Noncommutative field theories from a deformation point of view

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Abstract. In this review we discuss the global geometry of noncommutative field theories from a deformation point of view: The space-times under consideration are deformations of classical space-time manifolds using star products. Then matter fields are encoded in deformation quantizations of vector bundles over the classical space-time. For gauge theories we establish a notion of deformation quantization of a principal fiber bundle and show how the deformation of associated vector bundles can be obtained.

Mathematics Subject Classification (2000). Primary 53D55; Secondary 58B34, 81T75.

Keywords. Noncommutative field theory, Deformation quantization, Principal Bundles.

1. Introduction

Noncommutative geometry is commonly believed to be a reasonable candidate for the marriage of classical gravity theory in form of Einstein’s general relativity on one hand and quantum theory on the other hand. Both theories are experimentally well-established within large regimes of energy and distance scales. However, from a more fundamental point of view, the coexistence of these two theories becomes inevitably inconsistent when one approaches the Planck scale where gravity itself gives significant quantum effects.

Since general relativity is ultimately the theory of the geometry of space-time it seems reasonable to use notions of ‘quantum geometry’ known under the term noncommutative geometry in the sense of Connes [11] to achieve appropriate formulations of what eventually should become quantum gravity. Of course, this ultimate goal has not yet been reached but techniques of noncommutative geometry have been used successfully to develop models of quantum field theories on quantum space-times being of interest for their own. Moreover, a deeper understanding of ordinary quantum field theories can be obtained by studying their...
counterparts on ‘nearby’ noncommutative space-times. On the other hand, people started to investigate experimental implications of a possible noncommutativity of space-time in future particle experiments.

Such a wide scale of applications and interests justifies a more conceptual discussion of noncommutative space-times and (quantum) field theories on them in order to clarify fundamental questions and generic features expected to be common to all examples.

In this review, we shall present such an approach from the point of view of deformation theory: noncommutative space-times are not studied by themselves but always with respect to a classical space-time, being suitably deformed into the noncommutative one. Clearly, this point of view can not cover all possible (and possibly interesting) noncommutative geometries but only a particular class. Moreover, we focus on formal deformations for technical reasons. It is simply the most easy approach where one can rely on the very powerful machinery of algebraic deformation theory. But it also gives hints on approaches beyond formal deformations: finding obstructions in the formal framework will indicate even more severe obstructions in any non-perturbative approach.

In the following, we discuss mainly two questions: first, what is the appropriate description of matter fields on deformed space-times and, second, what are the deformed analogues of principal bundles needed for the formulation of gauge theories. The motivation for these two questions should be clear.

The review is organized as follows: in Section 2, we recall some basic definitions and properties concerning deformation quantizations and star products needed for the set-up of noncommutative space-times. We discuss some fundamental examples as well as a new class of locally noncommutative space-times. Section 3 is devoted to the study of matter fields: we use the Serre-Swan theorem to relate matter fields to projective modules and discuss their deformation theory. Particular interest is put on the mass terms and their positivity properties. In Section 4 we establish the notion of deformation quantization of principal fiber bundles and discuss the existence and uniqueness results. Finally, in Section 5 we investigate the resulting commutant and formulate an appropriate notion of associated (vector) bundles. This way we make contact to the results of Section 3.

The review is based on joint works with Henrique Bursztyn on one hand as well as with Martin Bordemann, Nikolai Neumaier and Stefan Weiß on the other hand.

2. Noncommutative space-times

In order to implement uncertainty relations for measuring coordinates of events in space-time it has been proposed already very early to replace the commutative algebra of (coordinate) functions by some noncommutative algebra. In [16] a concrete model for a noncommutative Minkowski space-time was introduced with commutation relations of the form

$$[\hat{x}^\mu, \hat{x}^\nu] = i\lambda\theta^\mu\nu, \quad \text{(2.1)}$$
where $\lambda$ plays the role of the deformation parameter and has the physical dimension of an area. Usually, this area will be interpreted as the Planck area. Moreover, $\theta$ is a real, antisymmetric tensor which in [16] and many following papers is assumed to be constant: in [16] this amounts to require that $\theta^\mu\nu$ belongs to the center of the new algebra of noncommutative coordinates.

Instead of constructing an abstract algebra where commutation relations like (2.1) are fulfilled, it is convenient to use a ‘symbol calculus’ and encode (2.1) already for the classical coordinate functions by changing the multiplication law instead. For functions $f$ and $g$ on the classical Minkowski space-time one defines the Weyl-Moyal star product by

$$f \ast g = \mu \circ e^{\lambda^\mu\nu \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}} (f \otimes g),$$

(2.2)

where $\mu(f \otimes g) = fg$ denotes the undeformed, pointwise product. Then (2.1) holds for the classical coordinate functions with respect to the $\ast$-commutator.

Clearly, one has to be slightly more careful with expressions like (2.2): in order to make sense out of the infinite differentiations the functions $f$ and $g$ first should be $C^\infty$. But then the exponential series does not converge in general whence a more sophisticated analysis is required. Though this can be done in a completely satisfying way for this particular example, we shall not enter this discussion here but consider (2.2) as a formal power series in the deformation parameter $\lambda$. Then $\ast$ becomes an associative $\mathbb{C}[[\lambda]]$-bilinear product for $C^\infty(\mathbb{R}^4)[[\lambda]]$, i.e. a star product in the sense of [3]. It should be noted that the interpretation of (2.2) as formal series in $\lambda$ is physically problematic: $\lambda$ is the Planck area and hence a physically measurable and non-zero quantity. Thus our point of view only postpones the convergence problem and can be seen as a perturbative approach.

With this example in mind, one arrives at several conceptual questions: The first is that Minkowski space-time is clearly not a very realistic background when one wants to consider quantum effects of ‘hard’ gravity. Here already classically nontrivial curvature and even nontrivial topology may arise. Thus one is forced to consider more general and probably even generic Lorentz manifolds instead. Fortunately, deformation quantization provides a well-established and successful mathematical framework for this geometric situation.

