Fidelity susceptibility and Loschmidt echo for generic paths in a three-spin interacting transverse Ising model

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Abstract. We study the effect of the presence of different types of critical points, such as ordinary critical points, multicritical points and quasicritical points, along different paths on the fidelity susceptibility and Loschmidt echo of a three-spin interacting transverse Ising chain using a method that does not involve the language of tensors. We find that the scaling of the fidelity susceptibility and Loschmidt echo with the system size at these special critical points of the model studied is in agreement with the known results, thus supporting our method.

Keywords: spin chains, ladders and planes (theory), quantum phase transitions (theory), entanglement in extended quantum systems (theory)
1. Introduction

Zero temperature quantum phase transitions have been an active area of research for more than two decades [1]–[3]. One of the main motives of these studies is to identify the quantum critical points (QCPs) where the ground state changes significantly from one form to another and to study the scaling behavior of various quantities around these critical points. Traditional methods like Landau–Ginzburg theory [4] and the relatively new methods of studying the fidelity and fidelity susceptibility [5, 6], which are derived from a completely different area of research, namely, quantum information theory, have provided a lot of insight into the physics of quantum phase transitions.

In this paper, we will be focusing on some of the quantum information theoretic measures like the fidelity susceptibility $\chi_F$, which defines the rate at which the fidelity changes in the limit when the two parameters are close to each other. Here, the fidelity is the measure of the overlap of the ground state wavefunction at two different values of the parameters of the Hamiltonian. A dip in the fidelity or a peak in the $\chi_F$ as a function of the system parameter signals the presence of a phase transition. These measures show interesting scaling behavior close to the critical point, attracting a great deal of attention from the scientific community. For example, the universal scaling relation of $\chi_F$ with the system size at the QCP ($\lambda = 0$) and with respect to finite but small $\lambda$ is given in terms of some of the critical exponents associated with the quantum critical point. It is well established that for a $d$-dimensional system of length $L$, the scaling form of $\chi_F$ (see [7]–[11]) at the critical point, say $\lambda = 0$, is given by $\chi_F \sim L^{2/\nu-d}$, whereas away from the QCP...
The scaling takes the form $\chi_F \sim |\lambda|^{\nu d - 2}$ with $\nu d < 2$ [6]. For $\nu d > 2$, contributions from high energy modes to the fidelity susceptibility cannot be ignored. However, it has been shown that for some models with $\nu d > 2$, the fidelity susceptibility can be used to determine the critical point provided one uses twisted boundary conditions [12]. On the other hand, in the marginal case $\nu d = 2$, $\chi_F$ shows logarithmic scaling with $L$ and $\lambda$ [13]; here, $\nu$ is the critical exponent associated with the divergence of the correlation length at the QCP. The fact that no previous knowledge about the order parameter or the symmetry of the system is required to locate the critical points adds to the popularity of these measures [14]. The success of these measures in detecting quantum critical points in a given system is remarkable. On the other hand, there are examples of quantum phase transitions that cannot be captured using the general definition of fidelity and fidelity susceptibility [12, 15].

We will be studying one more information theoretic measure in this paper, namely the Loschmidt echo (LE) [15]–[22]. The LE is the overlap of two wavefunctions; one is the ground state wavefunction $|\psi_G\rangle$ of a Hamiltonian $H(\lambda)$ which evolves as $e^{-iH(\lambda)t}|\psi_G\rangle$, and the other is the same state but evolving under the slightly different Hamiltonian $H' = H(\lambda + \delta)$. The LE also shows a dip at the QCP, thus enabling its detection. In the language of quantum information theory, it can be used to detect the quantum to classical transition of a spin-1/2 qubit coupled to a many body system undergoing a quantum phase transition [16, 17]. The notion of the LE was actually introduced in connection with the quantum to classical transition in quantum chaos [23]–[28] and has now been extended to various other systems undergoing a QPT like Ising model [16], Bose–Einstein condensate [29] or the Dicke model [30]. It has also been studied experimentally using NMR experiments [31]–[33].

In this paper, we study the scaling of $\chi_F$ and the LE along different paths of a three-spin interacting transverse Ising model which consists of ordinary critical points, multiritical points and quasicritical points. The method used for this path-dependent study is new and simpler than the conventional method adopted which involves tensors [9, 34]. To the best of our knowledge, none of the previous studies on $\chi_F$ or the LE have considered the path dependence in as much detail as in this paper.

