THE DONALDSON-THOMAS INVARIANTS UNDER BLOWUPs AND FLOPS

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Abstract. Using the degeneration formula for Donaldson-Thomas invariants, we proved formulae for blowing up a point and simple flops.

1. Introduction

Given a smooth projective Calabi-Yau 3-fold $X$, the moduli space of stable sheaves on $X$ has virtual dimension zero. Donaldson and Thomas \cite{D-T} defined the holomorphic Casson invariant of $X$ which essentially counts the number of stable bundles on $X$. However, the moduli space has positive dimension and is singular in general. Making use of virtual cycle technique (see \cite{B-F} and \cite{L-T}), Thomas showed in \cite{Thomas} that one can define a virtual moduli cycle for some $X$ including Calabi-Yau and Fano 3-folds. As a consequence, one can define Donaldson-type invariants of $X$ which are deformation invariant. Donaldson-Thomas invariants provide a new vehicle to study the geometry and other aspects of higher-dimensional varieties. It is important to understand these invariants.

Much studied Gromov-Witten invariants of $X$ are the counting of stable maps from curves to $X$. In \cite{MNOP1, MNOP2}, Maulik, Nekrasov, Okounkov, and Pandharipande discovered relations between Gromov-Witten invariants of $X$ and Donaldson-Thomas invariants constructed from moduli spaces of ideal sheaves of curves on $X$. They conjectured that these two invariants can be identified via the equations of partition functions of both theory. This suggests that many phenomena on Gromov-Witten theory have the counterparts in Donaldson-Thomas theory.

Donaldson-Thomas invariants are deformation independent. In the birational geometry of 3-folds, we have blowups and flops. Donaldson-Thomas invariants couldn’t be effective in studying birational geometry unless we understand how invariants change under birational operations. Li and Ruan in \cite{L-R} studied how Gromov-Witten invariants change under a flop for Calabi-Yau 3-fold. They proved that one can identify the 3-point functions of $X$ and the flop $X'$ of $X$ up to some transformation of the $q$ variables. The same question was also studied by Liu and Yau in \cite{L-Y} recently using the J. Li’s degeneration formula from algebraic geometry. In \cite{Hu1, Hu2}, the first author studied the change of Gromov-Witten

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invariants under the blowup. In this paper, we will study how Donaldson-Thomas invariants in \[\text{MNOP2}\] change under the blowup of a point and some flops.

The method we use is the degeneration formula for Donaldson-Thomas invariants studied in \[\text{Li1, Li2, MNOP2}\]. The blowup of \(X\) has a description in terms of a degeneration of \(X\). Then we can apply the degeneration formula. In the category of symplectic manifolds, one uses symplectic sum or symplectic cutting for the blowup operation on \(X\). The gluing formula for Gromov-Witten invariants in the symplectic setup is in \[\text{I-P1, I-P2, L-R}\]. Besides the difference of degeneration and symplectic cutting, the arguments used in \[\text{L-R, Hu1, Hu2, L-Y}\] rely on the fact that stable maps have connected domain, while the curves defined by ideal sheaves are in general not connected. Therefore the formula for the flop is a bit different from that of Gromov-Witten invariants in \[\text{L-R}\].

The organization of the paper is as follows. In section 2, we set up terminologies and notations, and list the basic results needed. The degeneration formula is discussed. In section 3, using J. Li’s degeneration formula, we prove a blowup formula for the blowup of \(X\) at a point. In section 4, we prove the equality of Donaldson-Thomas partition functions under a flop.

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2. Preliminaries

In this section, we shall discuss the basic materials on Donaldson-Thomas invariants studied by Maulik, Nekrasov, Okounkov and Pandharipande. For the details, one can consult \[\text{D-T, L-R, I-P1, I-P2, Li1, Li2, MNOP1, MNOP2, Thomas}\].

Let \(X\) be a smooth projective 3-fold and \(\mathcal{I}\) be an ideal sheaf on \(X\). Assume the sub-scheme \(Y\) defined by \(\mathcal{I}\) has dimension \(\leq 1\). Here \(Y\) is allowed to have embedded points on the curve components. Therefore we have the exact sequence

\[
0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0.
\]

The 1-dimensional components, with multiplicities taken into consideration, determine a homology class

\[
[Y] \in H_2(X, \mathbb{Z}).
\]

Let \(I_n(X, \beta)\) denote the moduli space of ideal sheaves \(\mathcal{I}\) satisfying

\[
\chi(\mathcal{O}_Y) = n, \quad [Y] = \beta \in H_2(X, \mathbb{Z}).
\]

\(I_n(X, \beta)\) is projective and is a fine moduli space. From the deformation theory, one can compute the virtual dimension of \(I_n(X, \beta)\) to obtain the following result

**Lemma 2.1.** The virtual dimension of \(I_n(X, \beta)\), denoted by \(\text{vdim}\), equals \(\int_\beta c_1(T_X)\).
Note that the actual dimension of the moduli space $I_n(X, \beta)$ is usually larger than the virtual dimension.

