Identification of perturbative ambiguity canceled against bion

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It was conjectured that the bion, a semi-classical object found in a compactified spacetime, is responsible to a cancellation of the so-called renormalon ambiguity. Contrary to the conjecture, we argue that the ambiguity due to the bion corresponds to the proliferation of Feynman diagrams. We point out that the amplitudes of almost all Feynman diagrams are enhanced due to modifications of the infrared structure of perturbation theory upon an $S^1$ compactification and twisted boundary conditions. Our findings clarify the role of the semi-classical object in resurgence structure, which has been a controversial issue in recent years.

INTRODUCTION AND OUTLINE

In quantum field theory, perturbation theory is extensively used as a basic and general method to analyze phenomena in particle physics. However, it is known that perturbation theory often possesses intrinsic errors in its predictions. This is caused by divergent behaviors of perturbation series, $\sum_k a_k g^k/(16\pi^2)^k$, due to factorial growth of perturbative coefficients $a_k$ at large orders. This poses a fundamental problem how one can achieve ultimate predictions. Recently, the so-called resurgence structure is believed where the perturbative ambiguities are eventually canceled against ambiguities of nonperturbative calculations, and there are vigorous discussions on its concrete structure [1–2].

It has been recognized that there are two origins of the factorial growth of the perturbative coefficients. One is caused by the proliferation of Feynman diagrams (PFD), and the other relates to renormalization properties and is known as the renormalon [3, 4]. They induce factorially growing perturbative coefficients [27] and result in perturbative ambiguities, $\delta$, typically as follows:

$$\text{PFD: } a_k \sim k!, \quad \delta \sim e^{-16\pi^2/g^2},$$

$$\text{Renormalon: } a_k \sim \beta_0^k k!, \quad \delta \sim e^{-16\pi^2/(\beta_0 g^2)},$$

where $\beta_0$ is the one-loop coefficient of the beta function [e.g. $\beta_0 = 11N/3$ for $SU(N)$ quenched QCD]. One can see that the perturbative ambiguities reduce to nonperturbative factors. It is convenient to quantitatively characterize the perturbative ambiguities by singularities of a Borel transform $B(t) = \sum_{k=0}^{\infty} (a_k/k!)^t$, the generating function of the perturbative coefficients. The PFD induces the singularities at $t = 1, 2, \ldots$, while the renormalon at $t = 1/\beta_0, 2/\beta_0, \ldots$. The PFD-type ambiguity is known to get canceled against the ambiguity of the semi-classical calculation for the instanton–anti-instanton amplitude $\sim e^{-2S_I}$ [5, 6], where $S_I = 8\pi^2/g^2$ is the one-instanton action. On the other hand, a cancellation of the renormalon ambiguity was shown only in a two dimensional non-linear sigma model [5].

In 2012, a conjecture was proposed [8–11] that the renormalon ambiguity is canceled by a new semi-classical object called the bion [12], a pair of a fractional instanton and anti-fractional instanton. This conjecture first requires an $S^1$ compactification of a spacetime as $\mathbb{R}^d \to \mathbb{R}^{d-1} \times S^1$ with a small $S^1$-radius $R$ such that $R \Lambda \ll 1$, where $\Lambda$ is a dynamical scale. The bion solution appears in this setup when twisted boundary conditions are imposed along the $S^1$-direction or equivalently a non-trivial holonomy exists. The bion action is given by $S_B = S_{I1}/N$, where $N$ is a parameter specifying the degree of freedom of dynamical variables. Accordingly, the ambiguities in bion calculus are typically given by $\sim e^{-2S_{I1}/N}$. The corresponding perturbative ambiguities are specified by the Borel singularities at $t = 1/N, 2/N, \ldots$. Due to similar $N$ dependence to the renormalon ambiguity, it was conjectured that the bion is responsible to the cancellation of the renormalon ambiguity. Since then, active discussions on the conjecture have been initiated.

However, recent studies have reported inconsistency of the conjecture. Two observations were made in the systems where the bion ambiguities exist. (i) In the $SU(2)$ and $SU(3)$ gauge theories with adjoint fermions on $\mathbb{R}^3 \times S^1$, renormalon ambiguities are absent [13]. (ii) In the $\mathbb{CP}^{N-1}$ models on $\mathbb{R} \times S^1$, the renormalon ambiguity is specified by the Borel singularity at $t = 3/(2N)$ [14–15], which conflicts with that of the bion ambiguity [16]. The both cases indicate that the bion ambiguities do not correspond to the renormalon ambiguities.