2. The model

The Hamiltonian of a one-dimensional three-spin interacting Ising system of length $L$ in the presence of a transverse field $h$ is given by [35, 36]

$$H = -\frac{1}{2} \sum_{n=1}^{L} \left[ \sigma_n^z (h + J_3 \sigma_{n-1}^x \sigma_{n+1}^x) + J_x \sigma_n^x \sigma_{n+1}^x \right]$$

(1)

where $\sigma^z$ and $\sigma^x$ are the usual Pauli spin matrices, $J_3$ is the three-spin coupling strength connecting spins at sites $n$, $n - 1$ and $n + 1$, and $J_x$ is the coupling constant of the nearest neighbor ferromagnetic interaction in the $x$ direction. Although the three-spin interaction term in the Hamiltonian makes it appear difficult to solve, the above Hamiltonian can be diagonalized using the standard Jordan–Wigner (JW) transformation [37]–[39], which maps an interacting spin-1/2 system to a system of spinless fermions. The Jordan–Wigner transformation maps an interacting spin-1/2 system to a system of spinless fermions.
transformation relations between spins and fermions are defined as

\[ c_n = \left( \prod_{j=1}^{n-1} \sigma_j^z \right) \sigma_n^- \]

\[ \sigma_n^z = 2c_n^\dagger c_n - 1 \]  \hspace{1cm} (2)

where \( \sigma_n^\pm = (\sigma_n^x \pm \sigma_n^y)/2 \), and \( c_n, c_n^\dagger \) are fermionic annihilation and creation operators respectively with usual anticommutation relations. Substituting the \( \sigma \)-operators by the JW fermions \( c_i \) and performing a Fourier transformation, the Hamiltonian (1) takes the form

\[
H = -\sum_k \left[ \left( h + J_x \cos k - J_3 \cos 2k \right) (c_k^\dagger c_k + c_{-k}^\dagger c_{-k}) \right. \\
+ \left. i \left( J_x \sin k - J_3 \sin 2k \right) (c_k^\dagger c_{-k} + c_k c_{-k}) \right] = \sum_k H_k. \]  \hspace{1cm} (3)

The Hamiltonian \( H_k \) is a \( 2 \times 2 \) matrix when written in the basis \( |0\rangle \) (with 0 \( c \)-fermions) and \( |k, -k\rangle \) (\( =c_k^\dagger c_{-k}^\dagger |0\rangle \)), and has the form

\[
H_k = \begin{bmatrix} h + J_x \cos k - J_3 \cos 2k & J_x \sin k - J_3 \sin 2k \\ J_x \sin k - J_3 \sin 2k & -(h + J_x \cos k - J_3 \cos 2k) \end{bmatrix}. \]  \hspace{1cm} (4)

The above Hamiltonian can be diagonalized after a rotation by an angle \( \theta_k/2 \), which is given by

\[
\tan \theta_k = \frac{J_x \sin k - J_3 \sin 2k}{h + J_x \cos k - J_3 \cos 2k} \]  \hspace{1cm} (5)

with the corresponding eigen energy for the \( k \)th mode as

\[
\varepsilon_k = \left( h^2 + J_3^2 + J_x^2 + 2hJ_x \cos k - 2hJ_3 \cos 2k - 2J_xJ_3 \cos k \right)^{1/2}. \]  \hspace{1cm} (6)

The diagonalized Hamiltonian can now be written as

\[
H = \sum_k \varepsilon_k (\eta_k^\dagger \eta_k - 1/2) \]  \hspace{1cm} (7)

where \( \eta_k \) is the quasiparticle corresponding to the Hamiltonian \( H \).

