THREE TYPES OF SELF-SIMILAR BLOW-UP FOR THE FOURTH-ORDER $p$-LAPLACIAN EQUATION WITH SOURCE: VARIATIONAL AND BRANCHING APPROACHES

V.A. GALAKTIONOV

Abstract. Self-similar blow-up behaviour for the fourth-order quasilinear $p$-Laplacian equation with source,

$$u_t = -\left(|u_{xx}|^n u_{xx} + |u|^{p-1}u\right)_{xx} + |u|^p \quad \text{in } \mathbb{R} \times \mathbb{R}_+,$$

where $n > 0$, $p > 1$, is studied. Using variational setting for $p = n + 1$ and branching techniques for $p \neq n + 1$, finite and countable families of blow-up patterns of the self-similar form

$$u_S(x, t) = (T - t)^{-\frac{1}{p-1}} f(y), \quad y = x/(T - t)^\beta, \quad \beta = -\frac{p-(n+1)}{2(n+2)(p-1)},$$

are described by an analytic-numerical approach. Three parameter ranges: $p = n + 1$ (regional), $p > n + 1$ (single point), and $1 < p < n + 1$ (global blow-up) are studied. This blow-up model is motivated by the second-order reaction diffusion counterpart

$$u_t = \left(|u_x|^n u_x\right)_x + u^p \quad (u \geq 0)$$

that was studied in the middle of the 1980s, while first results on blow-up of solutions were established by Tsutsumi in 1972.

This paper is an earlier extended preprint of [22].

1. Introduction: classic and recent blow-up reaction-diffusion models

1.1. Classic second-order model. The nonlinear $p$-Laplacian operator in $\mathbb{R}^N$,

$$(1.1) \quad \Delta_p u \equiv \nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad \text{with exponents } \ p > 1 \ (\nabla = \text{grad}_x),$$

which serves as a natural extension of the Laplacian

$$\Delta = \Delta_2, \quad \text{i.e., for } p = 2,$$

enters many classic PDEs of mathematical physics. One of the key mathematical advantages of the $p$-Laplacian (1.1) is that it is nonlinear and at the same time remains a monotone operator in the $L^2$-metric precisely as the linear Laplacian $\Delta$ does. Operators such as (1.1) appear in many works on nonlinear parabolic or elliptic PDEs since the 1950s; see various examples, references, and applications in Lions’ classic book [39]. Gradient-dependent nonlinear operators are typical for filtration, combustion (solid fuels), and non-Newtonian (dilatable, pseudo-plastic fluids) liquids theory; see [33] p. 428.

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Concerning parabolic PDEs admitting blow-up solutions that compose the main subject of the present paper, the \( p \)-Laplacian also appeared before other well-known nowadays porous medium type nonlinearities (see equation (1.4) below). Namely, it is remarkable that the first results on blow-up in quasilinear parabolic equations were obtained by Tsutsumi in 1972 \[50\] for the second-order \( p \)-Laplacian equation (pLE–2) with source posed in a bounded smooth domain \( \Omega \subset \mathbb{R}^N \) with the zero Dirichlet boundary condition:

\[
(1.2) \quad u_t = \nabla \cdot (|\nabla u|^n \nabla u) + u^p \quad \text{in} \quad \Omega \times \mathbb{R}_+ \quad (u \geq 0),
\]

where, in comparison with (1.1), we have renamed the exponents by setting \( n = p - 2 > 0 \) and write the source term as \( u^p \). Concerning the structure of blow-up singularities, various countable and finite families of self-similar blow-up patterns for the one-dimensional equation of (1.2) (and also for the radially symmetric version of (1.2)),

\[
(1.3) \quad u_t = (|u_x|^n u_x)_x + u^p,
\]

have been known since the middle of the 1980s; see \[26\] [27\] [7], and other related references therein. Surprisingly for the author who initiated the study in \[26\] [27\], it turned out that (1.3) generates much wider countable and even uncountable families of self-similar blow-up patterns than the porous medium equation with source (PME with source)

\[
(1.4) \quad u_t = (u^{n+1})_{xx} + u^p \quad (u \geq 0),
\]

which was studied by Kurdyumov’s Russian School on blow-up and localization since the beginning of the 1970s; see history, references, and basic results in \[47\, Ch. 4\].

It is worth mentioning that the set of blow-up similarity solutions of (1.3), to say nothing about non-radial patterns for (1.2), is rather complicated (e.g., contains infinite countable and even uncountable families of positive solutions for \( p > n + 1 \)), so there are still some difficult open mathematical problems concerning the structure of blow-up singularities for (1.3).

Blow-up results for the \( p \)-Laplacian equations with source (1.3) and (1.2) together with Fujita’s pioneering study of the semilinear heat equation (1966) \[18\],

\[
(1.5) \quad u_t = \Delta u + u^p \quad (p > 1),
\]

are crucial for modern singularity and blow-up theory of nonlinear evolution PDEs.

Nowadays, blow-up and other singularity formation phenomena for various classes of nonlinear evolution PDEs are rather popular in mathematical literature and applications in mechanics and physics. It is well established that blow-up phenomena in nonlinear PDEs not only present principal evolution patterns of interest in application, but also can give insight into the deep mathematical nature of nonlinear equations under consideration and describe general aspects of various fundamental problems of existence-nonexistence, uniqueness-nonuniqueness, optimal regularity classes, and admissible asymptotics of proper solutions.

To emphasize that this is not an exaggeration, let us mention that, according to the typical tools of the possible and already available analysis and proofs, that the two key open
PDE/geometry problems of the twentieth and twenty-first century are directly attributed to the area of PDE blow-up research:

**Problem (I): Poincaré Conjecture** (a closed connected 3D manifold is homeomorphic to $S^3$) and the general geometrization problem with Perelman’s recent proof by introducing two new monotonicity formulae and others to pass through blow-up singularities of Ricci flows with surgery (see [11] for a full account of history, references, and recent development); and

**Problem (II): Uniqueness or nonuniqueness** (and hence nonexistence or existence of local small-scale blow-up singularities) in the 3D Navier–Stokes equations\(^1\) (one of the Millennium Prize Problems for the Clay Institute; see Fefferman [17]).

There are several monographs [1, 47, 41, 34, 20, 46], which are devoted mainly to space-time stricture of blow-up singularities in second-order reaction-diffusion PDEs and explain the role of blow-up phenomena in general PDE theory. See also [33] presenting various exact solutions and some examples of partial singularity analysis of other classes of thin film, nonlinear dispersion, and hyperbolic PDEs. In the monograph [40], a nonlinear capacity approach was shown to be efficient to detect conditions of global nonexistence for a variety of nonlinear PDEs and systems of different orders and types.

The questions of the space-time structure and multiplicity of possible blow-up asymptotics represent problems of higher complexity that need another more involved mathematical treatment, which often and still cannot be fully justified rigorously, so a true combination of various approaches including enhanced numerics is in great demand.

### 1.2. Fourth-order reaction-diffusion equation

In this paper, we study self-similar blow-up for the following quasilinear parabolic fourth-order $p$-Laplacian equation with source (pLE–4 with source):

\[
(1.6) \quad u_t = A(u) \equiv -(|u_{xx}|^n u_{xx})_{xx} + |u|^{p-1}u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,
\]

where, as above, $n > 0$ and $p > 1$. Here, similar to (1.1), the fourth-order $p$-Laplacian operator, where we set $p = n + 2 > 1$ ($N = 1$ in (1.6)),

\[
\Delta_{p,2} u = -\Delta (|\Delta u|^{p-2} \Delta u)
\]

is monotone in the metric of $L^2(\mathbb{R}^N)$. For $n = 0$, (1.6) reduces to the semilinear equation

\[
(1.7) \quad u_t = -u_{xxxx} + |u|^{p-1}u,
\]

which describes single point blow-up only for all $p > 1$ and is already known to admit various similarity and other blow-up solutions, [8]. Moreover, it is curious that we have found quite fruitful to use the analogy with the linear bi-harmonic equation

\[
(1.8) \quad u_t = -u_{xxxx} + u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,
\]

which is obtained from (1.6) by both limits $n \to 0$ and $p \to 1$. A simple countable subset of exponential patterns for (1.8) is easy to describe on the basis of spectral theory\(^1\) See [24] as a most recent survey, where connections with reaction-diffusion theory are discussed.
presented in Section 3.3. Eventually, we will detect certain traces of such countable sets (the so-called \( p \)-branches) of similarity solutions in the nonlinear problem (1.6).

Being involved in the mathematical study of blow-up for the PME with source (1.4) from the middle of 1970s and for the \( p \)LE-2 with source (1.2) from the 1980s, the author must admit that the study of blow-up patterns for the proposed \( p \)LE-4 with source (1.6) was quite a challenge and the author did not expect that the necessary mathematics should be so dramatically changed to cover approximately the same concepts developed twenty or even thirty years earlier. Recall that in (1.6) we just increase by two the order of the diffusion operator in comparison with the standard model (1.3). However, this makes almost all mathematical tools applied before very successfully to (1.3) almost nonexistent.

Thus, we consider for (1.6) the Cauchy problem with given bounded compactly supported data

\[ u(x, 0) = u_0(x) \in C_0(\mathbb{R}). \]

Since the operator \( A \) in (1.6) is potential in the metric of \( L^2 \) and the \( p \)-Laplacian is also a monotone operator there, local existence and uniqueness of a unique weak (continuous) solution, which is defined in the standard manner, are not principal issues and follow from classic theory of monotone operators; see Lions [39, Ch. 2]. Finite propagation phenomena for the PDE (1.6) are proved by energy estimates via Saint–Venant’s principle; see [48], references therein, and a survey in [32]. Therefore, there exists the unique local solution of the Cauchy problem (1.6), (1.9), which is a compactly supported function \( u(x, t) \) that can blow up in finite time in the sense that

\[ \sup_{x \in \mathbb{R}} |u(x, t)| \to +\infty \quad \text{as} \quad t \to T^- < \infty. \]

Existence of blow-up in such higher-order quasilinear parabolic equations is a reasonably well-understood phenomenon; see references and approaches in [13, 19, 28] and Mitidieri–Pohozaev [40]. For instance (see [13] and references therein), it is known that, for the equation similar to (1.6) with the absolute value in the source-term,

\[ u_t = -\left(|u_{xx}|^n u_{xx}\right)_{xx} + |u|^p \quad \text{in} \quad \mathbb{R} \times \mathbb{R}+, \]

all nontrivial solutions with data having positive first Fourier coefficient,

\[ \int_{\mathbb{R}} u_0(x) \, dx > 0, \]

blow-up in finite time in the subcritical Fujita range

\[ n + 1 < p < p_0 = n + 1 + \frac{2(n+2)}{2n+5} \quad \text{if} \quad n \geq 0, \]

as well as, most probably, in the critical case \( p = p_0 \), which needs additional study.

