TROPICAL CONVEX HULLS OF INFINITE SETS

CVETELINA HILL, SARA LAMBOGLIA, AND FAYE PASLEY SIMON

Abstract. In this paper we study the interplay between tropical convexity and its classical counterpart. In particular, we focus on the tropical convex hull of convex sets and polyhedral complexes. We give a vertex description of the tropical convex hull of a line segment and of a ray in \( \mathbb{R}^n / \mathbb{R}1 \) and show that tropical convex hull and classical convex hull commute in \( \mathbb{R}^3 / \mathbb{R}1 \). Finally we prove results on the dimension of tropically convex fans and give a lower bound on the degree of a tropical curve under certain hypotheses.

Introduction

Tropical convexity is the analog of classical convexity in the tropical semiring \((\mathbb{R}, \oplus, \odot)\) where \( a \oplus b = \min(a, b) \), and \( a \odot b = a + b \). Tropical convexity has been primarily focused on tropical polytopes, or tropical convex hulls of finite sets. These are widely studied [DS04, CGQS05, CGQ04, GS07, Jos05, GM10, AGG10] and find applications in various areas of mathematics. Recently, techniques from tropical convexity have been applied to mechanism design [CT16], optimization [AGG12], and maximum likelihood estimation [RSTU18]. Some specific applications are the resolution of monomial ideals [DY07] and discrete event dynamic systems [BCOQ92]. Moreover, computational tools exist to aid in further study of tropical polytopes [Jos09, AGG10].

The goal of this paper is to investigate the relationship between tropical and classical convexity. The two do not coincide even for small examples. However, many properties and theorems valid in classical convexity are also valid in the tropical setting; for example, separation of convex sets [CGQ04, GS07], Minkowski-Weyl theorem [GK07, GK11, Jos05], Carathéodory and Helly theorems [DS04, GM10], and Farkas Lemma [DS04]. Here we consider the tropical convex hull of classically convex sets, polyhedral complexes, and other infinite sets.

A set \( U \subset \mathbb{R}^{n+1} \) is tropical convex if for every \( x, y \in U \) and \( a, b \in \mathbb{R} \) the tropical linear combination \( (a \odot x) \oplus (b \odot y) \) is in \( U \). It is customary to work with tropically convex sets in the tropical projective torus \( \mathbb{P}^n := \mathbb{R}^{n+1} / \mathbb{R}1 \) since any tropically convex set \( A \) satisfies \( A = A + \mathbb{R}1 \), where \( 1 = (1, \ldots, 1) \). Moreover, \( \mathbb{P}^n \) and \( \mathbb{R}^n \) are isomorphic as \( \mathbb{R} \)-vector spaces via the map

\[
\mathbb{P}^n \to \mathbb{R}^n, \quad (x_0, \ldots, x_n) \mapsto (x_1 - x_0, \ldots, x_n - x_0). \tag{1}
\]

The following theorem summarizes our main results. These results allow us to determine the dimension of \( \text{tconv} (\text{conv}(a, b)) \) and \( \text{tconv} (\text{pos}(a)) \) in Corollary 2.4 and bound the dimension of the tropical convex hull of a polyhedral complex in Lemma 2.5.

**Theorem (Theorem 2.1 [2.7])**. If \( a, b \in \mathbb{P}^n \) and \( V \subset \mathbb{P}^2 \), then

(i) \( \text{tconv} (\text{conv}(a, b)) = \text{conv}(\text{tconv}(a, b)) \).

(ii) \( \text{tconv} (\text{pos}(a)) = \text{pos}(\text{tconv}(0, a)) \).

(iii) \( \text{tconv} (\text{conv} V) = \text{conv}(\text{tconv} V) \).
A classical result in algebraic geometry (see for example [EH87]) bounds the degree of a projective variety \( X \) from below by
\[
\deg X \geq \dim(\text{span} \, X) - \dim X + 1.
\]

In Section 3 we study this inequality for tropical curves \( \Gamma \). We can substitute \( \text{span} \, X \) either with the tropical convex hull of \( \Gamma \) or with a tropical linear space containing \( \Gamma \) with smallest dimension. The latter may not be unique and it is not easy to determine. Thus, we choose to replace \( \text{span} \, X \) with \( t\text{conv} \, \Gamma \).

The tropical analogue of (2) we consider is
\[
\deg \Gamma \geq \dim(t\text{conv} \, \Gamma).
\]

If \( \Gamma \) is realizable, then this follows immediately from the classical inequality (2). In Section 3 we give a proof of (3) that relies entirely on tropical techniques. We restrict our attention to a class of tropical curves, not necessarily realizable, and provide insight on how this result may be extended to any tropical curve.

The structure of this paper is as follows. In Section 1 we recall basic definitions of tropical convexity and extend a few classical results. In particular, we prove that convexity and polyhedrality are preserved by taking the tropical convex hull in Corollary 1.5. Section 2 contains our main results, including the proof of Theorem 2.1, Corollary 2.4, and Theorem 2.7. Lastly, in Propositions 3.1 and 3.3 of Section 3 we give a tropical proof of (3) for a special class of tropical curves using results from Sections 1 and 2.

1. Tropical convex hulls of sets

Key definitions and results from tropical convexity are presented in the first part of this section. A description of the tropical convex hull of an arbitrary set is presented in Theorem 1.2. In Corollaries 1.3 and 1.5, we prove the tropical convex hull of a polyhedron is a polyhedron and the same holds for polytopes, cones, and polyhedral complexes.

