Poisson Geometry of Discrete Series Orbits, and Momentum Convexity for Noncompact Group Actions

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1 Introduction

The main result of this paper is a convexity theorem for momentum mappings \( J : M \to g^* \) of certain hamiltonian actions of noncompact semisimple Lie groups. The image of \( J \) is required to fall within a certain open subset \( D \) of \( g^* \) which corresponds roughly via the orbit method to the discrete series of representations of the group \( G \). In addition, \( J \) is required to be proper as a map from \( M \) to \( D \).

A related but quite different convexity theorem for noncompact groups may be found in [16].

Our result is a first attempt toward placing momentum convexity theorems in a Poisson-geometric setting. A momentum mapping is a Poisson mapping to the dual of a Lie algebra, but it takes more than the Poisson structure on the target manifold even to formulate a convexity theorem. As we explained last year’s Conference Moshé Flato Proceedings [20], it seems that a proper symplectic groupoid is the right extra structure to put on the

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\footnote{All momentum mappings in this paper will be coadjoint-equivariant, hence Poisson.}
target. This leads us to focus our attention on $D$, which is the largest subset of $g^*$ on which the (co)adjoint action is proper, so that the subgroupoid $G \times D$ of the symplectic groupoid $T^*G = G \times g^*$ is a proper groupoid. It also motivates our interest in “relative” convexity theorems as described in Section 3.

$D$ may be defined most simply as the set of elements in $g^*$ which have compact coadjoint isotropy group. To prove that $D$ is open and that the action of $G$ on it is proper, we give several other characterizations of this subset. These involve the identification of $g^*$ with $g$ via the Killing form and the Cartan decomposition with respect to a maximal compact subgroup $K$ of $G$. In particular, elements of $D$ correspond to Lie algebra elements for which the corresponding vector field on the symmetric space $G/K$ has a nondegenerate singular point. We also give a description of $D$ which shows that, as a Poisson manifold, it is Morita equivalent, in a sense close to that of Xu [21], to an open subset of $\mathfrak{k}^*$. This enables us to reduce the convexity theorem to Kirwan’s result [12] for the compact case.

This paper is in some sense a companion to an earlier one on the principal series [19]. I realize with hindsight that both papers owe a considerable amount to the paper [7] by Guillemin and Sternberg.

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2 The proper part of the (co)adjoint action

Let $G$ be a noncompact, semisimple reductive Lie group, $K$ a fixed maximal compact subgroup. Our main goal will be to relate the coadjoint representations of $G$ and $K$.

The Killing form of $g$ leads to the orthogonal (Cartan) decomposition $g = \mathfrak{k} \oplus \mathfrak{s}$, with the Killing form negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{s}$. This form and its restriction to $\mathfrak{k}$ (which is not in general the same
as the Killing form of \(\mathfrak{t}\) give equivariant identifications of the Lie algebras of \(G\) and \(K\) with their duals. For the rest of this section, we will work in the Lie algebras themselves. In Section 2.3 we will pass to the duals and consider their Poisson structures.

### 2.1 Stable and strongly stable elements of the Lie algebra

For \(\mu \in \mathfrak{g}\), its adjoint isotropy group \(G_\mu\) is also the centralizer of the closure \(T_\mu\) of the 1-parameter subgroup generated by \(\mu\). We shall say that \(\mu\) is **stable** if \(T_\mu\) is compact (in which case it is a torus) and **strongly stable** if \(G_\mu\) is compact. (The origin of this terminology is explained in Remark 2.3 below.) We denote the set of all strongly stable elements of \(\mathfrak{g}\) by \(\mathcal{D}\).

Since every compact subgroup of \(G\) is conjugate to a subgroup of \(K\) ([10], Thm.2.1, Ch. VI), and since \(\mu\) is in the Lie algebra of \(T_\mu\), the adjoint orbit of every stable \(\mu\) intersects \(\mathfrak{k}\). Since elements of \(\mathfrak{k}\) are obviously stable, we conclude that the set of stable elements of \(\mathfrak{g}\) is equal to the saturation \(\text{Ad}_G\mathfrak{k}\) of \(\mathfrak{k}\) by the adjoint action of \(G\), and that \(\mathcal{D}\) is a subset thereof. In fact \(\mathcal{D} = \text{Ad}_G\mathcal{E}\), where \(\mathcal{E}\) is the \(K\)-invariant subset \(\mathcal{D} \cap \mathfrak{k}\) of \(\mathfrak{k}\).

The following proposition characterizes the elements of \(\mathcal{E}\) and will lead to a description of \(\mathcal{D}\).

