New approaches to dynamic evaluation of quality indicators

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Abstract. The paper proposes a mathematical model for evaluating time-varying quality indicators of an arbitrary nature. The model is based on the Rasch’s method of estimating latent variables. Cases of single-criterion and multi-criteria evaluation are considered. The obtained estimates are measured on a linear interval scale and do not depend on a set of indicators, time periods and evaluation criteria. Comparison of results with traditional assessment methods.

1. Introduction

The problem of obtaining quantitative estimates of qualitative indicators is one of the most relevant in measurement theory, expert assessment and decision theory. The most popular in this direction are the development of methods for pairwise comparisons, in particular, the methods of the analytical hierarchy [1] (Analytic Hierarchy Process - AHP), developed by T. Saati, and its modification - multiplicative ANR [2]. Recently, papers [3–5] have appeared on the improvement of mathematical methods for obtaining estimates of AHP, which are based on the Rasch model for evaluating latent variables [6, 7]. Such assessments of quality indicators are characterized by linearity, high accuracy and objectivity. However, in all the works listed above, the assessment is performed at a certain point in time, that is, it is static. If a quality indicator changes in time, then for its assessment at the current moment, it is often not its current value that is important, but the dynamics of change, that is, a tendency to increase or decrease.

This paper is devoted to assessing the dynamics of changes in arbitrary quality indicators in the context of identifying trends of their change for one or several objects or systems.

We first consider the case when the criterion by which indicators are evaluated is one. Let there be n quality indicators \( A_1, A_2, \ldots, A_n \), which are monitored for change at the current moment and throughout m previous periods of time. Due to the fact that it is impossible to measure quality indicators directly, it can only be assessed whether it became better or worse, increased or decreased, i.e. to note its qualitative change, comparing the change in the indicator, it is possible by comparison to note the change in each quality indicator for each (except the last) period of time compared with the previous one. We introduce the matrix:
Consider two approaches to finding the integral assessment $X_i$ of the growth of a quality indicator over all periods.

2. Single Criteria Estimation by Additive Method

This is the simplest traditional method [8], based on the summation of partial estimates of the growth of all indicators for all periods, as a result of which the integral estimates of the growth of indicators $A_i$ are found using the formula:

$$X_i = \sum_{j=0}^{m-1} x_{ij}, \quad i=1,2,...,n.$$  

(2)

However, in calculations using formula (2), all time periods give the same contribution to the overall growth rate and do not take into account the fact that usually the importance of growth of a quality indicator in recent periods should be higher than in earlier ones. To account for this fact, you can enter the weight $w_j$ for the growth factors. If we use a linear scale of the scale, that is, the contribution of the private growth of the indicator to the integral should be proportional to the time interval between the measurement and the current moment, then to estimate the weight, you can use the formula:

$$w_j = \frac{m - j}{m},$$

(3)

where $j$ – lag or number of time period of measurement.

Then the formula (2) takes the form:

$$X_i = \sum_{j=0}^{m-1} w_j x_{ij} = \sum_{j=0}^{m-1} \frac{m - j}{m} x_{ij}, \quad i=1,2,...,n.$$  

(4)

This approach, despite its ease of use, has several disadvantages, the main of which is that the growth in each period of the indicator does not take into account what other indicators grew and if, for example, the period was long and increased almost all indicators, then the contribution of each the same as for the case, if the growth of one was accompanied by the invariance or fall of the others, and in this case, the growing indicators should make a greater contribution to the overall growth rate. In general, this disadvantage will give non-linearity of integral growth estimates.

To eliminate this drawback, an estimation model can be used, based on the Rasch’s method of evaluating latent variables.

