On the $q$-Charlier multiple orthogonal polynomials

J. Arvesú∗and A.M. Ramírez-Aberasturis
Department of Mathematics, Universidad Carlos III de Madrid,
Avenida de la Universidad, 30, 28911, Leganés, Spain

November 10, 2014

Abstract

We introduce a new family of special functions, namely $q$-Charlier multiple orthogonal polynomials. These polynomials are orthogonal with respect to $q$-analogues of Poisson distributions. We focus our attention on their structural properties. Raising and lowering operators as well as Rodrigues-type formulas are obtained. An explicit representation in terms of the second of $q$-Appell’s hypergeometric functions is given. A high-order linear $q$-difference equation with polynomial coefficients is deduced. We show how to obtain the nearest neighbor recurrence relation from some difference operators involved in the Rodrigues-type formula.

2010 Mathematics Subject Classification: 42C05, 33E30, 33C47, 33C65.
Keywords: Multiple orthogonal polynomials; Hermite-Padé approximation; Difference equations; Classical orthogonal polynomials of a discrete variable; Charlier polynomials; $q$-polynomials.

1 Introduction

Let $\vec{\mu} = (\mu_1, \ldots, \mu_r)$ be a vector of $r$ positive Borel measures on $\mathbb{R}$, and let $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ be a multi-index. By $\mathbb{N}$ we denote the set of all nonnegative integers. A type II multiple orthogonal polynomial $P_{\vec{n}}$, corresponding to the multi-index $\vec{n}$, is a polynomial of degree $\leq |\vec{n}| = n_1 + \cdots + n_r$ which satisfies the orthogonality conditions [16]

$$
\int_{\Omega_i} P_{\vec{n}}(x)^k d\mu_i(x) = 0, \quad k = 0, \ldots, n_i - 1, \quad i = 1, \ldots, r,
$$

where $\Omega_i$ is the smallest interval that contains $\text{supp}(\mu_i)$. In this paper we will consider the situation when $P_{\vec{n}}$ is a monic multiple orthogonal polynomial and has exactly degree $|\vec{n}|$. If the measures in (1.1) are discrete

$$
\mu_i = \sum_{k=0}^{N_i} \omega_{i,k} \delta_{x_{i,k}}, \quad \omega_{i,k} > 0, \quad x_{i,k} \in \mathbb{R}, \quad N_i \in \mathbb{N} \cup \{+\infty\}, \quad i = 1, 2, \ldots, r,
$$

where $\delta_{x_{i,k}}$ denotes the Dirac delta function and $x_{i_1,k} \neq x_{i_2,k}$, $k = 0, \ldots, N_i$, whenever $i_1 \neq i_2$, the corresponding polynomial solution is called discrete multiple orthogonal polynomial.

In [5] was studied some type II discrete multiple orthogonal polynomials on the linear lattice $x(s) = s$. In particular, multiple Charlier polynomials were considered (when the component measures of $\vec{\mu}$ are different Poisson distributions). In this paper we will introduce a $q$-analogue of such multiple orthogonal polynomials (when the component measures of $\vec{\mu}$ are different $q$-Poisson distributions) and study their algebraic properties (structural properties). We are motivated by the recent applications that have been found for their predecessors: Multiple Charlier polynomials. In particular, these polynomials appear in remainder Padé approximation for the exponential function [18] and as common eigenstate of a set of $r$ non-Hermitian oscillator Hamiltonians [13]. Furthermore, in [14] the authors pointed out a possible relationship of these polynomials to the orthogonal functions appearing in two speed totally asymmetric

∗The research of J. Arvesú was partially supported by the research grant MTM2012-36732-C03-01 of the Ministerio de Educación y Ciencia of Spain and mobility research grant from Fundación Caja Madrid.
simple exclusion process (TASEP) \cite{6}. Our new q-family of multiple orthogonal polynomials is likely to be of relevance in TASEP-like models as well as in q-extensions of the mathematical problems addressed in \cite{15, 18}.

Recently, two families of q-multiple orthogonal polynomials and some of their structural properties have been studied \cite{3, 17}. Moreover, in \cite{4, 12, 19} an (r + 1)-order difference equation for some discrete multiple orthogonal polynomials was obtained. An interesting fact is that these polynomials are common eigenfunctions of two distinct linear difference operators of order (r + 1) since multiple orthogonal polynomials also satisfy (r + 2)-term recurrence relations. The explicit expressions for the coefficients of this relation are the main ingredient for the study of some type of asymptotic behaviors for these polynomials. For instance, in \cite{2} the weak asymptotics was studied for multiple Meixner polynomials of the first and second kind, respectively. The zero distribution of multiple Meixner polynomials was also studied. In \cite{14} the recurrence relation is also used for studying the ratio asymptotic behavior of multiple Charlier polynomials (introduced in \cite{5}) and from it the authors obtain the asymptotic distribution of the zeros after a suitable rescaling. Furthermore, this recurrence relation is a key-ingredient for attaining a Christoffel-Darboux kernel \cite{7} among other applications, which plays important role in correlation kernel (for instance in the unitary random matrix model with external source).

In this paper, we will mainly focus on two structural properties satisfied by the multiple orthogonal polynomials introduced here, namely recurrence relations and the difference equation (with respect to the independent variable). The asymptotic analysis is out of the scope of this paper. Although we plan to address this question in a future publication.

The structure of the paper is as follows. Section 2 introduces the context and the background materials. In section 3 we will define the q-Charlier multiple orthogonal polynomials (new in the literature) and obtain raising operators and then the Rodrigues-type formula for these special functions when r orthogonality conditions are considered. The explicit series representation in terms of the second of q-Appell’s hypergeometric functions is given for \tilde{n} = (n_1, n_2). In section 4 an (r + 1)-order q-difference equation on a non-uniform lattice \(x(s)\) is obtained. Section 5 deals with the (r + 2)-term recurrence relations. In particular, the nearest neighbor recurrence relation is obtained. Explicit expressions for the recurrence coefficients are given. Finally, in section 6 some of our findings are summarized. A special limiting case when the parameter \(q\) involved in the components of the vector measure tends to one is pointed out. Under this limit the corresponding structural properties for the multiple Charlier polynomials are recovered.

## 2 Multiple Charlier polynomials

For the discrete measures \(\mu_1, \mu_2, \ldots, \mu_r\), we have that \(\text{supp}(\mu_i)\) is the closure of \(\{x_{i,k}\}_{k=0}^{N_i}\) and that \(\Omega_i\) is the smallest closed interval on \(\mathbb{R}\) which contains \(\{x_{i,k}\}_{k=0}^{N_i}\). Moreover, the above orthogonality conditions \(1.1\) give a linear system of \(|\tilde{n}|\) homogeneous equations for the \(|\tilde{n}| + 1\) unknown coefficients of \(P_{\tilde{n}}(x)\). This polynomial solution \(P_{\tilde{n}}\) always exists. We focus our attention on a unique solution (up to a multiplicative factor) with \(\text{deg} P_{\tilde{n}}(x) = |\tilde{n}|\). If this happen for every multi-index \(\tilde{n}\), we say that \(\tilde{n}\) is normal \(\cite{16}\). If the above system of measures forms an AT system \(\cite{16}\) then every multi-index is normal. Indeed, we will deal with such system of discrete measures, where \(\Omega_i = \Omega\) for each \(i = 1, 2, \ldots, r\).

