\( \mathcal{N} = 3 \) Harmonic Super-Wilson Loop

Yu-tin Huang\(^{\eta_1, \eta_2} \) and Dharmesh Jain\(^\psi \)

\(^{\eta_1} \)Department of Physics, National Taiwan University, No.1, Sec.4, Roosevelt Road, Taipei 10617, Taiwan

\(^{\eta_2} \)Physics Division, National Center for Theoretical Sciences, National Tsing-Hua University, No.101, Section 2, Kuang-Fu Road, Hsinchu 30013, Taiwan

\(^\psi \)Theory Division, Saha Institute of Nuclear Physics, 1/AF Bidhan Nagar, Kolkata 700064, India

ABSTRACT

We study supersymmetric Wilson loops in \( d = 3, \mathcal{N} = 3 \) harmonic superspace, leading to a construction of a supersymmetrized generalization of the \( \frac{1}{3} \)-BPS Wilson loop for \( \mathcal{N} = 3 \) gauge theories. This also includes a generalization of the \( \frac{1}{6} \)-BPS loop for ABJM theory. We perform a ‘one-loop’ computation of the vacuum expectation value of this operator directly in superspace and compare with the known \( \mathcal{N} = 2 \) localization results at large \( N \). This comparison also lets us identify certain fermionic contributions that do not receive any subleading corrections.

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\(^*\)ythuang@phys.ntu.edu.tw
\(^\dagger\)d.jain@saha.ac.in
1 Introduction

The power of supersymmetry to simplify computations and gain insights cannot be overstated. It sheds light on hidden structures and illuminates relationships among seemingly different objects. A perfect example of this power is given by the Wilson loops/Scattering amplitudes duality in $d = 4, \mathcal{N} = 4$ super-Yang Mills (SYM) theory. Even though evidence for such a duality existed [1–6], only after the construction of a supersymmetrized Wilson loop (WL) in superspace [7, 8] has the duality been confirmed for all helicity sectors of the amplitude. In three-dimensions, while similar evidence in the case of four-point amplitude/four-gon Wilson loop for $\mathcal{N} = 6$ ABJM theory [9] exists [10, 11], extending beyond four-points immediately forces us into the remaining sectors (in terms of R-symmetry instead of helicity) of the theory. This motivates us to construct supersymmetric Wilson loops in superspace.

After the introduction of ABJM theory, various Wilson loop operators with different amounts of preserved supersymmetry were studied extensively. Earlier efforts dealt with construction and perturbative computations of $\frac{1}{6}$-BPS WL [12–15]. Localization was applied to evaluate the vacuum expectation value (vev) of this WL in [16] and the results were found to match the perturbative calculations at large $N$ limit. $\frac{1}{2}$-BPS operators were constructed later in [17] and more calculations followed in [18, 19] where even finite $N$ contributions were computed. Being ‘cohomologically equivalent’ to the $\frac{1}{6}$-BPS operator, the localization results do not differ for these two operators. In [20], a classification was given for Wilson loops preserving various amounts of supersymmetry in $\mathcal{N} = 2, \cdots, 6$ Chern-Simons (CS) matter theories. New Wilson loops in $\mathcal{N} = 4$ theories have been constructed recently in [21].

In this ever-expanding literature of construction, classification and computation involving Wilson loops, we present here a supersymmetrization of the simplest WL operator in three-dimensional CS matter theories including ABJ(M) theories. Such an attempt has been made in [22] for ABJM theory in the framework of ‘ordinary’ $\mathcal{N} = 6$ superspace. It was also pointed out that there are at least three reasons why such a WL cannot be dual to the scattering amplitudes of ABJM theory. The main issue is the non-chiral nature of the superspace that leads to torsion, which does not allow a straightforward identification of the kinematics on the two sides of the duality [23]. So we content ourselves with the ‘well-studied’ framework of $\mathcal{N} = 3$ harmonic superspace [24, 25] to construct the supersymmetrized Wilson loop$^1$. This is to have as much manifest (off-shell) supersymmetry as possible along with a notion of chirality (or ‘harmonic analyticity’) built-in.

In the next section we consider a warm-up exercise of constructing a supersymmetrized WL in $\mathcal{N} = 2$ superspace and a sample localization computation. Then we review the $d = 3, \mathcal{N} = 3$ harmonic superspace in Section 3 before constructing the supersymmetrized $\frac{1}{3}$-BPS WL in Section 4. This leads to a generalization for $\frac{1}{6}$-BPS WL in ABJ(M) theories. In Section 5, we compute perturbatively the ‘one-loop’ vev of this new WL operator directly in harmonic superspace. Finally, we compare the perturbative result with localization computation and comment on future outlook in Section 6.

2 Warm-up

We construct here a supersymmetrized Wilson loop operator in $d = 3, \mathcal{N} = 2$ superspace with coordinates $\{x^\mu (x^{(\alpha \beta)}), \theta^\alpha, \bar{\theta}^{\dot{\alpha}}\}$, where the vector index $\mu = 0, 1, 2$ and spinorial index $\alpha = 1, 2$

$^1$Harmonic Superspace was originally constructed for $d = 4, \mathcal{N} = 2$ supersymmetric theories in [26].
The Jacobi identities give further relations among the field strengths \(D\) such as relation is 
\[
\{D_{\alpha}, D_{\beta}\} = iD_{\alpha\beta} + \epsilon_{\alpha\beta} W; \quad \{D_{\alpha}, D_{\beta}\} = 0
\]
(2.1)
The supersymmetry algebra has the following set of gauge-covariant superspace derivatives: \(\{D_\mu(D_{(a\beta)}, D_{a}, D_{a})\}\). These satisfy the following algebra:
\[
\{D_\alpha, D_{\beta}\} = iD_{\alpha\beta} + \epsilon_{\alpha\beta} W; \quad \{D_{\alpha}, D_{\beta}\} = 0
\]
(2.1)
The supersymmetrization of the familiar \(\frac{1}{2}\) BPS Wilson loop in chiral superspace then looks like
\[
W(x, \theta, \bar{\theta}) = \frac{1}{\dim R} \int d\tau \left( \frac{-1}{2} \bar{x}_A^{\alpha\beta} A_{\alpha\beta} + \theta^\alpha A_{\alpha} + |\bar{x}_A| W \right) \equiv \frac{1}{\dim R} \int d\tau \left( \frac{-1}{2} \bar{x}_A^{\alpha\beta} A_{\alpha\beta} + \theta^\alpha A_{\alpha} + |\bar{x}_A| W \right)
\]
(2.2)
where \(x_A^{\alpha\beta} = x^{\alpha\beta} + i\theta^{(\alpha}\bar{\theta}^{\beta)}\). We can do the component analysis of the connections and field strengths, leading to the fields of \(\mathcal{N} = 2\) vector multiplet \(\{a_{\alpha\beta}, \sigma, \lambda, \bar{\lambda}, D\}\) along with the field strength \(f_{\alpha\beta}\):
\[
W_1 = \sigma; \quad D_\alpha W_1 = \bar{\lambda}_\alpha; \quad \bar{D}_\alpha W_1 = \lambda_\alpha; \quad D_\alpha \bar{D}_\beta W_1 = f_{\alpha\beta} + \epsilon_{\alpha\beta} D
\]
\[
\bar{D}_{(a\beta)} A_{\alpha\beta} = a_{\alpha\beta}; \quad \bar{D} \cdot A_1 = \sigma; \quad \bar{D}^2 A_{\alpha} = \lambda_{\alpha}; \quad \bar{D}_a \bar{D} \cdot A_1 = \bar{\lambda}_{\alpha}; \quad \bar{D}^2 A_{\alpha\beta} = D
\]
(2.3)
Here \(|\) denotes that all \(\theta\)’s are set to vanish. Also relevant is \(\bar{D}_{(a\beta)} \bar{D}^2 A_{\alpha\beta} = f_{\alpha\beta}\). It is now trivial to verify that the \(\theta\)-independent piece of the exponent in (2.2) reduces to the well-known bosonic expression:
\[
\int d\tau (i\bar{x}_A^{\alpha\beta} A_{\alpha\beta} + |\bar{x}_A| W) = \int d\tau (i\bar{x}_A^{\alpha\beta} A_{\alpha\beta} + |\bar{x}_A| W).
\]
(2.4)
In arriving at the last step, we have used the algebra (2.1) to convert covariant derivatives acting on connections into the corresponding field strengths, and terms that look like field-dependent gauge transformations of the connections, i.e. \(\bar{x}_A^{\alpha\beta} D_{\alpha\beta}(e^\gamma A_\gamma)\), are dropped as \(W(x, \theta, \bar{\theta})\) is gauge invariant. The BPS condition for the purely bosonic WL requires \(x^\mu(\tau)\) to be an infinite line in Minkowski space or a great circle on \(S^3\) and one can choose it to satisfy \(|\bar{x}| = 1\) [16, 20]. Since (2.4) for the supersymmetrized case results in a similar equation, we will also consider \(|\bar{x}_A| = 1\). This does not determine \(\theta(\tau)\) completely but only up to a function of \(\tau\): \(\theta(\tau) = f(\tau) \theta_0, \bar{\theta}(\tau) = f^{-1}(\tau) \bar{\theta}_0\). Hence, constant solutions for \(\epsilon\) can still be found for