Let $\mathcal{J}$ be the universal family over $I_n(X, \beta) \times X$ and $\pi_i$ be the projection of $I_n(X, \beta) \times X$ to the $i$-th factor. For a cohomology class $\gamma \in H^i(X, \mathbb{Z})$, consider the operator

$$ch_{k+2}(\gamma) : H_* (I_n(X, \beta), \mathbb{Q}) \rightarrow H_{*-2k+2-\ell}(I_n(X, \beta), \mathbb{Q}),$$

$$ch_{k+2}(\gamma)(\xi) = \pi_{1*}(ch_{k+2}(\mathcal{J}) \cdot \pi_2^* (\gamma) \cap \pi_1^*(\xi)).$$

Descendent fields in Donaldson-Thomas theory are defined in [MNOP2], denoted by $\tilde{\tau}(\gamma)$, which correspond to the operations $(-1)^{k+1} ch_{k+2}(\gamma)$. The descendent invariants are defined by

$$<\tilde{\tau}_{k_1}(\gamma_{l_1}) \cdots \tilde{\tau}_{k_r}(\gamma_{l_r})>_{n, \beta} = \int_{[I_n(X, \beta)]^{vir}} \prod_{i=1}^{r} (-1)^{k_i+1} ch_{k_i+2}(\gamma_{l_i}),$$

where the latter integral is the push-forward to a point of the class $(-1)^{k_1+1} ch_{k_1+2}(\gamma_{l_1}) \circ \cdots \circ (-1)^{k_r+1} ch_{k_r+2}(\gamma_{l_r})([I_n(X, \beta)]^{vir}).$

The Donaldson-Thomas partition function with descendent insertions is defined by

$$Z_{DT}(X; q | \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}))_{\beta} = \sum_{n \in \mathbb{Z}} <\prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i})>_{n, \beta} q^n.$$

The degree 0 moduli space $I_n(X, 0)$ is isomorphic to the Hilbert scheme of $n$ points on $X$. The degree 0 partition function is $Z_{DT}(X; q)_0$.

The reduced partition function is obtained by formally removing the degree 0 contributions,

$$Z'_{DT}(X; q | \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}))_{\beta} = \frac{Z_{DT}(X; q | \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_{l_i}))_{\beta}}{Z_{DT}(X; q)_0}.$$

Relative Donaldson-Thomas invariants are also defined in [MNOP2]. Let $S$ be a smooth divisor in $X$. An ideal sheaf $\mathcal{I}$ is said to be relative to $S$ if the morphism $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_S \rightarrow \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_S$ is injective. A proper moduli space $I_n(X/S, \beta)$ of relative ideal sheaves can be constructed by considering the ideal sheaves relative to the expended pair $(X[k], S[k])$. For details, one can read [Li2] and [MNOP2].

Let $Y$ be the subscheme defined by $\mathcal{I}$. The scheme theoretic intersection $Y \cap S$ is an element in the Hilbert scheme of points on $S$ with length $[Y] \cdot S$. If we use $\text{Hilb}(S, k)$ to denote the Hilbert scheme of points of length $k$ on $S$, we have a map $\epsilon : I_n(X/S, \beta) \rightarrow \text{Hilb}(S, \beta \cdot [S])$.

The cohomology of the Hilbert scheme of points of $S$ has a basis via the representation of the Heisenberg algebra on the cohomologies of the Hilbert schemes.
Following Nakajima in [Nakajima], let $\eta$ be a cohomology weighted partition with respect to a basis of $H^\ast(S, \mathbb{Q})$. Let $\eta = \{\eta_1, \ldots, \eta_s\}$ be a partition whose corresponding cohomology classes are $\delta_1, \ldots, \delta_s$, let

$$C_\eta = \frac{1}{\mathfrak{z}(\eta)} P_{\delta_1}[\eta_1] \cdots P_{\delta_s}[\eta_s] \cdot 1 \in H^\ast(\text{Hilb}(S, |\eta|), \mathbb{Q}),$$

where

$$\mathfrak{z}(\eta) = \prod_i \eta_i |\text{Aut}(\eta)|,$$

and $|\eta| = \sum_j \eta_j$. The Nakajima basis of the cohomology of Hilb$(S, k)$ is the set,

$$\{C_\eta\}_{|\eta|=k}.$$

We can choose a basis of $H^\ast(S)$ so that it is self dual with respect to the Poincaré pairing, i.e., for any $i$, $\delta_i^\ast = \delta_j$ for some $j$. To each weighted partition $\eta$, we define the dual partition $\eta^\vee$ such that $\eta^\vee_i = \eta_i$ and the corresponding cohomology class to $\eta^\vee_i$ is $\delta_i^\ast$. Then we have

$$\int_{\text{Hilb}(S,k)} C_\eta \cup C_\nu = \frac{(-1)^{k-\ell(\eta)}}{\mathfrak{z}(\eta)} \delta_{\nu,\eta^\vee},$$

see [Nakajima].

The descendent invariants in the relative Donaldson-Thomas theory are defined by

$$<\bar{\tau}_{k_1}(\gamma_{l_1}) \cdots \bar{\tau}_{k_r}(\gamma_{l_r}) \mid \eta >_{n,\beta} = \int_{[\text{It}_n(X/S, \beta)]_{\text{vir}}} \prod_{i=1}^r (-1)^{k_i+1} \text{ch}_{k_i+2}(\gamma_{l_i}) \cap \epsilon^\ast(C_\eta),$$

Define the associated partition function by

$$Z_{DT}(X/S; q \mid \prod_{i=1}^r \bar{\tau}_{k_i}(\gamma_{l_i})_{\beta,\eta}) = \sum_{n \in \mathbb{Z}} <\prod_{i=1}^r \bar{\tau}_{k_i}(\gamma_{l_i}) \mid \eta >_{n,\beta} q^n.$$

The reduced partition function is obtained by formally removing the degree 0 contributions,

$$Z'_{DT}(X/S; q \mid \prod_{i=1}^r \bar{\tau}_{k_i}(\gamma_{l_i})_{\beta,\eta}) = \frac{Z_{DT}(X/S; q \mid \prod_{i=1}^r \bar{\tau}_{k_i}(\gamma_{l_i})_{\beta,\eta})}{Z_{DT}(X/S; q)_{0}}.$$

In the remaining of the section, we shall discuss the degeneration formula due to J. Li. It is the main tool employed in the paper.