**Definition 2.1.** The system of positive discrete measures \(\mu_1, \mu_2, \ldots, \mu_r\), given in \(1.2\), forms an AT system if there exist \(r\) continuous functions \(v_1, \ldots, v_r\) on \(\Omega\) with \(v_i(x_k) = \omega_{i,k}, k = 0, \ldots, N_i, i = 1, 2, \ldots, r\), such that the \(|\tilde{n}|\) functions

\[
\begin{align*}
v_1(x), x v_1(x), \ldots, x^{n_1-1} v_1(x), \ldots, v_r(x), x v_r(x), \ldots, x^{n_r-1} v_r(x),
\end{align*}
\]

form a Chebyshev system on \(\Omega\) for each multi-index \(\tilde{n}\) with \(|\tilde{n}| < N + 1\), i.e., every linear combination \(\sum_{i=1}^{r} Q_{n_i-1}(x) v_i(x)\), where \(Q_{n_i-1} \in \mathbb{P}_{n_i-1} \setminus \{0\}\), has at most \(|\tilde{n}| + 1\) zeros on \(\Omega\).

The monic multiple Charlier polynomials \(\cite{5}\) \(\tilde{C}_{\tilde{n}}(x)\), with multi-index \(\tilde{n} \in \mathbb{N}^r\) and degree \(|\tilde{n}|\) satisfy the following orthogonality conditions with respect to \(r\) Poisson distributions with different positive parameters \(\alpha_1, \ldots, \alpha_r\) (indexed by \(\tilde{\alpha} = (\alpha_1, \ldots, \alpha_r)\))

\[
\sum_{x=0}^{\infty} \tilde{C}_{\tilde{n}}(x) (-x)_j v^{\alpha_j}(x) = 0, \quad j = 0, \ldots, n_i - 1, \quad i = 1, \ldots, r,
\]

2
where
\[ u^{\alpha_i}(x) = \begin{cases} \frac{\alpha_i^x}{\Gamma(x+1)}, & \text{if } x \in \mathbb{R}\backslash \mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases} \]
and \((x)_j = (x)(x+1)\cdots(x+j-1), (x)_0 = 1, j \geq 1\), denotes the Pochhammer symbol.

In [5] the authors consider a normal multi-index \( \vec{n} \in \mathbb{N}^r \), whenever \( \alpha_i > 0, i = 1, 2, \ldots, r \), and with all the \( \alpha_i \) different. Moreover, it was found the following raising operators
\[ \mathcal{L}^{\alpha_i}_{\vec{n}} \left[ C^\vec{n}_{\vec{\alpha}}(x) \right] = -C^\vec{n}_{\vec{\alpha}+\vec{e}_i}(x), \quad i = 1, \ldots, r, \] where
\[ \mathcal{L}^{\alpha_i}_{\vec{n}} \equiv \frac{\alpha_i}{v^{\alpha_i}(x)} \bigtriangledown v^{\alpha_i}(x), \]
and \( \bigtriangledown f(x) = f(x) - f(x-1) \) denotes the backward difference operator. As a consequence of (2.1) there holds the Rodrigues-type formula
\[ C^\vec{n}_{\vec{\alpha}}(x) = (-1)^{|\vec{n}|} \left( \prod_{j=1}^{r} \alpha_j^{x_j} \right) \Gamma(x+1)C^\vec{n}_{\vec{\alpha}} \left( \frac{1}{\Gamma(x+1)} \right), \] (2.2)
where
\[ C^\vec{n}_{\vec{\alpha}} = \prod_{i=1}^{r} (\alpha_i^{-x} \sqcup \alpha_i^x). \]

Two important structural properties are known for multiple Charlier polynomials [5], namely the \((r+1)\)-order linear difference equation [12]
\[ \prod_{i=1}^{r} \mathcal{L}^{\alpha_i}_{\vec{n}} \left[ \Delta C^\vec{n}_{\vec{\alpha}}(x) \right] + \sum_{i=1}^{r} n_i \prod_{j=1}^{r} \mathcal{L}^{\alpha_j}_{\vec{n}} \left[ C^\vec{n}_{\vec{\alpha}}(x) \right] = 0, \quad \text{where} \quad \Delta f(x) = f(x+1) - f(x), \] (2.3)
and the recurrence relation [5] [10]
\[ xC^\vec{n}_{\vec{\alpha}+\vec{e}_i}(x) + (\alpha_k + |\vec{n}|) C^\vec{n}_{\vec{\alpha}}(x) + \sum_{j=1}^{r} \alpha_j n_j C^\vec{n}_{\vec{\alpha}-\vec{e}_i}(x), \] (2.4)
where the multi-index \( \vec{e}_i \) is the standard \( r \) dimensional unit vector with the \( i \)-th entry equals 1 and 0 otherwise. Interestingly, the multiple Charlier polynomials \( C^\vec{n}_{\vec{\alpha}}(x) \) are common eigenfunctions of the above two linear difference operators of order \((r+1)\), namely (2.3) and (2.4).

3 \ q-Charlier multiple orthogonal polynomials

Let us begin by recalling the definition of \( q \)-multiple orthogonal polynomials [2].

**Definition 3.1.** A polynomial \( P_{\vec{n}}(x(s)) \) on the lattice \( x(s) = c_1 q^s + c_3, q \in \mathbb{R}^+, s \in \mathbb{N} \), \( c_1, c_3 \in \mathbb{R} \), is said to be a \( q \)-multiple orthogonal polynomial of a multi-index \( \vec{n} \in \mathbb{N}^r \) with respect to positive discrete measures \( \mu_1, \mu_2, \ldots, \mu_r \) such that \( \text{supp}(\mu_i) \subset \Omega_i \subset \mathbb{R}, i = 1, 2, \ldots, r \), if there hold conditions
\[ \deg P_{\vec{n}}(x(s)) \leq |\vec{n}| = n_1 + n_2 + \cdots + n_r, \]
(b) \[ \sum_{s=0}^{N_i} P_{\vec{n}}(x(s)) x(s)^k d\mu_i = 0, \quad k = 0, \ldots, n_i - 1, \quad N_i \in \mathbb{N} \cup \{+\infty\}, \] (3.1)
Let us consider the following $r$ positive discrete measures on $\mathbb{R}^+$,

$$\mu_i = \sum_{s=0}^{\infty} \omega_i(k) \delta(k-s), \quad \omega_i > 0, \quad i = 1, 2, \ldots, r. \tag{3.2}$$

Here $\omega_i(s) = v_q^{\alpha_i}(s) \triangledown x(s-1/2)$, which involve the $q$-analogue of Poisson distributions

$$v_q^{\alpha_i}(s) = \begin{cases} \frac{\alpha_i}{\Gamma_q(s+1)}, & \text{if } s \in \mathbb{R}^+ \cup \{0\}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha_i > 0, i = 1, 2, \ldots, r$, with all the $\alpha_i$ different. Recall that the $q$-Gamma function is given by

$$\Gamma_q(s) = \begin{cases} f(s; q) = (1-q)^{1-s} \prod_{k=0}^{\infty} \frac{(1-q^{k+1})}{(1-q^{s+k})}, & 0 < q < 1, \\ q^\frac{(s-1)(s-2)}{2} f(s; q^{-1}), & q > 1. \end{cases} \tag{3.3}$$

See also [9, 15] for the above definition of the $q$-Gamma function.

