Infinite soliton and kink-soliton trains for nonlinear Schrödinger equations

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Received 14 October 2013, revised 23 July 2014
Accepted for publication 19 August 2014
Published 7 October 2014

Abstract
We look for solutions to general nonlinear Schrödinger equations built upon solitons and kinks. Solitons are localized solitary waves, and kinks are their non-localized counterparts. We prove the existence of infinite soliton trains, i.e. solutions behaving at large time as the sum of infinitely many solitons. We also show that one can attach a kink at one end of the train. Our proofs proceed by fixed point arguments around the desired profile. We present two approaches leading to different results, one based on a combination of \(L^p - L^{p'}\) dispersive estimates and Strichartz estimates, the other based only on Strichartz estimates.

Keywords: soliton train, multi-soliton, multi-kink, nonlinear Schrödinger equations
Mathematics Subject Classification: 35Q55(35C08,35Q51).

1. Introduction

We consider the nonlinear Schrödinger equation:

\[
i \partial_t u + \Delta u + f(u) = 0,
\]

(NLS)

where \(u = u(t, x)\) is a complex-valued function on \(\mathbb{R} \times \mathbb{R}^d, d \geq 1\).

Our goal in this paper is to advance a study initiated in [11] on the existence of exotic solutions to (NLS). We look for infinite soliton trains, i.e. solutions which behave asymptotically as the sum of infinitely many solitons, possibly attached to a kink at one end. We want to show that such behaviour is possible for general nonlinearities under mild hypotheses. A typical nonlinearity example is the double-power nonlinearity:

\[
f(u) = |u|^{\alpha} u - |u|^\beta u, \quad 0 < \alpha < \beta < \alpha_{\max}.
\]

(1.1)
Here and thereafter we denote the critical exponent by $\alpha_{\text{max}} = +\infty$ for $d = 1, 2$ and $\alpha_{\text{max}} = \frac{4}{d-2}$ for $d \geq 3$. Note that we are using the term *soliton* in its broader meaning of *solitary wave*.

Let us shortly review some results on multi-solitons, i.e. solutions to (NLS) behaving at large time as a finite sum of solitons. The inverse-scattering transform makes possible a precise long time description of the dynamics (see e.g. [8]), and provides a convenient way to build multi-solitons (see e.g. [18]); however it is limited to integrable equations (for Schrödinger equations, only the 1D cubic case is integrable). For non-integrable Schrödinger equations, one of the first results of the existence of multi-solitons was obtained by Merle [16] for $L^2$-critical equations, triggering a series of work on multi-solitons. For energy-subcritical nonlinearities, Côte, Martel and Merle [7,14] obtained the existence of multi-solitons built upon ground states, while the excited states case was treated by Côte and Le Coz [6] under a high speed assumption. Stability/instability results have been obtained by Côte and Le Coz [6], Martel, Merle, Tsai [12] and Perelman [17]. However, stability of multi-solitons for power-type nonlinearities is still an open issue.

The existence of objects like infinite soliton trains is of importance as they usually provide examples of extreme phenomena in the asymptotic behaviour of solutions of nonlinear dispersive equations. For example, for the Korteweg–de Vries equation, an infinite train of solitons was used in [13] as a counter example to show the optimality of an asymptotic stability statement. For nonlinear Schrödinger equations, the asymptotic stability results usually hold under assumptions (typically in weighted spaces) excluding the infinite train behaviour. To our knowledge, our previous work [11] was the first one to establish the existence of infinite soliton trains for non-integrable Schrödinger equations (for the integrable 1D cubic nonlinear Schrödinger equation, the existence of infinite soliton trains may be obtained via the inverse-scattering transform, see [10]).

Before stating our main results, let us give some preliminaries. To work in an energy-subcritical context, we first assume the following.

**Assumption (F0).** Let $d \geq 1$. Suppose $f(u) = g(|u|^2)u$ where $g \in C^0([0, \infty), \mathbb{R}) \cap C^2((0, \infty), \mathbb{R})$, $g(0) = 0$ and

$$|s g'(s)| + |s^2 g''(s)| \leq C_0(s^{\alpha_1/2} + s^{\alpha_2/2}), \quad \forall s > 0,$$

where $0 < \alpha_1 \leq \alpha_2 < \alpha_{\text{max}}$ and $C_0 > 0$.

A *bound state* is a nontrivial solution $\phi \in H^1(\mathbb{R}^d)$ of the elliptic equation:

$$\Delta \phi + f(\phi) = \omega \phi \quad (1.2)$$

for some frequency $\omega > 0$. We shall sometimes denote a bound state along with its frequency $(\phi, \omega)$ to emphasize the dependency of $\phi$ on $\omega$. Any bound state $\phi$ with frequency $\omega$ and parameters $x^0 \in \mathbb{R}^d$ (position), $v \in \mathbb{R}^d$ (velocity) and $\gamma \in \mathbb{R}$ (phase) corresponds to a solitary wave solution (soliton) of (NLS):

$$R_{\phi, \omega, x^0, v, \gamma}(t, x) = e^{i(\omega t + \frac{1}{2}v \cdot x - \frac{1}{4}v^2 t + \gamma)} \phi(x - x^0 - vt). \quad (1.3)$$

The profile of an infinite soliton train is a sum of the form:

$$R_\infty = \sum_{j=1}^{\infty} R_j, \quad R_j(t, x) = R_{\phi_j, \omega_j, x_j^0, v_j, \gamma_j}(t, x), \quad j \in \mathbb{N}, \quad (1.4)$$

where $(R_j)_j$ are given solitons with bound state profiles $(\phi_j, \omega_j)$ and parameters $x^0_j, v_j \in \mathbb{R}^d$ and $\gamma_j \in \mathbb{R}$. A solution $u(t)$ is called an *infinite soliton train* if, for some profile $R_\infty$:

$$u(t) - R_\infty(t) \to 0 \quad \text{as} \quad t \to \infty$$

in some space-time norm.
Constructing a solution to (NLS) around an infinite train profile as (1.4) is much trickier than when the profile is made with a finite number of solitons. First of all, we need to make sure that the profile is well defined, as the addition of infinitely many solitons may very well be infinite. We also have to take into account that it is very likely that the profile will not belong to the same functional spaces as the solitons. In order to deal with these issues we need a control on the growth of the solitons’ profiles (see (1.5)) and also to guarantee some space integrability of the train (see (1.6)).

We will assume the following for our infinite train:

Assumption (T1). For \( 0 < \alpha_1 < \alpha_{\text{max}} \) given, the sequence of bound states \( \{(\phi_j, \omega_j) : j \in \mathbb{N}\} \) satisfies, for some \( 0 < \alpha < 1 \) and \( D_\alpha \) independent of \( j \),

\[
|\phi_j(x)| + \omega_j^{1/2}|\nabla \phi_j(x)| \leq D_\alpha \omega_j^{1/\alpha} e^{-\omega_j/2|x|}, \quad \forall x \in \mathbb{R}^d, \forall j \in \mathbb{N},
\]

and, for some \( r_0 \geq 1, \frac{d_\alpha}{2} < r_0 < 2 + \alpha_1 \),

\[
A_1 := \sum_{j \in \mathbb{N}} \frac{\omega_j}{r_0} < \infty.
\]

We say a nonlinearity \( f \) satisfies (T1) if such an infinite sequence \( (\phi_j, \omega_j) \) exists for some \( r_0 \). Examples of such nonlinearities will be given in section 2.

Note that the set \([1, \infty) \cap (\frac{d_\alpha}{2}, 2 + \alpha_1)\) for \( r_0 \) is nonempty since \( 0 < \alpha_1 < \alpha_{\text{max}} \). The condition \( r_0 > \frac{d_\alpha}{2} \) ensures that the exponent \( \frac{1}{\alpha_1} - \frac{d}{2r_0} > 0 \). Thus \( \omega_j \rightarrow 0 \) as \( j \rightarrow \infty \), and (1.6) is a condition on how fast \( \omega_j \) goes to 0. The existence of sequences of bound states satisfying assumption (T1) is guaranteed by proposition 2.1, where bound states with small frequencies are constructed as bifurcation from 0 along a fixed radial bound state \( Q \) of the equation \( \Delta Q + |Q|^{\alpha_1} Q = Q \) together with the estimate (1.5). Note that the \( \phi_j \) may be arbitrary excited states solutions of (1.2); in particular they may be sign-changing, non-radial or complex-valued. Also note that we do not need the bound for \( \omega_j^{1/2}|\nabla \phi_j(x)| \) in (1.5) for theorems 1.2 and 1.14, but we assume it for all theorems for simplicity of presentation. For the same reason, we shall also set all initial positions \( x_0 \) to 0. Our assumption includes the finite multi-soliton case by setting \( (\phi_j, \omega_j) = (0, 0) \) for \( j \) sufficiently large.

We have followed two independent approaches for the study of this problem, leading to two different types of results with different assumptions and conclusions. Before stating our main results, we need a preliminary lemma which will be proved in section 4.

Lemma 1.1. Let \( d \geq 1 \). For any \( 0 < \alpha_1 < \alpha_2 < \alpha_{\text{max}} \) satisfying \( \frac{\alpha_2}{2\alpha_2} \leq \alpha_1 \), one can choose \( r_0 \) so that the following conditions hold.

\[
\max \left( 1, \frac{d \alpha_1}{2} \right) < r_0 < 2 + \alpha_1,
\]

\[
\frac{1}{2} \leq \frac{\alpha_1}{r_0} + \frac{1}{r_2},
\]

\[
1 < \frac{\alpha_1}{r_0} + \frac{1}{r_2},
\]

where \( r_2 = 2 + \alpha_2 \). Furthermore, if \( \alpha_1 < 4/d \), we can choose \( r_0 \leq 2 \).

