**Soft Closure Spaces**

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**Abstract.** In this paper, the concept of soft closure spaces is defined and studied its basic properties. We show that the concept soft closure spaces are a generalization to the concept of Čech soft closure spaces introduced by Krishnaveni and Sekar. In addition, the concepts of subspaces and product spaces are extended to soft closure spaces and discussed some of their properties.

1. **Introduction**

There are many mathematical tools obtainable for dealing with an imperfect knowledge or for modelling complex systems such as probability theory, fuzzy set theory, rough set theory and also in computer science, engineering, physics, social sciences, economics, and medical sciences, etc. All these tools require the pre-specification of some parameters to start with. To conquer these obstacles, in 1999 Molodtsov [12] proposed a new mathematical tool, namely soft set theory to model uncertainty, which associates a set with a set of parameters. After Molodtsove’s activity work, in 2003 Maji et al. [10] presented and studied several basic notions of soft set theory and some operation between two soft sets. The Applications of the theory of soft sets have been in many areas of mathematics. In 2011, Shahir and Naz [14] defined and studied the soft topological space. In 2014, El-Sheikh and Abd El-Latif [5] initiated the notion of supra soft topological spaces, which is wider and more general than the class of soft topological spaces.

The concept of closure space \((\mathcal{M}, \mathcal{U})\) were introduced by Čech [3] in 1968, where \(\mathcal{U}: P(\mathcal{M}) \to P(\mathcal{M})\) is a mapping defined on the power set \(P(\mathcal{M})\) of a set \(\mathcal{M}\) satisfying: (C1)\(\mathcal{U}(\emptyset) = \emptyset\), (C2)\(\mathcal{A} \subseteq \mathcal{U}(\mathcal{A})\) and (C3)\(\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{U}(\mathcal{A}) \subseteq \mathcal{U}(\mathcal{B})\), the mapping \(\mathcal{U}\) called closure operator on \(\mathcal{M}\). A closure operator \(\mathcal{U}\) is called Čech closure operator, if \(\mathcal{U}\) satisfies: (C4)\(\mathcal{U}(\mathcal{A} \cup \mathcal{B}) = \mathcal{U}(\mathcal{A}) \cup \mathcal{U}(\mathcal{B})\) and then \((\mathcal{M}, \mathcal{U})\) is called Čech closure space. Čech closure spaces studied by several authors and in several directions. In 1985, Mashhour and Ghanim [11] introduced the concept of Čech closure spaces in fuzzy setting. Independently, in 2014, Gowri and Jegadeesan [7] and Krishnaveni and Sekar [8] defined and studied Čech closure spaces in soft setting. Recently, Majeed [9] using fuzzy soft sets to define the concept of Čech fuzzy soft closure spaces.

In this work, motivated by the theory of soft sets we introduced the notion of soft closure spaces. In Section 3, the concept of soft closure spaces is defined. Also, the notion of closed (respectively, open) soft sets in soft closure spaces is defined and give the basic properties of them with several examples to explain these concepts. In addition, we show our notion of soft closure space in more general than
the notion of Čech soft closure spaces that defined by Krishnaveni and Sekar [8] (see Proposition 3.4). Moreover, we find for every soft closure space there exists a supra soft topology associative with it (see Remark 3.18). In Section 4, the soft closure subspace of a soft closure space is defined and studied with details. We discuss the relationships between the closed (respectively, open) soft sets in the soft-cs and its soft-c.subsp (see Proposition 4.7 and Theorems 4.10 and 4.12) Finally, Section 5 is devoted to introduce the notion of the product of soft closure spaces and studied its basic properties.

2. Preliminaries
In this section we recall some basic definitions and results of soft set theory defined and discussed by various authors. Throughout this paper, \( \mathcal{M} \) refers to the initial universe, \( P(\mathcal{M}) \) denote the power set of \( \mathcal{M} \) and \( R \) is the set of all parameters for \( \mathcal{M} \).

**Definition 2.1** [12] A soft set \( \mathbb{F}_R = (\mathcal{F}, R) \) over the universe set \( \mathcal{M} \) is defined by a mapping \( \mathcal{F} : R \rightarrow P(\mathcal{M}) \). Then \( \mathbb{F}_R \) can be represented by the set \( \mathcal{F}_R = \{(r, \mathcal{F}(r)) : r \in R \text{ and } \mathcal{F}(r) \in P(\mathcal{M})\} \). We denote the family of all soft sets over \( \mathcal{M} \) is denoted by \( SS(\mathcal{M}, R) \).

**Definition 2.2** [10] A null soft set, which denoted by \( \mathcal{F}_R \), is a soft set \( \mathbb{F}_R \) over \( \mathcal{M} \) such that for all \( r \in R, \mathcal{F}(R) = \emptyset \) (empty set).

**Definition 2.3** [10] An absolute soft set, which denoted by \( \mathcal{M} \), is a soft set \( \mathbb{F}_R \) over \( \mathcal{M} \) such that for all \( r \in R, \mathcal{F}(r) = R \).

**Definition 2.4** [6] Let \( \mathcal{F}_R \) and \( \mathcal{G}_R \) be two soft sets over \( \mathcal{M} \). Then, \( \mathcal{F}_R \subseteq \mathcal{G}_R \), if \( \mathcal{F}(r) \subseteq \mathcal{G}(r) \) for all \( r \in R \). \( \mathcal{F}_R \) equals \( \mathcal{G}_R \), denoted by \( \mathcal{F}_R = \mathcal{G}_R \) if \( \mathcal{F}_R \subseteq \mathcal{G}_R \) and \( \mathcal{G}_R \subseteq \mathcal{F}_R \).

**Definition 2.5** [10] The union of two soft sets \( \mathbb{F}_R \) and \( \mathbb{G}_R \) over \( \mathcal{M} \) is the soft set \( \mathcal{H}_R \) defined as \( \mathcal{H}(r) = \mathcal{F}(r) \cup \mathcal{G}(r) \) for all \( r \in R \). This is denoted by \( \mathcal{F}_R \sqcup \mathcal{G}_R \). And the soft intersection of \( \mathbb{F}_R \) and \( \mathbb{G}_R \) is the soft set \( \mathcal{H}_R \) given by \( \mathcal{H}(r) = \mathcal{F}(r) \cap \mathcal{G}(r) \) for all \( r \in R \) and denoted by \( \mathcal{F}_R \cap \mathcal{G}_R \).

**Definition 2.6** [14] Let \( \mathbb{F}_R \) and \( \mathcal{G}_R \) be two soft sets over \( \mathcal{M} \), the difference \( \mathcal{H}_R \) of \( \mathcal{F}_R \) and \( \mathcal{G}_R \) is denoted by \( \mathcal{F}_R - \mathcal{G}_R \), and defined as \( \mathcal{H}(r) = \mathcal{F}(r) - \mathcal{G}(r) \) for all \( r \in R \).

**Definition 2.7** [14] The relative complement of a soft set \( \mathbb{F}_R \) is denoted by \( \mathcal{F}_R^c \), where \( \mathcal{F}^c : R \rightarrow P(\mathcal{M}) \) defined as \( \mathcal{F}^c(r) = \mathcal{M} - \mathcal{F}(r) \), for all \( r \in R \). Clearly, \( \mathcal{F}_R^c = \mathcal{M} - \mathcal{F}_R \).

**Definition 2.8** [4, 15] The soft set \( \mathbb{F}_R \in SS(\mathcal{M}, R) \) is called soft point in \( \mathcal{M} \), denoted by \( x_r \), if for the element \( r \in R, \mathcal{F}(r) = \{x\} \) and \( \mathcal{F}(r') = \emptyset \) for every \( r' \in R - \{r\} \).

**Definition 2.9** [4, 15] The soft point \( x_r \) is said to be in the soft set \( \mathcal{G}_R \), denoted by \( x_r \in \mathcal{G}_R \), if for the element \( r \in R \), we have \( \{x\} \subseteq \mathcal{G}(r) \).

**Definition 2.10** [2] Let \( \mathbb{F}_R \in SS(\mathcal{M}, R) \) and \( \mathcal{G}_S \in SS(\mathcal{Y}, S) \). The Cartesian product \( \mathbb{F}_R \times \mathcal{G}_S \) is defined by \( \left( (\mathcal{F} \times \mathcal{G})_{R \times S} \right) \), where \( (\mathcal{F} \times \mathcal{G})_{R \times S}(r, s) = \mathcal{F}(r) \times \mathcal{G}(s) \), for all \( (r, s) \in R \times S \). According to this definition the soft set \( \mathbb{F}_R \times \mathcal{G}_S \) is a soft set over \( \mathcal{M} \times \mathcal{Y} \) and its parameter universe is \( R \times S \).

The pairs of projections \( p_M : \mathcal{M} \times \mathcal{Y} \rightarrow \mathcal{M}, q_R : R \times S \rightarrow R \) and \( p_Y : \mathcal{M} \times \mathcal{Y} \rightarrow \mathcal{Y}, q_S : R \times S \rightarrow S \) determine morphisms respectively \( (p_M, q_R) \) from \( \mathcal{M} \times \mathcal{Y} \) to \( \mathcal{M} \) and \( (p_Y, q_S) \) from \( \mathcal{M} \times \mathcal{Y} \) to \( \mathcal{Y} \), where
(p_M, q_R)(F_R \times G_S) = p_M(F \times G)_{q_R(R \times S)} \text{ and } (p_Y, q_S)(F_R \times G_S) = p_Y(F \times G)_{q_S(R \times S)} \text{.} [1].

