Some $p$-adic differential equations

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Abstract

We investigate various properties of $p$-adic differential equations which have as a solution an analytic function of the form

$$F_k(x) = \sum_{n \geq 0} n! P_k(n)x^n,$$

where $P_k(n) = n^k + C_{k-1}n^{k-1} + \cdots + C_0$ is a polynomial in $n$ with $C_i \in \mathbb{Z}$ (in a more general case $C_i \in \mathbb{Q}$ or $C_i \in \mathbb{C}_p$), and the region of convergence is $|x|_p < p^{-\frac{1}{d}}$. For some special classes of $P_k(n)$, as well as for the general case, the existence of the corresponding linear differential equations of the first- and second-order for $F_k(x)$, is shown. In some cases such equations are constructed. For the second-order differential equations there is no other analytic solution of the form $\sum a_n x^n$. Due to the fact that the corresponding inhomogeneous first-order differential equation exists one can construct infinitely many inhomogeneous second-order equations with the same analytic solution. Relation to some rational sums with the Bernoulli numbers and to $F_k(x)$ for some $x \in \mathbb{Z}$ is considered. Some of these differential equations can be related to $p$-adic dynamics and $p$-adic information theory.
1 Introduction

Some aspects of the $p$-adic series of the form

$$F_k(x) = \sum_{n \geq 0} n! P_k(n) x^n, \quad (1.1)$$

where $P_k(n) = n^k + C_{k-1} n^{k-1} + \cdots + C_0$ is a polynomial in $n$ with $C_i \in \mathbb{Z}$, have been considered in a few articles (see [1], [2] and references therein). It was noted in [1] that

$$F_0(x) = \sum_{n \geq 0} n! x^n \quad (1.2)$$

is an analytic solution of the following $p$-adic differential equation:

$$x^2 w''(x) + (3x - 1) w'(x) + w(x) = 0. \quad (1.3)$$

Here we investigate the existence, construction and various properties of the differential equations which have as an analytic solution $p$-adic power series of the form (1.1) with

$$P_k(n) = n^k + C_{k-1} n^{k-1} + \cdots + C_0, \quad C_i \in \mathbb{Q}. \quad (1.4)$$

In a sense we mainly consider an inverse problem related to differential equations, i.e. we are looking for differential equation for which a solution is known.

Recall that the power series (1.1) has $p$-adic region of convergence $D_p = \{ x \in \mathbb{C}_p : | x |_p < p^{-1/p^k} \}$, where $\mathbb{C}_p$ is the algebraic closure of $\mathbb{Q}_p$ [3]. In the case of restriction to $\mathbb{Q}_p$, we have $D_p = \mathbb{Z}_p$ for every $p$. Note that in the real case the series (1.1) is not convergent for any $0 \neq x \in \mathbb{Q}$.

A theory of the $p$-adic hypergeometric differential equations is presented in Dwork’s book [4].

2 Existence of some $p$-adic differential equations

It is not difficult to verify that expression (1.2), which is the simplest example of (1.1), satisfies not only equation (1.3) but also the first-order inhomogeneous differential equation

$$x^2 w' + (x - 1) w = -1. \quad (2.1)$$
Note that differentiation of (2.1) gives (1.3).
Combining (1.3) and (2.1) in the form
\[ x^2 w'' + (3x - 1)w' + w + R(x)[x^2 w' + (x - 1)w + 1] = 0, \] (2.2)
where \( R(x) \) is a rational function with integer coefficients, one can consider infinitely many second-order linear inhomogeneous \( p \)-adic differential equations with the same analytic solution (1.2). Generally, we will be interested in differential equations of the form
\[ (Polynomial)_1 w'' + (Polynomial)_2 w' + (Polynomial)_3 w = (Polynomial)_4, \] (2.3)
where the polynomials are in \( x \) with integer (or \( p \)-adic) coefficients, and \( w = F_k(x) \) with \( P_k(n) \) given by (1.4).

