Mechanics of Isolated Horizons

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Abstract

A set of boundary conditions defining an undistorted, non-rotating isolated horizon are specified in general relativity. A space-time representing a black hole which is itself in equilibrium but whose exterior contains radiation admits such a horizon. However, the definition is applicable in a more general context, such as cosmological horizons. Physically motivated, (quasi-)local definitions of the mass and surface gravity of an isolated horizon are introduced and their properties analyzed. Although their definitions do not refer to infinity, these quantities assume their standard values in the static black hole solutions. Finally, using these definitions, the zeroth and first laws of black hole mechanics are established for isolated horizons.

1 Introduction

The similarity between the laws of black hole mechanics and those of ordinary thermodynamics is one of the most remarkable results to emerge from classical general relativity \cite{1,2,3,4}. However, the original framework was somewhat restricted and it is of considerable interest to extend it in certain directions, motivated by physical considerations. The purpose of this paper is to present one such extension.

The zeroth and first laws refer to equilibrium situations and small departures therefrom. Therefore, in this context, it is natural to focus on isolated black holes. In the standard treatments, these are generally represented by stationary solutions of field equations, i.e, solutions which admit a time-translational Killing vector field everywhere, not just in a small neighborhood of the black hole. While this simple idealization is a natural starting point, it seems to be overly restrictive. Physically, it should be sufficient to impose boundary conditions at the horizon which ensure only that the black hole itself is isolated. That is, it should suffice
to demand only that the intrinsic geometry of the horizon be time independent, whereas the geometry outside may be dynamical and admit gravitational and other radiation. Indeed, we adopt a similar viewpoint in ordinary thermodynamics; in the standard description of equilibrium configurations of systems such as a classical gas, one usually assumes that only the system under consideration is in equilibrium and stationary, not the whole world. For black holes in realistic situations, one is typically interested in the final stages of collapse where the black hole is formed and has ‘settled down’ or in situations in which an already formed black hole is isolated for the duration of the experiment (see figure 1). In such situations, there is likely to be gravitational radiation and non-stationary matter far away from the black hole, whence the space-time as a whole is not expected to be stationary. Surely, black hole mechanics should incorporate such situations.

A second limitation of the standard framework lies in its dependence on event horizons which can only be constructed retroactively, after knowing the complete evolution of space-time. Consider for example, figure 2 in which a spherical star of mass $M$ undergoes a gravitational collapse. The singularity is hidden inside the null surface $\Delta_1$ at $r = 2M$ which is foliated by a family of marginally trapped surfaces and would be a part of the event horizon if nothing further happens. Suppose instead, after a very long time, a thin spherical shell of mass $\delta M$ collapses. Then $\Delta_1$ would not be a part of the event horizon which would actually lie slightly outside $\Delta_1$ and coincide with the surface $r = 2(M + \delta M)$ in the distant future. On physical grounds, it seems unreasonable to exclude $\Delta_1$ a priori from thermodynamical considerations. Surely one should be able to establish the standard laws of mechanics not
Figure 2: A spherical star of mass $M$ undergoes collapse. Later, a spherical shell of mass $\delta M$ falls into the resulting black hole. While $\Delta_1$ and $\Delta_2$ are both isolated horizons, only $\Delta_2$ is part of the event horizon.

only for the event horizon but also for $\Delta_1$.

Another example is provided by cosmological horizons in de Sitter space-time [5]. In this case, there are no singularities or black-hole event horizons. On the other hand, semi-classical considerations enable one to assign entropy and temperature to these horizons as well. This suggests the notion of event horizons is too restrictive for thermodynamical analogies. We will see that this is indeed the case; as far as equilibrium properties are concerned, the notion of event horizons can be replaced by a more general, quasi-local notion of ‘isolated horizons’ for which the familiar laws continue to hold. The surface $\Delta_1$ in figure 2 as well as the cosmological horizons in de Sitter space-times are examples of isolated horizons.

In addition to overcoming these two limitations, the framework presented here provides a natural point of departure for quantization and entropy calculations [6, 7, 8]. In contrast, standard treatments of black hole mechanics are often based on differential geometric identities and are not well-suited to quantization.

At first sight, it may appear that only a small extension of the standard framework based on stationary event horizons is needed to overcome the limitations discussed above. However, this is not the case. For example, in the stationary context, one identifies the black-hole mass with the ADM mass defined at spatial infinity. In the presence of radiation, this simple strategy is no longer viable since even radiation fields well outside the horizon also contribute to the ADM mass. Hence, to formulate the first law, a new definition of the black hole mass is needed. Similarly, in the absence of a globally defined Killing field, we need to generalize the notion of surface gravity in a non-trivial fashion. Indeed, even if space-time happens to be static in a neighborhood of the horizon — already a stronger condition than contemplated above — the notion of surface gravity is ambiguous because the standard expression fails to be invariant under constant rescalings of the Killing field. When a *global* Killing field exists, the ambiguity is removed by requiring the Killing field be unit at *infinity*. Thus, contrary to intuitive expectation, the standard notion of the surface gravity of a stationary black hole refers not just to the structure at the horizon, but also to infinity. This ‘normalization problem’ in the definition of the surface gravity seems especially difficult in the
case of cosmological horizons in (Lorentzian) space-times whose Cauchy surfaces are compact. Apart from these conceptual problems, a host of technical issues must also be resolved. In Einstein–Maxwell theory, the space of stationary black hole solutions is finite-dimensional whereas the space of solutions admitting isolated horizons is infinite-dimensional since these solutions also admit radiation. As a result, the introduction of a well-defined action principle is subtle and the Hamiltonian framework acquires qualitatively new features.

The organization of this paper is as follows. In Section 2 we recall the formulation of general relativity in terms of SL(2, C)-spin soldering forms and self-dual connections for asymptotically flat space-times without internal boundaries. In Section 3 we specify the boundary conditions which define non-rotating internal boundaries. The primary focus in Section 3, as in the rest of the paper, is on Einstein–Maxwell theory, although more general matter is also considered. The consequences of these boundary conditions which are needed to obtain the laws governing isolated horizons are discussed in Section 4. Using this structure, we introduce in Section 5 the notion of the surface gravity \( \kappa \) of an isolated horizon without any reference to infinity and prove the zeroth law.

The action principle and the Hamiltonian formulation are discussed in Section 6. The Hamiltonian has a bulk term and two surface terms, one at infinity and the other at the isolated horizon \( \Delta \). The bulk term is the standard linear combination of constraints and the surface term at infinity yields the ADM energy. In Section 7, we argue the horizon surface term in the Hamiltonian should be identified with the mass \( M_\Delta \) of the isolated horizon. In particular, in the situation depicted in figure 1(a) we show \( M_\Delta \) is the difference between the ADM energy and the energy radiated away through all of null infinity (provided certain assumptions on the structure at \( i^+ \) hold). That is, although the definition of \( M_\Delta \) uses only the structure available at the horizon, it equals the future limit of the Bondi mass defined entirely on \( \mathcal{I}^+ \) in the case under consideration. This is precisely what one might expect on physical grounds in the presence of radiation. Finally, having \( \kappa \) and \( M_\Delta \) at our disposal, we establish the first law in Section 8.

The overall viewpoint and the boundary conditions of Section 3 are closely related to those introduced by Hayward in an interesting series of papers [9] which also aims at providing a more physical framework for discussing black holes. There is also an inevitable overlap between this paper and [7] in which a Hamiltonian framework (in terms of real SU(2) connections) is constructed with an eye towards quantization. However, by and large the results presented here are complementary to those obtained in [7]. Even when there is an overlap, the material is presented from a different angle. The relation between these three papers is discussed in Section 9. Technical, background material needed at various stages is collected in the three appendices.

A brief summary of our main results was given in [10].
2 Mathematical Preliminaries

In this paper we will use the formulation of general relativity as a dynamical theory of connections [11] rather than metrics. Classically, the theory is unchanged by the shift to connection variables. However, at present the connection variables appear to be indispensable for non-perturbative quantization [12]. In particular, the entropy calculation for isolated horizons has been carried out only within this framework [8]. Therefore, for uniformity, we will use the connection variables in the main body of this paper. However, as indicated in Section 9, all these results can also be derived in the framework of tetrad dynamics.

In order to fix notation and conventions and to acquaint the reader with the basics of connection dynamics, we will provide here a brief review of this formulation in an asymptotically flat space-time without interior boundary. (The modifications required to accommodate a non-zero cosmological constant are discussed at the end.) For further details see, e.g., [12].

2.1 Connection Variables

Fix a four-dimensional manifold $\mathcal{M}$. In Einstein–Maxwell theory, the basic fields consist of the triplet of asymptotically flat, smooth fields $(\sigma_{a}^{AA'}, A_{a}^{AB}, A_{a})$. Here, $\sigma_{a}^{AA'}$ is an anti-Hermitian soldering form for $\text{SL}(2, \mathbb{C})$ spinors, $A_{a}^{AB}$ is a self dual $\text{SL}(2, \mathbb{C})$ connection acting only on unprimed spinors and $A_{a}$ is the $\text{U}(1)$ electro-magnetic connection. The action for the theory is given by

$$S(\sigma, A, A_{a}) = \frac{-i}{8\pi G} \int_{\mathcal{M}} \text{Tr}[\Sigma \wedge F] + \frac{1}{8\pi} \int_{\mathcal{M}} F \wedge \ast F + \frac{i}{8\pi G} \int_{C_{\infty}} \text{Tr}[\Sigma \wedge A]. \quad (2.1)$$

The 2-form fields $\Sigma$ are given by $\Sigma^{AB} = \sigma^{AA'} \wedge \sigma_{A'B'}$, $F$ is the curvature of the connection $A$, $\Pi$ is the electro-magnetic field strength, and $C_{\infty}$ is the time-like cylinder at spatial infinity. We define a metric $g_{ab}$ of signature $-+++$ on this manifold via $g_{ab} = \sigma_{a}^{AA'} \sigma_{b}^{AA'}$. With respect to this metric, the 2-form fields $\Sigma$ are self dual. (For details, see [12, 13].)

Let us consider the equations of motion arising from this action. Varying the action with respect to the connection $A$, one obtains

$$D\Sigma = 0. \quad (2.2)$$

This implies the connection $D$ defined by $A$ has the same action on unprimed spinors as the self dual part of the connection $\nabla$ compatible with the soldering form, $\nabla_{a} \sigma_{b}^{AA'} = 0$. When this equation is satisfied, the curvature $F$ is related to the Riemann curvature by:

$$F_{ab}^{AB} = -\frac{1}{4} \Sigma_{cd}^{AB} R_{ab}^{cd}. \quad (2.3)$$

Varying the action (2.1) with respect to $\sigma$ and taking into account the compatibility of $A$ with $\sigma$, we obtain a second equation of motion

$$G_{ab} = 8\pi G \Pi_{ab} \quad (2.4)$$
where $G_{ab}$ is the Einstein tensor of $g_{ab}$ and $T_{ab}$ is the standard stress-energy tensor of the Maxwell field $F$ \cite{4, 12}.

Next, let us consider the equations of motion for the Maxwell field,

$$dF = 0, \quad \text{and} \quad d^*F = 0.$$  \hfill (2.5)

Since $F$ is the curvature of the U(1) connection $A$, the first Maxwell equation $dF = 0$ is an identity. If one varies equation (2.1) with respect to $A$, one obtains the second Maxwell equation, $d^*F = 0$. Thus, the equations of motion which follow from the action (2.1) are the same as those given by the usual Einstein–Hilbert–Maxwell action; the two classical theories are equivalent.

### 2.2 Hamiltonian Formulation

To pass to the Hamiltonian description of the theory, it is necessary to re-express the action in terms of 3-dimensional fields. Let us assume the space-time $M$ is topologically $M \times \mathbb{R}$. Introduce a ‘time function’ $t$ which agrees with a standard time coordinate defined by the asymptotically Minkowskian metric at infinity. A typical constant $t$ leaf of the foliation will be denoted $M$. Fix a smooth time-like vector field $t^a$, transverse to the leaves $M$ such that:

(i) $t^a \nabla_a t = 1$ and

(ii) $t^a$ tends to the unit time translation at spatial infinity. We will denote the future directed, unit normal to the leaves $M$ by $\tau^a$. The intrinsic metric on the 3-surfaces $M$ is $g_{ab} := 4g_{ab} + \tau^a \tau^b$.

As usual, by projecting $t^a$ into and orthogonal to $M$, we obtain the lapse and shift fields, $N$ and $N^a$ respectively: $t^a = N \tau^a + N^a$.

We are now in a position to define the basic phase space variables of the theory. They are simply the pull backs to the space-like 3-surfaces $M$ of the space-time variables $A, \Sigma$ and $\mathcal{A}$, together with the electric field two-form $E$. To perform the Legendre transform, note the pull back of the soldering form $\sigma$ to $M$ induces an SU(2) soldering form and the pull back of the connection $A$ induces a complex-valued, SU(2) connection on spatial spinors. More precisely, we have

$$\sigma_a^{AB} = -i\sqrt{2}g^{a}_{b} \sigma_b^{AA'} \tau^{A'B} \quad \iff \quad \sigma_a^{AA'} = i\sqrt{2} \sigma_a^{AB} \tau^{BA'} - \tau^{A'A'} \tau^a$$

$$4A_a^B = A_0^B dt + \dot{A}_a^B$$

and

$$4\mathcal{A} = A_0 dt + \mathcal{A},$$

where $\tau^{AA'} := \sigma_a^{AA'} \tau^a, \tau^a A_a^{AB} = 0$ and $\tau^a 4A_a = 0$. (Note that this decomposition of connection 1-forms uses the space-time metric.) Using these definitions, one arrives at the following 3+1 decompositions of $\Sigma$ and the gravitational and electromagnetic field strengths:

$$4\Sigma = \Sigma - iN\sqrt{2} \sigma \wedge dt$$

$$4F = (\dot{A} + D(t.4A) + \vec{N} \bigwedge F) \wedge dt + F$$

$$4\mathcal{F} = (\dot{\mathcal{A}} + d(t.\mathcal{A}) + \vec{N} \bigwedge d\mathcal{A}) \wedge dt + d\mathcal{A},$$

\footnote{In the discussion of the Legendre transform and the Hamiltonian, both in this section and section 6, the four dimensional fields will carry a superscript $^4$ preceding the field and all other fields will be assumed to be three-dimensional, living on the space-like surface $M$.}
where $\vec{N}$ is the shift field. The electric field $E$ and magnetic field $B$ are defined as usual to be the pull backs to the space-like hypersurface $M$ of $^*E$ and $^*B$ respectively.

The phase space of the theory consists of quadruples $(A_a^{AB}, \Sigma_{ab}^{AB}, A_a, E_{ab})$. These fields satisfy the standard falloff conditions [12]. To specify them, let us fix an SU(2) soldering form $\tilde{\sigma}$ on $M$ such that the 3-metric $\tilde{g}_{ab}$ is flat outside of a compact set. Then the quadruple of fields is required to satisfy:

$$
\Sigma_{ab} - \left(1 + \frac{M(\theta, \phi)}{r}\right) \tilde{\sigma}_{ab} = O\left(\frac{1}{r^2}\right), \quad A = O\left(\frac{1}{r^2}\right),
$$

$$
A_a + \frac{1}{3} \text{Tr} [\tilde{\sigma}^m A_m] \tilde{\sigma}_a = O\left(\frac{1}{r^2}\right), \quad E = O\left(\frac{1}{r^2}\right),
$$

$$
\text{Tr} [\tilde{\sigma}^a A_a] = O\left(\frac{1}{r^3}\right),
$$

(2.8)

where $r$ is the radial coordinate defined by the flat metric $\tilde{g}_{ab}$.

The action can be re-expressed in terms of these fields using equations (2.6) and (2.7). From this action, it is straightforward to read off the Hamiltonian and symplectic structure.

$$
\tilde{H}_t = \int_M \frac{2i}{8\pi G} \text{Tr}[(t^A D\Sigma) + \frac{1}{4\pi} (t^A) dE] + \left[\frac{i}{8\pi G} \Sigma \wedge (\vec{N} \wedge F) - \frac{1}{4\pi} E \wedge (\vec{N} \wedge dA)\right]
$$

$$+ \frac{N}{8\pi G} \left[\text{Tr}[\sqrt{2}\sigma \wedge F] - G(E \wedge *E + dA \wedge *dA)\right]
$$

$$+ \frac{1}{8\pi G} \int_{S_\infty} \text{Tr}[\sqrt{2}N\sigma \wedge A + i(\vec{N} \wedge A)\Sigma]$$

(2.9)

As always in general relativity, the Hamiltonian takes the form of constraints plus boundary terms. The constraints consist of two Gauss law equations, one for the self dual two form $\Sigma$ and the other for the electric field $E$, together with the standard vector and scalar constraints. When the constraint equations are satisfied, the term at infinity equals $t^a P_a$ where $P_a$ is the ADM 4-momentum. In our signature, $t^a P_a$ is negative, so $-t^a P_a = E^{ADM}$, the standard ADM energy. The equations of motion are just Hamilton’s equations:

$$
\delta \tilde{H} = \tilde{\Omega}(\delta, X_{\tilde{H}}).
$$

(2.10)

These are the field equations (2.4) and (2.5) in a 3+1 form, expressed in terms of the canonical variables.

To conclude, we note two modifications which occur if the cosmological constant is non-zero. First, there is an extra term proportional to $\Lambda \text{Tr}[\Sigma \wedge \Sigma]$ in the action (2.4), where $\Lambda$ is the cosmological constant. This contributes a term proportional to $\Lambda \text{Tr}\sigma \wedge \sigma \wedge \sigma$ in the bulk term of the Hamiltonian (2.9). Second, the boundary conditions are modified. If $\Lambda$ is positive, it is natural to assume the Cauchy surfaces are compact, whence there are no falloff conditions or boundary terms in any of the expressions. If $\Lambda$ is negative, the dynamical fields approach asymptotically their values in the anti-de Sitter space-time [15].
3 Boundary Conditions

In this section, we specify the boundary conditions which define an *undistorted, non-rotating isolated horizon*. As explained in the Introduction, the purpose of these boundary conditions is to capture the essential features of a non-rotating, isolated black hole in terms of the intrinsic structure available at the horizon, without any reference to infinity or to a static Killing field in space-time. However, the boundary conditions model a larger variety of situations. For example, the cosmological horizons in de Sitter space-times [5] are isolated horizons, even though there is no sense in which they describe a black hole. As a result, the usual mechanics of cosmological horizons will be reproduced within the framework of isolated horizons. Other examples involve space-times admitting gravitational and electro-magnetic radiation such as those described in Section 3.2.

The physical situation we wish to model is illustrated by the example of figure 1(a). The late stages of the collapse pictured here should describe a non-dynamical, isolated black hole. However, a realistic collapse will generate gravitational radiation which must either be scattered back into the black hole or radiated to infinity. Physically, one expects most of the back-scattered radiation will be absorbed rather quickly and, in the absence of outside perturbations, the black hole will ‘settle down’ to a steady-state configuration. This picture is supported by numerical simulations. The continued presence of radiation elsewhere in space-time however implies $M$ cannot be stationary. As a result, the usual formulations of black hole mechanics in terms of stationary solutions cannot be easily applied to this type of physical black hole. Nevertheless, since the portion $\Delta$ of the horizon describes an isolated black hole one would hope to be able to formulate the laws of black hole mechanics in this context. We will see in Sections 5 and 8 that this is indeed the case.

3.1 Definition

We are now in a position to state our boundary conditions. Since this paper is concerned primarily with the mechanics of isolated horizons in Einstein–Maxwell space-times, conditions on gravitational and Maxwell fields are specified first and more general matter fields are treated afterwards.

A *non-rotating isolated horizon* is a sub-manifold $\Delta$ of space-time at which the following five conditions hold:

(I) $\Delta$ is topologically $S^2 \times \mathbb{R}$ and comes equipped with a preferred foliation by 2-spheres $S_\Delta$ and a ruling by lines transverse to those 2-spheres.

