The exact distribution of the largest eigenvalue of a singular beta $F$-matrix for Roy’s test

Koki Shimizu$^a$ Hiroki Hashiguchi$^b$

$^a$ Graduate School of Science, Tokyo University of Science
$^b$ Department of Applied Mathematics, Faculty of Science, Tokyo University of Science

Abstract: In this paper, the exact distribution of the largest eigenvalue of a singular random matrix for Roy’s test is discussed. The key to developing the distribution theory of eigenvalues of a singular random matrix is to use heterogeneous hypergeometric functions with two matrix arguments. In this study, we define the singular beta $F$-matrix and extend the distributions of a nonsingular beta $F$-matrix to the singular case. We also give the joint density function of eigenvalues and the exact distribution of the largest eigenvalue in terms of heterogeneous hypergeometric functions.

1 Introduction

The distribution of eigenvalues of an $F$-matrix plays an important role in multivariate analysis such as the test for equivalence of covariance matrices, multivariate analysis of variance (MANOVA), and discriminant analysis. The distributions of an $F$-matrix are known to be equal to the distributions of a ratio of two Wishart matrices. Khatri (1972) derived the exact distributions of the largest and smallest eigenvalues of a nonsingular ratio of two real Wishart matrices using a finite series of Laguerre polynomials of matrix arguments. Hashiguchi et al. (2018) suggested an alternative derivation approach and conducted a numerical experiment using the holonomic gradient method (HGM) for the hypergeometric functions of matrix arguments. There are various approaches to deriving the approximate distributions of a nonsingular real $F$-matrix. Johnstone’s results (2008, 2009), based on random matrix theory, showed that a Tracy-Widom distribution approximates the exact distribution for the largest eigenvalue. Matsubara and Hashiguchi (2016) derived the Laplace approximation via $F$ distributions for the nonsingular real $F$-matrix. Under the elliptical model, including the normal population, some results on the distributions of eigenvalues were discussed by Caro-Lopera et al. (2014) and Shinozaki et al. (2018). Caro-Lopera et al. (2014) presented the density of eigenvalues of a nonsingular ratio of two elliptical Wishart matrices for the moments of the modified likelihood ratio statistics. Díaz-García and Gutiérrez-Jáimez (2011) extended the nonsingular real Wishart distributions to the complex, quaternion and octonion cases under the normal population. These distributions are said to be nonsingular beta-Wishart distributions.

In the case of a singular random matrix, Uhlig (1994) derived the density of a singular real Wishart matrix and presented the Jacobian of the transformation to obtain the density of a singular real $F$-matrix as an open problem. The proof of Uhlig’s result was given by Díaz-García and Gutiérrez-Jáimez (1997). Shimizu and Hashiguchi (2019) established the distribution theory of eigenvalues of a singular random matrix and derived the exact distributions of the largest eigenvalues of a singular beta-Wishart matrix.

In this paper, we discuss the distributions of eigenvalues of a singular beta $F$-matrix on real finite-dimensional algebra. In Section 2, preliminary results and some notations are provided, which will be used throughout this paper. Furthermore, we introduce the heterogeneous hypergeometric functions of two matrix arguments, which were already defined by Shimizu and Hashiguchi (2019). These functions can be obtained using the integral formula for Jack polynomials over the Stiefel manifold. In Section 3, the density functions of a singular beta $F$-matrix and the joint density functions of its eigenvalues are given. We also show that the exact distribution of the largest eigenvalue can be expressed in terms of heterogeneous hypergeometric functions for Roy’s test. This derivation is similar to that of Shimizu and Hashiguchi (2019). Numerical computations on theoretical distributions are conducted using an algorithm presented by Hashiguchi et al. (2000) for zonal polynomials. In Section 4, we discuss the distribution of the largest eigenvalue of a ratio of two singular beta-Wishart matrices.
functions of two matrix arguments. Shimizu and Hashiguchi (2019) discussed the integral formula for Jack polynomials that differ from Xue (2009) proposed zonal polynomials and hypergeometric functions of quaternion matrix arguments for then the Jack polynomials are referred to as zonal polynomials and Shur polynomials, respectively Li and S

The eigenvalues of a Hermitian matrix are all real. If the eigenvalues of the parameter
of real and complex numbers, respectively, and

division algebra such that $F_1 = \mathbb{R}$, $F_2 = \mathbb{C}$, and $F_4 = \mathbb{H}$ for $\beta = 1, 2, 4$, where $\mathbb{R}$ and $\mathbb{C}$ are the fields of

and is normalized as $\int_{H_1 \in V_{n,m}^\beta} (dH_1) = 1$.

