AN IMPROVED A PRIORI ERROR ANALYSIS OF NITSCHE’S
METHOD FOR ROBIN BOUNDARY CONDITIONS

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Abstract. In a previous paper [6] we have extended Nitsche’s method [8] for
the Poisson equation with general Robin boundary conditions. The analysis
required that the solution is in $H^{s}$, with $s > 3/2$. Here we give an improved
error analysis using a technique proposed by Gudi [5].

1. The method and its consistency

In the article [6] a Nitsche-type method is introduced and analyzed for the
following model Poisson problem with general Robin boundary conditions: Find
$u \in H^{1}(\Omega)$ such that

$$-\Delta u = f \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial n} = \frac{1}{\epsilon} (u_0 - u) + g \quad \text{on } \Gamma,$$

where $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, is a bounded domain with polygonal or polyhedral
boundary $\Gamma$, $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$, $g \in L^2(\Gamma)$, and $\epsilon \in \mathbb{R}$, $0 \leq \epsilon \leq \infty$.
The limiting values of the parameter $\epsilon$ give the Dirichlet and Neumann problems,
respectively.

The error analysis presented was not entirely satisfactory. It assumed that the
solution is in $H^{s}(\Omega)$ with $s > 3/2$, which is the same condition that traditionally has
been needed for discontinuous Galerkin methods [4]. For discontinuous Galerkin
methods Gudi introduced a technique using a posteriori error analysis by which
this assumption could be avoided [5].

The purpose of this paper is to use these arguments to improve the analysis
of the Nitsche method for the above Robin problem. Below we start by recalling
the method of [6]. We first recall the derivation of the method in a way that
emphasizes the use of the residual, which will be crucial for the error analysis. The
same notation as in [6] will be used. The finite element partitioning into simplexes
is denoted by $T_h$. This induces a mesh, denoted by $\mathcal{G}_h$, on the boundary $\Gamma$. By
$K \in T_h$ we denote an element of the mesh and by $E$ we denote an edge or a face in
$\mathcal{G}_h$. By $h_K$ we denote the diameter of the element $K \in T_h$, and by $\rho_K$ the radius
of the biggest ball contained in $K$. The mesh is assumed to be regular, i.e. it holds

$$\sup_{K \in T_h} \frac{h_K}{\rho_K} = \kappa < \infty.$$

By $h_E$ we denote the diameter of $E \in \mathcal{G}_h$. The finite element subspace is denoted by

$$V_h := \{ v \in H^1(\Omega) : v|_K \in \mathcal{P}_p(K) \quad \forall K \in T_h \},$$

where $\mathcal{P}_p(K)$ is the space of polynomials of degree $p$. 

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The Nitsche method is obtained as follows. Multiplying the differential equation (1.1) with a test function $w \in V_h$ and integrating by parts we have

$$\left(\nabla u, \nabla w\right)_\Omega - \left(\frac{\partial u}{\partial n}, w\right)_\Gamma - (f, w)_\Omega = 0.$$  

(1.4)

Defining the residual

$$R_\Gamma(v) = \epsilon \left(\frac{\partial v}{\partial n} - g\right) + v - u_0,$$

the boundary condition is

$$R_\Gamma(u) = 0.$$  

(1.6)

Hence it holds

$$\sum_{E \in \mathcal{G}_h} \frac{1}{\epsilon + \gamma h_E} \langle R_\Gamma(u), w \rangle_E = 0$$

(1.7)

and

$$- \sum_{E \in \mathcal{G}_h} \frac{\gamma h_E}{\epsilon + \gamma h_E} \langle R_\Gamma(u), \frac{\partial w}{\partial n} \rangle_E = 0.$$  

(1.8)

