SLE with Jumps and Conformal Null Vectors

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Abstract

Ordinary SLE$_\kappa$ is defined using a Wiener noise and is related to CFT’s which have null vector at level two of conformal tower. In this paper we introduce stochastic variables which are made up of jumps and extend the ordinary SLE to have such stochastic variables. The extended SLE can be related to CFT’s which have null vectors in higher levels of Virasoro module.

Keywords: Conformal Field Theory, SLE, Stochastic Processes.

1 Introduction

Conformal field theories (CFT’s) [1] are powerful tools to analyze critical behavior of two dimensional systems. In addition, CFT has been remarkably successful in calculating the geometrical properties of some geometrical critical phenomena such as percolation, Brownian motion and so on. In recent years, another approach to study geometrical models has become popular, Schramm (Stochastic) Loewner evolution (SLE)[2]. SLE is a one-parameter family of stochastic equations which describes the growth of random curves. The parameter is usually named to be $\kappa$ and hence these evolution are sometimes referred to SLE$_\kappa$.

Some results, which were derived using CFT methods before, can be recalculated by means of SLE. Among them is Cardy’s formula[7], which can be re-derived in a more general form. As the two methods are able to calculate the same problems, one concludes that there should be a close relation between CFT and SLE. In [8] and later in [9] the relation between CFT and SLE was proposed. The method used by [8] can be simply stated as follows: using Ito’s formulas for Loewner’s equation and then extending the Witt algebra to Virasoro algebra, an equivalent stochastic equation in Virasoro group is found, and if the CFT model has null vector at level two, one is able to find the martingales of SLE. In this case the area taken into account is the upper half plane, but in [13] it was explained that if we work on some other domains, some special null vectors in higher levels could be produced.

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To find other null vectors, we introduce new stochastic variables, which are essentially made up of jumps, and extend the drift term of Schramm-Loewner equation to have these new variables. Doing this, we are able to produce all of the operators of the Witt algebra, so we can construct arbitrary null vectors.

In the following section we define SLE and then state its relation to CFT briefly, more details can be found in [8, 10, 11, 12]. In third section we introduce some stochastic variables which can be replaced by the Wiener noise present in ordinary SLE, and discuss their properties. The last section is devoted to differential equations corresponding to null vectors at arbitrary level.

2 Chordal Stochastic Loewner Evolutions and CFT

SLE, is a stochastic equation which reveals the evolution of mappings from upper half plane, $H$ to itself. Therefore, this equation could be regarded as the equation which gives the evolution of the boundary. Let the mapping at time $t$ be $g_t(z)$, then SLE evolution is written in the following form

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad (1)$$

where $\xi_t$ is a stochastic variable. In general, $\xi_t$ can be any function, but in standard SLE, it is taken to be Brownian motion, that is $\langle \xi_t \xi_s \rangle = k \min(t, s)$, and hence a one parameter family of such evolutions is defined, which usually are referred to SLE$_k$.

At any time, $t$, $g_t(z)$ has a domain of analyticity which is named $H_t$. This region is mapped to $H$ by the conformal mapping $g_t(z)$. The complement of $H_t$ in $H$ is called the hull in which the map is not well defined. As the times passes, the hull grows, so for a fixed $z$, the mappings are well defined up to the time $\tau_z \leq \infty$ with the property $g_{\tau_z}(z) = \sqrt{k} B_{\tau_z}$. At this point, the denominator of right hand side of (1) vanishes and the mapping becomes meaningless. The trace of SLE is defined to be the path of such singularities, that is, $\gamma(t) = \lim_{z \to 0} g_t^{-1}(z + \xi_t)$. SLE has been studied a lot in recent years and has found many different applications, for reviews see for example [3, 4, 5, 6].

To establish the connection between SLE and CFT, we first define the new series of maps $f_t(z) \equiv g_t(z) - \xi_t$ and write SLE in the Langvien form which is more convenient.

$$df_t = \frac{2 dt}{f_t} - d\xi_t. \quad (2)$$

This form of SLE helps us to write Ito’s formula :

$$d\gamma_f, F = (\gamma_f, \dot{F}) \left( \frac{2 dt}{f_t} - d\xi_t \right) + \frac{1}{2} (\gamma_f, F''),$$

where $\gamma_f, F \equiv F \circ f, \gamma_f$ belongs to the group $N_-$ of germs of holomorphic functions at $\infty$ of the form $z + \sum_{m \leq -1} f_m z^{m+1}$. We can write the equation(3) in the $N_-$ group space:

$$\gamma_f^{-1} d\gamma_f = 2 dt \left( \frac{2}{z} \partial_z + \frac{k}{2} \partial_z^2 \right) - d\xi_t \partial_z \quad (4)$$

