Identification and Estimation of Nonseparable Models with Multivalued Endogeneity and a Binary Instrument *

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Abstract

In this paper, I show that a nonseparable model where the endogenous variable is multivalued can be point-identified even when the instrument (IV) is only binary. Though the order condition generally fails in this case, I show that exogenous covariates are able to generate enough moment equations to restore the order condition as if enlarging the IV’s support under very general selection mechanisms for the endogenous variable. No restrictions are imposed on the way these covariates enter the model, such as separability or monotonicity. Further, after the order condition is fulfilled, I provide a new sufficient condition that is weaker than the existing results for the global uniqueness of the solution to the nonlinear system of equations. Based on the identification result, I propose a sieves estimator and uniform consistency and pointwise asymptotic normality are established under simple low-level conditions. A Monte Carlo experiment is conducted to examine its performance.

Keywords: Nonseparable models, instrumental variable, multivalued treatment, quantile treatment effect, monotonicity, sieves estimator.

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1 Introduction

Endogeneity is one of the central issues in econometrics. As a workhorse to resolve endogeneity, instrumental variable (IV) method is widely adopted in applied economics. It is well-known that in general, nonparametric identification of a model with multivalued endogeneity requires the support size of the IV to be no smaller than the endogenous variable, otherwise the order condition fails. However, in many applications, it is often to have a binary instrument while the endogenous variable takes on more values. In this paper, I show that when there is an exogenous covariate, under a general class of selection mechanisms for the endogenous variable, identification can be restored.

The model we focus on in this paper is a potential outcome function that is non-structurally separable in the endogenous variable, the covariates, and the scalar unobservable. In this way, the model is able to capture rich heterogeneous effects that is essential in a lot of applications such as quantile treatment effect.

When the endogenous variable is discrete, it is known that the IV is able to identify this model under monotonicity in the unobservable by normalizing its marginal distribution, provided that the IV takes on at least the same number of values as the endogenous variable. In particular, Chernozhukov and Hansen (2005) shows that conditional on the covariates and at a fixed value of the unobservable, each value the IV takes on induces a moment equation that the unknown outcome function satisfies. As the number of unknowns is then the support size of the endogenous variable, identification fails when the IV has a smaller support.

When identification fails due to the small support of the IV, denoted by $Z$, I develop a method to use an exogenous covariate $X$ to augment the existing moment equations by the matching points. Essentially, a matching point $X = x_1$ for a conditioning value of interest $X = x_0$, is such that paired with different values of the binary IV, $(z, x_0)$ and $(z', x_1)$ generate exactly the same selection patterns. Then I show the changes in the outcome function from $x_0$ to $x_1$ is identified. As a consequence, though in general conditioning on different values of $X$ does not help identification as the outcome changes as well, conditioning on the matching points is useful as such changes are, though non-zero, known. In other words, $x_1$ can be used as the third IV value and the support of the IV is effectively enlarged.

The existence and the finding of such matching points hinge on the selection model for the endogenous variable. It turns out a large class of threshold-crossing models meet the requirement, for example the selection mechanism introduced in Lee and Salanié (2018). Further, there is no need to fully specify and identify the selection model. Instead, under the high level condition I impose, finding the matching points is equivalent to matching the generalized propensity scores. Note that this method does not depend on whether the
potential outcome function is nonseparable or not, so it can also be easily applied to other models when order condition fails, for example additive nonparametric IV models.

After the order condition is fulfilled, I establish a new global uniqueness theorem to achieve identification. Instead of viewing the true outcome function as a solution to the moment equations at a fixed level of the unobservable, as in Chernozhukov and Hansen (2005), I consider it as a solution path to the system, i.e., a function of the realization of the unobservable that satisfies the system for all the realizations. Then monotonicity and continuity of the true potential outcome function can be exploited. I show that under regularity conditions, global uniqueness of the solution path is guaranteed when for every realization of the unobservable, the system is \textit{locally} invertible at the true outcome function. This result is much weaker than the traditional ones which usually require the system to be \textit{globally} invertible. Note this result does not depend on the moment equation augmentation process described earlier, so is also applicable in the cases where the IV has a large support.

Based on the identification result, I build a sieves estimator for the potential outcome function. Specifically, I estimate the function by minimizing the Euclidean distance between the empirical moment equations and zero evaluated at different values of the unobservable simultaneously, imposing monotonicity in the estimated nodes. As the number of nodes increase as the sample size goes to infinity, I show the estimator is consistent uniformly in the realization of the unobservable. It turns out monotonicity greatly simplifies the conditions needed for uniform consistency compared with the general theory in, for example Chen and Pouzo (2012). Also, asymptotic normality is straightforward to establish because pointwisely, the estimator is essentially equivalent as the standard GMM estimator where the moment conditions are estimated nonparametrically.

To examine the finite sample performance of our estimator, we conduct a Monte Carlo experiment. The data generating process is designed to match the key statistics in the Head Start Impact Study documented in Kline and Walters (2016). In Kline and Walters (2016), the endogenous treatment takes on three values: participating Head Start, participating a competing program, or not participating any preschool programs. The IV is the random assignment of a lottery granting access to the Head Start program so it is binary. The simulation results show the the estimator has small integrated mean squared error and squared bias, and captures the shape of the true potential outcome functions well.

My identification and estimation results extend the theories on the IVQR models developed and studied in Chernozhukov and Hansen (2005), Horowitz and Lee (2007), Chernozhukov, Imbens and Newey (2007), Chen and Pouzo (2012, 2015), etc. This line of research typically assume the IV has sufficient variation. Vuong and Xu (2017) studies pointidentification of a similar model for a binary treatment so the IV has enough variation even when it
is binary. Lee and Salanié (2018) considers multivalued treatment but the instrument essentially needs to be continuous. For small-support IVs, Torgovitsky (2015) and D’Haultfoeuille and Février (2015) show a binary IV is sufficient to identify the nonseparable model when the endogenous variable is continuous. However, their methods cannot be applied to discrete endogenous variables. Caetano and Escanciano (2018) allows for a discrete endogenous variable and a binary IV. Their strategy also needs a covariate, and the covariate can be endogenous, but they require the covariate to enter the model in a structurally separable way. In contrast, my strategy needs an exogenous covariate but does not put any restrictions on the separability of it. This is useful in many applications such as quantile regressions.

The rest of the paper is organized as follows. Section 2 introduces and discusses the model. In Section 3 identification results are given. Section 4 presents and discusses the estimator. The asymptotic theory of the estimator is shown in Section 5. Section 6 demonstrates the Monte Carlo experiment. Section 7 concludes. Proofs and additional results are presented in the Appendix.

2 The Model

The model we consider consists of an outcome function and a selection mechanism.

\[ Y = \sum_{d=1}^{K} 1(D = d)g_d^*(X, U_d) \]  \hspace{1cm} (1)

\[ D = d \iff h_d(\gamma(X, Z), V) = 1 \] \hspace{1cm} (2)

where \( Y \) is the continuous outcome. \( D \in \{1, 2, ..., K\} \) is a discrete treatment that can be ordered or unordered. To simplify the exposition, we focus on the case where \( K = 3 \). The more general case is discussed in the Appendix. \( \{g_d^*\} \) are real-valued functions and are the main objects of identification and estimation. \( h_d \) is a function that only takes on value from \( \{0, 1\} \). \( X \) is a \( q \)-vector of covariates and \( Z \in \{0, 1\} \) is a binary instrumental variable. \( \{U_d\} \) is a vector of scalar unobservables determining the outcome, and \( V \) is a vector of unobservables determining the treatment selection. When \( U \) and \( V \) are not independent, \( D \) is endogenous and this is the case we focus on. In the following subsections we discuss conditions imposed on the system to identify \( g_d^* \).

2.1 The Outcome Function

The conditions imposed on the outcome functions are standard in the literature of non-separable models, e.g. Chernozhukov and Hansen (2005), except we need the covariates to
be exogenous. For simplicity, all components in $X$ will be assumed to be exogenous but in fact, only the components that contribute to matching points need to be exogenous and all assumptions and results can be viewed as conditional on the rest of the components in $X$.

**Assumption 1.** Given a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$

  a. $g^*_d(x, \cdot)$ is Lipschitz continuous and strictly increasing for all $d$ and $x$ on $[0, 1]$.

  b. $\{U_d\}$ are identically distributed conditional on $V$ and are marginally distributed as $U[0, 1]$.

  c. The support of $U_d$ conditional on $V$ is equal to its unconditional support for all $d$.

  d. $(Z, X) \perp \perp (U_d, V)$ for all $d$.

Assumptions 1-a and 1-b together admit a quantile interpretation of the potential outcome functions. Denote the potential outcome with treatment $d$ conditional on observed characteristics $X = x$ by $Y_{dx}$, then from (1), $Y_{dx} = g^*_d(x, U_d)$, thus for any $u \in (0, 1)$,

$$P(Y_{dx} \leq g^*_d(x, u)) = P(g^*_d(x, U_d) \leq g^*_d(x, u)) = P(U_d \leq u) = u$$

$$\implies g^*_d(x, u) = Q_{Y_{dx}}(u)$$

where $Q_{Y_{dx}}(u)$ is the $u$-th quantile of $Y_{dx}$.

The first part of Assumption 1-b is called *rank similarity* by Chernozhukov and Hansen (2005). It is slightly weaker than *rank invariance* which require $U_d$s are equal across $d$. Rank invariance amounts to assuming agents’ rank in the counterfactual outcome remains the same across different treatment levels, while rank similarity allows the rank to vary across treatment, as long as they are identically distributed.

Assumption 1-c guarantees the support of potential outcome $Y_{dx}$, denoted by $S(Y_{dx})$, is identical with the support of $Y$ conditional on $D = d$ and $X = x$\(^1\). This condition ensures that all the potential outcomes are observable in population.

Assumption 1-d requires the covariates and the instrument are jointly independent of the unobservables in both the outcome and selection equations. Full independence of the instrument is typically required to identify nonseparable models (e.g., Imbens and Newey 2009, Torgovitsky 2015, etc), but we also require (some of) the covariates to be exogenous to help restore identification. As mentioned before, not all covariates need to be exogenous and the number needed depends on the selection model and how many matching points are needed to be found. This will be clear in the next section.

\(^1\)To see this, first note the support of $Y_{dx}$ is the range of $g^*_d(x, u)$ for $u \in [0, 1]$. The support of $Y$ conditional on $(D, X) = (d, x)$ is the range of $g^*_d(x, u)$ and $u$ is in the support set of $U_d$ conditional $V$. Since conditioning on $V$ does not change the support of $U_d$, the domain of the function $g^*_d(x, \cdot)$ remains to be $[0, 1]$ so the range is the same.
Under Assumption 1, the following equation hold for all $x$, $z$ and $u$, as shown in Chernozhukov and Hansen (2005):

$$\sum_{d=1}^{3} F_{Y|XZ}(g_{d}^{*}(x, u), d|x, z) = u$$

where $F_{Y|XZ}(g_{d}^{*}(x, u), d|x, z) \equiv P(Y \leq g_{d}^{*}(x, u), D = d|X = x, Z = z)$.

Conditional on a particular value of interest $X = x_{0}$, the functions $\{g_{d}^{*}(x_{0}, \cdot)\}$ are identified if for all $u \in [0, 1]$, $\{g_{d}^{*}(x_{0}, u)\}$ is the unique solution to $\sum_{d=1}^{3} F_{Y|XZ}(\cdot, d|x_{0}, z) = u$. However, when $Z \in \{0, 1\}$, for a fixed $u$, (3) only consists of two equations conditional on $(0, x_{0})$ and $(1, x_{0})$, but there are three unknowns $g_{1}^{*}(x_{0}, u)$, $g_{2}^{*}(x_{0}, u)$ and $g_{3}^{*}(x_{0}, u)$. The solution to such a system of equations is typically not unique and identification thus fails.

The idea to restore identification is to use variations in $X$. Notice that (3) holds for all $x$ since $X \perp U_d$ for all $d$. At a different value $X = x_{1}$, equation (3) induces two more equations by conditioning on $(Z, X) = (0, x_{1})$ and $(Z, X) = (1, x_{1})$, but at the same time three more unknowns, $g_{1}^{*}(x_{1}, u)$, $g_{2}^{*}(x_{1}, u)$ and $g_{3}^{*}(x_{1}, u)$, are introduced to the system. However, if we choose this $x_{1}$ carefully in a way that the mapping from $\{g_{d}^{*}(x_{1}, u)\}$ to $\{g_{d}^{*}(x_{0}, u)\}$ is identified, we can express $g_{1}^{*}(x_{1}, u)$ as a known function of $g_{2}^{*}(x_{0}, u)$, so the effective number of unknowns is not increased but more moment equations are added into the system.

### 2.2 The Selection Mechanism

In this section, we introduce and discuss conditions that the selection model given by equation (2) satisfies.

**Assumption 2.** For each $d$, $x$ and $z$,

1. $h_d(\gamma(x, z), \cdot)$ is Borel measurable.
2. $\sum_{d=1}^{3} h_d(\gamma(x, z), \cdot) = 1$.
3. For $x_0$, $\exists$ a matching point $x_1$, such that $\gamma(x_0, z) = \gamma(x_1, z')$ and $h_d(\gamma(x_0, 1 - z), \cdot) \neq h_d(\gamma(x_1, 1 - z'), \cdot)$ for some $d$.