Adding (1.4), (1.7) and (1.8) shows that the exact solution satisfies

$$\left(\nabla u, \nabla w\right)_\Omega - \left(\frac{\partial u}{\partial n}, w\right)_\Gamma - (f, w)_\Omega + \sum_{E \in \mathcal{G}_h} \frac{1}{\epsilon + \gamma h_E} \langle R_\Gamma(u), w \rangle_E$$

$$- \sum_{E \in \mathcal{G}_h} \frac{\gamma h_E}{\epsilon + \gamma h_E} \langle R_\Gamma(u), \frac{\partial w}{\partial n} \rangle_E = 0.$$  

(1.9)

Substituting the expression (1.5) for the boundary condition and rearranging the terms, we see that the exact solution satisfies

$$B_h(u, w) - F_h(w) = 0 \quad \forall w \in V_h$$  

(1.10)

where

$$B_h(v, w) = \left(\nabla v, \nabla w\right)_\Omega + \sum_{E \in \mathcal{G}_h} \left\{ - \frac{\gamma h_E}{\epsilon + \gamma h_E} \left[ \langle \frac{\partial v}{\partial n}, w \rangle_E + \langle v, \frac{\partial w}{\partial n} \rangle_E \right] + \frac{1}{\epsilon + \gamma h_E} \langle v, w \rangle_E - \frac{\epsilon}{\epsilon + \gamma h_E} \langle v, \frac{\partial w}{\partial n} \rangle_E \right\}$$

(1.11)

and

$$F_h(w) = \langle f, w \rangle_\Omega + \sum_{E \in \mathcal{G}_h} \left\{ \frac{1}{\epsilon + \gamma h_E} \langle u_0, w \rangle_E - \frac{\gamma h_E}{\epsilon + \gamma h_E} \langle u_0, \frac{\partial w}{\partial n} \rangle_E + \frac{\epsilon}{\epsilon + \gamma h_E} \langle g, w \rangle_E - \frac{\epsilon}{\epsilon + \gamma h_E} \langle g, \frac{\partial w}{\partial n} \rangle_E \right\}.$$  

(1.12)

The above derivation shows the consistency of the

**Nitsche Method** [6]. Find $u_h \in V_h$ such that

$$B_h(u_h, w) = F_h(w) \quad \forall w \in V_h.$$  

(1.13)
2. The new a priori error estimate

The estimate will be given in the mesh and problem dependent norm

\[ \|v\|_{h}^{2} := \|\nabla v\|_{0,\Omega}^{2} + \sum_{E \in \mathcal{G}_{h}} \frac{1}{\epsilon + h_{E}} \|v\|_{0,E}^{2}. \]  

We recall the following discrete trace inequality which is easily proved by scaling arguments.

Lemma 2.1. There is a positive constant \( C_{I} \) such that

\[ \sum_{E \in \mathcal{G}_{h}} h_{E} \|\frac{\partial v}{\partial n}\|_{0,E}^{2} \leq C_{I} \|\nabla v\|_{0,\Omega}^{2} \quad \forall v \in V_{h}. \]  

For the formulation we have the following stability result, cf. [6]. Here and in what follows \( C \) denotes a generic positive constant independent of both the mesh parameter \( h \) and the parameter \( \epsilon \).

Lemma 2.2. Suppose that \( 0 < \gamma < 1/C_{I} \). Then there exists a positive constant \( C \) such that

\[ \mathcal{B}_{h}(v, v) \geq C \|v\|_{h}^{2} \quad \forall v \in V_{h}. \]  

By \( f_{h} \in V_{h} \) and \( g_{h}, u_{0,h} \in V_{h}|\Gamma \) we denote the interpolants to the data. For \( E \in \mathcal{G}_{h} \) we denote by \( \mathcal{K}(E) \) the element with \( E \) as edge/face. In [6] we proved the following bound.

