ANALYTIC EXTENSION OF JORGE-MEEKS TYPE MAXIMAL SURFACES IN LORENTZ-MINKOWSKI 3-SPACE

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Abstract. The Jorge-Meeks $n$-noid ($n \geq 2$) is a complete minimal surface of genus zero with $n$ catenoidal ends in the Euclidean 3-space $\mathbb{R}^3$, which has $(2\pi/n)$-rotation symmetry with respect to its axis. In this paper, we show that the corresponding maximal surface $f_n$ in Lorentz-Minkowski 3-space $\mathbb{R}^3_1$ has an analytic extension $\tilde{f}_n$ as a properly embedded zero mean curvature surface. The extension changes type into a time-like (minimal) surface.

Introduction

A number of zero mean curvature surfaces of mixed type in Lorentz-Minkowski three-space $(\mathbb{R}^3_1; t, x, y)$ were found in [9], [5], [10], [8], [3], [1] and [2]. One of the main tools for the construction of such surfaces is based on the fact that fold singularities of space-like maximal surfaces have real analytic extensions to time-like minimal surfaces (cf. [5], [8], [7] and [2]). Some of the analytic extensions of such examples have neither singularities nor self-intersections. A typical such example is a space-like helicoid, which analytically extends to a time-like surface, and the entire surface coincides with the original helicoid as a minimal surface in $\mathbb{R}^3$. Also, the Scherk type surface

$$t(x, y) := \log \frac{\cosh y}{\cosh x}$$

(1)

gives an entire graph which changes type from a space-like maximal surface to a time-like zero mean curvature surface, as pointed out by Kobayashi [9]. Recently, it was shown in [3] that the space-like maximal analogues in $\mathbb{R}^3$ of the Schwarz D surfaces in $\mathbb{R}^3_1$ have analytic extensions as triply periodic embedded zero mean curvature surfaces. These examples caused the authors to be interested in space-like maximal analogues $f_n$ ($n = 2, 3, \ldots$) of Jorge-Meeks minimal surfaces with $n$ catenoidal ends. These surfaces have fold singularities, and have analytic extensions to time-like surfaces. We show in this paper that the analytic extension of $f_n$ is a proper embedding.

1. Preliminaries

We denote by $(\mathbb{R}^3_1; t, x, y)$ the Lorentz-Minkowski 3-space of signature $(-, +, +)$ and denote the Riemann sphere by $S^2 := \mathbb{C} \cup \{\infty\}$. 

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Figure 1. The Jorge-Meeks trinoid in $\mathbb{R}^3$ and the analytic exten-
sion of $f_3$ (in the figure on the right-hand side, the time-like parts
are indicated by black shading).

Definition 1.1. A pair $(g, \omega)$ consisting of a meromorphic function and a meromor-
phic 1-form defined on the Riemann sphere is called a Weierstrass data on $S^2$ if the
metric
$$ds_E^2 := (1 + |g|^2)^2 |\omega|^2$$
has no zeros on $S^2$. A point where $ds_E^2$ diverges is called an end of $ds_E^2$.

We now fix a Weierstrass data $(g, \omega)$ on $S^2$ and let $\{p_1, \ldots, p_n\}$ be the set of
ends of $ds_E^2$. Then the real part of the map
$$F := \int_{z_0}^z \left( -2g, 1 + g^2, i(1 - g^2) \right) \omega \quad (i = \sqrt{-1})$$
is a map
$$f_L = \text{Re}(F),$$
into $\mathbb{R}^3$ which is defined on the universal cover of $S^2 \setminus \{p_1, \ldots, p_n\}$. We call $f_L$
the maximal surface associated to $(g, \omega)$, and $F$ the holomorphic lift of $f_L$. If $f_L$ is
single-valued on $S^2 \setminus \{p_1, \ldots, p_n\}$, then we say that $f_L$ satisfies the period condition.
The first fundamental form of $f_L$ is given by
$$ds^2 = (1 - |g|^2)^2 |\omega|^2.$$
As pointed out in [11] Example 5.7, the companion of \( f_n \) is congruent to the well-known complete minimal surface with catenoidal ends, called a Jorge-Meeks surface (cf. [6]). In particular, the associated metric \( ds^2_E \) is complete on

\[ S^2 \setminus \{1, \zeta, \ldots, \zeta^{n-1}\} \]

where \( \zeta := e^{2\pi i/n} \).

This means that \( (g_n, \omega_n) \) is a Weierstrass data on \( S^2 \). It can be checked that \( f_n \) is single-valued on \( S^2 \setminus \{1, \zeta, \ldots, \zeta^{n-1}\} \), and the original Jorge-Meeks surface is as well. So \( f_n \) is a maxface, and we call \( \{f_n\}_{n=2,3,\ldots} \) the Jorge-Meeks type maximal surfaces. The singular set of \( f_n \) is the set \( |z| = 1 \), which consists of generic fold singularities in the sense of [2], that is, the image of the singular set consists of a union of non-degenerate null curves in \( \mathbb{R}^3 \).

We now observe that \( f_2 \) has a canonical analytic extension embedded in \( \mathbb{R}^3 \) (see Figure 1.2): By definition,

\[ f_2 = \text{Re} \left( \frac{i}{z^2 - 1}, -\frac{i z}{z^2 - 1}, \frac{1}{2} \log \frac{1 - z}{1 + z} \right). \]

If we set \( f_2 = (x_0, x_1, x_2) \) and \( z = r e^{i \theta} \), then

\[
\begin{align*}
x_0 &= \frac{r^2 \sin 2\theta}{r^4 - 2r^2 \cos 2\theta + 1}, \\
x_1 &= -\frac{r (r^2 + 1) \sin \theta}{r^4 - 2r^2 \cos 2\theta + 1}, \\
x_2 &= \frac{1}{4} \log \left( \frac{r^2 - 2r \cos \theta + 1}{r^2 + 2r \cos \theta + 1} \right).
\end{align*}
\]

In particular, it holds that

\[ \frac{x_0}{x_1} = -\frac{2r \cos \theta}{r^2 + 1} = \tanh 2x_2. \]

Thus, the image of \( f_2 \) is a subset of the graph \( t = x \tanh 2y \) (Figure 1.2 left), and it changes type on the set

\[ S := \left\{ \left( \pm \frac{\cosh 2y}{2}, y \right) ; y \in \mathbb{R} \right\}, \]

in the \( xy \)-plane, and the connected domain with boundary \( S \) consists of the image of the orthogonal projection of \( f_2 \) into the \( xy \)-plane (cf. Figure 1.2 right). This means that the image of \( f_2 \) has an analytic extension that coincides with a zero
mean curvature entire graph, like as in the case of the Scherk type surface (1) in the introduction.

In [9], the maximal surface $f_2$ is called a helicoid of the 2nd kind and it was already pointed out that the function $t = x \tanh 2y$ is an entire solution to the maximal surface equation for graphs. The conjugate surface of $f_2$ induces a singly periodic maxface, called the hyperbolic catenoid (see [2] for details).

2. Analytic extension of $f_n$

In this section, we will show that each $f_n$ has a canonical analytic extension for $n \geq 3$ as well. By definition, the Jorge-Meeks type maximal surface $f_n = (x_0, x_1, x_2)$ and its holomorphic lift $F = (X_0, X_1, X_2)$ are given by

$$f_n = \Re(F), \quad F = \int_0^z \alpha,$$

where we set

$$\alpha = \alpha(z) = a(z) \, dz = \left( a_0(z), a_1(z), a_2(z) \right) \, dz$$

$$= \left(-\frac{2iz^{n-1}}{(z^n - 1)^2}, \frac{i \left(1 + z^{2n-2}\right)}{(z^n - 1)^2}, -\frac{1 - z^{2n-2}}{(z^n - 1)^2}\right) \, dz.$$

Using these expressions, we show the following:

**Proposition 2.1.** Regarding $f_n$ as a column vector-valued function, the image of $f_n$ has the following two properties:

$$f_n(\bar{z}) = S f_n(z), \quad f_n(\zeta z) = R f_n(z),$$

where

$$S := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \frac{2\pi}{n} & \sin \frac{2\pi}{n} \\ 0 & -\sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix}$$

and

$$\zeta := e^{2\pi i/n}.$$
Proof. Since
\[ R = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & \zeta + \zeta^{-1} & -i(\zeta - \zeta^{-1}) \\ 0 & i(\zeta - \zeta^{-1}) & \zeta + \zeta^{-1} \end{pmatrix}, \]
the 1-form \( \alpha \) in (2.1) satisfies
\[ \overline{\alpha(z)} = S\alpha(z), \quad \alpha(\zeta z) = R\alpha(z), \]
where \( \zeta \) is considered as a column vector-valued 1-form. Since \( F(0) = 0 \), we have \( F(\zeta) = SF(z) \) and \( F(\zeta z) = RF(z) \). In particular, we have the relations \( f_n(\zeta) = Sf_n(z) \) and \( f_n(\zeta z) = Rf_n(z) \).

