Lelong numbers with respect to regular plurisubharmonic weights

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Abstract

Generalized Lelong numbers $\nu(T, \varphi)$ due to Demailly are specified for the case of positive closed currents $T = dd^c u$ and plurisubharmonic weights $\varphi$ with multicircled asymptotics. Explicit formulas for these values are obtained in terms of the directional Lelong numbers of the functions $u$ and the Newton diagrams of $\varphi$. An extension of Demailly’s approximation theorem is proved as well.

1 Introduction

A standard quantitative characteristic for singularity of a plurisubharmonic function $u$ at a point $x \in \mathbb{C}^n$ is its Lelong number

$$\nu(u, x) = \lim_{r \to 0} \int_{|z-x|<r} dd^c u \wedge (dd^c \log |z-x|)^{n-1};$$

here $d = \partial + \bar{\partial}$, $dd^c = (\partial - \bar{\partial})/2\pi i$. When $u = \log |f|$, $f$ being a holomorphic function with $f(x) = 0$, $\nu(u, x)$ is just the multiplicity of the zero of $f$ at the point $x$. The Lelong number can also be calculated as

$$\nu(u, x) = \lim_{r \to -\infty} r^{-1} \sup \{u(z) : |z-x| \leq e^r\} = \lim_{r \to -\infty} r^{-1} M(u, x, r), \quad (1.1)$$

where $M(u, x, r)$ is the mean value of $u$ over the sphere $|z-x| = e^r$, see [6]. Various results on Lelong numbers and their applications to complex analysis can be found in [4], [11], [3], [10].

A more detailed information on the behaviour of $u$ near $x$ can be obtained by means of the refined, or directional, Lelong numbers [7]

$$\nu(u, x, a) = \lim_{r \to -\infty} r^{-1} \sup \{u(z) : |z_k - x_k| \leq e^{ra_k}, 1 \leq k \leq n\}$$

$$\nu(u, x, a) = \lim_{r \to -\infty} r^{-1} \lambda(u, x, ra), \quad (1.2)$$
where \( a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n \) and \( \lambda(u, x, b) \) is the mean value of \( u \) over the set \( \{ z : |z_k - x_k| = \exp b_k, 1 \leq k \leq n \} \).

A general notion of the Lelong number with respect to a plurisubharmonic weight was introduced by J.-P. Demailly [2]. Let \( \varphi \) be a semiexhaustive plurisubharmonic function on a domain \( \Omega \subset \mathbb{C}^n \), that is, \( B_R^\varphi := \{ z : \varphi(z) < R \} \subset \subset \Omega \) for some real \( R \). The value

\[
\nu(u, \varphi) = \lim_{r \to -\infty} \int_{B_r^\varphi} dd^c u \wedge [dd^c \varphi]^{n-1}
\]

is called the generalized Lelong number of \( u \) with respect to the weight \( \varphi \). For a detailed study of this notion, see [4]. An analog of formula (1.1), if \( \varphi \) satisfies \( (dd^c \varphi)^n = 0 \) on \( \Omega \setminus \varphi^{-1}(-\infty) \), is the relation

\[
\nu(u, \varphi) = \lim_{r \to -\infty} \mu_r^\varphi(u)
\]

where \( \mu_r^\varphi \) is the swept out Monge-Ampère measure for \( (dd^c \varphi)^n \) on the pseudosphere \( S_r^\varphi := \{ z : \varphi(z) = r \} \), i.e.

\[
\mu_r^\varphi(u) = \int_{S_r^\varphi} u [(dd^c \varphi_r)^n - (dd^c \varphi)^n]
\]

(\( \varphi_r = \max\{\varphi, r\} \)). In particular, \( \nu(u, x) = \nu(u, \log |\cdot - x|) \) and

\[
\nu(u, x, a) = a_1 \ldots a_n \nu(u, \varphi_{a,x})
\]

with the weight

\[
\varphi_{a,x}(z) = \max_k a_k^{-1} \log |z_k - x_k|.
\]

Actually, the Lelong number \( \nu(u, \varphi) \) is a function of asymptotic behaviour of the functions \( u \) and \( \varphi \) near \( x \), that follows from the Comparison Theorem due to Demailly, which we state here in a form convenient for our purposes.

**Theorem A** ([2], Th. 5.9). Let \( u_1 \) and \( u_2 \) be plurisubharmonic functions on a neighbourhood of a point \( x \in \mathbb{C}^n \), \( \varphi_1 \) and \( \varphi_2 \) be plurisubharmonic weights with \( \varphi_1^{-1}(-\infty) = \varphi_2^{-1}(-\infty) = x \). Suppose that \( u_1(x) = -\infty \),

\[
\limsup_{z \to x} \frac{u_2(z)}{u_1(z)} \leq 1
\]

and

\[
\limsup_{z \to x} \frac{\varphi_2(z)}{\varphi_1(z)} \leq 1.
\]

Then \( \nu(u_2, \varphi_2) \leq \nu(u_1, \varphi_1) \).
The generalized Lelong numbers give a powerful and supple instrument for investigation of singularities of plurisubharmonic functions. Another thing is that one pays for the universality of such numbers with lack of explicit ways for their evaluation. The objectives for the present note are to look for a subclass of the weights which is wide enough and at the same time convenient for treatment. Theorem A suggests that one should try to consider weights with certain "regular" asymptotics. The condition \( \varphi(z) \sim \log |z - x| \) reduces the situation to the standard Lelong numbers and gives nothing new. To deal with more refined asymptotics, we consider here two classes of the weights.

