Modeling Fuzzy B-spline Interpolation Series using $\alpha$-cut operation for spatial earth surface problem

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Abstract. In this paper, the series of $\alpha$-cut is also known as the $\alpha$-cut operation used to construct a new model of surface called Fuzzy B-spline Interpolation Series surface. The surface is called Fuzzy B-spline Interpolation Series because of the various values of $\alpha$ chosen in the fuzzification method. Then, the operation is combined into a B-spline surface model. This new proposed model will be used to solve spatial earth surface problems in every collected data points. The spatial earth data have values in latitude, longitude and altitude. All these values are considered as uncertainty data because of the satellite’s measurement errors. Finally, constructed surfaces’ results will be compared between raw data’s surface and Fuzzy B-spline Interpolation Series surface.

Keywords: fuzzy set; uncertainty; series of alpha-cut; fuzzy b-spline interpolation series

1. Introduction
The concept of ambiguity was first started by Black [1], which was then expanded by Zadeh [2, 3, 4]. Zadeh classified the uncertainty into a set of values between 0 and 1 instead of classic set theory [5]. Then, the general operation of Fuzzy Set Theory (FST) [6, 7] was elaborated and used in computer decision-making [8].

Now, Fuzzy set's knowledge is spreading rapidly due to the technology industry’s growth, heading to the Fourth Industrial Revolution [9, 10, 11]. Besides the needs in computer decision-making [12] and pattern recognition [13], which is computer-based technology, it also helps in health, environmental management, [14] and road traffic safety [15].

The advantages of Fuzzy Set can be used to solve problems in constructing curves and surfaces for Computer-Aided Design (CAD). This new idea is necessary for constructing a better and smooth surfaces’ model. Fuzzy Set and CAD are two concepts that can be merged to solve the ambiguities in control or data points. Wahab [16, 17, 18] introduced the method for fuzzifying control points to construct Bezier and B-spline curves. Then, Zakaria extended the concept by applying the $\alpha$-cut operation to construct Fuzzified Bezier Curve [19]. Furthermore, the concept of $\alpha$-cut operation can be extended for producing surfaces and other types of CAD, such as B-spline that has never been published.

Therefore, this paper focuses on generalising the hybrid concept of FST and modifying the $\alpha$-cut into a Series concept. Then, the concepts are applied to the modeling of the Fuzzy B-spline Interpolation Series surface. The full steps of Fuzzy B-spline Interpolation Series modeling can also be referred to in [20].
2. Preliminaries
There are three main parts to constructing a fuzzified curve. The first part is fuzzifying the points. The uncertainty points are undergoing fuzzification into the membership function, where every point will be cut using α-cut operation. The second part is plotting the curve using fuzzified points in the first part. Then, the third part is defuzzifying the curve.

2.1. Classical fuzzification
In the classical fuzzification method, the fuzzy point of \( P \) can be defined as:

**Definition 1.** Let \( P \) be a fuzzy point in universal set \( S \). There exists \( \tilde{P}_{i} = \langle \tilde{P}_{i} \mid P_{i} \rangle \) with membership function \( \mu_{\tilde{P}_{i}} : S \rightarrow [0,1] \). \( \mu_{\tilde{P}_{i}}(P_{i}) = 1 \) denotes the crisp point, meanwhile the membership function for right-fuzzy \( \tilde{P}_{i} \) point and left-fuzzy \( \tilde{P}_{i} \) point can be represented by:

\[
\mu_{\tilde{P}_{i}}(\tilde{P}_{i}) = \begin{cases} 
0 & \text{for } \tilde{P}_{i} \notin S \\
1 & \text{for } \tilde{P}_{i} \in S 
\end{cases} \quad (1)
\]

and

\[
\mu_{\tilde{P}_{i}}(\tilde{P}_{i}) = \begin{cases} 
0 & \text{for } \tilde{P}_{i} \notin S \\
k & \text{for } \tilde{P}_{i} \in S 
\end{cases} \quad (2)
\]

Therefore, we can write the whole fuzzy points as:

\[
\tilde{P} = \{ \tilde{P}_{i} \}_{i=0}^{n} \text{ whereby } \tilde{P}_{i} = \langle \tilde{P}_{i} \mid P_{i} \rangle. \quad (3)
\]

**Example 1.** Given four crisp points \( \{P_{0}, P_{1}, P_{2}, P_{3}\} \) that will be fuzzified into fuzzy set \( \tilde{P} \), as in Definition 1. If the α-cut is chosen as 0.6, then all fuzzy points can be shown in Figure 1.