1.1. Infinite soliton trains

We now state our two results on the existence of infinite soliton trains. The first approach of the first theorem is based on \( L^p-L^q \) decay estimates for \( e^{it\lambda} \). The Strichartz space \( S(I, \infty) \) will be defined in section 3.
Theorem 1.2 (Infinite train of solitons (i)). Let \( d \geq 1 \) and assume assumption (F0) and
\[
\frac{\alpha_2}{2 + \alpha_2} \leq \alpha_1. \tag{1.10}
\]
Let \( r_2 = 2 + \alpha_2 \) and take any \( r_0 \) verifying (1.7), (1.8), and (1.9). Let \((\phi_j, \omega_j)_{j \in \mathbb{N}}\) be a sequence of bound states satisfying assumption (T1) with the chosen \( r_0 \). There exist constants \( c_1 > 0 \) and \( \gamma_j \geq 1 \) such that, for any infinite soliton train profile \( R_\infty \) given, as in (1.4) with parameters \( v_j \in \mathbb{R}^d, x_j^0 = 0, \gamma_j \in \mathbb{R} \) satisfying
\[
v_j = \inf_{j,k \in \mathbb{N}, j \neq k} \sqrt{\omega_j} |v_k - v_j| \geq v_1, \tag{1.11}
\]
there exists a solution \( u \) to (NLS) on \([0, \infty)\) satisfying
\[
\| (u - R_\infty)(t) \|_{L^2} + \| u - R_\infty \|_{S([t, \infty))} \leq e^{-c_1 v_1 t}, \quad \forall t \geq 0. \tag{1.12}
\]
It is unique in the class of solutions satisfying the above estimate.

Remark 1.3 (Nonlinearities verifying assumptions (F0) and (T1)). The double power nonlinearity (1.1) is a good example for which assumptions (F0) and (T1) are satisfied. More generally, in section 2, we introduce assumption (F2), under which we can perform a bifurcation analysis to obtain a sequence verifying assumption (T1). Assumption (F2) implies assumptions (F0) and (T1), but is much more restrictive: there may be many other examples of nonlinearities for which assumption (T1) can be verified and which would be excluded by assumption (F2) (e.g. nonlinearities having some oscillatory behaviour at 0).

Remark 1.4 (\(L^2\)-solutions). The solution \( u \) in theorem 1.2 (and those in the later theorems) is only in a distributional sense. When the profile has more integrability, a stronger notion of solution may hold. Indeed, by (1.12) and Hölder inequality,
\[
\| (u - R_\infty)(t) \|_{L^r} \leq e^{-c_1 v_1 t}, \quad \forall t \geq 0, \quad \forall r \in [2, r_2].
\]
As we will show that \( R_\infty \in L^\infty(0, \infty; L^\infty \cap L^\infty(R^d)) \) in (4.1), we have \( u \in L^\infty(0, \infty; L^\alpha \cap L^\infty(R^d)) \) where \( r_1 = \max(2, r_0) \). In the case \( \alpha_1 < 4/d \), we can choose \( r_0 \leq 2 \) by lemma 1.1, and thus \( u \in L^\infty(0, \infty; L^2(R^d)) \). Hence \( u \) is a localized solution in \( L^2(R^d) \) in the usual sense.

Remark 1.5 (Comparison to previous results). Theorem 1.2 contains the pure power case \( f(u) = |u|^\alpha u \) by writing \( f(u) = |u|^\alpha u - 0|u|^\alpha u \) for some small \( \epsilon > 0 \). It also includes the finite soliton train (multi-soliton) case by taking \((\phi_j, \omega_j) = (0, 0)\) for \( j \) sufficiently large. In addition the range of exponents is larger than in [11, theorem 6.4]. Hence theorem 1.2 extends theorems 1.1, 1.7, 6.3 and 6.4 in [11] in a unified approach (except that [11, theorem 6.3] does not require (1.10)).

Remark 1.6 (\(L^2\)-subcritical nonlinearities). If we use a pure Strichartz norm approach and do not use \( L^2 \) norm, we can construct infinite soliton trains for all \( L^2 \)-subcritical or critical exponents \( 0 < \alpha_1 < \alpha_2 \leq 4/d \) as in [11, theorem 6.3], without the restriction (1.10).

Remark 1.7 (Uniqueness). The uniqueness of the constructed infinite train holds in a class of exponentially decaying in time solutions of (NLS) with rate \( c_1 v_\ast \) (see (1.12)). Even for finite trains of solitons, it is an open question whether or not uniqueness holds in a less restrictive class of functions. For the nonlinear Schrödinger equation with a pure power nonlinearity, the conjecture is that, similar to what happens for the generalized Korteweg–de Vries multisolitons (see [3, 15]), uniqueness in \( H^1(R^d) \) of the \( N \)-soliton holds if the composing solitons are made of ground states and the nonlinearity is \( L^2 \)-subcritical. If the nonlinearity is supercritical, however, we have a classification in \( H^1(R^d) \) of the \( N \)-solitons as a \( N \)-parameters family (see partial steps towards this conjecture in [3, 6]).
In our second main result, we also control the train at the gradient level. The approach is based solely on Strichartz estimates. Here and thereafter, we use the Japanese bracket convention: \( \langle x \rangle := \sqrt{1 + |x|^2} \).

**Theorem 1.8 (Infinite train of solitons (ii)).** Let \( d \geq 1 \) and assume assumption (F0) with \( 0 < \alpha_1 < \frac{4}{d+2} \). Let \((\phi_j, \omega_j)\) j \in \mathbb{N} be a sequence of bound states satisfying assumption (T1) for some \( r_0 \). There exist constants \( C > 0, c_1 > 0, c_2 > 0 \), and \( v_2 \gg 1 \) such that, for any infinite soliton train profile \( R_\infty \) given as in (1.4) with parameters \( v_j \in \mathbb{R}^d, x_j^0 = 0, y_j \in \mathbb{R} \) satisfying

\[
v_* := \inf_{j,k \in \mathbb{N}, j \neq k} \sqrt{\eta_j} |v_k - v_j| \geq v_2,
\]

and

\[
V_* := \sum_{j \in \mathbb{N}} (v_j)_{|\omega_j|} < \infty,
\]

there exists a unique solution \( u \) to (NLS) satisfying, for some \( T_0 = T_0(V_\ast) \gg 1 \),

\[
e^{\varepsilon u_{\ast} t} \|u - R_\infty\|_{S(\mathbb{R},\mathbb{R}^d)} + e^{\varepsilon u_{\ast} t} \|\nabla (u - R_\infty)\|_{S(\mathbb{R},\mathbb{R}^d)} \leq C, \quad \forall t \geq T_0.
\]

**Remark 1.9 (Examples of parameters choices).** Condition (1.13) requires sufficiently large relative speed, while condition (1.14) puts an upper bound on the growth of \( \langle v_j \rangle \). By (1.14) we may assume \( r_0 \leq 2 \). One possible choice of parameters is:

\[
\omega_j = 4^{-j}, \quad v_j = 2^{j+1} \bar{v}, \quad |\bar{v}| \gg 1.
\]

Condition (1.14) can be satisfied (\( V_* \leq \sum_j (4^{-j})^{-\frac{d}{2} + \frac{1}{4}} \bar{v}^{\frac{1}{d} - \frac{1}{2}} < \infty \)) thanks to the assumption \( \alpha_1 < \frac{4}{d+2} \) (note this implies \( \alpha_1 < 1 \) unless \( d = 1 \)).

In the above choice \( V_* \) and \( v_* \) grow linearly in \( |\bar{v}| \). In the following choice \( V_* = O(h(|\bar{v}|)|\bar{v}|) \) while \( v_* = C|\bar{v}| \) for any function \( h > 1 \):

\[
\omega_j = 4^{-j}, \quad v_j = \begin{cases} 2^{|j+1|} h(|\bar{v}|) \bar{v}, & \text{if } j \text{ is odd} \\ 2^{-|j+1|} \bar{v}, & \text{if } j \text{ is even}, \end{cases} \quad |\bar{v}| \gg 1.
\]

**Remark 1.10 (Infinite train starting at time 0).** We use large \( T_0 \) to off-set the contribution of large \( V_* \). Note that the solution may very well exist before \( T_0 \). If we impose that \( V_* \) grows sub-exponentially in \( v_* \), e.g. \( V_* \leq C(1 + v_*)^M \) for some \( M \geq 1 \) (e.g. \( h(s) = (1 + s)^{M-1} \) in (1.17)), we may take \( T_0 = 0 \) as in [11, theorem 6.1].

**Remark 1.11 (Existence of infinite trains under (F0) and (T1)).** The proof of theorem 1.2 uses a combination of \( L^\xi \) norm and Strichartz norm. To estimate \( |\eta|^{\alpha_1} \) in \( L^\xi \) using \( L^\xi \)-\( L^\gamma \) decay estimates, a restriction like (1.10) is needed to avoid the limiting case \( \alpha_1 = 0+ \) and \( \alpha_2 = \alpha_{\max} - \). However, we claim that exponents excluded by (1.10) are covered by theorem 1.8 above. Indeed, let \( \bar{\alpha} = \sup_{\beta < \alpha < \alpha_{\max} - \frac{\mu}{2n}} \frac{\mu}{\alpha} \). We have \( \bar{\alpha} = 1 \) for \( d = 1, 2 \) and \( \bar{\alpha} = 2/d \) for \( d \geq 3 \).

One then verifies that \( \bar{\alpha} \leq \frac{\mu}{d+2} \) for all dimensions.

Hence we can construct infinite soliton trains for all energy-subcritical nonlinearities satisfying assumptions (F0) and (T1).

**Remark 1.12 (Comparison between theorems 1.2 and 1.8).** Theorem 1.2 applies for nonlinearities whose general form is not far from a power-type nonlinearity, no matter what this power is (\( \alpha_1 \) can be any \( H^{1} \)-subcritical power). Theorem 1.8 applies for nonlinearities that are sufficiently strong at 0 (\( \alpha_1 \) has to be small), but with any kind of growth possible away from 0. For the choice of the profile, theorem 1.2 is more flexible as it requires only some weak integrability condition (1.6), whereas theorem 1.8 requires \( L^{\gamma} \)-integrability of the profile (one take \( r_0 = 2 \) in (T1)) and its first derivative (1.14).
1.2. Infinite kink-soliton trains

In our next couple of theorems we let $d = 1$ and consider in $\mathbb{R}$ a train of the form

$$W = K + R_\infty$$

where $R_\infty$ is as in (1.4), and $K$ is a kink solution of (NLS) given by the same formula (1.3) but with the profile $\phi = \phi_K$ now being a half-kink satisfying the same equation (1.2) ($\phi'' = \omega \phi - f(\phi)$), $0 < \phi_K(s) < b$ for some $b > 0$, and

$$\lim_{s \to -\infty} \phi_K(s) = b, \quad \phi'_K(s) < 0 \quad \forall s \in \mathbb{R}, \quad \phi'_K(0) = \min \phi'_K, \quad \lim_{s \to +\infty} \phi_K(s) = 0. \quad (1.18)$$

A solution which converges to a profile $W$ as above at positive time infinity will be called an infinite kink-soliton train. We are going to give two results of the existence of infinite kink-soliton trains. Note that such an object was never exhibited before, even in integrable cases.

