Lepton number violation in a unified framework

Yuta Kawamura\textsuperscript{1,*}, Yamato Matsuo\textsuperscript{1†}, Takuya Morozumi\textsuperscript{2,3‡}, Apriadi Salim Adam\textsuperscript{1,4§}
Yusuke Shimizu\textsuperscript{2¶}, Yuya Tokunaga\textsuperscript{1}, and Naoya Toyota\textsuperscript{1∥}

\textsuperscript{1}Graduate School of Science, Hiroshima University,
Higashi-Hiroshima 739-8526, Japan

\textsuperscript{2}Physics Program, Graduate School of Advanced Science and
Engineering, Hiroshima University,
Higashi-Hiroshima 739-8526, Japan

\textsuperscript{3}Core of Research for the Energetic Universe, Hiroshima University,
Higashi-Hiroshima 739-8526, Japan

\textsuperscript{4}Gunung Sari RT.02, RW.24 Kelurahan Ngringo, Kecamatan Jaten, Kabupaten
Karanganyar, Jawa Tengah 57731, Indonesia

Abstract

We study the time evolution of lepton family number for neutrino which forms SU(2) doublet with charged lepton. The lepton family number is defined through a weak basis of SU(2) doublet in which the charged lepton mass matrix is a real and diagonal one. The lepton family number carried by the neutrino is defined with a left-handed current of the neutrino family. We study the time evolution of the lepton family number operator for Majorana neutrino. To be definite, we introduce the mass term at $t = 0$ and study the time evolution of the lepton family number for the later time. Since the operator in flavor eigenstate is continuously connected to that of the mass eigenstate, the creation and annihilation operators for flavor eigenstates are related to those of mass eigenstates. The total lepton number of the Majorana neutrino is conserved. By choosing a specific flavor eigenstate of neutrino as an initial state, we compute the time evolution of all lepton family numbers. They are sensitive to Majorana and Dirac phases and also are sensitive to the absolute mass and mass hierarchy of neutrinos.

\textsuperscript{*}E-mail address: yuta-kawamura@hiroshima-u.ac.jp
\textsuperscript{†}E-mail address: ya-matsuo@hiroshima-u.ac.jp
\textsuperscript{‡}E-mail address: morozumi@hiroshima-u.ac.jp
\textsuperscript{§}E-mail address: apriadi.adam@gmail.com
\textsuperscript{¶}E-mail address: yu-shimizu@hiroshima-u.ac.jp
\textsuperscript{∥}E-mail address: m191973@hiroshima-u.ac.jp
1 Introduction

Lepton number violation in the neutrino sector is classified into two categories. One is the flavor oscillation [1] and the other is the neutrinoless double $\beta$ decay [2] where in the former process, the total lepton number is conserved while in the latter process it is not. The flavor oscillation phenomena have been studied in an ultra-relativistic limit and the oscillation among neutrinos with an SU(2) charge and a definite chirality has been extensively studied. For Majorana neutrinos [3], neutrino-antineutrino oscillation has been also discussed [4, 5, 6].

In this paper, we develop a framework in which both phenomena, i.e., flavor oscillation and neutrino-antineutrino oscillation are formulated in a single framework. We treat them from the viewpoint of lepton family number non-conservation. We define the lepton family number of neutrino, such that the neutrino, an upper component of SU(2) doublet, has the same family number as that of the lower component, i.e., the charged lepton. We note that the lepton family number can be defined without introducing neutrino mass, since the lepton family is defined through charged lepton mass eigenstates. We also note that only this kind of family numbers are countable through the charged current weak interaction. When the mass of the neutrino is turned off, each of the three types of neutrinos with a specific lepton family number become massless and asymptotic state. Once the mass terms are turned on, they are no longer asymptotic state and each of the lepton family numbers is not conserved. The effect of the mass term depends on the types of mass. For Dirac neutrino, the flavor oscillation includes the transition from neutrino with SU(2) charge to singlet right-handed neutrino while neutrino-antineutrino oscillation triggered by Majorana mass is a transition among neutrinos in SU(2) doublet. When neutrinos are relativistic, as is in most cases, neutrino-antineutrino oscillation is suppressed. However, it can be enhanced when the energy carried by the neutrino is less than its rest mass. In fact, the cosmic neutrino background (CNB) may have such a property since its typical thermal energy is predicted as $O(10^{-4})$ (eV) for massless neutrinos. Therefore, the CNB can be studied within our present framework where neutrinos are considered to be Majorana particles.

In our work, the lepton family number carried by the neutrino is defined with a left-handed current of the neutrino family. We study the time evolution of the lepton family number operator for Majorana neutrino. To be definite, we introduce the mass term at $t = 0$ and study the time evolution of the lepton family number for the later time.

The paper is organized as follows. In section 2, we explain how the mass term is turned on. We also define the lepton family number of neutrinos and study their time evolution. In section 3, the time dependent expectation value is obtained by choosing a specific initial state with a definite lepton family number. The numerical results of the time evolution of the lepton family numbers are shown in section 4. Section 5 is devoted to summary and discussion.

2 Lepton family number

We start with the explanation of the measurement of lepton family number and the composition for our work. As an illustration, we consider the weak decay of a charged pion; $\pi^+ \to \pi^0 e^+ \nu_e$. As shown in Fig.1, when an electron neutrino is produced it is massless. Sup-
pose that the Majorana mass term is turned on at $t = 0$. Until then the neutrino continues to be an electron neutrino. We investigate the time evolution of the lepton family number after the neutrino acquires the mass term. We first discuss the one flavor case of Majorana neutrino and show how one can define the lepton number carried by left-handed neutrino. Corresponding to this situation, the Lagrangian for the left-handed neutrino $\nu_L$ is given as follows,

$$\mathcal{L} = \overline{\nu}_L i \gamma^\mu \partial_\mu \nu_L - \theta(t) \frac{m}{2} (\overline{\nu}_L)^\gamma \nu_L + h.c. \right).$$

(1)

We introduce the Majorana field as,

$$\psi = \nu_L + (\nu_L)^c,$$

(2)

where $\psi^c = \psi$. Using the Majorana field, one can rewrite the Lagrangian of Eq.(1) as,

$$\mathcal{L} = -\frac{1}{2} \overline{\psi} i \gamma^\mu \partial_\mu \psi - \theta(t) \frac{m}{2} (\overline{\psi} \psi).$$

(3)

For $t < 0$, the field of the neutrino is expanded with a massless on-shell spinor and can be written as Eq.(4).

$$\nu_L(t, x) = \int' \frac{d^3p}{(2\pi)^3 |p|} \left( a(p) e^{-i|p|t - ip \cdot x} u_L(p) + b^\dagger(p) e^{i|p|t - ip \cdot x} v_L(p) \right),$$

(4)

where $a(p)$ and $b(p)$ denote particle and anti-particle annihilation operators respectively and the neutrino field is expanded by the left-handed ($L = \frac{1 - \gamma^5}{2}$) spinor as,

$$u_L(p) = -v_L(p) = \sqrt{2|p|} \begin{pmatrix} 0 \\ \phi_- \end{pmatrix}, \quad \sigma \cdot \frac{p}{|p|} \phi_- = -\phi_-.$$