Definition 2.11 [5] A supra soft topological space is the triple \((M, T^*, R)\), where \(M\) is universe set, \(R\) is the fixed set of parameters and \(T^*\) is the collection of soft sets over \(M\), which are satisfies:

1. \(\Phi_R, \bar{M} \in T^*\),
2. The union of any number of soft sets in \(T^*\) belongs to \(T^*\).

The members of \(T^*\) are called supra open soft sets. A soft set \(F_R\) is called supra closed soft in \(M\) if, \(\bar{M} - F_R \in T^*\).

Definition 2.12 [14] Let \(Y\) be a non-empty subset of \(M\) and \(F_R\) be a soft set over \(M\). Then the subsoft set of \(F_R\) over \(Y\) denoted by \(F^Y_R\) is defined as follows \(F^Y_R(r) = Y \cap F_R\) for all \(r \in R\).

In other words that is \(F^Y_R = \bar{Y} \cap F_R\) where \(\bar{Y}\) denotes to the soft set \(Y_R\) over \(M\) for which \(Y(r) = Y\), for all \(r \in R\).

Definition 2.13 [13] Let \((M, T^*, R)\) be a supra soft topological space and \(Y\) be a non-empty subset of \(M\). Then, \(T^{\ast \cap Y} = \{F^Y_R : F_R \in T^*\}\) is called the supra soft relative topology on \(Y\) and \((Y, T^{\ast \cap Y}, R)\) is called a supra soft subspace of \((M, T^*, R)\).

3. The Basic Structures of Soft Closure Spaces

This section is devoted to introduce the notion of soft closure spaces and discussed the basic properties of these spaces.

Definition 3.1 An operator \(\bar{u}: SS(M, R) \to SS(M, R)\) is called a soft closure operator (soft-co, for short) on \(M\), if for all \(F_R, G_R \in SS(M, R)\) the following axioms are satisfied:

(C1) \(\Phi_R = \bar{u}(\Phi_R)\),
(C2) \(F_R \subseteq \bar{u}(F_R)\),
(C3) \(F_R \subseteq G_R \Rightarrow \bar{u}(F_R) \subseteq \bar{u}(G_R)\).

The triple \((M, \bar{u}, R)\) is called a soft closure space (soft-cs, for short).

Next, we give two examples to explain the notion in Definition 3.1.

Example 3.2 Let \(M = \{a, b, c\}\) and \(R = \{r_1, r_2\}\). Define a soft-co \(\bar{u}: SS(M, R) \to SS(M, R)\) as follows:

\[
\bar{u}(F_R) = \begin{cases} 
\Phi_R & \text{if } F_R = \Phi_R, \\
\{(r_1, \{c\}), (r_2, \{b\})\} & \text{if } F_R = \{(r_1, \{c\}), (r_2, \{b\})\}, \\
\{(r_1, \{b\}), (r_2, \{c\})\} & \text{if } F_R = \{(r_1, \{b\}), (r_2, \{c\})\}, \\
\bar{M} & \text{otherwise}.
\end{cases}
\]

Clearly, the soft-co \(\bar{u}\) satisfies the three axioms of Definition 3.1. Hence \((X, \bar{u}, R)\) is a soft-cs.

Example 3.3 Let \(M = \{a, b, c\}\) and \(R = \{r_1, r_2\}\). Define a soft-co \(\bar{u}: SS(M, R) \to SS(M, R)\) as follows:

\[
\bar{u}(F_R) = \begin{cases} 
\Phi_R & \text{if } F_R = \Phi_R, \\
\{(r_2, \{c\})\} & \text{if } F_R = \{(r_1, \{c\})\}, \\
\bar{M} & \text{otherwise}.
\end{cases}
\]

Then, it clear that the axiom \((C2)\) of Definition 3.1 is not hold because there exists \(F_R \in SS(M, R)\), where \(F_R = \{(r_1, \{c\})\}\) such that \(\{(r_1, \{c\})\} \not\subset \{(r_2, \{c\})\} = \bar{u}(F_R)\) and hence \((M, \bar{u}, R)\) is not soft-cs.
Now we give the relationship between our definition of soft-cs and the definition of Čech soft closure space introduced in [8].

**Proposition 3.4** Every Čech soft closure space is a soft-cs.

**Proof:** Let \((\mathcal{M}, \tilde{u}, R)\) be a Čech soft-cs. To show \((\mathcal{M}, \tilde{u}, R)\) is soft-cs, it is sufficient to prove the soft-co \(\tilde{u}\) satisfies the axioms (C3) in Definition 3.1. Now, let \(F_R, G_R \in SS(\mathcal{M}, R)\) such that \(F_R \subseteq G_R\). It is clear that \(\tilde{u}(F_R) \subseteq \tilde{u}(F_R) \cup \tilde{u}(G_R)\). By the axiom (C3) of definition Čech soft closure operator we get, \(\tilde{u}(F_R) \subseteq \tilde{u}(F_R \cup G_R) = \tilde{u}(G_R)\). This implies \(\tilde{u}(F_R) \subseteq \tilde{u}(G_R)\) and hence \(\tilde{u}\) is a soft-co and \((\mathcal{M}, \tilde{u}, R)\) is soft-cs.

**Remark 3.5** The convers of Proposition 3.4 is not true as the following example shows

**Example 3.6** Let \(\mathcal{M} = \{a, b\}\) and \(R = \{r_1, r_2\}\). Define a soft-co \(\tilde{u}: SS(\mathcal{M}, R) \rightarrow SS(\mathcal{M}, R)\) as follows:

\[
\tilde{u}(F_R) = \begin{cases}
\emptyset_R & \text{if } F_R = \emptyset_R, \\
\{r_1,\{a, b\}\} & \text{if } F_R = \{r_1,\{a\}\}, \\
\{r_2,\{b\}\} & \text{if } F_R = \{r_2,\{b\}\}, \\
\mathcal{M} & \text{otherwise}.
\end{cases}
\]

Then, \((\mathcal{M}, \tilde{u}, R)\) is a soft-cs, but it is not Čech soft closure space since there exist \(F_R, G_R \in SS(\mathcal{M}, R)\), where \(F_R = \{(r_1,\{a\})\}\) and \(G_R = \{(r_2,\{b\})\}\) such that \(\tilde{u}(F_R \cup G_R) \neq \tilde{u}(F_R) \cup \tilde{u}(G_R)\).

**Definition 3.7** Let \((\mathcal{M}, \tilde{u}, R)\) be a soft-cs. A soft subset \(F_R\) over \(\mathcal{M}\) is said to be a closed soft set, if \(\tilde{u}(F_R) = \emptyset_R\). A soft subset \(G_R\) over \(\mathcal{M}\) is called an open soft set if it is soft complement \(\mathcal{M} - F_R\) is closed soft set.

**Example 3.8** In Example 3.6, it is clear that \(F_R = \{(r_1,\{b\})\}\) is a closed soft set and its complement \(\mathcal{M} - F_R = \{(r_1,\{a\}), (r_2,\{a, b\})\}\) is an open soft set. While, the soft set \(F_R = \{(r_1,\{a\})\}\) is not a closed soft set neither open soft set.

**Proposition 3.9** Let \((\mathcal{M}, \tilde{u}, R)\) be a soft-cs and \(F_R \in SS(\mathcal{M}, R)\). If \(\tilde{u}(F_R) \subseteq F_R\), then \(F_R\) is a closed soft set in \((\mathcal{M}, \tilde{u}, R)\).

**Proof:** The proof obtained directly from hypothesis and Definition 3.1.

**Theorem 3.10** Let \((\mathcal{M}, \tilde{u}, R)\) be a soft-cs and let \(G_R \in SS(\mathcal{M}, R)\). Then, \(\tilde{u}(G_R) - G_R\) contains no non-empty open soft subset.

**Proof:** Let \(G_R\) be a soft subset in \((\mathcal{M}, \tilde{u}, R)\) and \(H_R\) be a nonempty open soft subset of \(\tilde{u}(G_R) - G_R\). Then, there exists a soft point \(x_R \in H_R\) such that \(x_R \notin \tilde{u}(G_R) - G_R\). This implies \(x_R \notin \tilde{u}(\mathcal{M} - H_R)\). Which is a closed soft set. Therefore, \(x_R \notin \tilde{u}(\mathcal{M} - H_R)\). That means, \(\tilde{u}(G_R)\) not contained in \(\tilde{u}(\mathcal{M} - H_R)\). Since \(H_R \subseteq \tilde{u}(G_R) - G_R\), then \(\tilde{u}(G_R) - H_R \subseteq \mathcal{M} - H_R\). From (C3), we get \(\tilde{u}(G_R) \subseteq \tilde{u}(\mathcal{M} - H_R)\) and this is a contradiction. Therefore, \(\tilde{u}(G_R) - G_R\) contains no non-empty open soft set.

