Learning Unstable Dynamical Systems with Time-Weighted Logarithmic Loss

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Abstract

When training the parameters of a linear dynamical model, the gradient descent algorithm is likely to fail to converge if the squared-error loss is used as the training loss function. Restricting the parameter space to a smaller subset and running the gradient descent algorithm within this subset can allow learning stable dynamical systems, but this strategy does not work for unstable systems. In this work, we look into the dynamics of the gradient descent algorithm and pinpoint what causes the difficulty of learning unstable systems. We show that observations taken at different times from the system to be learned influence the dynamics of the gradient descent algorithm in substantially different degrees. We introduce a time-weighted logarithmic loss function to fix this imbalance and demonstrate its effectiveness in learning unstable systems.

1 Introduction

Systems with memories that evolve over time require the use of a dynamical model for their representation. This model describes how the memory, or the state, of this system changes over time, how its state is affected by inputs to the system, and how it generates observable outputs. System identification corresponds to the task of learning the unknown parameters of this dynamical model from the known inputs and the observed outputs of the system.

Identification of dynamical systems from time-series data is an important problem for various applications, such as model prediction in reinforcement learning [Lambert et al., 2019; Zhang et al., 2016], analysis of medical health records [Rubanova et al., 2019] and prediction with financial time-series data [Tsay, 2014; Ganeshapillai et al., 2013]. However, the identification problems that arise in these applications pose some theoretical challenges:

1. Unless the state of the system is observed with a known noiseless mapping, the identification of the system model is coupled with the state estimation. Consequently, the system identification task is in general a nonconvex problem [Hardt et al., 2018]. To circumvent this nonconvexity, the initial state can be assumed to be zero in control settings, and a known input can be used to drive the state of the system [Sastry, 1984; Sastry and Bodson, 1989]. However, in medical and financial settings, the initial state of the system is typically not known a priori, and the deviations of the initial state from a nominal value cannot be neglected. Therefore, a joint and nonconvex optimization procedure is unavoidable in these settings to estimate the initial state of the system along with the unknown model parameters [Frigola et al., 2014; Duncker et al., 2019].
2. For control of a dynamical system in a reinforcement learning task, it is most critical that the unstable modes of the system be discovered and stabilized properly. Similarly, financial data and medical health records usually exhibit sudden changes in their pattern, which call for potentially unstable dynamics in their representation and estimation. However, the primary tools for nonconvex optimization, namely, the gradient methods, fail to converge and find an accurate model representation for unstable systems [Hardt et al., 2018].

3. Especially in medical and financial data sets, the data are sampled irregularly; that is, the observations are not periodically sampled. The common heuristic approach to handle this situation is imputing the absent observations by interpolating the observed values of the output [Che et al., 2018]. This approach, however, might fail to capture the correct dynamics of the underlying system. An alternative is to use a model that can take account for the evolution of the state of the system during unobserved intervals without requiring periodic observations [Chen et al., 2018].

In this work, we use the gradient descent algorithm to identify the unknown parameters of a linear dynamical system from its observed outputs. We look into the dynamics of this algorithm and try to pinpoint what causes the inability of the gradient methods to converge when they are used to identify an unstable dynamical system. Similar to the work of Chen et al. [2018], our analysis uses a continuous-time model so that it directly applies to irregularly sampled data sets with no need for imputation.

1.1 Our contributions

By analyzing the dynamics of the gradient descent algorithm during identification of a linear dynamical system, we achieve the following.

1. We obtain an upper bound on the learning rate of the gradient descent algorithm so that it can converge while learning a dynamical system with the squared-error loss. This upper bound explicitly depends on the eigenvalue of the system with the largest real part, and it shows that identifying a system becomes harder as the system becomes unstable. Furthermore, the upper bound on the learning rate shows that the samples taken at different times affect the convergence of the gradient descent algorithm in substantially different degrees.

2. To enable the convergence of the gradient descent algorithm even when learning unstable systems, we introduce a new loss function which balances the influence of the samples taken at different times on the convergence of the algorithm. Then we demonstrate the effectiveness of this loss function while estimating linear dynamical systems.

Note that the primary question our work addresses is about the use of the gradient descent algorithm while learning a dynamical system: can this algorithm converge at all while learning the parameters of a dynamical system model? This is a different problem than whether a specific algorithm, or a specific model can learn the dynamical system of interest more accurately than the state-of-the-art.

1.2 Related works

[Hardt et al.] [2018] studied the convergence of the gradient descent algorithm while learning linear dynamical systems. They demonstrated the failure of this algorithm to learn even stable systems, and proposed a projected gradient method that fixed the issue for linear stable systems. Learning an unstable system, however, was considered to be infeasible. In contrast, we retain the standard gradient descent algorithm in this work, and we introduce a new loss function that allows learning even unstable systems with no necessity for projection.

If the state of a linear system is directly accessed, that is, if the output of the system is equal to the state of the system possibly with some additive noise, learning the system parameters can be formulated as an ordinary least squares problem. [Alaeddini et al., 2018] and [Sarkar and Rakhlin, 2019] make this assumption and arrive at a convex optimization problem. By doing so, they avoid the use of gradient descent algorithm, and therefore, they do not suffer from the issues pointed out by

1The term stability refers to bounded-input bounded-output stability. For continuous-time linear time-invariant systems, this corresponds to the condition where the eigenvalues of the state transition matrix have strictly negative real parts.
Hardt et al. [2018]. However, as mentioned earlier, the assumption of having an access to the true internal state is unrealistic in many application domains, such as, health and finance.

Using variational inference is a common approach to estimate the initial state jointly with the dynamical model parameters in a Bayesian setting [Frigola et al., 2014] [Archer et al., 2015] [Krishnan et al., 2017] [Eleftheriadis et al., 2017] [Duncker et al., 2019] [Gregor et al., 2019]. In this approach, a separate model is employed to estimate the initial state from the whole observed trajectory. One of the models that we will consider in this work is a simpler, deterministic counterpart of this approach. We show that convergence issues of the gradient descent algorithm are also valid for this deterministic counterpart of variational inference.

Neural ordinary differential equations [Chen et al., 2018] [Rubanova et al., 2019] use a neural network to represent a continuous-time dynamical system. Since these models are also trained with the gradient descent algorithm, they also suffer from the stability issues of the gradient descent algorithm while learning the parameters of a dynamical model. Indeed, the training data of all the examples outlined in these works involve trajectories that converge to either a stable equilibrium or a stable limit cycle of the system.

