Matter with dilaton charge in Weyl–Cartan spacetime and evolution of the Universe

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Abstract. The perfect dilaton-spin fluid (as a model of the dilaton matter, the particles of which are endowed with intrinsic spin and dilaton charge) is considered as the source of the gravitational field in a Weyl–Cartan spacetime. The variational theory of such fluid is constructed and the dilaton-spin fluid energy-momentum tensor is obtained. The variational formalism of the gravitational field in a Weyl–Cartan spacetime is developed in the exterior form language. The homogeneous and isotropic Universe filled with the dilaton matter as the dark matter is considered and one of the field equations is represented as the Einstein-like equation which leads to the modified Friedmann–Lemaître equation. From this equation the absence of the initial singularity in the cosmological solution follows. Also the existence of two points of inflection of the scale factor function is established, the first of which corresponds to the early stage of the Universe and the second one corresponds to the modern era when the expansion with deceleration is replaced by the expansion with acceleration. The possible equations of state for the self-interacting cold dark matter are found on the basis of the modern observational data. The inflation-like solution is obtained.

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1. Introduction

The basis concept of the modern fundamental physics consists in the preposition that the spacetime geometrical structure is compatible with the properties of matter filling the spacetime. It means that the matter dynamics determines the metric and the connection of the spacetime manifold and in turn is determined by the spacetime geometric properties. Therefore the possible deviation from the geometrical structure of the General Relativity spacetime should be stipulated by the existence of matter with unusual properties, which fills spacetime, generates its structure and interacts with it. As examples of such matter there were considered the perfect media with intrinsic degrees of freedom, such as the perfect fluid with spin and non-Abelian colour charge [1], the perfect hypermomentum fluid (see [2] and the references therein), the perfect dilaton-spin fluid [3, 4, 5]. All these fluids are the generalization of the perfect Weyssenhoff–Raabe spin fluid [6].

The modern observations [7] lead to the conclusion about the existence of dark (non-luminous) matter with the density exceeding by one order of magnitude the density of baryonic matter, from which stars and luminous components of galaxies are formed. It is the dark matter interacting with the equal by order of magnitude positive vacuum energy or with quintessence [8, 9] that realizes the dynamics of the Universe in modern era. Another important consequence of the modern observations is the understanding of the fact that the end of the Friedmann era occurs when the expansion with deceleration is succeeded by the expansion with acceleration, the transition to the unrestrained exponential expansion being possible.

As dark matter we propose to consider the dilaton matter, the model of which is realized as the perfect dilaton-spin fluid. Every particle of such fluid is endowed with spin and dilaton charge. This type of matter generates in spacetime the Weyl–Cartan geometrical structure with curvature, torsion and nonmetricity of the Weyl type and interacts with it. Such hypothetical type of matter was proposed in [3, 4], where the variational theory of perfect fluid of this type has been constructed, and the peculiarities of motion of the particles with spin and dilaton charge in a Weyl–Cartan spacetime have been considered.

The importance of considering matter endowed with the dilaton charge is based on the fact that some variants of low-energy effective string theory are reduced to the theory of interacting metric and scalar dilaton field [10] represented by the gradient part of the Weyl connection. It should be pointed out that this gradient part was successfully used in [11] for explanation some effects of the modern cosmology. Therefore the dilaton gravity is one of the attractive approaches to the modern gravitational theory.

The Weyl–Cartan cosmology was considered in [12, 13, 14] as a particular case of the metric–affine theory of gravitation. In our approach we consider the Weyl–Cartan spacetime geometry from the very beginning, the Weyl’s restriction on a nonmetricity 2-form being imposed with a Lagrange multiplier. This approach in the exterior form language was developed in [15, 5]. These two approaches are not identical and yield the
different field equations in general, as it will be pointed out in section 4.

Our paper is organized as follows.

In section 2, the variational theory of the perfect dilaton-spin fluid is constructed in a Weyl–Cartan spacetime with torsion and nonmetricity with the help of the exterior form formalism. In section 3, the dilaton-spin fluid energy-momentum tensor and dilaton-spin current are obtained, and the energy conservation law along a streamline of the fluid is derived as a consequence of the generalized hydrodynamic Euler-type equation of the perfect fluid motion. In section 4, the variational formalism of the gravitational field in a Weyl–Cartan spacetime is developed and two gravitational field equations, \( \Gamma \)- and \( \theta \)-equations, are derived. In section 5, the general solution of the field \( \Gamma \)-equation is obtained. In section 6, the homogeneous and isotropic Universe is considered and the field \( \theta \)-equation is represented as the Einstein-like equation which leads to the modified Friedmann–Lemaître (FL) equation. From this equation it follows (section 7) that, firstly, under the certain conditions on the parameters of the gravitational Lagrangian this equation has a nonsingular solution and, secondly, two points of inflection of the scale factor function exist. The first one corresponds to the early stage of the Universe and the second one corresponds to the modern era when the expansion with deceleration is replaced by the expansion with acceleration. Here the possible equations of state for the self-interacting cold dark matter are found on the basis of the modern observational data. The inflation-like solution is obtained for the superrigid equation of state of the dilaton matter at the very early stage of the evolution of the Universe. Section 8 is devoted to the discussion.

Some facts of a Weyl–Cartan space and the notations used are stated in Appendix A, in particular, the signature of the metric is assumed to be \((+, +, +, -)\). In Appendices B and C the results of the intermediate calculations are stated. Throughout the paper the conventions \( c = 1, \hbar = 1 \) are used.

2. The variational theory of the dilaton-spin fluid

Let us develop the variational theory of the perfect dilaton-spin fluid in a Weyl–Cartan space. We shall describe the additional degrees of freedom, which a fluid element possesses, as it is accepted in the theory of the Weyssenhoff–Raabe fluid, with the help of the material frame attached to every fluid element and consisting of four vectors \( \mathbf{l}_p \) \((p = 1, 2, 3, 4)\) and inverse to them vectors \( \mathbf{l}^p \). The vectors \( \mathbf{l}_p \) are called directors. Three of the directors \((p = 1, 2, 3)\) are space-like and the fourth \((p = 4)\) is time-like. In the exterior form language the material frame of the directors turns into the coframe of 1-forms \( l^p \), which have dual 3-forms \( l_q \), the constraint \( l^p \wedge l_q = \delta^p_q \eta \) being fulfilled. This constraint means that \( l^p l^\beta = \delta^\beta_p \). Each fluid element possesses a 4-velocity vector \( \mathbf{u} = u^\alpha \mathbf{e}_\alpha \) which corresponds to a flow 3-form \( u := \mathbf{u} \wedge \eta = u^\alpha \eta_\alpha \) and a velocity 1-form \( *u = u_\alpha \theta^\alpha = g(\mathbf{u}, \cdots) \) with \(*u \wedge \eta = -\eta\) that means the usual condition \( g(\mathbf{u}, \mathbf{u}) = -1\).

In case of the dilaton-spin fluid the spin dynamical variable of the Weyssenhoff–Raabe fluid is generalized and becomes the new dynamical variable \( J^p_q \) named the
dilaton-spin tensor $\mathbb{J}$:

$$J^p_q = S^p_q + \frac{1}{4} \delta^p_q J, \quad S_{pq} = J_{[pq]}, \quad J = J^p_p.$$  

It is well-known due to Frenkel theory of the rotating electron [16] that the spin tensor $S^p_q$ of a particle is spacelike in its nature that is the fact of fundamental physical meaning. This leads to the classical Frenkel condition, which can be expressed in two equivalent forms: $S^p_q u^q = 0$, $u_p S^p_q = 0$ or in the exterior form language, $S^p_q l_p \wedge *u = 0$, $S^p_q l^q \wedge u = 0$. It should be mentioned that the Frenkel condition appears to be a consequence of the generalized conformal invariance of the Weyssenhoff perfect spin fluid Lagrangian [17].

It is important that only the first term of $J^p_q$ (the spin tensor) obeys to the Frenkel condition [3]. The second term is proportional to the specific (per particle) dilaton charge $J$ of the fluid element. The existence of the dilaton charge is the consequence of the extension of the Poincaré symmetry (with the spin tensor as the dynamical invariant) to the Poincaré–Weyl symmetry with the dilaton-spin tensor as the dynamical invariant.

The measure of intrinsic motion contained in a fluid element is the quantity $\Omega^p_q$ which generalizes the intrinsic ‘angular velocity’ of the Weyssenhoff spin fluid theory,

$$\Omega^p_q = u \wedge l^\alpha_p D_l^\alpha_p, \quad D_l^\alpha_p = d_l^\alpha_p + \Gamma^\alpha_{\beta\lambda} l^\beta_p l^\lambda_p.$$  

An element of the perfect dilaton-spin fluid possesses the additional intrinsic ‘kinetic’ energy density 4-form,

$$E = \frac{1}{2} n J^p_q \Omega^q_p u = \frac{1}{2} n S^p_q u \wedge l^\alpha_p D_l^\alpha_p + \frac{1}{8} n J u \wedge l^\beta_p D_l^\beta_p,$$

where $n$ is the fluid particles concentration.

The fluid element represents the quasi-closed statistical subsystem (with sufficiently large number of particles), the properties of which coincide with the macroscopic properties of the fluid. The internal energy density of the fluid $\varepsilon$ depends on the extensive (additive) thermodynamic parameters: the fluid particles concentration $n$ and the specific (per particle) entropy $s$ of the fluid in the rest frame of reference, the specific spin tensor $S^p_q$ and the specific dilaton charge $J$. The energy density $\varepsilon$ obeys to the first thermodynamic principle,

$$d\varepsilon(n, s, S^p_q, J) = \frac{\varepsilon + p}{n} dn + nT ds + \frac{\partial \varepsilon}{\partial S^p_q} dS^p_q + \frac{\partial \varepsilon}{\partial J} dJ,$$  

where $p$ is the hydrodynamic fluid pressure, the fluid particles number conservation law $d(nu) = 0$ being taken into account. The entropy conservation law $d(nsu) = 0$ is also fulfilled along the streamline of the fluid.

