An Efficient Algorithm for Designing Optimal CRCs for Tail-Biting Convolutional Codes

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Abstract—This paper proposes an efficient algorithm for designing the distance-spectrum-optimal (DSO) cyclic redundancy check (CRC) polynomial for a given tail-biting convolutional code (TBCC). Lou et al. proposed DSO CRC design methodology for a given zero-terminated convolutional code (ZTCC), in which the fundamental design principle is to maximize the minimum distance at which an undetectable error event of ZTCC first occurs. This paper applies the same principle to design the DSO CRC for a given TBCC. Our algorithm is based on partitioning the tail-biting trellis into several disjoint sets of tail-biting paths that are closed under cyclic shifts. This paper shows that the tail-biting path in each set can be constructed by concatenating the irreducible error events and/or circularly shifting the resultant path. This motivates an efficient collection algorithm that aims at gathering irreducible error events, and a search algorithm that reconstructs the full list of error events in the order of increasing distance, which can be used to find the DSO CRC for a given TBCC.

I. INTRODUCTION

Tail-biting convolutional codes (TBCCs) are simple and powerful codes for the short blocklength regime. Unlike the conventional zero-terminated convolutional code (ZTCC) whose trellis paths all begin and end in the zero state, a TBCC requires only that each trellis path ends in same state from which it began. This avoids the need for termination bits. TBCCs were first proposed by Ma and Wolf [1] as a modified version of the ZTCC to eliminate the rate loss caused by termination bits. They were recognized as quasi-cyclic block codes by Solomon and Tilborg [2] who demonstrated the intriguing connection that any ZTCC can be transformed into a quasi-cyclic code (or a TBCC) and conversely, many quasi-cyclic codes can be viewed as a TBCC with small constraint lengths. Subsequently, it was shown that any linear block code can correspond to a tail-biting trellis representation and the code represented by such trellis is called a tail-biting code [3], [4]. The significance of tail-biting codes lies in the fact that they achieve the best minimum distance of codes in the short-to-medium blocklength regime [1], [5], [6].

Since the advent of TBCCs and tail-biting codes, several authors proposed a variety of algorithms to decode a TBCC or a tail-biting code, e.g., [1], [7]–[12]. These algorithms are based on either maximum likelihood (ML) or maximum a posteriori (MAP) criteria. For the ML decoding algorithms, the wrap-around Viterbi algorithm (WAVA) [10] achieves the near-ML performance with the minimum complexity.

Cyclic redundancy checks (CRCs) are commonly used to detect whether a codeword is correctly received. Recently, with the development of 5G, CRC-aided list decoding of finite blocklength codes has received increasing popularity. CRC-aided list decoding can significantly help improve the code performance, e.g., [13]–[16]. Lou et al. [17] first designed the optimal CRC for a given ZTCC such that the concatenated CRC-ZTCC achieves the minimum FER. The CRC they designed can be referred to as the distance-spectrum-optimal (DSO) CRC in the sense that the minimum distance at which the undetectable error event of ZTCC first occurs. However, optimal CRC design for a given TBCC is still missing from the literature. It is remarkable that a simple suboptimal CRC design [16] can already nearly achieve the random coding union bound of Polyanskiy [18].

This paper provides an algorithm for DSO CRC design for TBCCs. The fundamental design principle parallels that of Lou et al., which is to maximize the undetectable minimum distance, i.e., the minimum distance at which an undetectable (tail-biting) error event first occurs. To this end, the first step is to gather a sufficient number of error events (or tail-biting paths) of distances less than some threshold, which is accomplished by the collection algorithm. Then, the search algorithm is employed to find the optimal CRC polynomial that maximizes the undetectable minimum distance. For TBCCs, one collection algorithm is to perform Viterbi search separately for each possible initial state to find all error events of a bounded distance. However, as the trellis depth increases, such an algorithm will be inefficient in determining the optimal CRC as the outputs will include tremendously many error events that could have been easily constructed by circularly shifting or concatenating irreducible error events that cannot be further decomposed.

