Waves in lattices with imperfect junctions and localized defect modes

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A correspondence between continuum periodic structures and discrete lattices is well known in the theory of elasticity. Frequently, lattice models are the result of the discretization of continuous mechanical problems. In this paper, we discuss the discretization of two-dimensional square thin-walled structures. We consider the case when thin-walled bridges have defects in the vicinity of junctions. At these points, the displacement satisfies an effective Robin-type boundary condition. We study a defect vibration mode localized in the neighbourhood of the damaged junction. We analyse dispersion diagrams that show the existence of standing waves in a structure with periodically distributed defects.

Keywords: inhomogeneous lattices with defects; band-gap diagram; Green’s function; vibration defect modes

1. Introduction

The analysis of localized vibration modes within continuous and discrete lattice structures has a long history, starting with Maradudin et al. (1963). The application of defect vibration modes was found initially in solid-state physics (Maradudin 1965; Mead 1973, 1996; Mead & Parthan 1979; Mead & Yaman 1991) and many other publications. In the two-component chains, defect modes were discussed in Bacon et al. (1962). In the scattering of waves in solids, this subject was studied by Callaway (1964). There are more examples of localized vibration modes in theoretical and experimental physics, especially in photonic crystal fibres (John 1987; Yablonovitch 1987, 1993). One aspect of the technological applications involves models of acoustics and elasticity with photonic crystal models based on ‘mass–spring’ interaction lattices (Jensen 2003; Martinson & Movchan 2003; Movchan & Slepyan 2007). The passband Green functions for two-dimensional lattices were discussed in the paper by Martin (2006).

It is well known that lattice structures possess phononic band gaps, i.e. intervals of frequencies for which there are no mechanical waves that could propagate through the structure. Neglecting the inertia of the thin-walled bridges in comparison with that of the masses at the junctions, we perform a discretization of the structure. Hence, we obtain a lattice with particles connected by springs. If the initial periodic structure has defects at the junctions...
such as cracks, at these points, the principal term of the displacement satisfies an effective Robin-type boundary condition characterized by a defect stiffness parameter (Zalipaev et al. 2007).

In the present paper, we study localized vibration modes for mechanical lattice structures of two types: a one-dimensional diatomic lattice and a two-dimensional square lattice. The defects, periodically distributed in the lattice, change the nature of the band gaps. For these lattices, we analyse dispersion diagrams and explore possible localized standing waves. Furthermore, we explore the existence of a localized vibration mode for a one-dimensional lattice with a single defect. We study how the behaviour of this mode depends on the defect stiffness parameter.

In §2, we consider the transition from two-dimensional wave propagation through the square periodic structure shown in figure 1a. The axis $x_3$ is orthogonal to the plane of the figure with in-plane axes $x_1$ and $x_2$. The displacement inside the walls of the structure $(0, 0, u(x_1, x_2))$ satisfies the wave equation

$$
\mu \Delta u(x) + \rho \omega^2 u(x) = 0,
$$

where $\mu$ is the shear modulus and $\rho$ is the density. The solution must satisfy the quasi-periodicity condition

$$
u(x_1 + n_1 L, x_2 + n_2 L) = \exp(iL(k_1 n_1 + k_2 n_2))u(x_1, x_2),
$$

where $L$ is the cell dimension; $k = (k_1, k_2) \in D_L$ is the Bloch vector (quasi-momentum free parameter); $n_{1,2} \in \mathbb{Z}$; and the domain $D_L$ is the square

$$
D_L = \left[-\frac{\pi}{L}, \frac{\pi}{L}\right] \times \left[-\frac{\pi}{L}, \frac{\pi}{L}\right].
$$

Let us assume that the periodic square structure is composed of two types of regions: the shaded regions in figure 1a having non-zero inertia, connected by thin-walled elements of negligible inertia.

We use a small parameter $\epsilon$, which is the ratio of the width to the length of a thin-walled bridge connecting the shaded regions in figure 1a. Hence, we obtain the following problem for a thin rectangle $\Pi = \{(x_1, x_2) : 0 < x_1 < L, |x_2| < \epsilon L\}$.

