INTERSECTION NUMBERS OF TWISTED HOMOLOGY AND COHOMOLOGY GROUPS ASSOCIATED TO THE RIEemann-WIRTingen INTEGRAL

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Abstract. The Riemann-Wirtinger integral is an analogue of the hypergeometric integral, which is defined as an integral on a one-dimensional complex torus. We study the intersection forms on the twisted homology and cohomology groups associated with the Riemann-Wirtinger integral. We derive explicit formulas of some intersection numbers, and apply them to study the monodromy representation, connection problems, and contiguity relations.

1. Introduction

The Gauss hypergeometric function \( \ hypergeometric integral is defined as an integral on a one-dimensional complex torus. We study the intersection forms on the twisted homology and cohomology groups associated with the Riemann-Wirtinger integral. We derive explicit formulas of some intersection numbers, and apply them to study the monodromy representation, connection problems, and contiguity relations.

The Riemann-Wirtinger integral \( \int_{\gamma} e^{2\pi \sqrt{-1} \theta_1(u)} \prod_{i=1}^{n} \theta_1(u-t_i) c_i \theta_1(u-t_n) \theta_1(\lambda) du \), where \( \gamma \) is a twisted cycle, and \( c_0, c_1, \ldots, c_n, \lambda \in \mathbb{C} \). For the theta function \( \theta_1 \), see \( \tau \). When \( \lambda \in \mathbb{Z} \), we replace \( \theta_1(u-t_j; \lambda) \) in \( \tau \) by 1. The Wirtinger integral (e.g., \( \tau \)) is obtained as an example by setting \( n = 4 \) and \( \lambda = 0 \). The integrand of \( \tau \) can be regarded as a multi-valued function on a complex torus minus \( n \) points for which we write \( M \). It defines a local system \( \mathcal{L}_{\lambda} \) on \( M \) and its dual \( \mathcal{L}_{\lambda}^\vee \). Thus, we can study the Riemann-Wirtinger integral in terms of the twisted cohomology group \( H^1(M; \mathcal{L}_{\lambda}) \) and the twisted homology group \( H_1(M; \mathcal{L}_{\lambda}^\vee) \). The structures of these groups are precisely studied in \( \tau \). Based on the results of \( \tau \), we study the intersection theory on these twisted homology and cohomology groups.

In this paper, we compute the intersection numbers for various types of twisted (co)cycles. Moreover, we show that the intersection form on the twisted homology (resp. cohomology) group is useful to
study the monodromy or connection problems (resp. contiguity relations) which are considered in [10]. We can reduce these problems to studying certain linear operators on twisted (co)homology groups. We focus on twisted (co)cycles whose changes by a corresponding operator are easily described. Then this operator can be expressed by using the intersection numbers with such (co)cycles. An advantage of our expressions is that they do not depend on the choice of a basis. By our expressions, we can recover the connection and contiguity matrices given in [10]. Our approaches are analogies of [12] and [6]. The idea of using the intersection form to study the connection problems is also provided in [15], which is slightly different from ours.

In fact, some intersection numbers for twisted homology groups are computed in [4]. Though we compute the intersection numbers of more cycles than [4], our results for the intersection numbers themselves are not essentially new. However, we show utility of the intersection form to study properties of the Riemann-Wirtinger integral, which is a different viewpoint from [4]. Further, we believe that it is the first time to study the intersection theory of twisted cohomology groups for our settings.

This paper is arranged as follows. In Section 2, we review basic properties of the theta function, and results of [14] for the twisted homology and cohomology groups associated with the Riemann-Wirtinger integral. In Section 3, we give the intersection numbers of some twisted cycles, and apply them to study of the monodromy and connection problems. In Section 4, we give the intersection numbers of some twisted cocycles, and apply them to study of the contiguity relations. Precise computation of these intersection numbers is given in Section 5.

2. Preliminaries

In this section, we review basic facts about a theta function and twisted homology and cohomology groups, which we will use throughout this paper.

2.1. Theta function. We define a theta function

\[ \vartheta_1(u) = \vartheta_1(u, \tau) = -\sum_{m \in \mathbb{Z}} \exp \left( \pi \sqrt{-1} \left( m + \frac{1}{2} \right)^2 \tau + 2 \pi \sqrt{-1} \left( m + \frac{1}{2} \right) \left( u + \frac{1}{2} \right) \right), \]

where \( z \in \mathbb{C} \) and \( \tau \in \mathbb{H} \). In this paper, we fix \( \tau \in \mathbb{H} \) and we frequently use the notation \( \vartheta_1(u) \). We note that \( \vartheta_1(u, \tau) \) defined here is equal to \( -\vartheta_1(u, \tau) \) which is introduced in [10]. It is well-known that \( \vartheta_1(u) \) is an odd function, has a simple zero at \( u = 0 \), and has the quasi-periodicity

\[ \vartheta_1(u + 1) = -\vartheta_1(u) = e^{\pi \sqrt{-1}} \vartheta_1(u), \quad \vartheta_1(u + \tau) = -e^{\pi \sqrt{-1} \tau} \vartheta_1(u) = e^{\pi \sqrt{-1} \tau + 2} \vartheta_1(u). \]

We also introduce the following two functions:

\[ \rho(u) = \frac{\vartheta_1'(u)}{\vartheta_1(u)}, \quad \mathfrak{s}(u; \lambda) = \frac{\vartheta_1(u - \lambda) \vartheta_1'(0)}{\vartheta_1(u) \vartheta_1(-\lambda)}, \]

where \( \vartheta_1'(u) = \frac{d}{du} \vartheta_1(u) \) and \( \lambda \in \mathbb{C} - \mathbb{Z} \). Note that \( \rho(u) \) is an odd function, and \( \mathfrak{s}(u; \lambda) \) has the quasi-periodicity

\[ \mathfrak{s}(u + 1; \lambda) = \mathfrak{s}(u; \lambda), \quad \mathfrak{s}(u + \tau; \lambda) = e^{2 \pi \sqrt{-1} \lambda} \mathfrak{s}(u; \lambda). \]

As a function in \( \lambda \), the Laurent expansion of \( \mathfrak{s}(u; \lambda) \) at \( \lambda = 0 \) is given as

\[ \mathfrak{s}(u; \lambda) = -\frac{1}{\lambda} + \rho(u) + \cdots. \tag{2.1} \]

The following formulas are not difficult, and can be regarded as analogies of the pole-zero cancellation and the partial fraction decomposition in the rational functions field.

Lemma 2.1 (cf. [10] (38),(45)]. Suppose that \( t_j, t_k, t_l \in \mathbb{C} \) are distinct points of \( \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) \). We have

\[ \mathfrak{s}(u - t_k; \lambda) \left( \rho(u - t_j) + \rho(t_j - t_k) - \rho(u - t_k - \lambda) - \rho(\lambda) \right) = \mathfrak{s}(u - t_j; \lambda) \mathfrak{s}(t_j - t_k; \lambda), \tag{2.2} \]
\[
\frac{\vartheta_1(u - t_k)}{\vartheta_1(u - t_i)}g(u - t_j; \lambda - t_k + t_i) = \frac{\vartheta_1(t_j - t_k)}{\vartheta_1(t_j - t_i)}g(u - t_j; \lambda) + \frac{\vartheta_1(t_k - t_i)}{\vartheta_1(t_j - t_i)}\vartheta_1(\lambda - t_k + t_j)g(u - t_i; \lambda).
\]

(2.3)

2.2. Twisted homology and cohomology groups. For a fixed \(\tau \in \mathbb{H}\), we set \(\Lambda_\tau = \mathbb{Z} + \tau \mathbb{Z}\) and \(E = \mathbb{C}/\Lambda_\tau\). For \(\lambda \in \mathbb{C}\), we can define a one-dimensional representation \(e_\lambda : \pi_1(E) \rightarrow \mathbb{C}^*\) of the fundamental group \(\pi_1(E)\) by the correspondence \(e_\lambda(1) = 1, e_\lambda(\tau) = e^{2\pi i \tau \lambda}\). Let \(R_\lambda\) be the local system on \(E\) determined by this representation \(e_\lambda\).

Let \(n \geq 2\) and let \(t_1, \ldots, t_n\) be distinct points in \(E\). We set \(D = \{t_1, \ldots, t_n\}\) and \(M = E - D\). Let \(\mathcal{O}_E(*D)\) (resp. \(\Omega^1_E(*D)\)) be the sheaf of functions (resp. 1-forms) meromorphic on \(E\) and holomorphic on \(M\). We set \(\mathcal{O}_\lambda(*D) = \mathcal{O}_E(*D) \otimes_{\mathbb{C}} R_\lambda\) and \(\Omega^1_\lambda(*D) = \Omega^1_E(*D) \otimes_{\mathbb{C}} R_\lambda\). We define a multi-valued function \(T(u)\) on \(M\) by

\[
T(u) = e^{2\pi \sqrt{-1} t_0} \vartheta_1(u - t_1)^{c_1} \cdots \vartheta_1(u - t_n)^{c_n},
\]

where \(t_0 \in \mathbb{C}\), and \(c_1, \ldots, c_n \in \mathbb{C} - \mathbb{Z}\) satisfy \(c_1 + \cdots + c_n = 0\). Let \(\mathcal{L}\) and \(\mathcal{L}'\) be the local systems on \(M\) defined by \(T(u)^{-1}\) and \(T(u)\), respectively: \(\mathcal{L} = \mathbb{C}T(u)^{-1}\) and \(\mathcal{L}' = \mathbb{C}T(u)\). We set \(L_\lambda = \mathcal{L} \otimes_{\mathbb{C}} R_\lambda\) and \(L_\lambda' = \mathcal{L}' \otimes_{\mathbb{C}} R_\lambda\). Note that if \(\lambda \in \mathbb{Z}\), we have \(L_\lambda = \mathcal{L}\) and \(L_\lambda' = \mathcal{L}'\). Let us consider the homology group \(H_1(M; L_\lambda)\) and cohomology group \(H^1(M; L_\lambda)\) which are called the twisted homology group and twisted cohomology group, respectively. Recall that the twisted homology group is defined as \(H_1(M; L_\lambda) = Z_1(M; L_\lambda')/B_1(M; L_\lambda')\), where \(Z_1\) and \(B_1\) are the kernel and image of the boundary operators for the twisted chains, respectively (e.g., [1]). We set \(\omega = d \log T(u) \in \Omega^1_M(M)\).

Fact 2.2 ([14]). If \(i \neq 1\), then \(H_i(M; L_\lambda') = 0\) and \(H^i(M; L_\lambda) = 0\). We have

\[
\text{dim } H_1(M; L_\lambda') = \text{dim } H^1(M; L_\lambda) = n, \quad H^1(M; L_\lambda) \cong \Omega^1_\lambda(*D)(E)/\mathcal{O}_\lambda(*D)(E)),
\]

where \(\nabla : \mathcal{O}_\lambda(*D) \rightarrow \Omega^1_\lambda(*D)\) is defined by \(\nabla f = df + f\omega\).

Hereafter, we identify \(H^1(M; L_\lambda)\) with \(\Omega^1_\lambda(*D)(E)/\mathcal{O}_\lambda(*D)(E))\). We call an element in \(Z_1(M; L_\lambda')\) (resp. \(\Omega^1_\lambda(*D)(E))\) a twisted cycle (resp. cocycle). The natural pairing \(H^1(M; L_\lambda) \times H_1(M; L_\lambda') \ni ([\varphi], [\gamma]) \mapsto \int_T u \varphi \in \mathbb{C}\) is non-degenerate and gives the Riemann-Wirtinger integral.

The structure of \(H^1(M; L_\lambda) = \Omega^1_\lambda(*D)(E)/\mathcal{O}_\lambda(*D)(E))\) is precisely studied in [14]. As mentioned in [14], we may assume that \(\lambda \in P = \{a + b\tau \mid 0 \leq a, b < 1\}\) without loss of generality. We set

\[
\varphi_0(u; \lambda) = -\lambda s(u - t_1; \lambda), \quad \varphi_1(u; \lambda) = \frac{\partial}{\partial u}(u - t_1; \lambda),
\]

\[
\varphi_j(u; \lambda) = s(u - t_j; \lambda) - s(u - t_1; \lambda) \quad (j = 2, \ldots, n).
\]

We can interpret \(\varphi_i(u; 0)\) as \(\lim_{\lambda \rightarrow 0} \varphi_i(u; \lambda)\) which converges because of (2.1).

Fact 2.3 ([14]). For any \(\lambda \in P\), the \(n+1\) classes \(\{[\varphi_i(u; \lambda) du]_{i=0, \ldots, n}\) generate \(H^1(M; L_\lambda)\) and satisfy a single relation:

\[
\left(2\pi \sqrt{-1} t_0 - c_1 \rho(\lambda) + \sum_{j=2}^n c_j (s(t_j - t_1; \lambda) - \rho(t_j - t_1))\right)[\varphi_0(u; \lambda) du] + [c_1 - 1]\lambda [\varphi_1(u; \lambda) du] - \sum_{j=2}^n c_j s(t_j - t_1; \lambda) [\varphi_j(u; \lambda) du] = 0.
\]

In particular, we have the following properties.

(i) If \(\lambda \in P - \{0\}\), then \(\{s(u - t_j; \lambda) du\}_{i=1, \ldots, n}\) form a basis of \(H^1(M; L_\lambda)\). Since \(s(u - t_j; \lambda) du\) has a simple pole at \(u = t_j\), each element in \(H^1(M; L_\lambda)\) is represented by a 1-form whose poles are of order 1.

\footnote{As mentioned in [4], the relation written in [14] p. 3876 is not correct. The differences are \(\sim -\rho(t_j - t_1)\) in the coefficient of \([\varphi_0(u; \lambda) du]\) and \(c_j\) in the coefficient of \([\varphi_j(u; \lambda) du]\).}
(ii) When \( \lambda = 0 \), we have
\[
\varphi_0(u; 0)du = du, \quad \varphi_1(u; 0)du = \rho'(u - t_1)du,
\]
\[
\varphi_j(u; 0)du = (\rho(u - t_j) - \rho(u - t_1))du \quad (j = 2, \ldots, n),
\]
and the 1-forms having only simple poles are not enough to generate \( H^1(M; L_\lambda) \). The 1-form \( \rho'(u - t_1; \lambda)du \) which has a pole of order 2 at \( u = t_1 \) is necessary to give a basis.

Generators of \( H_1(M; L'_\lambda) \) are also given in [14]. We recall them. By definition, each twisted cycle is loaded with the information of the branch of \( T(u)S(u) \) on its support, where \( S(u) \) is a section of \( R_\lambda \).