**Proposition 1** Let \( A(x) \) and \( B(x) \) be rational functions with rational coefficients. If there are differential equations
\[ A(x)F'_\nu(x) + B(x)F_\nu(x) = C, \quad C \in \mathbb{Q}, \] (2.4)
\[ A(x)F''_\nu(x) + (A'(x) + B(x))F'_\nu(x) + B'(x)F_\nu(x) = 0, \] (2.5)
with the analytic solution
\[ F_\nu(x) = \sum_{n \geq 0} n!P_\nu(n)x^n, \] (2.6)
then there exist also similar differential equations of the first- and second-order with the solution
\[ F_{\mu+\nu}(x) = \sum_{n \geq 0} n! \prod_{i=1}^{\mu} (n + i)^2 P_\nu(n + \mu)x^n. \] (2.7)

**Proof:** Rewriting eq. (2.4) in the form
\[ \frac{A(x)}{B(x)}F'_\nu(x) + F_\nu(x) = \frac{C}{B(x)} \]
and taking its derivative one obtains a new equation
\[ A_1(x)F''_\nu(x) + B_1(x)F'_\nu(x) = C, \]
which is of the same form as (2.4) but with new rational functions $A_1(x)$ and $B_1(x)$: $A_1(x) = -A(x)B(x)/B'(x)$, $B_1(x) = (B'(x)A(x) - A'(x)B(x) - B^2(x))/B'(x)$. Repeating this procedure $\mu$ times, we get

$$A_{\mu}(x)F_{\nu}^{(\mu+1)}(x) + B_{\mu}(x)F_{\nu}^{(\mu)} = C.$$ 

Taking into account that $F_{\nu}^{(\mu)}(x) = F_{\nu+\mu}(x)$, for functions $F_\nu(x)$ and $F_{\mu+\nu}(x)$ given by (2.6) and (2.7), respectively, we have differential equation for $F_{\mu+\nu}$:

$$A_{\mu}(x)F_{\mu+\nu}'(x) + B_{\mu}(x)F_{\mu+\nu}(x) = C,$$ 

which resembles equation (2.4). The corresponding second-order differential equation is

$$A_{\mu}(x)F_{\mu+\nu}''(x) + (A_{\mu}'(x) + B_{\mu}(x))F_{\mu+\nu}'(x) + B_{\mu}'(x)F_{\mu+\nu}(x) = 0.$$ 

From the proof of the Proposition 1 it also follows

**Corollary 1** Derivatives of any order of the function (2.6), which is related to equations (2.4) and (2.5), induce the corresponding first- and second-order differential equations.

**Proposition 2** If there are differential equations (2.4) and (2.5) with the analytic solution (2.6), then there exist also similar differential equations with the analytic solution

$$G_{\nu}(x) = x^m F_\nu(x) = x^m \sum_{n \geq 0} n! P_{\nu}(n)x^n, \quad m \in \mathbb{N}. \quad (2.10)$$

**Proof:** Differentiating (2.10), and replacing $F_\nu(x)$ and $F_\nu'(x)$ in (2.4) one gets similar equation

$$A_1(x)G_{\nu}'(x) + B_1(x)G_{\nu}(x) = C,$$ 

where $A_1(x) = A(x)/x^m$ and $B_1(x) = B(x)/x^m - mA(x)/x^{m+1}$. By differentiation of (2.11) one has the corresponding second-order differential equation.

**Proposition 3** There exist the first- and second-order differential equations with the analytic solution

$$F_k(x) = \sum_{n \geq 0} n! k^n x^n, \quad k = 1, 2, ... \quad (2.12)$$
Proof: Start with (1.2) which induces equations (2.1) and (2.2). According to the Corollary 1, \( F'(x) = \sum n!nx^n \) has its own differential equation. Due to the Proposition 2 it follows that there exist equations for \( F_1(x) = xF'(x) = \sum n!nx^n \). Performing this procedure \( k \) times we come to the Proposition 3.

**Proposition 4** There exist a first- and a second-order differential equation with the analytic solution
\[
\Phi_\alpha(x) = \sum_{n \geq 0} n!(n+\alpha)x^n, \quad \alpha \in \mathbb{Q}. \tag{2.13}
\]

Proof: Let us introduce \( G_\alpha(x) = x^\alpha F_0(x) = \sum_{n \geq 0} n!x^{n+\alpha} \). According to the Proposition 2, \( G_\alpha(x) \) is an analytic solution of a first- and second-order differential equation if \( \alpha \in \mathbb{N} \). In the same way one can show that \( G_\alpha(x) \) is also a solution of a first-order differential equation if \( \alpha \in \mathbb{Q} \), as well as if \( \alpha \in \mathbb{C}_p \). Differentiating equation for \( G_\alpha(x) \) in an appropriate way one can obtain the corresponding first-order differential equation for \( G'_\alpha(x) \) (see also Corollary 1). In an analogous way to the Proposition 2 it follows that \( \Phi_\alpha(x) = x^{-\alpha}G'_\alpha(x) = \sum_{n \geq 0} n!(n+\alpha)x^n \) is an analytic solution of some first- and second-order differential equations.

