Hadwiger number and the Cartesian Product of Graphs

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Abstract

The Hadwiger number $\eta(G)$ of a graph $G$ is the largest integer $n$ for which the complete graph $K_n$ on $n$ vertices is a minor of $G$. Hadwiger conjectured that for every graph $G$, $\eta(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number of $G$. In this paper, we study the Hadwiger number of the Cartesian product $G \circ H$ of graphs.

As the main result of this paper, we prove that $\eta(G_1 \boxtimes G_2) \geq h \sqrt{7(1-o(1))}$ for any two graphs $G_1$ and $G_2$ with $\eta(G_1) = h$ and $\eta(G_2) = l$. We show that the above lower bound is asymptotically best possible. This asymptotically settles a question of Z. Miller (1978).

As consequences of our main result, we show the following:

1. Let $G$ be a connected graph. Let the (unique) prime factorization of $G$ be given by $G_1 \boxtimes G_2 \boxtimes \ldots \boxtimes G_k$.
   Then $G$ satisfies Hadwiger’s conjecture if $k \geq 2 \log \log \chi(G) + c'$, where $c'$ is a constant. This improves the $2 \log \chi(G) + 3$ bound in [3].

2. Let $G_1$ and $G_2$ be two graphs such that $\chi(G_1) \geq \chi(G_2) \geq c \log^{1.5}(\chi(G_1))$, where $c$ is a constant.
   Then $G_1 \boxtimes G_2$ satisfies Hadwiger’s conjecture.

3. Hadwiger’s conjecture is true for $G^d$ (Cartesian product of $G$ taken $d$ times) for every graph $G$ and every $d \geq 2$. This settles a question by Chandran and Sivadasan [2]. (They had shown that the Hadwiger’s conjecture is true for $G^d$ if $d \geq 3$.)

Keywords: Hadwiger Number, Hadwiger’s Conjecture, Graph Cartesian Product, Minor, Chromatic Number

1 Introduction

1.1 General definitions and notation

In this paper we only consider undirected simple graphs i.e., graphs without multiple edges and loops. For a graph $G$, we use $V(G)$ to denote its vertex set and $E(G)$ to denote its edge set.

A $k$-coloring of a graph $G(V, E)$ is a function $f : V \rightarrow \{1, 2, \ldots, k\}$. A $k$-coloring $f$ is proper if for all edges $(x, y) \in E$, $f(x) \neq f(y)$. A graph is $k$-colorable if it has a proper $k$-coloring. The chromatic number $\chi(G)$ is the least $k$ such that $G$ is $k$-colorable.

Let $S_1, S_2 \subset V(G)$, such that $S_1 \neq \emptyset$, $S_2 \neq \emptyset$ and $S_1 \cap S_2 = \emptyset$. We say that $S_1$ and $S_2$ are adjacent in $G$ if and only if there exists an edge $(u, v) \in E(G)$ such that $u \in S_1$ and $v \in S_2$. The edge $(u, v)$ is said to connect $S_1$ and $S_2$.

Contraction of an edge $e = (x, y)$ is the replacement of the vertices $x$ and $y$ with a new vertex $z$, whose incident edges are the edges other than $e$ that were incident to $x$ or $y$. The resulting graph denoted by $G.e$ may be a multigraph, but since we are only interested in simple graphs, we discard any parallel edges.

A minor $M$ of $G(V, E)$ is a graph obtained from $G$ by a sequence of contractions of edges and deletions of edges and vertices. We call $M$ a minor of $G$ and write $M \preceq G$.

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It is not difficult to verify that $M \cong G$ if and only if for each vertex $x \in V(M)$, there exist a set $V_x \subseteq V(G)$, such that (1) every $V_x$ induces a connected subgraph of $G$, (2) all $V_x$ are disjoint, and (3) for each $(x, y) \in E(M)$, $V_x$ is adjacent to $V_y$ in $G$.

The Hadwiger number $\eta(G)$ is the largest integer $h$ such that the complete graph on $h$ vertices $K_h$ is a minor of $G$. Since every graph on at most $h$ vertices is a minor of $K_h$, it is easy to see that $\eta(G)$ is the largest integer such that any graph on at most $\eta(G)$ vertices is a minor of $G$. Hadwiger conjectured the following in 1943.

**Conjecture:** (Hadwiger [6]) For every graph $G$, $\eta(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number of $G$.

In other words, Hadwiger’s conjecture states that if $\eta(G) \leq k$, then $G$ is $k$-colorable. It is known to hold for small $k$. Graphs of Hadwiger number at most 2 are the forests. By a theorem of Dirac [5], the graphs with Hadwiger number at most 3 are the series-parallel graphs. Graphs with Hadwiger number at most 4 are characterized by Wagner [13]. The case $k = 4$ of the conjecture implies the Four Color Theorem because any planar graph has no $K_5$ minor. On the other hand, Hadwiger’s conjecture for the case $k = 4$ follows from the four color theorem and a structure theorem of Wagner [13]. Hadwiger’s conjecture for $k = 5$ was settled by Robertson et al. [14]. The case $k = 6$ onwards is still open.

