The speed of propagation for degenerate diffusion equations with time delay

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Abstract

We are concerned with a class of degenerate diffusion equations with time delay describing population dynamics with age structure. In our recent study [Nonlinearity, 33 (2020), 4013–4029], we established the existence and uniqueness of critical traveling wave for the time-delayed degenerate diffusion equations, and obtained the reducing mechanism of time delay on critical wave speed. In this paper, we now are able to show the asymptotic spreading speed and its coincidence with the critical wave speed \(c^*(m, r)\) of sharp wave, and prove that the initial perturbation or the boundary of the compact support of the solution propagates at the critical wave speed \(c^*(m, r)\) for the time-delayed degenerate diffusion equations. Remarkably, different from the existing studies related to spreading speeds, the time delay and the degenerate diffusion lead to some essential difficulties in the analysis of the spreading speed, because the time-delay makes the critical speed of traveling waves slow down, and the degenerate diffusion causes the loss of regularity for the solutions. By a new phase transform technique combining with the monotone method, we can determine the asymptotic spreading speed.

Keywords: Degenerate diffusion, time delay, spreading speed, sharp waves.

1 Introduction

We consider the following degenerate diffusion equation with time delay

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u^m - d(u) + b(u(t - r, x)), \quad x \in \mathbb{R}^n, \ t > 0, \\
u(s, x) &= u_0(s, x), \quad x \in \mathbb{R}^n, \ t \in [-r, 0],
\end{aligned}
\]  

(1.1)

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where the spatial dimension $n \geq 1$, $u(t, x)$ denotes the total mature population of the species at location $x$ and time $t > 0$, $r \geq 0$ is the maturation time, $b(u(t-r, x))$ is the birth function, $d(u)$ is the death rate function. The equation (1.1) describes the population dynamics of single species with age-structure and density-dependent diffusion ($m > 1$).

In the linear diffusion case without time delay, (i.e., $r = 0$ and $m = 1$), the equation (1.1) is reduced to the Fisher and Kolmogorov-Petrovsky-Piskunov (KPP) equation [14, 18]. It is well known that there exists a critical (minimal admissible) wave speed $c^* = 2 \sqrt{b'(0) - d'(0)} > 0$ (under certain conditions on the functions $b(\cdot)$ and $d(\cdot)$) for all the traveling waves connecting the two constant equilibria 0 and $\kappa > 0$, and the level set $\Gamma_\varepsilon(t) := \{x \in \mathbb{R}^n; u(t, x) = \varepsilon\}$ with $\varepsilon \in (0, \kappa)$ asymptotically propagates at the same speed $c^*$ [31]. It was Thieme and Zhao [28] who first established the theory of asymptotic spreading speed for a large class of nonlinear integral equations, which covers many time-delayed reaction and diffusion equations with linear diffusion (i.e., $r > 0$ and $m = 1$). Liang and Zhao [22] further developed the theory of spreading speeds to both discrete and continuous time monotone semiflows and investigated the application to a time-delayed evolution equation. In a series of works (e.g., [20, 38, 39]), Zhao and his collaborators investigated the traveling waves and spreading speeds of population dynamics model with nonlocal dispersal. Studies of the coincidence of the spreading speed with the critical wave speed for various evolution systems with linear diffusion systems can also be found in [11, 12, 19, 21, 22, 23, 25, 28, 37].

When the degenerate diffusion is included, the system can be used to describe biological population dynamics with density-dependent dispersal; see for instance [16, 26]. An interesting peculiarity of degenerate diffusion is the appearance of sharp type waves at the asymptotic speed [8, 29, 30]. For the case without time delay ($r = 0$), traveling wave solutions have been found by several authors [1, 2, 13, 15, 27]. Medvedev et al. [24] proved that the slowest traveling wave in the family yields the asymptotic speed of the propagation of disturbances in a class of degenerate Fisher-KPP equations. In recent works [5, 6, 7], more general cases of doubly nonlinear diffusion are considered, which includes both porous medium and $p$-Laplacian models.

An increasing attention has been paid to degenerate diffusion equations with time delay in order to study the effects of degenerate diffusion and time delay on the evolutionary behavior of biological systems with age structure (see [17, 33, 34, 36]). The existence of smooth monotone fronts for equations (1.1) with small time delay was proved by Huang et al. [17] based on a perturbation approach. In our recent works [33, 34], we proved that the time-delayed degenerate diffusion equation (1.1) admits a unique sharp type (semi-compactly supported) traveling wave $\phi(x + c^*t)$ for the one dimensional case, which corresponds to the minimal admissible (critical) wave speed $c^* = c^*(m, r)$. Moreover, the time delay slows down the minimal wave speed, i.e., $c^*(m, r) < c^*(m, 0)$ for $r > 0$. However, the asymptotic speeds of spreading for solutions with compactly supported initial data and the coincidence with the critical wave speed of sharp traveling wave still remain open. In this paper, we shall answer those unsettled questions on the spreading properties.

Time delay and degenerate diffusion lead to essential difficulties in the analysis of the spreading speed of (1.1). In the absence of time delay, the maximum principle and phase plane analysis proposed by Aronson and Weinberger [3, 4] yield conclusions about the asymptotic propagation speed of the linear diffusion and similar equations. This method was extended to cover degenerate diffusion equations of general Fisher-KPP sources without time delay by Medvedev et al. in [24], where all the trajectories in the phase plane are determined and correspond to special upper and lower solutions. However, time delay changes the situation dramatically. It is shown in [34] that the time delay reduces
the critical wave speed \( c^*(m, r) \) and the speed is not characterized by the classical phase plane analysis method. In order to construct upper and lower solutions with compact (or semi-compact) supports and with propagating speed approaching \( c^*(m, r) \), we employ a new phase transform technique developed in [34] and utilize the monotone dependence in the phase space, see the phase comparison principle Lemma [2,3]. Additionally, we also need to do two things for the lower solutions: first, we show that the support of \( u(t, \cdot) \) expands to including any given compact subset for large time; and secondly, the value of \( u(t, x) \) within given compact subset grows up as time increases.

The purpose of this paper is to show that the initial perturbation or the boundary of the compact support of the solution propagates at the critical wave speed \( c^*(m, r) \) for the time-delayed degenerate diffusion equation (1.1).

A function \( u(t, x) \) is said to be compactly supported for \( t \in [t_1, t_2] \), if supp \( u(t, \cdot) \) is compact for \( t \in [t_1, t_2] \). For the sake of convenience, we define the half space divided by the hyperplane through a point \( x_0 \) that has normal vector \( \nu \) as

\[
\Pi(x_0, \nu) := \{x \in \mathbb{R}^n; (x - x_0) \cdot \nu \geq 0\}. \tag{1.2}
\]

A function \( u(t, x) \) is said to be semi-compactly supported for \( t \in [t_1, t_2] \), if

\[
\text{supp } u(t, \cdot) \subset \Pi(x(t), \nu(t)) = \{x \in \mathbb{R}^n; (x - x(t)) \cdot \nu(t) \geq 0\}
\]

for \( t \in [t_1, t_2] \) with some points \( x(t) \) and vectors \( \nu(t) \). For any variable \( s \), the positive value of \( s \), denoted by \( s_0 \), is defined as \( s_0 = \max\{s, 0\} \).

