CONSTRUCTING ELLIPTIC CURVES OVER $\mathbb{Q}(T)$ WITH MODERATE RANK

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Abstract. We give several new constructions for moderate rank elliptic curves over $\mathbb{Q}(T)$. In particular we construct infinitely many rational elliptic surfaces (not in Weierstrass form) of rank 6 over $\mathbb{Q}$ using polynomials of degree two in $T$. While our method generates linearly independent points, we are able to show the rank is exactly 6 without having to verify the points are independent. The method generalizes; however, the higher rank surfaces are not rational, and we need to check that the constructed points are linearly independent.

1. Introduction

Consider the elliptic curve $\mathcal{E}$ over $\mathbb{Q}(T)$:

$$y^2 + a_1(T)xy + a_3(T)y = x^3 + a_2(T)x^2 + a_4(T)x + a_6(T),$$

where $a_i(T) \in \mathbb{Z}[T]$. By evaluating these polynomials at integers, we obtain elliptic curves over $\mathbb{Q}$. By Silverman’s Specialization Theorem, for large $t \in \mathbb{Z}$ the Mordell-Weil rank of the fiber $\mathcal{E}_t$ over $\mathbb{Q}$ is at least that of the curve $\mathcal{E}$ over $\mathbb{Q}(T)$.

For comparison purposes, we briefly describe other methods to construct curves with rank. Mestre [Mes1, Mes2] considers a 6-tuple of integers $a_i$ and defines $q(x) = \prod_{i=1}^6 (x - a_i)$ and $p(x, T) = q(x - T)q(x + T)$. There exist polynomials $g(x, T)$ of degree 6 in $x$ and $r(x, T)$ of degree at most 5 in $x$ such that $p(x, T) = g^2(x, T) - r(x, T)$. Consider the curve $y^2 = r(x, T)$ over $\mathbb{Q}(T)$. If $r(x, T)$ is of degree 3 or 4 in $x$, we obtain an elliptic curve with points $P_{\pm i}(T) = (\pm T + a_i, g(\pm T + a_i))$. If $r(x, T)$ has degree 4 we may need to change variables to make the coefficient of $x^4$ a perfect square (see [Mor], page 77). Two 6-tuples that work are $(-17, -16, 10, 11, 14, 17)$ and $(399, 380, 352, 47, 4, 0)$ (see [Na1]). Curves of rank up to 14 over $\mathbb{Q}(T)$ have been constructed this way, and using these methods Nagao [Na1] has found an elliptic curve of rank at least 21 and Fermigier [Fe2] one of rank at least 22 over $\mathbb{Q}$. Shiota [Sh] gives explicit constructions for not only rational curves of rank 2, 4, 6, 7 and 8 over $\mathbb{Q}(T)$, but generators of the Mordell-Weil groups.

We now describe the idea of our method. For $\mathcal{E}$ as in (1.1), define

$$A_\mathcal{E}(p) = \frac{1}{p} \sum_{t=0}^{p-1} a_i(p),$$

with $a_i(p) = p + 1 - N_i(p)$, where $N_i(p)$ is the number of points in $\mathcal{E}_t(\mathbb{F}_p)$ (we set $a_i(p) = 0$ when $p \mid \Delta(t)$). Rosen and Silverman [RS] prove a version of a conjecture of Nagao [Na1] which relates $A_\mathcal{E}(p)$ to the rank of $\mathcal{E}$ over $\mathbb{Q}(T)$.

Theorem 1.1 (Rosen-Silverman). Let $\mathcal{E} : y^2 = x^3 + A(T)x + B(T), A, B \in \mathbb{Z}[T]$, and assume Tate’s conjecture (known if $\mathcal{E}$ is a rational elliptic surface over $\mathbb{Q}$) for $\mathcal{E}$. Then

$$\lim_{X \to \infty} \frac{1}{X} \sum_{p \leq X} -A_\mathcal{E}(p) \log p = \text{rank } \mathcal{E}(\mathbb{Q}(T)).$$

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An elliptic curve $E$ (as in the previous theorem) is a rational elliptic surface over $\mathbb{Q}$ if and only if one of the following holds:

1. $0 < \max\{3\deg A(T), 2\deg B(T)\} < 12$.
2. $3\deg A(T) = 2\deg B(T) = 12$ and $\text{ord}_{T=0}T^{12}\Delta(T^{-1}) = 0$

(see $\text{[Min, RS]}$). In this paper we construct special rational elliptic surfaces where we are able to evaluate $A_E(p)$ exactly. For these surfaces, we have $A_E(p) = -r + O(1/p)$. By Theorem 1.1 and the Prime Number Theorem, we can conclude that the constant $r$ is the rank of $E$ over $\mathbb{Q}(T)$.

The novelty of this approach is that by forcing $A_E(p)$ to be essentially constant, provided $E$ is a rational elliptic surface over $\mathbb{Q}$, we can immediately calculate the Mordell-Weil rank without having to specialize points and calculate height matrices. Further, we obtain an exact answer for the rank, and not a lower bound.