Let \( l_\infty, l_0, l_2, \ldots, l_n, s_2, \ldots, s_n, m_0, m_1, m_2, m_3 \) be 1-chains drawn in Figure 1. For a fixed branch of \( T(u)S(u) \) at \( P_0 \), the branches on \( l_0, l_2, \ldots, l_n, m_0 \) and \( m_3 \cup m_2 \cup m_1 \) are naturally defined. The branch on \( s_j \) (resp. \( l_\infty \)) is defined so that the branch at the start point is same as that at the end point of \( l_j \) (resp. start point of \( m_1 \)). We note the images of some twisted 1-chains under the boundary operator:
\[
\partial(m_1 + m_2 + m_3 + m_0) = (e^{2\pi\sqrt{-1}c_1} - 1)P_1,
\]
\[
\partial(m_2 + m_3 + l_0) = (e^{2\pi\sqrt{-1}c_1} - 1)P_2,
\]
\[
\partial(-m_2 - m_1 + l_\infty) = (e^{-2\pi\sqrt{-1}c_\infty} - 1)P_3,
\]
where we put \( c_\infty = -\lambda - c_0 t_1 - c_1 t_1 - \cdots - c_n t_n \). We define \( n + 1 \) twisted cycles\(^2\) as the regularization of paths joining points in \( D \):

\[
\gamma_{1j} = \text{reg}(t_1, t_j) = \frac{m_0 + e^{2\pi\sqrt{-1}c_1}(m_1 + m_2 + m_3)}{e^{2\pi\sqrt{-1}c_1} - 1} + l_j - \frac{s_j}{e^{2\pi\sqrt{-1}c_j} - 1} \quad (j = 2, \ldots, n),
\]
\[
\gamma_{10} = \text{reg}(t_1, t_1 + 1) = \frac{m_0 + e^{2\pi\sqrt{-1}c_1}(m_1 + m_2 + m_3)}{e^{2\pi\sqrt{-1}c_1} - 1} + l_0 - e^{2\pi\sqrt{-1}c_0}(m_2 + m_3 + m_0) + e^{2\pi\sqrt{-1}c_1}m_1
\]
\[
\quad \quad \quad \quad \quad \quad = l_0 + \frac{(1 - e^{2\pi\sqrt{-1}c_0})m_0 + (1 - e^{2\pi\sqrt{-1}c_0})e^{2\pi\sqrt{-1}c_1}m_1 + (e^{2\pi\sqrt{-1}c_1} - e^{2\pi\sqrt{-1}c_0})(m_2 + m_3)}{e^{2\pi\sqrt{-1}c_1} - 1},
\]
\[
\gamma_{1\infty} = \text{reg}(t_1, t_1 + \tau) = \frac{m_1 + m_2 + m_3 + m_0}{e^{2\pi\sqrt{-1}c_1} - 1} + l_\infty - e^{2\pi\sqrt{-1}c_\infty}(m_3 + m_0) + e^{2\pi\sqrt{-1}c_1}(m_1 + m_2)
\]
\[
\quad \quad \quad \quad \quad \quad = l_\infty + \frac{(1 - e^{-2\pi\sqrt{-1}c_\infty})(m_3 + m_0) + (1 - e^{2\pi\sqrt{-1}c_1 - c_\infty})(m_1 + m_2)}{e^{2\pi\sqrt{-1}c_1} - 1}.
\]

\(^2\)As mentioned in [14], we note that the coefficient of \( (m_0 + e^{2\pi\sqrt{-1}c_1}m_1) \) in \( \gamma_0 \) of [13] p. 3877 should be \((1 - e^{2\pi\sqrt{-1}c_0})/(e^{2\pi\sqrt{-1}c_1} - 1)\).
Fact 2.4 ([14]). The twisted homology group \( H_1(M; \mathcal{L}_\lambda^\gamma) \) is generated by \( \{[\gamma_j]\}_{j=2,\ldots,n,0,\infty} \), and these generators satisfy a single \( \mathbb{C} \)-linear relation
\[
(e^{2\pi \sqrt{-1}c_0} - 1)[\gamma_1] + (1 - e^{-2\pi \sqrt{-1}c_\infty})[\gamma_{10}] - \sum_{j=2}^{n} e^{-2\pi \sqrt{-1}(c_1+\cdots+c_j)}(1 - e^{2\pi \sqrt{-1}c_j})[\gamma_j] = 0. \tag{2.4}
\]

3. Intersection Theory for Twisted Homology Group

3.1. Intersection form. The homology intersection form \( I_h \) is a non-degenerate bilinear form between \( H_1(M; \mathcal{L}_\lambda^\gamma) \) and \( H_1(M; \mathcal{L}_\lambda) \):
\[
I_h(\bullet, \bullet) : H_1(M; \mathcal{L}_\lambda^\gamma) \times H_1(M; \mathcal{L}_\lambda) \to \mathbb{C}.
\]

For a twisted cycle \( \gamma \in Z_1(M; \mathcal{L}_\lambda^\gamma) \), we can construct \( \gamma^\vee \in Z_1(M; \mathcal{L}_\lambda) \) by replacing \( (\lambda, c_0, c_1, \ldots, c_n) \) with \( (-\lambda, -c_0, -c_1, \ldots, -c_n) \) and hence \( c_\infty \) is also replaced by \( -c_\infty \).

According to [9], the intersection number \( I_h([\gamma], [\delta]) \) for \( [\gamma] \in H_1(M; \mathcal{L}_\lambda^\gamma) \) and \( [\delta] \in H_1(M; \mathcal{L}_\lambda) \) is evaluated as follows. We may assume that the twisted cycles \( \gamma \) and \( \delta \) are expressed as
\[
\gamma = \sum_i a_i \cdot \Delta_i \otimes (TS)_{\Delta_i}, \quad \delta = \sum_j a'_j \cdot \Delta'_j \otimes (TS)_{\Delta'_j}^{-1}, \quad (a_i, a'_j) \in \mathbb{C},
\]
where \( \Delta_i \) and \( \Delta'_j \) are simply connected and the intersection \( \Delta_i \cap \Delta'_j \) is at most one point at which they intersect transversally, and \( (TS)_{\Delta_i} \) denotes the branch of \( T(u)S(u) \) on \( \Delta_i \). Then the intersection number is evaluated as
\[
I_h([\gamma], [\delta]) = \sum_{\Delta_i \cap \Delta'_j = \{u_{ij}\}} a_i a'_j \cdot I_{\text{loc}}(\Delta_i, \Delta'_j) \cdot (TS)_{\Delta_i}(u_{ij}) \cdot (TS)_{\Delta'_j}(u_{ij})^{-1},
\]
where \( I_{\text{loc}} \) is the local intersection multiplicity.

Note that we often consider \( c_j \) as an indeterminate and we can regard an intersection number as an element in the field \( \mathbb{C}(e^{2\pi \sqrt{-1}c_0}, e^{2\pi \sqrt{-1}c_1}, \ldots, e^{2\pi \sqrt{-1}c_n}) \). In such a situation, for \( [\gamma], [\delta] \in H_1(M; \mathcal{L}_\lambda^\gamma) \), we have \( I_h([\gamma], [\delta^\vee]) = -I_h([\delta], [\gamma^\vee])^\vee \), where the last \( \vee \) stands for the involution on \( \mathbb{C}(e^{2\pi \sqrt{-1}c_0}, e^{2\pi \sqrt{-1}c_1}, \ldots, e^{2\pi \sqrt{-1}c_n}) \) given by \( (c_1, c_2, c_3, c_n) \mapsto (-c_1, -c_2, -c_3, -c_n, -c_\infty) \).

3.2. Intersection numbers. In this section, we give formulas of the intersection numbers for the twisted cycles introduced in \([2,2,2]\). Though they are computed in [14] essentially, we rewrite them in our notations. We also define other twisted cycles which will be used in \([3,3,3]\) and give their intersection numbers. Precise computations will be given in \([3,1,1]\).

Fact 3.1 ([14] Proposition 3.4.1). For \( j, k \in \{2, \ldots, n\} \), we have
\[
\begin{align*}
I_h([\gamma_{1j}], [\gamma_{1k}]) &= \begin{cases} 
\frac{e^{2\pi \sqrt{-1}c_1}}{1 - e^{-2\pi \sqrt{-1}c_1}} & (j < k) \\
\frac{1}{1 - e^{-2\pi \sqrt{-1}c_1}} & (j > k), 
\end{cases} \\
I_h([\gamma_{1j}], [\gamma_{10}]) &= \frac{e^{2\pi \sqrt{-1}c_0}}{1 - e^{-2\pi \sqrt{-1}c_1}}, \quad I_h([\gamma_{10}], [\gamma_{1j}]) = \frac{1 - e^{-2\pi \sqrt{-1}c_0}}{1 - e^{-2\pi \sqrt{-1}c_1}}, \\
I_h([\gamma_{1j}], [\gamma_{1\infty}]) &= \frac{e^{2\pi \sqrt{-1}c_\infty}}{1 - e^{-2\pi \sqrt{-1}c_1}}, \quad I_h([\gamma_{1\infty}], [\gamma_{1j}]) = \frac{1 - e^{-2\pi \sqrt{-1}c_\infty}}{1 - e^{-2\pi \sqrt{-1}c_1}}, \\
I_h([\gamma_{10}], [\gamma_{1\infty}]) &= -\frac{(e^{2\pi \sqrt{-1}c_0} - 1)(e^{2\pi \sqrt{-1}c_0} - e^{2\pi \sqrt{-1}c_1})}{e^{2\pi \sqrt{-1}c_0}(1 - e^{-2\pi \sqrt{-1}c_1})}, \\
I_h([\gamma_{1\infty}], [\gamma_{1\infty}]) &= \frac{(e^{2\pi \sqrt{-1}c_\infty} - 1)(e^{2\pi \sqrt{-1}c_\infty} - e^{2\pi \sqrt{-1}c_1})}{e^{2\pi \sqrt{-1}c_\infty}(1 - e^{-2\pi \sqrt{-1}c_1})}, \\
I_h([\gamma_{10}], [\gamma_{1\infty}]) &= \frac{e^{2\pi \sqrt{-1}c_1} - e^{2\pi \sqrt{-1}(c_0+c_1)} + e^{2\pi \sqrt{-1}(c_1+c_\infty)} + e^{2\pi \sqrt{-1}(c_0+c_\infty)}}{1 - e^{2\pi \sqrt{-1}c_1}},
\end{align*}
\]
with a tridiagonal matrix $H$.

In particular, the determinant of the intersection matrix $H_{11} = (I_h([\gamma_j], [\gamma_k^i]))_{j,k=2,\ldots,n-1,0,\infty}$ is equal to

$$1 - e^{-2\pi\sqrt{-1}c_n} \cdot (1 - e^{2\pi\sqrt{-1}c_1}) \cdots (1 - e^{2\pi\sqrt{-1}c_{n-1}}) \cdot \det H_1 = 1 - e^{-2\pi\sqrt{-1}c_n} \cdot (1 - e^{2\pi\sqrt{-1}c_1}) \cdots (1 - e^{2\pi\sqrt{-1}c_{n-1}}).$$

This determinant formula implies that $\{[\gamma_j]\}_{j=2,\ldots,n-1,0,\infty}$ give a basis of $H_1(M;\mathcal{L}_0^\chi)$ under the condition $c_1,\ldots,c_n \notin \mathbb{Z}$. This is another approach to \cite[Theorem 3.1]{14}.

Remark 3.2. Let $H_{11,0\infty} = (I_h([\gamma_j], [\gamma_k^i]))_{j,k=0,\infty}$ be a $2 \times 2$ submatrix of $H_{11}$. By straightforward calculation, we have $\det(H_{11,0\infty}) = 1$. Furthermore, if we put $c_0 = c_\infty = 0$, then we have $H_{11,0\infty} = (0 \ 1 \ 1 \ 0)$. Indeed, $\{\gamma_0, \gamma_\infty\}$ can be naturally identified with a symplectic basis of the usual homology group $H_1(E;\mathbb{Z})$, since the coefficients of $m_0, m_1$ (resp. $m_0, m_3$) in the definition of $\gamma_0$ (resp. $\gamma_\infty$) become zero.

For $2 \leq j < k \leq n$, we set $\gamma_{jk} = \text{reg}(t_j, t_k) = \gamma_{1k} - \gamma_{1j}$. As a corollary of Fact 3.1, we easily obtain the following.

**Corollary 3.3.** For $j,j',k,k' \in \{1,\ldots,n\}$ satisfying $j < k$, $j' < k'$, the intersection number $I_h([\gamma_{jk}], [\gamma_{j'k'}])$ is given as follows.

- If $(j,k) = (j',k')$, then $I_h([\gamma_{jk}], [\gamma_{j'k'}]) = e^{2\pi\sqrt{-1}(e_{jk} + e_{j'k'})}/(1 - e^{2\pi\sqrt{-1}e_{jk}}).$
- If $j = j'$, then $I_h([\gamma_{jk}], [\gamma_{j'k'}]) = \begin{cases} e^{2\pi\sqrt{-1}e_{jk}}/e_{jk} & (k < k') \\ 1 - e^{2\pi\sqrt{-1}e_{jk}}/e_{jk} & (k > k'). \end{cases}$
- If $k = k'$, then $I_h([\gamma_{jk}], [\gamma_{j'k'}]) = \begin{cases} e^{2\pi\sqrt{-1}e_{jk}}/e_{jk} & (j < j') \\ 1 - e^{2\pi\sqrt{-1}e_{jk}}/e_{jk} & (j > j'). \end{cases}$
- If $k = k'$, then $I_h([\gamma_{jk}], [\gamma_{j'k'}]) = e^{2\pi\sqrt{-1}e_{jk}}/e_{jk}$.
- If $j < j' < k < k'$ (resp. $j < j' < k < k'$), then $I_h([\gamma_{jk}], [\gamma_{j'k'}]) = -1$ (resp. 1).
- Otherwise, $I_h([\gamma_{jk}], [\gamma_{j'k'}]) = 0$.

If $2 \leq j < k \leq n$, then we have

$$I_h([\gamma_{jk}], [\gamma_{00}]) = I_h([\gamma_{0k}], [\gamma_{00}]) = I_h([\gamma_{00}], [\gamma_{jk}]) = I_h([\gamma_{00}], [\gamma_{j'k'}]) = 0.$$  

Let $H_1$ be the intersection matrix for $\gamma_{23}, \gamma_{34}, \ldots, \gamma_{n-1,n}, \gamma_{10}, \gamma_{1\infty}$. Then $H_1$ is of the form $(H'_1 \ 0 \ H_{11,0\infty})$ with a tridiagonal matrix $H'_1$. As evaluated in \cite[Remark 2.2]{13}, we have

$$\det H_1' = \frac{1 - e^{-2\pi\sqrt{-1}(c_2 + \cdots + c_n)}}{1 - e^{2\pi\sqrt{-1}c_1} \cdots (1 - e^{2\pi\sqrt{-1}c_{n-1}})} = \frac{1 - e^{-2\pi\sqrt{-1}c_1}}{1 - e^{2\pi\sqrt{-1}c_2} \cdots (1 - e^{2\pi\sqrt{-1}c_{n-1}})},$$

and hence, $\{\gamma_{23}, \gamma_{34}, \ldots, \gamma_{n-1,n}, \gamma_{10}, \gamma_{1\infty}\}$ also give a basis of $H_1(M;\mathcal{L}_0^\chi)$ under the condition $c_1,\ldots,c_n \notin \mathbb{Z}$.

Remark 3.4. (1) Note that $H'_1$ coincides with the intersection matrix for the basis $\{\text{reg}(x_j, x_{j+1})\}_{j=2,\ldots,n-1}$ of the twisted homology group associated with the multi-valued function $(z - x_2)^2 \cdots (z - x_n)^c$ on $\mathbb{P}^1_z - \{x_1 = \infty, x_2, \ldots, x_n\}$ (see, e.g., \cite[Theorem 2.1]{13}).