Since Hadwiger’s conjecture in the general case is still open, researchers have shown interest to derive lower bounds for Hadwiger number in terms of the chromatic number. Mader [12] showed (improving an earlier result of Wagner [19]) that for any graph $G$, $\eta(G) \geq \frac{\chi(G)}{\log(\chi(G))}$. Later, Kostochka [9] and Thomason [17] independently showed that there exists a constant $c$, such that for any graph $G$, $\eta(G) \geq \frac{\chi(G)}{c \sqrt{\log(\chi(G))}}$.

It is also known that Hadwiger’s conjecture is true for almost all graphs on $n$ vertices.

Improving on previous results, graph products can be other views, Kühn and Osthus [11] showed that if the girth (i.e., the length of a shortest cycle) is at least $g$ for some odd $g$ and the minimum degree $\delta$ is at least 3, then $\eta(G) \geq \frac{\sqrt{d}^{(\delta + 1)/4}}{\sqrt{\log(\delta)}}$. As a consequence of this result, Kühn and Osthus [11] showed that Hadwiger’s conjecture is true for $C_4$-free graphs of sufficiently large chromatic number. (Here $C_4$ denotes a cycle of length 4)

### 1.2 The Cartesian product of graphs

Let $G_1$ and $G_2$ be two undirected graphs, where the vertex set of $G_1$ is $\{0, 1, \ldots, n_1 - 1\}$ and the vertex set of $G_2$ is $\{0, 1, \ldots, n_2 - 1\}$. The **Cartesian product**, $G_1 \Box G_2$, of $G_1$ and $G_2$ is a graph with the vertex set $V = \{0, 1, \ldots, n_1 - 1\} \times \{0, 1, \ldots, n_2 - 1\}$ and the edge set defined as follows. There is an edge between vertices $(i, j)$ and $(i', j')$ of $V$ if and only if either $j = j'$ and $(i, i') \in E(G_1)$, or $i = i'$ and $(j, j') \in E(G_2)$.

In other words, graph products can be viewed in the following way: let the vertices of $G_1 \Box G_2$ be partitioned into $n_2$ classes $W_1, \ldots, W_{n_2}$, where $W_j = \{\langle 1, j \rangle, \ldots, \langle n_1, j \rangle\}$ induces a graph that is isomorphic to $G_1$, where the vertex $\langle i, j \rangle$ corresponds to vertex $i$ of $G_1$. If edge $(j, j')$ belongs to $G_2$ then the edges between classes $W_j$ and $W_{j'}$ form a matching such that the corresponding vertices, i.e., $\langle i, j \rangle$ and $\langle i, j' \rangle$, are matched. If edge $(j, j')$ is not present in $G_2$ then there is no edge between $W_j$ and $W_{j'}$.

It is easy to verify that the Cartesian product is a commutative and associative operation on graphs. Due to the associativity, the product of graphs $G_1, \ldots, G_k$ can be simply written as $G_1 \Box \ldots \Box G_k$ and has the following interpretation. If the vertex set of graph $G_1$ is $V_1 = \{1, \ldots, n_1\}$, then $G_1 \Box \ldots \Box G_k$ has the vertex set $V = V_1 \times V_2 \times \ldots \times V_k$. There is an edge between vertex $\langle i_1, \ldots, i_k \rangle$ and vertex $\langle i'_1, \ldots, i'_k \rangle$ of $V$ if and only if there is a position $t, 1 \leq t \leq k$, such that $i_t = i'_t$, $i_2 = i'_2 \ldots, i_{t-1} = i'_{t-1}$, $i_{t+1} = i'_{t+1}, \ldots, i_k = i'_k$, and the edge $\langle i_t, i'_t \rangle$ belongs to graph $G_t$.

We denote the product of graph $G$ taken $k$ times as $G^k$. It is easy to verify that if $G$ has $n$ vertices and $m$ edges, then $G^k$ has $n^k$ vertices and $m n^k - n^{k-1}$ edges.

Well known examples of Cartesian products of graphs are the $d$-dimensional hypercube $Q_d$, which is isomorphic to $K_2^d$, and a $d$-dimensional grid, which is isomorphic to $P_n^d$, where $P_n$ is a simple path on $n$ vertices.
**Unique Prime Factorization (UPF) of graphs:** A graph \( P \) is prime with respect to the Cartesian product operation if and only if \( P \) has at least two vertices and it is not isomorphic to the product of two non-identity graphs, where an identity graph is the graph on a single vertex and having no edge. It is well-known that every connected undirected graph \( G \) with at least two vertices has a UPF with respect to Cartesian product in the sense that if \( G \) is not prime then it can be expressed in a unique way as a product of prime graphs\(^1\). If \( G \) can be expressed as the product \( G_1 \square G_2 \square \ldots \square G_k \), where each \( G_i \) is prime, then we say that the product dimension of \( G \) is \( k \). The UPF of a given connected graph \( G \) can be found in \( O(m \log(n)) \) time, where \( m \) and \( n \) are the number of edges and number of vertices of \( G \) respectively \(^1\).