Recall that a star product on a manifold $M$ is an associative $\mathbb{C}[[\lambda]]$-bilinear multiplication $\ast$ for $f, g \in C^\infty(M)[[\lambda]]$ of the form

$$f \ast g = \sum_{r=0}^{\infty} \lambda^r C_r(f,g),$$

(2.3)

where $C_0(f,g) = fg$ is the undeformed, pointwise multiplication and the $C_r$ are bidifferential operators. Usually, one requires $1 \ast f = f = f \ast 1$ for all $f$. It is easy to see that $\{f,g\} = \frac{1}{\hbar}(C_1(f,g) - C_1(g,f))$ defines a Poisson bracket on $M$. Conversely, and this is the highly nontrivial part, any Poisson bracket $\{f,g\} = \theta(df, dg)$, where

$$\theta \in \Gamma^\infty(\Lambda^2 TM), \quad [\theta, \theta] = 0$$

(2.4)
is the corresponding Poisson tensor, can be quantized into a star product \[13, 28\]. Beside these existence results one has a very good understanding of the classification of such star products \[19, 28, 30\], see also \[14, 18\] for recent reviews and \[39\] for an introduction.

With this geometric interpretation the Weyl-Moyal star product on Minkowski space-time turns out to be a deformation quantization of the constant Poisson structure

\[
\theta = \frac{1}{2} \theta^{\mu \nu} \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu}.
\]  

(2.5)

On a generic space-time \(M\) there is typically no transitive action of isometries which would justify the notion of a ‘constant’ bivector field. Thus a star product \(\star\) on \(M\) is much more complicated than (2.2) in general: already the first order term is a (nontrivial) Poisson structure and for the higher order terms one has to invoke the (unfortunately rather inexplicit) existence theorems.

Thus answering the first question by using general star products raises the second: what is the physical role of a Poisson structure on space-time? While on Minkowski space-time with constant \(\theta\) we can view the finite number of coefficients \(\theta^{\mu \nu} \in \mathbb{R}\) as parameters of the theory this is certainly no longer reasonable in the more realistic geometric framework: there is an infinity of Poisson structures on each manifold whence an interpretation as ‘parameter’ yields a meaningless theory. Instead, \(\theta\) has to be considered as a field itself, obeying its own dynamics compatible with the constraint of the Jacobi identity \([\theta, \theta] = 0\). Unfortunately, up to now a reasonable ‘field equation’ justified by first principles seems to be missing.

This raises a third conceptual question, namely why should there by any Poisson structure on \(M\) and what are possible experimental implications? In particular, the original idea of introducing a noncommutative structure was to implement uncertainty relations forbidding the precise localization of events. The common believe is that such quantum effects should only play a role when approaching the Planck scale. Now it turns out that the quantum field theories put on such a noncommutative Minkowski space-time (or their Euclidean counterparts) suffer all from quite unphysical properties: Typically, the noncommutativity enters in long-distance/low-energy features contradicting our daily life experience. Certainly, a last word is not said but there might be a simple explanation why such effects should be expected: the global \(\theta\) (constant or not) yields global effects on \(M\). This was the starting point of a more refined notion of noncommutative space-times advocated in \[1, 22\] as locally noncommutative space-times. Roughly speaking, without entering the technical details, it is not \(M\) which should become noncommutative but \(TM\). Here the tangent bundle is interpreted as the bundle of all normal charts on \(M\) and for each normal chart with origin \(p \in M\) one constructs its own star product \(\star_p\). The crucial property is then that \(\star_p\) is the pointwise, commutative product outside a (small) compact subset around \(p\). This way, the long-distance behaviour (with respect to the reference point \(p\)) is classical while close to \(p\) there is a possibly even very strong noncommutativity. In some sense, this is an implementation of an idea of Julius Wess, proposing that the transition
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from classical geometry to quantum geometry should be understood as a kind of phase transition taking place at very small distances [41]. Of course, the conceptual question about the physical origin of the corresponding Poisson structure on $TM$ as well as the convergence problem still persists also in this approach.

Ignoring these questions about the nature of $\theta$, we shall assume in the following that we are given a star product $\star$ on a manifold $M$ which can be either space-time itself or its tangent bundle in the locally noncommutative case. Then we address the question how to formulate reasonable field theories on $(M, \star)$. Here we shall focus on classical field theories which still need to be quantized later on. On the other hand, we seek for a geometric formulation not relying on particular assumptions about the underlying classical space-time.

3. Matter fields and deformed vector bundles

In this section we review some results from [6, 9, 34, 38].

In classical field theories both bosonic and fermionic matter fields are given by sections of appropriate vector bundles. For convenience, we choose the vector bundles to be complex as also the function algebra $C^\infty(M)$ consists of complex-valued functions. However, the real case can be treated completely analogously. Thus let $E \rightarrow M$ be a complex vector bundle over $M$. Then the $E$-valued fields are the (smooth) sections $\Gamma^\infty(E)$ which form a module over $C^\infty(M)$ by pointwise multiplication. Thanks to the commutativity of $C^\infty(M)$ we have the freedom to choose this module structure to be a right module structure for later convenience.

It is a crucial feature of vector bundles that $\Gamma^\infty(E)$ is actually a finitely generated and projective module:

**Theorem 3.1 (Serre-Swan).** The sections $\Gamma^\infty(E)$ of a vector bundle $E \rightarrow M$ are a finitely generated and projective $C^\infty(M)$-module. Conversely, any such module arises this way up to isomorphism.

Recall that a right module $E_\mathcal{A}$ over an algebra $\mathcal{A}$ is called finitely generated and projective if there exists an idempotent $e^2 = e \in M_n(\mathcal{A})$ such that $E_\mathcal{A} \cong e\mathcal{A}^n$ as right $\mathcal{A}$-modules. More geometrically speaking, for any vector bundle $E \rightarrow M$ there is another vector bundle $F \rightarrow M$ such that their Whitney sum $E \oplus F$ is isomorphic to a trivial vector bundle $M \times \mathbb{C}^n \rightarrow M$. Note that the Serre-Swan theorem has many incarnations, e.g. the original version was formulated for compact Hausdorff spaces and continuous sections/functions. Note also that for our situation no compactness assumption is necessary (though it drastically simplifies the proof) as manifolds are assumed to be second countable.

