The Pfaff lattice and skew-orthogonal polynomials

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Abstract

Consider a semi-infinite skew-symmetric moment matrix, $m_\infty$ evolving according to the vector fields $\partial m/\partial t_k = \Lambda^k m + m \Lambda^\top k$, where $\Lambda$ is the shift matrix. Then The skew-Borel decomposition $m_\infty := Q^{-1} J Q^{-\top} - 1$ leads to the so-called Pfaff Lattice, which is integrable, by virtue of the AKS theorem, for a splitting involving the affine symplectic algebra. The tau-functions for the system are shown to be pfaffians and the wave vectors skew-orthogonal polynomials; we give their explicit form in terms of moments. This system plays an important role in symmetric and symplectic matrix models and in the theory of random matrices (beta=1 or 4).

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Lie algebra splitting. Throughout this paper the Lie algebra $\mathcal{D} = \text{gl}_\infty$ of semi-infinite matrices is viewed as composed of $2 \times 2$ blocks. It admits the natural decomposition into subalgebras:

$$\mathcal{D} = \mathcal{D}_- \oplus \mathcal{D}_0 \oplus \mathcal{D}_+ = \mathcal{D}_- \oplus \mathcal{D}_0^- \oplus \mathcal{D}_0^+ \oplus \mathcal{D}_+$$  \hspace{1cm} (0.1)

where $\mathcal{D}_0$ has $2 \times 2$ blocks along the diagonal with zeroes everywhere else and where $\mathcal{D}_+$ (resp. $\mathcal{D}_-$) is the subalgebra of upper-triangular (resp. lower-triangular) matrices with $2 \times 2$ zero matrices along $\mathcal{D}_0$ and zero below (resp. above). As pointed out in (0.1), $\mathcal{D}_0$ can further be decomposed into two Lie subalgebras:

$$\mathcal{D}_0^- = \{ \text{all } 2 \times 2 \text{ blocks } \in \mathcal{D}_0 \text{ are proportional to Id} \}$$

$$\mathcal{D}_0^+ = \{ \text{all } 2 \times 2 \text{ blocks } \in \mathcal{D}_0 \text{ have trace } 0 \}.$$  \hspace{1cm} (0.2)

Consider the semi-infinite skew-symmetric matrix $J$, zero everywhere, except for the following $2 \times 2$ blocks, along the “diagonal”,

$$J = \begin{pmatrix}
0 & 1 & \quad & 0 & 1 & \quad & \vdots \\
-1 & 0 & \quad & -1 & 0 & \quad & \ddots
\end{pmatrix} \in \mathcal{D}_0^+, \text{ with } J^2 = -I, \hspace{1cm} (0.3)
$$

and the associated Lie algebra order 2 involution

$$\mathcal{J} : \mathcal{D} \longrightarrow \mathcal{D} : a \mapsto \mathcal{J}(a) := Ja^\top J.$$  \hspace{1cm} (0.4)

The splitting into two Lie subalgebras

$$\mathcal{D} = k + n, \text{ with } k = \mathcal{D}_- + \mathcal{D}_0^- \text{ and } n = \{ a + \mathcal{J}a, a \in \mathcal{D} \} = \text{sp}(\infty), \hspace{1cm} (0.5)$$

\footnote{Note $n$ is the fixed point set of $\mathcal{J}$.}
with corresponding Lie groups $G_k$ and $G_n = Sp(\infty)$, will play a crucial role here. Let $\pi_k$ and $\pi_n$ be the projections onto $k$ and $n$. Notice that $n = sp(\infty)$ and $G_n = Sp(\infty)$ stand for the infinite rank affine symplectic algebra and group; e.g., see [10]; the exposition in this paper is totally self-contained and does not depend on any knowledge of the affine symplectic algebra.

**Pfaff Lattice.** As will be shown in this paper, the Lax pair, which we call the Pfaff lattice

$$\frac{\partial L}{\partial t_i} = [-\pi_k \nabla H_i, L] = [\pi_n \nabla H_i, L], \quad \text{with} \quad H_i = \frac{\text{tr} L_i}{i + 1},$$

\[(Pfaff \ Lattice)\]

on matrices $L = QAQ^{-1}$, with $Q \in G_k$ and $\Lambda$ the customary shift operator, is completely integrable, as a result of the AKS-theorem. The Lax pair, written in compact form in (0.6), is given explicitly in (1.4). In [1], this lattice was investigated from the point of view of the 2d-Toda lattice with special initial conditions. In [7], we developed the wave- and tau-function theory for this lattice and we exhibited a map from the Toda to the Pfaff lattice.

**Linearizing vector fields and Pfaffian $\tilde{\tau}$-functions.** The key to this system is its linearization (moment map): namely, if $L = QAQ^{-1}$ flows according to (0.6), then the following skew-symmetric matrix, constructed from $Q$,

$$m_\infty := Q^{-1}JQ^{-1},$$

flows as

$$\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty + m_\infty \Lambda^k, \quad k = 1, 2, 3, \ldots.$$ \[(0.7)\]

Having constructed the skew-symmetric matrix $m_\infty(t) = (\mu_{ij}(t))_{0 \leq i,j \leq \infty}$, we define the skew-symmetric submatrices

$$m_n(t) = (\mu_{ij}(t))_{0 \leq i,j \leq n-1}, \quad \text{with} \quad \tilde{\tau}_n(t) = \det^{1/2} m_2(t), \quad t \in \mathbb{C}^\infty.$$ \[(0.8)\]

---

1 $G_k$ is the group of invertible elements in $k$, i.e., lower-triangular matrices, with non-zero $2 \times 2$ blocks proportional to $\text{Id}$ along the diagonal.

2 The Hamiltonians $H_i$ are viewed as formal sums; the convergence of this formal sum would require some sufficiently fast decay of the entries of $L$. Since $\nabla H_i = L^i$, one does not need to be concerned about this point.
Note that, since $m_n$ is skew-symmetric, $\det m_{2n+1} = 0$, whereas $\det m_{2n}$ is a perfect square, with $\det^{1/2} m_{2n}$ being the Pfaffian; for a definition, specifying the sign, see section 3. The technology of letting $m_\infty$ flow in time and Borel decomposing $m_\infty$ has been used quite extensively in other situations, like the standard Toda lattice or the 2-Toda lattice [1].