The tropical convex hull of \( U \subset \mathbb{R}^{n+1} \) is the smallest tropically convex set that contains \( U \). Equivalently [GK07], this is defined by
\[
t\text{conv} \, U = \bigcup_{V \subset U : \|V\| < \infty} t\text{conv} \, V.
\]

For any set \( U \subset \mathbb{R}^{n+1} \) we have \( t\text{conv} \, U + \mathbb{R} \mathbf{1} = t\text{conv} \, U \). This implies \( t\text{conv} \, U = t\text{conv} \, U' \) where \( U' := \{(0, u_1 - u_0, \ldots, u_n - u_0) \mid (u_0, u_1, \ldots, u_n) \in U\} \). Hence, given a set \( V \subset \mathbb{P}^n \) we consider its tropical convex hull to be the image in \( \mathbb{P}^n \) of the tropical convex hull of \( \{(0, v_1, \ldots, v_n) \in \mathbb{R}^{n+1} : v + \mathbb{R} \mathbf{1} \subset V\} \). Additionally, for \( \text{conv}(t\text{conv} \, V) \), we first identify \( t\text{conv} \, V \) with its image under the projection from (1) and then work with its convex hull in \( \mathbb{R}^n \).

Notation 1.1. Let \( [n] = \{1, 2, \ldots, n\} \) and \([n]_0 = \{0\} \cup [n]\). The unit vectors in \( \mathbb{P}^n \) are denoted \( e_0, e_1, \ldots, e_n \) where \( (e_i)_j = 1 \) if \( j = i \) and \( (e_i)_j = 0 \) otherwise.

If \( V = \{v_1, \ldots, v_k\} \) is a finite set, then by [DS04, Proposition 4] its tropical convex hull is given by
\[
t\text{conv} \, V = \{a_1 \odot v_1 + \cdots + a_k \odot v_k : a_i \in \mathbb{R}\}.
\]
Furthermore, points in \( t\text{conv} \, V \) can be characterized by types as defined in [DS04]. Given a point \( x \in \mathbb{P}^n \), the type of \( x \) relative to \( V \), or covector as in [FR15, LS19], is the \((n+1)\)-tuple \( T_x = (T_0, T_1, \ldots, T_n) \) such that \( T_j \subseteq [k] \) for all \( j \), and \( i \in T_j \) if \( \min(v_i - x) \) is obtained in the
The polyhedron \( S_j \) is the closure of one of the \( n+1 \) connected components of \( \mathbb{P}^n \setminus L_{n-1} \). Here \( L_{n-1} \) denotes the max-standard tropical hyperplane whose cones are \( \text{pos}(e_i, \ldots, e_n) \) with lineality space \( \mathbb{R} \). Equivalently, it is the cone in \( \mathbb{R}^{n+1} \) spanned by \( \{-e_0, \ldots, -e_n\} \setminus \{-e_j\} \) with lineality space \( \mathbb{R} \). The proof of the Tropical Farkas Lemma \cite{DS04} states that \( x \in \text{tconv} V \) if and only if the \( j \)th entry of \( T_x \) is nonempty for all \( j \). That is, each sector \( x + S_j \) contains at least one \( v_i \). \cite{Jos05} Proposition 2.9.

Theorem 1.2 has been shown for a finite set \( U \) in different settings. In \cite{JL16} Lemma 27 the statement is shown for \( U \subset \mathbb{P}^{n+1} \) and in \cite{LS19} Proposition 7.3 it is done for \( U \subset (\mathbb{R} \cup \{\infty\})^{n+1} \). We give an explicit proof for any \( U \subset \mathbb{P}^n \) not necessarily finite.

**Theorem 1.2.** \cite{JL16} \cite{LS19} If \( U \subset \mathbb{P}^n \), then the tropical convex hull of \( U \) is equal to the intersection of the Minkowski sums of \( U \) with each of the sectors. That is, \( \text{tconv} U = \bigcap_{i=0}^n (U + S_i) \).

**Proof.** If \( x \in \text{tconv} U \), then \cite{JL16} Lemma 28 implies that \( x \in \text{tconv} V \) for some finite set \( V \subset U \). By \cite{JL16} Lemma 28 we obtain \( x \in \bigcap_{i=0}^n (V + S_i) \), hence \( x \in \bigcap_{i=0}^n (U + S_i) \). On the other hand, if \( x \in \bigcap_{i=0}^n (U + S_i) \), then there exist \( u_1, \ldots, u_n \in U \) such that \( x \in u_i + S_i \) for every \( i \). For \( V = \{u_1, \ldots, u_n\} \) it follows that \( x \in \bigcap_{i=0}^n (V + S_i) = \text{tconv} V \subset \text{tconv} U \). \( \square \)

Figure 1 gives an example of Theorem 1.2 in \( \mathbb{P}^2 \). To draw figures in \( \mathbb{P}^2 \) we use the isomorphism given by \cite{JL16} and draw the image in \( \mathbb{R}^2 \). Corollary 1.3 is a direct consequence of Theorem 1.2. It can also be proven using the definition of tropical convex hull. Corollary 1.4 shows that repeatedly taking the convex hull and tropical convex hull of a set stabilizes after one step.

**Corollary 1.3.** If \( P \subset \mathbb{P}^n \) is convex, then \( \text{tconv} P \) is convex.

**Corollary 1.4.** If \( V \subset \mathbb{P}^n \) is finite, then \( \text{tconv}(\text{conv} V) = \text{tconv}(\text{conv}(\text{tconv} V)) \).

**Proof.** The forward direction is immediate since \( \text{conv} V \subset \text{conv}(\text{tconv} V) \). The containment \( \text{tconv} V \subset \text{tconv}(\text{conv} V) \) and Corollary 1.3 imply \( \text{conv}(\text{tconv} V) \subset \text{tconv}(\text{conv} V) \), so its tropical convex hull is also contained in \( \text{tconv}(\text{conv} V) \). \( \square \)

**Corollary 1.5.** If \( P \subset \mathbb{P}^n \) is a polyhedron (resp. cone, polyhedral complex, fan, polytope), then \( \text{tconv} P \) is a polyhedron (resp. cone, polyhedral complex, fan, polytope).
Proof. If \( P \) is a polyhedron then \( \text{tconv} P \) is a polyhedron since it is the intersection of the finitely many polyhedra \( P + S_i \). If \( P \) is a cone then \( P + S_i \) is a cone for every \( i \) and Theorem 1.2 implies that \( \text{tconv} P \) is also a cone.

