Phase space constraints and statistical jet studies in heavy-ion collisions

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Abstract. The effect of the correlation induced by global momentum conservation on the two-particle distribution in nucleus-nucleus collisions is discussed, with a focus on the generic case of collisions with a non-vanishing impact parameter.

1. Introduction
As is well known, the distribution of particles in the final state of a collision between elementary particles results from the interplay of two ingredients [1]. First, there is a “phase space” part, which reflects the kinematic constraints arising from energy and momentum conservation. Then, there comes the dynamical component, namely the detailed interaction(s) between the “initial-state” particles. This is arguably the most interesting part, inasmuch as one does not expect any violation of energy-momentum conservation, where the modeling enters — either first-principle based modeling for the interaction of two elementary particles, or more effective descriptions as required when more particles are involved — together with its set of assumptions and parameters. In particular, the aim of experimental investigations is to determine, or at least to constrain, those possible models, taking the conservation of energy and momentum as granted. For that purpose, it is necessary to have a good control of the “physically trivial” kinematic component.

For single-particle distributions, the effect of energy-momentum conservation is typically to limit the value of the particle momentum and to result in a depletion in the yield close to the boundary: the effect amounts to a multiplicative factor. In the case of the joint distributions of two, three or more particles, the influence of the phase-space constraint is usually less straightforward. If there are only a few final-state particles, the joint distributions are quite constrained and the corresponding possible final states are conveniently represented on Dalitz plots [1]. However, in high-energy collisions, in particular of heavy nuclei, where many particles are emitted, such an approach becomes impractical. One has instead to estimate the effect of total momentum conservation on a statistical basis, be it for the induced correlation between two [2] or more than two [3] particles.

Here, I shall briefly recall how the multiparticle correlation induced by the conservation of total momentum can be derived in the limit of a large number of emitted particles [3]. Then I shall focus on a case of interest for heavy-ion collision studies, namely when the emission of particles in the plane transverse to the beam is anisotropic (i.e., in the presence of so-called “anisotropic flow”).

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2. Cumulants from momentum conservation

Let us consider a collision in which $N$ particles are emitted, viewed in the center-of-mass frame of the particles, so that the sum of their momenta vanishes $\mathbf{p}_1 + \cdots + \mathbf{p}_N = \mathbf{0}$. This relationship induces a correlation between the momenta of $M$ particles chosen arbitrarily among the $N$ ones: technically, the corresponding joint $M$-particle probability distribution does not factorize into the product of the $M$ single-particle probability distributions, but involves non-vanishing cumulants \(^4\) of all orders.\(^1\)

The strength of the $M$-particle correlation depends on both $M$ and the total number of particles $N$. In the simplest case of $N = 2$ final-state particles, the correlation is maximal, since $\mathbf{p}_2 = -\mathbf{p}_1$; the corresponding two-particle probability distribution $f(\mathbf{p}_1, \mathbf{p}_2) = f(\mathbf{p}_1)\delta(\mathbf{p}_1 + \mathbf{p}_2)$ is obviously not factorizable. In the large-$N$ limit, one can prove with the help of a saddle-point approximation that the $M$-particle cumulant scales like $1/N^{M-1}$ \(^3\). A convenient and systematic approach to show that, as well as to obtain the explicit expressions of the cumulants, consists in using a generating function of the joint multiparticle distributions, the logarithm of which generates the cumulants. A few steps allow one to write this generating function as the integral of the exponential of $N$ times some function $F(\mathbf{k})$ of the integration variable $\mathbf{k}$. The procedure to derive the successive cumulants then consists in computing to a given order in powers of $1/N$ the position of the maximum of $F$, i.e. the saddle point, then to calculate the value of $F$ at this maximum so as to perform the saddle-point integration \(^3\).

