From perturbative to non-perturbative in the $O(4)$ sigma model

Michael C. Abbott, Zoltán Bajnok, János Balog, Árpád Hegedűs

May 18, 2021

Wigner Research Centre for Physics
Konkoly-Thege Miklós u. 29-33, 1121 Budapest, Hungary

Abstract

We study the resurgent trans-series for the free energy of the two-dimensional $O(4)$ sigma model in a magnetic field. Exploiting integrability, we obtain very high-order perturbative data, from which we can explore non-perturbative sectors. We are able to determine exactly the leading real-valued exponentially small terms, which we check against the direct numerical solution of the exact integral equation, and find complete agreement.

1 Introduction

Perturbation theory in physics very seldom leads to a convergent Taylor series in the coupling, but the divergent tail of this series contains information about non-perturbative effects [1, 2]. For many systems, it is now possible to map out a web of relations between the original series expanding about the vacuum, and expansions around other saddle points of the path integral, and the set of tools for doing so is known as resurgence theory \(^1\). Quantum field theories certainly have badly behaved perturbation theory [7, 8, 9], and contain non-perturbative objects such as instantons and renormalons [10, 11]. But it is usually difficult to calculate enough terms to see the patterns connecting them in much detail.

In the standard model of particle physics, perturbation theory works extremely well for electroweak effects [12]. But it is less useful for the strong force, where non-perturbative effects such as quark confinement are unavoidable, and there is interest in using these to make better predictions from perturbation theory [13, 14]. The $O(N)$ sigma-model has often been used as a toy model for QCD, exhibiting asymptotic freedom and a dynamical mass gap [15]. We study in particular the $O(4)$ model, whose free energy (in terms of the running coupling $\alpha$) has the following leading power-series and exponential contributions, in a sense made precise below:

\[
f = 1 + \frac{\alpha}{2} + \frac{\alpha^2}{4} + \frac{10 - 3\zeta_3}{32}\alpha^3 + \chi_5\alpha^4 + \ldots + e^{-8/\alpha}\left(d_1 + d_2\alpha + d_3\alpha^2 + \ldots\right) + \ldots
\]  

Only the first three perturbative terms $\chi_n$ have been calculated by standard methods [16], but we can do much better by exploiting the integrability of the $O(N)$ model [17]. In particular, a method for using its thermodynamic Bethe ansatz (TBA) description [15] to calculate very high-order perturbative

\(^1\)For recent reviews see [3, 4, 5, 6].
coefficients $\chi_n$ was invented by Volin [18, 19]. His 26 terms were sufficient to see the structure of the Borel plane, where the leading singularities give rise to imaginary ambiguities of order $e^{-2/\alpha}$. We extend this work to calculate 2000 terms for the $O(4)$ case, for an energy $\epsilon$ and density $\rho$ separately, with $f \propto \epsilon/\rho^2$. From these we can map out the algebra of alien derivatives connecting different non-perturbative sectors [4]. These relations ultimately allow us to recover the real $e^{-8/\alpha}$ part of the free energy (1), and coefficients $d_n$, via median resummation [20, 21]. We can confirm this by comparing to a numerical solution of the TBA, which includes all exponential corrections.  

2 Perturbation theory and TBA

The $O(4)$ sigma model is a relativistic quantum field theory in two dimensions, with four scalar fields $\Phi_i(x,t)$ restricted to the unit sphere: $\sum_{i=1}^4 \Phi_i^2 = 1$. When a magnetic field $h$ is coupled to the conserved charge $Q_{12}$, the Lagrangian reads [15]

$$L = \frac{1}{2\lambda^2} \left\{ \partial_\mu \Phi_i \partial^\mu \Phi_i + 2i h (\Phi_i \partial_0 \Phi_2 - \Phi_2 \partial_0 \Phi_i) + h^2 (\Phi_3^2 + \Phi_4^2 - 1) \right\}.$$  

(2)

One of the scalar fields may be eliminated, say $\Phi_1 = 1 - \lambda^2 (\varphi_2^2 + \varphi_3^2 + \varphi_4^2)$ with $\lambda \varphi_i = \Phi_i$, and then the free energy density $F$ is given by the following path integral:

$$e^{-V F(h)} = \int D^3[\varphi] e^{-\int d^2 x L(x)}.$$  

(3)