2 Problem Formulation

For each \( k \in \mathcal{K} = \{1, \ldots, K\} \), let \( z_k : [0, \infty) \to \mathbb{R}^n \) denote a continuous-time process representing the state of a linear time-invariant dynamical system:

\[
\frac{dz_k(t)}{dt} = Az_k(t) \quad \forall t \geq 0, \forall k \in \mathcal{K},
\]

where \( A \in \mathbb{R}^{n \times n} \) denotes the state transition dynamics of the system. Then the evolution of the process is described by \( z_k(t) = e^{At}z_k(0) \) for all \( t \geq 0 \) for each \( k \in \mathcal{K} \) [Callier and Desoer, 1991]. Let \( \{x_k(t)\}_{t \in T_k} \) be the set of samples obtained from \( z_k \) at time instants \( t \in T_k \) via an observation matrix \( C \in \mathbb{R}^{m \times n} \):

\[
x_k(t) = Cz_k(t) = Ce^{At}z_k(0) \quad \forall t \in T_k, \forall k \in \mathcal{K}.
\]

Define the initial state of the trajectory of \( z_k \) as \( s_k \in \mathbb{R}^n \); that is, let \( s_k = z_k(0) \) for all \( k \in \mathcal{K} \). We will look for a linear dynamical system model that fits all the trajectories, and we will use the gradient descent algorithm to reveal the difficulty of its convergence. In particular, our goal is to study whether the gradient descent algorithm is able to converge to a solution while solving the problem

\[
\begin{align*}
\min_{A,C} & \sum_{k \in \mathcal{K}} \sum_{t \in T_k} \ell (x_k(t), Ce^{At}s_k) \\
\text{subject to} & \quad s_k \in \mathcal{S} 
\end{align*}
\]

where \( \ell \) is a differentiable loss function. We consider two choices for \( \ell \) in the following sections: the squared-error loss as it is used both in classical works [Åström and Eykhoff, 1971] and in recent works [Hardt et al., 2018], and the time-weighted logarithmic loss introduced in Section 4.

The set of initial states \( \{s_k\}_{k \in \mathcal{K}} \) is left arbitrary in the statement of (1); we consider three possible cases for these initial states, and our analysis in the following sections applies to all of these three cases.

1. Each \( s_k \) is known or has a fixed value. In other words, the set \( \{s_k\}_{k \in \mathcal{K}} \) is not updated by the gradient descent algorithm.

2. Each \( s_k \) is also a variable, and the gradient descent algorithm optimizes over \( \{s_k\}_{k \in \mathcal{K}} \) as well:

\[
\min_{A,C,\{s_k\}_{k \in \mathcal{K}}} \sum_{k \in \mathcal{K}} \sum_{t \in T_k} \ell (x_k(t), Ce^{At}s_k) 
\]

3. Each \( s_k \) is output of a state estimator:

\[
s_k = g_\phi (\{t, x_k(t)\}_{t \in T_k}) \quad \forall k \in \mathcal{K},
\]

where \( \phi \) is the parameters of this estimator, and the gradient descent algorithm solves the problem

\[
\begin{align*}
\min_{A,C,\phi} & \sum_{k \in \mathcal{K}} \sum_{t \in T_k} \ell (x_k(t), Ce^{At}s_k) + \mathcal{L}(\phi) \\
\text{subject to} & \quad s_k = g_\phi (\{t, x_k(t)\}_{t \in T_k}) \quad \forall k \in \mathcal{K},
\end{align*}
\]
where $L$ is an additional loss term associated with the estimation of the initial state, and it satisfies
\[ \frac{\partial L}{\partial A} = 0, \quad \frac{\partial L}{\partial C} = 0. \]

This case can be considered as the deterministic counterpart of the framework used in variational inference of state space models [Jordan et al., 1999; Archer et al., 2015]. This comparison is discussed further in Section 6.

In the following sections, we will demonstrate the analysis and state the theorems for problem (2) in the second case. The statements are identically valid for the other two cases, as explained in Appendix D.

3 Learning with Squared-Error Loss

In this section, we consider problem (2) with the squared-error loss:
\[
\min_{A, C, \{s_k\}_{k \in K}} \sum_{k \in K} \sum_{t \in T_k} \|x_k(t) - Ce^{At}s_k\|_2^2
\]

If we use the gradient descent algorithm to solve problem (2), the learning rate of the algorithm needs to be sufficiently small for the algorithm to converge [Bertsekas, 1999]. The next theorem gives an upper bound on the learning rate as a necessary condition for the convergence of the algorithm.

**Theorem 1.** Let \( \{z_k\}_{k \in K} \) be a set of trajectories, and let \( s_k \) denote the initial state for trajectory \( z_k \) for each \( k \in K \). Define the set of sampling instants of \( z_k \) as \( T_k \), and denote the samples taken from this trajectory by \( \{x_k(t)\}_{t \in T_k} \). Assume that the gradient descent algorithm is used to solve the problem
\[
\min_{A, C, \{s_k\}_{k \in K}} \sum_{k \in K} \sum_{t \in T_k} \|x_k(t) - Ce^{At}s_k\|_2^2.
\]

Then for almost every initialization, the learning rate of the gradient descent algorithm, \( \delta \), must satisfy
\[
\delta \leq \frac{2}{\lambda_{\min} \left( \rho^2 \sum_{k \in K} \sum_{t \in T_k} t^2 e^{2\text{Re}(\Lambda)^t} \hat{s}_k \hat{s}_k^\top \right)}
\]
so that the algorithm can converge to the solution \( (\hat{A}, \hat{C}, \{\hat{s}_k\}_{k \in K}) \) achieving zero training error, where \( \lambda_{\min}(\cdot) \) denotes the minimum eigenvalue of its argument, \( \Lambda \) is the eigenvalue of \( \hat{A} \) with the largest real part, \( \rho^2 = \max_{u \in \mathcal{U}} \|\hat{C}u\|_2^2 \), and \( \mathcal{U} \) is the set of eigenvectors of \( \hat{A} \) corresponding to \( \Lambda \).

**Proof.** See Appendix A.

Note that the eigenvalues of a linear dynamical system have a particular meaning in control theory: they describe the stability of the system [Callier and Desoer, 1991]. If any eigenvalue of \( \hat{A} \) has a real part that is strictly positive, then the state of the system will grow unboundedly large from almost all initial points; and the system is called unstable in this case. If, on the other hand, all eigenvalues of \( \hat{A} \) have a negative real part, then the state of the system will converge to a fixed point from all initial points, and the system will be stable.

The condition about reaching zero training error might be somewhat restrictive, but the main purpose of Theorem 1 is not to prescribe a learning rate for all possible cases; it is to reveal that the samples taken at different times affect the convergence of the algorithm very differently. Indeed, Theorem 1 shows that if the gradient descent algorithm is used to learn an unstable system, samples taken at later times impose a bound on the required learning rate exponentially more strict, which renders learning an unstable dynamical system infeasible.

Note that if the set of initial states \( \{\hat{s}_k\}_{k \in K} \) does not span the whole state space, then the bound given in Theorem 1 will be void. This suggests that it will be easier to train a dynamical model if the initial states of the trajectories given in the training data do not have a large variance. However, this does not mean the learned model will be accurate. Since there is no information available about how the
system evolves for the initial states in the nullspace of \( \sum_{k \in K} \hat{s}_k \hat{s}_k^T \), the model learned will fail to predict the behavior of the system for the initial states with a nonzero component in this unlearned subspace as well.