Now, from equation (6) one can easily verify that the low energy excitation gap vanishes on the critical lines \( h = J_3 + J_x \) and \( h = J_3 - J_x \) for the wavevectors \( k = \pi \) and 0, respectively. These two lines are the critical lines separating two phases, the ferromagnetically ordered phase and the paramagnetic phase. The long-range order in the ferromagnetic phase is present only for a weak transverse field lying in the range \( J_3 - J_x < h < J_3 + J_x \). The associated quantum critical exponents with these QPTs are the same as in the one-dimensional transverse Ising model with \( \nu = z = 1 \), where \( \nu \) and \( z \) are the correlation length and dynamical exponents, respectively [35]. There is also another phase transition at \( h = -J_3 \) between the three-spin dominated phase and the...
Figure 1. The phase diagram of the three-spin interacting transverse Ising model along with the various paths studied for approaching the MCPs. The point A corresponds to one of the multicritical points. The phase boundaries are marked by the three different lines, as shown in the key, whereas the paths studied in this paper are I, II, III and IV, as shown by the lines with arrows. Path IV is also the gapless line separating various phases. The shaded region corresponds to the region where quasicritical points exist.

The quantum paramagnetic phase. This phase transition is analogous to the anisotropic phase transition seen in the one-dimensional transverse XY model. The ordering wavevector \( k_0 \) in this case is parameter dependent and is given by

\[
\cos k_0 = \frac{J_x(h - J_3)}{4hJ_3}.
\]  

On the critical line \( h = -J_3 \), the incommensurate wavevector \( k_0 \) takes a value such that \( \cos k_0 = J_x/2J_3 \), which implies that the anisotropic transition cannot occur for \( J_3 < J_x/2 \). The two critical lines \( h = J_3 + J_x \) and \( h = -J_3 \) meet at a point, called the multicritical point (MCP), which will be the focus of attention in this paper. Another MCP occurs at the intersection of \( h = -J_3 \) and \( h = J_3 - J_x \). The critical exponents associated with these MCPs are given by \( z = 2 \) and \( \nu = 1/2 \). It can be shown that near the multicritical points, there may exist some special points called quasicritical points, which, although they do not play a role in determining the phase diagram of the model, do affect the scaling of various quantities. This is because, at these quasicritical points, the energy \( \epsilon_k \) for modes close to the critical mode has a local minimum shifted from the critical point.

The phase diagram of the model for \( J_3 = -1 \) is shown in figure 1. We study the scaling of \( \chi_F \) and the LE along different paths in this phase diagram containing ordinary critical points, multicritical points and quasicritical points, and in the process develop a scheme that can be extended to study any type of path.
3. Fidelity susceptibility

As defined before, the fidelity \( F \) is defined as the overlap of the ground state wavefunction at parameter values separated by a distance \( \delta \), whereas \( \chi_F \) is the rate at which the fidelity changes with a parameter of the Hamiltonian [5]–[13], [40, 41]. There are many mathematical forms for calculating \( \chi_F \) [5], [42]–[44]. We shall be focusing on one particular form given by

\[
\chi_F = \frac{1}{4L} \sum_k \left( \frac{d\theta_k}{d\lambda} \right)^2
\]

where \( \theta_k/2 \) is the angle by which the Hamiltonian needs to be rotated so that it is diagonalized; see the discussion around equation (5). We study the behavior of the fidelity susceptibility along four different paths, all of them crossing the multicritical point, with an approach that does not require the language of tensors as is used in previous studies [9, 34]. We chose these paths for specific reasons. Paths I and II cross quasicritical points along with the multicritical point, whereas path IV is a gapless line which does not have any quasicritical point. Path III is a special path containing quasicritical points but very close to the critical line which might show some interesting behavior. The Hamiltonian (1) has three parameters. For convenience we fix \( J_3 = -1 \) and work in the parameter space spanned by \( h \) and \( J_x \). We have repeated the calculations for \( J_3 = 1 \) and no major differences were observed. The paths studied in this paper are shown in figure 1.