These observations give rise to a new question: what cancels the bion ambiguities? If the renormalon does not correspond to the bion, there should be different type of perturbative ambiguities which cancel the bion ambiguities. Such a perturbative ambiguity has not been identified so far. This implies a lack of our understanding on the resurgence structure.

In this Letter, we argue that the bion ambiguity is canceled against the PFD-type ambiguity. Although this possibility was mentioned in Ref. [13], so far, it has not been understood whether a seemingly different singularity (by the factor $1/N$) can get compatible with the bion ambiguity. We clarify that it occurs by a non-trivial enhancement of the amplitudes of almost all Feynman diagrams as a consequence of the $S^1$ compactification and twisted boundary conditions. As a result, an $N$ times...
closer Borel singularity to the origin arises consistent with the bion. Our findings are helpful in giving a unified understanding to a series of controversial discussions.

We explain our setup. We consider $\mathbb{R}^{d-1} \times S^1$ spacetime with the $S^1$-radius $R$, and the $\mathbb{Z}_N$-twisted boundary conditions for an $N$-component field $\phi^A$:

$$\phi^A(x, x_d + 2\pi R) = e^{im_A 2\pi R} \phi^A(x, x_d).$$

(2)

Here $x$ denotes the coordinates $(x_1, \ldots, x_{d-1}, x_d)$ of $\mathbb{R}^{d-1}$, $x_d$ does that of $S^1$, and the twist angle $m_A$ is given by

$$m_A = \begin{cases} A/(NR) & \text{for } A = 1, 2, \ldots, N-1, \\ 0 & \text{for } A = N. \end{cases}$$

(3)

This setup allows us to find a bion solution with the bion action $S_B = (4\pi)^{d/2} m_A R / g^2$ with $A < N$, and the bion ambiguities $- e^{-S_B}$ appear. The corresponding perturbative ambiguities are specified by the Borel singularities at $t = 1/(m_A R) = A/N = 1/N, 2/N, \ldots$.

We explain the keys to understanding the enhancement of the PFD-type ambiguity under the above setup. First, the $S^1$ compactification reduces the dimension of a momentum integral from $d$ to $d-1$. Then, perturbation theory tends to suffer from severe infrared (IR) divergences. This is parallel to finite-temperature and low-dimensional super-renormalizable field theories with massless particles [20, 21]. Secondly, the twist angles play the role of mass terms for the zero Kaluza–Klein (KK) momentum, and hence work as IR regulators. As a result, an amplitude of a Feynman diagram is given by an inverse power of the twist angle as $[g^2/(m_A R)]^k$ at the $k$th order, as a signal of the IR divergence. (It diverges when $m_A \to 0$.) For small $A$, this behavior gives $\sim (Ng^2)^k$ due to Eq. (3). Thus, if we have the PFD exhibiting the IR divergences in the above sense, the $k$th order term of perturbation series behaves as $\sim k!(Ng^2)^k$ rather than $\sim k!(g^2)^k$. This causes the Borel singularities at $t = 1/N, 2/N, \ldots$, corresponding to the bion ambiguities.

In the following, we study the $CP^N$ model on $\mathbb{R} \times S^1$ as a concrete model, where an explicit calculation of the bion ambiguity has been performed [19].

**ENHANCEMENT OF PERTURBATIVE AMBIGUITY: THE $CP^{N-1}$ MODEL**

The $CP^{N-1}$ model in the two-dimensional Euclidean spacetime is given by

$$S = \frac{1}{g^2} \int d^2x \left[ \partial_\mu z^A \partial_\mu z^A - j_\mu j_\mu + f(z^A \bar{z} A - 1) \right],$$

(4)

in terms of homogeneous coordinates $z^A$ and $\bar{z}^A$, where $f$ is introduced as the Lagrange multiplier field to impose the constraint $\bar{z}^A z^A = 1$, and

$$j_\mu = \frac{1}{2i} \bar{z}^A \partial_\mu z^A, \quad \bar{j}_\mu = \partial_\mu - \bar{\partial}_\mu.$$

(5)