Now let \( P \) be a polyhedral complex, so \( P = \bigcup_{j=1}^{N} P_j \) where each \( P_j \) is a polyhedron. By Theorem 1.2 it follows that \( \text{tconv} P = \text{tconv} \left( \bigcup_{j=1}^{N} P_j \right) = \bigcap_{i=0}^{n} \bigcup_{j=1}^{N} (P_j + S_i) \). Observe that by distributing the intersection over the union of Minkowski sums we obtain the union of \( N^{n+1} \) sets. Each set in the union is an intersection of \( n+1 \) Minkowski sums of the form \( (P_{j_0} + S_0) \cap \ldots \cap (P_{j_n} + S_n) \), where \( (j_0, \ldots, j_n) \in \{N\}^{n+1} \), so

\[
\text{tconv} P = \bigcup_{(j_0, \ldots, j_n) \in \{N\}^{n+1}} ((P_{j_0} + S_0) \cap \ldots \cap (P_{j_n} + S_n)).
\]

It follows that \( \text{tconv} P \) is a polyhedral complex since the finite intersection of polyhedra is a polyhedron. In fact, the polyhedral structure may be given by a refinement of the polyhedral complex whose polyhedra are \( \{ (P_{j_0} + S_0) \cap \ldots \cap (P_{j_n} + S_n) \} \) if \( P \) is a fan, the results on polyhedral complexes and cones imply \( \text{tconv} P \) is also a fan.

Lastly, let \( P \) be a polytope. To show \( \text{tconv} P \) is a polytope it suffices to show it is bounded. Suppose \( \text{tconv} P \) is not bounded. Hence it contains a ray \( w + \text{pos}(v) \). Since \( P \) is bounded, Theorem 1.2 implies that \( \text{pos}(v) \) is contained in each sector \( S_i \). This is not possible because the intersection of all sectors is the origin.

\[\square\]

2. LINE SEGMENTS, RAYS AND \( \mathbb{P} \mathbb{T}^2 \)

In this section we describe explicitly the tropical convex hulls of one-dimensional polyhedra in \( \mathbb{P} \mathbb{T}^n \) for any \( n \), and state a result for any set in \( \mathbb{P} \mathbb{T}^2 \).

Let \( a \) and \( b \) be points in \( \mathbb{P} \mathbb{T}^n \). For the remainder of the section we assume that

\[ a = (0, \ldots, 0) \text{ and } 0 = b_0 < b_1 < \cdots < b_n. \]

In this case, using [DS04, Proposition 4], the tropical line segment \( \text{tconv}(a, b) \) is a concatenation of line segments with \( n+1 \) pseudovertices in \( \mathbb{P} \mathbb{T}^n \) given by \( p_0 = a \) and

\[ p_j = (b_0, b_1, \ldots, b_{j-1}, b_j, \ldots, b_j) \text{ for } j \in [n]. \]

The following theorem shows that \( \text{conv}(\text{tconv}(a, b)) = \text{tconv}(\text{conv}(a, b)) \) meaning the tropical convex hull and convex hull commute for two points in \( \mathbb{P} \mathbb{T}^n \) for any \( n \). In fact, this result holds for any two points in \( \mathbb{P} \mathbb{T}^n \). If \( a \) and \( b \) do not satisfy (5), we can apply first a linear transformation which translates \( a \) to the origin and then one that relabels the coordinates so that \( 0 = b_0 \leq b_1 \leq \ldots \leq b_n \). If \( b_i = b_j \) for some \( i \neq j \), then the pseudovertices of \( \text{tconv}(a, b) \) lie in the tropically convex hyperplane \( x_i = x_j \) and the same holds for \( \text{conv}(\text{tconv}(a, b)) \) [DS04 Theorem 2]. Thus \( \text{tconv}(\text{conv}(a, b)) \) and \( \text{conv}(\text{tconv}(a, b)) \) lie in the hyperplane \( x_i = x_j \) which is isomorphic to \( \mathbb{P} \mathbb{T}^{n-1} \). We can repeat this process until the appropriate projection of \( b \) has distinct coordinates.

**Theorem 2.1.** If \( a, b \) are points in \( \mathbb{P} \mathbb{T}^n \), then

\[
(i) \, \text{conv}(\text{tconv}(a, b)) = \text{tconv}(\text{conv}(a, b)); \quad (ii) \, \text{tconv}(\text{pos}(a)) = \text{pos}(\text{tconv}(0, a)).
\]

Corollary 1.3 implies the forward containment of Theorem 2.1(i). For the converse, we use an explicit description of \( \text{conv}(\text{tconv}(a, b)) \) given in the following lemma.
Lemma 2.2. If \( a, b \in \mathbb{P}T^n \) satisfy \( a = (0, \ldots, 0) \) and \( 0 = b_0 < b_1 < \cdots < b_n \), then \( \text{conv}(\text{tconv}(a, b)) \) is a full-dimensional simplex whose \( \mathcal{H} \)-representation is given by

\[
\begin{align*}
 b_1 - x_1 & \geq 0 \\
-(b_{j+1} - b_j)x_{j-1} + (b_{j+1} - b_{j-1})x_j - (b_j - b_{j-1})x_{j+1} & \geq 0 \quad \text{for } j \in [n-1]. \\
-x_{n-1} + x_n & \geq 0
\end{align*}
\]

Proof. Observe that the vertices of \( \text{conv}(\text{tconv}(a, b)) \) are the pseudovertices \( p_0, \ldots, p_n \) of \( \text{tconv}(a, b) \) as described in [6]. These are \( n+1 \) affinely independent points of \( \mathbb{P}T^n \cong \mathbb{R}^n \), since the vectors \( p_1 - a = p_1, \ldots, p_{n-1} - a = p_{n-1}, b - a = b \) are linearly independent. This implies \( \text{conv}(\text{tconv}(a, b)) \) is a simplex. Hence, each of its \( n+1 \) facets is the convex hull of \( n \) vertices. To show that \( \mathcal{H} \) is the \( \mathcal{H} \)-representation of \( \text{conv}(\text{tconv}(a, b)) \) we will show that the corresponding equation to each of the \( n+1 \) inequalities is the hyperplane supporting one of the facets of \( \text{conv}(\text{tconv}(a, b)) \).