Lemma 2.2. Up to a suitable translation in \( \mathbb{C}^3 \) by a vector in \( \mathbb{R}^3 \), the holomorphic lift \( F = (X_0, X_1, X_2) \) of the Jorge-Meeks type maximal surface \( f_n \) has the following expression:
\begin{align*}
(2.3) & \quad X_0 = \frac{2i}{n(z^n - 1)}, \\
(2.4) & \quad X_1 = -\frac{1}{n^2} \left( \frac{z(z^{n-2} + 1)}{n(z^n - 1)} + \sum_{j=1}^{n-1} (\zeta^j - \zeta^{-j}) \log(z - \zeta^j) \right), \\
(2.5) & \quad X_2 = -\frac{1}{n^2} \left( \frac{z(z^{n-2} - 1)}{n(z^n - 1)} + \sum_{j=0}^{n-1} (\zeta^j + \zeta^{-j}) \log(z - \zeta^j) \right). 
\end{align*}

Proof. The first identity (2.3) is obvious. To prove the second identity (2.4), we will show that differentiation of the right-hand side of (2.4) is equal to \( a_1(z) \). Denoting the right-hand side of (2.4) by \( \hat{X}_1 \), we have that
\[ \frac{d\hat{X}_1}{dz} - a_1(z) = \frac{d\hat{X}_1}{dz} - \frac{i(1 + z^{2n-2})}{(z^n - 1)^2} = -\frac{(n - 1) \varphi(z)}{n^2 (z^n - 1)}, \]
where we set
\[ \varphi(z) := n \left( z^{n-2} - 1 \right) + \sum_{j=1}^{n-1} \frac{(\zeta^j - \zeta^{-j})(z^n - 1)}{z - \zeta^j}. \]
For \( z = \zeta^k \) (\( k = 0, 1, 2, \ldots, n - 1 \)),
\[ \varphi(\zeta^k) = n \left( \zeta^{-2k} - 1 \right) + \sum_{j=1}^{n-1} \frac{(\zeta^j - \zeta^{-j})(z^n - 1)}{z - \zeta^j} \bigg|_{z = \zeta^k} \]
\[ = n \left( \zeta^{-2k} - 1 \right) + \left( \zeta^k - \zeta^{-k} \right) n\zeta^{-k} = 0. \]
Here we have used the following identity:
\[ \left. \frac{z^n - 1}{z - \zeta^j} \right|_{z = \zeta^k} = \begin{cases} 0 & \text{if } j \neq k \\ \frac{d}{dz} (z^n - 1) \bigg|_{z = \zeta^k} = n\zeta^{-k} & \text{if } j = k. \end{cases} \]
(2.6) means that the number of zeros for \( \varphi(z) \) is at least \( n \). However, \( \varphi(z) \) is a polynomial in \( z \) of degree at most \( n - 1 \). So we conclude that \( \varphi \) vanishes identically, and hence \( d\hat{X}_1/dz - a_1(z) = 0 \).
Proposition 2.3. The Jorge-Meeks type maximal surface\( \psi \) is obtained. Similarly, one can easily check that:

\[
\frac{dX_2}{dz} - a_2(z) = \frac{d\tilde{X}_2}{dz} + \frac{1 - z^{2n-2}}{(z^n - 1)^2} = -\frac{(n-1)\psi(z)}{n^2(z^n - 1)},
\]

where \( \psi(z) \) is a polynomial of degree at most \( n - 1 \) given by

\[
\psi(z) := n(z^{n-2} + 1) - \sum_{j=0}^{n-1} \frac{(\zeta^{-j} + \zeta^j)(z^n - 1)}{z - \zeta_j}.
\]

It can be easily checked that \( \psi(\zeta^k) = 0 \) for each \( k = 0, 1, 2, \ldots, n - 1 \). These prove that \( d\tilde{X}_2/dz - a_2(z) = 0 \), and thus (2.5) is verified. \( \square \)

Using Lemma 2.2, we obtain an integration-free formula of \( f_n \) as follows.

Proposition 2.3. The Jorge-Meeks type maximal surface \( f_n = (x_0, x_1, x_2) \) has the following expressions:

\[
\begin{align*}
(2.7) \quad x_0 &= \frac{2r^n \sin n\theta}{n(r^{2n} - 2r^n \cos n\theta + 1)}, \\
(2.8) \quad x_1 &= -\frac{(r^{2n-1} + r) \sin \theta + (r^{n+1} + r^{n-1}) \sin(n-1)\theta}{n(r^{2n} - 2r^n \cos n\theta + 1)} \\
&\quad + \frac{n-1}{n^2} \sum_{j=1}^{n-1} \log \left( r^2 - 2r \cos \left( \theta - \frac{2\pi j}{n} \right) + 1 \right) \frac{2\pi j}{n}, \\
(2.9) \quad x_2 &= -\frac{(r^{2n-1} + r) \cos \theta + (r^{n+1} + r^{n-1}) \cos(n-1)\theta}{n(r^{2n} - 2r^n \cos n\theta + 1)} \\
&\quad + \frac{n-1}{n^2} \sum_{j=0}^{n-1} \log \left( r^2 - 2r \cos \left( \theta - \frac{2\pi j}{n} \right) + 1 \right) \frac{2\pi j}{n},
\end{align*}
\]

where \( z = re^{i\theta} \).

Proof. Since

\[
x_0 = \Re X_0 = -\frac{2 \Im(z^n - 1)}{n(z^n - 1)(\bar{z}^n - 1)} = -\frac{2 \Im(r^n e^{-in\theta})}{n(r^{2n} - 2r^n \cos n\theta + 1)} = \frac{2r^n \sin n\theta}{n(r^{2n} - 2r^n \cos n\theta + 1)},
\]

the first identity (2.7) is obtained. Similarly, one can easily check that:

\[
\begin{align*}
\Re \left( \frac{-iz(z^{n-2} + 1)}{n(z^n - 1)} \right) &= -\frac{(r^{2n-1} + r) \sin \theta + (r^{n+1} + r^{n-1}) \sin(n-1)\theta}{n(r^{2n} - 2r^n \cos n\theta + 1)}, \\
\Re \left( \frac{-z(z^{n-2} - 1)}{n(z^n - 1)} \right) &= -\frac{(r^{2n-1} + r) \cos \theta + (r^{n+1} + r^{n-1}) \cos(n-1)\theta}{n(r^{2n} - 2r^n \cos n\theta + 1)}.
\end{align*}
\]

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On the other hand, 

\[
\Im \left( \sum_{j=1}^{n-1} (\zeta^j - \zeta^{-j}) \log(z - \zeta^j) \right) = \sum_{j=1}^{n-1} \frac{2\pi j}{n} \log|z - \zeta^j| \\
= \sum_{j=1}^{n-1} \sin \frac{2\pi j}{n} \log|z - \zeta^j|^2 = \sum_{j=1}^{n-1} \sin \frac{2\pi j}{n} \log((z - \zeta^j)(\bar{z} - \zeta^{-j})) \\
= \sum_{j=1}^{n-1} \log \left( r^2 - 2r \cos \left( \theta - \frac{2\pi j}{n} \right) + 1 \right) \sin \frac{2\pi j}{n},
\]

which proves (2.8). Similarly, we have (2.9). □

The following assertion is an immediate consequence of Proposition 2.3.