2. Formulation of the problem

Consider a two-dimensional wave propagation through the square periodic structure shown in figure 1a. The axis $x_3$ is orthogonal to the plane of the figure with in-plane axes $x_1$ and $x_2$. The displacement inside the walls of the structure $(0, 0, u(x_1, x_2))$ satisfies the wave equation

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where $\mu$ is the shear modulus and $\rho$ is the density. The solution must satisfy the quasi-periodicity condition

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u(x_1 + n_1 L, x_2 + n_2 L) = \exp(iL(k_1 n_1 + k_2 n_2))u(x_1, x_2),
$$

where $L$ is the cell dimension; $k = (k_1, k_2) \in D_L$ is the Bloch vector (quasi-momentum free parameter); $n_{1,2} \in \mathbb{Z}$; and the domain $D_L$ is the square

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representing a thin-walled bridge in the undamaged structure:
\[
\Delta u(x) = 0, \quad \text{(2.4)}
\]
\[
\frac{\partial u}{\partial x_2} \bigg|_{x_2=\pm \epsilon} = 0, \quad \text{(2.5)}
\]
\[
u(0, x_2) = q_1(x_2/(\epsilon L)), \quad u(1, x_2) = q_2(x_2/(\epsilon L)), \quad |x_2| < \epsilon L, \quad \text{(2.6)}
\]
where \(q_1\) and \(q_2\) are displacements on the interface with the junction region. In the case of a damaged junction, as in figure 1b, the first boundary condition in (2.6) has to be replaced by
\[
\frac{\partial u}{\partial x_1} \bigg|_{x_1=0} = 0, \quad \rho_\epsilon < |x_2| < \epsilon L, \quad u(0, x_2) = q_1(x_2/(\epsilon L)), \quad |x_2| < \rho_\epsilon, \quad \text{(2.7)}
\]
where \(\rho_\epsilon\) is the length of the intact part of the junction.

\[(a)\] Reduction to a lower dimensional model

To the leading order, the solution of (2.4)–(2.6) is given by
\[
u = u_0(x_1) + \epsilon^2 U(x_1, Y_2) + W(Y_1, Y_2) + O(\epsilon^4),
\]
where \(Y_1=x_1/(\epsilon L)\) and \(Y_2=x_2/(\epsilon L)\) are the scaled variables and the principal term satisfies
\[
u_0(0) = \tilde{q}_1 = \frac{1}{2} \int_{-1}^{1} q_1(Y_2) \, dY_2, \quad \nu_0(1) = \tilde{q}_2 = \frac{1}{2} \int_{-1}^{1} q_2(Y_2) \, dY_2. \quad \text{(2.8)}
\]
The boundary-layer term \(W\) decays exponentially away from the junctions. The term \(U\) satisfies the following problem:
\[
\frac{\partial^2}{\partial Y_2^2} U = -\frac{\partial^2 u_0}{\partial x_1^2}, \quad |Y_2| < 1, \quad \frac{\partial U}{\partial Y_2} \bigg|_{Y_2=\pm 1} = 0. \quad \text{(2.9)}
\]
The solvability condition for the problem (2.9)
\[
\frac{\partial^2 u_0}{\partial x_1^2} = 0,
\]
together with (2.8) gives a linear function for the solution \( u_0 \).

Thus, in this case, the principal term of the asymptotic approximation of the displacement within the thin-walled bridge is a linear function of the longitudinal coordinate. Hence, the thin-walled bridges in the periodic structure behave like springs connecting masses \( m \) located at the junction points. Thus, taking into account the approximations, we perform a discretization of the initial square periodic structure, and arrive at the equations of motion for a mass \( m \) in the two-dimensional undefected lattice. The displacement \( u_{i,j} \) of a mass located at the junction point with indices \((i, j)\) satisfies the equation
\[
-\omega^2 m u_{i,j} = \frac{f}{L} (u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}),
\]
where \( f \) is the magnitude of the tension force and \( L \) is the length of the thin-walled bridge. In addition, we must add the quasi-periodicity conditions for the displacement \( u_{i,j} \)
\[
u_{i+n_1,j+n_2} = \exp(iL(k_1n_1 + k_2n_2))u_{i,j},
\]
where \( k = (k_1, k_2) \in D_\epsilon \) is the Bloch vector.

If the initial square periodic structure has defects at the junction points similar to cuts (figure 1b), the system of the equations of motion for the two-dimensional lattice, given by (2.10), must be modified. In this case, we have to take into consideration an effective boundary condition between the thin-walled bridge and the junction domain. The asymptotic solution to the problem was obtained by Zalipaev et al. (2007), and a summary is given in §2b.

