The stability of the shell of D6-D2 branes in a $N = 2$ supergravity solution

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I. INTRODUCTION

One of the central issues in quantum gravity is the resolution of singularities, i.e., the initial singularity in our universe or a central singularity in black holes. It is expected that string theory, which is one of the most prominent theories including the quantum gravity, may resolve the issue of such singularities. From the general relativity theory point of view, the appearance of a naked singularity is a serious problem because we cannot predict what happens in the future. Therefore, it is particularly important to study whether the naked singularities are excised in string theory or not. An interesting example of the naked singularity is repulson singularity, which appears in supersymmetric solutions with spherical symmetry in supergravity theory [1–3]. Kallosh and Linde [2] showed that all test particles, either massless or massive, cannot touch the repulson singularity. This particular type of singularity is a globally naked singularity and the weak cosmic censorship [4] seems to be violated. This is essentially different from the extreme Reissner-Nordström black hole where a locally naked singularity exists inside the event horizon.

Recently, Johnson, Peet and Polchinski analyzed the motion of a D6-brane probe wrapped on a K3 manifold in the type IIA supergravity solution and proposed the enhançon mechanism which excises the repulson singularity [5,6]. The higher curvature corrections on the world volume of D6-brane induce a negative D2-charge and a negative D2-tension (see Refs. [7–10] for details), and the total tension of the wrapped D6-brane probe vanishes at a special radius $r = r_e$ larger than the radius of the repulson singularity $|r_2|$. This fact led the proposal that gauge symmetry is enhanced at the radius $r = r_e$, which is called the enhançon radius, and that a shell consisted of the wrapped D6-branes is located at the enhançon radius. As a result, the original repulson singularity can be removed due to the flat geometry for $r < r_e$.

Since there is no precise dual theory to this type of supergravity theory at the present stage, this motivates us to investigate the picture of the enhançon mechanism from supergravity side more deeply. It is especially interesting to examine whether or not the enhançon geometry is a solution of the supergravity theory including the self-gravity of the branes, and whether the shell of the wrapped D6-branes is stable or collapses by the gravitational effect.

So, in this paper, we first construct an effective energy-momentum tensor of the shell consisted of the wrapped D6-branes and study the static solutions by using Israel’s...
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By numerical analysis, it is found that there is no eigenmode whose frequency $\omega$ of the shell and the fields is imaginary. To our surprise, although the tension of the wrapped D6-brane is negative in the region $r < r_\epsilon$, the shell of the branes never has eigenmodes with $\omega^2 < 0$ if we take into account the interactions between the shell and the fields, which include the self-gravity of the shell.

Furthermore, it is shown that when the radius of the shell is less than $r_\epsilon$, resonances are caused at proper oscillation frequencies of the shell, namely the system becomes "unstable" in this region. On the other hand, when the radius of the shell is greater than or equal to $r_\epsilon$, the system is really stable, that is, there is no eigenmode with $\omega^2 < 0$ and no resonance is produced. These facts indicate that it may be able to explain even classically the dynamical reason why the shell is constructed at the enhançon radius.

The outline of this paper is as follows. In Sec. II, we briefly review the D6-D2 brane solution and show that the velocity of the D6-brane probe reaches the speed of light at the enhançon radius. In Sec. III, we derive the effective energy-momentum tensor of the shell of the D6-D2 branes under the reasonable conditions, and give the energy conservation law and the dilaton equation on the Gaussian normal coordinates. In Sec. IV, we show "the enhançon geometry" with the shell at any radius is the solution of the supergravity based on the Israel’s junction condition. In Sec. V, we first derive perturbed equations of the gravitational and matter fields inside the shell located at arbitrary radius. Next, we derive perturbed junction conditions of the fields on the shell, the equation of motion of the shell and perturbed equations of fields outside the shell. Finally, using numerical analysis, we study the stability of the shell and the fields. Sec. VI is devoted to conclusion and discussion. Throughout this paper, we set $\alpha' = 1$.

II. ENHANÇON GEOMETRY AND THE RELATIVISTIC MOTION OF A D6-BRANE

The low-energy IIA supergravity action is given by

$$S_{\text{sugra}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[ e^{-2\Phi} \left( \hat{R} + 4(\nabla\Phi)^2 \right) - \frac{1}{2} |\hat{H}_3|^2 - \frac{1}{2} |\hat{F}_4|^2 - \frac{1}{2} |\hat{F}_8|^2 - \frac{1}{2\sqrt{-G}} B_{(2)} \wedge F_{(4)} \wedge F_{(4)} \right],$$

(2.1)

where $\hat{G}_{MN}$ is the metric in the string frame, $\Phi$ is the dilaton field, $\hat{H}_3$ is the field strength of the anti-symmetric tensor field $B_{(2)}$, $\hat{F}_{(p+2)}$ is expressed by the $(p+1)$-form Ramond-Ramond (R-R) field $C_{(p+1)}$ as $F_{(p+2)} = dC_{(p+1)}$ and $\hat{F}_4 = dC_{(3)} + C_{(1)} \wedge dB_{(2)}$, where $C_{(7)}$ is the dual field of $C_{(1)}$. The hats on the fields and the differential denote that the contractions are done with the string metric $\hat{G}_{MN}$.

Let us consider a D6-D2 system in which there are N D2-branes and N D6-branes. The supersymmetric solution of the system is

$$ds^2 = \hat{G}_{MN} dx^M dx^N = \left( Z_2^2 Z_6^{-2} \eta_{\mu\nu} dx^\mu dx^\nu + Z_2^2 Z_6^{-2} \delta_{ij} dx^i dx^j \right) + V \hat{Z}_2^2 Z_6^{-2} G_{pq}^{(K3)} dx^p dx^q, \quad (2.2)$$

$$e^{2\Phi} = g^2 Z_2^2 Z_6^{-2}, \quad (2.3)$$

$$C_{(3)} = (g Z_2)^{-1} dx^0 \wedge dx^4 \wedge dx^5, \quad (2.4)$$

$$C_{(7)} = V (g Z_6)^{-1} dx^0 \wedge dx^4 \wedge dx^5 \wedge dV_{(K3)}, \quad (2.5)$$

where $\mu, \nu = (0, 4, 5)$, $i, j = (1, 2, 3)$, and $p, q = (6, 7, 8, 9)$. The $G_{pq}^{(K3)}$ denotes the metric of the K3 manifold with unit volume. The D2-branes are aligned along the 4, 5 directions, while the D6-branes are aligned to the 4, 5, 6, 7, 8, 9 directions.

The harmonic functions are expressed as

$$Z_2 = 1 + \frac{r_2}{r}, \quad Z_6 = 1 + \frac{r_6}{r}, \quad (2.6)$$

where $r := \sqrt{x^i x^i}$ and

$$r_2 := -\frac{g N V_*}{2V}, \quad r_6 := \frac{g N}{2}, \quad (2.7)$$

d$V_{(K3)}$ is the volume form of the K3 manifold of unit volume. To keep the tension of a wrapped D6-brane, which will be investigated in detail later, positive at infinity, we assume that $V > V_* := (2\pi)^4$.