**Corollary 2.4.** \( f_n \) satisfies the identity

(2.10) \( f_n(1/r, \theta) = f_n(r, \theta) \quad (r > 0, \ 0 \leq \theta < 2\pi). \)

Since \( f(r, \theta) \) is invariant under the symmetry \( r \mapsto 1/r \), the singular set \( \{|z| = 1\} \) of \( f \) coincides with the fixed point set under the symmetry. We remark that the set \( \{|z| = 1\} \) consists of non-degenerate fold singularities as in [2]. So, it is natural to introduce a new variable \( u \) by

(2.11) \[ u := \frac{r + r^{-1}}{2}, \]

which is invariant under the symmetry \( r \mapsto 1/r \). We set

\[ \tilde{D}_1^\ast := \{z \in \mathbb{C}; 0 < |z| \leq 1\}. \]

By Corollary 2.3 \( f(\tilde{D}_1^\ast \setminus \{1, \zeta, \ldots, \zeta^{n-1}\}) \) coincides with the whole image of \( f \). To obtain the analytic extension of \( f \), we define an analytic map

\[ \iota : \tilde{D}_1^\ast \ni z = re^{i\theta} \mapsto \left( \frac{r + r^{-1}}{2}, \theta \right) \in \mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}. \]

The image of the map \( \iota \) is given by

\[ \tilde{\Omega}_n := \{(u, \theta) \in \mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}; u \geq 1\}. \]

The map \( \iota \) is bijective, whose inverse is given by

\[ \iota^{-1} : \mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z} \ni (u, \theta) \mapsto (u - \sqrt{u^2 - 1}, \theta) \in \tilde{D}_1^\ast. \]

Using the Chebyshev polynomials, the formulas (2.7)–(2.9) can be rewritten in terms of \((u, \theta)\) as follows (see the appendix for the definition and basic properties of the Chebyshev polynomials).
Corollary 2.5. By setting \( \tilde{f}_n = f_n \circ \iota^{-1} \) and \( \tilde{f}_n = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2) \), it holds that

\[
\tilde{x}_0 = \frac{\sin n\theta}{n(T_n(u) - \cos n\theta)},
\]

\[
\tilde{x}_1 = -\frac{T_{n-1}(u) \sin \theta + u \sin (n-1)\theta}{n(T_n(u) - \cos n\theta)} + \frac{n-1}{n^2} \sum_{j=1}^{n-1} \log \left( u - \cos \left( \theta - \frac{2\pi j}{n} \right) \right) \sin \frac{2\pi j}{n},
\]

\[
\tilde{x}_2 = -\frac{T_{n-1}(u) \cos \theta + u \cos (n-1)\theta}{n(T_n(u) - \cos n\theta)} + \frac{n-1}{n^2} \sum_{j=0}^{n-1} \log \left( u - \cos \left( \theta - \frac{2\pi j}{n} \right) \right) \cos \frac{2\pi j}{n},
\]

where \( T_n(u) \), \( T_{n-1}(u) \) denote the first Chebyshev polynomials in the variable \( u \) of degree \( n \), \( n-1 \), respectively.

Proof. Since

\[
x_0 = \frac{2r^n \sin n\theta}{n(r^{2n} - 2r^n \cos n\theta + 1)} = \frac{\sin n\theta}{n \left( \frac{1}{2} (r^n + r^{-n}) - \cos n\theta \right)},
\]

(A.2) in the appendix yields (2.12). Similarly, the first terms of (2.8) and (2.9) are the same as the first terms of (2.13) and (2.14), respectively. On the other hand,

\[
\sum_{j=1}^{n-1} \log \left( r^2 - 2r \cos \left( \theta - \frac{2\pi j}{n} \right) + 1 \right) \sin \frac{2\pi j}{n}
\]

\[
= \sum_{j=1}^{n-1} \log \left( 2r \left( \frac{r + r^{-1}}{2} - \cos \left( \theta - \frac{2\pi j}{n} \right) \right) \right) \sin \frac{2\pi j}{n}
\]

\[
= \sum_{j=1}^{n-1} \log \left( u - \cos \left( \theta - \frac{2\pi j}{n} \right) \right) \sin \frac{2\pi j}{n} + \log 2r \sum_{j=1}^{n-1} \sin \frac{2\pi j}{n}.
\]

Then we have (2.13) because

\[
\sum_{j=1}^{n-1} \sin \frac{2\pi j}{n} = \Im \sum_{j=0}^{n} \zeta^j = 0.
\]

Similarly, we have (2.14). \( \square \)

If we consider \( \tilde{f}_n \) instead of \( f_n \), the origin \( z = 0 \) in the source space of \( f_n \) does not lie in that of \( f_n \). To indicate what the origin in the old complex coordinate \( z \) becomes in the new real coordinates \( (u, \theta) \), we attach a new point \( p_{\infty} \) to \( \tilde{\Omega}_n \) as the ‘point at infinity’, and extend the map \( \iota \) so that

\( \iota(0) = p_{\infty} \).

Hence we have a one-to-one correspondence between \( \{ |z| \leq 1 \} \) and \( \tilde{\Omega}_n \cup \{ p_{\infty} \} \). In particular, \( \tilde{\Omega}_n \cup \{ p_{\infty} \} \) can be considered as an analytic 2-manifold. We prove the following:
Proposition 2.6. The map \( \tilde{f}_n : \Omega_n \cup \{ p_\infty \} \rightarrow \mathbb{R}^3 \) can be analytically extended to the domain

\[
\Omega_n := \left\{ (u, \theta) \in \mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z} ; \ u > \max_{j=0,\ldots,n-1} \left[ \cos \left( \theta - \frac{2\pi j}{n} \right) \right] \right\} \cup \{ p_\infty \},
\]

Proof. In fact, (2.12)–(2.14) are meaningful if

\[
T_n(u) - \cos n\theta > 0 \text{ and } u > \cos \left( \theta - \frac{2\pi j}{n} \right) (j = 0, 1, \ldots, n - 1).
\]

Moreover, \( T_n(u) - \cos n\theta \) is factorized as (cf. Lemma A.1 in the appendix)

\[
T_n(u) - \cos n\theta = 2^{n-1} \prod_{j=0}^{n-1} \left( u - \cos \left( \theta - \frac{2\pi j}{n} \right) \right).
\]

So the condition (2.16) reduces to

\[
u > \cos \left( \theta - \frac{2\pi j}{n} \right) (j = 0, 1, \ldots, n - 1).
\]

Thus, the components \( \tilde{x}_j(u, \theta) \) of \( \tilde{f}_n \) given in (2.12), (2.13) and (2.14) can be extended to \( \Omega_n \). \( \square \)

By Proposition 2.6, we may assume that the map \( \tilde{f}_n \) is defined in \( \Omega_n \). From now on, we call this newly obtained analytic map

\[
\tilde{f}_n : \Omega_n \rightarrow \mathbb{R}^3
\]

the analytic extension of \( f_n \).

![Figure 2.1](image_url)

Figure 2.1. The domain \( \Omega_n \) and the fundamental domain \( \Omega_0^3 \) for \( n = 3 \), where the region of \( u > 1 \) is the space-like part and the regions of \( u < 1 \) are the time-like parts.

Proposition 2.1 and the real analyticity of \( \tilde{f}_n \) imply that

\[
\tilde{f}_n(u, -\theta) = S \tilde{f}_n(u, \theta), \quad \tilde{f}_n \left( u, \theta + \frac{2\pi}{n} \right) = R \tilde{f}_n(u, \theta),
\]
where $S$ and $R$ are the matrices given in (2.2), and $\tilde{f}_n$ is considered as a column vector-valued function. Let $G$ be the finite isometry group of $\mathbb{R}^3_1$ generated by $S$ and $R$. The subset
\begin{equation}
\Omega^0_n = \left\{ (u, \theta) : u > \cos \theta, \; 0 \leq \theta \leq \frac{\pi}{n} \right\}
\end{equation}
of $\Omega_n$ is called the fundamental domain of $\tilde{f}_n$. The equation (2.12) yields the following proposition.