**Proposition 2.1** For \(\mu \in \mathfrak{k}\), the following conditions are equivalent:

1. \(\mu \in \mathcal{E}\);
2. \(\mathfrak{g}_\mu \cap \mathfrak{s} = \{0\}\);
3. the endomorphism \(\mu_s\) of \(\mathfrak{s}\) given by the infinitesimal adjoint representation is a nonsingular map;
4. the vector field \(\mu_S\) on the symmetric space \(S = G/K\) given by the infinitesimal action of \(\mu\) has a nondegenerate zero at the coset \(eK\);
5. \(eK\) is the only zero point of \(\mu_S\).

**Proof.** (1) \(\Rightarrow\) (2): The 1-parameter subgroup generated by a nonzero element \(\mu\) of \(\mathfrak{s}\) goes to infinity in \(G\), so it cannot be contained in a compact subgroup. Thus \(\mu\) is unstable and hence cannot belong to \(\mathcal{E}\).

(2) \(\Leftrightarrow\) (3): An element \(w\) of \(\mathfrak{s}\) belongs to \(\mathfrak{g}_\mu\) if and only if \([\mu, w] = 0\).

(3) \(\Leftrightarrow\) (4): The tangent space to \(G/K\) at \(eK\) is isomorphic to \(\mathfrak{g}/\mathfrak{k}\). The linearization of \(\mu_S\) at \(eK\) is the adjoint action of \(\mu\) on \(\mathfrak{g}/\mathfrak{k}\), which is equivalent to its action on \(\mathfrak{s}\).
(4) ⇔ (5): The exponential map gives a $K$-equivariant diffeomorphism from $s$ to $S$ (Thm. 1.1, Ch. VI), so the vector field $\mu_S$ is equivalent to its linearization at $eK$.

(5) ⇒ (1) Any element of $G_\mu$ leaves the zero set of $\mu_S$ invariant. If this zero set reduces to the single point $eK$, then $G_\mu$ must be contained in the isotropy group $K$ and is therefore compact.

□

Corollary 2.2 $\mathcal{E}$ is an open subset of $\mathfrak{k}$.

Proof. The statement follows immediately from (3) or (4) in the proposition above.

□

We can now characterize the strongly stable elements of $\mathfrak{g}$ in several ways.

Proposition 2.3 For $\mu \in \mathfrak{g}$, the following conditions are equivalent:

(1) $\mu \in \mathcal{D}$;

(4) the vector field $\mu_S$ on the symmetric space $S = G/K$ given by the infinitesimal action of $\mu$ has a nondegenerate zero at some point;

(5) the vector field $\mu_S$ is zero at exactly one point of $S$;

(6) $\mu$ belongs to the Lie algebra of exactly one maximal compact subgroup of $G$;

(7) $\mu$ is an interior point of the set $\text{Ad}_G \mathfrak{k}$ of stable elements.

Proof. (1) ⇔ (4): If $\mu \in \mathcal{D}$, then $\text{Ad}_g \mu \in \mathcal{E}$ for some $g \in G$. The action of $g$ on $S$ gives an equivalence between the vector fields $(\text{Ad}_g \mu)_S$ and $\mu_S$. By (4) of Proposition 2.3, $(\text{Ad}_g \mu)_S$ has a nondegenerate zero at a point of $S$, hence so does $\mu_S$. Conversely, if $\mu_S$ has a zero at some point, then $\text{Ad}_g \mu \in \mathcal{E}$ for some $g \in G$. If the zero of $(\text{Ad}_g \mu)_S$ is nondegenerate, so is that of $\mu_S$, in which case $\text{Ad}_g \mu \in \mathcal{E}$, and hence $\mu \in \mathcal{D}$.

(1) ⇔ (5): The argument is just like the one which showed that (1) ⇔ (4). Or one can show that (4) ⇔ (5) by using the riemannian exponential map at the zero point.

\[^2\text{Some numbers are omitted to make the remaining numbers consistent with those in Proposition 2.1.}\]
(5) ⇔ (6): The maximal compact subgroups are the isotropy groups of the points of $S$.

(1) ⇔ (7): By Corollary 2.4 below, $\mathcal{D}$ is open in $\mathfrak{g}$. Since $\mathcal{D} \subset \text{Ad}_G \mathfrak{k}$, all points of $\mathcal{D}$ are interior points of $\text{Ad}_G \mathfrak{k}$. Conversely, suppose that $\mu$ is an interior point of $\text{Ad}_G \mathfrak{k}$. Then we have some $\nu = \text{Ad}_\mu \mu \in \mathfrak{k}$, and we need to show that $\nu$, which is also an interior point of $\text{Ad}_G \mathfrak{k}$, belongs to $\mathcal{E}$. If it did not, there would be a nonzero $w \in \mathfrak{s}$ with $[\nu, w] = 0$. But then, for all real $t$, $\nu + tw$ would be unstable, and hence $\nu$ could not be an interior point of $\text{Ad}_G \mathfrak{k}$.

