The $p$-Modular Descent Algebras

M.D. Atkinson
S.J. van Willigenburg
School of Mathematical and Computational Sciences
North Haugh, St Andrews, Fife KY16 9SS, UK

G. Pfeiffer
Department of Mathematics
University College, Galway, Eire

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Abstract

Solomon’s descent algebras are studied over fields of prime characteristic. Their radical and irreducible modules are determined. It is shown how their representation theory can be related to the representation theory in fields of characteristic zero.

1 Introduction

Descent algebras are non-commutative, non-semi-simple algebras associated with Coxeter groups. They were first discovered by Solomon in the 1970’s and for the last 10 years have been studied intensively. Previous work has concentrated on the case that the underlying field has characteristic zero. However, as we shall soon see, characteristic $p$ analogues exist and their structure is very sensitive to the value of $p$. This paper determines the radical of a descent algebra in characteristic $p$, and the irreducible modules. It also explains how the representation theory is connected to the representation theory in characteristic zero.

Let $W$ be a Coxeter group with generating set $S$ of fundamental reflections. Thus every element $w \in W$ can be written as a product of elements in $S$; we let $\lambda(w)$ denote the length of a shortest expression for $w$. If $L$ is any subset of $S$ let $W_L$ be the subgroup generated by $L$. $W_L$ is called a standard parabolic subgroup of $W$ and any subgroup conjugate to a standard parabolic subgroup is said to be parabolic. Let $X_L$ be the (unique) set of minimal length representatives of the left cosets of $W_L$ in $W$. Notice that $X_L^{-1} = \{g^{-1} | g \in X_L \}$ is then a
set of representatives (also of minimal length) for the right cosets of $W_L$ and that $X^{-1}_K \cap X_L$ is a set representatives for the double cosets corresponding to $W_L, W_K$.

Solomon proved the following remarkable theorem:

**Theorem 1** [19] For every subset $K$ of $S$ let

$$x_K = \sum_{w \in X_K} w.$$  

Then

$$x_J x_K = \sum a_{J KL} x_L$$

where $a_{J KL}$ is the number of elements $g \in X^{-1}_J \cap X_K$ such that $g^{-1} W_J g \cap W_K = W_L$ with $L = g^{-1} J g \cap K$.

The set of all $x_K$ is therefore a basis for an algebra $\Sigma_W$ over the field of rationals with integer structure constants $a_{J KL}$. This algebra is now known as the **descent algebra** of $W$ and much is known about its structure [1, 2, 3, 4, 5, 6, 8, 11, 14].

Solomon himself began the study of this algebra by determining its radical, $\text{rad}(\Sigma_W)$, and some properties of $\Sigma_W/\text{rad}(\Sigma_W)$. To describe his results let $\chi_K$ be the permutation character of $W$ acting on the right cosets of $W_K$ and let $G_W$ be the $Z$-module generated by all $\chi_K$. Note that each generalised character in $G_W$ has integer values on the elements of $W$.

**Theorem 2** [19]

1. $\text{rad}(\Sigma_W)$ is spanned by all differences $x_J - x_K$ where $J$ and $K$ are conjugate subsets of $S$.

2. The linear map $\theta$ defined by the images $\theta(x_K) = \chi_K$ is an algebra homomorphism, and $\ker \theta = \text{rad}(\Sigma_W)$.

Since the structure constants $a_{J KL}$ are integers the $Z$-module $Z_W$ spanned by all $x_K$ is a subring (an order) of $\Sigma_W$. This allows us to study the $p$-modular version of the descent algebra for any prime $p$. For any prime $p$, $pZ_W$ is an ideal of $Z_W$. We define $\Sigma(W, p) = Z_W/pZ_W$, the $p$-modular descent algebra of $W$. Obviously, $\Sigma(W, p)$ is an algebra over $F_p$, the field of order $p$.

Let $\rho_1$ be the natural projection $Z_W \to \Sigma(W, p)$ and let $\overline{\pi}_J = \rho_1(x_J)$. Then

$$\overline{\pi}_J \overline{\pi}_K = \sum \overline{a}_{J KL} \overline{\pi}_L$$

where $\overline{a}_{J KL}$ is the image of $a_{J KL}$ in $F_p$. Furthermore let $\rho_2$ be the map defined on $G_W$ which reduces character values modulo $p$, and let $G(W, p)$ be the image of $\rho_2$. 

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The map $\phi : \Sigma(W, p) \to G(W, p)$ defined by

$$\phi(\rho_1(x)) = \rho_2(\theta(x)) \text{ for all } x \in Z_W$$

is clearly well-defined and is an algebra homomorphism. In section 3 we shall give an analogue of Solomon’s theorem to describe the radical of $\Sigma(W, p)$ using the homomorphism $\phi$. In section 4 we build on this result by defining the irreducible modules of $\Sigma(W, p)$. Then we relate the representation theory of $\Sigma(W, p)$ to that of $\Sigma_W$ and give explicit details for each of the Coxeter types.

We begin, in section 2, by introducing a tool that we use throughout the paper: the parabolic table of marks. Many of our results were first suggested by computation with GAP [18].

## 2 The parabolic table of marks

We first recall the definition and basic properties of the Burnside ring (see [10, chapter 11] for details) and the table of marks of a finite group. In the case of a finite Coxeter group the parabolic table of marks is defined as a certain submatrix of the table of marks.

Let $G$ be a finite group and let $G_1(= 1), G_2, \ldots, G_r(= G)$ be representatives of the conjugacy classes of subgroups of $G$. Then each $G$-set decomposes as a disjoint union of transitive $G$-sets and each transitive $G$-set is isomorphic to $G/G_i$ for some $i$. Denote by $[X]$ the isomorphism type of the $G$-set $X$. The Burnside ring $\Omega(G)$ is the ring of formal integer linear combinations of isomorphism types of $G$-sets with addition defined by disjoint unions of $G$-sets

$$[X] + [Y] = [X \dot{\cup} Y]$$

and multiplication defined by the cartesian product

$$[X] \cdot [Y] = [X \times Y].$$

The transitive $G$-sets $G/G_i$ form a basis of $\Omega(G)$ so

$$\Omega(G) = \left\{ \sum_{i=1}^r a_i[G/G_i] \mid a_i \in \mathbb{Z} \right\}$$

the free abelian group generated by the isomorphism types of transitive $G$-sets.

The table of marks of $G$ is the $r \times r$-matrix

$$M(G) = (\lvert \text{Fix}_{G/G_i}(G_j) \rvert)_{i,j=1,\ldots,r}$$

which records for subgroups $G_i$, $G_j$ of $G$ the number of fixed points of $G_j$ in the action of $G$ on the cosets of $G_i$ (the mark of $G_j$ on $G/G_i$), and where both
$G_i$ and $G_j$ run through the (system of) representatives of conjugacy classes of subgroups of $G$.

We have

$$\left| \text{Fix}_{G/G_i}(G_j) \right| = |N_{G_i}(G_j) : G_i| \cdot |\{G^x_i \mid x \in G, G_j \leq G^x_i\}|$$

Thus, with a suitable ordering of the representatives $G_i$, we see that $M(G)$ is a lower triangular matrix with non-zero entries on the diagonal, and therefore invertible.

Each finite $G$-set $X$ has an associated vector of fixed point numbers

$$\beta_X = (|\text{Fix}_X(G_j)|)_{j=1}^r$$

and if $[X] = \sum a_i[G/G_i]$ in $\Omega(G)$ then $\beta_X = (\ldots a_i \ldots)M(G)$. Disjoint union and cartesian product of $G$-sets translate into componentwise addition and multiplication of fixed point vectors. We thus have

**Theorem 3 (Burnside 1911)** The map $\beta : \Omega(G) \to \mathbb{Z}^r$, $[X] \mapsto \beta_X$ is a well defined injective homomorphism of rings. In particular, $X$ and $Y$ are isomorphic as $G$-sets if and only if $\beta_X = \beta_Y$.