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2The vector \(x^\mu\) can be traded for a real second-rank symmetric tensor \(x^{\alpha\beta} \equiv x^\mu (\gamma_\mu)^{\alpha\beta}\) with the help of \(d = 3\) “gamma”-matrices. We do not need the explicit basis but the relation \(x^{\alpha\beta} x_{\alpha\beta} = -2x^\mu x_\mu \equiv -2|x|^2\) will be quite useful to know.

3It is most likely that one needs to consider superconformal transformations of the WL operator to fully
these configurations, where the condition $\epsilon^\alpha (\dot{x}^A_{\alpha\beta} + |\dot{x}^A|\epsilon_{\alpha\beta}) = 0$ projects half of the degrees of freedom, thus preserving two real degrees of freedom, i.e. $1/2$-BPS.

Given the Lagrangian and propagators of [27, 28], one should be able to compute the vev of the WL (2.2) perturbatively in superspace as well as in components at different $\theta$-orders. However, we will skip this analysis here and comment on the non-perturbative analysis instead. Using the localization results of [16] where a $\mathcal{N} = 2$ theory on $S^3$ (of radius $r$) is considered, we can obtain an ‘exact’ result for the vev of the supersymmetrized Wilson loop. Since the path integral is localized on the vector multiplet’s scalar field $\sigma = \text{constant}$ and $D = -\frac{r}{2}$, we have,

$$\mathcal{W}(x, \theta, \bar{\theta}) = \frac{1}{\text{dim } R} \text{tr}_R \mathcal{P} \exp \int d\tau \left[ \frac{i}{2} \dot{x}^A_{\alpha\beta} \theta^\alpha \bar{\theta}^\beta D + \theta \cdot (\bar{\theta} \sigma + \theta \bar{\theta}^2 D) + |\dot{x}_A| (\sigma + \theta \cdot \bar{\theta} D) \right]. \quad (2.5)$$

Even though we do not know $f(\tau)$ explicitly, we can evaluate $\langle \mathcal{W}(x, \theta, \bar{\theta}) \rangle$ formally. Let us denote everything in the exponent by $\Theta_\sigma$, with $\Theta = \frac{1}{2\pi} \int d\tau \left( 1 + i \dot{x}^A_{\alpha\beta} \theta^\alpha \bar{\theta}^\beta + \theta \cdot \bar{\theta} - \theta \cdot \bar{\theta} + \bar{\theta} \cdot \theta \bar{\theta}^2 \right)$ (also set $r = 1$). The path integral reduces to a matrix model in terms of eigenvalues $\lambda_i$ of $\sigma$ (we choose ABJM for concreteness, which has two $U(N)$'s as gauge groups and $\pm k$ as the two Chern-Simons levels):

$$\langle \mathcal{W}(x, \theta, \bar{\theta}) \rangle = \frac{1}{N! N Z} \int d\lambda_i d\bar{\lambda}_i \left( e^{-\frac{N}{2\pi} \lambda^2 - \frac{N}{2\pi} \bar{\lambda}^2} \right) \Delta(\lambda)^2 \Delta(\bar{\lambda})^2 \left( \sum_i e^{\Theta_\lambda} \right) \times Z_{1\text{-loop}}. \quad (2.6)$$

where $\alpha = -\hat{\alpha} = 2\pi i \frac{N}{k}$. We refer the reader to [16] for the definitions of various factors in the above result as we are interested in its perturbative limit only. To obtain a perturbative $\alpha$ expansion, we can expand $\langle \mathcal{W} \rangle$ in $\lambda$ and compute the vev using the orthogonal polynomials method (note that $\langle \lambda^{2k} \rangle = \mathcal{O}(\alpha^k)$):

$$\langle \mathcal{W}(x, \theta, \bar{\theta}) \rangle = 1 + \frac{1}{2} \Theta^2 \alpha - \left[ \frac{1}{6} \left( 1 + \frac{1}{2N^2} \right) \theta^2 - \frac{1}{24} \left( 2 + \frac{1}{N^2} \right) \Theta^4 \right] \alpha^2 + \mathcal{O}(\alpha^3). \quad (2.7)$$

Rewriting $\Theta = 1 + \frac{1}{2} \vartheta$, we get (note $\vartheta^3 = 0$)

$$\langle \mathcal{W}(x, \theta, \bar{\theta}) \rangle = 1 + \frac{1}{2} \left[ \vartheta + \frac{\vartheta^2}{4} \right] \alpha - \left[ \frac{1}{24} \left( 5 + \frac{1}{N^2} \right) + \frac{1}{4} \vartheta - \frac{1}{6} \left( \frac{1}{2} + \frac{1}{N^2} \right) \frac{\vartheta^2}{4} \right] \alpha^2 + \mathcal{O}(\alpha^3). \quad (2.8)$$

In the above expression, we have removed the bosonic term at $\mathcal{O}(\alpha)$ by multiplying the result by an overall phase $e^{-\frac{1}{2} \vartheta^3}$, which is necessary in matching the perturbative computation [16]. Note that we do not remove the whole $\vartheta$-dependent term at $\mathcal{O}(\alpha)$, since as we will see later there are indeed fermionic contributions at $\mathcal{O}(\alpha)$ in perturbative computation. We will return back to this result in Section 6.

### 3 Review of $\mathcal{N} = 3$ Harmonic Superspace

Now, we turn to $\mathcal{N} = 3$ supersymmetry. We collect here the necessary ingredients from three-dimensional $\mathcal{N} = 3$ harmonic superspace literature along with a few explicitly worked out details that will be relevant for us in later sections.

determine the $\theta(\tau)$ profile consistent with the circular bosonic WL. We do not pursue this exercise here. Thus, we will not evaluate the $\tau$-integrals explicitly and leave all the $\tau$-dependence of the coordinates intact.
3.1 $\mathcal{N} = 3$ Harmonic Superspace

The ‘ordinary’ $d = 3$, $\mathcal{N} = 3$ superspace with coordinates $\{x^{\alpha\beta}, \theta^{ij}_a\}$ has the following algebra of superspace derivatives:

$$\{D^i_{\alpha}, D^j_{\beta}\} = i(\epsilon^{ik} \epsilon^{jl} + \epsilon^{il} \epsilon^{jk}) \partial_{\alpha\beta}$$

$$D^i_{\alpha} = \partial^i_{\alpha} + i\theta^{ij}_{\alpha} \partial_{\alpha\beta}$$

(3.1)

To obtain constrained superfields in the form of $D^i_{\alpha}\Phi = 0$, it is useful to consider the case where $D^i_{\alpha}$ is given by a simple partial derivative, indicating the independence of $\Phi$ on certain variables. The obstacle to having a representation of $D^i_{\alpha}$ as a partial derivative is its anti-commutator algebra. This can be overcome by the introduction of $SU(2)/U(1)$ harmonics $u^\pm_i$. These bosonic variables satisfy

$$u^+ u^- = 1, \quad u^\pm u^\pm = 0,$$

(3.2)

where the raising and lowering of the $SU(2)$ index $i$ is done by contracting with the invariant tensor $\epsilon^{ij}$. (The contracted $i$ among $u$’s will be suppressed most of the time.) These new variables are to be integrated away using the following rules:

$$\int du = 1, \quad \int du^+ u^+ \cdots u^- = 0.$$ 

(3.3)