In addition to assumption (F0), we make the following assumption, which in particular ensures the existence of a half-kink satisfying (1.18) (see proposition 1.13).

**Assumption (F1).** For some $\omega_0 > 0$, there is a first $b > 0$ such that for $h(s) = \omega_0 s - f(s)$,

$$h(b) = 0, \quad \int_0^b h(s) \, ds = 0. \quad (1.19)$$

Moreover, $h'(b) > 0$, and for some $\tilde{\alpha} \in [0, \alpha_2]$,

$$|s'(b + s)| + |s||f''(b + s)| \leq C|s|^{\beta} + C|s|^\alpha, \quad \forall s \in \mathbb{R}. \quad (1.20)$$

The existence of half-kink profiles is guaranteed by the following result.

**Proposition 1.13.** Let $d = 1$ and assume assumptions (F0) and (F1). There is a solution $\phi_K(s)$ of

$$\phi''_K = \omega_0 \phi_K - f(\phi_K)$$

such that $0 < \phi_K(s) < b$,

$$\lim_{s \to -\infty} \phi_K(s) = b, \quad \phi'_K(s) < 0 \quad \forall s \in \mathbb{R}, \quad \phi'_K(0) = \min \phi'_K, \quad \lim_{s \to +\infty} \phi_K(s) = 0,$$

and that, for any $0 < a < \min(\omega_0, h'(b))$, there is $D_a > 0$ so that

$$\mathbf{1}_{s < 0}(b - \phi_K(s)) + \mathbf{1}_{s \geq 0} \phi_K(s) + |\phi'_K(s)| \leq D_a e^{-a|s|}, \quad \forall s \in \mathbb{R}.$$

Proposition 1.13 can be easily proved using classical ordinary differential equations techniques (see e.g. [11, proposition 1.12]).

Note (see example 5.2) that the double power nonlinearity (1.1) verifies assumption (F1) and admits a half-kink when $d = 1$ and, e.g.

$$f(u) = |u|u - |u|^2u.$$

Another example (see example 5.1) of a nonlinearity verifying assumption (F1) is

$$f(u) = u - \frac{\sin |u|}{|u|}u.$$

We now state our second set of results on the existence of infinite kink-soliton trains. Recall $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.
Theorem 1.14 (An infinite kink-soliton train (i)). Let \( d = 1 \) and assume assumptions (F0), (F1) and

\[
\frac{\alpha_2}{2 + \alpha_2} \leq \alpha_1.
\]  

Let \( r_2 = 2 + \alpha_2 \). Then we can find \( r_0 \) satisfying (1.7)–(1.9). Assume that \( \tilde{\alpha} \) is such that

\[
\frac{1}{2} \leq \frac{\tilde{\alpha} + 1}{r_0} + \frac{1}{r_2}.
\]  

Assume there is a sequence of bound states \((\phi_j, \omega_j)\) satisfying assumption (TI) with the chosen \( r_0 \). Let \( \phi_0 = \phi_k \) be the kink profile given in proposition 1.13. There exist constants \( c_1 > 0 \), and \( v_2 \gg 1 \) such that, for the infinite kink-soliton profile \( W = K + R_\infty \), given as in (1.4), with any parameters \( v_j \in \mathbb{R}, v_j < v_{j+1}, x_j^0 = 0, \gamma_j \in \mathbb{R} \) for \( j \in \mathbb{N}_0 \) satisfying

\[
v_\ast = \inf_{j \in \mathbb{N}, k \in \mathbb{N}_0, j \neq k} \sqrt{\omega_j} |v_k - v_j| \geq v_2,
\]

there exists a unique solution \( u \) to (NLS) for \( t \geq 0 \) satisfying

\[
\|u - W(t)\|_{L^2} + \|u - W\|_{S([t, \infty))} \leq e^{-c_1 v_\ast t}, \quad \forall t \geq 0.
\]

Theorem 1.15 (An infinite kink-soliton train (ii)). Let \( d = 1 \) and assume assumptions (F0) and (F1) with \( 0 < \alpha_1 < 4/3 \). Let \( (\phi_j, \omega_j), \ j \in \mathbb{N} \) be given and satisfying assumption (TI) for some \( r_0 \) which further satisfies

\[
r_0(\alpha_1 + 1) < (\tilde{\alpha} + 1)(\alpha_1 + 2).
\]

Let \( \phi_0 = \phi_k \) be the kink profile given in proposition 1.13. There exist constants \( C > 0, c_1 > 0 \), \( c_2 > 0, T_0 \gg 1 \) and \( v_2 \gg 1 \) such that, for the kink-soliton train profile \( W = K + R_\infty \) given as in (1.4) with any parameters \( v_j \in \mathbb{R}, v_j < v_0, x_j^0 = 0, \gamma_j \in \mathbb{R} \) for \( j \in \mathbb{N}_0 \) and sufficiently large relative speed

\[
v_\ast = \inf_{j \in \mathbb{N}, k \in \mathbb{N}_0, j \neq k} \sqrt{\omega_j} |v_k - v_j| \geq v_2,
\]

there exists a unique solution \( u \) to (NLS) for \( t \geq T_0 \) satisfying

\[
e^{c_1 v_\ast t} \|u - W\|_{S([t, \infty))} + e^{c_1 v_\ast t} \|\nabla(u - W)\|_{S([t, \infty))} \leq C, \quad \forall t \geq T_0.
\]

Remark 1.16. In theorems 1.14 and 1.15, the kink \( K \) is on the left in the profile and its velocity is less than the velocity of any soliton. This picture can be reversed by the symmetry \( u(x, t) \rightarrow \tilde{u}(x, t) = u(-x, t) \).

Remark 1.17. In theorem 1.15 we require upper bound \( \alpha_1 < 4/3 \) and lower bound (1.24) on \( \tilde{\alpha} \). The bound (1.24) is redundant if we choose a smaller \( r_0 \), e.g. \( r_0 = 1 \), but is nontrivial if we take \( r_0 = 2 \).
complicated estimates on the nonlinear and the source terms for general $f$ when infinitely many solitons are involved. The presence of the half-kinks also requires additional care of both nonlinear and source terms.

The rest of the paper is organized as follows. In section 2 we give an example of nonlinearity for which assumption (T1) is satisfied. In section 3 we give the general scheme of our proofs. In section 4 we prove theorems 1.2 and 1.8. In section 5 we give examples 5.1 and 5.2 for nonlinearities verifying assumption (F1) and we prove theorems 1.14 and 1.15.

2. Existence of a family of bound states satisfying (T1)

Assumption (T1) is satisfied for the nonlinearity $f$ if, for example, $f$ satisfies assumption (F2) below.

Assumption (F2). Suppose $f(u) = f_1(u) + f_2(u)$ where $f_1(u) = |u|^\alpha u$, $f_2(u) = g_2(|u|^\beta)u$, $g_2 \in C^0([0, \infty), \mathbb{R}) \cap C^2((0, \infty), \mathbb{R})$, $g_2(0) = 0$ and

$$|s^2 g_2'(s)| + |s^2 g_2''(s)| \leq C_0(s^{\beta_1/2} + s^{\beta_2/2}), \quad \forall s > 0,$$

where $0 < \alpha < \beta_1 \leq \beta_2 < \alpha_{\text{max}}$ and $C_0 > 0$.

This assumption is more specific about the small $u$ behaviour of $f(u)$ than those in assumption (F0) so that we can have more control on the bound states with respect to their frequencies. In particular, we do not consider $f_1(u)$ with an opposite sign.

The following proposition gives an existence result of bound states with small frequencies, obtained as the bifurcation from the radial ground state $Q$ of the pure power nonlinearity, together with uniform estimates.

**Proposition 2.1 (Bifurcation of solitons).** Let $d \geq 1$ and assume assumption (F2). Let $Q(x)$ be the unique positive radial solution of $\Delta Q + |Q|^\alpha Q = Q$ in $\mathbb{R}^d$. There is a small $\omega_* = \omega_0(d, \alpha, \beta_1, \beta_2, C_0) > 0$ so that for all $0 < \omega < \omega_*$ there is a solution $\phi = \phi_\omega$ of (1.2) of the form

$$\phi_\omega(x) = \omega^{1/\alpha}[Q(\omega^{1/\alpha}x) + \xi_\omega(\omega^{1/\alpha}x)],$$

(2.1)

where $||\xi_\omega||_{H^2} \leq C_0\omega^{\beta_1/\alpha - 1}$. Moreover, for any $0 < a < 1$ there is a constant $D_a > 0$ such that

$$|\phi_\omega(x)| + |\omega^{-1/2}\nabla \phi_\omega(x)| \leq D_a\omega^{1/\alpha} e^{-a(\omega^{1/2}|x|)}, \quad \forall x \in \mathbb{R}^d, \quad \forall \omega \in (0, \omega_*).$$

(2.2)

Note that we could allow $Q$ to be any radial excited state, provided we knew its non-degeneracy, i.e. invertibility of $L_a$ in the proof below (such a result should be a consequence of the classifications results [4, 5], however we did not pursue that direction).

Before proving proposition 2.1, we recall without proof the following classical lemma.