(b) Effective boundary conditions for a defected junction

Let \( L=1 \) and consider a thin rectangular domain \( \Omega_\epsilon = \{ x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, |x_2| < \epsilon \} \) in dimensionless coordinates representing a thin-walled bridge (the width of the domain is much less than its length), where \( \epsilon > 0 \) is a small non-dimensional parameter (figure 2). At the left-hand end of the domain, the displacement is specified on a smaller part of the boundary, whose length is \( 2\rho_\epsilon \). We denote the ratio of the two small parameters by \( C_\epsilon = \rho_\epsilon / \epsilon \). It is assumed that \( \rho_\epsilon \ll \epsilon \). The right-hand end of the domain \( \Omega_\epsilon \) is fixed. The remaining parts of the boundary (the upper and lower sides of \( \Omega_\epsilon \)) are traction free. We assume that the domain is occupied by an elastic isotropic homogeneous material with the shear modulus \( \mu \).
The displacement field has the form \((0, 0, u_\epsilon(x_1, x_2))\), where the function \(u_\epsilon\) satisfies the following mixed boundary-value problem:

\[
\Delta u_\epsilon(x) = 0, \quad x \in \Omega_\epsilon, \quad (2.12)
\]

\[
\partial_{x_2} u_\epsilon(x_1, \pm \epsilon) = 0, \quad 0 < x_1 < 1, \quad (2.13)
\]

\[
u_\epsilon(0, x_2) = q, \quad |x_2| < \rho_\epsilon, \quad (2.14)
\]

\[
\partial_{x_1} u_\epsilon(0, x_2) = 0, \quad \rho_\epsilon < |x_2| < \epsilon, \quad (2.15)
\]

\[
u_\epsilon(1, x_2) = Q, \quad |x_2| < \epsilon, \quad (2.16)
\]

where \(q\) and \(Q\) are assumed to be constant displacements.

The leading term \(U(x)\) of the asymptotic expansion for the solution of the problem (2.12)–(2.16) satisfies, at \(x_1 = 0\), one of the following boundary conditions (Zalipaev et al. 2007):

(i) \(U(0) = q\), (Dirichlet type) \(\quad (2.17)\)

if

\[
C_\epsilon > C \exp\left(-\frac{1}{\epsilon^\alpha}\right), \quad \alpha < 1,
\]

where \(C\) is a positive constant independent of \(\epsilon\).

(ii) \(U(0) - q + \frac{2}{\pi} U'(0) = 0\), (Robin type) \(\quad (2.18)\)

if

\[
C_\epsilon = C \exp\left(-\frac{1}{\epsilon}\right),
\]

(iii) \(U'(0) = 0\), (Neumann type) \(\quad (2.19)\)

if

\[
C_\epsilon < C \exp\left(-\frac{1}{\epsilon^\alpha}\right), \quad \alpha > 1.
\]

We are interested in case (ii) when \(\alpha = 1\). Now the effective junction conditions will be used in modelling defect modes within periodic structures.

3. Localization in a one-dimensional discrete system with imperfect junctions

(a) A periodic problem

Consider a one-dimensional lattice with a macro-cell consisting of two particles with masses \(m_1\) and \(m_2\) and with period \(2L\). Let the particles interact with their neighbours via thin strip connections. The displacements within thin ligaments could be modelled by a discrete one-dimensional harmonic oscillator equation (anti-plane shear case). This one-dimensional problem may be considered as a special case of wave propagation just in one direction for the two-dimensional...
problem described in (2.10) and (2.11). For a perfect connection between particles and strips, we have the following system of equations for the $n$th macro-cell for the case of a time-harmonic dependence (Movchan & Slepyan 2007):

$$-\omega^2 m_1 u_n^{(1)} = \frac{f}{L} \left( u_n^{(2)} + u_{n-1}^{(2)} - 2u_n^{(1)} \right),$$

$$-\omega^2 m_2 u_n^{(2)} = \frac{f}{L} \left( u_n^{(1)} + u_{n+1}^{(1)} - 2u_n^{(2)} \right),$$

where $u_n^{(1)}$ and $u_n^{(2)}$ are the displacements of the particles in the $n$th cell with masses $m_1$ and $m_2$, respectively; $\omega$ is the frequency of vibration of the particles; and $f$ is the magnitude of the tension force in the connectors.