The appearance of \( \rho \) in Theorem 1 reflects the notion of observability \cite{Callier_and_Desoeur}. Based on the relationship between the matrices \( \hat{A} \) and \( \hat{C} \), it may not be possible to observe certain eigenvalues, or modes, of the learned system in its output; these modes are called unobservable modes. As these modes do not appear in the output of the learned system, they cannot affect the gradient descent algorithm.

**Remark 1.** The analysis for Theorem 1 shows that, for the Hessian \( H \) of the loss function (4) at \((\hat{A}, \hat{C})\), the ratio of the largest eigenvalue to the smallest eigenvalue of \( H \) satisfies

\[
\frac{\lambda_{\text{max}}(H)}{\lambda_{\text{min}}(H)} \geq \frac{\lambda_{\text{max}}(\rho_1^2 \sum_{k \in K} \sum_{t \in T_k} t^2 e^{2\text{Re}(\lambda_1) t} \hat{s}_k \hat{s}_k^T)}{\lambda_{\text{max}}(\rho_2^2 \sum_{k \in K} \sum_{t \in T_k} t^2 e^{2\text{Re}(\lambda_2) t} \hat{s}_k \hat{s}_k^T)}
\]

for any pair of eigenvalues \((\lambda_1, \lambda_2)\) of \( \hat{A} \), where \( \rho_1 = \|\hat{C} u_1\|_2, \rho_2 = \|\hat{C} u_2\|_2 \), and \( u_1, u_2 \) are the right eigenvectors of \( \hat{A} \) corresponding to \( \lambda_1, \lambda_2 \), respectively. This implies that, if the loss function can be represented well by its second order approximation around \((\hat{A}, \hat{C})\), local convergence rate for estimating the eigenvalue \( \lambda_2 \) will require

\[
O \left( \left[ \log \left( 1 - \beta \frac{\sum_{k \in K} \sum_{t \in T_k} t^2 e^{2\text{Re}(\lambda_2) t}}{\sum_{k \in K} \sum_{t \in T_k} t^2 e^{2\text{Re}(\lambda_1) t}} \right) \right]^{-1} \right)
\]

iterations of the gradient descent algorithm, where \( \beta \) is some constant depending on \( \rho_1, \rho_2 \) and \( \sum_{k \in K} \hat{s}_k \hat{s}_k^T \). This shows that learning the stable modes of a system can become infeasible when the system is unstable. See Appendix C for more details.

The necessary condition given in Theorem 1 implies that the convergence of the algorithm gives us information about the rightmost eigenvalue of the dynamical system that is being estimated. This is stated in Corollary 1.

**Corollary 1.** Assume that the observation matrix \( C = I \), the gradient descent algorithm is used to solve the problem \( (4) \) and the algorithm has converged from a random initialization to the solution \((\hat{A}, \{\hat{s}_k\}_{k \in K})\) achieving zero training error. Then the eigenvalue of \( \hat{A} \) with the largest real part, \( \Lambda \), almost surely satisfies

\[
\text{Re}(\Lambda) \leq \inf_{\tau > 0} \frac{1}{2\tau} \log \frac{1}{2\tau^2} \left[ \frac{\lambda_{\text{min}} \left( \sum_{k \in K} \sum_{t \in T_k} \hat{s}_k \hat{s}_k^T \mathbf{1}(t \geq \tau) \right)}{\lambda_{\text{min}} \left( \sum_{k \in K} \sum_{t \in T_k} \hat{s}_k \hat{s}_k^T \mathbf{1}(t \leq \tau) \right)} \right] \]

if \( \text{Re}(\Lambda) > 0 \),

\[
\text{Re}(\Lambda) \leq \inf_{\tau_j > \tau_1 > 0} \frac{1}{2\tau_2} \log \frac{1}{2\tau_2^2} \left[ \frac{\lambda_{\text{min}} \left( \sum_{k \in K} \sum_{t \in T_k} \hat{s}_k \hat{s}_k^T \mathbf{1}(t \geq \tau_j) \right)}{\lambda_{\text{min}} \left( \sum_{k \in K} \sum_{t \in T_k} \hat{s}_k \hat{s}_k^T \mathbf{1}(t \leq \tau_1) \right)} \right] \]

if \( \text{Re}(\Lambda) < 0 \).

### 4 Learning with Time-Weighted Logarithmic Loss

Theorem 1 shows that when the gradient descent algorithm is used to learn the parameters of an unstable dynamical system, the effect of the samples taken at later times are exponentially more weighted around a global minimum. It is important to note that this is the case for the choice of squared-error loss as the training loss function. In this section, we introduce a new loss function in order to balance the effects of all samples on the dynamics of the algorithm. This new loss function greatly relaxes the necessary condition given in Theorem 1, and it enables training even unstable linear systems with the gradient descent algorithm.

For any \( \epsilon > 0 \), define \( F_\epsilon : \mathbb{R} \rightarrow \mathbb{R} \) as

\[
F_\epsilon(\xi) = \begin{cases} 
\log(\epsilon + \xi) & \xi \geq 0, \\
-\log(\epsilon - \xi) & \xi < 0.
\end{cases}
\] (5)

\footnote{The random distribution is assumed to assign zero probability to every set with Lebesgue measure zero.}
Given two trajectories \( \{ x(t) \}_{t \in T} \) and \( \{ y(t) \}_{t \in T} \) in \( \mathbb{R}^n \), consider the loss function defined as
\[
\ell(x, y) = \sum_{t \in T} \sum_{j=1}^n \frac{1}{t^2} \left( F_e(e_j^T x(t)) - F_e(e_j^T y(t)) \right)^2,
\]
where \( e_j \) denotes the \( j \)-th standard basis vector with a 1 in its \( j \)-th coordinate and 0 in all other coordinates. Note that \( \ell(x, y) \) is zero if and only if \( x(t) = y(t) \) for all \( t \in T \); and it is strictly positive otherwise. Similar to Section 3, we will analyze this loss functions for learning linear dynamical systems.