**Lemma 2.2.** Suppose $f(u) = g(|u|^\beta)u$, $g \in C^0([0, \infty), \mathbb{R})$, $f(0) = 0$ and

$$|sg'(s)| \leq C(s^{\beta_1/2} + s^{\beta_2/2}), \quad \forall s > 0.$$

For $W, \eta \in \mathbb{C}$ we have

$$|f(W + \eta) - f(W)| \lesssim |\eta|(|W|^\alpha + |W|^\beta) + |\eta|^{1+\alpha} + |\eta|^{1+\beta}.$$

**Proof of proposition 2.1.** Since $Q$ is real and radial, we will look for real and radial $\xi_\omega$. For the sake of simplicity in notation, we drop the subscript $\omega$ during the proof. Denoting $y = \omega^{1/2}x$ and substituting (2.1) in (1.2), we get

$$(-\Delta_y + 1)\xi = \omega^{-\frac{1}{\alpha} - 1} f(\omega^{1/\alpha}(Q + \xi)) - |Q|^\alpha Q.$$
It can be rewritten as
\[
L_+ \xi = N(\xi) = N_1(\xi) + N_2(\xi),
\]
(2.3)
where
\[
L_+ = -\Delta_x + 1 - (1 + \alpha)|Q|^a
\]
\[
N_1(\xi) = f_1(Q + \xi) - f_1(Q) - (1 + \alpha)|Q|^a \xi
\]
\[
N_2(\xi) = \omega^{-\frac{1}{a}} f_2(\omega^{\frac{1}{a}} (Q + \xi)).
\]
In the special case \( f_2(u) = -|u|^r u \), we have \( N_2(\xi) = -\omega^{-\frac{1}{a}} |Q + \xi|^b (Q + \xi) \).

Let \( X = H^2_{rad}([\mathbb{R}^d]) \). The properties of \( L_+ \) are well-known (see e.g. [2]). It has one negative eigenvalue, its kernel in \( L^2([\mathbb{R}^d]) \) is spanned by \((\partial_j Q) j \) and the rest of its spectrum is positive away from 0. Hence for radial functions \( L_+ : X \to L^2_{rad} \) is invertible and we have
\[
C_3 := \|(L_+)^{-1}\|_{B(H^2_{rad}; X)} < \infty.
\]
We have
\[
|N_1(\xi)| \lesssim 1_{a>1}|Q|^{a-1}|\xi|^2 + |\xi|^{1+a}
\]
(2.4)
\[
|N_1(\xi_1) - N_1(\xi_2)| \lesssim 1_{a>1}|Q|^{a-1}(|\xi_1| + |\xi_2|)|\xi_1 - \xi_2| + (|\xi_1| + |\xi_2|)^2|\xi_1 - \xi_2|.
\]
(2.5)
We also have, by assumption (F2) and lemma 2.2,
\[
|N_2(\xi)| \lesssim \omega^{-\frac{1}{a}} \sum_{j=1}^2 |\xi|^j (Q + \xi)^{1+\beta_j} = \sum_{j=1}^2 \omega^{-\frac{1}{a}} |Q + \xi|^{1+\beta_j},
\]
(2.6)
\[
|N_2(\xi_1) - N_2(\xi_2)| \lesssim \sum_{j=1}^2 \omega^{-\frac{1}{a}} (|Q| + |\xi_1| + |\xi_2|)^{\beta_j}|\xi_1 - \xi_2|.
\]
(2.7)
Denote \( B_r = \{ \xi \in X : \|\xi\|_X < r \} \) for 0 < r < 1 and 0 < \omega < 1. Because X is imbedded in \( L^{2+2a} \cap L^{2+2d} \) for any dimension d, we have, for some \( C_4 \),
\[
\|N(\xi_1)\|_L^2 \leq C_4 \left( \|\xi_1\|_X^{\min(1, a+1)} + \omega^{\frac{d}{a}} \right),
\]
\[
\|N(\xi_1) - N(\xi_2)\|_L^2 \leq C_4 \left( \|\xi_1\|_X + \|\xi_2\|_X \right)^{\min(1, a)} + \omega^{\frac{d}{a}} \left( \|\xi_1 - \xi_2\|_X \right),
\]
(2.8)
for any \( \xi_1, \xi_2 \in B_r \). Thus the map \( \xi \mapsto (L_+)^{-1} N(\xi) \) is a contraction map in \( B_r \subset X \) for any \( \omega \in (0, \omega_o) \) if we choose \( r = 2C_4 \omega^{\frac{d}{a}} \) and \( \omega_o \) sufficiently small.

Finally, standard argument for exponential decay (see [1] or [9, Appendix]) shows that for any \( a \in (0, 1) \)
\[
|\xi(x)| + |\nabla \xi(x)| \leq o(1) e^{-\alpha |x|}, \quad |Q(x)| + |\nabla Q(x)| \leq C e^{-\alpha |x|},
\]
using the uniform bound \( \|\xi\|_{H^2} \ll 1 \). We get (2.2) after rescaling. \( \square \)

3. The perturbation argument

We recall the definition of the Strichartz spaces \( S([t, \infty)) \) and \( N([t, \infty)) \) and the well-known dispersive and Strichartz estimates. A pair of exponents \( (q, r) \) is said to be (\( \text{Schr"odinger}\))-admissible if
\[
\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leq q, r \leq +\infty, \quad (d, q, r) \neq (2, 2, +\infty).
\]
Given a time \( t \in \mathbb{R} \), the Strichartz space \( S([t, \infty)) \) is defined via the norm
\[
\|u\|_{S([t, \infty))} = \sup_{(q, r) \text{admissible}} \|u\|_{L_t^q L_x^r([t, \infty) \times \mathbb{R}^d)}.
\]
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Above $r_{Str} = \infty$ for $d \neq 2$, but we choose $\alpha_2 + 2 < r_{Str} < \infty$ when $d = 2$ to stay away from the forbidden endpoint. We denote the dual space by $N((t, \infty)) = \mathcal{S}((t, \infty))^\ast$. Hence for any $(q, r)$ admissible, its norm verifies

$$\|u\|_{N((t, \infty))} \leq \|u\|_{L^q_t L^r_x((t, \infty) \times \mathbb{R}^d)}$$

where $q'$, $r'$ are the conjugate exponents of $q$ and $r$.

Let us recall the standard dispersive inequality

$$\|e^{it\Delta} u\|_p \lesssim |t|^{-d\left(\frac{1}{2} - \frac{1}{2}\right)}\|u\|_{p'} \quad \text{for } t \neq 0, \quad 2 \leq p \leq +\infty$$

from which one can deduce the usual Strichartz estimate:

$$\|u\|_{S(t_0, +\infty)} \lesssim \|u_0\|_{L^2} + \|F\|_{N(t_0, +\infty)}$$

where for $u_0 \in L^2(\mathbb{R})$ $u$ solves on $[t_0, \infty)$ the following equation

$$iu_t + \Delta u = F, \quad u(t_0) = u_0.$$ 

For the proof of the main theorems with a profile $W = R_\infty$ or $W = K + R_\infty$, we will consider the error term $\eta = u - W$, which satisfies

$$i\partial_t \eta + \Delta \eta = -|f(W + \eta) - f(W)| - H, \quad H = f(W) - \sum_{j \in \mathbb{N}_0} f(R_j). \quad (3.1)$$

Above $R_0 = 0$ if $W = R_\infty$ and $R_0 = K$ if $W = K + R_\infty$. In Duhamel form,

$$\eta(t) = -i \int_0^t e^{i(s-t)\Delta} \left[ f(W + \eta) - f(W) + H(s) \right] \, ds. \quad (3.2)$$

The proofs of theorems 1.2 and 1.14 given in sections 4 and 5 are self-contained. For the proofs of theorems 1.8 and 1.15, we rely on the following generic result proved in [11, proposition 2.4].

**Proposition 3.1.** Let $d \geq 1$ and assume assumption (F0). Let $H = H(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{C}$, $W = W(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{C}$ be given functions which satisfy for some $C_1 > 0$, $C_2 > 0$, $\lambda > 0$, $T_0 \geq 0$:

$$\|W(t)\|_{\infty} + e^{\lambda t} \|H(t)\|_2 \leq C_1, \quad \forall t \geq T_0;$$

$$\|\nabla W(t)\|_2 + \|\nabla W(t)\|_{\infty} + e^{\lambda t} \|\nabla H(t)\|_2 \leq C_2, \quad \forall t \geq T_0. \quad (3.3)$$

Consider the equation (3.2). There exists a constant $\lambda_\ast = \lambda_\ast(d, \alpha_1, \alpha_2, C_1) > 0$ independent of $C_2$, and a time $T_\ast = T_\ast(d, \alpha_1, \alpha_2, C_1, C_2) > 0$ sufficiently large such that if $\lambda \geq \lambda_\ast$ and $T_0 \geq T_\ast$, then there exists a unique solution $\eta$ to (3.2) on $[T_0, +\infty) \times \mathbb{R}^d$ satisfying

$$e^{\lambda t} \|\eta\|_{S(t, \infty)} + e^{\lambda t} \|\nabla \eta\|_{S(t, \infty)} \leq 1, \quad \forall t \geq T_0. \quad (3.4)$$

Here $c_1 > 0$ is a constant depending only on $(\alpha_1, d)$.

**4. Construction of infinite soliton trains**

**4.1. Proof of theorem 1.2**

In this section we prove theorem 1.2 and construct infinite soliton trains in $\mathbb{R}^d$, $d \geq 1$. Note that (1.6) in assumption (T1) implies $A_2 := \sum_{j \in \mathbb{N}} \alpha_j \frac{d}{m_j} < \infty$, and

$$\|R_\infty(t)\|_{L^\infty \cap L^0} \leq \sum_{j \in \mathbb{N}} \|R_j(t)\|_{L^\infty \cap L^0} \leq \sum_{j \in \mathbb{N}} \left( \omega_{j+}^{\frac{d}{m_j}} + \omega_{j-}^{\frac{d}{m_j}} \right) = A_2 + A_1. \quad (4.1)$$
We first show the existence of the exponent $r_0$ and prove lemma 1.1.

**Proof of lemma 1.1.** The idea is to choose $r_0 = \max(1, \frac{d\alpha}{2}) + \epsilon$ for some $0 < \epsilon \ll 1$. Clearly $r_0 < 2 + \alpha_1$ for sufficiently small $\epsilon > 0$ since $\alpha_1 < \alpha_{\text{max}}$. So (1.7) is satisfied.

In the case $\frac{d\alpha}{2} \geq 1$, we claim

$$\frac{\alpha_1}{d\alpha} + \frac{1}{r_2} > \frac{1}{2}, \quad \frac{\alpha_1 + 1}{r_2} > 1.$$  

Both are clear if $d \leq 2$. For $d \geq 3$, both left sides become strictly smaller if $\alpha_1$ is replaced by $\alpha_{\text{max}} = \frac{d\alpha}{2}$ and $r_2$ is replaced by $2 + \alpha_{\text{max}}$, but are no less than the right sides by direct computation. Thus (1.8) and (1.9) are satisfied for sufficiently small $\epsilon > 0$.