1.3. **Layout of the paper: three types of blow-up.** In Section 2 we describe some local and rather delicate oscillatory properties of travelling wave solutions near finite interfaces. This is the first time, where we face difficult and still non fully justified mathematics concerning higher-order degenerate \( p \)-Laplacians. Section 3 is devoted to the setting of blow-up self-similar solutions and some preliminaries concerning the linear
operator with \( n = 0 \) (even this issue is not that straightforward and demands essentially non self-adjoint theory).

Further principal difficulties and important mathematical problems in the study of such blow-up solutions concern the description and classification of possible types (the structure, stability, and multiplicity) of blow-up patterns occurring in finite time. Later on, we study three classes of similarity blow-up solutions of (1.6) in the ranges:

(i) Section 4: \( p = n + 1 \), regional blow-up, so the infinite limit (1.10) occurs on a bounded \( x \)-interval;

(ii) Section 5: \( p > n + 1 \), single point blow-up, so (1.10) happens at a single point, say, at \( x = 0 \), and then \( u(x, T^-) \) is bounded for any \( x \neq 0 \); and

(iii) Section 6: \( p \in (1, n + 1) \), global blow-up, and (1.10) happens for any \( x \in \mathbb{R} \) (and possibly uniformly on any bounded \( x \)-interval).

A similar classification and various single point blow-up patterns of the so-called \( P \)-, \( Q \)-, \( R \)-, and \( S \)-type for the second-order counterpart (1.3) have been known since 1980s; see [27] and more references and results in [7]. Actually, we show that some concepts of the methodology developed in [27, 7] for (1.3) also apply to the fourth-order reaction-diffusion equation (1.6), but indeed demand a different and more difficult mathematics. Several problems remain open still. It turns out that, in general, the PDE (1.6) admits more complicated sets of similarity patterns than the fourth-order porous medium equation (PME−4) with source [23].

\[
(1.12) \quad u_t = -(|u|^n u)_{xxxx} + |u|^{p-1}u.
\]

The general scheme of blow-up study via variational and branching approaches applies to higher-order \( p \)-Laplacian PDEs such as the pLE−6 with source (or any 2mth-order one)

\[
(1.13) \quad u_t = (|u_{xxx}|^n u_{xxx})_{xxx} + |u|^{p-1}u \quad \text{(or} \quad u_t = (-1)^{m+1}D_x^m(|D_x^m u|^n D_x^m u) + |u|^{p-1}u).\]

1.4. On some other higher-order PDEs with blow-up, extinction, and finite interfaces. Blow-up in parabolic PDEs with higher-order diffusion becomes much more difficult than for second-order reaction-diffusion equations. Even simpler PDEs such as the extended Frank-Kamenetskii equation in one dimension

\[
(1.14) \quad u_t = -u_{xxxx} + e^u \quad \text{or} \quad u_t = u_{xxxxx} + e^u,
\]

and their counterparts with power nonlinearities

\[
(1.15) \quad u_t = -u_{xxxx} + |u|^{p-1}u \quad \text{or} \quad u_t = u_{xxxxx} + |u|^{p-1}u,
\]

revealed several principally new asymptotic blow-up properties demanding novel mathematical approaches; see details in [8, 19]. Similar difficulties occur for the Semenov–Rayleigh–Benard problem with the leading operator of the form

\[
(1.15) \quad u_t = -u_{xxxx} + \beta [(u_x)^3]_x + e^u \quad (\beta \geq 0);
\]

see [35]. The mathematical difficulties in understanding the ODE and PDE blow-up patterns increase dramatically with the order of differential diffusion-like operators in the
equations. Interesting regional blow-up and oscillatory properties \cite{9} are exhibited by a semilinear diffusion equation with “almost linear” logarithmic source term

\begin{equation}
(1.16) \quad u_t = -u_{xxxx} + u \ln^4 |u|.
\end{equation}

The above models are semilinear and do not admit blow-up patterns with finite interfaces.

Concerning quasilinear higher-order PDEs, the interface and blow-up phenomena are natural and most well-known for the degenerate unstable thin film equations (TFEs) with lower-order terms such as

\begin{equation}
(1.17) \quad u_t = -(|u|^n u_{xxx})_x - (|u|^{p-1} u)_{xx}, \quad u_t = -\nabla \cdot (|u|^n \nabla u) \pm \Delta |u|^{p-1} u,
\end{equation}

where \( n > 0 \) and \( p > 1 \). Equations of this form are known to admit non-negative solutions constructed by special sufficiently “singular” parabolic approximations of nonlinear coefficients that lead to free-boundary problems. This direction was initiated by the pioneering paper \cite{3} and was continued by many researchers; we refer to \cite{38, 52} and the references therein. Blow-up similarity solutions of the fourth-order TFE (1.17) with the unstable sign “−”

\begin{equation}
(1.18) \quad u_t = -(u^n u_{xxx})_x - (u^p)_{xx} \quad (u \geq 0),
\end{equation}

have been also well studied and understood; see \cite{5, 6, 15, 49, 52}, where further references on the mathematical properties of the models can be found. Countable sets of blow-up patterns for this TFE were described in \cite{15}.

Interface and finite-time extinction behaviour, which is described by various similarity patterns, occur for other reaction-absorption PDEs such as

\begin{equation}
(1.19) \quad u_t = -u_{xxxx} - |u|^{p-1} u
\end{equation}

in the singular parameter range

\begin{equation}
(1.20) \quad p \in (-\frac{1}{3}, 1), \implies |u|^{p-1} u \text{ is not Lipschitz continuous at } u = 0,
\end{equation}

so that \( |u|^{p-1} u \) is not Lipschitz continuous at \( u = 0 \); see \cite{21} and references therein.

We have used a simply looking quasilinear model such as (1.6) to demonstrate various new aspects of higher-order reaction-diffusion blow-up phenomena. The mathematics then becomes more difficult than for the second-order PDEs in (1.3), where the Maximum Principle reveals its full capacity. We do not expect straightforward rigorous justifications of several our conclusions and results, and state key open problems when necessary.

2. LOCAL ASYMPTOTIC PROPERTIES OF SOLUTIONS NEAR INTERFACES

Here, we describe generic oscillatory behaviour of solutions of (1.6) close to finite interfaces.
2.1. **Local properties of travelling waves: oscillatory profiles for \( \lambda < 0 \).** We use simple TW solutions, 
\begin{equation}
(2.1) \quad u(x,t) = f(y), \quad y = x - \lambda t,
\end{equation}
to check generic propagation properties for reaction-diffusion equations involved. In a wide class of 1D second-order reaction-diffusion parabolic PDEs, the TWs rigorously describe the behaviour of finite interfaces for general classes of solutions; see [20, Ch. 7] and references therein.

We use this approach for the fourth-order PDE (1.6). The ODE for \( f \) takes the form
\begin{equation}
(2.2) \quad -\lambda f' = -(|f''|^n f'')'' + |f|^{p-1} f.
\end{equation}

By a local analysis near the singular point \( \{f = 0, f' = 0\} \), it is not difficult to show that the higher-order term \( |f|^{p-1} f \) on the right-hand side is negligible. Therefore, near interfaces, assuming that these are propagating, we can consider the simpler equation
\begin{equation}
(2.3) \quad (|f''|^n f'')' = -f \quad \text{for} \quad y > 0, \quad f(0) = 0,
\end{equation}
which is obtained on integration once. Here we set \( \lambda = -1 \) for propagating waves, by scaling. We need to describe its oscillatory solution of changing sign, with zeros concentrating at the given interface point \( y = 0^+ \). Oscillatory properties of solutions are a common feature of related higher-order degenerate ODEs; see pioneering paper by Bernis–McLeod [4] for similar fourth-order ODEs.

It follows from the scaling invariance of (2.3) that there exist solutions of the form
\begin{equation}
(2.4) \quad f(y) = y^\mu \varphi(s), \quad s = \ln y, \quad \text{where} \quad \mu = \frac{2n+3}{n} > 2 \quad \text{for} \quad n > 0,
\end{equation}
where \( \varphi(s) \) is called the oscillatory component of the given solution. Substituting (2.4) into (2.3) yields the following second-order equation for \( \varphi(s) \):
\begin{equation}
(2.5) \quad (n+1)|P_2(\varphi)|^n P_3(\varphi) = -\varphi,
\end{equation}
where \( P_k \) denote linear differential operators (see [33, p. 140]) given by the recursion
\begin{align*}
P_{k+1}(\varphi) &= P'_k(\varphi) + (\mu - k)P_k(\varphi), \quad k \geq 0; \quad P_0(\varphi) = \varphi, \quad \text{so that} \\
P_1(\varphi) &= \varphi' + \mu \varphi, \quad P_2(\varphi) = \varphi'' + (2\mu - 1)\varphi' + \mu(\mu - 1)\varphi, \\
P_3(\varphi) &= \varphi''' + 3(\mu - 1)\varphi'' + (3\mu^2 - 6\mu + 2)\varphi' + \mu(\mu - 1)(\mu - 2)\varphi, \\
P_4(\varphi) &= \varphi^{(4)} + 2(2\mu - 3)\varphi''' + (6\mu^2 - 18\mu + 11)\varphi'' \\
&\quad + 2(2\mu^3 - 9\mu^2 + 11\mu - 3)\varphi' + \mu(\mu - 1)(\mu - 2)(\mu - 3)\varphi, \quad \text{etc.}
\end{align*}

According to (2.4), we are interested in uniformly bounded global solutions \( \varphi(s) \) that are well defined as \( s = \ln y \to -\infty \), i.e., as \( y \to 0^+ \). The best candidates for such global orbits of (2.5) are periodic solutions \( \varphi_*(s) \) that are defined for all \( s \in \mathbb{R} \). These describe suitable (and, possibly, generic) connections with the interface at \( s = -\infty \). The following result is proved by shooting as in [16, § 7.1] and follows the arguments in [23, § 2].