These preferred structures give rise to a 1-form direction field $[n_a]$ and a (future-directed) vector direction field $[\ell^a]$ on $\Delta$. Furthermore, any $n_a \in [n_a]$ is normal to a

\[^2\] Throughout this paper, $\cong$ will denote equality at points of $\Delta$. For fields defined throughout space-time, a single left arrow below an index will indicate the pull-back of that index to $\Delta$, and a double arrow will indicate the pull-back of that index to the preferred 2-sphere cross-sections $S_\Delta$ of $\Delta$ introduced in condition [1]. For brevity of presentation, slightly stronger conditions were used in [11].
foliation of $\Delta$ by 2-spheres and we further impose $dn \equiv 0$. As a result, the equivalence class $[n_a]$ is defined with respect to rescaling only by functions which are constant on each $S_\Delta$. A function with this property will be said to be spherically symmetric. We ‘tie together’ the normalizations of the two direction fields by fixing $\ell^a n_a \equiv -1$ with $\ell^a \in [\ell^a]$. This leaves a single equivalence class $[\ell^a, n_a]$ of direction fields subject to the relation

$$ (\ell^a, n_a) \sim (F^{-1} \ell^a, F n_a), \quad (3.1) $$

where $F$ is any positive, spherically symmetric function on $\Delta$. 

(II) The soldering form $\sigma_a^{AA'}$ gives rise to a metric in which $\Delta$ is a null surface with $[\ell^a]$ as its null normal.

So far the 1-forms $n_a$ are defined intrinsically on the 3-manifold $\Delta$. They can be extended uniquely to 4-dimensional, space-time 1-forms (still defined only at points of $\Delta$) by requiring that they be null. We will do so and, for notational simplicity, denote the extension also by $n_a$. Given any null pair $(\ell^a, n_a)$ satisfying $\ell^a n_a \equiv -1$, it is easy to show there exists a spin basis $(\iota^A, o^A)$ satisfying $|\iota^A o^A| = 1$ such that

$$ \ell^a \equiv i \sigma_a^{AA'} o^A \bar{o}^A' \quad \text{and} \quad n_a \equiv i \sigma_a^{AA'} \iota_A \bar{\iota} A'. \quad (3.2) $$

We will work by fixing a spin dyad $(\iota^A, o^A)$ with $\iota^A o_A \equiv +1$ once and for all and regarding (3.2) as a condition on the field $\sigma_a^{AA'}$. Finally, using this spin dyad, we can complete $(\ell^a, n^a)$ to a null tetrad with the vectors

$$ m^a \equiv i \sigma_a^{AA'} o^A \bar{\iota}^A' \quad \text{and} \quad \bar{m}^a \equiv i \sigma_a^{AA'} \iota^A \bar{o}^A' \quad (3.3) $$

tangential to the 2-spheres $S_\Delta$.

(III) The derivatives of the spin dyad are constrained by

$$ o^A \nabla_a o_A \equiv 0 \quad \text{and} \quad \iota^A \nabla_a \iota_A \equiv \mu \bar{m}_a, \quad (3.4) $$

where $\mu$ is a real, nowhere vanishing, spherically symmetric function, and $\nabla_a$ denotes the unique torsion-free connection compatible with $\sigma_a^{AA'}$. The function $\mu$ is one of the standard Newman-Penrose spin coefficients.

(IV) All equations of motion hold at $\Delta$. In particular:

IVa. The $\text{SL}(2, \mathbb{C})$ connection is compatible with the soldering form: $D_a \lambda^A \equiv \nabla_a \lambda^A$.

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3Since the topology of $\Delta$ is $S^2 \times \mathbb{R}$, the tetrad vectors $m^a$ and $\bar{m}^a$ — and the spin-dyad $(\iota^A, o^A)$ — fail to be globally defined. Thus, when we refer to a fixed spin-dyad, we mean dyads which are fixed on two patches and related by a gauge transformation on the overlap. Our loose terminology is analogous to the one habitually used for (spherical) coordinates on a 2-sphere.
IVb. The Einstein equations hold: $G_{ab} + \Lambda g_{ab} \equiv 8\pi G T_{ab}$, where $T_{ab}$ is the stress-energy tensor of the matter fields under consideration and $G_{ab}$ is the Einstein tensor of the metric compatible with $\sigma_{AA'}$.

IVc. The electro-magnetic field strength $\mathbb{F}$ satisfies the Maxwell equations: $d\mathbb{F} = 0$ and $d^*\mathbb{F} = 0$.

(V) The Maxwell field strength $\mathbb{F}$ has the property that

$$\phi_1 := \frac{1}{2} m^a \bar{m}^b (\mathbb{F} - i^*\mathbb{F})_{ab}$$

is spherically symmetric.

These five conditions define a non-rotating isolated horizon. Each is imposed only at the points of $\Delta$. Let us now discuss the geometrical and physical motivations behind the boundary conditions, and see how they capture the intuitive picture outlined above.

Conditions I and II are straightforward. The first is primarily topological and fixes the kinematical structure of the horizon. (While the $S^2 \times \mathbb{R}$ topology is the most interesting one from physical considerations, most of our results go through if $S^2$ is replaced by a compact, 2-manifold of higher genus. This issue is briefly discussed at the end of Section 3.) The meaning of the preferred foliation of $\Delta$ will be made clear in the discussion of condition III below. The meaning of the preferred ruling can be seen immediately in condition II which simply requires that $\Delta$ be a null surface, and $[\ell^a]$ its null generator. Thus, the preferred ruling singles out the null generators of the horizon.

Condition IV is a fairly generic dynamical condition, completely analogous to the one usually imposed at null infinity: Any set of boundary conditions must be consistent with the equations of the motion at the horizon. It is likely that this condition can be weakened, e.g., by requiring only that the pull-backs to $\Delta$ of the equations of motion should hold. However, care would be needed to specify the precise form of equations which are to be pulled-back since pull-backs of two equivalent sets of equations can be inequivalent. We chose simply to avoid this complication.

Conditions I, II and IV are weak; in particular, they are satisfied by a variety of null surfaces in any solution to the field equations. It is condition III which endows $\Delta$ with the structure of an isolated horizon. Technically, this condition restricts the pull-back of the self-dual connection compatible with $\sigma_{AA'}$. (Note the pull-backs to $\Delta$ are important because we have introduced the dyad $(\iota^A, \sigma^A)$ only at the surface itself.) Geometrically, it is equivalent to requiring the pairs $(\ell^a, n_a) \in [\ell^a, n_a]$ to have the following properties:

1. $\ell^a$ is geodesic, twist-free, shear-free and divergence-free.

2. $n^a$ is twist-free, shear-free, has nowhere vanishing, spherically symmetric expansion

$$\theta_{(n)} \equiv 2\mu.$$  \hspace{1cm} (3.6)

and vanishing Newman-Penrose spin coefficient $\pi := l^a \bar{m}^b \nabla_a n_b$ on $\Delta$.  

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As we will see in the next section, the only independent consequence of condition \( \text{[I]} \) for \( \ell^a \) is its vanishing divergence; the rest then follows from simple geometry. The vanishing divergence of \( \ell^a \) is equivalent to the vanishing expansion of the horizon. We will see in Section \( \text{[L]} \) that this implies there is no flux of matter falling across \( \Delta \), which in turn captures the idea that the horizon is isolated. Collectively, the consequences of condition \( \text{[I]} \) for \( n^a \) imply the horizon is non-rotating and its intrinsic geometry is undistorted. Requiring \( \mu \) to be nowhere vanishing is equivalent to requiring the expansion of \( n^a \) to be nowhere vanishing. In black-hole space-times, we expect this expansion to be strictly negative on future horizons and strictly positive on past horizons. On cosmological horizons, both signs are permissible. In section \( \text{[6]} \) we will impose further restrictions tailored to different physical situations. Finally, it is easy to verify that these conditions on \( n^a \) suffice to single out the preferred foliation of \( \Delta \). Thus, we could have just required the existence of a foliation satisfying the first three conditions, and used \( \text{[II]} \) to conclude the foliation is unique.

Next, let us discuss the condition \( \text{[V]} \) on the Maxwell field. At first sight, this requirement seems to be a severe restriction. However, if the Newman-Penrose component \( \phi_0 \) of the electro-magnetic field, representing ‘the radiation field traversing \( \Delta \)’, vanishes in a neighborhood of \( \Delta \) and boundary conditions I through IV hold, then \( \phi_1 \) automatically satisfies \( \text{[V]} \). Heuristically, this feature can be understood as follows. From the definition of \( \phi_1 \) in (3.3), one can see condition \( \text{[V]} \) requires the electric and magnetic fluxes across the horizon to be spherically symmetric. If this condition did not hold, one would intuitively expect a non-rotating black hole to radiate away the asphericities in its electro-magnetic hair, giving rise to a non-vanishing \( \phi_0 \). Therefore, it is reasonable to expect \( \text{[V]} \) would hold once the horizon reaches equilibrium.

Let us now consider the generalization of the boundary conditions to other forms of matter. Conditions \( \text{[Vc]} \) and \( \text{[V]} \) refer to the Maxwell field; the rest involve only the gravitational degrees of freedom and are independent of the matter fields present at the horizon. We will now indicate how these two conditions must be modified in the presence of more general forms of matter. (For a discussion of dilatonic couplings, see [7, 16].) First, note that condition \( \text{[V]} \) is unambiguous. Hence, \( \text{[Vc]} \) would simply be replaced with the field equations of the relevant matter. Condition \( \text{[V]} \) on the other hand, is more subtle. In fact, there is no universal analog of (3.5) which applies to arbitrary matter fields; the boundary conditions used in place of condition \( \text{[V]} \) may vary from case to case. There are, however, two general properties which any candidate matter field and its associated boundary conditions must possess:

\( (V') \) For any \((\ell^a, n_a) \in [\ell^a, n_a] \), a matter field must satisfy

\[
V' a. \quad \text{The stress-energy tensor is such that}
\]

\[
k^a \triangleq -T^a_{\ b} \ell^b
\]  \hspace{1cm} (3.7)

\( \quad \text{is causal, i.e., future-directed, time-like or null.} \)
The quantity 

\[ e \equiv T_{ab} \ell^a n^b \]  

is spherically symmetric on \( \Delta \).

The first of these properties, \( \nabla a \), is an immediate consequence of the (much stronger) dominant energy condition which demands \(-T^a_k k^b\) be causal for any causal vector \( k^b \). Like any energy condition, this is a restriction on the types of matter which may be present near the horizon. On the other hand, the property \( \nabla b \) is a restriction on the possible boundary conditions which may be imposed on matter fields at the horizon.

We conclude this section with a remark on generalizations of these boundary conditions. Although our framework is geared to the undistorted, non-rotating case, only the requirements on \( n^a \) in condition \( \Box \Pi \) and the symmetry condition \( \nabla \) on the Maxwell field would have to be weakened to accommodate distortion and rotation \( \Pi \). Specifically, it appears that, in presence of distortion, \( \mu \) will not be spherically symmetric and in presence of rotation \( \pi \) will not vanish. However, it appears that a more general procedure discovered by Lewandowski will enable one to introduce a preferred foliation of \( \Delta \) and naturally extend the present framework to allow for distortion and rotation.

Finally, in light of the sphericity conditions on \( \theta_{(n)} \) and \( \phi_1 \), one may be tempted to call our isolated horizons 'spherical'. However, the Newman-Penrose curvature components \( \Psi_3 \) and \( \Psi_4 \) and the Maxwell field component \( \phi_2 \) need not be spherical on \( \Delta \) for our boundary conditions to be satisfied. Therefore, the adjectives 'undistorted and non-rotating' appear to be better suited to characterize our isolated horizons.

### 3.2 Examples

It is easy to check that all of these boundary conditions hold at the event horizons of Reissner-Nordström black hole solutions (with or without a cosmological constant). Similarly, they hold at cosmological horizons in de Sitter space-time. Furthermore, if one considers a spherical collapse, as in figure 2, they hold both on \( \Delta_1 \) and \( \Delta_2 \) (at suitably late times).

In the non-rotating context, these cases already include situations normally considered in connection with black hole thermodynamics. However, the isolated horizons in these examples are also Killing horizons for globally defined, static Killing fields. We will now indicate how one can construct more general isolated horizons.

First, an infinite-dimensional space of examples can be constructed using Friedrich’s results \( \{18\} \), and Rendall’s extension \( \{19\} \) thereof, on the null initial value formulation (see figure 3). In this framework, one considers two null hypersurfaces \( \Delta \) and \( \mathcal{N} \) with normals \( \ell^a \) and \( n^a \) respectively, which intersect in a 2-sphere \( S \). (At the end of the construction \( \Delta \) will turn out to be a non-rotating isolated horizon.) In a suitable choice of gauge \( \{18\} \), the free data for vacuum Einstein’s equations consists of \( \Psi_0 \) on \( \Delta \); \( \Psi_4 \) on \( \mathcal{N} \); and, the Newman Penrose coefficients \( \lambda, \sigma, \pi, \Re \[ \mu \], \Re \[ \rho \] \) as well as the intrinsic metric \( \gamma_{ab} \) on the 2-sphere \( S \). Given these fields, there is a unique solution (modulo diffeomorphisms) to the vacuum Einstein’s equations in a neighborhood of \( S \) bounded by (and including) the appropriate
Figure 3: Space-times with isolated horizons can be constructed by solving the characteristic initial value problem on two intersecting null surfaces, $\Delta$ and $\mathcal{N}$. The final solution admits $\Delta$ as an isolated horizon. Generically, there is radiation arbitrarily close to $\Delta$ and no Killing fields in any neighborhood of $\Delta$.

Figure 4: A space-time $\mathcal{M}$ with an isolated horizon $\Delta$ as internal boundary and radiation field in the exterior can be obtained by starting with an asymptotic region of Kruskal spacetime and modifying the initial data on the partial Cauchy surface $M$. While the new metric continues to be isometric with the Schwarzschild metric in a neighborhood of $\Delta$, it admits radiation in a neighborhood of infinity.
portions of ∆ and ℳ. Let us set Ψ₀ = 0 on ∆ and λ = σ = ρ = π = 0 on S, µ = const and $g_{ab}$ to be a (round) 2-sphere metric with area $a_{\Delta}$ on S. Lewandowski [17] has shown that, in the resulting solution, ∆ is an isolated horizon with area $a_{\Delta}$. Note that in the resulting solution Ψ₄ need not vanish in any neighborhood of ∆, or, indeed, even on ∆. Hence, in the vacuum case, there is an infinite-dimensional space of (local) solutions containing isolated horizons. It turns out that, in this setting, there is always a vector field $\xi^a$ in a neighborhood of ∆ with $\xi^a \equiv f^a$ and $\mathcal{L}_\xi g_{ab} \equiv 0$. However, in general $\mathcal{L}_\xi C_{abcd} \neq 0$, where $C_{abcd}$ is the Weyl tensor of $g_{ab}$. Hence, in general, $\xi^a$ cannot be a Killing field of $g_{ab}$ even in a neighborhood of ∆. (For details, see [17].) The Robinson–Trautman solutions provide interesting examples of exact solutions which bring out this point [20]: a sub-class of these solutions admit an isolated horizon but no Killing fields whatsoever. There is radiation in every neighborhood of the isolated horizon but, in a natural chart, the metric coefficients and several of their radial derivatives evaluated at ∆ are the same as those of the Schwarzschild metric at its event horizon.

A second class of examples can be constructed by starting with Killing horizons and ‘adding radiation’. To be specific, consider one asymptotic region of the Kruskal space-time (figure 4) and a Cauchy surface M therein. The idea is to change the initial data on this slice in the region $r \geq 3m$, say, where $m$ is the Schwarzschild mass of the initial space-time. In the Einstein–Maxwell case, one can use the strategy introduced by Cutler and Wald [21] in their proof of existence of solutions with smooth null infinity. In the vacuum case, one can use the more general ‘gluing methods’ recently introduced by Corvino and Schoen [22] to show that there exists an infinite-dimensional space of asymptotically flat initial data on M which agree with the data for a Schwarzschild space-time for $r < 3m$ but in which the evolved space-time admits radiation. Using these methods, one would be able to construct ‘triangular regions’ $\mathcal{M}$ bounded by $M$, ∆ and a partial Cauchy slice $M'$ in the future of $M$. If one takes $\mathcal{M}$ as the space-time of interest, then ∆ would serve as the isolated horizon at the inner boundary. Due to the presence of radiation, $\mathcal{M}$ will not admit any global Killing field. However, in a neighborhood of ∆, the 4-metric will be isometric to that of Schwarzschild space-time. Thus, in this case, there will in fact be four Killing fields in a neighborhood of the isolated horizon.

The two constructions discussed above are complementary. The first yields more general isolated horizons but the final result is local. The second would provide space-times which extend from the isolated horizon ∆ to infinity but in which there is no radiation in a neighborhood of ∆. We expect there will exist an infinite-dimensional space of solutions to the vacuum Einstein equations and Einstein–Maxwell equations which are free from both limitations, i.e., which extend to spatial infinity and admit isolated horizons with radiation arbitrarily close to them. However, a comprehensive treatment of this issue will be technically difficult. Given the current status of global existence and uniqueness results in the asymptotically flat contexts, the present limitations are not surprising. Indeed, the situation at null infinity is somewhat analogous: while the known techniques have provided several interesting partial results, they do not yet allow us to show that there exists a large class of solutions to Einstein’s vacuum equations which admit complete and smooth past and future
null infinities, \( \mathcal{I}^\pm \) and the standard structure at spatial infinity, \( i^0 \).

4 Consequences of the Boundary Conditions

In this section, we will discuss the rich structure given to the horizon by the boundary conditions. The discussion is divided into four subsections. The first describes the basic geometry of an isolated horizon. The next two subsections examine the restrictions on the space-time curvature and Maxwell field which arise from the boundary conditions. The last subsection contains a brief summary.

4.1 Horizon Geometry

Let us begin by examining the consequences of condition III in the main definition. The condition on \( o_A \) immediately implies

\[
\nabla_a \ell_b \equiv -2U_a \ell_b
\]

for some 1-form \( U_a \) on \( \Delta \). Hence, \( \ell^a \) is geodesic, twist-free, shear-free and divergence-free. We will denote the acceleration of \( \ell^a \) by \( \tilde{\kappa} \):

\[
\ell^a \nabla_a \ell^b \equiv \tilde{\kappa} \ell^b,
\]

so that \( \tilde{\kappa} \equiv -2\ell^a U_a \). Note that \( \tilde{\kappa} \) varies with the rescaling of \( \ell^a \).

Actually, the geodesic and the twist-free properties of \( \ell^a \) follow already from condition II which requires \( \Delta \) to be a null surface with \( \ell^a \) as its null normal. Furthermore, since \( T_{ab} \ell^a \ell^b \geq 0 \) by condition \( [V^a] \), and Einstein’s equations hold at \( \Delta \) (condition IVb), we can use the Raychaudhuri equation

\[
\mathcal{L}_\ell \theta(\ell) = -\frac{1}{2} \theta^2(\ell) - \sigma_{ab} \sigma^{ab} + \omega_{ab} \omega^{ab} - R_{ab} \ell^a \ell^b
\]

(4.3)

to conclude the shear \( \sigma_{ab} \) must vanish if the expansion \( \theta(\ell) \) vanishes. Thus, the only independent assumption contained in the first of equations (3.4) in condition III is that \( \ell^a \) is expansion-free, which captures the idea that \( \Delta \) is an isolated horizon. Finally, note that the Raychaudhuri equation also implies that

\[
R_{ab} \ell^a \ell^b \equiv 8\pi GT_{ab} \ell^a \ell^b \equiv 0
\]

(4.4)

e., that there is no flux of matter energy-momentum across the horizon.

Let us now consider the properties of the vector field \( n^a \). The second equation in (3.4) implies

\[
\nabla_a n_b \equiv (2U_a n_b + 2\mu m_{(a} \bar{m}_{b)}),
\]

(4.5)

with \( U_a \propto n_a \). Hence, \( n^a \) is twist-free, shear-free, has spherically symmetric expansion, \( 2\mu \), and vanishing Newman-Penrose coefficient \( \pi \equiv \bar{m}^a \ell^b \nabla_a n_b \). The twist-free property follows
from the very definition of \( n_a \), but the other three properties originate in condition III of the Definition.

