Stability Analysis for Stochastic Hybrid Inclusions

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Abstract—Stochastic hybrid inclusions (SHIs) address situations with the stochastic continuous evolution in a stochastic differential inclusions and random jumps in the difference inclusions due to the forced (the state reaching a boundary in the state space) and/or spontaneous (the state vector may occur spontaneously) transitions. An obvious characteristic of SHIs is the non-uniqueness of random solutions, which can be ensured by the mild regularity conditions, as well as nominal robustness. Basic sufficient conditions for stability/recurrence in probability are usually expressed based on different types of Lyapunov functions, including Lagrange/Lyapunov/Lyapunov-Forster functions respectively for Lagrange/Lyapunov/asymptotic stability in probability and Foster/Lagrange-Forster functions for recurrence, (weaker) relaxed Lyapunov-based sufficient conditions including Matrosov-Foster functions and the stochastic invariance principle, as well as Lyapunov-based necessary and sufficient conditions for asymptotic stability in probability or recurrence (i.e., converse theorems), etc. The converse theorems involving smooth Lyapunov functions are guaranteed by the sequential compactness and thus robustness. In addition, the uniformity property and causality are analyzed for the stabilities in probability. Hence, serving as a partial roadmap for the theoretical development of SHIs, also serving as inspiration, we anticipate that many of the open questions, including the prediction problem, the filtering problem and the control problem, will be resolved based on the techniques of SHIs.

Index Terms—Stochastic hybrid inclusions, stability, Lyapunov functions, converse theorems, robustness, sequential compactness.

I. OVERVIEW

After a series of mature theory results of deterministic hybrid systems [1], which specify a flow/jump set where continuous evolution/instantaneous change may occur and a flow/jump map that determines how continuous evolution/instantaneous change may occur, the research on corresponding stochastic hybrid systems (SHSs) is almost fully mature due to their potential applications in a variety of practical problems in engineering [2]. In SHS models, state changes are instantaneously triggered randomly in time and/or when states arrive a certain state space (i.e., forced transitions), where the jumps are determined via a probability distribution after instantaneous changes. A very general SHS model usually contains a diffusive component in the continuous evolutions [3], [4], or spontaneous transitions in the continuous evolutions [2], [5], [6].

This work surveys a very general SHS called stochastic hybrid inclusions (SHIs), where solutions are measurable mappings from topological space Ω to the set of not identically empty outer semicontinuous mappings (set-valued mappings) from $\mathbb{R}^2$ to $\mathbb{R}^n$ [7, pg. 182 & Thm 5.50], where the values of the mapping are hybrid arcs with closed graphs as defined in [1, Chap. 2] for example. These SHI models do not insist the boundedness of the expected value of the number of jumps in each finite time interval, i.e., Zeno ensues as depicted in [1, Fig. 2.3], where hybrid arcs may be Zeno, eventually discrete, discrete, eventually continuous and continuous.

There exist two types of SHIs with forced transitions, according to whether there are spontaneous transitions [2], [5], [6], or not [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [6]. There exist three types of SHIs with forced transitions, according to set-valued mapping, i.e., only with the jump set-valued mapping (purely discrete-time systems) [8], [9], [10], [12], [13], [15], [19], [20], [26], [22], [25], simultaneously with flow/jump set-valued mappings [2], [11], [14], [16], [17], [18], [21], [23], [24], [6], with jump set-valued mappings and diffusion terms appearing in the flow set-valued mapping [3], [4]. See Table I.

This survey serves as a partial roadmap for the theoretical development of SHIs, including the following properties: 1) A very general characteristic for SHIs, which may specify stochastic differential inclusions possibly with a diffusive component or spontaneous transitions, and difference inclusions. 2) Mild regularity conditions, in order to generate random solutions, robustness, the equivalence between non-uniform and uniform versions of stochastic stability properties, also to ensure the existence of a well-defined Lyapunov-based function for the stability in probability. 3) Non-unique random solutions, which are standard solutions as defined in [1, Chap. 2] with causal measurability properties that are adapted to the minimal filtration of stochastic variables. The non-uniqueness of random solutions comes from the potentially nontrivial overlap between the flow and jump sets, as well as the set-valued flow/jump mappings. 4) Stabilities in probability, such as Lyapunov-based sufficient conditions including Lagrange/Lyapunov/Lyapunov-Forster functions respectively for Lagrange/Lyapunov/asymptotic stability in probability and Foster/Lagrange-Forster functions for recurrence, (weaker) relaxed Lyapunov-based sufficient conditions including Matrosov-Foster functions and the stochastic invariance principle, as well as Lyapunov-based necessary and sufficient conditions for asymptotic stability in probability or recurrence (including converse theorems). 5) Sequential compactness for SHIs without diffusive component in the continuous evolutions, which is crucial to nominal robustness and thus converges Lyapunov theorems, also crucial to the connections between weakly totally recurrent in probability sets and invariant sets. 6) Nominal robustness of stability properties, i.e., stability.
in probability is robust to sufficiently small, positive definite state perturbations, which is crucial for developing converse Lyapunov theorems. There exist three kinds of inflations for SHI models, i.e., slowing down the convergence, perturbations of system data, perturbations in the open bounded set (i.e., a smaller open set within the original set). 7) Causality, where non-causality means that inside information is leaked. The stabilities in probability may not hold if the causality constraint is relaxed; the nominal robustness may not hold if the perturbations, although arbitrarily small, are not strictly causal. 8) Converse theorems, which address Lyapunov-based necessary and sufficient conditions for asymptotic stability in probability or recurrence. Nominal robustness is used to smooth the established Lyapunov-based function in the final step. Finally, we review a classical example on stochastic bouncing ball, in terms of the main theoretical results of stabilities in probability. Space constraints also limit our discussion on other SHs models’ as shown in [2].

The paper is organized as follows: Section II describes a very general model for SHIs, characterizes random solutions under mild regularity conditions. Section III describes stability concepts and a series of Lyapunov-based conditions, including the stochastic invariance principle and Matrosov-Foster functions. Section IV presents additional components of stability theory, as sequential compactness, nominal robustness, causality, and converse theorems. Section VI describes the applications and Section VII concludes the paper.

II. PRELIMINARIES:

A. Basic notation and definitions

We adopt the notation of [1] and [19]. \( \mathbb{R}_+ := [0, \infty) \); \( \mathbb{Z}_+ := \{0, 1, \ldots\} \). \( \mathbb{R}_0 = \mathbb{R}_+ \cup \{\infty\} \). \( \mathfrak{C} \) refers to the closed convex hull. For a closed set \( S \subset \mathbb{R}^n \) and \( x \in \mathbb{R}^n \), \( |x|_S := \inf_{y \in S} |x - y| \). \( \mathbb{B} \) (resp., \( \mathbb{B}^o \)) is the closed (resp., open) unit ball in \( \mathbb{R}^n \). For a closed set \( S \subset \mathbb{R}^n \) and \( \varepsilon > 0 \), \( S + \varepsilon \mathbb{B} \) (resp., \( S + \varepsilon \mathbb{B}^o \)) is the set \( \{ x \in \mathbb{R}^n : |x|_S \leq \varepsilon \} \) (resp., \( \{ x \in \mathbb{R}^n : |x|_S < \varepsilon \} \)). The indicator function \( I_S : \mathbb{R}^n \rightarrow \{0, 1\} \) is defined by \( I_S(x) = 1 \) if and only if \( x \in S \). A function \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \) is upper semicontinuous if \( \limsup_{i \rightarrow \infty} \phi(x_i) \leq \phi(x) \) whenever \( \lim_{i \rightarrow \infty} x_i = x \). \( I_S \) is upper semicontinuous for closed \( S \). Set-valued mappings (often just called “mappings”) are indicated by the symbol \( \exists \). A mapping \( M : \mathbb{R}^p \rightarrow \mathbb{R}^m \) is outer semicontinuous if, for each \( (x_i, y_i) \rightarrow (x, y) \in \mathbb{R}^p \times \mathbb{R}^m \) satisfying \( y_i \in M(x_i) \) for all \( i \in \mathbb{Z}_+ \), \( y \in M(x) \). A mapping \( M \) is locally bounded if, for each bounded set \( K \subset \mathbb{R}^p, M(K) := \bigcup_{x \in K} M(x) \) is bounded. \( B(\mathbb{R}^m) \) is the Borel \( \sigma \)-field on \( \mathbb{R}^m \). Given a measure space \( (\Omega, \mathcal{F}, \mathbb{P}) \), a set \( F \subset \Omega \) is \( \mathcal{F} \)-measurable if \( F \in \mathcal{F} \). Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a measure space, e.g., \( (\mathbb{R}^n, B(\mathbb{R}^m)) \) where \( B(\mathbb{R}^m) \) is the Borel field, i.e., the set of subsets of \( \mathbb{R}^m \) through the algebra of complements and finite and countable unions. A mapping \( M : \mathbb{R}^p \rightarrow \mathbb{R}^n \) is \( \mathcal{F} \)-measurable [27, Def. 14.1] if, for each open set \( O \subset \mathbb{R}^n \), \( M^{-1}(O) := \{ v \in \mathbb{R}^p : M(v) \cap O \neq \emptyset \} \subset \mathcal{F} \). When the values of \( M \) are closed, measurability is equivalent to \( M^{-1}(C) \) being measurable for each closed set \( C \subset \mathbb{R}^n \) [27, Thm. 14.3]. A function \( \alpha : \mathbb{R}_0 \rightarrow \mathbb{R}_0 \) is of class \( K \) if it is continuous, strictly increasing and \( \alpha(0) = 0 \). It is of class \( K_{\infty} \) if it is of class \( K \) and unbounded. A function \( \psi : \mathbb{R}_0 \rightarrow \mathbb{R}_0 \) is of class \( L \) if it is nonincreasing and \( \lim_{t \rightarrow \infty} \psi(t) = 0 \). A function \( \gamma : \mathbb{R}_0 \times \mathbb{Z}_0 \rightarrow \mathbb{R}_0 \) is of class \( K_{\infty}L \) if \( (\gamma(\cdot), k) \in K \) for each \( k \in \mathbb{Z}_0 \) and \( \gamma(s, \cdot) \in L \) for each \( s \in \mathbb{R}_0 \). Given a compact set \( A \subset \mathbb{R}^n, \rho : \mathbb{R}^n \rightarrow \mathbb{R}_0 \) is of class \( PD(A) \) if it is continuous and \( \rho(x) = 0 \Leftrightarrow x \in A \). A function \( \rho : \mathbb{R}_0 \rightarrow \mathbb{R}_0 \) is of class \( PD \) if it is continuous and \( \rho(s) = 0 \Leftrightarrow s = 0 \).

B. Models

We present a very general model for SHIs, usually containing diffusion terms and spontaneous transitions in the continuous evolutions by combining with [1] and [3], which can be specialized into the former two cases in Table I.

In order to cover all the types in Table I, a general model is written formally as

\[
\begin{align}
  x & \in C, \quad dx = F(x) + B(x) dw, \quad \uparrow \lambda(x) \quad \text{(1a)} \\
  x & \in D, \quad x^+ = G(x, v^+), \quad v \sim \mu(\cdot) \quad \text{(1b)}
\end{align}
\]

where \( x \in \mathbb{R}^n \) is the state, \( dx \) represents infinitesimal change in the state (flowing), \( x^+ \) represents instantaneous change in the state (jumping), \( C \) is the flow set (where flowing is allowed), \( F \) is called the drift map, \( B \) is called the dispersion map, \( D \) is the jump set (where jumps are allowed), and \( G \) is the jump map. The notation \( v^+ \) in (1b) is a placeholder for a sequence of independent, identically distributed (i.i.d.) input random variables \( v_i : \Omega \rightarrow \mathbb{R}^m, i \in \{1, 2, \ldots\} \), defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). We use \( \mu \) for the distribution function derived from these i.i.d. random variables. The variable \( w \) in (1a) represents an \( r \)-dimensional Brownian motion (or Weiner process), defined on the same probability space. A measurable jump-rate function \( \lambda : C \rightarrow \mathbb{R}_0 \) determines the probability of jumps at random times during flows, which is specialized to constant 1 in (6).
Often the data of the stochastic hybrid inclusion is written as \( \mathcal{H} = (C, (F, B), D, G, \lambda, \mu) \), which can be specialized into \( \mathcal{H} = (D, G, \lambda, \mu) \) for SDIs, \( \mathcal{H} = (C, (F, D, G, \lambda, \mu) \) for SDDIs and \( \mathcal{H} = (C, (F, B), D, G, \lambda, \mu) \) for SDDIs with \( \lambda \neq 0 \) and/or without \( \lambda = 0 \) spontaneous transitions, as shown in Table 1. Note that [2, Sec. 7] and [6] discuss the spontaneous transitions, while it is mentioned but not fully discussed in [3]. So the main results throughout this paper are the case without spontaneous transitions, unless otherwise indicated.