The Lagrangian density 4-form of the perfect dilaton-spin fluid can be constructed from the quantities $\varepsilon$ and $E$ with regard to the constraints which should be introduced into the Lagrangian density by means of the Lagrange multipliers $\varphi$, $\tau$, $\chi$, $\chi^q$ and $\zeta^p$,

$$\mathcal{L}_{\text{fluid}} = -\varepsilon(n, s, S^p_q, J) \eta + \frac{1}{2} n S^p_q u \wedge l^\alpha_p D_l^\alpha_p + \frac{1}{8} n J u \wedge l^\beta_p D_l^\beta_p + n u \wedge d\varphi + n \tau u \wedge ds + n \chi (\ast u \wedge u + \eta) + n \chi^q S^p_q l_p \wedge \ast u + n \zeta_p S^p_q l^q \wedge u.$$  

(2.2)
The fluid motion equations and the evolution equation of the dilaton-spin tensor are derived by the variation of (2.2) with respect to the independent variables \( n, s, S^p_{q}, J, u, l^l \) and the Lagrange multipliers, the thermodynamic principle (2.1) being taken into account and master-formula (B.1) (see Appendix B) being used. We shall consider the 1-form \( l^l \) as an independent variable and the 3-form \( l^l_p \) as a function of \( l^l \). One can verify that the Lagrangian density 4-form (2.2) is proportional to the hydrodynamic fluid pressure, \( L_{\text{fluid}} = p\eta \).

The variation with respect to the material coframe \( l^l_q \) yields the motion equations of the directors, which lead to the dilaton charge conservation law \( \dot{J} = 0 \) and to the evolution equation of the spin tensor,

\[
\dot{S}^\alpha_\beta + \dot{S}^\alpha_\gamma u^\gamma u_\beta + \dot{S}^\gamma_\beta u_\gamma u^\alpha = 0 , \tag{2.3}
\]

where the ‘dot’ notation for the tensor object \( \Phi \) is introduced, \( \dot{\Phi}^\alpha_\beta = \varepsilon (u \wedge D\Phi^\alpha_\beta) \). The equation (2.3) generalizes the evolution equation of the spin tensor of the Weyssenhoff fluid theory to a Weyl–Cartan space. With the help of the projection tensor \( \Pi^\sigma_\alpha \) these two equations can be represented in the equivalent form \[3\], \( \Pi^\sigma_\alpha \Pi^\rho_\beta J^\rho = 0 \).

3. The energy-momentum 3-form and the hydrodynamic Euler equation

By means of the variational derivatives of the matter Lagrangian density (2.2) one can derive the external matter currents which are the sources of the gravitational field. In case of the perfect dilaton-spin fluid the matter currents are the canonical energy-momentum 3-form \( \Sigma_\sigma \), the metric stress-energy 4-form \( \sigma^{\alpha\beta} \), the dilaton-spin momentum 3-form \( J^\alpha_\beta \), derived in \[3\].

The variational derivative of the Lagrangian density (2.2) with respect to \( \theta^\sigma \) yields the canonical energy-momentum 3-form,

\[
\Sigma_\sigma = \frac{\delta L_{\text{fluid}}}{\delta \theta^\sigma} = p\eta_\sigma + (\varepsilon + p)u_\sigma u + n\dot{S}_{\sigma\rho}u^\rho u \tag{3.1}.
\]

Here the Frenkel condition, the dilaton charge conservation law \( \dot{J} = 0 \) and the evolution equation of the spin tensor (2.3) have been used. In case of the dilaton-spin fluid the energy density \( \varepsilon \) in (3.1) contains the energy density of the dilaton interaction of the fluid.

The metric stress-energy 4-form can be derived in the same way,

\[
\sigma^{\alpha\beta} = 2\frac{\delta L_{\text{fluid}}}{\delta g^{\alpha\beta}} = T^{\alpha\beta}\eta ,
\]

\[
T^{\alpha\beta} = pg^{\alpha\beta} + (\varepsilon + p)u^\alpha u^\beta + n\dot{S}^{(\alpha}_{\gamma} u^{\beta)} u^\gamma . \tag{3.2}
\]

The dilaton-spin momentum 3-form can be obtained in the following way,

\[
J^\alpha_\beta = -\frac{\delta L_{\text{fluid}}}{\delta \Gamma^\beta_\alpha} = \frac{1}{2n} \left( S_{\alpha\beta} + \frac{1}{4} J_{\beta}^\alpha \right) u = S_{\alpha\beta} + \frac{1}{4} J_{\beta}^\alpha . \tag{3.3}
\]
In a Weyl–Cartan space the matter Lagrangian obeys the diffeomorphism invariance, the local Lorentz invariance and the local scale invariance that lead to the corresponding Noether identities [4]:

\[
D\Sigma^\sigma = (\bar{e}_\sigma | T^\alpha) \wedge \Sigma_\alpha - (\bar{e}_\sigma | R^\alpha_\beta) \wedge J^\beta_\alpha - \frac{1}{8}(\bar{e}_\sigma | Q) \sigma^\alpha_\alpha ,
\]

(3.4)

\[
\left(D + \frac{1}{4}Q\right) \wedge S_{\alpha\beta} = \theta_{\{\alpha} \wedge \Sigma_{\beta\}} ,
\]

(3.5)

\[
D J^\alpha = \theta^\alpha \wedge \Sigma^\alpha - \sigma^\alpha_\alpha .
\]

(3.6)

The Noether identity (3.4) represents the quasiconservation law for the canonical matter energy-momentum 3-form. This identity and also the identities (3.5) and (3.6) are fulfilled, if the equations of matter motion are valid, and therefore they represent in essence another form of the matter motion equations.

If one introduces a specific (per particle) dynamical momentum of a fluid element,

\[
\pi_\sigma = -\frac{1}{n} * u \wedge \Sigma_\sigma , \quad \pi_\sigma = \frac{\varepsilon}{n} u_\sigma - S_{\sigma\rho} [u] D u^\rho ,
\]

then the canonical energy-momentum 3-form reads,

\[
\Sigma_\sigma = p n_\sigma + n \left(\pi_\sigma + \frac{p}{n} u_\sigma\right) u .
\]

(3.7)

Substituting (3.7), (3.2) and (3.3) into (3.4), one obtains after some algebra the equation of motion of the perfect dilaton-spin fluid in the form of the generalized hydrodynamic Euler-type equation of the perfect fluid [4],

\[
u \wedge D \left(\pi_\sigma + \frac{p}{n} u_\sigma\right) = \frac{1}{n} \eta \bar{e}_\sigma | D p - \frac{1}{8n} \eta (\varepsilon + p) Q_\sigma - (\bar{e}_\sigma | T^\alpha) \wedge \left(\pi_\alpha + \frac{p}{n} u_\alpha\right) u
\]

\[
- \frac{1}{2}(\bar{e}_\sigma | R^\alpha_{\beta}) \wedge S_{\alpha\beta} u + \frac{1}{8}(\bar{e}_\sigma | R^\alpha_\alpha) \wedge J u .
\]

(3.8)

Let us evaluate the component of the equation (3.8) along the 4-velocity by contracting one with \( u^\sigma \). After some algebra we get the energy conservation law along a streamline of the fluid [4],

\[
d\varepsilon = \frac{\varepsilon + p}{n} d n .
\]

(3.9)

4. Variational formalism in a Weyl–Cartan space

We represent the total Lagrangian density 4-form of the theory as follows

\[
\mathcal{L} = \mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{fluid}} ,
\]

(4.1)

where the gravitational field Lagrangian density 4-form reads,

\[
\mathcal{L}_{\text{grav}} = 2f_0 \left(\frac{1}{2} R^\alpha_\beta \wedge \eta_\alpha^\beta - \Lambda \eta + \frac{1}{4} \lambda R^\alpha_\alpha \wedge * R^\beta_\beta + \varrho_1 T^\alpha \wedge * T^\alpha + \varrho_2 (T^\alpha \wedge \theta_\beta) \wedge * (T^\beta \wedge \theta_\alpha) + \varrho_3 (T^\alpha \wedge \theta_\alpha) \wedge * (T^\beta \wedge \theta_\beta)
\]

\[
+ \xi Q \wedge * Q + \zeta Q \wedge \theta^\alpha \wedge * T^\alpha \right) + \Lambda^\alpha_\beta \wedge \left(Q^\alpha_\beta - \frac{1}{4} g^\alpha_\beta Q\right) .
\]

(4.2)
Here $f_0 = 1/(2\pi) (\pi = 8\pi G)$, $\Lambda$ is the cosmological constant, $\lambda, \varrho_1, \varrho_2, \varrho_3, \xi, \zeta$ are the coupling constants, and $\Lambda_{\alpha\beta}$ is the Lagrange multiplier 3-form with the evident properties,

$$\Lambda_{\alpha\beta} = \Lambda_{\beta\alpha}, \quad \Lambda_{\gamma\gamma} = 0 , \quad (4.3)$$

which are the consequences of the Weyl’s condition (A.1).

In (4.2) the first term is the linear Gilbert–Einstein Lagrangian generalized to a Weyl–Cartan space. The second term is the Weyl quadratic Lagrangian, which is the square of the Weyl segmental curvature 2-form (see (C.1) and (A.4)),

$$R_{\alpha}\alpha = d\Gamma_{\alpha}\alpha = \frac{1}{2}dQ . \quad (4.4)$$

Here the Weyl 1-form $Q$, in contrast to the classical Weyl theory, represents the gauge field, which does not relate to an electromagnetic field, that has been firstly pointed out by Utiyama [18]. The field of the Weyl 1-form $Q$ we shall call a dilaton field. The term with the coupling constant $\zeta$ represents the contact interaction of the dilaton field with the torsion that can occur in a Weyl–Cartan space.

The gravitational field equations in a Weyl–Cartan spacetime can be obtained by the variational procedure of the first order. Let us vary the Lagrangian (4.1) with respect to the connection 1-form $\Gamma_{\alpha\beta}$ (\Gamma-equation) and to the basis 1-form $\theta_{\alpha}$ (\theta-equation) independently, the constraints on the connection 1-form in a Weyl–Cartan space being satisfied by means of the Lagrange multiplier 3-form $\Lambda_{\alpha\beta}$.

The including into the Lagrangian density 4-form the term with the Lagrange multiplier $\Lambda_{\alpha\beta}$ means that the theory is considered in a Weyl–Cartan spacetime from the very beginning [13, 14]. Another variational approach has been developed in [13, 14] where the field equations in a Weyl–Cartan spacetime have been obtained as a limiting case of the field equations of the metric-affine gauge theory of gravity. These two approaches are not identical in general and coincide only in case when $\Lambda_{\alpha\beta}$ is equal to zero as a consequence of the field equations.

