Endomorphism algebras of semi-tilting modules

Shunhua Zhang

School of Mathematics, Shandong University, Jinan, 250100,P.R.China

Abstract Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. We investigate the structure properties of the endomorphism algebras of semi-tilting $A$-modules, and prove that the endomorphism algebras arising from the mutations of semi-tilting $A$-modules can be realized as the endomorphism algebras of BB-tilting modules.

Key words and phrases: Semi-tilting modules, mutations, BB-tilting modules.

1 Introduction

Tilting theory plays an important role in the representation theory of Artin algebras, and the endomorphism algebras of tilting modules form a central class of Artin algebras. The tilting modules of finite projective dimension was defined by Miyashita in [9]. A further generalization of tilting modules to modules of possibly infinite projective dimension called semi-tilting modules was made by Koga in [8].

In [8], Koga proved that the class of basic semi-tilting modules is closed under mutations. In this paper we investigate the structure properties of endomorphism algebras of semi-tilting modules and prove that the endomorphism algebras of semi-tilting modules can be realised as the endomorphism algebras of BB-tilting modules. We state our main results as follows.

Theorem 1. Let $A$ be a finite dimensional algebra over an algebraically closed field $k$, and $T = U \oplus X$ be a basic semi-tilting $A$-modules with $X$ indecomposable and $X \in \text{gen } U$. Set $B = \text{End}_A T$. Then there exists an exact sequence $0 \rightarrow Y \xrightarrow{\mu} E \xrightarrow{\varepsilon} X \rightarrow 0$
in $A$-mod with $\varepsilon$ a minimal right add $U$-approximation, such that $T' = U \oplus Y$ is a basic semi-tilting $A$-modules and $M = \text{Hom}_A(T', T)$ is a $BB$-tilting $B^{op}$-module with $\text{End}_{B^{op}} M \simeq \text{End}_A T'$.

This paper is arranged as the following. In section 2, we fix the notations and recall some necessary facts needed for our further research. Section 3 is devoted to the proof of Theorem 1.

2 Preliminaries

Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. We denote by $\text{gl.dim} A$ the global dimension of $A$, and by $A$-mod the category of all finitely generated left $A$-modules. $D = \text{Hom}_k(-, k)$ is the standard duality between $A$-mod and $A^{op}$-mod. $\tau$ is the Auslander-Reiten translation of $A$ and $\tau^{-1}$ is its inverse.

Given a $A$-module $M$, we denote by $add M$ the full subcategory having as objects the direct sums of indecomposable summands of $M$ and by $pd_A M$ the projective dimension of $M$. Let $M = M_1^{n_1} \oplus \cdots \oplus M_t^{n_t}$ where the $M_i$ are pairwise non-isomorphic indecomposable modules and $n_i \geq 1$ is the multiplicity of $M_i$ in $M$. The module $M$ is called basic if $n_i = 1$ for all $i$. We denote by $\delta(M)$ the number of non-isomorphic indecomposable summands of $M$.

A module $T \in A$-mod is called a tilting module if the following conditions are satisfied:

1. $pd_A T = n < \infty$;
2. $\text{Ext}_A^i(T, T) = 0$ for all $i > 0$;
3. There is a long exact sequence

$$0 \rightarrow A \rightarrow T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_n \rightarrow 0$$

with $T_i \in add T$ for $0 \leq i \leq n$.

Let $S$ be a non-injective simple $A$-module with the following two properties:

(a) $pd_A \tau^{-1} S \leq 1$, and
(b) $\text{Ext}_A^1(S, S) = 0$.

We denote the projective cover of $S$ by $P(S)$, and assume that $A = P(S) \oplus P$ such that there is not any direct summand of $P$ isomorphic to $P(S)$. Let $T = \tau^{-1} S \oplus P$ and
Γ = \text{End}_A T$. Then $T$ is a tilting module with $\text{pd}_A T \leq 1$, which is called BB-tilting module associated to $S$, and $T_\Gamma$ is also a BB-tilting $\Gamma$-module, see Section 2.8 in [1] for details.