(2) If $c_1,\ldots,c_n, c_0, c_\infty \in \mathbb{R}$, then $\sqrt{-1}H_1$ is a monodromy invariant Hermitian matrix in the sense of \cite[§2.5]{13}. It is easy to see that the signature of $\sqrt{-1}H_{11,0\infty}$ is $(1,1)$. Thus, $\sqrt{-1}H_1$ is indefinite for any real parameters.
We next define twisted cycles \( \gamma_{j0} \) and \( \gamma_{j\infty} \) (\( 2 \leq j \leq n \)) in a similar manner to that for \( \gamma_{10} \) and \( \gamma_{1\infty} \), respectively. Let \( t_0^{(j)}, t_1^{(j)}, m_0^{(j)}, m_1^{(j)}, m_2^{(j)}, m_3^{(j)} \) be 1-chains drawn in Figure 2. Note that we have \( s_j = m_j^{(j)} + e^{2\pi \sqrt{-1} \epsilon_j} (m_1^{(j)} + m_2^{(j)} + m_3^{(j)}) \). We set

\[
\gamma_{j0} = \text{reg}(t_j, t_j + 1) = \frac{m_0^{(j)} + e^{2\pi \sqrt{-1} \epsilon_j} (m_1^{(j)} + m_2^{(j)} + m_3^{(j)})}{e^{2\pi \sqrt{-1} \epsilon_j} - 1} + t_0^{(j)}
\]

\[
- e^{2\pi \sqrt{-1} \epsilon_0} e^{2\pi \sqrt{-1} (-c_1 - \cdots - c_{j-1})} \cdot \frac{m_2^{(j)} + m_3^{(j)} + m_0^{(j)}}{e^{2\pi \sqrt{-1} \epsilon_j} - 1} + e^{2\pi \sqrt{-1} \epsilon_0} e^{2\pi \sqrt{-1} (-c_1 - \cdots - c_{j-1})} \cdot \frac{m_1^{(j)} + m_0^{(j)}}{e^{2\pi \sqrt{-1} \epsilon_j} - 1}
\]

\[
\gamma_{j\infty} = \text{reg}(t_j, t_j + \tau) = \frac{m_1^{(j)} + m_2^{(j)} + m_3^{(j)} + m_0^{(j)}}{e^{2\pi \sqrt{-1} \epsilon_j} - 1} + t_{\infty}^{(j)}
\]

\[
- e^{2\pi \sqrt{-1} \epsilon_0} e^{2\pi \sqrt{-1} (c_1 + \cdots + c_{j-1})} \cdot \frac{m_3^{(j)} + m_0^{(j)}}{e^{2\pi \sqrt{-1} \epsilon_j} - 1} + e^{2\pi \sqrt{-1} \epsilon_0} e^{2\pi \sqrt{-1} (c_1 + \cdots + c_{j-1})} \cdot \frac{m_1^{(j)} + m_2^{(j)}}{e^{2\pi \sqrt{-1} \epsilon_j} - 1}
\]

**Proposition 3.5.** We set \( H_{00} = (I_h([\gamma_{j0}], [\gamma_{j0}^\vee]))_{j,k=1,\ldots,n} \) and \( H_{0\infty} = (I_h([\gamma_{j\infty}], [\gamma_{j\infty}^\vee]))_{j,k=1,\ldots,n} \). These are diagonal matrices whose diagonal entries are

\[
I_h([\gamma_{j0}], [\gamma_{j0}^\vee]) = \frac{(e^{2\pi \sqrt{-1} \epsilon_0} - e^{2\pi \sqrt{-1} (c_1 + \cdots + c_{j-1})}) (e^{2\pi \sqrt{-1} \epsilon_0} - e^{2\pi \sqrt{-1} (c_1 + \cdots + c_j)})}{e^{2\pi \sqrt{-1} \epsilon_0} - e^{2\pi \sqrt{-1} (c_1 + \cdots + c_j)}}
\]

\[
I_h([\gamma_{j\infty}], [\gamma_{j\infty}^\vee]) = \frac{(e^{2\pi \sqrt{-1} \epsilon_0} - e^{2\pi \sqrt{-1} (c_1 + \cdots + c_{j-1})}) (e^{2\pi \sqrt{-1} \epsilon_0} - e^{2\pi \sqrt{-1} (c_1 + \cdots + c_j)})}{e^{2\pi \sqrt{-1} \epsilon_0} - e^{2\pi \sqrt{-1} (c_1 + \cdots + c_j)}}
\]

This implies that \( \{[\gamma_{j0}]\}_{j=1,\ldots,n} \) (resp. \( \{[\gamma_{j\infty}]\}_{j=1,\ldots,n} \)) form a basis of \( H_1(M; L_0 \chi) \) when we assume the conditions not only \( c_1, \ldots, c_n \not\in \mathbb{Z} \) but also \( c_0 - c_1 - \cdots - c_j \not\in \mathbb{Z} \) (resp. \( c_0 - c_1 - \cdots - c_j \not\in \mathbb{Z} \)) for \( 1 \leq j \leq n \).

**Remark 3.6.** These additional conditions are indispensable. For example, we can easily verify the relation \( |\gamma_{20}| - |\gamma_{10}| = (e^{2\pi \sqrt{-1} (c_0 - c_1)} - 1) |\gamma_{12}| \) which implies \( |\gamma_{20}| = |\gamma_{10}| \) if \( c_0 - c_1 \in \mathbb{Z} \).
induces a linear automorphism \( \ell \) by using intersection numbers:

\[
\gamma_{nj} = \frac{s_n}{e^{2\pi \sqrt{-1} c_n} - 1} + \ell^{(n)} - \frac{e^{2\pi \sqrt{-1} (c_{j+1}+\cdots+c_n)} s_j}{e^{2\pi \sqrt{-1} c_j} - 1}.
\]

Note that \([\gamma_{nj}] \neq -[\gamma_{jn}]\) in general.

**Proposition 3.7.** The matrix \( H_{1n} = (I_h([\gamma_{1j}],[\gamma_{nk}]))_{j=2,\ldots,n-1,0,\infty} \) is a diagonal one whose diagonal entries are given by

\[
I_h([\gamma_{1j}],[\gamma_{nj}]) = \frac{e^{2\pi \sqrt{-1} (c_1+\cdots+c_j)}}{1 - e^{2\pi \sqrt{-1} c_j}}, \quad (j = 2, \ldots, n - 1),
\]

\[
I_h([\gamma_{10}],[\gamma_{n\infty}]) = e^{2\pi \sqrt{-1} (c_0+c_{\infty})}, \quad I_h([\gamma_{1\infty}],[\gamma_{n0}]) = -e^{-2\pi \sqrt{-1} c_0}.
\]

Thus, we can express an arbitrary twisted cycle \( \gamma \) as a linear combination of the basis \([\gamma_{1j}]\) \( j = 2, \ldots, n-1, 0, \infty \) by using intersection numbers:

\[
[\gamma] = \sum_{k=2}^{n-1} \frac{I_h([\gamma],[\gamma_{nk}])}{I_h([\gamma],[\gamma_{nk}])} [\gamma_{1k}] + \frac{I_h([\gamma],[\gamma_{n\infty}])}{I_h([\gamma],[\gamma_{n\infty}])} [\gamma_{10}] + \frac{I_h([\gamma],[\gamma_{n0}])}{I_h([\gamma],[\gamma_{n0}])} [\gamma_{1\infty}]
\]

\[
= \sum_{k=2}^{n-1} \frac{1}{e^{2\pi \sqrt{-1} (c_1+\cdots+c_j)}} I_h([\gamma],[\gamma_{nk}]) [\gamma_{1k}] + e^{2\pi \sqrt{-1} (-c_0-c_{\infty})} I_h([\gamma],[\gamma_{n\infty}]) [\gamma_{10}] - e^{2\pi \sqrt{-1} c_0} I_h([\gamma],[\gamma_{n0}]) [\gamma_{1\infty}].
\]

For example, if we set \( \gamma = \gamma_{1n} \), then this expression yields the relation (2.4).

**3.3. Monodromy and connection problem.** In this section, we consider monodromy and connection problems by moving \( t_p \)'s. Though some of the representation matrices for the basis \([\gamma_{1j}]\) \( j = 2, \ldots, n-1, 0, \infty \) are given in [10] [8], we reconsider them by using the intersection form. To describe the move of \( t_p \)'s, we set

\[
\Xi = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \mid t_j - t_k \notin \Lambda_\tau \ (j \neq k)\}.
\]

We choose a base point \( t^0 = (t^0_1, \ldots, t^0_n) \in \Xi \) such that each \( t^0_j \) belongs to \( P = \{a + b \tau \mid 0 \leq a, b < 1\} \).

**3.3.1. Monodromy.** For \( 1 \leq p < q \leq n \), let \( t_{pq} \) be a loop in

\[
\{(t_{p1}, \ldots, t_{p-1}^0, t_{p+1}^0, \ldots, t_{n}^0) \in \Xi \mid s \in P, \ s \neq t_{pq}^0 \ (j \neq p)\} \subset \Xi
\]

with terminal \( t^0 \) starting from \( s = t^0_p \) approaching to \( t^0_q \) via the right side of the branch cut, turning around this point counterclockwisely, and tracing back to \( t^0_p \) (see Figure 4). The loop \( t_{pq} \) naturally induces a linear automorphism \( t_{pq} : H_1(M; \mathcal{L}^0_\lambda) \to H_1(M; \mathcal{L}^0_\lambda) \) which is called the circuit transformation. We give an expression of \( t_{pq} \) by using the intersection form.

**Theorem 3.8.** For an arbitrary \([\gamma] \in H_1(M; \mathcal{L}^0_\lambda)\), we have

\[
\ell_{pq}([\gamma]) = [\gamma] - (1 - e^{2\pi \sqrt{-1} \tau_p})(1 - e^{2\pi \sqrt{-1} \tau_q}) I_h([\gamma],[\gamma_{pq}]) [\gamma_{pq}].
\]
The proof of this theorem is quite similar to that of [12, Theorem 5.1], since there are few differences between $\mathbb{P}^1$ and $E$ as far as we consider the loop $\ell_{pq}$.

Lemma 3.9. (1) $[\gamma_{pq}]$ is an eigenvector of $\ell_{pq}$ with eigenvalue $e^{2\pi \sqrt{-1}(c_p+c_q)}$.

(2) We set $\gamma_{pq}^+ = \{[\gamma] \in H_1(M,\mathcal{L}_0^+) \mid I_0([\gamma],[\gamma_{pq}]) = 0\}$. Then $\dim(\gamma_{pq}^+) = n - 1$.

(3) Further, $\gamma_{pq}^+$ coincides with the eigenspace of $\ell_{pq}$, with eigenvalue one.

Proof. (1) We can prove this claim in a similar way to the proof of [12, Lemma 5.1].

(2) Since the non-degenerate property of the intersection form implies $\dim(\gamma_{pq}^+) \leq n - 1$, it suffices to find $n - 1$ linearly independent twisted cycles belonging to $\gamma_{pq}^+$. First, we assume $n \geq 3$. If $q \neq n$, then $n - 1$ cycles $\{[\gamma_{nj}]\}_{j=1, \ldots, n-1, 0, \infty}$ belong to $\gamma_{pq}^+$. If $q = n$ and $p \neq 1$, then $\{[\gamma_{n1} - \gamma_{nj}]\}_{j=2, \ldots, n-1} \cup \{[\gamma_{n0}, [\gamma_{n1}]\}}$ belong to $\gamma_{pq}^+$. Thus, in these cases, we can easily check $\dim(\gamma_{pq}^+) = n - 1$. We consider the case when $(p, q) = (1, n)$. We set $\gamma_{20}^+ = e^{2\pi \sqrt{-1}(c_1+c_2)} + (e^{2\pi \sqrt{-1}c_1} - 1)\gamma_{12}$ and $\gamma_{2\infty}^+ = e^{2\pi \sqrt{-1}c_1} (e^{2\pi \sqrt{-1}c_1} - 1)\gamma_{12}$ which are homologous to the twisted cycles drawn in Figure 5. It is not difficult to show that $\{[\gamma_{2j}]\}_{j=3, \ldots, n-1} \cup \{[\gamma_{20}^+], [\gamma_{2\infty}^+]\}$ are linearly independent and belong to $\gamma_{1n}^-$. Thus, we obtain $\dim(\gamma_{1n}^-) = n - 1$.

Next, we consider the case when $n = 2$ and $(p, q) = (1, 2)$. In this case, we have $e^{2\pi \sqrt{-1}(c_1+c_2)} = 1$, and hence $\gamma_{12}$ itself belongs to $\gamma_{12}^+$ by Fact 3.1. Thus, $\gamma_{12}^+$ has a positive dimension, which implies $\dim(\gamma_{12}) = 1$.

(3) If $n \geq 3$, it is clear that each of the above bases of $\gamma_{pq}^+$ is not changed under $\ell_{pq}$. Thus, $\gamma_{pq}^+$ is contained in the eigenspace of $\ell_{pq}$, with eigenvalue one. Since $\ell_{pq}$ is not the identity map, we obtain the claim.

When $n = 2$ and $(p, q) = (1, 2)$, we have $\gamma_{12}^+ = C \cdot [\gamma_{12}]$ by the proof of (2). Thus, $e^{2\pi \sqrt{-1}(c_1+c_2)} = 1$ and (1) shows the claim (3).

Proof of Theorem 3.8. By using Lemma 3.9 and Corollary 3.3, we can show the theorem in the same way as the proof of [12, Theorem 5.1].

Applying Theorem 3.8, we can obtain the representation matrix $M_{pq}$ of $\ell_{pq}$, with respect to the basis $\{[\gamma_{1j}]\}_{j=2, \ldots, n-1, 0, \infty}$. 

![Figure 4. The loop $\ell_{pq}$ (in the s-coordinate).](image1)

![Figure 5. The cycle drawn by solid (resp. dashed) line is homologous to $\gamma_{20}^+(1)$ (resp. $\gamma_{2\infty}^+(1)$).](image2)
Corollary 3.10. We set
\[ v_1 = t(0, \ldots, 0), \quad v_j = t(0, \ldots, 0, 1, 0, \ldots, 0) \quad (j \neq 1, n), \]
\[ v_n = \frac{1}{1 - e^{2\pi \sqrt{-1}c_n}} \left( -e^{-2\pi \sqrt{-1}(c_1+c_2)}(1 - e^{2\pi \sqrt{-1}c_2}), \ldots, -e^{-2\pi \sqrt{-1}(c_1+\cdots+c_{n-1})}(1 - e^{2\pi \sqrt{-1}c_{n-1}}), 1 - e^{2\pi \sqrt{-1}c_0} - 1 \right), \]
and \( v_{jk} = v_k - v_j \). Then \( M_{pq} := \text{id}_n - (1 - e^{2\pi \sqrt{-1}c_p})(1 - e^{2\pi \sqrt{-1}c_q})v_{pq} t^{v_q} t H_{11} \) satisfies
\[ \ell_{pq}([\gamma_{12}], \ldots, [\gamma_{1,n-1}], [\gamma_{10}], [\gamma_{1\infty}]) = ([\gamma_{12}], \ldots, [\gamma_{1,n-1}], [\gamma_{10}], [\gamma_{1\infty}]) M_{pq}, \]
\[ H_{11} = t M_{pq} H_{11} M_{pq}^{-1}, \]
where \( \text{id}_n \) is the unit matrix of size \( n \) and \( H_{11} \) is the intersection matrix defined in Fact 3.7.

