Thermocapillary rivulets in the uniformly heated liquid film

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Abstract. The set of equations which describes a non-steady 3D flow of non-isothermal liquid film in the presence of thermocapillary effect is deduced in the long-wave approach. The used model is applicable for moderate Reynolds's numbers Re~10 and does not imply in advance set the profile of temperature in a film. The linear analysis of a stability of a film in relation to perturbations in spanwise direction is carried out and dispersion relations are gained. Nonlinear development of instability is investigated numerically in cases of 2D and 3D flow and appearance of quasistationary rivulet structure. Influence of dimensionless parameteres on characteristic scales of rivulet structures is revealed.

1. Introduction
Non-isothermal flows of liquid film are used in many technological processes. Dependence of the surface tension on temperature (thermocapillary effect) leads to occurrence of the shear stress which influences velocity of a fluid. Owing to thermocapillarity the hydrodynamics and heat transfer appear being interconnected. One of types of flow of liquid layers with a free surface, close to film flow, is rivulet flow which occurs in many technological apparatus. Theoretical studies of stability of a heated film and thermocapillary rivulets are based, as a rule, on long-wave approximation, but use two different approaches to the description of dynamics of film. In the first approach one evolutionary equation for a film thickness of type of equation Benney [1] can be deduced using lubrication approximation and taking into account a shear stress on a film surface owing to thermocapillarity. While deducing the evolutionary equation the temperature profile in a film is considered linear, that implies only conductive heat transfer and convection is neglected. Models [2−5] using the mentioned approach, are applicable only at small values of Reynolds number. In the second approach the Reynolds number is not supposed small, and dynamics of a heated film is described by system of the equations concerning a film thickness, flow rate and temperature of liquid. The theoretical models based on the second approach [6−8], have essentially wider applicability as they are deduced directly from the Navier-Stokes equations using some suppositions concerning a velocity profile and a temperature profile. The detailed review of studies on a problem of a stability of a heated films is available in [9]. The presented paper investigates nonlinear development of small spanwise-directed perturbation in the uniformly heated film which leads to 3D rivulets. Equations of IBL model [10] modified taking into account thermocapillarity together with the equation for temperature of liquid are applied to describe the dynamics of non-isothermal film. The used model is applicable in a wide range of hydrodynamic and thermal parameters and does not imply in advance set the profile of temperature.

2. Theoretical model
Let's consider a three-dimensional film of a viscous liquid flowing on uniformly heated infinite plate inclined under an angle θ to horizon. The plate temperature is equal to $T_W$, the film contacts to the motionless gas which has temperature $T_g$, and heat exchange coefficient on interface is equal to $α$. 
Density $\rho$, kinematics viscosity $\nu$, liquid conductivity $\lambda$, thermal diffusivity $a$ of liquid suppose be constants, but a surface tension is linearly depending on temperature: $\sigma = \sigma_0 - \gamma(T - T_0)$. Condition of absence of slip is satisfied on a wall and shear stresses $\tau_s(x, z, t) = -\nu \frac{\partial T_s}{\partial z}$, $\tau_z(x, z, t) = -\nu \frac{\partial T_z}{\partial z}$ are applied on an interface due to non-uniformity of the temperature $T_s$ on a film surface. The perturbation of the film surface is considered to be long-wave (the typical length of the perturbation is much greater than the film thickness). We introduce a Cartesian coordinate system $Oxyz$ with the $Ox$ axis directed to streamwise direction, $Oy$ axis directed to spanwise direction and the $Oz$ axis directed along the normal to the plate. We choose the unperturbed film thickness $h_m$ as the scale and introduce the scales of velocity $u_m = gh_m^2 / 3\nu$, time $t_m = h_m / u_m$, flow rate $q_m = u_h h_m$, temperature $T_m = T_w - T_e$, and move on to the dimensionless variables $x / h_m$, $z / h_m$, $q / q_m$, $m / q_m$, $t / t_m$, $u / u_m$, $w / u_m$, $(T - T_e) / T_m$, keeping previous letter symbols for all values. In dimensionless variables the film flow is described by the set of equations with respect to film thickness $h(x, z, t)$, flow rate $q(x, z, t)$, $m(x, z, t)$ and equation for temperature $T(x, \eta, t)$ with matching boundary conditions:

$$
\begin{align*}
\frac{\partial q}{\partial t} + \frac{\partial J_{1,2}}{\partial x} + \frac{\partial J}{\partial z} &= \frac{3}{Re_m} \left( h^2 \sin \theta \cos \theta \frac{\partial h}{\partial x} - \frac{m}{2} \frac{\partial T}{\partial x} - \frac{q}{h^2} \right) + We \left( \frac{\partial^3 h}{\partial \eta^3} + \frac{\partial^3 h}{\partial \eta \partial x^2} \right), \\
\frac{\partial m}{\partial t} + \frac{\partial J_{1,2}}{\partial x} + \frac{\partial J}{\partial z} &= -\frac{3}{Re_m} \left( h^2 \cos \theta \frac{\partial h}{\partial x} + \frac{m}{2} \frac{\partial T}{\partial x} + \frac{m}{h^2} \right) + We \left( \frac{\partial^3 h}{\partial \eta^3} + \frac{\partial^3 h}{\partial \eta \partial x^2} \right), \\
\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} + \frac{\partial m}{\partial z} &= 0, \\
\frac{\partial T}{\partial t} + \frac{u}{h} \frac{\partial T}{\partial x} + \frac{w}{h} \frac{\partial T}{\partial z} + \frac{V}{h} \frac{\partial T}{\partial \eta} &= \frac{1}{h^2} \frac{\partial^3 T}{\partial \eta^2}, \\
\left[ \frac{\partial T}{\partial \eta} + Bih \right]_{\eta=0} &= 0, \\
T|_{t=0} &= 1.
\end{align*}
$$

Here $V = -\left( \frac{\partial h}{\partial t} \right) \left( \eta - \frac{3\eta^2}{2} \right) - \frac{Ma}{4} \left( \eta^3 - \eta^4 \right) \left( \frac{\partial h}{\partial \eta} \frac{\partial \partial T}{\partial \eta} \right) + \frac{\partial h}{\partial \eta} \left( \frac{\partial^2 T}{\partial \eta^2} \right), \quad \eta = y / h(x, z, t)$,

$$
\begin{align*}
\frac{u}{h} &= \frac{3\eta^2}{2} - \frac{Ma}{4} \left( \eta^3 - \eta^4 \right), \\
w &= \frac{m}{h} \left( 3\eta^2 - \eta^4 \right), \\
J &= \frac{6q^2}{5h} - \frac{Ma}{4} \frac{h^2 \frac{\partial T}{\partial x}}{20} + \frac{h^3 \left( \frac{Ma \frac{\partial T}{\partial x}}{120} \right)^2}, \\
J_{1,2} &= \frac{6m^2}{5h} - \frac{Ma}{4} \frac{h^2 \frac{\partial T}{\partial x}}{20} + \frac{h^3 \left( \frac{Ma \frac{\partial T}{\partial x}}{120} \right)^2}, \\
T_s(x, z) &= T|_{\eta=1}.
\end{align*}
$$

The heated film flow is determined by the following dimensionless criteria: $We = (3Fi / Re_m^{1/2})$ is the Weber number, $Fi = \sigma^2 / \rho^3 \nu^4$ is the Kapitza number, $Ma = \gamma T_m / \mu a m$ is the Marangoni number, $Bi = a h_m / \lambda$ is the Biot number, $Pe = Re_m Pr$ is the Peclet number. Parameter $Re_m = gh_m^2 / 3\nu^2$ characterizes the thickness of undisturbed film (Reynolds number $Re$, defined by flow rate $Re = Re_m \sin \theta$). Since $h_m = Re_m^{1/3}$, $u_m = Re_m^{2/3}$, then $Bi = Bi^* \cdot Re_m^{1/3}$, $Ma = Ma^* / Re_m^{2/3}$. Here dimensionless parameters $Bi^* = 3^{1/2} \alpha \nu^{1/3} / \lambda g^{1/3}$ and $Ma^* = 3^{1/2} \gamma (T_w - T_e) / \rho g^{1/3} \nu^{4/3}$ are determined only by the liquid properties and heating conditions. To unperturbed flow there corresponds solution

$$
h_0 = 1, \quad q_0 = \sin \theta, \quad m_0 = 0, \quad T_0(\eta) = 1 - \eta Bi / (1 + Bi).
$$
3. Stability and nonlinear development of perturbation in 2D statement

Let’s consider the perturbed flow independent on coordinate $x$. In this case the flow remains quasi-three-dimensional (the velocity component in the stream direction $u$ also undergoes to perturbation), but a velocity field is identical in all sections of a film. Let’s investigate stability of a base flow (2) concerning periodic on coordinate $z$ perturbations with the set period $L_z$. Let’s put $H^h = 1$, $Q^q = 10$, $m^m = 0$, $T^T = 0$. Linearizing the equations (1) relative to small perturbations $H(z,t)$, $Q(z,t)$, $m(z,t)$, $T(t,z)$, we obtain

\[
\frac{\partial H}{\partial t} + \frac{6}{5} \frac{\partial \bar{m}}{\partial z} - \frac{Ma}{40} \frac{\partial \bar{T}}{\partial z}^2 = \frac{3}{Re_n} (3H - Q),
\]

