Perturbed Iterate SGD for Lipschitz Continuous Loss Functions

Michael R. Metel1 · Akiko Takeda2,3

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Abstract
This paper presents an extension of stochastic gradient descent for the minimization of Lipschitz continuous loss functions. Our motivation is for use in non-smooth non-convex stochastic optimization problems, which are frequently encountered in applications such as machine learning. Using the Clarke $\epsilon$-subdifferential, we prove the non-asymptotic convergence to an approximate stationary point in expectation for the proposed method. From this result, a method with non-asymptotic convergence with high probability, as well as a method with asymptotic convergence to a Clarke stationary point almost surely are developed. Our results hold under the assumption that the stochastic loss function is a Carathéodory function which is almost everywhere Lipschitz continuous in the decision variables. To the best of our knowledge, this is the first non-asymptotic convergence analysis under these minimal assumptions.

Keywords Stochastic optimization · Lipschitz continuity · First-order method · Non-asymptotic convergence

Mathematics Subject Classification 62L20 · 68Q25 · 90C15 · 90C26

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Michael R. Metel
michael.metel@huawei.com
Akiko Takeda
takeda@mist.i.u-tokyo.ac.jp

1 Huawei Noah’s Ark Lab, Montréal, QC, Canada
2 Department of Creative Informatics, Graduate School of Information Science and Technology,
The University of Tokyo, Tokyo, Japan
3 RIKEN Center for Advanced Intelligence Project, Tokyo, Japan

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1 Introduction

The focus of this work is on unconstrained minimization problems of the form

$$\min_{w \in \mathbb{R}^d} f(w) := \mathbb{E}_{\xi}[F(w, \xi)],$$

(1)

where $\xi \in \mathbb{R}^p$ is a random vector from a probability space $(\Omega, \mathcal{F}, P)$. We assume that $F(w, \xi)$ is a Carathéodory function (see Sect. 3) which is Lipschitz continuous in $w$ for $\xi$ almost everywhere, implying that $f(w)$ is a Lipschitz continuous function. Given observed samples $\hat{\xi}_j$ of $\xi$ for $j = 1, 2, \ldots, n$, our work is also applicable when considering the expectation with respect to the samples’ associated empirical probability distribution,

$$f(w) := \frac{1}{n} \sum_{j=1}^{n} F(w, \hat{\xi}_j).$$

(2)

We do not assume the loss function to be differentiable nor convex. This class of functions is quite general and enables our work to be applicable for a wide range of loss functions used in practice. The lack of a smoothness assumption allows for functions used in deep learning models, such as the ReLU activation function, and functions used in sparse learning models, such as $L^1$-norm regularization. Further removing the requirement of convexity enables us to consider bounded loss functions, which are known to be more robust to outliers [30], including all bounded functions with Lipschitz continuous gradients, see Proposition A.1 in “Appendix.”

We propose a new first-order stochastic method with non-asymptotic convergence bounds for finding an approximate stationary point of problem (1) in expectation in terms of the Clarke $\epsilon$-subdifferential [17]. In this paper, $\partial f(w)$ will denote the Clarke subdifferential (see Eq. (4) in Sect. 3). A necessary condition for a point $\tilde{w}$ to be a minimizer of $f(w)$ is for it to be a stationary point, i.e., $0 \in \partial f(\tilde{w})$. It has been recently proven though that for $\epsilon \in [0, 1)$, there is no finite time algorithm which can guarantee an $\epsilon$-stationary point, $\text{dist}(0, \partial f(w)) \leq \epsilon$, for a directionally differentiable, bounded from below Lipschitz continuous function $f(w)$ using a family of 1-dimensional functions, see [33, Theorem 5]. Another difficulty is that without further assumptions placed on $f(w)$, $\partial f(w) \subset \mathbb{E}_\xi[\partial F(w, \xi)]$ [10, Theorem 2.7.2], so an unbiased estimate of a subgradient of $f(w)$ cannot be guaranteed by sampling stochastic subgradients of $\partial F(w, \xi)$. Motivated by the Gradient Sampling algorithm [6], these issues can be resolved by using a step direction computed by sampling the gradient of the stochastic function $F(w, \xi)$ at random points near each iterate. Our work is also motivated by papers such as [20, 23, 32], where the approximate function, $f_\sigma(w) = \mathbb{E}_z[f(w + \sigma z)]$, is shown to have a Lipschitz continuous gradient when $z$ follows a Normal distribution or a uniform distribution in a Euclidean ball. It has been shown in [19, Section 4.2] that this is an essentially optimal method of smoothing a non-smooth function.

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1 Any deterministic Lipschitz continuous function can be added to $f(w)$ without changing our analysis.
The next section contains a discussion about related works, where we highlight previous papers proving asymptotic convergence for algorithms minimizing Lipschitz continuous functions, as well as two papers which have proven non-asymptotic convergence results, with comparisons to this paper. Section 3 contains our assumptions and notation, the convergence criterion, as well as some necessary lemmas concerning Carathéodory functions which are Lipschitz continuous almost everywhere. Section 4 presents the algorithm Perturbed Iterate SGD (PISGD), with the main convergence result contained in Theorem 4.1, and accompanying computational complexity results for PISGD given in Corollaries 4.1, 4.2, and 4.3. In particular, Corollary 4.3 gives the computational complexity to find an approximate stationary point in high probability. A method to asymptotically converge to a Clarke stationary point almost surely is given in Corollary 4.4, and in Sect. 4.5, there is a discussion of applying PISGD for particular cases of $f(w)$. In Sect. 5, the developed convergence theory is applied to train a Lipschitz continuous feedforward neural network. The conclusion is given in Sect. 6, and “Appendix” contains proofs and some auxiliary results.

2 Related Works

2.1 Asymptotic Convergence Analysis

The Gradient Sampling algorithm [6] is proven to have asymptotic convergence to a stationary point for locally Lipschitz continuous functions which are continuously differentiable on an open set with full measure. In each iteration, the Gradient Sampling algorithm computes the minimum-norm vector $g$ in the convex hull of gradients sampled in a Euclidean ball near the current iterate, and uses a line-search method to determine the step size in the direction of $-g$, while also needing to ensure that the function is differentiable at the next iterate. The Gradient Sampling algorithm was inspired by the earlier analysis [7] of approximating the Clarke subdifferential by the convex hull of gradients sampled in a Euclidean ball.

In [12], the stochastic subgradient algorithm is studied, where it was proven that almost surely, every limit point is a Clarke stationary point for all locally Lipschitz continuous functions whose graphs are Whitney stratifiable, assuming iterates are bounded almost surely. This class of functions includes a wide range of applications, including standard architectures in deep learning.

Whereas the previous work’s results require vanishing step sizes, in [4] the convergence of constant step SGD is studied, where it is shown that SGD converges in probability to a Clarke stationary point when the constant step size approaches zero. Their problem setup is also similar to our own as they consider a Carathéodory function which is locally Lipschitz continuous in $w$ for each $\xi$.

Another interesting paper which studies non-convex, non-differentiable minimization is [3], which considers objective functions whose non-differentiability stems from simple kinks of the form $\max\{0, f_i(w)\}$ for a family of functions $\{f_i(w)\}$. A smoothing method is developed, inspired by penalty and multiplier methods for constrained optimization problems, and convergence to an optimal solution is established.
by minimizing increasingly accurate approximations of the original problem with a boundedness assumption on the partial derivatives of the objective.

2.2 Non-asymptotic Convergence Analysis

In [23], non-asymptotic convergence bounds for a zero-order stochastic algorithm are achieved for minimizing a Gaussian smoothed approximation, \( f_\sigma(w) = \mathbb{E}_z[f(w + \sigma z)] \), of a Lipschitz continuous function \( f(w) \), where \( z \) follows a Normal distribution and \( \sigma \geq 0 \) is a smoothing parameter. In each iteration, the algorithm takes a step in the direction of an unbiased estimate of \( \nabla f_\sigma(w) \). A computational complexity in terms of the number of iterations (or gradient estimates) of \( O\left(\frac{1}{\delta \epsilon^2}\right) \) is established for finding a solution satisfying \( \mathbb{E}(||\nabla f_\sigma(\bar{w})||^2) \leq \epsilon \), where \( |f_\sigma(w) - f(w)| \leq \delta \) for all \( w \in \mathbb{R}^d \). Gaussian smoothing is a class of mollifiers with unbounded support, see for example [13], where it is established that these types of averaged functions, when taking \( \sigma \to 0 \), converge to \( f \) when \( f \) is continuous, and can preserve infima even in cases where \( f \) is discontinuous.

The recent publication [33] seems to have been inspired by much of the same past research as our paper, resulting in similarities, most notably in the development of the same convergence criteria using the Clarke \( \epsilon \)-subdifferential. One of the biggest differences compared to our work is their assumption that the objective is directionally differentiable, resulting in a different analysis and algorithm design. In the stochastic setting, it is further assumed that an unbiased stochastic subgradient \( \hat{g} \) can be sampled, such that \( \mathbb{E}[\hat{g}] = g \in \partial f(w) \) and \( \langle g, d \rangle = f'(w, d) \), where \( f'(w, d) \) denotes the directional derivative. Two algorithms are presented in their work. In the deterministic setting where \( f(w) \) and a subgradient can be computed, an algorithm is presented with a convergence result in probability. In the stochastic setting, a convergence result in expectation is given. Using Markov’s inequality and rerunning the algorithm a logarithmic number of times, one of the solutions will be an approximate stationary point with high probability. A similar computational complexity to ours has been proven for their algorithm. It is also notable that unlike our convergence results and those of [23], their results are independent of the dimension size of the problem. A more detailed comparison of our convergence results is given in Sect. 4.3. Lipschitz continuous functions are generally not directionally differentiable, but with this further assumption directional stationarity could be considered, which gives a sharper definition of a stationary point than Clarke or even a Mordukhovich stationary point. For algorithms converging to directional stationary points for non-smooth non-convex optimization problems, see for example [24].

3 Preliminaries

3.1 Assumptions and Notation

We assume that the random vector \( \xi : \Omega \to \mathbb{R}^p \) is a \((\mathcal{F}, B_{\mathbb{R}^p})\)-measurable function where \( B_{\mathbb{R}^p} \) denotes the Borel \( \sigma \)-algebra on \( \mathbb{R}^p \), and that \( F(w, \xi) \) is a Carathéodory
function [1, Definition 4.50], which in our setting means that for each \( w \in \mathbb{R}^d \), \( F(w, \cdot) \) is \((\mathcal{B}_{\mathbb{R}^d}, \mathcal{B}_{\mathbb{R}})\)-measurable and for each \( \xi \in \mathbb{R}^p \), \( F(\cdot, \xi) \) is continuous. It follows that \( F(w, \xi) \) is a \((\mathcal{B}_{\mathbb{R}^d}, \mathcal{B}_{\mathbb{R}})\)-measurable function [1, Lemma 4.51]. The function \( F(w, \xi) \) is also assumed to be \( C(\xi) \)-Lipschitz continuous in \( w \) for almost every \( \xi \), implying that \( f(w) \) is \( L_0 := \mathbb{E}[C(\xi)] \)-Lipschitz continuous. In particular, the values of \( \xi \) for which \( F(w, \xi) \) is not Lipschitz continuous are contained in a Borel null set, whose complement of full measure is denoted as \( \Xi \). We further assume that \( Q := \mathbb{E}[C(\xi)^2] < \infty \). For a random variable \( X : \Omega \to \mathbb{R}^q \) for arbitrary \( q \in \mathbb{N} \), let \( P_X \) denote the image measure induced by the random variable \( X \), i.e. for a Borel set \( A \in \mathcal{B}_{\mathbb{R}^q} \), \( P_X(A) = P(\{\omega \in \Omega : X(\omega) \in A\}) \). Given that \( \mathbb{E}[C(\xi)^2] < \infty \), it holds that \( C(\xi) \in L^1(P_\xi) \) [15, Proposition 6.12].

Our analysis relies on randomly perturbed iterates, \( w = x + z \), where \( x \in \mathbb{R}^d \) represents the current iterate and \( z : \Omega \to \mathbb{R}^d \) is a random vector uniformly distributed in the \( d \)-dimensional Euclidean ball of radius \( \sigma > 0 \), \( B(\sigma) := \{z : \|z\|_2 \leq \sigma\} \), denoted as \( z \sim U(B(\sigma)) \). The probability density function of \( z \) is

\[
p(z) = \begin{cases} \frac{\Gamma\left(\frac{d}{2} + 1\right)}{(\sqrt{\pi} \sigma)^d} & \text{if } z \in B(\sigma) \\ 0 & \text{otherwise,} \end{cases}\]

where for \( d \in \mathbb{N} \), \( \Gamma\left(\frac{d}{2} + 1\right) = \left(\frac{d}{2}\right)! \) when \( d \) is even, \( \Gamma\left(\frac{d}{2} + 1\right) = 2^{-\left(\frac{d+1}{2}\right)} \sqrt{\pi} d!! \) when \( d \) is odd. The double factorial for \( d > 0 \) equals \( d!! = d(d-2)(d-4)\ldots(2 \text{ or } 1) \) for \( d \) even or odd, and \( 0!! = 1 \). The expected distance from the origin of \( z \) is

\[
\mathbb{E}[\|z\|_2] = \frac{\sigma d}{d + 1}.
\]

### 3.2 Convergence Criterion

The fact that \( f(w) \) is Lipschitz continuous implies that it is differentiable everywhere outside of a set of Lebesgue measure zero due to Rademacher’s theorem [14, Theorem 3.1.6]. This motivates the use of perturbed iterates as \( f(w) \) is then differentiable with probability 1 whenever its gradient is evaluated at \( w = x + z \). The proposed convergence criteria in this paper use the Clarke \( \epsilon \)-subdifferential. We first define the Clarke subdifferential, which for locally Lipschitz continuous functions on \( \mathbb{R}^d \) equals

\[
\partial f(w) := \text{co}\{ \lim_{i \to \infty} \nabla f(w_i) : w_i \to w, w_i \notin E \cup E_f \},
\]

where \( \text{co}\{\cdot\} \) denotes the convex hull, \( E \) is any set of Lebesgue measure 0, and \( E_f \) is the set of points at which \( f \) is not differentiable [10, Theorem 2.5.1]. The standard first-order convergence criterion for smooth non-convex functions is \( \epsilon \)-stationarity,

\[
\text{dist}(0, \partial f(w)) \leq \epsilon,
\]
but as highlighted in the introduction, proving a convergence rate to such a point for Lipschitz continuous functions is not possible [33, Theorem 5]. This leads us to consider the Clarke $\epsilon$-subdifferential,

$$\partial_{\epsilon} f(w) := \text{co}\{\partial f(\hat{w}) : \hat{w} \in w + B(\epsilon)\},$$

(6)

which is always a nonempty convex compact set with $\partial_{0} f(w) = \partial f(w)$ [17].

The gradient is always contained in the Clarke subdifferential wherever a Lipschitz continuous function is differentiable [10, Proposition 2.2.2], which motivates the use of the Clarke $\epsilon$-subdifferential, as for $z \sim U(B(\sigma))$ almost surely, $\nabla f(w) \in \partial_{\sigma} f(x)$ for $w = x + z$. Our focus is then on what we call $(\epsilon_{1}, \epsilon_{2})$-stationarity,

$$\text{dist}(0, \partial_{\epsilon_{1}} f(w)) \leq \epsilon_{2},$$

(7)

with the goal of designing a stochastic algorithm which can output a random solution $\hat{w}$ which is an $(\epsilon_{1}, \epsilon_{2})$-stationary point in expectation,

$$\mathbb{E}[\text{dist}(0, \partial_{\epsilon_{1}} f(\hat{w}))] \leq \epsilon_{2}.$$  

(8)

Relaxing the $\epsilon$-stationarity condition to finding a point which is a distance $\epsilon_{1}$ away from an $\epsilon_{2}$-stationary point, i.e., a point $\hat{w}$ such that

$$||\hat{w} - \hat{w}||_{2} \leq \epsilon_{1} \quad \text{and} \quad \text{dist}(0, \partial f(\hat{w})) \leq \epsilon_{2},$$

(9)

is an increasingly common convergence criterion for non-smooth non-convex objective functions, see for example [11, 29]. It has been shown though in [19, Proposition 1] that (7) does not imply (9), but we can see that (7) is a necessary condition for (9) to hold. In addition, (7) has recently been used as a convergence criteria in [33], where the equivalence between $(\epsilon_{1}, \epsilon_{2})$-stationarity and $\epsilon$-stationarity for functions with Lipschitz continuous gradients is shown in [33, Proposition 6]. For further discussion and examples of $(\epsilon_{1}, \epsilon_{2})$-stationary points, see [19].

### 3.3 Differentiability and Measurability Properties

The following lemma states that the gradient of $F(w, \xi)$ exists almost everywhere. Let $m^{d}$ denote the Lebesgue measure restricted to $(\mathbb{R}^{d}, B_{\mathbb{R}^{d}})$. The product measure $m^{d} \times P_{\xi}$ is unique given that $m^{d}$ and $P_{\xi}$ are $\sigma$-finite.

**Lemma 3.1** [4, Lemma 1] The stochastic function $F(w, \xi)$ is differentiable in $w$ almost everywhere on the product measure space $(\mathbb{R}^{d+p}, B_{\mathbb{R}^{d+p}}, m^{d} \times P_{\xi})$.