If the degrees of the defining polynomials of $E$ are too large, our results are conditional on Tate’s conjecture if we are able to evaluate $A_E(p)$. In many cases, however, we are unable to evaluate $A_E(p)$ to the needed accuracy. Our method does generate candidate points, which upon specialization yield lower bounds for the rank. In this manner, curves of rank up to 8 over $\mathbb{Q}(T)$ have been found.

Modifications of our method may yield curves with higher rank over $\mathbb{Q}(T)$, though to find such curves requires solving very intractable non-linear Diophantine equations and then specializing the points and calculating the height matrices to see that they are independent over $\mathbb{Q}(T)$.

For additional constructions, especially for lower rank curves over $\mathbb{Q}(T)$, see $\text{[Re2]}$. For a good survey on ranks of elliptic curves, see $\text{[RuS]}$.

2. Constructing Rank 6 Rational Surfaces over $\mathbb{Q}(T)$

2.1. Idea of the Construction. The main idea is as follows: we can explicitly evaluate linear and quadratic Legendre sums; for cubic and higher sums, we cannot in general explicitly evaluate the sums. Instead, we have bounds (Hasse, Weil) exhibiting large cancellation.

The goal is to cook up curves $E$ over $\mathbb{Q}(T)$ where we have linear and quadratic expressions in $T$. We can evaluate these expressions exactly by a standard lemma on Quadratic Legendre Sums (see Lemma A.2 of the appendix for a proof), which states that if $a$ and $b$ are not both zero mod $p$ and $p > 2$, then for $t \in \mathbb{Z}$

$$\sum_{t=0}^{p-1} \left( \frac{at^2 + bt + c}{p} \right) = \begin{cases} (p-1)\left(\frac{b}{p}\right) & \text{if } p | (b^2 - 4ac) \\ -\left(\frac{a}{p}\right) & \text{otherwise.} \end{cases}$$

(2.1)

Thus if $p | (b^2 - 4ac)$, the summands are $\left( \frac{a(t-t')^2}{p} \right) = \left( \frac{a}{p} \right) 0$, and the $t$-sum is large. Later when we generalize the method we study special curves that are quartic in $T$. Let

$$y^2 = f(x, T) = x^3 T^2 + 2g(x)T - h(x)$$
$$g(x) = x^3 + ax^2 + bx + c, \ c \neq 0$$
$$h(x) = (A-1)x^3 + Bx^2 + Cx + D$$
$$D_T(x) = g(x)^2 + x^3 h(x).$$

(2.2)

Note that $D_T(x)$ is one-fourth of the discriminant of the quadratic (in $T$) polynomial $f(x, T)$. We write $A-1$ as the leading coefficient of $h(x)$, and not $A$, to simplify future computations by making the coefficient of $x^6$ in $D_T(x)$ equal $A$.

Our elliptic curve $E$ is not written in standard form, as the coefficient of $x^3$ is $T^2$. This is harmless, and later we rewrite the curve in Weierstrass form. As $y^2 = f(x, T)$, for the fiber at $T = t$ we have

$$a_t(p) = -\sum_{x(p)} \left( \frac{f(x, t)}{p} \right) = -\sum_{x(p)} \left( \frac{x^3 T^2 + 2g(x)t - h(x)}{p} \right).$$

(2.3)
where \( \left( \frac{r}{p} \right) \) is the Legendre symbol. We study \(-pA(x) = \sum_{t=0}^{p-1} \sum_{i=0}^{p-1} \left( \frac{t+D}{p} \right) \). When \( x \equiv 0 \) the \( t \)-sum vanishes if \( c \neq 0 \), as it is just \( \sum_{t=0}^{p-1} \left( \frac{2ct-D}{p} \right) \). Assume now \( x \neq 0 \). By the lemma on Quadratic Legendre Sums (Lemma A.2)

\[
\sum_{t=0}^{p-1} \left( \frac{x^3t^2 + 2g(x)t - h(x)}{p} \right) = \begin{cases} (p-1)\left( \frac{t}{p} \right) & \text{if } p \mid D_t(x) \\ -\left( \frac{t}{p} \right) & \text{otherwise} \end{cases} \tag{2.4}
\]

Our goal is to find coefficients \( a, b, c, A, B, C, D \) so that \( D_t(x) \) has six distinct, non-zero roots. We want the roots \( r_1, \ldots, r_6 \) to be squares in \( \mathbb{Q} \), as their contribution is \((-1)\left( \frac{r}{p} \right) \). If \( r_i \) is not a square, \((-1)\left( \frac{r}{p} \right) \) will be 1 for half the primes and \(-1 \) for the other half, yielding no net contribution to the rank. Thus, for \( 1 \leq i \leq 6 \), let \( r_i = \rho_i^2 \).

