I analytically study the plateau of the magnetization curve at $M/M_S = 1/3$ (where $M_S$ is the saturation magnetization) of the one-dimensional $S = 1/2$ trimerized Heisenberg spin system with ferromagnetic ($J_F$)-ferromagnetic ($J_F$)-antiferromagnetic ($J_A$) interactions at $T = 0$. I use the bosonization technique for the fermion representation of the spin Hamiltonian through the Jordan-Wigner transformation. The plateau appears when $\gamma \equiv J_F/J_A < \gamma_C$, and vanishes when $\gamma > \gamma_C$, where the critical value $\gamma_C$ is estimated as $\gamma_C = 5 \sim 6$. The behavior of the width of the plateau near $\gamma_C$ is of the Kosterlitz-Thouless type. The present theory well explains the numerical result by Hida.
Recently Hida\(^1\) numerically studied the \(S = 1/2\) spin chain system in which the interaction between neighboring spins changes as ferromagnetic-ferromagnetic-antiferromagnetic (Fig.1). The Hamiltonian of this model is written by

\[
\mathcal{H} = J_A \sum_j (h_{3j-1,3j}^+ + \Delta h_{3j-1,3j}^z) - J_F \sum_j (h_{3j,3j+1}^+ + \Delta h_{3j,3j+1}^z + h_{3j+1,3j+2}^z + \Delta h_{3j+1,3j+2}^z),
\]

where \(J_A > 0\) and \(J_F > 0\) are the magnitudes of the antiferromagnetic and ferromagnetic couplings, respectively. I have introduced the anisotropy parameter \(\Delta > 0\) for later convenience, although Hida investigated only the \(\Delta = 1\) case. This model is not too artificial nor a theoretical toy, as noticed by Hida. In fact, the substance \(3\text{CuCl}_2 \cdot \text{dioxane}\), of which magnetization is measured by Ajiro \textit{et al.}\(^2\) in strong magnetic fields, is known to be a (quasi-)one dimensional magnet to which this model is applied.

Hida\(^1\) investigated this model (only the \(\Delta = 1\) case) by the numerical diagonalization method for finite systems (up to 24 spins) to find that there was a plateau in the magnetization curve as far as \(\gamma = J_F/J_A\) is small, and the width of the plateau decreases as the parameter \(\gamma\) increases (Fig.2). The location of the plateau was \(M_S/3\), where \(M_S\) is the saturation magnetization. He could not obtain a definite conclusion about the existence of the plateau for large \(\gamma\) case, because the magnetization varied stepwise for the finite systems. Since this model becomes \(S = 3/2\) antiferromagnetic chain model when \(\Delta = 1\) and \(\gamma \rightarrow \infty\) due to the formation of an \(S = 3/2\) quartet by spins \(S_{3j}, S_{3j+1}\) and \(S_{3j+2}\), it is expected that there is no plateau in this limit. It is believed that all the half-odd spin chains with antiferromagnetic nearest-neighbor interactions belong to the same universality class as that of \(S = 1/2\).

There arise several questions:

(i) Why does the plateau appear at \(M/M_S = 1/3\)? (ii) Does the plateau vanish at a finite value of \(\gamma\) or persist to \(\gamma = \infty\)? (iii) If the plateau vanishes at \(\gamma_C\), what is the behavior of the plateau near \(\gamma_C\)?

They are interesting questions not only from the standpoint of the statistical physics but also from that of explaining the properties of the existing materials. I note that the plateau was not observed in the report of Ajiro \textit{et al.}\(^2\).

In this paper I will analytically study the mechanism for the appearance of the plateau and also discuss whether the critical value \(\gamma_C\) exists or not.

Performing the spin rotation around the \(z\)-axis for the spins located at the sites \(3j\)

\[
S_{3j}^x \Rightarrow -S_{3j}^x, \quad S_{3j}^y \Rightarrow -S_{3j}^y, \quad S_{3j}^z \Rightarrow S_{3j}^z,
\]

Hamiltonian (1) is transformed into the form of a generalized version of the trimerized antiferromagnetic chain:

\[
\mathcal{H} = J_0(1 - 2\delta_\perp) \sum_j h_{3j-1,3j}^+ + J_0(1 + \delta_\perp) \sum_j (h_{3j,3j+1}^+ + h_{3j+1,3j+2}^+) + J_0(\Delta_0 - 2\delta_z) \sum_j h_{3j-1,3j}^z + J_0(\Delta_0 + \delta_z) \sum_j (h_{3j,3j+1}^z + h_{3j+1,3j+2}^z),
\]

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\]
In case of the XY model where \( \Delta = 0 \), we can solve Hamiltonian (4) by the use of the fermion representation through the Jordan-Wigner transformation. The dispersion relation is shown in Fig.3. When the magnetic field is applied, the dispersion curve shifts along the \( \omega \)-axis, which explains the existence of the plateau as well as its location \( M/M_S = 1/3 \). Note that the magnetization \( M \) is related to the number of occupied states as

\[
\frac{M}{N} = \frac{1}{2} - \frac{\text{occupied states}}{N},
\]

where \( N \) is the total number of spins.

