Feedback-control of quantum systems using continuous state-estimation

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We present a formulation of feedback in quantum systems in which the best estimates of the dynamical variables are obtained continuously from the measurement record, and fed back to control the system. We apply this method to the problem of cooling and confining a single quantum degree of freedom, and compare it to current schemes in which the measurement signal is fed back directly in the manner usually considered in existing treatments of quantum feedback. Direct feedback may be combined with feedback by estimation, and the resulting combination, performed on a linear system, is closely analogous to classical LQG control theory with residual feedback.

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I. INTRODUCTION

The continuous measurement of quantum systems has been a topic of considerable activity in recent years [1–10], and is particularly relevant at this time because experimental technology is now at the point where individual quantum systems can be monitored continuously [11]. With these developments it should be possible in the near future to control quantum systems in real time by using the results of the measurement in the process of continuous feedback. A theory describing the dynamics that results from feeding back the measurement signal (usually a photocurrent) at each instant to control the Hamiltonian of a quantum system has been developed by Wiseman and Milburn [12]. They have shown how to derive the resulting Stochastic Master Equation (SME) for the conditioned evolution, and the corresponding unconditioned master equation, both of which are Markovian in the limit of instantaneous feedback. This kind of feedback has already been used to reduce laser noise below the shot-noise level [13].

However, there are many ways in which the measurement signal may be fed back to affect the system. In general, at a given time, any integral of the measurement record up until that time may be used to alter the system Hamiltonian and affect the dynamics. This leads, however, to a non-Markovian master equation, and the resulting dynamics cannot therefore be easily investigated, and, more importantly, understood. Nevertheless, as we shall examine in this paper, certain integrals of the measurement record provide specific information, such as the best estimates of dynamical variables. These may be fed back to alter the system evolution in a desired way, and while the unconditioned evolution of the system is no longer Markovian, simple equations may be derived for the selective evolution of system variables, and correspondingly simple non-linear (but Markovian) SME’s describe the evolution of the system in the limit of instantaneous feedback. This approach to quantum feedback has close analogies to that used in classical control theory, in particular that control is broken down into a state-estimation step and a feedback step. Because of this, classical results regarding the design of feedback loops can be applied, opening up new possibilities for controlling quantum systems.

In this paper we consider a single quantum degree of freedom, which could be, for example, a trapped atom [14], ion [15] or a moving mirror forming one end of an optical cavity [16] subjected to continuous position observation. Naturally the continuous observation of position actually corresponds to a continuous joint measurement of both position and momentum, because momentum information is implicit in the observed change in the position over time. We show how the best estimate of both the position and momentum at each point in time may be obtained from an integral of the measurement signal when the initial state of the system is Gaussian.

We examine the dynamics which results from using the best estimates of the system variables in a feedback loop, and in particular investigate cooling and confinement using this mechanism. We also apply the Wiseman-Milburn direct feedback theory to investigate the implementation of cooling and confinement by feeding back the measurement signal at each time, a technique which has been considered by Dunningham et al. [17] and Mancini et al. [18], and contrast this with the method involving estimation.

In the next section we review briefly how the SME describing a continuous measurement of position results from real physical measurement schemes. In section IV we review the solution of this master equation, which may be obtained in a simple manner for initially Gaussian states, and give the integrals that are required to measure both position and momentum (ie. to obtain the best estimates of position and momentum at each time). We then examine the dynamical equations which result from using the best estimates for the purposes of feedback, and present the classical control theory which may applied to this quantum feedback process due to an equivalence with classical estimation theory. In section V we consider the problem of cooling and confinement using feedback. We apply both feedback by estimation and direct feedback to this problem. Section V concludes.
II. CONTINUOUS POSITION MEASUREMENTS

A. Two physical position measurement systems

A continuous measurement of the position of a macroscopic object may be obtained by observing continuously the phase of a light beam reflected from it. If we allow the object in question to form one end-mirror in an optical cavity, then in the limit in which one of the cavity mirrors is very lossy (the bad-cavity limit), the phase of the light spends little time in the cavity [19,20]. This is a simple way to treat position measurement by light reflection, and what is more, the position of a single atom may also be monitored continuously in the same manner. To monitor the position of a single atom, the atom is allowed to interact off-resonantly with the optical cavity mode, and this interaction is such that the atom generates a phase shift of the output light in a manner similar to the moving-end mirror. We now examine briefly these two situations, and derive the SME describing the measured systems.

The Hamiltonian describing an optical cavity in which one of the mirrors is free to move along the axis of the cavity is

\[ H = H_m - \hbar g_m a^\dagger a \sin x + H_d \]  

(1)

where \( a \) is the annihilation operator for the cavity mode, \( g_m = \omega_0 / L \) is the coupling constant describing the interaction between the cavity mode and the moving mirror (in which \( \omega_0 \) is the mode frequency, and \( L \) is the length of the cavity), \( H_m \) is the Hamiltonian for the mechanical motion of the mirror, and \( H_d \) describes the coherent driving of the cavity mode. Note that we have moved into the interaction picture with respect to the free Hamiltonian of the cavity mode. In deriving this Hamiltonian it is assumed that the cavity mode follows the motion of the mirror adiabatically, and in particular that the change in the cavity length due to the motion of the mirror is small compared to the cavity length itself [21]. That is, \( |\langle x \rangle| \ll L \). One of the end-mirrors is chosen to be lossy so as to provide output coupling, and the cavity is driven through this mirror. The part of the Hamiltonian which describes coherent driving of the cavity is given by

\[ H_d = i\hbar E(a - a^\dagger), \]  

(2)

where \( E \) is related to the laser power \( P \) by \( E = \sqrt{P/(\hbar \omega_0)} \), and \( \gamma \) is the decay rate of the cavity due to the output coupling mirror [22].

We see that the Hamiltonian describing the interaction between the cavity and the cavity field is of the form \( a^\dagger a ax \). This is exactly what we need in order to obtain a continuous measurement of the system by monitoring the phase of the output light. This is because \( a^\dagger a \) is the generator of a phase shift for the light, and therefore a Hamiltonian of this form produces a phase shift proportional to the position of the mirror, which is exactly what is required.

The Hamiltonian describing the off-resonant interaction between a two-level atom, and an optical cavity in which it is trapped, is

\[ H = H_a - \hbar \frac{g_0^2}{\Delta} a^\dagger a \cos^2(k_0x) + H_d \]  

(3)

where \( k_0 = \omega_0 / c \) is the wavenumber of the cavity mode, \( \Delta \) is the detuning between the cavity mode and the two level atom (\( \Delta = \omega_a - \omega_0 \), where \( \omega_a \) is the frequency of the atomic transition), \( x \) is the atomic position operator, \( g_0 \) is the cavity-QED coupling constant giving the strength of the interaction between the cavity mode and the atom and \( H_a \) is the Hamiltonian for the mechanical motion of the atom. We will assume the atom to be harmonically trapped, which might be achieved by using a second light field [1], or by ion tapping [23].