It is now obvious that any \( p \)-adic power series of the form
\[
F_k(x) = \sum_{n \geq 0} n! \prod_{i=1}^l (n + \alpha_i)^{k_i}x^n, \quad k_1 + k_2 + \cdots + k_l = k, \quad \alpha_i \in \mathbb{Q}, \tag{2.14}
\]
is an analytic solution of a first- and, consequently, of a second-order homogeneous differential equation.

We can take that in (2.14) some or all of \( \alpha_i \in \mathbb{Q}_p \) (or \( \mathbb{C}_p \)), but in such case there is restriction of our consideration to a definite \( \mathbb{Q}_p \) (or \( \mathbb{C}_p \)). However, taking \( \alpha_i \in \mathbb{Q} \) we have results valid in \( \mathbb{C}_p \) for every \( p \).

**Theorem 1** To each function of the form \( F_k = \sum_{n \geq 0} n!P_k(n)x^n \), where \( P_k(n) = n^k + C_{k-1}n^{k-1} + \cdots + C_0 \) is a polynomial in \( n \) with coefficients \( C_i \in \mathbb{Q} \) (or \( C_i \in \mathbb{C}_p \)), corresponds a first-order differential equation, and consequently the second-order one.

Proof: It follows from the fact that the above polynomial \( P_k(n) \) can be rewritten in the form
\[
P_k(n) = \prod_{i=1}^k (n + \alpha_i),
\]
where $\alpha_i \in \mathbb{C}_p$.

3 Construction of some $p$-adic differential equations

There are many ways to construct relevant differential equations for some $F_k(x) = \sum n!P_k(n)x^n$ with simple polynomials $P_k(n)$.

For functions $\sum n!n^kx^n$, where $k = 0, 1, 2, \cdots$, the relations [1] of the following form are valid:

$$x^k \sum n!n^kx^n + U_k(x) \sum n!x^n = V_{k-1}(x),\quad (3.1)$$

where $U_k(x)$ and $V_{k-1}(x)$ are certain polynomials in $x$ with integer coefficients. The first three of them are:

$$x \sum_{n\geq0} n!nx^n + (x - 1) \sum_{n\geq0} n!x^n = -1,\quad (3.2)$$

$$x^2 \sum_{n\geq0} n!n^2x^n + (-x^2 + 3x - 1) \sum_{n\geq0} n!x^n = 2x - 1,\quad (3.3)$$

$$x^3 \sum_{n\geq0} n!n^3x^n + (x^3 - 7x^2 + 6x - 1) \sum_{n\geq0} n!x^n = -3x^2 + 5x - 1.\quad (3.4)$$

We use the above relations for power series to construct differential equations for some simple cases of $F_k(x)$.

**Example 1:** $F_0(x) = \sum n!x^n$.

Starting with (3.2) one obtains

$$x^2F_0'(x) + (x - 1)F_0(x) = -1,\quad (3.5)$$

that is the equation (2.1). Differentiation of (3.5) gives

$$x^2F_0''(x) + (3x - 1)F_0'(x) + F_0(x) = 0,\quad (3.6)$$

which is just (1.3).

**Example 2:** $F_1(x) = \sum n!nx^n$.

Due to (3.2) and (3.3) one gets

$$x^2(x - 1)F_1'(x) + (x^2 - 3x + 1)F_1(x) = x.\quad (3.7)$$
Dividing (3.7) by \(x\) and performing derivation one has
\[
x^3 F''_1(x) + x(3x - 1)F'_1(x) + (x + 1)F_1(x) = 0. \quad (3.8)
\]

**Example 3:** \(F_1(x) = \sum n!(n + 1)x^n\).

Combining (3.2) and (3.3) we obtain:
\[
x^2 F'_1(x) + (2x - 1)F_1(x) = -1, \quad (3.9)
\]
\[
x^2 F''_1(x) + (4x - 1)F'_1(x) + 2F_1(x) = 0. \quad (3.10)
\]

**Example 4:** \(F_2(x) = \sum n!(n + 1)(n + 2)x^n\).

The corresponding differential equations are:
\[
x^2 F'_2(x) + (3x - 1)F_2(x) = -2, \quad (3.11)
\]
\[
x^2 F''_2(x) + (5x - 1)F'_2(x) + 3F_2(x) = 0, \quad (3.12)
\]

and can be obtained using equations (3.2), (3.3) and (3.4).

