Spherical designs via Brouwer fixed point theorem

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Abstract

For each $N \geq c_d n^{2d(d+1)/d+2}$ we prove the existence of a spherical $n$-design on $S^d$ consisting of $N$ points, where $c_d$ is a constant depending only on $d$.

Keywords: Spherical designs, Brouwer fixed point theorem, Marcinkiewich-Zygmund inequality, area-regular partitions.
1 Introduction

Let $S^d$ be the unit sphere in $\mathbb{R}^{d+1}$ with normalized Lebesgue measure $d\mu_d \left( \int_{S^d} d\mu_d(x) = 1 \right)$. The following concept of a spherical design was introduced by Delsarte, Goethals and Seidel [5]:

A set of points $x_1, \ldots, x_N \in S^d$ is called a spherical $n$-design if

$$\int_{S^d} P(x) d\mu_d(x) = \frac{1}{N} \sum_{i=1}^{N} P(x_i)$$

for all algebraic polynomials in $d + 1$ variables and of total degree at most $n$. For each $n \in \mathbb{N}$ denote by $N(d, n)$ the minimal number of points in a spherical $n$-design. The following lower bounds

$$N(d, n) \geq \binom{d + k}{d} + \binom{d + k - 1}{d}, \quad n = 2k,$$

$$N(d, n) \geq 2 \binom{d + k}{d}, \quad n = 2k + 1,$$

are also proved in [5].

Spherical $n$-designs attaining these bounds are called tight. Exactly eight tight spherical designs are known for $d \geq 2$ and $n \geq 4$. All such configurations of points are highly symmetrical and possess other extreme properties. For example, the shortest vectors in the $E_8$ lattice form a tight 7-design in $S^7$, and a tight 11-design in $S^{23}$ is obtained from the Leech lattice in the same way [4]. In general, lattices are a good source for spherical designs with small $(d, n)$ [7].

On the other hand construction of spherical $n$-design with minimal cardinality for fixed $d$ and $n \to \infty$ becomes a difficult analytic problem even for $d = 2$. There is a strong relation between this problem and the problem of finding $N$ points on a sphere $S^2$ that minimize the energy functional

$$E(\vec{x}_1, \ldots, \vec{x}_N) = \sum_{1 \leq i < j \leq N} \frac{1}{||\vec{x}_i - \vec{x}_j||},$$

see Saff, Kuijlaars [12].
Let us begin by giving a short history of asymptotic upper bounds on $N(d, n)$ for fixed $d$ and $n \to \infty$. First, Seymour and Zaslavsky [13] have proved that spherical design exists for all $d$, $n \in \mathbb{N}$. Then, Wagner [14] and Bajnok [2] independently proved that $N(d, n) \leq c_d n^{Cd}$ and $N(d, n) \leq c_d n^{Cd^2}$ respectively. Korevaar and Meyers have [8] improved this inequalities by showing that $N(d, n) \leq c_d n^{Cd^2/2}$. They have also conjectured that $N(d, n) \leq c_d n^d$. Note that (1) implies $N(d, n) \geq C_d n^d$. In what follows we denote by $b_d$, $c_d$, $c_{1d}$, etc., sufficiently large constants depending only on $d$. In [3] we proved the following

**Theorem BV.** Let $a_d$ be the sequence defined by

$$a_1 = 1, \quad a_2 = 3, \quad a_{2d-1} = 2a_{d-1} + d, \quad a_{2d} = a_{d-1} + a_d + d + 1, \quad d \geq 2.$$ 