Now let $G = (W, S)$ be a finite Coxeter group and let $E$ be a set of representatives of conjugate subsets of $S$. The intersection of any two parabolic subgroups of $W$ is a parabolic subgroup. Therefore, if $J, K \subseteq S$, the direct product $W/W_J \times W/W_K$ decomposes as a sum of transitive $W$-sets, each of which is isomorphic to $W/W_L$ for some $L \subseteq S$. Thus the coset spaces $W/W_J$, where $J$ runs through the set $E$, form the basis of a subring

$$\Omega^c(W) = \langle [W/W_J] \mid J \subseteq S \rangle = \left\{ \sum_{J \in E} \alpha_J [W/W_J] \mid \alpha_J \in \mathbb{Z} \right\}$$

of $\Omega(W)$, the parabolic Burnside ring of $(W, S)$, first introduced in [6].

We call the corresponding part of the table of marks of $W$ the parabolic table of marks of $W$, and denote it by

$$M^c(W) = \left( |\text{Fix}_{W/W_J}(W_K)| \right)_{J,K \in E}$$

where we also write $\beta_{JK} = |\text{Fix}_{W/W_J}(W_K)|$ for any $J, K \subseteq S$. Note that by the formula above for $|\text{Fix}_{G/G_i}(G_j)|$ we have the following result [6].

**Lemma 1**

$$\beta_{JK} = |N_W(W_J) : W_J| \cdot |\{W_J^w \mid w \in W, W_K \leq W_J^w\}|$$

$$= |\{w \in X^{-1}_J \cap X_K \mid J^w \cap K = K\}| = a_{JKK}.$$ 

In particular, $\beta_{JJ} = |N_W(W_J) : W_J| \neq 0$ and $\beta_{JJ}$ divides $\beta_{JK}$ for every $K \subseteq S$.  

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We have that the map $\beta^c : \Omega^c(W) \rightarrow \mathbb{Z}^E$, $[W/W_J] \mapsto (\beta_{JK})_{K \in E}$ is a well defined injective ring homomorphism.

The parabolic marks of an arbitrary subgroup of $W$ coincide with the marks of a particular parabolic subgroup associated to it. Let $U \leq W$ and define the parabolic closure $U^c$ of $U$ in $W$ as

$$U^c = \bigcap \{ W^c_J \mid J \subseteq S, w \in W, U \leq W^c_J \},$$

the intersection of all parabolic subgroups of $W$ that contain $U$. Then $U^c$ is a parabolic subgroup and conjugate to $W_K$ for some $K \in E$.

**Proposition 1** Let $U \leq W$ and let $K \subseteq S$. Then $U^c$ is conjugate to $W_K$ if and only if $|\text{Fix}_{W/W_J}(U)| = \beta_{JK}$ for all $J \subseteq S$.

**Proof** We have $\text{Fix}_{W/W_J}(U) \supseteq \text{Fix}_{W^c/J}(U^c)$ since $U \subseteq U^c$. Now let $x \in \text{Fix}_{W^c/J}(U)$. We may assume that $x = W_J$. But then, $U \leq W_J$, and by the definition of $U^c$, also $U^c \leq W_J$, whence $x \in \text{Fix}_{W^c/J}(U^c) = \text{Fix}_{W^c/J}(U)$.

The converse follows from the fact that $U^c$ is conjugate to some $W_K$, and that all the columns of the parabolic table of marks are different. ■

As a corollary we obtain complete information about the values of the permutation characters afforded by the $W$-sets $W/W_J$. Note that $\langle c_K \rangle^c = W_K$ for a Coxeter element $c_K$ of $W_K$. Furthermore note that a transitive permutation character value coincides with the mark of a cyclic subgroup, both being the same number of fixed points.

**Corollary 1** For $J \subseteq S$ let $\chi_J$ be, as in section 4, the permutation character of $W$ on $W/W_J$. Then $\chi_J(c_K) = \beta_{JK}$. Moreover, for $w \in W$ we have $\chi_J(w) = \beta_{JK}$ if $\langle w \rangle^c$ is conjugate to $W_K$.

### 3 The radical of $\Sigma(W, p)$

The main aim of this section is to prove the following $p$-modular analogue of Theorem 2.

**Theorem 4** $\text{rad}(\Sigma(W, p)) = \ker \phi$. Moreover, $\text{rad}(\Sigma(W, p))$ is spanned by all $\overline{x}_J - \overline{x}_K$ where $J, K$ are conjugate subsets of $S$, together with all $\overline{x}_J$ for which $p$ divides $[N_W(W_j) : W_j]$.

Let $r$ be the number of rows of $M^c(W)$ and let $s$ be the number of rows indexed by subsets $J$ with $p \nmid [N_W(W_J) : W_J]$.

**Lemma 2** 1. $M^c(W)$ is a lower triangular matrix of rank $r = \dim G_W$
2. The $p$-rank of $M^c(W)$ (i.e. the rank of $M^c(W)$ modulo $p$ or $\dim G(W,p)$) is $s$.

**Proof** The first part follows from Section 2. If $p$ divides a diagonal entry of $M^c(W)$ then, by Lemma 1 $p$ divides every entry of that row. Thus the rank of $M^c(W) \mod p$ (i.e. $\dim G(W,p)$) is the number of non-zero rows in $M^c(W) \mod p$ and this, by Lemma 1 again, is $s$.

**Lemma 3**

1. $\Sigma(W,p)/\text{rad}(\Sigma(W,p))$ is commutative.

2. Every nilpotent element of $\Sigma(W,p)$ lies in $\text{rad}(\Sigma(W,p))$

**Proof** Let $\theta_1$ be the restriction of $\theta$ to $Z_W$. Then $\theta_1$ maps $Z_W$ onto the commutative ring $G_W$. By Theorem 2, the kernel of $\theta_1$ is the $Z$-module $R_W$ spanned by all $x_J - x_K$ where $J$ and $K$ are conjugate subsets of $S$, and is a nilpotent ideal of $Z_W$. In particular $\rho_1(R_W)$ is a nilpotent ideal of $\Sigma(W,p)$, and therefore $\rho_1(R_W) \subseteq \text{rad}(\Sigma(W,p))$. Hence there exists an ideal $S_W$ of $Z_W$, the pre-image of $\text{rad}(\Sigma(W,p))$, such that $R_W \subseteq S_W$ and $S_W/P_W \cong \text{rad}(\Sigma(W,p))$. Since $\Sigma(W,p) \cong Z_W/P_W$, $\Sigma(W,p)/\text{rad}(\Sigma(W,p)) \cong Z_W/S_W$ is a homomorphic image of $Z_W/P_W \cong G_W$. Since the latter ring is commutative the first part follows.

If $x$ is any nilpotent element of $\Sigma(W,p)$ then the coset $x + \text{rad}(\Sigma(W,p))$ is a nilpotent element in the commutative semi-simple algebra $\Sigma(W,p)/\text{rad}(\Sigma(W,p))$ and so is zero. Therefore $x \in \text{rad}(\Sigma(W,p))$ proving the second part.

**Proof of Theorem 4**

First we note that $\text{rad}(\Sigma(W,p)) \subseteq \ker \phi$. This is because the image of $\phi$ is a space of functions defined over a field and is therefore semi-simple. Consequently the two-sided nilpotent ideal $\phi(\text{rad}(\Sigma(W,p)))$ must be zero.