The first one uses the notion of local indicator [12]. Let a plurisubharmonic function \( \Phi \) defined in the unit polydisk \( D = \{ z \in \mathbb{C}^n : |z_k| < 1, 1 \leq k \leq n \} \) be nonpositive there and satisfy the relation

\[
\Phi(z) = \Phi(|z_1|^c, \ldots, |z_n|^c) = c^{-1} \Phi(|z_1|^c, \ldots, |z_n|^c) \quad \forall c > 0.
\]

We will call such functions (abstract) indicators. The homogeneity of indicators implies \( (dd^c \Phi)^n = 0 \) outside the origin, provided \( \Phi^{-1}(-\infty) = 0 \). Such functions seem to be good candidates for the weights we are looking for. Besides, the indicators present a scale of plurisubharmonic characteristics for local behavior of plurisubharmonic functions near their singularity points. Namely, given a plurisubharmonic function \( v \), its local indicator at a point \( x \) is a plurisubharmonic function \( \Psi_{v,x} \) in the unit polydisk \( D \) such that

\[
\Psi_{v,x}(y) = -\nu(v, x, a), \quad a = -(\log |y_1|, \ldots, \log |y_n|).
\]

It is the largest negative plurisubharmonic function in \( D \) whose directional Lelong numbers at 0 coincide with those of \( v \) at \( x \), so

\[
\nu(z) \leq \Psi_{v,x}(z - x) + C
\]

near \( x \). Besides, as was shown in [14], \( \Psi_{v,x} \) can be described as the limit (in \( L^1_{loc} \)) of the sequence

\[
(T_{m,x} v)(y) = m^{-1} v(x + y^m)
\]

as \( m \to \infty \); here \( y^m = (y_1^m, \ldots, y_n^m) \), \( m \in \mathbb{Z}_+ \). Moreover, for a multicircled function \( v \) negative in the unit polydisk, the functions \( v_R(z) := R^{-1} v(|z_1|^R, \ldots, |z_n|^R) \) increase to a function \( V(z) \) as \( R \to +\infty \), and \( V^* = \Psi_{v,0} \).

Local indicators are obviously indicators, and due to relations (1.1) their use is quite efficient. Note that, in view of the definition of the local indicator and relation (1.9), we have

\[
\nu(u, \varphi_{a,x}) = \nu(\Psi_{u,x}, \varphi_{a,x}) = \nu(\Psi_{u,x}, \Psi_{\varphi_{a,x}}).
\]
The second equation is evident since \( \Psi_{\varphi, x} = \varphi_{a, x} \), while the first one is of some interest because it does not follow from Theorem A (no assumption on asymptotic behaviour of \( u \) is made).

We will say that a weight \( \varphi \) with \( \varphi^{-1}(-\infty) = x \) is *almost homogeneous* if it is asymptotically equivalent to its indicator \( \Psi_{\varphi, x} \), that is,

\[
\exists \lim_{z \to x} \frac{\varphi(z)}{\Psi_{\varphi, x}(z - x)} = 1.
\]

(1.12)

It is easy to see that if the limit exists, it necessarily equals 1. Besides, the residual Monge-Ampère measure of every almost homogeneous weight \( \varphi \) at \( x \), \( (dd^c \varphi)^n(x) \), coincides with that of its indicator. An example of such a weight is \( \varphi(z) = \log |F(z)| \) with a holomorphic mapping \( F : \Omega \to \mathbb{C}^m, m \geq n, F^{-1}(0) = x \), such that

\[
\lim_{z \to x} \frac{\log |F(z)|}{\sup_{J \in \omega_x} \log |(z - x)^J|} = 1,
\]

\( \omega_x \) being the collection of all multi-indices \( J \) satisfying \( \partial^J F/\partial z^J(x) \neq 0 \). Generalized pluri-complex Green functions with respect to given indicators \([12]\) give another example of almost homogeneous weights.

Note that every plurisubharmonic weight \( \varphi \) is the limit of a decreasing sequence of almost homogeneous weights with the same local indicator as \( \varphi \). Indeed, these are \( \varphi_N(z) = \sup\{\varphi(z), \Psi_{\varphi, x}(z - x) - N\}, N > 0 \).

Below we show that the generalized Lelong numbers with respect to almost homogeneous weights inherit some nice properties from the standard and directional Lelong numbers. In particular, calculation of \( \nu(u, \varphi) \) can be reduced to that for the indicators, both of \( u \) and \( \varphi \) (note that no regularity of \( u \) is assumed). Namely, let \( \varphi_x(z) := \varphi(z - x) \) and \( \varphi^{-1}(-\infty) = \{0\} \), then for any plurisubharmonic function \( u \) in a domain \( \Omega \subset \mathbb{C}^n \),

\[
\nu(u, \varphi_x) = \nu(\Psi_{u, x}, \varphi) = \nu(\Psi_{u, x}, \Psi_{\varphi, 0}) \quad \forall x \in \Omega
\]

(1.13)

(Theorem \([4]\)), which is an extension of relation (1.11). Further, by a slight modification of Demailly’s arguments \([3]\) we show that any plurisubharmonic function \( u \) in a bounded pseudoconvex domain \( \Omega \subset \mathbb{C}^n \) can be approximated by a sequence of functions

\[
u(u_m, \varphi_x) \leq \nu(u, \varphi_x) \leq \nu(u_m, \varphi_x) + \frac{A}{m} \quad \forall x \in \Omega
\]

(1.14)

with some constant \( A = A(\varphi) \) (Theorem \([3]\)).
Finally, we give a geometric description for the swept out Monge-Ampère measures $\mu^\Phi_r$ for indicators $\Phi$ (Theorem 4), which leads to explicit formulas for the numbers $\nu(u, \varphi)$ with almost homogeneous weights $\varphi$ in terms of the directional Lelong numbers of $u$ and $\varphi$ (Corollary 1). When $\varphi = \log |g|$ with a holomorphic mapping $g$, $g(0) = 0$, this reduces to computation on the Newton diagram of $g$ at the origin.