In the case $\frac{d\alpha}{2} < 1$, we claim

$$\frac{\alpha_1}{1} + \frac{1}{r_2} > \frac{1}{2}, \quad \frac{\alpha_1 + 1}{1} + \frac{1}{r_2} > 1.$$  

The first inequality is a consequence of the assumption $\alpha_1 \geq \alpha_2/(\alpha_2 + 2)$, while the second is trivial. Thus (1.8) and (1.9) are satisfied for sufficiently small $\epsilon > 0$.

Suppose $\alpha_1 < 4/d$. In the case $\frac{d\alpha}{2} > 1$, since $\frac{d\alpha}{2} = \frac{d\alpha}{2} + \epsilon < 2$ for sufficiently small $\epsilon > 0$. In the case $\frac{d\alpha}{2} < 1$, $r_0 = 1 + \epsilon < 2$. The proof of the lemma is complete. \(\square\)

**Remark 4.1.** Although we chose $r_0 = \max(1, \frac{d\alpha}{2}) + \epsilon$ in the proof of lemma 1.1, it is not necessary for theorem 1.2. We only need $r_0$ to satisfy (1.7)–(1.9).

We next estimate the source term in the equation for the error.

**Lemma 4.2.** Under the assumptions of theorem 1.2, the source term $H = f(R_\infty) - \sum_{j \in \mathbb{N}} f(R_j)$ satisfies, for some $c_1 \in (0, a/2)$,

$$\|H(t, \cdot)\|_{L^\infty L^2} \leq C e^{-c_1 t}.$$  

**Proof.** Fix $t > 0$. For any $x \in \mathbb{R}^d$, choose $m = m(x) \in \mathbb{N}$ so that $\phi_m$ is a nearest soliton, i.e. $|x - v_m t| = \min_{j \in \mathbb{N}} |x - v_j t|$.

For $j \neq m$, we have

$$|x - v_j t| \geq \frac{1}{2} |v_j t - v_m t| = \frac{1}{2} |v_j - v_m|.$$  

Thus, by (1.5), we have

$$|(R_\infty - R_m)(x, t)| \leq \sum_{j \neq m} |R_j(x, t)| \leq \delta_m(x, t) := \sum_{j \neq m} \delta_m \frac{1}{A \varepsilon} e^{-\frac{\alpha}{2}|x - v_j t|}.$$  

Hence, by (1.6), the definition of $v_m$ (1.11) and (4.2), we have

$$\delta_m(x, t) \leq \sum_{j \neq m} \delta_m \frac{1}{A \varepsilon} e^{-\frac{\alpha}{2}|x - v_m t|} = \frac{1}{A \varepsilon} e^{-\frac{\alpha}{2}|x - v_m t|}.$$  

Denote $A_3 = \sup_{|z| < |R_\infty|, z \in \mathbb{R}^2} |d f(z)|$ (identifying $\mathbb{C}$ with $\mathbb{R}^2$, we denote by $df$ the differential of $f$, that is $df(z) h = g(|z|^2) h + 2 z g'(|z|^2) \text{Re}(z \tilde{h})$). By lemma 2.2 and (4.1), we have

$$|H(t, x)| \leq |f(R_\infty) - f(R_m)| + \sum_{j \neq m} |f(R_j)|$$

$$\leq A_3 |R_\infty - R_m| + \sum_{j \neq m} A_3 |R_j| \leq 2 A_3 \sum_{j \neq m} |R_j| \leq 2 A_3 \delta_m(t, x).$$
In particular,
\[ \| H(t) \|_{L^\infty} \leq 2D A_2 A_3 e^{-\frac{t}{\alpha_1 + 1}}. \] (4.5)
Condition (1.9) is equivalent to \( \frac{1}{r_1} < \frac{1}{r_0} \). We can choose \( s \) so that
\[ \frac{1 + \alpha_1}{r_0} > \frac{1}{s} > \frac{1}{r_1}, \quad s > 1. \] (4.6)
The first inequality of (4.6) ensures that
\[ \frac{\alpha_1 + 1}{\alpha_1} - \frac{d}{2s} > \frac{\alpha_1 + 1}{\alpha_1} - \frac{d}{2r_0}, \]
and hence, using (1.6),
\[ \sum_{j \in \mathbb{N}} \| f(R_j) \|_{L^r} \leq \sum_{j \in \mathbb{N}} \left( \| R_j \|_{L^r} + \| R_j \|_{L^s} \right) \leq \sum_{j \in \mathbb{N}} \omega_j^{\alpha_1 + 1} < C < \infty. \]
Since \( r_0 < s(1 + \alpha_1) < s(1 + \alpha_2) < \infty \) by (4.6), we have by (4.1)
\[ \| f(R_\infty) \|_{L^r} \leq \| R_\infty \|_{L^s} + s(1 + \alpha_2) \| R_\infty \|_{L^s} < C < \infty. \]
Thus
\[ \| H(t) \|_{L^r} < \| f(R_\infty) \|_{L^r} + \sum_{j \in \mathbb{N}} \| f(R_j) \|_{L^r} < C < \infty. \] (4.7)
By Hölder inequality between \( L^\infty \) and \( L^r \) using (4.5) and (4.7), we have
\[ \| H(t) \|_{L^r} \leq Ce^{-(1-s/r)\frac{r}{2} v_{r,t}}, \quad \forall r \in (s, \infty). \]
Since \( s < r_2' < \infty \) by (4.6), we get the desired conclusion. \( \square \)

We now prove theorem 1.2.

**Proof of theorem 1.2.** The existence of \( r_0 \) has been shown in lemma 1.1. We now fix such a choice. The difference \( \eta = u - R_\infty \) satisfies equation (3.2) with \( W = R_\infty \) and \( H = f(R_\infty) - \sum_{j \in \mathbb{N}} f(R_j) \). Denote the right side of (3.2) as \( \Phi \eta \). We will show it is a contraction mapping and has a unique fixed point \( \eta = \Phi \eta \) in the class
\[ \| \eta(t) \|_{L^r} + \| \eta \|_{S^1, \infty} \leq e^{-c_1 v_{r,t}}, \quad \forall t \geq 0. \] (4.8)
We first show boundedness and suppose \( \eta \) satisfies (4.8). By Hölder inequality,
\[ \| \eta(t) \|_{L^r} \leq e^{-c_1 v_{r,t}}, \quad \forall r \geq 0, \quad \forall r \in [2, r_2]. \]
We have
\[ \| \Phi \eta(t) \|_{L^2} \leq C \int_t^\infty | t - \tau |^{-\theta} \left( \| f(W + \eta) - f(W) \|_{L^2} + \| H(\tau) \|_{L^2} \right) d\tau, \]
where \( \theta = d \left( \frac{1}{r_2'} - \frac{1}{r_2} \right) \), and \( 0 < \theta < 1 \) since \( 2 < r_2 < 2 + \alpha_{\text{max}} \).
By lemma 4.2 we have \( \| H(\tau) \|_{L^2} \leq Ce^{-c_1 v_{r,t}} \). By lemma 2.2,
\[ \| f(W + \eta) - f(W) \|_{L^2} \leq \| \eta \|_{S^1, \infty} \| W \|_{L^2} + \| W \|_{L^2} \| \eta \|_{L^2} \| W \|_{L^2} + \| \eta \|_{S^1, \infty} \| W \|_{L^2} \| \eta \|_{L^2} \| W \|_{L^2} \leq Ce^{-c_1 v_{r,t}}. \] (4.9)
The first term on the right side is bounded by Hölder inequality
\[ \| \eta \|_{S^1, \infty} \| W \|_{L^2} \| \eta \|_{L^2} \| W \|_{L^2} \leq \| W \|_{L^2} \| \eta \|_{L^2} \| W \|_{L^2} \| \eta \|_{L^2} \| W \|_{L^2} \leq Ce^{-c_1 v_{r,t}}. \]
if
\[ \frac{\alpha_1 + 1}{\alpha_1} - \frac{d}{2s} < \frac{\alpha_1 + 1}{\alpha_1} - \frac{d}{2r_0}. \]
The first inequality is always true since $r_2' \leq 2 \leq r_2$. The second inequality is correct if (1.8) holds. Thus this term can be estimated.

The last term of (4.9) is bounded by
$$
\|\eta^{\alpha_1+1}\|_{L^r} \leq \|\eta\|_{L^\infty}^{\alpha_1+1} + \|\eta\|_{L^\infty}^{\alpha_1+1},
$$
which is bounded by $Ce^{-\epsilon_1 v_* t}$ since
$$2 \leq r_2'(\alpha_1 + 1) < r_2'(\alpha_2 + 1) = r_2,$$
due to (1.10) and $r_2 = 2 + \alpha_2$.

Combining the above we have, assuming (4.8),
$$
\|\Phi_\eta(t)\|_{L^r} \leq \int_t^\infty |t - \tau|^{-\theta} Ce^{-\epsilon_1 v_* \tau} d\tau \leq C v_*^{-1+\theta} e^{-\epsilon_1 v_* t}
$$
for all $t \geq 0$, which is bounded by $\frac{1}{4} e^{-\epsilon_1 v_* t}$ if $v_*$ is sufficiently large.

For the Strichartz estimate, since $(2/\theta, r_2)$ is admissible, we have with $a = (2/\theta)'$
$$
\|\Phi_\eta\|_{S((t, \infty))} \lesssim \|f(W + \eta) - f(W) + H\|_{L^r(t, \infty; L^\infty)} \lesssim \|\eta\|_{L^\infty}^{\epsilon_1 v_* t} \lesssim v_*^{-1/\theta} e^{-\epsilon_1 v_* t},
$$
for all $t \geq 0$, which is bounded by $\frac{1}{4} e^{-\epsilon_1 v_* t}$ if $v_*$ is sufficiently large.