Next we turn to the intrinsic geometry of the horizon. Using the definitions (3.3) of the co-vectors \( m_a \) and \( \bar{m}_a \), one can show there exists a 1-form \( \nu_a \) on \( \Delta \) such that

\[
\begin{align*}
\text{d}m &\equiv -i\nu \wedge m \quad \text{and} \quad \text{d}\bar{m} \equiv i\nu \wedge \bar{m}.
\end{align*}
\]

One consequence of these relations is that the Lie derivative along \( \ell^a \) of the intrinsic metric on \( \Delta \) must vanish: \( \mathcal{L}_\ell g_{ab} \equiv 0 \). Thus, as mentioned in Section 3, the intrinsic geometry of an isolated horizon is ’time-independent’. In particular, the Lie derivative along \( \ell^a \) of the volume form on the foliation 2-spheres \( S_\Delta \) vanishes: \( \mathcal{L}_\ell \Omega_\Delta \equiv 0 \). Therefore, the areas of all the \( S_\Delta \) take the same value which we denote \( a_\Delta \). Finally, one can show \[23\] the scalar curvature \( 2R \) of the 2-sphere cross sections \( S_\Delta \) is related to the 4-dimensional, Newman-Penrose curvature scalars via:

\[
2R = -2\text{Re} [\Psi_2] + 2\Phi_{11} + R/12. \]

In Section \[3\], it is shown that the quantity on the right side of this equation is constant on \( S_\Delta \). Hence, the intrinsic metric \( 2m_a(\bar{m}_b) \) on \( S_\Delta \) is spherically symmetric. In this sense, the horizon geometry is undistorted. However, the discussion of Section \[3.2\] shows spherical symmetry will not extend, in general, to a neighborhood of \( \Delta \).

### 4.2 Form of the Curvature

Let us begin by exploring the effects of boundary condition \[\text{V} \] on the form of the Ricci tensor. Using the Raychaudhuri equation, we have derived (4.4). Whence \( k^a \) in (3.7) must be proportional to \( \ell^a \). Using the quantity \( e \) defined in (3.8), we then have

\[
(8\pi G)^{-1} \left( R_{ab} \ell^b - \frac{1}{2} R \ell_a + \Lambda \ell_a \right) = T_{ab} \ell^b \equiv -e \ell_a. \quad (4.7)
\]

where \( R \) is the scalar curvature. This formula yields a series of results for the Ricci tensor. In terms of Newman-Penrose components (A.14), these read

\[
\Phi_{00} \equiv \Phi_{01} \equiv \Phi_{10} \equiv 0 \quad \text{and} \quad \Phi_{11} + \frac{R}{8} - \frac{\Lambda}{2} \equiv 4\pi Ge. \quad (4.8)
\]

The first three results say, by way of the Einstein equation, there is no flux of matter radiation falling through the isolated horizon. The fourth result implies the combination \( \Phi_{11} + \frac{R}{8} \) is spherically symmetric.

We can now explore the consequences of the condition III for the full Riemann curvature. Since any \( SL(2, \mathbb{C}) \)-bundle over a 3-manifold is trivializable, and since our 4-manifold \( M \) has the topology \( M \times \mathbb{R} \) for some 3-manifold \( M \), the connection \( A \) can be represented globally as a Lie algebra-valued 1-form. Because of (3.4), in the \((t, o)\)-basis, the pull-back to \( \Delta \) of that connection must have the form

\[
A_a^{AB} \equiv -\left( \bar{\kappa} n_a + i\nu_a \right) \iota^A o^B - \mu \bar{m}_a o^A o^B. \quad (4.9)
\]

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on $\Delta$. (However, since the spin-dyad is defined only locally on $\Delta$, the decomposition (4.9) is also local.) The function $\mu$ appearing here is the same nowhere vanishing, spherically symmetric function introduced in (3.4) and $\tilde{\kappa}$ and $\nu$ are defined by (4.2) and (4.6), respectively. Now, using this expression for the pull-back of the self-dual connection, we can simply calculate the pull-back of its curvature to be

$$F_{AB} = -[d\tilde{\kappa} \wedge n + id\nu]_{\iota(AoB)} + [(\tilde{\kappa}\mu + \mathcal{L}_\mu)n \wedge \tilde{m}] o_Ao_B,$$

(4.10)

where we have suppressed all space-time indices for simplicity. On the other hand, we can also calculate the self-dual part of the Riemann spinor in terms of Newman-Penrose components (see Appendix B). Then, using the compatibility of the self-dual connection with the soldering form at the horizon, we get a second expression for the pulled-back curvature:

$$F_{AB} = \left[ \Phi_{00}n \wedge m + \Psi_{0}n \wedge \tilde{m} - (\Psi_{1} - \Phi_{01})m \wedge \tilde{m} \right] \iota_{A\iota_B}$$

$$- \left[ \Phi_{10}n \wedge m + \Psi_{1}n \wedge \tilde{m} - \left( \Psi_{2} - \Phi_{11} - \frac{R}{24} \right) m \wedge \tilde{m} \right] 2\iota_{(AoB)}$$

(4.11)

$$+ \left[ \Phi_{20}n \wedge m + \left( \Psi_{2} + \frac{R}{12} \right) n \wedge \tilde{m} - (\Psi_{3} - \Phi_{21})m \wedge \tilde{m} \right] o_Ao_B.$$

Equating this expression with (4.10) one arrives at a series of conclusions:

1. Since there is no $\iota_{A\iota_B}$ term in (4.10), we find

$$\Psi_{0} = \Psi_{1} = 0,$$

(4.12)

where we have used (4.8).

2. Equating the $\iota_{(AoB)}$ terms in the two expressions and using (4.12) and (4.8) then yields

$$n \wedge d\tilde{\kappa} = 0 \quad \text{and} \quad d\nu = 2i \left( \Psi_{2} - \Phi_{11} - \frac{R}{24} \right) m \wedge \tilde{m}.$$ 

(4.13)

The first expression here says the function $\tilde{\kappa}$ is spherically symmetric. In the second expression, the left side is real and the quantity $i m \wedge \tilde{m} = \tilde{\epsilon}$ on the right is real as well. As a result, the coefficient $\Psi_{2} - \Phi_{11} - \frac{R}{24}$ must be real. However, $\frac{R}{24}$ is manifestly real and, due to its definition (A.14), $\Phi_{11}$ is also real. Thus, the second expression in (4.13) implies the imaginary part of $\Psi_{2}$ vanishes. This encodes the property that $\Delta$ is non-rotating.

3. Equating the remaining $o_Ao_B$ terms similarly yields $\Phi_{20} = 0$ and $\Psi_{3} = \Phi_{21}$ as well as

$$\Psi_{2} + \frac{R}{12} \equiv \mathcal{L}_\mu + \tilde{\kappa}\mu.$$ 

(4.14)

Since $\mu$ and $\tilde{\kappa}$ are spherically symmetric, $\Psi_{2} + \frac{R}{12}$ must also have this property.
In the later sections of this paper, we will consider the action and phase space formulations of systems containing isolated horizons. In this discussion, it will be most useful to have a formula giving the relations which arise from the boundary conditions among the fundamental gravitational degrees of freedom in a simple, compact form. Using (4.11), the tetrad decomposition (A.9) of Σ^{AB} and the above restrictions on the Newman-Penrose curvature components, it is straightforward to demonstrate:

\[
F^{AB} \leftarrow \left[ \left( \Psi_2 - \Phi_{11} - \frac{R}{24} \right) \delta_C^A \delta_D^B - \left( \frac{3\Psi_2}{2} - \Phi_{11} \right) o^A o^B o_C o_D \right] \Sigma^{CD}.
\] (4.15)

In the phase space formalism, only the pull-back to \( S_\Delta \) of this formula will be relevant. This pull-back takes the simpler form

\[
F^{AB} \leftarrow \left( \Psi_2 - \Phi_{11} - \frac{R}{24} \right) \Sigma^{AB}.
\] (4.16)

Note that the essential content of this equation can be seen already in (4.13).

### 4.3 Form of the Maxwell Field

The stress-energy tensor of a Maxwell field is given by

\[
\mathbb{T}_{ab} = \frac{1}{4\pi} \left[ F_{ac} F_{b}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right].
\] (4.17)

This stress-energy tensor satisfies the dominant energy condition and hence, in particular, condition \( V_0 \). Furthermore, one can see from its definition that the trace of \( \mathbb{T}_{ab} \) is zero.

To see the restrictions which the boundary conditions place on the Maxwell field, it is useful to recast this discussion in terms of spinors as we did in the previous subsection for the self-dual curvature. The Maxwell spinor \( \phi_{AB} \) is defined in terms of the field strength via

\[
F_{ab} = \sigma_a^{\ A\prime} \sigma_b^{B\prime} (\phi_{AB} \epsilon_{A\prime B\prime} + \epsilon_{AB} \tilde{\phi}_{A\prime B\prime}).
\] (4.18)

It is then straightforward to show that the stress-energy (4.17) can be expressed in terms of the Maxwell spinor as

\[
\mathbb{T}_{ab} = -\frac{1}{2\pi} \sigma_a^{\ A\prime} \sigma_b^{B\prime} \phi_{AB} \phi_{A\prime B\prime}.
\] (4.19)

We have already seen in the previous subsections that the general matter field conditions, \( V_0 \), and the Raychaudhuri equation imply the stress energy tensor must satisfy (4.7). Using the spinorial definition (3.2) of \( \ell^a \), this restriction on the stress-energy tensor gives two important results for the Maxwell field:

\[
\phi_0 := \phi_{AB} o^A o^B \equiv 0 \quad \text{and} \quad \varphi \equiv \frac{1}{2\pi} \phi_{AB} l^A o^A \cdot \tilde{\phi}_{A\prime B\prime} l^{A\prime} o^{B\prime} \equiv \frac{1}{2\pi} |\phi_1|^2.
\] (4.20)

Here, \( \varphi \) is the quantity \( e \) introduced in (3.8) specialized to the Maxwell field. The second equation shows the spherical symmetry of \( \phi_1 \) we imposed in (3.5) does indeed guarantee
the spherical symmetry of \( \phi \) required by (3.8). However, the remaining Newman-Penrose component of the Maxwell field, \( \phi_2 \), is completely unconstrained. In particular, it need not be spherically symmetric.

Since \( \phi_1 \) is spherically symmetric, we can express it in terms of the electric and magnetic charges contained within the horizon. To do so, consider the general form of the Maxwell field compatible with (4.20):

\[
\mathbb{F} \equiv \ell \wedge (\phi_2 m + \bar{\phi}_2 \bar{m}) - 2 \left( \text{Im} [\phi_1] \wedge \bar{\epsilon} - \text{Re} [\phi_1] \wedge \ell \right). \tag{4.21}
\]

Here, \( \wedge \) denotes the volume form on \( S_\Delta \) and \( \wedge \) denotes its (space-time) dual. Now, since \( \bar{\epsilon} \) is defined with respect to the inward pointing unit space-like normal (see Appendix A), we have

\[
-4\pi Q_\Delta \equiv \oint_{S_\Delta} \star \mathbb{F} = -2 \oint_{S_\Delta} \text{Re} [\phi_1] \wedge \bar{\epsilon} \tag{4.22}
\]

and

\[
-4\pi P_\Delta \equiv \oint_{S_\Delta} \mathbb{F} = -2 \oint_{S_\Delta} \text{Im} [\phi_1] \wedge \bar{\epsilon}, \tag{4.23}
\]

where \( Q_\Delta \) and \( P_\Delta \) denote the electric and magnetic charges contained within \( S_\Delta \). Since \( \phi_1 \) is spherically symmetric, its real and imaginary parts can be pulled outside the integrals and we calculate it to be

\[
\phi_1 \equiv \frac{2\pi}{a_\Delta} (Q_\Delta + iP_\Delta). \tag{4.24}
\]

Here, \( Q_\Delta \) and \( P_\Delta \) are naturally spherically symmetric, but may as yet vary from one \( S_\Delta \) to another. However, the remaining boundary condition on the Maxwell field, (IVc), requires the Maxwell equations hold at the horizon. The Maxwell equations pulled-back to \( \Delta \) applied to the field strength (4.21) with \( \phi_1 \) given by (4.24) imply the charges \( Q_\Delta \) and \( P_\Delta \) must be constant over the entire horizon \( \Delta \). It should be noted that this constancy is caused by boundary conditions and not by equations of motion in the bulk. As a result, \( Q_\Delta \) and \( P_\Delta \) are constant over \( \Delta \) in any history satisfying our boundary conditions and not just ‘on-shell’.

### 4.4 Summary

As we have seen in this section, the boundary conditions place many restrictions on both the gravitational and electro-magnetic degrees of freedom. We will collect the results we have found here. These results use not only the boundary conditions, but also the fact that the only form of matter we consider is a Maxwell field.

The Newman-Penrose components of the Maxwell field at the horizon are constrained by

\[
\phi_0 \equiv 0 \quad \text{and} \quad \phi_1 \equiv \frac{2\pi}{a_\Delta} (Q_\Delta + iP_\Delta). \tag{4.25}
\]

The remaining component, \( \phi_2 \), is an arbitrary complex function over \( \Delta \).
The Newman-Penrose components of the Ricci tensor are
\[ R^i_j = 4\Lambda \quad \text{and} \quad \Phi_{ij} = 2G\phi_i\phi_j \quad \text{with} \quad i, j = 0, 1, 2. \] (4.26)

The second equation is simply a consequence of the Einstein equation and (4.19). The Newman-Penrose components of the Weyl tensor satisfy
\[ \Psi_0 = \Psi_1 = 0 \quad \text{and} \quad \Psi_3 = \Phi_{21}. \] (4.27)

The component \( \Psi_4 \) is an arbitrary complex function over \( \Delta \). The remaining component, \( \Psi_2 \), is related to the acceleration \( \tilde{\kappa} \) of \( \ell^a \) and the expansion \( 2\mu \) of \( n^a \) via:
\[ \Psi_2 + \frac{\Lambda}{3} = \mathcal{L}_\ell \mu + \tilde{\kappa} \mu, \] (4.28)

where \( \Lambda \) is the cosmological constant. It follows in particular that \( \Psi_2 \) is real and spherically symmetric. Finally, these components also satisfy
\[ d\nu = 2\left( \Psi_2 - \Phi_{11} - \frac{\Lambda}{6} \right) \tilde{\epsilon}, \] (4.29)

where \( \nu \) is a connection on the frame bundle of \( S_\Delta \) and \( \tilde{\epsilon} \) is its volume form.

5 Surface Gravity and the Zeroth Law

The zeroth law of black hole mechanics states that the surface gravity \( \kappa \) of a stationary black hole is constant over its horizon. In subsection 5.1, we will extend the standard definition of \( \kappa \) to arbitrary non-rotating isolated horizons using only structure available at the horizon. A key test of the usefulness of this definition comes from the zeroth law: Does the structure of \( \Delta \) enable us to conclude \( \kappa \) is constant on \( \Delta \) without reference to a static Killing field? In Section 5.2 we will show the answer is in the affirmative. Thus, our more general definition of surface gravity is consistent with our notion of isolation of the horizon. Furthermore, we will see that the structure of \( \Delta \) is rich enough to enable us to express \( \kappa \) in terms of the parameters \( r_\Delta, Q_\Delta \) and \( P_\Delta \) of the isolated horizon.

5.1 Gauge reduction and surface gravity

Our set-up suggests we define surface gravity using the acceleration, \( \tilde{\kappa} \), of the vector field \( \ell^a \). However, since the acceleration fails to be invariant under the rescalings of \( \ell^a \), we need to normalize \( \ell^a \) appropriately. As mentioned in the Introduction, the usual treatments of black hole mechanics in static space-times accomplish this by identifying \( \ell^a \) with the restriction to the horizon of that static Killing field which is unit at infinity. For a generic isolated horizon, there will be no such Killing field and our procedure can only involve structures defined on the horizon. Since \( \ell^a \) is null, and its expansion, twist and shear vanish, we cannot hope to
normalize it by fixing one of its own geometric characteristics. However, the normalizations of \( \ell^a \) and \( n^a \) are intertwined and we can hope to normalize \( n^a \) by fixing its expansion. The normalization of \( \ell^a \) will then be fixed.

To implement this strategy, let us begin by examining the available gauge freedom. The correspondence (3.2) between the fixed spin dyad \((\iota^A, o^A)\) and the preferred direction fields \([\ell^a, n^a]\) breaks the original \(SL(2, \mathbb{C})\) internal gauge group at \(\Delta\) to \(\mathbb{C}U(1)\). However, as we shall see shortly, the residual gauge invariance has a somewhat unusual structure. To see the nature of the residual gauge, it is simplest temporarily to consider a fixed soldering form and a variable spin dyad.

The most general transformation of the spin dyad which preserves the correspondence (3.2) is
\[
(\iota^A, o^A) \mapsto (e^{\Theta-i\theta} \iota^A, e^{-\Theta+i\theta} o^A).
\]
(5.1)

Here, \(\Theta\) and \(\theta\) are both real functions over \(\Delta\). Under these residual gauge transformations, one can show the null tetrad transforms as
\[
\ell^a \mapsto e^{-2\Theta} \ell^a \quad n^a \mapsto e^{2\Theta} n^a \\
m^a \mapsto e^{2i\theta} m^a \quad \bar{m}^a \mapsto e^{-2i\theta} \bar{m}^a.
\]
(5.2)

Thus, \(\Theta\) accounts for the rescaling freedom in \(\ell^a\) and \(n^a\) which must be eliminated to define the surface gravity unambiguously. \(\Theta\) is restricted to be spherically symmetric by (3.1). On the other hand, \(\theta\) is arbitrary and allows a general transformation on the frame bundle of each \(S_\Delta\). Furthermore, the function \(\mu\) appearing in (3.4) is not gauge invariant, but transforms according to
\[
\mu \mapsto e^{2\Theta} \mu.
\]
(5.3)

Note, however, that the spherical symmetry and nowhere-vanishing property of \(\mu\) are preserved by this transformation. Finally, the fields \(\tilde{\kappa}\) and \(\nu\) introduced in Section 4.1 transform as
\[
\tilde{\kappa} \mapsto e^{-2\Theta} (\tilde{\kappa} - 2 \mathcal{L}_\ell \Theta) \quad \text{and} \quad \nu_a \mapsto \nu_a - 2 \nabla_a \theta.
\]
(5.4)

The transformation of \(\tilde{\kappa}\) is the usual one for the acceleration of a vector field when that field is rescaled, and preserves the spherical symmetry of \(\tilde{\kappa}\). Meanwhile, the transformation of \(\nu_a\) is that of a \(U(1)\) connection.

We are now ready to fix the normalization of \(\ell^a\). The strategy outlined in the beginning of this subsection can be implemented successfully thanks to the following three non-trivial facts. First, the expansion of the properly normalized \(n^a\) in a Reissner–Nordström solution is ‘universal’: irrespective of the mass, electric and magnetic charges or cosmological constant, on the future, outer black hole event horizons in these solutions, \(\theta_{(n)} \equiv -2/r_0\) where \(r_0\) is the radius of the horizon. Motivated by this and the relation \(\theta_{(n)} = 2\mu\) (see (3.6)) we are led to set
\[
\mu \equiv -\frac{1}{r_\Delta}
\]
(5.5)

\footnote{To accommodate cosmological horizons, we will have to allow \(\mu\) to be strictly positive. This issue will be discussed in section \(\S\). For now, we focus on black hole horizons and assume \(\mu\) is strictly negative. Modifications required to accommodate positive \(\mu\) are straightforward.}
on a generic, non-rotating, Einstein–Maxwell isolated horizon of radius \( r_\Delta \) (i.e., of area \( a_\Delta = 4\pi r_\Delta^2 \)). Second, the gauge-freedom (5.2) available to us is such that we can always achieve the normalization (5.3) of \( \mu \). Third, it is obvious from (5.3) that this condition exhausts the freedom in \( \Theta \) completely. In particular, therefore, in a single stroke, this procedure fixes the normalization of \( n^a \) (and \( \ell^a \)) and gets rid of the awkwardness in the gauge freedom. The restricted gauge freedom is now given simply by:

\[
\ell^a \mapsto \ell^a, \quad n^a \mapsto n^a, \quad m^\alpha \mapsto e^{2i\theta} m^\alpha, \quad \bar{m}^\alpha \mapsto e^{-2i\theta}\bar{m}^\alpha, \quad (5.6)
\]

where \( \theta \) is an arbitrary real function on \( \Delta \); the local gauge group is reduced to \( U(1) \). We can now return to our usual convention wherein the spin dyad is fixed while the soldering form varies. The true residual gauge transformations are then the duals of (5.1), with \( \Theta = 0 \), acting on the soldering form. The effect of these residual transformations on the null tetrad and the other fields discussed here remain the same as those in (5.6).

From now on, we will assume that \( n^a \) and \( \ell^a \) are normalized via (5.5), denote the resulting acceleration of \( \ell^a \) by \( \kappa \) and refer to it as the surface gravity of the isolated horizon \( \Delta \). By construction, our general definition reduces to the standard one in Reissner–Nordström solutions.