In our approach to calculating \( \chi_F \) along a path, we rewrite the Hamiltonian in terms of only one variable using the equation of the path. We then rotate \( H_k \) by an angle \( \phi_k \) such that the path or the variable that is changed, say \( \lambda \), is brought to the diagonal term. We then evaluate the \( \chi_F \) using equation (9) after calculating the angle \( \theta_k \). We briefly mention the method of evaluating the angle \( \phi_k \) below. Let \( R \) be the rotation matrix with elements \( R(1, 1) = \cos(\phi_k) = R(2, 2) \) and \( R(2, 1) = -R(1, 2) = \sin(\phi_k) \). Rewriting \( H_k \) in terms of only one variable \( \lambda \) and performing the rotation by an angle \( \phi_k \) results in a matrix \( H'_k = R^T H_k R \). The angle \( \phi_k \) is then evaluated by demanding that the off-diagonal term in \( H'_k \) be \( \lambda \) independent. After substituting for the angle \( \phi_k \), the diagonal term in general will have the form \( a_k \lambda + b_k \) and the off-diagonal term will have the form \( c_k \). Let us assume \( b_k \sim k^{z_1} \) and \( c_k \sim k^{z_2} \) when expanded near the critical mode. When \( z_1 < z_2 \), the exponent \( z \) corresponding to the scaling of \( \varepsilon_k \) at the quantum critical point \( \lambda = 0 \) is equal to \( z_1 \), i.e., \( \varepsilon_k \sim k^{z_1} \) at \( \lambda = 0 \). On the other hand, there can arise situations where the path shows energy minima at \( a_k \lambda_0 + b_k = 0 \) such that \( \varepsilon_k \sim k^{z_2} \) at these special points \( \lambda_0 \), also called quasicritical points [45, 46]. It has been shown that it is the exponent \( z_2 \), which is different from the actual exponent \( z \) at the critical point, that will dominate the scaling of various quantities when quasicritical points exist. When there is no quasicritical point along the path, then only \( \lambda = 0 \) will be a minimum of the energy, and \( \varepsilon_k \) along with other quantities will scale as \( k^{z_1} \), \( z \) being the minimum of \( z_1 \) and \( z_2 \), as is the case in path IV. On the other hand, if \( z_1 > z_2 \), then the dynamics will always be governed by the exponent \( z_2 \), independent of the presence of a quasicritical point.

Below, we present our results on the fidelity susceptibility along the four paths crossing the multicritical point and discuss the effect of the presence of quasicritical points in each.
path. Using the definition of $\chi_F$ in equation (9), we get

$$\left( \frac{\partial \theta_k}{\partial \lambda} \right)^2 = \frac{a_k^2 c_k^2}{\varepsilon_k^4}$$

where $\theta_k = \tan^{-1}(c_k/(a_k \lambda + b_k))$. We shall write explicit expressions for $a_k$, $b_k$, $c_k$ and $\phi_k$ obtained by making the off-diagonal term in the Hamiltonian in equation (4) independent of $\lambda$ for each path.

### 3.1. Path I

In the first path that we consider, we fix $J_x = 2$ and approach the multicritical point $h = 1$ by varying $\lambda = h - 1$. In this case, the off-diagonal term in the Hamiltonian $H_k$ in equation (4) is already $\lambda$ independent, i.e., $H_k(1, 1) = \lambda + 2 \cos k + \cos 2k = -H_k(2, 2)$ and $H_k(1, 2) = H_k(2, 1) = 2 \sin k + \sin 2k$. Expanding around the critical mode $k_c = \pi$, we get $a_k \sim 1$, $b_k \approx -k^2$ and $c_k \approx -k^3$. With $\theta_k = \tan^{-1}(H_k(1, 2)/H_k(1, 1))$, the susceptibility is given by

$$\chi_F = \frac{1}{4L} \sum_{k > 0} \left( \frac{\partial \theta_k}{\partial \lambda} \right)^2 = \frac{1}{4L} \sum_{k > 0} \frac{(2 \sin k + \sin 2k)^2}{\varepsilon_k^4} \approx \frac{1}{4L} \sum_{k > 0} \frac{k^6}{\varepsilon_k^4}$$

which gives rise to $L^5$ scaling as $\varepsilon_k \sim k^3$ at the quasicritical point $\lambda \sim k^2$. Here, we have redefined $(k - k_c)$ as $k$ and expanded $a_k$, $b_k$ and $c_k$ around $k \to 0$, which will be followed throughout the paper. Note that here $z_1 = 2$ and $z_2 = 3$ such that $z = z_1$, but the scaling of $\chi_F$ is dictated by $z_2$.