The so-called Pfaffian $\tilde{\tau}$-function is not a KP $\tau$-function, but enjoys different bilinear identities and Hirota-type bilinear equations [1],

$$\left\{ \tilde{\tau}_{2n}(t-[u]), \tilde{\tau}_{2n}(t-[v]) \right\} + (u^{-1} - v^{-1})(\tilde{\tau}_{2n}(t-[u])\tilde{\tau}_{2n}(t-[v]) - \tilde{\tau}_{2n}(t)\tilde{\tau}_{2n}(t-[u] - [v])) = uv(u-v)\tilde{\tau}_{2n-2}(t-[u] - [v])\tilde{\tau}_{2n+2}(t), \text{ where } [u] = (u, u^2, \ldots).$$

In particular,

$$\left( p_{k+4}(\bar{\partial}) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}} \right) \tilde{\tau}_{2n} \circ \tilde{\tau}_{2n} = p_k(\bar{\partial}) \tilde{\tau}_{2n+2} \circ \tilde{\tau}_{2n-2},$$

for $k, n = 0, 1, 2, \ldots$. Both these equations are reminiscent of the differential Fay identities and the KP equations for usual $\tau$-functions [1, 13], but with an additional term on the right hand side, involving $\tilde{\tau}_{2n-2}$ and $\tilde{\tau}_{2n+2}$.

For $k = 0$, this KP-like equation has already appeared in the context of the charged BKP hierarchy, studied by V. Kac and van de Leur [11]; the precise relationship between the charged BKP hierarchy of Kac and van de Leur and the Pfaff Lattice, introduced here, deserves further investigation.

**Skew-Borel decomposition and skew-orthogonal polynomials.**

Starting with equations (0.7), we show they have a unique semi-infinite skew-symmetric matrix solution $m_\infty(t)$, provided the initial condition satisfies $\det m_{2n} \neq 0$, for $n = 0, 1, 2, \ldots$. It is given in terms of its (unique) “skew Borel decomposition” [15]

$$m_\infty = Q^{-1}JQ^{T-1}, \quad \text{with } Q \in G_k.$$  

(0.10)

Here, we view $m_\infty$ as a matrix, providing the “inner-product” between monomials,

$$\langle y^i, z^j \rangle := (m_\infty)_{ij} =: \mu_{ij}. \quad (0.11)$$

4unique, modulo a sequence of ±-signs.
Then the lower-triangular matrix $Q$ is conveniently described in terms of a sequence of so-called "skew-orthonormal polynomials" $q(z) = (q_0(z), q_1(z), ...)^\top = Q\chi(z)$ satisfying $\langle q_i, q_j \rangle_{i,j \geq 0} = J$.

Skew-orthonormal polynomials were first introduced by Mehta \[12\]; see also Brézin and Neuberger \[8\], where they appeared in the context of unoriented random surfaces. To the best of our knowledge, neither the form, nor the connection with the Pfaff lattice was known.

We show the polynomials $q(t, z)$ in $z$, depending on $t$, are explicitly given by pfaffians of skew-symmetric matrices, which in the even case ($q_{2n}$'s) are formed by replacing in $m_{2n+2}$ the $2n+2$th row and column by powers of $z$ and in the odd case ($q_{2n+1}$'s), by replacing the $2n+1$th row and column by the same powers of $z$, keeping the skewness of the matrices:

$$q_{2n}(t, z) = (\tilde{\tau}_{2n}\tilde{\tau}_{2n+2})^{-1/2} pf \begin{pmatrix}
0 & \mu_{01} & \ldots & \mu_{0,2n} & 1 \\
-\mu_{01} & 0 & \ldots & \mu_{1,2n} & z \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\mu_{0,2n} & -\mu_{1,2n} & \ldots & 0 & z^{2n} \\
-1 & -z & \ldots & -z^{2n} & 0 \\
\end{pmatrix}$$

$$q_{2n+1}(t, z) = (\tilde{\tau}_{2n}\tilde{\tau}_{2n+2})^{-1/2} pf \begin{pmatrix}
0 & \mu_{01} & \ldots & 1 & \mu_{0,2n+1} \\
-\mu_{01} & 0 & \ldots & z & \mu_{1,2n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\mu_{0,2n+1} & -\mu_{1,2n+1} & \ldots & 0 & -z^{2n+1} \\
-1 & -z & \ldots & 0 & 0 \\
\end{pmatrix}$$

$$= (\tilde{\tau}_{2n}\tilde{\tau}_{2n+2})^{-1/2} \left( z + \frac{\partial}{\partial t_1} \right) (\tilde{\tau}_{2n}\tilde{\tau}_{2n+2})^{1/2} q_{2n}(t, z).$$

(0.12)
Finally, the matrix
\[ L = Q\Lambda Q^{-1}, \] satisfying 
\[ zq(t, z) = L(t)q(t, z), \] (0.13)
forms a unique semi-infinite solution to the Lax pair (0.6).

It is interesting to point out the striking similarity with the orthogonal polynomials, the Toda lattice and the matrix models, described in [3]. Both, standard Toda and Pfaff lattices can be viewed as reductions of the 2-Toda lattice, where the initial condition \( m_\infty \) is a Hankel matrix for the standard Toda lattice and a skew-symmetric matrix for the Pfaff lattice. In the former case, the motion takes place within the big stratum and in the latter, within a deeper stratum.

1 The Pfaff Lattice and a Lie algebra splitting

Remember the decomposition (0.1) of the Lie algebra into subalgebras:
\[ \mathcal{D} = \mathfrak{gl}_\infty = \mathcal{D}_- \oplus \mathcal{D}^0 \oplus \mathcal{D}_+ = \mathcal{D}_- \oplus \mathcal{D}^-_0 \oplus \mathcal{D}^+_0 \oplus \mathcal{D}_+. \] (1.1)

Remember the element \( J \in \mathcal{D}_0^+ \), with \( J^2 = -I \) and the associated map:
\[ \mathcal{J} : \mathcal{D} \longrightarrow \mathcal{D} : a \longmapsto \mathcal{J}(a) := J a^\top J. \] (1.2)

**Theorem 1.1** The vector fields:
\[ \frac{\partial L}{\partial t_i} = [-\pi_k \nabla \mathcal{H}_i, L] = [\pi_n \nabla \mathcal{H}_i, L] \quad \text{with} \quad \mathcal{H}_i = \frac{\text{tr} L^{i+1}}{i+1}, \] (1.3)

or written out as:
\[ \frac{\partial L}{\partial t_i} = \left[ -\left( (L^i)_- - \mathcal{J}(L^i)_+ \right) - \frac{1}{2} \left( (L^i)_0 - \mathcal{J}(L^i)_0 \right) , L \right] \]
\[ = \left( (L^i)_+ + \mathcal{J}(L^i)_+ \right) + \frac{1}{2} \left( (L^i)_0 + \mathcal{J}(L^i)_0 \right) , L \], (1.4)

all commute.