**Remark 3.2.** The Serre-Swan theorem is the main motivation for noncommutative geometry to consider finitely generated and projective modules over a not necessarily commutative algebra $\mathcal{A}$ as ‘vector bundles’ over the (noncommutative) space described by $\mathcal{A}$ in general.
For physical applications in field theory one usually has more structure on \( E \) than just a bare vector bundle. In particular, for a Lagrangean formulation a ‘mass term’ in the Lagrangean is needed. Geometrically such a mass term corresponds to a Hermitian fiber metric \( h \) on \( E \). One can view a Hermitian fiber metric as a map
\[
 h : \Gamma^\infty(E) \times \Gamma^\infty(E) \longrightarrow C^\infty(M),
\]
which is \( \mathbb{C} \)-linear in the second argument and satisfies
\[
 h(\phi, \psi) = h(\psi, \phi),
\]
\[
 h(\phi, \psi f) = h(\phi, \psi) f
\]
for \( \phi, \psi \in \Gamma^\infty(E) \) and \( f \in C^\infty(M) \). The pointwise non-degeneracy of \( h \) is equivalent to the property that
\[
 h(\phi, \phi) \geq 0
\]
for \( \phi, \psi \in \Gamma^\infty(E) \) and \( f \in C^\infty(M) \).

In order to encode now the positivity (3.2) in a more algebraic way suitable for deformation theory, we have to consider the following class of algebras: First, we use a ring of the form \( \mathbb{C} = \mathbb{R}(i) \) with \( i^2 = -1 \) for the scalars where \( \mathbb{R} \) is an ordered ring. This includes both \( \mathbb{R} \) and \( \mathbb{R}[\![\lambda]\!] \), where positive elements in \( \mathbb{R}[\![\lambda]\!] \) are defined by
\[
 a = \sum_{r=r_0}^{\infty} \lambda^r a_r > 0 \quad \text{if} \quad a_{r_0} > 0.
\]
In fact, this way \( \mathbb{R}[\![\lambda]\!] \) becomes an ordered ring whenever \( \mathbb{R} \) is ordered. More physically speaking, the ordering of \( \mathbb{R}[\![\lambda]\!] \) refers to a kind of asymptotic positivity. Then the algebras in question should be \(^*\)-algebras over \( \mathbb{C} \). Indeed, \( C^\infty(M) \) is a \(^*\)-algebra over \( \mathbb{C} \) where the \(^*\)-involution is the pointwise complex conjugation. For the deformed algebras \( (C^\infty(M)[\![\lambda]\!], \ast) \) we require that the star product is Hermitian, i.e.
\[
 f \ast g = g \ast f
\]
for all \( f, g \in C^\infty(M)[\![\lambda]\!] \). For a real Poisson structure \( \theta \) this can be achieved by a suitable choice of \( \ast \).

For such a \(^*\)-algebra we can now speak of positive functionals and positive elements [7] by mimicking the usual definitions from operator algebras, see e.g. [32] for the case of (unbounded) operator algebras and [37] for a detailed comparison.

**Definition 3.3.** Let \( \mathcal{A} \) be a \(^*\)-algebra over \( \mathbb{C} = \mathbb{R}(i) \). A \( \mathbb{C} \)-linear functional \( \omega : \mathcal{A} \longrightarrow \mathbb{C} \) is called positive if \( \omega(a^* a) \geq 0 \) for all \( a \in \mathcal{A} \). An element \( a \in \mathcal{A} \) is called positive if \( \omega(a) \geq 0 \) for all positive functionals \( \omega \).

We denote the convex cone of positive elements in \( \mathcal{A} \) by \( \mathcal{A}^+ \). It is an easy exercise to show that for \( \mathcal{A} = C^\infty(M) \) the positive functionals are the compactly supported Borel measures and \( \mathcal{A}^+ \) consists of functions \( f \) with \( f(x) \geq 0 \) for all \( x \in M \).
Using this notion of positive elements and motivated by [29], the algebraic formulation of a fiber metric is now as follows [6, 9]:

**Definition 3.4.** Let \( \mathcal{E}_A \) be a right \( A \)-module. Then an inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{E}_A \) is a map
\[
\langle \cdot, \cdot \rangle : \mathcal{E}_A \times \mathcal{E}_A \rightarrow A,
\]
which is \( C \)-linear in the second argument and satisfies
\[
\langle x, y \rangle = \langle y, x \rangle^*, \quad \langle x, y \cdot a \rangle = \langle x, y \rangle a,
\]
and \( \langle x, y \rangle = 0 \) for all \( y \) implies \( x = 0 \). The inner product is called strongly non-degenerate if in addition
\[
\mathcal{E}_A \ni x \mapsto \langle x, \cdot \rangle \in \mathcal{E}_A^* = \text{Hom}_A(\mathcal{E}_A, A)
\]
is bijective. It is called completely positive if for all \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in \mathcal{E}_A \) one has \( (\langle x_i, x_j \rangle) \in M_n(A)^+ \).

Clearly, a Hermitian fiber metric on a complex vector bundle endows \( \Gamma^\infty(E) \) with a completely positive, strongly non-degenerate inner product in the sense of Definition 3.4.

With the above definition in mind we can now formulate the following deformation problem [6]:

**Definition 3.5.** Let \( \ast \) be a Hermitian star product on \( M \) and \( E \rightarrow M \) a complex vector bundle with fiber metric \( h \).

1. A deformation quantization \( \bullet \) of \( E \) is a right module structure \( \bullet \) for \( \Gamma^\infty(E)[[\lambda]] \) with respect to \( \ast \) of the form
\[
\phi \bullet f = \sum_{r=0}^{\infty} \lambda^r R_r(\phi, f)
\]
with bidifferential operators \( R_r \) and \( R_0(\phi, f) = \phi f \).

2. For a given deformation quantization \( \bullet \) of \( E \) a deformation quantization of \( h \) is a completely positive inner product \( h \) for \( (\Gamma^\infty(E)[[\lambda]], \bullet) \) of the form
\[
h(\phi, \psi) = \sum_{r=0}^{\infty} \lambda^r h_r(\phi, \psi)
\]
with (sesquilinear) bidifferential operators \( h_r \) and \( h_0 = h \).