Proof. By definition and (2.4), we have \( [\gamma_{pq}] = ([\gamma_{12}], \ldots, [\gamma_{1\infty}])v_{pq} \). If \( [\gamma] \) is expressed as \( [\gamma] = ([\gamma_{12}], \ldots, [\gamma_{1\infty}])v \) for some \( v \), then the intersection number \( I_h([\gamma],[\gamma_{pq}]) \) is given by \( t v_{pq} t H_{11} v \). This implies that \( I_h([\gamma],[\gamma_{pq}]) \) can be expressed as \( ([\gamma_{12}], \ldots, [\gamma_{1\infty}])v_{pq} t v_{pq} t H_{11} v \), and hence we obtain (3.1) by Theorem 3.8. The equality (3.2) follows from the monodromy invariant property \( I_h([\gamma],[\delta]) = I_h([\gamma_{pq}],[\gamma]), \) where \( c_{pq} : H_1(M; L_\lambda) \to H_1(M; L_\lambda) \) is the circuit transformation.

3.3.2. Translating \( t_p \) to \( t_p + 1 \). For \( p = 1, \ldots, n \), let \( \ell_{t_p} \) be a path
\[ [0,1] \ni s \mapsto (t_p^0, \ldots, t_{p-1}, t_p, s t_{p+1}, \ldots, t_n^0) \in \Sigma \]
from \( t_p^0 = (t_p^0, \ldots, t_{p-1}, t_p^0 + 1, t_{p+1}, \ldots, t_n^0) \) (see Figure 6). Following [10], we consider \( c_{\infty} \in C \) as a constant, and we move \( s \) from 0 to 1 keeping the constraint \( \lambda + c_0 \tau + c_1 t_1 + \cdots + c_n t_n + c_{\infty} = 0 \). When \( s = 1 \), the local system \( L_\lambda \) changes into \( L_\lambda^{(t_0^0)} V := CT(u)|_{t_p \to t_p + 1} \otimes R_{\lambda-c_p}^V \). Thus, the loop \( \ell_{t_p} \) induces a linear map \( \ell_{t_p} : H_1(M; L_\lambda^V) \to H_1(M; L_\lambda^{(t_0^0)} V) \). Since we have \( T(u)|_{t_p \to t_p + 1} = e^{-\pi \sqrt{-1} c_p T(u)}, \) the monodromy structure of \( L_\lambda^{(t_0^0)} V \) as a local system on \( M \) is same as that of \( L_\lambda^V |_{\lambda = -c_p} \). Note that the parameter \( c_{\infty} \) for \( L_\lambda^V |_{\lambda = -c_p} \) is \( c_{\infty}' = -(\lambda - c_p) - c_0 \tau - c_1 t_1 - \cdots - c_n t_n = c_{\infty} + c_p \).

For a twisted cycle \( \gamma \in Z_1(M; L_\lambda^V) \), we can construct a twisted cycle \( \gamma' \in Z_1(M; L_\lambda^{(t_0^0)} V) \) in the same manner as \( \gamma \). The “connection problem” in this paper is to express \( \ell_{t_p} ([\gamma]) \)'s in terms of \( [\delta]'s \).

Lemma 3.11. We have
\[ \ell_{t_p} ([\gamma_{j0}]) = \begin{cases} [\gamma_{j0}] & (j < p) \\ e^{2\pi \sqrt{-1}(c_0-c_1-\cdots-c_{p-1})} [\gamma_{j0}] & (j = p) \\ e^{2\pi \sqrt{-1} c_p} [\gamma'_{j0}] & (j > p) \end{cases} \]
Proof. It is clear that the support of \( \ell_{p0*}(\gamma_j) \) is same as that of \( [\gamma_j] \). Thus, it is sufficient to compare the branches at the point \( p_0^{(j)} \). This is not so difficult. □

**Theorem 3.12.** Assume \( c_j, c_0 - c_1 - \cdots - c_j \notin \mathbb{Z} \) \((j = 1, \ldots, n)\). The linear map \( \ell_{p0*} \) is expressed as

\[
\ell_{p0*}([\gamma]) = \sum_{j=1}^{p-1} \frac{I_h([\gamma], [\gamma_j])}{I_h([\gamma_j], [\gamma_j])} [\gamma_j] + \sum_{j=p+1}^{n} \frac{I_h([\gamma], [\gamma_j])}{I_h([\gamma_j], [\gamma_j])} [\gamma_j]
\]

\[
+ e^{2\pi \sqrt{-1}c_0 - c_1 \cdots - c_p - 1} \sum_{j=1}^{p-1} \frac{I_h([\gamma], [\gamma_j])}{I_h([\gamma_j], [\gamma_j])} [\gamma_j] + e^{2\pi \sqrt{-1}c_p} \sum_{j=p+1}^{n} \frac{I_h([\gamma], [\gamma_j])}{I_h([\gamma_j], [\gamma_j])} [\gamma_j]
\]

\[
= - \sum_{j=1}^{p-1} \frac{e^{2\pi \sqrt{-1}c_0 - c_1 \cdots - c_j - 1} (1 - e^{2\pi \sqrt{-1}c_j})}{e^{2\pi \sqrt{-1}c_0 - c_1 \cdots - c_j - 1} (1 - e^{2\pi \sqrt{-1}c_j})} I_h([\gamma], [\gamma_j]) [\gamma_j]
\]

\[
- e^{2\pi \sqrt{-1}c_0} \sum_{j=p+1}^{n} \frac{e^{2\pi \sqrt{-1}c_0 - c_1 \cdots - c_j - 1} (1 - e^{2\pi \sqrt{-1}c_j})}{e^{2\pi \sqrt{-1}c_0 - c_1 \cdots - c_j - 1} (1 - e^{2\pi \sqrt{-1}c_j})} I_h([\gamma], [\gamma_j]) [\gamma_j].
\]

**Proof.** By Proposition 3.5 any element \([\gamma] \in H_1(M; \mathcal{L}_\lambda^\vee)\) can be expressed as

\[
[\gamma] = \sum_{j=1}^{n} \frac{I_h([\gamma], [\gamma_j])}{I_h([\gamma_j], [\gamma_j])} [\gamma_j].
\]

Thus, Lemma 3.11 yields the theorem. □

As a corollary, we obtain the representation matrix with respect to the basis \( \{[\gamma_j]\}_{j=2,\ldots,n-1,0,\infty} \) and \( \{[\gamma'_j]\}_{j=2,\ldots,n-1,0,\infty} \). We use the intersection matrices \( H_{00} \) defined in Proposition 3.3 and \( H_{10} = (I_h([\gamma_j], [\gamma'_j]))_{j=2,\ldots,n-1,0,\infty} \), \( H_{0n} = (I_h([\gamma_j], [\gamma'_j]))_{j=1,\ldots,n} \). \( H_{1n} \) defined in Proposition 3.7. For a matrix \( H \) whose entries belong to \( \mathbb{C}(e^{2\pi \sqrt{-1}c}) \), we set \( H^{(p0)} = H|_{c\to c+ep} \). The matrix defined by

\[
M_{p0} = (H_{1n}^{-1} H_{0n})(p0) \cdot \text{diag}(1, \ldots, 1, e^{2\pi \sqrt{-1}c_0 - c_1 \cdots - c_{p-1}}, \ldots, e^{2\pi \sqrt{-1}c_p}) : H_{00}^{-1} H_{10}
\]

satisfies the following equalities:

\[
\ell_{p0*}([\gamma_{12}], \ldots, [\gamma_{1, n-1}], [\gamma_{10}], [\gamma_{1\infty}]) = ([\gamma_{12}], \ldots, [\gamma_{1, n-1}], [\gamma'_{10}], [\gamma'_{1\infty}]) M_{p0}. \quad (3.3)
\]

\[
H_{11} = \ell_{p0*} \cdot H_{1n}^{(p0)} \cdot M_{p0}. \quad (3.4)
\]

**Proof.** The diagonal matrix in the definition of \( M_{p0} \) is nothing but the representation matrix of \( \ell_{p0*} \) with respect to the basis \( \{[\gamma_j]\}_{j=1,\ldots,n} \) and \( \{[\gamma'_j]\}_{j=1,\ldots,n} \). Since \( H_{1n} \) and \( H_{00} \) are diagonal, it is easy to see that

\[
([\gamma_{10}], \ldots, [\gamma_{n0}]) = ([\gamma_{12}], \ldots, [\gamma_{1, n-1}], [\gamma_{10}], [\gamma_{1\infty}]) H_{1n}^{-1} H_{0n}
\]

and \((H_{1n}^{-1} H_{0n})^{-1} = H_{00}^{-1} H_{10} \). We thus obtain the equality (3.3). Next, we show (3.4). Let \( \ell_{p0} \) be the intersection form defined on \( H_1(M; \mathcal{L}_\lambda^{(p0)\vee}) \times H_1(M; \mathcal{L}_\lambda^{(p0)}) \), and \( \ell_{p0*} : H_1(M; \mathcal{L}_\lambda) \to H_1(M; \mathcal{L}_\lambda^{(p0)}) \) be the linear map induced by \( \ell_{p0} \). By definition of the intersection form, we have \( I_h([\gamma], [\delta]) = \ell_{p0*}([\gamma], [\delta]) \). This implies the equality (3.4). □

Explicit expressions of \( H_{10} \) and \( H_{0n} \) are given in (3.1-3.4). Since the inverse matrices of \( H_{1n} \) and \( H_{00} \) are easily obtained, each matrix in the definition of \( M_{p0} \) has an explicit formula.
Remark 3.14. On the level of integrals, the above discussion is interpreted as follows. We set $p = 1$ for simplicity. By deforming a twisted cycle $\gamma$, the integral (1) changes into
\[
\int_{I_{10}(\gamma)} e^{2\pi\sqrt{-1}c_0u} \vartheta_1(u - t_1 - 1) \cdots \vartheta_1(u - t_n) \cdot \cdot \cdot \vartheta_1(u - t_n) \cdot \cdot \cdot \gamma(u - t'_j; \lambda - c_1) du
= e^{-\pi\sqrt{-1}c_1} \int_{I_{10}(\gamma)} e^{2\pi\sqrt{-1}c_0u} \vartheta_1(u - t_1 - 1) \cdots \vartheta_1(u - t_n) \cdot \cdot \cdot \gamma(u - t'_j; \lambda - c_1) du,
\]
where $t'_j = t_j + \delta_j$. We rewrite the deformed cycle $\ell_{10}(\gamma)$ by using twisted cycles whose coefficients are in the local system defined by this integrand. To avoid the constant $e^{-\pi\sqrt{-1}c_1}$, Mano uses the integrand $\Phi_j(w)$ in [10] instead of $T(u)\vartheta(u - t; \lambda)$. For $\ell_{j\infty}$ in 3.3.3 we can also give a similar interpretation.

3.3.3. Translating $t_p$ to $t_p + \tau$. For $p = 1, \ldots, n$, let $\ell_{j\infty}$ be a path
\[
[0, 1] \ni s \mapsto (t^0_1, \ldots, t^0_{p-1}, t^0_p + s\tau, t^0_{p+1}, \ldots, t^0_n) \in \mathcal{T}
\]
from $t^0$ to $t^0_{\infty} = (t^0_1, \ldots, t^0_{p-1}, t^0_p + \tau, t^0_{p+1}, \ldots, t^0_n)$ (see Figure 6). Similarly to 3.3.2 by considering the constraint $\lambda + c_0\tau + c_1t_1 + \cdots + c_nt_n + c_\infty = 0$, we can obtain a linear map $\ell_{j\infty}: H_1(M; \mathcal{L}_\lambda) \to H_1(M; \mathcal{L}_\lambda^{(p\infty)}\nu)$, where $\mathcal{L}_\lambda^{(p\infty)}\nu := CT(u)|_{t_p\rightarrow t_p + \tau} \otimes R_{\lambda - c_p\tau}$. Since we have $T(u)|_{t_p\rightarrow t_p + \tau} = e^{2\pi\sqrt{-1}c_p(\tau - 2t_p + 1)}$, $e^{2\pi\sqrt{-1}c_u T(u)}$, the monodromy structure of $\mathcal{L}_\lambda^{(p\infty)}\nu$ as a local system on $M$ is same as that of $\mathcal{L}_\lambda|_{(c_0, \lambda) \rightarrow (c_0 + c_\ell, \lambda - c_\ell\tau)}$. Note that the parameter $c_\infty$ is not changed by this replacing.

For a twisted cycle $\gamma \in Z_1(M; \mathcal{L}_\lambda^\nu)$, we also write $\gamma' \in Z_1(M; \mathcal{L}_\lambda^{(p\infty)}\nu)$ for a twisted cycle constructed in the same manner as $\gamma$. The following lemma, theorem, and corollary can be shown similarly to 3.3.2.

Lemma 3.15. We have
\[
\ell_{j\infty}([\gamma_{j\infty}]) = \begin{cases} 
\gamma'_{j\infty} & (j < p) \\
\gamma_{j\infty} & (j = p) \\
\gamma'_{j\infty} & (j > p).
\end{cases}
\]

Theorem 3.16. Assume $c_j, c_\infty - c_1 - \cdots - c_j \notin \mathbb{Z}$ $(j = 1, \ldots, n)$. The linear map $\ell_{j\infty}$ is expressed as
\[
\ell_{j\infty}([\gamma]) = \sum_{j=1}^{p-1} I_h([\gamma], [\gamma'_{j\infty}]) \frac{I_h([\gamma], [\gamma'_{j\infty}])}{I_h([\gamma_{j\infty}], [\gamma'_{j\infty}])}[\gamma_{j\infty}]
+ e^{2\pi\sqrt{-1}(c_\infty + c_1 + \cdots + c_{p-1})} \frac{I_h([\gamma], [\gamma'_{j\infty}])}{I_h([\gamma_{j\infty}], [\gamma'_{j\infty}])}[\gamma_{j\infty}]
+ e^{-2\pi\sqrt{-1}c_p} \sum_{j=p+1}^{n} I_h([\gamma], [\gamma'_{j\infty}]) \frac{I_h([\gamma], [\gamma'_{j\infty}])}{I_h([\gamma_{j\infty}], [\gamma'_{j\infty}])}[\gamma_{j\infty}]
= - \sum_{j=1}^{p-1} \frac{e^{2\pi\sqrt{-1}(c_\infty + c_1 + \cdots + c_{j-1})}}{(e^{2\pi\sqrt{-1}c_{j-1}} - e^{2\pi\sqrt{-1}(c_1 + \cdots + c_{j-1})})} \frac{I_h([\gamma], [\gamma'_{j\infty}])}{I_h([\gamma_{j\infty}], [\gamma'_{j\infty}])}[\gamma_{j\infty}]
- \frac{e^{2\pi\sqrt{-1}(c_\infty + c_1 + \cdots + c_{p-1})}}{(e^{2\pi\sqrt{-1}c_{p-1}} - e^{2\pi\sqrt{-1}(c_1 + \cdots + c_{p-1})})} \frac{I_h([\gamma], [\gamma'_{j\infty}])}{I_h([\gamma_{j\infty}], [\gamma'_{j\infty}])}[\gamma_{j\infty}]
- e^{-2\pi\sqrt{-1}c_p} \sum_{j=p+1}^{n} \frac{e^{2\pi\sqrt{-1}(c_\infty + c_1 + \cdots + c_{j-1})}}{(e^{2\pi\sqrt{-1}c_{j-1}} - e^{2\pi\sqrt{-1}(c_1 + \cdots + c_{j-1})})} \frac{I_h([\gamma], [\gamma'_{j\infty}])}{I_h([\gamma_{j\infty}], [\gamma'_{j\infty}])}[\gamma_{j\infty}].
\]

We set $H_{j\infty} = (I_h([\gamma_{j\infty}], [\gamma'_{j\infty}]))_{j=2, \ldots, n-1, \infty}$, $H_{\infty\infty} = (I_h([\gamma_{j\infty}], [\gamma'_{j\infty}]))_{j=1, \ldots, n}$, $k=1, \ldots, n$, explicit expressions of which are given in 3.14. Recall that $H_{1n}$ and $H_{\infty \infty}$ are diagonal matrices.
Corollary 3.17 ([2], [10]). For a matrix $H$ whose entries belong to $\mathbb{C}(e^{2\pi \sqrt{t}}, e^{\frac{p-th}{1}}t)$, we set $H_{(p)} = H|_{t_0 - \alpha + c}$. The matrix defined by

$$M_{(p)} = (H_{11}^{-1} H_{21})_{(p)} \cdot \text{diag}(1, \ldots, 1, e^{2\pi \sqrt{t}}, -e^{2\pi \sqrt{t}}, \ldots, e^{2\pi \sqrt{t}}) \cdot H_{(p)}^{-1} \cdot H_{(p)}$$

satisfies the following equalities:

$$H_{11} = t M_{(p)} \cdot H_{11}^{(p)} \cdot M_{(p)}^{-1}.$$

4. Intersection theory for twisted cohomology group

As in §2.2, we assume $\lambda \in P = \{a + b \tau \mid 0 \leq a, b < 1\}$ when we discuss the twisted cohomology groups.