**Proposition 2.1.** For all \( n > 0 \), (2.5) has a periodic solution of changing sign \( \varphi_*(s) \).
There are two open problems:

(i) uniqueness of the periodic solution $\varphi_*(s)$, and
(ii) stability $\varphi_*(s)$ as $s \to +\infty$.

Numerical evidence answers positively to both questions. Then (i) and (ii) mean a unique (up to translation) periodic connection with $s = -\infty$, where the interface is situated.

The convergence to the unique stable periodic behaviour of (2.5) is shown in Figure 1 for various $n = 0.75$ (periodic oscillations are of order $10^{-7}$) and $n = 5$ (order is $10^{-2}$). Different curves therein correspond to different Cauchy data $\varphi(0), \varphi'(0), \varphi''(0)$ prescribed at $y = 0$. For $n < \frac{3}{4}$, the oscillatory component gets extremely small, so an extra scaling is necessary as explained in [16, § 7.3]. A more accurate passage to the limit $n \to 0$ in the degenerate ODEs such as (2.5) is presented there in Section 7.6 and in Appendix B.

Finally, given the periodic $\varphi_*(s)$ of (2.5), as a natural way to approach the interface point $y_0 = 0$ according to (2.4), we have that the ODE (2.3) and, asymptotically, (2.2),

**Figure 1.** Convergence to a stable periodic orbit of the ODE (2.5) for $n = \frac{3}{4}$, where $\varphi_* \sim 10^{-7}$, $n = 1$, $n = 3$, and $n = 5$, with $\varphi_* \sim 10^{-2}$.
admit at the singularity set \( \{ f = 0 \} \)

\begin{equation}
\tag{2.6}
\text{a } 2D \text{ local asymptotic family with parameters } y_0 \text{ and phase shift in } s \mapsto s + s_0.
\end{equation}

We also call (2.6) an asymptotic bundle of orbits.

2.2. Non-oscillatory case \( \lambda > 0 \). For \( \lambda = 1 \), we have the opposite sign in the ODE

\begin{equation}
\tag{2.7}
(n + 1)|P_2(\varphi)|^n P_3(\varphi) = \varphi,
\end{equation}

which admits two constant equilibria

\begin{equation}
\tag{2.8}
\varphi_\pm = \pm [(n + 1)(\mu - 2)]^{\frac{1}{n}} [\mu(n - 1)]^{\frac{n+1}{n}}.
\end{equation}

Figure 2(a) shows that as \( s \to +\infty \) the equilibria (2.8) are stable (easy to see by linearization). In (b), which gives the enlarged behaviour from (a) close to \( \varphi = 0 \), we observe a changing sign orbit, which is not periodic. This behaviour cannot be extended as a bounded solution up to the interface at \( s = -\infty \). In other similar ODEs, which are induced by other parabolic PDEs, such behaviour between two equilibria can be periodic; cf. [33, p. 143].

These results confirm that for \( \lambda > 0 \), the TWs are not oscillatory at interfaces, and actually such backward propagation via TWs is not possible for almost all (a.a.) initial data. More precisely, unlike (2.6), for \( \lambda > 0 \), the asymptotic family (a bundle) as \( s \to -\infty \) is 1D, which is not sufficient for matching purposes (see typical ideas of construction of similarity profiles below).
3. Blow-up similarity solutions: problem setting and preliminaries

3.1. ODE reduction. The parabolic PDE (1.6) formally possesses the following similarity solutions describing finite-time blow-up as \( t \to T^- \):

\[
(3.1) \quad u_S(x,t) = (T-t)^{-\frac{1}{n+1}} f(y), \quad y = x/(T-t)^\beta,
\]

with \( \beta = \frac{p-(n+1)}{2(n+2)(p-1)} \).

The rescaled blow-up profile \( f(y) \) satisfies the quasilinear fourth-order ODE

\[
(3.2) \quad A(f) \equiv -(|f''|^n f'')' - \beta y f' - \frac{1}{p-1} f + |f|^{p-1} f = 0 \quad \text{in} \quad \mathbb{R}.
\]

We impose at the origin \( y = 0 \) either the symmetry conditions,

\[
(3.3) \quad f'(0) = 0 \quad \text{and} \quad f'''(0) = 0,
\]

or the anti-symmetry ones,

\[
(3.4) \quad f(0) = 0 \quad \text{and} \quad f''(0) = 0.
\]

By a standard local analysis of (3.2) for small \( f \approx 0 \), and in view of general results on regularity [39, Ch. 1,2] and finite speed of propagation for such degenerate parabolic equations [48], a natural setting for the Cauchy problem assumes that, for \( p \in (1, n+1] \),

\[
(3.5) \quad f(y) \text{ is sufficiently smooth and compactly supported.}
\]

The actual regularity of \( f(y) \) close to interfaces has been determined in the previous section.

For \( p > n+1 \), the asymptotic analysis shows that the solutions are not compactly supported. Note that equation (3.2) possesses the constant equilibria

\[
(3.6) \quad \pm f_*(p) = \pm (p-1)^{-\frac{1}{p-1}}.
\]

3.2. Blow-up self-similar profiles: preliminaries. We next study solvability of the ODE (3.2) in \( \mathbb{R} \). First of all, the local interface analysis from Section 2 applies to (3.2).

Indeed, close to the interface point \( y = y_0 > 0 \) of the similarity profile \( f(y) \), the ODE (3.2) for \( p < n+1 \) contains the same leading terms as in (2.3) and other linear two are negligible as \( y \to y_0^- \).

For \( p = n+1 \), where \( \beta = 0 \), the leading terms close to the interface are

\[
-(|f''|^n f'')' - \frac{1}{n} f = 0.
\]

This gives solutions (2.4) with another exponent

\[
\mu = \frac{2(n+2)}{n},
\]

and a fourth-order ODE for \( \varphi(s) \), which admits a periodic solution \( \varphi_*(s) \); see examples in [33, Ch. 3-5].

It is important that, taking into account the local result (2.6) and bearing in mind the two conditions (3.3) or (3.4) yield two algebraic equations for two parameters \( \{y_0, s_0\} \) of the bundle. Therefore, we expect that

\[
(3.7) \quad \text{there exists not more than a countable set} \ \{f_k\} \ \text{of solutions}.
\]
Note that this assumes a certain analyticity of the dependence on parameters in the degenerate ODE (3.2), which is not easy to prove. In particular, relative to the parameter \( p > 1 \), we can expect at most a countable set of \( p \)-branches of solutions. This is true for the linear case \( n = 0 \) and \( p = 1 \); see below.

### 3.3. Fundamental solution and necessary spectral properties.

Here we review some properties of differential operators in the linear case \( n = 0 \). Consider the linear bi-harmonic equation

\[
(3.8) \quad u_t = -u_{xxxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+.
\]

Its fundamental solution has the form

\[
(3.9) \quad b(x,t) = t^{-\frac{1}{4}} F(y), \quad y = x/t^{\frac{1}{4}},
\]

where the rescaled kernel \( F \) is the unique radial solution of the ODE

\[
(3.10) \quad BF \equiv -F^{(4)} + \frac{1}{4} y F' + \frac{1}{4} F = 0 \quad \text{in} \quad \mathbb{R}, \quad \text{with} \quad \int_{\mathbb{R}} F \, dy = 1.
\]

On integration once, we obtain a third-order equation,

\[
(3.11) \quad -F''' + \frac{1}{4} y F = 0 \quad \text{in} \quad \mathbb{R}.
\]

The kernel \( F = F(\vert y \vert) \) is radial, has exponential decay, oscillates as \( \vert y \vert \rightarrow \infty \), and

\[
(3.12) \quad \vert F(y) \vert \leq D e^{-\frac{d}{2} \vert y \vert^{4/3}} \quad \text{in} \quad \mathbb{R},
\]

for a positive constant \( D \) and \( d = 3 \cdot 2^{-11/3} \); see [14, p. 46]. The necessary spectral properties of the linear non self-adjoint operator \( B \) and the corresponding adjoint operator \( B^* \) are of importance in the asymptotic analysis and are explained in [13] for general \( 2m \)-th-order operators (see also [15, §4]). In particular, \( B \) has a discrete (point) spectrum \( \sigma(B) \) in a weighted space \( L^2_\rho(\mathbb{R}) \), with \( \rho(y) = e^{\alpha \vert y \vert^{4/3}} \), \( \alpha \in (0, 2d) \) is a constant,

\[
(3.13) \quad \sigma(B) = \{\lambda_l = -\frac{1}{4}, \ l = 0, 1, 2, \ldots\}.
\]

The corresponding eigenfunctions are given by

\[
(3.14) \quad \psi_l(y) = \frac{(-1)^l}{\sqrt{l!}} F^{(l)}(y), \quad l = 0, 1, 2, \ldots.
\]

The adjoint operator

\[
(3.15) \quad B^* = -D^4_y - \frac{1}{4} y D_y
\]

has the same spectrum (3.13) and polynomial eigenfunctions

\[
(3.16) \quad \psi_l^*(y) = \frac{1}{\sqrt{l!}} \sum_{j=0}^{\lfloor -\lambda_l \rfloor} \frac{1}{j!} D^j_y y^l, \quad l = 0, 1, 2, \ldots,
\]

which form a complete subset in \( L^2_{\rho^*}(\mathbb{R}) \), where \( \rho^* = \frac{1}{\rho} \). As \( B \), the adjoint operator \( B^* \) has compact resolvent \( (B^* - \lambda I)^{-1} \). It is not difficult to see by integration by parts that the
eigenfunctions (3.14) are orthonormal to polynomial eigenfunctions \( \{ \psi_l^* \} \) of the adjoint operator \( B^* \), so

\[
(3.17) \quad \langle \psi_l, \psi_k^* \rangle = \delta_{lk},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the standard (dual) scalar product in \( L^2(\mathbb{R}) \).