---

6 Isospectral deformations on tridiagonal symmetric matrices.

7 \( \nabla \) denotes the gradient in the Lie algebra \( \mathcal{D} \) with respect to the natural pairing \( \mathcal{D} \ast \mathcal{D} \), i.e., \( \nabla \mathcal{H}_i = L_i \).

8 \( \mathcal{J}(L^i)_+ \) means \( \mathcal{J} \left( (L^i)_+ \right) \); the same for \( \mathcal{J}(L^i)_0 \).
The proof of this theorem hinges on a number of Lemmas and the Adler-Kostant-Symes theorem \[2\]:

**Lemma 1.2** \( J \) is a Lie algebra isomorphism and an order 2 involution on \( D \):

\[
J^2 = I, 
\] (1.5)

with \( N_+ \) and \( N_- \) being the \( \pm 1 \) eigenspaces of \( J \):

\[
N_\pm = \{ a \text{ such that } Ja = \pm a \} = \{ b \pm Jb \text{ with } b \in D \}. 
\] (1.6)

Then \( D \) is decomposed into the Lie algebra \( N_+ \) and the (symmetric) vector space \( N_- \):

\[
D = N_+ + N_- \quad \text{with [} N_+, N_\pm \text{] } \subset N_+ \text{ and [} N_+, N_- \text{] } \subset N_-,
\] (1.7)

with

\[
N_\pm = D_0^+ \oplus (N_\pm \setminus D_0),
\] (1.8)

and

\[
N_\pm \setminus D_0 = \{ a \pm Ja \text{ such that } a \in D_- \text{ or } D_+ \}
\]

\[
D_0^+ = D_0 \cap N_\pm = \{ a \pm Ja \text{ such that } a \in D_0 \}. 
\] (1.9)

**Proof**: The map \( J \) is a Lie algebra homomorphism, such that \( J^2 = I \); indeed, upon using \( J^2 = -I \),

\[
J [(a, b)] = J[a, b]^\top J = -J[a^\top, b^\top]J = [Ja^\top, Jb^\top J] = [Ja, Jb].
\]

In particular, \( J(a) = \pm a \) and \( J(b) = \pm b \) imply

\[
J[a, b] = [\pm a, \pm b] = [a, b],
\]

leading to the inclusions (1.7); so \( N_+ \) is a Lie subalgebra of \( D \), namely an infinite-dimensional version of the symplectic algebra \( \text{sp}(\infty) \). The second description (1.6) of \( N_\pm \) follows from

\[
J(b \pm Jb) = Jb \pm J^2b = \pm(b \pm Jb), \quad \text{for } b \in D;
\]

and the vector space decomposition \( D = N_+ + N_- \):

\[
a = \frac{1}{2}(a + Ja) + \frac{1}{2}(a - Ja).
\]
Since $JD_{\varepsilon} \in D_{\varepsilon}, D_{\varepsilon}^T = D_{-\varepsilon}$, for \( \varepsilon = +, -, 0 \), we have
\[
J : D_{\varepsilon} \rightarrow D_{-\varepsilon},
\]
and so, if \( a \in D_+ \), we have \( b := Ja \in D_- \) and
\[
N_{\pm} \supset a \pm Ja = J^2 a \pm Ja = \pm (b \pm Jb) \quad \text{with} \quad b = Ja \in D_-.
\]
Therefore
\[
\{ b \pm Jb \text{ such that } b \in D_- \} = \{ a \pm Ja \text{ such that } b \in D_+ \} = N_{\pm} \setminus D_0
\]
establishing the first relation (1.9). As to the second, one checks that
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\pm
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^T
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
= \begin{pmatrix}
a - d & 2b \\
2c & -(a - d)
\end{pmatrix}
\begin{pmatrix}
a + d
\end{pmatrix}
I
\]
(1.10)

**Lemma 1.3** The decomposition
\[
D = k \oplus n,
\]
with
\[
k = D_- \oplus D_0^- \quad \text{and} \quad n = N_+ = \{ a \text{ such that } Ja = a \} = sp(\infty),
\]
is a vector space splitting of $D$ into Lie subalgebras. Any element $a \in D$ decomposes uniquely into its projections onto $k$ and $n$:
\[
a = \pi_k a + \pi_n a \\
= \left\{ (a_- - J a_+) + \frac{1}{2} (a_0 - Ja_0) \right\} + \left\{ (a_+ + J a_+) + \frac{1}{2} (a_0 + Ja_0) \right\}.
\]

**Proof:** Since $[D_-, D_-] \subset D_-$, since $D_- D_0, \ D_0 D_- \subset D_-$ and $[D_0^-, D_0] = 0$, we have that $[k, k] = D_- \subset k$, so that $k = D_- \oplus D_0^-$ and $n = N_+$ are both Lie subalgebras.

But also $k \cap n = 0$; indeed, a typical element of $N_+$ has the form,
\[
(b_- + J(b_-)) + (d_0 + J(d_0)) \in (N_+ \setminus D_0) + (N_+ \cap D_0) = (N_+ \setminus D_0) + D_0^+,
\]

with \( b_- \in \mathcal{D}_- \), \( d_0 \in \mathcal{D}_0 \). If this element \( k = \mathcal{D}_- + \mathcal{D}_0^- \), then \( \mathcal{J}(b_-) = 0 \) and thus \( b_- = 0 \), but then, since \( d_0 + \mathcal{J}d_0 \in \mathcal{D}_0^+ \cap \mathcal{D}_0^- = 0 \). Finally, any element \( a \in \mathcal{D} \) can be decomposed into

\[
a = a_- + a_0 + a_+ = \left\{ (a_- - \mathcal{J}a_+) + \frac{1}{2}(a_0 - \mathcal{J}a_0) \right\} + \left\{ (a_+ + \mathcal{J}a_+) + \frac{1}{2}(a_0 + \mathcal{J}a_0) \right\} \in (\mathcal{D}_- + \mathcal{D}_0^-) \oplus ((\mathcal{N}_+ \mathcal{D}_0) + \mathcal{D}_0^+) = k \oplus n.
\]

We now state the R-matrix version of the AKS-theorem (see [2] and [13]). The substance of the R-matrix extension is that initial conditions \( \xi(0) \) can be taken in all of \( \mathcal{g} \), instead of merely in \( \mathcal{k} \).

