HEEKAARD FLOER HOMOLOGIES OF (+1) SURGERIES ON TORUS KNOTS

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Abstract. We compute the Heegaard Floer homology of $S^3_1(K)$ (the (+1) surgery on the torus knot $T_{p,q}$) in terms of the semigroup generated by $p$ and $q$, and we find a compact formula (involving Dedekind sums) for the corresponding Ozsváth–Szabó $d$-invariant. We relate the result to known knot invariants of $T_{p,q}$ as the genus and the Levine–Tristram signatures. Furthermore, we emphasize the striking resemblance between Heegaard Floer homologies of (+1) and (−1) surgeries on torus knots. This relation is best seen at the level of $\tau$ functions.

1. Introduction

Let $K \subset S^3$ be a knot. Let us consider $S^3_1(K)$, the (+1) surgery of $K$. The main goal of the present article is the determination of the Heegaard Floer homology $HF^+$ with $\mathbb{Z}$–coefficient of $S^3_1(K)$, when $K = T_{p,q}$ is the torus knot. Note that $S^3_1(K)$ is an integral homology sphere, hence its Heegaard Floer homology is concentrated in its unique $\text{spin}^c$–structure. Moreover, we provide several closed formulae for the correction term $d(K) := d(S^3_1(K))$ of Ozsváth–Szabó [OS2]. In fact, searching for such closed formulae, and the recent article of Peters [Pet] regarding several properties of $d(K)$ (see Section 1.2 below) was the motivation and the starting point of the present work.

1.1. The Heegaard Floer homology. Let us fix two relative prime integers $p$ and $q$. We set $\delta := \frac{(p-1)(q-1)}{2}$ and we define $\mathcal{S}_{pq}$ as the subsemigroup of $\mathbb{N}$ generated by $p$ and $q$ including 0 too. It is well–known that $\mathbb{N} \setminus \mathcal{S}_{pq}$ is finite of cardinality $\delta$, hence the integers

\[ \alpha_i := \# \{ s \notin \mathcal{S}_{pq} : s > i \} \quad \text{(1.1)} \]

are well defined. In fact, $\delta = \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{2\delta-2} = 1$, and $\alpha_i = 0$ for $i > 2\delta - 2$. These integers — or, equivalently, the semigroup $\mathcal{S}_{pq}$ — codifies the same amount of data as the Alexander polynomial $\Delta(t)$ of the knot, or the symmetrized Alexander polynomial $\Delta^\#(t) = t^{-\delta} \Delta(t)$ (see Section 6).

In the description of the Heegaard Floer homology we use (the already standard) notations of $\mathbb{Z}[U]$–modules, what is also recalled in Section 6.

Theorem 1.2. For the Seifert 3–manifold $-\Sigma = S^3_1(T_{p,q})$ one has

\[ HF^+_{odd}(-\Sigma) = 0, \quad HF^+_{even}(-\Sigma) = T^+_{d(-\Sigma)} \bigoplus_{k=0}^{\delta-2} T_{k(k+1)-2\alpha_{\delta+k}}(\alpha_{\delta+k})^{\oplus 2}, \]

\[ d(-\Sigma) = -2\alpha_{\delta-1}, \quad \text{and} \quad \text{rank}_\mathbb{Z} HF^+_{red}(-\Sigma) = \frac{1}{2}(\Delta^\#)'(1) - \alpha_{\delta-1}. \]
Remark 1.3. (a) If we write $\Delta^d(t) = a_0 + \sum_{j=1}^{\delta} a_j (t^j + t^{-j})$, then using the correspondence between the Alexander polynomial and the semigroup recalled in Section 7, one can show that $\alpha_{\delta-1} = \sum_{j \geq 1} j a_j$. In this way we recover the formula of Ozsváth and Szabó [OS1, Section 7]: $d(S^3_1(K)) = -2 \sum_{j \geq 1} j a_j$, but this time in terms of the semigroup $S_{pq}$, a fact which provides a new geometric interpretation. Note that the coefficients $\alpha_i$ enter in a substantial way in the reduced part of $HF^+_\infty$ as well.

(b) Since the Casson invariant $\lambda$ (or, the Seiberg–Witten invariant $sw$) of $-\Sigma$ satisfies $\lambda = sw = \text{rank}HF^\infty_{red} - d/2$, we get $\lambda(-\Sigma) = sw(-\Sigma) = \frac{1}{2}(\Delta^d)'(1)$, the classical surgery formula for $\lambda$, see [Le] and [AC].

1.2. The $d$–invariant. For any 3–manifold $M$, the collection of correction terms $d(M, \sigma)$ associated with its spin$^c$–structures capture very strong geometric information, and recently they provided many deep applications. If $K$ is a knot, and $M = S^3_1(K)$, then $d(K) := d(S^3_1(K))$ was intensively studied by Peters [Pet], who showed, among other things, that $d(K)$ is a concordance invariant and it provides a bound for four genus:

$$0 \leq -d(K) \leq 2g_4(K).$$

Furthermore, if $K'$ arises from $K$ by changing one negative crossing into a positive one on some diagram, one has the following relation

$$d(K) - 2 \leq d(K') \leq d(K).$$

It follows, in particular, that $-\frac{1}{2}(d(K) + d(mK)) \leq u(K)$, where $u(K)$ is the unknotting number and $mK$ is the mirror of $K$. Thus $d(K)$ bears a strong resemblance to the classical signature $\sigma(K)$ of $K$. Indeed, by [OS1], if $K$ is alternating then $d(K) = 2 \min \left(0, \left\lfloor \frac{\sigma(K)}{4} \right\rfloor \right)$.

In the present article we provide two further closed formulae of $d(T_{p,q})$. The first one is in terms of generalized Dedekind sums, the second one is in terms of Levine–Tristram signature of the torus knot evaluated at $\exp(2\pi i \frac{p}{pq})$.