Then for all $d$, $n \in \mathbb{N}$,

$$N(d, n) \leq c_d n^{a_d}.$$ 

**Corollary BV.** For each $d \geq 3$ and $n \in \mathbb{N}$ we have

$$N(d, n) \leq c_d n^{a_d}.$$ 

$a_3 \leq 4, \quad a_4 \leq 7, \quad a_5 \leq 9, \quad a_6 \leq 11, \quad a_7 \leq 12, \quad a_8 \leq 16, \quad a_9 \leq 19, \quad a_{10} \leq 22,$

and

$$a_d < \frac{d}{2} \log_2 2d, \quad d > 10.$$ 

In this paper we suggest a new nonconstructive approach for obtaining new upper bounds for $N(d, n)$. We will make extensive use of the Brouwer fixed point theorem (the source of nonconstructive nature of our method), the Marcinkiewich-Zygmund inequality on the sphere [10] and the notion of area-regular partitions [9]. The main result of this paper is

**Theorem 1.** For each $N \geq c_d n^{\frac{2d(d+1)}{d^2+1}}$ there exists a spherical $n$-design on $S^d$ consisting of $N$ points.
This result improves our previous estimate on $N(d, n)$ for all $d > 3$, $d \neq 7$, and in particular allows us to remove the "nasty" logarithm in the power in Corollary BV, so that the function in the power has a linear behavior, which confirms the conjecture of Korevaar and Meyers. Finally, Theorem 1 guaranties the existence of spherical $n$-design for each $N$ greater then our new existence bound.

2 Preliminaries

Let $\Delta$ be the Laplace operator in $\mathbb{R}^{d+1}$

$$\Delta = \sum_{j=1}^{d+1} \frac{\partial^2}{\partial x_j^2}.$$

We say that a polynomial $P$ in $\mathbb{R}^{d+1}$ is harmonic if $\Delta P = 0$. For integer $k \geq 1$, the restriction to $S^d$ of a homogeneous harmonic polynomial of degree $k$ is called a spherical harmonic of degree $k$. The vector space of all spherical harmonics of degree $k$ will be denoted by $\mathcal{H}_k$ (see [10] for details). The dimension of $\mathcal{H}_k$ is given by

$$\dim \mathcal{H}_k = \frac{2k + d - 1}{k + d - 1} \binom{d + k - 1}{k}.$$

The vector spaces $\mathcal{H}_k$ are invariant under the action of the orthogonal group $O(d + 1)$ on $S^d$ and are orthogonal to each other with respect to the scalar product

$$\langle P, Q \rangle := \int_{S^d} P(x)Q(x) d\mu_d(x).$$

Another remarkable property of harmonic polynomials is that the spaces $\mathcal{H}_k$ are eigenspaces of the spherical Laplacian (Laplace-Beltrami operator [6])

$$\Delta f(x) := \Delta f(\frac{x}{\|x\|}).$$

Thus, for a polynomial $P \in \mathcal{H}_k$ we have

$$\Delta P = -k(k + d - 1)P.$$
Here and below we use the notations $\|\cdot\|$ and $(\cdot, \cdot)$ for the Euclidean norm and usual scalar product in $\mathbb{R}^{d+1}$, respectively. For a twice differentiable function $f : \mathbb{R}^{d+1} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}^{d+1}$ denote by

$$\frac{\partial f}{\partial x}(x_0) := \left( \frac{\partial f}{\partial x_1}(x_0), \ldots, \frac{\partial f}{\partial x_{d+1}}(x_0) \right)$$

and

$$\frac{\partial^2 f}{\partial x^2}(x_0) := \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \right)_{i,j=1}^{d+1}$$

the gradient and the matrix of second derivatives of $f$ (Hessian matrix) at the point $x_0$ respectively. Analogously to (2) we will also define for a polynomial $Q \in \mathcal{P}_n$ the spherical gradient

$$\nabla Q(x) := \frac{\partial}{\partial x} Q\left( \frac{x}{\|x\|} \right)$$

and the Hessian matrix on the sphere

$$\nabla^2 Q(x) := \frac{\partial^2}{\partial x^2} Q\left( \frac{x}{\|x\|} \right).$$

We will also write

$$\nabla^2 Q \cdot x \cdot y := (\nabla^2 Q \cdot x, y) \quad \text{for } x, y \in \mathbb{R}^{d+1}.$$ 