$\square$

**Corollary 2.4** $\mathcal{D}$ is an open subset of $\mathfrak{g}$.

**Proof.** The statement follows directly from (4) in the proposition above, since the property of having a nondegenerate zero point is stable under small perturbations.

$\square$

**Remark 2.5** Part (7) of Proposition 2.3 says that a stable element of $\mathfrak{g}$ is strongly stable if and only if it remains stable under small perturbations. This property justifies the term “strongly stable” – the term was originally used in the case where $G$ is a symplectic group and $K$ a unitary group, elements of $\mathfrak{g}$ then being linear autonomous hamiltonian dynamical systems. See Section 2.3 below for further discussion of this example, with some references.

We may now identify those groups for which $\mathcal{D}$ is nonempty. Since every strongly stable element is stable, it is semisimple, and since $\mathcal{D}$ is open, it contains regular semisimple elements. The stabilizer of such an element is a compact Cartan subgroup of $G$ ([10] Thm. 3.1, Ch. III).

Conversely, if $G$ has a compact Cartan subgroup $T$, the stabilizer of any regular element of $\mathfrak{t}$ is equal to $T$, hence compact, so this element belongs to $\mathcal{D}$. Thus we have shown:

**Proposition 2.6** The set $\mathcal{D} \subset \mathfrak{g}$ of strongly stable elements is nonempty if and only if $G$ contains a compact Cartan subgroup, i.e. if and only if $G$ has the same rank as its maximal compact subgroup $K$. 

5
Remark 2.7 We note that the condition rank $K = \text{rank } G$ is precisely the condition for $G$ to admit discrete series representations \[9\] i.e. irreducible subrepresentations of its left regular representation. In fact, the discrete series representations correspond via character theory or the orbit method to certain orbits in $\mathcal{D}$. (See Remarks 2.9 and 2.17 below.)

2.2 Characterization in terms of roots

To give a more concrete description of the elements of $\mathcal{E}$, and hence those of $\mathcal{D}$, we choose a maximal torus $T \subset K$ (which is also a Cartan subgroup of $G$). Since $\text{Ad}_K \mathfrak{t}$ is all of $\mathfrak{t}$, we have $\mathcal{E} = \text{Ad}_K \mathcal{F}$, where $\mathcal{F} = \mathcal{E} \cap \mathfrak{t}$. Then $\mathcal{D} = \text{Ad}_G \mathcal{F}$.

To identify the elements of $\mathcal{F}$, we recall that the adjoint action of $T$ on $\mathfrak{t}$ leaves invariant the splitting $\mathfrak{k} \oplus \mathfrak{s}$, so that each root in $\mathfrak{t}^*$ can be designated as either “compact” or “noncompact” according to whether the corresponding eigenvector lies in $\mathfrak{k}$ or $\mathfrak{s}$. Then the following characterization of $\mathcal{F}$ follows immediately from (3) of Proposition 2.1.

Proposition 2.8 $\mathcal{D} = \text{Ad}_G \mathcal{F}$, where $\mathcal{F} \subset \mathfrak{t}$ is the complement of the zero-hyperplanes of the noncompact roots. In particular, $\mathcal{F}$ is dense in $\mathfrak{t}$ and is the disjoint union of finitely many convex open subsets.

Remark 2.9 According to Harish-Chandra \[9\], the discrete series representations are parametrized by the subset of $\mathcal{D}$ consisting of those (integral) orbits whose intersection with $\mathfrak{t}$ lies in the complement of the zero-hyperplanes of all the roots. The remaining integral orbits in $\mathcal{D}$ correspond to “limits of discrete series representations.”

The density of $\mathcal{F}$ in $\mathfrak{t}$ implies that the stable and strongly stable elements of $\mathfrak{g}$ have the same closure, i.e.:

Corollary 2.10 $\overline{\mathcal{D}} = \overline{\text{Ad}_G \mathfrak{t}}$.

The example of $SL(2, \mathbb{R})$ already shows that $\text{Ad}_G \mathfrak{t}$ itself is not closed–its closure includes the nilpotent cone.
2.3 Example: the symplectic group

Among the groups having the same rank as their maximal compact subgroups are the indefinite unitary groups $U(m,n)$, the indefinite orthogonal groups $SO(m,n)$ for which $m$ and $n$ are not both odd, and the (real) symplectic groups $Sp(2n)$. In this section, we will concentrate on the last example and will study the strongly stable elements of $sp(2n)$.

We use canonical coordinates $(q_j, p_j)$ for the symplectic structure $\omega = \sum dq_j \wedge dp_j$ on $\mathbb{R}^{2n}$. We also identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by using the complex coordinates $z_j = q_j + ip_j$. The maximal compact subgroup of $G = Sp(2n)$ is the unitary group $K = U(n)$; the Cartan subgroup $T = T^n$ consists of the diagonal unitary matrices. The symmetric space $S = Sp(2n)/U(n)$ may be identified with the set of positive polarizations on $\mathbb{R}^{2n}$, i.e. the almost complex structures $J$ on $\mathbb{R}^{2n}$ for which the bilinear form $\omega(x, Jy)$ is symmetric and positive-definite.