In contrast, this paper provides an efficient algorithm for designing DSO CRCs for a given TBCC based on partitioning the tail-biting trellis into several disjoint sets of tail-biting paths that are closed under cyclic shifts. Specifically, for a feedforward convolutional encoder with v memory elements and a specified blocklength, its tail-biting trellis is described entirely as the union of all tail-biting paths of the required length that start and end at any of the 2^v states. Each tail-biting path can be categorized by a state through which it traverses. Let TBP(0) be the set of tail-biting paths that traverse through state 0. Then, recursively define TBP(i), 1 ≤ i ≤ 2^v − 1, as...
the set of tail-biting paths that traverse through state $i$ but not through $0,1,\ldots,i-1$. Clearly, the $2^v$ sets are disjoint and collectively contain all tail-biting paths. Namely, they form a partition of the possible trajectories in the trellis. This paper shows that each path in $\text{TBP}(i)$ can be reconstructed from the irreducible error events. This is the essential motivation of our algorithm.

The paper is organized as follows. Sec. II reviews the preliminaries of the TBCC, trellises, and Lou et al.’s CRC design for ZTCCs. Sec. III introduces the partition of a tail-biting trellis, our DSO CRC design algorithm for TBCCs, and an illustrative example. Sec. IV concludes the entire paper.

II. PRELIMINARIES

A. Construction of the TBCC

We briefly follow [1] in describing a tail-biting convolutional code. For ease of understanding, consider a feedforward, $(n,1,v)$ convolutional code having rate $1/n$ and $v$ memory elements. Thus, for a binary information sequence of length $K$, $K \geq v$, we first use the last $v$ bits of the information sequence to initialize the convolutional encoder and ignore the outputs. Then the entire $K$-bit information sequence is fed into the encoder and the resultant $nK$-bit output is the TBCC. As can be seen, the initial and final end state are the same for the TBCC. In this way the rate loss required by a TBCC is called the time axis $T$ of depth $N$, a trellis section connecting time $i$ and $i+1$ is a subset $T_i \subseteq V_i \times A \times V_{i+1} \subseteq E$ that specifies the allowed combination $(s_i,a_i,s_{i+1})$ of state $s_i \in V_i$, output symbol $a_i \in A$, and state $s_{i+1} \in V_{i+1}$, $i = 0,1,\ldots,N-1$. Such allowed combinations are called trellis branches. A trellis path $(s,a) \in T$ is a state/output sequence pair, where $s \in V_0 \times V_1 \times \cdots \times V_N$, $a \in A^N$. The code represented by trellis $T$ is the set of all output sequences $a$ corresponding to all trellis paths $(s,a)$ in $T$.

For a tail-biting trellis $T$ of depth $N$, a tail-biting path $(s,a)$ of length $N$ on $T$ is a closed path through $N$ vertices. If $T$ is defined on a sequential time axis $I = \{0,1,\ldots,N\}$, then any tail-biting path $(s,a)$ of length $N$ satisfies $s_0 = s_N$.

In this paper, we only consider the tail-biting trellis $T$ of depth $N$ satisfying $V_0 = V_i$, $i = 1,2,\ldots,N-1$. Clearly, the tail-biting trellis generated by the feedforward, $(n,1,v)$ convolutional encoder $g(x)$ meets our condition.

B. Conventional and Tail-Biting Trellises

We follow [3, 4] in describing the conventional and tail-biting trellises. Let $V$ be a set of vertices (or states), $A$ the set of output alphabet, and $E$ the set of ordered triples or edges $(v,a,v')$, with $v,v' \in V$ and $a \in A$. In words, $(v,a,v') \in E$ denotes an edge that starts at $v$, ends at $v'$ and has label $a$.

**Definition 1** (Conventional trellises, [3, 4]). A conventional trellis $T = (V,E,A)$ of depth $N$ is an edge-labeled directed graph with the following property. The vertex set $V$ can be partitioned as

$$V = V_0 \cup V_1 \cup \cdots \cup V_N$$

such that every edge in $T$ begins at a vertex of $V_i$ and ends at a vertex of $V_{i+1}$, $i = 0,1,\ldots,N-1$. The sets $V_0,V_1,\ldots,V_N$ are called the vertex classes of $T$. The ordered index set $I = \{0,1,\ldots,N\}$ induced by the partition in (2) is called the time axis for $T$.

**Definition 2** (Tail-biting trellises, [4]). A tail-biting trellis $T = (V,E,A)$ of depth $N$ is an edge-labeled directed graph with the following property. The vertex set $V$ can be partitioned into $N$ vertex classes

$$V = V_0 \cup V_1 \cup \cdots \cup V_{N-1}$$

such that every edge in $T$ either begins at a vertex of $V_i$ and ends at a vertex of $V_{i+1}$ for some $i = 0,1,\ldots,N-2$, or begins at a vertex of $V_{N-1}$ and ends at a vertex of $V_0$.