**Proposition 2.7.** The whole image of $\tilde{f}_n$ can be generated by $\tilde{f}_n(\Omega^0_n)$ via the action of $G$, where $\Omega^0_n$ is the fundamental domain as in (2.19).

### 3. Properness of $\tilde{f}_n$

Firstly, we prepare some inequalities which will be necessary for proving that $\tilde{f}_n$ is a proper mapping.

By the definition (2.15) of $\Omega_n$, we have the following two inequalities on $\Omega_n$ (cf. (2.16), (2.17))
\begin{equation}
T_n(u) > \cos n\theta,
\end{equation}
and
\begin{equation}
u > \cos \frac{\pi}{n} \text{ on } \Omega_n,
\end{equation}
since the function $\max_{j=0, \ldots, n-1} [\cos (\theta - 2\pi j/n)]$ has a minimum value $\cos(\pi/n)$.

**Lemma 3.1.** On the fundamental domain $\Omega^0_n$, it holds that
\begin{equation}
u - \cos \left( \theta - \frac{2\pi j}{n} \right) \geq 2\sin^2 \frac{\pi}{n} \quad (j = 2, \ldots, n-1).
\end{equation}

**Proof.** Since $u > \cos \theta$ and $0 \leq \theta \leq \pi/n$ on $\Omega^0_n$,
\begin{align*}
v - \cos \left( \theta - \frac{2\pi j}{n} \right) &> \cos \theta - \cos \left( \theta - \frac{2\pi j}{n} \right) = 2\sin \left( \frac{\pi j}{n} - \theta \right) \sin \frac{\pi j}{n} \\
&\geq 2\sin \frac{\pi (j - 1)}{n} \sin \frac{\pi j}{n} \geq 2\sin^2 \frac{\pi}{n}
\end{align*}
for $2 \leq j \leq n - 1$, proving (3.3). \hfill \Box

Using these, we prove the following assertion:

**Proposition 3.2.** The analytic extension $\tilde{f}_n : \Omega_n \to \mathbb{R}^3_1$ is a proper mapping.

**Proof.** By Proposition 2.1 it is sufficient to show that the restriction of $\tilde{f}_n$ to $\Omega^0_n$ is a proper mapping. We set
\begin{equation}C := \overline{\Omega^0_n} \setminus \Omega^0_n = \left\{ (\cos \theta, \theta) : 0 \leq \theta \leq \frac{\pi}{n} \right\}.
\end{equation}
Consider a sequence $\{(u_k, \theta_k)\}_{k=1, 2, \ldots}$ in $\Omega^0_n$ such that
\begin{equation}\lim_{k \to \infty} (u_k, \theta_k) = (\cos \theta_\infty, \theta_\infty) \in C \quad \left(0 \leq \theta_\infty \leq \frac{\pi}{n}\right).
\end{equation}
It is sufficient to show that the sequence $\{\tilde{f}_n(u_k, \theta_k)\}$ is unbounded in $\mathbb{R}^3_1$. 

[10]
Case 1: We consider the case that \(0 \leq \theta_\infty < \pi/n\). The sequence \(\{(u_k, \theta_k)\}\) is bounded since it converges. So we can take positive numbers \(u_0\) and \(\delta\) such that 
\[
\begin{align*}
(u_k, \theta_k) &\in \Omega_{\delta, u_0} := \left\{ (u, \theta) \in \Omega^0_n; u \leq u_0, \theta \leq \frac{\pi}{n} - \delta \right\}.
\end{align*}
\]
We now set (cf. (2.14))
\[
\begin{align*}
\tilde{x}_2 &= \tilde{x}_{2,a} + \tilde{x}_{2,b}, \\
\tilde{x}_{2,a} &= -\frac{T_{n-1}(u) \cos \theta + u \cos(n-1)\theta}{n(T_n(u) - \cos n\theta)}, \\
\tilde{x}_{2,b} &= \frac{n-1}{n^2} \sum_{j=0}^{n-1} \log \left( u - \cos \left( \theta - \frac{2\pi j}{n} \right) \right) \cos \frac{2\pi j}{n}.
\end{align*}
\]
Since the numerator of \(\tilde{x}_{2,a}\) satisfies (cf. (A.1) in the appendix)
\[
-T_{n-1}(u) \cos \theta + u \cos(n-1)\theta \bigg|_{u=\cos \theta} = -\cos(n-1)\theta \cos \theta + \cos \theta \cos(n-1)\theta = 0,
\]
there exists a real analytic function \(\varphi(u, \theta)\) such that
\[
-T_{n-1}(u) \cos \theta + u \cos(n-1)\theta = (u - \cos \theta) \varphi(u, \theta).
\]
Since
\[
(u - \cos \left( \theta - \frac{2\pi}{n} \right)) \geq \cos \theta - \cos \left( \theta - \frac{2\pi}{n} \right) = 2 \sin \left( \frac{\pi}{n} - \theta \right) \sin \frac{\pi}{n} > 2 \sin \delta \sin \frac{\pi}{n}
\]
holds on \(\Omega_{\delta, u_0}\), (2.17) and (3.3) in Lemma 3.1 yield that there exist a real analytic function \(\psi(u, \theta)\) and a positive number \(\varepsilon\) such that
\[
T_n(u) - \cos n\theta = (u - \cos \theta) \psi(u, \theta), \quad \psi(u, \theta) \geq \varepsilon > 0 \quad \text{on} \quad \Omega_{\delta, u_0},
\]
Thus \(\tilde{x}_{2,a} = \varphi(u, \theta)/\psi(u, \theta)\) is bounded on \(\Omega_{\delta, u_0}\).

Since (3.3) in Lemma 3.1 and (3.4) imply that
\[
\log \left( u - \cos \left( \theta - \frac{2\pi j}{n} \right) \right) (j = 1, 2, \ldots, n-1)
\]
is bounded on \(\Omega_{\delta, u_0}\), we can write
\[
\tilde{x}_{2,b} = \frac{n-1}{n^2} \log(u - \cos \theta) + \beta(u, \theta),
\]
where \(\beta(u, \theta)\) is a real analytic function bounded on \(\Omega_{\delta, u_0}\). Thus, \(\tilde{x}_2(u_k, \theta_k) \to -\infty\) as \(k \to \infty\).
Case 2: We next consider the case that $\theta_\infty = \pi/n$. In other words, we suppose the sequence $\{(u_k, \theta_k)\}$ converges to $(\cos(\pi/n), \pi/n)$. In this case, we seek to prove

\[(3.5) \lim_{k \to \infty} \tilde{x}_1(u_k, \theta_k) = -\infty.\]

We may assume $\{(u_k, \theta_k)\} \subset \Omega_n^0 \cap \{u \leq u_0\}$ for some constant $u_0$. We set (cf. (2.13))

\[
\tilde{x}_1 = \tilde{x}_{1,a} + \tilde{x}_{1,b},
\]

\[
\tilde{x}_{1,a} := -\frac{T_{n-1}(u) \sin \theta + u \sin(n-1)\theta}{n(T_n(u) - \cos n\theta)},
\]

\[
\tilde{x}_{1,b} := \frac{n-1}{n^2} \sum_{j=1}^{n-1} \log \left(u - \cos \left(\theta - \frac{2\pi j}{n}\right)\right) \sin \frac{2\pi j}{n}.
\]

Let $(u, \theta) \in \Omega_n^0$. Then $u > \cos \theta$ and $\cos \theta \in (\cos(\pi/n), 1) \subset (\cos(\pi/n), \infty)$. So both $u$ and $\cos \theta$ belong to $[\cos(\pi/n), \infty)$. Since $T_{n-1}$ is monotone increasing on $[\cos(\pi/n), \infty) \cap (\cos(\pi/(n-1)), \infty))$ (cf. (A.1) and Proposition A.5 in the appendix), it holds that

\[T_{n-1}(u) > T_{n-1}(\cos \theta) = \cos(n-1)\theta \quad \text{on} \quad \Omega_n^0.\]