Because of the negative charge of the D2-branes, there is a repulsion singularity at $r = |r_2|$, where Ricci scalar curvature diverges as $(r - |r_2|)^{-3/4}$. To investigate the causal structure of this geometry with the repulsion singularity, let us solve a radial null geodesic in this solution. By using Eq. (2.2), the future-going null geodesics behaves near the singularity as

$$t = -\int^r (Z_2 Z_6)^{3/2} dr \propto (r + r_2)^{3/2}, \quad (2.8)$$

where $t := x^0$. This means that $t$ is finite when $r \to |r_2|$ and hence the repulsion singularity is a timelike singularity, as depicted in Fig. 1. To resolve the repulsion singularity, Johnson, Peet, and Polchinski have proposed

\[\text{\footnote{Although we cannot write down the metric explicitly, it is well known that the Ricci curvature is zero. For later convenience, here, we shall formally denote the metric as } G_{pq}^{(K3)} dx^p dx^q} \]
the enhançon geometry which is flat inside the unique enhançon radius $r_e$.

\[ +\mu_p \int_{p+1} e^{(B_{(2)}+2\pi F)} \wedge \sum_q C_{(q)}, \quad (2.9) \]

where $\mu_p = (2\pi)^{-p}$ is the charge of the Dp-brane and $\hat{g}_{ab}$ is the pull-back of the string metric $\hat{G}_{MN}$:

\[ \hat{g}_{ab} := \frac{\partial x^M}{\partial \xi^a} \frac{\partial x^N}{\partial \xi^b} \hat{G}_{MN}. \quad (2.10) \]

$B_{ab}$ is the component of the spacetime NS-NS fields $B_{(2)}$ parallel to the Dp-brane and $F_{ab}$ (and $F$) is the field strength of the gauge field living on the Dp-brane. In this paper, we consider the D6-D2 brane system with vanishing $B_{(2)}$ and $F_{ab}$, so that the action becomes

\[ S_{st}^{D_p} = -\mu_p \int d^{p+1}\xi e^{-\Phi} (\det \hat{g}_{ab})^{\frac{1}{2}} \]

\[ + \mu_p \int d^{p+1}\xi C_{(p+1)}. \quad (2.11) \]

Now, consider the motion of a wrapped D6-brane on the K3 manifold. If we take the higher-curvature correction terms in the K3 to the action of the D6-brane into account, a D2-brane with a negative D2-charge is induced (see [7–10] for details). Hence, the total action of the wrapped D6-brane on the K3 becomes $S_{st}^{D_6} = S_{st}^{D_2}$. After integration of the wrapped coordinates on the K3, we obtain the effective action of the brane which is just a membrane:

\[ S_{st}^{D_6-D_2} = -\mu_6 \int d^3\xi e^{-\Phi(r)}(V(r) - V_*) (\det \hat{g}_{ab})^{\frac{1}{2}} \]

\[ + \frac{\mu_6}{g} \int d^3\xi \left( \frac{V}{Z_6} - \frac{V_*}{Z_2} \right), \quad (2.12) \]

where $V(r)$ is defined by $V(r) := VZ_2/Z_6$. Note that the tension of the brane vanishes at the enhançon radius $r_e$ satisfying the following relation:

\[ \frac{V}{Z_6(r_e)} = \frac{V_*}{Z_2(r_e)}. \quad (2.13) \]

The explicit form of the enhançon radius is

\[ r_e = \frac{2V}{V - V_*}|r_2|. \quad (2.14) \]

If we take a static gauge\(^1\)

\[ x^0 = t = \xi^0, \quad x^4 = \xi^4, \quad x^5 = \xi^5, \quad (2.15) \]

the reduced Lagrangian density $\mathcal{L}$ is

\[ \text{Enhanc} \text{o}n \text{ radius} \]

\[ \text{Repulsion Singularity} \]

FIG. 1. The Penrose diagram of a D6-D2 supergravity solution is shown. The dashed line represents an orbit of a D6-brane wrapped on the K3, while the dotted line represents the orbit of an ingoing-radial null geodesic.

If we ignore higher order corrections including the curvature corrections, the action of a Dp-brane is given by

\[ S_{st}^{D_p} = -\mu_p \int d^{p+1}\xi e^{-\Phi} \left[ -\det(\hat{g}_{ab} + B_{ab} + 2\pi F_{ab}) \right]^{\frac{1}{2}} \]

\[^1\text{Since D2 brane is a compact surface, the space of }x^4 \text{ and } x^5 \text{ should be also compact in the static gauge, to be exact. Here, we take this space as two dimensional torus of unit volume.}\]
\[ \mathcal{L} = \frac{\mu_6}{g} \left( \frac{V}{Z_6} - \frac{V_a}{Z_2} \right) \left[ 1 - \left( 1 - Z_2 Z_6 \dot{x}^i \dot{x}^i \right)^{\frac{1}{2}} \right], \quad (2.16) \]

where dots denote the differentials with respect to \( t \).

The energy conservation law gives us the following equation

\[ \left( \frac{V}{Z_6} - \frac{V_a}{Z_2} \right) \left[ \left( 1 - Z_2 Z_6 \frac{dx^i}{dt} \frac{dx^i}{dt} \right)^{-\frac{1}{2}} - 1 \right] = E, \quad (2.17) \]

where \( E \) is a constant. If the probe has an ingoing-velocity at \( r = r_i \gg r_e \), \( E \) is strictly positive. If the brane probe stops at a radius \( r_m > r_e \), then the left hand side would be zero at \( r = r_m \). This leads to contradiction because \( E \) is positive. Hence the brane probe cannot stop outside of the enhançon radius \( r_e \). At the enhançon radius, the first term in the square bracket should diverge infinitely because \( V/Z_6 - V_a/Z_2 = 0 \) at \( r = r_e \). This means that the velocity of the brane probe reaches the speed of light at the enhançon radius (See Fig. 3).

It seems strange at first sight that according to the picture by Johnson, Peet and Polchinski in Ref. [5], the branes should collect near the enhançon radius and that they cannot leave the location of the enhançon radius, while we have just seen, the probe discussed above moves at a speed of light near the enhançon radius. Hence we need further analysis about this enhançon radius and the mechanism beyond the probe approximation. In next sections, we study the whole system including the self-gravity of the branes and the self-interactions between the fields and the branes.

### III. THE ENERGY-MOMENTUM TENSOR OF THE D6-D2 BRANES SHELL

By the conformal transformation such as \( G_{MN} = e^{-\Phi/2} \hat{G}_{MN} \), we obtain the low-energy effective action in the Einstein frame:

\[ S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-\tilde{G}} \left[ R - \frac{1}{2} (\nabla \Phi)^2 - \frac{e^\Phi}{2} |F_4|^2 \right. \]
\[ \left. - \frac{e^{-\Phi}}{2} |F_8|^2 \right] + \sum_{i=1}^{N} (S^{D6}_{(i)} - S^{D2}_{(i)}), \quad (3.1) \]

where \( G_{MN} \) is the Einstein metric and all the contractions are done with \( G_{MN} \). \( i \) represents the numbering of each wrapped D6-brane of which a shell consists. In [2, 3], the energy-momentum tensor is obtained when the shell is fixed at a radius. In this section we will obtain the tensor by the following covariant way so that it can be applied to more general cases.

In general relativity, it is well known that there is the following local (Gaussian normal) coordinates

\[ ds^2 = d\chi^2 + g_{ab} d\zeta^a d\zeta^b, \quad (a, b = 0, 2, ..., 9) \quad (3.2) \]

in the neighborhood of the D6-D2 branes shell such that it moves along \( \chi = \chi_0 \) constant.