In addition, we call two deformations \( \bullet \) and \( \tilde{\bullet} \) equivalent if there exists a formal series of differential operators
\[
T = \text{id} + \sum_{r=1}^{\infty} \lambda^r T_r : \Gamma^\infty(E)[[\lambda]] \rightarrow \Gamma^\infty(E)[[\lambda]],
\]
such that
\[
T(\phi \bullet f) = T(\phi)\tilde{\bullet}f.
\]
With other words, \( T \) is a module isomorphism starting with the identity in order \( \lambda^0 \) such that \( T \) is not visible in the classical/commutative limit. Conversely, starting with one deformation \( \bullet \) and a \( T \) like in (3.10), one obtains another equivalent
deformation \( \bullet \) by defining \( \bullet \) via (3.11). Similarly, we define two deformations \( h \) and \( \tilde{h} \) to be isometric if there exists a self-equivalence \( U \) with

\[
h(\phi, \psi) = \tilde{h}(U(\phi), U(\psi)).
\]

The relevance of the above notions for noncommutative field theories should now be clear: for a classical matter field theory modelled on \( E \rightarrow M \) we obtain the corresponding noncommutative field theory by choosing a deformation \( \bullet \) (if existing!) together with a deformation \( h \) (if existing!) in order to write down noncommutative Lagrangeans involving expressions like \( L(\phi) = h(\phi, \phi) + \cdots \).

Note that naive expressions like \( \phi \star \phi \) do not make sense geometrically, even on the classical level: sections of a vector bundle can not be ‘multiplied’ without the extra structure of a fiber metric \( h \) unless the bundle is trivial and trivialized. In this particular case we can of course use the canonical fiber metric coming from the canonical inner product on \( \mathbb{C}^n \). We refer to [34, 38] for a further discussion.

We can now state the main results of this section, see [6,9] for detailed proofs:

**Theorem 3.6.** For any star product \( \star \) on \( M \) and any vector bundle \( E \rightarrow M \) there exists a deformation quantization \( \bullet \) with respect to \( \star \) which is unique up to equivalence.

**Theorem 3.7.** For any Hermitian star product \( \star \) on \( M \) and any fiber metric \( h \) on \( E \rightarrow M \) and any deformation quantization \( \bullet \) of \( E \) there exists a deformation quantization \( h \) of \( h \) which is unique up to isometry.

The first theorem relies heavily on the Serre-Swan theorem and the fact that algebraic \( K_0 \)-theory is stable under formal deformations [31]. In fact, projections and hence projective modules can always be deformed in an essentially unique way. The second statement follows for much more general deformed algebras than only for star products, see [9].

**Remark 3.8.**

1. In case \( M \) is symplectic, one has even a rather explicit Fedosov-like construction for \( \bullet \) and \( h \) in terms of connections, see [35].
2. It turns out that also \( \Gamma^\infty(\text{End}(E)) \) becomes deformed into an associative algebra \( (\Gamma^\infty(\text{End}(E))[\lambda], \star') \) such that \( \Gamma^\infty(E)[\lambda] \) becomes a Morita equivalence bimodule between the two deformed algebras \( \star \) and \( \star' \). Together with the deformation \( h \) of \( h \) one obtains even a strong Morita equivalence bimodule [9].
3. Note also that the results of the two theorems are more than just the ‘analogy’ used in the more general framework of noncommutative geometry: we have here a precise link between the noncommutative geometries and their classical/commutative limits via deformation. For general noncommutative geometries it is not even clear what a classical/commutative limit is.

4. **Deformed principal bundles**

This section contains a review of results obtained in [5] as well as in [40].
In all fundamental theories of particle physics the field theories involve gauge fields. Geometrically, their formulation is based on the use of a principal bundle $\text{pr} : P \rightarrow M$ with structure group $G$, i.e. $P$ is endowed with a (right) action of $G$ which is proper and free whence the quotient $P/G = M$ is again a smooth manifold. All the matter fields are then obtained as sections of associated vector bundles by choosing an appropriate representation of $G$.

In the noncommutative framework there are several approaches to gauge theories: for particular structure groups and representations notions of gauge theories have been developed by Jurco, Schupp, Wess and coworkers [23–27]. Here the focus was mainly on local considerations and the associated bundles but not on the principal bundle directly. Conversely, there is a purely algebraic and intrinsically global formulation of Hopf-Galois extensions where not only the base manifold $M$ is allowed to be noncommutative but even the structure group is replaced by a general Hopf algebra, see e.g. [12] and references therein for the relation of Hopf-Galois theory to noncommutative gauge field theories. However, as we shall see below, in this framework which a priori does not refer to any sort of deformation, in general only very particular Poisson structures on $M$ can be used. Finally, in [36] a local approach to principal $\text{Gl}(n\mathbb{C})$ or $\text{U}(n)$ bundles was implicitly used via deformed transition matrices.

We are now seeking for a definition of a deformation quantization of a principal bundle $P$ for a generic structure Lie group $G$, arbitrary $M$ and arbitrary star product $\star$ on $M$ without further assumptions on $P$. In particular, the formulation should be intrinsically global.

The idea is to consider the classical algebra homomorphism
\[ \text{pr}^* : C^\infty(M) \rightarrow C^\infty(P) \] (4.1)
and try to find a reasonable deformation of $\text{pr}^*$. The first idea would be to find a star product $\star_P$ on $P$ with a deformation $\text{pr}^* = \sum_{r=0}^{\infty} \lambda^r \text{pr}^*_r$ of $\text{pr}^* = \text{pr}^*$ into an algebra homomorphism
\[ \text{pr}^*(f \star g) = \text{pr}^*(f) \star_P \text{pr}^*(g) \] (4.2)
with respect to the two star products $\star$ and $\star_P$. In some sense this would be the first (but not the only) requirement for a Hopf-Galois extension. In fact, the first order of (4.2) implies that the classical projection map $\text{pr}$ is a Poisson map with respect to the Poisson structures induced by $\star$ on $M$ and $\star_P$ on $P$. The following example shows that in general there are obstructions to achieve (4.2) already on the classical level:

**Example.** Consider the Hopf fibration $\text{pr} : S^3 \rightarrow S^2$ (which is a nontrivial principal $S^1$-bundle over $S^2$) and equip $S^2$ with the canonical symplectic Poisson structure. Then there exists no Poisson structure on $S^3$ such that $\text{pr}$ becomes a Poisson map. Indeed, if there would be such a Poisson structure then necessarily all symplectic leaves would be two-dimensional as symplectic leaves are mapped into symplectic leaves and $S^2$ is already symplectic. Fixing one symplectic leaf in $S^3$ one checks that $\text{pr}$ restricted to this leaf is still surjective and thus provides a
covering of $S^2$. But $S^2$ is simply connected whence the symplectic leaf is itself a $S^2$. This would yield a section of the nontrivial principal bundle $pr : S^3 \to S^2$, a contradiction.