One consequence of Stokes’s theorem is the first Green’s identity [15]

$$\int_{S^d} P(x) \nabla Q(x) \, d\mu_d(x) = - \int_{S^d} (\nabla P(x), \nabla Q(x)) \, d\mu_d(x).$$

Let $\mathcal{P}_n$ be the vector space of polynomials $P$ of degree $\leq n$ on $S^d$ such that

$$\int_{S^d} P(x) \, d\mu_d(x) = 0.$$ 

Each polynomial in $\mathbb{R}^{d+1}$ can be written as a finite sum of terms, each of which is a product of a harmonic and a radial polynomial (i.e. a polynomial
which depends only on \(\|x\|\)). Therefore the vector space \(\mathcal{P}_n\) decomposes into the direct sum \(\mathcal{H}_k\)
\[
\mathcal{P}_n = \bigoplus_{k=1}^n \mathcal{H}_k.
\]
For each vector of positive weights \(w = (w_1, \ldots, w_n)\) we can define a scalar product \(\langle \cdot, \cdot \rangle_w\) on \(\mathcal{P}_n\) invariant with respect to the action of \(O(d+1)\) on \(S^d\) by
\[
\langle P, Q \rangle_w := \sum_{k=1}^n w_k \langle P_k, Q_k \rangle,
\]
where \(P_k, Q_k \in \mathcal{H}_k\), \(P = P_1 + \ldots + P_n\) and \(Q = Q_1 + \ldots + Q_n\). For each \(Q \in \mathcal{P}_n\) denote by
\[
\|Q\|_w = \sqrt{\langle Q, Q \rangle_w}
\]
the norm corresponding to this scalar product. We will also define the operator
\[
\Delta_w P := \sum_{k=1}^n \frac{k(k+d-1)}{w_k} P_k, \quad P \in \mathcal{P}_n.
\]
Then from (3) and (5) we get
\[
(6) \quad \langle \Delta_w P, Q \rangle_w = \int_{S^d} \langle \nabla P(x), \nabla Q(x) \rangle d\mu_d(x).
\]
Now, for each point \(x \in S^d\) there exists a unique polynomial \(G_x \in \mathcal{P}_n\) (depending on \(w\)) such that
\[
\langle G_x, Q \rangle_w = Q(x) \quad \text{for all } Q \in \mathcal{P}_n.
\]
Then, the set of points \(x_1, \ldots, x_N \in S^d\) form a spherical design if and only if
\[
G_{x_1} + \ldots + G_{x_N} = 0.
\]
To construct the polynomials \(G_x\) explicitly we will use the Gegenbauer polynomials \(G^\alpha_k\) [1]. For a fixed \(\alpha\), the \(G^\alpha_k\) are orthogonal on \([-1, 1]\) with respect to the weight function \(\omega(t) = (1 - t^2)^{\alpha - \frac{1}{2}}\), that is
\[
\int_{-1}^1 G_m^\alpha(t)G_n^\alpha(t)(1 - t^2)^{\alpha - \frac{1}{2}} dt = \delta_{mn} \frac{\pi 2^{1-2\alpha} \Gamma(n + 2\alpha)}{n!(\alpha + n)\Gamma(2\alpha)}.
\]
Set $\alpha := \frac{d-1}{2}$, and let

$$G_x(y) := g_w((x, y)),$$

where

$$g_w(t) := \sum_{k=1}^{n} \frac{\dim \mathcal{H}_k}{w_k G_k^\alpha(1)} G_k^\alpha(t).$$