Theorem 1.6. Set $c := 0$ if $\delta - 1 \in S_{pq}$, otherwise take $c := 1$. Furthermore, for two integers $a$ and $b$, $a \neq 0$ we write

$$\varepsilon_a(b) = \begin{cases} 1 & \text{if } a \mid b \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad \varepsilon_{p,q}(b) = \varepsilon_p(b) + \varepsilon_q(b).$$

Then

$$d(S^3_1(T_{p,q})) = -1 - \frac{(\delta - 1)(\delta - 2)}{pq} - \frac{1}{6pq}(p^2 + q^2 - 3p - 3q - 2) -$$

$$- \left\lfloor \frac{\delta}{p} \right\rfloor - \left\lfloor \frac{\delta}{q} \right\rfloor + c + \frac{1}{2} \varepsilon_{p,q}(\delta - 1) + 2s(p, q; \frac{\delta - 1}{q}, 0) + 2s(q, p; \frac{\delta - 1}{p}, 0).$$

where $s(p, q; x, 0)$ are the generalized Dedekind sums (see Section 2 below).

(For the second formula see Section 7.)

For $q = p + 1$ we obtain

$$d(S^3_1(T_{p,p+1})) = - \left\lfloor \frac{p}{2} \right\rfloor \left( \left\lfloor \frac{p}{2} \right\rfloor + 1 \right).$$

For $p = 2$ we get $d(S^3_1(T_{2,2\delta+1})) = -2 \left\lfloor \frac{\delta}{2} \right\rfloor$; since in this case $\sigma(K) = -2\delta$, this identity is compatible with the above formula of Ozsváth and Szabó valid for alternating knots.

Our approach permits us to find sharp inequalities for $d(K)$ valid for $K = T_{p,q}$. Recall that in this case $g_4 = \delta [KM]$. 

Corollary 1.7. (a) 
\[ -d(S_1^3(T_{p,q})) \leq g_4 + 1 \leq -\sigma(K). \]
The first inequality is equality for \( p = 2 \) and \( \delta \) odd.
(b) Assume that \( p < q \) and \( 2q - \frac{5}{2} \leq c\sqrt{\delta} \) for some constant \( c > 0 \). Then
\[ -d(S_1^3(T_{p,q})) \leq 2q - 2 + \frac{\delta - 1}{2} \leq \frac{\delta}{2} + c\sqrt{\delta} \leq \frac{-\sigma - 1}{2} + c\sqrt{-\sigma}. \]

Part (b) is ‘strong’ whenever \( p \) ‘grows together with \( q \’\). In this case we reobtain at least asymptotically the growth \(-\sigma/2\) valid for alternating knots.

We expect similar inequalities for \( d(S_1^3(K)) \) for any algebraic knot \( K \).

1.3. The methods and the structure of the article. The main tool of the proof is the \( \tau \) function associated with any Seifert (or ‘almost rational’) negative definite plumbed manifold [Nem2]. For this, we interpret \( S_1^3(T_{p,q}) \) as a Seifert manifold and run the algorithm of [loc.cit.]. We study the corresponding \( \tau \) function in the same way, as in [Nem3, Section 7] and are able to obtain its local maxima and minima (cf. Section 4). This data provides the corresponding ‘graded roots’ (cf. Nem2, Nem5), hence all the information regarding the Heegaard Floer homology (see Section 4).

We recall that Peters studies \( d_-(K) = d(S_1^2(K)) \) too. But, for any positive knot, \( d_-(K) = 0 \). (See also Proposition 6.1(b) below.) [This also follows from the fact \( d_-(K) = d(mK) \). Indeed, as \( mK \) can be unknotted by changing only negative crossings to positive ones, by (1.5) we get \( d(mK) \geq d(U) = 0 \), where \( U \) is the unknot. But, by (1.4), \( d(mK) \leq 0 \) as well.]

Although the qualitative behavior of \( S_1^2(T_{p,q}) \) and \( -S_1^3(T_{p,q}) \) at the level of \( d \)-invariants is different, we wish to stress that the \( \tau \) functions related to these two manifolds turn out to be strongly related to each other. This relation can be clearly seen at the level of \( \tau \) functions, but becomes less transparent when we pass to graded roots or the Heegaard Floer homologies. For more details, see Section 8. We are wondering, whether this ‘duality’ holds for a larger group of knots, e.g. for all algebraic knots. [As for general algebraic knot \( K \), neither \( S_1^3(K) \) nor \( -S_1^3(K) \) has negative definite plumbing graph, a possible generalization of this ‘duality’ might use an extension of the results of Nem3 for not necessarily negative definite graphs.]

In Section 7 we relate the invariants to Levine–Tristram signatures. Finally, in the last section we establish the inequalities of Corollary 1.7.

2. Preliminary results

First we recall some results from [Nem2, Section 11]. Let \( \Sigma = \Sigma(e_0, (\alpha_1, \omega_1), \ldots, (\alpha_\nu, \omega_\nu)) \) be a plumbed negative definite three–manifold. The notation means that the plumbing diagram is star-shaped, the central vertex has weight \( e_0 \) and there are \( \nu \) arms stemming from it. Furthermore, \( 0 < \omega_l < \alpha_l \), \( \gcd(\alpha_l, \omega_l) = 1 \) (\( 1 \leq l \leq \nu \)), and if we write the continued fraction
\[ \frac{\alpha_l}{\omega_l} = k_{l1} - \frac{1}{k_{l2} - \frac{1}{\ldots - \frac{1}{k_{ls}}}}, \]
with all \( k_{ls} \geq 2 \) then the \( l \)-th arm consists of a chain of \( s_l \) vertices with weights \( -k_{l1}, \ldots, -k_{ls} \).
The weight of the vertex closest to the central one is \( -k_{l1} \).