Lemma 2.3. For an arbitrary \( v \in V_{h} \) and \( E \in \mathcal{G}_{h} \) it holds

\[ \frac{h_{E}^{1/2}}{\epsilon + h_{E}} \|R_{\Gamma}(v)\|_{0,E} \leq C \left( \|\nabla(u - v)\|_{0,K(E)} + h_{K} \|f - f_{h}\|_{0,K(E)} \right) \]

\[ + \frac{1}{(\epsilon + h_{E})^{1/2}} \|u - v\|_{0,E} + \frac{h_{E}^{1/2}}{\epsilon + h_{E}} \|\epsilon(g - g_{h}) + u_{0} - u_{0,h}\|_{0,E} \right). \]

We introduce the oscillation terms

\[ \text{osc}(f) = \left( \sum_{K \in \mathcal{T}_{h}} h_{K}^{2} \|f - f_{h}\|_{0,K}^{2} \right)^{1/2}, \]

\[ \text{osc}(\epsilon, u_{0}, g) = \left( \sum_{E \in \mathcal{G}_{h}} \frac{h_{E}}{(\epsilon + h_{E})^{2}} \|\epsilon(g - g_{h}) + u_{0} - u_{0,h}\|_{0,E}^{2} \right)^{1/2} \].

Lemma 2.3 then gives

Lemma 2.4. For \( v \in V_{h} \) it holds

\[ \left( \sum_{E \in \mathcal{G}_{h}} \frac{h_{E}}{(\epsilon + h_{E})^{2}} \|R_{\Gamma}(v)\|_{0,E}^{2} \right)^{1/2} \leq C \left\{ \|u - v\|_{h} + \text{osc}(f) + \text{osc}(\epsilon, u_{0}, g) \right\}. \]

We can now prove our new error estimate.

Theorem 2.1. Suppose that \( 0 < \gamma < 1/C_{I} \). Then there exists a positive constant \( C \) such that

\[ \|u - u_{h}\|_{h} \leq C \left\{ \inf_{v \in V_{h}} \|u - v\|_{h} + \text{osc}(f) + \text{osc}(\epsilon, u_{0}, g) \right\}. \]
Proof. We will divide the proof in 6 steps.

1. Treating the consistency by Gudi’s method.

Let \( v \in V_h \) be arbitrary. From the stability we have

\[
\|v - u_h\|_h^2 \leq B_h(v - u_h, v - u_h).
\]

Next, we denote \( w = v - u_h \) and use (1.13)

\[
B_h(v - u_h, v - u_h) = B_h(v - u_h, w) = B_h(v, w) - B_h(u_h, w)
= B_h(v, w) - F_h(w).
\]

Reversing the arguments leading from (1.9) to (1.10) we see that

\[
B_h(v, w) - F_h(w) = \langle \nabla v, \nabla w \rangle - \langle \frac{1}{\epsilon}(u_0 - u) + g, w \rangle \Gamma - \langle f, w \rangle \Omega - \sum_{E \in G_h} \frac{\gamma_h}{\epsilon + \gamma h^2} \langle R_\Gamma(v), \frac{\partial w}{\partial n} \rangle E
\]

Substituting the boundary condition (1.2) into (1.4) we get

\[
(\nabla u, \nabla w) - \langle \frac{1}{\epsilon}(u_0 - u) + g, w \rangle \Gamma - \langle f, w \rangle \Omega = 0.
\]

Subtracting this from the right hand side of (2.11) yields

\[
B_h(v, w) - F_h(w) = \langle \nabla(v - u), \nabla w \rangle - \langle \frac{1}{\epsilon}(u_0 - u) - \frac{1}{\epsilon}(u_0 - u) - g, w \rangle \Gamma
+ \sum_{E \in G_h} \frac{\gamma_h}{\epsilon + \gamma h^2} \langle R_\Gamma(v), \frac{\partial w}{\partial n} \rangle E
\]

\[
= R_1 + R_2 + R_3 + R_4.
\]

Next we estimate the terms in the right hand side above.