The variation of $\chi_F$ as a function of $h$ is shown in figure 2, whereas its behavior close to the multicritical point $h = 1$ shows oscillations, as is shown in the inset of the same figure. These oscillations can be explained as follows. The quasicritical point occurs when $\lambda = -b_k/a_k$. Since the momentum $k$ is quantized in units of $2\pi/L$, all the allowed momenta near the critical mode $k_c$, i.e. $k = k_c + 2\pi m/L$ for integer $m$, will also show $\varepsilon_k \sim k^3$ behavior. Each value of $k$ will give rise to a different value of $\lambda$ close to $\lambda = 0$, resulting in more than one quasicritical point near the multicritical point [9]. The scaling of $\chi_F$ with $L$ along this path is shown in figure 3(a) for the first two peaks occurring in the $\chi_F$–$\lambda$ plot (see the inset of figure 2).

### 3.2. Path II

We now consider the path $h + J_x = 3$, path II in figure 1, and approach the MCP by varying $\lambda = h - 1$. After performing a rotation by an angle $\phi_k$ to bring $\lambda$ to the diagonal term, we get

$$a_k = \cos 2\phi_k - \cos(k - 2\phi_k)$$
$$b_k = \cos 2\phi_k + 2 \cos(k - 2\phi_k) + \cos(2k - 2\phi_k)$$
$$c_k = -\sin 2\phi_k + 2 \sin(k - 2\phi_k) + \sin(2k - 2\phi_k)$$

and

$$\tan(2\phi_k) = \frac{\sin k}{-1 + \cos k}.$$
Figure 2. The variation of $\chi_F$ as a function of $h$ at $J_x = 2, J_3 = -1$ for a system size of $L = 100$. The first peak at $h = -3$ corresponds to the Ising critical point, showing linear scaling with $L$, and the second peak is at the MCP, i.e., at $h = 1$ where $L^5$ scaling is observed. The inset shows the oscillating fidelity susceptibility close to the multicritical point, pointing to the presence of quasicritical points.

When expanded around the critical mode, $a_k \approx 2, b_k \approx -k^2, c_k \approx -k^3/2$, resulting in a quasicritical point at $\lambda = k^2/2$ where $\varepsilon_k \sim k^3$. Since $\theta_k = \tan^{-1}(H_k(1, 2)/H_k(1, 1))$, $\chi_F$ is given by

$$\chi_F = \frac{1}{4L} \sum_{k > 0} a_k^2 c_k^2 \varepsilon_k^4 \approx \frac{1}{4L} \sum_{k > 0} k^6 \varepsilon_k^4.$$ (11)

This once again results in $L^5$ scaling, as also confirmed numerically in figure 3(b).

3.3. Path III

The Hamiltonian $H_k$ after rotation by an angle $\phi_k$ along the path $h - 0.9J_x = -0.8$ has the following elements:

$$a_k = \cos 2\phi_k + \frac{10}{9} \cos(k - 2\phi_k)$$

$$b_k = \cos 2\phi_k + 2 \cos(k - 2\phi_k) + \cos(2k - 2\phi_k)$$

$$c_k = -\sin 2\phi_k + 2 \sin(k - 2\phi_k) + \sin(2k - 2\phi_k)$$

with $\tan(2\phi_k) = \frac{10/9 \sin k}{1 + 10/9 \cos k}$. (12)

After expanding close to the critical mode $k_c = \pi$, we get $a_k \approx -1/9, b_k \approx -k^2$ and $c_k \approx 9k^3$ with a quasicritical point at $\lambda = -9k^2$. Since $\varepsilon_k \sim k^3$ at the quasicritical point, $\chi_F \sim L^5$, as also shown in figure 3(c). Along all the above three paths, quasicritical points
Figure 3. The scaling of $\chi_F$ along four different paths: (a), (b) and (c) correspond to paths I, II and III with $\chi_F \propto L^5$, showing the effect of the multicritical point, whereas path IV is linear in $L$ as there is no quasicritical point along this line or path.

exist either in the paramagnetic phase or in the three-spin dominated phase of the system close to the MCP. It can be seen in figure 1 that path III is very close to the critical line $h = J_x - 1$, which, as discussed in the next sub-section, does not have any quasicritical point. We chose this path to check the effect of this proximity on the scaling behavior of the fidelity susceptibility. Although we do observe $L^5$ scaling, it is present only for large $L$, and the small deviation for smaller $L$ which is not seen in paths I and II could be due to its proximity to the critical line. Let us try to explore this path further. The quasicritical point in this path exists at $\lambda = -9k^2$, where $k$ is inversely proportional to $L$. The factor of 9 compared to 1 in path I and $1/2$ in path II shifts the location of the quasicritical point farther away from the actual critical point. Since we expanded $a_k$, $b_k$ and $c_k$ around the critical mode and the critical point, which may not be correct in this path for small $L$, we observe a deviation from $L^5$ scaling for small $L$.