C. Stochastic hybrid basic conditions

Likewise, we present the mild regularity conditions for SHS [1], adopted from [3] and [4].

**Assumption 1** (Stochastic hybrid basic conditions).

A1) The sets \( C, D \subseteq \mathbb{R}^n \) are closed;
A2) The mapping \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is outer-semicontinuous, locally bounded with nonempty convex values on \( C \);
A3) 3a) The mapping \( G : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is locally bounded;
3b) The mapping \( G(v, x) \) is measurable (with respect to the Borel \( \sigma \)-algebra on \( \mathbb{R}^m \)) with closed-valued;
A4) \( B : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times r} \) is outer semicontinuous, locally bounded and, for each \( x \in C \), \( B(x) = \emptyset \) and \( x \in \mathbb{R}^n \), convex and diagonal convex, i.e., the set \( \{ \sigma \in \mathbb{R}^n \times \mathbb{R}^n : \sigma = b^T b, b \in B(x) \} \) is convex.

**Remark 1.** Now we explain the motivation of Assumption 1: The assumption of closed values in 3b) is equivalent to the outer semicontinuity of \( v \mapsto G(x,v) \) for each \( v \in \mathbb{R}^n \) according to [2, Thm. 5.7(a)]. These regularity conditions in Assumption 1 are used to: (i) generate random solutions to [1] as established in [4], Section IV], based on the conditions of local boundedness and measurability; (ii) guarantee robustness when \( B(x) = 0 \) and thus the equivalence between non-uniform and uniform versions of stochastic stability properties, based on the conditions of local boundedness and outer semicontinuity used in for example [19, Claim 1, Thms. 2-3] and [9, Thms. 4-5] for the difference inclusions, as well as [24] for (1). In particular, the existence of random solutions for SHS \( \mathcal{H} (1) \) can be ensured by the measurability in item 3b) and the locally bounded property in Assumption 1. The measurability in 3b) implies that \( v \mapsto G(x,v) \) is measurable for any \( x \in \mathbb{R}^n \) as analyzed in [13, Prop. 2] and [20, Remark 1] based on [2, Thm. 14.13(b)], which can be also used to establish certain well-posed integrals used later. Conversely, as analyzed in [2] and [10, item 3b) holds if \( G(x,v) \) is outer semicontinuous; holds if the domain of \( v \mapsto G(x,v) \) is countable; also holds if a single-valued mapping \( G \) (i.e., a function) is a Carathéodory mapping [7, Ex. 14.15], i.e., \( x \mapsto G(x,v) \) is continuous and \( x \mapsto G(x,v) \) is measurable. Moreover, local boundedness and outer semicontinuity play an important role in the converse theorem such as [19, Thm. 1] and [9, Thm. 1] for difference inclusion, i.e., to ensure the existence of a continuous Lyapunov-Foster function for the stability in probability of SHS \( \mathcal{H} (1) \) as envisioned in [24, Section 8], which, without those properties, may not exist even for non-stochastic systems [28]. Refer to [16, Exam. 2] that [23, Thm. 6] may fail if Assumption 1 does not hold.

A relationship route is presented in Fig. 1 for item 3b) in Assumption 1. For other features of Assumption 1 which are not used explicitly here, for example, the diagonal convexity in item A4) will eventually play a role in existence results and/or sequential compactness results, which is featured extensively in [29, Chapters 3-7, pg. 133-267]. Assumption 1 essentially agrees with the non-stochastic hybrid basic assumptions in [1] Assumption 6.5, if \( B(x) = 0 \) and \( G \) does not depend on \( v \).

![Fig. 1: A roadmap of the relationship about item 3b) in Assumption 1](image)

A function \( V : \text{dom} V \rightarrow \mathbb{R} \) is a certification candidate for \( \mathcal{H} = (C, (F, B), D, G, \lambda, \mu) \), denoted as \( V \in D(\mathcal{H}) \), i.e.,

**C1.** \( C \cup D \cup G(D \times V) \subset \text{dom} V \);
**C2.** \( 0 \leq V(x) \) for all \( x \in C \cup D \cup G(D \times V) \);
**C3.** the quantity \( \int_{\mathbb{R}^m} \sup_{v \in G(x,v)} V(g) \mu(dv) \) is well-defined for each \( x \in C \cup D \), using the convention that \( \sup_{v \in G(x,v)} V(g) = 0 \) when \( G(x,v) = \emptyset \), justified by the preceding item C2,

**C4.** \( V \) is twice continuously differentiable on an open set containing \( C \), where \( V := \cup_{\omega \in \Omega} \cup_{i \in \mathbb{Z}^n_\geq 1} v_i(\omega) \), and \( \{v_i\}_{i=1}^\infty \) is a sequence of i.i.d. random variables.

Given closed set \( S \subseteq \mathbb{R}^n \), let \( V \in D(\mathcal{H}_{\cap S}) \) be a candidate for \( \mathcal{H}_{\cap S} := (C \cap S, (F, B), D \cap S, G \cap S, \mu) \).

Based on Assumption 1 and [2, Lem. 4.1], any upper semicontinuous function for \( \mathcal{H} \) that satisfies C1-C2 is a certification candidate for \( \mathcal{H} \). Tighter than upper semicontinuity, the certification candidate \( V \) is a partially Lipschitz or partially C1

1Note that when spontaneous transitions exist (i.e., \( \lambda \neq 0 \)), Assumption 1 is not enough for ensuring the equivalence between non-uniform and uniform versions, see [5, Sec. 4.6.2].
for $\mathcal{H} = (C, F, D, G)$ if $V$ is locally Lipschitz (resp., smooth, or equivalently, continuously differentiable) on an open set containing $C \setminus L_V(0)$ as well as continuous on $C$, where $L_V(0) := \{ x \in \text{dom} V : V(x) = 0 \}$. Now we presented

sufficient conditions for the uniform global asymptotic stability in probability as shown in [2, Thm. 4.5]: the existence of a Lyapunov-Foster function implies uniform global asymptotic stability in probability.

D. Random solutions

As presented in [2, Chap.2], random solutions $x(\omega)$ to $\mathbf{1}$ have two features: 1) For each $\omega \in \Omega$, the sample path $x(\omega)$, appropriately coupled with the random input sequence $v(\omega)$, must be a standard solution; we adopt the solution concept of $\mathbf{1}$ in $[10, 19, 20]$ or $[9, 12]$ for more details on SDIs, $[23]$ for SDDIs with spontaneous transitions, and $[4]$ for SDDIs.

The potentially nontrivial overlap between the flow and jump sets, as well as the set-valued flow/jump mappings imply the non-uniqueness of solutions in these SHI models. Here we borrow the analysis in $[4]$ for SDDIs without spontaneous transitions (i.e., $\lambda = 0$).

1) Random inputs. Brownian motion, hybrid filtrations and hybrid stopping times: On a given probability space $(\Omega, F, P)$, define a sequence of i.i.d. random variables $\{v_i\}_{i=1}^{\infty}$ with $v_i : \Omega \to \mathbb{R}^m$ which drives the jumps of the hybrid system $\mathbf{1}$. Define $\tilde{F}_0 := \{0, \emptyset\}$. Denote $\{\tilde{F}_i\}_{i=1}^{\infty}$ as the minimal filtration associated to the random process $\{v_i\}_{i=1}^{\infty}$, $\tilde{F}_i$ is a sub-$\sigma$-field of $\mathcal{F}_i$, generated by $\{v(\omega), \ldots, v_i(\omega)\}$ ([30, Def. 4, pg. 171]). Define $\mathcal{F}_i := \{\tilde{F}_i\}_{i\in\mathbb{Z}}$. Independent of $\mathbb{F}$, the additional filtration $\tilde{F}_i := \{\tilde{F}_i\}_{i\in\mathbb{R}}$ is such that $\tilde{F}_i$ contains all $\mathbb{P}$-negligible sets in $\mathcal{F}_i$. Note that this requirement does not mean that $\tilde{F}_0$ is complete ([22, pg. 1], [31]), since some $\mathbb{P}$-negligible sets in $\mathcal{F}_i$ may not be in the completion of $\tilde{F}_0$ ([32, pg. 4]).

Given the filtered probability space $(\Omega, F, \tilde{F}, \mathbb{P})$, an $\tilde{F}$-adapted Brownian motion relative to this space is a continuous process $w = \{w(t, \cdot)\}_{t \geq 0}$ with the properties that $\sigma(w(s, \cdot), s \in [0, t]) \subset \tilde{F}_t$ for all $t \geq 0$, $w(0, \cdot) = 0$ almost surely and for $0 \leq t < s$ the increment $w(t, \cdot) - w(s, \cdot)$ is normally distributed with mean zero and variance $t - s$ and is independent of $\tilde{F}_s$ ([32, Def. 1.1, pg. 47]).

Simultaneously equipped with the two independent filtration, i.e., the discrete-time case $\tilde{F}$ and the continuous-time case $\mathbb{F}$, the hybrid filtration can be defined as

$$\mathbb{F} := \left\{ \tilde{F}_{t,j} := \bigcap_{\varepsilon > 0} \sigma(\tilde{F}_{t+\varepsilon}, \tilde{F}_j) \right\}_{(t,j) \in \mathbb{R}_0^+ \times \mathbb{Z}_{\geq 0}}$$

For the hybrid filtration, let $(s, i) \preceq (t, j)$ denote $s \leq t$ and $i \leq j$, respectively with $\tilde{F}_s \subset \tilde{F}_t$ ([33, pg. 7]) and $\tilde{F}_i \subset \tilde{F}_j$ ([30, Def. 4, pg. 171]), thus resulting in $\tilde{F}_{s,i} \subset \tilde{F}_{t,j}$ as defined in [4]. As stated in [3, Lemma 1], for the hybrid filtration $\mathbb{F}$ in $\mathbf{1}$, $\tilde{F}_0$ contains all $\mathbb{P}$-negligible events in $\mathbb{F}$ and is right-continuous in $t$ [4], i.e., $\tilde{F}_{t,j} = \cap_{n \geq 0} \tilde{F}_{t,j+n}$ for all $(t, j) \in \mathbb{R}_0^+ \times \mathbb{Z}_{\geq 0}$.

For each $\tilde{F}$-adapted Brownian motion, the increment $w(t, \cdot) - w(s, \cdot) (0 \leq s < t)$ is independent of $\tilde{F}_{s,i}$ for each $j \in \mathbb{Z}_{\geq 0}$.