Now we prove that, if $p|[N_W(W_j) : W_j]$, then $\pi_J \in \text{rad}(\Sigma(W,p))$. From the definition of $a_{JKL}$ in Theorem 1 $\pi_{JKL} = 0$ unless $L \subseteq K$ and, by Lemma 1 $\pi_{JKK} = 0$ also. Thus $\pi_J \pi_K$ is a linear combination of elements $\pi_L$ with $L \subseteq K$ (and so $|L| \leq |K| - 1$). Now, by induction, it follows that $\pi_J \pi_K$ is a linear combination of elements $\pi_L$ with $|L| \leq |K| - t$ and so $\pi_J^{K+1} \pi_K = 0$ for all $K$. In particular $\pi_J$ is nilpotent and so $\pi_J \in \text{rad}(\Sigma(W,p))$ by Lemma 3.

The elements $\pi_J - \pi_K$ where $J$ and $K$ are conjugate subsets of $S$ are all nilpotent and, by Lemma 4 lie in $\text{rad}(\Sigma(W,p))$. They span a space $U$ of dimension $\dim \text{rad}(\Sigma_W) = \dim \Sigma_W - \dim G_W = 2^{n-1} - r$. In addition there are $r - s$ elements $\pi_J$ corresponding to those rows of $M^c(W)$ for which $p|[N_W(W_j) : W_j]$ which also lie in $\text{rad}(\Sigma(W,p))$. These, together with $U$, span a space of dimension $2^{n-1} - r + (r - s) = 2^{n-1} - \dim G(W,p) = \dim \ker \phi$. Hence $\dim \text{rad}(\Sigma(W,p)) \geq \dim \ker \phi$. 

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This proves that \( \ker \phi = \text{rad}(\Sigma(W, p)) \) as required and that it is spanned by the desired set of elements.

4 Representation Theory of \( \Sigma(W, p) \)

The representation theory of \( \Sigma_W \) has not been much studied in general although some results for the Coxeter groups of types \( A \) and \( B \) have been found [2, 11]. In this section we show how the representation theory of \( \Sigma(W, p) \) depends on that of \( \Sigma_W \). Specifically, we shall be interested in the composition factors of the principal indecomposable modules (indecomposable summands of the regular module) for each of \( \Sigma_W \) and \( \Sigma(W, p) \). The first observation is straightforward: a representation of \( \Sigma_W \) over \( F_p \) necessarily has \( p\mathbb{Z}_W \) in its kernel and so induces a representation of \( \Sigma(W, p) \); moreover, every representation of \( \Sigma(W, p) \) arises in this way. Therefore we may study the representation theory of \( \Sigma(W, p) \) by examining the \( p \)-modular representations of \( \Sigma_W \). We do this in the manner pioneered in group theory: by relating the representations in characteristic zero to those in characteristic \( p \) via a decomposition matrix.

This approach is tractable because the irreducible representations are all 1-dimensional. In fact, since \( \Sigma_W/\text{rad}(\Sigma_W) \) and \( \Sigma(W, p)/\text{rad}(\Sigma(W, p)) \) are commutative of dimensions \( r \) and \( s \) respectively (where \( r \) and \( s \) have the meanings given in the previous section) \( \Sigma_W \) has \( r \) 1-dimensional irreducible representations over a field of characteristic zero and \( s \) 1-dimensional irreducible representations over a field of characteristic \( p \). It follows (see 54.16, [9]) that the multiplicities of the principal indecomposable modules as direct summands in the regular representation of both \( \Sigma_W \) and \( \Sigma(W, p) \) are all 1.

We can explicitly describe the irreducible representations. As in Section 2 let \( E \) denote the set of representatives of the subsets of \( S \) that index the rows and columns of \( M^c(W) \). For each \( K \in E \) define the map \( \lambda_K : \Sigma_W \to \mathbb{Q} \) by

\[
\lambda_K(x) = \theta(x)(c_K) \quad \text{for all } x \in \Sigma_W
\]

Since \( \theta \) is a homomorphism it follows readily that \( \lambda_K \) is also a homomorphism, therefore a 1-dimensional representation of \( \Sigma_W \). Notice that \( \lambda_K \) is completely determined by its values on basis elements \( x_J \), that \( \lambda(x_J) = \theta(x_J)(c_K) = \chi_J(c_K) \), and these values of \( \lambda_K \) comprise the column of the matrix \( M^c(W) \) indexed by \( K \). In particular, \( \lambda_K|_{\mathbb{Z}_W} \) takes integer values and reducing these values modulo \( p \) we shall obtain the irreducible representations in a field of characteristic \( p \). We already knew (Lemma 3) that the \( p \)-rank of \( M^c(W) \) was \( s \) and so the above arguments have now proved:

**Lemma 4**

1. The columns of \( M^c(W) \) define the irreducible representations of \( \Sigma_W \).
2. The columns of $M^c(W)$ modulo $p$ define the irreducible representations of $\Sigma(W,p)$ and $M^c(W)$ modulo $p$ has precisely $s$ distinct columns.

According to this lemma the set $E$ indexes the irreducible representations of $\Sigma_W$. We now select a subset $F \subseteq E$ to index the irreducible representations of $\Sigma(W,p)$. In principle any subset that indexes $s$ distinct columns of $M^c(W)$ mod $p$ will suffice but we shall make a specific choice so that our results are easier to state. In $M^c(W)$ mod $p$ there are exactly $s$ non-zero rows (see the proof of Lemma 4) and we let $F \subseteq E$ index this set of rows. Since $M^c(W)$ mod $p$ is lower triangular of rank $p$, $F$ also indexes a set of distinct columns of $M^c(W)$ mod $p$. We define a matrix $D = (d_{KL})$ whose rows and columns are indexed by the members of $E$ and $F$ respectively. If $K \in E, L \in F$ then $d_{KL} = 1$ if columns $K$ and $L$ of $M^c(W)$ are equal modulo $p$, $d_{KL} = 0$ otherwise. By the previous lemma, the sets $E$ and $F$ index the irreducible representations of $\Sigma_W$ and $\Sigma(W,p)$ respectively and, since $D$ determines the structure of each irreducible representation of $\Sigma_W$ when reduced modulo $p$, we have

**Proposition 2** $D$ is the decomposition matrix of the algebra $\Sigma_W$. 

Of course, the decomposition matrix can be defined in a much more general context. Whenever we have a finite dimensional algebra where reduction modulo $p$ makes sense we can let $\{\tau_i\}$ be its irreducible representations in characteristic zero, $\{\upsilon_j\}$ its irreducible representations in characteristic $p$, and define $d_{ij}$ to be the multiplicity of $\upsilon_j$ as a composition factor of $\tau_i$ when $\tau_i$ is reduced modulo $p$. In our case the situation is quite simple: as all the irreducible representations in question are 1-dimensional these multiplicities are either 0 or 1.

However, it is convenient to remain with the more general situation for a little longer. So let $E$ be an algebraically closed complete local field. Then $E$ is the field of fractions of a principal ideal domain $U$, $U$ has a maximal ideal $P$, and $\mathcal{F} = U/P$ is a field of prime characteristic $p$. Of course, for descent algebras we have been working over the rational field but, because their irreducible representations are 1-dimensional, we can extend to a larger field without any significant changes.

Let $A$ be an associative algebra over $E$ with an order $\mathcal{D} \subset A$. Then $\bar{\mathcal{D}} = \mathcal{D}/P\mathcal{D}$ is an algebra over the field $\mathcal{F}$. Moreover, for every $\mathcal{D}$-module $M$, $\bar{M} = M/P\bar{M}$ is a $\bar{\mathcal{D}}$-module in a natural way.

