Comment [1] claims that the laser threshold emerging from a new Coherent-Incoherent Model (CIM) [2] is “unattainable” when the term $\sum_{n \neq l} \delta(c_l^\dagger v_l^\dagger c_n)$ is added to the equation for the photon assisted polarization $\delta(bc^\dagger v)$. Moreover, it identifies the classical polarization $|P|^2$ with $\sum_{n,l} \langle c_l^\dagger v_l^\dagger c_n \rangle$, thus claiming that neglecting $\sum_{n \neq l} \delta(c_l^\dagger v_l^\dagger c_n)$ violates the quantum-classical correspondence.

Here we show that: 1) the threshold exists, persists and is attainable even with the wrong assumptions of [1]; 2) correctly taking into account terms of the order of $\sum_{n \neq l} \delta(c_l^\dagger v_l^\dagger c_n)$ and the sum’s spatial nonlocality confirms that CIM provides accurate values of the laser threshold.

In nanolasers, terms like $\sum_{n \neq l} \delta(c_l^\dagger v_l^\dagger c_n)$ are normally neglected. They represent collective effects, like superradiance, usually not observable in the presence of strong polarisation dephasing due to high carrier density screening [4, 5]. CIM [2] matches the parameters of standard GaAs-based QDs, with a very rapid decay [6] and negligible correlations of the intrinsic polarization.

Furthermore, $|P|^2$ does not correspond to $\sum_{n,l} \langle c_l^\dagger v_l^\dagger c_n \rangle$. Imposing operator normal ordering gives $\sum_{n,l} \langle c_l^\dagger v_l^\dagger c_n \rangle = \langle c_l^\dagger c_l \rangle - \sum_{n,l} \langle c_l^\dagger v_l^\dagger c_n \rangle \neq |P|^2$, where $\langle c_l^\dagger c_l \rangle$ is the excited state population and $\sum_{n,l} \langle c_l^\dagger v_l^\dagger c_n \rangle$ the sum of the expectation values of the product of polarisations between QDs placed at different positions: a spatially nonlocal term. This decomposition proves the point. The polarisation is local, does not depend on population, and is related to $\langle v | c \rangle^2$, included in CIM [2, Eq. (2)] but arbitrarily and inconsistently removed from Eq. (1) in [1].

The correct dynamical form for $\langle c_l^\dagger v_l^\dagger c_n \rangle$ is

$$
\left\{ d_t + 2 \gamma + i \Delta \varepsilon \right\} \langle c_l^\dagger v_l^\dagger c_n \rangle = g_{ls} \left[ \langle b_l^\dagger v_n^\dagger c_n \rangle (1 - 2 \langle c_l^\dagger c_l \rangle) - 2 \langle v_l^\dagger c_l \rangle (\langle b_l^\dagger c_n^\dagger c_l \rangle + \langle b_l^\dagger v_n^\dagger c_l \rangle) + g_{ns} \left( \langle b_n^\dagger c_l \rangle \langle c_n^\dagger v_l \rangle (1 - 2 \langle c_n^\dagger c_n \rangle) - 2 \langle c_n^\dagger v_l \rangle \langle b_n^\dagger c_n^\dagger \rangle \right) \right]
$$

where the coefficients $g_{ns}$ depend on the cavity-mode field at the QDs positions [3]. Spatial nonlocality introduces into Eq. (1) products of coupling coefficients, $g_{ls}$, and polarisation operators, $v_n^\dagger c_n$, from different QDs. Neglecting these phase differences [1], assumes that QDs and $g_{ls}$, which depend on the mode [3], are identical. These extremely strict conditions cannot be satisfied by all QDs for physically realistic boundary conditions.

Adding Eq. (1) of [1] to CIM [2] displaces the threshold (black star in Fig.1) from its original position [2] (blue star), rendering the post-bifurcation dynamics unstable due to the arbitrary removal of terms of comparable size. Consistently computing (as in [2]) the variables at the appropriate order (cf. Eq. (1) above), but keeping the unphysical assumption of identical QD coefficients [1] stabilizes the dynamics, moving the threshold to a lower pump (red star). Relaxing this unphysical condition returns the threshold to approximately the CIM value (red diamond and cross). In summary: thresholds leading to coherent fields can always be observed. Contrary to claims [1], the model of [2] is correct and widely applicable.

Note that neglecting $\delta(b^\dagger bc^\dagger c)$ and $\delta(b^\dagger b v)$ is standard procedure with cluster expansions [7] at the two-particle level [4]. Finally, there is a misinterpretation regarding the emission after the bifurcation in [1]: close to threshold only a fraction of the photon field is coherent and single-frequency.

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