Throughout this paper, we assume that the functions \( d(s) \) and \( b(s) \) satisfy the following conditions:

There exist \( u_1 = 0, u_2 > 0 \) such that \( d, b \in C^2([0, +\infty)), d(0) = b(0) = 0, d(u_2) = b(u_2) \),

\[
(b(s) - d(s)) \cdot (s - u_2) < 0, \quad \forall s \in (0, +\infty) \setminus \{u_2\}, b'(0) > d'(0) \geq 0, d'(s) \geq 0, b'(s) \geq 0. \tag{1.3}
\]

Here, \( u_1 = 0 \) and \( \kappa := u_2 > 0 \) are two constant equilibria of (1.1), and the functions \( b(u), d(u) \) are both non-decreasing.

We first recall the properties of the sharp type traveling wave obtained in our previous study [34].

**Theorem 1.1** ([34]) For any \( m > 1 \) and \( r \geq 0 \), the time-delayed degenerate diffusion equation (1.1) admits a unique (up to shift) sharp type (semi-compactly supported) traveling wave \( u^*(t, x) = \phi^*(x \cdot \nu + c^* t) \) with a unique speed \( c^* = c^*(m, r) > 0 \) and unit vector \( \nu \) such that:

(i) \( \text{supp } u^*(t, \cdot) = \Pi(0, +\infty), \phi^*(\xi) \) is monotone increasing and \( \phi^*(+\infty) = \kappa_0 \);

(ii) \( c^*(m, r) \) is the minimal admissible (or critical) traveling wave speed;

(iii) \( c^*(m, r) < c^*(m, 0) \), i.e., the time delay slows down the critical traveling wave speed.

If the initial data \( u_0(s, x) \) are compactly supported (or semi-compactly supported) in a direction \( \nu \in \mathbb{S}^{n-1} \), say \( \text{supp } u_0(s, \cdot) \subset \Pi(x_0, \nu) \) for example, it is expected that the solution \( u(t, x) \) is compactly supported (or semi-compactly supported) in this direction for all the time since the diffusion of the equation (1.1) is degenerate. The solution \( u(t, x) \) together with its support \( \text{supp } u(t, \cdot) \) expands toward opposite \( \nu \) direction. In what follows, we always assume that \( c^*(m, r) > 0 \) is the critical (minimal admissible) wave speed of traveling waves and also is the wave speed of the unique sharp type (semi-compactly supported) traveling wave of (1.1) shown by Theorem 1.1. Now we present the following large time propagation speed of the solution with compact (or semi-compact) support.
Theorem 1.2 Let $u(t, x)$ be the solution of (1.1) with initial data $u_0(s, x)$ for $s \in [-r, 0]$ satisfying

$$\text{supp } u_0(s, \cdot) \subset \Pi(x_0, r), \quad u_0(s, x) \geq \phi_0((x - x_0) \cdot \nu),$$

where $\phi_0(\eta)$ with $\eta = (x \cdot x_0) \cdot \nu$ is a non-negative, continuous and non-trivial function. Then for any $0 < c_1 < c^*(m, r) < c_2$, and any $0 < \kappa_1 < \kappa < \kappa_2$, there exist a time $T = T(c_1, c_2, \kappa_1, \kappa_2) > 0$ and two functions $\phi_1(\cdot)$ and $\phi_2(\cdot)$ with semi-compact supports, such that

$$\phi_1((x - x_0) \cdot \nu + c_1 t) \leq u(t, x) \leq \phi_2((x - x_0) \cdot \nu + c_2 t), \quad \forall (x - x_0) \cdot \nu \leq 0, \quad t \geq T,$$

and

$$\text{supp } \phi_1(\cdot) = \text{supp } \phi_2(\cdot) = \Pi(x_0, \nu), \quad \lim_{\eta \to +\infty} \phi_1(\eta) \geq \kappa_1, \quad \lim_{\eta \to +\infty} \phi_2(\eta) \leq \kappa_2.$$

Therefore,

$$\Pi(x_0 - c_1 t, \nu) \cap \Pi(x_0, -\nu) \subset \text{supp } u(t, \cdot) \subset \Pi(x_0 - c_2 t, \nu), \quad t \geq T.$$  (1.6)

Corollary 1.1 Under the condition of Theorem 1.2 there holds

$$\lim_{t \to +\infty} \inf_{x \in \mathbb{R}^n, \ u(t, x) > 0} \frac{(x - x_0) \cdot \nu}{t} = -c^*(m, r),$$

and

$$\lim_{t \to +\infty} \sup_{x \in \Pi(x_0, -\nu), \ u(t, x) = 0} \frac{(x - x_0) \cdot \nu}{t} = -c^*(m, r).$$

Additionally, for any $x \in \mathbb{R}^n$, it holds

$$\lim_{t \to +\infty} u(t, x - ct \nu) = \begin{cases} \kappa, & \text{if } c < c^*(m, r), \\ 0, & \text{if } c > c^*(m, r). \end{cases}$$  (1.7)

Theorem 1.2 implies that, the solution with non-trivial and non-negative initial data that are compactly supported in one direction (or compactly supported for the one dimensional case) propagates at the same speed as the sharp type traveling wave, which is the minimal wave speed of all the traveling waves. Particularly for the one dimensional case, any compactly supported initial perturbation remains compact and the boundary propagates at speed $c^*(m, r)$.

2 Proof of the main result

We are interested in the propagation speed of the solutions. We assume that $u_0(s, x)$ for $s \in [-r, 0]$ is non-trivial, non-negative, bounded and continuous, therefore the Cauchy problem (1.1) can be solved step by step such that $u(t, x)$ is non-negative, bounded and continuous on $x \in \mathbb{R}^n$ and $t \in [-r, +\infty)$ (see [29] for example). Moreover, the comparison principle holds for the Cauchy problem (1.1) and the initial boundary value problem on bounded domain since the time-delayed source $b(u(t - r, x))$ is monotone increasing with respect to $u(t - r, x)$. We shall prove that for large time scale the average speed of propagation is consistent with the sharp type traveling wave speed.
The sharp type traveling wave is the typical solution that is semi-compactly supported and propagates at a positive and finite speed \( c^*(m, r) \). In order to show the large time propagation speed of general solutions with compact (or semi-compact) supports, we need to construct upper and lower solutions with compact (or semi-compact) supports and propagates at speed near \( c^*(m, r) \). The case of \( r = 0 \) (no time delay) is proved by the phase-plane analysis, where all the trajectories are determined and correspond to special upper and lower solutions, see [24] and the references therein. Here for the time-delayed case \((r > 0)\), we need to employ a new phase transform approach developed in [34].

Consider the following “traveling wave” type special function defined for any \( c > 0 \) and \( k \geq k \)

\[
 u^k_c(t, x) := \phi^k_c(x \cdot v + ct) = \phi^k_c(\xi), \quad \text{with } \xi := x \cdot v + ct, \tag{2.1}
\]

such that

\[ \phi^k_c(\xi) = 0, \quad \forall \xi \leq 0, \quad \phi^k_c(\xi) \in (0, k), \quad \forall \xi \in (0, \xi^k_c), \tag{2.2} \]

for some \( \xi^k_c \in (0, +\infty] \). Note that the sharp type traveling wave \( \phi^* (\xi) \) in Theorem [1.1] is a traveling wave type special function \( \phi^*_c(\xi) \) corresponding to the critical traveling wave speed \( c^* = c^*(m, r) \). As proved in [34], \( c^* \) is the unique speed and \( \phi^* \) is the unique function such that \( \phi^* \) satisfies (2.1), (2.2), and the time-delayed degenerate diffusion equation (1.1). Therefore, we only expect that \( \phi^*_c \) is a local solution of (1.1) near the boundary \( \xi = 0 \) for \( c \neq c^* \). Actually, this is solved through a delayed iteration scheme as follows.