**Lemma 3.2.** The system of functions

$$\alpha_1, x(s) \alpha_1, \ldots, x(s)^{n_1-1} \alpha_1, \ldots, \alpha_r, x(s) \alpha_r, \ldots, x(s)^{n_r-1} \alpha_r, \tag{3.4}$$

with $\alpha_i > 0, (i = 1, 2, \ldots, r)$, and $(\alpha_i/\alpha_j) \neq q^k, (k \in \mathbb{Z}, \quad i, j = 1, \ldots, r, \quad i \neq j)$, form a Chebyshev system on $\mathbb{R}^+$ for every $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$.

**Proof.** This means that every linear combination $\sum_{i=1}^{r} Q_{n_i-1}(x(s)) \alpha_i^s$ has at most $|\vec{n}| - 1$ zeros on $\mathbb{R}^+$ for every $Q_{n_i-1}(x(s)) \in \mathbb{P}_{n_i-1} \setminus \{0\}$. Since $x(s) = c_1 q^s + c_3$, where $c_1, c_3$ are constants, then $\sum_{i=1}^{r} Q_{n_i-1}(x(s)) \alpha_i^s$ is equivalent to $\sum_{i=1}^{r} Q_{n_i-1}(q^s) \alpha_i^s$. Thus, the system (3.4) transforms into

$$a_{i,0}^s, a_{i,1}^s, \ldots, a_{i,n_i-1}^s, a_{r,0}^s, a_{r,1}^s, \ldots, a_{r,n_r-1}^s, \tag{3.5}$$

where $a_{i,k} = (q^k \alpha_i), \quad \text{with } k = 0, \ldots, n_i - 1, \quad i = 1, \ldots, r$. Observe that $a_{j,m} \neq a_{l,p}$ for $j \neq l, m \neq p$. Hence, identity $a_{i,k} = e^{s \log \alpha_i q^k}$ yields the well-known Chebyshev system (see [16, p. 138])

$$e^{s \log a_{1,0}}, e^{s \log a_{1,1}}, \ldots, e^{s \log a_{1,n_1-1}}, \ldots, e^{s \log a_{r,0}}, e^{s \log a_{r,1}}, \ldots, e^{s \log a_{r,n_r-1}}.$$

Then, we conclude that the functions (3.4) form a Chebyshev system on $\mathbb{R}^+$. 

As a consequence of Lemma 3.2, the system of measures $\mu_1, \mu_2, \ldots, \mu_r$ given in (3.2) forms an AT system on $\mathbb{R}^+$. Using this fact, we will rewrite Definition 3.1 for this system of measures. However, we first define the $q$-analogue of the Stirling polynomials denoted by $[s]_{q}^{(k)}$, which is a polynomial of degree $k$ in the variable $x(s) = (q^s - 1)/(q - 1)$, as follows

$$[s]_{q}^{(k)} = \prod_{j=0}^{k-1} \frac{q^{s-j} - 1}{q - 1} = x(s) x(s-1) \cdots x(s-k+1), \quad \text{for } k > 0, \quad \text{and } [s]_{q}^{(0)} = 1. \tag{3.6}$$

**Definition 3.3.** A polynomial $C_{q,\vec{n}}(s)$, with multi-index $\vec{n} \in \mathbb{N}^r$ and degree $|\vec{n}|$ that verifies the orthogonality conditions

$$\sum_{s=0}^{\infty} C_{q,\vec{n}}^s(s) [s]_{q}^{(k)} v_q^{\alpha_i}(s) \triangle x(s-1/2) = 0, \quad 0 \leq k \leq n_i - 1, \quad i = 1, \ldots, r, \tag{3.6}$$

(see (3.1) with respect to the measures (3.2)) is said to be the $q$-Charlier multiple orthogonal polynomial.
Let us point out some observations derived from the above definition. When \( r = 1 \) we recover the scalar \( q \)-Charlier polynomials [1]. The orthogonality conditions (3.1) have been written more conveniently as (3.6) since \([s]_q^{(k)} = q^{-\frac{k}{2}} x^k(s) + \text{lower terms} = O(q^k)\). Here, the symbol \( O(\cdot) \) stands for big-O notation. Indeed, when \( q \) goes to 1, the symbol \([s]_q^{(k)}\) converges to \((-1)^k(-s)_k\), where \((s)_k\) denotes the Pochhammer symbol. Hence, one can recover the multiple Charlier polynomials given in [5] as a limiting case, provided that the lattice \( x(s) = (q^s-1)/(q-1) \). Moreover, in the sequel we will only use this lattice. Finally, we have an AT-system of positive discrete measures, then the \( q \)-Charlier multiple orthogonal polynomial \( C_q^{\vec{\alpha}}(s) \) has exactly \(|\vec{n}|\) different zeros on \( \mathbb{R}^+ \) (see [2] Theorem 2.1, pp. 26-27).

For monic \( q \)-Charlier multiple orthogonal polynomials we have \( r \) raising operators

\[
D_q^{\vec{\alpha}} C_q^{\vec{\alpha},\vec{n}}(s) = -q^{1/2} C_q^{\vec{\alpha}+1,q,\vec{n}}(s), \quad i = 1, \ldots, r, \tag{3.7}
\]

where

\[
D_q^{\vec{\alpha}} \overset{\text{def}}{=} \left( \frac{\alpha_i q^{\vec{n}}}{v_q^{\alpha_i,q,q}} \nabla v_q^{\alpha_i,q}(s) \right), \quad \nabla \overset{\text{def}}{=} \nabla \frac{\nabla}{\nabla x(s)} \quad \text{and} \quad \vec{\alpha}/q = (\alpha_1/q, \ldots, \alpha_r/q) .
\]

Furthermore,

\[
q^{-|\vec{n}|/2} D_q^{\vec{\alpha}} f(s) = [\alpha_i - x(s)] f(s) + x(s) \nabla f(s),
\]

for any function \( f(s) \) defined on the discrete variable \( s \). Notice that we call \( D_q^{\vec{\alpha}} \) a raising operator since the \( i \)-th component of the multi-index \( \vec{n} \) in (3.7) is increased by 1. For finding this operator we have replace \([s]_q^{(k)}\) in (3.6) by the following finite-difference expression

\[
[s]_q^{(k)} = \frac{q^{(k-1)/2}}{[k+1]_q^{(k)}} \nabla [s+1]_q^{(k+1)}, \tag{3.8}
\]

and then used summation by parts along with conditions \( v_q^{\alpha_i}(-1) = v_q^{\alpha_i}(\infty) = 0 \).