3.4. **Countable set of similarity solutions for** \( n = 0, p = 1 \). Performing in the equation (1.8) the change

\[
(3.18) \quad u(x,t) = e^t w(x,t)
\]

reduces it to the pure bi-harmonic equation (3.8) for \( w(x,t) \). By the scaling as for the fundamental solution \( b(x,t) \) in (3.9),

\[
(3.19) \quad w(x,t) = t^{-\frac{1}{4}} v(y, \tau), \quad y = x/t^{\frac{1}{4}}, \quad \tau = \ln t,
\]

we obtain the rescaled equation with the \( B \) in (3.10) having eigenfunctions (3.14), so

\[
(3.20) \quad v_{\tau} = B v \quad \Rightarrow \quad \exists \ v_l(y, \tau) = e^{\lambda_l \tau} \psi_l(y).
\]

Setting \( \lambda_l = -\frac{l}{4} \) as in (3.13) and \( t = e^\tau \), we obtain a countable set of different asymptotic patterns for the linear PDE (1.8) corresponding to \( n = 0 \) and \( p = 1 \):

\[
(3.21) \quad u_l(x,t) = e^{-t} t^{-\frac{l+1}{4}} \psi_l(\frac{y}{t^{\frac{1}{4}}}), \quad l = 0, 1, 2, \ldots.
\]

It turns out that the blow-up similarity patterns (3.1) can be deformed as \( n \rightarrow 0 \) and \( p \rightarrow 1 \) to those in (3.21) (though entirely rigorous proof is very difficult and not fully completed for such degenerate equations, as will happen for some other related homotopy questions). Then (3.21) suggests that there exists a countable number of branches \( \{ f_l(y; n, p) \} \), which appear from the branching point \( \{ n = 0, p = 1 \} \) according to classic theory, [37, § 56]. We claim that the above two (linear for \( n = 0, p = 1 \) and nonlinear for \( n > 0, p > 1 \)) asymptotic problems admit a continuous homotopic connection as \( n \rightarrow 0, p \rightarrow 1 \), so that, after necessary scaling, (3.21) is obtained in the limit from nonlinear eigenfunctions. For such ODEs, this reduces to a matched asymptotic expansion analysis, which is rather technical and is not studied here.

What is key for the future study is that the oscillatory behaviour of linear patterns in (3.21) is then inherited by nonlinear blow-up patterns at least for small \( n > 0 \) and \( p > 1 \). This shows once more that similarity profiles \( f(y) \) corresponding to the Cauchy problem must be oscillatory near interfaces. Homotopy approaches can play a role for specifying correct settings of the Cauchy problem for variety of nonlinear PDEs with non-smooth or singular coefficients, if they share the same homotopy class with a well-posed linear equation; see [10] Ch. 8.

It follows from the ODE (3.2) that

\[
(3.22) \quad \| f \|_\infty \sim f_*(p) = (p - 1)^{-\frac{1}{p-1}} \rightarrow +\infty \quad \text{as} \quad p \rightarrow 1^+,
\]

so the divergence (in fact, towards the rescaled linear problem) is exponentially fast.
4. Regional blow-up profiles for \( p = n + 1 \): variational approach

We begin with the special case \( p = n + 1 \), where \( \beta = 0 \) in (3.1) (so \( y = x \)) and \( f(y) \) in (3.2) solves an autonomous fourth-order ODE of the form

\[
A(f) 
\equiv -(|f''|^n f'')'' - \frac{1}{n} f'' + |f|^n f = 0 \quad \text{in} \quad \mathbb{R}.
\]

This is a variational problem that can be studied in greater detail. Later on, we apply these patterns and classification for \( p = n + 1 \) in neighbouring parameter ranges \( p > n + 1 \) and \( p < n + 1 \) by using a natural idea of \( p \)-branches of solutions.

For convenience, we perform in (4.1) an extra scaling

\[
f = \left(\frac{1}{n}\right)^{\frac{1}{2}} F \quad \implies \quad -(|F''|^n F'')'' - F + |F|^n F = 0 \quad \text{in} \quad \mathbb{R}.
\]

For any \( n > 0 \), this equation admits three constant equilibria \( F \equiv -1, 0, 1 \).

4.1. Variational setting and compactly supported solutions. Operators involved in the ODE (4.2) are potential in \( L^2 \), so the problem admits a variational setting and solutions can be obtained as critical points of a \( C^1 \) functional of the form

\[
E(F) = -\frac{1}{n+2} \int |F''|^n + 2 \, dy - \frac{1}{2} \int F^2 \, dy + \frac{1}{n+2} \int |F|^n \, dy.
\]

Then we are looking for critical points in \( W^{n+2}_2(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^{n+2}(\mathbb{R}) \). For compactly supported solutions (see below), we choose a sufficiently large interval \( B_R = (-R, R) \) and consider the variational problem for (4.3) in \( W^{n+2}_{2,0}(B_R) \), assuming Dirichlet boundary conditions at the end points \( \partial B_R = \{ \pm R \} \). By Sobolev embedding theorem, \( W^{n+2}_{2,0}(B_R) \) is compactly embedded into \( L^2(B_R) \) and \( L^{n+2}(B_R) \). Continuity of any bounded solution \( F(y) \) is guaranteed by Sobolev embedding \( H^2(\mathbb{R}) \subset C(\mathbb{R}) \).

Thus, we will be looking for compactly supported solutions. This demand is associated with the well-known fact that the corresponding parabolic flow with the elliptic operator as in (4.2),

\[
w_t = -(|w_{xx}|^n w_{xx})_{xx} - w + |w|^n w,
\]

describes processes with finite propagation of interfaces. By energy estimates, such results have been proved for a number of quasilinear higher-order parabolic equations with potential \( p \)-Laplace-type operators; see [18]. Therefore, our blow-up patterns are indeed nontrivial compactly supported stationary solutions of (4.4). Examples of ODE proofs via typical energy estimates can be found in [4, § 7].

Thus, in what follows, to revealing compactly supported patterns \( F(y) \), we will pose the problem in bounded sufficiently large intervals \( (-R, R) \) with Dirichlet data at \( \pm R \).

4.2. L–S theory and direct application of fibering method. The functional (4.3) is \( C^1 \), uniformly differentiable, and weakly continuous, so we can apply classic Lusternik–Schnirel’man (L–S) theory of calculus of variations [37, § 57] in the form of the fibering method [44, 45].
According to L-S theory and the fibering approach, the number of critical points of the functional \((4.3)\) depends on the \textit{category} (or \textit{genus}) of functional subset on which the fibering is taking place. The critical points of \(E(F)\) are convenient to obtain by the \textit{spherical fibering} in the form

\[
F = r(v)v \quad (r \geq 0).
\]

Here \(r(v)\) is a scalar functional, and \(v\) belongs to a subset in \(W^{n+2}_{2,0}(B_R)\) given by

\[
\mathcal{H}_0 = \{ v \in W^{n+2}_{2,0}(B_R) : \ H_0(v) \equiv - \int |v''|^{n+2} \, dy + \int |v|^{n+2} \, dy = 1 \}.
\]

Then the new functional

\[
H(r, v) = E(rv) \equiv \frac{1}{n+2} r^{n+2} - \frac{1}{2} r^2 \int v^2 \, dy
\]

has the absolute minimum point, where

\[
H'_r \equiv r^{n+1} - r \int v^2 \, dy = 0 \quad \Rightarrow \quad r_0(v) = \left( \int v^2 \, dy \right)^{\frac{1}{n}},
\]

at which \(H(r_0(v), v) = -\frac{n}{2(n+2)} r_0^{n+2}(v)\).

Therefore, introducing

\[
\tilde{H}(v) = \left[ -\frac{2(n+2)}{n} H(r_0(v), v) \right]^{\frac{n}{n+2}} = \int v^2 \, dy,
\]

we arrive at the quadratic, even, non-negative, convex, and uniformly differentiable functional, to which L-S theory applies, \([37, \S 57]\). Searching for critical points of \(\tilde{H}\) in the set \(\mathcal{H}_0\), one needs to estimate the category-genus \(\rho\) of the set \(\mathcal{H}_0\). The details on this notation and basic results for semilinear equations can be found in Berger \([2, p. 378]\).

The Morse index \(q\) of the quadratic form \(Q\) in Theorem 6.7.9 therein is precisely the dimension of the space where the corresponding form is negatively definite. This includes all the multiplicities of eigenfunctions involved in the corresponding subspace. Note that Berger’s analysis and most of others are dealing with perturbation theory of linear operators, which makes it easier to get the genus of necessary functional sets involved. For the quasilinear operators that define the set \((4.6)\) by their potentials, an extra study of genus is needed (to be performed below).

For detecting geometric shapes of patterns, we recall that by the minimax analysis of L-S category theory \([37, p. 387]\), \([2, p. 368]\), the critical values \(\{c_k\}\) and the corresponding critical points \(\{v_k\}\) are given by

\[
c_k = \inf_{F \in \mathcal{M}_k} \sup_{v \in F} \tilde{H}(v),
\]

where \(F \subset \mathcal{H}_0\) are closed sets, and \(\mathcal{M}_k\) denotes the set of all subsets of the form

\[
BS^{k-1} \subset \mathcal{H}_0,
\]

where \(S^{k-1}\) is a suitable sufficiently smooth \((k-1)\)-dimensional manifold (say, sphere) in \(\mathcal{H}_0\) and \(B\) is an odd continuous map. Then each member of \(\mathcal{M}_k\) is of genus at least \(k\) (available in \(\mathcal{H}_0\)). It is also important to remind that the definition of genus \([37, p. 385]\) assumes that \(\rho(\mathcal{F}) = 1\), if no \textit{component} of \(\mathcal{F} \cup \mathcal{F}^*\), where

\[
\mathcal{F}^* = \{ v : -v \in \mathcal{F} \},
\]

available in \(\mathcal{H}_0\).
is the reflection of $F$ relative to 0, contains a pair of antipodal points $v$ and $v^* = -v$. Furthermore, $\rho(F) = n$ if each compact subset of $F$ can be covered by, minimum, $n$ sets of genus one.