This is a simple explanation for the question why the plateau appears at \( M/M_S = 1/3 \). However, there still remains questions. Why is there no plateau in the experimental results on 3CuCl\(_2\)·2dioxane? Is there any essential difference between the XY case (\( \Delta = 0 \)) and the isotropic Heisenberg case (\( \Delta = 1 \))? How can we explain Hida’s result, especially in \( \gamma \geq 5 \) case? To answer these questions, we have to consider the interactions between fermions when \( \Delta \neq 0 \).

To consider the effect of the trimerization and that of the interaction between fermions simultaneously, I use the method of the bosonization. The bosonization is one of the powerful methods in one-dimensional quantum problems. Here I do not enter into the details of the bosonization procedure. I note that the bosonization is usually done in case of no magnetization (half-filled in the language of the fermion), but in the present case, we have to perform the bosonization near \( M/M_S = 1/3 \) (1/3-filled in the language of the fermion).

After the bosonization, Hamiltonian (3) is transformed into a generalized sine-Gordon Hamiltonian:

\[
\tilde{H} = \frac{\sqrt{3}J_0a}{2} \int dx \left\{ A(\nabla \theta)^2 + CP^2 \right. \\
- B_\perp \cos \theta - B_z(\nabla \theta)^2 \cos \theta \left. \right\},
\]

\[
A = \frac{1}{8\pi} \left( 1 + \frac{2\sqrt{3}\Delta_0}{\pi} \right), \quad C = 2\pi \left( 1 - \frac{2\Delta_0}{\sqrt{3}\pi} \right),
\]

\[
B_\perp = \frac{2\delta_\perp}{\sqrt{3}a^2}, \quad B_z = \delta_z/\pi,
\]

\[
[\theta(x), \ P(x')] = i\delta(x - x'),
\]

where \( a \) is the distance between neighboring spins. The effect of the trimerization appears in the \( B_\perp \) and \( B_z \) terms. If the \( B_\perp \) term and/or the \( B_z \) term are relevant in the sense of the renormalization group, the spectrum of \( \tilde{H} \) has a gap, which brings about the plateau in the magnetization curve. If both of them are irrelevant, the spectrum of \( \tilde{H} \) is gapless, which results in no plateau. Which case is realized? — It depends on the magnitudes of \( B_\perp \) and \( B_z \) and also on the parameter

\[
\eta = 2\pi^{-1}\sqrt{C/A}.
\]

The renormalization group calculation shows that the \( B_\perp \) term and/or the \( B_z \) term are relevant as far as \( \eta < 4 \).

The value of \( \eta \) is slightly shifted through the bosonization procedure. The expressions of \( A \) and \( B \) in eq.(8) is considered to be the lowest order expansions with respect to \( \Delta_0 \). When \( \Delta_0 \to -1 \) (i.e. \( \gamma \to \infty \)),

\[
J_0 = \frac{2J_F + J_A}{3}, \quad \Delta_0 = \frac{-2\gamma + 1}{2\gamma + 1} \Delta,
\]

\[
\delta_\perp = \frac{\gamma - 1}{2\gamma + 1}, \quad \delta_z = -\frac{\gamma + 1}{2\gamma + 1} \Delta.
\]

\[
\Delta_0 = -\frac{2\gamma + 1}{2\gamma + 1} \Delta.
\]
the system becomes ferromagnetic. Therefore $\eta$ should diverge to $+\infty$ when $\Delta_0 \to -1$, although it seems to diverge at $\Delta_0 \to -\pi/2\sqrt{3} = -0.907$ from eq.(8).

The bosonized Hamiltonian (7) has the same form as that of the generalized version of the dimerized $XXZ$ model.

$$H^{(d)} = J \sum_l \left\{ (1 + (-1)^l)\delta_\perp^{(d)} h^{(d)}_l + \Delta^{(d)} \right\} H^{(d)}_{l,l+1} + J \sum_l \left\{ \delta_\perp^{(d)} + (1)^l\delta_z^{(d)} \right\} h^{(d)}_{l,l+1}, \quad J > 0,$$

(12)

In fact, if we perform the bosonization for Hamiltonian (12), we obtain

$$\tilde{H}^{(d)} = Ja \int dx \left\{ A^{(d)} (\nabla \theta)^2 + C^{(d)} p^2 - B_\perp^{(d)} \cos \theta - B_z^{(d)} (\nabla \theta)^2 \cos \theta \right\},$$