Note that both $\gamma$ and the unobservable $V$ in $h_d$ can be vectors. The key condition on the selection model is the weak separability between these two components. As $\gamma$ is not $d$ specific, this setup implies if for $(x_0, z)$ and $(x_1, z')$, $\gamma(x_0, z) = \gamma(x_1, z)$, then the $h_d$ functions at these two values are identical for all $d$.

Such separability is common in a large class of selection models. For example, Lee and Salanié (2018) proposes a similar selection mechanism and shows many popular models, with or without monotonicity, satisfies this representation. In the next section we will illustrate the
model by two examples. On the other hand, the selection model in this paper is more general than theirs in the sense that they mainly study the "rectangular" threshold, i.e., $h_d(\gamma(x,z), \cdot)$ is measurable with respect to the $\sigma-$field generated by the events $\{V_j < \gamma_j(X,Z)\}$ for all $j$ where $j$ is the index of the $\gamma$ and $V$ vectors. Here we only require $h_d$ to be Borel measurable because as will be clear later, the selection model in this paper is only to guarantee the existence of a matching point $x_1$.

Assumption 2-c is a new type of relevance condition. To see this, note that for the matching value $x_1$ to exist, $X$ is necessarily to have rich variations so that it can compensate the difference in $\gamma$ between $Z = 0$ and $Z = 1$. For example, suppose parents are considering whether to send children to a preschool program, and $Z$ is a randomly assigned lottery, winning which reduces cost of the program. If we assume parents are more likely to enroll children with lower baseline test scores, then the enrollment probability of a child who receives the lottery with high baseline scores may be the same as a child who does not receive the lottery but has lower baseline scores. This approach would fail when the impact of the instrument dominates. For example now if winning the lottery means compulsory enroll while losing it results in no access to the program, then the covariate $X$ is not "relevant" for the treatment selection and is actually excluded from $\gamma$.

The following theorem shows that between the matching points, the potential outcomes can be one-to-one linked and the link function is identified.

**Theorem 1.** Under Assumption 1 and 2, for $x_0$, if $x_1$ satisfies Assumption 2-c, then

$$F_{Y_d|XZ}(g^*_d(x_0), u|d|x_0, z) = F_{Y_d|XZ}(g^*_d(x_1), u|d|x_1, z'), \forall d$$

(4)

This theorem links the counterfactuals $Y_{dx_0}$ and $Y_{dx_1}$. Since the CDFs on both sides of the equation are directly identified from the population, this link is identified. Further, as $Y$ is continuously distributed, the mapping from $g^*_d(x_0)$ to $g^*_d(x_1)$ is one-to-one.

This idea is analogous to the *counterfactual mapping* introduced in Vuong and Xu (2017). In their case, the treatment is binary and the selection mechanism is simpler so they can identify the mapping from $Y_{dx}$ to $Y_{dx'}$. In contrast, in this paper, that counterfactual mapping is no longer identifiable because with a multivalued $D$ is, the selection mechanism is more complicated. Instead, we extrapolate information from agents with $(X,Z) = (x_1, z')$ for those with $(X,Z) = (x_0, z)$. Intuitively, these two groups are comparable because with any realization of $V$, they will make exactly the same selection, implied by $h_d(\gamma(x_0, z), V) = h_d(\gamma(x_1, z'), V)$ for all $d$.

Once the links between the $g^*_d(x_1, \cdot)$ and $g^*_d(x_0, \cdot) \forall d$ are established, two more equations in the form of equation (3) by conditioning on $(X,Z) = (x_1, z)$ and $(X,Z) = (x_1, z')$ can
be used as $g_d^*(x_1, u)$ are no longer free unknowns. However, as will be shown in the next section, the equation with $(x_1, z')$ is redundant with the link equations (4). Therefore, in the end, the matching point provides one additional moment equation with $(x_1, z)$, given the last part of Assumption 2-c holds so that it is not redundant again. This restriction is mild and is likely to hold in many applications. In the preschool example, now if we make a switch, it is reasonable to think that children who have the lottery with low baseline scores will be more likely to select into Head Start, compared to those who do not have the lottery with high scores.

3 Identification

From now on, all analysis are made for a given $x_0$. Let the matching point $x_1$ satisfying Assumption 2-c exists.

3.1 Identification of the Matching Point

So far we have only assumed the existence of a matching point $x_1$. To implement the idea, we need to find it. One way to find it is to specify and identify the selection model because when $\gamma$ is identified up to a monotone transformation and $\{h_d\}$ is also known, $x_1$ can be obtained by solving matching $\gamma$s. Identification of $\gamma$ in different classes of models can be found in for example Lee and Salanié (2018).

However, full identification of the selection model is not always necessary. In this paper we consider a class of models such that the following assumption is satisfied. Let $p_d(x, z) \equiv P(D = d|X = x, Z = z)$ denote the generalized propensity scores.

**Assumption 3.** $p_d(x_0, z) = p_d(x_1, z'), \forall d \in \{1, 2, 3\} \implies \gamma(x_0, z) = \gamma(x_1, z').$

Assumption 3 provides a very convenient tool to find $x_1$; fix $x_0$ and $z$, $x_1$ is the value that matches the generalized propensity scores, which are directly identified from the population.

There are many models widely applied in economics satisfy Assumption 2-a, 2-b and 3. The following are two examples.

**Example 1** (Generalized Ordered Choice Model). *Suppose $D$ is determined by the following rule:*

$$D = d \iff V_d < \gamma(X, Z) < V_{d+1}$$

*where $V_1$ and $V_4$ are normalized to be $-\infty$ and $+\infty$. $P(V_d < V_{d+1}) = 1, \forall d$, and $V_2$ and $V_3$ are continuously distributed.* Vytlacil (2006) shows this model is equivalent to the LATE ordered
treatment assumption (Angrist and Imbens 2005) and is strictly more general than ordered choice models with constant thresholds. Cunha, Heckman and Navarro (2007) provides economic models that imply it. In our notation, \( h_d(\gamma(X, Z), V) = 1(V_d < \gamma(X, Z) < V_{d+1}) \). It can shown that Assumption 3 holds for this model if \( 0 < p_d(x_0, z) < 1 \) (see proof in the Appendix).

**Example 2** (Additive Random Utility Model). Now we consider an unordered choice model. Suppose the treatment is selected by the following additive random utility model.

\[
D = d \iff R_d(X, Z) + \tilde{V}_d > R_{-d}(X, Z) + \tilde{V}_{-d}
\]

where \( R_d(X, Z) + \tilde{V}_d \) is the indirect utility of choosing treatment \( d \). \( R \) is an unknown function and \( \tilde{V}_d \)s are unobserved and continuously distributed. The subscript \(-d\) refers to any selection other than \( d \). Reparameterize the model by letting \( V_1 = \tilde{V}_2 - \tilde{V}_1, V_2 = \tilde{V}_3 - \tilde{V}_1, V_3 = \tilde{V}_3 - \tilde{V}_2, \gamma_1(X, Z) = R_1(X, Z) - R_2(X, Z) \) and \( \gamma_2(X, Z) = R_1(X, Z) - R_3(X, Z) \), then the model can be rewritten as

\[
D = 1 \iff V_1 < h_1(X, Z), V_2 < h_2(X, Z)
\]
\[
D = 2 \iff V_1 > h_1(X, Z), V_3 < h_2(X, Z) - h_1(X, Z)
\]
\[
D = 3 \iff V_2 > h_2(X, Z), V_3 < h_2(X, Z) - h_1(X, Z)
\]

Again, if \( 0 < p_d(x_0, z) < 1 \), we can show Assumption 3 is satisfied (see Appendix).

For models that do not satisfy Assumption 3, for example a two way flow model, we need to first identify \( \gamma \) (up to a monotone transformation) and use it to identify \( x_1 \).

### 3.2 Identification of the Potential Outcome Functions

Now we turn to identification of the potential outcome functions. To fix ideas, assume there is only one pair \((x_1, z')\) that satisfies Assumption 2-c. It will be straightforward to extend the analysis to the more general case. Without loss of generality, let \( z = 0 \) and \( z' = 1 \). Then combining all equations induced by (3) and (4) for \( X = x_0, x_1 \) and \( Z = 0, 1 \),
we have the following system of moment equations

\[
\begin{align*}
\sum_{d=1}^{3} F_{YD|XZ}(g_d^*(x_0, u), d|x_0, 0) &= u \\
\sum_{d=1}^{3} F_{YD|XZ}(g_d^*(x_0, u), d|x_0, 1) &= u \\
\sum_{d=1}^{3} F_{YD|XZ}(g_d^*(x_1, u), d|x_1, 0) &= u \\
F_{YD|XZ}(g_1^*(x_1, u), d|x, 1) - F_{YD|XZ}(g_1^*(x_0, u), d|x, 0) &= 0 \\
F_{YD|XZ}(g_2^*(x_1, u), d|x, 1) - F_{YD|XZ}(g_2^*(x_0, u), d|x, 0) &= 0 \\
F_{YD|XZ}(g_3^*(x_1, u), d|x, 1) - F_{YD|XZ}(g_3^*(x_0, u), d|x, 0) &= 0
\end{align*}
\]  

(5)

for all \( u \in [0, 1] \).

This system of equations consists of two parts. The first part, the first three equations, are induced from equation (3). As discussed before, for each value \( X \) takes on, equation (3) induces two equations with \( Z = 0 \) and \( Z = 1 \). However, the fourth equation with \((x_1, 1)\) is omitted here because it is implied by the first equation and the last three and thus is redundant. The second part consists of the last three link equations, resulted from equation (4) in Theorem 1.

From the system, it is clear that once the second part links \( g^*_d(x_1, \cdot) \) to \( g^*_d(x_0, \cdot) \), the first part effectively only have three unknowns with \( u \) fixed. Then the pair \((X, Z) = (x_1, 0)\), which provides the third equation, serves as the third instrument value.

Given the fulfilled order condition, denote \( g^*(u) \equiv (g_1^*(x_0, u), g_2^*(x_0, u), g_3^*(x_0, u), g_1^*(x_1, u), g_2^*(x_1, u), g_3^*(x_1, u))' \). If we fix \( u \), identification of \( g^*(u) \) boils down to the uniqueness of the solution to system (5).

However, this pointwise approach does not exploit known properties of \( g^* \). By construction, each component in \( g^* \) is continuous and strictly increasing. In order to utilize these properties, I develop a pathwise approach for identification. Specifically, we treat \( g^* \) as a continuous and increasing (componentwisely) solution path, and restrict our focus to the class of continuous or increasing functions.

Recall that the support of \( Y \) given \((D, X) = (d, x)\) is denoted by \( S(Y_{dx}) \). Let \( S(Y) \equiv \prod_{d=1,2,3} S(Y_{dx}) \) be the product of the six relevant support sets. Let \( \Pi(g^*(u), x_1) \) be a \( 6 \times 1 \) vector by stacking the left hand side of the six equations in (5), and let \( c(u) = (u, u, u, 0, 0, 0)' \). Finally let the \( ||\Pi(y(u), x_1) - c(u)||^2 = \int_0^1 [\Pi(y(u), x_1) - c(u)]' [\Pi(y(u), x_1) - c(u)] du \) where \( y(u) \) is a candidate solution path. Then we have the following theorem.

**Theorem 2.** If \( S(Y) \) is compact, \( \Pi(\cdot, x_1) \) is continuously differentiable on \( S(Y) \), and its
Jacobian is full rank at $g^*(u)$ for all $u \in [0, 1]$, then the following are true:

i). $g^*$ is the unique continuous function on $[0, 1]$ that satisfies $\|\Pi(y(u), x_1) - c(u)\| = 0$.

ii). $g^*$ is the unique increasing function on $(0, 1)$ that satisfies $\|\Pi(y(u), x_1) - c(u)\| = 0$, where increasing means every component in the vector of functions is weakly increasing.

The theorem states that as long as $Y$ is compactly supported (in fact in the proof, we only require the support is bounded on one side), the uniqueness of $g^*$ in the class of continuous or monotone functions is guaranteed solely by the compactness of the support, continuity of the Jacobian and its invertibility along the true solution path. From a pointwise point of view, this condition only implies local identification at each fixed $u$. Yet when we utilize the properties of the function $g^*$, the results are much sharper.

The intuition behind this theorem is as follows. Local invertibility ensures any solution path which departs from $g^*$ cannot do so continuously, otherwise there will be solutions so close to $g^*$ that lies in the neighborhood of it where $\Pi$ is locally injective. Therefore, if there exist multiple solution paths, they are either not equal to $g^*$ anywhere, or departing from $g^*$ abruptly with a jump. The first possibility is ruled out by compact support because by construction, at 0 and 1, all solution path must be equal to the boundary of the support. For the second possibility, since the departure is abrupt, the alternative solution path is no longer continuous, yielding the statement i) in the theorem. Also, abrupt departure can also be ruled out by focusing on the class of increasing functions, continuous or not. To see this, note the first three equations are formed by summing over CDFs which are all strictly increasing and continuous, implying that whenever a component in the solution path jumps up, there must be at least one component jumping down to make the equations still hold, violating the restriction of increasing functions.