To obtain a continuous position measurement by monitoring the phase of the output light, we require the interaction of the atomic motion and the cavity mode to be of the same form as that for the mirror. To achieve this we need simply ensure that the atom is trapped in a region small compared to the wavelength of the light, about a region halfway between a node and an anti-node so that we may approximate \( \cos^2(k_0x) = \cos^2(k_0x_0 + k_0x') \approx 1/2 + k_0x' \). Renaming \( x' \) as \( x \) (merely a shift in which the resulting extra term in the Hamiltonian is unimportant), we obtain the correct interaction Hamiltonian.

To realize a position measurement the phase quadrature of the output light must be monitored, and we choose homodyne detection since it provides the simplest treatment. Performing homodyne detection of the phase quadrature with a detector efficiency \( \eta \), the SME describing the evolution of the system conditioned on the continuous measurement record is [33] (see also [34,35,36]),

\[ d\rho_c = -\frac{i}{\hbar}[H, \rho_c]dt + \gamma D[a]\rho_c dt + \sqrt{\eta \gamma} \mathcal{H}[-ia]\rho_c dW, \]  

(4)

where \( \rho_c \) is the system density matrix conditioned on the measurement record, \( dW \) is the Wiener increment, satisfying the Ito calculus relation \( (dW)^2 = dt \), and the superoperators \( \mathcal{D} \) and \( \mathcal{H} \) are given by

\[ 2\mathcal{D}[c]\rho_c = 2c\rho_c e^c - e^c \rho_c e^c - \rho_c e^c e^c, \]  

(5)

\[ \mathcal{H}[c]\rho_c = c\rho_c + \rho_c e^c - \text{Tr}[c\rho_c + \rho_c e^c]\rho_c, \]  

(6)

for an arbitrary operator \( c \).

B. Adiabatic elimination of cavity dynamics

We are interested only in the dynamics of the atom (or equivalently the mirror), and we are also interested purely in the bad-cavity limit (large \( \gamma \)) which corresponds to good position measurement. In this limit, due directly
to the high cavity damping rate, the cavity mode is slaved to the atom dynamics, and can therefore be adiabatically eliminated to obtain a SME purely for the atom. To do this we proceed by following essentially the treatment in reference [1].

Noting first that in the absence of the interaction with the atom, the steady state of the cavity mode is the coherent state $|\alpha = -2E/\gamma\rangle$, we transform the system using $\rho_c' = D(-\alpha)\rho_c D(-\alpha)$

where $D(\alpha)$ is the displacement operator, such that $D(\alpha)|0\rangle = |\alpha\rangle$ [22]. In this ‘displacement picture’, the steady-state of the cavity is now close to the vacuum, with the SME being

$$d\rho'_c = -\frac{i}{\hbar}[H_m - \hbar g(a^\dagger a + \alpha(a + a^\dagger) + |\alpha|^2)x, \rho'_c]dt + \gamma D[a]\rho_c dt + \sqrt{\gamma}(|a\rangle\langle a| + |a^\dagger\rangle\langle a^\dagger|)\rho_c dW. \tag{8}$$

The regime required for adiabatic elimination is

$$\left|\frac{[H_m]}{\gamma}\right| \sim g(|\alpha|^2 + 1)|x\rangle \gamma = \epsilon \ll 1, \tag{9}$$

where $\epsilon$ will be our small parameter governing the approximation. Here $g = g_m$ for the case of the mirror, or $g = k_0g_0^2/\Delta$ for the atom. Relation [1] also implies that $g(|\alpha|^2)/\gamma < \epsilon$, so this quantity also serves as the small parameter. To proceed we assume that the elements of the cavity mode density matrix in the number basis, $\rho^{nm}_c$, scale with the small parameter $\epsilon$ as $\rho^{nm}_c \propto \epsilon^{(n+m)}$, and we will show that this is consistent with the regime [4]. Under this assumption, the state of the cavity+(atom/mirror) may then be expanded up to second order in $\epsilon$ as

$$\rho_c = \rho^{00}_c|0\rangle\langle 0| + (\rho^{10}_c|1\rangle\langle 0| + \text{H.c.}) + \rho^{11}_c|1\rangle\langle 1| + (\rho^{20}_c|2\rangle\langle 0| + \text{H.c.}) + O(\epsilon^3), \tag{10}$$

so that

$$\rho_n = \text{Tr}_c[\rho'_c] = \rho^{00}_c + \rho^{11}_c + O(\epsilon^3). \tag{11}$$

where $\text{Tr}_c$ represents a trace over the cavity mode.

The equations of motion for the density matrix elements $\rho^{nm}_c$ may then be obtained from the master equation for $\rho'_c$, giving

$$d\rho^{00}_c = \mathcal{L}^m_0\rho^{00}_c dt + ig\alpha(x\rho^{10}_c - \rho^{10}_c x)dt + \gamma\rho^{11}_c dt + i\sqrt{\gamma}\text{Tr}[\rho^{00}_c - \rho^{10}_c \rho^{10}_c]dW + O(\epsilon^3), \tag{12}$$

$$d\rho^{10}_c = \mathcal{L}^m_0\rho^{10}_c dt - \frac{\gamma}{2}\rho^{10}_c dt + ig(x\rho^{00}_c + \rho^{10}_c + \sqrt{2}\alpha\rho^{20}_c - \alpha\rho^{11}_c x)dt - i\sqrt{\gamma}(\sqrt{2}\rho^{20}_c - \rho^{11}_c) - \text{Tr}[\rho^{10}_c - \rho^{10}_c \rho^{10}_c]dW, + O(\epsilon^3), \tag{13}$$

$$d\rho^{11}_c = \mathcal{L}^m_0\rho^{11}_c dt + i\alpha\rho^{20}_c - \rho^{11}_c x)dt - \gamma\rho^{11}_c dt + i\sqrt{\gamma}\text{Tr}[\rho^{10}_c - \rho^{10}_c \rho^{10}_c]dW + O(\epsilon^3), \tag{14}$$

$$d\rho^{20}_c = \mathcal{L}^m_0\rho^{10}_c dt - \gamma\rho^{20}_c dt + igx(2\rho^{20}_c + \sqrt{2}\alpha\rho^{10}_c)dt + i\sqrt{\gamma}\text{Tr}[\rho^{10}_c - \rho^{10}_c \rho^{10}_c]dW + O(\epsilon^3), \tag{15}$$

where we have used

$$\text{Tr}[-i(a - a^\dagger)\rho'_c] = \text{Tr}[-i(\rho^{11}_c - \rho^{11}_c)] + O(\epsilon^3). \tag{16}$$

and defined

$$\mathcal{L}^m_0\rho^{nm}_c dt \equiv -\frac{i}{\hbar}[H_m - \hbar g(|\alpha|^2 + 1)x, \rho^{nm}_c] dt. \tag{17}$$

In order to write a SME for the motion of the atom we need to find a closed form equation for $\rho_n \approx \rho^{00}_n + \rho^{11}_n$, but the differential equations for the diagonal elements of the cavity mode density operator involve the off-diagonal elements. The adiabatic elimination exploits the difference in time-scales between the cavity and the motional dynamics by assuming that the heavily damped off-diagonal elements have reached steady state values determined by the motional state. This will allow the off-diagonal elements to be written in terms of the diagonal elements which will result in the desired SME. This is a little more complicated than the usual adiabatic elimination, because the off-diagonal elements are not merely strongly damped, but also contain a stochastic driving term.