In order to show that $\langle P_x, Q \rangle_w = G_x(Q) = Q(x)$ for each $Q \in \mathcal{P}_n$ we will use the following identity for Gegenbauer polynomials [10]

$$G_k^\alpha((x, y)) = \frac{G_k^\alpha(1)}{\dim \mathcal{H}_k} \sum_{j=1}^{\dim \mathcal{H}_k} Y_{jk}(x)Y_{jk}(y),$$

where $x, y \in S^d$ and $Y_{jk}$ are some orthonormal basis in the space $(\mathcal{H}_k, \mu_d)$. In particular, for a fixed $x \in S^d$, $G_k^\alpha((x, y)) \in \mathcal{H}_k$. Therefore, for a polynomial $Q \in \mathcal{P}_n$ we have

$$\langle G_x, Q \rangle_w = \sum_{k=1}^{n} w_k \langle G_k, Q_k \rangle = \sum_{k=1}^{n} \int_{S^d} G_k^\alpha((x, y))Q_k(y)d\mu_d(y) =$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{\dim \mathcal{H}_k} Y_{jk}(x) \int_{S^d} Q_k(y)Y_{jk}(y)d\mu_d(y) = \sum_{k=1}^{n} Q_k(x) = Q(x).$$

Fix the weight vector $w = (w_1, \ldots, w_n)$ such that $w_k = k(k + d - 1)$. Further we will use the following additional equalities for Gegenbauer polynomials [1]:

$$G_n^\alpha(1) = \binom{2\alpha + n - 1}{n},$$

and

$$\frac{d}{dt} G_n^\alpha(t) = 2\alpha G_{n-1}^{\alpha+1}(t), \quad \frac{d^2}{dt^2} G_n^\alpha(t) = 4\alpha(\alpha + 1)G_{n-2}^{\alpha+2}(t).$$

Applying Cauchy’s inequality to (7) we get, for all $k \in \mathbb{N}$ and $x, y \in S^d$,

$$|G_k^\alpha((x, y))|^2 \leq G_k^\alpha((x, x))G_k^\alpha((y, y)),$$

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and hence
\[
\max_{x \in [-1,1]} |g_w(x)| = g_w(1).
\]
Similarly, by (8) we obtain
\[
(9) \quad \max_{x \in [-1,1]} |g'_w(x)| = g'_w(1).
\]
Finally, let us estimate \(g'_w(1)\) and \(g''_w(1)\). We have
\[
(10) \quad g'_w(1) = \sum_{k=1}^{n} \frac{\dim \mathcal{H}^{\alpha}_k}{w_k G^\alpha_k(1)} G^\alpha_k(1) = \sum_{k=1}^{n} \frac{(2k + d - 1)(k + d - 2)!}{k!d!} \leq c_1 d^n.
\]
Hence, by (9) and Markov inequality we get
\[
(11) \quad g''_w(1) \leq n^2 \max_{x \in [-1,1]} |g'_w(x)| = n^2 g'_w(1) \leq c_1 d^n + 2.
\]

3 Proof of Theorem 1

Fix \(n \in \mathbb{N}\). As mentioned in section 2, points \(x_1, \ldots, x_N\) form a spherical \(n\)-design if and only if \(G_{x_1} + \ldots + G_{x_N} = 0\). First we will construct a set of points such that the norm \(||G_{x_1} + \ldots + G_{x_N}||_w\) is small, and then we will use the Brouwer fixed point theorem to show that there exists a collection of points \(\{y_1, \ldots, y_N\}\) “close” to \(\{x_1, \ldots, x_N\}\) with \(||G_{y_1} + \ldots + G_{y_N}||_w = 0\).

Let \(\mathcal{R} = \{R_1, \ldots, R_N\}\) be a finite collection of closed, non-overlapping (i.e., having no common interior points) regions \(R_i \subset S^d\) such that \(\bigcup_{i=1}^{N} R_i = S^d\). The partition \(\mathcal{R}\) is called area-regular if \(\text{vol}R_i := \int_{R_i} d\mu_d(x) = 1/N\), for all \(i = 1, \ldots, N\). The partition norm for \(\mathcal{R}\) is defined by
\[
\|\mathcal{R}\| := \max_{R \in \mathcal{R}} \text{diam } R.
\]
Now we will prove