In the sequel we will only consider monic \( q \)-Charlier multiple orthogonal polynomials. In addition, the following difference operators will appear regularly

\[
\Delta \overset{\text{def}}{=} \frac{\nabla}{\nabla x(s-1/2)}, \tag{3.9}
\]

\[
\nabla^{n_i} = \nabla \cdots \nabla, \quad n_i \text{-times} \tag{3.10}
\]

and \( \nabla x(s) = \nabla x(s+1/2) = \Delta x(s-1/2) = q^{s-1/2} \).

**Proposition 3.4.** There holds the following \( q \)-analogue of Rodrigues-type formula

\[
C_q^{\vec{\alpha},\vec{n}}(s) = K_q^{\vec{n}} q^{|\vec{n}|} \Gamma_q(s+1) C_q^{\vec{\alpha},\vec{n}} \left( \frac{1}{\Gamma_q(s+1)} \right), \tag{3.11}
\]

where

\[
C_q^{\vec{\alpha},\vec{n}} = \prod_{i=1}^r C_{q,\vec{n}_i}^{\alpha_i}, \quad C_{q,n_i}^{\alpha_i} = (\alpha_i q^{n_i})^{-s} \nabla^{n_i} (\alpha_i q^{n_i})^s, \tag{3.12}
\]

and

\[
K_q^{\vec{n}} = (-1)^{|\vec{n}|} q^{-|\vec{n}|} \left( \prod_{i=1}^r \alpha_i^{n_i} \right)^{r} \left( \prod_{i=1}^r n_i \sum_{j=1}^{n_i} \right). \tag{3.13}
\]

**Proof.** Replacing \([s]_q^{(k)}\) in (3.6) by the right-hand side of expression (3.8) one has

\[
\sum_{s=0}^{\infty} C_{q,\vec{n}}(s) \nabla [s+1]_q^{(k+1)} v_q^{\alpha_i}(s) \nabla x_1(s) = 0, \quad 0 \leq k \leq n_i - 1, \quad i = 1, \ldots, r.
\]

Hence, using summation by parts and the raising operators (3.7) in a recursive way one obtains the Rodrigues-type formula (3.11). \(\square\)
one obtains the following relation for the $q$-Charlier multiple orthogonal polynomials. Indeed, for $\bar{n} = (n_1, n_2)$ and using expression

$$\nabla^{n_i} f(s) = q^{(-s+1/2)n_i} \nabla^{n_i} f(s) = q^{(-s+1/2)n_i} \sum_{k=0}^{n_i} \binom{n_i}{k} (-1)^k f(s-k), \quad i = 1, 2,$$

or equivalently,

$$C_{q,n_1,n_2}^{\alpha_1, \alpha_2} (s) = \frac{(-\alpha_1)^{n_1} (-\alpha_2)^{n_2}}{(\alpha_2 q^{n_2})^s} \frac{\Gamma_q(s+1)}{\Gamma_q(s+1)} \frac{\nabla^{n_1} (\alpha_1 q^{n_1})^s}{\nabla^{n_2} (\alpha_2 q^{n_2})^s} \frac{\Gamma_q(s+1)}{\Gamma_q(s+1)} \sum_{k=0}^{n_1} \binom{n_1}{k} (-1)^k (\alpha_1 q^{n_1} - k) \nabla^{n_2} (\alpha_2 q^{n_2} - l) \frac{\Gamma_q(s-k-l+1)}{\Gamma_q(s-k-l+1)},$$

or equivalently,

$$C_{q,n_1,n_2}^{\alpha_1, \alpha_2} (s) = \frac{(-\alpha_1)^{n_1} (-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2}}{(\alpha_2 q^{n_2})^s} \sum_{k=0}^{n_1} \sum_{l=0}^{n_2} [n_1]_q^{(k)} [n_2]_q^{(l)} [s]_q^{(k+l)} \frac{(-1/\alpha_1 q^{n_1+(k-1)/4})^k}{[k]_q^l} \frac{(-1/\alpha_2 q^{n_2+(l-1)/4})^l}{[l]_q^l} \frac{[s]_q^{(j)}}{[j-l]_q^l}.$$

If we replace $k$ by $j-l$ in (3.13), we have

$$C_{q,n_1,n_2}^{\alpha_1, \alpha_2} (s) = \frac{(-\alpha_1)^{n_1} (-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2}}{(\alpha_2 q^{n_2})^s} \sum_{j=0}^{n_1+n_2} \sum_{l=0}^{\min(j,n_2)} [n_2]_q^{(l)} [n_1]_q^{(j-l)} \frac{(-1/\alpha_2 q^{n_2+(j-l)/4})^l}{[l]_q^l} \frac{[s]_q^{(j)}}{[j-l]_q^l}$$

or equivalently,

$$C_{q,n_1,n_2}^{\alpha_1, \alpha_2} (s) = \frac{(-\alpha_1)^{n_1} (-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2}}{(\alpha_2 q^{n_2})^s} \sum_{j=0}^{n_1+n_2} \sum_{l=0}^{j} [n_1]_q^{(l)} [n_2]_q^{(j-l)} \frac{(-1/\alpha_1 q^{n_1+(j-l)/4})^j}{[j]_q^l} \frac{[s]_q^{(j)}}{[j-l]_q^l}.$$

Alternatively, in (3.13) the $q$-falling factorials can be rewritten in terms of the $q$-analogue of the Pochhammer symbol [9, 11, 15]

$$(a; q)_k = \prod_{j=0}^{k-1} (1 - a q^j), \quad \text{for} \; k > 0, \quad \text{and} \; (a; q)_0 = 1,$$

by using relation

$$[s]_q^{(k)} = \frac{(q^{-s}; q)_k}{(q-1)_k^k} q^k (s-k+1), \quad k \geq 1.$$
Thus, (3.13) transforms into equation

\[
C^{\alpha_1, \alpha_2}_{q, \alpha_1, n_2}(s) = (-\alpha_1)^{n_1}(-\alpha_2)^{n_2}q^{n_1^2 + n_2^2 + n_1 n_2} \sum_{k=0}^{n_1} \sum_{\ell=0}^{n_2} (q^{-n_1}; q)_{\ell} (q^{-n_2}; q)_{k+l} (q; q)_{k+\ell} (q; q)_{\ell} \\
\left(-1/\alpha_1 (1 - q) q^{(k-1)/2+(k+1)/2-s}\right)^k \left(-1/\alpha_2 (1 - q) q^{(l-1)/2+(l+1)/2+n_1-s}\right)^l \frac{(q; q)_k (q; q)_l}{(q; q)} \\
= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2}q^{n_1^2 + n_2^2 + n_1 n_2} \\
\lim_{\gamma \to +\infty} \Phi_2 \left(q^{-s}, q^{-n_1}, q^{-n_2}, \gamma; \gamma; \frac{\gamma}{\alpha_1 (1 - q) q^{-(k+1)/2+(k+1)/2-s}}\right) \\
- \frac{\gamma}{\alpha_2 (1 - q) q^{l+1/2+(l+1)/2+n_1-s}}, \tag{3.15}
\]

where

\[
\Phi_2 \left(\zeta; \beta, \beta', \gamma, \gamma'; x, y\right) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} (\zeta; q)_{m+n} (\beta; q)_{m} (\beta'; q)_{n} (q; q)_{m} (q; q)_{n} x^m y^n,
\]

is the second of \(q\)-Appell’s hypergeometric functions of two variables [8].

