Configuration Spaces, Bistellar Moves, and Combinatorial Formulae for the First Pontryagin Class

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Abstract—The paper is devoted to the problem of finding explicit combinatorial formulae for the Pontryagin classes. We discuss two formulae, the classical Gabrielov–Gelfand–Losik formula based on investigation of configuration spaces and the local combinatorial formula obtained by the author in 2004. The latter formula is based on the notion of a universal local formula introduced by the author and on the usage of bistellar moves. We give a brief sketch for the first formula and a rather detailed exposition for the second one. For the second formula, we also succeed to simplify it by providing a new simpler algorithm for decomposing a cycle in the graph of bistellar moves of two-dimensional combinatorial spheres into a linear combination of elementary cycles.

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1. INTRODUCTION

The problem of finding combinatorial formulae for the Pontryagin classes of triangulated manifolds goes back to the remarkable work [1] by A.M. Gabrielov, I.M. Gelfand, and M.V. Losik, where they constructed the first explicit combinatorial formula for the first rational Pontryagin class. Later different combinatorial formulae were obtained in [2–8]. Let us also mention that N. Levitt and C. Rourke [9] proved the existence of local combinatorial formulae for all polynomials in rational Pontryagin classes without constructing explicit formulae. The simplest of all known combinatorial formulae for the first rational Pontryagin class was obtained by the author in 2004 [7]. The survey [10] is devoted to the comparison of different combinatorial formulae for the Pontryagin classes.

Before passing to a detailed discussion of combinatorial formulae for the Pontryagin classes, we shall briefly discuss a simpler problem of combinatorial computation of the Stiefel–Whitney classes. The following assertion was conjectured by E. Stiefel [11] and proved by H. Whitney [12].

**Theorem 1.1.** Suppose \( K \) is a closed \( n \)-dimensional combinatorial manifold and \( K' \) is its first barycentric subdivision. Denote by \( W_k \) the sum modulo 2 of all \( k \)-dimensional simplices of \( K' \). Then the simplicial chain \( W_k \) is a cycle with coefficients in \( \mathbb{Z}_2 \) and represents the Poincaré dual of the Stiefel–Whitney class \( w_{n-k}(K) \).

H. Whitney did not publish a detailed proof of this theorem. An accurate proof of it was published only in 1972 by S. Halperin and D. Toledo [13]. Their proof, as well as the original proof by H. Whitney, is based on an explicit construction of tangent vector fields \( F_1, \ldots, F_p \) on \( K \) such that the fields \( F_1, \ldots, F_p \) are linearly independent outside the \((n-p)\)-skeleton of \( K' \) and the index of the vector field \( F_p \) modulo \( F_1, \ldots, F_{p-1} \) at the barycentre of each \((n-p)\)-dimensional simplex.

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of $K'$ is equal to $\pm 1$. Another proof of Theorem 1.1, based on quite different ideas, was obtained by J. Cheeger [14].

In this paper we consider two combinatorial formulae for the first rational Pontryagin class, the classical Gabrielov–Gelfand–Losik formula [1] and the formula obtained by the author in [7]. In fact, these two formulae are at the moment the only known formulae for the Pontryagin classes that can be used for real computation. All known formulae for higher Pontryagin classes (see [4–6, 8]) cannot be used for real computation. The formula due to A.M. Gabrielov [4] can only be applied to a very small class of triangulations of manifolds. The formula due to I.M. Gelfand and R. MacPherson is not purely combinatorial [5], since it computes the Pontryagin classes of a manifold from a given triangulation with a given smoothing rather than from a given triangulation only. Though the formula due to the author [8] is purely combinatorial, it is extremely complicated and, hence, cannot be used for real computation even in the simplest cases. We shall separately mention an analytic approach developing the results of Atiyah–Patodi–Singer [15], which allowed J. Cheeger [6] to obtain formulae for the Pontryagin classes of triangulated manifolds in terms of the spectra of the Laplace operators in spaces of $L_2$-forms on incomplete Riemannian manifolds with locally flat metrics. These formulae can be applied to any combinatorial manifold. Nevertheless, the spectra of the Laplace operators also lack an explicit combinatorial description. Hence Cheeger’s formulae should be regarded as important relations between topological and analytic objects rather than as formulae for combinatorial computation of the Pontryagin classes.

Let us now describe the main ideas of the Gabrielov–Gelfand–Losik formula [1] and the author’s formula [7]. For the Gabrielov–Gelfand–Losik formula there are two almost equivalent approaches. The original approach in [1] is based on endowing a triangulated manifold with locally flat connections. Another approach due to R. MacPherson [3] is based on the construction of a homology Gaussian mapping for a combinatorial manifold. We shall describe the ideas of MacPherson’s approach, since it is more geometrical.

First, suppose that $M^m \subset \mathbb{R}^N$ is an oriented smooth closed manifold embedded in a Euclidean space of large dimension. Consider the Gaussian mapping $g: M \rightarrow G_m(\mathbb{R}^N)$ taking every point $x \in M$ to the linear subspace of $\mathbb{R}^N$ parallel to the tangent space to $M$ at $x$. Here $G_m(\mathbb{R}^N)$ is the Grassmannian manifold of $m$-dimensional subspaces of $\mathbb{R}^N$. For any homogeneous polynomial $F \in \mathbb{R}[p_1, p_2, \ldots, p_{m/4}]$ of degree $n = 4k$, there is a unique $O(N)$-invariant exterior $n$-form $P_F$ on $G_m(\mathbb{R}^N)$ that represents the class $F(p(\gamma)) = F(p_1(\gamma), p_2(\gamma), \ldots, p_{m/4}(\gamma))$ in the de Rham cohomology, where $\gamma$ is the tautological vector bundle over $G_m(\mathbb{R}^N)$. (Here and in what follows, by a homogeneous polynomial of degree $n$ in the Pontryagin classes we mean a polynomial all of whose monomials have degree $n$, where the degree of every variable $p_i$ is supposed to be $4i$.) Then the form $g^* P_F$ represents the class $F(p(M))$ in the de Rham cohomology.

Obviously, for combinatorial manifolds, there is no hope to construct a natural continuous mapping to $G_m(\mathbb{R}^N)$. Indeed, if such a mapping existed, we could define integral Pontryagin classes of a combinatorial manifold to be the pullbacks of the Pontryagin classes of the tautological bundle $\gamma$. This is certainly impossible, since integral Pontryagin classes are not invariant under piecewise linear homeomorphisms. However, for combinatorial manifolds, it is possible to construct a homology Gaussian mapping up to dimension 4, which yields the required combinatorial formula. It is interesting that the formula obtained is essentially nonlocal. The reason is that the definition of the homology Gaussian mapping is nonlocal, since it depends on the preliminarily chosen additional combinatorial structure on the combinatorial manifold. A local formula is obtained by certain special average procedure over different choices of this additional structure, which makes the formula much more complicated (see [2, 3]). Notice that the Gabrielov–Gelfand–Losik formula cannot be applied to an arbitrary combinatorial manifold. It can be applied only to a smaller class of so-called Brouwer manifolds (for definition, see Section 2).