**Lemma 1.** For each \(N \in \mathbb{N}\) there exists an area-regular partition \(\mathcal{R} = \{R_1, \ldots, R_N\}\) of \(S^d\) and a collection of points \(x_i \in R_i, i = 1, \ldots, N\) such that
\[
\left\| \frac{G_{x_1} + \ldots + G_{x_N}}{N} \right\|_w \leq \frac{b_d n^{d/2}}{N^{1/2+1/d}}.
\]
Proof. As shown in [9], for each \( N \in \mathbb{N} \) there exists an area-regular partition \( R = \{ R_1, \ldots, R_N \} \) such that \( \| R \| \leq c_{2d}N^{1/d} \) for some constant \( c_{2d} \). For this partition \( R \) we will estimate the average value of \( \left\| \frac{G_{x_1} + \ldots + G_{x_N}}{N} \right\|_w^2 \), when the points \( x_i \) are uniformly distributed over \( R_i \). We have

\[
\frac{1}{\text{vol} R_1 \cdots \text{vol} R_N} \int_{R_1 \times \cdots \times R_N} \left\| \frac{G_{x_1} + \ldots + G_{x_N}}{N} \right\|_w^2 d\mu_d(x_1) \cdots d\mu_d(x_N) =
\]

\[
= \frac{1}{\text{vol} R_1 \cdots \text{vol} R_N} \int_{R_1 \times \cdots \times R_N} \frac{1}{N^2} \sum_{i,j=1}^{N} \langle G_{x_i}, G_{x_j} \rangle_w d\mu_d(x_i) \cdots d\mu_d(x_N)
\]

\[
= \sum_{i \neq j} \int_{R_i \times R_j} \langle G_{x_i}, G_{x_j} \rangle_w d\mu_d(x_i) d\mu_d(x_j) + \sum_{i=1}^{N} \frac{1}{N} \int_{R_i} \langle G_{x_i}, G_{x_i} \rangle_w d\mu_d(x_i)
\]

\[
= \int_{S^d \times S^d} \langle G_{x_i}, G_{y} \rangle_w d\mu_d(x) d\mu_d(y) + \sum_{i=1}^{N} \left( \frac{1}{N} \int_{R_i} \langle G_{x_i}, G_{x_i} \rangle_w d\mu_d(x) - \int_{R_i \times R_i} \langle G_{x_i}, G_{y} \rangle_w d\mu_d(x) d\mu_d(y) \right)
\]

\[
= \int_{S^d \times S^d} g_w((x, y)) d\mu_d(x) d\mu_d(y) + \sum_{i=1}^{N} \int_{R_i \times R_i} g_w((x, y)) d\mu_d(x) d\mu_d(y).
\]

The first term of the sum is equal to zero because for each fixed \( x \in S^d \), the polynomial \( g_w((x, y)) \in \mathcal{P}_n \). We can estimate the second term by

\[
\sum_{i=1}^{N} \int_{R_i \times R_i} g_w((x, y)) d\mu_d(x) d\mu_d(y) \leq \frac{1}{N} \max_{R_i \in R, x, y \in R_i} |g_w(1) - g_w((x, y))| 
\]

\[
\leq \frac{1}{N} \max_{R_i \in R, x, y \in R_i} g_w'(1) \| x - y \|^2 \leq \frac{1}{N} c_{1d}N^d \| R \|^2 \leq c_1d \frac{c_{2d}N^d}{N^{1+2/d}}
\]

where in the last line we use [9] and [10]. This immediately implies the statement of the Lemma. \( \square \)

For a polynomial \( Q \in \mathcal{P}_n \) define the norm of the Hessian matrix on the sphere, as defined by [4], at the point \( x_0 \in S^d \) by

\[
\| \nabla^2 Q(x_0) \| = \max_{\| y \|=1} |\nabla^2 Q(x_0) \cdot y \cdot y|,
\]
where the maximum is taken over vectors \( y \) orthogonal to \( x_0 \). We will prove the following estimate

**Lemma 2.** For a polynomial \( Q \in \mathcal{P}_n \) and point \( x_0 \in S^d \)

\[
\|\nabla^2 Q(x_0)\| \leq (3g''_w(1) + g'_w(1))^{1/2}\|Q\|_w.
\]