One possible concern about the conditions in the theorem is the compactness of the support. There may be applications where this condition does not hold. Note that from the analysis above, the only place compactness is used is that it automatically provides a point at which all potential solution paths intersect. If we know there exists such a point for some $u \in (0, 1)$, i.e., if at some $u g^*(u)$ is known to be the unique solution, compactness can be removed and the conclusions in the theorem still holds. This suggests the pathwise approach and the pointwise approach as discussed in Chernozhukov and Hansen (2005) can be combined; we can first establish global identification at certain $u$, then by definition all solution paths must pass through $g^*_d(u)$ and Theorem 2 will follow. This case is presented and proved as Theorem B1 in the Appendix.
4 Estimation

In this section I discuss estimation of \( g^* \) and \( x_1 \). The estimator I propose follows the constructive identification strategy.

I first estimate \( x_1 \) for a given \( x_0 \) under Assumption 3, that is \((x_1, z')\) matches the generalized propensity scores of \((x_0, z)\). Denote \((p_1(x_1, 1) - p_1(x_0, 0), p_2(x_1, 1) - p_2(x_0, 0))\) by \( \Delta p(x_1) \), then

\[
x_1 \in \arg \min_{x \in S(X)} \Delta p(x_1)' W_0 \Delta p(x_1)
\]

(6)

where \( W_0 \) is a positive definite matrix and \( S(X) \) is the support of \( X \). Although not needed in identification, to obtain consistent estimator of \( x_1 \), we assume it is compact. Note that the minimizer of the problem may not be unique, i.e., there may exist multiple matching points due to the nonlinearity of \( \gamma \).

Based on (6), we estimate \( x_1 \) by minimizing the sample analogue of the population objective function:

\[
\hat{x}_1 \in \arg \min_{x \in \hat{S}(X)} \Delta \hat{p}(x_1)' W_n \Delta \hat{p}(x_1)
\]

(7)

where \( \Delta \hat{p}(x_1) \) is constructed by uniformly consistent estimators of the propensity scores that will be introduced later. \( W_n \) is a weighting matrix consistent of \( W_0 \). \( \hat{S}(X) \) is the estimated support of \( X \), which is simply the sample minimum and maximum of \( X \).

Next we discuss the estimator of the potential outcome functions \( \hat{g} \). From Theorem 2, the true function \( g^* \) is unique among certain classes of functions, but it does not imply the solution is unique at every \( u \). Therefore, based on this identification result, it is not suitable to estimate the structural function pointwisely. Instead, I directly impose monotonicity and estimate \( g^* \) using a sieves estimator as follows.

\[
\{ \hat{g}(u_j) \} = \arg \min_{\hat{g}} \frac{1}{J} \sum_{j=1}^{J} \left( \hat{\Pi}(y_j, \hat{x}_1) - c(u_j) \right)' \left( \hat{\Pi}(y_j, \hat{x}_1) - c(u_j) \right) + \lambda \sum_{j=2}^{J} (y_j - y_{j-1})' (y_j - y_{j-1})
\]

(8)

where \( J \to \infty \) and \( \lambda \to 0 \) as the sample size \( n \to \infty \). The first part in the objective function is the sample analogue of \(||\Pi(y(u), x_1) - c(u)||^2\), which is zero for \( y(u) = g^*(u) \). The integral is approximated at \( J \) nodes. The parameter space \( \hat{G} \) is the product of the six sets in the form of \( \{ y_{dx,1}, \ldots, y_{dx,J} : \hat{y}_{dx,1} \leq y_{dx,1} \leq \ldots \leq y_{dx,J} \leq \hat{y}_{dx} \} \) where \( \hat{y}_{dx} \) and \( \hat{y}_{dx} \) are consistent estimators of \( y_{dx} \) and \( \bar{y}_{dx} \), the lower and upper bound of \( S(Y_{dx}) \). This space automatically induces a finite dimensional functional space of piecewise affine, continuous and increasing functions, and \( \hat{g} \) is constructed by linearly connecting the estimated nodes. Note that If we have pointwise identification, we can estimate each \( \hat{g}(u_j) \) separately and obtain consistency. Under pathwise identification, we need to estimate them jointly under
monotonicity/continuity as required in Theorem 2.

The second part in the objective function is a smoothness penalty in order to control the magnitude of jumps between nodes in finite sample. $\lambda$ needs to converge to 0 fast enough so that it does not affect asymptotics.

To implement the estimator, we need preliminary estimators for the propensity scores, the CDFs in $\Pi$, and the boundaries of $S(Y)$. There are various options for estimating them. Here we use kernel estimators to illustrate. Specifically, let

$$\hat{p}_d(x, z) = \frac{\sum_{i=1}^{n} 1(D = d) 1(Z = z) K\left(\frac{x - X_i}{h}\right)}{\sum_{i=1}^{n} 1(Z = z) K\left(\frac{x - X_i}{h}\right)}$$

and

$$\hat{F}_{YD|XZ}(y, d | x, z) = \frac{\sum_{i=1}^{n} G\left(y - Y_i h_0\right) 1(D = d) 1(Z = z) K\left(\frac{x - X_i}{h}\right)}{\sum_{i=1}^{n} 1(Z = z) K\left(\frac{x - X_i}{h}\right)}$$

where $K$ is a $q$-dimensional kernel function where recall $q$ is the dimension of $X$. The function $G(\cdot)$ is a smooth CDF. $h_0$ and $h$ are bandwidth that converge to 0 as $n \to \infty$.

For the boundaries of the conditional support $y_{dx}$ and $\bar{y}_{dx}$, I adopt the boundary estimators developed by Guerre, Perrigne and Vuong (2000). Note that the upper/lower bounds need to be over/under estimated with probability approaching 1 (w.p.a.1) at appropriate rates so that the ranges of the estimated functions contain the ranges of the true functions w.p.a.1. Specifically, let

$$\hat{y}_{dx} = \max\{Y_i : D_i = d, X \in \pi_{k_1, \ldots, k_q}\} + \eta$$

$$\hat{\bar{y}}_{dx} = \min\{Y_i : D_i = d, X \in \pi_{k_1, \ldots, k_q}\} - \eta$$

where $x \in \pi_{k_1, \ldots, k_q} = [k_1 h_\theta, (k_1 + 1) h_\theta) \times \ldots \times [k_q h_\theta, (k_q + 1) h_\theta), (k_1, \ldots, k_q) \in \mathbb{Z}^q$. It can be seen that $\pi_{k_1, \ldots, k_q}$ induces a partition of $S(X)$, and as $h_\theta \to 0$, the partition is finer and finer. The sequence $\eta$ is a positive and converge to 0 at a slow rate.

To make these CDF and boundary estimators suitable for estimating the parameter of interest, we impose the following regularity conditions.

**Assumption 4.**

a. For each $d$ and $z$, $F_{YD|XZ}(\cdot, d | \cdot, z)$ and $p_{d|\cdot, z}$ are twice continuously differentiable with bounded partial derivatives. The conditional densities $f_{YD|XZ}(\cdot, d | x, z)$, $f_X(\cdot)$, $f_{XZ}(\cdot, z)$ and $f_{DXZ}(d, \cdot, z)$ are bounded away from 0 over the support.

b. $K(\cdot)$ is a density function supported on unit hypercube, twice continuously differentiable with bounded partial derivatives, symmetric at 0 with finite second moment. $G(\cdot)$ is a continuously differentiable CDF with bounded derivatives.
5 Asymptotic Theory

5.1 Consistency and Asymptotic Normality of $\hat{x}_1$

We first study asymptotic properties of $\hat{x}_1$. As the support estimator of $S(X)$ converges much faster than $\hat{p}_d(x,z)$, we treat the support as known. From (7) $\hat{x}_1$ is a nonparametric minimum distance estimator except the true $x_1$ may not be unique since we do not need uniqueness of it for identification. Therefore, consistency is first shown in terms of the Hausdorff distance between the set of $\hat{x}_1$ and the set of $x_1$. Then we pick any point in the set of $\hat{x}_1$ and show it must have a probability limit in the set of $x_1$.

**Theorem 3.** Under Assumption 3 and 4, $d_H(\{\hat{x}_1\},\{x_1\}) = o_p(1)$, where $d_H$ denotes the Hausdorff distance of two sets. Then for any $\hat{x}_1 \in \{\hat{x}_1\}$ with probability one, there exists $x_1 \in \{x_1\}$ such that $\hat{x}_1 = x_1 + o_p(1)$.

The Hausdorff distance between two subsets $A$ and $B$ of a metric space endowed with metric $d$ is defined as

$$d_H(A, B) = \max \{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}$$

Intuitively, if the Hausdorff distance between two sets are small, for any point in either of the set, there exists a point close to it from the (closure of the) other set. Under consistency in Hausdorff distance, we can pick any point from the estimated set of $\hat{x}_1$ and Theorem 3 guarantees there exists an $x_1$ that is close to it w.p.a.1.

Consistency in Hausdorff distance is broadly used in inference under partial identification (e.g., Chernozhukov, Hong and Tamer 2007, Chernozhukov, Lee and Rosen 2013). But note here $x_1$ is not unidentified. In the case of partial identification, the true parameter is a single point in the identified set, and when the set is sharp, no further information on where the point lies can be obtained. The difference here is that under Assumption 3, all $x_1$ in the identified set are "true" $x_1$s in the sense that equation (4) in Theorem 1 holds, therefore any pick from the set can be used in estimating $\hat{g}$.

From now on let $\hat{x}_1$ be the random point we pick from the random set of minimizers of (7), and let $x_1$ be its probability limit. We present its asymptotic distribution in Theorem 4. The result follows directly from theories of classical minimum distance estimator and asymptotic normality of kernel regressions.

**Theorem 4.** Under Assumption 3 and 4, if the set $\{x_1\}$ is in the interior of $S(X)$, and
\[ \partial_{x_1} \Delta p(x_1) W_0 \partial_{x_1} \Delta p(x_1) \] is nonsingular, \( h^2 \cdot \sqrt{nh^q} = o(1) \), \( \kappa \equiv \int v^2 K(v) dv \), we have

\[ \sqrt{nh^q} (\hat{x}_1 - x_1) \xrightarrow{p} N(0, V_{x_1}) \] (10)

where \( V_{x_1} = [\partial_{x_1} \Delta p(x_1) W_0 \partial_{x_1} \Delta p(x_1)]^{-1} \partial_{x_1} \Delta p(x_1) W_0 \Sigma_x W_0 \partial_{x_1} \Delta p(x_1) [\partial_{x_1} \Delta p(x_1) W_0 \partial_{x_1} \Delta p(x_1)]^{-1}, \)

and

\[ \Sigma_x = \kappa \left( f_{xZ}(x_0, 0) + f_{xZ}(x_1, 1) \right) \begin{pmatrix} p_1(x_0, 0)(1 - p_1(x_0, 0)) & 0 \\ 0 & p_2(x_0, 0)(1 - p_2(x_0, 0)) \end{pmatrix} \] (11)

From Theorem 4, it can be seen that to achieve efficiency for \( \hat{x}_1 \), we can set \( W_0 = \Sigma_x^{-1}, \)
then the asymptotic variance becomes

\[ [\partial_{x_1} \Delta p(x_1) \Sigma_x^{-1} \partial_{x_1} \Delta p(x_1)]^{-1} \]

Estimating the asymptotic variance is straightforward. We can simply plug in \( \hat{x}_1 \) and consistent estimators of the propensity scores as well as their derivatives.

### 5.2 Consistency of \( \hat{g} \)

From Theorem 2, \( g^* \) is the unique function such that

\[ g^* = \arg \min_{y \in G} Q_g(y, x_1) \] (12)

where \( Q_g(y, x_1) \equiv \int_0^1 [\Pi(y(u), x_1) - c(u)] [\Pi(y(u), x_1) - c(u)] du \) and \( G = \{ y : [0, 1] \mapsto \bar{S}(Y), y \text{ is continuous and increasing} \} \). Recall \( \bar{S}(Y) \) is any compact set contains \( S(Y) \).

We define the parameter space using \( \bar{S}(Y) \) instead of \( S(Y) \) because we need to estimate the boundaries of \( S(Y) \) and in order to guarantee the entire range of \( g^* \) lies in the estimated support w.p.a.1., we overestimate the size of the support. As a consequence, we enlarge the true parameter space so that the estimator \( \hat{g} \) is in it w.p.a.1.

To establish consistency, one crucial condition is the following:

\[ \inf_{g \in G : \|g - g^*\|_2 \geq \delta} \left| Q_g(g, x_1) - Q_g(g^*, x_1) \right| > 0 \] (13)

Note (13) holds if the parameter space is compact. When \( G \) is an infinite dimensional object, compactness in many cases does not hold. However, it turns out that equipped with monotonicity, \( G \) in our problem admits (13). This claim is shown as Lemma 1. We present the proof as it illustrates the central role of monotonicity.

**Lemma 1.** Under conditions in Theorem 2, (13) is true.
Proof. Suppose not, then by definition, there exists a sequence \( Q_g(y_j, x_1) \) such that

\[
\lim_{j \to \infty} Q_g(y_j, x_1) = 0
\]

where for every \( j \), \( y_j \in \mathcal{G} \). Since every element in the sequence \( \{y_j\} \) is monotone and the sequence is uniformly bounded, by Helly’s Selection Theorem, there exists a subsequence which pointwisely converges to an increasing function \( \tilde{g} \). Still denote the subsequence of \( \{y_j\} \).