To adiabatically eliminate $\rho^{20}_c$, we drop terms proportional to $\rho^{20}_c$ which are insignificant compared to the damping term, and obtain to leading order

$$d\rho^{20}_c = -\gamma\rho^{20}_c dt + i\sqrt{2}g\alpha\rho^{10}_c dt + i\sqrt{\gamma}\text{Tr}[\rho^{10}_c - \rho^{10}_c \rho^{10}_c]dW. \tag{18}$$

For the purposes of showing that the contribution from the stochastic driving is insignificant (ie. that it is not of leading order in $\epsilon$), we assume now that $\rho^{10}_c$ is constant. Since $\rho^{10}_c$ is actually stochastically driven as well, this is not exactly correct; in the steady state both off-diagonal elements will be fluctuating. However, the resulting analysis demonstrates that these fluctuations are higher order in $\epsilon$, so that setting $\rho^{10}_c$ to its mean value is self-consistent. With this assumption Eq. (18) is just linear multiplicative white noise [24] with the stead-state solution

$$\langle \rho^{20}_c \rangle = \frac{i\sqrt{\gamma}}{\hbar} \left( \frac{g\alpha}{\gamma} \right) \rho^{10}_c, \tag{19}$$

$$\sigma^2_20 = \frac{\sqrt{\gamma}}{\hbar} \left| \text{Tr}[\rho^{10}_c - \rho^{10}_c \rho^{10}_c] \right|, \tag{20}$$

where $\sigma^2_20$ is the standard deviation of $\rho_c$ in the steady-state. Since $\rho^{10}_c$ is first order in $\epsilon$, the fluctuations about the steady-state are of third order in $\epsilon$, while the average value is of second order. We can therefore ignore the fluctuations to leading order in $\epsilon$, and write the result of the adiabatic elimination as...
\[ \rho_{20} = i \sqrt{2} \left( \frac{g\alpha}{\gamma} \right) x \rho_{10}^{a} + O(\epsilon^{2}). \]  

(21)

Proceeding in the same way to adiabatically eliminate \( \rho_{10}^{c} \), we find again that the fluctuations may be ignored to leading order, and obtain,

\[ \rho_{10}^{a} = 2i \left( \frac{g\alpha}{\gamma} \right) [x \rho_{00}^{a} - \rho_{11}^{a} x] + O(\epsilon^{2}), \]  

(22)

Note that we have retained the term proportional to \( \rho_{11}^{a} \), which is only third order in \( \epsilon \). However, retaining this term is useful as this allows us to most easily recover the SME for the mirror. This result (Eq. (22)) confirms that \( \rho_{10}^{a} \) is indeed first order in \( \epsilon \). The fact that \( \rho_{20}^{a} \) is second order then follows immediately from Eq. (21), and that \( \rho_{11}^{a} \) is second order follows from the fact that \( \rho_{20}^{a} \) is first order. Thus the assertion made above regarding the scaling of the elements of \( \rho_{c} \) is seen to be consistent with the regime (3).

To obtain the stochastic master equation we now substitute Eqs. (21) and (22) into Eqs. (13) and (14) and combine them to obtain an equation for \( \rho_{a} \approx \rho_{00}^{a} + \rho_{11}^{a} \). To leading order in \( \epsilon \) the resulting stochastic master equation for the atom is

\[ d\rho_{a} = -\frac{i}{\hbar}[H_{m} - \hbar g|\alpha|^{2} x, \rho_{a}]dt + 2kD[x]|\rho_{a}dt + \sqrt{2\eta kH}[x]|\rho_{a}dW, \]  

(23)

This is the expected form for a process describing the continuous measurement of position \( \beta \). Note, however, that an extra term proportional to \( x \) appears in the effective Hamiltonian. This is just the radiation pressure force on the mirror or the dipole force on the atom. As this is simply a classical linear force on the mirror, it may be cancelled by applying an equal and opposite linear potential along with the trapping potential, and we will assume that this is the case in all further analysis. For the case of a measurement on an atom, \( k = 2k^{2}g_{0}^{a}|\alpha|^{2}/(\Delta \gamma) \), while for the moving mirror \( k = 2g_{2m}^{a}|\alpha|^{2}/\gamma \). This quantity may be referred to as the measurement constant, as it describes the rate at which information is obtained about the atomic position, and the corresponding rate at which noise is fed into the atomic momentum as a result of the measurement. Note that for unit efficiency detection and a pure initial state, the stochastic master equation is equivalent to a stochastic Schrödinger equation for the state vector \( \beta \).

The measurement signal is the photocurrent from the homodyne detection, being given by

\[ d\tilde{Q} = \beta[2\eta \sqrt{2}\gamma k(x)dt + \sqrt{\gamma\eta}dW], \]  

(24)

where \( \beta \) is determined by the strength of the local oscillator and the reflectivity of the beam splitter in the homodyne detection setup \( \beta \). Using Eqs. (13) and (22) in Eq. (24), we may write this as

\[ d\tilde{Q} = \beta[2\eta \sqrt{2}\gamma k(x)dt + \sqrt{\gamma\eta}dW] \]  

and defining a scaled measurement signal by \( dQ = d\tilde{Q}/(2\hbar k) \), we may write

\[ dQ = 4\eta k\langle x \rangle dt + \sqrt{2\eta k}dW. \]  

(25)

The scaled photocurrent, \( I(t) = dQ/dt \), may then be written as

\[ I(t) = 4\eta k\langle x \rangle + \sqrt{2\eta k}\varepsilon(t), \]  

(26)

where \( \varepsilon(t) \) is the delta correlated noise source corresponding to \( dW \).

III. ESTIMATION AND FEEDBACK

We come now to the central part of this paper, the question of how to employ a continuous measurement in the control of a quantum system. As we noted in the introduction an arbitrary integral over the whole photocurrent could be used to modulate an arbitrary feedback Hamiltonian. This large number of degrees of freedom makes it difficult to motivate any particular feedback scheme in real systems. Perhaps as a result both theoretical and experimental efforts in quantum feedback have focussed on feeding back the photocurrent at each moment in time. In this paper we refer to this as direct feedback. However, classical control theory \( \beta \) faces exactly the same problems and in this paper we propose a family of feedback algorithms which is adapted from strategies employed in analogous classical systems. Classical strategies often break the search for useful control down into an estimation step and a control step. The powerful technique of dynamic programming is able to find optimal algorithms for both tasks in sufficiently simple systems \( \beta \). In general we propose that the estimation step for a quantum mechanical system will involve the solution in real time of an appropriate SME which accounts for realistic levels of measurement and process noise, using an initially mixed state reflecting lack of knowledge of the system. The state estimate which results from the SME can then be used to modify the system Hamiltonian in order to achieve the desired control of the system.