Remark 4.1. Note that there are prominent examples of Hopf-Galois extensions using quantum spheres, see e.g. [20] and references therein. The above example shows that when taking the semi-classical limit of these $q$-deformations one obtains Poisson structures on $S^2$ which are certainly not symplectic. Note that this was a crucial feature in the above example. A further investigation of these examples is work in progress.

The above example shows that the first idea of deforming the projection map into an algebra homomorphism leads to hard obstructions in general, even though there are interesting classes of examples where the obstructions are absent. However, as we are interested in an approach not making too much assumptions in the beginning, we abandon this first idea. The next weaker requirement would be to deform $pr^*$ not into an algebra homomorphism but only turning $C^\infty(P)$ into a bimodule. This would have the advantage that there is no Poisson structure on $P$ needed. However, a more subtle analysis shows that again for the Hopf fibration such a bimodule structure is impossible if one uses a star product on $S^2$ coming from the symplectic Poisson structure. Thus we are left with a module structure: for later convenience we choose a right module structure and state the following definition [5]:

**Definition 4.2.** Let $pr : P \subset G \to M$ be a principal $G$-bundle over $M$ and $*$ a star product on $M$. A deformation quantization of $P$ is a right $*$-module structure $\bullet$ for $C^\infty(P)[[\lambda]]$ of the form

$$F \bullet f = Fpr^* f + \sum_{r=1}^{\infty} \lambda^r g_r(F, f),$$

(4.3)

where $g_r : C^\infty(P) \times C^\infty(M) \to C^\infty(P)$ is a bidifferential operator (along $pr$) for all $r \geq 1$, such that in addition one has the $G$-equivariance

$$g^*(F \bullet f) = g^* F \bullet f$$

(4.4)

for all $F \in C^\infty(P)[[\lambda]], f \in C^\infty(M)[[\lambda]]$ and $g \in G$.

Note that as $G$ acts on $P$ from the right, the pull-backs with the actions of $g \in G$ provide a left action on $C^\infty(P)$ in (4.4). Then this condition means that the $G$-action commutes with the module multiplications.

Note that the module property $F \bullet (f * g) = (F \bullet f) \bullet g$ implies that the constant function $1$ acts as identity. Indeed, since $1 * 1 = 1$ the action of $1$ via $\bullet$ is a projection. However, in zeroth order the map $F \mapsto F \bullet 1$ is just the identity and hence invertible. But the only invertible projection is the identity map itself. Thus

$$F \bullet 1 = F$$

(4.5)

for all $F \in C^\infty(P)[[\lambda]]$, so the module structure $\bullet$ is necessarily unital.
Finally, we call two deformation quantizations $\bullet$ and $\tilde{\bullet}$ equivalent, if there exists a $G$-equivariant equivalence transformation between them, i.e. a formal series of differential operators $T = \text{id} + \sum_{r=1}^{\infty} \lambda^r T_r$ on $C^\infty(P)[[\lambda]]$ such that
\[ T(F \bullet f) = T(F) \bullet f \quad \text{and} \quad g^* T = T g^* \] (4.6)
for all $F \in C^\infty(P)[[\lambda]]$, $f \in C^\infty(M)[[\lambda]]$ and $g \in G$.

We shall now discuss the existence and classification of such module structures. For warming up we consider the situation of a trivial principal fiber bundle:

**Example.** Let $P = M \times G$ be the trivial (and trivialized) principal $G$-bundle over $M$ with the obvious projection. For any star product $\star$ on $M$ we can now extend $\star$ to $C^\infty(M \times G)[[\lambda]]$ by simply acting only on the $M$-coordinates in the Cartesian product. Here we use the fact that we can canonically extend differential operators on $M$ to $M \times G$. Clearly, all algebraic properties are preserved whence in this case we even get a star product $\star_p = \star \otimes \mu$ with the undeformed multiplication $\mu$ for the $G$-coordinates. In particular, $C^\infty(M \times G)[[\lambda]]$ becomes a right module with respect to $\star$. So locally there are no obstructions even for the strongest requirement (4.2) and hence also for (4.3).

The problem of finding $\bullet$ is a global question whence we can not rely on local considerations directly. The most naive way to construct a $\bullet$ is an order-by-order construction: In general, one has to expect obstructions in each order which we shall now compute explicitly. This is a completely standard approach from the very first days of algebraic deformation theory [17] and will in general only yield the result that there are possible obstructions: in this case one needs more refined arguments to ensure existence of deformations whence the order-by-order argument in general is rather useless. In our situation, however, it turns out that we are surprisingly lucky.