In order to handle the non-differentiability of $f(w)$, we define an approximate gradient of $f(w)$, $\hat{\nabla} f(w)$, to be a Borel measurable function on $\mathbb{R}^{d}$ which equals the gradient of $f(w)$ almost everywhere it is differentiable. We also define the approximate stochastic gradient of $F(w, \xi)$, $\hat{\nabla} F : \mathbb{R}^{d+p} \to \mathbb{R}^{d}$, as a Borel measurable function which is equal to the gradient of $F(w, \xi)$ almost everywhere it exists. For
completeness, an example of how to generate a family of approximate stochastic gradient functions $\tilde{\nabla} F(w, \xi)$ is given in “Appendix,” see Example A.1. As the function $f(w)$ is continuous, it is Borel measurable and the approximate gradient $\tilde{\nabla} f(w)$ can be constructed in the same manner as presented in the example.

The following lemma shows that unbiased estimates of the approximate gradient of $f(w)$ can be obtained by sampling the approximate stochastic gradient of $F(w, \xi)$ for almost all $w \in \mathbb{R}^d$.

**Lemma 3.2** For almost every $w \in \mathbb{R}^d$

$$\tilde{\nabla} f(w) = \mathbb{E}_\xi[\tilde{\nabla} F(w, \xi)],$$

where $\tilde{\nabla} f(w)$ and $\tilde{\nabla} F(w, \xi)$ are approximate gradients of $f(w)$ and $F(w, \xi)$, respectively.

In the following lemma, the measurability of $\text{dist}(0, \partial_\epsilon f(w))$ is verified.

**Lemma 3.3** For any $\epsilon \geq 0$, $\text{dist}(0, \partial_\epsilon f(w))$ is a Borel measurable function in $w \in \mathbb{R}^d$.

Lemma 3.1 was proven in [4], but for completeness we give a proof in “Appendix,” along with the proofs of Lemmas 3.2 and 3.3.

## 4 Perturbed Iterate SGD

### 4.1 Algorithm Overview

We now present PISGD. In each iteration $k$, $S$ perturbed values, $w^k_l$, of the current iterate $x^k$ are generated, and $S$ samples $\xi^k_l$ are taken for $l = 1, \ldots, S$. The stochastic function’s approximate gradient is evaluated at each pair $(w^k_l, \xi^k_l)$ to generate the step direction, where all sampling is done independently. Our analysis assumes that the perturbation level $\sigma$ and step size $\eta$ are constant. As it is generally difficult to analyze the convergence of the last iterate of a stochastic algorithm, we use the standard technique [16] of analyzing the average performance of the algorithm, which is equivalent to examining the convergence of a randomly chosen iterate $R$ out of a predetermined total of $K$.

**Algorithm 1** Perturbed Iterate SGD (PISGD)

```
Input: $x^1 \in \mathbb{R}^d$, $K, S \in \mathbb{Z}_{>0}$; $\eta, \sigma > 0$
$R \sim \text{uniform}[1, 2, \ldots, K]$
for $k = 1, 2, \ldots, R - 1$
do
  Sample $\xi^k_l \sim U(B(\sigma))$ for $l = 1, \ldots, S$
  $w^k_l = x^k + \xi^k_l$ for $l = 1, \ldots, S$
  Sample $\xi^k_l \sim P_\xi$ for $l = 1, \ldots, S$
  $x^{k+1} = x^k - \frac{\eta}{S} \sum_{l=1}^S \tilde{\nabla} F(w^k_l, \xi^k_l)$
end for
Output: $x_R$
```
In the following subsections, we give a number of results concerning PISGD which are now summarized:

- Section 4.2 gives our main non-asymptotic convergence result to an expected \((\epsilon_1, \epsilon_2)\)-stationary point in Theorem 4.1.
- Section 4.3 provides the computational complexity in terms of the number of stochastic approximate gradient computations for a range of algorithm settings in Corollary 4.1, as well as for an optimized setting in Corollary 4.2 for an expected \((\epsilon_1, \epsilon_2)\)-stationary point. Corollary 4.3 provides the computational complexity for an \((\epsilon_1, \epsilon_2)\)-stationary point with probability \((1 - \gamma)\) for any \(\gamma \in (0, 1)\).
- Section 4.4 contains a description of a general algorithm using PISGD which has asymptotic convergence guarantees to a Clarke stationary point proven in Corollary 4.4.
- In Sect. 4.5, the problem setting of \(f(w)\) being a deterministic Lipschitz continuous function without any stochastic structure is considered, as well as the case where \(f(w)\) takes the form of a finite-sum problem as in (2).

### 4.2 Non-asymptotic Convergence to an Expected \((\epsilon_1, \epsilon_2)\)-Stationary Point

The following theorem provides guarantees for the values of \(\epsilon_1\) and \(\epsilon_2\) such that PISGD converges to an \((\epsilon_1, \epsilon_2)\)-stationary point in expectation. Making \(K\) large enough, \(\epsilon_1\) and \(\epsilon_2\) can be made arbitrarily small. There is also a parameter \(\beta \in (0, 1)\) which adjusts the rate of convergence in terms of \(\epsilon_1\) and \(\epsilon_2\). For example, taking \(\beta = \frac{1}{3}\) the guarantees for \(\epsilon_1\) and \(\epsilon_2\) both improve at the same rate of \(O(K^{-\beta})\).

**Theorem 4.1** Let \(K \in \mathbb{Z}_{>0}, S = \lceil K^{1-\beta} \rceil\) for \(\beta \in (0, 1)\), \(\sigma = \theta \sqrt{dK^{-\beta}}\) and \(\eta = \frac{\theta}{L_0} K^{-\beta}\) for \(\theta > 0\), \(Q = \mathbb{E}[C(\xi)^2]\), and \(\Delta = f(x^1) - f(x^*)\), where \(f(x^*)\) is the global minimum of \(f(x)\). After running PISGD using an approximate stochastic gradient \(\tilde{\nabla} F(w, \xi)\),

\[
\mathbb{E}[\text{dist}(0, \partial_\sigma f(x^R))] < K^{\frac{\beta-1}{2}} \sqrt{2 \left( \frac{L_0}{\theta} \Delta + L_0^2 \sqrt{dK^{-\beta}} + Q \right)}.
\]  

(10)

Considering a standard implementation of mini-batch SGD with any chosen \(K\), \(S > 1\), and \(\eta\), PISGD can be implemented by choosing \(\sigma = \eta L_0 \sqrt{d}\) (using \(\theta = L_0 \eta K^\beta\)), and convergence guarantees can be computed in terms of an expected \((\epsilon_1, \epsilon_2)\)-stationary point. The parameter \(\theta > 0\) allows for convergence guarantees to be made for any positive step size. The sample size has been fixed in the theorem, but the theorem holds for any \(S \geq K^{1-\beta}\) (see Eq. (23)), so a valid \(\beta\) always exists for any choice of \(S > 1\) and \(K\). We also mention that following Lemma 3.1 and the fact that the approximate stochastic gradient of \(F(w^k_i, \xi^k)\) is evaluated a countable number of times, the probability of encountering a point of non-differentiability running Algorithm 1 is zero. To gain some intuition of how the right-hand side of (10) is made small enough to ensure an expected \((\epsilon_1, \epsilon_2)\)-stationary point, it can be replaced by the right-hand side of inequality (22) in the proof, which shows that \(K\) needs to be made large enough to overcome problem specific constants and the choice of smoothing.
which is less than or equal to $\epsilon_1$, and the sample size $S$ needs to be made large enough to make the variance of the stochastic step direction sufficiently small.

The proof of Theorem 4.1 requires the following three lemmas. The proofs can be found in “Appendix.”

**Lemma 4.1** For $\{x, x', z\} \in \mathbb{R}^d$, let $w = x + z$ and $w' = x' + z$. For a Lipschitz continuous function $f(\cdot)$ with approximate gradient $\tilde{\nabla} f(\cdot)$, and any $x, x' \in \mathbb{R}^d$,

$$f(w) - f(w') - \langle \tilde{\nabla} f(w'), x - x' \rangle = \int_0^1 \langle \tilde{\nabla} f(w' + v(x - x')) - \tilde{\nabla} f(w'), x - x' \rangle dv$$

holds for almost all $z \in \mathbb{R}^n$.

**Lemma 4.2** The norms of the approximate gradients are bounded, with $\|\tilde{\nabla} f(w)\|_2 \leq L_0$ and $\|\tilde{\nabla} F(w, \xi)\|_2 \leq C(\xi)$ almost everywhere.

**Lemma 4.3** Let $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^d$ be random variables, where $z$ is absolutely continuous, and $x, z, \xi$ (as previously defined) are mutually independent. For any $S \in \mathbb{Z}_{>0}$, let

$$\nabla F := \frac{1}{S} \sum_{l=1}^S \tilde{\nabla} F(x + z_l, \xi_l),$$

where $z_l \sim P_z$ and $\xi_l \sim P_\xi$ for $l = 1, \ldots, S$. It holds that

$$E[\|\nabla F\|_2^2 - \|E[\nabla F|x]\|_2^2] = E[\|\nabla F - E[\nabla F|x]\|_2^2] \leq \frac{Q}{S},$$

where $Q := E[C(\xi)^2]$.

**Proof of Theorem 4.1** Assume PISGD is run for $K$ iterations instead of $R - 1$, and for simplicity let

$$\nabla F^k := \frac{1}{S} \sum_{l=1}^S \tilde{\nabla} F(w^k_l, \xi^k_l)$$

for $k = 1, \ldots, K$, which is the random direction PISDG moves in each iteration. For any two iterates $x^k$ and $x^{k+1}$, let $\hat{w}^k = x^k + \hat{z}^k$ and $\hat{w}^{k+1} = x^{k+1} + \hat{z}^{k+1}$ for a single sample $\hat{z}^k \sim U(B(\sigma))$. As $\hat{z}^k$ is sampled uniformly from a Euclidean ball in $\mathbb{R}^n$, Lemma 4.1 can be applied such that given $x^k$ and $x^{k+1}$, for almost all values of $\hat{z}^k$,

$$f(\hat{w}^{k+1}) - f(\hat{w}^k) - \langle \tilde{\nabla} f(\hat{w}^k), x^{k+1} - x^k \rangle$$

$$= \int_0^1 \langle \tilde{\nabla} f(\hat{w}^k + v(x^{k+1} - x^k)) - \tilde{\nabla} f(\hat{w}^k), x^{k+1} - x^k \rangle dv, \quad (11)$$
since the distribution of $\tilde{z}^k$ is absolutely continuous with respect to the Lebesgue measure. Considering now the expectation of (11),

\[
\mathbb{E}[f(\hat{w}^{k+1}) - f(\hat{w}^k) - (\nabla f(\hat{w}^k), x^{k+1} - x^k)] \\
= \mathbb{E}(\mathbb{E}[f(\hat{w}^{k+1}) - f(\hat{w}^k) - \langle \nabla f(\hat{w}^k), x^{k+1} - x^k\rangle | x^{k+1}, x^k]) \\
= \mathbb{E} \left( \mathbb{E} \left[ \int_0^1 \langle \nabla f(\hat{w}^k + v(x^{k+1} - x^k)) - \nabla f(\hat{w}^k), x^{k+1} - x^k \rangle dv \bigg| x^{k+1}, x^k \right] \right) \\
= \mathbb{E} \left( \int_0^1 \langle \nabla f(\hat{w}^k + v(x^{k+1} - x^k)) - \nabla f(\hat{w}^k), x^{k+1} - x^k \rangle dv \right).
\]

Applying $x^{k+1} - x^k = -\eta \nabla F^k$, and given that for any $l \in \{1, 2, \ldots, S\}$, $\mathbb{E}[f(\hat{w}^{k+1}) - f(\hat{w}^k)] = \mathbb{E}[f(w^{k+1}_l) - f(w^k_l)]$,

\[
\mathbb{E}[f(w^{k+1}_l) - f(w^k_l) + \eta \langle \nabla f(\hat{w}^k), \nabla F^k \rangle] \\
= \mathbb{E} \left[ \int_0^1 \langle \nabla f(\hat{w}^k - v\eta \nabla F^k) - \nabla f(\hat{w}^k), -\eta \nabla F^k \rangle dv \right]. \quad (12)
\]

Our first goal is to bound each side of (12) in terms of only $f(w^{k+1}_l) - f(w^k_l)$ and $\nabla F^k$. We will analyze each side of (12) separately and then combine the analysis to prove the convergence of the algorithm.

**Analysis of the left-hand side of (12):**

For all $k \in [1, \ldots, K]$,

\[
\mathbb{E}[f(w^{k+1}_l) - f(w^k_l) + \eta \langle \nabla f(\hat{w}^k), \nabla F^k \rangle] \\
= \mathbb{E}[f(w^{k+1}_l) - f(w^k_l)] + \eta \mathbb{E}(\langle \nabla f(\hat{w}^k), \nabla F^k \rangle | x^k) \\
= \mathbb{E}[f(w^{k+1}_l) - f(w^k_l)] + \eta \mathbb{E}(\langle \nabla f(\hat{w}^k) | x^k \rangle, \langle \nabla F^k | x^k \rangle)). \quad (13)
\]

The last equality holds since $\nabla f(\hat{w}^k)$ and $\nabla F^k$ are conditionally independent random variables with respect to $x^k$, so for all $j = 1, \ldots, d$, $\mathbb{E}[\hat{\nabla}_j f(\hat{w}^k) \cdot \nabla F^k | x^k] = \mathbb{E}[\hat{\nabla}_j f(\hat{w}^k) | x^k] \cdot \mathbb{E}[\nabla F^k | x^k]$. Focusing on $\mathbb{E}[\nabla F^k | x^k]$,

\[
\mathbb{E}[\nabla F^k | x^k] = \mathbb{E}[\hat{\nabla} F(w^k, 0)| x^k] \\
= \mathbb{E}[\hat{\nabla} f(w^k)| x^k] \quad (14) \\
= \mathbb{E}[\hat{\nabla} f(\hat{w}^k)| x^k], \quad (15)
\]

for an arbitrary $l \in [1, \ldots, S]$. The equality (14) follows from Lemma 3.2: First let

\[
g(y) := \mathbb{E}[\hat{\nabla} F(y + z^k_l, 0)| x^k] 
\]

\footnote{Equalities involving conditional expectations are to be interpreted as holding almost surely.}
then
\[ \mathbb{E}[\tilde{\nabla} F(w_k^i, \xi_k^i)|x^k] = g(x^k) \]
since \( z_k^i \) and \( \xi_k^i \) are independent of \( x^k \), see for example [28, Lemma 2.3.4]. From Lemma 4.2, \(|\tilde{\nabla} j F(y + z_k^i, \xi_k^i)| \leq C(\xi_k^i)\), which implies that \( \tilde{\nabla} j F(y + z_k^i, \xi_k^i) \in L^1(P_{w_k^i} \times P_{\xi_k^i}) \) for any \( y \in \mathbb{R}^d \). Given that \( z_k^i \) and \( \xi_k^i \) are independent, Fubini’s theorem can be applied with
\[
g(y) = \mathbb{E}_{z_k^i}[\mathbb{E}_{\xi_k^i}[\tilde{\nabla} F(y + z_k^i, \xi_k^i)]].
\]
From Lemma 3.2, \( \mathbb{E}_{z_k^i}[\tilde{\nabla} F(y + z_k^i, \xi_k^i)] = \tilde{\nabla} f(y + z_k^i) \) for almost all \( z_k^i \), hence
\[
g(y) = \mathbb{E}[\tilde{\nabla} f(y + z_k^i)] \text{ with } g(x^k) = \mathbb{E}[\tilde{\nabla} f(w_k^i)|x^k], \text{ again from the independence of } x^k \text{ and } z_k^i.
\]
Using (15) in (13),
\[
\mathbb{E}[f(w_k^{i+1}) - f(w_k^i) + \eta(\tilde{\nabla} f(\hat{w}^k), \nabla F^k)]
= \mathbb{E}[f(w_k^{i+1}) - f(w_k^i)] + \eta \mathbb{E}(\|\tilde{\nabla} F^k|x^k\|_2^2).
\]