The density $\rho$ and the ground-state energy density $\epsilon(\rho)$ are related to $F(h)$ by a Legendre transformation:

$$\rho = -\partial F/\partial h, \quad \epsilon(\rho) = F(h) - F(0) + \rho h.$$  

(4)

Instead of standard perturbation theory in the bare coupling $\lambda$, the expansion can be improved using the renormalization group, and the free energy is eventually expressed in terms of the running coupling $\alpha$, defined via

$$2/\alpha + 1 - \log \alpha = \log(\rho^2 32\pi/m^2).$$  

(5)

Direct perturbative results are available only for the first three terms [16], and technically it is very difficult to proceed to higher orders.

In the integrable description, the infrared degrees of freedom can be used to calculate the ground state energy. A large enough magnetic field forces these particles to condense into an interval $-B < \theta < B$ of rapidity, whose length depends on $h$. The thermodynamic limit of the Bethe ansatz equation then leads to a linear integral equation for the density of these particles, $\chi(\theta)$:

$$\chi(\theta) - \int_B^{-B} \frac{d\theta'}{2\pi} K(\theta - \theta') \chi(\theta') = m \cosh \theta.$$  

(6)

Here $K$ is the logarithmic derivative of the S-matrix

$$2\pi K(\theta) = -2\pi i \partial_\theta \log S(\theta)$$  

$$= 2 \left\{ \Psi(1 - i\theta/2\pi) - \Psi(1/2 - i\theta/2\pi) + c.c. \right\}.$$  

(7)

where $\Psi(\theta) = \partial_\theta \log \Gamma(\theta)$ is the digamma function. The density and energy are then

$$\rho = \int_B^{-B} \frac{d\theta}{2\pi} \chi(\theta), \quad \epsilon = m \int_B^{-B} \frac{d\theta}{2\pi} \cosh \theta \chi(\theta).$$  

(8)

These and related calculations are described at greater length in another article [22].
The parameter $B$ can be related to the magnetic field by $h = \partial_\rho \epsilon(\rho)$, which follows from minimizing the free energy over $\rho$. The large-$B$ expansion can thus be translated into a large-$h$ expansion, which then can be compared to the original perturbative expansion. Such a comparison was used to relate the dynamically generated $\Lambda_{\overline{MS}}$ scale to the masses of the particles [15].

Volin’s method to expand the TBA equation systematically works by solving the TBA both in the bulk $\theta \sim 0$ and near the edge $\theta \sim B$, and then matching these two expansions, order by order [18, 19]. Solving the recursion leads to a large-$B$ expansion of both the ground-state energy $\epsilon = \hat{\epsilon} m^2 e^{2B}/16$, $\hat{\epsilon} = 1 + \sum_{k=1}^{\infty} \xi_k/B^k$ (9) and the density $\rho = \hat{\rho} m e^{B} \sqrt{B/8\pi}$, $\hat{\rho} = 1 + \sum_{n=1}^{\infty} u_n/B^n$ (10) where we define $\hat{\epsilon}$ and $\hat{\rho}$ to standardise on expansions starting with 1.

He worked with generic $O(N)$ models, and was able to find the first 26 coefficients. These results were recently extended to 44 coefficients in [23], and to some non-relativistic theories in [24, 25, 26]. We decided to focus on the $O(4)$ model only, where we were able to solve the recursive equations in closed form. This allowed us to calculate $\sim 50$ coefficients analytically, and 2000 coefficients numerically with very high precision of 12 000 decimal digits.