A hybrid stopping time relative to $\mathbb{F}$ is a pair of $\mathbb{F}$-measurable mappings $T : \Omega \to (\mathbb{R}_0^+ \cup \{\infty\})$ and $J : \Omega \to (\mathbb{Z}_{\geq 0} \cup \{\infty\})$, if

$$(T, J) \preceq (t, j) \in \tilde{F}_{t,j}, \quad \forall (t, j) \in \mathbb{R}_0^+ \times \mathbb{Z}_{\geq 0}.$$
1) $\text{dom}(\phi)$ is a hybrid time domain (as defined in [1] Def. 3, pg. 27);
2) regarded as a set-valued mapping from $\mathbb{R}^2$ to $\mathbb{R}^n$, $\phi$ is outer semicontinuous (as defined in [1] Def. 5.9, pg. 102), i.e., for all $i \in \mathbb{Z}_{\geq 0}$ and for each sequence $(t_i, j_i, z_i) \to (t, j, z)$ satisfying $z_i = \phi(t_i, j_i)$, we have
\[
z = \phi(t, j).
\]
In fact, [3] Thm. 5.7(a), pg. 154) implies that the set graph($\phi$) is closed where graph($\phi$) := \{(t, j, z) \in \mathbb{R}^{n+2} : z = \phi(t, j)\}. For each $i \in \mathbb{Z}_{\geq 0}$, item 2) also implies that the mapping $\phi(\cdot, i) : \mathcal{I}_i \to \mathbb{R}^n$ is continuous if dom(\phi) \cap (\mathcal{R}_{\geq 0} \times \{i\}) := \mathcal{I}_i \times \{i\}$ is nonempty. In contrast, the modifier "relaxed" here is used to emphasize the difference from the fact that, a hybrid arc for deterministic hybrid systems is such that for each $i \in \mathbb{Z}_{\geq 0}$, $t \mapsto \phi(t, i)$ is locally absolutely continuous [1] Def. 2.4, pg. 28]. Thus, we can correspondingly define a stochastic hybrid arc, i.e., a mapping $x$ defined on $\Omega$ satisfying
\[
\text{• for each } \omega \in \Omega, x(\omega) (\text{abbr. } x_\omega) \text{ is a relaxed hybrid arc;}
\]
\text{• the set-valued mapping } x : \Omega \to \mathbb{R}^{n+2}, \text{i.e.,}
\[
\omega \mapsto \text{graph}(x(\omega)) := \\
\{ (t, j, z) \in \mathbb{R}^{n+2} : (t, j) \in \text{dom } x_\omega, z = x_\omega(t, j) \}
\]
is $\mathcal{F}$-measurable with closed values.
Correspondingly, an $\mathcal{F}$-adapted stochastic hybrid arc is a stochastic hybrid arc $x$ where the set-valued mapping
\[
\omega \mapsto \text{graph}(x(\omega)) \cap \{(0, t) \times \{0, 1, \ldots, j\} \times \mathbb{R}^n\}
\]
$s, w \mapsto x_\omega(s, i)$ is measurable.

A partial hybrid signal generator is a mapping $\phi : \text{dom}(\phi) \to \mathbb{R}^{n \times r}$ such that $\text{dom}(\phi)$ is a subset of a hybrid time domain [1] Def. 3, pg. 27]. Thus one can correspondingly define a stochastic partial hybrid signal generator for a given measure space $(\Omega, \mathcal{F})$, i.e., a mapping $f$ satisfying
\[
\text{• for each } \omega \in \Omega, f(\omega) (\text{abbr. } f_\omega) \text{ is a partial hybrid signal generator;}
\]
\text{• viewed as a set-valued mapping, } f : \Omega \times \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}^{n \times r} \text{ is measurable.}

Then, for a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}, \mathcal{P})$, an $\mathcal{F}$-adapted stochastic hybrid signal generator is a stochastic partial hybrid signal generator satisfying for each $(s, i) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, $w \mapsto f_\omega(s, i)$ is $\mathcal{F}_{s,i}$-measurable.

Following the above presented concepts, now we present the solution concept for SHS [1] Def. 5 starting at $x \in \mathbb{R}^n$ such that
\[
\text{• } x \text{ is an } \mathcal{F} \text{-adapted stochastic hybrid arc;}
\]
\text{• } (f, b, g) \text{ are } \mathcal{F} \text{-adapted stochastic partial hybrid signal generators;}
\text{• } w \text{ is an } \mathcal{F} \text{-adapted Brownian motion,}
\]
such that $x(\omega)$ is a solution to [1] for almost every $\omega \in \Omega$ selected by $(f(\omega), b(\omega), g(\omega))$, starting at $x$ with inputs $w(\cdot, \omega)$ and $v(\omega)$. Specifically, for almost every $\omega \in \Omega$,
\[
1) \quad x_\omega(0, 0) = x;
\]
\[
2) \quad \text{for each } j \in \mathbb{Z}_{\geq 0}, \text{ if } I^j(\omega) := \{ t : (t, j) \in \text{dom } x(\omega) \}
\]
has nonempty interiors then, for every $t \in I^j(\omega),$
\[
a) \quad x_\omega(t, j) \in C,
\]
\[
b) \quad (t, j) \in \text{dom } f_\omega \cap \text{dom } b_\omega
\]
\[
ii) \quad f_\omega(t, j) \in F(x_\omega(t, j)), \quad b_\omega(t, j) \in B(x_\omega(t, j)),
\]
\[
iii) \quad x_\omega(t, j) = x_\omega(t, j) + \int_0^t f_\omega(s, j)ds + \int_0^t b_\omega(s, j)dw(s), \text{ where } T_j
\]
is defined in [5] with $\mathcal{H} = \text{dom } x$;
\[
3) \quad \text{if } (t, j), (t, j + 1) \in \text{dom } x(\omega) \text{ then}
\]
\[
a) \quad x_\omega(t, j) \in D,
\]
\[
b) \quad (t, j) \in \text{dom } g(\omega)
\]
\[
c) \quad g_\omega(t, j) \in G(x_\omega(t, j), v_\omega(t+j+1))
\]
\[
d) \quad x_\omega(t, j + 1) = g_\omega(t, j).
\]
Denote $x \in \mathcal{S}_\mathcal{H}(K)$ as a solution $\{x, (f, b, g), w\}$ starting at certain point in $K$. Thus, as stated in [3] Lemma 5], we can get the following hybrid stopping times for a given solution $x$, i.e.,
\[\begin{align*}
\text{• the first time exiting from a closed set } C: \\
(T_C(\omega), J_C(\omega)) := \inf\{ (t, j) : x_\omega(t, j) \notin C \},
\end{align*}\]
\[\begin{align*}
\text{• and the smallest time greater than } \tau \geq 0: \\
(T_\tau(\omega), J_\tau(\omega)) := \inf\{ (t, j) : x_\omega(t, j) : t + j > \tau \},
\end{align*}\]
As stated in [3] Lemma 6], given solution $x$ with dom $x := H$, viewed as a set-valued mapping with the empty value for each $\omega \in \Omega$ and each $i \in \mathbb{Z}_{\geq 0}$ such that $T_i+1(\omega) = \infty$, both the (set-valued) mapping $\omega \mapsto \text{graph}(\omega(\cdot, i))(x(\omega))$ and the mapping $\omega \mapsto x_\omega(T_i+1(\omega), i)$ are $\mathcal{F}_\omega$-measurable. Moreover, $\{j \leq l\} \in \mathcal{F}_i$ if $(T, J)$ is an almost surely finite stopping time with $(T(\omega), J(\omega)) \in \text{dom } x(\omega)$ for almost all $\omega \in \Omega$.

3) Hybrid Itô’s Rule and Dynkin’s Formula: Given a solution with its hybrid stopping time $(T, J)$ (i.e., $(T(\omega), J(\omega))$ is a solution) for almost all $\omega \in \Omega$. Define $J_\omega(\cdot) := \sup\{ j \in \mathbb{Z}_{\geq 0} : x_\omega(s, j) \in \text{dom } x \}$ for each $s \in [0, T(\omega)]$. Then, we have presented Hybrid Itô’s rule and Dynkin’s formula borrowed from [4] Lemmas 7-9.

Hybrid Itô’s Rule [4] Lemma 7]: Let $S \subset \mathbb{R}^n$. Define $\mathcal{H}_{|S} := (C \cap S, (F, B), D \cap S, G \cap S)$, then $\mathcal{H} := \mathcal{H}_{|\mathbb{R}^n}$. Let $V \in \mathcal{D}(\mathcal{H}_{|S})$ when $V : \mathbb{R}^n \to \mathbb{R}$ is continuous on closed set $D \cap S$ and twice continuously differentiable on closed set $C \cap S$. Now we present hybrid Itô’s rule and Dynkin’s formula borrowed from [4] Lemmas 7-9.

Hybrid Dynkin’s formula, part I-II [4] Lemmas 8-9]: Let Assumption 1 hold, let $S \subset \mathbb{R}^n$ be closed, let $V \in \mathcal{D}(\mathcal{H}_{|S})$, let $(x, (f, b, g), w)$ be a solution of $\mathcal{H}_{|S}$ starting at $x$ and let $(T, J)$ be a hybrid stopping time of $x$. Suppose $E[T] < \infty$.
and that either $V$ has compact support or $(T,J)$ is bounded by the first exit time from a compact set. Also suppose that $J$ is uniformly bounded almost surely. Then,

$$
\mathbb{E}[V(x(T,J))] - V(x) = \mathbb{E}\left[\sum_{i=0}^{T-1} (V(g(T_{i+1}, i)) - V(T_{i+1}, i)))\right] + \mathbb{E}\left[\int_0^T \langle \nabla V(x(s,J_s)), f(s,J_s)\rangle ds\right]$$

$$+ \mathbb{E}\left[\int_0^T \frac{1}{2}\text{trace}(b(s,J_s)b^T(s,J_s)\nabla^2 V(x(s,J_s))) ds\right]$$

$$\leq \mathbb{E}\left[\sum_{i=0}^{T-1} \Delta s V(x(T_{i+1}, i)) + \int_0^T \mathcal{L}_s V(x(s,J_s)) ds\right],$$

where

$$\mathcal{L}_s V(x) := \sup_{f \in F(x), b \in B(x)} ((\nabla V(x), f) + \frac{1}{2}\text{trace}(bb^T\nabla^2 V(x)))$$

for all $x \in C \cap S$, and

$$\Delta s V(x) := \sup_{g \in \mathcal{G}(x) \cap S} V(g)\mu(dv) - V(x)$$

for all $x \in D \cap S$ are all well-defined and finite under Assumption 1.

### III. Stability

Based on the proposed stochastic hybrid models, stability results for SHS (1) are obtained via stochastic hybrid tools, such as Lyapunov-based sufficient conditions including Lagrange/Lyapunov/Lyapunov-Foster functions respectively for Lagrange/Lyapunov/asymptotic stability in probability and Foster/Lagrange-Foster functions for recurrence, (weaker) relaxed Lyapunov-based sufficient conditions including Matrosov-Foster functions and the stochastic invariance principle, as well as Lyapunov-based necessary and sufficient conditions for asymptotic stability in probability or recurrence (i.e.,converse theorems), etc. Moreover, before utilizing those techniques, the established models need to satisfy the mild regularity conditions (not necessarily $B(x) = 0$), under which, a sequential compactness property for solutions is preserved leading to the nominal robustness and converse theorems when $B(x) = 0$.

### A. Stability concepts

Essentially, uniform global asymptotic stability in probability (UGASp) is consisting of uniform global stability in probability (UGSp) and uniform global attractivity in probability (UGAp) (2). Before showing that the certification candidate $\mathcal{U}(\xi)$ satisfying Lyapunov-based sufficient conditions is related to the notion of UGSp based on (4) Thm. 3, the following states the definition adopted from (4) Section VI-A.

**Definition 1. [UGASp]** The closed set $\mathcal{A} \subseteq \mathbb{R}^n$ is said to be UGASp if the following conditions hold.

(a) **[UGSp]**

(i) **[Uniform Lyapunov stability in probability (ULySp)]** If there exists $\mu > 0$ such that, for each solution of $dx = F(x)dt + B(x)dw$ with $x \in C \cap (\mathcal{A} + \mu \mathbb{B})$, there are no finite escape times almost surely and for any $\varepsilon, \rho > 0$, there exists $\delta > 0$ such that

$$x \in \mathcal{S}_\varepsilon(\mathcal{A} + \delta \mathbb{B}) \implies \mathbb{P}(\mathcal{A} + \varepsilon \mathbb{B}) \geq 1 - \rho.$$ (9)

(ii) **[Uniform Lagrange stability in probability (ULaSp)]** If for each solution of (10) there are no finite escape times almost surely and for any $\delta, \rho > 0$ there exists $\varepsilon > 0$ such that

$$x \in \mathcal{S}_\varepsilon(\mathcal{A} + \delta \mathbb{B}) \implies \mathbb{P}(\mathcal{A} + \varepsilon \mathbb{B}) \geq 1 - \rho.$$ (10)

In inequality (10), the condition $x \in \mathcal{A} + \varepsilon \mathbb{B}$ is equivalent to $x \in \mathcal{A} + \varepsilon \mathbb{B}$ for all $(t,j) \in [0, T_\xi(\omega)] \times \{0,1, \ldots, J_n(\omega) - 1\}$, where $(T_\xi(\omega), J_n(\omega))$ is the same as that in (4). The set $\omega \in \mathcal{A}$ is measurable based on (7) Thm. 14.3(i) and the measurability of $\omega \mapsto \mathcal{A} \omega$ [7, Prop. 14.11 (b)&(d)], and additionally based on the measurability of $\omega \mapsto \Sigma(\omega)$ and $\omega \mapsto \Sigma(\omega)$ [7 Prop. 14.11 (a)].