For a positive integer $k$, let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_m)$ denote a partition of $k$ with $\kappa_1 \geq \cdots + \kappa_m \geq 0$ and $\kappa_1 + \cdots + \kappa_m = k$. The set of all partitions with lengths not longer than $m$ is denoted by $P_m^\kappa = \{ \kappa = (\kappa_1, \ldots, \kappa_m) \mid \kappa_1 + \cdots + \kappa_m = k, \kappa_1 \geq \kappa_2 \cdots \geq \kappa_m \geq 0 \}$. The $\beta$-generalized Pochhammer symbol of parameter $a > 0$ is defined as

$$
(a)_\kappa^\beta = \prod_{i=1}^{m} \left(a - \frac{i - 1}{2} \beta \right)^{\kappa_i}.
$$

The Jack polynomial $C_\kappa^\beta(X)$ is a symmetric polynomial in $x_1, \ldots, x_m$; these are eigenvalues of $X \in S^\beta(m)$. See Stanley (1989) and Koev and Edelman (2006) for the relevant detailed properties. If $\beta = 1, 2$, then the Jack polynomials are referred to as zonal polynomials and Shur polynomials, respectively Li and Xue (2009) proposed zonal polynomials and hypergeometric functions of quaternion matrix arguments for $\beta = 4$. Shimizu and Hashiguchi (2019) discussed the integral formula for Jack polynomials that differ from Theorem 3 in Díaz-García (2013a). This formula is needed to define the heterogeneous hypergeometric functions of two matrix arguments.
For $A \in S^\beta(m)$ and $B \in S^\beta(n)$, the integral formula for Jack polynomials over the Steifel manifold was given by Shimizu and Hashiguchi (2019) as

$$I_{H_1 \in V_{n,m}} C^\beta_k(AH_1BH_1^*) (dH_1) = \frac{C^\beta_k(A)C^\beta_k(B)}{C^\beta_k(I_m)},$$  \hspace{1cm} (1)

where $m \geq n$.

The hypergeometric functions of parameter $\beta > 0$ are defined as

$$pF_q^{(\beta;m)}(\alpha; \beta; A) = \sum_{k=0}^{\infty} \sum_{\kappa \in P^m_k} \frac{(\alpha_1)_m^\beta \cdots (\alpha_p)_m^\beta}{(\beta_1)_m^\beta \cdots (\beta_q)_m^\beta} \frac{C^\beta_k(A)}{k!},$$  \hspace{1cm} (2)

where $\alpha = (\alpha_1, \ldots, \alpha_p), \beta = (\beta_1, \ldots, \beta_q)$. The special case of (2) can be represented as $1F_0^{(\beta;m)}(\alpha_1, A) = |I_m - A|^{\alpha_1}$. If $\beta = 1$, we use $pF_q$ instead of $pF_q^{(1;m)}$. Shimizu and Hashiguchi (2019) defined the heterogeneous hypergeometric functions of two matrix arguments and provided some of their properties. Ratnarajah and Villancourt (2005a, 2005b) used their functions to derive the joint density of the eigenvalues of a singular complex Wishart matrix. From (1) and (2), the heterogeneous hypergeometric functions is defined as follows.

$$pF_q^{(\beta;m,n)}(\alpha; \beta; A, B) = \int_{H_1 \in V_{n,m}} pF_q^{(\beta;m)}(\alpha; \beta; AH_1BH_1^*) (dH_1)$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa \in P^m_k} \frac{(\alpha_1)_m^\beta \cdots (\alpha_p)_m^\beta C^\beta_k(A)C^\beta_k(B)}{k!C^\beta_k(I_m)},$$  \hspace{1cm} (3)

### 3 Singular beta F distribution

Suppose that an $m \times n$ beta-Gaussian random matrix $X$ is distributed as $X \sim N^\beta_{m,n}(M, \Sigma \otimes I_n)$, where $M$ is the $m \times n$ mean matrix, $\Sigma > 0$, and $\otimes$ is the Kronecker product. This means that the column vectors of $X$ are an i.i.d. sample of size $n$ from $N^\beta_m(\mu, \Sigma)$, where $\mu$ is the $m$-dimensional mean vector and $M = \mu 1'$ and $1 = (1, \ldots, 1)^t \in \mathbb{R}^n$. The density of $X$ is represented as

$$f(X) = \frac{1}{(2\pi)^{m+n/2} \Gamma(m/2)} \exp \left(-\frac{\beta}{2} \text{tr}(X - M)^*\Sigma^{-1}(X - M)\right),$$

