Cellularity of hermitian $K$-theory and Witt-theory

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Abstract

Hermitian $K$-theory and Witt-theory are cellular in the sense of stable motivic homotopy theory over any base scheme without points of characteristic two.

1 Introduction

The notion of a cellular object in motivic homotopy theory is intrinsically linked to the geometry of motivic spheres $S^{p,q}$ \cite{4}. Suppose the smooth scheme $X$ admits a filtration by closed subschemes

$$\emptyset \subset X_0 \subset \cdots \subset X_{n-1} \subset X_n = X,$$

where $X_i \setminus X_{i-1}$ is a disjoint union of affine spaces $\mathbb{A}^{n_{ij}}$. Examples of such filtrations arise in the context of Bialynicki-Birula decompositions for $G_m$-action on smooth projective varieties \cite{2}, cf. \cite{3} for a more recent implementation. By homotopy purity \cite[Theorem 3.2.23]{7} for Thom spaces of normal bundles of closed embeddings, there is a homotopy cofiber sequence

$$X \setminus X_i \longrightarrow X \setminus X_{i-1} \longrightarrow \text{Th}(N_i).$$

By assumption the normal bundle $N_i$ is trivial. Thus the splitting $\text{Th}(N_i) \cong \bigvee_j S^{2n_{ij},n_{ij}}$ and the two-out-of-three property for stably cellular objects \cite[Lemma 2.5]{4} imply inductively that $X$ is stably cellular in the sense of \cite[Definition 2.10]{4}.

In this paper we employ a similar strategy to prove cellularity for Thom spaces of direct sums of tautological sympletic bundles over quaternionic Grassmannians. This allows us to show cellularity of the motivic spectra representing hermitian $K$-theory and Witt-theory \cite{5}. By a base scheme we mean any regular noetherian separated scheme of finite Krull dimension.

Theorem 1.1. Suppose all points on the base scheme have residue characteristic unequal to two. Then hermitian $K$-theory $KQ$ and Witt-theory $KW$ are cellular motivic spectra.

For a related antecedent result showing cellularity of algebraic $K$-theory, see \cite[Theorem 6.2]{4}. The proof of Theorem 1.1 exploits the geometry of quaternionic Grassmannians and the explicit model for hermitian $K$-theory from \cite{9}.

Recent applications of $KQ$ and $KW$ concern computations of stable homotopy groups of motivic spheres \cite{6}, \cite{8}, \cite{12}, and a proof of the Milnor conjecture on quadratic forms \cite{11}. For cellular motivic spectra one has the powerful fact that stable motivic weak equivalences are detected by $\pi_{*,*}$-isomorphisms \cite[Corollary 7.2]{4}. Our main motivation for proving Theorem 1.1 is that it is being used in the computation of the slices of $KQ$ in \cite[Theorem 2.14]{12}. In terms of motivic cohomology with integral and mod-2 coefficients, the result is

$$s_q(KQ) \cong \begin{cases} \Sigma^{2q,q}\mathbb{M}Z \vee \bigvee_{i<q} \Sigma^{2i+q,q}\mathbb{M}Z/2 & q \text{ even} \\ \bigvee_{i<\frac{q+1}{2}} \Sigma^{2i+q,q}\mathbb{M}Z/2 & q \text{ odd.} \end{cases}$$
In turn, this is an essential ingredient in our proof of Morel’s $\pi_1$-conjecture in [12]. It is an interesting problem to make sense of Theorem 1.1 without any assumptions on the points of the base scheme.

This short paper is organized into Section 2 on basic properties of motivic cellular spectra, Section 3 on the geometry of quaternionic Grassmannians, and Section 4 on hermitian $K$-theory and Witt-theory.

2 Cellular objects

The subcategory of cellular spectra in the motivic stable homotopy category is the smallest full localizing subcategory that contains all suspensions of the sphere spectrum, cf. [4, §2.8]. For our purposes it suffices to know four basic facts about cellular motivic spectra. First we recall part (3) of Definition 2.1 in [4].

Lemma 2.1. The homotopy colimit of a diagram of cellular motivic spectra is cellular.

The second fact is a specialization of [4, Lemma 2.4].

Lemma 2.2. Let $E$ be a motivic spectrum and let $p$, $q$ be integers. Then $E$ is cellular if and only if its $(p,q)$-suspension $\Sigma^{p,q}E$ is cellular.

The third fact is a specialization of [4, Lemma 2.5].

Lemma 2.3. If $E \to F \to G$ is a homotopy cofiber sequence of motivic spectra such that any two of $E$, $F$, and $G$ are cellular, then so is the third.

Finally, we recall Lemma 3.2 in [4].