In the previous section we reviewed how the stochastic master equation (Eq. (23)) for the continuous position measurement of a single quantum degree of freedom may be derived from a real measurement process. Fortunately, if the Hamiltonian for the mechanical dynamics is no more than quadratic in the position and momentum, and the initial state of the system is Gaussian, the SME may be solved analytically, since it remains Gaussian at all times \( \beta \). Taking the initial state to be Gaussian is also sensible, because there is reason to believe that non-classical states evolve rapidly to Gaussian due
to environmental interactions, of which the measurement process is one example \cite{28,29}. A quantum mechanical Gaussian state is uniquely determined by its mean values and covariance matrix (see for example \cite{30}), just as is the case for classical probability distributions and so we only need to find equations for these variables in order to fully describe the evolution of the conditioned state. In what follows, we will refer to the elements of the covariance matrix, being the position and momentum variance and their joint covariance, simply as the covariances. The expectation values of operators are found from

$$d\langle c \rangle = \text{Tr}(cd\rho)$$

(28)
as for any master equation, see for example \cite{12}. The Itô rules for stochastic differential equations \cite{24} result in equations for the covariances. Performing this calculation gives the equations for the means as \cite{13,14,15,16}

$$d\langle x \rangle = -\frac{i}{\hbar} \langle [x, H_m] \rangle dt + 2\sqrt{2}\eta k V_x dW,$$

$$d\langle p \rangle = -\frac{i}{\hbar} \langle [p, H_m] \rangle dt + 2\sqrt{2}\eta k C dW,$$

(29)
and the equations for the covariances are

$$\dot{V}_x = -\frac{i}{\hbar} \langle [x^2, H_m] \rangle + \frac{2i}{\hbar} \langle x \rangle \langle [x, H_m] \rangle - 8\eta k V_x^2,$$

$$\dot{V}_p = -\frac{i}{\hbar} \langle [p^2, H_m] \rangle + \frac{2i}{\hbar} \langle p \rangle \langle [p, H_m] \rangle + 2\hbar^2 - 8\eta k C^2,$$

$$\dot{C} = -\frac{i}{2\hbar} \langle [xp + px, H_m] \rangle - 8\eta k V_x C$$

$$+ \frac{i}{\hbar} \langle [p, H_m] \rangle + \frac{i}{\hbar} \langle p \rangle \langle [x, H_m] \rangle.$$

(30)
In these equations, $V_x$ and $V_p$ are the variances in position and momentum respectively, and

$$C = \frac{1}{2} \langle xp + px \rangle - \langle x \rangle \langle p \rangle$$

(31)
is the symmetrized covariance. The Gaussian assumption is required to obtain Eqs. (\ref{30}) but not Eqs. (\ref{29}). These two systems of equations are precisely equivalent to the SME \cite{23} under the assumption that the initial state is Gaussian.

First of all it should be noted that, while it is not explicit, the equations for the covariances are closed, in that they do not depend on the means, and, in addition, they do not depend upon the measurement signal. As a consequence the covariances at any point in time depend only upon the duration of the measurement, and not the specific measurement record. These equations are instances of coupled Riccati equations, and may be solved analytically \cite{14}. The full solutions are fairly cumbersome, and we do not need to give them here. Once these equations have been solved, the solutions may be substituted into the equations for the means, and these are readily solved since they are merely linear equations with (stochastic) driving. Writing them in the form

$$d\langle x \rangle = A\langle x \rangle dt + 2\sqrt{2}\eta k dY(t)$$

(32)
where $\langle x \rangle = \langle (x, p) \rangle^T$ and $dY(t) = (V_x, C)^T dW$, the solution is naturally just

$$\langle x \rangle(t) = e^{At}\langle x \rangle(0) + 2\sqrt{2}\eta ke^{At} \int_0^t e^{-At'} dY(t').$$

(33)
During the measurement process two things happen. The first is that the mean position and momentum obey not only the evolution dictated by the Hamiltonian, but also suffer continual random kicks due to the measurement process. This is because, at each time step, the effect of the measurement process is to perform a ‘weak’ measurement of position, and since the result of the measurement is necessarily random, the position of the state in phase space changes in a random fashion \cite{9}. The second effect of the measurement process, and the part that is governed by the deterministic equations for the covariances, is to narrow the width of the state in phase space. For ideal (unit efficiency) detection, an initial mixture is reduced, over time, to a completely pure state. At such a time there is, in that sense, no uncertainty, as the quantum state is completely determined, and remains so. For inefficient detection, the degree to which the state is mixed is also reduced during the measurement, but, in general, to a non-zero level determined by the detection efficiency \cite{14}.

Now let us consider how we might control the system by the process of feedback. In classical control theory one attempts to obtain, at each point in time, the best possible estimate of the state of the system, and then uses the resulting estimates for the system variables in a feedback loop to control the dynamics. Now, in the quantum example we are considering here, since the distributions for all the variables are always Gaussian, the mean position and momentum are also our best estimates of these variables. In fact, it would be quite reasonable to define a continuous measurement of a system variable as a process by which we obtain an estimate of that variable at each point in time. Hence, by this definition, what we need to do to achieve a continuous measurement of a system variable, is to write down that integral of the measurement signal which gives us continuously our best estimate of that variable. The equations for the means are written above in terms of the Wiener process, rather than the actual measurement signal $dQ$. Rewriting them in terms of the measurement signal we have

$$d\langle x \rangle = -\frac{i}{\hbar} \langle [x, H_m] \rangle dt - 8\eta k \langle x \rangle V_x dQ + 2V_x dQ,$$

$$d\langle p \rangle = -\frac{i}{\hbar} \langle [p, H_m] \rangle dt - 8\eta k \langle x \rangle C dt + 2CdQ,$$

(34)
so that the continuous position measurement does indeed provide us with a continuous measurement of both position and momentum, in the sense introduced above. The strategy we have outlined here of only employing
the mean values and not the variances of the conditioned state in the control law turns out to be optimal in some classical systems which are termed separable.

The above equations require that we put in a priori estimates of the state as the initial conditions. While we expect the initial state to be highly mixed (so that in that sense we initially have very little knowledge regarding the state), it is assumed that one can obtain sensible estimates for the initial means and covariances. This assumption is certainly reasonable, for one can almost always obtain good estimates of the initial density matrix from a knowledge of the way in which the system is prepared. This is just as true in classical estimation theory. In that case the initial values of the variables will be, in general, poorly known, but good estimates for the initial probability distribution for the variables (being analogous to the density matrix in the quantum case) may be obtained from a knowledge of the initial preparation.