where $\text{etr}(\cdot) = \exp(\text{tr}(\cdot))$. Let $X \sim N^\beta_{m,n}(O, \Sigma \otimes I_n)$. If $n \geq m$, the random matrix $W = XX^*$ is called a nonsingular beta-Wishart matrix. The distributions of a nonsingular real Wishart matrix have been studied by some authors. See Muirhead (1982) and Gupta and Nagar (1999) for details. Ratnarajah et al. (2005) discussed the nonsingular complex Wishart distribution for MIMO communication system. Li and Xue (2009) also extended these results to a quaternion case. The density of a nonsingular beta-Wishart matrix that covers the real, complex, and quaternion Wishart matrices was given by Díaz-García and Gutiérrez-Jáimez (2011) as

$$f(W) = \frac{|\Sigma|^{-n\beta/2}}{(2\beta^{-1})^{mn|\beta/2}} |W|^{(n-m+1)\beta/2-1} \text{etr} \left(-\frac{\beta}{2} \Sigma^{-1}W \right).$$

On the other hand, if $m > n$, the matrix $W$ of order $m$ was said to be a singular beta-Wishart matrix. A singular beta-Wishart matrix has only $n$ eigenvalues. Shimizu and Hashiguchi (2019) gave the density of a singular beta-Wishart matrix as follows.

$$f(W) = \frac{\pi^{n(n-m)/2} |\Sigma|^{-n\beta/2}}{(2\beta^{-1})^{mn|\beta/2}} (\text{det}L_1)^{(n-m+1)\beta/2-1} \text{etr} \left(-\frac{\beta}{2} \Sigma^{-1}W \right),$$  \hspace{1cm} (4)

where $W^\beta_m(n, \Sigma)$ is referred to as either a nonsingular or singular case when $n \geq m$ or $n < m$, respectively.

For integers $p \geq m > n$, let $A \sim W^\beta_m(n, \Sigma_1)$ and $B \sim W^\beta_m(p, \Sigma_2)$, where $A$ and $B$ are independent. Put $B = T^*T$ where $T$ is an upper-triangular $m \times m$ matrix with the positive diagonal matrix. The singular
distribution as $F$. The singular beta F-matrix has the same distribution as $AB^{-1}$. The singular beta F distribution is denoted by $F^β_m(\frac{n}{r}, \frac{p}{r}, \Sigma)$, where $\Sigma = \Sigma_1 \Sigma_2^{-1}$. The nonsingular parts of the spectral decomposition can be represented as $F = H_1 Q H_1^*$ where $H_1 \in V^β_{n,m}$ and $Q = \text{diag}(q_1, \ldots, q_n)$ with $q_1 > \cdots > q_n > 0$. The following lemma was first given by Uhlig (1994) in a real case as a conjecture. Díaz-García and Gutiérrez-Jáimez (1997) gave a proof of Uhlig’s conjecture. Díaz-García (2013b) extended this result to complex, quaternion and octonion cases.

**Lemma 1.** For $X, Y \in S^β(m)$ with rank $n < m$. Let $T = T^{-1}X(T^{-1})^*$, where $T$ is a nonsingular $m \times m$ matrix. Let $X = G_1 \Lambda G_1^*$ and $Y = H_1 \Lambda H_1^*$, where $G_1, H_1 \in V^β_{n,m}$ and $\Lambda_1, \Lambda_2$ are $n \times n$ diagonal matrices. Then we have

$$(dX) = |\Lambda_1|^{(m-n-1)/2+1}|\Lambda_2|^{-(m-n-1)/2-1}|T|^{nβ}(dY).$$

The following theorem represents the density of a singular beta F distribution.

**Theorem 1.** Let $F \sim F^β_m(\frac{n}{r}, \frac{p}{r}, \Sigma)$, where $p \geq m > n$. Then the density of $F$ is given as

$$f(F) = \frac{n^{(n-m)/2}|\Sigma|^{nβ/2}}{(2β^{-1})^{mnβ/2}\Gamma^β_n(nβ/2)} |L_1|^{(n-m)/2}|etr\left(-\frac{β}{2}A\right),$$

and

$$f(B) = \frac{|\Sigma|^{-pβ/2}}{(2β^{-1})^{mpβ/2}\Gamma^β_m(pβ/2)} |B|^{(p-m)/2}|etr\left(-\frac{β}{2}\Sigma^{-1}B\right),$$

respectively. Then the joint density of $A$ and $B$ is given as

$$f(A, B) = \frac{n^{(n-m)/2}|\Sigma|^{-pβ/2}}{(2β^{-1})^{mnβ/2}\Gamma^β_n(nβ/2)\Gamma^β_m(pβ/2)} |L_1|^{(n-m-1)/2}|B|^{(p-m-1)/2}|etr\left(-\frac{β}{2}(\Sigma^{-1}B + A)\right).$$