Noticing this, we have

\[T_{n-1}(u) \sin \theta + u \sin(n-1)\theta \geq \cos(n-1)\theta \sin \theta + \cos \theta \sin(n-1)\theta = \sin n\theta \geq 0\]

on $\Omega_n^0$. By (3.1), the inequality $\tilde{x}_{1,a} \leq 0$ holds on $\Omega_n^0$. By (3.3) in Lemma 3.1

\[\log \left(u - \cos \left(\theta - \frac{2\pi j}{n}\right)\right) \quad (j = 2, \ldots, n-1)
\]

is bounded on $\Omega_n^0 \cap \{u \leq u_0\}$, and we can write

\[\tilde{x}_1 = \tilde{x}_{1,a} + \tilde{x}_{1,b} \leq \tilde{x}_{1,b} = \hat{\beta}(u, \theta) + \log \left(u - \cos \left(\theta - \frac{2\pi}{n}\right)\right) \sin \frac{2\pi}{n}
\]

on $\Omega_n^0$, where $\hat{\beta}(u, \theta)$ is a real analytic function bounded on $\Omega_n^0 \cap \{u \leq u_0\}$. Since the right-hand side tends to $-\infty$ as $(u, \theta) \to (\cos(\pi/n), \pi/n)$, (3.5) holds. \hfill \Box

4. IMMERSEDNESS OF $\tilde{f}_n$

**Proposition 4.1.** The analytic extension $\tilde{f}_n: \Omega_n \to R_1^n$ is an immersion.

**Proof.** In this proof, $f, \tilde{f}$ denote $f_n, \tilde{f}_n$, respectively, for notational simplicity.

Since $\partial/\partial z = (1/2z)(r\partial/\partial r - i\partial/\partial \theta)$ for $z = re^{i\theta}$, we have

\[\alpha = dF = F_z dz = (F + \bar{F})_z dz = 2f_z dz = \frac{1}{z}(rf_r - if_\theta) dz,
\]

where $\alpha = a(z) dz$ is as given in (2.1), that is,

\[(4.1) \quad za(z) = rf_r - if_\theta.
\]
On the other hand, we have (cf. (2.1))

\[(4.2) \quad za(z) = \left( \frac{-2ir^n e^{in\theta}}{(r^n e^{in\theta} - 1)^2}, \frac{iue^{i(2n-2)e^{i(2n-2)\theta}}}{(r^n e^{in\theta} - 1)^2}, \frac{re^{i(1-2n)e^{i(2n-2)\theta}}}{(r^n e^{in\theta} - 1)^2} \right). \]

We define by \(\xi^j := \iota(\xi^j, \xi^k)\) \((j < k)\) for \(\xi = (\xi^0, \xi^1, \xi^2) \in C^3\), and here \(\iota(*)\) means transposition. Then

\[2ir \det(f^j_r, f^j_\theta) = \det(rf^j_r - if^j_\theta, rf^j_r + if^j_\theta) = \det(za^j, za^j).\]

Using this and (4.3), one can arrive at

\[(4.3) \quad \det(f^0_1, f^0_\theta) = \frac{2(r^{2n} - r^2) \sin(n - 1) \theta}{(r^{2n} - 2r^n \cos n \theta + 1)^2},\]

\[(4.4) \quad \det(f'^0_1, f'^0_\theta) = \frac{2r^{2n} - 2r^n \cos(n - 1) \theta}{(r^{2n} - 2r^n \cos n \theta + 1)^2}.\]

Since we set \(u = (r + r^{-1})/2\), we have

\[(4.5) \quad f_u = \frac{2r^2}{r^2 - 1} f_r.\]

The equality (4.3) is equivalent to

\[
\det(f^0_1, f^0_\theta) = \frac{4r^n (r^{2n} - r^2) \sin(n - 1) \theta}{(r^2 - 1)(r^{2n} - 2r^n \cos n \theta + 1)^2} = \frac{4(2n - 1) n \sin(n - 1) \theta}{(r - r^{-1}) (r^n + r^{-n} - 2 \cos n \theta)^2} = \frac{U_{n-2}(u) \sin(n - 1) \theta}{(T_n(u) - \cos n \theta)^2},
\]

where \(U_{n-2}(u)\) denotes the second Chebyshev polynomial of degree \(n - 2\). (See (A.2), (A.4) and (A.3) in the appendix.) Similarly, by (4.4), we have

\[
\det(f'^0_1, f'^0_\theta) = -\frac{U_{n-2}(u) \cos(n - 1) \theta}{(T_n(u) - \cos n \theta)^2}.
\]

By the real analyticity, the identities

\[(4.6) \quad \det(f^0_1, f^0_\theta) = \frac{U_{n-2}(u) \sin(n - 1) \theta}{(T_n(u) - \cos n \theta)^2}, \quad \det(f'^0_1, f'^0_\theta) = -\frac{U_{n-2}(u) \cos(n - 1) \theta}{(T_n(u) - \cos n \theta)^2}\]

hold on \(\Omega_n\). Hence, it cannot occur that \(\det(\tilde{f}^0_1, \tilde{f}^0_\theta)\) and \(\det(\tilde{f}'^0_1, \tilde{f}'^0_\theta)\) vanish simultaneously, since \(U_{n-2}(u) > 0\) by (3.2) (cf. Corollary A.4 in the appendix). We conclude that \(\tilde{f}_n\) is an immersion.

\[
\boxed{5. \text{ Embeddedness of } \tilde{f}_n}
\]

5.1. **Outline.** We show that \(\tilde{f}_n = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2)\colon \Omega_n \to (R^3; t, x, y)\) is an embedding. The set

\[\tilde{x}_0^{-1}(h)\]

is called the **contour-line** of height \(t = h\), and

\[\Lambda_h := \tilde{f}_n(\tilde{x}_0^{-1}(h)) = \tilde{f}_n(\Omega_n) \cap \{t = h\}\]
is called the level curve set of height \( t = h \). To show the embeddedness of \( \tilde{f}_n \), it is sufficient to show that \( \tilde{f}_n : \tilde{x}_0^{-1}(h) \to \Lambda_h \) is injective at each height \( h \).

Since (cf. (2.7) or (2.12))

\[
(5.1) \quad \tilde{x}_0 = \frac{2n \sin n\theta}{n(2n^2 - 2n \cos n\theta + 1)} = \frac{\sin n\theta}{n(T_n(u) - \cos n\theta)},
\]

the contour-line of height \( h = 0 \) is given by

\[
(5.2) \quad \tilde{x}_0^{-1}(0) = \bigcup_{k=0}^{2n-1} \left\{ (u, \theta) \in \Omega_n : \theta = \frac{k}{n} \pi \right\} \cup \{ p_{\infty} \}.
\]

The figure of the contour-line \( \tilde{x}_0^{-1}(0) \) and its image (i.e. the level curve set of height \( h = 0 \)) are indicated in Figure 5.1.

![Figure 5.1](image)

**Figure 5.1.** Contour-lines \( \tilde{x}_0^{-1}(0) \) and the level set \( \Lambda_0 \) at height \( h = 0 \) in the case \( n = 6 \).

On the other hand, if \( h \neq 0 \), we have

\[
\tilde{x}_0^{-1}(h) = \left\{ (u, \theta) \in \Omega_n : T_n(u) = \cos n\theta + \frac{1}{nh} \sin n\theta \right\}.
\]

The following assertion is immediately obtained.

**Proposition 5.1.** The function \( \tilde{x}_0 \) (cf. (5.1)) is non-negative valued on \( \Omega_0^0 \), where \( \Omega_0^0 \) is the fundamental domain given by (2.19).