The following argument applies essentially to arbitrary algebras and module deformations and should be considered to be folklore. Suppose we have already found $\psi_0 = \text{pr}^*, \psi_1, \ldots, \psi_k$ such that
\[ F \bullet^{(k)} f = F \text{pr}^* f + \sum_{r=1}^{k} \lambda^r \psi_r(F, f) \] (4.7)
is a module structure up to order $\lambda^k$ and each $\psi_r$ fulfills the $G$-equivariance condition. Then in order to find $\psi_{k+1}$ such that $\bullet^{(k+1)} = \bullet^{(k)} + \lambda^{k+1} \psi_{k+1}$ is a module structure up to order $\lambda^{k+1}$ we have to satisfy
\[ \psi_{k+1}(F, f) \text{pr}^* g - \psi_{k+1}(F, fg) + \psi_{k+1}(F \text{pr}^* f, g) = \sum_{r=1}^{k} (\psi_r(F, C_{k+1-r} f, g) - \psi_r(g_{k+1-r} F, f, g)) = R_k(F, f, g), \] (4.8)
for all $F \in C^\infty(P)[[\lambda]]$ and $f, g \in C^\infty(M)[[\lambda]]$. Here $C_r$ denotes the $r$-th cochain of the star product $\star$ as in (2.3). In order to interpret this equation we consider...
the $g_r$ as maps
\[ g_r : \mathcal{C}^\infty(M) \ni f \mapsto g_r(\cdot, f) \in \text{Diffop}(P) \] (4.9)
and similarly
\[ R_k : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \ni (f, g) \mapsto R_k(\cdot, f, g) \in \text{Diffop}(P). \] (4.10)
Viewing $\text{Diffop}(P)$ as $\mathcal{C}^\infty(M)$-bimodule via $\text{pr}^*$ in the usual way, we can now re-interpret (4.8) as equation between a Hochschild one-cochain $\rho_k + 1$ and a Hochschild two-cochain $R_k$
\[ \delta \rho_k + 1 = R_k \] (4.11)
in the Hochschild (sub-)complex $\text{HC}^\bullet_{\text{diff}}(\mathcal{C}^\infty(M), \text{Diffop}(P))$ consisting of differential cochains taking values in the bimodule $\text{Diffop}(P)$. Here $\delta$ is the usual Hochschild differential. Using the assumption that the $\rho_0, \ldots, \rho_k$ have been chosen such that $\bullet^{(k)}$ is a module structure up to order $\lambda^k$ it is a standard argument to show
\[ \delta R_k = 0. \] (4.12)
Thus the necessary condition for (4.11) is always fulfilled by construction whence (4.11) is a cohomological condition: The equation (4.11) has solutions if and only if the class of $R_k$ in the second Hochschild cohomology $\text{HH}^2_{\text{diff}}(\mathcal{C}^\infty(M), \text{Diffop}(P))$ is trivial.

In fact, we have also to take care of the $G$-equivariance of $\rho_{k+1}$. If all the $\rho_0, \ldots, \rho_k$ satisfy the $G$-equivariance property then it is easy to see that also $R_k$ has the $G$-equivariance property. Thus we have to consider yet another subcomplex of the differential Hochschild complex, namely
\[ \text{HC}^\bullet_{\text{diff}}(\mathcal{C}^\infty(M), \text{Diffop}(P)^G) \subseteq \text{HC}^\bullet_{\text{diff}}(\mathcal{C}^\infty(M), \text{Diffop}(P)). \] (4.13)
Thus the obstruction for (4.11) to have a $G$-equivariant solution is the Hochschild cohomology class
\[ [R_k] \in \text{HH}^2_{\text{diff}}(\mathcal{C}^\infty(M), \text{Diffop}(P)^G). \] (4.14)
A completely analogous order-by-order construction shows that also the obstructions for equivalence of two deformations $\bullet$ and $\bullet$ can be formulated using the differential Hochschild complex of $\mathcal{C}^\infty(M)$ with values in $\text{Diffop}(P)^G$. Now the obstruction lies in the first cohomology $\text{HH}^1_{\text{diff}}(\mathcal{C}^\infty(M), \text{Diffop}(P)^G)$.

The following (nontrivial) theorem solves the problem of existence and uniqueness of deformation quantizations now in a trivial way [5]:

**Theorem 4.3.** Let $\text{pr} : P \longrightarrow M$ be a surjective submersion.

1. We have
\[ \text{HH}^k_{\text{diff}}(\mathcal{C}^\infty(M), \text{Diffop}(P)) = \begin{cases} \text{Diffop}_{\text{ver}}(P) & \text{for } k = 0 \\ \{0\} & \text{for } k \geq 1. \end{cases} \] (4.15)
2. If in addition $\text{pr} : P \cap G \longrightarrow M$ is a principal $G$-bundle then we have
\[ \text{HH}^k_{\text{diff}}(\mathcal{C}^\infty(M), \text{Diffop}(P)^G) = \begin{cases} \text{Diffop}_{\text{ver}}(P)^G & \text{for } k = 0 \\ \{0\} & \text{for } k \geq 1. \end{cases} \] (4.16)
The main idea is to proceed in three steps: first one shows that one can localize the problem to a bundle chart. For the local situation one can use the explicit homotopies from [4] to show that the cohomology is acyclic. This is the most nontrivial part. By a suitable partition of unity one can glue things together to end up with the global statement. For a detailed proof we refer to [5].

From this theorem and the previous considerations we obtain immediately the following result [5]:

**Corollary 4.4.** For every principal $G$-bundle $pr : P \cup G \rightarrow M$ and any star product $\star$ on $M$ there exists a deformation quantization $\bullet$ which is unique up to equivalence.

In particular, the deformation for the trivial bundle as in Example 4 is the unique one up to equivalence.

**Remark 4.5.**
1. It should be noted that the use of Theorem 4.3 gives existence and uniqueness but no explicit construction of deformation quantizations of principal bundles. Here the cohomological method is not sufficient even though in [5] rather explicit homotopies were constructed which allow to determine further properties of $\bullet$.
2. In the more particular case of a symplectic Poisson structure on $M$, Weiss used in his thesis [40] a variant of Fedosov’s construction which gives a much more geometric and explicit approach: there is a well-motivated geometric input, namely a symplectic covariant derivative on $M$ as usual for Fedosov’s star products and a principal connection on $P$. Out of this the module multiplication $\bullet$ is constructed by a recursive procedure. The dependence of $\bullet$ on the principal connection should be interpreted as a global version of the Seiberg-Witten map [33], now of course in a much more general framework for arbitrary principal bundles, see also [2, 23, 24].
3. For the general Poisson case a more geometric construction is still missing. However, it seems to be very promising to combine global formality theorems like the one in [15] or the approach in [10] with the construction [40]. These possibilities will be investigated in future works.

