_{ℓ=1, ε ≠ ε'}(H_{I,ℓ,ε,ε'}^{(α)}) + (H_{I,ℓ,ε,ε'}^{(2)})^*\]

is a symmetric operator defined on \(\mathcal{D}\).

We now want to prove that \(H\) is essentially self-adjoint on \(\mathcal{D}\) by showing that \(H_{I,ℓ,ε,ε'}^{(α)}\) and \((H_{I,ℓ,ε,ε'}^{(α)})^*\) are relatively \(H_0\)-bounded.

Once again, as above, for almost every \(ξ_3 ∈ Σ_2\), there exists closed operators in \(\mathfrak{F}_L\), denoted by \(B_{I,ℓ,ε,ε'}^{(α)}(ξ_3)\) and \((B_{I,ℓ,ε,ε'}^{(α)}(ξ_3))^*\) such that
\[B_{I,ℓ,ε,ε'}^{(1)}(ξ_3) = - \int G_{I,ℓ,ε,ε'}^{(1)}(ξ_1, ξ_2, ξ_3) b_{ℓ,ε}(ξ_1) c_{ε'}(ξ_2) dξ_1 dξ_2 ,\]
\[(B_{I,ℓ,ε,ε'}^{(1)}(ξ_3))^* = \int G_{I,ℓ,ε,ε'}^{(1)}(ξ_1, ξ_2, ξ_3) b_{ℓ,ε}(ξ_1) c_{ε'}^{*}(ξ_2) dξ_1 dξ_2 ,\]
\[B_{I,ℓ,ε,ε'}^{(2)}(ξ_3) = \int G_{I,ℓ,ε,ε'}^{(2)}(ξ_1, ξ_2, ξ_3) b_{ℓ,ε}^{*}(ξ_1) c_{ε'}^{*}(ξ_2) dξ_1 dξ_2,\]
\[(B_{I,ℓ,ε,ε'}^{(2)}(ξ_3))^* = - \int G_{I,ℓ,ε,ε'}^{(2)}(ξ_1, ξ_2, ξ_3) b_{ℓ,ε}^{*}(ξ_1) c_{ε'}^{*}(ξ_2) dξ_1 dξ_2\]

as quadratic forms on \(\mathcal{D}_L \times \mathcal{D}_L\).

We have that \(\mathcal{D}_L \subset \mathcal{D}(B_{I,ℓ,ε,ε'}^{(1)}(ξ_3))\) (resp. \(\mathcal{D}_L \subset \mathcal{D}(B_{I,ℓ,ε,ε'}^{(1)}(ξ_3))^*\) is a core for \(B_{I,ℓ,ε,ε'}^{(1)}(ξ_3)\) (resp. \((B_{I,ℓ,ε,ε'}^{(1)}(ξ_3))^*\)). We still denote by \(B_{I,ℓ,ε,ε'}^{(α)}(ξ_3)\) and \((B_{I,ℓ,ε,ε'}^{(α)}(ξ_3))^*\) their extensions to \(\mathfrak{F}_L\).

It then follows that the operator \(H_I\) with domain \(\mathcal{D}\) is symmetric and can be written in the following form
\[H_I = \sum_{α=1,2,}\sum_{ℓ=1, ε ≠ ε'}(H_{I,ℓ,ε,ε'}^{(α)}) + (H_{I,ℓ,ε,ε'}^{(1)})^*\]

\[= \sum_{α=1,2,}\sum_{ℓ=1, ε ≠ ε'}\int B_{I,ℓ,ε,ε'}^{(α)}(ξ_3) a_ε(ξ_3) dξ_3 + \sum_{α=1,2,}\sum_{ℓ=1, ε ≠ ε'}\sum_{ξ_3=α}(B_{I,ℓ,ε,ε'}^{(α)}(ξ_3))^* a_ε(ξ_3) dξ_3 .\]

Let \(N_ℓ\) denote the operator number of massive leptons \(ℓ\) in \(\mathfrak{F}_L\), i.e.,
\[N_ℓ = \sum_ε \int b_{ℓ,ε}^{*}(ξ_1) b_{ℓ,ε}(ξ_1) dξ_1 .\]

The operator \(N_ℓ\) is a positive self-adjoint operator in \(\mathfrak{F}_L\). We still denote by \(N_ℓ\) its extension to \(\mathfrak{F}_L\). The set \(\mathcal{D}_L\) is a core for \(N_ℓ\).

Then we have

**Proposition 2.4.** For a.e. \(ξ_3 ∈ Σ_2\), \(\mathcal{D}(B_{I,ε,ε'}^{(α)}(ξ_3))\), \(\mathcal{D}((B_{I,ε,ε'}^{(α)}(ξ_3))^*) \supset \mathcal{D}(N_ℓ^\mathfrak{F})\), and for \(Φ \in \mathcal{D}(N_ℓ^\mathfrak{F}) \subset \mathfrak{F}_L\) we have
\[\|B_{I,ε,ε'}^{(α)}(ξ_3)Φ\|_{\mathfrak{F}_L} \leq \|G_{I,ε,ε'}^{(α)}(\cdot, ξ_3)\|_{L^2(Σ_1 × Σ_1)} \cdot \|N_ℓ^{\mathfrak{F}} Φ\|_{\mathfrak{F}_L} .\]
(2.17) \[ \| (B^{(\alpha)}_{1, \ldots, \ell, -}(\xi))^{*} \Phi \|_{\tilde{F}_{L}} \leq \| G^{(\alpha)}_{1, \ldots, \ell} (\ldots, \xi) \|_{L^{2}(\Sigma_{1} \times \Sigma_{1})} \| N^{\frac{1}{4}}_{1} \Phi \|_{\tilde{F}_{L}}. \]

Proof. The estimates (2.10) and (2.17) are examples of $N_{r}$ estimates (see [16]). We give a proof for sake of completeness. We only consider $B^{(1)}_{1, +, -}$. The other cases are quite similar.

Let $\Phi = (\Phi^{(Q)})_{Q}$ and $\Psi = (\Psi^{(Q')})_{Q'}$ be two vectors in $D_{L}$, where we use the notations $Q = (q_{\ell}, \bar{q}_{\ell}, r_{\ell}, \bar{r}_{\ell})_{\ell = 1, 2, 3}$, and $Q' = (q'_{\ell}, \bar{q}'_{\ell}, r'_{\ell}, \bar{r}'_{\ell})_{\ell = 1, 2, 3}$. We have

(2.18) \[
(\Psi^{(Q')}, B^{(1)}_{1, +, -}(\xi)\Phi^{(Q)})_{\tilde{F}_{L}} = -\delta_{q_{1}} - \delta_{\bar{q}_{1}} - \delta_{r_{1}} - \delta_{\bar{r}_{1}} + \sum_{\ell = 2} \delta_{q_{\ell}} - \delta_{q'_{\ell}} - \delta_{r_{\ell}} - \delta_{r'_{\ell}} \int_{\Sigma_{1} \times \Sigma_{1}} (\Psi^{(Q')}, b_{1, +}(\xi)c_{1, -}(\xi)\Phi^{(Q)})_{\tilde{F}_{L}} G^{(1)}_{1, +, -}(\xi, \xi, \xi, \xi) d\xi_{1} d\xi_{2}.
\]

Here $Q = (q_{1} - 1, \bar{q}_{1}, r_{1} - 1, q_{2}, \bar{q}_{2}, r_{2}, \bar{r}_{2}, q_{3}, \bar{q}_{3}, r_{3}, \bar{r}_{3})$.

For each $Q$,

(2.19) \[
B^{(1)}_{1, +, -}(\xi)\Phi^{(Q)} \in \tilde{F}_{L}^{(q_{1} - 1, \bar{q}_{1}, r_{1} - 1)} \otimes \tilde{F}_{L}^{(q_{2}, \bar{q}_{2}, r_{2}, \bar{r}_{2})} \otimes \tilde{F}_{L}^{(q_{3}, \bar{q}_{3}, r_{3}, \bar{r}_{3})}.
\]

By the Fubini theorem we have

(2.20) \[
\left| (\Psi^{(Q)}, B^{(1)}_{1, +, -}(\xi)\Phi^{(Q)})_{\tilde{F}_{L}} \right| = \left| \int_{\Sigma_{1}} \left( \int_{\Sigma_{1}} G^{(1)}_{1, +, -}(\xi, \xi, \xi) c_{1, -}(\xi)\Phi^{(Q)} d\xi_{2}, b_{1, +}(\xi)\Phi^{(Q)} \right) d\xi_{1} \right|.
\]

By (2.9), and the Cauchy-Schwarz inequality we get

(2.21) \[
\left| (\Psi^{(Q)}, B^{(1)}_{1, +, -}(\xi)\Phi^{(Q)})_{\tilde{F}_{L}} \right|^{2} \leq \left( \int_{\Sigma_{1}} \| b_{1, +}(\xi)\Phi^{(Q)} \|^{2} \left( \int_{\Sigma_{1}} |G^{(1)}_{1, +, -}(\xi, \xi, \xi)|^{2} d\xi_{1} d\xi_{2} \right)^{\frac{1}{2}} d\xi_{1} \right)^{2} \| \Psi^{(Q)} \|^{2}.
\]

By the definition of $b_{1, +}(\xi)\Phi^{(Q)}$ and the Cauchy-Schwarz inequality we get

(2.22) \[
\left| (\Psi^{(Q)}, B^{(1)}_{1, +, -}(\xi)\Phi^{(Q)})_{\tilde{F}_{L}} \right|^{2} \leq q_{1} \left( \int_{\Sigma_{1}} \int_{\Sigma_{1}} |G^{(1)}_{1, +, -}(\xi, \xi, \xi)|^{2} d\xi_{1} d\xi_{2} \right) \| \Psi^{(Q)} \|^{2} \| \Phi^{(Q)} \|^{2} \| N^{\frac{1}{4}}_{1} \Phi^{(Q)} \|^{2} \| N^{\frac{1}{4}}_{1} \Phi^{(Q)} \|^{2}.
\]

By (2.19) we have

(2.23) \[
\| (\Psi, B^{(1)}_{1, +, -}(\xi)\Phi^{(Q)})_{\tilde{F}_{L}} \|^{2} \leq \| \Psi \|^{2} \| N^{\frac{1}{4}}_{1} \Phi^{(Q)} \|^{2} \int_{\Sigma_{1} \times \Sigma_{1}} |G^{(1)}_{1, +, -}(\xi, \xi, \xi)|^{2} d\xi_{1} d\xi_{2},
\]

for every $\Psi \in D_{L}$. Therefore we get

(2.24) \[
\| B^{(1)}_{1, +, -}(\xi)\Phi^{(Q)} \|^{2} \leq \left( \int_{\Sigma_{1} \times \Sigma_{1}} |G^{(1)}_{1, +, -}(\xi, \xi, \xi)|^{2} d\xi_{1} d\xi_{2} \right) \| N^{\frac{1}{4}}_{1} \Phi^{(Q)} \|^{2} \| N^{\frac{1}{4}}_{1} \Phi^{(Q)} \|^{2},
\]

and by (2.19) we finally obtain

(2.25) \[
\| B^{(1)}_{1, +, -}(\xi)\Phi^{(Q)} \|^{2} \leq \left( \int_{\Sigma_{1} \times \Sigma_{1}} |G^{(1)}_{1, +, -}(\xi, \xi, \xi)|^{2} d\xi_{1} d\xi_{2} \right) \| N^{\frac{1}{4}}_{1} \Phi^{(Q)} \|^{2} \| N^{\frac{1}{4}}_{1} \Phi^{(Q)} \|^{2}.
\]
for every $\Phi \in \mathcal{D}$.

Since $\mathcal{D}_L$ is a core for $N_{1}^\frac{1}{2}$ and $B_{1,+,-}^{(1)}$ with domain $\mathcal{D}_L$ is closable, $\mathcal{D}(B_{1,+,-}^{(1)}(\xi_3)) \supset \mathcal{D}(N_{1}^\frac{1}{2})$, and (2.10) is satisfied for every $\Phi \in \mathcal{D}(N_{1}^\frac{1}{2})$.

Let

$$H_{0,\epsilon}^{(3)} = \int w^{(3)}(\xi_3) a_\epsilon^*(\xi_3) a_\epsilon(\xi_3) d\xi_3.$$  

Then $H_{0,\epsilon}^{(3)}$ is a self-adjoint operator in $\mathcal{F}_W$, and $\mathcal{D}_W$ is a core for $H_{0,\epsilon}^{(3)}$.

We get

**Proposition 2.5.**

\begin{equation}
\| \int (B_{1,+,-}^{(a)}(\xi_3))^* \otimes a_\epsilon(\xi_3) d\xi_3 \Psi \|^2 \\
\leq \left( \int \frac{|G_{1,+,-}^{(a)}(\xi_3)|^2}{w^{(3)}(\xi_3)} d\xi_1 d\xi_2 d\xi_3 \right) \| (N_\epsilon + 1) \frac{1}{2} \otimes (H_{0,\epsilon}^{(3)})^{\frac{1}{2}} \Psi \|^2 \\
+ \left( \int \frac{|G_{1,+,-}^{(a)}(\xi_3)|^2}{w^{(3)}(\xi_3)} d\xi_1 d\xi_2 d\xi_3 \right) (\eta \| (N_\epsilon + 1) \frac{1}{2} \otimes 1 \Psi \|^2 + \frac{1}{4\eta} \| \Psi \|^2),
\end{equation}

for every $\Psi \in \mathcal{D}(H_0)$ and every $\eta > 0$.

