THE JET-SPACE OF A FROBENIUS MANIFOLD AND HIGHER-GENUS GROMOV-WITTEN INVARIANTS

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To Yuri I. Manin on his birthday

1. Introduction

The theory of genus zero Gromov-Witten invariants associates to a compact symplectic manifold $X$ a Frobenius manifold $H$ (also known as the small phase space of $X$) whose underlying flat manifold is the cohomology space $H^*(X, \mathbb{C})$. Higher genus Gromov-Witten invariants give rise to a sequence of generating functions $F_{g}^X$, one for each genus $g > 0$; these are functions on the large phase space

$$H_\infty = H^*(X \times \mathbb{C}P^\infty, \mathbb{C}).$$

The manifold $H_\infty$ has a rich geometric structure: it is the jet-space of curves in the Frobenius manifold $H$. (This identification is implicit in Dubrovin [9].)

Frobenius manifolds also arise naturally in a number of other geometric situations, such as singularity theory and mirror symmetry (see for example Dubrovin [9] and Manin [27]). The Frobenius manifolds of Gromov-Witten theory carry an additional geometric structure, a fundamental solution, given by the formula

$$\Theta^b_a(z) = \delta^b_a + \sum_{k=0}^\infty z^{k+1} \langle \tau_k(O_a)O^b \rangle_X.$$ 

In this paper, we formulate the differential equations satisfied by the potentials $F_{g}^X$, such as topological recursion relations and the Virasoro constraints, in an intrinsic fashion, that is, in such a way that the equations do not depend on the choice of fundamental solution. This effort is rewarded by a closer relationship between the resulting theory and the geometry of moduli spaces of stable curves.

A consequence of our analysis is the proof of a conjecture of Eguchi and Xiong [17]. (Here, we use the summation convention with respect to indices $a, b, \ldots$.) Introduce the coordinates

$$u^a = \partial \langle (O^a)_X \rangle_0^1.$$
on $H$, along with their derivatives
\[ \partial^k u^a = \partial^{k+1} \langle \langle \mathcal{O}^a \rangle \rangle_0, \]
which form a coordinate system on $H_\infty$; here, $\partial$ is the vector field on $H_\infty$ given by differentiation with respect to the puncture variable $t_0$.

**Theorem 1.1.** If $g > 0$, the Gromov-Witten potential $F^X_g$ has the form
\[
F^X_g = \sum_{n=0}^{3g-3} \frac{1}{n!} \sum_{k_i > 0, k_1 + \cdots + k_n \leq 3g-3} f_{k_1 \cdots k_n}^{a_1 \cdots a_n} (u^a, \partial u^a) \partial^{k_1+1} u^{a_1} \cdots \partial^{k_n+1} u^{a_n}.
\]

The coefficients $f_{k_1 \cdots k_n}^{a_1 \cdots a_n} (u^a, \partial u^a)$ are symmetric in the indices $(k_i, a_i)$, and homogeneous of degree $(2g-2) - (k_1 + \cdots + k_n + n)$ in the variables $\partial u^a$.

This theorem generalizes a well-known formula for the case of pure gravity: by the Kontsevich-Witten theorem, the functions $f_{a_1 \cdots a_n}^{k_1 \cdots k_n} (u, \partial u)$ are coefficients of the Gelfand-Dikii polynomials (Hamiltonians of the KdV hierarchy), and the theorem may be reinterpreted as a result in the theory of integrable hierarchies. Of course, it may be easily proved in this case by direct methods from the definition of the Gelfand-Dikii polynomials. Dubrovin and Zhang [13] have greatly generalized this observation, showing that Theorem 1.1 holds, for quite different reasons, in the theory of integrable systems.

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2. The jet-space of a Dubrovin manifold

2.1. Dubrovin connections. A large part of the theory of Frobenius manifolds does not require the existence of a metric or Euler vector field. In this section, we introduce the geometric structure which is the essential part of the definition of a Frobenius manifold, a Dubrovin connection.
As in Dubrovin [10], we work with connections on the cotangent bundle; of course, there is a bijection between connections on the tangent and cotangent bundles. Let
\[ \nabla : \Gamma(M, T^* M) \to \Omega^1(M, T^* M) \]
be a connection on the cotangent bundle of a manifold \( M \). Denote by
\[ \nabla_X = \iota(X) \nabla : \Gamma(M, T^* M) \to \Gamma(M, T^* M) \]
the operation of covariant differentiation along a vector field \( X \) on \( M \).

The connection \( \nabla \) is torsion-free if the torsion \( T \in \Omega^2(M, TM) \) vanishes, and flat if the curvature \( R \in \Omega^2(M, \text{End}(T^* M)) \) vanishes. A flat manifold is a manifold \( M \) with torsion-free and flat connection \( \nabla \) on the cotangent bundle \( T^* M \). A flat coordinate chart \( M \supset U \to \mathbb{R}^N \) on a manifold is a coordinate chart \( u^a : U \to M \) such that the one-forms \( du^a \) are parallel:
\[ \nabla(du^b) = 0. \]

A manifold is flat if and only if it has an atlas of flat coordinate charts.

A pencil of torsion-free affine connections is a family of connections \( \nabla^z \) depending on a complex parameter \( z \) such that
\[ \nabla^z = \nabla + z \mathcal{A}, \]
where \( \nabla = \nabla^{z=0} \) and \( \mathcal{A} \in \Omega^1(M, \text{End}(T^* M)) \). A Dubrovin connection is a pencil \( \nabla^z \) of torsion-free affine connections which is flat for all values of \( z \), together with a flat vector field \( e \) such that \( \mathcal{A}_e \) is the identity. A Dubrovin manifold \( (M, \nabla^z, e) \) is a manifold together with a Dubrovin connection.

Given a Dubrovin connection \( \nabla^z \), denote the vector field \( (\mathcal{A}_X)^* Y \) by \( X \circ Y \); this defines a product on the tangent bundle of \( M \). Denote the components of the tensor \( \mathcal{A} \) in flat coordinates \( (u^a) \) by \( \mathcal{A}^{c}_{ab} \):
\[ \partial_a \circ \partial_b = \mathcal{A}^{c}_{ab} \partial_c. \]

**Proposition 2.1.** The data \( (\nabla^z, e) \) form a Dubrovin connection if and only if the following conditions are satisfied:

1) the affine connection \( \nabla \) is flat;
2) the product \( X \circ Y \) is commutative and associative:
\[ (X \circ Y) \circ Z = X \circ (Y \circ Z); \]
3) the vector field \( e \) is flat, \( \nabla e = 0 \), and is an identity for the product \( X \circ Y \):
\[ e \circ X = X \circ e = X; \]
4) \( [\nabla_X, \mathcal{A}_Y] - [\nabla_Y, \mathcal{A}_X] = \mathcal{A}_{[X,Y]}. \)
Proof. If $\nabla$ is torsion-free, the commutativity of the product $X \circ Y$ is equivalent to the vanishing of the torsion of $\nabla^z$.

The curvature $R^z(X,Y)$ of the connection $\nabla^z$ is a quadratic polynomial in $z$:

$$\left[\nabla^z_X, \nabla^z_Y\right] - \left[\nabla^z_Y, \nabla^z_X\right] - \nabla^z_{[X,Y]} = R(X,Y) + z([\nabla_X, A_Y] - [\nabla_Y, A_X] - A_{[X,Y]}) + z^2[A_X, A_Y].$$

The constant term vanishes if and only if $\nabla$ is flat, the linear term if and only if 4) holds, and the quadratic term if and only if $X \circ Y$ is associative. □

The questions which we address in this paper are of local character; for this reason, we will work in the neighbourhood of a basepoint $p \in M$ of our Dubrovin manifold $(M, \nabla^z, e)$. Let $(u^a)$ be a flat coordinate system on $M$ centred at $p$: that is, $u^a(p) = 0$.

Since the identity vector field $e$ is flat, we may assume that the flat coordinate system $(u^a)$ is chosen in such a way that $e$ is differentiation with respect to $u^e$, where $e$ is one of the indices labelling the coordinates: in other words, $e = \partial/\partial u_e = \partial^e$.

The one-form

$$\omega = \text{Tr}(A_a)du^a \in \Omega^1(M)$$

plays a prominent role in the theory of Dubrovin manifolds; following Hertling [22], we call it the socle one-form.

2.2. Fundamental solutions. A fundamental solution of a Dubrovin manifold is a power series

$$\Theta = I + \sum_{n=0}^{\infty} z^{n+1} \Theta_n \in \Gamma(M, \text{End}(T^*M))[z]$$

such that

$$\nabla \cdot \Theta(z) = \Theta(z) \cdot \nabla^z.$$

This is equivalent to the sequence of equations

$$\nabla_X \Theta_n = \Theta_{n-1} A_X.$$

Since the endomorphism $\Theta_0$ plays a special role, we introduce the notation $\mathcal{M} = \Theta_0$. The case $n = 0$ of 2.2 shows that $\nabla_X \mathcal{M} = A_X$. In particular,

$$\partial_a \mathcal{M}_b^e = \delta_a^b.$$

It follows that $\mathcal{M}_e^a = u^a + c^a$, where $c^a$ is a constant.

Proposition 2.2. Let $p$ be a point of a Dubrovin manifold $(M, \nabla^z, e)$, and let $(u^a)$ be a system of flat coordinates which vanish at $p$. Then there exists a fundamental solution $\Theta(z)$ in a neighbourhood of $p$ such that $\mathcal{M}_e^a = u^a$. 
Proof. We construct endomorphisms $\Theta_n$ of the cotangent bundle $T^*M$ inductively in $n$, starting with $\Theta_{-1} = I$. To carry out the induction, we must show that the one-form $\Theta_{n-1}A \in \Omega^1(M, \text{End}(T^*M))$ is exact. To do this, we use condition 4) of Proposition 2.1: we have
\[
\nabla(\Theta_{n-1}A) = ([\nabla_a, \Theta_{n-1}A_b] - [\nabla_b, \Theta_{n-1}A_a]) \, du^a \wedge du^b
= (\Theta_{n-1}A_{[\partial_a, \partial_b]} + \Theta_{n-2}[A_a, A_b]) \, du^a \wedge du^b = 0.
\]
In particular, the existence of $\Theta_0$ is equivalent to condition 4) of Proposition 2.1, while granted the existence of $\Theta_0$, the existence of $\Theta_1$ is equivalent to the associativity of the product $X \circ Y$. \( \square \)

If $\Theta(z)$ and $\tilde{\Theta}(z)$ are two fundamental solutions of the Dubrovin connection $\nabla^z$, then $\tilde{\Theta}(z)^{-1}\Theta(z)$ is a flat section of $\Gamma(M, \text{End}(T^*M))[z]$, that is,
\[
\nabla(\tilde{\Theta}(z)^{-1}\Theta(z)) = 0.
\]
Conversely, if $\rho(z)$ is a flat section of $\Gamma(M, \text{End}(T^*M))[z]$ such that $\rho(0) = I$ and $\Theta(z)$ is a fundamental solution, then $\Theta(z)\rho(z)$ is again a fundamental solution.

2.3. The jet-space of a Dubrovin manifold. Let $n$ be a natural number and let $M$ be a manifold. An $n$-jet in $M$ is a map from the variety $\text{Spec}(\mathbb{C}[t]/(t^{n+1}))$ to $M$.

In local coordinates $(u^a)$, we may write an $n$-jet as
\[
u^a(t) = \sum_{k=0}^n \frac{t^k}{k!} u_k^a + O(t^{n+1}).
\]
In particular, $u_0^a$ are the coordinates of the origin of the jet, and $u_k^a \partial_a \in T_{u(0)}M$ is the velocity of the jet at $t = 0$.

The space of $n$-jets $J^nM$ in $M$ is a fibre bundle over $M$ whose fibres are affine spaces of dimension $n \dim M$. For example, a 0-jet is the same as a point of $M$, and a 1-jet is the same as a tangent vector; thus $J^0M = M$ and $J^1M = TM$. If $m < n$, denote the projection from $J^nM$ to $J^mM$ defined by reduction mod $t^{m+1}$ by $\rho_{n,m}$; we write $\rho_n : J^nM \to M$ instead of $\rho_{n,0}$.

Let $J^\infty M$ be the inverse limit
\[
\lim_{\longleftarrow n} J^nM.
\]
There is a fibration $\rho_{\infty,n}$ from $J^\infty M$ to $J^nM$.

If $\mathcal{V}$ is a vector bundle on $M$, denote the space of sections of the vector bundle $\rho_n^*\mathcal{V}$ on $J^nM$ by $\mathcal{V}_n$; in particular, the space of sections
\[
\mathcal{V}_\infty = \Gamma(J^\infty M, \rho_\infty^*\mathcal{V}) = \bigcup_{n=0}^\infty \mathcal{V}_n
\]
is filtered by subspaces
\[ \mathcal{V}_0 = \Gamma(M, \mathcal{V}) \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n \subset \mathcal{V}_\infty. \]
In the case where \( \mathcal{V} \) is the trivial bundle \( \mathcal{O} \), we see that \( \mathcal{O}_n \) is the algebra of functions on \( J^n M \).

Given a coordinate system \(( u^a )\) on \( M \), define vector fields
\[ \partial_{\ell,a} = \frac{\partial}{\partial u^a_\ell}, \quad 0 \leq \ell \leq n, \]
on the jet-space \( J^n M \). The differential operator
\[ \partial = \sum_{\ell=0}^\infty u^a_{\ell+1} \partial_{\ell,a} \]
is a vector field on \( J^\infty M \); formally speaking, \( \partial \) generates translation along the parameter \( t \) of the jet. Note that \( u^a_\ell = \partial^k u^a \), where \( u^a = u^a_0 \).

Another important vector field on the jet-space \( J^\infty M \) is the dilaton vector field
\[ \mathcal{Q} = \sum_{\ell=0}^\infty \ell u^a_\ell \partial_{\ell,a}, \]
which counts the coordinate \( u^a_\ell = \partial^\ell u^a \) with weight \( \ell \).

We now specialize to the case where \( M \) is a Dubrovin manifold. The Dubrovin connection \( \nabla^z \) on \( T^* M \) pulls back by \( \rho_\infty \) to a connection \( \nabla^z \) on \( \rho_\infty^* T^* M \). If \( X \) is a vector field on the jet-space \( J^\infty M \), denote by \( X \) the covariant derivative \( \nabla_X \) on \( \rho_\infty^* T^* M \) and its associated tensor bundles. For example, we write \( \partial \) instead of \( \nabla_\partial \).

Given a vector field \( X \) on \( J^\infty M \), define an endomorphism \( A_X \) of the bundle \( \rho_\infty^* \text{End}(T^* M) \) by the formula
\[ A_X = X(M). \]
This definition is independent of the fundamental solution \( \Theta(z) \), and consistent with the definition of \( A_X \) in the case that \( X \) is a vector field on \( M \).

We have the formula
\[ X(\Theta(z)) = z\Theta(z)A_X \in \text{End}(T^* M)_{\infty}. \]

If \( M \) has basepoint \( p \) and \( n > 0 \), let \( \pi \) be the basepoint of \( J^n M \) such that \( \pi(t) = p + te + O(t^{n+1}) \). At \( \pi \), the endomorphism \( \mathcal{X} = A_\partial \in \text{End}(T^* M)_{\infty} \) equals the identity; it follows that for \( n > 0 \), the open subset \( J^n_0 M \subset J^n M \) on which \( \mathcal{X} \) is invertible is a neighbourhood of \( \pi \in J^n M \). If \( \mathcal{V} \) is a vector bundle on \( M \), denote the space of sections of \( \rho^n_0 \mathcal{V} \) over \( J^n_0 M \) by \( \mathcal{V}_{n,0} \). In particular, we have
\[ \mathcal{V}_{\infty,0} = \bigcup_{n=0}^\infty \mathcal{V}_{n,0}. \]
2.4. A frame of vector fields on the jet-space. Let \((M, \nabla^z, e)\) be a Dubrovin manifold with basepoint \(p\), and let \(\Theta(z)\) be a fundamental solution such that the flat coordinates \(M^a = u^a\) vanish at \(p\). Consider the following noncommutative analogues of the Faà di Bruno polynomials:

\[ P_n(z) = z^{-1}\Theta^{-1}(z)\partial^n\Theta(z) \in \text{End}(T^*M)_n. \]

For example, we have \(P_0(z) = z - 1\) and \(P_1(z) = X\). In general, we see that for \(n > 0\),

\[ P_n(z) = (\partial + zX)^{n-1}X = \sum_{j=0}^{n-1} z^j P_{n,j}, \]

where \(P_{n,j} \in \text{End}(T^*M)_{n-j}\). The polynomials \(P_n(z)\) are intrinsic; they do not depend on the choice of fundamental solution.

Let \(\partial_\ell = du^a \partial_{\ell,a}\). We now introduce a generating function for vector fields on \(J^\infty M\), by the formula

\[ \sigma(z) = \sum_{k=0}^{\infty} z^k \sigma_k = \sum_{\ell=0}^{\infty} P_{\ell+1}(z)\partial_\ell : O_\infty \to T^*M[[z]]_\infty. \]

For example,

\[ \sigma_0 = \sum_{\ell=0}^{\infty} \partial^\ell X \partial_\ell, \quad \sigma_1 = \sum_{\ell=1}^{\infty} \sum_{k=1}^{\ell} \partial^{\ell-k}(X \partial^{k-1}X) \partial_\ell. \]

Observe that \([\partial + zX, \sigma(z)] = 0\); it follows that \([\partial, \sigma_k] = -X\sigma_{k-1}\). Together with the formula

\[ \sigma_a(z) u^b = X_a^b, \]

this characterizes \(\sigma(z)\).

If \(X\) is vector field on \(M\), define vector field \(\sigma_{k,X}\) on \(J^\infty M\) by the generating function

\[ \sigma_X(z) = \langle X, \sigma(z) \rangle = \sum_{k=0}^{\infty} z^k \sigma_{k,X}. \]

We abbreviate the vector fields \(\sigma_{k,\partial_\ell}\) to \(\sigma_{k,a}\). In particular, \(\sigma_{0,e} = \partial\).

Given of sequences \(k = (k_1, \ldots, k_n)\) and \(\ell = (\ell_1, \ldots, \ell_n)\), we say that \(k \geq \ell\) if \(k_i \geq \ell_i\) for all \(i\), and that \(k > \ell\) if in addition \(k_i > \ell_i\) for at least one \(i\).

Proposition 2.3. A function \(f \in O_\infty = O_{\infty,0}\) lies in \(O_{\ell,0}\) if and only if \(\sigma(z)f\) is a polynomial in \(z\) of degree at most \(\ell\).