Suppose that $P_1, P_2, \ldots, P_r$ are a full set of principal indecomposable modules for $\mathcal{D}$ over $E$ and that $T_1, T_2, \ldots, T_r$ are the associated irreducible modules (recall that $P_i$ has a unique maximal submodule $\text{rad}(P_i)$ and that $T_i \cong P_i/\text{rad}(P_i)$). The Cartan matrix of $\mathcal{D}$ is an $r \times r$ matrix $C = (c_{ij})$ whose $(i,j)$ entry is the number of times that $T_j$ occurs as a composition factor of $P_i$. It is known (see Theorem 11.10 in [7]) that $c_{ij}$ is the intertwining number $i(P_j, P_i) = \dim_E \text{Hom}(P_j, P_i)$. 

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In an exactly analogous way let $Q_1, Q_2, \ldots, Q_s$ be a full set of principal indecomposable modules for $\mathcal{D}$ over the field $F$ with associated irreducible modules $U_1, U_2, \ldots, U_s$ and let $\tilde{c}_{ij} = i(Q_j, Q_i) = \dim_F \text{Hom}(Q_j, Q_i)$ be the Cartan matrix $\tilde{C}$ of $\mathcal{D}$.

The algebras $\mathcal{D}$ and $\overline{\mathcal{D}}$ are related by the decomposition matrix $D$ which describes how each irreducible $\mathcal{D}$-module $T_i$ behaves when reduced “mod $p$”. Specifically, $D = (d_{ij})$ is an $r \times s$ matrix where $d_{ij}$ is the number of composition factors of $T_i$ which are isomorphic to $U_j$. Again by, Theorem 11.10 of [7], $d_{ij} = i(Q_j, T_i)$.

By [7] Theorem 37.4 there exist direct summands $R_i$ of the regular $\mathcal{D}$-module such that $\overline{R}_j = Q_j$ and we may write

$$R_j = \bigoplus_k h_{kj} P_k$$

where this equation signifies that $R_j$ can be expressed as a sum of principal indecomposable modules, and that the module $P_k$ occurs as an isomorphism type $h_{kj}$ times.

At this point we note two applications of [7] Lemma 38.1:

$$d_{ij} = i(Q_j, T_i) = i(\overline{R}_j, T_i) = i(R_j, T_i)$$

and

$$\tilde{c}_{ij} = i(Q_j, Q_i) = i(\overline{R}_j, R_i) = i(R_j, R_i)$$

From the first of these we have

$$d_{ij} = i\left(\bigoplus_k h_{kj} P_k, T_i\right)$$

$$= \sum_k h_{kj} i(P_k, T_i)$$

$$= h_{ij}$$

since $i(P_k, T_i) = 0$ unless $k = i$ and $i(P_k, T_k) = 1$.

From the second we obtain

$$\tilde{c}_{ij} = i\left(\bigoplus_k h_{kj} P_k, \bigoplus_l h_{li} P_l\right)$$

$$= \sum_{k,l} d_{kj} d_{li} i(P_k, P_l)$$

$$= \sum_{k,l} d_{kj} d_{li} c_{kl}$$

In other words we have
Theorem 5 \( \tilde{C} = D^T CD \)

A proof of this theorem in the language of Groethendieck groups has recently been given in [13] but it is likely that the result is not new. Clearly our proof has drawn heavily on the approach of Burrow [7] who considered the case that \( A \) was a group algebra and where \( C = I \) since group algebras are semi-simple. It is plausible that Burrow knew Theorem 5 over 30 years ago. Nevertheless the result deserves to be better known and we make no further apology for including it.

We return now to the special case of descent algebras. The decomposition matrix of a descent algebra has been defined in terms of a table of characters of the corresponding Coxeter group. We have the following easy result:

**Proposition 3** Let \( K \in E, L \in F \) head columns of the matrix \( M^c(W) \). Then, if \( c_K \) and \( c_L \) have conjugate \( p \)-regular parts, \( d_{KL} = 1 \).

**Proof.** By the arguments in §82 of [9] every character \( \chi_J \) takes equal values modulo \( p \) on \( c_K \) and \( c_L \). Thus, \( \lambda_K = \lambda_L \mod p \) and so \( d_{KL} = 1 \). \( \blacksquare \).

In the remainder of this section we shall consider descent algebras according to their Coxeter type. By a combination of theoretical argument and computer calculation we obtain a description of the decomposition matrix in all cases and this shows that, often, the converse of Proposition 3 is true.

We let \( \pi(n) \) denote the number of partitions of \( n \). This non-standard notation is necessary since we also define \( \pi(n, p) \) as the number of partitions of \( n \) in which no part has multiplicity \( p \) or more. We note the following result (see [16] p.41):

**Lemma 5** \( \pi(n, p) \) is the number of partitions of \( n \) into parts not divisible by \( p \).

4.1 Representation Theory of \( \Sigma(A_{n-1}, p) \)

In this subsection we let \( W = A_{n-1} \) which is best described as the symmetric group \( S_n \) acting in the usual way on \( \{1, 2, \ldots, n\} \) with generating set \( S = \{(i, i+1) | i = 1, \ldots, n-1\} \). If \( K \subseteq S \) then the Coxeter element \( c_K \) has cycles on sets \( [u, v] \) of consecutive integers. The ordered list of cycle lengths (one cycle appearing before another if it permutes integers with smaller values) determines and is determined by \( K \). Therefore the subsets of \( S \) can be parameterised by compositions of \( n \). The following lemma and corollary are easy consequences of this parameterisation and Lemmas 2 and 4.

**Lemma 6** 1. If \( K, L \subseteq S \) then \( K \) is conjugate to \( L \) if and only if the corresponding compositions determine the same partition of \( n \).
2. If \( K \subseteq S \) and its corresponding composition has \( a_i \) components equal to \( i \) (for \( i = 1, \ldots, n \)) then

\[
[N(W_K) : W_K] = a_1!a_2! \ldots a_n!
\]

Corollary 2

1. \( r = \pi(n) \)

2. \( s = \pi(n, p) \)

Theorem 6 Let \( W \) be one of the Coxeter groups \( A_{n-1} \) and let \( K \in E, L \in F \). Then \( d_{KL} = 1 \) if and only if \( c_K \) and \( c_L \) have conjugate \( p \)-regular parts.

Proof There are two equivalence relations \( \rho_1, \rho_2 \) on the set \( E \) (which indexes the columns of \( M^c(W) \)):

\[
(K, J) \in \rho_1 \text{ if } \lambda_K = \lambda_J \mod p
\]

\[
(K, J) \in \rho_2 \text{ if } c_K, c_J \text{ have conjugate } p \text{-regular parts}
\]

We have seen (Proposition 8) that \( \rho_2 \subseteq \rho_1 \). However, the number of \( \rho_1 \)-equivalence classes is \( s \) (Lemma 4) and this is \( \pi(n, p) \) (Corollary 2). By Lemma 5 this is also the number of partitions with no part divisible by \( p \) which is the number of equivalence classes of \( \rho_2 \). Hence \( \rho_1 = \rho_2 \) and the theorem follows. \( \blacksquare \)

4.2 Representation Theory of \( \Sigma(B_n, p) \)

It is convenient to represent \( B_n \) as a permutation group on \( \{\pm 1, \ldots, \pm n\} \) with block system \( \{i, -i\}_{i=1}^n \) on which it acts as the full symmetric group with kernel of order \( 2^n \). The set of Coxeter generators \( S = \{s_0, s_1, \ldots, s_{n-1}\} \) is defined as \( s_0 = (-1, 1) \) and \( s_i = (i, i+1)(-i, -i-1), 1 \leq i \leq n-1 \).