The traveling wave type function \( \phi^*_c(\xi) \) defined by (2.1) and (2.2) is a local solution of (1.1), on \( \xi \in (-\infty, \xi^*_c) \) with \( \xi^*_c \in (0, +\infty) \) if

\[
 \begin{cases} 
 c(\phi^*_c(\xi)) = ((\phi^*_c(\xi)) - d(\phi^*_c(\xi)) + b(\phi^*_c(\xi - cr)), \quad \xi \in (-\infty, \xi^*_c), \hfill \\
 \phi^*_c(\xi) = 0, \quad \forall \xi \leq 0, \quad \phi^*_c(\xi) \in (0, k), \quad \forall \xi \in (0, \xi^*_c). 
\end{cases} \tag{2.3}
\]

The local solvability of the degenerate second order differential equation (2.3) is proved in the following lemma.

**Lemma 2.1** For any \( c > 0 \) and \( k \geq k \), the degenerate problem (2.3) admits a unique local solution \( \phi^*_c(\xi) \) on \(( -\infty, \xi^*_c) \) with \( \xi^*_c \in (0, +\infty) \) (we may assume that \(( -\infty, \xi^*_c) \) is the maximal existence interval) such that

\[ \phi^*_c(\xi) = \left( \frac{(m - 1)c}{m} \right) \frac{\xi^{-1}}{c} + o(|\xi|^{-1}), \quad \xi \to 0. \]

Moreover, (a) \( \phi^*_c(\xi) \) is strictly increasing on \(( 0, \xi^*_c) \) and \( \phi^*_c(\xi^*_c) = k \), or (b) \( \phi^*_c(\xi) \) is not strictly increasing on \(( 0, \xi^*_c) \) and \( \phi^*_c(\xi^*_c) = 0 \). For the case (b), there holds \( \xi^*_c < +\infty \) and there exists a \( \xi^k_c \in (0, \xi^*_c) \) such that \( \phi^*_c(\xi) \) is strictly increasing on \(( 0, \xi^k_c) \) and decreasing on \(( \xi^k_c, \xi^*_c) \).

**Proof.** Note that \( \phi^*_c \) is semi-compactly supported and the time-delayed source function \( b(\phi^*_c(\xi - cr)) = 0 \) if \( \xi \leq cr \). Therefore, (2.3) is locally reduced to the following equation

\[
 \begin{cases} 
 c(\phi^*_c(\xi)) = ((\phi^*_c(\xi)) - d(\phi^*_c(\xi)), \quad \xi \in (-\infty, cr), \hfill \\
 \phi^*_c(\xi) = 0, \quad \forall \xi \leq 0, \quad \phi^*_c(\xi) > 0, \quad \forall \xi \in (0, cr), 
\end{cases} \tag{2.4}
\]

whose unique solvability is proved in [34] [36] and the solution is denoted by \( \phi^*_c \). Lemma 3.1 (and its proof) in [36] shows the locally asymptotical behavior of \( \phi^*_c(\xi) \) near zero and the strictly increasing
monotonicity of \( \phi_c(\xi) \) in \((0, cr)\). If \( \phi_c(cr) \geq k \), then \( \xi^k_c = \sup\{\xi \in (0, cr); \phi_c(\xi) < k\} \). If \( \phi_c(cr) < k \), then we solve (2.3) on \((cr, 2cr)\) as
\[
\begin{align*}
\begin{cases}
    c\phi'(\xi) = (\phi''(\xi))' - d(\phi(\xi)) + b(\phi(\xi) - cr), & \xi \in (cr, 2cr), \\
    \phi_c(cr), \phi'_c(cr) \text{ are determined from left side}.
\end{cases}
\end{align*}
\]

The problem (2.5) is locally solved near \( cr \) since \( \phi_c(cr) > 0 \) and \( b(\phi_c(\xi) - cr) \) is already known from (2.4). Then three cases may happen:

(i) \( \phi_c(\xi) \) is not strictly increasing on whole \((cr, 2cr)\), which means there exists a \( \xi_0 \in (cr, 2cr) \) such that \( \phi'_c(\xi_0) \leq 0 \). We employ Lemma 3.5 in [36] to derive that \( \phi_c(\xi) \) is always decreasing after \( \xi_0 \) until reaching zero for \( \xi > \xi_0 \). If \( \phi_c(\xi) > 0 \) for all \( \xi \in (cr, 2cr) \), then we solve (2.4) further on \((2cr, 3cr)\) and the intervals after this in a similar way as (2.5) until \( \phi_c(\xi_1) = 0 \) for some \( \xi_1 > \xi_0 > cr \). In this case, \( \xi^k_c = \sup\{\xi > cr; \phi_c(\xi) > 0\} \) and \( \phi_c(\xi^k_c) = 0 \). The assertion \( \xi^k_c < +\infty \) is proved in a similar way as the proof of Lemma 3.5 in [36].

(ii) \( \phi_c(\xi) \) is strictly increasing on \((cr, 2cr)\) and \( \phi_c(2cr) \geq k \), then \( \xi^k_c = \sup\{\xi \in (cr, 2cr); \phi_c(\xi) < k\} \) and \( \phi_c(\xi^k_c) = k \).

(iii) \( \phi_c(\xi) \) is strictly increasing on \((cr, 2cr)\) and \( \phi_c(2cr) < k \), then we solve (2.3) further on \((2cr, 3cr)\) and the intervals after this until (i) or (ii) happens. Otherwise, \( \phi_c(\xi) \) is strictly increasing and (2.5) is solved on \((-\infty, +\infty)\) such that \( \xi^k_c = +\infty \) and \( \phi_c(\xi^k_c) = k \). This happens for \( c = c^* \) and \( k = k^* \) since \( \phi^*(\xi) = \phi^*_c(\xi) \) is the unique sharp type traveling wave. □

In order to show more precise behavior of \( \phi^*_c \), we employ the following phase transform approach and formulate phase comparison principle. For any \( c > 0 \) and \( k \geq k^* \), let \( \phi^*_c(\xi) \) be the unique solution of the degenerate problem (2.3) on its maximal existence interval \((-\infty, \xi^*_c)\) with \( \xi^*_c \in (0, +\infty) \) as shown in Lemma 2.1 and let \( \xi^*_c \in (0, \xi^*_c) \) be the largest number (or equivalently, \( (0, \xi^*_c) \) be the maximal interval) such that \( \phi^*_c(\xi) \) is strictly increasing on \((0, \xi^*_c)\). Define (here we do not explicitly write down the dependence of \( \psi_c(\xi) \) on \( k \) for simplicity)

\[
\psi_c(\xi) := ((\phi^*_c(m(\xi)))')' = m(\phi^*_c(m(\xi)))' \cdot (\phi^*_c)'(\xi), \quad \xi \in (0, \xi^*_c).
\]