Let us introduce the quantities
\[ e = e_0 + \sum_{l=1}^{\nu} \frac{\omega_l}{\alpha_l} \text{ and } \varepsilon = \left( 2 - \nu + \sum_{l=1}^{\nu} \frac{1}{\alpha_l} \right)^{-1}. \]
and the notation $\alpha := \prod_l \alpha_l$ and $\hat{\alpha}_l := \alpha / \alpha_l$. We assume that $e < 0$. Observe that
\[ |H_1(\Sigma, \mathbb{Z})| = -e\alpha_1 \ldots \alpha_\nu, \]
in particular, $\Sigma$ is an integral homology sphere if and only if $e = -\frac{1}{\alpha_1 \ldots \alpha_\nu}$, that is
\[ (2.1) \quad -1 = e_0\alpha + \sum_l \omega_l\hat{\alpha}_l. \]
Hence the integers $\omega_l$‘s are determined by the $\alpha_l$‘s. Let us consider the canonical $spin^c$–structure $\sigma$ on $\Sigma$. We have the following fact.

**Proposition 2.2** (see [Nem2, Theorems 11.9, 11.12 and 11.13]). The $d$ invariant of $\Sigma = (\Sigma, \sigma)$ is given by the following formula

\[ (2.3) \quad d(\Sigma) = \frac{1}{4}(K^2 + s) - 2 \min_{m \geq 0} \tau(m), \]
where
\[ (2.4) \quad K^2 + s = e^2 e + e + 5 - 12 \sum_{l=1}^\nu s(\omega_l, \alpha_l), \]
and $\tau(m)$ is a function defined by
\[ (2.5) \quad \tau(m) = \sum_{j=0}^{m-1} \Delta_j, \quad \text{where} \quad \Delta_j := 1 - je_0 - \sum_{l=1}^\nu \left\lfloor \frac{j\omega_l}{\alpha_l} \right\rfloor. \]

$K^2 + s$ is an invariant of a plumbed 3–manifold computed from its plumbing graph: $K^2$ is the self–intersection of the canonical class and $s$ is the number of vertices of the graph; however, in this paper we only use formula (2.4). Note that the Heegaard Floer homology is also determined in terms of the $\tau$–function. For details, see e.g. [Nem2, Nem4].

We recall also a definition of the sawtooth function: for $x \in \mathbb{R}$ one sets
\[ \langle x \rangle = \begin{cases} 0 & x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac{1}{2} & x \notin \mathbb{Z}. \end{cases} \]
(We use the notation $\langle x \rangle$ and not $(x)$, because we think it is then easier to read formulae).

The generalized Dedekind sum is defined for two integers, $p, q \in \mathbb{Z} \setminus \{0\}$ and $x, y \in \mathbb{R}$ by
\[ (2.6) \quad s(p, q; x, y) = \sum_{i=0}^{q-1} \left( \frac{i+y}{q} \right) \left( \frac{p+i+y}{q} + x \right). \]

The classical Dedekind sum is $s(p, q) = s(p, q; 0, 0)$, cf. [RG]. We introduce one more notation. Recall that $\varepsilon_a(b)$ was defined in the statement of Theorem 1.6. For a set of Seifert invariants $\{\alpha_l\}_l$ we also write $\varepsilon_{\{\alpha\}}(b) := \sum_l \varepsilon_{\alpha_l}(b)$.

**3. A Compact Formula for $\tau(i)$**

Under conditions $H_1(\Sigma, \mathbb{Z}) = 0$ we provide a compact formula for $\tau(m)$.

**Proposition 3.1.** Let us write $m = d(m)\alpha + \sum_l a_l\hat{\alpha}_l$, for some integers $d(m)$ and $0 \leq a_l < \alpha_l$ for all $l$. Then
\[ (3.2) \quad \tau(m) = \sum_l \left( \frac{1}{2} \left\lfloor \frac{m-1}{\alpha_l} \right\rfloor - s(\hat{\alpha}_l, \alpha_l; \frac{m}{\alpha_l}, 0) + s(\hat{\alpha}_l, \alpha_l) \right) \]
\[ + \frac{m^2}{2\alpha} + m(1 - \frac{\nu}{2}) - \frac{d(m)}{2} + \frac{\nu}{4} + \frac{1}{4} \varepsilon_{\{\alpha\}}(m). \]
(For a slightly different formula — using the $\omega_l$‘s but no $d(m)$ — see (3.6) in the proof.)
Proof. After straightforward computations using (2.1) we get

\[
\Delta_j = \sum_l \left\langle \frac{j \omega_l}{\alpha_l} \right\rangle + 1 - \frac{\nu}{2} + \frac{j}{\alpha} + \frac{\varepsilon(\alpha)(j)}{2}.
\]

We need a following result whose proof is at the end of this section.

Lemma 3.4. Let \(a\) and \(b\) be two coprime integers, with \(b \neq 0\). Then, for any \(m \in \mathbb{Z}\),

\[
\sum_{j=0}^{m-1} \left\langle \frac{ja}{b} \right\rangle = - \frac{1}{2} \left\langle \frac{ma}{b} \right\rangle - s(a^*, b; \frac{m}{b}, 0) + s(a^*, b),
\]

where \(a^*\) is defined by the condition \(a^*a \equiv -1 \mod b\).

By this lemma applied for each \(\left\langle \frac{j \omega_l}{\alpha_l} \right\rangle\) in (3.3), and using \(\omega_l^* = \hat{\alpha}_l \mod \alpha_l\), cf. (2.1),

\[
\tau(m) = \sum_l \left( -\frac{1}{2} \left\langle \frac{m \omega_l}{\alpha_l} \right\rangle + \frac{1}{2} \left\lceil \frac{m}{\alpha_l} \right\rceil - s(\hat{\alpha}_l, \alpha_l; \frac{m}{\alpha_l}, 0) + s(\hat{\alpha}_l, \alpha_l) \right)
+ m(1 - \frac{\nu}{2}) + \frac{\nu}{2} + \frac{m(m-1)}{2 \alpha}.
\]

Then using the sum of

\[
\left\langle \frac{m \omega_l}{\alpha_l} \right\rangle = \left\langle \frac{a_l \hat{\alpha}_l \omega_l}{\alpha_l} \right\rangle = \left\langle \frac{-a_l}{\alpha_l} \right\rangle = -\frac{a_l}{\alpha_l} + \frac{1}{2} \frac{1}{2} \varepsilon_{\alpha_l}(m)
\]

we conclude the proof. \(\square\)