(b) **[UGAp]** There are no finite escape times for (10) and for each $\delta > 0, \varepsilon > 0$ and $\rho > 0$, there exists $\tau > 0$ such that

$$x \in \mathcal{S}_\varepsilon(\mathcal{A} + \delta \mathbb{B}) \implies \mathbb{P}(\mathcal{A} + \varepsilon \mathbb{B}) \geq 1 - \rho.$$ (10)

A. Stability concepts

Especially, uniform global asymptotic stability in probability (UGASp) is consisting of uniform global stability in probability (UGSp) and uniform global attractivity in probability (UGAp) (4). Before showing that the certification candidate $\mathcal{U}(\xi)$ satisfying Lyapunov-based sufficient conditions is related to the notion of UGSp based on (4) Thm. 3, the following states the definition adopted from (4) Section VI-A.

**Definition 1. [UGASp]** The closed set $\mathcal{A} \subseteq \mathbb{R}^n$ is said to be UGASp if the following conditions hold.

Note that this is not in contradiction to the fact that a compact set which is weakly totally recurrent in probability is also weakly forward invariant almost surely obtained in (23, Fact 4, 23, Coro. 4). Indeed, weak total recurrence in probability is stronger than recurrence in probability, which can be checked from their definitions in (23, Def. 1&20).
where $\Omega := \{ \omega \in \Omega : \text{graph}(x(\omega)) \subseteq (\Gamma_{<\tau} \times \mathbb{R}^n) \}$ and $\Omega_b := \{ \omega \in \Omega : \text{graph}(x(\omega)) \cap \Gamma_{\leq \tau} \times O \neq \emptyset \}$.

The above recurrence condition (11) where the probability is close to one, asks that the solution is either not defined at time $\tau$ or else it reaches within time $\tau$. Note that the recurrence condition (11) holds for initial conditions $x \in O$ as shown in (8) and so it needs to be checked only for the complement of $O$ denoted as $O^c$. Define the open set $O := \{ x \in \bar{X}, |x|_A \in [0, \delta) \cup (\Delta, +\infty) \}$ with compact set $O^c = \{ x \in \bar{X}, |x|_A \in [\delta, \Delta] \}$ for each $0 < \delta < \Delta < \infty$, where $A = \{ x \in \bar{X}, \hat{\theta} = \text{sgn}(h), \dot{\theta} = 0, c = 0 \}$. Now we present the probability relative to the open set $O$.

Note that we drop the phrase “relative to $\varpi$” in the above definitions with $\varpi(x) := |x|_A$ for all $x \in C \cup D$, see [2] or [4] for more details.

In the following we present two propositions for the connections between ULaSp and UGRp, as well as the connections between UGASp and UGR, which are borrowed from [2] Prop. 2.1.2 and [4]. In what follows, the results on UGASp and UGR are based on the following two propositions, besides the definition of UGASp, i.e., ULaSp, ULySp and UGAP.

**Proposition 1.** The compact set $A \subseteq \mathbb{R}^n$ is UGSp if it is UGRSp and for each $0 < \delta < \Delta < \infty$, the open set $\{ x \in \mathbb{R}^n, |x|_A \in [0, \delta) \cup (\Delta, +\infty) \}$ is UGRp.

**Proposition 2.** The open set $O \subseteq \mathbb{R}^n$ is UGRp if its closure is uniformly Lagrange stable in probability conditioned on not reaching $O$ and, for each $\Delta > 0$, the set $O \cup (\mathbb{R}^n \setminus O + \Delta \mathbb{R})$ is UGRSp.

The next result is used to establish uniform global recurrence, i.e., uniform global recurrence of open set $O \subseteq \mathbb{R}^n$ for $H$ is equivalent to uniform global recurrence of $O \subseteq \mathbb{R}^n$ for the truncated data $H_{\tau}$, where $H_{\tau} = (C \cap \Omega^c, F, D \cap \Omega^c, G \cap \Omega^c, \lambda, \mu)$.

**Proposition 3.** Let $\varpi : C \cup D \to \mathbb{R}_{\geq 0}$ and suppose there are no finite escape times for (1a). The open set $O \subseteq \mathbb{R}^n$ is UGR relative to $\varpi$ for $H$ if and only if it is UGR relative to $\varpi$ for $H_{\tau}$.

As shown in [2], since we do not necessarily assume that the open/closed set is bounded, we ask for some uniformity in probability over solutions from certain sets of initial conditions. As a result, sometimes we drop the word “uniform” for the compact set $A$ or the open, bounded set $O$ in the definitions of stability properties, as ULaSp, ULySp, UGAP, UGR and UGASp, see [14].

**B. Sufficient conditions**

Now we adopt these Lyapunov functions proposed in [2] to analyze stability in probability.

A certification candidate for $H$ (1) is a Lagrange function for $A$ relative to $\varpi$ for (1) if there exist $\alpha_1, \alpha_2 \in G_{\infty}$ such that

$$\alpha_1(|x|_A) \leq V(x), \quad x \in C \cup D \cup G(D \times V), \quad (12a)$$

$$V(x) \leq \alpha_2(\varpi(x)), \quad x \in C \cup D, \quad (12b)$$

$$V(\phi(t)) \leq V(x), \quad x \in C, \phi \in \mathcal{S}_{C,F}(x), \quad t \in \text{dom}(\phi). \quad (12c)$$

$$\int_{\mathbb{R}^m} \sup_{g \in G(x,v)} V(g)\mu(dv) \leq V(x), \quad x \in D. \quad (12d)$$

In addition, it is a Lyapunov function if the above conditions hold with $\alpha_1, \alpha_2 \in K_{\infty}$.

**Theorem 1** (Lagrange/Lyapunov function implies Lagrange/Lyapunov stability) [2 Thm. 4.1&4.2]. Let $A \subseteq \mathbb{R}^n$ be closed and $\varpi : C \cup D \to \mathbb{R}_{\geq 0}$. Suppose Assumption 1 holds and that either $A$ is bounded or there are no finite escape times for (1a) Under these conditions, if there exists a Lagrange/Lyapunov function for $A$ relative to $\varpi$ for (1) then $A$ is ULaSp/ULySp relative to $\varpi$ for (1).

**Theorem 2** (Lagrange-Foster function implies recurrence) [2 Thm. 4.3]. Let $O \subseteq \mathbb{R}^n$ be open and $\varpi : C \cup D \to \mathbb{R}_{\geq 0}$. Suppose Assumption 1 holds and that either $A$ is bounded or there are no finite escape times for (1a) Under these conditions, if there exists a Foster function for $A$ relative to $\varpi$ for (1) then $O$ is UGR relative to $\varpi$ for (1).

A partially Lipschitz certification candidate for $H_{\tau}$ is called a Foster function for $O$ relative to $\varpi$ for (1) if there exist $\rho > 0$ and $\alpha_1 \in G_{\infty}$ such that

$$V(x) \leq \alpha_2(\varpi(x)), \quad x \in C \cup D \cup G(D \times V), \quad (13a)$$

$$V(\phi(t)) \leq V(x) - \rho t, \quad x \in C \cup D, \phi \in \mathcal{S}_{C,F}(x), \quad t \in \text{dom}(\phi). \quad (13b)$$

$$\int_{\mathbb{R}^m} \sup_{g \in G(x,v)} V(g)\mu(dv) \leq V(x) - \rho, \quad x \in D. \quad (13c)$$

In addition, a partially Lipschitz certification candidate for $H_{\tau}$ is a regular Foster function for $O$ relative to $\varpi$ for (1) if it satisfies the conditions (13a) and (13b) and $V^\infty(x,F) \leq 0, \quad x \in C \setminus L_V(0), \quad f \in F(x)$. Note that the expression in (12c) and (13b) are very general, in order to respond to the spontaneous transition during flows. Even if certification candidate $V$ is continuous on $C$, the spontaneous transition of $x$ during $C$ renders that we could use the Clarke generalized directional derivative of $V$, i.e., the regular certification candidate $V$, which implies (12c) to (13b) according to [2] Props. 4.1&4.2&4.3, respectively for Lagrange function, Lyapunov function and Foster function.

**Theorem 3** (Lagrange-Foster function implies recurrence) [2 Thm. 4.4]. Let $O \subseteq \mathbb{R}^n$ be open and $\varpi : C \cup D \to \mathbb{R}_{\geq 0}$. Suppose Assumption 1 holds and that either $A$ is bounded or there are no finite escape times for (1a) Under these conditions, if there exists a regular Lagrange-Foster function $\alpha_1(|x|_A) \leq V(x), \quad x \in C \cup D \cup G(D \times V), \quad (14a)$

$$V(x) \leq \alpha_2(\varpi(x)), \quad x \in C \cup D, \quad (14b)$$

$$V^\infty(x,F) \leq -\rho(x), \quad x \in C \setminus L_V(0), \quad f \in F(x), \quad (14c)$$

$$\int_{\mathbb{R}^m} \sup_{g \in G(x,v)} V(g)\mu(dv) \leq V(x) - \rho(x), \quad x \in D. \quad (14d)$$
for $\mathcal{O}$ relative to $\varpi$ for (1) then $\mathcal{O}$ is UGR relative to $\varpi$ for (1).

A partially Lipschitz (respectively, partially $C^1$) certification candidate for $\mathcal{H}$ is called a regular (respectively, partially $C^1$) Lyapunov-Foster function for $\mathcal{A}$ with respect to $\varpi$ for (1) if there exist $\alpha_1, \alpha_2 \in K_\infty$ and $\rho \in \mathcal{P}(\mathcal{A})$ such that

$$\alpha_1([x]_A) \leq V(x), \quad x \in C \cup D \cup G(D \times V),$$

(15a)

$$V(x) \leq \alpha_2(\varpi(x)), \quad x \in C \cup D,$$

(15b)

$$V^0(x, F) \leq -\rho(x), \quad x \in C \setminus L_V(0), \quad f \in F(x),$$

(15c)

$$\int_{\mathbb{R}^n} \sup_{g \in G(x, x')} V(g) \mu(dv) \leq V(x) - \rho(x), \quad x \in D.$$  

(15d)

**Theorem 4** (Lyapunov-Foster function implies UGASp) \cite{2} Thm. 4.5. Let $\mathcal{A} \subset \mathbb{R}^n$ be closed, $\varpi : C \cup D \to \mathbb{R}_{\geq 0}$ be lower semicontinuous. Suppose that

$$\forall \delta > 0, \exists \delta A > 0 :$$

$$((A + \delta A B)^c \cap (C \cup D)) \subset \{x \in C \cup D : \varpi(x) > \delta A\}.$$  

(16)

Assumption 1 holds, and that either $A$ is bounded or there are no finite escape times for (13). Under these conditions, if there exists a regular Lyapunov-Foster function for $\mathcal{A}$ relative to $\varpi$ for (1), then $\mathcal{A}$ is UGASp relative to $\varpi$ for (1).

Note that the phrase relative to $\varpi$ is dropped in Definition 1 since $\varpi(x) := [x]_A$ for all $x \in C \cup D$. In this case, the mapping $\varpi(x)$ is continuous as it is the Euclidean distance to the closest set $\mathcal{A}$.