Now we have two functions \( \phi^*_c(\xi) \) and \( \psi_c(\xi) \) defined for \( \xi \in (0, \xi^*_c) \), and \( \phi^*_c(\xi) \) is strictly increasing on \((0, \xi^*_c)\), then we can interpret \( \psi = \psi_c(\xi) \) as a function of \( \phi = \phi^*_c(\xi) \) through the intermediate variable

\[
\xi = (\phi^*_c)^{-1}(\phi), \quad \phi \in (0, \phi^*_c(\xi^*_c)).
\]

That is, we rewrite

\[
\tilde{\psi}_c(\phi) := \psi_c(\xi) = \psi_c((\phi^*_c)^{-1}(\phi)), \quad \phi \in (0, \phi^*_c(\xi^*_c)).
\]

A key transform in dealing with the time delay in the degenerate diffusion equation (2.3) is to rewrite \( \phi^*_c(\xi - cr) \) as a function of \( \phi = \phi^*_c(\xi) \) depending on \( \tilde{\psi}_c(\phi) \) in a functional way:

\[
\phi_{c, cr}(\phi) := \phi^*_c(\xi - cr) = \phi^*_c((\phi^*_c)^{-1}(\phi) - cr) = \inf_{\phi \geq 0} \left\{ \int_0^\phi \frac{m s^{-1}}{\tilde{\psi}_c(s)} ds \leq cr \right\}, \quad \phi \in (0, \phi^*_c(\xi^*_c)).
\]

Lemma 2.2 (Phase transform) The functional interpretation (2.8) is well-defined for the sharp type functions \( \phi^*_c(\xi) \) for \( \phi \in (0, \phi^*_c(\xi^*_c)) \).
Proof. We divide the proof into two cases.

Case I. If for some \( \phi = \phi_k^\xi(\xi) \) with \( \xi \in (0, \xi_k^\xi) \) and \( \phi \in (0, \phi_k^\xi(\xi_k^\xi)) \) there holds

\[
\int_0^\phi \frac{m^{s-1}}{\psi_c(s)} \, ds > cr,
\]
then we rewrite the above integral through the method of substitution of \( s = \phi_k^\xi(t) \) for \( s \in (0, \phi) \) and \( t \in (0, \xi) \) to find

\[
cr < \int_0^\phi \frac{m^{s-1}}{\psi_c(s)} \, ds = \int_0^\xi \frac{m(\phi_k^\xi)^{m-1}(t)}{m(\phi_k^\xi)^{m-1}(t) \cdot (\phi_k^\xi)'(t)} (\phi_k^\xi)'(t) \, dt = \xi.
\]

Therefore, \( \xi - cr > 0 \) and \( \phi_{c,cr}(\phi) = \phi_k^\xi(\xi - cr) \) is the unique value such that

\[
\int_{\phi_{c,cr}(\phi)}^\phi \frac{m^{s-1}}{\psi_c(s)} \, ds = cr.
\]

Case II. If for some \( \phi = \phi_k^\xi(\xi) \) with \( \xi \in (0, \xi_k^\xi) \) and \( \phi \in (0, \phi_k^\xi(\xi_k^\xi)) \) there holds

\[
\int_0^\phi \frac{m^{s-1}}{\psi_c(s)} \, ds \leq cr,
\]
then \( \xi - cr \leq 0 \) and \( \phi_k^\xi(\xi - cr) = 0 \) since \( \phi_k^\xi \) is sharp type such that \( \phi_k^\xi(t) \equiv 0 \) for all \( t \leq 0 \).

\[\Box\]

Lemma 2.3 (Monotone dependence) For any \( c > 0 \) and \( k \geq \kappa \), let \( \phi_k^\xi(\xi) \) be the unique solution of the degenerate problem \((2.3)\) on its maximal existence interval \((-\infty, \xi_k^\xi)\) with \( \xi_k^\xi \in (0, +\infty) \) and let \( \tilde{\psi}_c(\phi) \) and \( \phi_{c,cr}(\phi) \) be the phase transform functions defined by \((2.7)\) and \((2.8)\). Then for \( c_1 > c_2 > 0 \), there holds

\[
\tilde{\psi}_{c_1}(\phi) > \tilde{\psi}_{c_2}(\phi), \quad \forall \phi \in (0, \min\{\phi_{c_1}^\xi(\xi_1), \phi_{c_2}^\xi(\xi_1)\}),
\]

and

\[
\phi_{c_1}^\xi(\xi) > \phi_{c_2}^\xi(\xi), \quad \forall \xi \in (0, \min\{\xi_1, \xi_2\}).
\]

Proof. The monotone dependence of \( \phi_k^\xi(\xi) \) with respect to \( c \) is proved in Lemma 3.6 in [34]. The monotone dependence \( \tilde{\psi}_{c_1}(\phi) > \tilde{\psi}_{c_2}(\phi) \) means that

\[
(\phi_{c_1}^\xi)'(\xi_1) > (\phi_{c_2}^\xi)'(\xi_2), \quad \text{at where } \phi_{c_1}^\xi(\xi_1) = \phi = \phi_{c_2}^\xi(\xi_2),
\]

or equivalently,

\[
(\phi_{c_1}^\xi)'(\phi_{c_1}^\xi(\xi_1)) > (\phi_{c_2}^\xi)'(\phi_{c_2}^\xi(\xi_2)). \tag{2.11}
\]

In contrast to the comparison between two functions \( \phi_{c_1}^\xi(\xi) \) and \( \phi_{c_2}^\xi(\xi) \) at the same point \( \xi \), \((2.11)\) shows the comparison of their derivatives at where they take the same value, hence we would call it the phase comparison principle. The prototype of \((2.11)\) (and \((2.9)\)) is already formulated in the proof of Lemma 3.6 in [34]. Here we omit the details.

The above monotone dependence is used to construct special upper and lower solutions at speed near the critical wave speed \( c^*(m,r) \).
Lemma 2.4 For any \( c > c^*(m, r) \), there exists a number \( k > \kappa \), such that

\[
\overline{u}(t, x) := \overline{\phi}^k_c(\xi) := \begin{cases} 
\phi^k_c(\xi), & \xi < \xi^k_c, \\
k, & \xi \geq \xi^k_c,
\end{cases} \quad \xi = x \cdot v + ct,
\]

is an upper solution of (1.1) with the initial data \( \overline{u}_0(s, x) := \overline{\phi}^k_c(x \cdot v + cs) \) for \( s \in [-r, 0] \), where \( \phi^k_c(\xi) \) is the unique solution of the degenerate problem (2.3) on its maximal existence interval \(( -\infty, \xi^k_c)\) with \( \xi^k_c \in (0, +\infty) \). Similarly, for any \( c \in (0, c^*(m, r)) \),

\[
u(t, x) := \phi^k_c(\xi) := \begin{cases} 
\phi^k_c(\xi), & \xi < \xi^k_c, \\
0, & \xi \geq \xi^k_c,
\end{cases} \quad \xi = x \cdot v + ct,
\]

is a lower solution of (1.1) with the initial data \( \nu_0(s, x) := \phi^k_c(x \cdot v + cs) \) for \( s \in [-r, 0] \), \( \xi^k_c < +\infty \), and \( \sup_{\xi \in \mathbb{R}} \phi^k_c(\xi) < \kappa \). Moreover, for any \( c \in (0, c^*(m, r)) \),

\[
\hat{u}(t, x) := \hat{\phi}^k_c(\xi) := \begin{cases} 
\phi^k_c(\xi), & \xi < \xi^k_c, \\
\phi^k_c(\xi^k_c), & \xi \geq \xi^k_c,
\end{cases} \quad \xi = x \cdot v + ct,
\]

also is a lower solution of (1.1) with initial data \( \hat{u}_0(s, x) := \hat{\phi}^k_c(x \cdot v + cs) \) for \( s \in [-r, 0] \). Additionally, it holds

\[
\lim_{c \to (c^*(m, r))^-} \sup_{\xi \in \mathbb{R}} \phi^k_c(\xi) = \kappa.
\]