In other words, only the $SU(2)$ invariant polynomial with vanishing $U(1)$ charge survives the integration. The harmonic variables allow us to linearly recombine the $3 \times 2$ fermionic coordinates into three new $SL(2, \mathbb{R})$ doublets $\theta^{\alpha, \pm \pm} \equiv u^{\pm \pm} \theta^{\alpha}_{ij}, \theta^{\alpha, 0} \equiv u^{i+} u^{-} \theta^{\alpha}_{ij}$. The upshot is that doing the same for the covariant derivatives, the supersymmetry algebra now reads,

$$\{D^{++}_{\alpha}, D^{-+}_{\beta}\} = 2i\partial^{\alpha\beta}, \quad \{D^0_{\alpha}, D^0_{\beta}\} = -i\partial_{\alpha\beta},$$

$$\{D^{\pm \pm}_{\alpha}, D^{\pm \pm}_{\beta}\} = 0, \quad \{D^{\pm}_{\alpha}, D^0_{\beta}\} = 0,$$

(3.4)

where one finds that we can $SU(2)$ covariantly isolate a doublet of commuting fermionic derivatives, for example $D^{++}_{\alpha}$. This implies that we can have a representation for the covariant derivatives where $D^{++}_{\alpha}$ is a simple partial derivative. This is referred to as the “analytic basis”, and it is given as the following:

$$\partial_{\alpha\beta} \rightarrow \partial^A_{\alpha\beta}$$

$$D^i_{\alpha} \rightarrow \left\{ \begin{array}{ll}
D^{++}_{\alpha} = \partial^{++}_{\alpha} \\
D^{-+}_{\alpha} = \partial^{-+}_{\alpha} + 2i\theta^{-+}_{\alpha} \partial^A_{\alpha\beta} \\
D^0_{\alpha} = -\frac{1}{2} \theta^0_{\alpha} + i\theta^{0\beta} \partial^A_{\alpha\beta}.
\end{array} \right.$$

(3.5)

We defined $x^A_{\alpha\beta} = x_{\alpha\beta} + i\theta^{++}_{\alpha} \theta^{-+}_{\beta}$. In the analytic basis, we obtain constrained superfields by imposing the ‘analytic’ constraint $D^{++}_{\alpha}\Phi = 0$, which now implies that $\Phi$ does not depend on $\theta^{-+}_{\alpha}$:

$$D^{++}_{\alpha}\Phi = 0 \quad \Rightarrow \quad \Phi \equiv \Phi(x^A_{\alpha\beta}, \theta^{++}_{\alpha}, \theta^0_{\alpha}, u).$$

(3.6)

The introduction of harmonic variables also introduces R-symmetry covariant derivatives,
and are given by:\footnote{\(D^0\) is strictly speaking not a covariant derivative on \(SU(2)/U(1)\). It should be treated as the subgroup generator that defines the \(U(1)\) charge for a given operator or field, as in \(D^0\Phi^{(q)} = q\Phi^{(q)}\).}
\[D^{\pm\pm} \equiv \partial^{\pm\pm} = u^{\pm}_i \frac{\partial}{\partial u^i}, \quad D^0 = [D^{++}, D^{--}] . \tag{3.7}\]
These have non-trivial commutator algebra with the fermionic derivatives:\footnote{For completeness, their explicit forms in the analytic basis is given as \(D^{\pm\pm} \rightarrow D^{\pm\pm} \equiv \partial^{\pm\pm} \pm 2i\theta^{\pm\pm} \alpha^0 \beta^A \partial_{\alpha \beta} + 2\theta^{\pm\pm} \alpha^0 \beta^A \partial_{\alpha} \pm \theta^{\pm\pm} \alpha^0 \beta^A \partial_{\alpha} \).}
\[ [D^{\pm\pm}, D^{\pm\pm}_\alpha] = 2D^0_\alpha, \quad [D^{\pm\pm}, D^0] = D^{\pm\pm}_\alpha. \tag{3.8}\]

### 3.2 Chern-Simons Matter Theories

To study gauge theories, we gauge-covariantize the full superspace derivatives \(D \rightarrow D = D + A\), which define the relevant field strengths:
\[
\{D^{++}_\alpha, D^{--}_\beta\} = 2iD_{\alpha \beta} + 2\epsilon_{\alpha \beta} W^0, \quad \{D^0_\alpha, D^0_\beta\} = -iD_{\alpha \beta}, \tag{3.9}\]
\[
\{D^{\pm\pm}_\alpha, D^{\pm\pm}_\beta\} = 0, \quad \{D^{++}_\alpha, D^{0}_\beta\} = \pm \epsilon_{\alpha \beta} W^{\pm\pm}. \tag{3.10}\]

The covariant derivatives, and the field-strengths, transforms as \(D \rightarrow e^\tau D e^{-\tau}\). Choosing a suitable ‘gauge-frame’ (from \(\tau \rightarrow \lambda\)) such that \(A^{++}_\alpha = 0\), allows us to define analytic super fields covariantly while maintaining its implication of independence on \(\theta^{--}_\alpha\): \(D^{++}_\alpha \Phi = D^{++}_\alpha \Phi = 0\).

Note that choosing such a gauge generates (new) harmonic connections \(A^{\pm\pm}\), from which all other connections can be obtained through Bianchi identities. In particular, \(A^{++}\) turns out to be the unique analytic \((D^{++}_\alpha A^{++} = 0)\) prepotential in this formalism. The prepotential transforms under a gauge variation as usual:
\[A^{++'} = e^\lambda D^{++} e^{-\lambda} \Rightarrow \delta \lambda A^{++} = -D^{++} \lambda, \tag{3.11}\]

where \(\lambda\) is an analytic gauge parameter. A convenient gauge is the Wess-Zumino gauge in which the prepotential has the following component expansion \[25\]:
\[
\frac{1}{2} D^0_\alpha D^{--}_\beta A^{++}_{\mid |} = a_{\alpha \beta}; \quad \frac{1}{2}(D^0_\beta)^2 D^{--}_\alpha A^{++}_{\mid |} = 2\lambda_\alpha; \quad \frac{1}{2} D^0_\alpha (D^{--}_\beta)^2 A^{++}_{\mid |} = 3\chi^{-\alpha}_{\mid |}; \quad \frac{1}{2}(D^0_\alpha)^2 (D^{--}_\beta)^2 A^{++}_{\mid |} = 3X^{-\alpha}. \tag{3.12}\]
This is clearly the \(\mathcal{N} = 3\) vector multiplet with fields \((a_\mu, \lambda_\alpha, \chi^{ij}_\alpha, \phi^{ij}, X^{ij})\). Of course, \(\phi^{-\alpha} = u^i_\alpha u^j_\beta \phi^{ij}\) and so on.