Imrich and Klavžar have published a book \(^7\), dedicated exclusively to the study of graph products. Readers who are interested to get an introduction to the wealth of profound and beautiful results on graph products are referred to this book.

We will use the following result by Sabidussi \(^4\) (which was rediscovered several times).

**Lemma 1.** \( \chi(G_1 \square G_2) = \max\{\chi(G_1), \chi(G_2)\} \).

### 1.3 Our results

The question of studying the Hadwiger number with respect to the Cartesian product operation was suggested by Miller in the open problems section of a 1978 paper \(^13\). He mentioned a couple of special cases (such as \( \eta(C_n \square K_2) = 4 \) and \( \eta(T \square K_n) = n + 1 \), where \( C_n \) and \( T \) denote a cycle and a tree respectively) and left the general case open. In this paper, we answer this question asymptotically. We give the following results.

**Result 1.** Let \( G_1 \) and \( G_2 \) be two graphs with \( \eta(G_1) = k_1 \) and \( \eta(G_2) = k_2 \). Then \( \eta(G_1 \square G_2) \geq k_1 \sqrt{k_2} (1 - o(1)) \). (Since the Cartesian product is commutative, we can assume without loss of generality that \( k_1 \geq k_2 \)). We demonstrate that this lower bound is asymptotically best possible.

We also show that in general, \( \eta(G_1 \square G_2) \) does not have any upper bound that depends only on \( \eta(G_1) \) and \( \eta(G_2) \), by demonstrating graphs \( G_1 \) and \( G_2 \) such that \( \eta(G_1) \) and \( \eta(G_2) \) are bounded, whereas \( \eta(G_1 \square G_2) \) grows with the number of vertices of \( G_1 \square G_2 \).

**Remark.** Note that if the average degrees of \( G_1 \) and \( G_2 \) are \( d_1 \) and \( d_2 \) respectively, then the average degree of \( G_1 \square G_2 \) is \( d_1 + d_2 \). In comparison, by Result 1, the Hadwiger number of \( G_1 \square G_2 \) grows much faster.

Hadwiger’s conjecture for Cartesian products of graphs was studied in \(^2\). There it was shown that if the product dimension (number of factors in the unique prime factorization of \( G \)) is \( k \), then Hadwiger’s conjecture is true for \( G \) if \( k \geq 2 \log \chi(G) + 3 \). As a consequence of Result 1, we are able to improve this bound. We show the following.

**Result 2.** Let the (unique) prime factorization of \( G \) be \( G = G_1 \square G_2 \square \cdots \square G_k \). Then Hadwiger’s conjecture is true for \( G \) if \( k \geq 2 \log(\log(\chi(G))) + c' \), where \( c' \) is a constant.

Another consequence of Result 1 is that if \( G_1 \) and \( G_2 \) are two graphs such that \( \chi(G_2) \) is not “too low” compared to \( \chi(G_1) \), then Hadwiger’s conjecture is true for \( G_1 \square G_2 \). More precisely:

**Result 3.** If \( \chi(G_2) \geq c \log^{1.5}(\chi(G_1)) \), where \( c \) is a constant, then Hadwiger’s conjecture is true for \( G_1 \square G_2 \).

It is easy to see that Result 3 implies the following: Let \( G_1 \) and \( G_2 \) be two graphs such that \( \chi(G_1) = \chi(G_2) \). (For example, as in the case \( G_1 = G_2 \)). Then \( G_1 \square G_2 \) satisfies Hadwiger’s conjecture if \( \chi(G_1) = \chi(G_2) = t \) is sufficiently large. (\( t \) has to be sufficiently large, because of the constant \( c \) involved in Result 3). For this special case, namely \( \chi(G_1) = \chi(G_2) \), we give a different proof (which does not depend on Result 1), to show that Hadwiger’s conjecture is true for \( G_1 \square G_2 \). This proof does not require that \( \chi(G_1) \) be sufficiently large.

**Result 4.** Let \( G_1 \) and \( G_2 \) be any two graphs such that \( \chi(G_1) = \chi(G_2) \). Then Hadwiger’s conjecture is true for \( G_1 \square G_2 \).
It was shown in [2] that Hadwiger’s conjecture is true for $G^d$, where $d \geq 3$, for any graph $G$. As a consequence of Result 4, we are able to sharpen this result.

Result 5. For any graph $G$ and every $d \geq 2$, Hadwiger’s Conjecture is true for $G^d$.

Another author who studied the minors of the Cartesian product of graphs is Kotlov [10]. He showed that for every bipartite graph $G$, the strong product $(G \boxtimes K_2)$ is a minor of $G \Box C_4$. ($K_2$ and $C_4$ are an edge and a 4-cycle respectively). As a consequence of this he showed that $\eta(K_2^d) \geq 2^{d-1}$. 

2 Hadwiger Number for $G_1 \square G_2$

2.1 Lower bound on $\eta(G_1 \square G_2)$

The following Lemma is not difficult to prove.