Proof. For the critical wave speed \( c = c^* \), the unique solution of the degenerate problem (2.3) is the sharp traveling wave \( \phi^*(\xi) = \phi^*_{c}((\xi)), \) and the phase transform function \( \hat{\psi}_{c^*}(\phi) \) satisfies \( \hat{\psi}_{c^*}(\phi) > 0 \) for \( \phi \in (0, \kappa) \) and \( \hat{\psi}_{c^*}(0) = \hat{\psi}_{c^*}(\kappa) = 0 \).

For any \( c > c^* \), according to the phase comparison principle (2.9) in Lemma 2.3, we see that for any \( k > \kappa, \hat{\psi}_{c^*}(k) > \hat{\psi}_{c^*}(\kappa) = 0 \), that is, \( (\phi^k_c)^{-1}(1(\kappa)) > 0 \). We choose \( k > \kappa \) such that case (a) in Lemma 2.1 occurs, i.e., \( \phi^k_c(\xi) \) is strictly increasing on \((0, \xi^k_c)\) and \( \phi^k_c(\xi^k_c) = k > \kappa \). Therefore, \( \phi^k_c \) is an upper solution of the second order differential equation (1.1).

For any \( 0 < c < c^* \), similar to the above analysis, according to the phase comparison principle (2.9), we have \( \hat{\psi}_{c^*}(\hat{\phi}) = 0 < \hat{\psi}_{c^*}(\hat{\phi}) \) for some \( \hat{\phi} \in (0, \kappa) \) since \( \hat{\psi}_{c^*}(\kappa) = 0 \). Therefore, \( \phi^k_c(\xi) \) is increasing up to \( \hat{\phi} < \kappa \) and then decreases to zero, which means Case (b) in Lemma 2.1 occurs. In this case, we have \( \xi^k_c < +\infty \) and \( \phi^k_c(\xi^k_c) = 0 \). Then it follows that \( \phi^k_c \) is a lower solution of the second order differential equation (1.1). Furthermore, \( \hat{u}(t, x) = \hat{\phi}^k_c(\xi) \) is a lower solution of (1.1) since \( \phi^k_c(\xi^k_c) = 0 \) at the cut-off edge. The limit of \( \lim_{c \to (c^*(m, r))^-} \sup_{\xi \in \mathbb{R}} \phi^k_c(\xi) = \kappa \) follows from the continuous dependence (see Lemma 3.4 in [34] for example), the monotone dependence Lemma 2.3 and the fact that \( \sup_{\xi \in \mathbb{R}} \phi^k_{c^*(m, r)}(\xi) = \kappa \). \( \square \)

Now we investigate the large-time evolution of the solution with semi-compact support.

Lemma 2.5 Let \( u(t, x) \) be the solution of (1.1) with the initial data \( u_0(s, x) \) semi-compactly supported and bounded

\[
\text{supp } u_0(s, \cdot) \subset \Pi(x_0, v), \quad u_0 \in L^\infty([-r, 0] \times \mathbb{R}^n).
\]

Then

\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}^n} u(t, x) \leq \kappa.
\]
Proof. Consider the following differential problem

\[
\begin{cases}
U'(t) = -d(U) + b(U(t - r)), & t > 0, \\
U(s) = U_0(s) \equiv \|u_0\|_{L^\infty([-r, 0] \times \mathbb{R}^n)}, & s \in [-r, 0].
\end{cases}
\] (2.13)

The large-time asymptotic analysis of the time-delayed ordinary differential equation (2.13) shows that \(\lim_{t \to +\infty} U(t) = \kappa\). Based on the comparison principle, and taking \(U(t)\) as an upper solution of (1.1), we have

\[
\limsup_{t \to +\infty} \sup_{x \in \mathbb{R}^n} u(t, x) \leq \limsup_{t \to +\infty} U(t) = \kappa.
\]

The proof is completed. \(\square\)

Lemma 2.6 Let \(u(t, x)\) be the solution of (1.1) with initial data \(u_0(s, x)\) satisfying

\[
u_0(s, x) \geq \phi_0((x - x_0) \cdot v),
\] (2.14)

where \(\phi_0(\eta)\) with \(\eta = (x - x_0) \cdot v\) is a non-negative, continuous and non-trivial function. Then for any compact subset \(K \subset \mathbb{R}^n\),

\[
\liminf_{t \to +\infty} \inf_{x \in K} u(t, x) \geq \kappa.
\]

Proof. The proof is divided into two steps. The first one is to show that the support of \(u(t, \cdot)\) expands to including any given compact subset for large time, and the second one is to show that the value of \(u(t, x)\) within given compact subset grows up as time increases.

Step I. Define a Barenblatt type function

\[
g(t, x) = e(\tau + t)^{-\sigma}[\left(\eta^2 - \frac{|x - x_0|^2}{\tau + t}\right)^d_+, \quad x \in \Omega, \ t \geq 0,
\]

where \(d = 1/(m - 1), \beta, \sigma, \varepsilon, \eta, \) and \(\tau\) are positive constants, \(x_0 \in \mathbb{R}^n\). Then by appropriately selecting \(\beta, \varepsilon, \tau, \sigma, \eta, \) and \(x_0\), the function \(g(t, x)\) is a weak lower solution of (1.1) for all the time \(t > 0\). The detailed calculations can be found in the proof of Lemma 4.4 in [35]. Although the value of \(g(t, x)\) is decaying, its support is expanding at a rate as \((\tau + t)^{\frac{\beta}{d}}\) for some \(\beta > 0\). Therefore, for any given compact subsets \(K_1 \subset K_2 \subset \mathbb{R}^n\), there exists a time \(t_1 > 0\) such that \(K_2 \subset \text{supp } u(t, \cdot)\) for any \(t \geq t_1\) and \(\inf_{x \in K_1} u(t_1, x) > 0\) for any \(t_2 > t_1\).