It is now possible to write every other connection and field strength in terms of the analytic prepotential \(A^{++}\). We start with the connections:
\[
D^0 = [D^{++}, D^{--}] \Rightarrow A^{--}(u) = \sum_{n=1}^{\infty} (-1)^n \int du_1, ..., u_\alpha \frac{A^{++}_1 \cdots A^{++}_n}{(u^+ u^1_\alpha)(u^1_\alpha u^2_\beta) \cdots (u^+ u^\alpha)}. \tag{3.13}\]

\[2D^0_\alpha = [D^{--}, D^{++}_\alpha] \Rightarrow A^0_\alpha = -\frac{1}{2} D_{\alpha}^{++} A^{--}, \tag{3.13}\]
\[D^{--}_\alpha = [D^{--}, D^0_\alpha] \Rightarrow A^{--}_\alpha = D^{--} A^0_\alpha - D^{--}_{\alpha} A^{--} + [A^{--}, A^0_\alpha]. \]
\[-iD_{\alpha \beta} = \{D^0_\alpha, D^0_\beta\} \Rightarrow A_{\alpha \beta} = 2i D^0_\alpha A^0_\beta - iD^0_\alpha D^{++}_\beta A^{--}. \tag{3.13}\]
Then the covariant field strengths can be derived from the connections as follows:

\[ D^{\alpha +} W^{++} = D^{++} W^{++} = 0 \quad \Rightarrow \quad W^{++} \text{ is analytic.} \]

\[ W^{++} = \frac{1}{2} D^{\alpha +} A^0_\alpha = -\frac{1}{4} (D^{++}_\alpha)^2 A^{-} . \tag{3.14} \]

\[ W^0 = \frac{1}{2} D^{-} W^{++} \quad \text{and} \quad W^{-} = D^{-} W^0 . \]

The \( \mathcal{N} = 3 \) matter multiplet consists of two complex scalars \( f^i \) transforming as a doublet under \( SU(2) \) and their fermionic partners \( \bar{\psi}_i^\alpha \), which are encoded in the following hypermultiplet superfield:

\[ q^+_i = f^i; \quad D^{-}_\alpha q^+_i = \bar{\psi}_i^\alpha; \quad \frac{1}{2} D^{\alpha 0}_\beta D^{-}_{\bar{\beta}} q^+_i = -i \partial_{\alpha \beta} f^i , \]

\[ \bar{q}^+_i = -\bar{f}^i; \quad D^{-}_{\bar{\alpha}} \bar{q}^+_i = \bar{\psi}_i^{\bar{\alpha}}; \quad \frac{1}{2} D^{\alpha 0}_{\bar{\beta}} D^{-}_{\bar{\beta}} \bar{q}^+_i = i \partial_{\alpha \bar{\beta}} \bar{f}^i . \tag{3.15} \]

where \( f^\pm \equiv u_i^{\pm} f^i, \bar{f}^\pm \equiv u_i^{\pm} \bar{f}^i \), and similarly for the fermions.

For the ABJM theory, we have two sets of \( q^{\pm a} \), with \( a = 1, 2 \). In this representation, the \( SO(6) \) R-symmetry is broken: \( SO(6) \to SU(2)_R \times SU(2)_{\text{ext}} \), and the ABJ(M) action for \( U_L(N) \times U_R(M) \) theory:

\[ S = S_{CS}[A_L^{++}] - S_{CS}[A_R^{++}] + \int d^4 \zeta (-4) q^{+a}_\alpha D^{++} q^{+a} , \]

\[ S_{CS}[A^{++}] = \frac{i k}{4 \pi} \text{tr} \sum_{n=2}^\infty \frac{(-1)^n}{n} \int d^3 x d^3 \theta d u_1 \cdots d u_n \frac{A_1^{++} \cdots A_n^{++}}{(u_1^+ u_2^+ \cdots (u_n^+ u_1^+))} , \tag{3.16} \]

\[ (D^{++} q^{+a})_A^B = D^{++}(q^{+a})_A^B + (A_L^{++})_A^B (q^{+a})_B^B - (q^{+a})_A^B (A_R^{++})_A^B , \tag{3.17} \]

where \( A \in U(N), \bar{A} \in U(M) \), and \( q^{+a} \) have ‘opposite’ gauge charges under the two gauge groups. From the action, one finds the following equations of motion:

\[ \frac{\delta S}{\delta q^{+a}_\alpha} = \nabla^{++} q^{+a} = 0 , \quad \frac{\delta S}{\delta A^{++}} = W^{++} + \frac{4 \pi i}{k} q^{+a}_\alpha q^{+a} = 0 . \tag{3.18} \]

with proper ordering of \( \bar{q} q \) to match the gauge indices of \( W^{++}_{L,R} \). The latter equation of motion implies that scalars from the vector multiplet get equated to bi-scalars of the matter multiplet. One such relation will be relevant for later use:

\[ W^0 = \frac{1}{2} D^{-} W^{++} \Rightarrow \phi^0 = -\frac{2 \pi i}{k} u_i^{-}\frac{\partial}{\partial u_i^+} ((-u_j^+ f_a^j)(u_k^+ f^ka)) = \frac{2 \pi i}{k} (u_j^- u_k^+ + u_j^+ u_k^-) \bar{f}_a^j f^ka . \tag{3.19} \]

This crucial relation is responsible for generating the well-known sextic potential involving \( f^i \)’s once \( \phi^0 \)’s are integrated out from the ABJM action.

The CS theories coupled to matter can be quantized directly in superspace [25] and the resulting propagators read:

\[ \langle q_1^+ q_2^+ \rangle = \frac{1}{2 \pi i} \frac{u_1^+ u_2^+}{\sqrt{2 \rho^2}} , \tag{3.20} \]

\[ \langle A_1^{++} A_2^{++} \rangle = -\frac{i}{2 \pi} \frac{1}{\sqrt{2 \rho^2}} \delta^{2}(\theta^{++}_{12}) \delta^{(-2,2)}(u_1, u_2) . \tag{3.21} \]
Simons theories $N$ There are two main types of Wilson loop operators that can be considered for $4$ Super-Wilson Loop

where $\rho^\alpha{}^\beta = x^\alpha_A - x^\alpha_B - 2i\theta^0(\alpha{}^\theta^0)\beta - \frac{2i}{u^1_u^2}(u^-_1u^-_2)\theta_1^{++}(\alpha{}^\theta^0)\beta - (u^-_1u^+_2)\theta_1^{++}(\alpha{}^\theta^0)\beta$\n
- $(u^+_1u^-_2)\theta_1^{++}(\alpha{}^\theta^0)\beta + (u^-_1u^+_2)\theta_1^{++}(\alpha{}^\theta^0)\beta + (u^-_1u^-_2)\theta_2^{0}(\alpha{}^\theta^0)\beta$.

The $\rho^\alpha{}^\beta$ has quite a complicated expression but in the presence of $\delta^2(\theta^+\theta^-)\delta(u_1, u_2)$ it simplifies in the vector propagator to the following:

$\rho^\alpha{}^\beta = (x^\alpha_A)_{12} - 2i\theta^0(\alpha{}^\theta^0)\beta \Rightarrow \rho^2 = -2|x_{12}^A|^2 - 4ix_{12}^A \cdot \theta^0\theta_2^2 + 4\theta^0\theta_2^2$.

The vertices are easily read from the relevant actions.

4 Super-Wilson Loop

There are two main types of Wilson loop operators that can be considered for $d = 3$ Chern-Simons theories $[12, 16, 17, 20]$; GY-type ($\frac{1}{2}$-BPS for $N = 2, 3, 4, 6$) and DT-type (still $\frac{1}{2}$-BPS for $N = 2, 3$ but $\frac{1}{3}$-BPS for $N = 4, 6$). We will focus only on the former case here. The $\frac{1}{3}$-BPS Wilson loop is usually written for $N = 3$ CS theory as follows:

$\mathcal{W}_{ij}(x) = \frac{1}{\dim R} \text{tr}_R \mathcal{P} \exp \int d\tau \left[ \frac{i}{2} \dot{x}^\alpha{}^\beta a_\alpha{}^\beta + \frac{1}{2} \dot{y}^{ij} \phi^{ij} \right]$, \n
(4.1)

where $y_{ij} = y_{ji}$ are $3$ $SU(2)$ ‘coordinates’. For this operator to locally preserve any supersymmetry, the susy parameter $\epsilon^a_i$ needs to be a solution of

$$\dot{x}^\alpha{}^\beta \epsilon^a_\beta + \dot{y}^{ij} \epsilon^{a, kj} = 0,$$

(4.2)

provided that $|\dot{x}| = |\dot{y}|$. To incorporate the condition on $|\dot{y}|$, we can rewrite the scalar term in WL as $\int d\tau |\dot{x}|(u^+_1u^-_2)\phi^{(ij)}$ using the harmonic coordinates on $SU(2)$.