2. Estimates for the terms \( R_1 \) and \( R_4 \).

The first and the last term are readily estimated. By Schwarz inequality and the definition (2.1) of the norm, we have

\[
R_1 = \langle \nabla(v - u), \nabla w \rangle \Omega \leq \|u - v\|_h \|w\|_h.
\]

Schwarz inequality, the discrete trace inequality (2.2), and Lemma 2.4 give

\[
R_4 \leq \left( \sum_{E \in G_h} \frac{\gamma h_E}{\epsilon + \gamma h^2} \|R_\Gamma(v)\|_{0,E}^2 \right)^{1/2} \left( \sum_{E \in G_h} h_E \|\frac{\partial w}{\partial n}\|_{0,E}^2 \right)^{1/2}
\]

\[
\leq C\|u - v\|_h + \text{osc}(f) + \text{osc}(\epsilon, u_0, g) \|w\|_h.
\]

3. Splitting the boundary.
To treat the two remaining terms $R_2$ and $R_3$, we have to separate the cases when the edge size $h_E$ is smaller or greater than $\epsilon$. To this end we denote the collection of edges of size greater than $\epsilon$ by

\[(2.16) \quad G'_h = \{ E \in G_h \mid \epsilon < h_E \}, \]

and the corresponding part of the boundary by

\[(2.17) \quad \Gamma_\epsilon = \bigcup_{E \in G'_h} E. \]

We then write

\[(2.18) \quad R_2 + R_3 = -\left\langle \frac{\partial v}{\partial n} - \frac{1}{\epsilon} (u_0 - u) - g, w \right\rangle_{\Gamma_\epsilon} + \sum_{E \in G_h} \frac{1}{\epsilon + \gamma h_E} \left\langle R_{\Gamma}(v), w \right\rangle_E \]

\[+ \sum_{E \subset \Gamma \setminus \Gamma_\epsilon} \left\{ -\left\langle \frac{\partial v}{\partial n} - \frac{1}{\epsilon} (u_0 - u) - g, w \right\rangle_E + \frac{1}{\epsilon + \gamma h_E} \left\langle R_{\Gamma}(v), w \right\rangle_E \right\}. \]

4. Estimation of the contribution to $R_2 + R_3$ from the part $\Gamma_\epsilon$.

On $E \subset \Gamma_\epsilon$ it holds $\epsilon < h_E$ and we estimate as follows, using Lemma 2.4,

\[(2.19) \quad \sum_{E \subset \Gamma_\epsilon} \frac{1}{\epsilon + \gamma h_E} \left\langle R_{\Gamma}(v), w \right\rangle_E \leq \sum_{E \subset \Gamma_\epsilon} \frac{(\epsilon + h_E)^{1/2}}{\epsilon + \gamma h_E} \| R_{\Gamma}(v) \|_{0,E} \cdot (\epsilon + h_E)^{-1/2} \| w \|_{0,E} \]

\[\leq \sum_{E \subset \Gamma_\epsilon} \frac{\sqrt{2} h_E^{1/2}}{\epsilon + \gamma h_E} \| R_{\Gamma}(v) \|_{0,E} \cdot (\epsilon + h_E)^{-1/2} \| w \|_{0,E} \]

\[\leq \left( \sum_{E \subset \Gamma_\epsilon} \frac{2h_E}{(\epsilon + \gamma h_E)^2} \| R_{\Gamma}(v) \|_{0,E}^2 \right)^{1/2} \left( \sum_{E \subset \Gamma_\epsilon} (\epsilon + h_E)^{-1} \| w \|_{0,E}^2 \right)^{1/2} \]

\[\leq C(\| u - v \|_h + \text{osc}(f) + \text{osc}(\epsilon, u_0, g)) \| w \|_h. \]

Next, we have to estimate

\[(2.20) \quad -\left\langle \frac{\partial v}{\partial n} - \frac{1}{\epsilon} (u_0 - u) - g, w \right\rangle_{\Gamma_\epsilon} \]