![Block diagram of a system employing CRC and convolutional codes.](image-url)

Geometrically, a tail-biting trellis can be viewed as cylinder of $N$ sections defined on some circular time axis. Alternatively, we can also define a tail-biting trellis on a sequential time axis $I = \{0,1,\ldots,N\}$ with the restriction that $V_0 = V_N$.

For a conventional trellis $T$ of depth $N$, a trellis section connecting time $i$ and $i+1$ is a subset $T_i \subseteq V_i \times A \times V_{i+1} \subseteq E$ that specifies the allowed combination $(s_i,a_i,s_{i+1})$ of state $s_i \in V_i$, output symbol $a_i \in A$, and state $s_{i+1} \in V_{i+1}$, $i = 0,1,\ldots,N-1$. Such allowed combinations are called trellis branches. A trellis path $(s,a) \in T$ is a state/output sequence pair, where $s \in V_0 \times V_1 \times \cdots \times V_N$, $a \in A^N$. The code represented by trellis $T$ is the set of all output sequences $a$ corresponding to all trellis paths $(s,a)$ in $T$.

For a tail-biting trellis $T$ of depth $N$, a tail-biting path $(s,a)$ of length $N$ on $T$ is a closed path through $N$ vertices. If $T$ is defined on a sequential time axis $I = \{0,1,\ldots,N\}$, then any tail-biting path $(s,a)$ of length $N$ satisfies $s_0 = s_N$.

In this paper, we only consider the tail-biting trellis $T$ of depth $N$ satisfying $V_0 = V_i$, $i = 1,2,\ldots,N-1$. Clearly, the tail-biting trellis generated by the feedforward, $(n,1,v)$ convolutional encoder $g(x)$ meets our condition.

C. The CRC Design for ZTCCs by Lou et al.

We briefly follow [17] in introducing their DSO CRC design scheme for a given ZTCC.

The basic system model Lou et al. considered is depicted as in Fig. 1. Let us consider a $K$-bit information sequence represented as a binary polynomial $u(x)$ of degree no greater than $K-1$. Then, the $m$ parity check bits are calculated as the remainder $r(x)$ of $x^m u(x)$ divided by a degree-$m$ CRC (generator) polynomial $p(x)$. Therefore, the $(K+m)$-bit sequence described by $x^m u(x) + r(x)$ is divisible by $p(x)$, i.e., there exists a unique polynomial $q(x)$ such that $x^m u(x) + r(x) = q(x)p(x)$. Hence, the CRC-coded sequence can be concisely expressed as $q(x)p(x)$. Let $g(x) = \{g^{(1)}(x),\ldots,g^{(m)}(x)\}$ be the generator polynomial of the feedforward, rate-$1/n$ convolutional encoder. After feeding the CRC-coded sequence $q(x)p(x)$ into the encoder, the output $q(x)p(x)g(x)$ is the final codeword of the ZTCC. The transmitter sends the BPSK-modulated codeword $q(x)p(x)g(x)$ through an additive white Gaussian noise (AWGN) channel. After receiving the channel outputs, the decoder reproduces the most likely message sequence of $g(x)$ via soft Viterbi decoding. An error $e(x)$ occurs if the soft Viterbi erroneously identified the path corresponding to input sequence $q(x)p(x)+e(x)$ as the maximum-likelihood (ML) path. If $e(x)$ is not divisible by $p(x)$, the decoder outputs a negative acknowledgement (NACK) and may ask for...
retransmission. Otherwise, the decoder declares the decoded message $\hat{u}(x)$ from the first $K$ bits of $q(x)p(x) + e(x)$.

We say that an undetectable error occurs if $e(x)$ is nonzero and is divisible by $p(x)$. Thus, a fundamental design challenge is to identify the optimal CRC for the ZTCC generated by $g(x)$ such that the probability of undetectable errors (or FER) is minimized. Lou et al. [17] showed that when FER is low, the problem is equivalent to designing a CRC polynomial $p(x)$ that maximizes the undetectable minimum distance, i.e., the minimum distance at which an undetectable error event first occurs. Therefore, they proposed an algorithm which first collects a sufficient number of error events (or zero-terminated paths) of distances less than some threshold $d$ by performing a Viterbi search at all-zero state. Note that the smaller the distance is, the more likely the error event occurs. They exhaustively enumerate each degree-$m$ candidate CRC polynomial and count the number of error events each candidate detects in the order of increasing distances. In the end, the one that maximizes the undetectable minimum distance is the DSO CRC for ZTCCs. By considering more terms in the distance spectrum and their multiplicities, Lou’s approach can also minimize FER in the more general case of noisier channels where minimizing FER is not necessarily equivalent to maximizing the undetectable minimum distance.