Note that in the above conditions \cite{12, 13, 14, 15}, there are two kinds of functions $K_\infty \subset \mathcal{G}^\infty$ for the radial unboundedness of those certification candidates, i.e., $\mathcal{G}^\infty$ for Lagrange function (13). Foster function (14) and Lyapunov-Foster function (15), while $K_\infty$ for Lyapunov function (12) and Lyapunov-Foster function (15). Based on the definitions of $K_\infty, \mathcal{G}^\infty : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we know that $K_\infty$-class function is continuous, strictly increasing, zero at zero and unbounded, while $\mathcal{G}^\infty$-class function is continuous, nondecreasing and unbounded. So it is easy to check that

1) $K_\infty$-class function in (12) and (15) implies $V(x) = 0$ iff $x \in A$ when we set $\varpi(x) := [x]_A$ for all $x \in C \cup D$, as shown in \cite{23} Thm. 8, however, $V(x) = 0$ if $x \in A$ for $\mathcal{G}^\infty$-class function;

2) the radial unboundedness with $K_\infty$-class function is much stronger than that with $\mathcal{G}^\infty$-class function, since $K_\infty$-class function is strictly increasing;

3) for ULYSp or UGASp in (12) and (15), we have to apply $K_\infty$-class function for the radial unboundedness, ULYSp \cite{5} as well as UGASp says that “no matter how close we want the solutions to remain to $A$ (within a distance $c > 0$) with probability, ...” \cite{3}, so that we have to stipulate that $V(x) = 0$ if and only if $x \in A$. Otherwise, if we apply $\mathcal{G}^\infty$-class function, it means $V(x) = 0$ when $x \in A$ or $[x]_A = c$ for certain $c > 0$, then $x$ stays in $(A + cB)$, so it is not possible to let the solutions remain sufficiently close to $A$, i.e., $\varepsilon$ has to be larger than or equal to $c$. However, for ULASP or UGR in (12), (13) and (14), we do not have to do like this and the mild $\mathcal{G}^\infty$-class function is enough, since ULA or UGR is about “no matter how far from $A$ or the closure of $\mathcal{O}$ the solutions start (within a distance $\delta > 0$), ...” \cite{3}.

### C. Relaxed sufficient conditions

As analyzed in \cite{3} Sec. 7, for the case of compact attractors, relaxed sufficient conditions include Matrosov-Foster functions \cite{2, 12, 3} and the stochastic invariance principle \cite{22, 23, 16, 18}.

1) Matrosov-Foster functions for UGR&UGASp: Sometimes finding a single certification candidate satisfying the conditions for UGASp is not easy. In this case, searching for a family of functions is useful as shown in (2), (12), (3) for ensuring UGR, where none of which satisfies the conditions presented in the previous sections, but if taken together, a strong conclusion about asymptotic stability is still permitted.

Given an open set $\mathcal{O} \subset \mathbb{R}^n$ whose complement is compact. Let $m \in \mathbb{Z}_{\geq 1}$. Following \cite{3} §5.2 for $\mathcal{H} = (C, (F, B), D, G, \mu)$ and \cite{2} Sec. 5.2 for $\mathcal{H} = (C, F, D, G, \mu)$, we say that the $m + 1$ functions $V_i$, $i \in \{0, 1, \ldots, m\}$ are Matrosov-Foster functions for $\mathcal{O}$ if there exist continuous functions $u_{c,i} : C \cap \mathcal{O}^c \to \mathbb{R}$ and $u_{d,i} : D \cap \mathcal{O} \to \mathbb{R}$, such that

1) for all $i \in \{0, 1, \ldots, m\}$, $V_i$ is a certification candidate for $(C \cap \mathcal{O}^c, (F, B), D \cap \mathcal{O}^c, G \cap \mathcal{O}^c, \mu)$ that is bounded on $(C \cup D) \cap \mathcal{O}^c$,\footnote{Note that \cite{3} (29) presents a more general form for $\mathcal{H} = (C, F, D, G, \mu)$, i.e., $V_i(\phi(b)) \leq V_i(x)$ and $V_i(x, f) \leq u_{c,i}(x)$ for $x \in C, \phi \in \mathcal{S}_{C, F}(x), f \in F(x), t \in \text{dom}(\phi)$ and $i \in \{1, 2, \ldots, m\}$. The two kinds of form are essentially the same, since generalized directional derivative is an upper bound for the usual directional derivative \cite{35} pg. 99].}

$$\mathcal{L}_{C^0} V_i(x) \leq u_{c,i}(x), \quad \forall x \in C \cap \mathcal{O}^c,$$

(17a)

$$\Delta_{C^0} V_i(x) \leq u_{d,i}(x), \quad \forall x \in D \cap \mathcal{O}^c,$$

(17b)

2a) $u_{c,0}(x) \leq 0$ for all $x \in C \cap \mathcal{O}^c$;

2b) $u_{d,0}(x) \leq 0$ for all $x \in D \cap \mathcal{O}^c$;

2c) with the functions $u_{c,m+1}, u_{d,m+1} : \mathbb{R}^n \to \{1\}$, for each $j \in \{0, 1, \ldots, m\}$ we have

i) if $x \in C \cap \mathcal{O}^c$ and $u_{c,i}(x) = 0$ for all $i \in \{0, 1, \ldots, j\}$ then $u_{c,i+1}(x) \leq 0$,

ii) if $x \in D \cap \mathcal{O}^c$ and $u_{d,i}(x) = 0$ for all $i \in \{0, 1, \ldots, j\}$ then $u_{d,i+1}(x) \leq 0$.

As pointed out in \cite{3} pg. 3131 or \cite{2} pg. 10, item 2) amounts to a nested negative semi-definite condition on functions $u_{c,i}$ and $u_{d,i}$. Items 2a)-2b) require that functions $u_{c,0}$ and $u_{d,0}$ are never positive; while item 2c) enforces that $u_{c,j+1}$ (or $u_{d,j+1}$) can be positive only when at least one of the functions $u_{c,i}$ (or $u_{d,i}$) is negative for $i \in \{0, 1, \ldots, j\}$. The definition of $u_{c,m+1}$ (or $u_{d,m+1}$) in item 2c) also implies that there are no points in $C \cap \mathcal{O}^c$ (or $D \cap \mathcal{O}^c$) such that all of $u_{c,i}$ (or $u_{d,i}$) are zero for each $i \in \{0, 1, \ldots, m\}$. See Fig. 2 for the evolitional relationship among $u_{c,i}$ or $u_{d,i}$ of a family of Matrosov-Foster functions.

Sometimes $u_{c,0}(x) \leq 0$ is written as $u_{c,0}(x) = 0$, as \cite{2} Sec. 5.2. This is because for UGASp or UGR, the precondition for using (relaxed) Matrosov-Foster functions is UGSp or LaSp as shown in Coro. 6&7, i.e., the first one of a family of certification candidates should satisfy $V_0^\infty(x; f) = 0$ or
Let $K$ be a Lyapunov function, as shown in [22], [23], [16] and [18]. Establishes Lyapunov and Lagrange stability in probability, the Feller property, when a Lyapunov function can be found that is stable in probability for (1). For each pair of positive real numbers $\delta \geq 0$ and $\Delta \geq 0$, the system (1) is UGSp if and only if there does not exist an almost surely complete solution $x$ that remains in a non-zero level set of the Lyapunov function almost surely.

For any $u_{c,0}$ and $u_{d,0}$ such that $u_{c,0} < 0$ and $u_{d,0} < 0$, with $i_0 = 1$ and $j_0 = 0$, thus $m = \min\{i_0, j_0\} = 0$.

Fig. 2: Relationship among $u_{c,i}$ and $u_{d,i}$ of a family of Matrosov-Foster functions. $u_{c,i}$ and $u_{d,i}$ share the same number $m$ which is negotiated by the condition $j \in \{1, \ldots, m\}$ in item 2c. More specifically, once $u_{c,i_0} < 0$ holds for certain $i_0$, then it stops at $m = i_0$ since $u_{c,i+1} > 0$ may appear; when once $u_{d,j_0} < 0$ holds for certain $j_0$, then it stops at $m = j_0$ since $u_{d,i+1} > 0$ may appear; and $m := \min\{i_0, j_0\}$, where item 2c implies that the constructed Matrosov-Foster functions are such that $i_0 - j_0 \leq 1$. For example in the stochastic bouncing ball [2 Prop. 6.1], one obtains $u_{c,0} = 0$ and $u_{d,0} < 0$, while $u_{c,1} < 0$ and $u_{d,1} > 0$, with $i_0 = 1$ and $j_0 = 0$, thus $m = \min\{i_0, j_0\} = 0$.

$\int_{g_m} \sup_{g \in \Gamma(x,v)} V_0(g) \mu(dv) = V_0(x)$ (i.e., UGSp or LaSp as shown in [12]). If the first certification candidate satisfies both of $V_0(x; f) < 0$ and $\int_{g_m} \sup_{g \in \Gamma(x,v)} V_0(g) \mu(dv) < V_0(x)$ (i.e., satisfying (15)), then UGSp ensues, so there is only one single function in a family of Matrosov-Foster functions, which is not wrong but defeats the purpose of (relaxed) Matrosov-Foster functions. In other words, at least one of $u_{c,0}(x)$ and $u_{d,0}(x)$ should be zero, when using Matrosov-Foster functions or stochastic invariance principle, as shown in [22]. Exm. 1, [16] Exm. 4, [18] Exm. A) or [2 Sec. 6.1].

Theorem 5 [2 Thm. 5.2]. Let $\mathcal{O} \subset \mathbb{R}^n$ be open and such that $\mathbb{R}^n \setminus \mathcal{O}$ is compact. If there exist Matrosov-Foster functions for $\mathcal{O}$ then there exists a Foster function for $\mathcal{O}$.

According to Thm. 2 where Foster function implies recurrence, Thm. 5 shows that Matrosov-Foster functions for $\mathcal{O}$ imply recurrence of $\mathcal{O}$. Following Prop. 1&2, we obtain the following corollaries.

Corollary 6 (UGR) [2 Coro. 5.3], [3 Thm. 13]. Let $\mathcal{O} \subset \mathbb{R}^n$ be open and bounded. For the system (1), suppose Assumption 1 holds and that $\mathcal{O}$ is ULaSp conditioned on stopping at $\mathcal{O}$. If for each $\Delta > 0$ there exist Matrosov-Foster functions for $\mathcal{O} \cup (\mathbb{R}^n \setminus (\mathcal{O} + \Delta \mathcal{B}))$ for (1) then $\mathcal{O}$ is UGR for (1).

Corollary 7 (UGASp) [2 Coro. 5.4], [3 Thm. 14]. Let $A \subset \mathbb{R}^n$ be compact. Suppose Assumption 1 holds and that $\mathcal{O}$ is UGSp for (1). If for each pair of positive real numbers $(\delta, \Delta)$, there exist Matrosov-Foster functions for $(\mathcal{A} + \delta \mathcal{B}^\circ) \cup (\mathbb{R}^n \setminus (\mathcal{A} + \Delta \mathcal{B}))$ for (1) then $\mathcal{A}$ is uniformly globally asymptotically stable in probability for (1).

2) Stochastic invariance principle for UGASp&UGR: For SHIs satisfying mild regularity conditions (not necessarily the Feller property), when a Lyapunov function can be found that establishes Lyapunov and Lagrange stability in probability, UGASp ensues when there are no almost surely complete solutions remains almost surely in a non-zero level set of the Lyapunov function, as shown in [22], [23], [16] and [18].

Now we present the results in [23] based on stochastic invariance principle for UGASp. Let $K_{\delta, \Delta} := \{x \in \mathbb{R}^n : |x| \in [\delta, \Delta]\}$.

Theorem 8 [23] Thm. 6. Suppose the compact set $\mathcal{A}$ is UGSp for SDDIs in (1). The set $\mathcal{A}$ is UGSp for SDDIs in (1) if for each $0 < \delta < \Delta < \infty$, the set $K_{\delta, \Delta}$ contains no compact set that is almost surely weakly forward invariant for SDDIs in (1). Theorem 9 [23] Thm. 8. Let $V \in \mathcal{A}_0$ be a Lyapunov function relative to a compact set $\mathcal{A} \subset \mathbb{R}^n$ for SDDIs in (1). Then, $\mathcal{A}$ is UGSp if and only if there does not exist an almost surely complete solution $x$ that remains in a non-zero level set of the Lyapunov function almost surely.