Let $\pi : \mathcal{X} \to C$ be a smooth 4-fold over a smooth irreducible curve $C$ with a marked point denoted by $0$ such that $\mathcal{X}_t = \pi^{-1}(t) \cong X$ for $t \neq 0$ and $\mathcal{X}_0$ is a union of two smooth 3-folds $X_1$ and $X_2$ intersecting transversely along a smooth surface $S$. We write $X_0 = X_1 \cup_S X_2$. Assume that $C$ is contractible and $S$ is simply-connected.

Consider the natural maps

$$i_t : X = \mathcal{X}_t \to \mathcal{X}, \quad i_0 : \mathcal{X}_0 \to \mathcal{X},$$
and the gluing map
\[ g = (j_1, j_2): X_1 \coprod X_2 \to X_0. \]

We have
\[ H_2(X) \xrightarrow{i^*} H_2(X) \xrightarrow{i_0^*} H_2(X_0) \xrightarrow{g_*} H_2(X_1) \oplus H_2(X_2), \]
where \( i_0^* \) is an isomorphism since there exists a deformation retract from \( X \) to \( X_0 \) (see [Clemens]) and \( g_* \) is surjective from Mayer-Vietoris sequence. For \( \beta \in H_2(X) \), there exist \( \beta_1 \in H_2(X_1) \) and \( \beta_2 \in H_2(X_2) \) such that
\[ i_1^*(\beta) = i_0^*(j_1^*(\beta_1) + j_2^*(\beta_2)). \]

For simplicity, we write \( \beta = \beta_1 + \beta_2 \) instead.

**Lemma 2.2.** With the assumption as above, given \( \beta = \beta_1 + \beta_2 \). Let \( d = \int_\beta c_1(X) \) and \( d_i = \int_{\beta_i} c_1(X_i), i = 1, 2 \). Then
\[ d = d_1 + d_2 - 2 \int_{\beta_1} [S], \quad \int_{\beta_1} [S] = \int_{\beta_2} [S]. \]

**Proof.** The formulae \((2.2)\) come from the adjunction formulae \( K_{X_i} = K_X|_{X_i} \) and \( K_{X_i} = (K_X + X_i)|_{X_i} \) for \( i = 1, 2 \), and \( X_1 \cdot (X_1 + X_2) = X_1 \cdot X_0 = 0 \).

Similarly for cohomology, we have the maps
\[ H^k(X) \xrightarrow{i^*} H^k(X) \xrightarrow{i_0^*} H^k(X_0) \xrightarrow{g^*} H^k(X_1) \oplus H^k(X_2), \]
where \( i_0^* \) is an isomorphism. Take \( \alpha \in H^k(X) \) and let \( \alpha(t) = i_t^* \alpha \).

There is a degeneration formula which takes the form
\[ Z'_{DT}(X_0; q | \prod_{i=1}^r \bar{\tau}_0(\gamma_i(t)))_\beta \]
\[ = \sum Z'_{DT}(X_1/S; q | \prod_{i=1}^r \bar{\tau}_0(\gamma_i(0)))_{\beta_1, \eta} \frac{(-1)^{|\eta| - \ell(\eta)} \delta(\eta)}{q^{\eta}} \cdot Z'_{DT}(X_2/S; q | \prod_{i=1}^r \bar{\tau}_0(\gamma_i(0)))_{\beta_2, \eta'}, \]
where the sum is over the splittings \( \beta_1 + \beta_2 = \beta \), and cohomology weighted partitions \( \eta \). \( \gamma_i \)'s are cohomology classes on \( X \). There is a compatibility condition
\[ |\eta| = \beta_1 \cdot [S] = \beta_2 \cdot [S]. \]

For details, one can see [Li1, Li2, MNOP2].

### 3. Blowup at a point and a Blowup formula

In [MNOP1, MNOP2], the authors discovered a correspondence between Gromov-Witten theories and Donaldson-Thomas theories. In [Hu1, Hu2], the first author studied the change of Gromov-Witten invariants under the blowup operation. In this section, we will study the change of Donaldson-Thomas invariants under the blowup along a point.

The key idea is that the blowup can be obtained via a semistable degeneration as follows. Let \( X \) be a smooth projective 3-fold and \( \bar{X} \) be the blowup of \( X \) at a
general point $x$. Denote by $p : \tilde{X} \rightarrow X$ the natural projection of the blowup. Let $\mathcal{X}$ be the blow up of $X \times \mathbb{C}$ at the point $(x, 0)$ and let $\pi$ be the natural projection from $\mathcal{X}$ to $\mathbb{C}$. It is a semistable degeneration of $X$ with the central fiber $X_0$ being a union of $X_1 \cong \tilde{X}$ and $X_2 \cong \mathbb{P}^2$, which is the exceptional divisor in $\mathcal{X}$. $X_1$ and $X_2$ intersect transversely along $E \cong \mathbb{P}^2$, which is the exceptional divisor in $X_1 = \tilde{X}$. As a divisor in $X_2$, $E$ is a hyperplane. $c_1(X_2) = 4E$.

**Theorem 3.1.** Let $X$ be a smooth projective 3-fold. Suppose that $\beta \in H_2(X, \mathbb{Z})$ and $\gamma_i \in H^*(X, \mathbb{R})$, $i = 1, \cdots , r$. Then

$$Z'_{DT}(X; q | \prod_{i=1}^r \tilde{\tau}_0(\gamma_i))_{\beta} = Z'_{DT}(\tilde{X}; q | \prod_{i=1}^r \tilde{\tau}_0(p^*\gamma_i))_{p'(\beta)},$$

(3.1)

where $p'(\beta) = PDp^*PD^{-1}(\beta)$.