**Analysis of the right-hand side of (12):**
We now analyze
\[
\mathbb{E} \left[ \int_0^1 \langle \tilde{\nabla} f(\hat{w}^k - v\eta\nabla F^k) - \tilde{\nabla} f(\hat{w}^k), -\eta\nabla F^k \rangle dv \right]
\]
for any \( k \in [1, \ldots, K] \). As the negation, addition, composition, and product of real-valued Borel measurable functions, \( \langle \tilde{\nabla} f(\hat{w}^k - v\eta\nabla F^k) - \tilde{\nabla} f(\hat{w}^k), -\eta\nabla F^k \rangle \) is a measurable function on \( \mathbb{R}^{d+S(d+p)} \times [0, 1], \mathcal{B}_{\mathbb{R}^{d+S(d+p)}} \otimes \mathcal{B}_{[0,1]} \), where \( (\hat{w}^k, w^i_k, \xi^i_k) \in \mathbb{R}^{d+S(d+p)} \) and \( v \in [0, 1] \). Given that \( \tilde{\nabla} f(w) \) and \( \tilde{\nabla} F(w, \xi) \) are bounded almost everywhere by Lemma 4.2, and the probability measure and the Lebesgue measure that the expectation and integral are with respect to are both finite, the function is integrable and Fubini’s theorem can be applied:
\[
\mathbb{E} \left[ \int_0^1 \langle \tilde{\nabla} f(\hat{w}^k - v\eta\nabla F^k) - \tilde{\nabla} f(\hat{w}^k), -\eta\nabla F^k \rangle dv \right]
= \int_0^1 \mathbb{E}[(\langle \tilde{\nabla} f(\hat{w}^k - v\eta\nabla F^k) - \tilde{\nabla} f(\hat{w}^k), -\eta\nabla F^k \rangle)]dv
= \int_0^1 \mathbb{E}[(\langle \tilde{\nabla} f(\hat{w}^k - v\eta\nabla F^k) - \tilde{\nabla} f(\hat{w}^k), -\eta\nabla F^k \rangle|\nabla F^k, x^k)]dv
\]
\[
\leq \int_0^1 \mathbb{E}[\|\langle \tilde{\nabla} f(\hat{w}^k - v\eta\nabla F^k) - \tilde{\nabla} f(\hat{w}^k)|\nabla F^k, x^k\|_2 \cdot \|\eta\nabla F^k\|_2]dv. \quad (17)
\]
Focusing on \( \tilde{\nabla} f(\hat{w}^k - v\eta \nabla F^k) - \tilde{\nabla} f(\hat{w}^k) \) within \( \mathbb{E}[\tilde{\nabla} f(\hat{w}^k - v\eta \nabla F^k) - \tilde{\nabla} f(\hat{w}^k)](\nabla F^k, x^k) \), and writing \( \hat{w}^k = x^k + \hat{\zeta}^k \), the only random variable which is not measurable with respect to the \( \sigma \)-algebra generated by \( \nabla F^k \) and \( x^k \), is \( \hat{\zeta}^k \), which is independent of \( \nabla F^k \) and \( x^k \). Similar to showing (14), letting

\[
g(y, y') := \mathbb{E}[\tilde{\nabla} f(y + \hat{\zeta}^k) - \tilde{\nabla} f(y' + \hat{\zeta}^k)].
\]  (18)

then

\[
\mathbb{E}[\tilde{\nabla} f(\hat{w}^k - v\eta \nabla F^k) - \tilde{\nabla} f(\hat{w}^k)](\nabla F^k, x^k) = g(x^k - v\eta \nabla F^k, x^k).
\]

Considering the norm of (18) for arbitrary \( y, y' \in \mathbb{R}^d \), and setting \( w = y + \hat{\zeta}^k \) and \( w' = y' + \hat{\zeta}^k \),

\[
\|\mathbb{E}[\tilde{\nabla} f(w) - \tilde{\nabla} f(w')]\|_2 = \|\int_{\mathbb{R}^d} \tilde{\nabla} f(w)p(w - y)dw - \int_{\mathbb{R}^d} \tilde{\nabla} f(w')p(w' - y')dw'\|_2
\]

\[
= \|\int_{\mathbb{R}^d} \tilde{\nabla} f(w)(p(w - y) - p(w - y'))dw\|_2
\]

\[
\leq \int_{\mathbb{R}^d} \|\tilde{\nabla} f(w)\|_2|p(w - y) - p(w - y')|dw
\]

\[
\leq L_0 \int_{\mathbb{R}^d} |p(w - y) - p(w - y')|dw
\]

\[
\leq L_0 \frac{\lambda(d)}{\sigma} \frac{d!!}{(d - 1)!!} \|y - y'\|_2
\]

\[
\leq L_0 \frac{\sqrt{d}}{\sigma} \|y - y'\|_2,
\]  (19)

where the second inequality follows from Lemma 4.2 and the third inequality bounding the integral \( \int_{\mathbb{R}^d} |p(w - y) - p(w - y')|dw \) can be found in the proof of Lemma 8 in [31] beginning at equation (33), where \( \lambda(d) = \frac{2}{\pi} \) when \( d \) is even and 1 when \( d \) is odd. The final inequality uses the bound \( \frac{\lambda(d)d!!}{(d - 1)!!} \leq \sqrt{d} \), which is proven in Lemma A.1 in “Appendix.” The bound (19) evaluated at \( y = x^k - v\eta \nabla F^k \) and \( y' = x^k \) gives

\[
\|\mathbb{E}[\tilde{\nabla} f(\hat{w}^k - v\eta \nabla F^k) - \tilde{\nabla} f(\hat{w}^k)](\nabla F^k, x^k)\|_2 \leq L_0 \frac{\sqrt{d}}{\sigma} \|v\eta \nabla F^k\|_2.
\]

Applying this bound in (17),

\[
\mathbb{E} \left[ \int_0^1 \left( \tilde{\nabla} f(\hat{w}^k - v\eta \nabla F^k) - \tilde{\nabla} f(\hat{w}^k), -\eta \nabla F^k \right) dv \right]
\]

\[
\leq \int_0^1 \mathbb{E} \left( L_0 \frac{\sqrt{d}}{\sigma} \|v\eta \nabla F^k\|_2 \cdot \|\eta \nabla F^k\|_2 \right) dv
\]

\[
= \int_0^1 L_0 \frac{\sqrt{d}}{\sigma} \eta^2 v \mathbb{E}[\|\nabla F^k\|_2^2] dv = L_0 \frac{\sqrt{d}}{\sigma} \frac{\eta^2}{2} \mathbb{E}[\|\nabla F^k\|_2^2].
\]  (20)
Proving the convergence of PISGD:

We now combine the analysis of Eq. (12), using (16) and (20) to get, for all $k \in [1, \ldots, K]$, the inequality

$$
E[f(w_{k+1}^l) - f(w_k^l)] + \eta E[(\|\nabla F^k(x^l)\|_2^2)]
= E[f(w_{k+1}^l) - f(w_k^l) + \eta(\tilde{\nabla}f(\hat{w}^k), \nabla F^k)]
= E \left[ \int_0^1 \langle \tilde{\nabla}f(\hat{w}^k - v\eta \nabla F^k) - \tilde{\nabla}f(\hat{w}^k), -\eta \nabla F^k \rangle dv \right]
\leq L_0 \frac{\sqrt{d} \eta^2}{\sigma} E[\|\nabla F^k\|_2^2].
$$

Adding $\eta E[\|\nabla F^k\|_2^2]$ to both sides and rearranging:

$$
E[f(w_{k+1}^l) - f(w_k^l)] + \eta E[\|\nabla F^k\|_2^2]
\leq L_0 \frac{\sqrt{d} \eta^2}{\sigma} E[\|\nabla F^k\|_2^2] + \eta E[\|\nabla F^k\|_2^2]
\leq L_0 \frac{\sqrt{d} \eta^2}{\sigma} E[\|\nabla F^k\|_2^2] + \eta \frac{Q}{S},
$$

(21)

using Lemma 4.3. Rearranging,

$$
\left( \eta - L_0 \frac{\sqrt{d} \eta^2}{\sigma} \right) E[\|\nabla F^k\|_2^2] \leq E[f(w_k^l) - f(w_{k+1}^l)] + \eta \frac{Q}{S}.
$$

Summing these inequalities for $k = 1, \ldots, K$,

$$
\left( \eta - L_0 \frac{\sqrt{d} \eta^2}{\sigma} \right) \sum_{k=1}^K E[\|\nabla F^k\|_2^2] \leq E[f(w_1^l) - f(w_K^{K+1})] + \eta K \frac{Q}{S}.
$$

As $R$ was sampled uniformly over $\{1, 2, \ldots, K\}$,

$$
\left( \eta - L_0 \frac{\sqrt{d} \eta^2}{\sigma} \right) \sum_{k=1}^K E[\|\nabla F^R\|_2^2] \leq \frac{1}{K} E[f(w_1^1) - f(w_K^{K+1})] + \eta \frac{Q}{S}
\leq \frac{1}{K} (E[f(w_1^1)] - f(x^*)) + \eta \frac{Q}{S}
\leq \frac{1}{K} (f(x^1) + L_0 E[\|z_1^1\|_2] - f(x^*)) + \eta \frac{Q}{S}
= \frac{1}{K} (\Delta + L_0 \frac{\sigma d}{d+1}) + \eta \frac{Q}{S}.
$$
The last inequality uses the Lipschitz continuity of \( f(w) \), and the equality uses (3) and sets \( f(x^1) - f(x^*) = \Delta \). Taking \( \eta = \frac{\sigma}{L_0 \sqrt{d}} \)

\[
E[\|\nabla F_R\|_2^2] \leq \frac{2}{K} L_0 \frac{\sqrt{d}}{\sigma} \left( \Delta + L_0 \frac{\sigma d}{d+1} \right) + 2 \frac{Q}{d} \\
< \frac{2 L_0}{K} \sqrt{d} \Delta + 2 \frac{L_0^2}{K} \sqrt{d} + 2 \frac{Q}{d}.
\] (22)

Setting \( \sigma = \theta \sqrt{d K^{-\beta}} \) and \( S \geq K^{1-\beta} \), e.g., \( S = [K^{1-\beta}] \),

\[
E[\|\nabla F_R\|_2^2] < 2K^{\beta-1} \frac{L_0}{\theta} \Delta + 2 \frac{L_0^2}{K} \sqrt{d} + 2K^{\beta-1} Q \\
= 2K^{\beta-1} \left( \frac{L_0}{\theta} \Delta + L_0^2 \sqrt{d} K^{-\beta} + Q \right),
\] (23)

and \( \eta = \frac{\theta}{L_0} K^{-\beta} \). In addition,

\[
E[\|\nabla F_R\|_2^2] = E \left[ \left\| \frac{1}{S} \sum_{l=1}^{S} \nabla F(w_R^l, \xi_R^l) \right\|_2^2 \right] \\
= E \left( E \left[ \left\| \frac{1}{S} \sum_{l=1}^{S} \nabla F(w_R^l, \xi_R^l) \right\|_2^2 \mid x_R \right] \right) \\
\geq E \left( \left\| \frac{1}{S} \sum_{l=1}^{S} \nabla f(w_R^l) \mid x_R \right\|_2^2 \right) \\
= E \left( \left\| \frac{1}{S} \sum_{l=1}^{S} \nabla f(w_R^l) \mid x_R \right\|_2^2 \right) \\
\geq E[\text{dist}(0, \partial \sigma f(x_R))^2].
\] (24)

The third equality follows from Lemma 3.2 like (14). For all \( l = 1, \ldots, S \), \( w_R^l = x_R + z_R^l \in x_R + B(\sigma) \). The gradient is always contained in the Clarke subdifferential wherever a Lipschitz continuous function is differentiable, so for almost every \( z_R^l \), the approximate gradient \( \nabla f(w_R^l) \in \partial \sigma f(x_R) \). The convex combination, \( \frac{1}{S} \sum_{l=1}^{S} \nabla f(w_R^l) \in \partial \sigma f(x_R) \) almost surely as well, given that \( \partial \sigma f(x_R) \) is a convex set, hence \( E[\frac{1}{S} \sum_{l=1}^{S} \nabla f(w_R^l) \mid x_R \] \( \in \partial \sigma f(x_R) \), resulting in the final inequality. Combining (23) and (25), and using Jensen’s inequality,

\[
E[\text{dist}(0, \partial \sigma f(x_R))] < K^{\frac{\beta-1}{2}} \sqrt{2 \left( \frac{L_0}{\theta} \Delta + L_0^2 \sqrt{d} K^{-\beta} + Q \right)}.
\]
4.3 Computational Complexity

The following corollary establishes computational complexities for finding an \((\epsilon_1, \epsilon_2)\)-stationary point in expectation in terms of the number of stochastic approximate gradient computations \(\tilde{\nabla} F(w, \xi)\), which we will simply refer to as gradient calls in this subsection. For example, choosing \(\beta = \frac{1}{3}\), the complexity is \(O(\min(\epsilon_1, \epsilon_2)^{-5})\) and for \(\beta = \frac{1}{2}\), it is \(O(\max(\epsilon_1^{-3}, \epsilon_2^{-6}))\).

**Corollary 4.1** For \(\beta \in (0, 1)\), an expected \((\epsilon_1, \epsilon_2)\)-stationary point (8) can be computed with \(O(\max(\epsilon_1^{-\beta}, \epsilon_2^{-2-\beta}))\) gradient calls.

The optimal choice for \(\beta\) is somewhat ambiguous as it depends on the importance placed on \(\epsilon_1\) and \(\epsilon_2\). This can be resolved by attempting to find the optimal \(\beta\), with respect to the bound (10) provided by Theorem 4.1, which minimizes the upper bound on the total number of gradient calls used in PISGD, \((K - 1)S = (K - 1)\lceil K^{1-\beta} \rceil\), assuming \(K - 1\) iterations are performed. By the Lipschitz continuity of \(f(w)\), every point is an \((\epsilon_1, L_0)\)-stationary point, so we will assume that \(\epsilon_2 < L_0\) for the rest of this subsection. In addition, the parameter \(\theta\) in Theorem 4.1 is included to allow the theorem to be applicable for any step size \(\eta > 0\), but it is redundant for the convergence of the algorithm, so for simplicity we fix \(\theta = 1\) for the rest of this subsection.

**Corollary 4.2** Assume \(\epsilon_2 < L_0\), \(\theta = 1\), and let

\[
K^* = \max \left( \frac{2}{\epsilon_2} \left( L_0 \Delta + Q + \sqrt{d} L_0^2 \right) + 1 \right), \quad \left[ \frac{2\sqrt{d}}{\epsilon_2^2} \left( \frac{L_0 \Delta + Q}{\epsilon_1} + L_0^2 \right) \right]
\]

and

\[
\beta^* = \frac{\log(K^* \epsilon_2^2 - 2\sqrt{d} L_0^2) - \log(2(L_0 \Delta + Q))}{\log(K^*)},
\]

then

1. an expected \((\epsilon_1, \epsilon_2)\)-stationary point (8) can be computed with \(O(\frac{1}{\epsilon_1 \epsilon_2})\) gradient calls,
2. \((K^*, \beta^*)\) has the same gradient call complexity as a minimizer of the number of gradient calls required to find an expected \((\epsilon_1, \epsilon_2)\)-stationary point using inequality (10) of Theorem 4.1, and
3. \((K^*, \beta^*)\) minimizes the number of iterations required to find an expected \((\epsilon_1, \epsilon_2)\)-stationary point using inequality (10) of Theorem 4.1.

Motivated by [16, Section 2.2] we present the computational complexity for finding an \((\epsilon_1, \epsilon_2)\)-stationary point with probability \(1 - \gamma\) for any \(\gamma \in (0, 1)\).

**Corollary 4.3** Let \(c \in (0, 1)\) and \(\phi > 1\) be arbitrary constants. For any \(\gamma \in (0, 1)\) and \(\epsilon_2 < L_0\), let PISGD be run \(R := \lceil -\ln(c \gamma) \rceil\) times using the parameter settings of \(\beta^*\).
Theorem 4.1 with \( \theta = 1 \),

\[
K = \max \left( \left\lceil \frac{2}{(\epsilon_2')}^2 \left( L_0 \Delta + Q + \sqrt{d} L_0^2 \right) \right\rceil, \left\lceil \frac{2\sqrt{d}}{(\epsilon_2')}^2 \left( \frac{L_0 \Delta + Q}{\epsilon_1} + L_0^2 \right) \right\rceil \right),
\]

and

\[
\beta = \log \left( \frac{K (\epsilon_2')^2 - 2\sqrt{d} L_0^2 - \log(2(L_0 \Delta + Q))}{\log(K)} \right),
\]

where \( \epsilon_2' = \sqrt{\frac{\epsilon_2^2 - 6\psi \frac{Q}{4\epsilon}}{4\epsilon}} \), \( \psi = \left\lceil \frac{-\ln(c\gamma)}{(1-c)\gamma} \right\rceil \), and \( T = \left\lceil 6\phi \psi \frac{Q}{\epsilon_2} \right\rceil \), outputting candidate solutions \( \bar{X} := \{ \bar{x}^1, \ldots, \bar{x}^R \} \). With \( T \) samples \((z_1, \xi_1), \ldots, (z_T, \xi_T)\), where \( z_i \sim U(B(\sigma)) \) and \( \xi_i \sim P_{\xi} \) for \( i = 1, \ldots, T \), let \( \bar{x}^* \in \bar{X} \) be chosen such that for \( \nabla F_T(x) := \frac{1}{T} \sum_{t=1}^T \tilde{\nabla} F(x + z_t, \xi_t) \),

\[
\bar{x}^* \in \arg\min \limits_{x \in \bar{X}} ||\nabla F_T(x)||_2.
\]

It follows that

1. \( \bar{x}^* \) is an \((\epsilon_1, \epsilon_2)\)-stationary point with a probability of at least \( 1 - \gamma \), and
2. the described method requires \( \tilde{O} \left( \frac{1}{\epsilon_1 \epsilon_2^2} + \frac{1}{\gamma \epsilon_2^2} \right) \) gradient calls.

The proofs of the three corollaries are contained in “Appendix.”