By continuity of \( \Pi(\cdot, x_1) \), \( [\Pi(y_j(u), x_1) - c(u)]/[\Pi(y_j(u), x_1) - c(u)] \) also converges pointwisely to \( [\Pi(\tilde{g}(u), x_1) - c(u)]/[\Pi(\tilde{g}(u), x_1) - c(u)] \). Since \( \Pi \) is uniformly bounded, it is dominated by a constant function, which is Lebesgue integrable. Therefore, by the Dominated Convergence Theorem (DCT),

\[
Q_g(\tilde{g}, x_1) = Q_g(\lim y_j, x_1) = \lim_{j \to \infty} Q_g(y_j, x_1) = 0
\]

Meanwhile, by applying DCT, we also have \( y_j \) converges to \( \tilde{g} \) in \( L^2 \). Therefore, \( ||\tilde{g} - g^*||_2 \geq \delta \). From Theorem 2.ii), \( g^* \) is unique except at the boundaries, so all solution paths have the same \( L^2 \) norm. This leads to a contradiction.

From the proof, we can see the key step is to invoke Helly’s Selection Theorem which establishes the existence of a convergent sequence and since the limit is different from \( g^* \) in \( L^2 \) norm, it contradicts with the uniqueness of \( g^* \). In fact Helly’s Selection Theorem implies the closure of \( \mathcal{G} \) (which includes discontinuous monotone functions) is sequentially compact. A space is sequentially compact if for every infinite sequence there exists a convergent subsequence. In metric spaces, it is equivalent to compactness. Since our identification results also hold in the closure of \( \mathcal{G} \), we have the desired result.

Under Lemma 1 and other conditions such as uniform consistency of the objective function proved in the Appendix, we have the following theorem.

**Theorem 5.** Under Assumption 4 and conditions in Theorem 2 and Lemma C1, if \( \lambda = o(1) \),

\[
\eta \cdot \left( \frac{n}{\log n} \right)^{1/(q+1)} \to \infty, \text{ then } ||\hat{g} - g^*||_2 = o_p(1).
\]

There are two conditions in Theorem 5 controlling the rate of the penalty parameter \( \lambda \) and the boundary parameter \( \eta \). As the penalty is only included for improving smoothness of the estimator in finite sample, it should not affect asymptotics therefore it needs to converge to 0 fast. In the next subsection we need to further increase its rate so that the asymptotic distribution is neither affected. On the other hand, \( \eta \) needs to converge to 0 slowly such that the estimated support contains the true conditional support \( S(Y) \) w.p.a.1..

Theorem 5 only shows \( L^2 \)-consistency. In general it does not even imply pointwise convergence. However, under monotonicity and convergence of the boundary estimators, \( L^2 \) convergence implies uniform convergence of \( \hat{g} \) over \([0, 1]\).
**Corollary 1.** Under the conditions in Theorem 5, \( \sup_{u \in [0,1]} |\hat{g}(u) - g^*(u)| = o_p(1) \).

Essentially, Corollary 1 holds because when the convergent sequence is monotone, the limiting function is continuous, and the two end points are converging to those of the true function, any discrepancy between the sequence and the limiting function at any point will generate a region with positive area that lies in between. The intuition behind this is that to maintain monotonicity, after a jump-up from the true function, the new function cannot immediately return to it as the it cannot go downward. The shortest path turn return is to go flat until reaching the true function again, but this generates an area with positive Lebesgue measure because the true function is strictly increasing and Lipschitz.

As we have a uniform consistent estimator for \( g^* \), we can construct consistent estimators for the quantile treatment effect (QTE) at \( u \) and the average treatment effect ATE conditional on \( X = x \). Specifically, let the estimated effect from \( d \) to \( d' \) be

\[
\hat{QTE}(x, u) = \hat{g}_{d'}(x, u) - \hat{g}_d(x, u)
\]

and

\[
\hat{ATE}(x) = \frac{1}{J} \sum_{j=1}^{J} \hat{g}_d(x, u_j) - \frac{1}{J} \sum_{j=1}^{J} \hat{g}_{d'}(x, u_j)
\]

then we have the following corollary.

**Corollary 2.** Under conditions in Theorem 5,

\[
\sup_{u \in [0,1]} |\hat{QTE}(x_0, u) - QTE(x_0, u)| \xrightarrow{p} 0
\]

\[
\hat{ATE}(x_0) \xrightarrow{p} ATE(x_0)
\]

### 5.3 Pointwise Asymptotic Normality

In this subsection we derive the asymptotic distribution for \( \hat{g}(u) \) at a given \( u_0 \in \{u_1, ..., u_J\} \).

From (8), it is clear that as long as all \( \hat{g}(u_j) \) lie in the interior of the space, i.e., none of the inequalities in the constraint is binding, then \( \hat{g}(u_j) \) satisfies the first order condition, which is analogous to a standard minimum distance estimator if the penalty converges to 0 fast enough.

For this idea to work, we first need to derive an upper bound of the convergence rate of \( \{\hat{g}(u_j)\} \), using which we can control the rate of \( J \) to guarantee the estimated nodes are in the interior w.p.a.1. Specifically, since the derivative of \( g^* \) is bounded away from 0, the distance between \( g^*(u_{j+1}) \) and \( g^*(u_j) \) for any \( j \) is of the order \( \frac{1}{J} \). Therefore, if \( J \) diverges slowly such
that every \( \hat{g}_d(u_j) \) will lie within a neighborhood of \( g^*_d(u_j) \) with radius of the order, say \( \frac{1}{\lambda_j} \), then they are strictly increasing thus none of the constraints are binding.

**Theorem 6.** Let \( \lambda_n = \left( \frac{\log n}{n \log} \right)^{\frac{1}{2}}. \) Under Assumption 4 and conditions in Theorem 4 and 5, if \( \lambda = o(r_n^2) \)

\[
\max_j |\hat{g}(u_j) - g^*(u_j)| = O_p(\sqrt{Jr_n}) \tag{14}
\]

Therefore, by imposing \( J^3r_n^2 \to 0 \), we can guarantee the sequence \( \{\hat{g}(u_j)\} \) is strictly increasing and none of the constraints is binding. This suggests that the estimator we obtain will be asymptotically equivalent as estimating \( g^* \) pointwisely without imposing monotonicity. This is intuitive because our estimator is minimizing an objective function which is essentially a summation of \( J \) separate problems. Imposing monotonicity is mainly to guarantee identification and if the model is in the first place pointwise identified, our estimation procedure is equivalent as estimate each point in \( J \) separate problems and the estimates will be asymptotically increasing.

Now we demonstrate the asymptotic expansion. To highlight the impact of estimating \( x_1 \) as well as the link equations, we present the asymptotic expansion of \( \{\hat{g}_{d_{00}}(u_0)\} \) only. By rearranging the first order condition and supposing \( \lambda \) satisfies the condition in Theorem 6, the following equation holds w.p.a.1.

\[
A \begin{pmatrix}
\hat{g}_1(x_0, u_0) - g^*_1(x_0, u_0) \\
\hat{g}_2(x_0, u_0) - g^*_2(x_0, u_0) \\
\hat{g}_3(x_0, u_0) - g^*_3(x_0, u_0)
\end{pmatrix} = - \begin{pmatrix}
\sum_{d=1}^{3} \hat{F}_{YD|XZ}(g^*_d(x_0, u_0), d|x_0, 0) - u_0 \\
\sum_{d=1}^{3} \hat{F}_{YD|XZ}(g^*_d(x_0, u_0), d|x_0, 1) - u_0 \\
\sum_{d=1}^{3} \hat{F}_{YD|XZ}(g^*_d(x_1, u_0), d|x_1, 0) - u_0
\end{pmatrix}

- \begin{pmatrix}
0 \\
0 \\
\sum_{d=1}^{3} B_d(\hat{F}_{YD|XZ}(g^*_d(x_0, u_0), d|x_0, 0) - \hat{F}_{YD|XZ}(g^*_d(x_1, u_0), d|x_1, 1))
\end{pmatrix}

+ o_p\left( \frac{1}{\sqrt{nh^q}} \right)
\]

where \( A \equiv (A_1, A_2, A_3) \), and for any \( d \),

\[
A_d = \begin{pmatrix}
f_{YD|XZ}(g^*_d(x_0, u_0), d|x_0, 0) \\
f_{YD|XZ}(g^*_d(x_0, u_0), d|x_0, 1) \\
f_{YD|XZ}(g^*_d(x_1, u_0), d|x_1, 0) \\
f_{YD|XZ}(g^*_d(x_1, u_0), d|x_1, 1)
\end{pmatrix}
\]

\[
B_d = \frac{f_{YD|XZ}(g^*_d(x_1, u_0), d|x_1, 0)}{f_{YD|XZ}(g^*_d(x_1, u_0), d|x_1, 1)}
\]

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and

\[
C \equiv \sum_{d=1}^{3} \left( f_{YD|XZ}(g_{d}^{*}(x_{1}, u_{0}), d|x_{1}, 0) \frac{\partial}{\partial x} F_{YD|XZ}(g_{d}^{*}(x_{1}, u_{0}), d|x_{1}, 1) - \frac{\partial}{\partial x} F_{YD|XZ}(g_{d}^{*}(x_{1}, u_{0}), d|x_{1}, 0) \right)
\]

\[
\cdot \left( \frac{\partial x}{\Delta p(x_{1})} W_{0} \frac{\partial x}{\Delta p(x_{1})} \right)^{-1} \frac{\partial x}{\Delta p(x_{1})}
\]

The derivation of (15) is in the proof of Theorem 7, presented in the Appendix. From (15), we can see the asymptotic variance is from three sources. The first term on the right hand side contains information if ideally we have a three-value instrument, conditioning on each of them generates a moment condition. The second and third terms are the impact of estimating the link equation (4) and \(x_{1}\), respectively. \(B\) and \(C\) are influence functions of these objects. For example, \(C\) consists of two parts. The first part is the impact of \(x_{1}\) through the link function. To see this, notice that the partial derivative \(\frac{\partial}{\partial x} F_{YD|XZ}(g_{d}^{*}(x_{1}, u_{0}), d|x_{1}, 1)\) is the impact of it on link function, and the density ratio in the front is equal to \(B\), the influence of the link function. The second part is the direct impact of \(x_{1}\). Since \(x_{1}\) only shows up in \(F_{YD|XZ}(g_{d}^{*}(x_{1}, u_{0}), d|x_{1}, 0)\) in the third moment condition, the direct impact of \(x_{1}\) is just the partial derivative of it.

Based on (15), by Lyapunov’s CLT, we can establish the asymptotic distribution:

**Theorem 7.** Under conditions of Theorem 6, if \(J^{3} r_{n}^{2} \rightarrow 0\) and \(\sqrt{nh} h_{0}^{2} = o(1)\), then

\[
\sqrt{nh} \begin{pmatrix} \hat{g}_{1}(x_{0}, u_{0}) - g_{1}^{*}(x_{0}, u_{0}) \\ \hat{g}_{2}(x_{0}, u_{0}) - g_{2}^{*}(x_{0}, u_{0}) \\ \hat{g}_{3}(x_{0}, u_{0}) - g_{3}^{*}(x_{0}, u_{0}) \end{pmatrix} \overset{d}{\rightarrow} N(0, A^{-1} \Sigma_{g^{*}} A^{-1}), \quad (16)
\]

Let \(\Sigma_{g^{*}}_{ij}\) be the \(i - j\)th entry in \(\Sigma_{g^{*}}\), then

\[
\Sigma_{g^{*}11} = \kappa f_{XZ}^{-1}(x_{0}, 0) u_{0} (1 - u_{0}),
\]

\[
\Sigma_{g^{*}22} = \kappa f_{XZ}^{-1}(x_{0}, 1) u_{0} (1 - u_{0}),
\]

\[
\Sigma_{g^{*}33} = \kappa f_{XZ}^{-1}(x_{1}, z) u_{0} (1 - u_{0})
\]

\[
+ \kappa (f_{XZ}^{-1}(x_{0}, 0) + f_{XZ}^{-1}(x_{1}, 1)) \left[ \sum_{d=1}^{3} B_{d}^{2} F_{YD|XZ}(g_{d}^{*}(x_{0}, u_{0}), d|x_{0}, 0) \right] - \left( \sum_{d=1}^{2} B_{d} F_{YD|XZ}(g_{d}^{*}(x_{0}, u_{0}), d|x_{0}, 0) \right)^{2}
\]

\[
+ \sum_{d=1}^{2} C_{d}^{2} p_{d}(x_{0}, 0) - \left( \sum_{d=1}^{2} C_{d} p_{d}(x_{0}, 0) \right)^{2} - \sum_{d=1}^{3} B_{d} C_{d} F_{YD|XZ}(g_{d}^{*}(x_{0}, u_{0}), d|x_{0}, 0)
\]

\[
+ \sum_{d=1}^{3} \sum_{d' = 1}^{2} B_{d} C_{d'} F_{YD|XZ}(g_{d}^{*}(x_{0}, u_{0}), d|x_{0}, 0) p_{d'}(x_{0}, 0)
\]
\[ \Sigma_{g^*13} = \Sigma_{g^*31} = \kappa f_{XZ}^{-1}(x_0, 0) \left[ (1 - u_0) \sum_{d=1}^{3} B_d F_{YD|XZ}(g_d^*(x_0, u_0), d|x_0, 0) \right] \\
- \sum_{d=1}^{2} C_d \left( F_{YD|XZ}(g_d^*(x_0, u_0), d|x_0, 0) - u_0 p_d(x_0, 0) \right) \]

and

\[ \Sigma_{g^*12} = \Sigma_{g^*21} = \Sigma_{g^*23} = \Sigma_{g^*32} = 0. \]

The asymptotic variance in (16) is messy, but it is straightforward to estimate. We can plug in consistent density and CDF estimators to form a consistent variance estimator.