To conclude this subsection, let us consider the gauge freedom in Maxwell theory. We just saw that a partial gauge fixing of the \( SL(2, \mathbb{C}) \) freedom in the gravitational sector is necessary to define the surface gravity in the absence of a static Killing field. The situation with the electric and magnetic potentials is analogous. In conventional treatments \[24\] these can be defined using the global static Killing field which, however, is unavailable for a generic isolated horizon. The idea again is to resolve this problem through a partial gauge fixing. Since the only available parameters are the radius \( r_\Delta \) and the charges \( Q_\Delta, P_\Delta \), dimensional considerations suggest the electric potential \( \Phi \triangleq A_a \ell^a \) is proportional to \( Q_\Delta/r_\Delta \) on the horizon. We fix the proportionality factor using the standard value of \( \Phi \) in Reissner–Nordström solutions. Thus, in the general case, we partially fix the gauge by requiring

\[
\ell^a A_a = \Phi = \frac{Q_\Delta}{r_\Delta} \quad (5.7)
\]

It turns out that this strategy of gauge-fixing also makes the variational principle well-defined for the Maxwell action.

The situation with the magnetic potential, however, is more subtle. Since there is no obvious expression for the magnetic potential in terms of \( A_a \), we cannot formulate a definition similar to (5.7). Instead, we will appeal to the well-known duality symmetry of the Maxwell field. Thus, in the remainder of the paper, we set \( P_\Delta = 0 \) in the main discussion. The results for isolated horizons with both electric and magnetic charge will follow from those including only electric charge by a duality rotation.
5.2 Zeroth law

We already know from (4.13) that \( \kappa \) is spherically symmetric. Therefore, to establish the zeroth law, it only remains to show that \( \mathcal{L}_\ell \kappa \) also vanishes. Recall first from (4.15) that on \( \Delta \), \( F \rightarrow -AB \) is given by:

\[
F \rightarrow -AB \hat{=} \left[ \left( \Psi_2 - \Phi_{11} - \frac{R}{24} \right) \delta^A_C \delta_D^B - \left( \frac{3}{2} \Psi_2 - \Phi_{11} \right) \sigma^A \sigma^B \epsilon_{CD} \right] \sum_{CD}.
\] (5.8)

Let us now consider the Bianchi identity \( D F \rightarrow -AB = 0 \). Transvecting it with \( \epsilon_A \epsilon_B \), we obtain

\[
\mathcal{L}_\ell \left( \Psi_2 - \Phi_{11} - \frac{R}{24} \right) \hat{=} 0.
\] (5.9)

In the Einstein–Maxwell system, \( \Phi_{11} \approx 2G|\phi_1|^2 \) and \( \phi_1 \) is constant on \( \Delta \) (see (4.24)). Similarly, since the stress-energy tensor is trace-free, \( R = 4\Lambda \) is also a constant. Hence it follows that \( \mathcal{L}_\ell \Psi_2 \hat{=} 0 \). Finally, (4.28) and our gauge condition (5.5) immediately imply:

\[
\kappa = -r_\Delta \left( \Psi_2 + \frac{\Lambda}{3} \right).
\] (5.10)

Hence, we conclude \( \mathcal{L}_\ell \kappa \hat{=} 0 \), as desired. This establishes the zeroth law of the mechanics of isolated horizons in the Einstein–Maxwell theory.

We will now obtain an explicit expression for \( \kappa \) in terms of the parameters of this isolated horizon. The key fact is that the first Chern number of the pull-back to \( S_\Delta \) of the connection \( \nu \) introduced in (4.6) is two. This can be seen in a number of ways, but it is essentially equivalent to the Gauss-Bonet theorem for a 2-sphere because \( \nu \) can be identified with a \( SO(2) \) connection in the frame bundle of \( S_\Delta \). Using this property, we have

\[
2 = -\frac{1}{2\pi i} \oint_{S_\Delta} id\nu = -\frac{1}{2\pi} \oint_{S_\Delta} 2 \left[ \left( \Psi_2 + \frac{R}{12} \right) - \left( \Phi_{11} + \frac{R}{8} \right) \right] \cdot \hat{\kappa},
\] (5.11)

where we have used (4.13) in the last equality here. Now, we have seen in (4.14) that the first term in the brackets here is spherically symmetric, and in (4.8) that the second term is proportional to the quantity \( e \) which has been restricted to be spherically symmetric. As a result, the entire integrand on the right side of (5.11) can be pulled through the integral. The remaining integral simply gives the area \( a_\Delta \) of \( S_\Delta \). Thus,

\[
\Psi_2 - \Phi_{11} - \frac{R}{24} \equiv -\frac{2\pi}{a_\Delta}\nabla \nabla,
\] (5.12)

Let us now specialize to the Einstein–Maxwell case. Then, \( R = 4\Lambda \) and \( \Phi_{11} = (G/2r_\Delta^2)(Q_\Delta^2 + P_\Delta^2) \). Therefore, using (5.10) we can express surface gravity in terms of \( r_\Delta, Q_\Delta \) and \( P_\Delta \):

\[
\kappa = \frac{1}{2r_\Delta} \left( 1 - \frac{G(Q_\Delta^2 + P_\Delta^2)}{r_\Delta^2} - \Lambda r_\Delta^2 \right).
\] (5.13)
We conclude with a few remarks.

1. The final expression (5.13) for $\kappa$ is formally identical to the expression for the surface gravity of a Reissner–Nordström black hole in terms of its radius and charge. This may be surprising since we have not restricted ourselves to static situations. However, if it is possible to express $\kappa$ in terms of the parameters $r_\Delta$, $Q_\Delta$, $P_\Delta$ and $\Lambda$ alone, this agreement must hold if the general expression is to reduce to the standard one on event horizons of static black holes. In our treatment, the agreement can be technically traced back to our general strategy for fixing the normalization of $\ell^a$.

2. It may also be surprising that, although we do not have a static Killing field at our disposal, it was possible to define $\kappa$ unambiguously and it turned out to satisfy the zeroth law. Furthermore, we could express $\kappa$ in terms of the parameters of the isolated horizon, irrespective of the details of gravitational and electro-magnetic radiation outside the horizon. This was possible because of two facts. First, the boundary conditions could successfully extract just that structure from static black holes which is relevant for these thermodynamical considerations. Second, at its core, the zeroth law is really local to the horizon; it does not know, nor care, about the space-time geometry away from the horizon. Physically, this meets one’s expectation that the degrees of freedom of a black hole in equilibrium should ‘decouple’ from the excitations present elsewhere in space-time.

3. Our expression (4.14) of surface gravity in terms of Weyl and scalar curvature is universal, i.e., holds independent of the particular matter sources so long as they satisfy the mild energy condition (3.7). Furthermore, in all these cases, $\kappa$ is spherically symmetric and the Bianchi identity ensures (5.9). The restriction to Maxwell fields as the only source has been used here only to show that $\frac{1}{4\pi G} \mathcal{L}_t \left( \Phi_{11} + \frac{R}{8} - \frac{\Lambda}{2} \right) = \mathcal{L}_a T_{ab} \ell^a n^b \equiv \mathcal{L}_e$ vanishes on $\Delta$. Thus, in the present setting of non-rotating isolated horizons, the zeroth law would hold for more general matter provided its stress energy tensor satisfies this last restriction. This condition is satisfied, for example, by dilatonic matter [4, 10].

4. In the main definition, we assumed $\Delta$ has topology $S^2 \times \mathbb{R}$. If $S^2$ is replaced by a compact 2-manifold of higher genus, the results of Section 4 and the proof of constancy of $\kappa$ on $\Delta$ remain unaffected. However, in obtaining the explicit expression (5.13) of surface gravity, we used the Gauss-Bonnet theorem. Hence this expression is not universal but depends on the genus of $S_\Delta$.

### 6 Action and Hamiltonian

As pointed out in the Introduction, to arrive at an appropriate generalization of the the first law of black hole mechanics, we first need to define the mass of an isolated horizon. The idea is to arrive at this definition through Hamiltonian considerations. In Section 6.1, we introduce an action principle which yields the correct equations of motion despite the presence of the internal boundary $\Delta$. In Section 6.2, we pass to the Hamiltonian theory by performing a Legendre transform and in 6.3 we discuss Hamilton’s equations of motion.
We will find that, due to the form of the boundary conditions, there are subtle differences between the Lagrangian and the Hamiltonian frameworks because the latter allows more general variations than the former.

As in previous sections, in the main discussion we assume the space-time under consideration is asymptotically flat with vanishing magnetic charge and discuss at the end the modifications required to incorporate non-zero $\Lambda$ and $P_\Delta$.

6.1 Action

Recall from Section 2 that, in absence of internal boundaries, the action for Einstein–Maxwell theory is given by:

$\tilde{S}(\sigma, A, \Lambda) = \frac{-i}{8\pi G} \int_M \text{Tr}[\Sigma \wedge F] + \frac{1}{8\pi} \int_M \mathcal{F} \wedge \ast \mathcal{F} + \frac{i}{8\pi G} \int_{C_\infty} \text{Tr}[\Sigma \wedge A]$  

(6.1)

In the presence of internal boundaries, however, this action need not be functionally differentiable. This is the case with our present boundary conditions at $\Delta$. In [7], the required modifications were discussed for the case when all histories $(\sigma, A, \Lambda)$ under consideration induce a fixed area $a_\Delta$ and electro-magnetic charges $Q_\Delta$ (and $P_\Delta$) on $\Delta$ and a differentiable action was obtained by adding to $\tilde{S}$ a surface term at $\Delta$. (For a general discussion of surface terms in the action, see [25].)

The strategy of fixing the parameters of the isolated horizon was appropriate in [7] because the aim of that analysis was to provide the classical framework for entropy calculations of horizons with specific values of their parameters. In this paper, on the other hand, we need a Hamiltonian framework which is sufficiently general for the discussion of the first law, in which one must allow the parameters to change. Therefore, we now need to allow histories with all possible values of parameters. Note, however, that a key consequence of the boundary conditions is that the area $a_\Delta$ and charge $Q_\Delta$ of the isolated horizon are constant in time. Therefore, the values of these parameters are fixed in any one history (although they may vary from one history to another). Now, since all the fields are kept fixed on the initial and final surfaces in the variational principle and the values of our dynamical fields on either of these surfaces determine $a_\Delta$ and $Q_\Delta$ for that history, one is only allowed to use those variations for which $\delta a_\Delta$ and $\delta Q_\Delta$ vanish identically. This fact simplifies the task of finding an appropriate action considerably: For example, as far as the action principle is concerned, one can continue to use the action used in [7].

There is however, a further subtlety which will lead us to use a more convenient boundary term in the action. Because of the nature of the variational principle discussed above, we are free to add any function of the horizon parameters $a_\Delta$ and $Q_\Delta$ without affecting the Lagrangian equations of motion. In the framework considered in [7], this just corresponds to the freedom of adding a constant (with appropriate physical dimension) to the Lagrangian. However, since the full class of histories now under consideration allows arbitrary areas $a_\Delta$ and charges $Q_\Delta$, the freedom is now more significant: it corresponds to changing the Lagrangian by a function of parameters $a_\Delta, Q_\Delta$. While the variational procedure used to derive
**Figure 5:** Region $\mathcal{M}$ of space-time considered in the variational principle is bounded by two partial Cauchy surfaces $M_1$ and $M_2$. They intersect the isolated horizon $\Delta$ in preferred 2-spheres $S_1$ and $S_2$ and extend to spatial infinity $i^\circ$.

the Lagrangian equations of motion is completely insensitive to this freedom, the Hamiltonians resulting from these Lagrangians will clearly be different. Can all these Hamiltonians yield consistent equations of motion? The answer is in the negative. The reason lies in subtle differences between the Lagrangian and Hamiltonian variations. In the Hamiltonian framework, the phase-space is based on a fixed space-like 3-surface $M$. Since the values of $a_\Delta$ and $Q_\Delta$ can vary from one history to another, they can also vary from one point of the phase space to another. In obtaining Hamilton’s equations, $\delta H = \Omega(\delta, X_H)$, we must now allow phase space tangent vectors $\delta$ which can change the values of parameters $a_\Delta, Q_\Delta$. Consequently, with boundary conditions such as ours, the burden on the Hamiltonian is greater than that on the Lagrangian. It turns out that these additional demands on the Hamiltonian suffice to eliminate the apparent functional freedom in its expression. More precisely, the requirement that Hamilton’s equations of motion be consistent for all variations $\delta$ in the phase space, including the ones for which $\delta a_\Delta$ and $\delta Q_\Delta$ do not vanish, determine the Hamiltonian completely. (There is no freedom to add a constant because, with only Newton’s constant $G$ and the speed of light $c$ at our disposal, there is no constant function on the phase space with dimensions of energy.) One can then work backwards and single out the expression of the action, which, upon Legendre transform, yields the correct Hamiltonian. We will follow this strategy.

Fix a region of space-time whose inner boundary is an undistorted, non-rotating isolated horizon, as depicted in figure 5. This region $\mathcal{M}$ is bounded in the past and future by space-like hypersurfaces $M_1$ and $M_2$ respectively which intersect the horizon $\Delta$ in preferred 2-sphere cross-sections $S_1$ and $S_2$ and extend to spatial infinity $i^\circ$. Since the space-time $\mathcal{M}$ is asymptotically flat at spatial infinity, the fields obey the standard falloff conditions at $i^\circ$. The interior boundary $\Delta$ is a non-rotating isolated horizon which satisfies the boundary conditions discussed in section 3. It turns out that, to obtain a well-defined action principle, we need to impose an additional condition at $\Delta$:

$$\oint_{S_\Delta} (\nu \cdot \ell) 2\epsilon = 0$$  \hspace{1cm} (6.2)
for any (preferred) 2-sphere cross-section $S_\Delta$ of $\Delta$, where $\hat{\epsilon}$ is the natural alternating tensor on $S_\Delta$ (see Appendix A.2). Note that this restriction is very mild since it only asks that the spherically symmetric part (with respect to $\hat{\epsilon}$) — or, equivalently, the zero mode — of $A \cdot \ell$ be real on $\Delta$. Then, the required action is given by:

$$S = \frac{i}{8\pi G} \int_M \text{Tr}[\Sigma \wedge F] + \frac{1}{8\pi} \int_M F \wedge ^* F + \frac{i}{8\pi G} \int_{C_\infty} \text{Tr}[\Sigma \wedge A] + \frac{1}{8\pi G} \int_\Delta (r_\Delta \Psi_2) \hat{\epsilon},$$

(6.3)

where $\hat{\epsilon}$ is the volume form on $\Delta$ compatible with our normalization for $\ell^a$ (see Appendix A). In Section 6.3, we find the resulting Hamiltonian does yield a consistent set of equations. That discussion will also show that the term added in the passage from (6.1) to (6.3) is uniquely determined by the consistency requirement.

Note that (6.3) does not have the Chern-Simons boundary term at $\Delta$ used in [7]. However, if one restricts oneself to histories with a fixed value of $a_\Delta$ as in [7], (6.3) is completely equivalent to the action used there. (The mild restriction (6.2) was not discussed in [7] but is needed also in the action principle used there.) In particular, as we will see in Section 6.2, the symplectic structure obtained from the present action (6.3) again has a boundary term at $\Delta$. If one works with a fixed $a_\Delta$, this term reduces to the Chern-Simons symplectic structure of [7]. For non-perturbative quantization [6, 8], it is this symplectic structure that plays the important role; the form of the action is not directly relevant.

In this paper we have chosen to work with (6.3) because it is more convenient for the Hamiltonian framework with variable $a_\Delta$, needed in the discussion of the first law. Furthermore, this form of the action appears to extend naturally to isolated horizons with distortion and rotation and also may be better suited for quantization in these more general contexts.

6.2 Hamiltonian Framework

To pass to the Hamiltonian framework, we need to perform the Legendre transform. As in Section 2.2, we begin by introducing the necessary structure on the 4-manifold $\mathcal{M}$. First, foliate $\mathcal{M}$ by partial Cauchy surfaces $M$ with the following properties: i) the 3-manifolds $M$ intersect the horizon at the preferred 2-spheres $S_\Delta$ such that the unit time-like normal $\tau^a$ to them coincides with the vector $(\ell^a + n^a)/\sqrt{2}$ at $S_\Delta$; and, ii) they extend to spatial infinity and intersect $C_\infty$ in 2-spheres $S_\infty$. Next, fix a smooth time-like vector field $t^a$, transverse to the leaves $M$ and a function $t$ such that: i) $t^a \nabla_a t = 1$ on $\mathcal{M}$; ii) $t^a$ tends to the unit time translation orthogonal to $M$ at spatial infinity; and, iii) $t^a$ tends to $\ell^a$ on the horizon $\Delta$. (The restriction on $t^a$ that it be orthogonal to the foliation at infinity has been made only for simplicity and can be removed easily by suitably modifying the discussion of the physical interpretation of surface terms in the Hamiltonian.) Finally, we will restrict ourselves to physically interesting situations in which $M$ are partial Cauchy surfaces for the space-time region $\mathcal{M}$ under consideration and adapt our orientations to the case in which the projection of $n^a$ into $M$ is a radial vector which points away from the region $\mathcal{M}$. (See figure 6 and the discussion that follows in Section 6.3).
The 3+1 decomposition of the space-time fields can now be performed exactly as in equations (2.6) and (2.7). Once again, the phase space consists of quadruples \((A^{AB}, \Sigma^{AB}, A_a, E_{ab})\) on the 3-manifold \(M\) satisfying appropriate boundary conditions. The conditions at infinity are the same as before, namely (2.8). However, there are additional boundary conditions at the horizon. First, because of (4.9), the form of the gravitational connection \(A\) is restricted at \(S_\Delta\):

\[
A^{AB}_{\infty} \equiv -i_\nu \ell^{(A}o^{B)} + \frac{1}{r_\Delta} m o^A o^B.
\]

Next, the curvature of \(A\) is restricted by (4.16) and (5.11) to satisfy \(F^{AB} = \hat{\Sigma} = -2\pi a_\Delta\), and the electro-magnetic field \(F\) is restricted by (4.24). Finally, the requirement that the action principle be well-defined imposes the mild restriction (6.2) at \(\Delta\).

The Legendre transform is straightforward using the procedure outlined in Section 2.2. The only new element is the treatment of boundary terms at the horizon which requires the use of the boundary conditions listed above. In order to state the final result, we have to introduce some notation. The space of our connections \(\nu\) on \(\Delta\) has the structure of an affine space. Let us choose any one of these connections \(\hat{\nu}\), satisfying \(\hat{\nu} \cdot \ell = 0\) as the ‘origin’. (For example, \(\hat{\nu}\) can be the ‘static magnetic monopole potential’ on every \(S_\Delta\).) Then using boundary conditions, it is easy to show that any other connection \(\nu\) can be expressed as:

\[
\nu = \hat{\nu} + \eta + d\psi
\]

where the 1-form \(\eta\) on \(\Delta\) satisfies: \(\ell \cdot \eta = 0\), \(\ell \cdot d\eta = 0\) and \(d^* \eta = 0\) where the dual is taken under the metric on each \(S_\Delta\). Note that there is the obvious gauge freedom \(\psi \mapsto \psi + \text{const}\) in the choice of the function \(\psi\). Let us eliminate it by requiring

\[
\oint_{S_\Delta} \psi^2 \epsilon = 0
\]

on any \(S_\Delta\). Then the Legendre transform of the action \(S\) of (6.3) is given by:

\[
S(\sigma, A, \mathcal{A}) = \int dt \left[ \frac{i}{8\pi G} \int_{M(t)} \mathcal{L}_t A \wedge \Sigma - \frac{i}{8\pi G} \oint_{S_\Delta(t)} \mathcal{L}_\ell \psi^2 \epsilon \right] - \int dt H_t,
\]

with

\[
H_t = \tilde{H}_t - \oint_{S_\Delta} \left( \frac{r_\Delta}{4\pi G} \Psi_2 - \frac{Q_\Delta}{2\pi r_\Delta} \phi_1 \right)^2 \epsilon,
\]

where \(\tilde{H}_t\) is defined in (2.9). Thus, the Hamiltonian has the familiar form: As in (2.9), the bulk term is a volume integral of constraints weighted by Lagrange multipliers determined by \(t^a\) and the surface term at infinity gives \(t^a P_a^{\text{ADM}} = -E^{\text{ADM}}\). However, now there is now a surface term at the horizon as well.