For the variational procedure it is efficiently to use the following general relations which can be obtained for the arbitrary 2-forms $\Phi_{\alpha\beta}, \Phi_{\alpha}$ and the arbitrary 3-form $\Psi_{\alpha\beta}$ with the help of the Cartan structure equations (A.3) and the structure equation for the nonmetricity 1-form $Q_{\alpha\beta}$ (A.1),

$$\delta R_{\alpha\beta} \wedge \Phi_{\alpha} = d(\delta \Gamma_{\alpha\beta} \wedge \Phi_{\alpha}) + \delta \Gamma_{\alpha\beta} \wedge D\Phi_{\alpha} , \quad (4.5)$$

$$\delta T_{\alpha} \wedge \Phi_{\alpha} = d(\delta \theta_{\alpha} \wedge \Phi_{\alpha}) + \delta \theta_{\alpha} \wedge D\Phi_{\alpha} + \delta \Gamma_{\alpha\beta} \wedge \theta_{\beta} \wedge \Phi_{\alpha} , \quad (4.6)$$

$$\delta Q_{\alpha\beta} \wedge \Psi_{\alpha\beta} = d(-\delta g_{\alpha\beta} \Psi_{\alpha\beta}) + \delta \Gamma_{\alpha\beta} \wedge 2\Psi_{(\alpha\beta)} + \delta g_{\alpha\beta} D\Psi_{\alpha\beta} . \quad (4.7)$$

The subsequent derivation of the variations of the Lagrangian density 4-form (4.2) is based on the master formula (B.1) derived in Lemma proved in [19] (see Appendix B). This master formula gives the rule how to compute the commutator of the variation operator $\delta$ and the Hodge star operator $\ast$. The result of the variation of every term of the gravitational field Lagrangian density 4-form (4.2) one can find in Appendix B.
The variation of the total Lagrangian density 4-form \( (4.1) \) with respect to the Lagrange multiplier 3-form \( \Lambda^{\alpha \beta} \) yields according to \((B.15)\) the Weyl’s condition \((A.1)\) on the nonmetricity 1-form \( Q_{\alpha \beta} \).

The variation of \((4.1)\) with respect to \( \Gamma^{\alpha \beta} \) can be obtained by combining all terms in \((3.7)\)–\((B.15)\) proportional to the variation of \( \Gamma^{\alpha \beta} \) and taking into account the same variation of the fluid Lagrangian density 4-form \((2.2)\), which is the dilaton-spin momentum 3-form \((3.3)\). This variation yields the field \( \Gamma \)-equation,

\[
\frac{1}{4} \lambda \delta_\alpha^\beta \, \mathrm{d} Q - \frac{1}{8} Q \wedge \eta_\alpha^\beta + \frac{1}{2} T_\gamma \wedge \eta_\alpha^\beta + 2 \varphi_1 \theta^\beta \wedge \ast T_\alpha \\
+ 2 \varphi_2 \theta^\beta \wedge \theta_\gamma \wedge \ast (T^\gamma \wedge \theta_\alpha) + 2 \varphi_3 \theta^\beta \wedge \theta_\alpha \wedge \ast (T^\gamma \wedge \theta_\gamma) \\
+ 4 \xi \delta_\alpha^\beta \ast Q + \zeta (\theta^\beta \wedge \ast (Q \wedge \theta_\alpha) + 2 \delta_\alpha^\beta \theta^\gamma \wedge \ast T_\gamma) - \frac{1}{f_0} \Lambda^{\alpha \beta} \\
= \frac{1}{4 f_0} n \left( S_\alpha^\beta + \frac{1}{4} J \delta_\alpha^\beta \right) u ,
\]

the condition \((A.1)\) being taken into account after the variational procedure has been performed.

The variation of \((4.1)\) with respect to the basis 1-form \( \theta^\gamma \) can be obtained in the similar way that gives the second field equation \((\theta \text{-equation})\),

\[
\frac{1}{2} R^{\gamma \beta}_{\alpha \sigma} \wedge \eta_\alpha^\beta - \Lambda \eta_\sigma + \varrho_1 \left( 2D \ast T_\sigma + T^\alpha \wedge \ast (T_\alpha \wedge \theta_\sigma) + \ast (T_\alpha \wedge \theta_\sigma) \wedge \ast T^\alpha \right) \\
+ \varrho_2 \left( 2D (\theta_\alpha \wedge \ast (T^\alpha \wedge \theta_\sigma)) - \ast (T^\beta \wedge \theta_\alpha \wedge \theta_\sigma)(T^\alpha \wedge \theta_\beta) \right) \\
+ 2 T^\alpha \wedge \ast (\theta_\alpha \wedge T_\sigma) - \ast (\ast (T^\beta \wedge \theta_\alpha) \wedge \theta_\gamma) \wedge (T^\gamma \wedge \theta_\beta) \right) \\
+ \varrho_3 \left( 2D (\theta_\sigma \wedge \ast (T^\alpha \wedge \theta_\alpha)) + 2 \ast (T^\alpha \wedge \theta_\alpha) \\
- \ast (T^\beta \wedge \theta_\beta \wedge \theta_\sigma) T^\alpha \wedge \theta_\sigma - \ast (\ast (T^\beta \wedge \theta_\beta) \wedge \theta_\sigma) \wedge (T^\gamma \wedge \theta_\gamma) \right) \\
+ \lambda \left( \frac{1}{4} R^{\beta \gamma}_{\alpha \sigma} \wedge \ast (R^\alpha_\alpha \wedge \theta_\sigma) + \frac{1}{4} \ast (\ast R^\alpha_\alpha \wedge \theta_\sigma) \wedge \ast R^\beta_\beta \right) \\
+ \zeta \left( (Q \wedge \theta_\sigma) + Q \wedge \theta^\sigma \wedge \ast (T_\alpha \wedge \theta_\sigma) \\
- Q \wedge \ast T_\sigma + \ast (T_\alpha \wedge \theta_\sigma) \wedge (Q \wedge \theta^\alpha) \right) \\
+ \xi \left( - Q \wedge \ast (Q \wedge \theta_\sigma) - \ast (\ast Q \wedge \theta_\sigma) * Q \right) = - \frac{1}{2 f_0} \Sigma_\sigma ,
\]

Here \( \Sigma_\sigma \) is the fluid canonical energy-momentum 3-form \((3.1)\). In \((4.9)\) the condition \((A.1)\) is used after the variational procedure has been performed.

The result of the variation of the total Lagrangian density 4-form \((4.1)\) with respect to the metric components \( g_{\alpha \beta} \) \((g \text{-equation})\) is not independent and is a consequence of the field \( \Gamma \)- and \( \theta \)-equations. For the metric-affine theory of gravitation it was pointed out in \((2.1)\). In the Weyl–Cartan theory of gravitation it can be justified as follows. In this theory, as the consequence of the scale invariance, the metric of the tangent space can be chosen in the form \((18)\), \( g_{ab} = \sigma(x) g^M_{ab} \), where \( g^M_{ab} \) is the metric tensor of the Minkowski space and \( \sigma(x) \) is an arbitrary function to be varied when the \( g \)-equation
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is derived. Therefore the $g$-equation appears only in the trace form. But the total Lagrangian density 4-form (4.1) also obeys the diffeomorphism invariance and therefore the Noether identity analogous to the identity (3.4) is valid, from which the trace of the $g$-equation can be derived via the $\Gamma$- and $\theta$-equations. The quantity $\vec{e}_\alpha \lbrack Q$ in (3.4) does not vanish identically in general, otherwise we should have a Riemann–Cartan spacetime, in which case we could choose $\sigma(x) = \text{const} = 1$ and the $g$-equation would not appear.

5. The analysis of the field $\Gamma$-equation

Let us give the detailed analysis of the $\Gamma$-equation (4.8). The antisym metric part of this equation determines the torsion 2-form $T^\alpha$. The symmetric part determines the Lagrange multiplier 3-form $\Lambda_\alpha$ and the Weyl 1-form $Q$.

After antisymmetrization the equation (4.8) gives the following equation for the torsion 2-form,

$$
-\frac{1}{2} T^\gamma \wedge \eta_{\alpha\beta\gamma} + \frac{1}{8} Q \wedge \eta_{\alpha\beta} + 2 \varepsilon_1 \theta_{[\alpha} \wedge *T_{\beta]} \\
+ 2 \varepsilon_2 \theta_{[\alpha} \wedge \theta_{\gamma]} \wedge *(T^{|\gamma}] \wedge \theta_{\beta]} + 2 \varepsilon_3 \theta_{\alpha} \wedge \theta_{\beta} \wedge *(T^\gamma \wedge \theta_{\gamma}) \\
+ \zeta \theta_{[\alpha} \wedge *(Q \wedge \theta_{\beta]}) = \frac{1}{2} \alpha n S_{\alpha\beta} u , \quad \alpha = \frac{1}{2} f_0 . \quad (5.1)
$$

The torsion 2-form can be decomposed into the irreducible pieces (the traceless 2-form $^{(1)}T^\alpha$, the trace 2-form $^{(2)}T^\alpha$ and the pseudotrace 2-form $^{(3)}T^\alpha$) [20], [13],

$$
T^\alpha = ^{(1)}T^\alpha + ^{(2)}T^\alpha + ^{(3)}T^\alpha . \quad (5.2)
$$

Here the torsion trace 2-form and the torsion pseudotrace 2-form of the pseudo-Riemannian 4-manifold are determined by the expressions, respectively,

$$
^{(2)}T^\alpha = \frac{1}{3} T \wedge \theta^\alpha , \quad T = *(\theta^\alpha \wedge *T^\alpha) = -(\vec{e}_\alpha \lbrack T^\alpha) , \quad (5.3)
$$

$$
^{(3)}T^\alpha = \frac{1}{3} * (P \wedge \theta^\alpha) , \quad P = *(\theta^\alpha \wedge T^\alpha) = \vec{e}_\alpha \lbrack *T^\alpha , \quad (5.4)
$$

where the torsion trace 1-form $T$ and the torsion pseudotrace 1-form $P$ are introduced.