According to (4.10),
\[
c_1 \leq c_2 \leq ... \leq c_0,
\]
where $l_0 = l_0(R)$ is the category of $H_0$ satisfying (see below)
\[
l_0(R) \rightarrow +\infty \text{ as } R \rightarrow \infty.
\]
Roughly speaking, since the dimension of the sets $F$ involved in the construction of $M_k$ increases with $k$, this guarantees that the critical points delivering critical values (4.10) are all different.

4.3. Category of $H_0$ gets arbitrarily large as $R \rightarrow +\infty$. It follows from [37, p. 385], [2, p. 376] (see also [15]) that according to (4.6), the category $l_0 = \rho(H_0)$ of the set $H_0$ can be associated with the maximal number $K = K(R)$ of nonlinear eigenvalues $\lambda_k < 1$ of the corresponding elliptic problem
\[
(4.12) \quad - (|\psi''|^{n^3} \psi'' + \lambda_k |\psi|^n \psi) = 0, \quad \psi \in W^{2,0}_2(B_R).
\]
This problem is solved by L–S theory and gives at least a countable set of critical values and different critical points of the positive homogeneous functional
\[
(4.13) \quad \int |v|^{n^2 + 2} dy \text{ on the unit sphere } S_1 = \{ \int |v''|^{n^2 + 2} dy = 1 \}.
\]
Indeed, given an eigenfunction $\psi_k \neq 0$ with $\lambda_k < 1$, multiplying (4.12) by $\psi_k$ yields
\[
- \int |\psi_k''|^{n^2 + 2} dy + \int |\psi_k|^n |\psi_k''|^{n^2 + 2} dy = (1 - \lambda_k) \int |\psi_k|^n |\psi_k''|^{n^2 + 2} dy > 0 \implies
\]
\[
\tilde{\psi}_k = B_k \psi_k \in H_0, \quad |B_k|^{n^2 + 2} = \left[(1 - \lambda_k) \int |\psi_k|^n |\psi_k''|^{n^2 + 2} dy \right]^{-1},
\]
where $B_k > 0$ is the necessary normalization factor. By L–S theory, all such nonlinear eigenfunctions are different (since correspond to different critical values of the functional), so that all of them $\{\tilde{\psi}_k, k = 1, ..., K\}$ are linearly independent. In order to estimate the genus of $H_0$, we take their linear combination
\[
(4.14) \quad v = C_1 \tilde{\psi}_1 + ... + C_K \tilde{\psi}_K \in H_0,
\]
so on substitution into the functional in (4.6) we get the following algebraic equation for the coefficients $C = \{C_1, ..., C_K\} \in \mathbb{R}^K$:
\[
(4.15) \quad G(C) \equiv - \int |C_1 \tilde{\psi}_1'' + ... + C_K \tilde{\psi}_K''|^{n^2 + 2} dy + \int |C_1 \tilde{\psi}_1 + ... + C_K \tilde{\psi}_K|^{n^2 + 2} dy = 1,
\]
which is an equation of a surface $L_K$ in $\mathbb{R}^K$ being symmetric under the reflection
\[
(4.16) \quad C \mapsto -C.
\]
One can see that, by construction of the normalized eigenfunctions $\tilde{\psi}_k$, for any fixed $k = 1, 2, ..., K$,
\[
(4.17) \quad G(C) = |C_k|^{n^2 + 2}(1 + o(1)) \text{ as } C_k \rightarrow \infty.
\]
It is not difficult to see (using the variational and extremal nature of nonlinear eigenfunctions) that $L_K$ contains a simple closed connected component, which, in view of (4.10), is homotopic to the unit sphere $S^{K-1}$ in $\mathbb{R}^K$. By the “additivity” properties of the genus, this implies that

\begin{equation}
\rho(\mathcal{H}_0) \geq K(R) - 1.
\end{equation}

We do not know whether this estimate is sharp: optimal estimates of the category (genus) of the sets and even multiplicity of nonlinear eigenfunctions for such functionals compose a difficult open problem, which persists even for classic $p$-Laplacian operators as in (1.1).

Since the dependence of the spectrum on the length $R$ for (4.12) is, by simple scaling,

\begin{equation}
\lambda_k(R) = R^{-4 + 2n} \lambda_k(1) \to 0^+ \text{ as } R \to \infty, \quad k = 0, 1, 2, \ldots,
\end{equation}

we have that the category $\rho(\mathcal{H}_0)$ can be arbitrarily large for $R \gg 1$, and (4.11) holds:

**Proposition 4.1.** The ODE problem (4.2) has at least a countable set of different solutions denoted by $\{F_l, l \geq 0\}$, and each one $F_l(y)$ is obtained as a critical point of the functional (4.3) in $W^{2,\infty}(0,1)$ with sufficiently large $R = R(l) > 0$.

### 4.4. First basic pattern and local structure of zeros.

Let us present numerical results concerning existence and multiplicity of solutions for equation (4.2). In Figure 3, we show the first basic pattern for (4.2) called the $F_0(y)$ for various $n \in [0.1, 0.7]$. These profiles are constructed by **MatLab** by using a natural regularization in the singular term,

\begin{equation}
-[(\varepsilon^2 + (F'')^2)^{\frac{\alpha}{2}} F'']'' - F + |F|^n F = 0 \quad \text{in } \mathbb{R} \quad (\varepsilon > 0).
\end{equation}

Here, the regularization parameter $\varepsilon$ and both tolerances in the bvp4c solver, typically, take the values

\begin{equation}
\varepsilon = 10^{-2} \text{ or } 10^{-3} \quad \text{and} \quad \text{Tols} = 10^{-3} \text{ or } 10^{-4}.
\end{equation}

For $n > 0.5$, convergence gets rather slow. For $n \leq 0.7$, the global structure of blow-up profiles (excluding their fine zero structure, see below) is stable with respect to reasonable variations of $\varepsilon$ and Tols. In fact, this reflects the structural stability of first basic blow-up patterns, which the author observed in dozens of other nonlinear parabolic models with blow-up. Note that proving stability even in the linearized setting involves non self-adjoint operators with non-constant coefficients that leads to several technical difficulties and remains open. On the other hand, for $n \geq 1$, i.e., for strongly nonlinear diffusion operators in (4.20), we did not get reliable enough numerical results with the necessary accuracy, so we will avoid using such cases for further illustrations.

Incidentally, this makes it possible to reveal some features of the local structure of multiple zeros close to the interface. Figure 4 shows how the zero structure of profiles $F_0$ from Figure 3 repeats itself in a “self-similar manner” from one zero to another in the usual linear scale. In Figure (b), a “discrete”, piece-wise continuous structure for $n = 0.5$ is already revealed, and this is the best we have been able to achieve numerically. However, this makes no problem, since the accuracy $10^{-3}$ achieved in (b) is already in agreement with parameters in (4.21), so further improvements make no practical sense. In addition, this shows that the discrete and continuous solutions of this difficult variational
problem remain very similar even for the present rough meshes, when the discrete features become clearly visually observable (as usual, it is a key fact for such numerics).

Further revealing zero structure and eventually the behaviour such as (2.4) as \( s = y_0 - y \to 0^+ \) cannot be reliably done in the parameter range (4.21). In [15, 16], for similar thin film models, this demanded \( \epsilon \) and Tols to achieve at least \( 10^{-12} \), which is not possible for the current model in view of slow convergence for higher-order \( p \)-Laplacians. It is also quite a challenge to detect numerically the free-boundary point. The main difficulty is to distinguish the nonlinear oscillations via (2.4) and the linear ones in the “linearized area”, where (4.20) implies an exponential behaviour for \( y \gg 1 \) governed by the ODE

\[
F^{(4)} = -\epsilon^{-n} F + \ldots \quad \implies \quad F(y) \sim e^{-\frac{\sqrt{2}}{2} \epsilon^{-n/4}} \cos \left( \frac{\sqrt{2}}{2} \epsilon^{-\frac{n}{4}} y + c \right),
\]

where \( c \) is a constant. Actually, we saw not more than first 1–3 nonlinear zeros of the type (2.4) and the rest of zeros corresponded to the linear behaviour (4.22).

4.5. **Basic countable family: approximate Sturm’s property.** In Figure 5 we show the basic family denoted by

\[
\{ F_l, \ l = 0, 1, 2, \ldots \}
\]

of solutions of (4.2) for \( n = 0.2 \). This family is connected with the application of L–S and fibering theory; see [29, 30]. Each profile \( F_l(y) \) has \( l + 1 \) “dominant” extrema and \( l \) “transversal” (not from the tail) zeros; see [29 § 5], [30 § 5], and [25 § 4] for further details. It is important that

all the internal zeros of \( F_l(y) \) are transversal,
excluding the oscillatory end points of the support. In other words, each profile $F_l$ is approximately obtained by a simple “interaction” (gluing together) of $l + 1$ copies of the first pattern $\pm F_0$ taking with necessary signs. Such a gluing of oscillatory tails is illustrated in Figures 6 and 7; see also further comments below. There is some analytic evidence [30, §5] that exactly this basic family $\{F_l\}$ is obtained by the classic Lusternik–Schnirel’man construction of critical points of the reduced functional (4.9). A rigorous definition of gluing assumes formation of all the internal transversal zeros while the outer ones at the end point of the support are the only ones that remain oscillatory according to the behaviour (2.4) with the periodic orbit $\varphi = \varphi_\ast (\ln(y_0 - y))$. Some question of global behaviour of such patterns $F_l(y)$ for large $l$ remain open, [29, 30].

Let us forget for a moment about the complicated oscillatory structure of solutions near interfaces, where an infinite number of extrema and zeros occur. Then the dominant geometry of profiles in Figure 5 looks like it approximately obeys Sturm’s classic zero set property, which is true rigorously for the case $m = 1$ only, i.e., for the second-order ODE

(4.23) \[ F'' = -F + |F|^{-\frac{n}{n+1}}F \quad \text{in} \quad \mathbb{R}. \]

For (4.23), the basic family $\{F_l\}$ is explicitly constructed by direct gluing together simple patterns $\pm F_0$ given explicitly; see [33, p. 168]. Therefore, each $F_l$ consists of precisely $l + 1$ patterns (with signs $\pm F_0$), so that Sturm’s property is clearly true by direct application of L–S category theory.