A defect in the continuous structure is a crack within the junction region, and it is modelled in the lattice approximation by a spring. In the lattice model, imperfect junctions are considered as additional spring connectors (figure 2) with the stiffness parameter $c$ as in the effective boundary condition (2.18). Namely, it includes the Robin-type boundary condition modelling a spring connection well known in mechanics. Thus, the displacements $u_n^{(1)}$ and $u_n^{(2)}$ of the particles with masses $m_1$ and $m_2$, respectively, satisfy the following system of equations:

$$-\omega^2 m_1 a_n^{(1)} = c \left( a_n^{(1)} + a_n^{(2)} - 2u_n^{(1)} \right),$$

$$-\omega^2 m_2 a_n^{(2)} = \frac{f}{L} \left( a_n^{(2)} + a_{n+1}^{(1)} - 2u_n^{(2)} \right),$$

where equations (3.4) and (3.5) describe the balance of forces at the junction points $A_1$ and $A_2$ in figure 3, and $a_n^{(1)}$ and $a_n^{(2)}$ represent the displacements of the points $A_1$ and $A_2$, respectively. Moreover, for a periodic structure, we impose the quasi-periodicity conditions

$$u_{n+m}^{(1)} = \exp(2ikLm) u_n^{(1)}, \quad u_{n+m}^{(2)} = \exp(2ikLm) u_n^{(2)},$$

Figure 3. The geometry of a one-dimensional lattice.
where \( k \in [-\pi/2L, \pi/2L] \) is the quasi-momentum or Bloch parameter. Eliminating the displacements \( a_n^{(1)} \) and \( a_n^{(2)} \) in the system (3.2)–(3.5) allows the equations to be written as

\[
\begin{align*}
-m_1 \Omega u_n^{(1)} &= \Gamma \left( u_n^{(2)} + u_{n-1}^{(2)} - 2u_n^{(1)} \right), \\
-m_2 \Omega u_n^{(2)} &= \Gamma \left( u_n^{(1)} + u_{n+1}^{(1)} - 2u_n^{(2)} \right),
\end{align*}
\]

where \( \Omega = \omega^2 L/f \) and \( \Gamma = cL/(cL + f) \).

The system (3.6) is similar to the one analysed in Movchan & Slepyan (2007), where \( \Gamma = 1 \). The transfer from perfect connections to imperfect ones as in the system (3.6) at the junction points \( A_1 \) and \( A_2 \) is obtained by the replacement of \( f/L \) in (3.1) by the parameter \( \Gamma \).

We use the result of Movchan & Slepyan (2007) to write the dispersion equation as

\[
\Omega_{1,2}(k) = \Gamma \left( \frac{(1 + r) \pm \sqrt{(1 + r)^2 - 4r \sin^2(kL)}}{m_1} \right),
\]

where \( r = m_1/m_2 \). This demonstrates that the dispersion relationship depends on the ratio of the masses, referenced to a value of \( m_1 \) and the degree of imperfection in the junction condition through the parameter \( c \) and hence \( \Gamma \). Note that the dispersion relationship is separable with respect to \( r \) and \( \Gamma \). The effect of the imperfect junction condition on the dispersion diagram is shown in figure 4 for \( m_1 = 1, L = 1 \) and \( r = 0.5 \). Frequencies within the band gap satisfy

\[
\Omega_2 < \Omega < \Omega_1, \quad \Omega_1 = \Gamma \frac{2}{m_1}, \quad \Omega_2 = \Gamma \frac{2}{m_2}.
\]

From figure 4, it is clearly seen that the width of the band gap is increasing and occurring at a higher frequency, with increasing \( \Gamma \).

\( \) (b) **Localized mode for a single defect**

Consider a non-periodic one-dimensional structure with masses \( m_1 \) and \( m_2 \) connected by springs. Let us assume that a defect is placed in the central \( n = 0 \)

---

*Fig. 4. Dispersion surfaces as a function of the Bloch parameter and the degree of imperfection in the junction condition for \( m_1 = 1 \) and \( r = 0.5 \).*
Figure 5. The geometry of a one-dimensional lattice with a single defect in the central macro-cell. macro-cell of the lattice (figure 5). Then, the particle displacements $u_n^{(1)}$ and $u_n^{(2)}$ for the $n$th macro-cell satisfy the unperturbed system of equations (see §3a)

$$-\Omega m_1 u_n^{(1)} = u_n^{(2)} + u_{n-1}^{(2)} - 2u_n^{(1)},$$

$$-\Omega m_2 u_n^{(2)} = u_n^{(1)} + u_{n+1}^{(1)} - 2u_n^{(2)},$$

except for $u_0^{(1)}$, $u_0^{(2)}$ and $u_{-1}^{(2)}$, which satisfy the following equations of motion:

$$-\Omega m_1 u_0^{(1)} = \alpha \left( a_1 + a_2 - 2u_0^{(1)} \right),$$

$$-\Omega m_2 u_0^{(2)} = a_2 + u_1^{(1)} - 2u_0^{(2)},$$

$$-\Omega m_2 u_{-1}^{(2)} = a_1 + u_{-1}^{(1)} - 2u_{-1}^{(2)},$$

where $a_1$ and $a_2$ are the displacements at the defect junctions $A_1$ and $A_2$, respectively (figure 5), and