### 4 High-order \(q\)-difference equation

In this section we will find a lowering operator for the \(q\)-Charlier multiple orthogonal polynomials. Then we will combine it with the raising operators \([4,6,7]\) to get an \((r + 1)\)-order \(q\)-difference equation (in the same fashion that \([4,12]\)).

#### Lemma 4.1

Let \(V\) be the linear subspace of polynomials \(Q(s)\) on the lattice \(x(s)\) of degree at most \(|\bar{n}| - 1\) defined by the following conditions

\[
\sum_{s=0}^{\infty} Q(s)[s]_q^{(k)} v^{\alpha_k/q^{-1}}_q(s) \nabla x_1(s) = 0, \quad 0 \leq k \leq n_j - 2 \quad \text{and} \quad j = 1, \ldots, r.
\]

Then, the system \( \left\{ C_{q, \alpha_i - \bar{n}, \bar{n}}^{\alpha_i/q^{-1}}(s) \right\}_{i=1}^{r} \) is a basis for \(V\).

#### Proof

From orthogonality relations

\[
\sum_{s=0}^{\infty} C^{\alpha_i/q^{-1}}_{q, \alpha_i - \bar{n}, \bar{n}}(s)[s]_q^{(k)} v^{\alpha_k/q^{-1}}_q(s) \nabla x_1(s) = 0, \quad 0 \leq k \leq n_j - 2, \quad j = 1, \ldots, r,
\]

we have that polynomials \( C^{\alpha_i/q^{-1}}_{q, \alpha_i - \bar{n}, \bar{n}}(s), i = 1, \ldots, r, \) belong to \(V\).

Now, aimed to get a contradiction, let us assume that there exists constants \(\lambda_i, i = 1, \ldots, r, \) such that

\[
\sum_{i=1}^{r} \lambda_i C^{\alpha_i/q^{-1}}_{q, \alpha_i - \bar{n}, \bar{n}}(s) = 0, \quad \text{where} \quad \sum_{i=1}^{r} |\lambda_i| > 0.
\]

Then, multiplying the previous equation by \([s]_q^{(n_k-1)} v^{\alpha_k/q^{-1}}_q(s) \nabla x_1(s)\) and then taking summation on \(s\) from 0 to \(\infty\), one gets

\[
\sum_{i=1}^{r} \lambda_i \sum_{s=0}^{\infty} C^{\alpha_i/q^{-1}}_{q, \alpha_i - \bar{n}, \bar{n}}(s)[s]_q^{(n_k-1)} v^{\alpha_k/q^{-1}}_q(s) \nabla x_1(s) = 0.
\]

Thus, taking into account relations

\[
\sum_{s=0}^{\infty} C^{\alpha_i/q^{-1}}_{q, \alpha_i - \bar{n}, \bar{n}}(s)[s]_q^{(n_k-1)} v^{\alpha_k/q^{-1}}_q(s) \nabla x_1(s) = c \delta_{i,k}, \quad c \in \mathbb{R} \setminus \{0\}, \tag{4.1}
\]

we deduces that \(\lambda_k = 0\) for \(k = 1, \ldots, r\). Here \(\delta_{i,k}\) represents the Kronecker delta symbol. Therefore, the \(\left\{ C^{\alpha_i/q^{-1}}_{q, \alpha_i - \bar{n}, \bar{n}}(s) \right\}_{i=1}^{r} \) is linearly independent in \(V\). Furthermore, we know that any polynomial of \(V\) can be determined with \(|\bar{n}|\) coefficients while \((|\bar{n}| - r)\) linear conditions are imposed on \(V\), consequently the dimension of \(V\) is at most \(r\). Hence, the system \( \left\{ C_{q, \alpha_i - \bar{n}, \bar{n}}^{\alpha_i/q^{-1}}(s) \right\}_{i=1}^{r} \) spans \(V\), which completes the proof. \(\Box\)
Now we will prove that operator (3.9) is indeed a lowering operator for the sequence of $q$-Charlier multiple orthogonal polynomials $C_{q,n}(s)$.

**Lemma 4.2.** There holds the following relation

\[
\Delta C_{q,n}(s) = \sum_{i=1}^{r} q^{[\frac{n_i}{2}]-n_i+1/2} |n_i| q^{1/2} C_{q,n-i}(s). \tag{4.2}
\]

**Proof.** Using summation by parts we have

\[
\sum_{s=0}^{\infty} \Delta C_{q,n}(s)[s]_q^{(k)} v_q^\alpha / q^{-1}(s) \nabla x_1(s) = -\sum_{s=0}^{\infty} C_{q,n}(s) \nabla \left[ [s]_q^{(k)} v_q^\alpha / q^{-1}(s) \right] \nabla x_1(s)
\]

\[
= -\sum_{s=0}^{\infty} C_{q,n}(s) \varphi_{j,k}(s) v_q^\alpha(s) \nabla x_1(s), \tag{4.3}
\]

where

\[
\varphi_{j,k}(s) = q^{1/2} [s]_q^{(k)} - q^{-1/2} x(s) \alpha_j / q - [s - 1]_q^{(k)},
\]

is a polynomial of degree $\leq k + 1$ in the variable $x(s)$. Consequently, from the orthogonality conditions we get

\[
\sum_{s=0}^{\infty} \Delta C_{q,n}(s)[s]_q^{(k)} v_q^\alpha / q^{-1}(s) \nabla x_1(s) = 0, \quad 0 \leq k \leq n_j - 2, \quad j = 1, \ldots, r.
\]

Hence, from Lemma 4.1 $\Delta C_{q,n}(s) \in \mathbb{V}$. Moreover, $\Delta C_{q,n}(s)$ can univocally be expressed as a linear combination of polynomials $\left\{ C_{q,n-i}(s) \right\}_{i=1}^{r}$, i.e.

\[
\Delta C_{q,n}(s) = \sum_{i=1}^{r} \beta_i C_{q,n-i}(s), \quad \sum_{i=1}^{r} |\beta_i| > 0. \tag{4.4}
\]

