Semi analytical study of lunar transferences using impulsive maneuvers and gravitational capture

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Abstract. In this work we have studied transferences between the Earth and the Moon considering bi- and tri-impulsive conventional maneuvers (like Hohmann) in order to make the acquisition of trajectories which are captured by the Moon’s gravitational field through the Lagrangian equilibrium point L1. Results show that these transfer models offer reduction in the $\Delta V_{\text{Total}}$ of the mission. However, they do not require long transfer times.

Keywords: Lunar transferences, mission design, restricted three-body problem, periodic orbits and gravitational capture.

1. Introduction
In the last twenty years, the activities related with the lunar exploration were intensified by the sending of several probes [1]. In this period we can highlight the detection of water in its solid state in the lunar poles [2] and the announcement of new lunar missions with several objectives, including studies leading to the establishment of permanent bases [1, 3]. Therefore, the need for many unmanned and manned transfer missions between the Earth and the Moon can be foreseen. In this scenery, the control and the reduction of fuel consumption required for transfer maneuvers are needed in order to increase the spacecraft payload. Techniques to transfer spacecrafts between the Earth and the Moon using gravitational capture have been studied and applied since the 1980’s [4, 5]. This work presents semi analytical studies for transferences which consider the gravitational capture phenomenon and bi- and tri-impulsive conventional maneuvers that do not require long transfer times.

2. Mathematical Models
The transfer maneuvers described here are divided into three parts. In the first part, a transfer from a circular parking orbit around the Earth with altitude $h_0$ up to a point in a capture trajectory is performed. This transference is done in two different ways: one based in a bi-impulsive Hohmann transfer and another one based in a tri-impulsive transfer [6]. The second part corresponds to the path up to the Moon by gravitational capture through the Lagrangian equilibrium point L1, and in the third part a maneuver to stabilize a spacecraft in an orbit around the Moon is performed. This latter maneuver may not be necessary for some missions. In the first and third parts the necessary $\Delta V$s can...
be analytically calculated considering the dynamics of the two-body problem. In the second part, the
dynamics of the Restricted Three-body Problem is considered, and the \( \Delta V \)s are calculated taking into
account numerical simulation results.

The Restricted Three-body Problem (R3BP) is well known in the literature [7] and the equations of
motion of a particle (a spacecraft) in the \( xy \) plane of a rotating and barycentric coordinate system, also
called synodic coordinate system, are given in components by

\[
\begin{align*}
\ddot{x} - 2\dot{y} - x &= - \left( \mu_1 \frac{x + \mu_2}{r_{13}^3} + \mu_2 \frac{x - \mu_1}{r_{23}^3} \right) \\
\ddot{y} + 2\dot{x} - y &= - \left( \mu_1 \frac{\mu_2}{r_{13}^3} + \mu_2 \frac{\mu_1}{r_{23}^3} \right) y
\end{align*}
\]

(1.a) (1.b)

\( \mu_1 \) and \( \mu_2 \) are the reduced masses of the Earth and the Moon, respectively, and their sum is 1, and the
mean motion, \( n \), of both around the common center of mass is also equal to 1. \( r_{13} = \sqrt{(x + \mu_2)^2 + y^2} \) and \( r_{23} = \sqrt{(x - \mu_1)^2 + y^2} \) are the distances between the Earth and the particle, and between the Moon
and the particle, respectively. In the R3BP, the angular momentum and the energy are not conserved.
However, it admits an integral of motion called Jacobi constant, \( C_j \), given by

\[
\begin{align*}
\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + 2\left( \frac{\mu_1}{r_{13}} + \frac{\mu_2}{r_{23}} \right) - \dot{x}^2 - \dot{y}^2 - \dot{z}^2 &= C_j
\end{align*}
\]

(2)

The value of \( C_j \) defines the surfaces of zero velocity. These structures delimit the boundaries of
regions in the space where the particle cannot move. The equations 1.(a) and 1.(b) have five particular
solutions which correspond to five points in the synodic coordinate system in which the velocity and
acceleration of a particle are null. These points are called Lagrangian equilibrium points (L1, L2, L3,
L4 and L5). When \( 3.17216 \leq C_j < 3.18834 \) a passage is opened around L1, allowing a particle to move
between the Earth and the Moon. This is the case studied in this work (Fig. 1.(a)). And if \( C_j < 3.17216 \)
another passage is opened around L2 (Fig. 1.(b)), and the particle can also move beyond the Earth-
Moon system. Being \( C_j(L1) = 3.18834 \) and \( C_j(L2) = 3.17216 \) [7, 8].