Lemma 2. [2] If $M_1 \subseteq G_1$ and $M_2 \subseteq G_2$, then $M_1 \square M_2 \subseteq G_1 \square G_2$.

Let $G_1$ and $G_2$ be two graphs such that $\eta(G_1) = h$ and $\eta(G_2) = l$, with $h \geq l$. In this section we show that $\eta(G_1 \square G_2) \geq h\sqrt{l}(1 - o(1))$. Since by Lemma 2, $K_h \square K_l \subseteq G_1 \square G_2$ it is sufficient to prove that $\eta(K_h \square K_l) \geq h\sqrt{l}(1 - o(1))$.

Definition 1. An affine plane $A$ of order $m$ is a family $\{A_{i,t} : \ i = 1, \ldots, m + 1, \ t = 1, \ldots, m\}$ of $m$-elements subsets of an $m^2$-element set $A$ such that

$$|A_{i,t} \cap A_{i',t'}| = \begin{cases} 1 & \text{if } i' \neq i; \\ 0 & \text{if } i' = i \text{ and } t' \neq t. \end{cases}$$

The sets $A_{i,t}$ are the lines of $A$. By definition, for each $i$, the sets $A_{i,1}, A_{i,2}, \ldots, A_{i,m}$ are disjoint and form a partition of $A$.

The following fact is widely known (see, e.g., [15]):

Lemma 1. If $m$ is a prime power, then there exists an affine plane of order $m$.

It is also known that the set of prime numbers is quite dense. The following is a weakening of the result of Iwaniwicz and Pintz [8].

Lemma 2. For every sufficiently large positive $x$, the interval $[x, x + x^{0.6}]$ contains a prime number.

Corollary 1. For every sufficiently large positive $x$, the interval $[x - 6x^{0.9}, x]$ contains a number of the form $(p(p + 1))^2$, where $p$ is some prime.

Theorem 1. $\eta(K_h \square K_l) \geq h\sqrt{l}(1 - o(1))$.

Proof. Consider $G = K_h \square K_l$, where $h \geq l$ and $l$ is large. Let $p$ be the maximum prime such that $l \geq (p(p + 1))^2$. We view $K_h \square K_l$ as a set of $h$ copies of $K_l$. Suppose that $s(p - 1)(2p + 1)/2 \leq h < (s + 1)(p - 1)(2p + 1)/2$. We neglect some $h - s(p - 1)(2p + 1)/2$ copies of $K_l$ and partition the remaining $s(p - 1)(2p + 1)/2$ copies into $s$ large groups of the same size, and each of these groups into $(p - 1)/2$ groups of size $2p + 1$. In other words, we consider $S = \{K_l(i,j,m) : i = 1, \ldots, s, j = 1, \ldots, (p - 1)/2, m = 1, \ldots, 2p + 1\}$, where each $K_l(i,j,m)$ is a copy of $K_l$.

For $i = 1, \ldots, s$, let $S_i = \{K_l(i,j,m) : j = 1, \ldots, (p - 1)/2, m = 1, \ldots, 2p + 1\}$.

In $S_i$, we will find $p^2(p - 1)(2p + 1)/2$ disjoint sets $M(1,j,m,t)$ ($j = 1, \ldots, (p - 1)/2, m = 1, \ldots, 2p + 1, t = 1, \ldots, p^2$) of size $(p + 1)^2$. These sets will have the property that

(a) the subgraph $G(M(1,j,m,t))$ induced by $M(1,j,m,t)$ is connected;

(b) for any two quadruples $(1,j,m,t)$ and $(1,j',m',t')$, there is a vertex $v$ in $K_l$ such that each of $M(1,j,m,t)$ and $M(1,j',m',t')$ contains a copy of $v$ (in different copies of $K_l$, since our sets are disjoint).

If we manage this, then copying these sets for every $i = 2, \ldots, s$, by (b), we will create $s p^2(p - 1)(2p + 1)/2 = p^2 h(1 - o(1))$ disjoint sets $M(i,j,m,t)$ that satisfy
(a') the subgraph $G(M(i, j, m, t))$ induced by $M(i, j, m, t)$ is connected;
(b') for any two quadruples $(i, j, m, t)$ and $(i', j', m', t')$, there is a vertex $v$ in $K_t$ such that each of $M(i, j, m, t)$ and $M(i', j', m', t')$ contains a copy of $v$

So, we go after (a) and (b).

To achieve this, we view the set of vertices of each $K_t$ as the disjoint union of a "big" square $Q_0$ of size $p^2 \times p^2$ with $2p + 1$ "small" squares $Q_k, k = 1, \ldots, 2p + 1$ of size $p \times p$ and the reminder $R$ (of size $l = p^4 - (2p + 1)p^2 = l - (p(p + 1))^2$) (see Fig. 1).