Proof of Lemma 3.4. Since

\[
\sum_{j=0}^{b-1} \left\langle \frac{ja}{b} \right\rangle = 0
\]

(see e.g. [RG Lemma 1]) we may assume that \(0 \leq m < b\). Then we can write

\[
\sum_{j=0}^{m-1} \left\langle \frac{ja}{b} \right\rangle = \sum_{j=0}^{m-1} \left\langle \frac{ja}{b} \right\rangle \left\lfloor \frac{m-j}{b} \right\rfloor = \sum_{j=0}^{b-1} \left\langle \frac{ja}{b} \right\rangle \left\lfloor \frac{m-j}{b} \right\rfloor.
\]

Upon using the definition of \(\langle \cdot \rangle\) the last sum becomes

\[-\frac{1}{2} \left\langle \frac{ma}{b} \right\rangle + \sum_{j=0}^{b-1} \left\langle \frac{ja}{b} \right\rangle \left( \frac{1}{2} + \left\langle \frac{-m+j}{b} \right\rangle + \frac{m-j}{b} \right).\]

Using (3.7) again, we can rewrite this as

\[-\frac{1}{2} \left\langle \frac{ma}{b} \right\rangle + \sum_{j=0}^{b-1} \left\langle \frac{ja}{b} \right\rangle \left\langle \frac{-m+j}{b} \right\rangle - \left\langle \frac{ja}{b} \right\rangle \left\langle \frac{j}{b} \right\rangle.\]

Now, the substitution \(j \mapsto -a^*j\) and identity \(\langle -x \rangle = -\langle x \rangle\) provide the statement. \(\square\)
4. Extrema of the function $\tau(i)$

It is well-known (see e.g. [Mo]), that if $K = T_{p,q}$ is a torus knot (for $p$ and $q$ relative prime positive integers), then

$$S_{+1}(K) = -\Sigma(e_0, (p, p'), (q, q'), (r, r')),$$

where $r = pq - 1$, $p'$, $q'$, $r'$ (the $\omega_i$'s) and $e_0$ satisfy $\tau$.' In fact, $e_0 = -2$ and $p'$, $q'$ and $r'$ are determine uniquely by

$$p'q \equiv 1 \mod p, \; pq' \equiv 1 \mod q, \; r' = pq - 2.$$

The minus sign in front of $\Sigma$ shows the change of orientations. We also write $\delta := (p-1)(q-1)/2$. Using [2.4] via $s(p', p) = s(q, p)$, the Dedekind reciprocity law and a direct computation we get for $\Sigma$

$$(K^2 + s)(\Sigma) = -4\delta(\delta - 3).$$

In this section we study the local extrema of the function $\tau(i)$ associated with $\Sigma =\Sigma(e_0, (p, p'), (q, q'), (r, r'))$. We find also the global minimum, which together with results from Section 3 gives the proof of Theorem 1.6.

**Proposition 4.2.** Let $S_{pq}$ be the subsemigroup of $\mathbb{N}$ generated by $p$ and $q$ and including $0$. Then the following facts hold:

(a) The function $\tau : \mathbb{N} \to \mathbb{Z}$ attains its local minima at values $m_n = n(pq - 1)$ for $0 \leq n \leq 2\delta - 2$, and local maxima at values $M_n = npq + 1$ for $0 \leq n \leq 2\delta - 3$. This means that $\tau$ is (not necessarily strict) increasing on any interval $[m_n, M_n]$ and $(m_{2\delta - 2}, \infty)$, and (not necessarily strict) decreasing on any interval $[M_n, m_{n+1}]$.

(b) The sequences $\{m_n\}_n$ and $\{M_n\}_n$ are minimal with these properties, that is:

$$\tau(M_n) > \max\{\tau(m_n), \tau(m_{n+1})\}.$$

In fact, for any $0 \leq n \leq 2\delta - 3$, one has

$$\tau(M_n) - \tau(m_{n+1}) = \#\{s \notin S_{pq} : s \geq n + 2\} > 0,$$

$$\tau(M_n) - \tau(m_n) = \#\{s \in S_{pq} : s \leq n\} > 0.$$

(c) The absolute minimum of $\tau$ occurs for $m_{\min} = m_{\delta - 1}$.

(d) For any $0 \leq n \leq 2\delta - 3$,

$$\tau(M_n) = \frac{n(n - 2\delta + 3)}{2} + 1.$$  

**Proof.** First note that $\tau(m_0) = \tau(0) = 0$ while $\tau(M_0) = \tau(1) = \Delta_0 = 1$.

We define $M_n = npq + 1$ for any $n$ and we will compute $\tau(j + 1) - \tau(j) = \Delta_j$ for any $j \in [M_n, M_{n+1}]$. Clearly, $M_n \leq j < M_{n+1}$ if and only if $0 \leq (n + 1)pq - j < pq$.

Note the following fact regarding the semigroup $S_{pq}$ and any integer $a \in [0, pq]$:

$$a \in S_{pq} \iff a = \alpha p + \beta q \quad (0 \leq \alpha < q, \; 0 \leq \beta < p),$$

$$a \notin S_{pq} \iff a + pq = \alpha p + \beta q \quad (0 \leq \alpha < q, \; 0 \leq \beta < p).$$

First we fix an interval $[M_n, M_{n+1}]$ for some $0 \leq n \leq 2\delta - 3$.

