On compactness in the Trudinger-Moser inequality

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Abstract. We show that the Moser functional $J(u) = \int_\Omega (e^{4\pi u^2} - 1) \, dx$ on the set $\mathcal{B} = \{ u \in H^1_0(\Omega) : \| \nabla u \|_2 \leq 1 \}$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain, fails to be weakly continuous only in the following exceptional case. Define $g_s w(r) = s^{-\frac{1}{2}} w(r^s)$ for $s > 0$. If $u_k \rightharpoonup u$ in $\mathcal{B}$ while $\lim \inf J(u_k) > J(u)$, then, with some $s_k \to 0$,

$$u_k = g_{s_k} \left[ (2\pi)^{-\frac{1}{2}} \min \left\{ 1, \log \frac{1}{|x|} \right\} \right],$$

up to translations and up to a remainder vanishing in the Sobolev norm. In other words, the weak continuity fails only on translations of concentrating Moser functions. The proof is based on a profile decomposition similar to that of Solimini [16], but with different concentration operators, pertinent to the two-dimensional case.

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1. Introduction

The purpose of this paper is to study weak continuity properties in the Trudinger-Moser inequality at the same level of detail as the better-understood weak continuity properties of the critical nonlinearity in higher dimensions. We draw comparisons between the Sobolev inequality that defines the continuous imbedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}$, with $p^* = \frac{pN}{N-p}$, when $N > p$, and the Trudinger-Moser inequality (see Yudovich [22], Peetre [14], Pohozaev [15], Trudinger [21] and Moser [13]):

$$\sup_{\mathcal{B}} \int_{\Omega} e^{\alpha_N |u|^N} \, dx < \infty, \quad \mathcal{B} = \{ u \in W^{1,N}_0(\Omega) : \| \nabla u \|_N \leq 1 \},$$  (1.1)

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N' = \frac{N}{N-1}$, the constant $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ is the optimal constant (due to Moser [13]), and $\omega_{N-1}$ is the area of the unit $(N-1)$-

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