A note on scaling asymptotics for 
Bohr-Sommerfeld Lagrangian submanifolds

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1 Introduction

The purpose of this note is to improve an expansion in [DP] for the asymptotics associated to Bohr-Sommerfeld Lagrangian submanifolds of a compact Hodge manifold, in the context of geometric quantization (see e.g. [BW], [BPU], [GS3], [W]). We adopt the general framework for quantizing Bohr-Sommerfeld Lagrangian submanifolds presented in [BPU], based on applying the Szegö kernel of the quantizing line bundle to certain delta functions concentrated along the submanifold.

Let $M$ be a $d$-dimensional complex projective manifold, with complex structure $J$; consider an ample line bundle $A$ on it, and let $h$ be an Hermitian metric on $A$ such that the unique compatible connection has curvature $\Omega = -2i\omega$, where $\omega$ is a Kähler form. Then the unit circle bundle $X \subseteq A^*$, endowed with the connection one-form $\alpha$, is a contact manifold. A Bohr-Sommerfeld Lagrangian submanifold of $M$ (or, more precisely, of $(M, A, h)$) is then simply a Legendrian submanifold $\Lambda \subseteq X$, conceived as an immersed submanifold of $M$.

In a standard manner, $X$ inherits a Riemannian structure for which the projection $\pi : X \rightarrow M$ is a Riemannian fibration. In view of this, in the following at places we shall implicitly identify (generalized) functions, densities and half-densities on $X$.

Referring to §2 of [DP] for a more complete description of the preliminaries involved, we recall that if $\Lambda \subseteq X$ is a compact Legendrian submanifold, and $\lambda$ is a half-density on it, there is a naturally induced generalized half-density $\delta_{\Lambda, \lambda}$ on $X$ supported on $\Lambda$; following [BPU], we can then define a sequence of CR functions

$$u_k =: P_k(\delta_{\Lambda, \lambda}) \in \mathcal{H}(X)_k,$$

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where $\mathcal{H}(X)_k$ is the $k$-th isotype of the Hardy space with respect to the $S^1$-action, and $P_k : L^2(X) \to \mathcal{H}(X)_k$ is the orthogonal projector (extended to $\mathcal{D}'(X) \to \mathcal{H}(X)_k$). In the present setting there are natural unitary structures on $\mathcal{H}(X)_k$ and the space of global holomorphic sections $H^0(M, A^{\otimes k})$, and a natural unitary isomorphism $\mathcal{H}(X)_k \cong H^0(M, A^{\otimes k})$. One thinks of the $u_k$’s as representing the quantizations of $(\Lambda, \lambda)$ at Planck’s constant $1/k$. It is easily seen that $u_k$ is rapidly decaying as $k \to +\infty$ on the complement of $S^1 \cdot \Lambda = \pi^{-1}(\pi(\Lambda))$; here $\pi : X \to M$ is the projection. On the other hand, the asymptotic concentration of the $u_k$’s along $S^1 \cdot \Lambda$ poses an interesting problem, already considered in Theorem 3.12 of [BPU].

This theme was revisited in [DP], in a somewhat different technical setting; in particular, Corollary 1.1 of [DP] shows that the scaling asymptotics of $u_k$ (to be defined shortly) near any $x \in S^1 \cdot \Lambda$ admit an asymptotic expansion, and explicitly computes the leading order term. We shall give presently a more precise description of this expansion, as a function on the tangent space of $M$ at $m = \pi(x)$. Namely, we shall show that this asymptotic expansion may be factored as an exponentially decaying term in the component $w^\perp$ of $w \in T_m M$ orthogonal to $\Lambda$, times an asymptotic expansion with polynomial coefficients in $w$ (more precisely, the expansion is generally given by a finite sum of terms of this form, one from each branch of $\Lambda$ projecting to $m$); the exponential term also contains a symplectic pairing between $w^\perp$ and the component of $w$ along $\Lambda$, $w^\parallel$. Furthermore, we shall give some relevant remainder estimates not mentioned in [DP].