3.4. Path IV

We finally consider the critical line $h - J_x = -1$ with $\lambda = h - 1$. Performing rotation by an angle $\phi_k$ to make the off-diagonal term of $H_k$ in equation (4) $\lambda$ independent, we get
the following form of the functions for $a_k$, $b_k$, $c_k$ and $\phi_k$:

\[ a_k = \cos 2\phi_k + \cos(k - 2\phi_k) \]
\[ b_k = \cos 2\phi_k + 2\cos(k - 2\phi_k) + \cos(2k - 2\phi_k) \]
\[ c_k = -\sin 2\phi_k + 2\sin(k - 2\phi_k) + \sin(2k - 2\phi_k) \] (13)

and \[ \tan(2\phi_k) = \frac{\sin k}{1 + \cos k} \]

which when expanded around the critical mode $k_c = \pi$ gives $a_k \approx k$, $b_k \approx k^3$ and $c_k \approx -k^2$.

We note that for paths I, II and III, the $a_k$s are independent of $k$ and we get quasicritical points with minimum energy. However, for path IV, $a_k$ is $k$-dependent and its exponent is less than $z_1$, so that we can ignore the term $b_k$ for non-zero $\lambda$. Thus, for all non-zero $\lambda$, $\varepsilon_k$ goes as $k$, i.e., path IV is a critical line, and we cannot get any quasicritical point near the MCP for which the energy is minimum. Since $\varepsilon_k \sim k^2$ at the MCP $\lambda = 0$, which is also the dominant point, $\chi_F$ scales linearly with $L$, as also confirmed numerically in figure 3(d).

4. Loschmidt echo (LE)

As mentioned above, the LE is defined as the overlap between two states differing from each other in the Hamiltonian with which they are evolving but both starting from the ground state of one of the Hamiltonians \[16, 21, 22\]. Mathematically, if $|\psi_G\rangle$ is the ground state of Hamiltonian $H(\lambda)$ with energy $E_g$, then the LE or $L$ is given by

\[ L(\lambda, t) = |\langle \psi(\lambda, t) | \psi(\lambda, t) \rangle|^2 = |\langle \psi(\lambda + \delta, t) | \psi_G \rangle|^2 \] (14)

where $|\psi(\lambda, t)\rangle = e^{-iH(\lambda)t}|\psi_G\rangle = e^{-iE_g t}|\psi_G\rangle$ and $|\psi(\lambda + \delta, t)\rangle = e^{-iH(\lambda + \delta)t}|\psi_G\rangle$, and $t$ corresponds to time. It is easier to calculate the LE in the momentum representation by noting the fact that the Hamiltonian is decoupled in the momentum space and hence the ground state wavefunction can be written as

\[ |\psi_G\rangle = \prod_k |\phi_k\rangle = \prod_k \cos(\theta_k^A/2)|0\rangle + \sin(\theta_k^A/2)|-k\rangle \] (15)

where $|\phi_k\rangle$ is the ground state of $H_k(\lambda)$, and $\theta_k^A/2$, as before, is the angle by which $H_k(\lambda)$ needs to be rotated to diagonalize it. Thus,

\[ L(\lambda, t) = \prod_k L_k(\lambda, t) = \prod_k |\langle \phi_k | e^{iH_0(\lambda + \delta)t} | \phi_k \rangle|^2. \]

To calculate the above expression, it is to be noted that $|\phi_k\rangle$ is not an eigenstate of $H_k(\lambda + \delta)$. Therefore, one needs to find an expression of $|\phi_k\rangle$ in terms of eigenstates of $H_k(\lambda + \delta)$ which we denote as $|1\rangle$ and $|2\rangle$. It can be shown that

\[ |\phi_k\rangle = \cos \alpha_k |1\rangle + \sin \alpha_k |2\rangle \] (16)

where $2\alpha_k = \theta_k^A - \theta_k^{A+\delta}$. Substituting this form of $|\phi_k\rangle$ in the expression of the LE, we get

\[ L(\lambda, t) = \prod_k L_k(\lambda, t) = \prod_k (1 - \sin^2 2\alpha_k \sin^2(\varepsilon_k(\lambda + \delta)t)). \] (17)
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Figure 4. The LE shows sharp dips at the Ising critical points \((h = -2, 0)\) and at the anisotropic critical point at \(h = 1\).