(5)

\(\int'\) implies that the zero momentum mode is excluded from the integration and shows the absence of the zero energy state for a massless particle. When the neutrino is massless, the following lepton number is a conserved quantity,

$$\int d^3x : \overline{\nu}_L \gamma^0 \nu_L := \int' \frac{d^3p}{|p|(2\pi)^3} \left( a^\dagger(p) a(p) - b^\dagger(p) b(p) \right).$$

(6)
Since the mass term is turned on by the step function with respect to time, the translational invariance of the space direction is maintained and the space momentum is conserved. Therefore the operator for \( t > 0 \) can be also expanded without the zero mode. This implies that the operator for \( t > 0 \) is given by the massive Majorana field of non-zero helicities,

\[
\psi(x, 0_+) = \int' \frac{d^3p}{(2\pi)^3 2E(p)} \sum_{\lambda=\pm 1} (a(p, \lambda)u(p, \lambda)e^{i\lambda p} + a^\dagger(p, \lambda)v(p, \lambda)e^{-i\lambda p}),
\]

where \( \lambda \) denotes the helicity state and the energy of massive particle is \( E(p) = \sqrt{p^2 + m^2} \).

The normalized spinors with definite helicities are given as,

\[
u_{\nu_L}(t = -\epsilon) - L\psi(t = +\epsilon) = O(\epsilon).
\]

By taking the limit \( \epsilon \to 0 \), one obtains the continuity condition at \( t = 0 \) that holds between the operator defined for \( t < 0 \) and the one for \( t > 0 \),

\[
\nu_L(t = 0_-) = L\psi(t = 0_+).
\]

The above condition leads to the relations,

\[
a(p) = \frac{\sqrt{N(p)}}{2E(p)} (a_M(p, -) + \frac{m}{E(p) + |p|} a^\dagger_M(-p, -)),
\]

\[
b(p) = \frac{\sqrt{N(p)}}{2E(p)} (a_M(p, +) - \frac{m}{E(p) + |p|} a^\dagger_M(-p, +)),
\]

where \( p \neq 0 \). We note that the relations are consistent with the anti-commutation relations both for the massless field on the left-hand side and the massive Majorana field on the right-hand side of Eq.\[12] and Eq.\[13],

\[
\{a(p), a^\dagger(q)\} = \{b(p), b(q)^\dagger\} = (2\pi)^3 2|q|\delta^3(p - q),
\]

\[
\{a_M(p, \lambda), a^\dagger_M(q, \lambda')\} = (2\pi)^3 2E(p)\delta^3(p - q)\delta_{\lambda\lambda'}.
\]

Because the annihilation operator \( a_M(p, \lambda) \) evolves as \( a_M(p, \lambda)e^{-iE(p)t} \), the evolved operators developed from \( a(p) \) and \( b(p) \) can be written as,

\[
a(p, t) = \frac{\sqrt{N(p)}}{2E(p)} (a_M(p, -)e^{-iE(p)t} + \frac{m}{E(p) + |p|} a^\dagger_M(-p, -)e^{+iE(p)t}),
\]

\[
b(p, t) = \frac{\sqrt{N(p)}}{2E(p)} (a_M(p, +)e^{-iE(p)t} - \frac{m}{E(p) + |p|} a^\dagger_M(-p, +)e^{iE(p)t}).
\]
Using the relations Eq. (12), Eq. (13), Eq. (16), and Eq. (17), one can write the evolved operators by \( a(p) \) and \( b(p) \).

\[
a(p, t) = \left[ \cos(E(p)t) - i \frac{E(p)}{|p|} \sin(E(p)t) \right] a(p) + i \frac{m}{|p|} \sin(E(p)t) a^\dagger(-p),
\]

\[
b(p, t) = \left[ \cos(E(p)t) - i \frac{E(p)}{|p|} \sin(E(p)t) \right] b(p) - i \frac{m}{|p|} \sin(E(p)t) b^\dagger(-p).
\]

(18)

(19)

The lepton number operator for arbitrary time \( t > 0 \) is written in terms of \( a(p, t) \) and \( b(p, t) \).

\[
L(t) = \int \frac{d^3p}{(2\pi)^3} \left( a^\dagger(p, t) a(p, t) - b^\dagger(p, t) b(p, t) \right).
\]

(20)

We substitute the relations Eqs. (18) and (19) in Eq. (20), one finds that the lepton number is conserved,

\[
L(t) = L(0).
\]

(21)

One can extend the above formulas to the multi-flavor case. We consider the following Lagrangian

\[
\mathcal{L} = \bar{\nu}_{\alpha} i \gamma^\mu \partial_\mu \nu_{\alpha L} - \Theta(t) \left\{ \frac{m_{\alpha\beta}}{2} (\nu_{\alpha L})^c \nu_{\beta L} + \text{h.c.} \right\},
\]

(22)

where \( \nu_{\alpha L} \) denotes \( \nu_{eL}, \nu_{\mu L} \) and \( \nu_{\tau L} \). Each forms a SU(2) doublet with a charged lepton. When \( t < 0 \), they are expanded with massless spinors just like Eq. (4),

\[
\nu_{L}(x, t) = \int \frac{d^3p}{(2\pi)^3} \left( a_{\alpha}(p) e^{-|p|x + ipt} + b_{\alpha}(p) e^{ipt} \right).
\]

(23)

The lepton family number for \( t < 0 \) is given by,

\[
L_{\alpha}(t) = \int d^3x : \bar{\nu}_{\alpha L} \gamma^0 \nu_{\alpha L} : = \int \frac{d^3p}{2|p|(2\pi)^3} \left( a_{\alpha}(p) a_{\alpha}(p) - b_{\alpha}(p) b_{\alpha}(p) \right).
\]

(24)

where we use Eq. (23). After the mass term is turned on, the family eigenstates are not mass eigenstates. The mass eigenstate operators denoted by \( \nu_{iL} \) \( (i = 1 \sim 3) \) are related to those of the family eigenstate through a unitary matrix \( V \),

\[
\nu_{\alpha L} = V_{\alpha i} \nu_{i L}.
\]

(25)

The mass matrix \( m_{\alpha\beta} \) in Eq. (22) is diagonalized as,

\[
m_{ij} \delta_{ij} = (V^T)_{\alpha \beta} m_{\alpha\beta} V_{\beta \beta}.
\]

(26)

By introducing Majorana fields,

\[
\psi_{i} = \nu_{iL} + (\nu_{iL})^c,
\]

(27)
one can rewrite the Lagrangian as,
\[ \mathcal{L} = \frac{1}{2} \bar{\psi}_i i \gamma^\mu \partial_\mu \psi_i - \theta(t) \frac{m_i}{2} (\bar{\psi}_i \psi_i). \] (28)

The continuity condition at \( t = 0 \) in Eq.(11) is extended to the multi-flavor case as,
\[ \nu_{L\alpha}(t = 0_-) = V_{ai} L \psi_i(t = 0_+). \] (29)