We substitute

\[(2.21) \quad \frac{1}{\epsilon} (u_0 - u) + g = \frac{\partial u}{\partial n}, \]

which gives

\[(2.22) \quad -\left\langle \frac{\partial v}{\partial n} - \frac{1}{\epsilon} (u_0 - u) - g, w \right\rangle_{\Gamma_\epsilon} = \left\langle \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n}, w \right\rangle_{\Gamma_\epsilon}. \]
Now we define the strip
\[(2.23) \quad \Omega_\epsilon = \bigcup_{\substack{K \in T_h \cap \Gamma_\epsilon \neq \emptyset}} K.\]
Following [1, 2, 7] we construct a linear finite element extension \( \mathcal{E}_h w \in V_h \) of \( w|_{\Gamma_\epsilon} \) such that
\[(2.24) \quad \mathcal{E}_h w|_{\Gamma_\epsilon} = w|_{\Gamma_\epsilon},\]
and
\[(2.25) \quad \mathcal{E}_h w = 0 \quad \text{in} \quad \Omega \setminus \Omega_\epsilon.\]
In [1, 2, 7] the following estimate is derived
\[(2.26) \quad \|\nabla \mathcal{E}_h w\|_{0, \Omega_\epsilon} \leq C \left( \sum_{E \subset \Gamma_\epsilon} h_E^{-1} \|w\|_{0, E}^2 \right)^{1/2}.\]
We denote
\[(2.27) \quad \Gamma^\oplus_\epsilon = \Omega_\epsilon \cap \Gamma.\]
and split the boundary of \( \Omega_\epsilon \) in three parts (cf. Figure 2)
\[(2.28) \quad \partial \Omega_\epsilon = \Gamma_\epsilon \cup \{\Gamma^\oplus_\epsilon \setminus \Gamma_\epsilon\} \cup \{\partial \Omega_\epsilon \setminus \Gamma^\oplus_\epsilon\}.\]
Note that \( \mathcal{E}_h w \neq w \) on \( \Gamma^\oplus_\epsilon \setminus \Gamma_\epsilon \). Since \( \mathcal{E}_h w|_{\partial \Omega_\epsilon \setminus \Gamma^\oplus_\epsilon} = 0 \), scaling and the estimate (2.26) show that
\[(2.29) \quad \left( \sum_{K \subset \Omega_\epsilon} h_K^{-2} \|\mathcal{E}_h w\|_{0, K}^2 \right)^{1/2} + \left( \sum_{E \subset \Omega_\epsilon \setminus \Gamma^\oplus_\epsilon} h_E^{-1} \|\mathcal{E}_h w\|_{0, E}^2 \right)^{1/2}
\leq C \|\nabla \mathcal{E}_h w\|_{0, \Omega_\epsilon} \leq C \left( \sum_{E \subset \Gamma_\epsilon} h_E^{-1} \|w\|_{0, E}^2 \right)^{1/2},\]
and also
\[(2.30) \quad \left( \sum_{E \subset \Gamma^\oplus_\epsilon \setminus \Gamma_\epsilon} h_E^{-1} \|\mathcal{E}_h w\|_{0, E}^2 \right)^{1/2} \leq C \|\nabla \mathcal{E}_h w\|_{0, \Omega_\epsilon} \leq C \left( \sum_{E \subset \Gamma_\epsilon} h_E^{-1} \|w\|_{0, E}^2 \right)^{1/2}.\]
Further, since \( \epsilon < h_E \), it holds
\[(2.31) \quad \left( \sum_{E \subset \Gamma_\epsilon} h_E^{-1} \|w\|_{0, E}^2 \right)^{1/2} \leq \sqrt{2} \left( \sum_{E \subset \Gamma_\epsilon} \frac{1}{h_E + \epsilon} \|w\|_{0, E}^2 \right)^{1/2} \leq \sqrt{2} \|w\|_{h}.\]
Next, integrating by parts and using (2.29)–(2.31) we estimate as follows

\[
\langle \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n}, \mathcal{E}_h w \rangle_{\Gamma^+} = \sum_{K \subset \Omega} \left[ - \left( f + \Delta v, \mathcal{E}_h w \right)_K + \left( \nabla (u - v), \nabla \mathcal{E}_h w \right)_K \right] 
+ \sum_{E \subset \Omega \setminus \Gamma^+} \left[ \left( \frac{\partial v}{\partial n} \right)_{\partial E} , \mathcal{E}_h w \right]_E
\]