Examples 3 and 4 are particular ones of \(F_1(x) = \sum n!(n + \alpha)x^n\), \(\alpha \in \mathbb{Q}\). Let us construct now the corresponding differential equations for any \(\alpha \in \mathbb{Q}\), which does exist according to the Proposition 4.

**Example 5:** \(\Phi_\alpha(x) = \sum n!(n + \alpha)x^n\), \(\alpha \in \mathbb{C}_p\).

It is worthwhile to start with \(G_\alpha(x) = x^\alpha F_0(x) = \sum n!x^{n+\alpha}\). Substituting \(F_0(x) = x^{-\alpha}G_\alpha(x)\) in its equation (3.5) one obtains
\[
x^2 G'_\alpha(x) - [(\alpha - 1)x + 1]G_\alpha(x) = -x^\alpha.
\]

Forming the second-order differential equation for \(G_\alpha(x)\) and taking \(G'_\alpha(x) = x^{\alpha-1}\Phi_\alpha(x)\) we have
\[
x^2[(\alpha - 1)x + 1]\Phi'_\alpha(x) + [(\alpha - 1)x^2 - (\alpha - 3)x - 1]\Phi_\alpha(x) = -(\alpha - 1)^2x - \alpha, \quad (3.13)
\]

and consequently
\[
x^2[(\alpha - 1)x + 1][(\alpha - 1)^2x + \alpha]\Phi''_\alpha(x)
+ [3(\alpha - 1)^3x^3 - (\alpha - 1)(\alpha^2 - 9\alpha + 4)x^2 - (2\alpha^2 - 7\alpha + 1)x - \alpha]\Phi'_\alpha(x)
+ [(\alpha - 1)^3x^2 + 2\alpha(\alpha - 1)x + (\alpha + 1)]\Phi_\alpha(x) = 0. \quad (3.14)
\]

**Example 6:** \(F_k(x) = \sum n! \prod_{i=1}^{k}(n + i)x^n\), \(k \in \mathbb{N}\).
Starting from the Example 3, using the method of mathematical induction and an analogous way to the Example 5, one can derive the following equations:

\[ x^2 F'_k(x) + [(k + 1)x - 1]F_k(x) = -k!, \quad (3.15) \]
\[ x^2 F''_k(x) + [(k + 3)x - 1]F'_k(x) + (k + 1)F_k(x) = 0. \quad (3.16) \]

Note that (3.15) and (3.16) hold for \( k = 0 \) as well.

**Example 7:** \( F_k(x) = \sum n! \prod_{i=1}^{k} (n + i)^2 x^n \).

Differentiation of equation (3.5) \( k \) times yields:

\[ x^2 F^{(k+2)}_0(x) + [(2k + 3)x - 1]F^{(k+1)}_0(x) + (k + 1)^2 F^{(k)}_0(x) = 0. \quad (3.17) \]

Since \( F^{(k)}_0(x) = F_k(x) \) we have

\[ x^2 F''_k(x) + [(2k + 3)x - 1]F'_k(x) + (k + 1)^2 F_k(x) = 0. \quad (3.17) \]

The corresponding first-order equation of (3.17) has a rather complex form.

**Example 8:** \( \Phi_{\alpha \beta}(x) = \sum n!(n + \alpha)(n + \beta)x^n, \quad \alpha, \beta \in C_p. \)

Denote \( G_{\alpha \beta}(x) = x^{\beta} \Phi_{\alpha}(x) = \sum n!(n + \alpha)x^{n+\beta} \) and note that \( G'_{\alpha \beta} = x^{\beta-1} \Phi_{\alpha \beta}(x) \). Using equation (3.13) for \( \Phi_{\alpha}(x) \) one can obtain the following differential equation:

\[
\begin{align*}
x^2[(\alpha - 1)x + 1][((\alpha - 1)(\beta - 1)x^2 + (\alpha + \beta - 3)x + 1)]
& + x[(\alpha - 1)(\beta - 1)x^2 + (\alpha + \beta - 3)x + 1]
& + x[3x(\alpha - 1) + 2][((\alpha - 1)(\beta - 1)x^2 + (\alpha + \beta - 3)x + 1]
& - x^2[(\alpha - 1)x + 1][((\alpha - 1)(\beta - 1)x + \alpha + \beta - 3]
& - [((\alpha - 1)(\beta - 1)x^2 + (\alpha + \beta - 3)x + 1)]^{\beta-1}] \Phi_{\alpha \beta}(x)