The left-hand side of the above equation is expanded by massless fields as in Eq.(23). On the other hand, \( \psi_i \) in the right-hand side are expanded by the massive spinors with non-zero helicity,
\[ V_{\alpha i} L \psi_i(x, 0_+) = \int' \frac{d^3p}{(2\pi)^3 2E_i(p)} \sum_{\lambda=\pm 1} (a_{Mi}(p, \lambda) L u_i(p, \lambda)e^{ip \cdot x} + a_{Mi}^\dagger(p, \lambda) L v_i(p, \lambda)e^{-ip \cdot x}). \] (30)

The continuity condition in Eq.(29) leads to the relations between the creation and annihilation operators of the family basis and those of the mass basis,
\[ \frac{a_\alpha(p)}{\sqrt{2|p|}} = \sum_{i=1}^3 \frac{V_{ai} \sqrt{N_i}(p)}{2E_i(p)} \left\{ a_{Mi}(p, -) + \frac{m_i}{E_i(p) + |p|} a_{Mi}^\dagger(-p, -) \right\}, \] (31)
\[ \frac{b_\alpha(p)}{\sqrt{2|p|}} = \sum_{i=1}^3 \frac{V_{ai}^* \sqrt{N_i}(p)}{2E_i(p)} \left\{ a_{Mi}(p, +) - \frac{m_i}{E_i(p) + |p|} a_{Mi}^\dagger(-p, +) \right\}. \] (32)

A relation between mass eigenstate operator and flavor eigenstate operator has been studied with Bogoliubov transformation [7]. The operators of the family basis at time \( t(\geq 0) \) are written with those of the initial time,
\[ a_\alpha(p, t) = \sum_{\beta, \iota} \left[ V_{\alpha i} V_{\beta i}^* \left\{ \cos(E_i(p)t) - i \frac{E_i(p)}{|p|} \sin(E_i(p)t) \right\} a_{\beta}(p) + iV_{\alpha i} V_{\beta i}^* \frac{m_i}{|p|} \sin(E_i(p)t) a_{\beta}^\dagger(-p) \right], \] (33)
\[ b_\alpha(p, t) = \sum_{\beta, \iota} \left[ V_{\alpha i} V_{\beta i}^* \left\{ \cos(E_i(p)t) - i \frac{E_i(p)}{|p|} \sin(E_i(p)t) \right\} b_{\beta}(p) - iV_{\alpha i} V_{\beta i}^* \frac{m_i}{|p|} \sin(E_i(p)t) b_{\beta}^\dagger(-p) \right]. \] (34)

In the expressions above, one finds the combinations such as \( V_{\alpha i} V_{\beta i}^* \) depend on Majorana
phases. The lepton family number at time $t \geq 0$ becomes,

$$L_\alpha(t) = \int' \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \{ a^\dagger_\alpha(p,t) a_\alpha(p,t) - b^\dagger_\alpha(p,t) b_\alpha(p,t) \}$$

$$= \int' \frac{d^3p}{(2\pi)^3} \sum_{\alpha,\beta,\gamma,i,j} \left[ \{ V_{\alpha i} V_{\beta j} V_{\gamma j} a_\beta(-p) a_\gamma(-p) - V_{\alpha i} V_{\beta i} V_{\gamma j} b_\beta(-p) b_\gamma(-p) \} \cos E_i(p)t \cos E_j(p)t \right.$$

$$\times \frac{m_i m_j}{|p|^2} \sin(E_i(p)t) \sin(E_j(p)t)$$

$$\left. + \{ V_{\alpha i} V_{\beta i} V_{\gamma j} a_\beta(p) a_\gamma(p) - V_{\alpha i} V_{\beta i} V_{\gamma j} b_\beta(p) b_\gamma(p) \} \cos E_i(p)t \cos E_j(p)t \sin E_i(p)t \sin E_j(p)t \right]$$

$$+ i \{ V_{\alpha i} V_{\beta i} V_{\gamma j} a_\beta(-p) a_\gamma(-p) + V_{\alpha i} V_{\beta i} V_{\gamma j} b_\beta(-p) b_\gamma(-p) \}$$

$$\times \frac{m_i}{|p|} \sin E_i(p)t \cos E_j(p)t + i \frac{E_i(p)}{|p|} \sin E_i(p)t$$

$$\left. - i \{ V_{\alpha i} V_{\beta i} V_{\gamma j} a_\beta(p) a_\gamma(p) - V_{\alpha i} V_{\beta i} V_{\gamma j} b_\beta(p) b_\gamma(p) \} \cos E_j(p)t \cos E_i(p)t \right.$$

$$\times \frac{m_i}{|p|} \sin E_i(p)t \cos E_j(p)t - i \frac{E_j(p)}{|p|} \sin E_j(p)t \right].$$

(35)

Using Eq. (35) and the unitarity of the mixing matrix $\sum \alpha V^*_{\alpha i} V_{\alpha j} = \delta_{ij}$, the total lepton number conservation follows as,

$$\sum_\alpha L_\alpha(t) = \sum_\alpha L_\alpha(0).$$

(36)

3 The time evolution of lepton family number for a specific flavor eigenstate

In this section, by choosing a specific flavor eigenstate, one computes the time evolution of lepton family number. As a specific flavor eigenstate, one chooses the following initial states,

$$|\sigma(q)\rangle = \frac{a^\dagger_\sigma(q)|0\rangle}{\sqrt{\langle 0|a_\sigma(q)a^\dagger_\sigma(q)|0\rangle}}, \quad |\bar{\sigma}(q)\rangle = \frac{b^\dagger_\sigma(q)|0\rangle}{\sqrt{\langle 0|b_\sigma(q)b^\dagger_\sigma(q)|0\rangle}},$$

(37)

where the states $|\sigma\rangle$ and $|\bar{\sigma}\rangle$ imply the neutrino of a specific family and the anti-neutrino, respectively. One can determine the matrix elements of a lepton family number $L_\alpha(t)$ for the
neutrino and the anti-lepton number $-L_a(t)$ for the anti-neutrino.

$$
\langle \sigma (q)|L_a(t)|\sigma (q)\rangle = \sum_i |V_{\alpha i}^*|^2 |V_{\sigma i}|^2 + \sum_{\{i,j\}\; \text{cyclic}} \text{Re}(V_{\alpha i}^* V_{\sigma i} V_{\alpha j} V_{\sigma j}^*)
$$

\begin{align}
&\times \left\{ \right.
&\left( 1 + \frac{E_{iq} E_{jq}}{|q|^2} - \text{Re} \left( \frac{V_{\alpha i}^* V_{\sigma j}}{V_{\sigma i} V_{\alpha j}^*} \right) \frac{m_i m_j}{|q|^2} \right) \cos(E_{iq} - E_{jq}) t + \\
&\left( 1 - \frac{E_{iq} E_{jq}}{|q|^2} + \text{Re} \left( \frac{V_{\sigma i}^* V_{\alpha j}}{V_{\alpha i} V_{\sigma j}^*} \right) \frac{m_i m_j}{|q|^2} \right) \cos(E_{iq} + E_{jq}) t \left. \right\}