Now, we are ready to write down the most general supersymmetrized expression for a Wilson loop (such that (4.1) is its bosonic component):

$\mathcal{W}(x, \theta^{\pm}, \theta^0) = \frac{1}{\dim R} \text{tr}_R \mathcal{P} \exp \int d\tau \left[ \frac{1}{4} \dot{x}^A{}^\alpha{}^\beta A_\alpha{}^\beta + \dot{\theta}^{\alpha+} \theta^\alpha A^0 + \sum \dot{u}_i^+ u_i^- A^{++}$$

$$+ \frac{1}{2} |\dot{x}^A|W^0 \right]$. \n
(4.3)

The usual BPS condition on the bosonic WL ($\epsilon^{(ij)}(Q^{(ij)}\mathcal{W}_{ij}(x) = 0$), which results in (4.2), translates to $\epsilon^{(ij)}(\gamma)D^{(ij)}\mathcal{W}_{ij} = 0$ (along with $\dot{x} \rightarrow \dot{x}^A$) in superspace for obvious reasons (see [29] for an explicit proof). Let us see what that implies for (4.3):

$$\epsilon^{(ij)}(\gamma)D^{(ij)}\mathcal{W}(x, \theta^{++}, \theta^0) \propto \int d\tau \left[ \frac{1}{4} \dot{x}^A{}^\alpha{}^\beta \epsilon^{(ij)}(\gamma) F^{(ij)}(\gamma)_{\gamma, \alpha} + \dot{\theta}^{\alpha+} \epsilon^{(ij)}(\gamma) F^{(ij)}(\gamma, \alpha)$$

$$+ \sum_0^{(ij), \gamma, \alpha = 0} + \frac{1}{2} |\dot{x}^A|\epsilon^{(ij)}(\gamma)D^{(ij)}W^0 \right]$$

where we use $F_{A,B}$ to represent the field strength arising from the (anti-)commutator of $\{D_A, D_B\}$. As we did for the case of $N = 2$ WL, we have ignored here terms that look like
field-dependent gauge transformations. Since we know that only \( F_{\gamma,\alpha}^{\pm,\pm} = F_{\gamma,\alpha}^{\pm,\mp} = 0 \), we can have only one of the \( \theta \) terms above in the Wilson loop. This means either \( \epsilon^{+} \) or \( \epsilon^{0} \) can be the only unbroken susy. However, choosing \( \epsilon^{0} \), we find that \( F_{\gamma,\alpha}^{0,0} \) contains not only the \( D_{\alpha}^{-} W^{0} \) term but also \( D_{\alpha}^{+} W^{--} \) so the above variation cannot vanish. Thus we are left with \( \epsilon^{++} \) and the remaining couple of terms do vanish in this case because

\[
F_{\gamma,\alpha}^{-} = -i \left( \epsilon_{\alpha} D_{\gamma}^{--} W^{0} + \epsilon_{\gamma} D_{\alpha}^{--} W^{0} \right),
\]

which implies

\[
- \frac{1}{2} i A_{\alpha}^{\pm} A_{\beta}^{\mp} F_{\gamma,\alpha}^{-} + |x^{4}| \epsilon^{++} D_{\gamma}^{--} W^{0} = \epsilon^{++} \left( i \dot{x}^{A,\alpha} \epsilon_{\gamma} D_{\beta}^{--} W^{0} + |x^{4}| D_{\gamma}^{--} W^{0} \right)
= i \epsilon^{++} \left( \dot{x}^{A,\alpha} - i |x^{4}| \epsilon_{\alpha}^{\beta} \right) D_{\beta}^{--} W^{0} = 0. \tag{4.5}
\]

This expression vanishes (for arbitrary \( W^{0} \)) in a way similar to the \( \mathcal{N} = 2 \) case, and we preserve half of the complex spinor \( \epsilon^{++} \). Thus, the final result for the supersymmetric generalization of the \( \frac{1}{6} \)-BPS Wilson loop is:

\[
W_{1/3} = \frac{1}{\dim R} \text{tr}_{R} \mathcal{P} \exp \int d\tau \left[ -\frac{1}{2} i A_{\alpha}^{\pm} A_{\beta}^{\mp} + \dot{\theta}^{++} \theta_{\alpha}^{--} + \sum_{\pm} u^{\pm}_{i} i u^{\pm}_{i} A^{\pm} + \frac{1}{2} |x^{4}| W^{0} \right]. \tag{4.6}
\]

To compare with the usual bosonic WL operator, we write the above in component fields

\[
W_{1/3} \sim \text{tr}_{R} \mathcal{P} \exp \int d\tau \left[ -\frac{1}{2} i \dot{\lambda}_{\alpha}^{\beta} + \dot{\theta}^{++} \theta_{\alpha}^{--} \phi^{0} + \frac{1}{2} |x^{4}| \phi^{0} + O(\theta^{2}) \right]. \tag{4.7}
\]

The difference starts at terms of order \( \theta \) containing fermionic fields \( \lambda_{\alpha}^{\beta} \) and at \( \theta^{2} \) order with bosonic fields \( \phi^{ij} \). Higher-order terms will contain \( \lambda_{\alpha}, X^{ij} \) fields too.

With this construction, we can readily give the supersymmetrized generalization of the \( \frac{1}{6} \)-BPS WL operator for \( \mathcal{N} = 6 \) ABJM theory in \( \mathcal{N} = 3 \) harmonic superspace:

\[
W_{1/6} = \frac{1}{\dim R} \text{tr}_{R} \mathcal{P} \exp \int d\tau \left[ -\frac{1}{2} i A_{\alpha}^{\pm} A_{\beta}^{\mp} + \dot{\theta}^{++} \theta_{\alpha}^{--} \phi^{0} + \frac{1}{2} |x^{4}| \phi^{0} + O(\theta^{2}) \right] \left[ L \rightarrow R \right]. \tag{4.8}
\]

This operator reduces to the canonical bosonic operator in ABJM theory with the matter coupling term \( M^{ij}_{C} C_{j} \bar{C}^{i} \) where \( M^{ij}_{C} = \text{diag}(-1, -1, 1, 1) \) (up to the factor \( \frac{2}{7} \)) if the \( u \)-matrix further satisfies \( u_{+}^{*} = (u_{-})^{-1} = u(\tau) \). To show this, we need to use the equation of motion for \( A^{++} \) (3.19) and (3.20) along with a change of notation from \( f \rightarrow C_{j} \) as discussed in [24]. (Without the constraint on \( u \), this operator has more content due to \( W^{0} \) containing not only \( \phi^{12} \equiv M^{ij}_{C} C_{j} \bar{C}^{i} \) but also \( \phi^{11} \) and \( \phi^{22} \).)

## 5 Computation

In this section, we will compute the ‘one-loop’ vacuum expectation value of the Wilson loop \( W_{1/3} \). The constraint on \( u \) will also be imposed so the operator and the expected vev slightly
simplify (with $R$ being the fundamental representation of $U(N)$ gauge group):

$$W_{ieta} = \frac{1}{N} \text{tr}_R \mathcal{P} \exp \int d\tau \left[ \frac{1}{2} \dot{x}^{A,\alpha\beta} A_{\alpha\beta} + \dot{\theta}^{+,\alpha} A_{\alpha}^- + \frac{1}{2} |\dot{x}^4|^2 W^0 \right]$$

$$\langle W_{i\beta} \rangle = 1 + \frac{1}{2N} \int d\tau_1 d\tau_2 \left( \left( \frac{1}{2} \dot{x}^A \cdot \dot{\theta}^{++} \cdot A^- + \frac{1}{2} |\dot{x}^4|^2 W^0 \right) \right) (\cdots)^2 \left( \cdots \right) + \cdots .$$

An important subtlety that occurs repeatedly in the computation is when $D^{++}(u) = u^i u^j D^i_j$ acts on an analytic superfield which depends on another harmonic variable, say $(\theta^{+,+}, \theta^0, u')$. The result is not the naive zero since using $(u^+ u^-) = 1$ and repeated Schouten identities, one can rewrite:

$$D^{++}(u) = (u^+ u^-)^2 u^i u^j D^i_j = ((u^+ u^-)^2 D^{++}(u')) + (u^+ u')^2 D^{--}(u') - 2(u^+ u^-)(u^+ u') D^0(u') \right),$$

where we have converted the harmonic dependence of the derivative from $u$ to $u'$. Note that the charges match on both sides separately for $u$'s and $u$'s. Thus for any analytic superfield $\Phi(\theta^{+,+}, \theta^0, u')$, we have:

$$D^{++}(u) \Phi(u') = (u^+ u')^2 \left[ D^{--} \Phi \right] (u') - 2(u^+ u^-)(u^+ u') \left[ D^0 \Phi \right] (u').$$