Hereafter, we shall impose spherical symmetry for simplicity. Hence, we can choose the Gaussian normal coordinates \((\chi, \zeta^a)\) as \( \chi = \chi(r, t) \) and \( \zeta^a = \zeta^a(r, t) \), while the other coordinates are same as \( \theta, \phi, x^m (m = 4, 5), x^p \).

The nine-dimensional embedded metric is rewritten by

\[ ds^2 = q_{ab} d\zeta^a d\zeta^b = -g_{00}(d\zeta^0)^2 + G_{0\phi} d\theta d\phi + G_{\phi\phi} d\phi^2 \]
\[ + G_{mn} dx^m dx^n + G_{pq} dx^p dx^q. \quad (3.3) \]

To derive the effective energy-momentum tensor coming from the wrapped D6-branes, we shall impose the following conditions:

1. we can take a static gauge such as \( \xi^0 = \zeta^0, \xi^m = x^m \) and \( \xi^p = \zeta^p \) for each D6 brane and \( \xi^0 = \zeta^0 \) and \( \xi^m = x^m \) for each D2 brane.

2. each D6 (D2) brane moves along the radial direction only.

3. each D6 (D2) brane is distributed uniformly on the shell.

The condition 2 implies, in other words, that each D6-D2 brane moves on the \( \chi = \chi_0, \zeta^a = \text{const} (a \neq 0) \).

In order to see the way how to derive the effective energy-momentum tensor of the D6-D2 brane shell, it would be enough to discuss \( \sum_{i=1}^{N} S^{D6}_{(i)} \) only. Using the conditions 1 and 2, the action can be reduced as follows:

\[ S_6 := \sum_{i=1}^{N} (S^{D6}_{(i)} - \mu_6 \int d^7 \xi C_{(7)}) |_{\chi = \text{const}} \]
\[ = -\mu_6 \sum_{i=1}^{N} \int d^7 \zeta e^{\frac{4\Phi}{2 \kappa_{10}^2}} \left( \frac{\partial x^M}{\partial \zeta^i} \frac{\partial x^N}{\partial \zeta^j} G_{MN} \right)^{\frac{1}{2}} \]
\[ \times (\det G_{mn} \cdot \det G_{pq})^{\frac{1}{2}} \]
\[ = -N \mu_6 \int d^7 \zeta e^{\frac{4\Phi}{2 \kappa_{10}^2}} (-q_{00} \cdot \det G_{mn} \cdot \det G_{pq})^{\frac{1}{2}}. \quad (3.4) \]

Because of the condition 3, each D6-brane is distributed uniformly over the two-sphere \((\theta, \phi)\). Introducing the number density \( \rho_6 \) satisfying

\[ \int d\theta d\phi (G_{0\phi} G_{\phi\phi})^{\frac{1}{2}} \rho_6 = N, \quad (3.5) \]

we can rewrite Eq. (3.4) as the following ten-dimensional form:

\[ S_6 = -\mu_6 \int d^9 \zeta e^{\frac{4\Phi}{2 \kappa_{10}^2}} (-\det q_{ab})^{\frac{1}{2}} \rho_6 \]
\[ = -\mu_6 \int dx^{10} \delta(\chi - \chi_0) e^{\frac{4\Phi}{2 \kappa_{10}^2}} (-\det G)^{\frac{1}{2}} \rho_6. \quad (3.6) \]
Note that $\rho_6$ is a function of $\zeta^0$ only. As concerns the subspace $(x^m, x^p)$, we can easily calculate the energy-momentum tensor by using the usual formula

$$T_{ab} = -\frac{2}{\sqrt{-q}} \frac{\delta(\sqrt{-q} L_m)}{\delta q^{ab}},$$

(3.7)

where $L_m$ is a matter Lagrangian. In the subspace $(\zeta^0, \theta, \phi)$, however, we should impose the constraint (3.3) when we derive $T_{ab}$ by using Eq. (3.5). Let us define the three-dimensional unit timelike flow vector along each D6-brane by $V^\alpha = (V^0, V^\theta, V^\phi) := (\sqrt{-q^{00}}, 0, 0)$. Then, in terms of the current vector $J^a = \rho_6 V^a$, Eq. (3.5) is rewritten in the following covariant form:

$$J^a \cdot \alpha = \frac{1}{\sqrt{-q_{3\gamma}}} \frac{\partial}{\partial x^\alpha}(\sqrt{-q_{3\gamma}} \rho_0) = 0,$$

(3.8)

where $\cdot$ denotes covariant derivative with respect to the subspace $x^\alpha = (\zeta^0, \theta, \phi)$. So, as seen on page 70 of Ref. [4],

$$\delta \rho_6 = \frac{\rho_6}{2} (V_\alpha V_\beta + q_{\alpha\beta}) \delta \rho_6 \rho_0.$$

(3.9)

Therefore, combining with $\delta \sqrt{-q}/\delta q^{ab} = -\sqrt{-q} \delta q_{ab}/2$, we get the tensor components as

$$T^{D6}_{ab} = \left\{ \begin{array}{ll}
\mu_6 e^{\frac{\phi}{\rho}} \rho_6 V_\alpha V_\gamma \delta(\chi - \chi_0), & (a, b = 0, \theta, \phi) \\
-\mu_6 e^{\frac{\phi}{\rho}} \rho_6 G_{ab} \delta(\chi - \chi_0), & (a, b = m, p) \\
0, & \text{(otherwise)}
\end{array} \right.$$  

(3.10)

Just like the case of D6 branes, $\sum_{i=1}^{N} S^{D2}_{(i)}$ can be also reduced as

$$S^{D2} := -\sum_{i=1}^{N} (S^{D2}_{(i)} - \mu_2 \int d^3 \xi C(3)_{(i)})$$

$$= \mu_2 \int d^1 x \delta(\chi - \chi_0) e^{-\frac{\phi}{\rho}} (- \det G)^{\frac{3}{2}} \rho_2,$$

(3.11)

where $\rho_2$ is a number density over the six-dimensional space, $K3 \times S^2$. It satisfies a conservation law as

$$\int d^5 x d\theta d\phi (G_{\theta\theta} G_{\phi\phi} \det G_{pq})^{\frac{3}{2}} \rho_2 = N.$$  

(3.12)

Defining the seven-dimensional timelike vector $U^\beta = (U^0, U^\theta, U^\phi, U^p) := (\sqrt{-q^{00}}, 0, \cdots, 0)$, the effective energy-momentum tensor is

$$T^{D2}_{ab} = \left\{ \begin{array}{ll}
\frac{\mu_2}{V_*} e^{-\frac{\phi}{\rho}} \rho_2 K_{\alpha} K_\beta \delta(\chi - \chi_0), & (a, b = 0, \theta, \phi, p) \\
\frac{\mu_2}{V_*} e^{-\frac{\phi}{\rho}} \rho_2 G_{ab} \delta(\chi - \chi_0), & (a, b = m) \\
0, & \text{(otherwise)}
\end{array} \right.$$  

IV. ISRAEL'S JUNCTION CONDITION AND ENAHNÇON GEOMETRY

It is interesting to note that the energy-momentum tensor has several particular features. Firstly, the dominant energy condition is violated near the enhançon radius $r = r_e$ because $T^{D2}_{00} + T^{D6}_{00}$ is proportional to $(V/Z_6 - V_*/Z_2) \sim 0$ and hence $|T^{D2}_{00} + T^{D6}_{00}| \ll |T^{D2}_{00} + T^{D6}_{ab}|$, where $\tilde{p}, \tilde{q}$ implies an orthogonal basis. Secondly, the energy-momentum tensor looks like dust in the three-dimensional subspace $(x^0, x^4, x^5)$, due to along which directions D6-D2 branes are aligned.

