Information Theoretical Methods as Discerning Quantifiers of the Equations of State of Neutron Stars

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Abstract. In this work we use the statistical measures of information entropy, disequilibrium and complexity to discriminate different approaches and parametrizations for different equations of state for quark stars. We confirm the usefulness of such quantities to quantify the role of interactions in such stars. We find that within this approach, a quark matter equation of state such as SU(2) NJL with vectorial coupling and phase transition is slightly favoured and deserves deeper studies.

1. Introduction

The basic concept of information theory is the Shannon Information, also known as Shannon Entropy or Information Entropy. Shannon defined, in 1948, an expression that measures the information (or randomness, or uncertainty, or ignorance) about a system. Obeying a set of mathematical properties defined by him, the information content of a system in terms of probabilities of a event to occur is:

\[ H = - K \sum_i p_i \log b_i \quad \text{or} \quad H = - K \int p(x) \log b[p(x)] dx, \]

 respectively for the discrete and continuous cases.

From this, people started thinking about how complex a system can be and how to measure this complexity calculating it from a mathematical definition. Thus, the statistical measure of complexity introduced by Lopez-Ruiz, Mancini and Calbet [2] relates the complexity of a system to the information stored in it and the distance to a situation in which all possible states of the system are equiprobable. This definition encodes the concepts of order and disorder of a given arrangement of the system.

It was assumed in the literature that the probability distribution is proportional to the energy density (or the mass density in the non-relativistic case) profile of the star, \( \epsilon(r) \), obtained when solving the equations of structure of the star (in the case of neutron stars, the TolmanOppenheimerVolkoff equations, the relativistic version of hydrostatic equilibrium). This
was justified by the statement that the energy/mass density is related to the probability of finding
some particles at a given defined location inside the star. Even though it may be criticized, it
proved difficult to suggest a better alternative from the available physical quantities.

To illustrate these concepts we may think about two ideal systems frequently used in physics:
the ideal gas and the perfect crystal, extremes in all aspects and opposites as well. Because they
both are idealized systems, they should be thought as minimally complex systems. However,
while the latter is totally ordered, the former is totally disordered; i.e. while for the perfect
crystal one state is more probable than the others, in the ideal gas all states are equiprobable.
Summarizing this intuitive view:

Perfect crystal: zero complexity by definition; strict symmetry rules ⇒ probability density
centered around the prevailing state of perfect symmetry ⇒ minimal information. Completely
ordered.

Ideal gas: zero complexity by definition; accessible states are equiprobable ⇒ maximal
information. Totally disordered.

Figure 1. Intuition of what should be complexity, at least asymptotically.

At this stage an suitable expression for complexity arises:

\[ C \equiv H \times D \quad \text{or} \quad C \equiv e^H \times D, \]  \hspace{1cm} (2)

where

\[ D = \sum_i \left[p_i - \frac{1}{N}\right]^2 \quad \text{or} \quad D = \int p^2(x)dx \]

is the distance to the (equilibrium) equiprobability of states.

If we measure the complexity of some system, and allow its proximity to a crystal or a gas to
encode the degree of order of the system under consideration, then one can wonder what state
of the system is preferable for Nature to realize. In particular, by calculating the information entropy of a star allowed by a chosen equation of state, we find a quite univocal behaviour of this quantity with the mass and radius of the stars in the stellar sequence.

2. The equations of state
To clarify the issue of the composition using these concepts, we have compared in Ref. [1] the information entropy, the disequilibrium and the complexity for two different sample cases: the quark stars constructed from a MIT Bag model equation of state (an admixture of quarks up, down and strange in approximately equal amounts and hadronic neutron stars from SLy4 equation of state (neutron rich in the core) [2]. We found that the information entropy trend for both cases is pretty much the same when viewed as a function of the stellar mass, but differs a lot when plotted as a function of the stellar radius.

If higher $D$ implies higher distance to the equiprobability and a trend to be more crystalline (and ordered), and additionally we assume that order has a cost, then for each EoS separately Nature prefers to form less massive objects, albeit more complex ones. This trend makes sense since ideal systems (in this case the perfect crystal and the ideal gas) have complexity zero by construction and should not form in Nature (because they are just idealizations). In other words, the complexity of any real system have to be greater than zero.

Here we extended that work by comparing a larger set of equations of state (Ref.[3]), enumerated and briefly described below:

(i) Original MIT Bag Model [4]: Neutron stars can be actually strange stars, stars in which the nuclei are subject to so high pressures that the nuclear boundaries dissolve, resulting in a soup of roughly equal numbers of quarks up, down and strange plus electrons to keep the star neutral.

(ii) Alford-Han-Reddy (AHR) [5]: The statement here is that at sufficiently low surface tension, the same strange quark matter of the above items will not be self-bound and at low pressure and temperature the matter will form a crystal of positively charged strangelets in a neutralizing background of electrons.

(iii) Dexheimer-Schramm (DS1 and DS2) [6]: the equation of state used here is actually a new parametrization of the equation of state presented in [6].

(iv) Coelho et al. [7, 8]: This equation of state is based on the SU(2) NJL model with vectorial coupling and a phase transition to a hadron phase described by the GM1 parametrization for different values of the ratio $g_v/g_s$ for the strength of the vectorial coupling.