4.1. Intersection form. The cohomology intersection form $I_c$ is a non-degenerate bilinear form between $H^1(M; \mathcal{L}_\lambda)$ and $H^1(M; \mathcal{L}_\lambda^*)$:

$$I_c(\bullet, \bullet) : H^1(M; \mathcal{L}_\lambda) \times H^1(M; \mathcal{L}_\lambda^*) \rightarrow \mathbb{C}.$$ 

By Fact 2.2, we have $H^1(M; \mathcal{L}_\lambda^*) \simeq \Omega^1_{\lambda}(\ast_D(E)) / \Omega^1(V(\ast_D(E)))$, where $\Omega^1 f = df - f \omega$. Thus, we also identify $H^1(M; \mathcal{L}_\lambda^*)$ with $\Omega^1_{\lambda}(\ast_D(E)) / \Omega^1(V(\ast_D(E)))$.

To define the intersection form, we introduce two de Rham cohomology groups. Let $\mathcal{E}_M$ and $\mathcal{E}_M^c$ be the sheaves of smooth $i$-forms on $M$ and those with compact support, respectively $(i = 0, 1, 2)$. We set $\mathcal{E}_\lambda^c = \mathcal{E}_M^c \otimes \mathcal{C}_R$ and $\mathcal{E}_\lambda^c = \mathcal{E}_{M,c}^c \otimes \mathcal{C}_R$. Then, $\nabla = d + \omega \wedge$ is naturally defined on $\mathcal{E}_\lambda^c$ and $\mathcal{E}_\lambda^c$.

There are two natural homomorphisms

$$\iota_1 : H^1(M; \mathcal{L}_\lambda) \rightarrow H^1(\mathcal{E}_\lambda^c(M), \nabla), \quad \iota_2 : H^1(\mathcal{E}_\lambda^c(M), \nabla) \rightarrow H^1(\mathcal{E}_\lambda^c(M), \nabla).$$

Proposition 4.1. The morphisms $\iota_1$ and $\iota_2$ are isomorphisms.

Proof. It suffices to show that the following claims hold:

1. $\iota_1$ is surjective.
2. $\iota_2$ is surjective.
3. $H^0(\mathcal{E}_\lambda^c(M), \nabla) = 0$ and $H^0(\mathcal{E}_\lambda^c(M), \nabla) = 0$.
4. $H^2(\mathcal{E}_\lambda^c(M), \nabla) = 0$ and $H^2(\mathcal{E}_\lambda^c(M), \nabla) = 0$.

First, we temporarily admit these claims and prove the proposition. The vanishing results (3), (4) imply $\dim H^1(\mathcal{E}_\lambda^c(M), \nabla) = -\chi(M) = n$ (similarly, $\dim H^1(\mathcal{E}_\lambda^c(M), \nabla) = n$), and hence $\iota_1$, $\iota_2$ are isomorphisms because of (1), (2). Now, let us prove the claims. We recall that for any $(0, 1)$-form $f$ on $M$, there exists a function $f \in \mathcal{E}_{M,c}^0(M)$ such that $\partial f = \psi$ (e.g., [13] Theorem 25.6]). Let $S$ be a holomorphic section of $\mathcal{O}_E \otimes R$ on $M$ which has no zeros (cf. [13] Lemma 30.2]).

1. Suppose $\psi = f_1 du + f_2 d\bar{u} \in \mathcal{E}_\lambda^c(M)$ satisfies $\nabla \psi = 0$. Since $f_2 \in \mathcal{E}_M^0(M)$, we have $f_2/S \in \mathcal{E}_M^0(M)$, and hence $(f_2/S)d\bar{u}$ is a $(0, 1)$-form on $M$. Thus, there exists a function $g \in \mathcal{E}_M^0(M)$ such that $\partial g = (f_2/S)d\bar{u}$. Since $S$ is holomorphic, we have $f_2d\bar{u} = \partial(gS)$. We set $\psi = \psi - \nabla(gS)$. Note that $[\psi] = [\psi]$ in $H^1(\mathcal{E}_\lambda^c(M), \nabla)$ because of $gS \in \mathcal{E}_\lambda^c(M)$. The $(1, 0)$-form $\psi = f_1 du - \partial(gS) - gS\omega$ is a holomorphic one. Indeed, we have

$$\partial \psi = \partial(f_1 du) + \partial \bar{u} (gS) + \omega \wedge \partial(gS) = \partial(f_1 du) + \partial(f_2 d\bar{u}) + \omega \wedge (f_2 d\bar{u}) = \nabla \psi = 0.$$ 

By [14] Proposition 2.5, there exists $\psi_2 \in \Omega^1(\ast_D(E))$ such that $\psi_1 - \psi_2 \in \nabla(O_\lambda(M))$. This implies $[\psi] = \iota_1([\psi_2])$, and hence the surjectivity of $\iota_1$ is proved.

2. It is sufficient to show that for any $\varphi \in \mathcal{E}_\lambda^c(M)$ satisfying $\nabla \varphi = 0$, there exist $\varphi_1 \in \mathcal{E}_{\lambda,c}^1(M)$ and $f \in \mathcal{E}_M^0(M)$ such that $\varphi - \nabla f = \varphi_1$. We will find such $\varphi_1$ and $f$ in a similar way to that in [13] Proposition 3. For $j = 1, \ldots, n$, let $U_j$ and $V_j$ be open neighborhoods of $t_j$ such that $U_j \subset V_j$ and
Since the global solution to \( \nabla \varphi = 0 \) and Stokes' theorem, \((4.1)\) is independent of the choice of the loop. It is not difficult to see that \( h_j f_j \in \mathcal{E}^*_\lambda(M) \) and \( \nabla(h_j f_j) = \varphi \) on \( U_j \). We set \( f = \sum_{j=1}^n h_j f_j \) and \( \varphi_1 = \varphi - \nabla f \). On each \( U_j \), we have \( \varphi_1 = \varphi - \nabla(h_j f_j) = 0 \), and hence \( \varphi_1 \) is an element in \( \mathcal{E}^1_{\lambda,c}(M) \).

(3) Since the global solution to \( \nabla f = 0 \) (\( f \in \mathcal{E}^0(M) \) or \( f \in \mathcal{E}^0_{\lambda,c}(M) \)) is only zero, the 0-th cohomology groups vanish.

(4) First, we show \( H^2(\mathcal{E}^*_{\lambda,c}(M), \nabla) = 0 \). Let us consider \( \eta = f du \wedge d\bar{u} \in \mathcal{E}^2_\lambda(M) \). Similarly to \((1.1)\), there exists \( g_1 \in \mathcal{H}^0(M) \) such that \( f d\bar{u} = \partial(g_1 S) \). We set \( \psi = -g_1 S d\bar{u} \). Then we have \( \psi \in \mathcal{E}^1_\lambda(M) \) and \( \nabla\psi = -\partial(g_1 S d\bar{u}) - \omega \wedge (g_1 S d\bar{u}) = -f d\bar{u} \wedge d\eta = 0 \), which implies \( H^2(\mathcal{E}^*_{\lambda,c}(M), \nabla) = 0 \). Next, we show \( H^2(\mathcal{E}^*_\lambda(M), \nabla) = 0 \). For any \( \eta \in \mathcal{E}^2_{\lambda,c}(M) \), we can find \( \psi_1 \in \mathcal{E}^1_\lambda(M) \) such that \( \nabla\psi_1 = \eta \) by the above discussion. Since \( \nabla\psi_1 = \eta \) has a compact support, we have \( \nabla\psi_1 = 0 \) on a small neighborhood of each \( t_j \). By the same manner as \((4.1)\), we can find \( f_j \) satisfying \( \nabla f_j = \psi_j \) around \( t_j \). Similarly to \((2.1)\), we can construct \( f \in \mathcal{E}^0(M) \) such that \( \psi_2 = \psi_1 - \nabla f \) belongs to \( \mathcal{E}^1_{\lambda,c}(M) \). Therefore we have \( \eta = \nabla\psi_2 \in \nabla(\mathcal{E}^1_{\lambda,c}(M)) \), and the proof is completed.

**Definition 4.2.** We set \( \text{reg}_{c} = \iota_{c}^{-1} o \iota_{t_1} : H^1(M; \mathcal{L}_{\lambda}) \rightarrow H^1(\mathcal{E}^*_{\lambda,c}, \nabla) \), and call it the regularization map on the twisted cohomology group.

Thanks to the discussion in \((1.1)\), we can evaluate the intersection number \( I_c([\varphi], [\varphi']) \) for \([\varphi] \in H^1(M; \mathcal{L}_{\lambda}) \) and \([\varphi'] \in H^1(\mathcal{E}^*_{\lambda,c}, \nabla) \) as follows. The intersection number is defined by

\[
I_c([\varphi], [\varphi']) = \int_M \varphi_1 \wedge \varphi' \quad (\varphi_1 \in \mathcal{E}^1_{\lambda,c}(M) \text{ satisfies } [\varphi_1] = \text{reg}_{c}([\varphi])),
\]

which converges and is well-defined. By using the expression \( \varphi_1 = \varphi - \nabla f = \varphi - \nabla(\sum_{j=1}^n h_j f_j) \) in the proof of Proposition \((1.1)\) (2), we obtain a formula

\[
I_c([\varphi], [\varphi']) = 2\pi \sqrt{-1} \sum_{j=1}^n \text{Res}_{u=t_j} (f_j \varphi')
\]

in a similar manner to \((1.1)\). Since \( \varphi \in \Omega^1_{\lambda}(\ast D)(E) \) and the expression \((1.1)\) of \( f_j \) imply \( f_j \in \mathcal{O}_{\lambda}(\ast D)(V_j) \), we have \( f_j \varphi' \in \Omega^1_{\lambda}(\ast D)(V_j) \). Thus, the residue in \((1.2)\) can be evaluated by using the Laurent expansions of \( f_j \) and \( \varphi' \).

Similarly to \((3.1)\) we often consider \( c_j \)'s and \( \lambda \) as indeterminates and we can regard an intersection number as an element in the field \( K(c_1, \ldots, c_n, \lambda, t_1, \ldots, t_n) \) of functions in \( c_1, \ldots, c_n, c_0, \lambda, t_1, \ldots, t_n \), which has an involution \( (c_1, \ldots, c_n, c_0, \lambda) \mapsto (-c_1, \ldots, -c_n, -c_0, -\lambda) \). For \( \varphi \in \Omega^1_{\lambda}(\ast D)(E) \), we set \( \varphi(u; \lambda) = \varphi(u; -\lambda) \in \Omega^1_{\lambda}(\ast D)(E) \). Thus, for a \( K(c_1, \ldots, c_n, \lambda, t_1, \ldots, t_n) \)-linear combination \( \varphi = \sum_i a_i \varphi_i \) \( (a_i \in K(c_1, \ldots, c_n, \lambda, t_1, \ldots, t_n), \varphi_i \in \Omega^1_{\lambda}(\ast D)(E)) \), we can naturally define \( \varphi' \in \Omega^1_{\lambda}(\ast D)(E) \) by \( \varphi' = \sum_i a_i' \varphi_i' \). For example, we have \( (c_1 s(u - t_1; \lambda) du)' = -c_1 s(u - t_1; -\lambda) du \). By using these notations, we have \( I_c([\varphi], [\varphi']) = -I_c(\varphi', [\varphi]) \), for \([\varphi], [\varphi] \in H^1(M; \mathcal{L}_{\lambda}) \).

### 4.2. Intersection numbers

In this section, we give formulas of the intersection numbers for the twisted cocycles introduced in \((2.2)\). We also define other twisted cocycles which will be used in \((4.3)\) and give their intersection numbers. Precise computations will be given in \((5.2)\).
4.2.1. The case when $\lambda \in P - \{0\}$. We set $\psi_j = s(u - t_j; \lambda)du \in \Omega^1_\lambda(*D)(E)$ which has a simple pole at $u = t_j$ and satisfies $\text{Res}_{u=t_j}(\psi_j) = 1$. By Fact 2.3 (i), $\{[\psi_j]\}_{j=1,\ldots,n}$ form a basis of $H^1(M; L_\lambda)$.

**Theorem 4.3.** We have
\[ I_c([\psi_j], [\psi_j]^\vee]) = 2\pi\sqrt{-1} \quad \text{and} \quad I_c([\psi_j], [\psi_k]^\vee]) = 0 \quad (j \neq k). \]

Note that the determinant of the intersection matrix $C_{\psi\psi} = (I_c([\psi_j], [\psi_k]^\vee]))_{j,k=1,\ldots,n}$ is equal to $(2\pi\sqrt{-1})^n/(c_1 \cdot \ldots \cdot c_n) \neq 0$. Thus, by using the intersection form, we can also verify that $\{[\psi_j]\}_{j=1,\ldots,n}$ become diagonal. For details, see \textsection 5.2. Note that the determinant of the intersection matrix $C_{\psi\psi} = (I_c([\psi_j], [\psi_k]^\vee]))_{j,k=1,\ldots,n}$. We set
\[ \phi_p \equiv \frac{\partial}{\partial u}(u - t_p; \lambda)du \in \Omega^1_\lambda(*D)(E) \]
which has a pole of order 2 at $u = t_p$. For the discussion in \textsection 4.3, we show that $\{[\phi_p]\} \cup \{[\psi_k]\}_{k \neq p}$ also form a basis, and we construct another basis dual to it.