Let \( K \subseteq S \) and consider the Coxeter element \( c_K \). If \( c_K \) has a cycle \( (a, b, \ldots) \) consisting of positive elements (a positive cycle) then it will also have a corresponding negative cycle \( (-a, -b, \ldots) \). Furthermore, at most one cycle of \( c_K \) can contain both positive and negative elements; such a cycle is present if and only if \( s_0 \in K \). We may write

\[
c_K = x_0x_1
\]

(1)

where \( x_0 \) is the cycle containing both positive and negative elements (or \( x_0 = 1 \) if there is no such cycle) and \( x_1 \) is the product of all the other cycles (positive and negative in matching pairs); note that \( x_0 \) commutes with \( x_1 \). Each positive cycle is on some range \([u..v]\) of consecutive integers and the list of lengths of positive cycles taken in the natural order (as in the previous subsection) determines and is determined by \( K \). In this way the subsets of \( S \) can be parameterised by compositions of integers \( m, 0 \leq m \leq n \). The following result is a consequence of the results of [15].
Lemma 7  1. If $K, L \subseteq S$ then $K$ is conjugate to $L$ if and only if the corresponding compositions determine the same partition.

2. If $K \subseteq S$ and the corresponding composition is a composition of $m$ with $a_i$ components of size $i$ and $t$ components in all then

$$[N(W_K) : W_K] = 2^t a_1! a_2! \ldots a_m!$$

Corollary 3  1. $r = \sum_{m=0}^{n} \pi(m)$

2. If $p \neq 2$ then $s = \sum_{m=0}^{n} \pi(m, p)$

Let $K$ be one of the subsets indexing the rows and columns of $M^c(W)$ and $c_K = x_0 x_1$ as in Equation 1. If $x_1$ is a $p$-regular element we say that $K$ is a $p$-special subset of $S$. Since the order of $x_1$ is the lowest common multiple of its cycle lengths, $K$ is $p$-special if and only if the partition corresponding to $K$ has no part divisible by $p$. By Lemma 5 and Corollary 3 there are precisely $s$ $p$-special subsets when $p \neq 2$.

Lemma 8 If $K \subseteq S$ there exists a $p$-special $K_1 \subseteq S$ such that $c_K$ and $c_{K_1}$ have conjugate $p$-regular parts.

Proof Let $c_K = x_0 x_1$ as in Equation 1 and let $x_2$ be the $p$-regular part of $x_1$. Since $x_2$ is a power of $x_1$, its cycles also come in matching positive, negative pairs. Therefore $x_2$ is conjugate, via a permutation in the centraliser of $x_0$, to a Coxeter element $x_3$ with this property. But then $x_0 x_3$ is also a Coxeter element $c_{K_1}$ whose $p$-regular part is conjugate to that of $x_0 x_1$.

Lemma 9 If $p \neq 2$ the columns of $M^c(W)$ which are indexed by the $p$-special subsets provide a full set of irreducible representations of $\Sigma(B_n, p)$.

Proof By the last lemma the columns of $M^c(W) \mod p$ indexed by $p$-special subsets contain a full set of distinct columns and since there are $s$ such columns they must yield a complete set of irreducible representations of $\Sigma(B_n, p)$.

Theorem 7 Let $W$ be one of the Coxeter groups $B_n$ and let $K \in E, L \in F$. If $p \neq 2$ then $d_{KL} = 1$ if and only if $c_K$ and $c_L$ have conjugate $p$-regular parts. If $p = 2$ then $F = \{S\}$ and $d_{KS} = 1$ for all $K$.

Proof Suppose first that $p \neq 2$. Proposition 8 has proved one implication already. For the other, suppose $d_{KL} = 1$ and let $K_1, L_1$ be the $p$-special subsets, guaranteed by Lemma 5 such that $K, K_1$ have conjugate $p$-regular parts and $L, L_1$ have conjugate $p$-regular parts. Then, by Proposition 8 $d_{K_1 L_1} = 1$ and Lemma 9 shows that $K_1 = L_1$.

If $p = 2$, Lemma 7 implies that the only $K \in E$ for which 2 does not divide $[N(W_K) : W_K]$ is the one with $t = 0$, namely $K = S$. Therefore $\Sigma(B_n, 2)$ has
just one irreducible representation and so $d_{KL} = 1$ for all $K \in E, L \in F = \{S\}$. 

4.3 Representation Theory of $\Sigma(D_n,p)$. 

The Coxeter group $(W,S)$ of type $D_n$ can be considered as a normal subgroup of index 2 in the Coxeter group $(\hat{W},\hat{S})$ of type $B_n$. As such it is the Coxeter group $S = \{u,s_1, \ldots, s_{n-1}\}$ where, as in the previous subsection, $\hat{S} = \{s_0,s_1, \ldots, s_{n-1}\}$ and $u = s_0s_1s_0 = (-1,1)(1,2)(-1,-2)(-1,1) = (-1,2)(1,-2)$.

For any $K \subseteq S$ the parabolic subgroup $W_K$ is isomorphic to $W_0 \times W_1$ where $W_0$ is of type $D_n$ for some $n_0 \leq n$, $n_0 \neq 1$ and $W_1 = \langle K_1 \rangle$ for some $K_1 \subseteq \{s_{n_0}, \ldots, s_{n-1}\}$. (Here the group of type $D_2$ is $\langle u,s_1 \rangle$ and isomorphic to a group of type $A_1 \times A_1$ and the group of type $D_3$ is $\langle u,s_1,s_2 \rangle$ and isomorphic to a group of type $A_3$.) If $n_0 = 0$ then $W_1$ is a subgroup of either the group $W'$ generated by $S' = \{s_1,s_2, \ldots, s_{n-1}\}$ or the group $W''$ generated by $S'' = \{u,s_2,s_3, \ldots, s_{n-1}\}$ which are both of type $A_{n-1}$.

Thus to each subset $K \subseteq S$ there is associated via $W_1$ a composition of $m \leq n$. Each composition occurs this way, except those of $n-1$. Conversely, for each composition $\lambda$ of $m \neq n-1$, there is a unique $K \subseteq S$, unless $\lambda$ is a composition of $n$ with $\lambda_1 > 1$. In that case there are two subsets with that label, each containing exactly one of $s_1$ and $u$.

Consider $K, L \subseteq S$. Then $K$ and $L$ are conjugate in $W$ if and only if their corresponding compositions determine the same partition, unless that partition is a partition of $n$ with all parts even. In that case $K$ and $L$ are conjugate only if they both lie in $S'$ or both in $S''$.

Consider $W_K = W_0 \times W_1$ with $W_0$ of type $D_{n_0}$ for some $n_0 \geq 2$. Then there is a parabolic subgroup $\hat{W}_K = W_0 \times W_1$ of $\hat{W}$ where $\hat{W}_0$ is of type $B_{n_0}$. We have $W_K = \hat{W}_K \cap W$ and $|\hat{W}_K : W_K| = 2$. Also $|N_W(W_K) : N_W(W_K)| = 2$ whence $\beta_{KK}$ is computed from the partition corresponding to $K$ in the same way as in case $B_n$.

Now let $n_0 = 0$ and let $W_K$ be a subgroup of $W'$ with corresponding partition $\mu$. Then $W_K$ is a parabolic subgroup of both $W$ and $\hat{W}$. We have $N_W(W_K) \subseteq W$ if and only if all parts of $\mu$ are even. We thus get the following formula.