Assume we can find such coefficients. Then

\[
-pA(x) = \sum_{x=0}^{p-1} \sum_{t=0}^{p-1} \left( \frac{t}{p} \right) \sum_{x=0}^{p-1} \sum_{t=0}^{p-1} \left( \frac{x^3t^2 + 2g(x)t - h(x)}{p} \right) = \begin{cases} (p-1)\left( \frac{t}{p} \right) & \text{if } p \mid D_t(x) \\ -\left( \frac{t}{p} \right) & \text{otherwise} \end{cases} \tag{2.5}
\]

We must find \( a, \ldots, D \) such that \( D_t(x) \) has six distinct, non-zero roots \( \rho_i^2 \):

\[
D_t(x) = g(x)^2 + x^3h(x) = Ax^6 + (B + 2a)x^5 + (C + a^2 + 2b)x^4 + (D + 2ab + 2c)x^3 \\
+ (2ac + b^2)x^2 + (2bc)x + c^2 \\
= A(x^6 + R_5x^5 + R_4x^4 + R_3x^3 + R_2x^2 + R_1x + R_0) \\
= A(x - \rho_1^2)(x - \rho_2^2)(x - \rho_3^2)(x - \rho_4^2)(x - \rho_5^2)(x - \rho_6^2). \tag{2.6}
\]

2.2. Determining Admissible Constants \( a, \ldots, D \). Because of the freedom to choose \( B, C, D \) there is no problem matching coefficients for the \( x^5, x^4, x^3 \) terms. We must simultaneously solve in integers

\[
2ac + b^2 = R_2A \\
2bc = R_1A \\
c^2 = R_0A. \tag{2.7}
\]

For simplicity, take \( A = 64R_0^3 \). Then

\[
c^2 = 64R_0^4 \implies c = 8R_0^2 \\
2bc = 64R_0^3R_1 \implies b = 4R_0R_1 \\
2ac + b^2 = 64R_0^3R_2 \implies a = 4R_0R_2 - R_1^2. \tag{2.8}
\]

For an explicit example, take \( r_i = \rho_i^2 = i^2 \). For these choices of roots,

\[
R_0 = 518400, \quad R_1 = -773136, \quad R_2 = 296296. \tag{2.9}
\]
Solving for \(a\) through \(D\) yields
\[
A = 64R_0^3 = 8916100448256000000
\]
\[
c = 8R_0^2 = 2149004800000
\]
\[
b = 4R_0R_1 = -1603174809600
\]
\[
a = 4R_0R_1 = 16660111104
\]
\[
B = R_0A - 2a = -81136514082461622208
\]
\[
C = R_4A - a^2 - 2b = 26494790347321493520384
\]
\[
D = R_3A - 2ab - 2c = -34310759434548813362300
\]  

We convert \(y^2 = f(x, t)\) to \(y^2 = F(x, t)\), which is in Weierstrass normal form. We send \(y \rightarrow \frac{x}{t^2 + 2t - A + 1}\), and then multiply both sides by \((t^2 + 2t - A + 1)^2\). For future reference, we note that
\[
t^2 + 2t - A + 1 = (t + 1 - \sqrt{A})(t + 1 + \sqrt{A}) = (t - t_1)(t - t_2) = (t - 2985983999)(t + 2985984001).
\]  

We have
\[
f(x, t) = t^2x^3 + (2x^3 + 2ax^2 + 2bx + 2c)t - (A - 1)x^3 - Bx^2 - Cx - D
\]
\[
F(x, t) = x^3 + (2at - B)x^2 + (2bt - C)(t^2 + 2t - A + 1)x + (2ct - D)
\]  

We now study the \(-pA_F(p)\) arising from \(y^2 = F(x, t)\). It is enough to show this is \(6p + O(1)\) for all \(p\) greater than some \(p_0\). Note that \(t_1, t_2\) are the unique roots of \(t^2 + 2t - A + 1 \equiv 0 \mod p\). We find
\[
-pA_F(p) = \sum_{t=0}^{p-1} \sum_{x=0}^{p-1} \left( \frac{F(x, t)}{p} \right) = \sum_{t \neq t_1, t_2} \sum_{x=0}^{p-1} \left( \frac{F(x, t)}{p} \right) + \sum_{t = t_1, t_2} \sum_{x=0}^{p-1} \left( \frac{F(x, t)}{p} \right).
\]  

For \(t \neq t_1, t_2\), send \(x \longrightarrow (t^2 + 2t - A + 1)x\). As \((t^2 + 2t - A + 1) \not\equiv 0\), \((t^2 + 2t - A + 1)^2 = 1\). Simple algebra yields
\[
-pA_F(p) = 6p + O(1) + \sum_{t = t_1, t_2} \sum_{x=0}^{p-1} \left( \frac{f_t(x)}{p} \right) + O(1)
\]
\[
= 6p + O(1) + \sum_{t = t_1, t_2} \sum_{x=0}^{p-1} \left( \frac{(2at - B)x^2 + (2bt - C)x + (2ct - D)}{p} \right).
\]  

The last sum above is negligible (i.e., is \(O(1)\)) if
\[
D(t) = (2bt - C)^2 - 4(2at - B)(2ct - D) \equiv 0(p).
\]  

Calculating
\[
D(t_1) = 4291243480243836561123092143580209905401856
\]
\[
= 2^{32} \cdot 3^{25} \cdot 7^5 \cdot 11^2 \cdot 13 \cdot 19 \cdot 29 \cdot 31 \cdot 47 \cdot 67 \cdot 83 \cdot 97 \cdot 103
\]
\[
D(t_2) = 429124381666245275189509325391719515488256
\]
\[
= 2^{34} \cdot 3^{12} \cdot 7 \cdot 11 \cdot 13 \cdot 41 \cdot 173 \cdot 17389 \cdot 805873 \cdot 9447850813
\]  

Hence, except for finitely many primes (coming from factors of \(D(t_i), \alpha, \ldots, D, t_1\) and \(t_2\)), \(-A_F(p) = 6p + O(1)\) as desired. We have shown the following result:
Remark 2.2. We can construct infinitely many $E$ over $\mathbb{Q}(T)$ with rank 6 using (2.10), as for generic choices of roots $\rho_1, \ldots, \rho_6$, (2.15) holds.

