Algebraic Numbers of the form $\alpha^T$ with $\alpha$ Algebraic and $T$ Transcendental

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**Abstract:** Let $\alpha \neq 1$ be a positive real number and let $P(x)$ be a non-constant rational function with algebraic coefficients. In this paper, in particular, we prove that the set of algebraic numbers of the form $\alpha^{P(T)}$, with $T$ transcendental, is dense in some open interval of $\mathbb{R}$.

**Keywords:** Gelfond–Schneider theorem; algebraic numbers; transcendence; Schanuel’s conjecture

**MSC:** Primary 11J81; Secondary 26CXX

**1. Introduction**

In 1900, in the International Congress of Mathematicians (in Paris), Hilbert provided a list of 23 problems and his seventh problem was about the arithmetic nature of the power $\alpha^\beta$ of two algebraic numbers $\alpha$ and $\beta$. In 1934–1935, Gelfond and Schneider [1] (p. 9) (independently) completely solved this problem: if $\alpha$ and $\beta$ are algebraic numbers, with $\alpha \neq 0$ or 1, and $\beta$ irrational. Then $\alpha^\beta$ is transcendental. This outstanding result is called the Gelfond–Schneider theorem.

As immediate consequences, we have the transcendence of the numbers $2\sqrt{2}$, $(-1)^{\sqrt{2}}$, and $e^{i\pi} = i^{-2i}$.

The Gelfond–Schneider theorem classifies the (arithmetic) nature of $x^y$, for algebraic numbers $x$ and $y$ (because $x^y$ is algebraic if $y$ is rational). However, when $\{x, y\} \not\subseteq \mathbb{Q}$, one may has all possibilities (see Table 1).

**Table 1.** Arithmetic possibilities for $z^w$ when either $z$ or $w$ is transcendental.

| Value of $z$ | Class of Numbers | Value of $w$ | Class of Numbers | Power $z^w$ | Class of Numbers |
|--------------|------------------|--------------|------------------|--------------|------------------|
| 2            | algebraic        | $\log 3/\log 2$ | transcendental   | 3            | algebraic        |
| 2            | algebraic        | $3/\log 2$    | transcendental   | $3^i$        | transcendental   |
| $e^i$        | transcendental   | $\pi$         | transcendental   | $-1$         | algebraic        |
| $e$          | transcendental   | $\pi$         | transcendental   | $e^\pi$      | transcendental   |
| $2\sqrt{2}$  | transcendental   | $\sqrt{2}$    | algebraic        | 4            | algebraic        |
| $2\sqrt[4]{2}$ | transcendental   | $i\sqrt{2}$   | algebraic        | $4^i$        | transcendental   |

The case $x = y$ may be of more interest: is it possible $T^T$ to be algebraic, for some transcendental number $T$? We point out that a negative answer for this question is expected (but still unproved), since the numbers $e^\pi$, $\pi^{\pi}$, and $(\log 2)^{\log 2}$ are expected to be transcendental. However, related to the previous question, Marques and Sondow [2] showed that its answer is Yes. In fact, they proved that...
the set of algebraic numbers of the form $T^T$ (where $T$ runs over all transcendental numbers) is dense in the interval $[e^{-1/e}, \infty)$.

In this direction, it may be interesting to consider the following generalization of the previous question: given arbitrary non-constant polynomials $P, Q \in \mathbb{Q}[x]$, is there always a transcendental number $T$, such that $P(T)Q(T)$ is algebraic? The answer for this question is also Yes and it was given by Marques [3]: the set of algebraic numbers of the form $P(T)Q(T)$, with $T$ transcendental, is dense in some connected subset of $\mathbb{R}$ or $\mathbb{C}$. See [4,5] for generalizations of this result and some new results in [6,7].

We still point out that, in 2020, Trojovský [8] proved some results related to the transcendence of numbers of the form $n^T$ for positive integers $n$ and complex numbers $\gamma$.

In this paper, we continue this program by studying the existence of algebraic numbers of the form $\alpha^T$, where $\alpha$ is algebraic and $T$ is transcendental. More precisely, our main result is the following:

**Theorem 1.** Let $\alpha \neq 1$ be a positive algebraic number and let $P(x) \in \overline{\mathbb{Q}}(x)$ be a non-constant rational function. Then the set of algebraic numbers of the form $\alpha^{P(T)}$, with $T$ transcendental, is dense in some open interval of $\mathbb{R}$.