Lemma 10 Let $K \subseteq S$ with corresponding partition $\mu = (1^{m_1},2^{m_2}, \ldots, n^{m_n})$. Then

$$[N_W(W_K) : W_K] = 2^{m_1}m_1! \cdots 2^{m_n}m_n! a$$

where $a = 1$ unless $\mu$ is a partition of $n$ and has at least one odd part. In that case $a = 1/2$. 

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Let $c_K$ be a Coxeter element of $W_K$. Again, we have a unique decomposition $c_K = x_0 x_1$ where $x_0 \in W_0$ and $x_1 \in W_1$. We call $K$ a $p$-special subset if $x_1$ is $p$-regular. And, by the same argument as for type $B_n$, we have that for each $K \subseteq S$ there is a $p$-special $K_1 \subseteq S$ such that $c_K$ and $c_{K_1}$ have conjugate $p$-regular parts.

Similar considerations as for type $B_n$ then lead to the following description of the decomposition matrix for type $D_n$.

**Theorem 8** Let $(W, S)$ be of type $D_n$ and let $K \in E$, $L \in F$. If $p \neq 2$ then $d_{KL} = 1$ if and only if $c_K$ and $c_L$ have conjugate $p$-regular parts. If $p = 2$ and $n$ is even then we have $F = \{ S \}$ and $d_{KS} = 1$ for all $K \in E$; if $n$ is odd then we have $F = \{ S', S \}$ and $d_{KL} = 1$ if and only if either $L = S$ and $K \neq S'$ or $L = S'$ and $K = S$.

**Proof** The theorem for $p \neq 2$ follows as in case $B_n$. For $p = 2$ we show that either $\beta_{KL} = 0 \mod 2$ for all $K, L \in E$ unless $K = S$, or $n$ is odd and $K = L = S'$. Note that by Lemma 10, $\beta_{KK} = [NW(W_K) : W_K]$ is odd only if all $m_i = 0$ (whence $\mu$ is the empty partition corresponding to $K = S$) or, if $\mu$ is a partition of $n$ with at least one odd part and at most one $m_i = 1$ (whence $n$ is odd and $\mu$ is the partition $[n]$ corresponding to $K = S'$). Thus, for $K \neq S$, $\beta_{KL}$ is even unless $n$ is odd and $K = S'$.

Finally, in order to see that $\beta_{KL}$ is even in the remaining cases (where $L = S'$ and $L$ not conjugate to $K$) we consider the following action on complementary pairs, first as a $B_n$ action. Let $I = \{1, \ldots, n\}$ and let

$$X = \{ \{P, Q\} \mid P, Q \subseteq I; P \cup Q = I; P \cap Q = \emptyset \}$$

(so we always have $Q = I \setminus P$). Then $B_n$ acts on $X$ as follows. The action of $s_i$ ($i \geq 1$) is induced from its action as $(i, i + 1)$ on $I$ and the action of $s_0$ is given by

$$\{P, Q\}^{s_0} = \{P \perp \{1\}, Q \perp \{1\}\},$$

where $A \perp B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of the sets $A, B$. Note that, if we define $t_i = s_i \cdots s_1 s_0 s_1 \cdots s_i$ then $t_i$ acts as

$$\{P, Q\}^{t_i} = \{P \perp \{i + 1\}, Q \perp \{i + 1\}\},$$

the symmetric difference with $\{i + 1\}$, and the longest element $w_0 = t_0 t_1 \cdots t_{n-1}$ of $W$ acts as symmetric difference with $I$ whence it fixes every point in $X$. The complementary pair $\{P, Q\}$ arises from $\{\emptyset, I\}$ by taking symmetric differences with $P$ (or $Q$). Thus the action of $W$ is transitive on all of the $2^{n-1}$ complementary pairs in $X$ and the stabiliser of $\{\emptyset, I\}$ is $\langle s_1, \ldots, s_{n-1}, w_0 \rangle$, a group of index $2^{n-1}$ in $W$.

Now let $n$ be odd and restrict the action to $W$. Then, since $w_0 \notin W$, the stabiliser of $\{\emptyset, I\}$ in $W$ is $W'$, which is of index $2^{n-1}$ in $W$. Hence $W$ acts
transitively on $X$ and the action is equivalent to the action on the cosets of $W'$, the one we are interested in.

We know that $\beta_{KL} = 0$ whenever $W_L$ is not conjugate to a subgroup of $W_K$. It remains to investigate the fixed points of parabolic subgroups of $W'$ which is of type $A_{n-1}$. Consider $s_1$ and its fixed points. If $n > 2$ then $\{P, Q\}$ is stable under $s_1$ if and only if $\{1, 2\} \subseteq P$ or $\{1, 2\} \subseteq Q$. In either case taking symmetric differences with $\{1, 2\}$ yields a different point $\{P', Q'\}$ which is also fixed by $s_1$. So the fixed points of $s_1$ come in pairs.

A similar argument applies to a Coxeter element $c_L$ of any parabolic subgroup $W_L$ of $W'$ unless $L = S'$. Here we denote by $J \subseteq I$ the set of points moved by $c_L$. Then we find that $\{P, Q\}$ is stable under $c_L$ if and only if $J \subseteq P$ or $J \subseteq Q$. Again, taking symmetric differences with $J$ produces a different fixed point $\{P', Q'\}$. This shows that $\beta_{KL}$ is even for all proper parabolic subgroups $W_L$ of $W'$.

\[\blacksquare\]

### 4.4 Representation Theory of Exceptional Types

The descriptions of the decomposition matrices in the case of the classical types in the previous subsections are special cases of a more general classification of columns of the parabolic table of marks that are equal if taken mod $p$.

For this more general classification we need to extend the notion of having the same $p$-regular part. Let $w'$ be the $p$-regular part of $w \in W$ and let $\rightarrow_p$ be the relation on $E$ defined by $J \rightarrow_p K$ if $\langle w' \rangle^c$ is conjugate to $W_K$ for some $w \in E$ such that $\langle w \rangle^c$ is conjugate to $W_J$.

**Theorem 9** Let $K \in E$ and $L \in F$. Then $d_{KL} = 1$ if and only if $K$ and $L$ lie in the same class of the equivalence generated by $\rightarrow_p$.

The following tables, which we have computed using the CHEVIE [12] package in GAP [18], describe the decomposition matrices for the exceptional types. The proof of the theorem follows by inspection of these tables and the parabolic tables of marks reduced mod $p$, together with theorems 6, 7, and 8. Note that the theorem is also true for the dihedral types $I_2(m)$ (see [20] for a full account of the representation theory in all characteristics in this case).