&- \text{Im}(V_{\alpha i} V_{\sigma i}^* V_{\alpha j} V_{\sigma j}^*) \sum_{\{i,j\}\; \text{cyclic}} \text{Im} \left( \frac{V_{\sigma i}^* V_{\alpha j}}{V_{\alpha i} V_{\sigma j}^*} \right) \frac{m_i m_j (\cos(E_{iq} - E_{jq}) - \cos(E_{iq} + E_{jq}) t)}{|q|^2}

&+ 2 \text{Im}(V_{\alpha i} V_{\sigma i}^* V_{\alpha j} V_{\sigma j}^*) \\
&\times \sum_{\{i,j\}\; \text{cyclic}} \left( \frac{E_{iq} - E_{jq}}{2|q|} \sin(E_{iq} + E_{jq}) t + \frac{E_{iq} + E_{jq}}{2|q|} \sin(E_{iq} - E_{jq}) t \right),
\end{align}

(38)

$$
- \langle \bar{\sigma} (q)|L_a(t)|\bar{\sigma} (q)\rangle = \sum_i |V_{\alpha i}^*|^2 |V_{\sigma i}|^2 + \sum_{\{i,j\}\; \text{cyclic}} \text{Re}(V_{\alpha i}^* V_{\sigma i} V_{\alpha j} V_{\sigma j}^*)
$$

\begin{align}
&\times \left\{ \right.
&\left( 1 + \frac{E_{iq} E_{jq}}{|q|^2} - \text{Re} \left( \frac{V_{\alpha i}^* V_{\sigma j}}{V_{\sigma i} V_{\alpha j}^*} \right) \frac{m_i m_j}{|q|^2} \right) \cos(E_{iq} - E_{jq}) t + \\
&\left( 1 - \frac{E_{iq} E_{jq}}{|q|^2} + \text{Re} \left( \frac{V_{\sigma i}^* V_{\alpha j}}{V_{\alpha i} V_{\sigma j}^*} \right) \frac{m_i m_j}{|q|^2} \right) \cos(E_{iq} + E_{jq}) t \left. \right\}

&- \text{Im}(V_{\alpha i} V_{\sigma i}^* V_{\alpha j} V_{\sigma j}^*) \sum_{\{i,j\}\; \text{cyclic}} \text{Im} \left( \frac{V_{\sigma i}^* V_{\alpha j}}{V_{\alpha i} V_{\sigma j}^*} \right) \frac{m_i m_j (\cos(E_{iq} - E_{jq}) - \cos(E_{iq} + E_{jq}) t)}{|q|^2}

&- 2 \text{Im}(V_{\alpha i} V_{\sigma i}^* V_{\alpha j} V_{\sigma j}^*) \\
&\times \sum_{\{i,j\}\; \text{cyclic}} \left( \frac{E_{iq} - E_{jq}}{2|q|} \sin(E_{iq} + E_{jq}) t + \frac{E_{iq} + E_{jq}}{2|q|} \sin(E_{iq} - E_{jq}) t \right),
\end{align}

(39)

where $E_{iq} = \sqrt{q^2 + m_i^2}$ and $\sum_{\{i,j\}\; \text{cyclic}} = \sum_{\{i,j\}=\{1,2\},\{2,3\},\{3,1\}}$. The difference between Eq. (38) and Eq. (39) is the sign in front of their last terms.
4 Numerical Analysis

In this section, we carry out the numerical analysis. We first write the sum of the contribution from the three mass eigenstates in Eq. (38) as follows,

\[
\langle \sigma(q)|L_\alpha(t)|\sigma(q)\rangle = |V_{\alpha 1}|^2 |V_{\alpha 1}|^2 + |V_{\alpha 2}|^2 |V_{\alpha 2}|^2 + |V_{\alpha 2}|^2 |V_{\alpha 3}|^2 \\
+ \Re \langle V_{\alpha 1}^* V_{\sigma 1} V_{\sigma 2} V_{\alpha 2}^* \rangle \left\{ \left( 1 + \frac{E_{1q} E_{2q}}{|q|^2} \right) - \Re \left( \frac{V_{\alpha 1} V_{\sigma 2}^*}{V_{\sigma 1} V_{\sigma 2}^*} \right) \right\} \cos(E_{1q} - E_{2q}) t \\
+ \left( 1 - \frac{E_{1q} E_{2q}}{|q|^2} \right) + \Re \left( \frac{V_{\alpha 1} V_{\sigma 2}^*}{V_{\sigma 1} V_{\sigma 2}^*} \right) \frac{m_1 m_2}{|q|^2} \cos(E_{1q} + E_{2q}) t \\
+ \Re \langle V_{\alpha 2} V_{\sigma 2} V_{\sigma 3} V_{\sigma 3}^* \rangle \left\{ \left( 1 + \frac{E_{2q} E_{3q}}{|q|^2} \right) - \Re \left( \frac{V_{\sigma 3} V_{\sigma 3}^*}{V_{\sigma 3} V_{\sigma 3}^*} \right) \right\} \cos(E_{2q} - E_{3q}) t \\
+ \left( 1 - \frac{E_{2q} E_{3q}}{|q|^2} \right) + \Re \left( \frac{V_{\sigma 3} V_{\sigma 3}^*}{V_{\sigma 3} V_{\sigma 3}^*} \right) \frac{m_2 m_3}{|q|^2} \cos(E_{2q} + E_{3q}) t \\
+ \Re \langle V_{\alpha 3} V_{\sigma 1} V_{\sigma 2} V_{\sigma 1}^* \rangle \left\{ \left( 1 + \frac{E_{3q} E_{1q}}{|q|^2} \right) - \Re \left( \frac{V_{\sigma 3} V_{\sigma 1}^*}{V_{\sigma 3} V_{\sigma 1}^*} \right) \right\} \cos(E_{3q} - E_{1q}) t \\
+ \left( 1 - \frac{E_{3q} E_{1q}}{|q|^2} \right) + \Re \left( \frac{V_{\sigma 3} V_{\sigma 1}^*}{V_{\sigma 3} V_{\sigma 1}^*} \right) \frac{m_3 m_1}{|q|^2} \cos(E_{3q} + E_{1q}) t \\
- \Im \langle V_{\alpha 1} V_{\sigma 1} V_{\sigma 2} V_{\sigma 2} \rangle \left\{ \Im \left( \frac{V_{\sigma 1} V_{\sigma 2}^*}{V_{\sigma 1} V_{\sigma 2}^*} \right) \right\} \frac{m_1 m_2 (\cos(E_{1q} - E_{2q}) t - \cos(E_{1q} + E_{2q}) t)}{|q|^2} \\
+ \Im \left( \frac{V_{\sigma 2} V_{\sigma 3}^*}{V_{\sigma 2} V_{\sigma 3}^*} \right) \frac{m_2 m_3 (\cos(E_{2q} - E_{3q}) t - \cos(E_{2q} + E_{3q}) t)}{|q|^2} \\
+ \Im \left( \frac{V_{\sigma 3} V_{\sigma 1}^*}{V_{\sigma 3} V_{\sigma 1}^*} \right) \frac{m_3 m_1 (\cos(E_{3q} - E_{1q}) t - \cos(E_{3q} + E_{1q}) t)}{|q|^2} \\
+ 2 \Im \langle V_{\alpha 1} V_{\sigma 1} V_{\sigma 2} V_{\sigma 2} \rangle \left\{ \left( \frac{E_{1q} - E_{2q}}{2 |q|} \sin(E_{1q} + E_{2q}) t + \frac{E_{1q} + E_{2q}}{2 |q|} \sin(E_{1q} - E_{2q}) t \right) \\
+ \left( \frac{E_{2q} - E_{3q}}{2 |q|} \sin(E_{2q} + E_{3q}) t + \frac{E_{2q} + E_{3q}}{2 |q|} \sin(E_{2q} - E_{3q}) t \right) \\
+ \left( \frac{E_{3q} - E_{1q}}{2 |q|} \sin(E_{3q} + E_{1q}) t + \frac{E_{3q} + E_{1q}}{2 |q|} \sin(E_{3q} - E_{1q}) t \right) \right\}. \tag{40}
\]