**Proof.** Suppose that $\Psi \in \mathcal{D}(N_{1}^\frac{1}{2}) \otimes \mathcal{D}((H_{0,\epsilon}^{(3)})^\frac{1}{2})$. Let

$$\Psi_\epsilon(\xi_3) = w^{(3)}(\xi_3) \frac{1}{2} ((N_\epsilon + 1) \frac{1}{2} \otimes a_\epsilon(\xi_3)) \Phi.$$  

We have

$$\int_{\Sigma_3} \| \Psi_\epsilon(\xi_3) \|^2 d\xi_3 = \| (N_\epsilon + 1) \frac{1}{2} \otimes (H_{0,\epsilon}^{(3)})^{\frac{1}{2}} \Psi \|^2.$$  

We get

$$\int (B_{1,+,-}^{(a)}(\xi_3))^* \otimes a_\epsilon(\xi_3) d\xi_3 \Psi = \int_{\Sigma_3} \left( \frac{1}{w^{(3)}(\xi_3)} \right)^{\frac{1}{2}} ((B_{1,+,-}^{(a)}(\xi_3))^* (N_\epsilon + 1)^{-\frac{1}{2}} \otimes 1) \Psi_\epsilon(\xi_3) d\xi_3.$$  

Therefore

$$\| \int (B_{1,+,-}^{(a)}(\xi_3))^* \otimes a_\epsilon(\xi_3) \Psi d\xi_3 \|^2$$

\begin{equation}
\leq \left( \int_{\Sigma_3} \frac{1}{w^{(3)}(\xi_3)} \| ((B_{1,+,-}^{(a)}(\xi_3))^* (N_\epsilon + 1)^{-\frac{1}{2}} \otimes 1) \Psi_\epsilon(\xi_3) \|^2 d\xi_3 \right)^2
\end{equation}

\begin{equation}
\leq \left( \int_{\Sigma_3} \frac{|G_{1,+,-}^{(a)}(\xi_2,\xi_3,\xi_3)|^2}{w^{(3)}(\xi_3)} d\xi_1 d\xi_2 d\xi_3 \right) \| (N_\epsilon + 1) \frac{1}{2} \otimes (H_{0,\epsilon}^{(3)})^{\frac{1}{2}} \Psi \|^2, 
\end{equation}

as it follows from Proposition 2.4.
We now have
\[
\| \int B_{\ell, c, e}^{(\alpha)}(\xi_3) \otimes a_e^*(\xi_3) \Psi \, d\xi_3 \|_\mathcal{F}^2
\]
\[
= \int (B_{\ell, c, e}^{(\alpha)}(\xi_3) \otimes a_e(\xi_3) \Psi, B_{\ell, c, e}^{(\alpha)}(\xi_3) \otimes a_e(\xi_3) \Psi) \, d\xi_3 \, d\xi_3' + \int \|(B_{\ell, c, e}^{(\alpha)}(\xi_3), 1)\Psi\|^2 \, d\xi_3,
\]
and
\[
\int_{\Sigma_2 \times \Sigma_2} (B_{\ell, c, e}^{(\alpha)}(\xi_3) \otimes a_e(\xi_3) \Psi, B_{\ell, c, e}^{(\alpha)}(\xi_3) \otimes a_e(\xi_3) \Psi) \, d\xi_3 \, d\xi_3'
\]
\[
= \int_{\Sigma_2 \times \Sigma_2} \frac{1}{w^3(\xi_3) \frac{1}{2} w^3(\xi_3') \frac{1}{2}} \left( (B_{\ell, c, e}^{(\alpha)}(\xi_3)(N_\ell + 1)^{-\frac{1}{2}} \otimes 1) \Psi_e(\xi_3),
\]
(2.23)
\[
(B_{\ell, c, e}^{(\alpha)}(\xi_3)(N_\ell + 1)^{-\frac{1}{2}} \otimes 1) \Psi_e(\xi_3) \right) \, d\xi_3 \, d\xi_3'
\]
\[\leq \left( \int_{\Sigma_2} \frac{1}{w^3(\xi_3) \frac{1}{2}} \|B_{\ell, c, e}^{(\alpha)}(\xi_3)(N_\ell + 1)^{-\frac{1}{2}} \otimes 1\| \Psi_e(\xi_3) \right) \|d\xi_3)^2
\]
\[\leq \left( \int_{\Sigma_1 \times \Sigma_1 \times \Sigma_2} |G_e^{(\alpha)}(\xi_1, \xi_2, \xi_3)^2 \frac{1}{w^3(\xi_3)} \, d\xi_1 \, d\xi_2 \, d\xi_3| \, ((N_\ell + 1)^{-\frac{1}{2}} \otimes (H_{0, \ell}^{(3)})) \Psi \|^2 .
\]
Furthermore (2.24)
\[
\int_{\Sigma_2} \|B_{\ell, c, e}^{(\alpha)}(\xi_3) \otimes 1\| \Psi \|^2 \, d\xi_3
\]
\[= \int_{\Sigma_2} \|B_{\ell, c, e}^{(\alpha)}(\xi_3)(N_\ell + 1)^{-\frac{1}{2}} \otimes 1\| ((N_\ell + 1)^{-\frac{1}{2}} \otimes 1) \Psi \|^2 \, d\xi_3
\]
\[\leq \left( \int_{\Sigma_1 \times \Sigma_1 \times \Sigma_2} |G_e^{(\alpha)}(\xi_1, \xi_2, \xi_3)^2 \frac{1}{w^3(\xi_3)} \, d\xi_1 \, d\xi_2 \, d\xi_3| \right) \| (N_\ell + 1) \Psi \|^2 + \frac{1}{4\eta} \| \Psi \|^2 ,
\]
for every $\eta > 0$.

By (2.22), (2.23), and (2.24), we finally get (2.20) and (2.21) for every $\Psi \in \mathcal{D}(N_{\ell}^{(3)} \otimes D(H_{0, e}^{(3)}))$. The set $\mathcal{D}(N_{\ell}^{(3)} \otimes D(H_{0, e}^{(3)}))$ is a core for $N_{\ell}^{(3)} \otimes H_{0, e}^{(3)}$, and $D(H_0) \subset \mathcal{D}(N_{\ell}^{(3)} \otimes H_{0, e}^{(3)})$. It then follows that (2.20) and (2.21) are verified for every $\Psi \in \mathcal{D}(H_0)$.

We now prove that $H$ is a self-adjoint operator in $\mathcal{F}$ for $g$ sufficiently small.

**Theorem 2.6.** Let $g_1 > 0$ be such that
\[
\frac{3g_1^2}{m_W} \left( \frac{1}{m_1^2} + 1 \right) \sum_{\alpha=1,2} \sum_{\ell=1}^3 \sum_{e \neq e'} \|G_{\ell, c, e'}^{(\alpha)}\|_{L^2(\Sigma_1 \times \Sigma_1 \times \Sigma_2)}^2 < 1 .
\]
Then for every $g$ satisfying $g \leq g_1$, $H$ is a self-adjoint operator in $\mathcal{F}$ with domain $\mathcal{D}(H) = \mathcal{D}(H_0)$, and $\mathcal{D}$ is a core for $H$.

**Proof.** Let $\Psi$ be in $\mathcal{D}$. We have
\[
\|H \Psi \|^2 \leq 12 \sum_{\alpha=1,2} \sum_{\ell=1}^3 \sum_{e \neq e'} \left\{ \left\| \int (B_{\ell, c, e}^{(\alpha)}(\xi_3))^* \otimes a_e(\xi_3) \Psi \, d\xi_3 \right\|^2
\]
\[
+ \left\| \int (B_{\ell, c, e}^{(\alpha)}(\xi_3)) \otimes a_e^*(\xi_3) \Psi \, d\xi_3 \right\|^2 \right\} ,
\]
(2.25)
Note that
\[ \|H^{(3)}_{0,\epsilon,\epsilon'}\| \leq \|H^{(3)}_{0,\epsilon}\| \leq \|H_{0}\|, \]
and
\[ \|N_\ell \psi\| \leq \frac{1}{m_\ell} \|H_{0,\ell}\psi\| \leq \frac{1}{m_1} \|H_{0,\ell}\psi\| \leq \frac{1}{m_1} \|H_{0}\psi\|, \]
where
\[ (2.26) \quad H_{0,\ell} = \sum_\epsilon \int w^{(1)}_\ell (\xi_1)b^{(1)}_{\epsilon,\epsilon}(\xi_1)b_{\epsilon,\epsilon}(\xi_1)d\xi_1 + \sum_\epsilon \int w^{(2)}_\ell (\xi_2)c^{*}_{\epsilon,\epsilon}(\xi_2)c_{\epsilon,\epsilon}(\xi_2)d\xi_2. \]

We further note that
\[ (2.27) \quad \| (N_\ell + 1)^{\frac{1}{2}} \otimes (H^{(3)}_{0,\epsilon,\epsilon'}) \psi \|^2 \leq \frac{1}{2} \left( \frac{1}{2} m_1 + 1 \right) \|H_0\psi\|^2 + \frac{\beta}{2 m_1^2} \|H_0\psi\|^2 + \left( \frac{1}{2} + \frac{1}{8}\beta \right) \|\psi\|^2, \]
for \( \beta > 0 \), and
\[ (2.28) \quad \eta \|(N_\ell + 1)\otimes1\|\psi\|^2 + \frac{1}{4\eta} \|\psi\|^2 \leq \frac{\eta}{m_1} \|H_0\psi\|^2 + \frac{\eta\beta}{m_1^2} \|H_0\psi\|^2 + \eta (1 + \frac{1}{4\beta}) \|\psi\|^2 + \frac{1}{4\eta} \|\psi\|^2. \]

Combining (2.25) with (2.20), (2.21), (2.27) and (2.28) we get for \( \eta > 0, \beta > 0 \)
\[ (2.29) \quad \|H_{1}\psi\|^2 \leq 6 \sum_{\alpha=1,2} \sum_{\ell=1}^3 \sum_{\epsilon \neq \epsilon'} \|G_{\ell,\epsilon,\epsilon'}^{(\alpha)}\|^2 \]
\[ \left( \frac{1}{2 m_W} \left( \frac{1}{m_1} + 1 \right) \|H_0\psi\|^2 + \frac{\beta}{2 m_W m_1^2} \|H_0\psi\|^2 + \frac{1}{2 m_W} \left( 1 + \frac{1}{4\beta} \right) \|\psi\|^2 \right) \]
\[ + 12 \sum_{\alpha=1,2} \sum_{\ell=1}^3 \sum_{\epsilon \neq \epsilon'} \|G_{\ell,\epsilon,\epsilon'}^{(\alpha)}\|^2 \left( \frac{\eta}{m_1} (1 + \beta) \|H_0\psi\|^2 + \eta (1 + \frac{1}{4\beta}) + \frac{1}{4\eta} \right) \|\psi\|^2, \]
by noting
\[ (2.30) \quad \int_{\Sigma_1 \times \Sigma_1 \times \Sigma_2} \frac{|G_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_1,\xi_2,\xi_3)|^2}{w^{(3)}(\xi_3)} d\xi_1 d\xi_2 d\xi_3 \leq \frac{1}{m_W} \|G_{\ell,\epsilon,\epsilon'}^{(\alpha)}\|^2. \]

By (2.29) the theorem follows from the Kato-Rellich theorem. \( \square \)

3. MAIN RESULTS

In the sequel, we shall make the following additional assumptions on the kernels \( G_{\ell,\epsilon,\epsilon'}^{(\alpha)} \).

**Hypothesis 3.1.**
(i) For \( \alpha = 1,2, \ell = 1,2,3, \epsilon, \epsilon' = \pm, \)
\[ \int_{\Sigma_1 \times \Sigma_1 \times \Sigma_2} \frac{|G_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_1,\xi_2,\xi_3)|^2}{|p_2|^2} d\xi_1 d\xi_2 d\xi_3 < \infty, \]
(ii) There exists \( C > 0 \) such that for \( \alpha = 1,2, \ell = 1,2,3, \epsilon, \epsilon' = \pm, \)
\[ \left( \int_{\Sigma_1 \times \{|p_2| \leq \sigma\} \times \Sigma_2} |G_{\ell,\epsilon,\epsilon'}^{(\alpha)}(\xi_1,\xi_2,\xi_3)|^2 d\xi_1 d\xi_2 d\xi_3 \right)^\frac{1}{2} \leq C \sigma^2. \]
(iii) For $\alpha = 1, 2, \ell = 1, 2, 3, \epsilon, \epsilon' = \pm$, and $i, j = 1, 2, 3$

\begin{align*}
(iii.a) & \int_{\Sigma_1 \times \Sigma_1 \times \Sigma_2} \left| (p_2 \cdot \nabla p_2) G_{\ell, \epsilon, \epsilon'}^{(a)}(\xi_1, \xi_2, \xi_3) \right|^2 d\xi_1 d\xi_2 d\xi_3 < \infty, \\
and
(iii.b) & \int_{\Sigma_1 \times \Sigma_1 \times \Sigma_2} p_{2,i}^2 p_{2,j}^2 \left| \frac{\partial^2 G_{\ell, \epsilon, \epsilon'}^{(a)}}{\partial p_{2,i} \partial p_{2,j}}(\xi_1, \xi_2, \xi_3) \right|^2 d\xi_1 d\xi_2 d\xi_3 < \infty.
\end{align*}

(iv) There exists $\Lambda > m_1$ such, that for $\alpha = 1, 2, \ell = 1, 2, 3, \epsilon, \epsilon' = \pm$,

\[ G_{\ell, \epsilon, \epsilon'}^{(a)}(\xi_1, \xi_2, \xi_3) = 0 \quad \text{if} \quad |p_2| \geq \Lambda. \]

Remark 3.2. Hypothesis (iii) (ii) is nothing but an infrared regularization of the kernels $G_{\ell, \epsilon, \epsilon'}^{(a)}$. In order to satisfy this hypothesis it is, for example, sufficient to suppose

\[ G_{\ell, \epsilon, \epsilon'}^{(a)}(\xi_1, \xi_2, \xi_3) = |p_2|^2 G_{\ell, \epsilon, \epsilon'}(\xi_1, \xi_2, \xi_3), \]

where $G_{\ell, \epsilon, \epsilon'}$ is a smooth function of $(p_1, p_2, p_3)$ in the Schwartz space.

The Hypothesis (iii) (iv), which is a sharp ultraviolet cutoff, is actually not necessary, and can be removed at the expense of some additional technicalities in Appendix A. However, in order to simplify the proof of Proposition 3.5 we shall leave it.

Our first result is devoted to the existence of a ground state for $H$ together with the location of the spectrum of $H$ and of its absolutely continuous spectrum when $g$ is sufficiently small.

Theorem 3.3. Suppose that the kernels $G_{\ell, \epsilon, \epsilon'}^{(a)}$, satisfy Hypothesis (iii) and Hypothesis (iii) (i). Then there exists $0 < g_2 \leq g_1$ such that $H$ has a unique ground state for $g \leq g_2$. Moreover

\[ \sigma(H) = \sigma_{ac}(H) = [\inf \sigma(H), \infty), \]

with $\inf \sigma(H) \leq 0$.

According to Theorem 3.3 the ground state energy $E = \inf \sigma(H)$ is a simple eigenvalue of $H$ and our main results are concerned with a careful study of the spectrum of $H$ above the ground state energy. The spectral theory developed in this work is based on the conjugated operator method as described in [23, 2] and [25]. Our choice of the conjugate operator denoted by $A$ is the second quantized dilation generator for the neutrinos.

Let $a$ denote the following operator in $L^2(\Sigma_1)$

\[ a = \frac{1}{2} (p_2 \cdot i\nabla p_2 + i\nabla p_2 \cdot p_2). \]

The operator $a$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$. Its second quantized version $d\Gamma(a)$ is a self-adjoint operator in \( \mathfrak{F}_a(L^2(\Sigma_1)) \). From the definition (2.4) of the space \( \mathfrak{F}_e \), the following operator in \( \mathfrak{F}_L \)

\[ A_L = 1 \otimes 1 \otimes d\Gamma(a) \otimes 1 + 1 \otimes 1 \otimes 1 \otimes d\Gamma(a) \]

is essentially self-adjoint on \( \mathfrak{D}_L \).

Let now $A$ be the following operator in \( \mathfrak{F}_L \)

\[ A = A_1 \otimes 1_2 \otimes 1_3 + 1_1 \otimes A_2 \otimes 1_3 + 1_1 \otimes 1_2 \otimes A_3. \]
Then $A$ is essentially self-adjoint on $\mathfrak{D}_L$.

We shall denote again by $A$ its extension to $\mathfrak{F}$. Thus $A$ is essentially self-adjoint on $\mathfrak{D}$ and we still denote by $A$ its closure.

We also set
\[
(A) = (1 + A^2)^{\frac{1}{2}}.
\]

We then have

**Theorem 3.4.** Suppose that the kernels $G_{t,\epsilon,\epsilon'}^{(\alpha)}$, satisfy Hypothesis 2.1 and 3.1. For any $\delta > 0$ satisfying $0 < \delta < m_1$ there exists $0 < g_\delta \leq g_2$ such that, for $0 < g \leq g_\delta$,

(i) The spectrum of $H$ in $(\inf \sigma(H), m_1 - \delta)$ is purely absolutely continuous.

(ii) Limiting absorption principle.

For every $s > 1/2$ and $\varphi, \psi$ in $\mathfrak{F}$, the limits
\[
\lim_{\varepsilon \to 0} (\varphi, (A)^{-s} (H - \lambda \pm i\varepsilon) (A)^{-s} \psi)
\]
exist uniformly for $\lambda$ in any compact subset of $(\inf \sigma(H), m_1 - \delta)$.