**Proposition 3.11** Let \((\mathcal{M}, \tilde{u}, R)\) be a soft-cs and \(\{(F_R)_\alpha : \alpha \in J\}\) be a family of soft subsets over \(\mathcal{M}\). Then:

1. \(\cup_{\alpha \in J} \tilde{u}((F_R)_\alpha) \subseteq \tilde{u}(\cup_{\alpha \in J} (F_R)_\alpha)\).
2. \(\tilde{u}(\cap_{\alpha \in J} (F_R)_\alpha) \subseteq \cap_{\alpha \in J} \tilde{u}((F_R)_\alpha)\).
Proof:

1- For all \(a \in J\) we have, \((\mathcal{F}_R)_a \subseteq \bigcup_{a \in J} (\mathcal{F}_R)_a\). From (C2) of Definition 3.1, we get for all \(a \in J\), \(\tilde{u}(\bigcup_{a \in J} (\mathcal{F}_R)_a) \subseteq \tilde{u}(\bigcup_{a \in J} (\mathcal{F}_R)_a)\). This implies, \(\bigcup_{a \in J} \tilde{u}(\mathcal{F}_R)_a \subseteq \tilde{u}(\bigcup_{a \in J} (\mathcal{F}_R)_a)\).

2- For all \(a \in J\), since \(\bigcap_{a \in J} (\mathcal{F}_R)_a \subseteq (\mathcal{F}_R)_a\). Then, by (C2) of Definition 3.1, we have \(\tilde{u}(\bigcap_{a \in J} (\mathcal{F}_R)_a) \subseteq \tilde{u}(\mathcal{F}_R)_a\) for all \(a \in J\). Hence, \(\tilde{u}(\bigcap_{a \in J} (\mathcal{F}_R)_a) \subseteq \bigcap_{a \in J} \tilde{u}(\mathcal{F}_R)_a\).

Remark 3.12 The inclusion of Proposition 3.11 cannot be replaced by equalities in general as the following example shows.

Example 3.13 Let \(\mathcal{M} = \{a, b, c\}\) and \(R = \{r_1, r_2\}\). Define a soft-co \(\tilde{u}: SS(\mathcal{M}, R) \rightarrow SS(\mathcal{M}, R)\) as follows:

\[
\tilde{u}(\mathcal{F}_R) = \begin{cases} 
\bar{\Phi}_R & \text{if } \mathcal{F}_R = \bar{\Phi}_R, \\
\{r_1, (a)\} & \text{if } \mathcal{F}_R = \{(r_1, (a))\}, \\
\{r_1, (b)\} & \text{if } \mathcal{F}_R = \{(r_1, (b))\}, \\
\{r_2, (c)\} & \text{if } \mathcal{F}_R = \{(r_2, (c))\}, \\
\bar{M} & \text{other wise.}
\end{cases}
\]

Then, \((\mathcal{M} , \tilde{u}, R)\) is a soft-cs. Let \(\mathcal{F}_R = \{(r_1, (a))\}\) and \(G_R = \{(r_1, (b))\}\), then it is clear that \(\tilde{u}(\mathcal{F}_R \cup G_R) = \bar{M} \neq \{(r_1, (a), (b))\} = \tilde{u}(\mathcal{F}_R) \cup \tilde{u}(G_R)\).

Also, if we take \(\mathcal{F}_R = \{(r_1, (a))\}\) and \(K_R = \{(r_1, (b), (c))\}\), then \(\tilde{u}(\mathcal{F}_R \cap K_R) = \bar{\Phi}_R \neq \{(r_1, (a))\} = \tilde{u}(\mathcal{F}_R) \cap \tilde{u}(K_R)\).

Proposition 3.14 The intersection of any collection of closed soft sets in a soft-cs is a closed soft set.

Proof: Let \(\{(\mathcal{F}_R)_a : a \in J\}\) be a family of closed sets in a soft-cs \((\mathcal{M}, \tilde{u}, R)\). We must prove \(\tilde{u}(\bigcap_{a \in J} (\mathcal{F}_R)_a) = \bigcap_{a \in J} \tilde{u}(\mathcal{F}_R)_a\). Since \(\bigcap_{a \in J} (\mathcal{F}_R)_a \subseteq (\mathcal{F}_R)_a\) for all \(a \in J\), then by (C3) of Definition 3.1, we get \(\tilde{u}(\bigcap_{a \in J} (\mathcal{F}_R)_a) \subseteq \tilde{u}(\bigcap_{a \in J} (\mathcal{F}_R)_a) = (\mathcal{F}_R)_a\) (by (C3) of Definition 3.1, we get \(\tilde{u}(\bigcap_{a \in J} (\mathcal{F}_R)_a) \subseteq \tilde{u}(\bigcap_{a \in J} (\mathcal{F}_R)_a)\)). This implies \(\tilde{u}(\bigcap_{a \in J} (\mathcal{F}_R)_a) = \bigcap_{a \in J} \tilde{u}(\mathcal{F}_R)_a\). On the other hand from (C2), it follows that \(\bigcap_{a \in J} (\mathcal{F}_R)_a \subseteq \tilde{u}(\bigcap_{a \in J} (\mathcal{F}_R)_a)\). Therefore, \(\tilde{u}(\bigcap_{a \in J} (\mathcal{F}_R)_a) = \bigcap_{a \in J} (\mathcal{F}_R)_a\). Hence, the result.

Corollary 3.15 The union of any collection of open soft sets in a soft-cs is an open soft set.

Proof: Let \(\{(\mathcal{F}_R)_a : a \in J\}\) be a family of open sets in a soft-cs \((\mathcal{M}, \tilde{u}, R)\). Clearly the complement of \(\bigcup_{a \in J} (\mathcal{F}_R)_a\) is \(\bar{M} = \bigcap_{a \in J} (\mathcal{F}_R)_a\). Since \(\bigcap_{a \in J} (\mathcal{F}_R)_a \subseteq (\mathcal{F}_R)_a\), then \(\bar{M} - (\mathcal{F}_R)_a\) is a closed soft set. By Proposition 3.14, we have \(\bigcap_{a \in J} \bar{M} - (\mathcal{F}_R)_a\). Therefore, \(\tilde{u}(\bigcap_{a \in J} (\mathcal{F}_R)_a) = \bigcap_{a \in J} (\mathcal{F}_R)_a\). Hence, the result.

Corollary 3.16 Let \(\{(\mathcal{F}_R)_a : a \in J\}\) be a collection of closed soft sets in a soft-cs \((\mathcal{M}, \tilde{u}, R)\). Then, \(\tilde{u}(\bigcap_{a \in J} (\mathcal{F}_R)_a) = \bigcap_{a \in J} \tilde{u}(\mathcal{F}_R)_a\).

Proof: The proof follows from Proposition 3.14 and definition of closed soft set.

Remark 3.17 The intersection (respectively, union) of any family of open (respectively, closed) soft sets in a soft-cs \((\mathcal{M}, \tilde{u}, R)\) need not to be an open (respectively, closed) soft set.

To explain that, in Example 3.6, there exist \(\mathcal{F}_R = \{(r_1, (b))\}\) and \(G_R = \{(r_2, (b))\}\) are closed soft sets but their union is not a closed soft set. In addition, there exist \(H_R = \{(r_1, (a)), (r_2, (a))\}\) and \(K_R = \{(r_1, (a), (b)), (r_2, (a))\}\) are open soft sets but \(H_R \cap K_R = \{(r_1, (a)), (r_2, (a))\}\) is not an open soft set in \((\mathcal{M}, \tilde{u}, R)\).
Remark 3.18 From Corollary 3.15 and Remark 3.17, it follows for each soft-cs there exists an underlying supra soft topological space that can be defined in a natural way:

Let $(\mathcal{M}, \tilde{u}, R)$ be a soft-cs, we denote the associative supra soft topology on $\mathcal{M}$ by $T_{\tilde{u}}$. That is $T_{\tilde{u}} = \{\overline{\tilde{u}}(R) = R\}$. The members of $T_{\tilde{u}}$ are called supra open soft sets and the complements are called supra closed soft sets.

i.e., $F_R$ is an open (respectively, closed) set in $(\mathcal{M}, \tilde{u}, R) \iff F_R$ is a supra open (respectively, closed) set in $(\mathcal{M}, T_{\tilde{u}}, R)$.

Example 3.19 In Example 3.2, the associative supra soft topology on $\mathcal{M}$ is $T_{\tilde{u}} = \{\overline{\tilde{u}}(R) = R\}$ which is a supra soft topology on $\mathcal{M}$. In addition, $T_{\tilde{u}}$ is not necessarily to be a soft topology on $\mathcal{M}$ since there exist $F_R, G_R \in T_{\tilde{u}}$, where $F_R = \{(r_1, \{a, 1\}), (r_2, \{b\})\}$ and $G_R = \{(r_1, \{a\}), (r_2, \{a, b\})\}$. However, $F_R \cap G_R = \{(r_1, \{a\}), (r_2, \{a\})\} \notin T_{\tilde{u}}$.