**Theorem 2.** Let \( \{ z_k \}_{k \in K} \) be a set of trajectories, and let \( s_k \) denote the initial state for trajectory \( z_k \) for each \( k \in K \). Define the set of sampling instants of \( z_k \) as \( T_k \), and denote the samples taken from this trajectory by \( \{ x_k(t) \}_{t \in T_k} \). Assume that the gradient descent algorithm is used to solve
\[
\min_{A,C,\{s_k\}_{k \in K}} \sum_{k \in K} \sum_{t \in T_k} \sum_{j=1}^n \frac{1}{t^2} \left( F_e(e_j^T x_k(t)) - F_e(\hat{e}_j^T C e^{A t} s_k) \right)^2,
\]
where \( F_e \) is as defined in \( \mathcal{S} \). Then, for almost every initialization, the learning rate \( \delta \) of the gradient descent algorithm must satisfy
\[
\delta \leq \frac{2}{\lambda_{\min} \left( \sum_{k \in K} \sum_{t \in T_k} \rho^2 e^{2t(\Lambda_t)} \parallel C e^{A t} s_k \parallel_{\infty} + \tau \parallel \hat{e}_j \parallel_2 \right)^2}
\]
so that the algorithm can converge to the solution \( (\hat{A}, \hat{C}, \{ \hat{s}_k \}_{k \in K}) \) achieving zero training error, where \( \Lambda \) is the eigenvalue of \( \hat{A} \) with the largest real part, \( \rho^2 = \max_{u \in \mathcal{U}} \parallel C u \parallel_2^2 \), and \( \mathcal{U} \) is the set of right-eigenvectors of \( \hat{A} \) corresponding to its eigenvalue \( \Lambda \).

**Proof.** See Appendix B.

The necessary conditions on the step size given in Theorem 2 and in Theorem 1 are obtained by following the identical analysis procedure. Theorem 2 shows that the loss function (6) substantially relaxes the necessary condition given in Theorem 1, and it balances the weights of all the sampling instants on the dynamics of the gradient descent algorithm. In other words, it makes it easier for the gradient descent algorithm to converge to the global minima. This is demonstrated in the next section.

## 5 Experiments

To check if the time-weighted logarithmic loss function introduced in Theorem 2 allows learning linear dynamical systems with the gradient descent algorithm, we generated a set of output trajectories from randomly generated linear systems and trained a linear model with this data set by using the logarithmic loss function. We also trained the model with the same data set by using the mean-squared-error loss to compare the two estimates.

For the experiments, we considered the discretized version of the dynamical systems. In other words, we used
\[
z_k(t) = A^t z_k(0) \quad \forall t \in \mathbb{N}, \forall k \in K.
\]
Note that with this discrete-time representation, the stability of the system is described based on the position of the eigenvalues relative to the unit circle. The system is stable if all of its eigenvalues are inside the unit circle.

We randomly generated \( A \in \mathbb{R}^{n \times n} \) and \( C \in \mathbb{R}^n \) to produce a set of observation sequences. In particular, we generated \( A \) as \( A = I + \Delta A \), where \( \Delta A \) is a matrix whose entries are independent and uniformly distributed between \([-0.5, 0.5]\). The elements of \( C \) were drawn from independent standard normal distributions. We obtained 50 trajectories from the generated system by providing different initial states, and each trajectory consisted of 50 observations.

For training a linear model on this data set, we used the stochastic gradient method with momentum. Both for the mean-squared-error loss and for the time-weighted logarithmic loss, the gradients were normalized to unit norm if their \( \ell_2 \) norm exceeded 1. Figure 1 shows a typical plot for the training
error of an unstable system for each of these loss functions. We observe that the gradient descent algorithm is not able to decrease the mean-squared error loss, whereas the time-weighted logarithmic loss function is diminished easily.

To check if this decrease in the loss function corresponds to an effective learning of the actual model, we computed the eigenvalues of the estimated system throughout training and compared them with the eigenvalues of the actual system. Figure 2 demonstrates an example of how the estimates for the eigenvalues evolve during training. The state space of the system in Figure 2 is three dimensional, and the system is unstable as one of its eigenvalues is outside of the unit circle. When the mean-squared-error loss is used, only the unstable mode of the system is estimated correctly. In contrast, the time-weighted logarithmic loss function is able to discover all three modes of the system. Additional plots are provided in Appendix E.

Figure 2: A linear system with three-dimensional state space is trained with mean-squared-error loss [left] and time-weighted logarithmic loss [right]. The red stars show the eigenvalues of the real system, whereas the green dots show the eigenvalues of the estimated system. Earlier estimates of the eigenvalues are depicted with faded colors. Mean-squared-error loss is able to find only the unstable mode, whereas the logarithmic loss function discovers all three modes correctly.

6 Discussion

Variational inference. Variational inference is a Bayesian approach to handle the unknown parameters and the unobserved states of a dynamical system simultaneously [Jordan et al., 1999, Archer]
et al., 2015]. For variational inference, the system is described by a generative model: \( p_{\theta}(x, z) \), where \( x = \{x(t)\}_{t \in \mathcal{T}} \) and \( z = \{z(t)\}_{t \in \mathcal{T}} \) are the sequence of observations and hidden states of the system, and \( \theta \) is the parameters of the model. Given the observations, the posterior is approximated by another model: \( g_{\phi}(\cdot|x) \). Then, the objective function to be minimized is described as [Archer et al., 2015]

\[
-H(g_{\phi}(z|x)) - \mathbb{E}_{g_{\phi}(z|x)}[\log(p_{\theta}(x, z))],
\]

where \( H \) is the entropy of its argument. Assume the stochasticity of the initial state and the state transitions is removed, and each observation \( x(t) \) is obtained through an observation mapping with an additive Gaussian noise:

\[
x(t) = c(z(t)) + \xi_t,
\]

where \( \{\xi_t\}_{t \in \mathcal{T}} \) is an independent and identically distributed sequence. Then the minimization of the loss function (7) reduces to the problem

\[
\begin{align*}
\min_{\theta, \phi} & \quad \sum_{t \in \mathcal{T}} \|x(t) - c(z(t))\|_2^2 \\
\text{subject to} & \quad z(0) = \text{argmax}_{\tilde{z}} g_{\phi}(\tilde{z}|x),
\end{align*}
\]

and the system identification problem becomes equivalent to problem (3). This is the reason why we referred to (3) as the deterministic counterpart of the variational inference formulation.

Random initial states. In our analyses, we treated the initial states as unknown but deterministic values that could be learned during training. With this deterministic viewpoint, Theorem 1 and Theorem 2 assumed that the observations could be matched to the latent state of the system perfectly and the training loss function could be made identically zero. It is not possible in general to satisfy this requirement with an expected loss over a set of random initial states. Therefore, Theorem 1 and Theorem 2 do not apply to the formulations with random initial states verbatim.

Convergence of policy gradient. Even though the focus of this work has been on system identification, the gradient descent algorithm will exhibit similar convergence problems when maximizing an objective over a time horizon while altering the dynamics of a dynamical system. Note that policy gradient methods in reinforcement learning [Sutton and Barto, 2018] fall into this category. This is why our analysis in this work can potentially be used for studying and improving the stability of policy gradient methods.

7 Conclusion

To understand the hardness of learning dynamical systems from observed trajectories, we analyzed the dynamics of the gradient descent algorithm while training the parameters of a dynamical model, and we observed that samples taken at different times affect the dynamics of the algorithm in substantially different degrees. To balance the effect of samples taken at different times, we introduced the time-weighted logarithmic loss function and demonstrated its effectiveness.