The irreducible pieces of torsion satisfy to the conditions [20],

$$
^{(1)}T^\alpha \wedge \theta^\alpha = 0 , \quad ^{(2)}T^\alpha \wedge \theta^\alpha = 0 , \quad (5.5)
$$

$$
\vec{e}_\alpha \lbrack^{(1)}T^\alpha = 0 , \quad \vec{e}_\alpha \lbrack^{(3)}T^\alpha = 0 . \quad (5.6)
$$

Using the computational rules (A.5)–(A.12) and (B.3) let us derive two efficient identities,

$$
T^\gamma \wedge \eta_{\alpha\beta\gamma} = T^\gamma \wedge (\vec{e}_\gamma \lbrack \eta_{\alpha\beta}) = \vec{e}_\gamma \lbrack (T^\gamma \wedge \eta_{\alpha\beta}) \wedge \eta_{\alpha\beta} \\
= \vec{e}_\gamma \lbrack (T^\gamma \wedge *(\theta^\alpha \wedge \theta^\beta)) + T \wedge \eta_{\alpha\beta} \\
= (\vec{e}_\gamma \lbrack (\theta^\alpha \wedge \theta^\beta)) \wedge *T^\gamma + \theta^\alpha \wedge \theta^\beta \wedge (\vec{e}_\gamma \lbrack *T^\gamma) + T \wedge \eta_{\alpha\beta}
$$
\[
\theta_\gamma \wedge (\dd T^\gamma \wedge \theta_\alpha) = * \left( \epsilon_\gamma \right) (\dd T^\gamma \theta_\alpha) = * (\dd \wedge \theta_\alpha + T_\alpha) = * T_\alpha - 3 * T_\alpha. \tag{5.8}
\]

Using the identities (5.7), (5.8), one can represent the field equation (5.1) as follows,
\[
(1 + 2 \rho_1 + 2 \rho_2) \theta_{[\alpha} \wedge \ast \dd T_{\beta]} + \left( \frac{1}{2} + 2 \rho_3 \right) \theta_{\alpha} \wedge \theta_\beta \wedge \dd - 6 \rho_2 \theta_{[\alpha} \wedge \ast \dd T_{\beta]} - \frac{1}{2} T \wedge \eta_{\alpha\beta} + \frac{1}{8} Q \wedge \eta_{\alpha\beta} + \zeta \theta_{[\alpha} \wedge \ast (Q \wedge \theta_\beta) = \frac{1}{2} \theta \eta_{\alpha\beta} u. \tag{5.9}
\]

Multiplying the equation (5.9) by $\theta^\gamma$ from the right externally, using the computation rules (B.2)–(B.6) and then the Hodge star operation, one gets in consequence of the Frenkel condition the relation between the torsion 1-form $T$ (5.3) and the Weyl 1-form $Q$,
\[
T = \frac{3(\frac{1}{4} + \zeta)}{2(1 - \rho_1 + 2 \rho_2)} Q. \tag{5.10}
\]

As a consequence of (5.10) and the relation (A.11) it can be proved equality for the trace 2-form,
\[
(1 + 2 \rho_1 + 4 \rho_2) \theta_{[\alpha} \wedge \ast \dd T_{\beta]} - \frac{1}{2} T \wedge \eta_{\alpha\beta} + \frac{1}{8} Q \wedge \eta_{\alpha\beta} + \zeta \theta_{[\alpha} \wedge \ast (Q \wedge \theta_\beta) = 0. \tag{5.11}
\]

Then as a consequence of (5.2), (5.4), (5.11) and the conditions (5.3) the field equation (5.9) is transformed as follows,
\[
(1 + 2 \rho_1 + 2 \rho_2) \left( \theta_{[\alpha} \wedge \ast \dd T_{\beta]} - \frac{1}{6} (1 - 4 \rho_1 - 4 \rho_2 - 12 \rho_3) \theta_{\alpha} \wedge \theta_\beta \wedge \dd = \frac{1}{2} \theta \eta_{\alpha\beta} u. \tag{5.12}
\]

Contracting this equation with $g^{\beta\gamma} \epsilon_\gamma$, we get with the help of the Leibnitz rule (A.7) the equation,
\[
(1 + 2 \rho_1 + 2 \rho_2) T_{\alpha} = \frac{2}{3} (1 - 4 \rho_1 - 4 \rho_2 - 12 \rho_3) \theta_{\alpha} \wedge \dd = \theta \eta_{\alpha\beta} u_{\gamma}. \tag{5.13}
\]

By contracting this equation with $g^{\alpha\beta} \epsilon_\beta$ we get the equation
\[
(1 - 4 \rho_1 - 4 \rho_2 - 12 \rho_3) \dd = \theta \eta_{\alpha\beta} u_{\gamma}. \tag{5.15}
\]

which represents the torsion pseudotrace 1-form $\dd$ via the Pauli–Lyubanski spin 1-form $\sigma$ of a fluid particle,
\[
\sigma = -\frac{1}{2} S^{\alpha\beta} u^\gamma \eta_{\alpha\beta\gamma} = \frac{1}{2} S^{\alpha\beta} u^\gamma \eta_{\alpha\beta\gamma} \theta^\lambda. \tag{5.14}
\]

As a consequence of (5.13) the field equation (5.12) yields the equation for the traceless piece of the torsion 2-form,
\[
(1 + 2 \rho_1 + 2 \rho_2) T_{\alpha} = \theta \left( S_{\alpha\beta} u_{\gamma} \theta^\beta \wedge \theta^\gamma + \frac{2}{3} \sigma^\beta \eta_{\beta\alpha} \right) = -\frac{2}{3} \theta \eta_{\alpha\beta} u_{\gamma} \theta^\beta \wedge \theta^\gamma. \tag{5.15}
\]

Now let us calculate the symmetric part of the $\Gamma$-equation (4.3). Because of (4.3) the result can be represented as follows,
\[
\alpha \Lambda_{\alpha\beta} = \theta_1 \theta_{(\alpha} \wedge \ast \dd T_{\beta)} + \theta_2 \theta_{[\alpha} \wedge \theta_{|\gamma} \wedge \ast (\dd T_{\beta]} \wedge \theta_\beta) + \frac{1}{8} \lambda g_{\alpha\beta} d^\ast d Q + 2 \zeta g_{\alpha\beta} * Q + \zeta \left( \frac{1}{2} \theta_{(\alpha} \wedge \ast (Q \wedge \theta_{|\beta)} + g_{\alpha\beta} \theta_{\gamma} \wedge \ast T_{\gamma}) - \frac{1}{32} \theta \eta_{\alpha\beta} J u. \tag{5.16}
\]
By contracting the equation (5.16) on the indices $\alpha$ and $\beta$ and after substituting (5.10) in the result, one finds the equation of the Proca type for the Weyl 1-form,

$$\ast d \ast d Q + m^2 Q = \frac{ae}{2\lambda} n J \ast u, \quad m^2 = 16 \frac{\xi}{\lambda} + \frac{3(\varrho_1 - 2\varrho_2 + 8(1 + 2\zeta))}{4\lambda(1 - \varrho_1 + 2\varrho_2)}.$$

(5.17)

The equation (5.17) shows that the dilaton field $Q$, in contrast to Maxwell field, possesses the non-zero rest mass and demonstrates a short-range nature, as it was pointed out by Utiyama [18] (see also [21]). In the component form the Proca type equation for Weyl vector was used in [18, 22] and in the exterior form language in [13].

By virtue of $d(\nu u) = 0$ and $\dot{J} = 0$ (see section 2) the equation (5.17) has the Lorentz condition as a consequence,

$$d \ast Q = 0, \quad \nabla_{\alpha} Q^\alpha = 0,$$

(5.18)

where $\nabla_{\alpha}$ is the covariant derivative with respect to the Riemann connection (see Appendix C). Here the latter relation is the component representation of the former one.

If we use (5.10) and (5.17), then the equation (5.16) takes the form,

$$ae\Lambda_{\alpha\beta} = \varrho_1 \theta_{(\alpha} \ast T_{\beta)} + \varrho_2 \theta_{(\alpha} \ast \theta_{\gamma)} \ast (T^{(\gamma)} \ast \theta_{\beta)})$$

$$+ \frac{4\zeta(1 - \varrho_1 + 2\varrho_2)}{3(1 + 4\zeta)} \theta_{(\alpha} \ast (T \ast \theta_{\beta)}) - \frac{4\zeta + \varrho_1 - 2\varrho_2}{4(1 + 4\zeta)} g_{\alpha\beta} \ast T.$$

(5.19)

This equation determines the Lagrange multiplier 3-form $\Lambda_{\alpha\beta}$. It is very important that $\Lambda_{\alpha\beta}$ is in general not equal to zero.

The equations (5.10), (5.13), (5.15) and (5.19) solve the problem of the evaluation the torsion 2-form and the Lagrange multiplier 3-form. With the help of the algebraic field equations (5.13) and (5.15) the traceless and pseudotrace pieces of the torsion 2-form are determined via the spin tensor and the flow 3-form $u$ of the perfect dilaton-spin fluid in general case, when the conditions $1 + 2\varrho_1 + 2\varrho_2 \neq 0$ and $1 - 4\varrho_1 - 4\varrho_2 - 12\varrho_3 \neq 0$ are valid. With the help of the equation (5.10) one can determine the torsion trace 2-form via the dilaton field $Q$, for which the differential field equation (5.17) is valid. Therefore the torsion trace 2-form can propagate in the theory under consideration.

### 6. The field $\theta$-equation in homogeneous and isotropic cosmology

Now we consider the homogeneous and isotropic Universe filled with the perfect dilaton-spin fluid, which realizes the model of the dark matter with $J \neq 0$ in contrast to the ordinary baryonic matter with $J = 0$. The metric of this cosmological model is the Robertson–Walker (RW) metric with scale factor $a(t)$,

$$ds^2 = \frac{a^2(t)}{1 - kr^2}dr^2 + a^2(t)r^2(d\theta^2 + (\sin \theta)^2 d\phi^2) - dt^2,$$

(6.1)

the comoving frame of reference being chosen,

$$u^1 = u^2 = u^3 = 0, \quad u^4 = 1.$$

(6.2)
As it was shown in [23, 12], in the spacetime with the RW metric (6.1) the only nonvanishing components of the torsion are $T_{141} = T_{242} = T_{343}$ and $T_{ijk}$ for $i = 1, 2, 3$. In this case from (5.3) we get that the only nonvanishing component of the trace 1-form is $T^4 = T^4(t)$ ($T_i = 0$ for $i = 1, 2, 3$). From (5.4) we also find, $P = 3T_{[123]}\eta^{1234}\theta_4$. But the field equation (5.13) yields, $P^4 \sim \sigma^4 = 0$, as a consequence of (5.14) and (6.2). Therefore the pseudotrace piece of the torsion 2-form vanishes. It is easy to calculate with the help of (5.2), (5.3) that the traceless piece also vanishes. Therefore for the RW metric (6.1) we have,

\begin{align}
^{(1)}T_\alpha &= 0 , \\
^{(3)}T_\alpha &= 0 , \end{align}

and the torsion 2-form consists only from the trace piece that in the component representation reads,

$$T_{\lambda\alpha\beta} = -\frac{2}{3}g_{[\alpha}T_{\beta]} .$$

As a consequence of (5.13)–(5.15) and the identity,

$$u_\lambda S_{\alpha\beta} \equiv u_{[\lambda}S_{\alpha\beta]} + \frac{2}{3}(u_{(\lambda}S_{\alpha)\beta} - u_{(\lambda}S_{\beta)\alpha}) ,$$

we have to conclude that the condition $S_{\alpha\beta} = 0$ is valid for the spin tensor of the matter source in the cosmological model considered. It can be understood in the sense that the mean value of the spin tensor is equal to zero under statistical averaging over all directions in the homogeneous and isotropic Universe. As a consequence of this fact in this section and in the sequential sections we shall simplify the equations of the theory by using the conditions (5.3) and $S_{\alpha\beta} = 0$. In case $S_{\alpha\beta} = 0$ dilaton-spin fluid becomes dilaton fluid.