From Lemma 1, the joint density of $F$ and $B$ is represented as

$$f(F, B) = \frac{n^{(n-m)/2}|\Sigma|^{-pβ/2}}{(2β^{-1})^{mnβ/2}\Gamma^β_n(nβ/2)\Gamma^β_m(pβ/2)} |Q|^{-(m-n)/2-1}|B|^{(n+p)/2-1}|etr\left(-\frac{β}{2}(\Sigma^{-1}F)B\right).$$

Integrating (7) respect to $B > 0$, we get

$$f(F) = \frac{n^{(n-m)/2}|\Sigma|^{-pβ/2}}{(2β^{-1})^{mnβ/2}\Gamma^β_n(nβ/2)\Gamma^β_m(pβ/2)} |Q|^{-(m-n-1)/2-1}|L_1|^{(n-m)/2}|etr\left(-\frac{β}{2}(\Sigma^{-1}F)\right).$$

**Theorem 2.** Let $F \sim F^β_m(\frac{n}{r}, \frac{p}{r}, \Sigma)$, where $p \geq m > n$. Then the joint density of eigenvalues $q_1, \ldots, q_n$ of $F$ is given as

$$f(q_1, \ldots, q_n) = C_1 |Q|^{(m-n)/2-1} \prod_{i<j}^n (q_i - q_j)^{\beta} I_1^{(\beta/2)} \left(\frac{p+n}{2}; -\Sigma, Q\right),$$

where $Q = \text{diag}(q_1, \ldots, q_n)$, $C_1 = \frac{n^{2β/2+r}|\Sigma|^{2β/2}\Gamma^β_n((n+p)/2)}{\Gamma^β_n(nβ/2)\Gamma^β_m(pβ/2)\Gamma^β_m(mβ/2)}$, and

$$r = \begin{cases} 0 & \beta = 1 \\ -nβ/2 & \beta = 2, 4 \end{cases}.$$
Proof. The Jacobian of the transformation $F = H_1 Q H_1^*$ given by Díaz-García (2013b) is

$$
(dF) = 2^{-n} \pi^2 \prod_{i=1}^n \delta_i \prod_{i<j}^n (q_i - q_j)^\beta (dQ) \wedge (H_1^* dH_1).
$$

Using equation (9) for the density of $F$, we have

$$
f(Q, H_1) = \frac{2^{-n} \pi^{n(n-m)\beta/2 + r} |\Sigma|^{n\beta/2} \Gamma_m^{\beta} \{(n + p)\beta/2\}}{\Gamma_m^{\beta}(n\beta/2) \Gamma_m^{\beta}(p\beta/2)} |Q|^{(m-n+1)\beta/2 - 1} \prod_{i<j}^n (q_i - q_j)^\beta |I_m + \Sigma H_1 Q H_1^*|^{-(n+p)\beta/2}.
$$

Moreover integrating $f(Q, H_1)$ with respect to $H_1$, the density of eigenvalues of $F$ is given as

$$
f(q_1, \ldots, q_n) = C_1 |Q|^{(m-n+1)\beta/2 - 1} \prod_{i<j}^n (q_i - q_j)^\beta \int_{H_1 \in V_{n,m}} |I_m + \Sigma H_1 Q H_1^*|^{-(n+p)\beta/2} (dH_1).
$$

From (8), the right hand side of the above equation can be evaluated as

$$
f(q_1, \ldots, q_n) = C_1 |Q|^{(m-n+1)\beta/2 - 1} \prod_{i<j}^n (q_i - q_j)^\beta 1F_0^{(\beta;m,n)} \left( \frac{(p + n)\beta}{2}; -Q, \Sigma \right).
$$

The joint density (8) when $\Sigma_1 = \Sigma_2$ is represented in Corollary 1.