In this work, we focused on learning linear dynamical systems. Whether a similar loss function improves training of nonlinear models is an important direction for future research. In addition, we considered a deterministic framework for our problem formulation with a dynamical system. An interesting question is whether allowing randomness in the state of the system or the state transitions could trade off the accuracy of the estimated model for the efficiency of the training procedure.

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A Proof of Theorem

To begin with, assume that \( C \) is a fixed matrix, and consider only one trajectory \( z \) with only one sample taken at time \( t \). Then the loss function to be minimized is

\[
\ell(A,s) = \frac{1}{2} \| x - Ce^{At}s \|^2_2,
\]

where \( s \) denotes the initial state of the trajectory. The update rule for the gradient descent algorithm gives

\[
\begin{align*}
A &\leftarrow A - \delta \frac{\partial}{\partial A}(Ce^{At}s - x, Ce^{At}s - x) \\
s &\leftarrow s - \delta \frac{\partial}{\partial s}(Ce^{At}s - x, Ce^{At}s - x)
\end{align*}
\]

This update rule creates a nonlinear dynamical system where the state of the system is the parameters \((A, s)\).

A dynamical system can converge to its equilibrium only if that equilibrium is stable in the sense of Lyapunov. A standard tool to analyze the stability for nonlinear systems is given by Lyapunov’s direct method: an equilibrium of a nonlinear system can be stable only if the linearization of the system around that equilibrium has no unstable mode [Khalil, 1996]. If, on the other hand, the linearized model has an eigenvalue larger than 1 in magnitude, then the nonlinear system is definitely unstable — which rules out the possibility of convergence to this equilibrium from its neighbors, except for a set on a low-dimensional manifold, which has Lebesgue measure zero. This shows that the system (8) can converge to an equilibrium only if all eigenvalues of the linearized model around that equilibrium are less than 1 in magnitude.

We can write the linearization of (8) around an equilibrium \((\hat{A}, \hat{s})\) as

\[
\begin{align*}
\hat{A} &\leftarrow \hat{A} - \delta f_1(\hat{A}) - \delta f_2(\hat{s}) \\
\hat{s} &\leftarrow \hat{s} - \delta f_3(\hat{A}) - \delta f_4(\hat{s})
\end{align*}
\]

where

- \( f_1 \) is the Jacobian with respect to \( A \) of the gradient with respect to \( A \) of the loss function \( \ell \) at \((A, s)\),
- \( f_2 \) is the Jacobian with respect to \( s \) of the gradient with respect to \( A \) of the loss function \( \ell \) at \((A, s)\),

and \( f_3 \) and \( f_4 \) are defined similarly. Note that \( f_2 \) and \( f_3 \) are the Jacobians of the gradients of the same function with respect to the same parameters in different orders; therefore, they are hermitian of each other:

\[
\langle \hat{A}, f_2(\hat{s}) \rangle = \langle f_3(\hat{A}), \hat{s} \rangle \quad \forall \hat{A}, \forall \hat{s}.
\]

This shows that the linearized model can be associated with a symmetric matrix; and consequently, all of its eigenvalues are real-valued, and its eigenvalues can be less than 1 only if all of its diagonal blocks have eigenvalues less than 1. In other words, a necessary condition for the solution \((\hat{A}, \hat{s})\) to be stable is that the mappings

\[
\begin{align*}
\hat{A} &\leftarrow \hat{A} - \delta f_1(\hat{A}) \\
\hat{s} &\leftarrow \hat{s} - \delta f_4(\hat{s})
\end{align*}
\]

have eigenvalues less than 1 in magnitude, or equivalently, the functions \( f_1 \) and \( f_4 \) have eigenvalues less than \( 2/\delta \). Note that this conclusion would be identical if \( C \) was also updated via the gradient descent algorithm. In particular, we would need the eigenvalue of the mapping \( f_1 \) to be less than 1 in magnitude around the equilibrium \((\hat{A}, \hat{C}, \{\hat{s}_k\}_{k \in K})\).

Finding a lower bound for the largest eigenvalue of the mapping \( f_1 \) will be easier with the following lemma.
Lemma 1. Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be a twice-differentiable function for all $i \in \mathcal{I}$, and define

$$F(x) = \frac{1}{2} \sum_{i \in \mathcal{I}} f_i^2(x).$$

If $F(x_0) = 0$, then the Hessian of $F$ at $x_0$ satisfies

$$\nabla^2 F(x_0) = \sum_{i \in \mathcal{I}} \nabla f_i(x_0) \nabla f_i(x_0)^\top.$$

Proof. We can write the gradient and the Hessian of $F$, respectively, as

$$\nabla F(x_0) = \sum_{i \in \mathcal{I}} (\nabla f_i(x_0)) f_i(x_0),$$

$$\nabla^2 F(x_0) = \sum_{i \in \mathcal{I}} \nabla f_i(x_0) \nabla f_i(x_0)^\top + f_i(x_0) \cdot \nabla^2 f_i(x_0).$$

Note that $F(x_0) = 0$ implies that $f_i(x_0) = 0$ for all $i \in \mathcal{I}$. Then we have

$$\nabla^2 F(x_0) = \sum_{i \in \mathcal{I}} \nabla f_i(x_0) \nabla f_i(x_0)^\top. \quad \blacksquare$$

Remember that $f_1(A)$ is the Jacobian with respect $A$ of the gradient with respect to $A$ of the loss function

$$\ell(A, C, s) = \frac{1}{2} \langle Ce^{At}s - x, Ce^{At}s - x \rangle.$$

Given $A \in \mathbb{R}^{n \times n}$, we can write

$$\ell(A, C, s) = \frac{1}{2} \sum_{j=1}^n (e_j^\top Ce^{At}s - e_j^\top x)^2,$$

where $e_j$ is the $j$-th standard basis vector with a 1 in its $j$-th coordinate and 0 in all other coordinates. Then, by using Lemma 1, the largest eigenvalue of the mapping $f_1$ can be lower bounded by

$$\max_{\|Y\|_p = 1} \sum_{j=1}^n |\langle Y, \nabla_A (e_j^\top Ce^{At}s - e_j^\top x) \rangle|^2. \quad (11)$$

To find the gradient, we can expand the matrix exponential:

$$\nabla_A \left( e_j^\top C \sum_{k=0}^\infty \frac{t^k}{k!} A^k s \right) = \sum_{k=1}^{\infty} \sum_{r=0}^{k-1} \frac{t^k}{k!} (A^\top)^r C^{\top} e_j s \top (A^\top)^{k-1-r}.$$