We identify elements of $sp(2n)$ with linear hamiltonian vector fields and, in turn, with the quadratic hamiltonian functions which generate them.

The equivalence of (5) and (7) in Proposition 2.3 tells us that a linear hamiltonian system is stable and remains so under small perturbations if and only if it is “uniquely unitarizable,” i.e. if an only if it leaves invariant a unique compatible complex structure. This type of result plays a basic role in I. Segal’s approach to quantum field theory. For instance, in [18], Segal proves in the infinite dimensional case that a stable linear symplectic map $T$ is “uniquely unitarizable” if and only if $T$ and $T^{-1}$ have disjoint spectra.

We can also use Proposition 2.8 to characterize strongly stable quadratic hamiltonians. The compact Cartan subalgebra consists of hamiltonians of the form

$$\sum_{j=1}^{n} \frac{\lambda_j}{2} (q_j^2 + p_j^2),$$

where the $\lambda_j$ are arbitrary real numbers whose absolute values are the frequencies of the normal modes of oscillation for the linear hamiltonian system.

The compact roots are the differences $\lambda_j - \lambda_k$ for all pairs $j < k$, while the noncompact roots are the sums $\lambda_j + \lambda_k$ for $j \leq k$. The strong stability condition – nonvanishing of the noncompact roots – means that all the normal mode frequencies are nonzero, and that, whenever there is a simple resonance $|\lambda_j| = |\lambda_k|$, the signs of $\lambda_j$ and $\lambda_k$ are the same. This criterion
for strong stability is well known in the theory of Hamiltonian systems.\footnote{Strong stability, also known as \textit{parametric stability}, was characterized in a similar way for elements of the symplectic group, rather than of its Lie algebra, by Krein \cite{krein} (who only announced results), and Gelfand and Lidskii \cite{lidskii}. The same characterization (definiteness of a quadratic form on eigenspaces) was given by Moser \cite{moser}.}

Note that all positive-definite and negative-definite Hamiltonians are strongly stable—these form two of the connected components of $\mathcal{D}$.

### 2.4 Bundle structure; properness of the action

We still owe the reader a proof of the property which originally motivated our definition of $\mathcal{D}$, namely the properness of the adjoint action. The proof will be based on the following description of $\mathcal{D}$.

**Proposition 2.11** $\mathcal{D}$ is $G$-equivariantly isomorphic to the associated bundle $(G \times \mathcal{E})/K$, which is an open subbundle of the homogeneous vector bundle $(G \times \mathfrak{t})/K$.

**Proof.** By (4) and (5) of Proposition 2.3, for each $\mu \in \mathcal{D}$ the vector field $\mu_S$ has a unique zero $\phi(\mu)$ in $S = G/K$, and this zero is nondegenerate. It follows from the implicit function theorem that $\phi$ is a smooth mapping from $\mathcal{D}$ to $G/K$. Since the map $\mu \mapsto \mu_S$ is $G$-equivariant, so is $\phi$. The fibre of $\phi$ over $eK$ consists of those $\mu \in \mathcal{D}$ which generate 1-parameter subgroups fixing $eK$, i.e. $\phi^{-1}(eK) = \mathcal{D} \cap \mathfrak{t} = \mathcal{E}$. The statement of the proposition follows. Concretely, we map $G \times \mathcal{E}$ to $\mathcal{D}$ by $(g, \nu) \mapsto \text{Ad}_g \nu$ and observe that the fibres of this map are the $K$-orbits.

\[\square\]

We will use the following simple lemma about proper actions.

**Lemma 2.12** If $\phi : X \to Y$ is a continuous equivariant map of $G$-manifolds, and if the action of $G$ on $Y$ is proper, so is the action on $X$.

**Proof.** By the definition of properness, the action map $\alpha_Y : (g, y) \mapsto (gy, y)$ from $G \times Y$ to $Y \times Y$ is proper. To check that the corresponding map $\alpha_X$ for the action on $X$ is proper, we let $K$ be an arbitrary compact subset of $X \times X$. Equivariance implies that $(\text{Id} \times \phi)(\alpha_X^{-1}(K)) \subseteq \alpha_Y^{-1}((\phi \times \phi)(K))$. Since $K$ is compact and $\phi$ is continuous, $(\phi \times \phi)(K)$ is compact; by the
properness of $\alpha_Y$, $\alpha_Y^{-1}((\phi \times \phi)(K))$ is compact as well, hence so is its closed subset $(\operatorname{Id} \times \phi)\alpha_Y^{-1}(K)$. Applying the projection to $G$, which is unaffected by $(\operatorname{Id} \times \phi)^{-1}$, we find that there is a compact subset $A \subseteq G$ such that $A \times X$ contains $\alpha_X^{-1}(K)$. On the other hand, applying the second projection from $K \subseteq X \times X$ into $X$, which is unaffected by $\alpha_X$, we find that $\alpha_X^{-1}(K)$ is contained in $G \times B$, where $B$ is a compact subset of $X$. Thus the closed set $\alpha_X^{-1}(K)$ is contained in $A \times B$, and so it is compact.