More generally, suppose \(K = (K_1, \ldots, K_n)\) is a sequence of integers such that \(K_i > 1\) for all \(i\), and suppose that \(\sigma_{k_1} \ldots \sigma_{k_n} f = 0\) for all \(k \geq K\); then \(\partial_{k_1} \ldots \partial_{k_n} f = 0\) for all \(k \geq K\).
Proof. Since \( \sigma_j \mathcal{X} = 0 \) for \( j > 1 \), we see that for \( k \geq K \),

\[
\sigma_{k_1,a_1} \cdots \sigma_{k_n,a_n}
= (\mathcal{X}^{k_1+1})_{a_1} \cdots (\mathcal{X}^{k_n+1})_{a_n} \partial_{k_1,b_1} \cdots \partial_{k_n,b_n} + \sum_{\ell > k} a(k,\ell) \partial_{\ell_1} \cdots \partial_{\ell_n}.
\]

Since \( \mathcal{X} \) is invertible on \( J_0^\infty M \), it follows that

\[
\partial_{k_1,a_1} \cdots \partial_{k_n,a_n}
= (\mathcal{X}^{-k_1-1})_{a_1} \cdots (\mathcal{X}^{-k_n-1})_{a_n} \sigma_{k_1,b_1} \cdots \sigma_{k_n,b_n} + \sum_{\ell > k} a(k,\ell) \sigma_{\ell_1} \cdots \sigma_{\ell_n}.
\]

Thus, \( \sigma_{k_1} \cdots \sigma_{k_n} f = 0 \) for \( k \geq K \) if and only if \( \partial_{k_1} \cdots \partial_{k_n} f = 0 \) for \( k \geq K \). \( \square \)

2.5. An affine structure on the jet-space. We now show how the choice of a fundamental solution \( \Theta(z) \) on a Dubrovin manifold \((M,\nabla^z,e)\) gives rise to an affine structure on the jet-space \( J_0^\infty M \), that is, a frame of the tangent bundle \( TJ_0^\infty M \) consisting of commuting vector fields. This affine structure identifies the formal neighbourhood of \( \partial \) with the large phase space of Gromov-Witten theory.

Introduce the generating function of vector fields on \( J^\infty M \),

\[
\tau(z) = \sum_{k=0}^{\infty} z^k \tau_k = \Theta(z) \sigma(z) = z^{-1} \sum_{\ell=0}^{\infty} \partial_{\ell+1} \Theta(z) \partial_{\ell}.
\]

The introduction of these vector fields is motivated by the fact that they are evolutionary:

\[
[\partial, \tau(z)] = 0.
\]

Lemma 2.4. \([\mathcal{Q}, \sigma(z)] = \sigma(z)\) and \([\mathcal{Q}, \tau(z)] = \tau(z)\)

Proof. Since \( \mathcal{Q} \Theta(z) = 0 \), it suffices to prove that \([\mathcal{Q}, \sigma(z)] = \sigma(z)\). Since \([\mathcal{Q}, \partial] = \partial\), it follows that \( \mathcal{Q} \mathcal{P}_\ell = \ell \mathcal{P}_\ell \), and hence that

\[
[\mathcal{Q}, \sigma(z)] = \sum_{\ell=0}^{\infty} [\mathcal{Q}, \mathcal{P}_{\ell+1}(z) \partial_{\ell}] = \sum_{\ell=0}^{\infty} ((\ell + 1) \mathcal{P}_{\ell+1}(z) \partial_{\ell} - \ell \mathcal{P}_{\ell+1}(z) \partial_{\ell}) = \sigma(z).
\]

If \( X \) is a vector field on \( M \), let \( \tau_{k,X} \) be the vector field on \( J^\infty M \) defined by the generating function

\[
\tau_X(z) = (X, \tau(z)) = \sum_{k=0}^{\infty} z^k \tau_{k,X},
\]
and let \( \tau_{k,a} = \tau_{k,\partial_a} \). Note that \( \tau_0 = \sigma_0 \); in particular, \( \tau_{0,c} = \partial_c \). Also, observe that if \( f \in \mathcal{O}_0 \) is a function only of the coordinates \( u^a \) on the Frobenius manifold \( M \), then
\[
\tau_{0,a} f = \sigma_{0,a} f = \mathcal{X}_a^b \partial_b f.
\]
In particular, we see that
\[
(2.3) \quad \partial_a f = (\mathcal{X}^{-1})^b_a \tau_{0,b} f.
\]

**Proposition 2.5.** The vector fields \( \tau_{k,a} \) form a frame of the tangent bundle of \( J_0^\infty M \).

**Proof.** We have
\[
\sigma_{k,a} = (\mathcal{X}^{k+1})_a^b \partial_{k,b} + \sum_{\ell > k} a(k,\ell) \partial_\ell,
\]
and the result follows, since \( \mathcal{X} \) is invertible on \( J_0^\infty M \). \( \Box \)

The vector fields \( \tau_{k,a} \) were shown by Dubrovin [9] to be the commuting flows of an integrable hierarchy. In this paper, these vector fields play an auxiliary role: the vector fields \( \sigma_{k,a} \) are a more natural frame for the tangent bundle of \( J_0^\infty M \), because they are intrinsic, that is, independent of the fundamental solution \( \Theta(z) \). For us, the fact that the vector fields \( \sigma_{k,a} \) do not commute is of secondary importance.

**Proposition 2.6.** \([\tau_a(z), \tau_b(y)] = 0\)

**Proof.** We have \( \mathcal{A}_{\sigma_a(y)} = \mathcal{A}_{\sigma_0,a} = \mathcal{W}_a = \mathcal{X} \mathcal{A}_a \); in other words, the components \( \mathcal{W}_{ab}^c \) of \( \mathcal{W}_a \) equal \( \mathcal{X}_a \mathcal{X}_c^b \), and are symmetric in \( a \) and \( b \). Since \( [\partial, \tau(z)] = 0 \), we see that
\[
[\tau_a(y), \tau_b(z)] = \sum_{k=0}^{\infty} \partial^{k+1}(z^{-1} \tau_a(y) \Theta_b^c(z) - y^{-1} \tau_b(z) \Theta_a^c(y)) \partial_k,c.
\]

\[
= \sum_{k=0}^{\infty} \partial^{k+1}(z^{-1} \Theta_a^{c'}(y) \sigma_{a'}(y) \Theta_b^c(z) - y^{-1} \Theta_b^{c'}(z) \sigma_{a}(z) \Theta_a^c(y)) \partial_k,c.
\]

\[
= \sum_{k=0}^{\infty} \partial^{k+1}(\Theta_a^{c'}(y) \Theta_b^{c'}(z) (\mathcal{W}_{a'b'}^c - \mathcal{W}_{b'a'}^c)) \partial_k,c = 0.
\]

**Corollary 2.7.** We have \([\sigma_a(y), \sigma_b(z)] = \mathcal{W}_{ab}^c(y \sigma_c(y) - z \sigma_c(z)); in other words,\)
\[
[\sigma_{k,a}, \sigma_{\ell,b}] = \begin{cases} -\mathcal{W}_{ab}^c \sigma_{\ell-1,c}, & k = 0, \\ 0, & k > 0. \end{cases}
\]
Proof. Since $[\tau_a(y), \tau_b(z)] = [\Theta^a_{a'}(y)\sigma_{a'}(y), \Theta^b_{b'}(z)\sigma_{b'}(z)]$ vanishes, we see that

$$\Theta^a_{a'}(y)\Theta^b_{b'}(z)[\sigma_{a'}(y), \sigma_{b'}(z)] = \Theta^b_{b'}(z)(\sigma_{b'}(y)\Theta^a_{a'}(y))\sigma_{a'}(y) - \Theta^a_{a'}(y)(\sigma_{a'}(y)\Theta^b_{b'}(z))\sigma_{b'}(z) = \Theta^a_{a'}(y)\Theta^b_{b'}(z)(y\mathcal{W}_{a'}^c\sigma_c(y) - z\mathcal{W}_{b'}^c\sigma_c(z)) \quad \Box$$

2.6. Flat coordinates on the jet-space. The results of this section are taken from [20]. The main result of this section is Theorem 6.1 of Coates and Givental [6]; it is interesting to compare this with Proposition 2.6 of Coates and Givental [6].

Since the vector fields $\tau_{k,a}$ are in involution, there is a coordinate system $t_k^a$ centered at $\pi \in J^\infty_0 M$ such that $\tau_{k,a} = \frac{\partial}{\partial t_k^a}$ and $t_k^a(\pi) = 0$; we call these the flat coordinates on the jet-space. Of course, these coordinates do not lie in $\mathcal{O}_{\infty,0}$; rather, they lie in the completion $\tilde{\mathcal{O}}$ of $\mathcal{O}_{\infty}$ at $\pi$, and they are only coordinates in a formal neighbourhood of $\pi$. This completion $\tilde{\mathcal{O}}$ may be identified with the algebra $\mathbb{C}[t_k^a \mid k \geq 0]$.

**Proposition 2.8.** Let $\Theta(z)$ be a fundamental solution on the Dubrovin manifold $(M, \nabla^z, e)$. There is a unique generating function

$$t(z) = \sum_{k=-\infty}^{\infty} z^{-k-1} t_k \in TM[z, z^{-1}]_{\infty,0}$$

such that $\Theta t(z) = -t(z)$ and

$$\tau(y)t(z) = \Theta^1(y)\Theta^{-1}(z) = \sum_{k=0}^{\infty} y^k z^{-k-1} \Theta(y)\Theta^{-1}(z). \quad (2.4)$$

For $k \geq 0$, we have $t_k^a = t_k^a - \delta_{k,1}\delta^a_c$.

**Proof.** We start with a lemma of Dubrovin [21].

**Lemma 2.9.** Let

$$\Omega(y, z) = \sum_{k,\ell=0}^{\infty} y^k z^\ell \Omega_{k,\ell}$$

be the generating function characterized by the equation

$$\quad (y - z)\Omega(y, z) = \Theta(y)\Theta^{-1}(z) - I. \quad (2.5)$$

Then $\tau_{k,a}\Omega_{\ell,m}^c = \tau_{\ell,b}\Omega_{k,m}^c$. 

**Proof.** We must prove that $\tau_{a}(x)\Omega_{b}^c(y, z) = \tau_{b}(y)\Omega_{a}^c(x, z)$ or equivalently, that

$$(x - z)\tau_{a}(x)(\Theta(y)\Theta^{-1}(z))_c^c = (y - z)\tau_{b}(y)(\Theta(x)\Theta^{-1}(z))_a^c.$$ 

But

$$\tau_{a}(x)(\Theta(y)\Theta^{-1}(z)) = (y - z)\Theta(y)\mathcal{A}_{a}(x)\Theta^{-1}(z).$$
Thus, it suffices to show that \( (\Theta(y)A_{\tau_a(x)})^c_b = (\Theta(x)A_{\tau_b(y)})^c_a \); this follows from the formula
\[
(\Theta(y)A_{\tau_a(x)})^c_b = \Theta_a^e(x)\Theta_b^d(y)A_{\sigma_{0,a}}^e_d = \Theta_a^e(x)\Theta_b^d(y)W_{d}^c,
\]
and the symmetry \( W_{ab}^c = W_{ba}^c \). □

To show the existence of \( t(z) \), we must solve the system of equations
\[
(2.6) \quad \tau_{j,a}t^b_k = \delta_{j,k}\delta_{d_a} - \Omega_{j,-k-1|a}^b,
\]
subject to the constraints \( Qt_k = t_k \). For \( k \geq 0 \), this system has the unique solution
\[
t^a_k = t^a_k - \delta_{k,1}\delta^a_e.
\]
For \( k < 0 \), the system is integrable by Lemma 2.9. Since \([Q,t(z)] = \tau(z)\) and \( Q\Omega(y,z) = 0 \), we see that \( Qt_k + t_k \) is a constant; replacing \( t_k \) by \(-Qt_k\), we obtain the desired solution. □

The vector fields \( Q \) and \( e \) have simple expressions in flat coordinates on the large phase space.

**Proposition 2.10.** The dilaton vector field \( Q \) and the identity vector field \( e \) are the residues of the generating functions \(-\langle t(z), \tau(z) \rangle\) and \(-\langle t(z), z\tau(z) \rangle\) respectively.

**Proof.** By Lemma 2.4, there are constants \( c^a_k \) such that
\[
Q = \sum_{k=0}^{\infty}(c^a_k - t^a_k)\tau_{k,a}.
\]
By the definition of \( \tau(z) \), we have
\[
\tau_{k,a}u^b_n = (\Theta_{k-n-1}^b a^{n+1})^b_a.
\]
Applying the equation for \( Q \) to the function \( u^b_n \) and evaluating at \( \pi \), we see that
\[
\delta_{n,1}\delta^b_e = \sum_{k=0}^{\infty}\Theta_{k-n-1,a}^b(\pi)c^a_k.
\]
This system of linear equations is upper triangular, with diagonal entries equal to 1, hence has a unique solution: it is easily checked that this solution is \( c^a_k = \delta_{k,1}\delta^a_e \), since \( \Theta^a_e(\pi) = u^a(\pi) = 0 \).

The proof of the formula for \( e \) in flat coordinates is similar. Since \( e(\Theta(z)) = z\Theta(z) \) and \([\partial, e] = 0 \), we see that
\[
[e, \tau(z)] = z\sum_{\ell=0}^{\infty}\partial^{\ell+1}e(\Theta(z))\partial_\ell = z\tau(z).
\]
In other words, \( [e, \tau_{k,a}] = \tau_{k-1,a} \). It follows that there are constants \( c_k^a \) such that
\[
e = \sum_{k=1}^{\infty} (c_k^a - t_k^a) \tau_{k-1,a}.
\]
Applying this equation to the function \( u_n^b \) and evaluating at \( \pi \), we see that
\[
\delta_{n,0} \delta_{e} = \sum_{k=1}^{\infty} \Theta_{k-n-2,a}(\pi) c_k^a.
\]
Again, this system has a unique solution \( c_k^a = \delta_{k,1} \delta_{e} \).

Just as the modification \( \sigma = \Theta(z)^{-1} \tau(z) \) of \( \tau(z) \) is intrinsic, so the modification \( s = \Theta^*(z)t(z) \) of \( t(z) \) is intrinsic. This modification was introduced in \([20]\), where it was denoted \( z^{-1}G^*(-z^{-1}) \).

**Theorem 2.11.** Let \((M, \nabla^z, e)\) be a Dubrovin manifold. The generating function
\[
s(z) = \Theta^*(z)t(z) = \sum_{k=-\infty}^{\infty} z^{-k-1} s_k \in TM[z^{-1}]_{\infty,0}
\]
is intrinsic, that is, independent of the fundamental solution \( \Theta(z) \), and
\[
s_k = \begin{cases} 
-\langle e, (\nabla^{-1} \partial)^k \nabla^{-1} \rangle, & k > 0, \\
0, & k \leq 0.
\end{cases}
\]

**Proof.** We first show that \( s_0 = 0 \). We have
\[
s_0^a = t_0^a + \sum_{k=1}^{\infty} t_k^a \Theta_{k-1,b}^a - \Theta_{0,e}^a.
\]
Since \( \Omega(y,0) = y^{-1}(\Theta(y) - I) \), we see that \( \Omega_{k,0} = \Theta_k \), and hence that \( \tau_k t_{-1} = -\Theta_k \). It follows that
\[
s_0^a = t_0^a + \left( \partial - \sum_{k=1}^{\infty} t_k^a \tau_{k-1,b} \right) t_{-1}^a = t_0^a + e(t_{-1}^a) = 0.
\]
A special case of (2.4) is \( \partial t(z) = z^{-1} \langle e, \Theta^{-1}(z) \rangle \); it follows that
\[
(2.7) \quad \partial s(z) = z^{-1} e + \nabla^*s(z).
\]
Taking the residue, we see that \( 0 = e + \nabla^*s_1 \), while taking the coefficient of \( z^{-k-1} \), we see that \( \partial s_k = \nabla^*s_{k+1} \) for \( k > 1 \); this establishes the formula for \( s_k, k \geq 0 \).

In order to prove that \( s_k \) vanishes for \( k < 0 \), we need two lemmas.

**Lemma 2.12.** A power series \( f \) in the coordinates \( t_k^a \) such that \( \partial f = ef = 0 \) is constant.
Proof. Since
\[(\partial - e)f = \sum_{k=1}^{\infty} t_k^* \tau_{k-1,a} f\]
vanishes, it follows that \(f\) is a constant. (See [20], Section 3, for more details of the proof.) \(\square\)

Lemma 2.13. \((e + z)t(z) = 0\)

Proof. The equations \([e, \tau(y)] = y\tau(y)\) and \((e + z - y)(\Theta(y)\Theta^{-1}(z)) = 0\) show that
\[(z - y)\tau(y)(e + z)t(z) = (z - y)(e + z - y)(\tau(y)t(z)) = (e + z - y)(\Theta(y)\Theta^{-1}(z)) = 0.\]
In other words, \(\tau(y)(e + z)t(z) = 0\), and hence \((e + z)t(z) \in \mathbb{C}[z, z^{-1}]\). Composing this equation with the dilaton vector field \(Q\), we see that \((e + z)t(z) = 0.\)

It follows from the equations \(Qt(z) = -t(z)\) and \((e + z)t(z) = 0\) that have \(Qs(z) = -s(z)\) and \(es(z) = 0.\)

Suppose that \(s_{-k}\) vanishes, for \(k > 0\). Taking the coefficient of \(z^{k+1}\) in \((2.7)\), we see that \(\partial s_{-k-1} = 0.\) Since we know that \(es_{-k-1} = 0\), it follows from Lemma 2.12 that \(s_{-k-1}\) is constant. Since \(Qs_{-k-1} = -s_{-k-1}\), we conclude that \(s_{-k-1}\) vanishes. Thus \(s_{-k} = 0\) for all \(k > 0\), by induction on \(k.\)

Using the series \(s(z)\), we may now rewrite Proposition 2.10 in an intrinsic fashion:
\[Q = -\sum_{k=0}^{\infty} \langle s_k, \sigma_k \rangle, \quad e = -\sum_{k=0}^{\infty} \langle s_{k+1}, \sigma_k \rangle.\]
Apart from being intrinsic, these formulas have the virtue that when applied to a function \(f \in \mathcal{O}_n\), they truncate to a sum over \(0 \leq k \leq n.\)

Relationship to prior results. The Dubrovin connection was introduced by Dubrovin [9]. He also initiated the study of the jet-space of a Frobenius manifold (which he calls the loop-space). Sections 2.4 and 2.5 are taken from our joint work with Eguchi and Xiong [14], and Section 2.6 from [20], Section 4. (Note that the generating function \(t(z)\) used here is related to the generating function \(\theta(\zeta)\) of [20] by the formula \(t(z) = z^{-1} \theta^*(-z^{-1}).\)
3. The jet-space of a conformal Dubrovin manifold

3.1. Conformal Dubrovin manifolds. A linear vector field on a flat manifold is a vector field \( E \) such that \( \nabla E \in \Gamma(M, \text{End}(TM)) \) is flat. In flat coordinates \((u^a)\), there are constants \( A^a_b \) and \( B^a \) such that

\[
E = (A^a_b u^b + B^a) \partial_a.
\]

An Euler vector field \( E \) for a Dubrovin connection is a linear vector field such that

\[
\left[ E, X \circ Y \right] = \left[ E, X \right] \circ Y + X \circ \left[ E, Y \right] + X \circ Y.
\]

Equivalently, \(
\left[ L_E, A_X \right] = A \left[ E, X \right] + A_X,
\)

where \( L_E \) is the Lie derivative of the vector field \( E \) acting on one-forms.

A conformal Dubrovin manifold \((M, \nabla^z, e, E, r)\) is a Dubrovin manifold \((M, \nabla, e)\) together with an Euler vector field \( E \) and a real number \( r \).

Let \( \mu \) be the endomorphism of the cotangent bundle defined by the formula

\[
\mu = 1 - \frac{r}{2} + \nabla_E - L_E.
\]

The adjoint of \( \mu \) is the endomorphism of the tangent bundle given by the formula

\[
\mu^*(X) = -\nabla_E X + \left[ E, X \right] + (1 - \frac{r}{2}) X = (1 - \frac{r}{2}) X - \nabla X E.
\]

Let \( \mathcal{U} \) be the endomorphism \( \mathcal{U} = A_E \) of the cotangent bundle.