III. OPTIMAL CRC DESIGN FOR THE TBCC

In this paper, we consider the same system model as in Fig. 1 except replacing ZTCCs with TBCCs. The primary distinction between the two types of CCs is that a tail-biting error event can start at a nonzero state and even remain in nonzero states for the entire error event.

The fundamental DSO CRC design principle is analogous to that of Lou et al., which is to maximize the minimum distance at which an undetectable tail-biting error event first occurs, (or undetectable minimum distance). Formally speaking, the degree-$m$ DSO CRC design for a given convolutional code (either zero-terminating or tail-biting) involves two steps. First, the collection algorithm gathers a sufficient number of error events of distances less than some threshold $d$ and stores them for future use. By sufficient, we mean that the number of error events is enough to sieve the unique, degree-$m$ CRC polynomial out of $2^{m-1}$ candidate CRC candidates. Next, the search algorithm enumerates $2^{m-1}$ CRC candidates. For each candidate, the search algorithm passes the error events in the order of increasing distance to the candidate CRC to perform a divisibility test and counts the total number of undetectable error events. In the end, the optimal CRC polynomial is the one that maximizes the undetectable minimum distance (or, more generally, minimizes the FER).

For TBCCs, one collection algorithm is to perform Viterbi search at each initial state and then aggregate the error events according to increasing distances. However, such an algorithm will be inefficient in designing high-degree CRCs since the list of error events includes tremendously many error events that could have been easily constructed by cyclic shifting and concatenating. Moreover, if the trellis depth changes, the error events found by the trivial collection algorithm are cumbersome to be adapted into other cases.

In this paper, we propose a novel collection algorithm based on the partitioning of tail-biting trellis and search of irreducible error events.

A. Partitioning of the Tail-Biting Trellis

For a given feedforward, $(n, 1, v)$ convolutional encoder $g(x)$, let us consider the corresponding tail-biting trellis $T = (V, E, A)$ defined on a given sequential time axis $\mathcal{I} = \{0, 1, \ldots, N\}$. Since $T$ can also be represented by the union of tail-biting paths (each corresponding to a TBCC codeword), we categorize each tail-biting path according to the states through which it traverses. Formally speaking, let

$$V_0^T = (\sigma_0, \sigma_1, \ldots, \sigma_{2^v-1})$$

be a predetermined permutation of $V_0 = \{0, 1, \ldots, 2^v - 1\}$. Define the set of tail-biting path w.r.t. $V_0^T$ as

$$\text{TBP}(\sigma_i) \triangleq \{(s, a) \in V_0^{N+1} \times \mathcal{A}^N : s_0 = s_N, \exists j \in \mathcal{I} \text{ s.t. } s_j = \sigma_i, \forall j \in \mathcal{I}, s_j \notin \{\sigma_0, \sigma_1, \ldots, \sigma_{i-1}\}, \forall i = 0, 1, \ldots, 2^v - 1\}.$$  (4)

In words, the set of $\text{TBP}(\sigma_0)$ only contains tail-biting paths that traverse through state $\sigma_0$; the set of $\text{TBP}(\sigma_1)$ contains tail-biting paths that traverse through state $\sigma_1$ but not $\sigma_0$; so on and so forth. Clearly, all sets $\text{TBP}(\sigma)$, $\sigma \in V_0^T$, form a partition of the tail-biting trellis $T$, i.e.,

$$\bigcup_{\sigma \in V_0^T} \text{TBP}(\sigma) = T.$$  (6)

An important property of the above decomposition is that each set $\text{TBP}(\sigma)$ is closed under cyclic shifts, stated in Theorem 1.

Theorem 1. Assume $\text{TBP}(\sigma)$ is nonempty. Then any cyclic shift of a tail-biting path $(s, a) \in \text{TBP}(\sigma)$ is also a tail-biting path in $\text{TBP}(\sigma)$.