By Lemma 2 there exists an affine plane of order $p^2$. We consider the lines $A_{i,t}$ of this plane as subsets of the big square $Q_0$. For $i = 1, \ldots, (p - 1)(2p + 1)/2$, we view $\{A_{i,1}, A_{i,2}, \ldots, A_{i,p^2}\}$ as a partition of the copy $Q_0(i)$ of the big square $Q_0$. If $i = (j - 1)(2p + 1) + m$, then the set $A_{i,t}$ will be the main part of the future set $M(1, j, m, t)$. All $A_{i,t}$ lies in one copy of $K_t$ and so $G(A_{i,t})$ is connected. By the definition of the affine plane, if $i' = (j' - 1)(2p + 1) + m'$ and $i' \neq i$, then the sets $A_{i,t}$ and $A_{i',t}$ intersect. Our goal now is to add $2p + 1$ vertices to each of $A_{(j-1)(2p+1)+m,t}$ to provide (b) for $M(1, j, m, t)$ and $M(1, j', m', t')$ only for the same $j$ and $m$.

Let us fix $j$ and $m$. For every $t = 1, \ldots, p^2$, the set $M(1, j, m, t)$ will be obtained from $A_{(j-1)(2p+1)+m,t}$ by adding a $(2p + 1)$-element subset of $\bigcup_{r=1}^{2p+1} Q_m(j, r)$, where $Q_m(j, r)$ is the copy of $Q_m$ that is contained in $K_t(1, j, r)$. Every $t = 1, \ldots, p^2$ can be written in the form $t = (a_1 - 1)p + a_2$, where $1 \leq a_1, a_2 \leq p$. So, we include into $M(1, j, m, t)$ the entry $(a_1, a_2)$ of the square $Q_m(j, m)$. We call this vertex $F(a_1, a_2, j, m)$. Since $F(a_1, a_2, j, m)$ is in the same copy of $K_t$ as $A_{(j-1)(2p+1)+m,t}$, it is adjacent to every vertex in this set. Let $C_a(j, r, m)$ and $R_a(j, r, m)$ denote the $r$th column and the $a$th row of the square $Q_m(j, r)$, respectively. If $t = (a_1 - 1)p + a_2$, then our set $M(1, j, m, t)$ will consist of $A_{(j-1)(2p+1)+m,t}$ the vertex $F(a_1, a_2, j, m)$, the row $R_{a_1}(j, m + a_2, m)$ and the column $C_{a_2}(j, m - a_1, m)$, where the values $m + a_2$ and $m - a_1$ are calculated modulo $2p + 1$. Since $F(a_1, a_2, j, m)$ is adjacent to the $a_2$-s entry of the row $R_{a_1}(j, m + a_2, m)$ and to the $a_1$-s entry of the column $C_{a_2}(j, m - a_1, m)$, condition (a) holds. Since the projection on $Q_m$ of $R_{a_1}(j, m + a_2, m) \cup C_{a_2}(j, m - a_1, m)$ is a cross, (b) also holds.

This finishes the construction. It implies that the Hadwiger number of $G = K_h \Box K_t$, with $h \geq l$ is at least

$$\frac{h}{(p - 1)(2p + 1)/2} p^2(p - 1)(2p + 1)/2 = (h - O(p^2))p^2.$$  

By Corollary 2, $p^2 = (1 - o(1))\sqrt{T}$. Hence the result.  

The following Theorem is an immediate consequence of Theorem 1

**Theorem 2.** Let $G_1, G_2$ be any two graphs with $\eta(G_1) = h, \eta(G_2) = l$ and $\eta(G_1) \geq \eta(G_2)$. Then $\eta(G_1 \Box G_2) = \eta(G_2 \Box G_1) \geq h\sqrt{T}(1 - o(1))$. 

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2.2 Tightness of the lower bound

Let $K_n$ and $K_m$ be the complete graphs on $n$ and $m$ vertices respectively ($n \geq m$) and let $h$ be the maximum number such that $K_h \leq K_n \sqcup K_m$. Let the sets $V_0, \ldots, V_{h-1} \subseteq V(K_n \sqcup K_m)$ be the pre-images of vertices of $K_h$ in $K_n \sqcup K_m$. Thus, the vertex sets $V_0, \ldots, V_{h-1}$ are pairwise disjoint and pairwise adjacent. Moreover, $V_i$ ($0 \leq i \leq h - 1$) induces a connected subgraph in $K_n \sqcup K_m$.

Without loss of generality let $n_0 = |V_0| = \min_{0 \leq i \leq h-1} |V_i|$. Thus $n_0 \leq \frac{nm}{h}$. For $S \subseteq V(K_m \sqcup K_n)$ let $N(S) = \bigcup_{u \in S} N(u) - S$. (Here $N(u)$ denote the neighbors of $u$ in $K_n \sqcup K_m$.) Since $K_h$ is a complete graph minor of $K_m \sqcup K_n$, we have:

$$|N(V_0)| \geq h - 1$$ (1)