For concreteness, we explicitly list a curve of rank at least 6. Doing a better job of choosing coefficients $a$ through $D$ (but still being crude) yields

**Theorem 2.3.** $y^2 = x^3 + Ax + B$ has rank at least 6, where

\[
\begin{align*}
A &= 1123187040185717205972 \\
B &= 50786893859117937639786031372848
\end{align*}
\]

Six points on the curve are:

\[
\begin{align*}
(67585071288, 20866449849961716) & \quad (60673071396, 18500949214922664) \\
(49153071576, 14991664661755236) & \quad (33025071828, 11131001682078096) \\
(12289072152, 8151425152633980) & \quad (13054927452, 5822267813027064)
\end{align*}
\]

As the determinant of the height matrix is approximately 880,000, the points are independent and therefore generate the group. A trivial modification of this procedure yields rational elliptic surfaces of any rank $r \leq 6$. For more constructions along these lines, see [Mil].

3. More Attempts for Curves with Rank 6, 7 and 8 over $\mathbb{Q}(T)$

3.1. **Curves of Rank 6.** We sketch another construction for a curve of rank 6 over $\mathbb{Q}(T)$ by modifying our previous arguments. We define a curve $E$ over $\mathbb{Q}(T)$ by

\[
y^2 = f(x, T) = x^4T^2 + 2g(x)T - h(x) \\
g(x) = x^4 + ax^3 + bx^2 + cx + d, \quad d \neq 0 \\
h(x) = -x^4 + Ax^3 + Bx^2 + Cx + D \\
D_T(x) = g(x)^2 + x^4h(x).
\]

We must find choices of the free coefficients such that $D_t(x) = \prod_{i=1}^7 (\alpha^2 x - \rho_i)$, with each root non-zero. For $x = 0$, we have $\sum_t (\frac{3d_t-D_t}{p}) = 0$. By Lemma [2.2] for $x$ a root of $D_t$, we have a contribution of $(p-1)(\frac{3d_t-D_t}{p}) = (p-1)(\rho_0^2\alpha^{-3}) = p-1; \text{ for all other } x \text{ a contribution of } -\left(\frac{x^4\alpha^{-3}}{p}\right) = -1$. Hence summing over $x$ and $t$ yields $7(p-1) + \sum_{x \neq \rho_i,0} -1 = 6p$. Similar reasoning as before shows we can find integer solutions (we included the factor of $\alpha^2$ to facilitate finding such solutions). We chose the coefficient of the $x^4$ term to be $t^2 + 2t + 1 = (t+1)^2$, as this implies each curve $E_t$ is isomorphic over $\mathbb{Q}$ to an elliptic curve $E'_t$ (see Appendix [3]). As $E$ is almost certainly not rational, the rank is exactly 6 if Tate’s conjecture is true for the surface. If we only desire a lower bound for the rank, we can list the 6 points and calculate the determinant of the height matrix and see if they are independent.
3.2. Probable Rank 7, 8 Curves. We modify the previous construction to

\[
\begin{align*}
y^2 &= x^3T^2 + 2g(x)T - h(x) \\
g(x) &= x^4 + ax^3 + bx^2 + cx + d, \quad d \neq 0 \\
h(x) &= Ax^4 + Bx^3 + Cx^2 + Dx + E
\end{align*}
\]

to obtain what should be higher rank curves over \(\mathbb{Q}(T)\). Choosing appropriate quartics for \(g(x), h(x)\) such that \(D_T(x) = g^2(x) + x^3h(x)\) has eight distinct non-zero perfect square roots should yield a contribution of \(8\). As the coefficient of \(T^2\) is \(x^3\), we do not lose \(p\) from summing over non-roots of \(D_T(x)\). By specializing to certain \(t = a_2s^2 + a_1s + a_0\) for some constants, we can arrange it so \(y^2 = k^2(s)x^4 + \cdots\), and by the previous arguments obtain a cubic. Unfortunately, we can no longer explicitly evaluate \(pA_E(p)\) (because of the replacement \(t \to a_2s^2 + a_1s + a_0\)). As the method yields eight points for all \(s\), we need only specialize and compute the height matrix.

As we construct a rank 8 curve over \(\mathbb{Q}(T)\) in \(\mathbb{Q}\) (when we generalize our construction), we do not provide the details here. Note, however, that sometimes there are obstructions and the rank is lower than one would expect (see [4]).