Definition 3.20 Let $\tilde{u}_1$ and $\tilde{u}_2$ be two soft-co’s on $\mathcal{M}$. Then $\tilde{u}_1$ is said to be finer than $\tilde{u}_2$, or equivalently, $\tilde{u}_2$ is coarser than $\tilde{u}_1$, if $\tilde{u}_1(F_R) \subseteq \tilde{u}_2(F_R)$ for all $F_R \in SS(\mathcal{M}, R)$.

Now, we give an example to explain the above definition.

Example 3.21 Let $\mathcal{M} = \{a, b, c\}$, and $R = \{r_1, r_2\}$. Define $\tilde{u}_1, \tilde{u}_2 : SS(\mathcal{M}, R) \rightarrow SS(\mathcal{M}, R)$ as follows:

$$\begin{align*}
\tilde{u}_1(F_R) &= \begin{cases} 
\overline{\tilde{u}}_R & \text{if } F_R = \overline{\tilde{u}}_R, \\
\{(r_1, \{a\})\} & \text{if } F_R = \{(r_1, \{a\})\}, \\
\{(r_2, \{b\})\} & \text{if } F_R = \{(r_2, \{b\})\}, \\
\{(r_2, \{c\})\} & \text{if } F_R = \{(r_2, \{c\})\}, \\
\mathcal{M} & \text{otherwise.}
\end{cases}
\end{align*}$$

And,

$$\begin{align*}
\tilde{u}_2(F_R) &= \begin{cases} 
\overline{\tilde{u}}_R & \text{if } F_R = \overline{\tilde{u}}_R, \\
\{(r_1, \{a, b\})\} & \text{if } F_R = \{(r_1, \{a\})\}, \\
\{(r_2, \{b, c\})\} & \text{if } F_R = \{(r_2, \{b\})\}, \\
\{(r_2, \{c\})\} & \text{if } F_R = \{(r_2, \{c\})\}, \\
\mathcal{M} & \text{otherwise.}
\end{cases}
\end{align*}$$

Then, it is easy to verify that $\tilde{u}_1$ and $\tilde{u}_2$ are soft-co’s on $\mathcal{M}$ and $\tilde{u}_1$ is finer than $\tilde{u}_2$ since for all $F_R \in SS(\mathcal{M}, R)$, $\tilde{u}_1(F_R) \subseteq \tilde{u}_2(F_R)$.

Theorem 3.22 Let $\tilde{u}_1$ and $\tilde{u}_2$ be two soft-co’s on $\mathcal{M}$. Define $\tilde{u}_1 \cup \tilde{u}_2, \tilde{u}_1 \cap \tilde{u}_2 : SS(\mathcal{M}, R) \rightarrow SS(\mathcal{M}, R)$ as follows: for all $F_R \in SS(\mathcal{M}, R)$, $\tilde{u}_1 \cup \tilde{u}_2(F_R) = \tilde{u}_1(F_R) \cup \tilde{u}_2(F_R)$ and $\tilde{u}_1 \cap \tilde{u}_2(F_R) = \tilde{u}_1(F_R) \cap \tilde{u}_2(F_R)$. Then, $\tilde{u}_1 \cup \tilde{u}_2$ and $\tilde{u}_1 \cap \tilde{u}_2$ are soft-co’s on $\mathcal{M}$.

Proof: We prove $\tilde{u}_1 \cup \tilde{u}_2$ is a soft-co on $\mathcal{M}$ and similarly one can prove $\tilde{u}_1 \cap \tilde{u}_2$ is soft-co on $\mathcal{M}$.

Now, we must prove $\tilde{u}_1 \cup \tilde{u}_2$ satisfies the axioms (C1), (C2) and (C3) of Definition 3.1.

(C1) $(\tilde{u}_1 \cup \tilde{u}_2)(\overline{\tilde{u}}_R) = \tilde{u}_1(\overline{\tilde{u}}_R) \cup \tilde{u}_2(\overline{\tilde{u}}_R) = \overline{\tilde{u}}_R \subseteq \overline{\tilde{u}}_R \subseteq \overline{\tilde{u}}_R = \overline{\tilde{u}}_R$.

(C2) For all $F_R \in SS(\mathcal{M}, R)$. Since $\tilde{u}_1$ and $\tilde{u}_2$ are soft-co’s on $\mathcal{M}$, then $F_R \subseteq \tilde{u}_1(F_R)$ and $F_R \subseteq \tilde{u}_2(F_R)$. This implies $F_R \subseteq \tilde{u}_1(F_R) \cap \tilde{u}_2(F_R)$.

(C3) Let $F_R, G_R \in SS(M, R)$ such that $F_R \subseteq G_R$. Since $\tilde{u}_1$ and $\tilde{u}_2$ are soft-co’s on $\mathcal{M}$, then $\tilde{u}_1(F_R) \subseteq \tilde{u}_1(G_R)$ and $\tilde{u}_2(F_R) \subseteq \tilde{u}_2(G_R)$. It follows, $\tilde{u}_1(F_R) \cup \tilde{u}_2(F_R) \subseteq \tilde{u}_1(G_R) \cup \tilde{u}_2(G_R)$ which implies, $(\tilde{u}_1 \cup \tilde{u}_2)(F_R) \subseteq (\tilde{u}_1 \cup \tilde{u}_2)(G_R)$. Hence, $\tilde{u}_1 \cup \tilde{u}_2$ is a soft-co on $\mathcal{M}$.
4. Soft closure subspaces

In this section we introduce the notion of soft closure subspace of a soft-cs and investigate some properties of its.

**Theorem 4.1** Let \((M, \tilde{u}, R)\) be a soft-cs and let \(\mathcal{Y} \subseteq M\). Let \(\tilde{u}_\mathcal{Y} : SS(\mathcal{Y}, R) \rightarrow SS(\mathcal{Y}, R)\) defined by \(\tilde{u}_\mathcal{Y}(F_R) = \mathcal{Y} \cap \tilde{u}(F_R)\). Then, \(\tilde{u}_\mathcal{Y}\) is a soft-co on \(\mathcal{Y}\).

**Proof:** We must prove \(\tilde{u}_\mathcal{Y}\) satisfying the axioms (C1) – (C3) of Definition 3.1.

- (C1) \(\tilde{u}_\mathcal{Y}(\Phi_R) = \mathcal{Y} \cap \tilde{u}(\Phi_R) = \mathcal{Y} \cap \Phi_R = \Phi_R\).
- (C2) For all \(F_R \in SS(\mathcal{Y}, R)\), we have \(F_R \subseteq \mathcal{Y}\) and \(F_R \subseteq \tilde{u}(F_R)\). This implies \(F_R \subseteq \mathcal{Y} \cap \tilde{u}(F_R) = \tilde{u}_\mathcal{Y}(F_R)\). Thus, \(F_R \subseteq \tilde{u}_\mathcal{Y}(F_R)\).
- (C3) Let \(F_R, G_R \in SS(\mathcal{Y}, R)\) such that \(F_R \subseteq G_R\). Since \(\tilde{u}\) is a soft-co, then \(\tilde{u}(F_R) \subseteq \tilde{u}(G_R)\). Therefore, \(\mathcal{Y} \cap \tilde{u}(F_R) \subseteq \mathcal{Y} \cap \tilde{u}(G_R)\) which means \(\tilde{u}_\mathcal{Y}(F_R) \subseteq \tilde{u}_\mathcal{Y}(G_R)\).

**Definition 4.2** Let \((M, \tilde{u}, R)\) be a soft-cs, and let \(\mathcal{Y} \subseteq M\). The soft closure operator \(\tilde{u}_\mathcal{Y}\) (defined in the Theorem 4.1) is called the relative soft closure operator on \(\mathcal{Y}\) induced by \(\tilde{u}\). The triple \((\mathcal{Y}, \tilde{u}_\mathcal{Y}, R)\) is called a soft closure subspace (soft-c.subsp, for short) of \((M, \tilde{u}, R)\).

**Remark 4.3** The soft-c.subsp \((\mathcal{Y}, \tilde{u}_\mathcal{Y}, R)\) is a closed (respectively, open) soft subspace if \(\tilde{u}(\mathcal{Y}) = \mathcal{Y}\) (respectively, \(\tilde{u}(\bar{M} - \mathcal{Y}) = (\bar{M} - \mathcal{Y})\)).

In the next we give an example to explain the notion of soft-c.subsp.

**Example 4.4** Let \((M, \tilde{u}, R)\) be a soft-cs as defined in Example 3.2, where \(M = \{a, b, c\}\), \(R = \{r_1, r_2\}\) and \(\tilde{u} : SS(M, R) \rightarrow SS(M, R)\) defined by

\[
\tilde{u}(F_R) = \begin{cases} 
\Phi_R & \text{if } F_R = \Phi_R, \\
\{(r_1, \{c\}), (r_2, \{b\})\} & \text{if } F_R \subseteq \{(r_1, \{c\}), (r_2, \{b\})\}, \\
\{(r_1, \{b\}), (r_2, \{c\})\} & \text{if } F_R \subseteq \{(r_1, \{b\}), (r_2, \{c\})\}, \\
\mathcal{M} & \text{other wise.}
\end{cases}
\]

Let \(\mathcal{Y} = \{a, b\} \subseteq M\), then \(\tilde{u}_\mathcal{Y} : SS(\mathcal{Y}, R) \rightarrow SS(\mathcal{Y}, R)\) defined as follows: for all \(G_R \in SS(\mathcal{Y}, R)\)

\[
\tilde{u}_\mathcal{Y}(G_R) = \begin{cases} 
\Phi_R & \text{if } G_R = \Phi_R, \\
\{(r_1, \{b\})\} & \text{if } G_R \subseteq \{(r_1, \{b\})\}, \\
\{(r_2, \{b\})\} & \text{if } G_R \subseteq \{(r_2, \{b\})\}, \\
\mathcal{Y} & \text{other wise.}
\end{cases}
\]

Then, \((\mathcal{Y}, \tilde{u}_\mathcal{Y}, R)\) is soft-c.subsp of \((M, \tilde{u}, R)\).