**Example 1.** In particular, for $\alpha = 2$ and $P(x) = x$, we have that the previous result implies in the existence of an open interval $I \subseteq \mathbb{R}$ such that the set of algebraic numbers of the form $2^T$ with $T$ transcendental, is dense in $I$. As we shall see in the next section, it is possible to make this interval explicit (if $\alpha$ and $P(x)$ are previously given). In the case of $\alpha = 2$ and $P(x) = x$, we infer that $I = (0, \infty)$.

**Example 2.** We remark that the transcendence of $2^T$ still remains as an open problem. However, as an application of the previous example, we have the existence of a sequence $(T_n)_n$ of transcendental numbers, such that $2^{T_n} \in \overline{\mathbb{Q}}$, for all $n \geq 1$, and $2^{T_n}$ tends to $2^\gamma$, as $n \to \infty$.

**Remark 1.** We recall that a rational function $f(x)$ is the quotient of two polynomials, say $P(x)/Q(x)$ (with no common zeros). This function is defined in the whole complex plane but the set of zeroes of $Q(x)$ (which are poles of $f(x)$). The coefficients of $f(x)$ are defined as the coefficients of $P(x)$ and $Q(x)$.

A well-known generalization of the notion of an algebraic number is a “like-algebraic” tuple: the complex numbers $\alpha_1, \ldots, \alpha_n$ are said to be algebraically dependent if there exists a nonzero polynomial $P \in \mathbb{Q}[x_1, \ldots, x_n]$ such that $P(\alpha_1, \ldots, \alpha_n) = 0$. Otherwise, $\alpha_1, \ldots, \alpha_n$ are called algebraically independent (in particular, they are all transcendental numbers).

A major open problem in transcendental number theory is a conjecture raised by Schanuel (in the 1960s) during a course at Yale given by Lang [9] (pp. 30–31).

**Conjecture 1** (Schanuel’s conjecture (SC)). Let $\alpha_1, \ldots, \alpha_n$ be complex numbers which are $\mathbb{Q}$-linearly independent. Then there are at least $n$ algebraically independent numbers among

$$\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}.$$

We remark that Schanuel’s conjecture is proved only for $n = 1$: Lindemann’s theorem asserts that $e^\alpha$ is a transcendental number, for any non-zero algebraic number $\alpha$. As an immediate consequence, we deduce that $\log \alpha$ is transcendental for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ (since $e^{\log \alpha} = \alpha$ is algebraic).

There are many research topics on the deep consequences of the veracity of the Schanuel’s conjecture (for some of them, we refer the reader to [10,11] and references therein). In the opposite direction of Theorem 1, here, we still prove the following conditional result:

**Theorem 2.** Assume Schanuel’s conjecture (SC) and let $P(x) \in \overline{\mathbb{Q}}(x)$ be a non-constant rational function. If $\alpha \notin \{0, 1\}$ is an algebraic number, then $\alpha^{P(x)}$ is transcendental.
Example 3. We point out that even the transcendence of $2^e$ is still unproved. Thus, the previous result gives (conditionally) the transcendence of this number (for the choice of $\alpha = 2$ and $P(x) = 1$).

We point out that the above fact is an open problem if (SC) is not assumed to be true (even for a particular rational function $f(x)$). The number $e$ can be replaced by some other known transcendental numbers, as $\pi$, log 2, etc.

2. Proof of the Theorem 1

Before proceeding further, we shall provide a field-theoretical result which will be an essential ingredient in the proof.

We know that the set of the real algebraic numbers is dense in $\mathbb{R}$ (because $\mathbb{Q} \subseteq \overline{\mathbb{Q}} \cap \mathbb{R}$). Indeed, we know that the set $\mathbb{Q}_m$ of the $m$-degree real algebraic numbers is dense in $\mathbb{R}$, for all $m \geq 1$, see [12] (Theorem 7.4). Before the proof of the theorem, we shall need a slightly stronger fact. More precisely,

Lemma 1. For any $m \geq 1$, the set

$$A_m = \mathbb{Q}_m \setminus \{ \beta \in \mathbb{Q}_m : \beta^n \in \mathbb{Q}, \text{ for some } n \in \mathbb{Z}_{>0} \}$$

is dense in $\mathbb{R}$.