Admittedly, the cumulants induced by total momentum conservation are in general small when $N$ is large. However it is worth keeping their existence in mind, since they can become significant in some regions of phase space. At the same time, the correlations induced by other more dynamical phenomena which one attempts to investigate might conversely be small, and thus not necessarily significantly larger than the “trivial” kinematic ones. Thus, it has been argued that some influence of momentum conservation on the measured two-particle short-range correlations of identical pions in $pp$ collisions at RHIC energies can be evidenced \(^6\). In heavy ion collisions, where $N$ is larger, momentum-conservation induced correlations are even smaller, yet they could play a non-negligible role in some studies of small signals. Since it is important to have a good idea of what their effect might look like, so as to try to identify and subtract similar patterns in the measured correlations, I shall now discuss further these cumulants.

3. Azimuthally-dependent cumulants and distributions

The most general expressions of the two- and three-particle cumulants due to the conservation of global momentum to leading order in powers of $1/N$, derived according to the method sketched above \(^3\), can be found in reference \(^7\) (equations (3.4) and (3.5)). Neglecting the components of the momenta along the beam direction\(^2\), I shall focus on the constraint from transverse momentum conservation.

In the heavy-ion context, the mean square momenta along the nucleus-nucleus impact parameter (i.e., in the reaction plane) $\langle p_x^2 \rangle$ and perpendicular to it in the transverse plane, $\langle p_y^2 \rangle$, are generally unequal: this is the celebrated anisotropic expansion. To account for the phenomenon, let me introduce the coefficient $v_2 \equiv \langle p_x^2 - p_y^2 \rangle/(p_x^2 + p_y^2)$.\(^3\) Note that while a

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\(^1\) Using probability distributions, instead of the distributions themselves, is much simpler, for it circumvents the issue of choosing a “proper” normalization — e.g. of particle-pair yields — when the multiplicity $N$ fluctuates from event to event (see the discussion in reference \(^5\), section III). Yet, one should not forget that the problem is always present in a real experimental correlation measurement.

\(^2\) In the absence of an experimental estimate of the mean square momentum along the beam as compared to along the transverse directions, this approximation can tentatively be justified by the reported quasi-independence of the average transverse momentum ($p_T$) on rapidity \(^8\).

\(^3\) This definition differs from that of the usual elliptic-flow coefficient $v_2 \equiv \langle (p_x^2 - p_y^2)/(p_x^2 + p_y^2) \rangle$, yielding values larger by about a factor 2.
fourth-harmonic modulation $v_4$ of the single-particle distribution has also been evidenced at RHIC. I need not consider it here, as it would affect the cumulants induced by momentum conservation only from the four-particle cumulant onwards. Introducing $\bar{v}_2$ into the two-particle cumulant expression, one can recast it as

$$f_c(p_{T1}, p_{T2}) = -\frac{2p_{T1}p_{T2}}{N(p_{T1})^2(1 - \bar{v}_2^2)} [\cos(\varphi_2 - \varphi_1) - \bar{v}_2 \cos(\varphi_1 + \varphi_2 - 2\Phi_R)],$$

(1)

where $\Phi_R$ is the reaction-plane azimuth. This two-particle cumulant depends not only on the relative angle $\Delta\varphi_{12} \equiv \varphi_2 - \varphi_1$ between the particles, but also on the absolute orientation of the particle pair with respect to the reaction plane. In turn, the joint probability distribution $f(p_{T1}, p_{T2}) = f(\varphi^{\text{pair}}, p_{T1}, p_{T2}, \Delta\varphi_{12})$ depends on the “pair angle” $\varphi^{\text{pair}} \equiv (\varphi_1 + \varphi_2)/2$ as well. Given equation (1), computing the two-particle distribution is straightforward. Keeping only the second-harmonic modulation (“elliptic flow”) of the single-particle distributions, one finds

$$f(\varphi^{\text{pair}}, p_{T1}, p_{T2}, \Delta\varphi_{12}) = \frac{1}{2\pi} \left[ 1 + 2v_{2,c}^{\text{pair}} \cos(\varphi^{\text{pair}} - \Phi_R) + 2v_{2,s}^{\text{pair}} \sin(2(\varphi^{\text{pair}} - \Phi_R) + \cdots \right],$$