Multiplying both sides of the equation (4.4) by $[s]_q^{(n_k-1)} v_q^\alpha / q^{-1}(s) \nabla x_1(s)$ and using relations (4.1) one has

\[
\sum_{s=0}^{\infty} \Delta C_{q,n}(s)[s]_q^{(n_k-1)} v_q^\alpha / q^{-1}(s) \nabla x_1(s) = \sum_{s=0}^{\infty} \beta_i \sum_{s=0}^{\infty} C_{q,n-i}(s)[s]_q^{(n_k-1)} v_q^\alpha / q^{-1}(s) \nabla x_1(s)
\]

\[
= \beta_k \sum_{s=0}^{\infty} C_{q,n-k}(s)[s]_q^{(n_k-1)} v_q^\alpha / q^{-1}(s) \nabla x_1(s). \tag{4.5}
\]

If we replace $[s]_q^{(k)}$ by $[s]_q^{(n_k-1)}$ in the left-hand side of equation (4.5), then left-hand side of equation (4.5) transforms into relation

\[
\sum_{s=0}^{\infty} \Delta C_{q,n}(s)[s]_q^{(n_k-1)} v_q^\alpha / q^{-1}(s) \nabla x_1(s) = -\sum_{s=0}^{\infty} C_{q,n}(s) \varphi_{k,n_k-1}(s) v_q^\alpha(s) \nabla x_1(s)
\]

\[
= q^{1/2} \sum_{s=0}^{\infty} C_{q,n}(s)[s]_q^{(n_k)} v_q^\alpha(s) \nabla x_1(s). \tag{4.6}
\]

Here we have used that $x(s) [s - 1]_q^{(n_k-1)} = [s]_q^{(n_k)}$ to get $\varphi_{k,n_k-1}(s) = -(q^{-1/2}/\alpha_k)[s]_q^{(n_k)} +$ lower terms.

On the other hand, from (3.7) one has that

\[
1/\alpha_k v_q^\alpha(s) C_{q,n}(s) = -q|\alpha|-1/2 \nabla \left[ v_q^\alpha / q^{-1}(s) C_{q,n-i}(s) \right]. \tag{4.7}
\]

Then, by conveniently substituting (4.7) in the right-hand side of equation (4.6) and using once more summation by parts, we get

\[
\sum_{s=0}^{\infty} \Delta C_{q,n}(s)[s]_q^{(n_k-1)} v_q^\alpha / q^{-1}(s) \nabla x_1(s) = -q|\alpha|-1 \sum_{s=0}^{\infty} [s]_q^{(n_k)} \nabla \left[ v_q^\alpha / q^{-1}(s) C_{q,n-i}(s) \right] \nabla x_1(s)
\]

\[
= q|\alpha|-1 \sum_{s=0}^{\infty} C_{q,n-i}(s) \Delta \left[ [s]_q^{(n_k)} \right] v_q^\alpha / q^{-1}(s) \nabla x_1(s). \tag{4.8}
\]
Since $\Delta_{[k]}(n_k) = q^{3/2-n_k} [n_k]_{q}^{(1)} [s]_{q}^{(n_k-1)}$ we finally have

$$
\sum_{s=0}^{\infty} \Delta C_{q,\bar{n}}(s)[s]^{(n_k-1)} \nu_{q}^{\alpha/k-1}(s) \nabla x_1(s)
= q^{[\bar{n}]-n_k+1/2} [n_k]_{q}^{(1)} \sum_{s=0}^{\infty} C_{q,\bar{n}}(s)[s]^{(n_k-1)} \nu_{q}^{\alpha/k-1}(s) \nabla x_1(s).
$$

Therefore, comparing this equation with (4.3) we obtain the coefficients in the expansion (4.4), i.e.

$$
\beta_k = q^{[\bar{n}]-n_k+1/2} [n_k]_{q}^{(1)},
$$

which proves relation (4.2). □

**Theorem 4.3.** The $q$-Charlier multiple orthogonal polynomial $C_{q,\bar{n}}(s)$ satisfies the following $(r+1)$-order $q$-difference equation

$$
\prod_{i=1}^{r} D_{q}^{\alpha_i} \Delta C_{q,\bar{n}}(s) = - \sum_{i=1}^{r} q^{[\bar{n}]-n_i+1/2} [n_i]_{q}^{(1)} \prod_{j=1, j \neq i}^{r} D_{q}^{\alpha_j} C_{q,\bar{n}}(s). \quad (4.8)
$$

**Proof.** Since operators (3.7) are commuting, we write

$$
\prod_{i=1}^{r} D_{q}^{\alpha_i} = \left( \prod_{i=1}^{r} D_{q}^{\alpha_i} \right) D_{q}^{\alpha_i}. \quad (4.9)
$$

Then, by acting on equation (4.2) with the product of operators (4.1), we obtain the following relation

$$
\prod_{i=1}^{r} D_{q}^{\alpha_i} \Delta C_{q,\bar{n}}(s) = - \sum_{i=1}^{r} q^{[\bar{n}]-n_i+1/2} [n_i]_{q}^{(1)} \prod_{j=1, j \neq i}^{r} D_{q}^{\alpha_j} C_{q,\bar{n}}(s)
\prod_{i=1}^{r} D_{q}^{\alpha_i} C_{q,\bar{n}}(s),
$$

which proves (4.8). □

### 5 Recurrence relation

In this section we will study two types of recurrence relations, namely the nearest neighbor recurrence relation for any multi-index $\bar{n}$ and a step-line recurrence relation for $\bar{n} = (n_1, n_2)$.

For the main result of this section we will make use of the following Lemma.

**Lemma 5.1.** Let $n_i$ be a positive integer and let $f(s)$ be a function defined on the discrete variable $s$. The following relation is valid

$$
\mathcal{C}_{q, n_i}^{n_i}(s) f(s) = q^{-n_i+1/2} x(n_i) (\alpha_i q^{n_i})^{-s} \nabla^{n_i-1} (\alpha_i q^{n_i})^s f(s) + q^{-n_i} [x(s) - x(n_i)] \mathcal{C}_{q, n_i}^{n_i} f(s), \quad (5.1)
$$

where difference operator $\mathcal{C}_{q, n_i}^{n_i}$ is given in (3.12).