Since \( \tilde{x}_0^{-1}(h) \cap \Omega_n^0 = \emptyset \) if \( h < 0 \), we may suppose \( h > 0 \) to prove the injectivity of \( \tilde{f}_n : \tilde{x}_0^{-1}(h) \to \Lambda_h \) for \( h \neq 0 \). Let \( \Lambda_h^0 \) be the level curve set of the image \( \tilde{f}_n(\Omega_n^0) \) of the fundamental domain \( \Omega_0 \), that is,

\[
\Lambda_h^0 := \tilde{f}_n(\Omega_n^0) \cap \{ t = h \} = \tilde{f}_n(\tilde{x}_0^{-1}(h) \cap \Omega_n^0).
\]

As a consequence of Proposition 2.7 we obtain the following:

**Corollary 5.2.**

\[
\Lambda_h = \bigcup_{k=0}^{n-1} R^k \Lambda_h^0, \quad \text{and} \quad \Lambda_{-h} = S \Lambda_h,
\]

where \( R \) and \( S \) are the matrices defined in (2.2) (cf. (2.18)).

Corollary 5.2 implies that we should seek to prove that
(1) the map $\tilde{f}_n$ restricted to $\tilde{x}_0^{-1}(h) \cap \Omega_n^0$, i.e., $\tilde{f}_n : \tilde{x}_0^{-1}(h) \cap \Omega_n^0 \to \Lambda_h^0$ is injective,

(2) $\bigcup_{k=0}^{n-1} R^k \Lambda_h^0$ is a disjoint union.

Figure 5.2. Contour-line $\tilde{x}_0^{-1}(h) \cap \Omega_n^0$ and the level curve set $\Lambda_h^0$ for $h = 0.01$ in the case $n = 6$.

Figure 5.3. The level curve set $\Lambda_h$ of $\tilde{f}_6(\Omega_6)$ for $h = 1$

5.2. Contour-lines in $\Omega_n^0$. Now we investigate $\tilde{x}_0^{-1}(h) \cap \Omega_n^0$. As mentioned above, we suppose $h > 0$.

**Proposition 5.3.** (1) Given $h > 0$ and $0 < \theta < \pi/n$, the equation $\tilde{x}_0(u, \theta) = h$ is uniquely solved for $u \in (\cos(\pi/n), \infty)$. Indeed, it determines the implicit function $u = u(h, \theta)$ defined on $A := \{(h, \theta) ; h > 0, \ 0 < \theta < \pi/n \}$ satisfying $\cos \theta < u(h, \theta)$. Moreover, the following hold:

(i) For a fixed $\theta_0 \in (0, \pi/n)$, the function $h \mapsto u(h, \theta_0)$ is monotone decreasing, and

$$\lim_{h \searrow 0} u(h, \theta_0) = \infty.$$

(ii) For a fixed $h_0 > 0$,

$$\lim_{\theta \searrow 0} u(h_0, \theta) = 1, \quad \lim_{\theta \nearrow \pi/n} u(h_0, \theta) = \cos(\pi/n).$$
Lemma 5.4. We show the following properties of the level curve set \( \Lambda \) which can be considered as a function of \( \theta \).

\[
\frac{(\tilde{x}_0)_u}{(\tilde{x}_0)_\theta} = -\frac{U_{n-1}(u) \sin n\theta}{(T_n(u) - \cos n\theta)^2}.
\]

Proof. 1. The equation \( \tilde{x}_0(u, \theta) = h(>0) \) is equivalent to

\[
T_n(u) = \cos n\theta + \frac{1}{nh} \sin n\theta,
\]

and

\[
-1 < \cos n\theta < \cos n\theta + \frac{1}{nh} \sin n\theta
\]

holds on \( A = \{(h, \theta); h > 0, 0 < \theta < \pi/n\} \). On the other hand, the Chebyshev polynomial \( T_n(u) \) is monotone increasing on the interval \([\cos(\pi/n), \infty)\) (see Proposition A.5 in the appendix), and hence it has the inverse function

\[
T_n^{-1}: [-1, \infty) \to [\cos(\pi/n), \infty),
\]

which is monotone increasing. Thus,

\[
u(h, \theta) := T_n^{-1}(\cos n\theta + \frac{1}{nh} \sin n\theta) \quad \text{on} \quad A
\]

is well-defined and the desired one. Obviously,

\[
\cos \theta = T_n^{-1}(\cos n\theta) < T_n^{-1}(\cos n\theta + \frac{1}{nh} \sin n\theta) = u(h, \theta)
\]

holds on \( A \).

Since \( T_n^{-1} \) is monotone increasing on \([-1, \infty)\), the formula (5.3) immediately implies the assertions (i) and (ii).

2. This can be determined directly from (2.12). \( \square \)

Hereafter, we set (cf. (5.3))

\[
u_h(\theta) := u(h, \theta)
\]

which can be considered as a function of \( \theta \) fixing \( h \). Proposition 5.3 implies that the contour-line \( \tilde{x}_0^{-1}(h) \cap \Omega_n^0 \) satisfies

\[
\tilde{x}_0^{-1}(h) \cap \Omega_n^0 = \{(u, \theta) \in \Omega_n^0; u = u_h(\theta)\} = \{(u_h(\theta), \theta) \in \Omega_n^0; 0 < \theta < \pi/n\}.
\]

The level curve set \( \Lambda^0_h \), i.e., \( \tilde{x}_0^{-1}(h) \cap \Omega_n^0 \) is given by

\[
\Lambda^0_h = \{(h, \tilde{x}_1(u_h(\theta), \theta), \tilde{x}_2(u_h(\theta), \theta)); 0 < \theta < \pi/n\}.
\]

We show the following properties of the level curve set \( \Lambda^0_h \).

Lemma 5.4.

1. \( \tilde{x}_1(u_h(\theta), \theta) \) is a monotone decreasing function of \( \theta \in (0, \pi/n) \), whose value is less than \( -h \).

2. \( \tilde{x}_2(u_h(\theta), \theta) \) attains a maximum at \( \theta = \frac{\pi}{2(n-1)} \in (0, \pi/n) \).

Proof. 1 By (4.6) and Proposition 5.3 2, we have

\[
\frac{d}{d\theta} \tilde{x}_1(u_h(\theta), \theta) = \frac{\partial \tilde{x}_1}{\partial u} \frac{du_h}{d\theta} + \frac{\partial \tilde{x}_1}{\partial \theta} = -\frac{\partial \tilde{x}_1}{\partial u} \frac{\partial x_0}{\partial u} + \frac{\partial \tilde{x}_1}{\partial \theta}
\]

\[
= \frac{1}{(\tilde{x}_0)_u} \det(\tilde{x}_0^{01}, \tilde{x}_0^{01}) = -\frac{U_{n-2}(u_h(\theta)) \sin(n-1)\theta}{U_{n-1}(u_h(\theta)) \sin n\theta},
\]

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which is negative for $0 < \theta < \pi/n$ (cf. Corollary A.4 in the appendix). Hence, 
$\tilde{x}_1(u_h(\theta), \theta)$ is a monotone decreasing function of $\theta$. Next, according to (2.13), we set

\[ \tilde{x}_1(u_h(\theta), \theta) = \tilde{x}_{1,a}(\theta) + \tilde{x}_{1,b}(\theta), \]

where

\[
\tilde{x}_{1,a}(\theta) = -h \left( T_{n-1}(u_h(\theta)) \frac{\sin \theta}{\sin n\theta} + u_h(\theta) \frac{\sin(n-1)\theta}{\sin n\theta} \right),
\]

\[
\tilde{x}_{1,b}(\theta) = \frac{n-1}{n} \sum_{j=1}^{n-1} \log \left( u_h(\theta) - \cos \left( \theta - \frac{2\pi j}{n} \right) \right) \sin \frac{2\pi j}{n}.
\]

These satisfy

\[
\lim_{\theta \to 0} \tilde{x}_{1,a}(\theta) = -h \left( T_{n-1}(1) \frac{1}{n} + 1 \frac{n-1}{n} \right) = -h \left( \frac{1}{n} + \frac{n-1}{n} \right) = -h,
\]

because of part (ii) of item (1) in Proposition 5.3. Moreover,

\[
\lim_{\theta \to 0} \tilde{x}_{1,b}(\theta) = \frac{n-1}{n} \sum_{j=1}^{n-1} \log \left( 1 - \cos \left( 0 - \frac{2\pi j}{n} \right) \right) \sin \frac{2\pi j}{n} = 0
\]

holds, since the terms in the summation cancel for each pair $(j, n-j)$. Therefore

\[
\lim_{\theta \to 0} \tilde{x}_1(u_h(\theta), \theta) = -h + 0 = -h.
\]