$$\alpha = c \frac{L}{f}.$$

The system (3.11) may be written as follows:

$$-\Omega m_1 u_0^{(1)} = u_0^{(2)} + u_{-1}^{(2)} - 2u_0^{(1)} + F_1 \left( u_0^{(1)}, u_0^{(2)}, u_{-1}^{(2)} \right),$$

$$-\Omega m_2 u_0^{(2)} = u_0^{(1)} + u_1^{(1)} - 2u_0^{(2)} + F_2 \left( u_0^{(1)}, u_0^{(2)}, u_{-1}^{(2)} \right),$$

$$-\Omega m_2 u_{-1}^{(2)} = u_0^{(1)} + u_{-1}^{(1)} - 2u_{-1}^{(2)} + F_3 \left( u_0^{(1)}, u_0^{(2)}, u_{-1}^{(2)} \right),$$

where

$$F_1 \left( u_0^{(1)}, u_0^{(2)}, u_{-1}^{(2)} \right) = \gamma \left( 2u_0^{(1)} - u_0^{(2)} - u_{-1}^{(2)} \right),$$

$$F_2 \left( u_0^{(1)}, u_0^{(2)}, u_{-1}^{(2)} \right) = \gamma \left( u_0^{(2)} - u_0^{(1)} \right),$$

$$F_3 \left( u_0^{(1)}, u_0^{(2)}, u_{-1}^{(2)} \right) = \gamma \left( u_{-1}^{(2)} - u_0^{(1)} \right),$$

$$\gamma = \frac{1}{1 + \alpha}.$$
Using the Green function for the unperturbed system of equations (3.9) and (3.10) (see the case of \( I=1 \) in Movchan & Slepyan (2007) and the case of arbitrary \( I \) in appendix A of the present paper) for the stopband

\[ \Omega_1 < \Omega < \Omega_2, \]

one can represent (3.12) as follows:

\[
\begin{align*}
u_0^{(1)} &= g_{11}^0(\Omega)F_1 + g_{12}^0(\Omega)F_2 + g_{12}^1(\Omega)F_3, \\
u_0^{(2)} &= g_{21}^0(\Omega)F_1 + g_{22}^0(\Omega)F_2 + g_{22}^1(\Omega)F_3, \\
u^{(2)}_1 &= g_{21}^{-1}(\Omega)F_1 + g_{22}^{-1}(\Omega)F_2 + g_{22}^0(\Omega)F_3.
\end{align*}
\]

Without loss of generality, we set \( m_1 + m_2 = 2 \), hence,

\[ m_1 = \frac{2r}{1+r}, \quad m_2 = \frac{2}{1+r}. \]

Thus, all the Green matrix elements depend on \( \Omega \), \( c \) and \( r \). The system (3.13) has a non-trivial solution if

\[ \det (S(\Omega, r) - \gamma^{-1}I) = 0, \]

where \( I \) is the identity matrix and

\[ S(\Omega, r) = \begin{pmatrix}
2g_{11}^0 - g_{12}^0 - g_{12}^1 & g_{12}^0 - g_{11}^0 & g_{12}^1 - g_{11}^1 \\
2g_{21}^0 - g_{22}^0 - g_{22}^1 & g_{22}^0 - g_{21}^0 & g_{22}^1 - g_{21}^1 \\
2g_{21}^{-1} - g_{22}^{-1} - g_{22}^0 & g_{22}^{-1} - g_{21}^{-1} & g_{22}^0 - g_{21}^0
\end{pmatrix}. \]

The existence of the localized defect mode in the current problem directly follows from the existence of the solutions of (3.14),

\[ \gamma^{-1} = \lambda_i(\Omega, r), \quad i = 1, 2, 3, \]

where \( \lambda_i(\Omega, r) \) are the eigenvalues of the matrix \( S(\Omega, r) \). In figure 6, we illustrate the localized defect modes by showing a family of curves representing frequencies of the localized defect modes versus \( r \) for fixed values of \( c = 1.5, 3, 7 \) and \( L = 1 \). In figure 7, we show for a localized defect mode, \( r \) parameter dependence on \( c \) taken for the frequency \( \Omega_0(r) \), which corresponds to the case of the extremal localization (see formula (A 3) in appendix A)

\[ \Omega_0(r) = \frac{\Omega_1 + \Omega_2}{2}. \]