4.6. Countable family of $\{F_0, F_0\}$-gluing. Further patterns to be introduced do not exhibit as clear a “dominated” Sturm property and are associated with a double fibering technique where both the Cartesian and spherical representations of critical points are used; see [29, §3], [30, §3]. Let us present some explanations.

The nonlinear interaction of the two first patterns $F_0(y)$ leads to a new family of profiles. In Figure 6 for $n = 0.2$, we show the first six profiles from this family denoted...
The first basic profile $F_0(y); \ n=0.2$

The second (dipole) basic profile $F_1(y); \ n=0.2$

The third basic profile $F_2(y); \ n=0.2$

The fourth basic profile $F_3(y); \ n=0.2$

The fifth basic profile $F_4(y); \ n=0.2$

The sixth basic profile $F_5(y); \ n=0.2$

Figure 5. The first six patterns of the basic family $\{F_i\}$ of the ODE (4.2) for $n = 0.2$. 
by \{F_{+2,k,+2}\}. In the refined zero structure of the last profile in (b), we already see some numerical effects of rather rough meshes, which again do not deny the sufficient overall quality of numerics. In each function \(F_{+2,k,+2}\) the multiindex
\[
\sigma = \{+2, k, +2\},
\]
from left to right denotes: +2 means two intersections with the equilibrium +1, then next \(k\) intersections with zero, and final +2 stands again for two intersections with +1. Later on, we will use such a multiindex notation to classify other patterns obtained.

As a general rule, we point out again that any finite gluing of a pair of patterns \(\pm F_0(y)\) actually means that all internal zeros become transversal. Note that this and all the Figures involved are not enough to explain the essence of this complicated and mathematically not fully understood procedure for non-homotopic variational problems [31]. The resulting patterns have zeros of infinite order only at the end points of its support.

In view of the infinite oscillatory character of \(F_0(y)\) at the interfaces, we expect that the family \(\{F_{+2,k,+2}\}\) is countable, and such functions exist for any even \(k = 0, 2, 4, \ldots\). Then \(k = +\infty\) corresponds to the non-interacting pair
\[
F_0(y + y_0) + F_0(y - y_0), \quad \text{where } \text{supp} F_0(y) = [-y_0, y_0].
\]

It is expected that there exist various triple \(\{F_0, F_0, F_0\}\) and any multiple interactions \(\{F_0, \ldots, F_0\}\) of \(k\) single profiles, with different distributions of zeros between any pair of neighbours (proof is an open problem).

4.7. Countable family of \(\{-F_0, F_0\}\)-gluing. We now describe the interaction of \(-F_0(y)\) with \(F_0(y)\). In Figure[7] for the case \(n = 0.2\) (which is convenient in terms of rather fast convergence of the numerical method employed), we show the first profiles from this family denoted by \(\{F_{-2,k,+2}\}\), where for the multiindex \(\sigma = \{-2, k, +2\}\), the first number \(-2\) means two intersections with the equilibrium \(-1\), etc. It can be seen that the first two
profiles belong to the same class $F_{-2,1,2}$, i.e., both have a single zero for $y \approx 0$. The last solution shown is $F_{-2,5,2}$. Again, we expect that the family $\{F_{-2,k,2}\}$ is countable, and such functions exist for any odd $k = 1, 3, 5, \ldots$, and $k = +\infty$ corresponds to the non-interacting pair
\begin{equation}
-F_0(y + y_0) + F_0(y - y_0).
\end{equation}

We expect that there exist families of an arbitrary number of gluing $\{\pm F_0, \pm F_0, \ldots, \pm F_0\}$ consisting of any $k \geq 2$ members (again an open problem).

4.8. Periodic solutions in $\mathbb{R}$ as new types of oscillations about $\pm 1$. Before introducing new types of patterns, we need to describe other non-compactly supported solutions in $\mathbb{R}$. As a variational problem, equation (4.2) admits infinitely many periodic solutions; see e.g., [40, Ch. 8]. Figure 8 for $n = 0.2$ reveals unstable periodic solutions obtained by shooting from the origin with various Cauchy data at $y = 0$. In (b), the periodic orbit $F_*(y)$ is oscillating about the equilibrium $F \equiv -1$. It turns out that precisely the periodic orbit $F_*(y)$ in (a) with the range
\begin{equation}
\min F_*(y) = 0.4135\ldots, \quad \max F_*(y) = 1.4085\ldots \quad (n = 0.2)
\end{equation}
plays an important part in the construction of other families of compactly supported patterns. Namely, all the variety of solutions of (4.2) that have oscillations about equilibria $\pm 1$ are close to $\pm F_*(y)$ there.

4.9. Family $\{F_{+2k}\}$. Such functions $F_{+2k}$ for $k \geq 1$ have $2k$ intersections with the single equilibrium $+1$ only and have a clear “almost” periodic structure of oscillations about. The number of intersections denoted by $+2k$ gives an extra Strum index to such a pattern. In this notation, for $k = 1$, we have
\begin{equation}
F_{+2} = F_0.
\end{equation}
Figure 8. Examples of convergence to periodic solutions of the ODE (4.2) for 
\( n = 0.2 \); about \( F \equiv 1 \) (a) and \( F \equiv -1 \) (b).

Two profiles \( F_{+4} \) and \( F_{+6} \) are shown in Figure 9 for \( n = 0.2 \). The further profile \( F_{+4,1,-2,1,4}(y) \) comprising two sub-structures \( F_{+4} \) from the family \{\( F_{+2k} \), \( k \geq 2 \}\} is shown in Figure 10 by the boldface line.

4.10. More complicated patterns: towards chaotic structures. By combining the above rather simple families of patterns, we claim that a pattern (more precisely, a class of patterns) with an arbitrary admissible multiindex of any length

\[
\sigma = \{\pm \sigma_1, \sigma_2, \pm \sigma_3, \sigma_4, \ldots, \pm \sigma_l\}
\]
can be constructed. For example, in Figure 11, a single complicated pattern with

\[ \sigma = \{-2, 2, -2, 1 + 2, 2, +2, 1, -8, 1, +2\} \]

is given. The computation (the convergence is rather slow for this type of \( p \)-Laplacian operators) is performed for \( n = 0.2 \) as usual. We must admit that, as it is seen from Figure 11, the iterations have not been properly converged within the parameter range \((4.21)\), but we can guarantee the convergence with the accuracy at least \( \sim 10^{-1} \). This is not that bad, since such patterns are not structurally stable and have multi-dimensional unstable manifolds, so the convergence must be extremely slow. It is worth mentioning that, using \texttt{bvp4c} solver, this computation took a few hours with the maximal number of 75000 points on the interval \((-50, 50)\). Nevertheless, regardless such a lack of accuracy, we are sure that such complicated critical point profiles really exist, since we have seen a lot of those in other similar (and simpler numerically) higher-order variational problems \([29, 30]\) that were not associated with such awkward and strongly degenerate operator as \( p \)-Laplacian ones. Special more conservative and divergent numerical techniques are necessary for tackling higher-order \( p \)-Laplacian operators in the ODEs, but here we demonstrate what an average (numerically, non-professional) PDE user can extract from standard \texttt{MatLab} codes. Theoretically, via the L-S/fiber theory, all those patterns are well defined.

We claim that the multiindex \((4.27)\) can be rather arbitrary taking finite parts of any non-periodic “fraction”. Actually, this means chaotic features of the whole family of solutions \( \{F_{\sigma}\} \). In fact, there is no any exiting news in such a chaotic proclamation: one can see that even the basic simple countable family \( \{F_{l}\} \) is indeed chaotic, since the choice of the sequence of elementary profiles \( \pm F_0 \) in \( F_l \) for \( l \gg 1 \) can be arbitrary long with an arbitrary sequence \( \{\pm\} \) of sign changes, thus exhibiting no finite periodic order in
the index $\sigma$. These chaotic types of behaviour are known for other simpler fourth-order ODEs with coercive operators and definite homotopic features, [42, p. 198].

A variety of complicated patterns of these types for different variational problems associated with the PME-type nonlinearities in (1.12) can be found in [29, §5.6], [30, §5.6] (see also §6 in [30] therein for sixth- and eighth-order models) and in [23, §5]. The convergence of standard numerical methods for these problems are much faster, with $\epsilon$, Tols up to $\sim 10^{-10}$.

5. Single point blow-up for $p > n + 1$: P- and Q-type profiles

We now return to the similarity ODE (3.2) in the case $p > n + 1$, which, in view of the spatial rescaled variable $y$ in (3.1), corresponds to single point blow-up. It is key that (3.2) for $p \neq n + 1$ is not variational. Formally, solutions of (3.2) can be traced out by shooting and matching procedures, which are too complicated. Instead, we will use a continuation in parameters approach, which allows us to predict solutions by using those in the variational case $p = n + 1$.

5.1. Asymptotics at infinity and single point blow-up. We begin with simpler asymptotics of the solutions of (3.2) as $y \to +\infty$. Unlike the previous case of regional blow-up for $p = n + 1$, for $p > n + 1$, equation (3.2) admits non-compactly supported solutions with the following behaviour:

$$f(y) \sim (C_0 y^\gamma + \ldots) + (C_1 e^{-b_0 y^\nu} + \ldots) \quad \text{as} \quad y \to +\infty,$$

where $C_0 \neq 0$ and $C_1 \in \mathbb{R}$ are arbitrary constants and

$$\gamma = -\frac{2(n+2)}{p-(n+1)} < 0, \quad \nu = \frac{2(n+2)(p-1)}{3[p-(n+1)]} > 0, \quad b_0 = \frac{1}{\nu} \left[ \beta \gamma^n (\gamma-1)^n C_0^{-n} \right]^{\frac{1}{4}} > 0.$$

The first term in (5.1) represents an “analytic” part (can be truly analytic for some parameters) of the expansion, while the second one gives the essentially “non-analytic”
part. Such a structure in (5.1) is usual for saddle-node-type equilibria [43, p. 311], but, for the fourth-order ODE (3.2), this expansion does not admit a simple phase-plane interpretation. However, existence of such asymptotic expansions can be justified by fixed point arguments, which becomes quite a technical issue and is not done here.