**Proof.** Let us act $n_i$-times with backward difference operators (3.10) on the product of functions $x(s) f(s)$. Assume momentarily that $n_i \geq N > 1$,

$$
\nabla^{n_i} x(s) f(s) = \nabla^{n_i-1} [\nabla x(s) f(s)] = \nabla^{n_i-1} [q^{-1/2} f(s) + x(s-1) \nabla f(s)]
= q^{-1/2} \nabla^{n_i-1} f(s) + \nabla^{n_i-1} [x(s-1) \nabla f(s)]
= q^{-1/2} \nabla^{n_i-1} f(s) + \nabla^{n_i-2} [\nabla x(s-1) \nabla f(s)]. \quad (5.2)
$$
Repeating this process -but on the second term of the right-hand side of equation (5.2)- yields
\[
\nabla^n_i x(s)f(s) = \left[ q^{1/2-n_i} + \ldots + q^{-5/2} + q^{-3/2} + q^{-1/2} \right] \nabla^{n_i-1}f(s) + x(s - n_i) \nabla^{n_i}f(s)
\]
\[= q^{1/2-n_i}x(n_i)\nabla^{n_i-1}f(s) + x(s - n_i) \nabla^{n_i}f(s),
\]
or equivalently,
\[
\nabla^n_i x(s)f(s) = q^{-n_i+1/2}x(n_i) \nabla^{n_i-1}f(s) + q^{-n_i} [x(s) - x(n_i)] \nabla^{n_i}f(s), \quad n_i \geq 1. \tag{5.3}
\]
Now, aimed to involve difference operator \(C_{q,n_i}^\alpha\) in the above equation, we multiply equation (5.3) from the left side by \((\alpha_i q^{n_i})^{-s}\) and replace \(f(s)\) by \((\alpha_i q^{n_i})^{s} f(s)\). Thus, equation (5.3) transforms into (5.4), which completes the proof.

**Theorem 5.2.** *The q-Charlier multiple orthogonal polynomials satisfy the following \((r + 2)\)-term recurrence relation*

\[
x(s)C_q^{\alpha}(s) = C_q^{\alpha}(x(s) + \sum_{i=1}^{r} q^{n_i} x(n_i) ((q - 1) (\alpha_i q^{n_i}) + 1) + \alpha q^{n_i+1} + 1) C_q^{\alpha}(s)
\]
[...]

**Proof.** We will use Lemma 5.1. First, let us consider equation

\[
(\alpha_k q^{n_k+1})^{-s} \nabla^{n_k+1} (\alpha_k q^{n_k+1})^{s} \frac{1}{\Gamma_q(s+1)} = (\alpha_k q^{n_k+1})^{-s} \nabla^{n_k} \left[ q^{-s+1/2} \nabla (\alpha_k q^{n_k+1})^{s} \frac{1}{\Gamma_q(s+1)} \right]
\]

\[
= q^{-s+1/2} (\alpha_k q^{n_k})^{-s} \nabla^{n_k} (\alpha_k q^{n_k})^{s} \frac{1}{\Gamma_q(s+1)}
\]

\[= - (\alpha_k q^{n_k+1})^{-1} q^{-s+1/2} (\alpha_k q^{n_k})^{-s} \nabla^{n_k} (\alpha_k q^{n_k})^{s} x(s) \frac{1}{\Gamma_q(s+1)},
\]

which can be rewritten in terms of difference operators (5.12) as follows

\[
C_{q,n_k+1}^{\alpha} \frac{1}{\Gamma_q(s+1)} = q^{-s+1/2} C_{q,n_k}^{\alpha} \frac{1}{\Gamma_q(s+1)} - (\alpha_k q^{n_k+1})^{-1} q^{-s+1/2} C_{q,n_k}^{\alpha} x(s) \frac{1}{\Gamma_q(s+1)}. \tag{5.5}
\]

Since operators (5.12) are commuting the multiplication of equation (5.5) from the left side by the product

\[
\prod_{j=1}^{r} C_{q,n_j}^{\alpha}
\]

yields

\[
C_{q,n}^{\alpha} x(s) \frac{1}{\Gamma_q(s+1)} = \alpha q^{n_k+1} \left( C_{q,n}^{\alpha} - q^{s-1/2} C_{q,n}^{\alpha} e_\alpha \right) \frac{1}{\Gamma_q(s+1)}. \tag{5.6}
\]

Second, let us recursively use expression (5.1) involving the product of \(r\) difference operators \(C_{q,n_1}^{\alpha}, \ldots, C_{q,n_r}^{\alpha}\), acting on the function \(f(s) = 1/\Gamma_q(s+1)\). Thus, we have

\[
q^{n_i} C_{q,n_i}^{\alpha} x(s) \frac{1}{\Gamma_q(s+1)} = q^{-s+1/2} \sum_{i=1}^{r} q^{n_i} x(n_i) [(q - 1) (\alpha_i q^{n_i}) + 1] \prod_{j=1}^{r} C_{q,n_j}^{\alpha} \frac{1}{\Gamma_q(s+1)}
\]

\[+ \left( x(s) - \sum_{i=1}^{r} q^{n_i} x(n_i) [(q - 1) (\alpha_i q^{n_i}) + 1] \right) C_{q,n}^{\alpha} \frac{1}{\Gamma_q(s+1)}. \tag{5.7}
\]
Hence, by using expressions (3.10)–(3.13) one gets
\[ x(s)C_{q, n, \vec{m}, \vec{m}}^\alpha (s + 1) = -\alpha_k q^{\vec{m}^2} + n_k + 1 q^{s-1/2}C_{q, \vec{m}}^{\alpha, \vec{m}} (s + 1) 
+ \left( \sum_{i=1}^{r} q^{\vec{m}^2} x(n_i) [(q - 1) (\alpha_k q^{n_i}) + 1] + \alpha_k q^{\vec{m}^2} + n_k + 1 \right) C_{q, \vec{m}, \vec{m}}^{\alpha, \vec{m}} (s + 1) \]
\[ - q^{-s+1/2} \sum_{i=1}^{r} q^{\vec{m}^2} x(n_i) [(q - 1) (\alpha_k q^{n_i}) + 1] \prod_{j=1}^{r} C_{q, n_i, \vec{m}, \vec{m}}^{\alpha, \vec{m}, \vec{m}} (s + 1) \].

Finally, multiplying both sides of the previous expression by \( K_q^{\vec{m}} \vec{m} \Gamma_q (s + 1) \) and using Rodrigues-type formula (3.11) we obtain (3.14), which completes the proof.

Observe that other recursion relations different from the above nearest neighbor recurrence relation (3.14) can be obtained. Indeed, from (3.13) a 4-term recurrence relation for \( \vec{m} = (n_1, n_2) \) can be obtained. In [5] it was given an approach for the recurrence relations of some discrete multiple orthogonal polynomials. This approach make use of the Rodrigues-type formulas along with some series representations (Kampé de Fériet series) for multiple orthogonal polynomials. Here we will proceed in the same fashion.