**Proposition 3.1.** Let \((M, \nabla^z, e, E, r)\) be a conformal Dubrovin manifold, and let \( \mathcal{H} \) be the bundle \( T^* M[z, z^{-1}] \). Let \( \delta_z \) be the endomorphism

\[
\delta_z = \partial_z + z^{-1}(\mu + \frac{r}{2}) + \mathcal{U}
\]

of \( \mathcal{H} \). Then \( \left[ \nabla^z, \delta_z \right] = 0 \).

**Proof.** If \( X \) is a vector field on \( M \), we have

\[
\left[ \nabla_X, \delta_z \right] = \left[ \nabla_X + z A_X, \partial_z + z^{-1} \mu + A_E \right]
= z^{-1} [\nabla_X, \mu] + (\left[ \nabla_X, A^X \right] - \left[ \nabla_X, A^E \right] + A_X + [L_E, A_X]) + z [A_X, A_E].
\]

Since \( E \) is linear, it follows that \( [\nabla_X, \mu] = 0 \); it is also clear that \( [A_E, A_X] = 0 \). By (3.1), we see that that

\[
[\nabla_X, A_E] = A [E, X] + A_X = [\nabla, A_X] - [\nabla_X, A_E] + A_X.
\]

Let \( \Theta(z) \) be a fundamental solution of the conformal Dubrovin manifold \((M, \nabla^z, e, E, r)\), and consider the conjugate of \( \delta_z \) by \( \Theta(z) \), defined by the formula

\[
\tilde{\delta}_z = \Theta(z) \cdot \delta_z \cdot \Theta^{-1}(z) = \partial_z + z^{-1} R(z),
\]

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where
\[ R(z) = \sum_{k=0}^{\infty} z^k R_k \in z\Gamma(M, \text{End}(T^*M))[z]. \]

In particular, \( R_0 = \mu + \frac{1}{2} \). It follows from Proposition 3.1 that \( R(z) \) is flat.

By the Jordan decomposition, we may write \( \mu \) as the sum of semisimple and nilpotent endomorphisms \( \mu_s \) and \( \mu_n \) of the cotangent bundle \( T^*M \); furthermore, both \( \mu_s \) and \( \mu_n \) are flat. (In many cases, such as the theory of Gromov-Witten invariants, \( \mu \) is semisimple.) Let \( \tau_\lambda \) be the projection onto the subbundle of \( T^*M \) on which \( \mu_s \) has eigenvalue \( \lambda \): we have
\[
I = \sum_{\lambda} \pi_\lambda, \quad \mu_s = \sum_{\lambda} \lambda \pi_\lambda.
\]

**Proposition 3.2.** There exists a fundamental solution such that \([\mu_s, R_k] = kR_k\).

**Proof.** If \( \lambda - \lambda' \neq k \), the endomorphism \( k - \text{ad}(\mu) \) is invertible on the space of endomorphisms of the form \( \pi_\lambda A \pi_{\lambda'} \): we have
\[
(k - \text{ad}(\mu))^{-1}(\pi_\lambda A \pi_{\lambda'}) = \sum_{i=0}^{\infty} (-\text{ad}(\mu))^i (k - \text{ad}(\mu_s)(\pi_\lambda A \pi_{\lambda'}))^{-i-1} = \sum_{i=0}^{\infty} (-\text{ad}(\mu))^i (\pi_\lambda A \pi_{\lambda'}) \frac{1}{(k - \lambda + \lambda')^{i+1}},
\]
where of course, the sum is finite since \( \mu_n \) is nilpotent.

Let \( \Theta(z) \) and \( \tilde{\Theta}(z) = \Theta(z)\rho(z) \) be a pair of fundamental solutions, where
\[ \rho(z) = I + \sum_{k=1}^{\infty} z^k \rho_k \]
is a flat endomorphism of \( \mathcal{H} = T^*M[z, z^{-1}] \). Let \( \partial_z + z^{-1}R(z) \) and \( \partial_z + z^{-1}\tilde{R}(z) \) be the associated endomorphisms of \( \mathcal{H} \). Then we have
\[ \tilde{R}_k = (k - \text{ad}(\mu))G_k + \sum_{\ell=1}^{k-1} (G_\ell R_{k-\ell} - \tilde{R}_{k-\ell}G_\ell) + R_k. \]

Setting
\[ G_k = - \sum_{\lambda - \lambda' \neq k} (k - \text{ad}(\mu))^{-1} \left( \sum_{\ell=1}^{k-1} \pi_\lambda (G_\ell R_{k-\ell} - \tilde{R}_{k-\ell}G_\ell) \pi_{\lambda'} + \pi_\lambda R_k \pi_{\lambda'} \right), \]
we obtain a fundamental solution \( \tilde{\Theta}(z) \) such that \([\mu_s, \tilde{R}_k] = k\tilde{R}_k \).

**Definition 3.1.** A **conformal** fundamental solution is a fundamental solution satisfying the condition \([\mu_s, R_k] = kR_k\).
Extracting the constant term of the equation $\Theta(z) \cdot \delta_z - \tilde{\delta}_z \cdot \Theta(z) = 0$, we see that

\[(3.2)\]
\[\mathcal{U} = [\mu, M] + M + R_1.\]

Applying $\partial$ to this equation, we see that

\[(3.3)\]
\[\partial \mathcal{U} = [\mu, \mathcal{X}] + \mathcal{X} .\]

3.2. Vector fields on the jet-space of a conformal Dubrovin manifold. Let $L_k$ be the vector field on the jet-space $J^\infty_0 M$ of a conformal Dubrovin manifold $(M, \nabla^z, e, E, r)$ defined by the formula

\[L_k = \text{Res}_{z=0} \langle s(z), z\delta^{k+1}_z \sigma(z) \rangle.\]

For example, $L_{-1} = -e$. By the work of Eguchi, Hori and Xiong [15] and Dubrovin and Zhang [12], these vector fields play a fundamental role in the theory of Gromov-Witten invariants. In this section, we give simple intrinsic formulas for them.

**Theorem 3.3.** Restricted to $\mathcal{O}_0 \subset \mathcal{O}_\infty$, the vector field $L_k$ equals $-E^{\sigma(k+1)}$.

We have

\[\text{Res}_{z=0} \langle e, \sigma^{k+1}_z \tau(z) \rangle ,\]

and

\[(3.4)\]
\[L_k = -\sum_{\ell=0}^{\infty} \left( \partial^\ell \langle e, \mathcal{U}^{k+1} \rangle + \sum_{j=1}^{\ell} \partial^{\ell-j} \text{Res}_{z=0} \langle e, \sigma^{k+1}_z (\partial + z\mathcal{X})^{\ell-j} \rangle \right) \partial \ell .\]

**Proof.** By Theorem 2.11, $s_\ell$ vanishes for $\ell \leq 0$. Since $\sigma(z) u^a = \mathcal{X} du^a$, it follows that

\[L_k u^a = \text{Res}_{z=0} \langle s(z), \sigma^{k+1}_z \mathcal{X} du^a \rangle = \langle s_1, L^{k+1} \mathcal{X} du^a \rangle = \langle \mathcal{X} s_1, L^{k+1} du^a \rangle = -\langle e, L^{k+1} du^a \rangle = -E^{\sigma(k+1)}(u^a).\]

Here, we have used that $[\mathcal{U}, \mathcal{X}] = 0$.

Choose a conformal fundamental solution $\Theta(z)$ on $M$. We may replace $s(z)$ and $\sigma(z)$ in the formula for $L_k$ by $\Theta^*(z) t(z)$ and $\Theta^{-1}(z) \tau(z)$, and rewrite $L_k$ as

\[L_k = \text{Res}_{z=0} \langle \Theta^*(z) t(z), z\sigma^{k+1}_z \Theta^{-1}(z) \tau(z) \rangle = \text{Res}_{z=0} \langle t(z), z\tilde{\sigma}^{k+1}_z \tau(z) \rangle.\]

From this formula, and the fact that $[\partial, \tilde{\delta}_z] = 0$ and $[\partial, \tau(z)] = 0$, we see that

\[\text{Res}_{z=0} \langle \partial t(z), z\tilde{\sigma}^{k+1}_z \tau(z) \rangle = \text{Res}_{z=0} \langle \Theta^*(z)^{-1} e, \tilde{\delta}^{k+1}_z \tau(z) \rangle = \text{Res}_{z=0} \langle e, \delta^{k+1}_z \sigma(z) \rangle.\]
If $X$ is a vector field on the jet space $J^\infty M$, we have

$$X(u_\ell^a) = \partial^\ell (X u^a) - \sum_{j=1}^\ell \partial^{j-1} \cdot [\partial_j, X] u_{\ell-j}^a.$$ 

In the special case $X = L_k$, this gives

$$L_k u_\ell^a = -\partial^\ell (L_k u^a) - \sum_{j=1}^\ell \partial^{j-1} \text{Res}_{z=0} \langle e, \delta_z^{k+1} \sigma(z) u_{\ell-j}^a \rangle,$$

from which (3.4) follows. □

Using the noncommutative Faà di Bruno polynomials $P_n(z)$, we may rewrite (3.4) in the elegant form

$$L_k = -\sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} \partial^j \text{Res}_{z=0} \langle e, \delta_z^{k+1} P_{\ell-j}(z) \rangle \partial_\ell.$$

The following corollary is an immediate consequence.

**Corollary 3.4.** The vector field $L_k$ preserves the filtration of $O_\infty$ by subalgebras $O_j$.

As an illustration of the utility of the formula (3.4), let us calculate $L_k X$; we will need this calculation later in the discussion of Virasoro constraints in genus 1.

**Proposition 3.5.**

$$L_k X = \sum_{\ell=0}^{k} [X, U^\ell \mu U^{k-\ell}] - \sum_{\ell=0}^{k} (U^\ell \mu U^{k-\ell})^a A_a X - \frac{3}{2} (k+1) X U^k$$

**Proof.** We have $X = u_1^a A_a$, hence $\partial_0 X = \partial A_a$, $\partial_1 X = A_a$, while $\partial_{n, a} X = 0$ for $n > 1$. It follows that

$$L_k X = -\langle e, U^{k+1} \rangle \partial_0 X - \partial \langle e, U^{k+1} \rangle \partial_1 X - \text{Res}_{z=0} \langle e, \delta_z^{k+1} X \rangle \partial_1 X$$

$$= -\partial (U^{k+1})^a e A_a) - \text{Res}_{z=0} (\delta_z^{k+1} X)^a e A_a$$

$$= -\partial (U^{k+1}) - \text{Res}_{z=0} (\delta_z^{k+1} X)^a e A_a,$$

since $[A_X, A_Y] = 0$. By (3.3), we have

$$\partial (U^{k+1}) = \sum_{\ell=0}^{k} U^\ell (X + [\mu, X]) U^{k-\ell} = -(k+1) X U^k + \sum_{\ell=0}^{k} [X, U^\ell \mu U^{k-\ell}].$$
We also have

\[\text{Res}_{z=0}(\delta_z^{k+1})_e^a A_a \mathcal{X} = \sum_{\ell=0}^{k} (\mathcal{U}^\ell (\mu + \frac{1}{2}) \mathcal{U}^{k-\ell})_e^a A_a \mathcal{X} = \frac{1}{2} (k + 1)(\mathcal{U}^k)_e^a A_a \mathcal{X} + \sum_{\ell=0}^{k} (\mathcal{U}^\ell \mu \mathcal{U}^{k-\ell})_e^a A_a \mathcal{X}.\]

The proposition follows, since \((\mathcal{U}^k)_e^a A_a = \mathcal{U}^k\). □

**Corollary 3.6.** In terms of the socle one-form \(\omega\) of (2.4),

\[\mathcal{L}_k \log \det(\mathcal{X}) = -\sum_{\ell=0}^{k} \langle e, \mathcal{U}^\ell (\mu + \frac{3}{2}) \mathcal{U}^{k-\ell} \omega \rangle.\]

**Proof.** We see from Proposition 3.5 that

\[\mathcal{L}_k \log \det(\mathcal{X}) = \text{Tr}(\mathcal{X}^{-1} \mathcal{L}_k \mathcal{X}) = \sum_{\ell=0}^{k} \text{Tr}[\mathcal{X}, \mathcal{X}^{-1} \mathcal{U}^\ell \mu \mathcal{U}^{k-\ell}] \rightleftharpoons -\sum_{\ell=0}^{k} (\mathcal{U}^\ell \mu \mathcal{U}^{k-\ell})_e \text{Tr}(A_a) - \frac{3}{2} (k + 1) \text{Tr}(\mathcal{U}^k).\]

The corollary follows, since \(\text{Tr}[\mathcal{X}, \mathcal{X}^{-1} \mathcal{U}^\ell \mu \mathcal{U}^{k-\ell}] = 0\) and \(\text{Tr}(\mathcal{U}^k) = \langle e, \mathcal{U}^k \omega \rangle\). □

The vector fields \(\mathcal{L}_k\) form a Lie algebra isomorphic to the Lie algebra of vector fields on the line, under the correspondence which associates to \(\mathcal{L}_k\) the vector field \(-z \delta_z^{k+1}\). These relations are sometimes called the Virasoro relations.

**Theorem 3.7.** \([\mathcal{L}_j, \mathcal{L}_k] = (j - k)\mathcal{L}_{j+k}\)

**Proof.** We have

\[
\left[ \langle t(y), y\delta_y^{j+1} \tau(y) \rangle, \langle t(z), z\delta_z^{k+1} \tau(z) \rangle \right]
\]

\[= \langle t(y), y\delta_y^{j+1} \cdot \frac{\Theta(y)\Theta^{-1}(z)}{z-y} \cdot z\delta_z^{k+1} \tau(z) \rangle - \langle t(z), z\delta_z^{k+1} \cdot \frac{\Theta(z)\Theta^{-1}(y)}{y-z} \cdot y\delta_y^{j+1} \tau(z) \rangle
\]

\[= \langle s(y), y\delta_y^{j+1} \cdot \frac{1}{z-y} \cdot z\delta_z^{k+1} \sigma(z) \rangle - \langle s(z), z\delta_z^{k+1} \cdot \frac{1}{y-z} \cdot y\delta_y^{j+1} \sigma(z) \rangle
\]

\[= \frac{1}{z-y} \langle (\delta_y^*)^{j+1} y s(y), z\delta_z^{k+1} \sigma(z) \rangle - \frac{1}{y-z} \langle (\delta_z^*)^{k+1} z s(z), y\delta_y^{j+1} \sigma(z) \rangle.
\]
The coefficients of $y^\ell$ and $z^\ell$ in $(\delta^*_y)^{j+1}ys(y)$ and $(\delta^*_z)^{k+1}zs(z)$ vanish for $\ell \geq 0$; it follows that

$$\text{Res}_{y=0} \frac{(\delta^*_y)^{j+1}ys(y)}{z-y} = (\delta^*_z)^{j+1}zs(z), \quad \text{Res}_{z=0} \frac{(\delta^*_z)^{k+1}zs(z)}{y-z} = (\delta^*_y)^{k+1}ys(y),$$

and hence that

$$[L_j, L_k] = \text{Res}_{y=0} \text{Res}_{z=0} \left[ \langle t(y), y\delta^*_y\tau(y) \rangle, \langle t(z), z\delta^*_z\tau(z) \rangle \right] - \text{Res}_{y=0} \text{Res}_{z=0} \left[ \langle z\delta^*_y\sigma(y), y\delta^*_y\sigma(y) \rangle, \langle z\delta^*_z\sigma(z), y\delta^*_z\sigma(z) \rangle \right] = \text{Res}_{z=0} \langle s(z), (z\delta^*_{y,j+1} - z\delta^*_z)(z) \rangle.$$

The result follows, since $z\delta^*_{y,j+1} - z\delta^*_{z,k+1} = z^2\delta^*_{y,j+k+2} + (j+1)z\delta^*_{y,j+k+1}$. □

As a corollary, we see that

$$[L^o_j, L^o_k] = (k-j)L^o_{j+k-1}.$$

This relation was first conjectured for Frobenius manifolds by Dubrovin and Zhang [12], and proved by Hertling and Manin [23], using their theory of $F$-manifolds.

**Relationship to prior results.** The theory of conformal Dubrovin connections is due to Dubrovin. The study of the Virasoro vector fields on the jet-space of a Frobenius manifold was initiated by Dubrovin and Zhang [12]; the presentation in Section 3.2 is based on the methods of Section 5 of [20] (for the action of these vector fields on functions of the underlying Dubrovin manifold) and of [14] (for the commutator with the vector field $\partial$).

### 4. Topological recursion relations for Gromov-Witten invariants and the jet-space

The genus 0 Gromov-Witten invariants of a smooth projective variety $X$ define a conformal Dubrovin manifold with conformal fundamental solution, called the small phase space of $X$. In this section, we recall the construction of the small phase space, following Dubrovin [9]. We then show the way in which the theory of topological recursion relations (Eguchi and Xiong [17], Kontsevich and Manin [25], and our own work [19]) takes a simpler form when rewritten using the intrinsic geometry of the jet-space of this Dubrovin manifold; this is the main new result of this paper.

#### 4.1. Stable maps and Gromov-Witten invariants.**

Let us recall the definition of the Gromov-Witten invariants of a projective manifold over $\mathbb{C}$; see Cox and Katz [5] and Manin [27] for more detailed expositions. The
definition which we outline is the one which works in the setting of alge-
braic geometry: Gromov-Witten invariants have also been de-

fined for compact symplectic manifolds, using entirely different tech-
quies. The Gromov-Witten invariants of a projective manifold reflect the in-

tersection theory of Kontsevich’s moduli spaces of stable maps $\overline{M}_{g,n}(X,\beta)$, whose definition we now recall.

Let $X$ be a projective manifold of dimension $r$. A **prestable map**

$$(f : C \rightarrow X, z_1, \ldots, z_n)$$

of genus $g \geq 0$ and degree $\beta \in H_2(X,\mathbb{Z})$ with $n$ marked points consists of the following data:

1) a connected projective curve $C$ of arithmetic genus $g = h^1(C,\mathcal{O}_C)$, whose only singularities are ordinary double points,
2) $n$ distinct smooth points $(z_1, \ldots, z_n)$ of $C$;
3) an algebraic map $f : C \rightarrow V$, such that the degree of $f$, that is, the cycle $f_![C] \in H^2(X,\mathbb{Z})$, equals $\beta$.

If $\tilde{C}$ is the normalization of $C$, the special points in $\tilde{C}$ are the inverse images of the singular and marked points of $C$. (Note that the degree of $f : C \rightarrow X$ equals 0 if and only if its image is a single point.)

A prestable map $(f : C \rightarrow X, z_1, \ldots, z_n)$ is **stable** if it has no infinitesimal automorphisms fixing the marked points. The condition of stability is equivalent to the following: each irreducible component of $\tilde{C}$ of genus 0 on which $f$ has degree 0 has at least 3 special points, while each irreducible component of $\tilde{C}$ of genus 1 on which $f$ has degree 0 has at least 1 special point. In particular, there are no stable maps of genus $g$ and degree 0 with $n$ marked points unless $2(g-1)+n > 0$.

The moduli stack of $n$-pointed stable maps $\overline{M}_{g,n}(X,\beta)$ is the classifying stack for stable maps of genus $g$ and of degree $\beta$; it is a complete Deligne-Mumford stack, though not in general smooth (Behrend and Manin [4]). The definition of Gromov-Witten invariants is based on the study of $\overline{M}_{g,n}(X,\beta)$.