*Proof.* Fix a unit vector \( y_0 \) orthogonal to \( x_0 \) and define a curve \( x(t) \) on the sphere \( S^d \) by

\[
x(t) = x_0 \cos(t) + y_0 \sin(t).
\]

For each \( t \in \mathbb{R} \) we consider the polynomial \( G_{x(t)}(y) = g_w((x(t), y)) \in \mathcal{P}_n \), which has the property \( \langle Q, G_{x(t)} \rangle_w = Q(x(t)) \) for all \( Q \in \mathcal{P}_n \). Setting \( G'' = \frac{d^2}{dt^2}G_{x(t)} \big|_{t=0} \), we have that

\[
\nabla^2 Q(x_0) \cdot y_0 = \frac{d^2}{dt^2}Q(x(t)) \big|_{t=0} = \langle Q, G'' \rangle_w.
\]

Hence

\[
\|\nabla^2 Q(x_0)\| \leq \|G''\|_w\|Q\|_w.
\]

It remains to show that \( \|G''\|_w = (3g''_w(1) + g'_w(1))^{1/2} \). Since

\[
\frac{d^2}{dt^2}G_{x(t)}(y) = \frac{d^2}{dt^2}g_w((x(t), y)),
\]

we obtain

\[
G''(y) = (y_0, y)^2 g''_w((x_0, y)) - (x_0, y)g'_w((x_0, y)).
\]

From (12) and (13) we get by direct calculation

\[
\langle G'', G'' \rangle_w = \frac{d^2}{dt^2}G''(x(t)) \big|_{t=0} = 3g''_w(1) + g'_w(1).
\]

Lemma 2 is proved. \( \Box \)

Denote by \( B^q \) the closed ball of radius 1 with center at 0 in \( \mathbb{R}^q \). To prove the following Lemma 3 we use the Brouwer fixed point theorem [11]

**Theorem B.** Let A be a closed bounded convex subset of \( \mathbb{R}^q \) and \( H : A \to A \) be a continuous mapping on A. Then there exists some \( z \in A \) such that \( H(z) = z \).
Lemma 3. Let \( F : B^q \to \mathbb{R}^q \) be a continuous map such that
\[
F(x) = A(x) + G(x),
\]
where \( A(x) \) is a linear map and for each \( x \in B^q \)
\[
\|A(x)\| \geq \alpha \|x\| \tag{14}
\]
and
\[
\|G(x)\| \leq \alpha \|x\|/2, \tag{15}
\]
for some \( \alpha > 0 \). Then, the image of \( F \) contains the closed ball of radius \( \alpha/2 \) with center at 0.

Proof. Take an arbitrary \( y \), with \( \|y\| \leq \alpha/2 \). It is sufficient to show that there exists \( x \in B^q \) such that \( F(x) = y \). The inequality (14) implies that \( \|A^{-1}(y)\| \leq 1/2 \). Denote by \( K \) the ball of radius 1/2 with center 0. Consider a map
\[
H_y(z) = -A^{-1}(G(A^{-1}(y) + z)).
\]
By (14) and (15) we obtain that \( H_y(K) \subset K \). Hence, by the Brouwer fixed point theorem, there exists \( z \in K \) such that \( H_y(z) = z \). This then implies that
\[
F(A^{-1}(y) + z) = y.
\]
\[\square\]

To prove the principal Lemma [4] we also need a result which is an easy corollary of Theorem 3.1 in [10]

