ON THE SPECIALIZATION THEOREM FOR ABELIAN VARIETIES

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ABSTRACT. In this note, we apply Moriwaki’s arithmetic height functions to obtain an analogue of Silverman’s Specialization Theorem for families of Abelian varieties over $K$, where $K$ is any field finitely generated over $\mathbb{Q}$.

Let $k$ be a number field and $\pi : A \to B$ be a proper flat morphism of smooth projective varieties over $k$ such that the generic fiber $A_\eta$ is an Abelian variety defined over $k(B)$. For almost all absolutely irreducible divisors $D/k$ on $B$, $A_D := A \times_B D$ is also a flat family of Abelian varieties, and our goal is to compare the Mordell-Weil rank of $A_\eta$ with the Mordell-Weil rank of the generic fiber of $A_D$. In fact, if we fix a projective embedding of $A, B$ into $\mathbb{P}^N$, and an integer $M > 0$, then we can show that for all but finitely-many divisors $D$ of degree less than $M$, the rank of the generic fiber of $A_D$ is at least the rank of $A_\eta$.

This result is an amusing example of the height machine in action. The main issue is to rephrase the problem in terms of Abelian varieties over function fields, and define the “right” height functions; the proofs then follow verbatim as in [?].

Assume now that we also have a proper, flat morphism $f : B \to X$ with generic fiber a smooth, irreducible curve $C$ defined over $K := k(X)$. Composition gives a flat morphism $g := f \circ \pi : A \to X$ whose generic fiber is a smooth, irreducible variety $A$ defined over $K$, and by base extension we have a flat morphism $\rho : A \to C$, whose generic fiber is $A_\eta$. Observe that in this setting, divisors $D \in \text{Div}(B)$ such that $f(D) = X$ correspond to points on $C$. The next proposition (based on a classical geometric argument, see [?, Proposition 5.1]) shows that, up to replacing $B$ and $A$ by birationally equivalent varieties, we can always reduce to this situation.

Proposition 0.1. Let $V$ be a smooth projective variety of dimension $n$ defined over $k$. Then, there exist finitely many birationally equivalent varieties $\nu_i : V_i \to V$ and proper flat morphisms $f_i : V_i \to \mathbb{P}^{n-1}$ such that:

(1) The generic fiber of $f_i$ is a smooth, irreducible curve $C_i$ defined over $k(\mathbb{P}^{n-1})$.
(2) Let $U_i$ be an affine open dense subset of $V$ such that $\nu_i$ is an isomorphism on $U_i$, and let $U := \bigcap U_i$. Then for all divisors $D \in \text{Div}(V)$ such that $D \cap U \neq \emptyset$, there exists an $i$ such that $f_i(D') = \mathbb{P}^{n-1}$, where $D' := \nu_i^{-1}(D \cap U_i)$.

Proof. Consider an embedding $V \hookrightarrow \mathbb{P}^N$, and take a linear projection inducing a finite morphism $\phi : V \to \mathbb{P}^n$. Let $p_0 \in \mathbb{P}^n$ be a point outside the ramification locus of $\phi$, let $\mu_0 : \tilde{V}_0 \to \mathbb{P}^n$ be the blow-up at $p_0$, and $\tilde{\mu}_0 : \tilde{V}_0 \to \mathbb{P}^{n-1}$ be the associated morphism. If $\nu_0 : V_0 \to V$ is the blow-up of $V$ at the points $\phi^{-1}(p_0)$, then we have a morphism $\tilde{\nu}_0 : \tilde{V}_0 \to \tilde{V}_0$ which, when composed with $\mu_0$, gives a fibration in curves $f_0 := \mu_0 \circ \tilde{\nu}_0 : V_0 \to \mathbb{P}^{n-1}$.

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Next, let $D'$ be the proper transform of $D$ under the blow-up $\mu_0$. Then $f_0(D') \neq \mathbb{P}^{n-1}$ if and only if $D$ is a cone with vertex $V_D \supset p_0$. Since the vertex of a cone is a linear subspace, the highest dimensional vertex is a hyperplane. Therefore, if we choose $n+1$ points $p_0, \ldots, p_n$ not all lying in a hyperplane, we see that for all divisors $D$ in $\mathbb{P}^n$, we can find a morphism $f_i$ such that $f_i(D') = \mathbb{P}^{n-1}$. Since $\phi$ is a finite morphism, the same holds for divisors in $\mathcal{V}$.