\[
\square
\]

**Corollary 2.13** The adjoint action of $G$ on $\mathcal{D}$ is proper.

**Proof.** The result follows from Lemma 2.12 and Proposition 2.11 once we observe that the action map $G \times G/K \to G/K \times G/K$ is a locally trivial fibration with compact fibres (essentially the isotropy groups of the action).

\[
\square
\]

### 2.5 Poisson geometry of $\mathcal{D}$

We will now relate the Poisson geometry of $\mathcal{D} \subset \mathfrak{g}^*$ to that of $\mathcal{E} \subset \mathfrak{t}^*$ by constructing a Morita equivalence, or dual pair, relating them. The construction is based on the slice method of [6], where an analogous relation is established between an open subset of $\mathfrak{t}^*$ (the regular elements) and the interior of the positive Weyl chamber in $\mathfrak{t}^*$.

The Killing form of $\mathfrak{g}$ gives equivariant identifications of $\mathfrak{g}^*$ with $\mathfrak{g}$ and of $\mathfrak{t}^*$ with $\mathfrak{t}$ and the annihilator of $\mathfrak{s}$ in $\mathfrak{g}^*$. This allows us to consider $\mathcal{E}$ and $\mathcal{D}$ as open Poisson submanifolds of $\mathfrak{g}^*$ and $\mathfrak{t}^*$ respectively.

We will call a submanifold $N$ of a Poisson manifold $P$ **cosymplectic** if the restriction of the Poisson tensor to each conormal space of $P$ is nondegenerate. Equivalently, if $P$ is defined locally by the vanishing of independent functions $f_1, \ldots, f_k$, $P$ is cosymplectic if the matrix of Poisson brackets $a_{ij} = \{f_i, f_j\}$ is nondegenerate along $P$. (In the language of Dirac [4], the $f_i$ are **second-class constraints**.) Another characterization is that $N$ is cosymplectic if it is transversal to each symplectic leaf of $P$, and if its intersection with each such leaf is a symplectic submanifold of the leaf. In particular, the cosymplectic submanifolds of a symplectic manifold are just the symplectic submanifolds.
Lemma 2.14 Let \( N \) be a cosymplectic submanifold of the Poisson manifold \( P \), and let \( \phi : Q \to P \) be a Poisson map. Then \( \phi \) is transverse to \( N \), and \( \phi^{-1}(N) \) is a cosymplectic submanifold of \( Q \).

Proof. Let \( \phi(x) \in N \) for some \( x \in Q \), and let \((f_1, \ldots, f_k)\) be independent functions defining \( N \) near \( \phi(x) \). Then the functions \((\phi^* f_1, \ldots, \phi^* f_k)\) define \( \phi^{-1}(N) \) near \( x \). The matrix of Poisson brackets

\[
\{ \phi^* f_i, \phi^* f_j \}(x) = \phi^* \{ f_i, f_j \}(x) = \{ f_i, f_j \}(\phi(x))
\]

is nondegenerate. It follows, first of all, that the functions \((\phi^* f_1, \ldots, \phi^* f_k)\) are independent near \( x \), so that \( \phi \) is transverse to \( N \). It then follows that \( \phi^{-1}(N) \) is cosymplectic.

\[\square\]

Proposition 2.15 The subset \( \mathcal{E} \subset \mathfrak{k}^* \) of strongly stable points is the set of points at which \( \mathfrak{k}^* \) is cosymplectic in \( \mathfrak{g}^* \). Hence \( \mathcal{E} \) is a cosymplectic submanifold of \( \mathcal{D} \).

Proof. The conormal space to \( \mathfrak{k}^* \subset \mathfrak{g}^* \) at \( \mu \) may be identified with \( \mathfrak{s} \subset \mathfrak{g} \), on which the Poisson tensor is the bilinear form \( (v, w) \mapsto \langle \mu, [v, w] \rangle \), where \( \langle \ , \ \rangle \) is the Killing form. By invariance of the Killing form, this can also be written as \( \langle [\mu, v], w \rangle \), so the Poisson tensor is nondegenerate exactly when the action of \( \mu \) on \( \mathfrak{s} \) is nonsingular. Now apply (3) of Proposition 2.1.

\[\square\]

We shall identify the symplectic manifold \( T^*G \) with \( G \times \mathfrak{g}^* \) by right translations. Then we have:

Corollary 2.16 The product \( M = G \times \mathcal{E} \subset G \times \mathfrak{g} \) is a symplectic submanifold of \( T^*G \).