In each case we give for any $K \in E$ and for any prime $p$ dividing the order of $W$ the list of $L \in E$ such that $K \rightarrow_p L$. The first entry in each list is determined by the $p$-regular part of a Coxeter element, and the number in parenthesis denotes the representative in the equivalence obtained as the closure of the relation $\rightarrow_p$ if different from the first entry of the list. If the list for $K$ consists of $K$ only and this is also the representative we just have a dot (.) as entry. Note that conversely, all the representatives have a dot entry.
For each $K \in E$ we also list its isomorphism type, possibly with dashes (' and '') to distinguish isomorphic parabolic subgroups, and the index $\beta_{KK}$ of $W_K$ in its normalizer in $W$.

|   | $\beta_{KK}$ | $p = 2$ | $p = 3$ | $p = 5$ |
|---|-------------|---------|---------|---------|
| 1 | 1           | 51840   | .       | .       |
| 2 | $A_1$       | 720     | 1       | .       |
| 3 | $A_1 \times A_1$ | 48 | 1       | .       |
| 4 | $A_2$       | 72      | 4 (1)   | 1       |
| 5 | $A_1 \times A_1 \times A_1$ | 12 | 1       | .       |
| 6 | $A_2 \times A_1$ | 6 | 4 (1)   | 2       |
| 7 | $A_3$       | 8       | 1       | .       |
| 8 | $A_2 \times A_1 \times A_1$ | 2 | 4 (1)   | 3       |
| 9 | $A_2 \times A_2$ | 12 | 1       | 1       |
| 10 | $A_3 \times A_1$ | 2 | 1       | .       |
| 11 | $A_4$       | 2       | .       | .       |
| 12 | $D_4$       | 6       | 4, 1 (1)| 12 (1)  |
| 13 | $A_2 \times A_2 \times A_1$ | 2 | 9       | 2       |
| 14 | $A_4 \times A_1$ | 1 | 11      | .       |
| 15 | $A_5$       | 2       | 9       | 5       |
| 16 | $D_5$       | 1       | 1, 4    | .       |
| 17 | $E_6$       | 1       | 17, 9 (9)| 12, 1 (1)|

Table 1: Decomposition matrix for $E_6$.

### 4.5 Cartan matrices

By Theorems 5 and 9 the Cartan matrix of $\Sigma(W, p)$ can be determined once it is known for $\Sigma_W$. Types $A$ and $B$ can therefore be handled by Theorem 5.4 of [11] and Theorem 3.3 of [5] which give the Cartan matrices in characteristic zero. Furthermore, the work of [20] allows the dihedral case to be solved. However, we have not calculated the Cartan matrix in characteristic zero in any other cases; such a calculation awaits a more detailed study of these algebras.

### References

[1] M.D. Atkinson: Solomon’s descent algebra revisited, Bull.London Math. Soc. 24 (1992) 545–551.

[2] M.D. Atkinson and S.J. van Willigenburg: The p-modular descent algebra of the symmetric group, Bull. London Math. Soc. 29 (1997), 407–414.
| $\beta_{KK}$ | $p = 2$ | $p = 3$ | $p = 5$ | $p = 7$ |
|-------------|---------|---------|---------|---------|
| 1 | 1 | 2903040 | . | . | . |
| 2 | $A_1$ | 23040 | 1 | . | . |
| 3 | $A_1 \times A_1$ | 768 | 1 | . | . |
| 4 | $A_2$ | 1440 | 4 (1) | 1 | . |
| 5 | $(A_1 \times A_1 \times A_1)'$ | 1152 | 1 | . | . |
| 6 | $(A_1 \times A_1 \times A_1)''$ | 96 | 1 | . | . |
| 7 | $A_2 \times A_1$ | 48 | 4 (1) | 2 | . |
| 8 | $A_3$ | 96 | 1 | . | . |
| 9 | $A_1 \times A_1 \times A_1 \times A_1$ | 48 | 1 | . | . |
| 10 | $A_2 \times A_1 \times A_1$ | 8 | 4 (1) | 3 | . |
| 11 | $A_2 \times A_2$ | 24 | 11 (1) | 1 | . |
| 12 | $(A_3 \times A_1)'$ | 48 | 1 | . | . |
| 13 | $(A_3 \times A_1)''$ | 8 | 1 | . | . |
| 14 | $A_4$ | 12 | 14 (1) | . | 1 |
| 15 | $D_4$ | 48 | 4, 1 (1) | 15 (1) | . |
| 16 | $A_2 \times A_1 \times A_1 \times A_1$ | 12 | 4 (1) | 5 | . |
| 17 | $A_2 \times A_2 \times A_1$ | 4 | 11 (1) | 2 | . |
| 18 | $A_3 \times A_1 \times A_1$ | 4 | 1 | . | . |
| 19 | $A_3 \times A_2$ | 4 | 4 (1) | 8 | . |
| 20 | $A_4 \times A_1$ | 2 | 14 (1) | . | 2 |
| 21 | $D_4 \times A_1$ | 8 | 4, 1 (1) | . | . |
| 22 | $A_5'$ | 12 | 11 (1) | 5 | . |
| 23 | $A_5''$ | 4 | 11 (1) | 6 | . |
| 24 | $D_5$ | 4 | 1, 4 | . | . |
| 25 | $A_3 \times A_2 \times A_1$ | 2 | 4 (1) | 12 | . |
| 26 | $A_4 \times A_2$ | 2 | 26 (1) | 14 | 4 |
| 27 | $A_5 \times A_1$ | 2 | 11 (1) | 9 | . |
| 28 | $D_5 \times A_1$ | 2 | 1, 4 | . | . |
| 29 | $A_6$ | 2 | 29 (1) | . | 1 |
| 30 | $D_6$ | 2 | 14, 1, 4, 11 (1) | . | . |
| 31 | $E_6$ | 2 | 31, 11 (1) | 15, 1 (1) | . |
| 32 | $E_7$ | 1 | 31, 1, 4, 11, 14, 26, 29 (1) | 32, 5 (5) | . |

Table 2: Decomposition matrix for $E_7$. 
| \( \beta_{KK} \) | \( p = 2 \) | \( p = 3 \) | \( p = 5 \) | \( p = 7 \) |
|---|---|---|---|---|
| 1 | 1 | 696729600 | . | . | . |
| 2 | \( A_1 \) | 2903040 | 1 | . | . | . |
| 3 | \( A_1 \times A_1 \) | 46080 | 1 | . | . | . |
| 4 | \( A_2 \) | 103680 | 4 (1) | 1 | . | . | . |
| 5 | \( A_1 \times A_1 \times A_1 \) | 2304 | 1 | . | . | . | . |
| 6 | \( A_2 \times A_1 \) | 1440 | 4 (1) | 2 | . | . | . |
| 7 | \( A_3 \) | 3840 | 1 | . | . | . | . |
| 8 | \( A_1 \times A_1 \times A_1 \times A_1 \) | 384 | 1 | . | . | . | . |
| 9 | \( A_2 \times A_1 \times A_1 \) | 96 | 4 (1) | 3 | . | . | . |
| 10 | \( A_2 \times A_2 \) | 288 | 10 (1) | 1 | . | . | . |
| 11 | \( A_3 \times A_1 \) | 96 | 1 | . | . | . | . |
| 12 | \( A_4 \) | 240 | 12 (1) | 1 | . | . | . |
| 13 | \( D_4 \) | 1152 | 4, 1 (1) | 13 (1) | . | . | . |
| 14 | \( A_2 \times A_1 \times A_1 \times A_1 \) | 24 | 4 (1) | 5 | . | . | . |
| 15 | \( A_2 \times A_2 \times A_1 \) | 24 | 10 (1) | 2 | . | . | . |
| 16 | \( A_3 \times A_1 \times A_1 \) | 16 | 4 (1) | 7 | . | . | . |
| 17 | \( A_3 \times A_2 \) | 16 | 12 (1) | . | 2 | . | . |
| 18 | \( A_4 \times A_1 \) | 48 | 4, 1 (1) | 19 (2) | . | . | . |
| 19 | \( D_4 \times A_1 \) | 48 | 10 (1) | 5 | . | . | . |
| 20 | \( A_5 \) | 24 | 4 (1) | 3 | . | . | . |
| 21 | \( D_5 \) | 48 | 1, 4 | . | . | . | . |
| 22 | \( A_2 \times A_2 \times A_1 \times A_1 \) | 8 | 10 (1) | 11 | . | . | . |
| 23 | \( A_3 \times A_2 \times A_1 \) | 4 | 4 (1) | 1 | . | . | . |
| 24 | \( A_2 \times A_1 \times A_1 \) | 4 | 12 (1) | . | 3 | . | . |
| 25 | \( A_3 \times A_3 \) | 8 | 1 | . | . | . | . |
| 26 | \( A_4 \times A_2 \) | 4 | 26 (1) | 12 | 4 | . | . |
| 27 | \( D_4 \times A_2 \) | 12 | 10, 4 (1) | 13 (1) | . | . | . |
| 28 | \( A_3 \times A_1 \) | 4 | 10 (1) | 8 | . | . | . |
| 29 | \( D_5 \times A_1 \) | 4 | 1, 4 | . | . | . | . |
| 30 | \( A_6 \) | 4 | 30 (1) | . | 1 | . | . |
| 31 | \( D_6 \) | 8 | 12, 1, 4, 10 (1) | . | . | . | . |
| 32 | \( E_6 \) | 12 | 32, 10 (1) | 13, 1 (1) | . | . | . |
| 33 | \( A_4 \times A_2 \times A_1 \) | 2 | 26 (1) | 18 | 6 | . | . |
| 34 | \( A_4 \times A_3 \) | 2 | 12 (1) | . | 7 | . | . |
| 35 | \( A_6 \times A_1 \) | 2 | 30 (1) | . | 2 | . | . |
| 36 | \( D_5 \times A_2 \) | 2 | 4, 10 (1) | 21 | . | . | . |
| 37 | \( A_7 \) | 2 | 1 | . | . | . | . |
| 38 | \( E_6 \times A_1 \) | 2 | 32, 10 (1) | 19, 2 (2) | . | . | . |
| 39 | \( D_7 \) | 2 | 10, 1, 4, 12 (1) | . | . | . | . |
| 40 | \( E_7 \) | 2 | 32, 1, 4, 10, 12, 26, 30 (1) | 40, 5 (5) | . | . | . |
| 41 | \( E_8 \) | 1 | 41, 1, 4, 10, 12, 26, 30, 32 (1) | 41, 1, 13 (1) | 41, 1 (1) | . | . |