Let $\text{ev}_i : \overline{M}_{g,n}(X,\beta) \rightarrow X, 1 \leq i \leq n$, be evaluation at the $i$th marked point:

$$\text{ev}_i : \overline{M}_{g,n}(X,\beta) \ni (f : C \rightarrow X, z_1, \ldots, z_n) \mapsto f(z_i) \in X.$$
In the case $N = 1$, this construction yields the universal curve
\begin{equation}
\pi = \pi_{n,1} : \overline{M}_{g,n+1}(X, \beta) \to \overline{M}_{g,n}(X, \beta).
\end{equation}
The fibre of $\pi$ at a stable map $(f : C \to X, z_1, \ldots, z_n)$ is the curve $C$; $f = \text{ev}_{n+1} : \overline{M}_{g,n+1}(X, \beta) \to X$ is the universal stable map.

The sheaf $R^1\pi_*f^*TX$ on $\overline{M}_{g,n}(X, \beta)$ is called the obstruction sheaf of $\overline{M}_{g,n}(X, \beta)$. If it vanishes, the Grothendieck-Riemann-Roch theorem implies that the stack $\overline{M}_{g,n}(X, \beta)$ is smooth, of dimension
\begin{equation}
\text{vdim} \overline{M}_{g,n}(X, \beta) = (3 - r)(g - 1) + \int_\beta c_1(X) + n;
\end{equation}
this is called the virtual dimension of $\overline{M}_{g,n}(X, \beta)$. This hypothesis is rarely true; however, there is an algebraic cycle $[\overline{M}_{g,n}(X, \beta)]_{\text{virt}} \in H_{\text{vdim} \overline{M}_{g,n}(X, \beta)}(\overline{M}_{g,n}(X, \beta), \mathbb{Q})$, the virtual fundamental class, which stands in for $[\overline{M}_{g,n}(X, \beta)]$ in the general case. The most important property of the virtual fundamental class is the formula
\begin{equation}
[\overline{M}_{g,n+1}(X, \beta)]_{\text{virt}} = \pi^*[\overline{M}_{g,n}(X, \beta)]_{\text{virt}}
\end{equation}
for integration over the fibres of the stabilization map.

If the obstruction sheaf $R^1\pi_*f^*TX$ is locally free of rank $e$, the moduli stack $\overline{M}_{g,n}(X, \beta)$ is smooth of dimension $\text{vdim} \overline{M}_{g,n}(X, \beta) + e$, and the virtual fundamental class satisfies the equation
\begin{equation}
[\overline{M}_{g,n}(X, \beta)]_{\text{virt}} = c_e(R^1\pi_*f^*TX) \cap [\overline{M}_{g,n}(X, \beta)].
\end{equation}

The universal curve (4.1) over $\overline{M}_{g,n}(X, \beta)$ has $n$ canonical sections
\[\sigma_i : \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n+1}(X, \beta),\]
corresponding to the $n$ marked points of the curve $C$. Consider the line bundles
\[\Omega_i = \sigma_i^*\omega, \quad 1 \leq i \leq n,\]
where $\omega = \omega_{\overline{M}_{g,n+1}(X, \beta)}/\overline{M}_{g,n}(X, \beta)$ is the relative dualizing sheaf of the universal curve. Thus, the fibre of $\Omega_i$ at the stable map $f : C \to X$ equals the cotangent line $T^*_{z_i}C$ of $C$ at the $i$th marked point $z_i$ of $C$. Let
\[\Psi_i = c_1(\Omega_i) \in H^2(\overline{M}_{g,n}(X, \beta), \mathbb{Z})\]
be the Chern class of $\Omega_i$.

The Gromov-Witten invariants of a projective manifold are defined by integrating tautological cohomology classes against the virtual fundamental
class $\widehat{\mathcal{M}}_{g,n}(X,\beta)^{\text{virt}}$. Given rational cohomology classes $x_1,\ldots,x_n$ on $X$, we define the Gromov-Witten invariants by the formula

$$\langle \tau_{k_1}(x_1)\ldots\tau_{k_n}(x_n) \rangle^X_{g,\beta} = \int_{\widehat{\mathcal{M}}_{g,n}(X,\beta)^{\text{virt}}} \Psi_1^{k_1} \ldots \Psi_n^{k_n} \ev_1^* x_1 \ldots \ev_n^* x_n \in \mathbb{Q}.$$  

Perhaps the most important property of the Gromov-Witten invariants is that they are invariant under deformation of the variety $X$.

The **Novikov ring** $\Lambda$ of $X$ is the commutative graded algebra consisting of all formal sums

$$\sum_{\beta \in H_2(X,\mathbb{Z})} a_{\beta} q^\beta$$

such that for all $C > 0$, the set of $\beta \in H_2(X,\mathbb{Z})$ such that $a_{\beta} \neq 0$ and $\int \beta \omega < C$ is finite; the product is defined by $q^\beta q^{\beta_2} = q^{\beta_1 + \beta_2}$, and the grading is defined by $|q^\beta| = -2c_1(X) \cap \beta$. For example, for $\mathbb{C}P^d$, we have $\Lambda = \mathbb{Q}[q]$, where $|q| = -2(d + 1)$. By working over the Novikov ring, we may combine the Gromov-Witten invariants in different degrees into a single generating function:

$$\langle \tau_{k_1}(x_1)\ldots\tau_{k_n}(x_n) \rangle^X_{g} = \sum_{\beta \in H_2(X,\mathbb{Z})} q^\beta \langle \tau_{k_1}(x_1)\ldots\tau_{k_n}(x_n) \rangle^X_{g,\beta}.$$  

**4.2. The small and large phase spaces.** Let

$$\{ \gamma_a \in H^{p_a,q_a}(X) \mid a \in A \}$$

be a homogeneous basis of the Dolbeault cohomology of $X$, such that for a distinguished element $e \in A$, $\gamma_e = 1$. The small phase space $H$ of $X$ is the formal neighbourhood of 0 in the vector space $H^\bullet(X,\mathbb{C})$. Denote by $u^a$ the coordinates on $H$ dual to the basis $\gamma_a$. Let $\eta$ be the flat metric on $H$ associated to the Poincaré form on $H^\bullet(X,\mathbb{C})$, with components

$$\eta_{ab} = \int_X \gamma_a \cup \gamma_b,$$

and by $\eta^{ab}$ the inverse matrix of $\eta_{ab}$.

Dubrovin [9] shows that $H$ is a conformal Dubrovin manifold, and construct from the genus 0 Gromov-Witten invariants of $X$ a conformal fundamental solution $\Theta(z)$ on $H$. The flat connection $\nabla$ of this Dubrovin manifold is the Levi-Civitá connection associated to the flat metric $\eta$. We defer the definition of the remaining geometric structures on $H$, namely the tensor $A$, the identity and Euler vector fields $e$ and $E$ and the fundamental solution $\Theta$, to below.

The large phase space $H_\infty$ of $X$ is the formal neighbourhood of 0 in the vector space $H^\bullet(X \times \mathbb{C}P^\infty,\mathbb{C})$; it is a formal manifold with coordinates $\{ t_k^a \mid a \in A, k \geq 0 \}$, where $t_k^a$ has degree $-p_a - 2k$. (If $X$ has cohomology...
of odd degree, $H$ and $H_\infty$ are actually supermanifolds; this detail does not materially change the theory.)

The genus $g$ potential $F_g$ of $X$ is the function on the large phase space $H_\infty$ given by the formula

\begin{equation}
F_g^X = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1 \ldots k_n} t_{k_1}^{a_1} \ldots t_{k_n}^{a_n} \langle \tau_{k_1, a_1} \ldots \tau_{k_n, a_n} \rangle_g,
\end{equation}

where we write $\tau_{k,a}$ in place of $\tau_k(\gamma_a)$.

Denote the constant coefficient vector field $\partial/\partial t_k^0$ on $H_\infty$ by $\tau_{k,a}$, and $\tau_{0,e}$ by $\partial$. The partial derivatives of the potential $F_g$ are denoted

\begin{equation}
\langle \langle \tau_{k_1, a_1} \ldots \tau_{k_n, a_n} \rangle \rangle_g = \partial_{\tau_{k_1, a_1}} \ldots \partial_{\tau_{k_n, a_n}} F_g^X.
\end{equation}

In particular, $\langle \langle \tau_{k_1, a_1} \ldots \tau_{k_n, a_n} \rangle \rangle_g$ is the value at $0 \in H_\infty$ of the function $\langle \langle \tau_{k_1, a_1} \ldots \tau_{k_n, a_n} \rangle \rangle_g$. Explicitly, we have

\begin{equation}
\langle \langle \tau_{k_1, a_1} \ldots \tau_{k_n, a_n} \rangle \rangle_g = \sum_{\beta \in H_2(X, \mathbb{Z})} q^\beta \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{b_1 \ldots b_N} t_{b_{k_n+1}} \ldots t_{b_{k_{n+N}}} \int_{\overline{M}_{g,n+N}(X, \beta)} \Psi_1^{\ell_1} \ldots \Psi_n^{\ell_n} \cdot \text{ev}_1^{*} \gamma_{a_1} \ldots \text{ev}_n^{*} \gamma_{a_n} \text{ev}_{n+1}^{*} \gamma_{b_1} \ldots \text{ev}_{n+N}^{*} \gamma_{b_N}.
\end{equation}

4.3. **Dual graphs and their associated cycles.** The moduli space of stable maps $\overline{M}_{g,n}(X, \beta)$ is stratified by boundary strata, which are most conveniently parametrized by dual graphs.

The dual graph $C(G)$ of a pointed prestable map $(f : C \to X, z_1, \ldots, z_n)$ is a graph $G = G(C)$ with one vertex $v$ for each component $C(v)$ of the normalization $\tilde{C}$ of the curve $C$, labelled by the genus $g(v)$ of this component and the degree $\beta(v)$ of the restriction $f(v) : C(v) \to X$ of $f$ to $C(v)$.

The edges of the dual graph $G(C)$ correspond to double points of the curve $C$; the two ends of an edge are attached to the vertices associated to the components on which the two branches of the double point lie. (If both branches lie in the same component of $C$, then the edge is a loop.)

Finally, to each marked point $z_i$ of the curve corresponds a leg of the graph, labelled by $i$, at the vertex corresponding to the component of $C$ on which $z_i$ lies (which is uniquely determined, since $z_i$ is a smooth point).

In drawing dual graphs, we denote vertices of genus 0 either by a solid circle $\bullet$ or leave them unmarked and vertices of genus $g > 0$ by $\circ$.

Define the genus $g(G)$ of a dual graph $G$ to be the sum of the genera $g(v)$ over the vertices $v \in \text{Vert}(G)$ of $G$ and the first Betti number of the graph $G$. Then $g(G)$ equals the arithmetic genus of the curve $C$. Define the degree
\( \beta(G) \) of a dual graph \( G \) to be the sum of the degrees \( \beta(v) \) over the vertices \( v \) of \( G \). The valence \( n(G) \) equals the number of external legs of \( G \).

A dual graph is stable if each vertex \( v \) such that \( \beta(v) = 0 \) satisfies the additional condition \( 2(g(v) - 1) + n(v) > 0 \); a pointed prestable curve is stable if and only if the associated dual graph is stable. The set of isomorphism classes of stable dual graphs \( G \) of fixed genus \( g(G) \), valence \( n(G) \), and degree \( \beta(G) \) is finite.

If \( G \) is a stable graph of genus \( g \), valence \( n \) and degree \( \beta \), let \( M(G) \subset \overline{\mathcal{M}}_{g,n}(X,\beta) \) be the moduli stack of stable maps with dual graph \( G \), and let \( \widetilde{\mathcal{M}}(G) \) be its closure in \( \overline{\mathcal{M}}_{g,n}(X,\beta) \). Let \( \widetilde{\mathcal{M}}(G) \) be the product

\[
\prod_{v \in \text{Vert}(G)} \mathcal{M}_{g(v),n(v)}(X,\beta(v)),
\]

with virtual fundamental class

\[
[\widetilde{\mathcal{M}}(G)]^{\text{virt}} = \prod_{v \in \text{Vert}(G)} [\mathcal{M}_{g(v),n(v)}(X,\beta(v))]^{\text{virt}}.
\]

There is a natural covering map

\[
\pi(G) : \widetilde{\mathcal{M}}(G) \longrightarrow \overline{\mathcal{M}}(G),
\]

with covering group \( \text{Aut}(G) \). Let \( [G] = [\overline{\mathcal{M}}(G)]^{\text{virt}} \) be the cycle

\[
[\overline{\mathcal{M}}(G)]^{\text{virt}} = \pi(G)_* [\widetilde{\mathcal{M}}(G)]^{\text{virt}} |_{\text{Aut}(G)}.
\]

In formulas, we will often symbolize the cycle \( [G] \) by the dual graph \( G \) itself.

A dual graph \( G \) with one edge determines a Cartier divisor \( D(G) \) supported by \( \overline{\mathcal{M}}(G) \). Axiom V for virtual fundamental classes in Behrend [3] implies that

\[
D(G) \cap [\overline{\mathcal{M}}_{g(G),n(G)}(X,\beta(G))]^{\text{virt}} = [G].
\]

The following formula is due to Witten [29].

**Proposition 4.1.** Let \( \pi : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta) \) be the universal curve \( \text{[1.1]} \). For \( 1 \leq i \leq n \), let \( D_i = D(G_i) \) be the Cartier divisor on \( \overline{\mathcal{M}}_{g,n+1}(X,\beta) \) associated to the dual graph

\[
G_i = \begin{array}{c}
\bullet \\
0 \\
i \\
n+1 \\
\beta
\end{array}
\]

Then on \( \overline{\mathcal{M}}_{g,n+1}(X,\beta) \), we have the formula \( \pi^* \Psi_i = \Psi_i + D_i \).

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4.4. The puncture, dilaton and divisor equations on the large phase space. For \( \omega \in H^2(X, \mathbb{C}) \), let \( R^b_0(\omega) \) be the matrix of multiplication by \( \omega \) on \( H(X) \):

\[
\omega \cup \gamma_a = R^b_0(\omega)\gamma_b.
\]

Combining Proposition 4.1 with (4.3), we obtain the puncture, divisor and dilaton equations:

\begin{align*}
(4.5) \quad & \langle \tau_{0,e}\tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \rangle^X_{g,\beta} = \sum_{i=1}^n \langle \tau_{k_1,a_1} \cdots \tau_{k_{i-1},a_{i-1}} \tau_{k_i,a_i} \cdots \tau_{k_n,a_n} \rangle^X_{g,\beta}, \\
(4.6) \quad & \langle \tau_0(\omega)\tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \rangle^X_{g,\beta} = \sum_{i=1}^n R^b_0(\omega)\langle \tau_{k_1,a_1} \cdots \tau_{k_{i-1},a_{i-1}} \tau_{k_i,a_i} \cdots \tau_{k_n,a_n} \rangle^X_{g,\beta} \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad + \int_\gamma \omega \cdot \langle \tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \rangle^X_{g,\beta}, \\
(4.7) \quad & \langle \tau_{1,e}\tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \rangle^X_{g,\beta} = (2g - 2 + n)\langle \tau_{k_1,a_1} \cdots \tau_{k_n,a_n} \rangle^X_{g,\beta}.
\end{align*}

Owing to the nonexistence of the stabilization maps \( \pi : \overline{\mathcal{M}}_{0,3}(X,0) \to \overline{\mathcal{M}}_{0,2}(X,0) \) and \( \pi : \mathcal{M}_{1,1}(X,0) \to \overline{\mathcal{M}}_{1,0}(X,0) \), we have the following exceptional cases:

\[
\begin{align*}
\langle \tau_{0,e}\tau_{0,a} \rangle^X_{0,0} &= \eta_{ab}, & \langle \tau_{1,e} \rangle^X_{1,0} &= \frac{1}{24}\chi(X), \\
\langle \tau_0(\omega)\tau_{0,a} \rangle^X_{0,0} &= R_{ab}(\omega), & \langle \tau_0(\omega) \rangle^X_{1,0} &= \frac{1}{24}\int_X \omega \cup c_{r-1}(X).
\end{align*}
\]

Introduce the vector fields

\[
e = -\sum_{k=0}^{\infty} t^a_{k+1} \partial_{k,a} = \partial - \sum_{k=0}^{\infty} t^a_{k+1} \partial_{k,a}, \quad Q = -\sum_{k=0}^{\infty} t^a_k \partial_{k,a} = \partial_{1,e} - \sum_{k=0}^{\infty} t^a_k \partial_{k,a}
\]

on the large phase space \( H_\infty \). The puncture and dilaton equations are equivalent to the differential equations

\[
e \mathcal{F}^X_g = \left\{ \begin{array}{ll}
\frac{1}{2} \eta^{ab} t^b_0, & g = 0, \\
0, & g > 0,
\end{array} \right. \quad Q \mathcal{F}^X_g = \left\{ \begin{array}{ll}
\frac{1}{24} \chi(X), & g = 1, \\
(2g - 2) \mathcal{F}^X_g, & g \neq 1,
\end{array} \right.
\]

for the Gromov-Witten potentials \( \mathcal{F}^X_g \).

We may identify the large phase space \( H_\infty \) with the jet space \( J^\infty_0 H \) of the small phase space \( H \). Consider the map \( u \) from the large phase space \( H_\infty \) to the small phase space \( H \) given by the formula

\[
u^a = \eta^{ab} \partial \langle (\tau_{0,b}) \rangle^X_0.
\]

Applying the vector field \( \partial_{0,b} \) to the genus 0 puncture equation \( e \mathcal{F}^X_0 = \frac{1}{2} \eta^{ab} t^b_0 \), we see that

\[
u^a = t^a_0 + \sum_{k=0}^{\infty} t^b_{k+1} \tau_{k,b} \nu^a.
\]
It follows that the map $u : H_\infty \to H$ is a submersion at 0; this submersion has a section, which identifies $H$ with the submanifold of $H_\infty$ along which the coordinates $t_k^a$, $k > 0$, vanish.

The following result shows that the jet coordinates $\partial^n u^a$ form a coordinate system on the large phase space, and that the origin of the large phase space corresponds to the basepoint of the jet-space $J^\infty_0 H$.

**Proposition 4.2.** We have $\partial^n u^a(0) = \delta_{n,1} \delta^a_e$, and $\tau_{k,b}(\partial^n u^a)(0) = \delta_{n,k} \delta^a_b$ if $k \leq n$.

**Proof.** We argue by induction on $n = 0$; we have already seen that the result is true for $n = 0$. Expanding the equation $e^a(u^a) = \delta_{n,1} \delta^a_e$, which holds for $n > 0$, we see that

\[(4.8) \quad \partial^n u^a = \delta_{n,1} \delta^a_e + \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \left( \sum_{k=0}^{\infty} t_k^b \tau_{k,b} \right)^{n-i} \partial^i u^a.\]

In particular, evaluating at $0 \in H_\infty$, we see that $\partial^n u^a(0) = \delta_{n,1} \delta^a_e$. Applying the vector field $\tau_{k,b}$ to (4.8) and evaluating at $0 \in H_\infty$, we see that

$$\tau_{k,b}(\partial^n u^a)(0) = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \tau_{k-n+i,b}(\partial^i u^a)(0).$$

The result now follows by the induction hypothesis. \qed

The following theorem is due to Hori \[24\]; we will only need it in genus 0, where it yields a construction of an Euler vector field on the small phase space.