**Remark 4.5** Let \((M, \tilde{u}, R)\) be a soft-cs and \((\mathcal{Y}, \tilde{u}_\mathcal{Y}, R)\) be a soft-c.subsp of \((M, \tilde{u}, R)\). If \((M, T_{\tilde{u}}, R)\) and \((\mathcal{Y}, T_{\tilde{u}_\mathcal{Y}}, R)\) be the supra soft topological spaces induced form \((M, \tilde{u}, R)\) and \((\mathcal{Y}, \tilde{u}_\mathcal{Y}, R)\) respectively. Then \((\mathcal{Y}, T_{\tilde{u}_\mathcal{Y}}, R)\) is a supra soft subspace of the supra soft topological space \((M, T_{\tilde{u}}, R)\).

We can use Example 4.4 to explain Remark 4.5. Therefore, \(T_{\tilde{u}} = \{\Phi_R, M, \{(r_1, \{a, b\}), (r_2, \{a, c\})\}, \{(r_1, \{a, c\}), (r_2, \{a, b\})\}\) and since \(T_{\tilde{u}_\mathcal{Y}} = \{F_R : F_R \in T_{\tilde{u}}\}\), then it follows \(T_{\tilde{u}_\mathcal{Y}} = \{\Phi_R, \mathcal{Y}, \{(r_1, \{a\}), (r_2, \{a, b\})\}, \{(r_1, \{a, b\}), (r_2, \{a\})\}\).
**Theorem 4.6** Let \( (\mathcal{M}, \tilde{u}, R) \) be a soft-cs and \( \mathcal{Y} \subseteq \mathcal{M} \). Then the relative supra soft topology \( (\tilde{T}_{\tilde{u}})_{\tilde{Y}} \) on \( \mathcal{Y} \) induced by \( \tilde{T}_{\tilde{u}} \) is coarser than the associative supra soft topology \( \tilde{T}_{\tilde{u}} \) on \( \mathcal{Y} \).

**Proof:** We must prove \( (\tilde{T}_{\tilde{u}})_{\tilde{Y}} \subseteq \tilde{T}_{\tilde{u}} \mathcal{Y} \). Let \( F_R \) be a \( (\tilde{T}_{\tilde{u}})_{\tilde{Y}} \)-closed soft set over \( \mathcal{Y} \). Then, there exists a \( \tilde{T}_{\tilde{u}} \)-supra closed soft set \( G_R \) such that \( F_R = \tilde{Y} \cap G_R \). Since \( F_R \subseteq G_R \), then \( \tilde{u}(F_R) \subseteq \tilde{u}(G_R) = G_R \). This implies \( \tilde{u}_Y(F_R) = \tilde{Y} \cap \tilde{u}(F_R) \subseteq \tilde{Y} \cap G_R = F_R \). Therefore, \( \tilde{u}_Y(F_R) = F_R \) and this implies \( F_R \) is a supra closed soft set in \( (\mathcal{Y}, \tilde{T}_{\tilde{u}}) \). Hence, \( (\tilde{T}_{\tilde{u}})_{\tilde{Y}} \subseteq \tilde{T}_{\tilde{u}} \mathcal{Y} \).

**Proposition 4.7** Let \( (\mathcal{Y}, \tilde{u}_y, R) \) be a soft-c.subsp of \( (\mathcal{M}, \tilde{u}, R) \). If \( F_R \in SS(\mathcal{M}, R) \), and \( F_R \) is a closed soft set in \( \mathcal{M} \), then \( F_R \) is a closed soft set in \( (\mathcal{Y}, \tilde{u}_y, R) \).

**Proof:** Let \( F_R \in SS(\mathcal{M}, R) \) such that \( \tilde{u}(F_R) = F_R \). Now, \( \tilde{u}_y(F_R) = \tilde{Y} \cap \tilde{u}(F_R) = \tilde{Y} \cap F_R = F_R \). Hence, \( F_R \) is a closed soft set in \( (\mathcal{Y}, \tilde{u}_y, R) \).

**Remark 4.8** The converse of Proposition 4.7 is not true as the following example shows.

**Example 4.9** In Example 4.4, consider \( G_R = \{(r_1, \{b\})\} \) which is a closed soft set in \( (\mathcal{Y}, \tilde{u}_y, R) \) but it is not a closed soft set in \( \mathcal{M} \) since \( \tilde{u}(G_R) = \{(r_1, \{b\}), (r_2, \{c\})\} \neq G_R \).

The following Theorem give the condition to be the converse of Proposition 4.7 is hold in general.

**Theorem 4.10** Let \( (\mathcal{M}, \tilde{u}, R) \) be a soft-cs and \( (\mathcal{Y}, \tilde{u}_y, R) \) be a closed soft subspace of \( (\mathcal{M}, \tilde{u}, R) \). If \( F_R \) is a closed soft set of \( \mathcal{Y} \) and \( \tilde{u}_y, R) \), then \( F_R \) is a closed soft set in \( (\mathcal{M}, \tilde{u}, R) \).

**Proof:** To prove \( F_R \) is a closed soft set of \( (\mathcal{M}, \tilde{u}, R) \) we must show \( \tilde{u}(F_R) = F_R \). Since \( F_R \) is a closed soft set of \( (\mathcal{Y}, \tilde{u}_y, R) \), then \( \tilde{u}_y(F_R) = F_R \), which means \( \tilde{Y} \cap \tilde{u}(F_R) = F_R \). From hypothesis we have \( \tilde{u}(\tilde{Y}) = \tilde{Y} \). Thus, it follows \( \tilde{u}(\tilde{Y}) \cap \tilde{u}(F_R) = F_R \). From Proposition 3.11(2), we have \( \tilde{u}(\tilde{Y} \cap F_R) \subseteq \tilde{u}(\tilde{Y}) \cap \tilde{u}(F_R) = F_R \). This yield, \( \tilde{u}(F_R) \subseteq F_R \). On the other hand, \( F_R \subseteq \tilde{u}(F_R) \). Therefore, we obtain \( \tilde{u}(F_R) = F_R \) and hence \( F_R \) is a closed soft set of \( (\mathcal{M}, \tilde{u}, R) \).

**Remark 4.11** In Theorem 4.10, the soft set \( \tilde{Y} \) is a closed soft set in \( \mathcal{M} \) is a necessary condition for this theorem. We can explain that in more details. In Example 4.4, \( \tilde{Y} = \{(r_1, \{a, b\}), (r_2, \{a, b\})\} \) is not a closed soft set in \( (\mathcal{M}, \tilde{u}, R) \) (because \( \tilde{u}(\tilde{Y}) \neq \tilde{Y} \)). Let \( G_R = \{(r_1, \{b\})\} \) be a closed soft set \( (\mathcal{Y}, \tilde{u}_y, R) \). Then, it is clear that \( G_R \) is not a closed soft set in \( (\mathcal{M}, \tilde{u}, R) \) since \( \tilde{u}(G_R) = \{(r_1, \{b\}), (r_2, \{c\})\} \neq G_R \).

**Theorem 4.12** Let \( (\mathcal{Y}, \tilde{u}_y, R) \) be a soft- c.subsp of a soft-cs \( (\mathcal{M}, \tilde{u}, R) \). If \( F_R \) is an open soft set in \( (\mathcal{Y}, \tilde{u}_y, R) \), then \( \tilde{Y} \cap G_R \) is an open soft set in \( (\mathcal{Y}, \tilde{u}_y, R) \).

**Proof:** To prove \( \tilde{Y} \cap G_R \) is an open soft set in \( (\mathcal{Y}, \tilde{u}_y, R) \), we must show \( \tilde{u}(\tilde{Y} \cap G_R) = \tilde{Y} \cap G_R \) is a closed soft set in \( (\mathcal{Y}, \tilde{u}_y, R) \). Now, \( \tilde{u}_y(\tilde{Y} \cap G_R) = \tilde{Y} \cap \tilde{u}(G_R) \subseteq \tilde{Y} \cap \tilde{u}(\tilde{M} - G_R) = \tilde{Y} \cap (\tilde{M} - G_R) \).