A further point to note is that after a sufficient time the best estimates of the variables actually do not depend upon the initial estimates, but only upon the measurement record. Hence, even an observer with very imprecise knowledge of the initial state will have obtained accurate information after a time, and the resulting feedback, while perhaps initially of no great advantage, will eventually produce the desired effect. This property of the SME is shown in reference [14], where the question of estimation is considered in detail. As a final point regarding the question of estimation and initial states, it is worth noting that one can always wait a sufficient time for the covariances to attain their steady-state values before initiating feedback, thus obviating to a certain extent the need to use initial estimates for the covariances. In classical control theory this is concern about the errors in state estimates is termed caution, and in linear systems with quadratic costs and Gaussian noises caution turns out not to be optimal.

Curiously enough, Eqs. (34) do not admit of an analytic solution in terms of $dQ$ unless the covariances have their steady-state values. This is because the linear equation now has an explicit time dependence due to the fact that the time-dependent covariances multiply the means. However, it is a simple matter to integrate these equations numerically, and a computer could perform the necessary calculations to obtain the best estimates of the variables in real time, and hence track the evolution of the system. This process of estimation is not only interesting because we can monitor the system evolution, but because the estimates may be used in a feedback loop to control the dynamics.

Now that we know how to obtain the best estimates of the system variables, the process of feedback involves continually adjusting the Hamiltonian so that one or more of its terms are proportional to some function of these estimates. In treating this process of feedback we must be careful to ensure that the act of feedback (the act of adjusting the Hamiltonian) happens after the measurement at each time step. This is obviously essential, due to the fact that the measurement must be obtained before any adjustment based on that measurement can take place. However, in the limit of instantaneous feedback this is very simple. First we consider the measurement step, in which the system evolves for a time $dt$, and the measurement signal is incremented by the amount $dQ$. At this point the feedback is allowed to act, and in the limit in which it is instantaneous (that is, much faster than any of the time scales which characterize the system dynamics), the Hamiltonian is updated before the next time step. At the next time step the equations of motion for the estimates (and, in fact, all system variables) have the new Hamiltonian with the new values for the estimates, so that, effectively, the Hamiltonian has the desired value at all times. In the limit of instantaneous feedback, the SME describing the evolution of the system is therefore simply just as it was before, but with the Hamiltonian, $H_m$, replaced with a new Hamiltonian, having specific dependencies on the estimates of $x$ and $p$, which are simply $\text{Tr}[x\rho_n]$ and $\text{Tr}[p\rho_n]$. The SME describing the conditional evolution of the system, for general instantaneous feedback via estimation from a continuous measurement of position, is, therefore given by Eq. (35), where $H_m$ is now a function of the average position and momentum:

$$H_m = f(x, p, \text{Tr}[x\rho_n], \text{Tr}[p\rho_n]).$$

While the SME for the conditional evolution is therefore rather simple, particularly in that it is Markovian, an equation that would describe the overall average (non-selective) evolution would not be. This is because the average evolution at any given time is not simply a function of the average density operator at that time, but depends on the previous history. We have however provided a recipe to calculate the unconditioned state at all times since this only requires averaging over all the trajectories generated by our conditional feedback SME.

We will now show that there is a precise analogy to be made between linear quantum mechanical systems (and in particular those subjected to continuous position measurement) and classical systems which are driven by a certain specified noise process. Once we define a quantum mechanical cost function, this precise analogy will allow us to identify the optimal feedback strategy by using classical LQG theory.

In the estimation step of classical control, a system called a Kalman filter is often used to obtain an estimate of the state of the system from the measurement record. Where the system is linear, and the noise on the system (often referred to as plant or process noise) and the noise on the measurement are both Gaussian, this is the optimal state observer. There is, in fact, a noise-driven classical system with noisy measurement for which the Kalman filter turns out to be precisely Eqs. (23) and Eqs. (30). Consider a classical harmonic oscillator with dynamical variables $x_c$ and $p_c$, obeying the equations [14]

$$\dot{x}_c = p_c/m$$

and
\[ \dot{p}_c = -m\omega^2 x_c + \sqrt{2\eta k} \zeta_1(t), \quad (37) \]

where the noise driving the classical system is delta correlated so that \( \langle \zeta_1(t) \zeta_1(t') \rangle = \delta(t - t') \), and the classical measurement result, being also noisy, is given by

\[ \dot{Q}_c = 4\eta k x_c + \sqrt{2\eta k} \zeta_2(t), \quad (38) \]

where \( \langle \zeta_2(t) \zeta_2(t') \rangle = \delta(t - t') \) and \( \zeta_1 \) and \( \zeta_2 \) are uncorrelated. The equations for the best estimates and their covariances provided by the Kalman filter are then exactly the same as for the quantum system, with the identification

\[ dW = 2\sqrt{2\eta k} (x_c - \langle x_c \rangle) dt + \zeta_2 dt, \quad (39) \]

which is referred to as the innovation or the residual. Hence, when the quantum states are Gaussian (and in that sense classical), the quantum measurement process may be viewed as a classical estimation process in which noise, \( \zeta_1(t) \), is continually fed into the system to maintain the uncertainty relations. It is important to note that the strength of the noise in this analogous classical process is determined by the accuracy of the measurement. Unlike classical systems where higher accuracy always results in better state estimation, high accuracy in the position measurement will result in large momentum variance, and hence large average energy, of the conditioned states.

The existence of an analogous classical system is very nice because it allows us to use results from classical control theory when considering the quantum system. In particular, when the cost function is quadratic in the system variables, a separation theorem applies to the classical system. This states that, given that it is the values of optimal estimates that being fed back, when calculating the feedback required for optimal control we may assume that the dynamical variables are known exactly: there is no advantage in considering the accuracy of our estimate. In this case the restriction of feeding back only best estimates is justified as leading to the optimal strategy. Moreover, another result which applies to the classical system states that the optimal control law will be the one which would be calculated by assuming there is no noise either on the plant or on the measurement. This stronger property is termed certainty equivalence. All that remains to be done to allow us to match the quantum with the classical theory is to consider a class of quantum mechanical feedback Hamiltonians and a quantum mechanical cost function, so as to complete the precise analogy between quantum and classical systems which exists for the Kalman filter and the SME.