**Proof.** Choose the support of $\gamma_i$ outside of $x$. Then we have $\gamma_i \in H^*(X_1)$ and no $\gamma_i$'s in $H^*(X_2)$. In fact, let $p_1 : \mathcal{X} \rightarrow X$ be the composition of the blowing-down map $\mathcal{X} \rightarrow X \times \mathbb{C}$ with the projection $X \times \mathbb{C} \rightarrow X$. One can check that $i_1^*p_1^*\gamma_i = \gamma_i$ and $j_1^*i_0^*p_1^*\gamma_i = p^*\gamma_i$ and $j_2^*i_0^*p_1^*\gamma_i = 0$. We apply the degeneration formula (2.3) to the cohomology classes $p_1^*\gamma_i$ on $\mathcal{X}$.

By the degeneration formula (2.3), we may express the absolute Donaldson-Thomas invariants of $X$ in terms of the relative Donaldson-Thomas invariants of $(X_1, E)$ and $(X_2, E)$ as follows:

$$Z'_{DT}(X; q | \prod_{i=1}^r \tilde{\tau}_0(\gamma_i))_{\beta}$$

(3.2)

$$= \sum_{\eta, \beta_1 + \beta_2 = \beta} Z'_{DT}(X_1/E; q | \prod_{i=1}^r \tilde{\tau}_0(p^*\gamma_i))_{\beta_1, \eta} \frac{(-1)^{|\eta| - \ell(\eta)} \delta(\eta)}{|q|^{|\eta|}} Z'_{DT}(X_2/E; q)_{\beta_2, \eta^\vee}.$$

Now we need to compute the summands in the right hand side of the degeneration formula. For this we have the following claim:

**Claim:** There are only terms with $\beta_2 = 0$.

In fact, if $|\eta| \neq 0$, then $\beta_2 \neq 0$ because $\beta_2 \cdot E = |\eta|$. By Lemma 2.1, we have

$$c_1(X_1) \cdot \beta_1 = \text{vdim}I_n(X_1/E, \beta_1) = \sum_{i=1}^r \deg ch_2(\gamma_i) + \deg \epsilon_1(C_n),$$

where $\epsilon_1 : I_n(X_1/E, \beta_1) \rightarrow \text{Hilb}(E, |\eta|)$ is the canonical intersection map, and

$$c_1(X_2) \cdot \beta_2 = \text{vdim}I_n(X_2/E, \beta_1) = 4E \cdot \beta_2 = 4|\eta|,$$

$$c_1(X) \cdot \beta = \text{vdim}I_n(X, \beta) = \sum_{i=1}^r \deg ch_2(\gamma_i).$$

We have the last equality above because, otherwise, the involved Donaldson-Thomas invariants of $X$ and $\tilde{X}$ will vanish and the theorem holds.

By (2.3), we have

$$c_1(X) \cdot \beta = c_1(X_1) \cdot \beta_1 + c_1(X_2) \cdot \beta_2 - 2|\eta|.$$
Combining all the four equations above, we obtain

\[ 0 = \deg C_\eta + 2|\eta|. \]

This is a contradiction. Therefore \(|\eta| = 0\). So the claim is proved.

Thus \(\beta_2 \cdot E = 0\). Since \(E\) is the hyperplane in \(X_2 \cong \mathbb{P}^3\), we must have \(\beta_2 = 0\). Also we have \(\beta_1 = p^!(\beta)\).

By the degeneration formula, we have

\[
Z'_{DT}(X; q | \prod_{i=1}^r \bar{\tau}_0(\gamma_{l_i}))_{\beta} = Z'_{DT}(X_1/E; q | \prod_{i=1}^r \bar{\tau}_0(p^*(\gamma_{l_i}))_{p^!(\beta)}. \tag{3.3}
\]

Now we want to use the degeneration formula one more time to study the Donaldson-Thomas invariants of \(\tilde{X}\). We blow up \(\tilde{X} \times \mathbb{C}\) along the surface \(E \times 0\) to get a 4-fold \(\tilde{X}\). There is a projection \(\tilde{\pi} : \tilde{X} \to \mathbb{C}\). The central fiber is a union of \(\tilde{X}_1 = \tilde{X}\) and \(\tilde{X}_2 = \mathbb{P}(O_E(-1) \oplus O_E)\) intersecting transversely along a smooth surface \(Z\), which is the surface \(E\) in \(\tilde{X}_1\) and the infinite section \(D_\infty\) in the projective bundle \(\tilde{X}_2\). Note that \(\tilde{X}_2 - D_\infty\) is the line bundle \(O_E(-1) \cdot p^!(\beta) \cdot E = 0\), and \(PD(\gamma_{l_i}) \cap E = \emptyset\). Let \(\tilde{\gamma}_1\) be the composition of the map \(\tilde{X} \to \tilde{X} \times \mathbb{C}\) and the map \(\tilde{X} \times \mathbb{C} \to \tilde{X}\). Applying the degeneration formula (2.3) to the cohomology classes \(\tilde{\pi}^* (\gamma_{l_i})\), we have

\[
Z'_{DT}(\tilde{X}; q | \prod_{i=1}^r \bar{\tau}_0(\gamma_{l_i}))_{p^!(\beta)} = \sum_{\beta_1 + \beta_2 = p^!(\beta), \eta} Z'_{DT}(\tilde{X}_1/Z; q | \prod_{i=1}^r \bar{\tau}_0(p^*(\gamma_{l_i}))_{\beta_1, \eta} Q_{\beta_1, \eta}^{(-1)\eta r - \ell(\eta)} Z'_{DT}(\tilde{X}_2/Z; q | \beta_2, \eta),
\]

where \(\beta_1 \cdot Z = |\eta|\).