**Theorem 4.3.** Let $\mathcal{L}_0$ be the vector field

$$\mathcal{L}_0 = \sum_{k=0}^{\infty} \left( k + p_a + \frac{1-r}{2} \right) t_k^a \tau_{k,a} + \sum_{k=0}^{\infty} R_a^b t_{k+1}^a \tau_{k,b}$$

$$= -\frac{1}{2} (3-r) \partial_{1,e} + \sum_{k=0}^{\infty} \left( k + p_a + \frac{1-r}{2} \right) t_k^a \tau_{k,a} - R_0^b \tau_{0,b} + \sum_{k=0}^{\infty} R_a^b t_{k+1}^a \tau_{k,b},$$

where $R_a^b$ is the matrix $R_a^b(c_1(X))$. We have

$$0 = \mathcal{L}_0 \mathcal{F}_g^X + \begin{cases} \frac{1}{2} R_{ab} t_0^a t_0^b, & g = 0, \\ \frac{1}{38} \int_X ((3-r)c_r(X) - 2c_1(X)c_{r-1}(X)), & g = 1, \\ 0, & g > 1. \end{cases}$$
Proof. The formula (4.2) for the dimension of the virtual fundamental class of \( \overline{M}_{g,n}(X, \beta) \) implies the following identity:

\[
\sum_{i=1}^{n} (p_{a_{i}} + k_{i}) \langle \tau_{k_{1}, a_{1}} \ldots \tau_{k_{n}, a_{n}} \rangle_{g, \beta}^{X} = \text{vdim} \overline{M}_{g,n}(X, \beta) \cdot \langle \tau_{k_{1}, a_{1}} \ldots \tau_{k_{n}, a_{n}} \rangle_{g, \beta}^{X} = (3 - r)(g - 1) + \int_{\beta} c_{1}(X) + n \langle \tau_{k_{1}, a_{1}} \ldots \tau_{k_{n}, a_{n}} \rangle_{g, \beta}^{X}.
\]

To eliminate the dependence on the genus \( g \), subtract \( \frac{1}{2}(3 - r) \) times the dilaton equation (4.6); after some rearrangement, this gives

\[
\sum_{i=1}^{n} (p_{a_{i}} + k_{i} + \frac{1}{2}) \langle \tau_{k_{1}, a_{1}} \ldots \tau_{k_{n}, a_{n}} \rangle_{g, \beta}^{X} = \frac{1}{2}(3 - r) \langle \tau_{1,e} \tau_{k_{1}, a_{1}} \ldots \tau_{k_{n}, a_{n}} \rangle_{g, \beta}^{X} + \int_{\beta} c_{1}(X) \cdot \langle \tau_{k_{1}, a_{1}} \ldots \tau_{k_{n}, a_{n}} \rangle_{g, \beta}^{X}.
\]

To eliminate the dependence on \( \beta \), apply the divisor equation (4.6) with \( \omega = c_{1}(X) \): this yields

\[
\sum_{i=1}^{n} \left( (p_{a_{i}} + k_{i} + \frac{1}{2}) \langle \tau_{k_{1}, a_{1}} \ldots \tau_{k_{n}, a_{n}} \rangle_{g, \beta}^{X} + \left. \frac{1}{2}(3 - r) \langle \tau_{1,e} \tau_{k_{1}, a_{1}} \ldots \tau_{k_{n}, a_{n}} \rangle_{g, \beta}^{X} \right) + \int_{\beta} c_{1}(X) \cdot \langle \tau_{k_{1}, a_{1}} \ldots \tau_{k_{n}, a_{n}} \rangle_{g, \beta}^{X}
\]

Taking into account the exceptional cases in degree 0, we obtain the theorem. \( \square \)

4.5. **Topological recursion relations.** A stable curve is a stable map with target a point; the moduli space \( \overline{M}_{g,n} = \overline{M}_{g,n}(\text{pt},0) \) of stable curves of genus \( g \) with \( n \) marked points is a smooth Deligne-Mumford stack of dimension \( 3g - 3 + n \), the Deligne-Mumford-Knudsen moduli stack of stable curves. Denote the line bundles \( \Omega_{i} \) on \( \overline{M}_{g,n} = \overline{M}_{g,n}(\text{pt},0) \) by \( \omega_{i} \), and their Chern classes \( c_{1}(\omega_{i}) \) by \( \psi_{i} \).

If \( n \) and \( N \) are nonnegative integers, let \( \overline{M}_{g,n,N}(X, \beta) \) be the moduli space of stable maps \( \overline{M}_{g,n+N}(\beta) \). If \( 2g - 2 + n > 0 \), the stabilization map

\[
\rho_{n,N}: \overline{M}_{g,n,N}(X, \beta) \rightarrow \overline{M}_{g,n}
\]

is the morphism which takes a stable map \( (f: C \rightarrow X, z_{1}, \ldots, z_{n}, x_{1}, \ldots, x_{N}) \) to the stabilization \( (C, z_{1}, \ldots, z_{n})^{\text{stab}} \). We abbreviate the pullback \( \rho_{n,N}^{*}\psi_{i} \) by \( \psi_{i} \).
Proposition 4.4. For $1 \leq i \leq n$, let $D_i = D(G_i)$ be the Cartier divisor on $\overline{M}_{g,n,N}(X,\beta)$ associated to the following dual graph:

\[
G_i = \begin{array}{c}
1 \cdots n \\
\downarrow \\
\beta \\
0 \\
i
\end{array}
\]

Then $\Psi_i = \psi_i + D_i$.

Proof. See the proof of Proposition 5 in [19]. □

Corollary 4.5.

\[
\frac{1}{1 - z\Psi_i} = \frac{1}{1 - z\psi_i} \left( \frac{1 + \frac{zD_i}{1 - z\psi_i}}{1 - z\Psi_i} \right)
\]

Proof. Divide the equation $1 - z\psi_i = 1 - z(\Psi_i - D_i)$ by $(1 - z\psi_i)(1 - z\Psi_i)$. □

From Proposition 4.3, we may derive many relations among Gromov-Witten invariants. For example, using the vanishing of $\psi_1^{k_1}\psi_2^{k_2}\psi_3^{k_3}$ on the zero-dimensional moduli space $\overline{M}_{0,3}$ when $k_1 > 0$, we obtain the genus 0 topological recursion relation

\[
\left< \left< \tau_{k_1,a_1} \tau_{k_2,a_2} \tau_{k_3,a_3} \right> \right>_0 = \eta^{AB} \left< \left< \tau_{k_1-1,a_1} \tau_{0,a} \right> \right>_0 \left< \left< \tau_{0,B} \tau_{k_2,a_2} \tau_{k_3,a_3} \right> \right>_0.
\]

The following result of Dijkgraaf and Witten [7], proved using the topological recursion relation in genus 0, may be viewed as an analogue of Theorem 1.1 in genus 0.

Proposition 4.6. The function $\left< \left< \tau_{k,a} \tau_{\ell,b} \right> \right>_0^X$ on the large phase space is the pullback of a function on the small phase space; that is,

\[
\left< \left< \tau_{k,a} \tau_{\ell,b} \right> \right>_0^X = \sum_{n=0}^{\infty} \frac{1}{n!} u^{a_1} \cdots u^{a_n} \left< \tau_{k,a} \tau_{\ell,b} \tau_{0,a_1} \cdots \tau_{0,a_n} \right>_0^X.
\]

Proof. Let $A_{k,a;\ell,b}$ and $B_{k,a;\ell,b}$ be the left and right-hand sides of (4.10). Since $u^a$ and $t_0^a$ are equal along $\mathcal{H} \subset \mathcal{H}_\infty$, it follows that $A_{k,a;\ell,b}$ and $B_{k,a;\ell,b}$ are too. We now calculate the derivatives of $A_{k,a;\ell,b}$ and $B_{k,a;\ell,b}$ with respect
to \( R_m^c \), \( m > 0 \), using (4.4). On the one hand,

\[
\tau_{m,c} B_{k,a;\ell,b} = (\tau_{m,c} u^a_0) \sum_{n=0}^{\infty} \frac{1}{n!} u^{a_1} \cdots u^{a_n} \langle \tau_{k,a} \tau_{\ell,b} \tau_{0,20} \tau_{0,a_1} \cdots \tau_{0,a_n} \rangle_0^X
\]

\[
= \eta^{AB} \langle \langle \tau_{m,c} \tau_{0,c} \tau_{0,A} \rangle_0^X \rangle^X_0 \sum_{n=0}^{\infty} \frac{1}{n!} u^{a_1} \cdots u^{a_n} \langle \tau_{k,a} \tau_{\ell,b} \tau_{0,B} \tau_{0,a_1} \cdots \tau_{0,a_n} \rangle_0^X
\]

\[
= \eta^{AB} \eta^{CD} \langle \langle \tau_{m-1,c} \tau_{0,c} \rangle_0^X \rangle^X_0 \langle \langle \tau_{0,D} \tau_{0,e} \tau_{0,A} \rangle_0^X \rangle^X_0 \sum_{n=0}^{\infty} \frac{1}{n!} u^{a_1} \cdots u^{a_n} \langle \tau_{k,a} \tau_{\ell,b} \tau_{0,B} \tau_{0,a_1} \cdots \tau_{0,a_n} \rangle_0^X
\]

\[
= \eta^{AB} \langle \langle \tau_{m-1,c} \tau_{0,A} \rangle_0^X \rangle^X_0 \tau_{0,B} B_{k,a;\ell,b}.
\]

On the other hand, we have

\[
\tau_{m,c} A_{k,a;\ell,b} = \langle \langle \tau_{k,a} \tau_{\ell,b} \tau_{m,c} \rangle_0^X \rangle^X_0 \eta^{AB} \langle \langle \tau_{m-1,c} \tau_{0,A} \rangle_0^X \rangle^X_0 \langle \langle \tau_{k,a} \tau_{\ell,b} \tau_{0,B} \rangle_0^X \rangle^X_0 \tau_{0,b} A_{k,a;\ell,b}.
\]

Induction in the order of vanishing of \( A_{k,a;\ell,b} - B_{k,a;\ell,b} \) in the variables \( \{ \ell^a_m | m > 0 \} \) shows that the two power series are equal. \( \square \)

4.6. The Dubrovin connection on \( \mathcal{H} \). Let \( \mathcal{M} \) be the endomorphism of the cotangent bundle of \( \mathcal{H} \) with components \( \mathcal{M}_a^b = \eta^{bc} \langle \langle \tau_{0,a} \tau_{0,c} \rangle_0^X \rangle^X_0 \); by Proposition 4.5 it is a function on \( \mathcal{H} \). We may now define the Dubrovin connection \( \nabla^z = \nabla + z A \) on the small phase space \( \mathcal{H} \): the tensor \( A \) is defined by the formula \( A_X = X(\mathcal{M}) \).

**Proposition 4.7.** \( A_X = X^{-1} \tau_{0,X} \mathcal{M} = (\tau_{0,c} X)^{-1}(\tau_{0,c} \mathcal{M}) \)

**Proof.** By (4.3), we have \( A_a = (X^{-1})^a_0 \tau_{0,b} \mathcal{M} = X^{-1} \tau_{0,a} \mathcal{M} \).

We now check that \( \nabla^z = \nabla + z A \) is a Dubrovin connection. The equation \( [\nabla_a, A_b] = [\nabla_b, A_a] \) follows from the fact that the tensor \( A \) is the covariant derivative of \( \mathcal{M} \). The equation \( [A_a, A_b] = 0 \) is equivalent to the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation

\[
\eta^{AB} \langle \langle \tau_{0,a} \tau_{0,b} \tau_{0,A} \rangle_0^X \rangle^X_0 \langle \langle \tau_{0,B} \tau_{0,c} \tau_{0,D} \rangle_0^X \rangle^X_0 = \eta^{AB} \langle \langle \tau_{0,a} \tau_{0,c} \tau_{0,A} \rangle_0^X \rangle^X_0 \langle \langle \tau_{0,B} \tau_{0,b} \tau_{0,d} \rangle_0^X \rangle^X_0.
\]

This equation is proved by applying the vector field \( \tau_{0,b} \) to the topological recursion relation (4.9):

\[
\langle \langle \tau_{1,a} \tau_{0,c} \tau_{0,d} \rangle_0 \rangle^X_0 = \eta^{AB} \langle \langle \tau_{0,a} \tau_{0,A} \rangle_0 \rangle^X_0 \langle \langle \tau_{0,B} \tau_{0,c} \tau_{0,d} \rangle_0 \rangle^X_0.
\]

This gives

\[
\eta^{AB} \langle \langle \tau_{0,a} \tau_{0,b} \tau_{0,A} \rangle_0 \rangle^X_0 \langle \langle \tau_{0,B} \tau_{0,c} \tau_{0,d} \rangle_0 \rangle^X_0
\]

\[
= \langle \langle \tau_{1,a} \tau_{0,b} \tau_{0,c} \tau_{0,d} \rangle_0 \rangle^X_0 - \eta^{AB} \langle \langle \tau_{0,a} \tau_{0,A} \rangle_0 \rangle^X_0 \langle \langle \tau_{0,B} \tau_{0,b} \tau_{0,c} \tau_{0,d} \rangle_0 \rangle^X_0;
\]
the left-hand side of this equation is symmetric in the indices $b$ and $c$, since
the right-hand side is, and this is precisely the WDVV equation.

The identity vector field $e$ of the Dubrovin connection is the constant vector field $\partial e$ in the direction of the coordinate $u^e$. The equation $\mathcal{A}_e = e \mathcal{M} = I$ is obtained by applying the differential operator $\tau_{0,a}^a t_0^b$ to the puncture equation $e \mathcal{F}_0^X = \frac{1}{2} \eta_{ab} t_0^a t_0^b = 0$.

4.7. **The Euler vector field on $H$.** Using Hori’s equation, we may construct an Euler vector field $E$ on $H$ which along with the Dubrovin connection $\nabla z$ and the identity vector field $e$ makes it into a conformal Dubrovin manifold. To see this, apply the differential operator $\tau_{0,b}^b \tau_{0,c}^c$ to the equation

$$L_0 F_{X_0} + \frac{1}{2} R_{ab} t_a^a t_b^b = 0;$$

we obtain

$$(4.11) \quad L_0 (\langle \tau_{0,b} \tau_{0,c} \rangle)_0^X + (p_b + p_c + 1 - r) \langle \tau_{0,b} \tau_{0,c} \rangle)_0^X + R_{bc} = 0.$$ 

In particular,

$$L_0 u^a + (1 - p_a) u^a + R_e^a = L_0 u^a + E u^a = 0,$$

where $E$ is the vector field

$$E = \sum_a (((1 - p_a) u^a + R_e^a) \partial_a).$$

Let $\mu$ be the endomorphism of the cotangent bundle of $H$ with components

$$\mu_b^a = \delta_b^a (p_a - r/2),$$

corresponding to the identification of the constant $r$ in the definition of the conformal Dubrovin manifold $M$ with the dimension of $X$. By (4.11), we see that

$$L_0 (\mathcal{M}) + [\mu, \mathcal{M}] + \mathcal{M} + R = 0.$$

Applying the vector field $\partial$ to (4.11), we see that $L_0 (\mathcal{X}) + [\mu, \mathcal{X}] + \frac{3-r}{2} \mathcal{X} = 0$, and hence that

$$L_0 (\mathcal{X}^{-1}) + [\mu, \mathcal{X}^{-1}] - \frac{3-r}{2} \mathcal{X}^{-1} = 0.$$

It follows that

$$E (\mathcal{A}_a) = -L_0 (X^{-1} \tau_{0,a} \mathcal{M})$$

$$= -L_0 (X^{-1}) \tau_{0,a} \mathcal{M} - X^{-1} \tau_{0,a} L_0 (\mathcal{M}) - X^{-1} [L_0, \tau_{0,a}] \mathcal{M}$$

$$= ([\mu, X^{-1}] \tau_{0,a} \mathcal{M} - \frac{3-r}{2} \mathcal{A}_a) + (X^{-1} \tau_{0,a} [\mu, \mathcal{M}] + \mathcal{A}_a) + (p_a + \frac{1-r}{2}) A_a$$

$$= [\mu, \mathcal{A}_a] + p_a \mathcal{A}_a.$$

In other words, $[E, A_{ab}^c] = (p_a + p_b - p_c) A_{ab}^c$; this implies (3.11), since on the one hand,

$$[E, \partial_a \circ \partial_b] = [E, A_{ab}^c \partial_c] = (E (A_{ab}^c) - (1 - p_c)) \partial_c.$$
while on the other hand,

\[ [E, \partial_a] \circ \partial_b + \partial_a \circ [E, \partial_b] + \partial_a \circ \partial_b = \left( -(1 - p_a) - (1 - p_b) + 1 \right) \mathcal{A}^c_{ab} \partial_c. \]

This establishes that \( E \) is an Euler vector field on the small phase space \( H \).

Observe that Hori’s vector field \( L_0 \) agrees with the vector field \( L_0 = \text{Res}(s(z), z\delta_z \sigma(z)) \) associated to the conformal Dubrovin manifold \((H, \nabla^z, e, E)\). Historically, Hori’s equation was an important step in the discovery of the Lie algebra of vector fields \( \mathcal{L}_k \).

4.8. The fundamental solution on \( H \). It is remarkable that the small phase space \( H \) has a canonical conformal fundamental solution; this was first shown by Dubrovin (see [9], Chapter 6).

Proposition 4.8. The matrix

\[
\Theta^b_a(z) = \delta^b_a + \eta^{bc} \sum_{k=0}^{\infty} z^{k+1} \langle \langle \tau_{k,a} \tau_0, b \rangle \rangle_0^X \]

is a conformal fundamental solution on \( H \).

Proof. The equation \( \partial_a \Theta_k = \Theta_{k-1} \mathcal{A}_a \) is an instance of the genus 0 topological recursion relation:

\[
\langle \langle \tau_{k,a} \tau_0, b \rangle \rangle_0^X = \eta^{AB} \langle \langle \tau_{k-1,a} \tau_0, A \rangle \rangle_0^X \langle \langle \tau_0, B \tau_0, b \rangle \rangle_0^X.
\]

It remains to show that the fundamental solution \( \Theta(z) \) is conformal. Applying the differential operator \( \tau_{k,a} \tau_0, b \) to Hori’s equation in genus 0, we see that

\[
\mathcal{L}_0 \langle \langle \tau_{k,a} \tau_0, b \rangle \rangle_0 + (k + 1 + p_a + p_b - r) \langle \langle \tau_{k,a} \tau_0, b \rangle \rangle_0 + \delta_{k,0} R_{ab} = 0.
\]

Multiplying by \( z^{k+1} \) and summing over \( k \), we see that

\[
E \Theta(z) = z \partial_z \Theta(z) + [\mu, \Theta(z)] + z R.
\]

On the other hand, \( E \Theta(z) = z \Theta(z) \mathcal{U} \), where \( \mathcal{U} = E \mathcal{M} \). From this equation, we see that the operator \( \tilde{\delta}_z \) equals

\[
\tilde{\delta}_z = \partial_z + z^{-1} (\mu + \frac{1}{2}) + R.
\]

Since \([\mu, R] = R\), we see that \( \Theta(z) \) is conformal; this equation is the statement that multiplication by \( c_1(X) \) in the Dolbeault cohomology raises degree by \( (1, 1) \). \( \square \)
4.9. Intrinsic formulation of topological recursion relations. We now come to the main result of this paper, an expression for the topological recursion relations among Gromov-Witten invariants in terms of the intrinsic vector fields $\sigma_{k,a}$ on the jet-space $J_0^\infty H$ of the small phase space $H$.

Introduce functions $\langle \langle \sigma_{k_1,a_1} \ldots \sigma_{k_n,a_n} \rangle \rangle^X_g$ on the large phase space by the formula

$$\langle \langle \sigma_{k_1,a_1} \ldots \sigma_{k_n,a_n} \rangle \rangle^X_g = \sum_{\beta \in H_2(X,\mathbb{Z})} q^\beta \sum_{N=0}^\infty \frac{1}{N!} \sum_{b_1 \ldots b_N} t_{b_1} \ldots t_{b_N}^N \int \psi_{a_1}^{k_1} \ldots \psi_{a_n}^{k_n} \psi_{n+1}^{f_1} \ldots \psi_{n+N}^{f_N} \left(1 - \psi_{1} \right) \ldots \left(1 - \psi_{n} \right).
$$

**Theorem 4.9.** If $k_i > 0$, then $\langle \langle \sigma_{k_1,a_1} \ldots \sigma_{k_n,a_n} \rangle \rangle^X_g = \sigma_{k_1,a_1} \ldots \sigma_{k_n,a_n} F_g^X$.