**Proposition 1.4 (Adler-Kostant-Symes, Russian flavor)** Let \( \mathcal{g} = \mathcal{k} + \mathcal{n} \) be a (vector space) direct sum of a Lie algebra \( \mathcal{g} \) in terms of Lie subalgebras \( \mathcal{k} \) and \( \mathcal{n} \), with \( \mathcal{g} \) paired with itself via a non-degenerate \( \text{ad} \)-invariant inner product \( \langle \cdot, \cdot \rangle \); this in turn induces a decomposition \( \mathcal{g} = \mathcal{k} \perp \mathcal{n} \perp \) and isomorphisms \( \mathcal{g} \simeq \mathcal{g}^* \), \( \mathcal{k} \perp \simeq \mathcal{n}^* \), \( \mathcal{n} \perp \simeq \mathcal{k}^* \). It also

leads to a second Lie algebra \( \mathcal{g}_R \simeq \mathcal{g}_R^* \) derived from \( \mathcal{g} \), namely:

\[
\mathcal{g}_R : [x, y]_R = \frac{1}{2}[Rx, y] + \frac{1}{2}[x, Ry] = [\pi_k x, \pi_k y] - [\pi_n x, \pi_n y],
\]

with \( R = \pi_k - \pi_n \). \( \text{Ad}^* \simeq \text{Ad} \)-invariant functions \( \varphi \) on \( \mathcal{g}^* \simeq \mathcal{g} \) Poisson commute for the Kostant-Kirillov Poisson structure

\[
\{ f, h \}_R(\xi) := \langle \xi, [\nabla h(\xi), \nabla f(\xi)]_R \rangle \quad \text{on} \quad \mathcal{g}^*_R \simeq \mathcal{g}_R.
\]

The associated Hamiltonian flows are expressed in terms of the Lax pair:

\[
\dot{\xi} = [-\pi_k \nabla \varphi(\xi), \xi] = [\pi_n \nabla \varphi(\xi), \xi] \quad \text{for} \quad \xi \in \mathcal{g}_R.
\]

The systems has a solution expressible in two different ways:

\[
\xi(t) = \text{Ad}_{K(t)}\xi_0 = \text{Ad}_{N^{-1}(t)}\xi_0 \quad (1.11)
\]

---

9. \( a_\pm = \pi_{\mathcal{D}_\pm} a, \quad a_0 = \pi_{\mathcal{D}_0} a \).
10. \( \langle \text{Ad}_g X; Y \rangle = \langle X, \text{Ad}_g Y \rangle, \quad g \in G \), and thus \( \langle [z, x], y \rangle = \langle x, -[z, y] \rangle \).
11. \( \nabla \varphi \) is defined as the element in \( \mathcal{g}^* \) such that \( d\varphi(\xi) = \langle \nabla \varphi, d\xi \rangle, \quad \xi \in \mathcal{g} \).
12. Naively written \( \text{Ad}_{K(t)}\xi_0 = K(t)\xi_0 K(t)^{-1}, \quad \text{Ad}_{N^{-1}(t)}\xi_0 = N^{-1}(t)\xi_0 N(t) \).
with

\[ K(t) = \pi G_k e^{t \nabla \varphi(\xi_0)}, \quad \text{and} \quad N(t) = \pi G_n e^{t \nabla \varphi(\xi_0)}. \]

**Proof of Theorem 1.1:** Identifying \( k \) and \( n \) of Lemma 1.3, with those of proposition 1.4 and unraveling the projection \( \pi_k \), according to Lemma 1.3, imply that the vector fields (1.3) and so (1.4) all commute. \( \blacksquare \)

## 2 The vector fields \( \partial m/\partial t_k = \Lambda^k m + m \Lambda^{\top k} \)

The main claim of this section can be summarized in the following statement:

**Theorem 2.1** Consider the skew-symmetric solution

\[ m_\infty(t) = e^{\sum t_k \Lambda^k} m_\infty(0) e^{\sum t_k \Lambda^{\top k}} \]

to the commuting equations

\[ \frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty + m_\infty \Lambda^{\top k}, \quad (2.1) \]

with skew-symmetric initial condition \( m(0) \) and its “skew-Borel decomposition” \[m_\infty = Q^{-1}JQ^{-1\top}, \quad \text{with} \quad Q \in G_k. \quad (2.2)\]

Then the matrix \( Q \) evolves according to the equations

\[ \frac{\partial Q}{\partial t_i} Q^{-1} = -\pi_k \left( Q \Lambda^i Q^{-1} \right) \quad (2.3) \]

and the matrix \( L := Q \Lambda Q^{-1} \) provides a solution to the Lax pair

\[ \frac{\partial L}{\partial t_i} = \left[ -\pi_k L^i, L \right] = \left[ \pi_n L^i, L \right]. \quad (2.4) \]

Conversely, if \( Q \in G_k \) satisfies (2.3), then \( m_\infty \), defined by (2.2), satisfies (2.1).

\[ ^{\text{13}}\text{with regard to the group factorization } A = \pi G_k A \pi G_n A. \]

\[ ^{\text{14}}\text{Remember } G_k \text{ denotes the group of invertible elements in } k. \]
The proof of this theorem hinges on the following proposition:

**Proposition 2.2** For the matrices

\[ L := QAQ^{-1} \quad \text{and} \quad m := Q^{-1}JQ^{-1\top}, \quad \text{with} \ Q \in G_k, \]

the following three statements are equivalent

(i) \[ \frac{\partial Q}{\partial t_i} Q^{-1} = -\pi_k L^i \]

(ii) \[ L^i + \frac{\partial Q}{\partial t_i} Q^{-1} \in \mathfrak{n} \]

(iii) \[ \frac{\partial m}{\partial t_i} = \Lambda^i m + m\Lambda^\top i. \]

Whenever the vector fields on \( Q \) or \( m \) satisfy (i), (ii) or (iii), then the matrix \( L = QAQ^{-1} \) is a solution of the Lax pair

\[ \frac{\partial L}{\partial t_i} = [-\pi_k L^i, L] = [\pi_n L^i, L]. \]

**Proof:** Written out, proposition 2.2 amounts to showing the equivalence of the three formulas:

(I) \[ \frac{\partial Q}{\partial t_i} Q^{-1} + \left((L^i)^{-} - J(L^i^+)^\top J\right) + \frac{1}{2} \left((L^i)_{0} - J((L^i)_{0})^\top J\right) = 0 \]

(II) \[ \left(L^i + \frac{\partial Q}{\partial t_i} Q^{-1}\right) - J \left(L^i + \frac{\partial Q}{\partial t_i} Q^{-1}\right)^\top J = 0 \]

(III) \[ \Lambda^i m + m\Lambda^\top i - \frac{\partial m}{\partial t_i} = 0. \]