The phenomena of escape and capture are temporary in the R3BP dynamics [4, 7]. Therefore, we
have considered this characteristic and also a family of periodic orbits around the Moon called H2
family [8]. The initial conditions of this family are given by [6, 8]

\[
(x_0; y_0; \dot{x}_0; \dot{y}_0) = (x_0; 0; 0; \dot{y}_0)
\]

(3)

\[
x_0 = \mu_1 - a_0^*(1 - e_0) \quad \Rightarrow \quad x(L1) \leq x_0 < (\mu_1 - R_{Moon}^*)
\]

(4)

\[
\dot{y}_0 = \left[ \frac{\mu_2 (1 + e_0)}{a_0 (1 - e_0)} \right]^{1/2} + a_0^*(1 - e_0)
\]

(5)

Where \( R_{Moon}^* = (R_{Moon}/384400 \text{ km}) \) is the Moon’s mean radius (\( R_{Moon} \approx 1738 \text{ km} \)). \( a_0^* = (a_0/384400 \text{ km}) \)
is the semimajor axis and \( e_0 \) is the eccentricity of the particle’s osculating initial orbit around the Moon.

The energy of two-bodies does not remain constant in the R3BP. However, its monitoring gives us
an idea about the effects of the Earth and the Moon’s gravitational fields on the motion of particles [9].
Thus, for each trajectory simulated and for each integration step, the Moon-particle two-body energy
has been measured. If this energy becomes positive during the integration, the trajectory is considered
an escape trajectory and it is classified taking into account the time at which the Moon-particle energy
remains negative, or simply by its capture time around the Moon, and also if this escape occurs
through L1 or L2.
In order to analyze the capture through the L1, we have adopted the integration for negative times. It is possible to accomplish this study for positive times, but each initial condition has a time in which it remains in orbit around the Earth, so each trajectory is captured by the Moon at different times. Because of this, the work would become extenuating.

![Figure 1. Lagrangian equilibrium points L1, L2 and L3 associated to the Earth-Moon system in the synodic coordinate system, and surfaces of zero velocities: (a) $C_j(L1) < C_j < C_j(L2)$, (b) $C_j > C_j(L2)$.](image)

From the initial conditions defined by equations (4) and (5) it is possible to generate trajectories and to investigate them before the capture by the Moon. Figure 2 shows one of these escape trajectories with $C_j = 3.1111$. While it remains captured, it reaches 118 km above the Moon’s surface. In this point, the osculating lunar orbital elements are: $a_0 = 23,200$ km, $e_0 = 0.920$, $\omega_0$ (argument of periapsis) = 270°, $\Omega_0$ (longitude of the ascending node) = 270º e $i_0$ (inclination) = 0. During its motion around the Earth, the same trajectory reaches a minimum distance of 144,795.55 km from the Earth and a maximum distance of 294,032.93 km.

3. Transfer Methods and Results
Two points can be also seen in Figure 2. They are B and B’. Point B corresponds to the point with lower distance from the Earth, and B’ to the maximum distance, both belonging to the trajectory whose initial conditions of the lunar osculating orbit are $a_0 = 23,200$ km, $e_0 = 0.920$, $\omega_0 = 270\degree$, $\Omega_0 = 270\degree$ and $i_0 = 0$. From an Earth circular parking orbit with a radius $R_C = h_0 + 6370$ km, two transfers are designed up to the point B and other two up to the point B’. For each point (B and B’) a bi-impulsive transfer (direct transfer) and a tri-impulsive transfer (bi elliptical transfer) are designed. Therefore, four cases are considered. In the first case, a transfer ellipse tangent to the Earth orbit at points P and to the capture trajectory at point B (Fig. 3(a)) is obtained, and the application of two $\Delta V$s is necessary. In the second case, two ellipses with the same apses line are obtained. The first one is tangent to the Earth orbit at P’ and it has an apogee at point A. The second ellipse also has its apogee at point A and its perigee at point B in the capture trajectory (Fig. 3(b)). In the third case (bi-impulsive transference), a transfer ellipse tangent to the Earth’s parking orbit at a point P and also tangent to the capture trajectory at the point B’ is obtained. By analogy, in the fourth case (tri-impulsive transfer), two ellipses are obtained – the first ellipse is tangent to the Earth’s parking orbit at a point P’ and its apogee is at point A’. The second ellipse also has its apogee at point A’ and it is tangent to the capture trajectory at point B’. In all cases, it is necessary to know the capture trajectories velocities at points B and B’. And these quantities are known by the numerical simulations of the capture trajectories.