Here we have the following claim as in the first part of our proof:

**Claim:** There are only terms with \(\beta_2 = 0\) and no \(\eta\).

It is easy to see that \(\tilde{X}_2\) is the blowup \(\mathbb{P}^3\) of \(\mathbb{P}^3\) at a point \(p_0\). Denote by \(\rho : \tilde{\mathbb{P}^3} \to \mathbb{P}^3\) the projection of the blowup. Let \(\ell \subset \mathbb{P}^3\) be the strict transform of a line in \(\mathbb{P}^3\) passing through the blown-up point \(p_0\), and \(S\) be the exceptional surface of the blowup \(\rho\). Denote by \(e\) a line in \(S\) which is an extremal ray. Since \(\ell\) is a line in \(S\) which is an extremal ray, by Mori’s theory, we have \(\beta_2 = a\ell + be\), \(a \geq 0\), \(b \geq 0\). Let \(H\) be the hyperplane class in \(\mathbb{P}^3\). Since \(\rho^* H \sim D_\infty\), we have \(a = \rho^* H \cdot \beta_2 = |\eta|\). One can show that \(2D_\infty \cdot \beta_2 = p^!(\beta) \cdot E\). Since \(p^!(\beta) \cdot E = 0\), we have \(a = |\eta| = \beta_2 \cdot D_\infty = 0\) and \(\beta_1 \cdot E = \beta_2 \cdot D_\infty = 0\).

We have

\[
p^!(\beta) = \tilde{\pi}_1^* (p^!(\beta)) = \tilde{\pi}_1^* (\beta_1 + \beta_2) = \beta_1 + be,
\]

where we still use the same \(e\) to represent a line in \(E \subset \tilde{X}\).
Since $E \cdot p^!(\beta) = 0$ and $E \cdot \beta_1 = 0$, we have $-b = b \cdot E = 0$. Thus $b = 0$ and hence $\beta_2 = 0$. The claim is proved.

We also see that $\beta_1 = p^!(\beta)$.

By the degeneration formula, we have

$$Z'_{DT}(\tilde{X}; q | \prod_{i=1}^{r} \tau_0(p^*\gamma_i))_{p^!(\beta)} = Z'_{DT}(\tilde{X}_1/Z; q | \prod_{i=1}^{r} \tau_0(p^*\gamma_i))_{p^!(\beta)} \cdot Z'_{DT}(\tilde{X}_2/Z; q)_{0}$$

$$= Z'_{DT}(\tilde{X}/E; q | \prod_{i=1}^{r} \tau_0(p^*\gamma_i))_{p^!(\beta)}. \quad (3.4)$$

Note that $\tilde{X}_1 \cong \tilde{X}$. Comparing (3.3) with (3.4), we proved the Theorem. \(\square\)

4. Blowup of $(-1,-1)$-curves and a flop formula

In this section, we will study how Donaldson-Thomas invariants change under some flops. The materials related to the birational geometry of 3-folds can be found in [Kollar], [Kawamata], [KMM], [K-M], [Matsuki].

Let $X$ be a smooth projective 3-fold, $D$ be an effective divisor on $X$. Suppose that $X$ admits a contraction of an extremal ray with respect to $K_X + \epsilon D$, where $0 < \epsilon \ll 1$,

$$\varphi: X \longrightarrow Y.$$

Assume furthermore that the exceptional locus $Exc(\varphi)$ of $\varphi$ consists of finitely many disjoint smooth rational $(-1,-1)$-curves $\Gamma_2, \ldots, \Gamma_{\ell}$. $Y$ is a normal projective variety, $-D$ is $\varphi$-ample, and all curves $\Gamma_i$ are numerically equivalent. Let’s use $[\gamma]$ to denote the numerically equivalent classes $\Gamma_i$, $i = 2, \ldots, \ell$. There exists a smooth projective 3-fold $X^f$ and a morphism

$$\varphi^f: X^f \longrightarrow Y,$$

which is the flop of $\varphi$. $X^f$ can be obtained as follows in our situation. We blow up $X$ along all the curves $\Gamma_i$, $i = 2, \ldots, \ell$ to get a smooth projective 3-fold $\tilde{X}$ with the exceptional divisors $E_i \cong \Gamma_i \times \mathbb{P}^1$, $i = 2, \ldots, \ell$. Let $\mu: \tilde{X} \rightarrow X$ be the blowup map. We can blow down $\tilde{X}$ along all the $\Gamma_i$-direction. The new 3-fold $X^f$ is smooth, projective and containing $(-1,-1)$-curves $\Gamma_i^f$ for $i = 2, \ldots, \ell$. $\Gamma_i^f$ is the image of $E_i$ under the blow down. $X$ and $X^f$ are birational and isomorphic in codimension one.

For any divisor $B$ on $X$, let $B^f$ be the strict transform of $B$ in $X^f$. We have an isomorphism $N^1(X) \cong N^1(X^f)$ and

$$N^1(X) \cong \varphi^*N^1(Y) \oplus \mathbb{R}[D], \quad N^1(X^f) \cong (\varphi^f)^*N^1(Y) \oplus \mathbb{R}[D^f].$$

Similarly we get an isomorphism $H_2(X) \rightarrow H_2(X^f)$, denoted by $\phi_*$, such that $\phi_*([\Gamma_i]) = -[\Gamma_i^f]$ (see [L-R]). The map $\phi_*$ induces isomorphisms $\phi^*: H^{2i}(X^f) \rightarrow H^{2i}(X)$.