If the shell sits on a radius $r = r_0$ ($> r_e$), timelike hypersurface $\chi = \chi_0$ corresponds to $r = r_0$ one. On the hypersurface, Israel's junction equations [11] are

$$\Pi_{ab} := [K_{ab} - q_{ab} K^c_{\cdot c}] = -\kappa_{10}^2 (S^{D6}_{ab} + S^{D2}_{ab}),$$  

(4.1)

where $S^{D6}_{ab}$ and $S^{D2}_{ab}$ are functions in front of $\delta$-function of $T^{D6}_{ab}$ and $T^{D2}_{ab}$ respectively. Using the unit normal vector $n_M = (d\chi)_M$, the second fundamental form $K_{ab}$ is defined by $K_{ab} := n_{ab}$. $[f]_{\pm}$ simply denotes $f_+ - f_-$, where $f_+$ and $f_-$ are quantities evaluated on the outside and inside of the $r = r_0$ timelike hypersurface, respectively.

According to the enhançon mechanism, the inside of the shell is replaced with a flat geometry. Reminding that $G_{MN} = e^{-\frac{2\phi}{\rho}} G_{MN}$, we obtain

$$d\chi = Z_2 \frac{\phi}{\rho} Z_6 \frac{\phi}{\rho} dr.$$  

(4.2)

Thus, we can easily calculate all components of $K_{ab}$ at any $r = r_0$:

$$K_{00} = 0,$$

(4.3)

$$K_{0\theta} = r e^{-\frac{\phi}{\rho}} (Z_2 Z_6)^\frac{1}{2},$$

(4.4)

$$K_{\theta\theta} = 0,$$

(4.5)

$$K_{pq} = 0,$$

(4.6)

and

$$K_{00} = \frac{1}{16} e^{-\frac{\phi}{\rho}} (Z_2 Z_6)^\frac{1}{2} \left( \frac{5Z_2'}{Z_2} + \frac{Z_6'}{Z_6} \right);$$

(4.7)

$$K_{\theta\theta} = \frac{r^2}{16} e^{-\frac{\phi}{\rho}} (Z_2 Z_6)^\frac{1}{2} \left( \frac{3Z_2'}{Z_2} + \frac{7Z_6'}{Z_6} + \frac{16}{r} \right),$$

(4.8)
K_{mn+} = -\frac{1}{16} e^{-\frac{1}{2} \Phi} (Z_2 Z_6)^{-\frac{1}{4}} \left( \frac{5 Z'_2}{Z_2} + \frac{Z'_6}{Z_6} \right) \delta_{mn}, \quad (4.9)

K_{pq+} = \frac{1}{16} e^{-\frac{1}{2} \Phi} V_2^2 Z_2^2 Z_6^{-\frac{1}{4}} \left( \frac{3 Z'_2}{Z_2} + \frac{Z'_6}{Z_6} \right) G^{(K3)}_{pq}. \quad (4.10)

We can check that Eq. (1.1) is satisfied at any radius \( r = r_0 \) with the help of Eqs. (3.5) and (3.12). So, we obtain a whole static solution without any repulsion singularity, taking the self-gravity of the shell into account. This is one of the main results in this paper [13].

Next, let us consider the whole energy-momentum tensor \( T_{MN} \) coming from both supergravity and the shell parts and derive the energy conservation on the shell. In terms of the unit timelike vector \( u^M \) on the shell, the energy conservation is written by \( u^M \nabla^N T_{MN} = 0 \). To derive the conservation equation effectively on the shell, we shall put a step function \( \chi \), and they can be calculated on the shell, the energy conservation is written by

\[
\nabla^2 \Phi = \frac{e^{\Phi}}{4 \cdot 4!} F_4^2 - \frac{3 e^{-\frac{3}{2} \Phi}}{4 \cdot 8!} F_8^2 - 2 \kappa_{10}^2 \delta \frac{d}{d \Phi} \sum_{i=1}^{S(D6-D2)}.
\]

As easily seen, this equation is always satisfied when the shell is static.

Finally, let us consider the dilaton equation. Varying the action (3.1) with respect to \( \Phi \), we get the dilaton field equation

\[
\nabla^2 \Phi = \frac{e^{\Phi}}{4 \cdot 4!} F_4^2 - \frac{3 e^{-\frac{3}{2} \Phi}}{4 \cdot 8!} F_8^2 - 2 \kappa_{10}^2 \delta \frac{d}{d \Phi} \sum_{i=1}^{S(D6-D2)}.
\]

In terms of Gaussian normal coordinates (3.3), the left hand side of Eq. (4.10) is expressed as

\[
\nabla^2 \Phi = \frac{1}{\sqrt{s}} \left[ \frac{\partial}{\partial \zeta} \left( \sqrt{-G} \Phi \right) + \frac{\partial}{\partial \zeta} \left( \sqrt{-G} q_{00} \Phi \right) \right].
\]

Since the second term is a regular function of \( \chi \), integrating Eq. (1.16) from \( \chi = 0 - \epsilon \) to \( \chi = 0 + \epsilon \), we get a junction condition for the dilaton field such as

\[
[\Phi_{\epsilon}]_\pm = \frac{\kappa_{10}^2 \rho_0}{2} \left( \nu \cdot \frac{e^{\frac{3}{2} \Phi}}{4} \rho_2 + 3 e^{\Phi} \rho_6 \right).
\]

In the following sections, we will consider the perturbation of the solution.

V. STABILITY ANALYSIS OF THE D6-D2 BRANE SHELL

To avoid complicity, let us focus on the following perturbed metric in Einstein frame.

\[
ds^2 = e^{-\frac{1}{2} \Phi} \left( Z_2 \mu \eta_{ab} dx^a dx^b + Z_6 \eta_{ijkl} dx^i dx^j \right)
+ V \left( Z_{2 \pm} \eta_{ab} dx^a dx^b \right) + V \left( Z_{6 \pm} \eta_{ijkl} dx^i dx^j \right),
\]

where

\[
\Psi_\pm(t, r) = \check{\Phi}_\pm(t, r) + \delta \Psi_\pm(t, r),
Z_{2 \pm}(t, r) = Z_{2 \pm}(r) \left[ 1 + \delta Z_{2 \pm}(t, r) \right],
Z_{6 \pm}(t, r) = Z_{6 \pm}(r) \left[ 1 + \delta Z_{6 \pm}(t, r) \right],
\]

where \( \pm \) denotes outside and inside of the shell. \( \check{\Phi} \) denotes an unperturbed background field. The dilaton field \( \Phi \) and Ramond-Ramond fields \( C_{(3)} \) and \( C_{(7)} \) are also written by

\[\text{We thank to Kouji Nakamura for suggesting this.}\]
\[ \Phi_{\pm}(t, r) = \Phi_{\pm}(r) + \delta \Phi_{\pm}(t, r), \]
\[ C_{3 \pm}(t, r) = \bar{C}_{3 \pm}(r) + \delta C_{3 \pm}(t, r), \]
\[ C_{7 \pm}(t, r) = \bar{C}_{7 \pm}(r) + \delta C_{7 \pm}(t, r). \]  
(5.3)