Proof: Since circularly shifting a tail-biting path $(s, a)$ on a tail-biting trellis $T$ defined on a given sequential time axis $\mathcal{I} = \{0, 1, \ldots, N\}$ is equivalent to circularly shifting $\mathcal{I}$ around $T$ defined on a circular time axis, this preserves the sequence of states (or vertices) through which the tail-biting path $(s, a)$ traverses. Hence, the statement in Theorem 1 holds. ■

Using the ideas of basis and linear combination for a vector space, we can consider the set of irreducible error events (IEEs) starting at state $\sigma$ as a basis from which each tail-biting path of length $N$ in $\text{TBP}(\sigma)$ may be constructed. The next section shows that this is accomplished by concatenating the irreducible error events and then circularly shifting the resultant tail-biting path.

1A CRC generator polynomial must have 1 as coefficients for both the scalar term and the degree-$m$ term.
**Algorithm 1** The Collection Algorithm

**Input:** The TB trellis $T$, threshold $d$, permutation $V_0^N$

**Output:** The list of IEEs $L = \{(s, a, u)\}$

1. Initialize lists $L$ to be empty for all $\sigma \in V_0^N$;
2. for $i \leftarrow 0, 1, \ldots, |V_0^N| - 1$ do
3. Perform Viterbi search at $\sigma_i$ on $T$ to collect list $L_{\sigma_i}$ of irreducible error events of distances less than $d$;
4. end for
5. return $L \leftarrow \bigcup_{\sigma \in V_0^N} L_{\sigma}$

**Algorithm 2** The Search Algorithm

**Input:** The degree $m$, threshold $d$, list of IEEs $L_{\text{IEE}}$

**Output:** The optimal degree-$m$ CRC gen. poly. $p(x)$

1. Initialize the list $L_{\text{CRC}}$ of $2^m - 1$ CRC candidates, the empty list $L_{\text{TBP}}$ of TBPs;
2. for $d \leftarrow 0, 1, \ldots, d - 1$ do
3. Construct new TBPs $(s, a, u)$ of length $N$ from $L_{\text{IEE}}$ s.t. $w_H(a) = d$ via concatenating or cyclic shifting;
4. $L_{\text{TBP}} \leftarrow L_{\text{TBP}} \cup \{(s, a, u)\}$;
5. end for
6. Sort $L_{\text{TBP}}$ in the order of increasing distances;
7. Candi$(1) \leftarrow L_{\text{CRC}}$;
8. for $d \leftarrow 0, 1, \ldots, d - 1$ do
9. for $i \in \text{Candi}(d)$ do
10. Pass all $u(x)$ of dist. $d$ to $p_i(x)$ for divisibility test;
11. $C_i \leftarrow$ the number of divisible $a(x)$ of dist. $d$;
12. end for
13. $C^* \leftarrow \min_{i \in \text{Candi}(d)} C_i$
14. Candi$(d+1) \leftarrow \{p_i(x) \in \text{Candi}(d) : C_i = C^*\}$;
15. if $|\text{Candi}(d+1)| = 1$ then
16. return Candi$(d+1)$
17. end if
18. end for

**Definition 3** (Irreducible Error Events). For a tail-biting trellis $T$ on sequential time axis $\mathcal{I} = \{0, 1, \ldots, N\}$, define the set of irreducible error events $(s, a)$ at state $\sigma$ w.r.t. $V_0^N = (\sigma_0, \sigma_1, \ldots, \sigma_{2^m - 1})$ as

$$ \text{IEE}(\sigma_i) = \bigcup_{j=1,2,\ldots,N} \text{IEE}(\sigma_{i,j}), \forall i = 0, 1, \ldots, 2^m - 1, \quad (7) $$

where

$$ \text{IEE}(\sigma_{i,j}) \triangleq \{(s, a) \in V_{0}^{j+1} \times A^j : s_0 = s_j = \sigma_i, s_j \notin \{\sigma_0, \sigma_1, \ldots, \sigma_i\} \text{ for all } j', 0 < j' < j\}. \quad (8) $$

**Theorem 2.** Assume TBP$(\sigma)$ is nonempty for a given tail-biting trellis $T$ generated by feedforward, $(n, 1, v)$ convolutional encoder $g(x)$. Then, the tail-biting path $(s, a) \in \text{TBP}(\sigma)$ can be constructed from the irreducible error event in IEE$(\sigma)$ via concatenation and/or cyclic shifting operations.