The map $\phi_*$ can also be seen as follows (see [L-R]). There is an injection $\iota$ from $H_2(X)$ to $H_2(\tilde{X})$ such that the image of $\iota$ is the set $\{\beta \in H_2(\tilde{X}) | \beta \cdot E = 0\}$ where
Lemma 4.1. The power series $\sum_{d>0} d^k x^d$ has an analytic continuation $f_k(x)$ in the domain $\mathbb{C} - \{1\}$ such that

$$f_k(x^{-1}) = (-1)^{k+1} f_k(x).$$

Proof. From the geometric series formula $1 + x + \ldots + x^d + \ldots = (1 - x)^{-1}$, we get

$$x + 2x^2 + \ldots + dx^d + \ldots = x \cdot (1 + x + \ldots + x^d + \ldots)' = \frac{x}{(1 - x)^2}.$$

Let $f_1(x) = \frac{x}{(1 - x)^2}$. One can check that $f_1(x^{-1}) = f_1(x)$.

Assume that the statement in the Lemma holds for $k$. Then

$$x + 2^{k+1}x^2 + \ldots + d^{k+1}x^d + \ldots = x \cdot (x + \ldots + d^k x^d + \ldots)'$$

has an analytic continuation $f_{k+1}(x) = f_k'(x) \cdot x$. From the chain rule, one has

$$f_{k+1}(x^{-1}) = (-1)^{k+1} f_k'(x).$$

Therefore

$$f_{k+1}(x^{-1}) = x^{-1} f_k'(x^{-1}) = (-1)^{k+2} x f_k'(x) = (-1)^{k+2} f_{k+1}(x).$$

By the mathematical induction, we proved the Lemma. \qed

Theorem 4.2. Suppose cohomology classes $\gamma_i \in H^{2k}(X^f)$, $i = 1, \ldots, r$ and $k = 1, 2, 3$, have supports away from all the exceptional curve $\Gamma_i$.

(i) If $\beta = m[\gamma]$, we have

$$Z'_{DT}(X; q)_{\beta} = Z'_{DT}(X^f; q)_{-\phi_*(\beta)}.$$

(ii) There exist power series

$$\Phi_X(q, v|\{\phi^*\gamma_i\}) = \sum_{\beta \in \langle H_2(X) \rangle} \Phi_X(q|\{\phi^*\gamma_i\})_{\beta} \cdot v^\beta,$$

$$\Phi_{X^f}(q, v|\{\gamma_i\}) = \sum_{\beta \in \langle H_2(X^f) \rangle} \Phi_{X^f}(q|\{\gamma_i\})_{\beta} \cdot v^\beta,$$
and $G(q, v, \Gamma)$ such that
\[ \Phi_X(q, v|\{\phi^{*}\gamma_\ell\}) = \Phi_X(q, v|\{\gamma_\ell\}), \]
$G(q, v, \Gamma_i)/g(q, v, \Gamma_i)$ and $G(q, v^{-1}, \Gamma_i^\ell)/g(q, v^{-1}, \Gamma_i^\ell)$ are equivalent under analytic continuation, and
\[
Z'_{DT}(X; q, v | \prod_{i=1}^{r} \tau_0(\phi^{*}\gamma_{\ell_i})) = \Phi_X(q, v|\{\phi^{*}\gamma_\ell\}) \cdot \prod_{i=1}^{r} G(q, v, \Gamma_i), \tag{4.2}
\]
\[
Z'_{DT}(X^f; q, v | \prod_{i=1}^{r} \tau_0(\gamma_{\ell_i})) = \Phi_{X^f}(q, v|\{\gamma_\ell\}) \cdot \prod_{i=1}^{\ell} G(q, v, \Gamma_i^\ell). \tag{4.3}
\]

Proof. There is a degeneration formula similar to (2.3) (see [4]) for the degeneration $X$ described above. For simplicity, we shall prove the case when there is only one $\Gamma_i$, denoted by $\Gamma$. The proof for the general case is similar.

By the degeneration formula (2.3), we have
\[
Z'_{DT}(X; q | \prod_{i=1}^{r} \tau_0(\phi^{*}\gamma_{\ell_i}))_{\beta} = \sum_{\eta, \beta_1 + \beta_2 = \beta} Z'_{DT}(X_1/E; q | \prod_{i=1}^{r} \tau_0(\mu^{*}\phi^{*}\gamma_{\ell_i}))_{\beta_1, \eta} \frac{(-1)^{|\eta| - l(\eta)}}{q^{|\eta|}} Z'_{DT}(X_2/E; q)_{\beta_2, \eta^v},
\]
where $E$ is the intersection of $X_1$ with $X_2$, which is also the exceptional divisor in $X_1$.

Similar to the proof of Theorem 3.1, we need to study the summands in RHS. Therefore, we also need to compute the virtual dimensions of involved moduli spaces. About the contributions of each term in RHS, we have the following claim:

Claim: There are only terms without $\eta$.

In fact, suppose that $|\eta| \neq 0$. First of all, we want to compute the first Chern class of $X_2$.