Here the fields \( C_3 \) and \( C_7 \) are components of the Ramond-Ramond fields \( \bar{C}_{(3)} \) and \( \bar{C}_{(7)} \) respectively:

\[ C_{(3)} = C_3 \ dt \wedge dx^4 \wedge dx^5. \]  
(5.4)

\[ C_{(7)} = VC_7 \ dt \wedge dx^4 \wedge dx^5 \wedge dV_{(K3)}. \]  
(5.5)

The above unperturbed fields are defined as

\[ Z_2(r) := \begin{cases} 
\bar{Z}_{2-}(r) = 1 + r_2/r_0 \\
\bar{Z}_{2+}(r) = 1 + r_2/r 
\end{cases} \]  
(5.6)

\[ Z_6(r) := \begin{cases} 
\bar{Z}_{6-}(r) = 1 + r_6/r_0 \\
\bar{Z}_{6+}(r) = 1 + r_6/r 
\end{cases} \]  
(5.7)

and \( \Phi_{\pm}(r) \), \( \bar{C}_{3 \pm}(r) \) and \( \bar{C}_{7 \pm}(r) \) are given by \( \bar{Z}_{2 \pm}(r) \) and \( \bar{Z}_{6 \pm}(r) \).

Assume that the shell sits on \( r = r_0 \) before it is perturbed. Since the inside of the geometry is just flat, all functions \( \bar{f} \) are constants determined by the continuity of the outside and inside of the metric, i.e., \( \bar{f} = \bar{f}_+(r = r_0) \).

A. The perturbation of the flat geometry inside the metric

The perturbed energy-momentum tensor \( \delta T_{MN} \) does not include any linear terms of \( \delta \Phi_- \), \( \delta C_3 \) or \( \delta C_7 \) because the background energy-momentum tensor \( T_{MN} \) is zero. Hence, the perturbed Einstein equation is simply \( \delta R_{MN} = 0 \). Defining derivative operator \( \nabla_-^2 := \partial \partial^t + \partial \partial^i \), the equations are written by

\[ \nabla_-^2 (\delta \Psi_- + \delta Z_{2-} + \delta Z_{6-}) - \partial_t^2 (8 \delta \Psi_- - 6 \delta Z_{2-} + 2 \delta Z_{6-}) = 0, \]  
(5.8)

\[ \partial_t \partial_i (2 \delta \Psi_- - 6 \delta Z_{2-} + 2 \delta Z_{6-}) = 0, \]  
(5.9)

\[ \nabla_-^2 (\delta \Psi_- + \delta Z_{2-} + \delta Z_{6-}) = 0, \]  
(5.10)

\[ \nabla_-^2 (\delta \Psi_- - \delta Z_{2-} - \delta Z_{6-}) \delta_{ij} + \partial_i \partial_j (8 \delta \Psi_- - 2 \delta Z_{2-} + 6 \delta Z_{6-}) = 0, \]  
(5.11)

\[ \nabla_-^2 (\delta \Psi_- - \delta Z_{2-} + \delta Z_{6-}) = 0. \]  
(5.12)

If we take

\[ \delta \Psi_\pm = \xi_\pm(r) e^{i\omega t}, \]  
(5.13)

\[ \delta Z_{2\pm} = \zeta_{2\pm}(r) e^{i\omega t}, \]  
(5.14)

\[ \delta Z_{6\pm} = \zeta_{6\pm}(r) e^{i\omega t}, \]  
(5.15)

\[ \delta \Phi_\pm = \eta_\pm(r) e^{i\omega t}, \]  
(5.16)

Eq. (5.9) indicates

\[ \xi_- = \frac{1}{2} (\zeta_2 - \zeta_6 - \zeta_6 - \zeta_2). \]  
(5.17)

Substituting this into Eqs. (5.8)-(5.12), we obtain the following equations

\[ \nabla_-^2 \delta \Psi_- = 0, \]  
(5.18)

\[ \xi_- = \zeta_2 - \zeta_6 = -\zeta_6 - \zeta_2. \]  
(5.19)

The perturbed field equation of the dilaton is same as Eq. (5.18):

\[ \nabla_-^2 \delta \Phi_- = 0. \]  
(5.20)

Imposing a regular boundary condition at \( r = 0 \), the solution of Eqs. (5.18) and (5.20) are

\[ \xi_-(r) = A_\Psi \frac{\sin(\Omega r)}{\Omega r}, \]  
(5.21)

\[ \eta_-(r) = A_\Phi \frac{\sin(\Omega r)}{\Omega r}, \]  
(5.22)

where \( \Omega = \sqrt{Z_{2-} Z_{6-}} \), \( A_\Psi \) and \( A_\Phi \) are integral constants. Note that when the shell is absent, the frequency \( \omega \) must be real. Otherwise, i.e., when \( \omega \) is imaginary, the eigenfunctions diverge at infinity. However, when the shell exists at some radius, the gradient of eigenfunctions becomes discontinuous at the shell and there is a possibility that the exponential growth of the eigenfunction may be changed by the exponentially decaying function at the shell. In this sense, the stability of the shell is not trivial.

B. The perturbation of the junction conditions on the shell

Suppose that the orbit of the shell is \( r = r_0 + \delta R(t) \) in the metric (5.1). In this and next subsections, we mainly discuss the values around the shell. Hence all the variables are evaluated on the shell if we do not mention in these subsections. We also drop the index \( \pm \) of the continuous variables on the shell. Then, the (linear order) continuity of the metric and dilaton field yields the following equations,

\[ \delta \Phi_+ = \delta \Phi_- - \bar{\Phi}_{\pm}^t \delta R, \]

\[ \delta Z_{2+} = \delta Z_{2-} - \frac{Z_{2+}}{Z_{2-}} \delta R, \]

\[ \delta Z_{6+} = \delta Z_{6-} - \frac{Z_{6+}}{Z_{6-}} \delta R, \]  
(5.23)

where a prime denotes differential with respect to \( r \).
Firstly, let us solve Eq. (5.13) and show that the dilaton field $\Phi$ is constant along the shell, i.e., $d\Phi/d\ell = 0$. The components of the vectors $u^M = \tilde{u}^M + \delta u^M$ and $v^M = \tilde{v}^M + \delta v^M$ are expressed as

$$\tilde{u}^M = (\tilde{U}^i, \tilde{U}^r, 0, \ldots, 0)$$

$$\delta u^M = \left( \tilde{Z}_2^+ \frac{\tilde{Z}_6^2}{\tilde{Z}_6^3} \delta \tilde{R}, \right.$$

$$\left. \frac{1}{4} \tilde{Z}_2^+ \frac{\tilde{Z}_6^2}{\tilde{Z}_6^3} \delta \tilde{R}, \delta Z_3 - \delta Z_5, \ldots, 0 \right).$$

Finally, substituting Eqs. (5.25) and (5.29) into Eq. (5.17), we get the following simple equation

$$\delta \Phi_\gamma = 0,$$  

(5.30)

with help of Eqs. (5.19) and (5.24). Finally, substituting Eqs. (5.25) and (5.29) into Eq. (5.17), we get the following simple equation

$$\delta \Phi_\gamma = 0,$$  

(5.30)

which means that the dilaton field is constant along the shell in the first order of the perturbation. It should be noted, however, that it does not necessarily imply $\delta \Phi_\omega = 0$. For example, the general solution of the perturbed dilaton field equation is (5.22). When the frequency $\Omega$ (or $\omega$) is real, the condition (5.30) means that the $\delta \Phi_\gamma$ becomes node at $r = r_0$. The frequency of the perturbation becomes discrete to satisfy this condition when the perturbation of the dilaton field has nontrivial configuration. Of course, there may exist a solution with $A_\Phi = 0$. In this case, the frequency shows continuous spectrum. On the other hand, when $\omega$ is imaginary, the eigenfunction grows exponentially with $r$, and it cannot become zero unless $A_\Phi = 0$. Hence $\delta \Phi_\gamma = 0$ for the imaginary frequency, i.e., unstable modes.