One can see passing to the limit $t \to T^-$ in (3.1) that the first term in the asymptotic expansion (5.1) gives the following final-time profile of this single point blow-up for even patterns $f = f(|y|)$:

$$u_S(x, T^-) = C_0 |x|^{-\frac{2(n+2)}{p-(n+1)}} < \infty \quad \text{for all} \quad x \neq 0.$$  (5.2)

Returning to the asymptotic expansion, we conclude that (5.1) represents a 2D asymptotic family (bundle) of solutions. Hence, the family (5.1) is well suitable for matching with also two symmetry conditions at the origin (3.3), so we expect not more than a countable set of solutions. For first patterns, we keep the same notation $f_1(y)$ as in Section 4 for $p = n + 1$.

5.2. Oscillatory behaviour about constant equilibrium $f_*$. In order to predict the multiplicity of solutions of (3.2), we need to study its oscillatory properties. To this end, we perform the linearization about the constant equilibrium $f_*$ in (3.6) of the ODE (3.2),

$$f = f_* + Y$$  (5.4)

formally assuming that $|Y| \ll 1$ on some bounded intervals. This yields the “linearized” nonlinear equation

$$B_n(Y) \equiv -(|Y''|^n Y'')'' - \beta Y' y + Y = 0 \quad (\beta = \frac{p-(n+1)}{2(n+2)(p-1)}).$$  (5.5)

We are going to study oscillatory, sign-changing properties of solutions of (5.5) for various $n > 0$. Notice that the “linearized” ODE (5.5) remains a difficult fourth-order equation. Indeed, in view of the invariance with respect to the group of scalings

$$Y \mapsto \varepsilon^{-\frac{2(n+2)}{n}} Y, \quad y \mapsto \varepsilon y \quad (\varepsilon > 0)$$

the transformation

$$Y(y) = y^{-\frac{2(n+2)}{n}} \varphi(s), \quad s = \ln y,$$

reduces (5.5) to an autonomous fourth-order ODE. Setting $P(\varphi) = \varphi'$ yields a third-order ODE, but further reductions are impossible. Thus, (5.5) cannot be studied on the phase-plane in principle; cf. [7], where, for (1.3), oscillatory analysis on the phase-plane is a convenient and exhaustive tool.

Therefore, we will need another further investigation of (5.5), and we begin with the following useful comment:

**Linear case** $n = 0$. Then the quasilinear operator $B_n$ in (5.5) becomes linear,

$$B_0 Y = 0, \quad \text{where} \quad B_0 = -D_y^4 - \frac{1}{4} y D_y + I \equiv B_* + I \quad (\beta = \frac{1}{4})$$  (5.6)
and $B^*$ is the adjoint linear operator (3.15). According to its point spectrum (3.13), equation (5.5) for $n = 0$ has a non-oscillatory solution, being the eigenfunction $\psi_4^*(y)$ for $l = 4$, i.e.,

\[(5.7) \quad Y(y) = \psi_4^*(y) = \frac{1}{\sqrt{24}} (y^4 + 24) \quad (n = 0),\]

which is an example of a non-oscillatory solution. Nevertheless, one can see from the operator (5.6) that the linear ODE (5.5) for $n = 0$ has other oscillatory solutions with an increasing envelope as $y \to +\infty$; see below.

**Quasilinear case** $n > 0$. In Figure 12(a)–(d), we show that, for any $p \geq n + 1$, the ODE (5.5) admits infinitely oscillatory solutions with increasing amplitude of oscillations.

Here, (a) shows linear increasing amplitude of oscillations for $n = 0$. It is curious that such behaviour persists in the nonlinear range $n > 0$, $p \geq n + 1$, so that $n = 0$ is a branching point for (5.5) from solutions of the linear equation (5.6). In (b), we show the bounded periodic solution for the variational case $p = n + 1$ (cf. Figure 8), which generates a $p$-branch of non-periodic patterns for $p > n + 1$; see (c). This suggests that basic blow-up similarity patterns $\{f_l(y)\}$ are expected to exist for $p > n + 1$ sufficiently close to $n + 1$. Figure 12(d) shows that for $p < n + 1$, the amplitude of oscillations becomes decreasing, so we expect a single P-type profile $f_0(y)$ for $p \in (1, n + 1)$.

5.3. **Basic patterns: first numerical conclusions.** For convenience, similar to the change in (4.2), we perform the following scaling in (3.2) for $p \neq n + 1$:

\[(5.8) \quad f = AF, \quad y \mapsto ay, \quad \text{where} \quad A = f_*(p - 1)^{-\frac{1}{p - 1}}, \quad a = (p - 1)^\beta.\]

Then $F = F(y)$ solves the equation

\[(5.9) \quad -(|F''|^n F')' - \tilde{\beta} F' y - F + |F|^{p - 1} F = 0, \quad \text{with} \quad \tilde{\beta} = (p - 1)\beta = \frac{p - (n + 1)}{2(n + 2)},\]

which has the scaled equilibria $F_* = \pm 1$ that are convenient for numerical experiments for small $n > 0$ and $p$ close to $1^+$.

In Figure 13 we present the first pattern $F_0(y)$ for $n = 0.2$ for $p = n + 1 = 1.2$ (the dotted line for comparison), 1.4, 1.8, 2.2, 2.6, 3, 4, and 6. In particular, it is clearly seen that, for larger $p$, the profiles approach the positive asymptotic behaviour (5.1) for $y \gg 1$ with $C_0 > 0$, and become strictly positive in $\mathbb{R}$.

Figure 14 shows the first $F_0(y)$, the third $F_2(y)$, and the fifth $F_4(y)$ P-type patterns for $n = 0.2$ and $p = 1.5$. In Figure 15 we demonstrate two profiles $F_0(y)$ and $F_2(y)$ for the same $n = 0.2$ and $p = 2.6$. In the last case, the next even profile $F_4(y)$ was not detected numerically, and this nonexistence will be confirmed later by the $p$-branching approach.

5.4. **Q-type profiles.** It follows from (3.2) that an oscillatory expansion can be started at any finite point $y = y_0 > 0$, at which

\[(5.10) \quad f(y_0) = f_* \quad \implies \quad f(y) \equiv f_* \quad \text{for} \quad y \leq y_0.\]

This yields the so-called Q-type solutions; see classification in [27, 7]. Then setting

\[y \mapsto y_0 + y, \quad y > 0,\]
we again arrive asymptotically at a linearized equation similar to (5.5),

\[ \mathcal{B}_n(Y) \equiv -(|Y'|^n Y'')'' - \lambda_0'Y' = 0, \quad \lambda_0 = \beta y_0 > 0, \]

so on integration we obtain the TW equation (2.3), where the constant \( \lambda_0 \) is scaled out. Therefore, we use the change (2.3) to get the oscillatory ODE (2.5). According to (2.6), this gives a 2D asymptotic family to be matched with the bundle (5.1) at infinity.

Analytically, as well as numerically, the problem of existence of a countable subset of such Q-type similarity profiles is more difficult. Figure 16 shows the first Q-type profile \( F_Q^0(y) \) for \( n = 0.2 \) for \( p = 1.5 \). The convergence here is slower and we do not succeed in getting other Q profiles. For the sake of comparison, we also present here P solutions \( F_0(y) \) and \( F_2(y) \).

5.5. **On branching of solutions from variational critical points.** Consider the ODE (5.9) from the point of view of a perturbation approach. For \( p = n + 1 \), i.e., for \( \beta = 0 \),
the ODE has been studied in Section 4. Setting $\varepsilon = p - (n + 1)$ and assuming that $|\varepsilon| > 0$ is sufficiently small, we write (3.2) in the form

$$F(f) \equiv -(|f''|^n f''')'' - \frac{1}{n} f + |f|^n f = \frac{\varepsilon}{2(n+2)(p-1)} y f' - \frac{\varepsilon}{n(n+\varepsilon)} f + |f|^n f (1 - |f|^{p-1}) \quad (5.12)$$
On the right-hand side, the key perturbation term satisfies

\begin{equation}
    g(f, \varepsilon) = |f|^n f (1 - |f|\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0
\end{equation}

and is at least continuous at the singular point $f = 0$, $\varepsilon = 0$. On the left-hand side of (5.12), we have the variational operator from Section 4. Thus, the non-perturbed problem
for \( \varepsilon = 0 \) admits families of solutions described in Section 4 (see also extra details on other families of solutions in [29, § 5] and [30, § 5]). Therefore, classic perturbation and branching theory [10, 51, 37] suggests (and does not prove since \( g \notin C^1 \) at \((0,0)\)) that, under natural hypotheses, the variational problem for \( \varepsilon = 0 \) generates a countable family of \( p \)-branches, which can be extended for some sufficiently small \( |\varepsilon| > 0 \).

The analysis of bifurcation, branching, and continuous extensions should be performed for the equivalent to (5.12) integral equation with Hammerstein compact operators; see typical examples in [8, 25, 36], where similar perturbation problems for blow-up and global patterns were investigated. Namely, denoting by \( D = (D^2)^{-1} \) the inverse of \( D_y^2 \) in a sufficiently large interval \((-R,R)\) (for \( p \leq n + 1 \)) and in \( \mathbb{R} \) (for \( p > n + 1 \)), we write (5.12) as follows:

\[
(5.14) \quad f = A(f, \varepsilon) \equiv D \left[ |Dh|^{n+1} D h \right],
\]

where

\[
(5.15) \quad \gamma \neq 0.
\]

For differentiable operators, where the index \( \text{ind}(f_0, I - A'(f_0, 0)) \) is equal to the rotation of the vector field \( I - A'(f_0, 0) \), there are special techniques for its calculations; see [37, Ch. 20, 24].