Considering the expansion
\[ C_{q, n_1, n_2}^{\alpha_1, \alpha_2} (s) = \sum_{j=0}^{n_1 + n_2} c_{q, n_1, n_2}^{(j)} [s]_q, \]
the coefficients \( c_{q, n_1, n_2}^{(j)} \) can be used to compute the recurrence coefficients in
\[ x(s)P_{q,n_1,n_2}(s) = P_{q,n_1,n_2}(s) + b_{q,n_1,n_2} P_{q,n_1,n_2}(s) + c_{q,n_1,n_2} P_{q,n_1,n_2-1}(s) + d_{q,n_1,n_2} P_{q,n_1,n_2-1}(s), \]
where \( P_{q,n_1,n_2}(s) = C_{q,n_1,n_2}^{\alpha_1,\alpha_2}(s) \). Indeed,
\[ b_{q,n_1,n_2} = x(n_1) + q^{n_1} x(n_2) - (-1)^{n_1+n_2} \left( c_{q,n_1,n_2}^{(n_1+n_2-1)} - c_{q,n_1,n_2}^{(n_1+n_2+1)} \right), \]
\[ c_{q,n_1,n_2} = (-1)^{n_1+n_2} \left( c_{q,n_1,n_2}^{(n_1+n_2-2)} - c_{q,n_1,n_2}^{(n_1+n_2+1)} \right) \]
\[ + (b_{q,n_1,n_2} - x(n_1) - q^{n_1} x(n_2) + q^{n_1+n_2-1}) c_{q,n_1,n_2}^{(n_1+n_2-1)} \],
\[ d_{q,n_1,n_2} = (-1)^{n_1+n_2} \left( c_{q,n_1,n_2}^{(n_1+n_2-2)} - c_{q,n_1,n_2}^{(n_1+n_2+1)} \right) \]
\[ + (b_{q,n_1,n_2} - x(n_1) - q^{n_1} x(n_2) + q^{n_1+n_2-1}) c_{q,n_1,n_2}^{(n_1+n_2-2)} \].

From the explicit expression (3.14) we then get, after some calculations, that
\[ b_{q,n_1,n_2} = x(n_1) [(q - 1) \alpha_1 q^{n_1} + 1] + q^{n_1} x(n_2) [(q - 1) \alpha_2 q^{n_2} + 1] + \alpha_2 q^{n_1+2n_2+1}, \]
\[ c_{q,n_1,n_2} = x(n_1) [(q - 1) \alpha_1 q^{n_1} + 1] \alpha_1 q^{2n_1+n_2-1} + q^{n_1} x(n_2) [(q - 1) \alpha_2 q^{n_2} + 1] \alpha_2 q^{n_1+2n_2-1}, \]
\[ d_{q,n_1,n_2} = q^{n_1} x(n_2) [(q - 1) \alpha_2 q^{n_2} + 1] \alpha_2 q^{n_1+2n_2-1} - \alpha_1 q^{2n_1+n_2-1} \alpha_1 q^{n_1+2n_2-1}. \]

Here, when \( q \to 1 \) we recover the recurrence coefficients given in [5].

6 Conclusions

In summary, let us recall some of our results. We have introduced a new family of special functions, i.e. q-Charlier multiple orthogonal polynomials. For these polynomials we have obtained a Rodrigues-type formula (3.11) as well as their explicit representation in terms of the second of q-Appell’s hypergeometric functions of two variables (3.15). Moreover, these polynomials are common eigenfunctions of two different \((r+1)\)-order difference operators, namely (4.18) and (5.4). In the limiting situation \( q \to 1 \) we recover the corresponding structural relations for multiple Charlier polynomials [2]. Indeed, our relations (3.11), (4.18), and (5.4) transform into (2.2), (2.3), and (2.4), respectively.

Our algebraic approach for the nearest neighbor recurrence relation (5.4) does not require to introduce type I multiple orthogonality [20]. Indeed, the q-difference operators involved in the Rodrigues-type formula constitute the key-ingredient in our approach.
A description of the main term of the logarithm asymptotics of $q$-Charlier multiple orthogonal polynomials deserves our future attention. We expect to give it in terms of an algebraic function (see [2]). These results yield the Cauchy transform of the weak-star limit for scaling zero counting measure of the polynomials. In addition, the zero distribution of these type II multiple orthogonal polynomials will be studied.

References

[1] R. Álvarez-Nodarse and J. Arvesú, *On the $q$-polynomials in the exponential lattice*, Integr. Transf. Spec. F., 8, (1999), 299–324.

[2] A.I. Aptekarev and J. Arvesú, *Asymptotics for Multiple Meixner Polynomials*, J. Math. Anal. Appl., 411 (2014), 485–505.

[3] J. Arvesú, *On some properties of $q$-Hahn multiple orthogonal polynomials*, J. Comput. Appl. Math., 233, (2010), 1462-1469.

[4] J. Arvesú and C. Esposito, *A high order $q$-difference equation for $q$-Hahn multiple orthogonal polynomials*, J. Diffe. Equ. Appl., 18, (2012), 833-847.

[5] J. Arvesú, J. Coussement and W. Van Assche, *Some discrete multiple orthogonal polynomials*, J. Comput. Appl. Math., 153, (2003), 19–45.

[6] A. Borodin, P.L. Ferrari, and T. Sasamoto, *Two speed TASEP*, J. Stat. Phys., 137 (2009), 936–977.

[7] E. Daems and A.B.J. Kuijlaars, *A Christoffel-Darboux formula for multiple orthogonal polynomials*, J. Approx. Theory, 130 (2004), 188–200.

[8] T. Ernst, *On the $q$-Analogues of Srivastava's Triple Hypergeometric Functions*, Axioms, 2, (2013), 85-99.

[9] G. Gasper and M. Rahman, *Basic Hypergeometric Series. Cambridge University Press*, Cambridge, (2004).

[10] M. Haneczok and W. Van Assche, *Interlacing properties of zeros of multiple orthogonal polynomials*, J. Math. Anal. Appl., 389, (2012), 429–438.

[11] R. Koekoek, R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their $q$-Analogues*, Springer-Verlag Berlin Heidelberg, (2010).

[12] D. W. Lee, *Difference equations for discrete multiple orthogonal polynomials*, J. Approx. Theory, 150, (2008), 132–152.

[13] H. Miki, L. Vinet, and A. Zhedanov, *Non-Hermitian oscillator Hamiltonians and multiple Charlier polynomials*, Phys. Lett. A, 376(2) (2011), 65–69.

[14] F. Ndayiragije and W. Van Assche, *Asymptotics for the ratio and the zeros of multiple Charlier polynomials*, J. Approx. Theory, 164 (2012), 823–840.

[15] A. F. Nikiforov, S. K. Suslov and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer-Verlag, Berlin, (1991).

[16] E.M. Nikishin, V.N. Sorokin, *Rational Approximations and Orthogonality*, Translations of Mathematical Monographs, vol. 92, Amer. Math. Soc., Providence, RI, 1991.

[17] K. Postelmans and W. Van Assche, *Multiple little $q$-Jacobi polynomials*, J. Comput. Appl. Math., 178 (2005), 361–375.

[18] M. Prévost and T. Rivoal, *Remainder Padé approximants for the exponential function*, Constr. Approx., 25 (2007), 109–123.
[19] W. Van Assche, *Difference equations for multiple Charlier and Meixner polynomials*, Proceedings of the Sixth International Conference on Difference Equations, CRC Press, Boca Raton, FL (2004), 549–557.

[20] W. Van Assche, *Nearest neighbor recurrence relations for multiple orthogonal polynomials*, J. Approx. Theory, 163, (2011), 1427–1448.