**Case 1.** Assume that $(n + 1)pq - j \in S_{pq}$. Then, by (4.5)

$$j = (n + 1)pq - \alpha p - \beta q \quad (0 \leq \alpha < q, \; 0 \leq \beta < p).$$

In particular, $jp' \equiv -\beta \mod p$, $jq' \equiv -\alpha \mod q$, and in general, $jr' \equiv -j \mod r$. Using $-\beta/p = \beta/p - 1/2 + \varepsilon_p(\beta)/2 = \beta/p - 1/2 + \varepsilon_p(j)/2$, and similarly for $\alpha/q$, by a computation the value of $\Delta_j$ from (3.3) transforms into $\Delta_j = [j/r] - n$. Therefore

$$\Delta_j = \left\lfloor \frac{j}{r} \right\rfloor - n = \left\{ \begin{array}{ll} 0 & \text{if } M_n \leq j < m_{n+1} \\ 1 & \text{if } m_{n+1} \leq j < M_{n+1}. \end{array} \right.$$
Case 2. Assume that \((n + 1)pq - j \not\in \mathcal{S}_{pq}\). Then, by (4.5)

\[ j = (n + 2)pq - \alpha p - \beta q \quad (0 \leq \alpha < q, \ 0 \leq \beta < p). \]

Similar computation as in case 1 provides

\[ \Delta_j = \left\lfloor \frac{j}{r} \right\rfloor - n - 1 = \begin{cases} -1 & \text{if } Mn \leq j < Mn+1 \\ 0 & \text{if } mn+1 \leq j < Mn+1. \end{cases} \]  

Case 3. Assume that \(j \in [Mn, M_{n+1}]\) with \(n \geq 2\delta - 2\). If \((n + 1)pq - j \in \mathcal{S}_{pq}\), then by a similar argument as in Case 1 we get \(\Delta_j = \lfloor j/n \rfloor - n\). But \(j \geq Mn \geq n\), hence \(\Delta_j \geq 0\).

If \((n + 1)pq - j \not\in \mathcal{S}_{pq}\), then \((n + 1)pq - j \leq 2\delta - 1\) (the conductor of \(\mathcal{S}_{pq}\)), hence automatically \(j \geq (n + 1)pq - 2\delta + 1\). But \((n + 1)pq - 2\delta + 1 \geq (n + 1)r\) whenever \(n \geq 2\delta - 2\). Hence \(\Delta_j = \lfloor j/r \rfloor - n - 1 \geq 0\) again. This ends the proof of (a).

Moreover, equations (4.6) and (4.7) provide (4.3) too. Since \(2\delta - 1 \not\in \mathcal{S}_{pq}\) and \(0 \in \mathcal{S}_{pq}\), both differences are strictly positive. This proves (b).

In order to prove (c), let us find the difference between two subsequent local minima. Note that \(s \in \mathcal{S}_{pq}\) if and only if \(2\delta - 1 - s \not\in \mathcal{S}_{pq}\). Therefore, via (4.3), the difference equals

\[ \tau(m_{n+1}) - \tau(m_n) = \#\{s \not\in \mathcal{S}_{pq} : s \geq 2\delta - 1 - n\} - \#\{s \not\in \mathcal{S}_{pq} ; s \geq n + 2\}. \]

For \(n \leq \delta - 2\) one gets \(2\delta - 1 - n \geq n + 2\), hence this difference is \(\leq 0\). Similarly,

\[ \tau(m_{n+1}) - \tau(m_n) = \#\{s \in \mathcal{S}_{pq} : s \leq n\} - \#\{s \in \mathcal{S}_{pq} ; s \leq 2\delta - 1 - (n + 2)\}, \]

which is \(\geq 0\) for \(n \geq \delta - 1\). This ends part (c) too.

Part (d) can be verified in two ways. First, by (4.3) we deduce that \(\tau(M_{n+1}) - \tau(M_n)\) is \(\#\{s \in \mathcal{S}_{pq} : s \leq n + 1\} - \#\{s \not\in \mathcal{S}_{pq} ; s \geq n + 2\} = -\#\{s \not\in \mathcal{S}_{pq}\} + \#\{0 \leq s \leq n + 1\}\) which is \(n + 2 - \delta\). Then one proceeds by induction. Alternatively, by a direct computation one gets that \(\tau(npq) = n(n - 2\delta + 3)/2\) (here the Dedekind reciprocity law is used) and \(\Delta_{npq} = 1\).

5. PROOF OF THEOREM 1.6

Having studied the monotonicity properties of the \(\tau\)-function, we compute the value \(\tau(m_{\text{min}})\) via (3.6). By Proposition 4.2 \(m_{\text{min}} = (\delta - 1)r\). We write \(\delta - 1 = ap + bq - cpq\), with \(0 \leq a < q, \ 0 \leq b < p\) and \(c \in \{0, 1\}\); cf. (4.5). We have

\[ \left\lfloor \frac{m - 1}{p} \right\rfloor + \left\lfloor \frac{m - 1}{q} \right\rfloor = \left\lfloor \frac{m - 1}{r} \right\rfloor = (\delta - 1)(p + q) - \left\lfloor \frac{\delta}{p} \right\rfloor - \left\lfloor \frac{\delta}{q} \right\rfloor + \delta - 2. \]

Moreover \(\langle \frac{m}{p} \rangle = \langle \frac{m}{q} \rangle = \langle \frac{m}{r} \rangle\). Their sum is

\[ \langle \frac{-a}{q} \rangle + \langle \frac{-b}{p} \rangle = 1 - \frac{a}{q} - \frac{b}{p} - \frac{1}{2} \varepsilon_{p,q}(\delta - 1) = 1 - \frac{1}{2} \varepsilon_{p,q}(\delta - 1) - \frac{\delta - 1}{pq} - c. \]

As for the Dedekind sums, we observe that \(s(pq, r; \frac{m}{r}, 0) = s(pq, r)\) so these terms cancel each other in (3.6). Moreover, \(s(pr, q; \frac{m}{r}, 0) = s(-p, q; -\frac{\delta - 1}{q}, 0) = -s(p, q; \frac{\delta - 1}{q}, 0)\), and by the reciprocity law

\[ s(p, q) + s(q, p) = \frac{1}{12pq}(p^2 + q^2 + 1 - 3pq) = \frac{1}{12pq}(p^2 + q^2 - 3p - 3q - 2) - \frac{\delta - 1}{2pq}. \]