Step II. We assert that for any \(\hat{k} \in K\) and any \(\hat{k} < \kappa\), there exist a time \(\hat{t}\) and an open neighbourhood \(B(\hat{x})\) of \(\hat{x}\) such that \(u(t, x) \geq \hat{k}\) for all \(t \geq \hat{t}\) and \(x \in B(\hat{x})\). Then the assertion \(\liminf_{t \to +\infty} \inf_{x \in K} u(t, x) \geq \kappa\) follows from the finite covering theorem. For any given \(\hat{k} < \kappa\), we define

\[
\hat{d}(s) := d(s) + \lambda_0 s
\]

with \(\lambda_0 > 0\) sufficiently small such that \(b(s) > \hat{d}(s)\) for all \(s \in (0, \hat{k}]\) due to \(b(s) > d(s)\) for all \(s \in (0, \kappa)\). That is, the minimal positive equilibrium for \(b(s)\) and \(\hat{d}(s)\) is located in \((\hat{k}, \kappa)\). Consider the following separated variable function

\[
U(t, x) := (\cos(\mu_0(x - \hat{x})))^{\frac{1}{\lambda}} \cdot g(t),
\] (2.15)
with \( \mu_0 > 0 \) and function \( g(t) > 0 \) to be determined. We have

\[
\Delta U^m(t, x) = -\mu_0^2 (\cos(\mu_0(x - \hat{x})))_+ \cdot g^m(t) = -\mu_0^2 U^m(t, x), \quad \forall x \in B_{R_0}(\hat{x}), \quad \text{with} \ R_0 := \frac{\pi}{2\mu_0},
\]

and the generalized derivative (which is not Lebesgue integrable) satisfies

\[
\Delta U^m(t, x) \geq -\mu_0^2 (\cos(\mu_0(x - \hat{x})))_+ \cdot g^m(t) \cdot \chi_{B_0}(x) \geq -\mu_0^2 U^m(t, x),
\]

in the sense of distributions, where \( \chi_B(x) \) is the characteristic function of the set \( B \).

Let us choose \( g(t) \in (0, \kappa) \) and \( \mu_0 = \sqrt{\lambda_0/\kappa^{m-1}} \), then

\[
\Delta U^m(t, x) \geq -\mu_0^2 U^m(t, x) \geq -\mu_0^2 \kappa^{m-1} U(t, x) = -\lambda_0 U(t, x).
\]

In order to construct \( U(t, x) \) as a lower solution of (1.1) for \( t > t_1 \) with some \( T > 0 \), it suffices to set

\[
\begin{cases}
\frac{\partial U}{\partial t} \leq -\Delta(U) + b(U(t-r, x)), & x \in \mathbb{R}^n, \ t > T, \\
U(s, x) \leq u(s, x), & x \in \mathbb{R}^n, \ s \in [T-r, T].
\end{cases}
\] (2.16)

Therefore, we have

\[
\frac{\partial U}{\partial t} \leq -\Delta(U) + b(U(t-r, x)) = -\lambda_0 U(t-r) + b(U(t-r, x)) \leq \Delta U^m(t, x) - d(U) + b(U(t-r, x)),
\]

which is the differential inequality in the definition of lower solutions. As to the comparison of the initial data, we have

\[
U(s, x) \leq g(s) \cdot \chi_{B_0}(x).
\]

According to Step I, by setting \( K_1 = B_{R_0}(\hat{x}) \) and \( T = t_1 + r, \ t_2 = T \), we further have

\[
u(s, x) \geq \inf_{t \in [T-r, T]} u(t, x) := e_0 > 0, \quad \forall x \in B_{R_0}(\hat{x}), \ s \in [T-r, T].
\]

It follows that a sufficient condition for (2.16) is

\[
\begin{cases}
(c(\mu_0(x - \hat{x})))_{+} \cdot g'(t) \\
\leq -\hat{d}(c(\mu_0(x - \hat{x})))_{+} \cdot g(t) + b(c(\mu_0(x - \hat{x})))_{+} \cdot g(t-r), & x \in \mathbb{R}^n, \ t > T, \\
g(s) = e_0, & s \in [T-r, T], \quad g(t) \in (0, \kappa), \ t > T,
\end{cases}
\] (2.17)

or alternatively,

\[
\begin{cases}
g'(t) \leq \inf_{t \in (0, 1)} \frac{b(t) - \hat{d}(t)}{\lambda}, & t > T, \\
g(s) = e_0, & s \in [T-r, T], \quad g(t) \in (0, \kappa), \ t > T.
\end{cases}
\] (2.18)

Note that

\[
\lim_{\lambda \to 0^+} \frac{b(\lambda s) - \hat{d}(\lambda s)}{\lambda} = b'(0)s - \hat{d}'(0)s = b'(0)s = (d'(0) + \lambda_0)s,
\]
which is strictly increasing for all \( s > 0 \) since \( b'(0) > d'(0) \geq 0 \) (\( \lambda_0 \) is sufficiently small), and \( b(s) > \hat{d}(s) \) for all \( s \in (0, \hat{k}] \). There exists a constant \( \delta_0 > 0 \) such that

\[
\inf_{\lambda \in (0,1)} \frac{b(\lambda s) - \hat{d}(\lambda s)}{\lambda} \geq \delta_0 > 0, \quad \forall s \in [\epsilon, \hat{k}].
\]

We now solve the following time-delayed ordinary differential equation step by step:

\[
g'(t) = \inf_{\lambda \in (0,1)} \frac{b(\lambda g(t - r)) - \hat{d}(\lambda g(t))}{\lambda}, \quad t > T,
\]

\[
g(s) = \epsilon_0, \quad t \in [T - r, T].
\]

Firstly, for \( t \in [T, T + r) \), we have

\[
g'(T) = \inf_{\lambda \in (0,1)} \frac{b(\lambda \epsilon_0) - \hat{d}(\lambda \epsilon_0)}{\lambda} \geq \delta_0 > 0,
\]

which means \( g(t) \) is strictly increasing until \( t \geq T + r \) or

\[
\inf_{\lambda \in (0,1)} \frac{b(\lambda \epsilon_0) - \hat{d}(\lambda g(t))}{\lambda} = 0.
\]

Since there exist two constants \( C_2 \geq C_2 > 0 \) such that \( C_1 g(t) \leq \hat{d}(\lambda g(t))/\lambda \leq C_2 g(t) \), the asymptotic analysis of linear differential inequality shows that (2.20) cannot happen in finite time. Therefore, \( g(t) \) is strictly increasing on \([T, T + r]\) and

\[
\inf_{\lambda \in (0,1)} \frac{b(\lambda \epsilon_0) - \hat{d}(\lambda g(t))}{\lambda} > 0, \quad \forall t \in [T, T + r].
\]

Secondly, for \( t \in [T + r, T + 2r) \), we have

\[
g'(T + r) = \inf_{\lambda \in (0,1)} \frac{b(\lambda g(T)) - \hat{d}(\lambda g(T + r))}{\lambda} = \inf_{\lambda \in (0,1)} \frac{b(\lambda \epsilon_0) - \hat{d}(\lambda g(T + r))}{\lambda} > 0,
\]

due to (2.21). It follows that \( g(t) \) is strictly increasing until \( t \geq T + 2r \) or

\[
\inf_{\lambda \in (0,1)} \frac{b(\lambda g(t - r)) - \hat{d}(\lambda g(t))}{\lambda} = 0,
\]

where \( g(t - r) \) is already known as \( t - r \in [T, T + r] \). An asymptotic analysis shows that (2.22) cannot happen in finite time, especially, in \([T + r, T + 2r]\). Otherwise, let \( t^* \in (T + r, T + 2r] \) be the minimal time such that (2.22) is valid. Then \( g'(t) = 0 \) and \( b(\lambda g(t - r))/\lambda \) is strictly increasing. Hence there exists a \( \hat{r}^* \in (T + r, t^*) \) such that