(iii) Pointwise decay in time.

Suppose $s \in (\frac{1}{2}, 1)$ and $f \in C_0^\infty(\mathbb{R})$ with $\text{supp} f \subset (\inf \sigma(H), m_1 - \delta)$. Then
\[
|| (A)^{-s} e^{-itH} f(H) (A)^{-s} || = O(t^{\frac{1}{2} - s}),
\]
as $t \to \infty$.

The proof of Theorem 3.4 is based on a positive commutator estimate, called the Mourre estimate and on a regularity property of $H$ with respect to $A$ (see [23], [3] and [25]). According to [13], the main ingredient of the proof are auxiliary operators associated with infrared cutoff Hamiltonians with respect to the momenta of the neutrinos that we now introduce.

Let $\chi_0(\cdot), \chi_\infty(\cdot) \in C^\infty(\mathbb{R}, [0, 1])$ with $\chi_0 = 1$ on $(-\infty, 1]$, $\chi_\infty = 1$ on $[2, \infty)$ and $\chi_0^2 + \chi_\infty^2 = 1$.

For $\sigma > 0$ we set
\[
\begin{align*}
\chi_\sigma(p) &= \chi_0(|p|/\sigma), \\
\chi^\sigma(p) &= \chi_\infty(|p|/\sigma), \\
\tilde{\chi}^\sigma(p) &= 1 - \chi_\sigma(p),
\end{align*}
\]
where $p \in \mathbb{R}^3$.

The operator $H_{1,\sigma}$ is the interaction given by (2.10), (2.11) and (2.12) and associated with the kernels $\tilde{\chi}^\sigma(p_2) G_{t,\epsilon,\epsilon'}^{(\alpha)}(\xi_1, \xi_2, \xi_3)$. We then set
\[
H_\sigma := H_0 + gH_{1,\sigma}.
\]

Let
\[
\begin{align*}
\Sigma_{1,\sigma} &= \Sigma_1 \cap \{(p_2, s_2); |p_2| < \sigma\}, \\
\Sigma_{1}^\sigma &= \Sigma_1 \cap \{(p_2, s_2); |p_2| \geq \sigma\}, \\
\mathfrak{F}_{t,2,\sigma} &= \mathfrak{F}_a(L^2(\Sigma_1, \sigma)) \otimes \mathfrak{F}_a(L^2(\Sigma_{1,\sigma})), \\
\mathfrak{F}_{t,2} &= \mathfrak{F}_a(L^2(\Sigma_1)) \otimes \mathfrak{F}_a(L^2(\Sigma_1)), \\
\mathfrak{F}_{t,2}^\sigma &= \mathfrak{F}_a(L^2(\Sigma_1^\sigma)) \otimes \mathfrak{F}_a(L^2(\Sigma_1^\sigma)), \\
\mathfrak{F}_{t,2}^\sigma &= \mathfrak{F}_a(L^2(\Sigma_1)) \otimes \mathfrak{F}_a(L^2(\Sigma_1)).
\end{align*}
\]
The space $\mathcal{F}_{\ell,1}$ is the Fock space for the massive leptons $\ell$ and $\mathcal{F}_{\ell,2}$ is the Fock space for the neutrinos and antineutrinos $\ell$.

Set
\[ \mathcal{F}_\ell^\sigma = \mathcal{F}_{\ell,1} \otimes \mathcal{F}_{\ell,2}^\sigma , \]
\[ \mathcal{F}_\ell,\sigma = \mathcal{F}_{\ell,2}^\sigma . \]

We have
\[ \mathcal{F}_\ell \simeq \mathcal{F}_\ell^\sigma \otimes \mathcal{F}_\ell,\sigma . \]

Set
\[ \mathcal{F}_L^\sigma = \bigotimes_{\ell=1}^3 \mathcal{F}_\ell^\sigma , \]
\[ \mathcal{F}_L,\sigma = \bigotimes_{\ell=1}^3 \mathcal{F}_\ell,\sigma . \]

We have
\[ \mathcal{F}_L \simeq \mathcal{F}_L^\sigma \otimes \mathcal{F}_L,\sigma . \]

Set
\[ \mathcal{F}_L^\sigma = \mathcal{F}_L^\sigma \otimes \mathcal{F}_W , \]

We have
\[ \mathcal{F} \simeq \mathcal{F}_L,\sigma \otimes \mathcal{F}_W . \]

Set
\[ H_{0}^{(1)} = \sum_{\ell=1}^{3} \sum_{\epsilon=\pm} \int w_{\ell}^{(1)}(\xi_1) b_{\ell,\epsilon}(\xi_1) b_{\ell,\epsilon}(\xi_1) d\xi_1 , \]
\[ H_{0}^{(2)} = \sum_{\ell=1}^{3} \sum_{\epsilon=\pm} \int w_{\ell}^{(2)}(\xi_2) c_{\ell,\epsilon}(\xi_2) c_{\ell,\epsilon}(\xi_2) d\xi_2 , \]
\[ H_{0}^{(3)} = \sum_{\epsilon=\pm} \int w_{\ell}^{(3)}(\xi_3) a_{\epsilon}(\xi_3) a_{\epsilon}(\xi_3) d\xi_3 , \]

and
\[ H_{0}^{(2)\sigma} = \sum_{\ell=1}^{3} \sum_{\epsilon=\pm \ |p_2| > \sigma} \int w_{\ell}^{(2)}(\xi_2) c_{\ell,\epsilon}(\xi_2) c_{\ell,\epsilon}(\xi_2) d\xi_2 , \]
\[ H_{0,\sigma}^{(2)} = \sum_{\ell=1}^{3} \sum_{\epsilon=\pm \ |p_2| \leq \sigma} \int w_{\ell}^{(2)}(\xi_2) c_{\ell,\epsilon}(\xi_2) c_{\ell,\epsilon}(\xi_2) d\xi_2 . \]

We have on $\mathcal{F}^\sigma \otimes \mathcal{F}_\sigma$
\[ H_{0}^{(2)} = H_{0}^{(2)\sigma} \otimes 1 + 1 \otimes H_{0,\sigma}^{(2)} . \]

Here, $1^\sigma$ (resp. $1_\sigma$) is the identity operator on $\mathcal{F}^\sigma$ (resp. $\mathcal{F}_\sigma$).

Define
\[ (3.2) \quad H^\sigma = H_{\sigma}|_{\mathcal{F}^\sigma} \quad \text{and} \quad H_{0}^{\sigma} = H_{0}|_{\mathcal{F}^\sigma} . \]

We get
\[ H^\sigma = H_{0}^{(1)} + H_{0}^{(2)\sigma} + H_{0}^{(3)} + gH_{I,\sigma} \quad \text{on} \quad \mathcal{F}^\sigma , \]

and
\[ H_{0}^{(2)\sigma} = \sum_{\ell=1}^{3} \sum_{\epsilon=\pm \ |p_2| > \sigma} \int w_{\ell}^{(2)}(\xi_2) c_{\ell,\epsilon}(\xi_2) c_{\ell,\epsilon}(\xi_2) d\xi_2 , \]
\[ H_{0,\sigma}^{(2)} = \sum_{\ell=1}^{3} \sum_{\epsilon=\pm \ |p_2| \leq \sigma} \int w_{\ell}^{(2)}(\xi_2) c_{\ell,\epsilon}(\xi_2) c_{\ell,\epsilon}(\xi_2) d\xi_2 . \]

We have on $\mathcal{F}^\sigma \otimes \mathcal{F}_\sigma$
\[ H_{0}^{(2)} = H_{0}^{(2)\sigma} \otimes 1 + 1 \otimes H_{0,\sigma}^{(2)} . \]
and

\[ H_\sigma = H^\sigma \otimes 1_\sigma + 1^\sigma \otimes H^\text{(2)}_{1,\sigma} \] on \( \mathcal{F}^\sigma \otimes \mathcal{F}_\sigma \).

In order to implement the conjugate operator theory we have to show that \( H_\sigma \) has a gap in its spectrum above its ground state.

We now set, for \( \beta > 0 \) and \( \eta > 0 \),

\[ C_{\beta \eta} = \left( \frac{3}{m_W} (1 + \frac{1}{m_1}) + \frac{3\beta}{m_W m_1} + \frac{12\eta (1 + \beta)}{m_1^2} \right)^{\frac{1}{2}}, \]

and

\[ B_{\beta \eta} = \left( \frac{3}{m_W} (1 + \frac{1}{4\beta}) + 12(\eta (1 + \frac{1}{4\beta}) + \frac{1}{4\eta}) \right)^{\frac{1}{2}}. \]

Let

\[ G = \left( G^{(\alpha)}_{\ell, \ell', \ell''} \right)_{\alpha=1,2, \ell=1,2,3, \ell', \ell'' = \pm, \ell \neq \ell'} \]

and set

\[ \tilde{K}(G) = \left( \sum_{\alpha=1,2} \sum_{\ell=1}^3 \sum_{\ell'=\ell} \sum_{\ell'' \neq \ell'} \| G^{(\alpha)}_{\ell, \ell', \ell''} \|_{L^2(\Sigma_1 \times \Sigma_1 \times \Sigma_2)}^2 \right)^{\frac{1}{2}}. \]

Let

\[ \tilde{C}_{\beta \eta} = C_{\beta \eta} \left( 1 + \frac{g_1 K(G) C_{\beta \eta}}{1 - g_1 K(G) C_{\beta \eta}} \right), \]

\[ \tilde{B}_{\beta \eta} = \left( 1 + \frac{g_1 K(G) C_{\beta \eta}}{1 - g_1 K(G) C_{\beta \eta}} \left( 2 + \frac{g_1 K(G) B_{\beta \eta} C_{\beta \eta}}{1 - g_1 K(G) C_{\beta \eta}} \right) \right) B_{\beta \eta}. \]

Let

\[ \hat{K}(G) = \left( \sum_{\alpha=1,2} \sum_{\ell=1}^3 \sum_{\ell'=\ell} \sum_{\ell'' \neq \ell'} \int_{\Sigma_1 \times \Sigma_2} \frac{|G^{(\alpha)}_{\ell, \ell', \ell''}(\xi_1, \xi_2, \xi_3)|^2}{|p_2|^2} d\xi_1 d\xi_2 d\xi_3 \right)^{\frac{1}{2}}. \]

Let \( \delta \in \mathbb{R} \) be such that

\[ 0 < \delta < m_1. \]

We set

\[ \tilde{D} = \sup\left( \frac{4\Lambda \gamma}{2m_1 - \delta}, 1 \right) \hat{K}(G) \left( 2m_1 \tilde{C}_{\beta \eta} + \tilde{B}_{\beta \eta} \right), \]

where \( \Lambda > m_1 \) has been introduced in Hypothesis 3.1(iv).

Let us define the sequence \( (\sigma_n)_{n \geq 0} \) by

\[ \sigma_0 = \Lambda, \]

\[ \sigma_1 = m_1 - \frac{\delta}{2}, \]

\[ \sigma_2 = m_1 - \delta = \gamma \sigma_1, \]

\[ \sigma_{n+1} = \gamma \sigma_n, \quad n \geq 1, \]

where \( \gamma = 1 - \delta/(2m_1 - \delta) \).

Let \( g_3^{(1)} \) be such that

\[ 0 < g_3^{(1)} < \inf(1, g_1, \frac{\gamma - \gamma^2}{3D}). \]
For $0 < g \leq g_0^{(1)}$ we have

$$0 < \gamma < (1 - \frac{3g\hat{D}}{\gamma}) ,$$

and

$$0 < \sigma_{n+1} < (1 - \frac{3g\hat{D}}{\gamma}) \sigma_n, \quad n \geq 1 .$$

Set

$$H^n = H^n_\sigma; \quad H^n_0 = H^n_0 \sigma, \quad n \geq 0 \quad E^n = \inf \sigma(H^n) , \quad n \geq 0 .$$

We then get

**Proposition 3.5.** Suppose that the kernels $G_{(\cdot,\cdot)}^{(\cdot)}$ satisfy Hypothesis (2.1), Hypothesis (3.1)(i) and (3.1)(iv). Then there exists $0 < \hat{g} \leq g_0^{(1)}$ such that, for $g \leq \hat{g}$ and $n \geq 1$, $E^n$ is a simple eigenvalue of $H^n$ and $H^n$ does not have spectrum in $(E^n, E^n + (1 - \frac{3g\hat{D}}{\gamma}) \sigma_n)$.

The proof of Proposition 3.5 is given in Appendix A.

We now introduce the positive commutator estimates and the regularity property of $H$ with respect to $A$ in order to prove Theorem 3.4.

The operator $A$ has to be split into two pieces depending on $\sigma$.

Let

$$\eta_\sigma(p^2) = \chi_{2\sigma}(p^2) ,$$

$$\eta^\sigma(p^2) = \chi_{2\sigma}(p^2) ,$$

$$a_\sigma = \eta_\sigma(p^2) a \eta_\sigma(p^2) ,$$

$$a^\sigma = \eta^\sigma(p^2) a \eta^\sigma(p^2) .$$

Since $\eta_\sigma^2 + (\eta^\sigma)^2 = 1$, and $[\eta_\sigma, [\eta_\sigma, a]] = 0 = [\eta^\sigma, [\eta^\sigma, a]]$, we obtain (see [13])

$$a = a^\sigma + a_\sigma .$$

Note that we also have

$$a_\sigma = \frac{1}{2} (\eta_\sigma(p^2)^2 p^2 \cdot i\nabla p^2 + i\nabla w_\sigma(p^2)^2 p^2) ,$$

$$a^\sigma = \frac{1}{2} (\eta^\sigma(p^2)^2 p^2 \cdot i\nabla p^2 + i\nabla w_\sigma(p^2)^2 p^2) .$$

The operators $a$, $a_\sigma$ and $a^\sigma$ are essentially self-adjoint on $C_0^\infty(\mathbb{R}^3, C^2)$ (see [8], Proposition 4.2.3). We still denote by $a$, $a_\sigma$ and $a^\sigma$ their closures. If $\hat{a}$ denotes any of the operator $a$, $a_\sigma$ and $a^\sigma$, we have

$$D(\hat{a}) = \{u \in L^2(\Sigma_1); \hat{a}u \in L^2(\Sigma_1)\} .$$

The operators $d\Gamma(a)$, $d\Gamma(a^\sigma)$, $d\Gamma(a_\sigma)$ are self-adjoint operators in $\mathcal{F}_a(L^2(\Sigma_1))$ and we have

$$d\Gamma(a) = d\Gamma(a^\sigma) + d\Gamma(a_\sigma) .$$

By [24], the following operators in $\mathcal{F}_\ell$, denoted by $A_\ell^\sigma$ and $A_{\sigma\ell}$ respectively,

$$A_\ell^\sigma = 1 \otimes 1 \otimes d\Gamma(a^\sigma) \otimes 1 + 1 \otimes 1 \otimes d\Gamma(a^\sigma) ,$$

$$A_{\sigma\ell} = 1 \otimes 1 \otimes d\Gamma(a_\sigma) \otimes 1 + 1 \otimes 1 \otimes d\Gamma(a_\sigma) ,$$
are essentially self-adjoint on $\mathcal{D}_\ell$.