6 Monte Carlo

We conduct a Monte Carlo experiment to examine the finite sample performance of the estimator. The data generating process is specified to roughly match the main statistics in Head Start Impact Study (HSIS) dataset documented in Kline and Walters (2016, henceforth denoted by KW). The treatment takes on three values (participating Head Start, denoted by \( h \), participating any other competing program, denoted by \( c \), and not participating any programs, denoted by \( n \)), and is unordered. In their sample, two continuous covariates can be observed: household income (denoted by \( X_1 \), fraction of the federal poverty line, ranging from 0 to 1.3) and kids’ baseline test scores (\( X_2 \)). The instrument (\( Z \)) is a binary lottery receiving which would grant the children access to Head Start.

First, following KW, the treatment selection is modeled by an additive random utility model. Let \( \mu_h + V_h \), \( \mu_h + V_c \) and 0 be the utility obtained by choosing Head Start (\( h \)), competing program (\( c \)), and not participating \( n \), where \( \mu \) is a function of the instrument and covariates and \( v \) is unobserved, then

\[ D = h \text{ if } \mu_h + V_h \geq \mu_c + V_c, \mu_h + V_h \geq 0 \]
\[ D = c \text{ if } \mu_h + V_h < \mu_c + V_c, \mu_h + V_c \geq 0 \]
\[ D = n \text{ if } \mu_h + V_h < 0, \mu_c + V_c < 0 \]

We specify \( \mu \) to be a linear function of the instruments, covariates, and their interaction:

\[ \mu_h = -2.1 + 11.46Z + 2.2X_1 - 11Z \cdot X - 0.02X_2 \]
\[ \mu_c = -0.1 + 0.02X_1 - 0.03X_2 \]

In KW, they have more covariates. Also, they discretize \( X_1 \) to make it binary, I do not adopt
their parameter estimates. Instead, I calibrate the model to match the summary statistics of all the relevant variables, while keeping the signs of the parameters the same as in KW.

This model captures some features of program selection. First, when $Z = 0$, higher income leads to higher utility for either choices but when $Z = 1$, lower income households obtain higher utility in choosing Head Start. Also, a higher baseline score makes households find not choosing any program slightly more preferable.

To match the moments in KW,

- $X_1$ is drawn from a truncated normal with mean 0.9, standard error 0.2, truncated from 0 to 1.3 (130% of the federal poverty line is the highest income level eligible for Head Start).
- $X_2$ is drawn from a truncated normal with mean 0.008, standard error 1, and truncated from $-5$ to $5$. The range is obtained by using the same method to standardize the test scores.
- $Z$ is drawn from a Bernoulli distribution with $P(Z = 1) = 0.63$.
- The unobservable vector $(V_h, V_c)$ follows $N(0, \begin{bmatrix} 1 & 0.3 \\ 0.3 & 1 \end{bmatrix})$. The covariance is set to match the estimated value in KW.

Table 1 presents a comparison of all the available relevant moments documented in KW and the average of the 400 simulated sample.

The outcome equation is specified as:

$$ Y = 2 \cdot \left[ \mathbb{1}(D = h) \left( \frac{e^U - 1}{e - 1} + 0.15 \right) + \mathbb{1}(D = c) \left( \frac{e^{2U} - 1}{e^2 - 1} + \mathbb{1}(D = n) \frac{e^{3U} - 1}{e^{3U} - 1} \right) + (X_1 - 0.9)U + 0.5X_2 \right] $$

where $U$ is first drawn jointly with $V_h$ and $V_c$ from $N(0, \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.3 \\ 0.5 & 0.3 & 1 \end{bmatrix})$. Then transform $U$ by applying the standard normal CDF on it so that the resulting variable is marginally distributed as $U[0, 1]$. This DGP is specified to roughly match the LATE estimates in KW. Table 2 presents the results: Significance is marked using the average of the t statistics from the 400 simulated samples.

We examine the potential outcome at the average income and test score by setting $x_0 = (0.9, 0.008)$. By simple calculation, the $x_1$ satisfying Assumption 2-c is $x_1 = (1.0767, 0.1258)$.

To implement the estimators, we use the biweight kernel function to estimate the CDF and densities. $h$ is chosen by Silverman’s rule of thumb, and $h_0 = h^{1.5}$, $J = 19$, $\lambda \in$
Table 1: Key Moments in the Simulated Sample and in KW

|               | Simulated Sample | KW   |
|---------------|------------------|------|
| Sample size (n) | 3600             | 3571 |
| $P(Z = 1)$    | 0.6299           | 0.63 |
| $P(D = h|Z = 1)$ | 0.7191           | 0.7395 |
| $P(D = c|Z = 1)$ | 0.1447           | 0.1725 |
| $P(D = h|Z = 0)$ | 0.3250           | 0.3150 |
| $P(D = c|Z = 0)$ | 0.3367           | 0.3470 |
| $E(X_1)$      | 0.8889           | 0.896 |
| $E(X_1|D = h)$ | 0.8467           | 0.892 |
| $E(X_1|D = c)$ | 0.9467           | 0.983 |
| $E(X_1|D = n)$ | 0.9444           | 0.851 |
| $E(X_2)$      | 0.0076           | 0.008 |
| $E(X_2|D = h)$ | 0.0044           | -0.001 |
| $E(X_2|D = c)$ | -0.015           | 0.1  |
| $E(X_2|D = n)$ | 0.0394           | -0.04 |

Table 2: LATE using the Simulated Sample and in KW

|               | Simulated Sample | KW   |
|---------------|------------------|------|
|               | $LATE_h$ | $LATE_c$ | $LATE_h$ | $LATE_c$ |
| Z as IV       | 0.3439*** | 0.247*** |
| Z and interactions | 0.6374* | 0.6006 | 0.384*** | 0.419 |

$\{0, 10^{-6}, 10^{-5}\}$. For the $\lambda = 0$ case we estimated the model using both $x_1$ and $\hat{x}_1$. Table 3 presents the integrated mean squared error (IMSE) and the integrated squared bias $\hat{g}$. In Table 3, in the $\hat{x}_1$ columns $g^*$ is estimated using the estimated $x_1$ while the $x_1$ column means $\hat{g}$ is obtained using the true $x_1$.

First if we compare the $\hat{x}_1$ and $x_1$ columns under $\lambda = 0$, we can see estimating $x_1$ both increase bias and variance. Now compare all the $\hat{x}_1$ columns across $\lambda$, bias increases slightly as $\lambda$ increases, but the IMSE, thus the variance, decrease a lot, so the finite sample estimates are indeed smoother with penalty.

Figures 1-4 show a comparison of the estimates and the true functions. In all the figures, the black solid lines are true functions. The left panel shows the average of the 400 simulations, while the right panel contains 5 individual estimates randomly drawn from the 400 estimates. These figures reflect the patterns in the bias and variance of different specifications in Table 3, and shows the shape of the true functions are well captured by the
estimated functions.

### 7 Conclusion

This paper provides a new method for identification of nonseparable models with a multivalued discrete treatment. I show that if the potential outcome function is strictly monotone in the scalar disturbance, which is distributed as \( U[0, 1] \), pointidentification can be achieved by using a binary IV and exogenous covariates via a general class of selection mechanisms. The selection mechanism induces link equations that identify the changes in the potential outcome caused by shifting the covariates between some special matching points, which can consequently be used as additional IV, and thus fulfilling the order condition. Given the fulfilled order condition, I also prove a result on the global uniqueness of the solution path to a nonlinear system of equations by exploiting continuity and monotonicity. Based on the identification results, I construct a sieves estimator which is uniformly consistent and asymptotic normal. The Monte Carlo results suggest the estimator performs well in finite sample.
Figure 1: $\lambda = 0$, $\hat{x}_1$
Figure 2: $\lambda = 0, x_1$
Figure 3: \( \lambda = 10^{-6}, \hat{x}_1 \)
Figure 4: $\lambda = 10^{-5}$, $\hat{x}_1$
Appendix

A General Multivalued Endogenous Variables and Instruments

In this section we discuss the general case when $D \in \{1, 2, ..., K\}$ and $Z \in \{1, 2, ..., L\}$ where $K > 2$.

First we assume $Z$ is still binary. Under Assumption 2, equation (4) still holds for both values of $z$ and all $x$. Then for $x_0$ we now have two equations and $K > 2$ unknowns for a fixed $u$. We now use the same idea to augment the system of equation.

Suppose for $Z = 0$ we find $x_1$ such that $\gamma(x_1, 1) = \gamma(x_0, 0)$. Then $(x_1, 0)$ will serve as the third instrument value. Now if we need more instrument, say $K > 3$, then we can try to find links for $x_1$. For example, if there exists an $x_2 \in S(X)$ such that $\gamma(x_2, 1) = \gamma(x_1, 0)$, then similarly $(x_2, 0)$ is another instrument value. And we can continue this process to generate more and more instrument values. In the mean time, if in the beginning there also exists $x_1'$ that can match $(x_0, 1)$, this process can also be extended to the other direction. Therefore if $X$ has rich variation and effective impact on $D$, even if $Z$ is binary and $D$ has a much larger support, identification is possible.

Then when $Z$ has richer variation, each match can provide more instrumental values. For example when $Z \in \{0, 1, 2\}$, if $\gamma(x_1, 1) = \gamma(x_0, 0)$, then both $(x_1, 0)$ and $(x_1, 2)$ can be used as instrument values.

B Additional Identification Results

In this section we present two more theorems on identification. The first one states that as long as any potential solution paths at least intersect $g^*$ once at the same point, the conclusions in Theorem 2 holds. This theorem relaxes the compact support restriction so it can entertain applications where the outcome is unbounded. For the intersection point to exist, we can invoke the pointwise identification theorem in Chernozhukov and Hansen (2005).

The second theorem states that under the conditions of Theorem 2, in fact there are regions near the end points where the solution is even pointwisely globally unique. This results is useful because it suggests for estimation we can drop the regions too close to the boundaries to improve the nonparametric estimators’ performance while still maintain identification.
Theorem B1. If $\Pi(\cdot, x_1)$ is continuously differentiable on $S(Y)$, and the Jacobian is full rank along the solution path $g^*(u)$ for all $u \in [0, 1]$, and $\exists u_0$ such that $g^*(u_0)$ is the unique solution to $\Pi(y, x_1) = c(u_0)$, then the following are true:

i). $g^*$ is the unique continuous function on $[0, 1]$ that satisfies $\|\Pi(y(u), x_1) - c(u)\|_2 = 0$.

ii). $g^*$ is the unique increasing function on $(0, 1)$ that satisfies $\|\Pi(y(u), x_1) - c(u)\|_2 = 0$, where increasing means every component in the vector of functions is weakly increasing.

Proof. The proof is very similar to the proof of Theorem 2, thus omitted. \hfill \Box

Theorem B2. Under the conditions in Theorem 2, there exists $u_1$ and $u_2$ such that for every $u \in [0, u_1) \cup (u_2, 1]$, $g^*(u)$ is the globally unique solution to $\Pi(y, x_1) = c(u)$.

Proof. We only show uniqueness for $u \in [0, u_1)$ as the other case is symmetric.

Being a solution path to $\Pi(y(u), x_1) - c(u) = 0$, $g^*(u)$ is equivalently equal to

$$g^*(u) = \arg \min_{y(u) \in S(Y)} [\Pi(y(u), x_1) - c(u)]$$

As the objective function is continuous and the correspondence from $u$ to $S(Y)$ is both upper- and lower-continuous (since $S(Y)$ is not $u$ dependent and is compact), by the Maximum Theorem, the solution correspondence is upper-hemicontinuous. Therefore, let $u_1 = \inf \{u : \Pi(y(u), x_1) - c(u) = 0, y(u) \neq g^*(u)\}$, then $u_1 > 0$. Otherwise, there exists a sequence $\{u_j\}$ in the set that converges to 0. By boundedness, a convergence subsequence of $y(u_j)$ exists. Still denote it by $y(u_j)$. Then by upper-hemicontinuity, $\lim_{u_j \to 0} y(u_j)$ must also be a minimizer, or equivalently, satisfies the equation. Since at 0 the only solution is $y$, $\lim_{u_j \to 0} y(u_j) = g^*(0)$, violating local injectivity of $\Pi(\cdot, x_1)$ at $g^*(0)$. \hfill \Box

C Proofs of the Theorems, Corollaries and Statements in Examples

Proof of Theorem 1. By Assumption 1 and 2, we have

$$F_{YD|XZ}(g^*_d(x_0, u), d|x_0, z) = P(U \leq u, h_d(\gamma(x_0, z), V) = 1|X = x_0, Z = z)$$

$$= P(U \leq u, h_d(\gamma(x_0, z), V) = 1)$$

$$= P(U \leq u, h_d(\gamma(x_1, z'), V) = 1)$$

$$= P(U \leq u, h_d(\gamma(x_1, z'), V) = 1|X = x_1, Z = z')$$

$$= F_{YD|XZ}(g^*_d(x_1, u), d|x_1, z')$$

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where the first equality follows monotonicity of $g_d^*$ and the selection equation, the second equality follows independence, and the third holds because $x_1$ satisfies Assumption 2-c.

Proof for Example 3. By construction, the generalized propensity scores satisfy

$$p_3(x_0, z) - p_3(x_1, z') = P(V_3 < \gamma(x_0, z)) - P(V_3 < \gamma(x_1, z')) = 0$$

which implies $\gamma(x_0, z) = \gamma(x_1, z')$ since $0 < p_3(x_0, z) < 1$.