**Proof.** Form the generating function

$$F_{g|a_1 \ldots a_n}^X(z_1, \ldots, z_n) = \sum_{k_1, \ldots, k_n=0}^\infty z_1^{k_1} \ldots z_n^{k_n} \langle \langle \sigma_{k_1,a_1} \ldots \sigma_{k_n,a_n} \rangle \rangle^X_g$$

$$= \sum_{\beta \in H_2(X,\mathbb{Z})} q^\beta \sum_{N=0}^\infty \frac{1}{N!} \sum_{b_1 \ldots b_N} t_{b_1} \ldots t_{b_N}^N \int \psi_{a_1}^{k_1} \ldots \psi_{a_n}^{k_n} \psi_{n+1}^{f_1} \ldots \psi_{n+N}^{f_N} \left(1 - \psi_{1} \right) \ldots \left(1 - \psi_{n} \right).
$$

Corollary 4.5 implies that

$$\Theta_{a_1}^{b_1}(z_1) \ldots \Theta_{a_n}^{b_n}(z_n) F_{g|b_1 \ldots b_n}^X(z_1, \ldots, z_n) = \tau_{a_1}(z_1) \ldots \tau_{a_n}(z_n) F_g^X(z_1, \ldots, z_n).
$$

By the relation $\sigma_a(z) = \Theta_a^b(z) \tau_a(z)$, it follows that

$$\Theta_{a_1}^{b_1}(z_1) \ldots \Theta_{a_n}^{b_n}(z_n) F_{g|a_1 \ldots a_n}^X(z_1, \ldots, z_n) = \sigma_{a_1}(z_1) \tau_{a_2}(z_2) \ldots \tau_{a_n}(z_n) F_g^X(z_1, \ldots, z_n).
$$

Since

$$\sigma_{a_1}(z_1) \cdot \Theta_{a_2}^{b_2}(z_2) = \Theta_{a_2}^{b_2}(z_2) \cdot (\sigma_{a_1}(z_1) + z_2 A_{a_1}),$$

we see that

$$\Theta_{a_1}^{b_1}(z_3) \ldots \Theta_{a_n}^{b_n}(z_n) F_{g|a_1 \ldots a_n}^X(z_1, \ldots, z_n) = (\sigma_{a_1}(z_1) + z_2 A_{a_1}) \sigma_{a_2}(z_2) \tau_{a_3}(z_3) \ldots \tau_{a_n}(z_n) F_g^X(z_1, \ldots, z_n).
$$

Continuing in the same vein, we see that

$$F_{g|a_1 \ldots a_n}^X(z_1, \ldots, z_n) = (\sigma_{a_1}(z_1) + (z_2 + \ldots + z_n) A_{a_1}) (\sigma_{a_n-1}(z_{n-1}) + z_n A_{a_{n-1}}) \sigma_{a_n}(z_n) F_g^X(z_1, \ldots, z_n).$$

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The proposition follows on expanding in the parameters \( z_i \).

This theorem immediately implies Theorem 1.1. Let \( g > 0 \), so that the moduli space \( \mathcal{M}_{g,n} \) has dimension \( 3g - 3 + n \); it follows that when \( k_1 + \cdots + k_n > 3g - 3 + n \),

\[
\sigma_{k_1} \cdots \sigma_{k_n} F_g^X = \langle \langle \sigma_{k_1,a_1} \cdots \sigma_{k_n,a_n} \rangle \rangle_g^X = 0.
\]

Hence, by Proposition 2.3, we see that when \( k_1 + \cdots + k_n > 3g - 3 + n \),

\[
\tau_{k_1,a_1} \cdots \tau_{k_n,a_n} F_g^X = 0.
\]

Theorem 4.9 gives rise to a simple procedure for translating identities among cycles on Deligne-Mumford moduli spaces into differential equations among Gromov-Witten potentials. For example, the identity on \( \mathcal{M}_{1,1} \)

\[
\psi_1 \cap [\mathcal{M}_{1,1}] = \frac{1}{24} \quad \Box
\]

where the vertices denoted by open circles are of genus 1, translates into the topological recursion relation

\[
\langle \langle \sigma_{1,a} \rangle \rangle_1^X = \frac{1}{24} \langle \langle O_a O_b O_b \rangle \rangle_0^X.
\]

Here, \( O_a \) is an abbreviation for \( \sigma_{0,a} = \tau_{0,a} \). This is the case \( k = 1 \) of the following equation:

\[
\sigma_{k,a} F_1 = \begin{cases} 
\frac{1}{24} \sigma_{0,a} \text{Tr}(\mathcal{M}) & k = 1, \\
0 & k > 1.
\end{cases}
\]

(The cases \( k > 1 \) reflect the fact that \( \mathcal{M}_{1,1} \) has dimension 1.)

**Lemma 4.10.** The equation (4.13) has a particular solution \( \frac{1}{24} \log \det(\mathcal{X}) \in O_1 \).

**Proof.** Clearly, \( \sigma_{k,a} \log \det(\mathcal{X}) = 0 \) for \( k > 1 \), while

\[
\sigma_{1,a} \log \det(\mathcal{X}) = \text{Tr}(\mathcal{X}^{-1}(\mathcal{X}^2)_{a}^{b} \partial_{1,b} \mathcal{X}) = \text{Tr}(\mathcal{X}^{-1}(\mathcal{X}^2)_{a}^{b} A_b) = \text{Tr}(\mathcal{X} A_a) = \sigma_{0,a} \text{Tr}(\mathcal{M}).
\]

Let \( \mathcal{G} = F_1 - \frac{1}{24} \log \det(\mathcal{X}) \); we see that \( \sigma_{k,a} \mathcal{G} = 0 \) for all \( k > 0 \); hence, by Corollary 2.3, we recover a result of Dijkgraaf and Witten \[7\]: there is a function \( \mathcal{G} \in O_0 \) such that

\[
F_1 = \mathcal{G} + \frac{1}{24} \log \det(\mathcal{X}).
\]

The dilaton equation \( Q F_1 = \frac{1}{24} \chi(\mathcal{X}) \) follows automatically from the formulas \( Q \mathcal{G} = 0 \) and

\[
Q \log \det(\mathcal{X}) = \text{Tr}(\mathcal{X}^{-1} Q \mathcal{X}) = \chi.
\]

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To give a second example, the identity on $\overline{\mathcal{M}}_{2,1}$

$$\psi^2_1(\overline{\mathcal{M}}_{2,1}) = \frac{7}{5} \begin{array}{c} \circ \end{array} + \frac{1}{5} \begin{array}{c} \circ \end{array} - \frac{1}{120} \begin{array}{c} \circ \end{array} + \frac{13}{120} \begin{array}{c} \circ \end{array} + \frac{1}{120} \begin{array}{c} \circ \end{array}$$

translates into the topological recursion relation

$$\langle\langle \sigma_{2,a} \rangle\rangle_2^X = \frac{7}{5} \langle\langle (\mathcal{O}_a \mathcal{O}^b \mathcal{O}^c)_0 \rangle\rangle_1^X \langle\langle \mathcal{O}_b \rangle\rangle_1^X + \frac{1}{5} \langle\langle (\mathcal{O}_a \mathcal{O}^b \mathcal{O}^c)_0 \rangle\rangle_1^X \langle\langle \mathcal{O}_b \mathcal{O}_c \rangle\rangle_1^X$$

$$- \frac{1}{120} \langle\langle (\mathcal{O}_a \mathcal{O}_b)_0 \rangle\rangle_1^X \langle\langle (\mathcal{O}^b \mathcal{O}^c \mathcal{O}_c)_0 \rangle\rangle_0^X + \frac{13}{120} \langle\langle (\mathcal{O}_a \mathcal{O}^b \mathcal{O}^c \mathcal{O}_c)_0 \rangle\rangle_0^X \langle\langle (\mathcal{O}^b)_0 \rangle\rangle_0^X$$

$$+ \frac{1}{120} \langle\langle (\mathcal{O}_a \mathcal{O}^b \mathcal{O}^c \mathcal{O}_c)_0 \rangle\rangle_0^X.$$ 

5. **Virasoro constraints on the jet-space of a Frobenius manifold**

5.1. **Frobenius manifolds.** A Frobenius manifold is a conformal Dubrovin manifold

$$(M, \nabla^*, e, E, r)$$

together with a non-degenerate symmetric bilinear form $\eta$ on the tangent bundle (also know as a pseudo-Rimannian metric) such that

1. $\nabla$ is the Levi-Civitã connection associated to $\eta$;
2. $\eta(X \circ Y, Z) = \eta(X, Y \circ Z)$, or equivalently, $A^*_X = A_X$;
3. $E\eta(X, Y) = \eta([E, X], Y) + \eta(X, [E, Y]) + (2 - r)\eta(X, Y)$, that is, $\mathcal{L}_E\eta = (2 - r)\eta$.

On a Frobenius manifold, the endomorphism $\mu$ of the cotangent bundle is skew-adjoint:

$$\eta(\mu^*(X), Y) + \eta(X, \mu^*(Y)) = (2 - r)\eta(X, Y) + \eta(\nabla X E, Y) + \eta(X, \nabla Y E)$$

$$= (2 - r)\eta(X, Y) + \eta(\nabla E X, Y) + \eta(X, \nabla E Y) - \eta([E, X], Y) - \eta(X, [E, Y]) = 0.$$ 

Here, we have used that $\nabla$ is torsion-free and compatible with the metric $\eta$.

If $A$ is a linear operator on the bundle $\mathcal{H} = T^* M[z, z^{-1}]$, denote by $A^+$ its adjoint with respect to the symmetric bilinear form $\eta$

$$\eta(f, g) = \text{Res}_{z=0} \eta(f(-z), g(z)) \frac{dz}{z}.$$ 

For example, $\Theta^+(z) = \Theta^*(-z)$, $\partial_z^+ = \partial + z^{-1}$ and $z^+ = -z$. Since $\mu^* = -\mu$ and $\mathcal{U}^+ = \mathcal{U}$, it follows that $\delta_z^+ = (\partial_z + z^{-1}) - z^{-1}(\mu^* + \frac{1}{2}) + \mathcal{U}^* = (\partial_z + z^{-1}) + z^{-1}(\mu - \frac{1}{2}) + \mathcal{U} = \delta_z$.

If $\Theta(z)$ is a fundamental solution, recall that the endomorphism $R(z)$ of $\mathcal{H} = T^* M[z]$ is defined by the formula

$$\partial_z \Theta(z) + z^{-1} R(z) \Theta(z) = z^{-1}(\mu + \frac{1}{2}) \Theta(z) + \mathcal{U} \Theta(z).$$
**Definition 5.1.** A fundamental solution $\Theta(z)$ on a Frobenius manifold $M$ is symmetric if it is conformal and $\Theta^+(z)\Theta(z) = I$.

It follows from the condition $\Theta^+(z)\Theta(z) = I$ that $R(z) + R^+(z) = 0$, in other words, $R_k^* = (-1)^{k+1}R_k$ for all $k > 0$.

**Proposition 5.1.** There exists a symmetric fundamental solution on any Frobenius manifold.

**Proof.** Let $\Theta(z)$ be a conformal fundamental solution. Since $\nabla \cdot \Theta(z) = \Theta(z) \cdot \nabla z$, we have

$$0 = \eta(\alpha(-z), \Theta(z) \cdot \nabla z \beta(z)) - \eta(\alpha(-z), \nabla \cdot \Theta(z) \beta(z))$$

$$= \eta(\Theta^*(z)\alpha(-z), \nabla z \beta(z)) - d\eta(\alpha(-z), \Theta(z)\beta(z)) + \eta(\nabla \alpha(-z), \Theta(z) \beta(z))$$

$$= -\eta(\nabla^{-z} \cdot \Theta^*(z)\alpha(-z), \beta(z)) + \eta(\Theta(z) \cdot \nabla \alpha(-z), \beta(z)).$$

This shows that $\nabla^{-z} \cdot \Theta^*(z) = \Theta^*(z) \cdot \nabla^{-z}$; conjugating by $\Theta^*(z)$, we obtain the equation

$$\nabla \cdot \Theta^+(-z) = \Theta^+(z) \cdot \nabla^{-z}.$$

Replacing $z$ by $-z$, we see that $\Theta^+(z)^{-1}$ is also a fundamental solution.

It follows that $\rho(z) = \Theta^+(z)\Theta(z)$ is a flat section of $\Gamma(M, \text{End}(TM))[z]$. Replacing $\Theta(z)$ by $\tilde{\Theta}(z) = \Theta(z)\rho(z)^{-1/2}$, we obtain a new fundamental solution. Since $\rho^+(z) = \rho(z)$, we see that

$$\tilde{\Theta}^+(z)\tilde{\Theta}(z) = \rho^+(z)^{-1/2}\Theta^+(z)\Theta(z)\rho(z)^{-1/2} = I,$$

hence $\tilde{\Theta}(z)$ is symmetric. $\square$

In the remainder of this section, we only consider symmetric fundamental solutions.

5.2. **Gromov-Witten invariants and Frobenius manifolds.** In Section 4, we showed that the small phase space $H$ associated to the genus 0 Gromov-Witten invariants of a projective manifold $X$ is conformal Dubrovin manifold. It also carries a flat metric $\eta$; with respect to this metric, it is a Frobenius manifold. The equation $\mathcal{L}_E \eta = (2 - r)\eta$ is equivalent to $(p_a + p_b)\eta_{ab} = r\eta_{ab}$, which is a basic property of the Poincaré form of a projective manifold of dimension $r$.

**Proposition 5.2.** The conformal fundamental solution

$$\Theta^b_a(z) = \delta^b_a + \eta^{bc} \sum_{k=0}^{\infty} z^{k+1} \langle\tau_{k,a} \tau_{0,c}\rangle_X$$

on the small phase space is symmetric.
Proof. Let $\rho(z) = \Theta^+(z)\Theta(z)$. We have
\[
\partial_a \rho^+(z) = \partial_a \Theta^+(z)\Theta(z) + \Theta^+(z)\partial_a \Theta(z) = -z\Theta^+(z)A_a \Theta(z) + z\Theta^+(z)A_a \Theta(z) = 0.
\]
Thus $\rho(z)$ is constant on $H$. It remains to show that it equals $I$ at the origin $0 \in H$. We may write
\[
\rho(z)(0) = I + \sum_{k=0}^{\infty} z^{k+1} \sum_{\beta \neq 0} q^\beta \rho_k,\beta.
\]
For $\omega \in H^2(X, \mathbb{C})$, the divisor equation implies that
\[
\int \beta^\omega \rho_k,\beta = \left[R(\omega), \rho_{k-1,\beta}\right],
\]
from which it follows that $\rho_{k,\beta} = 0$ for $\beta \neq 0$. On the other hand, in the limit $q \to 0$ in the Novikov ring, the value of $\Theta_k$ at the origin of $H$ is an integral over $\mathcal{M}_{0,2}(X, 0)$, hence vanishes, since $\mathcal{M}_{0,2}(X, 0)$ is empty. \hfill \Box

5.3. The genus 0 potential on the large phase space of a Frobenius manifold. On a Frobenius manifold, we may define a power series $F_0$ on
the large phase space by integrating the power series $t(z)$:
\[
\tau_{k,a} F_0 = (-1)^k \eta_{ab} t^{b}_{-k-1}.
\]
Denote $\eta_{bc} \Omega^{c}_{j,k|a}$ by $\Omega_{j,k|a,b}$; taking another derivative, we have
\[
\tau_{j,a} \tau_{k,b} F_0 = (-1)^k \Omega_{j,k|a,b}.
\]
This determines $F_0$ up to an affine function on the large phase space, which may be fixed by the dilaton equation
\[
Q F_0 = -2 F_0.
\]
The following formula $F_0$ is due to Dubrovin [9].

Proposition 5.3. $F_0 = \frac{1}{2} \sum_{\ell,m=0}^{\infty} (-1)^m t^p_{\ell} t^q_{m} \Omega_{\ell,m|p,q}$

Proof. Since the fundamental solution $\Theta(z)$ is symmetric, we see that
\[
(y-z)\Theta^*(y, z) = \Theta^*(z)^{-1}\Theta^*(y) - I = \Theta(-z)\Theta^{-1}(-y) - I = (y-z)\Omega(-z, -y),
\]
in other words, $\Omega_{j,k|a,b} = (-1)^{j+k} \Omega_{k,j|b,a}$. It follows by Lemma 2.9 that
\[
\tau_{k,b} \Omega_{\ell,m|p,q} = (-1)^{\ell+m} \tau_{k,b} \Omega_{m,\ell|q,p} = (-1)^{\ell+m} \tau_{m,q} \Omega_{k,\ell|b,p} = (-1)^{k+m} \tau_{m,q} \Omega_{k,\ell|p,b}.
\]
We have
\[ \tau_{k,b} \mathcal{F}_0 = \frac{1}{2} \sum_{\ell,m} (-1)^m t_{\ell}^p \tau_{\ell,m} \Omega_{\ell,m}|p,q \]
\[ + \frac{1}{2} (-1)^k \sum_{\ell} t_{\ell}^p \Omega_{\ell,k}|p,b \]
\[ + \frac{1}{2} \sum_{m} (-1)^m t_{m}^q \Omega_{k,m}|b,q \]
\[ = \frac{1}{2} (-1)^k \sum_{\ell,m} t_{\ell}^p \tau_{m,q} \Omega_{\ell,k}|p,b \]
\[ + \frac{1}{2} \sum_{m} (-1)^m t_{m}^q \Omega_{k,m}|b,q \]
\[ = \frac{1}{2} (-1)^k \sum_{\ell} t_{\ell}^p (1 - Q) \Omega_{\ell,k}|p,b \]
\[ + \frac{1}{2} \sum_{m} (-1)^m t_{m}^q \Omega_{k,m}|b,q \]
\[ = \frac{1}{2} (-1)^k \sum_{\ell} t_{\ell}^p \Omega_{\ell,k}|p,b \]
\[ + \frac{1}{2} \sum_{m} (-1)^m t_{m}^q \Omega_{k,m}|b,q \]
\[ = (1 - Q) \Omega_{j,a} = 0. \]

It follows that
\[ \tau_{j,a} \tau_{k,b} \mathcal{F}_0 = \frac{1}{2} (-1)^k \Omega_{j,k}|a,b \]
\[ + \frac{1}{2} (-1)^j \Omega_{j,k}|a,b \]
\[ + \frac{1}{2} \sum_{m} (-1)^m t_{m}^q \Omega_{k,m}|b,q \]
\[ = \frac{1}{2} (-1)^k \Omega_{j,k}|a,b \]
\[ + \frac{1}{2} (-1)^j \Omega_{j,k}|a,b \]
\[ + \frac{1}{2} \sum_{m} (-1)^m t_{m}^q \Omega_{k,m}|b,q \]
\[ = \frac{1}{2} (-1)^k (1 - Q) \Omega_{j,k}|a,b \]
\[ + \frac{1}{2} (-1)^j (1 - Q) \Omega_{j,k}|a,b \]
\[ + \frac{1}{2} \sum_{m} (-1)^m t_{m}^q \Omega_{k,m}|b,q \]
\[ = \frac{1}{2} (-1)^k \Omega_{j,k}|a,b \]
\[ + \frac{1}{2} (-1)^j \Omega_{j,k}|a,b \]
\[ + \frac{1}{2} \sum_{m} (-1)^m t_{m}^q \Omega_{k,m}|b,q \]
\[ = (-1)^k \Omega_{j,k}|a,b. \]

This shows that \( \mathcal{F}_0 \) satisfies \( 5.11 \).