Comparison with the computational complexity in [33]:

For the deterministic setting, Interpolated Normalized Gradient Descent (INGD) is developed which has a gradient call complexity of \( \tilde{O}(\frac{1}{\epsilon_1 \epsilon_2}) \) to achieve an \((\epsilon_1, \epsilon_2)\)-stationary point with arbitrarily high probability. For the stochastic setting, Stochastic INDG finds an expected \((\epsilon_1, \epsilon_2)\)-stationary point with \( \tilde{O}(\frac{1}{\epsilon_1 \epsilon_2}) \) stochastic subgradient calls. In particular, their algorithm finds an expected \((\epsilon_1, \epsilon_2)\)-stationary point. Running the algorithm \( - \log(\gamma) \) times, one of the solutions will be an \((\epsilon_1, \epsilon_2)\)-stationary point with a probability of at least \( 1 - \gamma \). Our computational complexities are similar, with the extra \( \tilde{O} \left( \frac{1}{\gamma \epsilon_2} \right) \) term in our high probability convergence result coming from returning an \( \bar{x}^* \in \bar{X} \) instead of simply \( X \). We also point out that we have omitted the problem dimension \( d \) in our computational complexity, where the convergence result of [33] is dimension-free, which is a sought-after property when studying the computational complexity of algorithms, see for example [8].

4.4 Convergence to a Clarke Stationary Point Almost Surely

Inspired by the discussion of an algorithm with asymptotic convergence to an \( \epsilon \)-stationary point (5) with high probability on page 4 of [33], in the following corollary,
we prove that if PISDG is run for a sequence of increasing iteration sizes, any accumulation point of the solutions is a Clarke stationary point almost surely.

**Corollary 4.4** Let \( \{K_1, K_2, \ldots \} \) be a strictly increasing sequence of positive integers. Let \( \beta \in (0, 1) \) and a finite \( \theta > 0 \) be fixed, with \( (S_i, \sigma_i, \eta_i) \) equal to \( (S, \sigma, \eta) \) as described in Theorem 4.1 given \( \beta, \theta, K = K_i \). Assume for \( i = 1, 2, \ldots \), PISGD is run with the indexed parameters, outputting solutions \( \{x_1, x_2, \ldots \} \), i.e., \( x_i = x^R \) for the \( i \)th instance of running PISGD. Any accumulation point \( x^* \) of the solutions \( \{x_1, x_2, \ldots \} \) is a Clarke stationary point almost surely.

**Proof** Assuming there exists an accumulation point \( x^* \) of \( \{x_i\} \), let \( \{x_i\} \) be redefined as a subsequence of \( \{x_i\} \) such that \( \lim_{i \to \infty} x_i = x^* \). Let

\[
B_i := K_i^{\beta-1} \sqrt{2 \left( \frac{L_0}{\theta} \Delta + L_0^2 \sqrt{d} K_i^{-\beta} + Q \right)},
\]

which is the right-hand side of (10) with \( K = K_i \). As \( i \to \infty \), \( \sigma_i \to 0 \) and \( B_i \to 0 \). For any \( i \in \mathbb{N} \) there exists an \( I \in \mathbb{N} \) such that for all \( j > I \), \( x_j \) is an expected \( \left( \sigma_i, \frac{B_i}{i} \right) \)-stationary point with \( ||x_j - x^*||_2 \leq \frac{Q}{\sqrt{2}} \). For such solutions \( x_j \),

\[
\{ \partial f(x) : x \in x_j + B(\sigma_i/2) \} \subseteq \{ \partial f(x) : x \in x^* + B(\sigma_i) \},
\]

hence \( \partial_{\sigma_i} f(x_j) \subseteq \partial_{\sigma_i} f(x^*) \) and given that \( x_j \) is an expected \( \left( \sigma_i, \frac{B_i}{i} \right) \)-stationary point,

\[
\mathbb{E}[\text{dist}(0, \partial_{\sigma_i} f(x^*))] \leq \frac{B_i}{i}.
\]

Using Markov’s inequality,

\[
\mathbb{P} \left[ \text{dist}(0, \partial_{\sigma_i} f(x^*)) \geq \frac{1}{i} \right] \leq B_i.
\]

Given that \( \text{dist}(0, \partial_{\sigma_i} f(x^*)) \leq \text{dist}(0, \partial_{\sigma_{i+1}} f(x^*)) \) and \( \frac{1}{i} > \frac{1}{i+1} \), the sets

\[
V_i := \left\{ x^* \in \mathbb{R}^d : \text{dist}(0, \partial_{\sigma_i} f(x^*)) \geq \frac{1}{i} \right\}
\]

are monotonically increasing, \( V_i \subseteq V_{i+1} \), and the limit \( \lim_{i \to \infty} V_i = \bigcup_{i \geq 1} V_i \) exists [2, Exercise 2.F]. Since the functions \( \text{dist}(0, \partial_{\sigma_i} f(x^*)) \) are Borel measurable from Lemma 3.3, each \( V_i \) is Borel measurable, as is \( \lim_{i \to \infty} V_i \) as a countable union of Borel measurable sets.
The next step is to prove that $\lim_{i \to \infty} V_i = \{x^* \in \mathbb{R}^d : \text{dist}(0, \partial f(x^*)) > 0\}$. For an $x \in \bigcup_{i \geq 1} V_i$ there exists an $i \geq 1$ such that $\text{dist}(0, \partial f(x)) \geq \text{dist}(0, \partial f_i(x)) \geq \frac{1}{i} > 0$, hence $x \in \{x^* \in \mathbb{R}^d : \text{dist}(0, \partial f(x^*)) > 0\}$.

For an $x \in \{x^* \in \mathbb{R}^d : \text{dist}(0, \partial f(x^*)) > 0\}$, let $\omega = \text{dist}(0, \partial f(x))$. The Clarke subdifferential is an upper semicontinuous set valued mapping [10, Proposition 2.1.5 (d)], which means that for all $\omega_1 > 0$, there exists an $\omega_2 > 0$ such that $\partial f(\hat{x}) \subseteq \partial f(x) + B(\omega_1)$ for all $\hat{x} \in x + B(\omega_2)$, hence $\partial_{\omega_2} f(x) \subseteq \text{co}\{\partial f(x) + B(\omega_1)\} = \partial f(x) + B(\omega_1)$, and

$$\text{dist}(0, \partial_{\omega_2} f(x)) \geq \text{dist}(0, \partial f(x) + B(\omega_1)). \quad (26)$$

The function $\text{dist}(0, \partial f(x) + B(\omega_1))$ can be bounded below as

$$\text{dist}(0, \partial f(x) + B(\omega_1)) = \min_{z \in \partial f(x), \ y \in B(\omega_1)} ||z + y||_2 \geq \min_{z \in \partial f(x), \ y \in B(\omega_1)} ||z||_2 - ||y||_2 = \omega - \omega_1. \quad (27)$$

Choosing $\omega_1 = \frac{\omega}{2}$, there then exists an $\omega_2 > 0$ such that $\text{dist}(0, \partial_{\omega_2} f(x)) \geq \frac{\omega}{2}$ from (26) and (27). A $J \in \mathbb{N}$ exists such that for all $i \geq J$, $\sigma_i \leq \omega_2$, and setting $I \geq \max\{J, \lceil \frac{2}{\omega_1} \rceil\}$, $x \in V_i$ for all $i \geq I$, proving that $x \in \bigcup_{i \geq 1} V_i$ and $\lim_{i \to \infty} V_i = \{x^* \in \mathbb{R}^d : \text{dist}(0, \partial f(x^*)) > 0\}$.

It follows that $\mathbb{P}[\text{dist}(0, \partial f(x^*)) = 0] = 1$ as

$$\mathbb{P}[\text{dist}(0, \partial f(x^*)) > 0] = \lim_{i \to \infty} \mathbb{P}[\text{dist}(0, \partial f_i(x^*)) \geq \frac{1}{i}] \leq \lim_{i \to \infty} B_i = 0,$$

where the equality holds since $V_i \subseteq V_{i+1}$ [28, Theorem A.1.1].

\[ \square \]

### 4.5 PISGD and Theorem 4.1 in Particular Cases

In this subsection, we look at how the convergence result of Theorem 4.1 changes for particular forms of $f(w)$.

**Deterministic** $f(w)$: In the case where $f(w)$ does not have the structure of (1) and is simply a deterministic $L_0$-Lipschitz continuous function, there will be no sampling of $\xi$ with the algorithm update rule being

$$x^{k+1} = x^k - \frac{\eta}{S} \sum_{l=1}^{S} \tilde{\nabla} f(w_l^k). \quad (28)$$

\[ ^3 \text{For our setting, using closed balls is equivalent to using open balls in the definition.} \]
The only change in Theorem 4.1 is that $Q$ is no longer needed, with $L_0^2$ replacing it in inequality (10):

\[
\mathbb{E}[\text{dist}(0, \partial_{\sigma} f(x^R))] \leq K^{-\frac{1}{2}} \left( \frac{\Delta}{\delta} + L_0^2 K^{-\beta} + L_0 \right).
\]

This comes from changing inequality (21) in the proof of Theorem 4.1 after replacing

\[
\frac{1}{S} \mathbb{E}[\|\bar{\nabla} F(w_k \xi_k^l)\|_2^2] \leq \frac{Q}{S}
\]

with

\[
\frac{1}{S} \mathbb{E}[\|\bar{\nabla} f(w_k^l)\|_2^2] \leq \frac{L_0^2}{S},
\]

at inequality (39) in the proof of Lemma 4.3, given that $\|\bar{\nabla} f(w_k^l)\|_2 \leq L_0$ almost surely by Lemma 4.2.

**Finite-sum $f(w)$:** For the case where $f(w)$ takes the finite-sum structure of (2), in mini-batch SGD, when the required sample size $S \geq n$, it is better to switch to gradient descent. In PISGD, if $S \geq n$, one could use the update rule (28), but this would result in $n$ times the number of approximate stochastic gradient computations compared to continuing to use

\[
x^{k+1} = x^k - \frac{\eta}{S} \sum_{i=1}^{S} \bar{\nabla} F(w_k^i, \xi^i_k).
\]

Using the update (28) would replace $Q$ with $L_0^2$ in the bound of $\epsilon_2$, and it would no longer be required to uniformly sample $\{\xi\}$ each iteration, but the increased number of approximate stochastic gradient computations required is likely to outweigh these benefits.

**5 Application: Feedforward Neural Network**

In this section, we consider training a fully connected feedforward neural network with one hidden layer using MNIST data, with $N = [68, 9, 3]$ nodes in each layer, respectively. The MNIST training dataset consists of image data $v_i$ for $i = 1, \ldots, 60,000$, of the digits 0, 1, $\ldots$, 9, of dimension 784 and one-hot encoded labels $y_i$ of dimension 10. The neural network trained on the digits $[0, 1, 2]$, which consisted of $n = 18,624$ samples. PCA was applied to $v$ with 90% explained variance,\(^4\) which reduced the dimension of each $v_i$ to $p = 68$.

---

\(^4\) This modified MNIST dataset is available from the corresponding author on reasonable request.
The decision variables of the model are $x = [W, b]$, where for $l = 2, 3$, $W_{jk}^l$ is the weight for the connection between the $k$th neuron in the $(l - 1)$th layer and the $j$th neuron in the $l$th layer, and $b_j^l$ is the bias of the $j$th neuron in the $l$th layer. The input and output of the activation functions in each layer are denoted as $z^l_j$ and $\alpha^l_j$, respectively. ReLU-m activation functions were used in the hidden layer,

$$\alpha^2_j(z^2_j) := \min(\max(z^2_j, 0), m),$$

with $m > 0$, and softmax functions were used in the output layer,

$$\alpha^3_j(z^3) := \frac{e^{z^3_j}}{\sum_{k=1}^{N_3} e^{z^3_k}},$$

with a cross-entropy loss function,

$$\mathcal{L}(\alpha^3, y^i) := -\sum_{j=1}^{N_3} y^i_j \log(\alpha^3_j).$$

All of the weights in $W^3$ were put through hard tanh activation functions,

$$H_{jk}(W_{jk}^3) := \min(\max(W_{jk}^3, -1), 1).$$

The optimization problem is then

$$\min_{W, b} \frac{1}{n} \sum_{i=1}^{n} \mathcal{L}(\alpha^3(H(W^3)\alpha^2(W^2v^i + b^2) + b^3), y^i).$$

Applying hard tanh activation functions directly to weights is similar to ideas such as weight normalization [26] and using bounded-weights [21]. Our motivation to include these activation functions was to be able to compute a Lipschitz constant for $\mathcal{L}$ and objectively test PISDG with parameters computed using Theorem 4.1. The proof of the following proposition is given in “Appendix.”

**Proposition 5.1** Each function $\mathcal{L}(\alpha^3(H(W^3)\alpha^2(W^2v^i + b^2) + b^3), y^i)$ is $L_i := 2 \max(\sqrt{N_2N_3} ||[(v^i)^T, 1]]_2, \sqrt{(N_2m^2 + 1)})$-Lipschitz continuous.

By the Lipschitz continuity proved in Proposition 5.1, $\mathcal{L}(\alpha^3(H(W^3)\alpha^2(W^2v^i + b^2) + b^3), y^i)$ is differentiable almost everywhere in $x = [W, b]$. However, applying the chain rule as done in backpropagation for deep learning models, even when Lipschitz continuous, is not generally valid for almost all $x$. For an in depth analysis of the validity of using auto differentiators, and in particular the backpropagation algorithm, see [5]. In the following subsection, we show that using the chain rule to compute $\nabla F(u_i^k, \xi_i^k)$ for the current application outputs the gradient with probability 1.
### 5.1 Using PISGD with the Chain Rule for Minimizing $\mathcal{L}$

Let

$$
\mathcal{L}_i := \mathcal{L}(\alpha^3 (H(W^3 + zW^3)\alpha^2 ((W^2 + zW^2)v^l + b^2 + zb^2) + b^3 + zb^3), y^i)
$$

where $z = [z_{W^3}, z_{b^3}, z_{W^2}, z_{b^2}] \sim U(B(\sigma))^{N_3 \times N_2 + N_3 + N_2 \times N_1 + N_2}$ is the required iterate perturbation used in PISGD. In this application, $\xi \in \mathbb{R}^P$ is uniformly drawn from $(v, y)$, i.e., $\xi = (v^l, y^l)$ for any $i \in \{1, 2, \ldots, n\}$ with probability $\frac{1}{P}$. For simplicity, we omit iteration or sample notation. Our analysis holds considering $[W, b] = x^k$, $[z_{W^3}, z_{b^3}, z_{W^2}, z_{b^2}] = z^k_i$ and $(v^l, y^l) = (v^k_l, y^k_l) = \xi^k_l$ for any $k \in \{1, \ldots, K\}$ and $l \in \{1, \ldots, S\}$. Our approach is to show that using the chain rule for each decision variable $x^l \in x$ produces the partial derivative $\frac{\partial \mathcal{L}_i}{\partial x^l}(x + z)$ with probability 1. Given that $\mathcal{L}_i$ is Lipschitz continuous, this implies that the chain rule outputs $\nabla x^l \mathcal{L}_i(x + z)$ with probability 1 given that $\mathcal{L}_i$ is differentiable almost everywhere. Over the course of running PISGD, there are a countable number of partial derivatives to be approximated, hence $\nabla \tilde{F}(w^k_l, \xi^k_l)$ equals the gradient for $k \in \{1, \ldots, K\}$ and $l \in \{1, \ldots, S\}$ with probability 1 using the chain rule. In our implementation of backpropagation, the following formulas for computing an approximate gradient were used. All arguments have been omitted to make the formulas simpler.

$$
\begin{align*}
\tilde{\nabla}_{b^3} \mathcal{L}_i &= \frac{\partial \mathcal{L}_i}{\partial z^3_j} = (a^3_j - y^l_j) \\
\tilde{\nabla}_{w^3_{jk}} \mathcal{L}_i &= \frac{\partial \mathcal{L}_i}{\partial z^3_j} \frac{\partial z^3_j}{\partial h_{jk}} \frac{\partial h_{jk}}{\partial w^3_{jk}} = (a^3_j - y^l_j) \alpha^2_k \frac{\partial h_{jk}}{\partial w^3_{jk}} \\
\tilde{\nabla}_{b^2} \mathcal{L}_i &= \sum_{h=1}^{N_3} \frac{\partial \mathcal{L}_i}{\partial z^3_h} \frac{\partial z^3_h}{\partial \alpha^2_j} \frac{\partial \alpha^2_j}{\partial z^2_j} = \sum_{h=1}^{N_3} (a^3_h - y^l_h) H_{hj} \frac{\partial \alpha^2_j}{\partial z^2_j} \\
\tilde{\nabla}_{w^2_{jk}} \mathcal{L}_i &= \sum_{h=1}^{N_3} \frac{\partial \mathcal{L}_i}{\partial z^3_h} \frac{\partial z^3_h}{\partial \alpha^2_j} \frac{\partial \alpha^2_j}{\partial z^2_j} \frac{\partial z^2_j}{\partial w^2_{jk}} = \sum_{h=1}^{N_3} (a^3_h - y^l_h) H_{hj} \frac{\partial \alpha^2_j}{\partial z^2_j} v^l_k
\end{align*}
$$

(29)

The proof of the following proposition is given in “Appendix.”

**Proposition 5.2** The approximate stochastic gradient $\tilde{\nabla} \mathcal{L}_i$, computed using the formulas (29), equals the gradient of $\mathcal{L}_i$ with probability 1.