Theorem MNW. There exist constants \( r_d \) and \( N_d \) such that for each area-regular partition \( \mathcal{R} = \{R_1, \ldots, R_N\} \) with \( \|\mathcal{R}\| < \frac{4}{m} \), each collection of points \( x_i \in R_i, i = 1, \ldots, N \) and each algebraic polynomial \( P \) of total degree \( m > N_d \) the following inequality
\[
\frac{1}{2} \int_{S_d} |P(x)| d\mu_d(x) < \frac{1}{N} \sum_{i=1}^N |P(x_i)| < \frac{3}{2} \int_{S_d} |P(x)| d\mu_d(x) \tag{16}
\]
Consider the map \( \Phi : (S^d)^N \to \mathcal{P}_n \) defined by
\[
(x_1, \ldots, x_N) \xrightarrow{\Phi} \frac{G_{x_1} + \ldots + G_{x_N}}{N}.
\]

Lemma 4. Let \( x_1, \ldots, x_N \in S^d \) be the collection of points and \( \mathcal{R} = \{ R_1, \ldots, R_N \} \) an area-regular partition such that \( x_i \in R_i \) and \( \| \mathcal{R} \| \leq \frac{d}{2n} \). Then the image of the map \( \Phi \) contains a ball of radius \( \rho \geq A_d n^{-(d-2)/2} \) with center at the point \( G = \frac{G_{x_1} + \ldots + G_{x_N}}{N} \), where \( A_d \) is a sufficiently small constant, depending only on \( d \).

Proof. For each polynomial \( P \in \mathcal{P}_n \) consider the circles on \( S^d \) given by
\[
\tilde{x}_i(t) = x_i \cos(\| \nabla P(x_i) \| t) + y_i \sin(\| \nabla P(x_i) \| t),
\]
where \( y_i = \frac{\nabla P(x_i)}{\| \nabla P(x_i) \|} \), \( i = 1, \ldots, N \). Define the map \( X : \mathcal{P}_n \to (S^d)^N \) by
\[
X(P) = (x_1(P), \ldots, x_N(P)) := (\tilde{x}_1(1), \ldots, \tilde{x}_N(1)).
\]
Now we will consider the composition \( L = \Phi \circ X : \mathcal{P}_n \to \mathcal{P}_n \) which takes the form
\[
L(P) = \frac{G_{x_1(P)} + \ldots + G_{x_N(P)}}{N}.
\]
For each \( Q \in \mathcal{P}_n \) one can take the Taylor expansion
\[
\langle G_{x_i(t)}, Q \rangle_w = Q(\tilde{x}_i(t)) = Q(x_i) + \frac{d}{dt} Q(\tilde{x}_i(0)) t + \frac{1}{2} \frac{d^2}{dt^2} Q(\tilde{x}_i(t)) t^2, \quad t_i \in [0, t].
\]
Hence, we can represent the function \( L(P) \) in the form
\[
L(P) = L(0) + L'(P) + L''(P).
\]
Here \( L'(P) \) is the unique polynomial in \( \mathcal{P}_n \) satisfying
\[
\langle L'(P), Q \rangle_w = \frac{1}{N} \sum_{i=1}^{N} \langle \nabla Q(x_i), \nabla P(x_i) \rangle \text{ for all } Q \in \mathcal{P}_n,
\]
\[ L''(P) = L(P) - L(0) - L'(P). \]

First, for each \( P \in \mathcal{P}_n \) we will estimate the norm of \( L'(P) \) from below. We have

\[ \|L'(P)\|_w \geq \frac{1}{\|P\|_w} \cdot \langle L'(P), P \rangle_w = \frac{1}{\|P\|_w} \cdot \frac{1}{N} \sum_{i=1}^{N} (\nabla P(x_i), \nabla P(x_i)). \]

Applying (16) to the polynomial \((\nabla P, \nabla P)\) of degree \( \leq 2n \), we get

\[ \frac{1}{N} \sum_{i=1}^{N} (\nabla P(x_i), \nabla P(x_i)) \geq \frac{1}{2} \int_{S^d} (\nabla P(x), \nabla P(x)) d\mu_d(x). \]

On the other hand, by (6) we have

\[ \int_{S^d} (\nabla P(x), \nabla P(x)) d\mu_d(x) = \langle P, \Delta_w P \rangle_w = \|P\|_w^2. \]