The action of \( G \) on \( M \) by right translations \((g \cdot h = hg^{-1}\) is hamiltonian with momentum map equal to \( \psi : (g, \mu) \mapsto -\text{Ad}_g \mu \), which is a Poisson submersion from \( M \) to \( \mathcal{D} \). \( M \) is also invariant under the hamiltonian action of \( K \) by left translations, for which the momentum map is right translation, i.e. the projection from \( M = G \times \mathcal{E} \) to \( \mathcal{E} \). The two momentum maps form a symplectic dual pair relating the Poisson manifolds \( \mathcal{E} \) and \( \mathcal{D} \).
Proof. The momentum map for the cotangent lift of the action of $G$ on itself by right translations is the negative of the left translation map. In our right trivialization, this appears as $\psi: (g, \mu) \mapsto -\text{Ad}_g \mu$. It follows from Lemma 2.14 and Proposition 2.15 that $\mathcal{M}$ is a symplectic submanifold of $T^*G$. In general, the momentum map for the restriction of a hamiltonian action to an invariant symplectic submanifold is the restriction of the original momentum map, since the momentum map for the right action of $G$ on $\mathcal{M}$ is the negative of the projection, hence $\psi: (g, \mu) \mapsto -\text{Ad}_g \mu$ is a Poisson map from $\mathcal{M}$ to $\mathfrak{g}$. It is a submersion because the action of $G$ on $\mathcal{M}$ is free, and its image is $\text{Ad}_G \mathcal{E} = \mathcal{D}$.

Invariance of $\mathcal{M}$ under left translations by $K$ follows from the $\text{Ad}_K$-invariance of $\mathcal{E}$. The momentum map for the left action of $K$ on $T^*G = G \times \mathfrak{g}^*$ is given by right translation, which is just the projection. Again, it is a submersion because the action of $K$ is free.

To see that the pair of maps $\mathfrak{g} \leftarrow \mathcal{M} \rightarrow \mathfrak{k}$ form a symplectic dual pair, i.e. that they are submersions whose fibres have symplectically orthogonal tangent spaces, we simply note that they are the momentum maps of commuting free actions of groups whose dimensions add up to that of $\mathcal{M}$.

\[\square\]

Remark 2.17 Assuming that $G$ (and hence $K$) is connected, the dual pair of Proposition 2.15 has connected fibres. By Proposition 9.2 of [2], this gives a bijection between the symplectic leaves of $\mathcal{D}$ and $\mathcal{E}$. Here, this bijection is simply given by intersecting the leaves with $\mathfrak{k}$.

An important property of dual pairs of group actions is that the orbit space of one action is Poisson-isomorphic to the image of the momentum map of the other. Using this fact one way gives the uninteresting isomorphism between $(G \times \mathcal{E})/G$ and $\mathcal{E}$. But in the other direction we obtain an isomorphism between $(G \times \mathcal{E}/K)$ and $\mathcal{D}$. This is a symplectic version of the description of $\mathcal{D}$ as an $\mathcal{E}$ bundle over $G/K$ given in 2.11.

A symplectic dual pair, i.e. a pair of Poisson submersions with connected, symplectically orthogonal fibres, is called a Morita equivalence if the fibres are simply-connected and if the submersions satisfy a completeness condition. Xu [21], has shown that Morita equivalence of Poisson manifolds implies representation equivalence, i.e. an equivalence between
their symplectic realizations. Although in some examples the dual pair of $P$ does not have simply-connected fibres, one may think of the representation equivalence between $\mathcal{D}$ and $\mathcal{E}$ as a classical analogue of the equivalence between the discrete series representations of $G$ and the representations of the maximal compact subgroup $K$.

3 Actions and convexity

We will begin this section by defining properness of momentum maps relative to open subsets of the dual of a Lie algebra. After recalling the known convexity theorems for compact group actions, we will prove the central theorem of this paper, a convexity theorem for certain actions of noncompact groups.

3.1 Proper hamiltonian spaces

Let $G$ be a Lie group, and let $\mathcal{U} \subseteq \mathfrak{g}^*$ be a coadjoint-invariant open subset. We define a hamiltonian $(G,\mathcal{U})$-space $(M,J)$ to be a symplectic manifold $M$ with a symplectic $G$-action and a coadjoint-equivariant momentum map $J : M \to \mathcal{U} \subseteq \mathfrak{g}^*$. We shall consider $J$ as a map to $\mathcal{U}$ rather than to $\mathfrak{g}^*$ and will call the $(G,\mathcal{U})$ space proper if $J$ is a proper mapping and if the action of $G$ on $M$ is proper. By Lemma 2.12, the second condition follows from the first if the coadjoint action of $G$ on $\mathcal{U}$ is proper, e.g. when $G$ is semisimple and $\mathcal{U}$ consists of strongly stable elements.