Proof. Let $p$ be a prime number. It suffices to prove that the set

$$\mathcal{P}_m := \{ Q(1 + \sqrt[p]{p}) : Q \in \mathbb{Q} \setminus \{0\} \} \subseteq A_m,$$

because clearly $\mathcal{P}_m$ is dense in $\mathbb{R}$. First, note that $Q(1 + \sqrt[p]{p})$ is an $m$-degree algebraic number, for any non-zero rational number $Q$. In fact, $1 + \sqrt[p]{p}$ is a root of $P(x) = (x-1)^m - p$ which is irreducible over $\mathbb{Q}$ (since $P(x+1)$ is irreducible by the Eisenstein’s irreducibility criterion).

Now, we must prove that

$$\mathcal{P}_m \cap \{ \beta \in \mathbb{Q}_m : \beta^n \in \mathbb{Q}, \text{ for some } n > 0 \} = \emptyset.$$

Towards a contradiction, we suppose the existence of an integer number $n > 0$ such that $(Q(1 + \sqrt[p]{p}))^n$ is a rational number. Thus, so is $(1 + \sqrt[p]{p})^n$. We know that $B = \{1, \sqrt[p]{p}, \ldots, (\sqrt[p]{p})^{m-1}\}$ is a basis of the field extension $\mathbb{Q}(\sqrt[p]{p})/\mathbb{Q}$. By the Binomial Theorem, $(1 + \sqrt[p]{p})^n = \sum_{k=0}^{n} \binom{n}{k}(\sqrt[p]{p})^k$.

If $n < m$, then $(1 + \sqrt[p]{p})^n \notin \mathbb{Q}$, by the $\mathbb{Q}$-linear independence of $B$. When $n \geq m$, we get

$$(1 + \sqrt[p]{p})^n = \sum_{k=0}^{m-1} \binom{n}{k}(\sqrt[p]{p})^k + \sum_{k=m}^{n} \binom{n}{k}(\sqrt[p]{p})^k.$$

We then rewrite the second summatory as

$$\sum_{k=m}^{n} \binom{n}{k}(\sqrt[p]{p})^k = \sum_{t=0}^{n-m} \binom{n}{t+m}(\sqrt[p]{p})^{m+t} = \sum_{t=0}^{n-m} \binom{n}{t+m}p(\sqrt[p]{p})^t.$$

If $n - m < m$, the result follows again by the $\mathbb{Q}$-linear independence of $B$. Otherwise, we have $n - m \geq m$ and so
\[
\sum_{j=0}^{n-m} \binom{n}{m+j} p(\sqrt[n]{P})^j = \sum_{j=0}^{m-1} \binom{n}{m+j} p(\sqrt[n]{P})^j + \sum_{s=0}^{n-2m} \binom{n}{2m+s} p^2(\sqrt[n]{P})^s.
\]

Repeating this procedure \(\ell = \lfloor n/m \rfloor\) times, we arrive at

\[(1 + \sqrt[n]{P})^n = \sum_{j=0}^{m-1} a_j(\sqrt[n]{P})^j,
\]

where \(a_j \in \mathbb{Z}_{\geq 0}\) and we used that \(0 \leq n - \ell m < m\). Thus \((1 + \sqrt[n]{P})^n \notin \mathbb{Q}\), again because \(\mathcal{B}\) is linearly independent over \(\mathbb{Q}\). \(\Box\)

Now, we are ready to deal with the proof of theorem.

**Proof.** Set \(P(x) = P_1(x)/P_2(x)\) and let us consider an open interval \(\mathcal{J} \subseteq \mathbb{R}\) such that \(P_2(x) \neq 0\), for all \(x \in \mathcal{J}\). Now, the function \(f : \mathcal{J} \to \mathbb{R}\) given by \(f(x) := a^{P(x)}\) is well-defined and analytical in \(\mathcal{J}\). We claim that \(f(x)\) is a non-constant function. Indeed, on the contrary, \(f'(x) = 0\), for all \(x \in \mathcal{J}\). However, this would imply that \(P(x)\) is constant (contradicting the hypothesis) since \(f'(x) = a^{P(x)}P'(x)\log a\). Thus, there exists \(x_0 \in \mathcal{J}\) with \(f'(x_0) \neq 0\). Therefore, by the Inverse Function Theorem, there are open connected intervals \(\mathcal{I}\) and \(\mathcal{I}_0\) (with \(x_0 \in \mathcal{I}\)) such that \(f : \mathcal{I} \to \mathcal{I}_0\) is a diffeomorphism (i.e., a differentiable function which has a differentiable inverse). In particular, \(f : \mathcal{I} \to \mathcal{I}_0\) is an open map (i.e., it maps open sets to open sets). In particular, \(\mathcal{I}_0 = f(\mathcal{I})\) is an open interval of \(\mathbb{R}\).