3 Resurgence in $B$

To explore the non-perturbative sectors and to reveal the resurgence structure, we start with the density $\hat{\rho}$, whose first two coefficients are

$$u_1 = -\frac{3}{8} + \frac{\ell}{2}, \quad u_2 = -\frac{15}{128} + \frac{3\ell}{16} - \frac{\ell^2}{8}$$

where $\ell = \ln 2$. From the first few coefficients, calculated analytically, we observe that $u_n$ is a polynomial up to $\ell^n$, and may contain zeta-functions, of odd order no higher than $n$. At large $n$, we see that $u_n$ grows factorially, such that the following $c_n$ approaches a constant:

$$c_n = 2^{n+1}u_{n+1}/n!.$$ (12)

We have seen this with very high precision numerically [22]. Technically we introduced $c_n$ by (12) and analysed their large $n$ behavior via high order Richardson transform [6]. The tail of the 100th order Richardson transform was constant to 150 digits precision. The coefficients $\xi_k$ from the energy $\hat{\epsilon}$ behave analogously.

To see the analytic structure on the Borel plane (i.e. of the function $\sum_n c_n t^n$) we plot the poles of the Padé approximant corresponding to $\hat{\epsilon}$ in Fig. 1. It shows a cut starting at $t = -1$, a pole at 1, and another cut starting at 2. The analytic structure of the Borel transform of $\hat{\rho}$ is similar, except without the pole at $t = 1$. These agree with the findings of [18, 23] for the free energy, who established the factorial growth, and determined the location of the cuts. They attributed this behaviour to UV and IR renormalons.

In order to see if our Borel transformed functions are simple resurgent functions with logarithmic cuts we applied a two-step procedure as in [27]. We first changed the asymptotics to ensure a square root branch cut, then we used a conformal type mapping to transform it into a pole singularity. The changes in the analytical structures were followed by high order Pade approximants. These numerical results convinced us that we are dealing with simple resurgent functions and all cuts are logarithmic, which we assume from now on.

The notion of an alien derivative for simple resurgent functions is a concise and elegant way to characterize the logarithmic cut (and pole) structure of the Borel transform. It is related to the logarithm of the Stokes authormorphism, see later for our convention. We refer to [4] for the definition,
and here merely summarize the connection to asymptotic coefficients. Consider the formal asymptotic expansion

\[ \Psi(z) = 1 + \sum_{n=1}^{\infty} s_n / z^n, \quad z = 2B \]

whose Borel transform is \( B(t) = \sum_{n=0}^{\infty} c_n t^n \) with \( c_n = s_{n+1}/n! \) which behaves asymptotically as

\[
\begin{align*}
c_n &= \left( p^+ + \frac{p^+_0}{n} + \frac{p^+_1}{n(n-1)} + \ldots \right) \\
&\quad + (-1)^n \left( p^- + \frac{p^-_0}{n} + \frac{p^-_1}{n(n-1)} + \ldots \right)
\end{align*}
\]

Then the alien derivatives at \( t = \pm 1 \) are given by

\[ \Delta_{\pm 1} \Psi(z) = \mp i2\pi \left\{ p^+ \pm \sum_{m=0}^{\infty} \frac{(\pm 1)^m p^+_n}{z^{m+1}} \right\}. \]

Treating \( \hat{\rho} \) first, using a version of the Richardson transform we could see with about 150 digits precision that all the \( p^+ \) coefficients vanish [22]. Technically we defined the even \( c_{2n} + c_{2n-1} \) and the odd \( c_{2n} - c_{2n-1} \) combinations and used a high order Richardson transform to read off \( p^+ \) and \( p^- \) with 150 digits precision, respectively. We then subtracted \( p^+ + (-1)^n p^- \) from \( c_n \) and multiplied the result with \( n \) and repeated the analysis. Using these high precision numerical values we found analytic expressions for the first 8 coefficients \( p^+_m \), with similar structure to the original \( u_n \) and \( \xi_k \) coefficients\(^3\). We then determined the next 42 terms with high, but decreasing numerical precision. After repeating the same analysis for \( \hat{\epsilon} \), and investigating the obtained coefficients of the alien derivatives \( \Delta_{\pm 1} \) we observed that they can be written in terms of the original functions:

\[
\begin{align*}
\Delta_{1} \hat{\rho} &= 0, & \Delta_{-1} \hat{\rho} &= i\hat{\epsilon}\hat{\rho}, \\
\Delta_{1} \hat{\epsilon} &= -4i, & \Delta_{-1} \hat{\epsilon} &= i\hat{\epsilon}^2.
\end{align*}
\]

Although we obtained this result for the first 40 coefficients we believe it is true in general, which is a beautiful manifestation of resurgence, and allows us to calculate the result of all combinations of \( \Delta_{\pm 1} \) in terms of \( \hat{\rho} \) and \( \hat{\epsilon} \).