Table 3: Decomposition matrix for \( E_8 \).
| $\beta_{KK}$ | $p = 2$ | $p = 3$ |
|---|---|---|
| 1 | 1 | 1152 | . | . |
| 2 | $A_1'$ | 48 | 1 | . |
| 3 | $A_1''$ | 48 | 1 | . |
| 4 | $A_1 \times A_1$ | 4 | 1 | . |
| 5 | $A_2'$ | 12 | 5 (1) | 1 |
| 6 | $A_2''$ | 12 | 6 (1) | 1 |
| 7 | $B_2$ | 8 | 1 | . |
| 8 | $(A_2 \times A_1)'$ | 2 | 5 (1) | 2 |
| 9 | $(A_2 \times A_1)''$ | 2 | 6 (1) | 3 |
| 10 | $B_3'$ | 2 | 5, 1 (1) | . |
| 11 | $B_3''$ | 2 | 6, 1 (1) | . |
| 12 | $F_4$ | 1 | 12, 1, 5, 6 (1) | 12, 1 (1) |

Table 4: Decomposition matrix for $F_4$.

| $\beta_{KK}$ | $p = 2$ | $p = 3$ | $p = 5$ |
|---|---|---|---|
| 1 | 1 | 120 | . | . | . |
| 2 | $A_1$ | 4 | 1 | . | . |
| 3 | $A_1 \times A_1$ | 2 | 1 | . | . |
| 4 | $A_2$ | 2 | 4 (1) | 1 | . |
| 5 | $I_2(5)$ | 2 | 5 (1) | . | 1 |
| 6 | $H_3$ | 1 | 5, 1, 4 (1) | . | . |

Table 5: Decomposition matrix for $H_3$.

| $\beta_{KK}$ | $p = 2$ | $p = 3$ | $p = 5$ |
|---|---|---|---|
| 1 | 1 | 14400 | . | . | . |
| 2 | $A_1$ | 120 | 1 | . | . |
| 3 | $A_1 \times A_1$ | 8 | 1 | . | . |
| 4 | $A_2$ | 12 | 4 (1) | 1 | . |
| 5 | $I_2(5)$ | 20 | 5 (1) | . | 1 |
| 6 | $A_2 \times A_1$ | 2 | 4 (1) | 2 | . |
| 7 | $I_2(5) \times A_1$ | 2 | 5 (1) | . | 2 |
| 8 | $A_3$ | 2 | 1 | . | . |
| 9 | $H_3$ | 2 | 5, 1, 4 (1) | . | . |
| 10 | $H_4$ | 1 | 10, 1, 4, 5 (1) | 10, 1 (1) | 10, 1 (1) |

Table 6: Decomposition matrix for $H_4$.  

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[3] F. Bergeron and N. Bergeron: Symbolic manipulation for the study of the
descent algebra of finite Coxeter groups, J. Symbolic Comp. 14 (1992),
127–139.

[4] F. Bergeron and N. Bergeron: A decomposition of the descent algebra of
the hyperoctahedral group 1, J. Algebra 148 (1992), 86–97.

[5] N. Bergeron: A decomposition of the descent algebra of the hyperoctahe-
dral group 2, J. Algebra 148 (1992), 98–122.

[6] F. Bergeron, N. Bergeron, R.B. Howlett, and D.E. Taylor, A decomposition
of the descent algebra of a finite Coxeter group, J. Algebraic Combinatorics
1 (1992), 23–44

[7] M. Burrow: Representation Theory of Finite Groups, Academic Press (New
York – London) 1965.

[8] P. Cellini: A general commutative descent algebra, J. Algebra 175 (1995),
990–1014.

[9] C.W. Curtis and I. Reiner: Representation of finite groups and associative
algebras, Interscience Publishers, New York (1962).

[10] C. W. Curtis and I. Reiner: Methods of representation theory, vol. II, Wiley,
New York, 1987, reprinted 1994 as Wiley Classics Library Edition.

[11] A.M. Garsia and C. Reutenauer: A decomposition of Solomon’s descent
algebra, Adv. Math. 77 (1989) 189–262.

[12] M. Geck, G. Hiß, F. Lübeck, G. Malle, and G. Pfeiffer: CHEVIE – A system
for computing and processing generic character tables, Applicable Algebra
in Engineering, Communication and Computing 7 (1996), 175–210.

[13] M. Geck and R. Rouquier, Centers and simple modules for Iwahori-Hecke
algebras: in Finite Reductive Groups, Related Structures and Representa-
tions, Progress in Mathematics, volume 141 (Ed. M. Cabanes), Birkhäuser
(Boston–Basel–Berlin) 1997, 251–272.

[14] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh and J.-Y. Thibon:
Non-commutative symmetric functions, Adv. Math. 112 (1995) 218–348.

[15] R. B. Howlett: Normalizers of parabolic subgroups of reflection groups: J.
London Math. Soc. (Series 2) 21 (1980), 62–80.

[16] G. James and A. Kerber: Encyclopedia of mathematics and its applications;
v.16. Section:Algebra; The representation theory of the symmetric group,
Massachusetts, Addison-Wesley (1981).

[17] P. Orlik and L. Solomon: Coxeter arrangements, Proceedings of Symposia
in Pure Mathematics 40 (1983), 269–291.
[18] M. Schönert et al., GAP – Groups, Algorithms, and Programming: Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, fifth edition, 1995

[19] L. Solomon: A Mackey formula in the group ring of a Coxeter group, J. Algebra 41 (1976), 255–268.

[20] S. van Willigenburg: The descent algebras of Coxeter groups, St Andrews University 1997.