\[
\leq C \left( \sum_{K \subset \Omega} h_K^2 \| f + \Delta v \|_{0,K}^2 \right)^{1/2} \left( \sum_{K \subset \Omega} h_K^{-2} \| \mathcal{E}_h w \|_{\partial K}^2 \right)^{1/2}
+ \| \nabla (u - v) \|_{0,\Omega} \| \nabla \mathcal{E}_h w \|_{0,\Omega}
+ \left( \sum_{E \subset \Omega \setminus \Gamma^+} h_E \left\| \frac{\partial v}{\partial n} \right\|_{0,E}^2 \right)^{1/2} \left( \sum_{E \subset \Omega \setminus \Gamma^+} h_E^{-1} \| \mathcal{E}_h w \|_{E}^2 \right)^{1/2}
\]

\[
\leq C \left\{ \left( \sum_{K \subset \Omega} h_K^2 \| f + \Delta v \|_{0,K}^2 \right)^{1/2} + \| \nabla (u - v) \|_{0,\Omega} \right. 
\left. + \left( \sum_{E \subset \Omega \setminus \Gamma^+} h_E \left\| \frac{\partial v}{\partial n} \right\|_{0,E}^2 \right)^{1/2} \right\} \| w \|_h.
\]

From a posteriori error analysis [3, 9] we know that

\[
\left( \sum_{E \subset \Omega \setminus \Gamma^+} h_E \left\| \frac{\partial v}{\partial n} \right\|_{0,E}^2 \right)^{1/2}
\leq C \left( \sum_{K \subset \Omega} h_K^2 \| f + \Delta v \|_{0,K}^2 \right)^{1/2} + \| \nabla (u - v) \|_{0,\Omega},
\]

and

\[
\left( \sum_{K \subset \Omega} h_K^2 \| f + \Delta v \|_{0,K}^2 \right)^{1/2} \leq C \left( \| \nabla (u - v) \|_{0,\Omega} + \text{osc}(f) \right).
\]

Hence we have

\[
\langle \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n}, \mathcal{E}_h w \rangle_{\Gamma^+} \leq C \left( \| u - v \|_h + \text{osc}(f) \right) \| w \|_h.
\]

Since \( \mathcal{E}_h w = w \) on \( \Gamma_e \), we get

\[
\langle \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n}, w \rangle_{\Gamma_e} \leq C \left( \| u - v \|_h + \text{osc}(f) \right) \| w \|_h - \langle \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n}, \mathcal{E}_h w \rangle_{\Gamma_e \setminus \Gamma_e}.
\]
Thus we have
\begin{equation}
(2.37)
\end{equation}
Thus we can estimate
\begin{equation}
(2.38)
\end{equation}
\begin{align}
\sum_{E \subset \Gamma^+ \setminus \Gamma} - \langle \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n}, \mathcal{E}_h w \rangle_{\Gamma^+ \setminus \Gamma} &= \sum_{E \subset \Gamma^+ \setminus \Gamma} -\frac{1}{\epsilon} \langle R_1(v), \mathcal{E}_h w \rangle_E - \frac{1}{\epsilon} \langle u - v, \mathcal{E}_h w \rangle_E \\
&\leq C \left( \sum_{E \subset \Gamma^+ \setminus \Gamma} \frac{1}{\epsilon + h_E} \| R_1(v) \|_{0,E} \| \mathcal{E}_h w \|_{0,E} + \frac{1}{\epsilon + h_E} \| u - v \|_{0,E} \mathcal{E}_h w \|_{0,E} \right) \\
&\leq C \left( \sum_{E \subset \Gamma^+ \setminus \Gamma} \frac{h_E}{(\epsilon + h_E)^2} \| R_1(v) \|_{0,E}^2 + \frac{1}{h_E + \epsilon} \| u - v \|_{0,E}^2 \right)^{1/2} \\
&\times \left( \sum_{E \subset \Gamma^+ \setminus \Gamma} \frac{1}{h_E + \epsilon} \| \mathcal{E}_h w \|_{0,E}^2 \right)^{1/2}.
\end{align}