The geometric proposition above allows us to reduce our problem to the following situation: Let $K$ be a field finitely generated over $\mathbb{Q}$, and let $C$ be a smooth projective curve defined over $K$. Suppose $A$ is a smooth projective variety equipped with a proper flat morphism $\rho : A \to C$, such that the generic fiber $A_\rho$ is an Abelian variety with Chow trace $(\tau, B)$. Let $x \in C(K)$ be a point for which the fiber $A_x$ is nonsingular. Then the specialization map

$$\sigma_x : A(C/K) \to A_x(\overline{K})$$

is a homomorphism from the group of sections, to the group of points on the fiber.

**Theorem 0.1.** Let $\Gamma$ be a finitely generated free subgroup of $A(C/K)$ which injects in $A(C/K)/\tau B(K)$. Then the set

$$\{ x \in C(\overline{K}) : \sigma_x \text{ is not injective on } \Gamma \}$$

is a set of bounded height in $C(\overline{K})$. In particular, if $d \geq 1$, then $\sigma_x$ is injective for all but finitely many $x \in \bigcup_{[L:K] \leq d} C(L)$. Furthermore, if $A_\rho$ has trivial Chow trace, this shows that, excluding a finite number of points $x \in C$, the Mordell-Weil rank of the special fibers $A_x$ is at least that of the generic fiber $A_\rho$.

The proof of this theorem is based on being able to measure the variation of certain height functions in a family of Abelian varieties. We briefly describe the necessary height functions below, before outlining the major steps in the proof.

**Arithmetic Height.** We consider the arithmetic height functions on $K$ introduced by Moriwaki [?], and, as much as possible, stick to Moriwaki’s notation and terminology. Let $Z$ be a normal projective arithmetic variety whose function field is $K$, and fix nef $C^\infty$-hermitian line bundles $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_d$ on $Z$. The collection $(Z; \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_d)$ is called a polarization of $Z$, and will be denoted by $\mathcal{Z}$.

Now suppose that $X$ is a smooth projective variety over $K$, and $L$ a line bundle on $X$. If $\mathcal{X}$ is a projective arithmetic variety over $Z$ and $\mathcal{Z}$ a hermitian line bundle on $\mathcal{X}$ with $\mathcal{X}_K = X$ and $\mathcal{L}_K = L$ then the pair $(\mathcal{X}, \mathcal{Z})$ is a model for $(X, L)$, and we can define a height function using Arakelov intersection theory, as follows: $h^Z_{(\mathcal{X}, \mathcal{Z})} : X(\overline{K}) \to \mathbb{R}$ via

$$h^Z_{(\mathcal{X}, \mathcal{Z})}(P) := \deg \left( \hat{c}_1(\mathcal{Z}|_{\Delta_P}) \cdot \hat{c}_1(f^*\mathcal{H}_1|_{\Delta_P}) \cdots \hat{c}_1(f^*\mathcal{H}_d|_{\Delta_P}) \right) / [K(P) : K],$$

where $\Delta_P$ is the Zariski closure of the point $P$ in $\text{Spec}(\overline{K}) \to X$, and $f : X \to Z$ is the canonical morphism. Furthermore, if the polarization $Z$ is big (as defined in [?, Section 2]), then $h^Z_{(\mathcal{X}, \mathcal{Z})}$ is an arithmetic height function, and denoted by $h^{\text{arith}}_{(\mathcal{X}, \mathcal{Z})}$. In fact, [?, Corollary 3.3.5] shows that, up to a bounded function, the height $h^Z_{(\mathcal{X}, \mathcal{Z})}$ does not depend on the choice of model $(\mathcal{X}, \mathcal{Z})$ for $(X, L)$, and hence we will denote the arithmetic height function simply by $h^{\text{arith}}_{(X, L)}$, or $h^L_{\text{arith}}$ when $X$ is understood.
Proposition 0.2 (Height Machine). Let $X$ be a smooth projective variety over $K$, and $L$ a line bundle on $X$. Then the arithmetic height function $h^\text{arith}_L$ satisfies the following properties:

1. If $M$ is another line bundle on $X$, then $h^\text{arith}_{L\otimes M} = h^\text{arith}_L + h^\text{arith}_M + O(1)$, while $h^\text{arith}_{L\otimes -1} = -h^\text{arith}_L + O(1)$.
2. Let $BS(L)$ denote the base locus of $L$, and set $SBs(L) := \cap_{n>0} BS(L^n)$. Then $h^\text{arith}_L$ is bounded below on $(X \setminus SBs(L))(\overline{K})$, and in particular:
   - (a) If $L$ is ample, then $h^\text{arith}_L$ is bounded below.
   - (b) If $L = O_X$ then $h^\text{arith}_L = O(1)$.
3. If $N$ is any number, and $R$ a positive integer, then the set
   \[
   \left\{ P \in X(\overline{K}) | h^\text{arith}_L(P) \leq N \text{ and } [K(P) : K] \leq R \right\}
   \]
   is finite.
4. Let $q : X \to Y$ be a morphism of smooth, projective varieties over $K$, and $M$ a line bundle on $Y$; then
   \[ h^\text{arith}_{(X,q^*(M))}(P) = h^\text{arith}_{(Y,M)}(q(P)) \quad \text{for all } P \in X(\overline{K}). \]
5. If $L$ is ample, and $M$ is any line bundle on $X$, then there is a constant $c$ such that
   \[ h^\text{arith}_M < ch^\text{arith}_L + O(1). \]
6. Suppose $X$ is a smooth projective curve over $K$, and if $D$ is a divisor write $h^\text{arith}_{L(D)}$ for $h^\text{arith}_{L(D)}$, where $L(D)$ is the line bundle associated to $D$. If $D$ is of degree $d > 0,$ and $E$ is a divisor of degree $e,$ then
   \[ \lim_{h^\text{arith}_{L(D)}(t) \to \infty} \frac{h^\text{arith}_{E}(t)}{h^\text{arith}_{L(D)}(t)} = \frac{e}{d}. \]

Proof. Properties (1)-(3) are [?, Proposition 3.3.7], while Property (4) is [?, Theorem 4.3]. To prove Property (5), we start with the definition of arithmetic height, and note that if $f : Y \to B$ is the canonical morphism on $Y$, then the canonical morphism on $X$ is given by $f' = f \circ q$:

\[
\begin{align*}
  h_{(X,q^*(M))}(P) &:= \frac{\hat{\text{deg}}((f')^*(\hat{c}_1(\overline{P})) \cdots \hat{c}_1(\overline{P}_d)) \cdot \hat{c}_1(q^*(\overline{M})) \cdot (\Delta_P, 0))}{[K(P) : K]} \\
  &= \frac{\hat{\text{deg}}((q^* \circ f^*)^*(\hat{c}_1(\overline{P})) \cdots \hat{c}_1(\overline{P}_d)) \cdot q^*\hat{c}_1(\overline{M}) \cdot (\Delta_P, 0))}{[K(P) : K]} \\
  &= \frac{\text{deg}(\Delta_P \to \Delta_{q(P)})\hat{\text{deg}}((f^*)^*(\hat{c}_1(\overline{P})) \cdots \hat{c}_1(\overline{P}_d)) \cdot \hat{c}_1(\overline{M}) \cdot (\Delta_{q(P)}, 0))}{[K(P) : K]} \\
  &= h_{(Y,M)}(q(P)),
\end{align*}
\]

where the third equation follows from the Projection formula ([?, Proposition 1.3]), and the final step is again by definition.

To prove Property (6), we note that there is some constant $a$ such that $L^{\otimes a} \otimes M^{\otimes -1}$ is ample. Hence, by Property (3.a) we have $h_{L^{\otimes a} \otimes M^{\otimes -1}} > O(1)$. The result now follows by Properties (1) and (2).

Finally, the proof of Property (7) is very similar to the proof for heights over number fields (see for example [?, Chapter 4, Corollary 3.5]). ☐
If $A/K$ is an Abelian variety, then we can also define a canonical height $\hat{h}_L^{\text{arith}}$, which has the following properties:

**Proposition 0.3** (Canonical Height Machine).

1. If $M$ is another line bundle on $A$, then $\hat{h}_L^{\text{arith}} = \hat{h}_M^{\text{arith}} + \hat{h}_L^{\text{arith}}$, while $\hat{h}_{L^\otimes -1}^{\text{arith}} = -\hat{h}_L^{\text{arith}}$.
2. If we assume furthermore that $L$ is ample, then we have:
   - There is a constant $c$, such that if $M$ is another line bundle on $A$, then $\hat{h}_M^{\text{arith}} \leq c \hat{h}_L^{\text{arith}}$.
   - $\hat{h}_L^{\text{arith}}(x) \geq 0$ for all $x \in A(K)$, and equality holds if and only if $x$ is a torsion point.