In order to bring out the similarity and differences in the two surface terms, let us express the term at infinity using the Weyl curvature. Assuming the field equations hold near infinity
and with the shift \( \tilde{N} \) set to zero at \( S_\infty \) because of our current assumptions on the asymptotic value of \( t^a \), we have [27]

\[
H_t = \int_M \text{constraints} + \lim_{r_o \to \infty} \oint_{S_{r_o}} \left( \frac{r_o}{4\pi G} \Psi_2 \right) \tilde{\chi} - \oint_{S_\Delta} \left( \frac{r_\Delta}{4\pi G} \Psi_2 - \frac{Q_\Delta}{2\pi r_\Delta} \phi_1 \right) \tilde{\chi} \quad (6.9)
\]

Thus, only the ‘Coulombic parts’ of the two curvatures enter the expressions of the two surface terms. However, while the surface term at infinity depends only on the gravitational curvature, the term at the horizon depends also on the Maxwell curvature.

The symplectic structure also acquires a term at the horizon.

\[
\Omega(\delta_1, \delta_2) = \tilde{\Omega}(\delta_1, \delta_2) - \frac{i}{8\pi G} \int_{S_\Delta} [\delta_1 \psi \delta_2 (\tilde{\chi}) - \delta_2 \psi \delta_1 (\tilde{\chi})] \quad (6.10)
\]

where \( \tilde{\Omega} \) is defined in (2.9). (As mentioned in Section 5.4, if we restrict ourselves to the phase space corresponding to a fixed value of \( a_\Delta \), the surface term in (6.10) reduces to the Chern-Simons symplectic structure for the connection \( A \) on \( S_\Delta \).) The presence of the surface term in the symplectic structure is rather unusual. For instance, although there is a boundary term in the action at infinity, the symplectic structure does not acquire a corresponding boundary term. Also note that we have not added new ‘surface degrees of freedom’ at the horizon (in contrast with, e.g., [27, 28]). Indeed, our phase space consists only of the standard bulk fields on \( M \) which normally arise in Einstein–Maxwell theory (see Section 2.2) and whose values on \( S_\Delta \) are determined by their values in the bulk by continuity. If anything, the boundary conditions restrict the degrees of freedom on \( \Delta \) by relating fields which are independent in the absence of boundaries. The symplectic structure on the space of these bulk fields simply acquires an extra surface term. In the classical theory, while this term is essential for consistency of the framework, it does not play a special role. In the description of the quantum geometry of the horizon and the entropy calculation [3, 8], on the other hand, this term turns out to be crucial.

### 6.3 Hamilton’s Equations

Hamilton’s equations are

\[
\delta H_t = \Omega(\delta, X_H), \quad (6.11)
\]

where \( X_H \) is the Hamiltonian vector field associated with the given time evolution vector field \( t^a \) and \( \delta \) is an arbitrary variation of the fields. Unlike in the discussion of the action, all fields appearing in the Hamiltonian are defined only on the space-like surface \( M \). Hamilton’s equations describe the time evolution of these fields. In particular, there is no a priori reason to expect the area or charge of the isolated horizon to be constant in time. The constancy of the area and charge must arise, if at all, as equations of motion of the theory. As we already noted, since the linearized fields \( \delta \) in (6.11) can have \( \delta a_\Delta \neq 0 \) and \( \delta Q_\Delta \neq 0 \), there are ‘more’ Hamilton’s equations than what one would have naively thought from the Lagrangian
perspective. The question is whether the additional equations ensure the area and charge are conserved and if the full set of equations is self-consistent.

Let us summarize the consequences of Hamilton’s equations for the Hamiltonian and symplectic structure introduced in the last subsection. The bulk equations of motion give the standard Einstein–Maxwell equations expressed in terms of connection variables. As usual, the variation of the term at infinity in the Hamiltonian cancels the surface term at infinity arising from the variation of the bulk term. Also, the equations of motion preserve the boundary conditions at infinity.

Thus, it only remains to examine the horizon terms. Using the relation (5.10) among $\Psi_2$, $\kappa$ and $r_\Delta$, and equating the horizon terms on the two sides of Hamilton’s equations, we obtain:

\[ 2\dot{\epsilon} = 0 \quad \text{and} \quad \dot{\psi} = \nu \cdot \ell. \quad \text{(6.12)} \]

The first of these equations in particular guarantees that the horizon area does not change under time-evolution. The second equation follows from (6.5) which defines $\psi$ and is thus a consistency condition. Note also that the restriction (6.6) on $\psi$ is preserved in time because of (6.2).

Finally, it is natural to ask whether Hamilton’s equations imply $\dot{Q}_\Delta = 0$. The answer is in the affirmative. However, this result is a consequence of a bulk equation of motion which guarantees $\int_S \dot{E} = 0$ where $S$ is any closed two surface. If we take $S = S_\Delta$, the obvious consequence is

\[ \dot{Q}_\Delta = 0. \quad \text{(6.13)} \]

To summarize, there exists a unique consistent Hamiltonian formulation in the presence of inner boundaries which are isolated horizons. The symplectic structure is given by (6.10), and the Hamiltonian by (6.9). The bulk equations of motion are the standard 3+1 versions of the Einstein–Maxwell equations. There are, however, additional equations on the horizon 2-sphere $S_\Delta$ which guarantee that $r_\Delta$ is a constant of motion.

For simplicity, in the main discussion we restricted ourselves to zero magnetic charge and cosmological constant, only one asymptotic region and only one inner boundary. However, these restrictions can be easily removed. If there is more than one asymptotic region and/or isolated horizon inner boundary, one need only include surface terms for each of these boundary 2-spheres. The incorporation of a non-zero magnetic charge and cosmological constant has a slightly more significant effect. As in Section 2.2, the presence of a cosmological constant changes the boundary conditions and the surface terms at infinity. The symplectic structure is unchanged. But, as discussed in Section 7.1, the surface term at the horizon in the expression of the Hamiltonian acquires additional terms involving $P_\Delta$ and $\Lambda$.

### 7 Physics of the Hamiltonian

In this section, we will examine the expression (6.3) of the Hamiltonian in some detail and extract physical information from it. In Section 7.1, we will argue that the surface term at $S_\Delta$ should be identified with the mass of the isolated horizon. In Section 7.2, we will
show that the numerical value of the Hamiltonian in any static solution vanishes identically so that the mass of the isolated horizon in such a space-time reduces to the ADM mass at infinity. Finally, in Section 7.3, we will argue that, in any solution to the field equations which is asymptotically flat at null and spatial infinity and asymptotically Schwarzschild at future time-like infinity, the Hamiltonian (6.9) equals the total energy radiated away through future null infinity, $\mathcal{I}^+$. In these space-times, the mass of the isolated horizon then equals the future limit of the Bondi mass, exactly as one would intuitively expect. We should emphasize, however, that the argument rests on assumptions on the asymptotic behavior of various fields (particularly near $i^+$) and we do not prove the existence of solutions to field equations with this behavior. Therefore, the discussion of Section 7.3 has a different status from the rest of the paper. Its primary purpose is to strengthen our intuition about the Hamiltonian and the mass of the isolated horizon.

### 7.1 Isolated Horizon Mass

For any physical system, energy can be identified with the numerical, on-shell value of the generator of the appropriate time translation. In Minkowskian field theories, for example, it is the generator of motions on phase space which correspond to space-time diffeomorphisms along a constant time-like vector field. Consider, as a second example, the theory of gravitational and electro-magnetic radiation in general relativity. In this case, one can construct a phase space of radiative modes at null infinity and the total radiated energy is the numerical value of the Hamiltonian generating a time translation in the BMS group at null infinity [29, 30]. Finally, in the physics of fields which are asymptotically flat at spatial infinity, the ADM energy arises as the on-shell, numerical value of the Hamiltonian generating an asymptotic time-translation. If the space-time under consideration admits several asymptotic regions (as, for example, in the Kruskal picture) then the energy in any one asymptotic region is given by the generator of a diffeomorphism which is an asymptotic time-translation in the region under consideration and asymptotically identity in all other regions.

These considerations suggest we define the energy associated with a given isolated horizon $\Delta$ to be the numerical, on-shell value of the generator of a diffeomorphism which is a time translation at $\Delta$ and asymptotically identity. To obtain an expression for this energy, let us examine the expression (6.9) of the Hamiltonian $H_t$. The bulk term vanishes on shell and the term at infinity does not contribute if the vector field $t^a$ vanishes at spatial infinity. Thus, the required expression is given simply by the surface term at $S_{\Delta}$. Furthermore, since the vector field $t^a$ tends to $\ell^a$ on $\Delta$ and, by construction, $\ell^a$ defines the ‘rest-frame’ of the isolated horizon, this energy can be identified with the mass $M_{\Delta}$ of $\Delta$. Thus, the Hamiltonian considerations lead us to set

$$M_{\Delta} = -\oint_{S_{\Delta}} \left[ \frac{r_{\Delta}}{4\pi G} \left( \Psi_2 + \frac{\Lambda}{3} \right) - \frac{Q_{\Delta} - iP_{\Delta}}{2\pi r_{\Delta}} \phi_1 \right] \, d\phi,$$

where we have now allowed for a non-zero cosmological constant $\Lambda$ and magnetic charge $P_{\Delta}$.

For purposes of the first law, it will be useful to rewrite this expression by eliminating the curvatures $\Psi_2$ and $\phi_1$ in favor of surface gravity $\kappa$ and electro-magnetic scalar potential
Φ. Using (4.24), (5.4) and (5.10), when $P_\Delta = 0$, we have:

$$M_\Delta = \frac{1}{4\pi G} a_\Delta + \Phi Q_\Delta$$

(7.2)

Note that the expression is formally identical with the familiar Smarr formula \[31\] for the mass of a Reissner–Nordström black hole. One also knows directly (i.e., without making an appeal to black hole uniqueness theorems) that the ADM mass of any static black hole in the Einstein–Maxwell theory is given by (7.2) \[24\]. Thus, as with our definition of surface gravity $\kappa$, although $M_\Delta$ is defined using only the structure at the isolated horizon $\Delta$, it agrees with the standard definition of black hole mass for static solutions. The reason behind this agreement will become clear in the next sub-section. However, in general (non-static) space-times, due to the presence of radiation, the ADM mass at infinity is quite distinct from the isolated horizon mass $M_\Delta$. When constraints are satisfied, we have

$$H_t = M_\Delta - E^{\text{ADM}},$$

(7.3)

and we will see in Section \[7.3\] that the numerical value of $-H_t$ can be identified with the total energy radiated through future null infinity. Finally, note that $M_\Delta$ has a specific contribution from the Maxwell field. We will see that this contribution is rather subtle but quite crucial to adequately handle charged processes in the first law. As far as we are aware, none of the general, quasi-local expressions of mass contain this precise contribution from the Maxwell field. Thus, in the charged case, it appears that $M_\Delta$ does not agree with any of the proposed quasi-local mass expressions.

A natural question is whether $M_\Delta$ is positive. Let us first consider the case with zero cosmological constant. Then, by fixing the value of the charge $Q_\Delta$ and minimizing $M_\Delta$ with respect to $r_\Delta$, one finds $M_\Delta$ is not only positive, but also bounded below: $M_\Delta^2 \geq GQ_\Delta^2$. At the minimum, i.e. when $M_\Delta^2 = GQ_\Delta^2$, the surface gravity $\kappa$ vanishes. However, unlike $M_\Delta$, $\kappa$ can be negative. This structure is familiar from Reissner–Nordström solutions, where the same inequality holds, $\kappa$ vanishes at extremality, is positive on the outer horizon and negative on the inner. However, it was not obvious that this entire structure would remain intact on general isolated horizons.

Let us now consider the case when the cosmological constant $\Lambda$ is non-zero. If $\Lambda$ is negative, $M_\Delta$ is again positive and $M_\Delta^2 > GQ_\Delta^2$. In this case, is natural to impose asymptotically anti-de Sitter boundary conditions, whence one only expects ‘black-hole type’ horizons. If $\Lambda$ is positive, one also has cosmological horizons and the situation becomes more involved. The resulting complications are illustrated by the Schwarzschild-de Sitter space-time (see figure 6). The Hamiltonian framework is physically useful only in those situations in which $M$ is a partial Cauchy surface for the space-time region $\mathcal{M}$ under consideration. The isolated horizons in this case are future boundaries of space-time such as (a) and (b) in the figure. Surface gravity as well as mass are positive on the black hole horizon (a). The case of the cosmological horizon requires a reconsideration of the sign conventions we previously adopted (see footnote 4). Specifically, in the construction of the Hamiltonian framework of Section \[6.1\], we chose our orientations by assuming the projection of $n^a$ into $\bar{M}$ is ‘outward pointing’
at $S_\Delta$ relative to $M$. With this choice, the expansion of $n^a$ (and hence the Newman–Penrose coefficient $\mu$) is negative on the black hole horizon (a), but positive on the cosmological horizon (b). Since we assumed, again for definiteness, that $\mu$ is negative in Sections 5 and 6, certain trivial modifications are needed to accommodate cosmological future horizons of the type (b). With these changes, the surface gravity and mass are again positive on (b).

To summarize, it is future horizons of type (a) and (b) that are of physical interest in our Hamiltonian framework. For them, the surface gravity and mass are positive in Schwarzschild–de Sitter space-time and we expect the situation to be similar for general isolated horizons with positive cosmological constant. More detailed considerations suggest that the interpretation of $a_\Delta$ as entropy and $\kappa$ as temperature are applicable only to such horizons.

### 7.2 Static solutions

The phase space now under consideration admits a 2-parameter family of static solutions, labeled by $M$ and $Q$ — the Reissner–Nordström solutions. Let us begin by evaluating the Hamiltonian $H_t$ (of (6.9)) on these solutions using for $t^a$ the static Killing field. Then, the volume integrals will vanish since the constraints are satisfied and only contributions from $S_\infty$ and $S_\Delta$ will remain. The term at infinity equals the negative of the ADM mass while, as noted above, the horizon term is given by $\frac{k a_\Delta}{4\pi G} + \Phi Q_\Delta$. Now, it is well known from the Smarr formula that on a Reissner–Nordström space-time, the value of the ADM mass is precisely
\[ \frac{\kappa_{\alpha}}{4\pi G} + \Phi Q_\Delta.\] Therefore the value of the Hamiltonian \( H_t \) when evaluated on static space-times is zero. (The same reasoning extends to the case when \( \Lambda \) and \( P_\Delta \) are non-zero.)

This feature is not accidental; there is a general argument from symplectic geometry which 'explains' this vanishing of \( H_t \). We will conclude this sub-section by presenting the argument.

In symplectic geometry, Hamilton’s equations are \( \delta H = \Omega(\delta, X_H) \), where \( \delta \) is an arbitrary variation and \( X_H \) is the Hamiltonian vector field. A stationary solution (such as a Reissner–Nordström solution in the sector of Einstein–Maxwell theory now under consideration) is one at which the Hamiltonian vector field either vanishes or generates pure gauge evolution. In either case, the symplectic structure evaluated on \( X_H \) and any other vector field \( \delta \) vanishes. Therefore, at these points of the phase space, \( \delta H = 0 \) for any variation \( \delta \). In particular \( \delta H = 0 \) for variations relating two nearby stationary solutions. Let us suppose the phase space is such that the space of stationary solutions is connected (an assumption satisfied by the Reissner–Nordström family in our case). Then, the Hamiltonian must take some constant ‘preferred value’ on all stationary solutions. Now, let us suppose there is no natural energy scale in the theory. (This assumption is satisfied in our case because \( M_\Delta \) and \( Q_\Delta \) are not fixed on our phase space and because one cannot construct a quantity with dimensions of mass from \( G \) and \( c \) alone.) Then, this ‘preferred value’ must be zero.

### 7.3 Hamiltonian equals Radiative Energy

We now present a result which provides an intuitive interpretation of the Hamiltonian \( H_t \) and a further motivation for our definition of the isolated horizon mass \( M_\Delta \). More precisely, using suitable regularity assumptions, we will show that, when the equations of motion (with \( \Lambda = 0 \)) are satisfied and \( t^a \) is adapted to the natural rest frames at the horizon, the numerical value of \( H_t \) equals \( t^a P^\text{rad}_a = -E^\text{rad}_\infty \), where \( E^\text{rad}_\infty \) is flux of energy radiated across \( \mathcal{I}^+ \). However, as explained in the beginning of Section 6, because we will need a number of new assumptions, the considerations of this sub-section are not as self-contained as those of the rest of the paper.

Let us assume that the underlying space-time \( \mathcal{M} \) is of the type indicated in figure (1.a). That is, we assume i) the space-time is asymptotically flat at future null infinity \( \mathcal{I}^+ \) and asymptotically Schwarzschild at time-like infinity \( i^+ \); ii) the Bondi news tensor on \( \mathcal{I}^+ \) tends to zero as one approaches \( i^0 \) and \( i^+ \); iii) the isolated horizon \( \Delta \) extends to \( i^+ \); and, iv) the boundary of \( \mathcal{M} \) consists of \( \Delta, \mathcal{I}^+ \cup i^+ \cup i^0 \) and a partial Cauchy surface \( \mathcal{M} \).

Fix a conformal completion \((\hat{\mathcal{M}}, \hat{g}_{ab})\), of \((\mathcal{M}, g_{ab})\) which has \( \mathcal{I}^+ \) as its (future) null boundary. Appendix C summarizes the structure available at \( \mathcal{I}^+ \). Let us begin by recalling that part of its structure which we will need in this subsection. The conformal factor \( \Omega \) vanishes at \( \mathcal{I}^+ \) and \( \hat{n}_a := \nabla_a \Omega \) is the null normal to \( \mathcal{I}^+ \). The conformally rescaled metric \( \hat{g}_{ab} \) induces a degenerate metric \( \hat{q}_{ab} \) at \( \mathcal{I}^+ \) which satisfies \( \hat{q}_{ab} \hat{V}^a = 0 \) on \( \mathcal{I}^+ \) if and only if the tangent vector \( \hat{V}^a \) to \( \mathcal{I}^+ \) is proportional to \( \hat{n}^a \). Although \( \hat{q}_{ab} \) is degenerate, we can define its ‘inverse’ \( \hat{q}^{ab} \) via \( \hat{q}_{ab} \hat{q}^{bc} \hat{q}_{cd} = \hat{q}_{ad} \). This ‘inverse’ is unique up to additions of terms of the form \( \hat{n}^{(a} \hat{V}^{b)} \) for some vector field \( \hat{V}^a \) tangent to \( \mathcal{I}^+ \). The phase space of radiative
modes of the gravitational and electro-magnetic fields at $\mathcal{J}^+$ can be coordinatized by pairs of fields $(\gamma_{ab}, A_a)$ defined intrinsically on $\mathcal{J}^+$. The fields $\gamma_{ab}$ code the gravitational degrees of freedom; they are symmetric, transverse (i.e., $\gamma_{ab}n^b = 0$) and trace-free (i.e., $\gamma_{ab}\hat{\gamma}^{ab} = 0$). These properties imply that $\gamma$ has precisely two independent components which represent the two radiative modes of the gravitational field. The Maxwell degrees of freedom are coded in the 1-form fields $A_a$ on $\mathcal{J}^+$, satisfying $\hat{n}^a = 0$, with $\hat{n}$ tending to zero at $\mathcal{I}^0$. Again, $A_a$ has two independent components which capture the two radiative modes of electro-magnetic field. The symplectic structure can be written as:

\[
\Omega^{\text{rad}}(\delta_1, \delta_2) := \frac{1}{32\pi G} \int_{\mathcal{J}^+} \hat{q}^{ac} \hat{q}^{bd} [\delta_1 \gamma_{ab} \mathcal{L}_{\hat{n}}(\delta_2 \gamma_{cd}) - \delta_2 \gamma_{ab} \mathcal{L}_{\hat{n}}(\delta_1 \gamma_{cd})] \hat{\epsilon} + \frac{1}{8\pi} \int_{\mathcal{J}^+} \hat{\gamma}^{ab} [\delta_1 A_a \mathcal{L}_{\hat{n}}(\delta_2 A_b) - \delta_2 A_a \mathcal{L}_{\hat{n}}(\delta_1 A_b)] \hat{\epsilon}
\]

(7.4)

(For details, see Appendix C.1 and [29, 30].)