### 5.2 Numerical Substantiation

The described neural network was trained using PISGD as well as with Algorithm 1 implemented with $w^k_l = x^k$, i.e., no iterate perturbation, which matches how neural networks are generally trained using a mini-batch stochastic gradient descent algorithm, which will be referred to as SGD. We ran both algorithms under a typical implementation of SGD: The step size was chosen as $\eta = 0.01$, which is a default setting when using, for example, Keras 2.3.0 [9], as well as $\eta = 0.02$ and $\eta = 0.005$. 

\[\text{Springer}\]
For each $\eta$, following Theorem 4.1, $\sigma = \eta L_0 \sqrt{d}$, where $d$ is the number of decision variables and $L_0$ is the mean value of $\{L_i\}$. The mini-batch size is generally chosen between 32-512 samples [18], so we chose $S = 250$, which is roughly the average. From Theorem 4.1, the number of iterations should satisfy $S = \lceil K^{1-\beta} \rceil$ for $\beta \in (0, 1)$. We chose $K = 62,500$, inferring a choice of $\beta = 0.5$. The neural network was trained five times with each algorithm, and the function values at each iteration were averaged together. Both algorithms were implemented in Python 3.6 on a server running Ubuntu 16.04 with an Intel Xeon E5-2698 v4 processor. Examining Fig. 1, we can see that both algorithms performed similarly, with PISGD producing a slightly lower loss function than SGD in this application.

6 Conclusion

In this paper, a new variant of stochastic gradient descent, PISGD, was developed which contains two forms of randomness in the step direction from sampling the stochastic function’s approximate gradient at randomly perturbed iterates. Using this methodology, non-asymptotic convergence to an expected $(\epsilon_1, \epsilon_2)$-stationary point was proven for minimizing stochastic Lipschitz continuous loss functions. From this result, the computational complexities for finding an expected $(\epsilon_1, \epsilon_2)$-stationary point and an $(\epsilon_1, \epsilon_2)$-stationary point with high probability were given, as well as a method to obtain a Clarke stationary point almost surely.
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Appendix Proofs and Auxiliary Results

A.1 Section 1

Proposition A.1 A bounded function $f(w)$ such that $|f(w)| \leq R$ for all $w \in \mathbb{R}^d$, with a Lipschitz continuous gradient with parameter $L_1$, is Lipschitz continuous with parameter $L_0 = 2R + \frac{L_1}{2}d$.

Proof A function has a Lipschitz continuous gradient if there exists a constant $L_1$ such that for all $x, w \in \mathbb{R}^d$, $\|\nabla f(x) - \nabla f(w)\|_2 \leq L_1 \|x - w\|_2$, which is equivalent to (see [22, Lemma 1.2.3])

$$|f(x) - f(w) - \langle \nabla f(w), x - w \rangle| \leq \frac{L_1}{2} \|x - w\|_2^2.$$  \hfill (30)

By the mean value theorem, if a differentiable function has a bounded gradient such that $\|\nabla f(w)\|_2 \leq L_0$ for all $w \in \mathbb{R}^n$, then it is Lipschitz continuous with parameter $L_0$. Using (30) with $x = w - y$ for any $y \in \mathbb{R}^d$,

$$f(w - y) - f(w) + \langle \nabla f(w), y \rangle \leq \frac{L_1}{2} \|y\|_2^2.$$

Taking $y_j = \text{sgn}(\nabla_j f(w))$ for $j = 1, \ldots, d$, and using the boundedness of $f(w)$,

$$\|\nabla f(w)\|_2 \leq \|\nabla f(w)\|_1 \leq 2R + \frac{L_1}{2}d.$$

\hfill \Box

A.2 Section 3

Proof of Lemma 3.1 Let $D \subseteq \{(w, \xi) : w \in \mathbb{R}^d, \xi \in \Xi\} =: \overline{\Xi}$ be the set of points where $F(w, \xi)$ is differentiable in $w$ within the set $\overline{\Xi}$ where it is Lipschitz continuous in $w$. For $F(w, \xi)$ to be differentiable at a point $(w, \xi)$, there exists a unique $g \in \mathbb{R}^d$ such that for any $\omega > 0$, there exists a $\delta > 0$ such that for all $h \in \mathbb{R}^d$ where $0 < \|h\|_2 < \delta$, it holds that

$$\frac{|F(w + h, \xi) - F(w, \xi) - \langle g, h \rangle|}{\|h\|_2} < \omega.$$
For simplicity let \( H(w, \xi, h, g) := \frac{|F(w + h, \xi) - F(w, \xi) - \langle g, h \rangle|}{||h||_2} \). The set \( D \) can be represented as
\[
\bigcup_{g \in \mathbb{R}^d} \bigcap_{\omega \in \mathbb{Q} > 0} \bigcap_{\delta \in \mathbb{Q} > 0} \bigcap_{0 < ||h||_2 < \delta} \bigcap_{h \in \mathbb{Q}^d} \{(w, \xi) \in \mathbb{S} : H(w, \xi, h, g) < \omega \},
\]
where \( h \) can be restricted to be over \( \mathbb{Q}^d \) as \( H(w, \xi, h, g) \) is continuous in \( h \) when \( ||h||_2 > 0 \) and \( \mathbb{Q}^d \) is dense in \( \mathbb{R}^d \). We want to prove that the set \( \hat{D} \) defined as
\[
\bigcap_{\omega \in \mathbb{Q} > 0} \bigcup_{(g, \delta) \in \mathbb{Q}^d \times \mathbb{Q} > 0} \bigcap_0 \bigcap_{||h||_2 < \delta} \bigcap_{h \in \mathbb{Q}^d} \{(w, \xi) \in \mathbb{S} : H(w, \xi, h, g) < \omega \},
\]
is equal to \( D \), proving that \( D \) is an element of \( B_{\mathbb{R}^d+p} \).

For an element \((w', \xi') \in D\) with \( g' \) being the gradient at \((w', \xi')\), for any \( \omega > 0 \), take \( \delta(\frac{\omega}{2}) > 0 \) such that \((w', \xi') \in \bigcap_0 \bigcap_{||h||_2 < \delta(\frac{\omega}{2})} \bigcap_{h \in \mathbb{Q}^d} \{(w, \xi) \in \mathbb{S} : H(w, \xi, h, g') < \frac{\omega}{2} \} \), and take \( g \in \mathbb{Q}^d \) such that \( ||g' - g||_2 < \frac{\omega}{2} \). It follows that
\[
H(w', \xi', h, g') < \frac{\omega}{2}
\]
\[
\implies \frac{|F(w' + h, \xi') - F(w', \xi') - \langle g, h \rangle - \langle g' - g, h \rangle|}{||h||_2} < \frac{\omega}{2}
\]
\[
\implies H(w', \xi', h, g) - \frac{|\langle g' - g, h \rangle|}{||h||_2} < \frac{\omega}{2}
\]
\[
\implies H(w', \xi', h, g) < \omega
\]
when \( 0 < ||h||_2 < \delta(\frac{\omega}{2}) \), using the reverse triangle inequality for the third inequality, proving that \((w', \xi') \in \hat{D}\).

Considering now an element \((w', \xi') \in \hat{D}\), let \( \{\omega_i\} \subset \mathbb{Q} > 0 \) be a non-increasing sequence approaching zero in the limit, with \( \{g_i\} \subset \mathbb{Q}^d \), and let \( \{\delta_i\} \subset \mathbb{Q} > 0 \) be a non-increasing sequence such that for all \( i \in \mathbb{N} \), \( H(w', \xi', h, g_i) < \omega_i \) when \( 0 < ||h||_2 < \delta_i \). The sequence \( \{g_i\} \) is bounded as
\[
H(w', \xi', h, g_i) < \omega_i
\]
\[
\implies \frac{|F(w' + h, \xi') - F(w', \xi') - \langle g_i, h \rangle|}{||h||_2} < \omega_i
\]
\[
\implies \frac{|\langle g_i, h \rangle|}{||h||_2} < \omega_i \implies |\langle g_i, h \rangle| < \omega_i + C(\xi')
\]
(31)
for all $0 < ||h||_2 < \delta_i$, using again the reverse triangle inequality and the Lipschitz continuity of $F(w, \xi')$. Taking $h = \delta'_i \frac{g_i}{||g||_2}$ for any $\delta'_i < \delta_i$ in (31),

$$||g_i||_2 < \omega_i + C(\xi') \leq \omega_i + C(\xi').$$

Given that the sequence $\{g_i\}$ is bounded, it contains at least one accumulation point $g'$. There then exists a subsequence $\{i_j\} \subset \mathbb{N}$ such that for any $\omega \in \mathbb{Q}_{>0}$, there exists a $J \in \mathbb{N}$ such that for $j > J$, $\omega_{i_j} < \frac{\omega}{2}$ and $||g_{i_j} - g'||_2 < \frac{\omega}{2}$, from which it holds that $H(w', \xi', h, g') \leq H(w', \xi', h, g_{i_j}) + ||g' - g_{i_j}||_2 < \omega$ when $0 < ||h||_2 < \delta_{i_j}$, proving $g'$ is the gradient of $F(w, \xi)$ at $(w', \xi')$ and $(w', \xi') \in D$.

We now want to establish that $F(w, \xi)$ is differentiable almost everywhere in $w$. Let $\mathbb{1}_{D^c}(w, \xi)$ be the indicator function of the complement of $D$. The set $D^c$ is the set of points $(w, \xi)$ where $F(w, \xi)$ is not differentiable or not Lipschitz continuous in $w$. Showing that $D^c$ is a null set is then sufficient. Given that the function $\mathbb{1}_{D^c}(w, \xi) \in L^+(\mathbb{R}^d \times \mathbb{R}^p)$, and $m^d$ and $P$ are $\sigma$-finite, the measure of $D^c$ can be computed by the iterated integral

$$E_{\xi} \left[ \int_{w \in \mathbb{R}^d} \mathbb{1}_{D^c}(w, \xi)dw \right]$$

by Tonelli’s theorem [15, Theorem 2.37 a.]. Let $\bar{\xi} \in \mathbb{R}^p$ be chosen such that $F(w, \bar{\xi})$ is Lipschitz continuous in $w$. By Rademacher’s theorem, $F(w, \bar{\xi})$ is differentiable in $w$ almost everywhere, which implies that $\int_{w \in \mathbb{R}^d} \mathbb{1}_{D^c}(w, \bar{\xi})dw = 0$. As this holds for almost every $\xi$, $E_{\xi} \left[ \int_{w \in \mathbb{R}^d} \mathbb{1}_{D^c}(w, \xi)dw \right] = 0$ [15, Proposition 2.16].

**Example A.1** Let $e_j$ for $j = 1, \ldots, d$ denote the standard basis of $\mathbb{R}^d$. For $i \in \mathbb{N}$, let

$$h^+_j(w, \xi) := i(F(w + i^{-1} e_j, \xi) - F(w, \xi))$$

define a sequence $\{h^+_j(w, \xi)\}_{i \in \mathbb{N}}$ of real-valued Borel measurable functions. It holds that $h^+_j(w, \xi) := \limsup_{i \to \infty} h^+_j(w, \xi)$ and $h^-_j(w, \xi) := \liminf_{i \to \infty} h^+_j(w, \xi)$ are extended real-valued Borel measurable functions [2, Lemma 2.9]. For $\xi \in [0, 1]$ and $a \in \mathbb{R}$, a family of candidate approximate gradients can be defined as having components

$$\tilde{\nabla} F_j(w, \xi) = \begin{cases} \xi h^+_j(w, \xi) + (1 - \xi) h^-_j(w, \xi) & \text{if } [h^+_j(w, \xi), h^-_j(w, \xi)] \in \mathbb{R} \\ a & \text{otherwise.} \end{cases}$$

(32)

The function $\tilde{\nabla} F(w, \xi)$ will equal $\nabla F(w, \xi)$ wherever it exists. The set

$$A = \{(w, \xi) : |h^+_j(w, \xi)| < \infty \} \cap \{(w, \xi) : |h^-_j(w, \xi)| < \infty \}$$

is measurable given that for an extended real-valued measurable function $h$, the set $\{|h| = \infty\}$ is measurable [2, Page 11]. Let $\mathbb{1}_A(w, \xi)$ and $\mathbb{1}_{A^c}(w, \xi)$ denote

\[ Springer \]
the indicator functions of $A$ and its complement. The product of extended real-valued functions is measurable [2, Page 12-13], implying that $\zeta h^+_j(w, \xi) 1_{A}(w, \xi), (1 - \zeta)h^-_j(w, \xi) 1_{A}(w, \xi),$ and $a 1_{A^c}(w, \xi)$ are all measurable. Given that all three functions are real-valued, their sum is measurable, implying the measurability of (32).

Proof of Lemma 3.2 Following the proof of Lemma 3.1, let $D^c \subset \{ (w, \xi) : w \in \mathbb{R}^d, \xi \in \mathbb{R}^p \}$ be the same Borel measurable set containing the points where $F(w, \xi)$ is not differentiable or not Lipschitz continuous in $w.$ By Tonelli’s theorem, it was established that

$$\int_{w \in \mathbb{R}^d} \mathbb{E}_\xi [1_{D^c}(w, \xi)] dw = 0.$$ (33)

The function $G(w) := \mathbb{E}_\xi [1_{D^c}(w, \xi)]$ is measurable in $(\mathbb{R}^d, B_{\mathbb{R}^d}),$ hence the set $D_w := \{ w \in \mathbb{R}^d : G(w) = 0 \}$ is measurable with full measure by (33). As in Example A.1, let $h^+_j(w, \xi) := i(F(w + i^{-1} e_j, \xi) - F(w, \xi))$ for $i \in \mathbb{N}.$ For all $w \in D_w,$ $\lim_{i \to \infty} h^+_j(w, \xi) = \tilde{\nabla} F_j(w, \xi)$ for almost all $\xi,$ by the assumption that $\tilde{\nabla} F(w, \xi) = \nabla F(w, \xi)$ almost everywhere $F(w, \xi)$ is differentiable. Given the Lipschitz continuity condition of $F(w, \xi),$ for all $i \in \mathbb{N},$

$$|h^+_j(w, \xi)| \leq i C(\xi) |i^{-1}| = C(\xi)$$

for almost all $\xi.$ Given that $C(\xi) \in L^1(P_\xi),$ the dominated convergence theorem can be applied for all $w \in D_w.$ It follows that

$$\mathbb{E}_\xi[\tilde{\nabla} F_j(w, \xi)] \overset{a.e.}{=} \lim_{i \to \infty} \mathbb{E}_\xi[h^+_j(w, \xi, i)]$$

$$= \lim_{i \to \infty} i(f(w + i^{-1} e_j) - f(w))$$

$$\overset{a.e.}{=} \nabla f_j(w)$$

$$\overset{a.e.}{=} \tilde{\nabla} f_j(w),$$

where the first equality holds for all $w \in D_w,$ the third equality holds for almost all $w$ due to Rademacher’s theorem, and the last equality holds almost everywhere by assumption.

Proof of Lemma 3.3 The set valued function (6) is outer semicontinuous [17, Proposition 2.7]. The function $\text{dist}(0, \partial_{w} f(w))$ is then lower semicontinuous [25, Proposition 5.11 (a)], hence Borel measurable.

5 Using the standard convention that $0 \cdot \infty = 0.$
A.3 Section 4

**Proof of Lemma 4.1** Throughout the proof let \( \{x, x'\} \in \mathbb{R}^d \) be fixed. Consider the function in \( v \in [0, 1] \),

\[
\hat{f}_z(v) = f(x' + z + v(x - x')),
\]

for any \( z \in \mathbb{R}^d \). Where it exists,

\[
\hat{f}'_z(v) = \lim_{h \to 0} \frac{f(x' + z + (v + h)(x - x')) - f(x' + z + v(x - x'))}{h}
\]

is equal to the directional derivative of \( f(\hat{w}) \) at \( \hat{w} = x' + z + v(x - x') \) in the direction of \( x - x' \). Let \( \mathbb{1}_{Dc}(\cdot) \) be the indicator function of the complement of the set where \( f(\cdot) \) is differentiable, which is a Borel measurable function from the continuity of \( f(w) \) [14, Page 211]. Its composition with the continuous function \( \hat{w} \) in \( (z, v) \in \mathbb{R}^d \times [0, 1] \) is then as well. Similar to the proof of Lemma 3.1, using Tonelli’s theorem, the measure of where \( f(\hat{w}) \) is not differentiable can be computed as

\[
\int_0^1 \int_{z \in \mathbb{R}^d} \mathbb{1}_{Dc}(x' + z + v(x - x')) \, dz \, dv. \tag{34}
\]

For any \( v \in [0, 1] \), \( \int_{z \in \mathbb{R}^d} \mathbb{1}_{Dc}(x' + z + v(x - x')) \, dw = 0 \) by Rademacher’s theorem, implying that (34) equals 0, and \( f(\hat{w}) \) is differentiable for almost all \((z, v)\). It follows that for almost all \((z, v)\), the directional derivative exists, the approximate gradient \( \tilde{\nabla} f(\hat{w}) \) is equal to the gradient, and

\[
\hat{f}'_z(v) = \langle \tilde{\nabla} f(x' + z + v(x - x')), x - x' \rangle. \tag{35}
\]

In addition \( \hat{f}_z(v) \) is Lipschitz continuous,

\[
|\hat{f}_z(v) - \hat{f}_z(v')| = |f(x' + z + v(x - x')) - f(x' + z + v'(x - x'))| \\
\leq L_0 \|x - x'\|_2 |v - v'|.
\]

Choosing \( z = \bar{z} \) such that (35) holds for almost all \( v \in [0, 1] \), by the fundamental theorem of calculus for Lebesgue integrals,

\[
f(x + \bar{z}) = \hat{f}_{\bar{z}}(1) = \hat{f}_{\bar{z}}(0) + \int_{0}^{1} \hat{f}'_z(v) \, dv \\
= f(x' + \bar{z}) + \int_{0}^{1} \langle \tilde{\nabla} f(x' + \bar{z} + v(x - x')), x - x' \rangle \, dv.
\]
Rearranging and subtracting \( \langle \nabla f(x' + z), x - x' \rangle \) from both sides,

\[
f(x + z) - f(x' + z) - \langle \nabla f(x' + z), x - x' \rangle = \int_0^1 (\nabla f(x' + z + v(x - x'))) - \nabla f(x' + z), x - x' \rangle dv.
\]

As for almost all \( z \in \mathbb{R}^d \), (35) holds for almost all \( v \in [0, 1] \),

\[
f(w) - f(w') - \langle \nabla f(w'), x - x' \rangle = \int_0^1 (\nabla f(w' + v(x - x'))) - \nabla f(w'), x - x' \rangle dv
\]

holds for almost all \( z \in \mathbb{R}^n \).