Lemma 2.4. If $E_i$ is a cellular motivic spectrum for all $i \in I$, then $\coprod_{i \in I} E_i$ is cellular.

3 Quaternionic Grassmannians

The quaternionic Grassmannian $\text{HGr}(r, n)$ is the open subscheme of the ordinary Grassmannian $\text{Gr}(2r, 2n)$ parametrizing $2r$-dimensional subspaces of the trivial vector bundle $\mathcal{O}^\oplus_{2n}$ on which the standard symplectic form is nondegenerate. It is smooth affine of dimension $4r(n - r)$ over the base scheme. Let $\mathcal{U}_{r,n}$ be short for the tautological symplectic subbundle of rank $2r$ on $\text{HGr}(r, n)$. It is the restriction to $\text{HGr}(r, n)$ of the tautological subbundle of $\text{Gr}(2r, 2n)$ together with the restriction to $\mathcal{U}_{r,n}$ of the standard symplectic form on $\mathcal{O}^\oplus_{2n}$.

More generally, to every symplectic bundle $(\mathcal{E}, \phi)$ one associates the quaternionic Grassmannian $\text{HGr}(r, \mathcal{E}, \phi)$; it is the open subscheme of the Grassmannian $\text{Gr}(2r, \mathcal{E})$ parametrizing $2r$-dimensional subspaces of the fibers of $\mathcal{E}$ on which $\phi$ is nondegenerate. Associated to the trivial rank $2n - 2$ symplectic bundle $(\mathcal{E}, \psi)$ is the bundle $\mathcal{F} = \mathcal{O} \oplus \mathcal{E} \oplus \mathcal{O}$ equipped with the direct sum of $\psi$ and the hyperbolic symplectic form, i.e.,

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & \psi & 0 \\
-1 & 0 & 0
\end{pmatrix}.
\]

For simplicity we write $\text{HGr}(\mathcal{E})$ for $\text{HGr}(r, \mathcal{E}, \psi)$ and likewise for $\mathcal{F}$.

The normal bundle $N$ of the embedding $\text{HGr}(\mathcal{E}) \subset \text{HGr}(\mathcal{F})$ is the tensor product $\mathcal{U}_{\mathcal{E}}^\vee \otimes \mathcal{O}^\oplus_{2n}$ for the dual of the tautological symplectic subbundle of rank $2r$ on $\text{HGr}(\mathcal{E})$. Theorem 4.1 in [10] shows that $N$ is naturally isomorphic to an open subscheme of $\text{Gr}(2r, \mathcal{F})$ and there is a decomposition $N = N^+ \oplus N^-$; here, $N^+ = \text{HGr}(\mathcal{F}) \cap \text{Gr}(2r, \mathcal{O} \oplus \mathcal{E})$ and $N^- = \text{HGr}(\mathcal{F}) \cap \text{Gr}(2r, \mathcal{E} \oplus \mathcal{O})$ have
intersection $\text{HGr}(E)$. Thus there are natural vector bundle isomorphisms $N^+ \cong N^- \cong U_{r,n-1}$ and the normal bundle $N$ of $N^+$ in $\text{HGr}(F)$ is isomorphic to $\pi^+_1 U_{r,n-1}$ for the bundle projection $\pi+: N^+ \to \text{HGr}(E)$. Moreover, there is a vector bundle isomorphism between the restriction $U_{r,n}|N^+$ of $U_{r,n}$ to $N^+$ and $\pi^+_1 U_{r,n-1}$. For $r \leq n-1$, let $Y$ denote the complement of $N^+$ in $\text{HGr}(F)$ [10] (5.1).

**Proposition 3.1.** For $m \geq 0$ the suspension spectrum of the Thom space of the vector bundle $U_{r,n}^{\oplus m}$ on $\text{HGr}(r, n)$ is a finite cellular spectrum. In particular, $\Sigma^\infty \text{HGr}(r,n)_+$ is a cellular spectrum.

**Proof.** The proof proceeds by a double induction argument on $r$ and $n \geq r$. The base cases $\text{HGr}(0, n)$ and $\text{HGr}(n, n)$ are clear, so we may assume $0 < r < n$. Define the motivic space $Z$ by the homotopy cofiber sequence

$$\text{Th}(U_{r,n}^{\oplus m}|Y) \longrightarrow \text{Th}(U_{r,n}^{\oplus m}) \longrightarrow Z.$$  \hfill (1)