This gives us the estimate

(18) \[ \|L'(P)\|_w \geq \frac{1}{2} \|P\|_w. \]

Now we will estimate the norm of \( L''(P) \) from above. By (17) we have

\[ \langle L''(P), Q \rangle_w = \frac{1}{2N} \sum_{i=1}^{N} \frac{d^2}{dt^2} Q(\tilde{x}_i(t_i)), \]

for some \( t_i \in [0, 1] \). Since the following equality holds

\[ \frac{d^2}{dt^2} Q(\tilde{x}_i(t)) = \nabla^2 Q \cdot \frac{d\tilde{x}_i(t)}{dt} \cdot \frac{d\tilde{x}_i(t)}{dt}, \]

Lemma 2 implies that

\[ \left| \frac{d^2}{dt^2} Q(\tilde{x}_i(t)) \right| \leq (3g''(1) + g'(1)^2)^{1/2} \left\| \frac{d\tilde{x}_i}{dt} \right\|^2 \cdot \|Q\|_w. \]

It follows from the identity

\[ \left\| \frac{d\tilde{x}_i}{dt}(t) \right\| = \|\nabla P(x_i)\| \]

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and estimates (10), (11) that
\[\frac{d^2}{dt^2} Q(\tilde{x}_i(t)) \leq c_{3d} t^{(d+2)/2} \|\nabla P(x_i)\|^2 \cdot \|Q\|_w.\]

This inequality yields immediately
\[|\langle L''(P), Q \rangle_w | = \left| \frac{1}{2N} \sum_{i=1}^{N} \frac{d^2}{dt^2} Q(\tilde{x}_i(t)) \right| \leq \frac{c_{3d} t^{(d+2)/2} \|Q\|_w}{N} \sum_{i=1}^{N} \|\nabla P(x_i)\|^2.\]

Applying again (10), we obtain
\[\frac{1}{N} \sum_{i=1}^{N} \|\nabla P(x_i)\|^2 \leq \frac{3}{2} \|P\|_w^2.\]

So, for each \(Q \in \mathcal{P}_n\) we have that
\[|\langle L''(P), Q \rangle_w | \leq \frac{3}{2} c_{3d} t^{(d+2)/2} \|P\|_w^2 \cdot \|Q\|_w.\]

Thus, we get
\[\|L''(P)\|_w \leq \frac{3}{2} c_{3d} t^{(d+2)/2} \|P\|_w^2.\]

Lemma 3 combined with inequalities (18) and (19) implies that the image of \(L\), and hence the image of \(\Phi\), contains a ball of radius \(\rho \geq A_d t^{(-d-2)/2}\) around \(L(0) = G\), where \(A_d = 1/(6c_{3d})\), proving the lemma.

**Proof of Theorem 1.** By Lemma 1, there exists an area-regular partition \(R = \{R_1, \ldots, R_N\}\) such that \(\|R\| \leq c_{2d} N^{1/d}\), and a collection of points \(x_i \in R_i\), \(i = 1, \ldots, N\) such that
\[\left\| \frac{G_{x_1} + \ldots + G_{x_N}}{N} \right\|_w \leq \frac{b_d t^{d/2}}{N^{1/2+1/d}}.\]

Take \(N\) large enough such that \(N > N_d\) and \(c_{2d} N^{1/d} < \frac{r_d}{2n}\), where \(N_d\) and \(r_d\) are defined by Theorem MNW. Applying Lemma 2 to the partition \(R\) and the collection of points \(x_1, \ldots, x_N\), we obtain immediately that \(G_{y_1} + \ldots + G_{y_N} = 0\) for some \(y_1, \ldots, y_N \in S^d\) if
\[\frac{b_d t^{d/2}}{N^{1/2+1/d}} < \frac{A_d}{N} t^{(-d-2)/2}.\]

So, we can choose a constant \(c_d\) such that the last inequality holds for all \(N > c_d t^{2d/(d+1)}\). Theorem 1 is proved. \(\square\)
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