To study higher alien derivatives, we begin by observing that the Borel transform of \( 1/\hat{\epsilon} \) has only a pole singularity at \( t = -1 \), whose residue is exactly known. After removing this pole, no singularity remains between \( -2 \) and \( 1 \), thus we continue analytically this function using Padé approximation and re-expand it around \( t = -1/2 \). By this trick the large \( n \) asymptotics of the new coefficients will carry information not only about the singularity at 1, but also at \( -2 \) (being \( 3/2 \) and \( -3/2 \) in the new

---

\(^3\)This can be done e.g. by using the \texttt{FindIntegerNullVector} function in Mathematica or by the EZ-Face website, CECM, Simon Fraser University: http://wayback.cecm.sfu.ca/projects/EZFace/
variable). We then perform a (rescaled by 3/2) asymptotic analysis of these coefficients. We found that the analogous \( p^- \) coefficients vanished for 60 digits, implying \( \Delta_{-2}(1/\ell) = 0 \), i.e. there is no cut there, and since alien derivatives obey the Leibniz rule, this means that \( \Delta_{-2}\ell = 0 \).

Next define \( G = (\ell + \ell')/\rho^2 \), where prime denotes \( d/dz \). It is easy to see that \( \Delta_{\pm 1} G = 0 \), hence its expansion around \( t = 0 \) has radius of convergence 2. Applying the (rescaled by 2) asymptotic analysis we found numerically that \( \Delta_{-2} G = 0 \), which implies \( \Delta_{-2}\hat{\rho} = 0 \). Thus the point \( t = -2 \) does not seem to be singular for our model. Using similar analysis we calculated

\[
\Delta_{2}\hat{\rho} = iR/2, \quad R = 1 + \sum_{n=1}^{\infty} r_n/z^n.
\]

Here we can fix the first 50 coefficients \( r_n \) numerically, with gradually decreasing precision. For the first five of these, we found analytic expressions in terms of \( \ell = \ln 2 \) and zeta-functions using EZ-Face.

The first three are:

\[
\begin{align*}
r_1 &= 1/2 + \ell, \quad r_2 = -\ell/2 - \ell^2/2, \\
r_3 &= \frac{21}{64} + \frac{3}{4}\ell^2 + \frac{3}{8}\zeta_3.
\end{align*}
\]

Similar analyses also gave

\[
\Delta_{2}\hat{\ell} = 2iE, \quad E = 1 + \sum_{n=1}^{\infty} e_n/z^n
\]

and again we fixed five coefficients exactly, including

\[
\begin{align*}
e_1 &= 1/4, \quad e_2 = 5/32 - \ell/2, \\
e_3 &= \frac{57}{128} - \frac{5}{8}\ell + \ell^2.
\end{align*}
\]

There appears to be no simple relation between these coefficients \( r_n, e_n \) and those of the original functions, \( \hat{\rho} \) and \( \hat{\ell} \). Thus unlike the resurgence at \( \pm 1 \), which is trivial in the sense that the functions resurge to themselves, the resurgence at \( t = 2 \) is non-trivial. In order to refer to this fact we call \( R \) and \( E \) as second generation functions, while \( \hat{\rho} \) and \( \hat{\ell} \) as first generation ones. The naming also refers to the fact that \( \hat{\rho} \) and \( \hat{\ell} \) can be obtained directly from the expansion of the TBA equation (6), while this is not so clear for \( R \) and \( E \). The trivial resurgence at \( t = \pm 1 \) is manifested also for \( E \) and \( R \).

Investigating \( \eta = E/\rho^2 \), we find numerically that \( \Delta_{\pm 1}\eta = 0 \), implying that

\[
\Delta_{1} E = 0, \quad \Delta_{-1} E = 2i\hat{E}.
\]

Similar analysis gives

\[
\Delta_1 R = 0, \quad \Delta_{-1} R = i(4\hat{\rho}E + \hat{\ell}R).
\]

There is also some evidence (a few digits) of the vanishing of \( \Delta_{-2}\eta \), leading to

\[
\Delta_{-2} R = \Delta_{-2} E = 0.
\]

which again demonstrates that the point is not singular.