We note the combinations of PMNS matrix which have the form \(V_{\alpha i} V_{\sigma 1} V_{\alpha j} V_{\sigma j}^*\) are independent of Majorana phases, while the other combinations of the form \(V_{\alpha i} V_{\sigma j}^*\) depend on the Majorana phases. To clarify this point, we substitute \(e\) for \(\sigma\) and write the factor \(\frac{V_{\sigma i} V_{\sigma j}^*}{V_{\sigma i} V_{\sigma j}^*}\) in terms of CP violating phases,

\[
\frac{V_{\alpha 1} V_{\alpha 2}}{V_{\alpha 1} V_{\alpha 2}^*} = e^{i \alpha_{21}}, \quad \frac{V_{\alpha 2} V_{\alpha 3}}{V_{\alpha 2} V_{\alpha 3}^*} = e^{i (\alpha_{32} - \alpha_{21} - 2 \delta)}, \quad \frac{V_{\alpha 3} V_{\alpha 1}}{V_{\alpha 3} V_{\alpha 1}^*} = e^{i (-\alpha_{31} + 2 \delta)}. \tag{41}
\]

where \(\alpha_{21}\) (\(i = 2, 3\)) are Majorana phases \[8\] and \(\delta\) is a Dirac phase \[9\].
As for lepton family numbers, one computes the electron number ($\alpha = e$) as,

$$
\langle e(q)|L_e(t)|e(q)\rangle = |V^*_{e1}|^2|V_{e1}|^2 + |V^*_{e2}|^2|V_{e2}|^2 + |V^*_{e3}|^2|V_{e3}|^2
+ |V_{e1}|^2|V_{e2}|^2 \left\{ \left( 1 + \frac{E_{1q}E_{2q}}{|q|^2} - \cos(\alpha_{21})\frac{m_{1m_2}}{|q|^2} \right) \cos(E_{1q} - E_{2q})t \right.
+ \left( 1 - \frac{E_{1q}E_{2q}}{|q|^2} + \cos(\alpha_{21})\frac{m_{1m_2}}{|q|^2} \right) \cos(E_{1q} + E_{2q})t \right\}
+ |V_{e2}|^2|V_{e3}|^2 \left\{ \left( 1 + \frac{E_{2q}E_{3q}}{|q|^2} - \cos(-\alpha_{21} + \alpha_{31} - 2\delta)\frac{m_{2m_3}}{|q|^2} \right) \cos(E_{2q} - E_{3q})t \right.
+ \left( 1 - \frac{E_{2q}E_{3q}}{|q|^2} + \cos(-\alpha_{21} + \alpha_{31} - 2\delta)\frac{m_{2m_3}}{|q|^2} \right) \cos(E_{2q} + E_{3q})t \right\}
+ |V_{e3}|^2|V_{e1}|^2 \left\{ \left( 1 + \frac{E_{3q}E_{1q}}{|q|^2} - \cos(2\delta - \alpha_{31})\frac{m_{3m_1}}{|q|^2} \right) \cos(E_{3q} - E_{1q})t \right.
+ \left( 1 - \frac{E_{3q}E_{1q}}{|q|^2} + \cos(2\delta - \alpha_{31})\frac{m_{3m_1}}{|q|^2} \right) \cos(E_{3q} + E_{1q})t \right\},
$$

(42)

where the relevant elements of the PMNS matrix are,

$$
V_{e1} = c_{12}c_{13}, \quad V_{e2} = s_{12}c_{13}e^{i\frac{\alpha_{21}}{2}}, \quad V_{e3} = s_{13}e^{-i\delta}e^{i\frac{\alpha_{31}}{2}}.
$$

(43)
One also can compute the muon number ($\alpha = \mu$) as,

$$
\langle e(q)|L_\mu(t)|e(q)\rangle = |V_{\mu_1}|^2|V_{e_1}|^2 + |V_{\mu_2}|^2|V_{e_2}|^2 + |V_{\mu_3}|^2|V_{e_3}|^2
$$

+ \text{Re}(V_{\mu_1}^*V_{e_1}V_{\mu_2}V_{e_2}) \left\{ \left( 1 + \frac{E_1qE_2q}{|q|^2} - \cos(\alpha_{21}) \frac{m_1m_2}{|q|^2} \right) \cos(E_{1q} - E_{2q})t \right. \\
+ \left. \left( 1 - \frac{E_1qE_2q}{|q|^2} + \cos(\alpha_{21}) \frac{m_1m_2}{|q|^2} \right) \cos(E_{1q} + E_{2q})t \right\} \\
+ \text{Re}(V_{\mu_2}^*V_{e_3}V_{\mu_3}V_{e_3}^*) \left\{ \left( 1 + \frac{E_2qE_3q}{|q|^2} - \cos(-\alpha_{21} + \alpha_{31} - 2\delta) \frac{m_2m_3}{|q|^2} \right) \cos(E_{2q} - E_{3q})t \right. \\
+ \left. \left( 1 - \frac{E_2qE_3q}{|q|^2} + \cos(-\alpha_{21} + \alpha_{31} - 2\delta) \frac{m_2m_3}{|q|^2} \right) \cos(E_{2q} + E_{3q})t \right\} \\
+ \text{Re}(V_{\mu_3}^*V_{e_3}V_{\mu_1}V_{e_1}^*) \left\{ \left( 1 + \frac{E_3qE_{1q}}{|q|^2} \right) \cos(E_{3q} - E_{1q})t \right. \\
+ \left. \left( 1 - \frac{E_3qE_{1q}}{|q|^2} \right) \cos(E_{3q} + E_{1q})t \right\} \\
- \text{Im}(V_{\mu_1}V_{\mu_2}^*V_{e_2}V_{e_2}^*) \left\{ \sin(\alpha_{21}) \frac{m_1m_2(\cos(E_{1q} - E_{2q})t - \cos(E_{1q} + E_{2q})t)}{|q|^2} \right. \\
+ \left. \sin(-\alpha_{21} + \alpha_{31} - 2\delta) \frac{m_2m_3(\cos(E_{2q} - E_{3q})t - \cos(E_{2q} + E_{3q})t)}{|q|^2} \right. \\
+ \left. \sin(2\delta - \alpha_{31}) \frac{m_3m_1(\cos(E_{3q} - E_{1q})t - \cos(E_{3q} + E_{1q})t)}{|q|^2} \right\} \\
+ 2 \text{Im}(V_{\mu_1}V_{\mu_2}^*V_{\mu_2}V_{e_2}) \left\{ \left( \frac{E_1q - E_{2q}}{2|q|} \sin(E_{1q} + E_{2q})t + \frac{E_1q + E_{2q}}{2|q|} \sin(E_{1q} - E_{2q})t \right) \right. \\
+ \left( \frac{E_{2q} - E_{3q}}{2|q|} \sin(E_{2q} + E_{3q})t + \frac{E_{2q} + E_{3q}}{2|q|} \sin(E_{2q} - E_{3q})t \right) \\
+ \left( \frac{E_{3q} - E_{1q}}{2|q|} \sin(E_{3q} + E_{1q})t + \frac{E_{3q} + E_{1q}}{2|q|} \sin(E_{3q} - E_{1q})t \right) \right\}. \tag{44}
$$