(13)

where

$$A^{(d)} = \frac{a}{\delta_\pi} \left( 1 + \frac{3\Delta^{(d)}}{\pi} \right), \quad C^{(d)} = 2\pi a \left( 1 - \frac{\Delta^{(d)}}{\pi} \right).$$

(14)

$$B_\perp^{(d)} = \delta_\perp^{(d)}/a^2, \quad B_z^{(d)} = \delta_z^{(d)}/\pi.$$

(15)

Also in this case, the expression of $A^{(d)}$ and $C^{(d)}$ should be considered to be the lowest order expansions near $\Delta^{(d)} = 0$. In fact, in the absence of the dimerization, the exact form of $\eta^{(d)} \equiv (2\pi)^{-1}\sqrt{C^{(d)}/A^{(d)}}$ is obtained from the application of the exact solution of the eight-vertex model as

$$\eta^{(d)} = 2/[1 + (2/\pi) \sin^{-1} \Delta^{(d)}].$$

(16)

If we expand eq.(16) near $\Delta^{(d)} = 0$, we can see that it agrees with the expression of $\eta^{(d)}$ from eq.(11) and (14).

$\tilde{H}$ has the same form as $\tilde{H}^{(d)}$ with the identification of the parameters

$$\Delta^{(d)} = \frac{2\Delta_0}{\sqrt{3}}, \quad \delta_\perp^{(d)} = \frac{2\delta_\perp}{\sqrt{3}}, \quad \delta_z^{(d)} = \delta_z.$$

(18)

Then we can use the knowledge on the dimerized $XXZ$ model. The phase diagram$^{6-8}$ of the dimerized $XXZ$ model when $\delta_\perp^{(d)} = \delta_z^{(d)} = \delta^{(d)}$ is shown in Fig.4. This phase diagram was obtained by the renormalization group calculation,$^6$ by the high temperature series expansion after mapping $H^{(d)}$ onto the finite-temperature classical 2D model (modified Ashkin-Teller model),$^7$ and by the numerical diagonalization of the original spin Hamiltonian for finite systems.$^8$ When $\delta_\perp^{(d)} = \delta_z^{(d)}$, a naive consideration leads to the effective dimerization parameter

$$\delta^{(d)}_{\text{eff}} = (2\delta_\perp^{(d)} + \delta_z^{(d)})/3,$$

(18)

because $\delta_\perp^{(d)}$ is related to $S^x$ and $S^y$, whereas $\delta_z^{(d)}$ to $S^z$. However, the renormalization group method and the variational method bring about

$$\delta^{(d)}_{\text{eff}} = (2\delta_\perp^{(d)} + \eta \delta_z^{(d)})/(2 + \eta).$$

(19)

From eqs.(5), (17) and (19), we obtain the mapping of the present model onto the dimerized $XXZ$ model, as shown in Fig.4. The case $\gamma = 1$ of the present model corresponds to the case $\Delta^{(d)} = -2/3\sqrt{3} = -0.385$, $\delta^{(d)} = 0.374$ of the dimerized model, and the case $\gamma = \infty$ to the case $\Delta^{(d)} = -1$, $\delta^{(d)} = 1/2$. Therefore there exists
the critical value $\gamma_C$, where the transition from the plateau state to the no-plateau state. This transition is of the Kosterlitz-Thouless type, as known from the critical properties of the sine-Gordon Hamiltonian.\textsuperscript{9}

The above discussion is based on the bosonization method, which make it difficult to estimate the value of $\gamma_C$ itself. It is because the parameters are slightly shifted through the bosonization procedure, as already explained. A rough estimation of the value of $\gamma_C$ is

$$\gamma_C = 4 \sim 5 .$$ (20)

I have analytically investigated the plateau in the magnetization curve of the $S = 1/2$ ferromagnetic-ferromagnetic-antiferromagnetic spin chain, which is first pointed out by Hida\textsuperscript{3} by the use of the numerical diagonalization. The present analytical study semi-qualitatively explains the numerical result of Hida.
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Figure Captions:

Fig.1. Ferromagnetic-ferromagnetic-antiferromagnetic spin chain. Solid lines and dotted lines represent ferromagnetic couplings $J_F$ and antiferromagnetic couplings $J_A$, respectively. Three spins in an ellipse make an $S = 3/2$ quartet when $\Delta = 1$ and $\gamma \rightarrow \infty$.

Fig.2. Sketch of Hida’s numerical result for the magnetization curve. The case (a) is for smaller $\gamma$ and (b) for larger $\gamma$.

Fig.3. Dispersion relation of $H$ of eq.(4) in case of $\Delta = 0$. (a): $\delta > 0$ case. (b): $\delta = 0$ case.

Fig.4. Mapping of the present model onto the dimerized $XXZ$ chain. The open circle corresponds to the $\gamma = 1$ case and the closed circle to the $\gamma = \infty$ case.