Next we present the similar results for UGR.

Theorem 10 [23] Thm. 7. Suppose the compact set $\mathcal{O}$ is ULaSp for SDDIs $\mathcal{H} = (\mathcal{O} \cap \mathcal{A}^\circ, F, \mathcal{D} \cap \mathcal{D}^\circ, G \cap \mathcal{D}^\circ, \lambda, \mu)$ in (1). Then, $\mathcal{O}$ is UGR for SDDIs $\mathcal{H}$ if for each $\Delta > 0$, there does not exist an almost surely weakly forward invariant set contained in the compact set $(\mathcal{O} + \Delta \mathcal{B}) \setminus \mathcal{O}$ for SDDIs $\mathcal{H}$.

Theorem 11 [23] Thm. 9. Let $V \in \mathcal{A}_0$ be a Foster function relative to an open, bounded set $\mathcal{O} \subset \mathbb{R}^n$ for SDDIs $\mathcal{H}$. Then, $\mathcal{O}$ is UGR if and only if there does not exist an almost surely complete solution $x$ that remains almost surely in the set $L_V(c) \cap (\mathbb{R}^n \setminus \mathcal{O})$ for every $c \geq 0$ for which $L_V(c) \cap (\mathbb{R}^n \setminus \mathcal{O})$ is non-empty.

Recurrence principles and the stochastic invariance principle are weak sufficient conditions for stability properties in probability [1 Chap. 8], [22], [23], [23], as shown in Fig. 3 where ‘weak’ means that Lyapunov functions satisfy non-increasing conditions along solutions (as opposed to strong Lyapunov functions whose derivatives along the system solutions are strictly negative outside the attractor). Unlike the invariance principles [34], which can be used to assert convergence of random solutions for time-invariant systems as introduced in [23] Introduction), UGRp can be applied to assert the UGAp of the attractor for SHS $\mathcal{H}$ (1) as shown in Prop. 1, essentially from the point of view of the Matrosov’s theorem [3 Section X-E] or [2 Sec. 5.2]. Indeed, the sufficient recurrence-Lyapunov function in [4 Thm. 5] obviously satisfies the Matrosov property presented in [3] Section X-
E], which thus connects UGAp to UGRp in terms of [4, Thm. 1]. Compared with the invariance principle, Matrosov’s theorem does not necessarily require the knowledge about the system solutions to analyze the asymptotically convergence properties, as analyzed in [10] and [12] for the time-varying, stochastic difference inclusions. When specialized to time-invariant or almost periodic cases, Matrosov properties provide an alternative to the invariance principle for deterministic SHSs [1 Chapter. 8.2] or stochastic systems [16 Chapter 9]. The readers are referred to [22 Section IV] and [23 Section VIII] for comparisons between invariance principle and recurrence. See Fig. 3 for a summary of results on stability properties.

IV. ADDITIONAL COMPONENTS OF STABILITY THEORY: SEQUENTIAL COMPACTNESS, NOMINAL ROBUSTNESS, CAUSALITY AND CONVERSE THEOREMS

A. Sequential compactness

The set of hybrid arcs with closed graphs can be thought of as a subset in the space of not-identically empty-valued outer semicontinuous set-valued mappings from $\mathbb{R}^2$ to $\mathbb{R}^n$. As established in [2 Thm. 5.50, pg. 110&134&138&183], equipped with the metric of graph distance, this space is a separable, locally compact, complete (and $\sigma$-compact) metric space, which we denote $(X, d)$, as presented in [23] for SDDIs, [2, 17] for SDDIs, so the case for SDs obviously holds.

For each $j \in \mathbb{Z}_{>0}$, we define $E_j := \mathbb{R}_{>0} \times \{0, \ldots, j\}$ and for each hybrid arc $\phi$, we define $|\phi|_{E_j}$ to be the hybrid arc with domain $\phi : \mathbb{R}_{>0} \times \{j\}$ such that $\phi|_{E_j}(t, j) = \phi(t, j)$ for all $(t, j) \in \text{dom} \phi|_{E_j}$. For each $S \in X$ we use $S|_{E_j}$ to refer to the outer semicontinuous set-valued mapping from $\mathbb{R}^2$ to $\mathbb{R}^n$ with domain $\text{dom} S \cap E_j$ such that $S|_{E_j}(t, j) = S(t, j)$ for all $(t, j) \in \text{dom} S \cap E_j$.

**Assumption 2** [17]. The functions $\varphi, \varphi_i : X \to \mathbb{R}_{>0}, \ i \in \mathbb{Z}_{\geq 0}$, are upper semicontinuous and bounded, and for each $\epsilon > 0$ there exist $i^*, j^* \in \mathbb{Z}_{\geq 0}$ such that, for each $i \geq i^*$ and each hybrid arc $\phi$, $\varphi_i(\phi) \leq \varphi_i(|\phi|_{E_j}) + \epsilon, \ \forall \ j \geq j^*$ (18) and, for each unbounded $N \subset \mathbb{Z}_{\geq 0}$, and each sequence of locally eventually bounded hybrid arcs $\{S_i\}_{i=0}^\infty$ and each $j \geq j^*$ $\lim_{i \to \infty, i \in N} S_i = S \implies \limsup_{i \to \infty, i \in N} \varphi_i(S_i|_{E_j}) \leq \varphi(S|_{E_j})$, (19)

**Theorem 12** [17, Thm. 1]. Let Assumptions 1 and 2 hold. Let $\{x_i\}_{i=0}^\infty$ be an almost surely locally eventually bounded sequence of solutions, let $\{\Delta_i\}_{i=0}^\infty$ be a sequence of nonnegative real numbers, and suppose $\Delta_i \leq E[\varphi_i(x_i)]$ for each $i \in \mathbb{Z}_{\geq 0}$. Then there exists a solution $x$ belonging to the pointwise outer limit of the sequence $\{x_i\}_{i=0}^\infty$ such that $\lim_{i \to \infty} \Delta_i \leq E[\varphi(x)]$.

**Corollary 13** [17, Coro. 1]. Under the conditions of Thm. 12, if the sequence $\{x_i\}_{i=0}^\infty$ approaches completeness in probability then the solution $x$ in the conclusion of Thm. 12 can, in addition, be chosen to be complete almost surely.

Non-causality means that inside information is leaked, so that exact analysis is based on “−”, compared with the causal case where analysis is based on “−”. See [15, Exm. 1] for that fact that robustness does not holds if the perturbations, although arbitrarily small, are not strictly causal. [2, Sec. 6.2.3] and [14, Exm. 1] for the fact that the stabilities in probability (UGAp or UGR) may not hold if the causality constraint on solutions is relaxed.

A sequential compactness property for solutions is key to nominal robustness [24] and thus converse Lyapunov theorems as analyzed in [5, 13]. Also, the results in [17] for SDDIs as presented above, which use Assumption 1, are crucial for developing the connections between weakly totally recurrent in probability sets and invariant sets, see [23 Thms. 5&8, Prop. 2], [16, Thm. 3, Prop. 2] and [18, Thm. 2] for SDDIs. However, such a result for the case where Brownian motion is involved remains an open problem, as analyzed in [8 Sec. XI], so the nominal robustness and the converse theorems for SDDIs also remain an open problem.

B. Nominal robustness

Nominal robustness, also called inherent robustness asks how large a disturbance a closed-loop system, designed ignoring uncertainty, can tolerate and still maintain stability. Nominal robustness of stability properties, i.e., stability in probability is robust to sufficiently small, positive definite state perturbations, is crucial in developing converse Lyapunov theorems as shown in [9] and [19]. There exist three kinds of inflations for stochastic hybrid inclusion in [1], as shown in [24], [13], [2], [3], [9] and [19].

Before smoothing the constructed Lyapunov-Foster/Lagrange-Foster/Foster function for $\mathcal{H}$, one initially needs to construct such a function from solutions to an inflation of $\mathcal{H}$.

**Case 1: The data of the SHS is modified to slow down recurrence.**

The first inflation is denoted as $\mathcal{H}_\nu$ for SDIs, $x^+ \in G_\nu(x, v) := G(x, v) \cup M_\nu(x)$, (20) where $\nu \in \mathcal{K}_{\infty}$, and for compact set $\mathcal{A}$ in [19], $M_\nu(x) = \begin{cases} 0, & x \in \mathcal{A} + 2\mathbb{B}; \\ \mathcal{A} + \nu(|x|_\mathcal{A})\mathbb{B}, & \text{otherwise}; \end{cases}$, (21) and for open bounded set $\mathcal{O}$ in [9] satisfying $\mathcal{O} + \epsilon \mathbb{B} \subset \mathcal{O}$, $M_\nu(x) = \{x_0\} + \nu(x - x_0)\mathbb{B}, \ x_0 \in \mathcal{O}$. (22)

The first inflation guarantees radial unboundedness of the constructed Lagrange-Foster/Foster function.

Similarly, the first inflation of $\mathcal{H} = (C, F, D, G, \lambda, \mu)$ for SDDIs in [24, Sec. 7] could be denoted as $\mathcal{H}_\nu = (C, F, \mathbb{R}^n, G_\nu, \mu)$ with $M_\nu(x)$ satisfying (22) for any $x \in \mathbb{R}^n$, without loss of generality.

The inflation in Case I with (21) is to possibly slow down but not destroy the convergence to $\mathcal{A}$ of the solutions that start far from $\mathcal{A}$, and finally, to result in a radically unbounded Lyapunov-Foster function in the proof of converse theorem, as
Definition

ULaSp for $A$:
LaF (12) for $\mathcal{H}$

Definition

ULySp for $A$:
LyF (12) for $H$

LyF (15) for $H$:
for $A$

UGAp for $A$:
for $H$

UGSp for $A$:
LaSp&LySp for $H$

UGR:
$O_s$ for $H$

UGASp:
$A$ for $H$

Prop. 1

Thm. 17
Thm. 4 or 17

Prop. 2.4 or
Prop. 3

Thm. 18
Thm. 2 or 18

Fig. 3: Summary of stability/recurrence results, where $O_b := (A + \varepsilon B) \cup (\mathbb{R}^n \setminus (A + \Delta B))$ and $O_s := A + \varepsilon B$ for positive real numbers $\varepsilon$ and $\Delta$. “IP” denotes “invariance principle”, “Laf ” denotes “Lagrange function”, “Lyf ” denotes “Lyapunov function”, “Fof” denotes “Foster function”, “LaF” denotes “Lagrange-Foster function”, and “LyF” denotes “Lyapunov-Foster function”. See [5, Fig. 8&9] for other summaries of stability/recurrence results.

shown in [19] about the converse theorem for UGASp. Similarly, the inflation in Case I with (22) is to preserve recurrence, meanwhile, to make certain quantities grow unbounded in $|x|_\mathcal{O}$ since the certification candidate is related to the expected value of worst case first hitting time to $O$, as shown in [24], which is key to develop converse theorems for recurrence with radially unbounded Foster/Lagrange-Foster functions, see [9].

Anyway, the inflation in Case I is to prevent a function from not being radially unbounded in developing converse theorem for stabilities. As a result, the nominal robustness is key to converse theorems.

**Case II: Perturbing the system data in a sufficiently small manner.**

The second inflation relative to (20) in [9, 19] is, denoted as $H_{\rho,\nu}$ for SDIs,

$$ x^+ \in G_{\rho,\nu}(x,v) := \{ \omega \in \mathbb{R}^n : \omega \in g + \rho(g)B, \ g \in G_{\rho,\nu}(x + \rho(x)B, v) \}, $$

where sufficiently small $\rho \in \mathcal{P}(A)$ or $\rho \in \mathcal{P}(O)$ respectively for compact set $A$ or open bounded set $O$. Sometimes, $A$ is defined as the closure of open bounded set $\hat{O}$. The second inflation guarantees the smoothness of the constructed Lyapunov function. Such perturbations might include passing from constant to slowly-varying parameters or allowing persistent but small measurement errors in a control system as shown in [13]. Specifically, the practical stochastic stability properties ensue in the presence of small constant perturbations, i.e., when $\rho \in \mathcal{P}(A)$ is a small non-zero constant.