The result in figure 7 was obtained as the intersection of two surfaces

\[ u = \det \left( S(\Omega_0(r), r) - \left( 1 + c \frac{L}{J} \right) I \right) \]

and \( u = 0 \). This intersection is shown in figure 8. The extremal localization for a defect mode was noted in the paper by Movchan & Slepyan (2007), where a similar result was obtained for the corresponding frequency inside the band gap.
4. Bloch–Floquet waves and standing modes in two-dimensional lattices with imperfect junctions

Consider a two-dimensional square lattice with a macro-cell consisting of four particles with period $2L$. Let the first particle with mass $m_1$ ($j=0$) of the macro-cell interact with its neighbours and particles with mass $m_2$ ($j=1, 2, 3$) via
spring connections (figure 9). Similar to (3.6), within such a macro-cell, the particle displacements $u_{n_1,n_2}^{(j)}$, $j=0, 1, 2, 3$, satisfy the following system of equations of motion:

$$
\begin{align*}
-\Omega m_1 u_{n,m}^{(0)} &= \Gamma \left( u_{n,m}^{(1)} + u_{n+1,m}^{(1)} + u_{n,m}^{(2)} + u_{n,m+1}^{(2)} - 4u_{n,m}^{(0)} \right), \\
-\Omega m_2 u_{n,m}^{(1)} &= \Gamma \left( u_{n-1,m}^{(0)} + u_{n,m}^{(0)} - 2u_{n,m}^{(1)} + u_{n,m}^{(3)} + u_{n,m+1}^{(3)} - 2u_{n,m}^{(1)} \right), \\
-\Omega m_2 u_{n,m}^{(2)} &= \Gamma \left( u_{n,m-1}^{(0)} + u_{n,m}^{(0)} - 2u_{n,m}^{(2)} + u_{n,m}^{(3)} + u_{n,m+1}^{(3)} - 2u_{n,m}^{(2)} \right), \\
-\Omega m_2 u_{n,m}^{(3)} &= u_{n,m-1}^{(1)} + u_{n,m}^{(1)} + u_{n-1,m}^{(2)} + u_{n,m}^{(2)} - 2u_{n,m}^{(3)}. \\
\end{align*}
$$

(4.1)

The particle displacements $u_{n_1,n_2}^{(j)}$, $j=0, 1, 2, 3$, satisfy the quasi-periodicity conditions

$$
\begin{align*}
&u_{n+N,m+M}^{(j)} = \exp(2iL(k_1 N + k_2 M)) u_{n,m}^{(j)}, \\
&\text{where } k = (k_1, k_2) \in D_{2L} \text{ is the Bloch vector (quasi-momentum); } N, M \in \mathbb{Z}; \text{ and the domain } D_{2L} \text{ is the square}
\end{align*}
$$

$$D_{2L} = \left[ -\frac{\pi}{2L}, \frac{\pi}{2L} \right] \times \left[ -\frac{\pi}{2L}, \frac{\pi}{2L} \right].$$

This system of equations for the macro-cell with indices $n, m$ may be written in the usual form

$$
(\Omega M - \sigma(k)) \begin{pmatrix} u_{n,m}^{(0)} \\ u_{n,m}^{(1)} \\ u_{n,m}^{(2)} \\ u_{n,m}^{(3)} \end{pmatrix} = 0,
$$

(4.2)
where the mass matrix is
\[
M = \begin{pmatrix}
m_1 & 0 & 0 & 0 \\
0 & m_2 & 0 & 0 \\
0 & 0 & m_2 & 0 \\
0 & 0 & 0 & m_2
\end{pmatrix}
\]
and the stiffness matrix is
\[
\sigma(k) = \begin{pmatrix}
4\Gamma & -\Gamma(1 + e^{2ik_1L}) & -\Gamma(1 + e^{2ik_2L}) & 0 \\
-\Gamma(1 + e^{-2ik_1L}) & 2 + 2\Gamma & 0 & -1 - e^{2ik_1L} \\
-\Gamma(1 + e^{-2ik_2L}) & 0 & 2 + 2\Gamma & -1 - e^{2ik_2L} \\
0 & -1 - e^{-2ik_2L} & -1 - e^{-2ik_1L} & 4
\end{pmatrix}
\]
As before, the dispersion equation of the lattice
\[
\det[\Omega M - \sigma(k)] = 0,
\]
gives four non-negative roots $\Omega_{1,2,3,4}(k)$. They are shown as the dispersion diagram in figures 10 and 11 for the parameters $m_1=1$, $m_2=2$, $L=1$, $\Gamma=0.5$ and 0.1, respectively. Making the parameter $\Gamma$ smaller, we make the band gap wider. Moreover, the new low-frequency band gap appears. The decrease of $\Gamma$ results in the appearance of flat parts of the dispersion curves representing standing waves.