We observe that some differential operators in the form $\ell_n = -z^{n+1} \partial_z$ have emerged. These operators form an algebra called Witt algebra

$$[l_n, l_m] = (m - n) l_{n+m} \quad (5)$$
These operators are the generators of the conformal mapping in the complex plane. On the quantum level, the corresponding algebra is the well known Virasoro algebra with corresponding generators $L_n$'s. So, the equation (4) in the Hilbert space turns out to be

$$G_{f_t}^{-1}dG_{f_t} = dt \left( -2L_{-2} + \frac{k}{2}L_{-1}^2 \right) + d\xi_t L_{-1}. \quad (6)$$

This equation defines stochastic trajectories on manifold of the group produced by elements of Virasoro algebra which have some martingales in verma module. To find these martingales we can make both side of equation (6) operate on the highest weight vector of a representation of Virasoro algebra, $|\omega\rangle$, with conformal weight $h = \frac{6-k}{2k}$ and central charge $c = \frac{(6-k)(3k-8)}{2k}$. As the vector $-2L_{-2} + \frac{k}{2}L_{-1}^2 |\omega\rangle$ is a null vector and hence is orthogonal to all vectors in Verma module, one finds

$$E[G_{f_s}|\omega\rangle] = G_{f_s}|\omega\rangle, \quad (7)$$

where the time averaging is for all times less than $s$. This means that correlation functions of the conformal field theory in $H_t$ are time independent and equal to their value at $t = 0$. Let’s see what the state $G_t|\omega\rangle$ means. Suppose $|\omega\rangle$ be a boundary changing operator in $H$, then one can show that the equivalent operator in $H_t$ is just $G_t|\omega\rangle$. In fact $G_t|\omega\rangle$ is a generating function for all conserved quantities in chordal SLE.

We can repeat this calculation for SLE in $\pi/n$ space, $\frac{H}{n}$, with the modified Loewner’s equation[13]:

$$\partial_t g_t(z) = \frac{2}{g_t(z)^{n-1}(g_t(z)^n - \xi_t)}. \quad (8)$$

gt(z) maps the hull of gt(z), $\frac{H_t}{n}$, to the whole $\frac{H}{n}$. To connect this modified equation to CFT, one defines $f_t^n = g_t(z)^n - \xi_t$, and with similar steps to derive martingales in Virasoro group for some n’s.

### 3 Stochastic Jump Variables

In the previous section, we introduced a class of stochastic equation where the stochastic part came from a Wiener noise. In this section, we’ll introduce a new stochastic variable, which is basically produced by a series of jumps, and show that in a special case, this new noise will be identical with Brownian motion. This helps us extend the stochastic Loewner equations using the new stochastic variables.

Consider the following stochastic equation

$$dx = J \, dN. \quad (9)$$

The variable $dN$ is always zero except at some points, say at $t = t_i$, where it takes the value 1. One can state this equation in another form which may be more understandable

$$\frac{dx}{dt} = \sum_i J_i \delta(t - t_i). \quad (10)$$
Now it is clear that the variable $x$, is constant. It has only some jumps at times equal with $t_i$’s with magnitude $J_i$. We have supposed that the magnitude of jumps can be different, in general we can consider that jumps have the distribution function $\rho(J)$. Also we will assume that the distribution of jumps in time is a Poison distribution.

Now we will make a special choice for $\rho(J)$ and see how it is related to Wiener noise. Take $\rho(J) = \delta(J) + (1/2)\delta''(J)$. Though it seems to be a very ridiculous choice (for example, it can not be positive everywhere), but as we shall see it lead to a well-defined noise for $x$.

Let’s focus on the properties of $x$, such as its expectation value and correlation functions. The equation (10) can be solved to find $x(t)$:

$$x(t) = \sum_i J_i \theta(t - t_i). \quad (11)$$

It is clear from form of the distribution function we have chosen, that the mean value of $x$ vanishes.

Now consider the two point correlation function $\langle x(t)x(t') \rangle$. Using the solution (11), we see

$$\langle x(t)x(t') \rangle = \left( \sum_{i,j} \langle J_i J_j \rangle J_i \theta(t - t_i) \theta(t' - t_j) \right)_t, \quad (12)$$

where the inner averaging is on different jump magnitudes and the outer one is on distribution of time of jumps. As we have assumed that magnitude of jumps are independent of one another, the inner averaging vanishes unless if $i = j$. In this case, one should compute the integral

$$\int J^2 [\delta(J) + (1/2)\delta''(J)]$$

which is equal to one. So, the inner averaging yields $\delta_{ij}$. Taking the time scale of the poison distribution of time intervals of jumps to be unity, it is easy to show the two point function (12) turns out to be

$$\langle x(t)x(t') \rangle = \min(t, t'), \quad (13)$$

which is the same as two point function of Wiener noise. It is also easy to show that one can use Wick theorem to derive n-point correlation functions. To have a more complete proof of equivalence of the two noises, one can look at their Fokker-Plank equation. Ito’s formula for the general process (10) with $J$’s having distribution function $\rho(J)$, has the following form:

$$\frac{dF(x, t)}{dt} = \frac{\partial F(x, t)}{\partial t} + \int [(F(x + J, t) - F(x, t)) \rho(J)] dJ \quad (14)$$