Similar manipulations lead to the following list of identities:

$$(D^{++}(u))^2 \Phi(u') = (u^+ u')^2 \left( (u^+ u')^2 \left[ (D^{--})^2 \Phi \right] (u') - 4(u^+ u^-)(u^+ u') \left[ D^{--} D^0 \Phi \right] (u') + (u^+ u^-)^2 \left[ (D^0)^2 \Phi \right] (u') \right),$$

$$D^0(u) \Phi(u') = (u^+ u')^2 (u^- u') \left[ D^{--} \Phi \right] (u') + (1 - 2(u^+ u')(u^- u')) \left[ D^0 \Phi \right] (u') \right),$$

$$D^{-}(u) \Phi(u') = (u^- u')^2 \left[ D^{--} \Phi \right] (u') - 2(u^- u')(u^- u') \left[ D^0 \Phi \right] (u').$$

For the sake of convenience, we list generating expressions for component expansions of some connections and field strengths below (that is, keeping only a single $A^{++}$ in (3.13) and (3.14)):

$$A_{\alpha\beta}(u) = -i \left[ D^0_{(\alpha} D^{++}_{\beta)} A^{-} \right] (u) = i \int du' \left[ D^0_{(\alpha} D^{--}_{\beta)} A^{++} \right] (u') \right),$$

$$A^-_{\alpha}(u) = -\frac{1}{2} \left[ (D^- - D^+ + 2D^0) A^- \right] (u) = \frac{1}{2} \int du' \left[ \frac{u^- u'}{u^+ u^+} \left[ D^- A^{++} \right] (u') + 2 \frac{u^- u'}{u^+ u^+} \left[ D^0 A^{++} \right] (u') \right) \right),$$

$$W^{++}(u) = -\frac{1}{4} \left[ (D^{++})^2 A^- \right] (u) = -\frac{1}{4} \int du' \left[ (u^+ u')^2 \left[ (D^{-})^2 A^{++} \right] (u') - 4(u^+ u^-)(u^+ u') \left[ D^- D^0 A^{++} \right] (u') \right).$$

Note that all the fields depend on the same $\theta$-coordinate. The components can be obtained from the above expressions by using (3.12) and performing not only the $D$-algebra but some harmonic algebra too. The simplest component to obtain is the vector: $A_{\alpha\beta} = 2ia_{\alpha\beta}$. To get the scalars, we need to perform slightly more involved algebra:

$$W^{++} = -\frac{1}{4} \int du' (u^+ u')^2 \left[ (D^{-})^2 A^{++} \right] (u') = 3 \int du' (u^+ u')^2 \phi^-(u') = u^+_j u^-_k \delta^{(jk)} = \phi^{++};$$

$$W^0 = \frac{1}{2} D^- W^{++} = \frac{1}{2} (u^- u^+_k + u^+_j u^-_k) \delta^{(jk)} = \phi^0;$$

$$W^- = D^- W^0 = \phi^-.$$  

(5.7)
Other components can be similarly obtained, which we leave as an exercise and refer the reader to [30] for useful identities involving harmonic variables.

Now we turn to evaluating various contributions to $\langle \mathcal{W}_{1/2} \rangle$. First let us consider the contribution from the vector connection. In general, we have from (3.13):

$$
x_{A,1}^\alpha x_{A,2}^\delta \langle A_{1,\alpha \beta} A_{2,\gamma \delta} \rangle = \frac{i}{2} \varepsilon_{\alpha \beta \gamma \delta} \langle D_{1\alpha}^{0} D_{1\beta}^{0} A_{-}^{-} D_{2\gamma}^{0} D_{2\delta}^{0} A_{+}^{+} \rangle
$$

Using (5.6), we find

$$
x_{A,1}^\alpha x_{A,2}^\delta \langle A_{1,\alpha \beta} A_{2,\gamma \delta} \rangle = \frac{i}{2} \varepsilon_{\alpha \beta \gamma \delta} \int du \left[ D_{1\alpha}^{0} D_{1\beta}^{0} A_{-}^{-} \right] \left( \frac{u}{u_{1}} \right) \int dv \left[ D_{2\delta}^{0} D_{2\gamma}^{0} A_{+}^{+} \right] \left( \frac{v}{v_{2}} \right) (u_{1} v_{2})^{2} + \cdots
$$

We used here $D_{a}^{-} D_{b}^{-} = \frac{1}{2} \varepsilon_{ab} D^{2}$, $D_{-}^{-} \delta^{2}(\theta^{+}) = 4$, $x_{A,1}^\alpha x_{A,2}^\delta \varepsilon_{\beta \gamma} x_{A,12,\alpha \gamma} \sim \varepsilon_{mnp} x_{A,1}^n x_{A,2}^p x_{A,12} \to 0$, and expanded $\frac{1}{\rho^2}$ in powers of $\theta^0$'s. The next term (quadratic in $A^{++}$) in the expansion of $A^{-}$ also contributes:

$$
x_{A,1}^\alpha x_{A,2}^\delta \langle A_{1,\alpha \beta} A_{2,\gamma \delta} \rangle = \frac{x_{A,1} \cdot x_{A,2}}{4\pi^{2}|x|^{6}} \frac{i x_{A,1} \cdot \theta_{12}^{0} (v_{1}^{0} + v_{2}^{0})^{2}}{4\pi^{2}|x|^{6}}.
$$

Let us sketch how we got this result. We require that all $\delta^{2}(\theta_{12}^{+})$ be cancelled so higher orders of $A^{++}$ cannot contribute as there are not enough $D_{a}^{-}$ derivatives in $\langle A_{1,\alpha \beta} A_{2,\gamma \delta} \rangle$ to cancel more than two such $\delta$-functions. After expanding $D_{1\alpha}^{0} D_{1\beta}^{0}$ using the identities given above, doing two harmonic integrals using the harmonic $\delta$-functions in the two propagators and then hitting the two $\delta^{2}(\theta_{12}^{+})$ with correct $D^{-}$'s, we are left with:

$$
\langle A_{1,\alpha \beta} A_{2,\gamma \delta} \rangle \sim \int dudv \frac{(-u_{1}^{+} v_{2}^{+})^{2}(w^{+} u_{1}^{+})(w^{+} u_{1}^{+}) + (u_{1}^{+} w^{+})^{2}(v^{+} u_{1}^{+})(v^{+} u_{1}^{+})}{(u_{1}^{+} v_{2}^{+})(w^{+} u_{1}^{+})(w^{+} u_{2}^{+})(w^{+} u_{2}^{+})} \frac{i}{2\rho^{2}}
$$

$$
\sim \int dudv \frac{(-u_{1}^{+} v_{2}^{+})^{2}(w^{+} u_{1}^{+})(w^{+} u_{2}^{+})}{(u_{1}^{+} v_{2}^{+})(w^{+} u_{1}^{+})(w^{+} u_{2}^{+})(w^{+} u_{2}^{+})} \frac{i}{2\rho^{2}}
$$

$$
\sim \int dudv \frac{(-u_{1}^{+} v_{2}^{+})^{2}(w^{+} u_{1}^{+})(w^{+} u_{2}^{+})}{(u_{1}^{+} v_{2}^{+})(w^{+} u_{1}^{+})(w^{+} u_{2}^{+})(w^{+} u_{2}^{+})} \frac{i}{2\rho^{2}}
$$

$$
\sim \int dudv \frac{(-u_{1}^{+} v_{2}^{+})^{2}(w^{+} u_{1}^{+})(w^{+} u_{2}^{+})}{(u_{1}^{+} v_{2}^{+})(w^{+} u_{1}^{+})(w^{+} u_{2}^{+})(w^{+} u_{2}^{+})} \frac{i}{2\rho^{2}}
$$

$$
\sim \frac{\varepsilon_{\alpha \gamma} \varepsilon_{\beta \delta}}{\rho^{2}} = \frac{\varepsilon_{\alpha \gamma} \varepsilon_{\beta \delta}}{|x_{12}|^{2}} \frac{|x_{A,1}^{2}|^{2} - i x_{A,1}^{A} \cdot \theta_{12}^{0} \theta_{12}^{0} + \frac{1}{3} \theta_{12}^{0} \theta_{12}^{0} |x_{A,1}^{2}|^{2}}{|x_{A,1}^{2}|^{6}}.
$$

Keeping track of various signs and numerical factors above, we get (5.10).