Let $\mathcal{C}$ be a full subcategory of $A$-mod, $C_M \in \mathcal{C}$ and $\varphi : C_M \to M$ with $M \in A$-mod. Recall from [3] that the morphism $\varphi$ is a right $\mathcal{C}$-approximation of $M$ if the induced morphism $\text{Hom}_A(C, C_M) \to \text{Hom}_A(C, M)$ is surjective for any $C \in \mathcal{C}$. A minimal right $\mathcal{C}$-approximation of $M$ is a right $\mathcal{C}$-approximation which is also a right minimal morphism, i.e., its restriction to any nonzero summand is nonzero. The subcategory $\mathcal{C}$ is called contravariantly finite if any module $M \in \mathcal{A}$-mod admits a (minimal) right $\mathcal{C}$-approximation. The notions of (minimal) left $\mathcal{C}$-approximation and of covariantly finite subcategory are dually defined. It is well known that add $M$ is both a contravariantly finite subcategory and a covariantly finite subcategory.

The following Lemma is taken from [8].

**Lemma 2.1.** Let $T = U \oplus X \in A$-mod with $X$ indecomposable, $X \not\in \text{add } U$ and $\text{Ext}_A^i(T, T) = 0$ for $i \geq 0$. Assume that there exists an exact sequence $0 \to Y \to E \xrightarrow{\varepsilon} X \to 0$ with $\varepsilon$ a right add $U$-approximation. Set $B = \text{End}_A T$. Then $\text{Hom}_A(U \oplus Y, T)$ is a tilting $B$-module with $\text{pd}_B \text{Hom}_A(U \oplus Y, T) = 1$.

Recall from [8], a module $T \in A$-mod is said to be a semi-tilting module if the following conditions are satisfied:
(i) $\text{Ext}_A^i(T, T) = 0$ for all $i > 0$;
(ii) There is a long exact sequence $0 \to A \to T_0 \to T_1 \to \cdots \to T_n \to 0$ with $T_i \in \text{add } T$ for $0 \leq i \leq n$.

Throughout this paper, we follow the standard terminologies and notations used in the representation theory of algebras, see [2, 4, 5, 10].

## 3 Endomorphism algebras of semi-tilting modules

In this section, we prove our main theorem.

**Theorem 3.1.** Let $A$ be a finite dimensional algebra over an algebraically closed field $k$, and $T = U \oplus X$ be a basic semi-tilting $A$-modules with $X$ indecomposable and $X \in \text{gen } U$. Set $B = \text{End}_A T$. Then there exists an exact sequence $0 \to Y \xrightarrow{\mu} E \xrightarrow{\varepsilon}$
$X \to 0$ in mod-$A$ with $\varepsilon$ a minimal right add $U$-approximation, such that $T' = U \oplus Y$ is a basic semi-tilting $A$-modules and $M = \text{Hom}_A(T', T)$ is a $BB$-tilting $B^{\text{op}}$-module with $\text{End}_{B^{\text{op}}} M \simeq \text{End}_A T'$.

**Proof.** According to Lemma 2.1, there exists an exact sequence

\[(\dag) \quad 0 \to Y \xrightarrow{\mu} E \xrightarrow{\varepsilon} X \to 0\]

in mod-$A$ with $\varepsilon$ a minimal right add $U$-approximation, such that $T' = U \oplus Y$ is a basic semi-tilting $A$-modules and $M = \text{Hom}_A(T', T)$ is a tilting $B^{\text{op}}$-module with $\text{pd}_B M = 1$.

Note that $M = \text{Hom}_A(U \oplus Y, T) = P_B \oplus L$ with $P_B$ is a projective $B^{\text{op}}$-module and $L = \text{Hom}_A(Y, T)$.