However, in view of non-potential structure in (5.14), and of a non-sufficient (in the usual sense) differentiability of (5.13) at \((0,0)\), it is convenient to use other alternatives of bifurcation-branching theory without direct differentiability hypotheses that assume sufficient regularity of the perturbations; cf. [12], § 28. Namely, in view of our difficulties with the differentiability, using Theorem 28.1 in [12, p. 381] replaces differentiability by a slightly weaker control of smallness of nonlinear terms in a neighbourhood of \((0,0)\). As usual, the key principle of branching is that it occurs if the corresponding eigenvalue has odd multiplicity (the even multiplicity case needs an additional treatment, which is also a routine procedure not to be treated here); see further comments below. Justification of branching phenomena for such quasilinear degenerate \( p \)-Laplacian operators (including also questions of compactness) in the present problem needs further deeper analysis and more involved functional topology/constraints. Therefore, most of further analytical conclusions remain formal and are open problems. Nevertheless, it turns out and will be
checked numerically, the predicted branching behaviour from $p = n + 1$ actually occurs, and thus becomes a key tool of the proposed study of non-variational problems at hand.

Thus, anyway, for actual applications, one needs to know the spectrum of the self-adjoint operator $F'(f_0)$. We have that (5.14) contains nonlinearities that are hardly differentiable at 0, so these applications lead to difficult technical problems, where branching analysis from oscillatory profiles $f_0(y)$ with interior transversal zeros needs taking into account more complicated “functional topology”. Anyway, continuing the application, it is worth mentioning that

$$\lambda = 0 \quad (\text{with the eigenfunction } \psi_0 \sim f'_0)$$

is always an eigenvalue of $F'(f_0)$. Actually, this corresponds to the invariance of the original PDE (1.6) relative to the group of translations with the infinitesimal generator $D_y$. In particular, this implies that $\lambda = 1$ is an eigenvalue of $A'(f_0, 0)$, so this corresponds to the critical case, where computing of the index is more difficult and is performed as in [37, § 24]. It is more important that

$$\lambda = 1 \quad (\text{with the eigenfunction } \psi_1 \sim f_0)$$

is also an eigenvalue of $F'(f_0)$ (this is associated with the group of translations with the generator $D_y$). In other words, the index condition (5.15) or smallness assumption via [12, Th. 28.1] need special additional treatment associated with spectral properties of the linearized operator $F'(f_0)$. In this connection, we conjecture that branching at $p = n + 1$ is valid in appropriate functional setting, and there exist continuous $\varepsilon$-curves, from any profiles from the family of basic patterns $\{f_l, l \geq 0\}$ constructed in Section 4.5.

Finally, let us note another convenient (but not that efficient) way to use Schauder’s Theorem applied to (5.14) to get solutions of (5.12) for small $\varepsilon > 0$ and to trace out $p$-branches of the suitable profiles. This approach effectively applies in the case of porous medium operators as in [12], [23, § 6].

5.6. Numerical construction of $p$-branches. We recall that (5.9) is not variational, and we are going to use a certain continuity feature concerning the limit $p \to n + 1$.

The basic $p_0$-branch of $F_0(y)$ of the simplest shape (as well as the $p_1$-one for $l = 1$) exists for all $p > 1$. In Figure 17, we show the first $p$-branch of $F_0$ in (a) and the deformation of the profiles $F_0(y)$ in (b), for $n = 0.2$ and $p \in (1.05, 6.15)$. This branch is extended to the global blow-up case $p < n + 1$, to be discussed next in Section 6. We expect that, as usual in blow-up analysis, this first $p_0$ branch is composed from structurally stable solutions and hence represents the generic blow-up behaviour for the parabolic PDE (1.6).

Deformation with $p$ of $F_1(y)$ (the part for $y > 0$ is shown only) for $p$ slightly above the variational exponent $p = 1 + n = 1.2$ for $n = 0.2$ is presented in Figure 18. Further extension of this branch beyond $p = 1.218$ leads to strong numerical instabilities that possibly reflects the actual nonexistence of such solutions far away from $p = n + 1$.

Concerning other, more complicated profiles for $p = n + 1$, such as $F_{+4}(y)$ and others containing structures shown in Figures 6, 9 and 10 numerical results suggest that some of them cannot be extended for $p > n + 1$ (then any version of (5.15) is not valid).
\[ p\text{-branch of } F_0(y) \text{ for } n=0.2, \ p \in (1.05, 6.15) \] (a); corresponding deformation of \( F_0 \) (b).

Figure 17. The first \( p_0 \)-branch of solutions \( F_0(y) \) of equation (5.9) for \( n = 0.2, \ p \in (1.05, 6.15) \) (a); corresponding deformation of \( F_0 \) (b).

Figure 18. Deformation of the dipole \( F_1(y) \) of equation (5.9) for \( p \in [1.2, 1.218], \ n = 0.2 \).

Nevertheless, for \( F_{+4} \) this is not the case; see Figure 19, where the extension is shown to exist for all \( p > n + 1 \) and that

\begin{equation}
\|F_{+4}\|_\infty \to 1^+ \quad \text{as} \quad p \to +\infty.
\end{equation}

It seems that all the global \( p \)-branches satisfy (5.16); cf. an analogous result in [25] for global similarity solutions. We expect that a similar \( p \)-branch of \( F_{+4} \) is originated at a saddle-node bifurcation for some \( p_* \in (1, n + 1) \), at which it appears together with the \( p \)-branch of the profiles \( F_{+2,2+2} \); see further comments below.
The $p$-branches can connect various profiles, with rather obscure understanding of possible geometry of such branches and their saddle-node bifurcation (turning) points. For $p = n + 1$, the questions on connections with respect to regularization parameters as in (4.20) are addressed in [31] posing problems of homotopy classification of patterns in variational problems and approximate “Sturm’s Index” of solutions.

6. ON GLOBAL BLOW-UP SIMILARITY PROFILES FOR $p \in (1, n + 1)$

6.1. Local oscillatory behaviour close to interfaces remains the same. Indeed, the ODE (3.2) now reads for $f \approx 0$ as

$$-(|f'''|^n f''')' - \beta y f' + ... = 0 \quad (\beta < 0),$$

and reflecting near interface $y \rightarrow y - y_0$, on integration for small $y > 0$, we have

$$(|f'''|^n f'')' = \beta y_0 f + ....$$

This is precisely (2.3) with $\lambda = \beta y_0 < 0$ to be reduced to $-1$ by scaling. Hence, for $p \in (1, n + 1)$, the similarity profiles are equally oscillatory near interfaces as for $p = n + 1$. Therefore, according to (2.6), this local 2D asymptotic family looks sufficient to be matched with two symmetry boundary conditions (3.3) (or (3.4)) at the origin, though the proof of existence remains open.

6.2. On similarity profiles and $p$-branches. For $1 < p < n + 1$, the rescaled ODE (5.9) is more difficult to solve numerically than for $p \geq n + 1$. Figure 20 shows deformation of similarity profiles $F_0(y)$ for $n = 0.2$ and $p \in [1.05, 1.2]$. We observe an easy visible growth of solutions as $p \rightarrow 1^+$. Structurally, the first basic profile $F_0(y)$ remains of a similar geometric shape as in the variational case $p = n + 1 = 1.2$. 

Figure 19. The $p$-branch of solutions $F_{+4}(y)$ of equation (5.9) for $n = 0.2$ (a); corresponding deformation of $F_{+4}(y)$ (b).
Figure 20. Deformation of the first basic profile $F_0(y)$ of the ODE (5.9) for $n = 0.2$ and $p < n + 1 = 1.2$.

A part of the corresponding $p$-branch of profiles $F_0$ was shown earlier in Figure 17(a). More detailed and sharp results are presented in Figure 21 for $p \in [1.023, 1.2]$, $n = 0.2$, where we used branching from the variational profile for $p = 1.2$ (with the step size $\Delta p = -10^{-3}$). From (a), we definitely observe that this $p_0$-branch is going to blow-up as $p \to 1^-$, as suggested before. Note that the $p_0$-branch is expected to consist of asymptotically (structurally) stable blow-up profiles $F_0(y)$, but we cannot prove this even in the linearized approximation. The linearized operator is a difficult non-self-adjoint one with unknown spectrum and proper functional setting.

The next $p$-branch of dipole-like profiles $F_1(y)$ is shown in Figure 22(a), together with the deformation (b) of the functions $F_1(y)$. It is seen that this $p$-branch is global and blows up as $p \to 1^-$. We claim that the $p$-branches of the basic similarity profiles $\{f_l(y)\}$ (q.v. (5.8)) are extended up to $p = 1^-$, with a blow-up behaviour as in (3.22). We expect that these $p$-branches can be connected as $n \to 0$ with those predicted by the linear problem with patterns (3.21). We refer to [36, p. 1090] for an example of such an analysis. For instance, in Figure 23, we present the $p$-branch and the corresponding deformation of the third basic profile $F_2(y)$, which for $p = n + 1$ is given in Figure 5(c) (with the opposite sign).

Also, a principal fact of existence of the $p$-branch of the non-basic profiles is explained in Figure 24 where a local $p$-branch of $F_{+4}(y)$ (see Figure 9 for $p = n + 1$) is shown to exist for $p < n + 1 = 1.2$ for $n = 0.2$.

It is key that this branch cannot be extended for all $1 < p < n + 1$. We expect that, as $p < n + 1$ decreases, the $p$-branch of $F_{+4}$ meets the $p$-branch of the “geometrically similar” profile $F_{+2,2,2}$ shown in Figure 6(a) (both have two dominant maxima and a
single minimum in between) in a turning saddle-node bifurcation point (another branch scenario is also possible, [23, §7]). Such scenarios were detected in variational problems; see [31]. In the present non-variational case, both analytic and even a reliable numerical description of such bifurcations become much more difficult and still obscure.

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Figure 23. The third $p$-branch of profiles $F_2(y)$ of equation (5.9) for $n = 0.2$ (a); the corresponding deformation of $F_2(y)$ (b).

Figure 24. The deformation of $F_{+4}(y)$ of the ODE (5.9) for $n = 0.2$ and $p = 1.2, 1.195, 1.19, 1.185$.

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DEPARTMENT OF MATH. SCI., UNIVERSITY OF BATH, BATH, BA2 7AY, UK

E-mail address: vag@maths.bath.ac.uk
$m=2$: basic pattern $F_0(y)$ for various $n>0$