Let us now evaluate the second contribution to $\langle \mathcal{W} \rangle$ due to the ‘charged’ fermionic connection. Using the fact that we need enough $D_{a}^{-}$ to get relevant terms, we ignore terms with $D_{a}^{0}$
in the expansion of $A^{--}$ in (5.6):

$$
\dot{\theta}_1^{++} \dot{\theta}_2^{++} \left\langle A_1^{--} A_2^{--}\right\rangle = \dot{\theta}_1^{++} \dot{\theta}_2^{++} \int du \left( \frac{u^+ u^1}{u^+ u_1^+} \right) D_{1}^{--} \left( \frac{u^+ u_2^+}{u^+ u_1^+} \right) D_{2}^{--} \frac{i \delta^2(\theta^{++})}{2\pi \sqrt{2\rho^2}}
$$

$$
= \dot{\theta}_1^{++} \dot{\theta}_2^{++} \left( \frac{u^+_1 u^+_2}{u^+_1 u_2^+} \right) \left| x_{12}^{A^2} - i x_{12}^{A^2} \cdot \theta_1^0 \theta_2^0 + \frac{1}{4} \theta_1^0 \theta_2^0 \theta_3^0 \theta_4^0 \right|.
$$

(5.11)

The $u$-factor in parentheses might look divergent upon imposing the constraint on $u$-matrix discussed in the previous section but using an explicit parameterization, one can show that it instead limits to unity up to a $U(1)$ ‘charge factor’. We will, however, leave this factor as it is to account for the correct $U(1)$ charges along with an understanding that there is no non-trivial $u$-dependence.

The third contribution to $\langle W \rangle$ due to mixed contraction of the two connections vanishes:

$$
\langle A_1,\alpha A_2^{--}\rangle = -i \int du \left[ D_{0}^{--}(u^+ u^1) \right] \frac{i \delta^2(\theta^{++})}{2\pi \sqrt{2\rho^2}} = 0.
$$

(5.12)

The fourth contribution to $\langle W \rangle$ due to the scalar field strength is

$$
|x_{A1}||x_{A2}| \langle W^0 W^0 \rangle^{(1)} = \frac{1}{12} |x_{A1}| |x_{A2}| \left( \frac{\theta^0_1 \theta^0_2 + \theta^0_1 \theta^0_2 + \theta^0_2 \theta^0_2}{2\pi |x_{12}^{A^2}|^3} \right)(1 - 2(u^+_1 u^+_2)(u^-_1 u^-_2)).
$$

(5.13)

This is a contribution from the linear term in $A^{--}$ and is straightforward to compute. Like $\langle A_0 A_0 \rangle$, we get a second contribution from the contraction of quadratic terms in $A^{--}$ here too:

$$
|x_{A1}||x_{A2}| \langle W^0 W^0 \rangle^{(2)} = \frac{1}{4\pi^2 |x_{12}^{A^2}|^6} \left( u_{12}^{A^2} - 2(u^+_1 u^+_2)(u^-_1 u^-_2) \right).
$$

(5.14)

This computation proceeds very similarly to the case of the vector connection but there are more terms; we sketch them below (again, various signs and numerical factors need to be tracked):

$$
\langle W^0 W^0 \rangle^{(2)} \sim \int dv_{1,2} \frac{D_{1}^{-+} D_{1}^{++}}{u^+_1 v^+_1 (v^+_1 v^-_2)(v^-_2 u^+_1)} \int dv_{1,2} \frac{D_{2}^{-+} D_{2}^{++}}{u^+_2 v^+_1 (v^+_1 v^-_2)(v^-_2 u^+_2)}
$$

$$
\sim D_{1}^{-+} D_{2}^{-+} \int dv_{1,2} \frac{D_{1}^{--} D_{2}^{--}}{u^+_1 v^+_1 (v^+_1 v^-_2)(v^-_2 u^+_1)(u^-_2 v^+_1)(v^-_2 u^-_2)}
$$

$$
\sim D_{1}^{-+} D_{2}^{-+} \int dv_{1,2} \frac{D_{1}^{--} D_{2}^{--}}{u^+_1 v^+_1 (v^+_1 v^-_2)(v^-_2 u^+_1)(u^-_2 v^+_1)(v^-_2 u^-_2)}
$$

$$
\sim D_{1}^{-+} D_{2}^{-+} \int dv_{1,2} \frac{D_{1}^{--} D_{2}^{--}}{(u^+_1 v^+_1)^2 (u^+_1 v^-_2)^2 (v^-_2 u^+_1)^2 (v^-_2 u^-_2)^2}
$$

$$
\sim (2 - 4(u^+_1 u^+_2)(u^-_1 u^-_2)) \frac{1}{\rho^2}.
$$

Similarly, we can compute two more mixed contractions between the two connections and $W^0$,
but only one is non vanishing:

\[
\langle A^{\alpha}_{1\omega} W^0_2 \rangle = - \frac{(ix^1_2 \cdot \theta^1_1 - \frac{i}{2} \theta^0_1 \theta^0_2)^a}{2\pi |x^A_{12}|^3} (u^+_1 u^+_2) (u^-_1 u^-_2).
\]

(5.15)

One more contribution to \( \langle W \rangle \) needs to be considered (at the order being studied) and this one includes a 3-point vertex insertion:

\[
\begin{align*}
\langle d\tau_1 \hat{x}_{A,1} \cdot A_1 d\tau_2 \hat{x}_{A,2} \cdot A_2 d\tau_3 \hat{x}_{A,3} \cdot A_3 \rangle \\
= -i \int d\tau_1,2,3 \hat{x}_{A,1}^\alpha \hat{x}_{A,2}^\beta \hat{x}_{A,3}^\lambda \langle d\tau_{1,2,3} (D^n_0 D^\beta_\gamma - A^{++})_1 (D^n_0 D^\gamma_\delta - A^{++})_2 (D^n_0 D^\lambda_\kappa - A^{++})_3 \\
\approx i \int d^3 \tau \cdot \int d^3 v (D^n_0 D^\beta_\gamma - A^{++})_1 (D^n_0 D^\gamma_\delta - A^{++})_2 (D^n_0 D^\lambda_\kappa - A^{++})_3 \int d^3 x_0 \delta(v_0, x_0) \\
\approx \int d^3 \tau \cdot \int d^3 v \left( \frac{v^1_3 v^2_3}{v^1_0 v^2_0} \right) \left( \frac{v^3_0 v^3_0}{v^3_0 v^3_0} \right) \left( \frac{v^3_0 v^3_0}{v^3_0 v^3_0} \right) \int d^3 x_0 \delta(v_0, x_0) \\
\times \int d^3 x_0 \left( \frac{i (x_{01})_0 (x_{02})_0 (x_{03})_0}{|x_0|^3} + \frac{i (x_{01})_1 (x_{02})_1 (x_{03})_2}{|x_0|^3} + \cdots + \frac{i (x_{01})_3 (x_{02})_3 (x_{03})_0}{|x_0|^3} \right).
\end{align*}
\]

(5.16)

The second to last line is obtained after performing the \( \int d^3 \theta_0 \) in the previous line with the help of three \( \delta^2(\theta_0^{+}) \)'s, cancelling the divergent harmonic denominator. The last line then follows by converting \( D^n_0 \rightarrow D_{+1,1} \) in cyclic order and acting on the numerator, thus picking up eight terms. The \( \int d^3 v \) integral produces only a numerical factor. Note that the first term in the integral \( \int d^3 x_0 \) is the only ‘bosonic’ piece given by the well-known integral (6.12) of [15].