Before stating the results of this paper, let us recall that for any $x \in X$ we can find a Heisenberg local chart for $X$ centered at $x$,

$$\rho : B_{2d}(\epsilon) \times (-\pi, \pi) \to X, \quad (p, q, \theta) \mapsto r_{e^{i\theta}}(\epsilon(\rho(p, q)));$$

here $B_{2d}(\epsilon) \subseteq \mathbb{R}^{2d}$ is a ball of radius $\epsilon$ centered at the origin, $\rho : B_{2d}(\epsilon) \to M$ is a preferred local chart for $M$ centered at $m =: \pi(x)$, meaning that it trivializes the holomorphic and symplectic structures at $m$, and $\epsilon$ is a unitary local frame of $A^*$, given by the unitarization of a preferred local frame (complete definitions are in [SZ]). Finally, $r_{e^{i\theta}} : X \to X$ is the diffeomorphism induced by the circle action. It is in this kind of local coordinates that the scaling limits of Szegő kernels exhibit their universal nature (Theorem 3.1 of [SZ]). If $\rho$ is a system of Heisenberg local coordinates centered at $x$, and $p, q \in \mathbb{R}^d$, $w = p + iq$, one poses

$$x + w =: \rho((p, q), 0).$$

For any $\theta$, we have

$$u_k(\rho(p, q, \theta)) = e^{ik\theta} u_k(\rho(p, q, 0)) = e^{ik\theta} u_k(x + w).$$
Given that a system of Heisenberg local coordinates induces a unitary isomorphism of $T_m M$ and $\mathbb{C}^d$, with this understanding we can also consider the expression $x + w$ with $w \in T_m M$.

If $x \in S^1 \cdot \Lambda$, there are only finitely many elements $h_1, \ldots, h_{N_x} \in S^1$ such that $x_j := r_{h_j}(x) \in \Lambda$. Since $\Lambda$ is Legendrian, hence horizontal for the connection, for any $j$ we may naturally identify the tangent space $T_{x_j} \Lambda \subseteq T_{x_j} X$ with a subspace of $T_{\pi(x)} M$. With this in mind, if $w \in T_{\pi(x)} M$ we can write $w = w_j^{\perp} + w_j^{\|}$ for unique $w_j^{\|} \in T_{x_j} \Lambda$ and $w_j^{\perp} \in T_{x_j} \Lambda^{\perp}$; the latter denotes the orthocomplement of $T_{x_j} \Lambda$ in $T_{\pi(x)} M$ in the Riemannian metric of $M$.

Finally, let $\text{dens}_{\Lambda}^{(1/2)}$ be the Riemannian half-density on $\Lambda$ (for the induced metric); thus if $\lambda$ is a $C^\infty$ half-density on $\Lambda$ we can write $\lambda = F_\lambda \cdot \text{dens}_{\Lambda}^{(1/2)}$ for a unique $F_\lambda \in C^\infty(\Lambda)$.

**Theorem 1.** Let $\Lambda \subseteq X$ be a compact Legendrian submanifold, and suppose $\lambda$ is a smooth half-weight on it. For every $k = 1, 2, \ldots$, let $u_k := P_k(\delta_{\Lambda, \lambda})$. Suppose $x \in S^1 \cdot \Lambda$, and choose a system of Heisenberg local coordinates for $X$ centered at $x$. Let $h_1, \ldots, h_{N_x} \in S^1$ be the finitely many elements such that $r_{h_j}(x) \in \Lambda$. Then:

1. Suppose $a > 0$. Uniformly for $\min_j \{\|w_j^{\perp}\|\} \gtrsim k^a$, we have
   $$u_k \left( x + \frac{w}{\sqrt{k}} \right) = O(k^{-\infty}).$$