We now examine briefly the relevant results from classical control theory. We note that classical optimal control theory has been applied in the past to closed (unmonitored) quantum systems by Rabitz and co-workers. The classical system for which the Kalman filter is equivalent to the SME given by Eqs. may be written as

\[ dx_c = Ax_c + \sqrt{2\eta k} (0, 1)^T \zeta_1(t) dt + Bu, \quad (40) \]

where \( x_c = (x_c, p_c)^T \) are the classical variables. Here we have added feedback variables \( u \), which will be chosen to be some function of the dynamical variables \( x_c \) so as to implement control. Note that ‘optimal’ control is defined as a feedback algorithm which minimizes a cost function, which is usually a function of the system state. The role of the cost function is to define how far the system state is from the desired state during the process of feedback-control. Classical LQG control theory tells us that for linear systems, driven by Gaussian noise, and in which the cost function is quadratic in the system variables (LQG stands for Linear, Quadratic, Gaussian), the optimal feedback is obtained by choosing \( u = -K(x) \). The form of the cost function is chosen to be

\[ I = \int_o^t \langle x_c^T P x_c + u^T Q u \rangle \ dt', \quad (41) \]

and the optimal solution is the one that minimizes the expectation value \( J = E[I] \) of this cost function over the time that the feedback acts. It turns out that \( K = Q^{-1}B^TU \) and although \( U \) is time-dependent, the steady state value is often all that is used in practice, and obeys the equation

\[ 0 = P + A^T U + U A - U B Q^{-1} B^T U. \quad (42) \]

We will use these results in the next section when we consider cooling a single quantum particle. Note that the choice of cost function is crucial in determining the optimal strategy. For example, placing boundaries on the available strength of feedback, rather than using a quadratic cost function typically implies that some form of bang-bang control (a control algorithm in which \( u \) takes one of two values) is optimal. In this case the optimal strategy is therefore not linear in the estimated state.

So long as the feedback Hamiltonian given by Eq. is linear in the position and momentum operators, so that it has the form

\[ H_m = f(\text{Tr}[x \rho_a], \text{Tr}[p \rho_a]) x + g(\text{Tr}[x \rho_a], \text{Tr}[p \rho_a]) p \quad (43) \]

where \( f \) and \( g \) are arbitrary functions, then the dynamic equations for the covariances remain decoupled from the equations for the means, and remain deterministic. If this is not the case, then the equations for the means become coupled to those for the covariances, and the situation becomes more complex. Furthermore, if the feedback Hamiltonian has terms that are higher than quadratic in \( x \) and \( p \) then the correspondence between quantum feedback and LQG control will be lost since it will not preserve the Gaussian property of the states and since the quantum Hamiltonian will affect the state in a way that is distinct from the evolution of a classical probability distribution.
It remains to define a physically motivated quantum mechanical cost function which maps onto the classical system we are considering. Clearly the part of the cost function which refers to the performance of the system should be an expectation value of the unconditioned density operator. It should also be quadratic in the position and momentum, since this ensures that it is straightforward to minimize a typical measure of control such as the average energy, just as it does for a classical system. On the other hand, when assessing the cost of control, it will be sufficient to consider classical quantities since the feedback Hamiltonian will typically be modulated by essentially classical quantities such as electric currents or lasers powers. With these considerations, we can define a sensible quantum mechanical cost function as

$$J_q = \int_0^t \left( \text{Tr} \left( x^T P x \rho \right) + \langle u^T Q u \rangle_c \right) \, dt \tag{44}$$

Where $\langle \cdot \rangle_c$ indicates an average over the classical random variables $u$. As noted above the density matrix average can be performed by first taking expectation values over the conditioned states given by the SME and then averaging over the trajectories. The cost function for a given trajectory is

$$J_q = \int_0^t \left( \langle x^T P x \rangle_c + \text{Tr} \left( PV \right) + \langle u^T Q u \rangle_c \right) \, dt \tag{45}$$

where the angle brackets indicate that we are talking about the mean values and $V$ is the covariance matrix of the conditioned states. The second term in the integral is the expectation value of $P$ for the conditioned state if the mean $x$ is zero. This term is independent of both the mean values and the feedback, so long as we use a linear feedback Hamiltonian as discussed above. It represents a minimum cost due to the finite width of the conditioned states. For a given trajectory, $u$ is not random, since it is a deterministic function of the current, and possibly of the past values of $x$. We can now reexpress the quantum cost by averaging Eq.(45) over the trajectories, which gives

$$J_q = \int_0^t \left( \langle x^T P x \rangle_c + \text{Tr} \left( PV \right) + \langle u^T Q u \rangle_c \right) \, dt. \tag{46}$$

On the other hand, we have identified a classical problem for which $x$ and $V$ obey the Kalman filter equations for the estimate of the noisy classical state $x_c$. Since the Kalman filter is merely sufficient statistics for a posterior probability distribution for $x_c$, we can write the average over $x_c$ in $J$ in terms of this mean and covariance, giving

$$J = \int_0^t \left( \langle x^T P x \rangle_c + \text{Tr} \left( PV \right) + \langle u^T Q u \rangle_c \right) \, dt. \tag{47}$$

With this we see that the classical and quantum cost functions are identical. Thus the quantum mechanically optimal strategy (given that the feedback Hamiltonians are no more than quadratic) for the cost function we have introduced will be the classically optimal strategy for the fictitious classical system, whose Kalman filter equations reproduce the SME, under the analogous classical cost function. Moreover, it is clear that for each SME describing a linear quantum system subjected to a linear measurement, there will be some classical SME model which can be constructed to find the optimal feedback algorithm for a similarly defined quadratic cost.

It is important to note that this is merely the optimal strategy for a given strength of measurement. For example, if the aim is to minimize the energy of an oscillator, overly strong position measurement will result in states of high average momentum and therefore energy. It is, however, straightforward to find the optimal measurement strength, if desired. One simply uses the procedure above to find the optimal strategy for a given measurement strength, $k$, and takes the extra step of optimizing the result over $k$. For the physical position measurements we discuss in section I, changing the measurement strength corresponds to changing the laser power driving the cavity.

Up to this point we have not considered how particular feedback Hamiltonians could be implemented, and so we complete this section with a discussion of this important question. Clearly terms in the feedback Hamiltonian proportional to functions of $x$ are implemented by applying the required force to the system. By the use of estimation the forces can be adjusted so that they are proportional to any particular function of the average momentum and position as indicated above. This allows terms to be added to the dynamical equation for the momentum, but not to those for the position. We will show in the following section that in order to achieve the best results for phase-space localization we must add terms to the dynamical equation for the position, and therefore it is important to be able to implement a term in the feedback Hamiltonian proportional to momentum. This is not so straightforward, but we suggest two possible ways in which it might be achieved. If the exact location of the trap is not an important consideration, then shifts in the position (being strictly equivalent to a linear momentum term in the Hamiltonian), are achieved simply by shifting all the position dependent terms in the Hamiltonian, in particular the trapping potential. This is a shift in the origin of the coordinates, and, being a virtual shift in the position, produces a term in the dynamical equation for the position proportional to the rate at which the trap is being shifted. When the experimental arrangement is such that the distance covered by the particle during the cooling is negligibly small compared to the trapping apparatus this may prove to be a very effective way of implementing a feedback Hamiltonian linear in momentum. A second method would be to apply a large impulse to the particle so that during one feedback time-step the particle is moved the desired distance, and an equal and opposite impulse is then applied to reset the momentum. Naturally the feasibility of this method will also depend.
upon the practicalities of a given experimental arrangement.