4. Using Cubics and Quartics in \(T\)

Previously we used \(y^2 = f(x, T)\), with \(f\) quadratic in \(T\). The reason is that, for special \(x\), we obtain \(y_i^2 = s_i(x_i)^2(T - t_i)^2\). For such \(x\), the \(t\)-sum is large (of size \(p\)); we then show for other \(x\) that the \(t\)-sum is small.

4.1. Idea of Construction. The natural generalization of our Discriminant Method is to consider \(y^2 = f(x, T)\), with \(f\) of higher order in \(T\). We first consider polynomials cubic in \(T\). For a fixed \(x_i\), we have the \(t\)-sum \(\sum_{t(p)} (T(x_i, t))^3\), and there are several possibilities:

1. \(f(x_i, T) = a(T - t_1)^3\). In this case, the \(t\)-sum will vanish, as \((\frac{t - t_1}{p})^3 = (\frac{t - t_1}{p})\).
2. \(f(x_i, T) = a(T - t_1)^2(T - t_2)\). The \(t\)-sum will be \(O(1)\), as for \(t \neq t_1\) we have \((\frac{t - t_1}{p})(t - t_2) = (\frac{t - t_2}{p})\).
3. \(f(x_i, T) = a(T - t_1)(T - t_2)(T - t_3)\). This will in general be of size \(\sqrt{p}\).
4. \(f(x_i, T) = a(T - t_1)(T^2 + bt + c)\), with the quadratic irreducible over \(\mathbb{Z}/p\mathbb{Z}\). This happens when \(b^2 - 4c\) is not a square mod \(p\). This will in general be of size \(\sqrt{p}\).
5. \(f(x_i, T) = aT^3 + bT^2 + cT + d\), with the cubic irreducible over \(\mathbb{Z}/p\mathbb{Z}\). Again, this will in general be of size \(\sqrt{p}\).

Thus, our method does not generalize to \(f(x, T)\) cubic in \(T\). The problem is we cannot reduce to \(\left(\frac{t - t_1}{p}\right)^{2n-1} \left(\frac{t - t_2}{p}\right)\). We therefore investigate \(f(x, T)\) quartic in \(T\). Consider, for simplicity, a curve \(E\) over \(\mathbb{Q}(T)\) of the form:

\[y^2 = f(x, T) = A(x)T^4 + B(x)T^2 + C(x),\]

\(A(x), B(x), C(x) \in \mathbb{Z}[x]\) of degree at most 4. The polynomial \(AT^4 + BT^2 + C\) has discriminant \(16AC(AAC - B^2)^2\). There are several possibilities for special choices of \(x\) giving rise to large \(t\)-sums (sums of size \(p\)):

1. \(A(x_i), B(x_i) \equiv 0 \mod p\), \(C(x_i)\) a non-zero square mod \(p\). Then the \(t\)-summand is of the form \(c^2\), contributing \(p\).
2. \(A(x_i), C(x_i) \equiv 0 \mod p\), \(B(x_i)\) a non-zero square mod \(p\). Then the \(t\)-summand is of the form \((bt)^2\), contributing \(p - 1\).
3. \(B(x_i), C(x_i) \equiv 0 \mod p\), \(A(x_i)\) a non-zero square mod \(p\). Then the \(t\)-summand is of the form \((at^2)^2\), contributing \(p - 1\).
4. \(A(x_i)\) is a non-zero square mod \(p\) and \(B(x_i)^2 - 4A(x_i)C(x_i) \equiv 0 \mod p\). Then the \(t\)-summand is of the form \(a^2(t^2 - t_1)^2\), contributing \(p - 1\).
In the above construction, we are no longer able to calculate \( A_E(p) \) exactly. Instead, we construct curves where we believe \( A_E(p) \) is large. This is accomplished by forcing points to be on \( E \) which satisfy any of (1) through (4) above. As we are unable to evaluate the \( A_E(p) \) sums, we specialize and calculate height matrices to show the points are independent. Unfortunately, some of our constructions yielded 9 and 10 points on \( E \), but some of these points were linearly dependent on the others, or torsion points (see \[\text{Ref} \]).

This method, with a quartic in \( T \), can force a maximum number of 12 points on \( E \). It is possible to have 8 points from the vanishing of the discriminant (in \( t \)), and an additional 6 points from the simultaneous vanishing of pairs of \( A(x), B(x), C(x) \); however, any common root of \( A \) or \( C \) with \( B \) is also a root of \( B^2 - 4AC \), so there are at most 4 new roots arising from simultaneous vanishing, for a total of 12 possible points.