Let \( x = (0, x_1, \ldots, x_n) \) be a point in \( \text{conv}(\text{tconv}(a, b)) = \text{conv}(a, p_1, \ldots, p_{n-1}, b) \). The \( j \)th coordinate of \( x \) is given by \( x_j = \lambda_1 b_1 + \cdots + \lambda_{j-1} b_{j-1} + (\lambda_j + \lambda_{j+1} + \cdots + \lambda_n) b_j \) where \( \lambda_1 + \cdots + \lambda_n \leq 1 \) and \( \lambda_i \geq 0 \) for every \( i \). Substituting the coordinates of \( x \) into the first linear form of (7) we obtain \((1 - \lambda_1 - \cdots - \lambda_n) b_1 \). Since \( \lambda_1 + \cdots + \lambda_n \leq 1 \) and \( b_1 \geq 0 \) it follows that \( b_1 - x_1 \geq 0 \). Note that equality occurs if and only if \( x \) is in the facet \( \text{conv}(p_1, \ldots, p_{n-1}, b) \). Thus, \( b_1 - x_1 = 0 \) defines this facet of \( \text{conv}(\text{tconv}(a, b)) \), that is \( \{b_1 - x_1 = 0\} \cap \text{conv}(\text{tconv}(a, b)) = \text{conv}(p_1, \ldots, p_{n-1}, b) \).

After substituting into the second linear form of (7) we have that

\[
-(b_{j+1} - b_j)x_{j-1} + (b_{j+1} - b_{j-1})x_j - (b_j - b_{j-1})x_{j+1} = \lambda_j(b_{j+1} - b_j)(b_j - b_{j+1}).
\]

Since \( \lambda_j \geq 0 \) and \( b_j \geq b_{j-1} \) for each \( j \), we know \( x \) satisfies the second inequality. Here equality occurs if and only if \( x \) is in the facet \( \text{conv}(a, p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_{n-1}, b) \), so \(- (b_{j+1} - b_j)x_{j-1} + (b_{j+1} - b_{j-1})x_j - (b_j - b_{j-1})x_{j+1} = 0 \) defines this facet of \( \text{conv}(\text{tconv}(a, b)) \) for each \( j \in [n-1] \).

Lastly, we have that \(- x_{n-1} + x_n = \lambda_n(b_n - b_{n-1}) \geq 0 \). Equality holds if and only if \( x \) is in the facet \( \text{conv}(a, p_1, \ldots, p_{n-1}) \), and hence this facet is defined by \(- x_{n-1} + x_n = 0 \). \( \square \)

Lemma 2.3. If \( a, b \in \mathbb{P}T^n \) and \( V \) is a finite set in \( \text{conv}(a, b) \), then \( \text{tconv}(V) \subseteq \text{conv}(\text{tconv}(a, b)) \).

Proof. Without loss of generality, assume \( a = (0, \ldots, 0) \) and \( 0 = b_0 < b_1 < \cdots < b_n \). Let \( V = \{t_1 b, t_2 b, \ldots, t_r b\} \subseteq \text{conv}(a, b) \) for some parameters \( t_i \in [0, 1] \). We may assume the parameters \( t_i \) are ordered \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_r \leq 1 \). Take \( x \in \text{tconv} V \) and let \( T_x \) be the type of \( x \) relative to \( V \). By [DS04, Lemma 10], the point \( x \) satisfies

\[
 x_k - x_j \leq t_i(b_k - b_j) \quad \text{for } j, k \in [n]_0 \text{ with } i \in T_j.
\]

We will show that \( x \) satisfies the \( \mathcal{H} \)-representation of \( \text{conv}(\text{tconv}(a, b)) \) given in Lemma 2.2. Since the union of all coordinates \( T_j \) of \( T_x \) covers \([r]\), (8) implies that

\[
0 \leq \frac{x_{j+1} - x_j}{b_{j+1} - b_j} \leq \frac{x_j - x_{j-1}}{b_j - b_{j-1}} \leq 1 \quad \text{for all } j \in [n-1].
\]

For \( j = 1 \), this implies \( \frac{x_1}{b_1} \leq 1 \), so \( b_1 - x_1 \geq 0 \). For \( j \in [n-1] \), rewriting the inequality

\[
\frac{x_{j+1} - x_j}{b_{j+1} - b_j} \leq \frac{x_j - x_{j-1}}{b_j - b_{j-1}}
\]

shows that \(- (b_{j+1} - b_j)x_{j+1} + (b_{j+1} - b_{j-1})x_j - (b_j - b_{j-1})x_{j+1} \geq 0 \).

Lastly, if \( j = n-1 \), then \( 0 \leq \frac{x_n - x_{n-1}}{b_n - b_{n-1}} \), so \(- x_{n-1} + x_n \geq 0 \). \( \square \)
Proof of Theorem 2.7. For part (i), assume without loss of generality that \( a = (0, \ldots, 0) \) and \( 0 = b_0 < \cdots < b_n \). Corollary 1.3 and the containment \( \text{tconv}(a, b) \subset \text{tconv}(\text{conv}(a, b)) \) imply that \( \text{conv}(\text{tconv}(a, b)) \subseteq \text{tconv}(\text{conv}(a, b)) \). Now take \( x \in \text{tconv}(\text{conv}(a, b)) \). Since the tropical convex hull of a set is the union of the tropical convex hulls of all of its subsets, it follows that there is a finite set \( V \subseteq \text{conv}(a, b) \) such that \( x \in \text{tconv}(V) \). Lemma 2.3 implies \( \text{tconv}(V) \subset \text{conv}(\text{tconv}(a, b)) \), so \( x \in \text{conv}(\text{tconv}(a, b)) \).