IV. COOLING AND CONFINEMENT VIA FEEDBACK

A. Using feedback by estimation

Cooling and localization of individual quantum systems is an important first step in the process of control. This is certainly true for trapped atoms, ions, and cavity mirrors which we used as our examples in section [1]. By *cooling*, we mean localization in momentum space, and by *confinement* we mean localization in position space. When these two processes are combined, then we may speak of phase-space localization. We now apply the formulation of quantum feedback introduced in the previous section to the problem of phase space localization. As indicated in that section, we can use classical control theory to find the optimal feedback. First however, let us examine the steady state solutions for the covariances in the absence of feedback. For a harmonically trapped particle the equations for the covariances become [6, 14]

\[
\begin{align*}
V_x &= (2/m)C - 8knV_x^2 \\
V_p &= -2m\omega^2C - 8knC^2 + 2kh^2 \\
C &= V_p/m - m\omega^2V_x - 8knCV_x \\
\end{align*}
\]

The resulting steady-state covariances are

\[
\begin{align*}
V_x &= \left(\frac{h}{\sqrt{2\eta m\omega}}\right) \frac{1}{\sqrt{\xi + 1}} \\
V_p &= \left(\frac{h\eta m\omega}{\sqrt{2\eta}}\right) \frac{\xi}{\sqrt{\xi + 1}} \\
C &= \left(\frac{h}{2\sqrt{\eta}}\right) \frac{\sqrt{\xi - 1}}{\sqrt{\xi + 1}}. \\
\end{align*}
\]

where

\[
\xi = \sqrt{1 + \frac{4}{\eta k}}, \quad r = \frac{m\omega^2}{2\eta k}
\]

Clearly the final state is in general a mixed Gaussian state, with the exact orientation and squeezing determined by the measurement constant, oscillation frequency, particle mass and detection efficiency. From these covariances the purity of the final state is readily obtained by using [29]

\[
Tr[\rho^2] = (h/2)(V_xV_p - C^2)^{-1/2}.
\]

For perfect detection efficiency ($\eta = 1$), the state is therefore pure, and perfectly determined at each point in time. For imperfect detection the state is not completely pure, and is increasingly mixed as the detection becomes less efficient. Inefficient detection also models environmental noise, which in the case of a cavity mirror would be coupling to a thermal bath, and in the case of a atom would be spontaneous emission [14]. The equations of motion for the conditioned covariances are unchanged by linear feedback, whether direct or using estimation, and Eq. (49) therefore gives the lower limit on the purity of the final cooled state achievable for a given detector efficiency.

Now we know the covariances and resulting purity of the conditioned state, we want to know how well we can localize the mean position and momentum of this state in phase space, by feeding back the estimated values. The stochastic equations for the means are

\[
\begin{align*}
d(x) &= (p/m)dt + 2\sqrt{2\eta k}V_xdW, \\
d(p) &= -m\omega^2(x)dt + 2\sqrt{2\eta k}CdW. \\
\end{align*}
\]

We wish to minimize the distribution of $\langle x \rangle$ and $\langle p \rangle$ about the origin, and so it is sensible to take the cost function to be minimized as in the previous section

\[
J_\eta = \int_0^t \left(\text{Tr}(x^TPx\rho) + q^2(u^TQu)_x\right)dt
\]

where $q$ is a weighting constant which in this case has units of time, and

\[
P = Q = \left(\begin{array}{cc}
m\omega^2 & 0 \\
0 & 1/m \end{array}\right).
\]

With this choice of $P$ and $Q$ the cost function is a weighted sum of the energy of the oscillator and a fictitious energy one could associate with the feedback variable $u$. In general one can choose any quadratic function of the feedback variables, and a particular choice would be made to suit a given situation. Note that the feedback variable $u$ appears in the cost function to reflect the fact that we are not unrestricted in the magnitude of the feedback we bring to bear. If this consideration is relatively unimportant, $q$ is chosen to be small, and so the cost function reduces essentially to the energy of the oscillator, which is certainly the quantity we wish to minimize in the process of phase-space localization.

With this form for the cost function, classical LQG control theory tells us that linear feedback will provide optimal control. Choosing $B = I$ in the feedback equation, so as to allow feedback in the dynamical equations for both variables (the most general case), we need merely solve Eq. (12) for $V$ to find the optimal value of the feedback matrix $K$. Performing this calculation we find that an optimal solution is $K = (1/q)I$. That is, feedback to provide an equal damping rate on both the position and momentum. Note that the smaller we make the weighting constant $q$, the larger the damping rate, being
\[ \Gamma_x = \Gamma_p = \Gamma = 1/q. \]

With this feedback, the dynamical equations for the means become

\[ d\langle x \rangle/dt = -\Gamma_x \langle x \rangle dt + (1/m)\langle p \rangle dt + 2\sqrt{2nk}V_x dW, \]

\[ d\langle p \rangle/dt = -m\omega^2 \langle x \rangle dt - \Gamma_p \langle p \rangle dt + 2\sqrt{2nk}CdW. \]

It is now the mean and variance of the conditioned means which are of interest, as they tell us how well localized the particle is, and about what point in phase-space. We will denote the means of the conditioned means as \( \langle \langle x \rangle \rangle \) and \( \langle \langle p \rangle \rangle \), and the covariances of the means as \( V_x^e, V_p^e \) and \( C^e \), where the ‘e’ refers to the fact that they are excess to the quantum conditional covariances resulting from the measurement process. Clearly the steady state values for the means \( \langle \langle x \rangle \rangle \) and \( \langle \langle p \rangle \rangle \) is the origin of phase space, while the equations for the covariances are

\[ \dot{V}_x^e = \frac{2Q}{r(1 + 4Q^2)} \left[ (1 + 2Q^2) \dot{V}_x^2 + 2Q^2C^2 + 2QC\dot{V}_x \right] \]

\[ \dot{V}_p^e = \frac{2Q}{r(1 + 4Q^2)} \left[ 2Q^2\dot{V}_x^2 + (1 + 2Q^2)C^2 - 2QC\dot{V}_x \right] \]

\[ \dot{C}^e = -\left( \Gamma_x + \Gamma_p \right) \dot{C}^e - \omega(\dot{V}_x^e - \dot{V}_p^e) + \frac{2\omega}{r} \dot{C}\dot{V}_x, \]

where \( Q \equiv \omega/(2\Gamma) \).

The total average covariances resulting from the localization process are simply the sum of the conditional covariances and these excess covariances. The overall resulting purity may then be calculated using Eq. (63), if so desired.