2.2. The α-cut operation
Fuzzy Series is a fuzzification method whereby the classic α-cut is modified by cutting it into a series of numbers. The α-cut is also known as the Degree of Confidence, which can be chosen as desired by an expert. Let a fuzzy set \( \tilde{P} = [a, b] \), then there exists fuzzy point data \( \tilde{P}_{i}^{n} \) within every chosen α-cut, as shown in Equation 4 and Figure 2, given by

\[
\tilde{P}_{i}^{0.1} = [a^{0.1}, b^{0.1}], \\
\tilde{P}_{i}^{0.3} = [a^{0.3}, b^{0.3}], \\
\vdots \\
\tilde{P}_{i}^{0.9} = [a^{0.9}, b^{0.9}].
\]

(4)
**Figure 1.** Fuzzy points after fuzzification method.

**Figure 2.** Fuzzy number is under series of $\alpha$-cut.

**Definition 2.** If fuzzy series set of points $\vec{P}^{\alpha} = \{\vec{P}^{\alpha}_{i}\}_{i=0}^{n}$ is in the universal set $S$, then there exists $\vec{P}^{\alpha}_{i} = \langle \vec{P}^{\alpha}_{i}, |P_{i}| \vec{P}^{\alpha}_{i} \rangle$ and membership function $\mu_{\vec{P}^{\alpha}} : S \to [0,1]$, whereby the membership degree of crisp point is $\mu_{\vec{P}^{\alpha}}(P_{i}) = 1$. Therefore, the set of fuzzy series data points $\vec{P}^{\alpha}$ can be represented as $\vec{P}^{\alpha} = \{(x,\mu_{\vec{P}^{\alpha}}(x)) | x \in \mathbb{R}\}$ for $0 \leq \mu_{\vec{P}^{\alpha}}(x) \leq 1$.

Given the cuts $\alpha_{i} < \alpha_{j}$. Therefore, the interval support of $\vec{P}^{\alpha_{j}} \subset \vec{P}^{\alpha_{i}}$ for every $\forall \alpha_{j}, \alpha_{i} \in (0,1]$ can be represented as in Figure 3.

**Definition 3.** Let $\alpha_{r}$ be a $\alpha$-cut series with $r = 1, 2, \ldots, s$ for every fuzzy data point $\vec{P}_{i}$, with $s$ an infinity and the value of $\forall \alpha_{r} \in (0,1]$. Therefore, the cut of $\alpha_{r}$ approaches 1 and the left-right fuzzy data $\vec{P}_{i}$ approaches crisp point as $\lim_{x \to \vec{P}_{i}} \mu_{\vec{P}^{\alpha}}(x) = \alpha$, whereby $0 < \alpha \leq 1$. If points approach the crisp $P_{i}$, then $\alpha = 1$ is defined as $\lim_{x \to P_{i}} \mu_{\vec{P}^{\alpha}}(x) = 1$. 


2.3. Defuzzification method

In order to achieve a single data point after being fuzzificated, the centroid method [21] is used. For the \(\alpha\)-cut series, the solution for the centroid method is shown in Definition 4.

**Definition 4.** If \(\alpha_r\) is an \(\alpha\)-cut series with \(r = 1, 2, \ldots, s\) for every fuzzy point \(\tilde{P}_i\), then \(P\) represents the defuzzified data point for \(\tilde{P}_i^{\alpha_r}\) that can be summarised as follows:

\[
P_i = P_i + \sum_{r=1}^{s} \left( \tilde{P}_i^{\alpha_r} + \tilde{P}_i^{\alpha_r} \right) \left/ \left(2s + 1\right) \right.
\]

3. Modeling of Fuzzy B-spline Interpolation Series using \(\alpha\)-cut operation

If a set of fuzzy points of B-spline curve is fuzzified as in Definition 1, then there are 3 curves that can be plotted, as shown in Figure 4. Each curve is stated as either Left Curve (\(\tilde{C}\)), Crisp Curve, or Right Curve (\(\overline{C}\)), respectively.