Since $V_0$ induces a connected graph in $K_n \sqcup K_m$, the vertices of $V_0$ can be ordered as $v_1, \ldots, v_{n_0}$ such that for $2 \leq j \leq n_0$, $v_j$ is adjacent to at least one of the vertices in $\{v_1, \ldots, v_{j-1}\}$. Let us define a sequence of sets $\emptyset = X_0, X_1, \ldots, X_{n_0} = V_0$ by setting $X_j = X_{j-1} \cup \{v_j\}$, for $1 \leq j \leq n_0$. Clearly, $|N(X_j)| = n + m - 2$. We claim that $|N(X_j)| \leq |N(X_{j-1})| + n - 2$, for $2 \leq j \leq n_0$. To see this, recall that $v_j$ is adjacent to at least one vertex $v_k \in X_{j-1}$. Clearly, out of the $n + m - 2$ neighbors of $v_j$, at least $m - 2$ are neighbors of $v_k$ also, and thus are already in $N(X_{j-1})$. Now, accounting for $v_j$ and $v_k$ also, we have $|N(X_j)| \leq |N(X_{j-1})| + n - 2$, as required. Thus we get $|N(V_0)| = |N(X_{n_0})| \leq n + m - 2 + (n_0 - 1)(n - 2) \leq n + m - 2 + (\frac{nm}{h} - 1)(n - 2)$. Combining this with Inequality (1) we get:

$$n + m - 2 + \left(\frac{nm}{h} - 1\right)(n - 2) \geq h - 1$$

It is easy to verify that if $h > n\sqrt{m} + m$, the above inequality will not be satisfied. So we infer that $h \leq n\sqrt{m} + m$. (Recalling $n \geq m$, the upper bound tends to $n\sqrt{m}$ asymptotically.)

2.3 Nonexistence of an upper bound that depends only on $\eta(G_1)$ and $\eta(G_2)$

We have seen that if we take $G_1$ and $G_2$ as the complete graphs on $k_1$ and $k_2$ vertices respectively ($k_1 \geq k_2$), then $\eta(G_1 \sqcup G_2) \leq k_1\sqrt{k_2} + k_2$ for some constant $c$. It is very natural to ask the following question. Let $G_1$ and $G_2$ be two arbitrary graphs with $\eta(G_1) = k_1$ and $\eta(G_2) = k_2$. Then does there exists a function $f : N \times N \rightarrow N$, such that $\eta(G_1 \sqcup G_2) \leq f(k_1, k_2)$? In this section we demonstrate that in general such a function cannot exist.

Definition 2. (Grid) An $n \times n$ grid is a graph with the vertex set $V = \{1, \ldots, n\} \times \{1, \ldots, n\}$. Nodes $(i, j)$ and $(i', j')$ are adjacent if and only if $|i - i'| + |j - j'| = 1$. Note that, an $n \times n$ grid (which can be viewed as the adjacency graph on an $n \times n$ chessboard) has $n$ rows and $n$ columns, where $i$th row is the induced path on the vertex set $\{i, 1, \ldots, i, n\}$ and $j$th column is the induced path on the vertex set $\{1, j, \ldots, i, n\}$.

Definition 3. (Double-grid) An $n \times n$ double-grid is obtained by taking two $n \times n$ grids and connecting the identical vertices (vertices with identical labels) from the two grids by an edge.

Let $R_n$ be an $n \times n$ grid. It is easy to see that $R_n$ is a planar graph and hence $\eta(R_n) \leq 4$. By the definition of Cartesian product, $R_n \sqcup K_2$ is an $n \times n$ Double-grid. It was proved in [2] that the Hadwiger number of an $n \times n$ double-grid is at least $n$. (We give here a sketch of their proof.) Let $G_1$ and $G_2$ be the two grids of the double grid $R_n \sqcup K_2$. Observe that there is an edge between any “row” of $G_1$ and any “column” of $G_2$. Contracting all the rows of $G_1$ and all the columns of $G_2$ we get a complete bipartite graph $K_{n,n}$, from which we easily obtain a $K_n$ minor

Thus, $\eta(R_n \sqcup K_2) \geq n$, while $\eta(R_n) \leq 4$ and $\eta(K_2) = 2$. This example shows that in general there is no upper bound on $\eta(G_1 \sqcup G_2)$ which depends only on $\eta(G_1)$ and $\eta(G_2)$. 

6
2.4 Consequences of Theorem 2. Hadwiger’s conjecture for graph products

2.4.1 In terms of chromatic number

Theorem 2 naturally leads us to the following question: Let $G_1$ and $G_2$ be any two graphs with $\chi(G_1) = k_1$ and $\chi(G_2) = k_2$, where $k_1 \geq k_2$. Let $f(k_1)$ be such that if $k_2 \geq f(k_1)$, Hadwiger’s conjecture is true for $G_1 \Box G_2$. In fact Hadwiger’s conjecture states that $f(k_1) = 1$. Since Hadwiger’s conjecture in the most general case, seems to be hard to prove, it is interesting to explore how small we can make $f(k_1)$, so that the conjecture can still be verified, for $G_1 \Box G_2$. To obtain a bound on $f(k_1)$, we need the following result, proved by Kostochka [8] and Thomason [16], independently.

**Lemma 3.** For any graph $G$, $\eta(G) \geq \frac{2^{2\chi(G)}}{\sqrt{\log \chi(G)}}$ where $c_2$ is a constant.

As a consequence of Theorem 2, we have the following result.