**Proposition 4.4.** For $p, q \in \{1, \ldots, n\}$ with $p \neq q$, we set
\[ \varphi_j^{(pq)} = \begin{cases} \phi_p & (j = q) \\ \psi_j & (j \neq q), \end{cases} \]
\[ \eta_k^{(pq)} = \begin{cases} \psi_p + \frac{1}{\lambda + q - \lambda \lambda}(\frac{1}{c_p} 2\pi\sqrt{-1} \sum_{l \neq p} c_l \rho(t_p - t_l) - \rho(-\lambda)) \psi_q & (k = p) \\ \psi_q & (k = q) \\ \psi_k - \frac{1}{\lambda + q - \lambda \lambda} \psi_q & (k \neq p, q). \end{cases} \]

Then we have $I_c([\varphi_j^{(pq)}], [\eta_k^{(pq)}]) = 0$ if $j \neq k$, and
\[ I_c([\varphi_q^{(pq)}], [\eta_k^{(pq)}]) = I_c([\phi_p], [\psi_q]) = -2\pi\sqrt{-1} \cdot \frac{s(t_p - t_q; -\lambda)}{c_p - 1}, \]
\[ I_c([\varphi_j^{(pq)}], [\eta_j^{(pq)}]) = I_c([\psi_j], [\psi_j]) = 2\pi\sqrt{-1} \quad (j \neq q). \]

We have constructed $\{\eta_k^{(pq)}\}_{k=1,\ldots,n}$ so that the intersection matrix $C_{\psi\eta} = (I_c([\varphi_j^{(pq)}], [\eta_k^{(pq)}]))_{j,k=1,\ldots,n}$ becomes diagonal. For details, see \textsection 4.4. Note that $\eta_k^{(pq)}$ can be defined under a condition $t_p - t_q + \lambda \notin \Lambda_\tau$. We thus conclude that $\{[\varphi_j^{(pq)}]\}_{j=1,\ldots,n} = \{[\phi_p]\} \cup \{[\psi_k]\}_{k \neq p}$ form a basis of $H^1(M; L_\lambda)$ under this condition.

4.2.2. The case when $\lambda = 0$. As mentioned in Fact 2.3 (ii), to obtain a basis of $H^1(M; L_\lambda)$ for $\lambda = 0$, we need to use a 1-form having a pole of order 2. Though $t_1$ is specified in Fact 2.3 (ii), there is no difficulty caused by specifying another $t_i$. Thus, in this section, we consider
\[ \varphi_0 = du, \quad \varphi_{ii} = \rho'(u - t_i)du, \quad \varphi_{ij} = (\rho(u - t_j) - \rho(u - t_i))du \quad (j \in \{1, \ldots, n\} - \{i\}). \]
We often write $\varphi_{i0}$ for $\varphi_0$.

**Theorem 4.5.** For $j, k \in \{1, \ldots, n\} - \{i\}$, we have
\[ I_c([\varphi_0], [\varphi_j]) = I_c([\varphi_j], [\varphi_0]) = 0, \quad I_c([\varphi_{ij}], [\varphi_k]) = 2\pi\sqrt{-1} \left( \frac{1}{c_i} + \frac{\delta_{jk}}{c_j} \right), \]
\[ I_c([\varphi_{ij}], [\varphi_0]) = -\frac{2\pi\sqrt{-1}}{c_i - 1}, \]
\[ I_c([\varphi_{ij}], [\varphi_k]) = \frac{2\pi\sqrt{-1}}{(c_i - 1)(c_i + 1)} \left( \frac{1}{c_i} \left( 2\pi\sqrt{-1}c_0 + \sum_{l \neq i} c_l \rho(t_i - t_l) \right)^2 - c_i \frac{\partial^2 \varphi_i(0)}{\partial (t_i - t_l)} - \sum_{l \neq i} c_l \rho(t_i - t_l) \right), \]
\[ I_c([\varphi_{ij}], [\varphi_{ij}]) = -\frac{2\pi\sqrt{-1}}{c_i(c_i - 1)} \left( 2\pi\sqrt{-1}c_0 + \sum_{l \neq i} c_l \rho(t_i - t_l) + c_i \rho(t_i - t_l) \right), \]
and the other intersection numbers are obtained by applying the formula $I_c([\varphi], [\psi]) = -I_c([\psi], [\varphi])$.\]
Since the determinant of the intersection matrix $(I_c([\varphi_{ij}], [\varphi_{ik}]))_{j,k=0,\ldots,n-1}$ is equal to

\[-(2\pi \sqrt{-1})^n \frac{c_1 + \cdots + c_{n-1}}{(c_1 - 1)(c_1 + 1)c_1 \cdots c_{n-1}} = \frac{(2\pi \sqrt{-1})^nc_n}{(c_1 - 1)(c_1 + 1)c_1 \cdots c_{n-1}},\]

we can verify that $([\varphi_{ij}])_{j=0,\ldots,n-1}$ form a basis of $H^1(M; \mathcal{L}_\lambda)$ under the condition $c_j \notin \mathbb{Z}$. Similarly, $([\varphi_{ij}])_{j=0,\ldots,k-1}$ with $k \neq 0$ also form a basis.

### 4.3. Contiguity relations

In the case when $\lambda \in P - \{0\}$, the contiguity relations for the Riemann-Wirtinger integral can be expressed in terms of intersection forms on twisted cohomology groups.

Basic idea is same as that in [6, Section 5].

We consider the shift of the parameters $(c_p, c_q) \to (c_p + 1, c_q - 1)$. The corresponding local system, denoted by $\mathcal{L}_\lambda^{(p,q)}$, is obtained by replacing $(c_p, c_q, \lambda)$ by $(c_p + 1, c_q - 1, \lambda - t_p + t_q)$, because of the constraint $\lambda + c_0 + c_1 t_1 + \cdots + c_n t_n + c_\infty = 0$. Here, we assume $\lambda - t_p + t_q \notin \Lambda_\tau$ which is also implicitly assumed in [10, Section 5].

For $\varphi \in \Omega^1_0(\ast D)(E)$, we define $\varphi^{(p,q)} \in \Omega^1_{\lambda - t_p + t_q}(\ast D)(E)$ by replacing $(c_p, c_q, \lambda)$ by $(c_p + 1, c_q - 1, \lambda - t_p + t_q)$. The twisted cohomology group $H^1(M; \mathcal{L}_\lambda^{(p,q)})$ is identified with the de Rham cohomology group with respect to $\nabla^{(p,q)}$, where $\nabla^{(p,q)} \varphi = \nabla \varphi + d \log(\vartheta_1(u - t_p)/\vartheta_1(u - t_q)) \wedge \varphi$. Let $I_c^{(p,q)}$ denote the intersection form on $H^1(M; \mathcal{L}_\lambda^{(p,q)}) \times H^1(M; \mathcal{L}_\lambda^{(p,q)})$. If we regard the parameters as indeterminates, we have $I_c^{(p,q)}([\varphi^{(p,q)}], [\varphi^{(p,q)}]) = I_c([\varphi], [\varphi])$, where the last $(p,q)$ means replacing $(c_p, c_q, \lambda)$ with $(c_p + 1, c_q - 1, \lambda - t_p + t_q)$, and $[\varphi^{(p,q)}] = (\varphi^{(p,q)})^\vee$.

By the same discussion of [3, Proposition 5.2], we obtain the following.

**Proposition 4.6.** The map defined by

$$S_q^p : H^1(M; \mathcal{L}_\lambda^{(p,q)}) \ni [\varphi] \mapsto \left[ \frac{\vartheta_1(u - t_p)}{\vartheta_1(u - t_q)} \cdot \varphi \right] \in H^1(M; \mathcal{L}_\lambda)$$

is a well-defined linear map.

For a fixed twisted cycle $\gamma \in Z_1(M; \mathcal{L}_\lambda^\vee)$, we set

$$f = \left( \int_\gamma T(u) \psi_1, \ldots, \int_\gamma T(u) \psi_n \right),$$

and $f^{(p,q)}$ denotes the vector obtained by replacing $(c_p, c_q, \lambda)$ with $(c_p + 1, c_q - 1, \lambda - t_p + t_q)$. A relation between $f$ and $f^{(p,q)}$ is called the contiguity relation. For $\varphi \in \Omega^1_0(\ast D)(E)$, we have

$$\int_\gamma T(u) \cdot \left( \frac{\vartheta_1(u - t_p)}{\vartheta_1(u - t_q)} \varphi^{(p,q)} \right) = \int_\gamma \left( T(u) |_{(c_p,c_q) \to (c_p+1,c_q-1)} \right) \cdot \varphi^{(p,q)} = \left( \int_\gamma T(u) \varphi \right) |_{(c_p,c_q,\lambda) \to (c_p+1,c_q-1,\lambda - t_p + t_q)}.$$

Therefore, if $S_q^p$ denotes the representation matrix of $S_q^p$ with respect to the bases $\{[\psi^{(p,q)}]\} \subset H^1(M; \mathcal{L}_\lambda^{(p,q)})$ and $\{[\psi_j]\} \subset H^1(M; \mathcal{L}_\lambda)$, the contiguity relation is obtained as $f^{(p,q)} = S_q^p \cdot f$. We will give an explicit formula of $S_q^p$ in Corollary 4.10.

Since the expression of $S_q^p([\psi^{(p,q)}])$ as a linear combination of $\{[\psi_j]\}$ is complicated, we first consider the basis $\{(\varphi_{[pq]})^{(p,q)}\} \subset H^1(M; \mathcal{L}_\lambda^{(p,q)})$ defined in 4.2.1. The image of this basis under $S_q^p$ can be obtained by the following lemma which is proved by straightforward calculation.

**Lemma 4.7.** The following equalities hold as elements in $\Omega^1_0(\ast D)(E)$:

$$\frac{\vartheta_1(u - t_p)}{\vartheta_1(u - t_q)} \psi_j^{(p,q)} = \frac{\vartheta_1(t_j - t_p)}{\vartheta_1(t_j - t_q)} \psi_j + \frac{\vartheta_1(t_p - t_q)}{\vartheta_1(t_j - t_q)} \vartheta_1(\lambda - t_p + t_q) \psi_q \quad (j \neq p, q),$$

(4.3)
\[
{\psi_p}^{(p+q-)}(u-t_p) = \frac{\vartheta_1(\lambda)}{\vartheta_1(\lambda-t_p+q)} \psi_q, \\
{\psi_p}^{(p+q-)}(u-t_q) = \frac{\vartheta_1(\lambda)}{\vartheta_1(\lambda-t_p+q)}(\rho(t_p-t_q) - \rho(\lambda))\psi_q - \frac{\vartheta_1'(0)}{\vartheta_1'(t_p-t_q)} \psi_p.
\]

**Proof.** The equality (4.4) is obvious. The equalities (4.5) and (4.6) follow from (2.3) and (2.2), respectively. As an example, we show (4.5):

\[
\begin{align*}
&\frac{\vartheta_1'(u-t_p)}{\vartheta_1'(u-t_q)} \vartheta(u-t_p; \lambda-t_p+t_q) \\
&= \frac{(\vartheta_1'(u-\lambda-t_q)\vartheta_1(u-t_p) - \vartheta_1(u-\lambda-t_q)\vartheta_1'(u-t_p))\vartheta_1'(0)}{\vartheta_1(u-t_p)\vartheta_1(u-t_q)} - \frac{\vartheta_1'(u-\lambda-t_q)}{\vartheta_1(u-t_p)\vartheta_1(u-t_q)}(\vartheta_1'(u-\lambda-t_q) - \vartheta_1'(u-t_p)) \\
&= \frac{\vartheta_1'(0)}{\vartheta_1(-\lambda)} \vartheta(u-t_q; \lambda)(\rho(u-\lambda-t_q) - \rho(u-t_p)) \\
&= \frac{\vartheta_1(-\lambda)}{\vartheta_1(-\lambda-t_p+t_q)} \big(\vartheta(u-t_q; \lambda)(\rho(u-\lambda-t_q) - \rho(u-t_p)) - \vartheta(u-t_p; \lambda)\vartheta(t_p-t_q; \lambda)\big).
\end{align*}
\]

By using the basis \(\{[\eta_j^{(pq)}]\}_{j=1}^n\) defined in (4.2.1) we can express \(S_q^p\) in terms of the intersection form.

**Theorem 4.8.** Suppose \(t_p - t_q \pm \lambda \notin \Lambda_r\). For any \(\varphi \in \Omega_1^1(*D)(E)\), we have

\[
S_q^p(\varphi^{(p+q-)}) = \sum_{j=1}^n I_c^{(p+q-)}(\varphi^{(p+q-)}, [\eta_j^{(pq)}]) S_q^p(\varphi^{(p+q-)})
\]

\[
= \sum_{j \neq p,q} \frac{I_c^{(p+q-)}(\varphi^{(p+q-)}, [\eta_j^{(pq)}])}{2\pi \sqrt{-1}} c_j \vartheta_1(t_j - t_p) \psi_j + \frac{I_c^{(p+q-)}(\varphi^{(p+q-)}, [\eta_j^{(pq)}])}{2\pi \sqrt{-1}} c_p \vartheta_1(\lambda-t_p+t_q) \psi_p
\]

\[
+ \left(\frac{I_c^{(p+q-)}(\varphi^{(p+q-)}, [\eta_j^{(pq)}])}{2\pi \sqrt{-1}} c_p \vartheta_1(\lambda) \frac{1}{\vartheta_1'(0)} + \frac{I_c^{(p+q-)}(\varphi^{(p+q-)}, [\eta_j^{(pq)}])}{2\pi \sqrt{-1}} c_p \vartheta_1'(t_p-t_q) \frac{1}{\vartheta_1'(0)} \right) \psi_q.
\]

Here, we omit \([\ ]\) and use notations \(\varphi_j^{(pq)}(p+q-) = (\varphi_j^{(pq)})^{(p+q-)}\) and \(\eta_j^{(pq)}(p+q-) = ([\eta_j^{(pq)}])^{(p+q-)}\) for simplicity.

**Proof.** If we set \(\varphi = \varphi_j^{(pq)}\) in the right-hand side of (4.6), then it coincides with \(S_q^p(\varphi_j^{(pq)}(p+q-))\). Since \(\{([\varphi_j^{(pq)}])^{(p+q-)}\}\) form a basis, the equality (4.6) holds. The expression (4.7) follows from Proposition 4.4 and Lemma 4.7. 

Though our expression seems to be complicated, we can treat with not only \(\psi_j\), but also any \(\varphi \in \Omega_1^1(*D)(E)\). Once we obtain the intersection numbers \(I_c(\varphi, [\eta_j^{(pq)}])\)'s, we can express \(S_q^p(\varphi^{(p+q-)})\) as a linear combination of \(\psi_j\)'s.
Remark 4.9. In general, to derive the contiguity relations in terms of twisted cohomology groups, we need to find some \( f \in \Omega^*(\kappa(D)) \) such that \( S^p_\eta \varphi - \nabla f \) becomes a linear combination of the basis \( \{ |\psi_j\rangle \}_{j=1,\ldots,n} \) (e.g., [10] Theorem 5.1). Our method can avoid this difficulty.

Similarly to the connection problems, we can obtain the representation matrix of \( S^p_\eta \). We use the diagonal matrices \( C_{\psi_\eta} \) and \( C_{\varphi_\eta} \) whose diagonal entries are given in Theorem 4.3 and Proposition 4.4, and we set \( C_{\psi_\eta} = (I_c(|\psi_j\rangle, |\eta_j\rangle))_{j,k=1,\ldots,n} \). By definition of \( \eta_j^{(pq)} \), we have a simple relation

\[
\psi(t(\eta_1^{(pq)}, \ldots, \eta_n^{(pq)})) = A \cdot \psi(\eta_1, \ldots, \eta_n), \quad A = \text{id}_n + A',
\]

where \( A' \) is an \( n \times n \) matrix whose entries are zero except for the \((p,q)\)-entry

\[
\frac{1}{a(t_p - t_q, \lambda)} \left( \frac{1}{c_p} \left( 2\pi \sqrt{-1} c_0 + \sum_{k \neq p} c_k \rho(t_p - t_k) \right) - \rho(-\lambda) \right)
\]

and the \((j,q)\)-entry \( -\frac{a(t_p - t_q, \lambda)}{a(t_p - t_j, \lambda)p} \) for \( j \neq p, q \). An explicit formula of \( C_{\psi_\eta} \) is given by \( C_{\psi_\eta} = C_{\psi} \cdot tA' \).