(2)

where the Fourier “pair-flow” coefficients [5] that characterise the azimuthal dependence of $f(p_{T1}, p_{T2})$ depend on $p_{T1}, p_{T2}$, and $\Delta\varphi_{12}$. Considering only terms up to $O(v_2/N)$ and $O(v_2^3)$:

$$v_{2,c}^{\text{pair}} (p_{T1}, p_{T2}, \Delta\varphi_{12}) \approx [1 - 2v_2(1)v_2(2) \cos(4\Delta\varphi_{12})] [v_2(1) + v_2(2)] \cos(2\Delta\varphi_{12}) \frac{p_{T1}p_{T2}}{N(p_{T1})^2} \bar{v}_2,$$

(3)

$$v_{2,s}^{\text{pair}} (p_{T1}, p_{T2}, \Delta\varphi_{12}) \approx [1 - 2v_2(1)v_2(2) \cos(4\Delta\varphi_{12})] [v_2(1) - v_2(2)] \sin(2\Delta\varphi_{12}),$$

(4)

where $v_2(1), v_2(2)$ are shorthand notations for $v_2(p_{T1})$ and $v_2(p_{T2})$, respectively. There are also higher harmonic terms ($v_{4,c}^{\text{pair}}, v_{4,s}^{\text{pair}}, \cdots$), which are however smaller by at least a factor $v_2$. The effect of momentum conservation is actually subleading in the sine term (4), which reflects the non-invariance of the system under the $\varphi^{\text{pair}} - \Phi_R \rightarrow -(\varphi^{\text{pair}} - \Phi_R)$ symmetry when the particles in the pair are different and/or have different transverse momenta [5], and is mostly due to anisotropic flow. On the other hand, momentum conservation affects the cosine term (3), which is non-zero even for particles with vanishing elliptic flow. Since $\bar{v}_2 > 0$ at relativistic energies, $v_{2,c}^{\text{pair}}$ is positive for small-angle ($|\Delta\varphi_{12}| \leq \pi/4$) or large-angle ($|\Delta\varphi_{12}| \geq 3\pi/4$) pairs. This means that the yield of such pairs is larger in the reaction plane ($\varphi^{\text{pair}} \approx \Phi_R$) than perpendicular to it. Since $v_2(p_{T1})$ grows with increasing transverse momentum, as does the second term in the right-hand of equation (3), $v_{2,c}^{\text{pair}}$ increases with both $p_{T1}$ and $p_{T2}$, i.e. the anisotropy in the pair yield increases with the particle transverse momenta. Stated differently, for a pair of particles close in azimuth ($\Delta\varphi_{12}$ close to 0), the anticorrelation (1) is smaller (resp. larger) when the pair azimuth is along (resp. perpendicular to) $\Phi_R$, so that the pair yield is less (resp. more) “suppressed” by momentum conservation: there are overall more particles to balance the pair momentum along $\Phi_R$ than out-of-plane. Conversely, for a pair of back-to-back particles ($\Delta\varphi_{12} \approx \pi$) momentum correlation induces a positive correlation (1), which is larger if both particles lie along the reaction plane (resulting in $\varphi^{\text{pair}} \approx \Phi_R \pm \pi$) than perpendicular to it.

Rephrasing the above in yet another manner, one can investigate the conditional probability to find an “associated” particle (transverse momentum $p_{T2}$) given a “trigger” particle ($p_{T1}$), by dividing the pair distribution $f(p_{T1}, p_{T2})$ by the single-particle distribution $f(p_{T1})$. One then finds that for a trigger along the reaction plane ($\varphi_1 \approx \Phi_R$), there is a higher probability for associated particles close or away ($\Delta\varphi_{12} \approx 0$ or $\pi$) in azimuth than around $\Delta\varphi_{12} \approx \pm \pi$. On the opposite, if the trigger escapes the system perpendicular to $\Phi_R$, the conditional probability