Since the function $\theta \mapsto \tilde{x}_1(u_h(\theta), \theta)$ is monotone decreasing, we conclude that 
$\tilde{x}_1(u_h(\theta), \theta) < -h$ for all $\theta \in (0, \pi/n)$.

\[(2)\] Similarly to (5.5), we have

\[
\frac{d}{d\theta} \tilde{x}_2(u_h(\theta), \theta) = \frac{\partial \tilde{x}_2}{\partial u} \frac{du_h}{d\theta} + \frac{\partial \tilde{x}_2}{\partial \theta} = -\frac{\partial \tilde{x}_2}{\partial u} \frac{\partial u_h}{\partial u} + \frac{\partial \tilde{x}_2}{\partial \theta} = \frac{1}{(\tilde{x}_0)_{\theta}} \det(j_0, j_0) = \frac{U_{n-2}(u_h(\theta))}{U_{n-1}(u_h(\theta))} \frac{\cos(n-1)\theta}{\sin n\theta},
\]

which is

\[
\left\{
\begin{array}{ll}
\text{positive if } & 0 < \theta < \pi/2(n-1), \\
\text{zero if } & \theta = \pi/2(n-1), \\
\text{negative if } & \pi/2(n-1) < \theta < \pi/n.
\end{array}
\right.
\]

This proves the assertion (2). \hfill \square

**Proposition 5.5.** The restriction of the map $\tilde{f}_n$ given by

\[
(5.7) \quad \tilde{f}_n : \tilde{x}_0^{-1}(h) \cap \Omega_0 \ni (u_h(\theta), \theta) \mapsto (h, \tilde{x}_1(u_h(\theta), \theta), \tilde{x}_2(u_h(\theta))) \in \Lambda_h
\]

is injective.

**Proof.** (5.4) and Lemma 5.4 (1) imply that the above correspondence (5.7) gives a regular curve without self-intersection. \hfill \square
5.3. **Level curve sets.** Firstly, we deal with the level curve set $\Lambda_{0}$ of height $h = 0$ (cf. Figure 5.1).

**Proposition 5.6.** The map $\tilde{f}_{n}$ restricted to $\tilde{x}_{0}^{-1}(0)$ is injective.

To prove the assertion, we prepare the following lemma:

**Lemma 5.7.**

\[
\begin{align*}
\tilde{x}_{k} &= \frac{\sin(2n - 1)\theta + 2U_{n-2}(u)\sin(n-1)\theta + U_{2n-2}(u)\sin\theta}{2(T_{n}(u) - \cos n\theta)^{2}}, \\
\tilde{y}_{k} &= \frac{-\cos(2n - 1)\theta - 2U_{n-2}(u)\cos(n-1)\theta + U_{2n-2}(u)\cos\theta}{2(T_{n}(u) - \cos n\theta)^{2}}.
\end{align*}
\]

**Proof.** By (4.1), (4.2) and (4.5), we obtain that

\[
\begin{align*}
\frac{(\tilde{x}_{k})_{u}}{2} &= \frac{r}{r^{2} - 1} \text{Re}(\tilde{z}_{1}(z)) \\
&= \frac{2(r^{3n} - r^{n+2})\sin(n-1)\theta + (r^{2} - 1) r^{2n}\sin(2n - 1)\theta + (r^{4n} - r^{2})\sin\theta}{(r^{2} - 1)(r^{2n} - 2r^{n}\cos n\theta + 1)^{2}}, \\
\frac{(\tilde{y}_{k})_{u}}{2} &= \frac{r}{r^{2} - 1} \text{Re}(\tilde{z}_{2}(z)) \\
&= -\frac{2(r^{3n} - r^{n+2})\cos(n-1)\theta + (r^{2} - 1) r^{2n}\cos(2n - 1)\theta + (r^{2} - 4n)\cos\theta}{(r^{2} - 1)(r^{2n} - 2r^{n}\cos n\theta + 1)^{2}}.
\end{align*}
\]

So $(r + r^{-1})/2 = u$ proves (5.8) and (5.9). \qed

**Proof of Proposition 5.6.** Recall the equality (5.2) which asserts that

\[
\tilde{x}_{0}^{-1}(0) = \bigcup_{k=0}^{2n-1} B_{k} \cup \{p_{\infty}\},
\]

where

\[
B_{k} := \{(u, \theta) \in \Omega_{n}; \; \theta = \frac{k}{n}\pi\}.
\]

Consider the map

\[
\tilde{f}_{n}|_{B_{k}} = (\tilde{x}_{0}, \tilde{x}_{1}, \tilde{x}_{2})|_{\theta = k\pi/n} = (0, \tilde{x}_{1}(u, k\pi/n), \tilde{x}_{2}(u, k\pi/n)).
\]

It follows from (5.8), (5.9) and (A.6) that

\[
\frac{d}{du} \left( \tilde{f}_{n}|_{B_{k}} \right) = V(u) \left( 0, \sin \frac{k}{n}\pi, \cos \frac{k}{n}\pi \right),
\]

where

\[
V(u) := \frac{U_{n-2}(u)}{T_{n}(u) - (-1)^{k}}.
\]

This implies that $\tilde{f}_{n}|_{B_{k}}$ parametrizes a straight half-line with the velocity $V(u)$. If $k$ is even, $\tilde{f}_{n}|_{B_{k}}$ is defined on the interval $(1, \infty)$ and $V(u)$ is positive on $(1, \infty)$. If $k$ is odd, $\tilde{f}_{n}|_{B_{k}}$ is defined on the interval $(\cos(\pi/n), \infty)$ and $V(u)$ is positive on $(\cos(\pi/n), \infty)$. Hence, for any $k$, the map $\tilde{f}_{n}|_{B_{k}}$ is injective. Moreover, the monotonicity of $\tilde{f}_{n}|_{B_{k}}$ and the equality

\[
\lim_{u \to \infty} \tilde{f}_{n}|_{B_{k}}(u) = \tilde{f}_{n}(p_{\infty}) = (0, 0, 0)
\]

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imply that the point $p_{\infty}$ is the unique inverse-image of $(0,0,0)$. Therefore we conclude the map $\tilde{f}_n: \tilde{x}_0^{-1}(0) \to \mathbb{R}^3_1$ is injective. \hfill \Box \\

We next consider the case where the height $h$ is not equal to 0. For a fixed $h$, let $P_h$ denote a plane in $(\mathbb{R}^3_1; t, x, y)$ defined by the equation $t = h$, with coordinate system $(x, y)$.

**Proposition 5.8.** For any fixed $h > 0$, the level curve set $\Lambda^0_h$ of height $h$ lies in the region

$$D_h := \{(x, y); x < -h, \ x \cos(2\pi/n) - y \sin(2\pi/n) + h > 0\} \subset P_h.$$ 

**Proof.** We parametrize $\Lambda^0_h$ so that (cf. Proposition 5.5)

$$(x_h(\theta), y_h(\theta)) := (\tilde{x}_1(u_h(\theta), \theta), \tilde{x}_2(u_h(\theta), \theta)) \ (0 < \theta < \pi/n).$$

We have already determined that $x_h(\theta) < -h$ (cf. Lemma 5.4 (1)). It remains to show

$$\varphi_h(\theta) := x_h(\theta) \cos(2\pi/n) - y_h(\theta) \sin(2\pi/n) + h > 0$$

for $0 < \theta < \pi/n$. Using (5.5), (5.6), we have

$$\frac{d}{d\theta} \varphi_h(\theta) = -\frac{U_{n-2}(u_h(\theta))}{U_{n-1}(u_h(\theta))} \sin n\theta \left((n-1)\theta + \frac{2\pi}{n}\right).$$