Figures 3(a) and 3(b) correspond to the first and second cases, respectively, and both represent the capture trajectory’s acquisition at point B (the shorter distance from the Earth). The geometry transfers of the third and fourth cases are similar to the ones in these figures.
Figure 2. Escape and capture trajectory by the Moon through L1, whose orbital elements of the initial osculating lunar orbit are: $a_0 = 23,200$ km, $e_0 = 0.920$, $\omega_0 = 270^\circ$, $\Omega_0 = 270^\circ$ and $i_0 = 0$.

Figure 3. Transfer schemes for the acquisition of the capture trajectories through the L1 at point B. In the first and third cases (bi-impulsive transfers), $\Delta V_1$ is analytically determined and the $\Delta V_2$ is defined by the difference between the velocity of the capture trajectory at point B or B’, and the apogee velocity of the transfer ellipse (Table 1). In second and fourth cases, that is, in the tri-impulsive transfers (bi-elliptic), $\Delta V_1$ is also analytically determined. The determination of ellipses apogee radius, $R_{A12}$ (Fig. 3.(b)), it is linked to the total time of transfer between the Earth and the Moon. In general, the farther from the Earth the apogee of an ellipse is, the lower its velocity. This means, in terms of orbital maneuvers, a smaller $\Delta V_2$ to jump from the first transfer ellipse to the second one, since $\Delta V_2$ is the difference between the apogee velocities of the two transfer ellipses (analytically calculated). Though, starting from certain values of $R_{A12}$ the observed variation for $\Delta V_2$ is small, however, the time of transfer becomes large. Still in the second and fourth cases, there is a $\Delta V_3$ and it corresponds to the difference between the velocities of the capture trajectory in B or B’ and the perigee of the second transfer ellipse. Figure 4.(a) shows the variation of $\Delta V_{\text{Total}}$ ($\Delta V_{\text{Total}} = \Delta V_1 + \Delta V_2 + \Delta V_3$) for one bi-elliptical transfer as a function of the apogee distance from the Earth, $R_{A12}$, and in Figure 4.(b), $\Delta V_{\text{Total}}$ is given as function of the time of transfer. The values are obtained for the trajectory shown in Figure 2 considering that the acquisition of the capture trajectory occurs at point B’. For times of transfers larger than 120 days, which corresponds to $R_{A12} > 1.5 \times 10^6$ km, $\Delta V_{\text{Total}}$ is at 3.785 km/s, a value having little variation. Thus, starting from a certain point, the time of transfer becomes more important in order to determine the transfer parameters than $\Delta V_{\text{Total}}$.
Figure 4. (a) \( \Delta V_{\text{Total}} = \Delta V_1 + \Delta V_2 + \Delta V_3 \) for a bi-elliptical transfer among P' and B' as function of the apogee distance from the Earth, \( R_{A12} \). (b) \( V_{\text{Total}} \) among P' and B' as function of the time of transfer.

Tables 1 and 2 show the results for the capture trajectory in Figure 2. But the technique for calculating \( \Delta V_{\text{Total}} \) can be considered for any trajectories whose initial conditions satisfy eq. (4) and (5). Table 1 brings the values for the bi-impulsive transfer for cases 1 and 3 that correspond to a direct transfer between points P and B, and P and B'. Table 2 brings some values for the tri-impulsive transfer (bi-elliptical) for \( \Delta V_{\text{Total}} \) and makes the relationship between them and the time of transfer, and distance from the apogee to the Earth \( (R_{A12}) \) for case 4. It is possible to observe that for \( \Delta V_{\text{Total}} = 3.800 \text{ km/s} \), the time of transfer is 82 days, and for \( \Delta V_{\text{Total}} = 3.785 \text{ km/s} \), 123 days are needed to complete the maneuver. In other words, there is an increase of 41 days for a reduction of only 15 m/s in the \( \Delta V_{\text{Total}} \). For a \( \Delta V_{\text{Total}} = 3.900 \text{ km/s} \), the time of transfer is of 33.8 days. This value is practically similar to the one in the third case (direct transfer among the points P' and B). An analysis of the second case is similar to the fourth case, however, \( \Delta V_{\text{Total}} \) below 4.000 km/s is obtained only for time of transfers longer than 100 days. This makes any application of this maneuver unfeasible, since there are faster and more economical alternatives.