The dispersion surfaces as a function of the Bloch vector around the contour ABCA and $\Gamma$ are shown in figure 12, for $m_1=1$, $m_2=2$ and $L=1$. It shows the presence of standing waves. The eigenvectors indicate that the lowest frequency of standing waves ($\sqrt{\Omega} \approx 0.6$) corresponds to the relatively large amplitude.

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Figure 9. The geometry of a two-dimensional lattice with four particles: (a) the macro-cell of the periodic structure, (b) expanded view—the circles show the damaged junctions within the periodic structure and (c) the elementary cell in the reciprocal space.
oscillations of the mass $m_1$ when compared with the other masses. For small values of $\Gamma$, the spring constant is also small compared with the force per unit length in the connecting links.

In figures 13 and 14, we show the displacement within the lattice fragment for $m_1=1$, $m_2=2$, $L=1$ and $\Gamma=0.1$ for the first and the second eigenvectors, respectively, with $k_1=0$, $k_2=0$ (see points P and Q in figure 11). In the case of the point P, for small $\Gamma$, the amplitude of vibration of particle 0 is much larger compared with amplitudes for other particles. This is a localized vibration mode with respect to the macro-cell. A simple asymptotic approximation for the first non-zero eigenfrequency can be obtained by assuming that all the particles, except for the particle 0, within the macro-cell are stationary. This situation may
Figure 12. The dispersion diagram for a lattice with periodically distributed defects: $m_1=1$ and $r=0.5$.

Figure 13. The fragment of the displacements (a localized vibration mode with respect to the macro-cell) for the eigenvector with $k_1=0$, $k_2=0$, $\Gamma=0.1$, corresponding to the point P.

Figure 14. The fragment of the displacements for the eigenvector with $k_1=0$, $k_2=0$, $\Gamma=0.1$, corresponding to the point Q.

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be modelled as follows:

\[-\Omega m_1 u = \Gamma (a_1 + a_2 + a_3 + a_4 - 4u),\]

\[\Gamma (a_i - u) + \frac{f}{L} a_i = 0, \quad i = 1, 2, 3, 4,\]

where \(u\) is the displacement of the particle 0. Eliminating \(a_i\), \(i = 1, 2, 3, 4\), and using the asymptotic approximation \(\Omega_{1\text{asym}}\) for \(\Gamma \ll 1\), we obtain

\[-\Omega_{1\text{asym}} m_1 u = 4u\Gamma.\]

Hence,

\[\Omega_{1\text{asym}} = 4 \frac{\Gamma}{m_1}.\]

In figure 15, we show that the difference \(\Omega_1(\Gamma) - \Omega_{1\text{asym}}(\Gamma)\) tends to 0 as \(\Gamma \to 0\).

In the second case, we have only particles 1 and 2 vibrating, whereas the other particles 0 and 3 of the macro-cell are stationary. This situation may be modelled as follows:

\[-\omega_2^2 m_2 u = (-2u - 2\Gamma u) \frac{f}{L},\]

where \(u\) is the displacement of the particle 1 or 2 as the behaviour of both the particles is identical. Hence,

\[-\Omega_2 m_2 u = -2u - 2\Gamma u.\]

It is worth noting that, for both the cases of the vibrating standing waves, the centre of mass of four particles of the macro-cell is zero.

5. Conclusion

In this paper, we analyse vibrations of two types of mechanical lattices with defects: one-dimensional and two-dimensional discrete structures possessing band gaps and the localized oscillation modes. The existence of a localized defect
mode for a one-dimensional lattice, having a single imperfect junction in the central macro-cell, was considered. We have derived the representation for the lattice Green function for frequencies within the stopband. The presence of a crack in the junction region leads to a localized defect mode. We have introduced the notion of extremal localization for a lattice with an imperfect junction, and we have studied the corresponding vibration modes.

For a two-dimensional lattice, we presented the analysis of dispersion surfaces and showed the existence of standing waves within a periodic system of defects. It is shown that the increase in the crack length leads to a formation of a low-frequency stopband on the dispersion diagram. We also note that a standing wave of a low frequency appears within periodic structures with cracked junctions.