Another choice of a class of the weights are those whose behaviour near $x$ is asymptotically independent of the arguments of $z_k - x_k$, $1 \leq k \leq n$. Namely, we will say that a plurisubharmonic function $\varphi$ on a domain $\Omega \subset \mathbb{C}^n$ has a multicircled singularity at a point $x \in \Omega$ (or that $\varphi$ is almost multicircled near a point $x \in \Omega$) if there exists a multicircled plurisubharmonic function $\lambda$ (i.e. $\lambda(z) = \lambda(|z_1|, \ldots, |z_n|)$) in a neighbourhood of the origin, such that

$$\exists \lim_{z \to x} \frac{\varphi(z)}{\lambda(z - x)} = 1.$$  

(1.15)

In the terminology of [17], it means that $\varphi$ has a standard singularity generated by a multicircled function. It is easy to see that $\varphi$ has multicircled singularity at $x$ if and only if it satisfies relation (1.15) with $\lambda$ equal to some "circularization" of $\varphi$, say, to the mean value

$$\lambda = \lambda_{\varphi, x}(z) = (2\pi)^{-n} \int_{[0, 2\pi]^n} \varphi(x_1 + z_1 e^{i\theta_1}, \ldots, x_n + z_n e^{i\theta_n}) d\theta$$

(1.16)

or to its maximum on the same set.

Evidently, every almost homogeneous weight is almost multicircled, however the converse is not true. On the other hand, it can be seen that each multicircled weight $\varphi$ has the same residual Monge-Ampère measure at $x$ as its indicator has at the origin [15], so one might hope that the above results could be extended to the whole class of almost multicircled weights. This turns to be really the case, but only when it concerns plurisubharmonic functions $u$ whose $-\infty$ sets do not contain lines parallel to the coordinate axes. Namely, we prove (1.13) and (1.14) for every almost multicircled weight $\varphi$ and functions $u$ with the above extra condition. And, surprisingly, it fails to be true, for instance, when $u(z_1, 0, \ldots, 0) \equiv -\infty$ and $x = 0$.

Note. Throughout the paper, $D$ is the unit polydisk in $\mathbb{C}^n$, $\Omega$ is a domain in $\mathbb{C}^n$, and $\text{PSH}(\Omega)$ is the collection of all plurisubharmonic functions on $\Omega$. If $x \in \Omega$ and a function $u \in \text{PSH}(\Omega)$ is such that its restriction to each line $\{z : z_j = x_j \forall j \neq k\}$, $1 \leq k \leq n$, is not identically $-\infty$, then we will write $u \in \text{PSH}_x(\Omega, x)$. The Lelong number of $u$ at $x$ is denoted $\nu(u, x)$, and $\nu(u, x, a)$ and $\nu(u, \varphi)$ are its directional (1.2) and generalized Lelong numbers (1.3). Any
plurisubharmonic function $\Phi$ in $D$ satisfying (1.7) will be called an indicator, and
the function $\Psi_{v,x}$ defined by (1.8) is the (local) indicator of $v$ at $x$. The class of
all plurisubharmonic weights $\varphi$ in a neighborhood of the origin, $\varphi^{-1}(-\infty) = \{0\}$,
will be denoted by $W$, and its subclasses consisting of almost homogeneous and
almost multicircled weights, in the sense of (1.12) and (1.15), will be denoted by
$W^h$ and $W^m$, respectively.

2 Reduction to indicators

Proposition 1 Let $u \in PSH(\Omega)$, $x \in \Omega$, and the functions $T_{m,x}u$ be defined by
(1.10). Then for each weight $\varphi \in W$ there exists the limit
$$\lim_{m \to \infty} \nu(T_{m,x}u, \varphi) = \nu(\Psi_{u,x}, \varphi).$$

Proof. As was mentioned in Introduction, $T_{m,x}u \to \Psi_{u,x}$ in $L^1_{loc}(D)$ (14, Theorem 8). Semicontinuity theorem for generalized Lelong numbers (Proposition 3.12 of 14) then implies
$$\limsup_{m \to \infty} \nu(T_{m,x}u, \varphi) \leq \nu(\Psi_{u,x}, \varphi).$$
On the other hand, the functions $T_{m,x}u$ have the same indicator at the origin for
all $m$, and it coincides with $\Psi_{u,x}$. By (1.8), $T_{m,x}u \leq \Psi_{u,x} + C_m$ near the origin,
so $\nu(T_{m,x}u, \varphi) \geq \nu(\Psi_{u,x}, \varphi) \forall m$, and the proof is complete.