If we choose $\tilde{Y} = uv^\top$, where $u$ and $v$ are the unit-norm right and left eigenvectors of $A$ corresponding to its eigenvalue $\Lambda$ with the largest real part, we obtain

$$\langle \tilde{Y}, \nabla_A \left( e_j^\top C \sum_{k=0}^\infty \frac{t^k}{k!} A^k s \right) \rangle = \sum_{k=1}^\infty \sum_{r=0}^{k-1} \frac{t^k}{k!} \Lambda^{k-1} \langle u, C^{\top} e_j \rangle \langle v, s \rangle$$

$$= \sum_{k=1}^\infty \frac{t^k}{(k-1)!} \Lambda^{k-1} \langle u, C^{\top} e_j \rangle \langle v, s \rangle$$

$$= t \Lambda \langle u, C^{\top} e_j \rangle \langle v, s \rangle$$

$$= t \Lambda \langle Cu, e_j \rangle \langle v, s \rangle.$$

Remember that (11) is a lower bound for the largest eigenvalue of $f_1$, and so is

$$\sum_{j=1}^n \left| \langle \tilde{Y}, \nabla_A (e_j^\top Ce^{At}s - e_j^\top x) \rangle \right|^2 = \sum_{j=1}^n t^2 e^{2\text{Re}(\Lambda)t} |\langle Cu, e_j \rangle|^2 |\langle v, s \rangle|^2$$

$$= \rho^2 t^2 e^{2\text{Re}(\Lambda)t} |\langle v, s \rangle|^2,$$
where Re(Λ) is the largest real part of the eigenvalues of Λ and \( \rho^2 = \|Cu\|^2 \). If we have multiple trajectories, this lower bound will become

\[
\sum_{k \in K} \sum_{t \in T_k} \rho^2 t^2 e^{2\Re(\Lambda)t} |\langle v, s_k \rangle|^2,
\]

where \( \{s_k\}_{k \in K} \) is the set of initial states of the trajectories.

As a result, for convergence of the gradient descent algorithm to a solution \( (\hat{\Lambda}, \hat{C}, \hat{s}) \), it is necessary that

\[
\sum_{k \in K} \sum_{t \in T_k} \rho^2 t^2 e^{2\Re(\Lambda)t} |\langle v, s_k \rangle|^2 \leq \frac{2}{\delta}.
\]

Without making any assumptions about the eigenvectors of \( \hat{\Lambda} \), we can write a further lower bound for this expression as

\[
\sum_{k \in K} \sum_{t \in T_k} \rho^2 t^2 e^{2\Re(\Lambda)t} |\langle v, s_k \rangle|^2 = \frac{2}{\lambda_{\min} (\rho^2 \sum_{k \in K} \sum_{t \in T_k} t^2 e^{2\Re(\Lambda)t} (s_k)^\top)}.
\]

This completes the proof. \( \Box \)

### B Proof of Theorem 2

Similar to the proof of Theorem 1, we will use Lemma 1 to find a lower bound for the largest eigenvalue of the linearized system around \( (\hat{\Lambda}, \hat{C}, \{\hat{s}_k\}_{k \in K}) \). Without loss of generality, assume \( e_j \hat{C} e^{\hat{A}t} s > 0 \). Then,

\[
\nabla_A \log \left( e_j^\top \hat{C} e^{\hat{A}t} s + \epsilon \right) = \nabla_A \log \left( e_j^\top \hat{C} \sum_{k=0}^\infty \frac{t^k}{k!} \hat{A}^k s + \epsilon \right)
\]

\[
= \frac{1}{e_j^\top \hat{C} e^{\hat{A}t} s + \epsilon} \sum_{k=1}^\infty \sum_{r=0}^{k-1} \frac{t^k}{k!} \hat{A}^k \langle u, C^\top e_j \rangle \langle v, s \rangle.
\]

For the matrix \( \hat{Y} = uv^\top \), where \( u \) and \( v \) are the right and left eigenvectors of \( \hat{A} \) corresponding to its eigenvalue \( \lambda \) with the largest real part, we have

\[
\left\langle \hat{Y}, \nabla_A \log \left( e_j^\top \hat{C} e^{\hat{A}t} s + \epsilon \right) \right\rangle = \frac{1}{e_j^\top \hat{C} e^{\hat{A}t} s + \epsilon} \sum_{k=1}^\infty \sum_{r=0}^{k-1} \frac{t^k}{k!} \lambda^{k-1} \langle u, C^\top e_j \rangle \langle v, s \rangle
\]

\[
= \frac{t^e \hat{A}^t}{e_j^\top \hat{C} e^{\hat{A}t} s + \epsilon} \langle u, C^\top e_j \rangle \langle v, s \rangle.
\]

By using Lemma 1, we obtain a lower bound for the largest eigenvalue of the linearization of the gradient descent algorithm around \( (\hat{\Lambda}, \hat{C}, \{\hat{s}_k\}_{k \in K}) \) as

\[
\sum_{k \in K} \sum_{t \in T_k} \sum_{j=1}^n \frac{1}{t^2} \left| \frac{t^e \hat{A}^t}{e_j^\top \hat{C} e^{\hat{A}t} s_k + \epsilon} \langle C u, e_j \rangle \langle v, s_k \rangle \right|^2.
\]

We can write a further lower bound for this expression as

\[
\sum_{k \in K} \sum_{t \in T_k} \sum_{j=1}^n \frac{e^{2\Re(\Lambda) t}}{2} \left| \langle C u, e_j \rangle \right|^2 \left| \langle v, s_k \rangle \right|^2 = \sum_{k \in K} \sum_{t \in T_k} \frac{\rho^2 e^{2\Re(\Lambda) t}}{2} \left| \langle v, s_k \rangle \right|^2,
\]

where \( \rho^2 = \|Cu\|^2 \). If we have multiple trajectories, this lower bound will become

\[
\sum_{k \in K} \sum_{t \in T_k} \rho^2 t^2 e^{2\Re(\Lambda) t} |\langle v, s_k \rangle|^2 = \frac{2}{\lambda_{\min} (\rho^2 \sum_{k \in K} \sum_{t \in T_k} t^2 e^{2\Re(\Lambda) t} (s_k)^\top)}.
\]

This completes the proof. \( \Box \)
and finally,
\[ \lambda_{\min} \left( \sum_{k \in K} \sum_{t \in T_k} \frac{\rho^2 e^{2\text{Re}(\Lambda)t}}{\left(\| CeA^t s_k \|_\infty + \epsilon \right)^2 s_k s_k^T} \right), \]
where \( \rho^2 = \| \dot{\mathbf{C}} u \|_2^2 \) and \( u \) is the right-eigenvector of \( \mathbf{A} \) corresponding to its eigenvalue \( \Lambda \). For stability of the algorithm around the equilibrium point \( (\mathbf{A}, \{ \mathbf{s}_k \}_{k \in K}) \), we need
\[ \lambda_{\min} \left( \sum_{k \in K} \sum_{t \in T_k} \frac{\rho^2 e^{2\text{Re}(\Lambda)t}}{\left(\| CeA^t \mathbf{s}_k \|_\infty + \epsilon \right)^2 \mathbf{s}_k \mathbf{s}_k^T} \right) \leq \frac{2}{\delta}, \]
where \( \delta \) is the step size of the algorithm.