**Theorem 3.** Let $G_1$ and $G_2$ be any two graphs. There exists a constant $c'$ such that if $\chi(G_1) \geq \chi(G_2) \geq c' \log^{1.5}(\chi(G_1))$, then Hadwiger’s conjecture is true for $G_1 \Box G_2$.

**Proof.** Let $k_1 = \chi(G_1)$ and $k_2 = \chi(G_2)$. Applying Lemma 3 and Theorem 2 and noting that $\sqrt{\log(k_2)} \leq (\sqrt{\log(k_1)})^{0.5}$, we have

$$\eta(G_1 \Box G_2) \geq c_1 c_2 \frac{k_1}{(\sqrt{\log(k_1)})^{1.5}}$$

Now taking $c' = \frac{1}{c_1 c_2^{1.5}}$, we get $\eta(G_1 \Box G_2) \geq k_1 = \chi(G_1 \Box G_2)$. The latter equality follows from Lemma 2.

2.4.2 In terms of product dimension

Recall that the product dimension of a connected graph $G$ is the number of prime factors in its (unique) prime factorization. It was shown in [2] that if the product dimension of $G$ is at least $2 \log \chi(G) + 3$, then Hadwiger’s conjecture is satisfied for $G$. Using Theorem 2, we can bring this bound to $2 \log \chi(G) + c'$, where $c'$ is a constant. The following Lemma proved in [2] gives a lower bound for the Hadwiger number of the $d$-dimensional hypercube, $H_d$.

**Lemma 4.** $\eta(H_k) \geq 2^{(k-1)/2} \geq 2^{(k-2)/2}$

**Theorem 4.** Let $G$ be a connected graph and let the (unique) prime factorization of $G$ be $G = G_1 \Box G_2 \Box \ldots \Box G_k$. Then there exists a constant $c'$, such that Hadwiger’s conjecture is true for $G$, if $k \geq 2 \log \chi(G) + c'$

**Proof.** Let $c' = 4 \log_{\frac{1}{c_1 c_2^{1.5}}}$, where $c_1$ and $c_2$ are the constants from Theorem 2 and Lemma 3 respectively.

We may assume that $\chi(G_1) \geq \chi(G_i)$, for all $i > 1$. By Lemma 2 $\chi(G) = \max\{\chi(G_1), \chi(G_2), \ldots, \chi(G_k)\}$.

Let $X = G_2 \Box G_3 \Box \cdots \Box G_k$. Since $G$ is connected, each $G_i$ is also connected. Moreover, $G_i$ has at least two vertices (and hence at least one edge) since $G_i$ is prime. It follows that the $(k - 1)$-dimensional hypercube is a minor of $X$. Thus by Lemma 2, $\eta(X) \geq \eta(H_{k-1}) \geq 2^{(k-3)/2} \geq 2^{\log \chi(G) + 2 \log \frac{1}{c_1 c_2^{1.5}}}$.

Applying Theorem 2 to $G_1 \Box X$, we get

$$\eta(G) = \eta(G_1 \Box X) \geq c_1 \eta(G_1) \sqrt{\eta(X)}$$

Recalling that (by Lemma 3), $\eta(G_1) \geq \frac{c_2 \chi(G_1)}{\sqrt{\log \chi(G_1)}}$, we get $\eta(G) \geq \chi(G)$.  

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1. From Theorem 2 we have $\eta(G_1 \Box G_2) \geq c_1 \chi(G_1)$, where $c_1$ is a constant.
2. $c_1, c_2 \leq 1$. So $\frac{1}{c_1 c_2} \geq 1$.
3 Hadwiger’s Conjecture for $G_1 \square G_2$ when $\chi(G_1) = \chi(G_2)$

Theorem 3 implies the following. Let $G_1$ and $G_2$ be two graphs such that $\chi(G_1) = \chi(G_2)$. Then $G_1 \square G_2$ satisfies Hadwiger’s conjecture if $\chi(G_1) = \chi(G_2) = t$ is sufficiently large. ($t$ has to be sufficiently large, because of the constant $c'$ involved in Theorem 3). In this section we give a different proof for this special case. We show that irrespective of the value of $t (= \chi(G_1))$, $G_1 \square G_2$ satisfies Hadwiger’s conjecture if $\chi(G_1) = \chi(G_2)$.

A graph $G$ is said to be $k$-critical if and only if $\chi(H) < \chi(G)$ for every proper subgraph $H$ of $G$. Every $k$-chromatic graph has a $k$-critical subgraph in it, obtained by greedily removing as many vertices and edges as possible from $G$, such that the chromatic number of the resulting graph remains the same.

We need the following two Lemmas, the proofs of which can be found in [20].

Lemma 5. If $G$ is a $k$-critical graph, then the minimum degree of $G$, $\delta(G) \geq k - 1$

Lemma 6. Let $G$ be a graph with minimum degree $\delta$. Then $G$ contains a simple path on at least $\delta + 1$ vertices.