**Proof of Lemma 4.2** As \( f(w) \) is differentiable almost everywhere, and \( \nabla f(w) \) is equal to the gradient of \( f(w) \) almost everywhere it is differentiable, using the directional derivative and Lipschitz continuity of \( f(w) \),

\[
\| \nabla f(w) \|^2_2 = \lim_{h \to 0} \frac{f(w + h \nabla f(w)) - f(w)}{h} \leq L_0 \| \nabla f(w) \|^2_2
\]

holds almost everywhere. Similarly, by assumption and Lemma 3.1, \( F(w, \xi) \) is Lipschitz continuous and differentiable almost everywhere, with \( \nabla F(w, \xi) \) equal to the gradient almost everywhere \( F(w, \xi) \) is differentiable. It follows that almost everywhere,

\[
\| \nabla F(w, \xi) \|^2_2 = \lim_{h \to 0} \frac{F(w + h \nabla F(w, \xi), \xi) - F(w, \xi)}{h} \leq C(\xi) \| \nabla F(w, \xi) \|^2_2.
\]

**Proof of Lemma 4.3** We first show that

\[
\mathbb{E}[\| \nabla F \|^2_2 - \| \nabla F | x \|_2^2] = \mathbb{E}[\| \nabla F - \nabla F | x \|_2^2];
\]

\[
\mathbb{E}[\| \nabla F \|^2_2 - \| \nabla F | x \|_2^2] = \mathbb{E}[\| \nabla F \|^2_2 - 2\langle \nabla F, \nabla F | x \rangle + \| \nabla F | x \|_2^2]
\]

\[
= \mathbb{E}[\| \nabla F \|^2_2] - 2\mathbb{E}[\langle \nabla F, \nabla F | x \rangle] + \mathbb{E}[\| \nabla F | x \|_2^2]
\]

\[
= \mathbb{E}[\| \nabla F \|^2_2] - 2\mathbb{E}[\| \nabla F \|^2] + \mathbb{E}[\| \nabla F | x \|_2^2]
\]

\[
= \mathbb{E}[\| \nabla F \|^2_2] - 2\mathbb{E}[\| \nabla F \|^2] + \mathbb{E}[\| \nabla F | x \|_2^2]
\]

\[
= \mathbb{E}[\| \nabla F \|^2_2] - \mathbb{E}[\| \nabla F | x \|_2^2].
\]

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Let \( w_l := x + z_l \) for \( l = 1, \ldots, S \). Analyzing now \( \mathbb{E} [\| \nabla F - \mathbb{E} [\nabla F | x] \|^2]\),

\[
\begin{align*}
\mathbb{E} [\| \nabla F - \mathbb{E} [\nabla F | x] \|^2] &= \mathbb{E} \left[ \sum_{j=1}^{d} (\nabla_j F - \mathbb{E} [\nabla_j F | x])^2 \right] \\
&= \mathbb{E} \left[ \sum_{j=1}^{d} \left( \frac{1}{S} \sum_{l=1}^{S} (\widetilde{\nabla}_j F (w_l, \xi_l) - \mathbb{E} [\nabla_j F | x]) \right)^2 \right] \\
&= \frac{1}{S^2} \sum_{j=1}^{d} \mathbb{E} \left( \left( \sum_{l=1}^{S} (\widetilde{\nabla}_j F (w_l, \xi_l) - \mathbb{E} [\nabla_j F | x]) \right)^2 \right) \\
&= \frac{1}{S^2} \sum_{j=1}^{d} \mathbb{E} \left( \sum_{l=1}^{S} \mathbb{E} [((\widetilde{\nabla}_j F (w_l, \xi_l) - \mathbb{E} [\nabla_j F | x])^2 | x] \right) \\
&= \frac{1}{S^2} \sum_{j=1}^{d} \mathbb{E} \left( \sum_{l=1}^{S} \mathbb{E} [((\widetilde{\nabla}_j F (w_l, \xi_l) - \mathbb{E} [\nabla_j F | x])^2 | x] \right), \quad (36)
\end{align*}
\]

where (37) holds since \( \widetilde{\nabla}_j F (w_l, \xi_l) - \mathbb{E} [\nabla_j F | x] \) for \( l = 1, \ldots, S \) are conditionally independent random variables with conditional expectation of zero with respect to \( x \):

Considering the cross terms of \( \mathbb{E} [\sum_{l=1}^{S} (\widetilde{\nabla}_j F (w_l, \xi_l) - \mathbb{E} [\nabla_j F | x])^2 | x] \) in (36) with \( l \neq m \),

\[
\begin{align*}
\mathbb{E} [((\widetilde{\nabla}_j F (w_l, \xi_l) - \mathbb{E} [\nabla_j F | x]) (\widetilde{\nabla}_j F (w_m, \xi_m) - \mathbb{E} [\nabla_j F | x])) | x] &= \mathbb{E} [\widetilde{\nabla}_j F (w_l, \xi_l) \widetilde{\nabla}_j F (w_m, \xi_m) | x] - \mathbb{E} [\widetilde{\nabla}_j F (w_l, \xi_l) \mathbb{E} [\nabla_j F | x] | x] \\
&\quad - \mathbb{E} [\mathbb{E} [\widetilde{\nabla}_j F | x] \widetilde{\nabla}_j F (w_m, \xi_m) | x] + \mathbb{E} [\mathbb{E} [\widetilde{\nabla}_j F | x]^2 | x] \\
&= \mathbb{E} [\widetilde{\nabla}_j F (w_l, \xi_l) \widetilde{\nabla}_j F (w_m, \xi_m) | x] - \mathbb{E} [\widetilde{\nabla}_j F (w_l, \xi_l) \mathbb{E} [\nabla_j F | x] | x] \\
&\quad - \mathbb{E} [\mathbb{E} [\widetilde{\nabla}_j F | x] \widetilde{\nabla}_j F (w_m, \xi_m) | x] + \mathbb{E} [\mathbb{E} [\widetilde{\nabla}_j F | x]^2 | x] \\
&= 0.
\end{align*}
\]

Continuing from (37),

\[
\begin{align*}
\frac{1}{S^2} \sum_{j=1}^{d} \mathbb{E} \left( \sum_{l=1}^{S} \mathbb{E} [((\widetilde{\nabla}_j F (w_l, \xi_l) - \mathbb{E} [\nabla_j F | x])^2 | x] \right) &= \frac{1}{S^2} \sum_{j=1}^{d} \sum_{l=1}^{S} \mathbb{E} [((\widetilde{\nabla}_j F (w_l, \xi_l) - \mathbb{E} [\nabla_j F | x])^2] \\
&= \frac{1}{S} \sum_{j=1}^{d} \mathbb{E} [((\widetilde{\nabla}_j F (w_l, \xi_l) - \mathbb{E} [\nabla_j F | x])^2] \quad (38)
\end{align*}
\]
where the last equality holds for any \( l \in \{1, \ldots, S\} \). Continuing from (38),

\[
\frac{1}{S} \sum_{j=1}^{d} \mathbb{E}[(\nabla_j F(w_l, \xi_l) - \mathbb{E}[\nabla_j F|x])^2] \\
= \frac{1}{S} \sum_{j=1}^{d} \mathbb{E}(\mathbb{E}[(\nabla_j F(w_l, \xi_l) - \mathbb{E}[\nabla_j F|x])^2|x]) \\
= \frac{1}{S} \sum_{j=1}^{d} \mathbb{E}(\mathbb{E}[\nabla_j F(w_l, \xi_l)^2 - 2\nabla_j F(w_l, \xi_l)\mathbb{E}[\nabla_j F|x] + \mathbb{E}[\nabla_j F|x]^2] \\
= \frac{1}{S} \sum_{j=1}^{d} \mathbb{E}(\mathbb{E}[\nabla_j F(w_l, \xi_l)^2|x] - 2\mathbb{E}[\nabla_j F(w_l, \xi_l)]\mathbb{E}[\nabla_j F|x] + \mathbb{E}[\nabla_j F|x]^2) \\
= \frac{1}{S} \sum_{j=1}^{d} \mathbb{E}(\mathbb{E}[\nabla_j F(w_l, \xi_l)^2|x] - \mathbb{E}[\nabla_j F|x]^2) \\
= \frac{1}{S} \sum_{j=1}^{d} \mathbb{E}[\nabla_j F(w_l, \xi_l)^2] - \mathbb{E}[\mathbb{E}[\nabla_j F|x]^2] \\
\leq \frac{1}{S} \sum_{j=1}^{d} \mathbb{E}[\nabla_j F(w_l, \xi_l)^2] \\
= \frac{1}{S} \mathbb{E}[\|\nabla F(w_l, \xi_l)\|^2] \\
\leq \frac{Q}{S},
\]

(39)

where the final inequality uses Lemma 4.2 and the definition \( Q := \mathbb{E}[C(\xi)^2] \); Similar to showing (14), since \( z_l \) and \( \xi_l \) are independent of \( x_l \), \( \mathbb{E}[\|\nabla F(x + z_l, \xi_l)\|^2|x] = g(x) \), where \( g(y) := \mathbb{E}[\|\nabla F(y + z_l, \xi_l)\|^2] \). By the absolute continuity of \( z_l \), for almost every \((z_l, \xi_l)\) from Lemma 4.2, hence \( g(y) \leq \mathbb{E}[C(\xi)^2] \) for all \( y \in \mathbb{R}^d \), and in particular, \( \mathbb{E}[\|\nabla F(w_l, \xi_l)\|^2] = \mathbb{E}[g(x)] \leq \mathbb{E}[C(\xi)^2] \). \( \square \)

**Lemma A.1** For \( d \in \mathbb{N} \),

\[
\frac{\lambda(d)d!!}{(d-1)!!} \leq \sqrt{d}.
\]

**Proof** For \( d = 1 \), \( \frac{\lambda(d)d!!}{(d-1)!!} = 1 \), and for \( d = 2 \), \( \frac{\lambda(d)d!!}{(d-1)!!} = \frac{4}{\pi} < \sqrt{2} \). For \( d \geq 2 \), we will show that the result holds for \( d + 1 \) assuming that it holds for \( d - 1 \), proving the result
by induction.

\[
\frac{\lambda(d + 1)(d + 1)!}{d!!} = \frac{\lambda(d - 1)(d + 1)(d - 1)!}{d(d - 2)!!} = \frac{\lambda(d - 1)(d - 1)!}{(d - 2)!!} \frac{(d + 1)}{d} \\
\leq \sqrt{d - 1} \frac{(d + 1)}{d} \\
= \sqrt{\frac{(d - 1)(d + 1)^2}{d^2}} \\
= \sqrt{\frac{d^3 + d^2 - d}{d^2}} \\
< \sqrt{d + 1}.
\]

\[\square\]

**Proof of Corollary 4.1** From Theorem 4.1, \(\sigma = \theta \sqrt{d} K^{-\beta}\), and requiring \(\sigma \leq \epsilon_1\) implies

\[
\left(\frac{\theta \sqrt{d}}{\epsilon_1}\right)^{1/\beta} \leq K.
\]

Taking \(K_{\epsilon_1} = \left\lceil \left(\frac{\theta \sqrt{d}}{\epsilon_1}\right)^{1/\beta} \right\rceil + 1\) and \(S_{\epsilon_1} = \left\lceil K_{\epsilon_1}^{1-\beta} \right\rceil < K_{\epsilon_1}^{1-\beta} + 1\),

\[
S_{\epsilon_1} < \left(\frac{\theta \sqrt{d}}{\epsilon_1}\right)^{1/\beta} + 1 < \left(\frac{\theta \sqrt{d}}{\epsilon_1}\right)^{1-\beta} + 2,
\]

where the second inequality follows from a general result for \(a_i > 0\) for \(i = 1, \ldots, n\) and \(\beta \in (0, 1)\): \((\sum_{i=1}^n a_i)^{1-\beta} = \frac{\sum_{i=1}^n a_i}{(\sum_{i=1}^n a_i)^{\beta}} < \sum_{i=1}^n \frac{a_i}{a_i^{\beta}} = \sum_{i=1}^n a_i^{1-\beta}\). An upper bound on the total number of gradient calls required to satisfy \(\epsilon_1\), considering up to \(K_{\epsilon_1} - 1\) iterations of PISGD is then

\[
(K_{\epsilon_1} - 1)S_{\epsilon_1} < \left(\frac{\theta \sqrt{d}}{\epsilon_1}\right)^{1/\beta} \left(\left(\frac{\theta \sqrt{d}}{\epsilon_1}\right)^{1-\beta} + 2\right) = O\left(\frac{\beta-2}{\epsilon_1^p}\right).
\]

Choosing \(K\) such that

\[
\mathbb{E}[\text{dist}(0, \partial_{\sigma} f(x^R))] < K^{\frac{\beta-1}{2}} \sqrt{2 \left(\frac{L_0}{\theta \Delta} + L_0^2 \sqrt{d} K^{-\beta} + Q\right)}
\]

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\[
\leq K^{\frac{\beta - 1}{2}} \sqrt{2 \left( \frac{L_0 \Delta}{\theta} + L_0^2 \sqrt{d} + Q \right)} \\
\leq \epsilon_2
\]
gives the bound
\[
\left( \frac{2}{\epsilon_2^2} \left( \frac{L_0}{\theta} \Delta + L_0^2 \sqrt{d} + Q \right) \right)^{\frac{1}{1 - \beta}} \leq K.
\]
Taking \( K_{\epsilon_2} = \left\lceil \left( \frac{2}{\epsilon_2^2} \left( \frac{L_0}{\theta} \Delta + L_0^2 \sqrt{d} + Q \right) \right)^{\frac{1}{1 - \beta}} \right\rceil \) and \( S_{\epsilon_2} = \lceil K_{\epsilon_2}^{1 - \beta} \rceil \),
\[
S_{\epsilon_2} < \left( \left( \frac{2}{\epsilon_2^2} \left( \frac{L_0}{\theta} \Delta + L_0^2 \sqrt{d} + Q \right) \right)^{\frac{1}{1 - \beta}} + 1 \right)^{1 - \beta} + 1
\]
\[
< \frac{2}{\epsilon_2^2} \left( \frac{L_0}{\theta} \Delta + L_0^2 \sqrt{d} + Q \right) + 2,
\]
and the number of gradient calls required to satisfy \( \epsilon_2 \) is bounded by
\[
(K_{\epsilon_2} - 1)S_{\epsilon_2} < \left( \frac{2}{\epsilon_2^2} \left( \frac{L_0}{\theta} \Delta + L_0^2 \sqrt{d} + Q \right) \right)^{\frac{1}{1 - \beta}} \left( \frac{2}{\epsilon_2^2} \left( \frac{L_0}{\theta} \Delta + L_0^2 \sqrt{d} + Q \right) + 2 \right)
\]
\[
= O \left( \epsilon_2^{-2 \frac{2-\beta}{1-\beta}} \right).
\]
The number of gradient calls required to satisfy both \( \epsilon_1 \) and \( \epsilon_2 \) is then
\[
\max((K_{\epsilon_1} - 1)S_{\epsilon_1}, (K_{\epsilon_2} - 1)S_{\epsilon_2}) = O \left( \max \left( \epsilon_1^{\frac{\beta - 2}{\beta}}, \epsilon_2^{-2 \frac{2-\beta}{1-\beta}} \right) \right).
\]

Proof of Corollary 4.2 Based on Theorem 4.1, an optimal choice for \( K \in \mathbb{Z}_{>0} \) and \( \beta \in (0, 1) \) can be written as the following optimization problem for the minimization of the number of gradient calls,
\[
\min_{K, \beta} \ (K - 1) \lceil K^{1 - \beta} \rceil \quad (40)
\]
s.t. \( \sqrt{d K^{-\beta}} \leq \epsilon_1 \\
K^{\frac{\beta - 1}{2}} \sqrt{2 \left( \frac{L_0 \Delta}{\theta} + L_0^2 \sqrt{d K^{-\beta}} + Q \right)} \leq \epsilon_2 \\
K \in \mathbb{Z}_{>0}, \quad \beta \in (0, 1), \]
requiring \( \sigma \leq \epsilon_1 \) and the right-hand side of (10) to be less than or equal to \( \epsilon_2 \). Rearranging the inequalities and adding the valid inequality \( 1 \leq K^\beta \) given the constraints on \( K \) and \( \beta \), (40) can be rewritten as

\[
\min_{K, \beta} (K - 1)[K^{1 - \beta}]
\]

s.t. \[
\begin{align*}
1, \frac{\sqrt{d}}{\epsilon_1} \leq K^\beta \\
K^\beta \leq \frac{K\epsilon_2^2 - 2L_0^2\sqrt{d}}{2(L_0\Delta + Q)} \\
K \in \mathbb{Z}_{>0}, \quad \beta \in (0, 1). 
\end{align*}
\]