Proof for Example 4. By construction, the generalized propensity scores for $D = 1$ and $D = 2$ are

$$p_1(X, Z) = P(V_1 < \gamma_1(X, Z), V_2 < \gamma_2(X, Z))$$

and

$$p_2(X, Z) = P(V_1 > \gamma_1(X, Z), V_3 < \gamma_2(X, Z) - \gamma_2(X, Z))$$

Suppose Assumption 3 does not hold. Without loss of generality let $\gamma_1(x_0, z) < \gamma_1(x_1, z')$. Then to make $p_1(x_0, z) = p_1(x_1, z')$, it is necessary to have $\gamma_2(x_0, z) > \gamma_2(x_1, z')$. However, this implies $p_2(x_0, z) > p_2(x_1, z')$, a contradiction.

Proof of Theorem 2-i). We prove the statement in two steps. First we show $g^*$ is unique among continuous functions whose codomain is $S(Y)$. Then we show it is unique among all continuous functions.

Step 1. Suppose there exists another $\tilde{g}(u) \in \{y(u) : [0, 1] \mapsto S(Y), y(u) \in \mathbb{C}\}$. By continuity, both $g^*$ and $\tilde{g}$ satisfies $\Pi(y(u), x_1) = c(u)$ for all $u \in [0, 1]$. Note that by construction, $\tilde{g}(0) = g^*(0) = y$ where $y$ is the vector of lower bounds of $S(Y)$. Then let $\tilde{u} = \sup\{u : \tilde{g}(t) = g^*(t), 0 \leq t \leq u\}$. If $\tilde{u} = 1$, we are done. If not, by continuity, for any sequence $u_j \to \tilde{u}$ from the right, $\lim \tilde{g}(u_j) = \lim g^*(u_j) = g^*(\tilde{u})$, which contradicts with the fact that $\Pi(\cdot, x_1)$ is locally injective implied by the Jacobian being full rank.

Step 2. Now we show that in fact such uniqueness holds among all continuous functions. Let $\tilde{g} : [0, 1] \mapsto \mathbb{R}^6$ be a continuous function such that $\Pi(\tilde{g}(u), x_1) = c(u)$ for all $u \in [0, 1]$. Now construct $g^\dagger$ such that each component follows

$$g^\dagger_{dx}(u) = \begin{cases} 
    y_{dx}, & \text{if } \tilde{g}_{dx}(u) < y_{dx} \\
    \tilde{g}_{dx}(u), & \text{if } \tilde{g}_{dx}(u) \in S(Y_{dx}) \\
    \tilde{g}_{dx}, & \text{if } \tilde{g}_{dx}(u) > \tilde{g}_{dx}
\end{cases}$$

Apparently $g^\dagger$ is still continuous. Then by uniqueness obtained in Step 1, $g^\dagger = g^*$, which further implies the only two possible $u$ at which $\tilde{g} \neq g^*$ are 0 and 1, which is ruled out by continuity of $\tilde{g}$.
Proof of Theorem 2-ii). Without continuity, \(||\Pi(y(u), x_1) - c(u)|| = 0\) implies there exist at most countable \(u\) s at which \(\Pi(y(u), x_1) - c(u) \neq 0\). We will first show uniqueness of \(g^*\) among increasing functions that satisfy the equation for all \(u\). Similar to the proof before, we prove it in two steps.

Step 1. Suppose \(\tilde{g} : [0, 1] \rightarrow S(Y)\) is componentwisely increasing and satisfies the norm equation. Again, at \(u = 0\), \(\tilde{g}(u) = g^*(u)\) so the set \(\{u : \tilde{g}(t) = g^*(t), 0 \leq t \leq u\}\) is nonempty. Let \(\bar{u}\) be the supremum of it. When \(\bar{u} < 1\), there must be components in \(\tilde{g}\) jumping up from \(g^*\) such that the right limit of \(\tilde{g}\) at \(\bar{u}\) is larger than \(g^*(u)\) (the limit exists because \(\tilde{g}\) is monotone). This is because Theorem 2.i) rules out the case where \(\tilde{g}\) converges to \(g^*\) from the right at \(\bar{u}\), and jumping up is necessary because \(\tilde{g}\) has to be increasing. However, from (5), since all CDFs in \(\Pi\) are continuous and strictly increasing, if one component in \(\tilde{g}\) jumps up, there must be at least a component jumps down at the same point to make the equation still hold. Then this component is no longer increasing.

Step 2. Now we consider any increasing functions. Notice that if \(\tilde{g}_{dx}\) now have regions that are outside the support \(S(Y_{dx})\), and let \(u_{dx,1} = \sup\{u : \tilde{g}(u) \leq y_{dx}\}\) and \(u_{dx,2} = \inf\{u : \tilde{g}_{dx}(u) \geq y_{dx}\}\), then it must be the case that \(\tilde{g}_{dx}(u) \leq y_{dx}\) for all \(u \leq u_{dx,1}\) and \(\tilde{g}_{dx}(u) \geq y_{dx}\) for all \(u \geq u_{dx,2}\) by monotonicity. Therefore, constructing \(g^\dagger\) the same way as before, \(g^\dagger\) is increasing. Thus by Step 1, \(g^\dagger = g^*\). Therefore, \(\tilde{g}(u) = g^*(u)\) for all \(u \in (0, 1)\), and \(\tilde{g}_{dx}(0) \leq y_{dx}\), \(\tilde{g}_{dx}(1) \geq y_{dx}\).

Finally, we show for any increasing functions, if \(\exists u\) such that \(\Pi(y(u), x_1) \neq c(u)\), the set of such \(u\) is uncountable. Suppose not, let the function be \(\tilde{g}\) and \(u_0\) be such that \(\Pi(\tilde{g}(u_0), x_1) \neq c(u_0)\). We assume \(\tilde{g} : [0, 1] \rightarrow S(Y)\) because the general case can be shown by reconstructing \(\tilde{g}\) using the same way before. Then by monotonicity,

\[
\lim_{u \rightarrow u_0^-} \tilde{g}(u) \leq \tilde{g}(u_0) \leq \lim_{u \rightarrow u_0^+} \tilde{g}(u)
\]

When \(u_0 = 0\) or 1, set \(\lim_{u \rightarrow u_0^-} \tilde{g}(u) = y\) or \(\lim_{u \rightarrow u_0^-} \tilde{g}(u) = \bar{y}\) respectively. Since there are only at most countable points of \(u\) such that the equations do not hold, we can always construct sequences of \(u\) on both sides of \(u_0\) such that \(\tilde{g}(u)\) satisfies the system of equations on every \(u\) in the sequence. The sequences of \(\tilde{g}(u)\) are monotone and bounded above (the sequence to the left of \(u_0\)) and below (the sequence to the right of \(u_0\)), so limits exist which are equal to \(\lim_{u \rightarrow u_0^-} \tilde{g}(u)\) and \(\lim_{u \rightarrow u_0^+} \tilde{g}(u)\), respectively. Then by continuity of \(\Pi(\cdot, x_1)\), the equation holds at the right and left limits, which implies there must be components in \(\tilde{g}\) such that the inequalities above hold strictly because \(\tilde{g}(u_0)\) is not a solution. But as in Step 1, whenever a component jumps up, there must be components jumping down to make the equation hold at the same \(u_0\), a contradiction. \(\square\)
Proof of Theorem 3. First, from Lemma 1, it is easy to see the objective function is uniform consistent. For simplicity denotes the population objective function by \( Q_x(\cdot) \) and the sample analogue by \( \hat{Q}_x(\cdot) \). Suppose \( d_H(\{\hat{x}_1\}, \{x_1\}) > \varepsilon \), then by the Maximum Theorem both of the sets are compact (with probability 1), so at least one of the following cases are true:

i) there exists \( \hat{x}_1 \in \{\hat{x}_1\} \) such that \( |\hat{x}_1 - x_1| > \varepsilon \) for all \( x_1 \in \{x_1\} \).

ii) there exists \( x_1 \in \{x_1\} \) such that \( |\hat{x}_1 - x_1| > \varepsilon \) for all \( \hat{x}_1 \in \{\hat{x}_1\} \). where \( |\cdot| \) denotes the Euclidean distance.

Suppose case i) is true. Then \( \hat{x}_1 \) lies \( \varepsilon \) away from the set \( \{x_1\} \). By compactness of \( \{x_1\} \) and the continuity of \( Q_x(\cdot) \), there exists \( \delta > 0 \) such that \( Q_x(\hat{x}_1) - Q_x(x_1) > \delta \) for any \( x_1 \). Thus \( [Q_x(\hat{x}_1) - \hat{Q}_x(\hat{x}_1)] + [\hat{Q}_x(\hat{x}_1) - Q_x(x_1)] > \delta \). By uniform consistency the first bracket is \( o_p(1) \). For the second bracket, since \( \hat{x}_1 \) minimizes \( \hat{Q}_x \), it is no greater than \( \hat{Q}_x(x_1) - Q_x(x_1) \), which is again \( o_p(1) \). Contradiction.

Now suppose ii) is true. Symmetrically, it implies \( \exists \delta > 0 \) such that \( \hat{Q}_x(x_1) - \hat{Q}_x(\hat{x}_1) > \delta \) for all \( \hat{x}_1 \) with probability 1, given \( \hat{Q}_x \) is continuous with probability one. Then \( [\hat{Q}_x(x_1) - Q_x(x_1)] + [Q_x(x_1) - \hat{Q}_x(\hat{x}_1)] > \delta \). The first bracket is \( o_p(1) \). For the second, since \( Q_x(x_1) = 0 \), it is smaller than \( Q_x(\hat{x}_1) - \hat{Q}_x(\hat{x}_1) \), which is again \( o_p(1) \) by uniform consistent of \( \hat{Q}_x \).

Finally, rejection of case i) implies all \( \hat{x}_1 \) lies with an \( \varepsilon \) ball of the set \( \{x_1\} \) w.p.a.1. And since the latter is compact, there always exists an \( x_1 \) such that \( \hat{x}_1 \) is within \( \varepsilon \) distance from it w.p.a.1.

Next we show the proof of Lemma 4, before which we introduce the following lemma and since the results are standard, the proof is omitted.

\( \square \)
Lemma C1. Under Assumption 4, for all \( d \) and \( z \), if \( h_0 = \left( \frac{\log(n)}{n} \right)^{\frac{1}{q+1}} \), we have

\[
\sup_{y \in \bar{S}(Y)} \sup_{x \in S(X)} \left| \hat{F}_{Y|D|XZ}(y, d|x, z) - F_{Y|D|XZ}(y, d|x, z) \right| = o_p(1)
\]

\[
\sup_{y \in S(Y)} \sup_{x \in S^0(X)} \left| \hat{F}_{Y|D|XZ}(y, d|x, z) - F_{Y|D|XZ}(y, d|x, z) \right| = O_p\left( \left( \frac{\log(n)}{nh^q} \right)^{\frac{1}{2}} + h^2 \right)
\]

\[
\sup_{y \in S^0(Y)} \sup_{x \in S^0(X)} \left| \partial_y \hat{F}_{Y|D|XZ}(y, d|x, z) - f_{Y|D|XZ}(y, d|x, z) \right| = o_p(1)
\]

\[
\sup_{x \in S(X)} \left| \hat{p}_d(x, z) - p_d(x, z) \right| = o_p(1)
\]

\[
\sup_{x \in S^0(X)} \left| \partial_{x_s} \hat{p}_d(x, z) - \partial_{x_s} p_d(x, z) \right| = o_p(1)
\]

\[
\sup_{x \in S(X)} \left| \hat{y}_{dx} - \bar{y}_{dx} - \eta \right| = O_p\left( \left( \frac{\log(n)}{n} \right)^{\frac{1}{q+1}} \right)
\]

\[
\sup_{x \in S(X)} \left| \hat{y}_{dx} - y_{dx} + \eta \right| = O_p\left( \left( \frac{\log(n)}{n} \right)^{\frac{1}{q+1}} \right)
\]

where \( \bar{S}(Y) \) is any compact set that contains \( S(Y) \) as a proper subset.

Note we need uniform consistency holds over a region larger than the support set because w.p.a.1, our boundary estimators will in the end admit a larger set than the true support by design. Meanwhile, as for convergence rate, we only need the rate for an interior compact set of \( S(X) \), denoted by \( S^0(X) \). Some other results in the lemma are for uniform convergence of the partial derivatives of the CDF and the propensity score estimators, either with respect to \( y \) or respect to the \( s \)-th component in \( x \). These results are needed when deriving the asymptotic distribution. The last two results are shown by Guerre, Perrigne and Vuong (2000). We need the convergence rate of the main part of the boundary estimators to determine the rate of \( \eta \) to guarantee the support estimate contains the true support w.p.a.1.