Let $A^\sigma$ and $A_\sigma$ be the following two operators in $\mathfrak{F}_L$:

$$A^\sigma = A^\sigma_1 \otimes 1_2 \otimes 1_3 + 1_1 \otimes A^\sigma_2 \otimes 1_3 + 1_1 \otimes 1_2 \otimes A^\sigma_3,$$

$$A_\sigma = A_\sigma_1 \otimes 1_2 \otimes 1_3 + 1_1 \otimes A_\sigma_2 \otimes 1_3 + 1_1 \otimes 1_2 \otimes A_\sigma_3.$$

The operators $A^\sigma$ and $A_\sigma$ are essentially self-adjoint on $\mathfrak{F}_L$. Still denoting by $A^\sigma$ and $A_\sigma$ their extensions to $\mathfrak{F}$, $A^\sigma$ and $A_\sigma$ are essentially self-adjoint on $\mathcal{D}$ and we still denote by $A^\sigma$ and $A_\sigma$ their closures.

We have

$$A = A^\sigma + A_\sigma.$$

The operators $a$, $a^\sigma$ and $a_\sigma$ are associated to the following $C^\infty$-vector fields in $\mathbb{R}^3$ respectively,

$$v(p_2) = p_2,$$

$$v^\sigma(p_2) = \eta^\sigma(p_2)^2 p_2,$$

$$v_\sigma(p_2) = \eta_\sigma(p_2)^2 p_2.$$

Let $\mathcal{V}(p)$ be any of these vector fields. We have

$$|\mathcal{V}(p)| \leq \Gamma |p|,$$

for some $\Gamma > 0$ and we also have

$$\mathcal{V}(p) = \tilde{v}(|p|)p,$$

where the $\tilde{v}$'s are defined by (3.11) and (3.12), and fulfill $|p|^\alpha \frac{d}{d|p|} \tilde{v}(|p|)$ bounded for $\alpha = 0, 1, 2$.

Let $\psi_t(\cdot): \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the corresponding flow generated by $\mathcal{V}$:

$$\frac{d}{dt} \psi_t(p) = \mathcal{V}(\psi_t(p)),$$

$$\psi_0(p) = p.$$

$\psi_t(p)$ is a $C^\infty$-flow and we have

$$e^{-\Gamma|t|} |p| \leq |\psi_t(p)| \leq e^{\Gamma|t|} |p|.$$

$\psi_t(p)$ induces a one-parameter group of unitary operators $U(t)$ in $L^2(\Sigma_1) \simeq L^2(\mathbb{R}^3, \mathbb{C}^2)$ defined by

$$(U(t)f)(p) = f(\psi_t(p))(\det \nabla \psi_t(p))^{\frac{1}{2}}$$

Let $\phi_t(\cdot)$, $\phi^\sigma_t(\cdot)$ and $\phi_\sigma(\cdot)$ be the flows associated with the vector fields $v(\cdot)$, $v^\sigma(\cdot)$ and $v_\sigma(\cdot)$ respectively.

Let $U(t)$, $U^\sigma(t)$ and $U_\sigma(t)$ be the corresponding one-parameter groups of unitary operators in $L^2(\Sigma_1)$. The operators $a$, $a^\sigma$, and $a_\sigma$ are the generators of $U(t)$, $U^\sigma(t)$ and $U_\sigma(t)$ respectively, i.e.,

$$U(t) = e^{-iat},$$

$$U^\sigma(t) = e^{-ia^\sigma t},$$

$$U_\sigma(t) = e^{-ia_\sigma t}.$$

Let

$$w^{(2)}(\xi_2) = (w^{(2)}_t(\xi_2))_{t=1,2,3}$$
and
\[ d\Gamma(w^{(2)}) = \sum_{\ell=1}^{3} \sum_{\epsilon} \int w^{(2)}_\ell(\xi_2)c_{\ell,\epsilon}(\xi_2)c_{\ell,\epsilon}(\xi_2)d\xi_2. \]

Let \( V(t) \) be any of the one-parameter groups \( U(t) \), \( U^\sigma(t) \) and \( U_{\sigma}(t) \). We set
\[ V(t)w^{(2)}V(t)^* = (V(t)w^{(2)}_\ell V(t)^*)_{\ell=1,2,3}, \]
and we have
\[ V(t)w^{(2)}V(t)^* = w^{(2)}(\psi_t). \]

Here \( \psi_t \) is the flow associated to \( V(t) \).

This yields, for any \( \varphi \in \mathcal{D} \), see Lemma 2.8])
\[ e^{-itA}H_0 e^{itA}\varphi - H_0\varphi = (d\Gamma(e^{-itw^{(2)}e^{itA}}) - d\Gamma(w^{(2)}))\varphi = (d\Gamma(w^{(2)} \circ \phi_t - w^{(2)}))\varphi, \]
\[ e^{-itA^\sigma t}H_0 e^{itA^\sigma t}\varphi - H_0\varphi = (d\Gamma(e^{-i\alpha\sigma t}w^{(2)} e^{i\alpha t}) - d\Gamma(w^{(2)}))\varphi = (d\Gamma(w^{(2)} \circ \phi_{\sigma t}^\sigma - w^{(2)}))\varphi, \]
\[ e^{-itA^\tau t}H_0 e^{itA^\tau t}\varphi - H_0\varphi = (d\Gamma(e^{-i\alpha \tau t}w^{(2)} e^{i\alpha t}) - d\Gamma(w^{(2)}))\varphi = (d\Gamma(w^{(2)} \circ \phi_{\sigma t}^\tau - w^{(2)}))\varphi. \]

Proposition 3.6. Suppose that the kernels \( G^{(s)}_{\ell,\epsilon,\epsilon'} \) satisfy Hypothesis 2.7.

For every \( t \in \mathbb{R} \) we have, for \( g \leq g_1 \),
\[ (i) \quad e^{itA}D(H_0) = e^{itA}D(H) \subset D(H_0) = D(H), \]
\[ (ii) \quad e^{itA^\tau}D(H_0) = e^{itA^\tau}D(H) \subset D(H_0) = D(H), \]
\[ (iii) \quad e^{itA^\sigma}D(H_0) = e^{itA^\sigma}D(H) \subset D(H_0) = D(H). \]

Proof. We only prove \( i) \), since \( ii) \) and \( iii) \) can be proved similarly. By (3.14) we have, for \( \varphi \in \mathcal{D} \),
\[ e^{-itA}H_0 e^{itA}\varphi = (H_0^{(1)} + H_0^{(3)} + d\Gamma(w^{(2)} \circ \phi_t))\varphi. \]

It follows from (3.13) and (3.17) that
\[ \|H_0 e^{itA}\varphi\| \leq e^{\Gamma|t|}\|H_0\varphi\|. \]

This yields \( i) \) because \( \mathcal{D} \) is a core for \( H_0 \). Moreover we get
\[ \|H_0 e^{itA}(H_0 + 1)^{-1}\| \leq e^{\Gamma|t|}. \]

In view of \( D(H_0) = D(H) \), the operators \( H_0(H+i)^{-1} \) and \( H(H+i)^{-1} \) are bounded and there exists a constant \( C > 0 \) such that
\[ \|H e^{itA}(H+i)^{-1}\| \leq Ce^{\Gamma|t|}. \]

Similarly, we also get
\[ \|H_0 e^{itA^\sigma}(H_0 + 1)^{-1}\| \leq e^{\Gamma|t|}, \]
\[ \|H_0 e^{itA^\tau}(H_0 + 1)^{-1}\| \leq e^{\Gamma|t|}, \]
\[ \|H e^{itA^\sigma}(H+i)^{-1}\| \leq Ce^{\Gamma|t|}, \]
\[ \|H e^{itA^\tau}(H+i)^{-1}\| \leq Ce^{\Gamma|t|}. \]
Let $H_I(G)$ be the interaction associated with the kernels $G = (G_{t,e,e'})_{\alpha=1,2;\ e=1,2,3;\ e'\neq\pm}$, where the kernels $G_{t,e,e'}^{(\alpha)}$ satisfy Hypothesis 2.1.

We set
$$V(t)G = (V(t)G_{t,e,e'})_{\alpha=1,2;\ e=1,2,3;\ e'\neq\pm}$$
We have for $\varphi \in \mathcal{D}$ (see [9] Lemma 2.7),
$$e^{-itA}H_I(G)e^{itA}\varphi = H_I(e^{-ita}G)\varphi,$$
$$e^{-iA^\sigma t}H_I(G)e^{iA^\sigma t} \varphi = H_I(e^{-ita^\sigma}G)\varphi.$$  
(3.18)

According to [3] and [25], in order to prove Theorem 3.3 we must prove that $H$ is locally of class $C^2(A^\sigma)$, $C^2(A_\sigma)$ and $C^2(A)$ in $(-\infty, m_1 - \tfrac{\delta}{2})$ and that $A$ and $A_\sigma$ are locally strictly conjugate to $H$ in $(E, m_1 - \tfrac{\delta}{2})$.

Recall that $H$ is locally of class $C^2(A)$ in $(-\infty, m_1 - \tfrac{\delta}{2})$ if, for any $\varphi \in C^\infty_0((-\infty, m_1 - \tfrac{\delta}{2}))$, $\varphi(H)$ is of class $C^2(A)$, i.e., $t \rightarrow e^{-itA}\varphi(H)e^{itA}\psi$ is twice continuously differentiable for all $\varphi \in C^\infty_0((-\infty, m_1 - \tfrac{\delta}{2})$ and all $\psi \in \mathcal{F}$.

Thus, one of our main results is the following one

**Theorem 3.7.** Suppose that the kernels $G_{t,e,e'}^{(\alpha)}$ satisfy Hypothesis 2.1 and 3.1(iii).

(a) $H$ is locally of class $C^2(A)$, $C^2(A^\sigma)$ and $C^2(A_\sigma)$ in $(-\infty, m_1 - \delta/2)$.

(b) $H^\sigma$ is locally of class $C^2(A^\sigma)$ in $(-\infty, m_1 - \delta/2)$.

It follows from Theorem 3.7 that $[H, iA]$, $[H, iA_\sigma]$, $[H, iA^\sigma]$ and $[H^\sigma, iA^\sigma]$ are defined as sesquilinear forms on $\bigcup_K E_K(H)\mathcal{F}$, where the union is taken over all the compact subsets $K$ of $(-\infty, m_1 - \delta/2)$.

Furthermore, by Proposition 3.6, Theorem 3.7 and [13] Lemma 29, we get for all $\varphi \in C^\infty_0((E, m_1 - \delta/2))$ and all $\psi \in \mathcal{F}$,

$$\varphi(H)[H, iA]\varphi(H)\psi = \lim_{t \to 0} \varphi(H)\left[H, \frac{e^{itA} - 1}{t}\right] \varphi(H)\psi,$$
$$\varphi(H)[H, iA_\sigma]\varphi(H)\psi = \lim_{t \to 0} \varphi(H)\left[H, \frac{e^{itA_\sigma} - 1}{t}\right] \varphi(H)\psi,$$
$$\varphi(H)[H, iA^\sigma]\varphi(H)\psi = \lim_{t \to 0} \varphi(H)\left[H, \frac{e^{itA^\sigma} - 1}{t}\right] \varphi(H)\psi,$$
$$\varphi(H^\sigma)[H^\sigma, iA^\sigma]\varphi(H^\sigma)\psi = \lim_{t \to 0} \varphi(H^\sigma)\left[H^\sigma, \frac{e^{itA^\sigma} - 1}{t}\right] \varphi(H^\sigma)\psi.$$  
(3.19)

The following proposition allows us to compute $[H, iA]$, $[H, iA_\sigma]$, $[H, iA^\sigma]$ and $[H^\sigma, iA^\sigma]$ as sesquilinear forms. By Hypothesis 2.1 and 3.1(iii), the kernels $G_{t,e,e'}^{(\alpha)}(\xi_1, .., \xi_3)$ belong to the domains of $a$, $a^\sigma$, and $a_\sigma$.

**Proposition 3.8.** Suppose that the kernels $G_{t,e,e'}^{(\alpha)}$ satisfy Hypothesis 2.1 and 3.1(ii.iii). Then

(a) For all $\psi \in \mathcal{D}(H)$ we have

(i) $\lim_{t \to 0} [H, \frac{e^{itA} - 1}{t}]\psi = (dG(w^{(2)}) + gH_I(-iaG)\psi,$

(ii) $\lim_{t \to 0} [H, \frac{e^{itA_\sigma} - 1}{t}]\psi = (dG((\eta^\sigma)^2w^{(2)}) + gH_I(-ia^\sigma G)\psi,$

(iii) $\lim_{t \to 0} [H, \frac{e^{itA^\sigma} - 1}{t}]\psi = (dG((\eta^\sigma)^2w^{(2)}) + gH_I(-ia^\sigma G)\psi,$
(iv) \( \lim_{t \to 0} \left[H^\sigma, \frac{e^{itA^\sigma}}{t} \right] \psi = (\, d\Gamma((\eta^\sigma)^2w(2)) + gH_1(-i\sigma^G(\chi^\sigma(p_2)G)) \, )\psi. \)

(b) (i) \( \sup_{0 < |t| \leq 1} \left\| \left[H, \frac{e^{itA}}{t} \right] (H + i)^{-1} \right\| < \infty, \)

(ii) \( \sup_{0 < |t| \leq 1} \left\| \left[H, \frac{e^{itA^\sigma}}{t} \right] (H + i)^{-1} \right\| < \infty, \)

(iii) \( \sup_{0 < |t| \leq 1} \left\| \left[H, \frac{e^{itA^\sigma}}{t} \right] (H + i)^{-1} \right\| < \infty, \)

(iv) \( \sup_{0 < |t| \leq 1} \left\| \left[H^\sigma, \frac{e^{itA}}{t} \right] (H + i)^{-1} \right\| < \infty. \)

Proof. Part (b) follows from part (a) by the uniform boundedness principle. For part (a), we only prove (a)(i), since other statements can be proved similarly.

By (3.14), we obtain
\[
\left| \frac{1}{|t|} \left[ \psi_1^{(2)}(p_2) - \psi_0^{(2)}(p_2) \right] \right| \leq \frac{1}{|t|} (e^{t|\sigma|} - 1) \left| \psi_0^{(2)}(p_2) \right|
\]
for \( t, \ell = 1, 2, 3. \)

By (3.13), and the Lebesgue’s Theorem we then get for all \( \psi \in D(H_0) \)
\[
\lim_{t \to 0} \left[H_0, \frac{e^{itA}}{t} \right] \psi = \lim_{t \to 0} \frac{1}{t} \left[ e^{-itA^\sigma} H_0 e^{itA^\sigma} - H_0 \right] \psi = d\Gamma((\eta^\sigma)^2w(2)) \psi,
\]
for \( \ell = 1, 2, 3. \)

By (3.12), we obtain for all \( \psi \in D(H), \)
\[
\lim_{t \to 0} \left[H_1(G), \frac{e^{itA}}{t} \right] \psi = \lim_{t \to 0} \frac{1}{t} \left[ e^{-itA} H_1(G) e^{itA} - H_1(G) \right] \psi = H_1(-i(a^G)) \psi,
\]
and
\[
\lim_{t \to 0} \left[H_1(G), \frac{e^{itA^\sigma}}{t} \right] \psi = \lim_{t \to 0} \frac{1}{t} \left[ e^{-itA^\sigma} H_1(G) e^{itA^\sigma} - H_1(G) \right] \psi = H_1(-i(a^\sigma)) \psi,
\]
This concludes the proof of Proposition 3.8.