We use $W_n$ to denote the graph whose vertex set is $\{0, 1, ..., n - 1\}$ with an edge defined between two vertices $i$ and $j$ (assuming $i < j$) if and only if either $i = 0$ or $j = i + 1$. $W_n$ is essentially a simple path on $n$ vertices, with the extra property that vertex 0 is adjacent to all the other vertices. An illustration of $W_n$ is given in Figure 1.

![Illustration of $W_n$](image)

**Figure 1: Illustration of $W_n$**

Lemma 7. Every $k$-chromatic graph $G$ has $W_k$ as a minor.

**Proof.** Let $H$ be a $k$-critical subgraph of $G$. By Lemma 5 $\delta(H) \geq k - 1$. Let $P = (v_0, v_1, \ldots, v_{l-1})$ be the longest simple path in $H$. By Lemma 6 $l \geq k = \chi(G)$. Let $N(v_0)$ denote the set of neighbors of $v_0$ in $H$. i.e., $N(v_0) = \{u \in V(H) - v_0 : (u, v_0) \in E(H)\}$. Since $P$ is the longest simple path, $N(v_0) \subseteq V(P) - \{v_0\} = \{v_1, v_2, \ldots, v_{l-1}\}$. Otherwise if $w \in N(v_0)$ and $w \notin V(P) - \{v_0\}$, then $(w, v_0, v_1, \ldots, v_{l-1})$ will be a longer simple path in $G$, contradicting the assumption that $P$ is the longest.

Let $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{k-1}}\} \subseteq V(P)$ be any $k - 1$ neighbors of $v_0$ in $H$, where $i_1 \leq i_2 \leq \ldots \leq i_{k-1}$. Consider the $k - 1$ sub-paths of $P$, from $v_0$ to $v_{i_1}$, from $v_{i_1}$ to $v_{i_2}$, \ldots, from $v_{i_{k-2}}$ to $v_{i_{k-1}}$. Contracting each sub-path to a single edge, we get $W_k$ as a minor of $G$. \hfill \Box

$W_n \square W_n$ is the graph with vertex set $V = \{0, 1, ..., n - 1\} \times \{0, 1, ..., n - 1\}$. By the definition of graph Cartesian product, vertices $(i, j)$ and $(i', j')$ are adjacent in $W_n \square W_n$ if and only if either $i = i'$ and $(j, j') \in E(W_n)$ or $j = j'$ and $(i, i') \in E(W_n)$. Thus $(i, j)$ and $(i', j')$ in $W_n \square W_n$ are adjacent if and only
if at least one of the following conditions hold.

(1) $i = i'$ and $j = j' ± 1$

(2) $i = i'$ and $j = 0$

(3) $i = i'$ and $j' = 0$

(4) $j = j'$ and $i = i' ± 1$

(5) $j = j'$ and $i = 0$

(6) $j = j'$ and $i' = 0$

**Lemma 8.** $K_n \preceq W_n \square W_n$.

**Proof.** For $0 \leq i \leq n - 1$, let $B_i \subseteq V(W_n \square W_n)$ be defined as $B_i = \{(i, 0), (i, 1), \ldots, (i, i - 1), (i, i), (i - 1, i), \ldots, (i, 0), (i, 1)\}$. The following properties hold for $B_i$.

1. For $i \neq j$, $B_i \cap B_j = \emptyset$. This follows from the definition of $B_i$.
2. Each $B_i$ induces a connected graph. This follows from the fact that $\langle (i, j), (i, j + 1) \rangle \in E(W_n \square W_n)$ and $\langle (j, i), (j - 1, i) \rangle \in E(W_n \square W_n)$, by the definition of $W_n \square W_n$.
3. For $i < j$, $B_i$ and $B_j$ are adjacent. This is because, $\langle (i, 0), (i, j) \rangle \in B_i$ and $\langle (i, 0), (i, j) \rangle \in B_j$ and $\langle (i, 0), (i, j) \rangle \in E(W_n \square W_n)$.

In other words, the sets $B_i$ are connected, disjoint and are pair-wise adjacent. Thus contracting each $B_i$ to a single vertex we get a $K_n$ minor.

**Theorem 5.** If $\chi(G) = \chi(H)$, then Hadwiger’s conjecture is true for $G \square H$.

**Proof.** Let $\chi(G) = \chi(H) = n$. By Lemma 8 we have $W_n \preceq G$ and $W_n \preceq H$. Now Lemma 2 implies $W_n \square W_n \preceq G \square H$. Since by Lemma 3 $K_n \preceq W_n \square W_n$, we have $K_n \preceq G \square H$. This together with Lemma 4 gives $\eta(G \square H) \geq n = \chi(G \square H)$, proving the Theorem.

It was shown in [2] that if a graph $G$ is isomorphic to $F_d^d$, for some graph $F$ and $d \geq 3$ then Hadwiger’s conjecture is true for $G$. The following improvement is an immediate consequence of Theorem 5 and Lemma 4.

**Theorem 6.** Let a graph $G$ be isomorphic to $F_d^d$ for some graph $F$ and for $d \geq 2$. Then Hadwiger’s conjecture is true for $G$.

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