In Eq. (44), the time independent part is given by the following combinations of PMNS matrix.

$$
|V_{\mu_1}|^2|V_{e_1}|^2 + |V_{\mu_2}|^2|V_{e_2}|^2 + |V_{\mu_3}|^2|V_{e_3}|^2
$$

$$
|V_{\mu_1}|^2(|V_{e_1}|^2 - |V_{e_2}|^2) + |V_{e_2}|^2 + |V_{\mu_3}|^2(|V_{e_3}|^2 - |V_{e_2}|^2)
$$

$$
|V_{\mu_1}|^2(c_{13}^2 \cos 2\theta_{12}) + c_{13}^2 s_{13}^2 + c_{13}^2 s_{23}^2 (s_{13}^2 - c_{13}^2 s_{12}^2), \tag{45}
$$

with

$$
|V_{\mu_1}|^2 = s_{12}^2 c_{23}^2 + c_{12}^2 s_{23}^2 s_{13}^2 + 2 s_{12} c_{23} c_{12} s_{23} s_{13} \cos \delta. \tag{46}
$$

One also substitutes Eq. (43) and the following elements in Eq. (44),

$$
V_{\mu_1} = -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta}, \quad V_{\mu_2} = (c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta}) e^{\frac{i\pi}{2}}, \quad V_{\mu_3} = s_{23} c_{13} e^{\frac{-i\pi}{2}}, \tag{47}
$$

10
and one can write the coefficients of time dependent terms in Eq. (44),

\[
\begin{align*}
\text{Re}(V_{e2}V_{e3}^*V_{\mu 2}^*V_{\mu 3}) &= s_{12}c_{13}s_{13}s_{23}(c_{12}c_{23} \cos \delta - s_{12}s_{23}s_{13}), \\
\text{Re}(V_{\mu 1}V_{\mu 3}^*V_{e1}^*V_{e3}) &= c_{12}c_{13}^2s_{13}s_{23}^2(-s_{12}c_{23} \cos \delta - c_{12}s_{23}s_{13}), \\
\text{Im}(V_{\mu 1}V_{\mu 3}^*V_{e2}V_{e1}^*) &= -c_{12}c_{13}^2s_{13}s_{12}s_{23}c_{23} \sin \delta, \\
\text{Re}(V_{\mu 1}V_{\mu 2}^*V_{e3}V_{e1}^*) &= c_{12}s_{12}^2c_{13}^2(-s_{12}c_{12}c_{23}^2 - \cos 2\theta_1s_{23}s_{13}s_{23} \cos \delta + c_{12}s_{12}s_{23}^2s_{13}).
\end{align*}
\]

The imaginary part in Eq. (50) is proportional to the Jarlskog invariant \textbf{10}. The \(\tau\) lepton number \((\alpha = \tau)\) can be written in terms with the other lepton family numbers because the total lepton number is conserved,

\[
\langle e(q)|L_\tau(t)|e(q)\rangle = 1 - \langle e(q)|L_e(t)|e(q)\rangle - \langle e(q)|L_\mu(t)|e(q)\rangle.
\]

It can be also derived from Eq.(38) and the explicit expression is given in Eq. (60) of Appendix A. With preparation, one carries out the numerical analysis and the results are shown in Fig.2 and Fig.3. In the numerical calculations, the lightest neutrino mass is assumed to be 0.01(eV). The other parameters depend on the mass hierarchies. We adopt the following data from Ref.\textbf{11}. For the normal mass hierarchy case, we use \(\Delta m_{21}^2 = 7.37 \times 10^{-5}\text{(eV}^2)\), \(\Delta m_{31}^2 = 2.56 \times 10^{-3}\text{(eV}^2)\). The Dirac phase and sine of the mixing angles are chosen as, \(\delta = 1.38\pi, s_{12} = 0.545, s_{13} = 0.147, s_{23} = 0.652\). For the inverted mass hierarchy case, we adopt \(\Delta m_{21}^2 = 7.37 \times 10^{-5}\text{(eV}^2), \Delta m_{23}^2 = 2.54 \times 10^{-3}\text{(eV}^2)\). The Dirac phase and sine of the mixing angles are chosen as, \(\delta = 1.31\pi, s_{12} = 0.545, s_{13} = 0.147, s_{23} = 0.767\).

We summarize our observation of numerical results. In Fig.2 we find that the lepton family number oscillates with the long period and the large amplitude. In addition to the long period, there is a rapid oscillation with the small amplitude. The rapid oscillation
The electron number for the normal hierarchy  The muon number for the normal hierarchy

Figure 3: Time dependence of lepton family numbers $\langle e|L_e(\hat{t})|e\rangle$ (left figure) and $\langle e|L_\mu(\hat{t})|e\rangle$ (right figure) for the normal hierarchy case. The momentum of the neutrino is $|q| = 0.02$ (eV) and the other parameters are the same as those used for Fig.2. The black and red lines show the cases of Majorana phases $(\alpha_{21}, \alpha_{31}) = (0, 0)$ and $(\frac{\pi}{2}, \frac{\pi}{3})$ respectively.

The electron number for the normal hierarchy  The muon number for the normal hierarchy

Figure 4: Time dependence of electron family number $\langle e|L_e(\hat{t})|e\rangle$ (left figure) and muon family number $\langle e|L_\mu(\hat{t})|e\rangle$ (right figure) for the normal hierarchy case. The momentum of the neutrino is $|q| = 0.0002$ (eV) and the lightest neutrino mass is 0.01 (eV). The black and red lines show the cases of Majorana phases $(\alpha_{21}, \alpha_{31}) = (0, 0)$ and $(\frac{\pi}{2}, \frac{\pi}{3})$ respectively.

shows the behavior of the beat. As shown in Fig.2 for the momentum $q = 0.2$ (eV) which is larger than the neutrino masses $m_3 \sim 0.05$ (eV), $m_1 \sim m_2 \sim 0.01$ (eV), the coefficients of $\cos(E_i + E_j)t$ and $\sin(E_i + E_j)t$ are suppressed compared to those of $\cos(E_i - E_j)t$ and $\sin(E_i - E_j)t$. Therefore the oscillation with longer period $T_L$ originates from the smallest energy difference $E_2 - E_1$.

$$T_L = \frac{2\pi}{E_2 - E_1}.$$  \hspace{1cm} (53)

The oscillation with shorter periods comes from the larger energy differences $E_3 - E_2$ and $E_3 - E_1$. Their superposition leads to the beat like behavior because the difference of the
Figure 5: Time dependence of $\tau$ lepton family number $\langle e|L_\tau(\hat{t})|e \rangle$ for normal hierarchy case (left) and for the inverted hierarchy case (right). The lightest neutrino mass is 0.01 (eV) and the momentum of neutrino is $|q| = 0.0002$ (eV). The black and red lines show the case of Majorana phases $(\alpha_{21}, \alpha_{31}) = (0, 0)$ and $(\frac{\pi}{2}, \frac{\pi}{3})$.