\[
\inf_{\lambda \in (0,1)} \frac{b(\lambda g(t - r)) - \hat{d}(\lambda g(t))}{\lambda} < 0, \quad t \in (\hat{r}^*, t^*), \quad \text{and} \quad \inf_{\lambda \in (0,1)} \frac{b(\lambda g(\hat{r}^* - r)) - \hat{d}(\lambda g(\hat{r}^*))}{\lambda} = 0,
\]

which contradicts to the minimality of \( t^* \). Repeating the above arguments, we see that \( g(t) \) is increasing and the minimal positive equilibrium of \( \inf_{\lambda \in (0,1)} (b(\lambda s) - \hat{d}(\lambda s))/\lambda \) is greater than \( \hat{k} \). There exists a time \( \hat{t} > T \) such that \( g(t) > \hat{k} \) for all \( t \geq \hat{t} \). Furthermore, by the comparison principle,

\[
u(t, x) \geq U(t, x) = (\cos(\mu_0(x - \hat{x})))_{+} \cdot g(t) > (\cos(\mu_0(x - \hat{x})))_{+} \cdot \hat{k}, \quad t \geq \hat{t}.
\]
That is,
$$u(t, \hat{x}) \geq U(t, \hat{x}) = g(t) > \hat{k}, \quad t \geq \hat{t}.$$ 

Based on the uniformly continuity of \((\cos(\mu_0(x - \hat{x})))\), near \(\hat{x}\) with respect to \(t\), we can find a neighborhood \(B(\hat{x})\) of \(\hat{x}\), independent of time, such that \(u(t, x) \geq \hat{k}\), for all \(x \in B(\hat{x})\) and \(t \geq \hat{t}\). The proof is complete. \(\square\)

Next, in order to get the large time speed of propagation, we are going to prove it by combining the large time evolution of the solution proved in Lemma 2.5, Lemma 2.6, and the special upper and lower solutions in Lemma 2.4.

**Proof of Theorem 1.2** First of all, from Lemma 2.5 and Lemma 2.6, we have the large time evolution such that

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}^n} u(t, x) \leq \kappa,$$

and for any compact subset \(K \subset \mathbb{R}^n\)

$$\liminf_{t \to +\infty} \inf_{x \in K} u(t, x) \geq \kappa.$$ 

Note that the initial condition \((1.1)\) is translation invariant in the direction perpendicular to \(v\), similar to the proof of Lemma 2.6 for any finite numbers \(s_1 < s_2\), there holds

$$\liminf_{t \to +\infty} \inf_{(x-x_0) \cdot \nu \in [s_1, s_2]} u(t, x) \geq \kappa.$$ 

Without loss of generality, we may assume that \(\phi_0(\eta)\) is symmetric (after shifting if necessary) with respect to \(\eta = 0\). Otherwise, we can choose another function with symmetry and smaller than \(\phi_0(\eta)\). Then the lower solution \(u^*(t, x)\) with the initial data given by \(u_0^*(s, x) = \phi_0((x-x_0) \cdot \nu)\) is also symmetric with respect to \(\eta = 0\). The propagation properties of \(u^*(t, x)\) at one side of \(\eta < 0\) is equivalent to the initial boundary value problem with homogeneous Neumann condition in the half space according to the reflection principle.

For any \(c > c^*(m, r)\), let \(\hat{c} \in (c^*(m, r), c)\) and \(\overline{u}(t, x) = \overline{\phi}^c_\xi(\xi)\) with \(\xi = x \cdot \nu + \hat{c} t\) be the upper solution of \((1.1)\) corresponding to \(\hat{c} > c^*(m, r)\) as proved in Lemma 2.4. Note that \(\lim_{\xi \to +\infty} \overline{\phi}^c_\xi(\xi) = k > \kappa\), we can change the initial time to some \(T > 0\) such that

$$\sup_{x \in \mathbb{R}^n} u(t, x) < \frac{k + \kappa}{2} < k, \quad \forall t > T,$$

and then shift \(\overline{u}(t, x) = \overline{\phi}^c_\xi(\xi)\) such that the comparison of the initial data is valid. The comparison principle shows that

$$\lim_{t \to +\infty} \overline{u}(t, x - cv) \leq \lim_{t \to +\infty} \overline{u}(t, x - cv) = \lim_{t \to +\infty} \overline{\phi}^c_\xi((x - cv) \cdot \nu + \hat{c} t) = \lim_{t \to +\infty} \overline{\phi}^c_\xi(x \cdot \nu - (c - \hat{c})t) = 0,$$

since \(c > \hat{c}\) and \(\overline{\phi}^c_\xi(\xi) = 0\) for \(\xi \leq \xi_0\), where \(\xi_0\) is given after the shifting.

Similarly, for any \(c < c^*(m, r)\), let \(\check{c} \in (c, c^*(m, r))\) and \(\underline{u}(t, x) = \underline{\phi}^c_\xi(\xi)\) with \(\xi = x \cdot \nu + \check{c} t\) be the lower solution of \((1.1)\) corresponding to \(\check{c} < c^*(m, r)\) constructed in Lemma 2.4. Since \(\sup_{\xi \in \mathbb{R}} \underline{\phi}^c_\xi(\xi) < \kappa\), we change the initial time to some \(T > 0\) such that

$$\inf_{x \in [-\xi_0^c + \check{c} t, 0]} u(t, x) > \sup_{\xi \in \mathbb{R}} \underline{\phi}^c_\xi(\xi), \quad t \geq T,$$
Note that $\hat{\phi}^t(\xi) = 0$ for $\xi \leq 0$ and $\hat{\phi}^t(\xi) = \phi^t(\xi^{+})$ for $\xi \geq \xi^+$, we shift $\hat{\phi}^t(\xi)$ such that $\hat{\phi}^t(\xi) = 0$ for $\xi \leq -(\xi^+ + 1)$ and $\hat{\phi}^t(\xi) = 0$ for $\xi \geq -1$. Therefore, $\tilde{u}(t, x) = \hat{\phi}^t(\xi)$ is a lower solution of the corresponding homogeneous Neumann problem (1.1) on the half space. According to the comparison principle, we have

$$\lim_{t \to +\infty} u(t, x - ctv) \geq \lim_{t \to +\infty} \tilde{u}(t, x - ctv) = \lim_{t \to +\infty} \hat{\phi}^t((x - ctv) \cdot \nu + \hat{c} t) = \lim_{t \to +\infty} \hat{\phi}^t(x \cdot \nu + (\hat{c} - c) t) = \hat{\phi}^t(\xi^+),$$

since $\hat{c} > c$. According to Lemma 2.4

$$\lim_{t \to +\infty} \sup_{\xi \in \mathbb{R}} \phi^t(\xi) = \lim_{t \to +\infty} \hat{\phi}^t(\xi^+) = \kappa,$$

and $\hat{c} \in (c, c^+(m, r))$ is arbitrary, we see that $\lim_{t \to +\infty} u(t, x - ctv) \geq \kappa$. Combining this with the fact that $\limsup_{t \to +\infty} \sup_{\xi \in \mathbb{R}} u(t, x) \leq \kappa$, we have $\lim_{t \to +\infty} u(t, x - ctv) = \kappa$. The proof is completed. \hfill \Box

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**References**

[1] D.G. Aronson, The porous medium equation, in: A. Fasano, M. Primicerio (Eds.), Some Problems in Nonlinear Diffusion, in: Lecture Notes in Math., Springer-Verlag, New York/Berlin, 1986.