**Proof:** Let us consider $T$ as a tail-biting trellis defined on a sequential time axis $\mathcal{I} = \{0, 1, \ldots, N\}$. For any tail-biting path $(s, a) \in \text{TBP}(\sigma)$ of length $N$ on $T$, we can first circularly shift it to some other tail-biting path $(s^{(0)}, a^{(0)}) \in \text{TBP}(\sigma)$ on $T$ such that $s^{(0)}_0 = s^{(0)}_N = \sigma_i$.

Now, we examine $s^{(0)}$ over $\mathcal{I}$. If $s^{(0)}$ is already an element of IEE$(\sigma)$, then there is nothing to prove. Otherwise, there exists a time index $j$, $0 < j < N$, such that $s_j = \sigma_i$. In this case, we break the tail-biting path $(s^{(0)}, a^{(0)})$ at time $j$ into two sub-paths $(s^{(1)}, a^{(1)})$ and $(s^{(2)}, a^{(2)})$, where

$$ s^{(1)} = (s_0, s_1, \ldots, s_j), \quad a^{(1)} = (a_0, a_1, \ldots, a_j), $$
$$ s^{(2)} = (s_j, s_{j+1}, \ldots, s_N), \quad a^{(2)} = (a_{j+1}, a_{j+2}, \ldots, a_{N-1}). $$

Note that after segmentation of $(s^{(0)}, a^{(0)})$, the resultant two sub-paths, $(s^{(1)}, a^{(1)})$ and $(s^{(2)}, a^{(2)})$, still meet the tail-biting condition. Repeat the above procedures on $(s^{(1)}, a^{(1)})$ and $(s^{(2)}, a^{(2)})$. Since the length of new sub-path is strictly decreasing after each segmentation, the boundary case is the atomic sub-path $(s, a)$ satisfying $s = (\sigma, \sigma)$, which is clearly an element of IEE$(\sigma)$. Thus, we end up obtaining sub-paths that are all elements of TBP$(\sigma)$. Concatenating them yields the circularly shifted tail-biting path $(s^{(0)}, a^{(0)})$.

Theorem 2 indicates that collecting irreducible error events starting at state $\sigma$ is enough to reconstruct the tail-biting path in set TBP$(\sigma)$, which underlies the collection and search algorithm we are about to propose. In fact, we can also prove Theorem 2 in the perspective of the corresponding state diagram, by demonstrating that each cycle traversing through state $\sigma$ can be decomposed into one or several irreducible cycles traversing through state $\sigma$.

**B. The CRC Design Algorithm for the TBCC**

Assume the tail-biting trellis $T$ of a feedforward, $(n, 1, v)$ convolutional encoder $g(x)$ is given, let $(s, a, u)$ denote the triple of states $s$, outputs $a$ and inputs $u$, where the inputs $u$ are uniquely determined by state transitions $s_i \rightarrow s_{i+1}, i = 0, 1, \ldots, N - 1$. Motivated by the partitioning of $T$ and irreducible error events in Sec. III-A, we propose the collection algorithm and search algorithm to design the degree-$m$ DSO CRC polynomial $p(x)$, as demonstrated in Algorithm 1 and 2 respectively. In the pseudo-code description, we use $u$ and $u_i(x)$ interchangeably to denote the sequence and the corresponding polynomial, respectively.

To visualize the process of the collection algorithm, we can think of the state diagram of the convolutional code, where each cycle in the state diagram with a length equal to the trellis depth represents a tail-biting path. For a given ordering of states $V_0^N = (\sigma_0, \sigma_1, \ldots, \sigma_{2^m - 1})$, once the algorithm finds all irreducible error events starting from $\sigma_0$, the state diagram is reduced by pruning $\sigma_0$ and the incoming and outgoing edges associated with it. The algorithm then finds the irreducible error events starting at $\sigma_1$ on the reduced state diagram. Repeating the above process, the collection algorithm is able to find all sets of irreducible error events.

The search algorithm first constructs each length-$N$ tail-biting path of bounded distance by concatenating irreducible error events of length at most $N$ together and then finds the DSO CRC polynomial. This can be accomplished via
dynamic programming, for which the algorithm enumerates
the concatenations of irreducible error events in the order of
increasing lengths and only reserves concatenated tail-biting
paths of length equal to \( N \) and of distance less than \( d \).