\section{Remarks on Convergence Rate}

In the proof of Theorem \( \text{I} \) we considered the mapping
\[ \mathbf{A} \leftarrow \mathbf{A} - \delta f_1(\mathbf{A}), \]
where \( f_1 \) is the Jacobian of the gradient of the loss function
\[ \ell(A, s) = \frac{1}{2} \| x - CeA^t s \|_2^2 \]
with respect to \( A \) at the point \( (\mathbf{A}, \mathbf{C}, \mathbf{s}) \). For Theorem \( \text{I} \) we computed the largest learning rate at which the algorithm can still converge to the specified equilibrium. Note that this was equivalent to computing a lower bound for the largest eigenvalue of the mapping \( f_1 \). Similar to the proof of Theorem \( \text{I} \), we can compute an upper bound for the smallest eigenvalue of \( f_1 \) around the solution \( (\mathbf{A}, \mathbf{C}, \mathbf{s}) \).

By using Lemma \( \text{I} \) the smallest eigenvalue of the mapping \( f_1 \) can be upper bounded by
\[ \min_{Y: \| Y \|_2 = 1} \sum_{j=1}^n \left| \mathbf{Y}, \mathbf{\nabla}_A \left( e_j^T CeA^t s - e_j^T x \right) \right|^2. \tag{12} \]
Similar to the proof of Theorem \( \text{I} \) we can expand the matrix exponential:
\[ \mathbf{\nabla}_A \left( e_j^T CeA^t \sum_{k=0}^\infty \frac{t^k}{k!} A^k s \right) = \sum_{k=1}^{\infty} \sum_{r=0}^{k-1} \frac{t^k}{k!} (A^T)^r C^T e_j s^T (A^T)^{k-1-r}. \]
If we choose \( \tilde{Y} = uv^T \), where \( u \) and \( v \) are the unit-norm right and left eigenvectors of \( A \) corresponding to its eigenvalue \( \lambda_2 \), we obtain
\[ \mathbf{\langle Y, \nabla}_A \left( e_j^T CeA^t \sum_{k=0}^\infty \frac{t^k}{k!} A^k s \right) \mathbf{\rangle} = \sum_{k=1}^{\infty} \sum_{r=0}^{k-1} \frac{t^k}{k!} \lambda_2^{k-1} \mathbf{\langle u, C^T e_j \rangle} \mathbf{\langle v, s \rangle} = \sum_{k=1}^{\infty} \frac{t^k}{(k-1)!} \lambda_2^{k-1} \mathbf{\langle u, C^T e_j \rangle} \mathbf{\langle v, s \rangle} = te^{\lambda_2 t} \mathbf{\langle u, C^T e_j \rangle} \mathbf{\langle v, s \rangle}. \]
Remember that \( \text{I} \) is an upper bound for the smallest eigenvalue of \( f_1 \), and so is
\[ \sum_{j=1}^n \left| \mathbf{\langle Y, \nabla}_A \left( e_j^T CeA^t s - e_j^T x \right) \mathbf{\rangle} \right|^2 = \sum_{j=1}^n t^2 e^{2\text{Re}(\lambda_2)t} |\mathbf{\langle Cu, e_j \rangle}|^2 |\mathbf{\langle v, s \rangle}|^2 = \rho^2 t^2 e^{2\text{Re}(\lambda_2)t} |\mathbf{\langle v, s \rangle}|^2, \]
As a result, convergence of the gradient descent algorithm to the minimum of $H$ where
\[
\sum_{k \in K} \sum_{t \in T_k} \rho_2^t e^{2Re(\lambda_2) t} \langle (v, s_k) \rangle^2,
\]
where $\{s_k\}_{k \in K}$ is the set of initial states of the trajectories. We can bring this upper bound into a form independent of $v$:
\[
\lambda_{\text{max}} \left( \sum_{k \in K} \sum_{t \in T_k} \rho_2^t e^{2Re(\lambda_2) t} s_k s_k^\top \right).
\]
This shows that the ratio of the largest eigenvalue to the smallest eigenvalue of $f_1$ satisfies
\[
\frac{\lambda_{\text{max}}(f_1)}{\lambda_{\text{min}}(f_1)} \geq \frac{\lambda_{\text{max}}(f_1)}{\lambda_{\text{min}}(f_1)} \left( \frac{\rho_2^t \sum_{k \in K} \sum_{t \in T_k} t^2 e^{2Re(\lambda_2) t} s_k s_k^\top}{\rho_2^t \sum_{k \in K} \sum_{t \in T_k} t^2 e^{2Re(\lambda_2) t} s_k s_k^\top} \right).
\]
for any pair of eigenvalues $(\lambda_1, \lambda_2)$ of $\hat{A}$, where $\rho_1 = \|Cu_1\|_2$, $\rho_2 = \|Cu_2\|_2$, and $u_1, u_2$ are the right eigenvectors of $\hat{A}$ corresponding to $\lambda_1, \lambda_2$. If $H$ denotes the Hessian of the loss function $\ell$ at the point $(\hat{A}, \hat{C}, \{s_k\}_{k \in K})$, we have $\lambda_{\text{max}}(H) \geq \lambda_{\text{max}}(f_1)$ and $\lambda_{\text{min}}(H) \leq \lambda_{\text{min}}(f_1)$. Therefore, we also have
\[
\frac{\lambda_{\text{max}}(H)}{\lambda_{\text{min}}(H)} \geq \frac{\lambda_{\text{max}}(f_1)}{\lambda_{\text{min}}(f_1)} \left( \frac{\rho_2^t \sum_{k \in K} \sum_{t \in T_k} t^2 e^{2Re(\lambda_2) t} s_k s_k^\top}{\rho_2^t \sum_{k \in K} \sum_{t \in T_k} t^2 e^{2Re(\lambda_2) t} s_k s_k^\top} \right).
\]
To understand the relationship of (13) to the convergence rate, consider a quadratic function $h : \mathbb{R}^n \mapsto \mathbb{R}$ defined as
\[
h(w) = \frac{1}{2} (w - w^*)^\top H (w - w^*),
\]
where $H$ is the Hessian of $h$ and $w^*$ is the point where $h$ attains its minimum. For the gradient descent algorithm
\[
w \leftarrow w - \delta H (w - w^*)
\]
to converge to the minimum of $h$ from arbitrary initializations, we need the learning rate $\delta$ to be smaller than $\frac{2}{\lambda_{\text{max}}(H)}$. Assume $(w_0 - w^*)$, where $w_0$ is the initial point where the algorithm starts, is in the direction of the eigenvector of $H$ corresponding to its minimum eigenvalue. In other words,
\[
H(w_0 - w^*) = \lambda_{\text{min}}(H)(w_0 - w^*).
\]
Then the iterations of the gradient descent algorithm becomes
\[
(w_k - w^*) \leftarrow (w_{k-1} - w^*) - \delta H (w_{k-1} - w^*)
\]
\[
\leftarrow (w_{k-1} - w^*) - \delta \lambda_{\text{min}}(H)(w_{k-1} - w^*)
\]
\[
\leftarrow (1 - \delta \lambda_{\text{min}}(H))(w_{k-1} - w^*)
\]
\[
\leftarrow (1 - \delta \lambda_{\text{min}}(H))^k (w_0 - w^*).
\]
Attaining $\|w_k - w^*\|_2 \leq \epsilon$ for any $\epsilon > 0$ will require
\[
(1 - \delta \lambda_{\text{min}}(H))^k \|w_0 - w^*\|_2 \leq \epsilon \implies k \log (1 - \delta \lambda_{\text{min}}(H)) + \log (\|w_0 - w^*\|_2) \leq \log (\epsilon),
\]
which gives a lower bound for the number of iterations needed:
\[
k \geq \frac{1}{\log \left( \frac{1}{1 - \delta \lambda_{\text{min}}(H)} \right)} \left( \log \left( \frac{1}{\epsilon} \right) + \log (\|w_0 - w^*\|_2) \right).
\]
As a result, convergence of the gradient descent algorithm to the minimum of $h$ in the direction of the bottom eigenvector of $H$ requires
\[
O \left( \left[ \log \left( (1 - \delta \lambda_{\text{min}}(H))^{-1} \right) \right]^{-1} \right)
\]
iterations. Remember that for convergence of the algorithm, we require $\delta < \frac{2}{\lambda_{\text{max}}(H)}$; therefore, $\delta \lambda_{\text{min}}(H) < 2 \lambda_{\text{min}}(H)$. Combining (13) and (14) gives the local convergence rate for the loss function $\ell$, if we assume the second approximation of $\ell$ represents it well around $(\hat{A}, \hat{C}, \{s_k\}_{k \in K})$. 