The asymptotic symmetry group at $\mathcal{I}^+$ is the BMS group [32] which admits a preferred four-dimensional Abelian sub-group of translations. Let us suppose that the conformal factor is chosen such that $\gamma_{ab}$ is the unit 2-sphere metric. Then, $\hat{n}^a$ is a BMS time-translation. Diffeomorphisms generated by $\hat{n}^a$ induce motions on the radiative phase-space. As one might expect, they preserve $\Omega^{\text{rad}}$ and the corresponding Hamiltonian is given by [30]:

\[
H_{\hat{n}}^{\text{rad}} = -\frac{1}{32\pi G} \int_{\mathcal{J}^+} N_{ab} N_{cd} \hat{q}^{ac} \hat{q}^{bd} \hat{\epsilon} - \frac{1}{8\pi} \int_{\mathcal{J}^+} \left( F_{ac} F^c_b + F_{ac} * F^c_b \right) \hat{n}^a \hat{n}^b \hat{\epsilon}
\]

(7.5)

where $N_{ab} = -2\mathcal{L}_{\hat{n}} \gamma_{ab}$ is the Bondi News tensor at the point in the radiative phase space labeled by $\gamma_{ab}$. Thus, $\delta H^{\text{rad}} = \Omega^{\text{rad}}(\delta, X_{\hat{n}})$ for any tangent vector $\delta$ to the radiative phase space. Using the asymptotic form of the space-time metric in suitable coordinates, Bondi and Sachs [32] identified the right side of (7.5) as

\[
P^{\text{rad}} \cdot t = -E^{\text{rad}}
\]

(7.6)

where $P^{\text{rad}}$ is the 4-momentum radiated across $\mathcal{J}^+$ and $t$ represents the BMS time translation defined by $\hat{n}^a$. Thus, $E^{\text{rad}}$ is the flux of energy across $\mathcal{J}^+$ carried by gravitational and electro-magnetic waves. (Again, the negative sign arises in (7.6) because our signature is $-+++$.)

The Hamiltonian formulation at null infinity [30] provides a general conceptual setting in support of this interpretation.

We wish to relate these structures on the radiative phase space with those on the canonical phase space introduced in Section 6. Fix a point on the constraint surface of the canonical phase space and evolve it using field equations. Consider tangent vectors satisfying linearized

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5More precisely, the radiative phase space consists of certain equivalence classes of connections on $\mathcal{J}^+$. Thus it has a natural affine space structure. In introducing this coordinatization, an equivalence class of ‘trivial’ connections has been chosen as the origin. The fact that the phase space is an affine — rather than a vector — space has some subtle but important consequences. These will be ignored here as they do not affect the issues now under discussion. For details, see [29, 30].
constraints at this point and evolve them using linearized field equations. In appendix C, assuming the background and linearized solutions satisfy certain falloff conditions, we find

\[ \Omega(\delta_1, \delta_2) = \Omega^{\text{rad}}(\delta_1^{\text{rad}}, \delta_2^{\text{rad}}). \]

(7.7)

where \( \Omega^{\text{rad}} \) is evaluated at the point in the radiative phase space defined by the background solution and \( \delta^{\text{rad}} \) is the tangent vector in the radiative phase space defined by the linearized solution \( \delta \). The idea is to let \( \delta_1 \) be arbitrary, choose for \( \delta_2 \) the Hamiltonian vector field defined by a time translation, and use (7.7) to relate the canonical Hamiltonian (6.9) to the radiative Hamiltonian (7.5).

Let us choose a vector field \( t^a \) on \( M \) such that: i) \( t^a = \ell^a \) on \( \Delta \); ii) \( t^a \) is unit at spatial infinity and defines an asymptotic time-translation; and, iii) \( t^a \) is a BMS time translation at \( I^+ \) and the conformal factor is so chosen that \( t^a = \hat{n}^a \) at \( I^+ \). As in Section 6, leaves \( M \) of our foliation will be assumed to be asymptotically orthogonal to \( t^a \). Then, from Section 6 we have \( \delta H_t = \Omega(\delta, X_t) \), with \( X_t \equiv (\mathcal{L}_t A, \mathcal{L}_t \Sigma; \mathcal{L}_t \Lambda, \mathcal{L}_t E) \). From the above discussion of the radiative phase space, we have: \( \delta^{\text{rad}} H_t^{\text{rad}} = \Omega^{\text{rad}}(\delta^{\text{rad}}, X_t^{\text{rad}}) \) where \( X_t^{\text{rad}} \equiv (\mathcal{L}_{\hat{n}} \gamma; \mathcal{L}_{\hat{n}} \Lambda). \) Therefore, using the equality (7.7) of the two symplectic structures, we conclude:

\[ \delta H_t = \delta^{\text{rad}} H_t^{\text{rad}} = -\delta^{\text{rad}} E_{\infty}^{\text{rad}} \]

(7.8)

for all linearized solutions \( \delta \) satisfying the asymptotic conditions. Let us assume such solutions span the tangent space at every point in the portion of the phase space under consideration. Then, we conclude that \( H_t \) and \( H_t^{\text{rad}} \) differ by a constant. To fix the value of that constant, let us examine the Reissner–Nordström space-times. We already saw in Section 7.2 that in these space-times \( H_t \) vanishes. Since they are static, the Bondi News tensor and the electro-magnetic radiation vanish on \( \mathcal{I}^+ \). Hence, \( H_t^{\text{rad}} \) also vanishes in a Reissner–Nordström solution. Thus, the value of the constant is zero and

\[ H_t = -E_{\infty}^{\text{rad}}. \]

(7.9)

As one might have intuitively expected, the canonical Hamiltonian equals the time component of the flux of the 4-momentum that is radiated across \( \mathcal{I}^+ \).

Recall from (7.3) that, when the constraints are satisfied, the value of the canonical Hamiltonian \( H_t \) is equal to the isolated horizon mass \( M_\Delta \) minus the ADM energy \( E^{\text{ADM}} \) in the rest frame of the isolated horizon. Therefore, it now follows that \( M_\Delta = E^{\text{ADM}} - E_{\infty}^{\text{rad}} \). It is well-known that the difference of the ADM energy and the flux of energy through \( \mathcal{I}^+ \), denoted \( E^{\text{Bondi}}(i^+) \), is the future limit of the Bondi energy (in the rest frame defined by \( t^a \) [33]. Hence, we conclude:

\[ M_\Delta = E^{\text{Bondi}}(i^+) \]

(7.10)

Thus, \( M_\Delta \) can be thought of as the mass remaining in the space-time after all radiation has escaped to infinity, or, equivalently, the mass of the black hole with its static hair. This simple interpretation provides additional support for our definition of the horizon mass.

As emphasized earlier, the discussion of this subsection is based on a number of technical assumptions (which are stated in Appendix C.2). We will conclude with a summary.
of their physical content. Apart from asymptotic flatness at spatial and null infinity, the key assumptions involve the structure of $i^+$. We assume there is only one bound state in asymptotic future, represented by the isolated horizon. This reflects the expectation that, in the Einstein–Maxwell theory, there would be no gravitational or electro-magnetic radiation hovering around the horizon at late times. Multiple black hole solutions which reach equilibrium asymptotically are excluded, as their structure at $i^+$ would be quite different from that of the Schwarzschild space-time. If the black holes do not reach asymptotic equilibrium but accelerate away from each other, the structure at $i^+$ may be similar to (or even simpler than) that in the Schwarzschild space-time. An example is provided by the C-metric, where, if the parameters are adjusted suitably, the structure at $i^+$ as well as that at $i^o$ is regular as in Minkowski space-time. However, the accelerating black holes pierce $I^+$ which is now singular. Therefore, as it stands, our analysis is not applicable to this case either. Although our analysis could conceivably be generalized to cover these two types of situation, its current form is primarily applicable to the situation depicted in figure 1(a) in which a single gravitational collapse occurs.

8 First Law

Since we now have well-defined notions of surface gravity $\kappa$, electric potential $\Phi$ and horizon mass $M_\Delta$, we are ready to examine the question of whether the first law holds. In section 8.1, we will consider the equilibrium version of the law in which the horizon observables of two nearby space-times are compared. This version is closer in spirit to the treatment of the first law of thermodynamics in which one compares the values of macroscopic, thermodynamic quantities associated with two nearby equilibrium configurations, without reference to the process which causes the transition between them. In section 8.2, we will consider the physical process version of the first law in which one explicitly considers the process responsible for the transition. This version will bring out certain subtleties.

8.1 Equilibrium version

Denote by $I^H$ the (infinite-dimensional) space of space-times admitting (one or more) isolated horizons. In this section, we will be concerned only with the structure near isolated horizons. In particular, we will not have to refer at all to the boundary conditions at infinity or to the precise nature of the matter outside the isolated horizon. We will simply assume the surface gravity $\kappa$, the potential $\Phi$ and the mass $M_\Delta$ of the isolated horizon are determined by its intrinsic parameters $r_\Delta$ and $Q_\Delta$ via (6.13), (5.7) and (7.2):

$$M_\Delta = \frac{r_\Delta}{2G} \left(1 + \frac{GQ_\Delta^2}{r_\Delta^2} - \Lambda r_\Delta^2 \right) \quad a_\Delta = 4\pi r_\Delta^2$$

$$\kappa = \frac{1}{2r_\Delta} \left(1 - \frac{GQ_\Delta^2}{r_\Delta^2} - \Lambda r_\Delta^2 \right) \quad \Phi = \frac{Q_\Delta}{r_\Delta},$$

(8.1)
and allow general matter fields in the exterior. (Recall from Section 3 that the notion of isolated horizons is not tied down to Einstein–Maxwell theory.) This viewpoint is similar to that normally adopted for the ADM 4-momentum and angular momentum defined at spatial infinity. These quantities are first derived from Hamiltonian considerations adapted to specific matter sources (e.g., Klein Gordon, Maxwell, Dirac and Yang-Mills fields) but then used also for total 4-momentum and angular momentum for more general forms of matter (e.g. fluids) for which a satisfactory initial value formulation and Hamiltonian framework may not exist. Thus, one often uses the expressions of 4-momentum and angular momentum at infinity without specifying the precise matter content, assuming only that the stress-energy tensor satisfies physically reasonable conditions and falls off appropriately. In the same spirit, we now assume that $M_\Delta$, $\kappa$ and $\Phi$ of an isolated horizon $\Delta$ of radius $r_\Delta$ and charge $Q_\Delta$ are given by (8.1), irrespective of the matter content outside, so long as that matter does not endow the horizon with additional intrinsic parameters (such as a dilatonic charge or a new $U(1)$ charge).

Given a space-time $(\mathcal{M}, g_{ab})$ in $\mathcal{I}H$ and a tangent vector $\delta$ at this point, we can consider a smooth curve in $\mathcal{I}H$ passing through this point with $\delta$ as a tangent vector there and examine how $M_\Delta$ and $\kappa$ associated with the isolated horizon $\Delta$ in the background change. Straightforward algebra yields:

$$\delta M_\Delta = \frac{1}{8\pi G} \kappa \delta a_\Delta + \Phi \delta Q_\Delta.$$  

(8.2)

This equation tells us the relation between infinitesimal changes in the mass, area, and charge of two nearby, non-rotating isolated horizons. It is our generalization to isolated horizons of the equilibrium version of the first law of black hole mechanics.

We will conclude this sub-section with a few remarks.

1. The above calculation leading to the first law is trivial. The non-trivial part of the analysis was to arrive at expressions (8.1) of $\kappa$ and $M_\Delta$ in absence of a static Killing field. Again, although our boundary conditions allow the presence of radiation arbitrarily close to the horizon, they successfully extract the structure from event horizons of static black holes that is relevant for thermodynamic considerations. As with the zeroth law, the veracity of the first law can be taken as additional support for our definitions of $\kappa$ and $M_\Delta$.

2. The laws of black hole mechanics were first derived by Bardeen, Carter and Hawking [2, 3]. They considered stationary black holes possibly surrounded by a perfect fluid in a circular flow and arrived at the first law by comparing two nearby stationary solutions. For purposes of comparison, it is more convenient to use an extension of that work to more general matter fields discussed by Heusler [24]. In the non-rotating case, their main results can be summarized as follows. Identities governing the Komar integral of the static Killing vector imply the mass $M_{ADM}$ measured at spatial infinity is given by

$$M_{ADM} = \frac{\kappa}{4\pi G} a_h + \int_M (2T_{ab} - T g_{ab}) K^a dS^b.$$  

(8.3)

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where \( a_h \) is the area of the horizon, \( M \) is a partial Cauchy surface from the horizon to spatial infinity, \( T_{ab} \) is the stress-energy of the matter field, and \( K^a \) is the static Killing field. Under variations \( \delta \) from one such stationary black hole solution to another, it was then shown that

\[
\delta M_{\text{ADM}} = \frac{\kappa}{8\pi G} \delta A + \frac{1}{16\pi} \int_M G_{ab}(\delta g_{ab})^* K - \frac{1}{8\pi} \int_M \delta^*(G.K),
\]

(8.4)

where \(^*K\) is the 3-form dual to \( K^a \) and \(^*(G.K)\), the 3-form dual to \( G^{ab}K_b \). In the Einstein–Maxwell case, it turns out that the two volume integrals in (8.4) collapse to a single term at the horizon which is precisely \( \Phi \delta Q \Delta \), where \( \Phi \equiv A.K \). Thus, in this case, the first law relates the change in the ADM mass to changes in quantities at the horizon. The final form is the same as (8.2) (with \( M_{\Delta} \) replaced by \( M_{\text{ADM}} \)). Similarly the final expression of \( M_{\text{ADM}} \) is just the right side of (7.2).

However, there are some important differences from our approach. First, in the above derivation, one restricts oneself to static solutions of field equations and makes a heavy use of the Komar integral associated with the Killing vector \( K^a \). Second, the permissible variations \( \delta \) are only those which relate nearby static solutions. Third, the term \( \Phi \delta Q \Delta \) at the horizon arose from the volume terms in (8.4); that its value depends only on the fields evaluated at the horizon is not fundamental to this derivation. The whole calculation of [24] is based on an interplay between infinity and the horizon which is possible only because the assumption of staticity makes the problem elliptic and field variations ‘rigid’. The black hole is not studied as an isolated, separate entity; the quantities defined at the horizon are tightly tied to the exterior fields. Indeed, there is no useful analog of the mass \( M_{\Delta} \), associated with the horizon. Perhaps the closest analog is the Komar integral evaluated at the horizon, sometimes denoted \( M_H \) [3]. However, in the charged case, this integral does not include the Maxwell contribution \( \Phi Q \Delta \) and cannot therefore be used directly in the first law. Our derivation, by contrast, makes no reference to a Killing vector and allows radiation fields outside the horizon. All our considerations are local to the isolated horizon; in the variation, we did not have to refer to bulk fields on \( M \) at all. Finally, the mass which appears in our first law is the isolated horizon mass, \( M_{\Delta} \), and not the ADM mass \( M_{\text{ADM}} \).

3. A treatment of the first law based on Hamiltonian considerations was given by Wald and collaborators [34, 35, 36]. The final result of this work is more general than that of Bardeen-Carter-Hawking type of analyses. The background space-time is again a stationary black hole, possibly with matter fields in the exterior. However, the perturbations \( \delta \) are no longer required to be stationary; they can relate the background stationary solution to any nearby solution. Since our approach is also based on the Hamiltonian framework, the two treatments share a number of common features.

However, there are also a number of important differences, both in methodology and final results. While boundary conditions play a key role in our analysis, their analogs are not specified in references [34, 35, 36]. Consequently, the issue of differentiability of action is not discussed. In particular, while our action \( \tilde{S} \) of (6.3) cannot be written as a pure bulk term, in [34, 35, 36] there is no surface term in the action either at infinity or at the horizon. Consequently, the Hamiltonian contains only bulk terms and there is no analog of
our horizon mass $M_\Delta$. When restricted to non-rotating black holes, the first law [34, 35, 36] has the same form as (8.2). However, as in the Bardeen-Carter-Hawking approach, our $M_\Delta$ is replaced by the ADM mass $M_{ADM}$ and the background space-time is assumed to be a stationary black hole solution.

4. As indicated in Section 3.2, the space $\mathcal{IH}$ of space-times admitting isolated horizons in Einstein–Maxwell theory is infinite-dimensional. The space $\mathcal{S}$ of static solutions is a finite-dimensional subspace of $\mathcal{IH}$. In the Bardeen-Carter-Hawking type approach, the first law holds only at points of $\mathcal{S}$ and for tangent vectors $\delta$ to $\mathcal{S}$. In the Wald approach, it holds again only at points of $\mathcal{S}$ but the variations $\delta$ need not be tangential to $\mathcal{S}$. In the approach developed in this paper, the first law holds at all points of $\mathcal{IH}$ and for all tangent vectors to $\mathcal{IH}$. Thus, the generalization involved is very significant. However, since our boundary conditions imply the intrinsic geometry of $S_\Delta$ is spherically symmetric, distorted black holes are excluded from our analysis. By contrast, the other two approaches can handle static, distorted black hole solutions. In Einstein–Maxwell theory, there are no such solutions. However, if we allow charged fluid sources, such solutions presumably exist. Therefore, in the general context, our framework misses out certain situations which are encompassed by the other two approaches. It would be interesting to generalize our framework to overcome this limitation.

8.2 Physical process version

Let us now consider the situation depicted in figure 1(b). We are given a space-time with a non-rotating, isolated horizon $\Delta_1$ (with parameters $r_1$ and $Q_1$). Suppose a small amount of matter falls in to the horizon and after a brief dynamical period the horizon settles down to a new equilibrium configuration $\Delta_2$ (with parameters $r_2$ and $Q_2$). The question is: How do the observables associated with the horizons change in this physical process? The difference from the situation considered in the previous subsection is that one is now considering a physical process occurring in a single space-time rather than comparing two nearby space-times.

It is completely straightforward to analyze the process in our framework since the two masses and surface gravities are determined by their intrinsic parameters via (8.1). The actual algebraic calculation is the same as in the last sub-section; only the physical meaning of the variation $\delta$ is different. Hence, we again find the changes in mass, area and charge are governed by (8.2).

It is instructive to analyze this relation in terms of properties of the matter which fell across the horizon. Using the Raychaudhuri equation, and keeping only first order terms in variations, it is straightforward to show [30]

$$\kappa \delta a_\Delta = 8\pi G \delta E^{\text{flux}} \equiv 8\pi G \int_{\mathcal{H}} * (\delta T \cdot \ell),$$

(8.5)

where $\mathcal{H}$ is the portion of the horizon (between $\Delta_1$ and $\Delta_2$) crossed by the matter and $*(\delta T \cdot \ell)$ is the 3-form dual to $\delta T^a_{\ \ b} \ell^b$. Let us first suppose $Q_2 = Q_1$, i.e., the charge of the horizon did not change in this physical process. Then, comparing (8.5) with (8.2), we arrive
at a simple physical picture: the change in the mass of the horizon is equal to the total energy flux $E^{\text{flux}}$ across the horizon. However, it is interesting to note that, if $Q_2 \neq Q_1$, there is an extra contribution, $\Phi \delta Q_\Delta$, to $\delta M_\Delta$. What is the origin of this term? It arises due to ‘book-keeping’ in the following sense. As we observed in Section 7, $M_\Delta$ contains not just the ‘raw energy of the content of the horizon’, but also the energy of the electro-magnetic hair outside the horizon. (Recall that in static solutions, $M_\Delta$ equals the ADM mass and, more generally, it equals the future limit of the Bondi mass, both of which include the contribution from energy in the Coulombic electro-magnetic field outside the horizon.) Before the physical process began, the charge $\delta Q_\Delta = Q_2 - Q_1$ is outside the horizon and the energy in its Coulomb field does not contribute to $M_1$. At the end of the process, however, the black hole charge changes by $\delta Q_\Delta$ and the energy in the corresponding Coulombic field does contribute to $M_2$. This accounts for the term $\Phi \delta Q_\Delta$ in the expression of $\delta M_\Delta = M_2 - M_1$. Thus, the physical process version of the first law is subtle. The first order change in the mass of the horizon has a two-fold origin: a contribution due to flux of energy across horizon and another contribution from book-keeping of the energy in the Coulombic hair of the horizon.