[2] D.G. Aronson, Density-dependent interaction-diffusion systems, in: Proc. Adv. Seminar on Dynamics and Modeling of Reactive System, Academic Press, New York, 1980.

[3] D.G. Aronson, H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. Partial differential equations and related topics, Lecture Notes in Math., 446, Springer-Verlag, Berlin, 1975, 5–49.

[4] D.G. Aronson, H.F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics. *Adv. Math.* 30 (1978), 33–76.

[5] A. Audrito, Bistable reaction equations with doubly nonlinear diffusion, *Discrete Contin. Dyn. Syst.*, 39 (2019), 2977–3015.

[6] A. Audrito, J.L. Vázquez, The Fisher-KPP problem with doubly nonlinear diffusion, *J. Differential Equations*, 263 (2017), 7647–7708.

[7] A. Audrito, J.L. Vázquez, Travelling wave behaviour arising in nonlinear diffusion problems posed in tubular domains, *J. Differential Equations*, 269 (2020), 2664–2696.
[8] R.D. Benguria, M.C. Depassier, A variational principle for the asymptotic speed of fronts of the density dependent diffusion-reaction equation, *Phys. Rev. E*, **52** (1995), 3285–3287.

[9] R.D. Benguria, M.C. Depassier, Variational characterization of the speed of reaction diffusion fronts for gradient dependent diffusion, *Ann. Henri Poincaré*, **19** (2018), no. 9, 2717–2726.

[10] H. Berestycki, F. Hamel, G. Nadin, Asymptotic spreading in heterogeneous diffusive excitable media, *J. Funct. Anal.*, **255** (2008), 2146–2189.

[11] H. Berestycki, F. Hamel, N. Nadirashvili, The speed of propagation for KPP type problems. II. General domains. *J. Amer. Math. Soc.*, **23** (2010), 1–34.

[12] I-L. Chern, M. Mei, Q. Zhang, and X. Yang, Stability of non-montone critical traveling waves for reaction-diffusion equations with time-delay, *J. Differential Equations*, **259** (2015), 1503–1541.

[13] A. de Pablo, J.L. Vázquez, Travelling waves and finite propagation in a reaction-diffusion equation, *J. Differential Equations*, **93** (1991), 19–61.

[14] R.A. Fisher, The wave of advance of advantageous genes, *Ann. Hum. Genet.*, **7** (1937), 353–369.

[15] B.H. Gilding, R. Kersner, A Fisher/KPP-type equation with density-dependent diffusion and convection: travelling-wave solutions, *J. Phys. A*, **38** (2005), 3367–3379.

[16] W.S.C. Gurney, R.M. Nisbet, The regulation of inhomogeneous population, *J. Theors. Biol.*, **52** (1975), 441–457.

[17] R. Huang, C.H. Jin, M. Mei, J.X. Yin, Existence and stability of traveling waves for degenerate reaction-diffusion equation with time delay, *J. Nonlinear Sci.*, **28** (2018), 1011–1042.

[18] A. Kolmogorov, I. Petrovskii, N. Piscounov, Étude de l’équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Bull. Univ. Etat Moscou, Ser. Int., Sect. A, Math. et Mecan.*, **1** (1937), 1–25.

[19] B. Li, H.F. Weinberger, M.A. Lewis, Spreading speeds as slowest wave speed for cooperative systems, *Math. Biosci.*, **196** (2005), 82–89.

[20] W.-T. Li, J.-B. Wang, X.-Q. Zhao, Propagation dynamics in a time periodic nonlocal dispersal model with stage structure, *J. Dynam. Differential Equations*, **32** (2020), 1027–1064.

[21] X. Liang, Y. Yi, X.-Q. Zhao, Spreading speeds and traveling waves for periodic evolution systems, *J. Differential Equations*, **231** (2006), 57–77.

[22] X. Liang, X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, *Commun. Pure Appl. Math.*, **60** (2007), 1–40.

[23] X. Liang, X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, *J. Funct. Anal.*, **259** 2010, 857–903.
[24] G.S. Medvedev, K. Ono, P.J. Holmes, Travelling wave solutions of the degenerate Kolmogorov-Petrovski-Piskunov equation, *European J. Appl. Math.*, **14** (2003), 343–367.

[25] M. Mei, C.K. Lin, C.T. Lin and J.W.-H. So, Traveling wavefronts for time-delayed reaction-diffusion equation: (i) local nonlinearity, *J. Differential Equations*, **247** (2009), 495–510.

[26] J.D. Murry, Mathematical Biology I: An Introduction, Springer, New York, USA, 2002.

[27] F. Sánchez-Garduño, P.K. Maini, M.E. Kappos, A shooting argument approach to a sharp-type solution for nonlinear degenerate Fisher-KPP equations, *IMA J. Appl. Math.*, **57** (1996), 211-221.

[28] H.R. Thieme, X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models, *J. Differential Equations*, **195** (2003), 430–470.

[29] J.L. Vázquez, *The Porous Medium Equation: Mathematical Theory*, Oxford Univ. Press, 2006.

[30] J.L. Vázquez, Smoothing and Decay Estimates for Nonlinear Diffusion Equations. Equations of Porous Medium Type, Oxford University Press, Oxford, 2006.

[31] H.F. Weinberger, Long-time behavior of a class of biological models, *SIAM J. Math. Anal.*, **13** (1982), 353–396.

[32] Z. Wu, J. Zhao, J. Yin, H. Li. *Nonlinear diffusion equations*, World Scientific Publishing Co. Put. Ltd., 2001.

[33] T.Y. Xu, S.M. Ji, M. Mei, J.X. Yin, Traveling waves for time-delayed reaction diffusion equations with degenerate diffusion, *J. Differential Equations*, **265** (2018), 4442–4485.

[34] T.Y. Xu, S.M. Ji, M. Mei, J.X. Yin, Variational approach of critical sharp front speeds in degenerate diffusion model with time delay, *Nonlinearity*, **33** (2020), 4013–4029.

[35] T.Y. Xu, S.M. Ji, M. Mei, J.X. Yin, On a chemotaxis model with degenerate diffusion: initial shrinking, eventual smoothness and expanding, *J. Differential Equations*, **268** (2020), 414–446.

[36] T.Y. Xu, S.M. Ji, M. Mei, J.X. Yin, Sharp oscillatory traveling waves of structured population dynamics model with degenerate diffusion, *J. Differential Equations*, **269** (2020), 8882–8917.

[37] L. Zhang, Z.-C. Wang, X.-Q. Zhao, Propagation dynamics of a time periodic and delayed reaction-diffusion model without quasi-monotonicity, *Trans. Amer. Math. Soc.*, **372** (2019), 1751–1782.

[38] G.-B. Zhang, X.-Q. Zhao, Propagation phenomena for a two-species Lotka-Volterra strong competition system with nonlocal dispersal, *Calc. Var. Partial Differential Equations*, **59** (2020), 1–34.

[39] G.-B. Zhang, X.-Q. Zhao, Propagation dynamics of a nonlocal dispersal Fisher-KPP equation in a time-periodic shifting habitat, *J. Differential Equations*, **268** (2020), 2852–2885.