2. There exists polynomials $a_{ij}$ on $\mathbb{C}^d$ such that the following holds: for $w \in T_{\pi(x)} M$ and $k, \ell = 1, 2, \ldots$, let us define
   $$R_{k, \ell}(x, w) := u_k \left( x + \frac{w}{\sqrt{k}} \right) - \left( \frac{2k}{\pi} \right)^{d/2} \sum_{j=1}^{N_x} h_j^{-k} e^{-\|w_j^{\perp}\|^2 - i\omega_\pi(x)(w_j^{\perp}, w_j^{\|})} F_\lambda(x_j) \cdot \left( 1 + \sum_{l=1}^{\ell} k^{-l/2} a_{lj}(w) \right).$$
   Then uniformly for $\|w\| \lesssim k^{1/6}$ we have
   $$\left| R_{k, \ell}(x, w) \right| \leq C_{\ell} k^{(d-\ell-1)/2} \sum_{j=1}^{N_x} e^{-\frac{1}{k} \|w_j^{\perp}\|^2}. \tag{1}$$

**Corollary 1.** $\forall w \in T_{\pi(x)} M$, the following asymptotic expansion holds as $k \to +\infty$:
   $$u_k \left( x + \frac{w}{\sqrt{k}} \right) \sim \left( \frac{2k}{\pi} \right)^{d/2} \sum_{j=1}^{N_x} h_j^{-k} e^{-\|w_j^{\perp}\|^2 - i\omega_\pi(x)(w_j^{\perp}, w_j^{\|})} F_\lambda(x_j) \cdot \left( 1 + \sum_{l=1}^{\ell} k^{-l/2} a_{lj}(w) \right).$$
This agrees with Corollary 1.1 of [DP] to leading order, but gives a clearer picture of the asymptotic expansion, as well as an explicit estimate on the remainder.

The proof of Theorem 1 is based on the scaling asymptotics of Szegő kernels in Theorem 3.1 of [SZ], whereas the proofs in [DP] are based on microlocal arguments that encompass the equivariant setting, involving a direct use of the parametrix developed in [BS]. In view of the scaling asymptotics of equivariant Szegő kernels proved in [P], factorizations akin to Theorem 1 also hold in the equivariant setting; we shall not discuss this here.

2 Proof of Theorem 1

Let us first prove 2. Thus, we want to investigate the asymptotics of

$$u_k \left( x + \frac{w}{\sqrt{k}} \right)$$

as $k \to +\infty$, assuming that $w \in T_{\pi(x)} M$, $\|w\| \leq C k^{1/6}$ for some fixed $C > 0$.

Let $\Pi_k \in C^\infty(X \times X)$ be the Schwartz kernel of $P_k$; explicitly, if \( \{ s_r^{(k)} \} \) is an orthonormal basis of $H_k(X)$, then

$$\Pi_k(y, y') = \sum_r s_r^{(k)}(y) \cdot \overline{s_r^{(k)}(y')} \quad (y, y' \in X).$$

Let $\text{dens}_X$ and $\text{dens}_\Lambda$ denote, respectively, the Riemannian density on $X$ and $\Lambda$. Then, in standard distributional short-hand, by definition of $\delta_{\Lambda, \lambda}$ for any $x' \in X$ we have

$$u_k(x') = \int_X \Pi_k(x', y) \delta_{\Lambda, \lambda}(y) \text{dens}_X(y)$$

$$= \langle \delta_{\Lambda, \lambda}, \Pi_k(x', \cdot) \rangle = \int_\Lambda \Pi_k(x', y) F_\lambda(y) \text{dens}_\lambda(y). \quad (2)$$