The point is to show that

\[ (I)_+ = 0, \quad (I)_- = (II)_- = -J (II_+)^\top J, \quad (I)_0 = \frac{1}{2} (II)_0, \]

\[ Q^{-1}(II)JQ^{-1\top} = (III). \quad (2.5) \]

The fact that \( Q \in G_k \subset k = \mathcal{D}_- \oplus \mathcal{D}_0^- = \mathcal{D}_- \oplus (\mathcal{D}_0 \cap N^-) \) amounts to

\[ Q_+ = 0 \quad \text{and} \quad (Q + JQ^\top J)_0 = 0. \]

The latter statement implies that \( Q \) is lower-triangular and, along the diagonal, is composed of \( 2 \times 2 \) blocks proportional to the identity. Hence

\[ Q_0, \dot{Q}_0 \in \mathcal{D}_0^- \quad \text{and} \quad (\dot{Q}Q^{-1})_0 \in \mathcal{D}_0^- \]
and thus $J$ commutes with $(\dot{Q}Q^{-1})_0$, yielding:

$$J(\dot{Q}Q^{-1})_0J = J^2(\dot{Q}Q^{-1})_0 = -(\dot{Q}Q^{-1})_0.$$ (2.6)

Also notice that for any matrix $A$

$$(JAJ)_\pm = JA_\pm J \quad \text{and} \quad (JAJ)_0 = JA_0J.$$ 

At first, one observes that

$$(I)_+ = 0. \quad (2.7)$$

Also

$$\begin{align*}
(II)_- &= \left( L^i + \dot{Q}Q^{-1} - J(L^i + \dot{Q}Q^{-1})^\top J \right)_-
\equiv \left( L^i + \dot{Q}Q^{-1} - J((L^i)_+)^\top J \right)_-, \quad \text{using } (\dot{Q}Q^{-1})_+ = 0 \\
&= (I)_-
\end{align*}$$

$$\begin{align*}
(II)_0 &= \left( L^i + \dot{Q}Q^{-1} - J(L^i + \dot{Q}Q^{-1})^\top J \right)_0
\equiv \left( L^i \right)_0 - J \left( L^i \right)^\top_0 J + \left( \dot{Q}Q^{-1} \right)_0 - J \left( (\dot{Q}Q^{-1})^\top \right)_0 J \\
&= (L^i)_0 - J(L^\top)_0 J + 2(\dot{Q}Q^{-1})_0, \quad \text{using } (2.6)
\equiv 2(I)_0 \quad (2.9)
\end{align*}$$

and

$$\begin{align*}
(II)_+ &= \left( L^i + \dot{Q}Q^{-1} - J(L^i + \dot{Q}Q^{-1})^\top J \right)_+
\equiv (L^i)_+ - J((L^i)_-)^\top J - J((\dot{Q}Q^{-1})_-)^\top J \\
&= \left( ((L^i)_+)^\top - J(L^i)_- J - J(\dot{Q}Q^{-1})_- J \right)^\top \\
&= J \left( (L^i)_+ ^\top J - J^2(L^i)_- J^2 - J^2(\dot{Q}Q^{-1})_- J^2 \right)^\top J \\
&= -J \left( (L^i + \dot{Q}Q^{-1} - J(L^\top)_+ J)_- \right)^\top J \\
&= -J((I)_-)^\top J. \quad (2.10)
\end{align*}$$
Using the definitions \( L = QAQ^{-1}, m = Q^{-1}JQ^{-1\top} \) and \( J = -I \), one finds

\[
\text{(III)} = \Lambda^\top \frac{\partial m}{\partial t_i} + \partial m_i + \Lambda^\top Q^{-1}(JQ^{-1\top})_{\partial t_i} + \partial m_i + \Lambda^\top (JQ^{-1\top})_{\partial t_i} = \Lambda^\top \frac{\partial m}{\partial t_i} + \partial m_i + \Lambda^\top Q^{-1}(JQ^{-1\top})_{\partial t_i} + \partial m_i + \Lambda^\top (JQ^{-1\top})_{\partial t_i}.
\]

The five facts (2.7) up to (2.11) imply (2.5) and so the desired equivalences.

3 Skew-orthogonal polynomials

As before, consider the skew-symmetric solution

\[
m_\infty(t) = e^{\sum t_k \Lambda^k} m_\infty(0) e^{\sum t_k \Lambda^k}
\]

to the commuting equations (2.1). We now perform the “Borel decomposition” of \( m_\infty \):

\[
m_\infty = Q^{-1}JQ^{-1\top},
\]

where \( Q \in \mathcal{G}_k \) (lower-triangular invertible matrices in \( k \)). To carry out this factorization, we found the following useful algorithm, based on the observation that the skew-Borel decomposition is tantamount to finding skew-orthogonal polynomials, in the same way that Borel decomposing a Hänkel matrix is tantamount to finding orthogonal polynomials.

Define the skew-inner product \( \langle f, g \rangle \) between formal Taylor series, induced by

\[
\langle y^i, z^j \rangle := \mu_{ij}(t) = (m_\infty(t))_{ij}.
\]

Monic polynomials \( p_k \) are said to be skew-orthogonal, when

\[
\langle (p_i, p_j) \rangle_{0 \leq i, j < \infty} = J\tilde{h}, \text{ with } \tilde{h} \in D_0^\top.
\]
They, in turn, lead to skew-orthonormal polynomials $q_k(z)$,

$$q_{2n}(t, z) = \frac{1}{\sqrt{\hat{h}_{2n}}} p_{2n}(t, z) \quad \text{and} \quad q_{2n+1}(t, z) = \frac{1}{\sqrt{\hat{h}_{2n}}} p_{2n+1}(t, z),$$

(3.3)

thus satisfying

$$\langle q_i, q_j \rangle_{0 \leq i, j < \infty} = J.$$  
(3.4)

Indeed, finding a set of skew-orthonormal polynomials is tantamount to the “skew-Borel decomposition” (3.1). Using the lower-triangular matrix $Q$ of coefficients of the polynomials $q(z) = Q\chi(z)$, we have

$$J = \left( \langle q_i, q_j \rangle \right)_{i, j \geq 0} = \left( \langle (Q\chi)_i, (Q\chi)_j \rangle \right)_{i, j \geq 0} = Q \left( \langle y^i, z^j \rangle \right)_{i, j \geq 0} Q^\top = Qm_\infty Q^\top,$$

(3.5)

from which (3.1) follows, and vice-versa.