## 5. The commutant and associated bundles

Theorem 4.3 gives in addition to the existence and uniqueness of deformation quantizations of $P$ also a description of the differential commutant of the right multiplications by functions on $M$ via $\bullet$; we are interested in those formal series $D = \sum_{r=0}^{\infty} \lambda^r D_r \in \text{Diffop}(P)[[\lambda]]$ of differential operators with the property

$$D(F \bullet f) = D(F) \bullet f$$

(5.1)

for all $F \in C^\infty(P)[[\lambda]]$ and $f \in C^\infty(M)[[\lambda]]$. In particular, if $D_0 = \text{id}$ then (5.1) gives a self-equivalence. Clearly, the differential commutant

$$\mathcal{K} = \{ D \in \text{Diffop}(P)[[\lambda]] \mid D \text{ satisfies (5.1)} \} \subseteq \text{Diffop}(P)[[\lambda]]$$

(5.2)
is a subalgebra of $\text{Diffop}(P[[\lambda]])$ over $\mathbb{C}[[\lambda]]$.

Note that there are other operators on $C^\infty(P[[\lambda]])$ which commute with all right multiplications, namely the highly non-local pull-backs $g^* \text{ with } g \in G$. This was just part of the Definition [4],[2] of a deformation quantization of a principal bundle. However, in this section we shall concentrate on the differential operators with (5.1) only.

Before describing the commutant it is illustrative to consider the classical situation. Here the commutant is simply given by the vertical differential operators $\text{Diffop}_{\text{ver}}(P) = \left\{ D \in \text{Diffop}(P) \mid D(F \text{pr}^* f) = D(F) \text{pr}^* f \right\}$ (5.3)

by the very definition of vertical differential operators. Alternatively, the commutant is the zeroth Hochschild cohomology. More interesting is now the next statement which gives a quantization of the classical commutant, see [5].

**Theorem 5.1.** There exists a $\mathbb{C}[[\lambda]]$-linear bijection $g' : \text{Diffop}_{\text{ver}}(P)[[\lambda]] \rightarrow \mathcal{K} \subseteq \text{Diffop}(P)[[\lambda]]$ (5.4)

of the form $g' = \text{id} + \sum_{r=1}^{\infty} \lambda^r g'_r$ (5.5)

which is $G$-equivariant, i.e.

$$g^* g' = g' g^*$$ (5.6)

for all $g \in G$. The choice of such a $g'$ induces an associative deformation $\ast'$ of $\text{Diffop}_{\text{ver}}(P)[[\lambda]]$ which is uniquely determined by $\ast$ up to equivalence. Finally, $g'$ induces a left $(\text{Diffop}_{\text{ver}}(P)[[\lambda]], \ast')$-module structure $\bullet'$ on $C^\infty(P)[[\lambda]]$ via

$$D \bullet' F = g'(D)F.$$ (5.7)

The proof relies on an adapted symbol calculus for the differential operators $\text{Diffop}(P)$: using an appropriate $G$-invariant covariant derivative $\nabla^P$ on $P$ which preserves the vertical distribution and a principal connection on $P$ one can induce a $G$-equivariant splitting of the differential operators $\text{Diffop}(P)$ into the vertical differential operators and those differential operators which differentiate at least once in horizontal directions. Note that this complementary subspace has no intrinsic meaning but depends on the choice of $\nabla^P$ and the principal connection. A recursive construction gives the corrections terms $g_r'(D)$ for a given $D \in \text{Diffop}_{\text{ver}}(P)$, heavily using the fact that the first Hochschild cohomology $\text{HH}_1^{\text{diff}}(C^\infty(M), \text{Diffop}(P))$ vanishes. Since the commutant itself is an associative algebra the remaining statements follow.

**Corollary 5.2.** For the above choice of $g'$ the resulting deformation $\ast'$ as well as the module structure are $G$-invariant, i.e. we have

$$g^* (D \ast' \tilde{D}) = g^* D \ast' g^* \tilde{D} \quad \text{and} \quad g^* (D \bullet' F) = g^* D \bullet' g^* F$$ (5.8)

for all $D, \tilde{D} \in \text{Diffop}_{\text{ver}}(P)[[\lambda]]$ and $F \in C^\infty(P)[[\lambda]]$. 
This follows immediately from the $G$-equivariance of $\bullet$ and the $G$-equivariance of $\varrho'$.

Remark 5.3. A simple induction shows that the commutant of $(\text{Diffop}_\text{ver}(P)[[\lambda]], \star')$ inside all differential operators $\text{Diffop}(P)[[\lambda]]$ is again $(\mathcal{C}^\infty(M)[[\lambda]], \star)$, where both algebras act by $\bullet'$ and $\bullet$, respectively. This way $\mathcal{C}^\infty(P)[[\lambda]]$ becomes a $(\star', \star)$-bimodule such that the two algebras acting from left and right are mutual commutants inside all differential operators. Though this resembles already much of a Morita context, it is easy to see that $\mathcal{C}^\infty(P)[[\lambda]]$ is not a Morita equivalence bimodule, e.g. it is not finitely generated and projective. However, as we shall see later, there is still a close relation to Morita theory to be expected.

Remark 5.4. Note that classically $\text{pr}^*: \mathcal{C}^\infty(M) \longrightarrow \text{Diffop}(P)$ is an algebra homomorphism, too. Thus the questions raised at the beginning of Section 4 can now be rephrased as follows: for a bimodule deformation of $\mathcal{C}^\infty(P)$ into a bimodule over $\mathcal{C}^\infty(M)[[\lambda]]$ equipped with possibly two different star products for the left and right action, one has to deform $\text{pr}^*$ into a map

$$\text{pr}^*: \mathcal{C}^\infty(M)[[\lambda]] \longrightarrow (\text{Diffop}_\text{ver}(P)[[\lambda]], \star') \quad (5.9)$$

such that the image is a subalgebra. In this case, we can induce a new product $\star'_{\mathcal{M}}$ also for $\mathcal{C}^\infty(M)[[\lambda]]$ making $\mathcal{C}^\infty(P)[[\lambda]]$ a bimodule for the two, possibly different, star product algebras $(\mathcal{C}^\infty(M)[[\lambda]], \star'_{\mathcal{M}})$ from the left and $(\mathcal{C}^\infty(M)[[\lambda]], \star)$ from the right. Note that this is the only way to achieve it since $\star'$ is uniquely determined by $\star$. Thus it is clear that we have to expect obstructions in the general case as there might be no subalgebra of $(\text{Diffop}_\text{ver}(P)[[\lambda]], \star')$ which is in bijection to $\mathcal{C}^\infty(M)[[\lambda]]$. Even if this might be the case, the resulting product $\star'_{\mathcal{M}}$ might be inequivalent to $\star$. Note however, that we have now a very precise framework for the question whether $\text{pr}^*$ can be deformed into a bimodule structure.