**Remark 4.13** The converse of Theorem 4.12 is not true as the following example shows.
Example 4.14 In Example 4.4, consider the soft set \( G_R = \{(r_1, \{b\}), (r_2, \{a, b\})\} \) is an open soft set in \((\tilde{Y}, \tilde{u}_Y, R)\) since \( \tilde{u}_Y(\tilde{Y} - G_R) = \tilde{Y} - G_R \). But \( G_R \) is not an open soft set in \((\mathcal{M}, \tilde{u}, R)\). Because \( \mathcal{M} - G_R \) is not a closed soft set in \( \mathcal{M} \).

5. The product of soft closure spaces

In this section, we define the product of a collection of soft-cs’s and give the properties of open and closed soft sets in the product soft-cs.

**Theorem 5.1** Let \( \{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha) : \alpha \in J\} \) be a family of soft-cs’s. Define a soft operator \( \bigotimes \tilde{u} : SS(\prod_{\alpha \in J} \mathcal{M}_\alpha, \prod_{\alpha \in J} R_\alpha) \rightarrow SS(\prod_{\alpha \in J} \mathcal{M}_\alpha, \prod_{\alpha \in J} R_\alpha) \), where \( \prod_{\alpha \in J} \mathcal{M}_\alpha \) and \( \prod_{\alpha \in J} R_\alpha \) denotes to the Cartesian product of the sets \( \mathcal{M}_\alpha \) and \( R_\alpha \), \( \alpha \in J \), respectively as follows:

\[
\bigotimes \tilde{u}(F_{\prod_{\alpha \in J} R_\alpha}) = \prod_{\alpha \in J} \tilde{u}_\alpha \left((p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(F_{\prod_{\alpha \in J} R_\alpha}\right)\right), \forall F_{\prod_{\alpha \in J} R_\alpha} \in SS(\prod_{\alpha \in J} \mathcal{M}_\alpha, \prod_{\alpha \in J} R_\alpha).
\]

Then, the operator \( \bigotimes \tilde{u} \) is a soft closure operator on \( \prod_{\alpha \in J} \mathcal{M}_\alpha \).

**Proof:** We must prove \( \bigotimes \tilde{u} \) satisfies the axioms (C1)-(C3) of Definition 3.1.

(C1) Let \( \bigotimes \tilde{u}(\Phi_{\prod_{\alpha \in J} R_\alpha}) = \prod_{\alpha \in J} \tilde{u}_\alpha \left((p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(\Phi_{\prod_{\alpha \in J} R_\alpha}\right)\right) = \prod_{\alpha \in J} \tilde{u}_\alpha \left(\Phi_{R_\alpha}\right) = \prod_{\alpha \in J} \tilde{u}_\alpha \left(\Phi_{\prod_{\alpha \in J} R_\alpha}\right). \)

This implies \( \prod_{\alpha \in J} \left((p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(\Phi_{\prod_{\alpha \in J} R_\alpha}\right)\right) = \prod_{\alpha \in J} \tilde{u}_\alpha \left((p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(\Phi_{\prod_{\alpha \in J} R_\alpha}\right)\right). \)

(C2) Let \( F_{\prod_{\alpha \in J} R_\alpha} \in SS(\prod_{\alpha \in J} \mathcal{M}_\alpha, \prod_{\alpha \in J} R_\alpha) \). For all \( \alpha \in J \), since \( \tilde{u}_\alpha \) is a soft-co on \( \mathcal{M}_\alpha \), then it follows \( (p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(F_{\prod_{\alpha \in J} R_\alpha}\right) \subseteq \tilde{u}_\alpha \left((p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(F_{\prod_{\alpha \in J} R_\alpha}\right)\right) \).

Since \( F_{\prod_{\alpha \in J} R_\alpha} \subseteq \prod_{\alpha \in J} \left((p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(F_{\prod_{\alpha \in J} R_\alpha}\right)\right) \), then we have \( F_{\prod_{\alpha \in J} R_\alpha} \subseteq \prod_{\alpha \in J} \tilde{u}_\alpha \left((p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(F_{\prod_{\alpha \in J} R_\alpha}\right)\right) = \bigotimes \tilde{u}(F_{\prod_{\alpha \in J} R_\alpha}). \)

Therefore, \( F_{\prod_{\alpha \in J} R_\alpha} \subseteq \bigotimes \tilde{u}(F_{\prod_{\alpha \in J} R_\alpha}). \)

(C3) Let \( F_{\prod_{\alpha \in J} R_\alpha} \in G_{\prod_{\alpha \in J} R_\alpha} \). Then, for all \( \alpha \in J \), \( (p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(F_{\prod_{\alpha \in J} R_\alpha}\right) \subseteq \prod_{\alpha \in J} \tilde{u}_\alpha \left((p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(G_{\prod_{\alpha \in J} R_\alpha}\right)\right) = \bigotimes \tilde{u}(G_{\prod_{\alpha \in J} R_\alpha}). \)

This implies \( \prod_{\alpha \in J} \tilde{u}_\alpha \left((p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(F_{\prod_{\alpha \in J} R_\alpha}\right)\right) \subseteq \tilde{u}_\alpha \left((p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(F_{\prod_{\alpha \in J} R_\alpha}\right)\right) \).

Thus, \( \prod_{\alpha \in J} \tilde{u}_\alpha \left((p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(F_{\prod_{\alpha \in J} R_\alpha}\right)\right) \subseteq \prod_{\alpha \in J} \tilde{u}_\alpha \left((p_{\mathcal{M}_\alpha}, q_{R_\alpha}) \left(G_{\prod_{\alpha \in J} R_\alpha}\right)\right) \). And that means \( \bigotimes \tilde{u}(F_{\prod_{\alpha \in J} R_\alpha}) \subseteq \bigotimes \tilde{u}(G_{\prod_{\alpha \in J} R_\alpha}). \) Hence, we get the result.

**Definition 5.2** Let \( \{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha) : \alpha \in J\} \) be a family of soft-cs’s, and let \( \bigotimes \tilde{u} \) be the soft-co defined as in Theorem 5.1. Then the triple \( (\prod_{\alpha \in J} \mathcal{M}_\alpha, \bigotimes \tilde{u}, \prod_{\alpha \in J} R_\alpha) \) is said to be the product soft-cs of the family \( \{(\mathcal{M}_\alpha, \tilde{u}_\alpha, R_\alpha) : \alpha \in J\} \).

**Example 5.3** Let \( \mathcal{M}_1 = \{a, b\}, \mathcal{M}_2 = \{x, y, z\}, R_1 = \{r_1, r_2\} \) and \( R_2 = \{k_1, k_2\} \). Define soft-co’s \( \tilde{u}_1 \) and \( \tilde{u}_2 \) on \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) respectively as follows:

\[
\tilde{u}_1(\Phi_{R_1}) = \begin{cases} \Phi_{R_1} & \text{if } \Phi_{R_1} = \Phi_{r_1}, \\ \{(r_1, \{a, b\})\} & \text{if } \Phi_{R_1} = \{(r_1, \{a\})\}, \\ \{\{(r_1, \{b\})\} & \text{if } \Phi_{R_1} = \{(r_1, \{b\})\}, \\ \mathcal{M}_1 & \text{otherwise.} \end{cases}
\]

And, \( \tilde{u}_2(\Phi_{R_2}) = \begin{cases} \Phi_{R_2} & \text{if } \Phi_{R_2} = \Phi_{k_1}, \\ \{(k_1, \{x\})\} & \text{if } \Phi_{R_2} = \{(k_1, \{x\})\}, \\ \mathcal{M}_2 & \text{otherwise.} \end{cases} \)

Then, \( (\mathcal{M}_1, \tilde{u}_1, R_1) \) and \( (\mathcal{M}_2, \tilde{u}_2, R_2) \) are soft-cs’s. Let \( p_{\mathcal{M}_1} : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathcal{M}_1, q_{R_1} : R_1 \times R_2 \rightarrow R_1 \) and \( p_{\mathcal{M}_2} : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathcal{M}_2, q_{R_2} : R_1 \times R_2 \rightarrow \mathcal{R}_2 \) be the projection maps. Then, \( (\mathcal{M}_1 \times \mathcal{M}_2, \bigotimes \tilde{u}, R_1 \times R_2) \)
is the product soft-cs of $\tilde{u}_1$ and $\tilde{u}_2$, where $\tilde{u}: SS(M_1 \times M_2, R_1 \times R_2) \rightarrow SS(M_1 \times M_2, R_1 \times R_2)$ defined as: for all $F_{R_1 \times R_2} \in SS(M_1 \times M_2, R_1 \times R_2)$, $\tilde{u}(F_{R_1 \times R_2}) = \tilde{u}_1((p_{M_1}, q_{R_1})(F_{R_1 \times R_2})) \times \tilde{u}_2((p_{M_2}, q_{R_2})(F_{R_1 \times R_2}))$.

For example, if we take $F_{R_1 \times R_2} = \{(r_1, k_1), ((a, x))\}$. Then,

$$\tilde{u}(F_{R_1 \times R_2}) = \tilde{u}_1((p_{M_1}, q_{R_1})(F_{R_1 \times R_2})) \times \tilde{u}_2((p_{M_2}, q_{R_2})(F_{R_1 \times R_2}))$$

$$= \{(r_1, k_1), ((a, x))\} \times \{(k_1, \{x\})\}$$

$$= \{((r_1, k_1), ((a, x), (b, x)))\}$$

It is clear that, $F_{R_1 \times R_2} \subseteq \tilde{u}(F_{R_1 \times R_2})$.