Let us now decompose the field $\theta$-equation (4.9) into Riemannian and non-Riemannian parts using the formulae (C.1)–(C.6) and then transform the result to the component form. In Appendix C one can find the decomposition of the each part of the field equation (4.9). After gathering all expressions (C.7)–(C.11) together and substituting the relation (5.10) we receive the following results.

The terms with the derivatives of the dilaton field, like $R_{\alpha} Q^\alpha$ and $R_\sigma Q^\alpha$, and the same derivatives of the torsion trace 1-form in a remarkable manner mutually compensate each other and vanish as a consequence of (5.10),

$$\frac{2}{3}(1 - q_1 + 2g_2) \left( \eta_\sigma R_\rho T^\rho - \eta_\rho R_\sigma T^\rho \right) - \left( \frac{1}{4} + \zeta \right) \left( \eta_\sigma R_\rho Q^\rho - \eta_\rho R_\sigma Q^\rho \right) = 0 .$$

The terms with $dQ$ also vanish, as the equality $dQ = 0$ is valid identically for the RW metric (6.1) that can be easy verified in the holonomic basis, when $\theta^\alpha = dx^\alpha$,

$$dQ = \partial_\beta Q_\alpha dx^\beta \wedge dx^\alpha = \partial_4 Q_4 dx^4 \wedge dx^4 = 0 .$$

This follows from the fact that for this metric one has $Q_4 = Q_4(t)$, $Q_i = 0$ ($i = 1, 2, 3$) as a consequence of (5.10) and the values of the trace torsion for the metric (6.1).
The remainder terms of the equation (6.3) after some algebra can be represented as follows,
\[
\left( R^\rho_\sigma - \frac{1}{2} \delta^\rho_\sigma R \right) \eta_\rho + \Lambda \eta_\sigma + \alpha (2Q_\sigma Q^\rho_\eta_\rho - Q_\rho Q^\rho_\eta_\sigma) = \alpha \epsilon \Sigma_\sigma ,
\]
(6.7)
\[
\alpha = \frac{3 \left( \frac{1}{4} + \zeta \right)^2}{4(1 - \varrho_1 + 2\varrho_2)} + \xi - \frac{3}{64} .
\]
(6.8)

Then we shall derive the Weyl 1-form \( Q_\alpha \) algebraically as a consequence of (6.6) from the equation (5.17) via the right side of this equation,
\[
Q_\alpha = \alpha \left( \frac{2}{\lambda m^2} \right) n J u^\alpha ,
\]
(6.9)
which is in accordance with the conditions (6.2) for the comoving system of reference.

After substituting (6.9) and (3.1) to the equation (6.7), the condition \( S_{\alpha \beta} = 0 \) being used, we can represent the field equation (4.9) as an Einstein-like equation,
\[
R^\rho_\sigma - \frac{1}{2} g^\rho_\sigma R = \alpha \epsilon \left( \frac{2J}{\lambda m^2} \right)^2 ,
\]
(6.10)
where \( R^\rho_\sigma, R \) are a Ricci tensor and a curvature scalar of a Riemann space, respectively, \( \epsilon_\epsilon \) and \( p_\epsilon \) are an energy density and a pressure of an effective perfect fluid:
\[
\epsilon_\epsilon = \epsilon + \epsilon_\nu - \alpha \epsilon \left( \frac{J}{2\lambda m^2} \right)^2 , \quad p_\epsilon = p + p_\nu - \alpha \epsilon \left( \frac{J}{2\lambda m^2} \right)^2 ,
\]
(6.11)
and \( \epsilon_\nu = \Lambda / \alpha \) and \( p_\nu = -\Lambda / \alpha \) are an energy density and a pressure of a vacuum with the equation of state, \( \epsilon_\nu = -p_\nu > 0 \).

7. Evolution of the Universe with dilaton matter

The field equation (6.10) yields the modified Friedmann–Lemaitre (FL) equation,
\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{\alpha \epsilon}{3} \left( \epsilon + \epsilon_\nu - \alpha \epsilon \left( \frac{J}{2\lambda m^2} \right)^2 \right) .
\]
(7.1)

The integration of the continuity equation \( d(nu) = 0 \) (\( d \) – the operator of exterior differentiation) for RW metric (5.1) yields the matter conservation law \( na^3 = N = \text{const} \). As an equation of state of the dilaton fluid we choose the equation of state
\[
p = \gamma \epsilon , \quad 0 \leq \gamma < 1 .
\]
(7.2)

Then the integration of the energy conservation law (3.9) for RW metric (5.1) yields the condition,
\[
\epsilon a^{3(1+\gamma)} = \mathcal{E}_\gamma = \text{const} , \quad \mathcal{E}_\gamma > 0 .
\]
(7.3)

By virtue of these relations the modified FL equation (7.1) takes the form,
\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{\alpha \epsilon}{3a^6} \left( \epsilon_\nu a^6 + \epsilon_\gamma a^{3(1-\gamma)} - \mathcal{E} \right) , \quad \mathcal{E} = \alpha \epsilon \left( \frac{J}{2\lambda m^2} \right)^2 .
\]
(7.4)

From this equation one can see that the influence of dilaton matter is most essential at the early stage of the Universe evolution (when \( a << 1 \)), and that the vacuum
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energy contribution dominates at the last stage $a \gg 1$, when the size of the Universe will exceed some specific magnitude defined by the parameters of the dilaton fluid and dilaton field.

Now we put $k = 0$ in (7.4) in accordance with the modern observational evidence \[7, 8, 9\], from which one should conclude that the Universe is spatially flat in cosmological scale. In this case we can conclude from the equation (7.4) that extremum points of the scale factor ($\dot{a} = 0$) correspond to zero points of the equation,

$$ a^6 + \frac{E_\gamma}{E_v} \left(a^{3(1-\gamma)} - \frac{E}{E_\gamma} \right) = 0 \, . $$ (7.5)

We are finding zero points corresponding to the positive values of the scale factor, $a > 0$. It is easy to see that, if the condition,

$$ 0 < E << 1 \, , $$ (7.6)

is valid, then in case $a << 1$ the equation (7.5) has one such zero point,

$$ a_0 \approx \left( \frac{E}{E_\gamma} \right)^{\frac{1}{3(1-\gamma)}} \left( \frac{\alpha \varepsilon}{E_\gamma} \left( \frac{J N}{2 \lambda m^2} \right)^2 \right)^{\frac{1}{3(1-\gamma)}} \, , $$ (7.7)

and none zero points in case $a \gg 1$ (as $E_\gamma > 0$).

In order to clarify, whether there is a minimum or a maximum in the extremum point, one can use the other components of the equation (6.10). By virtue of (7.3) and (7.7) the result reads,

$$ \frac{\ddot{a}}{a} = \frac{\alpha E_\gamma}{3a^6} \left( \varepsilon_v + \frac{3(1 - \gamma)}{2} \frac{E}{a_0^6} \right) \, . $$ (7.8)

At the extremum point (7.7) (in case $a << 1$) one gets

$$ \left. \left( \frac{\ddot{a}}{a} \right) \right|_{a=a_0} = \frac{\alpha E_\gamma}{3} \left( \varepsilon_v + \frac{3(1 - \gamma)}{2} \frac{E}{a_0^6} \right) > 0 \, ,$$

Therefore the value $a_0$ corresponds to the minimum point of the scale factor $a(t)$.

We conclude that under the some conditions on the parameters of the Lagrangian density 4-form (4.2) there exists a nonsingular solution of the homogeneous and isotropic cosmological model of the Universe to which the minimum value of the scale factor $a_0$ and the maximum value of the matter density of the Universe correspond,

$$ \varepsilon_{\text{max}} \approx \frac{E_\gamma}{3(1+\gamma)} \left( \frac{E}{a_0^6} \right)^{\frac{1}{3(1+\gamma)}} \, . $$ (7.9)

In case $\gamma = 0$ the value (7.9) corresponds to that one which was established in \[8\], where the influence of the vacuum energy contribution was neglected.

In case $a \gg 1$ we can neglect the last term in (7.8) and get the equation,

$$ \frac{\ddot{a}}{a} = \frac{\alpha E_\gamma}{3} \left( \varepsilon_v - \frac{1}{2} (1 + 3\gamma) \varepsilon(t) \right) \, , $$ (7.10)

where $\varepsilon(t)$ is the current value of the dilaton fluid energy density. This equation is valid for the most part of the history of the Universe.
Consider now the conditions under which the points of inflection of the function $a(t)$ can exist. To this end let us equate the right side of (7.8) to zero ($\ddot{a} = 0$) and find zero points of the equation,

$$
a^6 - \frac{(1 + 3\gamma)\varepsilon \gamma}{2\varepsilon_v} \left( a^{3(1-\gamma)} - \frac{4}{1 + 3\gamma} a_0^{3(1-\gamma)} \right) = 0 .
$$

(7.11)

As the last term in (7.11) is very small, then this equation has two types of zero points: very small by magnitude with the value,

$$
a_1 \approx a_0 \left( \frac{4}{1 + 3\gamma} \right)^{\frac{4}{3(1-\gamma)}} ,
$$

(7.12)

and large by magnitude with the value,

$$
a_2 \approx \left( \frac{(1 + 3\gamma)\varepsilon \gamma}{2\varepsilon_v} \right)^{\frac{1}{3(1+\gamma)}} .
$$

(7.13)

As both solutions correspond to the positive values of the scale factor ($a > 0$), then two points of inflection exist. As $\gamma < 1$, the first point of inflection (7.12) occurs at once after the minimum $a_0$ of the scale factor. Then up to the second point of inflection $a_2$ one has $\ddot{a} < 0$ and the expansion with deceleration occurs up to the end of the Friedmann era. The point of inflection $a_2$ corresponds to the modern era when the Friedmann expansion with deceleration is replaced by the expansion with acceleration that means the beginning of the ”second inflation” era.