Let us write $\text{dist}_M$ for the Riemannian distance function on $M$, pulled-back to a smooth function on $X \times X$ by the projection $\pi \times \pi$. Let us set:

$$V_k =: \{ x' \in X : \text{dist}_M(x, x') < 4C k^{-1/3} \},$$

$$V'_k =: \{ x' \in X : \text{dist}_M(x, x') > 3C k^{-1/3} \}.$$

If $y \in V'_k$ and $\|w\| \leq C k^{1/6}$, then $\text{dist}_M \left( x + \frac{w}{\sqrt{k}}, y \right) \geq C k^{-1/3}$ for $k \gg 0$; by the off-diagonal estimates on Szegő kernels in [C], therefore, $\Pi_k \left( x + \frac{w}{\sqrt{k}}, y \right) = O (k^{-\infty})$ uniformly for $y \in V'_k$.\
For \( k \gg 0 \), \( \Lambda \cap V_k \) has \( N_x \) connected components:

\[
\Lambda \cap V_k = \bigcup_{j=1}^{N_x} \Lambda_{kj},
\]

where \( \Lambda_{kj} \) is the connected component containing \( x_j \). Let \( \{s_k, s'_k\} \) be an \( S^1 \)-invariant partition of unity on \( X \), subordinate to the open cover \( \{V_k, V'_k\} \). In view of (2) and the previous discussion, we obtain

\[
u_k \left( x + \frac{w}{\sqrt{k}} \right) = \int_{\Lambda} \Pi_k \left( x + \frac{w}{\sqrt{k}}, y \right) F_{\lambda}(y) \text{dens}_{\Lambda}(y)
\]

\[
\sim \sum_{j=1}^{N_x} \int_{\Lambda_{kj}} \Pi_k \left( x + \frac{w}{\sqrt{k}}, y \right) F_{\lambda}(y) s_k(y) \text{dens}_{\Lambda}(y),
\]

where \( \sim \) means that the two terms have the same asymptotics. Let us now evaluate the asymptotics of the \( j \)-th summand in (3).

To this end, recall that the Heisenberg local chart \( \rho \) centered at \( x \) depends on the choice of the preferred local chart \( \rho \) at \( \pi(x) \), and of the local frame \( e \) of \( A^* \). We obtain a Heisenberg local chart \( \rho_j \) centered at \( x_j \) by setting \( \rho_j(p, q, \theta) =: r_{hj}(\rho(p, q, \theta)) \). By the discussion in §2 of [DP] and (48) of the same paper, we can compose \( \rho_j \) with a suitable transformation in \( (p, q) \) (that is, a change of preferred local chart for \( M \)) so as to obtain a Heisenberg local chart \( \rho_j(p, q, \theta) \) centered at \( x_j \) with the following property: \( \Lambda \) is locally defined near \( x_j \) by the conditions \( \theta = f_j(q) \) and \( p = 0 \), where \( f_j \) vanishes to third order at the origin. By construction, we have \( \rho_j(p, q, \theta) = r_{hj}(\rho(p', q', \theta)) \)

for a certain local diffeomorphism \( (p, q) \mapsto (p', q') \).

Thus \( \Lambda \) is locally parametrized, near \( x_j \) and in the chart \( \rho_j \), by the imaginary vectors \( iq \); viewing the \( q \)'s as local coordinates on \( \Lambda \) near \( x_j \), locally we have \( \text{dens}_{\Lambda} = D_{\Lambda} \cdot |dq| \), for a unique locally defined smooth function \( D_{\Lambda} \). By construction of Heisenberg local coordinates, \( D_{\Lambda}(0) = 1 \).

Applying a rescaling by \( k^{-1/2} \), we obtain

\[
\int_{\Lambda_{kj}} \Pi_k \left( x + \frac{w}{\sqrt{k}}, y \right) F_{\lambda}(y) s_k(y) \text{dens}_{\Lambda}(y)
\]

\[
= k^{-d/2} \int_{\mathbb{R}^d} \Pi_k \left( x + \frac{w}{\sqrt{k}}, \frac{r_{e^{if_j(q/\sqrt{k})}}(x_j + \frac{iq}{\sqrt{k}})}{\sqrt{k}} \right) F_{\lambda} \left( \frac{q}{\sqrt{k}} \right) s_k \left( \frac{iq}{\sqrt{k}} \right) D_{\Lambda} \left( \frac{q}{\sqrt{k}} \right) dq
\]