What is the situation in the standard framework, where one uses $M_{\text{ADM}}$ in place of $M_\Delta$? To our knowledge, the physical process version of the first law has been discussed only in the uncharged case [36]. One assumes that the background space-time is globally static and considers a (non-static) matter perturbation which falls across the horizon. The ADM mass of the unperturbed space-time is taken to be $M_1$ and the ADM mass of the background plus perturbation is taken to be $M_2$. Then, using the reasoning given above, one arrives at the first law $\delta M = (\kappa \delta a)/8\pi G$ and interprets $\delta M$ as the change in the mass of the black hole due to the energy flux across the horizon. However, in the charged case, if $\delta Q_\Delta$ is not equal to zero, it seems difficult to account for the term $\Phi \delta Q_\Delta$ which also contributes to $\delta M$ without bringing in $M_\Delta$. It is interesting to note that, in this respect, there is a key difference between the angular momentum work term $\Omega \delta J$ and the electro-magnetic term $\Phi \delta Q_\Delta$: While the angular momentum contribution is coded easily in the flux of the stress-energy across the horizon, the electro-magnetic contribution is not. This is why, unlike the electro-magnetic work term, the angular momentum work term can be easily incorporated in the physical process version in the standard approach [36].

9 Discussion

Let us begin with a summary of the main ideas and results.

In Section 3 we introduced the notion of a non-rotating isolated horizon $\Delta$. While one needs access to the entire space-time to locate an event horizon, isolated horizons can be located quasi-locally. Event horizons of static black holes in Einstein–Maxwell theory do...
qualify as isolated horizons. However, the definition does not require the presence of a Killing field even in a neighborhood of $\Delta$. Rather, physically motivated, geometric conditions are imposed on the null normal $\ell^a$ to $\Delta$ and on an associated inward pointing null vector $n^a$ at $\Delta$. These conditions imply the Lie derivative along $\ell^a$ of the intrinsic (degenerate) metric of $\Delta$ vanishes which in turn implies the area of an isolated horizon is constant in time. In this sense, the horizon itself is isolated or ‘in equilibrium’. However, the space-time may well admit electro-magnetic and/or gravitational radiation. The quasi-local nature of the definition of $\Delta$ and the possibility of the presence of radiation suggest the space of solutions to the Einstein–Maxwell equations admitting isolated horizons would be infinite-dimensional, in striking contrast to the space of static black holes which is only three-dimensional. Recent mathematical results by a number of workers [17, 18, 19, 22] show this expectation is indeed correct.

While the conditions used in the definition seem mild, they lead to a surprisingly rich structure. In particular, the intrinsic metric of $\Delta$, the shear, twist and expansion of $\ell^a$ and $n^a$ and several of the Newman-Penrose gravitational and electro-magnetic curvature scalars at $\Delta$ have the same functional dependence on the radius $r_{\Delta}$ and charges $Q_{\Delta}, P_{\Delta}$ as in the Reissner–Nordström solutions. This rich structure enables one to fix naturally the scaling of the null normal $\ell^a$ and leads to an unambiguous definition of surface gravity, $\kappa$. Furthermore, using only the structure available at $\Delta$, one can show that $\kappa$ is constant on $\Delta$; the zeroth law is thus extended from static black holes to isolated horizons.

To formulate the first law, we need a notion of the mass $M_{\Delta}$ of the isolated horizon. Since we allow for the presence of radiation outside $\Delta$, we cannot use the ADM mass $M_{\text{ADM}}$ as $M_{\Delta}$, nor do we have a static Killing field to perform a Komar integral at $\Delta$. Fortunately, we can use the Hamiltonian framework. Although the presence of the internal boundary introduces several subtleties, a satisfactory Hamiltonian framework can be constructed. When the constraints are satisfied, the Hamiltonian turns out to be a sum of two surface terms, one at infinity and one at $\Delta$. As usual, the term at infinity yields the ADM energy and we define $M_{\Delta}$ to be the surface term at $\Delta$. This definition is supported by several independent considerations. In particular, under suitable conditions, $M_{\Delta}$ turns out to be the future limit of the Bondi mass. Having expressions for both $\kappa$ and $M_{\Delta}$ at our disposal, we ask if the first law holds. The answer is in the affirmative for both the ‘equilibrium state’ and the ‘physical process’ versions. This provide a significant generalization of the first law of mechanics of static black holes in the Einstein–Maxwell theory. Furthermore, in the charged case, this analysis brings out some subtleties associated with the ‘physical process’ version. However, since our framework focuses on isolated horizons and small perturbations thereof, it does not shed new light on the second law of black hole mechanics which refers to fully dynamical situations.

These underlying ideas overlap with those introduced in references [9, 7]. In [9], Hayward introduced, and very effectively used, the notion of ‘trapping horizons’. Our isolated horizons are a special case of trapping horizons, the most important restriction being our assumption that the expansion of the horizon is zero. This assumption is essential to capture the notion that the horizon is in equilibrium, which underlies the zeroth and the first law. Furthermore,
our method of defining the surface gravity $\kappa$ and the mass $M_\Delta$ of isolated horizons differ from those used by Hayward for trapping horizons and consequently our treatment of the two laws is also different. (To our knowledge, in the context of trapping horizons, a satisfactory definition of surface gravity is available only for spherically symmetric space-times.) However, the notion of isolated horizon is clearly inadequate for the treatment of dynamical situations which are considered, for example, in the second law and it is these situations that provide a primary motivation in the analysis of trapping horizons.

The relation between the ideas discussed in this paper and those introduced in [7] is closer. Both papers deal with isolated horizons. However, while the focus of reference [7] is on constructing a Hamiltonian framework suitable for quantization and entropy calculations, the focus of the present paper is on the mechanics of isolated horizons. The two overlap in their constructions of Hamiltonian frameworks. However, as explained in Section 4, reference [7] only considers isolated horizons with fixed parameters $r_\Delta, Q_\Delta, P_\Delta$ and therefore ignores several subtleties which are critical to our present treatment of the first law. Reciprocally, in [7], significant effort went into the construction of a Hamiltonian framework in terms of real variables which is necessary for quantization but not for the laws of mechanics. Finally, in Sections 3 and 4 and in Appendices A and B, we took the opportunity to present the necessary background material from a perspective which is different from but complementary to that adopted in [7].

We will conclude by indicating a few avenues to extend the present work.

1. Let us begin with the non-rotating case. Although we did not explicitly require the intrinsic metric of an isolated horizon to be spherically symmetric, our assumptions on properties of the null vector fields $\ell^a$ and $n^a$ at the horizon led us to this conclusion. The discussion of Section 2.2 shows the assumptions are not overly restrictive: the class of space-times satisfying them is infinite-dimensional. Typically, these space-times will admit radiation and will not be spherically symmetric in the bulk. Nonetheless, it is of interest to weaken our assumptions to allow space-times with ‘distorted’ horizons on which the intrinsic geometry will not be spherical. For simplicity, consider the case in which there is no matter in a small neighborhood of $\Delta$. Then, we would expect only to have to weaken the conditions on $n^a$ and allow $\mu$ to be non-spherical. The structure at event horizons of static, distorted black-holes has been recently examined by Fairhurst and Krishnan and their analysis confirms this hypothesis. The extension of the framework presented here to incorporate distortion should be fairly straightforward.

2. Inclusion of rotation would provide an even more interesting extension. Again, conditions

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7This may seem surprising at first since most of the current intuition comes from static black holes and, in the static context, generalizations of Reissner–Nordström solutions naturally lead to distorted horizons. However, this is because static problems are governed by elliptic equations and generic perturbations in the exterior then force the horizon itself to be distorted. Radiative space-times provide generalizations in quite a different direction. Now, the equations are hyperbolic and, as the ‘gluing methods’ of Corvino and Schoen [22] show, the geometry can be spherical even in a neighborhood of the horizon without being spherical everywhere. More generally, as the Robinson–Trautman solutions illustrate [20], the rotational Killing fields may not extend even to a neighborhood of the horizon.
would remain unchanged. Only the conditions on \( n^a \) and the sphericity requirement on
the component \( T_{ab} \ell^a \ell^b \) of stress-energy tensor at \( \Delta \) will have to be weakened. In particular
the Newman-Penrose spin-coefficient \( \pi \) can no longer be zero since it is a potential for the
imaginary part of \( \Psi_2 \) which carries the angular momentum information. Work is already in
progress on this generalization.

3. In the stationary context, using Hamiltonian methods and Noether charges, Wald [35]
has extended the notion of entropy and discussed the first law in a wide variety of gravita-
tional theories, possibly coupled to bosonic fields, in any space-time dimension. It would be
very interesting to extend the present framework for isolated horizons in a similar fashion.
As a first step, one would recast the framework in terms of tetrads \( e^I_\ell \) and the associated
real, Lorentz connections \( A_{aI}^\ell \). The extension of the resulting (Einstein-matter) action and
Hamiltonian framework to higher dimensions should then be straight-forward. The first step
is easy to carry out since the tetrads can be easily obtained from soldering forms and the
Lorentz connection is just the real part of our self-dual connection. Thus, the 4-dimensional
tetrad action is, in effect, just the real part of (6.3) and the corresponding Hamiltonian is
just the real part of (6.9). Hence, it should be rather easy to extend the present results to
higher-dimensional general relativity, possibly coupled to matter. Furthermore, since the ba-
sic variables are tetrads rather than metrics, it should be straightforward to allow fermionic
matter as well. Incorporating general gravitational theories, on the other hand, could be
highly non-trivial.

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A Conventions

In this paper, capital primed and unprimed indices represent \( \text{SL}(2, \mathbb{C}) \) spinors fields. As
usual, spinors with only unprimed indices are also interpreted as \( \text{SU}(2) \) spinors in the phase
space framework. The spinor conventions are largely those of [23], but with minor modifi-
cations to replace the \(+ − −−\) signature of [23] with the \(− + ++\) signature used here. We
describe these modifications here.

A.1 Metric and Null Tetrad

The metric is given in terms of the soldering form by
\[
g_{ab} = \sigma_a^{AA'} \sigma_b^{AA'}. \tag{A.1}
\]
Because of our choice of signature, however, the soldering form must be taken to be anti-Hermitian:  $\sigma_a^{AA'} = -\sigma_a^{AA'}$.

A spin dyad $(\iota^A, o^A)$, satisfying $\iota^A o_A = 1$, defines a null tetrad as follows:

$$\ell^a = i\sigma_A^{AA'} o^A o^{A'} \quad m^a = i\sigma_a^{AA'} o^A \iota^{A'}$$

$$n^a = i\sigma_a^{AA'} \iota^A \bar{o}^{A'} \quad \bar{m}^a = i\sigma_a^{AA'} \iota^A \bar{o}^{A'}.$$  \hspace{1cm} (A.2)

This tetrad obeys the usual inner product conventions in the $-+++$ signature: the only non-vanishing inner products are $\ell^a n_a = -1$ and $m^a \bar{m}_a = 1$. (Note that the definitions of $m$ and $\bar{m}$ differ from those used in [7]. This change was necessary to make the values of our Newman-Penrose curvature components the same as those found in the literature even though our signature is $-+++$.)

### A.2 Volume Forms and Orientations

The volume form on space-time is defined by its spinor expression, which is the same as that used in [23]:

$$4 \epsilon_{abcd} = \sigma_a^{AA'} \sigma_b^{BB'} \sigma_c^{CC'} \sigma_d^{DD'} [-i\epsilon_{AB} \epsilon_{CD} \epsilon_{A'C'} \epsilon_{B'D'} + c.c.] .$$  \hspace{1cm} (A.3)

This volume form can be expressed in terms of the null tetrad as

$$4 \epsilon_{abcd} = 24i \ell[a] n_b m_c \bar{m}_d .$$  \hspace{1cm} (A.4)

The conventions for inducing volume forms on sub-manifolds of space-time are designed to be compatible with those used in [12] and with the usual orientation conventions used in Stokes’ theorem on Riemannian manifolds. Specifically, this means that a volume form is induced on a space-like sub-manifold of space-time by contracting its future-directed, unit normal with the last index of $4 \epsilon$. Then, within a space-like hypersurface, a volume form is induced on a two-dimensional sub-manifold by contracting its outward-bound, unit normal with the first index of the volume form on the hypersurface. All other orientation conventions can be determined from these two. In particular,

$$3 \epsilon_{abc} = 4 \epsilon_{abcd} \hat{r}^d$$ \hspace{1cm} (A.5)

$$2 \epsilon_{bc} = \hat{r}_{in}^a 3 \epsilon_{abc} = 2i [n_{[b} \bar{m}_{c]}]$$ \hspace{1cm} (A.6)

$$\Delta \epsilon_{abc} = -3 2 \epsilon_{[ab] c} = -6i [n_{[a} m_{b} \bar{m}_{c}].$$ \hspace{1cm} (A.7)

Here, $3 \epsilon$ denotes the induced volume form on a space-like hypersurface, $2 \epsilon$ denotes the volume form on one of the $S_\Delta$, and $\Delta \epsilon$ denotes the preferred alternating tensor on the null surface $\Delta$. Meanwhile, $\hat{r}$ denotes the future-directed future normal to the space-like hypersurface and $\hat{r}_{in}^a$ denotes the unit radial vector directed inward at the horizon. Note that the inward normal is appropriate because $S_\Delta$ is the inner boundary of the space-like surface $M$. 

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A.3 Self-Dual Basis

The soldering form defines a basis of self-dual 2-forms on space-time via

$$\sum_{ab}^{AB} = 2\sigma_a^{AA'}\sigma_b^{B'B}A' = 2\sigma_a^{A'}\sigma_b^{B}A.\quad (A.8)$$

Using the spin dyad, these self-dual 2-forms can be expressed as

$$\sum_{ab}^{AB} = 4\ell_{[a}m_{b]}A^A_{AB} + 4(m_{[a}\bar{m}_{b]} - \ell_{[a}m_{b]})\ell^{(A}_{B'}B') - 4\bar{m}_{[a}m_{b]}A^B_{AB}.\quad (A.9)$$

One can check that these are, in fact, self-dual in that $^*\Sigma^{AB} = i\Sigma^{AB}$.

A.4 Newman-Penrose Components

We define the Riemann curvature tensor to be

$$R_{abcd}^d k_d = 2\nabla_{[a}k_{b]}k_c.\quad (A.10)$$

The space-time curvature spinors are defined by decomposing the Riemann tensor as

$$R_{abcd} = \sigma_a^{AA'}\sigma_b^{B'B'}\sigma_c^{CC'}\sigma_d^{DD'}$$

$$\times \left\{ \epsilon_{A'B'C'D}^a\Phi_{A'B'C'D} + \epsilon_{A'B'C'D}^a\epsilon_{C'D}^c [\Psi_{ABCD} - \frac{1}{12}R\epsilon_{A(C}\epsilon_{D)B}] + c.c. \right\}, \quad (A.11)$$

leading to the expression for the Ricci tensor:

$$R_{ab} = \sigma_a^{AA'}\sigma_b^{B'B'}\left\{ -2\Phi_{ABA'B'} + \frac{1}{4}R\epsilon_{AB}\epsilon_{A'B'} \right\}.\quad (A.12)$$

Since the Ricci tensor is real, $\Phi_{ABA'B'}$ is Hermitian.

In terms of the curvature spinors, we define the Newman-Penrose components of the Weyl tensor $C_{abcd}$ by

$$\Psi_0 = \Psi_{ABCD}A^A_{AB}C^C_{CD} = C_{abcd}\ell^a m^b\bar{m}^c n^d$$

$$\Psi_1 = \Psi_{ABCD}A^A_{AB}\ell^c = C_{abcd}\ell^a m^b\epsilon^c n^d$$

$$\Psi_2 = \Psi_{ABCD}A^A_{AB}\ell^c = C_{abcd}\ell^a m^b\bar{m}^c n^d$$

$$\Psi_3 = \Psi_{ABCD}A^A_{AB}\ell^c = C_{abcd}\ell^a n^b\bar{m}^c n^d$$

$$\Psi_4 = \Psi_{ABCD}A^A_{AB}\ell^c = C_{abcd}\bar{m}^a n^b\bar{m}^c n^d.\quad (A.13)$$

Note that these definitions are the same as those found in the literature [23, 37] and, despite the difference in signature, the functions $\Psi_n$ take their usual values in specific space-times. Similarly, the expressions for the Newman-Penrose components of the Ricci tensor read

$$\Phi_{00} = \Phi_{ABA'B'}A^A_{AB}B^B_{BB'} = \frac{1}{2}R_{ab}\ell^a\ell^b$$

$$\Phi_{22} = \Phi_{ABA'B'}A^A_{AB}B^B_{BB'} = \frac{1}{2}R_{ab}n^a n^b$$

$$\Phi_{01} = \Phi_{ABA'B'}A^A_{AB}\ell^b = \frac{1}{2}R_{ab}\ell^a m^b$$

$$\Phi_{21} = \Phi_{ABA'B'}A^A_{AB}\ell^b = \frac{1}{2}R_{ab}\bar{m}^a n^b$$

$$\Phi_{02} = \Phi_{ABA'B'}A^A_{AB}n^b\bar{m}^b = \frac{1}{2}R_{ab}m^a \bar{m}^b$$

$$\Phi_{20} = \Phi_{ABA'B'}A^A_{AB}n^b\bar{m}^b = \frac{1}{2}R_{ab}m^a \bar{m}^b$$

$$\Phi_{10} = \Phi_{ABA'B'}A^A_{AB}n^b\ell^b = \frac{1}{2}R_{ab}n^a m^b$$

$$\Phi_{12} = \Phi_{ABA'B'}A^A_{AB}n^b\ell^b = \frac{1}{2}R_{ab}n^a m^b$$

$$\Phi_{11} = \Phi_{ABA'B'}A^A_{AB}n^b\ell^b = \frac{1}{4}R_{ab}(\ell^a n^b + m^a \bar{m}^b).$$
As before, these are the standard spinorial definitions for the $\Phi_{ij}$. However, their expressions in terms of the Ricci tensor differ from those of [23] by a minus sign. This difference occurs because our Ricci tensor is the negative of the one used in [23].

A similar decomposition can be performed on the electro-magnetic field. The Maxwell spinor is defined by expressing the field strength as

$$ F_{ab} = \sigma^A_a \sigma^B_b (\phi_{AB} \epsilon_{A'B'} + \epsilon_{AB} \bar{\phi}_{A'B'} ) $$ \hspace{1cm} (A.15)

Then, the Newman-Penrose components of the Maxwell field are defined by

$$ \phi_0 = \phi_{AB} o^{A} o^{B} = -\ell^a m^b \Phi_{ab} $$

$$ \phi_1 = \phi_{AB} \iota^{A} o^{B} = -\frac{1}{2}(\ell^a n^b - m^a \bar{m}^b) \Phi_{ab} = \frac{1}{2} m^a \bar{m}^b (F - i \bar{F})_{ab} $$ \hspace{1cm} (A.16)

As with the gravitational field, the values of these functions will be the same as those found in the literature.

Finally, the Newman-Penrose spin-coefficients used in this paper are given by:

$$ \mu = m^a \bar{m}^b \nabla^a n^b, \quad \lambda = \bar{m}^a m^b \nabla^a n^b, \quad \pi = \ell^a m^b \nabla^a \ell^b, $$

$$ \sigma = -m^a m^b \nabla^a \ell^b, \quad \rho = -\bar{m}^a \bar{m}^b \nabla^a \ell^b, \quad \epsilon + \bar{\epsilon} = -\ell^a n^b \nabla^a \ell^b. $$ \hspace{1cm} (A.17)

Note that, as is common in the black-hole literature, we denote the surface gravity by $\kappa$ (so that our $\kappa$ equals $(\epsilon + \bar{\epsilon})$ in the Newman-Penrose notation.) We never need to refer to the Newman-Penrose spin coefficient $\kappa$.

**B Newman-Penrose Components and Self-Dual Curvature**

The purpose of this appendix is to establish the relation between the self-dual SL$(2, \mathbb{C})$ curvature used in [12] and the Newman-Penrose curvature components described in [23].