By equating to zero the equation (7.10) we get the correlation between the vacuum energy density $\varepsilon_v$ and the dilaton matter energy density $\varepsilon$ at the point of inflection $a_2$,

$$
\varepsilon = \frac{2\varepsilon_v}{1 + 3\gamma} .
$$

(7.14)

After the second point of inflection $a_2$ the dilaton matter energy density $\varepsilon$ diminishes and the inequality is valid,

$$
\varepsilon < \frac{2\varepsilon_v}{1 + 3\gamma} ,
$$

that corresponds to the condition $\ddot{a} > 0$ and therefore to the expansion with acceleration.

For the nonrelativistic cold matter ($\gamma = \frac{2}{3}$) the formula (7.14) yields,

$$
\Omega_{cdm} = \frac{2}{3} \Omega_{\Lambda} , \quad \Omega_{cdm} = \frac{\varepsilon}{\varepsilon_{tot}} , \quad \Omega_{\Lambda} = \frac{\varepsilon_v}{\varepsilon_{tot}} , \quad \varepsilon_{tot} = \frac{3H^2}{8\pi G} ,
$$

that fits to the boundary of the modern observational data [7],

$$
\Omega_{\Lambda} = 0.66 \pm 0.06 , \quad \Omega_{cdm} h_0^2 = 0.17 \pm 0.02 ,
$$

with $H_0 = 100 h_0 = 65 \text{ km s}^{-1} \text{ Mpc}^{-1} [8]$.

Curiously, that if one takes the generally accepted data of [8], $\Omega_{cdm} = \frac{1}{3}$, $\Omega_{\Lambda} = \frac{2}{3}$, and substitute to (7.14), then one gets the value $\gamma = 1$, that corresponds to the equation of state of the superrigid matter. Therefore we can conclude that the theory together
with the observational data gives the approximate range for the viable values of the factor $\gamma$ in the equation of state of the dilaton matter,

$$\frac{2}{3} \leq \gamma \leq 1.$$ (7.15)

It is interesting to investigate the limiting case $\gamma = 1$. For this case the equations (7.3)–(7.5) are valid, but the zero point of the equation (7.5) in case $\gamma = 1$ is

$$a_e = \left(\frac{\mathcal{E} - \mathcal{E}_1}{\varepsilon_v}\right)^{\frac{1}{6}},$$ (7.16)

where $\mathcal{E}_1 = \mathcal{E}_{\gamma=1}$ is the integration constant of the equation (7.3).

In case $\gamma = 1$ the equation (7.8) at the extremum point (7.16) yields

$$\left(\frac{\ddot{a}}{a}\right)_{a=a_e} = \frac{\alpha}{3a_e^6} \left(\varepsilon_v a_e^6 - 2\mathcal{E}_1 + 2\mathcal{E}\right) = 3\Lambda > 0.$$ (7.17)

Therefore the value $a_e$ corresponds to the minimum point of the scale factor, $a_e = a_{\text{min}}$.

In case $\gamma = 1$, $k = 0$ the equation (7.4) reads,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\alpha}{3a^6} \left(\varepsilon_v a^6 - \mathcal{E}_1 + \mathcal{E}\right) = \frac{\Lambda}{3a^6} \left(a^6 - a_{\text{min}}^6\right).$$ (7.18)

This equation can be exactly integrated. The solution corresponding to the initial data $t = 0$, $a = a_{\text{min}}$ reads,

$$a = a_{\text{min}} \left(\cosh \sqrt{3\Lambda} t\right)^{\frac{1}{3}}, \quad a_{\text{min}} = \left(\frac{\alpha \varepsilon_v^2}{\Lambda} \left(\frac{JN}{2\lambda m^2}\right)^2 - \frac{\alpha \mathcal{E}_1}{\Lambda}\right)^{\frac{1}{6}}.$$ (7.19)

This solution describes the inflation-like stage of the evolution of the Universe, which continues until the equation of state of the dilaton matter will change and will become differ from the equation of state of the superrigid matter.

8. Discussion

We come to the conclusion that the matter with dilaton charge can play the essential role in the whole dynamic of the Universe. The existence of such matter at the beginning of the Universe leads, if the condition (7.6) is fulfilled, to the absence of the initial singularity in the cosmological solution of the gravitational theory in a Weyl–Cartan spacetime and to the existence of the maximum value of the matter density of the Universe, in contrast to the opinion of [13]. In [12] the nonsingular solution of the cosmological problem was established in the metric–affine theory of gravitation with quadratic Lagrangians. In the theory under consideration the nonsingular solution appears not as a consequence of the quadratic Lagrangians by itself, but because of the existence of the dilaton matter at the early stage of the Universe.

The condition (7.6) means that $\alpha > 0$ and the quantity $|\lambda m^2|$ is very large by magnitude. From relation (5.17) it follows that $m^2$ is determined by the quantities $\xi/\lambda$ and $\zeta^2/\lambda$ (if $\rho_i \approx 1$, $i = 1, 2, 3$). Therefore the mass $m$ of a quantum of the dilaton field and at least one of the coupling constants, $\xi$ or $\zeta$, has to be very large. As $\alpha > 0$,
then we see from (6.8) that it is $\zeta$. From (5.10) it follows that the large value of $\zeta$ will intensify to a great extent the value of a torsion field, the coupling constant of which with matter is very small in the absence of the dilaton field. In this case torsion can have any influence on Particle Physics. In other words, the great dilaton field mass specifies the extent of the influence of the torsion field on the matter dynamics that finally determines the minimal radius of the Universe, the initial point of the Friedmann era and maybe some properties of elementary particles behavior.

The condition (7.6) also guarantees the existence of two points of inflection of the scale factor function $a(t)$, between which the Friedmann era of evolution of the Universe settles. The first point of inflection $a_1$ corresponds to the early stage of the Universe. The second point of inflection $a_2$ (the end of the Friedmann era and the beginning of the "second inflation" era) is determined by the condition (7.14). This condition on the basis of the observational data yields the viable range of the possible forms of the dilaton dark matter equation of state, $p = \gamma \epsilon$, from the cold dark matter (with $\gamma = \frac{2}{3}$) up to the superrigid matter (with $\gamma = 1$). Neither the relativistic matter ($\gamma = \frac{4}{3}$) nor the dust matter ($\gamma = 0$) does not be allowed to belong to this range. Therefore the theory predicts the dark matter to be cold and self-interacting that coincides with the astrophysical data [24].

It should be emphasized that when the equation of state of the dilaton dark matter is superrigid ($\gamma = 1$) at the very early stage of the Universe then the inflation-like stage of the evolution of the Universe occurs until the equation of state of the dilaton matter will change.

The hypothesis on the dilaton matter as the dark self-interacting (by means of the dilaton charge) matter explains, why the dark matter is detected only as a consequence of gravitational effects and cannot be discovered by means of the nongravitational interaction with particles for which the dilaton charge $J = 0$. This hypothesis also leads to the conclusion that the expansion with acceleration begins when the dark matter energy density becomes equal by order of magnitude to the vacuum energy density. This is the answer on the Steinhardt’s question: "why is the quintessence dominating after 15 billion year and not, say, 1.5 billion years or 150 billion years" [3].

Appendix A

Let us consider a connected 4-dimensional oriented differentiable manifold $\mathcal{M}$ equipped with a metric $\tilde{g}$ of index 1, a linear connection $\Gamma$ and a volume 4-form $\eta$. Then a Weyl–Cartan space $CW_4$ is defined as the space equipped with a curvature 2-form $\mathcal{R}^\alpha_\beta$ and a torsion 2-form $\mathcal{T}^\alpha$ with the metric tensor and the connection 1-form obeying the condition

\[-Dg_{\alpha\beta} = \mathcal{Q}_{\alpha\beta} = \frac{1}{4}g_{\alpha\beta} \mathcal{Q}, \quad \mathcal{Q} := g^{\alpha\beta} \mathcal{Q}_{\alpha\beta} = Q_\alpha g^\alpha, \quad (A.1)\]

Our notations differ by some details from the notations accepted in [24].
where $Q_{\alpha\beta}$ - a nonmetricity 1-form, $Q$ - a Weyl 1-form and $D := d + \Gamma \wedge \ldots$ - the exterior covariant differential. Here $\theta^\alpha (\alpha = 1, 2, 3, 4)$ - cobasis of 1-forms of the $CW_4$-space ($\wedge$ - the exterior product operator).

A curvature 2-form $R^\alpha_{\beta\gamma}$ and a torsion 2-form $T^\alpha$, 

$$R^\alpha_{\beta\gamma} = \frac{1}{2} R^\alpha_{\beta\gamma\lambda} \theta^\gamma \wedge \theta^\lambda, \quad T^\alpha = \frac{1}{2} T^\alpha_{\beta\gamma} \theta^\beta \wedge \theta^\gamma, \quad T^\alpha_{\beta\gamma} = -2 \Gamma^\alpha_{[\beta\gamma]}.$$  \hspace{1cm} (A.2)

are defined by virtue of the Cartan’s structure equations,

$$R^\alpha_{\beta} = d\Gamma^\alpha_{\beta} + \Gamma^\alpha_{\gamma} \wedge \Gamma^\gamma_{\beta}, \quad T^\alpha = D\theta^\alpha = d\theta^\alpha + \Gamma^\alpha_{\beta} \wedge \theta^\beta.$$  \hspace{1cm} (A.3)

The Bianchi identities for the curvature 2-form, the torsion 2-form and the Weyl 1-form are valid \[20\],

$$D R^\alpha_{\beta} = 0, \quad D T^\alpha = R^\alpha_{\beta} \wedge \theta^\beta, \quad D Q = 2 R^\gamma_{\gamma}.$$  \hspace{1cm} (A.4)

It is convenient to use the auxiliary fields of 3-forms $\eta_{\alpha}$, 2-forms $\eta_{\alpha\beta}$, 1-forms $\eta_{\alpha\beta\gamma}$ and 0-forms $\eta_{\alpha\beta\gamma\lambda}$ with the properties,

$$\eta_{\alpha} = \theta^\alpha | \eta = *\theta_{\alpha}, \quad \eta_{\alpha\beta\gamma} = \theta^\gamma | \eta_{\alpha\beta} = *(\theta_{\alpha} \wedge \theta_{\beta} \wedge \theta_{\gamma}), \quad \eta_{\alpha\beta} = \theta^\beta | \eta_{\alpha} = *(\theta_{\alpha} \wedge \theta_{\beta}), \quad \eta_{\alpha\beta\gamma\lambda} = \theta^\lambda | \eta_{\alpha\beta\gamma} = *(\theta_{\alpha} \wedge \theta_{\beta} \wedge \theta_{\gamma} \wedge \theta_{\lambda}).$$  \hspace{1cm} (A.5)

Here $*$ is the Hodge operator and $| \eta$ is the operation of contraction (interior product) which obeys to the Leibnitz antidifferentiation rule,

$$\vec{v} | (\Phi \wedge \Psi) = (\vec{v} | \Phi) \wedge \Psi + (-1)^{p} \Phi \wedge (\vec{v} | \Psi), \quad (A.7)$$

where $\Phi$ is a $p$-form.