& = x[(\alpha - 1)^2x + \alpha][2(\alpha - 1)(\beta - 1)x + \alpha + \beta - 3] - [((\alpha - 1)^2(\beta + 1)x + \alpha \beta]
& \times [(\alpha - 1)(\beta - 1)x^2 + (\alpha + \beta - 3)x + 1]. \quad (3.18)
\end{align*}
\]

The corresponding homogeneous second-order differential equation exists, but it is more complex than (3.18).

**Example 9:** \( F_2(x) = \sum n!n^2 x^n \).
This can be considered as special case of the Example 9 for \( \alpha = \beta = 0 \). From (3.18) it follows

\[
x^2(x^2 - 3x + 1)F_2''(x) + (x^3 - 7x^2 + 6x - 1)F_2(x) = -x(x + 1). \tag{3.19}
\]

The corresponding homogeneous second-order differential equation is

\[
x^3(x + 1)(x^2 - 3x + 1)F_2''(x) + x(3x^4 - 6x^3 - 7x^2 + 6x - 1)F_2'(x)
+ (x^4 + 2x^3 - 13x^2 + 2x + 1)F_2(x) = 0. \tag{3.20}
\]

It is obvious that using the above procedures one can construct differential equation for any function of the form \( F_k(x) = \sum n! \prod_{i=1}^k (n + \alpha_i)x^n \), where \( \alpha_i \in \mathbb{C}_p \).

### 4 On other solutions

It seems that the homogeneous second-order differential equation for analytic function \( F_k(x) = \sum n!P_k(n)x^n \) has no another analytic solution in the region containing point \( x = 0 \). Namely, in any particular case of the above examples one can start with power series expansion and conclude that only \( F_k(x) = \sum n!P_k(n)x^n \) is the corresponding analytic solution. However, the corresponding general statement needs a clear rigorous proof.

Note that the solution \( F_0(x) = \sum_{n \geq 0} n!x^n \) can be presented in the form

\[
F_0(x) = \sum_{n \geq 0} b_n(x - \beta)^n, \tag{4.1}
\]

where coefficients \( b_n \) satisfy conditions

\[
\sum_{n \geq k} b_n \binom{n}{k} \beta^{n-k} = k!, \quad k = 0, 1, 2, \ldots \tag{4.2}
\]

Solution of the system of equations (4.2) yields

\[
b_n = \sum_{k \geq n} (-1)^{k-n}k! \binom{k}{n} \beta^{k-n}. \tag{4.3}
\]
One can easily verify that in the simplest case, given by the Example 1 and equation (1.3), one has the following two new solutions (see also [5]):

\[
\begin{align*}
  w_1(x) &= \frac{1}{x} \exp \left( -\frac{1}{x} \right), \\
  w_2(x) &= \frac{1}{x} \exp \left( -\frac{1}{x} \right) \int_{x_0}^{x} \frac{1}{t} \exp \left( \frac{1}{t} \right) dt,
\end{align*}
\]  

where the region of \( p \)-adic convergence of \( w_1(x) \) and \( w_2(x) \) in (4.4) is \( \Delta_p = \{ x \in \mathbb{C}_p : |x|_p > p^{-\frac{1}{p-1}} \} \). Thus \( \mathbb{C}_p = D_p \cup S_p \cup \Delta_p \), where \( D_p \) is the region of convergence of analytic solution (1.2) and \( S_p \) is the sphere \( S_p = \{ x \in \mathbb{C}_p : |x|_p = p^{-\frac{1}{p-1}} \} \). Note that \( D_p, S_p, \) and \( \Delta_p \) are mutually disjoint subsets of \( \mathbb{C}_p \).

Using a reasoning analogous to the preceding section, one can show that all homogeneous second-order differential equations for \( F_k(x) = \sum n! P_k(n)x^n \) have the corresponding two other solutions which are connected with (4.4) in the similar way as analytic solutions \( F_k(x) \) are related to \( F_0(x) \).