**Corollary 1.** Let $F \sim F_m^{\beta}(\frac{p}{2}, \frac{p}{2}, I_m)$, where $p \geq m > n$. Then the joint density of eigenvalues of $F$ is given as

$$
f(q_1, \ldots, q_n) = \frac{\pi^{n^2\beta/2 + r} \Gamma_m^{\beta} \{(n + p)\beta/2\}}{\Gamma_m^{\beta}(n\beta/2) \Gamma_m^{\beta}(p\beta/2) \Gamma_m^{\beta}(m\beta/2)} |Q|^{(m-n+1)\beta/2 - 1} \prod_{i<j}^n (q_i - q_j)^\beta |I_n + Q|^{-(n+p)\beta/2}.
$$

**Proof.** Put $\Sigma = I_m$ for the density function (8), the functions $1F_0^{(\beta;m,n)}$ can be represented as

$$
1F_0^{(\beta;m,n)} \left( \frac{(p + n)\beta}{2}; I_m, -Q \right) = |I_n + Q|^{-(n+p)\beta/2}.
$$

The result for $\beta = 1$ in Corollary 1 coincides with Theorem 4 (i) of Díaz-García and Gutiérrez-Jáimez (1997). The next result was proposed by Sugiyama (1967) in the real case. Lemma 2 is required in order to integrate (8) with respect to $q_2, \ldots, q_n$. Shimizu and Hashiguchi (2019) generalized this result for complex and quaternion cases.

**Lemma 2.** Let $X_1 = \text{diag}(1, x_2, \ldots, x_n)$ and $X_2 = \text{diag}(x_2, \ldots, x_n)$ with $x_2 > \cdots > x_n > 0$; then the following equation holds

$$
\int_{1 > x_2 \cdots > x_n > 0} |X_2|^{a - (n-1)\beta/2 - 1} C_\kappa^{\beta}(X_1, a) \prod_{i=2}^n (1 - x_i)^\beta \prod_{i<j}^n (x_i - x_j)^\beta \prod_{i=2}^n dx_i
$$

$$
= (na + k)(\Gamma_n^{\beta}(n\beta/2)/\pi^{n^2\beta/2 + r}) \frac{\Gamma_n^{\beta}(a, \kappa) \Gamma_n^{\beta}\{(n - 1)\beta/2 + 1\} C_\kappa^{\beta}(I_n)}{\Gamma_n^{\beta}(a + (n - 1)\beta/2 + 1, \kappa)},
$$

where $\Re(a) > (n - 1)\beta/2$ and $\Gamma_n^{\beta}(a, \kappa) = (a)_n \Gamma_n^{\beta}(a)$.
Theorem 3. Let $F \sim F^\beta_m(\frac{n}{2}, \frac{p}{2}, \Sigma)$, where $p \geq m > n$. The probability density function of the largest eigenvalue $q_1$ of $F$ is given as

$$\Pr(q_1 < x) = C_2 |\Sigma|^{n\beta/2} m^{n\beta/2} 2F_1(\beta; m, n) \left( \frac{(n+p)\beta}{2}; \frac{m\beta}{2}; \frac{m+n-1}{2}; -x\Sigma, I_n \right),$$  (10)

where $C_2 = \frac{\Gamma^\beta_m((n+p)\beta/2)\Gamma^\beta_m((n-1)\beta/2+1)}{\Gamma^\beta_m(\beta)\Gamma^\beta_m(m+n-1)\beta/2+1}$

Proof. We first start with the joint density (8)

$$f(q_1, \ldots, q_n) = C_1 |Q|^{(m-n+1)\beta/2-1} \prod_{i<j} (q_i - q_j)^\beta 1F_0(\beta; m, n) \left( \frac{(p+n)\beta}{2}; -\Sigma, Q \right)$$

$$= C_1 |Q|^{(m-n+1)\beta/2-1} \prod_{i<j} (q_i - q_j)^\beta \sum_{k=0}^{\infty} \sum_{\kappa \in P_n^k} \frac{(p+n)\beta/2)\kappa C_\kappa^\beta(-\Sigma)C_\kappa^\beta(Q)}{k!C_\kappa^\beta(I_m)}.$$

Translating $q_i$ to $x_i = q_i/q_1$ for $i = 2, \ldots, n$ and using Lemma 2, the joint density of eigenvalues $f(q_1, \ldots, q_n)$ can be evaluated as

$$f(q_1) = C_1 \int_{x_2 > x_3 > \ldots > x_n} |X_2|^{(m-n+1)\beta/2-1} C_\kappa^\beta(X_1) \prod_{i=2}^n (1-q_i)^\beta \prod_{2 \leq i < j} (q_i - q_j)^\beta$$

$$\sum_{k=0}^{\infty} \sum_{\kappa \in P_n^k} \frac{(p+n)\beta/2)\kappa C_\kappa^\beta(-\Sigma)}{k!C_\kappa^\beta(I_m)}.$$

$$= C_1 (mn\beta/2 + k)(\Gamma^\beta_n(\beta/2)/\pi^2\beta/2 + r) \frac{\Gamma^\beta_m((n+p)\beta/2)\Gamma^\beta_m((n-1)\beta/2+1)C_\kappa^\beta(I_n)}{\Gamma^\beta_m(\beta)\Gamma^\beta_m(m+n-1)\beta/2+1}$$

$$\sum_{k=0}^{\infty} \sum_{\kappa \in P_n^k} \frac{(p+n)\beta/2)\kappa C_\kappa^\beta(-\Sigma)}{k!C_\kappa^\beta(I_m)}.$$

Finally, integrating $f(q_1)$ with respect to $q_1$, we have the desired result (10).