Combining (3.19) with Proposition 3.8, we finally get for every \( \varphi \in C_0^\infty((-\infty, m_1 - \delta/2)) \) and every \( \psi \in \mathfrak{F} \)
\[
(3.20) \quad \varphi(H) \left[H, iA \right] \varphi(H) \psi = \varphi(H) \left[ d\Gamma(w(2)) + gH_1(-i(a^G)) \right] \varphi(H) \psi,
\]
\[
(3.21) \quad \varphi(H) \left[H, iA^\sigma \right] \varphi(H) \psi = \varphi(H) \left[ d\Gamma((\eta^\sigma)^2w(2)) + gH_1(-i(a^\sigma)) \right] \varphi(H) \psi,
\]
\[
(3.22) \quad \varphi(H) \left[H, iA^\sigma \right] \varphi(H) \psi = \varphi(H) \left[ d\Gamma((\eta^\sigma)^2w(2)) + gH_1(-i(a^\sigma)) \right] \varphi(H) \psi,
\]
and
\[
(3.23) \quad \varphi(H^\sigma) \left[H^\sigma, iA^\sigma \right] \varphi(H^\sigma) \psi = \varphi(H^\sigma) \left[ d\Gamma((\eta^\sigma)^2w(2)) + gH_1(-i(a^\sigma(\chi^\sigma(p_2)G))) \right] \varphi(H^\sigma) \psi.
\]

We now introduce the Mourre inequality.
Let $N$ be the smallest integer such that $N\gamma \geq 1$.

We have, for $g \leq g^1_b$,

$$
\begin{align*}
\gamma < \gamma + \frac{1}{N}(1 - \frac{3g\tilde{D}}{\gamma} - \gamma) < 1 - \frac{3g\tilde{D}}{\gamma}, \\
\frac{\gamma}{N} \leq \gamma - \frac{1}{N}(1 - \frac{3g\tilde{D}}{\gamma} - \gamma) < \gamma.
\end{align*}
$$

(3.24)

Let

$$
\epsilon_\gamma = \frac{1}{2N} (1 - \frac{3g^1_b\tilde{D}}{\gamma} - \gamma).
$$

We choose $f \in C^\infty_0(\mathbb{R})$ such that $1 \geq f \geq 0$ and

$$
f(\lambda) = \begin{cases}
1 & \text{if } \lambda \in [(\gamma - \epsilon_\gamma)^2, \gamma + \epsilon_\gamma], \\
0 & \text{if } \lambda > \gamma + \frac{1}{N}(1 - \frac{3g^1_b\tilde{D}}{\gamma} - \gamma) = \gamma + 2\epsilon_\gamma, \\
0 & \text{if } \lambda < (\gamma - \frac{1}{N}(1 - \frac{3g^1_b\tilde{D}}{\gamma} - \gamma))^2 = (\gamma - 2\epsilon_\gamma)^2.
\end{cases}
$$

(3.25)

Note that $\gamma + 2\epsilon_\gamma < 1 - 3g\tilde{D}/\gamma$ for $g \leq g^1_b$ and $\gamma - \epsilon_\gamma > \gamma/N$.

We set, for $n \geq 1$,

$$
f_n(\lambda) = f\left(\frac{\lambda}{\sigma_n}\right).
$$

Let

$$
\begin{align*}
H_n &= H_{\sigma_n}, \\
E_n &= \inf \sigma(H_n), \\
H^{(2)}_{0,n} &= H^{(2)}_{0,\sigma_n}.
\end{align*}
$$

Let $P^n$ denote the ground state projection of $H^n$. It follows from proposition 3.5 that, for $n \geq 1$ and $g \leq \tilde{g}_b \leq g^1_b$,

$$
f_n(H_n - E_n) = P^n \otimes f_n(H^{(2)}_{0,n}).
$$

(3.26)

Note that

$$
E_n = E^n = \inf \sigma(H^n).
$$

(3.27)

Set

$$
\begin{align*}
a^n &= a^{\sigma_n}, \\
\bar{a}_n &= a_{\sigma_n}, \\
A^n &= A^{\sigma_n}, \\
A_n &= A_{\sigma_n}, \\
\tilde{\mathcal{S}}^n &= \tilde{\mathcal{S}}^{\sigma_n}, \\
\tilde{\mathcal{S}}_n &= \tilde{\mathcal{S}}_{\sigma_n}.
\end{align*}
$$

We have

$$
\tilde{\mathcal{S}} \simeq \tilde{\mathcal{S}}^n \otimes \tilde{\mathcal{S}}_n, \\
A = A^n + A_n.
$$

We further note that

$$
a^n \chi^{\sigma_n}(p_2) = a^n.
$$

(3.28)
By (3.21), (3.23) and (3.28), we obtain

\[ [H, iA^n] = [H^n, iA^n] \otimes 1, \]

as sesquilinear forms with respect to \( F = F_n \otimes F_n \).

Furthermore, it follows from the virial Theorem (see [25, Proposition 3.2] and Proposition 6.1) that

\[ P^n[H^n, iA^n]P^n = 0. \]

By (3.26) and (3.29) we then get, for \( g \leq \tilde{g}_\delta \leq \tilde{g}_\delta^{(1)} \),

\[ f_n(H_n - E_n)[H, iA^n]f_n(H_n - E_n) = 0. \]

We then have

**Proposition 3.9.** Suppose that the kernels \( G^{(\alpha)}_{\ell, \epsilon, \epsilon'} \) satisfy Hypothesis (2.1) and (3.1).

Then there exists \( \tilde{C}_\delta > 0 \) and \( \tilde{g}_\delta^{(1)} > 0 \) such that \( \tilde{g}_\delta^{(1)} \leq \tilde{g}_\delta \) and

\[ f_n(H_n - E_n)[H, iA^n]f_n(H_n - E_n) \geq \tilde{C}_\delta \frac{1}{N^2} \sigma_n f_n(H_n - E_n)^2 \]

for \( n \geq 1 \) and \( g \leq \tilde{g}_\delta^{(1)} \).

Let \( E_\Delta(H - E) \) be the spectral projection for the operator \( H - E \) associated with the interval \( \Delta \), and let

\[ \Delta_n = \left[ (\gamma - \epsilon_\gamma)^2 \sigma_n, (\gamma + \epsilon_\gamma)\sigma_n \right], \quad n \geq 1. \]

Note that

\[ \left[ \sigma_{n+2}, \sigma_{n+1} \right] \subset \left[ (\gamma - \epsilon_\gamma)^2 \sigma_n, (\gamma + \epsilon_\gamma)\sigma_n \right], \quad n \geq 1. \]

**Theorem 3.10.** Suppose that the kernels \( G^{(\alpha)}_{\ell, \epsilon, \epsilon'} \) satisfy Hypothesis (2.1) and (3.1).

Then there exists \( C_\delta > 0 \) and \( \tilde{g}_\delta^{(2)} > 0 \) such that \( \tilde{g}_\delta^{(2)} \leq \tilde{g}_\delta^{(1)} \) and

\[ E_{\Delta_n}(H - E)[H, iA]E_{\Delta_n}(H - E) \geq C_\delta \frac{1}{N^2} \sigma_n E_{\Delta_n}(H - E), \]

for \( n \geq 1 \) and \( g \leq \tilde{g}_\delta^{(2)} \).

4. Existence of a ground state and location of the absolutely continuous spectrum

We now prove Theorem 3.3. The scheme of the proof is quite well known (see [5], [20]). It follows from Proposition 3.5 that \( H^n \) has an unique ground state, denoted by \( \phi^n \), in \( \mathfrak{F}^n \),

\[ H^n \phi^n = E^n \phi^n, \quad \phi_n \in \mathcal{D}(H^n), \quad ||\phi^n|| = 1, \quad n \geq 1. \]

Therefore \( H_n \) has an unique normalized ground state in \( \mathfrak{F} \), given by \( \tilde{\phi}_n = \phi^n \otimes \Omega_n \), where \( \Omega_n \) is the vacuum state in \( \mathfrak{F}_n \),

\[ H_n \tilde{\phi}_n = E^n \tilde{\phi}_n, \quad \tilde{\phi}_n \in \mathcal{D}(H_n), \quad ||\tilde{\phi}_n|| = 1, \quad n \geq 1. \]

Since \( ||\tilde{\phi}_n|| = 1 \), there exists a subsequence \( (n_k)_{k \geq 1} \), converging to \( \infty \) such that \( (\tilde{\phi}_{n_k})_{k \geq 1} \) converges weakly to a state \( \tilde{\phi} \in \mathfrak{F} \). We have to prove that \( \tilde{\phi} \neq 0 \). By
adapting the proof of Theorem 4.1 in [2] (see also [7]), the key point is to estimate \( \|c_{\ell,\epsilon}(\xi_2)\Phi_n\|_\phi \) in order to show that

\[
(4.1) \quad \sum_{\ell=1}^{3} \sum_{\epsilon} \int \|c_{\ell,\epsilon}(\xi_2)\phi_n\|^2 d\xi_2 = O(g^2),
\]

uniformly with respect to \( n \).

The estimate (4.1) is a consequence of the so-called “pull-through” formula as it follows.

Let \( H_{I,n} \) denote the interaction \( H_I \) associated with the kernels \( 1_{\{|p_2|\geq s_n\}}(p_2)G^{(\alpha)}_{\ell,\epsilon,\epsilon'} \).

We thus have

\[
H_0c_{\ell,\epsilon}(\xi_2)\phi_n = c_{\ell,\epsilon}(\xi_2)H_0\phi_n - w_{\ell}^{(2)}(\xi_2)c_{\ell,\epsilon}(\xi_2)\phi_n
\]

\[
gH_{I,n}c_{\ell,\epsilon}(\xi_2)\phi_n = c_{\ell,\epsilon}(\xi_2)gH_{I,n}\phi_n + gV_{\ell,\epsilon,\epsilon'}(\xi_2)\phi_n,
\]

with

\[
V_{\ell,\epsilon,\epsilon'}(\xi_2) = g \int G^{(1)}_{\ell,\epsilon,\epsilon'}(\xi_2,\xi_3)b_{\ell,\epsilon,\epsilon'}(\xi_1)a_\epsilon(\xi_3)d\xi_1 d\xi_3
\]

\[
+ g \int G^{(2)}_{\ell,\epsilon,\epsilon'}(\xi_2,\xi_3)b_{\ell,\epsilon,\epsilon'}(\xi_1)a_\epsilon^*(\xi_3)d\xi_1 d\xi_3.
\]

This yields

\[
(4.2) \quad \left( H_n - E_n + w_{\ell}^{(2)}(\xi_2) \right) c_{\ell,\epsilon}(\xi_2)\phi_n = V_{\ell,\epsilon,\epsilon'}(\xi_2)\phi_n.
\]

By adapting the proof of Propositions 2.4 and 2.5 we easily get

\[
\|V_{\ell,\epsilon,\epsilon'}\|_\phi \leq \frac{|g|}{m^W} \left( \sum_{\alpha=1,2} \|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\cdot,\xi_2,\cdot)\|_{L^2(S_1 \times S_2)} \right) \|H_0^\phi \|\psi\|
\]

\[
+ g \|G^{(2)}_{\ell,\epsilon,\epsilon'}(\cdot,\xi_2,\cdot)\|_{L^2(S_1 \times S_2)} \|\psi\|,
\]

where \( \psi \in \mathcal{D}(H_0) \).

Let us estimate \( \|H_0\phi_n\| \). By (2.29), (2.30), (3.3), (3.4) and (3.6) we have

\[
g\|H_{I,n}\phi_n\| \leq gK(G)(C_{\beta_0}\|H_0\phi_n\| + B_{\beta_0})
\]

and

\[
\|H_0\phi_n\| \leq |E_n| + g\|H_{I,n}\phi_n\|.
\]

Therefore

\[
(4.4) \quad \|H_0\phi_n\| \leq \frac{|E_n|}{1 - g_1 K(G)C_{\beta_0}} + \frac{gK(G)B_{\beta_0}}{1 - g_1 K(G)C_{\beta_0}}.
\]

By (3.27), (4.3) and (4.4), there exists \( C > 0 \) such that

\[
(4.5) \quad \|H_0\phi_n\| \leq C,
\]

uniformly in \( n \) and \( g \leq g_1 \).

By (1.2), (1.3) and (1.5) we get

\[
\|c_{\ell,\epsilon}\phi_n\| \leq \frac{g}{|p_2|} \left( C^2 \sum_{\alpha=1}^{2} \|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\cdot,\xi_2,\cdot)\|_{L^2(S_1 \times S_2)} + \|G^{(2)}_{\ell,\epsilon,\epsilon'}(\cdot,\xi_2,\cdot)\|_{L^2(S_1 \times S_2)} \right).
\]
By Hypothesis 3.1(i), there exists a constant \( C(G) > 0 \) depending on the kernels \( G = (G^{(i)}_{\ell,\ell',\alpha})_{\ell=1,2,\alpha=1,2,\pm} \) and such that

\[
\sum_{\ell=1}^{3} \sum_{\epsilon} \int \| c_{\ell,\epsilon}(\xi_2) \tilde{\phi}_n \|^2 d\xi_2 \leq C(G)^2 g^2 .
\]

The existence of a ground state \( \tilde{\phi} \) for \( H \) follows by choosing \( g \) sufficiently small, i.e. \( g \leq g_2 \), as in [2] and [7]. By adapting the method developed in [19] (see [19, Corollary 3.4]), one proves that the ground state of \( H \) is unique. We omit here the details.

Statements about \( \sigma(H) \) are consequences of the existence of a ground state and follows from the existence of asymptotic Fock representations for the CAR associated with the \( c^\dagger_{\ell,\epsilon}(\xi_2) \)'s. For \( f \in L^2(\mathbb{R}^3, \mathbb{C}^2) \), we define on \( D(H_0) \) the operators

\[
c^\dagger_{\ell,\epsilon}(f) = e^{-itH} c^\dagger_{\ell,\epsilon}(f) e^{itH} .
\]

By mimicking the proof given in [20, 28] one proves, under the hypothesis of Theorem 3.3 and for \( f \in C^\infty(\mathbb{R}^3, \mathbb{C}^2) \), that the strong limits of \( c^\dagger_{\ell,\epsilon}(f) \) when \( t \to \pm\infty \) exist for \( \psi \in D(H_0) \),

\[
\lim_{t \to \pm\infty} c^\dagger_{\ell,\epsilon}(f) \psi := c^\dagger_{\ell,\epsilon}(f) \psi .
\]

The operators \( c^\dagger_{\ell,\epsilon}(f) \) satisfy the CAR and we have

\[
c^\dagger_{\ell,\epsilon}(f) \tilde{\phi} = 0, \quad f \in C^\infty(\mathbb{R}^3, \mathbb{C}^2) ,
\]

where \( \tilde{\phi} \) is the ground state of \( H \).

It then follows from (4.6) and (4.7) that the absolutely continuous spectrum of \( H \) equals to \( [\inf \sigma(H), \infty) \). We omit the details (see [20, 28]).

### 5. Proof of the Mourre Inequality

We first prove Proposition 3.9. In view of Proposition 3.8(a) (iii) and (3.22), we have, as sesquilinear forms,

\[
[H, iA_\sigma] = (1 - g) d\Gamma((\eta_\sigma)^2 w^{(2)}) + g d\Gamma((\eta_\sigma)^2 w^{(2)}) + g H_T (-i(a_\sigma G)) .
\]

Let \( \mathfrak{F}^{(1)}_F \) (respectively \( \mathfrak{F}^{(2)}_F \)) be the Fock space for the massive leptons \( \ell \) (respectively the neutrinos and antineutrinos \( \ell \)).