It is first shown that \( K^* \) is a lower bound for a feasible \( K \) to problem (41). For the case \( \epsilon_1 < \sqrt{d} \), minimizing the gap between \( \frac{\sqrt{d}}{\epsilon_1} \) and \( \frac{K\epsilon_2^2 - 2L_0^2\sqrt{d}}{2(L_0\Delta + Q)} \) sets \( K \) equal to

\[
K_l^* = \left[ \frac{2\sqrt{d}}{\epsilon_2^2} \left( \frac{L_0\Delta + Q}{\epsilon_1} + L_0^2 \right) \right],
\]

i.e., \( K_l^* \) is the minimum \( K \in \mathbb{Z}_{>0} \) such that \( \frac{\sqrt{d}}{\epsilon_1} \leq \frac{K\epsilon_2^2 - 2L_0^2\sqrt{d}}{2(L_0\Delta + Q)} \). When \( \epsilon_1 \geq \sqrt{d} \), minimizing the gap between 1 and \( \frac{K\epsilon_2^2 - 2L_0^2\sqrt{d}}{2(L_0\Delta + Q)} \) requires

\[
K \geq \frac{2}{\epsilon_2^2} \left( \frac{L_0\Delta + Q + \sqrt{d}L_0^2}{\epsilon_1} \right) > 2\sqrt{d},
\]

given that \( \epsilon_2 < L_0 \). A valid lower bound on \( K \) equals

\[
K_g^* = \left[ \frac{2}{\epsilon_2^2} \left( \frac{L_0\Delta + Q + \sqrt{d}L_0^2}{\epsilon_1} \right) + 1 \right].
\]

The use of \( \lfloor \cdot + 1 \rfloor \) instead of \( \lceil \cdot \rceil \) in this case takes into account of the possibility that \( \frac{2}{\epsilon_2^2} \left( \frac{L_0\Delta + Q + L_0^2\sqrt{d}}{\epsilon_1} \right) \in \mathbb{Z}_{>0} \). Trying to use \( K = \left[ \frac{2}{\epsilon_2^2} \left( \frac{L_0\Delta + Q + L_0^2\sqrt{d}}{\epsilon_1} \right) \right] \) would set \( \frac{K\epsilon_2^2 - 2L_0^2\sqrt{d}}{2(L_0\Delta + Q)} = 1 \), which would imply \( \beta = 0 \) as \( K > 2\sqrt{d} \). Comparing \( K_l^* \) and \( K_g^* \), for the case when \( \epsilon_1 < \sqrt{d} \), since \( K_l^* > \frac{2}{\epsilon_2^2} \left( \frac{L_0\Delta + Q + \sqrt{d}L_0^2}{\epsilon_1} \right) \), it follows that \( K_l^* \geq K_g^* \), and when \( \epsilon_1 \geq \sqrt{d} \),

\[
K_g^* \geq \left[ \frac{2\sqrt{d}}{\epsilon_2^2} \left( \frac{L_0\Delta + Q}{\epsilon_1} + L_0^2 \right) + 1 \right] \geq K_l^*,
\]

hence \( K^* \) is a valid lower bound for the number of iterations over all values of \( \epsilon_1 \).
For any fixed $K$, the objective of (41) is minimized by maximizing $\beta$, so we would want to set $\beta = \beta^*_K$ such that

$$K^{\beta_K} = \frac{K\epsilon^2_2 - 2L^2_0\sqrt{d}}{2(L_0\Delta + Q)},$$

which equals

$$\beta^*_K = \frac{\log(K\epsilon^2_2 - 2L^2_0\sqrt{d}) - \log(2(L_0\Delta + Q))}{\log(K)}.$$

We will show the validity of this choice of $\beta$ for all $K \geq K^*_g$, implying the validity for all $K \geq K^*$. For any $K \geq K^*_g > 2\sqrt{d}$, the division by $\log(K)$ in $\beta^*_K$ is defined.

We now verify that $\beta^*_K \in (0, 1)$ for $K \geq K^*$. To show that $\beta^*_K < 1$, isolating $\epsilon^2_2$, we require

$$\epsilon^2_2 < 2(L_0\Delta + Q) + \frac{2L^2_0\sqrt{d}}{K},$$

which holds given that $Q \geq L^2_0$ by Jensen’s inequality and $\epsilon_2 < L_0$. The bound $\beta^*_K > 0$ is equivalent to

$$0 < K\epsilon^2_2 - 2L^2_0\sqrt{d} - 2(L_0\Delta + Q).$$

For $K \geq K^*_g$,

$$K\epsilon^2_2 - 2L^2_0\sqrt{d} - 2(L_0\Delta + Q) > 2\left(L_0\Delta + Q + L^2_0\sqrt{d}\right) - 2L^2_0\sqrt{d} - 2(L_0\Delta + Q) \geq 0,$$

hence for all $K \geq K^*$, $\beta^*_K$ is feasible. This also proves that $K^*$ is the minimum feasible value for $K$, with $\beta^* = \beta^*_K$, proving statement 3.

We now consider the minimization of a relaxation of (41), allowing the number of samples $S \in \mathbb{R}$. As statements 1 and 2 concern the computational complexity of $(K^*, \beta^*)$ we will also now assume $\epsilon_1 < 1$ for simplicity.

$$\min_{K, \beta} (K - 1)K^{1-\beta} (42)$$

s.t. $\frac{\sqrt{d}}{\epsilon_1} \leq K^\beta$

$$K^\beta \leq \frac{K_2^2 - 2L_0^2\sqrt{d}}{2(L_0\Delta + Q)}$$

$\beta \in (0, 1), \quad K \in \mathbb{Z}_{>0}$.
Plugging in $K_{\beta}^* = \frac{K \epsilon_1^2 - 2L_0^2 \sqrt{d}}{2(L_0 \Delta + Q)}$, the optimization problem becomes

$$\min_{K \in \mathbb{Z}_{>0}} (K - 1) \frac{K(2(L_0 \Delta + Q))}{K \epsilon_2^2 - 2L_0^2 \sqrt{d}}$$

s.t. $K \geq K^*$.

For simplicity let

$$a := 2(L_0 \Delta + Q) \quad \text{and} \quad b := 2L_0^2 \sqrt{d}.$$ 

The problem is now

$$\min_{K \in \mathbb{Z}_{>0}} \frac{aK(K - 1)}{\epsilon_2^2 K - b}$$

s.t. $K \geq K^*$.

We will prove that $K^*$ is optimal for problem (43) by showing that the derivative of the objective function with respect to $K$ is positive for $K \geq K^*$.

$$\frac{d}{dK} \frac{aK(K - 1)}{\epsilon_2^2 K - b} = \frac{a(\epsilon_2^2 K^2 - 2bK + b)}{(\epsilon_2^2 K - b)^2}$$

is non-negative for $K \geq \frac{b + \sqrt{b(b - \epsilon_2^2)}}{\epsilon_2^2}$ and positive for $K \geq \frac{2b}{\epsilon_2^2}$ by removing $\epsilon_2^2$ in the numerator. Written in full form, the objective is increasing in $K$ for

$$K \geq \frac{4L_0^2 \sqrt{d}}{\epsilon_2^2}.$$ 

Comparing this inequality with $K^* = K_i^*$ given that $\epsilon_1 < 1$,

$$K^* > \frac{2\sqrt{d}}{\epsilon_2^2} \left( L_0 \Delta + Q + L_0^2 \right) \geq \frac{2\sqrt{d}}{\epsilon_2^2} \left( Q + L_0^2 \right) \geq \frac{4L_0^2 \sqrt{d}}{\epsilon_2^2}$$

using $Q \geq L_0^2$, hence over the feasible $K \geq K^*$, the objective (43) is increasing, and $(K^*, \beta^*)$ is an optimal solution of (42).

Writing $K^* = K_i^* = \left[ \frac{1}{\epsilon_2^2} \left( \frac{a \sqrt{d}}{\epsilon_1} + b \right) \right]$, a bound on the optimal value of the relaxed problem (43) gives

$$(K^* - 1) \left( \frac{aK^*}{\epsilon_2^2 K^* - b} \right) < \frac{1}{\epsilon_2^2} \left( \frac{a \sqrt{d}}{\epsilon_1} + b \right) \left( \frac{aK^*}{\epsilon_2^2 K^* - b} \right)$$
where for the second inequality \( \frac{aK}{\epsilon_2 K - b} \) is decreasing in \( K \), \( \frac{d}{dK} \frac{aK}{\epsilon_2 K - b} = \frac{-ab}{(\epsilon_2 K - b)^2} \). This bound cannot be improved as \( (K^* - 1) \left( \frac{aK^*}{\epsilon_2 K^* - b} \right) \geq \left( \frac{1}{\epsilon_2} \left( \frac{a \sqrt{d}}{\epsilon_1} + b \right) - 1 \right) \left( \frac{a}{\epsilon_2} \right) = O \left( \frac{1}{\epsilon_1 \epsilon_2} \right) \). A bound on the gradient call complexity of \((K^*, \beta^*)\) for finding an expected \((\epsilon_1, \epsilon_2)\)-stationary point is then \( (K^* - 1) \left( \frac{aK^*}{\epsilon_2 K^* - b} + 1 \right) = O \left( \frac{1}{\epsilon_1 \epsilon_2} \right) \), proving statement 1.

Let \((\hat{K}, \hat{\beta})\) be an optimal solution to the original problem (41) with \( \epsilon_1 < 1 \). The inequalities

\[
(K^* - 1) \frac{aK^*}{\epsilon_2 K^* - b} \leq (\hat{K} - 1) \left[ \hat{K}^{1 - \hat{\beta}} \right] \leq (K^* - 1) \left[ \frac{aK^*}{\epsilon_2 K^* - b} \right]
\]

hold since restricting \( S \in \mathbb{Z}_{>0} \) cannot improve the optimal objective value of (42), and by the optimality of \((\hat{K}, \hat{\beta})\) for problem (41), respectively. This proves that using \((\hat{K}, \hat{\beta})\) will result in a gradient call complexity of \( O \left( \frac{1}{\epsilon_1 \epsilon_2^2} \right) \), proving statement 2. \( \Box \)

**Proof of Corollary 4.3** Let \( \nabla f(x) := \mathbb{E}[\tilde{\nabla} f(x + z) | x] \) for \( z \sim U(B(\sigma)) \). Following [16, Eq. 2.28] for the first inequality,\(^6\)

\[
\|
\nabla f(\tilde{x}^*)\|^2 \leq 4 \min_{i=1, \ldots, \mathcal{R}} \|
\nabla f(\tilde{x}^i)\|^2 + 4 \max_{i=1, \ldots, \mathcal{R}} \|
\nabla F_T(\tilde{x}^i) - \nabla f(\tilde{x}^i)\|^2 
\]

\[
+ 2\|
\nabla F_T(\tilde{x}^*) - \nabla f(\tilde{x}^*)\|^2 
\]

\[
\leq 4 \min_{i=1, \ldots, \mathcal{R}} \|
\nabla f(\tilde{x}^i)\|^2 + 6 \max_{i=1, \ldots, \mathcal{R}} \|
\nabla F_T(\tilde{x}^i) - \nabla f(\tilde{x}^i)\|^2. \quad (44)
\]

where the second inequality holds since \( 2\|
\nabla F_T(\tilde{x}^*) - \nabla f(\tilde{x}^*)\|^2 \leq 2 \max_{i=1, \ldots, \mathcal{R}} \|
\nabla F_T(\tilde{x}^i) - \nabla f(\tilde{x}^i)\|^2 \). We will compute an upper bound in probability of the left-hand side of (44) using the two terms of the right-hand side. Let \( B \) equal the right-hand

\(^6\) The derivation of this bound is independent of how \( \nabla f(\cdot) \) and \( \nabla F_T(\cdot) \) are defined.
side of (10),

\[ B^2 = 2K^{\beta-1} \left( L_0 \Delta + L_0^2 \sqrt{d} K^{-\beta} + Q \right). \]  

(45)

Since \( \nabla f(x) = \mathbb{E} \left[ \frac{1}{S'} \sum_{j=1}^{S'} \nabla f(x + z_j) | x \right] \) for any number of samples \( S' \in \mathbb{Z}_{>0} \) of \( z \), using inequalities (23) and (24) of the proof of Theorem 4.1, for all \( i \in \{1, \ldots, R\} \),

\[ \mathbb{E}[||\nabla f(\tilde{x}^i)||_2^2] \leq B^2. \]

From Markov’s inequality,

\[ \mathbb{P}(4 \min_{i=1,\ldots,R} ||\nabla f(\tilde{x}^i)||_2^2 \geq 4eB^2) = \mathbb{P}(4||\nabla f(\tilde{x}^i)||_2^2 \geq 4eB^2) \leq e^{-R}. \]

Given that also \( \nabla f(x) = \mathbb{E} \left[ \frac{1}{T'} \sum_{l=1}^{T'} \nabla F(x + z_l, \xi_l) | x \right] \) for any number of samples \( T' \in \mathbb{Z}_{>0} \) of \( z \) and \( \xi \), \( \mathbb{E}[||\nabla F_T(\tilde{x}^i) - \nabla f(\tilde{x}^i)||_2^2] \leq \frac{Q}{T} \) from Lemma 4.3. For \( \psi > 0 \), it holds that

\[
\mathbb{P} \left( 6 \max_{i=1,\ldots,R} ||\nabla F_T(\tilde{x}^i) - \nabla f(\tilde{x}^i)||_2^2 \geq 6\psi \frac{Q}{T} \right) \\
\leq \mathbb{P} \left( \bigcup_{i=1}^{R} \left\{ 6||\nabla F_T(\tilde{x}^i) - \nabla f(\tilde{x}^i)||_2^2 \geq 6\psi \frac{Q}{T} \right\} \right) \\
\leq \sum_{i=1}^{R} \mathbb{P}(6||\nabla F_T(\tilde{x}^i) - \nabla f(\tilde{x}^i)||_2^2 \geq 6\psi \frac{Q}{T}) \\
\leq \sum_{i=1}^{R} \frac{1}{\psi} = \frac{R}{\psi},
\]

using Boole’s and Markov’s inequalities for the first and second inequalities, respectively. Using the fact that \( \nabla f(\tilde{x}^*) \in \partial_{\sigma} f(\tilde{x}^*) \) for the first inequality,

\[
\mathbb{P} \left( \text{dist}(0, \partial_{\sigma} f(\tilde{x}^*))^2 \geq 4eB^2 + 6\psi \frac{Q}{T} \right) \\
\leq \mathbb{P} \left( ||\nabla f(\tilde{x}^*)||_2^2 \geq 4eB^2 + 6\psi \frac{Q}{T} \right) \\
\leq \mathbb{P} \left( 4 \min_{i=1,\ldots,R} ||\nabla f(\tilde{x}^i)||_2^2 + 6 \max_{i=1,\ldots,R} ||\nabla F_T(\tilde{x}^i) - \nabla f(\tilde{x}^i)||_2^2 \geq 4eB^2 + 6\psi \frac{Q}{T} \right) \\
\leq \mathbb{P} \left( \bigcup_{i=1}^{R} \left\{ 4||\nabla f(\tilde{x}^i)||_2^2 \geq 4eB^2 \right\} \cup \left\{ 6 \max_{i=1,\ldots,R} ||\nabla F_T(\tilde{x}^i) - \nabla f(\tilde{x}^i)||_2^2 \geq 6\psi \frac{Q}{T} \right\} \right) \\
\leq e^{-R} + \frac{R}{\psi}.
\]  

(46)
where the second inequality uses inequality (44), and the third inequality holds given that the event of the left-hand-side is a subset of the right-hand-side: considering the contraposition, if

\[
\left\{ 4 \min_{i=1,\ldots,R} ||\nabla f(\bar{x}^i)||_2^2 < 4eB^2 \right\} \cap \left\{ 6 \max_{i=1,\ldots,R} ||\nabla F_T(\bar{x}^i) - \nabla f(\bar{x}^i)||_2^2 < 6\psi \frac{Q}{T} \right\}
\]

occurs then

\[
4 \min_{i=1,\ldots,R} ||\nabla f(\bar{x}^i)||_2^2 + 6 \max_{i=1,\ldots,R} ||\nabla F_T(\bar{x}^i) - \nabla f(\bar{x}^i)||_2^2 < 4eB^2 + 6\psi \frac{Q}{T}.
\]

The final inequality holds using Boole’s inequality, as the right-hand-side is the sum of the derived upper bounds of the probabilities of the two events of the union.