To show part (ii), take \( x \in \text{tconv}(\text{pos}(a)) \). There exist \( \lambda_0, \ldots, \lambda_n \geq 0 \) such that \( \lambda_j a \in \text{pos}(a) \) for each \( j \in [n]_0 \) and \( x \in \text{tconv}(0, \lambda_0 a, \ldots, \lambda_n a) \). Assume \( \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \), so \( x \in \text{tconv}(\text{conv}(0, \lambda_n a)) \). By Theorem 2.1(i) it follows that \( x \in \text{conv}(\text{tconv}(0, \lambda_n a)) \). Furthermore, this means \( x \in \text{pos}(\text{tconv}(0, \lambda_n a)) \). The pseudovertices of \( \text{tconv}(0, \lambda_n a) \) and \( \text{tconv}(0, a) \) are scalar multiples of one another meaning \( x \in \text{pos}(\text{tconv}(0, a)) \). The other inclusion \( \text{pos}(\text{tconv}(0, a) \subset \text{tconv}(\text{pos}(0, a)) \) follows from Corollary 1.3.

\[
\begin{align*}
\text{Corollary 2.4.} & \text{ If } a \text{ and } b \text{ are points in } \mathbb{P}\mathbb{P}^n, \text{ then } \\
(i) & \dim(\text{tconv}(\text{conv}(a, b))) \text{ is one less than the number of the distinct coordinates of } a - b; \\
(ii) & \dim(\text{tconv}(\text{pos}(a))) \text{ is one less than the number of the distinct coordinates of } a.
\end{align*}
\]

Proof. Part (i) follows from the proof of Lemma 2.2 since \( \text{tconv}(\text{conv}(a, b)) \) is a full-dimensional simplex in \( \mathbb{R}^d \) where \( d \) is one less than the number of distinct coordinates in \( a - b \). For part (ii) observe that the generators of \( \text{pos}(\text{tconv}(0, a)) \) are the pseudovertices of \( \text{tconv}(0, a) \) which are vertices of \( \text{tconv}(\text{conv}(0, a)) \).

As a consequence of Corollary 2.4 we have the following result for tropically convex fans. An application of this lemma appears in Section 3.

\[
\begin{align*}
\text{Lemma 2.5.} & \text{ If } F \text{ is a tropically convex fan in } \mathbb{P}\mathbb{P}^n, \text{ then } \dim F \text{ is equal to one less than the maximum number of distinct coordinates of a point in } F.
\end{align*}
\]

Proof. Let \( d + 1 \) be the maximum number of distinct coordinates of any point in \( F \), and let \( x \) be one such point in \( F \). Since \( F \) is a tropically convex fan it contains \( \text{tconv}(\text{pos}(x)) \). Corollary 2.4 implies that \( \dim(\text{tconv}(\text{pos}(x))) = d \). Hence \( \dim F \geq d \). Suppose that \( \dim F > d \). Let \( C \) be a cone contained in \( F \) such that \( \dim C = \dim F \). By hypothesis, each point in \( C \) has at most \( d + 1 \) distinct coordinates. This implies that \( C \) is contained in the union of finitely many linear spaces in \( \mathbb{P}\mathbb{P}^n \) of dimension at most \( d \). This contradicts the assumption that \( \dim C = \dim F > d \).

Now we consider arbitrary sets in \( \mathbb{P}\mathbb{P}^2 \) and give a generalization of Theorem 2.1. Given any set of points in \( \mathbb{P}\mathbb{P}^2 \), its tropical convex hull and convex hull commute. First we show this result for 3 points using the combinatorial classification of tropical polytopes with 3 vertices in general position in \( \mathbb{P}\mathbb{P}^2 \). As shown in Figure 2, there are 5 distinct combinatorial types [DS04, San05]. Three points in \( \mathbb{P}\mathbb{P}^2 \) are in general position if there is no tropical line containing all of them, and the tropical line with apex at one of the points does not contain any of the other two.

\[
\begin{align*}
\text{Lemma 2.6.} & \text{ If } V = \{v_1, v_2, v_3\} \subset \mathbb{P}\mathbb{P}^2, \text{ then } \text{tconv}(\text{conv}(V)) = \text{conv}(\text{tconv}(V)).
\end{align*}
\]

Proof. Each of the 5 combinatorial types in Figure 2 can be characterized by a system of strict inequalities in the coordinates of the three points. The corresponding inequalities parametrize the cases when the points are not in general position. Using these inequalities it
can be shown directly that the vertex sets of $t\text{conv}(\text{conv} V)$ and $\text{conv}(t\text{conv} V)$ are identical in each of the five cases, hence the two polytopes coincide.

We may assume that all coordinates $v_{ij}$ are nonnegative and set $v_i = (0, v_{i1}, v_{i2})$ for $i \in [3]$. We will show that for the first combinatorial type shown in Figure 2 the two polytopes $t\text{conv}(\text{conv} V)$ and $\text{conv}(t\text{conv} V)$ coincide. The proofs for the other four cases are analogous.

Without loss of generality, assume $v_1 = (0, 0, 0)$ and $v_2$ and $v_3$ are as shown in the first combinatorial type of Figure 2. Suppose further that $v_{22}v_{31} < v_{21}v_{32}$ and let $P = \text{conv} V$. Note that $P + S_0 = v_3 + S_0$ and $P + S_1 = v_1 + S_1$. Applying Theorem 1.2 we have $t\text{conv} P = (v_3 + S_0) \cap (v_1 + S_1) \cap (P + S_2)$. This polytope is described by

$$
\begin{align*}
-x_j &\leq 0, \quad j = 1, 2 \\
x_j - v_{3j} &\leq 0, \quad j = 1, 2 \\
v_{12}x_1 - v_{11}x_2 &\leq 0, \quad i = 2, 3 \\
x_1 + x_2 &\leq 0 \\
(v_{32} - v_{22})(x_1 - v_{21}) - (v_{31} - v_{21})(x_2 + v_{22}) &\leq 0.
\end{align*}
$$