Now, let us focus on the perturbation of the junction equations (5.1). It is noteworthy that the right hand side does not include any $\delta \Phi_\gamma$ terms, provided that we calculate it by using the inside metric (5.1), as shown above. So, all non-vanishing equations are:

$$\delta Z_2' - \delta Z_6' - 2\delta \Psi'_\gamma = \frac{1}{8\rho_6^2} \left( \frac{3 r^2 - r_6}{Z_2 - r_6} \right) \delta Z_2 -$$

$$+ \frac{1}{2 r_0^2} \left( \frac{r^2 + r_6^2}{Z_2^2} \right) \delta R, \quad (5.31)$$

$$\delta Z_2' = 2 \delta Z_6' - 4 \delta \Psi'_\gamma$$

$$= - \frac{1}{4 r_0^2} \delta Z_2 - \delta Z_2' + \frac{1}{r_0^2} \delta R, \quad (5.32)$$

$$\delta Z_2' = 3 \delta Z_6' - 4 \delta \Psi'_\gamma = \left. - \frac{1}{8 r_0^2} \left( \frac{3 r^2}{Z_2 + \frac{7 r_6}{Z_6}} \right) \delta R. \quad (5.33)$$

By solving these equations with respect to $\delta \Psi'_\gamma$, $\delta Z_2'$, and $\delta Z_6'$, we obtain

$$\delta Z_2' = 2 \delta Z_6' = \frac{3 r^2}{4 r_0^2 Z_2} \delta Z_2 + \frac{1}{r_0^2} \frac{r_6^2}{Z_2^2} \delta R, \quad (5.34)$$

$$\delta Z_2' = 3 \delta Z_6' = - \frac{1}{4 r_0^2} \frac{r_6}{Z_2} \delta Z_2 -$$

$$+ \frac{1}{8 r_0^3} \left( \frac{3 r^2}{Z_2} + \frac{7 r_6}{Z_6} + 8 \frac{r_6^2}{r_0 Z_6} \right) \delta R, \quad (5.35)$$

$$\delta \Psi'_\gamma - \delta \Psi'_\gamma = \left. \frac{3}{16 \rho_6^2} \left( \frac{r^2}{Z_2 - \frac{7 r_6}{Z_6}} - \frac{4 r_6^2}{Z_2^2} \right) \delta Z_2 - \right.$$

$$- \frac{1}{16 r_0^3} \left[ \frac{3 r^2}{Z_2} + \frac{7 r_6}{Z_6} - \frac{4}{r_0} \frac{r_6^2}{Z_2^2} \right] \delta R. \quad (5.36)$$
The left hand side in Eqs. (5.34)-(5.36) are interpreted as a difference between inside and outside quantities. So, if the number densities \( \rho_2 \) and \( \rho_6 \) of D2 and D6 branes on the shell approach zero, the difference also approaches zero.

Similarly, we can obtain \( \delta \Phi'_6 \) from the junction condition of the dilaton field Eq. (4.18) as follows:

\[
\delta \Phi'_6 - \delta \Phi'_6' = -\frac{3}{16r_0^2} \left( \frac{r_2}{Z_2} + \frac{r_6}{Z_6} \right) \delta Z_2 - \frac{1}{4r_0} \left( \frac{r_2^2}{Z_2^2} - \frac{3r_6^2}{Z_6^2} \right) \delta R. \tag{5.37}
\]

We also have the following equations from the RR field equations using Eq. (7.23):

\[
\delta C'_{7+} = \frac{r_6}{2gr_0^2Z_6^2} \left( \delta \Psi - 3\delta \Phi \right) - \frac{2r_6}{gr_0^2Z_6^2} \delta R, \tag{5.38}
\]

\[
\delta C'_{3+} = -\frac{r_2}{2gr_0^2Z_2^2} \left( 3\delta \Psi + \delta \Phi \right) - \frac{2r_2}{gr_0^2Z_2^2} \delta R. \tag{5.39}
\]

**C. Equation of motion of the shell**

When the locus of the shell made of the wrapped D6-branes shifts from \( r = r_0 \) to \( r = R(t) = r_0 + \delta R(t) \), where \( |\delta R(t)| \ll r_0 \), the effective action of the shell up to the second order of \( \delta R(t) \) is given by

\[
S^{D6-D2} = -N \int d^3 \xi \left[ \mu_6 V e^{\Phi - \Phi} (\Phi - \bar{\Phi}) Z_2 \frac{1}{2} Z_6 \right] - \frac{1}{2} \left( \frac{\mu_6 V}{Z_6} e^{-\phi} - \frac{\mu_2}{Z_2} e^{-\phi} \right) \left[ 1 - \frac{1}{2} Z_2 Z_6 (\delta \tilde{R})^2 \right] + N \int d^3 \xi \left( \mu_6 V C_7 - \mu_2 C_3 \right). \tag{5.40}
\]

From the action (5.40), we have the following equation of motion of the shell:

\[
\left[ \mu_6 V \mathcal{Z}_2 (R) - \mu_2 \mathcal{Z}_6 (R) \right] \delta \tilde{R} = -\frac{\mu_6 V}{4Z_6 (R)} \left[ 3\delta \Phi - 7\delta \Psi + \delta Z_2 - 7\delta Z_6 \right] \tag{5.41}
\]

\[
\text{where the primes denote differentials with respect to } R. \delta C_3 \text{ and } \delta C_7 \text{ are the perturbed parts of the Ramond-Ramond fields respectively.}
\]

The fields in Eq. (5.41) are discontinuous at the shell \( r = R(t) \), so that the equation needs to be averaged near the shell. Recalling that the average of \( A(R) \) at the shell, \( \langle A \rangle \), is defined by \( \langle A \rangle = \frac{1}{2} (A_- + A_+) \), the equation of motion of the shell up to the first order of the perturbation can be written as

\[
(\mu_6 V \mathcal{Z}_2 - \mu_2 \mathcal{Z}_6) \delta \tilde{R} = \frac{\mu_6 V}{8Z_6} \left( 3\delta \Phi - 7\delta \Psi + \delta Z_2 - 7\delta Z_6 \right) \tag{5.42}
\]

where we have taken \( \mathcal{Z}_2 = -r_2/r_0^2 \), \( \mathcal{Z}_6 = -r_6/r_0^2 \). Here we have also used \( \mathcal{Z}_6^{\prime} (R) = \mathcal{Z}_2^{\prime} (R) = 0 \). Using the relations between the inside and the outside quantities near the shell, that is Eqs. (5.19) and (5.34)-(5.36), we finally obtain the equation of motion of the shell:

\[
A \delta \tilde{R} = B + C \delta R, \tag{5.43}
\]

where the definitions of \( A, B, C \) are as follows

\[
A = \mu_6 V \mathcal{Z}_2 - \mu_2 \mathcal{Z}_6 \tag{5.44}
\]

\[
B = \frac{1}{8} \left( \frac{\mu_2}{Z_2} + \frac{3\mu_6 V}{Z_6} \right) \delta \Phi - \frac{1}{4} \left( \frac{3\mu_2}{Z_2} + \frac{\mu_6 V}{Z_6} \right) \delta \Psi \tag{5.45}
\]

\[
C = \left( \frac{1}{128r_0^2} \left( \frac{3\mu_2}{Z_2} + \frac{7\mu_6 V}{Z_6} \right) \left( \frac{3r_2}{Z_2} + \frac{7r_6}{Z_6} \right) \right). \tag{5.46}
\]