5 Relation to rational summation of \( p \)-adic series

The above differential equations may be used to obtain various expressions for sums of some \( p \)-adic series.

For example, from (3.15) one can rederive (3.2)-(3.4), as well as any other sum of the form

\[
\sum n! [n^k + u_k(x)] x^n = v_k(x),
\]  

where \( u_k(x) \) and \( v_k(x) \) are rational functions of variable \( x \). Any other possible rational sum can be generated from (5.1) multiplying it by rational numbers and performing the corresponding summation. For \( k = 1, ..., 5 \) we calculated (5.1) in the explicit form:

\[
\begin{align*}
  \sum_{n \geq 0} n! \left( n + \frac{x - 1}{x} \right) x^n &= \frac{-1}{x}, \\
  \sum_{n \geq 0} n! \left( n^2 + \frac{-x^2 + 3x - 1}{x^2} \right) x^n &= \frac{2x - 1}{x^2},
\end{align*}
\]
\[ \sum_{n \geq 0} n! \left( n^3 + \frac{x^3 - 7x^2 + 6x - 1}{x^3} \right) x^n = \frac{-3x^2 + 5x - 1}{x^3}, \quad (5.4) \]

\[ \sum_{n \geq 0} n! \left( n^4 + \frac{-x^4 + 15x^3 - 25x^2 + 10x - 1}{x^4} \right) x^n = \frac{4x^3 - 17x^2 + 9x - 1}{x^4}, \quad (5.5) \]

\[ \sum_{n \geq 0} n! \left( n^5 + \frac{x^5 - 31x^4 + 90x^3 - 65x^2 + 15x - 1}{x^5} \right) x^n = \frac{-5x^4 + 49x^3 - 52x^2 + 14x - 1}{x^5}. \quad (5.6) \]

Taking \( x = t \in \mathbb{Z} \) in (5.2)-(5.6) we obtain \( p \)-adic sums valid in all \( \mathbb{Q}_p \). The case \( x = 1 \) and \( k = 1, \ldots, 11 \) is presented in [1]. For some evaluation of \( \sum n! \) one can see Schikhof’s book ([3], p. 17). We write down sums for \( x = -1 \) and \( k = 1, \ldots, 5 \):

\[ \sum_{n \geq 0} (-1)^n n!(n + 2) = 1, \quad \sum_{n \geq 0} (-1)^n n!(n^2 - 5) = -3, \]

\[ \sum_{n \geq 0} (-1)^n n!(n^3 + 15) = 9, \quad \sum_{n \geq 0} (-1)^n n!(n^4 - 52) = -31, \]

\[ \sum_{n \geq 0} (-1)^n n!(n^5 + 203) = 121. \quad (5.7) \]

Note also that putting \( x = 1/(1-\alpha), \ x = 1 \) and \( x = -1 \) successively in (3.13) we have:

\[ \sum_{n \geq 0} n!(n + \alpha) \left( \frac{1}{1-\alpha} \right)^n = \alpha - 1, \quad |1 - \alpha|^p^{-1} < p^{-1}, \quad (5.8) \]

\[ \sum_{n \geq 0} n!(n + \alpha)(\alpha n + 1) = -\alpha^2 + \alpha - 1, \quad (5.9) \]

\[ \sum_{n \geq 0} (-1)^n n!(n + \alpha)[(\alpha - 2)n + 2\alpha - 5] = \alpha^2 - 3\alpha + 1. \quad (5.10) \]

The sum (5.10) can be easily verified employing (5.7).

Since the \( p \)-adic sums (5.2)-(5.6) are convergent in \( \mathbb{Z}_p \) one can use them to obtain a new kind of \( p \)-adic sums with the Bernoulli numbers \( B_n \), which may be regarded as [3]

\[ B_n = \int_{\mathbb{Z}_p} x^n dx, \quad n = 0, 1, 2, \ldots, \]
where \( f_{z_p} f(x) dx \) denotes the Volkenborn integral. Recall that expressions
\[
B_0 = 1, \quad \sum_{i=1}^{n-1} \binom{n}{i} B_i = 0, \quad n \geq 2
\]
determine all Bernoulli numbers. Rewriting (5.2)-(5.6) in the form (3.1) and performing the Volkenborn integration, we get the first five sums:

\[
\sum_{n \geq 0} n![(n + 1)B_{n+1} - B_n] = -1,
\]
\[
\sum_{n \geq 0} n![(n^2 - 1)B_{n+2} + 3B_{n+1} - B_n] = -2,
\]
\[
\sum_{n \geq 0} n![(n^3 + 1)B_{n+3} - 7B_{n+2} + 6B_{n+1} - B_n] = -4,
\]
\[
\sum_{n \geq 0} n![(n^4 - 1)B_{n+4} + 15B_{n+3} - 25B_{n+2} + 10B_{n+1} - B_n] = -\frac{25}{3},
\]
\[
\sum_{n \geq 0} n![(n^5 + 1)B_{n+5} - 31B_{n+4} + 90B_{n+3} - 65B_{n+2} + 15B_{n+1} - B_n] = -\frac{33}{2}.
\]

The termwise integration of an analytic function is provided by the Proposition 55.4 of [3]. If we first make transformation \( x \to -x \) and then apply the Volkenborn integral we can obtain the corresponding sums with \((-1)^n\) factors. As an illustration we give the following two sums:

\[
\sum_{n \geq 0} (-1)^n n![(n + 1)B_{n+1} + B_n] = 1,
\]
\[
\sum_{n \geq 0} (-1)^n n![(n^2 - 1)B_{n+2} - 3B_{n+1} - B_n] = -2.
\]

Since \( |B_n|_p \leq p \) (see [3], p. 172), there are no problems with the convergence of the above series in \( \mathbb{Q}_p \) for every \( p \) and results are valid in all \( \mathbb{Q}_p \). Multiplying the series (3.1) by \( x^m \) before integration, one can generalize the above formulas involving the Bernoulli numbers.
6 Possible physical applications

Since 1987, when a notion of $p$-adic strings [6] was introduced for the first time, there have been exciting investigations in application of $p$-adic numbers in many parts of modern theoretical and mathematical physics (for a review, see, e.g. Refs. [7],[8] and [9]). One of the very perspective approaches is related to adeles [10], which unify $p$-adic and real numbers. So, adelic quantum theory (see [11]-[13]) seems to be a more complete theory then the ordinary one based on real and complex numbers only.

Some of the above $p$-adic differential equations may be regarded as classical equations of motion in the Lagrangian formalism. Recall that for a given Lagrangian $L(\dot{q}, q, t)$, the classical equation of motion is the Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0,$$

where $\dot{q}$ denotes derivative of $q$ with respect to the time variable $t$. In the case of quadratic Lagrangians, i.e.

$$L(\dot{q}, q, t) = a(t)\dot{q}^2 + 2b(t)\dot{q}q + c(t)q^2 + 2d(t)\dot{q} + 2e(t)q + f(t),$$

the classical equation of motion reads:

$$a(t)\ddot{q} + \dot{a}(t)\dot{q}(t) + [\dot{b}(t) - c(t)]q(t) = e(t) - \ddot{d}(t).$$

Let us consider the simplest case of our $p$-adic differential equations presented in the form (2.2), where $R(x)$ is a rational function with integer coefficients. According to (6.3), a second-order differential equation can be an equation of motion if there is a definite relation between the coefficients of the terms with $\ddot{q}$ and $\dot{q}$. One can easily see that the case $R(x) = 0$ does not lead to an equation of motion. However, if $R(x) = (-x + 1)/x^2$ then (2.2) becomes equation of motion in the following form:

$$t^4\ddot{q} + 2t^3\dot{q} + (2t - 1)q = t - 1,$$

One of the possible Lagrangians which give (6.4) is

$$L(\dot{q}, q, t) = \frac{t^2}{2} \dot{q}^2 + \left(\frac{t^3}{3} + 2 \log t + \frac{1}{t} + C\right)\dot{q}q + \frac{t^2}{2}q^2 - \frac{1}{t}q + \frac{1}{t}q,$$

where $C$ is a constant. Other Lagrangians, which lead to (6.4), have less symmetric coefficients than (6.5). A solution of (6.4) is $q(t) = \sum n!t^n$
and represents $p$-adic classical trajectory. In virtue of the Proposition 5 this is a unique $p$-adic analytic solution around $x = 0$ and there is not the corresponding real analytic counterpart.