3.2 The compact case

The first convexity theorem applies to torus actions.

**Theorem 3.1** Let $\mathcal{U}$ be a disjoint union of convex open subsets of the dual of the Lie algebra of a torus $T$, and let $(M,J)$ be a connected, proper, hamiltonian $(T,\mathcal{U})$-space. Then $J(M)$ is a closed, convex locally polyhedral subset of $\mathcal{U}$, and the inverse image $J^{-1}(\mu)$ is connected for each $\mu \in \mathcal{U}$.

Theorem 3.1 was originally proved by Guillemin and Sternberg \[\text{[6]}\] and Atiyah \[\text{[1]}\] in the case where $M$ is compact and $\mathcal{U}$ is all of $\mathfrak{t}^*$. Extensions to noncompact $M$ were given by Prato \[\text{[17]}\] and, using methods of Condevaux, Dazord, and Molino \[\text{[3]}\], by Hilgert, Neeb, and Plank \[\text{[11]}\].
An extension of the convexity theorem to actions of nonabelian compact groups on compact manifolds was conjectured and partially proved by Guillemin and Sternberg [8]. The proof was completed by Kirwan [12]. The result was extended to proper actions in [11], and the “relative” version, where the momentum map is proper as a map into an open subset containing its image, was obtained by Lerman, Meinrenken, Tolman, and Woodward [14].

In the statement of the following theorem, we have identified the Lie algebras with their duals by using a bi-invariant metric on the group.

**Theorem 3.2** Let $K$ be a compact Lie group, let $t^*_+$ be a positive Weyl chamber in $t^* \subseteq g^*$, and let $U \subseteq g$ be a coadjoint-invariant open subset such that each component of $U \cap t^*_+$ is convex. If $(M, J)$ is a connected, proper, hamiltonian $(G, U)$-space, then $J(M) \cap t^*_+$ is a closed, convex, locally polyhedral subset of $U \cap t^*_+$, and $J^{-1}(\mu)$ is connected for each $\mu \in U$.

### 3.3 The noncompact case

We will now state and prove the main result of this paper.

**Theorem 3.3** Let $G$ be a semisimple Lie group, let $t^*_+$ be a positive Weyl chamber for a maximal compact subgroup $K$ of $G$, and let $U$ be a coadjoint-invariant open subset of the set $D \subseteq g^*$ of strongly stable elements such that $U \cap t^*_+$ is convex. If $(M, J)$ is a connected, proper, hamiltonian $(G, U)$-space, then $J(M) \cap t^*_+$ is a closed, convex, locally polyhedral subset of $t^*_+ \cap U$, and $J^{-1}(\mu)$ is connected for each $\mu \in U$.

**Proof.** Just as the convexity theorem for nonabelian compact $K$ is proved by reduction to the abelian case, our theorem for noncompact $G$ will be proved by reduction to the compact case. To this end, we let $N = J^{-1}(U \cap E) = J^{-1}(U \cap t^*) \subseteq M$. Since $E$ is symplectic in $D$, $U \cap E$ is symplectic in $U$; by Lemma 2.14, $J$ is transverse to $U \cap E$, and $N$ is a symplectic submanifold of $M$. Thus, $N$ is a symplectic manifold, and, since $J$ is equivariant, $N$ is $K$-invariant. By basic facts about the behavior of momentum maps under restriction (to subgroups and submanifolds), $J|_N : N \to U \cap E$ is an equivariant momentum map for the action of $K$ on $N$, making $N$ into a hamiltonian $(K, U \cap E)$-space. The properness of $J$ implies immediately that $J|_N$ is proper as well.
To apply Theorem 3.2 to this $K$-space, we just need to know that $N$ is connected. To prove this, we compose $J$ with the equivariant map $\phi : D \to G/K$ of Proposition 2.11. Since $\mathcal{E} = \phi^{-1}(eK)$, $\phi \circ J$ makes $M$ into a bundle over $G/K$ with typical fibre $J^{-1}(\mathcal{E}) = N$. Now $G/K$ is contractible, hence simply connected, and $M$ is connected, so $N$ must be connected as well. It follows from Theorem 3.2 that $J|_N(N) \cap t^*_+ \cap t^* \cap \mathbb{U} \cap E$ is a closed, convex, locally polyhedral subset of $t^* \cap \mathbb{U} \cap E$ and that $J|_N^{-1}(\mu)$ is connected for each $\mu \in \mathbb{U} \cap E$. But $J|_N(N) \cap t^*_+$ is equal to $J(M) \cap t^*_+$ also, for any $\nu$ in the invariant subset $\mathbb{U} \subseteq \mathcal{D} \subseteq \text{Ad}_G \mathcal{E}$, so $\nu = g\mu$ for some $\mu$ in $\mathbb{U} \cap E$, so $J^{-1}(\nu) = gJ^{-1}(\mu)$ is connected. This completes the proof of the theorem.