Remark 5.5. As a last remark we note that changing $\star$ to an equivalent $\tilde{\star}$ via an equivalence transformation $\Phi$ yields a corresponding right module structure $\tilde{\bullet}$ by

$$F \tilde{\bullet} f = F \bullet \Phi(f), \quad (5.10)$$

which is still unique up to equivalence by Theorem 4.3. It follows that the commutants are equal (for this particular choice of $\tilde{\bullet}$) whence the induced deformations $\star'$ and $\tilde{\star}'$ coincide. An equivalent choice of $\tilde{\bullet}$ would result in an equivalent $\tilde{\star}'$. This shows that we obtain a well-defined map

$$\text{Def}(\mathcal{C}^\infty(M)) \longrightarrow \text{Def}(\text{Diffop}_\text{ver}(P)) \quad (5.11)$$

for the sets of equivalence classes of associative deformations. In fact, the resulting deformations $\star'$ are even $G$-invariant, whence the above map takes values in the smaller class of $G$-invariant deformations $\text{Def}_G(\text{Diffop}_\text{ver}(P))$.

To make contact with the deformed vector bundles from Section 3 we consider now the association process. Recall that on the classical level one starts with a
(continuous) representation $\pi$ of $G$ on a finite-dimensional vector space $V$. Then the associated vector bundle is

$$E = P \times_G V \rightarrow M,$$  \hspace{1cm} (5.12)

where the fibered product is defined via the equivalence relation $(p \cdot g, v) \sim (p, \pi(g)v)$ as usual. As the action of $G$ on $P$ is proper and free, $E$ is a smooth manifold again and, in fact, a vector bundle over $M$ with typical fiber $V$. Rather tautologically, any vector bundle is obtained like this by association from its own frame bundle. For the sections of $E$ one has the canonical identifications

$$\Gamma^\infty(E) \cong C^\infty(P, V)^G$$  \hspace{1cm} (5.13)

as right $C^\infty(M)$-modules, where the $G$-action of $C^\infty(P, V)$ is the obvious one.

After this preparation it is clear how to proceed in the deformed case. From the $G$-equivariance of $\bullet$ we see that $\Gamma^\infty(E)[[\lambda]] \cong C^\infty(P, V)^G[[\lambda]] \subseteq C^\infty(P, V)[[\lambda]]$ is a $*$-submodule with respect to the restricted module multiplication $\bullet$. It induces a right $*$-module structure for $\Gamma^\infty(E)[[\lambda]]$ which we still denote by $\bullet$. This way we recover the deformed vector bundle as in Section 6.

Moreover, we see that the $\text{End}(V)$-valued differential operators $\text{Diffop}(P) \otimes \text{End}(V)$ canonically act on $C^\infty(P, V)[[\lambda]]$ in such a way that the action commutes with the $\bullet$-multiplications from the right. By the $G$-invariance of $\bullet'$ we see that the invariant elements $(\text{Diffop}_{\text{ver}}(P) \otimes \text{End}(V))^G[[\lambda]]$ form a $\bullet'$-subalgebra which preserves (via $\bullet'$) the $\bullet$-submodule $C^\infty(P, V)^G[[\lambda]]$. Thus we obtain an algebra homomorphism

$$((\text{Diffop}_{\text{ver}}(P) \otimes \text{End}(V))^G[[\lambda]], \bullet') \rightarrow (\Gamma^\infty(\text{End}(E))[\lambda], \bullet')$$  \hspace{1cm} (5.15)

where $\bullet'$ on the left hand side is the deformation from Remark 5.8 part 2.

We conclude this section with some remarks and open questions:

**Remark 5.6.**

1. The universal enveloping algebra valued gauge fields of [23, 24] can now easily be understood. For two vertical vector fields $\xi, \eta \in \text{Diffop}_{\text{ver}}(P)$ we have an action on $C^\infty(P, V)[[\lambda]]$ via $\bullet'$-left multiplication. In zeroth order this is just the usual Lie derivative $L_\xi$. Now the module structure says that

$$\xi \bullet' (\eta \bullet' F) - \eta \bullet' (\xi \bullet' F) = ([\xi, \eta]_\bullet') \bullet' F$$  \hspace{1cm} (5.16)

for all $F \in C^\infty(P)[[\lambda]]$. Here $[\xi, \eta]_\bullet' = \xi \bullet' \eta - \eta \bullet' \xi \in \text{Diffop}_{\text{ver}}(P)[[\lambda]]$ is the $\bullet'$-commutator. In general, this commutator is a formal series of vertical differential operators but not necessarily a vector field any more. Note that (5.10) holds already on the level of the principal bundle.

2. For noncommutative gauge field theories we still need a good notion of gauge fields, i.e. connection one-forms, and their curvatures within our global approach. Though there are several suggestions from e.g. [26] a conceptually clear picture seems still to be missing.
3. In a future project we plan to investigate the precise relationship between $(\text{Diffop}_{\text{ver}}(P)[[\lambda]], \star')$ and the Morita theory of star products [6–8]. Here (5.15) already suggests that one can re-construct all algebras Morita equivalent to $(C^{\infty}(M)[[\lambda]], \star)$ out of $\star'$.

Acknowledgment

It is a pleasure for me to thank the organizers Bertfried Fauser, Jürgen Tolksdorf and Eberhard Zeidler for their invitation to the very stimulating conference “Recent Developments in Quantum Field Theory”. Moreover, I would like to thank Rainer Matthes for valuable discussions on Hopf-Galois extensions and Stefan Weiß for many comments on the first draft of the manuscript.

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