The properties (A.3), (A.6) lead to the following useful relations,

$$\theta^\sigma \wedge \eta_{\alpha} = \delta^\sigma_{\alpha} \eta, \quad \theta^\sigma \wedge \eta_{\alpha_1 \ldots \alpha_p} = (-1)^{p-1} p \delta^\sigma_{[\alpha_1} \eta_{\alpha_2 \ldots \alpha_p]}, \quad (A.8)$$

$$\theta^\sigma \wedge \theta^\rho \wedge \eta_{\alpha\beta} = 2 \delta^\sigma_{[\alpha} \delta^\rho_{\beta]} \eta, \quad (A.9)$$

$$\theta^\sigma \wedge \theta^\rho \wedge \eta_{\alpha\beta\gamma} = 3 \theta^\sigma \wedge \delta^\rho_{[\alpha} \eta_{\beta\gamma]} + 6 \delta^\rho_{[\alpha} \delta^\sigma_{\beta]} \eta_{\gamma]}, \quad (A.10)$$

$$\theta_{[\alpha} \wedge *(\theta_{\beta]} \wedge \theta_{\gamma]) = -\frac{1}{2} \eta_{\alpha\beta\gamma}, \quad (A.11)$$

$$\vec{e}_{\lambda} \theta_{[\alpha_1} \wedge \ldots \wedge \theta_{\alpha_p]} = p \delta^\alpha_{[\alpha_1} \eta_{\alpha_2 \ldots \alpha_p]}.$$  \hspace{1cm} (A.12)

In space $CW_4$ the equality $D\eta = 0$ is fulfilled and the following formulae are valid \[20\],

$$D \eta_{\alpha\beta\gamma\lambda} = -\frac{1}{2} Q \eta_{\alpha\beta\gamma\lambda}, \quad D \eta_{\alpha\beta\gamma} = -\frac{1}{2} Q \wedge \eta_{\alpha\beta\gamma} + T^\lambda \eta_{\alpha\beta\gamma\lambda}, \quad (A.13)$$

$$D \eta_{\alpha\beta} = -\frac{1}{2} Q \wedge \eta_{\alpha\beta} + T^\gamma \wedge \eta_{\alpha\beta\gamma}, \quad D \eta_{\alpha} = -\frac{1}{2} Q \wedge \eta_{\alpha} + T^\beta \wedge \eta_{\alpha\beta}, \quad (A.14)$$

Appendix B

The variational procedure in the exterior form language is based on the master formula derived the following Lemma, proved in \[13\] (see also \[23\]).
Lemma. Let $\Phi$ and $\Psi$ be arbitrary $p$-forms defined on an $n$-dimensional manifold. Then the variational identity for the commutator of the variation operator $\delta$ and the Hodge star operator $\ast$ is valid,

$$
\Phi \wedge \delta * \Psi = \delta \Psi \wedge \ast \Phi
$$

$$
+ \delta g_{\sigma \rho} \left( \frac{1}{2} g^{\sigma \rho} \Phi \wedge \ast \Psi + \left(-1\right)^{p(n-1)+s+1} \theta^\sigma \wedge (*) \Psi \wedge \theta^\rho \wedge \ast \Phi \right)
$$

$$
+ \delta \theta^\alpha \wedge \left( \left(-1\right)^{p} \Phi \wedge (*) (\Psi \wedge \theta_\alpha) + \left(-1\right)^{p(n-1)+s+1} (*) \Psi \wedge \theta_\alpha \wedge \ast \Phi \right).
$$

The variational procedure is realized with the help of the computation rules \[26\],

$$
* * \Psi = \left(-1\right)^{p(n-1)+s} \Psi, \quad \Phi \wedge * \Psi = \Psi \wedge \ast \Phi,
$$

$$
\varepsilon_\alpha \wedge * \Psi = \ast (\Psi \wedge \theta_\alpha), \quad \theta^\alpha \wedge (\varepsilon_\alpha \wedge \Psi) = p \Psi,
$$

where $\Psi$ and $\Phi$ are $p$-forms and $s = \text{Ind} (\tilde{g})$ is the index of the metric $\tilde{g}$, which is equal to the number of negative eigenvalues of the diagonalized metric. The relations (B.2) and (B.3) lead to the consequences,

$$
\varepsilon_\alpha \wedge \Psi = \left(-1\right)^{n(p-1)+s} \left( \theta_\alpha \wedge * \Psi \right),
$$

$$
* (\varepsilon_\alpha \wedge \Psi) = \left(-1\right)^{p-1} \theta_\alpha \wedge * \Psi,
$$

$$
* (\varepsilon_\alpha \wedge * \Psi) = \left(-1\right)^{(n-1)(p+1)+s} \Psi \wedge \theta_\alpha.
$$

Let apply the master formula (B.1) to the variation of the Lagrangian density 4-form (4.2), the general relations (4.3)–(4.7) and the computation rules (A.8)–(A.14), (B.2)–(B.3) being used. The results of the variational procedure for every term of (4.2) have the following form, the exact forms being omitted,

$$
2f_0 : \delta \Gamma^\alpha_\beta \wedge \left( \frac{1}{4} \Theta \wedge \eta^\beta + \frac{1}{2} \cal{T}_\lambda \wedge \eta^\beta_\lambda + \frac{1}{2} \eta^\beta_\gamma \wedge \Theta^\gamma \right)
$$

$$
+ \delta g_{\sigma \rho} \left( \frac{1}{2} g^{\sigma \rho} \cal{R}^\alpha_\beta \wedge \eta^\beta + \frac{1}{2} \eta^\beta \wedge \theta^\beta \wedge \ast \cal{R}^\beta \rho \right)
$$

$$
+ \delta \theta^\sigma \wedge \left( \frac{1}{2} \cal{R}^\alpha_\beta \wedge \eta^\beta_\sigma \right),
$$

$$
2f_0 \varrho_1 : \delta \Gamma^\alpha_\beta \wedge 2 \theta^\beta \wedge \ast \varrho_1
$$

$$
+ \delta g_{\sigma \rho} \left( \cal{T}^\sigma \wedge \ast \cal{T}^\rho + \frac{1}{2} g^{\sigma \rho} \varrho_1 \wedge \ast \cal{T}^\sigma \wedge \theta^\sigma \wedge \ast (\ast \cal{T}^\alpha \wedge \theta^\rho) \wedge \ast \varrho_1 \right)
$$

$$
+ \delta \theta^\sigma \wedge \left( 2 \cal{D} \ast \varrho_1 + \cal{T}^\alpha \wedge \ast (\varrho_1 \wedge \theta_\sigma) + \ast (\ast \varrho_1 \wedge \theta_\sigma) \wedge \ast \varrho_1 \right),
$$

$$
2f_0 \varrho_2 : \delta \Gamma^\alpha_\beta \wedge 2 \theta^\beta \wedge \theta_\gamma \wedge \ast (\varrho_1 \wedge \theta_\alpha)
$$

$$
+ \delta g_{\sigma \rho} \left( \frac{1}{2} g^{\sigma \rho} \left( \varrho_1 \wedge \theta^\beta \wedge \theta_\gamma \right)
$$

$$
+ 2 \delta_\beta \varrho_1 \wedge \theta^\sigma - \theta^\sigma \wedge \ast (\varrho_1 \wedge \theta_\beta \wedge \theta^\sigma) \wedge \ast \varrho_1 \wedge \theta_\alpha
$$

$$
+ \delta \theta^\sigma \wedge \left( 2 \varrho_1 \left( \theta_\alpha \wedge \ast (\varrho_1 \wedge \theta_\sigma) \right) - \ast (\varrho_1 \wedge \theta_\alpha \wedge \theta_\sigma) \wedge \ast \varrho_1 \wedge \theta_\beta \wedge \theta_\alpha \right)
$$

$$
+ 2 \varrho_1 \wedge \ast (\varrho_1 \wedge \theta_\beta \wedge \theta_\alpha) - \ast \varrho_1 \wedge \ast (\ast \varrho_1 \wedge \theta_\alpha \wedge \theta_\beta \wedge \theta_\alpha)
$$

$$
+ 2 \varrho_1 \wedge \ast (\varrho_1 \wedge \theta_\alpha \wedge \theta_\beta \wedge \theta_\alpha)
$$

$$
+ 2 \varrho_1 \wedge \theta^\sigma - \theta^\sigma \wedge \ast (\varrho_1 \wedge \theta_\beta \wedge \theta^\sigma) \wedge \ast (\varrho_1 \wedge \theta_\alpha \wedge \theta^\sigma)
$$

$$
+ 2 \varrho_1 \wedge \ast (\ast \varrho_1 \wedge \theta_\alpha \wedge \theta_\beta \wedge \theta_\alpha)
$$

$$
+ 2 \varrho_1 \wedge \theta^\sigma - \theta^\sigma \wedge \ast (\ast \varrho_1 \wedge \theta_\alpha \wedge \theta_\beta \wedge \theta_\alpha)
$$

$$
+ 2 \varrho_1 \wedge \ast (\varrho_1 \wedge \theta_\alpha \wedge \theta_\beta \wedge \theta_\alpha)
$$

$$
+ 2 \varrho_1 \wedge \theta^\sigma - \theta^\sigma \wedge \ast (\ast \varrho_1 \wedge \theta_\alpha \wedge \theta_\beta \wedge \theta_\alpha)
$$

$$
+ 2 \varrho_1 \wedge \ast (\varrho_1 \wedge \theta_\alpha \wedge \theta_\beta \wedge \theta_\alpha)
$$

$$
+ 2 \varrho_1 \wedge \theta^\sigma - \theta^\sigma \wedge \ast (\ast \varrho_1 \wedge \theta_\alpha \wedge \theta_\beta \wedge \theta_\alpha)
$$

$$
+ 2 \varrho_1 \wedge \ast (\varrho_1 \wedge \theta_\alpha \wedge \theta_\beta \wedge \theta_\alpha)
$$

$$
+ 2 \varrho_1 \wedge \theta^\sigma - \theta^\sigma \wedge \ast (\ast \varrho_1 \wedge \theta_\alpha \wedge \theta_\beta \wedge \theta_\alpha)
$$