Let $V = \mathcal{O}_V(-1) \oplus \mathcal{O}_V(-1) \oplus \mathcal{O}_V$ and $p : \mathbb{P}(V) \to \Gamma$ be the projection. $X_2 = \mathbb{P}(V)$. For this projective bundle, we have the Euler exact sequence
\[
0 \to \mathcal{O}_{\mathbb{P}(V)} \to p^*V \to \mathcal{O}_{\mathbb{P}(V)(1)} \to T_{\mathbb{P}(V)/\Gamma} \to 0.
\]
We also have
\[
0 \to p^*\Omega^1_V \to \Omega^1_{\mathbb{P}(V)} \to \Omega^1_{\mathbb{P}(V)/\Gamma} \to 0.
\]
Therefore, we have
\[
c_1(\Omega^1_{\mathbb{P}(V)}) = p^*c_1(\Omega^1_V) + c_1(\Omega^1_{\mathbb{P}(V)/\Gamma}) = p^*c_1(\Omega^1_V) - c_1(p^*V \otimes \mathcal{O}_{\mathbb{P}(V)}(1)) = p^*c_1(K_V) - p^*c_1(V) - 3c_1(\mathcal{O}_{\mathbb{P}(V)}(1)) = -3c_1(\mathcal{O}_{\mathbb{P}(V)}(1)),
\]
where \( c_1(\mathcal{O}_{\mathbb{P}(V)}(1)) = [E] \) is the hyperplane at the infinity in \( \mathbb{P}(V) \) due to the inclusion \( \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \to \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma \). Therefore we have

\[
c_1(X_2) \cdot \beta_2 = 3|\eta|.
\]

By the definition of absolute Donaldson-Thomas invariants, we may assume that

\[
c_1(X) \cdot \beta = \text{vdim}I_n(X, \beta) = \sum_{i=1}^r \deg c_2(\gamma_i).
\]

Otherwise, the involved Donaldson-Thomas invariants of \( X \) and \( \tilde{X} \) will vanish and the theorem holds.

We also have

\[
c_1(X_1) \cdot \beta_1 = \text{vdim}I_n(X_1/E, \beta) = \sum_{i=1}^r \deg c_2(\gamma_{\ell_i}) + \deg \epsilon_1^*\eta.
\]

By Lemma 2.2 we have

\[
c_1(X) \cdot \beta = c_1(X_1) \cdot \beta_1 + c_1(X_2) \cdot \beta_2 - 2|\eta|.
\]

Combining all the four equalities above, we have

\[
0 = \deg \epsilon_1^*C_n + |\eta|
\]

Hence \( |\eta| = 0 \).

(i) Suppose that \( \beta = m[\Gamma] \). Notice that the virtual dimension of the moduli space will be zero since \( c_1(X) \cdot \beta = 0 \). Let \( \Gamma_\infty \) be the curve coming from the inclusion \( \mathcal{O}_\Gamma \to V, F \cong \mathbb{P}^2 \) be a fiber of \( \rho, f \) be a line in \( F \). Then one can compute easily that

\[
E \cdot \Gamma_\infty = 0, \quad F \cdot \Gamma_\infty = 1, \quad f \cdot F = 0, \quad f \cdot E = 1.
\]

Therefore we can write \( \beta_2 = af + m[\Gamma_\infty] \). Since \( E \cdot \beta_2 = 0 \), we have \( a = 0 \). Therefore \( \beta_2 = m[\Gamma_\infty] \) for some \( m \geq 0 \). Under the morphism \( \sigma \)

\[
\sigma: X \to X \times C \to X,
\]

we have \( \beta = \sigma(\beta_1) + m[\Gamma] \) in \( NE(X) \). \( \beta_1 \) can only be a union of curves \( C_i \)'s not lying on \( E \) and curves \( D_j \)'s on \( E \). Since \( \mathbb{R}[\Gamma] \) is a ray, we must have \( C_i = 0 \). For effective curves \( D_j \) on \( E \), \( D_j \cdot E \neq 0 \). However since \( \beta_1 \cdot E = 0 \), we must have \( D_j = 0 \). Thus \( \beta_1 = 0 \). Therefore, by the degeneration formula, we have

\[
Z'_{DT}(X; q)_{m[\Gamma]} = Z'_{DT}(\tilde{X}/E; q)_{m[\Gamma]} \cdot Z'_{DT}(X_2/E; q)_{m[\Gamma_\infty]} = Z'_{DT}(X_2/E; q)_{m[\Gamma_\infty]} (4.4)
\]

\[
Z'_{DT}(X^f; q)_{m[\Gamma]} = Z'_{DT}(\tilde{X}^f/E; q)_{m[\Gamma]} \cdot Z'_{DT}(X_2^f/E; q)_{m[\Gamma_\infty]} = Z'_{DT}(X_2^f/E; q)_{m[\Gamma_\infty]}.
\]

Observe that \( (\tilde{X}_2, E) \) and \( (\tilde{X}_2^f, E) \) are isomorphic. Therefore, we have

\[
Z'_{DT}(X; q)_{m[\Gamma]} = Z'_{DT}(X^f; q)_{m[\Gamma]}.
\]

To write in another way for \( \beta = m[\Gamma] \), we have

\[
Z'_{DT}(X; q)_\beta = Z'_{DT}(X^f; q)_{\phi_*(\beta)}.
\]
To prove (ii), by the similar argument as in (i), we have \( \beta = \beta_1 + m[\Gamma_\infty] \) with \( m \geq 0 \) and \( \beta_1 \cdot E = 0 \).