**Figure 3.** The interval of support for every cut \(\alpha_i, \alpha_j \in (0, 1]\).

**Figure 4.** Fuzzification of B-spline curves using the classical fuzzy set.
Definition 5. If $\alpha_r$ is $\alpha$-cut series with $r = 1, 2, \ldots, s$ for each curve $\tilde{C}^{\alpha_r}$, whereby $s$ is infinite and a serial of $\alpha$-cut values is $\forall \alpha_r \in (0, 1]$. Thus, a series of curves, $SC(t)$, as in Figure 5, can be defined as:

$$SC(t) = \{\tilde{C}^{\alpha_1} + \tilde{C}^{\alpha_2} + \tilde{C}^{\alpha_3} + \cdots + \tilde{C}^{\alpha_s}\}.$$  \hfill (6)

**Figure 5.** A series of curves.

Thus, Fuzzy B-spline Interpolation Series (FBSIS) curve can be defined as:

$$FBSISC(t) = \sum_{r=1}^{s} \tilde{C}^{\alpha_r}, \alpha_r \in (0, 1],$$  \hfill (7)

whereby $\tilde{C}^{\alpha_r} = \sum_{i=0}^{n} \beta_i N_{i,d}(t)$ is Fuzzy B-spline’s interpolation curve for each data point $\tilde{P}_{i}^{\alpha_r}$.

Through this proposed method, more curves can be created as more values of $\alpha_r$ were defined as in Example 2 and Example 3. This will give users an advantage to make a flexible curve.

Example 2. Given $\alpha_r$ with $r = \{1, 2, 3, 4, 5\}$ and $\alpha$-series $\alpha_r = \{0.25, 0.3, 0.4, 0.5, 0.7\}$. Thus, the series of curves can be plotted as in Figure 6.

Example 3. Given $\alpha_r$ with $r = \{1, 2, 3, 4, 5, \ldots, s\}$, whereby $s \in N$ is infinite and $\alpha$-cut series $\alpha_r \in (0, 1]$. Thus, the series of curves can be plotted as in Figure 7.

In order to apply spatial earth data problem with 3 variables, we need to model a surface. From Definition 3, we can define Fuzzy B-spline Interpolation Series surface as follows.

Definition 6. Let $\alpha_r$ be a $\alpha$-cut series with $r = \{1, 2, 3, 4, 5, \ldots, s\}$ for each fuzzy control point $\tilde{P}_{i,j}^{\alpha_r}$ and $(2s + 1)$ is the total of surfaces. Thus, the equation for Fuzzy B-spline Interpolation Series (FBSIS) surface for $\alpha_r = \{\alpha_1, \alpha_2, \ldots, \alpha_s\} \in (0, 1]$ can be defined as:

$$FBSIS^{\alpha_r}(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \tilde{\beta}_{i,j}^{\alpha_r} N_{i,p}(u) N_{j,q}(v).$$  \hfill (8)

4. Results and discussions

In this section, we will use the real spatial earth data to test the Fuzzy B-spline Interpolation Series surface model. There are 400 data points collected from Gua Musang, Malaysia using Google Maps application.
Figure 6. A series of curves for $\alpha_r = \{0.25, 0.3, 0.4, 0.5, 0.7\}$.

Figure 7. A series of curves for an infinite $r$.

as in Figure 8. According to Berry et. al [22], the values of altitude have an error in measurement because of the altimeter device for measuring air pressure. Thus, from the collected data, we assume that altitude values are uncertainty data that need to be fuzzified.

Figure 8. 400 spatial data of Gua Musang, Malaysia.

After the fuzzification of data as in Section 5, the surface of the Fuzzy B-spline Interpolation Series was plotted, as shown in Figure 9. This Series of surfaces were then defuzzified, as discussed in Section 4, to compile them into a single surface. The final surface was compared with raw data plotted as in Figure 10. We can see that some missing data in raw surface was filled to fix the surface highlighted in the red circle from the figure. We can conclude that the Fuzzy B-spline Interpolation Series surface is smoother than using the raw data.
Figure 9. Surfaces of Fuzzy B-spline Interpolation Series.

Figure 10. Comparison between raw data surface and Fuzzy B-spline Interpolation Series surface.

5. Conclusion
This study successfully developed a model of Fuzzy B-spline Interpolation Series surface with \( \alpha \)-cut Series to solve the spatial earth data problem. This method gives flexibility and a more defined surface in constructing the earth surface. The modeling method consists of three steps: data collection, data
fuzzification, and Computer-Aided Design (CAD) model construction. Further research can be done on different kinds of CAD surface models such as Non-uniform Rational B-Spline or different fuzzy numbers. For advanced study, this hybrid concept of FST and CAD modeling can also be extended using the Pythagorean fuzzy set, which would give more flexibilities and new approaches in developing fuzzy curves and surfaces.

Acknowledgement
The first author would like to acknowledge Universiti Sultan Zainal Abidin (UniSZA) and Universiti Malaysia Terengganu (UMT) for providing the facilities in conducting this research.

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