In the previous section we noted that terms in the feedback Hamiltonian proportional to momentum are harder to generate than those proportional to position, and since the optimal feedback we have used above requires both kinds of terms, it is of interest to examine what may be achieved with a position term alone. This imposes the condition that \( K_{11} = K_{12} = 0 \), where \( K_{ij} \) are the elements of the feedback matrix \( K \). To derive the optimal solution under this condition we solve Eq. (42) as before, but this time set

\[ B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

This time taking the small \( q \) limit, \( q\omega = 2Q \ll 1 \), an optimal feedback strategy is given by \( K_{21} = m\omega/q \) and \( K_{22} = 1/q \). In this case the steady state solution for the excess covariances are

\[ \dot{V}_x^e = \frac{1}{r} \left[ V_x^2 + 4Q^2\dot{C}^2 + 4QC\dot{V}_x \right] \]

\[ \dot{V}_p^e = \frac{1}{r} \left[ V_x^2 + 2QC\dot{C}^2 \right] \]

\[ \dot{C}^e = -\frac{1}{r} \dot{V}_x^2, \]

We see, therefore, that using feedback by estimation, it is indeed possible to obtain phase-space localization with only a position dependent term in the feedback Hamiltonian, although this is clearly not as good as using a combination of position and momentum damping.

To summarize our results so far, we see that when using feedback by estimation, and when we average over the conditional evolution, there is an additional uncertainty in the final localized state over that due to measurement inefficiency, and that this excess uncertainty decreases with the magnitude of the feedback. This additional uncertainty is due to the noise which is continually fed into the system as the result of the measurement. The effect of this noise is decreased as the damping constant, \( \Gamma \), is increased. However, there is ultimately a limit upon the magnitude of the feedback, and hence upon \( \Gamma \), and this is reflected in the choice of the weighting constant \( q \) in the cost function. We will see in the next section that direct feedback provides an alternative strategy for dealing with the measurement noise.

### B. Adding direct feedback

The beauty of direct quantum feedback, formulated by Wiseman and Milburn [12], is that it may be used to cancel the noise which drives the mean values of the dynamical variables. This is possible because the noise in the measurement signal is the same noise that drives the system. Feeding back the measurement signal itself (by choosing a feedback Hamiltonian directly proportional to this signal) essentially allows the noise driving the system to drive it twice at each step. If the feedback Hamiltonian is chosen in the right way, then the effect of the noise at the first step may be canceled by that at the second step.
The result is that the steady state of the feedback master equation can have the same variances as the conditioned states since all of the fluctuations in the mean values are overcome. We note that direct feedback would be analogous to the use of residual feedback in classical control theory (in which the innovation is fed back to drive the system) to cancel the noise driving the Kalman filter.

As in the previous section, we choose the feedback Hamiltonian to be linear in $x$ and $p$, as this is sufficient for our purposes. For direct feedback, the feedback Hamiltonian is proportional to the measurement signal $I(t)$, so we may write

$$H_D = I(t)(\alpha x + \beta p).$$

(69)

The stochastic master equation that results is [11]

$$d\rho_a = -\frac{i}{\hbar} [H_m, \rho_a]dt + 2kD[x]\rho_a dt + \frac{1}{\eta} D[F]\rho_a dt$$

$$-i\sqrt{2k}[F, x\rho_a + \rho_a x]dt$$

$$+ H(\sqrt{2\eta k}x - \frac{i}{\sqrt{\eta}}F)\rho_a dW, \quad (70)$$

where $F = (\sqrt{2\eta k})(\alpha x + \beta p)/\hbar$. This is precisely the model that has been used previously to discuss the manipulation of the motion of atoms and mirrors through feedback [17-19]. Applied to a harmonic oscillator and initially Gaussian states it is possible to rewrite this master equation in terms of the mean values and covariances exactly as was done above. The equations for the covariances are just as before (Eqs.(50)), but the equations for the means are now

$$d\langle x \rangle = (\langle p \rangle/m)dt + 4\eta k\beta\langle x \rangle dt + \sqrt{2\eta k}(2V_x + \beta)dW,$$

$$d\langle p \rangle = -ma^2\langle x \rangle dt - 4\eta k\alpha\langle x \rangle dt + \sqrt{2\eta k}(2C - \alpha)dW.$$

We see that in order to cancel the noise driving the means we merely need choose the feedback Hamiltonian such that $\alpha = 2C$ and $\beta = -2V_x$. However, it is also clear that direct feedback from a continuous position measurement is limited in a way in which feedback by estimation is not. Using direct feedback alone is it not possible to provide a damping term for the mean momentum, or, in fact, any term in the equations for the mean values which is proportional to the mean momentum. This is because we are using continuous position measurement, so that the measurement signal is proportional to the mean position, and not the mean momentum. It is this limitation that feedback using estimation allows us to overcome. Further, it is also clear that while feedback by estimation allows us to achieve phase space localization even in cases where it was not possible to provide the feedback Hamiltonian with a term involving the momentum operator, direct feedback alone will not provide either cooling or confinement without the use of a momentum term. Consequently, a momentum term in the Hamiltonian is crucial for the cooling achieved in this system by the scheme of Mancini et al. [19]. Alternatively, in the absence of the momentum term, these equations are clearly well adapted to modifying the effective potential seen by the atom, and this is discussed at length by Dunningham et al. [21].

It is important to note that the feedback Hamiltonian Eq. (69) could only be realized in the limit of an ideal (infinitely broad band) feedback signal, due to the fact that the measurement noise is white noise (at least to an excellent approximation). The cost, $J_\rho$, of such a signal is therefore also infinite. The use of linear direct feedback cannot improve the results obtained using feedback by estimation (since this is already optimal), and this is reflected in the fact that it eliminates the noise only in the limit of infinite cost, a statement which is also true of the optimal algorithm using feedback by estimation. However, direct feedback does provide an alternative strategy, and, depending on the limitations imposed by a specific implementation, it might well prove advantageous to use it in combination with feedback by estimation.

To summarize, the lowest temperature available is given by the steady-state covariance matrix of the conditional states (Eqs.(53)), and in the limit in which the cost of control can be disregarded, either direct feedback or LQG control, or some combination (being analogous to classical LQG control plus residual feedback), give a means of achieving this as a limiting case. The result is that the final cooled, localized state has the covariances given by Eqs.(54), and the resulting purity, given by Eq.(54), would be limited only by the detection efficiency and environmental noise.

V. CONCLUSION

In this paper we have shown that it is possible to formulate, in a simple manner, feedback in linear quantum systems such that the best estimates of system variables are used to control the system. This significantly extends the range of available possibilities for the control of quantum systems using feedback. Due to the fact that in linear systems the estimation process may be modeled by its classical analogue, Kalman filtration, classical LQG control theory may be applied to quantum feedback by estimation.

While we have focused on applying results from LQG theory to linear systems, there are many other techniques from classical state observer based control which could be applied to control quantum systems. For example, the techniques of adaptive control, where system parameters are estimated on-line in order to cope with non-linearities or uncertainties about the system to be controlled, could be employed. Another problem to be faced in more complicated systems is the computational overhead in propagating the state estimate. Linear systems are tractable classically because the mean and covariance matrix provide all the necessary information about the posterior...
probability distribution with the result that the whole distribution need not be propagated. Propagating the SME’s of non-linear quantum systems in real time will require extensive computational resources. Approximations to the full SME, perhaps along the lines of classical extended Kalman filters, may well be useful or necessary in real near-future experiments.

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