Putting this together we get

\[ 2\tau(m_{\text{min}}) = -\delta^2 + 3\delta - 1 + \frac{-\delta^2 + 3\delta - 2}{pq} - \frac{1}{6pq}(p^2 + q^2 - 3p - 3q - 2) \]

\[ - \left\lfloor \frac{\delta}{p} \right\rfloor - \left\lfloor \frac{\delta}{q} \right\rfloor + c + \frac{1}{2} \varepsilon_{p,q}(\delta - 1) + 2s(p, q; \frac{\delta - 1}{q}, 0) + 2s(q, p; \frac{\delta - 1}{p}, 0). \]
Hence, using (2.3), (4.11) and \(d(S_0^1(T_{pq})) = -d(\Sigma)\) we obtain formula (a) of Theorem 1.6. Formula (b) of 1.6 follows from the first equation of (4.3) written for \(n = \delta - 2\), (4.4) and (4.1).

6. The Heegaard Floer homology of \(\Sigma = -S_0^1(T_{pq})\)

Let \(\Delta(t)\) be the Alexander polynomial of the knot \(K = T_{pq}\) normalized in such a way that \(\Delta(1) = 1\). One has the following identity connecting \(\Delta\) and \(S_{pq}\), cf. [GDC]:

\[
\frac{\Delta(t)}{1-t} = \sum_{s \in S_{pq}} t^s.
\]

Since \(\Delta(1) = 1\) and \(\Delta'(1) = \delta\), one gets

\[
\Delta(t) = 1 + \delta(t-1) + (t-1)^2 \cdot Q(t)
\]

for some polynomial \(Q(t) = \sum_{i=0}^{2\delta-2} \alpha_i t^i\) of degree \(2\delta - 2\) with integral coefficients. In fact, all the coefficients \(\{\alpha_i\}_i\) are strictly positive, and:

\[
\delta = \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{2\delta-2} = 1.
\]

Indeed, by the above identity \(\delta + (t-1)Q(t) = \sum_{s \in S_{pq}} t^s\), or \(Q(t) = \sum_{s \in \Gamma}(t^{s-1} + \cdots + t + 1)\). Therefore

\[
\alpha_i = \# \{ s \notin S_{pq} : s > i \}.
\]

Sometimes it is convenient to consider the symmetrized Alexander polynomial \(\Delta'(t) = t^{-\delta} \Delta(t)\) too. Its second derivative at 1 satisfies

\[
(\Delta')''(1) = 2Q(1) + \delta - \delta^2.
\]

Regarding Heegaard Floer homologies we will use the following notations, cf. [OS3, Nem2]. Consider the \(\mathbb{Z}[U]\)-module \(\mathbb{Z}[U, U^{-1}]\), and denote by \(T_0^+\) its quotient by the submodule \(U \cdot \mathbb{Z}[U]\). It is a \(\mathbb{Z}[U]\)-module with grading \(\text{deg}(U^{-h}) = 2h\). Similarly, for any \(n \geq 1\), define the \(\mathbb{Z}[U]\)-module \(T_0(n)\) as the quotient of \(\mathbb{Z}(U^{-n-1}, U^{-n+1}, \ldots, U, U^{-2}, U^{-1})\) by \(U \cdot \mathbb{Z}[U]\) (with the same grading). Hence, \(T_0(n)\), as a \(\mathbb{Z}\)-module, is the free \(\mathbb{Z}\)-module \(\mathbb{Z}(U^{-n-1}, U^{-n+1}, \ldots, U^{-1})\) with finite \(\mathbb{Z}\)-rank \(n\).

More generally, fix an arbitrary \(\mathbb{Q}\)-graded \(\mathbb{Z}[U]\)-module \(P\) with \(h\)-homogeneous elements \(P_h\). Then for any \(x \in \mathbb{Q}\) we denote by \(P[x]\) the same module graded in such a way that \(P[x]h+x = P_h\). Then set \(T_x^+ := T_0^+[x]\) and \(T_x(n) := T_0(n)[x]\).

Now we are ready to prove Theorem 1.2 regarding the integral Heegaard Floer homology and the Casson (or Seiberg–Witten) invariant of \(-\Sigma = S_0^3(T_{pq})\), cf. Remark 1.3.

**Proof of Theorem 1.2** The statements are direct consequences of Proposition 1.2 and Theorem 8.3 of [Nem2]. We only have to check the numerical data; for this see also Proposition 3.5.2 and Corollary 3.7 of [Nem2] for the lattice cohomology associated with graded roots.

In the second identity the torsion modules appear symmetrically, their lengths are respectively \(\tau(M_{\delta-2-k}) - \tau(M_{\delta-2-k})\) and \(\tau(M_{\delta-1+k}) - \tau(M_{\delta+k})\) \((0 \leq k \leq \delta - 2)\), both equal to \(\alpha_{\delta+k}\). The weight of elements in \(\ker U\) are \(2\tau(M_{\delta-2-k} - (K^2 + s)/4\) \(= 2\tau(M_{\delta-2-k} - 2\alpha_{\delta+k} + \delta^2 - 3\delta = k^2 + k - 2\alpha_{\delta+k}\). The third identity is exactly this identity for \(k = -1\); it is the statement of Theorem 1.6 as well. For the fourth identity we use either the second one combined with the identities \(\alpha_i - \alpha_{2\delta-2-i} = \delta - i - 1\) or, directly Corollary 3.7 of [Nem2].