Note that $G(x,v) \subset G_{\nu}(x,v) \subset G_{\rho,\nu}(x,v)$ for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^m$ implies that the constructed Lyapunov construction to (23) is radially unbounded.

The second inflation of $\mathcal{H}$ for SDIs [13] or SDDIs [2, 3]
not relative to (20), denoted as $H_{\delta}$, could be written as

\[ C_\delta := \{ x \in \mathbb{R}^n : (x + \delta(x)) \in C \neq \emptyset \}, \]

\[ F_\delta := \text{con} F(x + \delta(x)) + \rho(x), \]

\[ D_\delta := \{ x \in \mathbb{R}^n : (x + \delta(x)) \in D \neq \emptyset \}, \]

\[ G_\delta := \{ g \in \mathbb{R}^n : g = \delta(x) \} \in \mathbb{R}^n \}

where $\delta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is continuous.

**Case III: Robustness to perturbations of the set.**

In fact, according to (24) and (9), there is another type of inflation, i.e., robust recurrence of an open bounded set to sufficient small perturbations in the set, i.e., a smaller open set within the original set.

Note that the three kinds of perturbations are proved to satisfy regularity property, refer to (2), Fact 3.2, (19) Prop. 2&3, (9) Prop. 2&3, (13) Prop. 2] and [6, Sec. 7]. Next we derive the robust stabilities in probability in the following.

Slowing down but not destroying the convergence to $\mathcal{A}$ establishes the robustness of GASp to the inflated data in Case I with (21), as follows borrowed from (19) Thm. 2. See (9) Thm. 4] and (24) Thm. 2 for the robustness of recurrence to the inflated data in Case I with (22), i.e., making the expected value of worst case first hitting time to $\mathcal{O}$ grow unbounded.

**Theorem 14** (Case I: Slowing down the convergence) (19) Thm. 2]. If the compact set $\mathcal{A} \subset \mathbb{R}^n$ is GASp for SDIs/SDDIs in (1) then there exists $\nu \in K_{\infty}$ such that $\nu$ is GASp for (20) and (21).

Perturbing the system data in a sufficiently small manner establishes the robustness of GASp to measurement noise, additive disturbances and parametric uncertainty in the system data, as follows borrowed from (19) Thm. 3]. Refer to (3) Thm. 5] and (24) Thm. 3 for the similar results of robust recurrence.

**Theorem 15** (Case II: Perturbations of system data) (19) Thm. 3]. If the compact set $\mathcal{A} \subset \mathbb{R}^n$ is GASp for SDIs/SDDIs in (1) then there exists $\rho \in PD(A)$ such that $\mathcal{A}$ is GASp for (23).

we now show the recurrence of an open bounded set implies recurrence of a smaller open set within the original set, i.e., robustness to perturbations in the set, borrowed from (24) Thm. 1]. See (9) Thm. 3 for similar results of robust recurrence.

**Theorem 16** (Case III: Perturbations in the set) (24) Thm. 1]. Let the open bounded set $\mathcal{O} \subset \mathbb{R}^n$ be globally recurrent for SDIs/SDDIs in (1). Then, there exists an open bounded set $\hat{\mathcal{O}}$ and $\varepsilon > 0$ such that $\hat{\mathcal{O}} + \varepsilon \mathbb{B} \subset \mathcal{O}$ and $\hat{\mathcal{O}}$ is globally recurrent for SDIs/SDDIs in (1).

**C. Converse theorems**

Converse theorem means Lyapunov-based necessary and sufficient conditions for asymptotic stability in probability or recurrence in (19, 9).

**Theorem 17** (19) Thm. 1]. If the compact set $\mathcal{A} \subset \mathbb{R}^n$ is GASp for $\mathcal{H} = (C \cup D, G, \mu)$ then there exists a smooth on $\mathbb{R}^n \setminus \mathcal{A}$, continuous, (regular) Lyapunov-Foster function relative to $\mathcal{A}$ for (1).

Thm. 17 is a converse of (16) Thm. 1&2] or (2) Thm. 4.5] with the same hybrid model, establishing that Lyapunov-based conditions (Lyapunov-Foster function in (15)) imply GASp for (1) as shown in Thm. 1 and that a continuous Lyapunov-Foster function is a sensible stability analysis tool for SHIs. Also, it shows in (13, Thm. 1&3], (15, Th. 1&2] and (2) Prop. 1] that how to pass from a continuous Lyapunov/Foster function $V$ for $\mathcal{H} = (D, G, \mu)$ to a continuous Lyapunov/Foster function $\Gamma(V)$, where $\Gamma \in K_{\infty}$ is an appropriate concave function, for the sufficiently small, state-dependent, strictly causal, worst-case perturbations. In fact, an appropriate concave function $\Gamma \in K_{\infty}$ is necessary to avoid the integral of function $V$ being unbounded, as shown in (13, Exm. 3).

Next, we present the converse theorem for UGR, as shown in (9).

**Theorem 18** (9, Thm. 2]. The open bounded set $\mathcal{O} \subset \mathbb{R}^n$ is strongly globally recurrent for $\mathcal{H} = (D, G, \mu)$ if and only if there exists a smooth Foster function relative to $\mathcal{O}$ for $\mathcal{H} = (C \cup D, G, \mu)$.

Note that here we did not distinguish the sufficient recurrence-Lyapunov function (i.e., Foster function) and recurrence-Lyapunov function as analyzed in (9), since the latter can be regarded as an (tighter/stronger) extension of Foster function, in order to uniformly bound the expected value of the function for solutions starting from $\mathcal{O}$. Moreover, (9] shows that recurrence-Lyapunov function implies sufficient recurrence-Lyapunov function, but not vice versa (3] Exm. 1); similarly, Lagrange-Foster function implies Foster function (2] Thm. 4.4], but not vice versa. In the above theorems, nominal robustness is used to smooth the established function in the final step, refer to (13] Thm. 1] and (9, Th. 2].

**V. APPLICATIONS**

Stochastic hybrid inclusion (1), including linear time-triggered SHs and state-driven SHs, can be used to model sampled-data stochastic control systems, average dwell-time switching among stochastic differential equations, model spontaneous transitions and a stochastic bouncing ball system as shown in (3]. On the other hand, the stochastic hybrid approach can be used to achieve robustness, i.e., designing hybrid control law, for example, the robust synchronization on the circle in the presence of adversaries while maintaining good recurrence rates in (11, 14], which is extended to robust global attitude control in (37]. Those reasons inspire researchers to delve more deeply into SHs, including the models and potentially significant advantages for further applications.

Next, we present two typical examples for a stochastic bouncing ball system and the robust synchronization on the circle to show the advantages and necessity of SHIs.

**A. Stochastic bouncing ball**

First, we present the stability in probability using the relax sufficient conditions in Sec. III-C i.e., a family of Matrosov-Foster functions (2] Sec. 6.1], (3] XII-C] and stochastic invariance principle (10] Exm. 4], respectively.
The bouncing ball is a system of the form $H = (C, F, D, G, \mu)$ for SDDIs \cite{1} with data

$$
C := \{x \in \mathbb{R}^2 : x_1 \geq 0\},
F(x) := (x_2, -\gamma)^T, \quad \forall x \in C \setminus \{(0,0)\},
F(0) := \{0\} \times [-\gamma, 0],
D := \{x \in \mathbb{R}^2 : x_1 = 0, \ x_2 \leq 0\},
G_g(x, v) := -\sigma(v) x + (0, [0, \gamma])^T,
$$

where the $\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is measurable and $\eta \mathbb{R}_{\geq 0}$. Note that the interval $[0, \eta]$ is added to $-\sigma(v)x_2$ in the $x_2^+$ inclusion, while $x_1^+ = 0$ for $x \in D$. The data $(C, F, D, G_g)$ satisfies Assumption 1 for all $\eta \mathbb{R}_{\geq 0}$. The case $\eta = 0$ we call the bouncing ball without offset. We assume there exists $r > 0$ such that

$$
\lambda := \int_{\mathbb{R}} \sigma(v) \mu(dv) < 1, \quad (26)
$$

where $r$ can be chosen as 1 or 2 as analyzed in \cite[Sec. 6.1]{2}.

Next we analyze the stabilities in probability: UGASp without offset and UGR with offset. Compared with the deterministic bouncing ball in \cite{1} pg. 2\&35\&54\&61\&82\&125\&178 using deterministic invariance principle \cite{1} pg. 174, here we could also apply the Matrosov-Foster functions, besides the stochastic invariance principle \cite{16} Exm. 4, to verify UGASp for \eqref{25}-\eqref{26}.

1) **UGASp of origin without offset.** Let $A$ be the origin in $\mathbb{R}^2$ and consider the certification candidate $V(x) := (x_2^2 + 2\gamma x_1)^\gamma$ for all $x \in \mathbb{R}^2$. It is easy to check that $V(x)$ satisfy \eqref{12a} and \eqref{12b} with $\varpi(x) = |x|_{\mathcal{A}}$ for some functions $\ln(V(x) + 1) =: \alpha_1 \in K_{\infty}$ and $\exp(V(x)) - 1 =: \alpha_2 \in K_{\infty}$.

For all $x \in C$,

$$
\langle V(x), f(x) \rangle = \frac{r}{2} (x_1^2 + 2\gamma x_1)^\gamma \cdot (2x_2 x_2 + 2\gamma x_1) = \frac{r}{2} (x_1^2 + 2\gamma x_1)^\gamma \cdot (2x_2(-\gamma) + 2\gamma x_2) = 0,
$$

while for $x \in D$

$$
\int_{\mathbb{R}} \sup_{g \in G_0(x,v)} V(g) \mu(\sigma(v)) = \lambda \int_{\mathbb{R}} \sup_{g \in G_0(x,v)} \tilde{\Phi}(0) \mu(\sigma(v)) = \lambda \gamma \int_{\mathbb{R}} \sigma(v) \mu(\sigma(v)) = \lambda |x_2|^2 \gamma \leq \lambda (x_2^2 + 2\gamma |x_2|) = \lambda V(x).
$$

So $\langle V(x), f(x) \rangle = 0$ for all $x \in C$ satisfying \eqref{12b} and $\int_{\mathbb{R}} \sup_{g \in G_0(x,v)} V(g) \mu(\sigma(v)) \leq \lambda V(x)$ for $x \in D$ imply that $V$ is a Lyapunov function for the origin. According to Thm. 4, the origin is UGSp for $(C, F, D, G_0, \mu)$.