Furthermore, by the degeneration formula, we have

\[
Z'_{DT}(X; q | \prod_{i=1}^{r} \tau_0(\phi^* \gamma_{\ell_i})) = \sum_{\beta=\beta_1 + m[\Gamma_\infty], \beta_1 \in (H_2(X))} Z'_{DT}(\tilde{X}/E; q | \prod_{i=1}^{r} \tau_0(\mu^* \phi^* \gamma_{\ell_i}))_{\beta_1} \cdot Z'_{DT}(X_2/E; q)_{m[\Gamma_\infty]}.
\]  

(4.5)

Consider the map \( c_* : H_2(X) = H_2(\mathcal{X}) \xrightarrow{i_*} H_2(\mathcal{X}) \xrightarrow{i_0^*} H_2(\mathcal{X}_0) \). From Lemma 2.11 in [L-R], \( c_* \) is injective. Therefore we have

\[
Z'_{DT}(X; q, v | \prod_{i=1}^{r} \tau_0(\phi^* \gamma_{\ell_i})) = \sum_{\beta \in H_2(X)} Z'_{DT}(X; q | \prod_{i=1}^{r} \tau_0(\phi^* \gamma_{\ell_i}))_{\beta} v^\beta.
\]

Define a function \( \Phi_X(q, v|\{\phi^* \gamma_{\ell_i}\}) \) as follows

\[
\Phi_X(q, v|\{\phi^* \gamma_{\ell_i}\}) = \sum_{\beta_1 \in (H_2(X))} Z'_{DT}(\tilde{X}/E; q | \prod_{i=1}^{r} \tau_0(\mu^* \phi^* \gamma_{\ell_i}))_{\beta_1} v^{\beta_1}.
\]

Apply the formula (4.5) to \( X = X_2 \), we get \( Z'_{DT}(X_2/E; q)_{m[\Gamma_\infty]} = Z'_{DT}(X_2; q)_{m[\Gamma_\infty]} \).

We define a function \( G(q, v, \Gamma_\infty) \) as follows:

\[
G(q, v, \Gamma_\infty) = \sum_{m \geq 0} Z'_{GW}(\mathcal{O}_{\Gamma_\infty}(-1) \oplus \mathcal{O}_{\Gamma_\infty}(-1); u, v).
\]

The last equality is the Theorem 3 in [MNOP2] for local Calabi-Yau \( \mathcal{O}_{\Gamma_\infty}(-1) \oplus \mathcal{O}_{\Gamma_\infty}(-1) \).

From [MNOP2], we have

\[
Z'_{GW}(\mathcal{O}_{\Gamma_\infty}(-1) \oplus \mathcal{O}_{\Gamma_\infty}(-1); u, v) = \exp\{F'_{GW}(\mathcal{O}_{\Gamma_\infty}(-1) \oplus \mathcal{O}_{\Gamma_\infty}(-1); u, v)\},
\]
\[ F'_{GW} = \sum_{d > 0} \sum_{g \geq 0} N_{g,d} u^{2g-2} v^d, \]

where \( N_{g,d} \) is computed in [F-P]:

\[ N_{0,d} = \frac{1}{d^3}, \quad N_{1,d} = \frac{1}{12d}, \quad N_{g,d} = \frac{|B_{2g}|}{2g \cdot (2g - 2)!} \]

for \( g \geq 2 \).

Therefore, we have

\[ F'_{GW} = u^{-2} \sum_{d > 0} \frac{1}{d^3} (v^{[\Gamma_\infty]})^d + \sum_{d > 0} \frac{1}{12d} (v^{[\Gamma_\infty]})^d + \sum_{g \geq 2} \frac{|B_{2g}|}{2g \cdot (2g - 2)!} u^{2g-2} \sum_{d > 0} d^{2g-3} (v^{[\Gamma_\infty]})^d. \]

Now \( G(q,v,\Gamma_\infty)/g(u,v,\Gamma_\infty) \) has the analytic continuation

\[ \exp \left\{ \sum_{g \geq 2} \frac{|B_{2g}|}{2g \cdot (2g - 2)!} u^{2g-2} f_{2g-3}(v^{[\Gamma_\infty]}) \right\} \]

where \( f_{2g-3}(x) \) is defined in the Lemma 4.1.

Applying the same argument above for \( X_f \), we also have

\[ Z'_{DT}(X_f; q \mid \prod_{i=1}^{r} \tilde{\tau}_0(\gamma_{i})), \]

where \( \nu: \tilde{X} \to X_f \) is the blowup map, \( \tilde{X} \cong \tilde{X}_f \).

Applying the same argument above for \( X \) to \( X_f \), define a function \( \Phi_X(q,v \mid \{\gamma_i\}) \) as follows

\[ \Phi_X(q,v \mid \{\gamma_i\}) = \sum_{\beta_1 \in (H_2(X))} Z'_{DT}(\tilde{X}/E; q \mid \prod_{i=1}^{r} \tilde{\tau}_0(\nu^* \gamma_i))_{\beta_1} v^{\beta_1}. \]

We have (4.3).

The function \( G(q,v,\Gamma_f^\infty)/g(q,v,\Gamma_f^\infty) \) has the analytic continuation

\[ \exp \left\{ \sum_{g \geq 2} \frac{|B_{2g}|}{2g \cdot (2g - 2)!} u^{2g-2} f_{2g-3}(v^{[\Gamma_\infty]}) \right\} \]

From the Lemma 4.1 and the fact that \( \mu^* \phi^* = \nu^* \), we proved (ii). \qed

One should compare the Theorem 4.2 with Definition 1.1, Theorem A and Corollary A.2 in [L-R]. There, Li and Ruan studied the question of naturality of quantum cohomology under birational operations such as flops. They observed that one must use analytic continuation to compare the quantum cohomology of two Calabi-Yau 3-folds which are flop equivalent. The similar phenomenon occurs for Donaldson-Thomas invariants. However, there is a slight complexity due to the
function $g(q,v,\Gamma)$ coming from genus zero and genus one contributions. It is possible that genus zero and genus one create an anomaly.

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