$$
+ 2 \varrho_1 \wedge \ast (\varrho_1 \wedge \theta_\alpha \wedge \theta_\beta \wedge \theta_\alpha)
$$
\[
+ \delta \theta^\alpha \land \left( 2D \left( \theta_\sigma \land * (T^\alpha \land \theta_\alpha) \right) - * (T^\beta \land \theta_\beta \land \theta_\sigma) T^\alpha \land \theta_\alpha \right)
+ 2T_\sigma \land * (T^\alpha \land \theta_\alpha) - * \left( * (T^\beta \land \theta_\beta) \land \theta^\alpha \right) \land * (T^\alpha \land \theta_\alpha) \right), \tag{B.10}
\]

\[
2f_0 \lambda : \delta \Gamma^\alpha_\beta \land \left( \frac{1}{2} \delta^\alpha_\alpha D \ast R^\gamma_\gamma \right) + \delta g_{\sigma \rho} \left( \frac{1}{8} \sigma^\rho \rho^\alpha \land \ast R^\beta_\beta + \frac{1}{4} \theta^\alpha \land * (\ast R^\alpha_\alpha \land \theta^\rho) \land \ast R^\beta_\beta \right)
+ \delta \theta^\alpha \land \left( \frac{1}{4} R^\beta_\beta \land * (R^\alpha_\alpha \land \theta_\sigma) + \frac{1}{4} \ast (R^\alpha_\alpha \land \theta_\sigma) \land \ast R^\beta_\beta \right) \tag{B.11}
\]

\[
2f_0 \xi : \delta \Gamma^\alpha_\beta \land \left( 4 \delta^\beta_\alpha \ast Q \right)
+ \delta g_{\sigma \rho} \left( 2 g^\sigma_\rho D \ast Q + \frac{1}{2} \theta^\alpha Q \land \ast Q - \ast (Q \land \theta^\rho) \land \ast Q \right)
+ \delta \theta^\alpha \land \left( - Q \land \ast (Q \land \theta_\sigma) - \ast (Q \land \theta_\sigma) \land \ast Q \right) \tag{B.12}
\]

\[
2f_0 \zeta : \delta \Gamma^\alpha_\beta \land \left( 2 \delta^\beta_\alpha \ast T_\gamma + \theta^\alpha \land \ast (Q \land \theta_\sigma) \right)
+ \delta g_{\sigma \rho} \left( g^\sigma_\rho T^\alpha \land \ast T_\alpha - g^\sigma_\rho \theta^\alpha \land D \ast T_\alpha + \frac{1}{2} g^\sigma_\rho Q \land \theta^\alpha \land \ast T_\alpha \right)
+ \theta^\alpha \land \ast (T_\alpha \land \theta_\sigma) \land \ast (Q \land \theta^\alpha) + T^\alpha \land \ast (Q \land \theta^\alpha) \right)
+ \delta \theta^\alpha \land \left( D \ast (Q \land \theta_\sigma) + Q \land \theta^\alpha \land \ast (T_\alpha \land \theta_\sigma) \right)
+ \ast (T_\alpha \land \theta_\sigma) \land \ast (Q \land \theta^\alpha) - Q \land \ast T_\sigma \right), \tag{B.13}
\]

\[
2f_0 \Lambda : \delta g_{\sigma \rho} \left( \frac{1}{2} g^\rho_\rho \eta \right) + \delta \theta^\sigma \land \eta_\sigma \tag{B.14}
\]

The variation of the term with the Lagrange multiplier in (B.12) has the form,
\[
\delta \Lambda^{\alpha \beta} \land \left( Q_{\alpha \beta} - \frac{1}{4} g_{\alpha \beta} Q \right) + \delta \Gamma^\alpha_\beta \land \left( -2 \Lambda^\beta_\gamma \right) + \delta g_{\sigma \rho} \left( - D \Lambda^\sigma_\rho - \frac{1}{4} \Lambda^\sigma_\rho \land Q \right). \tag{B.15}
\]

Appendix C

In a Weyl–Cartan space the following decomposition of the connection 1-form is valid,
\[
\Gamma^\alpha_\beta = \tilde{\Gamma}^\alpha_\beta + \Delta^\alpha_\beta, \quad \Delta^\alpha_\beta = \frac{1}{8} \left( 2 \theta_{[\alpha} Q_{\beta]} + g_{\alpha \beta} Q \right), \tag{C.1}
\]

where $\tilde{\Gamma}^\alpha_\beta$ denotes a connection 1-form of a Riemann–Cartan space $U_4$ with curvature, torsion and a metric compatible with a connection. This decomposition of the connection induces corresponding decomposition of the curvature 2-form [4],
\[
R^\alpha_\beta = \tilde{R}^\alpha_\beta + \tilde{D} \Delta^\alpha_\beta + \Delta^\alpha_\gamma \land \Delta^\beta_\gamma = \tilde{R}^\alpha_\beta + \frac{1}{4} \delta^\alpha_\beta R^\gamma_\gamma + P^\alpha_\beta, \tag{C.2}
\]

\[
P_{\alpha \beta} = \frac{1}{4} \left( T_{[\alpha} Q_{\beta]} - \theta_{[\alpha} \land \tilde{C} Q_{\beta]} + \frac{1}{8} \theta_{[\alpha} Q_{\beta]} \land Q - \frac{1}{16} \theta_{\alpha} \land \theta_{\beta} Q \land Q^\gamma \right), \tag{C.3}
\]

where $\tilde{D}$ is the exterior covariant differential with respect to the Riemann–Cartan connection 1-form $\tilde{\Gamma}^\alpha_\beta$ and $\tilde{R}^\alpha_\beta$ is the Riemann–Cartan curvature 2-form. The decomposition (C.2) contains the Weyl segmental curvature 2-form $\tilde{R}^\gamma_\gamma$ [4].
The Riemann–Cartan connection 1-form can be decomposed as follows,\(^\text{[20]}\),

\[
\Gamma^\alpha_{\beta\gamma} = R^\alpha_{\beta\gamma} + K^\alpha_{\beta\gamma}, \quad T^\alpha = K^\alpha_{\beta\gamma} \wedge \theta^\beta, \quad (C.4)
\]

\[
K_{\alpha\beta} = 2\tilde{e}_{[\alpha} \tau_{\beta]} - \frac{1}{2} \tilde{e}_{\alpha} \tilde{e}_{\beta} (\tau_{\gamma} \wedge \theta^\gamma), \quad (C.5)
\]

where \(R^\alpha_{\beta\gamma}\) is a Riemann (Levi–Civita) connection 1-form and \(K^\alpha_{\beta\gamma}\) is a contortion 1-form.

The decomposition \((C.4)\) of the connection induces the decomposition of the curvature as follows,

\[
\mathcal{R}^\alpha_{\beta\gamma} = R^\alpha_{\beta\gamma} + \mathcal{D} K^\alpha_{\beta\gamma} + K^\alpha_{\beta\gamma} \wedge K^\gamma_{\beta\gamma}, \quad (C.6)
\]

where \(\mathcal{R}^\alpha_{\beta\gamma}\) is the Riemann curvature 2-form and \(\mathcal{D}\) is the exterior covariant differential with respect to the Riemann connection 1-form \(R^\alpha_{\beta\gamma}\).

Let us substitute the decomposition \((C.3)\) into the equation \((1.9)\) and after this use the decomposition \((C.9)\), the relation \((6.4)\) being taken into account. We get the following results for the every term of the equation \((1.9)\). The linear term or this equation decomposes as follows,

\[
-\left( \frac{R^\alpha_{\sigma}}{2} - \frac{1}{2} \delta^\alpha_{\sigma} \frac{R}{R} \right) \eta_\alpha + \frac{2}{3} \left( \frac{R}{R} T^\alpha \right) \eta_\sigma - \frac{2}{3} \left( \frac{R}{R} T^\alpha \right) \eta_\alpha - \frac{1}{9} T_\alpha T^\alpha \eta_\sigma - \frac{2}{9} T_\alpha T^\alpha \eta_\sigma
\]

\[
\quad + \frac{1}{12} T_\alpha Q^\alpha \eta_\sigma + 1 \frac{1}{12} T_\sigma Q^\alpha \eta_\alpha + \frac{1}{12} T_\alpha Q^\alpha \eta_\sigma
\]

\[
\quad + 1 \frac{1}{4} \left( \frac{R}{R} Q^\alpha \right) \eta_\sigma - 1 \frac{1}{4} \left( \frac{R}{R} Q^\alpha \right) \eta_\sigma - \frac{1}{64} Q_\alpha Q^\alpha \eta_\sigma - \frac{1}{32} Q_\alpha Q^\alpha \eta_\alpha. \quad (C.7)
\]

The other terms decompose as follows,

\[
\varrho_1: \quad \frac{2}{3} \left( \frac{R}{R} T^\alpha \right) \eta_\alpha - \frac{2}{3} \left( \frac{R}{R} T^\alpha \right) \eta_\sigma + \frac{2}{9} T_\sigma T^\alpha \eta_\sigma + \frac{1}{9} T_\alpha T^\alpha \eta_\sigma
\]

\[
\quad - \frac{1}{4} Q_\alpha T^\alpha \eta_\sigma - \frac{1}{12} T_\sigma Q^\alpha \eta_\alpha + \frac{1}{12} Q_\alpha T^\alpha \eta_\sigma, \quad (C.8)
\]

\[
\varrho_2: \quad - \frac{4}{3} \left( \frac{R}{R} T^\alpha \right) \eta_\alpha + \frac{4}{3} \left( \frac{R}{R} T^\alpha \right) \eta_\sigma - \frac{4}{9} T_\sigma T^\alpha \eta_\sigma - \frac{2}{9} T_\alpha T^\alpha \eta_\sigma
\]

\[
\quad + \frac{1}{2} Q_\sigma T^\sigma \eta_\alpha + \frac{1}{6} T_\sigma Q^\alpha \eta_\alpha - \frac{1}{6} Q_\alpha T^\sigma \eta_\sigma, \quad (C.9)
\]

\[
\zeta: \quad - \left( \frac{R}{R} Q^\alpha \right) \eta_\sigma + \left( \frac{R}{R} Q^\alpha \right) \eta_\alpha + \frac{2}{3} T_\alpha Q^\alpha \eta_\sigma
\]

\[
\quad - \frac{2}{3} Q_\alpha T^\alpha \eta_\sigma + \frac{1}{8} Q_\alpha Q^\alpha \eta_\sigma - \frac{1}{2} Q_\sigma Q^\alpha \eta_\alpha, \quad (C.10)
\]

\[
\xi: \quad Q_\alpha Q^\alpha \eta_\sigma - 2 Q_\sigma Q^\alpha \eta_\alpha. \quad (C.11)
\]

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