4.2. **Rank (at least) 7 Curve.** For appropriate choices of the parameters, the curve \( E : y^2 = A(x)T^4 + 4B(x)T^2 + 4C(x) \) over \( \mathbb{Q}(T) \) with

\[
A(x) = a_1a_2a_3a_4(x - a_1)(x - a_2)(x - a_3)(x - a_4) \\
C(x) = a_1a_2c_1c_2(x - a_1)(x - a_2)(x - c_1)(x - c_2) \\
B(x) = a_1^2a_2^2(x - c_1)(x - c_2)(x - a_3)(x - a_4)
\]

has rank at least 7. We get 6 points from the common vanishing of \( A, B, C \) in pairs and an additional point from a factor of \( B^2 - AC \). Choosing \( a_1 = -25, a_2 = -5, a_3 = -10, a_4 = -1, c_1 = -9, c_2 = 15 \) we find that the points

\[
(-25, 120000T), (-5, 10000T), (-10, 11250), (-1, 28800), \\
(-9, 800T^2), (15, 20000T^2), (65/7, (540000^2 - 2880000)/49)
\]

all lie on \( E \). Upon transforming to a cubic (see Appendix \[\text{Ref} \]), specializing to \( T = 20 \), and considering the minimal model, we found that these points are linearly independent (PARI calculates the determinant of the height matrix is approximately 37472). Note this is not a rational surface, as the coefficient of \( x \) in Weierstrass form is of degree 8.

4.3. **Rank (at least) 8 Curve.** For appropriate choices of the parameters, the curve \( E : y^2 = A(x)T^4 + B(x)T^2 + C(x) \) over \( \mathbb{Q}(T) \) with

\[
A(x) = x^4, \quad B(x) = 2b_3x^3 + b_2x^2 + b_1x + b_0 + b^2, \quad C(x) = x(b_3^2x^3 + c_2x^2 + c_1x + c_0)
\]

has rank at least 8. As the coefficient of \( x^4 \) is \( T^4 + 2b_3T^2 + b_1^2 \), a perfect square, \( E \) can easily be transformed into Weierstrass form (see Appendix \[\text{Ref} \]). The common vanishing of \( A \) and \( C \) at \( x = 0 \) produces a point \( S_0 = (0, bT) \) on \( E/\mathbb{Q}(T) \). Also notice that as before, if \( B^2 - 4AC \) vanishes at \( x = x_i \) then we can rewrite:

\[
A(x_i)T^4 + B(x_i)T^2 + C(x_i) = A(x_i) \left( T^2 + \frac{B(x_i)}{2A(x_i)} \right)^2 = x_i^2 \left( T^2 + \frac{B(x_i)}{2x_i^2} \right)^2
\]

Thus we obtain a point \( P_{x_i} = (x_i, x_i^2(T^2 + B(x_i)/2x_i^2)) \) on \( E \). We chose constants \( b_1, b_2 \) an \( c_i \) so that

\[
B^2 - 4AC = (x - 1)(x + 1)(x - 4)(x + 4)(x - 9)(x + 9)(x - 16)
\]

and obtain a curve \( E \) over \( \mathbb{Q}(T) \) with coefficients:

\[
A = x^4, \quad B(x) = 382205952x^4 + 89233x^3 - 923x^2 - 92x + 144, \\
C(x) = 34254919166180065369x^4 - 528356915749387x^3 \\
+ 58432555897695216x^2 - 28179280429056x \\
+ 880602531408x - 169869312
\]

As discussed above, the curve \( E \) given by (4.3) has 8 rational points over \( \mathbb{Q}(T) \), namely \( S_0 \) and \( P_{x_i} \), for \( x_i = \pm 1, \pm 4, \pm 9, 16 \). As \( E \) is not a rational surface, and as we cannot evaluate \( A_E(p) \) exactly, we need to
make sure the points are linearly independent. Specializing to $T = 1$ yields the elliptic curve with minimal model

$$E_1: y^2 = x^3 - x^2 - \alpha x + \beta$$

$$\alpha = 357917711928106838175050781865$$

$$\beta = 8790806811671574287759999288018136706011725.$$  

The eight points of $E_T$ at $T = 1$ are linearly independent on $E_1/Q$ (PARI calculates the determinant of the height matrix to be about 124079248627.08), proving $E$ does have rank at least $8$ over $Q(T)$.

### 5. Linear Dependencies Among Points

Not all choices of $A(x), B(x), C(x)$ which yield $r$ points on the curve $E : y^2 = A(x)T^4 + 4B(x)T^2 + 4C(x)$ actually give a curve of rank at least $r$ over $Q(T)$. We found many examples giving 9 and 10 points by choosing $A(x) = C(x)$ so that $B^2 - AC$ factors nicely, and then searching through prospective roots of this quantity as well as roots of $A(x) = C(x)$. One such curve giving 10 points arises from

$$A(x) = C(x) = (x - 1)^2(2x - 1)^2$$

$$B(x) = 12316x^4 + 2346x^3 - 239x^2 - 24x + 1,$$  

and has the following points on it

$$(0, T^2 + 2), \left( -\frac{1}{19}, \frac{420}{361}(T^2 + 2) \right), \left( -\frac{1}{4}, \frac{15}{8}(T^2 + 2) \right), \left( \frac{1}{5}, \frac{56}{81}(T^2 + 2) \right), \left( -\frac{1}{7}, \frac{72}{49}(T^2 - 2) \right), \left( -\frac{1}{5}, \frac{42}{25}(T^2 - 2) \right), \left( \frac{1}{11}, \frac{90}{121}(T^2 - 2) \right), \left( \frac{1}{16}, \frac{105}{128}(T^2 - 2) \right), (1, 240T), \left( \frac{1}{2}, 63T \right).$$

It can be shown, however, that upon translating to a cubic only the (translated versions of the) second, third, fifth, sixth, and ninth of these points are independent over $Q(T)$. While the contribution from these points makes $A_E(p)$ want to be large, this is not reflected by a large rank.