Some remarks are given as follows. First, the collection
contains all length-\( N \) tail-biting paths in the search algorithm.
For example, if the CRC polynomial \( p(x) \) is a minimal polynomial
and inputs \( u(x) \) of a length-\( N \) tail-biting path is divisible by
\( p(x) \), so does its cyclic shift. Thus, we can directly calculate
the number of distinct tail-biting paths in this case.

C. An Illustrative Example

In this section, we consider the feedforward, (2, 1, 3) con-
volutional encoder \( g = (13, 17)_8 \), to exemplify our algorithm.

As discussed in Sec III-A, an alternative way to visualize
the tail-biting path is to consider the tail-biting path as a cycle in
the state diagram depicted in Fig. 2. In Fig. 2, assume that the
first element in \( \mathcal{V}_0^\sigma \) is \( \sigma_0 = (110) \), as marked in yellow. A
tail-biting path \( (s, a) \) starting at \( s_0 = \sigma_0 = (110) \) for trellis
depth \( N = 10 \), is given by
\[
\begin{align*}
    s &= (110, 100, 000, 001, 011, 110, 101, 010, 101, 011, 111, 110) \\
    a &= (10, 11, 11, 00, 10, 01, 00, 10, 11, 01, 01).
\end{align*}
\]

As can be seen, this path can be constructed by concatenating
two irreducible error events (or irreducible cycles) starting at
state \( s_0 = (110) \), \( (s^{(1)}, a^{(1)}) \) in blue and \( (s^{(2)}, a^{(2)}) \) in red,
\[
\begin{align*}
    s^{(1)} &= (110, 100, 000, 001, 011, 110) \\
    a^{(1)} &= (10, 11, 11, 00, 10) \\
    s^{(2)} &= (110, 101, 010, 101, 011, 111, 110) \\
    a^{(2)} &= (01, 00, 10, 11, 01, 01).
\end{align*}
\]

Assume the tail-biting trellis depth \( N = 12 \). Let \( \mathcal{V}_0^\sigma = \{0, 1, \ldots, 7\} \) be a predetermined ordering of states, we obtain sets
\( \text{IEE}(\sigma) \) and \( \text{TBP}(\sigma) \). The cardinality of each set is given in Table I. One can see that the number of irreducible error events
decreases exponentially as the collection algorithm proceeds.

### Table I

| Type of Sets | Cardinality of IEE(\( \sigma \)) and \( \text{TBP}(\sigma) \) |
|--------------|---------------------------------------------------------|
| IEE(\( \sigma \)) | \[ 178, 88, 19, 8, 0, 0, 1 \] |
| TBP(\( \sigma \)) | \[ 2597, 1175, 295, 27, 0, 2, 0 \] |

Since the minimum distance of the TBCC \( C_{TB} \) given by the
tail-biting trellis of depth \( N = 12 \) is \( d_{\text{min}}(C_{TB}) = 6 \), we also present the degree-3 up to 7 DSO CRC generator polynomials
and their associated undetectable error distance spectra of
distance up to 12, presented in Table II. Each CRC generator
polynomial is represented in octal. For example, \((75)_8\) denotes
the polynomial \( p(x) = x^5 + x^4 + x^3 + x^2 + 1 \). For a given
CRC polynomial \( p(x) \), the corresponding row represents the
sequence of numbers of length-\( N \) undetectable tail-biting error
events of increasing distances. Let \( d_{\text{min}}(C_{TB}, p(x)) \) be the
undetectable minimum distance, i.e., the minimum distance
at which an undetectable error event first occurs for a given
TBCC \( C_{TB} \) and CRC generator polynomial \( p(x) \). Table II indicates that, \( d_{\text{min}}(C_{TB}, p(x)) \) is improved as the CRC degree
increases. If \( d_{\text{min}}(C_{TB}, p(x)) \) remains the same for two distinct
degrees, the larger degree CRC polynomial typically would
produce a smaller number of undetectable error events of
distance \( d_{\text{min}}(C_{TB}, p(x)) \).

### IV. Conclusion

In this paper, we propose an efficient algorithm for de-
signing DSO CRC generator polynomials for a given TBCC.
The algorithm is based on decomposing the tail-biting trellis
into several disjoint sets of tail-biting paths that are closed
under cyclic shifts. We also showed that the tail-biting path
in each set can be easily constructed from the irreducible
error events via circularly shifting and/or concatenating, which
greatly facilitates the search of DSO CRCs. Future work
includes finding DSO CRCs for the existing best TBCCs
and comparing the code performance with other state-of-the-art
short blocklength codes.

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