\[
= k^{-d/2} \int_{\mathbb{R}^d} e^{-ikf_j(q/\sqrt{k})} \Pi_k \left( x + \frac{w}{\sqrt{k}}, x_j + \frac{iq}{\sqrt{k}} \right) F_{\lambda} \left( \frac{q}{\sqrt{k}} \right) s_k \left( \frac{iq}{\sqrt{k}} \right) D_{\Lambda} \left( \frac{q}{\sqrt{k}} \right) dq.
\]
Here, \( x + \frac{w}{\sqrt{k}} = \rho \left( \frac{\mathcal{R}(w)}{\sqrt{k}}, \frac{\mathcal{I}(w)}{\sqrt{k}}, 0 \right) \) (we use the Heisenberg chart to unitarily identify \( T \mathcal{M} \) with \( \mathbb{C}^d \), and \( x + \frac{iq}{\sqrt{k}} = \rho_j \left( 0, \frac{q}{\sqrt{k}}, 0 \right) \). Notice that \( s_k \left( \frac{iq}{\sqrt{k}} \right) = 1 \) for \( \|q\| \lesssim k^{1/6} \), \( s_k \left( \frac{iq}{\sqrt{k}} \right) = 0 \) for \( \|q\| \gtrsim k^{1/6} \). In particular, integration takes place over a ball of radius \( \sim k^{1/6} \). Also, Taylor expanding \( F_\lambda \) and \( f_j \) at the origin we have asymptotic expansions

\[
F_\lambda \left( \frac{q}{\sqrt{k}} \right) \sim F_\lambda (x_j) + \sum_{r \geq 1} k^{-r/2} b_r(q), \quad D_\lambda \left( \frac{q}{\sqrt{k}} \right) \sim 1 + \sum_{r \geq 1} k^{-r/2} c_r(q),
\]

and, since \( f_j \) vanishes to third order at the origin,

\[
f_j \left( \frac{q}{\sqrt{k}} \right) \sim \sum_{r \geq 0} k^{-(3+r)/2} d_r(q), \quad e^{-ikf_j \left( \frac{q}{\sqrt{k}} \right)} \sim 1 + \sum_{r \geq 1} k^{-r/2} e_r(q),
\]

for suitable polynomials \( b_r, c_r, d_r, e_r \).

Let \( w_j \in \mathbb{C}^d \) correspond to \( w \) in the Heisenberg local coordinates \( \rho_j \). By the above, Taylor expanding the transformation \( (p, q) \mapsto (p', q') \), we obtain \( x + \frac{w}{\sqrt{k}} = r_{h_j^{-1}} \left( x_j + \frac{w_j}{\sqrt{k}} + H(w, k) \right) \), where \( H(w, k) \sim \sum_{j \geq 2} k^{-f_j/2} h_j(w) \). Without affecting the leading order term of the resulting asymptotic expansion, we may pretend for simplicity that \( x + \frac{w}{\sqrt{k}} = r_{h_j^{-1}} \left( x_j + \frac{w_j}{\sqrt{k}} \right) \).

Write \( w_j = p_j + iq_j \), with \( p_j, q_j \in \mathbb{R}^d \). Thus \( w_j^\perp = p_j, w_j^\parallel = iq_j \). In view of Theorem 3.1 of [SZ], we have

\[
\Pi_k \left( x + \frac{w}{\sqrt{k}}, x_j + \frac{iq}{\sqrt{k}} \right) = \Pi_k \left( r^{-1}_{h_j} \left( x_j + \frac{w_j}{\sqrt{k}} \right), x_j + \frac{iq}{\sqrt{k}} \right) = h_j^{-k} \Pi_k \left( x_j + \frac{w_j}{\sqrt{k}}, x_j + \frac{iq}{\sqrt{k}} \right) \sim h_j^{-k} \left( \frac{k}{\pi} \right)^d e^{-ip_j \cdot q - \frac{1}{2} \|p_j\|^2 - \frac{1}{2} \|q_j\|^2} \left( 1 + \sum_{r \geq 1} k^{-r/2} R_j(w, q) \right),
\]