Finally, collecting all the results at ‘one-loop’ order (we suppress \( u \)-dependent factors from \( \langle W^0 W^0 \rangle \) to keep the expression below manageable), we have

\[
\langle W_{i/\beta}(x, \theta^{+\pm}, \theta^{0}) \rangle = 1 + \frac{14 \pi N}{2} \frac{N}{k} \int d\tau_1 d\tau_2 \left\{ \frac{\hat{x}_{A,1}^\alpha \hat{x}_{A,2}^\beta \hat{x}_{A,3}^\gamma \epsilon_{\beta\gamma} \theta^0_1 \theta^0_2 + |\hat{x}_{A,1}| |\hat{x}_{A,2}| (\theta^0_1 \theta^0_2 + \theta^0_1 \theta^0_2 + \theta^0_2 \theta^0_2)}{2\pi |x^A_{12}|^3} \\
+ \frac{(u^+_1 u^+_2)}{(u^-_1 u^-_2)} \left( \frac{|x^A_{12}|^2 - ix_{A,2} \cdot \theta^0_1 \theta^0_2 + i \theta^0_1 \theta^0_2 \theta^0_2}{2\pi |x^A_{12}|^3} - \frac{i \theta^0_1 \theta^0_2 + i \theta^0_2 \theta^0_2}{2\pi |x^A_{12}|^3} \right) \hat{x}_{A,2} + |\hat{x}_{A,1}| \hat{x}_{A,2} \cdot (i x_{A,2} \cdot \theta^0_1 - \frac{i}{2} \theta^0_1 \theta^0_2) \right\} \\
- \frac{1}{2} \frac{16 \pi^2 N^2}{k^2} \int d\tau_1 d\tau_2 \left\{ \frac{|\hat{x}_{A,1} \cdot \hat{x}_{A,2} - |\hat{x}_{A,1}| \hat{x}_{A,2}|}{4\pi^2 |x^A_{12}|^2} \left( 1 - \frac{2ix_{A,2} \cdot \theta^0_1 \theta^0_2}{|x^A_{12}|^2} \right) \right\} + \int d\tau_3 \int d^3 x_0 \frac{\hat{x}_{A,1}^\alpha \hat{x}_{A,2}^\beta \hat{x}_{A,3}^\gamma (x_{01})_0 (x_{02})_0 (x_{03})_0}{4 \cdot 16 \pi^2 |x^A_{12}|^3 |x^A_{12}|^3}.
\]

(5.17)

6 Comments

We have constructed a \( \frac{1}{3} \)-BPS supersymmetrized Wilson loop operator in \( d = 3 \), \( \mathcal{N} = 3 \) harmonic superspace for CS theories. This operator readily generalizes the \( \frac{1}{6} \)-BPS operator for ABJM theories. We were also able to use the power of harmonic superspace to compute the ‘one-loop’ perturbative corrections directly in superspace.

Using the component expansion of \( \mathcal{N} = 3 \) connections and field strengths, and just focussing
on the localization locus discussed in Section 2 \((σ \equiv \phi^0 \text{ and } D \equiv X^0 = -\frac{2}{N})\), we can see that the \(\mathcal{W}_{i/β} \) given in (5.1) reduces to (2.5) once we identify \(θ, \bar{θ} \) with \(θ^{++}, θ^{--} \). Then one can expect that

\[
\langle \mathcal{W}_{i/β} \rangle = 1 + \frac{1}{2} \left[ \vartheta + \frac{\vartheta^2}{4} \right] α - \left[ \frac{1}{24} \left( 5 + \frac{1}{N^2} \right) + \frac{1}{4} \vartheta - \frac{1}{6} \left( \frac{5}{2} + \frac{1}{N^2} \right) \frac{\vartheta^2}{4} \right] α^2 + O(α^3). \tag{6.1}
\]

At order \(α^2 \), we can directly compare the ‘bosonic’ factor \(-\frac{5α^2}{24} \equiv \frac{5α^2N^2}{24}\) above to the corresponding perturbative expression in (5.17). They exactly match once we perform the integrals in the latter case for a circular WL, i.e., \(x^θ = (0, \sin(τ), \cos(τ))\) as one might expect. \(^6\)

Formally, both (5.17) and (6.1) have nonvanishing ‘fermionic’ contributions at \(O(α)\) and \(O(α^2)\). However, without knowing the explicit profile functions of \(θ(τ)\) and \(u(τ)\) we cannot proceed further. However, we can identify a contribution at \(O(α^2)\) that does not receive any \(O(\frac{1}{N})\) corrections! \(^7\) These are the \(O(θ\bar{θ})\) terms in the \(θ\) piece of (6.1) and comparing with (5.17), we can give an explicit expression for these terms:

\[
\vartheta |_{O(θ\bar{θ})} = \frac{2i}{π^2} \int dτ_1 dτ_2 \frac{(x_1 \cdot \dot{x}_2 - |\dot{x}_1||\dot{x}_2|) x_1^{αβ} x_2^{αβ} (θ_{1,α} \bar{θ}_{1,β} - θ_{2,α} \bar{θ}_{2,β})}{|x_1|^4} - \frac{i}{16π^3} \int dτ_{1,2,3} \dot{x}_1^{αβ} \dot{x}_2^{γδ} \dot{x}_3^{κλ} \int d^3 x_0 \frac{(x_{01})_{βγ} (x_{02})_{δκ} \bar{θ}_{3,α} \bar{θ}_{3,α} + 2 \text{ more terms}}{|x_0|^3 |x_0|^3 |x_0|^3}. \tag{6.2}
\]

The fermionic pieces from the term proportional to \((\dot{x}_{A,1} \cdot \dot{x}_{A,2} - |\dot{x}_{A,1}||\dot{x}_{A,2}|)\) do not contribute above because the combination \(θ\bar{θ}\) is independent of \(τ\) as discussed in Section 2. Such fermionic contributions to the Wilson loop operators do not seem to have been considered in \(d = 3\) but similar terms have appeared in the \(d = 4, N = 4\) SYM literature, specifically in the study of supersymmetrized Maldacena-Wilson loops \([29, 31]\). Thus, a careful study of the \(τ\)-dependence of the \(θ\) and \(u\) coordinates that is consistent with the ‘bosonic’ circular WL is required to understand how the general perturbative result (5.17) reduces to the simpler localization result (6.1) at various \(θ\) orders. \(^8\) We leave this exercise for future work.

One can also ask whether the construction of a \(\frac{1}{2}\)-BPS WL with ‘supermatrix’ structure \([20]\) is feasible in harmonic superspace. Our preliminary analysis suggests that the \(U(1)\) charge structure of the supersymmetry parameters \(ε^{±±}, ε^0\) and the matter superfield \(q^+\) is an obstruction for constructing a straightforward generalization. As mentioned in the Introduction, a motivation to study such supersymmetrized WL operators is to probe Wilson loops/Scattering amplitudes duality in ABJM theory. The expectation is that polygonal WL operators with certain bifundamental vertex insertions would be dual to the ABJM scattering amplitudes. The matter superfield \((q^{±α})^B_A\) in bifundamental representation provides a natural candidate for such insertions. However, this leads to some superficial divergences that need to be tamed. Progress on these aspects will be reported elsewhere.

\(^6\)We refer the readers to [15] for evaluation of the relevant integrals.

\(^7\)We do not have \(O(\frac{1}{N})\) terms at \(O(α)\) either, but that could still be treated as a phase factor. The striking feature of the term at \(O(α^2)\) is that it remains unchanged even after the removal of the \(O(α)\) phase:

\[
\langle \mathcal{W}_{i/β} \rangle = 1 - \left[ \frac{1}{24} \left( 5 + \frac{1}{N^2} \right) + \frac{1}{4} \vartheta + \frac{1}{6} \left( \frac{5}{2} + \frac{1}{N^2} \right) \frac{\vartheta^2}{4} \right] α^2 + O(α^3).
\]

\(^8\)This is the case when the conjectured equality (6.2) would hold and we assume that the consistency would require \(θ^0(τ)\) to vanish.
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