The above results — namely (4.4), Proposition 1.2 and Theorem 1.2 — bear strong resemblance to the case of negative surgeries [Nem4]. We now cite these results.
Proposition 6.1. Consider the negative definite Seifert 3–manifold $\Sigma' := S^3_{−1}(T_{p,q})$. Then one has the following facts:

(a) $K^2 + s$ and the $\tau$ function associated with $\Sigma'$ satisfy:

$$\tau(K^2 + s)(\Sigma') = −4\delta(\delta − 1).$$

$\tau$ has $2\delta − 1$ local maxima at $M_n$ $(0 ≤ n ≤ 2\delta − 2)$ and $2\delta$ local minima at $m_n$ $(0 ≤ n ≤ 2\delta − 1)$, where

$$\tau(m_n) = \frac{n(n − 2\delta + 1)}{2}, \quad \min \tau = \tau(m_{δ−1}) = \tau(m_δ) = −\delta(\delta − 1)/2,$$

$$\tau(M_n) − \tau(m_n) = \alpha_{2\delta−2−n},$$

$$\tau(M_n) − \tau(m_{n+1}) = \alpha_n.$$

(b) The Heegaard Floer homology of $−\Sigma'$ satisfies:

$$HF^+_{odd}(−\Sigma') = 0, \quad d(−\Sigma') = 0,$$

$$HF^+_{even}(−\Sigma') = T_0^+ \oplus \bigoplus_{k=0}^{\delta−2} T_{(k+1)(k+2)}(\alpha_{δ+k}) ⊕ T_0(\alpha_{δ−1}),$$

$$\lambda(−\Sigma') = sw(−\Sigma') = \text{rank} HF^+_{red}(−\Sigma') = Q(1) − \frac{δ(δ − 1)}{2} = \frac{1}{2}(δ#)'(1).$$

In the case of $\Sigma' = S^3_{−1}(T_{p,q})$ all the local minima are easily determinable numbers depending only on $\delta$, while the local maxima depend essentially on the coefficients $\alpha_i$. In contrast, for $\Sigma = S^3_1(T_{p,q})$ all the local maxima depend only on $\delta$, while the local minima on the coefficients $\alpha_i$.

In order to explain more deeply this parallelism, we consider the following construction. Assume that we have a sequence of $v + 1$ integers $\{\beta_0, \ldots, \beta_v\}$ with $\beta_0 ≥ \beta_1 ≥ \cdots ≥ \beta_v ≥ 1$. Then, using this sequence one can construct a continuous sawtooth diagram with $v$ teeth — as the graph of a function $τ : [0, 2v + 2] \to \mathbb{R}$ — as follows.

We begin with $τ(0) = 0$, then $τ$ is affine on each interval $[\beta_i, \beta_{i+1}]$; and for each $i$ it increases by $\beta_{i−1}$ on the interval $[2i, 2i+1]$, while it decreases by $\beta_i$ on $[2i+1, 2i+2]$.

Let us now consider the coefficients $\{\alpha_0, \alpha_1, \ldots, \alpha_{2\delta−2}\}$ of $Q(t)$ associated with the semi-group $\mathcal{S}_{pq}$, or with the Alexander polynomial $\Delta$. Then the $τ$ function of $\Sigma' = S^3_{−1}(T_{p,q})$ is associated exactly with this sequence $\{\alpha_0, \alpha_1, \ldots, \alpha_{2\delta−2}\}$ (and it has in the Heegaard Floer theory the normalization term $K^2 + s = −4\delta(\delta − 1)$), while the $τ$ function of $\Sigma = S^3_1(T_{p,q})$ is associated with the shorter sequence $\{\alpha_1, \ldots, \alpha_{2\delta−2}\}$, hence it has one tooth less (and $K^2 + s = −4\delta(\delta − 3)$). See Figure 1 for $(p, q) = (3, 4)$.

From these data all the numerical properties of the corresponding Heegaard Floer homologies follow by the general theory of graded roots, see [Nem2]. Note that in the case of $\Sigma$, the missing entry is $\alpha_0 = δ$, hence once we know $δ$, the two sequences determine each other!

We expect that this ‘duality’ is valid in a more general situation.

7. Relation to the Levine–Tristram signature.

Let $p$ and $q$ be two coprime integers as above. For $y ∈ [0, 1]$ we denote by $σ(y)$ the Levine–Tristram signature of the (positive) torus knot $T_{p,q}$ evaluated at $e^{2πiy}$.

Moreover, let $Sp := \{\frac{i}{p} + \frac{j}{q} : 1 ≤ i ≤ p − 1; 1 ≤ j ≤ q − 1\}$ be the spectrum of the local plane curve singularity $x^p + y^q = 0, (x, y) ∈ (\mathbb{C}, 0)$.

Proposition 7.1. With the above notations, for any integer $a ∈ [0, pq]$ one has

$$\#\{s ∈ S_{pq} : s ≥ a\} = \#\{[1 + \frac{a}{pq}, 2) \cap Sp\},$$
Figure 1. The $\tau$ function for $S_{3,1}^2(T_{3,4})$ on the left and $-S_{3,1}^2(T_{3,4})$ on the right. Below we show the increment and the decrement of the $\tau$ function on each interval. $S_{3,4} = N \setminus \{1, 2, 5\}$, hence $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (3, 2, 1, 1, 1)$.

and

$$\# \{s \not\in S_{p,q} : s \geq a\} = \delta + \frac{1}{4} \cdot \sigma\left(\frac{a}{pq}\right) - \frac{1}{2} (a - \left\lfloor \frac{a}{p} \right\rfloor - \left\lfloor \frac{a}{q} \right\rfloor - \tilde{c}(a)),$$

where

$$\tilde{c}(a) = \begin{cases} 
\frac{1}{2} & \text{if } 1 + \frac{a}{pq} \in Sp, \\
-\frac{1}{2} & \text{if } \frac{a}{pq} \in Sp, \\
0 & \text{otherwise.}
\end{cases}$$

Proof. If $s \not\in S_{pq}$ and $s \geq a$, then $s + pq = \alpha p + \beta q$ for some $\alpha \in [0, q)$ and $\beta \in [0, p)$ (see e.g. (4.3)). But $\alpha \beta \neq 0$ (otherwise $s \in S_{pq}$), hence $1 + \frac{a}{pq} \in Sp \cap \{1 + \frac{a}{pq}, 2\}$. This proves the first identity.