Now we specify the weight $\varphi$ to be almost multicircled. For a function $f$
defined on a subset of $\mathbb{C}^n$, $f_x(z)$ will denote $f(z - x)$, $x \in \mathbb{C}^n$. Then

\begin{equation}
\nu(u, \varphi_x) = \nu(u - x, \varphi) = \nu(u - x, \lambda)
\end{equation}

with multicircled $\lambda$ from (1.13), the second equation being a consequence of
relation (1.15) in view of Theorem A. Actually, a stronger relation takes place.

Theorem 1 a) If $\varphi \in W^m$, $x \in \Omega$, then for every function $u \in PSH_*(\Omega, x)$

\begin{equation}
\nu(u, \varphi_x) = \nu(\Psi_{u,x}, \varphi) = \nu(\Psi_{u,x}, \Psi_{\varphi,0});
\end{equation}

b) if $\varphi \in W^h$ then (2.2) holds for every function $u \in PSH(\Omega)$;
c) there exist a multicircled function $u$ and a weight $\varphi \in W^m$ such that
$\nu(u, \varphi) > \nu(\Psi_{u,0}, \Psi_{\varphi,0})$.  

Proof. When \( \varphi \) is an indicator (i.e., \( \Psi_{\varphi,0} = \varphi \)), relation (2.2) follows from Proposition 1:

\[
\nu(T_{m,x}u, \varphi) = \lim_{r \to -\infty} \int_{B_r^e} d\ell T_{m,x}u \wedge (d\ell^e \varphi)^{n-1} \]

\[
= \lim_{r \to -\infty} \int_{B_{nr}^e} d\ell u_{-x} \wedge (d\ell^e \varphi)^{n-1}
\]

by the homogeneity of indicators (see (1.7)). The right-hand side equals \( \nu(u_{-x}, \varphi) \), and the statement follows from (2.1) and Proposition 1.

By Theorem A, this implies b).

To prove a), we need the following

**Lemma 1** Let \( v \in \text{PSH}_*(D,0) \) be multicircled in the unit polydisk, then for every \( r \in (0,1) \) there exists a constant \( A > 0 \) such that

\[
v(z) \geq A \sup_j \log |z_j|
\]

for all \( z, |z_k| \leq r, 1 \leq k \leq n \).

**Proof of Lemma 1.** Consider the function \( v_1(\zeta) = v(\zeta, 0, \ldots, 0) - C_1 \) with \( C_1 \) such that \( \sup \{v_1(\zeta) : |\zeta| < 1\} = 0 \). Since the ratio \( v_1(\zeta)/\log |\zeta| \) decreases to \( v_1 \geq 0 \) as \( |\zeta| \to 0 \), we have \( v_1(\zeta) \geq A_1 \log |\zeta| \) for some \( A_1 > 0 \) and all \( \zeta \) with \( |\zeta| \leq r \). Therefore, \( v(z) \geq v_1(z_1) \geq A_1 \log |z_1| + C_1 \geq A_1 \log |z_1|, |z_1| < r \). The same arguments for \( j = 2, \ldots, n \) complete the proof of the lemma.

Let now \( \varphi \in W^m \) and \( u \in \text{PSH}_*(\Omega,0) \). By (2.1) we may assume \( \varphi \) to be multicircled. In this case, \( \nu(u, \varphi_x) = \nu(\lambda_{u,x}, \varphi) \) with the function \( \lambda_{u,x} \) defined by (1.13). So, it suffices to prove the assertion for nonpositive, multicircled functions \( u \in \text{PSH}_*(D_r,0) \) in a polydisk \( D_r = \{z : |z_k| < 2r, 1 \leq k \leq n\}, r \in (0,1/2) \).

As was mentioned in Introduction, the functions

\[
u R(z) = R^{-1} u(|z_1|^R, \ldots, |z_n|^R) \nearrow u(z)
\]

and

\[
\varphi_R(z) = R^{-1} \varphi(|z_1|^R, \ldots, |z_n|^R) \nearrow \Phi(z)
\]

as \( R \to +\infty \), with \( U^* = \Psi_{u,0} \) and \( \Phi^* = \Psi_{\varphi,0} \). Let \( A > 0 \) be chosen as in Proposition 1 for both \( u \) and \( \varphi \), and \( L = 2A \log r \). Denote \( v_R(z) = \max \{u_R(z), L\}, w_R(z) = \max \{\varphi_R(z), L\}, V(z) = \max \{U(z), L\}, W(z) = \max \{\Phi(z), L\}, P(z) = \max \{\Psi_{u,0}(z), L\}, Q(z) = \max \{\Psi_{\varphi,0}(z), L\} \) by the choice of \( L, v_R = u_R \) and \( w_R = \varphi_R \) near \( \partial D_r \) for all \( R \geq 1 \), as well as \( P = \Psi_{u,0} \) and \( Q = \Psi_{\varphi,0} \) there. Then

\[
(2.3) \int_{D_r} d\ell^e v_R \wedge (d\ell^e w_R)^{n-1} = \int_{D_r} d\ell^e u_R \wedge (d\ell^e \varphi_R)^{n-1} \geq \nu(u_R, \varphi_R) = \nu(u, \varphi)
\]
since
\[
\nu(u_R, \varphi_R) = \lim_{r \to 0} \int_{D_r} dd^c u_R \wedge (dd^c \varphi_R)^{n-1} = \lim_{r \to 0} \int_{D_r} dd^c u \wedge (dd^c \varphi)^{n-1} = \nu(u, \varphi).
\]

On the other hand, \(v_R \searrow V, w_R \nearrow V, V^* = P,\) and \(W^* = Q,\) so by the convergence theorem for increasing sequences of bounded plurisubharmonic functions [1],

\[
\lim_{R \to \infty} \int_{D_r} dd^c v_R \wedge (dd^c w_R)^{n-1} = \int_{D_r} dd^c P \wedge (dd^c Q)^{n-1} = \int_{D_r} dd^c \Psi_{u,0} \wedge (dd^c \Psi_{\varphi})^{n-1}.
\]

Being compared with (2.3), it gives us the relation \(\nu(u, \varphi) \leq \nu(\Psi_{u,x}, \Psi_{\varphi,0}).\) As the opposite inequality is true due to Theorem A, the proof of a) is complete.