Given a skew-symmetric matrix

$$A = (a_{ij})_{0 \leq i, j \leq n-1}, \quad \text{even } n,$$

we define the Pfaffian by means of the formula\footnote{In the formula below $(i_0, i_1, ..., i_{n-2}, i_{n-1}) = \sigma(0, ..., n-1)$, where $\sigma$ is a permutation and $\varepsilon(\sigma)$ its parity.}

$$\text{pf}(A)dx_0 \wedge ... \wedge dx_{n-1}$$

$$= \frac{1}{(n/2)!} \left( \sum_{0 \leq i < j \leq n-1} a_{ij} dx_i \wedge dx_j \right)^{n/2}$$

$$= \frac{1}{2^{n/2}(n/2)!} \left( \sum_{\sigma} \varepsilon(\sigma) a_{i_0, i_1} a_{i_2, i_3} ... a_{i_{n-2}, i_{n-1}} \right) dx_0 \wedge ... \wedge dx_{n-1},$$

(3.6)

so that $\text{pf}(A)^2 = \det A$, but with a sign specification for $\text{pf}(A)$. Henceforth, we shall be using the notation

$$\text{pf}(i_0, i_1, ..., i_{\ell-1}), \quad 0 < i_0 < i_1 < ... < i_{\ell-1}, \quad \ell \text{ even},$$

to denote the pfaffian of the skew-symmetric matrix, formed by the $i_0, i_1, ..., i_{\ell-1}$th rows and columns of $m_\infty$.\footnote{In the formula below $(i_0, i_1, ..., i_{n-2}, i_{n-1}) = \sigma(0, ..., n-1)$, where $\sigma$ is a permutation and $\varepsilon(\sigma)$ its parity.}
Theorem 3.1 Consider a semi-infinite skew-symmetric matrix $m_\infty$, satisfying (2.1); then the monic polynomials $p_k$

$$p_{2n}(t, z) = \frac{1}{\tilde{T}_{2n}(t)} pf \begin{pmatrix}
0 & \mu_{01} & \ldots & \mu_{0,2n} & 1 \\
-\mu_{01} & 0 & \ldots & \mu_{1,2n} & z \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\mu_{0,2n} & -\mu_{1,2n} & \ldots & 0 & z^{2n} \\
-1 & -z & \ldots & -z^{2n} & 0
\end{pmatrix}$$

$$p_{2n+1}(t, z) = \frac{1}{\tilde{T}_{2n}} pf \begin{pmatrix}
0 & \mu_{01} & \ldots & 1 & \mu_{0,2n+1} \\
-\mu_{01} & 0 & \ldots & z & \mu_{1,2n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\mu_{0,2n+1} & -\mu_{1,2n+1} & \ldots & 0 & -z^{2n+1} \\
-1 & -z & \ldots & -z^{2n+1} & 0
\end{pmatrix}$$

$$= \frac{1}{\tilde{T}_{2n}(t)} \left( z + \frac{\partial}{\partial t_1} \right) \tilde{T}_{2n}(t)p_{2n}(t, z).$$

(3.7)

form a skew-orthogonal sequence, $\langle p_i, p_j \rangle_{0 \leq i, j < \infty} = J\tilde{h}$, where

$$\tilde{h} = (\tilde{h}_0, \tilde{h}_0, \tilde{h}_2, \tilde{h}_2, \ldots) \in D_0^\infty, \text{ with } \tilde{h}_{2n} = \frac{\tilde{T}_{2n+2}}{\tilde{T}_{2n}}, \quad \tilde{T}_{2n} = pf(0, \ldots, 2n-1).$$

whereas the $q_k$, defined in (3.3) from the $p$'s, form an orthonormal sequence. The matrix $Q$ defined by $q(z) = QX(z)$ is the unique solution (modulo signs) to the skew-Borel decomposition of $m_\infty$:

$$m_\infty(t) = Q^{-1}JQ^{-1}, \quad \text{with } Q \in k. \quad (3.8)$$

The matrix $L = QAQ^{-1}$, also defined by

$$zq(t, z) = Lq(t, z),$$

and the diagonal matrix $\tilde{h}$ satisfy the equations

$$\frac{\partial L}{\partial t_i} = \left[ -\pi_k L^i, L \right]. \quad \text{and} \quad \tilde{h}^{-1}\frac{\partial \tilde{h}}{\partial t_i} = 2\pi_k (L^i)_0. \quad (3.9)$$

Remark: It is of interest to mention that, except for the second expression for $p_{2n+1}$ in (3.7), the first part of theorem 3.1 is actually $t$-independent. Namely, (3.7) gives a geometric map, but also an effective algorithm, to perform the skew-Borel decomposition of a skew-symmetric matrix $m_\infty$, by applying formulae (3.7).
Corollary 3.2  The polynomials \( (p_0(t, z), p_1(t, z), ...) \) have the following explicit form in terms of Pfaffians of the moment matrix \( m \) and the Pfaffian \( \tilde{\tau} \)-functions:

\[
p_{2n}(z) = \sum_{0 \leq k \leq 2n} (-z)^k \frac{pf(0, ..., \hat{k}, ..., 2n)}{pf(0, ..., 2n - 1)}
\]

\[
p_{2n+1}(z) = z^{2n+1} + \sum_{0 \leq k \leq 2n-1} (-z)^k \frac{pf(0, ..., \hat{k}, ..., \hat{2n}, 2n + 1)}{pf(0, ..., 2n - 1)}.
\]

(3.10)

Remark: Taking into account the identity \( pf(\text{odd set}) = 0 \) and the convention

\[
pf(0, ..., \hat{k}, ..., \hat{2n}, 2n + 1) = -pf(0, ..., 2n - 1) \quad \text{for} \quad k = 2n + 1,
\]

the polynomials can be written as follows:

\[
p_{2n}(t, z) = \frac{1}{\tilde{\tau}_{2n}(t)} \sum_{0 \leq k \leq 2n} (-z)^k pf(0, ..., \hat{k}, ..., 2n, 2n + 1)
\]

\[
p_{2n+1}(t, z) = \frac{1}{\tilde{\tau}_{2n}(t)} \sum_{0 \leq k \leq 2n+1} (-z)^k pf(0, ..., \hat{k}, ..., \hat{2n}, 2n + 1).
\]

(3.11)

The proof of this theorem hinges on a number of lemmas, involving properties of Pfaffians:

Lemma 3.3  Consider an arbitrary skew-symmetric matrix \( a = (a_{ij})_{i,j \geq 0} \). For odd \( \ell \) and fixed \( i \geq \ell \), the following holds:

\[
\sum_{0 \leq k \leq \ell - 1} (-1)^k a_{ki} pf(0, ..., \hat{k}, ..., \ell - 1) = pf(0, ..., \ell - 1, i).
\]

Proof: On the one hand

\[
\det(a_{ij})_{0 \leq i,j \leq \ell} = pf^2(0, 1, ..., \ell)
\]