We shall be using the above expression for the calculation of the LE, taking into account the effect of the path in \(\alpha_k\) and \(\varepsilon_k\). To analyze the behavior of the LE, we define a partial sum \(S = \ln \mathcal{L}\) along lines similar to [16].

We first demonstrate the applicability of the LE as a tool to detect the presence of a critical point by taking a path parallel to path I of the previous section at \(J_x = 1\) that has three critical points as \(h\) is varied for a fixed time \(t\). Figure 4 shows that the LE can successfully detect all the critical points in its path. It was shown in [16] that at the critical point, the LE shows decay and revival as a function of time, which is an indicator of the presence of a critical point. The time period of the oscillations is proportional to \(L\) in the case of an Ising critical point but can vary in a non-linear way for other types of critical points [21]. We shall demonstrate the difference in the LE behavior between an anisotropic critical point (or any critical point), the multicritical point and also the quasicritical point by studying the behavior of the short time decay and the time period of LE oscillations [16] for the three-spin interacting model discussed in section 2. In this case also, we use the same method of including the effect of the path as discussed in section 3.

4.1. Anisotropic critical point (ACP)

Short time decay: on the ACP line (see the discussion around equation (8)), the energy gap vanishes for the critical mode \(k_c = \cos^{-1}(-J_x/2)\). Now, to study the short time behavior of the LE, we fix \(J_x = 1\) and change \(h\). Note that the transverse field \(h\) in the Hamiltonian \(H_k\) is already in the diagonal term and hence need not be rotated similarly to path I in section 3. We now expand equation (17) around the critical mode \(k_c = \cos^{-1}(-1/2)\) to obtain

\[
\sin^2 \varepsilon_k(h + \delta)t \approx (h + \delta - 1)^2t^2 \text{ and } \sin^2(2\alpha_k) \approx 9k^2\delta^2/4(h - 1)^2(h + \delta - 1)^2.
\]

For small time \(t\), this results in \(S \propto -\Gamma t^2\), i.e.,

\[
\mathcal{L}(h, t) \approx \exp(-\Gamma t^2)
\]
**Figure 5.** The variation of the LE as a function of the scaled time $t/L^\alpha$ to highlight the scaling of the time period with the system size $L$, with $\alpha$ being the scaling exponent ($T \propto L^\alpha$), as obtained in the text for various types of critical points, i.e., (a) the anisotropic critical point where $T \propto L$, (b) the multicritical point where $T \propto L^2$ and (c) the quasicritical point with $T \propto L^3$. In (d), we present the almost linear variation of $\ln L$ with $t^2$ for small times with $L = 100$ at the various critical points, confirming the general small time behavior given by $L \sim e^{-\Gamma t^2}$.

where the decay constant $\Gamma \propto \delta^2/(h-1)^2L^2$. Using the expression of $L(h,t)$, one can easily show that it remains invariant under the transformation $L \rightarrow L\alpha$ and $t \rightarrow t\alpha$ for fixed $\delta$, $\alpha$ being some integer. These scaling relations are also confirmed by the collapse and revival of the LE (see figure 5(a)).