\[
\frac{\partial \bar{m}}{\partial t} = -\frac{3}{Re_n} \left( \frac{\cos \theta \partial H}{\partial z} + \frac{Ma}{2} \frac{\partial \bar{T}}{\partial z} + \bar{m} \right) + We \frac{\partial \bar{H}}{\partial z},
\]

\[
\frac{\partial \bar{T}}{\partial t} + \frac{\partial \bar{m}}{\partial z} = 0,
\]

\[
\frac{\partial \bar{T}}{\partial t} + \frac{Bi}{1 + Bi} \left( \left( \eta - \frac{3\eta^2}{2} + \eta^4 \right) \frac{\partial H}{\partial t} + \frac{Ma}{4} \left( \eta^2 - \eta^4 \right) \frac{\partial ^2 \bar{T}}{\partial z^2} \right) = \frac{1}{Pe} \frac{\partial ^2 \bar{T}}{\partial z^2}.
\]

(3)

Like it is made in [11], we present perturbations as follows

\[
H(z,t) = H_o \exp(i k (z - ct)), \quad Q(z,t) = Q_o \exp(i k (z - ct)), \quad \bar{m}(z,t) = M_o \exp(i k (z - ct)), \quad \bar{T}(\eta,z,t) = \Phi(\eta) H(z,t) Bi / (1 + Bi).
\]

(4)

(5)

Here $k = 2 \pi h_o / L_z$ is real wave number, $\Phi(\eta) = \Phi_R + i \Phi_I$ is a complex-valued function of coordinate $\eta$. $c = c_R + i c_I$ is a complex velocity of wave (real part $c_R$ determine velocity of propagation and imaginary one $c_I$ determine the time increment $\beta = kc_I$). Substitution (4) and (5) in (3) leads to dispersion relations which were solved by numerical method. All calculations are made for a vertical film of water ($Fr^* = 3418$, $Pr = 7$).

The curves $\beta(k)$ are shown in figure 1 at $Re_n = 5$, $Bi^* = 0.2$ and various values of $Ma^*$. From figure it is visible, that the increase in parameter $Ma^*$ leads to growth of the maximum increment and range extension of a wave number of instability. In figure 2 the neutral wave number $k_n$ is shown depending

![Figure 1](image1.png)

**Figure 1.** Increment in dependence of wave number at $Re_n = 5$, $Bi^* = 0.2$ and different $Ma^*$: 50 (1), 75 (2), 100 (3).

![Figure 2](image2.png)

**Figure 2.** Neutral wave number versus $Ma^*$ at $Bi^* = 0.2$ and different $Re_n$: 1 (1), 5 (2), 10 (3).
on $Ma^*$ at $Bi^* = 0.1$ and different values of $Re_m$. From figure it is visible, that growth of parameter $Re_m$ leads to expansion of area of instability.

To investigate a nonlinear stage of instability development the equations (1) are solved numerically (for 2D flows at $\partial / \partial x = 0$) by finite difference method described in [11, 12]. In an initial instant a perturbed film thickness $h(z,0)=1 + 0.05 \cos(kz)$ and unperturbed values $q_0$, $m_0$, $T_0$ were set.

Calculations were fulfilled for an interval $0 \leq z \leq L_z$, at interval boundaries the periodicity conditions were set. The development of perturbation is shown in figures 3 and 4 at $L_z = 0.03 \text{ m}$, $Re_m = 5$, $Ma^* = 70$, $Bi^* = 0.2$. From figure 3 it is visible, that two humps on interval edges grow in an initial stage of development, and in the middle of an interval (in an initial trough) additional hump appears with two new small humps from both sides (a curve 4 on fig. 3). In regions between humps new local minimums of film thickness appear (see fig. 4). Let's note, that the same development of instability was observed in 2D full-scale direct numerical simulation [13] fulfilled for a horizontal heated film.

Further perturbation development transfers in an asymptotic stage, on which a film thickness between humps and amplitude of humps do not change essentially (see curves on fig. 4 at instants of time 31.25, 32.5, 33.75, 35). The height of humps increases a little, and the minimal film thickness $h_{\text{min}}$ on segments between humps decreases owing to gradual draining of a liquid in a direction of humps. The effect of unlimited decreasing of $h_{\text{min}}$ should be interpreted eventually as a thermocapillary rupture of a film.