(3.12)
and, on the other hand, using

\[
\begin{vmatrix}
0 & ... & a_{0r} & | & a_{0k} \\
... & & ... & | & ...
\end{vmatrix}
\]

\[
\begin{vmatrix}
-a_{0r} & ... & 0 & | & a_{r} \\
... & & ... & | & ...
\end{vmatrix}
\]

\[-a_{00} & ... & a_{\ell r} & | & \ast
\]

\[
= pf(0, \ldots, r, k) pf(0, \ldots, r, \ell),
\]

and expanding according to the last row and column

\[
det(a_{ij})_{0 \leq i,j \leq \ell}
\]

\[
= - \sum_{0 \leq i,j \leq \ell - 1} (-1)^{i+j} a_{i\ell} a_{j\ell} pf(0, \ldots, \hat{i}, \ldots, \ell - 1) pf(0, \ldots, \hat{j}, \ldots, \ell - 1)
\]

\[
= \left( \sum_{0 \leq i \leq \ell - 1} (-1)^{i} a_{i\ell} pf(0, \ldots, \hat{i}, \ldots, \ell - 1) \right) \left( \sum_{0 \leq j \leq \ell - 1} (-1)^{j} a_{\ell j} pf(0, \ldots, \hat{j}, \ldots, \ell - 1) \right)
\]

\[
= \left( \sum_{0 \leq i \leq \ell - 1} (-1)^{i} a_{i\ell} pf(0, \ldots, \hat{i}, \ldots, \ell - 1) \right)^2.
\]

Comparing the two expressions (3.12) and (3.13) and taking into account the sign, lead to the claim of lemma 3.3.

**Remark:** Lemma 3.3 will be applied as follows (for odd \(\ell\)):

\[
\sum_{0 \leq k \leq \ell - 1} (-1)^{k} a_{i_{k}i_{\ell}} pf(i_{0}, \ldots, \hat{i}_{k}, \ldots, i_{\ell-1}) = pf(i_{0}, \ldots, i_{\ell-1}, i_{\ell}).
\]

**Lemma 3.4** The polynomials \(p_k(z)\) defined by the determinantal expression (3.7) equal the expressions (3.10) of Corollary 3.2 and satisfy

\[
\langle p_i(z), p_j(z) \rangle_{i,j \geq 0} = J \tilde{h}
\]

**Proof:** At first notice that the leading coefficients of the pfaffians in the polynomials (3.7) equal \(pf(0, 1, \ldots, 2n - 1) = \tilde{\tau}_{2n}(t)\). Then apply Lemma 3.3 (remark following the lemma) to the first case, setting the
a's equal to μ's from 0 to 2n and \( a_{k,2n+1} = (1 - \delta_{k,2n+1})z^k \). Also apply Lemma 3.3 to the second case, by setting the a's equal to μ, from 0 to 2n + 1, skipping 2n, and \( a_{k,2n} = (1 - \delta_{k,2n})z^k \). Then, we find:

\[
\tilde{\tau}_{2n}(t)p_{2n}(t, z) = \sum_{0 \leq k \leq 2n} (-z)^k pf(0, \ldots, \hat{k}, \ldots, 2n - 1, 2n, \overline{2n+1})
\]

\[
= \sum_{0 \leq k \leq 2n} (-z)^k pf(0, \ldots, \hat{k}, \ldots, 2n)
\]

\[
\tilde{\tau}_{2n}(t)p_{2n+1}(t, z) = \sum_{0 \leq k \leq 2n+1} (-z)^k pf(0, \ldots, \hat{k}, \ldots, 2n - 1, \overline{2n}, 2n + 1)
\]

\[
= \sum_{0 \leq k \leq 2n-1} (-z)^k pf(0, \ldots, \hat{k}, \ldots, \overline{2n}, 2n + 1) + z^{2n+1} pf(0, \ldots, 2n - 1).
\]

(3.15)

Using the expressions (3.15) and Lemma 3.3, one computes the inner-product

\[
\langle p_{2n}(z), z^i \rangle = \sum_{0 \leq k \leq 2n} (-1)^k \mu_{ki} \frac{pf(0, \ldots, \hat{k}, \ldots, 2n)}{pf(0, \ldots, 2n - 1)}
\]

\[
= \frac{pf(0, \ldots, 2n, i)}{pf(0, \ldots, 2n - 1)},
\]

which vanishes for \( 0 \leq i \leq 2n \), and so

\[
\langle p_{2n}(z), z^i \rangle = 0 \quad \text{for} \quad 0 \leq i \leq 2n;
\]

therefore

\[
\langle p_{2n}(z), p_i(z) \rangle = 0 \quad \text{for} \quad 0 \leq i \leq 2n.
\]

Moreover, using the expression for \( p_{2n+1} \) and Lemma 3.3,

\[
\langle p_{2n+1}(z), z^i \rangle = \mu_{2n+1,i} + \sum_{0 \leq k \leq 2n-1} (-1)^k \mu_{ki} \frac{pf(0, \ldots, \hat{k}, \ldots, 2n)}{pf(0, \ldots, 2n - 1)}
\]

\[
= \frac{pf(0, \ldots, 2n, 2n + 1, i)}{pf(0, \ldots, 2n - 1)}
\]

\[
= \begin{cases} 
0 & \text{for } 0 \leq i \leq 2n + 1, i \neq 2n \\
\frac{pf(0, \ldots, 2n, 2n + 1, i)}{pf(0, \ldots, 2n - 1)} & \text{for } i = 2n.
\end{cases}
\]

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and so
\[ \langle p_{2n+1}(z), p_i(z) \rangle = -\delta_{i,2n} \tilde{h}_2 \quad \text{for} \quad 0 \leq i \leq 2n + 1; \]
the skew-symmetry of \( \langle \cdot, \cdot \rangle \) does the rest. \hfill \Box

**Lemma 3.5** We have
\[ \frac{\partial}{\partial t_k} p_f(i_0, \ldots, i_\ell) = \sum_{r=0}^\ell p_f(i_0, \ldots, i_r + k, \ldots, i_\ell). \]

**Proof:** The differential equations (2.1) for \( m_\infty(t) \) read in terms of the elements \( \mu_{ij} \):
\[ \frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k,j} + \mu_{i,j+k}; \quad (3.16) \]
hence, setting \( \sigma(i) = i' \), we compute
\[ \frac{\partial}{\partial t_k} p_f(i_0, \ldots, i_{n-1}) \]
\[ = \frac{1}{(n/2)!2^{n/2}} \sum_\sigma \varepsilon(\sigma) \sum_{r \in \{0, \ldots, n-2\} \atop r \text{ even}} \mu_{i_0,i'_1} \cdots (\mu_{i_{r+k},i'_{r+1}} + \mu_{i'_{r},i'_{r+1}+k}) \cdots \mu_{i_{n-2},i'_{n-1}} \]
\[ = \sum_{r=0}^{n-1} p_f(i_0, \ldots, i_r + k, \ldots, i_{n-1}), \]
on the right hand side appropriately. \hfill \Box