5.4. **The Virasoro operators on the large phase space of a Frobenius manifold.** Let \( \Phi(z) \) be the generating function (or free field) whose coefficients are linear differential operators acting on functions on the large phase space
\[ \Phi_a(z) = h \sum_{k=0}^{\infty} z^k \tau_{k,a} + \sum_{k=0}^{\infty} (-z)^{k-1} \eta_{ab} t^b_k. \]

The conjugate of \( \Phi(z) \) by \( Z_0 = e^{\mathcal{F}_0/h} \) is given by the formula
\[ Z_0^{-1} \cdot \Phi_a(z) \cdot Z_0 = h \tau_a(z) + \eta_{ab} t^b(-z). \]

The normal ordering of quadratic expressions in the field \( \Phi(z) \) is defined by the formulas
\[ :t_{k,a}^b t_{\ell}^b: = t_{k,a}^b t_{\ell}^b, \quad :t_{k,a}^b \tau_{\ell,b}: = t_{k,a}^b \tau_{\ell,b}. \]
If \( k \geq -1 \), let \( L_k \) be the second-order differential operator on the large phase space

\[
L_k = \frac{1}{\hbar} \text{Res}_{z=0} : \eta(\Phi(-z), z\delta_z^{k+1}\Phi(z)) : - \frac{1}{4}\delta_{k,0} \text{Tr}(\mu^2 - \frac{1}{4}).
\]

These operators were introduced in the theory of Gromov-Witten invariants by Eguchi, Hori and Xiong [15], who proved the following result.

**Proposition 5.4.** The operators \( L_k \) satisfy the commutation relations

\[
[L_j, L_k] = (j - k)L_{j+k}.
\]

**Proof.** We give an outline of the proof; see [20], Section 2, for the complete details.

Using the canonical commutation relations among the coefficients of \( \Phi(z) \), the formula \( [L_k, \Phi(z)] + z\delta_z^{k+1}\Phi(z) = 0 \) is easily proved. It follows that

\[
[[L_j, L_k], \Phi(z)] = [L_j, [L_k, \Phi(z)]] - [L_k, [L_j, \Phi(z)]]
\]

\[
= -[L_j, z\delta_z^{k+1}\Phi(z)] + [L_k, z\delta_z^{j+1}\Phi(z)]
\]

\[
= -z\delta_z^{k+1}[L_j, \Phi(z)] + z\delta_z^{j+1}[L_k, \Phi(z)]
\]

\[
= [z\delta_z^{k+1}, z\delta_z^{j+1}]\Phi(z) = (k - j)\delta_z^{j+k+1}\Phi(z),
\]

and hence that \( [L_j, L_k] - (j - k)L_{j+k} \) commutes with \( \Phi(z) \).

Any differential operator commuting with \( \Phi(z) \) must lie in the centre of the algebra of differential operators, and hence

\[
[L_j, L_k] = (j - k)L_{j+k} + c(j, k)1
\]

for some two-cocyle \( c(j, k) \). Evaluating both sides of this equation at the basepoint of the large phase space, the result follows. \( \square \)

In the following theorem, we show how the Virasoro operators \( L_k \) give rise to intrinsic differential operators on the jet-space of the Frobenius manifold. The explicit formula for \( \mathcal{H}_k \) is equivalent to a formula of Liu (26, Theorem 4.4).

**Theorem 5.5.** We have

\[
L_k \cdot Z_0 = Z_0 \cdot (\hbar \Delta_k + L_k + \mathcal{H}_k),
\]

where \( \Delta_k = \frac{1}{2} \text{Res}_{z=0} \eta(\sigma(-z), z\delta_z^{k+1}\sigma(z)) \), and

\[
\mathcal{H}_k = -\frac{1}{4} \sum_{\ell=0}^k \text{Tr}(\mu - \frac{1}{2})U^\ell(\mu + \frac{1}{2})U^{k-\ell}.
\]

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Proof. We have
\[
Z_0^{-1} \cdot \frac{1}{2\pi} \eta(\Phi(-z), z\delta^{k+1}_z \Phi(z)) \cdot Z_0 = \frac{1}{2} h \eta(\tau(-z), z\delta^{k+1}_z \tau(z)) \\
+ \frac{1}{2} \langle t(z), z\delta^{k+1}_z \tau(z) \rangle + \frac{1}{2} \langle z\delta^{k+1}_z t_-(z), \tau(-z) \rangle + \frac{1}{2} \langle \tau(-z), z\delta^{k+1}_z t_+(z) \rangle \\
+ \frac{1}{2\pi} \eta(t(z), z\delta^{k+1}_z t(-z)).
\]
On taking the residue, the first term on the right-hand side yields
\[
\frac{1}{2} \text{Res}_{z=0} \eta(\tau(-z), z\delta^{k+1}_z \tau(z)) = \frac{1}{2} \text{Res}_{z=0} \eta(\Theta^{-1}(-z)\sigma(-z), \Theta^{-1}(z)z\delta^{k+1}_z \sigma(z)) = \Delta_k.
\]
The residue of the last term on the right-hand side vanishes by Theorem \ref{2.11} (this is the genus 0 Virasoro constraint), since
\[
\eta(t(z), z\delta^{k+1}_z t(-z)) = \eta(\Theta(z)t(z), \Theta^{-1}(z)z\delta^{k+1}_z t(-z)) = \eta(s(z), z\delta^{k+1}_z s(-z)) = O(z^{-3}).
\]
It follows that
\[
Z_0^{-1} \cdot L_k \cdot Z_0 - h \Delta_k + \frac{1}{4} \delta_{k,0} \text{Tr}(\mu^2 - \frac{1}{4}) \\
= \frac{1}{2} \text{Res}_{z=0} \langle t(z), z\delta^{k+1}_z \tau(z) \rangle \\
+ \frac{1}{2} \text{Res}_{z=0} \left( \langle z\delta^{k+1}_z t_-(z), \tau(-z) \rangle + \langle \tau(-z), z\delta^{k+1}_z t_+(z) \rangle \right) \\
= \frac{1}{2} \text{Res}_{z=0} \left( \langle t(z), z\delta^{k+1}_z \tau(z) \rangle + \langle t_-(z), z\delta^{k+1}_z \tau(z) \rangle + \langle z\delta^{k+1}_z \tau(z), t_+(z) \rangle \right) \\
= \text{Res}_{z=0} \langle t(z), z\delta^{k+1}_z \tau(z) \rangle - \frac{1}{2} \text{Res}_{z=0} \text{Tr}(y\delta^{k+1}_y \Omega(y, z))_{y=z},
\]
since \(\tau(y)t_+(z) = -\Omega(y, z)\). It remains to show that
\[
\text{Res}_{z=0} \text{Tr}(y\delta^{k+1}_y \Omega(y, z))_{y=z}
= \begin{cases} 
0, & k \leq 0, \\
-\frac{1}{2} \sum_{\ell=0}^{k} \text{Tr}((\mu - \frac{1}{2})U^\ell (\mu + \frac{1}{2})U^{k-\ell}), & k > 0.
\end{cases}
\]
Applying the operator \(y\delta^{k+2}_y\) to \ref{2.11}, we see that
\[(k + 2)y\delta^{k+1}_y \Omega(y, z) + (y - z)y\delta^{k+2}_y \Omega(y, z) = y\delta^{k+2}_y (\Theta(y)\Theta^{-1}(z) - I).
\]
Taking the trace, we see that
\[
(k + 2) \text{Tr}(y\delta^{k+1}_y \Omega(y, z)) + (y - z) \text{Tr}(y\delta^{k+2}_y \Omega(y, z)) \\
= \text{Tr}(\Theta^{-1}(z)y\delta^{k+2}_y \Theta(y)) - \text{Tr}(y\delta^{k+2}_y I).
\]
Next, we restrict to the diagonal \( y = z \):

\[
(k + 2) \text{Tr}(y^{k+1}_y \Omega(y, z))_{y = z} = \text{Tr}(\Theta^{-1}(z) z^{k+2} \Theta(z)) - \text{Tr}(z^{k+2} I) = \text{Tr}(z^{k+2} I) - \text{Tr}(z^{k+2} I).
\]

Taking the residue, we see that

\[
(k + 2) \text{Res}_{z=0} \text{Tr}(y^{k+1}_y \Omega(y, z))_{y = z} = \sum_{0 \leq i + j \leq k} \text{Tr}(U^i(\mu - \frac{1}{2}) U^j(\mu + \frac{1}{2}) U^{k-i-j}) - \text{Res}_{z=0} \text{Tr}(z^{k+2} I).
\]

Since \([\mu_s, R_k] = k R_k\), we see that

\[
\text{Tr}(z^{k+1} I) = z^{-k-1} \text{Tr}((\mu - \frac{1}{2} - k) \ldots (\mu - \frac{1}{2}) (\mu + \frac{1}{2})),
\]

and hence \(\text{Res}_{z=0} \text{Tr}(z^{k+1} I) = \delta_{k,0} \text{Tr}(\mu^2 - \frac{1}{4})\). We have

\[
\sum_{0 \leq i + j \leq k} \text{Tr}(U^i(\mu - \frac{1}{2}) U^j(\mu + \frac{1}{2}) U^{k-i-j}) = \sum_{0 \leq i + j \leq k} \text{Tr}(U^i U^j) - \frac{1}{4} \sum_{0 \leq i + j \leq k} \text{Tr}(U^k) = \left(\frac{k}{2} + 1\right) \sum_{\ell=0}^k \text{Tr}(U^\ell U^{k-\ell}) - \frac{1}{4} \left(\frac{k+2}{2}\right) \text{Tr}(U^k)
\]

\[
= \left(\frac{k}{2} + 1\right) \sum_{\ell=0}^k \text{Tr}(U^\ell U^{k-\ell}),
\]

and the formula for \(\mathcal{H}_k\) follows.

As a corollary of this theorem and the Virasoro relations for \(L_k\), we obtain the following formula of Liu [25], Section 6).

**Corollary 5.6.** \(\mathcal{L}_j \mathcal{H}_k - \mathcal{L}_k \mathcal{H}_j = (j - k) \mathcal{H}_{j+k}\)

5.5. **The Virasoro constraints on a Frobenius manifold.** The Virasoro constraints are differential equations among a sequence of functions \(F_g, g > 0\), on the large phase space of a Frobenius manifold \(M\) such that the dilaton equation holds:

\[
(5.2) \quad Q F_g = (2g - 2) F_g + \frac{1}{24} \delta_{g,1} \chi,
\]

were \(\chi = \text{Tr}(I)\). As originally introduced by Eguchi, Hori and Xiong [15], in the case where \(M\) is the Frobenius manifold associated to the genus 0 Gromov-Witten theory of a projective manifold \(X\) and \(F_g\) is the genus \(g\)
Gromov-Witten potentials $X$, these differential equations are given by the vanishing of functions $z_{k,g}$ on the large phase space defined as follows:

$$\sum_{g=0}^{\infty} \hbar^{g-1} z_{k,g} = \exp\left(-\sum_{g=0}^{\infty} \hbar^{g-1} F_g\right) \cdot L_k \cdot \exp\left(\sum_{g=0}^{\infty} \hbar^{g-1} F_g\right).$$

Using Theorem 5.5, we may write the vanishing of $z_{k,g}$ intrinsically on the jet-space of the Frobenius manifold $M$: in fact, we have

$$\sum_{g=1}^{\infty} \hbar^{g-1} z_{k,g} = \exp\left(-\sum_{g=1}^{\infty} \hbar^{g-1} F_g\right) \cdot h\Delta_k \cdot \exp\left(\sum_{g=1}^{\infty} \hbar^{g-1} F_g\right) + \sum_{g=1}^{\infty} \hbar^{g-1} L_k F_g + \mathcal{H}_k.$$

This proves the following theorem.

**Theorem 5.7.** For genus $g > 1$, the Virasoro constraint $z_{k,g} = 0$ is equivalent to the equation

$$L_k F_g + \Delta_k F_{g-1} + \frac{1}{2} \sum_{i=1}^{g-1} \text{Res}_{z=0} \eta(z)F_i, z\delta_{z}^{k+1} \sigma(z) F_{g-i} = 0,$$

while the Virasoro constraint $z_{k,1} = 0$ is equivalent to the equation

$$L_k F_1 = \frac{1}{k} \sum_{\ell=0}^{k} \text{Tr}((\mu - \frac{1}{2}) U^{\ell} (\mu + \frac{1}{2}) U^{k-\ell}).$$

Using Proposition 3.5, we see that the genus 1 Virasoro constraint $z_{k,1} = 0$ is equivalent to the equation

$$(5.3) \quad L_k G = \sum_{\ell=0}^{k} \left(\frac{1}{2} \text{Tr}(\mu^{\ell} U^{k-\ell} U^{k-\ell}) + \frac{1}{24} \langle e, U^{\ell} U^{k-\ell} \omega \rangle\right)$$

for $G = F_1 X - \frac{1}{24} \log \det(X)$. This equation was first proved in the special case where $M$ is a semisimple Frobenius manifold by Dubrovin and Zhang [12], and extended to general Frobenius manifolds by Liu [26].

Let $M$ be a Frobenius manifold. There is a second-order differential operator

$$\Psi : \Gamma(M, \mathcal{O}) \to \Gamma(M, S^4(T^*M)),$$

given by the explicit formula

$$(5.4) \quad \Psi(X_1, X_2, X_3, X_4) = \frac{1}{24} \sum_{\pi \in S_4} \sum_{a_1, a_2, a_3, a_4} X_{\pi_1}^{a_1} X_{\pi_2}^{a_2} X_{\pi_3}^{a_3} X_{\pi_4}^{a_4}$$

$$\left(3 A_{\mu_{a_1 a_2}} A_{\nu_{a_3 a_4}} \partial_\mu \partial_\nu - 4 A_{\mu_{a_1 a_2}} A_{\nu_{a_3 a_4}} \partial_\mu A_{a_3 \mu} \partial_\nu + 2 \partial_{a_4} A_{\mu_{a_1 a_2}} A_{\nu_{a_3 a_4}} A_{a_1 \mu} \partial_\nu - A_{\mu_{a_1 a_2}} \partial_\mu A_{\nu_{a_3 a_4}} \partial_\nu\right)$$

$$+ \frac{1}{6} \partial_{a_3} A_{a_1 a_2} A_{\mu_{a_4 \nu}} + \frac{1}{24} \partial_{a_3} A_{a_1 a_2} A_{\mu_{a_3 a_4}} A_{\nu_{a_1 a_2}} A_{\nu_{a_3 a_4}} - \frac{1}{4} \partial_{a_3} A_{a_1 a_2} \partial_\mu A_{\nu_{a_3 a_4}},$$

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such that the generating function $G$ of genus 1 Gromov-Witten invariants of a compact symplectic manifold $X$ satisfies the equation $\Psi G = 0$ (see [18]).

Let

$$y_k = L_k G - \sum_{\ell=0}^{k} \left( \frac{1}{4} \text{Tr}(\mu U^\ell \mu U^{k-\ell}) + \frac{1}{24} (e, U^\ell \mu U^{k-\ell} \omega) \right).$$

By a lengthy calculation, Liu [26] proves the formula

$$y_k = \frac{1}{2} \sum_{i=1}^{k} \Psi(E^{*k-i+1}, E^{*i}, E, E) G - \frac{k+1}{2} L_{k-1} y_1.$$

In this way, he obtains the following theorem.

**Theorem 5.8.** If $\Psi G = 0$ and $G$ satisfies (5.3) for $k = 1$, then $G$ satisfies (5.3) for all $k > 1$.

**Relationship to prior results.** The results of Sections 5.1 and 5.2 are due to Dubrovin. The formula for the Virasoro operators on the large phase space in terms of free fields (Section 5.3) is a modification of the formalism of Section 2 of [20]. (Related formulas have been used by Givental [21].)

6. **The jet-space of a semisimple Dubrovin manifold**

6.1. **Semisimple Dubrovin manifolds.** A Dubrovin manifold is **semisimple** if the subset $M_0 \subset M$ on which the commutative algebra $(T_x M, \circ)$ is semisimple is dense in $M$. These manifolds were extensively studied by Dubrovin [9], who proved the following theorem.

**Theorem 6.1.** Around each point semisimple point in a semisimple Dubrovin manifold, there are coordinates $v^i$ such that the vector fields $\pi_i = \partial/\partial v^i$ satisfy

$$\pi_i \circ \pi_j = \delta_{ij} \pi_i.$$

**Proof.** Locally in $M_0$, there is a frame $\{\pi_1, \ldots, \pi_n\}$ of the tangent bundle $TM$ such that $\pi_i \circ \pi_j = \delta_{ij} \pi_i$. We must show that the vector fields $\pi_i$ satisfy $[\pi_i, \pi_j] = 0$. If $i$ and $j$ are distinct, the equation

$$\nabla_i (\pi_j \circ \pi_k) - \pi_j \circ (\nabla_i \pi_k) - \nabla_j (\pi_i \circ \pi_k) + \pi_i \circ (\nabla_j \pi_k) = [\pi_i, \pi_j] \circ \pi_k$$

is a consequence of condition 4) of Proposition 2.1. If $k$ is not equal to $i$ or $j$, (6.1) implies that $[\pi_i, \pi_j] \circ \pi_k$ vanishes. On the other hand, if $k$ equals $i$, we see that

$$-\pi_j \circ \nabla_i \pi_i - \nabla_j \pi_i + \pi_i \circ (\nabla_j \pi_i) = [\pi_i, \pi_j] \circ \pi_i,$$

from which we conclude that $[\pi_i, \pi_j] \circ \pi_i$ vanishes. This shows that $[\pi_i, \pi_j] = 0$. It follows that there are coordinates $v^i$ defined locally on $M_0$ such that $\pi_i = \partial/\partial v^i$. □
The canonical coordinates \((v^i)\) are seen to be defined up to addition of a constant and permutation. Let \(J = (J^a_i)\) be the Jacobian
\[
J^a_i = \frac{\partial u^a}{\partial v^i},
\]
and let \(J^{-1} = (J^{-i}_a)\) be its inverse
\[
J^{-i}_a = \frac{\partial v^i}{\partial u^a}.
\]

Denote by \(v^i_n\) the functions
\[
v^i_n = \partial^n v^i = \partial^n (J^i_au^a_1) = \sum_{k=1}^{n-1} (n-1) \left( \partial^{n-k} J^i_a u^a_k \right).
\]
The identity vector field on \(M_0\) is given in canonical coordinates by the formula
\[
e = \sum_i \pi_i.
\]

**6.2. Conformal semisimple Dubrovin manifolds.** Let \((M, \nabla, e, E, r)\) be a semisimple Dubrovin manifold which is conformal. Taking \(X = Y = \pi_i\) in (3.1), we see that \([E, \pi_i] = 2\pi_i \circ [E, \pi_i] + \pi_i\), hence \([E, \pi_i] = -\pi_i\). This shows that, after shifting the canonical coordinates \(v_i\) by constants \(c^i\), the Euler vector field on \(M_0\) is given by the formula
\[
E = \sum_i v^i \pi_i.
\]

In other words, the canonical coordinates are the eigenvalues of \(U\).