**Proof.** This is [? , Propositions 3.4.1 and 3.4.2]. He proves these under the condition that $L$ is a symmetric line bundle, but the proofs hold verbatim for $L$ anti-symmetric, hence for all line bundles $L$. \(\Box\)

**Geometric Height.** Since $K$ is a global field, and $C/K$ is a smooth projective curve, $K(C)$ is also a global field, and we can define the geometric height $\hat{h}_K^{\text{geom}}$ so that, for any $x \in K(C)$, $\hat{h}_K^{\text{geom}}([x, 1])$ is the degree of the morphism $[x, 1] : C \to \mathbb{P}^1$. If $A$ is an Abelian variety defined over $K$, then we also have a canonical height $\hat{h}_K^{\text{geom}}$, which is non-degenerate on $A(K)/ (\tau B(K) + A_{\text{tors}})$ [?, Chapter 6, Theorem 5.4]. Both $\hat{h}_K^{\text{geom}}$ and $\hat{h}_K^{\text{geom}}$ satisfy the usual properties of heights/canonical heights, see [?, Chapter 3].

Equipped with these height functions, the proof of the following theorems then proceeds exactly as in [?], and leads to a proof of Theorem 0.1. Fix a line bundle $L$ on $A$, and let $\hat{h}_{(A,L)}^{\text{arith}}$ be the associated arithmetic height function. For each $t \in C(K)$ let $L_t$ be its restriction to $A_t$, and $D_\rho$ be its restriction to the generic fiber $A_\rho$. Let $C^0 \subset C$ be an affine open subset such that for all $t \in C^0(K)$, the fiber $A_t$ is an Abelian variety. Then there is a geometric canonical height

$$\hat{h}_{\rho}^{\text{geom}} : A_\rho(K(C)) \to \mathbb{R},$$

and for each $t \in C^0(K)$, an arithmetic canonical height

$$\hat{h}_{(A_t,L_t)}^{\text{arith}} : A_t(K) \to \mathbb{R}.$$

Set $U := \pi^{-1}(C^0)$. The canonical heights $\hat{h}_{(A_t,L_t)}^{\text{arith}}$ on the good fibers $A_t$ can be fitted together to give a ”canonical height” on $U(K)$, $\hat{h}_L : U(K) \to \mathbb{R}$. If we fix also a line bundle $M$ on $C$, then

**Theorem 0.2.** There is a constant $c$, depending on $M$, $L$, and the family $A \to C$, such that

$$\left| \hat{h}_L(P) - \hat{h}_L^{\text{arith}}(P) \right| < c \hat{h}_M^{\text{arith}}(P) + O(1) \quad \text{for all } P \in U(K).$$

(The $O(1)$ depends on the choice of heights $\hat{h}_L^{\text{arith}}$ and $\hat{h}_M^{\text{arith}}$ but not on $P$.)

Consider now the case where $C$ is a smooth projective curve over $K$, and fix an arithmetic height on $C$ as follows: Let $D$ be a divisor on $C$ with $\deg D > 0$, and define

$$h_C^{\text{arith}} := \frac{1}{\deg D} h_{(C,D)}^{\text{arith}}.$$
Theorem 0.3. With notation as above, fix a section $P \in A(C)$. Then
\[
\lim_{t \in C^0(\overline{K}) \atop h_{\text{arith}}(t) \to \infty} \frac{\hat{h}_{(A_t, L_t)}(P_t)}{h_{\text{arith}}(t)} = \hat{h}_{(A_\rho, D_\rho)}(P_\rho).
\]

Note that by Property (7) of arithmetic heights, this result does not depend on the choice of arithmetic height on $C$.

Finally, we observe that in the case that $A$ is an elliptic surface, another crank of the height machine yields the following sharpened estimate à la Tate [?] for Theorem 0.3:

Theorem 0.4. Assume that $\pi : A \to C$ is an elliptic surface, and assume that $P \in A(C/K)$ is a section. Then
\[
\hat{h}_{(A_t, L_t)}(P_t) = \hat{h}_{(A_\rho, D_\rho)}(P_\rho)h_{\text{arith}}(t) + O\left(\sqrt{h_{\text{arith}}(t)} + 1\right).
\]

Tate’s proof requires the existence of a good compactification of the Neron model, hence is not known to apply to families of higher dimensional Abelian varieties. However, in the case of number fields, Call[?], Theorem I], using a different approach, generalized the result to the case of a one-parameter family of Abelian varieties. It remains an interesting question whether an analogous estimate exists for a family of Abelian varieties over $K$.

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