3. Single Criteria Rasch’s Method

Latent (implicit, hidden) variables are variables that cannot be directly measured, but their estimates can be obtained using measured indicator variables. To assess the dynamics of changes in quality indicators, the following will be used as latent variables: $\theta_i$ – is the assessment of the $i$-th quality indicator (analogous to $X_i$ assessment from the additive method), and $\beta_j$ – is a certain indicator characterizing the cumulative growth of all quality indicators for the $j$- time period , and the smaller

$$x_{ij} = \begin{cases} 1, & \text{if the index } A_i \text{ increased in the period } j \\ 0, & \text{if the index } A_i \text{ did not increase in the period } j \end{cases} \quad \text{compared to the period } (j - 1),$$

(1)

$i=1,2,...,n; \quad j=0,1,...,m-1.$
the value of $\beta$, the more indicators showed growth at this time stage. In such a model, the probability $p_{ij}$ of the fact that the $j$-th indicator increased in the $i$-th time period will be determined by the logistic function of the form:

$$p_{ij} = \frac{e^{\theta_i - \beta_j}}{1 + e^{\theta_i - \beta_j}}.$$  \hfill (5)

To find the latent variables $\theta_i$ and $\beta_j$ without taking into account the weights, it is necessary to solve an optimization problem of the form [4, 9]:

$$S(\theta_i, \beta_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} - p_{ij})^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} \left( x_{ij} - \frac{e^{\theta_i - \beta_j}}{1 + e^{\theta_i - \beta_j}} \right)^2 \rightarrow \min.$$  \hfill (6)

Due to the fact that in (6) the latent variables $\theta_i$ and $\beta_j$ are present as a difference, their origin is not defined. In the case of normalizing the estimates of latent variables and setting the origin to zero to the smaller of the estimates, the optimization problem (6) will be complemented by a system of constraints:

$$\theta_i \geq 0; \quad \beta_j \geq 0; \quad i = 1, 2, \ldots, n; \quad j = 0, 1, \ldots, m - 1.$$  \hfill (7)

If it is necessary to take into account the weights $w_j$, obtained, for example, by the formula (3), then instead of the optimization problem (6), we need to solve the following:

$$S(\theta_i, \beta_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} (w_j x_{ij} - p_{ij})^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} \left( w_j x_{ij} - \frac{e^{\theta_i - \beta_j}}{1 + e^{\theta_i - \beta_j}} \right)^2 \rightarrow \min.$$  \hfill (8)

Problem solving (6) - (7) or (8) - (7) can be performed using computer technology, for example, in MS Excel with the Solver add-in (Search for a solution).

The authors carried out computational experiments, which consisted in generating random data for matrices (1) of different sizes and analyzing the properties of estimates obtained by different methods. In all cases, adequate estimates were obtained with good correlation. The average correlation matrix between the estimates obtained using computational experiments by different methods with and without weights is given in table 1. As can be seen from the table, the method based on the Rasch’s model is much more responsive to the introduction of weights than the additive method, which is a more adequate result when evaluating.

### Table 1. Correlation matrix between estimates obtained by different methods.

| Pearson Correlation | Additive method without weights | Rasch’s method without weights | Additive weights method | Rasch’s method with weights |
|---------------------|--------------------------------|--------------------------------|------------------------|---------------------------|
| Additive method without weights | 1 | 0.910 | 0.915 | 0.731 |
| Rasch’s method without weights | 0.910 | 1 | 0.806 | 0.717 |
| Additive weights method | 0.915 | 0.806 | 1 | 0.888 |
| Rasch’s method with weights | 0.731 | 0.717 | 0.888 | 1 |

Another advantage of the method based on latent variables is that, in addition to evaluations of quality indicators, it is possible to find a latent indicator $\beta_j$, which characterizes the cumulative growth of indicators at the $j$-th time period.
4. Approach to multi-criteria evaluation

Let us now consider a model that allows for multi-criteria evaluation of quality indicators in the dynamics of their changes. In this case, for evaluating quality indicators at each stage $l$ criteria are used: $K_1, K_2, \ldots, K_l$, and at each stage a matrix of the form is formed:

$$
\begin{align*}
X^{(j)}_{ik} &= \begin{cases} 
1, & \text{if the index } A_i \text{ increased in the period } j \\
0, & \text{if the index } A_i \text{ did not increase in the period } j \\
& \text{compared to the period } (j - 1) \text{ by criterion } k,
\end{cases} \\
& \quad i = 1, 2, \ldots, n; \quad j = 0, 1, \ldots, m - 1; \quad k = 1, 2, \ldots, l.
\end{align*}
$$