**Proposition 6.2.** The semisimple locus \(M_0 \subset M\) contains the tame locus \(M_{00}\) of points at which the eigenvalues of \(U\) are distinct and nonzero.

**Proof.** Locally in \(M_{00}\), the endormorphism \(U^*\) has eigenvectors \(\pi_i\) with eigenvalues \(v_i\). We have
\[
U^*(\pi_i \circ \pi_j) = (U^* \pi_i) \circ \pi_j = v^i \pi_i \circ \pi_j.
\]
Antisymmetrizing in \(i\) and \(j\) gives \((v^i - v^j)\pi_i \circ \pi_j = 0\), showing that \(\pi_i \circ \pi_j = 0\) if \(i\) and \(j\) are distinct. Since \(E \circ \pi_i = v^i \pi_i\) is nonzero, we see that \(\pi_i \circ \pi_i = f_i \pi_i\), where \(f_i\) is nowhere vanishing, hence \((T_x M, \circ)\) is semisimple in \(M_{00}\). \(\square\)

We will restrict attention to conformal semisimple Dubrovin manifolds satisfying the following condition.

**Condition 6.1.** There is a section \(\gamma \in \Gamma(M_0, \text{End}(T^*M))\) such that
\[
\mu = [\gamma, U].
\]

Let \(\mu^i_j\) and \(\gamma^i_j\) be the matrix elements of the tensors \(\mu\) and \(\gamma\) in the frame \((\pi_i)\); Condition 6.1 amounts to the relation \(\mu^i_j = (v^i - v^j)\gamma^i_j\).
6.3. **Semisimple Frobenius manifolds.** A Frobenius manifold is semisimple if the underlying Dubrovin manifold is semisimple. We now show, borrowing from the exposition of Manin [27], that semisimple Frobenius manifolds satisfy Condition 6.1.

In Gromov-Witten theory, it is known that the Frobenius manifolds associated to Grassmannians [1], to Del Pezzo varieties [2], and to arbitrary products of these, are semisimple. Another class of examples are the quotients $V/W$ of a Euclidean vector space $V$ by a Coxeter group $W$ of reflections, considered by Saito [28] (see also Dubrovin [8]).

Let $M$ be a semisimple Frobenius manifold. If $\varepsilon \in \Omega^1(M)$ is the one-form on $M$ defined by the formula

$$\varepsilon(X) = \eta(e, X),$$

then

$$\eta(X, Y) = \eta(e \circ X, Y) = \varepsilon(X \circ Y);$$

thus $\eta(X, Y) = 0$ if $X \circ Y = 0$. It follows that the idempotent eigenvectors $\pi_i$ on $M_0$ are orthogonal. Let $\eta_i = \eta(\pi_i, \pi_i)$, let $\eta_{ij} = \pi_j(\eta_i)$, and let $\eta_{ijk} = \pi_k(\eta_{ij})$. We now list the main properties of these functions; these are known as the Darboux-Egoroff equations.

**Lemma 6.3.**

1) $\eta_{ij} = \eta_{ji}$
2) $\eta_{ijk} = \frac{1}{2} \left( \frac{\eta_{ki} \eta_{ij}}{\eta_i} + \frac{\eta_{ij} \eta_{jk}}{\eta_j} + \frac{\eta_{jk} \eta_{ki}}{\eta_k} \right)$ when $i$, $j$ and $k$ are distinct
3) $e(\eta_i) = 0$
4) $E(\eta_i) = -r \eta_i$

**Proof.** The coefficients $\Gamma_{ijk} = \eta(\nabla_i \pi_j, \pi_k)$ of the Levi-Civita connection $\nabla$ are given by the formula

$$\Gamma_{ijk} = \frac{1}{2} \left( \delta_{ik} \eta_{ij} + \delta_{jk} \eta_{ji} - \delta_{ij} \eta_{ik} \right).$$

Taking the inner product of (6.1) with $\pi_\ell$, we obtain the equation

$$\frac{1}{4} (\delta_{\ell i} \delta_{jk} + \delta_{j \ell} \delta_{ik}) (\eta_j - \eta_{ji}) = 0,$$

which shows that $\eta_{ij} = \eta_{ji}$.

A lengthy calculation shows that

$$R_{ijk\ell} = \frac{1}{4} (\delta_{\ell i} - \delta_{j \ell}) \left( \frac{\eta_{ki} \eta_{ij}}{\eta_i} + \frac{\eta_{ij} \eta_{jk}}{\eta_j} + \frac{\eta_{jk} \eta_{ki}}{\eta_k} - 2 \eta_{ijk} \right)$$

$$+ \frac{1}{4} (\delta_{jk} - \delta_{ik}) \left( \frac{\eta_{i\ell} \eta_{ij}}{\eta_i} + \frac{\eta_{ij} \eta_{j\ell}}{\eta_j} + \frac{\eta_{j\ell} \eta_{i\ell}}{\eta_{j\ell}} - 2 \eta_{ij\ell} \right)$$

$$+ \frac{1}{4} (\delta_{j\ell} \delta_{ik} - \delta_{i\ell} \delta_{jk}) \sum_m \eta_{im} \eta_{mj} \eta_{im}.$$

Part 2) follows, by the flatness of the connection $\nabla$. 44
Since the vector field $e$ is flat, we have
\[0 = \eta(\pi_i, \nabla_i e) = \sum_j \eta(\pi_i, \nabla_i \pi_j) = \sum_j \Gamma_{iji}\]
\[= \frac{1}{2} \sum_j \left( \eta_{ij} + \delta_{ij} \eta_{ji} - \delta_{ij} \eta_{ii} \right) = \frac{1}{2} e(\eta).\]

Since $\mathcal{L}_E \eta = (2 - r) \eta$, we see that
\[E(\eta_i) = (\mathcal{L}_E \eta)(\pi_i, \pi_i) - 2\eta([E, \pi_i], \pi_i) = (2 - r)\eta(\pi_i, \pi_i) - 2\eta(\pi_i, \pi_i),\]
which shows that $E(\eta_i) = -r \eta_i$. □

**Proposition 6.4.** We have $\mu = [\gamma, \mathcal{U}]$, where $\gamma \in \Gamma(M_0, \text{End}(T^*M))$ is the endomorphism of the tangent bundle of $M_0$ defined by
\[
\gamma dv^i = \frac{1}{2} d \log \eta_i.
\]

**Proof.** By the definition of $\gamma$,
\[\eta(\pi_i, \gamma^* \pi_j) = \frac{1}{2} \eta_{ij}.\]

We must show that
\[\eta(\pi_i, \mu^* \pi_j) = \eta(\mathcal{U}^* \pi_i, \gamma^* \pi_j) - \eta(\gamma^* \pi_i, \mathcal{U}^* \pi_j) = \frac{1}{2} (v^i - v^j) \eta_{ij}.\]

We have
\[\eta(\pi_i, \mu^* \pi_j) = (1 - \frac{r}{2}) \eta(\pi_i, \pi_j) - \eta(\pi_i, \nabla_j E)\]
\[= (1 - \frac{r}{2}) \delta_{ij} \eta_i - \eta(\pi_i, [\pi_j, E]) - \eta(\pi_i, \nabla E \pi_j)\]
\[= -\frac{r}{2} \delta_{ij} \eta_i - \sum_k v^k \Gamma_{kji}\]
\[= -\frac{1}{2} \left( r \delta_{ij} \eta_i + \sum_k v^k (\delta_{ik} \eta_{kj} + \delta_{ij} \eta_{jk} - \delta_{jk} \eta_{ik}) \right)\]
\[= -\frac{1}{2} (\delta_{ij} (r \eta_i + E(\eta_i)) + (v^i - v^j) \eta_{ij}),\]
and (6.2) follows. □

6.4. The vector fields $\mathcal{L}_k$ on a semisimple Dubrovin manifold. We now give a new proof of an important recent theorem of Dubrovin and Zhang ([13], Theorem 3.10.20). This proof is simpler than theirs, though it is in the same spirit.

**Theorem 6.5.** Let $M$ be a conformal semisimple Dubrovin manifold satisfying Condition 6.1. If $f \in \mathcal{O}_\infty$ satisfies $\mathcal{L}_k f = 0$ for all $k \geq -1$, then $f$ is a constant.
Proof. The theorem is proved by induction: if $f \in \mathcal{O}_n$ satisfies $L_k f = 0$ for all $k \geq -1$, then $f \in \mathcal{O}_{n-1}$. In order to carry out this induction, we must calculate the action of $L_k$ on $\mathcal{O}_n$ modulo $\mathcal{O}_{n-1}$. By (3.4), we see that

$$L_k u_n^a = -\sum_{j=1}^{n} \partial^{j-1} \cdot \text{Res}_{z=0} \langle e, \delta_z^{k+1}(\partial + z\mathcal{X})^{n-j}\mathcal{X}du^a \rangle - \partial^n \langle e, \mathcal{U}^{k+1}du^a \rangle$$

$$= -n \text{Res}_{z=0} \langle e, \delta_z^{k+1} \cdot \partial^{n-1}\mathcal{X}du^a \rangle - \partial^n \langle e, \mathcal{U}^{k+1}du^a \rangle \pmod{\mathcal{O}_{n-1}}$$

$$= -n \sum_{\ell=0}^{k} \langle e, \mathcal{U}^{\ell}(\mu + \frac{1}{2})\mathcal{U}^{k-\ell}\partial^{n-1}\mathcal{X}du^a \rangle - \partial^n \langle e, \mathcal{U}^{k+1}du^a \rangle \pmod{\mathcal{O}_{n-1}}.$$ Form the generating function

$$\sum_{k=-1}^{\infty} \lambda^{-k-2} L_k u_n^a = -n \langle e, (\lambda - \mathcal{U})^{-1}(\mu + \frac{1}{2})(\lambda - \mathcal{U})^{-1}\partial^{n-1}\mathcal{X}du^a \rangle$$

$$- \partial^n \langle e, (\lambda - \mathcal{U})^{-1}du^a \rangle \pmod{\mathcal{O}_{n-1}}.$$ We now invoke the hypothesis that $\mathcal{M}$ is semisimple. Since $\mathcal{U}dv^i = v^i dv^i$, we see that

$$\langle e, (\lambda - \mathcal{U})^{-1}du^a \rangle = \sum_{i,j} \langle \pi_j, (\lambda - v^i)^{-1}J_i^a dv^i \rangle = \sum_i \frac{J_i^a}{\lambda - v^i}.$$ It follows that

$$\partial^n \langle e, (\lambda - \mathcal{U})^{-1}du^a \rangle = \sum_i \frac{v_i^j J_i^a}{(\lambda - v^i)^2} + \sum_i \frac{\partial^n J_i^a}{\lambda - v^i} \pmod{\mathcal{O}_{n-1}}.$$ We also have

$$\partial^{n-1}\mathcal{X} = \partial^{n-1}(u_n^e A_{bc}^a) \partial_a \otimes du^b$$

$$= u_n^e A_{bc}^a \partial_a \otimes du^b = \sum_i v_i^a \pi_i \otimes dv^i \pmod{\mathcal{O}_{n-1}},$$ hence

$$\partial^{n-1}\mathcal{X}du^a = \sum_i v_i^a J_i^a dv^i \pmod{\mathcal{O}_{n-1}}.$$ It follows that

$$\langle e, (\lambda - \mathcal{U})^{-1}(\mu + \frac{1}{2})(\lambda - \mathcal{U})^{-1}\partial^{n-1}\mathcal{X}du^a \rangle = \sum_{i,j} \frac{v_i^j J_i^a(\mu_j + \frac{1}{2} \delta_j^i)}{(\lambda - v^i)(\lambda - v^j)}$$

$$= \frac{1}{2} \sum_i \frac{v_i^j J_i^a}{(\lambda - v^i)^2} + \sum_{i,j} v_i^j J_i^a \gamma_j^i \left( \frac{1}{\lambda - v^i} - \frac{1}{\lambda - v^j} \right) \pmod{\mathcal{O}_{n-1}}.$$
Assembling these results, we see that
\[ \sum_{k=-1}^{\infty} \lambda^{-k-2} \mathcal{L}_k u_n^a = -\left(\frac{n}{2} + 1\right) \sum_i v_i^a \mathcal{J}_i^a \left(\lambda - v^i\right)^2 - \sum_i \frac{c_i^a}{\lambda - v^i} \mod \mathcal{O}_{n-1} \]
for some coefficients \(c_i^a \in \mathcal{O}_n\).

Now suppose that \(f \in \mathcal{O}_n\) satisfies \(\mathcal{L}_k f \in \mathcal{O}_{n-1}\) for all \(k \geq 0\). Extracting the coefficient of \((\lambda - v^i)^{-2}\) in
\[ \sum_{k=-1}^{\infty} \lambda^{-k-2} \mathcal{L}_k f = \partial_{n,a} f \sum_{k=-1}^{\infty} \lambda^{-k-2} \mathcal{L}_k u_n^a \mod \mathcal{O}_{n-1}, \]
we see that \(\left(\frac{n}{2} + 1\right) v_i^a \mathcal{J}_i^a \partial_{n,a} f = 0\), and hence that \(\partial_{n} f = 0\). In other words, \(f \in \mathcal{O}_{n-1}\). □

**Corollary 6.6.** On a semisimple Frobenius manifold, any solution \(\{\mathcal{F}_g \in \mathcal{O}_\infty \mid g \geq 1\}\) of the Virasoro constraints (Theorem 5.7) and the dilaton equation (5.2) is unique (except in genus 1, where it is determined up to an additive constant).

**Proof.** Consider two solutions \(\{\mathcal{F}_g\}\) and \(\{\tilde{\mathcal{F}}_g\}\) of the Virasoro constraints; we argue by induction on \(g\) that \(\mathcal{F}_g = \tilde{\mathcal{F}}_g\). Suppose that \(\mathcal{F}_h = \tilde{\mathcal{F}}_h\) for \(h < g\). We see that \(\mathcal{L}_k(\mathcal{F}_g - \tilde{\mathcal{F}}_g) = 0\) for all \(k \geq -1\), hence by Theorem 6.5 \(\mathcal{F}_g - \tilde{\mathcal{F}}_g\) is a constant. If \(g > 1\), the constant vanishes by the dilaton equation, which implies that
\[ Q(\mathcal{F}_g - \tilde{\mathcal{F}}_g) = (2g - 2)(\mathcal{F}_g - \tilde{\mathcal{F}}_g). \] □

**6.5. The function \(\mathcal{G}\) on a semisimple Frobenius manifold.** A solution of the genus 1 Virasoro constraints (5.3) on a semisimple Frobenius manifold was found by Dubrovin and Zhang [12]; by Theorem 6.5, their solution is the unique one. Their formula for \(\mathcal{G}\) involves a function on the semisimple locus \(M_0\) called the isomonodromic \(\tau\)-function \(\tau_I\); in defining it, we follow the exposition of Hertling [22].

**Lemma 6.7.** The differential form
\[ \alpha = \frac{1}{8} \sum_{i,j} (v^i - v^j) \eta_{ij}^2 \frac{\eta_{ij}}{\eta_{ij}} \Omega^1(M_0) \]
is closed, and \(\alpha(X) = -\frac{1}{4} \text{Tr}(\mu[\gamma, A_X])\)

**Proof.** We have
\[ d\alpha = \frac{1}{8} \sum_{i,j,k} \frac{\eta_{ij}}{\eta_{ij}} \left( \frac{1}{2} \eta_{ijk} \eta_{jk} \eta_{ik} - \frac{1}{2} \eta_{ijk} \eta_{jk} \eta_{ik} \right) \left( v^i - v^j \right) dv^k \wedge dv^i = \frac{1}{8} \sum_{i,j,k} \frac{\eta_{ijk}}{\eta_{ijk}} \eta_{ijk} \left( v^i - v^j \right) dv^k \wedge dv^i. \]

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This vanishes, since \((v^i - v^j) dv^k \wedge dv^i\) is antisymmetric in the indices \(i, j\) and \(k\).

The matrix \(\mathcal{A}_i\) has components \((\mathcal{A}_i)^k_j = \delta^k_i \delta^i_j\). Since the matrices \(\mathcal{U}\) and \(\mathcal{A}_i\) are diagonal in the frame \(dv^i\) of \(T^* M_0\), we have
\[
\partial_i \log \tau_I = -\frac{1}{4} \text{Tr}([\gamma, \mathcal{U}][\gamma, \mathcal{A}_i])
\]
\[
= -\frac{1}{4} \text{Tr}(\gamma \mathcal{U} \gamma \mathcal{A}_i - \gamma \mathcal{A}_i \gamma - \mathcal{U} \gamma^2 \mathcal{A}_i + \mathcal{U} \gamma \mathcal{A}_i \gamma)
\]
\[
= -\frac{1}{4} \sum_j (\gamma^i_j \mathcal{U}_j \gamma^j_i - \gamma^j_i \mathcal{U}_i \gamma^i_j - \mathcal{U}_i \gamma^i_j \gamma^j_i + \mathcal{U}_j \gamma^j_i \gamma^i_j)
\]
\[
= \frac{1}{2} \sum_j (v^i - v^j) \gamma^i_j \gamma^j_i = \langle \pi_i, \alpha \rangle.
\]
□

The isomonodromic tau-function \(\tau_I\) is defined by the equation
\[
d \log \tau_I = \alpha.
\]
Note that just like \(\mathcal{G}\), \(\log \tau_I\) is only determined up to an additive constant.

**Theorem 6.8.** \(\mathcal{G} = \log \tau_I - \frac{1}{48} \sum_i \log \eta_i\)

**Proof.** We calculate the derivative of the two terms contributing to \(L_k \mathcal{G}\) separately: we will see that they respectively contribute the two terms of \((5.3)\). We have
\[
L_k \log \tau_I = \frac{1}{4} \text{Tr}(\mu[\gamma, \mathcal{A}_{E^\gamma(k+1)}]) = \frac{1}{4} \text{Tr}(\mu[\gamma, \mathcal{U}^{k+1}])
\]
\[
= \frac{1}{4} \sum_{\ell=0}^k \text{Tr}(\mu \mathcal{U}^{\ell}[\gamma, \mathcal{U}] \mathcal{U}^{k-\ell}) = \frac{1}{4} \sum_{\ell=0}^k \text{Tr}(\mu \mathcal{U}^{\ell} \mu \mathcal{U}^{k-\ell}).
\]
Further, since \(e(\eta_i) = 0\), we have
\[
\frac{1}{2} L_k (\log \eta_i) = \frac{1}{2} (L_k + (v^i)^{k+1} e)(\log \eta_i) = \sum_j \left((v^j)^{k+1} - (v^j)^{k+1}\right) \gamma^i_j
\]
\[
= \sum_j \frac{(v^j)^{k+1} - (v^j)^{k+1}}{v^i - v^j} \mu^i_j = \sum_{\ell=0}^k \sum_j (v^j)^{\ell} \mu^i_j (v^j)^{k-\ell}
\]
\[
= \sum_{\ell=0}^k \left(e, \mathcal{U}^{\ell} \mu \mathcal{U}^{k-\ell} dv^i\right).
\]
Summing over \(i\), the Virasoro constraints \(z_{k,1}\) follows, since \(\omega = \sum_i dv^i\). □

It is proved in \([11]\) that \(\mathcal{G}\) automatically satisfies the differential equation \(\Psi \mathcal{G} = 0\) of \((5.4)\). Ideally, this should be part of a larger phenomenon, whereby all of the known differential equations satisfied by the Gromov-Witten potentials \(\{\mathcal{F}^{X_g}\}_{g \geq 1}\), namely the Virasoro constraints and the topological recursion relations, have a (necessarily unique) solution on every semisimple Frobenius manifold.
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