We implement the $\mathbb{Z}_N$-twisted boundary conditions [2] for $z^A$ along the $x_2$-direction. Then the propagator of $z^A$ is obtained as

$$\langle z^A(x) \bar{z}^B(y) \rangle = g^2 \delta^{AB} \int_0^\infty \frac{dp}{2\pi} \frac{1}{(p_2 + m_A)^2 + f_0} \equiv g^2 \delta^{AB} \Box^{-1} \delta (x - y),$$

(6)

with a vacuum expectation value $f_0$, which is determined by the tadpole condition that a linear term of the fluctuation $\delta f$ vanishes. Here and hereafter, we use the abbreviation

$$\langle \int \frac{dp}{2\pi} \frac{1}{2\pi R} \sum_{p_2 < 2\pi R} \rangle.$$

(7)

Note that, in the denominator of Eq. (6), the second and third terms serve as a mass parameter for $p_2 = 0$.

We consider the partition function in perturbation theory,

$$Z = \int DF \int Dz^A D\bar{z}^A e^{-S}$$

$$= \int DF e^{(1/2)g^2} \int d^2x \int Z[f]$$

(8)

where

$$Z[f] = Z[f]_{f = f_0 + \delta f}$$

$$\equiv \exp \left( g^2 \oint_{\partial A} \delta \bar{z}^A - \delta \bar{z}^A \right) \times \exp \left[ \frac{1}{g^2} \int d^2x \left( j_\mu j_\mu - \delta f \bar{z}^A z^A \right) \right]_{z=0}.$$

(9)

In the following, we consider only the $A = 1$ flavor. As we shall see, this contribution is significant for enhancement of the amplitude. (Note that a flavor symmetry is broken by the twisted boundary conditions.) We study the $O(\delta f^0)$ term in $Z[f]$ for simplicity.

We first give a review on the PFD in the non-compactified case. We estimate the $k$th order perturbative contribution by equally assigning one to every possible diagram. For this purpose, it is suitable to consider the replacements $\Box^{-1} \to 1$ and $j_\mu \to \bar{z}$. This approximately corresponds to employing the zero-dimensional version of the model. Then $Z[f] \equiv \sum_{k=0}^\infty T_k (g^2)^k$ is estimated as

$$T_k \sim \frac{1}{(2k)!} \left( \frac{\delta \bar{z}}{\delta z} \right)^{2k} \frac{1}{k!} \left( \bar{z} z \right)^k \approx \frac{(2k)!}{k!} \sim 4^k \Gamma(k + 1/2).$$

(10)

This factorial growth is interpreted as the source of the Borel singularities at $t = 1, 2, \ldots$ in the non-compactified spacetime, corresponding to the ambiguity in the instanton–anti-instanton calculus [2, 3].

Now we explain how the enhancement in terms of $N$ occurs under the $S^1$ compactification and twisted boundary
conditions. For this purpose, the above naive counting is not sufficient and we need to have a closer look at the structure of loop integrals. The $k$th order perturbative contribution to $Z'[f]$ is given by

$$Z'[f]_{k\text{th order}} = \frac{1}{(2k)!} \begin{pmatrix} g^2 \frac{\delta}{\delta z^A} \cdot \Delta_1 \cdot \frac{\delta}{\delta z^A} \end{pmatrix}^{2k} \times \frac{1}{k!} \left[ \left( -\frac{1}{4} \right) \int d^2x \left( \bar{z}^A \mu A z^A \right)^2 \right]^k. \quad (11)$$

The number of vertices, $V$, is identical to the perturbation order, $V = k$. We first consider connected Feynman diagrams, for which the following relations follow:

$$P = 2V, \quad L = V + 1, \quad (12)$$

where $P$ and $L$ denote the numbers of propagators and loops, respectively. Noting these relations, an amplitude of a Feynman diagram is written as

$$V_2(g^2)^k \int_{p_1,\ldots,p_{k+1}} \frac{F^{(2k)}(p_{i,\mu} + m_A \delta_{i2})}{\prod_{i=1}^{2k}[q_{i1} + (q_{i2} + m_A)^2 + f_0]}, \quad (13)$$

where $V_2$ is the volume factor of the two-dimensional spacetime, $p_i$ denotes a loop momentum, and $q_i$ is the momentum of a propagator, which is given by a linear combination of $(p_1, \ldots, p_{k+1})$. In the numerator, we have a $(2k)$th-order homogeneous polynomial $F^{(2k)}$, originating from the derivative in the interaction term.