Applying $\text{Hom}_A(-, T)$ to $(\dag)$ yields an exact sequence

\[(\ddagger) \quad 0 \to \text{Hom}_A(X, T) \xrightarrow{\varepsilon_*} \text{Hom}_A(E, T) \xrightarrow{\mu_*} L \to 0.\]

Note that $(\ddagger)$ is the minimal projective resolution of $L$. Applying $\text{Hom}_{B^{\text{op}}}(-, B)$ to $(\ddagger)$ we get an exact sequence of $B$-modules

$$\text{Hom}_{B^{\text{op}}} (\text{Hom}_A(E, T), B^{\text{op}}) \xrightarrow{\varepsilon_*} \text{Hom}_{B^{\text{op}}} (\text{Hom}_A(X, T), B^{\text{op}}) \to \text{Tr}_{B^{\text{op}}} L \to 0,$$

which is isomorphic to the following exact sequence

$$\text{Hom}_A(T, E) \xrightarrow{\varepsilon_*} \text{Hom}_A(T, X) \to \text{Tr}_{B^{\text{op}}} L \to 0,$$

where $\varepsilon_* = \text{Hom}_A(T, \varepsilon)$.

We claim that $\text{Im} \varepsilon_*$ is the radical of the indecomposable projective $B^{\text{op}}$ module $\text{Hom}_A(T, X)$.

Indeed, $\varepsilon_* = \text{Hom}_A(U \oplus X, \varepsilon) = \text{Hom}_A(U, \varepsilon) \oplus \text{Hom}_A(X, \varepsilon)$.

Since $\varepsilon$ is a minimal left add$U$-approximation of $X$, by applying $\text{Hom}_A(U, -)$ to $(\dag)$ we get an exact sequence

$$0 \to \text{Hom}_A(U, Y) \to \text{Hom}_A(U, E) \xrightarrow{\text{Hom}_A(U, \varepsilon)} \text{Hom}_A(U, X) \to 0.$$

Hence $\text{Hom}_A(U, \varepsilon)$ is surjective.

By applying $\text{Hom}_A(X, -)$ to $(\dag)$ we have an exact sequence

$$\text{Hom}_A(X, E) \xrightarrow{\text{Hom}_A(X, \varepsilon)} \text{Hom}_A(X, X) \to \text{Ext}_A^1(X, Y) \to 0,$$

it forces that $\text{Im} \text{Hom}_A(X, \varepsilon) = \text{rad} \text{Hom}_A(X, X)$ since $\text{dim}_k \text{Ext}_A^1(X, Y) = 1$. It follows that $\text{Im} \varepsilon_* = \text{rad} \text{Hom}_A(T, X)$, and our claim is true.
It follows from our claim that $\mathrm{Tr}_{B^\text{op}} L$ is a simple $B^\text{op}$-module and $\tau_{B^\text{op}} L$ is the simple socle $S$ of the indecomposable injective $B^\text{op}$-module $D\mathrm{Hom}_A(T, X)$, and we know that $L \cong \tau_{B^\text{op}}^{-1} S$. Hence $M$ is a BB-tilting $B^\text{op}$-module. By Lemma 3.4 in [7], we have that $\mathrm{End}_{B^\text{op}} M = \mathrm{Hom}_{B^\text{op}}(\mathrm{Hom}_A(U \oplus Y, T), \mathrm{Hom}_A(U \oplus Y, T)) \cong \mathrm{End}_A(U \oplus Y)$. $\Box$

We state the dual result as follows.

Theorem 3.2. Let $A$ be a finite dimensional algebra over an algebraically closed field $k$, and $T = U \oplus Y$ be a basic semi-tilting $A$-modules with $Y$ indecomposable and $Y \in \text{cogen} U$. Set $B = \mathrm{End}_A T$. Then there exists an exact sequence $0 \to Y \xrightarrow{\mu} E \xrightarrow{\epsilon} X \to 0$ in $\text{mod-} A$ with $\mu$ a minimal left add $U$-approximation, such that $T' = U \oplus X$ is a basic semi-tilting $A$-modules and $M = \mathrm{Hom}_A(T', T)$ is a BB-tilting $B^\text{op}$-module with $\mathrm{End}_{B^\text{op}} M \cong \mathrm{End}_A T'$.

Remark. Let $T = U \oplus X$ be a basic semi-tilting $A$-modules with $X$ indecomposable and $X \in \text{gen} U$. We denote by $\mu_X(T)$ the module $T' = U \oplus X$ in the case of Theorem 3.1.