4. Formation of 3D rivulet structure

The space development of periodic on $z$ perturbations with wave length $L_z$ was simulated by means of the solution of the equations (1) by finite difference method. The numerical algorithm is analogous one applied in [11, 12] for two-dimensional waves in the non-isothermal film. Initial conditions were set as small perturbation of a film thickness but unperturbed values of the flow rate and temperature

$$h = 1 + H_a \cos(kz), \quad q = q_0, \quad m = m_0, \quad T = T_0$$ (6)

Here $H_a$ is small amplitude of initial perturbation. The calculation area represented a rectangle $0 \leq x \leq X_{\text{end}}, \quad 0 \leq z \leq L_z$. The size of calculation area in stream direction $X_{\text{end}}$ was large enough that it was possible to trace perturbation development downwards on a stream. On an inlet (i.e. at $x = 0$) the conditions (6) were supported, and on boundaries $z = 0$ and $z = L_z$ the periodicity conditions were set. Such statement of problem corresponds to real conditions of flow for a locally heated film in experiments [14-16] where developed stationary thermocapillary rivulets are observed. On an exit (at

![Figure 3](image-url)  
**Figure 3.** Development of perturbation at $L_z = 3 \cdot 10^{-2} \text{ m}$, $Re_m = 5$, $Ma^* = 70$, $Bi^* = 0.2$. Curves: $t = 22.5$ (1), 25 (2), 27.5 (3), 30 (4)).

![Figure 4](image-url)  
**Figure 4.** Development of perturbation at $L_z = 3 \cdot 10^{-2} \text{ m}$, $Re_m = 5$, $Ma^* = 70$, $Bi^* = 0.2$ on an asymptotic stage
during calculation the three-dimensional waves appear in a film in an initial stage and then escape downwards on a stream, leaving the calculation area. The solution of the equations (1) which corresponds to a stationary flow was found by means of a transient method. Calculation was terminated if the three-dimensional structure appears which does not change during time.

In fig. 5 the development of perturbation with a wavelength $0.01 \text{ m}$ is shown (two periods are shown) at $\text{Re}_m = 5$, $\text{Ma}^* = 100$, $\text{Bi}^* = 0.2$. Change in a film section downwards on a stream qualitatively corresponds to evolution in time of a film in 2D simulation. From a figure it is visible, that a region on which there is a growth of amplitude of initial perturbation, has length approximately 16 sm. Downwards on a stream (approximately from 16 sm to 20 sm) the high amplitude rivulets are formed, separated by a segments of thin film where additional rivulet of small amplitude appears. Rivulets formation occurs because of reallocations of flow rate $q$ owing to the thermocapillary shear stress acting in the transverse direction. Owing to inflow of a liquid to humps from thin bridges between rivulets the thickness of bridges gradually decreases downwards on a stream. Nevertheless, because of small value of this inflow the amplitude of developed rivulets increases inappreciably.

Figure 5. Development of 3D rivulets down the stream at $L_z = 1.5 \cdot 10^{-2} \text{ m}$, $\text{Re}_m = 5$, $\text{Ma}^* = 100$, $\text{Bi}^* = 0.2$

Figure 6. Development of 3D rivulets down the stream at $L_z = 2 \cdot 10^{-2} \text{ m}$, $\text{Re}_m = 10$, $\text{Ma}^* = 70$, $\text{Bi}^* = 0.2$
increment for conditions in fig. 6 almost same as for conditions in fig. 5, therefore qualitatively and quantitatively the rivulet structure on fig. 6 slightly differs from that in fig. 5. The increase twice values of $R_{em}$ (in comparison with conditions in fig. 5) leads to increase in length of region of a linear development of perturbation (also approximately twice). This effect is quite clear, as parameter $R_{em}$ characterizes the inertia properties of a film flow.

5. Conclusions
In long-wave approach the system of equations which describes a non-stationary three-dimensional flow of heated liquid film at moderate Reynolds number $Re\sim 10$ is deduced. On the basis of these equations the linear analysis of stability of 2D film flow relative to perturbations in the transverse direction is carried out. The simulation of a nonlinear development of small transverse perturbation having wavelength $L_z$ is fulfilled. In 2D calculations it is shown, that except the basic humps corresponding to wavelength $L_z$ the additional humps separated by most thin segments develop in the film. Further an asymptotic stage occurs, on which humps grow insignificantly, and the minimal film thickness $h_{min}$ monotonously decreases and asymptotically tends to be zero. Space development of perturbation on an inlet downwards on a stream was investigated in 3D calculations. It is shown, that downwards on a stream from region of linear development, where the amplitude of initial perturbation grows, the rivulets with high amplitude are formed. Development of a film cross-section in 3D simulations qualitatively matches to a film evolution in a time obtained by 2D simulations. Developed 3D rivulets have quasisteady character which corresponds to an asymptotic stage in 2D simulations. The amplitude of rivulets on an asymptotic region grows slightly with distance, and the minimum film thickness $h_{min}$ between humps gradually decreases and asymptotically tends to be zero. The effect of unlimited decreasing of $h_{min}$ is interpreted as a thermocapillary rupture of a film.

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7. References
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