**Theorem 5.4** Let $\{(M_\alpha, \tilde{u}_\alpha, R_\alpha) : \alpha \in J\}$ be a family of soft-cs’s. Then, $F_{R_\alpha}$ is a closed soft set in $(M_\alpha, \tilde{u}_\alpha, R_\alpha)$ for all $\alpha \in J$ if and only if $\prod_{\alpha \in J} F_{R_\alpha}$ is a closed soft set in $(\prod_{\alpha \in J} M_\alpha, \tilde{u}, \prod_{\alpha \in J} R_\alpha)$.

**Proof:** Let $\alpha \in J$ and $F_{R_\alpha}$ be a closed soft set of $(M_\alpha, \tilde{u}_\alpha, R_\alpha)$. Then, $\tilde{u}_\alpha(F_{R_\alpha}) = F_{R_\alpha}$ for all $\alpha \in J$.

From the definition of soft projection map, it follows, $(p_{M_\alpha}, q_{R_\alpha})(\prod_{\alpha \in J} F_{R_\alpha}) = F_{R_\alpha}$. Hence, $\prod_{\alpha \in J} F_{R_\alpha} = \prod_{\alpha \in J} \tilde{u}_\alpha(F_{R_\alpha}) = \prod_{\alpha \in J} \tilde{u}_\alpha((p_{M_\alpha}, q_{R_\alpha})(\prod_{\alpha \in J} F_{R_\alpha})) = \tilde{u}(\prod_{\alpha \in J} F_{R_\alpha})$. That means, $\prod_{\alpha \in J} F_{R_\alpha}$ is a closed soft set in $(\prod_{\alpha \in J} M_\alpha, \tilde{u}, \prod_{\alpha \in J} R_\alpha)$.

Conversely, Let $\alpha \in J$ and $F_{R_\alpha} \in SS(M_\alpha, R_\alpha)$, to prove $\tilde{u}_\alpha(F_{R_\alpha}) = F_{R_\alpha}$. From hypothesis we have $\prod_{\alpha \in J} F_{R_\alpha}$ is a closed soft set in $(\prod_{\alpha \in J} M_\alpha, \tilde{u}, \prod_{\alpha \in J} R_\alpha)$. This means $\tilde{u}(\prod_{\alpha \in J} F_{R_\alpha}) = \prod_{\alpha \in J} \tilde{u}_\alpha((p_{M_\alpha}, q_{R_\alpha})(\prod_{\alpha \in J} F_{R_\alpha}))$. By compute the soft projection, we get $(p_{M_\alpha}, q_{R_\alpha})(\prod_{\alpha \in J} F_{R_\alpha}) = (p_{M_\alpha}, q_{R_\alpha})(\prod_{\alpha \in J} \tilde{u}_\alpha((p_{M_\alpha}, q_{R_\alpha})(\prod_{\alpha \in J} F_{R_\alpha})))$. It follows,

$F_{R_\alpha} = \tilde{u}_\alpha((p_{M_\alpha}, q_{R_\alpha})(\prod_{\alpha \in J} F_{R_\alpha})) = \tilde{u}_\alpha(F_{R_\alpha})$. Therefore, $F_{R_\alpha}$ is a closed soft set in $(M_\alpha, \tilde{u}_\alpha, R_\alpha)$ for all $\alpha \in J$.

**Lemma 5.5** Let $\{(M_\alpha, \tilde{u}_\alpha, R_\alpha) : \alpha \in J\}$ be a collection of soft-cs’s and $\nu \in J$. If $G_{\prod_{\alpha \in J} R_\alpha} \subseteq \prod_{\alpha \in J} M_\alpha$ and $((x_\alpha)_{(\alpha)})_{\alpha \in J} \in G_{\prod_{\alpha \in J} R_\alpha}$, then $x_{\nu_{\alpha \nu}} \times \prod_{\alpha \in J} \tilde{u}_{\alpha \nu}((p_{X_\nu}, q_{R_\nu})(((x_\alpha)_{(\alpha)})_{\alpha \in J})) \subseteq \prod_{\alpha \in J} M_\alpha - G_{\prod_{\alpha \in J} R_\alpha}$ for all $x_{\nu_{\alpha \nu}} \subseteq M_{\nu}, (p_{X_\nu}, q_{R_\nu}) \in G_{\prod_{\alpha \in J} R_\alpha}$.

**Proof:** Let $G_{\prod_{\alpha \in J} R_\alpha} \subseteq \prod_{\alpha \in J} M_\alpha$ and $((x_\alpha)_{(\alpha)})_{\alpha \in J} \in G_{\prod_{\alpha \in J} R_\alpha}$. Let $\nu \in J$ and $x_{\nu_{\alpha \nu}} \subseteq M_{\nu} - (p_{M_{\nu}}, q_{R_{\nu}})(G_{\prod_{\alpha \in J} R_\alpha})$. Then, $x_{\nu_{\alpha \nu}} \subseteq \tilde{M}_{\nu} - (p_{M_{\nu}}, q_{R_{\nu}})(G_{\prod_{\alpha \in J} R_\alpha})$. Since $((x_\alpha)_{(\alpha)})_{\alpha \in J} \subseteq G_{\prod_{\alpha \in J} R_\alpha}$, then $(p_{M_\alpha}, q_{R_\alpha})((x_\alpha)_{(\alpha)})_{\alpha \in J} \subseteq (p_{M_\alpha}, q_{R_\alpha})(G_{\prod_{\alpha \in J} R_\alpha})$ for all $\alpha \in J$. That means, $\prod_{\alpha \in J}((p_{M_\alpha}, q_{R_\alpha})((x_\alpha)_{(\alpha)})_{\alpha \in J}) \subseteq \prod_{\alpha \in J} G_{\prod_{\alpha \in J} R_\alpha}$. Thus, $x_{\nu_{\alpha \nu}} \times \prod_{\alpha \in J} \tilde{u}_{\alpha \nu}((p_{X_\nu}, q_{R_\nu})(((x_\alpha)_{(\alpha)})_{\alpha \in J})) \subseteq \prod_{\alpha \in J} M_\alpha - G_{\prod_{\alpha \in J} R_\alpha}$.

But, $x_{\nu_{\alpha \nu}} \times \prod_{\alpha \in J} \tilde{u}_{\alpha \nu}((p_{X_\nu}, q_{R_\nu})(((x_\alpha)_{(\alpha)})_{\alpha \in J})) \notin \prod_{\alpha \in J} M_\alpha - G_{\prod_{\alpha \in J} R_\alpha}$.

Thus, $x_{\nu_{\alpha \nu}} \times \prod_{\alpha \in J} \tilde{u}_{\alpha \nu}((p_{X_\nu}, q_{R_\nu})(((x_\alpha)_{(\alpha)})_{\alpha \in J})) \notin \prod_{\alpha \in J} M_\alpha - G_{\prod_{\alpha \in J} R_\alpha}$. ■

**Lemma 5.6** Let $\{(M_\alpha, \tilde{u}_\alpha, R_\alpha) : \alpha \in J\}$ be a collection of soft-cs’s and let $\nu \in J$. If $G_{\prod_{\alpha \in J} R_\alpha} \subseteq \prod_{\alpha \in J} M_\alpha$, then $\tilde{M}_{\nu} - (p_{M_{\nu}}, q_{R_{\nu}})(G_{\prod_{\alpha \in J} R_\alpha}) \subseteq (p_{M_{\nu}}, q_{R_{\nu}})(\prod_{\alpha \in J} M_\alpha - G_{\prod_{\alpha \in J} R_\alpha})$. 


Proof: Let $v \in \mathcal{J}$ and $G_{\alpha \in \mathcal{J}} R_{\alpha} \subseteq \prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha}$. If $G_{\alpha \in \mathcal{J}} R_{\alpha} = \Phi_{\alpha \in \mathcal{J}} R_{\alpha}$, then $(p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha}) = 0$. Since $\bar{M}_v = (p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha})$, then $\bar{M}_v - (p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha}) \subseteq \bar{M}_v$. If $G_{\alpha \in \mathcal{J}} R_{\alpha} \neq \Phi_{\alpha \in \mathcal{J}} R_{\alpha}$, then there exists a soft point $(x_{(\alpha)}(r_{(\alpha)}))_{\alpha \in \mathcal{J}} \notin G_{\alpha \in \mathcal{J}} R_{\alpha}$. Let $x_{y_{\alpha}} \subseteq \bar{M}_{\alpha} - (p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha})$. Then by Lemma 5.5 we have $x_{y_{\alpha}} \times \prod_{\alpha \in \mathcal{J}} ((p_{M'_{\alpha}}, q_{R_{\alpha}})((x_{(\alpha)}(r_{(\alpha)}))_{\alpha \in \mathcal{J}})) \subseteq \prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha}$. It follows that $(p_{M'_{\alpha}}, q_{R_{\alpha}})((x_{y_{\alpha}})_{\alpha \in \mathcal{J}}(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha})) \subseteq (p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha})$. This implies $x_{y_{\alpha}} \subseteq (p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha})$. Therefore, $\bar{M}_v - (p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha}) \subseteq (p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha})$.