**1.1 Using Matrosov-Foster functions** \cite[Sec. 6.1]{2}, \cite[XII-C]{3}.

Let $K \subset \mathcal{A}^c$ be compact. Let candidate functions be $V_0(x) = V(x)$ and $V_1(x) = \beta_1 + x_2$ where $\beta_1 + x_2 \geq 0$ for all $x_2 \in K$. Let $\beta_2 > 0$ be such that $\beta_1 + x_2 \leq \beta_2$ for all $x_2 \in K$. Based on \eqref{27}-\eqref{28}, we have

$$
\langle V_0(x), f(x) \rangle = 0,
\int_{\mathbb{R}} \sup_{g \in G_0(x,v)} V_0(g) \mu(dv) \leq \lambda V_0(x) \leq V(x) + u_{d,0},
$$

where $u_{e,0} = 0$ and $u_{d,0} = (\lambda - 1)V_0(x) < 0$. Similarly, choosing $r = 1$, we get

$$
\langle V_1(x), f(x) \rangle = \dot{x}_2 = -\gamma < 0,
\int_{\mathbb{R}} \sup_{g \in G_0(x,v)} V_1(g) \mu(dv) = \int_{\mathbb{R}} \sup_{g \in G_0(x,v)} (\beta_1 + (-\sigma(v)x_2)) \mu(dv) \leq \beta_1 + \lambda |x_2| \leq \beta_1 + |x_2|\gamma.
$$

\begin{align*}
\text{where } u_{e,1} = -\gamma < 0 \text{ and } u_{d,0} = \beta_2 > 0. \text{ As a result, } (V_0, V_1) \text{ are Foster-Matrosov functions for } \mathbb{R}^2 \setminus K, \text{ so Prop. 1 implies that } A + \varpi \mathbb{B}^0 \text{ is UGR since } K_{\infty} \subset G_{\infty}. \text{ According to Coro. 7, the origin is UGSp for } (C, F, D, G_0, \mu) \text{ in } \eqref{25}-\eqref{26}. \text{ 1.2 Using invariance principle } \cite[Exm. 4]{16}. \text{ Instead of Foster-Matrosov functions, we use Thm. 8 to establish UGASp of the origin, i.e., ruling out the existence of a compact subset of } K_{\delta, \Delta} \text{ that is weakly forward invariant in probability. The analysis in } \cite[p. 35]{1} \text{ implies that for the bouncing ball, every complete solution is Zeno, i.e., having an infinite number of jumps. The above analysis for } \langle V(x), f(x) \rangle = 0 \text{ in } \eqref{27} \text{ shows that the Lyapunov function } V(x) \text{ remains constant during jumps and } \eqref{28} \text{ implies that } \int_{\mathbb{R}} \sup_{g \in G_0(x,v)} V(g) \mu(dv) \leq \lambda V(x) \text{ for all } x \in D, \text{ where } \lambda < 1. \text{ Therefore, classical results for discrete-time stochastic systems give that } V \text{ almost surely converges to zero. So, the set } K_{\delta, \Delta} \text{ for each } 0 < \delta < \Delta < \infty \text{ contains no compact set that is almost surely weakly forward invariant for } (C, F, D, G_0, \mu) \text{ in } \eqref{25}-\eqref{26}. \text{ Consequently, Thm. 8 implies that the origin is UGASp for } (C, F, D, G_0, \mu) \text{ in } \eqref{25}-\eqref{26}. \text{ Next we present the UGR of origin with offset } \cite[Sec. 6.1]{2}. \text{ 2) UGR of origin with offset.} \text{ According to above analysis, the same function } V \text{ is a Lagrange function for } \mathcal{O} \text{ where } \mathcal{O} := \bar{\gamma} \mathbb{B}^0 \text{ for } (C, F, D_\gamma, G_\gamma, \mu), \text{ so } \mathcal{O} \text{ is LaSp since } K_{\infty} \subset G_{\infty}. \text{ Based on the same Foster-Matrosov functions } (V_0, V_1) \text{ for } \mathbb{R}^2 \setminus K, \text{ Prop. 1 implies that } \mathcal{O} \text{ is UGR for } (C, F, D, G_\gamma, \mu) \text{ in } \eqref{25}-\eqref{26}. \text{ B. Robust global almost sure synchronization on the circle} \text{ Nominal global almost sure synchronization result is established in } \cite{14}, \text{ where a robustness property is established with respect to sufficiently small, worst-case perturbations, based}
on the stochastic, hybrid state feedback algorithm. In contrast, non-stochastic almost globally asymptotically synchronizing algorithms do not confer this robustness property.

Continuous stabilizers for almost global synchronization and discontinuous stabilizers for global synchronization on the circle exhibit non-robust stability behavior in the presence of adversaries. Stochastic gossip algorithms to achieve almost sure asymptotic stability tend to have slow convergence rates. This motivates the stochastic hybrid approach in [11] and [14] to achieve robust synchronization on the circle in the presence of adversaries while maintaining good convergence rates. In [14] a stochastic hybrid algorithm for synchronizing two agents on the circle is proposed, while an extension of the algorithm to a finite number of agents under all-to-all communication is discussed and proved in [11] by providing a complete Lyapunov function, where robustness margins are also established.

C. Discussions for potential applications

Besides the above mentioned applications, SHIs can be used to 1) model and analyze the networked security problems using event-triggering sampling control strategy under the digital communication channel, including but not limited to multi-agent systems; 2) design hybrid control law for networked rigid-body attitude motions, where one needs to simultaneously consider the hybrid control law and the discontinuous communications; 3) model and analyze other systems, including financial systems, air traffic management systems, biological systems and power systems, etc, as analyzed in [5].

VI. CONCLUSION

For SHIs, besides the possibly stochastic continuous evolution via a stochastic differential inclusions, randomness is mainly restricted to the jump map based on forced transitions while sometimes also to the spontaneous transitions in the difference inclusions. Mild regularity conditions on the system data are summarized to guarantee existence of random solutions, the equivalence between non-uniform and uniform versions of stochastic stability properties, and the nominal robustness of Lyapunov conditions. We have presented several properties for SHIs, including random solutions, stabilities and recurrence as well as corresponding (relaxed) Lyapunov-based sufficient/necessary conditions, sequential compactness, nominal robustness and converse theorems. Besides the sufficient Lyapunov conditions, other weaker conditions are summarized, as stochastic invariance principle and Matrosov-Fosters functions. The Lyapunov conditions are similar to classical Lyapunov conditions, rather than based on the derivative of the expected value of a function. An explicit example for stochastic bouncing ball system is present to show the stability results.

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REFERENCES

[1] R. Goebel, R. G. Sanfelice, and A. R. Teel, Hybrid Dynamical Systems: modeling, stability, and robustness. Princeton University Press, 2012.
[2] A. R. Teel, “Lyapunov conditions certifying stability and recurrence for a class of stochastic hybrid systems,” Annual Reviews in Control, vol. 37, no. 1, pp. 1–24, 2013.
[3] A. R. Teel and J. P. Hespanha, “Stochastic hybrid systems: A modeling and stability theory tutorial,” in Proceedings of the 54th IEEE Conference on Decision and Control, Osaka, Japan, December 15-18, 2015, pp. 3116–3136.
[4] A. R. Teel, “Stochastic hybrid inclusions with diffusive flows,” in Proceedings of the 53rd IEEE Conference on Decision and Control, 2014, pp. 3071–3076.
[5] A. R. Teel, A. Subbaraman, and A. Sferlazza, “Stability analysis for stochastic hybrid systems: A survey,” Automatica, vol. 50, no. 10, pp. 2435–2456, 2014.
[6] A. R. Teel, “A class of stochastic hybrid inclusions with spontaneous jumps,” in Proc. 56th IEEE Conf. Decision and Control. 2017, pp. 5475–5480.
[7] R. T. Rockafellar and R. J.-B. Wets, Variational Analysis. Berlin, Germany: Springer, 1998, vol. 317.
[8] A. R. Teel, J. P. Hespanha, and A. Subbaraman, “Stochastic difference inclusions: Results on recurrence and asymptotic stability in probability,” in Proc. 51st IEEE Conf. Decision and Control, Maui, HI, Dec. 2012, pp. 4051–4056.
[9] A. Subbaraman and A. R. Teel, “A converse Lyapunov theorem for strong global recurrence,” Automatica, vol. 49, no. 10, pp. 2963–2974, Oct. 2013.
[10] A. R. Teel, “A Matrosov theorem for adversarial Markov decision processes,” IEEE Trans. Autom. Control, vol. 58, no. 8, pp. 2142–2148, Aug. 2013.
[11] A. Subbaraman, M. Hartman, and A. R. Teel, “A stochastic hybrid algorithm for robust global almost sure synchronization on the circle: All-to-all communication,” in Proc. 52nd IEEE Conf. Decision Control, 2013, pp. 600–605.
[12] A. Subbaraman and A. R. Teel, “A Matrosov theorem for strong global recurrence,” Automatica, vol. 49, no. 11, pp. 3390–3395, 2013.
[13] S. Grammatico, A. Subbaraman, and A. R. Teel, “Discrete-time stochastic control systems: A continuous Lyapunov function implies robustness to strictly causal perturbations,” Automatica, vol. 49, no. 10, pp. 2939–2952, 2013.
[14] A. R. Teel, “Sure almost global vs. global almost sure synchronization on the circle: the virtues of stochastic hybrid feedback,” Proceedings of the 9th IFAC Symposium on Nonlinear Control Systems, pp. 217–222, 2013.
[15] S. Grammatico, A. Subbaraman, and A. R. Teel, “Discrete-time stochastic control systems: examples of robustness to strictly causal perturbations,” in 52nd Conference on Decision and Control, pp. 6403–6408, IEEE, December 2013.
[16] A. R. Teel, “On a recurrence principle for a class of stochastic hybrid systems,” in *American Control Conference*, 2014, pp. 4518–4523.

[17] A. R. Teel, “On sequential compactness of solutions for a class of stochastic hybrid systems,” in *Proceedings of the American Control Conference (ACC)*, pp. 4512–4517, 2014.

[18] A. Subbaraman and A. R. Teel, “A Krasovskii-LaSalle function based recurrence principle for a class of stochastic hybrid systems,” in *Proc. 53rd IEEE Conf. Decision and Control*. 2014, pp. 83–92.

[19] A. R. Teel, J. P. Hespanha, and A. Subbaraman, “A converse Lyapunov theorem and robustness for asymptotic stability in probability,” *IEEE Transactions on Automatic Control*, pp. 2426–2441, 2014.

[20] A. R. Teel, J. P. Hespanha, and A. Subbaraman, “Equivalent characterizations of input-to-state stability for stochastic discrete-time systems,” *IEEE Trans. Automatic Control*, vol. 59, no. 2, pp. 516–522, 2014.

[21] A. Subbaraman and A. R. Teel, “Robustness of recurrence for a class of stochastic hybrid systems,” in *Proceedings of the IFAC Conference on Analysis and Design of Hybrid Systems*, October 2015.

[22] A. R. Teel, “A recurrence principle for stochastic difference inclusions,” *IEEE Trans. Autom. Control*, vol. 60, no. 2, pp. 420–435, 2015.

[23] A. Subbaraman and A. Teel, “Recurrence principles and their application to stability theory for a class of stochastic hybrid systems,” *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3477–3492, 2016.

[24] A. Subbaraman, A. R. Teel, “Robust global recurrence for a class of stochastic hybrid systems,” *Nonlinear Analysis: Hybrid Systems*, vol. 01, no. 25, pp. 283–297, Aug. 2017.

[25] C. Possieri and A. R. Teel, “A Lyapunov theorem certifying global weak reachability for stochastic difference inclusions with random inputs,” *Syst. Control Lett.*, vol. 109, pp. 37–42, 2017.

[26] J. I. Poveda, A. R. Teel, and D. Nešić, “Flexible Nash seeking using stochastic difference inclusions,” in *Proc. Amer. Control Conf.*, 2015, pp. 2236–2241.

[27] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*. Springer, 1998.

[28] C. M. Kellett and A. R. Teel, “Smooth Lyapunov functions and robustness of stability for differential inclusions,” *Systems & Control Letters*, vol. 52, pp. 395–405, 2004.

[29] M. Kisielewicz, *Stochastic Differential Inclusions and Applications*. Springer, 2013.

[30] B. Fristedt and L. Gray, *A modern approach to probability theory*. Birkhäuser, 1997.

[31] X. Mao, “Stochastic versions of the LaSalle theorem,” *J. Differential Equations*, vol. 153, no. 1, pp. 175–195, 1999.

[32] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*. Springer, 1988.

[33] K. L. Chung, R. J. Williams, *Introduction to stochastic integration* (Vol. 2). Boston: Birkhäuser, 1990.

[34] J. P. LaSalle, *An Invariance Principle in the Theory of Stability*, in *Differential Equations and Dynamical Systems*. New York: Academic, 1967.

[35] A. R. Teel and L. Praly, “On assigning the derivative of a disturbance attenuation control Lyapunov function,” *Mathemat. Control, Signals and Syst.*, vol. 13, no. 2, pp. 95–124, 2000.

[36] G. G. Yin and C. Zhu, *Hybrid Switching Diffusions*. Springer, 2010.

[37] C. G. Mayhew, R. G. Sanfelice, and A. R. Teel, “Quaternion-based hybrid control for robust global attitude tracking”, *IEEE Trans. Autom. Control*, vol. 56, no. 11, pp. 2555–2566, November 2011.

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