Set \(\mathcal{A}\) as the union of all sets \(\mathcal{A}_m\), where \(m\) runs over all positive integers coprime to \(\deg(a)\) (the degree of \(a\) over \(\mathbb{Q}\)). That is, \(\mathcal{A} = \bigcup_{m \geq 1, \gcd(m,\deg(a))=1} \mathcal{A}_m\).

By Lemma 1, the set \(\mathcal{A}\) is dense in \(\mathbb{R}\) and therefore \(\mathcal{A} \cap \mathcal{I}_0\) is dense in the interval \(\mathcal{I}_0\). Now, it suffices to prove that every element of this intersection has the desired form. In fact, if \(a \in \mathcal{A} \cap \mathcal{I}_0\), then \(A = a^{P(T)}\), for some \(T \in \mathcal{I}\) (since \(\mathcal{I}_0 = f(\mathcal{I})\)). We claim that \(T\) is a transcendental number. Indeed, assume that \(T\) is algebraic. Since \(a \notin \{0,1\}\), we use the Gelfand–Schneider theorem to conclude that \(P(T)\) is a rational number, say \(m/n\) with \(n > 0\). Thus, we can write \(A^n = a^m\). Hence \(\deg(A^n)\) divides \(\deg(a)\) (since \(\mathbb{Q}(a^n) \subseteq \mathbb{Q}(a)\)). Since \(\deg(A^n) | \deg(A)\) and \(\gcd(\deg(A),\deg(a)) = 1\), we conclude that \(\deg(A^n) = 1\) yielding that \(A^n\) is a rational number which contradicts the choice of \(A \in \mathcal{A}\). This contradiction implies that \(T\) is transcendental and the proof is complete. \(\Box\)

### 3. Proof of the Theorem 2

As usual, we denote by \(\text{trdeg}_\mathbb{Q}\{x_1, \ldots, x_n\}\) (the transcendence degree) the cardinality of the largest algebraically independent subset of \(\{x_1, \ldots, x_n\}\). Thus Schanuel’s conjecture can be restated as: if \(\alpha_1, \ldots, \alpha_n\) are linearly independent over \(\mathbb{Q}\), then

\[\text{trdeg}_\mathbb{Q}\{\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}\} \geq n.\]

By Lindemann’s theorem, the number \(\log a\) is transcendental and so \(1, \log a\) are \(\mathbb{Q}\)-linearly independent. We then use (SC) to obtain

\[\text{trdeg}_\mathbb{Q}\{1, \log a, e, a\} \geq 2.\]

On the other hand,
trdeg\(\mathbb{Q}(1, \log a, e, \alpha) = \text{trdeg}\(\mathbb{Q}(\log a, e) \leq 2\).

Thus \(e, \log a\) are algebraically independent which, in particular, implies in the \(\mathbb{Q}\)-linear independence of \(1, \log a, P(e) \log a\). In fact, any non-trivial \(\mathbb{Q}\)-relation
\[
a + b \log a + cP(e) \log a = 0,
\]
leads to the contradiction that \(R(\log a, e) = 0\), for \(R(x, y) = a + bx + cP(y)x\). So, we can apply (SC) again to obtain that
\[
\ell := \text{trdeg}\(\mathbb{Q}(1, \log a, P(e) \log a, e, \alpha, \alpha P(e)) \geq 3\).
\]
Since
\[
\ell = \text{trdeg}\(\mathbb{Q}(\log a, e, \alpha P(e)) \leq 3,
\]
we infer that \(\ell = 3\) and so \(\log a, e, \alpha P(e)\) are algebraically independent. This yields, in particular, the transcendence of \(\alpha P(e)\). The proof is then complete. \(\square\)

4. Conclusions

In this paper, we study the arithmetic nature of complex numbers of the form \(\alpha^T\). In particular, we prove that if \(\alpha \neq 1\) is a positive algebraic number, then \(\alpha^T\) is an algebraic number for infinitely many transcendental numbers \(T\). Moreover, we provide a conditional result (by assuming the veracity of Schanuel’s conjecture) about the transcendence of \(\alpha^{P(e)}\), for all non-constant rational function \(P(x)\) with algebraic coefficients. The proof combines classical results of transcendental theory (such as, Lindemann and Gelfond–Schneider theorem) together with tools from real analysis (such as the Inverse Function Theorem) and algebraic number theory (such as some properties of fields extensions).

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