Finally, it remains to analyze the one which will happen to the spacecraft after it is captured by the Moon. Naturally, it will remain in an orbit around the Moon for some time until a new escape happens. This time may be enough for the development of a lot of missions. The trajectory shown in Figure 2, for instance, this time is 37.4 days. But, if this time is not enough to accomplish a mission. The spacecraft can be stabilized in a keplerian orbit around the Moon. To stabilize the spacecraft in an orbit around the Moon, it would be necessary to apply an extra \( \Delta V \) in a point that corresponds to the larger distance from the Moon (in the Moon’s sphere of influence), or in the point of smaller distance, both extra \( \Delta V \) are analytically calculated. If this maneuver is necessary, the extra \( \Delta V \) should be added to the \( \Delta V_{\text{Total}} \) of tables 1 and 2.

4. Conclusions
The transfers presented in this work are planed starting from the two- and three-body problem dynamics, and they correspond to alternative ways to study Earth-Moon transfers, and they are perfectly feasible. The results, analyzed in terms of \( \Delta V_{\text{Total}} \) and the time of transfer (here presented for the purpose of an example) are attractive; especially if they are compared to the Patched-conic approximation [10] used in the Apollo missions, for instance, whose \( \Delta V_{\text{Total}} \) ranged between 4.100 and 4.500 km/s. The choice, or study, of these alternative transfers necessarily involves the analysis of the time of transfers. In the cases of directed bi-impulsive transfers, for example, between points P and B’, they present \( \Delta V_{\text{Total}} = 3.900 \text{ km/s} \) and the time of transfer is not very long. In this case, 33.26 days. This is a long time for manned missions, but taking into account unmanned missions of many natures, as supply transport, given the \( \Delta V_{\text{Total}} \) reduction; these transferences represent consistent alternative.
Table 1. First and third cases, bi-impulsive transfers between P and B and P and B’ for the trajectory of Figure 2.

| Earth parking orbit and transfer ellipse variables | Case 1 | Case 3 |
|--------------------------------------------------|--------|--------|
| Radius of Earth parking orbit \( R_c = h_0 + 6,370 \text{ km, } (h_0 = 200 \text{ km}) \) | 6,570 km | 6,570 km |
| Velocity of the Earth parking orbit \( V_c = \left[ \frac{GM_E}{R_c} \right]^{1/2} \) | 7.788 km/s | 7.788 km/s |
| Distance Earth of points B and B' \( R_A \) | 144,795.55 km | 294,032.93 km |
| Semimajor axis \( a = \frac{(R_c + R_A)}{2} \) | 75,682.17 km | 150,301.5 km |
| Eccentricity \( e = 1 - \frac{R_c}{a} \) | 0.9131 | 0.9562 |
| Perigee velocity \( V_p = \left[ \frac{GM_E}{(1/a + 2/R_c)} \right]^{1/2} \) | 10.772 km/s | 10.893 km/s |
| Apogee velocity \( V_A = \left[ \frac{GM_E}{(1/a + 2/R_A)} \right]^{1/2} \) | 0.489 km/s | 0.244 km/s |
| \( \Delta V_1 \) \( \Delta V_1 = V_p - V_c \) | 2.984 km/s | 3.104 km/s |
| \( \Delta V_2 \) \( \Delta V_2 = |V_A - V_p| \) or \( |V_A - V_B| \) | 1.431 km/s | 0.696 km/s |
| \( \Delta V_{Total} \) \( \Delta V_{Total} = \Delta V_1 + \Delta V_2 \) | 4.415 km/s | 3.900 km/s |
| Time of transfer \( T_{of} = \frac{2\pi}{\sqrt{\left( a^3 / GM_E \right)}} \) | 1.2 days | 3.66 days |
| Time of periselenium trajectory \( T_1 \) numerically calculated | 23.8 days | 29.6 days |
| Total time of transfer \( T_{total} = T_{of} + T_1 \) | 25 days | 33.26 days |

\( M_E (= 5.9742 \times 10^{24} \text{ kg}) \) is the Earth’s mass, \( G (= 6.67 \times 10^{-20} \text{ km}^3/\text{kg s}^{-2}) \) is the Gravitational constant, \( V_p \) (= 1.920 km/s) is the capture trajectory velocity at the points B and \( V_{B'} \) (= 0.940 km/s) at the point B’.

Table 2. Case 4: tri-impulsive (bi elliptical between P’ and B’).

| Apogee distance (km) | \( \Delta V_{Total} \) (km/s) | Total time of transfer (days) |
|----------------------|-----------------------------|-----------------------------|
| 1,743,349            | 3.785                       | 123.78                      |
| 1,193,109            | 3.800                       | 82.07                       |
| 314,205              | 3.900                       | 33.83                       |

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