We have

\[
\mathfrak{F} \simeq \mathfrak{F}^{(1)}_F \otimes \mathfrak{F}^{(2)}_F .
\]

Let

\[
\mathfrak{F}^{(1)} = \mathfrak{F}_W \otimes (\otimes_{\ell=1}^3 \mathfrak{F}^{(2)}_F) \quad \text{and} \quad \mathfrak{F}^{(2)} = (\otimes_{\ell=1}^3 \mathfrak{F}^{(2)}_F) .
\]

We have

\[
\mathfrak{F} \simeq \mathfrak{F}^{(1)} \otimes \mathfrak{F}^{(2)} ,
\]

\( \mathfrak{F}^{(1)} \) is the Fock space for the massive leptons and the bosons \( W^\pm \), and \( \mathfrak{F}^{(2)} \) is the Fock space for the neutrinos and antineutrinos.
We have, as sesquilinear forms and with respect to (3.2),
\[
\begin{aligned}
&d\Gamma((\eta_\sigma)^2(p_2)w_\ell^{(2)}) + H_1(-i(a_\sigma G)) \\
&= \sum_{\ell=1}^3 \sum_{e} \int \eta_\sigma(p_2)^2 |p_2| c^{*}_{\ell,e}(\xi_2) c_{\ell,e}(\xi_2) d\xi_2 \\
&+ \sum_{\ell=1}^3 \sum_{e \neq e'} \int |p_2| \left( 1 \otimes \eta_\sigma(p_2) c^{*}_{\ell,e}(\xi_2) + \sum_{\alpha=1,2} \frac{\mathcal{M}_{\ell,e,e',\sigma}^{(\alpha)}(\xi_2)}{|p_2|^2} \otimes 1_2 \right) \\
&\quad \left( 1 \otimes \eta_\sigma(p_2) c_{\ell,e}(\xi_2) + \sum_{\alpha=1,2} \frac{\mathcal{M}_{\ell,e,e',\sigma}^{(\alpha)}(\xi_2)}{|p_2|^2} \otimes 1_2 \right) d\xi_2 \\
&- \sum_{\ell=1}^3 \sum_{e \neq e'} \left( \sum_{\alpha=1,2} \frac{\mathcal{M}_{\ell,e,e',\sigma}^{(\alpha)}(\xi_2)}{|p_2|^2} \otimes 1_2 \right) d\xi_2,
\end{aligned}
\]

where
\[
\mathcal{M}_{\ell,e,e',\sigma}(\xi_2) = i \int \left( \sum_{\alpha=1,2} (a \eta_\sigma(p_2) G_{\ell,e,e',\sigma}^{(\alpha)}(\xi_2, \xi_2, \xi_2, \xi_2)) \right) b^*_{\ell,e}(\xi_1) a_{e'}(\xi_3) d\xi_1 d\xi_3,
\]
and where $1_j$ is the identity operator in $\mathfrak{H}^{(j)}$.

By mimicking the proofs of Proposition 2.4 and 2.5, we get, for every $\psi \in \mathcal{D}$,
\[
\begin{aligned}
&\sum_{\ell=1}^3 \sum_{e \neq e'} \left( \psi, \int \left( \sum_{\alpha=1,2} \frac{\mathcal{M}_{\ell,e,e',\sigma}^{(\alpha)}(\xi_2)}{|p_2|^2} \otimes 1_2 \right) \left( \sum_{\alpha=1,2} \frac{\mathcal{M}_{\ell,e,e',\sigma}^{(\alpha)}(\xi_2)}{|p_2|^2} \otimes 1_2 \right) \psi d\xi_2 \right) \\
&= \sum_{\ell=1}^3 \sum_{e \neq e'} \left\| \int \left( \sum_{\alpha=1,2} \frac{\mathcal{M}_{\ell,e,e',\sigma}^{(\alpha)}(\xi_2)}{|p_2|^2} \otimes 1_2 \right) \psi d\xi_2 \right\|^2 \\
&\leq \left( \int \left| \sum_{\alpha=1,2} (a \eta_\sigma(p_2) G_{\ell,e,e',\sigma}^{(\alpha)}(\xi_2, \xi_2, \xi_2, \xi_2)) \right|^2 w^{(3)}(\xi_1)|p_2| d\xi_1 d\xi_2 d\xi_3 \right) \| H_0^{(3)} \| \psi \|.
\end{aligned}
\]

Noting that $|a \eta_\sigma(p_2)| \leq C$ uniformly with respect to $\sigma$, it follows from hypothesis 2.1 and 3.1 that there exists a constant $C(G) > 0$ such that
\[
\int \left| \sum_{\alpha=1,2} (a \eta_\sigma(p_2) G_{\ell,e,e',\sigma}^{(\alpha)}(\xi_1, \xi_2, \xi_3)) \right|^2 w^{(3)}(\xi_1)|p_2| d\xi_1 d\xi_2 d\xi_3 \leq C(G)\sigma.
\]

This yields
\[
\begin{aligned}
&\sum_{\ell=1}^3 \sum_{e \neq e'} \left( \sum_{\alpha=1,2} \frac{\mathcal{M}_{\ell,e,e',\sigma}^{(\alpha)}(\xi_2)}{|p_2|^2} \otimes 1_2 \right) \left( \sum_{\alpha=1,2} \frac{\mathcal{M}_{\ell,e,e',\sigma}^{(\alpha)}(\xi_2)}{|p_2|^2} \otimes 1_2 \right) d\xi_2 \\
&\geq -C(G)\sigma.
\end{aligned}
\]

Combining (5.1), (5.3) with (5.4), we obtain
\[
[H, iA_n] \geq (1 - g) d\Gamma((\eta_\sigma)^2 w_\ell^{(2)}) + g C(G)\sigma_n.
\]

We have
\[
d\Gamma((\eta_\sigma)^2 w_\ell^{(2)}) \geq H_0^{(2)}.
\]
By \(3.24\), \(3.26\) and \(5.6\) we get
\[
f_n(H_n - E_n)\mathrm{d}^\pi (\eta \sigma_n^2 w^{(2)} \ell) f_n(H_n - E_n) \geq P_n \otimes f_n(H_0^{(2)} H_0^{(2)} f_n(H_0^{(2)}) \\
\geq \frac{\gamma^2}{N^2} \sigma_n f_n(H_n - E_n)^2 ,
\]
for \(g \leq g_\delta^{(1)}\).

This, together with \(5.5\), yields for \(g \leq g_\delta^{(1)}\)
\[
f_n(H_n - E_n) [H, i A_n] f_n(H_n - E_n) \\
\geq (1 - g_\delta^{(1)}) \frac{\gamma^2}{N^2} \sigma_n f_n(H_n - E_n)^2 - g C(G) \sigma_n f_n(H_n - E_n)^2 .
\]
Setting
\[
g_\delta^{(2)} = \inf \left( g_\delta^{(1)}, \frac{1 - g_\delta^{(1)}}{2 C(G) N^2} \right) ,
\]
we get
\[
f_n(H_n - E_n) [H, i A_n] f_n(H_n - E_n) \geq \frac{1 - g_\delta^{(1)}}{2} \sigma_n f_n(H_n - E_n)^2 ,
\]
for \(g \leq g_\delta^{(2)}\).

Proposition 3.9 is proved by setting \(\tilde{g}_\delta^{(1)} = g_\delta^{(2)}\) and \(\tilde{\beta}_\delta = \frac{1 - g_\delta^{(1)}}{2}\).

The proof of Theorem 3.10 is the consequence of the following two lemmata.

**Lemma 5.1.** Assume that the kernels \(G_{\eta, \xi, \xi'}^{\alpha}\) satisfy Hypothesis \(2.1\) and \((3.1)\)(ii).
Then there exists a constant \(D > 0\) such that
\[
|E - E_n| \leq g D \sigma_n^2 ,
\]
for \(n \geq 1\) and \(g \leq g^{(2)}\).

**Proof.** Let \(\phi\) (respectively \(\tilde{\phi}_n\)) be the unique normalized ground state of \(H\) (respectively \(H_n\)). We have
\[
(5.7) \quad E - E_n \leq (\tilde{\phi}_n, (H - H_n) \tilde{\phi}_n)
\]
\[
E_n - E \leq (\phi, (H_n - H) \phi) ,
\]
with
\[
(5.8) \quad H - H_n = g H I (\chi_{\sigma_n}(p_2) G) .
\]
Combining \(2.29\) and \(2.30\) with \(3.3\)-\(3.6\) and \(5.8\), we get
\[
(5.9) \quad \| (H - H_n) \tilde{\phi}_n \| \leq g K(\chi_{\sigma_n}(p_2) G) (C_{\beta_0} \| \tilde{\phi}_n \| + B_{\beta_0})
\]
and
\[
(5.10) \quad \| (H - H_n) \phi \| \leq g K(\chi_{\sigma_n}(p_2) G) (C_{\beta_0} \| \phi \| + \beta_{\beta_0})
\]
It follows from Hypothesis \(3.1\)(ii), \(4.3\), \(5.9\) and \(5.10\) that there exists a constant \(D > 0\) such that
\[
\max (\| (H - H_n) \tilde{\phi}_n \|, \| (H - H_n) \phi \| \leq g D \sigma_n^2 ,
\]
for \(n \geq 1\) and \(g \leq g^{(2)}\).

By \(5.7\), this proves Lemma 5.1. \(\square\)
Lemma 5.2. Suppose that the kernels $G^{(n)}_{\ell, \epsilon', \epsilon}$ satisfy Hypothesis 2.1 and 3.1(ii). Then there exists a constant $C > 0$ such that
\[
\|f_n(H - E) - f_n(H_n - E_n)\| \leq g C \sigma_n,
\]
for $n \geq 1$ and $g \leq g^{(2)}$.

Proof. Let $\tilde{f}(.)$ be an almost analytic extension of $f(.)$ given by (3.25) satisfying
\[
\|\partial_z \tilde{f}(x + iy)\| \leq Cy^2.
\]
Note that $\tilde{f}(x + iy) \in C^\infty_0(\mathbb{R}^2)$. We thus have
\[
f(s) = \int \frac{df(z)}{z - s}, \quad d\tilde{f}(z) = -\frac{1}{\pi} \frac{\partial \tilde{f}}{\partial \bar{z}} d\mu d\nu.
\]
Using the functional calculus based on this representation of $f(s)$, we get
\[
f_n(H - E) - f_n(H_n - E_n) = \sigma_n \int \frac{1}{H - E - z\sigma_n} (H - H_n + E_n - E) \frac{1}{H_n - E_n - z\sigma_n} \tilde{f}(z).
\]
Combining (2.29) and (2.30) with (3.3)-(3.6) and Hypothesis 3.1(ii), we get, for every $\psi \in \mathcal{D}(H_0)$ and for $g \leq g^{(2)}$,
\[
\|H_I(\chi_{\sigma_n}G)\psi\| \leq 2 g C \sigma_n^2 K(G) (C_{\beta \eta} \|H_0 + 1\| \psi\| + (C_{\beta \eta} + B_{\beta \eta}) \|\psi\|).
\]
This yields
\[
\|H_I(\chi_{\sigma_n}(p_2)^n(G)(H_0 + 1)^{-1}\| \leq g C_1 \sigma_n^2,
\]
for some constant $C_1 > 0$ and for $g \leq g^{(2)}$.

By mimicking the proof of (A.12) we show that there exists a constant $C_2 > 0$ such that
\[
\|(H_0 + 1)(H_n - E_n - z\sigma_n)^{-1}\| \leq C_2(1 + \frac{1}{\text{Im} z |\sigma_n|}),
\]
for $g \leq g^{(1)}$.

Combining Lemma 5.1 and (5.14) with (5.15)-(5.17) we obtain
\[
\|f_n(H - E) - f_n(H_n - E_n)\| \leq g C \sigma_n \int \frac{|\partial \tilde{f}(x + iy)|}{y^2} d\mu d\nu,
\]
for some constant $C > 0$ and for $g \leq g^{(2)}$.

Using (5.12) and $\tilde{f}(x + iy) \in C^\infty_0(\mathbb{R}^2)$ one concludes the proof of Lemma 5.2. □

We now prove Theorem 3.10.

Proof. It follows from Proposition 3.9 that
\[
f_n(H_n - E_n)[H, iA]f_n(H_n - E_n)
\]
\[= f_n(H_n - E_n)[H, iA]f_n(H_n - E_n) \geq C_\delta \gamma_n \sigma_n f_n(H_n - E_n)^2,
\]
for $n \geq 1$ and $g \leq g^{(1)}$. 

This yields

\[ f_n(H - E)[H, iA_n]f_n(H - E) \geq \tilde{C}_\delta \frac{\gamma^2}{N^2} \sigma_n f_n(H - E)^2 - f_n(H - E)[H, iA](f_n(H_n - E_n) - f_n(H - E)) \\
+ \tilde{C}_\delta \frac{\gamma^2}{N^2} \sigma_n (f_n(H_n - E_n) - f_n(H - E))^2 \\
+ \tilde{C}_\delta \frac{\gamma^2}{N^2} \sigma_n (f_n(H_n - E_n) - f_n(H - E)) f_n(H - E) \\
+ \tilde{C}_\delta \frac{\gamma^2}{N^2} \sigma_n (f_n(H_n - E_n) - f_n(H - E)) f_n(H - E). \]

Combining Proposition 3.8 (i) and (5.13) with (5.16) and (5.17) we show that \([H, iA]f_n(H_n - E_n)\) and \(f_n(H - E)[H, iA]\) are bounded operators uniformly with respect to \(n\). This, together with Lemma 5.2, yields

\[ f_n(H - E)[H, iA]f_n(H - E) \geq \tilde{C}_\delta \frac{\gamma^2}{N^2} \sigma_n f_n(H - E)^2 - \tilde{C}_\delta \sigma_n, \]

for some constant \(\tilde{C} > 0\) and for \(g \leq \inf(\tilde{C}_\delta, \tilde{C}_\delta^2, \tilde{C}_\delta^2 / N^2, \tilde{C}_\delta \tilde{C}_\delta^2 \sigma_n)\).

Multiplying both sides of (5.18) with \(E_n(H - E)\) we then get

\[ E_n(H - E)[H, iA]E_n(H - E) \geq \tilde{C}_\delta \frac{\gamma^2}{N^2} \sigma_n E_n(H - E)^2 - \tilde{C} \sigma_n E_n(H - E). \]

Setting

\[ \tilde{g}_\delta = \inf \left( \frac{\tilde{C}_\delta \gamma^2}{C \tilde{C}_\delta^2 \sigma_n}, g^{1(2)} \sigma_n \right), \]

Theorem 3.10 is proved with \(C_\delta = \tilde{C}_\delta - \tilde{C}_\delta \sigma_n / N^2 > 0\). \(\square\)

### 6. Proof of Theorem 3.7

We set

\[ A_t = e^{itA} - 1 \]
\[ \text{ad}_{A_t} \cdot [A_t, \cdot], \]
\[ A_t^\sigma = \frac{e^{itA^\sigma} - 1}{t}, \]
\[ A_{\sigma t} = \frac{e^{itA_{\sigma}} - 1}{t}. \]

The fact that \(H\) is of class \(C^1(A), C^1(A^\sigma)\) and \(C^1(A_{\sigma})\) in \((\infty, m_1 - \frac{\delta}{2})\) is the consequence of the following proposition.
Proposition 6.1. Suppose that the kernels $G^{(n)}_{\ell, \epsilon, \epsilon'}$ satisfy Hypothesis 2.1 and 3.1(iii.a). For every $\varphi \in C_0^\infty((-\infty, m_1 - \frac{\delta_2}{2}))$ and $g \leq g_1$, we then have
\[
\sup_{0 < |t| \leq 1} \| [\varphi(H), A_t] \| < \infty,
\sup_{0 < |t| \leq 1} \| [\varphi(H), A_t^\sigma] \| < \infty,
\sup_{0 < |t| \leq 1} \| [\varphi(H), A_{\sigma t}] \| < \infty,
\sup_{0 < |t| \leq 1} \| [\varphi(H^\sigma), A_t^\sigma] \| < \infty.
\]

Proof. We use the representation
\[
\varphi(H) = \int d\phi(z) (z - H)^{-1},
\]
where $\phi(z)$ is an almost analytic extension of $\varphi$ with
\[
|\partial_z \phi(x + iy)| \leq C|y|^2 \quad \text{and} \quad d\phi(z) = -\frac{1}{\pi} \frac{\partial}{\partial \bar{z}} \phi(z) dz dy.
\]
Note that $\phi(x + iy) \in C_0^\infty(\mathbb{R}^2)$.