T_{\text{beat}}$ is the same as $T_L$. One can also understand the reason why the muon number does show the larger amplitude for the beat compared to the case of the electron family number. In electron family number, the terms proportional to $\cos(E_3 - E_2)t$ and $\cos(E_3 - E_1)t$ are suppressed by a factor of $s_{13}^2$. For muon family number case, there are terms which are proportional to $\sin(E_3 - E_2)t$ and $\sin(E_3 - E_1)t$. Their coefficients are proportional to $s_{13}$. Therefore the amplitude of the beat for muon number is larger than that of the electron number.

In Fig.3, we show the lepton family numbers with the smaller momentum value, $q = 0.02$ (eV) compared to that of Fig.2. The momentum is the same order of the masses of the three neutrinos in the normal hierarchical case, $m_1 \sim m_3 = 0.01$ (eV) $\sim 0.05$ (eV). In contrast to Fig.2, the dependence on the values of Majorana phases is visible in the amplitudes of the oscillation with the short period. We also note that the muon family number has negative values.

Fig.4 also shows the lepton family number for the case with much smaller momentum ($q = 0.0002$ (eV)) than that of Fig.3. The momentum is as small as the cosmic microwave background (CMB) temperature $T_\gamma = 2.7$ (K) and the relic neutrino temperature for massless neutrinos $T_{\text{relic}} = (\frac{4}{11})^{\frac{1}{3}} T_\gamma \simeq 1.9$ (K). In this figure, we can see the lepton family numbers are more sensitive to Majorana phases compared to the case in Fig.3 where the momentum is larger. In particular, the muon family number depends on the choice of the Majorana phases more strongly than the electron family number does. The periodicity of the muon family number and electron family number is obscure. Since even the lightest mass $m_1 = 0.01$ (eV) which we have chosen is larger than the momentum, all the neutrinos are non-relativistic.

\begin{equation}
T_{\text{beat}} = \frac{2\pi}{E_2 - E_1}.
\end{equation}
The dominant contribution to the electron family number can be written approximately as,
\[
\langle e(q)|L_e(t)|e(q)\rangle \simeq |V_{e1}|^2|V_{e2}|^2 \frac{m_1 m_2}{|q|^2} (1 - \cos(\alpha_{21}))(\cos(m_1 - m_2)t - \cos(m_1 + m_2)t)
\]
\[+ |V_{e2}|^2|V_{e3}|^2 \frac{m_2 m_3}{|q|^2} (1 - \cos(-\alpha_{21} + \alpha_{31} - 2\delta))(\cos(m_2 - m_3)t - \cos(m_2 + m_3)t)
\]
\[+ |V_{e3}|^2|V_{e1}|^2 \frac{m_3 m_1}{|q|^2} (1 - \cos(2\delta - \alpha_{31}))(\cos(m_3 - m_1)t - \cos(m_3 + m_1)t).
\] (55)

The black line in Fig.4 corresponds to the case that the Majorana phases are zero and the Dirac phase \(\delta\) is \(\pi\). With this choice, the first line of in Eq.(55) vanishes. The maximum amplitude is approximately equal to
\[
(|V_{e2}|^2|V_{e3}|^2 \frac{m_2 m_3}{|q|^2} + |V_{e3}|^2|V_{e1}|^2 \frac{m_3 m_1}{|q|^2})(1 - \cos(2\delta)) \times 2 \simeq 4|V_{e3}|^2 \frac{m_3 m_1}{|q|^2}
\] (56)
and it is numerically about \(50^2 \times 5 \times 0.02 \times 4 = 1000\). As for time dependence, there are four cosine functions with angular frequencies of \(m_3 \pm m_2\) and \(m_3 \pm m_1\). The interference of them leads to beat like behaviour. The longest period of the beat is given by \(\frac{2\pi}{m_2 - m_1} \simeq 2 \times 10^3 (1/\text{eV})\). It corresponds to \(\hat{t} = 0.02 \text{(eV)} \times 2 \times 10^3 = 40\) in the unit of the dimensionless time adopted in the horizontal axis of Fig.4. It agrees with the period of the beat which can be seen in the figure. In Fig.5, the momentum is the same as that of Fig.4. We study the tau family number for two different neutrino mass hierarchies, normal and inverted. For the inverted hierarchy, the normal hierarchy leads to smaller amplitude than that of the inverted hierarchy. For this case, all the terms equally contribute to the tau family number. The periods of sine and cosine functions are given by
\[
T_{ij\pm} = \frac{2\pi}{m_j \pm m_i} \quad (j > i).
\] (59)
For the normal hierarchy, $T_{13^+} \simeq T_{23^+} = 100 \ (1/eV)$, $T_{13^-} \simeq T_{23^-} = 150 \ (1/eV)$, $T_{12^+} = 300 \ (1/eV)$ and $T_{12^-} = 2000 \ (1/eV)$. In the dimensionless time, they correspond to 2, 3, 6 and 40, respectively. They can be read from the left figure of Fig.5.

5 Summary and Discussion

We formulate the time evolution of lepton family numbers under the presence of Majorana mass terms. The lepton family number operator for arbitrary time is derived. We also study the time variation of its expectation values. When the momentum is large and the neutrino is relativistic, the expectation values are reduced to the time dependent probabilities of neutrino flavor oscillations. However, when the momentum of neutrinos is smaller than their rest masses, the lepton family numbers undergo completely different behaviors. Starting with a single neutrino with a definite family number, the absolute value of the corresponding family number can take a huge value at later time. Depending on the time, it can also take a negative value. The negative value implies that the transition from neutrino to antineutrino occurs. We study the dependence on Majorana phases and the neutrino mass hierarchies. The dependence on the Majorana phases is more pronounced when the momentum of the neutrinos is smaller than their rest masses. The terms which depend on the phases are proportional to the product, $\frac{m_i m_j}{|q|^2}$. For the normal hierarchy case, the largest product is $\frac{m_2 m_3}{|q|^2}$ while it is $\frac{m_1 m_2}{|q|^2}$ for inverted hierarchy case. The latter is larger than the former. Therefore the Majorana phase dependence is more significant for the inverted hierarchy case.

We comment on the implication for the study of CNB. In the early universe, the neutrinos decouple from the weak interaction, they are relativistic and the lepton family numbers are approximately conserved. As the universe expands, the neutrinos’ momenta decrease because of the redshift. At some stage, the neutrinos momenta are comparable to their rest mass. This situation could be mimicked by switching on the Majorana mass term with the step function. As time passes, eventually they become non-relativistic. In contrast to the above history of the CNB, we have considered the time evolution of lepton family numbers for neutrinos with a fixed momentum.