Here, we have used \( \delta C_{3-} = \delta C_{7-} = 0 \) inside the shell. Moreover, using \( a := V/V_\ast \) and \( x := r_0/|r_2| \), \( A, B \) and \( C \) have the following simple expressions:

\[
A = \frac{\mu_2}{x} [(a - 1)x - 2a] \tag{5.47}
\]

\[
B = \frac{\mu_2}{8} \left[ \frac{1}{x - 1} \left( \frac{3a}{x + a} \right) \delta \Phi - \frac{x}{4} \left( 3 + \frac{a}{x + a} \right) \delta \Psi \right]. \tag{5.48}
\]
so that the resonance is not caused.

so that there is necessarily a real frequency \( \omega \)

shell, where resonances are produced, is

stable" inside the enhançon radius.

amplitude at this radius. In that sense, the shell is "un-

(ii) When the shell is constructed at the enhançon radius.

of the occurrence of the resonances under small fluctu-

From the above analysis, the "unstable" region of the

\[
C = \frac{\mu^2}{128r_2^2x} \left( \frac{7a}{x + a} - \frac{3}{x - 1} \right)^2. 
\]

If we take

\[
\delta R = \delta \tilde{R} e^{i\omega t} \quad \text{and} \quad B = \tilde{B} e^{i\omega t}, \tag{5.50}
\]

the equation of motion of the shell becomes

\[
\alpha \delta \tilde{R} = -\tilde{B} \quad \text{and} \quad \alpha = \omega^2 A + C. \tag{5.53}
\]

Therefore, we obtain the amplitude of the forced oscillation:

\[
\tilde{\delta} R = -\frac{\tilde{B}}{\alpha}. \tag{5.54}
\]

Using the above equation, we can investigate whether resonances are produced by the forced oscillation or not. If \( \alpha \) can be taken 0, then the amplitude \( \delta \tilde{R} \) becomes infinity and a resonance is caused. This indicates a kind of "instability" of the shell within our perturbation theory. Furthermore, from the numerical analysis of the Subsec. \( \sqrt{\delta} \), the square of the frequency \( \omega \) cannot take negative values. This fact and the behavior of the \( \alpha \), taking notice that \( C \geq 0 \) at any radius, lead us to classify the region of the radius of the shell into the following cases.

(i) When \( r_e < r_0, A > 0 \) (positive tension) and \( C > 0 \), so that the resonance is not caused.

(ii) When \( r_0 = r_e, A = 0 \) (tensionless) and \( \alpha = C > 0 \), so that the resonance is not caused.

(iii) When \( r_0 < r_e, A < 0 \) (negative tension) and \( C \geq 0 \), so that there is necessarily a real frequency \( \omega \) satisfying \( \alpha = 0 \) at an arbitrary radius of the shell. Therefore, a resonance is produced and the shell oscillates with huge amplitude at this radius. In that sense, the shell is "unstable" inside the enhançon radius.

From the above analysis, the "unstable" region of the shell, where resonances are produced, is \( r_0 < r_e \). If the shell of the wrapped D6-branes enters the inside of the enhançon radius \( r_e \), the shell becomes "unstable" because of the occurrence of the resonances under small fluctuations of the fields, and the shell may push back to the enhançon radius, where it is stable. This can explain why the shell is constructed at the enhançon radius.

D. The perturbation of the geometry outside the shell

The calculation of the perturbation of the outer region is tedious but straightforward. First we examine the matter field equations. The perturbed equations of the RR fields become

\[
\delta C_{3+} = -\frac{1}{2}(4 \delta Z_{2+} - \delta \Psi_+ + \delta \Phi_+) \bar{C}_3', \tag{5.55}
\]

\[
\delta C_{7+} = -\frac{1}{2}(4 \delta Z_{6+} + 3 \delta \Psi_+ - 3 \delta \Phi_+) \bar{C}_7'. \tag{5.56}
\]

The R-R fields are expressed by the perturbations of other fields and obtained just by integrating these equations after we determine other field. The perturbed equation of the dilaton field becomes

\[
-\dot{Z}_{2+} + \dot{Z}_{6+} + \delta \Psi_+ + \delta \Phi_+ + \frac{2}{r} \delta \Phi_+ = \frac{\Phi_+'}{2} (4 \delta \Psi_+ - \delta Z_{2+} + 3 \delta Z_{6+})' \\
+ \pi \frac{1}{4} \left( \frac{Z_{2+}'}{Z_{2+}} + \frac{Z_{6+}'}{Z_{6+}} \right) (\delta \Psi_+ - \delta Z_{2+} - \delta Z_{6+}) \\
- \frac{Z_{2+}'}{8Z_{2+}} (4 \delta \Psi_+ + 3 \delta Z_{2+} - 5 \delta Z_{6+}) \\
+ \frac{3}{8Z_{6+}} (4 \delta \Psi_+ - \delta Z_{2+} - \delta Z_{6+}). \tag{5.57}
\]

Next we turn to the gravitational equations. The first order perturbation of the Ricci curvature becomes

\[
\delta R_{00+} = \frac{1}{32Z_{2+}Z_{6+}} \left[ 8 \dot{Z}_{2+} \dot{Z}_{6+} (9 \delta \Psi_+ - 5 \delta \dot{Z}_{2+} + 3 \delta \dot{Z}_{6+}) \\
- 8(\delta \Psi_+'' + \delta \dot{Z}_{2+}' + 3 \delta \dot{Z}_{6+}'') \\
- \left( \frac{5Z_{2+}'}{Z_{2+}} + \frac{Z_{6+}'}{Z_{6+}} + \frac{16}{r} \right) \delta Z_{2+}' \\
+ \left( \frac{15Z_{2+}'}{Z_{2+}} + \frac{3Z_{6+}'}{Z_{6+}} - \frac{16}{r} \right) \delta Z_{6+}' \\
+ 4 \left( \frac{5Z_{2+}''}{Z_{2+}} + \frac{Z_{6+}''}{Z_{6+}} - \frac{4}{r} \delta \Psi_+ \\
+ 2 \left( \frac{5Z_{2+}''}{Z_{2+}} - \frac{Z_{6+}''}{Z_{6+}} + \frac{10}{r} \frac{Z_{2+}''}{Z_{2+}} + \frac{Z_{6+}''}{Z_{6+}} - \frac{Z_{2+}''}{Z_{6+}} \right) \right) \right]. \tag{5.58}
\]

\[
\delta R_{0r+} = \frac{1}{8} \left[ 8(2 \delta \Psi_+ - \delta Z_{2+} + \delta \dot{Z}_{6+}) \\
+ \left( \frac{5Z_{2+}'}{Z_{2+}} + \frac{Z_{6+}'}{Z_{6+}} \right) (\delta \dot{Z}_{2+}) \\
+ \left( \frac{3Z_{2+}'}{Z_{2+}} - \frac{Z_{6+}'}{Z_{6+}} \right) \delta \dot{Z}_{6+} \right]. \tag{5.59}
\]

\[
\delta R_{rr+} = \frac{1}{32} \left[ -\dot{Z}_{2+} \dot{Z}_{6+} (\delta \Psi_+ - \delta \dot{Z}_{2+} - \delta \dot{Z}_{6+}) \right].
\]
On the other hand, the energy momentum tensor is
$$\delta T_{pq} = \frac{1}{64Z_6^+} \left[ -8 \left( \frac{5Z_2^+ + Z_6^+}{Z_2^+} \right) \delta Z_2^+ + \left( \frac{Z_2^+}{Z_2^+} + \frac{Z_6^+}{Z_6^+} \right) \delta Z_6^+ \right], \quad \delta T_{pq} = \left( \frac{Z_2^+}{Z_2^+} + \frac{Z_6^+}{Z_6^+} \right) \delta Z_2^+ - \left( \frac{Z_2^+}{Z_2^+} + \frac{Z_6^+}{Z_6^+} \right) \delta Z_6^+ \right],$$