Analysing the \( t = 2 \) singularity we again observe a non-trivial resurgence with new functions, which we call third generations

\[
\Delta_2 E = -i/2 \hat{E}, \quad \Delta_2 R = -i/2 \hat{R}
\]

where the leading expansions are

\[
\begin{align*}
\hat{R} &= 1 + \left(\frac{1}{4} + \ell\right)/z + \left(\frac{\zeta_2}{8} - \ell - 2\ell^2\right)/4z^2 + \mathcal{O}(1/z^3) \\
\hat{E} &= 1 + 3/8z^2 + \mathcal{O}(1/z^3).
\end{align*}
\]

At this point we have run out of precision and we were not able to calculate alien derivatives \( \Delta_{\pm 3} \) or higher, but based on the available data we can conjecture a pattern, shown in Fig. 2. Defining \( \phi_- = \ell/\hat{\rho} \) and \( \phi_+ = 1/\hat{\rho} \) as representatives of the first generation, we observe that they are exchanged by the action of \( i/4\Delta_1 \) and \( i\Delta_{-1} \). Mapping them to the second generation as \( \Delta_2\phi_{\pm} \), we notice that these functions are again exchanged by \( \Delta_{\pm 1} \), with precisely the same coefficients. If this pattern persists, then it fixes the third generation’s \( \Delta_{\pm 1}\hat{R} \) and \( \Delta_{\pm 1}\hat{E} \).
The second generation contains theory, (1). While expansions in large demonstrated in (16) to (24); dashed lines are conjectures, which allow us to fix downwards, while Figure 2: Resurgence pattern, in which alien derivative

Due to the special form of the relation between and the third generation is the alien derivative with respect to related to by [18]

Since is the alien derivative with respect to above, we write for alien derivatives of functions expanded in . We can translate between them using a formula for function composition [28]:

Applying this to gives

where the dot indicates . We are interested in the alien derivatives of and we obtained

Due to the special form of the relation between and , (26), the singularities in the variable are at the same positions as they are in . After a long calculation we obtained that

\[ D_{1} f = -16i/e, \]
\[ D_{2} f = \frac{16i}{e^2} F, \quad F = \frac{2z\rho^2}{x} E - \frac{z^2\rho^3 (f + \dot{f})}{x(1 + x)} R. \]
Using (28), it follows that $D_1 F = D_{-2} F = 0$. We can similarly calculate $D_{-1} f$ and $D_{-2} f = 0$, and then $D_{-1} F$. In order to make comparison with the TBA result we will also need

$$D_2 F = \frac{16i}{e^2} \left( 1 - \frac{5}{4e} - \frac{1}{2e^2} + \ldots \right).$$

(31)

## 5 Median resummation

Using these alien derivatives of $f = \sum_{n=1}^{\infty} \chi_n e^n$, we can now propose an ambiguity-free resummation of the perturbative series. Clearly the two lateral Borel resummations

$$S_\pm(f) = \chi_1 + \alpha \chi_2 + \int_0^{\infty} dt e^{-tx} B(t)$$

(32)

are different, due the singularities on the positive real line. They are related by the Stokes automorphism $\mathcal{S}$, which can be written in terms of the alien derivatives as$^4$

$$S_+(f) = S_-(\mathcal{S} f), \quad \mathcal{S} = \exp \left( - \sum_{n=1}^{\infty} e^{-nx} D_n \right).$$

(33)

The median resummation arises from demanding that the lateral resummations for the trans-series are the same, and it involves the square root of the Stokes automorphism $[20, 21, 4]$. It was formulated originally for a one-parameter trans-series, but we assume that it works also for our more complicated case. In terms of the alien derivatives the proposed median resummation reads as:

$$S_{\text{med}}(f) = S_+(\mathcal{S}^{-\frac{1}{2}} f) = S_+ \left( e^{\frac{i}{2} \sum_{n=1}^{\infty} e^{-nx} D_n f} \right).$$

(34)