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A The expectation value of $\tau$ lepton family number

Below we show the expectation value of $\tau$ lepton family number which is derived from Eq.\([38]\) by setting $\alpha = \tau$ and $\sigma = e$.

\[
\begin{align*}
\langle e(q)|L_\tau(t)|e(q)\rangle &= |V_{t1}|^2|V_{e1}|^2 + |V_{t2}|^2|V_{e2}|^2 + |V_{t3}|^2|V_{e3}|^2 \\
&+ \text{Re}(V_{t1}^*V_{e1}V_{t2}^*V_{e2}) \left\{ 1 + \frac{E_{1q}E_{2q}}{|q|^2} - \cos(\alpha_{21})\frac{m_{1m_2}}{|q|^2} \right\} (E_{1q} - E_{2q})t \\
&+ \left( 1 - \frac{E_{1q}E_{2q}}{|q|^2} + \cos(\alpha_{21})\frac{m_{1m_2}}{|q|^2} \right) (E_{1q} + E_{2q})t \\
&+ \text{Re}(V_{t2}^*V_{e2}V_{t3}^*V_{e3}) \left\{ 1 + \frac{E_{2q}E_{3q}}{|q|^2} - \cos(\alpha_{21} + \alpha_{31} - 2\delta)\frac{m_{2m_3}}{|q|^2} \right\} (E_{2q} - E_{3q})t \\
&+ \left( 1 - \frac{E_{2q}E_{3q}}{|q|^2} + \cos(\alpha_{21} + \alpha_{31} - 2\delta)\frac{m_{2m_3}}{|q|^2} \right) (E_{2q} + E_{3q})t \\
&+ \text{Re}(V_{t3}^*V_{e3}V_{t1}^*V_{e1}) \left\{ 1 + \frac{E_{3q}E_{1q}}{|q|^2} - \cos(2\delta - \alpha_{31})\frac{m_{3m_1}}{|q|^2} \right\} (E_{3q} - E_{1q})t \\
&+ \left( 1 - \frac{E_{3q}E_{1q}}{|q|^2} + \cos(2\delta - \alpha_{31})\frac{m_{3m_1}}{|q|^2} \right) (E_{3q} + E_{1q})t \\
&- \text{Im}(V_{t1}^*V_{t2}^*V_{e1}^*V_{e2}) \left\{ \sin(\alpha_{21})\frac{m_{1m_2}(\cos(E_{1q} - E_{2q})t - \cos(E_{1q} + E_{2q})t)}{|q|^2} \\
&+ \sin(-\alpha_{21} + \alpha_{31} - 2\delta)\frac{m_{2m_3}(\cos(E_{2q} - E_{3q})t - \cos(E_{2q} + E_{3q})t)}{|q|^2} \\
&+ \sin(2\delta - \alpha_{31})\frac{m_{3m_1}(\cos(E_{3q} - E_{1q})t - \cos(E_{3q} + E_{1q})t)}{|q|^2} \right\} \\
&+ 2\text{Im}(V_{t1}^*V_{t2}^*V_{e1}^*V_{e2}) \left\{ \left( \frac{E_{1q} - E_{2q}}{2|q|} \right) \sin(E_{1q} + E_{2q})t + \frac{E_{1q} + E_{2q}}{2|q|} \sin(E_{1q} - E_{2q})t \\
&+ \left( \frac{E_{2q} - E_{3q}}{2|q|} \right) \sin(E_{2q} + E_{3q})t + \frac{E_{2q} + E_{3q}}{2|q|} \sin(E_{2q} - E_{3q})t \\
&+ \left( \frac{E_{3q} - E_{1q}}{2|q|} \right) \sin(E_{3q} + E_{1q})t + \frac{E_{3q} + E_{1q}}{2|q|} \sin(E_{3q} - E_{1q})t \right\} \right). \quad (60)
\end{align*}
\]

For inverted hierarchical case with $|q| \ll m_3 < m_1 < m_2$, the terms proportional to $\frac{m_{1m_2}}{|q|^2}$ give rise to the dominant contribution,

\[
\begin{align*}
\langle e(q)|L_\tau(t)|e(q)\rangle &\simeq \frac{m_{1m_2}}{|q|^2} (\cos(m_1 - m_2)t - \cos(m_1 + m_2)t) \\
&\times \left\{ \text{Re}(V_{t1}^*V_{e1}V_{t2}^*V_{e2})(1 - \cos(\alpha_{21})) - \text{Im}(V_{t1}^*V_{e1}^*V_{t2}^*V_{e2}) \sin(\alpha_{21}) \right\} \\
&\simeq \frac{m_{1m_2}}{|q|^2} (\cos(m_1 - m_2)t - \cos(m_1 + m_2)t) \\
&\times \left\{ -e_{12}^2 e_{12}^2 s_{23}^2 (1 - \cos(\alpha_{21})) + c_{12} s_{12} s_{23} c_{23} s_{13} (\cos 2\theta_{12} \cos \delta (1 - \cos(\alpha_{21})) + \sin \delta \sin(\alpha_{21})) \right\}. \quad (61)
\end{align*}
\]
For normal hierarchical case with $|q| \ll m_1 < m_2 < m_3$,

$$
\langle e(q)|L_r(t)|e(q)\rangle \simeq \text{Re}(V^*_{\tau_1}V_{e_1}V_{\tau_2}V^*_{e_2}) \frac{m_1m_2}{|q|^2} (1 - \cos(\alpha_{21})) (\cos(m_1 - m_2)t - \cos(m_1 + m_2)t)
+ \text{Re}(V^*_{\tau_2}V_{e_2}V_{\tau_3}V^*_{e_3}) \frac{m_2m_3}{|q|^2} (1 - \cos(-\alpha_{21} + \alpha_{31} - 2\delta)) (\cos(m_2 - m_3)t - \cos(m_2 + m_3)t)
+ \text{Re}(V^*_{\tau_3}V_{e_3}V_{\tau_1}V^*_{e_1}) \frac{m_3m_1}{|q|^2} (1 - \cos(2\delta - \alpha_{31})) (\cos(m_3 - m_1)t - \cos(m_3 + m_1)t)
- \text{Im}(V^*_{\tau_1}V^*_{e_1}V^*_{\tau_2}V_{e_2}) \left\{ \sin(\alpha_{21}) \frac{m_1m_2(\cos(m_1 - m_2)t - \cos(m_1 + m_2)t)}{|q|^2} 
+ \sin(-\alpha_{21} + \alpha_{31} - 2\delta) \frac{m_2m_3(\cos(m_2 - m_3)t - \cos(m_2 + m_3)t)}{|q|^2} 
+ \sin(2\delta - \alpha_{31}) \frac{m_3m_1(\cos(m_3 - m_1)t - \cos(m_3 + m_1)t)}{|q|^2} \right\}.
$$

In contrast to the inverted hierarchy, all the terms proportional to $\frac{m_im_j}{|q|^2}$ ($i,j = (1, 2), (2, 3), (3, 1)$) can equally contribute to the lepton family number for this case.

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