This implies that $\varphi_h(\theta)$ has a minimum at

$$\theta_0 = \frac{n - 2}{(n-1)n} \pi.$$ 

Hence, we have only to prove that

$$\varphi_h(\theta_0) > 0.$$ 

Indeed,

$$\Phi(h) := \varphi_h(\theta_0) = \tilde{x}_1(u_h(\theta_0), \theta_0) \cos(2\pi/n) - \tilde{x}_2(u_h(\theta_0), \theta_0) \sin(2\pi/n) + h$$

satisfies

$$\lim_{h \to 0} \Phi(h) = 0 \cdot \cos(2\pi/n) - 0 \cdot \sin(2\pi/n) + 0 = 0,$$

because of part (i) of item (1) in Proposition 5.3. Moreover, a straightforward computation using Proposition 5.3 (2), (5.8) and (5.9) leads us to

$$\frac{d\Phi}{dh}(h) = 1 + \frac{U_{2n-2}(u_h(\theta_0))}{2U_{n-1}(u_h(\theta_0))} + 1 = 1 + \frac{U_{2n-2}(u_h(\theta_0)) + 2U_{n-1}(u_h(\theta_0))}{2U_{n-1}(u_h(\theta_0))}.$$ 

We wish to know the sign of $d\Phi/dh$. Note that $u_h(\theta_0) \in (\cos \theta_0, \infty)$ for $h \in (0, \infty)$. For this purpose, we set

$$\Upsilon(u) := \frac{1 + U_{2n-2}(u) + 2U_{n-1}(u)}{2U_{n-1}(u)} \text{ for } u \in (\cos \theta_0, \infty).$$

Then, it is obvious $\Upsilon(u) > 0$ for $u \in [1, \infty)$ (cf. Proposition 5.3 in the appendix). For $u \in (\cos \theta_0, 1)$, it is also obvious that the denominator of $\Upsilon(u)$ is positive. Since
there exists a unique $\alpha \in (0, \theta_0)$ with $u = \cos \alpha$, the numerator is computed as

$$1 + U_{2n-2}(\cos \alpha) + 2U_{n-1}(\cos \alpha) = \frac{\sin \alpha + \sin(2n - 1)\alpha + 2 \sin n\alpha}{\sin \alpha} \cdot \frac{2 \sin n\alpha}{\sin \alpha} = \frac{2 \sin n\alpha (\cos (n-1)\alpha + 1)}{\sin \alpha}.$$  

So the numerator is positive because $0 < \alpha < \theta_0 = \frac{n-2}{(n-1)n} \pi$.

Thus, $\Upsilon(u) > 0$ for all $u \in (\cos \theta_0, \infty)$. Hence we obtain

$$\frac{d\Phi}{dh}(h) = \Upsilon(u_h(\theta_0)) > 0 \text{ for } h \in (0, \infty).$$  

(5.12)

It follows from (5.11) and (5.12) that $\Phi(h) > 0$ for all $h \in (0, \infty)$, that is, $\varphi_h(\theta_0) > 0$ for all $h \in (0, \infty)$. We have now proved (5.10).  

We are in a position to complete a proof of the embeddedness of $\tilde{f}_n$.

**Theorem 5.9.** For any integer $n \geq 2$, the analytic extension $\tilde{f}_n : \Omega_n \to R^3_1$ is a proper embedding.

**Proof.** The assertion for $n = 2$ is trivial, as stated in Section 1. We have already proved that $\tilde{f}_n$ is a proper immersion (cf. Propositions 3.2 and 4.1). So it is sufficient to show that $\tilde{f}_n$ is injective for each $n \geq 3$. For this purpose, we will show that $\tilde{f}_n$ restricted to each contour-line $\tilde{x}_0^{-1}(h)$ is injective. We have already done this for $h = 0$ in Proposition 5.6. For $h \neq 0$, it suffices to show $\Lambda^0_h$ never intersects the other $R^k \Lambda^0_h$ ($k = 1, 2, \ldots, n - 1$), since we have already seen $\tilde{f}_n : \tilde{x}_0^{-1}(h) \cap \Omega_n^0 \to \Lambda^0_h \subset D_h$ is injective (cf. Proposition 5.8). In fact, the region $D_h$ of Proposition 5.8 does not intersect the other $R^k(D_h)$ ($k = 1, 2, \ldots, n - 1$) (see Figures 5.4 and 5.5), thus, $\Lambda^0_h$ never intersects the other $R^k \Lambda^0_h$. Therefore, we conclude that $\tilde{f}_n : \Omega_n \to R^3_1$ is an injective proper immersion, i.e., a proper embedding. \[\square\]
The first Chebyshev polynomial $T_n(x)$ ($n = 1, 2, \ldots$) is, by definition, the polynomial of degree $n$ such that
\begin{equation}
T_n(\cos \theta) = \cos n\theta.
\end{equation}
It holds that
\begin{equation}
T_n(u) = \frac{r^n + r^{-n}}{2} \quad \left( u := \frac{r + r^{-1}}{2} \right).
\end{equation}

Lemma A.1. The following identity holds:
\begin{equation}
\Psi_n(u) := T_n(u) - \cos n\theta = 2^{n-1} \prod_{j=0}^{n-1} \left( u - \cos \left( \theta - \frac{2\pi j}{n} \right) \right).
\end{equation}

Proof. By (A.1), we have
\[
\Psi_n \left( \cos \left( \theta - \frac{2\pi j}{n} \right) \right) = \cos \left( n \left( \theta - \frac{2\pi j}{n} \right) \right) - \cos n\theta = 0.
\]
Since $\Psi_n(u)$ is a polynomial in $u$ of degree $n$ and the highest coefficient of $T_n(u)$ is equal to $2^{n-1}$, we obtain the assertion. □

The second Chebyshev polynomial $U_n(x)$ ($n = 1, 2, \ldots$) is, by definition, the polynomial of degree $n$ such that
\begin{equation}
\sin(n + 1)\theta = U_n(\cos \theta) \sin \theta.
\end{equation}
It holds that
\begin{equation}
U_{n-1}(u) = \frac{r^n - r^{-n}}{r - r^{-1}} \quad \left( u = \frac{r + r^{-1}}{2} \right).
\end{equation}

The first and the second Chebyshev polynomials are related as follows:
\[
\frac{d}{dx} T_n(x) = nU_{n-1}(x).
\]

Proposition A.2. For $m \geq 1$, it holds that
\begin{equation}
U_{2m}(x) - 1 = 2T_{m+1}(x)U_{m-1}(x).
\end{equation}
Proof. It is sufficient to show the identity for $x = \cos \theta$ ($\theta \in [0, 2\pi]$). Then
\[
U_{2m}(\cos \theta) - 1 = \frac{\sin(2m + 1)\theta}{\sin \theta} - 1 = \frac{\sin(2m + 1)\theta - \sin \theta}{\sin \theta} = 2\cos(m + 1)\theta \sin m\theta \sin \theta = 2T_{m+1}(\cos \theta)U_{m-1}(\cos \theta).
\]

Proposition A.3. Let $n$ be an integer greater than 2 (resp. $n = 2$). Then $y = U_{n-1}(x)$ is monotone increasing on the interval $\{x; \cos \frac{\pi}{n} \leq x < \infty\}$ and the range is $\{y; -1 \leq y < \infty\}$ (resp. $\{y; -2 \leq y < \infty\}$). Furthermore, $U_{n-1}(\cos(\pi/n)) = 0$ and $U_{n-1}(1) = n$ hold.

Corollary A.4. For arbitrary $m \leq n - 1$,
\[
U_{m}(x) > 0 \text{ for } \cos(\pi/n) < x < \infty.
\]

Proposition A.5. Let $n$ be an integer greater than or equal to 2. Then $y = T_{n}(x)$ is monotone increasing on the interval $\{x; \cos \frac{\pi}{n} \leq x < \infty\}$ and the range is $\{y; -1 \leq y < \infty\}$. Furthermore, $T_{n}(\cos(\pi/2n)) = 0$ and $T_{n}(1) = 1$ hold.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Chebyshev polynomials are monotone increasing on the interval toward the right.}
\end{figure}

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