The total number of gradient calls required for computing \(X\) and then \(\nabla F_T(x)\) for all \(x \in X\) to find \(\bar{x}^*\) is equal to \(R((K-1)[K^{1-\beta}] + T)\). We can write its minimization requiring that \(\mathbb{P}(\text{dist}(0, \partial_{\epsilon_1} f(\bar{x}^*)) > \epsilon_2) \leq \gamma\) using (46) with (45) similarly to how (40) was derived, as

\[
\begin{align*}
\min_{K,\beta, R, T, \psi} \quad & \mathcal{R}((K-1)[K^{1-\beta}] + T) \\
\text{s.t.} \quad & \sqrt{d} K^{\beta-1} \leq \epsilon_1 \\
& 8eK^{\beta-1} \left( L_0 \Delta + L_0^2 \sqrt{d} K^{\beta-1} + Q \right) + 6\psi \frac{Q}{T} \leq \epsilon_2^2 \\
& e^{-\mathcal{R}} + \frac{\mathcal{R}}{\psi} \leq \gamma \\
& K, R, T \in \mathbb{Z}_{>0}, \quad \beta \in (0, 1), \quad \psi > 0.
\end{align*}
\]

Inequality (47) can be rewritten as

\[
2K^{\beta-1} \left( L_0 \Delta + L_0^2 \sqrt{d} K^{\beta-1} + Q \right) \leq \frac{\epsilon_2^2 - 6\psi \frac{Q}{T}}{4e}.
\]

We apply the choice of \(K\) and \(\beta\) from Corollary 4.2 for finding an expected \((\epsilon_1, \epsilon_2')\)-stationary point, where \(\epsilon_2' = \sqrt{\frac{\epsilon_2^2 - 6\psi \frac{Q}{T}}{4e}}\):

\[
K^* = \max \left( \left\lfloor \frac{2}{(\epsilon_2')^2} \left( L_0 \Delta + Q + \sqrt{d} L_0^2 \right) + 1 \right\rfloor, \left\lceil \frac{2\sqrt{d}}{(\epsilon_2')^2} \left( \frac{L_0 \Delta + Q}{\epsilon_1} + L_0^2 \right) \right\rceil \right)
\]

and

\[
\beta^* = \frac{\log(K^*(\epsilon_2')^2 - 2\sqrt{d} L_0^2) - \log(2(L_0 \Delta + Q))}{\log(K^*)}.
\]
In order to ensure the validity of these choices for $K$ and $\beta$, we require that $0 < \epsilon'_2 < L_0$. The optimization problem then becomes

$$\min_{R, T, \psi} \mathcal{R}((K^* - 1) [(K^*)^{1-\beta^*}] + T)$$

s.t. $e^{-\mathcal{R}} + \frac{\mathcal{R}}{\psi} \leq \gamma$ \hspace{1cm} (48)

$$\epsilon_2^2 - 6\psi \frac{Q}{T} \in (0, 4eL_0)$$ \hspace{1cm} (49)

$$\mathcal{R}, T \in \mathbb{Z}_{>0}, \quad \psi > 0,$$

where (49) ensures that $0 < \epsilon'_2 < L_0$. Choosing $R = \lceil -\ln(c\gamma) \rceil$ and $\psi = \lceil -\ln(c\gamma) \rceil (1 - c)\gamma$ for any $c \in (0, 1)$ ensures that (48) holds by satisfying the inequalities

$$e^{-\mathcal{R}} \leq c\gamma \quad \text{and} \quad \frac{\mathcal{R}}{\psi} \leq (1 - c)\gamma.$$ 

Choosing $T = \lceil 6\phi \psi \frac{Q}{\epsilon_2^2} \rceil$ is feasible for (49):

$$4eL_0^2 > \epsilon_2^2 \geq \epsilon_2^2 - 6\psi \frac{Q}{T} = \epsilon_2^2 - 6\psi \frac{Q}{6\phi \psi \frac{Q}{\epsilon_2^2}} \geq \epsilon_2^2 - 6\psi \frac{Q}{6\phi \psi \frac{Q}{\epsilon_2^2}} = \epsilon_2^2 - \frac{\epsilon_2^2}{\phi} > 0,$$

(50)

given the assumptions that $\epsilon_2 < L_0$ and $\phi > 1$. We have verified that the choices for $K, \beta, \mathcal{R},$ and $T$ ensure that the output $\tilde{x}^*$ of the proposed method is an $(\epsilon_1, \epsilon_2)$-stationary point with a probability of at least $1 - \gamma$. What remains is the computational complexity. The total number of gradient calls equals

$$\lceil -\ln(c\gamma) \rceil \left( (K^* - 1) [(K^*)^{1-\beta^*}] + \lceil 6\phi \frac{[-\ln(c\gamma)]}{(1 - c)\gamma} \frac{Q}{\epsilon_2^2} \rceil \right), \hspace{1cm} (51)$$

From Corollary 4.2, $(K^* - 1) [(K^*)^{1-\beta^*}] = O\left(\frac{1}{\epsilon_1 (\epsilon_2)^{4\epsilon}}\right)$ and from (50) $\epsilon'_2 = \sqrt{\frac{\epsilon_2^2 - 6\psi \frac{Q}{4e^2}}{2\epsilon}} \geq \frac{\epsilon_2^2}{2} \sqrt{\frac{(1 - \phi^{-1})^{-1}}{e}},$ hence $(K^* - 1) [(K^*)^{1-\beta^*}] = O\left(\frac{1}{\epsilon_1 \epsilon_2^4}\right).$ The total computational complexity from (51) then equals $\tilde{O}\left(\frac{1}{\epsilon_1 \epsilon_2^4} + \frac{1}{\gamma \epsilon_2^2}\right).$ \qed

**A.4 Section 5**

**Proof of Proposition 5.1** We will ultimately consider the decision variables in a vector form $\overline{w} := [(\overline{w}^2)^T, (\overline{w}^3)^T]^T$, where $\overline{w}^l := [W^l_1, b^l_1, W^l_2, b^l_2, \ldots, W^l_{N_l}, b^l_{N_l}]^T$ for $l = 2, 3$, where $W^l_j$ is the $j$th row of $W^l$. Springer
The partial derivative of $L$ with respect to $z^3_j$ is

$$\frac{\partial L}{\partial z_j^3} = \alpha^3_j - y_j^i.$$ 

Given that $y^i$ is one-hot encoded, and the $\alpha^3_j$ take the form of probabilities, $\|\nabla z^3 L\|_2 \leq \sqrt{2}$, and $L$ as a function of $z^3$ is $\sqrt{2}$-Lipschitz continuous. Considering $z^3 = H(W^3)\alpha^2 + b^3$ as a function of $\widebar{w}^3$, let

$$\overline{h}(\overline{w}^3) := [H(W^3_1), b^3_1, H(W^3_2), b^3_2, \ldots, H(W^3_{N_3}), b^3_{N_3}] \in \mathbb{R}^{N_3(N_2+1)},$$

$$\overline{\alpha} := [(\alpha^2)^T, 1] \in \mathbb{R}^{N^2+1}, \ 0 := [0, \ldots, 0] \in \mathbb{R}^{N^2+1},$$

and the matrix

$$A := \begin{bmatrix}
\overline{\alpha} & 0 & 0 & \ldots & 0 \\
0 & \overline{\alpha} & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & \overline{\alpha}
\end{bmatrix} \in \mathbb{R}^{N_3 \times (N_3(N_2+1))},$$

so that $z^3(\overline{w}^3) = A\overline{h}(\overline{w}^3)$. The Lipschitz constant of $z^3(\overline{w}^3)$ is found by first bounding the spectral norm of $A$, which equals the square root of the largest eigenvalue of

$$AA^T = \text{diag}(\|\overline{\alpha}\|_2^2, \|\overline{\alpha}\|_2^2, \ldots) \in \mathbb{R}^{N_3 \times N_3},$$

hence $\|A\|_2 = \|(\alpha^2)^T, 1\|_2 \leq \sqrt{N_2 m^2 + 1}$. The function $\overline{h}(\overline{w}^3)$ is 1-Lipschitz continuous, and the composition of $L_i$-Lipschitz continuous functions is $\prod_i L_i$-Lipschitz continuous [27, Claim 12.7], therefore $L$ is $\sqrt{2(N_2 m^2 + 1)}$-Lipschitz continuous in $\overline{w}^3$.

Considering now $z^3$ as a function of $\alpha^2$, $z^3(\alpha^2)$ is $\|H(W^3)\|_2$-Lipschitz continuous. Given the boundedness of the hard tanh activation function, $\|H(W^3)\|_2 \leq \|H(W^3)\|_F \leq \sqrt{N_2 N_3}$. The ReLU-m activation functions are 1-Lipschitz continuous. As was done when computing a Lipschitz constant for $z^3(\overline{w}^3)$, to do so for $z^2(\overline{w}^3)$, let $\overline{v} := [(v^i)^T, 1]$, redefine $\mathbf{0} := [0, \ldots, 0] \in \mathbb{R}^{N^1+1}$, and let

$$V := \begin{bmatrix}
\overline{v} & 0 & 0 & \ldots & 0 \\
0 & \overline{v} & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & \overline{v}
\end{bmatrix} \in \mathbb{R}^{N_2 \times (N_2(N_1+1))}.$$ 

The Lipschitz constant for $z^2(\overline{w}^3) = V\overline{w}^2$ is then $\|(v^i)^T, 1\|_2$. In summary, $L(z^3)$ is $\sqrt{2}$-Lipschitz, $z^3(\alpha^2)$ is $\sqrt{N_2 N_3}$-Lipschitz, $\alpha^2(z^2)$ is 1-Lipschitz, and $z^2(\overline{w}^3)$ is $\|(v^i)^T, 1\|_2$-Lipschitz continuous, hence $L$ is $\sqrt{2N_2 N_3}\|(v^i)^T, 1\|_2$-Lipschitz continuous in $\overline{w}^2$. 

\[ \text{Springer} \]
Computing the Lipschitz constant for all decision variables,

$$\| \mathcal{L}(\bar{w}) - \mathcal{L}(\bar{w}') \|_2$$

$$= \| \mathcal{L}(w^2, \bar{w}^3) - L(w^2', \bar{w}'^3) + L(w^2, \bar{w}'^3) - \mathcal{L}(w^2', \bar{w}'^3) \|_2$$

$$\leq \| \mathcal{L}(w^2, \bar{w}^3) - \mathcal{L}(w^2', \bar{w}'^3) \|_2 + \| L(w^2', \bar{w}'^3) - \mathcal{L}(w^2', \bar{w}'^3) \|_2$$

$$\leq 2N_2N_3\| (v^i)^T, 1 \|_2 \| \bar{w}^2 - \bar{w}'^2 \|_2 + \sqrt{2(N_2m^2 + 1)} \| \bar{w}'^3 - \bar{w}'^3 \|_2$$

$$\leq \max(\sqrt{2N_2N_3\| (v^i)^T, 1 \|_2, \sqrt{2(N_2m^2 + 1)} \| \bar{w}'^3 - \bar{w}'^3 \|_2)$$

$$\leq 2 \max(\sqrt{N_2N_3\| (v^i)^T, 1 \|_2, \sqrt{(N_2m^2 + 1)} \| \bar{w}^3 - \bar{w}'^3 \|_2),$$

where the last inequality uses Young’s inequality:

$$2\| \bar{w}^2 - \bar{w}'^2 \|_2 \| \bar{w}^3 - \bar{w}'^3 \|_2 \leq \| \bar{w}^2 - \bar{w}'^2 \|_2 + \| \bar{w}^3 - \bar{w}'^3 \|_2$$

$$\implies \| \bar{w}^2 - \bar{w}'^2 \|_2 + 2\| \bar{w}^2 - \bar{w}'^2 \|_2 \| \bar{w}^3 - \bar{w}'^3 \|_2 + \| \bar{w}^3 - \bar{w}'^3 \|_2$$

$$\leq 2(\| \bar{w}^2 - \bar{w}'^2 \|_2 + \| \bar{w}^3 - \bar{w}'^3 \|_2)$$

$$\implies (\| \bar{w}^2 - \bar{w}'^2 \|_2 + \| \bar{w}^3 - \bar{w}'^3 \|_2)^2 \leq 2(\| \bar{w}^2, \bar{w}^3 \| - (\bar{w}'^2, \bar{w}'^3))_-^2$$

$$\implies \| \bar{w}^2 - \bar{w}'^2 \|_2 + \| \bar{w}^3 - \bar{w}'^3 \|_2 \leq \sqrt{2}(\| \bar{w}^2, \bar{w}^3 \| - (\bar{w}'^2, \bar{w}'^3))_-.$$

\[ \square \]

**Proof of Proposition 5.2** The problematic terms within (29) for which the chain rule does not necessarily apply are

$$\frac{\partial H_{jk}}{\partial W_{jk}^3} (t) = 1_{\{ t \geq -1 \}} 1_{\{ t \leq 1 \}} \quad \text{and} \quad \frac{\partial \alpha_j^2}{\partial z_j^2} (t) = 1_{\{ t \geq 0 \}} 1_{\{ t \leq m \}}.$$

Using PISGD, \( \frac{\partial H_{jk}}{\partial W_{jk}^3} (t) \) is evaluated at \( t = W_{jk}^3 + z_{W_{jk}^3} \). The probability that \( \frac{\partial H_{jk}}{\partial W_{jk}^3} (t) \) is evaluated at a point of non-differentiability, \( |W_{jk}^3 + z_{W_{jk}^3}| = 1 \), is zero:

$$\mathbb{E}[\{ \frac{\partial H_{jk}}{\partial W_{jk}^3} (t) = 1_{\{ t \geq -1 \}} 1_{\{ t \leq 1 \}} \} \big| |W_{jk}^3 + z_{W_{jk}^3}| = 1 \}$$

$$= \mathbb{E}[\mathbb{E}[\{ \frac{\partial H_{jk}}{\partial W_{jk}^3} (t) = 1_{\{ t \geq -1 \}} 1_{\{ t \leq 1 \}} \} | |W_{jk}^3 + z_{W_{jk}^3}| = 1 \}] W_{jk}^3].$$

Defining \( g(y) := \mathbb{E}[\{ y + z_{W_{jk}^3} = 1 \} + \{ y + z_{W_{jk}^3} = -1 \}] \),

$$g(W_{jk}^3) = \mathbb{E}[\{ W_{jk}^3 + z_{W_{jk}^3} = 1 \} + \{ W_{jk}^3 + z_{W_{jk}^3} = -1 \}| W_{jk}^3].$$

given the independence of \( z_{W_{jk}^3} \) with \( W_{jk}^3 \). Since \( z_{W_{jk}^3} \) is an absolutely continuous random variable, for any \( y \in \mathbb{R} \), \( g(y) = 0 \), hence

\[ \square \]
\[ \mathbb{E}[\mathbb{I}\{W_{jk}^3+z_{W_{jk}}^3=1\} + \mathbb{I}\{W_{jk}^3+z_{W_{jk}}^3=-1\} | W_{jk}] = 0. \]

The partial derivative \( \frac{\partial F_t}{\partial z} \) is evaluated at \( t = (W_j^2 + z_{W_j})v' + b_j^2 + z_{b_j}^2 \), which we rearrange as \( t = z_{W_j}v' + z_{b_j}^2 + W_j^2v' + b_j^2 \) for convenience. The points of non-differentiability are when \( z_{W_j}v' + z_{b_j}^2 + W_j^2v' + b_j^2 \in \{0, m\} \). Computing the probability of this event,

\[
\mathbb{E}[\mathbb{I}\{z_{W_j}v' + z_{b_j}^2 = -W_j^2v' - b_j^2\} + \mathbb{I}\{z_{W_j}v' + z_{b_j}^2 = m-W_j^2v' - b_j^2\}] \\
= \mathbb{E}[\mathbb{E}[\mathbb{I}\{z_{W_j}v' + z_{b_j}^2 = -W_j^2v' - b_j^2\} + \mathbb{I}\{z_{W_j}v' + z_{b_j}^2 = m-W_j^2v' - b_j^2\}] | W_j, v', b_j^2].
\]

Defining \( g(Y^1, y^2, y^3) := \mathbb{I}\{z_{W_j}y^2 + z_{b_j}^2 = y^1y^2 - y^3\} + \mathbb{I}\{z_{W_j}y^2 + z_{b_j}^2 = -y^1y^2 + y^3\} \),

\[ g(W_j, v', b_j^2) = \mathbb{E}[\mathbb{I}\{z_{W_j}v' + z_{b_j}^2 = -W_j^2v' - b_j^2\} + \mathbb{I}\{z_{W_j}v' + z_{b_j}^2 = m-W_j^2v' - b_j^2\} | W_j, v', b_j^2], \]

since \( (z_{W_j}^2, z_{b_j}^2) \) are independent of \( (W_j, v', b_j^2) \). For any \( (Y^1, y^2, y^3) \in \mathbb{R}^{2N_1+1} \),

\[ z_{W_j}y^2 + z_{b_j}^2 \text{ is an absolutely continuous random variable, hence the probability} \]

\[ \mathbb{E}[\mathbb{I}\{z_{W_j}v' + z_{b_j}^2 = -W_j^2v' - b_j^2\} + \mathbb{I}\{z_{W_j}v' + z_{b_j}^2 = m-W_j^2v' - b_j^2\}] = 0. \]

Given that the formulas (29) evaluated at \( x + z \) produce the partial derivatives of \( \mathcal{L}_i \) with probability 1, and \( \mathcal{L}_i \) is differentiable at \( x + z \) with probability 1, the result follows. \( \square \)

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