(v) Douchin-Haensel (SLy4) [2]: Based on the effective nuclear interaction SLy of the Skyrme type, which is particularly suitable for the application to the calculation of the properties of very neutron rich matter.

3. Results and Discussion
We have compared our statistical measures for all the EoSs. Notice that the quantity which we have to compare, in terms of what Nature would want to form, is the complexity. Recalling that complexity is a composite quantity that encodes a “distance” of the real object to the two extremes and opposites ideal systems, the optimal complexity (the preferred object) is the one that at the same time maximizes the information entropy while keeping the disequilibrium at a safe distance of the extremes.

As an example, we show here the analysis of the information entropy, disequilibrium and complexity for the SLy4 EoS against the SU(2) NJL EoS. In Figure 2 we note that from the complexity measures we can also have different regimes: for masses $0.109M_\odot \leq M \leq 0.74M_\odot$, a slight preference for the SLy4 EoS; for masses $\geq 0.74M_\odot$ Nature would prefer to make star with the SU(2) NJL EoS.
From the complexity measures, we see that we can also have different regimes: for masses $0.109 M_\odot \leq M \leq 0.74 M_\odot$, a slight preference for the SLy4 EoS; for masses $\geq 0.74 M_\odot$, Nature would prefer to make a star with the SU(2) NJL EoS.

From the point of view of complexity, as stated earlier, although all sequences of neutron stars in our study are similar, there is a noticeable preference for the SU(2) NJL equation of state with phase transition and vectorial coupling for masses $\geq 0.75 M_\odot$.

In Figure 3 we present the complexity versus information entropy for all of our equations of state. Allowing then the possibility of a phase transition hadrons $\rightarrow$ quarks and considering that the information entropy evolution is a good indicator of the time evolution of the system, then we see that for a given mass, such 1.73 $M_\odot$, for example, the object will start as a SLy4 star and end up as a SU(2) NJL star (see the arrow). The higher the mass, more attractive and pronounced is the transition.

Finally, from Figure 3 we see that the SU(2) NJL would be the final state of a neutron star that is born as a SLy4 star. Actually, all neutron stars would be SU(2) NJL stars. On the other hand, the threshold of 1.06 $M_\odot$ also marks the possibility of coexistence of two families of neutron stars: below this threshold down to $\sim 0.85 M_\odot$, SLy4 stars; above it AHS or DS1 stars. Besides, SU(2) NJL is almost in all circumstances preferred over the other quark stars. This does not mean, however, that the SU(2) NJL equation of state with phase transition and vectorial coupling is the “true” equation of state of neutron stars. The real meaning is that our results suggest a direction where our studies should go to, they suggest that this kind of equation of state deserves more studies. Another outcome is this kind of quantifiers is that they suggest also that there could be two or more families of neutron stars in Nature depending on the masses.
Figure 3. Complexity versus information entropy for all EoSs used in this work. The straight lines delimit the region of observed masses; the black squares represent the position of the given mass for each EoS. The arrow shows the trend of evolution in time for a given mass.

4. Conclusions
We have used the definitions of information content, disequilibrium and complexity to discriminate among the equations of state for white dwarfs and, more interestingly, neutron stars (Reference [3]). However, while this statement (jump from a low density WD to a high density NS) seems to be valid from the information theoretic point of view, we cannot neglect the fact that an analogous energy barrier exist when considering the formation of neutron stars with quarks. The pressure of the collapse must be high enough to attain central densities at least few times $10^{14}$ g/cm$^3$ in order to be entropically favourable to form quarks, at least in principle. Unfortunately, the specific mechanism(s) of formation of neutron stars are still poorly understood to answer the issue of the presence of quarks. For example, the mass of the core that will become a neutron star is a major source of uncertainty the stellar evolution theory for progenitor masses between $7.5 - 11 M_\odot$, and with it the explosion mechanism itself for progenitor masses between $11 - 25 M_\odot$ [9]. It may happen that the energy (and entropy) barrier to form quarks is never attained, or rather that it is a trivial fact for all explosive formation scenarios [10].

References
[1] M. G. B. de Avellar, J. E. Horvath, Physics Letters A 376 (2012) 1085.
[2] F. Douchin, P. Haensel, Astronomy and Astrophysics 380 (2001) 151.
[3] M. G. B. de Avellar, R.A. de Souza, J. E. Horvath, D.M. Paret, Physics Letters A 378 (2014) 47.
[4] E. Witten, Physical Review D 30 (1984) 272.
[5] M. Alford, S. Han, S. J. Reddy, Phys. G: Nucl. Part. Phys. 39 (2012) 065201.
[6] J. Steinheimer, S. Schramm, H. J. Stöcker, Phys. G: Nucl. Part. Phys. 38 (2011) 035001.
[7] J. Coelho, C. H. Lenzia, M. Malheiro, R. M. Marinho Jr., C. Providência, M. Fiolhais, Nuclear Physics B (Proc. Suppl.) 199 (2010) 325.
[8] J. Coelho, Ph.D. Thesis, Instituto de Tecnologia Aeronáutica (ITA), São José dos Campos/SP, Brazil, 2009.
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