Finally, consider the function \(\varphi(z_1, z_2) = \max\{-|\log |z_1||^{1/2}, \log |z_2|\}.\) Clearly, \(\Psi_{\varphi,0} \equiv 0.\) At the same time, for the function \(u(z) = \log |z_1|,\) we have \(\nu(u, \varphi) = 1.\)

Remark. If \(u\) is a semiexhaustive plurisubharmonic function in \(\Omega, u^{-1}(-\infty) = x,\) its residual Monge-Ampère measure \((dd^c u)^{n}(\{x\})\) at \(x\) is just its generalized Lelong number \(\nu(u, \varphi)\) with respect to the weight \(\varphi = u,\) while the residual measure of its indicator \(\Psi_{u,x}\) is, by the definition, the Newton number of \(u\) at \(x\) [14]. Theorem [1] then implies in particular that for every almost multicircular function \(u \in \text{PSH}_s(\Omega, x),\) its residual measure equals its Newton number, the result proved earlier in [14].

3 Approximation theorems

An important theorem on approximation of plurisubharmonic functions was obtained by J.-P. Demailly ([3], Proposition 3.1). Let \(\Omega\) be a bounded pseudoconvex domain, \(u \in \text{PSH}(\Omega),\) and \(\{\sigma_{ml}\}_l\) be an orthonormal basis of the Hilbert space

\[
H_m := H_m, u(\Omega) = \{f \in Hol(\Omega) : \int_\Omega |f|^2 e^{-2mu} \beta_n < \infty\}.
\]

Consider the functions

\[
(3.1) \quad u_m = \frac{1}{2m} \log \sum_l |\sigma_{ml}|^2 \in \text{PSH}(\Omega).
\]

Then there are constants \(C_1, C_2 > 0\) such that for any point \(z \in \Omega\) and every \(r < \text{dist}(z, \partial \Omega),\)

\[
u(z) - \frac{C_1}{m} \leq u_m(z) \leq \sup_{\zeta \in B_r(z)} u(\zeta) + \frac{1}{m} \log \frac{C_2}{r^n}.
\]
In particular, \( u_m \to u \) pointwise and in \( L^1_{loc}(\Omega) \), and
\[
\nu(u, x) - \frac{n}{m} \leq \nu(u_m, x) \leq \nu(u, x) \quad \forall x \in \Omega.
\]

Here we will show that actually the singularities of the functions \( u_m \) are almost the same that of \( u \) not only in the sense of the standard Lelong numbers but also with respect to arbitrary almost homogeneous weights, as well as with almost multicircled weights (subject to the same restriction as in Theorem [4]).

We start with the following modification of Demailly’s result.

**Theorem 2** Given a bounded pseudoconvex domain \( \Omega \), there are constants \( C_1, C_2 > 0 \) such that for any function \( u \in PSH(\Omega) \) and every \( z \in \Omega \), the functions \( u_m \) defined by (3.1) satisfy the relations
\[
(3.2) \quad u(z) - \frac{C_1}{m} \leq u_m(z) \leq \sup_{\zeta \in D_r(z)} u(\zeta) + \frac{1}{m} \log \frac{C_2}{r_1 \ldots r_n} m
\]
for all \( r = (r_1, \ldots, r_n) \) such that \( D_r(z) = \{ \zeta : |z_k - \zeta_k| < r_k, 1 \leq k \leq n \} \subset \subset \Omega \), and
\[
(3.3) \quad \Psi_{u,x}(y) \leq \Psi_{u_m,x}(y) \leq \Psi_{u,x}(y) - m^{-1} \log |y_1 \ldots y_n| \quad \forall x \in \Omega, \forall y \in D.
\]

**Proof.** The first inequality in (3.2) is the same as in Demailly’s Approximation theorem, and we repeat its proof here for completeness. Let \( u(z) \neq -\infty \). By the Ohsawa-Takegoshi extension theorem [13] applied to the 0-dimensional subvariety \( \{ z \} \subset \Omega \), for any \( a \in \mathbb{C} \) there exists a holomorphic function \( f \) on \( \Omega \) such that
\[
f(z) = a \quad \text{with a constant } C = C(n, \text{diam } \Omega).
\]
Choosing \( a \) such that the right-hand side equals 1 we get \( f \in B_m \), the unit ball in the space \( H_m \). Since
\[
(3.4) \quad u_m(\zeta) = \sup_{g \in B_m} \log \frac{|g(\zeta)|}{m} \quad \forall \zeta \in \Omega,
\]
it gives us
\[
u_m(z) \geq \frac{\log |f(z)|}{m} = \frac{\log |a|}{m} = u(z) - \frac{\log C}{2m}.
\]