**Lemma 3.6** The polynomials \( p_k \) of Theorem 3.1 satisfy:
\[ \frac{1}{\tilde{r}_{2n}(t)} \left( \frac{\partial}{\partial t_1} + z \right) \tilde{r}_{2n}(t)p_{2n}(z) = p_{2n+1}(z). \]

**Proof:** Using the explicit form (3.10) of \( p_{2n} \) and Lemma 3.5, one computes
\[ \left( z + \frac{\partial}{\partial t_1} \right) \tilde{r}_{2n}(t)p_{2n}(z) \]
\[ = -\sum_{k=0}^{2n} (-z)^{k+1} p_f(0,1,\ldots,\hat{k},\ldots,2n) \]
+ \sum_{k=0}^{2n} (-z)^k(p_f(0, \ldots, k-1, \ldots, 2n) + p_f(0, \ldots, \hat{k}, \ldots, 2n, 2n+1))
\right)
+ \sum_{k=1}^{2n-1} (-z)^k(p_f(0, \ldots, \hat{k}, \ldots, 2n, 2n+1) + p_f(1, \ldots, \hat{2n}, 2n+1)
= \hat{r}_{2n}(t)p_{2n+1}(z),
\end{align*}

ending the proof.

Lemma 3.7 The decomposition
\begin{equation}
m_\infty = Q^{-1}JQ^\top^{-1}, \quad with \ Q \in \mathcal{G}_k, \tag{3.17}
\end{equation}
is unique (modulo signs) and is completely given by the skew-orthonormal polynomials \(q_k(z)\).

\textbf{Proof:} In view of relations (3.5), the skew-orthonormal polynomials (3.3) provide a solution to the Borel decomposition \(m_\infty = Q^{-1}JQ^\top^{-1}\), with \(Q \in \mathcal{G}_k\). We now show this \(Q \in \mathcal{G}_k\) is unique! To do so, we proceed by induction on \(q_k = (Q \chi(z))_k\). Assuming the existence of another matrix \(\tilde{Q} \in \mathcal{G}_k\), such that \(m_\infty = \tilde{Q}^{-1}J\tilde{Q}^\top^{-1}\), we must show \(q = \tilde{q}\). Indeed \(m_\infty = Q^{-1}JQ^\top^{-1}\) leads unambiguously (except for the sign) to
\begin{align*}
q_0 &= \mu_0^{-1/2}, \quad and \quad q_1 = \mu_0^{-1/2}z.
\end{align*}
But, for some \(\alpha_i\) and \(\beta_i\), we would have, taking into account the absence of the \(z^{2n}\)-term in \(\tilde{q}_{2n+1}\),
\begin{equation}
\tilde{q}_{2n} = \sum_{i=0}^{2n} \alpha_i q_i \quad and \quad \tilde{q}_{2n+1} = \alpha_2 q_{2n+1} + \sum_{i=0}^{2n-1} \beta_i q_i. \tag{3.18}
\end{equation}
Then using the inductive step, namely \(\tilde{q}_k = q_k\) for \(0 \leq k \leq 2n-1\), and the skew-orthonormality, we compute for \(\ell < n\),
\begin{align*}
0 &= \langle \tilde{q}_{2n}, \tilde{q}_{2\ell} \rangle = \langle q_{2n}, q_{2\ell} \rangle = -\alpha_{2\ell+1}
0 &= \langle \tilde{q}_{2n}, \tilde{q}_{2\ell+1} \rangle = \langle q_{2n}, q_{2\ell+1} \rangle = \alpha_{2\ell}
0 &= \langle \tilde{q}_{2n+1}, \tilde{q}_{2\ell} \rangle = \langle q_{2n+1}, q_{2\ell} \rangle = -\beta_{2\ell+1}
0 &= \langle \tilde{q}_{2n+1}, \tilde{q}_{2\ell+1} \rangle = \langle q_{2n+1}, q_{2\ell+1} \rangle = \beta_{2\ell},
\end{align*}
and so, since also $\beta_{2n} = 0$,

$$\tilde{q}_{2n} = \alpha_{2n}q_{2n} \quad \text{and} \quad \tilde{q}_{2n+1} = \alpha_{2n}q_{2n+1}.$$  

Finally

$$1 = \langle \tilde{q}_{2n}, \tilde{q}_{2n+1} \rangle = \alpha_{2n}^2 \langle q_{2n}, q_{2n+1} \rangle = \alpha_{2n}^2,$$

leading to $\alpha_{2n}^2 = 1$.

Proof of Theorem 3.1 and Corollary 3.2: The proof follows at once from lemmas 3.3 up to 3.7. The fact that $Q$ and $L$ satisfy the differential equations (2.3) and (2.4) follows from the fact that the matrix $m_\infty$ satisfies the differential equations

$$\frac{\partial m}{\partial t_k} = \Lambda^k m + m \Lambda^\top k,$$

according to Theorem 2.1.

The second relation in (3.9) is established as follows: defining the semi-infinite lower-triangular matrix $P$ (with 1’s along the diagonal) by means of the monic skew-orthogonal polynomials $p(z) = P\chi(z)$ and differentiating $P = \tilde{h}^{1/2}Q$ with regard to $t_k$, one obtains

$$\tilde{h}^{1/2}Q \tilde{Q}^{-1} \tilde{h}^{-1/2} = -\frac{1}{2}(\log \tilde{h})^* + \tilde{P} P^{-1}.$$

Then, using proposition 2.2, (formula (II)), one finds:

$$0 = \tilde{h}^{1/2}(\text{II})_0 \tilde{h}^{-1/2} = \tilde{h}^{1/2} \left( (L_0^k - J(L_0^k)J + 2(\tilde{Q}Q^{-1})_0 \right) \tilde{h}^{-1/2} = \tilde{h}^{1/2}2\pi_k(L_0^k - (\log \tilde{h})^* + 2(\tilde{P} P^{-1})_0 = 2\pi_k(L_0^k - (\log \tilde{h})^* ,$$

using the commutation of $\pi_k(L_0^k)$ with $\tilde{h}^{1/2} \in D_0^-$, together with $(\tilde{P} P^{-1})_0 = 0$.

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