The enhancement of the amplitude is caused by the zero KK-modes where $p_{i2} = 0$ for $i$, and hence, $q_{i2} = 0$ for $i$. For this part, we have

$$\sim \frac{V_2 (g^2)^k}{(2\pi R)^{k+1}} \int \left( \prod_{i=1}^{k+1} \frac{dp_{i1}}{2\pi} \right) \frac{F^{(2k)}(p_{i1}, m_A)}{\prod_{i=1}^{2k}[q_{i1} + m_A^2 + f_0]^2}. \quad (14)$$

Note that each loop integral becomes a single integral due to the $S^1$ compactification. Accordingly, this expression possesses an IR divergence in the massless limit $m_A^2 + f \to 0$ with the degree of divergence $k - 1$; the mass dimension of the integration measures is $(k+1)$, whereas that of the integrand is $-2k$. Then, if $f_0$ is negligible compared with $m_A = 1/(NK)$, we obtain

$$\sim \frac{V_2 (g^2)^k}{R^2(m_A R)^{k-1}} \left( \frac{g^2}{4\pi} \right)^k = \frac{V_2}{R^2} \frac{1}{N^2} \left( \frac{Ng^2}{4\pi} \right)^k. \quad (15)$$

because $m_A$ works as an IR regulator. (See below for discussion on the size of $f_0/(m_A^2)$.) We note that this argument indicates that a general connected diagram has the contribution naturally specified by $N^k g^{2k}$ rather than $g^{2k}$.

We see that disconnected diagrams show weaker enhancement. As an example, let us consider a $k$th order disconnected diagram given as a product of two connected diagrams. Since each connected diagram satisfies

$$\sim \left( \frac{V_2}{f_0} \right) \sum_{\alpha \geq 0} \frac{(m_A R)^{\alpha}}{(f_0 R^2)^{(k+\alpha-1)/2}} \left( m_A R \right)^{k-1} \frac{g^2}{4\pi}. \quad (16)$$

from Eq. (15). (Note that the total number of vertices is fixed as $k$.) This is suppressed by the inverse power of $N$ compared with the contribution of the connected diagram (15).

We can show that the connected diagrams, which have the strongest enhancement, increase factorially. For the present purpose, it is sufficient to assign one to every connected diagram. One can use

$$C = \sum_{k=0}^{\infty} C_k (g^2)^k = \ln \left[ \sum_{k=0}^{\infty} T_k (g^2)^k \right], \quad (17)$$

since the partition function $Z'$ is given by the exponential of the total sum of connected diagrams $C$ (i.e. $Z' = e^C$). From the estimate of $T_k$ in Eq. (10) and this formula, we estimate $C_k$ in Eq. (11). One sees that $C_k$ grows factorially at the almost same rate as $T_k$. Thus, it is plausible that we have the perturbation series as $\sum_k k!(Ng^2)^k$ and the Borel singularity at $t = 1/N$, corresponding to the bion ambiguity.

We emphasize that both the bion ambiguity and the position of the Borel singularity are uniformly controlled by the twist angles. This indicates validity of the resurgence structure between the bion and enhanced perturbative contributions.

We note that the above enhancement is specific to the case of $m_A^2 \gg f_0$. In contrast, if $m_A^2 \ll f_0$, since $f_0$ works as an IR regulator, we have

$$\sim \frac{V_2}{R^2} \sum_{\alpha \geq 0} C_\alpha \left( \frac{m_A R}{f_0 R^2} \right)^{(k+\alpha-1)/2} \left( \frac{g^2}{4\pi} \right)^k. \quad (18)$$

The enhancement does not occur in this case.

The size of $m_A^2/f_0$, which is critical for the existence of the $N$ times closer Borel singularity, is determined
depending on the magnitude of $NRA$. To determine the vacuum expectation value $f_0$, we consider the tadpole condition at the one-loop level. For $NRA \ll 1$, one finds a perturbative solution $\sqrt{f_0 R} \sim g^2((NRA)^{-1})/(4\pi)$ in terms of the renormalized coupling $g^2(\mu)$. Hence $m^2_A \gg f_0$ is satisfied within the perturbative expansion. Thus, the Borel singularity at $t = 1/N$ is likely to arise. It is worth noting that $NRA \ll 1$ also ensures validity of the bion analyses. This shows overall consistency.