### 6. Using Higher Degree Polynomials

Let $f(x, T)$ be a polynomial of degree 3 or 4 in $x$ and arbitrary degree in $T$ and let $E$ be the elliptic curve over $Q(T)$ given by $y^2 = f(x, T)$ (with the coefficient of $x^4$ a perfect square or zero). The remarks at the beginning of Section 4 about cubics suggest that we should look for polynomials $f(x, T)$ with even degree in $T$, say $\deg_T(f) = 2n$.

The nice feature of quadratics and biquadratics that we used in the previous constructions was the fact that a zero of the discriminant indicates that the polynomial $f(x, T)$ factors as a perfect square. However, when $f$ is of arbitrary degree $2n$ in $T$ this is no longer true: a zero of the discriminant $D_T(x)$ indicates just a multiple root. However, in the most general case, there exist $n$ quantities $D_i(T)$ such that their common vanishing at $x = x_0$ implies that $f(x, T)$ factors as a perfect square. As an example we look at a quartic of the form $f(x, T) = A^2T^4 + BT^3 + CT^2 + DT + E^2$, where $\deg_x(A, E) \leq 2$ and $\deg_x(B, C, D) \leq 4$. This can be rewritten as:

$$A^2T^4 + 2AT^2\left(\frac{B}{A} + \frac{C}{2A} - \frac{B^2}{8A^2}\right) + \left(\frac{B^2}{2A} + \frac{C}{2A} - \frac{B^2}{8A^2}\right)^2 + (D - \frac{B}{A}(\frac{C}{2A} - \frac{B^2}{8A^2}))T + (\frac{C}{2A} - \frac{B^2}{8A^2})^2 + E^2.$$

The last two terms are the ones which are keeping the polynomial from being a perfect square. Thus, if

$$D - \frac{B}{A} \left( \frac{C}{2A} - \frac{B^2}{8A^3} \right) = 0, \quad E^2 - \left( \frac{C}{2A} - \frac{B^2}{8A^3} \right)^2 = 0$$  

(6.12)
then the polynomial $f$ will be a square. This is equivalent to
\[
D_{1,T} = 8A^4D - 4A^2BC + B^3 = 0
\]
\[
D_{2,T} = 64A^6E^2 - 16A^4C^2 - B^4 + 8A^2CB^2 = 0.
\]
(6.13)

Note that if $B=D=0$, the conditions that these polynomials impose reduce to the usual discriminant. Also, \(\deg_x(D_{1,T}) \leq 12\), \(\deg_x(D_{2,T}) \leq 16\), so we could get up to 12 points of common vanishing of the $D_i$. The authors have tried to find suitable constants without success, due to the complexity of the Diophantine equations.

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**Appendix A. Sums of Legendre Symbols**

For completeness, we provide proofs of the quadratic Legendre sums that are used in our constructions.

**A.1. Factorizable Quadratics in Sums of Legendre Symbols.**

**Lemma A.1.** For $p > 2$

\[
S(n) = \sum_{x=0}^{p-1} \left( \frac{n+x}{p} \right) \left( \frac{x}{p} \right) = \begin{cases} 
  p - 1 & \text{if } p | (n_1 - n_2) \\
  -1 & \text{otherwise.}
\end{cases}
\]

(A.14)

**Proof.** Translating $x$ by $-n_2$, we need only prove the lemma when $n_2 = 0$. Assume $(n, p) = 1$ as otherwise the result is trivial. For $(a, p) = 1$ we have:

\[
S(n) = \sum_{x=0}^{p-1} \left( \frac{n+x}{p} \right) \left( \frac{x}{p} \right) = \sum_{x=0}^{p-1} \left( \frac{n + a^{-1}x}{p} \right) \left( \frac{a^{-1}x}{p} \right)
\]

\[
= \sum_{x=0}^{p-1} \left( \frac{an + x}{p} \right) \left( \frac{x}{p} \right) = S(an)
\]

(Hence)

\[
S(n) = \frac{1}{p-1} \sum_{a=1}^{p-1} \sum_{x=0}^{p-1} \left( \frac{an + x}{p} \right) \left( \frac{x}{p} \right)
\]

\[
= \frac{1}{p-1} \sum_{a=0}^{p-1} \sum_{x=0}^{p-1} \left( \frac{an + x}{p} \right) \left( \frac{x}{p} \right) - \frac{1}{p-1} \sum_{x=0}^{p-1} \left( \frac{x}{p} \right)^2
\]

\[
= \frac{1}{p-1} \sum_{x=0}^{p-1} \left( \frac{x}{p} \right) \sum_{a=0}^{p-1} \left( \frac{an + x}{p} \right) - 1
\]

\[
= 0 - 1 = -1.
\]

(A.16)
We note that $\sum_{r=0}^{p-1} \left( \frac{r^2 + \delta}{p} \right) = 0$ for $(n, p) = 1$. This is true for all odd primes (as there are $\frac{p-1}{2}$ quadratic residues, $\frac{p-1}{2}$ non-residues, and 0); for $p = 2$, there is one quadratic residue, no non-residues, and 0.