Consider now the difference estimate. Suppose both $\eta_1$ and $\eta_2$ satisfy (4.8). Denote $\eta = \eta_1 - \eta_2$ and
$$
\delta = \sup_{t \geq 0} e^{\epsilon_1 v_* t} \left( \|\eta(t)\|_{L^r} + \|\eta\|_{S((t, \infty))} \right) \leq 2.
$$
We have
$$
\|\Phi_\eta(t) - \Phi_{\eta_2}(t)\|_{L^r} \leq C \int_t^\infty |t - \tau|^{-\theta} \|f(W + \eta_1) - f(W + \eta_2)\|_{L^\infty} d\tau.
$$
By lemma 2.2 again with $W$ replaced by $W + \eta_2$,
$$
\|f(W + \eta_1) - f(W + \eta_2)\|_{L^r} \lesssim \|\eta\|_{L^\infty} (\|W + \eta_1\|^{\alpha_1} + \|W + \eta_2\|^{\alpha_2}) + \|\eta\|^{\alpha_1+1} + \|\eta\|^{\alpha_2+1}
$$
$$
\lesssim \|\eta\| (\|W\|^{\alpha_1} + \|W\|^{\alpha_2}) + \|\eta\| (E^{\alpha_1}_t + E^{\alpha_2}_t)\|_{L^\infty}
$$
$$
(4.10)
$$
where $E = |\eta_1| + |\eta_2|$. The first term is already bounded above
$$
\|\eta\| (\|W\|^{\alpha_1} + \|W\|^{\alpha_2})\|_{L^\infty} \leq C \|\eta\|_{L^r(t, \infty)} \lesssim C \delta e^{-\epsilon_1 v_* t}.
$$
The last term of (4.10) is bounded similarly as above
$$
\|\eta\| (E^{\alpha_1}_t + E^{\alpha_2}_t)\|_{L^\infty} \lesssim \|\eta\|_{L^r(t, \infty)} (\|E^{\alpha_1}_t\|_{L^\infty} + \|E^{\alpha_2}_t\|_{L^\infty}) \lesssim C \delta e^{-\epsilon_1 (v_*) t}.
$$
Thus
$$
\|\Phi_\eta(t) - \Phi_{\eta_2}(t)\|_{L^r} \leq \int_t^\infty |t - \tau|^{-\theta} C \delta e^{-\epsilon_1 v_* \tau} d\tau \lesssim C \delta v_*^{-1+\theta} e^{-\epsilon_1 v_* t}
$$
for all $t \geq 0$, which is bounded by $\frac{1}{4} \delta e^{-\epsilon_1 v_* t}$ if $v_*$ is sufficiently large.

We also have (recall $a = (2/\theta)'$)
$$
\|\Phi_\eta - \Phi_{\eta_2}\|_{S((t, \infty))} \lesssim \|f(W + \eta_1) - f(W + \eta_2)\|_{L^\infty(t, \infty; L^\infty)} \lesssim \delta v_*^{-1/\theta} e^{-\epsilon_1 v_* t},
$$
for all $t \geq 0$, which is bounded by $\frac{1}{4} \delta e^{-\epsilon_1 v_* t}$ if $v_*$ is sufficiently large.

We have shown that $\Phi$ is a contraction mapping and hence has a unique fixed point in the set (4.8). The proof of theorem 1.2 is complete. \qed
Remark 4.3. The assumption (1.10) is used to estimate $L^s$. To estimate $|\eta|^{\alpha_1+1}$ in $L^{r_0}$ using $L^s$-$L^s$ decay estimates, a restriction like (1.10) is needed to avoid the limiting case $\alpha_1 = 0+$ and $\alpha_2 = \alpha_{\text{max}}$.

The condition (1.8) is used to bound the linear term in $\eta$, while (1.9) is used to bound the source term (it ensures the existence of $s$ in the proof of lemma 4.2).

In (1.7), we need $r_0 \geq 1$ for (4.1). We need $r_0 > \frac{d}{\alpha_1}$ so that the exponent in (1.6) is positive. The condition $r_0 < \alpha_1 + 2$ in (1.7) is redundant and follows from (1.9).

4.2. Proof of theorem 1.8

In this section we prove theorem 1.8 and construct infinite soliton trains in $\mathbb{R}^d$, $d \geq 1$. All along this section, we assume that we are under the assumptions of theorem 1.8, in particular we suppose that we are given a sequence of bound states $(\phi_j, \omega_j)$ for $j \in \mathbb{N}$ satisfying assumptions (T1), (1.13) (with $v_2$ to be determined later) and (1.14).

We first prove the following lemma.

Lemma 4.4. Let $a \in (0, 1)$ be given by assumption (T1). For $\lambda = \min(1, 2a)v_4/4 > 0$, we have

$$
\|R_\infty(t)\|_\infty + e^{\lambda t} \|H(t)\|_2 \leq C, \quad \forall t \geq 0;
$$

$$
\|\nabla R_\infty(t)\|_2 + \|\nabla R_\infty(t)\|_\infty + e^{\lambda t} \|\nabla H(t)\|_2 \leq C(1 + V_\infty), \quad \forall t \geq 0.
$$

Equation (1.6) in assumption (T1) implies $A_2 := \sum_{j \in \mathbb{N}} \omega_j^\frac{1}{\alpha_1} < \infty$, and

$$
\|R_\infty(t)\|_{L^\infty \cap L^\infty} \leq \sum_{j \in \mathbb{N}} \|R_j(t)\|_{L^\infty \cap L^\infty} \leq \sum_{j \in \mathbb{N}} (\omega_j^\frac{1}{\alpha_1} + \omega_j^\frac{1}{\alpha_2} - \frac{d}{2}) = A_2 + A_1.
$$

We also have for $1 \leq r \leq \infty$

$$
\|\nabla R_\infty(t)\|_r \leq \sum_{j \in \mathbb{N}} \|\nabla R_j(t)\|_r \leq \sum_{j \in \mathbb{N}} \omega_j^\frac{1}{\alpha_1} + \omega_j^\frac{1}{\alpha_2} + \sum_{j \in \mathbb{N}} |v_j| \omega_j^\frac{1}{\alpha_1}.
$$

(4.12)

If we take $r = 2$, we have $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} - \frac{d}{\alpha_1} \geq \frac{1}{\alpha_1} - \frac{d}{\alpha_1}$ for all dimensions since $r_0 < 2 + \alpha_{\text{max}}$. Thus the first sum of the right hand side of (4.12) is finite for $r \in [2, \infty]$ by (1.6). The second sum is also finite for $r \in [2, \infty]$ by (1.14). Thus

$$
\|\nabla R_\infty(t)\|_{L^2 \cap L^\infty} \leq A_1 + V_\infty.
$$

We next consider the estimates of $H = f(R_\infty) - \sum_{j \in \mathbb{N}} f(R_j)$. Fix $t > 0$. As in the proof of lemma 4.2, take any $x \in \mathbb{R}^d$ and choose $m = m(x) \in \mathbb{N}$ so that $\phi_m$ is a nearest soliton, i.e.

$$
|x - v_mm| = \min_{j \in \mathbb{N}} |x - v_jt|.
$$

Since $\alpha_1 < \alpha_{\text{max}}$ and $r_0 < 2 + \alpha_1$, there exists $s = \frac{\alpha_1 + 1}{\alpha_1 - \epsilon}$ with $0 < \epsilon \ll 1$ such that

$$
r_0 < 2 + \alpha_1 - \epsilon, \quad \alpha_1 + 1 - \frac{d}{\alpha_1} \geq \frac{1}{\alpha_1} - \frac{d}{2r_0}.
$$

From arguments identical to those of the proof of lemma 4.2, we have

$$
\|H(t)\|_r \leq C e^{\epsilon(1 - 1/r)x_0}, \quad \forall r \in (s, \infty),
$$

with acceptable $r$ including $\frac{\alpha_1 + 2}{\alpha_1 + 1}$ and 2.
To estimate $\|\nabla H(t)\|_{L^2}$, recall that by the Chain Rule we have
\[
\nabla H = \nabla (f(R_\infty)) - \sum_{j \in \mathbb{N}} \nabla (f_j(R_j))
= \sum_{j \in \mathbb{N}} (f_j(R_\infty) - f_j(R_j)) \nabla R_j + \sum_{j \in \mathbb{N}} (f_j(R_\infty) - f_j(R_j)) \nabla R_j.
\] (4.13)

Here, we denoted $f_j(x) = \frac{a}{2\pi} f(x)$ and $f_j(x) = \frac{a}{2\pi} f(x)$ the Wirtinger derivatives of $f$. Thus (here $x$ and $m = m(x)$ are still as above), we have
\[
|\nabla H(t,x)| \lesssim \sum_{j \neq m} \|\nabla R_j\| + (|f_j(R_\infty) - f_j(R_m)| + |f_j(R_\infty) - f_j(R_m)|) \|\nabla R_m\|
\]
\[
\lesssim \sum_{j \neq m} \langle v_j \rangle \omega_j^{1/2} e^{-\langle v_j \rangle |x-v_j| t} + (\delta_m(t,x)) \min(1,\alpha) \langle v_m \rangle \omega_m^{1/2} e^{-\langle v_m \rangle |x-v_m| t}
\]
\[
\lesssim \sum_{j \neq m} \langle v_j \rangle \omega_j^{1/2} e^{-\langle v_j \rangle |x-v_j| t} e^{-\frac{\lambda}{2} \min(1,\alpha) v_j t} + e^{-\frac{\lambda}{2} \min(1,\alpha) v_j t} \langle v_m \rangle \omega_m^{1/2} e^{-\langle v_m \rangle |x-v_m| t}
\]
\[
\lesssim e^{-\lambda t} \sum_{j \neq m} \langle v_j \rangle \omega_j^{1/2} e^{-\frac{\lambda}{2} \langle v_j \rangle |x-v_j| t}
\]
where $\delta_m(t,x)$ is defined and estimated in (4.3)-(4.4), and $\lambda = \frac{\pi}{4} \min(1,2\alpha) v_x$. Thus
\[
\|\nabla H(t)\|_{L^2} \lesssim e^{-\lambda t} \sum_{j \neq m} \langle v_j \rangle \omega_j^{1/2} e^{-\frac{\lambda}{2} \langle v_j \rangle |x-v_j| t} \lesssim e^{-\lambda t} v_x
\]
by assumption (1.14). The proof of lemma 4.4 is complete. \hfill \square

We now prove theorem 1.8.