Corollary 2. Let $F \sim F^\beta_m(\frac{p}{2}, \frac{p}{2}, I_m)$, where $p \geq m > n$. Then the probability density function of the largest eigenvalue $q_1$ of $F$ is given as

$$\Pr(q_1 < x) = \frac{\Gamma_m((n+p)/2)\Gamma_n((n+1)/2)(x/1+x)^{mn/2}}{\Gamma_m(p/2)\Gamma_n(m+n+1/2)} 2F_1 \left( \frac{m-p+1}{2}; \frac{m+n+1}{2}; -xI_n \right).$$  (11)

Proof. If $\Sigma = I_m$ and $\beta = 1$, the function (10) is given as

$$\Pr(q_1 < x) = \frac{\Gamma_m((n+p)/2)\Gamma_n((n+1)/2)}{\Gamma_m(p/2)\Gamma_n(m+n+1/2)} 2F_1 \left( \frac{n+p}{2}; \frac{n+m+1}{2}; -xI_n \right).$$  (12)

The well-known Kummer’s relation for $2F_1$ is

$$2F_1 \left( a; b; c; X \right) = \det(I_n - X)^{-b} 2F_1 \left( c - a; b; c; -X(I_n - X)^{-1} \right),$$  (13)

where $O < X < I_m$. From (13), the function $2F_1$ on the right hand side of (12) is represented as

$$2F_1 \left( \frac{n+p}{2}; \frac{n+m+1}{2}; -xI_n \right) = (1+x)^{-mn/2} 2F_1 \left( \frac{m-p+1}{2}; \frac{m+n+1}{2}; -xI_n \right).$$

□
4 Numerical experiments

In this section, we discuss the numerical experiments conducted for the distribution function (11). In general (11) is an infinite series, but, if \( r = \frac{p - m - 1}{2} \) is a non-negative integer, it is a finite series represented by

\[
F(x) = \frac{\Gamma_m \{(n + p)/2\} \Gamma_n \{(n + 1)/2\} (\frac{x}{x + r})^{mn/2}}{\Gamma_m \{p/2\} \Gamma_n \{(m + n + 1)/2\}} \sum_{k=0}^{r} \sum_{\kappa} \frac{\{(m - p + 1)/2\}_\kappa \{m/2\}_\kappa \ C_n \left( \frac{x}{x + k} I_n \right)}{k! \{(m + n + 1)/2\}_\kappa}, \quad (14)
\]

where \( \sum_{\kappa} \) is the sum of all partitions of \( k \) with \( \kappa_1 \leq r \). The function (14) for \( p = 20, m = 15 \) and \( n = 3 \) is given as

\[
\frac{128877}{8} \left( \frac{x}{x + 1} \right)^{45/2} \left\{ \frac{260x^6}{1083(x + 1)^6} - \frac{104x^5}{57(x + 1)^5} + \frac{110x^4}{19(x + 1)^4} - \frac{500x^3}{51(x + 1)^3} + \frac{3725x^2}{399(x + 1)^2} - \frac{90x}{19(x + 1) + 1} \right\}.
\]

We find that \( rn = 6 \) in the series (14) and that the finite series (14) can be calculated using small terms when the parameters \( p \) and \( m \) are the value of the close. The empirical distribution based on a \( 10^6 \)-trial Monte Carlo simulation is denoted by \( F_{sim}(x) \). In Table 1, the percentile points of the functions (14) and \( F_{sim}(x) \) are shown.

| \( \alpha \)  | \( F_{sim}^{-1}(\alpha) \) | \( F^{-1}(\alpha) \) |
|--------------|-----------------|-----------------|
| 0.01         | 0.267           | 0.267           |
| 0.05         | 0.372           | 0.372           |
| 0.50         | 0.819           | 0.818           |
| 0.95         | 1.861           | 1.856           |
| 0.99         | 2.679           | 2.669           |
| \( \alpha \) | \( F_{sim}^{-1}(\alpha) \) | \( F^{-1}(\alpha) \) |
| 0.01         | 2.326           | 2.324           |
| 0.05         | 3.154           | 3.155           |
| 0.50         | 7.517           | 7.513           |
| 0.95         | 24.41           | 24.38           |
| 0.99         | 45.36           | 45.57           |