In any putative space-time where only the Gauss law (2.2) is solved, the self-dual curvature $F^{AB}$ is equal to the self-dual portion of the Riemann curvature defined by

$$ +R^{AB}_{\ cd} = \frac{1}{2} \sigma^{CC'}_a \sigma^{DD'}_b R_{CC'DD'}^{AB} \bar{\phi}_{A'B'}^{A'B'} $$ \hspace{1cm} (B.1)

If we now substitute for the Riemann spinor in this expression using (A.11), one can rearrange the terms to yield

$$ +R^{AB}_{\ cd} = -\frac{1}{2} \sum^{A'B'}_{ab} \Phi_{A'B'}^{AB} - \frac{1}{2} \sum^{CD}_{ab} \Phi_{CD}^{AB} - \frac{R}{24} \sum^{AB}_{ab}. $$ \hspace{1cm} (B.2)
It is now a long, but straightforward, process to break this formula into spinor components using (A.9). Then, using the definitions (A.13) and (A.14) of the Newman-Penrose components, one can express the result as

$$^+R_{AB} = \left[ (\Psi_3 + \Phi_{21}) \ell \wedge n - \Psi_4 \ell \wedge m - \Phi_{22} \ell \wedge \bar{m} + \Phi_{20} n \wedge m + \left( \Psi_2 + \frac{R}{12} \right) n \wedge m - (\Psi_3 - \Phi_{21}) m \wedge \bar{m} \right] o_A o_B$$

$$- \left[ \left( \Psi_2 + \Phi_{11} - \frac{R}{24} \right) \ell \wedge n - \Psi_3 \ell \wedge m - \Phi_{12} \ell \wedge \bar{m} + \Phi_{10} n \wedge m + \Psi_1 n \wedge \bar{m} - \left( \Psi_2 - \Phi_{11} - \frac{R}{24} \right) m \wedge \bar{m} \right] 2\iota_{(A} o_{B)}$$

$$+ \left[ (\Psi_1 + \Phi_{01}) \ell \wedge n - \left( \Psi_2 + \frac{R}{12} \right) \ell \wedge m - \Phi_{02} \ell \wedge \bar{m} + \Phi_{00} n \wedge m + \Psi_0 n \wedge \bar{m} - (\Psi_1 - \Phi_{01}) m \wedge \bar{m} \right] \iota_{A} \iota_{B}.$$  \hspace{1cm} (B.3)

This expresses the self-dual curvature in terms of the Newman-Penrose components in a spin dyad satisfying $\iota^A o_A = 1$. The null tetrad here is defined, of course, by the same dyad.

C. Symplectic Structure at Null Infinity

In section 6 we used the Legendre transform to introduce a symplectic structure (6.10) on the canonical phase space. On the other hand, there is also a natural symplectic structure on the space of radiative modes of the Einstein–Maxwell system, defined intrinsically at (future) null infinity $\mathcal{I}^+$. In this appendix, using field equations, we will show the two symplectic structures are equal in an appropriate sense, provided the fields under consideration have suitable asymptotic behavior.

Throughout this discussion, we will restrict ourselves to the region $\mathcal{M}$ of figure 1(a) which has $\mathcal{I}^+$ as its future boundary and which admits partial Cauchy surfaces $M$ which extend from the isolated horizon $\Delta$ to spatial infinity $i^\circ$. As in Section 7.3, we will set the cosmological constant $\Lambda$ to zero.

C.1 Phase Space of Radiative Modes at $\mathcal{I}^+$

Fix an asymptotically flat space-time $(\mathcal{M}, g_{ab})$ and consider its Penrose completion $(\hat{\mathcal{M}}, \hat{g}_{ab})$. As usual, $\hat{g}_{ab} = \Omega^2 g_{ab}$ is the conformally rescaled metric and $\mathcal{I}^+$ is the future null boundary of $\mathcal{M}$ where the conformal factor $\Omega$ vanishes. All fields appearing with a ‘hat’ will refer to the geometry defined by the conformally rescaled metric $\hat{g}_{ab}$ which is smooth at $\mathcal{I}^+$.

Let us begin by recalling the ‘universal structure’ at null infinity of asymptotically flat space-times. First, $\mathcal{I}^+$ is topologically $S^2 \times \mathbb{R}$. Second, the conformally rescaled metric naturally defines an intrinsic, degenerate metric $\hat{g}_{ab} = \hat{\nabla}_a \hat{\nabla}_b$ and a null normal field $\hat{n}_a = \hat{\nabla}_a \Omega$. 

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on \( \mathcal{I}^+ \). We shall assume the conformal factor \( \Omega \) is so chosen as to make \( \mathcal{I}^+ \) divergence free in the sense that \( \nabla_a \hat{n}^a = 0 \) on \( \mathcal{I}^+ \). By construction, \( \hat{n}^a \) defines the unique degenerate direction of the intrinsic metric \( q_{ab} \): \( \hat{n}^a \hat{q}_{ab} = 0 \). Therefore, the ‘inverse metric’ is unique only up to the addition of a term of the type \( \hat{n}^a \hat{v}^b \) where \( \hat{v}^b \) is an arbitrary vector field on \( \mathcal{I}^+ \). (Irrespective of the choice of \( v^a \), we have \( \hat{q}_{ab} \hat{q}^{bc} \hat{q}_{cd} = \hat{q}_{ad} \)). Finally, the volume 3-form \( \mathcal{I}^+ \hat{\epsilon} \) on \( \mathcal{I}^+ \) can be defined as

\[
\mathcal{I}^+ \hat{\epsilon} = 4 \hat{\epsilon}^{abcd} \hat{n}^d \tag{C.1}
\]

where \( \hat{\epsilon}^{abcd} \) is the volume 4-form defined by the rescaled metric \( \hat{g}^{ab} \). These structures are universal in the sense that they are common to all asymptotically flat space-times; they do not carry any information about, e.g., the radiation field which can vary from one space-time to another.

Note however that there remains a conformal freedom in the rescaled metric \( \hat{g}^{ab} \). If \( \Omega \) is an allowable conformal factor which makes \( \mathcal{I}^+ \) divergence-free, so is by \( \Omega' = \omega \Omega \), where \( \omega \) is nowhere vanishing on \( \mathcal{I}^+ \) and \( \mathcal{L}_{\hat{n}} \omega = 0 \) at \( \mathcal{I}^+ \). Under this transformation, the conformal metric is rescaled as \( \hat{g}^{ab} \rightarrow \omega^2 \hat{g}^{ab} \). As a consequence, the pairs, \( (\hat{q}_{ab}, \hat{n}^a) \) and \( (\omega^2 \hat{q}_{ab}, \omega^{-1} \hat{n}^a) \) are to be regarded as (conformally) equivalent at \( \mathcal{I}^+ \).

We can now turn to the dynamical structures and introduce the radiative modes. Note first that the derivative operator \( \nabla \) defined by the metric \( \hat{g}^{ab} \) on \( \mathcal{M} \) naturally induces a derivative operator \( \hat{D} \) defined intrinsically on \( \mathcal{I}^+ \) via the pull-back

\[
\hat{D}_a \hat{K}_b = \nabla_a \hat{K}_b, \tag{C.2}
\]

where \( \hat{K}_a \) is an arbitrary co-vector field defined intrinsically at \( \mathcal{I}^+ \) and \( \hat{K}_b \) is any extension of \( \hat{K}_a \) to \( \mathcal{M} \). Since \( \nabla \) is metric compatible, it follows that \( \hat{D}_a \hat{q}_{bc} = 0 \) and \( \hat{D}_a \hat{n}^b = 0 \). The radiative modes of the gravitational field in general relativity are fully encoded in connections \( \hat{D} \) on \( \mathcal{I}^+ \) satisfying the above conditions. Recall, however, that there is a residual conformal freedom at \( \mathcal{I}^+ \). As a consequence, one is led to introduce an equivalence relation between connections. The phase space of radiative modes consists of these equivalence classes. It thus has the structure of an affine space. The difference between any two connections in different equivalence classes can be encoded in a tensor field \( \gamma_{ab} \) which satisfies

\[
\gamma_{ab} \hat{q}^{ab} = 0 \quad \gamma_{ab} \hat{n}^b = 0 \quad \gamma_{ab} = (\gamma_{ab}). \tag{C.3}
\]

Therefore, by fixing a point in the phase space as the ‘origin’, we can label any other point by the corresponding tensor field \( \gamma_{ab} \). It is easy to see that \( \gamma_{ab} \) has two independent components which represent the two physical degrees of freedom of gravitational radiation.

The radiative degrees of freedom of the electro-magnetic field can also be described by fields intrinsic to \( \mathcal{I}^+ \). It turns out that \( F_{ab} \) is completely characterized by the unique connection \( A_a \) at \( \mathcal{I}^+ \) satisfying

\[
A_a \hat{n}^a = 0 \quad \text{and} \quad \lim_{u \to -\infty} A_a = 0 \tag{C.4}
\]

where \( u \) is the affine parameter along \( \hat{n}^a \). The connection \( A_a \) satisfying the above conditions has two independent components. These represent the two radiative degrees of freedom of the Maxwell field.
Thus, the phase space of radiative modes at $\mathcal{I}^+$ consists of pairs $(\gamma_{ab}, \partial_t A)$ satisfying the conditions (C.3) and (C.4) respectively. The symplectic structure on this phase space is

$$\Omega^{\text{rad}}(\delta_1^{\text{rad}}, \delta_2^{\text{rad}}) = \frac{1}{32\pi G} \int_{\mathcal{I}^+} \tilde{q}^{ab} \delta_2 \gamma^{cd} \mathcal{L}_n(\delta_1 \gamma^{cd}) - \delta_2 \gamma^{ab} \mathcal{L}_n(\delta_1 \gamma^{ab}) \right] \mathcal{I} \hat{\epsilon}.$$  

For further details, see [29, 30].

### C.2 Equality of Symplectic Structures

Let us now return to the canonical phase space of Section 6. Fix a point on the constraint hypersurface and consider tangent vectors which satisfy the linearized constraints. Evolve these fields using the appropriate field equations. Then, assuming the resulting 4-geometry and the linearized fields thereon satisfy appropriate falloff conditions, they would provide a point in the radiative phase space at $\mathcal{I}^+$ and tangent vectors at that point. Using this correspondence, we will now show the canonical symplectic structure (6.10) associated with a partial Cauchy surface $M$ (of figure 1(a)) equals the radiative symplectic structure (C.5) at $\mathcal{I}^+$. (The calculation is modeled after [38] which discussed the relation between the two symplectic structures in the absence of internal boundaries within the framework of geometrodynamics.) For simplicity of presentation, we will just make assumptions on the asymptotic behavior of fields as they are needed in the intermediate stages of the calculation and collect our assumptions at the end.

The canonical symplectic structure (6.10) can be obtained by integrating a symplectic current $\omega$ on the partial Cauchy surface $M$. This 3-form $\omega$ is given by

$$\omega = \frac{-i}{8\pi G} \text{Tr}[\delta_1 A \wedge \delta_2 \Sigma - \delta_2 A \wedge \delta_1 \Sigma] - \frac{i}{8\pi G} d[\delta_1 \psi \delta_2 (\hat{\epsilon}) - \delta_2 \psi \delta_1 (\hat{\epsilon})]$$

Note that the exact differential in the expression of $\omega$ vanishes at infinity due to the fall-off conditions on $A$. Hence, on integrating over $M$, it provides just the surface term at the horizon in the expression (6.10) of the symplectic structure. The expression of the 3-form $\omega$ involves 4-dimensional fields. However, when we integrate it over $M$ to obtain the symplectic structure, only the pull-backs to $M$ of these fields contribute.

The main idea behind our calculation can be summarized as follows. When equations of motion are satisfied, the 3-form $\omega$ is curl-free. Therefore, the integral of $d\omega$ trivially vanishes over $M$, provided all fields remain regular (in a conformal completion in which $i^+$ is a single point). Hence, provided all fields remain regular, the integral of $\omega$ on $M$ equals the sum of the integrals over $\Delta$ and $\mathcal{I}^+$. Let us first consider the integral over $\Delta$. Using the isolated horizon boundary conditions one can show the sum of the first and last terms can be expressed as an exact
differential which is precisely the negative of the one appearing in the second term of \( \omega \). Thus, the integral of the symplectic current over \( \Delta \) vanishes. Hence, the integral of the symplectic current over \( M \) equals that over \( I^+ \). The former is just the canonical symplectic structure \( (6.10) \). The idea now is to show that the latter is the radiative symplectic structure at \( I^+ \).

Let us therefore evaluate \( \int_{I^+} \omega \). It is immediate from the falloff condition \( A \sim O(\frac{1}{r^2}) \) that the exact part (i.e., the second term in the expression) of \( \omega \) does not contribute at \( I^+ \). Next, using the conformal invariance of Maxwell’s equations, it is fairly straightforward to evaluate the electro-magnetic part (i.e. the third term) of this integral. Since \( \tilde{A} = A \) satisfies Maxwell’s equations on \( (\tilde{M}, \tilde{g}_{ab}) \), the Maxwell potentials \( \tilde{A} \) are well-behaved at \( I^+ \). Therefore, we impose the gauge condition \( (C.4) \) at \( I^+ \) and evaluate the electro-magnetic contribution to the integral of \( \omega \) over \( I^+ \). It equals precisely the electro-magnetic part of the symplectic structure \( (C.5) \) at \( I^+ \).

Thus, the non-trivial part of the calculation lies in integrating the first term in the symplectic current over \( I^+ \). We will now sketch the main steps.

The gravitational symplectic structure \( (6.10) \) is expressed in terms of the fields \( \Sigma \) and \( A \). To compare it with the symplectic structure at \( I^+ \), we first need to re-express it in terms of the metric \( \hat{g}_{ab} \) and its variations. Since we are assuming the equations of motion, and in particular Gauss’ law \( D\Sigma = 0 \), the connection \( A \) can be expressed in terms of the soldering form \( \sigma \) as

\[
A_a^{AB} = -\frac{1}{2} \sigma^{bAA'} \nabla_a \sigma_b^{B A'}.
\]

An arbitrary variation to the soldering form \( \sigma \) can be written as

\[
\delta \sigma_a^{AA'} = \frac{1}{2} \left( \delta g_{ab} \sigma^{bAA'} + \mu_{ab} \sigma^{bAA'} \right)
\]

where \( \delta g_{ab} \) is symmetric and \( \mu_{ab} \) is antisymmetric. It is easy to check that the above variation in \( \sigma \) induces a variation \( \delta g_{ab} \) in the metric. Also, by performing an internal gauge transformation on \( \delta \sigma \) without changing the background field \( \sigma \), \( \mu_{ab} \) can be set equal to zero. This gauge transformation leaves the variation of the metric \( \delta g_{ab} \) unchanged. Hence, from now on, without loss of generality, we assume we are in a gauge in which \( \mu_{ab} \) has been set to zero. This choice of internal gauge will simplify our calculations considerably but is not essential since the symplectic structure is gauge invariant. We can now express \( \delta \sigma \) in terms of the conformally rescaled soldering form \( \hat{\sigma}_a = \Omega \sigma_a \) as

\[
\delta \hat{\sigma}_a^{AA'} = \frac{1}{2\Omega} \delta g_{ab} \sigma^{bAA'}
\]

Now, one would naively expect \( \Omega^2 \delta g_{ab} \) to be finite at \( I^+ \). However, in the context of vacuum general relativity, Geroch and Xanthopoulos \( [39] \) have shown that perturbations with \( C^\infty \) initial data of compact support have a better behavior: in a suitable gauge, \( \Omega^2 \delta g_{ab} \) in fact vanishes at \( I^+ \). Furthermore,

\[
h_{ab} := \Omega \delta g_{ab}
\]
is well defined and \( C^\infty \) at \( \mathcal{I}^+ \) and satisfies the following conditions:

\[
\Omega^{-1} \hat{n}^a h_{ab} \quad \text{and} \quad \Omega^{-2} \hat{n}^a \hat{n}^b h_{ab} \quad \text{are} \quad C^\infty, \quad \text{and} \quad \hat{g}^{ab} \mathcal{L}_n h_{ab} = 0 \quad (C.11)
\]
at \( \mathcal{I}^+ \). Finally, the trace-free part of the field \( h_{ab} \) at \( \mathcal{I}^+ \) is precisely the field \( \delta \gamma_{ab} \) representing the change in the equivalence class of connections at \( \mathcal{I}^+ \), i.e. the tangent vector to the phase space of radiative modes induced by \( \delta g_{ab} \):

\[
\delta \gamma_{ab} = h_{ab} - \frac{1}{2} \hat{q}^{mn} h_{mn} \hat{q}_{ab}. \quad (C.12)
\]

We will assume the tangent vectors \( \delta \) under consideration have this asymptotic behavior.

With this structure at hand, a straightforward but lengthy calculation enables one to express the variations of \( A \) and \( \Sigma \) in terms of fields which are smooth at \( \mathcal{I}^+ \):

\[
\delta A_{c AB} = \frac{1}{4} \Sigma^{ef AB} [\Omega (\hat{\nabla}^e h_{cf}) + \hat{g}_{cf} \hat{h}^k h_{ek}]
\]

\[
\delta \Sigma_{ab AB} = \Omega^{-1} \Sigma_{[a} d^{AB} h_{b]d} \quad (C.13)
\]

Using the identity

\[
\Sigma_{ab} A^B \Sigma_{cdAB} = 2(g_{ac} g_{bd} - g_{ad} g_{bc}) - 2i \epsilon_{abcd} \quad (C.14)
\]

and simple consequences of the Geroch-Xanthopoulos asymptotic behavior (C.11), one can now express the symplectic current in terms of the fields \( h_{1ab} \) and \( h_{2ab} \). Assuming that at least one of the two perturbations, \( h_{1ab} \) and \( h_{2ab} \), vanishes at \( i^- \) and \( i^+ \), we can therefore write the gravitational part of the canonical symplectic structure as

\[
\Omega(\delta_1, \delta_2) = \frac{1}{32 \pi G} \int_{\mathcal{I}^+} (h_{1ab} \hat{n}^c \hat{\nabla}^e h_{2cd} - h_{2ab} \hat{n}^c \hat{\nabla}^e h_{1cd}) \hat{g}^{ac} \hat{g}^{bd} \hat{\epsilon}, \quad (C.15)
\]

where the volume form \( \hat{\epsilon} \) on \( \mathcal{I}^+ \) is given by (C.4).

To bring this expression to the same form as appears in (C.5) it is necessary to replace the fields \( h_{ab} \) with their trace-free parts \( \delta \gamma_{ab} \) in the first integral. This will not introduce any additional terms because of the properties (C.11) of \( h_{ab} \). Also, since \( \mathcal{I}^+ \) is divergence-free, we can replace \( n^c \hat{\nabla} c h_{ab} \) with \( \mathcal{L}_n h_{ab} \).

In summary, we have shown that, when the equations of motion hold, both the gravitational and electro-magnetic parts of the symplectic structure can be rewritten in terms of fields living at \( \mathcal{I}^+ \). Combining these results, it follows that

\[
\Omega(\delta_1, \delta_2) = \Omega^{\text{rad}}(\delta_{1}^{\text{rad}}, \delta_{2}^{\text{rad}}), \quad (C.16)
\]

provided the background and the tangent vectors have certain asymptotic properties.

To conclude, let us collect the assumptions on the behavior of various fields that were necessary to arrive at (C.16). The background solution is assumed to be asymptotically flat at spatial and future null infinity and asymptotically Schwarzschild at future time-like infinity. In a conformal frame in which \( \mathcal{I}^+ \) is divergence-free, the linearized fields \( h_{ab} \) are assumed to
satisfy the Geroch-Xanthopoulos conditions \((C.10), (C.11)\) and the Maxwell potential \(\delta A\) is assumed to satisfy \((C.4)\) at \(\mathcal{I}^+\). Next, at least one of \(h_{1ab}\) and \(h_{2ab}\) has to vanish at \(i^+\) and at least one of them has to vanish at \(i^o\). This last assumption can easily be met in the actual application of \((C.16)\) in the main text (Section 7.3). There, \(\delta_2\) is the Hamiltonian vector field associated with a BMS time translation \(\dot{n}^a\); \(\delta_2 = (\mathcal{L}_{\dot{n}}\gamma_{ab}, \mathcal{L}_{\dot{n}}A)\). Now, up to numerical factors, the total energy radiated across \(\mathcal{I}^+\) in the (background) space-time is given by the integral of squares of these two fields. Hence, it is physically reasonable to restrict oneself to space-times in which the two fields go to zero as one approaches \(i^+\) and \(i^o\) along \(\mathcal{I}^+\). In this case, \(h_{2ab}\) will automatically satisfy the last requirement. Finally, for the main result (7.3) of Section 7.3 to hold, an additional condition must be satisfied: the linearized fields \(h_{ab}\) and \(\delta A\) satisfying \((C.10), (C.11)\) and \((C.4)\) should span the tangent space at each point of the sector of phase space considered.

While these assumptions seem plausible, we do not know of general results which will ensure that a ‘sufficient number’ of such background solutions exist or that they will admit a ‘sufficient number’ of linearized fields satisfying our conditions. Indeed, at this stage, one does not even have a conclusive proof of existence of a ‘sufficient number’ of radiating solutions which have smooth and complete \(\mathcal{I}^+\).

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