After removing redundant inequalities the irredundant representation of $t\text{conv} P$ is given by

$$
\begin{align*}
x_2 - v_{32} &\leq 0 \\
x_1 + x_2 &\leq 0 \\
(v_{32} - v_{22})(x_1 - v_{21}) - (v_{31} - v_{21})(x_2 + v_{22}) &\leq 0,
\end{align*}
$$

where each inequality defines a facet. Observe that $t\text{conv} V$ consists of the six pseudovertices $v_1, v_2, v_3, p_1 = (0, v_{32} - v_{22} + v_{21}, v_{22}), p_2 = (0, v_{32}, v_{32}),$ and $p_3 = (0, v_{22}, v_{22})$. Since $p_1$ and $p_3$ are contained in $t\text{conv}(v_1, v_3)$ it follows that $\text{conv}(t\text{conv} V) = \text{conv}(v_1, v_2, v_3, p_2)$. To show that the polytope defined by (9) and $\text{conv}(v_1, v_2, v_3, p_2)$ coincide, it suffices to show they have the same vertex sets. The vertices of $t\text{conv} P$ are points where pairs of inequalities in (9) are satisfied at equality. Solving all pairwise systems we obtain the points $v_1, v_2, v_3,$ and $p_2$, all of which are vertices of $\text{conv}(v_1, v_2, v_3, p_2)$. Hence, $t\text{conv}(\text{conv} V) = \text{conv}(t\text{conv} V)$.

Figure 2. Combinatorial types of the tropical convex hull of three points in PT$^2$ [DS04] and a description of their vertices. We assume all vertices are nonnegative.
Similarly, in the case when $v_{22}v_{31} > v_{21}v_{32}$ we apply Theorem 1.2 and eliminate redundancies. This results in the facet-defining inequalities for $t_{\text{conv}} P$ given by

\[
-x_1 + x_2 \leq 0 \\
x_2 - v_{32} \leq 0 \\
v_{32}x_1 - v_{31}x_2 \leq 0.
\]

This polytope has vertices $v_1, v_3,$ and $p_2 = (0, v_{32}, v_{32})$. Note that $p_2$ is the third pseudovertex of $t_{\text{conv}}(v_1, v_3)$. The additional pseudovertices obtained from $t_{\text{conv}}(v_1, v_2)$ and $t_{\text{conv}}(v_2, v_3)$ are also contained in $t_{\text{conv}}(v_1, v_3)$. Moreover, $v_2 \in \text{conv}(v_1, v_3, p_2)$ based on the choice of its coordinates. □

**Theorem 2.7.** If $V \subseteq \mathbb{PT}^2$, then $t_{\text{conv}}(\text{conv} V) = \text{conv}(t_{\text{conv}} V)$.

**Proof.** The containment $t_{\text{conv}} V \subset t_{\text{conv}}(\text{conv} V)$ and Corollary 1.3 imply $\text{conv}(t_{\text{conv}} V) \subset t_{\text{conv}}(\text{conv} V)$. For the forward inclusion, suppose $x \in t_{\text{conv}}(\text{conv} V)$. By (4) and Tropical Charathéodory Theorem [DS04, GK07] we may assume $x \in t_{\text{conv}} W$, for $W \subset \text{conv} V$ containing at most 3 points. Therefore, it suffices to show $t_{\text{conv}}(\text{conv}(v_1, v_2, v_3)) \subset \text{conv}(t_{\text{conv}}(v_1, v_2, v_3))$ for $v_i \in \mathbb{PT}^2$ which follows immediately from Lemma 2.6. □

Theorem 2.7 does not hold when $n \geq 3$. It is not difficult to find examples for which $\text{conv}(t_{\text{conv}} V)$ is not tropically convex.

**Example 2.8.** Let $P \subset \mathbb{PT}^3$ be the triangle in Figure 3 with vertices $v_1 = (0, 0, 0, 0)$, $v_2 = (0, 1, 2, 3)$, and $v_3 = (0, 4, 1, 7)$. The convex hull of $t_{\text{conv}}(v_1, v_2, v_3)$ has 7 vertices and is not tropically convex. In fact, it is possible to find a point $x$ in the classical line segment $v_1v_3$ such that the tropical convex hull of $x$ and the midpoint of the line segment $v_2v_3$ is not contained in $t_{\text{conv}}(v_1, v_2, v_3)$. Using Theorem 1.2 we compute the tropical convex hull of $P$ which is a polytope with 7 vertices strictly containing $\text{conv}(t_{\text{conv}}(v_1, v_2, v_3))$. △

3. **LOWER BOUND ON THE DEGREE OF A TROPICAL CURVE**

In Proposition 3.1 and Lemma 3.3 of this section, we prove

\[
\deg \Gamma \geq \dim (t_{\text{conv}} \Gamma)
\]

for tropical curves $\Gamma$ under specific hypotheses. The proofs rely entirely on tropical and combinatorial techniques and utilize results from Sections 1 and 2.
Let $\Gamma$ be a tropical curve in $\mathbb{P}^n$. This is a weighted balanced rational polyhedral complex of dimension one. The degree of $\Gamma$ is defined to be the multiplicity at the origin of the stable intersection between $\Gamma$ and the standard tropical hyperplane. For realizable curves, this is equal to the degree of any classical curve which tropicalizes to $\Gamma$ [MS15, Corollary 3.6.16].

Let $r_1, \ldots, r_k$ be the rays of a tropical curve $\Gamma$ where $r_i = w_i + \text{pos}(v_i)$ for some $w_i \in \mathbb{P}^n$. Since $\Gamma \subset \mathbb{P}^n$ we can choose each $v_i \in \mathbb{P}^n$ to be the minimal nonnegative integer vector that generates $r_i$. If the multiplicity of the ray $r_i$ in $\Gamma$ is $m_i$, then by [BGS17, Lemma 2.9] we have

$$\deg(\Gamma) = \sum_{i=1}^k m_i v_i.$$ \hspace{1cm} (10)

The first result of this section states that (3) holds for a special class of tropical curves.