We discuss the distribution of the ratio of two Wishart matrices when they are both singular. In this case, their distributions have often used the test of equalities for covariance matrices. Let \( A \sim W^1_m(n, I_p) \) and \( B \sim W^1_p(n, I_p) \) where \( A \) and \( B \) are independent. Assuming \( p > m > n \), the product \( AB^{-1} \) is not defined, and some authors have provided ideas for this. Srivastava (2007) defined the product \( AB^+ \) instead of \( AB^{-1} \), where \( A^+ \) is a Moore-Penrose inverse matrix of \( A \). The density of eigenvalues of \( AB^+ \) is approximately equivalent to \( \frac{1}{2} \mathcal{U} \) for a large \( m \) where \( \mathcal{U} \) is distributed as a singular real Wishart distribution \( W^1_m(n, I_m) \). Furthermore, the probability density functions of \( AB^+ \) are also equivalent to \( UW^{-1} \) where \( W \sim W^1_m(p, I_m) \). See Srivastava (2007) and Grinek (2019) for details. We extend this result to the complex and quaternion cases. The next theorem implies that the exact distribution of the largest eigenvalue of \( AB^+ \) is equivalent to the distribution of (11).

**Theorem 4.** Let \( A \sim W^\beta_p(n, I_p) \) and \( B \sim W^\beta_p(m, I_p) \) where \( A \) and \( B \) are independent. If \( p > m > n \), then the distribution of the largest eigenvalue \( q_1(AB^+) \) is equivalent to the largest eigenvalue \( q_1(UW^{-1}) \), where \( U \sim W^\beta_m(n, I_m) \), \( W \sim W^\beta_m(p, I_m) \).

**Proof.** The singular beta Wishart matrix \( A \) and \( B \) can be written as \( A = ZZ^* \) and \( B = H_1 LH_1^* \) where \( Z \sim N^\beta_p(O, I_p \otimes I_n) \), \( L \) is the \( m \times m \) diagonal matrix, and \( H_1 \in V^\beta_{m,p} \). Then

\[
q_1(AB^+) = q_1(Z^*B^+Z) = q_1(Z^*H_1L^{-1}H_1^*Z),
\]
where $V = H^*_1 Z$ is distributed as matrix variate beta normal distributions $N^\beta_{m,n}(O, I_m \otimes I_n)$. We consider the nonsingular random matrix $HL^{-1}H^*$, where $H \in U^\beta_m$. Then we have

$$q_1(AB^+) = q_1(V^*L^{-1}V) = q_1(V^*H^*HL^{-1}H^*V) = q_1(HVV^*H^*HL^{-1}H^*) = q_1(UW^{-1}).$$

The exact computation of the largest eigenvalue of $AB^+$ using distribution [14] for $p = 10, m = 5$ and $n = 3$ is represented as

$$\frac{693}{4} \left( \frac{x}{x+1} \right)^{15/2} \left\{ \frac{5x^6}{198(x+1)^6} - \frac{3x^5}{11(x+1)^5} + \frac{85x^4}{66(x+1)^4} - \frac{250x^3}{77(x+1)^3} + \frac{50x^2}{11(x+1)^2} - \frac{10x}{3(x+1)} + 1 \right\}.$$  

(15)

Fig 1 shows the graph of the exact distribution [15], where the 95 percentage point is 7.63. We obtain the upper 5% probability of the empirical distribution using a $10^6$-trial Monte Carlo simulation for the largest eigenvalue of $AB^+$, which is 0.050.

Fig. 1: $p = 10, m = 5, n = 3$

5 Conclusion

In this study, we discussed the exact distribution of the largest eigenvalue of a singular random matrix for Roy’s test. The exact distribution of the largest eigenvalue of a singular beta $F$-matrix was derived in terms of heterogeneous hypergeometric functions. Numerical experiments were performed for the theoretical distributions of [10] when the covariance matrices are identity. We also considered the distribution of the largest eigenvalue of the ratio of two singular beta-Wishart matrices. This distribution for $\beta = 1$ could be reduced in the form of [11].

References

[1] Caro-Lopera, F.J., González-Farías, G., and Balakrishnan, N. (2014). On Generalized Wishart Distribution -I: Likelihood Ratio Test for Homogeneity of Covariance Matrices, Sankhyā: The Indian Journal of Statistics, 76, 179-194.