Here we have used Eqs. (5.55) and (5.56).

By assuming the time dependence of these perturbed variables as harmonic oscillator as Eqs. (5.13)-(5.14), we find
$$\xi_+ = \frac{3}{4} \tilde{\xi}_2^+ - \frac{1}{4} \tilde{\xi}_6^+.$$
By using the relation (5.69), the dilaton field equation becomes

$$
\eta''_+ + \frac{2}{r} \eta'_+ = \Phi'_+ (\zeta_2' + \zeta_6') - \frac{\Phi'_+}{16} \left( \frac{Z_{2+}}{Z_{2+}} + \frac{Z_{6+}}{Z_{6+}} \right) (\zeta_2 + 5 \zeta_6) - \frac{3}{4} \left( \frac{Z_{2+}}{Z_{2+}} - \frac{Z_{6+}}{Z_{6+}} \right) (\zeta_2 - \zeta_6) - \omega^2 Z_{2+} \zeta_6 + \eta_+ = 0.
$$

(5.72)

E. Numerical analysis

Now, everything has been prepared for the perturbation analysis, i.e., the perturbed solutions of the geometry inside the shell (5.13) and (5.21), the matching and the junction conditions (5.29) and (5.34)-(5.37), the equation of motion for the shell (5.43), and the perturbed equations in the outer geometry (5.69)-(5.74). Since we know the solutions inside the shell, we can integrate from the \( r_0 \) to outside region with the boundary value and the junction conditions of the perturbed functions at the shell. If there is an eigenmode with a negative eigenvalue \( \omega^2 \), the background solution is unstable against the perturbations.

Our numerical calculations show, surprisingly, that no eigenmodes with negative \( \omega^2 \) can be found for any radii (\( r_0 > |r_2| \)) of the shell. Furthermore, as shown in Subsec. V C, we found resonances for the case (iii). The followings are two typical examples for the numerical calculations.

Fig. 2 shows configurations of the stable modes when the shell is located at \( r = r_0 > r_e \). The enhançon radius of the parameters in Fig. 2 is \( r_e = 10.8123 \). Outside of the shell, the perturbed functions couple each other complicatedly and they show disordered behaviors. The stable modes show a continuous spectrum as usual.

When the shell is located inside of the enhançon radius, the configurations of the perturbed functions are similar to the previous case \( r_0 > r_e \) except for the existence of the resonance radii. For the same parameter in Fig. 2, the resonance radii are \( r_{\text{reso}} = 2.74099, 10.8071 \) in \( |r_2| < r_{\text{reso}} < r_e \), where \( |r_2| = 2.59758 \) and \( r_e = 10.8123 \). Fig. 3 shows configurations when the shell is located near the resonance radius. The perturbed functions become very large at the shell by the resonance.

When the shell crosses the enhançon radius, the total tension of the shell becomes negative. So we intuitively guess that the shell has eigenmodes with negative \( \omega^2 \). Our numerical calculations show, however, that there is no such eigenmode. Since the shell we are considering couples to the bulk fields, it is considered that the lost of the energy by moving the shell is compensated by the increase of the bulk fields energy.
FIG. 3. The typical configuration of the perturbed functions with the parameter $g = 0.1$, $N = 100$, $V = 3000$. The radius of the shell is $r_0 = 10.8$ and the eigenvalue is $\omega_0^2 = 1.0$. The solid and the dotted lines are $\delta Z_2$ and $\delta Z_0$, respectively. Since the shell locates near the resonance radius $r_{reso} = 10.8071$ the amplitudes of the perturbed functions become very large at the shell.

VI. CONCLUSION AND DISCUSSIONS

We have investigated the stability of the shell of wrapped D6-branes on K3 from the point of view of supergravity. As shown in Sec. [1], if a probe D6-brane (wrapped on K3) is kicked inward slightly at the outside of the enhançon radius $r = r_e$, the velocity reaches the speed of light at the radius. This behavior seems to be in contradiction to the behavior of the enhançon mechanism, which states that the D6-branes shell sits around the enhançon radius. So, what mechanism stabilizes the non-probe D6-brane?

In the probe case, the background spacetime is fixed as a supersymmetric D6-D2 brane supergravity solution. Therefore, there is no obstacle to stop the motion of the probe since no force acts on it. In the non-probe case, the background spacetime gets excited by the motion of the shell and becomes non-supersymmetric. Then, new forces act on the D6 branes shell, as seen in Eq. (5.43). In other words, the kinematic energy of the shell could be lost through the waves such as gravitational and the dilaton waves. This is one of the reasons why the non-probe D6-branes shell is stable.

Quite interestingly, it is found numerically that there is no eigenmode with negative $\omega^2$ for any radius of the shell. One may doubt it when the shell has negative tension, because its energy falls as it moves. This naive picture is not applicable to our case because the bulk energy is also changed. For example, let us consider the shell located inside of the enhançon radius. Because of the negative tension, the energy $E$ falls as $E \propto -\omega^2 \delta \tilde{R}^2$ when it moves slowly. On the other hand, the bulk energy of the dilaton field and the R-R fields may increase as the shell moves. So, we cannot discuss the stability of the whole system in terms of $E$ only. Indeed, the D6-D2 system under consideration provides a notable example for demonstrating that D-branes shell with negative tension is stable in the sense that there is no eigenmode with negative $\omega^2$.

In Sec. V, we found that there is a resonant frequency mode whenever the shell is located at $r_0 < r_e$. Because every time-dependent function $f(t)$ contains this mode as a Fourier mode, the shell would oscillate with huge amplitude. In this sense the shell is “unstable” generically whenever the shell is located inside the enhançon radius. This fact suggests that the shell located inside of the enhançon radius is pushed back to the enhançon radius, where it is stable. Therefore, we may say that the enhançon radius is naturally selected as a suitable position of the wrapped D6-branes shell even from the viewpoint of the “classical” supergravity with the D-branes. This coincides with the enhançon picture stated in the introduction.

It is interesting to investigate the stability of other models such as brane world scenario [14]. There is a common belief that the brane with negative tension is unstable against the perturbation. By analogy with the present analysis, we may expect that it can be stabilized if one considers the perturbation of the whole system including the self-gravity of the brane. It is also interesting to investigate the non-commutative effect of N D6-branes or non-linear effect of the D6-D2 branes system. After such steady works, we could see new physics of gravity combined with string theory.

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