To derive this expression, first we should note that at any jump $x(t)$ goes to $x(t) + J$ and also the fact that the the jump magnitudes are independent of the poison jump occurrence process, that is

$$\langle [F(x + J, t) - F(x, t)] dN \rangle = \langle [F(x + J, t) - F(x, t)] \rangle dt \quad (15)$$

where the averaging is over the probability distribution of jumps. Also note that in equation (14), we have taken the time scale of the poison distribution to be unity.

Let’s consider our specific choice of $\rho(J)$. Doing the integration in equation (14) one can easily find the related Fokker-Plank function

$$\frac{\partial P(x, t)}{\partial t} + \frac{1}{2} \frac{\partial^2 P(x, t)}{\partial x^2} = 0 \quad (16)$$
which is the same as Fokker-Plank equation of Wiener noise.

One would criticize the equivalence of the two noises, telling Wiener noise is a continuous one, but a noise having jumps is not. However with the choice we have made, the process produces a continuous noise, as one can examine that it satisfies Lindberg’s condition [14]. The other point is that the probability distributions we have chosen, are somehow odd, as they have negative values at some points. Perhaps one can think of \( \rho(J) \) as not a probability distribution, rather as a measure defined in this way. This will also apply to other choices we’ll mention in below.

Before examining other choices for \( \rho(J) \), we would like to emphasize that, in SLE equation, one can use jumps with the distribution mentioned above instead of Wiener noise and derive the very same results, e. g. one can find the same Ito’s formula and the same null vectors, that is, all the results derived using equation (2) could be re-derived using the same equation, replacing Wiener noise with jumps which obey the above distribution.

Now let’s see what happens if we consider other distribution functions such as third or higher derivatives of delta functions. Take \( \rho(J) = \delta(J) + \frac{1}{n!}\delta^{(n)}(J) \), where \( \delta^{(n)}(J) \) is the \( n \)’th derivative of delta function. The solution to stochastic equation (10) is again given by equation (11), but now all the \( m \)-point functions \( \langle x(t_1)x(t_2)\ldots x(t_m) \rangle \) vanish for \( m < n \) and for \( m = n \) one has

\[
\langle x(t_1)x(t_2)\ldots x(t_n) \rangle = \min(t_1,t_2,\ldots,t_n). \tag{17}
\]

The Fokker-Plank of such evolution is just like equation (16), with the second order derivation being replaced by \( n \)’th order and the prefactor a half being replaced by \( \frac{(-1)^n}{n!} \). This allows us to extend the ordinary SLE to a more general family of stochastic evolution. In the next section we will see how this extension helps us to find relations between SLE and CFT’s with null vectors of third or higher ranks.

### 4 SLE and Higher Level Null Vectors

In section 2, we saw that if \( \xi_t \) be a Brownian motion then we were able to define an infinite set of SLE zero modes, or martingales, whose existence is a consequence of the existence of a null vector at level two. Some CFT’s do not have null vectors at level two, instead they have null vectors at higher levels. The first question that may arise is that is it possible to find some martingales which are related to higher null vectors so that SLE could be related to such models. The ordinary SLE cannot produce such null vectors, because Brownian motion is only able to produce one and second order differentiations. But as we saw in the previous section, replacing \( \xi_t \) with jumps, which have some specific distributions, then higher differentiations, and hence the required operators to produce higher level null vectors, appear.

Consider the following stochastic differential equation with respect to a compensated poisson process:

\[
df_t = a(f_t)dt + b(f_t)dB_t + c(f_t)JdN \tag{18}
\]

Taking \( a(f) = 2/f \) and \( b(f) = \sqrt{k} \) and \( c(f) = 0 \), one arrives at the ordinary SLE\(_k\) process. The last term is the one which produces jumps. Note that the magnitude of the jump is given by \( c(f_t) \times J \).
Ito’s formula for the process (18) has the following form:

\[
\begin{align*}
    dF(f_t) &= \left( a(f_t)\partial F(f_t) + \frac{b(f_t)^2}{2}\partial^2 F(f_t) + \mathcal{L}[F(f_t)] \right) dt + b(f_t)\partial F(f_t) dB_t \\
    \mathcal{L}[F(f)] &= \int \left[ (F(f + c(f)J) - F(f)) \rho(J) \right] dJ
\end{align*}
\]  

(19) (20)

The steps are the same as the ones in previous section, again the time distribution is a poisson one with time scale equal to unity, and the magnitude distributions is given by \( \rho(J) \).