[2] Díaz-García, J.A. and Gutiérrez-Jáimez, R. (1997). Proof on the conjectures of H. Uhlig on the singular multivariate beta and the Jacobian of a certain matrix transformation, The Annals of Statistics, 25, 2018–2023.
Díaz-García, J.A. and Gutiérrez-Jáimez, R. (2011). On Wishart distribution: Some extensions, *Linear Algebra and its Applications*, **435**, 1296–1310.

Díaz-García, J.A. (2013a). Distribution theory of quadratic forms for matrix multivariate elliptical distribution, *Journal of Statistical Planning and Inference*, **143**, 1330–1342.

Díaz-García, J.A. and Gutiérrez-Sánchez, R. (2013b). Distributions of singular random matrices: Some extensions of Jacobians, *South African Statistical Journal*, **47**, 111–121.

Li, F. and Xue, Y. (2009). Zonal Polynomials and Hypergeometric Functions of Quaternion Matrix Argument, *Communications in Statics–Theory and Methods*, **35**, 1184–1206.

Grinek, S. (2019). Exact Largest Eigenvalue Distribution for Doubly Singular Beta Ensemble, *arXiv:1905.01774*.

Gupta, A.K. and Nager, D.K. (1999). Matrix Variate Distributions. *Chapman & Hall/ CRC*.

Johnstone, I, M. (2008). Multivariate Analysis and Jacobi Ensembles: Largest Eigenvalue, Tracy-Widom Limits and Rates of Convergence, *The Annals of Statistics*, **36**, 2638-2716.

Johnstone, I, M. (2009). Approximate Null Distribution of the Largest Root in Multivariate Analysis, *The Annals of Applied Statistics*, **3**, 1616-1633.

Hashiguchi, H., Nakagawa, S. and Niki, N. (2000). Simplification of the Laplace Beltrami operator, *Mathematics and Computers in Simulation*, **51**, 489–496.

Hashiguchi, H., Takayama, N. and Takemura, A. (2018). Distribution of the ratio of two Wishart matrices and cumulative probability evaluation by the holonomic gradient method, *Journal of Multivariate Analysis*, **165**, 270–278.

Khatri, C. G. (1972). On the exact finite series distribution of the smallest or the largest root of matrices in three situations, *Journal of Multivariate Analysis*, **2**, 201–207.

Koev, P. and Edelman, A. (2006). The efficient evaluation of the hypergeometric function of a matrix argument, *Mathematics of Computation*, **75**, 833-846.

Mathai, A.M. (1997). Jacobians of Matrix Transformations and Functions of Matrix Argument, World Scientific, London.

Matsubara, S. and Hashiguchi, H. (2016). Approximate eigenvalue distribution for the ratio of Wishart matrices, *SUT Journal of Mathematics*, **52**, 141-158.

Muirhead, R. J. (1982). Aspects of multivariate statistical theory, *John Wiley*.

Shimizu, K. and Hashiguchi, H. (2019). Heterogeneous hypergeometric functions with two matrix arguments and the exact distribution of the largest eigenvalue of a singular beta-Wishart matrix. *arXiv:1912.03903*, submitted for publication.

Shinozaki, A., Hashiguchi, H. and Iwashita, T. (2018). Distribution of the largest eigenvalue of an elliptical Wishart matrix and its simulation, *Journal of the Japanese Society of Computational Statistics*, **19**, 45–56.

Srivastava, M. S. (2007). Multivariate theory for analyzing high dimensional data, *Journal of the Japan statistical society*, **37**, 53-86.

Stanley, R. P. (1989). Some combinatorial properties of Jack symmetric functions, *The Annals of Statistics*, **77**, 76-115.

Sugiyama, T. (1967). On the distribution of the largest latent root of the covariance matrix, *The Annals of Mathematical Statistics*, **38**, 1148–1151.

Ratnarajah, T. and Vaillancourt, R. (2005a). Quadratic forms on complex random matrices and multiple-antenna systems, *IEEE Transactions on Information Theory*, **51**, 2976–2984.
[24] Ratnarajah, T. and Vaillancourt, R. (2005b). Complex singular Wishart matrices and applications, *Computers and Mathematics with Applications*, **50**, 399–411.

[25] Ratnarajah, T., Vaillancourt, R. and Alvo, M. (2005). Eigenvalues and condition numbers of complex random matrices, *SIAM Journal on Matrix Analysis and Applications*, **26**, 441–456.

[26] Uhlig, H. (1994). On singular Wishart and singular multivariate beta distributions, *The Annals of Statistics*, **22**, 395–405.