Thus we have
\begin{equation}
(2.39)
\end{equation}
which together with (2.19) gives
\begin{equation}
(2.40)
\end{equation}
\begin{align}
\sum_{E \subset \Gamma^+ \setminus \Gamma} \left\{ - \langle \frac{\partial u}{\partial n} - \frac{1}{\epsilon} (u_0 - u) - g, w \rangle_E + \frac{1}{\epsilon + h_E} \langle R_1(v), w \rangle_E \right\} \\
&\leq C \left( \| u - v \|_h + \text{osc}(f) + \text{osc}(\epsilon, u_0, g) \right) \| w \|_h.
\end{align}

5. Estimation of the contribution to \( R_2 + R_3 \) from the part \( \Gamma \setminus \Gamma_\epsilon \).

It now holds \( \epsilon \geq h_E \). First write
\begin{equation}
(2.41)
\end{equation}
Hence, on \( E \) it holds
\begin{align}
- \langle \frac{\partial v}{\partial n} - \frac{1}{\epsilon} (u_0 - u) - g, w \rangle_E + \frac{1}{\epsilon + h_E} \langle R_1(v), w \rangle_E \\
&= \left( \frac{1}{\epsilon + h_E} - \frac{1}{\epsilon} \right) \langle R_1(v), w \rangle_E - \frac{1}{\epsilon} \langle u - v, w \rangle_E \\
&= \left( \frac{\gamma h_E}{\epsilon + h_E} \right) \langle R_1(v), w \rangle_E - \frac{1}{\epsilon} \langle u - v, w \rangle_E \\
&\leq \left( \frac{\gamma h_E}{\epsilon + h_E} \right) \epsilon \| R_1(v) \|_{0,E} \| w \|_{0,E} + \frac{1}{\epsilon} \| u - v \|_{0,E} \| w \|_{0,E}.
\end{align}

Since \( \epsilon + h_E \leq 2\epsilon \), it holds
\begin{equation}
(2.42)
\end{equation}
Since $h_E/\epsilon \leq 1$ we estimate as follows

\[
\frac{\gamma h_E}{(\epsilon + \gamma h_E)\epsilon} \| R_\Gamma (v) \|_{0,E} \| w \|_{0,E} = \frac{\gamma h_E (\epsilon + h_E)^{1/2}}{(\epsilon + \gamma h_E)\epsilon} \| R_\Gamma (v) \|_{0,E} \cdot (\epsilon + h_E)^{-1/2} \| w \|_{0,E}
\]

\[
\leq h_E \frac{h_E^{1/2}}{\epsilon^{1/2} (\epsilon + \gamma h_E)^{1/2}} \| R_\Gamma (v) \|_{0,E} \cdot (\epsilon + h_E)^{-1/2} \| w \|_{0,E}
\]

\[
(2.43)
\]

Combining (2.41)–(2.43) yields

\[
(2.44) \quad \sum_{E \subseteq \Gamma_*} \left( -\langle \frac{\partial v}{\partial n}, \frac{1}{\epsilon} (u_0 - u), g \rangle_E + \frac{1}{\epsilon + \gamma h_E} \langle R_\Gamma (v), w \rangle_E \right) \leq C \left( \| u - v \|_h + \text{osc}(f) + \text{osc}(\epsilon, u_0, g) \right) \| w \|_h.
\]

6. Collecting the estimates.

Adding (2.39) and (2.44) gives

\[
(2.45) \quad R_2 + R_3 \leq C \left( \| u - v \|_h + \text{osc}(f) + \text{osc}(\epsilon, u_0, g) \right) \| w \|_h.
\]

The assertion then follows from this and (2.13), (2.14), and (2.15).

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