**Proof of theorem 1.8.** By lemma 4.4, there exists $v_x$ such that if $v_x > v_x$, then the hypothesis (3.3) of proposition 3.1 is satisfied under the assumptions of theorem 1.8, with $W = R_\infty$ and $H = f(R_\infty) - \sum_{j \in \mathbb{N}} f_j(R_j)$. By proposition 3.1, there exists $T_0$ large enough and $\eta \in C([T_0,\infty),H^1)$ with $\|\nabla \eta\|_{S([T_0,\infty)})$ (in particular $\|\nabla(t)\|_{H^2}$) decaying exponentially in $t$. \hfill \square

**Remark 4.5.** Assumption (1.14) guarantees that $\|\nabla H\|_{L^2} < \infty$. One may tend to relax the exponent $2$ in the norm $\|\nabla W\|_{L^2}$ so that $\nabla W$ is not that localized. However, $\|\nabla W\|_{L^2}$ with $\beta_1 < 0.1$ is used in the proof of proposition 3.1. It would not gain much trying to optimize it.

### 5. Construction of infinite kink-soliton trains

In this section we prove theorems 1.14 and 1.15, and construct a train made of infinitely many solitons and a half-kink for space dimension 1.

We first examine assumption (F1) and give some examples. Estimate (1.20) is natural since $f'$ is Hölder continuous. If $f'(b) \neq 0$, we can only take $\tilde{\alpha} = 0$. Otherwise, we may take $\tilde{\alpha} = 1$ if $f$ is locally $C^{1,1}$ near $b$. For certain $f(b)$ we have $\tilde{\alpha} > 1$.

**Example 5.1.** Let $f(s) = s - \sin \left( \frac{|s|}{2} \right) s$. If we write $f(s) = f_1(s) + f_2(s)$ with $f_1(s) = \frac{1}{2} |s|^2 s$ and $f_2(s) = s - \sin \left( \frac{|s|}{2} \right) s - \frac{1}{3} |s|^3 s = O(s^5)$, $f$ satisfies assumptions (F0) and (F2) with $\alpha_1 = 2$ and $\alpha_2 = 4$. We can choose $R_0 = 1 + \epsilon$, $0 < \epsilon \ll 1$, for assumption (F1). The function $f(s)$ also satisfies assumption (F1) with $\alpha = 1$, $b = 2\pi$, $h(s) = \sin s$ and $h'(b) = 1$. Moreover, $f(2\pi) = 2\pi \neq 0$, $|f'(2\pi + s)| = |1 - \cos s| \leq C \delta^{\tilde{\alpha}}$, $\tilde{\alpha} = 2$.

Hence conditions (1.21)–(1.22) are satisfied. Thus we can construct infinite kink-soliton trains using theorem 1.14. Since $\alpha_1 > 4/3$, theorem 1.15 does not apply to this example. \hfill \square
Example 5.2. Let \( f(s) = |s|^\alpha s - |s|^\beta s, 0 < \alpha < \beta < \infty \). Clearly \( f \) satisfies assumptions (F0) and (F2) with \( \alpha_1 = \alpha \) and \( \alpha_2 = \beta \). The conditions \( h(b) = \int_0^b h(s) \, ds = 0 \) give
\[
\omega = b^\alpha - b^\beta = \frac{2}{2 + \alpha} b^\alpha - \frac{2}{2 + \beta} b^\beta.
\]
Thus
\[
b^{\beta - \alpha} = \frac{\alpha (2 + \beta)}{(2 + \alpha)\beta} \in \left( \frac{\alpha}{\beta}, 1 \right), \quad \omega = b^\alpha (1 - b^{\beta - \alpha}) > 0,
\]
and
\[
h'(b) = \omega - (1 + \alpha)b^\alpha + (1 + \beta)b^\beta = -\alpha b^\alpha + \beta b^\beta > 0.
\]
Thus (1.19) can be always satisfied by unique \( \omega > 0 \) and \( b > 0 \). For (1.20), we have \( \tilde{\alpha} = 0 \) for most pair \((\alpha, \beta)\). Theorem 1.14 is not applicable in those cases. The exception is when
\[
0 = f'(b) = (1 + \alpha)b^\alpha - (1 + \beta)b^\beta = \frac{\alpha (2 + \beta)}{(2 + \alpha)\beta} - \frac{1}{1 + \gamma}, \quad \text{or} \quad \alpha \beta = 2.
\]
Thus the exceptional case is
\[
\tilde{\alpha} = 1 \quad \text{if} \quad 0 < \alpha < \sqrt{2}, \quad \beta = \frac{2}{\alpha}.
\]
Since \( d\alpha/2 < 1 \), we can take \( r_0 = 1 \). Conditions (1.21)-(1.22) imply
\[
\frac{\sqrt{5} - 1}{2} \leq \alpha < \sqrt{2}, \quad \beta = \frac{2}{\alpha}.
\]
Thus for \( \alpha \) satisfying (5.1), using theorem 1.14 we can construct infinite kink-soliton trains for the nonlinearity \( f(u) = (|u|^\alpha - |u|^\beta)u \). On the other hand, by theorem 1.15 we can construct infinite kink-soliton trains if
\[
0 < \alpha < 4/3, \quad \alpha < \beta < \infty.
\]
We do not need \( \alpha \beta = 2 \). Indeed, since \( d\alpha/2 < 1 \) for \( \alpha < 4/3 \), we can take \( r_0 = 1 \), and condition (1.24) is satisfied for any \( \tilde{\alpha} \geq 0 \). We can choose \( \omega_j \) and \( v_j \) as in (1.16) or (1.17). In comparison, theorem 1.15 covers more exponents than theorem 1.14 except when
\[
\frac{4}{3} \leq \alpha < \sqrt{2} \quad \text{and} \quad \beta = 2/\alpha.
\]

As mentioned in section 1, a kink solution of (NLS) with parameters \((v_0, \gamma)\) is (setting the spatial translation to \(x_0 = 0\))
\[
K(t, x) = \phi_K(x - v_0 t) e^{i(\omega_0 t + \frac{1}{4}v_0 x - \frac{1}{2}v_0^2 t^2 + \gamma)}.
\]
For notational simplicity, we denote \( K = R_0 \) and we consider the kink-soliton train profile
\[
W = K + R_\infty = \sum_{j=0}^{\infty} R_j
\]
where \( R_\infty \) and \( R_j, j > 0 \), are given in (1.4).

5.1. Proof of theorem 1.14

We will solve the difference \( \eta = u - W \) in the class (1.23). To start the proof, we note that, because \( \tilde{\alpha} \) satisfies the same conditions as \( \alpha_1 \), we can choose \( r_0 \) as in lemma 1.1 to satisfy (1.22) in addition to (1.7)-(1.9). From now on we fix \( r_0 \).

We start by estimating the source term.
Lemma 5.3. Under the assumptions of theorem 1.14, the source term $H = f(W) - \sum_{j=0}^{\infty} f(R_j)$ satisfies, for some $c_1 > 0$,
$$
\|H(\cdot, t)\|_{L^{\infty}\cap L^2} \leq C e^{-c_1 v_1 t}.
$$

Proof. By (1.22), we have $\frac{2r+1}{2r} > \frac{1}{f_2}$. We can choose $s$ as in the proof of lemma 4.2 to satisfy (4.6) and $\frac{2s+1}{2s} > \frac{1}{f_2}$.

For $x > \frac{1}{2}(v_0 + v_1)t$, the contribution from $R_0$ is the same as if $R_0$ were a soliton. Thus the estimate follows from lemma 4.2.

For $x \leq \frac{1}{2}(v_0 + v_1)t$, we have $H = (f(W) - f(K)) - \sum_{j=1}^{\infty} f(R_j)$. In the proof of lemma 4.2 we have shown

$$
|R_\infty(t, x)| \leq C e^{-\frac{1}{2}v_1 t}, \quad \|R_\infty\|_{L^m} \leq C, \quad \sum_{j=1}^{\infty} f(R_j)(t, x) \leq C e^{-\frac{1}{2}v_1 t}, \quad \left|\sum_{j=1}^{\infty} f(R_j)\right| \leq C.
$$

(5.3)

For simplicity in notations, we assume now that $v_0 = \gamma_0 = 0$. This causes no loss of generality since (NLS) is invariant under a Galilean transform and it guarantees that the left part of the kink is approximately $b$ without correction by a phase factor containing $e^{i\frac{1}{2}v_1 t}$. By assumption (F1), the mean value theorem, and since $K, R_\infty \in L^\infty$, we have

$$
|f(W) - f(K)| = |f(b + K - b + R_\infty) - f(b + K - b)| \lesssim (|K| - b) + |R_\infty| |\tilde{v}||R_\infty|.
$$

We first derive

$$
|f(W) - f(K)| \leq C e^{-\frac{1}{2}v_1 t}.
$$

Because $r_0 < (1 + \tilde{a})\delta x$,

$$
\|f(W) - f(K)\|_{L^t} \leq \|\|K| - b\|_{L^1} + \|R_\infty\|_{L^1} + \|R_\infty\|_{L^1} \leq \|\theta\|_{L^1} + \|\theta\|_{L^1} \leq C.
$$

Summing these estimates, we have

$$
\|H(t)\|_{L^\infty} \leq C e^{-\frac{1}{2}v_1 t}, \quad \|H(t)\|_{L^t} \leq C.
$$

The lemma follows by Hölder inequality between $L^\infty$ and $L^t$. \qed

Proof of theorem 1.14. Fix a choice of $r_0$ satisfying (1.7)–(1.9) and (1.22). Let $\chi_1 = \chi_1(x, t) = 1_{t \leq \frac{1}{(v_0 + v_1)n}}$ and $\chi_2 = 1 - \chi_1$. Using (5.3), we have

$$
\|\chi_1(W - b)\|_{L^\infty \cap L^\infty} + \|\chi_2 W\|_{L^\infty \cap L^\infty} \lesssim 1.
$$

Assume

$$
\|\eta(t)\|_{L^2} + \|\eta\|_{S(t, \infty)} \leq e^{-c_1 v_1 t}, \quad \forall t \geq 0.
$$

(5.4)

Note

$$
|f(W + \eta) - f(W)| \lesssim \chi_1|W - b|\tilde{v}|\eta| + \chi_2|W|^q + |\eta|^q + 1 + |\eta|^{q+1} + |\eta|^{q^2 + 1}.
$$

Thus by (1.8) and (1.22) we have

$$
\|f(W + \eta) - f(W)\|_{L^2} \lesssim (1 + \|\chi_1(W - b)\|_{L^\infty \cap L^\infty} + \|\chi_2 W\|_{L^\infty \cap L^\infty} + \|\eta\|_{L^\infty \cap L^2} + \|\eta\|_{L^2 \cap L^2} + \|\eta\|_{L^2 \cap L^2} + \|\eta\|_{L^2 \cap L^2}
\lesssim C e^{-c_1 v_1 t}.
$$
Denote the right side of (3.2) as $\Phi \eta$. The same argument as for theorem 1.2 shows that

$$\|\Phi \eta(t)\|_{L^2} \leq C v^4 e^{-\epsilon_1 v t}, \quad \|\Phi \eta\|_{S_{1(t, \infty)}} \leq C v^{1+\epsilon} e^{-\epsilon_1 v t}.$$  

Thus $\| \Phi \eta(t)\|_{L^2} + \| \Phi \eta\|_{S_{1(t, \infty)}} \leq e^{-\epsilon_1 v t}$ for $v_\ast$ sufficiently large.