The proof of the second inequality in (3.2) is a slight modification of the corresponding arguments from [3]. For any \( g \in H_m \),
\[
|g(z)|^2 \leq \frac{1}{\pi^n r_1^2 \ldots r_n^2} \int_{D_r(z)} |g(\zeta)|^2 dV
\]
\[
\leq \frac{1}{\pi^n r_1^2 \ldots r_n^2} \exp\{2m \sup_{\zeta \in D_r(z)} u(\zeta)\} \int_{D_r(z)} \sup_\zeta u(\zeta) |g|^2 \exp\{-2mu\} dV,
\]

and so by (3.4)
\[ u_m(z) \leq \sup_{\xi \in D_r(z)} u(\xi) + \frac{1}{m} \log \frac{\pi^{n/2}}{r_1 \cdots r_n}. \]

The first inequality in (3.2) implies \( \Psi_{u,x} \leq \Psi_{u_m,x} \) \( \forall x \in \Omega \). To get the other bound, take any \( y \in D \), \( y_1 \ldots y_n \neq 0 \), and \( R > 0 \). Then for \( r_k = |y_k|^R < \dist(x, \partial \Omega) \),
\[ \frac{1}{R} \sup_{D_r(x)} u_m(\zeta) \leq \frac{1}{R} \sup_{D_{2r}(x)} u_m(\zeta) - \frac{1}{m} \log |y_1 \cdots y_n| + \frac{C}{R^m}, \]
and the limit transition as \( R \to \infty \) gives us the desired inequality in view of the definition of the indicator (1.8) and (1.2).

Remark. In terms of the directional Lelong numbers, relations (3.3) have the form
\[ \nu(u, x, a) \leq \nu(u_m, x, a) \leq \nu(u, x, a) + m^{-1} \sum_{1 \leq k \leq n} \tau_k(\varphi) \] \( \forall x \in \Omega \), \( \forall a \in \mathbb{R}^n_+ \), so
\[ (3.5) \lim_{m \to \infty} \nu(u_m, x, a) = \nu(u, x, a) \quad \forall a \in \mathbb{R}^n_+. \]

For the indicators, the relation corresponding to (3.5) is true for all \( y \) with \( y_1 \ldots y_n \neq 0 \), while when \( y_1 \ldots y_n = 0 \) the regularization is needed:
\[ \Psi_{u,x}(y) = \lim_{y' \to y} \lim_{m \to \infty} \Psi_{u_m,x}(y'). \]

**Theorem 3** In the conditions of Theorem 2,
\[ \nu(u_m, \varphi_x) \leq \nu(u, \varphi_x) \leq \nu(u_m, \varphi_x) + m^{-1} \sum_{1 \leq k \leq n} \tau_k(\varphi) \] \( \forall x \in \Omega \)
for any almost homogeneous weight \( \varphi \) and \( \tau_k(\varphi) = \nu(\log |z_k|, \varphi) \).

The same is true for every weight \( \varphi \in W^m \) provided \( u \in PSH_*(\Omega, x) \).

**Proof.** Let \( \varphi \in W^h \). Denote \( \Phi = \Psi_{\varphi,0} \). By Theorems 1 and 2,
\[ \nu(u_m, \varphi_x) = \nu(\Psi_{u_m,x}, \Phi) \leq \nu(\Psi_{u,x}, \Phi) = \nu(u, \varphi_x). \]

Similarly,
\[ \nu(u, \varphi_x) = \nu(\Psi_{u,x}, \Phi) \leq \nu(\Psi_{u_m,x}, \Phi) + m^{-1} \sum_{1 \leq k \leq n} \nu(\log |y_k|, \Phi) \]
\[ = \nu(u_m, \varphi_x) + m^{-1} \sum_{1 \leq k \leq n} \tau_k(\varphi). \]

If \( u \in PSH_*(\Omega, x) \), then (3.3) implies \( u_m \in PSH_*(\Omega, x) \) for each \( m \), so all the above arguments work with arbitrary weights \( \varphi \in W^m \), too. The only exception is the relation \( \nu(\log |y_k|, \Phi) = \tau_k(\varphi) \) which is to be replaced with the inequality \( \nu(\log |y_k|, \Phi) \leq \tau_k(\varphi) \).
4 Swept out Monge-Ampère measures

The role of the swept out Monge-Ampère measures $\mu^r_\varphi$ (1.4) is demonstrated by the Lelong-Jensen-Demailly formula [4]: if $u$ is a plurisubharmonic function in the pseudoball $B^\varphi_R$, then for any $r < R$,

$$\mu^r_\varphi(u) - \int_{B^\varphi_r} u(dd^c \varphi)^n = \int_{-\infty}^r \left[ \int_{B^\varphi_t} dd^c u \wedge (dd^c \varphi)^{n-1} \right] dt.$$ 

When $(dd^c \varphi)^n = \tau \delta_x$, $\delta_x$ being the Dirac $\delta$-function at $x$, the Lelong-Jensen-Demailly formula leads to the representation formula for plurisubharmonic functions:

$$u(x) = \tau^{-1} \mu^r_\varphi(u) + \tau^{-1} \int_{-\infty}^r \left[ \int_{B^\varphi_t} dd^c u \wedge (dd^c \varphi)^{n-1} \right] dt,$$

and $\nu(u, \varphi) = \lim_{r \to -\infty} r^{-1} \mu^r_\varphi(u)$.