Based on the data (9), a matrix of private assessments of quality indicators is formed according to evaluation criteria: $\tilde{x}_{ik} = \sum_{j=0}^{m-1} x^{(k)}_{ij}$ - without weights, and $\tilde{x}_{ik} = \sum_{j=0}^{m-1} w_j x^{(k)}_{ij}$ - taking into account weights.

To calculate the integral estimates of qualitative indicators, we use formulas (2) for the additive model, and (6), (7) for the model based on latent variables, but using the matrix as input data $\tilde{x}_{ik}$. The method of obtaining estimates for the Rush model is the same as in the example described above. The only difference is that the estimates of the latent variables $\beta_k$ obtained according to the Rasch’s method should be interpreted as the degree of impracticability of the criteria. That is, the smaller the value of $\beta_k$, the more positive ratings for the growth of the $k$-th criterion were given for the entire set of quality indicators over the entire set of time periods.

5. Approach for quantitative evaluation criteria

In the previously considered models, the source data matrix (1) or (9) is discrete: its value is “1” if the growth of the corresponding quality indicator at the appropriate stage is expertly determined, and “0” if growth is not defined.

This limitation is connected exclusively with the specifics of measuring a quality indicator: in most cases it is impossible to determine how much the indicator has grown or decreased, one can only fix the fact of its change at the current stage compared to the previous one. However, the mathematical model for solving the problem does not impose any restrictions on the discreteness of the data. In view of this, if there is an opportunity to determine the degree of growth of the indicator $A_i$ at the $j$-th stage, you can use continuous baseline data to get a more accurate assessment of the quality indicators. In the case of a single-criterion assessment, we introduce an indicator variable $d_{ij}$ - the relative change in the qualitative indicator $A_i$ on the time period $j$ compared to the period $j-1$: $d_{ij} = \frac{a_{ij} - a_{ij-1}}{a_{ij-1}}$.

We present the normalization of this indicator on a single scale, the necessary condition imposed on the source data for the Rasch model. As the rationing algorithm, we take the one for which the largest relative gain gives a single value, the smallest (possibly negative) gives zero, and all the rest will be proportional to the relative gains. With this rationing [10], instead of (1) we will use the initial data of the form:

$$
x_{ij} = \frac{d_{ij} - \min_i(d_{ij})}{\max_i(d_{ij}) - \min_i(d_{ij})} = \frac{\frac{a_{ij} - a_{ij-1}}{a_{ij-1}} - \min_i \left( \frac{a_{ij} - a_{ij-1}}{a_{ij-1}} \right)}{\max_i \left( \frac{a_{ij} - a_{ij-1}}{a_{ij-1}} \right) - \min_i \left( \frac{a_{ij} - a_{ij-1}}{a_{ij-1}} \right)},
$$

$$
i = 1, 2, \ldots, n; \quad j = 0, 1, \ldots, m - 1.
$$
All other calculations are similar to the above. In multi-criteria evaluation, the same algorithm is used.

6. Conclusion
Thus, the presented methodology for assessing a quality indicator in the dynamics of its change allows to obtain integral assessments of changes in indicators over time, both in single criteria and in multi criteria. In this case, you can enter weighting factors that characterize the different importance of the time periods during which the measurement was carried out.

In addition to assessments of the cumulative growth of quality indicators, the method allows to analyze the cumulative growth of the entire group of quality indicators in each time period and the properties of criteria impracticability in multi-criteria evaluation.

The estimates obtained will be measured on a linear interval scale and are objective [11], that is, independent of the set of quality indicators, the number of time intervals and evaluation criteria.

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