It is worth noting also that other analytic solutions of the form $F_k(t) = \sum n!P_k(n)x^n$ have no real counterparts and may describe some dynamical systems for which real numbers are useless. As a possible application of these analytic solutions one can consider dynamics on information spaces introduced in [14].

7 Concluding remarks

When the coefficients $C_i$ of the polynomials $P_k(n)$ in (1.1) are rational numbers and $x \in \mathbb{Q}_p$, then all the obtained results for $F_k(x)$ are valid in $\mathbb{Z}_p$ for every $p$. Taking into account solutions (see Section 4) which have real counterparts, we can construct also some adelic [10] solutions. Namely, an adelic solution for the case $k = 1$ in the form (6.4) is:

$$q(t) = (q_\infty(t_\infty), F_0(t_2), F_0(t_3), ..., F_0(x_p), ...),$$

(7.1)

where the index $\infty$ denotes real case, $F_0(t) = \sum n!t^n$, and

$$q_\infty(t) = \frac{1}{t} \exp\left(-1/t\right)$$

\times \left(A_1 + A_2 \int_{t_0}^t \exp\left(2/y\right)dy + \int_{t_0}^t dy \exp\left(2/y\right) \int_{y_0}^y dz \frac{z-1}{z^3} \exp\left(-1/z\right) \right),$$

(7.2)

where $A_1$ and $A_2$ are arbitrary integration constants.

All the above considered differential equations are linear. Some of them are homogeneous and the others are inhomogeneous. Rewriting all equations in the form

$$D_k\left(\frac{d^2}{dx^2}, \frac{d}{dx}, x\right)F_k(x) = 0,$$

(7.3)

where operator $D_k$ linearly depends on derivatives $\frac{d^2}{dx^2}$ and $\frac{d}{dx}$, one can construct many non-linear differential equations taking various products of $D_k$. For example, according to (3.5) and (3.9), we have

$$[x^2u' + (x - 1)u + 1][x^2u' + (2x - 1)u + 1] = 0$$

(7.4)
with solutions: \( u_1(x) = \sum n!x^n \), \( u_2(x) = \sum n!(n + 1)x^n \).

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References

[1] B Dragovich. On \( p \)-adic power series. In: WH Schikhof, C Perez-Garcia, J Kakol, eds. \( p \)-Adic Functional Analysis. Lecture Notes in Pure and Applied Mathematics. Vol. 207. New York: Marcel Dekker, 1999. pp 65-75.

[2] B Dragovich. On some \( p \)-adic series with factorials. In: WH Schikhoh, C Perez-Garcia, J Kakol, eds. \( p \)-Adic Functional Analysis. Lecture Notes in Pure and Applied Mathematics. Vol. 192. New York: Marcel Dekker, 1997. pp 95-105.

[3] WH Schikhof. Ultrametric Calculus: An Introduction to \( p \)-Adic Analysis. Cambridge: Cambridge University Press, 1984.

[4] B Dwork. Lectures on \( p \)-Adic Differential Equations. New York: Springer-Verlag, 1982.

[5] E Kamke, Handbook on Ordinary Differential Equations. (Russian Edition). Moscow: Nauka, 1976.

[6] IV Volovich. \( p \)-adic string. Class Quantum Grav 4:L83-L87, 1987

[7] VS Vladimirov, IV Volovich, EI Zelenov. \( p \)-Adic Analysis and Mathematical Physics. Singapore: World Scientific, 1994.

[8] A Khrennikov. \( p \)-Adic Valued Distributions in Mathematical Physics. Dordrecht: Kluwer Academic Publishers, 1994.

[9] A Khrennikov. Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models. Dordrecht: Kluwer Academic Publishers, 1997.
[10] A Weil. Adeles and Algebraic Groups. Boston: Birkhäuser, 1982.

[11] B Dragovich. Adelic harmonic oscillator. Int J Mod Phys A10:2349-2365, 1995.

[12] GS Djordjević, B Dragovich. \( p \)-Adic path integrals for quadratic actions. Mod Phys Lett A12:1455-1463, 1997.

[13] B Dragovich, Lj Nešić. \( p \)-Adic and adelic generalization of quantum cosmology. Grav Cosm 5:222-228, 1999.

[14] A Khrennikov. Classical and quantum mechanics on information spaces with applications to cognitive, psychological, social, and anomalous phenomena. Found Phys 29:1065-1097, 1999.