Corollary 4.10 ([10]). For a matrix \( C \) whose entries belong to \( K(c_\ast, \lambda, t_\ast) \), we set \( C^{(p+q, -)} = C |c_p, c_q, \lambda, t_p, t_q \rangle \mapsto (c_{p+1}, c_{q-1}, \lambda - t_p + t_q) \rangle \). Let \( (S^p_\eta)^{\vee} \) be the matrix satisfying

\[
S^p_\eta \psi(\psi(\psi_1^{(pq)}, \ldots, \psi_n^{(pq)})) = (S^p_\eta)^{\vee} \psi(\psi(\psi_1, \ldots, \psi_n)),
\]

the entries of which are given in Lemma 4.3. The matrix defined by

\[
S^p_\eta = (C_{\psi_\eta} C_{\psi_\eta}^{\vee})^{(p+q, -)} \cdot (S^p_\eta)^{\vee}
\]

satisfies the following equalities:

\[
S^p_\eta \psi(\psi(\psi_1, \ldots, \psi_n)) = (C_{\psi_\eta} C_{\psi_\eta}^{\vee})^{(p+q, -)} \cdot (S^p_\eta)^{\vee} \psi(\psi(\psi_1, \ldots, \psi_n)), \tag{4.8}
\]

\[
S^p_\eta \cdot C_{\psi_\eta} = (C_{\psi_\eta} \cdot (S^p_\eta)^{\vee})^{(p+q, -)}. \tag{4.9}
\]

Proof. Since we have \( (\psi_1, \ldots, \psi_n) = C_{\psi_\eta} C_{\psi_\eta}^{\vee} \cdot (\psi_1^{(pq)}, \ldots, \psi_n^{(pq)}) \), the equality (4.8) is easily obtained. The equality (4.8) follows from the property \( I_c(S^p_\eta(\psi)) C_{\psi_\eta} \cdot I_c^{(p+q, -)}([\psi], (S^p_\eta)^{\vee}(\psi)) \), where \( (S^p_\eta)^{\vee} : H^1(M; \mathcal{L}_\lambda) \ni \langle \varphi \rangle \mapsto \left( \frac{\partial_1 (u - t_p) \partial_1 (u - t_q)}{\partial_1 (u - t_j)} \right) \in H^1(M; \mathcal{L}_\lambda^{(p+q, -)^\vee}) \). This property can be shown in the same manner as [6] Proposition 5.8.

Example 4.11. As in [10] Theorem 5.1, the expression of \( S^p_\eta \psi(\psi_1^{(pq)}) \) is complicated. According to [10], the coefficient of \( \psi_1 \) should be

\[
\frac{\partial_1 (t_p - t_q)}{\partial_1 (0)} \left( \rho(t_p - t_q) - \rho(\lambda - t_p + t_q) + \frac{1}{c_p} \left( 2\pi \sqrt{-1} c_0 + \sum_{j \neq q} c_j \rho(t_p - t_j) - c_q \rho(\lambda) \right) \right). \tag{4.10}
\]

Let us verify that it coincides with the \((q, q)\)-entry of \( S^p_\eta \). Since the entries of the \( q \)-th row of \( C_{\psi_\eta} \) are

\[
\frac{(C_{\psi_\eta})_{qq}}{2\pi \sqrt{-1}} = \frac{1}{c_q}, \quad \frac{(C_{\psi_\eta})_{qj}}{2\pi \sqrt{-1}} = -\frac{1}{c_q} \frac{\partial_1 (t_p - t_j + \lambda) \partial_1 (t_p - t_q)}{\partial_1 (t_p - t_j) \partial_1 (t_p - t_q + \lambda)} \quad (j \neq p, q),
\]

\[
\frac{(C_{\psi_\eta})_{pq}}{2\pi \sqrt{-1}} = \frac{1}{c_p} \frac{\partial_1 (t_p - t_q)}{\partial_1 (0)} \left( \frac{1}{c_q} \left( 2\pi \sqrt{-1} c_0 + \sum_{j \neq p, q} c_j \rho(t_p - t_j) - c_p \rho(\lambda) \right) + \rho(t_p - t_q) \right) \frac{\partial_1 (\lambda)}{\partial_1 (t_p - t_q + \lambda)},
\]

the \((q, q)\)-entry of \( S^p_\eta \) is equal to

\[
\frac{\partial_1 (t_p - t_q)}{\partial_1 (0)} \left( - \frac{c_p}{c_q - 1} \rho(t_p - t_q) + \frac{c_p}{c_q - 1} \rho(\lambda) \right) + \frac{1}{c_q - 1} \left( 2\pi \sqrt{-1} c_0 + \sum_{j \neq p, q} c_j \rho(t_p - t_j) - (c_p + 1) \rho(\lambda - t_p + t_q) \right) + \rho(t_p - t_q) \right) \frac{\partial_1 (\lambda)}{\partial_1 (t_p - t_q + \lambda)}.
\]

\[
\sum_{j \neq p, q} c_j \frac{\partial_1 (0) \partial_1 (\lambda - t_p + t_j) \partial_1 (\lambda - t_j + t_q) \partial_1 (t_p - t_q)}{\partial_1 (\lambda) \partial_1 (\lambda - t_p + t_q) \partial_1 (t_p - t_j) \partial_1 (t_j - t_q)}.
\]

\[
\frac{\partial_1 (0)}{\partial_1 (0)} \left( - \frac{c_p}{c_q - 1} \rho(t_p - t_q) + \frac{c_p}{c_q - 1} \rho(\lambda) \right) + \frac{1}{c_q - 1} \left( 2\pi \sqrt{-1} c_0 + \sum_{j \neq p, q} c_j \rho(t_p - t_j) - (c_p + 1) \rho(\lambda - t_p + t_q) \right) + \rho(t_p - t_q) \right) \frac{\partial_1 (\lambda)}{\partial_1 (t_p - t_q + \lambda)}.
\]
By (2.2), we have
\[
\frac{\vartheta_1'(0) \vartheta_1(\lambda - t_p + t_j) \vartheta_1(\lambda - t_j + t_q) \vartheta_1(t_p - t_q)}{\vartheta_1(\lambda) \vartheta_1(\lambda - t_p + t_q) \vartheta_1(t_p - t_j) \vartheta_1(t_j - t_q)} = \rho(t_p - t_j) + \rho(t_j - t_q) - \rho(t_p - t_q - \lambda) - \rho(\lambda).
\]
Using this relation and \(\sum_{j \neq p,q} c_j = -c_p - c_q\), we can show that (4.11) coincides with (4.10). Note that our computation does not require the relation \(\nabla f = 0\) in \(H^1(M; \mathcal{L}_\lambda)\) for some \(f \in \mathcal{O}_\lambda(\ast D)(E)\).

5. Computation of intersection numbers

We give precise computations of the intersection numbers.

5.1. Intersection numbers of twisted cycles. Since the intersection numbers that we use in this paper are so many, we cannot explain all of them.

5.1.1. Fact 3.3 and Corollary 3.3. Fact 3.1 is computed in [3, Proposition 3.4.1]. A detailed computation of \(I_h([\gamma_{1\infty}], [\gamma_{1\infty}^\vee])\) is given in [4]. For readers’ convenience, we explain \(I_h([\gamma_{1j}], [\gamma_{1j}^\vee])\) for \((j = 2, \ldots, n)\) and \(I_h([\gamma_{10}], [\gamma_{1\infty}^\vee])\) in detail.

First of all, we consider the intersection numbers \(I_h([\gamma_{jk}], [\gamma_{jk}^\vee])\) for \(j, j', k, k' \in \{1, \ldots, n\}\) satisfying \(j < k, j' < k'\). As mentioned above Corollary 3.3, the cases of \(j, j' \geq 2\) follow from that of \(j, j' = 1\). In fact, it is not difficult to compute all the cases directly. A method to compute is quite similar to [9], see also [12, Fact 4.2]. For example, we have
\[
I_h([\gamma_{12}], [\gamma_{1\infty}^\vee]) = \frac{-1}{e^{2\pi \sqrt{-1} c_2} - 1} \cdot (-1) \cdot 1 = \frac{-1}{1 - e^{2\pi \sqrt{-1} c_2}}.
\]
Here, the first factor is the coefficient of \(s_2\) in \(\gamma_{12}\), the second “\((-1)\)” is the local intersection multiplicity, and the third “\(1\)” indicates the difference of the branches (see Figure 7).

Next, we compute \(I_h([\gamma_{1j}], [\gamma_{1\infty}^\vee])\) for \(j = 2, \ldots, n\). By Figure 8, we can compute it as
\[
I_h([\gamma_{1j}], [\gamma_{1\infty}^\vee]) = \frac{1 - e^{2\pi \sqrt{-1} c_1} (1 - e^{2\pi \sqrt{-1} c_1})}{e^{2\pi \sqrt{-1} c_1} - 1} \cdot 1 = \frac{e^{2\pi \sqrt{-1} c_1} (1 - e^{2\pi \sqrt{-1} c_1})}{1 - e^{2\pi \sqrt{-1} c_1}}.
\]
Note that since the coefficient of \(m_0\) in \(\gamma_{1\infty}\) is \(\frac{1 - e^{2\pi \sqrt{-1} c_1}}{e^{2\pi \sqrt{-1} c_1} - 1}\), that in \(\gamma_{1\infty}^\vee\) is obtained by operating \(\vee\) to it.
We now compute the self-intersection numbers. By Figures 10 and 11, we obtain

\[
I_h([\gamma_10], [\gamma_{1\infty}]) = \frac{1 - e^{2\pi\sqrt{-1}c_0}}{e^{2\pi\sqrt{-1}c_0} - 1} \cdot (-1) \cdot 1
\]

\[
+ \frac{e^{2\pi\sqrt{-1}c_j} - e^{2\pi\sqrt{-1}(c_0 - c_1 - \ldots - c_j + c_{j+1})}}{e^{2\pi\sqrt{-1}c_j} - 1} \cdot 1 \cdot e^{2\pi\sqrt{-1}(c_0 + c_1 + \ldots + c_{j+1})}
\]

\[
= - \frac{(e^{2\pi\sqrt{-1}c_0} - e^{2\pi\sqrt{-1}(c_1 + c_2 + \ldots + c_j)}) (e^{2\pi\sqrt{-1}(c_0 + c_1 + \ldots + c_j + c_{j+1})})}{e^{2\pi\sqrt{-1}(c_0 + c_1 + \ldots + c_j)} (1 - e^{2\pi\sqrt{-1}c_j})},
\]

and

\[
I_h([\gamma_{j0}], [\gamma_{1\infty}]) = \frac{1 - e^{2\pi\sqrt{-1}(-c_1 + c_2 + \ldots + c_j + c_{j+1})}}{e^{2\pi\sqrt{-1}c_j} - 1} \cdot (-1) \cdot e^{2\pi\sqrt{-1}c_j}
\]

\[
+ \frac{1 - e^{2\pi\sqrt{-1}(-c_1 + c_2 + \ldots + c_j + c_{j+1})}}{e^{2\pi\sqrt{-1}c_j} - 1} \cdot 1 \cdot e^{2\pi\sqrt{-1}(-c_0 - c_1 - \ldots - c_{j+1})}
\]

\[
= - \frac{(e^{2\pi\sqrt{-1}c_0} - e^{2\pi\sqrt{-1}(c_1 + c_2 + \ldots + c_j)}) (e^{2\pi\sqrt{-1}(c_0 + c_1 + \ldots + c_j + c_{j+1})})}{e^{2\pi\sqrt{-1}(c_0 + c_1 + \ldots + c_j)} (1 - e^{2\pi\sqrt{-1}c_j})}.
\]
5.1.3. Proposition [3.7] We compute the intersection numbers $I_h([\gamma_{1j}], [\gamma^\vee_{nj}])$ ($j = 2, \ldots, n-1, \infty; k = 2, \ldots, n-1, \infty, 0$). It is clear that the intersection matrix becomes diagonal. The non-zero intersection numbers are computed as follows:

$$I_h([\gamma_{1j}], [\gamma^\vee_{nj}]) = -\frac{1}{e^{2\pi \sqrt{-1}c_j} - 1} \cdot 1 \cdot e^{-2\pi \sqrt{-1}(c_j + \cdots + c_n)} = \frac{e^{2\pi \sqrt{-1}(c_1 + \cdots + c_j)}}{1 - e^{2\pi \sqrt{-1}c_j}},$$

$$I_h([\gamma_{10}], [\gamma^\vee_{n\infty}]) = 1 \cdot 1 \cdot e^{2\pi \sqrt{-1}(c_n + c_\infty)} = e^{2\pi \sqrt{-1}(c_n + c_\infty)},$$

$$I_h([\gamma_{1j}], [\gamma^\vee_{n0}]) = 1 \cdot (-1) \cdot e^{-2\pi \sqrt{-1}c_0} = -e^{-2\pi \sqrt{-1}c_0},$$

where $j = 2, \ldots, n-1$.

5.1.4. Some intersection matrices. To obtain explicit formulas of connection matrices in [3.3.2] and [3.3.3] we need four intersection matrices $H_{10} = (I_h([\gamma_{1j}], [\gamma^\vee_{k0}]), H_{0n} = (I_h([\gamma_{j0}], [\gamma^\vee_{nk}]), H_{1\infty} = (I_h([\gamma_{1j}], [\gamma^\vee_{n\infty}])$, and $H_{0\infty} = (I_h([\gamma_{j\infty}], [\gamma^\vee_{nk}]))$. The entries of them are computed similarly as above (in fact, some entries have been computed). We list them. In the following formulas, $j, k$ belong to \{1, \ldots, n\} or its subset.