$^4$Observe that we introduced an unconventional extra sign in the relation between the alien derivative and the Stokes automorphism. This is also consistent with the large order relations (15).
Expanding this, notice that $D_1f = -16i/e$ implies that all higher terms $D_kD_1f$ vanish. We also found that $D_1D_2f = 0$, and so the leading terms are:

$$S_{med}(f) = S_+(f + \frac{e^{-x}}{2}D_1f + \frac{e^{-2x}}{2}D_2f + \frac{e^{-3x}}{2}D_3f + \frac{e^{-4x}}{2}D_4f + \ldots$$

$$+ \frac{e^{-4x}}{8}D_2D_2f + \frac{e^{-4x}}{8}D_1D_3f + \ldots),$$

(35)

where the dots stand for contributions proportional to $e^{-5x}$ and higher.

On the other hand, using the definition of the Stokes automorphism we write

$$S_-(f) = S_+(S^{-1}f) = S_+\left(\sum_{n=1}^{\infty} e^{-nx}D_n f\right)$$

(36)

and expanding this up to $O(e^{-4x})$ we obtain

$$S_-(f) = S_+\left(f + e^{-x}D_1f + e^{-2x}D_2f + e^{-3x}D_3f + e^{-4x}D_4f + \ldotsight.$$

$$+ \frac{e^{-4x}}{2}D_2D_2f + \frac{e^{-4x}}{2}D_1D_3f + \ldots).$$

(37)

We have not calculated $D_3f$ and $D_4f$, however, these unknown quantities drop out from the formula if we express $S_{med}(f)$ using the averaged lateral resummation

$$\text{Re}(S_+(f)) = \frac{1}{2}S_+(f) + \frac{1}{2}S_-(f).$$

(38)

The median resummation formula simplifies to

$$S_{med}(f) = \text{Re}(S_+(f)) - \frac{e^{-4x}}{8}D_2D_2f - \frac{e^{-4x}}{8}D_1D_3f + \ldots$$

(39)

We have not (yet) calculated $D_1D_3$ either, but since we have seen that for almost all of our functions the alien derivative at $+1$ vanishes, at this point we make the bold assumption

$$D_1D_3f = 0.$$  

(40)

With this assumption the final result reads

$$S_{med}(f) = \text{Re}(S_+(f)) - \frac{e^{-4x}}{8}D_2D_2f + \ldots$$

(41)

We have investigated this formula (and hence our assumption) numerically, by comparing it to the result obtained from numerically solving the TBA equations. To calculate the integral $S_+(f)$, we used the conformal mapping method to obtain a precise enough analytical continuation of $B(t)$, which we integrated numerically with high (at least 20 digits) precision [22]. We see convincingly good agreement, shown in Fig. 3, which indicates that our assumption (40) might be correct. It would be very nice to calculate also these missing terms in the future. Our numerical precision is sufficient to clearly see some deviation between the exact TBA result and that given by (41) but this can be attributed to the missing $O(\alpha^3)$ (and higher) terms in (31).

6 Conclusion

We investigated the free energy of the two dimensional $O(4)$ sigma model in a magnetic field. Using the integrability of the model we determined 2,000 perturbative coefficients with very high precision, which enabled us to investigate the analytic structure of the density and energy on the Borel plane.
Using asymptotic analyses we revealed a nice resurgence structure and determined via the median resummation the leading exponentially small corrections to the free energy (1), with $d_0 = \frac{32}{e^4}$, $d_1 = -\frac{20}{e^4}$ and $d_2 = -\frac{4}{e^4}$, which agreed with the numerical solution of the TBA equation. These results fit into a trans-series of the form

$$f = \sum_{m=0}^{\infty} e^{-2m/\alpha} \left( \sum_{n=1}^{\infty} \chi_n^{(m)} \alpha^{n-1} \right)$$

(42)

where $\chi_n^{(0)} = \chi_n$ are the perturbative coefficients, $\chi_n^{(1)} = -\frac{8i}{e^2} \delta_{n,1}$ are related to $D_1 f$, and $\chi_n^{(2)}$ to $D_2 f$. Similar formulations are possible for each of $\rho$ and $\epsilon$. For more general observables and complex couplings we expect a trans-series with exponents $e^{2m/\alpha}$ corresponding to the negative alien derivatives. These cannot fit into the best-studied case of a one-parameter trans-series, because of the more complicated pattern of resurgence relations we have discovered. It would be interesting to formulate Écalle’s bridge equations for this theory.