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4 This truncation of the expansion is driven by the respective values of $v_2$ and $N$ at SPS and RHIC, and is by no means mandatory.
to find an associated particle close or back-to-back to the trigger ($\Delta \phi_{12} \approx 0$ or $\pi$) is smallest. These rough features, entirely dictated by elliptic flow, are illustrated in figure 1, where the values $p_{T1} = 6$ GeV/$c$, $p_{T2} = 4$ GeV/$c$, $v_2(p_{T2}) = 0.2$, $\bar{v}_2 = 0.1$, $\langle p_T^2 \rangle = (500$ MeV/$c)^2$ and $N = 8000$ have been used. In further detail, for both in-plane or out-of-plane trigger particles, the probability for associated particles is higher away ($\Delta \phi_{12} \approx \pi$) than close ($\Delta \phi_{12} \approx 0$) to the trigger: this is the trademark of the back-to-back correlation (1) induced by momentum conservation.

Now, the aim of a correlation study would be to investigate this finer structure. For that purpose, one would like to “remove” the anisotropic-flow-induced pattern. Given the general expression of the conditional probability distribution (resp. joint probability distribution) as a function of the two-particle cumulant [4], the most natural — although experimentally highly challenging — recipe would be to divide $f(p_{T2}|p_{T1})$ [resp. $f(p_{T1}, p_{T2})$] by the single-particle distribution $f(p_{T2})$ [resp. by the product $f(p_{T1})f(p_{T2})$], so as to isolate the correlation. Yet the usual procedure is rather to subtract a flow-modulated background, whose normalization is for instance fixed by requiring that the yield vanish at its minimum (“ZYAM” [9]). Following this approach, the “flow-subtracted distribution of associated particles” is pictured in figure 2 for both in- and out-of-plane trigger particles. As anticipated, this distribution is larger away from the trigger than close to it. Furthermore, the away-side probability is smaller when the trigger points perpendicular to the reaction plane than when it points along $\Phi_R$: this was also not unexpected given the shape of the two-particle probability distribution (2). However, the detailed structures are quite non-trivial, in particular the dip at $\Delta \phi_{12} = \pi$ in the case of an out-of-plane trigger, which both is a remnant of the “incompletely” subtracted anisotropic flow pattern.

5 This procedure does not entirely suppress the influence of the anisotropic expansion, since the latter affects the strength of the correlation according to the pair azimuth, see $\bar{v}_2$ in equation (1). However, inasmuch as it is experimentally feasible, it would indeed cancel the single-particle modulation induced by anisotropic flow, while leaving intact the azimuthal dependence of the two-particle cumulant.

6 If $f(p_{T2})$ had been divided from the conditional probability, instead of being subtracted from it, the resulting quotient would have been a smooth first-harmonic sinusoid, rather than the curve in figure 2.
The azimuthally-dependent behaviours described above follow solely from assuming 1. that the particle transverse momenta sum up to 0 and 2. that the particle emission is anisotropic: \( \langle p_T^2 \rangle > \langle p_T^y \rangle \). Those mere two ingredients are enough to give rise to non-trivial structures, which then have to be disentangled from those arising from additional physical effects. Two such patterns, whose analogues\(^7\) have reportedly been observed at RHIC, were identified: (a) a lower away-side \((\Delta \phi_{12} \approx \pi)\) probability of associated particles in the case of an out-of-plane trigger compared to an in-plane trigger \([10]\), and (b) a dip in the away-side yield of associated particles \([11, 12]\). These or similar structures have been predicted in models of parton energy loss that involve some path-length dependence \([13]\) [pattern (a)], or as signature of the interaction (“Mach cone”, “gluon Bresstrahlung”, “Cerenkov ring”, “jet deflection”, see e.g. reference \([12]\)) between the away-side parton and the medium through which it propagate [structure (b)]. If such models are to yield quantitative results, the possible contribution to the data of “trivial” correlations induced by momentum-conservation has to be investigated seriously.

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\(^7\) The qualitative behaviours discussed here are similar to those observed experimentally, but the quantitative aspects differ.