Let us define the following four quantities $S_i := \#Sp \cap I_i \ (1 \leq i \leq 4)$, where

$I_1 = (0, \frac{a}{pq}]$, $I_2 = (\frac{a}{pq}, 1)$, $I_3 = (1, 1 + \frac{a}{pq}]$, $I_4 = [1 + \frac{a}{pq}, 2)$.

Then $S_1 + S_2 = S_3 + S_4 = \delta$. Moreover, by [BN, Corollary 4.4.9] (see also [Lith]), one has

$$\sigma\left(\frac{a}{pq}\right) = S_1 + S_4 - S_2 - S_3 + \#(Sp \cap \{\frac{a}{pq}, 1 + \frac{a}{pq}\}).$$

On the other hand, we claim that

$$S_1 + S_3 + \#(Sp \cap \{1 + \frac{a}{pq}\}) = a - \left\lfloor \frac{a}{p} \right\rfloor - \left\lfloor \frac{a}{q} \right\rfloor.$$

Indeed, the left hand side is $((0, \frac{a}{pq}] \cup (1, 1 + \frac{a}{pq}]) \cap Sp$. Then $\frac{a}{pq}$ is in this set if and only if $s$ is an integer in $[0, x]$ which is not divisible either by $p$ or by $q$. Their number is the right hand side of the identity.

In order to end the proof, we write

$$4S_4 = S_1 + S_4 - S_3 - S_2 + 3(S_4 + S_3) + (S_2 + S_1) - 2(S_1 + S_3)$$

and substitute the above identities. \qed

Substituting $a = \delta$ into (7.2) we obtain

Corollary 7.3.

$$d(S_{1}^{3}(T_{p,q})) = -2\# \{s \not\in S_{p,q} : s \geq \delta\} = -\delta - \frac{1}{2} \cdot \sigma\left(\frac{\delta}{pq}\right) - \left\lfloor \frac{\delta}{p} \right\rfloor - \left\lfloor \frac{\delta}{q} \right\rfloor - \tilde{c}(\delta).$$
8. Inequalities

Fix $p < q$. Consider the surface Brieskorn singularity $(u, v, w) \in (\mathbb{C}^3, 0): u^p + v^q + w^2 = 0$, i.e. the double suspension of the plane curve singularity $x^p + y^q = 0$. Its Milnor number is $\mu = 2\delta$; let $\mu_+$ and $\mu_-$ be the dimensions of maximal subspaces of the vanishing homology where the intersection form is positive/negative definite. (Note that the intersection form is non-degenerate, hence the dimension $\mu_0$ of its kernel is zero.) Therefore, $2\delta = \mu_+ + \mu_-$. 

**Lemma 8.1.** For $p$ and $q$ relative primes one has:

(a) $\# \{ s \not\in S_{pq} : s \geq \delta \} \leq \frac{\delta + 1}{2}$

(b) $\delta + 1 \leq -\sigma$.

The inequality (a) is sharp for $p = 2$ and $\delta$ odd, while (b) is equality for $(p, q) = (2, 3)$.

**Proof.** (a) Set $S^* := \{ s \in S_{pq} : 0 < s < \delta \}$, and let $s^*$ be its maximal element. Then $\{0\} \cup S^* \cup (s^* + S^*) \subset S_{pq} \cap [0, 2\delta - 1]$, hence $1 + 2#S^* \leq #\{0, 2\delta - 1\} - #\{\mathbb{N} \setminus S_{pq}\} = \delta$. Therefore, $\# \{ s \not\in S_{pq} : s \geq \delta \} = \# \{ s \in S_{pq} : s \leq \delta - 1 \} = 1 + #S^* \leq (\delta + 1)/2$.

(b) By Theorem 4.1 of [Nem1], $-6\sigma \geq 6\delta + q - 1$, hence the inequality follows for $q \geq 7$. For the other pairs when $p < q \leq 6$ can be checked case by case.

Since the four genus $g_4$ of $T_{p,q}$ is $\delta$, we get the statement of Corollary 1.7(a).

Having the equality $-d(K) = 2\left[\frac{-\sigma(K)}{4}\right]$ for alternating knots, we might wonder if for any torus knot $-d \leq 2\left[\frac{-\sigma}{4}\right]$ is valid. However, this is not the case: for the pair $(p, q) = (4, 5)$ one has $-d = 6$ and $\sigma = -8$.

Nevertheless, asymptotically for ‘most’ of the pairs $(p, q)$, $-d$ grows like $-\sigma/2$.

**Lemma 8.2.** Recall that $p < q$. Then

$$\# S_p \cap \left[1 + \frac{\delta}{pq}\right] \leq q - 1 + \# S_p \cap \left[\frac{3}{2}, 2\right].$$

**Proof.** We need to show that $\# S_p \cap [1 + \delta/pq, 3/2] \leq q - 1$. For this notice that if $s = \alpha/p + \beta/q$ is in the interval $[1 + \delta/pq, 3/2]$, then $s + \frac{1}{p} \geq 3/2$. Hence, $1 \leq \beta \leq q - 1$ determines any $s$ in that interval.

Note that $\# S_p \cap \left[3/2, 2\right] = \mu_+ / 2$ by Thom–Sebastiani type theorem for the spectrum of the suspension and by relationship between the spectral number of surfaces and the intersection form $[S]$. Note also that the inequality $\delta \leq -\sigma - 1$ of [S.L]b can be rewritten as $\mu_+ \leq (\delta - 1)/2$. Therefore, we obtain:

$$-d(S^1_3(T_{p,q})) \leq 2q - 2 + \mu_+ \leq 2q - 2 + \frac{\delta - 1}{2}.$$  

(8.3)

If $p$ is ‘small’ with respect to $q$, then the $q$ term at the right hand side makes the inequality weak. Nevertheless, if $p$ ‘grows together with $q$’, then one can find a positive constant $c$ such that $2q - \frac{5}{2} \leq c\sqrt{q}$. For example, if $p = q - 1$ then $2q - \frac{5}{2} \leq 8\sqrt{q} + 1$. Hence (8.3) together with Lemma 5.1b provides Corollary 1.7(b).

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