It would also be interesting to extend this work to other $O(N)$ models, or to similar theories [23, 29]. The method of [18] for calculating $\chi_n^{(0)}$ works for all $O(N)$, but is more complicated. It may also be possible to extract information about higher $\chi_n^{(m)}$ directly from the TBA, instead of starting only from perturbative data.

Finally (and more importantly), it would be interesting to know what semi-classical configurations (instantons, renormalons, bions [30, 31]) are responsible for the resurgence structure found here if there is any.

**Acknowledgements**

We thank Ines Aniceto and Daniel Nogradi for useful discussions. Our work was supported by ELKH, with infrastructure provided by the Hungarian Academy of Sciences. This work was supported in part by NKFIH grant K134946. MCA was also supported by NKFIH grant FK128789.

**References**

[1] C. M. Bender, T. T. Wu, Anharmonic oscillator, Phys. Rev. 184 (1969) 1231–1260. doi:10.1103/PhysRev.184.1231.

[2] E. Brezin, J.-C. Le Guillou, J. Zinn-Justin, Perturbation theory at large order. 2. Role of the vacuum instability, Phys. Rev. D 15 (1977) 1558–1564. doi:10.1103/PhysRevD.15.1558.

[3] M. Marino, Lectures on non-perturbative effects in large $N$ gauge theories, matrix models and strings, Fortsch. Phys. 62 (2014) 455–540. arXiv:1206.6272, doi:10.1002/prop.201400005.

[4] D. Dorigoni, An introduction to resurgence, trans-series and alien calculus, Annals Phys. 409 (2019) 167914. arXiv:1411.3585, doi:10.1016/j.aop.2019.167914.

[5] G. V. Dunne, M. Ünsal, What is QFT? Resurgent trans-series, Lefschetz thimbles, and new exact saddles, PoS LATTICE2015 (2016) 010. arXiv:1511.05977, doi:10.22323/1.251.0010.

[6] I. Aniceto, G. Basar, R. Schiappa, A primer on resurgent transseries and their asymptotics, Phys. Rept. 809 (2019) 1–135. arXiv:1802.10441, doi:10.1016/j.physrep.2019.02.003.

[7] F. J. Dyson, Divergence of perturbation theory in quantum electrodynamics, Phys. Rev. 85 (1952) 631–632. doi:10.1103/PhysRev.85.631.

[8] C. A. Hurst, The enumeration of graphs in the Feynman-Dyson technique, Proc. Roy. Soc. Lond. A 214 (1952) 44. doi:10.1098/rspa.1952.0149.
[9] L. Lipatov, Divergence of the perturbation theory series and the quasiclassical theory, Sov. Phys. JETP 45 (1977) 216–223.

[10] G. ’t Hooft, Can we make sense out of quantum chromodynamics?, Erice Subnucl. Ser. 15 (1979) 943.

[11] M. Beneke, Renormalons, Phys. Rept. 317 (1999) 1–142. arXiv:hep-ph/9807443, doi:10.1016/S0370-1573(98)00130-6.

[12] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Complete tenth-order QED contribution to the muon g-2, Phys. Rev. Lett. 109 (2012) 111808. arXiv:x:1205.5370, doi:10.1103/PhysRevLett.109.111808.

[13] C. Bauer, G. S. Bali, A. Pineda, Compelling evidence of renormalons in QCD from high order perturbative expansions, Phys. Rev. Lett. 108 (2012) 242002. arXiv:1111.3946, doi:10.1103/PhysRevLett.108.242002.

[14] I. Caprini, Conformal mapping of the Borel plane: going beyond perturbative QCD, Phys. Rev. D 102 (2020) 054017. arXiv:2006.16605, doi:10.1103/PhysRevD.102.054017.