for certain polynomials \( R_j \) in \( w \) and \( q \). Furthermore, by the large ball estimate on the remainder discussed in §5 of [SZ], the remainder after summing over \( 1 \leq r \leq R \) is bounded by

\[
C_R k^{d-(R+1)} e^{-\frac{1}{4R} (\|p_j\|^2 + \|q_j\|^2)}.
\]

It follows that the product of these asymptotic expansions can be integrated term by term; given this, we only lose a contribution which is \( O(k^{-\infty}) \) by setting \( s_k = 1 \) and integrating over all of \( \mathbb{R}^d \).
We have
\[ \int_{\mathbb{R}^d} e^{-ip_j \cdot q - \frac{1}{2}||q||^2} dq = e^{-ip_j \cdot q_j} \int_{\mathbb{R}^d} e^{-ip_j \cdot s - \frac{1}{2}||s||^2} ds = (2\pi)^{d/2} e^{-ip_j \cdot q_j - \frac{1}{2}||p_j||^2}. \]

Given (5), this implies that (4) is given by an asymptotic expansion, with leading order term
\[ h_j^{-k} \left( \frac{2k}{\pi} \right)^{d/2} e^{-\|w_j^+\|^2 - i\omega_x(x)(w_j^+ \cdot w_j^+)} F_\lambda(x_j). \]

To determine the general term of the expansion, on the other hand, we are led to computing integrals of the form
\[ \int_{\mathbb{R}^d} q^\beta e^{-ip_j \cdot q - \frac{1}{2}||q||^2} dq = e^{-ip_j \cdot q_j} \int_{\mathbb{R}^d} (s + q_j)^\beta e^{-ip_j \cdot s - \frac{1}{2}||s||^2} ds. \]

where \( q^\beta \) is some monomial. Thus we led to a sum of terms of the form
\[ e^{-ip_j \cdot q_j} C_\gamma(q_j) \int_{\mathbb{R}^d} s^\gamma e^{-ip_j \cdot s - \frac{1}{2}||s||^2} ds, \]
and the integral is the evaluation at \( p_j \) of the Fourier transform of \( s^\gamma e^{-\frac{1}{2}||s||^2} \). Up to a scalar factor, the latter is an iterated derivative to \( e^{-\frac{1}{2}||s||^2} \); therefore we are left with a summand of the form \( e^{-ip_j \cdot q_j} T(q_j, q_j) e^{-\frac{1}{2}||p_j||^2} ds \), where \( T \) is a polynomial in \( p_j, q_j \). Given (5), this implies that the general term of the asymptotic expansion for (4) has the form
\[ h_j^{-k} \left( \frac{2k}{\pi} \right)^{d/2} k^{-l/2} e^{-\|w_j^+\|^2 - i\omega_x(x)(w_j^+ \cdot w_j^+)} F_\lambda(x_j) \cdot a_{ij}(w) \]
for an appropriate polynomial \( a_{ij}(w) \). Finally, (11) (at \( x_j \)) follows by integrating (11).

To complete the proof of 2., we need only sum over \( j \).

Turning to the proof of 1., by definition of preferred local coordinates, if \( w_j^+ \geq C k^a \), say, then
\[ \text{dist}_M \left( x + \frac{w}{\sqrt{k}}, \Lambda_{kj} \right) \geq C' k^{a-\frac{1}{2}}, \]
for all \( k \gg 0 \). By the off-diagonal estimates of [C], \( \Pi_k \left( x + \frac{w}{\sqrt{k}}, y \right) = O(k^{-\infty}) \) uniformly for \( y \in \Lambda_{kj} \).

Q.E.D.
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