We get
\[
ad_{A_t} \varphi(H) = \int d\phi(z) (z - H)^{-1} [A_t, H] (z - H)^{-1}.
\]

This yields
\[
\| \ad_{A_t} \varphi(H) \| \leq \sup_{0 < |t| \leq 1} \| [A_t, H] (i - H)^{-1} \| \int |d\phi(z)| \| (z - H)^{-1} \| \| (i - H)(z - H)^{-1} \|.
\]

It is easy to prove that
\[
\| [A_t, H] (i - H)^{-1} \| \leq C \int \frac{|d\phi(z)|}{|\text{Im} z|^2} < \infty.
\]

By Proposition 3.8(b)(i) and (6.1) we finally get, for $g \leq g_1$
\[
\sup_{0 < |t| \leq 1} \| \ad_{A_t} \varphi(H) \| < \infty.
\]

In a similar way we obtain, for $g \leq g_1$
\[
\sup_{0 < |t| \leq 1} \| [A_t^\sigma, \varphi(H)] \| < \infty,
\sup_{0 < |t| \leq 1} \| [A_{\sigma t}, \varphi(H)] \| < \infty,
\sup_{0 < |t| \leq 1} \| [A_t^\sigma, \varphi(H^\sigma)] \| < \infty.
\]

The proof of Theorem 3.7 is the consequence of the following proposition
**Proposition 6.2.** Suppose that the kernels \( G^{(s)}_{t, t', t''} \) satisfy Hypothesis \( (i) \)-\( (iii) \). We then have, for \( g \leq g_1 \),

\[
\sup_{0 < |t| \leq 1} \| [A_t, [A_t, H]](H + i)^{-1} \| < \infty ,
\]

\[
\sup_{0 < |t| \leq 1} \| [A_t^\sigma, [A_t^\sigma, H]](H + i)^{-1} \| < \infty ,
\]

\[
\sup_{0 < |t| \leq 1} \| [A_{\sigma t}, [A_{\sigma t}, H]](H + i)^{-1} \| < \infty ,
\]

\[
\sup_{0 < |t| \leq 1} \| [A_t^\sigma, H^\sigma](H^\sigma + i)^{-1} \| < \infty ,
\]

**Proof.** We have, for every \( \psi \in \mathcal{D}(H) \),

\[
[A_t, [A_t, H]]\psi = \frac{1}{t^2} e^{2itA} (e^{-2itA} H e^{2itA} - 2 e^{-itA} H e^{itA} + H)\psi .
\]

By (3.14) we get

\[
[A_t, [A_t, H_0]]\psi = \frac{1}{t^2} e^{2itA} (d\Gamma(w^{(2)} \circ \phi_{2t} - 2w^{(2)} \circ \phi_t + w^{(2)}))\psi ,
\]

where, for \( \ell = 1, 2, 3 \),

\[
(w^{(2)} \circ \phi_{2t})(p_2) - 2(w^{(2)} \circ \phi_t)(p_2) + w^{(2)}(p_2) = |\phi_{2t}(p_2)| - 2|\phi_t(p_2)| + |p_2| .
\]

We further note that

\[
\frac{1}{t^2} \left| |\phi_{2t}(p_2)| - 2|\phi_t(p_2)| + |p_2| \right| \leq \sup_{|s| \leq 2|t|} \left| \frac{\partial^2}{\partial s^2} \phi_s(p_2) \right| ,
\]

and

\[
\frac{\partial^2}{\partial s^2} \phi_s(p_2) = |\phi_s(p_2)| \leq e^{\Gamma|s|}|p_2| .
\]

Combining (6.3) with (6.4)\- (6.6) we get

\[
\| [A_t, [A_t, H_0]](H_0 + 1)^{-1} \| \leq e^{2\Gamma|t|} ,
\]

and

\[
\sup_{0 < |t| \leq 1} \| [A_t, [A_t, H_0]](H_0 + 1)^{-1} \| \leq e^{2\Gamma} .
\]

In a similar way we obtain

\[
\sup_{0 < |t| \leq 1} \| [A_t^\sigma, [A_t^\sigma, H_0]](H_0 + 1)^{-1} \| \leq C e^{2\Gamma} ,
\]

\[
\sup_{0 < |t| \leq 1} \| [A_{\sigma t}, [A_{\sigma t}, H_0]](H_0 + 1)^{-1} \| \leq C e^{2\Gamma} .
\]

Here \( C \) is a positive constant.

Let us now prove that

\[
\sup_{0 < |t| \leq 1} \| [A_t, [A_t, H_t(G)]](H + i)^{-1} \| < \infty
\]
By (3.18) and (6.2) we get, for every \( \psi \in D(H) \),
\[
[A_t, [A_t, H_t(G)]]\psi = \sum_{\alpha=1,2} \sum_{\ell=1,2,3} \sum_{\ell' \neq \ell'} \frac{\varepsilon^{2itA}}{t^2} \left( e^{-2itA} H_t(G) H_t(G) e^{-2itA} - 2e^{-itA} H_t(G) e^{itA} \right) \psi
\]
(6.7)
\[+ H_t(G) \psi \]
\[
= \sum_{\alpha=1,2} \sum_{\ell=1,2,3} \sum_{\ell' \neq \ell'} \frac{\varepsilon^{2itA}}{t^2} \left( H_t(G) (G + \ell, \ell') \psi \right) ,
\]
where
\[
G_t(G) (\xi_1, \xi_2, \xi_3) = (D\phi_t(p_2))^2 G_t(G) (\xi_1; \phi_t(p_2), s_2; \xi_3)
\]
\[
eq (e^{-itA} G_t(G)) (\xi_1, \xi_2, \xi_3) .
\]
Combining (2.29) and (2.30) with (3.3)-(3.6) and (6.7) we get
\[
(6.8) \quad \| [A_t, [A_t, H_t(G)]] \psi \| \leq g K(G_t) \| (H_0 + I) \psi \| + (C_{\beta \eta} + B_{\beta \eta}) \| \psi \| .
\]
Here \( K(G_t) > 0 \) and
\[
(6.9) \quad K(G_t)^2 = \sum_{\alpha=1,2} \sum_{\ell=1,2,3} \sum_{\ell' \neq \ell'} \frac{1}{2} \| G_t(G) \|^2 - 2G_t(G) + G_t(G)^2 \| L^2(\Sigma_1 \times \Sigma_1 \times \Sigma_2) .
\]
We further note that, for \( 0 < |t| \leq 1 \),
\[
(6.10) \quad K(G_t) \leq \sup_{0 < |t| \leq 1} \left( \sum_{\alpha=1,2} \sum_{\ell=1,2,3} \sum_{\ell' \neq \ell'} \left\| \frac{\partial}{\partial s} G_t(G) \right\| L^2(\Sigma_1 \times \Sigma_1 \times \Sigma_2) \right) .
\]
We get
\[
(6.11) \quad \frac{\partial}{\partial t} G_t(G) = \frac{3}{2} (e^{-ita} G_t(G) + (e^{-ita} \cdot \nabla p_2 G_t(G))) ,
\]
and
\[
(6.12) \quad \frac{\partial^2}{\partial s^2} G_t(G) = \frac{9}{4} (e^{-ita} G_t(G) + \frac{3}{2} (e^{-ita} \cdot \nabla p_2 G_t(G)) + \sum_{i,j=1,2,3} e^{-ita} (p_2, p_2, \nabla p_2 G_t(G)) .
\]
Recall that \( e^{-ita} \) is an one parameter group of unitary operators in \( L^2(\Sigma_1 \times \Sigma_1 \times \Sigma_2) .
\)
Combining Hypothesis (3.1)(ii.a) and (iii.b), with (6.8)-(6.12) we finally get
\[
\sup_{0 < |t| \leq 1} \| [A_t, [A_t, H_t(G)]](H_0 + 1)^{-1} \| < \infty .
\]
In view of \( D(H) = D(H_0) \) the operators \( H_0(H + i)^{-1} \) and \( H(H_0 + 1)^{-1} \) are bounded and we obtain
\[
\sup_{0 < |t| \leq 1} \| [A_t, [A_t, H_0]] (H + i)^{-1} \| < \infty ,
\]
(6.13)
\[
\sup_{0 < |t| \leq 1} \| [A_t, [A_t, H_t(G)]] (H + i)^{-1} \| < \infty .
\]
This yields
\[(6.14)\quad \sup_{0 < |t| \leq 1} \| [A_t, [A_t, H]] (H + i)^{-1} \| < \infty ,\]
for $g \leq g_1$.

Let $V(p_2)$ denote any of the two $C^\infty$-vector fields $v^\sigma(p_2)$ and $v_\sigma(p_2)$ and let $\tilde{a}$ denote the corresponding $\sigma^\alpha$ and $a_\sigma$ operators. We get
\[
\left( \frac{\partial^2}{\partial t^2} (e^{-i\tilde{a}t} G_{\ell,\ell,\ell}') \right) (\xi_1, \xi_2, \xi_3)
= \frac{1}{4} \left( e^{-i\tilde{a}t} ((\text{div} V(p_2))^2 G_{\ell,\ell,\ell}') \right) (\xi_1, \xi_2, \xi_3)
+ \frac{1}{2} \left( e^{-i\tilde{a}t} ((\text{div} V(p_2))V(p_2) \cdot \nabla_p G_{\ell,\ell,\ell}') \right) (\xi_1, \xi_2, \xi_3)
+ \frac{1}{2} \left( e^{-i\tilde{a}t} \left( \sum_{i,j=1}^3 V_i(p_2) (\partial_{p_{2,i}} V_j(p_2)) G_{\ell,\ell,\ell}') \right) \right) (\xi_1, \xi_2, \xi_3)
+ \frac{1}{2} \left( e^{-i\tilde{a}t} \left( \sum_{i,j=1}^3 V_i(p_2) \frac{\partial V_j}{\partial p_{2,i}}(p_2) \frac{\partial}{\partial p_{2,j}} G_{\ell,\ell,\ell}') \right) \right) (\xi_1, \xi_2, \xi_3)
+ \frac{1}{2} \left( e^{-i\tilde{a}t} \left( \sum_{i,j=1}^3 V_i(p_2) V_j(p_2) \frac{\partial^2}{\partial p_{2,i} \partial p_{2,j}} G_{\ell,\ell,\ell}') \right) \right) (\xi_1, \xi_2, \xi_3) .
\]

Combining the properties of the $C^\infty$ fields $v^\sigma(p_2)$ and $v_\sigma(p_2)$ together with Hypothesis 2.1 and 3.1 we get, from (6.13) and by mimicking the proof of (6.14),
\[(6.15)\quad \sup_{0 < |t| \leq 1} \| [A^\sigma_t, [A^\sigma_t, H]] (H + i)^{-1} \| < \infty ,\]
for $g \leq g_1$.

Similarly, by mimicking the proof of (6.14), we easily get, for $g \leq g_1$,
\[(6.16)\quad \sup_{0 < |t| \leq 1} \| [A^\sigma_t, [A^\sigma_t, \varphi(H)]] (H^\sigma + i)^{-1} \| < \infty .\]

This concludes the proof of Proposition [6.2] \(\square\)

We now prove Theorem [3.7].

\textbf{Proof of Theorem [3.7]} In view of [3 Lemma 6.2.3] (see also [13 Proposition 28]), the proof of Theorem 3.7 will follow from Proposition 6.1 and the following estimates
\[(6.16)\quad \sup_{0 < |t| \leq 1} \| [A_t, [A_t, \varphi(H)]] \| < \infty ,\]
\[(6.17)\quad \sup_{0 < |t| \leq 1} \| [A^\sigma_t, [A^\sigma_t, \varphi(H)]] \| < \infty ,\]
\[(6.18)\quad \sup_{0 < |t| \leq 1} \| [A_\sigma t, [A_\sigma t, \varphi(H)]] \| < \infty ,\]
\[(6.19)\quad \sup_{0 < |t| \leq 1} \| [A_t^\sigma, [A_t^\sigma, \varphi(H^\sigma)]] \| < \infty ,\]
for every \( \varphi \in C_0^\infty((-\infty, m_1 - \delta/2)) \) and for \( g \leq g_1 \).

Let us prove (6.16). The inequalities (6.17) - (6.19) can be proved similarly.

To this end, let \( \phi \) be an almost analytic extension of \( \varphi \) satisfying

\[
|\partial_x \varphi(x + iy)| \leq C|y|^3,
\]

and

\[
\varphi(H) = \int (z - H)^{-1} d\phi(z), \quad d\phi(z) = -\frac{1}{\pi} \partial_x \varphi(z) dz dy.
\]

It follows that

\[
[A_t[A_t, \varphi(H)]] = \int \left( (z - H)^{-1}[A_t[A_t, H]](z - H)^{-1} + 2(z - H)^{-1}[A_t, H](z - H)^{-1}\right) d\phi(z)
\]

We note that

\[
\sup_{0 < |t| \leq 1} \| (H^2 + i)(H - z)^{-1} \| \leq C \frac{|z|}{|\mathrm{Im} z|}, \text{ for } z \in \text{supp}\phi.
\]

We also have

\[
(6.20) \quad \sup_{0 < |t| \leq 1} \| \int (z - H)^{-1}[A_t[A_t, H]](z - H)^{-1} d\phi(z) \|
\]

\[
(6.21) \quad \leq C \sup_{0 < |t| \leq 1} \| [A_t[A_t, H]](H + i)^{-1} \| \| (H + i)(z - H)^{-1} \| \frac{|d\phi(z)|}{|\mathrm{Im} z|^2}
\]

Therefore, combining Proposition 3.3 (b)(i) and (6.20) we obtain

\[
\sup_{0 < |t| \leq 1} \| \int d\phi(z)(H - z)^{-1}[A_t, H](H + i)^{-1}(H + i)(H - z)^{-1} \|
\]

\[
(6.22) \quad = \sup_{0 < |t| \leq 1} \| \int (H - z)^{-1}[A_t, H](H + i)^{-1}(H + i)(H - z)^{-1} \|
\]

\[
\leq C \left( \int \frac{|d\phi(z)|}{|y|^3} \right) \sup_{0 < |t| \leq 1} \| [A_t, H](H + i)^{-1} \|^2 < \infty.
\]

Inequality (6.22) together with (6.21) yields (6.16), and \( H \) is locally of class \( C^2(A) \) on \((-\infty, m_1 - \delta/2)\) for \( g \leq g_1 \).

In a similar way it follows from Proposition 3.3 (b), Proposition 6.1 and Proposition 6.2 that \( H \) is locally of class \( C^2(A^\sigma) \) and \( C^2(A_{\sigma}) \) in \((-\infty, m_1 - \delta/2)\) and that \( H^\sigma \) is locally of class \( C^2(A^\sigma) \) in \((-\infty, m_1 - \delta/2)\), for \( g \leq g_1 \). This ends the proof of Theorem 3.7.

\[\square\]

7. Proof of Theorem 3.11

By (3.31), \( U_{n_i} \subseteq ((\gamma - \epsilon_i)^2 \sigma_n, (\gamma + \epsilon_i)\sigma_n) \) is a covering by open sets of any compact subset of \( (E, m_1 - \delta) \) and of the interval \( (E, m_1 - \delta) \) itself. Theorem 3.3 (i) and (ii) follow from Theorems 0.1 and 0.2 in [25] and Theorems 3.7 and 3.10 above with \( g_0 = \tilde{g}_0 (2) \), where \( \tilde{g}_0 (2) \) is given in Theorem 3.10. Theorem 3.11 (iii) follows from Theorem 25 in [23].
Appendix A.