**Proposition 3.1.** Let $\Gamma$ be a tropical curve in $\mathbb{P}^n$ with rays $r_1, \ldots, r_k$. If $\dim(t\text{conv} \Gamma) = \max_{i \in \{1, \ldots, k\}} \dim(t\text{conv} r_i)$, then $\deg \Gamma \geq \dim(t\text{conv} \Gamma)$.

**Proof.** Let $\max_{i \in \{1, \ldots, k\}} (\dim(t\text{conv} r_i)) = d$ and $v_1, \ldots, v_k \in \mathbb{P}^n$ be the minimal nonnegative integer vectors such that $r_i = w_i + \text{pos}(v_i) \subset \mathbb{P}^n$ for $i \in \{1, \ldots, k\}$. Then there exists some $j \in \{1, \ldots, k\}$ such that $\dim(t\text{conv} r_j) = d$, so $v_j$ has $d + 1$ distinct entries by Corollary 2.4. Hence the maximum component of $v_j$ is at most $d$. Now (10) implies that $\deg \Gamma \geq d$. \hspace{1cm} \square

The following examples show that Proposition 3.1 does not hold for all tropical curves.

**Example 3.2.** Let $\Gamma$ be the tropical curve in $\mathbb{P}^2$ with rays spanned by $(0, 1, 0), (0, 0, 1), (0, -1, 0),$ and $(0, -1, 0)$ emanating from the origin. Each ray $r \in \Gamma$ is tropically convex so $\max_{r \in \Gamma}(\dim(t\text{conv} r)) = 1$. However, $\dim(t\text{conv} \Gamma) = 2$. In fact, $t\text{conv}(\text{pos}(0, -1, 0), \text{pos}(0, 0, 1))$ is the 2-dimensional cone spanned by $(0, -1, 0)$ and $(0, 0, 1)$. \hspace{1cm} \triangle

Lemma 3.3 shows the result of Proposition 3.1 holds in the case of the Fano curve even though this curve does not satisfy the hypotheses. The cocircuit matrix $M_F$ \hspace{1cm} (11) associated with the Fano matroid has tropical rank 2 and Kapranov rank 3 [MS15, Section 5.3]. By [MS15, Theorem 5.2.23] the tropical convex hull of the columns of $M_F$ has dimension 2. Moreover, over a field of characteristic 0, the dimension of the smallest tropical linear space containing the columns of $M_F$ is at least 3 [MS15, Section 5.3]. This matrix is given by

$$M_F = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}.$$ \hspace{1cm} (11)

Let $\Gamma_F$ be the tropical curve whose rays are spanned by the columns of $M_F$ emanating from the origin. We call $\Gamma_F$ the Fano curve and show that it satisfies inequality (3).

**Lemma 3.3.** If $\Gamma_F$ is the Fano curve, then $\deg \Gamma_F \geq \dim(t\text{conv} \Gamma_F) = 2$.

**Proof.** Let $v_1, v_2, \ldots, v_7 \in \mathbb{P}^6$ denote the columns of $M_F$. Since $t\text{conv}(v_1, \ldots, v_7) \subset t\text{conv} \Gamma_F$ and $\dim(t\text{conv}(v_1, \ldots, v_7)) = 2$, it follows that $\dim(t\text{conv} \Gamma_F) \geq 2$. We will show that
dim(tconv V) ≤ 2 for any finite V ⊂ Γ_F. By Tropical Charathéodory Theorem [DS04, GK07], we only need to prove this for subsets in Γ_F of at most 7 points.

A set of V ⊂ Γ_F of 7 points can be such that each point is on a distinct ray, all 7 points are on the same ray, or the points belong to less than 7 rays. In the first case, suppose each point of V ⊂ Γ_F is on a distinct ray. We use Macaulay2 [GS02] to check that every 4 × 4 submatrix of M_F is tropically singular and has tropical determinant zero. Hence, the tropical rank of M_F is at most 3 and invariant under positive scaling of the columns of M_F, which implies dim(tconv(λ_1 v_1, ..., λ_7 v_7)) ≤ 2 for any λ_i > 0. In the second case, suppose all 7 points are on the same ray. Since each ray is tropically convex, we have that tconv Γ_F contains 7 points from at most 6 rays. For each i ∈ [7] let V_i = {λ_i v_1, ..., λ_i k_i v_i} ⊂ W and λ_{max,i} = max{λ_i 1, ..., λ_i k_i}. Since V_i lies on a tropically convex ray, then V_i ⊆ tconv(0, λ_{max,i} v_i) ⊂ tconv(λ_{max,i} v_1, ..., λ_{max,i} v_7) for each i ∈ [7]. This implies tconv(W) ⊂ tconv(λ_{max,i} v_1, ..., λ_{max,i} v_7). The dimension of the tropical convex hull of any choice of the columns of M_F is at most 2, hence dim(tconv W) ≤ 2.

Next, we show dim(tconv Γ_F) ≤ 2. Suppose that dim(tconv Γ_F) = 3. By Corollary 2.4, tconv Γ_F contains a point ˜p with 4 distinct coordinates. Since Γ_F is a fan, Corollary 1.5 implies that tconv Γ_F contains the ray pos( ˜p). Let p be the minimal nonnegative integer vector such that pos( ˜p) = pos(p) and p has 4 distinct coordinates. We may assume that 0 = p_0 < p_1 < p_2 < p_3. Let a_1 p, a_2 p, a_3 p, and a_4 p be points on pos(p) with 0 < a_1 < a_2 < a_3 < a_4. Let M_p be the matrix with columns a_i p for i ∈ [4] containing the 4 × 4 submatrix

\[
D = \begin{pmatrix}
0 & 0 & 0 & 0 \\
 a_1 p_1 & a_2 p_1 & a_3 p_1 & a_4 p_1 \\
 a_1 p_2 & a_2 p_2 & a_3 p_2 & a_4 p_2 \\
 a_1 p_3 & a_2 p_3 & a_3 p_3 & a_4 p_3
\end{pmatrix}.
\]

The tropical determinant of D is a_1 p_3 + a_2 p_2 + a_3 p_1, and D is tropically nonsingular. Hence, the tropical rank of M_p is at least 4 and dim(tconv(a_1 p, ..., a_4 p)) ≥ 3. Each a_i p ∈ tconv Γ_F can be written as a tropical linear combination of a finite number of points on Γ_F. Hence, tconv(a_1 p, ..., a_4 p) ⊂ tconv W for a finite W ⊂ Γ_F. This is a contradiction because dim(tconv W) ≤ 2 for all finite W ⊂ Γ_F. Thus, dim(tconv Γ_F) = 2 and dim(tconv Γ_F) ≤ deg Γ_F = 3.