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Appendix A

Here, we give a detailed description of the Green function for a one-dimensional lattice with periodically distributed defects. This solution was obtained in Movchan & Slepyan (2007) for the case $\Gamma=1$. From §3a, consider a one-dimensional lattice with a macro-cell of two particles with masses $m_1$ and $m_2$ and with period $2L$. The first particle of the macro-cell interacts with its neighbour via a spring connection as described in §3a. Then, the particle displacements $u_n$ and $v_n$ for the $n$th macro-cell satisfy the following system of equations (see §3a):

$$-\Omega m_1 u_n = \Gamma (v_n + v_{n-1} - 2u_n) + F_1 \delta_{n0},$$

$$-\Omega m_2 v_n = \Gamma (u_n + u_{n+1} - 2v_n) + F_2 \delta_{n0},$$

where $F_1$ and $F_2$ are two external forces applied only to particles of zero number macro-cell.

Using the Fourier transforms,

$$u_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikn} U(k) \, dk, \quad v_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikn} V(k) \, dk,$$

we obtain

$$(\Omega m_1 - 2\Gamma) U(k) + \Gamma (1 + e^{ik}) V(k) = -F_1,$$

$$\Gamma (1 + e^{-ik}) U(k) + (\Omega m_2 - 2\Gamma) V(k) = -F_2.$$
\[
\begin{pmatrix}
u_n \\ v_n
\end{pmatrix} = g^n \begin{pmatrix} F_1 \\ F_2 \end{pmatrix},
\]
\[g^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikn} G(k) \, dk,
\]
\[G(k) = -\left( \begin{array}{cc} A & E \\ E^* & B \end{array} \right)^{-1},
\]
where
\[A = \Omega m_1 - 2\Gamma, \quad E = \Gamma (1 + e^{ik}), \quad B = \Omega m_2 - 2\Gamma.
\]
Next, we evaluate the Fourier transforms for the elements of the Green matrix using the theorem of residue over the complex plane with respect to a new complex variable \(z = e^{ik}\). Thus, for example, for the element \(g^n_{11}\), we have
\[g^n_{11} = \frac{B}{2\pi i \Gamma^2} \int_{|z|=1} \frac{z^n}{z^2 - 2pz + 1} \, dz, \quad p = \frac{AB}{2\Gamma^2} - 1.
\]
If \(|p| > 1\), we obtain the Green matrix elements
\[g^n = \frac{\lambda}{\Gamma (\lambda^2 - 1)} \begin{pmatrix} \left( \frac{\Omega m_2}{\Gamma} - 2 \right) \lambda^{|n|} & -\lambda^{|n|} - \lambda^{|n-1|} \\ -\lambda^{|n|} - \lambda^{|n+1|} & \left( \frac{\Omega m_1}{\Gamma} - 2 \right) \lambda^{|n|} \end{pmatrix}, \quad (A 2)
\]
which decay exponentially with \(n\).
For \(p > 1\), when the frequency exceeds the upper passband,
\[\Omega > 2\Gamma \left( \frac{1}{m_1} + \frac{1}{m_2} \right),
\]
we have
\[\lambda = p - \sqrt{p^2 - 1}, \quad 0 < \lambda < 1.
\]
For \(p < -1\), when the frequency is within the band gap,
\[\frac{2\Gamma}{m_1} < \Omega < \frac{2\Gamma}{m_2},
\]
we have
\[\lambda = p + \sqrt{p^2 - 1}, \quad -1 < \lambda < 0.
\]
The most rapid decay with \(n\) within the band gap for the Green matrix elements occurs for
\[\Omega_0 = \Gamma \frac{m_1 + m_2}{m_1 m_2}, \quad (A 3)
\]
as
\[|\lambda|^{|n|}(\Omega) = \exp(|n| \log (|\lambda(\Omega)|)), \quad |\lambda(\Omega)| = -p - \sqrt{p^2 - 1}
\]
and function \(\lambda(\Omega)\) has its minimum at \(\Omega_0\).
The case $|p| \leq 1$ takes place only for the passband, for which we have oscillatory behaviour of the Green matrix elements

$$
  g^n = \begin{pmatrix}
    \frac{B \sin |n|\Phi}{2T^2 \sin \Phi} & -\frac{1}{2T} \sin |n|\Phi + \sin |n-1|\Phi \\
    -\frac{1}{2T} \sin |n|\Phi + \sin |n+1|\Phi & A \frac{\sin |n|\Phi}{2T^2 \sin \Phi}
  \end{pmatrix},
$$

where

$$
  \Phi = \arg (p + i\sqrt{1-p^2}).
$$

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