\[\square\]

4 Products of coadjoint orbits and sums of positive definite hamiltonians

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be coadjoint orbits in $\mathfrak{g}^*$. Then $G$ acts diagonally on the product $\mathcal{O}_1 \times \mathcal{O}_2$ with momentum map given by the addition map $A : \mathcal{O}_1 \times \mathcal{O}_2 \to \mathfrak{g}^*$. There are certain cases in which the image of $A$ lies in $\mathcal{D}$. For instance, in the case of the symplectic groups (see Section 2.3), we may let $\mathcal{O}_1$ and $\mathcal{O}_2$ be orbits consisting of positive-definite hamiltonians, since the sum of two positive functions is positive. In this case, our convexity theorem has the following interesting consequence in hamiltonian dynamics.

**Theorem 4.1** For any positive-definite quadratic hamiltonian function $H$ on $\mathbb{R}^{2n}$, let $F(H)$ be the $n$-tuple $(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \leq \cdots \leq \lambda_n$ are the frequencies of the normal modes of oscillation for the linear hamiltonian system generated by $H$; i.e. $F(H)$ are the coefficients of the normal form $\sum_{j=1}^n \frac{\lambda_j}{2}(q_j^2 + p_j^2)$ for $H$ in suitably chosen canonical coordinates. If $\lambda$ and $\mu$ are nondecreasing $n$-tuples of positive real numbers, then

$$\{F(H_1 + H_2) | F(H_1) = \lambda \text{ and } F(H_2) = \mu\}$$

is a closed, convex, locally polyhedral set.

**Proof.** We follow a standard approach, used already to study the spectrum of sums of hermitian matrices with prescribed eigenvalues. Let $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$ be
the (co)adjoint orbits of \( \sum_{j=1}^{n} \frac{\lambda_j}{2} (q_j^2 + p_j^2) \) and \( \sum_{j=1}^{n} \frac{\mu_j}{2} (q_j^2 + p_j^2) \) respectively.

Theorem 4.1 will follow from Theorem 3.3 once we have proven that the addition operation \((H_1, H_2) \mapsto H_1 \times H_2\) is a proper map from \(\mathcal{O}_\lambda \times \mathcal{O}_\mu\) to \(\mathcal{D}\).

Suppose, then, that \(\{H_{1,i}\}\) and \(\{H_{2,i}\}\) are sequences in \(\mathcal{O}_\lambda\) and \(\mathcal{O}_\mu\) respectively such that \(H_{1,i} + H_{2,i}\) converges to an element \(H\) of \(\mathcal{D}\). We must show that \(\{(H_{1,i}, H_{2,i})\}\) has a convergent subsequence.

We will use the “diagonal” projection \(\pi\) from \(\mathfrak{sp}(2n)\) to its Cartan subalgebra \(\mathbb{R}^n\), which selects the coefficients of the terms \(\frac{1}{2}(q_j^2 + p_j^2)\) in a quadratic hamiltonian, ignoring the coefficients of \(\frac{1}{2}(q_j^2 - p_j^2)\) and all “cross terms”. The map \(\pi\) is the momentum map for the action of the Cartan subgroup and, according to Proposition 2.2 in [17], its restriction to any coadjoint orbit of positive-definite hamiltonians is proper.

By applying a preliminary transformation in \(Sp(2n)\), we may assume that \(H\) is in the normal form \(\sum_{j=1}^{n} \frac{\nu_j}{2} (q_j^2 + p_j^2)\). Since \(\pi\) is linear and takes positive definite matrices to positive \(n\)-tuples, the sequences \(\{\pi(H_{1,i})\}\) and \(\{\pi(H_{2,i})\}\) are bounded and, hence, after we pass to subsequences, may be assumed convergent. But now the properness of \(\pi\) on the coadjoint orbits implies that \(\{(H_{1,i}, H_{2,i})\}\) has a convergent subsequence, and the theorem is proven.

\[
\rho = 2\pi \frac{\text{energy}}{\text{action}}
\]

for the simple periodic trajectories of the hamiltonian dynamical system. We may ask, then, about the possible values of \(\rho\) for periodic orbits for \(\sum_{j=1}^{n} \frac{\lambda_j}{2} (q_j^2 + p_j^2) \circ \phi_1 + \sum_{j=1}^{n} \frac{\mu_j}{2} (q_j^2 + p_j^2) \circ \phi_2\), where \(\phi_1\) and \(\phi_2\) are homogeneous, but not necessarily linear, symplectic diffeomorphisms of \(\mathbb{R}^{2n}\setminus\{0\}\).

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