In the case of $\varphi = \varphi_{a,x}$ given by (1.6), $\mu^r_\varphi = (a_1 \ldots a_n)^{-1} m_{ra}$ with $m_{ra}$ the normalized Lebesgue measure on $\{|z_k| = \exp\{ra_k\}, 1 \leq k \leq n\}$. No explicit formulas are available in the general situation. However when studying singularities, one interests mainly in asymptotic behavior of $\mu^r_\varphi$ as $r \to -\infty$. For regular weights $\varphi$ it means that we may restrict ourselves to study of the swept out measures for their indicators $\Phi = \Psi_{\varphi,x}$.

Since $(dd^c \Phi)^n = 0$ on $D \setminus \{0\}$, $\mu^r_\varphi = (dd^c \Phi_r)^n$ for each $r < 0$. The function $\Phi_r$ is invariant under the rotations

$$(z_1, \ldots, z_n) \mapsto (z_1 e^{i\omega_1}, \ldots, z_n e^{i\omega_n}),$$

and so is $\mu^r_\varphi$. Therefore we can write

$$\mu^r_\varphi = (2\pi)^{-n} d\theta \otimes d\rho^\varphi_r$$

with some measure $\rho^\varphi_r$ defined on the set $\{a \in \mathbb{R}_+^n : \Phi(a) = r\}$. Moreover, since $\mu^r_\varphi$ has no masses on the pluripolar set $S^\varphi_r \cap \{z : z_1 \ldots z_n = 0\}$, we can pass to the coordinates $z_k = \exp\{t_k + i\theta_k\}, 1 \leq k \leq n$. The functions

$$f(t) = \Phi(e^{t_1}, \ldots, e^{t_n})$$

and

$$f_r(t) = \Phi_r(e^{t_1}, \ldots, e^{t_n}) = \max\{f(t), r\}$$

are convex in $\mathbb{R}^n = -\mathbb{R}_+^n$ and increasing in each $t_k$. Simple calculations show that in these coordinates $\rho^\varphi_r$ transforms to

$$\gamma^\varphi_r = n! \mathcal{MA}[f_r]$$
where $\mathcal{MA}$ is the real Monge-Ampère operator, see the details in [14]. We recall that for smooth functions $v$,

$$\mathcal{MA}[v] = \det \left( \frac{\partial^2 v}{\partial t_j \partial t_k} \right) dt,$$

and it can be extended as a positive measure to any convex function (see [16]). So,

$$\mu_r^\Phi(u) = \int_{\Phi(u)} (2\pi)^{-n} \int_{[0,2\pi]^n} u(z_1 e^{i\theta_1}, \ldots, z_n e^{i\theta_n}) d\theta \, d\rho_r^\Phi(|z_1|, \ldots, |z_n|)$$

$$(\lambda(u, 0, t)$$ is the mean value of $u$ over $\{|z_k| = e^{t_k}, 1 \leq k \leq n\}).$

Since $f_r(t) = |r|f_{-1}(t/|r|),$$$
\mu_r^\Phi(u) = n! \int_{\mathbb{R}^n} \lambda(u, 0, |r|t) \, d\gamma_r^\Phi(t),$$

So, we only have to find an explicit expression for the measure $\gamma_r^\Phi.$

First of all, supp $\gamma_r^\Phi \subset L^\Phi$ with

$$(4.1) \quad L^\Phi = \{t \in \mathbb{R}^n : f(t) = -1\}.$$

If $\Phi = \Psi_{u,x}$, then $L^\Phi = -\{b \in \mathbb{R}^n_+ : \nu(u, x, b) = 1\}.$

As was shown in [16], for any convex function $v$ in a domain $G \subset \mathbb{R}^n$,

$$(4.2) \quad \int_F \mathcal{MA}[v] = \text{Vol} \omega(F, v) \quad \forall F \subset G,$$

where

$$\omega(F, v) = \bigcup_{t^0 \in F} \{a \in \mathbb{R}^n : v(t) \geq v(t^0) + \langle a, t - t^0 \rangle \quad \forall t \in G\}$$

is the gradient image of the set $F$ for the surface $\{y = v(x), \ x \in G\}.$

Given a subset $F$ of $L^\Phi$, we put

$$(4.3) \quad \Gamma^\Phi_F = \{a \in \mathbb{R}^n_+ : \sup_{t \in F} \langle a, t \rangle = \sup_{t \in L^\Phi} \langle a, t \rangle = -1\}$$

and

$$(4.4) \quad \Theta^\Phi_F = \{\lambda a : 0 \leq \lambda \leq 1, \ a \in \Gamma^\Phi_F\}.$$

If $\Phi = \Psi_{u,x}$, then

$$\Theta^\Phi_{L^\Phi} = \{a \in \mathbb{R}^n_+ : \sup_b [\nu(u, x, b) - \langle a, b \rangle] \geq 0\}.$$
Note that $\Gamma^\Phi_{L^\Phi}$ is an unbounded convex subset of $\mathbb{R}^n_+$ and $f$ is the restriction of its supporting function to $\mathbb{R}^n_+$. When $\Phi$ is the indicator of $\log |g|$ for a holomorphic mapping $g = (g_1, \ldots, g_n)$, $g(0) = 0$, the set $\Gamma^\Phi_{L^\Phi}$ is the Newton diagram for $g$ and $\mathbb{R}^n_+ \setminus \Theta^\Phi_{L^\Phi}$ is the Newton polyhedron for $g$ at $0$ as defined in [8]. In this case, $\Gamma^\Phi_F$ is the union of bounded faces of the polyhedron corresponding to $F$.