We conjecture the proof techniques of Lemma 3.3 can be extended to prove (3) for all tropical curves in P^T^n with rays generated by vectors with entries of only 0 and 1. Proving the inequality in this special case would allow us to better understand how it can be proven for any tropical curve.

Acknowledgements. This material is based upon work supported by NSF-DMS grant #1439786 while the authors were in residence at the Fall 2018 Nonlinear Algebra program at the Institute for Computational and Experimental Research in Mathematics in Providence, RI as well as their time spent at the Summer 2019 Collaborate@ICERM program. Cvetelina Hill was partially supported by NSF-DMS grant #1600569. Sara Lamboglia was supported by the LOEWE research unit Uniformized Structures in Arithmetic and Geometry. Faye Pasley Simon was partially supported by NSF-DMS grant #1620014. The authors are particularly grateful to Josephine Yu for motivating this project, helpful discussions, and
a close reading. The authors would also like to thank Marvin Hahn, Georg Loho, Diane Maclagan, and Ben Smith for useful feedback during the development of the project.

REFERENCES

[AGG10] Xavier Allamigeon, Stéphane Gaubert, and Eric Goubault. The tropical double description method. arXiv preprint arXiv:1001.4119, 2010.

[AGG12] Marianne Akian, Stéphane Gaubert, and Alexander Guterman. Tropical polyhedra are equivalent to mean payoff games. International Journal of Algebra and Computation, 22(01):1250001, 2012.

[BCOQ92] François Baccelli, Guy Cohen, Geert Jan Olsder, and Jean-Pierre Quadrat. Synchronization and linearity: an algebra for discrete event systems. 1992.

[BGS17] Anna Lena Birkmeyer, Andreas Gathmann, and Kirsten Schmitz. The realizability of curves in a tropical plane. Discrete & Computational Geometry, 57(1):12–55, 2017.

[CGQ04] Guy Cohen, Stéphane Gaubert, and Jean-Pierre Quadrat. Tropical polyhedra are equivalent to mean payoff games. International Journal of Algebra and Computation, 22(01):1250001, 2012.

[CGQS05] Guy Cohen, Stéphane Gaubert, Jean-Pierre Quadrat, and Ivan Singer. Max-plus convex sets and functions. Contemporary Mathematics, 377:105–130, 2005.

[CT16] Robert Alexander Crowell and Ngoc Mai Tran. Tropical geometry and mechanism design. arXiv preprint arXiv:1606.0880, 2016.

[DS04] Mike Develin and Bernd Sturmfels. Tropical convexity. Doc. Math, 9(1-27):7–8, 2004.

[DY07] Mike Develin and Josephine Yu. Tropical polytopes and cellular resolutions. Experimental Mathematics, 16(3):277–291, 2007.

[EH87] David Eisenbud and Joe Harris. On varieties of minimal degree. In Proc. Sympos. Pure Math, volume 46, pages 3–13, 1987.

[FR15] Alex Fink and Felipe Rincón. Stiefel tropical linear spaces. J. Combin. Theory Ser. A, 135:291–331, 2015.

[GK07] Stéphane Gaubert and Ricardo D Katz. The minkowski theorem for max-plus convex sets. Linear Algebra and its Applications, 421(2-3):356–369, 2007.

[GK11] Stéphane Gaubert and Ricardo D Katz. Minimal half-spaces and external representation of tropical polyhedra. Journal of Algebraic Combinatorics, 33(3):325–348, 2011.

[GM10] Stéphane Gaubert and Frédéric Meunier. Carathéodory, helly and the others in the max-plus world. Discrete & Computational Geometry, 43(3):648–662, 2010.

[GS02] Daniel R Grayson and Michael E Stillman. Macaulay2, a software system for research in algebraic geometry, 2002.

[GS07] S. Gober and S. N. Sergeeev. Cyclic projections and separability theorems in idempotent semimodules. Fundam. Prikl. Mat., 13(4):31–52, 2007.

[JL16] Michael Joswig and Georg Loho. Weighted digraphs and tropical cones. Linear Algebra and its Applications, 501:304–343, 2016.

[Jos05] Michael Joswig. Tropical halfspaces. Combinatorial and computational geometry, 52:409–431, 2005.

[Jos09] Michael Joswig. Tropical convex hull computations. Contemporary Mathematics, 495:193, 2009.

[LS19] Georg Loho and Ben Smith. Face posets of tropical polyhedra and monomial ideals. arXiv preprint arXiv:1909.01236, 2019.

[MS15] Diane Maclagan and Bernd Sturmfels. Introduction to tropical geometry, volume 161. American Mathematical Soc., 2015.

[RSTU18] Elina Robeva, Bernd Sturmfels, Ngoc Tran, and Caroline Uhler. Maximum likelihood estimation for totally positive log-concave densities. arXiv preprint arXiv:1806.10120, 2018.

[San05] Francisco Santos. The Cayley trick and triangulations of products of simplices. Contemporary Mathematics, 374:151–178, 2005.
Cvetelina Hill, Sara Lamboglia, and Faye Pasley Simon

Georgia Institute of Technology, North Ave NW, Atlanta, GA, USA 30332
E-mail address: cvetelina.hill@gatech.edu

Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 6-8, 60325 Frankfurt a. M., Germany
E-mail address: lamboglia@math.uni-frankfurt.de

North Carolina State University
Greensboro College, 815 West Market Street, Greensboro, NC, USA 27401
E-mail address: faye.simon@greensboro.edu