Proof of Theorem 4. By Theorem 3, \( \hat{x}_1 \) will be in the interior of \( S(X) \) w.p.a.1. Therefore, the first order condition holds for it. By standard theory of minimum distance estimators (see e.g., Newey and McFadden 1994),

\[
\sqrt{n h^q} (\hat{x}_1 - x_1) = -\left( \partial_{x_1} \Delta \hat{p}(\hat{x}_1) W_n \partial_{x_1} \Delta \hat{p}(\bar{x}_1) \right)^{-1} \partial_{x_1} \Delta \hat{p}(\bar{x}_1) W_n \Delta \sqrt{n h^q} \bar{p}(x_1)
\]
By Lemma C1, and Slutsky's theorem,

\[-(\partial_{x_1'} \Delta \hat{p}(\hat{x}_1)]W_n \partial_{x_1} \Delta \hat{p}(\hat{x}_1)]W_n \overset{p}{\to} -(\partial_{x_1'} \Delta \hat{p}(x_1)]W_0 \partial_{x_1} \Delta \hat{p}(x_1)]W_0^{-1} \partial_{x_1} \Delta \hat{p}(x_1)]W_0 \]

As for \(\sqrt{n}h \Delta \hat{p}(x_1)\), it is equal to

\[\sqrt{n}h^q \Delta \hat{p}(x_1) = \sqrt{n}h^q \left( \sum_{i=1}^n 1(D=1)1(Z=1)K(\frac{x_i - x_i}{h}) - \sum_{i=1}^n 1(D=2)1(Z=0)K(\frac{x_i - x_i}{h}) \right) \]

Since all the expectations of the cross product term are 0 because all of them involve exclusive events like \(1(D=1)1(D=2)\) and \(1(Z=1)1(Z=0)\), the variance of it is equal to

\[\Sigma_x = \kappa \left( \begin{array}{cc} \frac{p_1(x_1,1)(1-p_1(x_1,1))}{f_{xZ}(x_1,1)} & \frac{p_1(x_0,0)(1-p_1(x_0,0))}{f_{xZ}(x_0,0)} \\ 0 & \frac{p_2(x_1,1)(1-p_1(x_1,1))}{f_{xZ}(x_1,1)} + \frac{p_2(x_0,0)(1-p_1(x_0,0))}{f_{xZ}(x_0,0)} \end{array} \right) \]

\[= \kappa \left( \begin{array}{cc} 1 & 1 \\ f_{xZ}(x_0,0) & f_{xZ}(x_1,1) \end{array} \right) \left( \begin{array}{cc} p_1(x_0,0)(1-p_1(x_0,0)) & 0 \\ 0 & p_2(x_0,0)(1-p_2(x_0,0)) \end{array} \right) \]

To prove Theorem 5, we need the following lemmas. The proofs of them are in Appendix D.

**Lemma C2.** Under conditions in Lemma C1, if \(\eta \cdot \left( \frac{n}{\log n} \right)^{1/(q+1)} \to \infty\), then

\[p \lim \mathbb{I}(\hat{y}_{dx} > \bar{y}_{dx}) = 1\]

\[p \lim \mathbb{I}(\hat{y}_{dx} < \bar{y}_{dx}) = 1\]

**Lemma C3.** Under Assumption 4 and conditions in Lemma C1 and C2, if \(\lambda J \to 0\),

\[\sup_{y \in \mathcal{Y}} \left| \hat{Q}(y, \hat{x}_1) - Q_{g^*}(x_1) \right| = o_p(1) \quad (17)\]

**Lemma C4.** Under conditions in Lemma C2, for every function \(g\) in \(\mathcal{G}\), there exists a function \(\hat{g} \in \hat{\mathcal{G}}\) such that \(||\hat{g} - g||_2 = o_p(1)\).

**Proof of Theorem 5.** By Lemma C2, w.p.a.1 the range of \(\hat{g}\) is contained in \(\hat{S}(Y)\). Therefore by Lemma C1, \(||\hat{g} - g^*||_2 \geq \delta > 0\) implies there exists \(\varepsilon > 0\) such that \(Q_{g^*}(\hat{g}, x_1) - Q_{g^*}(x_1) \geq \varepsilon\).
\[ Q_g(\hat{g}, x_1) - Q_g(\hat{g}, \hat{x}_1) + Q_g(\hat{g}, x_1) - Q_g(\hat{g}, \hat{x}_1) + Q_g(\hat{g}, \hat{x}_1) - Q_g(g^*, x_1) \geq \varepsilon \]

where \( \hat{g}^* \in \hat{G} \) converges to \( g^* \) and the existence is guaranteed by Lemma C4.

For the first bracket,
\[
Q_g(\hat{g}, x_1) - Q_g(\hat{g}, \hat{x}_1) = Q_g(\hat{g}, x_1) - Q_g(\hat{g}, \hat{x}_1) + Q_g(\hat{g}, \hat{x}_1) - Q_g(g^*, x_1) \leq \sup_{y \in \hat{G}} |Q_g(y, \hat{x}_1) - Q_g(y, x_1)| + \sup_{y \in \hat{G}} \sup_{x \in S(X)} |\hat{Q}_g(y, x_1) - Q_g(y, x_1)|
\]

The first term on the right hand side is \( o_p(1) \) by continuous mapping theorem and uniform boundedness of the partial derivative of \( Q \) with respect to \( x_1 \). The second term is also \( o_p(1) \) by Lemma C3.

For the second term, since \( \hat{g}^* \in \hat{G} \), by definition \( \hat{Q}_g(\hat{g}, \hat{x}_1) \leq \hat{Q}_g(\hat{g}, \hat{x}_1) \), then by Lemma C3, the second bracket is \( o_p(1) \).

For the third bracket is also \( o_p(1) \) by continuity of \( Q_g \) and continuous mapping theorem.

Proof of Corollary 1. Suppose not. Consider an arbitrary component in \( \hat{g} \), denoted by \( \hat{g}_{dx} \), such that \( \sup_{u \in [0,1]} |\hat{g}_{dx}(u) - g^*_{dx}(u)| \geq \delta > 0 \). Let \( u_0 \) be the value of \( u \) at which the supremum is taken, which is random because it is a function of the estimator. We consider the case when \( \hat{g}_{dx}(u_0) - g^*_{dx}(u_0) \geq \delta \) because the other case is completely symmetric. Let \( u^* = g^*_{dx}(\bar{y}_{dx} - \delta) \).

We will first show that given \( u_0 > u^* \), \( u_0 \not\to u^* \). Suppose not, for any given \( \varepsilon > 0 \), the event \( u_0 \geq u^* + \varepsilon \) implies there exists \( \rho > 0 \) such that \( g^*_{dx}(u_0) \geq g^*_{dx}(u^*) + \rho \). Then by monotonicity,
\[
\hat{y}_{dx} - \bar{y}_{dx} = \hat{g}_{dx} - g^*_{dx}(u^*) - \delta \\
\geq \hat{g}_{dx}(u_0) - g^*_{dx}(u^*) - \delta \\
\geq \rho + \hat{g}_{dx}(u_0) - g^*_{dx}(u_0) - \delta \\
\geq \rho
\]

By consistency of \( \hat{y}_{dx} \), we obtain the desired result.

Therefore, for any \( \nu > 0 \), \( u_0 \leq u^* + \nu \) w.p.a.1. Let \( \delta' = \delta - \nu C_{g_{dx}} \), where \( C_{g_{dx}} \) is the Lipschitz constant of \( g^*_{dx} \). Then for \( \nu \) sufficient small,
\[
\int_0^1 (\hat{g}_{dx}(u) - g^*_d(u))^2 du \geq \int_0^1 (\hat{g}_{dx}(u) - g^*_d(u))^2 du \\
\geq \int_{u_0}^{g^*_d(u_0)+\delta'} (\hat{g}_{dx}(u) - g^*_d(u))^2 du \\
\geq \int_{u_0}^{g^*_d(u_0)+\delta'} (g^*_d(u_0) + \delta' - g^*_d(u))^2 du \\
\geq \delta'^2 (g^*_d(u_0) + \delta' - u_0) \\
\geq \delta'^3/C_g^* > 0
\]

This completes the proof since by \(L^2\) consistency the event has probability converging to 0.

**Proof of Corollary 2.** The first statement directly follows uniform consistence of \(\hat{g}\). To prove the second statement, note that

\[
\frac{1}{J} \sum_{j=1}^{J} \hat{g}_d(x_0, u_j) - \int_0^1 g^*_d(x_0, u_j) du = o_p(1)
\]

By uniform convergence, for every \(u\), the difference between \(\hat{g}_d(x_0, u)\) and \(g^*_d(x_0, u)\) is \(o_p(1)\) uniform over \(u\), so all the \(\hat{g}\) on the left hand side can be replaced by \(g^*\). Then following the proof of Lemma B3 we have the desired result.

**Proof of Theorem 6.** Note that \(\max_j |\hat{g}(u_j) - g^*(u_j)| \leq \sqrt{\sum_{j=1}^{J} [\hat{g}(u_j) - g^*(u_j)] [\hat{g}(u_j) - g^*(u_j)]}\). Now we derive the convergence rate of the sum on the right hand side.

For each summand, by the mean value theorem,

\[
\Pi(\hat{g}(u_j), x_1) - \Pi(g^*(u_j), x_1) = \Pi^T(\hat{g}(u_j), x_1)(\hat{g}(u_j) - g^*(u_j))
\]

where \(\Pi^T\) is the transpose of the Jacobian and \(\tilde{g}(u)\) is a mean value. By Corollary 2, all \(\hat{g}(u_j)\) uniformly converges to \(g^*(u_j)\)s. Therefore, with probability approaching 1, all the Jacobians are invertible, under conditions in Theorem 2. Therefore,

\[
\sum_{j=1}^{J} [g^*(u_j) - g^*(u_j)] [\hat{g}(u_j) - g^*(u_j)] = \sum_{j=1}^{J} [\Pi(\hat{g}(u_j), x_1) - \Pi(g^*(u_j), x_1)] [\Pi(\hat{g}(u_j), x_1) - \Pi(g^*(u_j), x_1)]
\]

where \(\Psi_j = [\Pi^T(\tilde{g}(u_j), x_1)\Pi'(\tilde{g}(u_j), x_1)]^{-1}\) is uniformly bounded over \(S(Y)\) and \(S(X)\), so
there exists a constant $C$ such that

$$\sum_{j=1}^{J}[\hat{g}(u_j)-g^*(u_j)]'[\hat{g}(u_j)-g^*(u_j)] \leq C \sum_{j=1}^{J}[\Pi(\hat{g}(u_j), x_1)-\Pi(g^*(u_j), x_1)]'\Pi(\hat{g}(u_j), x_1)-\Pi(g^*(u_j), x_1)]$$

By adding and subtracting $\hat{\Pi}(\hat{g}(u_j), \hat{x}_1)$,

$$\Pi(\hat{g}(u_j), x_1) - \Pi(g^*(u_j), x_1) = [\Pi(\hat{g}(u_j), x_1) - \hat{\Pi}(\hat{g}(u_j), \hat{x}_1)] + [\hat{\Pi}(\hat{g}(u_j), \hat{x}_1) - \Pi(g^*(u_j), x_1)]$$

Then

$$\sum_{j=1}^{J}[\hat{g}(u_j)-g^*(u_j)]'[\hat{g}(u_j)-g^*(u_j)] \leq C \sum_{j=1}^{J}[\Pi(\hat{g}(u_j), x_1) - \Pi(g^*(u_j), x_1)]'\Pi(\hat{g}(u_j), x_1)-\Pi(g^*(u_j), x_1)]$$

$$+ C \sum_{j=1}^{J}[\hat{\Pi}(\hat{g}(u_j), \hat{x}_1) - c(u_j)]'[\hat{\Pi}(\hat{g}(u_j), \hat{x}_1) - c(u_j)]$$

$$+ 2C \sum_{j=1}^{J}[\Pi(\hat{g}(u_j), x_1) - \Pi(g^*(u_j), x_1)]'[\hat{\Pi}(\hat{g}(u_j), \hat{x}_1) - c(u_j)]$$

For the first bracket,

$$|\Pi(\hat{g}(u_j), x_1) - \hat{\Pi}(\hat{g}(u_j), \hat{x}_1)| \leq |\Pi(\hat{g}(u_j), x_1) - \Pi(\hat{g}(u_j), \hat{x}_1)| + |\Pi(\hat{g}(u_j), \hat{x}_1) - \hat{\Pi}(\hat{g}(u_j), \hat{x}_1)|$$

$$= O_p \left( \frac{1}{\sqrt{(nh)^q}} + r_n \right)$$

where the first item is $O_p \left( \frac{1}{\sqrt{(nh)^q}} \right)$ because the partial derivative of $\Pi$ with respect to $x_1$ is uniformly bounded thus the convergence depends linearly on $\hat{x}_1$, then by Theorem 4 we have the results. The second term is from Lemma C1 since it is dominated by the supremum of the difference over the $S(Y)$ and $S^o(X)$. Note it is independent of $j$, so the first bracket is $O_p(Jr^2_n)$