**Proposition 2** For any compact subset $F$ of $L^\Phi$, $\Theta^\Phi_F = \omega(F, f_{-1})$.

**Proof.** If $a \in \omega(F, f_{-1})$ then for some $t^0 \in F$,

(4.5) $\langle a, t^0 \rangle \geq \langle a, t \rangle - f_{-1}(t) - 1 \quad \forall t \in \mathbb{R}^n_+$.

In particular,

(4.6) $\langle a, t^0 \rangle \geq \langle a, t \rangle \quad \forall t \in L^\Phi$.

When $t \to 0$, (4.3) implies $\langle a, t^0 \rangle \geq -1$. In view of (4.6) it means that $a \in \Theta^\Phi_F$.

Let now $a = \lambda a^0$, $a^0 \in \Gamma^\Phi_F$, $0 \leq \lambda \leq 1$. Then there is a point $t^0 \in F$ such that

$\langle a, t^0 \rangle = \sup_{t \in F} \langle a, t \rangle = \sup_{t \in L^\Phi} \langle a, t \rangle = -\lambda$.

For any $t \in \mathbb{R}^n_+$, $t/|f(t)| \in L^\Phi$. If $f(t) \leq -1$, then

$\langle a, t^0 \rangle \geq \langle a, t/|f(t)| \rangle \geq \langle a, t^0 \rangle = \langle a, t \rangle - f_{-1}(t) - 1$.

If $f(t) = -\delta > -1$, then $t/\delta \in L^\Phi$ and

\[
\langle a, t \rangle - f_{-1}(t) - 1 = \delta \langle a, t/\delta \rangle - 1 + \delta \leq \delta \sup_{s \in L^\Phi} \langle a, s \rangle - 1 + \delta = \delta \langle a, t^0 \rangle - 1 + \delta = -\delta \lambda - 1 + \delta \leq \lambda = \langle a, t^0 \rangle.
\]

The proposition is proved.

**Proposition 3** The measure $\gamma^\Phi_{-1}$ is supported by $E^\Phi$, the set of extreme points of the convex set $\{t : f(t) \leq -1\}$.

**Proof.** As

$$\sup_{t \in L^\Phi} \langle a, t \rangle = \sup_{t \in E^\Phi} \langle a, t \rangle \quad \forall a \in \mathbb{R}^n_+,$$

$\Theta^\Phi_L = \Theta^\Phi_E$. Hence $\gamma^\Phi_{-1}(L^\Phi) = \gamma^\Phi_{-1}(E^\Phi)$ and thus $\gamma^\Phi_{-1}(F) = 0$ for every $F \subset \subset L^\Phi \setminus E^\Phi$.

We have thus obtained the following
Theorem 4 For any plurisubharmonic function $u$ in a neighborhood of the origin and for any indicator weight $\Phi$, the swept out Monge-Ampère measure $\mu^\Phi_r(u)$ on the set $\{\Phi(z) = r\}$, $r < 0$, is determined by the formula

$$
\mu^\Phi_r(u) = n! \int_{E^\Phi} \lambda(u, 0, |r|t) d\gamma_{\Phi_{-1}}(t)
$$

where $\lambda(u, 0, |r|t)$ is the mean value of $u$ over the distinguished boundary of the polydisk $\{|z_k| < \exp(|r|t_k), 1 \leq k \leq n\}$ and the measure $\gamma_{\Phi_{-1}}$ on the set $E^\Phi$ of extreme points of the convex set $\{t \in \mathbb{R}^n : \Phi(e^{t_1}, \ldots, e^{t_n}) \leq -1\}$ is given by the relation $\gamma_{\Phi_{-1}}(F) = \text{Vol} \Theta_F^\Phi$ for compact subsets $F$ of $E^\Phi$, $\Theta_F^\Phi$ being defined by (4.3), (4.4), and (4.1).

Corollary 1 If $\varphi$ is an almost homogeneous weight, $\varphi^{-1}(-\infty) = \{x\}$, then for any plurisubharmonic function $u$ near $x$,

$$
\nu(u, \varphi) = n! \int_{E^\Phi} \nu(u, x, -t) d\gamma_{\Phi_{-1}}(t)
$$

with $\Phi = \Psi_{\varphi,x}$ and the measure $\gamma_{\Phi_{-1}}$ the same as in Theorem 4. If $u \in PSH_*(\Omega, x)$ then the formula takes place with arbitrary almost multicircled weight $\varphi$.

So, Corollary 1 gives a quantative expression for the fact that the generalized Lelong number of a plurisubharmonic function $u$ with respect to a regular weight $\varphi$ is completely determined by the directional Lelong numbers of $u$ and $\varphi$.

Note that, under no regularity condition on a weight $\varphi$, we have always the inequality

$$
\nu(u, \varphi) \geq n! \int_{E^\Phi} \nu(u, x, -t) d\gamma_{\Phi_{-1}}(t).
$$

Of course, when $\varphi = \log |g|$ with $g$ a holomorphic mapping, $g(0) = 0$, the set $E^\Phi$ is finite and the measure $\gamma_{\Phi_{-1}}$ charges it with the volumes of the cones generated by the corresponding $(n-1)$-dimensional faces of the Newton polyhedron of $g$.

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