[15] P. Hasenfratz, M. Maggiore, F. Niedermayer, The Exact mass gap of the O(3) and O(4) nonlinear sigma models in d = 2, Phys. Lett. B 245 (1990) 522–528. doi:10.1016/0370-2693(90)90685-Y.

[16] Z. Bajnok, J. Balog, B. Basso, G. Korchemsky, L. Palla, Scaling function in AdS/CFT from the O(6) sigma model, Nucl. Phys. B 811 (2009) 438–462. arXiv:0809.4952, doi:10.1016/j.nuclphysb.2008.11.023.

[17] A. B. Zamolodchikov, A. B. Zamolodchikov, Relativistic factorized S matrix in two-dimensions having O(N) isotopic symmetry, JETP Lett. 26 (1977) 457. doi:10.1016/0550-3213(78)90239-0.

[18] D. Volin, From the mass gap in O(N) to the non-Borel-summability in O(3) and O(4) sigma-models, Phys. Rev. D 81 (2010) 105008. arXiv:0904.2744, doi:10.1103/PhysRevD.81.105008.

[19] D. Volin, Quantum integrability and functional equations: Applications to the spectral problem of AdS/CFT and two-dimensional sigma models, J. Phys. A 44 (2011) 124003, PhD thesis, 2009. arXiv:1003.4725, doi:10.1088/1751-8113/44/12/124003.

[20] M. Mariño, Nonperturbative effects and nonperturbative definitions in matrix models and topological strings, JHEP 12 (2008) 114. arXiv:0805.3033, doi:10.1088/1126-6708/2008/12/114.

[21] I. Aniceto, R. Schiappa, Nonperturbative ambiguities and the reality of resurgent transseries, Commun. Math. Phys. 335 (2015) 183–245. arXiv:1308.1115, doi:10.1007/s00220-014-2165-z.

[22] M. C. Abbott, Z. Bajnok, J. Balog, A. Hegedus, S. Sadeghian, Resurgence in the O(4) sigma model, arXiv:2006.05131.

[23] M. Mariño, T. Reis, Renormalons in integrable field theories, JHEP 04 (2020) 160. arXiv:1909.12134, doi:10.1007/JHEP04(2020)160.

[24] M. Mariño, T. Reis, Exact perturbative results for the Lieb-Liniger and Gaudin-Yang models, J. Stat. Phys. 177 (2019) 1148–1156. arXiv:1905.09575, doi:10.1007/s10955-019-02413-1.

[25] M. Mariño, T. Reis, Resurgence and renormalons in the one-dimensional Hubbard model arXiv:2006.05131.

[26] M. Marino, T. Reis, Resurgence for superconductors, Journal of Statistical Mechanics: Theory and Experiment 2019 (12) (2019) 123102. doi:10.1088/1742-5468/ab4802. URL http://dx.doi.org/10.1088/1742-5468/ab4802
[27] I. Aniceto, B. Meiring, J. Jankowski and M. Spaliński, “The large proper-time expansion of Yang-Mills plasma as a resurgent transseries,” JHEP 02 (2019), 073 doi:10.1007/JHEP02(2019)073 [arXiv:1810.07130 [hep-th]].

[28] E. Delabaere, F. Pham, Resurgent methods in semi-classical asymptotics, Annales de l’I.H.P. Physique théorique 71 (1999) 1–94.

[29] V. Kazakov, E. Sobko, K. Zarembo, Double-scaling limit in the principal chiral model: A new noncritical string?, Phys. Rev. Lett. 124 (2020) 191602. arXiv:1911.12860, doi:10.1103/PhysRevLett.124.191602.

[30] T. Fujimori, S. Kamata, T. Misumi, M. Nitta, N. Sakai, Nonperturbative contributions from complexified solutions in $\mathbb{C}P^{N-1}$ models, Phys. Rev. D 94 (10) (2016) 105002. arXiv:1607.04205, doi:10.1103/PhysRevD.94.105002.

[31] T. Fujimori, S. Kamata, T. Misumi, M. Nitta, N. Sakai, Bion non-perturbative contributions versus infrared renormalons in two-dimensional $\mathbb{C}P^{N-1}$ models, JHEP 02 (2019) 190. arXiv:1810.03768, doi:10.1007/JHEP02(2019)190.