14
D Alternatives for Initial States

For the proof of Theorem 1 and Theorem 2, we considered the loss function

\[ \ell(A, C, s) = \frac{1}{2} \sum_{t \in \mathcal{T}} \| x(t) - Ce^{At}s \|^2, \]

and analyzed the linearization of the dynamics of the gradient descent algorithm around the solution \((\hat{A}, \hat{C}, \hat{s})\):

\[ \begin{align*}
\hat{A} &\leftarrow f_{1,1}(\hat{A}) + f_{1,2}(\hat{C}) + f_{1,3}(\hat{s}) \\
\hat{C} &\leftarrow f_{2,1}(\hat{A}) + f_{2,2}(\hat{C}) + f_{2,3}(\hat{s}) \\
\hat{s} &\leftarrow f_{3,1}(\hat{A}) + f_{3,2}(\hat{C}) + f_{3,3}(\hat{s}),
\end{align*} \]

(15)

where \( \{f_{i,j}\}_{i \in [3], j \in [3]} \) are the Jacobians of the partial derivatives of \( \ell \) with respect to \( A, C \) and \( s \), evaluated at the point \((\hat{A}, \hat{C}, \hat{s})\). We used the fact that system (15) can be represented by a symmetric matrix to use only the eigenvalues of \( f_{1,1} \) in order to obtain a lower bound for the largest eigenvalue of the system (15).

Note that fixing the initial state \( s \) and not updating it with the gradient descent algorithm will not affect the eigenvalues of \( f_{1,1} \). Therefore, the results for Theorem 1 and Theorem 2, which only depend on the largest eigenvalues of \( f_{1,1} \), will still hold when the initial state is fixed.

Now assume the initial state is obtained via a state estimator:

\[ s = g_{\phi}(\{t, x(t)\}_{t \in \mathcal{T}}), \]

where \( \mathcal{T} \) is the set of sampling instants for the trajectory and \( \{x_t\}_{t \in \mathcal{T}} \) is the set of samples obtained. While solving the problem

\[ \minimize_{A, C, \phi} \sum_{t \in \mathcal{T}} \ell(x(t), Ce^{At}g_{\phi}(\{t, x(t)\}_{t \in \mathcal{T}})) + L(\phi), \]

the linear approximation to the gradient descent algorithm can be written as

\[ \begin{align*}
\hat{A} &\leftarrow \hat{f}_{1,1}(\hat{A}) + \hat{f}_{1,2}(\hat{C}) + \hat{f}_{1,3}(\hat{s}), \\
\hat{C} &\leftarrow \hat{f}_{2,1}(\hat{A}) + \hat{f}_{2,2}(\hat{C}) + \hat{f}_{2,3}(\hat{s}), \\
\hat{s} &\leftarrow \hat{f}_{3,1}(\hat{A}) + \hat{f}_{3,2}(\hat{C}) + \hat{f}_{3,3}(\hat{s}),
\end{align*} \]

(16)

where \( \{\hat{f}_{i,j}\}_{i \in [3], j \in [3]} \) are the Jacobians of the partial derivatives of \( \ell \) with respect to \( A, C \) and \( \phi \), evaluated at the point \((\hat{A}, \hat{C}, \hat{s})\). Note that system (16) can still be represented by a symmetric matrix; therefore, the largest eigenvalues of \( \hat{f}_{1,1} \) can be used to obtain an upper bound on the learning rate of the algorithm. Furthermore, given that \( \frac{\partial C}{\partial A} = 0 \), \( f_{1,1} \) in (15) and \( \hat{f}_{1,1} \) in (16) are identical, with the substitution \( s = g_{\phi}(\{t, x(t)\}_{t \in \mathcal{T}}) \). For this reason, the results of Theorem 1 and Theorem 2 still hold for systems with a state estimator \( g_{\phi} \), provided that the estimation error at equilibrium is zero; that is,

\[ \sum_{t \in \mathcal{T}} \ell(x(t), Ce^{At}g_{\phi}(\{t, x(t)\}_{t \in \mathcal{T}})) = 0, \]

which is needed only to allow the use of Lemma 1.

E Additional Experiments

In this section, we provide additional experimental results to show that the comparison in Figure 2 is not incidental. Figure 3 and Figure 4 demonstrate the comparison of the estimated eigenvalues for a different initialization and for a system with a four-dimensional state space, respectively. Note that we were not able to enable the gradient descent algorithm to learn any of the eigenvalues correctly when the training loss is mean-squared error despite the fact that we used various learning rates for these experiments.
Figure 3: A linear system with three-dimensional state space is trained with mean-squared-error loss [left] and time-weighted logarithmic loss [right]. The red stars show the eigenvalues of the real system, whereas the green dots show the eigenvalues of the estimated system. Earlier estimates of the eigenvalues are depicted with faded colors.

Figure 4: A linear system with four-dimensional state space is trained with mean-squared-error loss [left] and time-weighted logarithmic loss [right]. The red stars show the eigenvalues of the real system, whereas the green dots show the eigenvalues of the estimated system. Earlier estimates of the eigenvalues are depicted with faded colors.