Examining different distributions for \( J \) and different dependencies of \( c \) on \( f \), one is able to produce many different \( \mathcal{L} \)’s. For example take \( c(f_t) = 1 \) and \( \rho(J) = \delta(J) \) then \( \mathcal{L}F(f) = 0 \), that is we have not added any jumps. But if we take \( \rho(J) = \delta(J) + \frac{\partial^n}{\partial J^n}\delta(J) \) keeping \( c(f_t) = 1 \), the resulting operator would be

\[
\mathcal{L}F(f) = (-1)^n \frac{\partial^n}{\partial f^n} F(f)
\]

(21)

This distribution helps us to produce the \( L_{n-1} \) operators in Virasoro algebra. Another interesting choice, which is more general, is \( c(f_t) = f_t^m \) and \( \rho(J) = \delta(J) + \frac{\partial^n}{\partial J^n}\delta(J) \), which leads to the operators

\[
\mathcal{L}F(f) = (-1)^n f^m \frac{\partial^n}{\partial f^n} F(f)
\]

(22)

which are the most general form of the operators needed to construct higher level null vectors, as they can produce the general form of operators in Virasoro algebra \( L_{n_1}L_{n_2}...L_{n_k} \).

Let’s try to connect the modified Loewner’s equation to CFT’s having null vector at level 3. Consider the following stochastic evolution

\[
df_t(z) = \frac{2}{f_t(z)^2} dt - \frac{2\sqrt{k}}{3f_t(z)} dB + \alpha J dN
\]

(23)

Note that the maps are not defined in the upper half plane, they are defined in only two third of the whole complex plane, say from \( \theta = 0 \) to \( \theta = 3\pi/2 \) [13]. Let \( \rho(J) = \delta(J) + \frac{\partial^n}{\partial J^n}\delta(J) \) and \( k = \frac{18}{3\sin(\theta)} \), \( \alpha = \frac{-2}{(h+2)} \). Writing Ito’s formula (equation(19)) and doing the same steps as in the ordinary SLE case, one arrives at

\[
\gamma_t^{-1}d\gamma_t = c \left( l_{-3} - \frac{2}{h+1}l_{-1}l_{-2} + \frac{l_{-1}^3}{(h+1)(h+2)} \right) dt + \frac{2\sqrt{k}}{3f_t(z)} l_{-1} dB
\]

(24)

where \( c = 2(h+1)/(1-h) \). Averaging over Brownian motion and going to quantum level, will lead us to an operator which produces a level three null vector

\[
\left( L_{-3} - \frac{2}{h+1}L_{-1}L_{-2} + \frac{L_{-1}^3}{(h+1)(h+2)} \right) |\omega_{1,3}\rangle = 0
\]

(25)

where \( |\omega_{1,3}\rangle \) is the highest weight vector with weight equal to \( h \) in a CFT which has a null vector at level 3.

One can do the same manipulations to relate the null vector at level four to a modified SLE. In this case, one should take into account the following stochastic equation:

\[
df_t(z) = \frac{2}{f_t(z)^3} dt - \frac{\sqrt{k}}{2f_t(z)} dB - \frac{p}{2f_t(z)} J_1 dN_1 + q J_2 dN_2
\]

(26)
Together with the following conditions, the vector \( G|\omega_{1,4}\rangle \) is a martingale, where \( G \) is the proper operator derived from this evolution.

\[
\rho(J_1) = \delta(J_1) + \frac{\partial^3}{\partial J_1^3}\delta(J_1), \quad \rho(J_2) = \delta(J_2) + \frac{\partial^4}{\partial J_2^4}\delta(J_2)
\]

\[
q = \frac{-9}{2(4h^2 + 9h + 9)}, \quad p = \frac{-6(2h + 3)}{4h^2 + 9h + 9} \quad k = \frac{8(8h^2 + 12h + 9)}{4h^2 + 9h + 9}
\]

This means that we have a null vector at level four of the form

\[
\left(L_{-4} - \frac{4h}{9}L_{-2}^2 - \frac{4h + 15}{6h + 18}L_{-1}L_{-3} + \frac{2h + 3}{3h + 9}L_{-1}^2L_{-2} - \frac{1}{4h + 12}L_{-1}^4\right)|\omega_{2,2}\rangle = 0
\]  

(28)

Note that at this case, the evolution is defined in the quarter of complex plane.

Now it is clear that for any null vector at any level, we are able to define a stochastic evolution, whose martingales could be found using the properties of the null vector and the stochastic evolution on Virasoro group.

Note that it is not the only possible way to produce desired operators. We have had several assumptions which can be modified, for example the assumption that the time distribution of jumps is poisson distribution or the magnitude of jumps are independent. In general, one can define stochastic differential equations with poisson point form jumps, with a specified form for the distribution of jumps, which produce the same differential operators (see [15], and references therein).

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