Theorem 5.7 Let $\{(\mathcal{M}_{\alpha}, \bar{u}_{\alpha}, R_{\alpha}); \alpha \in \mathcal{J}\}$ be a family of soft-cs's. If $G_{\alpha \in \mathcal{J}} R_{\alpha}$ is an open soft set in the product soft closure space $(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha}, \bar{u}_{\alpha}, R_{\alpha})$, then $(p_{M'_{\alpha}}, q_{R_{\alpha}})(G_{\alpha \in \mathcal{J}} R_{\alpha})$ is an open soft set in $(\mathcal{M}_{\alpha}, \bar{u}_{\alpha}, R_{\alpha})$ for all $\alpha \in \mathcal{J}$.

Proof: Let $G_{\alpha \in \mathcal{J}} R_{\alpha}$ be an open soft set of $(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha}, \bar{u}_{\alpha}, R_{\alpha})$. Then, $\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha}$ is a closed soft set in $(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha}, \bar{u}_{\alpha}, R_{\alpha})$. That is mean, $\bar{u}_{\alpha}(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha}) = \prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha}$. From the definition of $\bar{u}_{\alpha}$ we obtain, $\prod_{\alpha \in \mathcal{J}} \bar{u}_{\alpha}(p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha}) = \prod_{\alpha \in \mathcal{J}} \bar{u}_{\alpha}(G_{\alpha \in \mathcal{J}} R_{\alpha})$. Hence, there exists soft point $x_{y_{\alpha}}$ such that $x_{y_{\alpha}} \subseteq \bar{M}_v - (p_{M'_{\alpha}}, q_{R_{\alpha}})(G_{\alpha \in \mathcal{J}} R_{\alpha})$. Therefore, $\bar{M}_v - (p_{M'_{\alpha}}, q_{R_{\alpha}})(G_{\alpha \in \mathcal{J}} R_{\alpha})$ is not contained in $\bar{M}_v - (p_{M'_{\alpha}}, q_{R_{\alpha}})(G_{\alpha \in \mathcal{J}} R_{\alpha})$. Hence, there exists soft point $x_{y_{\alpha}}$ such that $x_{y_{\alpha}} \subseteq \bar{M}_v - (p_{M'_{\alpha}}, q_{R_{\alpha}})(G_{\alpha \in \mathcal{J}} R_{\alpha})$. If $\alpha_{y_{\alpha}}(x_{(\alpha)}(r_{(\alpha)}))_{\alpha \in \mathcal{J}} \notin G_{\alpha \in \mathcal{J}} R_{\alpha}$ such that $(p_{M'_{\alpha}}, q_{R_{\alpha}})((x_{(\alpha)}(r_{(\alpha)}))_{\alpha \in \mathcal{J}}) \subseteq x_{y_{\alpha}}$. For all $\alpha_{y_{\alpha}} \subseteq \bar{M}_v - (p_{M'_{\alpha}}, q_{R_{\alpha}})(G_{\alpha \in \mathcal{J}} R_{\alpha})$ we have $\alpha_{y_{\alpha}}(x_{(\alpha)}(r_{(\alpha)}))_{\alpha \in \mathcal{J}} \subseteq G_{\alpha \in \mathcal{J}} R_{\alpha}$. From the projection for the last inclusion we get $(p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha}) \subseteq (p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha})$. This yields $(p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha}) \subseteq (p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha})$. From Lemma 5.6, we have $\bar{M}_v - (p_{M'_{\alpha}}, q_{R_{\alpha}})(G_{\alpha \in \mathcal{J}} R_{\alpha}) \subseteq (p_{M'_{\alpha}}, q_{R_{\alpha}})(\prod_{\alpha \in \mathcal{J}} \bar{M}_{\alpha} - G_{\alpha \in \mathcal{J}} R_{\alpha})$.
\( \{x_{r_1}^a\} \times \prod_{a \in J} \left\{ ((x_a)_a)_{a \in J} \right\} \)
\( = \left\{ (p_{M_a}, q_{R_a}) \left( ((x_a)_a)_{a \in J} \right) \right\} \times \prod_{a \in J} \left\{ (p_{M_a}, q_{R_a}) \left( ((x_a)_a)_{a \in J} \right) \right\} \)
\( = \prod_{a \in J} \left\{ (p_{M_a}, q_{R_a}) \left( ((x_a)_a)_{a \in J} \right) \right\} \)
\( = \left\{ (x_a)_a \right\}_{a \in J} \)
Consequently, \( \prod_{a \in J} \tilde{u}_a(p_{M_a}, q_{R_a}) \left( \prod_{a \in J} M_a - G_{\prod a \in J} R_a \right) \). Therefore, \( \left( (x_a)_a \right)_{a \in J} \notin \prod_{a \in J} M_a - G_{\prod a \in J} R_a \).
\( \left( (x_a)_a \right)_{a \in J} \notin \prod_{a \in J} \tilde{u}_a(p_{M_a}, q_{R_a}) \left( \prod_{a \in J} M_a - G_{\prod a \in J} R_a \right) \).
\( \left( (x_a)_a \right)_{a \in J} \notin \prod_{a \in J} \tilde{u}_a(p_{M_a}, q_{R_a}) \left( \prod_{a \in J} M_a - G_{\prod a \in J} R_a \right) \).
\( = \prod_{a \in J} \left\{ (x_a)_a \right\}_{a \in J} \).
\( \prod_{a \in J} \tilde{u}_a(p_{M_a}, q_{R_a}) \left( \prod_{a \in J} M_a - G_{\prod a \in J} R_a \right) \).
\( \prod_{a \in J} \tilde{u}_a(p_{M_a}, q_{R_a}) \left( \prod_{a \in J} M_a - G_{\prod a \in J} R_a \right) \).
\( \prod_{a \in J} \tilde{u}_a(p_{M_a}, q_{R_a}) \left( \prod_{a \in J} M_a - G_{\prod a \in J} R_a \right) \).
\( \prod_{a \in J} \tilde{u}_a(p_{M_a}, q_{R_a}) \left( \prod_{a \in J} M_a - G_{\prod a \in J} R_a \right) \).
Remark 5.8 The converse of Theorem 5.7 is not hold in general as the following example shows:

Example 5.9 Let \( M_1 = \{ a, b, c \} \), \( M_2 = \{ x, y, z \} \) and \( R_1 = \{ r_1, r_2 \} \), \( R_2 = \{ r_3, r_4 \} \). Define soft-co’s \( \tilde{u}_1 \) and \( \tilde{u}_2 \) on \( M_1 \) and \( M_2 \) respectively as follows:
\( \tilde{u}_1 \left( (F_{R_1}) \right) = \begin{cases} \Phi_{R_1} & \text{if } F_{R_1} = \Phi_{R_1}, \\ F_{R_1} & \text{if } F_{R_1} = \left\{ (r_1, c) \right\}, \left\{ (r_2, c) \right\}, \\ M_1 & \text{otherwise.} \end{cases} \)
And
\( \tilde{u}_2 \left( (F_{R_2}) \right) = \begin{cases} \Phi_{R_2} & \text{if } F_{R_2} = \Phi_{R_2}, \\ F_{R_2} & \text{if } F_{R_2} = \left\{ (r_3, x) \right\}, \left\{ (r_4, x) \right\}, \\ M_2 & \text{otherwise.} \end{cases} \)
Then, \( (M_1, \tilde{u}_1, R_1) \) and \( (M_2, \tilde{u}_2, R_2) \) are soft-co’s. Let \( \left( p_{M_1}, q_{R_1} \right) \) and \( \left( p_{M_2}, q_{R_2} \right) \) be the soft projection maps. Consider \( G_{R_1 \times R_2} \subseteq SS (M_1 \times M_2, R_1 \times R_2) \), where \( G_{R_1 \times R_2} = \{ (r_1, r_2), (a, x), (a, y), (b, x), (b, y) \}, \left\{ (r_1, r_2), (a, x), (a, y), (b, x), (b, y) \right\}, \left\{ (r_2, r_3), (a, x), (a, y), (b, x), (b, y) \right\}, \left\{ (r_2, r_4), (a, x), (a, y), (b, x), (b, y) \right\} \).
Then, \( \left( p_{M_1}, q_{R_1} \right) \left( G_{R_1 \times R_2} \right) = \{ (r_1, a, b), (r_2, (a, b)) \} \), and \( \left( p_{M_2}, q_{R_2} \right) \left( G_{R_1 \times R_2} \right) = \{ (r_3, x, y), (r_4, (x, y)) \} \) are open soft sets in \( (M_1, \tilde{u}_1, R_1) \) and \( (M_2, \tilde{u}_2, R_2) \), respectively. But \( G_{R_1 \times R_2} \) is not an open in \( (M_1 \times M_2, \otimes \tilde{u}, R_1 \times R_2) \). Since \( M_1 \times M_2 - G_{R_1 \times R_2} \) is not closed soft set in \( (M_1 \times M_2, \otimes \tilde{u}, R_1 \times R_2) \).

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