Recall from [8], the semi-tilting quiver $K$ is defined as follows: The vertices of $K$ are isomorphism classes of basic semi-tilting modules and there is an arrow $V \to W$ if $W$ and $V$ are represented by basic semi-tilting modules $T$ and $\mu_X(T)$ with $X$ a non-projective indecomposable direct summand of $T$, respectively.

Corollary 3.3. Let $A$ be a finite dimensional algebra over an algebraically closed field $k$, and $C$ be a connected component of the semi-tilting quiver $K$ of $A$. Then all the endomorphism algebras of semi-tilting modules in $C$ can be realized as the iterated endomorphism algebras of BB-tilting modules.

Proof Let $T_1$ and $T_2$ be a pair of basic semi-tilting modules in $C$. Then there is a path $T_1 = W_0 \to W_1 \to \cdots \to W_s \to T_2 = W_{s+1}$ in $C$. Set $\Lambda_i = \mathrm{End}_A W_i$ for $0 \leq i \leq s+1$. According to Theorem 3.2, for each arrow $W_i \to W_{i+1}$ there is a BB-tilting $\Lambda_i^\text{op}$-module $M_i$ such that $\mathrm{End}_{\Lambda_i^\text{op}} M_i = \Lambda_{i+1}$. In particular, $\mathrm{End}_A T_2 \cong \mathrm{End}_{\Lambda_s^\text{op}} M_s$. This completes the proof. $\Box$

Note that a semi-tilting module is a tilting module if and only if its projective dimension is finite. According to [6], the tilting quiver $\mathcal{T}$ of $A$ is defined as follows. The vertices of $\mathcal{T}$ are isomorphism classes of basic tilting modules, and there is an arrow
$T' \to T$ if $T' = T \oplus X$ and $T = T \oplus Y$ with $X,Y$ indecomposable and there is a short exact sequence $0 \to X \xrightarrow{f} E \xrightarrow{g} Y \to 0$ with $f$ (resp.$g$) being a minimal left (resp.right) add $T$-approximation.

Let $C$ be a connected component of be the semi-tilting quiver $K$ of $A$. According to Theorem 3.11 in \cite{Koga13} we know that $C$ is a connected component of the tilting quiver $T$ of $A$ if and only if $C$ containing a tilting $A$-module. The following result is a consequence of Corollary 3.3.

**Corollary 3.4.** Let $A$ be a finite dimensional algebra over an algebraically closed field $k$, and $C$ be a connected component of the tilting quiver $T$ of $A$. Then all the endomorphism algebras of tilting modules in $C$ can be realized as the iterated endomorphism algebras of BB-tilting modules.

References

[1] I. Assem, Tilting theory an introduction, in: Topics in Algebra, Part 1, Warsaw, 1988, in: Banach Center Publ., vol. 26, 1990, 127-180.

[2] I. Assem, D. Simson, A. Skowronski, Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory. London Mathematical Society Student Texts, 65. Cambridge University Press, Cambridge, 2006.

[3] M. Auslander, I. Reiten, Applications of contravariantly finite subcategories. Adv. Math., 86(1991), 111-152.

[4] M. Auslander, I. Reiten, S. O. Smalø, Representation Theory of Artin Algebras. Cambridge Univ. Press, 1995.

[5] D. Happel, C. M. Ringel, Tilted algebras. Trans. Amer. Math. Soc., 274(1982), 399-443.

[6] D. Happel, L. Unger, On the quiver of tilting modules, J. Algebra, 284(2005), 857-868.

[7] W. Hu, C. Xi, $\mathcal{D}$-split sequences and derived equivalences. Adv. Math., 227(2011), 292-318.

[8] H. Koga, Semi-tilting Modules and Mutation. Algebra Represent Theory, 16(2013), 1469C1487.
[9] Y. Miyashita, Tilting modules of finite projective dimension, Math. Z., 193 (1986), 113-146.

[10] C.M. Ringel, Tame algebras and integral quadratic forms. Lecture Notes in Math., 1099. Springer-Verlag, Berlin, Heidelberg, New York, 1984.