On the other hand, for $NRA \gg 1$, $f_0$ is given by $f_0 = A^2$ as in the non-compactified case due to the so-called large $N$ volume independence. Then since $m^2_A/f_0 = 1/(NRA)^2 \ll 1$, we do not have the above enhancement or a Borel singularity at $t = 1/N$ from the PFD. In fact, since we have inverse powers of $A$ in this case, genuine perturbative analysis cannot be considered reasonably. For $NRA \gg 1$, the bion analyses are not always valid.

We give supplementary explanations in the case $NRA \ll 1$, where we have the enhancement. First, we mention effects of other flavor contributions than $A = 1$. From the zero KK modes, we have $k!/[m_A R]^{k-1}[g^2/(4\pi)]^k \sim k!(N/A)^{k-1}[g^2/(4\pi)]^k$ when we consider the $A$ flavor alone. Here, $A/N$ corresponds to the position of the Borel singularity. Then, the closest singularity is given by $A = 1$, and so is the asymptotic behavior of the perturbative coefficients. This is the reason why we focused on this contribution. Secondly, we clarify difference in power counting of $N$ from the non-compactified case. So far, we have not considered sums over flavor indices and limited the flavor to $A = 1$. Nevertheless, $N$ dependence appears as a consequence of loop momentum integrations, where, in particular, the IR structure of loop integrands is essential. This is in contrast to the non-compactified case, where $N$ dependence arises exclusively from the sums over flavor indices, and momentum integrations are irrelevant. When one takes into account the sums over indices as well in the compactified case, one needs to pay attention to both the degree of IR divergence and structure of flavor indices. For instance, the $N$ dependence of the diagram depicted in Fig. 2 is given by $N^{2k-1}g^{2k}$ at the $k$th order, provided that flavor indices for individual loops are independent. Here, we note that the loops outside a central loop do not possess IR divergences. Thirdly, we note that, in fact, the $A = N - 1$ sector also has the strongest enhancement. This is because $p_2 + m_A$ can be $-1/(NR)$ for $p_2 = -1/R$. Thus, this part has the same order contribution as the $A = 1$ sector.

Although we have considered the $O(\delta f^0)$ term so far, we can repeat a similar analysis, for instance, for the quadratic term of $\delta f$ in the effective action for this field, which has a clearer physical meaning.

We finally point out that there is a problematic sector in defining fixed order perturbation theory. For $A = N$, the IR regulator is given by $f_0$ since $m_A = 0$. Namely, $z^N$ is subject to the periodic boundary condition and we partially have the same situation as finite-temperature field theory. For $NRA \ll 1$, $f_0$ is given by $f_0 = g^4/(4\pi R)^2$ and plays the role of a screening mass. It should be kept in the denominator of Eq. 14, otherwise calculations break down due to IR divergences. In this treatment, we have the term with $\alpha = 0$ in Eq. 18, and the $k$th order perturbative contribution is partially given by $g^{2k}/(f_0 R^2)^{(k-1)/2} = O(g^2)$. It is irrelevant to the perturbation order $k$. Therefore, higher-loop diagrams can contribute at the same order. This is known as the Linde problem. Then, it is practically impossible to systematically obtain a series expansion in $g^2$ even at relatively low orders.

**CONCLUSIONS**

For nearly a decade, there have been vigorous discussions on the conjecture that the bion is responsible to the cancellation of the renormalon ambiguity, and a unified interpretation has not been established. In this Letter, we argued convincingly that the bion cancels the perturbative ambiguity caused by the proliferation of Feynman diagrams. In particular, we demonstrated how a Borel singularity appears at $t = 1/N$ upon the $S^1$ compactification and twisted boundary conditions, which is located at $t = 1$ in the non-compactified spacetime. Here, the IR structure of loop integrals is crucial, and the $N$ times closer singularity is regarded as a signal of IR divergences in perturbation theory. This observation settles a recent controversy about the role of the bion and deepens our understanding on the resurgence structure.

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[28] The renormalon analyses have been performed for $N R A \gg 1$, in which the bion analyses are not always valid. An important point is that it was shown that the $S^1$ compactification affects renormalon structure even in this so-called large-$N$ volume independence domain. This is against an expectation of the conjecture.