According to [13] Lemma 3.5 there is a canonical isomorphism in the motivic homotopy category

$$Z \cong \text{Th}(U_{r,n}^{\oplus m}|N^+ \oplus N).$$

Using the above we note $U_{r,n}^{\oplus m}|N^+ \oplus N \cong \pi^+_1 U_{r,n-1}^{\oplus (m-1)}$ and hence there are canonical isomorphisms

$$Z \cong \text{Th}(\pi^+_1 U_{r,n-1}^{\oplus (m+1)}) \cong \text{Th}(U_{r,n-1}^{\oplus (m+1)}).$$

By induction hypothesis $\Sigma^\infty Z$ is a finite cellular spectrum. Thus Lemma 2.3 and (1) reduce the proof to showing that $\Sigma^\infty \text{Th}(U_{r,n}^{\oplus m}|Y)$ is a finite cellular spectrum. To this end we recall parts of Theorem 5.1 in [10]: There exists maps

$$Y \xleftarrow{g_1} Y_1 \xleftarrow{g_2} Y_2 \xrightarrow{q} \text{HGr}(r-1, E, \psi),$$

where $g_1$ and $q$ are Zariski locally trivial torsors over vector bundles of rank $2r-i$ and $4n-3$, respectively. Moreover, $g^*_2 g^*_1 U_{r,n}$ is isomorphic to $O_{Y_2}^2 \oplus q^* U_{r-1,n}$. Invoking [7, §3.2, Example 2.3] this implies the canonical isomorphisms

$$\Sigma^\infty \text{Th}(U_{r,n}^{\oplus m}|Y) \cong \Sigma^\infty \text{Th}(g^*_2 g^*_1 U_{r,n}^{\oplus m}|Y) \cong \Sigma^\infty \text{Th}(O_{Y_2}^{2m} \oplus q^* U_{r-1,n}^{\oplus m}) \cong \Sigma^{2m, m} \Sigma^\infty \text{Th}(U_{r-1,n}^{\oplus m}).$$

Here, the suspension spectrum of $\text{Th}(U_{r-1,n}^{\oplus m})$ is finite cellular by the induction hypothesis. This finishes the proof using Lemma 2.2. \hfill \Box

4 Hermitian $K$-theory and Witt-theory

In this section we finish the proof of Theorem 1.1 stated in the introduction.

The quaternionic plane $\text{HP}^1$ is the first quaternionic Grassmannian $\text{HGr}(1, 2)$. In the pointed unstable motivic homotopy category, $(\text{HP}^1, x_0)$ is isomorphic to the two-fold smash product of the Tate object $T \equiv A^1/A \setminus \{0\}$. It follows that the $A^1$-mapping cone $\text{HP}^1+$ of the rational point $x_0: S \to \text{HP}^1$ is isomorphic to $T^n2$. Hence the stable homotopy category of $\text{HP}^1+$-spectra is equivalent to the standard model for the stable motivic homotopy category [9 Theorem 12.1).

Theorem 12.3 in [9] shows there is an isomorphism between hermitian $K$-theory $\text{KQ}$ and an $\text{HP}^1+$-spectrum $\text{BO}_{\text{geom}}$. For $n$ odd, $\text{BO}_{\text{geom}} = Z \times \text{HGr}$ [9 (12.5)]. Here $\text{HGr}$ denotes the infinite quaternionic Grassmannian, i.e., the sequential colimit

$$\lim_{n} \text{HGr}(n, 2n).$$
We note that the transition maps in the colimit are defined in [9, (8.1)]. The motivic space $\mathbb{Z} \times \text{HGr}$ is pointed by $(0, \text{HGr}(0,0))$. Thus $KQ$ is isomorphic to the homotopy colimit

$$\text{hocolim}_{n \text{ odd}} \Sigma^{4n,2n} \Sigma^\infty \mathbb{Z} \times \text{HGr}.$$ (2)

It remains to show cellularity of (2). Note that $\Sigma^\infty \mathbb{Z} \times \text{HGr}$ is a homotopy colimit of cellular spectra by Lemma 2.4 and Proposition 3.1. It follows that $\Sigma^{4n,2n} \Sigma^\infty \mathbb{Z} \times \text{HGr}$ is cellular according to Lemmas 2.1 and 2.2. We conclude the proof for $KQ$ by applying Lemma 2.1.

Cellularity of $KW$ follows from that of $KQ$ via Lemma 2.1 and the description of $KW$ as the homotopy colimit of the diagram

$$KQ \xrightarrow{\eta} \Sigma^{-1,-1}KQ \xrightarrow{\Sigma^{-1,-1}\eta} \Sigma^{-2,-2}KQ \xrightarrow{\Sigma^{-2,-2}\eta} \cdots$$

given in [1, Theorem 6.5]. Here, $\eta$ is the first stable Hopf map induced by the canonical map $\mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$.

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