**Time period analysis:** collapse and revival has been seen, setting the parameter values $h = 1 - \delta$, $J_x = 1.0$ and $\delta = 0.01$. We can expand $\varepsilon_k(h+\delta)$ close to the critical mode $k_c$, which gives $\varepsilon_k(h+\delta) \approx \sqrt{4 - J_x^2}(k-k_c)$. The dominant contribution to $L(h,t)$ comes from the mode $k = k_c + 2\pi/L$ in the limit of large $L$. One can see from the expression of the LE in equation (17) that the time dependence comes from the term $\sin^2(\varepsilon_k(h+\delta)t)$. Therefore, the quasi period of this collapse and revival is given by

$$T = \frac{1}{2} \frac{L}{\sqrt{4 - J_x^2}}.$$  \hspace{1cm} (19)

This is presented in figure 5(a) for three different system sizes.
4.2. Multicritical point (MCP)

In this case we consider the critical line \( h - J_x = -1 \) and approach the MCP \((J_x = 2\) and \( h = 1\)) by changing \( \lambda = h - 1 \). This is identical to path IV studied in section 3. This path is chosen to study the effect of an absence of QCPs on the LE. We perform a similar rotation of the Hamiltonian in equation (4) to that in the case of path IV to shift the \( \lambda \) dependence solely to the diagonal term.

**Short time decay:** expanding around the critical mode \( k_c = \pi \) and assuming short time, we get

\[
\sin^2 \varepsilon_k (\lambda + \delta) t \approx (\lambda + \delta)^2 k^2 t^2 \quad \text{and} \quad \sin^2(2\alpha_k) \approx k^2 \delta^2/\lambda^2 (\lambda + \delta)^2,
\]

resulting in an exponential decay of the LE, as also observed in the anisotropic critical point studied in section 4.1. In this case, \( \Gamma \propto \delta^2/\lambda^2 L^4 \), so that \( \mathcal{L}(\lambda, t) \) is invariant under the transformation \( L \rightarrow L\alpha \) and \( t \rightarrow t\alpha^2 \) with fixed \( \delta \). Again these scalings can be verified by using the collapse and revival of the LE as a function of time (see figure 5(b)).

**Time period analysis:** the collapse and revival of the LE as a function of time at the MCP can be seen by setting \( \lambda = -\delta \) and \( \delta = 0.01 \) (path IV \( h - J_x = -1 \)). At this point, \( \varepsilon_k (\lambda + \delta) \approx 4\pi^2/L^2 \), which gives time period of oscillation \( T \) as

\[
T = \frac{1}{2} \frac{L^2}{2\pi}.
\]

This is also verified numerically in figure 5(b).

4.3. Quasicritical point

**Short time decay:** we once again use the path I of section 3 with \( J_x = 2 \) so that the path contains quasicritical points in addition to the multicritical point, as discussed in section 3. Expanding around the critical mode \( k_c = \pi \) and assuming short time, we get

\[
\sin^2 \varepsilon_k (h + \delta) t \approx (h + \delta - 1)^2 t^2 \quad \text{and} \quad \sin^2(2\alpha_k) \approx k^6 \delta^2/(h - 1)^2 (h + \delta - 1)^2.
\]

Once again

\[
\mathcal{L}(\lambda, t) \approx \exp (-\Gamma t^2)
\]

where \( \Gamma \propto \delta^2/(h - 1)^2 L^6 \). With \( L \rightarrow L\alpha \), \( t \rightarrow t\alpha^3 \) and fixed \( \delta \), equation (21) remains invariant.

**Time period analysis:** the collapse and revival of the LE as a function of time at a quasicritical point is obtained for \( h = 1 - \delta + 4\pi^2/L^2 \), \( J_x = 2.0 \) and \( \delta = 0.01 \) (path I), where \( \varepsilon_k (h + \delta) \approx 8\pi^3/L^3 \). The time period \( T \) of oscillation is then given by

\[
T = \frac{1}{2} \frac{L^3}{4\pi^2}
\]

as also confirmed numerically in figure 5(c).

5. Conclusions

In this paper, we have proposed a method that can be used to study the fidelity susceptibility and Loschmidt echo for a generic path and verified our method by studying a three-site interacting transverse Ising model. Using this method, we studied the scaling of the fidelity susceptibility and Loschmidt echo with the system size along different paths
Fidelity susceptibility and Loschmidt echo for generic paths in a three-spin interacting transverse Ising model which consisted of ordinary critical points, quasicritical points and multicritical points. We discussed in detail how the scaling changes due to the presence and absence of quasicritical points. We also studied the system size dependence of the time period of oscillations in the case of the Loschmidt echo at critical points, multicritical points and quasicritical points for different paths. To the best of our knowledge, there is no previous study in such detail as is given here, involving the effect of various paths on $\chi_F$ and the LE.

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