In this appendix, we will prove Proposition A.3.5. We apply the method developed in [4] because every infrared cutoff Hamiltonian that one considers has a ground state energy which is a simple eigenvalue.

Let, for \( n \geq 0 \),
\[
\begin{align*}
\tilde{\mathcal{F}}^{\sigma_n} &= \mathcal{F}^n, \\
\Sigma_n^{n+1} &= \Sigma_n \cap \{ p_2 ; |p_2| < \sigma_n \}, \\
\tilde{\mathcal{F}}_{\ell,2,n}^{n+1} &= \mathcal{F}(L^2(\Sigma_n^{n+1})) \otimes \mathcal{F}(L^2(\Sigma_n^{n+1})), \\
\tilde{\mathcal{F}}^{n+1} &= \otimes_{\ell=1}^3 \tilde{\mathcal{F}}_{\ell,2,n}^{n+1}.
\end{align*}
\]

We have
\[
\tilde{\mathcal{F}}^{n+1} \simeq \mathcal{F}^n \otimes \tilde{\mathcal{F}}^{n+1}.
\]

Let \( \Omega_n \) (respectively \( \Omega_n^{n+1} \)) be the vacuum state in \( \mathcal{F}^n \) (respectively in \( \tilde{\mathcal{F}}^{n+1} \)). We now set
\[
H_n^{n+1} = H_0^{n+1} + \sum_{\ell=1}^3 \sum_{\ell=\pm} \int \sigma_{n+1} \leq |p_2| < \sigma_n w^{(2)}(\xi_2) c_{\ell,e}^{*}(\xi_2) c_{\ell,e}(\xi_2) d\xi_2.
\]
The operator \( H_0^{n+1} \) is a self-adjoint operator in \( \mathcal{F}^{n+1} \).

Let us denote by \( H_n^I \) and \( \tilde{H}_n^{n+1} \) the interaction \( H_I \) given by (2.10)-(2.12) but associated with the following kernels
\[
\tilde{\chi}^{\sigma_n}(p_2) G^{(\alpha)}_{\ell,e,e} (\xi_1, \xi_2, \xi_3),
\]
and
\[
(\tilde{\chi}^{\sigma_n+1}(p_2) - \tilde{\chi}^{\sigma_n}(p_2)) G^{(\alpha)}_{\ell,e,e} (\xi_1, \xi_2, \xi_3),
\]
respectively, where \( \tilde{\chi}^{\sigma_n+1} \) is defined by (3.1).

Let for \( n \geq 0 \),
\[
H_+ = H_n - E^n, \\
\tilde{H}_+ = H_+^n \otimes 1_{n+1} + 1_n \otimes H_0^{n+1}.
\]
The operators \( H_+^n \) and \( \tilde{H}_+^n \) are self-adjoint operators in \( \tilde{\mathcal{F}}^{n+1} \) and \( \mathcal{F}^{n+1} \) respectively.

Here \( 1_n \) and \( 1_{n+1} \) are the identity operators in \( \mathcal{F}^n \) and \( \tilde{\mathcal{F}}^{n+1} \) respectively.

Combining (2.29) and (2.30) with (3.3)-(3.6) we obtain for \( n \geq 0 \),
\[
(A.1) \quad g\|H_+^n \psi\| \leq gK(G)(C_{\beta n}\|H_0 \psi\| + B_{\beta n}\|\psi\|),
\]
for every \( \psi \in \mathcal{D}(H_+^n) \subset \mathcal{F}^n \).

It follows from [22, §V, Theorem 4.11] that
\[
H^n \geq -gK(G)B_{\beta n} 1 - gK(G)C_{\beta n} \geq -\frac{g_1 K(G)B_{\beta n}}{1 - g_1 K(G)C_{\beta n}},
\]
and
\[
E^n \geq -\frac{gK(G)B_{\beta n}}{1 - g_1 K(G)C_{\beta n}},
\]

We have
\[
(A.2) \quad (\Omega^n, H^n \Omega^n) = 0.
\]
Therefore
\[
E^n \leq 0,
\]
and

(A.3) \[ |E^n| \leq \frac{gK(G)B_{\beta\eta}}{1 - g_1K(G)C_{\beta\eta}}. \]

Let

(A.4) \[ K_{n+1}^n(G) = K(1_{\sigma_{n+1} \leq |p_2| \leq |2\sigma_n|} G). \]

Combining (2.29) and (2.30) with (3.3), (3.4) and (A.4) we obtain for \( n \geq 0 \)

(A.5) \[ g\|H_{n+1}^n\psi\| \leq gK_{n+1}^n(G)(C_{\beta\eta}\|H_0^1\psi\| + B_{\beta\eta}\|\psi\|), \]

for \( \psi \in D(H_0^{n+1}) \subset \mathfrak{F}^{n+1} \), where we remind that \( H_0^{n+1} = H_0|_{\mathfrak{F}^{n+1}} \) as defined in (3.2).

We have for every \( \psi \in D(H_0^{n+1}) \),

(A.6) \[ H_0^{n+1}\psi = \tilde{H}_n^+\psi + E^n\psi - g(H_1^n \otimes 1_{n+1}^n)\psi, \]

and by (A.4)

(A.7) \[ g\|(H_1^n \otimes 1_{n+1}^n)\psi\| \leq gK(G)(C_{\beta\eta}\|H_0^1\psi\| + B_{\beta\eta}\|\psi\|). \]

In view of (A.3) and (A.6) it follows from (A.7) that

(A.8) \[ g\|(H_1^n \otimes 1_{n+1}^n)\psi\| \leq \frac{gK(G)C_{\beta\eta}}{1 - g_1K(G)C_{\beta\eta}}\|\tilde{H}_n^+\psi\| + \frac{gK(G)B_{\beta\eta}}{1 - g_1K(G)C_{\beta\eta}}(1 + \frac{gK(G)B_{\beta\eta}}{1 - g_1K(G)C_{\beta\eta}})\|\psi\|. \]

By (3.7), (3.8), (A.5), (A.6), (A.8) we finally get

(A.9) \[ g\|H_{n+1}^n\psi\| \leq gK_{n+1}^n(G)(\tilde{C}_{\beta\eta}\|\tilde{H}_n^+\psi\| + \tilde{B}_{\beta\eta}\|\psi\|). \]

For \( n \geq 0 \), a straightforward computation yields

(A.10) \[ K_{n+1}^n(G) \leq \sigma_n\tilde{K}(G) \leq \sup(\frac{4\lambda_\gamma}{2m_1 - \delta}, 1) \tilde{K}(G)\sigma_{n+1}. \]

Recall that for \( n \geq 0 \),

(A.11) \[ \sigma_{n+1} < m_1. \]

By (A.9), (A.10) and (A.11), we get, for \( \psi \in D(H_0) \),

\[ g\|H_{n+1}^n\psi\| \leq gK_{n+1}^n(G)(\tilde{C}_{\beta\eta}\|\tilde{H}_n^+ + \sigma_{n+1}\psi\| + (\tilde{C}_{\beta\eta}m_1 + \tilde{B}_{\beta\eta})\|\psi\|), \]

and for \( \phi \in \mathfrak{G} \),

(A.12) \[ g\|H_{n+1}^n(\tilde{H}_n^+ + \sigma_{n+1})^{-1}\phi\| \leq gK_{n+1}^n(G)(\tilde{C}_{\beta\eta} + \frac{m_1\tilde{C}_{\beta\eta} + \tilde{B}_{\beta\eta}}{\sigma_{n+1}})\|\phi\| \]

\[ \leq g\sup(\frac{4\lambda_\gamma}{2m_1 - \delta}, 1) \tilde{K}(G)(2m_1\tilde{C}_{\beta\eta} + \tilde{B}_{\beta\eta})\|\phi\|. \]

Thus, by (A.12), the operator \( H_{n+1}^n(\tilde{H}_n^+ + \sigma_{n+1})^{-1} \) is bounded and

\[ g\|H_{n+1}^n(\tilde{H}_n^+ + \sigma_{n+1})^{-1}\| \leq \frac{\tilde{D}}{\gamma}, \]

where \( \tilde{D} \) is given by (see (3.9))

\[ \tilde{D} = \sup(\frac{4\lambda_\gamma}{2m_1 - \delta}, 1) \tilde{K}(G)(2m_1\tilde{C}_{\beta\eta} + \tilde{B}_{\beta\eta}). \]
This yields, for $\psi \in D(\tilde{H}_+^n)$,
\[ g\|H_{I_n}^{n+1}\psi\| \leq g\frac{\tilde{D}}{\gamma}\|\tilde{H}_+^n + \sigma_{n+1}\psi\|. \]
Hence it follows from \[22, \S V, \text{Theorems 4.11 and 4.12}\] that
\[ g\|H_{I_n}^{n+1}\psi, \psi\| \leq g\frac{\tilde{D}}{\gamma}\|\tilde{H}_+^n + \sigma_{n+1}\psi, \psi\|. \quad (A.13) \]
Let $g_2(2) > 0$ be such that
\[ g_2(2)\frac{\tilde{D}}{\gamma} < 1 \quad \text{and} \quad g_2(2) \leq g_1(1). \]
By (A.13) we get, for $g \leq g_2(2)$,
\[ H_{I_n}^{n+1} = \tilde{H}_+^n + E^n + gH_{I_n}^{n+1} \geq E^n - \frac{g\tilde{D}}{\gamma}\sigma_{n+1} + (1 - \frac{g\tilde{D}}{\gamma})\tilde{H}_+^n. \quad (A.14) \]
Because $(1 - g\tilde{D}/\gamma)\tilde{H}_+^n \geq 0$ we get from (A.14)
\[ E_{n+1} \geq E^n - \frac{g\tilde{D}}{\gamma}\sigma_{n+1}, \quad n \geq 0. \quad (A.15) \]
Suppose that $\psi^n \in \mathfrak{F}^n$ satisfies $\|\psi^n\| = 1$ and for $\epsilon > 0$,
\[ (\psi^n, H^n\psi^n) \leq E^n + \epsilon. \quad (A.16) \]
Let
\[ \tilde{\psi}^{n+1} = \psi^n \otimes \Omega_{n+1} \in \mathfrak{F}^{n+1}. \quad (A.17) \]
We obtain
\[ E_{n+1} \leq (\tilde{\psi}^{n+1}, H_{n+1}^{n+1}\tilde{\psi}^{n+1}) \leq E^n + \epsilon + g(\tilde{\psi}^{n+1}, H_{I_n}^{n+1}\tilde{\psi}^{n+1}) \quad (A.18) \]
By (A.13), (A.16), (A.17) and (A.18) we get, for every $\epsilon > 0$,
\[ E_{n+1} \leq E^n + \epsilon(1 + g\frac{\tilde{D}}{\gamma}) + g\frac{\tilde{D}}{\gamma}\sigma_{n+1}, \]
where $g \leq g_2(2)$.
This yields
\[ E_{n+1} \leq E^n + g\frac{\tilde{D}}{\gamma}\sigma_{n+1}, \quad (A.19) \]
and by (A.15), we obtain
\[ |E^n - E_{n+1}| \leq g\frac{\tilde{D}}{\gamma}\sigma_{n+1}. \]
For $n = 0$, since $\sigma_0 = \Lambda$, remind that $H_0^0 = H_0^{0|0} = H_0^{0|\Lambda} = H_0|\Lambda$. Thus, the ground state energy of $H_0^0$ is 0 and it is a simple isolated eigenvalue of $H_0^0$ with $\Omega^0$, the vacuum in $\mathfrak{F}^0$, as eigenvector. Moreover, since $\Lambda > m_1$,
\[ \inf(\sigma(H_0^0) \setminus \{0\}) = m_1, \]
thus $(0, m_1)$ belongs to the resolvent set of $H_0^0$. 

By Hypothesis 3.1(iv) we have $H^0 = H_0^0$. Hence $E^0 = \{0\}$ is a simple isolated eigenvalue of $H^0$ and $H^0 = \hat{H}_+^0$. We finally get

\[ \inf \left( \sigma(H_+^0) \setminus \{0\} \right) = m_1 \geq m_1 - \frac{\delta}{2} = \sigma_1. \]

We now prove Proposition 3.5 by induction in $n \in \mathbb{N}^*$. Suppose that $E^n$ is a simple isolated eigenvalue of $H^n$ such that

\[ \inf \left( \sigma(H^n_+) \setminus \{0\} \right) \geq (1 - 3g\tilde{D}\gamma)\sigma_n, \quad n \geq 1. \]

Since (3.10) gives $\sigma_{n+1} < (1 - 3g\tilde{D}\gamma)\sigma_n$ for $g \leq g^{(2)}$, 0 is also a simple isolated eigenvalue of $\hat{H}_+^n$ such that

\[ \inf \left( \sigma(\hat{H}_+^n) \setminus \{0\} \right) \geq \sigma_{n+1}. \]

We must now prove that $E^{n+1}$ is a simple isolated eigenvalue of $H^{n+1}$ such that

\[ \inf \left( \sigma(H_+^{n+1}) \setminus \{0\} \right) \geq (1 - 3g\tilde{D}\gamma)\sigma_{n+1}. \]

Let

\[ \lambda^{(n+1)} = \sup_{\psi \in \mathfrak{F}^{n+1}; \psi \neq 0} \inf_{\phi \in D(H^{n+1}); \|\phi\| = 1} (\phi, H_+^{n+1}\phi). \]

By (A.20) and (A.21), we obtain, in $\mathfrak{F}^{n+1}$

\[ H_+^{n+1} = E^n - E^{n+1} - \frac{g\tilde{D}}{\gamma}\sigma_{n+1} + (1 - \frac{g\tilde{D}}{\gamma})\hat{H}_+^n \]

\[ \geq (1 - \frac{g\tilde{D}}{\gamma})\hat{H}_+^n - 2g\tilde{D}\gamma\sigma_{n+1}. \]

By (A.17), $\hat{\psi}^{n+1}$ is the unique ground state of $\hat{H}_+^n$ and by (A.21) and (A.22), we have, for $g \leq g^{(2)}$,

\[ \lambda^{(n+1)} \geq (\phi, \hat{H}_+^{n+1}\phi) \]

\[ \geq (1 - \frac{g\tilde{D}}{\gamma})\sigma_{n+1} - 2g\tilde{D}\gamma\sigma_{n+1} = (1 - \frac{3g\tilde{D}}{\gamma})\sigma_{n+1} > 0. \]

This concludes the proof of Proposition 3.5 by choosing $g_\delta = g^{(2)}$, if one proves that $H^1$ satisfies Proposition 3.5. By noting that 0 is a simple isolated eigenvalue of $\hat{H}_+^0$ such that $\inf(\sigma(\hat{H}_+^0) \setminus \{0\}) = \sigma_1$, we prove that $E^1$ is indeed an isolated simple eigenvalue of $H^1$ such that $\inf(\sigma(H_+^1) \setminus \{0\}) \geq (1 - \frac{3g\tilde{D}}{\gamma})\sigma_1$ by mimicking the proof given above for $H_+^{n+1}$.

\[ \square \]

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