A.2. General Quadratics in Sums of Legendre Symbols.

**Lemma A.2** (Quadratic Legendre Sums). Assume $a$ and $b$ are not both zero mod $p$ and $p > 2$. Then

$$\sum_{t=0}^{p-1} \left( \frac{at^2 + bt + c}{p} \right) = \begin{cases} \frac{(p-1)(b^2 - 4ac)}{p} & \text{if } p \mid (b^2 - 4ac) \\ \left( -\left( \frac{a}{p} \right) \right) & \text{otherwise.} \end{cases}$$  \hspace{1cm} (A.17)

**Proof.** Assume $a \not\equiv 0(p)$ as otherwise the proof is trivial. Let $\delta = 4^{-1}(b^2 - 4ac)$. Then

$$\sum_{t=0}^{p-1} \left( \frac{at^2 + bt + c}{p} \right) = \sum_{t=0}^{p-1} \left( \frac{a^{-1}}{p} \right) \left( \frac{a t^2 + b t + c}{p} \right) = \sum_{t=0}^{p-1} \left( \frac{a}{p} \right) \left( \frac{t^2 + b t + ac}{p} \right) = \sum_{t=0}^{p-1} \left( \frac{a}{p} \right) \left( \frac{(t - 2^{-1}b - 4^{-1}(b^2 - 4ac)}{p} \right) = \left( \frac{a}{p} \right) \sum_{t=0}^{p-1} \left( \frac{t^2 - \delta}{p} \right)$$  \hspace{1cm} (A.18)

If $\delta \equiv 0(p)$ we get $p - 1$. If $\delta \equiv \eta^2, \eta \not\equiv 0$, then by Lemma A.1

$$\sum_{t=0}^{p-1} \left( \frac{t^2 - \delta}{p} \right) = \sum_{t=0}^{p-1} \left( \frac{t - \eta}{p} \right) \left( \frac{t + \eta}{p} \right) = -1.$$  \hspace{1cm} (A.19)

We note that $\sum_{\delta=0}^{p-1} \left( \frac{t^2 - \delta}{p} \right)$ is the same for all non-square $\delta$’s (let $g$ be a generator of the multiplicative group, $\delta = g^{2k+1}$, change variables by $t \rightarrow g^{k}t$). Denote this sum by $S$, the set of non-zero squares mod $p$ by $R$, and the non-squares mod $p$ by $N$. Since $\sum_{\delta=0}^{p-1} \left( \frac{t^2 - \delta}{p} \right) = 0$ we have

$$\sum_{\delta=0}^{p-1} \sum_{t=0}^{p-1} \left( \frac{t^2 - \delta}{p} \right) = \sum_{t=0}^{p-1} \left( \frac{t^2}{p} \right) + \sum_{\delta \in N} \sum_{t=0}^{p-1} \left( \frac{t^2 - \delta}{p} \right) + \sum_{\delta \in R} \sum_{t=0}^{p-1} \left( \frac{t^2 - \delta}{p} \right)$$

$$\quad = (p-1) + \frac{p-1}{2}(-1) + \frac{p-1}{2}S = 0.$$  \hspace{1cm} (A.20)

Hence $S = -1$, proving the lemma. \hfill \square

**Appendix B. Converting from Quartics to Cubics**

We record two useful transformations from quartics to cubics. In all theorems below, all quantities are rational.

**Theorem B.1.** If the quartic curve $y^2 = x^4 - 6cx^2 + 4dx + e$ has a rational point, then it is equivalent to the cubic curve $Y^2 = 4X^3 - g_2X - g_3$, where

\[ g_2 = e + 3c^2, \quad g_3 = -ce - d^2 + c^3, \]  \hspace{1cm} (B.21)

and

\[ 2x = (Y - d)/(X - c), \quad y = -x^2 + 2X + c. \]  \hspace{1cm} (B.22)
CONSTRUCTING ELLIPTIC CURVES OVER $\mathbb{Q}(T)$ WITH MODERATE RANK

See [Mor], page 77. Note that if the leading term of the quartic is $a^2x^4$, one can send $y \to y/a$ and $x \to x/a$.

**Theorem B.2.** The quartic $v^2 = au^4 + bu^3 + cu^2 + du + q^2$ is equivalent to the cubic $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, where

$$a_1 = d/q, \quad a_2 = c - (d^2/4q^2), \quad a_3 = 2qb, \quad a_4 = -4q^2a, \quad a_6 = a_2a_4$$

(B.23)

and

$$x = \frac{2q(v + q) + du}{u^2}, \quad y = \frac{4q^2(v + q) + 2q(du + cu^2) - (d^2u^2/2q)}{u^3}$$

(B.24)

The point $(u, v) = (0, q)$ corresponds to $(x, y) = \infty$ and $(u, v) = (0, -q)$ corresponds to $(x, y) = (-a_2, a_1a_2 - a_3)$.

See [Wa], page 37.

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