- $H_{10}$:

$$I_h([\gamma_{1\infty}], [\gamma^\vee_{k0}]) = \begin{cases} \frac{1 - e^{-2\pi \sqrt{-1}c_\infty} - e^{-2\pi \sqrt{-1}c_0} + e^{2\pi \sqrt{-1}(c_0 + c_1 - \infty)}}{1 - e^{2\pi \sqrt{-1}c_1}} & (k = 1) \\ \frac{-e^{-2\pi \sqrt{-1}c_0}}{1 - e^{2\pi \sqrt{-1}c_1}} & (k = 2, \ldots, n), \end{cases}$$

$$I_h([\gamma_{10}], [\gamma^\vee_{k0}]) = \begin{cases} \frac{e^{2\pi \sqrt{-1}(c_0 - 1)}(e^{2\pi \sqrt{-1}c_0} - e^{2\pi \sqrt{-1}c_1})}{e^{2\pi \sqrt{-1}c_0}(1 - e^{2\pi \sqrt{-1}c_1})} & (k = 1) \\ 0 & (k = 2, \ldots, n), \end{cases}$$

$$I_h([\gamma_{1j}], [\gamma^\vee_{k0}]) = \begin{cases} -1 & (1 \neq k < j) \\ \frac{1 - e^{2\pi \sqrt{-1}(c_0 + c_1 + \cdots + c_j)}}{1 - e^{2\pi \sqrt{-1}c_j}} & (j = k(> 1)) \\ 0 & (k > j), \end{cases}$$

- $H_{0n}$:

$$I_h([\gamma_{j0}], [\gamma^\vee_{n\infty}]) = \begin{cases} e^{2\pi \sqrt{-1}(c_n + c_\infty)} & (j = 1, 2, \ldots, n - 1) \\ e^{2\pi \sqrt{-1}c_0(1 - e^{2\pi \sqrt{-1}c_n})} + e^{2\pi \sqrt{-1}(c_0 + c_n + c_\infty)} & (j = n), \end{cases}$$

$$I_h([\gamma_{j0}], [\gamma^\vee_{n0}]) = \begin{cases} 0 & (j = 1, 2, \ldots, n - 1) \\ \frac{e^{2\pi \sqrt{-1}(c_0 + c_n - 1)}(e^{2\pi \sqrt{-1}c_0 - 1})}{e^{2\pi \sqrt{-1}c_0(1 - e^{2\pi \sqrt{-1}c_n})}} & (j = n), \end{cases}$$

- $H_{1\infty}$:

$$I_h([\gamma_{1\infty}], [\gamma^\vee_{j0}]) = \begin{cases} \frac{1 - e^{2\pi \sqrt{-1}(c_1 + c_\infty)}}{1 - e^{2\pi \sqrt{-1}c_1}} & (j = 1, 2, \ldots, n - 1) \\ \frac{e^{2\pi \sqrt{-1}(c_0 + c_\infty)} - e^{2\pi \sqrt{-1}c_0} - e^{2\pi \sqrt{-1}c_1} + e^{2\pi \sqrt{-1}(c_0 + c_1 + c_\infty)}}{1 - e^{2\pi \sqrt{-1}c_1}} & (j = n), \end{cases}$$

- $H_{0\infty}$:

$$I_h([\gamma_{j\infty}], [\gamma^\vee_{nk}]) = \begin{cases} 0 & (j = 1, 2, \ldots, n - 1) \\ \frac{e^{2\pi \sqrt{-1}(c_0 + c_n - 1)}(e^{2\pi \sqrt{-1}c_0 - 1})}{e^{2\pi \sqrt{-1}c_0(1 - e^{2\pi \sqrt{-1}c_n})}} & (j = n), \end{cases}$$

5.1.5. The twisted cycles $\gamma_{j\infty}$ and $\gamma^\vee_{j\infty}$ ($\bullet$ and $\circ$ are the intersection points.)
\[
I_h([\gamma_j], [\gamma^\vee_{nk}]) = \begin{cases} 
-\frac{e^{2\pi \sqrt{-1}c_0}}{e^{2\pi \sqrt{-1}c_1} (e^{2\pi \sqrt{-1}c_1} - e^{2\pi \sqrt{-1}(c_1 + \cdots + c_{j-1})})} & (k < j \neq n) \\
\frac{1}{1-e^{2\pi \sqrt{-1}c_0}} & (k = j (< n)) \\
0 & (k > j) \\
\frac{1}{1-e^{2\pi \sqrt{-1}c_0}} & (j = n).
\end{cases}
\]

- \( H_{1\infty} \):

\[
I_h([\gamma_{1\infty}], [\gamma^\vee_{k\infty}]) = \begin{cases} 
\frac{(-e^{2\pi \sqrt{-1}c_0} - e^{2\pi \sqrt{-1}c_1} - e^{2\pi \sqrt{-1}(c_1 + \cdots + c_{k-1})})}{e^{2\pi \sqrt{-1}c_1} (1-e^{2\pi \sqrt{-1}c_1})} & (k = 1) \\
0 & (k = 2, \ldots, n),
\end{cases}
\]

\[
I_h([\gamma_{0\infty}], [\gamma^\vee_{k\infty}]) = \begin{cases} 
\frac{e^{2\pi \sqrt{-1}c_0} (c_0 - c_1 - \cdots - c_{k-1})}{e^{2\pi \sqrt{-1}c_1} (1-e^{2\pi \sqrt{-1}c_1})} & (1 \neq k < j) \\
0 & (j = k (> 1)) \\
\frac{e^{2\pi \sqrt{-1}c_1 (1-e^{2\pi \sqrt{-1}c_1})}}{1-e^{2\pi \sqrt{-1}c_1}} & (k = 1).
\end{cases}
\]

- \( H_{\infty\infty} \):

\[
I_h([\gamma_{j\infty}], [\gamma^\vee_{0\infty}]) = \begin{cases} 
0 & (j = 1, 2, \ldots, n - 1) \\
\frac{(-e^{2\pi \sqrt{-1}c_0} - c_1 - \cdots + c_{j-1})}{e^{2\pi \sqrt{-1}c_0} (1-e^{2\pi \sqrt{-1}c_0})} & (j = n),
\end{cases}
\]

\[
I_h([\gamma_{j\infty}], [\gamma^\vee_{00}]) = \begin{cases} 
\frac{e^{2\pi \sqrt{-1}(c_1 + \cdots + c_{j-1})} - e^{2\pi \sqrt{-1}(c_0 + c_1 + \cdots + c_{j-1})}}{e^{2\pi \sqrt{-1}(c_1 + \cdots + c_{j-1})} (1-e^{2\pi \sqrt{-1}(c_1 + \cdots + c_{j-1})})} & (k < j \neq n) \\
\frac{e^{2\pi \sqrt{-1}(c_1 + \cdots + c_{j-1})} - e^{2\pi \sqrt{-1}(c_0 + c_1 + \cdots + c_{j-1})}}{e^{2\pi \sqrt{-1}(c_1 + \cdots + c_{j-1})} (1-e^{2\pi \sqrt{-1}(c_1 + \cdots + c_{j-1})})} & (k = j (< n)) \\
0 & (k > j)
\end{cases}
\]

5.2. Intersection numbers of twisted cocycles. Thanks to the formula \([\ref{12}]\), we can obtain the intersection number \(I_h([\varphi], [\varphi'])\) by

- finding a Laurent series solution \(f_l\) to the equation \(\nabla f_l = \varphi\) around \(u = t_l\), and
- evaluating the residue \(\text{Res}_{u=t_l} (f_l \varphi')\),
for each \(l = 1, \ldots, n\). If it is clear that \(f_l \varphi'\) is holomorphic around \(u = t_l\), an explicit form of \(f_l\) is not needed.

We first give some computation in a general setting. The logarithmic 1-form \(\omega\) has a Laurent series expansion \(\omega/du = \sum_{m=1}^{\infty} \alpha^{(l)}_m (u - t_l)^m\), where

\[
\alpha_{-1}^{(l)} = c_l, \quad \alpha_0^{(l)} = 2\pi \sqrt{-1}c_0 + \sum_{k \neq l} c_k \rho(t_l - t_k), \quad \alpha_1^{(l)} = \frac{c_l \varphi' (0)}{3}, \quad \alpha_2^{(l)} = \sum_{k \neq l} c_k \rho'(t_l - t_k), \ldots.
\]

We assume that \(\varphi\) has a Laurent expansion\(^3\)

\[
\frac{\varphi}{du} = \frac{a_{-2}}{(u - t_l)^2} + \frac{a_{-1}}{u - t_l} + a_0 + a_1(u - t_l) + \cdots
\]

\(^3\)Though \(a_m\) should be written as, for example, \(a^{(l)}_m\), we use this notation for simplicity.
To compute $I$, the property $s$ differentiation of the Laurent series of $f$ and hence the Laurent expansion of $f$

On the other hand, the Laurent expansion of $5.2.2$. calculation, we have

Theorem 4.3.

Thus, Lemma 5.1 shows

Lemma 5.1. If $ord_t(\varphi) + ord_t(\varphi') \geq -1$, then $Res_{u=t}(f_1u) = 0$.

5.2.1. Theorem [2,3] First, we compute the intersection number $I_c([\psi_j], [\psi_k^\prime])$. The differential form $\psi_j = s(u-t_j; \lambda)du$ has a pole at $u = t_j$ with residue 1, and it is holomorphic around $u = t_l$ if $l \neq j$. Thus, Lemma 5.1 shows $I_c([\psi_j], [\psi_k]) = 0$ for $k \neq j$ and

$\sum_{t \in S} f_1 \left(\frac{1}{u- \ldots} + \frac{1}{u-t_j} + \ldots\right)du = \frac{2\pi \sqrt{-1}}{c_j}$. 

5.2.2. Proposition [3,4] Next, we consider $I_c([\varphi_j (pq)], [\eta_k^{(pq)}])$. Since we have

$\frac{\varphi_{pq}}{\eta_{pq}} = \frac{\eta_{pq}}{\eta_{pq}} \left(\frac{1}{1} + \ldots + \frac{1}{1} + \ldots\right)du = \frac{2\pi \sqrt{-1}}{c_j}$. 

Thus, Lemma 5.1 shows $I_c([\varphi_j (pq)], [\eta_k^{(pq)}]) = 0$ for $j \neq k$ follows from Lemma 5.1 except for $(j, k) = (q, p)$. To compute $I_c([\varphi_q(pq)], [\eta_p^{(pq)}])$, we solve $\nabla f_p = \varphi(pq)$ around $u = t_p$. By considering the termwise differentiation of the Laurent series of $s(u-t_p; \lambda)$, we have

and hence the Laurent expansion of $f_p$ has the form of

$\frac{\varphi_{pq}}{\eta_{pq}} = \frac{\partial s}{\partial u}(u-t_p; \lambda) = \frac{-1}{(u-t_p)^2} + (\text{constant}) + \ldots$. 

On the other hand, the Laurent expansion of $\eta_{pq}$ is

$\frac{\eta_{pq}}{du} = s(u-t_p; \lambda) + \frac{1}{s(t_p-t_q)} \left(\frac{\alpha_0^{(p)}}{c_p} - \rho(-\lambda)\right)s(u-t_q; \lambda)

= \left(\frac{1}{u-t_p} + \rho(-\lambda) + \ldots\right) + \left(\frac{\alpha_0^{(p)}}{c_p} - \rho(-\lambda)\right) + \ldots = \frac{1}{u-t_p} + \frac{\alpha_0^{(p)}}{c_p} + \ldots$. 

Therefore, the intersection number $I_c([\varphi_q^{(pq)}], [\eta_p^{(pq)}])$ is
\[
I_c([\varphi_q^{(pq)}], [\eta_p^{(pq)}]) = 2\pi \sqrt{-1} \Res_{u=t_p} (f_p \eta_q^{(pq)})
\]
\[
= 2\pi \sqrt{-1} \left( \frac{-1}{c_p-1} \cdot \left( \frac{\alpha_0^{(p)}}{c_p} \right)^\vee + \frac{\alpha_0^{(p)}}{c_p(c_p-1)} \cdot 1 \right) = 0,
\]
because of the property $(\alpha_0^{(p)})^\vee = -\alpha_0^{(p)}$.

Let us compute $I_c([\varphi_q^{(pq)}], [\eta_p^{(pq)}])$. The intersection numbers for $j \neq q$ are obtained by Theorem 4.3 and $I_c([\varphi_q^{(pq)}], [\eta_p^{(pq)}])$ is computed as follows:
\[
I_c([\varphi_q^{(pq)}], [\eta_p^{(pq)}]) = 2\pi \sqrt{-1} \Res_{u=t_p} (f_p \eta_q^{(pq)})
\]
\[
= 2\pi \sqrt{-1} \Res_{u=t_p} \left( \frac{-1}{c_p-1} \cdot \frac{1}{u-t_p} + \frac{\alpha_0^{(p)}}{c_p(c_p-1)} + \cdots \right) \left( s(t_p-t_q; -\lambda) + \cdots \right) du
\]
\[
= -2\pi \sqrt{-1} \cdot \frac{s(t_p-t_q; -\lambda)}{c_p-1}.
\]

5.2.3. Theorem 4.5 Finally, we compute the intersection numbers when $\lambda = 0$. Except for $I_c([\varphi_{ii}], [\varphi_{ij}^\vee])$ $(j = 0, 1, \ldots, n)$, the intersection numbers in Theorem 4.5 can be obtained similarly to the above discussion. By Lemma 5.1 we have $I_c([\varphi_{ii}], [\varphi_{ij}^\vee]) = 2\pi \sqrt{-1} \Res_{u=t_i} (f_i \varphi_{ij}^\vee)$, where $f_i$ is a solution to $\nabla f_i = \varphi_{ii}$ around $u = t_i$. Since we have
\[
\frac{\varphi_{ii}}{du} = \rho'(u-t_i) = -\frac{1}{(u-t_i)^2} + \frac{\varphi'''_{ii}(0)}{3\varphi'_i(0)} + \cdots,
\]
the Laurent expansion of $f_i$ has the form of
\[
\frac{-1}{c_i-1} \cdot \frac{1}{u-t_i} - (-1) \cdot \frac{\alpha_0^{(i)}}{c_i(c_i-1)} + \left( \frac{\varphi'''_{ii}(0)}{3\varphi'_i(0)} - (-1) \cdot \frac{1}{c_i(c_i-1)(c_i+1)} \right) (u-t_i) + \cdots
\]
\[
= -\frac{-1}{c_i-1} \cdot \frac{1}{u-t_i} + \frac{\alpha_0^{(i)}}{c_i(c_i-1)} + \frac{1}{(c_i-1)(c_i+1)} \left( (c_i-1) \frac{\varphi'''_{ii}(0)}{3\varphi'_i(0)} + \alpha_1^{(i)} - \frac{(\alpha_0^{(i)})^2}{c_i} \right) (u-t_i) + \cdots.
\]
Therefore, by (5.1) and
\[
\varphi_{ii} = du, \quad \varphi_{ij} = \left( -\frac{1}{u-t_i} + \rho(t_i-t_j) + \cdots \right) du \quad (j \neq 0, i),
\]
the intersection numbers are computed as follows:
\[
\frac{I_c([\varphi_{ii}], [\varphi_{ij}])}{2\pi \sqrt{-1}} = -\frac{1}{c_i-1},
\]
\[
\frac{I_c([\varphi_{ii}], [\varphi_{ij}^\vee])}{2\pi \sqrt{-1}} = \frac{1}{(c_i-1)(c_i+1)} \left( (c_i-1) \frac{\varphi'''_{ii}(0)}{3\varphi'_i(0)} + \alpha_1^{(i)} - \frac{(\alpha_0^{(i)})^2}{c_i} \right) \cdot (-1) + \frac{-1}{c_i-1} \cdot \frac{\varphi'''_{ij}(0)}{3\varphi'_i(0)}
\]
\[
= \frac{1}{(c_i-1)(c_i+1)} \left( \frac{1}{c_i} \left( 2\pi \sqrt{-1} c_0 + \sum_{k \neq i} c_k \rho(t_i-t_k) \right)^2 - c_i \frac{\varphi'''_{ii}(0)}{\varphi'_i(0)} - \sum_{k \neq i} c_k \rho'(t_i-t_k) \right),
\]
\[
\frac{I_c([\varphi_{ii}], [\varphi_{ij}'])}{2\pi \sqrt{-1}} = \frac{\alpha_0^{(i)}}{c_i(c_i-1)} \cdot (-1) + \frac{-1}{c_i-1} \cdot \rho(t_i-t_j)
\]
\[
= -\frac{1}{c_i(c_i-1)} \left( 2\pi \sqrt{-1} c_0 + \sum_{k \neq i} c_k \rho(t_i-t_k) + c_i \rho(t_i-t_j) \right),
\]
where $j \neq 0, i$. 

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Acknowledgments. The author is grateful to Professors Toshiyuki Mano and Humihiko Watanabe for their helpful advice. This work was supported by JSPS KAKENHI Grant Number JP20K14276.

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