For the difference estimate, for $\eta_1$ and $\eta_2$ satisfying (5.4), we use

$$|f(W + \eta_1) - f(W + \eta_2)| \lesssim \left( x_1 |W - b|^{\tilde{\alpha}} + x_2 |W|^{\tilde{\alpha}} + \sum_{j=1,2} (|\eta_j|^{\tilde{\alpha}} + |\eta_j|^2) \right) |\eta|$$

where $\eta = \eta_1 - \eta_2$, and follow the same argument for theorem 1.2 to derive, for $v_\ast$ sufficiently large,

$$\|\Phi \eta_1 - \Phi \eta_2\| \lesssim \frac{1}{2} \|\eta_1 - \eta_2\|$$

where $\|\eta\| = \sup_{t>0} e^{itv_\ast} (\|\eta(t)\|_{L^2} + \|\eta\|_{S_{1(t, \infty)}})$. We have shown that $\Phi$ is a contraction mapping in the class (5.4). The proof of theorem 1.14 is complete. \(\square\)

5.2. Proof of theorem 1.15

In this section we prove theorem 1.15 and use proposition 3.1 to construct a train of infinitely many solitons and a half-kink for space dimension 1.

We assume throughout this section that the assumptions of theorem 1.15 hold. In particular, $(\phi_j, \omega_j)$ for $j \in \mathbb{N}$ denotes a sequence of bound states satisfying assumptions (T1), (1.25) (with $v_\ast$ to be determined later) and $\phi_0 = \phi_K$ is the kink profile given in proposition 1.13.

As in section 4.2, our main task is to prove that the profile $W = K + R_\infty$ and the source term $H = f(W) - f(K) - \sum_{j \in \mathbb{N}} f(R_j)$ satisfy the hypotheses of proposition 3.1.

**Lemma 5.4.** Let $a \in (0,1)$. For $\lambda = a \min(1, 2a) v_\ast / 4 > 0$, we have

$$\|W(t)\|_\infty + e^{\lambda t} \|H(t)\|_2 \leq C_1, \quad \forall t \geq 0;$$

$$\|\nabla W(t)\|_2 + \|\nabla W(t)\|_\infty + e^{\lambda t} \|\nabla H(t)\|_2 \leq C(1 + V_\ast), \quad \forall t \geq 0.$$

**Proof.** Since $R_\infty$ satisfies the same hypotheses as in lemma 4.4, we only have to treat the addition of the kink. We have, by lemma 4.4 and proposition 1.13

$$\|W\|_\infty + \|\nabla W\|_\infty \leq \|K\|_\infty + \|R_\infty\|_\infty + \|\nabla K\|_\infty + \|\nabla R_\infty\|_\infty \leq C.$$

Note that by exponential decay $\nabla K \in L^2(\mathbb{R})$, therefore, combined with lemma 4.4 this gives

$$\|\nabla W\|_2 \leq \|\nabla K\|_2 + \|\nabla R_\infty\|_2 \leq C.$$

We now estimate the source term $H$. As in the proof of lemma 4.4, we fix $t > 0$, take any $x \in R$ and choose $m = m(x)$ corresponding to the nearest profile, i.e.

$$|x - v_m t| = \min_{j \in \mathbb{N}} |x - v_j t|.$$

If $m \geq 1$, then as in the proof of lemma 4.2, we still have

$$|(R_\infty - R_m)(t, x)| \leq C e^{-\frac{1}{2} \alpha v_m t},$$

and by proposition 1.13 it holds

$$|K(t, x)| \leq D_0 e^{-|x - v_m t|} \leq D_0 e^{-\frac{1}{2} \alpha v_m t}.$$

Therefore, if $m \geq 1$ we have

$$H(t, x) \lesssim |f(R_\infty) - \sum_{j \in \mathbb{N}} f(R_j)| + A_4 |K| + |f(K)| \lesssim e^{-\frac{1}{2} \alpha v_m t},$$

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where $A_4 = \max_{s \in [0,|W|\infty]} f'(s)$. If $m = 0$, we replace the previous estimate by

$$H(t,x) \leq A_4 |R_\infty| + \sum_{j \in \mathbb{N}} |f'(R_j)| \lesssim e^{-\frac{1}{2}a_v t}.$$ 

This implies that

$$\|H(t)\|_{\infty} \lesssim e^{-\frac{1}{2}a_v t}.$$ 

With $x$ and $m$ as above, if $m = 0$, we have (using a similar expression as (4.13))

$$|\nabla H(t,x)| \lesssim (\|f_z(K + R_\infty) - f_z(K)\| + \|f_z(K + R_\infty) - f_z(K)\|) |\nabla K| + \sum_{j \in \mathbb{N}} |\nabla R_j|.$$ 

Since we are close to the kink ($m = 0$), the last sum will be small:

$$\sum_{j \in \mathbb{N}} |\nabla R_j| \lesssim e^{-\frac{4}{v_j}} \sum_{j \in \mathbb{N}} (v_j) \omega_j^{1/4} e^{-\frac{1}{4}a_j |x-v_j t|}.$$ 

In addition we have

$$|\nabla H(t,x)| \lesssim e^{-\frac{1}{2}a_v t} \sum_{j \in \mathbb{N}} (v_j) \omega_j^{1/4} e^{-\frac{1}{4}a_j |x-v_j t|}.$$ 

Therefore

$$|\nabla H(t,x)| \lesssim e^{-\frac{1}{2}a_v t} \sum_{j \in \mathbb{N}} (v_j) \omega_j^{1/4} e^{-\frac{1}{4}a_j |x-v_j t|}.$$ 

The estimate for the case $m \geq 1$ is similar as in lemma 4.4 and we can conclude by (1.26) that

$$\|\nabla H\|_{2} \lesssim e^{-\lambda t} \sum_{j \in \mathbb{N}} (v_j) \omega_j^{1/4 - d/4} \lesssim e^{-\lambda t} V_\ast.$$ 

Now let $s$ be defined as in the proof of lemma 4.4. By (1.24), we can further assume

$$r_0 \leq s(\tilde{\alpha} + 1).$$ 

(5.5)

For simplicity in notations, assume that the kink is not moving, i.e. $v_0 = 0$. Therefore the main contribution will come from the kink for $x < 0$ and the soliton train for $x > 0$. We have on the right

$$\|H\|_{L^t(x<0)} \leq \|f(K + R_\infty) - f(R_\infty)\|_{L^t(x>0)} + \|f(K)\|_{L^t(x>0)} + \|f(R_\infty)\|_{L^t} + \sum_{j \in \mathbb{N}} \|f(R_j)\|_{L^t} \leq A_4 \|K\|_{L^t(x>0)} + C < +\infty,$$

where the last inequality is due to exponential decay to 0 on the right for the kink. On the left, we have

$$\|H\|_{L^t(x<0)} \leq \|f(K + R_\infty) - f(K)\|_{L^t(x<0)} + \sum_{j \in \mathbb{N}} \|f(R_j)\|_{L^t}.$$ 

The first term cannot be treated as previously (unless $R_\infty \in L^t(\mathbb{R})$, which is a priori not the case). Since $f$ verifies (1.20), by the mean value theorem we have

$$|f(K + R_\infty) - f(K)| \lesssim \left(|K - b| + |R_\infty|\right)^{\tilde{\alpha}} + \left(|K - b| + |R_\infty|\right)^{\alpha_0} |R_\infty|.$$ 

Hence,

$$\|f(K + R_\infty) - f(K)\|_{L^t(x<0)} \lesssim \left(\|K - b\|_{L^t(x<0)}^{\tilde{\alpha}} + \|K - b\|_{L^t(x<0)}^{\alpha_0}\right) \|R_\infty\|_{L^\infty} + \|R_\infty\|_{L^\infty}^{\frac{1}{2}}.$$
The right hand side is finite since $K$ converges exponentially to $b$ and the $L^{s(G+1)}$-norm of $R_\infty$ is finite thanks to our choice of $r_0$ and (5.5). In conclusion,

$$\|H\|_{L^s} \leq \|H\|_{L^s(x<0)} + \|H\|_{L^s(x>0)} < +\infty.$$  

By interpolation between $s < 2$ and $\infty$ we get

$$\|H\|_{L^s} \lesssim e^{-\lambda t}.$$  

This concludes the proof. \hfill \Box

**Proof of theorem 1.15.** By lemma 5.4, there exists $v_\#$ such that if $v^* > v_\#$, then the hypothesis (3.3) of proposition 3.1 is satisfied under the assumptions of theorem 1.15. The conclusion of the theorem then follows immediately from the conclusion of proposition 3.1. \hfill \Box

**Acknowledgments**

The authors are grateful to Dong Li for stimulating discussions at the origin of this work. We thank Liren Lin for helpful comments about proposition 2.1. We also thank the anonymous reviewers for their careful reading of the manuscript and for helpful comments which improved the presentation of the paper. The research of S. Le Coz is supported in part by the french ANR through project ESONSE. The research of Tsai is supported in part by NSERC grant 261356-13 (Canada).

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