For the second bracket, we can add and subtract the penalty term, then

$$\sum_{j=1}^{J}[\hat{\Pi}(\hat{g}(u_j), \hat{x}_1) - c(u_j)]'[\hat{\Pi}(\hat{g}(u_j), \hat{x}_1) - c(u_j)]$$

$$\leq \sum_{j=1}^{J}[\hat{\Pi}(g^*(u_j), \hat{x}_1) - \Pi(g^*(u_j), x_1)]'[\hat{\Pi}(\hat{g}(u_j), \hat{x}_1) - \Pi(g^*(u_j), x_1)]$$

$$+ J\lambda \sum_{j=2}^{J}(g^*(u_j) - g^*(u_{j-1}))'(g^*(u_j) - g^*(u_{j-1})) - J\lambda \sum_{j=2}^{J}((\hat{g}(u_j) - \hat{g}(u_{j-1})))'((\hat{g}(u_j) - \hat{g}(u_{j-1})))$$

$$= O_p(Jr^2_n)$$

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For the third bracket, note that \( \hat{\Pi}(\hat{g}(u_j), \hat{x}_1) - c(u_j) = \hat{\Pi}(\hat{g}(u_j), \hat{x}_1) - \Pi(g^*(u_j), x_1) \) which in turn equals \( \hat{\Pi}(\hat{g}(u_j), \hat{x}_1) - \Pi(\hat{g}(u_j), x_1) + \Pi(\hat{g}(u_j), x_1) - \Pi(g^*(u_j), x_1) \), where the rate of the first term is already derived. The second term is equal to \( \Pi^T(\hat{g}(u_j), x_1)[\hat{g}(u_j) - g^*(u_j)] \), then by Cauchy-Schwartz:

\[
\sum_{j=1}^{J} [\Pi(\hat{g}(u_j), x_1) - \Pi(\hat{g}(u_j), \hat{x}_1)]' [\hat{\Pi}(\hat{g}(u_j), \hat{x}_1) - c(u_j)]
\]

\[
= O_p(Jr_n^2) + O_p(\sqrt{Jr_n}) \sqrt{\sum_{j=1}^{J} [\hat{g}(u_j) - g^*(u_j)]' [\hat{g}(u_j) - g^*(u_j)]}
\]

Putting all these pieces together, we obtain

\[
\sum_{j=1}^{J} [\hat{g}(u_j) - g^*(u_j)]' [\hat{g}(u_j) - g^*(u_j)] = O_p(Jr_n^2) + O_p(\sqrt{Jr_n}) \sqrt{\sum_{j=1}^{J} [\hat{g}(u_j) - g^*(u_j)]' [\hat{g}(u_j) - g^*(u_j)]}
\]

Therefore, \( \max_j |\hat{g}(u_j) - g^*(u_j)| \leq \sqrt{\sum_{j=1}^{J} [\hat{g}(u_j) - g^*(u_j)]' [\hat{g}(u_j) - g^*(u_j)]} = O_p(\sqrt{Jr_n}) \)

Proof of Theorem 7. By Theorem 6, the following first order condition holds w.p.a.1.

\[ \hat{\Pi}(\hat{g}(u_0), \hat{x}_1) = c(u_0) + o_p\left( \frac{1}{\sqrt{nh^q}} \right) \]

By the mean value theorem, the equation can be rewritten as follows:

1. The first two rows in \( \hat{\Pi} \). For \( z = 0, 1 \),

\[
\sum_{d=1}^{3} \hat{F}_{Y|D|XZ}(\hat{g}_d(x_0, u_0), d|x_0, z)\left( \hat{g}_d(x_0, u_0) - g^*_d(x_0, u_0) \right) = -\left( \sum_{d=1}^{3} \hat{F}_{Y|D|XZ}(g^*_d(x_0, u_0), d|x_0, z) - u_0 \right)
\]

2. The third row in \( \hat{\Pi} \).

\[
\sum_{d=1}^{3} \hat{F}_{Y|D|XZ}(\hat{g}_d(x_1, u_0), d, \hat{x}_1, 0)\left( \hat{g}_d(x_1, u_0) - g^*_d(x_1, u_0) \right)
\]

\[
= -\left( \sum_{d=1}^{3} \hat{F}_{Y|D|XZ}(g^*_d(x_1, u_0), d|x_1, 0) - u_0 \right) - \left( \sum_{d=1}^{3} \frac{\partial}{\partial x} \hat{F}_{Y|D|XZ}(\hat{g}_d(x_1, u_0), d|x_1, 0) \right)(\hat{x}_1 - x_1)
\]
3. The last three rows in $\hat{\Pi}$. For $d = 1, 2, 3,$

$$\hat{f}_{YD|XZ}(\tilde{g}_d(x_1, u_0), d|\tilde{x}_1, 1)(\hat{g}_d(x_1, u_0) - g^*_d(x_1, u_0))$$

$$= \hat{f}_{YD|XZ}(\tilde{g}_d(x_0, u_0), d|x_0, 0)(\hat{g}_d(x_0, u_0) - g^*_d(x_0, u_0))$$

$$+ \hat{F}_{YD|XZ}(g^*_d(x_0, u_0), d|x_0, 0) - \hat{F}_{YD|XZ}(g^*_d(x_1, u_0), d|x_1, 1)$$

$$- \frac{\partial}{\partial x}\hat{F}_{YD|XZ}(\tilde{g}_d(x_1, u_0), d|\tilde{x}_1, 1))(\hat{x}_1 - x_1)$$

where $\tilde{g}_d(x, u_0)$ and $\tilde{g}_d(x, u_0)$ are between $g^*_d(x, u_0)$ and $\hat{g}_d(x, u_0)$. $\tilde{x}_1$ and $\tilde{x}_1$ are between $x_1$ and $\hat{x}_1$. By consistency of $\hat{g}_d(x, u_0)$ and $x_1$ and Lemma C1, $f_{YD|XZ}(\tilde{g}_d(x_1, u_0), d|\tilde{x}_1, 1)$ is consistent for $f_{YD|XZ}(g^*_d(x_1, u_0), d|x_1, 1) > 0$. So we can solve for $\hat{g}_d(x_1, u_0) - g^*_d(x_1, u_0)$. Substitute it back and we get

$$\sum_{d}^{3} \hat{f}_{YD|XZ}(\tilde{g}_d(x_1, u_0), d, \tilde{x}_1, 0)$$

$$- \sum_{d=1}^{3} \hat{F}_{YD|XZ}(\tilde{g}_d(x_1, u_0), d|x_1, 0) - \sum_{d=1}^{3} \frac{\partial}{\partial x}\hat{F}_{YD|XZ}(\tilde{g}_d(x_1, u_0), d|\tilde{x}_1, 1))(\hat{x}_1 - x_1)$$

$$+ \sum_{d=1}^{3} \hat{F}_{YD|XZ}(\tilde{g}_d(x_1, u_0), d|x_1, 0) - \hat{F}_{YD|XZ}(g^*_d(x_1, u_0), d|x_1, 1)$$

$$+ \sum_{d=1}^{3} \frac{\partial}{\partial x}\hat{F}_{YD|XZ}(\tilde{g}_d(x_1, u_0), d|\tilde{x}_1, 1))(\hat{x}_1 - x_1)$$

(18)

Finally we can substitute the asymptotic expansion of $(\hat{x}_1 - x_1)$ from the proof of Theorem 4, and we can obtain (15).

\[\square\]

### D Proofs of Lemmas in Appendix C

**Proof of Lemma C2.** We only present the proof of the first statement as the second is completely symmetric.

By construction,

$$P(\tilde{y}_{dx} > \tilde{y}_{dx}) = P(\eta > a_n)$$

where by Lemma C1 $a_n = O_p\left(\frac{\log n}{n}\right)^{1/(q+1)}$. Hence $a_n/\eta = o_p(1)$, which completes the proof.

\[\square\]

**Proof of Lemma C3.** First, by Lemma C2, the penalty part is bounded w.p.a.1, so $\lambda J \to 0$.
implies this part is $o_p(1)$ uniformly.

Then by triangular inequality,

$$
\sup_{\substack{y \in \hat{G} \\ x \in S(X)}} \left| \hat{Q}(y, \hat{x}_1) - Q_g(g^*, x_1) \right|
= \sup_{\substack{y \in \hat{G} \\ x \in S(X)}} \left( \frac{1}{J} \sum_{j=1}^{J} \left[ \hat{\Pi}(y_j, x_1) - c(u_j) \right]' \left[ \hat{\Pi}(y_j, x_1) - c(u_j) \right] - \left[ \Pi(y_j, x_1) - c(u_j) \right]' \left[ \Pi(y_j, x_1) - c(u_j) \right] \right)
+ \sup_{\substack{y \in \hat{G} \\ x \in S(X)}} \left( \frac{1}{J} \sum_{j=1}^{J} \left[ \Pi(y_j, x_1) - c(u_j) \right]' \left[ \Pi(y_j, x_1) - c(u_j) \right] - \int_{0}^{1} \left[ \Pi(y_j, x_1) - c(u_j) \right]' \left[ \Pi(y_j, x_1) - c(u_j) \right] du \right)
+ o_p(1)
$$

By Lemma C2, for any $\tilde{S}(Y)$, all changing points in $\hat{G}$ are in $\tilde{S}(Y)$ w.p.a.1. Therefore, the first term on the right hand side is smaller than

$$
\sup_{\substack{y \in \tilde{S}(Y) \\ x \in S(X)}} \left( \frac{1}{J} \sum_{j=1}^{J} \left[ \hat{\Pi}(y_j, x_1) - c(u_j) \right]' \left[ \hat{\Pi}(y_j, x_1) - c(u_j) \right] - \left[ \Pi(y_j, x_1) - c(u_j) \right]' \left[ \Pi(y_j, x_1) - c(u_j) \right] \right)
= \frac{1}{J} \sum_{j=1}^{J} \sup_{\substack{y \in \tilde{S}(Y) \\ x \in S(X)}} \left[ \hat{\Pi}(y, x_1) - c(u_j) \right]' \left[ \hat{\Pi}(y, x_1) - c(u_j) \right] - \left[ \Pi(y, x_1) - c(u_j) \right]' \left[ \Pi(y, x_1) - c(u_j) \right]
$$

Since $c(u)$ is bounded, the right hand side is $o_p(1)$ by Lemma C1.
For the second term,

\[
\sup_{y \in \hat{G}} \left| \frac{1}{J} \sum_{j=1}^{J} \left[ \Pi(y_j, x_1) - c(u_j) \right] \left[ \Pi(y_j, x_1) - c(u_j) \right] - \int_{0}^{1} \left[ \Pi(y(u), x_1) - c(u) \right] \left[ \Pi(y(u), x_1) - c(u) \right] du \right|
\]

\[
= \sup_{y \in \hat{G}} \left| \sum_{j=1}^{J} \int_{\frac{j-1}{J+1}}^{\frac{j}{J+1}} \left( \frac{J+1}{J} \left[ \Pi(y_j, x_1) - c(u_j) \right] \left[ \Pi(y_j, x_1) - c(u_j) \right] - \left[ \Pi(y(u), x_1) - c(u) \right] \left[ \Pi(y(u), x_1) - c(u) \right] \right) du \right|
\]

\[
+ \sup_{y \in \hat{G}} \int_{\frac{j-1}{J+1}}^{\frac{j}{J+1}} \left[ \Pi(y(u), x_1) - c(u) \right] \left[ \Pi(y(u), x_1) - c(u) \right] du
\]

\[
\leq \sup_{y \in \hat{G}} \left| \sum_{j=1}^{J} \int_{\frac{j-1}{J+1}}^{\frac{j}{J+1}} A(\tilde{y}(u))(y_j - y(u)) du \right| + \frac{1}{J} \sup_{y \in \hat{G}} \left| \frac{1}{J+1} \sum_{j=1}^{J} \Pi(y_j, x_1) - c(u_j) \right| \left[ \Pi(y_j, x_1) - c(u_j) \right] + o(1)
\]

\[
\leq \sup_{y \in \hat{G}} \sum_{j=1}^{J} \int_{\frac{j-1}{J+1}}^{\frac{j}{J+1}} |A(\tilde{y}(u))| \cdot |y_j - y(u)| du + o(1)
\]

\[
\leq \sup_{y \in \hat{G}} \sum_{j=1}^{J} \int_{\frac{j-1}{J+1}}^{\frac{j}{J+1}} |A(\tilde{y}(u))| \cdot |y_j - y_{j-1}| du + o(1)
\]

\[
\leq C \sup_{x_1 \in S(X)} (\tilde{y} - y) \cdot J + 1 + o(1)
\]

where the first inequality is from boundedness of \( \Pi \) and \( c \), and the mean value theorem. \( A \) is the Jacobian which is uniformly bounded over \( S(X) \) by boundedness of conditional densities of \( Y \), and \( \tilde{y}(u) \) is the mean value. Note that from the first inequality the space is changed from \( \hat{G} \) to \( G \) because all \( y(u) \) lies outside \( S(Y) \) admit the same \( \Pi \) as replacing them to the nearest boundaries. Then within the support, mean value theorem can be applied. The third inequality follows from monotonicity of \( y \) and the last inequality is by construction of \( y \), where \( C \) is an upper bound of elements in \( A \).

\( \Box \)

**Proof of Lemma C4.** By Lemma C2, w.p.a.1 we can construct \( \tilde{g} \) by setting it equal to \( g \) at
every $u_j$. Then
\[
\int_0^1 (\tilde{g}(u) - g(u))(\tilde{g}(u) - g(u))\,du = \sum_{j=1}^{J+1} \int_{\frac{j-1}{J+1}}^{\frac{j}{J+1}} (\tilde{g}(u) - g(u))(\tilde{g}(u) - g(u))\,du \\
\leq \sum_{j=1}^{J+1} \int_{\frac{j-1}{J+1}}^{\frac{j}{J+1}} (g(u_j) - g(u_{j-1}))(g(u_j) - g(u_{j-1}))\,du \\
\leq \sum_{j=1}^{J+1} (\bar{y} - y)\,g(u_j) - g(u_{j-1})\,\int_{\frac{j-1}{J+1}}^{\frac{j}{J+1}} \,du \\
\leq \frac{(\bar{y} - y)(\bar{y} - y)}{J+1} \to 0
\]

\[\square\]

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